Creation of persistent current and vortex in a Bose-Einstein condensate of alkali-metal atoms

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It is shown theoretically that a persistent current can be continuously created in a Bose-Einstein condensate (BEC) of alkali atoms confined in a multiply connected region by making use of a spin-degree of freedom of the order parameter of a BEC. We demonstrate that this persistent current is easily transformed into a vortex. Relaxation processes of these BEC after the confining field is turned off are also studied so that our analyses are compared with time of flight experiments. The results are shown to clearly reflect the existence of a persistent current.

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1. INTRODUCTION

Since the discovery of Bose-Einstein condensation (BEC) in alkali atoms \( ^6\text{Li} \), numerous attempts are made to show that the system exhibits superfluidity. One of these attempts is to create and observe quantized vortices by confining the system in a rotating anisotropic trap. Recently the vortex is created \( ^{\text{3}}\text{He}-\text{A} \) using the two-component BEC.

In the present paper, we propose a novel method to create a state with a persistent current or a vortex, where the hyperfine spin \( F \) of alkali atoms is fully utilized. This method has been briefly reported in \( ^{\text{3}}\text{He}-\text{A} \), and more detailed theoretical and numerical analyses are made in the present paper. Although we restrict ourselves mainly to the case \( F = 1 \) to simplify our discussions, our method is also applicable to cases with an arbitrary \( F \).

BEC with \( F = 1 \) may be expressed in terms of a three-component order parameter, similarly to the spin or the orbital part of superfluid \( ^{\text{3}}\text{He} \). In particular, a spin-polarized BEC has the same order parameter as that of \( ^{\text{3}}\text{He}-\text{A} \). In the case where the spin-exchange interaction is ferromagnetic, the BEC is spin-polarized even in the absence of an external magnetic field. Even when the spin-exchange interaction is antiferromagnetic, it may be also spin-polarized provided that the Zeeman energy is larger than the spin-exchange energy. Accordingly, each of the weak field seeking state and the strong field seeking state has the same order parameter as in \( ^{\text{3}}\text{He}-\text{A} \).

In contrast with \( ^{\text{3}}\text{He}-\text{A} \), the local order parameter configuration, known as a texture in \( ^{\text{3}}\text{He} \), of a spin-polarized BEC may be easily controllable by a magnetic field. Making use of this property, a BEC in a vortex state or with a persistent current can be continuously created from a BEC without circulation by adiabatically changing an external magnetic field as shown below. The general theoretical framework for describing a spinor BEC \( ^{\text{3}}\text{He} \) was given by Ohmi and Machida \( ^{\text{3}}\text{He} \) and independently by Ho \( ^{\text{3}}\text{He} \), which turned out to be equivalent. This framework is based on the Bogoliubov theory extended to a vectorial order parameter with three components, corresponding to \( m_F = 1, 0, -1 \) of the \( F = 1 \) atomic hyperfine state. As a result, the generalized Gross-Pitaevskii (GP) equation has been constructed. They calculated low-lying collective modes such as sound wave, spin wave, and their coupled modes and predicted various topological defects or spin textures.

This paper is organized as follows. In the next section, the order parameter of a BEC with \( F = 1 \) is discussed. We employ two sets of basis vectors and their transformations are also considered. Then the generalized Gross-Pitaevskii equation is introduced. In Section III, we consider the cross disgyration texture which is expected to appear in an Ioffe-Pritchard trap. Section IV is the main part of the present paper. We first consider an axially symmetric BEC without a current confined in an Ioffe-Pritchard trap with an optical plug. A strong magnetic field is applied along the axis of the BEC. Then the sign of this axial magnetic field is adiabatically changed so that the local magnetization vector flips in the end of this process. Then it is shown that a persistent current with two units of circulation is created. If the optical plug may be turned off at this stage, we are left with a vortex line. The topological justification of this behavior is also given. Observational consequences of the existence of a quantized vortex or a persistent current are discussed in Section V, where the relaxation of the order parameter after the confining fields are turned off is studied. Section VI is devoted to summary and discussions.

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II. SPINOR BOSE-EINSTEIN CONDENSATE

A. Spinor order parameter

Let $F_\alpha (\alpha = x, y, z)$ be the angular momentum operators with $F = 1$. The eigenvalues of $F_\alpha$ are $1, 0, -1$ and their corresponding eigenvectors, that satisfy $F_\alpha |i\rangle = i|i\rangle$, are

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |0\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |-1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (1)$$

In this basis, called the ZQ basis, $F_\alpha$ are represented as

$$F_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$F_y = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$F_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

which satisfy the commutation relation $[F_\alpha, F_\beta] = i\epsilon_{\alpha\beta\gamma} F_\gamma$. The order parameter $|\Psi\rangle$ is expanded in terms of $|i\rangle$ as

$$|\Psi\rangle = \sum_{i=0,\pm1} \Psi_i |i\rangle. \quad (3)$$

It is convenient for our purposes to introduce another set of basis vectors $|\alpha\rangle (\alpha = x, y, z)$ called the XYZ basis which satisfy $F_\alpha |\alpha\rangle = 0$. Note that $|z\rangle = |0\rangle$ by definition. The vectors $|x\rangle$ and $|y\rangle$ are obtained by rotating $|z\rangle$ around the $y$ axis and the $x$ axis by $\pm \pi/2$ as

$$|x\rangle = \exp\left(-i\frac{\pi}{2} F_y\right) |z\rangle = \frac{1}{\sqrt{2}} (-|1\rangle + |1\rangle) \quad (4)$$

$$|y\rangle = \exp\left(i\frac{\pi}{2} F_x\right) |z\rangle = \frac{i}{\sqrt{2}} (1|1\rangle - |1\rangle) \quad (5)$$

Then the order parameter $|\Psi\rangle$ may be decomposed in terms of $|\alpha\rangle$ as

$$|\Psi\rangle = \sum_{\alpha = x,y,z} \Psi_\alpha |\alpha\rangle. \quad (6)$$

The components $\Psi_i$ and $\Psi_\alpha$ are related with each other as

$$\begin{pmatrix} \Psi_1 \\ \Psi_0 \\ \Psi_{-1} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & i & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & i & 0 \end{pmatrix} \begin{pmatrix} \Psi_x \\ \Psi_y \\ \Psi_z \end{pmatrix}. \quad (7)$$

In a spin-polarized BEC, the weak or strong field seeking state is represented by an order parameter:

$$\Psi = \frac{\psi}{\sqrt{2}} e^{i\alpha} (\hat{m} + i\hat{n}) \quad (8)$$

in the XYZ basis, where $\psi = \sqrt{\sum_k |\Psi_k|^2}$ and $\hat{m}$ and $\hat{n}$ are real unit vectors. We also define

$$\hat{l} = \hat{m} \times \hat{n}, \quad (9)$$

which represents the direction of the atomic hyperspin. The three real vectors $(\hat{m}, \hat{n}, \hat{l})$ form a triad.

In the above explanation the direction of the axis of quantization is named $z$. We may take this direction arbitrary. When the axis is parallel to the direction of the magnetic field (B in this paper), the Zeeman energy term is written most simply. We call this $B$-quantized (BQ) notation. When the direction of the axis does not vary spatially and parallel to the $z$ axis, we call this $z$-quantized (ZQ) notation. The kinetic energy term is written simply in this way. The numerical details are given in Appendix [A].

B. Gross-Pitaevskii equations

The time-dependent form of the Gross-Pitaevskii (GP) equation with a spin-degree of freedom obtained by Ohmi and Machida [6], originally in the XYZ basis, can be rewritten in the ZQ notation as

$$i \frac{\partial}{\partial t} |\Psi\rangle = \{h_{jk} + g_{n}\delta_{jk} \sum_{l} |\Psi_i|^2 + g_s \sum_{\alpha} \sum_{lp} (\Psi_l (F_\alpha)_{lp} \Psi_p) (F_\alpha)_{jk} \Psi_k \} \quad (10)$$

where

$$h_{jk}(r) = \left( \begin{array}{c} \frac{\hbar^2 \nabla^2}{2m} - \mu + V(r) \end{array} \right) \delta_{jk} - B_{jk}, \quad (11)$$

$$B = \left( \begin{array}{ccc} B_z & B_x - iB_y & 0 \\ B_x + iB_y & B_z & \frac{B_x - iB_y}{\sqrt{2}} \\ 0 & \frac{B_x + iB_y}{\sqrt{2}} & -B_z \end{array} \right), \quad (12)$$

$m$ is the mass of an atom and $B = (B_x, B_y, B_z)$ is a magnetic field scaled so that the amplitude is the Zeeman energy of the atom. The potential $V(r)$ is spin-independent and $i, j = 0, \pm 1$ are the spin indices in the ZQ basis. The parameters $g_n$ and $g_s$ denote the strength of the interactions. The relationship between $(B_x, B_y, B_z)$ and $B$ is explained in Appendix [A]. Time-independent solutions of GP equation are obtained by solving

$$0 = \{h_{jk} + g_{n}\delta_{jk} \sum_{l} |\Psi_i|^2 + g_s \sum_{\alpha} \sum_{lp} (\Psi_l (F_\alpha)_{lp} \Psi_p) (F_\alpha)_{jk} \Psi_k \}. \quad (13)$$
The above equations are derived from the Hamiltonian
\[
H = \int dr \sum_j \Psi_j^\dagger(r) h_{jk}(r) \Psi_k(r)
+ \frac{g_n}{2} \sum_{jk} \Psi_j^\dagger(r) \Psi_k^\dagger(r) \Psi_k(r) \Psi_j(r)
+ \frac{g_s}{2} \sum_\alpha \left( \sum_{jk} \Psi_j^\dagger(r) (F_\alpha)_{jk} \Psi_k(r) \right)^2. \tag{14}
\]

III. STRONG AND WEAK FIELD SEEKING STATES IN IOFFE-PRITCHARD TRAP

We consider a system of BEC which is uniform along the \( z \) axis. The cylindrical coordinates \( r = (r, \phi, z) \) are introduced. Suppose that an Ioffe-Pritchard field
\[
B = (B_\perp(r) \cos \phi, -B_\perp(r) \sin \phi, B_z)
\tag{15}
\]
is applied to the system. We treat two-dimensional system (uniform along the \( z \) axis) in the following calculations and \( B_z \) is treated as a constant, which differs from the usual Ioffe-Pritchard field. There should be a blue-detuned laser beam penetrating along the \( z \) axis to prevent the atoms from escaping from the trap by spin-flipping. Ohmi and Machida have shown that there appears the cross disgyration when \( B_z = 0 \).

Let us derive the configuration of the condensate in this system. The \( B \)-matrix Eq. (12) becomes
\[
\left( \begin{array}{ccc}
B_z & B_\perp e^{i\phi} & 0 \\
B_\perp e^{-i\phi} & 0 & B_\perp e^{i\phi} \\
0 & B_\perp e^{-i\phi} & -B_z
\end{array} \right). \tag{16}
\]
The eigenvalues of \( B \) are \( \pm B \) and 0 where \( B = \sqrt{B_\perp^2 + B_z^2} \), and the corresponding eigenvectors, denoted by \( | \psi \rangle_{BQ} \), are
\[
| \pm 1 \rangle_{BQ} = \frac{1}{2B} \left( \begin{array}{c}
(B \mp B_z) e^{i\phi} \\
\pm \sqrt{2} B_\perp e^{-i\phi} \\
(B \mp B_z) e^{-i\phi}
\end{array} \right), \tag{17}
\]
\[
| 0 \rangle_{BQ} = \frac{1}{\sqrt{2B}} \left( \begin{array}{c}
-B_\perp e^{i\phi} \\
\sqrt{2} B_z e^{-i\phi} \\
B_\perp e^{-i\phi}
\end{array} \right). \tag{18}
\]
The vectors \( |1\rangle_{BQ} \) and \( |-1\rangle_{BQ} \) are identified with the strong field seeking state and the weak field seeking state respectively. Accordingly when the whole system is in the strong or the weak field seeking state, the order parameter is written in terms of these vectors as \( |\Psi\rangle = |\Psi(r)| \pm 1\rangle_{BQ} \). The \( B\Psi \) term in Eqs. (14) and (15) is then simplified to \( \pm B(r)\Psi \) so that the GP equation takes the form
\[
\frac{\partial \Psi}{\partial t} = \left\{ -\frac{\hbar^2 \nabla^2}{2m} - \mu + V(r) \mp B(r) + (g_n + g_s) |\Psi|^2 \right\} \Psi. \tag{19}
\]

We have ignored the small corrections, which comes from the spatial derivative, to the kinetic energy term. This is the usual GP equation without the spin degrees of freedom.

When \( B_z = 0 \), the order parameter with the highest eigenvalue corresponds to the strong field seeking state is
\[
\frac{\Psi_1}{\Psi_0} = \psi \left( \begin{array}{c}
\frac{1}{2} e^{i\phi} \\
\frac{1}{\sqrt{2}} e^{-i\phi}
\end{array} \right) e^{iw\phi}, \tag{20}
\]
where \( w \) is an integer and \( \psi \) is the amplitude of \( \Psi \). In the XYZ basis, this is rewritten as
\[
\frac{\Psi_x}{\Psi_z} = \psi \left( \begin{array}{c}
-i\frac{1}{2} \sin \phi \\
-i\frac{1}{\sqrt{2}} \cos \phi
\end{array} \right) e^{iw\phi}. \tag{21}
\]

The corresponding \( \mathbf{m}, \mathbf{n} \) and \( \mathbf{l} \) vectors are
\[
\mathbf{m} = (\sin \phi \sin(w\phi), \cos \phi \sin(w\phi), \cos(w\phi)),
\mathbf{n} = (-\sin \phi \cos(w\phi), -\cos \phi \cos(w\phi), \sin(w\phi)), \tag{22}
\mathbf{l} = (\cos \phi, -\sin \phi, 0).
\]

In the weakening field seeking state, the order parameter is written as
\[
\frac{\Psi_x}{\Psi_z} = \psi \left( \begin{array}{c}
-i\frac{1}{2} \sin \phi \\
-i\frac{1}{\sqrt{2}} \cos \phi
\end{array} \right) e^{iw\phi}. \tag{23}
\]
or
\[
\frac{\Psi_x}{\Psi_y} = \psi \left( \begin{array}{c}
-i\frac{1}{2} \sin \phi \\
-i\frac{1}{\sqrt{2}} \cos \phi
\end{array} \right) e^{iw\phi}. \tag{24}
\]

The corresponding triad is
\[
\mathbf{m} = (\sin \phi \sin(w\phi), \cos \phi \sin(w\phi), -\cos(w\phi)),
\mathbf{n} = (-\sin \phi \cos(w\phi), -\cos \phi \cos(w\phi), -\sin(w\phi)), \tag{25}
\mathbf{l} = (-\cos \phi, \sin \phi, 0).
\]

This \( \mathbf{l} \) vector field in both (22) and (25) clearly represents the cross disgyration.
IV. CREATION OF VORTEX

In the present Section, we propose a simple method to create a persistent current (and also a vortex state) in a torus-shaped BEC from a state with no persistent current. Then it will be shown that this persistent current is easily transformed to a vortex.

In the following we consider two cases separately. In Case I, the condensate is confined optically and the spin of each atom points the direction of the magnetic field. Namely, atoms are strong field seekers. In Case II, the magnetic field is also used to confine the condensate. The spin of atoms points antiparallel to the magnetic field and the atoms are weak field seekers. The former case has an advantage in its theoretical simplicity. On the other hand, the latter case does not require apparatus for optical confinement except for the repulsive plug around $r = 0$ and can be realized more easily.

In the following discussions we consider a BEC of Na atoms with $F = 1$. The mass of the atom $m = 2.41 \times 10^{-26}$kg, the interaction parameter $g_n = 4\pi\hbar^2a/m$, and the scattering length $a = 2.75 \times 10^{-9}$m are employed. We ignore the other interaction parameter $g_s$. This is possible since the whole condensate is assumed to be in either strong or weak field seeking state as a whole in the following. In those states the interaction terms in the GP equation are reduced as shown in Eq. (20).

The particle density is taken to be around $10^{19}$m$^{-3}$ and the detailed density profile is given in each figure. The time span of the persistent current creation process $T = 30$ms is chosen since this is between (the Larmor frequency)$^{-1} \sim 1$m sec and the life time of the condensate $\sim 1$sec.

A. Case I: Optical confinement

The external magnetic field $B(r, \phi, z)$ takes the form

$$
(B_x, B_y) = B_{\perp}(\cos \phi, -\sin \phi),
B_z = B_{z0}\cos(\pi(1 - t/T)),
B_{\perp} = B_{\perp0}\sin(\pi(1 - t/T)),
$$

(26)

where $t$ is the time. The factors of the field are taken to be $B_{\perp0} = 200\hbar$ [J/$\mu$m] and $B_{z0} = 2 \times 10^4\hbar$ [J] (note that the scaled magnetic field represents the Zeeman energy in fact). One finds from Eq. (23) that $B_z$ flips from $-B_{z0}$ to $B_{z0}$ so that $\hat{l}$ also flips in the end of the evolution. The Larmor frequency is $\omega_L \sim 0.6 \times 10^4[HHz]$ for $B \sim 2 \times 10^4\hbar[J]$. Thus $\omega_L \times T \sim 18$. The spin-independent potential is

$$
V(r) = \frac{m(2\pi\nu)^2}{2}r^2 + U \exp \left( -\frac{r^2}{2r_0^2} \right),
$$

(27)

with $\nu = 200[Hz]$, $U = 1 \times 10^4\hbar[J]$ and $r_0 = 5[\mu$m]. The first term of Eq.(27) is the confining potential while the second term is the potential produced by the optical plug.

We obtained the order parameter profile by numerical integration of the time dependent GP equation (10). The initial state is taken to be the ground state with no circulation. The magnetic field changes the direction slowly from upward to downward according to Eq. (26) as shown in Fig. 1(a) so that the atoms remain in the strong field seeking state. The change in the number of the $k$-th component

$$
N_k(t) = \int |\Psi_k(r, t)|^2 d^2r \quad (k = -1, 0, 1)
$$

(28)

is shown in Fig. 1(b). The total particle density

$$
n(r, t) = \sum_k |\Psi_k(r, t)|^2
$$

(29)

changes with the magnetic field as shown in Fig. 1(c).

The resulting triad configurations are shown in Figs. 2 for $t = 0, 15, 30$ [ms]. Figure 2(a) shows the initial vector configurations. Figure 2(b) shows the vector configurations when $t = 15$ms. The $l$-texture is nothing but the cross disgyration explained in the previous Section since $B_z$ vanishes now. We see that the vectors $\hat{m}$ and $\hat{n}$ rotates around $\hat{l}$ by $2\pi$ as we go around the $z$ axis once. Finally when $t = 30$ms, we obtain a texture with $\hat{l}$ almost points up everywhere. The vectors $\hat{m}$ and $\hat{n}$ rotates around $\hat{l}$ by $4\pi$ as one goes around the $z$ axis once in this case and one finally obtained a uniform $l$-texture with a circulation with the winding number 2.

Now that a persistent current is created, it is easy to transform this into a vortex. The BEC has the $k = 1$ component only at $t = 30$ms. Then there are no atoms near the axis $r = 0$ since the centrifugal force prevents the atoms to come close to the axis. Thus one may simply turn off the optical plug to obtain a vortex. The details are analyzed in next section.
FIG. 1. The process of the persistent current creation in Case I. (a) Magnetic field at $r = 10\mu m$ as a function of time. Because $B_\perp \propto r$, the total magnetic field varies slightly at $r \neq 10\mu m$. (b) Particle numbers $N_k$ as a function of time. The condensate has $\Psi_{-1}$ component only at $t = 0$ and $\Psi_1$ component only at $t = T$. (c) Total number density distribution. The condensate is almost fixed at around $r = 9\mu m$ by the optical potential $V(r)$. However the change of the total magnetic field [see caption (a)] causes the change in the radial distribution as shown here.

FIG. 2. The triad configurations for $t = 0, 15$, and $30$ ms. The arrows denote $\hat{l}$ while $\hat{m}$ and $\hat{n}$ are on the disk. The line on the disk is $\hat{m}$. Note that there is no need to draw $\hat{n}$ since it is uniquely given by $\hat{l} \times \hat{m}$. When $t = 0$ and $30$ ms, $\hat{l}$ point down and up respectively. When $t = 15$ ms, $\hat{l}$ lies almost on the $xy$ plane.

B. Case II: Magnetic confinement

In Case I, the condensate is confined with spin-independent optical trap. Here in Case II, we consider the situation where the quadrupole magnetic field Eq. (15) always exists and the additional field $B_z$ change from a large positive value to a large negative value as shown in Fig. 3(a):

\[
(B_x, B_y) = B_\perp (\cos \phi, -\sin \phi),
\]
\[
B_z = B_{z0}(1 - 2t/T),
\]
\[
B_\perp = B'_\perp r.
\]

Here we take $B'_\perp = 400h [1/\mu m]$, $B_{z0} = 2 \times 10^4 h$ and $T = 30 [ms]$. Contrary to Case I, the atoms are in the weak field seeking state and the gradient of $B_r$ is responsible for the confinement of the condensate. The optical plug produces a spin-independent potential
V(r) = U \exp \left( -\frac{r^2}{2r_0^2} \right), \quad (31)

where we take \( U = 1 \times 10^4 \hbar [\text{J}] \) and \( r_0 = 5 [\mu \text{m}] \).

The evolution of the order parameter field is analyzed numerically and it was found that the order parameter configurations in this case is essentially the same as in Case I. Figure 3(b) shows the temporal evolution of the components \( N_k(t) \) and Fig. 3(c) shows the total density profile at various time.

The persistent current created here may also be transformed into a vortex. The BEC is mostly made of the \( k = 1 \) component at \( t = 30 \text{ms} \) and there are only a small number of atoms near \( r = 0 \). Suppose the optical plug is turned off. Then atoms will escape but this process should be very slow. Thus one expects that the vortex is stable for a considerable period of time.

**C. Mathematical analysis of continuous creation of circulation**

It may be surprising that we have continuously created a persistent current (a vortex) from a system without circulation. Mathematically this is justified by invoking homotopy theory. Let us denote a rotation around direction \( \mathbf{n} \) by an angle \( \alpha \) by a “vector” \( \alpha \mathbf{e} \). This rotation is expressed as a rotation matrix

\[
R(\mathbf{e}, \alpha) = (1 - \cos \alpha)\hat{n}_{i}\hat{n}_{j} + \cos \alpha \delta_{ij} - \sin \alpha \varepsilon_{ijk}\hat{n}_{k}. \quad (32)
\]

Since \( \alpha \) may be restricted within the region \( 0 \leq \alpha \leq \pi \), the set of all the rotations is represented by a ball \( B^3 \) with the radius \( \pi \). Note however that the points \( \pi \mathbf{e} \) and \( -\pi \mathbf{e} \) corresponds to equivalent rotations. Thus all the antipodal points on the surface of the ball must be identified. This space \( B^3/Z_2 \) is called the three-dimensional real projective space, denoted by \( RP^3 \).

Let us take a “standard” triad \((\mathbf{m}_0, \mathbf{n}_0, \mathbf{l}_0)\) shown in Fig. 4. Then an arbitrary triad is obtained by applying a certain rotation \( \alpha \) \( uni\mathbf{e} \) to the standard triad. Thus the local vector configuration is in one-to-one correspondence with a point in \( RP^3 \).

Consider an order parameter configuration shown in Fig. 3(a). When one circumnavigates the circle, one finds that all the triads along the circle are obtained from the standard one by applying no rotations, namely \( \alpha = 0 \). Thus this circle is mapped to the origin of \( RP^3 \). Next consider the triads in Fig. 3(b). As one goes along the circle, one finds that the standard triad is rotated by an angle \( \pi/2 \) around the axis.
The continuous current creation can be explained more generally using the eigenvectors of the $B$-matrix with the highest and lowest eigenvalues in the BQ basis. The magnetic field expressed in $B$-matrix is the dominant factor to determine the behavior of the order parameter $\Psi$. We first discuss the BEC with $F = 1$, which we have analyzed in the present paper.

The magnetic fields Eq. (26) in Case I and Eq. (30) in Case II are of the form Eq. (15). The corresponding $B$-matrix is given by Eq. (14). Because the magnetic fields are sufficiently strong, we may assume that the order parameter $\Psi$ is proportional to the highest (upper sign) or lowest (lower sign) eigenvector

$$| \pm 1 \rangle_{BQ} = \frac{1}{2|B|} \begin{pmatrix} (B \pm B_z)e^{i\phi} \\ \pm\sqrt{2}B_\perp \\ (B \mp B_z)e^{-i\phi} \end{pmatrix}.$$ (33)

of the $B$-matrix. Case I uses the highest (upper sign) and Case II uses the lowest (lower sign) eigenvector. The order parameter is

$$\begin{pmatrix} \Psi_1 \\ \Psi_0 \\ \Psi_{-1} \end{pmatrix} = C \begin{pmatrix} (B \pm B_z)e^{i\phi} \\ \pm\sqrt{2}B_\perp \\ (B \mp B_z)e^{-i\phi} \end{pmatrix} e^{iw\phi},$$ (34)

where $C$ is a complex number independent of $\phi$ and $w$ is an integer.

Let us consider Case I. (Case II can be analyzed similarly.) When $t = 0$, the magnetic field is $(B_1, B_z) = (0, \pm B\omega)$. As shown in Eq. (15), the condensate behaves as $\Psi_{-1} \propto e^{i(w-1)\phi}$ and $\Psi_0 = \Psi_1 = 0$. Because we start with a state with no circulation, the integer $w$ must be 1.

The components $B_z$ and $B_\perp$ change as shown in, for example, Fig. 2(a) from $t = 0$ to $t = T$. Both $B_\perp$ and $(B \pm B_z)$ have finite values during the change and the condensate $\Psi$ stays in the strong field seeking state. The phase factor $w$ will not change during the process for the order parameter to be defined uniquely. Thus we take $w \equiv 1$ throughout the process. When $t = T$, the magnetic field is $(B_\perp, B_z) = (0, \pm B\omega)$. Since $w = 1$, Eq. (15) leads to the conclusion that we have a condensate with $\Psi_1 \propto e^{2i\phi}$ and $\Psi_0 = \Psi_{-1} = 0$.

Accordingly a vortex with the winding number 2 has been created.

A similar discussion is possible in the system with $F = 2$ atoms using the eigenvector given in Eq. (15). Starting from a state with no winding number, we eventually obtain a state with the winding number 4.

V. DETECTION OF VORTEX: TIME OF FLIGHT IMAGING

The detection of a vortex (or persistent current) has been a problem as difficult as their creation. We consider the relaxation of the spinor texture after the confining field and the optical plug are turned off to facilitate the comparison between our theory and experiments, in particular the time of flight (TOF) analysis.

The temporal evolution of the BEC is described by the time-dependent GP equation (10). We consider Case I and Case II separately.
A. Case I

We consider three cases where the confining potentials are turned off separately at $t = 0, T/2$ and $T$ ms. There is no vortex at $t = 0$ while there is a vortex with the winding number 2 at $t = T$. The comparison between these two cases is essential to observe our vortex.

(i) $t = 0$: The condensate has a component $\Psi_{-1}$ only. The density profile at $t = 0$ is determined by solving the time-independent GP equation. The relaxation process after the potentials are turned off is found by solving the time-dependent GP equation Eq. (10), whose result is shown in Fig. 5(a). Since $\Psi_{-1}$ has no singularity at the origin, the condensate fills the central region ($r \sim 0$) in later time. It is interesting to note that the components $\Psi_0$ and $\Psi_1$ do not appear in later time since the total spin must be conserved.

(ii) $t = T/2$: The cross disgyration appears in this stage. The order parameter of this texture is given by Eq. (20) with $w = 1$. Thus all the components are non-vanishing in this case. After the potentials are turned off at $t = 0$, the order parameter relaxes as shown in Fig. 5(b). The component $\Psi_0$ has winding number 1 while $\Psi_1$ has winding number 2, and hence they cannot fill the central region. The central region near $r = 0$ may be filled only with $\Psi_{-1}$ component since it has vanishing winding number. Note also that the $\Psi_0$ component is dominant in the vicinity of $r \sim 1 \mu m$.

(iii) $t = T$: The vector $\hat{l}$ points up now and hence $\Psi_{-1} = \Psi_0 = 0$ while $\Psi_1 \neq 0$. Figure 5(c) shows the temporal evolution of the order parameter after the potentials are turned off. Since the order parameter has a nontrivial phase factor, it cannot fill the central region at all in later times. Similarly to the case (i), the components $\Psi_0$ and $\Psi_{-1}$ do not appear in the relaxation process. The absence of the condensate at $r = 0$ at an arbitrary time is a clear distinction between the case (iii) and the rest, which may be used to show the existence of the vortex or the persistent current experimentally.

The vacuum region around $r = 0$ which shows the existence of the vortex have length scale around $10 \mu m$ after $7 ms$ relaxation, and the length will be sufficient to observe experimentally. Because the length scale of the central vacuum region is almost same as that of the density waves at larger $r$ (outer), the resolution of the imaging will be checked by the outer density waves.
FIG. 5. Temporal evolution of the condensate. (a) The potentials are turned off at $t = 0$. The solid line is the total density purely made of $|\Psi_{-1}|^2$. (b) The potentials are turned off at $t = T/2$. The solid line is the total density $\sum_k |\Psi_k|^2$ while the dashed line, the dotted line, and the dashed dotted line are components $|\Psi_1|^2$, $|\Psi_0|^2$ and $|\Psi_{-1}|^2$, respectively. The corresponding winding numbers are 2, 1 and 0. (c) The potentials are turned off at $t = T$. The solid line is the total number density made of $|\Psi_1|^2$ only. The condensate has the winding number 2 and cannot fill the central region. This should be compared with (a) and (b).

B. Case II

Figure 6(a) shows the relaxation process when the potentials are turned off at $t = 0$ while they are turned off at $t = T$ in Fig. 6(b). Because the magnetic field is not exactly parallel or antiparallel to the $z$ axis, the non-dominant component of the condensate appears slightly in both cases.

VI. SUMMARY AND DISCUSSIONS

In summary, we have proposed a new method to create a persistent current and a vortex with the winding number 2 in a Bose-Einstein condensate of alkali atoms. The dynamics of vortex creation are simulated by solving the time-dependent Gross-Pitaevskii (GP) equation. The continuity of this process is justified by invoking homotopy theory and also by the angular momentum analysis. The existence of the vortex may be demonstrated by comparing the time of flight (TOF) data before and after the vortex creation.

It is also possible to create a persistent current or a vortex with the winding number 1. Suppose one prepares the ground state in the Ioffe-Pritchard field. The resulting texture is the cross disgyration with no winding of the $\hat{m}$-vector around the $\hat{l}$-vector as shown in [6]. Then apply a strong magnetic field $B_z$ either parallel to or antiparallel to the $z$ axis. The $l$-vector in the resulting texture points up or down, depending on the direction of $B_z$ or whether the state is weak field seeking or strong field seeking. In any case, the $\hat{m}$-vector rotates around $\hat{l}$ once as one circumnavigates around the $z$ axis once. Thus one obtains a persistent current or a vortex of the winding number 1 by simply preparing a sample in the Ioffe-Pritchard trap and applying a strong magnetic field along the $z$ axis.

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APPENDIX A: MAGNETIC FIELD MATRIX

We consider a system of atoms with spin $F = f$. The $F$-matrices, which are the angular momentum operators, are given by
\[ F_x = \begin{pmatrix} 0 & \frac{\sqrt{2f+1}}{2} & \frac{\sqrt{(2f-1)^2}}{2} \\ \frac{\sqrt{2f+1}}{2} & 0 & \frac{\sqrt{(2f-1)^2}}{2} \\ \frac{\sqrt{(2f-1)^2}}{2} & \frac{\sqrt{(2f-1)^2}}{2} & 0 \end{pmatrix}, \]

\[ F_y = \begin{pmatrix} 0 & -\frac{\sqrt{2f+1}}{2i} & -\frac{\sqrt{(2f-1)^2}}{2i} \\ -\frac{\sqrt{2f+1}}{2i} & 0 & -\frac{\sqrt{(2f-1)^2}}{2i} \\ -\frac{\sqrt{(2f-1)^2}}{2i} & -\frac{\sqrt{(2f-1)^2}}{2i} & 0 \end{pmatrix} \]  

\[(F_z)_{jj} = f + 1 - j \quad (j = 1, 2, \cdots, 2f + 1).\]

They satisfy the commutation relation \([F_x, F_\beta] = iF_\gamma \varepsilon_{\alpha \beta \gamma}.\)

We write the magnetic field with the \(B\) vector

\[
\begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = B \begin{pmatrix} \sin \theta_y \cos \theta_z \\ \sin \theta_y \sin \theta_z \\ \cos \theta_y \end{pmatrix} = \begin{pmatrix} B_\perp \cos \theta_z \\ B_\perp \sin \theta_z \\ B_z \end{pmatrix}
\]

where \(0 \leq \theta_y \leq \pi\) and \(0 \leq \theta_z < 2\pi\). The amplitude \(|\tilde{B}|\) is scaled so that it represents the Zeeman energy.

The order parameter of a BEC is written with a vector of \(2f + 1\) components and operators are expressed in terms of \((2f + 1) \times (2f + 1)\) square matrices. We call the operator for the Zeeman energy \(B\)-matrix. We can choose the quantization axis so that the \(B\)-matrix is proportional to the matrix \(F_z\). We call this the \(B\) quantized (BQ) notation because \(B\) is proportional to the quantization axis.

The \(B\) matrix in the ZQ notation is obtained by successive spatial rotations \(\theta_y\) along the \(y\) axis and \(\theta_z\) along the \(z\) axis as

\[
B_{\text{ZQ}} = U^\dagger B_{\text{BQ}} U,
\]

\[U^\dagger = \exp(-iF_\beta \theta_z) \exp(-iF_y \theta_y).\]  

Let us study a few examples. When \(f = 1\),

\[
B_{\text{ZQ}} = \begin{pmatrix} B_z & \frac{B_\perp}{\sqrt{2}} \cos \theta_z & 0 \\ \frac{B_\perp}{\sqrt{2}} e^{i\theta_z} & 0 & \frac{B_\perp}{\sqrt{2}} e^{-i\theta_z} \\ 0 & \frac{B_\perp}{\sqrt{2}} e^{i\theta_z} & -B_z \end{pmatrix}.
\]  

The eigenvector with the lowest eigenvalue is

\[
U^\dagger \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{B_\perp e^{-i\theta_z}}{B_\perp^2 - \frac{B_\perp}{\sqrt{2}} e^{-i\theta_z}} \\ -\frac{B_\perp e^{-i\theta_z}}{B_\perp^2 - \frac{B_\perp}{\sqrt{2}} e^{i\theta_z}} \\ \frac{B_\perp e^{-i\theta_z}}{B_\perp^2 - \frac{B_\perp}{\sqrt{2}} e^{i\theta_z}} \end{pmatrix}.
\]  

When \(f = 2\),

\[
B_{\text{ZQ}} = \begin{pmatrix} 2B_z & B_\perp e^{-i\theta_z} & 0 & 0 & 0 \\ B_\perp e^{i\theta_z} & B_z & \sqrt{\frac{3}{2}} B_\perp e^{-i\theta_z} & 0 & 0 \\ 0 & \sqrt{\frac{3}{2}} B_\perp e^{i\theta_z} & 0 & \sqrt{\frac{3}{2}} B_\perp e^{-i\theta_z} & 0 \\ 0 & 0 & \sqrt{\frac{3}{2}} B_\perp e^{i\theta_z} & -B_z & B_\perp e^{-i\theta_z} \\ 0 & 0 & 0 & B_\perp e^{i\theta_z} & -2B_z \end{pmatrix}.
\]  

The eigenvector with the lowest eigenvalue is
\[
U^1 \begin{pmatrix} 0 \\
0 \\
0 \\
1 
\end{pmatrix} = C \begin{pmatrix}
B(1 - \cos \theta_y)^2 e^{-2i\theta_z} \\
-B(1/2)(1 - \cos \theta_y)e^{-i\theta_z} \\
B(\sin \theta_y)^2 \sqrt{3/8} \\
-B(1/2)(1 + \cos \theta_y)e^{i\theta_z} \\
\end{pmatrix}.
\]
(A7)

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