Enhancement of quantum correlations between two particles under decoherence in finite-temperature environment

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Abstract – Enhancing the quantum correlations in realistic quantum systems interacting with the environment of finite temperature is an important subject in quantum information processing. In this paper, we use weak measurement and measurement reversal to enhance the quantum correlations in a quantum system consisting of two particles. The transitions of the quantum correlations measured by the local quantum uncertainty of qubit-qubit and qutrit-qutrit quantum systems under generalized amplitude damping channels are investigated. We show that, after the weak measurement and measurement reversal, the joint system shows more robustness against decoherence.

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Introduction. – Decoherence in realistic quantum systems severely affects quantum features in quantum information processing (QIP) [1,2]. Thus, protecting quantum states under decoherence is an important subject in QIP tasks. Many schemes have been put forward to achieve this purpose, including dynamical decoupling [3–5], decoherence free subspaces [6–8], the quantum error correction code [9–11], the environment-assisted error correction scheme [12,13], quantum Zeno dynamics [14,15], etc. A novel idea for protecting quantum states by weak measurement and measurement reversal has been proposed theoretically [16,17], and it has been experimentally implemented in the last few years [18–20]. The research focuses on the fidelity and quantum entanglement of a quantum system protected by weak measurement and measurement reversal under decoherence [21–24].

It is widely believed that quantum entanglement is only one of the ingredients of quantum features [25]. As a larger family, quantum correlations are believed to reflect more about the quantumness in QIP [26]. Explicitly, quantum entanglement is a subset of quantum correlations for mixed states. In most QIP tasks, we always face the situation that the quantum system is a mixed state, especially when the quantum system suffers from decoherence. Therefore, it is desirable to study, and to protect, the quantum correlations in the realistic quantum systems under decoherence. There are many kinds of quantifiers of quantum correlations, we adopt the local quantum uncertainty [27,28] for its operability.

We study the enhancement of quantum correlations for qubit-qubit and qutrit-qutrit quantum systems. It needs to be noted that, in the three-dimensional case, we suppose that each of the two particles has V-configuration energy levels, as illustrated in fig. 1. The extension to A-configuration can be naturally done by our approach. In this case, only the transitions from |2⟩ and |1⟩ to |0⟩ are allowed, which simplifies our analysis. In order to characterize decoherence in a finite-temperature environment, we use the generalized amplitude damping channel.

In this paper, we study the enhancement of quantum correlations by weak measurement and measurement reversal. We show that under decoherence, the quantum correlations between two particles can be enhanced. The remainder of this paper is organized as follows: In the next section, we give the preliminaries needed in the following
The temperature is non-zero.

parts. We will introduce the local quantum uncertainty and its closed form. The Kraus operators of the generalized amplitude damping for two- and three-dimensional quantum states having $V$-configuration energy levels are given. We have also shown the mathematical expressions for the weak measurement and measurement reversal operators. In the third section, we investigate the enhancement of quantum correlations using the weak measurement and measurement reversal for the qubit-qubit Bell state with white noise, a non-symmetrical qubit-qubit mixed state, and the qutrit-qutrit Bell state with white noise. We have shown that the approach can be used to enhance the quantum correlations under decoherence. In the fourth section, we will discuss the fidelity of the final output state, and give some conclusions.

**Basic theory.**

**Local quantum uncertainty.** The local quantum uncertainty (LQU) is defined as

$$U_A = \min_{K^A} I(\rho_{AB}, K^A),$$  

where we have denoted the two particles as $A$ and $B$, the minimum is optimized over all the non-degenerate operators on $A$: $K^A = \Lambda^A \otimes I_B$, and

$$I(\rho, K) = -\frac{1}{2} \text{Tr} \left\{ [\sqrt{\rho}, K^A]^2 \right\}$$

is the skew information [29]. It has been shown that the closed form of the LQU for quantum states in $\mathcal{H}^2 \otimes \mathcal{H}^d$ is [27]

$$U_A = 1 - \lambda_{\text{max}}(W),$$

where $\lambda_{\text{max}}$ is the maximum eigenvalue of the $3 \times 3$ matrix $W$ with elements $W_{ij} = \text{Tr} \left\{ \sqrt{\rho} (\sigma_i \otimes I) \sqrt{\rho} (\sigma_j \otimes I) \right\}$ and $\sigma_i \ (i = 1, 2, 3)$ represents the Pauli matrices. The closed form of the LQU for a large class of high-dimensional quantum states in $\mathcal{H}^{d_1} \otimes \mathcal{H}^{d_2}$ is [28]

$$U_A = \frac{2}{d_1} - \lambda_{\text{max}}(W),$$

where $W$ is a $(d_1^2 - 1) \times (d_1^2 - 1)$ matrix with elements

$$W_{ij} = \text{Tr} \left\{ \sqrt{\rho} (\lambda_i \otimes I_{d_2}) \sqrt{\rho} (\lambda_j \otimes I_{d_2}) \right\} - G_{ij}L,$$

with

$$G_{ij} = (g_{ij1}, \cdots, g_{ijd^2-1}),$$

$$L = (\text{Tr}(\rho\lambda_1 \otimes I_{d_2}), \cdots, \text{Tr}(\rho\lambda_k \otimes I_{d_2}), \cdots, \text{Tr}(\rho\lambda_{d^2-1} \otimes I_{d_2}))^T,$$

and $\lambda_j \ (j = 1, \cdots, d^2 - 1)$ are the generators of $SU(d)$, namely

$$\lambda_j = \begin{cases} \sqrt{\frac{2}{j(j+1)}} \left( \sum_{k=1}^{j} |k\rangle \langle k| - |j\rangle |j+1\rangle \right), \\
|k\rangle \langle k| + |m\rangle \langle k| \ (1 \leq k < m \leq d), \\
i(|k\rangle \langle m| - |m\rangle \langle k|) \ (1 \leq k < m \leq d), \\
\frac{d^2 + 1}{2} \end{cases}$$

and $g_{ijk} = \frac{4}{d^2} \text{Tr} \{ (\lambda_i, \lambda_j, \lambda_k) \}$. It needs to be noted that the definition of the LQU requires $A^d$ not being degenerate; therefore the results after the simulation should be re-checked to make sure $A^d$ is non-degenerate when the LQU is maximized. This can be realized by the approach given in ref. [28]. In the following, unless noted, the results are checked to be valid.

**Generalized amplitude damping.** For zero-temperature environment, there only exist transitions from higher energy levels to lower ones, in other words, the loss of excitations. This kind of transition is characterized by the amplitude damping (AD). In the two-dimensional case, the AD can be mathematically expressed by Kraus operators as

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix},$$

where $p$ represents the transition probability from quantum state $|1\rangle$ to state $|0\rangle$. When the temperature of the environment is non-zero, the situation turns out to be more complicated since except for the loss of excitations, there exists the gain of excitations. This process can be characterized by the generalized amplitude damping (GAD). Suppose the probability of losing the excitation $|1\rangle$ is $r$, then the probability of gaining the excitation is $1-r$. Therefore, in two-dimensional quantum systems, the Kraus operators of the GAD are [1]

$$E_0 = \sqrt{r} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, \quad E_1 = \sqrt{r} \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix},$$

$$E_2 = \sqrt{1-r} \begin{pmatrix} \sqrt{1-p} & 0 \\ 0 & 1 \end{pmatrix}, \quad E_3 = \sqrt{1-r} \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}. $$

It needs to be noted that when $r = 1$, the GAD reduces to the AD case.
For quantum systems consisting of three energy levels of $V$-configuration, the derivation of the Kraus operators of the GAD can be done naturally following the approach we have given. The results are

$$E_0 = \sqrt{r}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{1-p_1} & 0 \\ 0 & 0 & \sqrt{1-p_2} \end{pmatrix},$$

$$E_1 = \sqrt{r}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{p_1} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$E_2 = \sqrt{r}\begin{pmatrix} 0 & 0 & \sqrt{1-p_1-p_2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$E_3 = \sqrt{r}\begin{pmatrix} \sqrt{1-p_1-p_2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$E_4 = \sqrt{r}\begin{pmatrix} 0 & 0 & 0 \\ \sqrt{p_1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$E_5 = \sqrt{r}\begin{pmatrix} 0 & 0 & \sqrt{1-p_1-p_2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $p_1$ and $p_2$ are the transition probabilities from $|1\rangle$ and $|2\rangle$ to $|0\rangle$, respectively.

In our study, we let the two particles undergo different GAD channels as illustrated in fig. 2. We assume that the initial state is $\rho_i$, the state after decoherence is

$$\rho_f = \sum_{i,j=0}^{n-1} (E_i \otimes E_j) \rho_i (E_i \otimes E_j)^\dagger,$$

where $n$ is the number of the Kraus operators.

**Weak measurement and measurement reversal.** The basic approach of enhancing quantum correlations by weak measurement and measurement reversal for two-partite quantum systems is illustrated in fig. 3, where we call this scheme two-step enhancement of quantum correlations. First, we apply weak measurement $M$ to the quantum system in order to push the initial state to a space with less decoherence effect. Then the two particles are put in the finite-temperature environment characterized by GAD channels. After the decoherence, we apply the reversal measurement $N$ to recover the quantum correlations.

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & m(1) & 0 \\ 0 & 0 & m(2) \end{pmatrix},$$

where $m_1, m_2 \in [0, \infty)$.

The measurement reversal operators for qubit and qutrit quantum systems are

$$N^{(2)} = \begin{pmatrix} n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$N^{(3)} = \begin{pmatrix} n^{(1)} & 0 & 0 \\ 0 & n^{(2)} & 0 \\ 0 & 0 & n^{(3)} \end{pmatrix},$$

where $n \in [0, \infty)$, and $0 \leq n^{(1)}, n^{(2)}, n^{(3)} \leq 1$. It needs to be noted here that for qutrit quantum systems, we have constructed the operators $M^{(3)}$ and $N^{(3)}$ in more general forms than in ref. [23].

**Enhancing quantum correlations.**

**Qubit-qubit Bell state.** To demonstrate the approach, we will give our analysis in an explicit manner. The initial quantum state of the two particles is chosen as the qubit-qubit Bell state,

$$\rho_0 = \frac{1}{2}(|00\rangle + |11\rangle)(|00\rangle + |11\rangle),$$

and it is no surprise that $U_A(\rho_0) = 1$.

As we have stated above, the weak measurement is

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & m(1) & 0 \\ 0 & 0 & m(2) \end{pmatrix}.$$

After the weak measurement $M$ performed on $\rho_0$, the state becomes

$$\rho_1 = \frac{M \rho_0 M^\dagger}{\text{Tr}(\rho_0 M^\dagger M)}.$$
Then we put the particles in a finite-temperature environment to study the quantum correlations under decoherence. Without loss of generality, we choose \( r_1 = r_2 = 0.5, p_1 = p_2 = 0.5 \) in the GAD channels (eq. (9)) and assume that the two particles suffer from the same quantum noises (see fig. 2). It can be calculated that without the weak measurement and measurement reversal, the LQU reduces to 0.134.

As the last step, we need to perform the measurement reversal

\[
N = \begin{pmatrix} n_1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} n_2 & 0 \\ 0 & 1 \end{pmatrix},
\]

then the state is

\[
\rho_3 = \frac{Np_2N^\dagger}{\text{Tr}(p_2N^\dagger N)}.
\]

Because of the large number of the parameters, we use the genetic algorithm in our simulation. By optimizing upon \( m_1, m_2, n_1, n_2 \), the LQU of \( \rho_3 \) is maximized when \( m_1 = 1.285, m_2 = 0.760, n_1 = 1.606, n_2 = 0.830 \), in this case, \( \mathcal{U}_A(\rho_1) = 0.218 \). The dependence of \( \mathcal{U}_A(\rho_3) \) on \( n_1 \) and \( n_2 \) with fixed \( m_1 = 1.285, m_2 = 0.760 \) is shown in fig. 4.

Therefore, we have shown that the weak measurements and measurement reversal have enhanced the quantum correlations between the particles under decoherence.

**Non-symmetrical qubit-qubit mixed state.** We consider a general case in which the qubit-qubit state has no symmetry under the permutation of the two particles. The quantum state is chosen as

\[
\rho_0 = \frac{1}{2} |\psi\rangle\langle\psi| + \frac{1}{8} I,
\]

where \( |\psi\rangle = \frac{1}{\sqrt{3}} |01\rangle + \frac{1}{\sqrt{2}} |10\rangle + \frac{1}{\sqrt{2}} |11\rangle \), and \( \mathcal{U}_A(\rho_0) = 0.096 \).

First we perform the weak measurement (see eq. (16)), then the two particles decoherence under different GAD channels where \( r_1 = r_2 = 0.5, p_1 = p_2 = 0.5 \). As the last step, the measurement reversal is performed. It can be optimized that the maximum of the LQU is given when \( m_1 = 1.65, m_2 = 1.20, n_1 = 0.85, n_2 = 0.90 \), and, in this case, the LQU is 0.031. The LQU against \( N \) with fixed \( m_1 = 1.65, m_2 = 1.20 \) is shown in fig. 5.

Now we compare our results with and without weak measurement and measurement reversal. It can be easily calculated that if \( M \) and \( N \) are omitted, the decoherence of the two particles causes the quantum correlations to drop rapidly. The LQU reduces to 0.019. We can see that the weak measurement and measurement reversal have enhanced the quantum system’s ability against decoherence.

**Non-symmetrical qutrit-qutrit mixed state.** To illustrate our approach of enhancing the quantum correlations between qutrits, we consider

\[
\rho_0 = \frac{1}{2} |\psi\rangle\langle\psi| + \frac{1}{18} I,
\]

where \( |\psi\rangle = \frac{1}{\sqrt{2}} |101\rangle + \frac{1}{\sqrt{2}} |012\rangle + \frac{1}{\sqrt{2}} |211\rangle \), and \( \mathcal{U}_A(\rho_0) = 0.130 \).

In this case, the weak measurement and measurement reversal operators are

\[
M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & m_1^{(1)} & 0 \\ 0 & 0 & m_2^{(1)} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & m_2^{(1)} & 0 \\ 0 & 0 & m_2^{(2)} \end{pmatrix},
\]

\[
N = \begin{pmatrix} n_1^{(1)} & 0 & 0 \\ 0 & n_1^{(2)} & 0 \\ 0 & 0 & n_1^{(3)} \end{pmatrix} \otimes \begin{pmatrix} n_2^{(1)} & 0 & 0 \\ 0 & n_2^{(2)} & 0 \\ 0 & 0 & n_2^{(3)} \end{pmatrix}.
\]

In this case, we consider a more general case in which \( r = 0.5, p_1 = 0.1, p_2 = 0.4 \) in the GAD channels. As the weak measurement and measurement reversal operators are performed, and the LQU of the quantum state is maximized when \( m_1^{(1)} = 1.2745, m_1^{(2)} = 1.29, m_1^{(3)} = 1.1175, m_2^{(2)} = 0.939, n_1^{(1)} = 0.751, n_1^{(2)} = 0.564, n_1^{(3)} = 0.480 \),
Table 1: The enhancement of the quantum correlations under decoherence by weak measurement and measurement reversal.

| (LQU of) quantum states | 2D Bell | Non-symmetrical 2D | 3D |
|------------------------|---------|-------------------|----|
| Initial state          | 1.0     | 0.096             | 0.130 |
| Perform M and N        | 0.218   | 0.031             | 0.081 |
| Without M and N        | 0.134   | 0.019             | 0.072 |

\(n_1^{(1)} = 0.954, n_2^{(2)} = 0.884, n_2^{(3)} = 0.759\), where the LQU is 0.081. It needs to be noted that without \(M\) and \(N\), the LQU after decoherence is 0.072.

To summarize, we list our results in table 1, where we have used “2D Bell”, “Non-symmetrical 2D”, and “3D” to represent “qubit-qubit Bell state”, “non-symmetrical qubit-qubit mixed state”, and “non-symmetrical qudit-qudit mixed state”, respectively.

**Conclusion.** — We use weak measurement and measurement reversal to enhance the quantum correlations in a quantum system consisting of two particles. The transitions of the quantum correlations of two- and three-dimensional quantum states during decoherence under generalized amplitude damping are investigated. We show that, after the weak measurement and measurement reversal, the joint system becomes robust against decoherence. Except for the quantum correlations, we also care about the fidelity of the final output state. The fidelity of the final state \(\rho_f\) is defined as [30]

\[
F(\rho_i, \rho_f) = \left[ \text{Tr} \sqrt{\sqrt{\rho_i} \rho_f \sqrt{\rho_i}} \right]^2,
\]

where \(\rho_i\) is the initial state. In the first place we consider the case of the qubit-qubit Bell state. It can be calculated that after decoherence, the fidelity is reduced to 0.56. When \(M\) and \(N\) are performed, the fidelity of the final output is 0.52. As for the qutrit-quqrit mixed state, the fidelity is 0.925 and 0.964 with and without \(M\) and \(N\), respectively. However, the fidelity for the non-symmetrical qubit-qubit mixed state has been improved from 0.960 to 0.964 with weak measurement and measurement reversal. To summarize, in most of the cases, due to the different physical meanings of the quantum correlations and fidelity, we can enhance the quantum correlations by sacrificing the fidelity [24]. But in some quantum states, it is still possible to improve (or keep) both the quantum correlations and the fidelity of the final state by using weak measurement and measurement reversal.

It needs to be noted that some of our examples can be implemented in nuclear magnetic resonance systems [31], linear photon systems [32], nitrogen-vacancy centres [33], etc. We expect that our work may find further theoretical and experimental applications.

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