Solvable Systems of Linear Differential Equations

Katherine M. Robertson and Nasser Saad
Department of Mathematics and Statistics, University of Prince Edward Island Charlottetown, Prince Edward Island C1A 4P3, Canada

Abstract: The asymptotic iteration method (AIM) is an iterative technique used to find exact and approximate solutions to second-order linear differential equations. In this work, we employed AIM to solve systems of two first-order linear differential equations. The termination criteria of AIM will be re-examined and the whole theory is re-worked in order to fit this new application. As a result of our investigation, an interesting connection between the solution of linear systems and the solution of Riccati equations is established. Further, new classes of exactly solvable systems of linear differential equations with variable coefficients are obtained. The method discussed allow to construct many solvable classes through a simple procedure.

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I. INTRODUCTION

There are a limited number of exactly solvable systems of differential equations that are known in the literature [1]-[2]. For example, the system of two constant-coefficient differential equations:

\begin{align}
\phi'_1 &= a\phi_1 + b\phi_2 \\
\phi'_2 &= c\phi_1 + d\phi_2
\end{align}

(1)

has a general solution depends on the roots of the characteristic equation [1]-[2]:

\[\lambda^2 - (a + d)\lambda + (ad - bc) = 0.\]

(2)

The purpose of the present work is two-fold: (1) to introduce a new technique based on the asymptotic iteration method [3]-[6] to analyze the exact solutions of linear systems

\begin{align}
\phi'_1 &= \lambda_0(x)\phi_1 + s_0(x)\phi_2, \\
\phi'_2 &= \omega_0(x)\phi_1 + \rho_0(x)\phi_2,
\end{align}

(3)

where \(\lambda_0(x), s_0(x), \omega_0(x)\) and \(\rho_0(x)\) are continuous and differentiable functions, and (2) to add new classes of exactly solvable systems to the known ones [1]-[2].

The asymptotic iteration method (AIM) is an iterative technique introduced originally [3] to find the exact and approximate solutions of second-order linear homogeneous differential equations [5]. Since then, the method has enjoyed a number of interesting applications, particularly, in relativistic and non-relativistic quantum mechanics [3]-[11]. In their study of Dirac equation, Hakan et al [4] modified AIM to study Dirac equation by analyzing a system of linear differential equations, however, their work was limited to this particular application. In the present work, we extend AIM to deal with different systems of first-order linear differential equations in general, and to give a detail structure useful in generating exactly solvable systems.

The paper is organized as follows: in next section, we re-examine AIM to deal with a system of linear differential equations. In section III, we used AIM to provide a simple recipe to find the exact solution of a system of two constant-coefficient differential equations [11]. In section IV, we give a necessary condition for polynomial solutions of linear systems of differential equations [3]. In section V, we establish an interesting connection between the exact solution of linear system [3] and the exact solution of a Riccati equation; namely, for every exactly solvable Riccati equation, there exist at least one system of linear differential equations that is exactly solvable. In section VI, some applications and tables of results are reported. The conclusion is given in section VII.
II. ITERATIVE SOLUTION OF LINEAR SYSTEMS

We consider a system given by (3), direct differentiation with respect to \( x \) yields

\[
\begin{align*}
\phi''_1 &= \lambda_1 \phi_1 + s_1 \phi_2 \\
\phi''_2 &= \omega_1 \phi_1 + \rho_1 \phi_2
\end{align*}
\]

where

\[
\begin{align*}
\lambda_1 &= \lambda'_0 + \lambda_0^2 + s_0 \omega_0 \\
s_1 &= s'_0 + \lambda_0 s_0 + s_0 \rho_0 \\
\omega_1 &= \omega'_0 + \omega_0 \lambda_0 + \rho_0 \omega_0 \\
\rho_1 &= \rho'_0 + \omega_0 s_0 + \rho_0^2
\end{align*}
\]

(5)

In general, the \( n^{th} \) and \((n+1)^{th}\) derivative of (4) are:

\[
\begin{align*}
\phi^{(n+1)}_1 &= \lambda_n \phi_1 + s_n \phi_2 \\
\phi^{(n+1)}_2 &= \omega_n \phi_1 + \rho_n \phi_2
\end{align*}
\]

and

\[
\begin{align*}
\phi^{(n+2)}_1 &= \lambda_{n+1} \phi_1 + s_{n+1} \phi_2 \\
\phi^{(n+2)}_2 &= \omega_{n+1} \phi_1 + \rho_{n+1} \phi_2
\end{align*}
\]

(7)

respectively, where \( \lambda_{n+1}, s_{n+1}, \omega_{n+1} \) and \( \rho_{n+1} \) are computed recursively by:

\[
\begin{align*}
\lambda_{n+1} &= \lambda'_n + \lambda_n \lambda_0 + s_n \omega_0, \\
s_{n+1} &= s'_n + \lambda_n s_0 + s_0 \rho_0, \\
\omega_{n+1} &= \omega'_n + \omega_n \lambda_0 + \rho_n \omega_0, \\
\rho_{n+1} &= \rho'_n + \omega_n s_0 + \rho_0 \rho_n.
\end{align*}
\]

(8)

The ratio \( \phi^{(n+2)}_1/\phi^{(n+1)}_1 \) then reads

\[
\frac{d}{dx} \ln(\phi^{(n+1)}_1) = \frac{\lambda_{n+1}(\phi_1 + \frac{s_{n+1}}{\lambda_{n+1}} \phi_2)}{\lambda_n(\phi_1 + \frac{s_n}{\lambda_n} \phi_2)}.
\]

(9)

If, for sufficiently large \( n > 0 \), we have

\[
\frac{s_n}{\lambda_n} = \frac{s_{n+1}}{\lambda_{n+1}} \equiv \alpha
\]

then (9) reads

\[
\frac{d}{dx} \ln(\phi^{(n+1)}_1) = \frac{\lambda_{n+1}}{\lambda_n}
\]

(11)

and consequently, we have

\[
\phi^{(n+1)}_1 = C_1 \exp \left( \int^x \frac{\lambda_{n+1}}{\lambda_n} dt \right) = C_1 \lambda_n \exp \left( \int^x (\alpha \omega_0 + \lambda_0) dt \right)
\]

(12)

where we used (8) and (10). Further, using (6), we have

\[
\phi_1 + \alpha \phi_2 = C_1 \exp \left( \int^x (\alpha \omega_0 + \lambda_0) dt \right)
\]

(13)

which can be solve for \( \phi_1 \) to yields, using (3), the first-order differential equation of \( \phi_2 \)

\[
\phi'_2 + \left( \omega_0 \alpha - \rho_0 \right) \phi_2 = \omega_0 C_1 \exp \left( \int (\omega_0 \alpha + \lambda_0) dx \right).
\]

(14)
The exact solution of this first-order differential equation is easily obtained; namely

\[ \phi_2 = \exp \left( \int^\tau (\rho_0 - \omega_0 \alpha) \, dt \right) \left[ C_2 + C_1 \int^\tau \left( \omega_0 \exp \left( \int^\tau \left( \lambda_0 - \rho_0 + 2\omega_0 \alpha \right) \, d\tau \right) \right) \, dt \right] \]  

(15)

where \( C_1 \) and \( C_2 \) are the constant of integration. Consequently, we have

\[ \phi_1 = C_1 \exp \left( \int^\tau (\alpha \omega_0 + \lambda_0) \, dt \right) - \alpha \phi_2. \]  

(16)

Equivalently, if we consider instead the ratio \( \phi_2^{(n+2)} / \phi_2^{(n+1)} \) and for sufficiently large \( n > 0 \), we have

\[ \frac{\omega_{n+1}}{\rho_{n+1}} = \frac{\omega_n}{\rho_n} \equiv \beta \]  

(17)

that

\[ \phi_1 = \exp \left( \int (\lambda_0 - s_0 \beta) \, dx \right) \left[ C'_2 + C'_1 \int \left( s_0 \exp \left( \int (\rho_0 - \lambda_0 + 2s_0 \beta) \, dx \right) \right) \, dx \right] \]  

(18)

and

\[ \phi_2 = -\beta \phi_1 + C'_1 \exp \left( \int (\beta s_0 + \rho_0) \, dx \right) \]  

(19)

where again \( C'_1 \) and \( C'_2 \) are constant. The above discussion establish the following theorem:

**Theorem 1** The general solution of the system:

\[ \begin{align*}
\phi'_1 &= \lambda_0(x) \phi_1 + s_0(x) \phi_2 \\
\phi'_2 &= \omega_0(x) \phi_1 + \rho_0(x) \phi_2
\end{align*} \]  

(20)

is given by:

\[ \begin{align*}
\phi_2(x) &= \exp \left( \int^x (\rho_0(t) - \alpha \omega_0(t)) \, dt \right) \left[ C'_2 + C'_1 \int^x \omega_0(t) \exp \left( \int^t \left( \lambda_0(\tau) - \rho_0(\tau) + 2\omega_0(\tau) \alpha \right) \, d\tau \right) \, dt \right] \\
\phi_1(x) &= C'_1 \exp \left( \int^x (\alpha \omega_0(t) + \lambda_0(t)) \, dt \right) - \alpha \phi_2(x)
\end{align*} \]  

(21)

if for sufficiently large \( n \geq 0 \)

\[ \alpha \equiv \frac{s_n}{\lambda_n} = \frac{s_{n+1}}{\lambda_{n+1}} \]  

(22)

Equivalently, if for some \( n \geq 0 \),

\[ \beta \equiv \frac{\omega_n}{\rho_n} = \frac{\omega_{n+1}}{\rho_{n+1}} \]  

(23)

then the general solution is given by:

\[ \begin{align*}
\phi_1(x) &= \exp \left( \int^x (\lambda_0(t) - \beta s_0(t)) \, dt \right) \left[ C'_2 + C'_1 \int^x \left( s_0(t) \exp \left( \int^t \left( \rho_0(\tau) - \lambda_0(\tau) + 2s_0(\tau) \beta \right) \, d\tau \right) \right) \, dt \right], \\
\phi_2(x) &= -\beta \phi_1(x) + C'_1 \exp \left( \int^x (\beta s_0(t) + \rho_0(t)) \, dt \right)
\end{align*} \]  

(24)

where

\[ \begin{align*}
\lambda_{n+1} &= \lambda'_n + \lambda_n \lambda_0 + s_n \omega_0, \\
s_{n+1} &= s'_n + \lambda_n s_0 + s_n \rho_0, \\
\omega_{n+1} &= \omega'_n + \omega_n \lambda_0 + \rho_n \omega_0, \\
\rho_{n+1} &= \rho'_n + \omega_n s_0 + \rho_n \rho_0.
\end{align*} \]  

(25)
TABLE I: Exactly solvable systems \( \phi_1' = \lambda_0(x)\phi_1 + s_0(x)\phi_2, \) and \( \phi_2' = \omega_0(x)\phi_1 + \rho_0(x)\phi_2, \) for \( \lambda_0(x), s_0(x), \omega_0(x) \) and \( \rho_0(x) \) arbitrary functions. Here, \( a, b, c, \) and \( d \) are constants and \( n = 1, 2, \ldots \)

| \( \lambda_0(x) \) | \( s_0(x) \) | \( \omega_0(x) \) | \( \rho_0(x) \) | Solution |
|-----------------|------------|----------------|------------|----------|
| \( \frac{bc-(n-1)^2+(n-1)d}{(d-n+1)x} \) | \( \frac{a}{x} \) | \( \frac{\omega_0}{\rho} \) | \( \frac{\rho_0}{\rho} \) | \( \phi_1 = C_1 \left[ \frac{bc}{bc+(1+d-n)^2} \right] x^{d-1} + C_2 \left[ \frac{n-d-1}{c} \right] x^{n-1} \) |
| \( \frac{a}{x} \) | \( \frac{\omega_0}{\rho} \) | \( \frac{\rho_0}{\rho} \) | \( \phi_2 = C_1 \left[ \frac{c(1+d-n)}{bc+(1+d-n)^2} \right] x^{d-1} + C_2 x^{n-1} \) |

Note that (22) and (23) can be written in more convenient way as

\[
\frac{s_{n+1}}{\lambda_{n+1}} = \frac{s_n}{\lambda_n} \iff \delta_{n+1} = \lambda_{n+1}s_n - \lambda_n s_{n+1} = 0, \quad (26)
\]

and

\[
\frac{\omega_{n+1}}{\rho_{n+1}} = \frac{\omega_n}{\rho_n} \iff \Delta_{n+1} = \omega_{n+1}\rho_n - \omega_n\rho_{n+1} = 0. \quad (27)
\]

**Example 1:** Consider the following linear system of differential equations

\[
\begin{align*}
\phi_1' &= \frac{a}{x}\phi_1 + \frac{b}{x}\phi_2 \\
\phi_2' &= \frac{c}{x}\phi_2 + \frac{d}{x}\phi_2
\end{align*}
\]

where \( a, b, c, \) and \( d \) are arbitrary constants with \( ad - bc \neq 0 \). It is straightforward, using (26), to show that

\[
\delta_n = -\frac{b}{x^{2n+1}} \prod_{m=0}^{n-1} (m^2 - m(a + d) + ad - bc), \quad n = 1, 2, \ldots
\]

Clearly, \( \delta_n = 0 \), if \( (n-1)^2 - (n-1)(a+d) + ad - bc = 0 \) or \( a = \frac{bc-(n-1)^2+(n-1)d}{d-n+1} \). Furthermore, we have, using (22), \( \alpha = \frac{d-(n-1)}{c} \) and consequently the system, for \( n = 1, 2, \ldots \)

\[
\begin{align*}
\phi_1' &= \left[ \frac{bc-(n-1)^2+(n-1)d}{(d-n+1)x} \right] \phi_1 + \frac{b}{x}\phi_2 \\
\phi_2' &= \frac{c}{x}\phi_2 + \frac{d}{x}\phi_2
\end{align*}
\]

has the general solution

\[
\begin{align*}
\phi_1 &= C_1 \left[ \frac{bc}{bc+(1+d-n)^2} \right] x^{d-1} + C_2 \left[ \frac{n-d-1}{c} \right] x^{n-1} \\
\phi_2 &= C_1 \left[ \frac{c(1+d-n)}{bc+(1+d-n)^2} \right] x^{d-1} + C_2 x^{n-1}
\end{align*}
\]

for \( n = 1, 2, \ldots \). Similar cases are reported in Table 1. Further examples of exactly solvable systems are reported in Table II.
TABLE II: Exactly solvable systems \([1]\) for different \(\lambda_0(x), s_0(x), \omega_0(x)\) and \(\rho_0(x)\) using Theorem 1 where \(a, b, c, d, \) and \(a_i, i = 1, \ldots, 4\) are arbitrary real numbers. Here \(n = 1, 2, \ldots\) and \(E_a[z]\) is the exponential integral function \(E_a[z] = \int_1^\infty t^{-a}e^{-zt}dt\).

| \(a = \frac{d x+n-1}{x}\) | \(\frac{d}{x}\) | \(\frac{d}{x}\) | \(\frac{d}{x}\) | \(\frac{d}{x}\) | Solution |
|---|---|---|---|---|---|
| \(a + \frac{b}{x}\) | \(\frac{b-n+1}{d x}\) | \(\frac{b-n+1}{d x}\) | \(\frac{b-n+1}{d x}\) | \(\frac{b-n+1}{d x}\) | \(\phi_1 = C_1 \left[ \frac{x^n + \rho x^n - c dx x^{n-1} \int (cd+ax^2) dx}{(cd+ax^2) dx} \right] - C_2 e^{\rho x^n}, \)  
\(\phi_2 = (cd+ax^2) dx + C_2).\)  
\(\phi_1 = C_1 \left[ \frac{x^n + \rho x^n - c dx x^{n-1} \int (cd+ax^2) dx}{(cd+ax^2) dx} \right] - C_2 e^{\rho x^n}, \)  
\(\phi_2 = (cd+ax^2) dx + C_2).\)  
\(\phi_1 = C_1 \left[ \frac{x^n + \rho x^n - c dx x^{n-1} \int (cd+ax^2) dx}{(cd+ax^2) dx} \right] - C_2 e^{\rho x^n}, \)  
\(\phi_2 = (cd+ax^2) dx + C_2).\)  
\(\phi_1 = C_1 \left[ \frac{x^n + \rho x^n - c dx x^{n-1} \int (cd+ax^2) dx}{(cd+ax^2) dx} \right] - C_2 e^{\rho x^n}, \)  
\(\phi_2 = (cd+ax^2) dx + C_2).\)  
\(\phi_1 = C_1 \left[ \frac{x^n + \rho x^n - c dx x^{n-1} \int (cd+ax^2) dx}{(cd+ax^2) dx} \right] - C_2 e^{\rho x^n}, \)  
\(\phi_2 = (cd+ax^2) dx + C_2).\) |

### III. SOLUTION OF LINEAR CONSTANT-COEFFICIENT SYSTEMS

**Theorem 2** The constant-coefficient first-order linear system

\[ \begin{align*}  
\dot{\phi}_1' &= \lambda_0 \phi_1 + s_0 \phi_2 \\
\dot{\phi}_2' &= \omega_0 \phi_1 + \rho_0 \phi_2  
\end{align*} \]

where \(\lambda_0, s_0, \omega_0\) and \(\rho_0\) are constants, has the general solution

\[ \begin{align*}  
\phi_1 &= C_1 \left[ 1 - \frac{\omega_0 \alpha}{\lambda_0 - \rho_0 + 2 \omega_0 \alpha} \right] e^{(\lambda_0 + \omega_0) x} - \alpha C_2 e^{(\rho_0 - \omega_0) x}  \\
\phi_2 &= C_1 \frac{\omega_0}{\lambda_0 - \rho_0 + 2 \omega_0 \alpha} e^{(\lambda_0 + \omega_0) x} + C_2 e^{(\rho_0 - \omega_0) x}  
\end{align*} \]

where \(\alpha\) is given by:

\[ \alpha = \frac{(\rho_0 - \lambda_0) \pm \sqrt{(\lambda_0 - \rho_0)^2 + 4 \omega_0 s_0}}{2 \omega_0} \quad \text{or} \quad \alpha = \frac{(\rho_0 - \lambda_0) - \sqrt{(\lambda_0 - \rho_0)^2 + 4 \omega_0 s_0}}{2 \omega_0}. \]

**Proof:** From the AIM sequence \([24]\), we have for the constant-coefficient linear system that

\[ \begin{align*}  
\lambda_{n+1} &= \lambda_n \lambda_0 + s_n \omega_0, \\
s_{n+1} &= \lambda_n s_0 + s_n \rho_0, \\
\omega_{n+1} &= \omega_n \lambda_0 + \rho_n \omega_0, \\
\rho_{n+1} &= \omega_n s_0 + \rho_n \rho_0  
\end{align*} \]

Consequently, the ratio \([22]\) now reads

\[ \alpha = \frac{s_0 + \alpha \rho_0}{\lambda_0 + \alpha \omega_0}. \]

which yield the quadratic equation

\[ \omega_0 \alpha^2 + (\lambda_0 - \rho_0) \alpha - s_0 = 0 \]

with solutions given by \(\alpha = \frac{(\rho_0 - \lambda_0) \pm \sqrt{(\lambda_0 - \rho_0)^2 + 4 \omega_0 s_0}}{2 \omega_0}\). The exact solutions \([33]\), then, follows immediately from \([21]\). \(\square\)
Example 2: Consider the following elementary system:

\[
\begin{align*}
\phi'_1 &= \phi_1 + 2\phi_2 \\
\phi'_2 &= 3\phi_1 + 2\phi_2 
\end{align*}
\]  
(37)

we have, using (34), that \( \alpha = 1, -\frac{2}{3} \). Thus, for \( \alpha = 1 \), the exact solution of the given system is given by

\[
\begin{align*}
\phi_1 &= \frac{2}{5}C_1e^{4x} - C_2e^{-x} \\
\phi_2 &= \frac{3}{5}C_1e^{4x} + C_2e^{-x} 
\end{align*}
\]  
(38)

while for \( \alpha = -\frac{2}{3} \), we obtain the same solution up to a constant.

Example 3: Consider the following system

\[
\begin{align*}
\phi'_1 &= 6\phi_1 - \phi_2 \\
\phi'_2 &= 5\phi_1 + 4\phi_2 
\end{align*}
\]  
(39)

we have, using (34) that \( \alpha = -\frac{1+2\sqrt{-1}}{3} \) where \( i = \sqrt{-1} \). The general solution, using (33) is given by

\[
\begin{align*}
\phi_1 &= \left( \frac{1-2i}{4} \right)C_1e^{(5+2i)x} + C_2\left( \frac{1-2i}{5} \right)e^{(5-2i)x} \\
\phi_2 &= -\frac{5i}{4}C_1e^{(5+2i)x} + C_2e^{(5-2i)x} 
\end{align*}
\]  
(40)

which is again agree (up to a constant) with the exact solution obtain by the standard method.

IV. A CRITERION FOR POLYNOMIAL SOLUTIONS

In Ref. [4], Saad et al give a sufficient and necessary condition for a second-order linear homogeneous differential equation to have a polynomial solution. Although a similar criterion for the existence of a polynomial solution of a linear system such as (3) is not possible in general, the following theorem gives a necessary condition.

Theorem 3 If \( \phi_1(x) \) and \( \phi_2(x) \) are \( n \)-degree polynomial solutions of a linear system with variable coefficients (3), then

\[
\eta_n(x) \equiv s_n(x)\omega_n(x) - \lambda_n(x)\rho_n(x) = 0, \quad n = 1, 2, \ldots
\]

(41)

where \( \lambda_n, s_n, \omega_n \) and \( \rho_n \) are given recursively by (25).

Proof: The existence of nontrivial solutions \( \phi_1 \) and \( \phi_2 \) of (6) required the vanishing of the determinant

\[
\left| \begin{array}{cc}
\lambda_n(x) & s_n(x) \\
\omega_n(x) & \rho_n(x)
\end{array} \right| = 0.
\]

(42)

Note that the converse of this theorem is not true in general, for example, consider the system \( \phi'_1 = 5x(\phi_1 + \phi_2), \phi'_2 = 3x(\phi_1 + \phi_2) \), clearly \( \eta_1(x) = 0 \), however the system has a non-polynomial general solution given by \( \phi_1 = \frac{5}{6}C_1e^{4x^2} - C_2 \), and \( \phi_2 = \frac{1}{5}C_1e^{4x^2} + C_2 \).

V. ELEMENTARY SYSTEMS OF DIFFERENTIAL EQUATIONS

By means of the iteration sequence (25), it is interesting to note that the termination condition (22) can be written, equivalently, as Riccati equation,

\[
\left( \frac{\lambda_n}{s_n} \right)' + (\lambda_0 - \rho_0) \left( \frac{\lambda_n}{s_n} \right) - s_0 \left( \frac{\lambda_n}{s_n} \right)^2 = -w_0
\]

(43)
where the solution of this equation yields the exact analytic expression of the ratio \( \alpha \equiv \frac{\omega}{\lambda} \) by which the exact solution of the corresponding system follows immediately using (41). Thus, we have a useful connection between the exact solution of a linear-system of differential equations and the exact solution of a Riccati equation; namely,

**Theorem 4** The general solution of the linear system:

\[
\begin{align*}
\phi_1' &= \lambda_0(x)\phi_1 + s_0(x)\phi_2 \\
\phi_2' &= \omega_0(x)\phi_1 + \rho_0(x)\phi_2
\end{align*}
\]

is given by:

\[
\begin{align*}
\phi_2(x) &= \exp \left( \int^x \left( \rho_0(t) - \alpha\omega_0(t) \right) dt \right) \left[ C_2 + C_1 \int^x \omega_0(t) \exp \left( \int^t \left( \lambda_0(\tau) - \rho_0(\tau) + 2\omega_0(\tau)\alpha \right) d\tau \right) dt \right] \\
\phi_1(x) &= C_1 \exp \left( \int^x \left( \alpha\omega_0(t) + \lambda_0(t) \right) dt \right) - \alpha\phi_2(x)
\end{align*}
\]

where \( \alpha \equiv \alpha(x) \) is the solution of the Riccati equation

\[
\frac{d\alpha}{dx} = w_0\alpha^2 + (\lambda_0 - \rho_0)\alpha - s_0.
\]

Clearly, for linear constant-coefficient systems, (44) is equivalent to (41) because \( \frac{d\alpha}{dx} = 0 \). Recently, Saad et al. [6] studied the exact solutions of certain classes of such Riccati equations which allow us to obtain exact solutions to many systems of linear differential equations. We, first, consider the simpler case, namely, the first iteration using Theorem 1, we obtain:

**Theorem 5** For the following system of differential equations:

\[
\begin{align*}
\phi_1' &= \lambda_0(x)\phi_1 + s_0(x)\phi_2, \\
\phi_2' &= \omega_0(x)\phi_1 + \rho_0(x)\phi_2,
\end{align*}
\]

if the functions \( \lambda_0, s_0, \omega_0, \) and \( \rho_0 \) satisfy:

\[
\left( \frac{\lambda_0}{s_0} \right)' - \left( \frac{\lambda_0}{s_0} \right) \rho_0 = -\omega_0
\]

then the general solution to the system is given by:

\[
\begin{align*}
\phi_2 &= \frac{\lambda_0(x)}{s_0(x)} \left[ C_1 \int^x \frac{w_0(t)s_0(t)}{\lambda_0(t)} \exp \left( \int^t \left( \lambda_0(\tau) + \frac{s_0(\tau)w_0(\tau)}{\lambda_0(\tau)} \right) d\tau \right) dt + C_2 \right] \\
\phi_1 &= C_1 \exp \left( \int^x \left( \lambda_0(t) + \frac{s_0(t)w_0(t)}{\lambda_0(t)} \right) dt \right) - \frac{s_0(x)}{\lambda_0(x)} \phi_2
\end{align*}
\]

Proof: Equation (46) can be written as

\[
\frac{d}{dx} \left( -\frac{1}{\alpha} \right) = w_0 + \frac{(\lambda_0 - \rho_0)}{\alpha} - \frac{s_0}{\alpha^2}
\]

For \( n = 0, \alpha \equiv \frac{\omega}{\lambda_0} \), and (50) reduces to (48).

As direct examples of Theorem 5, we have the following two corollaries.

**Corollary 1** The following system of differential equations:

\[
\begin{align*}
\phi_1' &= f(x)(\phi_1 + \phi_2) \\
\phi_2' &= g(x)(\phi_1 + \phi_2)
\end{align*}
\]

where \( f(x) \) and \( g(x) \) are arbitrary functions, has the general solution given by:

\[
\begin{align*}
\phi_1 &= C_1 \left[ \exp \left( \int^x (f(t) + g(t)) dt \right) - \int^x \left( g(t) \exp \left( \int^t (f(\tau) + g(\tau)) d\tau \right) \right) dt \right] - C_2, \\
\phi_2 &= C_1 \int^x \left( g(t) \exp \left( \int^t (f(\tau) + g(\tau)) d\tau \right) \right) dt + C_2.
\end{align*}
\]
TABLE III: Exact solutions of the system $\phi_1' = \lambda_0(x)\phi_1 + s_0(x)\phi_2$, $\phi_2' = w_0(x)\phi_1 + \rho_0(x)\phi_2$ for certain functions $\lambda_0(x), s_0(x), w_0(x),$ and $\rho_0(x)$.

| $\lambda_0(x)$ | $s_0(x)$ | $\omega_0(x)$ | $\rho_0(x)$ | Solution |
|----------------|----------|----------------|-------------|----------|
| $f(x)$         | $g(x)$   | $g(x)$         | $f(x)$      | $\phi_1 = C_1e^{f(t)\alpha(t)+f(t)dt} - \alpha(x)\phi_2$, $\phi_2 = e^{f(t)\alpha(t)+f(t)dt}[C_2 + C_1\int^x g(t)e^{f(t)\alpha(t)\rho(t)\alpha(t)\rho(t)dt}]$, where $\alpha(x) = \tanh(-\int^x g(t)dt)$. |
| $f(x)$         | $ag(x)$  | $bg(x)$        | $f(x)$      | $\phi_1 = C_1e^{f(t)\alpha(t)+f(t)dt} - \alpha(x)\phi_2$, $\phi_2 = e^{f(t)\alpha(t)+f(t)dt}[C_2 + C_1\int^x g(t)e^{f(t)(2\alpha(t)\rho(t)\alpha(t)\rho(t)dt}]$, where $\alpha(x) = -\sqrt[4]{\tanh(\sqrt{ab} \int^x g(t)dt)}$. |
| $f(x)$         | $-g(x)$  | $g(x)$         | $f(x)$      | $\phi_1 = C_1e^{f(t)\alpha(t)+f(t)dt} - \alpha(x)\phi_2$, $\phi_2 = e^{f(t)\alpha(t)+f(t)dt}[C_2 + C_1\int^x g(t)e^{f(t)(2\alpha(t)\rho(t)\alpha(t)\rho(t)dt}]$, where $\alpha(x) = \tanh(\int^x g(t)dt)$. |
| $f(x)$         | $-ag(x)$ | $bg(x)$        | $f(x)$      | $\phi_1 = C_1e^{f(t)\alpha(t)+f(t)dt} - \alpha(x)\phi_2$, $\phi_2 = e^{f(t)\alpha(t)+f(t)dt}[C_2 + C_1\int^x g(t)e^{f(t)(2\alpha(t)\rho(t)\alpha(t)\rho(t)dt}]$, where $\alpha(x) = \sqrt[4]{\tanh(\sqrt{ab} \int^x g(t)dt)}$. |

Corollary 2 The following system of differential equations:

$$\begin{align*}
\phi_1' &= f(x) (\phi_1 - \phi_2) \\
\phi_2' &= g(x) (\phi_1 - \phi_2) (53)
\end{align*}$$

where $f(x)$ and $g(x)$ are arbitrary functions, has the general solution:

$$\begin{align*}
\phi_1 &= C_1 \left[ \exp \left( \int^x (f(t) - g(t)) dt \right) + \int^x g(t) \exp \left( \int^t (f(\tau) - g(\tau)) d\tau \right) \right] - C_2 \\
\phi_2 &= C_1 \int^x g(t) \exp \left( \int^t (f(\tau) - g(\tau)) d\tau \right) dt - C_2 (54)
\end{align*}$$

Using theorem (4), we can now prove the following.

Theorem 6 The system of differential equations:

$$\begin{align*}
\phi_1' &= \lambda_0(x)\phi_1 + s_0(x)\phi_2, \\
\phi_2' &= w_0(x)\phi_1 + \rho_0(x)\phi_2, (55)
\end{align*}$$

where the functions $\lambda_0$, $s_0$, and $\rho_0$ are assumed to be continuous and differentiable functions, is analytically solvable if the quantity $(\rho(x) - \lambda_0(x))/s_0(x)$ is independent of $x$.

Proof: If the quantity $(\rho(x) - \lambda_0(x))/s_0(x)$ is independent of $x$, then the Riccati equation [46] becomes a separable equation.

Theorem 7 The system of differential equations:

$$\begin{align*}
\phi_1' &= \lambda_0(x)\phi_1 + s_0(x)\phi_2, \\
\phi_2' &= w_0(x)\phi_1 + \rho_0(x)\phi_2, (56)
\end{align*}$$

where the functions $\lambda_0$, $s_0$, $w_0$ and $\rho_0$ are assumed to be continuous and differentiable functions, is analytically solvable if the quantities $w_0(x)/(\rho(x) - \lambda_0(x))$ and $s_0(x)/(\rho(x) - \lambda_0(x))$ are independent of $x$.
TABLE IV: Exact expressions of the ratio \( \alpha \equiv \frac{s_0}{\lambda_0} \), \( n = 1, 2, \ldots \) for the system \( \phi' = \lambda_0(x)\phi_1 + s_0(x)\phi_2 \), \( \phi'' = u_0(x)\phi_1 + \rho_0(x)\phi_2 \), for different \( \lambda_0(x), s_0(x), \omega_0(x) \) and \( \rho_0(x) \). The exact solutions of the system then follows by direct substitution of \( \alpha \) in (44). Here \( R(x) \) is an arbitrary differentiable function.

| \( \lambda_0(x) \) | \( s_0(x) \) | \( \omega_0(x) \) | \( \rho_0(x) \) | \( \alpha \equiv \frac{s_0}{\lambda_0} \) |
|------------------|------------------|------------------|------------------|------------------|
| \( R(x) \)       | \( \frac{y_0}{\lambda_0} \) | \( \frac{y_0}{\lambda_0} \) | \( \frac{y_0}{\lambda_0} \) | \( \frac{y_0}{\lambda_0} \) |
| \( R(x) \)       | \( \frac{y_0}{\lambda_0} \) | \( \frac{y_0}{\lambda_0} \) | \( \frac{y_0}{\lambda_0} \) | \( \frac{y_0}{\lambda_0} \) |

Proof: If the quantities \( u_0(x)/(\rho(x) - \lambda_0(x)) \) and \( s_0(x)/(\rho(x) - \lambda_0(x)) \) are independent of \( x \), then the Riccati equation (40) becomes a separable equation.

VI. SOLVABLE LINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS

For higher iteration levels, \( n = 1, 2, \ldots \), the results of the earlier work of Saad et al. [6] on Riccati equation [12]-[14] can be used to obtain exact analytical solutions to different classes of the linear system [6]. Under a certain conditions on the functions \( \lambda_0, s_0, u_0 \) and \( \rho_0 \), we may express [15] some of these exact solutions in terms of generalized hypergeometric functions \( qF_p(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; x) \)

\[
pFq(\alpha_1, \alpha_2, \ldots, \alpha_p; \beta_1, \beta_2, \ldots, \beta_q; x) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k(\alpha_2)_k \ldots (\alpha_p)_k}{(\beta_1)_k(\beta_2)_k \ldots (\beta_q)_k} \frac{x^k}{k!}
\]

where \( p \) and \( q \) are nonnegative integers and no \( \beta_k, k = 1, 2, \ldots, q \) is zero or a negative integer. Clearly, (57) includes the special cases of the confluent hypergeometric function \( _1F_1 \) and the classical ‘Gaussian’ hypergeometric function \( _2F_1 \). The Pochhammer symbol \( (\alpha)_k \) is defined in terms of Gamma function as

\[
(\alpha)_k = \frac{\Gamma(\alpha + k)}{\Gamma(\alpha)} \quad k = 0, 1, 2, \ldots
\]

If \( \alpha \) is a negative integer \( -n \), we have

\[
(\alpha)_k = \begin{cases} 
(-1)^k \frac{n!}{(n-k)!} & 0 \leq k \leq n \\
0 & k > n 
\end{cases}
\]

in which case, the generalized hypergeometric series reduces to a polynomial of degree \( n \) in its variable \( x \). In Table IV, we report the exact expression of \( \alpha \) for a given system, by which we can easily obtain the exact solutions through a direct substitution of \( \alpha \) in (21).

**Theorem 8** If functions \( F_0 \equiv F_0(x) \) and \( G_0 \equiv G_0(x) \) satisfy the recursive relation:

\[
\delta_n = F_nG_{n-1} - F_{n-1}G_n = 0, \quad \text{for some} \quad n = 1, 2, \ldots
\]
where
\[
\begin{align*}
F_n &= F'_{n-1} + G_{n-1} + F_0 F_{n-1} , \\
G_n &= G'_{n-1} + G_0 F_{n-1} , \\
\end{align*}
\] (61)

then the linear system of differential equations:
\[
\begin{align*}
\phi'_1 &= \lambda_0 \phi_1 + G_0 \exp \left( - \int^x (F_0 + \rho_0 - \lambda_0) dt \right) \phi_2 \\
\phi'_2 &= \exp \left( \int^x (F_0 + \rho_0 - \lambda_0) dt \right) \phi_1 + \rho_0 \phi_2
\end{align*}
\] (62)

has a general solution given by (45) with
\[
\alpha = \frac{G_{n-1}}{F_{n-1}} \exp \left( - \int (F_0 + \rho_0 - \lambda_0) dx \right) ,
\] (63)

where \( \lambda_0 \equiv \lambda_0(x) \) and \( s_0 \equiv s_0(x) \) are arbitrary functions.

Proof: The substitution \( \alpha = - \frac{u'}{w_0 u} \) transfer the Riccati equation (46) into the second-order differential equation
\[
u'' = \left[ \frac{u'}{w_0} + \lambda_0 - \rho_0 \right] u' + s_0 w_0 u
\] (64)

Using the standard theorem of the asymptotic iteration method \cite{3}, the solution of (64) satisfies \cite{5}
\[
\frac{u'}{u} = - \frac{G_{n-1}}{F_{n-1}}
\] (65)

if, for some \( n > 0 \),
\[
\delta_n \equiv F_n G_{n-1} - F_{n-1} G_n = 0,
\]

where \( F_n \) and \( G_n \) are computed using the recursive relations
\[
\begin{align*}
F_n &= F'_{n-1} + G_{n-1} + F_0 F_{n-1} , \\
G_n &= G'_{n-1} + G_0 F_{n-1} .
\end{align*}
\]

Thus, the general solution of (68), using Theorem 4, is given by (45) with
\[
\alpha = \frac{G_{n-1}}{w_0 F_{n-1}} ,
\] (66)

which complete the prove of the theorem.

\[\square\]

**Example 4:** Let \( F_0 = 2x \) and \( G_0 = -2n , \quad n = 1, 2, \ldots \) Using (61), see also \cite{3}, we know that \( \delta_n = F_n G_{n-1} - F_{n-1} G_n = 0 \) for \( n = 1, 2, \ldots \) with
\[
\frac{G_{n-1}}{F_{n-1}} = - \frac{1}{H_n(x)} \frac{dH_n(x)}{dx}
\] (67)

where \( H_n(x) \) is the well-known Hermite polynomials. Thus, for \( \rho_0 \equiv \rho_0(x) \) and \( \lambda_0 \equiv \lambda_0(x) \) arbitrary functions, the system
\[
\begin{align*}
\phi'_1 &= \lambda_0 \phi_1 - 2n \exp \left( - \int^x (2x + \rho_0 - \lambda_0) dt \right) \phi_2 \\
\phi'_2 &= \exp \left( \int^x (2x + \rho_0 - \lambda_0) dt \right) \phi_1 + \rho_0 \phi_2
\end{align*}
\] (68)
TABLE V: Exact expressions of the ratio \( \alpha \equiv \frac{G_{n+1}}{w_0(x)\lambda_{n+1}}, \) \( n = 1, 2, \ldots \) for the system \( \phi'_1 = \lambda_0(x)\phi_1 + s_0(x)\phi_2, \) \( \phi'_2 = w_0(x)\phi_1 + \rho_0(x)\phi_2, \) for arbitrary functions \( \lambda_0(x) \) and \( \rho_0(x) \). The exact solutions of the system then follow by direct substitution of \( \alpha \) in (45).

\[
\begin{align*}
\phi_2(x) &= H_n(x)e^{\int_0^x \rho_0(t)dt} \left[ C_2 + C_1 \int_0^x e^{t^2 \frac{e^{t^2}}{H_n(t)^2}} dt \right], \\
\phi_1(x) &= \frac{C_1}{H_n(x)} e^{\int_0^x \lambda_0(t)dt} - \alpha \phi_2(x). \tag{69}
\end{align*}
\]

In Table V, we gave the exact expressions of \( \alpha \equiv \frac{G_{n+1}}{w_0(x)\lambda_{n+1}}, \) \( n = 1, 2, \ldots \) for different systems. The general solutions can be found by direct substitution of \( \alpha \) in (45).

Theorem 9 If functions \( F_0 = F_0(x) \) and \( G_0 = G_0(x) \) satisfy the recursive relation:

\[
\delta_n \equiv F_n G_{n-1} - F_{n-1} G_n = 0, \quad \text{for some} \quad n = 1, 2, \ldots \tag{70}
\]

where

\[
\begin{align*}
F_n &= F_n' + G_{n-1} - F_0 F_{n-1}, \\
G_n &= G_n' + G_0 F_n - F_0. \tag{71}
\end{align*}
\]

then the linear system of differential equations:

\[
\begin{align*}
\phi'_1 &= \lambda_0(x)\phi_1 + \exp \left( \int_0^x (F_0 - \rho_0 + \lambda_0)dx \right) \phi_2, \\
\phi'_2 &= G_0(x) \exp \left( \int_0^x -(F_0 - \rho_0 + \lambda_0)dx \right) \phi_1 + \rho_0 \phi_2. \tag{72}
\end{align*}
\]
has a general solution given by (72) with $s_0 = \exp \left( \int^x (F_0 - \rho_0 + \lambda_0)dx \right)$, $w_0 = \exp \left( - \int^x (F_0 - \rho_0 + \lambda_0)dx \right)$, and

$$\alpha = \frac{F_{n-1}}{G_{n-1}} \exp \left( \int (F_0 + \rho_0 - \lambda_0)dx \right),$$

(73)

where $\lambda_0 \equiv \lambda_0(x)$ and $s_0 \equiv s_0(x)$ are arbitrary functions.

Proof: The proof of this theorem follows similarly to the proof of Theorem 6 using the substitution $\alpha = -\frac{s_0}{\rho_0}$. □

In Table VI, we gave the exact expressions of $\alpha = \frac{s_0 F_{n-1}}{G_{n-1}}$, $n = 1, 2, \ldots$ for different systems. The general solutions can be found by direct substitution of $\alpha$ in (73).

VII. CONCLUSION

Guided by the recent applications of the asymptotic iteration method on Dirac equation and Riccati equation [3, 4], we extend AIM to solve analytically different systems of linear differential equations with variable coefficients. This allow us to provide a simple recipe for solving any linear constant-coefficient systems in few steps. Further, we have extended the standard list of exactly solvable system known in the literature (see [1], Chapter 7, Section 7.6.1.1, Page 1229). Furthermore, the work presented here established a connection between the solution of arbitrary linear system of differential equation and the solution of Riccati equation. Indeed, one of the main results in the present work, is to have this connection written explicitly in terms of a theorem; namely Theorem 4. Another interesting conclusion of the present work is that it provides a set of conditions on $\lambda_0$, $s_0$, $w_0$, $\rho_0$ which govern the complete solvability of the system [3]. It should be clear the results of our investigation are not limited to the exact solutions reported in tables I-VI but it can easily extended to many other solvable systems. General speaking, the results obtained here are useful in the sense that they can set up a more rational base to approach to solve coupled differential equations...
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