A recursive construction of projective cubature formulas and related isometric embeddings

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Abstract. A recursive construction is presented for the projective cubature formulas of index \( p \) on the unit spheres \( S(m, K) \subset K^m \) where \( K \) is \( \mathbb{R} \) or \( \mathbb{C} \), or \( \mathbb{H} \). This yields a lot of new upper bounds for the minimal number of nodes \( n = N_{K}(m, p) \) in such formulas or, equivalently, for the minimal \( n \) such that there exists an isometric embedding \( \ell^m_{2,K} \rightarrow \ell^m_{p,K} \).

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1 Introduction and overview

Let \( K \) be one of three classical fields: \( \mathbb{R} \) (real), \( \mathbb{C} \) (complex), \( \mathbb{H} \) (quaternion). Its real dimension is

\[
\delta = \delta(K) = \begin{cases} 
1 & (K = \mathbb{R}) \\
2 & (K = \mathbb{C}) \\
4 & (K = \mathbb{H}). 
\end{cases}
\]

We consider the right \( K \)-linear space \( K^m \) consisting of the columns \( x = [\xi_i]^m_1, \xi_i \in K, 1 \leq i \leq m \). This becomes an Euclidean space being provided with the inner product

\[
\langle x, y \rangle = \sum_{i=1}^{m} \bar{\xi}_i \eta_i, \quad x = [\xi_i]^m_1, \quad y = [\eta_i]^m_1,
\]

where the bar means the standard conjugation in \( K \). Obviously,

\[
\langle x, y \rangle = \overline{\langle x, y \rangle}, \quad \langle x \alpha, y \beta \rangle = \overline{\alpha} \langle x, y \rangle \beta.
\]

The corresponding Euclidean norm is the case \( p = 2 \) in the family

\[
\|x\|_p = \left( \sum_{i=1}^{m} |\xi_i|^p \right)^{1/p}, \quad 1 \leq p \leq \infty.
\]

With the latter the space \( K^m \) is denoted by \( \ell^m_{p,K} \), so the Euclidean space \( K^m \) is just \( \ell^m_{2,K} \). In this case we will omit the subindex 2 in the notation of the norm.
Since $S(m, K) \equiv S(\delta m, R)$, the topological dimension of $S(m, K)$ is equal to $\delta m - 1$. In particular, $S(1, K) = U(K) \equiv \{ \alpha \in K, |\alpha| = 1 \}$. This is a multiplicative group acting as $x \mapsto x\alpha$ on $S(m, K)$. The corresponding quotient space is the projective space $KP^{m-1}$. Its topological dimension is equal to $\delta(m - 1)$. The space $KP^0$ is a singleton.

**DEFINITION 1.1.** [14] Let $p$ be an integer even, $p \geq 2$. A function $\phi : K^m \to C$ belongs to the class $\Phi_K(m, p)$ if

a) $\phi$ is a homogeneous polynomial of degree $p$ on the real space $R^{\delta m} \equiv (K^m)_R$ and

b) $\phi$ is $U(K)$-invariant in the sense that $\phi(x\alpha) = \phi(x)$, $x \in K^m$, $|\alpha| = 1$, or equivalently,

\[
\phi(x\alpha) = \phi(x)|\alpha|^p, \quad x \in K^m, \quad \alpha \in K.
\]

As a result, the restriction $\phi|S(m, K)$ is well defined on $KP^{m-1}$. Accordingly, it is called a polynomial function on $KP^{m-1}$ [16]. For simplicity we preserve the notation $\phi$ for the projective image of $\phi \in \Phi_K(m, p)$. This is acceptable since the projectivization is one-to-one.

The simplest example of $\phi \in \Phi_K(m, p)$ is $\phi(x) = \|x\|^p$. Every $U(K)$-invariant (thus even) polynomial $\psi$ of degree $\leq p$ can be included into $\Phi_K(m, p)$ by multiplying each of its homogeneous component $\psi_d$ by $\|\cdot\|^{p-d}$, $d = \deg \psi_d = 0, 2, \ldots, p - 2, p$. Since its transformation does not change the restriction $\psi|S(m, K)$, we have the inclusions

\[
\Phi_K(m, d)|S(m, K) \subset \Phi_K(m, p)|S(m, K) \quad (d = 0, 2, \ldots, p - 2).
\]

For $K = R$ the $U(K)$-invariance reduces to the central symmetry, $\phi(-x) = \phi(x)$, since $U(R) = Z_2$. On the other hand, $Z_2 \subset U(K)$, hence

\[
\Phi_K(m, p) \subset \Phi_K(\delta m, p).
\]

Obviously, $\Phi_K(m, p)$ is a finite-dimensional complex linear space. For $K = R$ this space consists of all complex-valued homogeneous polynomials of degree $p$ on $R^m$. The monomials

\[
\xi_1^{i_1} \cdots \xi_m^{i_m}, \quad (\xi_k)^{i_k} \in R^m,
\]

with $i_1 + \ldots + i_m = p$ form a basis of $\Phi_R(m, p)$. Accordingly,

\[
\dim \Phi_R(m, p) = \binom{m + p - 1}{m - 1}.
\]

In the space $\Phi_C(m, p)$ a natural basis consists of all monomials

\[
\xi_1^{i_1} \cdots \xi_m^{i_m} \xi_1^{j_1} \cdots \xi_m^{j_m}, \quad (\xi_k)^{i_k} \in C^m,
\]

where $(i_1, \ldots, i_m)$ and $(j_1, \ldots, j_m)$ independently run over all nonnegative $m$-tuples such that $i_1 + \cdots + i_m = j_1 + \cdots + j_m = p/2$. Thus, the space $\Phi_C(m, p)$ coincides with that of [11]. We have

\[
\dim \Phi_C(m, p) = \binom{m + p/2 - 1}{m - 1}^2.
\]

The structure of $\Phi_H(m, p)$ is much more complicated because of the non-commutativity of the field $H$. The point is that the quaternion monomials are not $U(H)$-invariant, in general. However, there exists an alternative way to calculate $\dim \Phi_K(m, p)$ for all fields $K$ at once, see [11]. In particular,

\[
\dim \Phi_H(m, p) = \frac{1}{2m - 1} \binom{2m + p/2 - 2}{2m - 2} \binom{2m + p/2 - 1}{2m - 2}.
\]
DEFINITION 1.2 ([14], [16]). A projective cubature formula of index \( p \) in \( \mathbb{K} \mathbb{P}^{m-1} \) is an identity
\[
\int_{S(m, \mathbb{K})} \phi \, d\sigma_{\delta_{m-1}} = \sum_{k=1}^{n} \phi(x_k) \rho_k, \quad \phi \in \Phi_{\mathbb{K}}(m, p),
\]
(1.7)
where \( \sigma_{\delta_{m-1}} \) is the normalized measure on \( S(m, \mathbb{K}) \) induced by the volume in \( \mathbb{R}^{\delta_{m}} \), the nodes \( x_k \in \mathbb{K} \mathbb{P}^{m-1} \), all weights \( \rho_k > 0 \) and their sum is equal to \( 1 \).

In an equivalent setting all \( x_k \in S(m, \mathbb{K}) \) and \( x_i \neq x_j \alpha \) for \( \alpha \in U(\mathbb{K}) \) and \( i \neq k \). In this sense \( x_k \) are pairwise projectively distinct.

For \( \mathbb{K} = \mathbb{R} \) the identity (1.7) is a spherical cubature formula of index \( p \) [6], [17]. In the case of equal weights the set of nodes of a spherical cubature formula is a spherical design [5] of the same index. Similarly, a projective design over any field \( \mathbb{K} \) can be defined as the set of nodes of a projective cubature formula with equal weights, c.f. [10]. Note that a spherical cubature formula is projective if and only if it is podal [17], i.e. there are no pairs of antipodal nodes.

For our purposes it is important that every projective cubature formula of index \( p \) is also of all indices \( d = 0, 2, \ldots, p-2 \). This immediately follows from (1.2) [14, 16]. Hence, a natural symmetrization of a podal spherical cubature formula of index \( p \) is an isometric embedding of degree \( p + 1 \) that means its validity for all polynomials on \( \mathbb{R}^m \) of degrees \( \leq p + 1 \).

Now note that the space \( \Phi_{\mathbb{K}}(m, p) \) contains all elementary polynomials \( \phi_{p, p}(x) = |(x, y)|^p, \ y \in \mathbb{K}^m \). Moreover, any function \( \phi \in \Phi_{\mathbb{K}}(m, p) \) is a linear combination of elementary polynomials [16]. For this reason the projective cubature formula (1.7) is equivalent to the identity
\[
\int_{S(m, \mathbb{K})} |(x, y)|^p \, d\sigma_{\delta_{m-1}}(x) = \sum_{k=1}^{n} |(x_k, y)|^p \rho_k, \quad y \in \mathbb{K}^m.
\]
(1.8)
On the other hand,
\[
\int_{S(m, \mathbb{K})} |(x, y)|^p \, d\sigma_{\delta_{m-1}}(x) = \gamma_{m, p, \mathbb{K}} \|y\|^p, \quad \gamma_{m, p, \mathbb{K}} = \text{const}, \quad y \in \mathbb{K}^m,
\]
(1.9)
see [14]. For \( \mathbb{K} = \mathbb{R} \) this is the identity applied by Hilbert [9] to solve the Waring problem in the number theory. Irrespective to \( \mathbb{K} \), we call (1.9) the Hilbert identity.

Comparing (1.9) to (1.8) we obtain
\[
\sum_{k=1}^{n} |(u_k, y)|^p = \|y\|^p, \quad y \in \mathbb{K}^m,
\]
(1.10)
where \( u_k = x_k \alpha_k \) with some \( \alpha_k > 0 \). This just means that the linear mapping \( y \mapsto ((u_k, y))_{k=1}^{n} \) is an isometric embedding \( \ell^m_{2, \mathbb{K}} \rightarrow \ell^m_{p, \mathbb{K}} \). Moreover, this one is irreducible in the sense that every pair of the vectors \( u_k, u_k \) is linearly independent, in particular, all \( u_k \neq 0 \). With any \( u_k \)'s the identity (1.10) can be reduced to a similar identity with some \( \tilde{u}_k \)'s, \( 1 \leq k \leq n \leq m \), such that the corresponding isometric embedding is irreducible.

Conversely, every irreducible isometric embedding \( \ell^m_{2, \mathbb{K}} \rightarrow \ell^m_{p, \mathbb{K}} \) is generated by a projective cubature formula since (1.10) \& (1.10) \rightarrow (1.8) with \( x_k = u_k/\|u_k\| \) and \( \rho_k = \gamma_{m, p, \mathbb{K}}/\|u_k\|^p \). Thus, we have a 1-1 correspondence between projective cubature formulas of index \( p \) with \( n \) nodes on \( S(m, \mathbb{K}) \) and irreducible isometric embeddings \( \ell^m_{2, \mathbb{K}} \rightarrow \ell^m_{p, \mathbb{K}} \).

Note that the image of any isometric embedding \( \ell^m_{2, \mathbb{K}} \rightarrow \ell^m_{p, \mathbb{K}} \) is an Euclidean subspace of \( \ell^m_{p, \mathbb{K}} \), and all Euclidean subspace are of this origin.

For any \((m, p)\) and large \( n \) an identity of form (1.10) can be derived from the Hilbert identity directly (i.e. without (1.8), see [14] and the references therein. Accordingly, an isometric embedding \( \ell^m_{2, \mathbb{K}} \rightarrow \ell^m_{p, \mathbb{K}} \) exists with such \( m, p, n \).

The minimal \( n \) such that an isometric embedding \( \ell^m_{2, \mathbb{K}} \rightarrow \ell^m_{p, \mathbb{K}} \) exists is denoted by \( N_{\mathbb{K}}(m, p) \). Every minimal isometric embedding \( \ell^m_{2, \mathbb{K}} \rightarrow \ell^m_{p, \mathbb{K}} \) (i.e. such that \( n = N_{\mathbb{K}}(m, p) \)) is irreducible, obviously. Thus, \( N_{\mathbb{K}}(m, p) \) is also the minimal number of nodes in the projective cubature formulas of index \( p \) on \( S(m, \mathbb{K}) \).

It is known that
\[
N_{\mathbb{K}}(m, p) \leq \dim \Phi_{\mathbb{K}}(m, p) - 1,
\]
(1.11)
see [13] and the references therein. For any fixed \( m \) and \( p \rightarrow \infty \) the inequality (1.11) combined with the formulas (1.4), (1.5) and (1.6) yields the asymptotical upper bound
\[
N_{\mathbb{K}}(m, p) \lesssim \frac{p^{\gamma(m-1)}}{c_m(\mathbb{K})},
\]
(1.12)
where
\[ c_m(R) = (m-1)!, \quad c_m(C) = 4^{m-1}(m-1)!^2, \quad c_m(H) = 16^{m-1}(2m-1)(2m-2)!. \] (1.13)

The exact values \( N_K(m,p) \) are unknown, except for some special cases, see [10], [11], [12], [15], [17], [20]. The trivial examples are
\[ N_K(1,p) = 1, \quad N_K(m,2) = m. \] (1.14)
The simplest nontrivial example is \( N_K(2,4) = 3 \), see [12]. More generally,
\[ N_K(2,p) = p/2 + 1, \] (1.15)
see [17], [20].

From (1.2) it follows that
\[ N_K(m,p) \leq N_K(m,p). \] (1.16)
Another useful inequality is
\[ N_K(m,p) \leq N_R(\delta m,p) \leq N_R(\delta,p)N_K(m,p). \] (1.17)
Here the left-hand side follows from (1.3) immediately. With \( K = C \) the right-hand side of (1.17) follows from [11].

In the present paper we construct a recursion with respect to \( m \) for the projective cubature formulas of index \( p \) in \( K \mathbb{P}^{m-1} \). For a large set of pairs \( m,p \) this yields the upper bounds for \( N_K(m,p) \) which are effective in the sense that they are better than (1.11). Later on we call the right-hand side of (1.11) the General Upper Bound, briefly GUB. This is a polynomial in \( p \) of degree \( \delta(m-1) \). It is an open problem to improve (1.11) in general.

Our Main Theorem is

**Theorem 1.3.** Let \( m \geq 2, p \geq 4 \). Any projective cubature formula of index \( p \) with \( n \) nodes on \( S(m-1,K) \) determines a projective cubature formula of the same index with \( n' \) nodes on \( S(m,K) \) where
\[ n' = \begin{cases} \nu_K(p)(p/2+1)n, & p \equiv 2 \pmod{4} \\ \nu_K(p)((p/2)n+1), & p \equiv 0 \pmod{4} \end{cases} \] (1.18)
and
\[ \nu_K(p) = N_R(\delta,2[p/4]) = \begin{cases} N_R(\delta,p/2-1), & p \equiv 2 \pmod{4} \\ N_R(\delta,p/2), & p \equiv 0 \pmod{4}. \end{cases} \] (1.19)

In fact, \( \nu_R(p) = 1 \) and \( \nu_C(p) = [p/4] + 1 \) according to (1.11) and (1.15), respectively. In contrast, for \( \nu_H(p) \) we only have an upper bound (see (1.17)), except for \( \nu_H(4) = N_R(4,2) = 4 \), see (1.14), and \( \nu_H(8) = N_R(4,4) = 11 \), see [20], Proposition 9.26.

In terms of isometric embeddings the Theorem 1.3 is reformulated as follows.

**Theorem 1.4.** Let \( m \geq 2, p \geq 4 \). Any irreducible isometric embedding \( \ell_2^{m-1} \rightarrow \ell_p^{n'}_{p,K} \) determines an irreducible isometric embedding \( \ell_2^{m}_{2,K} \rightarrow \ell_p^{n'}_{p,K} \) where \( n' \) is that of (1.18).

Taking \( n = N_K(m-1,p) \) in (1.18) we obtain

**Corollary 1.5.** The inequality
\[ N_K(m,p) \leq \begin{cases} N_R(\delta,p/2-1)(p/2+1)N_K(m-1,p), & p \equiv 2 \pmod{4} \\ N_R(\delta,p/2)((p/2)N_K(m-1,p)+1), & p \equiv 0 \pmod{4} \end{cases} \] (1.20)

holds.

The inequality (1.20) being combined with the left-hand side of (1.17) yields

**Corollary 1.6.** The inequality
\[ N_K(m,p) \leq \begin{cases} N_R(\delta,p/2-1)(p/2+1)N_R(\delta(m-1),p), & p \equiv 2 \pmod{4} \\ N_R(\delta,p/2)((p/2)N_R(\delta(m-1),p)+1), & p \equiv 0 \pmod{4} \end{cases} \] (1.21)

holds.
We prove the Main Theorem in Section 3 using a series of lemmas from Section 2. The recursion (1.18) corresponds to a partial separation of spherical coordinates and subsequent applying of some relevant cubature (in particular, quadrature) formulas for the partial integrals. For the spherical cubature formulas and designs this way is well known [2], [3], [4], [10], [19], [22], [24]. The lemmas mentioned above allow us to realize the recursion in the projective context. For the projective designs our proof can be adapted by using of a quadrature formula of Chebyshev type of degree $p/2$ instead of Gauss-Jacobi. This yields a counterpart of Corollary 1.5 with an upper bound for the number of nodes instead of $2$.

In Section 4 we reformulate the Main Theorem for each of three fields separately and, as a result, explicitly. Then in each case we specify the range of $m$ where the corresponding upper bound $N_K(m, p) \leq n$ is effective for all $p$. In addition, the Main Theorem yields a lot of “sporadic” numerical upper bounds arising from some known ones. In Section 5 these results are presented in form of tables.

2 The lemmas

Lemma 2.1. Denote by $\hat{S}_{r-1}$ the (non-normalized) surface area on $S(r, R)$, $r \geq 2$. Let $1 \leq l \leq r - 1$, and let $x = [\xi]_r \in S(r, R)$, $y = [\xi]_l$, $z = [\xi]_{l+1}$, $\rho = ||z||$. With $\hat{y} = y/||y||$ and $\hat{z} = z/||z||$ ($y, z \neq 0$) the formula

$$d\hat{S}_{r-1}(x) = (1 - \rho^2)^{l-1} \rho^{r-l-1} d\rho d\hat{S}_{l-1}(\hat{y}) d\hat{S}_{r-l-1}(\hat{z})$$

(2.1)

holds (under agreement $d\hat{S}_0(\cdot) = 1$).

Proof. The column $x$ can be written in the form

$$x = h(\rho, \hat{y}, \hat{z}) = \left[ \sqrt{1 - \rho^2 \hat{y}^2} \right].$$

(2.2)

Denote by $\theta = (\theta_1, \ldots, \theta_{l-1})$ and $\varphi = (\varphi_1, \ldots, \varphi_{r-l-1})$ where $\theta_i$ and $\varphi_j$ are the spherical coordinates of $\hat{y} \in S(l, R)$ and $\hat{z} \in S(r-l, R)$, respectively. (For $l = 1$ there is no $\theta$, for $l = r - 1$ there is no $\varphi$.) From (1.16) we obtain the Jacobi matrix

$$J = \frac{Dh(\rho, \hat{y}, \hat{z})}{D(\rho, \theta, \varphi)} = \left[ \begin{array}{ccc} \frac{\partial h}{\partial \rho} & 0 & \sqrt{1 - \rho^2} \hat{y}^2 \\ 0 & \rho Y & 0 \\ 0 & 0 & \rho Z \end{array} \right],$$

where $\left[ \tilde{\xi} \right]_1^l = \tilde{y}$, $\left[ \tilde{\xi} \right]_{l+1}^r = \hat{z}$,

$$Y = \left[ \frac{\partial \tilde{\xi}_i}{\partial \theta_k} \right]_{1 \leq i \leq l, 1 \leq k \leq l-1}, \quad Z = \left[ \frac{\partial \tilde{\xi}_i}{\partial \varphi_j} \right]_{l+1 \leq i \leq r, 1 \leq j \leq r-l-1}.$$

(There is no $Y$ for $l = 1$, no $Z$ for $l = r - 1$.)

The corresponding Gram matrix is

$$\Gamma = J' J = \left[ \begin{array}{ccc} (1 - \rho^2)^{-1} & 0 & 0 \\ 0 & (1 - \rho^2) Y' Y & 0 \\ 0 & 0 & \rho^2 Z' Z \end{array} \right].$$

(2.3)

where dash means conjugation. Indeed, $||\tilde{y}||^2 = ||\tilde{z}||^2 = 1$ and

$$\tilde{y}' Y = \sum_{i=1}^l \hat{y}_i \frac{\partial \tilde{\xi}_i}{\partial \theta_k} = \frac{1}{2} \frac{\partial}{\partial \theta_k} \left( \sum_{i=1}^l \hat{y}_i^2 \right) = 0, \quad 1 \leq k \leq l - 1,$$

and

$$\tilde{z}' Z = \sum_{i=l+1}^r \hat{z}_i \frac{\partial \tilde{\xi}_i}{\partial \varphi_j} = \frac{1}{2} \frac{\partial}{\partial \varphi_j} \left( \sum_{i=l+1}^r \hat{z}_i^2 \right) = 0, \quad 1 \leq j \leq r - l - 1.$$

Note that $G \equiv Y' Y$ and $H \equiv Z' Z$ are the Gram matrices for the Jacobi matrices $Y$ and $Z$ of the mappings $(\theta_1, \ldots, \theta_{l-1}) \mapsto (\tilde{\xi}_1, \ldots, \tilde{\xi}_l)$ and $(\varphi_1, \ldots, \varphi_{r-l-1}) \mapsto (\hat{\xi}_{l+1}, \ldots, \hat{\xi}_r)$, respectively. From (2.3) it follows that

$$\det \Gamma = (1 - \rho^2)^{(l-2)} \rho^{2(r-l-1)} \det G \det H.$$
This results in (2.4) since
\[ d\tilde{\sigma}_{\ell-1}(x) = \sqrt{\det\Gamma} \, d\rho d\theta_1 \ldots d\theta_{\ell-1} d\varphi_1 \ldots d\varphi_{\ell-1} \]
and
\[ d\tilde{\sigma}_{\ell-1}(y) = \sqrt{\det G} \, d\theta_1 \ldots d\theta_{\ell-1}, \quad d\tilde{\sigma}_{\ell-1}(z) = \sqrt{\det H} \, d\varphi_1 \ldots d\varphi_{\ell-1}. \]

Now let \( x \in S(m, K) \), \( m \geq 2 \). Then \( x = \eta \oplus z \) where \( \eta \in K \) and \( z \in K^{m-1} \), and then
\[ x = \sqrt{1 - \rho^2} \oplus \rho w, \quad \rho \in [-1, 1], \quad \theta \in S(1, K) \equiv S(\delta, R), \quad w \in S(m - 1, K) \equiv S(\delta(m - 1), R). \] (2.4)

Accordingly, we set
\[ \phi(\rho, \theta, w) = \phi(\sqrt{1 - \rho^2} \oplus \rho w) \] (2.5)
for a continuous function \( \phi(x) \). Obviously, \( \phi(-\rho, \theta, -w) = \phi(\rho, \theta, w) \). If \( \phi(x) \) is central symmetric, i.e. \( \phi(-x) = \phi(x) \), then \( \phi(\rho, -\theta, -w) = \phi(\rho, \theta, w) \). As a result, \( \phi(\rho, -\theta, w) = \phi(\rho, -\theta, w) \). Therefore, the \( Z_2 \)-average with respect to \( \rho \), i.e.
\[ \tilde{\phi}(\rho, \theta, w) = \frac{1}{2} (\phi(\rho, \theta, w) + \phi(-\rho, \theta, w)), \] (2.6)
coincides with the \( Z_2 \)-average with respect to \( \theta \):
\[ \tilde{\phi}(\rho, \theta, w) = \frac{1}{2} (\phi(\rho, \theta, w) + \phi(\rho, -\theta, w)). \] (2.7)

Now we consider the integral
\[ I_{\phi}(w) = \int_{S(1, K)} d\sigma_{\ell-1}(\theta) \int_0^1 \phi(\rho, \theta, w) \pi(\rho) \, d\rho \] (2.8)
with any integrable \( \pi(\rho) \).

**Lemma 2.2.** If \( \phi(x) \) is central symmetric then \( I_{\phi}(w) = I_{\tilde{\phi}}(w) \).

**Proof.** This follows from (2.7) since the measure \( \sigma_{\ell-1}(\theta) \) is central symmetric.

**Lemma 2.3.** If \( \phi(x) \) is \( U(K) \)-invariant then \( I_{\phi}(w) \) is also \( U(K) \)-invariant.

**Proof.** From (2.5) it follows that \( \phi(\rho, \theta, w) = \phi(\rho, \theta, w) \) for all \( \alpha \in U(K) \). On the other hand, the measure \( \sigma_{\ell-1}(\theta) \) is \( U(K) \)-invariant.

Actually, only the functions \( \phi(x) \) from \( \Phi_K(m, p) \) are needed for our purposes.

**Lemma 2.4.** If \( \phi(x) \) belongs to \( \Phi_K(m, p) \) then the function \( I_{\phi}(w) \) belongs to \( \Phi_K(m - 1, p)|S(m - 1, K) \).

**Proof.** In view of the Lemma 2.3 and inclusion (1.2) we only have to prove that \( I_{\phi}(w) \) is the restriction to the unit sphere of a polynomial of degree \( \leq p \) on \( R^{d(m-1)} \). Since \( \Phi_K(m, p) = \text{Span}\{\phi_{\nu, p} : \nu \in K^m\} \) and since the mapping \( \phi \mapsto I_{\phi} \) is linear, we can assume that \( \phi(x) = \phi_{\nu, p}(x) = |\langle x, \nu \rangle|^p, \nu \in K^m \). Let \( y = \xi \oplus v \) where \( \xi \in K, v \in K^{m-1} \). Then by (2.5)
\[ \phi_{\nu, p}(\rho, \theta, w) = \sqrt{1 - \rho^2} \theta \xi + \rho \langle w, v \rangle \]
\[ = \left( (1 - \rho^2) \left| \theta \xi \right|^2 + \rho^2 \left| \langle w, v \rangle \right|^2 + 2 \rho \sqrt{1 - \rho^2} \Re \left( \theta \xi \langle w, v \rangle \right) \right)^{p/2}. \] (2.9)

With fixed \( \rho \) and \( \theta \) let us consider the right-hand side of (2.9) as a function of \( w \in R^{d(m-1)} \). This is a polynomial of degree \( \leq p \). Therefore, such is \( I_{\phi}(w) \) obtained by substitution of (2.9) into the integral (2.8).

The last lemma we need is
LEMMA 2.5. If \( \phi(x) \) belongs to \( \Phi_K(m,p) \) then with a fixed \( w \) the function \( \tilde{\phi}(\rho,\theta,w) \) defined by (2.6) is a linear combination of functions of form \( f(\rho^2)((\theta,\zeta)_R)^q \) where \( f \) is a polynomial of degree \( \leq p/2 \), \( 0 \leq q \leq [p/4] \), \( \zeta \in K \), \( (\theta,\zeta)_R = \Re(\theta \zeta) \).

**Proof.** As before, it suffices to consider \( \phi = \phi_{r,p} \), so we can use (2.4). Note that

\[
\Re \left( \xi (\theta,\nu,\omega) \right) = \Re \left( (\nu,\omega) \tilde{\phi} \right) = \Re \left( \tilde{\theta} \xi (\nu,\omega) \right) = (\theta,\zeta)_R
\]

where \( \zeta = \xi (\nu,\omega) \). We have

\[
\phi(\rho,\theta,w) = (A(\rho^2) + B(\rho^2)\text{sign}(\rho) (\zeta R)_R)^{p/2}
\]

where

\[
A(t) = |\xi|^2 (1-t) + |(\nu,\omega)|^2 t, \quad B(t) = \sqrt{4t(1-t)}.
\]

Hence,

\[
\phi(\rho,\theta,w) = \sum_{k=0}^{p/2} \binom{p/2}{k} A(\rho^2)^{p/2-k} B(\rho^2)^k \left( \text{sign}(\rho) \right)^k ((\theta,\zeta)_R)^k.
\]

and then (2.6) yields

\[
\tilde{\phi}(\rho,\theta,w) = \sum_{q=0}^{[p/4]} \binom{p/2}{2q} \left( \sum_{k=1}^{p/2} \binom{p/2}{k} A(\rho^2)^{p/2-2q} B(\rho^2)^{2q} \right) ((\theta,\zeta)_R)^{2q}.
\]

It remains to note that \( A(t)^{p/2-2q} B(t)^{2q} \) is a polynomial of degree \( \leq p/2 \) for every \( q \leq [p/4] \). \( \square 

3 Proof of the Main Theorem

Let \( \phi \in \Phi_K(m,p), x \in S(m,K), \phi(x) = \phi(\rho,\theta,w) \) as in (2.5). According to Lemma 2.1 with \( r = \delta m \) and \( l = \delta \), we have

\[
\int_{S(m,K)} \phi(x) \, d\sigma_{r-1}(x) = \int_{S(m-1,K)} d\sigma_{r-\delta-1}(w) \int_{S(1,K)} d\sigma_{\delta-1}(\theta) \int_{S(1,K)} \phi(\rho,\theta,w) \pi_{\alpha,\beta}(\rho) \, d\rho
\]

where

\[
\pi_{\alpha,\beta}(\rho) = C \rho^{2\alpha+1}(1-\rho^2)^\beta, \quad \alpha = \delta(m-1)/2, \quad \beta = \delta/2 - 1,
\]

the constant \( C = C_{r,\delta} \) comes from the normalization of the areas in (2.1):

\[
\int_0^1 \pi_{\alpha,\beta}(\rho) \, d\rho = 1.
\]

By (2.8) and Lemma 2.2 we get

\[
\int_{S(m,K)} \phi(x) \, d\sigma_{r-1}(x) = \int_{S(m-1,K)} d\sigma_{r-\delta-1}(w) \int_{S(1,K)} d\sigma_{\delta-1}(\theta) \int_0^1 \tilde{\phi}(\rho,\theta,w) \pi_{\alpha,\beta}(\rho) \, d\rho.
\]

Lemma 2.4 allows us to apply a projective cubature formula of index \( p \) on \( S(m-1,K) \) existing by assumption. If its nodes and weights are \( w_i \) and \( \lambda_i \), \( 1 \leq i \leq n \), respectively, then

\[
\int_{S(m,K)} \phi(x) \, d\sigma_{r-1}(x) = \sum_{i=1}^n \lambda_i \int_{S(1,K)} d\sigma_{\delta-1}(\theta) \int_0^1 \tilde{\phi}(\rho,\theta,w_i) \pi_{\alpha,\beta}(\rho) \, d\rho.
\]

(3.1)

By Lemma 2.4 the integrals against \( d\sigma_{\delta-1}(\theta) \) in (3.1) can be calculated by a podal spherical cubature formula of index \( 2[p/4] \) on \( S(1,K) \equiv S(\delta,R) \). The minimal number of nodes in such a formula is

\[
\nu = N_K(\delta,2[p/4]) = \begin{cases} N_K(\delta,p/2-1), & p \equiv 2 \pmod{4} \\ N_K(\delta,p/2), & p \equiv 0 \pmod{4}. \end{cases}
\]

(3.2)

As a result,

\[
\int_{S(m,K)} \phi(x) \, d\sigma_{r-1}(x) = \sum_{i=1}^n \sum_{j=1}^{\nu} \lambda_i \mu_j \int_0^1 \tilde{\phi}(\rho,\theta_j,w_i) \pi_{\alpha,\beta}(\rho) \, d\rho
\]

(3.3)
where \( \theta_j \) and \( \mu_j \) are the corresponding nodes and weights.

Now we consider the integral
\[
\int_0^1 f(\rho^2) \pi_{\alpha,\beta}(\rho) \, d\rho = \int_0^1 f(\tau) \chi_{\alpha,\beta}(\tau) \, d\tau
\]
where \( f \) is a polynomial of degree \( \leq p/2 \) and
\[
\chi_{\alpha,\beta}(\tau) = \frac{\pi_{\alpha,\beta}(\sqrt{\tau})}{2\sqrt{\tau}} = \frac{1}{2} C_\tau^\alpha (1 - \tau)^\beta, \quad \int_0^1 \chi_{\alpha,\beta}(\tau) \, d\tau = 1.
\]
Assume that \( p \equiv 2 \) (mod 4), i.e. \( p/2 \) is odd. Since \( \deg f \leq p/2 = 2(p+2)/4 - 1 \), the classical Gauss-Jacobi quadrature formula yields
\[
\int_0^1 f(\tau) \chi_{\alpha,\beta}(\tau) \, d\tau = \sum_{k=1}^{(p+2)/4} \omega_k f(\tau_k) \tag{3.4}
\]
with relevant nodes and weights, see [23], Theorems 3.4.1 and 3.4.2. Therefore,
\[
\int_0^1 f(\rho^2) \pi_{\alpha,\beta}(\rho) \, d\rho = \sum_{k=1}^{(p+2)/4} \omega_k f(\rho_k), \quad \rho_k = \sqrt{\tau_k}.
\]
By Lemma 2.5
\[
\int_0^1 \tilde{\phi}(\rho, \theta_j, \omega_i) \pi_{\alpha,\beta}(\rho) \, d\rho = \sum_{k=1}^{(p+2)/4} \omega_k \tilde{\phi}(\rho_k, \theta_j, \omega_i) = \frac{1}{2} \sum_{k=1}^{(p+2)/4} \omega_k \left( \phi(\rho_k, \theta_j, \omega_i) + \phi(\rho_k, -\theta_j, \omega_i) \right) \tag{3.5}
\]
for all \( 1 \leq i \leq n, 1 \leq j \leq \nu \). The substitution from (3.5) into (3.3) yields
\[
\int_{X(m, K)} \phi(x) \, d\sigma_{r-1}(x) = \sum_{i=1}^{n} \sum_{j=1}^{\nu} \sum_{k=1}^{(p+2)/4} \varphi_{ijk} \left( \phi(x^+_{ijk}) + \phi(x^-_{ijk}) \right) \tag{3.6}
\]
where
\[
x^\pm_{ijk} = \pm \theta_j \sqrt{1 - \rho^2_k \pm \rho_k \omega_i}, \quad \varphi_{ijk} = \frac{1}{2} \lambda_j \mu_j \omega_k. \tag{3.7}
\]
The number of nodes \( x^+_{ijk} \) is
\[
n' = (p/2 + 1)\nu n = N_R(\delta, p/2 - 1)(p/2 + 1)n \tag{3.8}
\]
according to (3.2).

Now let \( p \equiv 0 \) (mod 4), i.e. let \( p/2 \) be even. In this case, instead of (3.4), we use its Markov’s modification (see [18], formula (1.16)):
\[
\int_0^1 f(\tau) \chi_{\alpha,\beta}(\tau) \, d\tau = \omega_0 f(0) + \sum_{k=1}^{p/4} \omega_k f(\tau_k). \tag{3.9}
\]
This is valid for all polynomials \( f \) of deg \( f \leq 2(p/4) = p/2 \). (Of course, the nodes and the weights in (3.9) are different from those of (3.4)) As before,
\[
\int_0^1 \tilde{\phi}(\rho, \theta_j, \omega_i) \pi_{\alpha,\beta}(\rho) \, d\rho = \omega_0 \phi(0, \theta_j, \omega_i) + \frac{1}{2} \sum_{k=1}^{p/4} \omega_k \left( \phi(\rho_k, \theta_j, \omega_i) + \phi(\rho_k, -\theta_j, \omega_i) \right)
\]
and then
\[
\int_{X(m, K)} \phi(x) \, d\sigma_{r-1}(x) = \sum_{j=1}^{\nu} \varphi_j \phi(x_j) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{\nu} \sum_{k=1}^{p/4} \varphi_{ijk} \left( \phi(x^+_{ijk}) + \phi(x^-_{ijk}) \right) \tag{3.10}
\]
where
\[
x_j = \theta_j \oplus 0, \quad \varphi_j = \mu_j \omega_0 \sum_{i=1}^{n} \lambda_i = \mu_j \omega_0, \tag{3.11}
\]
the rest of nodes and weights is determined as in (3.7). Now the total number of nodes is
\[
n' = \nu + (p/2)\nu n = N_R(\delta, p/2)((p/2)n + 1) \tag{3.12}
\]
according to (3.2) again.

It remains to note that in each of formulas (3.6) and (3.10) the nodes are projectively distinct. \( \square \)
4 Some applications

Further \( m \geq 2, p \geq 4 \) as in the Main Theorem. It is convenient to set \( p = 2s \), so \( s \) is an integer, \( s \geq 2 \).

Let us start with \( \mathbf{K} = \mathbb{C} \). In this case the Main Theorem takes the form of

**Theorem 4.1.** Any projective cubature formula of index \( 2s \) with \( n \) nodes on \( S(m-1, \mathbb{C}) \) determines a projective cubature formula of the same index with \( n' \) nodes on \( S(m, \mathbb{C}) \) where

\[
n' = \begin{cases} 
\frac{(s+1)^2}{2}n, & s \equiv 1 \pmod{2} \\
\frac{s+2}{2}(sn+1), & s \equiv 0 \pmod{2}.
\end{cases}
\]

(4.1)

**Proof.** By (1.19) and (1.15)

\[
\nu_{\mathbb{C}}(2s) = N_{\mathbb{R}}(2, 2 \lfloor s/2 \rfloor) = \lfloor s/2 \rfloor + 1 = \begin{cases} 
\frac{s+1}{2}, & s \equiv 1 \pmod{2} \\
\frac{s+2}{2}, & s \equiv 0 \pmod{2}.
\end{cases}
\]

The corollary (4.2) reduces to

**Corollary 4.2.** The inequality

\[
N_{\mathbb{C}}(m, 2s) \leq \begin{cases} 
\frac{(s+1)^2}{2}N_{\mathbb{C}}(m-1, 2s), & s \equiv 1 \pmod{2} \\
\frac{s+2}{2}(sN_{\mathbb{C}}(m-1, 2s) + 1), & s \equiv 0 \pmod{2}.
\end{cases}
\]

(4.2)

holds.

In particular,

\[
N_{\mathbb{C}}(2, 2s) \leq \begin{cases} 
\frac{(s+1)^2}{2}, & s \equiv 1 \pmod{2} \\
\frac{(s+2)(s+1)}{2}, & s \equiv 0 \pmod{2}.
\end{cases}
\]

(4.3)

since \( N_{\mathbb{C}}(1, 2s) = 1 \). Asymptotically,

\[
N_{\mathbb{C}}(2, 2s) \lesssim \frac{1}{2}s^2, \quad s \to \infty.
\]

(4.4)

Taking \( m = 3 \) in (4.2) and using (4.3) we obtain

\[
N_{\mathbb{C}}(3, 2s) \leq \begin{cases} 
\frac{(s+1)^4}{4}, & s \equiv 1 \pmod{2} \\
\frac{s+2}{2} \left( \frac{(s+2)(s+1)s}{2} + 1 \right), & s \equiv 0 \pmod{2}.
\end{cases}
\]

(4.5)

whence

\[
N_{\mathbb{C}}(3, 2s) \lesssim \frac{1}{4}s^4, \quad s \to \infty.
\]

(4.6)

The upper bounds (4.3) and (4.5) are effective. Indeed, for \( \mathbf{K} = \mathbb{C} \) the cases \( m = 2, 3 \) in GUB (i.e., in (1.11)) are

\[
N_{\mathbb{C}}(2, 2s) \leq (s+1)^2, \quad N_{\mathbb{C}}(3, 2s) \leq \frac{(s+2)^2(s+1)^2}{4}.
\]

(4.7)
that is worse than (4.3) and (4.5), respectively. Asymptotically, (4.4) also remains effective, i.e. better than what the first inequality (4.7) implies. However, (4.6) coincides with the corresponding consequence of (4.7). (Clearly, it cannot be worse.)

The next iteration of (4.2) yields an ineffective upper bound for $N_{C}(m, 2s), m \geq 4$. However, for some $s$ the effectiveness may be reached by using a more precise bound (or an exact value, if any) for $N_{C}(m - 1, 2s)$ in (4.4). Also, some effective bounds can be improved in this way. In Section 5 the reader can find a lot of examples of this approach (for all three fields). One of them is below.

EXAMPLE 4.3. From the known (see [15]) equality $N_{C}(2, 8) = 10$ it follows that

$$N_{C}(3, 8) \leq 3(4N_{C}(2, 8) + 1) = 123,$$

while (4.5) yields $N_{C}(3, 8) \leq 183$.

The following is the iterated form of Theorem 4.1.

THEOREM 4.4. Any projective cubature formula of index $2s$ with $n$ nodes on $S(m - 1, C)$ determines a projective cubature formula of the same index with $n^{(l)}$ nodes on $S(m + l - 1, C)$, where $l \geq 0$ and

$$n^{(l)} = \begin{cases} \frac{(s + 1)^{2l}}{2^{l}}n, & s \equiv 1 \pmod{2}, \\ A(s + 2)^{l}s^{l} + B, & s \equiv 0 \pmod{2}, \end{cases} \tag{4.8}$$

with

$$A = \frac{(s + 3)s}{(s + 2)s - 2}, \quad B = -\frac{s + 2}{(s + 2)s - 2}. \tag{4.9}$$

Proof. For any $s$ the sequence on the right-hand side of (4.8) satisfies the recurrent relation (4.1), and $A + B = 1$.

COROLLARY 4.5. The inequality

$$N_{C}(m + l - 1, 2s) \leq \begin{cases} \frac{(s + 1)^{2l}}{2^{l}}N_{C}(m - 1, 2s), & s \equiv 1 \pmod{2}, \\ A(s + 2)^{l}s^{l} + B, & s \equiv 0 \pmod{2}. \end{cases} \tag{4.10}$$

holds.

Now let us proceed to $K = R$. In this case we have

THEOREM 4.6. Any podal spherical cubature formula of index $2s$ with $n$ nodes on $S(m - 1, R)$ determines a podal spherical cubature formula of the same index with $n'$ nodes on $S(m, R)$ where

$$n' = \begin{cases} (s + 1)n, & s \equiv 1 \pmod{2} \\ sn + 1, & s \equiv 0 \pmod{2}. \end{cases} \tag{4.11}$$

Proof. $\nu_{R}(2s) = N_{R}(1, 2 \lfloor s/2 \rfloor) = 1$.

COROLLARY 4.7.

$$N_{R}(m, 2s) \leq \begin{cases} (s + 1)N_{R}(m - 1, 2s), & s \equiv 1 \pmod{2} \\ sN_{R}(m - 1, 2s) + 1, & s \equiv 0 \pmod{2}. \end{cases} \tag{4.12}$$
For $m = 2$ both inequalities (4.12) reduce to $N_R(2, 2s) \leq s + 1$. (In fact, $N_R(2, 2s) = s + 1$, see (1.15).) Hence,

$$N_R(3, 2s) \leq \begin{cases} 
(s + 1)^2, & s \equiv 1 \pmod{2} \\
2s + s + 1 & s \equiv 0 \pmod{2},
\end{cases}$$

thus

$$N_R(3, 2s) \lesssim s^2, \quad s \to \infty.\quad (4.14)$$

The next iteration yields

$$N_R(4, 2s) \leq \begin{cases} 
(s + 1)^3, & s \equiv 1 \pmod{2} \\
2(s + 1)(s + 1) & s \equiv 0 \pmod{2}.
\end{cases}$$

However, the latter can be improved by means of the inequality

$$N_R(2m, 2s) \leq (s + 1)N_C(m, 2s)\quad (4.16)$$

which is just the case $\delta = 2$ on the right-hand side of (1.17). Indeed,

$$N_R(4, 2s) \lesssim \frac{s^3}{2}, \quad s \to \infty,\quad (4.17)$$

instead of $N_R(4, 2s) \lesssim s^3$ that follows from (4.15).

Similarly,

$$N_R(6, 2s) \leq (s + 1)N_C(3, 2s) \leq \begin{cases} 
\frac{(s + 1)^2}{4}, & s \equiv 1 \pmod{2} \\
2(s + 2)(s + 1) \frac{(s + 2)(s + 1)s}{2} + 1 & s \equiv 0 \pmod{2},
\end{cases}$$

by (4.23). Hence,

$$N_R(6, 2s) \lesssim \frac{s^6}{4}, \quad s \to \infty.\quad (4.20)$$

In addition, from (4.12) and (4.17) it follows that

$$N_R(5, 2s) \leq \begin{cases} 
\frac{(s + 1)^3}{2}, & s \equiv 1 \pmod{2} \\
2(s + 2)(s + 1)^2s^2 + 1 & s \equiv 0 \pmod{2},
\end{cases}$$

hence,

$$N_R(5, 2s) \lesssim \frac{s^4}{2}, \quad s \to \infty.\quad (4.22)$$

All upper bounds for $N_R(m, 2s)$, $3 \leq m \leq 6$, obtained above are effective, even asymptotically, c.f. (1.12).

The $\mathbf{R}$-counterpart of Theorem 4.4 looks simpler.

**Theorem 4.8.** Any nodal spherical cubature formula of index $2s$ with $n$ nodes on $S(m - 1, \mathbf{R})$ determines a nodal spherical cubature formula of the same index with $n^{(l)}$ nodes on $S(m + l - 1, \mathbf{R})$ where $l \geq 0$ and

$$n^{(l)} = \begin{cases} 
(s + 1)^ln, & s \equiv 1 \pmod{2} \\
2s'(n - 1) & s \equiv 0 \pmod{2},
\end{cases}$$

(4.23)
Proof. Induction on \( l \).

COROLLARY 4.9.

\[
N_R(m + l - 1, 2s) \leq \begin{cases} 
(s + 1)^l N_R(m - 1, 2s), & s \equiv 1 \pmod{2} \\
N_R(m - 1, 2s) + \frac{s^l - 1}{s - 1}, & s \equiv 0 \pmod{2}.
\end{cases}
\]

(4.24)

It remains to consider the case \( K = H \).

THEOREM 4.10. Any projective cubature formula of index \( 2s \) with \( n \) nodes on \( S(m - 1, H) \) determines a projective cubature formula of the same index with \( n' \) nodes on \( S(m, H) \) where

\[
n' = \begin{cases} 
N_R(4, s - 1)(s + 1)n, & s \equiv 1 \pmod{2} \\
N_R(4, s)(sn + 1), & s \equiv 0 \pmod{2}.
\end{cases}
\]

(4.25)

Proof. We have

\[
\nu_H(2s) = N_R(4, 2[s/2]) = \begin{cases} 
N_R(4, s - 1), & s \equiv 1 \pmod{2} \\
N_R(4, s), & s \equiv 0 \pmod{2}.
\end{cases}
\]

(4.26)

COROLLARY 4.11. The inequality

\[
N_H(m, 2s) \leq \begin{cases} 
N_R(4, s - 1)(s + 1)N_H(m - 1, 2s), & s \equiv 1 \pmod{2} \\
N_R(4, s)(sn_H(m - 1, 2s) + 1), & s \equiv 0 \pmod{2}.
\end{cases}
\]

(4.27)

holds.

The exact values of \( N_R(4, 2[s/2]) \) are unknown, except for the cases \( s = 2 \) and \( s = 4 \) when \( N_R(4, 2) = 4 \) and \( N_R(4, 4) = 11 \), respectively. However, we can use the upper bound (4.17).

THEOREM 4.12. Any projective cubature formula of index \( 2s \) with \( n \) nodes on \( S(m - 1, H) \) determines a projective cubature formula of the same index with \( n' \) nodes on \( S(m, H) \) where

\[
16n' \leq \begin{cases} 
(s + 1)^4n, & s \equiv 3 \pmod{4} \\
(s + 3)(s + 1)^3n, & s \equiv 1 \pmod{4} \\
(s + 3)^2(sn + 1), & s \equiv 2 \pmod{4} \\
(s + 4)(s + 2)^2(sn + 1), & s \equiv 0 \pmod{4}.
\end{cases}
\]

(4.28)

Proof. If \( s \equiv 0 \pmod{4} \) then \( s/2 \equiv 0 \pmod{2} \) and (4.17) yields

\[
N_R(4, s) \leq \frac{(s/2 + 2)(s/2 + 1)^2}{2} = \frac{(s + 4)(s + 2)^2}{16}.
\]

(4.29)

Now let \( s \equiv 1 \pmod{4} \). Then \( s - 1 \equiv 0 \pmod{2} \) and (4.29) turns into

\[
N_R(4, s - 1) \leq \frac{(s + 3)(s + 1)^2}{16}.
\]

(4.30)

Similarly, if \( s \equiv 2 \pmod{4} \) then \( s/2 \equiv 1 \pmod{2} \), hence

\[
N_R(4, s) \leq \frac{(s/2 + 1)^3}{2} = \frac{(s + 2)^3}{16}.
\]

(4.31)

by (4.17). Finally, if \( s \equiv 3 \pmod{4} \) then \( s - 1 \equiv 2 \pmod{2} \), hence

\[
N_R(4, s - 1) \leq \frac{(s + 1)^3}{16}.
\]

(4.32)

by (4.31). It remains to substitute the inequalities (4.29)-(4.32) into (4.25).
COROLLARY 4.13. The inequality
\[
16N_H(m, 2s) \leq \begin{cases} 
(s + 1)^4N_H(m - 1, 2s), & s \equiv 3 \pmod{4} \\
(s + 3)(s + 1)^3N_H(m - 1, 2s), & s \equiv 1 \pmod{4} \\
(s + 2)^3(sN_H(m - 1, 2s) + 1), & s \equiv 2 \pmod{4} \\
(s + 4)(s + 2)^2(sN_H(m - 1, 2s) + 1), & s \equiv 0 \pmod{4} 
\end{cases} \tag{4.33}
\]
holds.

In particular,
\[
16N_H(2, 2s) \leq \begin{cases} 
(s + 1)^4, & s \equiv 3 \pmod{4} \\
(s + 3)(s + 1)^3, & s \equiv 1 \pmod{4} \\
(s + 2)^3(s + 1), & s \equiv 2 \pmod{4} \\
(s + 4)(s + 2)^2(s + 1), & s \equiv 0 \pmod{4} 
\end{cases} \tag{4.34}
\]
Asymptotically,
\[
N_H(2, 2s) \lesssim \frac{1}{16}s^4, \quad s \to \infty. \tag{4.35}
\]

The upper bounds \((4.34)\) are effective, even asymptotically.

5 The numerical results

In this section we present the tables of effective numerical upper bounds for \(N_K(m, p)\) obtained by the recursion combined with other tools, if any. We do not include those of bounds which are worse than known once. Of course, it would be meaningless to tabulate the general inequalities like \((4.3)\). However, some their numerical consequences are presented for the reader convenience.

The tables are organized as follows. The Table 1 contains those known equalities of form \(n = N_K(m, p)\) which are used as the starting data (the input) for the recursion. The equalities are enumerated as \(e_1, e_2, \ldots\) Similarly, in the Table 2 the input inequalities \(N_K(m, p) \leq n\) are enumerated as \(i_1, i_2, \ldots\) The Tables 3, 4, 5 contain the resulting upper bounds for \(K = R, C, H\), respectively, enumerated as \(r_0, r_1, \ldots\) within each table. In every of these tables the enumeration is established in ascending order of \(m\). The effectiveness of all results is demonstrated by including of the corresponding GUB \((1.11)\) into the tables. Several cases of known upper bounds which are weaker than ours are mentioned after the tables.

All input data are provided with the bibliographic references. For all results we refer to the input data and to the general facts from Section 4 and, sometimes, from Section 1. Also, there are some cross-references between the Tables of results.

Let us remember three equivalent interpretations of the inequality \(N_K(m, p) \leq n\).

a) There exists a projective cubature formula of index \(p\) with \(n\) nodes on the sphere \(S(m, K)\).

b) There exists an isometric embedding \(l^n_{2, K} \to l^n_{p, K}\).

c) There exists an \(m\)-dimensional Euclidean subspace in the normed space \(l^n_{p, K}\).

Thus, each row of our tables is an existence theorem which can be formulated in any of equivalent form a), b), c) with some concrete values \(m, p, n\).
Table 1: Input equalities \( n = N_K(m, p) \)

| K   | m  | p  | n   | References |
|-----|----|----|-----|------------|
| e1  | R  | 4  | 11  |            |
| e2  | R  | 23 | 6   | 2 300      |
| e3  | R  | 24 | 10  | 98 280     |
| e4  | C  | 2  | 8   | 10         |
| e5  | C  | 2  | 10  | 12         |
| e6  | C  | 4  | 6   | 40         |
| e7  | C  | 6  | 6   | 126        |
| e8  | H  | 5  | 6   | 165        |

Table 2: Input inequalities \( N_K(m, p) \leq n \)

| K   | m  | p  | n   | References |
|-----|----|----|-----|------------|
| i1  | R  | 4  | 6   | 23         |
| i2  | R  | 4  | 10  | 60         |
| i3  | R  | 4  | 18  | 360        |
| i4  | R  | 8  | 10  | 1200       |
| i5  | R  | 8  | 12  | 12 120     |
| i6  | R  | 8  | 14  | 13 200     |
| i7  | R  | 12 | 6   | 756        |
| i8  | R  | 12 | 8   | 4 032      |
| i9  | R  | 12 | 10  | 25 200     |
| i10 | R  | 14 | 4   | 378        |
| i11 | R  | 14 | 6   | 756        |
| i12 | R  | 14 | 8   | 44 982     |
| i13 | R  | 14 | 10  | 53 718     |
| i14 | R  | 16 | 6   | 2 160      |
| i15 | R  | 16 | 8   | 32 780     |
| i16 | R  | 16 | 10  | 65 760     |
| i17 | R  | 16 | 12  | 2 277 600  |
| i18 | R  | 20 | 4   | 1 980      |
| i19 | R  | 20 | 8   | 172 920    |
| i20 | R  | 20 | 10  | 2 263 800  |
| i21 | R  | 24 | 14  | 8 484 840  |
| i22 | R  | 24 | 16  | 207 501 840|
| i23 | R  | 24 | 18  | 2 522 192 400|
| i24 | R  | 26 | 4   | 10 920     |
| i25 | R  | 26 | 6   | 21 840     |
| i26 | R  | 32 | 6   | 73 440     |
| i27 | R  | 36 | 6   | 164 160    |
| i28 | C  | 9  | 4   | 90         |
| i29 | C  | 12 | 10  | 32 760     |
| i30 | C  | 28 | 4   | 4 060      |
| i31 | H  | 3  | 10  | 315        |

Table 3: Results \( N_R(m, p) \leq n \)

| m  | p  | n   | GUB | References |
|----|----|-----|-----|------------|
| r0 | 4  | 14  | 256 | 679        | (4.14)     |
| r1 | 4  | 16  | 360 | 968        | (1.16), i3|
| r2 | 5  | 10  | 360 | 1000       | (4.12), i2|
| r3 | 5  | 14  | 2 048| 3 059    | (4.21)     |
| m   | p   | n     | GUB   | References |
|-----|-----|-------|-------|------------|
| r4  | 5   | 16    | 2881  | 4844       | (4.13), r1 |
| r5  | 5   | 18    | 3600  | 7314       | (4.12), i3 |
| r6  | 6   | 8     | 615   | 1286       | (4.12), (4.13), r1(C) |
| r7  | 6   | 10    | 1296  | 3002       | (4.12), (4.13), r2(C) |
| r8  | 8   | 8     | 1200  | 6434       | (4.12), i4 |
| r9  | 9   | 8     | 4801  | 12869      | (4.12), r8 |
| r10 | 9   | 10    | 7200  | 43757      | (4.12), i5 |
| r11 | 9   | 12    | 72721 | 125969     | (4.12), i5 |
| r12 | 9   | 14    | 105600| 319769     | (4.12), i6 |
| r13 | 10  | 6     | 1280  | 5004       | (4.12), r1(C) |
| r14 | 10  | 8     | 19205 | 24309      | (4.12), i9 |
| r15 | 10  | 10    | 43200 | 92377      | (4.12), r10 |
| r16 | 11  | 6     | 5120  | 8007       | (4.12), r13 |
| r17 | 13  | 6     | 3024  | 18563      | (4.12), i7 |
| r18 | 13  | 8     | 16129 | 125969     | (4.12), i8 |
| r19 | 13  | 10    | 151200| 646645     | (4.12), i9 |
| r20 | 15  | 4     | 757   | 3059       | (4.12), i10 |
| r21 | 15  | 6     | 3024  | 38759      | (4.12), r11 |
| r22 | 15  | 8     | 179929| 319769     | (4.12), r12 |
| r23 | 15  | 10    | 322308| 1961255    | (4.12), i13 |
| r24 | 17  | 6     | 8640  | 74612      | (4.12), i14 |
| r25 | 17  | 8     | 131121| 735470     | (4.12), i15 |
| r26 | 17  | 10    | 394560| 5311734    | (4.12), i16 |
| r27 | 17  | 12    | 13665601| 30421754  | (4.12), i17 |
| r28 | 18  | 6     | 34560 | 100946     | (4.12), r24 |
| r29 | 18  | 8     | 524485| 1081574    | (4.12), r25 |
| r30 | 18  | 10    | 2367360| 9436284   | (4.12), r26 |
| r31 | 20  | 6     | 3795  | 177099     | (4.12), i1, e8 |
| r32 | 21  | 4     | 3961  | 10625      | (4.12), i18 |
| r33 | 21  | 6     | 15180 | 230229     | (4.12), r31 |
| r34 | 21  | 8     | 691681| 3108104    | (4.12), i19 |
| r35 | 21  | 10    | 13582800| 30045014  | (4.12), r20 |
| r36 | 22  | 4     | 7923  | 12649      | (4.12), r32 |
| r37 | 22  | 6     | 60721 | 296009     | (4.12), r33 |
| r38 | 22  | 8     | 2766725| 4292144   | (4.12), r34 |
| r39 | 24  | 4     | 9200  | 17549      | (4.12), r40 |
| r40 | 24  | 6     | 9200  | 475019     | (4.12), e2 |
| r41 | 24  | 8     | 98280 | 7888724    | (4.12), e3 |
| r42 | 25  | 6     | 36800 | 593774     | (4.12), r40 |
| r43 | 25  | 8     | 393121| 10518299   | (4.12), r41 |
| r44 | 25  | 10    | 589680| 131128139  | (4.12), e3 |
| r45 | 25  | 12    | 67878720| 1251677699| (4.12), r46 |
| r46 | 25  | 14    | 67878720| 9669554099| (4.12), i21 |
| r47 | 25  | 16    | 1660014721| 62852101649| (4.12), i22 |
| r48 | 25  | 18    | 252219240000| 353697121049| (4.12), i23 |
| r49 | 26  | 8     | 1572485| 13884155   | (4.12), r43 |
| r50 | 26  | 10    | 3538080 | 183579395 | (4.12), r44 |
| r51 | 26  | 12    | 543029760| 1852492995| (4.12), r52 |
| r52 | 26  | 14    | 543029760| 15084504385| (4.12), r46 |
| r53 | 26  | 16    | 13280117760| 103077446705| (4.12), r47 |
| r54 | 26  | 18    | 252219240000| 608359048205| (4.12), r48 |
| r55 | 27  | 4     | 21841 | 27404      | (4.12), i24 |
| r56 | 27  | 6     | 87360 | 906191     | (4.12), i25 |
| r57 | 27  | 8     | 6289941| 18156203   | (4.12), r49 |
Continued from previous page

Table 4: Results $N_C(m, p) \leq n$

| $m$ | $p$ | $n$ | GUB | References |
|-----|-----|-----|-----|------------|
| r0  | 2   | 18  | 50  | 99         | (4.3)       |
| r1  | 3   | 8   | 123 | 224       | (4.2), e4   |
| r2  | 3   | 10  | 216 | 440       | (4.2), e5   |
| r3  | 3   | 18  | 2 500 | 3 024        | (4.2), r0   |
| r4  | 5   | 6   | 320 | 1 224     | (4.2), e6   |
| r5  | 7   | 6   | 1 008 | 7 055        | (4.2), e7   |
| r6  | 8   | 6   | 2 160 | 14 399       | (4.2), r4   |
| r7  | 9   | 6   | 17 280 | 27 224       | (4.2), r6   |
| r8  | 10  | 4   | 362 | 3 024     | (4.2), e8   |
| r9  | 11  | 4   | 1450 | 4 355        | (4.2), r8   |
| r10 | 12  | 4   | 5802 | 6 083        | (4.2), r9   |
| r11 | 12  | 6   | 32 760 | 132 495      | (1.16), r12 |
| r12 | 12  | 8   | 32 760 | 1 863 224    | (1.16), r12 |
| r13 | 13  | 6   | 73 600 | 207 024      | (1.21), (1.14), r1 |
| r14 | 13  | 8   | 393 123 | 3 312 399    | (1.21), r2   |
| r15 | 13  | 10  | 589 680 | 38 291 343   | (1.21), i20 |
| r16 | 14  | 6   | 174 720 | 313 599      | (1.21), (1.14), i25 |
| r17 | 14  | 8   | 4 717 479 | 5 664 399   | (1.21), r14 |
| r18 | 14  | 10  | 63 685 440 | 73 410 623 | (1.21), (1.15), r50(R) |
| r19 | 17  | 6   | 587 520 | 938 960     | (1.21), (1.14), i26 |
| r20 | 19  | 6   | 1 313 280 | 1 768 899    | (1.21), (1.14), i27 |
| r21 | 20  | 4   | 16 242 | 189 224    | (4.2), i30   |
| r22 | 30  | 4   | 64 970 | 216 224    | (4.2), r21   |

Table 5: Results $N_H(m, p) \leq n$

| $m$ | $p$ | $n$ | GUB | References |
|-----|-----|-----|-----|------------|
| r1  | 4   | 10  | 20 790 | 60 983     | (1.20), e1, i31 |
| r2  | 5   | 4   | 165  | 824       | (1.16), e8   |
| r3  | 6   | 4   | 1 324 | 1 715       | (1.20), (1.14), r2 |
| r4  | 6   | 6   | 2 640 | 26 025     | (1.20), (1.14), e8 |
| r5  | 7   | 6   | 42 240 | 63 699      | (1.20), (1.14), r4 |
| r6  | 7   | 10  | 6 486 480 | 8 836 463  | (1.21), i2, e3   |

In conclusion let us note that

- $r_0(R)$ improves $N_R(4, 14) \leq 264$ from [8].
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- $r_{31}(R)$ improves $N_R(20, 6) \leq 3960$ from [1].
- $r_{39}(R)$ improves $N_R(24, 4) \leq 13104$ from [1].
- $r_{40}(R)$ improves $N_R(24, 6) \leq 26213$ from [1].
- $r_{0}(C)$ improves $N_C(2, 18) \leq 60$ from [15].

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