KNTZ trick from arborescent calculus and the structure of differential expansion

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ABSTRACT

The recently suggested KNTZ trick completed the lasting search for exclusive Racah matrices \( \bar{S} \) and \( S \) for all rectangular representations and has a potential to help in the non-rectangular case as well. This was the last lacking insight about the structure of differential expansion of (rectangularly-)colored knot polynomials for twist knots – and the resulting success is a spectacular achievement of modern knot theory in a classical field of representation theory, which was originally thought to be a tool for knot calculus but instead appeared to be its direct beneficiary. In this paper we explain that the KNTZ ansatz is actually a suggestion to convert the arborescent evolution matrix \( \bar{S} T^2 \bar{S} \) into triangular form \( B \) and demonstrate how this works and what is the form of the old puzzles and miracles of the differential expansions from this perspective. The main new fully result is the conjecture for the triangular matrix \( B \) in the case of non-rectangular representation \([3,1]\). This paper does not simplify any calculations, but highlights the remaining problems, which one needs to overcome in order to prove that things really work. We believe that this discussion is also useful for further steps towards non-rectangular case and the related search of the gauge-invariant arborescent vertices. As an example we formulate a puzzling, still experimentally supported conjecture, that the study of twist knots only is sufficient to describe the shape of the differential expansion for all knots.

1 Introduction

This paper is about the existence and implications of the differential expansion \([1,4]\) for colored knot polynomials \([5,8]\)

\[
H^K_R(A, q) = \sum_{X \in R_X} Z^K_R(A, q) F^K_X(A, q)
\]

which separate their dependencies on knots \( K \) and representations \( R \). Since the use of this expansion for evaluation of exclusive Racah matrices \([9]\) and for arborescent calculus \([10]\) along the lines, originally suggested in \([11]\), was reviewed very recently in \([12,14]\), we avoid doing this once again and directly proceed to considerations, outlined in the abstract of this paper. They touch three main issues.

The first is the KNTZ claim \([12]\) that the switch from a diagonal evolution matrix \( T^2 \) to triangular \( B \), though looks like a complication, actually reveals the hidden structure of the differential expansion for twist knots and somehow trades the sophisticated Racah matrix \( \bar{S} \) for a much simpler and universal (representation-independent) \( B \). Following \([15]\), we suggest that the reason for this can be that the actual evolution matrix was not the simple diagonal \( T^2 \), but rather a sophisticated symmetric \( \bar{S} T^2 \bar{S} \), and then the switch to triangular \( B \) is indeed a simplification. In this approach the crucial role is played by the switching matrix \( U \), and the central phenomenon is a drastic simplification of the first line in a peculiar matrix \( U T^2 U^{-1} B^{-1} \): for rectangular representations its entries are just products of the differentials. Better understanding of the phenomenon can help to explain what are the linear combinations of those, which emerge for non-rectangular representations.

The second issue is to explain the main differences in non-rectangular case. The crucial one is emergency of multiplicities – and we consider explicit examples of representations \([2,1]\) and \([3,1]\) to demonstrate what is their role. In twist knot polynomials the multiplicities can be largely ignored, in the sense that there exists a version of triangular \( B \) without multiplicities – as already argued in \([14]\). However, already to tame the \( Z \)-factors and, further, to find the Racah matrices \([9]\), the full \( B \) in the full multiplicity space are needed. An amusing fact is that still some pieces of \( B \) and \( \bar{S} \) decouple. In numbers this looks as follows: for representation \([2,1]\) the full matrices \( B \) and \( \bar{S} \) are \( 10 \times 10 \), but they split into the relevant \( 8 \times 8 \) blocks \( B^{rel} \), \( \bar{S}^{rel} \) and irrelevant \( 2 \times 2 \) blocks, while to describe knots it is enough to look at the reduced \( 6 \times 6 \) matrix \( B^{rel} \). We also list multiplicities for more complicated representations, but building up the corresponding matrices \( B \) etc requires new techniques and will be discussed elsewhere.

The third issue is the structure of the differential expansion for arbitrary knots, where nothing like matrix \( B \) exists. Still one can assume that the number of items and the corresponding \( Z \)-factors are always the same as for the twist knots – and we explicitly formulate this conjecture at the end of this paper.
Unifying the three topics is the question of how much of exclusive Racah matrices $\tilde{S}$ and $S$ is contained in $B$. The point is that $B$ looks much simpler, moreover it is explicitly known from [12,13] for all rectangular representations and even for their Macdonald deformations, which lead to hyper- and, presumably, superpolynomials [16,22]. Even more than that, it looks like reduced matrix $B^\text{red}$ can be also found for non-rectangular representations. Therefore it is important to understand, if and how one can reconstruct $\tilde{S}$ from the knowledge of $B$ – and what substitutes $B$ or what survives from the entire structure (if anything) for non-twist knots, at least arborescent. There is still no satisfactory answer for all these questions, and in this paper we review various attempts and concrete examples for small representations – which involve additional matrices $\text{at least arborescent.}$

**2 $U$-matrix and its properties**

Arborescent formula [10] for the normalized HOMFLY-PT polynomial of the twist knot

$$H^\text{twist}_R = d_R \cdot \left( \tilde{S}_R \tilde{T}^2 \tilde{S}_R \tilde{S}_R \tilde{T}^{2m} \tilde{S}_R \right)_{\emptyset \emptyset}$$

with symmetric and orthogonal Racah matrix $\tilde{S}_R$, $\tilde{S}_R^2 = I$, can be identically transformed into

$$H^\text{twist}_R = d_R \cdot \left( \tilde{S}_R \tilde{T}^2 \left( \tilde{S}_R \tilde{T}^2 \tilde{S}_R \right)^m \right)_{\emptyset \emptyset} = d_R \cdot \left( \tilde{S}_R \tilde{T}^2 \tilde{S}_R \tilde{T}^{-2} \tilde{S}_R \left( \tilde{S}_R \tilde{T}^2 \tilde{S}_R \right)^{m+1} \right)_{\emptyset \emptyset}$$

and then rewritten in the KNTZ form [12,15] of the differential expansion [1,4], [11], [23]–[29]:

$$H^\text{twist}_R = \sum_X d_R \cdot \left( \tilde{S}_R \tilde{T}^2 \tilde{S}_R \tilde{T}^{-2} \tilde{S}_R \tilde{U}^{-1} \right)_{\emptyset \emptyset} \cdot \sum_Y \left( \tilde{U}_R \tilde{S}_R \tilde{T}^2 \tilde{S}_R \tilde{U}^{-1} \right)_{\emptyset \emptyset} \cdot \sum_Y \left( \tilde{B}^{(m+1)} \right)_{XY} \cdot U_{Y \emptyset} = \sum_X Z^X_R \cdot \sum_Y \left( \tilde{B}^{(m+1)} \right)_{XY}$$

This is achieved by additional insertion of the unity decompositions $I = U_R^{-1} U_R$ and $I = S_R U_R^{-1} U_R S_R$ between all the squares $\tilde{T}^2$. The last transition in (4) requires that the auxiliary matrix $U_R$, which is one of the main heroes of the present paper, has units everywhere in the first column,

$$U_{Y \emptyset} = 1$$

while its other elements are adjusted to make the KNTZ matrix [12]

$$B := U_R \tilde{S}_R \tilde{T}^2 \tilde{S}_R U_R^{-1}$$

triangular and satisfying the constraints

$$\sum_Y B_{XY} = \delta_{X,\emptyset} \quad \forall X$$

and

$$\sum_Y (B^2)_{XY} = \sum_{Y,\tilde{Z}} B_{XZ} B_{\tilde{Z}Y} \quad \forall X$$

with $B_{X,\emptyset} = \Lambda_X$ and $B_{\emptyset,\emptyset} = \Lambda'^\emptyset_X$ being the known monomials, see [23] below. Remarkably, after $U_R$ is adjusted to convert symmetric $\tilde{S}_R \tilde{T} \tilde{S}_R$ into triangular $B$, the matrix elements

$$Z^X_R := d_R \cdot \left( \tilde{S}_R \tilde{T}^2 \tilde{S}_R \tilde{T}^{-2} \tilde{S}_R \tilde{U}^{-1} \right)_{\emptyset X}$$

appear to be factorized for all rectangular representations $R$ (for non-rectangular $R$ they are factorized in a special basis for $X$, otherwise they are sums of several factorized expressions, see [27]) and reproduce the hook formulas for the $Z$-factors in the differential expansions, in particular

$$Z^\emptyset_R = d_R \cdot \left( \tilde{S}_R \tilde{T}^2 \tilde{S}_R \tilde{T}^{-2} \tilde{S}_R \tilde{U}^{-1} \right)_{\emptyset \emptyset} = 1 \quad \forall R$$
and can serve as a prototype of the ("gauge invariant") arborescent vertex \[30\]

Question marks here and below express not a doubt in the validity of the statements, but the lack of explanation/proof why they are always true. One more impressive fact is the factorization property, which was the starting observation of \[11\]:

\[
H_{\text{double braid},m,n}^{R \rightarrow} \left( S_R T^{2m} S_R T^{2n} \right)_{\emptyset \emptyset} = \sum_X Z^X_R \cdot \frac{F^{(m)}_X \cdot F^{(n)}_X}{F^{(1)}_X} \tag{11}
\]

It is now equivalent to a mysterious identity

\[
\left( S_R T^{2n} S_R T^{2m} \right)_{\emptyset \emptyset} \equiv \sum_X \left( S_R T^2 S_R T^{-2} S_R U_R^{-1} \right)_{\emptyset X} \frac{U_R S_R T^{2(m+1)} S_R}{X \emptyset} \left( U_R S_R T^{2(n+1)} S_R \right)_{X \emptyset} \frac{U_R S_R T^2 S_R U_R^{-1}}{X \emptyset} \tag{12}
\]

and can serve as a prototype of the ("gauge invariant") arborescent vertex \[30\]

\[
\mathcal{V} := \sum_X \frac{S_R U_R^{-1} |X\rangle \otimes \langle X| U_R S_R \otimes \langle X| U_R S_R}{\Lambda_X} \tag{13}
\]

or, perhaps,

\[
\mathcal{V} := \sum_X \frac{S_R T^{-2} S_R U_R^{-1} |X\rangle \otimes \langle X| U_R S_R T^2 S_R \otimes \langle X| U_R S_R T^2 S_R}{\Lambda_X} \tag{14}
\]

Matrices \( \bar{T} \) and \( B \), as well as the first column \( U_{0Y} = 1 \) of \( U_R \), are universal – do not actually depend on \( R \). What happens is that for a particular \( R \) only finite blocks of these semi-infinite matrices contribute to the formulas.

Note the presence of quantum dimension \( d_R \) in eq. \[2\], despite it describes normalized HOMFLY polynomial, the un-normalized one would be proportional to \( d_R^2 \). As usual in the theory of HOMFLY polynomials parameter \( N \) of \( SL_N \) gauge algebra is analytically continued to arbitrary non-integer values and often substituted by \( A = q^N \). Also, we include the combinatorial numbers into \( Z \)-factors. In original \[11\] they were considered independently, but then \[26\] expressed them explicitly for rectangular representations (thus no need to distinguish and study them separately in this case), while for non-rectangular representations it turns impractical to separate them from \( Z \) at all.

**Example** \( R = [1] \): From

\[
\bar{S}_{[1]} = \frac{1}{[N]} \left( \begin{array}{ccc} 1 & \sqrt{[N+1][N-1]} & \sqrt{[N+1][N-1]} \\ \sqrt{[N+1][N-1]} & 1 & -1 \\ \sqrt{[N+1][N-1]} & -1 & 1 \end{array} \right), \quad \bar{T}^2 = \left( \begin{array}{cc} 1 & 0 \\ 0 & A^2 \end{array} \right) \tag{15}
\]

we get

\[
U_{[1]} = \left( \frac{1}{[q^2-q^{-2}]^{1/2}} \frac{\sqrt{[N+1][N-1]}}{[N][N-1]} \right)^{1/2} \left( \begin{array}{ccc} 1 & \sqrt{[N+1][N-1]} & \sqrt{[N+1][N-1]} \\ \sqrt{[N+1][N-1]} & 1 & -1 \\ \sqrt{[N+1][N-1]} & -1 & 1 \end{array} \right), \quad B = \left( \begin{array}{cc} 1 & 0 \\ -A^2 & A^2 \end{array} \right) \tag{16}
\]

and

\[
Z^0_{[1]} = 1, \quad Z^1_{[1]} = \{ Aq \} \{ A/q \} = D_1 D_{-1} = \{ q \}^2 [N+1][N-1] \tag{17}
\]

with \( \{ x \} := x - x^{-1}, \ D_n := \{ Aq^n \} \), so that

\[
H^{\text{twist},m}_{[1]} = 1 + Z^1_{[1]} \left( B^{m+1}_{[1],\emptyset} + B^{m+1}_{[1],[1]} \right) = 1 - \{ Aq \} \{ A/q \} \sum_{i=1}^m A^{2i} \tag{18}
\]

To compare, before the \( U \)-"rotation", which converted it into triangular \( B \), the symmetric matrix was

\[
\bar{S}_{[1]} \bar{T}^2 \bar{S}_{[1]} = \frac{1}{[2][N]} \left( \begin{array}{ccc} A^2 \cdot (q^{-2}[N+1] + q^2[N-1]) & -A \{ q^2 \} \sqrt{[N+1][N-1]} \\ -A \{ q^2 \} \sqrt{[N+1][N-1]} & q^2[N+1] + q^{-2}[N-1] \end{array} \right) \tag{19}
\]
Note that the first line is \( U \) is the same as in \( S \) (consists of the square roots of quantum dimensions of the relevant representations from \( R \otimes \hat{R} \)). However, while \( S \) is finite, some entries of \( U \) are singular in the double-scaling limit when \( q, A \to 1 \) and \( N \) is fixed. This is because \( T \) and thus \( STS \) in this limit tend to a unit matrix, which is preserved by any \( U \)-rotation, but does not satisfy (7) – thus, when approaching the limit, \( U \) develops a singularity.

**Example** \( R = [2] \): Likewise from

\[
S_{[2]} = \frac{[2]}{[N][N+1]} \begin{pmatrix}
1 & \sqrt{[N+1][N-1]} & \frac{[N]}{[N+2]} \sqrt{[N+3][N-1]} \\
\sqrt{[N+1][N-1]} & \frac{[N+1]}{[N+2]} \left( [N+3][N-1] - 1 \right) & -\frac{[N]}{[N+2]} \sqrt{[N+3][N+1]} \\
\frac{[2]}{[N+2]} \sqrt{[N+3][N-1]} & -\frac{[N]}{[N+2]} \sqrt{[N+3][N+1]} & \frac{[N]}{[N+2]}
\end{pmatrix}
\]

it follows that

\[
U_{[2]} = \begin{pmatrix}
1 & \sqrt{[N+1][N-1]} & \frac{[N]}{[N+2]} \sqrt{[N+3][N-1]} \\
\frac{[N]}{[N+2]} \sqrt{[N+3][N-1]} & \frac{[N+1]}{[N+2]} \left( [N+3][N-1] - 1 \right) & -\frac{[N]}{[N+2]} \sqrt{[N+3][N+1]} \\
\frac{[N+1]}{[N+2]} \sqrt{[N+3][N-1]} & -\frac{[N]}{[N+2]} \sqrt{[N+3][N+1]} & \frac{[N]}{[N+2]}
\end{pmatrix} \quad \text{and} \quad
B = \begin{pmatrix}
1 & 0 & 0 \\
0 & A^2 & 0 \\
0 & 0 & q^4 A^4
\end{pmatrix}
\]

and

\[
Z_{[2]}^0 = 1, \quad Z_{[2]}^1 = [2] \{ Aq^2 \} \{ A/q \} = \{ q^2 \} [N+2][N-1], \\
Z_{[2]}^2 = [2] \{ Aq^3 \} \{ Aq^2 \} = \{ q^4 \} [N+3][N+2][N][N-1]
\]

We see that, unlike \( S_R \) and \( U_R \), the matrices \( \bar{T} \) and \( B \) for \( R = [2] \) contain those for \( R = [1] \) as sub-matrices – this is a manifestation of their universality.

These examples have a far-going generalization – to arbitrary rectangular representations \( R = [r^s] \). Relevant in this case are composite representations of peculiar diagonal form, which we denote by \( X = (\lambda, \lambda) \) or \( Y = (\mu, \mu) \) – only they appear in the product

\[
R = [r^s] \implies R \otimes \hat{R} = \oplus_{\lambda \in R} (\lambda, \lambda)
\]

For such diagonal \( X, Y \) a general expression is known for \( B_{XY} \) through the skew Schur functions \( \chi_{\lambda/\mu} \), evaluated at the peculiar ”unit locus” \( p^k = \frac{\{ q^k \}}{\{ q \}} \xrightarrow{q \to 1} \delta_{k,1} \):

\[
B_{XY} = B_{(\lambda, \lambda), (\mu, \mu)} = (-)^{|\lambda|-|\mu|} \cdot A_{(\lambda, \lambda)} \cdot \chi_{\lambda/\mu} \cdot \chi^\circ_{\lambda} \cdot \chi^\circ_{\lambda},
\]

\[
(B^{-1})_{XY} = (B^{-1})_{(\mu, \mu), (\lambda, \lambda)} = \frac{\chi^\circ_{\mu} \cdot \chi^\circ_{\mu}}{\chi^\circ_{\lambda}} \cdot \frac{1}{A_{(\mu, \mu)}},
\]

where \( \chi^\circ \) denotes the transposition of the Young diagram \( \lambda \), we remind that \( \chi_{\lambda} \{ p^k \} = (-)^{|\lambda|} \chi_{\lambda} \{ -p^k \} \). Inverse matrix \( B^{-1} \) does not contain transpositions and looks even simpler than \( B \) itself. The evolution eigenvalues \( \bar{T}^2_{\lambda} = A_{(\lambda, \lambda)} \) and the trefoil \( F \)-function \( A'_{(\lambda, \lambda)} = F^{(1)}_{(\lambda, \lambda)} \) are monomials, expressed through the hook parameters of \( \lambda = (a_1, b_1|a_2, b_2|, \ldots:) \):

\[
A_{(\lambda, \lambda)} = \prod_{i=1}^{\#_{\text{hooks}(\lambda)}} \left( q^{2(a_i-b_i)} A^2 \right)^{a_i+b_i+1} = A^2|\lambda| \prod_{i=1}^{\#_{\text{hooks}(\lambda)}} q^{2(a_i-b_i)(a_i+b_i+1)}
\]

\[
A'_{(\lambda, \lambda)} = \prod_{i=1}^{\#_{\text{hooks}(\lambda)}} \left( -q^{a_i-b_i} A^2 \right)^{a_i+b_i+1} = (-A^2)^{|\lambda|} \prod_{i=1}^{\#_{\text{hooks}(\lambda)}} q^{(a_i-b_i)(a_i+b_i+1)}
\]

(1) If we manage to guess a general formula for \( U \), then \( S \) will be directly extractable from (6). Also we can hope to guess what are \( B \) and \( U \) for no-rectangular \( R \), when multiplicities occur and \( S \) is not well defined without additional ”gauge-fixing” requirements.
3 The \( E \)-based approach

Another possibility is just the opposite: if we know \( B \), then

\[
E^{-1} := S_R^{-1}U_R^{-1}
\]  

(24)

is its diagonalizing matrix, obtained by solving a linear system

\[
\bar{T}^2(\bar{S}_RU_R^{-1}) = (\bar{S}_RU_R^{-1})B \quad \iff \quad E\bar{T}^2 = BE
\]

(25)

Since \( B \) is triangular and universal, one can expect the same from \( E \) (though the truth will be a little more involved, see \( s^\text{[II]} \) below). Moreover, the double-braid factorization, discovered in \( [11] \), allows one to express \( \bar{S} \) through \( E \) only (and generally-known \( T, \Lambda' \) and \( Z \)) – see eq. \( (29) \) above:

\[
\bar{S} = \frac{1}{d_R^2}T^2E^trZ\frac{1}{\Lambda'}E\bar{T}^2
\]

(26)

Together with

\[
B = E\bar{T}^2E^{-1}, \quad U = E\bar{S}
\]

(27)

this provides complete description of the pentad in terms of \( E \).

Actually, \( (26) \) can serve as expression for \( \bar{S} \) through \( E = \frac{U\bar{S}}{d_R} \), alternative to

\[
\bar{S}^{(r'|r)}_{\mu\nu} = \frac{\chi^{(r')}_{\mu\nu}}{D_{\mu\nu}^{r'}, \lambda \in [r]} \sum Z^{(r')}_{\lambda} f_{\lambda\mu} f_{\lambda\nu}, \quad f_{\lambda\mu} = E_{\lambda\mu}\Lambda_{\mu} \sum_{\lambda'} E_{\mu\lambda'}^{-1}
\]

(28)

Note also that decomposition

\[
\bar{T}^{-2}\bar{S}\bar{T}^{-2} = E^{tr}Z\frac{1}{\Lambda'}E
\]

(29)

is a kind of a dual to expression through the second exclusive Racah matrix \( S \)

\[
\bar{T}\bar{S}\bar{T} = ST^{-1}S^{-1} \quad \iff \quad \sum_{\mu} \bar{T}_{\lambda\mu}\bar{S}_{\mu\nu}\bar{T}_{\nu\lambda} = S_{\lambda\nu}
\]

(30)

\( S \) is the standard diagonalizing matrix for \( \bar{S} \), while \( E \) diagonalizes it as a quadratic form (i.e. the two decompositions correspond to treating symmetric \( \bar{S} \) as a tensor with respectively one covariant and one contravariant index or with two covariant indices.

4 Rectangular case, of \textit{Examples}

For \( R = [r] \) the matrix elements of \( U \) are:

\[
U_{\emptyset,[j]} = \left( \prod_{k=1}^{j-1} \{ Aq^{k-2} \} \right) \sqrt{ \{ Aq^{2j-3} \} / \{ A / q \} }
\]

(31)

\[
U_{[1],[j]} = U_{\emptyset,[j]} + \frac{\{ q^j \} A \{ Aq^r \} - \{ q^r \} A \{ Aq^j \}}{A \{ Aq^r \} - \{ q^r \} A \{ Aq^j \}} \left( \prod_{k=1}^{j-1} \{ Aq^{k-1} \} / \{ q^k \} \right) \sqrt{ \{ Aq^{2j-1} \} / \{ A / q \} }
\]

(32)

\[
U_{[j],[1]} = U_{\emptyset,[1]} + \frac{\{ q^j \} A \{ Aq^r \} - \{ q^r \} A \{ Aq^j \}}{q^{r-1} \{ q \} A \{ Aq^r \} - \{ q^r \} A \{ Aq^j \}} \sqrt{ \{ Aq \} / \{ A / q \} }
\]

(33)

In fact, not only \( B = US\bar{T}^2S^{-1}U = (U\bar{S})\bar{T}^2(U\bar{S})^{-1} \), but its constituent \( U\bar{S} \) is triangular and universal – and can be explicitly evaluated. However, in variance from \( B \), the entries of \( E = U\bar{S} \) depend on \( A \) in a rather complicated way. For symmetric \( R = [r] \):

\[
(U\bar{S})_{ij} = d_{[r]} \cdot \frac{\left( \begin{array}{c} j - 1 \\ j \end{array} \right) \frac{Aq^{(j-1)(j-2) - 2(j-1)(j-2)}}{\prod_{k=0}^{j+3-3} \{ Aq^k \}} \cdot \frac{\{ q^{j-1} \} A \{ Aq^r \} - \{ q^r \} A \{ Aq^j \}}{\left( \begin{array}{c} j - 3 \\ j \end{array} \right) \frac{q^{j-3}}{\prod_{k=0}^{j+3-3} \{ Aq^k \}} \cdot \frac{\{ q^j \} A \{ Aq^r \} - \{ q^r \} A \{ Aq^j \}}{\left( \begin{array}{c} j - 1 \\ j \end{array} \right) \frac{Aq^{2j-3}}{\{ A / q \}}} }
\]

(34)
Because of factorial \([i - j]!\) in denominator this expression vanishes for \(j > i\). It is convenient to introduce a condensed notation \(D_n := \{ Ag^n\} \) and \(D_n! := \prod_{i=0}^{n} D_i\). With this definition \(D_{-1}! = 1\) and in the final formulas we will always write the ratio \(D_n!/D_{-1}!\), which is independent of the lower boundary in the product.

For \(R = [1, R] = 2\) and \(R = [1, 1]\) we have respectively

\[
\mathcal{E}_{[1]} = U \bar{S} = d_{[1]} \cdot \left( \frac{1}{D_0} - \frac{0}{DA_0 \sqrt{D_1 D_{-1}}} \right) \tag{35}
\]

\[
\mathcal{E}_{[2]} = U \bar{S} = d_{[2]} \cdot \left( \begin{array}{ccc}
\frac{1}{D_0} & - \frac{0}{DA_0 \sqrt{D_1 D_{-1}}} & 0 \\
\frac{A^2}{qA^2} \frac{D_1}{qD_0 D_{-1}} & \frac{qA^2}{q^2 A^2 D_1 D_{-1}} & 0 \\
\frac{q^3 A^2 D_1 D_{-1}}{q^3 A^2 D_0 D_{-1}} & \frac{q^3 A^2 D_0 D_{-1}}{q^3 A^2 D_1 D_{-1}} & 0 \\
\end{array} \right) \tag{36}
\]

and

\[
\mathcal{E}_{[1,1]} = U \bar{S} = d_{[1,1]} \cdot \left( \begin{array}{ccc}
\frac{1}{D_0} & - \frac{0}{DA_0 \sqrt{D_1 D_{-1}}} & 0 \\
\frac{A^2}{qA^2} \frac{D_1}{qD_0 D_{-1}} & \frac{qA^2}{q^2 A^2 D_1 D_{-1}} & 0 \\
\frac{q^3 A^2 D_1 D_{-1}}{q^3 A^2 D_0 D_{-1}} & \frac{q^3 A^2 D_0 D_{-1}}{q^3 A^2 D_1 D_{-1}} & 0 \\
\end{array} \right) \tag{37}
\]

For \(R = [2, 2]\)

\[
\frac{\mathcal{E}_{[2,2]} \cdot U \bar{S}}{d_{[2,2]} = \bar{d}_{[2,2]} =} \tag{38}
\]

Dimensions of the composite representations, defining the first line of Racah matrix, \(\bar{S}_\theta \lambda = \frac{\sqrt{d_{\theta \lambda \eta}}}{d_{\theta \lambda}}\) and partly recognizable also in the entries of \(\mathcal{E}\), are

\[
d_{(\theta, \theta)} = 1, \quad d_{([1],[1])] = D_1 D_{-1}, \quad d_{([2],[2])] = \frac{D_1 D_2^2 D_{-1}}{[2]^2}, \quad d_{([1,1],[1,1])] = \frac{D_1 D_2^2 D_{-3}}{[2]^2}, \quad d_{([2,1],[2,1])] = \frac{D_1 D_2^2 D_{-1}}{[3]^2 D_{-3}}.
\]

\[
d_{([2,2],[2,2])] = \frac{D_1 D_2^2 D_{-1} D_{-3}^2 D_{-3}}{[3]^2 [2]^4}
\]

5 A universal version of \(\mathcal{E}\)

Examples in the previous section demonstrate that, against expectation, \(\mathcal{E}\) is not fully universal: it contains factors \(d_R\), which explicitly depend on \(R\). This is because solution to the system (25) does not immediately reproduce \(\mathcal{E}\), instead it is ambiguous – defined up to right multiplication by a diagonal matrix,

\[
\mathcal{E} \rightarrow \mathcal{E} \cdot D
\]

In the true \(\mathcal{E}\) the freedom is fixed by the request that \(\mathcal{E}_\lambda^\theta = 1\), which is important to reproduce the Z-factors:

\[
U_{X^0} = 1 \quad \Rightarrow \quad Z^Y_R = d_R \cdot \left( \bar{S}_R T^2 \bar{S}_R \bar{T}^{-2} \bar{S}_R U_R^{-1} \right)_{\theta \lambda \eta} = d_R \cdot \sum_Y \left( \bar{S}_R T^2 \bar{S}_R \bar{T}^{-2} \right)_{\theta \lambda \eta} \mathcal{E}_Y^{-1}
\]

\[
\frac{X^0}{X^0} \tag{40}
\]
This is not quite a simple condition on the diagonal matrix \( D \) in \( \mathcal{E} \). One way to express it is through the sum rules for every given sub-representation \( \lambda \subset R \):

\[
\sum_{\mu \subset \lambda} \mathcal{E}_{\lambda \mu} \Lambda_{\mu} \bar{S}_{\delta_{0} \mu} = \delta_{\lambda \emptyset}, \quad \sum_{\mu \subset \lambda} \mathcal{E}_{\lambda \mu} \Lambda_{\mu}^{2} \bar{S}_{\delta_{0} \mu} = \Lambda'_{\lambda},
\]

for example, for \( R = [1] \),

\[
\left( \frac{1}{A_{\delta_{0}}} - \frac{0}{A_{D_{2} \sqrt{D_{1} D_{-1}}}} \right) \left( \frac{1}{D_{1} D_{-1}} \Lambda_{1} \right) = \left( \frac{1}{A_{\delta_{0}}} \right) = \left( \frac{1}{A_{\delta_{0}}} \right) = \left( \frac{1}{0} \right)
\]

and

\[
\left( \frac{1}{A_{\delta_{0}}} - \frac{0}{A_{D_{2} \sqrt{D_{1} D_{-1}}}} \right) \left( \frac{1}{D_{1} D_{-1}} \Lambda_{2}^{2} \right) = \left( \frac{1}{A_{\delta_{0}}} \right) = \left( \frac{1}{A_{\delta_{0}}} \right) = \left( 1 - A^{2} \right)
\]

We can fix the ambiguity in an alternative way – by requiring that diagonal elements are unities. The resulting matrix is not exactly \( \mathcal{E}^{-1} := \bar{S}_{R} U_{R}^{-1} \), but it is fully universal, i.e. independent of representation \( R \), like \( B \):

\[
\tilde{\mathcal{E}}^{-1} = \begin{pmatrix}
\emptyset & [1] & [2] & [3] & \ldots \\
\emptyset & 1 & 0 & 0 & 0 \\
[1] & \frac{-A}{\Lambda_{1}} & 1 & 0 & 0 & \ldots \\
[2] & \frac{q A^{2}}{A q^{2}} \Lambda_{2}^{2} \Lambda_{1}^{2} - \frac{q^{2} A}{A q^{2}} & \frac{q^{2} A^{2}}{A q^{2} A q^{2}} & 1 & 0 \\
[3] & -\frac{q^{2} A^{2}}{A q^{2} A q^{2}} & \frac{q^{2} A^{2}}{A q^{2} A q^{2}} & -\frac{q^{2} A}{A q^{2}} & 1 \\
\ldots
\end{pmatrix}
\]

In general the properly normalized \( \mathcal{E}_{\mu \nu} \) (which gives rise to \( U_{\nu \emptyset} = 1 \text{ or } 0 \)) differs from \( \tilde{\mathcal{E}} \) with unit non-vanishing diagonal elements, \( \tilde{\mathcal{E}}_{\mu \nu} = 1 \text{ or } 0 \), by a rather sophisticated Abelian factor \( K \):

\[
K_{\nu} = \left\{ (-q)^{\lambda_{\nu}} \right\} \frac{\Lambda_{\nu}^{2}}{\Lambda_{\nu}^{2}} \cdot \prod_{\text{hooks } x \in \nu} \frac{D_{-2 b_{x} - 1}!}{D_{2 a_{x}}!} \cdot \frac{1}{\sqrt{D_{2 a_{x} + 1} D_{-2 b_{x} - 1}}}.
\]

\[
K_{\nu}^{-1} = \left\{ (-q)^{\lambda_{\nu}} \right\} \frac{\Lambda_{\nu}^{2}}{\Lambda_{\nu}^{2}} \cdot \prod_{\text{hooks } x \in \nu} \frac{D_{2 a_{x}}!}{D_{-2 b_{x} - 1}!} \cdot \sqrt{D_{2 a_{x}} + 1} D_{-2 b_{x} - 1}.
\]

In hook variables the Young diagram \( \nu = (a_{1}, b_{1} | a_{2}, b_{2} | \ldots) \), e.g. \( \nu = [5, 4, 1, 1] = (4, 3 | 2, 0) \). Comparing \( \mathcal{E} \) with the similar examples of \( U \) and \( \mathcal{E} \) in the previous sections, we see, that \( \tilde{\mathcal{E}}^{-1} \) can be simpler to find in a general form.

### 6 Non-rectangular case, generalities

The main difficulty when one passes from rectangular to non-rectangular representations \( R \) is that

\[
R \otimes \bar{R} = \sum_{X} c_{R}^{X} \cdot X = \sum_{\lambda, \lambda' \in \mathcal{R}} c_{R}^{\lambda \lambda'} (\lambda, \lambda')
\]

now contains a whole variety of composite representations \( X = (\lambda, \lambda') \), where, first, \( \lambda' \) can be different from \( \lambda \) and, second, they can come with non-trivial multiplicities \( c_{R}^{\lambda \lambda'} > 1 \). This makes exclusive Racah matrix \( S \) and other members of the pentad \( (\bar{S}, S, B, U, \mathcal{E}) \) much bigger and more complicated. For example, for \( R = [2, 1] \)

\[
[2, 1] \otimes [2, 1] = (\emptyset, \emptyset) + 4 \cdot ([1, [1]) + ([2, [2])) + ([1, [1]), [1, 1]) + ([2, [1, 1]) + ([1, [1]), [2]) + ([2, [1], [2, 1])
\]
and all matrices are $10 \times 10$. However, as explained in [14], one can actually handle twist knot polynomials in terms of just $6 \times 6$ "reduced" matrices $\mathcal{B}^{\text{red}}$, with all the four $([1],[1])$ shrunk to one, and two underlined non-diagonal representations also substituted by one. This, however, is not enough for description of Racah matrices and thus for handling arbitrary arborescent knots by the technique of [10]. The brute-force extraction of Racah matrices from twist-knot calculus is described in [28] for representation $R = [3,1]$, and it is quite difficult to use for bigger non-rectangular $R$. Clearly, we need a more systematic approach to the problem.

As we explain in this paper, the true situation is more interesting. For generic non-rectangular representation $R$ symmetric Racah matrix $\mathcal{S}_R$ can have blocks of the type

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{2t} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}^2$$

which are separated from the relevant part of the matrix $\mathcal{S}^{\text{rel}}$ and do not actually contribute to arborescent knot calculus, because the corresponding $Z$-factors are vanishing. In above example of $R = [2,1]$ the "relevant" part $\mathcal{S}^{\text{rel}}$ is $8 \times 8$, and the decoupling $2 \times 2$ block consists of a different $([2],[1,1]) - ([1],[2])$ and a single linear combination of the four copies of $([1],[1])$. The further reduction $\mathcal{B}^{\text{rel}} \rightarrow \mathcal{B}^{\text{red}}$ from $8 \times 8$ to $6 \times 6$ is made possible by the vanishing of one extra $Z$-factor for the three remaining $([1],[1])$ and by existence of a simple combination of the other two – which causes the $Z^{[1]}$-factor to be non-factorized in the $6 \times 6$ formalism (while it is factorized, as all other $Z$-factors) when we stay with the $8 \times 8$ matrices). One more general remark is that because of [15] in irrelevant $2 \times 2$ sectors the $E$ is not triangular, but rather has the form

$$\frac{1}{\sqrt{t}} \begin{pmatrix} \sqrt{t} & \sqrt{t} \\ \sqrt{t} & \sqrt{t} \end{pmatrix}$$

In what follows we ignore these irrelevant pieces and identify $E$ with triangular $E^{\text{rel}}$.

In what follows we are going to describe all these structures in some detail for the case of $R = [2,1]$ and provide the next example of $R = [3,1]$, but only for the $B$ matrix. We also present the decompositions like [17] for some more complicated non-rectangular representations $R$ – also a necessary step for further generalizations.

### 7 Reduced version of $B$

As already mentioned, one can use the technique of this paper for two purposes: calculation of knot polynomials for twist knots and calculation of Racah matrices. For non-rectangular representations the first purpose is significantly simpler, because for twist knot polynomials we need a little less than $\tilde{S}$ – just its particular matrix elements, which are captured by the reduced matrix $\mathcal{B}^{\text{red}}$. This was already explained in [14], and we briefly repeat this description to make the story complete. Like in [14] we consider the simplest case of $R = [r,1]$, for which the answers are already known from [27].

In this case in addition to the $2r+1$ diagrams $X = (\lambda, \lambda)$ with $\lambda \subset R = [r,1]$, i.e. $\lambda = \emptyset, [i], [i,1], i = 1, \ldots, r$ there are $r - 1$ additional composite pairs $\tilde{X}_i = ([i-1,1],[i]) \oplus ([i],[i-1,1])$ with the same dimensions and eigenvalues

$$\tilde{\Lambda}_i = (q^{-2}A)^{2i}, \quad i = 2, \ldots, r$$

each contributing once to the differential expansion. These $\tilde{X}_i$ contribute $r - 1$ additional lines to the reduced matrix $\mathcal{B}^{\text{red}}$, which thus becomes of the size $2r+1 + r - 1 = 3r$. Remarkably, $\mathcal{B}^{\text{red}}$ remains triangular, though a notion of ordering for generic composites $X$, appearing in [10], gets somewhat more subtle than [21]. The first $2r+1$ lines remain as they were in [22]. The new entries in the new $r - 1$ lines $\tilde{X}_i$ with $i = 2, \ldots, r$ are:

$$\mathcal{B}^{\text{red}}_{\tilde{X}_i,0} = \frac{(r^{-1})^{i-1}\tilde{\Lambda}_i}{q^{i-1}(r-1)^i} \cdot A^2$$

$$\mathcal{B}^{\text{red}}_{\tilde{X}_i,[j]} = \frac{(r^{-1})^{i-1}\tilde{\Lambda}_i}{q^{i-1}(r-1)^i} \cdot \frac{[i-2]!}{[i-j]![j-1]!} \cdot \frac{[i-1]!q^{i+j}-2A^2 - [i-j]q^{-3}A^2 - [j-1]}{q^{j-1}}$$

$$\mathcal{B}^{\text{red}}_{\tilde{X}_i,[i,1]} = \frac{(r^{-1})^{i-1}\tilde{\Lambda}_i}{q^{i-1}(r-1)^i} \cdot \frac{[i-2]!}{[i-j]![j-1]!} \cdot \frac{(A^2 - q^2)(A^2 - q^6)}{(q^{j-1} - A)(q^{j-1} - A^2)}$$

$$\mathcal{B}^{\text{red}}_{\tilde{X}_i,\tilde{X}_j} = \frac{(r^{-1})^{i-1}\tilde{\Lambda}_i}{q^{i-1}(r-1)^i} \cdot \frac{[i-2]!}{[i-j]![j-2]!} \cdot \frac{A^2 q^{i+j} - 1}{A^2 q^{i+j} - 1}$$
In particular,

\[ B^\text{red}_{\tilde{X}_1, [1]} = (-i) \hat{A}_1 \cdot \frac{[i+1] A^2}{q^{[i-2])}} \]

\[ B^\text{red}_{\tilde{X}_1, [1, 1]} = (-i) \hat{A}_1 \cdot \frac{A^2 - q^6}{q^{[i-2])} \cdot (q^4 - 1)} \]

\[ B^\text{red}_{\tilde{X}_1, [i]} = -i \hat{A}_i \cdot \frac{A^2 - q^6}{q^{[i-2])} \cdot (q^4 - 1)} \]

\[ B^\text{red}_{\tilde{X}_1, [i, 1]} = 0 \]

(51)

\[
\begin{align*}
B^\text{red}_{\tilde{X}_1, 0} &= B_{[1], 0} \\
B^\text{red}_{\tilde{X}_1, [j]} &= B_{[j, [j]} + \frac{(-i)^{i+j-1}}{q^{i+j-2i-j-2}} \cdot \frac{[i-2)!}{[j-2)!} \cdot \frac{A^{2i+1} D_{j-2}}{(q^4)} j=1,\ldots,i \\
B^\text{red}_{\tilde{X}_1, [1, 1]} &= B_{[1, 1]} + (-i) q^2 - 2 A^{2i+1} \cdot \frac{D_{j-2}}{(q^4)} \\
B^\text{red}_{\tilde{X}_1, [i, 1]} &= (-i)^{i+j+1} q^{i+j-2i-j-2} A^{2i+1} \cdot \frac{[i-2)!}{[j-2)!} \cdot \frac{D_{j-2}}{(q^4)^2} j=1,\ldots,i-1 \\
B^\text{red}_{\tilde{X}_1, [i, 1]} &= 0 \\
B^\text{red}_{\tilde{X}_1, [j]} &= (-i)^{i+j} A^{2i} q^i + 1) j=2,\ldots,i \\
\end{align*}
\]

In the simplest case of \( R = [2, 1] \) the matrix is

\[
\begin{pmatrix}
\emptyset & [1] & [1, 1] & [2] & [2, 1] & \tilde{X}_2 \\
0 & 1 & 0 & 0 & 0 & 0 \\
[1] & -A^2 & A^2 & 0 & 0 & 0 \\
[1, 1] & \frac{A^4}{q^4} - \frac{[2] A^4}{q^2} & \frac{A^4}{q^2} & 0 & 0 & 0 \\
[2] & q^2 A^4 - [2] q^3 A^4 & 0 & q^4 A^4 & 0 & 0 \\
[2, 1] & -A^6 & [3] A^6 & \frac{[3] A^6}{q^2} - \frac{[3] q A^6}{[2]} & \frac{[3] q A^6}{[2]} & 0 \\
\tilde{X}_2 & -A^6 & [3] A^6 & \frac{[3] A^6}{q^2} - \frac{[3] q A^6}{[2]} & \cdot \frac{[3] q A^6}{[2]} & A^6 & 0 \\
\end{pmatrix}
\]

(52)

The new one – revealed by consideration of the non-rectangular \( R \) – is the last line.

For \( R = [3, 1] \) the line \( \tilde{X}_2 \) remains the same – this is the universality property of \( B \) – and there is one more line for \( \tilde{X}_3 \), new as compared to (22):

| \( \emptyset \) | [1] | [1, 1] | [2] | [2, 1] | [3] | [3, 1] | \( \tilde{X}_2 \) | \( \tilde{X}_3 \) |
|---|---|---|---|---|---|---|---|---|
| \( q^4 A^8 \) | \(-4 q^5 A^8 \) | \( -q^4 (A^2 - q^6) A^6 \) | \( q^4 (A^2 - q^6) A^6 \) | \( q^4 (A^2 - q^6) A^6 \) | \( q^4 (A^2 - q^6) A^6 \) | \( q^6 A^6 \) | \( q^6 A^6 \) |

As another manifestation of universality, the matrix \( E \) for \( R = [2, 1] \) is the same as (38) for \( R = [2, 2] \), except for the very last line, associated with the composite representation \(( [2], [1, 1]) \oplus ([1, 1], [2]) \) instead of \(( [2, 2], [2, 2]) \):

\[
\frac{E_{[2, 1]}^\text{red}}{d_{[2, 1]}} = \frac{U_{[2, 1]}}{d_{[2, 1]}} = 0
\]

(53)
The next step after \([27]\) and \([14]\) in direction of this section would be development of twist-knot calculus for arbitrary non-rectangular representations, i.e. fixing the notion of reduced and finding its non-trivial matrix elements. In the next section we just described the decomposition of more complicated representations, leaving construction of associated \(\mathcal{B}^{red}\) for the future work.

## 8 Decomposition of \(R \otimes \bar{R}\)

For arbitrary \(N\) we have the following decomposition of the product – either of representations or of the corresponding characters (Schur functions)

\[
[r] \otimes [r^{-1}] = \oplus_{i=0}^{r}[N+i,r^{-2},r-i] \longrightarrow \chi_{[r]}\{p\} \cdot \chi_{[r^{-1}]}\{p\} = \sum_{i=0}^{r} \chi_{[N+i,r^{-2},r-i]}\{p\} \tag{54}
\]

Representations appearing at the r.h.s. are called composite.

\[
\begin{array}{c}
\underbrace{\underbrace{\lambda}_{l_\lambda}}_{\text{any line}} \quad \underbrace{\overbrace{\bar{\mu}}^\mu}_{h_\mu = l_\mu \vee = \mu_1} \\
\end{array}
\]

\[
(\lambda, \mu)_N = \left[ l_\lambda + \mu_1, \ldots, l_\lambda + \mu_1, \mu_2, \ldots, \mu_2, \mu_1 - \mu_1, \ldots, \mu_1 - \mu_2 \right]
\]

\[
\chi_{(\lambda, \mu)}[\lambda] = \det \lambda^{\mu\nu} \cdot \sum_{\eta} (-)^{\eta | \eta} \cdot \chi_{\lambda/\eta}[\lambda] \cdot \chi_{\mu/\eta}[\lambda^{-1}] 
\]

and we can rewrite the statement as

\[
\chi_{[r]}\{p\} \cdot \chi_{[0]}\{p\} = \sum_{i=0}^{r} \chi_{[i]}\{i\} \{p\} \tag{55}
\]

where the \(i = 0\) terms at the r.h.s. is understood as \(\chi_{[0]}\{p\}\), not \(\chi_{[0]}\{p\} = 1\). Here and almost everywhere below we are suppressing the index \(N\) – but remember that composite representations are explicitly \(N\)-dependent. The next step is to restrict \(p\)-variables to the Miwa locus \(p_k = \sum_{i=0}^{N} x_i^k = \text{tr} \lambda^k\) with the matrix \(\lambda\) of the same size \(N\). The corresponding restriction of Schur functions is denoted by square brackets: \(\chi_R[p_k = \text{tr} \lambda^k] = \chi_R[\lambda]\). Immediate corollaries are the simple expression for conjugate representations

\[
\chi_{(\mu, \lambda)}[\lambda] = \det \lambda^{\mu\nu} \cdot \chi_{(\lambda, \mu)}[\lambda^{-1}] \tag{56}
\]

\((\lambda_1\) is the length of the longest row in \(\lambda\)) and vanishing of \(\chi_{(\lambda)}[\lambda]\) for Young diagrams \(\lambda\) with more than \(N\) lines, \(l_\lambda > N\). Also, one can eliminate any full line of length \(N\), but multiply by an extra factor of \(\det \lambda\). For the particular case of symmetric representations we now get:

\[
\det \lambda^T \cdot \chi_{[r]}[\lambda] \cdot \chi_{[r]}[\lambda^{-1}] = \sum_{i=0}^{r} \chi_{([i], [i])}[\lambda] \tag{57}
\]
Expression like \[55\] through composite representations are not obligatory existing. The simplest example appears already for \(R = [2,1]\):

\[
[2,1]^2 = [2,2,2] + 2 \cdot [3,2,1] + [4,2] + [4,1,1] + [3,3] + [3,1,1,1] + [2,2,1,1]
\]

\[
[2,1] \otimes [2,2,1] = [2,2,2,2] + 2 \cdot [3,2,1,1] + [4,2,2] + [4,3,1] + [4,1,1,1] + [3,3,2] + [3,2,1,1,1] + [2,2,2,1,1]
\]

\[
[2,1] \otimes \begin{array}{c}
{1} \\
{(0,[2,1])}
\end{array} = [2N] + 2 \cdot ([1],[1])_N + ([2],[2])_N + ([1],[1],[1])_N + ([2],[1],[2])_N + ([2],[1],[1],[2])_N + 3 \cdot [2,2N-3,1^4] + [2N-1,1,1]
\]

The first two lines are particular examples of the third one at \(N = 3\) and \(N = 4\). Underlined items can be treated as composite representations, the very first item is a rather innocent deviation, but the last two are more serious – they contain a line of the length \(N + 1\). Both problems can be eliminated by rewriting the representation product in terms of Schur functions and restricting to the Miwa locus:

\[
\det \chi^2 \cdot \chi_{[2,1]}[\chi^\lambda] \cdot \chi_{[2,1]}[\chi^{-1}] = \det \chi^2 + 2 \chi_{([1],[1])}[\chi^\lambda] + \chi_{([2],[2])}[\chi^\lambda] + \chi_{([1],[1],[1])}[\chi^\lambda] + \chi_{([2],[1],[1])}[\chi^\lambda] + \chi_{([1],[1],[2])}[\chi^\lambda] + \sum_{\lambda,\lambda' \subset [2,1]} c^{[2,1]}_{\lambda,\lambda'} \cdot \delta_{\lambda,\lambda'} \cdot \chi_{(\lambda,\lambda')}[\chi^\lambda]
\]

The two non-underlined terms disappeared, moreover the two non-diagonal composites in the second line, which were double-underlined, are now equal to each other. The formula is further simplified if Miwa locus is further restricted to \(\det \chi^\lambda = 1\) (i.e. from \(GL_N\) to \(SL_N\)). The resulting expression is very similar to \(57\), only in general the sum is diagonal only in the size of the sub-diagrams and there can be non-trivial multiplicities \(c^{[2,1]}_{\lambda,\lambda'}\):

\[
\chi_R[\chi^\lambda] \cdot \chi_R[\chi^{-1}] = \sum_{\lambda,\lambda' \subset R} c^{[2,1]}_{\lambda,\lambda'} \cdot \chi_{(\lambda,\lambda')}[\chi^\lambda] \quad \text{for} \quad \det \chi^\lambda = 1
\]

This formula is a kind of a dual or inverse to the Kojke formula for the composite Schur functions \[31\[32\]:

\[
\chi_{(\lambda,\mu)}[\chi^\lambda] = \det \chi^{|\mu|} \cdot \sum_{\eta} (-)^{|\eta|} \cdot \chi_{\lambda/\eta}[\chi^\lambda] \cdot \chi_{\mu/\eta'}[\chi^{-1}]
\]

where multiplicities do not show up explicitly: they are hidden/incorporated in the skew characters.

We now describe these multiplicities in some particular cases. To make formulas better readable we omit \(\chi^\lambda\) and write \(\chi_R = \chi_R[\chi^{-1}]\).

- For arbitrary rectangular \(R = [r^s]\) there are only diagonal contributions, \(\lambda' = \lambda\), all with multiplicities one:

\[
\chi_{[r^s]} \cdot \chi_{[r^s]} = \sum_{\lambda \subset [r^s]} \chi_{(\lambda,\lambda)}
\]

- For the simplest double-line diagrams \(R = [r,1]\) non-diagonal terms are present, and single-line sub-diagrams, besides two, come with multiplicities 2

\[
\chi_{[r,1]} \cdot \chi_{[r,1]} = 1 + 2 \sum_{i=1}^{r-1} \chi_{([i],[i])} + \chi_{([r],[r])} + \sum_{i=1}^{r} \chi_{([i],[i],[i],[i])} + \sum_{i=1}^{r-1} \left( \chi_{([i+1],[i],[i+1])} + \chi_{([i],[i],[i],[i+1])} \right)
\]

The two items in the last sum are pairwise equal. In the particular case of \(r = 1\) we get just the particular case of \(61\): \(\chi_{[1,1]} \cdot \chi_{[1,1]} = 1 + \chi_{([1],[1])} + \chi_{((1,1],[1,1])}\).

- For the next – two-line – example \(R = [r,2]\) the decomposition gets a little trickier (diagonal composites
are collected in the first two lines):

\[
\chi_{[r,2]} \cdot \chi_{[r,2]} = 1 + \left( -\chi_{([1,1])} + 3 \sum_{i=1}^{r-2} \chi_{([i,1])} + 2 \chi_{([r-1],[r-1])} + \chi_{([r],[r])} \right) + \\
+ \left( \chi_{([1,1],[1,1])} + 2 \sum_{i=2}^{r-1} \chi_{([i-1],[1,1])} + \chi_{([r],[r])} \right) + \sum_{i=2}^{r} \chi_{([i,2],[i,2])} + \\
+ \left( \chi_{([2],[1,1])} + \chi_{([1,1],[2])} \right) + 2 \sum_{i=2}^{r-2} \left( \chi_{([i+1],[1,1])} + \chi_{([i,1],[i+1])} \right) + \left( \chi_{([r],[r-1,1])} + \chi_{([r-1],[r,1])} \right) \cdot (1 - \delta_{r,2}) + \\
+ \sum_{i=2}^{r-1} \left( \chi_{([i+1,1],[1,2])} + \chi_{([i,2],[1,1])} \right) + \sum_{i=2}^{r-2} \left( \chi_{([i+2,2],[2,2])} + \chi_{([i,2],[2,2])} \right) \\
\] (63)

in particular,

\[
\chi_{[2,2]} \cdot \chi_{[2,2]} = 1 + \chi_{([1,1])} + \chi_{([2,2])} + \chi_{([1,1],[1,1])} + \chi_{([2,1],[2,1])} + \chi_{([2,2],[2,2])} \\
\chi_{[3,2]} \cdot \chi_{[3,2]} = 1 + 2 \cdot \chi_{([1,1])} + 2 \cdot \chi_{([2,2])} + \chi_{([1,1],[1,1])} + \chi_{([2,2],[2,2])} + \\
+ \chi_{([3,3])} + 2 \cdot \chi_{([1,1],[2,1])} + \chi_{([3,2],[2,1])} + \chi_{([2,1],[3,1])} + \\
+ \chi_{([3,1],[3,1])} + \chi_{([2,2],[2,2])} + \chi_{([3,1],[3,1])} + \chi_{([4,4])} + 2 \cdot \chi_{([1,1],[3,1])} + \chi_{([2,2],[2,2])} + \\
+ \chi_{([1,1],[4,1])} + \chi_{([3,1],[4,1])} + \chi_{([2,2],[4,1])} + \chi_{([3,1],[2,1])} + \chi_{([2,2],[3,1])} + \\
+ \chi_{([4,1],[4,1])} + \chi_{([3,2],[3,2])} + \chi_{([3,2],[4,1])} + \chi_{([4,2],[4,2])} \\
\chi_{[4,2]} \cdot \chi_{[4,2]} = 1 + 2 \cdot \chi_{([1,1])} + 3 \cdot \chi_{([2,2])} + \chi_{([1,1],[1,1])} + \chi_{([2,2],[2,2])} + \\
+ 2 \cdot \chi_{([3,3])} + 2 \cdot \chi_{([1,1],[2,1])} + 2 \cdot \chi_{([3,2],[2,1])} + 2 \cdot \chi_{([2,1],[3,1])} + \\
+ 2 \cdot \chi_{([3,1],[3,1])} + 2 \cdot \chi_{([2,2],[2,2])} + \chi_{([1,1],[4,1])} + \chi_{([4,1],[4,1])} + \chi_{([3,2],[3,2])} + \\
+ \chi_{([5,2],[4,1])} + \chi_{([4,2],[4,2])} + \chi_{([5,1],[4,2])} + \chi_{([4,2],[5,1])} + \chi_{([5,1],[5,2])} \\
\chi_{[5,2]} \cdot \chi_{[5,2]} = 1 + 2 \cdot \chi_{([1,1])} + 3 \cdot \chi_{([2,2])} + \chi_{([1,1],[1,1])} + \chi_{([2,2],[2,2])} + \\
+ 3 \cdot \chi_{([3,3])} + 2 \cdot \chi_{([1,1],[2,1])} + 2 \cdot \chi_{([3,2],[2,1])} + 2 \cdot \chi_{([2,1],[3,1])} + \\
+ 2 \cdot \chi_{([4,1],[4,1])} + \chi_{([3,2],[2,2])} + \chi_{([1,1],[4,1])} + \chi_{([4,1],[5,1])} + \chi_{([5,2],[5,1])} + \\
+ \chi_{([5,1],[5,1])} + \chi_{([4,2],[4,2])} + \chi_{([5,1],[4,2])} + \chi_{([4,2],[5,1])} + \chi_{([5,1],[5,2])} + \chi_{([6,1],[5,2])} \\
\chi_{[6,2]} \cdot \chi_{[6,2]} = 1 + 2 \cdot \chi_{([1,1])} + 3 \cdot \chi_{([2,2])} + \chi_{([1,1],[1,1])} + \chi_{([2,2],[2,2])} + \\
+ 3 \cdot \chi_{([3,3])} + 2 \cdot \chi_{([1,1],[2,1])} + 2 \cdot \chi_{([3,2],[2,1])} + 2 \cdot \chi_{([2,1],[3,1])} + \\
+ 2 \cdot \chi_{([4,1],[4,1])} + \chi_{([3,2],[2,2])} + \chi_{([1,1],[4,1])} + \chi_{([4,1],[5,1])} + \chi_{([5,2],[5,1])} + \\
+ \chi_{([5,1],[5,1])} + \chi_{([4,2],[4,2])} + \chi_{([5,1],[4,2])} + \chi_{([4,2],[5,1])} + \chi_{([5,1],[5,2])} + \chi_{([6,1],[5,2])} + \\
+ \chi_{([6,1],[6,1])} + \chi_{([5,2],[5,2])} + \chi_{([6,1],[5,2])} + \chi_{([5,2],[6,1])} + \chi_{([6,2],[6,2])} \\
\]

• The starting points for two other series:

\[
[4,3] \otimes [4,3] = ([4,3], [4,3]) + ([4,2],[4,2]) + ([3,3],[3,3]) + ([4,2],[3,3]) + ([3,3],[4,2]) + \\
+([4,1],[4,1]) + 2 \cdot ([3,2],[3,2]) + ([4,1],[3,2]) + ([3,2],[4,1]) + \\
+([4,4]) + 2 \cdot ([3,1],[3,1]) + ([2,2],[2,2]) + ([4],[3,1]) + ([3,1],[4]) + 0 \cdot ([4],[2,2]) + ([2,2],[4]) + ([3,1],[2,2]) + ([2,2],[3,1]) + \\
+([4],[3,1]) + ([3,2],[2,2]) + ([3,2],[3,1]) + ([3,2],[4,1]) + \\
+ ([3,1],[2,2]) + ([3,2],[2,2]) + ([3,2],[3,1]) + ([3,2],[4,1]) + \\
+ 2 \cdot ([2],[2]) + ([1,1],[2,1]) + ([2],[1,1]) + ([1,1],[2]) + ([2],[1,1]) + ([1,1],[2]) + 2([1],[1]) + (0,0) \\
\]

and
With some abuse of terminology we refer to these formulas by saying that representation products are decomposed in this way – this simplifies the formulations, but, as we explained in this section, this is not exactly true: representations contain contributions of diagrams with extra line length, and accurate statements are about the Schur functions at the \( SL_N \) Miwa locus.

9 Conjecture about the summation domain in differential expansion

Now, after we got some impression of what the product of representation and its conjugate, \( R \otimes \bar{R} \) can look like, it is time to formulate our expectation about the general structure of differential expansion. The conjecture is that HOMFLY and other knot polynomials always decompose in exactly this space:

\[
H_K^R = \sum_{X \in R \otimes \bar{R}} Z^K_X f^K_X
\]

with, roughly, the same set of \( Z^K_X \) for all knots \( K \). We make this claim despite it is anyhow justified only for \( K \) which are twist knots, where one indeed deals with the two upcoming braids and needs to now the decomposition of \( R \otimes \bar{R} \). For all other knots natural decompositions are very different – for example, \( R^{\otimes m} \) for an \( m \)-braid knot. Still, if one believes in universality of differential expansion, i.e. that it looks the same for all knots, then one can use the formulas for the twist family as a prototype. Surprisingly or not, such decompositions indeed exist in many examples, with the only correction that \( Z \)-factors actually depend on the defect, see \[33\] – and are exactly the same as for twist knots only in the case of non-positive defects (zero and minus one), while get somehow truncated for positive defects, still remaining the same for all knots with the given defect. Evidence in support of this mysterious conjecture will be presented elsewhere \[34\].

10 Pentad of matrices in non-rectangular case, examples

Finally we return from the twist-knot polynomials and relatively simple reduced matrices \( B^{\text{red}} \) to description of the full-fledged Racah matrices \( \bar{S} \) and \( S^{\text{rel}} \) and some of their pentad components. First we study the case of \( R = [2, 1] \), where the full \( \bar{S} \) is explicitly known, and then provide a new conjecture for \( R = [3, 1] \).

10.1 \( R = [2, 1] \)

For \( R = [2, 1] \) we get from explicitly known \( \bar{S} \) of \[35\] and \[11\]:

\[
[3, 2, 1] \otimes [3, 2, 1] = ([3, 2, 1], [3, 2, 1]) + ([3, 2], [3, 2]) + ([3, 1, 1], [3, 1, 1]) + +([2, 2, 1], [2, 2, 1]) + ([3, 2], [3, 1, 1]) + ([3, 1, 1], [3, 2, 1]) + ([3, 2, 1], [3, 2, 1]) + ([3, 1, 1], [2, 2, 1]) + ([2, 2, 1], [3, 1, 1]) + +2 \cdot \left( ([3, 1], [3, 1]) + ([2, 2], [2, 2]) + ([2, 1, 1], [2, 1, 1]) + ([3, 1], [2, 2]) + ([2, 2], [3, 1]) + ([2, 1, 1], [3, 1]) + ([2, 2], [2, 1, 1]) + ([2, 1, 1], [2, 2]) \right) + +([3], [3]) + 6 \cdot ([2, 1], [2, 1]) + ([1, 1, 1], [1, 1, 1]) + 2 \cdot ([3], [2, 1]) + 2 \cdot ([2, 1], [3]) + ([3], [1, 1, 1]) + ([1, 1, 1], [3]) + 2 \cdot ([2, 1], [1, 1, 1]) + 2 \cdot ([1, 1, 1], [2, 1]) + +3 \cdot \left( ([2, 2], [(1, 1), [1, 1]) + ([2, 1, 1], [1, 1]) + ([1, 1, 1], [2]) \right) + 3 \cdot ([1], [1]) + (0, 0)
\]
\[
\frac{1}{d_{[2,1]}} \sum_{\mu,\nu=1}^{10} (\tilde{T}^{-2} \tilde{S}^{-2})_{\mu \nu} x_\mu x_\nu = x_1^2 + \frac{\{q\}^2(q)}{A^2 D_0 D_{-1}} \left( \frac{x_4 - x_5}{2} \right)^2 + \frac{[3]}{2} \frac{D_0^2}{A^2} \left( \frac{A}{D_0} x_1 - \frac{\{q\}}{AD_0 \sqrt{D_1 D_{-1}}} x_3 \right)^2 + \frac{[3]}{2} D_2 D_{-2} \cdot \left( \frac{A}{D_0} x_1 - \frac{\{q\}}{AD_0 \sqrt{D_1 D_{-1}}} \right) \frac{x_2}{2} + \frac{\{q\}^2(A + A^{-1})}{[3] AD_0 \sqrt{D_2 D_1 D_{-1} D_{-2}}} \left( x_4 - x_5 \right)^2 + \frac{[3]}{2} D_2 D_0 D_{-2} D_{-3} \cdot \left( \frac{q^{-1} A^2}{D_0 D_{-1}} x_1 - \frac{q^{-1}(q^2)}{D_0 \sqrt{D_2 D_1 D_{-1}}} x_2 - \frac{q^{-1}(q)}{D_0 \sqrt{D_2 D_1 D_{-1} D_{-2}}} x_3 + \frac{q^3(q)}{A^2 D_0 D_{-1} D_{-2} \sqrt{D_1 D_{-1}}} x_6 \right) + \frac{[3]}{2} q^2 A^4 \cdot \left( \frac{q^{-1} d_{[2,1]}}{D_0 D_{-1}} x_1 - \frac{q^{-1}(q^2)}{D_0 \sqrt{D_2 D_1 D_{-1}}} x_2 + \frac{q(q)}{D_0 \sqrt{D_2 D_1 D_{-1} D_{-2}}} x_3 + \frac{q^{-3}(q)}{A^2 D_0 D_{-1} D_{-2} \sqrt{D_1 D_{-1}}} x_7 \right) + D_1 D_0 D_{-2} D_{-3} \cdot \left( \frac{A^3}{D_1 D_0 D_{-1}} x_1 - \frac{\{q\}^3 A^2}{D_2 D_1 D_0 \sqrt{D_1 D_{-1}}} x_2 + \frac{\{q\}^3(q^3)}{A^2 D_2 D_1 D_{-1} \sqrt{D_1 D_{-1}}} x_6 + \frac{\{q\}^4(q^3)}{A^2 D_2 D_1 D_{-1} \sqrt{D_1 D_{-1}}} x_7 - \frac{A^3 q^4}{D_2 D_2 \sqrt{D_1 D_{-1}}} x_8 \right) + \frac{q^{-3}(q)}{[2] A D_0 D_{-1} D_{-2} \sqrt{D_1 D_{-1}}} x_6 - \frac{q^4}{[2] A D_0 D_{-1} D_{-2} \sqrt{D_1 D_{-1}}} x_7 + \frac{q^{-4}}{[2] A D_0 D_{-1} D_{-2} \sqrt{D_1 D_{-1}}} x_8 \right) + \frac{d_{[2,1]}^{-1}}{A^6} (x_4 + x_5)(x_9 - x_{10})
\]

which we can now compare with (20) to get the fully-factorized Z-factors and

\[
\mathcal{E} = U \tilde{S} = d_{[2,1]}^{-1}
\]

Here the obvious reordering was made w.r.t. conventions of [11][35] to obtain a triangular matrix:
the first line/column corresponds to \( x_1 \),
the second one – to the combination \( x_4 - x_5 \),
the third one – to \( x_3 \),
the fourth one – to \( x_2 \),
the fifth one – to \( x_6 \),
the sixth one – to \( x_7 \),
the seventh one – to \( x_8 \),
the eighth one – to the combination \( x_9 + x_{10} \).
We omitted the last two lines, associated to \( x_4 + x_5 \) and \( x_9 - x_{10} \), where we have an "irrelevant" non-triangular
\( 2 \times 2 \) block, decoupled from the rest of the matrix.
Suppressed (to save the space) is also the second column, which is actually

\[
\begin{pmatrix}
0 \\
1 \\
0 \\
* \\
* \\
0 \\
* \\
* \\
\end{pmatrix} = \begin{pmatrix}
0 \\
1 \\
0 \\
\frac{(q^2)^2(A + A^{-1})}{\sqrt{2|3|AD_0\sqrt{D_2D_1D_{-1}D_{-2}}}} \\
\frac{1}{q}\frac{(q^2)}{\sqrt{2|3|AD_0\sqrt{D_2D_1D_{-1}D_{-2}}}} \\
0 \\
\frac{1}{q}\frac{(q^2)}{\sqrt{2|3|D_0\sqrt{D_2D_1D_{-1}D_{-2}}}} \\
\frac{1}{q}\frac{(q)}{\sqrt{2|3|D_0\sqrt{D_2D_1D_{-1}D_{-2}}}} \\
\frac{A(A + A^{-1})}{\sqrt{2|3|D_0\sqrt{D_2D_1D_{-1}D_{-2}}}}
\end{pmatrix}
\] (65)

It is easy to observe that the sum of the entries in the third and the fourth columns reproduce the answers in the second column of \[3\] describing reduced $E_{\text{red}}^\text{red}$ in the $6 \times 6$ formalism. More accurately, reduction from $10 \times 10$ to $6 \times 6$ matrices with omitted lines and columns # 2, 3, 9, 10 and the entries of columns 3 and 4 added, is provided by multiplication from the left and from the right by respectively

\[
P_L = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
\quad \text{and} \quad
P_R = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\] (66)

The product $P_L P_R$ equals the $6 \times 6$ unit matrix, but the product $P_R P_L$ is not quite a projection matrix in 10-dimensional space, however, it acts as unit matrix in the 6-dimensional one.

For example,

\[
P_L(U \bar{S})P_R = E_{\text{red}}^\text{red} = \begin{pmatrix}
\frac{1}{D_0} & \frac{-q}{D_0\sqrt{D_1D_{-1}}} & 0 & 0 & 0 & 0 \\
\frac{-q^2}{D_0^2\sqrt{D_1D_{-1}}} & \frac{q^2}{D_0\sqrt{D_1D_{-1}}} & 0 & 0 & 0 & 0 \\
\frac{-q^2}{D_0\sqrt{D_1D_{-1}}} & \frac{q^2}{D_0\sqrt{D_1D_{-1}}} & 0 & 0 & 0 & 0 \\
\frac{-q^2}{D_0\sqrt{D_1D_{-1}}} & \frac{q^2}{D_0\sqrt{D_1D_{-1}}} & 0 & 0 & 0 & 0 \\
\frac{-q^2}{D_0\sqrt{D_1D_{-1}}} & \frac{q^2}{D_0\sqrt{D_1D_{-1}}} & 0 & 0 & 0 & 0 \\
\frac{-q^2}{D_0\sqrt{D_1D_{-1}}} & \frac{q^2}{D_0\sqrt{D_1D_{-1}}} & 0 & 0 & 0 & 0
\end{pmatrix}
\]

which is exactly the reduced \[3\]. We see that the first five lines are in close correspondence with the table for $R = [2, 2]$, and that the $Z$-factors are properly reproduced. Moreover, the non-factorized $Z$-factor for $\lambda = [1]$ (originally suggested in \[3\]) is naturally decomposed into two contributions:

\[
Z_{[2, 1]}^{[2, 3]} = D_3D_{-3} + D_2D_0 + D_0D_{-2} = \frac{3}{[2]^2}D_0^2 + \frac{3}{[2]^2}D_2D_{-2}
\] (67)

From our experience with the case of rectangular $R = [r^s]$ we could expect that this is decomposed as

\[
\frac{1}{d_{[2, 1]}} \sum_{\mu, \nu = 1}^{10} (T^{-2}S\bar{T}^{-2})_{\mu\nu} x_\mu x_\nu = \sum_{\lambda = 1}^{\lambda} Z_\lambda \frac{X_\lambda}{A_\lambda} \left( \sum_{\mu = 1}^{\lambda} \frac{(U \bar{S})\lambda_\mu x_\mu}{d_{[2, 1]}} \right)^2
\] (68)
but this is not fully true: the term in the box is not of this form. What happens is that the $10 \times 10$ matrix $S$ has a block form, with non-trivial $8 \times 8$ block and a trivial, but non-unit $2 \times 2$:

$$S = S_8 \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$  \hfill (69)$$

The same is true for $\bar{T} - 2\bar{T}^{-2}$. The point is that the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ does not possess Gauss decomposition. However, since it is a separate block, it does not interfere with the $8 \times 8$ sector and vanishes after projection onto the state $|0\rangle$, which belongs to the 8-sector. This means that we actually expect (68) to be true in the $8$ sector, orthogonal to $x_4 + x_5$ and $x_9 - x_{10}$ – where it is indeed true. Note that in these coordinates $\bar{T}$ is not diagonal, only $\bar{T}^2$ is. The relevant fragment in diagonal $\bar{T}$, was $\begin{pmatrix} A^2 & -A^2 \\ -A^2 & A^4 \end{pmatrix}$, and it is now substituted by $\begin{pmatrix} 0 & A^2 \\ A^2 & 0 \\ 0 & A^4 \\ A^4 & 0 \end{pmatrix}$.

The full $10 \times 10$ matrix $B$ in these coordinates is also of the block form:

$$B_{[2,1]} = \begin{pmatrix} \emptyset & [1]_x & [1]_y & [1]_z & [2] & [1,1] & [2,1] & X_2 \\ \emptyset & 1 & 0 & A^2 \\ [1]_x & 0 & -A^2 & 0 & A^2 \\ [1]_y & -A^2 & 0 & -A^2 & A^2 \\ [1]_z & -A^2 & 0 & 0 & A^2 \\ [2] & q^2A^4 & -q^3A^4[N-2] & -\frac{q^3A^4}{q^2} & -\frac{q^3A^4}{q^2} & q^4A^4 \\ [1,1] & \frac{4}{q^2} - \frac{A^4[N+2]}{q^2} & -\frac{4}{q^2} & 0 & \frac{A^4}{q^2} \\ [2,1] & -A^6 & -\frac{3A^6}{2} & \frac{3A^6}{2} & 0 & A^6 \\ X_2 & -A^{6} -\frac{3A^{6}[N]}{2} & -\frac{3A^{6}[N]}{2} & -\frac{3A^{6}[N]}{2} & 0 & A^4 \\ 0 & 0 & 0 & 0 & 0 & 0 & A^4 & 0 \end{pmatrix}$$

and it also reproduces reduced $B^{\text{red}}$ from (52) after rejecting the third, ninth and tenth lines/columns, and adding the entries in the second and forth columns and lines. Overlined items in the line $X_2$ coincide with the entries of the previous line $[2,1]$ – this is one of the structures, convenient in the search for generalizations to higher representations $R$.

Also, to see the universality it is useful to compare this $B_{[2,1]}$ to a simpler matrix in the case of $R = [1,1]$ without multiplicities:
Appendix

To conclude, we made a new small step in the difficult study of differential expansions for knot polynomials. They are now very well described (though not fully understood) for the intersection of two domains: of twist knots and of rectangular representations. To be precise, summation set \( \mathcal{R}_R \), \( Z \)-factors and \( F \)-coefficients in

\[
H^K_R(A, q) = \sum_{X \in \mathcal{R}_R} Z_X^K(A, q) F_X^K(A, q)
\]

are explicitly known for \( K \subset \) twist & double braid knots and for \( R = [r^r] \). The two obvious directions to generalize are to arbitrary \( \mathcal{R}_R \) and to arbitrary knots. For the first task we suggest to simultaneously deform the entire pentad structure \( \mathcal{E} \), which complements Racah matrices \( S \) and \( S \) by \( U \) and, most important, by two triangular and universal (\( R \)-independent) matrices \( \mathcal{B} \) and \( \mathcal{E} \). As a new step in this direction we lift the previously known result for the simplest non-rectangular \( R = [2, 1] \) to \( R = [3, 1] \). Concerning generalization to other knots, we begin with conjecturing in [30] the universality (\( K \)-independence) of the domain \( \mathcal{R}_R \), to begin with: \( \mathcal{R}_R \supseteq R \otimes \tilde{R} \). Now we have all the notions and means to proceed for technical calculations, which are going to cover more general representations and more general knows.

10.2 \( R = [3, 1] \)

In this case

\[
[3, 1] \otimes \overline{[3, 1]} = (0, 0) + 4 \cdot ([1], [2]) + (2, [2]) + ([1], [1, 1]) + ([2], [2, 1]) + ([1, 1], [2]) + ([3], [2, 1]) + ([2, 1], [3]) + ([3, 1], [3, 1])
\]

contains two representations with quadruple multiplicities and two pairs of non-diagonal composites (underlined), this the full matrices of the pentad will be 17 \( \times \) 17. However, the two \( 2 \times 2 \) blocks decouple, so that relevant matrices will be 13 \( \times \) 13, and if one is interested in twist-knot polynomials only, \( \mathcal{B} \) can be further reduced to 9 \( \times \) 9. In this reduced case we will get two non-factorized \( Z \)-factors, which are the linear combinations of factorized ones from the relevant level of 13 \( \times \) 13.

Since compact notation is not yet invented to describe the complicated formulas in the \( R = [r, 1] \) case, we present at the end of this paper just the simplest of the relevant matrices: \( \mathcal{B}^{rel}_{[3, 1]} \). Reduced 9 \( \times \) 9 matrix \( \mathcal{B}^{rel}_{[3, 1]} \), needed for the twist-knot calculus and explicitly provided in [14] and in s 8 above, is obtained from it by omission of the lines/columns \( [1]_z \) and \( [2]_z \) and by summing up the entries of the columns \( [1]_x + [1]_y \) and \( [2]_x + [2]_y \). To obtain Racah matrices \( \bar{S} \) and \( S \), needed for arborescent calculus and originally found in [28] one should follow the sequence of steps, described in s3 solve [25] to obtain \( \mathcal{E} \), then normalize it properly to satisfy (41) and use it to define \( \bar{S} \) as a quadratic form (26), and finally find \( S \) from its diagonalization [30]. All these are straightforward linear-algebra operations, moreover, they remain just the same for all other representations \( R \) – in variance of artistic result of [28], which is very difficult to generalize. Thus what we now need for other non-rectangular \( R \) are the bigger pieces of the universal \( \mathcal{B}^{rel} \), which would include all representations from \( R \otimes \tilde{R} \).

11 Conclusion

In this appendix we present the fragment of \( \mathcal{B}^{rel} \), including all the representations from \( [3, 1] \times \overline{[3, 1]} \). Its role and applications are described in s.10.2.
|         | [0] | [1]x | [2]x | [1]y | [2]y | [1]z | [2]z |
|---------|-----|------|------|------|------|------|------|
| [0]     | 0   | 1    |      |      |      |      |      |
| [1]x    | 0   | $A^2$ | 0   | 0   | $A^2$ | 0   | 0   |
| [2]x    | 0   | $-q^2 A^3$ | 0   | 0   | $q^4 A^4$ | 0   | 0   |
| [1]y    | 0   | $-A^2$ | 0   | 0   | 0   | $-A^2$ | 0   |
| [2]y    | 0   | $q^2 A^4$ | 0   | 0   | $q^4 A^4$ | 0   | 0   |
| [1]z    | 0   | $-A^2$ | 0   | 0   | 0   | $A^2$ | 0   |
| [2]z    | 0   | $-q^2 A^4$ | 0   | 0   | 0   | $q^4 A^4$ | 0   |
| [3]     | 0   | 0   | 0   | $q^8 A^6$ | $q^{11} A^7 (N - 2)$ | $q^{10} A^6$ | $-q^{42} A^6 (N - 2)$ |
| [1, 1]  | 0   | 0   | 0   | 0   | 0   | $-A^2$ | 0   |
| [2, 1]  | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| [1, 1]  | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| [1, 1]  | 0   | 0   | 0   | 0   | 0   | 0   | 0   |
| $x_2$   | 0   | $-A^6 - A^2 (N + 3)$ | 0   | $-A^6 (N - 2)$ | 0   | $A^6 (N + 3)$ | 0   |
| $x_3$   | 0   | $q^6 A^6$ | 0   | $q^{11} A^7 (N + 1)$ | 0   | $q^{10} A^6$ | 0   |

Continued...
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