Newton-Hooke type symmetry of anisotropic oscillators

P. M. Zhang\textsuperscript{1†}, P. A. Horvathy\textsuperscript{1,2‡}, K. Andrzejewski\textsuperscript{3§}, J. Gonera\textsuperscript{3¶}, P. Kosiński\textsuperscript{3∗∗} 

\textsuperscript{1}Institute of Modern Physics, Chinese Academy of Sciences, Lanzhou, China

\textsuperscript{2}Laboratoire de Mathématiques et de Physique Théorique, Université de Tours, France

\textsuperscript{3}Faculty of Physics and Applied Informatics, University of Lodz, Poland,

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Abstract

The rotation-less Newton–Hooke - type symmetry found recently in the Hill problem and instrumental for explaining the center-of-mass decomposition is generalized to an arbitrary anisotropic oscillator in the plane. Conversely, the latter system is shown, by the orbit method, to be the most general one with such a symmetry. Full Newton-Hooke symmetry is recovered in the isotropic case. Star escape from a Galaxy is studied as application.

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* Dedicated to the memory of J.-M. Souriau, deceased on March 15 2012, at the age of 90.
† email:zhpm@impcas.ac.cn
‡ email:horvathy@lmpt.univ-tours.fr
§ email:k-andrzejewski@uni.lodz.pl
¶ email:jgonera@uni.lodz.pl
∗∗ email: pkosinsk@uni.lodz.pl
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1. INTRODUCTION

Renewed interest in Kohn’s theorem on decomposing a system of charged particles in a magnetic field into center-of-mass and relative coordinates stems from relating it to the Newton-Hooke symmetry of the Landau problem [1–5].

Yet another example was found recently, however, namely for Hill’s equations [6, 7]. Remember that Hill originally devised his method for finding approximate solutions of the three-body problem, and in particular for the Moon-Earth-Sun system [8, 9]. Further applications involve stellar dynamics [10–13].

Hill’s equations also admit a center-of-mass decomposition, but no full Newton-Hooke symmetry. Rotations are broken, but translations and (generalized) boosts are still symmetries, hinting at that it is the subgroup spanned by the latter which is important for the purpose; additional symmetries like rotations are secondary.

In this paper, when referring to Newton-Hooke type symmetry it is Newton-Hooke with or without rotations that we have in mind.

The possibility of decomposing an isolated system into center-of-mass and relative coordinates has been linked to Galilean symmetry: Souriau [14] argued, in fact, that this property depends on the Galilei group having an invariant Abelian subgroup — namely the one spanned by translations and (Galilean) boosts.

Remarkably, the cohomological structure which determines the existence of a central extension originates precisely in this subgroup [14, 15]. Remember that, in dimension $d \geq 3$, both the Newton-Hooke and the Galilei groups have a one-parameter central extension, but in the plane they both admit an “exotic”, two-parameter central extension [16–19].

In this paper, we focus our attention at the Newton-Hooke case, the Galilean one being rather well-known.

The ordinary Landau problem admits the one-parameter centrally extended Newton-Hooke group as symmetry [1], but the “exotic” [non-commutative] version has indeed the two-parameter version [4]. In the Hill case, rotation-less “Newton-Hooke type” symmetry, with one (or, in the “exotic” case, with two) central extensions could be established [6, 7].

Then Our main result, proved in Section 6, says:
Theorem 1: The most general planar system with Newton–Hooke-type symmetry is a (possibly non-commutative) anisotropic oscillator in a uniform magnetic background. The symmetry extends to full Newton-Hooke symmetry in the isotropic case.

The proof will be accomplished by applying the orbit method, which provides us indeed with all systems upon which the symmetry group acts transitively [14, 20, 22]. From the technical point of view, we will find it convenient to use chiral decomposition [6, 7, 19, 23–25]. The motion in the (ordinary) Hall effect can in fact be decomposed into two uncoupled chiral oscillators with opposite chirality [23]. Conversely, combining two 1d chiral oscillators may yield the non-commutative Landau problem [24–27]; then the chiral method allows for an elegant derivation of the (Newton-Hooke) symmetry.

Recently, the method was extended to the Hill problem [6, 7] which is in fact a “maximally anisotropic oscillator”; here we further extend it to arbitrary anisotropy.

Our paper is organized as follows.

In section 2 chiral oscillators are reviewed.

Then, after outlining the Landau problem and its Newton-Hooke symmetry, we turn to the Hill problem. After some remark on its application to stellar dynamics, we present its the rotation-less Newton–Hooke type symmetry.

In Section 5 we generalize to an arbitrary, possibly anisotropic, oscillator.

In Section 6 we proceed conversely: applying the orbit method we describe all systems with Newton-Hooke type symmetry acting transitively.

We also study the arising of further symmetries and explain the difference between symmetry with or without rotations. Our results allow us to deduce:

Theorem 2: The system is either a truly anisotropic oscillator with Newton-Hooke-type symmetry only and no rotations, or it is isotropic with full Newton-Hooke symmetry, including rotations.

Moreover, in the first case, it can be brought into a “Hill-type form”, and in the second one it can be transformed into a free particle, cf. Sections 6D and 7.

An outlook is presented in the Conclusion, section 8.
2. CHIRAL OSCILLATORS

Chiral oscillators arise owing to the ambiguity of the phase-space description of a harmonic oscillator. In detail, let us consider a one-dimensional harmonic oscillator of unit mass \( m = 1 \) and frequency \( \omega \). Viewing the position and velocity, \( x \) and \( \dot{x} \), simply as coordinates on the phase space, we write

\[
y_1 = x, \quad y_2 = \dot{x},
\]

and consider the two first-order phase space Lagrangians

\[
L_\pm = \pm \frac{1}{2} \epsilon_{ij} y_i \dot{y}_j - \frac{\omega}{2} y^2.
\]

The associated (Euler-Lagrange) equations read

\[
\dot{y}_i = \mp \omega \epsilon_{ij} y_j.
\]

Our clue is that, for both signs in eqn. (2.2), eliminating either \( y_1 \) or \( y_2 \) yields, for the remaining variable, the same equation, namely that of a 1d harmonic oscillator,

\[
\ddot{y}_i + \omega^2 y_i = 0, \quad i = 1, 2.
\]

The solutions of (2.3) are simple rotations in phase space – but in opposite directions, depending on the sign \([36]\). (This is indeed the very meaning of the word “chiral”). Then we note that both types of motions project into configuration space according to the same motion \( x(t) \), as illustrated on Fig. 1.

We note that the same conclusion can be reached using a Hamiltonian framework. The eqns. (2.3) are indeed those of the symplectic structure and Hamiltonian

\[
\Omega_\pm = \pm \frac{i}{2} \epsilon_{ij} dy_i \wedge dy_j, \quad H = \frac{i}{2} \omega y^2,
\]

namely

\[
\dot{y}_i = \{ y_i, H \}_\pm, \quad \text{where the Poisson brackets} \quad \{ \cdot, \cdot \}_\pm \quad \text{are those associated with the chosen symplectic structure} \quad \Omega_\pm. \quad \text{The coordinates} \quad y_i \quad \text{are non-commuting,}
\]

\[
\{ y_1, y_2 \}_\pm = \mp 1,
\]

— as it is natural for position and momentum on the phase space. We mention for completeness that the Lagrangians (2.2) are the Cartan forms of the Souriau forms \([14, 28]\),

\[
L_\pm dt = \lambda_\pm, \quad d\lambda_\pm = \Omega_\pm - dH \wedge dt.
\]
FIG. 1: The phase-space trajectory of a chiral oscillator turns clockwise or anti-clockwise, depending on the sign of the frequency. Both trajectories project, however, onto the same motion in configuration space.

3. LANDAU PROBLEM

The classical example of a system with one-parameter-centrally-extended Newton-Hooke symmetry is provided by the “ordinary” [meaning commutative] Landau problem [1]. Generalizing the latter, we consider $N$ “exotic” particles endowed with masses, charges and non-commutative parameters $m_a$, $e_a$ and $\theta_a$, respectively, moving in a planar electromagnetic field $B, E$ [4]. Following [18, 27], we describe our system by

$$m_a^* \dot{x}_a = p_a^i - m_a e_a \theta_a \varepsilon^{ij} E^j,$$

$$\dot{p}_a^i = e_a B \varepsilon^{ij} \dot{x}_a^j + e_a E^i,$$

where

$$m_a^* = \Delta_a m_a \quad \text{with} \quad \Delta_a = 1 - e_a \theta_a B \quad (3.2)$$

is the effective mass of the particle labeled by $a = 1, \ldots, N$. Note, in the first relations, also the “anomalous velocity terms” perpendicular to the electric field, $E$. The variables $p_a$ here could be called “momenta” – but to avoid confusion with the conserved quantities, we simply consider them as coordinates on the phase space.

Although our theory works for any $B$ and $E$, we assume, for simplicity, that the magnetic
field is constant, \( B = \text{const} \), and that the electric field is that of an isotropic harmonic trap, \(-k\mathbf{x}\), augmented with an interparticle force coming from some two-body potential, \( V = \sum_{a \neq b} V_{ab}(\mathbf{x}_a - \mathbf{x}_b) \).

For \( \theta_a = 0 \), the ordinary Landau problem is recovered.

Summing over all particles, we find that when \( e_a/m_a \) and \( e_a \theta_a \) are both constants i.e. when the generalized Kohn conditions [4],

\[
\kappa_a \equiv \frac{e_a}{m_a} = \frac{e}{M} \equiv \kappa, \quad e_a \theta_a = e \Theta, \quad \Theta = \frac{\sum_a m_a^2 \theta_a}{M^2}
\]

(3.3)

hold (where \( M = \sum_a m_a \) and \( e = \sum_a e_a \) are the total mass and charge), then the center-of-mass, \( \mathbf{X} = \sum_a m_a \mathbf{x}_a/M \), splits off,

\[
M^* \dot{X}^i = P^i - Me \Theta \varepsilon^{ij} E^j, \\
\dot{P}^i = eB \varepsilon^{ij} \dot{X}^j + e E^i,
\]

(3.4)

where

\[
M^* = \Delta M, \quad \Delta = 1 - e \Theta B, \quad P = \sum_a p_a.
\]

(3.5)

The center-of-mass behaves hence as a single “exotic” particle carrying the total mass, charge and non-commutative parameter, \( M, e \) and \( \Theta \), respectively.

We note that eqns. (3.4) are in fact Hamilton’s equations for the Poisson structure [27],

\[
H = \frac{P^2}{2M} + V(\mathbf{X}),
\]

(3.6)

\[
\{X^i, X^j\} = \frac{\Theta}{\Delta} \varepsilon^{ij}, \quad \{X^i, P^j\} = \delta^{ij} \frac{\Delta}{\Delta}, \quad \{P^i, P^j\} = \frac{eB}{\Delta} \varepsilon^{ij}.
\]

(3.7)

When the [generalized] Kohn condition (3.3) is satisfied, then, for identical initial velocities, all individual particles move in the same way, — and this motion is shared by their center of mass, cf. Fig. [2]

The best way to understand the intuitive content of the Kohn condition is, however, to consider what happens when the Kohn condition is not satisfied. Consider, for example, two particles in a pure magnetic field such that

\[
\kappa_2 \equiv \frac{e_2}{m_2} = 2\kappa_1 \equiv 2 \frac{e_1}{m_1}.
\]

(3.8)

Then, assuming identical initial velocities, each of them performs a rotational motion but with different radii,

\[
R = (m/e) \frac{v}{B} \quad \Rightarrow \quad R_2 = \frac{1}{2} R_1,
\]

(3.9)
FIG. 2: If the Kohn conditions (3.3) are satisfied, all particles turn along circles of equal radii with common angular velocity. Their motion is shared by their center of mass (in dashed).

FIG. 3: If the Kohn conditions (3.3) are not satisfied, \( \kappa_2 \equiv e_2/m_2 = 2e_1/m_1 \equiv 2\kappa_1 \), for example, the individual radii and the angular velocities are different. The motion is not more collective, and the center of mass describes a complicated (dashed) curve.

and with different frequencies:

\[
\omega = \frac{v}{R} \quad \Rightarrow \quad \omega_2 = 2\omega_1 \quad (3.10)
\]

[so that \( \omega_1 R_1 = v = \omega_2 R_2 \)]. Their center-of-mass would then clearly not move on a circle, rather on some complicated curve, cf. Fig. 3. The 3-body situation is illustrated on Fig. 4.

Symmetries.

Let us restrict ourselves henceforth to the purely magnetic case, \( E = 0 \) and to the center-of-mass motion. The coordinate \( P \) is not conserved; one readily shows, however, that the “magnetic momentum” (which can also be derived by Noether’s theorem as the conserved quantity associated with the translational symmetry \[25\]) \[37\] and “magnetic
FIG. 4: The behavior of a 3-body system. (a) If the Kohn conditions (3.3) are satisfied, all particles move collectively, along with their center-of-mass. (b) If (3.3) is not satisfied, $\kappa_1 = 1, \kappa_2 = 2, \kappa_3 = 3$, for example, the motion is not more collective, and the center of mass (in dashed) describes a complicated curve.


center-of-mass”,

\[ \Pi_i = M\Delta(\dot{X}_i - \omega^* \varepsilon_{ij} X_j), \]
\[ K = M\Delta^2 R(\omega^* t) \dot{X}, \]

respectively, where $\omega^* = eB/M^* = \omega/\Delta$, are both conserved [38], and span indeed two uncoupled Heisenberg algebras with central charges $-M\omega$ and $\Delta M\omega$,

\[ \{\Pi^i, \Pi^j\} = -M\omega \varepsilon^{ij}, \quad \{K^i, K^j\} = \Delta M\omega \varepsilon^{ij}, \quad \{\Pi^i, K^j\} = 0. \]

Time translations and rotations are plainly symmetries also, and the associated conserved quantities, namely the Hamiltonian $H$ in [39], augmented with the total angular momentum [18, 27],

\[ J = X \times P + \frac{eB}{2} x^2 + \frac{\Theta}{2} P^2, \]

have commutation relations

\[ \{H, \Pi^i\} = 0, \quad \{H, K^i\} = -\frac{\omega}{\Delta} \varepsilon^{ij} K^j, \]
\[ \{J, X^i\} = \varepsilon^{ij} X^j, \quad \{J, P^i\} = \varepsilon^{ij} P^j, \quad \{J, H\} = 0. \]
In conclusion, the exotic Landau problem [with or without an isotropic harmonic trapping force] admits an “exotic” i.e. two-parameter centrally extended Newton-Hooke symmetry [4, 25]. In the commutative case $\Theta = 0$, the central charges are correlated, $\mp M\omega$, and the symmetry reduces to the one-parameter extension studied in [1].

We record for further use that the total angular momentum, $J$ in (3.13), can also be presented in a number of different ways. Firstly, we note that the new variables [18]

$$Q_i = x_i + \frac{1}{eB} \left( 1 - \sqrt{1 - \theta eB} \right) \varepsilon_{ij} p_j,$$

(3.16)

$$P_i = \frac{1 + \sqrt{1 - \theta eB}}{2} p_i - \frac{1}{2eB} \varepsilon_{ij} x_j,$$

(3.17)

are canonical, and in their terms the total angular momentum is simply

$$J = Q \times P.$$

(3.18)

Here we just mention that using chiral coordinates (sect. 5C), the angular momentum can further be decomposed, see (5.22).

From now on the generalized Kohn conditions (3.3) will always be assumed, allowing us to consider the center-of-mass alone. Coordinates will again be denoted by lower-case letters, as for a one-particle theory.

4. THE HILL PROBLEM

Hill’s original aim has been to study the Moon-Earth-Sun system [8, 9]. Later, his technique has been applied to stellar dynamics [10–13], and it is this second context that we have in mind here. “Moon and Earth” will become a “star cluster”, and the role of the “Sun” will be played by the “Galactic Center”.

Assuming, for simplicity, that the motion is in the plane, the $z$ coordinate can be dropped. Then, for approximately circular orbits, the first-order approximation to Newton’s gravitational equations provides us with Hill’s equations [6, 12, 29],

$$m_a \left( \ddot{x}_a - 2\omega \dot{y}_a - 3\omega^2 x_a \right) = \sum_{b \neq a} \frac{G m_a m_b (x_b - x_a)}{|x_a - x_b|^3},$$

$$m_a \left( \ddot{y}_a + 2\omega \dot{x}_a \right) = \sum_{b \neq a} \frac{G m_a m_b (y_b - y_a)}{|x_a - x_b|^3},$$

(4.1)
In these equations $\mathbf{x}_a = (x_a, y_a)$ are the coordinates of star No $a$ measured in a rotating coordinate system whose origin lies on the Keplerian orbit with $r = R, \theta = \omega t$. The $x$-axis is radial so that $r = R + x$, and the $y$ axis is tangent to the orbit. $\omega^2 = GM/R^3$ [where $M$ is the mass of the Galaxy] is the angular velocity of a circular Keplerian orbit with radius $R$. The origin $(x = y = 0)$ of this frame represents hence the reference orbit; our investigations concern the behavior in its neighborhood.

The linear-in-velocity terms in (4.1) correspond to the Coriolis force induced in a rotating coordinate system.

The motion of the “Galactic Center” is neglected. The only remnant of the influence of the Galaxy on the star cluster corresponds, in the first-order approximation, to the repulsive anisotropic harmonic term in the (radial) $x$-equation, which arises from balancing attractive gravitational and repulsive centrifugal forces.

The right hand sides represent the mutual gravitational interactions between the stars.

The Coriolis force plays a role analogous to a uniform magnetic field, turning our study here analogous to the one in the Landau problem of the preceding section.

In the stellar context, a particularly interesting question is that of escape from the Galaxy \[12, 13, 29, 30\]. For individual stars the answer is complicated, only allowing for a numerical treatment. The motion of the center of mass (COM),

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \frac{\sum_a m_a \mathbf{x}_a}{\sum_a m_a},$$

is much simpler, though: inter-stellar interactions drop out by Newton’s third law, leaving us with the simple planar system,

$$\ddot{x} - 2\omega \dot{y} - 3\omega^2 x = 0,$$

$$\ddot{y} + 2\omega \dot{x} = 0.$$  \hspace{1cm} (4.3)

These equations describe the oscillations of the center of mass of the considered star cluster in the neighborhood of a reference Keplerian circle, represented here by $x = 0, y = 0$.

The interest of studying the COM-problem is underlined by the fact that a cluster is formed by a huge number of stars, — in fact, of the order of a million \[13\].
FIG. 5: Trajectory of the center of mass in the Hill problem (in blue) in the rotating coordinates $x,y$. The red dotted straight horizontal line indicates the trajectory of the guiding center about which the center of mass performs “flattened elliptic motion”. The heavy (green) dot in the origin stands for the reference Keplerian orbit. Note the unconventional orientation of the axes.

A. Star escape: Hall motions in the Sky

For the COM, the problem of escape can also be reduced to that of a guiding center. Equations (4.3) are readily solved [6] as

$$
x(t) = \frac{A}{\omega} \sin \omega t - \frac{B}{\omega} \cos \omega t + x_0,

y(t) = 2\frac{A}{\omega} \cos \omega t + 2\frac{B}{\omega} \sin \omega t - \frac{3}{2} \omega t x_0 + y_0.
$$

(4.4)

where $A, B, x_0, y_0$ are integration constants. (4.4) is an ellipse centered at $(x_0, y_0 - \frac{3}{2} \omega x_0 t)$ with major axes lying along the $y$ direction, whose centers drift along $y$ with constant speed $-\frac{3}{2} \omega x_0$, see Fig. 5.

- For initial condition $x_0 = y_0 = 0$, the trajectory is an ellipse centered at the origin and oriented along the $y$ direction,

$$
X_+(t) = \begin{pmatrix} X^1_+(t) \\ X^2_+(t) \end{pmatrix} = \begin{pmatrix} \frac{A}{\omega} \sin \omega t - \frac{B}{\omega} \cos \omega t \\ 2\frac{A}{\omega} \cos \omega t + 2\frac{B}{\omega} \sin \omega t \end{pmatrix}.
$$

(4.5)

- For the particular, “Hall” initial conditions,

$$
X^1_-(0) = x_0, \quad X^2_-(0) = y_0,

\dot{X}^1_-(0) = 0, \quad \dot{X}^2_-(0) = -\frac{3}{2} \omega x_0,
$$

(4.6)

we get, instead,

$$
X_-(t) = \begin{pmatrix} X^1_-(t) \\ X^2_-(t) \end{pmatrix} = \begin{pmatrix} x_0 \\ -\frac{3}{2} \omega t x_0 + y_0 \end{pmatrix},
$$

(4.7)
identified with the motion the of guiding center. The latter solution arises when the harmonic and the inertial forces cancel, so that the COM drifts with the Hall velocity perpendicularly to the harmonic field,

\[ \dot{X}_i^+ = \varepsilon^{ij} \frac{E_j}{B}, \quad \text{where} \quad eE_1^+ = 3\omega^2 X_1^- \quad \text{and} \quad E_2^+ = 0, \quad eB = 2\omega. \quad (4.8) \]

i.e., it performs a Hall motion.

The general solution, (4.4), is the sum of two particular solutions,

\[ x(t) = X_+(t) + X_-(t). \quad (4.9) \]

The coordinate \( X_+(t) \) describes, in particular, the relative motion about the guiding center. Note that an initial condition \( x(0) = 0, y(0) = y_0 \neq 0 \) yields shifted elliptic trajectories with the fixed point \( x(t) = 0, y(t) = y_0 \neq 0 \) on the \( y \)-axis as guiding center.

**Translation into fixed coordinates**

So far we worked in a rotating coordinate system. Putting

\[
\begin{align*}
u(t) &= (R + x(t)) \cos \omega t - y(t) \sin \omega t, \\
v(t) &= (R + x(t)) \sin \omega t + y(t) \cos \omega t,
\end{align*}
\]

allows us to express our results in the fixed coordinate system \((u, v)\), as shown on Figs. [6](#) and [7](#). As said before, our Keplerian reference circle corresponds to the origin of the \( x - y \) plane, \( x(t) = y(t) = 0 \).

Our clue now is that the motion is only bounded when that of the guiding center is. We focus therefore our attention to \( X_-(t) \). By eqn. (4.7) the guiding centers move forcelessly, governed by the Hall law. Then, although the coordinate \( x(t) \) remains bounded for any choice of the parameters, motion in the tangential (\( y \)) direction increases the distance from the galactic center in all cases. All motions with initial condition \( x_0 \neq 0 \) [i.e., \( u_0 \neq R \)] are, therefore unbounded.

Bounded motions only arise for initial condition on the \( y \)-axis \( x_0 = 0 \), when, in the co-moving frame, \((x, y) = (0, y_0)\) is fixed. Then the expansion is stopped and the guiding center trajectory is in fact a Keplerian circle. As a result of the oscillatory motion of \( X_+(t) \) the final trajectories are however, quite complicated, as illustrated on Fig. [7](#).
FIG. 6: For all initial conditions \( u_0 = R + x_0 \), with \( x_0 \neq 0 \), \( v_0 = y_0 \) the motions (in blue) are unbounded: our star cluster escapes. The dotted red line is the guiding center, and the dashed green circle is the reference Keplerian trajectory, which corresponds to the heavy dot at the origin on Fig. 5.

The distinction between bounded and unbounded motions corresponds to the negative semi-definite nature of the effective potential energy in the co-moving frame,

\[
V_{eff} = -3m\omega^2x^2,
\]

for which only \( x = 0 \) is a neutral direction. For all initial conditions with \( x_0 \neq 0 \) the repulsive harmonic potential is indeed of the tidal nature, implying that all such solutions are unstable.

Looking at Figs. 6, it is tempting to think at those celebrated spiral arms of galaxies. One should keep in mind, however, that our investigations are only valid in the neighborhood of the Keplerian orbit characterized by \( x = y = 0 \). For large values of \( x \) and \( y \) our linear ap-
FIG. 7: Bounded trajectories (in blue) in the fixed reference frame \((u, v)\) arise for initial condition \(u_0 = R \) i.e. \(x_0 = 0 \) (only), indicated by a dot. The shape of the trajectory strongly depends on the initial condition \(y_0\) and on the choice of the Keplerian circle (in green). The red circle is the guiding center.

proximation breaks down, and it would plainly be abusive to draw any conclusion about the long-range behavior from studying the linearized equations \([4.3]\). Our linear approximation is only justified when all coordinates \(x_a, y_a\) are small when compared to the Keplerian radius \(R\). The tendency to escape is, however, reflected by their initial behavior studied here.
B. Newton-Hooke-type symmetry in the Hill problem

Having understood the importance of the center-of-mass decomposition, we turn to study the symmetry which makes it work – namely that of Newton-Hooke with no rotations.

The coordinates $X_{\pm}$ introduced above can, indeed, be completed to chiral coordinates, namely by putting

$$x = X_+ + X_-, \quad p^1 = \frac{1}{2} \omega X_+^2, \quad p^2 = -2\omega X_+^1 - \frac{3}{2} \omega X_-^1. \quad (4.12)$$

Our investigations can in fact be generalized to exotic particles i.e. to non-commutative particles, see [7]. Skipping details, we state that, in all cases one finds that ordinary translations and certain “time dependent translations” (also called “generalized boosts”),

$$\Pi = \begin{pmatrix} X_+^1(t) \\ X_-^2(t) + \frac{3}{2} \omega t X_-^1(t) \end{pmatrix}, \quad (4.13)$$

$$K = \begin{pmatrix} X_+^1(t) \cos(\omega/\Delta) t - \frac{1}{2\Gamma} X_-^2(t) \sin(\omega/\Delta) t \\ 2\Gamma X_+^1(t) \sin(\omega/\Delta) t + X_-^2(t) \cos(\omega/\Delta) t \end{pmatrix},$$

are conserved, where

$$\Delta = 1 - 2m\omega \theta, \quad \Gamma = 1 - 3m\theta \omega / 2. \quad (4.14)$$

Their commutation relations are, once again, those of two exotic Heisenberg algebras with central charges $-(2/m\omega)$ and $(\Gamma/\Delta)(2m\omega)$, respectively,

$$\{\Pi^1, \Pi^2\} = -\frac{2}{m\omega}, \quad \{K^1, K^2\} = \frac{\Gamma}{\Delta} \frac{2}{m\omega}, \quad \{\Pi_i, K_j\} = 0. \quad (4.15)$$

In the commutative case $\theta = 0$ so that $\Gamma = \Delta = 1$, and the one-parameter centrally extended symmetry found in [6] is recovered. The Hamiltonian,

$$H = H_+ + H_- = \frac{m\omega^2}{2} \left( X_+^1 X_+^1 + \frac{1}{4\Gamma^2} X_-^2 X_-^2 \right) - \frac{3m\omega^2}{8} X_-^1 X_+^1, \quad (4.16)$$

is also conserved. Its commutation relations with translations and boosts read

$$\{H, \Pi^1\} = 0, \quad \{H, \Pi^2\} = \frac{3}{2} \omega \Pi^1, \quad (4.17)$$

$$\{H, K^1\} = -\frac{\omega^*}{2\Gamma} K^2, \quad \{H, K^2\} = 2\Gamma \omega^* K^1.$$

As rotational symmetry is plainly broken, the total symmetry of the Hill problem is *Newton-Hooke without rotations.*
5. ANISOTROPIC HARMONIC OSCILLATOR

We note that the Hill problem is in fact a maximally anisotropic “one sided” oscillator. The case of a general anisotropic oscillator is worth studying in some detail therefore.

A. Chiral coordinates

Consistently with the general theory sketched in Section 3, an “exotic” [i.e., non-commutative] charged harmonic oscillator in the plane in a homogenous magnetic field $B$ is described by the symplectic form and Hamiltonian,

$$\Omega = dp^i \wedge dx^i + \frac{\theta}{2} \varepsilon^{ij} dp^i \wedge dp^j + \frac{eB}{2} \varepsilon^{ij} dx^i \wedge dx^j, \quad (5.1)$$

$$H = \frac{p^2}{2m} + V, \quad V = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 x_2^2, \quad (5.2)$$

respectively, with the parameters having the same physical interpretation as before. The spring constants $k_1$ and $k_2$ may or may not be identical.

The idea of Alvarez et al. [24] has been to combine chiral oscillators. Multiplying both the symplectic form and the Hamiltonian (or, alternatively, the Lagrangian) by the same overall constant $\mu$,

$$\Omega \rightarrow \mu \Omega, \quad H \rightarrow \mu H \quad i.e. \quad L \rightarrow \mu L,$$

would not change the equations of motion. But what happens, if we multiply them with different coefficients before adding them? Conversely, can we decompose a given system into two chiral parts? To answer these questions we introduce, following [6, 24, 25], new coordinates on the phase space,

$$x^i = X^i_+ + X^i_-,$$

$$p^1 = \alpha_+ X^2_+ + \alpha_- X^2_-, \quad p^2 = -\beta_+ X^1_+ - \beta_- X^1_-,$$  

where the coefficients $\alpha_\pm$ and $\beta_\pm$ will be determined from the requirement that both the symplectic form and the Hamiltonian should split into two uncoupled one-dimensional sub-systems we shall call chiral components. Inserting (5.3) into (5.1) shows that the symplectic form splits as $\Omega = \Omega_+ + \Omega_-$, whenever

$$\alpha_- + \beta_+ - \theta \alpha_+ \beta_+ = eB, \quad \alpha_+ + \beta_- - \theta \alpha_- \beta_- = eB. \quad (5.4)$$
Similarly, inserting (5.3) into (5.2) yields that the Hamiltonian splits as $H = H_+ + H_-$ when

$$\alpha_+ \alpha_- + mk_2 = 0, \quad \beta_+ \beta_- + mk_1 = 0. \quad (5.5)$$

Then a tedious calculation allows choosing

$$\alpha_+ = -\frac{1}{2(eB + \theta mk_1)} \left( -e^2 B^2 + m(k_2 - k_1) + \theta^2 m^2 k_1 k_2 
+ \sqrt{4mk_1(eB + \theta mk_2)^2 + (e^2 B^2 - m(k_1 - k_2) - \theta^2 m^2 k_1 k_2)^2} \right), \quad (5.6)$$

$$\alpha_- = \frac{1}{2(eB + \theta mk_1)} \left( e^2 B^2 - m(k_2 - k_1) - \theta^2 m^2 k_1 k_2 
+ \sqrt{4mk_1(eB + \theta mk_2)^2 + (e^2 B^2 - m(k_1 - k_2) - \theta^2 m^2 k_1 k_2)^2} \right), \quad (5.7)$$

and

$$\beta_+ = -\frac{1}{2(eB + \theta mk_2)} \left( -e^2 B^2 + m(k_1 - k_2) + \theta^2 m^2 k_1 k_2 
+ \sqrt{4mk_1(eB + \theta mk_2)^2 + (e^2 B^2 - m(k_1 - k_2) - \theta^2 m^2 k_1 k_2)^2} \right), \quad (5.8)$$

$$\beta_- = \frac{1}{2(eB + \theta mk_2)} \left( e^2 B^2 - m(k_1 - k_2) - \theta^2 m^2 k_1 k_2 
+ \sqrt{4mk_1(eB + \theta mk_2)^2 + (e^2 B^2 - m(k_1 - k_2) - \theta^2 m^2 k_1 k_2)^2} \right). \quad (5.9)$$

provides us with decomposed symplectic form and the Hamiltonian,

$$\Omega = \Omega_+ + \Omega_- = \quad (5.10)$$

$$\left( -\alpha_+ - \beta_+ + \theta \alpha_+ \beta_+ + eB \right)_\mu dX_+^1 \wedge dX_+^2 + \left( -\alpha_- - \beta_- + \theta \alpha_- \beta_- + eB \right)_\mu dX_-^1 \wedge dX_-^2,$$

and

$$H = H_+ + H_- = \frac{1}{2m} \times \quad (5.11)$$

$$\left[ (\beta_+^2 + mk_1) X_+^1 X_+^1 + (\alpha_+^2 + mk_2) X_+^2 X_+^2 
+ (\beta_-^2 + mk_1) X_-^1 X_-^1 + (\alpha_-^2 + mk_2) X_-^2 X_-^2 \right],$$

respectively.
• For $\theta = 0$ the commutative cases $[6, 24]$ are recovered;

• For $k_1 = -3m\omega^2, k_2 = 0$ and $B = 2\omega$, we get

$$
\alpha_+ = 0, \quad \alpha_- = \frac{m\omega}{2\Gamma}, \quad \beta_+ = \frac{3}{2}m\omega, \quad \beta_- = 2m\omega,
$$
and the results found before in the Hill Problem $[6, 7]$ are obtained [up to interchanging $X_+$ and $X_-]$;

• When $k_1 = k_2$ our oscillator is isotropic. Then $\alpha_+ = \beta_+$, and (5.10) reduce to the chiral decomposition for the [exotic] Landau problem with harmonic force, studied in $[25]$;

• For $k_1 = k_2 = 0$ the oscillator is switched off, and the system reduces to the purely-magnetic non-commutative Landau problem $[24, 25, 27]$.

B. Motions

Let us assume that none of the coefficients $\mu_\pm$ vanishes. Then it follows from (5.11) that our chiral coordinates satisfy the Poisson bracket relations

$$
\{X^i_+, X^j_+\} = -\frac{1}{\mu_+} \varepsilon^{ij}, \quad \{X^i_+, X^j_-\} = 0 \quad \{X^i_-, X^j_-\} = -\frac{1}{\mu_-} \varepsilon^{ij}.
$$

The equations of motion read therefore

$$
m\mu_\pm \dot{X}^1_\pm = - (\alpha^2_\pm + mk_2)X^2_\pm, \quad m\mu_\pm \dot{X}^2_\pm = (\beta^2_\pm + mk_1)X^1_\pm.
$$

Both chiral components $X_\pm$ are governed, hence, by uncoupled equations which are reminiscent of those of $1d$ harmonic oscillators, to which they reduce, however, only in the isotropic case, $k_1 = k_2$.

Assuming $\alpha^2_\pm + mk_2 \neq 0$ [40], eqns. (5.14) are solved by

$$
X^1_\pm = A_\pm \cos \omega_\pm t + B_\pm \sin \omega_\pm t, \quad X^2_\pm = F_\pm \left( A_\pm \sin \omega_\pm t - B_\pm \cos \omega_\pm t \right), \quad F_\pm = \sqrt{\frac{\beta^2_\pm + mk_1}{\alpha^2_\pm + mk_2}},
$$

(5.15)
where the frequencies read

$$\omega_{\pm} = \sqrt{\left(\alpha_{\pm}^2 + mk_2\right)\left(\beta_{\pm}^2 + mk_1\right)} \, m\mu_{\pm}. \quad (5.16)$$

Both $X_{\pm}$-trajectories are ellipses, as illustrated in Fig. 8. Note that the frequencies, $\omega_+$ and $\omega_-$ are in general different even in the isotropic case, and the curves do not close therefore.

### C. Symmetries

Eqns. (5.15) allow us to infer that

$$A_{\pm} = X_{\pm}^1 \cos \omega_{\pm} t + \frac{1}{F_{\pm}} X_{\pm}^2 \sin \omega_{\pm} t,$$

$$B_{\pm} = X_{\pm}^1 \sin \omega_{\pm} t - \frac{1}{F_{\pm}} X_{\pm}^2 \cos \omega_{\pm} t$$

are conserved. A direct calculation yields, furthermore, for both labels $\pm$, the uncoupled Heisenberg algebra relations

$$\{A_{\pm}, B_{\pm}\} = -\frac{1}{F_{\pm} \mu_{\pm}}, \quad \{ (\cdot)_+, (\cdot)_- \} = 0. \quad (5.18)$$

Adding the Hamiltonian (5.11), the doubly-centrally-extended rotation-less Newton-Hooke algebra is obtained.
Both sets of chiral coordinates $X_\pm$ describe 2d symplectic vectorspaces. The symplectic forms $\Omega_\pm$ are plainly symmetric under phase-space chiral rotations, $X_\pm \rightarrow R(X_\pm)$. None of the Hamiltonians $H_\pm$ is symmetric in general, though. The natural diagonal action,

$$x = X_+ + X_- \rightarrow R(X_+) + R(X_-) = R(x),$$

is not a symmetry therefore: *rotations are broken by the anisotropy.*

In the isotropic case,

$$k_1 = k_2,$$

however, we have $\alpha_\pm = \beta_\pm$ and the coefficients of the quadratic terms both in $H_+$ and $H_-$ are hence identical, so that the chiral rotations $X_\pm \rightarrow R(X_\pm)$ do act as symmetries for the components: *rotational symmetry is restored.* The square-root factors in (5.15) become unity, $F_\pm = 1$, and the trajectories become circles. The frequencies,

$$\omega_\pm = \frac{\alpha_\pm^2 + mk}{m\mu_\pm},$$

are not identical, though, since $\alpha_+ \neq \alpha_-$ and $\mu_+ \neq \mu_-$ in general, cf. (5.6) – (5.7) and (5.10).

It is worth recording that, in terms of chiral coordinates, the total angular momentum, (3.13), is also decomposed, as

$$J = J_+ + J_-, \quad J_+ = \frac{eB}{2} \left( \vec{X}_+ \right)^2, \quad J_- = -\Delta \frac{eB}{2} \left( \vec{X}_- \right)^2,$$

where $\Delta = 1 - eB\theta$, as before. Its conservation, $\dot{J} = 0$, can also be checked directly, using the equations of motion.

We just mention that the singular case $\mu_+ = 0$ or $\mu_- = 0$, leading to Hall-type motion, can be dealt with as in the previous occasions, $[4, 7, 18, 25, 27]$.

6. SYSTEMS WITH PRESCRIBED NH-TYPE SYMMETRY

Any physical theory consists of some mathematical structure together with a set of operational rules relating the abstract notions entering this structure to physically measurable quantities. It happens quite often that various theories share the same mathematics and differ only in their interpretation. It is, therefore, interesting to analyse the mathematical theories commonly appearing in different physical contexts. In most cases (if not all, at least
as far as the basic microscopic theories are concerned) the choice of the dynamical equations
is based on symmetry principles. If the symmetry group is selected the form of dynamics is
strongly restricted; in some cases all admissible dynamics can be even fully classified. Once
this is done one may look for theories sharing the same formal structure but differing in the
operational meaning of formal notions used.

In the present paper we are interested in physical systems exhibiting Newton-Hooke type
symmetry. This section is devoted to the classification and analysis of formal properties of
dynamical systems possessing such type of symmetry. We wish to show that the results of
the previous Sections, which provide physical examples of the systems under consideration,
fit, in fact, into some general framework. To this end, we assume that the dynamics under
consideration is invariant under the transitive action of some Lie group \( G \), and then classify
all such symplectic manifolds upon which \( G \) acts by symplectically. The proper tool for
doing this is provided by the orbit method \[14, 20, 21\].

Our choice for the group \( G \) is dictated by the following considerations. As far as possible,
we would like to allow for a generalization which includes both the Galilei and the Newton-
Hooke groups, and also the “rotation-free part” of the latter. The main characteristic
features are therefore the following:

(i) there exists generators (namely of boosts and momenta) which, via the orbit method,
yield the basic canonical variables;

(ii) The Hamiltonian equations of motion are linear in the latter variables; the Hamilto-
nian belongs therefore to the Lie algebra itself, and acts linearly on the remaining variables.

We want our generalization to be a minimal one in that no further symmetry generators
beyond the above ones should be included. Such generators will appear later however for
specific values of the structure constants.

Guided by these considerations, we start with the following Lie algebra commutation
relations,

\[
\begin{align*}
[\xi_i, \xi_j] &= i\omega_{ij} M, \quad i, j = 1, \ldots, 2N \\
[M, \xi_i] &= 0, \\
[M, H] &= 0, \\
[H, \xi_i] &= iA_{ij}\xi_j,
\end{align*}
\]

(6.1)
where \( \omega = (\omega_{ij}) \) is a non-singular antisymmetric matrix. The only non-trivial Jacobi identity,

\[
\left[ \left[ H, \xi_i \right], \xi_j \right] + \text{(cyclic)} = 0,
\]

yields the constraint \( A_{ik} \omega_{kj} - A_{jk} \omega_{ki} = 0 \), i.e., that \( B = A \omega \) is a symmetric matrix, \( B^T = B \).

The algebra (6.1) admits the Casimir operator of the form

\[
C = MH - \frac{1}{2} X_{ij} \xi_i \xi_j,
\]

where without loss of generality we can assume that \( X = (X_{ij}) \) is symmetric. \( C \) commutes with all generators, provided \( A = -\omega X \).

Collecting our results, our algebra reads

\[
\begin{align*}
\left[ \xi_i, \xi_j \right] & = i \omega_{ij} M, \\
\left[ M, (\cdot) \right] & = 0, \\
\left[ H, \xi_i \right] & = -i \omega_{ik} X_{kj} \xi_j, \\
C & = MH - \frac{1}{2} X_{ij} \xi_i \xi_j,
\end{align*}
\]

and is uniquely defined by choosing the non-singular antisymmetric matrix \( \omega \) and the symmetric matrix \( X \).

The next step is to classify the inequivalent algebras (6.4). Under the invertible transformation

\[
\xi_i' = D_{ij} \xi_j, \quad \det (D_{ij}) \neq 0,
\]

The matrices \( \omega \) and \( X \) transform according to

\[
\begin{align*}
\omega' & = D \omega D^T, \\
X' & = (D^{-1})^T X D^{-1}.
\end{align*}
\]

Using the latter we can find the “canonical” form in any class of equivalent algebras (6.4).

In what follows we shall restrict ourselves to the case \( 2N = 4 \), the generalization to arbitrary \( N \) being straightforward.

To complete our classification scheme some further assumptions on the matrix \( X \) have to be made. The existence of the Casimir operator \( C \) implies that, on each orbit, the Hamiltonian is a quadratic function of the basic canonical variables, to which a trivial term, representing the internal energy, has been added (see Appendix B). Whether the energy is positive definite or not depends, therefore, on the choice of \( X \).

The following cases will be considered separately.
A. X Positive definite

Consider first the case of a positive definite matrix $X$. By an appropriate choice of $D$ in eqns. (6.6), $X = I$ can be achieved. In fact, $X$, being symmetric, can be diagonalized by a suitable orthogonal transformation. Then an additional diagonal transformation reduces $X$ to the unit matrix.

Assuming $X = I$, we still have some residual transformations left at our disposition. Namely, as it is seen from eqns. (6.6), $D$ can be taken to be an arbitrary orthogonal matrix, without spoiling the condition $X = I$. The question is now to classify all antisymmetric $4 \times 4$ matrices $\omega$ up to an orthogonal transformation. This problem is solved in Appendix A (which is actually the Euclidean version of the classification problem for electromagnetic field configurations under the action of the Lorentz group). As shown in Appendix A, $\omega$ can be put into the form

$$
\omega = \begin{pmatrix}
0 & \Omega_1 & 0 \\
-\Omega_1 & 0 & \Omega_2 \\
0 & -\Omega_2 & 0
\end{pmatrix}, \quad \Omega_{1,2} > 0.
$$

(6.7)

Defining

$$
B_1 = \Omega_1^{-1}\xi_1, \quad B_2 = \Omega_2^{-1}\xi_2, \quad P_1 = \xi_3, \quad P_2 = \xi_4,
$$

(6.8)

one finds the following non-trivial commutators:

$$
[B_i, P_k] = i\delta_{ik}M,
$$

$$
[H, B_i] = -iP_i,
$$

$$
[H, P_i] = i\Omega_i^2B_i,
$$

(6.9)

together with

$$
C = MH - \frac{1}{2}(P_1^2 + P_2^2 + \Omega_1^2B_1^2 + \Omega_2^2B_2^2).
$$

(6.10)

Orbits

We can now apply the orbit method (Appendix B). Consider the coadjoint orbit parametrized by $m > 0$, the coordinate in dual space corresponding to the Casimir operator $M$ and interpreted as the mass, and by $\epsilon m$, corresponding to the Casimir operator $C$.
and interpreted as the *internal energy*. Let \( p_i, b_i, h, i = 1, 2 \) be the relevant coordinates in the space dual to the Lie algebra \( (6.9) \). As shown in Appendix B, the points of the coadjoint orbit are parametrized by \( p_i \) and \( b_i \). Defining

\[
q_i = b_i/m,
\]

we find

\[
\{ q_i, p_k \} = \delta_{ik}, \quad h = \left( \frac{p_1^2}{2m} + \frac{m\Omega_2^2}{2} q_1^2 \right) + \left( \frac{p_2^2}{2m} + \frac{m\Omega_2^2}{2} q_2^2 \right) + \epsilon.
\]

Hence, we arrive at an in general anisotropic oscillator, as the most general system with positive definite energy, admitting the symmetry defined by the rotation-less Newton-Hooke commutation relations \( (6.1) \).

**B. X semi-positive**

Let us consider the case when the matrix \( X \) is semidefinite. We restrict ourselves to \( X \) having a single zero eigenvalue (as in the Hill case). Then one can select the matrix \( D \) in \( (6.6) \) in such a way that \( X \) acquires the form

\[
X = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

(6.13)

One can show again (see Appendix A) that the residual freedom in the choice of the basis of our algebra allows us to put \( \omega \) into the form \( (6.7) \). Using again eqns \( (6.8) \), we find therefore

\[
[H, B_1] = -iP_1, \\
[H, P_1] = i\Omega_1^2 B_1, \quad [H, P_2] = 0, \\
C = MH - \frac{1}{2}(P_1^2 + P_2^2 + \omega_1^2 B_1^2)
\]

(6.14)

The orbit method yields, in this case, the dynamics describing a harmonic oscillator in one direction, and free motion in the second one — as in the Hill problem \( (6.3) \).

The case of multiple null eigenvalues of \( X \) can be dealt with similarly.
C. \textit{X indefinite}

Let us drop, finally, the assumption of positive (semi)definiteness of \textit{X}. We consider in more detail the cases of two positive – one negative – one null eigenvalues. By an appropriate choice of \textbf{D} one can achieve

\[
X = \begin{pmatrix}
0 & 0 \\
0 & G
\end{pmatrix}, \quad \text{where} \quad G = \text{diag}(-1, 1, 1).
\]  

(6.15)

According to the results in Appendix A, the symplectic form \(\omega\) can acquire three canonical forms, namely those presented in eqns (A.5) - (A.16) - (A.17). Then the orbit method gives the following dynamical systems:

(i)

\[
\{q_i, p_k\} = \delta_{ik}, \quad h = \frac{p_1^2}{2m} + \left(\frac{p_2^2}{2m} - \frac{m\Omega^2}{2}q_2^2\right) + \epsilon; \tag{6.16}
\]

\[
h = \frac{p_1^2}{2m} + \left(\frac{p_2^2}{2m} - \frac{q_2^2}{2}\right) + \epsilon; \tag{6.17}
\]

(ii)

\[
\{q_i, q_j\} = \sigma \epsilon_{ij}, \quad \{p_i, p_j\} = \tau \epsilon_{ij}, \tag{6.18}
\]

\[
h = \frac{p_1^2}{2m} + \left(\frac{p_2^2}{2m} - \frac{q_2^2}{2}\right) + \epsilon; \tag{6.19}
\]

(iii)

\[
\{q_i, q_j\} = \sigma \epsilon_{ij}, \quad \{q_2, p_2\} = 1, \quad \{p_i, p_j\} = \tau \epsilon_{ij}, \tag{6.20}
\]

\[
h = \frac{p_1^2}{2m} + \left(\frac{p_2^2}{2m} - \tau q_2^2\right) + \epsilon. \tag{6.21}
\]

The parameter \(\sigma\) here is a clear indication of \textit{non-commuting nature} of the coordinates \(q_1\) and \(q_2\).\[41\]

The case of non(semi)definite Hamiltonian is the most involved one. Unlike in the previous cases, after the “canonical” Hamiltonian is fixed, there still remain three inequivalent forms of the basic Poisson brackets.

The reason for that is clearly seen from the derivation given in Appendix A. The \(3 \times 3\) submatrix \(\omega_g\) of the matrix \(\omega\) transforms, under the transformations leaving the form of
the Hamiltonian invariant, as an $O(2, 1)$ antisymmetric tensor. Its canonical form depends therefore on the value of the “electromagnetic” invariant

$$\sum_{i=1}^{2} (\omega_{0i})^2 - (\omega_{12})^2. \quad (6.22)$$

Depending on its value, the basic Poisson brackets can take different, inequivalent forms (assuming the form of Hamiltonian is fixed). The labeling of variables in equations (6.16) - (6.21) is dictated by our preference for the form of the Hamiltonian, rather than that of the Poisson brackets. It must be stressed, however, that the final choice of appropriate variables should be dictated by additional assumptions, not resulting from symmetry considerations only.

As an example, let us consider the planar Hill equations, as presented in Refs. [6, 7]. The Hamiltonian reads

$$H = \frac{1}{2m} (p_1^2 + p_2^2) - \frac{3m\omega^2}{2} q_2^2, \quad (6.23)$$

and yields Hill’s equations for the following Poisson brackets,

$$\{q_i, p_j\} = \delta_{ij}, \quad \{p_i, p_j\} = 2m\omega \epsilon_{ij}, \quad (6.24)$$

where the commutative case, $\sigma = 0$, has been chosen for simplicity. The other parameter is $\tau^2 = 3m^2 \omega^2$, and $B = 2m\omega$ is the effective “magnetic” field. Let us put

$$\xi_1 = \lambda q_1, \quad \xi_2 = \sqrt{3m\omega} q_2, \quad \xi_3 = \frac{p_1}{\sqrt{m}}, \quad \xi_4 = \frac{p_2}{\sqrt{m}} \quad (6.25)$$

with $\lambda \neq 0$ arbitrary. Then $H$ acquires the standard form

$$H = \frac{1}{2} (\xi_3^2 + \xi_4^2 - \xi_2^2), \quad (6.26)$$

and the relevant Poisson brackets read

$$\{\xi_2, \xi_4\} = \sqrt{3} \omega, \quad \{\xi_2, \xi_4\} = 2\omega. \quad (6.27)$$

Therefore one finds, with the notations of Appendix A,

$$\omega_{01} = 0, \quad \omega_{02} = \sqrt{3} \omega, \quad \omega_{12} = 2\omega, \quad \bar{\omega}^2 - \omega_{12}^2 = -\omega^2 < 0. \quad (6.28)$$

According to the classification given in Appendix A, we are dealing with the case (A.14), and the “canonical” form of the Poisson brackets is given by eqn. (A.16), in full agreement with the results of Refs. [6, 7].
D. Additional symmetries

We now study the question of additional symmetries. Consider the case of a positive definite Hamiltonian. As it has been shown in the previous Section, the initial algebra can be put into the form

\[
\begin{align*}
[\xi_i, \xi_j] &= i\omega_{ij}M, \quad (6.29) \\
[H, \xi_i] &= -i\omega_{ij}\xi_j, \quad (6.30) \\
[M, \cdot] &= 0, \quad (6.31)
\end{align*}
\]

where \(\omega\) is given by eqn. (6.7). We add a new generator \(J\) which is assumed to obey

\[
\begin{align*}
[J, M] &= 0, \quad [J, H] = 0, \quad [J, \xi_i] = ij_k\xi_k, \quad (6.32)
\end{align*}
\]

where \(j = (j_{ik})\) is an appropriate matrix. The two additional Jacobi identities

\[
\begin{align*}
[J, [H, \xi_i]] + (\text{cycl}) &= 0 \quad [\xi_i, [J, \xi_j]] + (\text{cycl}) = 0 \quad (6.33)
\end{align*}
\]
yield \(j\omega + \omega j^T = 0, j\omega - \omega j = 0\). Hence \(j = -j^T\), and the general solution reads

(i) \(\Omega_1 \neq \Omega_2\),

\[
\begin{align*}
j &= \alpha \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
-1 & 0 & 0
\end{pmatrix} + \beta \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & -1 & 0
\end{pmatrix}, \quad (6.34)
\end{align*}
\]

(ii) \(\Omega_1 = \Omega_2\),

\[
\begin{align*}
j &= \alpha \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
-1 & 0 & 0
\end{pmatrix} + \beta \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & -1 & 0
\end{pmatrix} + \gamma \begin{pmatrix}
0 & 0 & 1 \\
-1 & 0 & 0 \\
0 & -1 & 0
\end{pmatrix} + \delta \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-1 & 0 & 0
\end{pmatrix}. \quad (6.35)
\end{align*}
\]

Before explaining the meaning of the particular solutions, let us note that, once the equations (6.33) are obeyed, there exists a second Casimir operator, namely

\[
\tilde{C} = MJ - \frac{1}{2}Y_{ij}\xi_i\xi_j, \quad (6.36)
\]
where $Y = -\omega^{-1} j$. If the coadjoint orbit is parametrized by the value $m\tilde{\sigma}$ of the Casimir $\tilde{C}$, eqn. (6.36) yields the expression for $J$ which, as in the case of the Hamiltonian, consists of the sum of the quadratic term plus the “internal” contribution to $J$,

$$mJ = \frac{1}{2}Y_{ij}z_iz_j + m\tilde{\sigma}, \quad (6.37)$$

where the $z$’s are the basic variables parametrizing the points on the orbit (cf. (B.18) in Appendix B). Eqn. (6.37) allows us interpret $\tilde{\sigma}$ as the internal angular momentum, analogous to the internal energy, $\epsilon$, in our previous formulæ, cf. [14]. Using the general solution for $j$, eqns. (6.34) and (6.35), one finds the following generators as functions on the coadjoint orbit (up to an internal part):

(i) for $\Omega_1 \neq \Omega_2$,

$$H_1 = \frac{p_1^2}{2m} + \frac{m\Omega_1^2}{2}q_1^2, \quad H_2 = \frac{p_2^2}{2m} + \frac{m\Omega_2^2}{2}q_2^2; \quad (6.38)$$

(ii) for $\Omega_1 = \Omega_2 = \Omega$,

$$H_1 = \frac{p_1^2}{2m} + \frac{m\Omega^2}{2}q_1^2, \quad H_2 = \frac{p_2^2}{2m} + \frac{m\Omega^2}{2}q_2^2,$$

$$J = q_1p_2 - q_2p_1, \quad Z = p_1p_2 + m^2\Omega^2q_1q_2. \quad (6.39)$$

The meaning of the above expressions is clear. First of all, for an anisotropic oscillator we have two integrals, corresponding to the partial energies; our system is integrable.

For equal frequencies, the dynamics is superintegrable: there are three functionally independent integrals. One can choose the angular momentum as the third one. The four integrals in eqn. (6.39) are linearly independent but they are functionally dependent. Note also that our integrals (6.39), being quadratic in canonical variables, form the $u(2)$ Lie algebra — the well-known dynamical algebra of a two-dimensional isotropic harmonic oscillator. In fact, if one defines

$$V_1 \equiv \frac{1}{2}J, \quad V_2 \equiv \frac{1}{2\Omega}(H_2 - H_1), \quad V_3 \equiv \frac{1}{2m\Omega}(p_1p_2 + m^2\Omega^2q_1q_2), \quad (6.40)$$

the resulting Poisson brackets algebra reads

$$\{V_i, V_j\} = \epsilon_{ijk}V_k, \quad (6.41)$$

i.e., span the $su(2)$ algebra. The fourth generator, namely the Hamiltonian,

$$V_0 \equiv H, \quad (6.42)$$
can also be added \([24]\). \(V_0\) commutes with all other \(V\)'s, completing the \(\text{su}(2)\) algebra into the unitary algebra \(\text{u}(2)\).

Let us remark that even for \(\Omega_1 \neq \Omega_2\) there exists an additional integral, provided the ratio of the frequencies is rational,

\[
\rho = \frac{\Omega_1}{\Omega_2} = \frac{m}{n}.
\]

(6.43)

It is, however, no longer quadratic in the canonical variables, yielding a \(W\)-algebra, instead of a Lie algebra \([31]\). In fact, an additional integral of the motion which yields our system superintegrable can be constructed as follows. One defines the classical counterparts of the creation/annihilation operators by

\[
a_i = q_i - \frac{ip_i}{m\Omega_i}, \quad \bar{a}_i = q_i + \frac{ip_i}{m\Omega_i}.
\]

(6.44)

It is then easy to check that

\[
C^{n,m} = (a_1)^n(a_2)^m
\]

(6.45)

is an integral of the motion. In the isotropic case \(n = m = 1\), for example,

\[
C \equiv C^{1,1} = \frac{Z}{m^2\omega^2} + \frac{i}{m\Omega} L
\]

(6.46)

is a combination of those conserved quantities in the second line of (6.39), namely of the angular momentum and the “mixed” quantity denoted by \(Z\).

The integral \(C^{n,m}\) is functionally independent of the partial energies, \(H_{1,2}\). Moreover, there are no further independent (and explicitly time-independent) integrals; therefore, the Poisson bracket between \(H_{1,2}\) and \(C^{n,m}\) are functionally expressible in terms of them, and form a finite \(W\)-algebra \([32]\).

Let us conclude this section with some remarks. We have shown, at least in the case of (semi)definite hamiltonian, that there exists a unique “canonical” form of the underlying dynamics. However, the choice of this canonical form is dictated by mathematical simplicity rather than by physical requirements which are, in fact, additional assumptions. It seems reasonable to assume, generally, that the physical variables are those which convert the system into (non-commutative) anisotropic oscillator in a uniform magnetic background. This can be always done because our canonical form may be converted back into any other hamiltonian form obeying the symmetry assumptions. Therefore, we end up with Theorem 1, as stated in the Introduction.
7. THE BARGMANN POINT OF VIEW

The NH symmetry of an isotropic oscillator can conveniently be derived by “importing” the Galilei symmetry of a free particle using Niederer’s transformation, which maps every half period of the oscillator onto a free particle \[3, 33\]. One way of seeing this is to work within Duval’s “Bargmann” framework, where classical non-relativistic motions are null geodesics of a suitable relativistic spacetime \[34, 35\]. Null geodesics are invariant w.r.t. conformal transformations, and Niederer’s transformation,

$$T = \frac{\tan \omega t}{\omega}, \quad \vec{X} = \frac{\vec{x}}{\cos \omega t}, \quad S = s - \frac{\omega r^2}{2} \tan \omega t \quad (7.1)$$

maps indeed every half oscillator period conformally onto the space-time which describes a free particle,

$$d\vec{X}^2 + 2dT dS = \frac{1}{\cos^2 \omega t} (dx^2 + 2dtds - \omega^2 r^2 dt^2). \quad (7.2)$$

This trick can not work for an anisotropic oscillator, though, otherwise the latter would also carry a full NH symmetry including rotation.

An anisotropic oscillator is described by the metric \[43\]

$$dx^2 + 2dtds - (\omega_1^2 x_1^2 + \omega_2^2 x_2^2) dt^2. \quad (7.3)$$

Applying Niederer’s transformation \((7.1)\) i.e.

$$t = \frac{\arctan \omega T}{\omega}, \quad \vec{x} = \frac{\vec{X}}{\sqrt{1 + \omega^2 T^2}}, \quad s = S + \frac{1}{2} \frac{\omega^2 X^2 T}{1 + \omega^2 T^2} \quad (7.4)$$

with some \(\omega\) then yields

$$\frac{1}{1 + \omega^2 T^2} \left( d\vec{X}^2 + 2dT dS - \frac{\omega_1^2 - \omega^2}{(1 + \omega^2 T^2)^2} X_1^2 dT^2 - \frac{\omega_2^2 - \omega^2}{(1 + \omega^2 T^2)^2} X_2^2 dT^2 \right).$$

Now choosing either \(\omega = \omega_1\) or \(\omega = \omega_2\) eliminates one, but not both oscillator terms, leaving us with

$$ds^2 = \frac{1}{1 + \omega_2^2 T^2} \left( d\vec{X}^2 + 2dT dS - \frac{\omega_1^2 - \omega_2^2}{(1 + \omega_2^2 T^2)^2} X_1^2 dT^2 \right)$$

$$= \frac{1}{1 + \omega_1^2 T^2} \left( d\vec{X}^2 + 2dT dS - \frac{\omega_2^2 - \omega_1^2}{(1 + \omega_1^2 T^2)^2} X_2^2 dT^2 \right). \quad (7.5)$$

[where we should have put indices 1 or 2 on \(X\), depending on our choice of \(\omega\)]. For both choices we get, hence, a maximally anisotropic “one-sided” “Hill-type” system, with Newton-Hooke symmetry — except in the isotropic case

$$\omega_1 = \omega_2, \quad (7.6)$$
when both oscillator terms drop out, leaving us with a free system carrying its full Galilei symmetry. The latter can then be “re-imported” through the inverse of the Niederer transformation (7.1) to yield full Newton-Hooke symmetry.

In conclusion, the “prototype system” is of the “Hill type”, with its rotation-less Newton-Hooke symmetry — which, in the isotropic case, degenerates to a free particle with restored rotational symmetry.

8. CONCLUSION

Souriau [14] attributes the center-of-mass decomposition of a free non-relativistic system to Galilei symmetry, more precisely, to an invariant Abelian subgroup of it, whose existence is rooted in turn in the cohomology of the Galilei group [14]. Remarkably, it is this same cohomology which rules central extensions [16].

In this paper we performed an analogous study in the Landau problem, based on the Newton-Hooke group. The clue is that Newton-Hooke and Galilean symmetries are indeed “hiddenly the same” [9], and have therefore identical cohomological structures [19].

The intuitive content of Kohn’s theorem, i.e., the relation between [Newton-Hooke] symmetry and center-of-mass, is now clear: each particle, taken individually, would carry such a symmetry; Kohn’s condition is precisely what is needed to extend this symmetry to the center-of-mass, which will hence represent the motion of all particles collectively.

A method for finding approximate solutions of the 3-body problem of Celestial Mechanics, also used in Galactic Dynamics [12], is referred to as the Hill Problem. The latter also has a symmetry reminiscent of the Newton-Hooke one, except for rotations, which are missing.

In Section 4 we applied, for the first time in our knowledge, the center-of-mass decomposition to study of the star escape problem in Hill’s framework. But as mentioned above, the very possibility of such a decomposition relies on the existence of an invariant Abelian subgroup of the symmetry group (which can either be the Galilei group or the rotation-less Newton-Hooke group). Our main result here is to find, conversely, the most general mechanical system with the latter symmetry, namely the anisotropic harmonic oscillator in a uniform magnetic background.

At the technical level, the Hill Problem is a particular case of an anisotropic harmonic oscillator in an effective magnetic field.
In this paper, we performed a similar study for a *general anisotropic harmonic oscillator*. All our investigations here have been purely classical. The decomposition of Newton-Hooke symmetry into Heisenberg algebras is, however, particularly useful for the quantum description, see [25, 26] for details.

**Appendix A**

We find here the canonical form of the $4 \times 4$ antisymmetric nonsingular matrix $\omega$ undergoing the transformation

$$
\omega \to D\omega D^T, \quad \text{where } D \text{ obeys } D^T XD = X,
$$

(A.1)

$X$ being the matrix defined in eqn. (6.3).

As it has been noted in the main text, $X$ can be put into canonical form, which depends on the assumption concerning the eigenvalues of $X$.

Consider first $X$ positive definite; then we can put $X = I$. As a result $D$ is orthogonal and we have to find the canonical form of $\omega$ under $o(4)$ transformations (A.1). This resembles the problem of classifying the electromagnetic field configurations under the Lorentz group $O(3,1)$. Guided by this analogy, we define

$$
\begin{align*}
    f_i &= \omega_{0i}, \\
    g_i &= \frac{1}{2} \epsilon_{jkl} \omega_{jk}.
\end{align*}
$$

(A.2)

Note that $f_i$ and $g_i$ transform like vectors under $SO(3)$ transformations acting on the last three coordinates. Moreover, $\det \omega \sim (f \cdot g)^2$, so that $f \cdot g \neq 0$, i.e., $f \neq 0$, $g \neq 0$ and $\vec{f}$ is not perpendicular to $\vec{g}$.

Let us consider the rotation in the plane spanned by the $O$-axis, and the axis which is orthogonal to it and defined by the unit vector $\vec{n}$. The transformation rules under such a rotation read

$$
\begin{align*}
    \vec{f}'_\parallel &= \vec{f}_\parallel, \\
    \vec{f}'_\perp &= \vec{f}_\perp \cos \varphi + (\vec{n} \times \vec{g}_\perp) \sin \varphi, \\
    \vec{g}'_\parallel &= \vec{g}_\parallel, \\
    \vec{g}'_\perp &= \vec{g}_\perp \cos \varphi - (\vec{n} \times \vec{f}_\perp) \sin \varphi,
\end{align*}
$$

(A.3)

where $\parallel (\perp)$ denotes the component parallel (orthogonal) to $\vec{n}$. If $\vec{f} \parallel \vec{g}$ we put

$$
\vec{n} = \frac{\vec{f} \times \vec{g}}{|\vec{f} \times \vec{g}|} \quad \text{and} \quad \sin 2\varphi = \frac{2|\vec{f} \times \vec{g}|}{f^2 + g^2}
$$

(A.4)
to achieve $f \parallel g$. Then by $SO(3)$ rotation one gets further $f_i = \Omega_1 \delta_{i2}$, $\Omega_1 > 0$, $g_i = -\Omega_2 \delta_{i2}$. Renumbering, if necessary, $1 \leftrightarrow 3$ (which is an $O(3)$ transformation) we let $\Omega_2 > 0$. Due to definition (A.2),

$$\omega = \begin{pmatrix} 0 & f_1 & f_2 & f_3 \\ -f_1 & 0 & g_3 & -g_2 \\ -f_2 & g_3 & 0 & g_1 \\ -f_3 & g_2 & -g_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \Omega_1 & 0 \\ 0 & 0 & 0 & \Omega_2 \\ -\Omega_1 & 0 & 0 & 0 \\ 0 & -\Omega_2 & 0 & 0 \end{pmatrix}. \tag{A.5}$$

Consider next the case of semidefinite $X$ with one null eigenvalue. Then $X$ can be put in the form $X = \begin{pmatrix} 0 & 0 \\ 0 & I_3 \end{pmatrix}$. Put

$$D = \begin{pmatrix} d & A \\ B & U \end{pmatrix}. \tag{A.6}$$

Eqns. (A.1) implies $B = 0$, $U \in O(3)$, so $D$ acquires the form $D = \begin{pmatrix} d & A \\ 0 & U \end{pmatrix}$, $d \neq 0$. Then, with $\omega_{ij} = \varepsilon_{ijk} g_k$,

$$D \omega D^T = \begin{pmatrix} 0 & df^T + A \omega g U^T \\ -dU f^T + U \omega g A^T & U \omega g U^T \end{pmatrix}. \tag{A.7}$$

Here $\omega_g$ is an antisymmetric matrix, so it belongs to the algebra $so(3)$. One can choose therefore $U \in SO(3)$ such that

$$U \omega_g U^T = \begin{pmatrix} 0 & 0 & \Omega_2 \\ 0 & 0 & 0 \\ -\Omega_2 & 0 & 0 \end{pmatrix}, \quad \Omega_2 > 0. \tag{A.8}$$

Consider now the elements $df^T + A \omega g U^T = df^T + A U^T U \omega g U^T$. Call

$$df^T \equiv (\tilde{f}_1, \tilde{f}_2, \tilde{f}_3), \quad AU^T \equiv (\tilde{a}_1, \tilde{a}_2, \tilde{a}_3). \tag{A.9}$$

Then

$$df^T + A U^T U \omega g U^T = (\tilde{f}_1, \tilde{f}_2, \tilde{f}_3) + (\tilde{a}_1, \tilde{a}_2, \tilde{a}_3) \begin{pmatrix} 0 & 0 & \Omega_2 \\ 0 & 0 & 0 \\ -\Omega_2 & 0 & 0 \end{pmatrix}$$

$$= (\tilde{f}_1, \tilde{f}_2, \tilde{f}_3) + (-\Omega_2 \tilde{a}_3, 0, \Omega_2 \tilde{a}_1). \tag{A.10}$$
Knowing \( f, U \), and \( d \) one determines \( \tilde{f}_{1,2,3} \) and chooses \( \tilde{a}_{1,3} \) in such a way that

\[
df U^T + AU^T U \omega g U^T = (0, \tilde{f}_2, 0), \quad \tilde{f}_2 \neq 0.
\]  
(A.11)

By an appropriate choice of \( d \) we get \( 0 < \tilde{f}_2 \equiv \Omega_1 \); so (A.7) acquires the form (A.5).

Finally, let \( X \) have two positive, one negative and one zero eigenvalue. Without losing generality, we put

\[
X = \begin{pmatrix} 0 & 0 \\ 0 & G \end{pmatrix}, \quad \Omega \equiv \text{diag}(-1, 1, 1)
\]  
(A.12)

With \( D \) of the form (A.6) eqn. (A.1) yields \( B = 0, U \in O(2, 1) \); \( D \omega D^T \) has the same form (A.7).

Consider now \( U \omega g U^T \). Again proceeding along the same lines as in the classification of electromagnetic field configurations, we find that \( U \omega g U^T \) can acquire three “canonical” forms:

\[
U \omega g U^T = \begin{pmatrix} 0 & 0 & \Omega_2 \\ 0 & 0 & 0 \\ -\Omega_2 & 0 & 0 \end{pmatrix}, \quad \Omega_2 > O
\]  
(A.13)

\[
U \omega g U^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \Delta \\ 0 & -\Delta & 0 \end{pmatrix}, \quad \Delta > O
\]  
(A.14)

\[
U \omega g U^T = \begin{pmatrix} 0 & 0 & \Sigma \\ 0 & 0 & \Sigma \\ -\Sigma & -\Sigma & 0 \end{pmatrix}, \quad \Sigma \neq O.
\]  
(A.15)

If (A.13) holds the same reasoning as previously leads to eqn. (A.5). In the second case

\[
\omega = \begin{pmatrix} 0 & \Omega & 0 & 0 \\ -\Omega & 0 & 0 & 0 \\ 0 & 0 & 0 & \Delta \\ 0 & 0 & -\Delta & 0 \end{pmatrix}, \quad \Omega > O, \quad \Delta > 0.
\]  
(A.16)
Finally, if \([A.15]\) holds,

\[
\omega = \begin{pmatrix}
0 & \Omega & 0 & 0 \\
-\Omega & 0 & 0 & \Sigma \\
0 & 0 & 0 & \Sigma \\
0 & -\Sigma & -\Sigma & 0
\end{pmatrix}.
\]  

(A.17)

Appendix B

We consider here the orbit method for the Lie algebra \([6.1]\). The general element of the dual space can be written as

\[
h \tilde{H} + m \tilde{M} + z_i \tilde{\xi}^i.
\]  

(B.1)

Consider the coadjoint action of \(g = \exp (iy^k \xi_k)\). It reads

\[
m' = m, \\
z'_i = z_i + \omega_{ki}y^km, \\
h' = h + y^k \omega_{kl}X_{lj}z_j + \frac{1}{2}y^ky^l\omega_{lm}\omega_{kj}X_{mj}m.
\]  

(B.2)

Assuming \(m \neq 0\) and using the fact that \(\omega\) is invertible, we conclude that each orbit contains the points corresponding to \(z_i = 0\). The set of these points forms the coadjoint orbit of the stability subgroup of the relations \(z_i = 0\). However, the latter is generated by \(M\) and \(H\), so the coadjoint orbits are trivial. We conclude that \(z_i = 0\) define exactly one point on coadjoint orbit. Therefore, generating the whole orbit by the action of our group on that point we conclude that the orbit can be parametrized by the variables \(z_i\) and

\[
h = \epsilon + \frac{1}{2m}X_{ij}z_iz_j,
\]  

(B.3)

where \(\epsilon\) is the value of \(h\) at the point \(z_i = 0\) (internal energy). The basic Poisson bracket reads

\[
\{z_i, z_j\} = \omega_{ij}m,
\]  

(B.4)

which completes the description.

The additional symmetry generators can be dealt with in a similar way.

In the case of two degrees of freedom and (semi)definite \(H\) it is convenient to identify the “physical” generators as described by eqns. \([6.8]\) (i.e. to single out the boosts and momenta). In this basis the counterparts of dual coordinates \(z_i\) are denoted by \(p_i\) and \(b_i\) (cf. eqns. \([6.11]\) and \([6.12]\)).
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[36] This is particularly clear if we use the equivalent Lagrangian

$$\tilde{L}_\pm = \frac{1}{2} \epsilon_{ij} \dot{y}_i \dot{y}_j \mp \frac{\omega}{2} y^2_i.$$ 

[37] For the record, \(\Pi\) and \(P\) are related as \(P^i = \Pi^i + M \epsilon^{ij} X^j\).

[38] Note that \(dK^i/dt = \partial_i K^i + \{H, K^i\} = 0\) as it should.

[39] An isotropic harmonic electric force can be freely added, cf. [25].

[40] In the Hill case \(\omega_+ = 0\) and the \(X_+\)-dynamics is free, while \(\omega_- = \omega/\Delta\), cf. [24].

[41] In fact, \(\sigma = \theta/(1 - \theta \tau), \theta = \sigma/(1 + \sigma \tau)\), where \(\theta\) is the non-commutativity parameter.

[42] \(m\) is the eigenvalue of the operator \(M\).

[43] The Bargmann space (7.3) is not conformally flat as its Weyl tensor does not vanish, unless \(\omega_1 = \omega_2\).