Second order quantum corrections to the classical reflection factor of the sinh-Gordon model

A. Chenaghlou

Physics Department, Faculty of Science, Sahand University of Technology, P O Box 51335-1996, Tabriz, Iran

Abstract

The sinh-Gordon model on a half-line with integrable boundary conditions is considered in low order perturbation theory developed in affine Toda field theory. The quantum corrections to the classical reflection factor of the model are studied up to the second order in the difference of the two boundary parameters and to one loop order in the bulk coupling. It is noticed that the general form of the second order quantum corrections are consistent with Ghoshal’s formula.
1 Introduction

Affine Toda field theory [1, 2] is one of the most successful examples of the two-dimensional quantum field theory since it can be solved exactly. This theory possesses remarkable properties including classical and quantum integrability. An interesting review of the recent developments in Affine Toda field theory is presented in Ref. [3]. The classical affine Toda field theories remain integrable in the presence of certain boundary conditions restricting them to a half-line, or to an interval [4–9]. Indeed, Corrigan et.al [4–6] have classified the boundary conditions which preserve classical integrability. However, quantum integrability are hardly explored in the presence of a boundary although there has been progress for models based on $a_n^{(1)}$ class of theories [4, 10–13]. The simplest of these (n=1) is the sinh-Gordon model. In fact, this model is the only example in the ade series of affine Toda field theory which allows continuous boundary parameters. Although the sinh-Gordon model has been studied much more than other models in the integrable quantum field theories there still remains much to be studied in connection with the quantum integrability of the model with a boundary. In particular, one of the interesting problems is to see how the boundary parameters could relate to the quantum reflection factor of the theory.

In 1993 Ghoshal and Zamolodchikov [14] studied the behaviour of the sine-Gordon model restricted to a half-line. They found the soliton reflection factors in the model and discovered a two parameter family of boundary conditions which preserved integrability. Then, Ghoshal [15] computed the reflection factors of the soliton-antisoliton bound states (the breathers) of the model. The pioneering ideas of Ghoshal and Zamolodchikov motivated many researchers to investigate integrable quantum field theory with a boundary. Corrigan [16] was the first to discover that the lightest breather reflection factor of the sine-Gordon model is identical to the reflection factor of the sinh-Gordon model followed by an analytic continuation in the coupling constant.

Recently, Corrigan and Delius [17] calculated the bound-state spectrum of the sinh-Gordon model on a half-line in two different ways, firstly by using a boundary bootstrap principal, and secondly by quantizing the classical boundary breather states using a WKB technique (for further discussion see Ref. [18]). Comparing these two calculations provides a relationship between the boundary parameters, the bulk coupling constant and the parameters in Ghoshal’s formula. They performed the calculations in the special case where the bulk $Z_2$ symmetry is conserved at the boundary, by requiring the two boundary parameters to be equal. Then, Corrigan and Taormina [19] studied and extended the idea of [17] in the intricate case where the two boundary parameters are different and the bulk symmetry is violated. Their calculations lead to a determination of the relationship between the Lagrangian boundary parameters and the two parameters appearing in the reflection factor of the sinh-Gordon particle. Moreover, their results clarify the weak-strong coupling duality of the sinh-Gordon model with integrable boundary conditions.

The one loop quantum corrections to the classical reflection factor of the sinh-Gordon model have been found [10] when the boundary parameters are equal. If the two boundary parameters are different and the bulk $Z_2$ symmetry of the model is broken then, the lowest energy static background configuration will not be a trivial
solution \((\phi = 0)\). In this intricate case, the perturbation theory must be developed within a complicated background. Meanwhile, the corresponding calculations involve a significantly more difficult propagator as well as intricate coupling constants. In a recent paper \cite{20} the quantum reflection factor has been found in one loop order up to the first order in the difference of the boundary parameters. The calculations provide a further verification of Ghoshal’s formula up to the first order. The quantum reflection factor up to the second order in the difference of the boundary data is treated in this article. The result could test more deeply the expression for the sinh-Gordon model reflection factor given in Ref. \cite{13}.

2 The sinh-Gordon model on the half-line

The sinh-Gordon theory describes a real massive scalar quantum field theory in \(1+1\) dimensions with exponential self-interaction. The Lagrangian density of the theory in the presence of a boundary is

\[
\bar{L} = \theta(-x) L - \delta(x) B
\]

where the bulk Lagrangian density of the model is

\[
L = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{2m^2}{\beta^2} \cosh(\sqrt{2} \beta \phi), \tag{2.2}
\]

and the boundary potential \(B\) has the following generic form \cite{14}

\[
B = \frac{m}{\beta^2} \left( \sigma_0 e^{-\frac{\beta}{\sqrt{2}} \phi} + \sigma_1 e^{\frac{\beta}{\sqrt{2}} \phi} \right). \tag{2.3}
\]

Note, in the above expressions \(m\) and \(\beta\) are a mass scale and a coupling constant of the theory and we have used normalizations customary in affine Toda field theory. The two real coefficients \(\sigma_0\) and \(\sigma_1\) are essentially free which indicate \cite{3,4} the extra parameters permitted at the boundary \(x = 0\). Meanwhile, the boundary potential is required to satisfy the following equation as a consequence of maintaining integrability on the half-line:

\[
\frac{\partial \phi}{\partial x} = -\frac{\partial B}{\partial \phi} \quad \text{at} \quad x = 0, \tag{2.4}
\]

or

\[
\frac{\partial \phi}{\partial x} = -\frac{\sqrt{2}}{\beta} \left( \sigma_1 e^{\beta \phi/\sqrt{2}} - \sigma_0 e^{-\beta \phi/\sqrt{2}} \right) \quad \text{at} \quad x = 0, \tag{2.5}
\]

where we have rescaled the mass scale to unity in the integrable boundary condition. It is also convenient to use an alternative expression for the boundary parameters, i.e. \(\sigma_i = \cos a_i \pi\).

The sinh-Gordon model is integrable classically which implies there are infinitely many mutually commuting, independent conserved charges \(Q_{\pm s}\), where \(s\) is an arbitrary
odd integer. On the other hand, the model is integrable in the context of quantum field theory which means the S-matrix describing the n-particles scattering factorises into a product of two-particles scattering amplitudes. The S-matrix describing the elastic scattering of two sinh-Gordon particles with relative rapidity $\theta$ is conjectured to be given by [1, 21, 22]

$$S(\theta) = -\frac{1}{(B)(2 - B)}$$

where we have used the hyperbolic building block notation

$$(x) = \frac{\sinh(\theta/2 + \frac{i\pi x}{4})}{\sinh(\theta/2 - \frac{i\pi x}{4})},$$

and the quantity B is related to the coupling constant $\beta$ by $B = \frac{2\beta^2}{4\pi + \beta^2}$.

For the boundary sinh-Gordon model, the most important physical quantity is the boundary S-matrix or the reflection factor which describes the reflection of a single particle from the boundary. Firstly, Ghoshal and Zamolodchikov [14] calculated the soliton reflection factors for the sine-Gordon model by solving the boundary Yang-Baxter equation and using general constraints implementing unitarity and a form of crossing symmetry. Then, Ghoshal [15] calculated the soliton-antisoliton bound states reflection factor by using the boundary bootstrap equation. Finally, following the idea discovered by Corrigan [16], the quantum reflection factor for the lightest breather in the sine-Gordon model is supposed to be identical with the quantum reflection factor of the sinh-Gordon particle after doing analytic continuation in the coupling constant to derive

$$K_q(\theta) = \frac{(1)(2 - B/2)(1 + B/2)}{(1 - E(\sigma_0, \sigma_1, \beta))(1 + E(\sigma_0, \sigma_1, \beta))(1 - F(\sigma_0, \sigma_1, \beta))(1 + F(\sigma_0, \sigma_1, \beta))},$$

(2.8)

When the bulk reflection symmetry is preserved which means $\sigma_0 = \sigma_1$, then one of two parameters $E$ or $F$ vanishes.

Recently, on the basis of a perturbative calculation, the form of the $E$ and $F$ was conjectured to be [20],

$$E = (a_0 + a_1)(1 - B/2), \quad F = (a_0 - a_1)(1 - B/2).$$

(2.9)

In fact, Corrigan and Taormina [19] gave further evidence for these formulae. They computed the energy spectrum of the quantized boundary states, firstly by using a bootstrap technique and, secondly using a WKB approximation. Requiring that the two methods agree with each other provides further support to the above conjecture. Meanwhile, they did the non-perturbative calculations in the significantly more difficult case when $\sigma_0 \neq \sigma_1$.

### 3 Low order perturbation theory

Since perturbative expansion will be developed around an $x$-dependent static background field configuration, it is possible to use standard Feynman rules in configuration
space. We will find that the theory needs three- and four-point couplings which depend upon the $x$-dependent static background. In other words, the interaction vertices in the Feynman diagrams will be position dependent. In order to derive the three and four-point couplings, we need to expand the bulk and boundary potentials in terms of the coupling constant $\beta$ around the background solution. We obtain the bulk three- and four-point couplings:

\[ C^{(3)}_{\text{bulk}} = \frac{2\sqrt{2}}{3} \beta \sinh(\sqrt{2}\beta \phi_0) \]  
\[ C^{(4)}_{\text{bulk}} = \frac{1}{3} \beta^2 \cosh(\sqrt{2}\beta \phi_0), \]  

where $\phi_0$ is the background solution to the equation of motion of the model. Similarly, the boundary three and four-point couplings may be found as:

\[ C^{(3)}_{\text{boundary}} = \frac{\sqrt{2} \beta}{12} \left( \sigma_1 e^{\beta \phi_0 / \sqrt{2}} - \sigma_0 e^{-\beta \phi_0 / \sqrt{2}} \right) \]  
\[ C^{(4)}_{\text{boundary}} = \frac{\beta^2}{48} \left( \sigma_1 e^{\beta \phi_0 / \sqrt{2}} + \sigma_0 e^{-\beta \phi_0 / \sqrt{2}} \right). \]

Meanwhile, we may consider the linear perturbation of the equation of the motion and the boundary condition around the background field \[ [5, 16] \] to yield

\[ e^{\beta \phi_0 / \sqrt{2}} = \frac{1 + e^{2(x - x_0)}}{1 - e^{2(x - x_0)}}, \]

where $x_0$ is related to the boundary data and satisfies

\[ \coth x_0 = \sqrt{\frac{1 + \sigma_0}{1 + \sigma_1}}. \]  

So, after some manipulation, the bulk three- and four-point couplings become

\[ C^{(3)}_{\text{bulk}} = \frac{4\sqrt{2}}{3} \beta \cosh 2(x - x_0) \left( \coth^2 2(x - x_0) - 1 \right), \]  
\[ C^{(4)}_{\text{bulk}} = \frac{1}{3} \beta^2 \left( 2 \coth^2 2(x - x_0) - 1 \right) \]  

and similarly,

\[ C^{(3)}_{\text{boundary}} = \frac{\sqrt{2} \beta}{12} \left( \sigma_1 \coth x_0 - \sigma_0 \tanh x_0 \right), \]  
\[ C^{(4)}_{\text{boundary}} = \frac{\beta^2}{48} \left( \sigma_1 \coth x_0 + \sigma_0 \tanh x_0 \right). \]  

We also need to find the Green’s function for the theory. The two-point propagator for the boundary sinh-Gordon model is known \[ [16] \] to have the form

\[ G(x, t; x', t') = \int \int \frac{d\omega}{2\pi} \frac{dk}{2\pi} \frac{ie^{-i\omega(t-t')}}{\omega^2 - k^2 - 4 + i\rho} \left( f(k, x)f(-k, x')e^{ik(x-x')} + K_c f(-k, x)f(-k, x')e^{-ik(x+x')} \right), \]  

(3.11)
where
\[ f(k, x) = \frac{ik - 2 \coth 2(x - x_0)}{ik + 2} \] (3.12)

and \( K_c \) represent the classical reflection factor of the model which is given by
\[ K_c = \left( \frac{(ik)^2 + 2ik \sqrt{1 + \sigma_0 \sqrt{1 + \sigma_1} + 2(\sigma_0 + \sigma_1)}}{(ik)^2 - 2ik \sqrt{1 + \sigma_0 \sqrt{1 + \sigma_1} + 2(\sigma_0 + \sigma_1)}} \right) \left( \frac{ik - 2}{ik + 2} \right). \] (3.13)

Indeed, the quantum reflection factor (2.8) reduces to the classical reflection (3.13) in the classical limit \( \beta \to 0 \) provided \( E(\sigma_0, \sigma_1, 0) = a_0 + a_1 \) and \( F(\sigma_0, \sigma_1, 0) = a_0 - a_1 \).

In this paper it is intended to calculate the quantum corrections to the classical reflection factor of the sinh-Gordon model at one loop order and up to the second order in the difference of the boundary parameters. Hence, it is necessary to expand the bulk and the boundary couplings. In fact,
\[ C^{(3)}_{\text{bulk}} = \frac{2\sqrt{2}}{3} \beta \frac{\epsilon}{1 + \sigma_1} e^{2x} + ... \] (3.14)
\[ C^{(4)}_{\text{bulk}} = \frac{1}{3} \beta^2 \left( 1 + \frac{\epsilon^2}{2(1 + \sigma_1)^2} e^{4x} + ... \right), \] (3.15)

where \( \epsilon = \sigma_0 - \sigma_1 \). Similarly,
\[ C^{(3)}_{\text{boundary}} = \frac{\sqrt{2}\beta}{12} \left( -\frac{\epsilon}{1 + \sigma_1} \right) + ... \] (3.16)

and
\[ C^{(4)}_{\text{boundary}} = \frac{\beta^2}{48} \left( 2\sigma_1 + \epsilon - \frac{\sigma_1 + 2}{4(1 + \sigma_1)^2} \epsilon^2 \right) + .... \] (3.17)

Now let us find the expansions of \( f(k, x) \) and the classical reflection factor \( K_c \), both of them appear in the Green’s function (3.11). These can be shown to be
\[ f(k, x) = 1 + \frac{\epsilon^2}{4(1 + \sigma_1)^2} \frac{1}{ik + 2} e^{4x} + ... \] (3.18)

and
\[ K_c = \frac{ik + 2\sigma_1}{ik - 2\sigma_1} + \frac{2ik}{(ik - 2\sigma_1)^2} \epsilon \]
\[ + \frac{i k (ik^3 - 4k^2 - 6k^2 \sigma_1 - 4ik \sigma_1 + 8\sigma_1^2 - 16 \sigma_1)}{2(1 + \sigma_1)(ik - 2)(ik + 2)(ik - 2\sigma_1^3)} \epsilon^2 \] (3.19)

or in a compact form
\[ K = K_0 + K_1 \epsilon + K_2 \epsilon^2 + .... \] (3.20)

Here, \( K_0 \) is the classical reflection factor when the two boundary parameters are equal. To calculate the one loop \( O(\beta^2) \) quantum corrections to the classical reflection factor, we use the idea introduced by Kim [23] and developed by Corrigan [16]. In other words,
after perturbative calculation of the propagator the $O(\beta^2)$ corrections to the classical reflection factor can be made by looking at the coefficient of $e^{-ik(x+x')}$ in the residue of the on-shell pole in the asymptotic region $x, x' \to -\infty$.

The computations corresponding to the one loop quantum corrections to the classical reflection factor may be performed by the standard perturbation theory which has been generalised to the affine Toda field theory \[16, 23–25\] in the presence of an integrable boundary. There are three basic types of Feynman diagrams which contribute to the two-point Green’s function of affine Toda field theory in one loop order. These are shown in figure 1. Now by looking at the three-point and four-point couplings, we realize that all types of these diagrams are concerned in our problem (up to the second order in the difference of the boundary data). In what follows we calculate the contributions of types I and III Feynman diagrams to the reflection factor. The remaining diagrams will be treated elsewhere.

![Figure 1: Three basic Feynman diagrams in one loop order.](image)

4 Type III (boundary-boundary)

Clearly, the type III Feynman diagram includes four different configurations depending on the fact that the interaction vertices to be located in the bulk region or at the boundary. This section deals with the calculations corresponding to the contribution of the type III Feynman diagram to the reflection factor when both vertices are placed at the boundary. The associated contribution may be given by

$$
- \frac{\beta^2}{4} \frac{e^2}{(1 + \sigma_1)^2} \int dt dt' G(x_1, t_1; 0, t) G(0, t; 0, t') G(0, t'; 0, t') G(0, t; x_2, t_2). \tag{4.1}
$$

Here, the related three point couplings and the combinatorial factor have been considered. It is evident that in this case the two-point Green function has the simplest form as

$$
G(x, t; x', t') = i \int \frac{d\omega}{2\pi} \frac{dk}{2\pi} \frac{e^{-i\omega(t-t')}}{\omega^2 - k^2 - 4 + i\rho} \left( e^{ik(x-x')} + K_0(k) e^{-ik(x+x')} \right), \tag{4.2}
$$
where
\[ K_0(k) = \frac{ik + 2\sigma_1}{ik - 2\sigma_1}. \] (4.3)
Moreover, the loop propagator has been found in Ref. [13] and so,
\[ G(0, t'; 0, t') = -a_1 \frac{\cos a_1 \pi}{\sin a_1 \pi}. \] (4.4)
Now it is convenient to calculate the time integral of the other middle propagator in (4.1), i.e.
\[ \int dt' G(0, t; 0, t') \] (4.5)
which is equal to
\[ i \int \frac{dk''}{2\pi} \left( \frac{1}{-k''^2 - 4} \right) \left( \frac{2ik''}{ik'' - 2\sigma_1} \right). \] (4.6)
Integrating over \( t' \) gives us a Dirac delta function so, the above relation reduces to
\[ i \int \frac{dk''}{2\pi} \left( \frac{1}{-k''^2 - 4} \right) \left( \frac{2ik''}{ik'' - 2\sigma_1} \right). \] (4.7)
Hence,
\[ \int dt' G(0, t; 0, t') = -\frac{i}{2(1 + \sigma_1)}. \] (4.8)
Therefore, up to now the contribution (4.1) has the form
\[ \frac{i \beta^2 a_1 \cos a_1 \pi}{8 \sin a_1 \pi} \frac{\epsilon^2}{(1 + \cos a_1 \pi)^3} \int dt \int \frac{d\omega}{2\pi} \int \frac{dk}{2\pi} \frac{e^{-i\omega(t_1 - t)}}{2\pi \omega^2 - k^2 - 4 + i\rho} \left( e^{ikx_1} + K_0(k)e^{-ikx_1} \right) \]
\[ \times \int \frac{d\omega'}{2\pi} \int \frac{dk'}{2\pi} \frac{e^{-i\omega'(t_2 - t)}}{2\pi \omega'^2 - k'^2 - 4 + i\rho} \left( e^{-ik'x_2} + K_0(k')e^{-ik'x_2} \right). \] (4.9)
First of all, it is understood to perform the transformation \( k \to -k \) in the first term of the first propagator. Secondly, after integration over \( t \) which ensures energy conservation at the interaction vertex, the result will be a Dirac delta function which immediately gives rise to the substitution \( \omega = \omega' \). Finally, the momenta of the two propagator can be integrated out by taking the contours to be closed in the upper half-plane and considering the pole at \( \hat{k} = k = k' = \sqrt{\omega^2 - 4} \) and ignoring the other pole i.e. \( -2i\sigma_1 \) (when \( \sigma_1 < 0 \)) due to the fact that its residue will vanish in the limit \( x_1, x_2 \to -\infty \). Thus the type III (boundary-boundary) contribution becomes
\[ \frac{i \beta^2 a_1 \cos a_1 \pi}{8 \sin a_1 \pi} \frac{\epsilon^2}{(1 + \cos a_1 \pi)^3} \int \frac{d\omega}{2\pi} e^{-i\omega(t_1 - t_2)} e^{-ik(x_1 + x_2)} \frac{1}{(i\hat{k} - 2\sigma_1)^2}, \] (4.10)
where \( \hat{k} = 2 \sinh \theta. \)
5 Type III (boundary-bulk)

Now let us calculate the contribution of the type III Feynman diagram to the reflection factor when the interaction vertex corresponding to the loop is placed inside the bulk region and the other vertex is at the boundary. As before, the combinatorial factor of the diagram should be considered as a coefficient factor. Moreover, in this case we must take into account the bulk three-point coupling in the loop vertex as well as the boundary three-point coupling in the other vertex. The contribution this time is

$$\frac{2\beta^2}{(1 + \cos a_1 \pi)^2} e^2 \int_0^0 \! dx' \int dt dt' G(x_1, t_1; 0, t) G(0, t; x', t') G(x', t'; x', t') G(0, t; x_2, t_2) e^{2x'}. \quad (5.1)$$

In fact, in the previous section we found the following relation which is some part of the contribution (5.1):

$$\int dtG(x_1, t_1; 0, t) G(0, t; x_2, t_2) = - \int \frac{d\omega}{2\pi} e^{-i\omega(t-t_2)} e^{-i(k_1 + x_2)} \frac{1}{(ik - 2\sigma_1)^2}. \quad (5.2)$$

Clearly, in the boundary-bulk contribution (5.1), it is seen that the $t'$ dependence is involved only in the propagator $G(0, t; x', t')$. So, it is convenient to compute the following relation

$$\int dt' G(0, t; x', t') = i \int dt' \int \frac{d\omega'}{2\pi} \frac{dw'}{2\pi} \left( \frac{e^{-i\omega'(t-t')}}{\omega'^2 - k'^2 - 4 + i\rho} \right) \left( \frac{2ik'}{ik' - 2\sigma_1} \right) e^{-ik'x'}. \quad (5.3)$$

Integrating over $t'$ generates a Dirac delta function which allows the substitution $\omega' = 0$ and hence,

$$\int dt' G(0, t; x', t') = i \int \frac{dk'}{2\pi} \left( \frac{e^{-ik'x'}}{-k'^2 - 4} \right) \left( \frac{2ik'}{ik' - 2\sigma_1} \right). \quad (5.4)$$

Therefore, we obtain

$$\int dt' G(0, t; x', t') = - \frac{i}{2(1 + \sigma_1)} e^{2x'}. \quad (5.5)$$

Now let us calculate the loop propagator which has the form

$$G(x', t'; x', t') = i \int \frac{d\omega''}{2\pi} \frac{dk''}{2\pi} \frac{1}{\omega''^2 - k''^2 - 4 + i\rho} \left( 1 + K_0(k'') e^{-2ik''x'} \right). \quad (5.6)$$

In fact, the above integral is divergent. However, this divergence can be removed by an infinite renormalization of the mass parameter in the bulk potential. In other words, a minimal subtraction of the divergent part can be made by adding an appropriate counterterm to the bulk potential. Meanwhile, if we integrate over $\omega''$ then, we will find

$$G(x', t'; x', t') = \frac{1}{2} \int \frac{dk''}{2\pi} \frac{1}{\sqrt{k''^2 + 4i\rho - 2\sigma_1}} e^{-2ik''x'}. \quad (5.7)$$
Hence, the remaining part of the contribution (5.1) is simplified as (apart from the coefficients)

\[
\int \int dt' dx' G(0,t;x',t')G(x',t';x',t')e^{2x'}
= -\frac{i}{4(1 + \sigma_1)} \int_{-\infty}^{0} dx' \int \frac{dk''}{2\pi} \frac{1}{\sqrt{k''^2 + 4i k'' - 2\sigma_1}} e^{(4-2ik'')x'}
\] (5.8)

In order to integrate over \(x'\), one may use the following formula

\[
\int_{-\infty}^{0} dx' e^{ik''x' + \tau x'} = -\frac{i}{k'' - i\tau}
\] (5.9)

where the quantity \(\tau\) must be positive. The \(k''\) integration can be evaluated by closing the contour in the upper (lower) half-plane depending on whether \(\sigma_1 > 0\) (\(\sigma_1 < 0\)). However, due to the branch cut the contour has to run around the cut line. Note, the cut line stretches from \(k'' = 2i\) to infinity along the imaginary axis when \(\sigma_1 > 0\). Otherwise, the cut line run from \(k'' = -2i\) to \(-\infty\) along the imaginary axis when \(\sigma_1 < 0\). So, after performing the required integrations

\[
\int \int dt' dx' G(0,t;x',t')G(x',t';x',t')e^{2x'}
= \frac{i}{16\pi(1 + \cos a_1\pi)} \left( \frac{2\pi a_1 \cos a_1\pi}{(1 - \cos a_1\pi) \sin a_1\pi} - \frac{1 + \cos a_1\pi}{1 - \cos a_1\pi} \right). \quad (5.10)
\]

Therefore, the contribution of the type III (boundary-bulk) diagram to the reflection factor is

\[
-\frac{i\beta^2 \epsilon^2}{8\pi(1 + \cos a_1\pi)^3} \left( \frac{2\pi a_1 \cos a_1\pi}{(1 - \cos a_1\pi) \sin a_1\pi} - \frac{1 + \cos a_1\pi}{1 - \cos a_1\pi} \right) \int \frac{d\omega}{2\pi} e^{-i\omega(t_1-t_2)} e^{-i k(x_1+x_2)} \frac{1}{(ik - 2\sigma_1)^2}.
\] (5.11)

6 Type III (bulk-boundary)

Let us consider the type III Feynman diagram when the loop vertex is at the boundary and the other interaction vertex is inside the bulk region. So, in this case the contribution is

\[
\frac{2\beta^2}{(1 + \cos a_1\pi)^2} \epsilon^2 \int_{-\infty}^{0} dx \int \int dt dt' G(x_1, t_1; x, t) G(x, t; 0, t') \times G(0, t'; 0, t') G(x, t; x_2, t_2) e^{2x}.
\] (6.1)

In fact, we have obtained the two middle propagators in the previous sections. Hence,
\[ G(0, t'; 0, t') = -a_1 \frac{\cos a_1 \pi}{\sin a_1 \pi} \] (6.2)

and

\[ \int dt' G(x, t; 0, t') = -\frac{i}{2(1 + \sigma_1)} e^{2x}. \] (6.3)

So, the contribution (6.1) can be shown in detail as (after doing the integration over \( t \))

\[
\frac{i\beta^2 \epsilon^2}{(1 + \cos a_1 \pi)^3} \frac{a_1 \cos a_1 \pi}{\sin a_1 \pi} \times \int_{-\infty}^{0} dx \int d\omega \frac{dk}{2\pi} \frac{dk'}{2\pi} \frac{ie^{-i\omega(t_1-t_2)}}{\omega^2 - k^2 - 4 + i\rho} \left( e^{-ik(x_1-x)} + K_0(k)e^{-ik(x_1+x)} \right) \times \int \frac{d\omega'}{2\pi} \frac{dk'}{2\pi} \frac{ie}{{\omega'}^2 - k'^2 - 4 + i\rho} \left( e^{ik'(x-x_2)} + K_0(k')e^{-ik'(x+x_2)} \right) e^{4x}. \] (6.4)

The integration over \( x \) may be done by using the device (5.9). Meanwhile, it is necessary to perform a transformation \( k \to -k \) in the first term of the first propagator. Therefore, the above expression becomes

\[
\frac{i\beta^2 \epsilon^2}{(1 + \cos a_1 \pi)^3} \frac{a_1 \cos a_1 \pi}{\sin a_1 \pi} \int \int \int d\omega d\omega' \frac{dk}{2\pi} \frac{dk'}{2\pi} e^{-i\omega(t_1-t_2)} e^{-ik(x_1+k'x_2)} \frac{i}{\omega^2 - k^2 - 4 + i\rho} \frac{i}{\omega'^2 - k'^2 - 4 + i\rho} \times \left\{ -\frac{iK_0(k')}{k+k'-4i} - \frac{iK_0(k)}{-k+k'-4i} - \frac{iK_0(k)K_0(k')}{-k-k'-4i} \right\}. \] (6.5)

Clearly, all that remains is to integrate over the momenta \( k, k' \). This can be achieved by completing the contours in the upper half-plane and picking up the pole at \( \hat{k} = k = k' = \sqrt{\omega^2 - 4} \) and regarding the fact that all the other poles have no contributions because their residues yield exponentially damped terms in the asymptotic region \( x_1, x_2 \to -\infty \). Hence, the contribution of the type III (bulk-boundary) Feynman diagram to the reflection factor is

\[
\frac{\beta^2 \epsilon^2}{(1 + \cos a_1 \pi)^3} \frac{a_1 \cos a_1 \pi}{\sin a_1 \pi} \int d\omega \frac{e^{-i\omega(t_1-t_2)} e^{-ik(x_1+x_2)}}{(2k)^2} \frac{1}{(2k)^2} \times \left\{ \frac{1}{2k-4i} + \frac{i}{2} K_0(\hat{k}) - \frac{1}{2k+4i} K_0^2(\hat{k}) \right\}. \] (6.6)

7 Type III (bulk-bulk)

In this section we study the type III (bulk-bulk) Feynman diagram which contribute to the quantum correction to the classical reflection factor. It is evident that in this case
both interaction vertices are located inside the bulk region. So, in this configuration the corresponding contribution is given by

$$-16 \frac{\beta^2}{(1 + \cos a_1 \pi)^2} e^2 \int_{-\infty}^{0} dx \int_{-\infty}^{0} dx' \int dt \int dt' G(x_1, t_1; x, t) G(x, t; x', t') G(x', t'; x', t') G(x, t; x_2, t_2) e^{2x} e^{2x'}.$$ \quad (7.1)$$

As it was shown in the type III (boundary-bulk) case, the loop propagator can be simplified to obtain

$$G(x', t'; x', t') = \frac{1}{2} \int \frac{dk_1}{2\pi} \frac{1}{\sqrt{k_1^2 + 4}} \left( \frac{i k_1 + 2 \sigma_1}{i k_1 - 2 \sigma_1} \right) e^{-2ik_1 x'}.$$ \quad (7.2)$$

Moreover, the time variable $t'$ appears only in the other middle propagator. Therefore, it is appropriate to calculate the following relation

$$\int dt' G(x, t; x', t') = \int \int \int \int dt'' dt''' \frac{d\omega''}{2\pi} \frac{dk''}{2\pi} \frac{e^{-i\omega''(t-t')}}{\omega'' - k''^2 - 4 + i\rho} \left( e^{ik''(x-x')} + K_0(k'') e^{-ik''(x+x')} \right).$$ \quad (7.3)$$

The integration over $t'$ gives us a Dirac delta function which replaces $\omega''$ by zero and hence,

$$\int dt' G(x, t; x', t') = \int \frac{dk''}{2\pi} \left( \frac{i}{-k''^2 - 4} \right) e^{-ik'' x'} \left( e^{ik'' x} + K_0(k'') e^{-ik''(x+x')} \right).$$ \quad (7.4)$$

Let us rewrite the contribution (7.1) in the expanded form in order to find how we may carry out the required integrations ( first setting $k \rightarrow -k$ in the first term of the first propagator)

$$\int dt' G(x, t; x', t') = \int \int \int \int dt \frac{d\omega}{2\pi} \frac{dk}{2\pi} \frac{e^{-i\omega(t_1-t)}}{\omega^2 - k^2 - 4 + i\rho} (e^{-ik(1-x)})$$

$$\times \int \frac{dk''}{2\pi} \left( \frac{i}{-k''^2 - 4} \right) e^{-ik'' x'} \left( e^{ik'' x} + K_0(k'') e^{-ik''(x+x')} \right) e^{2x} e^{2x'}$$

$$\times \frac{1}{2} \int \frac{dk_1}{2\pi} \frac{1}{\sqrt{k_1^2 + 4}} \frac{i k_1 + 2 \sigma_1}{ik_1 - 2 \sigma_1} e^{-2ik_1 x'}$$

$$\times \int \int \frac{d\omega'}{2\pi} \frac{dk'}{2\pi} \frac{e^{-i\omega(t-t_2)}}{\omega^2 - k'^2 - 4 + i\rho} (e^{ik'(x-x_2)} + K_0(k') e^{-ik'(x+x_2)}).$$ \quad (7.5)$$

The starting point is to perform the $t$ integration which immediately removes the energy variable $\omega'$. Moreover, by multiplying the first and the third propagator then we obtain four pole pieces. If we perform the calculations corresponding to one of them ( for example the first one which involves $e^{i(k+k')x}$ term), then the calculations for the
remaining three pole pieces may be done in the same manner, except that \( k + k' \) is replaced by one of \( k - k' \), \(-k + k' \) and \(-k - k' \). So, in what follows we continue the computations in detail only for one pole piece. Now, the integration over \( k' \) replaced by one of \( k'' \). Similarly, the \( x \) plane in order to become free of all extra poles except \( \pm k'' \). The above integral may be solved by closing the contours in the upper or lower half-plane in order to become free of all extra poles except \( \pm 2i \). Therefore, after some manipulation we obtain

\[
\int \frac{dk''}{2\pi} \frac{i}{k''^2 + 4 - k'' - 2k_1 - 2i} \left( \frac{1}{k + k' + k'' - 2i} + \frac{1}{k + k' - k'' - 2i} K_0(k'') \right). 
\]

(7.9)

The above integral may be solved by closing the contours in the upper or lower half-plane in order to become free of all extra poles except \( \pm 2i \). Therefore, after some manipulation we obtain

\[
\int \frac{dk''}{2\pi} \frac{i}{k''^2 + 4 - k'' - 2k_1 - 2i} \left( \frac{1}{k + k' + k'' - 2i} + \frac{1}{k + k' - k'' - 2i} K_0(k'') \right) = \frac{1}{4i} \left\{ \frac{1 - \sigma_1}{2} \frac{1}{k + k' + k'' - 2i} + \frac{1}{2k_1 + 4i} \frac{1}{k + k' - k'' - 2i} 
\]

\[
- \frac{1}{k + k' + k'' - 2i} \frac{1}{k + k' - 2k_1 - 4i} \right\}. 
\]

(7.10)

Next, we need to calculate the integration over \( k_1 \). In order to calculate the loop momentum \( k_1 \) integral, we will have different kinds of integrals however, all of them can be solved similarly. For example, consider the integral

\[
\int \frac{dk_1}{\sqrt{k_1^2 + 4k + k' - 2k_1 - 4i}}. 
\]

(7.11)
Let us close the contour in the upper half-plane and due to the branch cut which extends from $2i$ to infinity along the imaginary axis, the contour has to turn around the cut line. Therefore, the integral (7.11) reduces to

$$2 \int_{2}^{\infty} \frac{dy}{\sqrt{y^2 - 4} k + k' - 2iy - 4i}$$

and the above integral gives the following result

$$\int \frac{dk_1}{\sqrt{k_1^2 + 4} k + k' - 2k_1 - 4i} \left\{ \frac{1}{4} \ln \left\{ \frac{1 + i \frac{k + k' - 4i}{4} + i \frac{(k + k' - 4i)^2 + 4}{4}}{1 + i \frac{k + k' - 4i}{4} - \frac{i (k + k' - 4i)^2 + 4}{4}} \right\} \right\}$$

Now we can write down the solution of the $k_1$ integration as

$$\frac{1}{8i} \int \frac{dk_1}{2\pi} \frac{1}{\sqrt{k_1^2 + 4}} \left( \frac{ik_1 + 2\sigma_1}{ik_1 - 2\sigma_1} \right) \left\{ \frac{1 - \sigma_1}{1 + \sigma_1} - \frac{1}{2k_1 + 4i k + k' - 2k_1 - 4i} - \frac{1}{k + k' - 4i k + k' - 2k_1 - 4i} \right\}$$

$$= -\frac{16\pi (1 - \cos a_1 \pi) (k + k')(k + k' - 4i)}{2k + 2k' + 4i \cos a_1 \pi - 12i} + \frac{a_1 \cos a_1 \pi}{16 \sin^3 a_1 \pi} \frac{1}{k + k' - 4i \cos a_1 \pi - 4i} \sqrt{\frac{(k + k' - 4i)^2 + 4}{4}}$$

$$\times \ln \left\{ \frac{1 + i \frac{k + k' - 4i}{4} + i \frac{(k + k' - 4i)^2 + 4}{4}}{1 + i \frac{k + k' - 4i}{4} - \frac{i (k + k' - 4i)^2 + 4}{4}} \right\}$$

Finally, regarding the type III (bulk-bulk) contribution (7.5), all that remains is to integrate over the momenta $k, k'$. This can be achieved by means of the contours in the upper half-plane and considering the pole at $\hat{k} = k = k' = \sqrt{\omega^2 - 4}$ and ignoring all the other extra poles because their contributions will vanish when $x_1, x_2 \to -\infty$. Hence, the relation (7.5) has the solution

$$-\frac{\beta^2}{(1 + \cos a_1 \pi)^2} e^2 \int \frac{d\omega}{2\pi} e^{-i\omega(t_1 - t_2)} e^{-ik(x_1 + x_2)} \frac{1}{(2\hat{k})^2}$$

$$\left\{ -\frac{\hat{k} - i - i \cos a_1 \pi}{\pi(1 - \cos a_1 \pi)\hat{k}(2\hat{k} - 4i)} + \frac{a_1 \cos a_1 \pi}{\sin^3 a_1 \pi} \frac{\hat{k} + i \cos a_1 \pi - 3i}{(\hat{k} - 2i)(\hat{k} + 2i \cos a_1 \pi - 2i)} \right\}$$
paper. So, the type III (bulk-bulk) contribution takes the simple Feynman diagram. This is one of the interesting results which may be found in this paper. Fortunately, these terms will be cancelled by matching terms in the contribution corresponding to the type I (bulk) Feynman diagram. Notice that the above relation depends on the \( \ln \) terms in a manner which is very inconvenient for a comparison with Ghoshal’s formula. Fortunately, these terms will be cancelled by matching terms in the contribution corresponding to the type I (bulk) Feynman diagram. This is one of the interesting results which may be found in this paper. So, the type III (bulk-bulk) contribution takes the simple form

\[
-\frac{1}{\pi k(k-2i)} \frac{\hat{k} - 2i \cos a_1 \pi - 2i}{\hat{k} + 2i \cos a_1 \pi - 2i} \frac{1}{\sqrt{(k-2i)^2 + 4}}
\times \ln \left\{ \frac{1 + \frac{i}{2}(\hat{k} - 2i) + \frac{i}{2} \sqrt{(k-2i)^2 + 4}} {1 + \frac{i}{2}(\hat{k} - 2i) - \frac{i}{2} \sqrt{(k-2i)^2 + 4}} \right\}
\]

\[+ K_0(\hat{k}) \left( -\frac{i}{2\pi(1 - \cos a_1 \pi)^2} - \frac{ia_1 \cos a_1 \pi (\cos a_1 \pi - 3)}{2 \sin^3 a_1 \pi (1 - \cos a_1 \pi)} + \frac{i(1 + \cos a_1 \pi)}{12\pi(1 - \cos a_1 \pi)} \right) \]

\[+ K_0^2(\hat{k}) \left( \frac{\hat{k} + i + i \cos a_1 \pi}{(1 - \cos a_1 \pi)k(2k + 4i)} \frac{\hat{k} - i \cos a_1 \pi + 3i}{\sin^3 a_1 \pi} \frac{1}{\sqrt{(k+2i)^2 + 4}} \right) \times \ln \left\{ \frac{1 - \frac{i}{2}(\hat{k} + 2i) + \frac{i}{2} \sqrt{(k+2i)^2 + 4}} {1 - \frac{i}{2}(\hat{k} + 2i) - \frac{i}{2} \sqrt{(k+2i)^2 + 4}} \right\} \]. (7.15)

\[\frac{\beta^2}{(1 + \cos a_1 \pi)^2} \epsilon^2 \int \frac{d\omega}{2\pi} e^{-i\omega(t_1 - t_2)} e^{-ik(x_1 + x_2)} \frac{1}{(2k)^2} \]

\[\left\{ \begin{array}{c}
-\frac{\hat{k} - i - i \cos a_1 \pi}{\pi(1 - \cos a_1 \pi)k(2k - 4i)} + \frac{a_1 \cos a_1 \pi}{\sin^3 a_1 \pi} \frac{\hat{k} + i \cos a_1 \pi - 3i}{(k - 2i)(k + 2i \cos a_1 \pi - 2i)} \\
+ K_0(\hat{k}) \left( -\frac{i}{2\pi(1 - \cos a_1 \pi)^2} - \frac{ia_1 \cos a_1 \pi (\cos a_1 \pi - 3)}{2 \sin^3 a_1 \pi (1 - \cos a_1 \pi)} + \frac{i(1 + \cos a_1 \pi)}{12\pi(1 - \cos a_1 \pi)} \right) \\
+ K_0^2(\hat{k}) \left( \frac{\hat{k} + i + i \cos a_1 \pi}{(1 - \cos a_1 \pi)k(2k + 4i)} \frac{\hat{k} - i \cos a_1 \pi + 3i}{\sin^3 a_1 \pi} \frac{1}{\sqrt{(k+2i)^2 + 4}} \right) \times \ln \left\{ \frac{1 - \frac{i}{2}(\hat{k} + 2i) + \frac{i}{2} \sqrt{(k+2i)^2 + 4}} {1 - \frac{i}{2}(\hat{k} + 2i) - \frac{i}{2} \sqrt{(k+2i)^2 + 4}} \right\} \right\} \]. (7.16)

### 8 Type I (boundary)

In fact in the case of the type I Feynman diagram, there are two possible configurations depending on the fact that whether the interaction vertex is located at the boundary or inside the bulk region. From now on we call these two configurations the type I (boundary) and the type I (bulk) respectively. Considering the boundary four point coupling (3.17) and the associated combinatorial factor as well, the contribution
corresponding to the type I (boundary) diagram is described by

\[-i\beta^2 \left(2\sigma_1 + \epsilon - \frac{\sigma_1 + 2}{4(1 + \sigma_1)^2} \epsilon^2 \right) \int_{-\infty}^{\infty} dt'' G(x, t; 0, t'') G(0, t''); 0, t'') G(0, t''; x', t'). \tag{8.1}\]

Let us find the appropriate form of the two-point Green function which will be used many times throughout this and next sections. Now by looking at the general form of the propagator (3.11) and considering the expansions of the classical reflection factor (3.20) and the function \(f(k, x)\) i.e. (3.18) as well, the required form of the two-point Green function will be

\[
G(x, t; x', t') = i \int \frac{d\omega}{2\pi} \frac{dk}{2\pi} \frac{e^{-i\omega(t-t')}}{\omega^2 - k^2 - 4 + i\rho} \left\{ \left(1 + \frac{\epsilon^2}{4(1 + \sigma_1)^2} \frac{1}{(ik + 2)} e^{4\sigma_1} \right) \right.
\]

\[
- \frac{\epsilon^2}{4(1 + \sigma_1)^2} \frac{1}{(ik - 2)} e^{4\sigma_1} e^{ik(x-x')}
\]

\[
+ \left( K_0 - \frac{\epsilon^2}{4(1 + \sigma_1)^2} \frac{1}{(ik - 2)} e^{4\sigma_1} K_0 - \frac{\epsilon^2}{4(1 + \sigma_1)^2} \frac{1}{(ik - 2)} e^{4\sigma_1} K_0 \right.
\]

\[
+ \epsilon K_1 + \epsilon^2 K_2 \right\}. \tag{8.2}\]

So, in our problem the loop propagator takes the following form

\[
G(0, t''; 0, t'') = \int \frac{d\omega''}{2\pi} \frac{dk''}{2\pi} \frac{e^{-i\omega''(t-t'')}}{\omega'' - k''^2 - 4 + i\rho} \left(1 + K_0 + \epsilon K_1 + \epsilon^2 K_2 \right)
\]

\[
+ \frac{\epsilon^2}{4(1 + \sigma_1)^2} \frac{1}{(ik'' + 2)} - \frac{\epsilon^2}{4(1 + \sigma_1)^2} \frac{1}{(ik'' - 2)} - \frac{\epsilon^2}{2(1 + \sigma_1)^2} \frac{1}{(ik'' - 2)} K_0. \tag{8.3}\]

First of all it is necessary to perform a minimal subtraction in order to remove the divergence of the above integral. Secondly, we have solved the integral in Ref. [20] up to the first order in \(\epsilon\). Hence, what we need to do is to solve the part which is proportional to \(\epsilon^2\) including four terms. Indeed, three of them can be simply manipulated and because of this reason we solve the first one in detail which is given by

\[
i \int \frac{d\omega''}{2\pi} \frac{dk''}{2\pi} \frac{1}{\omega'' - k''^2 - 4 + i\rho} \epsilon^2 K_2(k'') \tag{8.4}\]

or

\[
\frac{1}{2} \int \frac{dk''}{2\pi} \frac{1}{\sqrt{k''^2 + 4}} \epsilon^2 \left( \frac{i}{4(\sigma_1^2 - 2\sigma_1 + 1)(k'' + 2i)} + \frac{i}{4(1 + \sigma_1)^2(k'' - 2i)} \right)
\]

\[
+ \frac{i\sigma_1(\sigma_1^2 - 2\sigma_1 - 1)}{(1 + \sigma_1)(\sigma_1^2 - \sigma_1 + 1)} \frac{1}{(k'' + 2i\sigma_1)} - \frac{\sigma_1^2 - 2}{(1 + \sigma_1)(\sigma_1 - 1)} \frac{1}{(k'' + 2i\sigma_1)^2} \right) + \frac{4i\sigma_1}{(k'' + 2i\sigma_1)^3} \right). \tag{8.5}\]
Fortunately, we have already calculated these kinds of integrals in Ref. [20] except the last one which is (apart from the coefficients)

$$\int \frac{dk''}{\sqrt{k''^2 + 4 (ik'' - 2\sigma_1)^3}}. \quad (8.6)$$

The above integral can be converted to a complex one and regarding the branch cut, it reduces to

$$-2 \int_2^{\infty} \frac{dy}{\sqrt{y^2 - 4 (y + 2\sigma_1)^3}} \quad (8.7)$$

and after some manipulation we obtain

$$\int \frac{dk''}{\sqrt{k''^2 + 4 (ik'' - 2\sigma_1)^3}} = \frac{3 \cos a_1\pi}{8 \sin^4 a_1\pi} - \frac{1 + 2 \cos^2 a_1\pi}{8 \sin^5 a_1\pi} a_1\pi. \quad (8.8)$$

Now using the above formula and the required formulae in Ref. [20], we may find the following result

$$i \int \int \frac{d\omega'' \, dk''}{2\pi \, 2\pi} \frac{1}{\omega'' - k''^2 - 4} \epsilon^2 K_2(k'') \quad = \frac{\epsilon^2}{4\pi} \left( \frac{1 + \cos a_1\pi + \cos^2 a_1\pi}{\sin^3 a_1\pi} - \frac{\cos a_1\pi (2 + \cos a_1\pi)}{\sin^5 a_1\pi} a_1\pi \right). \quad (8.9)$$

Finally after doing the necessary computations and simplifications in connection with (8.3), we obtain the loop propagator as

$$G(0, t''; 0, t''') = -a_1 \frac{\cos a_1\pi}{\sin a_1\pi} + \epsilon \left( \frac{\cos a_1\pi}{2\pi \sin^2 a_1\pi} - \frac{a_1}{2 \sin^3 a_1\pi} \right) + \epsilon^2 \left( \frac{2 + \cos^2 a_1\pi}{4\pi \sin^4 a_1\pi} - \frac{3 \cos a_1\pi}{4\pi \sin^5 a_1\pi} a_1\pi \right). \quad (8.10)$$

Let us consider the boundary contribution (8.1). The integration over $t''$ creates a Dirac delta function which means we must set $\omega' = \omega$. As before in previous sections, the remaining integral over the momenta $k$ and $k'$ can be achieved by closing the contours in the upper half-plane and taking into account the pole at $k = k' = \hat{k} = \sqrt{\omega^2 - 4}$. Note, all the other poles will give damped contributions in the asymptotic region $x, x' \to -\infty$. So, the type I (boundary) contribution is

$$-\frac{i\beta^2}{4} \left( 2\sigma_1 + \epsilon - \frac{\sigma_1 + 2}{4(1 + \sigma_1)^2} \epsilon^2 \right) \times \left\{ -\frac{a_1 \cos a_1\pi}{\sin a_1\pi} + \epsilon \left( \frac{\cos a_1\pi}{2\pi \sin^2 a_1\pi} - \frac{a_1}{2 \sin^3 a_1\pi} \right) + \epsilon^2 \left( \frac{2 + \cos^2 a_1\pi}{4\pi \sin^4 a_1\pi} - \frac{3 \cos a_1\pi}{4\pi \sin^5 a_1\pi} a_1\pi \right) \right\} \times \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} e^{-ik(x+x')} \frac{1}{(2\hat{k})^2}.$$

16
\[ \times \left\{ \frac{2i\hat{k}}{ik - 2\sigma_1} + \frac{2i\hat{k}}{(ik - 2\sigma_1)^2} \epsilon \right. \\
\left. + \frac{i\hat{k}(-i\hat{k}^3 + 2\hat{k}^2 - 4i\hat{k}\sigma_1 + 6\hat{k}^2\sigma_1 + 16 + 16\sigma_1)}{2(1 + \sigma_1)(i\hat{k} - 2)(i\hat{k} + 2)(-i\hat{k} + 2\sigma_1)^3} \epsilon^2 \right\}. \quad (8.11) \]

\section{Type I (bulk)}

Now let us evaluate the contribution of the type I (bulk) Feynman diagram to the classical reflection factor when the interaction vertex is placed inside the bulk region. It is clear that in this case we have to consider the bulk four point coupling (3.15) in the interaction vertex however, the related combinatorial factor is the same as the boundary case. The corresponding contribution may be written as

\[ -4i\beta^2 \int_{-\infty}^{\infty} dt'' \int_{-\infty}^{0} dx'' G(x, t; x'', t'')G(x'', t''; x'', t'') \]

\[ G(x'', t''; x', t') \left( 1 + \frac{\epsilon^2}{2(1 + \sigma_1)^2} e^{4x''} \right). \quad (9.1) \]

As it was shown in the previous section, the loop propagator is given by

\[ G(x'', t''; x'', t'') = i \int \int \frac{d\omega'' dk''}{2\pi 2\pi} \frac{1}{\omega''^2 - k''^2 - 4 + i\rho} \left\{ 1 + \frac{\epsilon^2}{4(1 + \sigma_1)^2} \frac{1}{(ik'' + 2)} e^{4x''} \right. \]

\[ -\frac{\epsilon^2}{4(1 + \sigma_1)^2} \frac{1}{(ik'' - 2)} e^{4x''} \]

\[ + \left( K_0 - \frac{\epsilon^2}{2(1 + \sigma_1)^2} \frac{1}{(ik'' - 2)} e^{4x''} K_0 + \epsilon K_1 + \epsilon^2 K_2 \right) e^{-2ik''x''}. \quad (9.2) \]

The above integral is logarithmically divergent. Nevertheless, the divergence of the loop integral can be removed by performing a minimal subtraction. Moreover, doing the integration over \( \omega'' \) then, we obtain

\[ G(x'', t''; x'', t'') = \frac{\epsilon^2}{8\pi(1 + \sigma_1)^2} e^{4x''} \]

\[ + \frac{1}{2} \int \frac{dk''}{2\pi} \sqrt{k''^2 + 4} \left( K_0 - \frac{\epsilon^2}{2(1 + \sigma_1)^2} \frac{1}{(ik'' - 2)} e^{4x''} K_0 \right. \]

\[ + \epsilon K_1 + \epsilon^2 K_2 \left) e^{-2ik''x''}. \quad (9.3) \]

Now regarding the bulk contribution (9.1), the next step is to set \( k \rightarrow -k \) in the first term of the first propagator. Secondly, the integration over \( t'' \) leads to the fact that the energy variables of the first and the third propagators are equal. Hence, the type I (bulk) contribution takes the following form:

\[ -4i\beta^2 \int_{-\infty}^{0} dx'' \int \int \frac{d\omega dk}{2\pi 2\pi} \frac{ie^{-i(\omega - t')}}{\omega^2 - k^2 - 4 + i\rho} \left\{ 1 + \frac{\epsilon^2}{4(1 + \sigma_1)^2} \frac{1}{(ik + 2)} e^{4x} \right. \]
\[
\int \frac{dk''}{2\pi} \left\{ \frac{\epsilon^2}{8\pi(1+\sigma_1)^2} e^{4\varepsilon''} + \frac{1}{2} \int \frac{dk''}{2\pi} \frac{1}{\sqrt{k''^2 + 4}} \left( K_0 - \frac{\epsilon^2}{4(1+\sigma_1)^2} e^{4\varepsilon''} K_0(k) - \frac{\epsilon^2}{4(1+\sigma_1)^2} e^{4\varepsilon''} K_0 \right) + \epsilon K_1 + \epsilon^2 K_2 \right\} e^{-\frac{\epsilon^2}{(ik')^2} e^{4\varepsilon''}} K_0(k')
\]

The integration over \(x''\) may be performed by the formula (5.9) and therefore, (9.4) converts to

\[
-2\beta^2 \int \int \frac{d\omega}{2\pi} \frac{dk'}{2\pi} \frac{dk''}{2\pi} e^{-i\omega(t-t')} e^{-i(kx+k'x')} \frac{i}{\omega^2 - k^2 - 4 + i\rho} \frac{i}{\omega^2 - k'^2 - 4 + i\rho}
\]

\[
\int \frac{dk''}{2\pi} \left\{ \frac{\epsilon^2}{2(1+\sigma_1)^2} e^{4\varepsilon''} \right\}
\]

\[
\left( K_0 - \frac{\epsilon^2}{4(1+\sigma_1)^2} e^{4\varepsilon''} K_0(k') - \frac{\epsilon^2}{4(1+\sigma_1)^2} e^{4\varepsilon''} K_0 \right) + \epsilon K_1 + \epsilon^2 K_2 \right\} e^{-\frac{\epsilon^2}{(ik')^2} e^{4\varepsilon''}} K_0(k')
\]

\[
\times \left\{ 1 + \frac{\epsilon^2}{2(1+\sigma_1)^2} e^{4\varepsilon''} \right\}
\]

(9.4)
Next, we need to integrate over $k''$ however, the corresponding calculations are tedious. We will discover two important facts during the computations. Firstly, all the terms involving the following kind of integral

$$\int \frac{1}{k - k'' - 2k'''} K_1(k') - \frac{1}{k + k' + 2k'''} K_0(k) K_1(k')$$

$$+ \frac{1}{k' - k - 2k''} K_1(k) - \frac{1}{k + k' + 2k'''} K_0(k') K_1(k)$$

$$- \frac{1}{2(1 + \sigma)^2 i k'' - 2} \left( \frac{1}{k + k' - 2k''' - 4i} + \frac{1}{k - k' - 2k''' - 4i} K_0(k') \right)$$

$$+ \frac{1}{k' - k - 2k''} K_0(k) - \frac{1}{k + k' + 2k''} K_0(k) K_0(k')$$

$$+ \frac{1}{2(1 + \sigma)^2} K_2(k'') \left( \frac{1}{k + k' - 2k''' - 4i} + \frac{1}{k - k' - 2k''' - 4i} K_0(k') \right)$$

$$+ \frac{1}{k' - k - 2k''} K_0(k) - \frac{1}{k + k' + 2k''} K_0(k) K_0(k') \right) \}.$$  \quad (9.5)

Meanwhile, there is another term which is the remaining part of the contribution i.e.:

$$\frac{\beta^2 e^2}{2\pi (1 + \sigma)^2} \int \int \frac{d\omega \, dk \, dk'}{2\pi^2 2\pi} e^{-i(\omega(t-t')} e^{-i(kx+k'x')}} \frac{1}{\omega^2 - k^2} \frac{1}{(1 + \sigma)^2} \right)$$

$$\left( \frac{1}{k + k' - 4i} + \frac{1}{k - k' - 4i} K_0(k') \right) + \frac{1}{k' - k - 4i} K_0(k) + \frac{1}{-k - k' - 4i} K_0(k) K_0(k') \right) \}.$$  \quad (9.6)

Next, we need to integrate over $k''$ however, the corresponding calculations are tedious. We will discover two important facts during the computations. Firstly, all the terms involving the following kind of integral

$$\int \frac{1}{\sqrt{k''^2 + 4k + k' - 2k''}}$$

which is proportional to $\theta$ (after doing the integrations over $k$ and $k'$ and using the fact that $k = k' = 2 \sinh \theta$) will be cancelled. This is consistent for a comparison with Ghoshal's formula. Secondly, there are many other terms which contain this type of integral

$$\int \frac{1}{\sqrt{k''^2 + 4k + k' - 2k'' - 4i}}$$

and all of them (after assembling) will be canceled with the counterpart terms in the type III (bulk-bulk) Feynman diagram as we mentioned in section seven. This result is also compatible with Ghoshal's formula. Meanwhile, further simplification can be made by using the values of $k$ and $k'$ given by their poles and simplifying the integrand.
So the type I (bulk) Feynman diagram has the following contribution:

\[
-i\beta^2e^2 \frac{1}{\pi} \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} e^{-ik(x+x')} \frac{1}{(2\hat{k})^2} \left\{ \begin{array}{l}
\frac{\cos a_1\pi}{(1 + \cos a_1\pi)^2 \sin a_1\pi} \left( \frac{1}{ik} - \frac{1}{i\hat{k}} - \frac{2}{i\hat{k} - 2 \cos a_1\pi} + \frac{2}{i\hat{k} + 2 \cos a_1\pi - 2} \right) \\
- \frac{2a_1\pi \cos a_1\pi}{\sin a_1\pi} \frac{1}{i\hat{k} - 2 \cos a_1\pi + 2} - \frac{2a_1\pi \cos a_1\pi}{\sin a_1\pi} \frac{1}{i\hat{k} + 2 \cos a_1\pi - 2} \\
+ \left( \frac{1}{4} + \frac{a_1\pi}{\sin a_1\pi} \frac{2i\hat{k}}{i\hat{k} - 2 \cos a_1\pi} + \frac{\pi}{\sqrt{k^2 + 4}} \right) K_2(\hat{k}) \\
- \left( \frac{1}{2 \sin^2 a_1\pi} - \frac{a_1\pi \cos a_1\pi}{2 \sin^3 a_1\pi} \right) \frac{2i\hat{k}}{i\hat{k} - 2 \cos a_1\pi} K_1(\hat{k}) + \frac{a_1\pi}{\sin a_1\pi} \frac{i\hat{k}}{i\hat{k} - 2 \cos a_1\pi + 2} K_0(\hat{k}) + \frac{a_1\pi}{\sin a_1\pi} \frac{i\hat{k}}{i\hat{k} + 2 \cos a_1\pi - 2} K_0(\hat{k}) \\
+ \left( \frac{7}{12 \sin^2 a_1\pi} - \frac{\cos a_1\pi}{2 \sin^3 a_1\pi} \right) K_0(\hat{k}) - \frac{2 \cos a_1\pi}{\sin^2 a_1\pi} \frac{1}{(i\hat{k} - 2 \cos a_1\pi)^2} \\
+ \frac{a_1\pi}{2 \sin a_1\pi} \left( K_2(\hat{k}) + K_0(\hat{k}) K_2(-\hat{k}) \right) \\
+ \frac{1}{2 \sin^4 a_1\pi} \left( 2 \cos a_1\pi - a_1\pi \left( 1 + 3 \cos^2 a_1\pi \right) \right) \frac{\hat{k}^2}{(i\hat{k} - 2 \cos a_1\pi)^2} \right\}.
\]

(9.7)

10 Ghoshal’s formula up to the second order

Ghoshal’s formula (2.8) for the sinh-Gordon quantum reflection factor up to one loop order is given by:

\[
K_q(\theta) \sim K_c(\theta) \left( 1 - i\beta^2 \frac{1}{8} \sin \theta \mathcal{F}(\theta) \right),
\]

where

\[
\mathcal{F}(\theta) = \frac{1}{\cosh \theta + 1} - \frac{1}{\cosh \theta} + \frac{e_1}{\cosh \theta + \sin(e_0\pi/2)} - \frac{e_1}{\cosh \theta - \sin(e_0\pi/2)} + \frac{f_1}{\cosh \theta + \sin(f_0\pi/2)} - \frac{f_1}{\cosh \theta - \sin(f_0\pi/2)}.
\]

(10.2)

In calculating (10.2) we have made use of the expansions of \(E\) and \(F\) to \(O(\beta^2)\):

\[
E \sim e_0 + e_1 \frac{\beta^2}{4\pi} \quad F \sim f_0 + f_1 \frac{\beta^2}{4\pi}.
\]

(10.3)
where
\[ e_0 = a_0 + a_1 \quad \text{and} \quad f_0 = a_0 - a_1. \] (10.4)

Since \( K_q = K_c + \delta K_c \), we find that
\[ \frac{\delta K_c}{K_c} = -\frac{i\beta^2}{8} \sinh \theta \mathcal{F}(\theta). \] (10.5)

Hence, expanding to \( O(\epsilon^2) \), we obtain
\[
\mathcal{F}(\theta) = \left\{ \frac{1}{\cosh \theta + 1} - \frac{1}{\cosh \theta} + \frac{e_1}{\cosh \theta + \sin a_1 \pi} - \frac{e_1}{\cosh \theta - \sin a_1 \pi} \right. \\
+ \frac{e_1 \epsilon \cos a_1 \pi}{2 \sin a_1 \pi} \left( \frac{1}{(\cosh \theta + \sin a_1 \pi)^2} + \frac{1}{(\cosh \theta - \sin a_1 \pi)^2} \right) \\
+ \frac{e_1 \epsilon^2}{8 \sin a_1 \pi} \left( \frac{1}{(\cosh \theta + \sin a_1 \pi)^2} + \frac{1}{(\cosh \theta - \sin a_1 \pi)^2} \right) \\
+ \frac{e_2 \epsilon^2 \cos^2 a_1 \pi}{4 \sin^2 a_1 \pi} \left( \frac{1}{(\cosh \theta + \sin a_1 \pi)^3} - \frac{1}{(\cosh \theta - \sin a_1 \pi)^3} \right) \\
\left. + \frac{\epsilon f_1}{\sin a_1 \pi \cosh^2 \theta} \right\}, \] (10.6)

where
\[ e_1 = -2a_1 + \frac{\epsilon}{\pi \sin a_1 \pi} + O(\epsilon^2) \] (10.7)

and \( f_1 \) is proportional to \( \epsilon \).

On the other hand, the relative correction to the classical reflection factor \( K_c \) is given by (using (3.20))
\[
\frac{\delta K_c}{K_c} = K_0^{-1} \delta K_0 + \epsilon \left( K_0^{-1} \delta K_1 - K_0^{-2} K_1 \delta K_0 \right) \\
+ \epsilon^2 \left( K_0^{-1} \delta K_2 - K_0^{-2} K_1 \delta K_1 - K_0^{-2} K_2 \delta K_0 + K_0^{-3} K_1^2 \delta K_0 \right). \] (10.8)

In fact, the corrections to \( K_0(\delta K_0) \) and \( K_1(\delta K_1) \) have been calculated in Ref. [16] and Ref. [20], respectively. Clearly, all the contributions that have been obtained in this paper are involved in the correction to \( K_2 \) i.e. \( \delta K_2 \). It is seen that the functional form of the second order quantum corrections to the classical reflection factor corresponding to the type I and type III Feynman diagrams are consistent with Ghoshal’s formula.

11 Discussion

In this paper we studied the sinh-Gordon model restricted to a half-line by boundary conditions which are compatible with integrability. We calculated the second order quantum corrections to the classical reflection factor of the model. The principal purpose of the perturbative calculation is to test the quantum reflection factor of the sinh-Gordon particle given by Ghoshal up to the second order in the difference of the
two boundary parameters. In fact, Ghoshal found the general form of the quantum reflection factor of the sinh-Gordon model. However, apart from two special cases (Neumann and Dirichlet boundary conditions) Ghoshal’s formula could not provide a complete relationship between the reflection factor and the boundary data. As we mentioned before, on the basis of a perturbative calculation, the relationship between the various parameters was conjectured to be

\[ E = (a_0 + a_1)(1 - B/2), \quad F = (a_0 - a_1)(1 - B/2). \] (11.1)

Corrigan et.al \[19\] tried to give further support for these formulae. They used two different technique i.e. the boundary bootstrap method and the WKB semi-classical approximation approach. Their non-perturbative calculations have been performed for the complicated case of the general boundary conditions in which the bulk symmetry is violated. Alternatively, the conjecture (11.1) may be checked on the basis of perturbative calculations up to the second order. It is understood that this paper provides most part of the collection of ingredients one need.

The computations associated with the type II Feynman diagram have not been performed in this article. In fact, in this diagram the middle momenta will be related to each other in an intricate manner due to the double Green functions. Nevertheless, it is expected that the same procedure will be applicable to the remaining diagram as well. Moreover, many formulae of this paper will be helpful.

All the affine Toda field theories whose corresponding Lie algebras are simply laced or non-simply laced algebras have known exact quantum S-matrices. However, in the presence of a boundary the boundary S-matrices or the reflection factors of the theories are largely unknown. In other words, the analogue of Ghoshal’s formula has not been formulated for most of the models. Besides, classifying the quantum integrability of the affine Toda field theories has not been completed.

12 Acknowledgment

The author wishes to thank E. Corrigan and P. Bowcock for discussions and fruitful comments.

References

[1] A.E. Arinshtein, V.A. Fateev and A.B. Zamolodchikov, ‘Quantum S-matrix of the 1+1 dimensional Toda chain’, Phys.Lett. B87 (1979) 389-392.

[2] A.V. Mikhailov, M.A. Olshanetsky and A.M. Perelomov, ‘Two-dimensional generalized Toda lattice’, Comm.Math.Phys. 79 (1981) 43

[3] E. Corrigan, ‘Recent developments in affine Toda quantum field theory’, CRM-CAP Summer School on Particles and Fields ’94, Banff, eds. G. Semenoff, L. Vinet (Springer New York) 1999, hep-th/9412213
[4] E. Corrigan, P.E. Dorey, R.H. Rietdijk and R. Sasaki, ‘Affine Toda field theory on a half-line’ Phys.Lett. B333 (1994) 83.

[5] E. Corrigan, P.E. Dorey, R.H. Rietdijk, ‘Aspects of affine Toda field theory on a half-line’, Prog.Theor.Phys.Suppl. 118:143-164, 1995, hep-th/9407148.

[6] P. Bowcock, E. Corrigan, P.E. Dorey and R.H. Rietdijk, ‘Classically integrable boundary conditions for affine Toda field theory’, Nucl.Phys B445 (1995) 469.

[7] A. Fujii and R. Sasaki, ‘Boundary effects in integrable field theory on a half-line’, Prog.Theor.Phys. 93:1123-1134, 1995, hep-th/9503083.

[8] A. Fring and R. Koberle, ‘Factorized scattering in the presence of reflecting boundaries’, Nucl.Phys. B421 (1994) 159.

[9] A. Fring and R. Koberle, ‘Affine Toda field theory in the presence of reflecting boundaries’, Nucl.Phys. B419 (1994) 647.

[10] G.M. Gandenberger, ‘On a_2^{(1)} reflection matrices and affine Toda theories’, Nucl. Phys. B542:659-693,1999, hep-th/9806003

[11] G.W. Delius and G.M. Gandenberger, ‘Particle reflection amplitudes in a_n^{(1)} Toda field theories’, Nucl. Phys. B554:325-364,1999, hep-th/9904002

[12] M. Perkins and P. Bowcock, ‘Quantum corrections to the classical reflection factor in a_2^{(1)} Toda field theory’, Nucl.Phys. B538 (1999) 612-630.

[13] A. Chenaghlou, ‘On the quantum reflection factor for the sinh-Gordon model with general boundary conditions’, Int.J.Mod.Phys. A15:4623-4654, 2000, hep-th/0004121.

[14] S. Ghoshal and A. Zamolodchikov, ‘Boundary S matrix and boundary state in two-dimensional integrable quantum field theory, Int.J.Mod.Phys. A9 (1994) 3841.

[15] S. Ghoshal, ‘Boundary state boundary S-matrix of the sine-Gordon model’, Int.J.Mod.Phys. A9 (1994) 4801.

[16] E. Corrigan, ‘On duality and reflection factors for the sinh-Gordon model with a boundary’, Int.J.Mod.Phys. A13:2709-2722, 1998, hep-th/9707235.

[17] E. Corrigan and G.W. Delius, ‘Boundary breathers in the sinh-Gordon model’, J.Phys. A32:8601-8614, 1999, hep-th/9909143.

[18] E. Corrigan, ‘Boundary bound states in integrable quantum field theory’, to be published in the proceedings of 4th Annual European TMR Conference on ”Integrability, Non-perturbative Effects and Symmetry in Quantum Field Theory”, Paris, France, 7-13 Sep 2000, hep-th/0010094.

[19] E. Corrigan and A. Taormina, ‘Reflection factors and a two-parameter family of boundary bound states in the sinh-Gordon model’, J.Phys. A33:8739-8754, 2000, hep-th/0008237.
[20] A. Chenaghlou and E. Corrigan, ‘First order quantum corrections to the classical reflection factor of the sinh-Gordon model’, *Int.J.Mod.Phys.* A15:4417-4432, 2000, [hep-th/0002065](https://arxiv.org/abs/hep-th/0002065).

[21] L.D. Faddeev and V.E. Korepin, ‘Quantum theory of solitons, *Phys.Rep.* 42 (1978) 1-87.

[22] A.B. Zamolodchikov and Al.B. Zamolodchikov, ‘Factorized S-matrices in two dimensions as the exact solutions of certain relativistic quantum field theory models’ *Ann.Phys.* 120 (1979) 253-291.

[23] J.D. Kim, ‘Boundary reflection matrix in perturbative quantum field theory’, *Phys.Lett.* B353 (1995) 213.

[24] J.D. Kim, ‘Boundary reflection matrix for A-D-E affine Toda field theory’, *J.Phys.* A29:2163-2174, 1996, [hep-th/9506031](https://arxiv.org/abs/hep-th/9506031).

[25] N. Topor, ‘Perturbation method for boundary S-matrix in 2D quantum field theory’, *Mod.Phys.Lett.* A12 (1997) 2951-2962.