Tautological rings of Hilbert modular varieties

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Abstract

We compute the tautological ring for a Hilbert modular variety at an unramified prime. The method generalises that of van der Geer from the Siegel case.

1 Introduction

In this note we shall compute the tautological ring of Hilbert modular varieties in characteristic $p$.

For a Shimura variety $S_K$, its tautological ring $T^\bullet(S_K)$ is the subring of its Chow ring with rational coefficients generated by the Chern classes of automorphic vector bundles. This ring was first defined and studied by van der Geer [vdG99] in the Siegel case and later by Wedhorn-Ziegler [Zie18] for Shimura varieties admitting an embedding into a Siegel Shimura variety (i.e. Hodge-type).

Given an integral canonical model of a Shimura variety, conjectured to exist at hyperspecial level by Milne [Mil92] (and proven to exist for Hodge- and abelian-type by Kisin [Kis10] and Vasiu [Vas99]), one can also define the tautological ring for the reduction of a Shimura variety to characteristic $p$. When suitable toroidal compactifications $S_K^{tor}$ exist there is the folklore conjecture that

$$T^\bullet(S_K^{tor}) \cong H^{2\bullet}(X^\vee(C), \mathbb{Q})$$

where $X^\vee$ is the compact dual. Van der Geer proved this in the Siegel case in all characteristics and Wedhorn-Ziegler proved that it holds for Shimura varieties of Hodge-type in characteristic $p$ (where suitable toroidal compactifications of integral models exist by work of Madapusi-Pera [MP19]). We are interested in the related question:

**Question 1.1.** Is there a similar description of $T^\bullet(S_K)$?

In the Siegel case the answer is yes. Indeed, van der Geer proved that $T^\bullet(A_g) \cong H^{2\bullet}(X_{g-1}^\vee)$, where $X_{g-1}$ is the double Siegel half-space. In this note we compute $T^\bullet(S_K)$ for a Hilbert modular variety in characteristic $p$ (see the beginning of section 5 for a precise statement).
Theorem 1. Let $X_K$ be a Hilbert modular variety at an unramified prime.

$$T^\bullet(X_K) \cong H^2\bullet(X^\vee)/\text{(top degree part)}$$

In particular, the only extra relation is the vanishing of the top Chern class of the Hodge vector bundle pulled back from the Siegel case. It is not clear what a general description for Hodge-type Shimura varieties could be. For example, $S_K$ could itself be compact.

The proof of Theorem 1 generalises the method of van der Geer from the Siegel case, using properness of subvarieties to obtain positivity of certain classes. Pullback gives a surjective map of graded rings

$$H^2\bullet(X^\vee) \cong T^\bullet(S^\text{tor}_K) \longrightarrow T^\bullet(S_K)$$

Our aim is to determine the kernel of this morphism of graded rings. Section 2 recalls the Hilbert modular datum and in section 3 we review Wedhorn-Ziegler’s definition of $T^\bullet(S^\text{tor}_K)$ and computation of $T^\bullet(S^\text{tor}_K)$ for the Hilbert modular datum. In section 4 we choose an explicit Siegel embedding and explain that the pullback of the Hodge vector bundle splits into line bundles whose Chern classes generate $T^\bullet(S^\text{tor}_K)$. This automatically gives a new relation coming from the vanishing of the top Chern class of the Hodge vector bundle in the Siegel case [vdG99, 1.2]. The final section tackles the problem of showing that there are no further relations. This reduces to an analysis of the highest degree part, which is at most $d$-dimensional. There are $d$ elements given by multiples of $d-1$ generators and we show that they are linearly independent. This requires the properness of certain strata closures with a description of their cycle classes and is the novel difficulty in the Hilbert modular case: in the Siegel case the highest degree part is 1-dimensional so it sufficed to find a non-zero class.

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2 Hilbert modular datum

We begin by recalling the Shimura datum which gives a Hilbert modular variety and describing the cocharacter datum one obtains from this in characteristic $p$.

Let $F$ be a totally real field of degree $d > 1$ over $\mathbb{Q}$. Define $\mathbf{G}$ as the algebraic group

$$
\begin{array}{ccc}
\mathbf{G} & \longrightarrow & \mathbb{G}_{m,\mathbb{Q}} \\
\downarrow & & \downarrow \\
\text{Res}_{F/\mathbb{Q}}GL_2 & \longrightarrow & \text{Res}_{F/\mathbb{Q}}\mathbb{G}_{m}
\end{array}
$$

$$
\begin{array}{cc}
\text{det} & \\
\downarrow & \\
\text{Res}_{F/\mathbb{Q}}\mathbb{G}_{m}
\end{array}
$$

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Define $X$ as the $G(R)$-conjugacy class of the cocharacter $h: S \to G_R$ given on $R$-points $C^\times \to GL_2(R)^d$ by
\[
x + iy \mapsto \begin{pmatrix} x & y \\ y & -x \end{pmatrix}_{i \leq d}
\]

$(G, X)$ is a Shimura datum, henceforth referred to as the Hilbert modular datum given by $F$. The associated Shimura variety is the Hilbert modular variety and is of PEL-type. Indeed, Raynaud constructed a model over $\mathbb{Z}$ as a moduli space of principally-polarised abelian varieties with real multiplication by $O_F$.

**Definition 2.1.** Let $p$ be a rational prime unramified in $F$. Fix an algebraic closure $k$ of $\mathbb{F}_p$ and a hyperspecial level $K = K^pK_p$. Denote $X_K$ the special fibre of the integral model of the Hilbert modular variety at level $K$ and $X_{K, k}$ its base change to $k$. Fix a smooth proper toroidal compactification $X_{K, \text{tor}}$, which exists by [MP19].

Let $G$ be the special fibre of a smooth reductive $\mathbb{Z}_p$-model of $G$. We obtain a cocharacter datum $(G, \mu)$ as in [GK19, 1.2.2]. If $p$ is inert then $G$ is the fibre product
\[
G \xrightarrow{\text{Res}_{F_p d/F_p}} GL_2 \xrightarrow{\text{det}} G_{m, F_p}
\]
and if $p$ splits then $G$ is the fibre product
\[
(GL_2, F_p)^d \xrightarrow{\text{det}} (G_{m, F_p})^d
\]

Fix an embedding $F_p \hookrightarrow k$ so we can identify $\{\tau: F_p \hookrightarrow k\} \cong \mathbb{Z}/d\mathbb{Z}$. For any unramified prime, the group $G_{F_p d}$ can be described as the $d$-tuples in $GL_2^d$ which have the same determinant in each factor. Any field extension $F_{p^d} \subset K$ contains the field of definition $\kappa = F_p$ of $\mu$ and

$\mu_K: G_{m, K} \to G_K$

is given by $t \mapsto \left( \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \right)_{i \leq d}$, $t$.

Let $P$ be the parabolic subgroup associated with $\mu$ given on $k$-valued points $P(k) = \{ \begin{pmatrix} a_i & b_i \\ 0 & d_i \end{pmatrix}, a, d \}$ $\subset G(k)$ (this is called $P_{\mu}$ in [Zie18], $P^-$ in [GK19]). This has a Levi subgroup $L = Cent(\mu) = \{ \begin{pmatrix} a_i & 0 \\ 0 & d_i \end{pmatrix}, a, d \}$ which is a maximal torus $T$. 3
3 Tautological ring of a toroidal compactification

In this section we review the computation of \( T^\bullet(X_{K,K}^{\text{tor}}) \).

For a Hodge-type Shimura variety \( S_K \) in characteristic \( p \), Wedhorn-Ziegler [Zie18] proved that \( T^\bullet(S_{K}^{\text{tor}}) \cong A^\bullet(G-\text{Zip}^\mu) \), where \( G-\text{Zip}^\mu \) is the stack introduced in [PWZ15] defined from the cocharacter datum \((G, \mu)\) and classifying certain semi-linear algebraic data. In characteristic \( p \) the morphism \( \sigma: S_K \to [P\setminus\ast] \) given by the \( P \)-torsor on \( S_K \) appearing in the \( G \)-zip \( H^{1\text{dR}} \) factors as:

\[
\begin{array}{ccc}
S_{K}^{\text{tor}} & \xrightarrow{\zeta^{\text{tor}}} & G-\text{Zip}^\mu \\
\downarrow j & & \downarrow \beta \\
S_K & \xrightarrow{\zeta} & [P\setminus\ast]
\end{array}
\]

**Definition 3.1.** Let \( S_K \) be a Hodge-type Shimura variety in characteristic \( p \). Its tautological ring is defined \( T^\bullet(S_K) := \text{Im}(\sigma^*) = \text{Im}(\zeta^* \circ \beta^*) \subseteq A^\bullet(S_K) \). Similarly, for a toroidal compactification \( T^\bullet(S_{K}^{\text{tor}}) := \text{Im}(\zeta_{\text{tor},*} \circ \beta^*) \).

In general, \( \zeta \) is smooth ([Zha18, 3.1.2]) while smoothness of \( \zeta_{\text{tor}} \) is expected to follow from [LS18] (and is known in the PEL case). Wedhorn-Ziegler proved that \( \beta^* \) is surjective and \( \zeta_{\text{tor,*}} \) is injective, giving \( T^\bullet(S_{K}^{\text{tor}}) := \text{Im}(\zeta_{\text{tor,*}} \circ \beta^*) \cong A^\bullet(G-\text{Zip}^\mu) \).

Now let \((G, \mu)\) be the cocharacter datum associated with \( X_K \) introduced in section 2. Then \( \pi: G-\text{ZipFlag}^\mu \to G-\text{Zip}^\mu \) is an isomorphism, where \( G-\text{ZipFlag}^\mu \) is the stack of zip flags defined in [GK19, 2.1]. Thus, on Chow rings

\[
A^\bullet(G-\text{Zip}^\mu_k) \cong A^\bullet(G-\text{ZipFlag}^\mu_k)
\]

The Weyl group \( W = W_{G,T} = \{ \pm 1 \}^{\mathbb{Z}/d\mathbb{Z}} \) acts on the abelian group \( X^\ast(T_k) \) via its action on \( T_k \). This extends to an action on the symmetric algebra \( S := \text{Sym}(X^\ast(T_k) \otimes_{\mathbb{Z}} \mathbb{Q}) \) and we denote \( I = S_I^W \). Then [Zie18, 8.2] gives

\[
A^\bullet(G-\text{ZipFlag}^\mu_k) = S/IS \cong \mathbb{Q}[z_1, \ldots, z_d]/(z_1^2, \ldots, z_d^2)
\]

These generators \( z_i \in S \) are given by characters

\[
z_i: \left( \begin{array}{cc}
a_j & 0 \\
0 & d_j
\end{array} \right) \to d_i
\]

in \( X^\ast(T_k) \). Thus,

\[
T^\bullet(X_{K,K}^{\text{tor}}) \cong \mathbb{Q}[z_1, \ldots, z_d]/(z_1^2, \ldots, z_d^2)
\]
Remark 3.1. This description is essentially independent of the prime. However, one has an action of Frobenius $\sigma$ on $S$ and this depends on the prime. For example, if $p$ is inert then $\sigma \cdot z_i = z_{i+1}$ for all $i$, whereas if $p$ is split then the action is trivial. Use $\sigma$ to also refer to the corresponding permutation of \{1, \ldots, d\}.

Remark 3.2. It is possible to interpret the relation $z_i^2$ geometrically. In the notation of section 5, there is a now-where vanishing section in $H^0(\mathcal{Z}_{w}, L_i \otimes L^p(\sigma(i)))$ giving $0 = c_1(L_i \otimes L^p(\sigma(i))|_{\mathcal{Z}_{w}}) = (z_i + pz_{\sigma(i)})(z_i - pz_{\sigma(i)})$ for each $i$.

4 A Siegel embedding

Notation. Denote $x_i := j^*(z_i) \in T^*(X_{K,k})$. Denote $V(\rho)$ the vector bundle on $[G\backslash \ast]$ given by a representation $\rho$ of $G$.

In this section we prove that the relation $x_1 \ldots x_d = 0$ holds in $T^*(X_{K,k})$. Fixing a Siegel embedding gives a description of $z_i$ in terms of the Hodge vector bundle. Namely, the pullback of the Hodge vector bundle $E$ splits on $X_{K,k}$ into line bundles $\omega_i$ with $x_i = c_1(\omega_i)$. The relation follows because $c_d(E) = 0$ \cite[1.2]{vdG99}. Note that we have made a judicious choice of embedding to give precisely $x_i = c_1(\omega_i)$; this is simply to ease computation. We will work with the following explicit description of the general symplectic group involved in the Siegel Shimura datum.

Example 4.1. Consider the symplectic vector space $V = F_{2d}$ with symplectic form given by $J := \begin{pmatrix} 0 & -I_d \\ I_d & 0 \end{pmatrix}$. Then $GSp(V) = \{M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid M^TJM = \lambda J \text{ for some } \lambda \in k\}$. Let $A_{d,K}$, the moduli of principally-polarised abelian varieties with level $K$ hyperspecial at $p$. Then as in section 3 there is a morphism $\sigma_d: A_{d,K} \to GSp(V) - \text{Zip} \to [P_d\backslash \ast]$.

The Siegel parabolic $P_d = \{\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}\}$ has a Levi quotient isomorphic to $GL_d \times \mathbb{G}_m$. The Hodge vector bundle on $A_{d,K}$ is $E := \sigma_d^*V(\rho_d)$ given by the representation $\rho_d: P_d \to GL_d \times \mathbb{G}_m \xrightarrow{p_1} GL_d$. Explicitly

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \mapsto D$$

Now consider the standard embedding $\iota: G \hookrightarrow GSp(V) = GSp_{2d}$. This can be described by "embedded squares" over $k$:

$$\begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}, \lambda) \mapsto \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where $A = \text{diag}(a_1, \ldots, a_d)$, $B = \text{diag}(b_1, \ldots, b_d)$ etc. The parabolic $P \subset G$ maps into the Siegel parabolic $P_d \subset GSp(V)$.
We can explicitly study the pullback of the Hodge vector bundle to $X_{K,k}$.
The following diagram commutes, giving $\iota^*E = \sigma^*V(\rho)$.

\[
\begin{array}{ccc}
A_{d,K'} & \longrightarrow & [P_d^\ast] \\
\uparrow & & \uparrow \\
X_{K,k} & \longrightarrow & [P^\ast]
\end{array}
\]

Explicitly, $\rho: P \to P_d \to GL_d$ is given by

\[
\begin{pmatrix}
a_i & b_i \\
0 & d_i
\end{pmatrix} \mapsto D
\]

where $D = \text{diag}(d_1, \ldots, d_d)$. Consider the characters of $P$ given by

\[
\rho_j: \begin{pmatrix}
a_i & b_i \\
0 & d_i
\end{pmatrix} \mapsto d_j
\]

Then $\rho = \rho_1 \oplus \ldots \oplus \rho_d$ and so $V(\rho) \simeq V(\rho_1) \oplus \ldots \oplus V(\rho_d)$ as vector bundles on $[P^\ast]$. Considering the commutative diagram

\[
\begin{array}{ccc}
X^\ast(P) & \longrightarrow & X^\ast(T) \\
\downarrow & & \downarrow \\
A^\ast([P^\ast]) & \longrightarrow & S \longrightarrow S/IS \longrightarrow T^\ast(X_{K,k})
\end{array}
\]

we see that $x_i = \sigma^*c_1(V(\rho_i))$.

**Proposition 4.1.** $x_1 \cdot \ldots \cdot x_d = 0$ in $T^\ast(X_{K,k})$.

**Proof.** On $X_{K,k}$ we have $\iota^*E = \sigma^*V(\rho)$ so

\[
x_1 \ldots x_d = c_d(\sigma^*V(\rho)) = c_d(\iota^*E) = \iota^*c_d(E) = 0
\]

with the last equality given in [vdG99, 1.2].

### 5 Positivity of classes: proof of Theorem 1

In this section we prove that there are no further relations.

**Theorem 1.** Let $F$ be a totally real field, $p$ an unramified prime, $k$ an algebraic
closure of $\mathbb{F}_p$, $K$ a hyperspecial level and $X_{K,k}$ the Hilbert modular variety at
level $K$ in characteristic $p$.

\[
T^\ast(X_{K,k}) = \frac{\mathbb{Q}[x_1, \ldots, x_d]}{(x_1^2, \ldots, x_d^2, x_1 \cdot \ldots \cdot x_d)}
\]

The proof of Theorem 1 follows from:
Proposition 5.1. Let $p$ be a prime unramified in $F$. The $d$ classes 

$$\gamma_i := x_1 \ldots \hat{x}_i \ldots x_d$$

are linearly independent in $A^{d-1}(X_{K,k})$.

The closures of Ekedahl-Oort strata $\overline{X_{K,w}} = \zeta^{-1}(\mathbb{Z}_w)$ give cycles in $T^\bullet(X_K)$ [Zie18 6.1]. The codimension 1 strata are those indexed by $w_i = (\epsilon_j) \in \{\pm\}^d$ with $\epsilon_j = (-1)^{\delta_{ij} + 1}$ (length $d-1$). There are sections $s_i$ of $L_i \otimes L_{\sigma(i)}^p$, where $L_i := \beta^*V(\rho_i)$ on $\text{G-Zip}^\mu$, such that $Z(s_i) = \overline{Z_{w_i}}$. To be precise, there is a separating system of partial Hasse invariants for the ordinary locus $Z_{\text{ord}}$ as defined in [GK18 3.4.2]. The divisors $D_i := Z(\zeta^*s_i) = \overline{X_{K,w_i}}$ are the Goren-Oort strata closures defined and shown to be smooth and proper in [GO00] and [Gor01]. Indeed, one can see by the modular description of $X_{K}^{\text{tor}}$ that $X_{K,w_i}$ doesn’t intersect the toroidal boundary: a semi-abelian variety with real multiplication is either abelian or ordinary.

Proof of Proposition 5.1. Denote $\gamma_{i}^{\text{tor}} := z_1 \ldots \hat{z}_i \ldots z_d$ in $A^{d-1}(X_{K,k}^{\text{tor}})$ so that $\gamma_i = j^*\gamma_{i}^{\text{tor}}$. We obtain the following commutative diagram.

\[
\begin{array}{ccc}
X_{K,k}^{\text{tor}} & \xrightarrow{\zeta} & \text{G-Zip}^\mu_k \\
\downarrow D_i & & \downarrow \\
X_{K,k} & \xrightarrow{f_i \circ j} & \text{G-Zip}^\mu_k
\end{array}
\]

Assume that $\sum_{i=1}^d a_i \gamma_i = 0$ in $A^{d-1}(X_{K,k})$. For each $i$ the projection formula [Ful98 2.6(b)] for the morphism $f_i \circ j$ applied to the class $\sum_{i=1}^d a_i \gamma_{i}^{\text{tor}}$ in $A^{d-1}(X_{K,k}^{\text{tor}})$ gives

\[(f_i \circ j)_*(f_i \circ j)^* \sum_{i=1}^d a_i \gamma_{i}^{\text{tor}} = [D_i]. \sum_{i=1}^d a_i \gamma_{i}^{\text{tor}}\]

\[(f_i \circ j)^* \sum_{i=1}^d a_i \gamma_{i}^{\text{tor}} = f_i^* \sum_{i=1}^d a_i \gamma_i = 0\] by assumption. Since $[D_i] = \zeta_{i}^{\text{tor} \circ \sigma} \cdot \overline{Z_i} = z_i - pz_{\sigma(i)}$ this implies that for all $i \in \mathbb{Z}/d\mathbb{Z}$

\[0 = [D_i]. \sum_{i=1}^d a_i \gamma_{i}^{\text{tor}}\]

\[= (z_i - pz_{\sigma(i)}) . \sum_{i=1}^d a_i \gamma_{i}^{\text{tor}}\]

\[= (a_i - pa_{\sigma(i)}) (z_1 \ldots z_d)\]
Since \( \text{deg}(z_1 \ldots z_d) = d > 0 \) we have that \( a_i - pa_{\sigma(i)} = 0 \) for all \( i \). It remains to show that the only solution is \( a_i = 0 \) for all \( i \). For \( p \) inert, \( \sigma(i) = i + 1 \) (modulo \( d \)) and this system of linear equations is given by the circulant matrix

\[
A := \begin{pmatrix}
1 & -p & 0 & \cdots & 0 \\
0 & 1 & -p \\
\vdots & \ddots & \ddots \\
-p & \cdots & 1 & -p \\
-p & 1 & 1 \\
\end{pmatrix}
\]

A finite sequence of row operations gives an upper triangular matrix with diagonal entries \( 1, \ldots, 1 \) for some \( n \neq 0 \). The determinant of this matrix is non-zero, so the only solution to the system of linear equations is \( a_1 = \ldots = a_d = 0 \).

A similar argument yields the other cases. Hence, \( \gamma_1, \ldots, \gamma_d \) are linearly independent. \( \square \)

The proof of Theorem 1 can now be completed by showing that there are no relations in lower degrees (below \( d \)).

**Proof of Theorem 1.** \( T^\bullet(X_{K,k}) \) is a quotient of \( \frac{\mathbb{Q}[x_1,\ldots,x_d]}{(x_1^2,\ldots,x_d^2,x_1 \cdots x_d)} \). The degree of the generators \( x_i \) is 1 so the maximum degree is degree \( d - 1 \). Suppose that there is a relation

\[
\sum_{I \subseteq \{1,\ldots,d\}, |I| = l} a_I z_I = 0
\]

in \( T^l(X_{K,k}) \), indexed over multisets, with \( l < d \). For any \( I_0 \) there exists some \( i \notin I_0 \). Let \( J = \{1,\ldots,i-1,i+1,\ldots,d\} \setminus I_0 \). Multiplying by \( z_J \) gives \( \sum_I a_I z_{I \cup J} = 0 \) in \( T^{d-1}(X_{K,k}) \). This can be rewritten uniquely as \( \sum_I a_I z_{I \cup J} = \sum b_I \gamma_I \) by Proposition 5.1. If \( i \) is not in \( I \cup J \) then either \( I = I_0 \) or there is a repeated index so \( z_{I \cup J} = 0 \). Hence, \( a_{I_0} \) is the only potentially non-zero coefficient of \( \gamma_i \) and by linear independence in \( T^{d-1}(X_{K,k}) \) we have \( a_{I_0} = 0 \). Since this holds for each \( I_0 \) we have that the relation is trivial.

Thus, there are no relations in lower degree and \( T^\bullet(X_{K,k}) \cong \frac{\mathbb{Q}[x_1,\ldots,x_d]}{(x_1^2,\ldots,x_d^2,x_1 \cdots x_d)} \).

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