Non-conservative Noether’s theorem for fractional action-like variational problems with intrinsic and observer times*

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ABSTRACT

We extend Noether’s symmetry theorem to fractional action-like variational problems with higher-order derivatives.

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1 Introduction

Symmetries play an important role both in physics and mathematics. They are described by transformations leaving structural relations unchangeable. Their importance range from fundamental and theoretical aspects to concrete applications, having profound implications in the dynamical behavior of the systems and in their basic qualitative properties. Knowledge of symmetries result on a deep insight about the inner structure of a system, and permits to apply the conservation laws to the investigation of the objects, i.e. to link the invariance principles with the conservation laws. This interrelation includes three classes of the most fundamental principles of physics: symmetry, conservation, and extremality.

When a closed system is characterized by a quantity which remains unchangeable in the course of time, no matter what kind of processes take place in the system, such quantity is said to be a conservation law. Some fundamental conservation laws include the conservation of energy, impulse, momentum impulse, motion of the centre of gravity, electrical charge, and others. All physical laws are described in terms of differential equations (equations of motion). The conservation laws represent the first integrals of the equations of motion and are important.

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for three reasons. Firstly, the task of solving the equations of motion explicitly is not always possible and knowledge of the first integrals may considerably simplify that task. Secondly, often there is no more necessity to solve the equation of motion, as the useful information is contained in the conservation laws. Thirdly, the conservation laws have a deep physical meaning and can be measured directly.

All basic differential equations of physics (i.e. the equations of motion of physical systems) have a variational structure. In other words, the equations of motion of a physical system are the Euler-Lagrange equations of a certain variational problem. It turns out that the conservation laws are the result of the invariance of the action with respect to a continuous group of transformations, given by some symmetry principle. The more general expression of the interrelation symmetry/variational structure/conservation, is given by Noether’s theorem. Noether’s theorem asserts that the conservation laws for a system of differential equations which correspond to the Euler-Lagrange equations of a certain variational problem, come from the invariance of the variational functional with respect to a one-parameter continuous group of transformations. The group of symmetry transformations requested by Noether’s theorem depend, of course, on the physical properties of the system. We refer the interested reader to [7].

Conservative physical systems imply frictionless motion and are a simplification of the real dynamical world. Almost all systems contain internal damping and are subject to external forces. For non-conservative dynamical systems, i.e. in the presence of non-conservative forces (forces that do not store energy and which are not equivalent to the gradient of a potential), the conservation laws are broken so that the standard Lagrangian or Hamiltonian formalism is no longer valid for describing the behavior of the system. Methodologically, Newtonian dissipative dynamical systems are a complement to conservative systems, because not only energy, but also other physical quantities as linear or angular momentums, are not conserved. In this case the classical Noether’s theorem ceases to be valid. However, it is still possible to obtain a Noether-type theorem which covers both conservative (closed system) and nonconservative cases [1, 5]. Roughly speaking, one can prove that Noether’s conservation laws are still valid if a new term, involving the nonconservative forces, is added to the standard constants of motion.

In order to better model non-conservative dynamical systems, a novel approach entitled Fractional Action-Like Variational Approach (FALVA) has been recently introduced [2, 3]. This approach is based on the concept of fractional integration. Fractional theory plays an important role in the understanding of both conservative and non-conservative behaviors of complex dynamical systems, and has important physical applications in various fields of science, e.g. physics, material sciences, chemistry, biology, scaling phenomena, etc. In [2, 3] Riemann-Liouville fractional integral functionals, depending on a parameter $\alpha$ but not on fractional-order derivatives of order $\alpha$, are introduced and respective fractional Euler-Lagrange type equations obtained. In [8], Jumarie uses the variational calculus of fractional order to derive an Hamilton-Jacobi equation and a Lagrangian variational approach to the optimal control of one-dimensional fractional dynamics with fractional cost functional. In this paper we extend the results of [4] to more general FALVA problems with higher-order derivatives.

2 Preliminaries

We begin by collecting the necessary results from [2, 6]. In 2005 El-Nabulsi (cf. [2]) introduced the FALVA problem as follows:
Problem 2.1. Find the stationary points of the integral functional

\[ I[q(\cdot)] = \frac{1}{\Gamma(\alpha)} \int_{a}^{t} L(\theta, q(\theta), \dot{q}(\theta)) (t - \theta)^{\alpha-1} d\theta \]  

(2.1)

under the initial condition \( q(\alpha) = q_a, \) where \( \dot{q} = \frac{dq}{d\theta}, \) is the Euler gamma function, \( 0 < \alpha \leq 1, \) \( \theta \) is the intrinsic time, \( t \) is the observer time, \( t \neq \theta, \) and the Lagrangian \( L : [a, b] \times \mathbb{R}^{n} \times \mathbb{R}^{n} \rightarrow \mathbb{R} \) is a \( C^2 \)-function with respect to all its arguments.

Along all the work we denote by \( \partial_{i} L, \) \( i = 1, 2, 3, \) the partial derivative of \( L(\cdot, \cdot, \cdot) \) with respect to its \( i \)th argument.

Theorem 2.2 summarizes one of the main results of [2].

Theorem 2.2 ([2]). If \( q(\cdot) \) is a solution to Problem 2.1 (i.e., \( q(\cdot) \) is a critical point of the functional (2.1)), then \( q(\cdot) \) satisfy the following Euler-Lagrange equation:

\[ \partial_{2} L(\theta, q(\theta), \dot{q}(\theta)) - \frac{d}{d\theta} \partial_{3} L(\theta, q(\theta), \dot{q}(\theta)) = \frac{1 - \alpha}{t - \theta} \partial_{3} L(\theta, q(\theta), \dot{q}(\theta)) . \]  

(2.2)

In [3] the authors introduced the following FALVA problem with higher-order derivatives:

Problem 2.3. Find the stationary points of the integral functional

\[ I^{m}[q(\cdot)] = \frac{1}{\Gamma(\alpha)} \int_{a}^{t} L(\theta, q(\theta), \dot{q}(\theta), \ldots, q^{(m)}(\theta)) (t - \theta)^{\alpha-1} d\theta , \]  

(2.3)

\( m \geq 1, \) under the initial conditions \( q^{(i)}(a) = q_{a}, \) \( i = 0, \ldots, m, \)

(2.4)

where \( q^{0}(\theta) = q(\theta), \) \( q^{(i)}(\theta) \) is the derivative of \( q(\theta) \) of order \( i, \) \( \Gamma \) is the Euler gamma function, \( 0 < \alpha \leq 1, \) \( \theta \) is the intrinsic time, \( t \) is the observer time, \( t \neq \theta, \) and the Lagrangian \( L : [a, b] \times \mathbb{R}^{n} \times (m+1) \rightarrow \mathbb{R} \) is a function of class \( C^{2m} \) with respect to all its arguments.

Remark 2.1. In the particular case where \( m = 1, \) Problem 2.3 reduces to Problem 2.1.

Theorem 2.4 generalizes Theorem 2.2 to the higher-order case.

Theorem 2.4 ([5]). If \( q(\cdot) \) is a stationary point of (2.3), then \( q(\cdot) \) satisfy the following higher-order Euler-Lagrange equation:

\[ \sum_{i=0}^{m} (-1)^i \frac{d^i}{d\theta^i} \partial_{i+2} L(\theta, q(\theta), \dot{q}(\theta), \ldots, q^{(m)}(\theta)) = F\left(\theta, q(\theta), \dot{q}(\theta), \ldots, q^{(2m-1)}(\theta)\right), \]  

(2.5)

\( m \geq 1, \) where

\[ F\left(\theta, q(\theta), \dot{q}(\theta), \ldots, q^{(2m-1)}(\theta)\right) = \frac{1 - \alpha}{t - \theta} \sum_{i=1}^{m} i(-1)^{i-1} \frac{d^{i-1}}{d\theta^{i-1}} \partial_{i+2} L(\theta, q(\theta), \dot{q}(\theta), \ldots, q^{(m)}(\theta)) \]

\[ + \sum_{k=2}^{m} \sum_{i=1}^{k} (-1)^{i-1} \frac{\Gamma(i - \alpha + 1)}{(t - \theta)^i \Gamma(1 - \alpha)} \left(\begin{array}{c} k \\ k - i \end{array}\right) \frac{d^{k-i}}{d\theta^{k-i}} \partial_{k+2} L(\theta, q(\theta), \dot{q}(\theta), \ldots, q^{(m)}(\theta)) . \]  

(2.6)
We borrow from [9] the notation

\[ \psi^j = \sum_{i=0}^{m-j} (-1)^i \frac{d^i}{d\theta^i} \partial_{i+j+2} L \left( \theta, q(\theta), \dot{q}(\theta), \ldots, q^{(m)}(\theta) \right), \quad j = 1, \ldots, m, \tag{2.7} \]

which is useful for our purposes because of the following property:

\[ \frac{d}{d\theta} \psi^j = \partial_{j+1} L \left( \theta, q(\theta), \dot{q}(\theta), \ldots, q^{(m)}(\theta) \right) - \psi^{j-1}, \quad j = 1, \ldots, m. \tag{2.8} \]

Remark 2.2. One can write equations (2.5) in the following form:

\[ \partial^2 L \left( \theta, q(\theta), \dot{q}(\theta), \ldots, q^{(m)}(\theta) \right) - \frac{d}{d\theta} \psi^1 = F \left( \theta, q(\theta), \dot{q}(\theta), \ldots, q^{(2m-1)}(\theta) \right). \tag{2.9} \]

Theorem 2.5 ([6]). If \( q(\cdot) \) is a solution of Problem 2.3, then it satisfy the following higher-order DuBois-Raymond condition:

\[ \frac{d}{d\theta} \left\{ \partial \left( \theta, q(\theta), \dot{q}(\theta), \ldots, q^{(m)}(\theta) \right) - \sum_{j=1}^{m} \psi^j \cdot q^{(j)}(\theta) \right\} = \partial_1 L \left( \theta, q(\theta), \dot{q}(\theta), \ldots, q^{(m)}(\theta) \right) + F \left( \theta, q(\theta), \dot{q}(\theta), \ldots, q^{(2m-1)}(\theta) \right) \cdot \dot{q}(\theta), \tag{2.10} \]

where \( F \) and \( \psi^j \) are defined as in (2.6) and (2.7) respectively.

3 Main Result

In this work we generalize the Noether-type theorem proved in [4] to the more general FALVA problem with higher-order derivatives.

3.1 Noether’s theorem for higher-order FALVA problems

In order to generalize the Noether’s theorem to Problem 2.3 (see Theorem 3.2 below) we use the DuBois-Reymond necessary stationary condition (2.10) and the following invariance definition.

Definition 3.1. (Invariance of (2.3)) The functional (2.3) is said to be invariant under the infinitesimal transformations

\[ \begin{aligned}
\dot{\theta} &= \theta + \varepsilon \tau(\theta, q) + o(\varepsilon) \\
\dot{q}(\theta) &= q(\theta) + \varepsilon \xi(\theta, q) + o(\varepsilon)
\end{aligned} \tag{3.1} \]

if

\[ \begin{aligned}
L \left( \dot{\theta}, \dot{q}(\theta), \dot{q}^2(\theta), \ldots, \dot{q}^{(m)}(\theta) \right) (t - \theta)^{1-a-1} \frac{d\theta}{d\theta} \\
&= L \left( \theta, q(\theta), \dot{q}(\theta), \ldots, q^{(m)}(\theta) \right) (t - \theta)^{1-a} \\
&+ \varepsilon (t - \theta)^{1-a-1} \frac{dA}{d\theta} \left( \theta, q(\theta), \dot{q}(\theta), \ldots, q^{(2m-1)}(\theta) \right) + o(\varepsilon). \tag{3.2} \end{aligned} \]
Remark 3.1. Expressions $\bar{q}^{(i)}$ in equation (3.2), $i = 1, \ldots, m$, are interpreted as

$$\bar{q}' = \frac{d\bar{q}}{d\theta}, \quad \bar{q}^{(i)} = \frac{d^i\bar{q}}{d\theta^i}, \quad (i = 2, \ldots, m).$$

(3.3)

Next theorem gives a necessary and sufficient condition for invariance of (2.3). Theorem 3.1 is useful to check invariance and also to compute the infinitesimal generators $\tau$ and $\xi$.

Theorem 3.1. (Necessary and sufficient condition for invariance of (2.3)) The integral functional (2.3) is invariant in the sense of Definition 3.1 if and only if

$$\partial_1 L \left( \theta, q(\theta), \dot{q}(\theta), \ldots, q^{(m)}(\theta) \right) \tau + \sum_{i=0}^{m} \partial_{i+2} L \left( \theta, q(\theta), \dot{q}(\theta), \ldots, q^{(m)}(\theta) \right) \cdot \rho^i + L \left( \theta, q(\theta), \dot{q}(\theta), \ldots, q^{(m)}(\theta) \right) \dot{\tau} = \hat{\Lambda} \left( \theta, q(\theta), \dot{q}(\theta), \ldots, q^{(2m-1)}(\theta) \right),$$

(3.4)

where

$$\begin{cases}
\rho^0 = \xi, \\
\rho^i = \frac{d}{d\theta} \left( \rho^{i-1} \right) - q^{(i)}(\theta) \dot{\tau}, \quad i = 1, \ldots, m. \quad (3.5)
\end{cases}$$

Remark 3.2. If $\alpha = 1$, condition (3.4) gives the higher-order necessary and sufficient condition of invariance proved in [9]:

$$\partial_1 L \left( \theta, q(\theta), \dot{q}(\theta), \ldots, q^{(m)}(\theta) \right) \tau + \sum_{i=0}^{m} \partial_{i+2} L \left( \theta, q(\theta), \dot{q}(\theta), \ldots, q^{(m)}(\theta) \right) \cdot \rho^i + L \left( \theta, q(\theta), \dot{q}(\theta), \ldots, q^{(m)}(\theta) \right) \dot{\tau} = \hat{\Lambda} \left( \theta, q(\theta), \dot{q}(\theta), \ldots, q^{(2m-1)}(\theta) \right).$$

Proof. (of Theorem 3.1) Differentiating equation (3.2) with respect to $\varepsilon$, then setting $\varepsilon = 0$, we obtain:

$$\partial_1 L \tau + \sum_{i=0}^{m} \partial_{i+2} L \cdot \frac{\partial}{\partial \varepsilon} \left( \frac{d\bar{q}}{d\theta} \right) \bigg|_{\varepsilon=0} + L \left( \dot{\tau} + \frac{1}{t - \theta} \tau \right) = \hat{\Lambda}.$$

The intended conclusion follows from (3.3):

$$\frac{\partial}{\partial \varepsilon} \left( \frac{d\bar{q}}{d\theta} \right) \bigg|_{\varepsilon=0} = \xi - \dot{\bar{q}} \dot{\tau},$$

$$\frac{\partial}{\partial \varepsilon} \left( \frac{d^i\bar{q}}{d\theta^i} \right) \bigg|_{\varepsilon=0} = \frac{d}{d\theta} \left[ \frac{\partial}{\partial \varepsilon} \left( \frac{d^{i-1}\bar{q}}{d\theta^{i-1}} \right) \bigg|_{\varepsilon=0} \right] - q^{(i)} \dot{\tau}, \quad i = 2, \ldots, m. \quad \square$$

Definition 3.2. (Higher-order conservation law) A quantity $C \left( \theta, q(\theta), \dot{q}(\theta), \ldots, q^{(2m-1)}(\theta) \right)$ is said to be a conservation law if

$$\frac{d}{d\theta} C \left( \theta, q(\theta), \dot{q}(\theta), \ldots, q^{(2m-1)}(\theta) \right) = 0$$

along all the solutions $q(\cdot)$ of the higher-order Euler-Lagrange equation (2.5).
Theorem 3.2. (Higher-order Noether’s theorem) If the integral functional (2.3) is invariant in the sense of Definition 3.1 and \( \tau(\theta, q) \) and \( \xi(\theta, q) \) satisfy the condition

\[
G \left( \theta, q(\theta), \dot{q}(\theta), \ldots, q(2^{m-1})(\theta) \right) \cdot \Omega = -L \left( \theta, q(\theta), \dot{q}(\theta), \ldots, q^{(m)}(\theta) \right) \tau,
\]

where

\[
G \left( \theta, q(\theta), \dot{q}(\theta), \ldots, q(2^{m-1})(\theta) \right) = \sum_{i=1}^{m} (-1)^{i-1} \frac{d^{i-1}}{d\theta^{i-1}} \partial_{i+2} L \left( \theta, q(\theta), \dot{q}(\theta), \ldots, q^{(m)}(\theta) \right)
\]

\[
+ \sum_{k=2}^{m} \sum_{i=2}^{k} (-1)^{i} \frac{\Gamma(i - \alpha + 1)}{\Gamma(2 - \alpha)(t - \theta)^{i-1}} \left( \frac{k}{k - i} \right) \frac{d^{k-i}}{d\theta^{k-i}} \partial_{k+2} L \left( \theta, q(\theta), \dot{q}(\theta), \ldots, q^{(m)}(\theta) \right)
\]

and \( \Omega = \xi - \dot{\xi} \tau \), then

\[
C \left( \theta, q(\theta), \dot{q}(\theta), \ldots, q(2^{m-1})(\theta) \right)
\]

\[
= \sum_{j=1}^{m} \psi^{j} \cdot \rho^{j-1} + \left( L \left( \theta, q(\theta), \dot{q}(\theta), \ldots, q^{(m)}(\theta) \right) - \sum_{j=1}^{m} \psi^{j} \cdot q^{(j)}(\theta) \right) \tau
\]

\[
- \Lambda \left( \theta, q(\theta), \dot{q}(\theta), \ldots, q(2^{m-1})(\theta) \right)
\]

is a conservation law.

Remark 3.3. Under hypothesis (3.6), the necessary and sufficient condition of invariance (3.4) takes the following form:

\[
\partial_{i} L \left( \theta, q(\theta), \dot{q}(\theta), \ldots, q^{(m)}(\theta) \right) \tau + \sum_{i=0}^{m} \partial_{i+2} L \left( \theta, q(\theta), \dot{q}(\theta), \ldots, q^{(m)}(\theta) \right) \cdot \rho^{i}
\]

\[
+ L \left( \theta, q(\theta), \dot{q}(\theta), \ldots, q^{(m)}(\theta) \right) \dot{\tau} - F \left( \theta, q(\theta), \dot{q}(\theta), \ldots, q(2^{m-1})(\theta) \right) \cdot \Omega
\]

\[
= \dot{\Lambda} \left( \theta, q(\theta), \dot{q}(\theta), \ldots, q(2^{m-1})(\theta) \right) .
\]

Proof. (of Theorem 3.2) We begin by writing the Noether’s conservation law (3.3) in the form

\[
C = \psi^{1} \cdot \rho^{0} + \sum_{j=2}^{m} \psi^{j} \cdot \rho^{j-1} + \left( L - \sum_{j=1}^{m} \psi^{j} \cdot q^{(j)}(\theta) \right) \tau - \Lambda.
\]

Differentiation of equation (3.10) with respect to \( \theta \) gives

\[
\dot{\Lambda} = \rho^{0} \cdot \frac{d}{d\theta} \psi^{1} + \psi^{1} \cdot \frac{d}{d\theta} \rho^{0} + \sum_{j=2}^{m} \left( \rho^{j-1} \cdot \frac{d}{d\theta} \psi^{j} + \psi^{j} \cdot \frac{d}{d\theta} (\rho^{j-1}) \right)
\]

\[
+ \tau \frac{d}{d\theta} \left( L - \sum_{j=1}^{m} \psi^{j} \cdot q^{(j)}(\theta) \right) + \left( L - \sum_{j=1}^{m} \psi^{j} \cdot q^{(j)}(\theta) \right) \frac{d}{d\theta} \tau.
\]
Using the Euler-Lagrange equation (2.5), the DuBois-Reymond condition (2.10), and relations (2.8) and (3.5) in (3.11), we obtain:

\[
\dot{\Lambda} = \left( \partial^2 L - \mathbf{F} \right) \cdot \mathbf{\xi} + \sum_{j=2}^{m} \left[ \left( \partial_{j+1} L - \psi^{j-1} \right) \cdot \rho^{j-1} + \psi^{j} \cdot \left( \rho^{j} + q^{(j)}(\theta) \right) \right]
\]

\[
+ \left( \partial_{1} L + \mathbf{F} \cdot \dot{q} \right) \tau + \left( L - \sum_{j=1}^{m} \psi^{j} \cdot q^{(j)}(\theta) \right) \dot{\tau}
\]

\[
= \partial_{1} L \tau + L \dot{\tau} + \partial_{2} L \cdot \mathbf{\xi} + \psi^{1} \cdot \left( \rho^{1} + q \dot{\tau} \right) - \psi^{1} \cdot \rho^{1}
\]

\[
- \psi^{1} \cdot \dot{q} \dot{\tau} + \psi^{m} \cdot \rho^{m} + \sum_{j=2}^{m} \partial_{j+1} L \cdot \rho^{j-1} .
\] (3.12)

Simplification of (3.12) lead us to the necessary and sufficient condition of invariance (3.9). \[\square\]

In the particular case \( m = 1 \) we obtain from our Theorem 3.2 the main result of [4].

**Corollary 3.3.** (cf. [3]) If the integral functional (2.1) is invariant under the infinitesimal transformations (3.1), and \( \tau(\theta, q) \) and \( \xi(\theta, q) \) satisfy the condition

\[
\partial_{3} L(\theta, q, \dot{q}) \cdot \Omega = -L(\theta, q, \dot{q}) \tau,
\] (3.13)

then

\[
C(\theta, q, \dot{q}) = \partial_{3} L(\theta, q, \dot{q}) \cdot \xi(\theta, q) + (L(\theta, q, \dot{q}) - \partial_{3} L(\theta, q, \dot{q}) \cdot \dot{q}) \tau(\theta, q) - \Lambda(\theta, q, \dot{q}) \] (3.14)

is a conservation law (i.e., (3.14) is constant along all the solutions \( q(\cdot) \) of the Euler-Lagrange equation (2.2)).

**Proof.** For \( m = 1 \) we obtain from (3.7) and (3.8) that

\[
G(\theta, q, \dot{q}) = \partial_{3} L(\theta, q, \dot{q})
\] (3.15)

and

\[
C(\theta, q, \dot{q}) = \psi^{1} \cdot \rho^{0} + \left( L(\theta, q, \dot{q}) - \psi^{1} \cdot \dot{q}(\theta) \right) \tau - \Lambda(\theta, q, \dot{q}) .
\] (3.16)

Having in mind the equations (2.7) and (3.5), we conclude that

\[
\begin{cases}
\psi^{1} = \partial_{3} L(\theta, q, \dot{q}) , \\
\rho^{0} = \xi .
\end{cases}
\] (3.17)

We obtain the intended result substituting (3.17) into (3.16). \[\square\]

### 3.2 Example

In order to illustrate our result, we consider an example for which the Lagrangian \( L \) do not depend explicitly on the intrinsic time \( \theta \).
Example 3.4. Let us consider the following second-order \((m = 2)\) FALVA problem: to find a stationary function \(q(\cdot)\) for the integral functional

\[
I_2[q(\cdot)] = \frac{1}{2} \int_0^t \left(aq^2 + bq^2 + \ddot{q}^2\right) (t - \theta)^{\alpha-1} d\theta, \tag{3.18}
\]

where \(a\) and \(b\) are arbitrary constants. In this case the Euler-Lagrange equation \((2.5)\) reads

\[
-aq + \frac{b(1 - \alpha)}{t - \theta} \dot{q} + \left(b - \frac{(1 - \alpha)(2 - \alpha)}{(t - \theta)^2}\right) \ddot{q} - \left(1 + \frac{2(1 - \alpha)}{(t - \theta)}\right) \dddot{q} = 0. \tag{3.19}
\]

Since the Lagrangian \(L\) do not depend explicitly on the independent variable \(\theta\), the necessary and sufficient invariance condition \((3.9)\) is satisfied with

\[
\tau = 1, \quad \xi = 0, \tag{3.20}
\]

\[
\dot{\Lambda} = F\dot{q} \Rightarrow \Lambda = \int F\dot{q} d\theta, \tag{3.22}
\]

where

\[
F = \frac{1 - \alpha}{t - \theta} (b\dot{q} - 2\ddot{q}) - \frac{(1 - \alpha)(2 - \alpha)}{(t - \theta)^2} \dddot{q}. \tag{3.23}
\]

The conservation law \((3.8)\) with \(m = 2\) takes the following form:

\[
C(\theta, q, \dot{q}, \ddot{q}, \dddot{q}) = L(\theta, q, \dot{q}, \ddot{q}) \tau + \left(\partial_3 L(\theta, q, \dot{q}, \ddot{q}) - \frac{d}{d\theta} \partial_4 L(\theta, q, \dot{q}, \ddot{q})\right) \cdot \Omega
+ \partial_4 L(\theta, q, \dot{q}, \ddot{q}) \cdot \dddot{\Omega} - \Lambda(\theta, q, \dot{q}, \ddot{q}, \dddot{q}). \tag{3.24}
\]

Substituting the quantities \(L = \frac{1}{2} \left(aq^2 + bq^2 + \ddot{q}^2\right)\), \((3.20)\), \((3.21)\) and \((3.22)\) into \((3.24)\), we conclude that

\[
\frac{1}{2} \left(aq^2 - bq^2 + 3\ddot{q}^2\right) - \dot{q} \ddot{q} - \int F\dot{q} d\theta \tag{3.25}
\]

is constant along any solution \(q\) of \((3.19)\).

If \(\alpha = 1\), then one see from \((3.23)\) that \(F = 0\), and \((3.25)\) gives the classical result in [1]:

\[
\frac{1}{2} \left(aq^2 - bq^2 + 3\ddot{q}^2\right) - \dot{q} \ddot{q}
\]

is a conservation law.

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