Quasi-local characteristics of dynamical extreme black holes

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Introducing the concept of the extreme trapping horizon, we discuss geometric features of dynamical extreme black holes in four dimensions and then derive the integral identities which hold for the dynamical extreme black holes. We address the causal/geometrical features too.

I. INTRODUCTION

Recently it has been shown that the extreme Reissner-Nordstrom black holes are unstable under linear perturbations [1, 2] (See also Refs. [3, 4] for the extreme Kerr spacetime). To show this, the conservation feature under perturbations played a central role, called Aretakis’ constant. Using coordinate transformations, one may identify the horizon with a set in an artificial null infinity and then the conservation law can be written as an asymptotic quantity conserved on the artificial null infinity [5]. But, the meaning of those quantities remains unclear. In Ref. [6], using non-linear numerical analysis of the back-reaction to the spacetimes with the spherical symmetry, dynamical extreme black holes under a specific initial condition has been created.

In this paper, we will discuss the features of the dynamical extreme black holes and we will define extreme trapping horizon in doing so. We will present a pair of integral formulas which are valid on 2-surfaces corresponding to horizons of the extreme black holes. We will derive them in two different forms. The tool we use is the second variation of the area of minimal surface, as in [8, 9] (See also Refs. [10, 11] where the same observation was used to discuss an inequality for charged black holes.) The integral identities presented in this article may give us extra information for understanding the origin of Aretakis’ constant, as both observations rely on the extremality of the black hole spacetimes. For the moment, however, we cannot find direct relations between them. We also discuss some general features of the extreme trapping horizon.

The article is organized as follows. In Sec.II, we derive a surface integral identity on momentarily static slices. Such surfaces correspond to the event horizon of the extreme Reissner-Nordstrom black hole, for example. In Sec.III, we give a definition of the extreme trapping horizon in a general set-up and another surface integral identity. We also address the properties of the extreme trapping horizon. Finally we will give the summary and discussion in Sec.IV.

II. DYNAMICAL EXTREME BLACK HOLES IN MOMENTARILY STATIC SLICES

Let us consider a 3-dimensional spacelike hypersurface Σ and a compact 2-dimensional submanifold S in Σ. The following Riemannian geometric identity then holds unconditionally,

\[ \mathcal{L}_r k = -\varphi^{-1}D^2\varphi + \frac{1}{2}R - \frac{1}{2}(3R + k^2 + k_{ab}k^{ab}), \]  

where \( r^a \) is the unit outward normal vector of \( S \), \( \varphi \) is the lapse function, \( k_{ab} \) is the extrinsic curvature of the surface \( S \) in \( \Sigma \) and \( \mathcal{L}_r \) is the Ricci scalar of \( S \). \( \mathcal{D}_a \) is the covariant derivative with respect to the induced metric of \( S \). Recall that this identity is closely related to the second variation formula of the area functional of \( S \subset \Sigma \).

In general the expansion rate of the null geodesic congruence is given by

\[ \theta = k + K_{ab}a^a a^b + K, \]  

where \( K_{ab} \) is the extrinsic curvature of \( \Sigma \) and \( K \) is the trace part of \( K_{ab} \). If one considers the momentarily static (or, equivalently, time symmetric) slice \( \Sigma \), that is, the extrinsic curvature of \( \Sigma \) vanishes \( (K_{ab} = 0) \), the vanishing of \( \theta \) corresponds to the vanishing of \( k \), so that \( S \) is a minimal surface in \( \Sigma \). The surface where the expansion vanishes is defined as the cross section of the trapping horizon, which is nearly the same as the apparent horizon [12]. The first variation of \( k \) does not vanish in general. In the dynamical extreme black holes, however, we consider the “horizon” \( S_H \subset \Sigma \) where the first variation of the expansion also vanishes as

\[ \mathcal{L}_r k = 0. \]

We may call this surface \( S_H \) the extreme minimal surface. This definition of the “horizon” is consistent with the one given by Israel [13]. Note that the trapped surfaces do not exist inside of the horizon \( S_H \). Then the surface integral on \( S_H \) of Eq. [10] implies

\[ \int_{S_H} (2) \mathcal{D} S = \int_{S_H} \left( (3R + k_{ab}k^{ab} + 2\varphi^{-2}(\mathcal{D} \varphi)^2 \right) dS. \]

On the momentarily static slices, the Hamiltonian constraint becomes

\[ (3R) = 2T_{ab}a^a a^b, \]
where $T_{ab}$ is the energy-momentum tensor and $n^a$ is the future directed unit normal vector of the current time slice $\Sigma$. If the energy condition is satisfied so that $T_{ab}n^an^b > 0$ (the inequality is strict because we consider the extreme Reissner-Nordstrom-type black hole and there is always a non-trivial contribution from the Maxwell field to the energy-momentum tensor), the right-hand side is positive. Then, the Gauss-Bonnet theorem tells us the left-hand side becomes $8\pi$. We then have

$$\int_{S_H} \left( 2T_{ab}n^an^b + k_{ab}k^{ab} + 2\varphi^{-2}(\mathcal{D}\varphi)^2 \right) dS = 8\pi. \quad (6)$$

This is an integral identity which holds on any momentarily static slices. Note that we cannot apply it to the static slice of the extreme Reissner-Nordstrom black hole because the static slice crosses the bifurcation surface and $n^a$ cannot be kept to be a timelike vector field.

One may, however, apply the above equality to cases with the massless scalar fields on the spherically symmetric case gives the following identity;

$$\int_{S_H} \mathcal{g}^{rr} \left( 2E_r^2 + \phi'^2 \right) dS = 8\pi, \quad (7)$$

where $E_r = F_{r\alpha}n^\alpha$ is the radial component of the electric field, $\phi$ is a massless scalar field and the prime stands for the derivative along the radial direction and $\mathcal{g}^{rr}$ is determined by solving the Hamiltonian constraint. Here we have used that momentary staticity implies $\phi' = 0$ on $\Sigma$ (the dot stands for the time derivative) through the momentum constraint.

When there is a sequence of the momentarily static initial data, which suggests an underlying time evolution, the left-hand side in Eq. $(7)$ can be regarded as a conserved quantity. We remark that this is consistent with the fact that $\phi'$ becomes constant at a later time in the linear perturbation level $[1, 2]$.

### III. GENERAL CASES

In this section, we will consider more general cases. We will define the extreme trapping horizon and give the surface integral identity at the extreme trapping horizon. We also discuss the causal/geometrical properties.

#### A. Extreme trapping horizon

We first derive the basic equation, and then define the extreme trapping horizon. Here we will employ the double null coordinate. In the double null decomposition, the metric of spacetimes is written as $g_{ab} = h_{ab} - e^{-f}(n_+a n_-b + n_-a n_+b)$, where $n_\pm$ are outgoing/ingoing null vectors. The 2-surfaces with the induced metric $h_{ab}$ have the canonical parameters $(\xi_+, \xi_-)$, so that $n_\pm^a = e^f((\partial_\pm)^a - r_\pm^a)$ with $r_\pm^a$ is the shift vector. Then we have $[2]$

$$e^f \mathcal{L}_{\theta_+} + e^f \theta_+ \mathcal{L}_{\theta_-} + \frac{1}{2} R - \tau_a \tau^a = - D_a r^a,$$

where $\theta_\pm = (1/2)h_{ab} L_\pm h_{ab}$, $L_\pm$ is the Lie derivative with respect to $e^{-f} n_\pm^a$ and $\tau_a = \omega_a - D_a f/2$. $D_a$ is the covariant derivative with respect to $h_{ab}$ and $\omega_a = (1/2)e^{-f} h_{ab} (n_+^c \nabla_c n_-^b - n_-^c \nabla_c n_+^b)$.

If

$$\theta_+ = e^f \mathcal{L}_{\theta_+} = 0$$

is satisfied on a $S_{\text{eth}}$, we call $S_{\text{eth}}$ the extreme trapping horizon$^1$. In addition, inspired by the model of extreme Reissner-Nordstrom spacetime and the recent numerical study $[2]$, we require that the region inside $S_{\text{eth}}$ are not trapped in the definition.

We are interested in the geometrical feature of the time “development” $\mathcal{H}_{\text{ext}} := \cup_{t \in S_{\text{eth}}(t)} S_{\text{eth}}$. Let us suppose $z$ to be the tangent vector of $\mathcal{H}_{\text{ext}}$ written as $z = e^{-f}(\alpha n_+ + \beta n_-)$. From the definitions, $\mathcal{L}_{\pm} \theta_+ |_{\mathcal{H}_{\text{ext}}} = (\alpha \mathcal{L}_+ \theta_+ + \beta \mathcal{L}_- \theta_+) |_{\mathcal{H}_{\text{ext}}} = 0$. The last equality comes from the definition of $\mathcal{H}_{\text{ext}}$. Since $\mathcal{L}_+ \theta_+ |_{\mathcal{H}_{\text{ext}}} = 0$, we see that $\alpha = 0$ or $\mathcal{L}_+ \theta_+ |_{\mathcal{H}_{\text{ext}}} = 0$ holds. In the former case, $z = e^{-f} \beta n_-$. This cannot be the case because $z \propto e^{-f} n_+$ for the future extreme trapping horizon, which is the future event horizon, of a certain domain of outer communications in the extreme Reissner-Nordstrom spacetime. Then, it is natural to expect the latter, namely $\mathcal{L}_+ \theta_+ |_{\mathcal{H}_{\text{ext}}} = 0$, holds for the extreme trapping horizon. This would imply a strong constraint on the induced geometry of $\mathcal{H}_{\text{ext}}$ and matters, that is, the shear $\sigma_{ab}$ of outgoing null geodesic congruences and $T_{abh} n_+^a n_-^b$ vanish on $\mathcal{H}_{\text{ext}}$. Indeed, while we do not have the causal feature of $\mathcal{H}_{\text{ext}}$ unlike the case for the non-extremal cases, we could instead show the presence of the Killing vector along $\mathcal{H}_{\text{ext}}$, from which the symmetry of the extreme trapping horizon would follow: a strong geometric consequence.

#### B. The surface integral at the extreme trapping horizon

Integrating Eq. $(8)$ over the extreme trapping horizon gives us

$$4\pi = \int_{S_{\text{eth}}} \left( e^{-f} T_{abh} n_+^a n_-^b + \tau_a \tau^a \right) dS. \quad (10)$$

When the surface develops such that Eq. $(10)$ holds, the above quantity is conserved.

$^1$ Originally the definition of the trapping horizon has the form of the “time development” $[2]$. 


As before, we consider cases with the massless scalar fields on spherical symmetric and dynamical extreme black hole. The energy-momentum tensor is \( T_{ab}^{\text{scalar}} = \partial_a \phi \partial_b \phi - (1/2)g_{ab} (\partial \phi)^2 \). In the double null coordinate, the metric is
\[
d s^2 = -2e^{-f(u,v)} du dv + (r(u,v))^2 d\Omega^2.
\] (11)

Then we see \( T_{ab}^{\text{scalar}} n^a n^b \propto T_{uv}^{\text{scalar}} = 0 \). Therefore, there is no contribution of the probe massless scalar fields into Eq. (10). Hence, in this setting, it is unlikely that there is no contribution of the probe massless scalar fields on spherical symmetric and dynamical extreme trapping horizon, and we can check that
\[
4\pi = \int_{S_{\text{eth}}} e^{-f} T_{ab}^{(\text{Maxwell})} n^a n^b dS
\] (12)
holds, where \( T_{ab}^{(\text{Maxwell})} = 2(F_{ac}F^c_b - \frac{1}{4}g_{ab} F^2) \). We used the fact that \( \tau^b \) vanishes for spherical symmetric spacetimes.

**C. Properties of extreme trapping horizon**

There is an inequality \( A_{\text{eth}} \geq 4\pi Q^2 \), where \( A_{\text{eth}} \) is the area of extreme trapping horizon and \( Q \) is the charge defined on \( S_{\text{eth}} \) \( Q = \frac{1}{4\pi} \int_{S_{\text{eth}}} F \), as shown in Ref. [16]. On the other hand, it is reported in the numerical study that the area of the event horizon (\( A_{\text{EH}} \)) is less than \( 4\pi Q^2 \), \( A_{\text{EH}} \leq 4\pi Q^2 \) [8]. Note that the numerical study is restricted to the spherically symmetric case. In order to understand these mutually incompatible inequalities (usually we expect \( A_{\text{eth}} \leq A_{\text{EH}} \)), we shall consider three possible cases separately: (i) the extreme trapping horizon is inside of the event horizon or (ii) the extreme trapping horizon is outside of the event horizon or (iii) the extreme trapping horizon coincides with the event horizon. Of course, the case (ii) is unlikely, but it is non-trivial to remove it if the extreme trapping horizon indeed exists.

In the case (i), if the spacetime becomes to be stationary at a sufficiently later time, the extreme trapping horizon will approach the event horizon. Then we see that the expansion rate of the null geodesic congruences should be negative there because of the inequalities \( A_{\text{EH}} \leq 4\pi Q^2 \leq A_{\text{eth}} \). Since there are not trapped surfaces in the current cases, this is impossible. Therefore, the extreme trapping horizon will not approach the event horizon if \( A_{\text{EH}} \leq 4\pi Q^2 \) holds. This observation is consistent with the existing results in the numerical study [7].

The case (ii) is possible in principle if the extreme trapping horizon exists. However, one thinks that the perturbations would easily destroy the extreme trapping horizon by the following argument 2. Indeed, it is easy to realize situations with non-zero shear \( \sigma_{ab}^+ \) (we suppose that the generic condition [14] is satisfied) near the extreme trapping horizon so that there is a 2-surface with a very small yet positive expansion. Then the Raychaudhuri equation tells us that the expansion will be negative at a sufficiently later time due to the presence of the shear. This means the formation of the trapped surface outside of the event horizon. Following the standard argument on black holes [8], this is impossible if the cosmic censorship conjecture holds [15].

The case (iii) is expected to be the stationary or static in asymptotically flat spacetimes. Then the spacetime will be the extreme Reissner-Nordstrom one if the black hole is single and non-rotating one.

There are two remarks. In a reality, the case (i) is more probable than (ii). This is because the perturbations which induce a nontrivial shear seem to exist in general except for spherical symmetric cases 3. In the case (i), it is expected that the trapped surface is easily formed by the same argument as in the case (ii) and it then contradicts the presence of the extreme trapping horizon. This remains a speculation, however, as we cannot be certain of the presence of trapped surface inside of the event horizon due to the possible spacetime singularities, which could precede the formation of trapped surfaces.

**IV. SUMMARY AND DISCUSSION**

We derived a pair of surface integral identities on the extreme minimal surface and the extreme trapping horizon in the dynamical extreme black hole geometry. They may give us a new constraint on the dynamical extreme black holes. For example, we can see that the area of the extreme trapping horizon is larger than \( 4\pi Q^2 \), where \( Q \) is the total charge defined at the extreme trapping horizon. We also examined the general feature of the extreme trapping horizon. Then it was shown that the shear of null geodesic congruence and a component of the energy-momentum tensor vanish on the extreme trapping horizon. From these facts it follows that the extreme trapping horizon has the symmetry. And we observed that the extreme trapping horizon seems to be inside of the event horizon and both horizons will not approach each other even at a sufficiently later time if the area of the cross section of the event horizon is less than \( 4\pi Q^2 \) (this is indicated through the recent numerical study [8]). This means that the spacetime could be dynamical forever. We also had the comment on the linear instability founded recently. However, we cannot show the direct relation between our integral identities and the conserved quantities of the perturbations on the event horizon if the trapped surfaces exist.

2 The similar argument to show that trapped surfaces are inside

3 Note that the shear vanishes if one considers the spherical symmetric cases, and then the case (ii) may occur.
extreme black holes. Since the extremality of the geometry is common in both settings, one may hope that they are related somehow. Nevertheless, our identities will be useful for the estimation of the error of numerical study.

Finally we list a set of remaining issues and open questions. Firstly, the geometrical/causal aspects of the extreme trapping horizon should be further clarified. Next, our argument relied on the two-dimensionality as we used the Gauss-Bonnet theorem, while the instability of the extreme black hole has been observed regardless of the spacetime dimensions [5]. Thus we pose the problem of extending our argument to higher dimensions. Finally, it is natural to consider the extreme trapping horizon corresponding to the extreme Kerr black hole (See Ref. [10]), where instead of charge, the contribution of the angular momentum needs to be taken into account.

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