EXTENSION OF THE UNIT NORMAL VECTOR FIELD FROM A HYPERSURFACE

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Abstract. It is important in many applications to be able to extend the (outer) unit normal vector field from a hypersurface to its neighborhood in such a way that the result is a unit gradient field. The aim of the paper is to provide an elementary proof of the existence and uniqueness of such an extension.

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INTRODUCTION

It is important in many applications to be able to extend the (outer) unit normal vector field $\nu$ from a hypersurface $S$ to a neighborhood of $S$ in such a way that the result is a unit gradient field (see, e.g., [1]–[4] and the references therein). We call such extensions proper.

Definition 1. Let $S$ be a hypersurface in $\mathbb{R}^n$ and $\nu$ be the unit normal vector field on $S$. A vector field $N \in C^1(\Omega_S)$ in a neighborhood $\Omega_S$ of $S$ is referred to as a proper extension if $N|_S = \nu$ and

$$ |N(x)| = 1, \quad \partial_j N_k(x) = \partial_k N_j(x) \quad \text{for all} \quad x \in \Omega_S, \quad j, k = 1, \ldots, n. $$

The existence of such an extension follows from the well known existence and uniqueness result for the following boundary value problem for the eikonal equation: for a given hypersurface $S$, find a function $\Phi_S$ such that

$$ |\nabla \Phi_S(x)| = 1 \quad \forall x \in \Omega_S, \quad \Phi_S(x) = 0 \quad \text{and} \quad \nabla \Phi_S(x) = \nu(x) \quad \text{for} \quad x \in S, $$

where $\nu(x), x \in S$ is the unit normal vector field on the hypersurface $S$ (see, e.g., [5] page 88-89]). Indeed, if $\Phi_S$ is a solution to the problem (2), the gradient $\nabla \Phi_S(x) = \nabla \Phi_S(x), x \in \Omega_S$, is a proper extension of the unit normal vector field $\nu(x), x \in S$.

The aim of this paper is to present an elementary proof of the existence and uniqueness result for the proper extension problem and for (2), which does not rely on the theory of Hamilton-Jacobi equations, and to provide a streamlined presentations of some results discussed in [1]–[3].

The paper is organized as follows. In Section 1, we recall some definitions and introduce basic notation from the theory of hypersurfaces. In Section 2, we present some useful properties of a proper extension of a unit normal vector field to a hypersurface. The main result of the paper is proved in Section 3.

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1. Hypersurfaces and their normal vectors

**Definition 2.** A subset \( S \subset \mathbb{R}^n \) of the Euclidean space is called a **hypersurface** if there exist an open covering \( S = \bigcup_{j=1}^M S_j \) and coordinate mappings

\[
\Theta_j : \omega_j \to S_j : = \Theta_j(\omega_j) \subset \mathbb{R}^n, \quad \omega_j \subset \mathbb{R}^{n-1}
\]

such that the corresponding differentials

\[
D\Theta_j(p) := \text{matr} [\partial_1 \Theta_j(p), \ldots, \partial_{n-1} \Theta_j(p)],
\]

have the full rank

\[
\text{rank} \ D\Theta_j(p) = n - 1, \quad \forall p \in \omega_j, \quad j = 1, \ldots, M,
\]

i.e., \( \Theta_j \) are regular over the domains \( \omega_j \) for all \( j = 1, \ldots, M \).

The hypersurface is called **smooth** (or \( k \)-smooth) if the corresponding coordinate diffeomorphisms \( \Theta_j \) in (1) are \( C^\infty \)-smooth (\( k \)-smooth respectively).

**Remark 1.** Defining the smoothness of a manifold one needs to consider transition maps like of the atlas \( \Theta_i^{-1} \circ \Theta_j : \omega_i \cap \omega_j \to \omega_i \cap \omega_j \) when defining a general manifold that is not assumed a priori to be embedded into a Euclidean space. The ambient Euclidean space allows one to define hypersurfaces without recourse to transition maps. The latter can be proved to have the necessary smoothness with the help of the rank condition and the implicit function theorem.

A closed hypersurface (without boundary) in \( \mathbb{R}^n \) is orientable. An elementary proof of this can be found in [6].

**Definition 3.** Let \( k \geq 1 \) and \( \Omega \subset \mathbb{R}^n \) be a bounded domain. An **implicit \( C^k \)-smooth** hypersurface in \( \mathbb{R}^n \) is defined as the set

\[
S = \{ t \in \Omega : \Psi_S(t) = 0 \},
\]

where \( \Psi_S : \Omega \to \mathbb{R} \) is a \( C^k \)-mapping which is regular \( \nabla \Psi(t) \neq 0, \forall t \in \Omega \).

**Lemma 1** ((see, e.g., [2, §1])). **Definition** 2 and **Definition** 3 of a \( k \)-smooth hypersurface \( S \) are equivalent.

**Remark 2.** For a given hypersurface, an implicit surface function is defined with the help of the signed distance

\[
\Psi_S(x) := \pm \text{dist}(x, S), \quad x \in \Omega_S,
\]

where the signs “+” and “−” are chosen for \( x \) “above” in the direction of the unit normal vector and “below” \( S \), respectively (see [1, §3]).

We will need the following textbook result.

**Lemma 2.** Let \( S \subset \mathbb{R}^n \) be a \( k \)-smooth hypersurface, \( k = 2, 3, \ldots \), given implicitly \( \Psi_S(x) = 0 \) by the function \( \Psi_S \in C^k(\Omega_S) \).

The \( C^{k-1} \)-smooth unit vector field

\[
\nu(x) := \frac{\nabla \Psi_S(x)}{|(\nabla \Psi_S(x))|}, \quad x \in S
\]

is normal (orthogonal) to the surface \( S \).
2. Properties of a proper extension

First note that the extension

\[ \mathbf{v}(x) := \frac{(\nabla \Psi_S)(x)}{|(\nabla \Psi_S)(x)|}, \quad x \in \Omega_S \]

of the normal vector field \( \mathbf{v}(x) \) (see (1)) is not in general a proper one. Indeed, let \( n = 2 \) and let \( S \) be the ellipse

\[ \{ x = (x_1, x_2) \in \mathbb{R}^2 | \Psi_S(x_1, x_2) := x_1^2 + 2x_2^2 - 1 = 0 \} . \]

Then

\[ N(x) := \frac{(\nabla \Psi_S)(x)}{|(\nabla \Psi_S)(x)|} = \frac{(x_1, 2x_2)}{\sqrt{x_1^2 + 4x_2^2}} , \]

\[ \partial_1 N_2(x) = -\frac{2x_2}{(x_1^2 + 4x_2^2)^{3/2}} , \]

\[ \partial_2 N_1(x) = -\frac{4x_1}{(x_1^2 + 4x_2^2)^{3/2}} . \]

Hence \( \partial_1 N_2(x) \neq \partial_2 N_1(x) \) unless \( x_1 = 0 \) or \( x_2 = 0 \).

**Lemma 3.** Gunter’s derivatives

\[ (2) \hspace{1cm} \mathcal{D}_k := \partial_k - \nu_k \partial_N \]

satisfy the following equalities:

\[ (3) \hspace{1cm} \mathcal{D}_k \nu_j(x) = \mathcal{D}_j \nu_k(x) \quad \text{for all} \quad x \in S, \quad j, k = 1, 2, \ldots, n. \]

**Proof.** Since (see (4))

\[ \nu_k^2 := \frac{\partial_k \Psi_S}{|(\nabla \Psi_S)|} , \]

a routine calculation gives

\[ \mathcal{D}_j \nu_k = \partial_j \nu_k - \nu_j \partial_N \nu_k = \partial_j \frac{\partial_k \Psi_S}{|(\nabla \Psi_S)|} - \frac{\partial_j \Psi_S}{|(\nabla \Psi_S)|} \sum_{m=1}^{n} \frac{\partial_m \Psi_S}{|\nabla \Psi_S|^3} \partial_m \frac{\partial_k \Psi_S}{|\nabla \Psi_S|^3} \]

\[ = \frac{\partial_j \partial_k \Psi_S}{|(\nabla \Psi_S)|} - \sum_{l=1}^{n} \frac{\partial_l \Psi_S \partial_k \Psi_S \partial_l \partial_j \Psi_S}{|\nabla \Psi_S|^3} - \sum_{m=1}^{n} \frac{\partial_j \Psi_S \partial_m \Psi_S \partial_m \partial_k \Psi_S}{|\nabla \Psi_S|^3} \]

\[ + \sum_{m=1}^{n} \sum_{l=1}^{n} \frac{\partial_l \Psi_S \partial_m \Psi_S \partial_l \partial_j \Psi_S \partial_m \partial_k \Psi_S}{|\nabla \Psi_S|^5} = \mathcal{D}_k \nu_j \quad \text{for all} \quad j, k = 1, 2, \ldots, n. \]

The last equality holds because the expression \( \mathcal{D}_j \nu_k \) turns out to be symmetric with respect to the indices \( j \) and \( k \).

One can give an alternative proof that avoids the above calculations, if one assumes the existence of a proper extension of \( \mathbf{v} \) to a neighborhood \( \Omega_S \) of \( S \) (see the proof of (5) in Lemma 5 below).

**Lemma 4.** For a unitary (not necessarily proper) extension \( \mathcal{N}(x) \in C^1(\Omega_S), |\mathcal{N}(x)| \equiv 1 \) of \( \mathbf{v}(t) \) to a neighborhood \( \Omega_S \) of \( S \), the following conditions are equivalent:

(i) \( \partial_{\mathcal{N}}|_S = 0 \),
(ii) \[ \partial_k N_j - \partial_j N_k \big|_S = 0 \quad \text{for } k, j = 1, 2, \ldots, n. \]

**Proof.** Suppose (i) holds. Then one has on \( S \)
\[
\partial_k N_j = D_k N_j \quad \text{(due to (i) and the definition (2) of \( D_k \))}
\]
\[
= D_k \nu_j \quad \text{(since \( D_k \) is a tangent derivative and \( N = \nu \) on \( S \))}
\]
\[
= D_j \nu_k \quad \text{(by (3))}
\]
\[
= D_j N_k = \partial_j N_k \quad \text{(as above).}
\]

Suppose now (ii) holds. Then one has on \( S \)
\[
\partial N_j = \sum_{k=1}^{n} N_k \partial_k N_j = \sum_{k=1}^{n} N_k \partial_j N_k = \frac{1}{2} \sum_{k=1}^{n} \partial_j N_k^2 = \frac{1}{2} \partial j 1 = 0.
\]

\[ \square \]

**Corollary 1.** Let \( N \in C^1(\Omega_S), |N(x)| \equiv 1, \) be a unitary (not necessarily proper) extension of \( \nu \) to a neighborhood \( \Omega_S \) of \( S. \)

If one of conditions (i) or (ii) of Lemma 4 holds, then
\[
D_k N_j(x) = \partial_k N_j(x) = \partial_j N_k(x) = D_j N_k(x) \quad \text{for all } x \in S.
\]

**Proof.** The claimed equalities follow from conditions (i) and (ii) of Lemma 4 and the definition of Gunter’s derivative \( D_k. \)

**Lemma 5.** Any proper extension \( N(x), x \in \Omega_S \subset \mathbb{R}^n \) of the unit normal vector field \( \nu \) to the surface \( S \subset \Omega_S \) satisfies the equality
\[
\partial_N N(x) = 0, \quad \text{for all } x \in \Omega_S.
\]

Moreover, for the extensions of Gunter’s derivatives \( D_k = \partial_k - N_k \partial_N \) to the neighborhood \( \Omega_S \) of the surface \( S\)
\[
D_k = \partial_k - N_k \partial_N, \quad k = 1, \ldots, n,
\]
the following equalities hold
\[
D_k N_j(x) = \partial_k N_j(x) = \partial_j N_k(x) = D_j N_k(x) \quad \text{for all } x \in \Omega_S
\]
and, in particular,
\[
D_k \nu_j(x) = D_j \nu_k(x) \quad \text{for all } x \in S, \ j, k = 1, 2, \ldots, n.
\]

**Proof.** Equality (6) is proved in exactly the same way as (1). Since \( D_k = \partial_k - N_k \partial_N, \) (8) and (9) are a direct consequence of (6).

\[ \square \]

3. **Existence of a proper extension**

We prove in this section that the formula
\[
N(x + t \nu(x)) = \nu(x), \quad x = x + t \nu(x), \quad x \in S, \quad -\varepsilon < t < \varepsilon
\]
defines a unique proper extension \( N(x) \) of the unit normal vector field \( \nu(x) \) from the hypersurface \( S \subset \mathbb{R}^n \) into a neighborhood \( \Omega_S \)
\[
\Omega_S := \{ x = x + t \nu(x) : x \in S, \quad -\varepsilon < t < \varepsilon \}
\]
Theorem 1. Let $S \subset \mathbb{R}^n$ be a hypersurface given by an implicit function

$$S = \{ x \in \mathbb{R}^n : \Psi_S(x) = 0 \}.$$ 

Then the function

$$\Phi_S(x + t \nu(x)) := t, \quad x + t \nu(x) \in \Omega_S$$

represents a unique solution to the eikonal boundary value problem (2), while its gradient

$$\nabla \Phi_S(x + t \nu(x)) = \nu(x), \quad x = x + t \nu(x), \quad x \in S, \quad -\varepsilon < t < \varepsilon$$

is a unique proper extension of the the unit normal vector field $\nu$ to the surface $S \subset \Omega_S$.

Proof. Uniqueness. Let $N$ be a proper extension of $\nu$ and let $\tau \mapsto \gamma(\tau)$ be an integral curve of $N$ starting at $x \in S$. Then $\gamma(0) = x$, $\frac{d\gamma}{d\tau}(0) = \nu(x)$, and it follows from (6) that

$$\frac{d^2\gamma}{d\tau^2}(\tau) = \frac{dN(\gamma(\tau))}{d\tau} = (\partial_N N)(\gamma(\tau)) = 0.$$ 

Hence $\frac{d\gamma}{d\tau}(\tau) = \text{const} = \frac{d\gamma}{d\tau}(0) = \nu(x)$ and $\gamma(\tau) = \gamma(0) + \tau \nu(x) = x + \tau \nu(x)$. Therefore, $N(x + \tau \nu(x)) = N(\gamma(\tau)) = \frac{d\gamma}{d\tau}(t) = \nu(x)$, i.e. (1) holds, which proves the uniqueness of a proper extension of $\nu$. The uniqueness of a solution of (2) is now immediate. Indeed, if $\Phi_1$ and $\Phi_2$ are solutions of (2), then $\Phi_1 - \Phi_2 = 0$ on $S$, and it follows from the uniqueness of a proper extension that $\nabla(\Phi_1 - \Phi_2) = \nabla \Phi_1 - \nabla \Phi_2 = 0$. So, $\Phi_1 - \Phi_2 = 0$ in $\Omega_S$.

Existence. Our aim here is to prove that the gradient of the function $\Phi_S$ defined by (3) is an extension we need and that (1) holds. Let $S_t, t \in (-\varepsilon, \varepsilon)$ be the $t$-level set of $\Phi_S$:

$$S_t = \{ y \in \Omega_S | \Phi_S(y) = t \} = \{ x + \tau \nu(x) | x \in S \}.$$ 

Let us show that $\nu(x)$ is normal to $S_t$ at the point $x + \tau \nu(x)$. Using the local coordinates (1) we see that any tangential to $S_t$ vector at $\Theta_j(p)$ is a linear combination of the vectors

$$\partial_p(\Theta_j(p) + \nu(\Theta_j(p))), \quad k = 1, \ldots, n - 1.$$ 

Taking the scalar product with $\nu(\Theta_j(p))$ we get

$$\langle \partial_p(\Theta_j(p) + \nu(\Theta_j(p))), \nu(\Theta_j(p)) \rangle$$

$$= \langle \partial_p \Theta_j(p), \nu(\Theta_j(p)) \rangle + t \langle \partial_p \nu(\Theta_j(p)), \nu(\Theta_j(p)) \rangle$$

$$= 0 + \frac{t}{2} \partial_p|\nu(\Theta_j(p))|^2 = \frac{t}{2} \partial_p 1 = 0, \quad k = 1, \ldots, n - 1.$$ 

Hence $\nu(x)$ is indeed normal to $S_t$ at the point $x + \tau \nu(x)$. Since $S_t$ is defined by the equation $\Phi_S(y) = t$, the gradient $\nabla \Phi_S(x + \tau \nu(x))$ is normal to $S_t$ at the point $x + \tau \nu(x)$. So, there exists $\rho = \rho(x) \in \mathbb{R}$ such that $(\nabla \Phi_S)(x + \tau \nu(x)) = \rho \nu(x)$, and it is easy to see that $\rho \geq 0$. It is left to prove that $\rho = 1$. Since all tangential derivatives of $\Phi_S$ on $S_t$ are equal to 0, we have

$$|\nabla \Phi_S(x + \tau \nu(x))| = |\partial_{\nu} \Phi_S(x + \tau \nu(x))|$$

$$= \lim_{h \to 0} \frac{\Phi_S(x + (t + h) \nu(x)) - \Phi_S(x + t \nu(x))}{h}$$

$$= \lim_{h \to 0} \frac{(t + h) - t}{h} = 1.$$ 

Hence $\rho = 1$, i.e. $(\nabla \Phi_S)(x + \tau \nu(x)) = \nu(x), x \in S$. \qed
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