Kerr-Schild Structure and Harmonic 2-forms on (A)dS-Kerr-NUT Metrics

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ABSTRACT

We demonstrate that the general (A)dS-Kerr-NUT solutions in $D$ dimensions with \((D/2,[(D+1)/2])\) signature admit \([D/2]\) linearly-independent, mutually-orthogonal and affinely-parameterised null geodesic congruences. This enables us to write the metrics in a multi-Kerr-Schild form, where the mass and all of the NUT parameters enter the metrics linearly. In the case of $D = 2n$, we also obtain $n$ harmonic 2-forms, which can be viewed as charged (A)dS-Kerr-NUT solution at the linear level of small-charge expansion, for the higher-dimensional Einstein-Maxwell theories. In the BPS limit, these 2-forms reduce to $n - 1$ linearly-independent ones, whilst the resulting Calabi-Yau metric acquires a Kähler 2-form, leaving the total number the same.
1 Introduction

One intriguing feature of General Relativity is that, despite its high degree of non-linearity, many exact solutions can be cast into a Kerr-Schild form \[1\] where non-trivial parameters such as mass, charge, or cosmological constant enter the metrics as a linear perturbation of flat spacetime. A simple example is the (A)dS metric, which can be written as

\[
\begin{align*}
    ds^2 &= -dt^2 + dr^2 + r^2 d\Omega_n^2 + \Lambda r^2 (dt - dr)^2,
\end{align*}
\]  

where the first three terms describe the \((n+2)\)-dimensional Minkowski spacetime and the cosmological constant enters the last term linearly. More complicated examples include the Plebanski metric \[2\]; in \((2,2)\) signature, the Plebanski metric can have a double Kerr-Schild form where both the mass and the NUT charge enter the metric linearly \[3\].

The most general higher-dimensional (A)dS-Kerr-NUT solutions, which can be viewed as higher-dimensional generalisations of the Plebanski metric, were recently obtained in \[4\]. The solutions are parameterised by the mass, multiple NUT charges and arbitrary orthogonal rotations. The metrics have \(U(1)^n\) isometries, where \(n = [(D+1)/2]\). They are demonstrated \[5\] to be of type D in the higher-dimensional generalisation \[6\] of the Petrov classification.

Many further interesting properties of the metrics were obtained, such as the separability of the Hamiltonian-Jacobi and Klein-Gordon equations \[7\], and the existence of Killing-Yano tensors \[8\]. The metrics also admit BPS limits where the Killing spinors can emerge \[4\]. In the odd \(2n+1\) dimensions, this leads to a large class of Einstein-Sasaki metrics with \(U(1)^n\) isometry, generalising the previously known \(Y^{p,q}\) \[9\] and \(L^{pqr}\) \[10\] spaces. In the even \(2n\) dimensions, this leads to the non-compact Calabi-Yau metrics that can provide a resolution of the cone over the Einstein-Sasaki metrics constructed in the odd dimensions \[11\, 12\].

In this letter, we demonstrate in section 2 that the \(D\)-dimensional (A)dS-Kerr-NUT solution admits \([D/2]\) linearly-independent, mutually-orthogonal and affinely parameterised null geodesic congruences upon Wick-rotation of the metric to \([(D/2), [(D+1)/2)]\) signature. This enables us to cast the metric into the multi-Kerr-Schild form, where the mass and all of the NUT parameters enter the metric linearly. In section 3, we obtain \(n\) harmonic 2-forms on the (A)dS-Kerr-NUT metrics in \(D = 2n\) dimensions. In the BPS limit, these \(n\) harmonic 2-forms becomes linearly dependent, and the number of linearly-independent ones becomes \(n - 1\). However, a Kähler 2-form emerges under the BPS limit, and hence the total number of harmonic 2-forms remains \(n\). We conclude the letter in section 4.
2 Multi-Kerr-Schild structure

Let us first consider the case of $D = 2n + 1$ dimensions, for which the metric was given in [4]. In order to put the metric in a Kerr-Schild form, it is necessary to Wick rotate to $(n, n + 1)$ signature. This can be easily achieved by Wick rotating all the spatial $U(1)$ coordinates. The corresponding metric is then given by

$$ds^2 = \sum_{\mu=1}^{n} \left\{ \frac{dx_\mu^2}{Q_\mu} - Q_\mu \left( \sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k \right)^2 \right\} + \frac{c}{(\prod_{\nu=1}^{n} x_\nu^2)} \left( \sum_{k=0}^{n} A^{(k)} d\psi_k \right)^2,$$

where

$$Q_\mu = \frac{X_\mu}{U_\mu}, \quad U_\mu = \prod_{\nu=1}^{n} (x_\nu^2 - x_\mu^2), \quad X_\mu = \sum_{k=1}^{n} c_k x_\mu^{2k} + \frac{c}{x_\mu^2} - 2b_\mu,$$

$$A_\mu^{(k)} = \sum_{\nu_1 < \nu_2 < \cdots < \nu_k} x_{\nu_1}^2 x_{\nu_2}^2 \cdots x_{\nu_k}^2, \quad A^{(k)} = \sum_{\nu_1 < \nu_2 < \cdots < \nu_k} x_{\nu_1}^2 x_{\nu_2}^2 \cdots x_{\nu_k}^2.$$

The prime on the summation and product symbols in the definition of $A_\mu^{(k)}$ and $U_\mu$ indicates that the index value $\mu$ is omitted in the summations of the $\nu$ indices over the range $[1, n]$. Note that $\psi_0$ was denoted as $t$ in [4], playing the role of the time-like coordinate in the $(1, 2n)$ spacetime signature. In this way of writing the metric, all of the integration constants of the solution enter only in the functions $X_\mu$. The constant $c_n = (-1)^n \Lambda$ is fixed by the value of the cosmological constant, with $R_{\mu\nu} = 2n \Lambda g_{\mu\nu}$. The other $2n$ constants $c_k$, $c$ and $b_\mu$ are arbitrary. These are related to the $n$ rotation parameters, the mass and the $(n - 1)$ NUT parameters, with one parameter being trivial and removable through a scaling symmetry [4]. Note that in $(n, n + 1)$ signature, the NUT charges are really masses with respect to different time-like Killing vectors. However, we shall continue to refer them as NUT charges.

We now re-arrange the metric (2) into the form

$$ds^2 = -\sum_{\mu=1}^{n} \frac{X_\mu}{U_\mu} \left[ \sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k + \frac{U_\mu}{X_\mu} dx_\mu \right] \left[ \sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k - \frac{U_\mu}{X_\mu} dx_\mu \right]$$

$$+ \frac{c}{(\prod_{\nu=1}^{n} x_\nu^2)} \left( \sum_{k=0}^{n} A^{(k)} d\psi_k \right)^2.$$

If we perform the following coordinate transformation,

$$d\hat{\psi}_k = d\psi_k + \sum_{\mu=1}^{n} \frac{(-x_\mu^2)^{n-k-1}}{X_\mu} dx_\mu, \quad k = 0, \ldots, n,$$

the metric can then be cast into the n-Kerr-Schild form, namely

$$ds^2 = ds^2 + \sum_{\mu=1}^{n} \frac{2b_\mu}{U_\mu} \left[ \sum_{k=0}^{n-1} A_\mu^{(k)} d\hat{\psi}_k \right]^2.$$
where

\[
 ds^2 = - \sum_{\mu=1}^{n} \left\{ \frac{\bar{X}_\mu}{U_\mu} \left[ \sum_{k=0}^{n-1} A^{(k)}_\mu d\hat{\psi}_k \right]^2 - 2 \sum_{k=0}^{n-1} A^{(k)}_\mu d\hat{\psi}_k dx_\mu \right\} \\
+ \frac{c}{(\prod_{\nu=1}^{n} x_\nu^2)} \left( \sum_{k=0}^{n} A^{(k)} d\hat{\psi}_k \right)^2 ,
\]

\[
 \bar{X}_\mu = \sum_{k=1}^{n} c_k x_\mu^{2k} + \frac{c}{x_\mu^2} .
\]

(7)

It is straightforward to verify that the metric \( ds^2 \) is that of pure (A)dS spacetime. The mass and NUT parameters \( b_\mu \) appear linearly in the metric \( ds^2 \). It should be emphasised that although the constants \( c \) and \( c_k \) with \( k < n \) are trivial in the metric \( ds^2 \), they provide non-trivial angular momentum parameters in the metric \( ds^2 \). It is interesting to note that all of the constants \( c_k \), including \( c_n \) that is related to the cosmological constant, appear linearly in the metric, and can all be extracted from \( ds^2 \) and grouped in the second term of (6). This implies that all the parameters, the mass, NUTs and angular momenta and cosmological constant can enter the metric linearly as a perturbation of flat spacetime. In this letter, we shall consider in detail only the Kerr-Schild form where the mass and NUT parameters enter the metric linearly as a perturbation of pure (A)dS spacetime.

The (A)dS metric (7) can be diagonalised, in a way that the second term of (6) remains simple. To do so, let us first rewrite the \( \bar{X}_\mu \) as follows

\[
 \bar{X}_\mu = \frac{(1 + A x_\mu^2)}{x_\mu^2} \prod_{k=1}^{n} (a_k^2 - x_\mu^2) .
\]

(8)

Then we complete the square in \( ds^2 \):

\[
 ds^2 = \sum_{\mu=1}^{n} \left\{ \frac{U_\mu}{X_\mu} d\tilde{x}_\mu - \frac{\bar{X}_\mu}{U_\mu} \left[ \sum_{k=0}^{n-1} A^{(k)}_\mu d\tilde{\psi}_k - \frac{U_\mu}{X_\mu} dx_\mu \right]^2 \right\} + \frac{c}{(\prod_{\nu=1}^{n} x_\nu^2)} \left( \sum_{k=0}^{n} A^{(k)} d\tilde{\psi}_k \right)^2 ,
\]

(9)

and make the coordinate transformation,

\[
 d\tilde{\psi}_k = -d\hat{\psi}_k + \sum_{\mu=1}^{n} \left( \frac{(-x_\mu^2)^{n-k-1}}{X_\mu} \right) dx_\mu , \quad k = 0 , \cdots , n .
\]

(10)

The metric can be put into a new form,

\[
 ds^2 = ds^2 + \sum_{\mu=1}^{n} \frac{2b_\mu}{U_\mu} \left[ \sum_{k=0}^{n-1} A^{(k)}_\mu d\tilde{\psi}_k - \frac{U_\mu}{X_\mu} dx_\mu \right]^2 ,
\]

(11)

where

\[
 ds^2 = \sum_{\mu=1}^{n} \left\{ \frac{U_\mu}{X_\mu} d\tilde{x}_\mu - \frac{\bar{X}_\mu}{U_\mu} \left[ \sum_{k=0}^{n-1} A^{(k)}_\mu d\tilde{\psi}_k \right]^2 \right\} + \frac{c}{(\prod_{\nu=1}^{n} x_\nu^2)} \left( \sum_{k=0}^{n} A^{(k)} d\tilde{\psi}_k \right)^2 .
\]

(12)
Performing a recombination of the \( U(1) \) coordinates, namely
\[
\tau = \sum_{k=0}^{n} B^{(k)} \frac{d \tilde{\psi}_k}{k}, \quad \frac{\varphi^i}{a_i} = \sum_{k=1}^{n} B^{(k-1)} \frac{d \tilde{\psi}_k}{k} - \Lambda \sum_{k=0}^{n-1} B^{(k)} \frac{d \tilde{\psi}_k}{k}, \quad i = 1, \ldots, n ,
\]
(13)
where
\[
B^{(k)} = \sum_{j_1 < j_2 < \cdots < j_k} a_{j_1}^2 a_{j_2}^2 \cdots a_{j_k}^2 , \quad B^{(k)} = \sum_{j_1 < j_2 \cdots < j_k} a_{j_1}^2 a_{j_2}^2 \cdots a_{j_k}^2 ,
\]
(14)
the odd dimensional (A)dS-Kerr-NUT metrics can be expressed as
\[
ds^2 = ds^2 + \sum_{\mu=1}^{n} 2b_\mu (k(\mu)\alpha dx^\alpha)^2 ,
\]
(15)
\[
ds^2 = \frac{W}{\prod_{i=1}^{n-1} \Xi_i} d\tau^2 + \sum_{\mu=1}^{n} \frac{U_\mu}{X_\mu} dx^2_\mu - \sum_{i=1}^{n} \Xi_i \prod_{k=1}^{n} (a_i^2 - a_k^2) d\varphi_i^2 ,
\]
(16)
\[
k(\mu)\alpha dx^\alpha = \frac{W}{1 + \Lambda x_\mu^2 \prod_{i=1}^{n-1} \Xi_i} - \sum_{i=1}^{n} \frac{a_i \gamma_i d\varphi_i}{(a_i^2 - a_\mu^2) \Xi_i \prod_{k=1}^{n} (a_i^2 - a_k^2)} ,
\]
(17)
where
\[
\Xi_i = 1 + \Lambda a_i^2 , \quad \gamma_i = \prod_{\nu=1}^{n} (a_\nu^2 - x_\nu^2) , \quad W = \prod_{\nu=1}^{n} (1 + \Lambda x_\nu^2) .
\]
(18)
If we set all but one of the \( b_\mu \) to zero, the result reduces to the Kerr-Schild form for rotating (A)dS black holes obtained previously in [13].

We now turn our attention to the case of \( D = 2n \) dimensions. The corresponding (A)dS-Kerr-NUT metrics were obtained in [4]. After performing Wick rotations, the metric with \( (n, n) \) signature is given by
\[
ds^2 = \sum_{\mu=1}^{n} \left\{ \frac{dx^2_\mu}{Q_\mu} - Q_\mu \left( \sum_{k=0}^{n-1} A^{(k)}_\mu \frac{d \psi_k}{k} \right)^2 \right\} ,
\]
(19)
where we \( Q_\mu, U_\mu \) and \( A^{(k)}_\mu \) have the same form as those in the even dimensions, given in (3). The functions \( X_\mu \) are given by
\[
X_\mu = \sum_{k=0}^{n} c_k x_\mu^{2k} + 2b_\mu x_\mu .
\]
(20)
The constants \( c_k \) and \( b_\mu \) are arbitrary, except for \( c_n = (-1)^n \Lambda \), which is fixed by the value of the cosmological constant, \( R_{\mu\nu} = (2n - 1)\Lambda g_{\mu\nu} \). The metric can be re-arranged into the form
\[
ds^2 = -\sum_{\mu=1}^{n} \frac{X_\mu}{U_\mu} \left[ \sum_{k=0}^{n-1} A^{(k)}_\mu \frac{d \psi_k}{k} + \frac{U_\mu}{X_\mu} dx_\mu \right] \left[ \sum_{k=0}^{n-1} A^{(k)}_\mu \frac{d \psi_k}{k} - \frac{U_\mu}{X_\mu} dx_\mu \right] ;
\]
(21)
After performing the coordinate transformation
\[ d\hat{\psi}_k = d\psi_k + \sum_{\mu=1}^{n} \frac{(-x_\mu^2)^{n-k-1}}{X_\mu} dx_\mu, \quad k = 0, \ldots, n-1, \tag{22} \]
the metric can be cast into the \( n \)-Kerr-Schild form,
\[ ds^2 = ds^2 - \sum_{\mu=1}^{n} \frac{2b_\mu x_\mu}{\bar{U}_\mu} \left[ \sum_{k=0}^{n-1} A^{(k)}_\mu d\hat{\psi}_k \right]^2 \tag{23} \]
where
\[ \bar{X}_\mu = \sum_{k=0}^{n} c_k x_\mu^{2k}. \tag{24} \]

It is straightforward to verify that \( ds^2 \) is the metric for pure (A)dS spacetime. As in the odd dimensions, this metric can be put into a diagonal form, while keeping the second term of (23) simple. To do that, we first reparameterise \( X_\mu \) as
\[ \bar{X}_\mu = -(1 - g^2 x_\mu^2) \prod_{k=1}^{n-1} (a_k^2 - x_\mu^2). \tag{25} \]

We then complete the square in \( ds^2 \), i.e.
\[ ds^2 = \sum_{\mu=1}^{n} \left\{ \frac{U_\mu}{\bar{X}_\mu} dx_\mu^2 - \frac{\bar{X}_\mu}{U_\mu} \left[ \sum_{k=0}^{n-1} A^{(k)}_\mu d\hat{\psi}_k - \frac{U_\mu}{\bar{X}_\mu} dx_\mu \right]^2 \right\} \tag{26} \]
and make the coordinate transformation
\[ d\tilde{\psi}_k = -d\hat{\psi}_k + \sum_{\mu=1}^{n} \frac{(-x_\mu^2)^{n-k-1}}{X_\mu} dx_\mu, \quad k = 0, \ldots, n-1. \tag{27} \]

The metric (23) can then be put into a new form:
\[ ds^2 = ds^2 - \sum_{\mu=1}^{n} \frac{2b_\mu x_\mu}{\bar{U}_\mu} \left[ \sum_{k=0}^{n-1} A^{(k)}_\mu d\tilde{\psi}_k - \frac{U_\mu}{\bar{X}_\mu} dx_\mu \right]^2, \tag{28} \]
where
\[ ds^2 = \sum_{\mu=1}^{n} \left\{ \frac{U_\mu}{\bar{X}_\mu} dx_\mu^2 - \frac{\bar{X}_\mu}{U_\mu} \left[ \sum_{k=0}^{n-1} A^{(k)}_\mu d\tilde{\psi}_k \right]^2 \right\}. \tag{29} \]

The \( ds^2 \) metric can now straightforwardly be diagonalised by means of the coordinate transformation
\[ \tau = \sum_{k=0}^{n-1} B^{(k)} d\tilde{\psi}_k, \quad \frac{\varphi_i}{a_i} = \sum_{k=1}^{n-1} B_i^{(k-1)} d\tilde{\psi}_k + g^2 \sum_{k=0}^{n-2} B_i^{(k)} d\tilde{\psi}_k \quad i = 1, \ldots, n-1, \tag{30} \]
where
\[
B_i^{(k)} = \sum_{j_1 < j_2 < \cdots < j_k} a_{j_1}^2 a_{j_2}^2 \cdots a_{j_k}^2, \quad B^{(k)} = \sum_{j_1 < j_2 < \cdots < j_k} a_{j_1}^2 a_{j_2}^2 \cdots a_{j_k}^2.
\]

(31)
The even dimensional (A)-dS Kerr-NUT metrics can now be expressed as
\[
ds^2 = \bar{d}s^2 - \sum_{\mu=1}^{n} \frac{2b_\mu x_\mu}{U_\mu} (k(\mu)\alpha dx^\alpha)^2, \]
where
\[
ds^2 = \frac{W}{\prod_{i=1}^{n-1} \Xi_i} d\tau^2 + \sum_{\mu=1}^{n} \frac{U_\mu}{X_\mu} dx_\mu^2 - \sum_{i=1}^{n-1} \frac{\gamma_i}{a_i^2 \Xi_i \prod_{k=1}^{n-1} (a_i^2 - a_k^2)} d\phi_i^2,
\]
\[
k(\mu)\alpha dx^\alpha = \frac{W}{1 - g^2 x_\mu^2 \prod_{i=1}^{n-1} \Xi_i} - \frac{U_\mu}{X_\mu} dx_\mu - \sum_{i=1}^{n-1} \frac{\gamma_i d\phi_i}{(a_i^2 - x_\mu^2) a_i \Xi_i \prod_{k=1}^{n-1} (a_i^2 - a_k^2)},
\]
where \(\Xi_i, \gamma_i\) and \(W\) have the same structure as that in the even dimensions, given by (18). When all but one of the \(b_\mu\) vanishes, the metric reduces to the Kerr-Schild form of the rotating (A)dS black hole obtained in [13].

To summarise, we find that in both even and odd dimensions, the (A)dS-Kerr-NUT solution can be cast into the following multi-Kerr-Schild form:
\[
ds^2 = \bar{d}s^2 + \sum_{\mu=1}^{n} \frac{2b_\mu f(x_\mu)}{U_\mu} (k(\mu)\alpha dx^\alpha)^2,
\]
where \(f(x_\mu) = 1\) for odd dimensions and \(f(x_\mu) = x_\mu\) for even dimensions. The vectors \(k(\mu)\alpha\) are \(n\) linearly-independent, mutually-orthogonal and affinely-parameterised null geodesic congruences, satisfying
\[
k(\mu)\alpha k(\mu)\beta = 0, \quad k(\mu)\alpha \bar{\nabla}_\alpha k(\mu)\beta = 0.
\]
Note that the index \(\alpha\) in \(k(\mu)\alpha\) can be raised with either \(g^{\alpha\beta}\) or \(\bar{g}^{\alpha\beta}\) for the above conditions to be satisfied.

3 Harmonic 2-forms in \(D = 2n\) dimensions

In this section, we find \(n\) harmonic 2-forms \(G^{(\mu)} = dB^{(\mu)}\) on the \(2n\)-dimensional (A)dS-Kerr-NUT metric (19), where we use the index \(\mu = 1, 2, \ldots n\) to label the harmonic 2-forms. The potentials have a rather simple form, given by
\[
B^{(\mu)} = \frac{x_\mu}{U_\mu} \left( \sum_{k=0}^{n-1} A^{(k)} d\psi_k \right).
\]
(37)
The metric (19) admits a natural vielbein basis, namely
\[ e^\mu = \frac{dx^\mu}{\sqrt{Q_\mu}}, \quad \tilde{e}^\mu = \sqrt{Q_\mu} \left( \sum_{k=0}^{n-1} A^{(k)}_\mu d\psi_k \right). \tag{38} \]

In this vielbein basis, the harmonic 2-forms \( G^{(2)}_\mu \) are given by
\[ G^{(2)}_\mu = \sum f^{(\mu)}_\nu e^\nu \wedge \tilde{e}^\nu, \tag{39} \]
where the coefficients are
\[ f^{(\mu)}_\nu = \frac{1}{U_\mu} \left[ A^{(n-1)} + \sum_{k=1}^{n-2} (-1)^k (2k+1) x^2_{2(2k+1)} A^{(n-k-2)}_\mu \right], \]
\[ f^{(\mu)}_\nu = -\frac{2x_\mu x_\nu}{U_\mu^2} \prod_{\rho \neq \mu, \nu} (x^2_\rho - x^2_\mu), \quad \text{with } \mu \neq \nu. \tag{40} \]

We verify with low-lying examples that all of the \( G^{(2)}_\mu \) are harmonic, i.e. \( dG^{(2)}_\mu = 0 = d \ast G^{(2)}_\mu \). It is worth observing that these 2-forms are harmonic regardless of the detailed structure of the functions \( X_\mu \).

It was shown in [4] that the BPS limit of the metric (19) gives rise to the non-compact Calabi-Yau metric that can provide a resolutions of the cone over the Einstein-Sasaki spaces. Under suitable coordinate transformation, the metric is given by
\[ ds^2 = \sum_{\mu=1}^{n} \left\{ \frac{dx^2_\mu}{Q_\mu} + Q_\mu \left( \sum_{k=0}^{n-1} A^{(k)}_\mu d\psi_k \right)^2 \right\}, \tag{41} \]
where we define
\[ Q_\mu = \frac{4X_\mu}{U_\mu}, \quad U_\mu = \prod_{\nu=1}^{m} (x_\nu - x_\mu), \quad X_\mu = x_\mu \prod_{k=1}^{n-1} (x_\mu + \alpha_k) + 2b_\mu, \]
\[ A^{(k)}_\mu = \sum_{\nu_1 < \nu_2 < \cdots < \nu_k} x_{\nu_1} x_{\nu_2} \cdots x_{\nu_k}. \tag{42} \]

Note that we have Wick rotated the metric to have Euclidean signature. We can choose the same form of the vielbein basis as in (38). The Kähler 2-form is then given by
\[ J^{(2)} = \sum_{\mu=1}^{n} e^\mu \wedge \tilde{e}^\mu. \tag{43} \]

The 1-form potentials for the harmonic 2-forms are given by
\[ B^{(1)}_\nu = \frac{1}{U_\mu} \left( \sum_{k=0}^{n-1} A^{(k)}_\mu d\psi_k \right). \tag{44} \]
The corresponding harmonic 2-forms $G_{(2)}^{(\mu)}$ have the same form as in [29], with the functions $f_{\nu}^{(\mu)}$ are given by

$$f_{\nu}^{(\mu)} = \frac{2}{U^{2}} \prod_{\rho \neq \mu, \nu} (x_{\rho} - x_{\mu}), \text{ with } \mu \neq \nu, \quad f_{\mu}^{(\mu)} = - \sum_{\nu \neq \mu} f_{\nu}^{(\mu)}. \quad (45)$$

Note that $G_{(2)}^{(\mu)}$ satisfy the linear relation $\sum_{\mu=1}^{n} G_{(2)}^{(\mu)} = 0$. Thus, in the BPS limit, there are $(n - 1)$ linearly independent such harmonic 2-forms. Together with the Kähler 2-form, the total number of harmonic 2-forms is $n$ again.

4 Conclusion

In this letter, we explicitly express the general (A)dS-Kerr-NUT metrics in Kerr-Schild form for both even and odd dimensions. We demonstrate that, in a suitable coordinate system the mass, NUT and angular momentum parameters enter linearly in the metric, and hence they can be viewed as a linear perturbation of pure (A)dS spacetime.

We also obtain $n$ harmonic 2-forms on the $2n$-dimensional (A)dS-Kerr-NUT metrics. An interesting property of these harmonic 2-forms is that the closure and co-closure do not depend on the detailed structure of the functions $X_{\mu}$. This provides a potential ansatz for charged (A)dS-Kerr-NUT solutions for pure Einstein-Maxwell theories in higher dimensions, whose explicit analytical solutions remain elusive. In the case of four dimensions, the back-reaction of the gauge field to the Einstein equations gives precisely the charged Plebanski metric [2], where only the functions $X_{\mu}$ in the metric have extra contributions from the electric and magnetic charges. However, the same phenomenon does not occur in higher dimensions; nevertheless, the harmonic 2-forms we constructed can be viewed as charged (A)dS-Kerr-NUT solutions at the linear level for small-charge expansion. Together with the charged slowly-rotating black holes obtained in [14, 15], our results may lead to the general charged (A)dS-Kerr-NUT solutions.

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