Solving a problem of theory of viscoelasticity for continuously accreted solids with the satisfaction of static boundary conditions in the integral sense

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Abstract. The paper is devoted to the study of a non-classical mechanical problem for solids additively formed in the process of continuous accreting. The problem is being formulated within the framework of the linear mathematical theory of surface growth of ageing viscoelastic solids developed by Professor A. V. Manzhirov. It is considered a quasi-static problem in which the static boundary conditions on some parts of the boundary surface of the being formed solid are satisfied in the integral sense. The main obstacle to construct the solution of such problems is the expansion of these surface parts due to influx of the additional material. The paper provides a technique allowing to construct the analytical solution of the problem in question and to trace herewith the evolvement of the stress-strain state of the considered additive-formed solids in the course and on completion of the processes of their continuous accretion.

1. Introduction
Technological processes of additive manufacturing materials, elements of structures and machine parts are accompanied by an increase in the size and shape of the solids obtained in these processes due to the addition of new material layers to their surface. A distinctive feature of this kind of processes is the inherent fact that the considered solid does not exist in its final form before the beginning of its deformation process, as it is always considered in the classical mechanics of deformable solids, and is formed already in the course of this process. Therefore, in the mechanical analysis of additive processes, it is necessary to simultaneously take into account both external loads acting in the simulated technological process on the solid under consideration, including loads acting on the additional material being attached to it, and the geometric, kinematic and force characteristics of the process of gradual attachment of new material elements to the solid. Such an account cannot be correctly carried out within the framework of classical mechanics of solid, even if the traditional equations and conditions for a time variable spatial domain are formulated. This is due to the fundamental absence for the entire additive-formed solid as a whole such a configuration in which there would be no any stresses and from which it would be possible to count out traditional strain measures. Thus, the problems associated with the study of the mechanical behavior of additive-formed solids, having a number of specific features, constitute a special class of problems of solid mechanics, in contrast, for instance, from the problems associated with the study of the processes of deformation of...
solids subjected to gradual removal of the material. The state-of-the-art of the mechanics of additive processes can be examined, for example, on the basis of works [1–14].

Many artificial and natural materials used in additive manufacturing processes exhibit pronounced rheological properties, in particular, the properties of deformation heredity (viscoelasticity) and ageing (weakening over time the deformation properties of the material regardless of the mechanical loads acting on it). Additive processes in which solids are formed from such materials are quite difficult to simulate, since the time-evolving deformation reaction of such a solid to the impacts applied to it continuously interacts with its mechanical response to the process of adding new material elements, and during pauses in the growth process, as well as after the final termination of this process, the formed solid continues to change its stress-strain state. However, the study of this kind of processes is relevant from the point of view of a variety of engineering and physical applications.

The present paper is devoted to the study of a non-classical mechanical problem for solids additively formed in the process of their continuous accretion. The problem is being formulated within the framework of the linear mathematical theory of surface growth of ageing viscoelastic solids developed by Professor A. V. Manzhirov. It is considered a quasi-static problem in which the static boundary conditions on some parts of the boundary surface of the being formed solid are satisfied in the integral sense. The main obstacle to construct the solution of such problems is the expansion of these surface parts due to influx of the additional material. The paper provides a technique allowing to construct the analytical solution of the problem in question. The constructed solution allows to trace the evolution of the stress-strain state of the considered additive-formed solids during and after the processes of continuous accreting.

2. Solution of classical elastic problem

The elastic isotropic homogeneous material is described in the framework of linear solid mechanics by the equation of state

$$T(r) = G[2E(r) + (\nu - 1)1 \text{tr} E(r)]$$

where \(T\) and \(E\) are the stress and small strain tensors at the point of the solid with the radius-vector \(r\), \(1\) is the unit tensor of the second rank, \(G = \text{const}\) is the shear modulus, \(\nu = (1 - 2\nu)^{-1}\), and \(\nu = \text{const}\) is the Poisson’s ratio.

Let us consider a classical (non-accreted) elastic truncated circular cone of length \(l > 0\) with the ends of radii \(a > 0\) and \(b > 0\) (it can be \(a > b\) or \(a < b\)). Suppose that a central axial tensile force of magnitude \(P\) is applied to the end surface with the radius \(b\) (it can be \(P > 0\), in the case of true tension, or \(P < 0\), in the case of axial compression of the cone). The cone end with the radius \(a\) is rigidly fixed.

We introduce the polar cylindrical coordinate system \((\rho, \varphi, z)\) in the region occupied by the cone by the following way. The axial coordinate \(z\) is measured from the cone end of the radius \(a\) in the direction to the cone end of the radius \(b\). The radial coordinate \(\rho\) is measured from the symmetry axis of the cone in its any cross-section. The angle coordinate \(\varphi\) determines the angular position in a cross-section of the cone. Let \(\{e_\rho(\varphi), e_\varphi(\varphi), e_z\}\) be the normalized local basis of the introduced cylindrical coordinate system \((\rho, \varphi, z)\): the variable unit vector \(e_\rho(\varphi)\) indicates the radial direction from each point in the cross-section, i.e. the inverse direction to the direction from the point to the centre of the cross-section, the constant unit vector \(e_\varphi(\varphi)\) indicates the circumferential direction in the cross-section, and the constant unit vector \(e_z\) indicates the direction along the symmetry axis of the cone from the cone end \(\{z = 0\}\) of the radius denoted by \(a\) to its end \(\{z = l\}\) of the radius denoted by \(b\). So the radius-vector \(r\) of an arbitrary point of the considered conical solid can be represented in the form \(r = e_\rho(\varphi)\rho + e_z z\).

A cone is considered to be sufficiently long in the axial direction as compared with its transverse dimensions, i.e. the cone length \(l\) is sufficiently large in comparison with the maximum
of the ends diameters 2a and 2b. In this case, according to the Saint-Venant principle, the particular distribution of the stresses acting on the cone end does not influence the stress-strain state of the greater part of the cone, and the static conditions on the end can be stated in the integral sense, because the stress distributions inside the cone far from its ends is determined only by the resultant force vector and the resultant moment of the end stresses. To construct these distributions we can use the known solution of the problem of tensioning an infinitely long pointed cone with an axial force \( P \) applied to its vertex \([15]\).

To do it we introduce the auxiliary spherical coordinate system \((r, \vartheta, \psi)\) with the center at the pointed cone vertex: \( r \) is the distance from the vertex to the considered arbitrary point of the pointed cone, \( \vartheta \in [0, 2\pi) \) is the longitudinal angle counted around the axis of symmetry of the cone, \( \vartheta \in (0, \vartheta_0) \) is the pole angle counted from the axis of symmetry inside the solid, where \( \vartheta_0 \in (0, \pi/2) \) is the taper angle of the pointed cone. Let \( \{e_r(\vartheta, \psi), e_\vartheta(\vartheta, \psi), e_\psi(\psi)\} \) be the normalized local basis of the spherical coordinate system. Relative to this basis the stress tensor corresponding to the mentioned solution has the form

\[
T = e_r e_r \sigma_r + e_\vartheta e_\vartheta \sigma_\vartheta + e_\psi e_\psi \sigma_\psi + (e_r e_\vartheta + e_\vartheta e_r) \tau_{r\vartheta}
\]

(2)

with the components equaling to

\[
\begin{align*}
\sigma_r &= \frac{C}{r^2} \left[ 1 + \cos \vartheta_0 - (3\varrho + 1) \cos \vartheta \right], \\
\sigma_\vartheta &= \frac{C}{r^2} \left( 1 - \frac{1 + \cos \vartheta_0}{1 + \cos \vartheta} \right) \cos \vartheta, \\
\sigma_\psi &= \frac{C}{r^2} \left( \cos \vartheta - \frac{1 + \cos \vartheta_0}{1 + \cos \vartheta} \right), \\
\tau_{r\vartheta} &= \frac{C}{r^2} \left( 1 - \frac{1 + \cos \vartheta_0}{1 + \cos \vartheta} \right) \sin \vartheta.
\end{align*}
\]

(3)

We have used here the constant

\[
C = \frac{P}{2\pi (\varrho \cos^3 \vartheta_0 - \cos^2 \vartheta_0 + \cos \vartheta_0 - \varrho)}.
\]

To apply the written solution to the formulated problem of theory of elasticity for a truncated conical piece it is necessary to extend the lateral surface of the considered truncated cone of length \( l \) into both sides along the axial direction so as to obtain infinitely long cone with a vertex lied at the axis. Denote this vertex as \( O' \). After this it is necessary to analyze separately the cases \( a < b \) and \( a > b \).

In the case \( a < b \) the reference end \( \{z = 0\} \) of the truncated cone lies closer to the vertex \( O' \) of a pointed cone than another one \( \{z = l\} \). In the case \( a > b \) the end \( \{z = 0\} \) lies farther from the vertex \( O' \) than the end \( \{z = l\} \). Therefore, the above introduced axial unit vector \( e_z \) is codirected, in the case \( a < b \), or oppositely directed, in the case \( a > b \), to the vector \( e_r(0, \psi) \) defining the ray \( \{\vartheta = 0\} \), and the vector \( e_\psi \) at any point is respectively codirected or oppositely directed to the corresponding vector \( e_\vartheta \). Thus, the transition from the additionally introduced spherical coordinate system to the original cylindrical one is maintained by means of the following transformation of the local bases:

\[
\begin{align*}
e_r &= e_\rho \sin \vartheta \pm e_z \cos \vartheta \\
e_\vartheta &= e_\rho \cos \vartheta \mp e_z \sin \vartheta \\
e_\psi &= \pm e_\varphi
\end{align*}
\]

(4)
Here and everywhere below the upper signs (“+” or “−”) correspond to the case \(a < b\), the lower ones — to the case \(a > b\). Meanwhile, we also need to put

\[
\begin{align*}
\vartheta_0 &= \pm \omega, \quad \cos \vartheta_0 = \cos \omega, \quad \omega = \arctan \frac{b - a}{l}
\end{align*}
\]

where \(\omega\) is, accurate to sign, the taper angle of the considered truncated conical piece, \(\omega > 0\) if \(a < b\), and \(\omega < 0\) if \(a > b\). In any case we have a sufficiently small value of \(|\omega|, |\omega| \ll \arctan(1/2)\), so long as

\[
|\tan \omega| = \frac{|b - a|}{l} < \max\{a, b\} = \frac{1}{2} \cdot \max\{2a, 2b\} \ll \frac{1}{2}.
\]

Taking into account representation (2) and transformation (4) we have, obviously, the following representation of the stress tensor in the originally polar coordinate system:

\[
\mathbf{T} = e_\rho e_\rho \sigma_\rho + e_\varphi e_\varphi \sigma_\varphi + e_z e_z \sigma_z + (e_z e_\rho + e_\rho e_z) \tau_{z\rho}
\]

where the corresponding stresses are equal to

\[
\begin{align*}
\sigma_\rho &= \sigma_r \sin^2 \vartheta + \sigma_\varphi \cos^2 \vartheta + 2\tau_{r\varphi} \sin \vartheta \cos \vartheta, \\
\sigma_\varphi &= \sigma_\psi, \\
\sigma_z &= \sigma_r \cos^2 \vartheta + \sigma_\varphi \sin^2 \vartheta - 2\tau_{r\varphi} \sin \vartheta \cos \vartheta, \\
\tau_{z\rho} &= \pm(\sigma_r - \sigma_\varphi) \sin \vartheta \cos \vartheta \pm \tau_{r\varphi}(\cos^2 \vartheta - \sin^2 \vartheta).
\end{align*}
\]

We obtain the following expressions for these stresses substituting (3) into (5):

\[
\begin{align*}
\sigma_\rho(z, \rho) &= \frac{C}{r^2(z, \rho)} \left\{ \frac{1 + \cos \omega}{1 + \cos \vartheta(z, \rho)} - \frac{\cos \omega + 3\varphi \sin^2 \vartheta(z, \rho)}{1 + \cos \vartheta(z, \rho)} \right\}, \\
\sigma_\varphi(z, \rho) &= \frac{C}{r^2(z, \rho)} \left( \frac{\cos \vartheta(z, \rho)}{1 + \cos \vartheta(z, \rho)} - \frac{1 + \cos \omega}{1 + \cos \vartheta(z, \rho)} \right), \\
\sigma_z(z, \rho) &= \frac{C}{r^2(z, \rho)} \left[ \cos \omega - 3\varphi \cos^2 \vartheta(z, \rho) \right] \cos \vartheta(z, \rho), \\
\tau_{z\rho}(z, \rho) &= \pm \frac{C}{r^2(z, \rho)} \left[ \cos \omega - 3\varphi \cos^2 \vartheta(z, \rho) \right] \sin \vartheta(z, \rho), \\
C &= \frac{P}{2\pi \left( \varphi \cos^2 \omega - \cos^2 \omega + \cos \omega - \varphi \right)}.
\end{align*}
\]

It is easily seen that, if \(M\) is an arbitrary point with the coordinates \(z\) and \(\rho\) inside the truncated cone, and \(M'\) is the axis projection of this point, than

\[
\pm \tan \vartheta(z, \rho) = \frac{MM'}{\pm O'M'} = \frac{\rho}{F(z)} \tan \omega, \quad F(z) = \left( 1 - \frac{z}{l} \right) \cdot a + \frac{z}{l} \cdot b
\]

where \(a \leq F(z) \leq b\) is the shape function of the cone lateral surface: this surface is described by the equation \(\rho = F(z)\). So in any case \(a \leq b\) we can write

\[
\begin{align*}
\cos \vartheta(z, \rho) &= \frac{1}{\sqrt{\tan^2 \vartheta(z, \rho) + 1}} = \frac{1}{\sqrt{\frac{\rho^2 \tan^2 \omega}{F^2(z)} + 1}}, \\
\pm \sin \vartheta(z, \rho) &= \pm \tan \vartheta(z, \rho) \cos \vartheta(z, \rho) = \frac{\rho \tan \omega}{F(z)} \sqrt{\frac{\rho^2 \tan^2 \omega}{F^2(z)} + 1}, \\
r(z, \rho) &= \sqrt{(MM')^2 + (O'M')^2} = \sqrt{\rho^2 + \frac{F^2(z)}{\tan^2 \omega}} = \frac{F(z)}{\tan \omega} \sqrt{\frac{\rho^2 \tan^2 \omega}{F^2(z)} + 1}.
\end{align*}
\]
3. Viscoelastic constitutive relations and corresponding problem solution

Let us consider the same problem of classical solid mechanics as in the previous Section but for the ageing viscoelastic material. We will consider the viscoelastic ageing material described by the homogeneous isotropic constitutive equation [16,17]

\[ T(r,t) = H_{\tau_0}^{-1}[2E(r,t) + (\kappa - 1)1 \text{ tr } E(r,t)] \]

being the operator analogue of equation (1). In (9) the stress tensor \( T \) and the small strain tensor \( E \) are already depending on the timing \( t \) which is counted from the material nucleation moment. The value \( \tau_0 \) is the time of occurrence of stresses at the solid points. The linear viscoelasticity operator \( H_{\tau_0} \) and the inverse to it operator \( H_{\tau_0}^{-1} \) are defined by the following integral representations:

\[ H_{\tau_0} g(t) = \frac{g(t)}{G(t)} - \int_{\tau_0}^{t} \frac{g(\tau)}{G(\tau)} K(t,\tau) \, d\tau, \quad H_{\tau_0}^{-1} h(t) = G(t) \left[ h(t) + \int_{\tau_0}^{t} h(\tau) R(t,\tau) \, d\tau \right], \]

where \( g(t) \) and \( h(t) \) are arbitrary functions depending on time, \( K(t,\tau) \) and \( R(t,\tau) \) are the kernels of creep and relaxation, \( G(t) \) is the elastic shear modulus varying in time due to material ageing. The creep kernel can be expressed through various characteristics of the material, describing its behaviour in this or that elementary stress state. For example, using the characteristics for the pure shear state it will be

\[ K(t,\tau) = G(\tau) \frac{\partial \Delta(t,\tau)}{\partial \tau}, \quad \Delta(t,\tau) = \frac{1}{G(\tau)} + \Omega(t,\tau), \]

where \( \Omega(t,\tau), t \geq \tau \geq 0 \), is the so called creep measure for the pure shear. The function \( \Delta(t,\tau) \), named the specific strain function, describes the evolution over time \( t \) of specific (per unit of the acting shear stress) shear strain caused by the constant stress state of pure shear created at the time moment \( \tau \); immediately at the loading time \( t = \tau \) we have only the elastic material response \( \Delta(\tau,\tau) = 1/G(\tau) \), so, by definition, \( \Omega(\tau,\tau) \equiv 0 \).

Remark that constitutive relations similar to (9) are widely used to describe the mechanical behaviour of various natural and artificial stone (in particular, concrete), polymers, soil, ice, wood. Typical experimental creep curves for such materials represent the evolution with time \( t \) of the specific longitudinal strain

\[ \varepsilon(t,\tau) = \left[ \mathcal{H}_{\tau_0} \sigma_0 \right](t) / \sigma_0 = \frac{\Delta(t,\tau)}{2(1 + \nu)} \]

at the uniaxial tension of a specimen by a constant stress \( \sigma_0 \) applied to it at the different time moments \( \tau \). Similar kind of curves can be borrowed, for example, from [18].

It is evident that the elastic state equation (1) follows from more general state equation (9) by taking \( \Omega(t,\tau) \equiv 0 \), \( G(t) = \text{const.} \).

One can prove correspondence principles in the linear theory of viscoelasticity [17] according to which the stresses inside the considered ageing viscoelastic solid under the stipulated static and kinematic conditions coincide at each time instant with the same for the analogous purely elastic solid and consequently are given by (6), (7), (8). In the viscoelastic case the quantity \( P \) in (7) is to be understood as the tensile force value acting on the cone end at the current time instant \( t \).

4. Accretion problem statement

Let \( t = t_0 > 0 \) be the instant of the loading application to the considered in the previous Section ageing viscoelastic truncated cone under axial force \( P(t) \). Further we will deem that at the time
\(t = t_1 \geq t_0\) the process of gradual axisymmetric thickening of the considered conical solid by accreting the additional material to its lateral surface starts. Thickening occurs in such a way that in each time moment the accreted solid maintains the shape of a circular truncated cone of length \(l\). This process is continuous in time, i.e. an infinitely thin layer of material attaches to the solid each infinitely small period of time, and goes on up to the time moment \(t = t_2 > t_1\). The added material is supposed identical to the original one and is being added to the lateral surface of the cone initially free of any stresses. In other words, we believe that the process of attaching layers of additional material to the surface of the considered solid does not cause the appearance of nonzero stresses in this solid near its lateral surface. That means that

\[
\mathbf{T}(\mathbf{r}, \tau_a(\mathbf{r})) = 0
\]

(11)

where \(t = \tau_a(\mathbf{r})\) is the moment of attaching the particle \(\mathbf{r}\) of the additional material to the solid surface, \(t_1 \leq \tau_a(\mathbf{r}) \leq t_2\). Due to the axial symmetry of the considered accreting process

\[
\tau_a(\mathbf{r}) = \tau_a(z, \rho).
\]

(12)

Note that condition (11) provides amongst other things the equality to zero of the stress vector \(\mathbf{n} \cdot \mathbf{T}\) at the current lateral surface of the cone, i.e. absence of any loading onto this surface during the accretion process.

Changing the geometry of the considered conical solid due to its continuous accreting is completely defined by defining laws of increasing the radii \(a(t)\) and \(b(t)\). For any \(t \geq t_0\) these functions are continuous and non-decreasing.

In the process of continuous accreting and after its completion the time-varying central axial force \(F(t)\) continues to act on the end surface of the cone with the radius \(b(t)\), and the end with the radius \(a(t)\) is permanently rigidly fixed.

The lateral surface of the cone under consideration is moving in space due to the influx of additional material (accreting). Its current shape is described by the equation \(\rho = F(z, t)\) where

\[
F(z, t) = \left(1 - \frac{z}{l}\right) \cdot a(t) + \frac{z}{l} \cdot b(t)
\]

is the shape function of the accreted cone current lateral surface, \(a(t) \leq F(z, t) \leq b(t)\). The trace of its passing in space forms the additional part of the considered solid. At all time moments \(t \in (t_1, t_2)\) the lateral surface represents the actual growing surface of the considered accreted conical solid and is the \(l\)-level surface of the function \(\tau_a(\mathbf{r})\). Therefore the unit vector directing the external normal to this surface at each point \(\mathbf{r}\) equals to

\[
\mathbf{n}(\mathbf{r}) = \frac{\nabla \tau_a(\mathbf{r})}{|\nabla \tau_a(\mathbf{r})|}.
\]

(13)

Also we can use the following representation for this vector:

\[
\mathbf{n}(\mathbf{r}) = \mathbf{e}_{\rho}(\varphi) \cos \omega(\tau_a(\mathbf{r})) - \mathbf{e}_z \sin \omega(\tau_a(\mathbf{r})), \quad \omega(t) = \arctan \frac{b(t) - a(t)}{l}
\]

where \(\omega(t)\) is the current taper angle of the accreted conical piece.

Note that for any \(t \in (t_1, t_2)\) and \(z \in (0, l)\) the functions \(\tau_a(\mathbf{r}) = \tau_a(z, \rho)\) and \(F(z, t)\) satisfy the identity

\[
\tau_a(z, F(z, t)) = t.
\]

(14)

For \(z = 0\) and \(z = l\) identity (14) is valid on the time intervals of strict monotonicity of the function \(a(t)\) or \(b(t)\) respectively.

We will investigate the quasi-static evolution of the stress-strain state of the considered ageing viscoelastic conical solid under specified conditions of loading after the start and upon the completion of the described continuous process of its accreting.
5. Features in viscoelastic material description for accreted solids

If the state equation (9) is used to describe the mechanical behaviour of a growing solid which is formed additively by attaching the additional material to the current solid surface then the time \( \tau_0 \) of occurrence of stresses at solid points in (9) must depend on the variable radius-vector \( r \) of these points: \( \tau_0 = \tau_0(r) \) in the viscoelasticity operator \( \mathcal{H}_{\tau_0} \). As in the considered accreting process the additional material is being loaded simultaneously with its attaching to the solid, the corresponding function \( \tau_0(r) \) is to be determined in the following way. In the original part of the considered conical solid the function \( \tau_0(r) \) should be identically equal to the time moment \( t_0 \) of loading of this part, and in the formed due to accreting process additional part of the solid under consideration the function \( \tau_0(r) \) should coincide with the distribution \( \tau^*(r) \) of moments of attaching particles \( r \) of the additional material to the solid. Because of (12) we have \( \tau_0(r) = \tau_0(z, \rho) \).

We denote for the further text the concise notation

\[
g^0(r, t) = \mathcal{H}_{\tau_0(r)} g(r, t), \quad g^0(t) = \mathcal{H}_{t_0} g(t) \tag{15}
\]

for an arbitrary function \( g(r, t) \) of solid point \( r \) and time \( t \), and for an arbitrary function of time \( g(t) \) which is not associated with specific points of considered solid.

6. Correlations for accretion process

The standard for the classical solid mechanics kinematic description of the deformation process is not suitable for any accreted solid. That is due to the objective lack for such a solid of natural (unstressed) configuration. However, it is clear that the particles of the new material after the attaching to the surface of the solid continue to move as a part of cohesive continuum. This means that in the region of space occupied by the whole accreted solid at any time moment a smooth velocity field \( v(r, t) \) of the motion of its particles is uniquely determined. Therefore, the problem on such a solid deformation can be stated in terms of velocities. In this case the strain velocity tensor \( D(r, t) = (\nabla v^T + \nabla v) / 2 \) may play a part of the deforming process characteristic in the formulation of the defining relations of the material. The adopted equation of state (9) can be rewritten by using this tensor in the form [19]:

\[
S = 2D + (\chi - 1) \text{tr} D, \tag{16}
\]

where we have introduced the so-called operator stress velocity tensor \( S(r, t) = \partial T^0 / \partial t \). State equation (16) is mathematically similar to elastic equation (1) in which the relative stress tensor \( T/G \) is replaced with the operator stress velocity tensor \( S \) and the small strain tensor \( E \) is replaced with the strain velocity tensor \( D \).

The described approach requires knowing the whole history of changing the state of additional material elements up to their inclusion in the composition of the solid considered. In the here studied accreting process the additional material is supposed to be initially free of stresses, so the mentioned history is reduced to the specific initial condition (11).

The equation manifested instantaneous local equilibrium of an accreted solid has evidently the same form as in the classical solid mechanics for the solid of permanent composition. In the considered case of mass forces absence it is the form

\[
\nabla \cdot T(r, t) = 0. \tag{17}
\]

One managed to prove [19] that for the simulated accreting process (in the case of loading absence on any future and actual accreting surface of the considered accreted solid during the
whole process of its deformation) equation (17) induces the similar differential equations for the tensors $T^0$ and $S$:

$$\nabla \cdot T^0(r, t) = 0, \quad (18)$$

$$\nabla \cdot S(r, t) = 0. \quad (19)$$

Equations (18) and (19) are fair at every moment of time $t > t_1$ in the whole region of space occupied by the considered solid at this moment. It should be emphasized that these equations are not a trivial consequence of the equilibrium equation (17) as in the case of accreting the solid the integral operator $H_{\tau_0(r)}$ and the operator of divergence $\nabla \cdot$ do not commute in general because of the principal dependence of time $\tau_0$ of the occurrence of stresses at a solid point on the position $r$ of this point.

We can also show that the specific initial condition (11) implies the following condition on the components of the tensor $S$ on the moving due to material inflow lateral cone surface:

$$n \cdot S = 0, \quad \rho = F(z, t). \quad (20)$$

Remark that this condition is mathematically similar to the standard in solid mechanics boundary condition for a stress-free surface (the tensor $S$ takes place of the stress tensor $T$ in such classical condition).

Indeed, for any point $r$ in the accreted part of the cone we have

$$T^0(r, t) = T^0(r, \tau_*(r)) + \int_{\tau_*(r)}^t S(r, \tau) d\tau = \int_{\tau_*(r)}^t S(r, \tau) d\tau \quad (21)$$

as according to (15), (10), and (11) it must be

$$T^0(r, \tau_*(r)) = T(r, \tau_*(r))/G(\tau_*(r)) = 0.$$

Let us calculate the divergence of the operator stress tensor $T^0$ in representation (21):

$$\nabla \cdot T^0(r, t) = \int_{\tau_*(r)}^t \nabla \cdot S(r, \tau) d\tau - \nabla \tau_*(r) \cdot S(r, \tau_*(r)) = -|\nabla \tau_*(r)| n(r) \cdot S(r, \tau_*(r))$$

where we have taken equation (19) and representation (13) into account. Proceeding from the latter result and equation (18), with identity (14) in mind, we arrive at (20).

After the completion of accreting, when $t > t_2$, we are to set the traditional condition of the stress vector nonentity on the lateral surface of the completely formed truncated cone: $n \cdot T = 0$.

Acting on this condition with the operator $H_{\tau_0(r)}$ and differentiating the result with respect to time $t$, we will see that the boundary condition (20) saves force also after the time period $(t_1, t_2)$ of the accretion process duration.

7. Transformation of integral static conditions on expanding surface

The above stipulated integral static conditions on the accreted cone end $\{z = l\}$ — for the resultant vector and the resultant moment of the corresponding surface loading distribution — have the form

$$\int_{\{z=l\}} e_z \cdot T dS = e_z P(t), \quad \int_{\{z=l\}} e_\rho \rho \times (e_z \cdot T) dS = 0. \quad (22)$$

Remember that the classical elastic problem for the non-accreted truncated cone was solved in Section 2 under the same integral static conditions.
Let us consider the integral of the tensor \( S(r, t) = S(\rho, \varphi, z, t) \) over the cone end surface \( \{z = l\} \) expanding due to the additional material inflow. This integral can be written as follows:

\[
\int_{\{z=l\}} S(r, t) dS = \int_0^{2\pi} d\varphi \int_0^{b(t)} S(\rho, \varphi, l, t) \rho d\rho. \tag{23}
\]

We can transform the inner integral in (23) by using the rule of differentiation of an integral over the coordinate \( \rho \) with respect to the time \( t \) as a parameter:

\[
\int_0^{b(t)} S(\rho, \varphi, l, t) \rho d\rho = \int_0^{b(t)} \frac{\partial T^\circ(\rho, \varphi, l, t)}{\partial t} \rho d\rho = -\frac{\partial}{\partial t} \int_0^{b(t)} T^\circ(\rho, \varphi, l, t) \rho d\rho - T^\circ(b(t), \varphi, l, t) b(t) b'(t). \tag{24}
\]

We reveal now the symbolic notation \((\ )^\circ\) in the integral standing in the right hand side of (24), see (15) and (10):

\[
\int_0^{b(t)} T^\circ(\rho, \varphi, l, t) \rho d\rho = \frac{1}{G(t)} \int_0^{b(t)} T(\rho, \varphi, l, t) \rho d\rho - \int_0^{b(t)} \rho d\rho \int_{\tau_0(t, \rho)}^t \frac{T(\rho, \varphi, l, \tau)}{G(\tau)} K(t, \tau) d\tau. \tag{25}
\]

And after that we change the order of integration in the repeated integral in (25):

\[
\int_0^{b(t)} \rho d\rho \int_{\tau_0(t, \rho)}^t \frac{T(\rho, \varphi, l, \tau)}{G(\tau)} K(t, \tau) d\tau = \int_{\tau_0(t, \rho)}^t \frac{K(t, \tau)}{G(\tau)} d\tau \int_0^{b(\tau)} T(\rho, \varphi, l, \tau) \rho d\rho. \tag{26}
\]

We consider here that \( b(t) \equiv b(t_1) \) if \( t_0 \leq t \leq t_1 \), and \( b(t) \equiv b(t_2) \) if \( t \geq t_2 \). From (25) and (26) we conclude that

\[
\int_{\{z=l\}} T^\circ(r, t) dS = \left[ \int_{\{z=l\}} T(r, t) dS \right]^\circ. \tag{27}
\]

Let us treat also the term beyond the integrals in (24). We have

\[
T^\circ(b(t), \varphi, l, t) = \frac{T(b(t), \varphi, l, t)}{G(t)} - \int_{\tau_0(l, b(t))}^t \frac{T(b(t), \varphi, l, \tau)}{G(\tau)} K(t, \tau) d\tau. \tag{28}
\]

On the intervals of strict monotonicity of the function \( b(t) \) we have identity (14) and therefore it will be \( \tau_0(l, b(t)) \equiv t \) at the lower integration limit. So the integral in (28) becomes zero. Meanwhile for such time values we have

\[
T(b(t), \varphi, l, t) = T(\rho, \varphi, l, \tau_*(l, \rho)) \equiv 0
\]

in accordance with condition (11). So the whole right hand side in (28) is zeroth.

On the intervals of the function \( b(t) \) constancy (before the continuous accretion and after its completing, and also on those time periods in the course of the continuous accretion where only the cone end \( \{z = 0\} \) expands) it will be \( b'(t) \equiv 0 \), therefore the term beyond the integrals in (24) vanishes as well.

Eventually, from (24) and (27) we get

\[
\int_{\{z=l\}} S(r, t) dS = \frac{d}{dt} \left[ \int_{\{z=l\}} T(r, t) dS \right]^\circ.
\]
This equality allows to rewrite the integral static conditions (22) for the velocities of operator stresses keeping their mathematical form in force:

\[ \int_{\{z=l\}} e_z \cdot S dS = e_z \frac{dP^o(t)}{dt}, \quad \int_{\{z=l\}} e_\rho \rho \times (e_z \cdot S) dS = 0. \]

8. Solution of accretion problem

As we can see, it appears that in Sections 4–7 formulated mechanical problem for an accreted conical piece under integral static conditions is mathematically similar to the corresponding classical problem of theory of elasticity for a non-accreted truncated cone solved in Section 2. The latter problem turns into the former one by replacing the relative stress tensor \( T \) with the operator stress tensor \( S \), the small strain tensor \( E \) with the strain velocity tensor \( D \), and the tensile end force \( P(t) \) with the rate \( dP^o(t)/dt \) of the operator force changing. As the result we will have the time evolution of the stress tensor \( S(r, t) \) at each point \( r \) on the time beam \( t > \max\{t_1, \tau_0(r)\} \) covering the entire deformation history of the neighborhood of this point in the composition of the accreted conical solid after the accretion beginning:

\[
S(r, t) = e_\rho(\varphi)e_\rho(\varphi)\zeta_\rho(z, \rho, t) + e_\varphi(\varphi)e_\varphi(\varphi)\zeta_\varphi(z, \rho, t) + e_\rho e_\rho(\varphi) + e_\rho(\varphi)e_\rho \eta_\varphi(z, \rho, t)
\]

where the components of the tensor are changing in compliance with the laws:

\[
\zeta_\rho(z, \rho, t) = \frac{C(t)}{r^2(z, \rho, t)} \left\{ \frac{1 + \cos \omega(t)}{1 + \cos \vartheta(z, \rho, t)} - \frac{\cos \omega(t) + 3\varphi \sin^2 \vartheta(z, \rho, t)}{1 + \cos \vartheta(z, \rho, t)} \right\}
\]

\[
\zeta_\varphi(z, \rho, t) = \frac{C(t)}{r^2(z, \rho, t)} \left[ \cos \vartheta(z, \rho, t) - \frac{1 + \cos \omega(t)}{1 + \cos \vartheta(z, \rho, t)} \right]
\]

\[
\zeta_z(z, \rho, t) = \frac{C(t)}{r^2(z, \rho, t)} \left[ \cos \omega(t) - 3\varphi \cos^2 \vartheta(z, \rho, t) \right] \cos \vartheta(z, \rho, t)
\]

\[
\eta_\varphi(z, \rho, t) = \pm \frac{C(t)}{r^2(z, \rho, t)} \left[ \cos \omega(t) - 3\varphi \cos^2 \vartheta(z, \rho, t) \right] \sin \vartheta(z, \rho, t)
\]

\[
C(t) = \frac{\frac{dP^o(t)}{dt}}{2\pi \left[ \varphi \cos^3 \omega(t) - \cos^2 \omega(t) + \cos \omega(t) - \varphi \right]}
\]

Here the following geometrical functions are used:

\[
\cos \vartheta(z, \rho, t) = 1 + \frac{\rho^2 \tan^2 \omega(t)}{F^2(z, t)} + 1,
\]

\[
\pm \sin \vartheta(z, \rho, t) = \frac{\rho \tan \omega(t)}{F(z, t)} + \frac{\rho^2 \tan^2 \omega(t)}{F^2(z, t)} + 1,
\]

\[
r(z, \rho, t) = \frac{F(z, t)}{\tan \omega(t)} + \frac{\rho^2 \tan^2 \omega(t)}{F^2(z, t)} + 1.
\]

After we have found the evolution of the operator stress velocity tensor \( S \) at each point \( r \), we can reconstruct the evolution of the operator stress tensor \( T^o \) at each point in the following way. If the point \( r \) belongs to the additional, formed in the course of accreting, part of the solid then we are to use formula (21). And at the points \( r \) of the originally existing, before the
accretion, part of the solid we know the distribution \( T^0(\mathbf{r}, t_1) \) of the operator stresses straight before the accretion beginning — from the solution of corresponding classical problem of theory of viscoelasticity discussed in Section 3. So we can calculate

\[
T^0(\mathbf{r}, t) = T^0(\mathbf{r}, t_1) + \int_{t_1}^{t} S(\mathbf{r}, \tau) d\tau, \quad \rho < F(z, t_1), \quad t \geq t_1.
\]

Suppose we have found out the complete evolution of the tensor \( T^0 \) at a considered point \( \mathbf{r} \) of the completely formed conical solid, i.e. the values of this tensor for all time moments since the moment \( t = \tau_0(\mathbf{r}) \) of the stress occurrence at the point \( \mathbf{r} \). Then we can find the complete evolution of the stress tensor \( T \) at this point by using the integral transformation \( \mathcal{H}^{-1}_{\tau_0(\mathbf{r})} \) inverse to the operator \( \mathcal{H}_{\tau_0(\mathbf{r})} \), see (10):

\[
\frac{T(\mathbf{r}, t)}{G(t)} = T^0(\mathbf{r}, t) + \int_{\tau_0(\mathbf{r})}^{t} T^0(\mathbf{r}, \tau) R(t, \tau) d\tau.
\]

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