Corrections to Berry’s phase in a solid-state qubit due to low frequency noise

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We present a quantum open system approach to analyze the non-unitary dynamics of a superconducting qubit when it evolves under the influence of external noise. We consider the presence of longitudinal and transverse environmental fluctuations affecting the system’s dynamics and model these fluctuations by defining their correlation function in time. By using a Gaussian like noise-correlation, we can study low and high frequency noise contribution to decoherence and implement our results in the computation of geometric phases in open quantum systems. We numerically study when the accumulated phase of a solid-state qubit can still be found close to the unitary (Berry) one. Our results can be used to explain experimental measurements of the Berry phase under high frequency fluctuations and design experimental future setups when manipulating superconducting qubits.

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I. INTRODUCTION

Geometric phases are closely linked to the classical concept of parallel transport of a vector on a curved surface. This analogy is particularly clear in the case of a two-level system (a qubit) in the presence of a biased field that changes in time. Take for example a spin-1/2 particle in a changing magnetic field. The general Hamiltonian for such a system is $H = \hbar/2 \vec{R} \cdot \vec{\sigma}$, where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli operators and $\vec{R}$ is the biased field vector. The qubit state can be represented by a point on a sphere of unit radius, called Bloch sphere. This sphere can be embedded in a three dimensional space of Cartesian coordinates, and hence the Bloch vector $\vec{R}$ is a vector whose components ($x, y, z$) single out a point on the sphere. This representation offers a particularly well-suited framework to visualize the dynamics of the qubit, which consists in the qubit state continually precessing about the vector $\vec{R}$, acquiring a dynamical phase $\gamma(t)$. If the evolution is done adiabatically, the qubit also acquires a geometric phase (GP), sometimes called Berry phase.

It is known that the system can retain the information of its motion in the form of this GP, which was first put forward by Pancharatman in optics [1] and later studied explicitly by Berry in a general quatum system [2]. Since then, great progress has been achieved in this field. The application of the GP has been proposed in many fields, such as the geometric quantum computation. Due to its global properties, the GP is propitious to construct fault tolerant quantum gates. In this line of work, many physical systems have been investigated to realize geometric quantum computation, such as NMR (Nuclear Magnetic Resonance) [3], Josephson junction [4], Ion trap [5] and semiconductor quantum dots [6]. The quantum computation scheme for the GP has been proposed based on the Abelian or non-Abelian geometric concepts, and the GP has been shown to be robust against faults in the presence of some kind of external noise due to the geometric nature of Berry phase [7,8]. Then, for isolated quantum systems, the GP is theoretically perfectly understood and experimentally verified. However, it was seen that the interactions play an important role for the realization of some specific operations. As the gates operate slowly compared to the dynamical time scale, they become vulnerable to open system effects and parameters’ fluctuations that may lead to a loss of coherence. Consequently, the study of the GP was soon extended to open quantum systems. Following this idea, many authors have analyzed the correction to the GP under the influence of an external thermal or non-equilibrium environments, using different approaches (see [10–15] and references therein). In all cases, the purely dephasing model considered was a spin-1/2 particle coupled to the environment’s degrees of freedom through a $\sigma_z$ coupling. The interest on the GP in open systems has also been extended to some experimental setups [16].

The GP is a promising building block for noise-resilient quantum operations. Lately, the GP has also been observed in a variety of superconducting systems [17,18]. Superconducting circuits are good candidates to potentially manipulate efficiently quantum information. Current circuit technology allows scaling to large and more complex circuits [19,20]. Several experiments with superconducting Josephson-junction circuits have demonstrated quantum coherent oscillations with a long decay time, probing coherent properties of Josephson qubits and positioning them as useful candidates for applications in quantum computing and quantum communication. Despite the long coherence times of the quantum state, the decoherence induced process still deserves study for using these circuits for the development of a quantum processor. When the two lowest energy lev-
els of a current-biased Josephson junction are used as a qubit, the qubit state can be fully manipulated with low and microwave frequency control currents. Circuits presently being explored combine in variable ratios the Josephson effect and single Cooper-pair charging effects. In all cases the Hamiltonian of the system can be written

\[ H = \hbar \omega_z \sigma_z + h \Omega_R \cos(\omega t + \varphi_R) \sigma_x, \quad (1) \]

where \( h \Omega_R \) is the dipole interaction amplitude between the qubit and the microwave field of frequency \( \omega \) and phase \( \varphi_R \). \( \Omega_R/2\pi \) is the Rabi frequency. This Hamiltonian can be transformed to a rotating frame at the frequency \( \omega \) by means of an unitary transformation, resulting in a new effective Hamiltonian of the form

\[ H_{\text{eff}} = \frac{\hbar}{2} (\Delta \sigma_z + \Omega_x \sigma_x + \Omega_y \sigma_y), \quad (2) \]

where \( \Omega_x = \Omega_R \cos \varphi_R \) and \( \Omega_y = \Omega_R \sin \varphi_R \). This model is similar to the generic situation of a qubit in a changing magnetic field, where \( \mathbf{R} = (\Omega_x, \Omega_y, \Delta) \) and \( \Delta = \omega_q - \omega \) is the detuning between the qubit transition frequency and the applied microwave frequency. In an experimental situation \( \Delta \) can be kept fixed and one can control the biased field to trace circular paths of different radii \( \Omega_R \).

The same physical structures that make these superconducting qubits easy to manipulate, measure, and scale are also responsible for coupling the qubit to other electromagnetic degrees of freedom that can be a source of decoherence via noise and dissipation. Thus, a detailed mechanism of decoherence and noise due to the coupling of Josephson devices to external noise sources is still required. It has been shown that low frequency noise is an important source of decoherence for superconducting qubits. Generally, this noise is described by fluctuations in the effective magnetic field which are directed either in the \( z \)-axis-longitudinal noise- or in a transverse direction -transversal noise. Both types of noise have been phenomenologically modeled by making different assumptions on these fluctuations, such as being due to a stationary, Gaussian and Markovian process \[17\]. Others, have considered that the \( 1/f \) noise must be rooted in a non Gaussian long-time correlation stochastic process. In the context of quantum information, the implication of long-time correlations of stochastic processes is that the effects suffered by the system’s evolution due to the \( 1/f \) noise are protocol or measurement dependent. Apparently, some protocols clearly reveal a non Gaussian nature while others Gaussian approximations attain the main effects in a short-time scale \[21\].

In this manuscript, we shall present a fully quantum open system approach to analyze the non-unitary dynamics of the solid-state qubit when it is considered evolving under the influence of external fluctuations. We consider the qubit coupled in a longitudinal and transversal directions. As a physical example, we study the dynamics and decoherence induced process on the superconducting qubit. We further analyze when the accumulated phase gained by the system after one period can still be found close to the unitary (Berry) one and focus on the importance of the longitudinal coupling as a source of decoherence. The paper is organized as follows: in Section II, we develop a general quantum open system model in order to consider different type of fluctuations (longitudinal and/or transverse) that induce decoherence on the main system. By means of a general master equation for the reduced density matrix of the qubit, we follow the non-unitary evolution characterized by fluctuations, dissipation and decoherence. This gives us a complete insight into the state of the system: complete knowledge of different dynamical time-scales and analysis of the effective role of noise sources inducing decoherence. Section III contains the numerical evaluation of the geometric phase and its noise induced corrections for the several scenarios considered. We shall emphasize the effect of longitudinal and transversal noise on the global geometric phase. Comparison between theory and experiment verify our understanding of the physics underlying the system as a dissipative two-level device. Berry’s phase measurements provide an important constraint to take into account about noise models and their correction induced over the GP, at least, at the times in which the experiments can be performed. The comprehension of the decoherence and dissipative processes should allow their further suppression in future qubits designs or experimental setups. In Section IV we summarize our findings.

**II. MASTER EQUATION APPROACH TO DECOHERENCE IN A SUPERCONDUCTING QUBIT**

We shall begin by deriving a general master equation for the reduced density matrix for the qubit (obtained after tracing out all the environmental degrees of freedom). The dynamics of a generic two-level system steered by a system’s Hamiltonian of the type (where we have set \( h = 1 \) all along the paper)

\[ H_{\text{Total}} = H_q + H_{\text{int}} + H_E, \quad \text{with} \quad (3) \]

\[ H_q = \frac{1}{2}(\Omega_x \sigma_x + \Omega_y \sigma_y), \quad (4) \]

where we have defined a qubit Hamiltonian \( H_q \) similar to that of a solid-state qubit Eq.\[2\] - setting \( \varphi_R = 0 \) for simplicity-, and \( H_E \) is the Hamiltonian of the bath. The interaction Hamiltonian is thought as some longitudinal and transverse noise coupled to the main system:

\[ H_{\text{int}} = \frac{1}{2}(\delta \dot{\omega}_1 \sigma_z + \delta \dot{\omega}_0 \sigma_x). \quad (5) \]

We must note that the system’s unitary dynamics and coupling to the environment is different from the usual purely dephasing models proposed to study geometric
phases in open systems Refs. [10–16]. We shall derive
the master equation in the Born-Markov approximation,
for general noise terms \( \delta \omega_1 \) and \( \delta \omega_0 \) interacting with the system in the \( x \) and \( z \) directions, respectively. We will
consider a weak coupling between system and environ-
ment and that the bath is sufficiently large to stay in a
stationary state. In other words, the total state \( \rho_{\text{SE}} \)
(system and environment) can be split as

\[
\rho_{\text{SE}} \approx \rho(t) \times \rho_{\text{E}},
\]

(6)

for all times. It is important to stress that due to the
Markov regime, we will restrict to cases for which the
self-correlation functions generated at the environment
due to the coupling interaction) would decay faster than
typical variation scales in the system. In this way, the
evolution equation for \( \rho(t) \) is local in time [22]. In the in-
teraction picture, the evolution of the total state is ruled by
the Liouville equation

\[
\dot{\rho}_{\text{SE}} = -i [H_{\text{int}}, \rho_{\text{SE}}],
\]

(7)

where we have denoted the state \( \rho_{\text{SE}} \) in the interaction
picture in the same way than before, just in order to sim-
plify notation. A formal solution of the Liouville equation

\[
\rho_{\text{SE}}(t) = \sum_{n \geq 0} \int_0^t ds_1 \int_0^{s_1} ds_2 \ldots \int_0^{s_n} ds_n \left( \frac{1}{i} \right) \times
\]

\[
\left[ H_{\text{int}}(s_1), [H_{\text{int}}(s_2), \ldots, [H_{\text{int}}(s_n), \rho_{\text{SE}}(0)] \ldots] \right] .
\]

From this expansion, one can obtain a perturbative
master equation, up to second order in the coupling con-
tant between system and environment for the reduced
density matrix \( \rho = \text{Tr}_E \rho_{\text{SE}} \). In the interaction picture
the formal solution reads as

\[
\rho(t) \approx \rho(0) - i \int_0^t ds \text{Tr}_E ([H_{\text{int}}(s), \rho_{\text{SE}}(0)])
\]

\[
- \int_0^t ds_1 \int_0^{s_1} ds_2 \text{Tr}_E ([H_{\text{int}}(s), [H_{\text{int}}(t), \rho_{\text{SE}}(0)])].
\]

Taking the temporal derivative of the previous equa-
tion, and assuming that system and bath are not cor-
related at the initial time, the master equation can be
written as

\[
\dot{\rho} = -i \text{Tr}_E [H_{\text{int}}(t), \rho(t) \times \rho_{\text{E}}(0)]
\]

\[
- \int_0^t ds \text{Tr}_E [H_{\text{int}}(t), [H_{\text{int}}(s), \rho(t) \times \rho_{\text{E}}(0)]]
\]

\[
+ \int_0^t ds \text{Tr}_E ([H_{\text{int}}(t), \text{Tr}_E ([H_{\text{int}}(s), \rho(t) \times \rho_{\text{E}}(0)]) \times \rho_{\text{E}}(0)]) .
\]

Considering that the \( \hat{\omega}_i \) of the \( H_{\text{int}} \) (Eq. [5]) are operators
acting only on the Hilbert space of the environment
and the Pauli matrices applied on the system Hilbert
space), the master equation, in the Schrödinger picture,
can be written as

\[
\dot{\rho} = -i \int_0^t ds \text{Tr}_E [H_{\text{int}}(t), [H_{\text{int}}(s), \rho(t) \times \rho_{\text{E}}(0)]].
\]

(10)

The master equation explicitly reads

\[
\dot{\rho} = -i [H_q, \rho] - D_{xx}(t) [\sigma_x, [\sigma_x, \rho]] - f_{xy}(t) [\sigma_x, [\sigma_y, \rho]]
\]

\[
- f_{xz}(t) [\sigma_x, [\sigma_z, \rho]] - f_{zx}(t) [\sigma_z, [\sigma_x, \rho]]
\]

\[
- f_{yz}(t) [\sigma_z, [\sigma_y, \rho]] - D_{zz}(t) [\sigma_z, [\sigma_z, \rho]],
\]

(11)

where the noise coefficients are given by

\[
D_{xx}(t) = \int_0^t ds \langle \delta \omega_1(0) \delta \omega_1(-s) \rangle X_1(-s)
\]

\[
f_{xy}(t) = \int_0^t ds \langle \delta \omega_1(0) \delta \omega_1(-s) \rangle Y_1(-s)
\]

\[
f_{xz}(t) = \int_0^t ds \langle \delta \omega_1(0) \delta \omega_1(-s) \rangle Z_1(-s)
\]

\[
f_{zx}(t) = \int_0^t ds \langle \delta \omega_2(0) \delta \omega_0(-s) \rangle X_0(-s)
\]

\[
f_{yz}(t) = \int_0^t ds \langle \delta \omega_2(0) \delta \omega_0(-s) \rangle Y_0(-s)
\]

\[
D_{zz}(t) = \int_0^t ds \langle \delta \omega_2(0) \delta \omega_0(-s) \rangle Z_0(-s).
\]

(12)

It is possible to recognize \( D_{ab} \) and \( f_{ab} \) as normal and anomalous
diffusion coefficients, respectively (\( a, b = x, y, z \)). The functions \( X_{0,1}, Y_{0,1}, \) and \( Z_{0,1} \) are derived by
obtaining the temporal dependence of the Pauli operators
\( \sigma_i \) in the Heisenberg representing through the differential
equations,

\[
\frac{d\sigma_k(t)}{dt} = i [H_q, \sigma_k(t)],
\]

(13)

with \( k = x, y, z \) and \( H_q \) as in Eq. [4]. The solution can be
expressed as a linear combination of the Pauli matrices
(in the Schrödinger representation),

\[
\sigma_{x,z}^{0,1} = X_{0,1}(t) \sigma_x + Y_{0,1}(t) \sigma_y + Z_{0,1}(t) \sigma_z.
\]

(14)

The solution can be easily written as

\[
X_1(t) = \frac{\Omega^2 + \Delta^2 \cos(2t \sqrt{\Omega^2 + \Delta^2})}{\Omega^2 + \Delta^2},
\]

\[
Y_1(t) = \frac{\Delta \sin(2t \sqrt{\Omega^2 + \Delta^2})}{\sqrt{\Omega^2 + \Delta^2}},
\]

\[
Z_1(t) = \frac{\Delta \Omega [1 - \cos(2t \sqrt{\Omega^2 + \Delta^2})]}{\Omega^2 + \Delta^2},
\]

\[
Y_0(t) = \frac{-\Omega \sin(2t \sqrt{\Omega^2 + \Delta^2})}{\sqrt{\Omega^2 + \Delta^2}},
\]

\[
Z_0(t) = \frac{-\Omega \sin(2t \sqrt{\Omega^2 + \Delta^2})}{\sqrt{\Omega^2 + \Delta^2}},
\]

\[
Z_0(t) = 1 - \frac{\Omega^2 [1 - \cos(2t \sqrt{\Omega^2 + \Delta^2})]}{\Omega^2 + \Delta^2}.
\]
It is easy to check that if the Rabi frequency is zero and \( \delta \omega_i = 0 \), we recover the dynamics of a spin-1/2 precessing a bias field vector \( \mathbf{R} \).

The idea is to use different noise correlation functions to model different types of noise that can be found in solid-state qubits. Once the coefficients in Eqs. [12] are defined, we can numerically solve the master equation and obtain the evolution in time of the reduced density matrix. Once this quantity is known, we can further obtain interesting features of the qubit dynamics such as the biased vector \( \mathbf{R} \) and the decoherence induced on the superconducting qubit.

The noise correlations can be defined by their spectral density \( J_i(\omega) = 1/(2\pi) \int dt e^{i\omega t} \langle \delta \omega_i(0) \delta \omega_i(-s) \rangle_c \) with \( i = 0, 1 \). Herein, we shall focus on the long and short-correlated noise (slow and sharp decay of \( \langle \delta \omega_i(0) \delta \omega_i(-s) \rangle_c \)), i.e. on the noise power peaked at low or high frequencies. We will describe different types of noise as

\[
\langle \delta \omega_i(0) \delta \omega_i(-s) \rangle_c = \gamma_i F(\alpha_i, t)
\]

(15)

(where \( \gamma_i \) is a dissipative constant that includes the coupling strength between system and bath, and \( \alpha_i \) is a parameter with frequency units). This function \( F \) keeps the information about the correlation times and couplings in the environment. Phenomenologically, \( F \) can be thought of as a Dirac delta functional for short-correlations in time-domain, or a Gaussian-like function of time for a more general scenario. In solid-state systems decoherence is potentially strong due to numerous microscopic modes. Noise is dominated by material-dependent sources, such as background-charge fluctuations or variations of magnetic fields and critical currents, with given power spectrum, often known as \( 1/f \). This noise is difficult to suppress and, since the dephasing is generally dominated by the low-frequency noise, it is particularly destructive (though it is said that can be reduced by tuning the linear longitudinal qubit-noise coupling to zero [24]). A further relevant contribution is the electromagnetic noise of the control circuit, typically Ohmic at low frequencies.

**Gaussian noise.** An interesting way to model the fluctuations is through a Gaussian-correlated noise. We assume that the operator \( \delta \omega_i(t) \), is given by a random function \( \delta \omega_i(t) \) with \( \langle \delta \omega_i(t) \rangle_c = 0 \) and its correlation between the values of \( \delta \omega_i(t) \) at two different times is non-zero only for this time interval. Explicitly,

\[
\langle \delta \omega_i(t_1) \delta \omega_i(t_2) \rangle_c = \Phi_i(t_1 - t_2),
\]

(16)

where \( \Phi_i(t) \) is a function sharply peaked at \( t = 0 \) and vanishing for \( t > \tau_c \) for a critical time-scale \( \tau_c \). We have set \( \Phi_i(t) = \gamma_i F(\alpha_i, t) \) as defined in Eq. [15], where \( F \) is a Gaussian-like function of time. By setting the parameter \( \alpha_i \) \( (\alpha_0 \) for the longitudinal noise since it affects the coupling in \( z \)-axis and \( \alpha_1 \) the transverse noise -coupling in \( \hat{z} \)-) of the model, we can study low or high frequency noise contribution to decoherence. Therefore, in this case, decoherence depends on the interplay of \( \alpha_0 \) and \( \alpha_1 \) and the value of the dissipation constants \( \gamma_0 \) and \( \gamma_1 \). For example, in Fig. we present the trajectory of the Bloch vector during a cyclic (or quasicyclic) evolution. The black circle on the surface of the Bloch sphere is the evolution of the vector \( \mathbf{R} \) in the unitary case, i.e. \( \gamma_0 = 0 = \gamma_1 \). Herein, we see that in absence of environment the qubit performs a closed trajectory in a period \( \tau_c \), acquiring the known GP, \( \phi_c = \pi(1 - \cos(\theta)) \) with \( \theta = \Delta/(\sqrt{\Delta^2 + \Omega^2}) \). By considering different values for the parameters of our noise model: \( \gamma_i \) and \( \alpha_i \), we can evaluate how the distinct environments affect the system’s dynamics. In Fig. we also present the different trajectories of the Bloch vector \( \mathbf{R} \) for a value of \( \gamma_0 = 0.03\Delta \) and \( \gamma_1 = 0.03\Delta \). As \( \gamma_i \) are related to the square of the coupling constant, these values for \( \gamma_i \) represent a significant environment within the weak coupling approximation. The blue dotted line is the trajectory of the Bloch vector when \( \alpha_0 = 30\Delta \) and \( \alpha_1 = 30\Delta \). This trajectory is very similar to the unitary one, which means that the environment has little influence on the systems’ dynamics. The blue arrow line that starts in the center of the sphere and goes to the surface.
indicates the position of the Bloch sphere after one cycle \( \tau = 2\pi/\Omega, \Omega = \Delta/\sqrt{\Delta^2 + \Omega^2} \). The red solid arrow line is the trajectory for a low value of \( \alpha_0 = 0.03\Delta = \alpha_1 \). This is what we shall call low frequency noise. In this case, we can note that the trajectory differs substantially from the unitary one, meaning the system’s dynamics is affected by the decoherence process. Qualitatively, decoherence can be thought of as the deviation of probabilities measurements from the ideal intended outcome. Therefore, decoherence can be understood as fluctuations in the Bloch vector \( R \) induced by noise. Since decoherence rate depends on the state of the qubit, we will represent decoherence by the change of \(|R|\) in time, starting from \(|R| = 1\) for the initial pure state, and decreasing as long as the quantum state loses purity. The red dashed Bloch vector after a cycle is not longer on the surface of the sphere as can be seen in Fig.1. The module of the red dashed Bloch vector has been reduced 16% after one cycle with respect to the module of the unitary Bloch vector.

As a particular case, we can mention a noise correlation function given by a function \( F = \delta(s) \). If the general environment considered in this approach is a bath of harmonic oscillators with a delta-correlation function \((J_0(\omega) \sim \omega)\), then we will be modeling an ohmic bath in the limit of finite temperature \( T = 0 \). This assumption implies that the only coefficients in Eq. (11) which are constant and non zero are \( D_{zz} = \gamma_0 k_B T \) and \( D_{xx} = \gamma_1 k_B T \). This model is commonly known as dephasing. This modeling of the environment is also included in Fig.1 for \( \gamma_0 = \gamma_1 = 0.03\Delta \) with an orange line. It is easy to see that the Bloch vectors decays to the center of the sphere loosing purity faster than in the Gaussian model. In the latter, due to the presence of more terms in the master equation, the Bloch vector does not decay to the center of the sphere \( T = 0 \).

In Fig.2 we present a different scenario since the trajectories presented correspond to a very weak environment \( \gamma_0 = \gamma_1 = 0.03\Delta \). Once again, the black solid line is the reference for the unitary case while the blue line (almost coincident with the black) is for high frequency \( \alpha_0 = \alpha_1 = 30\Delta \) and the red one for low frequency noise \( \alpha_0 = \alpha_1 = 0.003\Delta \). Here, the Bloch vector for the low frequency noise (red) is 5% reduced with respect to the unitary Bloch vector after one cycle \( \tau \).

Finally we can comment the \( 1/f \) noise mentioned above. This noise can be modeled by a bath composed of an infinite set of harmonic oscillators (similarly to what has been done in the spin boson model \[12\]). At \( T = 0 \), the noise kernel \( \nu(t) \) can be evaluated when \( J_0(\omega) \sim A/\omega \). Then, the \( 1/f \) noise is determined by a correlation function \( \nu(t) = -\gamma_0 A S(t) \), where \( A(x) \) is the cosine integral function, and \( \Lambda \) is the typical infrared cutoff for the \( 1/f \) noise. In the high temperature limit, this kernel is given by \( \nu(t) = T \gamma_0 \Lambda \cos(\Lambda t)/\Lambda + t S(t) \), with \( S(t) \) the sine integral function. This quantitative modeling of the \( 1/f \) noise through a master equation approach is somewhat analogous to the effect of the phenomenological modeling of the noise through an ensemble of “spin-fluctuators” \[21\]. In Fig.3 we effectively note how harmful this type of noise is for the dynamics of the qubit, even in the very low temperature limit. Therein, the black solid line represents the unitary trajectory of the Bloch vector. In this model, the relevant parameter is the infrared frequency cutoff \( \Lambda \). The blue dotted line is for a big value of the infrared cutoff \( \Lambda = 1\Delta \), while the red solid line is for a low frequency cutoff \( \Lambda = 0.001\Delta \). Both cases are affected by decoherence. In the low frequency cutoff case the module of the Bloch vector -indicated as
a dashed red arrow from the center of the sphere is reduced 20% in a cycle \( \tau \).

### III. APPLICATION: GEOMETRIC PHASE OF A SOLID-STATE QUBIT IN A NON-UNITARY EVOLUTION

Practical implementations of quantum computing are always done in the presence of decoherence. Thus, a proper generalization for the geometric phase to non-unitary evolutions is central in the evaluation of the robustness of geometric quantum computation. This generalization has been done in [11], where a functional representation of GP was proposed, after removing the dynamical phase from the total phase acquired by the system under a gauge transformation.

The GP for a mixed state under nonunitary evolution is then defined as

\[
\Phi = \arg \left\{ \sum_k \sqrt{\varepsilon_k(0)\varepsilon_k(\tau)} \langle \Psi_k(0) | \Psi_k(\tau) \rangle \right\} \times e^{-\int_0^\tau dt \langle \Psi(t) | \dot{H} | \Psi(t) \rangle}, \tag{17}
\]

where \( \varepsilon_k(t) \) are the eigenvalues and \( | \Psi_k \rangle \) the eigenstates of the reduced density matrix \( \rho \), solution of the master equation. In the last definition, \( \tau \) denotes a time after the total system completes a cyclic evolution when it is isolated from the environment. Taking the effect of the environment into account, the system no longer undergoes a cyclic evolution. However, we will consider a quasicyclic path \( \mathcal{P} : t \in [0, \tau] \) with \( \tau = 2\pi/\Omega \) [10]. It is worth noting that the phase in Eq. (17) is manifestly gauge invariant, since it only depends on the path in the state space, and that this expression, even though is defined for non-degenerate mixed states, corresponds to the unitary geometric phase in the case that the state is pure (closed system).

It is expected that Berry’s phase can be only observed in experiments carried out in a time scale slow enough to ignore nonadiabatic corrections, but rapid enough to avoid destructive decoherence [12]. The noise induced corrections to the GP depend on the value of parameters present in the noise model, for example \( \alpha_i \) and \( \gamma_i \) used in the above section. The purpose of this section is twofold: study how the GPs are affected by the different models of noise and explain some recent experimental setups where the GP has been measured in presence of noise [17, 18]. In the mentioned works, authors observed the Berry’s phase in a superconducting qubit by different approaches. However, both experiments agree on the fact that the longitudinal noise affects the system’s dynamics in a clearer way that the transversal noise. Another important fact is that in [18] authors claimed to have observed the Berry phase under high-frequency fluctuations. They considered that this robustness of the GPs to high-frequency noise may be exploitable in the realization of logic quantum gates for quantum computation. Therefore, we aim to explain these features of the GP for our model from a primary derivation of a master equation approach. In the following, we shall use the Gaussian model of noise for the study of the GP since it is widely said that the \( 1/\Omega \) can be reduced in spin-echo experiments, by tuning the linear longitudinal qubit-noise to zero [23]. In our gaussian model, we have shown that the decoherence process was very dependent on the value of the \( \alpha_i \) parameter, which we associated to a frequency. In all cases shown, decoherence was enhanced in the low frequency case (small values of \( \alpha \), see Figs. 1 and 2).

In Fig. 4 we present the ratio between the GP \( \Phi \) computed for a system evolving under a noisy environment after a cycle \( \tau \) and the unitary one \( \Phi_U \), for different values of \( \alpha_i \), having \( \gamma_i \) fixed as \( \gamma_0 = \gamma_1 = 0.001\Delta \). We show how this ratio varies once you have a fixed environment and a tunable frequency. Herein, we can note that the ratio does not practically change for different values of \( \alpha \), meaning that the transversal fluctuations are not relevant. However, we can see that the ratio varies considerably in the \( \alpha_0 \) direction. The GP is visibly corrected for small values of \( \alpha_0 \), i.e., for low frequency noise in the longitudinal coupling of the qubit. This correction means that the Bloch vector has a relevant difference with the initial unitary Bloch vector since the environment induces more decoherence in the low frequency case (see Fig. 1).

![FIG. 4: (Color online) Ratio between the computed GP in presence of noise and the one computed in the isolated case \( \Phi_U \), as function of \( \alpha_0 \) and \( \alpha_1 \) (in units of \( \Delta \)), with \( \gamma_0 = \gamma_1 = 0.001\Delta \). The GP is more affected by the presence of longitudinal noise frequency \( \alpha_0 \) since the rate is bigger. The GP does not considerably depend on the transversal noise \( \alpha_1 \). We have set \( \Omega = 0.5\Delta \).](attachment:image.png)

In Fig. 4 we again present the ratio between the GP \( \Phi \) computed for a system evolving under a noisy environment after a cycle \( \tau \) and the one unitary computed \( \Phi_U \). This time we show how this ratio varies for different values of \( \gamma_0 \) and \( \gamma_1 \) for small values of \( \alpha_i \), say \( \alpha_0 = \alpha_1 = 0.01\Delta \). It is easy to note that the GP \( \Phi \) is very similar to the unitary GP \( \Phi_U \), in absence of longitudinal noise \( \gamma_0 = 0 \), which means that the evolution is not considerably affected by the transverse noise. How-
ever, we can see a different behavior if we consider longitudinal noise ($\gamma_1 = 0$). The GP varies perceptibly as the environment is coupled in the longitudinal direction is stronger (bigger values of $\gamma_0$). It is important to say that the relevant role of the tunable frequency $\alpha_i$ makes sense if we are dealing with a considerable environment which can effectively induce noise into our system’s dynamics. For very small values of $\gamma_0$, Fig. 5 shows that the GP computed is similar to the unitary GP, independently of low or high frequency fluctuations.

In Fig. 5, we present the ratio between the GP $\Phi$ computed for a system evolving under a noisy environment after a cycle $\tau$ and the one unitary computed $\Phi_U$ as a function of $\gamma_0$ and $\gamma_1$ (in units of $\Delta$), for a fixed value of $\alpha_0 = \alpha_1 = 0.01\Delta$. The ratio is more affected by the longitudinal noise. We have set $\Omega = 0.5\Delta$.

**FIG. 5:** (Color online) Rate between the computed GP $\Phi$ in presence of noise and the one computed in the isolated case $\Phi_U$ as function of the dissipative constants $\gamma_0$ and $\gamma_1$ (in units of $\Delta$), for a fixed value of $\alpha_0 = \alpha_1 = 0.01\Delta$. The ratio is more affected by the longitudinal noise. We have set $\Omega = 0.5\Delta$. 

Finally, in Fig. 7 we quantitatively show how the GP is affected by the longitudinal and transverse fluctuations separately. We present the ratio between the observed GP $\Phi$ after a cycle $\tau$ and the unitary GP $\Phi_U$ as a function of both dissipative constants, $\gamma_i$. We consider that the qubit is coupled to only one noise, i.e. that when we show how the ratio varies as function of $\gamma_0$, the qubit is evolving only under a longitudinal noise and $\gamma_1 = 0$ (black circled-line). If the ratio varies as a function of $\gamma_1$, then the qubit is suffering the presence of transversal fluctuations only $\gamma_0 = 0$ (blue squared-lines). We have also add the $\alpha_i$ parameter to have the full scenario. The correction to the GP is almost imperceptible to low and high frequency transversal fluctuations (full and empty squares with $\alpha_1 = 0.03\Delta$ and $\alpha_1 = 10\Delta$ respectively), at least in the weak coupling limit. On the contrary, if the fluctuations of the environment are longitudinal, only those high-frequency ones does not considerably affect the measurement of the geometric phase. It is evident that low frequency longitudinal noise induces a bigger correction to the phase (as can be seen from the full black-circled line with $\alpha_0 = 0.03\Delta$). These results agree

**FIG. 7:** (Color online) Ratio of the computed GP $\Phi$ in presence of a noisy environment and the unitary GP $\Phi_U$ as a function of the dissipative constants $\gamma_0$ and $\gamma_1$ (in units of $\Delta$), for a fixed value of $\alpha_0 = \alpha_1 = 10\Delta$. The correction to the GP is almost negligible for higher values of $\alpha_i$, for weak coupling with the environment. We have set $\Omega = 0.5\Delta$. 

**FIG. 6:** (Color online) Rate between the computed GP in presence of noise and the one computed in the isolated case $\Phi_U$ as function of the dissipative constants $\gamma_0$ and $\gamma_1$ (in units of $\Delta$), for a fixed value of $\alpha_0 = \alpha_1 = 10\Delta$. The correction to the GP is almost negligible for higher values of $\alpha_i$, for weak coupling with the environment. We have set $\Omega = 0.5\Delta$. 

In Fig. 6 we present the ratio between the GP $\Phi$ computed for a system evolving under a noisy environment after a cycle $\tau$ and the one unitary computed $\Phi_U$ as a function of $\gamma_0$ and $\gamma_1$ for bigger values of $\alpha_i$, say $\alpha_0 = \alpha_1 = 10\Delta$. Herein, we see that the system evolution in the presence of an environment with high frequency fluctuations is very similar to the unitary evolution, since the GP acquired is practically similar to the $\Phi_U$, for almost all values of $\gamma_0$. If we get a closer look, we can note that the difference between both phases becomes slowly to increase for stronger values of $\gamma_0$. We believe that the situation depicted in Fig. 6 is very similar to the experimental situation reported in [18] where authors have measured the Berry phase for a superconducting qubit under high frequency fluctuations.

Finally, in Fig. 7 we quantitatively show how the GP is affected by the longitudinal and transverse fluctuations separately. We present the ratio between the observed GP $\Phi$ after a cycle $\tau$ and the unitary GP $\Phi_U$ as a function of both dissipative constants, $\gamma_i$. We consider that the qubit is coupled to only one noise, i.e. that when we show how the ratio varies as function of $\gamma_0$, the qubit is evolving only under a longitudinal noise and $\gamma_1 = 0$ (black circled-line). If the ratio varies as a function of $\gamma_1$, then the qubit is suffering the presence of transversal fluctuations only $\gamma_0 = 0$ (blue squared-lines). We have also add the $\alpha_i$ parameter to have the full scenario. The correction to the GP is almost imperceptible to low and high frequency transversal fluctuations (full and empty squares with $\alpha_1 = 0.03\Delta$ and $\alpha_1 = 10\Delta$ respectively), at least in the weak coupling limit. On the contrary, if the fluctuations of the environment are longitudinal, only those high-frequency ones does not considerably affect the measurement of the geometric phase. It is evident that low frequency longitudinal noise induces a bigger correction to the phase (as can be seen from the full black-circled line with $\alpha_0 = 0.03\Delta$). These results agree
with the previous analysis done on decoherence induced in the qubit and with the experimental setups reported of the observed geometric phase [17, 18]. It is important to emphasize that our approach is general and allows several ways of modeling the environment coupled to the main system.

IV. FINAL REMARKS

We have considered the effective two-state Hamiltonian for the current-biased Josephson junction. The qubit has been shown to be fully manipulated with the control currents. Like any other quantum object, the qubit is subject to decoherence due to the interaction with uncontrolled degrees of freedom in its environment, including those in the device itself. These degrees of freedom appear as noise induced in the parameters entering the qubit Hamiltonian and also as noise in the control currents. These noise sources produce decoherence in the qubit, with noise, mainly, at microwave frequencies affecting the relative population between the ground and excited state, and noise or low-frequency fluctuations affecting the phase of the qubit. It is important to study the physical origins of decoherence by means off noise spectral densities and noise statistics.

We have derived a master equation for the two-level system including the combined effect of noise in the longitudinal and transversal directions. We considered different types of noise by defining their correlation function in time. We have mainly analyzed a Gaussian-like correlated type of noise, with low and fast decaying times that induce different decoherence processes in the low or high frequency parts of the environmental spectrum. We have even presented very correlated noise, where the noise kernel is proportional to a Dirac delta function in time and the $1/f$ known commonly used in spin fluctuators environments. For each type of noise presented, we numerically solved the master equation and obtained the system’s dynamics. Qualitatively, decoherence can be thought of as the deviation of probabilities measurements from the ideal intended outcome. Therefore, decoherence can be understood as fluctuations in the Bloch vector $\mathbf{R}$ induced by noise. Since decoherence rate depends on the state of the qubit, we have represented decoherence by the change of $|\mathbf{R}|$ in time, starting from $|\mathbf{R}| = 1$ for the initial pure state, and decreasing as long as the quantum state loses purity.

We have extended our analysis of decoherence to understand the corrections induced in the geometric phase, when the qubit evolves in time under fluctuations of the environment. Within the general picture of the master equation, we provide a framework to understand when the accumulated phase can still be found close to the unitary (Berry) one. We have focused on the effect of longitudinal and transversal noise on the global geometric phase. It is important to note that the relevant role of the tunable frequency $\alpha_i$ in our gaussian model makes sense if we are dealing with a considerable environment which can effectively induce noise into our system’s dynamics. For very small values of $\gamma_0$, we have shown that the GP computed is similar to the unitary GP, independently of low or high frequency fluctuations. We have also noted that the difference between both phases increases for stronger values of $\gamma_0$, becoming important when there are low frequency longitudinal fluctuations in the environment. The difference among the phases are not considerable if the fluctuations of low frequency are originated in a transversal noise ($\gamma_1$). The correction to the GP is almost imperceptible to the transversal fluctuations, at least in the weak coupling limit.

It is important to recall that the results presented show that the system evolution in the presence of an environment with high frequency fluctuations is very similar to the unitary evolution, since the GP acquired is practically similar to the $\Phi_U$, for almost all values of $\gamma_0$. We believe that these results show a very similar scenario to that of the experimental situation reported in [13] where they have measured the Berry phase for a superconducting qubit under high frequency fluctuations. In addition, we have checked that noise in the $\hat{z}$-direction induces a bigger correction to the phase than the noise in the transversal components. This correction agrees with the previous analysis done on decoherence induced in the qubit and with the experimental setups reported of the observed geometric phase. Comparison between theory and experiment verifies our understanding of the physics underlying the system as a dissipative two-level device. The analysis of the dephasing time-scales may provide additional information about the statistical properties of the noise. Berry’s phase measurements provide an important constraint to take into account about noise models and their correction induced over the GP, at least, at the times in which the experiments can be performed. The comprehension of the decoherence and dissipative processes should allow their further suppression in future qubits designs or experimental setups.

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