DISPERSE ESTIMATE FOR THE SCHRODINGER EQUATION WITH
POINT INTERACTIONS

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ABSTRACT. We consider the Schrödinger operator in $\mathbb{R}^3$ with $N$ point interactions placed at $Y = (y_1, \ldots, y_N)$, $y_j \in \mathbb{R}^3$, of strength $\alpha = (\alpha_1, \ldots, \alpha_N)$, $\alpha_j \in \mathbb{R}$. Exploiting the spectral theorem and the rather explicit expression for the resolvent we prove the (weighted) dispersive estimate for the corresponding Schrödinger flow.

In the special case $N = 1$ the proof is directly obtained from the unitary group which is known in closed form.

1. INTRODUCTION

In this paper we study the time dependent Schrödinger equation in $\mathbb{R}^3$ with a finite number of point interactions and, in particular, we shall prove a dispersive estimate for the solution.

At a formal level the Schrödinger operator with point interactions can be written as

\begin{equation}
H = -\Delta + \sum_{j=1}^{N} \mu_j \delta_{y_j}
\end{equation}

where $\delta_{y_j}$ is the Dirac measure placed at $y_j \in \mathbb{R}^3$ and the parameters $\mu_j$ are coupling constants.

As a matter of fact the Dirac measure in $\mathbb{R}^3$ is not a small perturbation of the Laplacian, even in the sense of quadratic forms.

As a consequence a self-adjoint Hamiltonian in $L^2(\mathbb{R}^3)$ cannot be defined as the sum of a kinetic plus an interaction part and this explains why (1.1) is only a formal expression.

In order to obtain a rigorous counterpart of (1.1) one considers the following restriction of the free Laplacian

\begin{equation}
\hat{H} = -\Delta, \quad D(\hat{H}) = C^\infty_0(\mathbb{R}^3 \setminus Y)
\end{equation}

where $Y = (y_1, \ldots, y_N)$. The operator (1.2) is symmetric but not self-adjoint in $L^2(\mathbb{R}^3)$ and, obviously, one possible self-adjoint extension is trivial, i.e. it coincides with the free Laplacian $H_0 = -\Delta$, $D(H_0) = H^2(\mathbb{R}^3)$.

Using the theory of self-adjoint extensions of symmetric operators, developed by von Neumann and Krein, one can show that the operator (1.2) has $N^2$ (non trivial) self-adjoint extensions which, by definition, are all the possible Schrödinger operators with point interactions at $Y$ (for a comprehensive treatment we refer to the monograph [1]).

Any such extension can be considered as a Laplace operator with a singular boundary condition satisfied at each point $y_j \in Y$.

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In the following we shall only consider the case of local boundary conditions which are more relevant from the physical point of view. More precisely, we shall restrict to the self-adjoint extensions $H_{\alpha,Y}$ parametrized by $\alpha = (\alpha_1, \ldots, \alpha_N), \alpha_j \in \mathbb{R},$ and corresponding to the singular boundary condition at $Y$

\begin{equation}
\lim_{r_j \to 0} \left[ \frac{\partial (r_j u)}{\partial r_j} - 4\pi \alpha_j (r_j u) \right] = 0, \quad r_j = |x - y_j|, \quad j = 1, \ldots, N
\end{equation}

In the next section we shall give the precise definition and the main properties of the Hamiltonian $H_{\alpha,Y}$.

We only notice that the domain of $H_{\alpha,Y}$ contains functions with a singularity of the type $|x - y_j|^{-1}$ at each point $y_j \in Y$ and this explains why in the boundary condition (1.3) the behaviour of the function $u$ near $y_j$ must be regularized.

We also recall that the physical meaning of the parameters $\alpha_j$ is connected with the scattering length of the scatterer in $y_j$ and their relation with the formal coupling constants $\mu_j$ introduced in (1.1) can be understood via a suitable renormalization procedure (see e.g. [1]).

In the following we shall be concerned with the unitary evolution group generated by $H_{\alpha,Y}$

\begin{equation}
e^{itH_{\alpha,Y}} f
\end{equation}

which gives the solution of the Cauchy problem

\begin{equation}
iu_t + H_{\alpha,Y} u = 0, \quad u(0, x) = f(x).
\end{equation}

It is well known that for the free Schrödinger group in dimension three the following dispersive estimate holds

\begin{equation}
\| e^{-it\Delta} f \|_{L^\infty(\mathbb{R}^3)} \leq \frac{C}{t^{3/2}} \| f \|_{L^1(\mathbb{R}^3)}
\end{equation}

The estimate (1.6) can be generalized to smooth perturbation of the Laplacian, i.e. the l.h.s. can be replaced by $P_{ac} e^{it(-\Delta + V)} f$, where $V$ is a smooth real-valued potential satisfying some decay condition at infinity, $P_{ac}$ is the projection onto the absolutely continuous spectrum of $-\Delta + V$ and, moreover, it is assumed that zero is not an eigenvalue nor a resonance for $-\Delta + V$ (see e.g. [3], [4] [6], [7] and the references therein).

Here we want to extend the validity of the dispersive estimate to the case of the Schrödinger equation with point interactions.

Due to the presence of the (unavoidable) singularity at the points where the interaction is placed, we are forced to introduce the following weight function

\begin{equation}
w(x) = \sum_{j=1}^{N} \left( 1 + \frac{1}{|x - y_j|} \right).
\end{equation}

Moreover, for $z \in \mathbb{C}$, we define the matrix

\begin{equation}
[\Gamma_{\alpha,Y}(z)]_{j,\ell} = \left[ \left( \alpha_j - \frac{iz}{4\pi} \right) \delta_{j,\ell} - \tilde{G}_z(y_j - y_\ell) \right]_{j,\ell=1}^{N},
\end{equation}
The role of the matrix $\Gamma_{\alpha,Y}(z)$ for the characterization of the main properties of $H_{\alpha,Y}$ will be explained in the next section. Here we only notice that our basic assumption is the invertibility of $\Gamma_{\alpha,Y}(\mu)$, for $\mu \in [0, +\infty)$. Such assumption in particular implies that zero is not an eigenvalue nor a resonance for $H_{\alpha,Y}$.

In the special case $N = 1$, i.e. for the Schrödinger operator $H_{\alpha,y}$ with a point interaction in $y \in \mathbb{R}^3$ of strength $\alpha \in \mathbb{R}$, the assumption simply means $\alpha \neq 0$, which is precisely the condition for the absence of a zero-energy resonance.

Our main result is the following theorem.

**Theorem 1.1.** Assume that the matrix $\Gamma_{\alpha,Y}(\mu)$ is invertible for $\mu \in [0, +\infty)$ with a locally bounded inverse. Then the following dispersive estimate holds

$$\|w^{-1}e^{itH_{\alpha,Y}}P_{ac}f\|_{L^\infty(\mathbb{R}^3)} \leq C \frac{t^{3/2}}{t^{1/2}} \|w \cdot f\|_{L^1(\mathbb{R}^3)}$$

for any $f \in L^2(\mathbb{R}^3)$ such that $w \cdot f \in L^1(\mathbb{R}^3)$.

In the special case $N = 1$, estimate (1.10) holds for all $\alpha \neq 0$; moreover, when $\alpha > 0$ the projection $P_{ac}$ can be replaced by the identity. Finally, in the resonant case $\alpha = 0$ we have the slower decay estimate

$$\|w^{-1}e^{itH_{0,Y}}f\|_{L^\infty(\mathbb{R}^3)} \leq C \frac{t^{1/2}}{t^{1/2}} \|w \cdot f\|_{L^1(\mathbb{R}^3)}$$

We remark that the result for $N = 1$ is more detailed due to the fact that the unitary propagator is explicitly known (see e.g. [5], [2]) and then the estimate can be obtained by a straightforward computation.

The rest of the paper is organized as follows.

In section 2 we recall the precise definition and the main properties of the Hamiltonian $H_{\alpha,Y}$.

In section 3 we give the proof of the theorem in the special case $N = 1$, exploiting the knowledge of the kernel of the Schrödinger group.

In section 4 we give the proof in the general case using the spectral calculus and the explicit expression for the resolvent of $H_{\alpha,Y}$.

**2. The Schrödinger operator with point interactions**

In this section we review the definition and some basic properties of the Schrödinger operator with point interactions, referring to [1] for more details. We start considering the perturbation of $-\Delta$ by a zero-range (singular) potential supported at the origin of $\mathbb{R}^3$. As we pointed out in the introduction, such perturbation is by definition a non trivial self-adjoint extension of the symmetric operator (1.2) when $N = 1$.

We recall that the trivial extension $-\Delta$ is the (self-adjoint) Laplace operator in $L^2(\mathbb{R}^3)$ (with Lebesgue measure), with domain $D(-\Delta) = H^2(\mathbb{R}^3) = W^{2,2}(\mathbb{R}^3)$, where $W^{p,k}(\mathbb{R}^3)$ denotes the Sobolev space of all functions belonging to $L^p(\mathbb{R}^3)$ whose weak derivatives of order smaller or equal to $k$ belong to $L^p(\mathbb{R}^3)$.

Using the theory of self-adjoint extensions, one can show that the non trivial self-adjoint extensions $H_{\alpha}$, parametrized by $\alpha \in \mathbb{R}$, can be completely characterized.
Indeed, the domain $\mathcal{D}(H_\alpha)$ consists of all elements $\psi \in L^2(\mathbb{R}^3)$ of the type

$$
(2.1) \quad \psi(x) = \phi_\alpha(x) + \left(\alpha - \frac{i\xi}{4\pi}\right)^{-1} \phi_\alpha(0) \frac{e^{-iz|x|}}{4\pi|x|}
$$

where $\phi_\alpha \in H^2(\mathbb{R}^3)$ and $\Im z > 0$. The decomposition (2.1) is unique and with $\psi \in \mathcal{D}(H_\alpha)$ one has

$$
(2.2) \quad (H_\alpha - z^2)\psi = (-\Delta - z^2)\phi_\alpha.
$$

We notice that the trivial extension is recovered in the limit $\alpha \to \infty$.

A remarkable property of the Hamiltonian $H_\alpha$ is that the integral kernel of the resolvent $R_\alpha(z) = (H_\alpha - z)^{-1}$, $\Im z > 0$, can be explicitly computed by the Krein’s formula. In fact

$$
(2.3) \quad (R_\alpha(z^2)f)(x) = (R_0(z^2)f)(x) + \left(\alpha - \frac{i\xi}{4\pi}\right)^{-1} \frac{e^{-iz|x|}}{4\pi|x|} \int_{\mathbb{R}^3} \frac{e^{iz|y|}}{4\pi|y|} f(y)dy.
$$

where the free resolvent is given by

$$
(R_0(z^2)f)(x) = \int_{\mathbb{R}^3} \frac{e^{iz|x-y|}}{4\pi|x-y|} f(y)dy,
$$

By an elementary computation we obtain that for any $\lambda \in \mathbb{R}$ and $\varepsilon > 0$

$$
(2.4) \quad (R_0(\lambda \pm i\varepsilon)f)(x) = \int_{\mathbb{R}^3} \frac{e^{\pm i\sqrt{\lambda}|x-y|}}{4\pi|x-y|} e^{-\varepsilon|x-y|/2\sqrt{\lambda}} f(y)dy,
$$

where

$$
(2.5) \quad \lambda_\varepsilon = \lambda + \frac{\lambda^2 + \varepsilon^2}{2} > 0.
$$

From the last inequality it is easy to derive the limiting absorption principle for the free resolvent, i.e. for $\lambda > 0$

$$
(2.6) \quad (R_0(\lambda + i0)f)(x) = \int_{\mathbb{R}^3} \frac{e^{i\sqrt{\lambda}|x-y|}}{4\pi|x-y|} f(y)dy.
$$

We can also deduce the limiting absorption principle for the resolvent (2.3)

$$
(2.7) \quad (R_\alpha(\lambda + i0)f)(x) = (R_0(\lambda + i0)f)(x) + \left(\alpha - \frac{i\sqrt{\lambda}}{4\pi}\right)^{-1} \frac{e^{-i\sqrt{\lambda}|x|}}{4\pi|x|} \int_{\mathbb{R}^3} \frac{e^{i\sqrt{\lambda}|y|}}{4\pi|y|} f(y)dy.
$$

The spectral properties of $H_\alpha$ are easily derived from (2.3). The continuous spectrum of $H_\alpha$ is purely absolutely continuous and covers the nonnegative real axis $[0, \infty)$ while the point spectrum is empty if $\alpha \geq 0$ and $\{(-4\pi\alpha)^2\}$ if $\alpha < 0$. The normalized eigenfunction associated with the only negative eigenvalue is given by

$$
(2.8) \quad \Psi_\alpha(x) = \sqrt{-2\alpha} \frac{e^{4\pi\alpha|x|}}{|x|}, \quad \alpha < 0.
$$

For $\alpha = 0$ the Hamiltonian has a zero-energy resonance.

In analogous way we introduce the properties of the Schrödinger operator $H_{\alpha,Y}$ with point interactions located at $Y = (y_1, \ldots, y_N)$ with strength $\alpha = (\alpha_1, \ldots, \alpha_N)$.

The domain $\mathcal{D}(H_{\alpha,Y})$ is the set of all functions $\psi \in L^2(\mathbb{R}^3)$ of following type

$$
(2.9) \quad \psi(x) = \phi_\alpha(x) + \sum_{j=1}^N a_j \frac{e^{-iz|x-y_j|}}{4\pi|x-y_j|}, \quad x \in \mathbb{R}^3 \setminus Y,
$$
where 

$$a_j = \sum_{i=1}^{N} [\Gamma_{\alpha,Y}(z)]_{j,i}^{-1} \phi_z(y), \quad j = 1, \ldots, N,$$

$$\phi_z \in H^2(\mathbb{R}^3), \ \exists \ z > 0$$

and the matrix $\Gamma_{\alpha,Y}(z)$ has been defined in (1.8). The decomposition (2.9) is unique and with $\psi \in \mathcal{D}(H_{\alpha,Y})$ we obtain

$$(H_{\alpha,Y} - z^2)\psi = (-\Delta - z^2)\phi_z.$$ 

It is an easy computation to verify that each element of $\mathcal{D}(H_{\alpha,Y})$ satisfies the local boundary condition (1.3).

Also in the $N$-centers case the resolvent of these operators is explicitly given by Krein’s formula. In fact $R_{\alpha,Y}(z) = (H_{\alpha,Y} - z)^{-1}$ is given by the following formula for $\exists \ z > 0$

$$R_{\alpha,Y}(z^2)f(x) = (R_0(z^2)f)(x) + \sum_{j,l=1}^{N} \frac{e^{-iz|x-y|} e^{-iz|x-y|}}{4\pi|x-y|} \int_{\mathbb{R}^3} \frac{e^{iz|y-y'|} f(y)dy}{4\pi|y-y'|}.$$

Moreover we deduce the validity of the limiting absorption principle, i.e. for $\lambda > 0$

$$R_{\alpha,Y}(\lambda + i0)f(x) = (R_0(\lambda + i0)f)(x) + \sum_{j,l=1}^{N} \frac{e^{-i\sqrt{\lambda}|x-y|} e^{-i\sqrt{\lambda}|y-y'|}}{4\pi|x-y|} \int_{\mathbb{R}^3} \frac{e^{i\sqrt{\lambda}|y-y'|} f(y)dy}{4\pi|y-y'|}.$$

We conclude summarizing the spectral properties of $H_{\alpha,Y}$.

The essential spectrum of $H_{\alpha}$ is purely absolutely continuous and coincides with the real axis $[0, \infty)$, indeed the singular continuous spectrum is empty. Moreover, the operator $H_{\alpha,Y}$ has the point spectrum included in $(0, -\infty)$, i.e. there are no positive embedded eigenvalues; in particular $H_{\alpha,Y}$ has at most $N$ (negative) eigenvalues counting multiplicity. In addition, the eigenvalues can be determined computing the zeros of the determinant of an $N \times N$ matrix, i.e.

$$z^2 \in \sigma_p(H_{\alpha,Y}) \iff \det (\Gamma_{\alpha,Y}(z)) = 0,$$

and the multiplicity of eigenvalue $z^2$ equals the multiplicity of the eigenvalue zero of the matrix $\Gamma_{\alpha,Y}(z)$.

3. Proof of the result for $N = 1$

In the case of a single point interaction considered in in the second part of Theorem 1.1, the proof is quite easy. We believe it is of some interest, both because the proof is much easier, and it is possible to give a complete description of the dispersive behaviour of the propagator, including the case when a resonance at zero occurs. The main advantage is that, as proved in [5] (see also [2]), it is possible to give an explicit representation of the Schrödinger propagator. If we denote by

$$S(x; t) = \frac{e^{i|x|^2/4it}}{(4\pi it)^{3/2}}, \quad t > 0, \quad x, y \in \mathbb{R}^3$$

the free Schrödinger propagator in $\mathbb{R}^3$, i.e.

$$(e^{-it\Delta} f)(x) = S(x; t) * f(x),$$

the kernel of the propagator of $e^{itH_{\alpha}}$ can be written as follows:

1) for $\alpha > 0$

$$S_\alpha(x, y; t) = S(x - y; t) + \frac{1}{|x||y|} \int_0^\infty e^{-4\pi \alpha s} (s + |x| + |y|)S(s + |x| + |y|; t)ds$$

2) for $\alpha = 0$

$$S_0(x, y; t) = S(x - y; t) + \frac{2it}{|x||y|} S(|x| + |y|; t)$$
3) for $\alpha < 0$

\begin{equation}
S_\alpha(x,y;t) = S(x-y;t) + \Psi_\alpha(x)\Psi_\alpha(y)e^{it(4\pi \alpha)^2} + \frac{1}{|x||y|} \int_0^\infty e^{4\pi \alpha s} (s - |x| - |y|)S(s - |x| - |y|;t)ds\end{equation}

Then we have the representation

\begin{equation}(e^{itH_\alpha}f)(x) = \int_{\mathbb{R}^3} S_\alpha(x,y;t)f(y)dy.\end{equation}

Notice that in the case $\alpha < 0$ we can easily distinguish the continuous part from the standing wave: indeed, the continuous part can be expressed as

\begin{equation}(P_{ac}e^{itH_\alpha}f)(x) = (e^{-it\Delta}f)(x) + \int_{\mathbb{R}^3} \int_0^\infty \frac{1}{|x||y|} e^{4\pi \alpha s} (s - |x| - |y|)S(s - |x| - |y|;t)dsdy.\end{equation}

The standing wave is the function

\begin{equation}C_1 \Psi_\alpha(x)e^{it(4\pi \alpha)^2}, \quad C_1 = \int_{\mathbb{R}^3} \Psi_\alpha(y)f(y)dy\end{equation}

which can be written explicitly using (2.8) as

\begin{equation}e^{it(4\pi \alpha)^2}(-2\alpha)\frac{e^{4\pi \alpha|x|}}{|x|} \int_{\mathbb{R}^3} \frac{e^{4\pi \alpha|y|}}{|y|} f(y)dy.\end{equation}

Consider the case $\alpha > 0$ first. From the trivial estimate

\begin{equation}|S(s + |x| + |y|;t)| \leq \frac{C}{t^{3/2}}\end{equation}

and

\begin{equation}\int_0^\infty e^{-4\pi \alpha s} (s + |x| + |y|)ds \leq C(1 + |x| + |y|)\end{equation}

and using also the standard dispersive estimate for the free solution, we obtain immediately the estimate

\begin{equation}\left|\int_{\mathbb{R}^3} S_\alpha(x,y;t)f(y)dy\right| \leq C \int_{\mathbb{R}^3} \left(1 + \frac{1 + |x| + |y|}{|x||y|}\right) |f(y)|dy.\end{equation}

Since we have

\begin{equation}1 + \frac{1 + |x| + |y|}{|x||y|} = \left(1 + \frac{1}{|x|}\right) \left(1 + \frac{1}{|y|}\right) = w(x)w(y)\end{equation}

we easily conclude that

\begin{equation}|(w^{-1}e^{itH_\alpha}f)(x)| \leq \frac{C}{t^{3/2}} \int_{\mathbb{R}^3} w(y)|f(y)|dy\end{equation}

as claimed.

In the case $\alpha < 0$, we can estimate the continuous part (3.5) exactly in the same way, obtaining

\begin{equation}|(w^{-1}P_{ac}e^{itH_\alpha}f)(x)| \leq \frac{C}{t^{3/2}} \int_{\mathbb{R}^3} w(y)|f(y)|dy.\end{equation}

Finally, when $\alpha = 0$ we can see the effect of the resonance at zero in the slower rate of decay: from (3.3), proceeding exactly as before, we obtain

\begin{equation}|(w^{-1}e^{itH_\alpha}f)(x)| \leq \frac{C}{t^{1/2}} \int_{\mathbb{R}^3} w(y)|f(y)|dy\end{equation}

since an additional factor $t$ is present in the propagator.
We shall now prove the main part of Theorem 1.1 concerning the general case \( N > 1 \), where we do not have an explicit formula for the propagator. Thus we resort to the spectral calculus and we represent the (continuous part of the) solution as follows:

\[
(4.1) \quad P_{ac}e^{itH_{ac}}f = \int_0^{\infty} e^{it\lambda} \mathfrak{A}R_{\alpha,Y}(\lambda + i0)f d\lambda.
\]

Recall now the integral expression (2.12). The first term reproduces the free solution \( e^{-it\Delta}f \) for which we already know the decay estimate. Thus we are reduced to estimate the integrals

\[
(4.2) \quad I = \int_0^{\infty} \int_{\mathbb{R}^3} e^{it\lambda} \frac{f(y)}{|x - y_j||y - y_\ell|} dyd\lambda
\]

where

\[
(4.3) \quad c_{j\ell}(\mu) = (4\pi)^{-2}[\Gamma_{\alpha,Y}(\mu)]^{-1}_{j,\ell} \quad \text{for} \quad \mu \geq 0, \quad j, \ell = 1, \ldots, N.
\]

To make the following computation rigorous, we introduce a cutoff function. Let \( \psi(s) \) be a non-negative function in \( C_0^\infty(\mathbb{R}) \) equal to 1 on \([0, 1]\) and vanishing on \([2, \infty)\); it is not restrictive to assume that \( \psi \) is an even function \( \psi(-s) = \psi(s) \). Then we can approximate \( I \) with

\[
(4.4) \quad I_M = \int_0^{\infty} \int_{\mathbb{R}^3} e^{it\lambda} \mathfrak{A} \left( e^{-i\sqrt{\lambda}|x - y_j|} e^{i\sqrt{\lambda}|y - y_\ell|} c_{j\ell}(\sqrt{\lambda}) \right) \frac{f(y)}{|x - y_j||y - y_\ell|} \psi(\sqrt{\lambda}/M) dyd\lambda
\]

as \( M \to +\infty \). After the change of variables \( \lambda = \mu^2 \) this can be written also as

\[
(4.5) \quad I_M = \int_0^{\infty} \int_{\mathbb{R}^3} 2ie^{it\mu^2} \mathfrak{A} \left( e^{-i\mu|x - y_j|} e^{i\mu|y - y_\ell|} c_{j\ell}(\mu) \right)^2 \frac{f(y)}{|x - y_j||y - y_\ell|} \psi(\mu/M) dyd\mu
\]

if we can prove a dispersive estimate for \( I_M \) which is uniform in \( M \), this will also give an estimate for \( I \). Since \( c_{j\ell}(\mu) \) is bounded near \( \mu = 0 \) by assumption and \( c_{j\ell}(0) \) is real, while

\[
\mathfrak{A}(e^{-i\mu|x - y_j|} e^{i\mu|y - y_\ell|}) = \sin(\mu(|x - y_j| - |y - y_\ell|))
\]

vanishes for \( \mu = 0 \), we can integrate by parts with respect to \( \mu \) and we obtain

\[
(4.6) \quad I_M = i \int_0^{\infty} \int_{\mathbb{R}^3} e^{it\mu^2} \frac{\partial}{\partial \mu} \mathfrak{A} \left( e^{-i\mu|x - y_j|} e^{i\mu|y - y_\ell|} c_{j\ell}(\mu)\psi(\mu/M) \right) \frac{f(y)}{|x - y_j||y - y_\ell|} dyd\mu.
\]

For reasonable data \( f \) the two integrals can be swapped using Fubini’s theorem and we arrive at

\[
(4.7) \quad I_M = \int_0^{\infty} \int_{\mathbb{R}^3} e^{it\mu^2} \frac{\partial}{\partial \mu} \mathfrak{A} \left( e^{-i\mu|x - y_j|} e^{i\mu|y - y_\ell|} c_{j\ell}(\mu)\psi(\mu/M) \right) \psi(\mu/M) dyd\mu.
\]

The core of our proof will be a suitable estimate of the inner integral in the variable \( \mu \); expanding the derivative we obtain three terms

\[
\int_0^{\infty} e^{it\mu^2} \frac{\partial}{\partial \mu} \mathfrak{A} \left( e^{-i\mu|x - y_j|} e^{i\mu|y - y_\ell|} c_{j\ell}(\mu)\psi(\mu/M) \right) d\mu = I_1 + I_2 + I_3
\]

and precisely

\[
(4.8) \quad I_1 = (|y - y_\ell| - |x - y_j|) \int_0^{\infty} e^{it\mu^2} \Re \left( e^{i\mu|y - y_\ell| - i\mu|x - y_j|} c_{j\ell}(\mu)\psi(\mu/M) \right) d\mu,
\]

\[
(4.9) \quad I_2 = \int_0^{\infty} e^{it\mu^2} \mathfrak{A} \left( e^{-i\mu|x - y_j|} e^{i\mu|y - y_\ell|} c_{j\ell}'(\mu)\psi(\mu/M) \right) d\mu,
\]

and

\[
(4.10) \quad I_3 = \int_0^{\infty} e^{it\mu^2} \mathfrak{A} \left( e^{-i\mu|x - y_j|} e^{i\mu|y - y_\ell|} c_{j\ell}(\mu)\psi'(\mu/M) \frac{1}{M} \right) d\mu.
\]
Consider the first integral $I_1$, or rather the integral

\begin{equation}
\tilde{I}_1 = \int_0^{+\infty} e^{it\mu^2} \Re \left( e^{i\mu |y-y_1|} e^{-i\mu |x-y_1|} c_{j\ell}(\mu) \psi(\mu/M) \right) d\mu.
\end{equation}

Inspired by a clever idea of Rodnianski and Schlag ([4]), we remark that an integral of the form

\[ u(t, A) = \int_0^{+\infty} e^{it\mu^2} \Re \left( e^{iA\mu} c_{j\ell}(\mu) \psi(\mu/M) \right) d\mu \]

is a solution of the one dimensional Schrödinger equation $i\partial_t u = \partial_{xx} u$. Thus we can apply the classical dispersive estimate

\[ |u(t, A)| \leq Ct^{-1/2} \int |u(0, A)| dA \]

which in our case gives

\begin{equation}
|\tilde{I}_1| \leq \frac{C}{\sqrt{t}} \cdot \int |J_1(A)| dA
\end{equation}

with

\begin{equation}
J_1(A) = \int_0^{+\infty} \Re \left( e^{iA\mu} c_{j\ell}(\mu) \psi(\mu/M) \right) d\mu.
\end{equation}

The same argument, used for $I_3$, shows that

\begin{equation}
|I_3| \leq \frac{C}{\sqrt{t}} \cdot \int |J_3(A)| dA
\end{equation}

where

\begin{equation}
J_3(A) = \int_0^{+\infty} J_3(\mu, A) = \int_0^{+\infty} \Re \left( e^{iA\mu} c_{j\ell}(\mu) \psi(\mu/M) \frac{1}{M} \right) d\mu.
\end{equation}

A similar argument can be applied to $I_2$. First of all we recall that the matrix $\Gamma_{\alpha,Y}(\mu)$ for $\mu \in [0, +\infty)$ has the form (see (1.8))

\[ \Gamma_{\alpha,Y}(\mu) = -\frac{i\mu}{4\pi} + \text{diag}[\alpha_1, \ldots, \alpha_N] + A(\mu) \]

where $A(\mu)$ has coefficients

\begin{equation}
[A(\mu)]_{jj} = 0, \quad [A(\mu)]_{j\ell} = -\frac{e^{i\mu |y_j-y_\ell|}}{4\pi |y_j-y_\ell|} \quad \text{for } j \neq \ell.
\end{equation}

Hence the derivative $\partial_\mu \Gamma_{\alpha,Y}(\mu)$ is simply

\begin{equation}
[\partial_\mu \Gamma_{\alpha,Y}(\mu)]_{j\ell} = -\frac{i}{4\pi} e^{i\mu |y_j-y_\ell|}.
\end{equation}

Since

\[ \partial_\mu [\Gamma_{\alpha,Y}(\mu)^{-1}] = \frac{1}{2} \partial_\mu \Gamma_{\alpha,Y}(\mu) \Gamma_{\alpha,Y}(\mu)^{-1}, \quad c_{j\ell}(\mu) = [\Gamma_{\alpha,Y}(\mu)^{-1}]_{j\ell}, \]

we obtain the representation

\begin{equation}
c'_{j\ell}(\mu) = -\frac{i}{4\pi} \sum_{k_1, k_2=1}^N c_{j\ell_k}(\mu) e^{i\mu |y_1-y_{k_2}|} c_{k_2\ell}(\mu)
\end{equation}

If we plug this into $I_2$ we obtain

\begin{equation}
I_2 = \frac{1}{4\pi} \sum_{k_1, k_2=1}^N \int_0^{+\infty} e^{it\mu^2} \Re \left( c_{k_1\ell}(\mu) c_{k_2\ell}(\mu) e^{i\mu |y-y_1|} e^{-i\mu |x-y_1|} e^{i\mu |y_1-y_{k_2}|} \psi(\mu/M) \right) d\mu.
\end{equation}

The same argument used above for $I_1$ and $I_3$ gives that

\begin{equation}
|I_2| \leq \frac{C}{\sqrt{t}} \cdot \int |J_2(A)| dA
\end{equation}
where

\begin{equation}
J_2(A) = \int_{0}^{\infty} \Re \left( e^{i\mu A} c_{jk_1}(\mu)c_{k_2\ell}(\mu)\psi(\mu/M) \right) d\mu.
\end{equation}

We shall now prove the estimate

\begin{equation}
\int (|J_1(A)| + |J_2(A)| + |J_3(A)|)dA < C
\end{equation}

for some constant independent of \( M \), from which the Theorem will follow immediately. To this end we shall need to study the coefficients \( c_{j\ell}(\mu) \) closer.

**Lemma 4.1.** Let \( \Gamma_{\alpha,Y}(\mu) \) be the matrix defined in (4.16), (4.17) and assume it is invertible with inverse locally bounded for \( \mu \in [0, +\infty) \).

Then the coefficients \( c_{j\ell}(\mu) \) of the inverse matrix \( \Gamma_{\alpha,Y}(\mu)^{-1} \) satisfy the following properties:

(i) The coefficients are holomorphic on a neighbourhood of the positive real axis; their derivatives satisfy the estimates

\begin{equation}
|c_{j\ell}(\mu)| \leq C(\mu)^{-1}, \quad |c'_{j\ell}(\mu)| \leq C(\mu)^{-2}, \quad |c''_{j\ell}(\mu)| \leq C(\mu)^{-2}; \quad \mu \geq 0
\end{equation}

where we used the notation \( \langle \mu \rangle = (1 + \mu^2)^{1/2} \).

(ii) The coefficients can be written as

\begin{equation}
c_{j\ell}(\mu) = 4\pi i \delta_{j\ell}(\mu)^{-1} + d_{j\ell}(\mu),
\end{equation}

or equivalently

\begin{equation}
c_{j\ell}(\mu) = 4\pi i \delta_{j\ell} \mu \langle \mu \rangle^{-2} + \tilde{d}_{j\ell}(\mu),
\end{equation}

where the holomorphic functions \( d_{j\ell}, \tilde{d}_{j,\ell} \) satisfy the inequalities

\begin{equation}
|d_{j\ell}| + |d'_{j\ell}| + |d''_{j\ell}| \leq C(\mu)^{-2}, \quad |\tilde{d}_{j\ell}| + |\tilde{d}'_{j\ell}| + |\tilde{d}''_{j\ell}| \leq C(\mu)^{-2}; \quad \mu \geq 0.
\end{equation}

**Proof.** It is clear from the definition that \( c_{j\ell}(\mu) \) are holomorphic (as soon as they are defined); moreover, by formula (4.19) proved above we see that the second and third estimates in (4.24) are immediate consequences of the first one in (4.24).

In order to prove the first inequality in (4.24), it is sufficient to recall formulas (4.16), (4.17), i.e.,

\[ \Gamma_{\alpha,Y}(\mu) = C_0 \mu I + D + A(\mu), \]

with \( C_0 = -i/4\pi, D \) a constant diagonal matrix, and \( A(\mu) \) a matrix with bounded coefficients. From this expression it is clear that the entries of the inverse matrix \( \Gamma_{\alpha,Y}(\mu)^{-1} \) are of the form

\[ c_{j\ell}(\mu) = \frac{\pm(C_0\mu)^{N-1} + P(\mu)}{\pm(C_0\mu)^{N} + Q(\mu)}, \]

where \( P, Q \) are functions of \( \mu \) of order \( N-2 \) and \( N-1 \) respectively:

\[ |P(\mu)| \leq C(\mu)^{N-2}, \quad |Q(\mu)| \leq C(\mu)^{N-1}. \]

Taking into account the assumption that \( c_{j\ell}(\mu) \) are locally bounded functions, estimate (4.24) follows easily.

In order to prove the asymptotic expansion (4.25), we write

\[ I = \Gamma_{\alpha,Y}(\mu)^{-1} \Gamma_{\alpha,Y}(\mu) = \Gamma_{\alpha,Y}(\mu)^{-1}(C_0 \mu I + D + A(\mu)) = C_0 \mu \Gamma_{\alpha,Y}(\mu)^{-1} + \Gamma_{\alpha,Y}(\mu)^{-1}[D + A(\mu)] \]

which implies \( (C_0^{-1} = 4\pi i) \)

\[ \Gamma_{\alpha,Y}(\mu)^{-1} = 4\pi i \mu^{-1} I - [D + A(\mu)] \cdot \Gamma_{\alpha,Y}(\mu)^{-1} 4\pi i \mu^{-1} \]
(it is sufficient to prove the estimates for \( \mu > 1 \) since we already know that the functions are smooth near 0). Notice that the last term is bounded by \( C(\mu)^{-2} \). Now, using the elementary identity
\[
\frac{1}{\mu} = \frac{1}{(1 + \mu^2)^{1/2}} + \frac{1}{\mu(1 + \mu^2)^{1/2}[\mu + (1 + \mu^2)^{1/2}]}
\]
and estimate (4.24) already proved, we easily obtain (4.25) and the estimate
\[
|d_{j\ell}(\mu)| \leq C(\mu)^{-2}.
\]
The remaining estimates on the derivatives of \( d_{j\ell} \) follow immediately from (4.25) and (4.24).

The second expansion (4.26) is proved in an identical way; indeed,
\[
\frac{1}{\mu} = \frac{\mu}{1 + \mu^2} + \frac{1}{\mu(1 + \mu^2)}.
\]

\[\Box\]

**Remark 4.1.** The reason for the two different asymptotic expansions in the Lemma is the following: the main term decays like \( \mu^{-1} \) and must be treated with care, using explicit formulas for the Fourier transform. Since the Fourier transform here appears as a sinus or cosinus transform only, it is quite useful to have its expression both as an even and an odd function of \( \mu \). The remainder terms are harmless since they give rise to integrals which can be easily estimated uniformly in \( M \).

We are ready to estimate the \( L^1 \) norms on \( \mathbb{R} \) of the three quantities \( J_1(A), J_2(A), J_3(A) \). Consider the quantity \( J_1(A) \); using the expansion (4.26) we have
\[\label{eq:4.28}
J_1(A) = J_1^a(A) + J_1^b(A)
\]
where
\[
J_1^a(A) = 4\pi \int_0^{+\infty} \Re \left( e^{iA\mu} 4\pi i \mu(\mu)^{-2} \psi(\mu/M) \right) d\mu = -(4\pi)^2 \int_0^{+\infty} \sin(A\mu) \mu(\mu)^{-2} \psi(\mu/M) d\mu
\]
and
\[
J_1^b(A) = \int_0^{+\infty} \Re \left( e^{iA\mu} \tilde{d}_{j\ell}(\mu) \psi(\mu/M) \right) d\mu.
\]
The remainder term \( J_1^b(A) \) can be estimated directly. Indeed, by (4.27) it is easy to see that \( J_1^b(A) \) is bounded
\[
|J_1^b(A)| \leq C \int_0^{+\infty} |\mu|^{-2} d\mu \leq C
\]
with \( C \) independent of \( M, A \). Moreover, writing \( e^{iA\mu} = (iA)^{-1} \partial_{\mu} e^{iA\mu} \) we have
\[
J_1^b(A) = \frac{1}{A} \int_0^{+\infty} \Im \left( \partial_{\mu} e^{iA\mu} \tilde{d}_{j\ell}(\mu) \psi(\mu/M) \right) d\mu
\]
but \( c_{j\ell}(0) \) is real, so integrating by parts we get
\[
J_1^b(A) = -\frac{1}{A} \int_0^{+\infty} \Im \left( e^{iA\mu} \tilde{d}_{j\ell}(\mu) \psi(\mu/M) \right) d\mu - \frac{1}{A} \int_0^{+\infty} \Im \left( e^{iA\mu} \tilde{d}_{j\ell}(\mu) \psi'(\mu/M) \frac{1}{M} \right) d\mu.
\]
As before, using (4.27) we see that both integrals are bounded by a constant \( C' \) independent of \( M, A \) and this implies
\[
|AJ_1^b(A)| \leq C'.
\]
Finally, we write \( e^{iA\mu} = (iA)^{-1} \partial_{\mu} e^{iA\mu} \) one more time in the last formula:
\[
J_1^b(A) = \frac{1}{A^2} \int_0^{+\infty} \Re \left( \partial_{\mu} e^{iA\mu} \tilde{d}_{j\ell}(\mu) \psi(\mu/M) \right) d\mu + \frac{1}{A^2} \int_0^{+\infty} \Re \left( \partial_{\mu} e^{iA\mu} \tilde{d}_{j\ell}(\mu) \psi'(\mu/M) \frac{1}{M} \right) d\mu.
\]
We integrate again by parts; this time we obtain a boundary term
\[\label{eq:4.29}
\frac{1}{A^2} \Re \tilde{d}_{j\ell}(0) \psi(0) + \frac{1}{A^2} \Re \tilde{d}_{j\ell}(0) \psi'(0) \frac{1}{M}
\]
and a few integrals
\begin{align}
(4.30) \quad \frac{1}{A^2} \int_0^{+\infty} \Re \left( e^{i A \mu} \hat{J}_0(\mu) \psi(\mu/M) \right) d\mu + \frac{2}{A^2} \int_0^{+\infty} \Re \left( e^{i A \mu} \hat{d}_{J_0}(\mu) \psi'(\mu/M) \frac{1}{M} \right) d\mu
\end{align}
\begin{align}
(4.31) \quad + \frac{1}{A^2} \int_0^{+\infty} \Re \left( e^{i A \mu} \hat{d}_{J_0}(\mu) \psi''(\mu/M) \frac{1}{M^2} \right) d\mu.
\end{align}

Applying as above the estimates (4.27) of the Lemma we obtain
\[ |A^2 J^b_1(A)| \leq C' \]
for a constant independent of \( M, A \). Then we conclude
\[ \int |J^b_1(A)| dA \leq \int (1 + A^2)^{-1} dA \leq C \]
for some bound \( C \) uniform in \( M \).

Let us now consider the main term \( J^b_1(A) \) in (4.28). It is a standard sinus-transform, which can be rewritten as a Fourier transform as follows (recall \( \psi \) is an even function)
\[ J^b_1(A) = 8\pi^2 i \int_{-\infty}^{+\infty} e^{i A \mu} \frac{\mu}{1 + \mu^2} \psi(\mu/M) d\mu = C \cdot (\mathcal{F}(g_1) * \mathcal{F}(g_2))(A) \]
where we have used the convolution theorem and we have denoted
\begin{align}
(4.32) \quad g_1(\mu) = \frac{\mu}{1 + \mu^2}, \quad g_2(\mu) = \psi(\mu/M)
\end{align}
Since
\[ (\mathcal{F}(g_1))(A) = C \frac{A}{|A|} e^{-|A|} \]
and
\[ (\mathcal{F}(g_2))(A) = M(\mathcal{F}\psi)(AM) \]
we obtain
\[ \|J^b_1\|_{L^1(\mathbb{R})} \leq C \|\mathcal{F}(g_1)\|_{L^1(\mathbb{R})} \|\mathcal{F}(g_2)\|_{L^1(\mathbb{R})} \leq C \]
for some constant independent of \( M \). In conclusion we have proved the bound \( \|J_1\|_{L^1(\mathbb{R})} < C \), uniformly in \( M \).

The corresponding estimate for \( J^c_1(A) \), see (4.15), is almost identical; the only modification is the use of the explicit formula for the Fourier transform of \( \langle \mu \rangle^{-1} \), which is
\[ \mathcal{F}(\langle \mu \rangle^{-1}) = C K_0(|A|). \]
Here \( K_0(s) \) is the modified Bessel function of order 0, whose behaviour is the following:
\[ |K_0(s)| \leq C |\log s|, \quad 0 < s \leq 1; \quad |K_0(s)| \leq C s^{-\frac{1}{2}} e^{-s}, \quad s \geq 1. \]
In particular, it is clear that \( \mathcal{F}(\langle \mu \rangle^{-1}) \) belongs to \( L^1(\mathbb{R}) \) which is what is needed in our computation.

On the other hand the estimate for \( J^c_2(A) \) (see (4.22)) is immediate since the integrand decays as \( \langle \mu \rangle^{-2} \). This concludes the proof of (4.23).

Recalling now (4.8), (4.9), (4.10) and (4.12), (4.21), (4.14) we have
\[ |I_1| \leq \frac{C}{t^{1/2}} \left( |y - y| + |x - y| \right), \quad |I_2| + |I_3| \leq \frac{C}{t^{1/2}} \]
with \( C \) independent of \( M \) and of \( x \). Thus, recalling (4.6) and taking the limit as \( M \to \infty \), we obtain the following estimate for the integral \( I \) in (4.2):
\[ |I| \leq \frac{C}{t^{3/2}} \int_{\mathbb{R}^3} |f(y)| \frac{1 + |y - y| + |x - y|}{|y - y| |x - y|} dy \]
Since $P_{ac}e^{itH_{\alpha,Y}}f$ is the sum of the integrals $I$ for all couples $j,\ell = 1,\ldots,N$, plus the free wave $e^{-it\Delta}f$, we obtain

$$|(P_{ac}e^{itH_{\alpha,Y}}f)(x)| \leq \frac{C}{t^{3/2}} \int |f(y)| \sum_{j,\ell=1}^{N} \left( 1 + \frac{1 + |y - y_\ell| + |x - y_j|}{|y - y_\ell||x - y_j|} \right) dy.$$ 

The inner sum gives

$$\sum_{j,\ell=1}^{N} \left( 1 + \frac{1 + |y - y_\ell| + |x - y_j|}{|y - y_\ell||x - y_j|} \right) = \sum_{j,\ell=1}^{N} \left( 1 + \frac{1}{|y - y_\ell|} \right) \cdot \left( 1 + \frac{1}{|x - y_j|} \right) = w(x)w(y)$$

where $w(x)$ is the weight function (1.7).

In conclusion we have proved

$$w(x)^{-1}|(P_{ac}e^{itH_{\alpha,Y}}f)(x)| \leq \frac{C}{t^{3/2}} \int |f(y)|w(y)dy$$

as claimed.

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