A1. Analyzing peer effects without Assumptions 1 or 2

A1.1. Randomization tests

We use randomization tests for the significance of peer effects for the following reasons. First, they provide additional evidence for the significance of peer effects. Second, randomization tests are exact and valid for finite samples. Third, randomization tests do not require Assumptions 1–3, as long as the assignment mechanism is known. Fourth, as Fisher (1935) suggested, we can use randomization tests to check the Normal approximations.

We can use randomization tests for the sharp null hypothesis for all units:

\[ H_0 : Y_i(z) = Y_i(z'), \text{ for all } z, z' \text{ and for all unit } i, \]

or the null hypothesis for units with attribute \( a \):

\[ H_{0,a} : Y_i(z) = Y_i(z'), \text{ for all } z, z' \text{ and for all unit } i \text{ such that } A_i = a. \]

\( H_{0,a} \) is not sharp. But we can still conduct randomization test for \( H_{0,a} \). We choose test statistics depending only on the outcomes of units with attribute \( a \), and their randomization distributions are known under \( H_{0,a} \). We can then obtain exact \( p \)-values under \( H_{0,a} \).

We first discuss the choices of test statistics for the subgroup null \( H_{0,a} \), and then the test statistics for \( H_0 \). A choice of test statistic for the subgroup null \( H_{0,a} \) is

\[ T_a = \max_{r, r'} \hat{T}_a(r, r') = \max_r \hat{Y}_a(r) - \min_r \hat{Y}_a(r). \tag{A1} \]

Another choice of test statistic, \( F_a \), is the F statistic from the analysis of variance of the linear regression of \( Y_i \) on \( R_i \) among units with attribute \( a \). For the sharp null \( H_0 \), we can use \( T = \max_{r, r'} \hat{\tau}(r, r') \), the F statistic \( F \) from the linear model (16), \( \max_a T_a \), and \( \max_a F_a \).

For the motivating application, Table A1 shows the \( p \)-values from randomization tests for \( H_0 \) and \( H_{0,a} \) with different test statistics. At significance level 0.05, the peer effects are not significant.
Table A1: p values of randomization tests for the subgroup null $H_{0,[a]}$ and the sharp null $H_0$, with test statistics for $H_0$ shown in the parentheses.

| Department | Null hypothesis | Test statistic for $H_{0,[a]}$ ($H_0$) | $T_{[a]}$ (max$_a T_{[a]}$) | $F_{[a]}$ (max$_a F_{[a]}$) | (T) | (F) |
|------------|----------------|----------------------------------------|-----------------------------|-----------------------------|-----|-----|
| Informatics| $H_0$          | 0.2678                                 | 0.3640                      | 0.1092                      | 0.1468 |
|            | $H_{0,[1]}$     | 0.2390                                 | 0.3384                      |                             |       |
|            | $H_{0,[2]}$     | 0.0615                                 | 0.2050                      |                             |       |
| physics    | $H_0$          | 0.4553                                 | 0.0719                      | 0.3431                      | 0.0366 |
|            | $H_{0,[1]}$     | 0.3545                                 | 0.0360                      |                             |       |
|            | $H_{0,[2]}$     | 0.6994                                 | 0.7232                      |                             |       |

for students in the informatics department; the subgroup average peer effects are significant for students from Gaokao in the physics department if we use the F statistic as the test statistic, and the subgroup average peer effects are not significant for students from recommendation. We ignore the multiple testing issue. Compared to Table 2, randomization tests reject only $H_{0,[1]}$ for students from Gaokao in the physics department, which may be due to the lack of power of randomization tests (Ding 2017).

Moreover, we check the randomization distributions of the $\hat{\tau}_{[a]}(r, r')$’s under the sharp null hypothesis. Under the sharp null hypothesis that the potential outcomes are not affected by the treatment assignment, all potential outcomes are known and identical to the observed outcomes. Therefore, the distributions of the subgroup peer effect estimators are known under complete randomization. For students in the informatics and physics departments graduating in 2013, Figures A1(a) and A1(b) show, respectively, the histograms of the subgroup peer effect estimators under the sharp null hypothesis based on $10^5$ treatment assignments from complete randomization. From Figures A1(a) and A1(b), the Normal approximations work fairly well. For our application, we do not know all potential outcomes and thus can not directly check the Normal approximations over repeated sampling of the treatment assignments. However, we view Figures A1(a) and A1(b) as intuitive justifications for the Normal approximation of $\hat{\tau}_{[a]}(r, r')$ in the Neymanian inference.

A1.2. Estimands, unbiased estimators, and optimal assignment mechanism

Even if Assumptions 1 or 2 fails, the estimators in (5) and (6) are still meaningful in the sense of unbiasedly estimating some average potential outcomes. Recall the definitions in Section 7:

\[
Y_i^D(r) = \sum_{z \in A} \Pr(Z = z \mid R_i = r)Y_i(z), \quad \tau_i^D(r, r') = Y_i^D(r) - Y_i^D(r'),
\]

\[
\bar{Y}_{[a]}^D(r) = n_{[a]}^{-1} \sum_{i : A_i = a} Y_i^D(r), \quad \tau_{[a]}^D(r, r') = n_{[a]}^{-1} \sum_{i : A_i = a} \tau_i^D(r, r'), \quad \tau^D(r, r') = n^{-1} \sum_{i=1}^n \tau_i^D(r, r').
\]
Figure A1: Histograms of subgroup peer effect estimators under the sharp null hypothesis, based on $10^5$ draws from complete randomization. The lines are densities of Normal approximations.
Proposition A1. Under Assumption 3, $\hat{Y}_{[a]}(r)$ has mean $\tilde{Y}_{[a]}(r)$.

The conclusion follows from

$$
E \left\{ \hat{Y}_{[a]}(r) \right\} = \{ n_{[a]} \pi_{[a]}(r) \}^{-1} \sum_{i: A_i = a} E \left\{ I(R_i = r) Y_i \right\} 
$$

$$
= \{ n_{[a]} \pi_{[a]}(r) \}^{-1} \sum_{i: A_i = a} \left( I(R_i = r) \sum_z I(Z = z) Y_i(z) \right) 
$$

$$
= \{ n_{[a]} \pi_{[a]}(r) \}^{-1} \sum_{i: A_i = a} \sum_z \Pr(R_i = r) \Pr(Z = z \mid R_i = r) Y_i(z) 
$$

$$
= \{ n_{[a]} \pi_{[a]}(r) \}^{-1} \sum_{i: A_i = a} \sum_z \Pr(Z = z \mid R_i = r) Y_i(z) = n_{[a]}^{-1} \sum_{i: A_i = a} Y_{D}(r) = \tilde{Y}_{[a]}(r). 
$$

By the linearity of the expectation, $\hat{\tau}_{[a]}(r, r')$ and $\hat{\tau}(r, r')$ are unbiased for $\tau_{D}(r, r')$ and $\tau_{D}(r, r')$, respectively. However, without Assumptions 1 and 2, for a given treatment we do not have replications of units, making it difficult to evaluate the uncertainty of these estimators. Similarly, for the first type of interference, Hudgens and Halloran (2008) discussed the expectations of the point estimators under general settings, but invoked “stratified interference” (analogous to Assumption 2) to evaluate the uncertainty.

Moreover, the estimands $\tau_{D}(r, r')$ and $\tau_{D}(r, r')$ are meaningful in many situations. In the expression of $Y_i^{D}(r)$, the weight $\Pr(Z = z \mid R_i = r)$ is nonzero only if $R_i(z_i) = r$, i.e., the attributes of unit $i$’s peers constitute $r$. Therefore, $Y_i^{D}(r)$ summarizes unit $i$’s potential outcomes when he/she has $K$ peers with attributes $r$. Consequently, $\tau_{D}(r, r')$ measures the difference when unit $i$ has $K$ peers with attributes $r$ rather than $r'$. Thus we can view $\tau_{D}(r, r')$ as the individual peer effect comparing treatments $r$ and $r'$, and $\tau_{D}(r, r')$ and $\tau_{D}(r, r')$ as the corresponding average peer effects. Below we further simplify $Y_i^{D}(r)$ under some special cases, making its meaning more intuitive. When Assumption 1 holds, $Y_i^{D}(r)$ reduces to $\sum_{z_i} \Pr(Z_i = z_i \mid R_i = r) Y_i(z_i)$; if further the treatment assignment mechanism is random partitioning or complete randomization, then the weight $\Pr(Z_i = z_i \mid R_i = r)$ is a nonzero constant for $z_i$ such that $R_i(z_i) = r$, and $Y_i^{RP}(r) = Y_i^{CR}(r)$ further reduces to the average of $Y_i(z_i)$’s for $z_i$ such that $R_i(z_i) = r$. When both Assumptions 1 and 2 hold, $Y_i^{D}(r)$ is the same as $Y_i(r)$ in the main paper, which does not depend on the assignment mechanism $D$. In sum, $Y_i^{D}(r)$ is an extension of $Y_i(r)$.

We now consider the optimal complete randomization mechanism for the assignment of a new population of size $n'$ without Assumptions 1 or 2. Define similarly $Y_i^{D'}(r)$ and $\tilde{Y}_{[a]}^{D'}(r)$ for the new population. By the same logic as (18), the expected total outcome under complete
randomization with \( L'(z) = l' \) for the new population is

\[
E \left( \sum_{i=1}^{n'_i} Y_i' \right) = \sum_{a=1}^{H} E \left( \sum_{i:A_i = a} Y_i' \right) = \sum_{a=1}^{H} \sum_{r \in R} n'_a r \bar{Y}_{a}^{CR'}(r).
\]  
(A2)

To estimate the maximizer \( l'_{opt} \) of (A2), we need to assume the new population is similar to the one in our data. Let \( D_0 \) denote the assignment mechanism for our observed data. The following assumption is similar to the one in the main paper.

**Assumption A7.** \( \bar{Y}_{a}^{CR'}(r) = c \bar{Y}_{a}^{D_0}(r) + \xi_a \) for some constants \( c > 0 \) and \( \xi_a \), for all \( 1 \leq a \leq H \) and \( r \in R \).

Under Assumption A7, (A2) reduces to

\[
E \left( \sum_{i=1}^{n'_i} Y_i' \right) = \sum_{a=1}^{H} \sum_{r \in R} n'_a r \bar{Y}_{a}^{CR'}(r) = c \sum_{a=1}^{H} \sum_{r \in R} n'_a r \bar{Y}_{a}^{D_0}(r) + \sum_{a=1}^{H} n'_a \xi_a,
\]

implying that the maximizer \( l'_{opt} \) of (A2) is the same as that of \( \sum_{a=1}^{H} \sum_{r \in R} n'_a r \bar{Y}_{a}^{D_0}(r) \). We can unbiasedly estimate \( \bar{Y}_{a}^{CR}(r) \) by \( \bar{Y}_{a}^{D_0}(r) \), and then estimate \( l'_{opt} \) by simply plugging in the estimators \( \hat{Y}_{a}^{D_0}(r) \)'s. Again, it is difficult to evaluate the uncertainty of the estimator for \( l'_{opt} \) for the average peer effect estimators.

Below we give some comments on Assumption A7, which is key for inferring the optimal complete randomization. Assumption A7 is a strong requirement of the similarity between the new population and the one in our data, due to the dependence of \( \bar{Y}_{a}^{CR}(r) \) and \( \bar{Y}_{a}^{D_0}(r) \) on the designs. Even if we assume that the new population is the same as the one in our data, Assumption A7, or, equivalently, \( \bar{Y}_{a}^{CR}(r) = c \bar{Y}_{a}^{D_0}(r) + \xi_a \), may fail because the values of \( \bar{Y}_{a}^{CR}(r) \) and \( \bar{Y}_{a}^{D_0}(r) \) depend on the assignment mechanisms, CR and \( D_0 \), respectively. If the new population is different from the one in our data, then Assumption A7 is even less plausible.

We summarize several concerns for inferring the optimal complete randomization in the absence of Assumptions 1 or 2. First, it is unnatural to infer the optimal peer assignment for a new population, because the treatment is the set of the identity numbers of units varying across populations. Second, the similarity assumption becomes stronger due to the dependence of the estimands on the design. Third, it is difficult to evaluate the uncertainty of the point estimator.

### A1.3. Distributional assumptions on potential outcomes

Under Assumption 4, we decompose \( \hat{Y}_{a}(r) \) into two parts:

\[
\hat{Y}_{a}(r) = n_{a}^{-1} \sum_{i:A_i = a, R_i = r} Y_i = n_{a}^{-1} \sum_{i:A_i = a, R_i = r} \{ Y_i(r) + \varepsilon_i(Z) \} \\
= n_{a}^{-1} \sum_{i:A_i = a, R_i = r} Y_i(r) + n_{a}^{-1} \sum_{i:A_i = a, R_i = r} \varepsilon_i(Z) \equiv \hat{Y}_{a}(r) + \hat{\varepsilon}_{a}(r),
\]
where \( \tilde{Y}_a(r) \equiv n_{[a]r}^{-1} \sum_{i:A_i=a,R_i=r} Y_i(r) \) and \( \tilde{\xi}_a(Z) \equiv n_{[a]r}^{-1} \sum_{i:A_i=a,R_i=r} \xi_i(Z) \) are the main part and deviance part, respectively. Let \( \tilde{\tau}_{[a]}(r,r') = \tilde{Y}_a(r) - \tilde{Y}_a(r') \) and \( \tilde{\delta}_{[a]}(r,r') = \tilde{\xi}_a(r) - \tilde{\xi}_a(r') \). Correspondingly, we can decompose the subgroup peer effect estimator \( \tilde{\tau}_{[a]}(r,r') \) into two parts:

\[
\tilde{\tau}_{[a]}(r,r') = \tilde{\tau}_{[a]}(r,r') + \tilde{\delta}_{[a]}(r,r').
\]

First, we discuss the sampling mean and variance of \( \tilde{\tau}_{[a]}(r,r') \). Under Assumption 4, we have, for any \( 1 \leq a \leq H \) and \( r \neq r' \in R \),

\[
E \left\{ \tilde{\tau}_{[a]}(r,r') \mid Z \right\} = E \left\{ \tilde{\tau}_{[a]}(r) \mid Z \right\} = E \left\{ \tilde{\tau}_{[a]}(r',r') \mid Z \right\} = 0,
\]

\[
\text{Var} \left\{ \tilde{\tau}_{[a]}(r,r') \mid Z \right\} = \text{Var} \left\{ \tilde{\tau}_{[a]}(r) \mid Z \right\} = \text{Var} \left\{ \tilde{\tau}_{[a]}(r',r') \mid Z \right\} = n_{[a]r}^{-1} \sigma_{[a]r}^2,
\]

\[
\text{Cov} \left\{ \tilde{\tau}_{[a]}(r,r'), \tilde{\tau}_{[a]}(r,r') \right\} = \text{Cov} \left\{ \tilde{\tau}_{[a]}(r) \mid Z \right\} = \text{Cov} \left\{ \tilde{\tau}_{[a]}(r',r') \mid Z \right\} = 0,
\]

which immediately imply that \( E \{ \tilde{\delta}_{[a]}(r,r') \mid Z \} = 0 \) and \( \text{Var} \{ \tilde{\delta}_{[a]}(r,r') \mid Z \} = n_{[a]r}^{-1} \sigma_{[a]r}^2 + n_{[a]r}^{-1} \sigma_{[a]r'}^2 \). Thus, marginally, \( \tilde{\delta}_{[a]}(r,r') \) has mean 0 and variance \( n_{[a]r}^{-1} \sigma_{[a]r}^2 + n_{[a]r'}^{-1} \sigma_{[a]r'}^2 \). Because \( \tilde{\tau}_{[a]}(r,r') \) is constant given \( Z \), we have

\[
\text{Cov} \left\{ \tilde{\tau}_{[a]}(r,r'), \tilde{\tau}_{[a]}(r,r') \right\} = E \left[ E \left\{ \tilde{\tau}_{[a]}(r,r') \tilde{\tau}_{[a]}(r,r') \mid Z \right\} \right] = E \left[ \tilde{\tau}_{[a]}(r,r') E \left\{ \tilde{\tau}_{[a]}(r,r') \mid Z \right\} \right] = 0,
\]

which implies that \( \text{Var} \{ \tilde{\tau}_{[a]}(r,r') \} = \text{Var} \{ \tilde{\tau}_{[a]}(r,r') \} + \text{Var} \{ \tilde{\delta}_{[a]}(r,r') \} \). Because the sampling variance of \( \tilde{\tau}_{[a]}(r,r') \) is the same as that in Corollary 2 with Assumption 2, we can derive that

\[
\text{Var} \left\{ \tilde{\tau}_{[a]}(r,r') \right\} = \text{Var} \left\{ \tilde{\tau}_{[a]}(r,r') \right\} + \text{Var} \left\{ \tilde{\delta}_{[a]}(r,r') \right\} = \frac{S_{[a]r}^2(r)}{n_{[a]r}} + \frac{S_{[a]r'}^2(r')}{n_{[a]r'}} - \frac{S_{[a]r}^2(r')}{n_{[a]r}} - \frac{S_{[a]r'}^2(r)}{n_{[a]r}} - \frac{\sigma_{[a]r}^2(r)}{n_{[a]r}} - \frac{\sigma_{[a]r'}^2(r)}{n_{[a]r}}.
\]

Second, we discuss the variance estimator for \( \tilde{\tau}_{[a]}(r,r') \). We decompose the sample variance of observed outcomes for units with attribute \( a \) receiving treatment \( r \) as

\[
s_{[a]r}^2(r) = (n_{[a]r} - 1)^{-1} \sum_{i:A_i=a,R_i=r} \left( Y_i - \tilde{Y}_a(r) \right)^2 = (n_{[a]r} - 1)^{-1} \sum_{i:A_i=a,R_i=r} \left( Y_i(r) + \tilde{\xi}_a(r) \right)^2 - \tilde{Y}_a(r) - \tilde{\xi}_a(r) \right)^2
\]

\[
= (n_{[a]r} - 1)^{-1} \sum_{i:A_i=a,R_i=r} \left( Y_i(r) - \tilde{Y}_a(r) \right)^2 + (n_{[a]r} - 1)^{-1} \sum_{i:A_i=a,R_i=r} \left( \tilde{\xi}_a(r) \right)^2 - \tilde{Y}_a(r) - \tilde{\xi}_a(r) \right)^2
\]

\[
+ (n_{[a]r} - 1)^{-1} \sum_{i:A_i=a,R_i=r} \left( Y_i(r) - \tilde{Y}_a(r) \right) \left( \tilde{\xi}_a(r) \right).
\]

Below we discuss the expectation of the three terms in (A4) separately. The expectation of the first term is the same as that in Theorem 2 with Assumption 2. The expectation of the second term is \( \sigma_{[a]r'}^2 \), because, conditional on \( Z \), it is the sample variance of independent zero-mean random
variables with variance $\sigma_{a|r}^2$. The expectation of the third term is zero, because, conditioning on $Z$, $Y_i(r) - \bar{Y}_{a|(r)}(r)$ is a constant and $\varepsilon_i(Z) - \bar{\varepsilon}_{a|(r)}$ has mean zero. Above all, $E\{s_{a|(r)}^2\} = S_{a|(r)}^2(r) + \sigma_{a|r}^2$. The variance estimator is conservative because

$$E \{ \hat{V}_{a|(r,r')} \} = \frac{E\{s_{a|(r)}^2\}}{n_{a|r}} + \frac{E\{s_{a|(r')}^2\}}{n_{a|r'}} = \frac{S_{a|(r)}^2(r) + \sigma_{a|r}^2}{n_{a|r}} + \frac{S_{a|(r')}^2(r') + \sigma_{a|r'}^2}{n_{a|r'}} \geq \text{Var}\{ \hat{V}_{a|(r,r')} \}.$$ 

A1.4. Peer effects for a target subpopulation

First, we study the sampling variance and its estimator for the difference-in-means estimator $\hat{\tau}_{tg}(r,r')$. For any $r,r' \in \mathcal{R}$ and units in the target subpopulation, let $\bar{Y}_{tg}(r) = \frac{1}{m - 1} \sum_{i=1}^{m} Y_i(r)$ and $\mathcal{S}_{2}(r) = (m - 1)^{-1} \sum_{i=1}^{m} \{ Y_i(r) - \bar{Y}_{tg}(r) \}^2$ be the finite population average and variance of individual potential outcome $Y_i(r)$'s, and $\mathcal{S}_{2}(r-r')$ be the finite population variance of individual peer effect $\tau_{r,r'}$'s. The number of units in the target subpopulation receiving treatment $r \in \mathcal{R}$ is $m_r = \sum_{k=1}^{m} \{ \{ A_{j} : j \in \zeta_{k} \} = r \}$. Following Neyman (1923), the sampling variance of $\hat{\tau}_{tg}(r,r')$ is

$$\text{Var}\{ \hat{\tau}_{tg}(r,r') \} = \frac{\mathcal{S}_{2}(r)}{m_r} + \frac{\mathcal{S}_{2}(r')}{m_r} - \frac{\mathcal{S}_{2}(r-r')}{m}.$$ 

For any $r \in \mathcal{R}$, let $s^2(r)$ be the sample variance of observed outcome $Y_i$'s for units receiving treatment $r$. We can show that $s^2(r)$ is an unbiased estimator for $\mathcal{S}_{2}(r)$. Therefore, a conservative sampling variance estimator for $\hat{\tau}_{tg}(r,r')$ is

$$\text{Var}\{ \hat{\tau}_{tg}(r,r') \} = \frac{s^2(r)}{m_r} + \frac{s^2(r')}{m_r}.$$ 

Moreover, under some regularity conditions, the Wald-type confidence interval is asymptotically conservative.

Second, we show that, averaging over all possible constructions, the point estimator for $\tau_{tg}(r,r')$ is the same as $\hat{\tau}_{a|(r,r')}$ in (13). Because the point estimator for $\tau_{tg}(r,r')$ is the standard difference-in-means for units receiving treatments $r$ and $r'$, it suffices to show that the average of $m_r^{-1} \sum_{i=1}^{m} I(R_i = r) Y_i$ over all configurations of the target subpopulation is the same as $n_{a|r}^{-1} \sum_{i : A_i = a, R_i = r} Y_i$, the average observed outcome for units with attribute $a$ receiving treatment $r$. This is true because (i) a unit with attribute $a$ receiving treatment $r$ must be in a group with group attribute $\{a\} \cup r$, (ii) any group with group attribute $\{a\} \cup r$ has the same number of units with attribute $a$, and (iii) each configuration randomly picks one unit with attribute $a$ in these groups with group attribute $\{a\} \cup r$ and calculates their average observed outcome to get $m_r^{-1} \sum_{i=1}^{m} I(R_i = r) Y_i$. 

7
A2. Technical details for general treatment assignments

A2.1. Lemmas

Recall that \(S^2_{[a]}(r)\) and \(S^2_{[a]}(r' r')\) are the finite population variances of potential outcomes \(Y_i(r)'s\) and individual peer effects \(\tau_i(r, r')'s\) among units with attribute \(a\). We further define \(S_{[a]}(r, r')\) as the finite population covariance between the \(Y_i(r)'s\) and the \(Y_i(r')'s\) among units with attribute \(a\), and \(\bar{Y}_i(r) = Y_i(r) - \bar{Y}_{[A_i]}(r)\) as the centered potential outcome of unit \(i\) by subtracting the average potential outcome among units with the same attribute as unit \(i\). We can then rewrite

\[
S^2_{[a]}(r) = (n_{[a]} - 1)^{-1} \sum_{i: A_i = a} \bar{Y}_i^2(r), \quad S^2_{[a]}(r, r') = (n_{[a]} - 1)^{-1} \sum_{i: A_i = a} \bar{Y}_i(r) \bar{Y}_i(r'), \quad (A5)
\]

and decompose the subgroup average potential outcome estimator as

\[
\hat{Y}_{[a]}(r) = \{n_{[a]} \pi_{[a]}(r)\}^{-1} \sum_{i: A_i = a} I(R_i = r) \bar{Y}_i(r) + \{n_{[a]} \pi_{[a]}(r)\}^{-1} \sum_{i: A_i = a} I(R_i = r) \bar{Y}_{[a]}(r) \equiv B_{[a]}(r) + C_{[a]}(r),
\]

and decompose the subgroup average peer effect estimator as

\[
\hat{\tau}_{[a]}(r, r') = \hat{Y}_{[a]}(r) - \hat{Y}_{[a]}(r') \equiv B_{[a]}(r) + C_{[a]}(r) - B_{[a]}(r') - C_{[a]}(r''). \quad (A6)
\]

The following three lemmas characterize the covariances of the terms in (A6).

**Lemma A1.** For \(1 \leq a, a' \leq H\) and \(r, r' \in \mathcal{R}\),

\[
\text{Cov}\{B_{[a]}(r), B_{[a']} (r')\} = \begin{cases} 
0, & \text{if } a \neq a'; \\
-(n_{[a]} - 1) \frac{\pi_{[a]}(r) \pi_{[a']} (r')} {n_{[a]} \pi_{[a]}(r) \pi_{[a']} (r')} S_{[a]}(r, r'), & \text{if } a = a', r \neq r'; \\
(n_{[a]} - 1) \frac{\pi_{[a]}(r') - \pi_{[a]}(r) \pi_{[a]}'(r)} {n_{[a]} \pi_{[a]}'(r)} S^2_{[a]}(r), & \text{if } a = a', r = r'.
\end{cases}
\]

**Lemma A2.** For \(1 \leq a, a' \leq H\) and \(r, r' \in \mathcal{R}\), \(\text{Cov}\{B_{[a]}(r), C_{[a']} (r')\} = 0.

**Lemma A3.** For any \(1 \leq a, a' \leq H\) and \(r, r' \in \mathcal{R}\),

\[
\text{Cov}\{C_{[a]}(r), C_{[a']} (r')\} = (n_{[a]} n_{[a']})^{-1/2} c_{[a][a']} (r, r') \bar{Y}_{[a]}(r) \bar{Y}_{[a']} (r'),
\]

where \(c_{[a][a']} (r, r')\) is defined in (8) in the main text.

Recall that \(\bar{Y}_{[a]}(r) \bar{Y}_{[a]}(r') = \{n_{[a]}(n_{[a]} - 1)^{-1} \sum_{i: A_i = a} Y_i(r) Y_j(r')\) is the average of the products of the potential outcomes for pairs of different units with the same attribute \(a\). The following lemma represents the finite population covariance \(S^2_{[a]}(r, r')\) and the product of average potential outcomes \(\bar{Y}_{[a]}(r) \bar{Y}_{[a]}(r')\) as functions of \(S^2_{[a]}(r), S^2_{[a]} (r'), S^2_{[a]}(r' r')\) and \(\bar{Y}_{[a]}(r) \bar{Y}_{[a]}(r')\).

**Lemma A4.** For \(1 \leq a \leq H\), and \(r \neq r' \in \mathcal{R}\),

(a) $2S_{[a]}(r, r') = S^2_{[a]}(r) + S^2_{[a]}(r') - S^2_{[a]}(r-r')$;

(b) $\bar{Y}_{[a]}(r) \bar{Y}_{[a]}(r') = \bar{Y}_{[a]}(r) \bar{Y}_{[a]}(r') + (2n_{[a]})^{-1} \{S^2_{[a]}(r) + S^2_{[a]}(r') - S^2_{[a]}(r-r')\}$.

A2.2. Proofs of the lemmas

Proof of Lemma A1. Based on the definitions of $\pi_{[a]}(r)$ and $\pi_{[a][a']}_{[r, r']}$, for units $i$ and $j$ such that $A_i = a$ and $A_j = a'$, the covariance between their treatment indicators is

$$\text{Cov}\{I(R_i = r), I(R_j = r')\} = \text{pr}(R_i = r, R_j = r') - \text{pr}(R_i = r)\text{pr}(R_j = r')$$

$$= \begin{cases} \pi_{[a][a']}_{(r, r')} - \pi_{[a]}(r)\pi_{[a']}(r'), & \text{if } i \neq j, \\ -\pi_{[a]}(r)\pi_{[a']}(r'), & \text{if } i = j, r \neq r', \\ \pi_{[a]}(r) - \pi_{[a]}^2(r) & \text{if } i = j, r = r'. \end{cases} \quad (A7)$$

Below we discuss three cases separately. (1) When $a \neq a'$, any units $i$ and $j$ such that $A_i = a$ and $A_j = a'$ must satisfy $i \neq j$. Therefore,

$$\text{Cov}\{B_{[a]}(r), B_{[a']}(r')\} = \text{Cov}\left\{\left\{n_{[a]}\pi_{[a]}(r)\right\}^{-1} \sum_{i:A_i = a} I(R_i = r)\bar{Y}_i(r), \left\{n_{[a']}',\pi_{[a']}(r')\right\}^{-1} \sum_{j:A_j = a'} I(R_j = r')\bar{Y}_j(r')\right\}$$

$$= \left\{\left\{n_{[a]}n_{[a']}',\pi_{[a]}(r)\pi_{[a']}_{[a']}(r')\right\}^{-1} \sum_{i:A_i = a} \sum_{j:A_j = a'} \bar{Y}_i(r)\bar{Y}_j(r') \text{Cov}\{I(R_i = r), I(R_j = r')\}\right\}$$

$$= \frac{\pi_{[a][a']}_{(r, r')} - \pi_{[a]}(r)\pi_{[a']}(r')} {n_{[a]}n_{[a']}',\pi_{[a]}(r)\pi_{[a']}(r')} \sum_{i:A_i = a} \sum_{j:A_j = a'} \bar{Y}_i(r)\bar{Y}_j(r'),$$

where the last equality follows from (A7). We can further simplify $\text{Cov}\{B_{[a]}(r), B_{[a']}(r')\}$ as

$$\text{Cov}\{B_{[a]}(r), B_{[a']}(r')\} = \frac{\pi_{[a][a']}_{(r, r')} - \pi_{[a]}(r)\pi_{[a']}(r')} {n_{[a]}n_{[a']}',\pi_{[a]}(r)\pi_{[a']}(r')} \sum_{i:A_i = a} \sum_{j:A_j = a'} \bar{Y}_i(r)\bar{Y}_j(r') = 0.$$

(2) When $a = a'$ and $r \neq r'$, we need to consider the covariances between the treatment indicators of two different units and those of the same unit:

$$\text{Cov}\{B_{[a]}(r), B_{[a]}(r')\} = \text{Cov}\left\{\left\{n_{[a]}\pi_{[a]}(r)\right\}^{-1} \sum_{i:A_i = a} I(R_i = r)\bar{Y}_i(r), \left\{n_{[a]}\pi_{[a]}(r)\right\}^{-1} \sum_{j:A_j = a} I(R_j = r')\bar{Y}_j(r')\right\}$$

$$= \left\{\left\{n_{[a]}^2\pi_{[a]}(r)\pi_{[a]}(r')\right\}^{-1} \sum_{i:j \neq i:A_i = a, A_j = a} \bar{Y}_i(r)\bar{Y}_j(r') \text{Cov}\{I(R_i = r), I(R_j = r')\}\right\}$$

$$+ \left\{\left\{n_{[a]}^2\pi_{[a]}(r)\pi_{[a]}(r')\right\}^{-1} \sum_{i:A_i = a} \bar{Y}_i(r)\bar{Y}_j(r') \text{Cov}\{I(R_i = r), I(R_j = r')\}\right\}$$

$$= \frac{\pi_{[a][a']}_{(r, r')} - \pi_{[a]}(r)\pi_{[a]}(r')} {n_{[a]}^2\pi_{[a]}(r)\pi_{[a]}(r')} \sum_{i:j \neq i:A_i = a, A_j = a} \bar{Y}_i(r)\bar{Y}_j(r') - \frac{\pi_{[a]}(r)\pi_{[a]}(r')} {n_{[a]}^2\pi_{[a]}(r)\pi_{[a]}(r')} \sum_{i:A_i = a} \bar{Y}_i(r)\bar{Y}_j(r').$$
where the last equality follows from (A7). We can further simplify \( \text{Cov}\{B_{[a]}(r), B_{[a]}(r')\} \) as

\[
\text{Cov}\{B_{[a]}(r), B_{[a]}(r')\} = \frac{\pi_{[a][a]}(r, r') - \pi_{[a][a]}(r)\pi_{[a][a]}(r')}{n_{[a]}^2\pi_{[a]}(r)\pi_{[a]}(r')} \sum_{i:A_i=a} \sum_{j:A_j=a} \bar{Y}_i(r) \bar{Y}_j(r') \\
- \left( \frac{\pi_{[a][a]}(r, r') - \pi_{[a][a]}(r)\pi_{[a][a]}(r')}{n_{[a]}^2\pi_{[a]}(r)\pi_{[a]}(r')} + \frac{\pi_{[a][a]}(r)\pi_{[a][a]}(r')}{n_{[a]}^2\pi_{[a]}(r)\pi_{[a]}(r')} \right) \sum_{i:A_i=a} \bar{Y}_i(r) \bar{Y}_j(r') \\
= 0 - \frac{(n_{[a]} - 1)\pi_{[a][a]}(r, r')}{n_{[a]}^2\pi_{[a]}(r)\pi_{[a]}(r')} (n_{[a]} - 1)^{-1} \sum_{i:A_i=a} \bar{Y}_i(r) \bar{Y}_j(r') \\
= -\frac{(n_{[a]} - 1)\pi_{[a][a]}(r, r')}{n_{[a]}^2\pi_{[a]}(r)\pi_{[a]}(r')} S_i(a)(r, r'),
\]

where the last equality follows from (A5).

(3) When \( a = a' \) and \( r = r' \), we similarly consider two cases with \( i = j \) and \( i \neq j \):

\[
\text{Var}\{B_{[a]}(r)\} = \text{Cov} \left[ \{n_{[a]}\pi_{[a]}(r)\}^{-1} \sum_{i:A_i=a} I(R_i = r) \bar{Y}_i(r), \{n_{[a]}\pi_{[a]}(r)\}^{-1} \sum_{j:A_j=a} I(R_j = r) \bar{Y}_j(r) \right] \\
= \left\{n_{[a]}^2\pi_{[a]}^2(r)\right\}^{-1} \sum_{i \neq j:A_i=a, A_j=a} \bar{Y}_i(r) \bar{Y}_j(r) \text{Cov}\{I(R_i = r), I(R_j = r)\} \\
+ \left\{n_{[a]}^2\pi_{[a]}^2(r)\right\}^{-1} \sum_{i:A_i=a} \bar{Y}_i(r) \bar{Y}_i(r) \text{Cov}\{I(R_i = r), I(R_i = r)\} \\
= \frac{\pi_{[a][a]}(r, r) - \pi_{[a][a]}^2(r)}{n_{[a]}^2\pi_{[a]}^2(r)} \sum_{i \neq j:A_i=a, A_j=a} \bar{Y}_i(r) \bar{Y}_j(r) + \frac{\pi_{[a][a]}(r) - \pi_{[a][a]}^2(r)}{n_{[a]}^2\pi_{[a]}^2(r)} \sum_{i:A_i=a} \bar{Y}_i^2(r)
\]

where the last equality follows from (A7). We can further simplify \( \text{Var}\{B_{[a]}(r)\} \) as

\[
\text{Var}\{B_{[a]}(r)\} = \frac{\pi_{[a][a]}(r, r) - \pi_{[a][a]}^2(r)}{n_{[a]}^2\pi_{[a]}^2(r)} \sum_{i:A_i=a} \sum_{j:A_j=a} \bar{Y}_i(r) \bar{Y}_j(r) \\
+ \left( \frac{\pi_{[a][a]}(r) - \pi_{[a][a]}^2(r)}{n_{[a]}^2\pi_{[a]}^2(r)} - \frac{\pi_{[a][a]}(r, r) - \pi_{[a][a]}^2(r)}{n_{[a]}^2\pi_{[a]}^2(r)} \right) \sum_{i:A_i=a} \bar{Y}_i^2(r) \\
= 0 + \frac{\pi_{[a][a]}(r) - \pi_{[a][a]}^2(r)}{n_{[a]}^2\pi_{[a]}^2(r)} \sum_{i:A_i=a} \bar{Y}_i^2(r) \\
= (n_{[a]} - 1)\frac{\pi_{[a][a]}(r) - \pi_{[a][a]}^2(r)}{n_{[a]}^2\pi_{[a]}^2(r)} S_i^2(a)(r),
\]

where the last equality follows from (A5). \( \square \)

Proof of Lemma A2. Similarly to the proof of Lemma A1, we discuss three cases, and use (A7) to
calculate the covariance between two treatment indicators. (1) When \(a \neq a'\),

\[
\text{Cov}\{B_{[a]}(r), C_{[a']} (r')\} = \text{Cov} \left[\frac{n_{[a]} \pi_{[a]}(r)}{n_{[a']} \pi_{[a']}(r')} \sum_{i: A_i = a} I(R_i = r) \bar{Y}_i(r), \frac{n_{[a]} \pi_{[a]}(r)}{n_{[a']} \pi_{[a']}(r')} \sum_{j: A_j = a'} I(R_j = r') \bar{Y}_j(r')\right]^{-1} \sum_{i: A_i = a} I(R_i = r) \bar{Y}_i(r) \sum_{j: A_j = a'} I(R_j = r') \bar{Y}_j(r')
\]

where the last equality follows from (A7). We can further simplify \(\text{Cov}\{B_{[a]}(r), C_{[a']} (r')\}\) as

\[
\text{Cov}\{B_{[a]}(r), C_{[a']} (r')\} = \frac{\pi_{[a']} (r, r') - \pi_{[a]} (r) \pi_{[a']} (r')}{n_{[a']} \pi_{[a']} (r')} \bar{Y}_{[a']} (r') \cdot n_{[a']} \sum_{i: A_i = a} \bar{Y}_i (r) = 0.
\]

(2) When \(a = a'\) and \(r \neq r'\),

\[
\text{Cov}\{B_{[a]}(r), C_{[a]} (r')\} = \text{Cov} \left[\frac{n_{[a]} \pi_{[a]}(r)}{n_{[a]} \pi_{[a]}(r')} \sum_{i: A_i = a} I(R_i = r) \bar{Y}_i(r), \frac{n_{[a]} \pi_{[a]}(r)}{n_{[a]} \pi_{[a]}(r')} \sum_{j: A_j = a} I(R_j = r') \bar{Y}_j(r')\right]^{-1} \sum_{i: A_i = a} I(R_i = r) \bar{Y}_i(r) \sum_{j: A_j = a} I(R_j = r') \bar{Y}_j(r')
\]

where the last equality follows from from (A7). We can further simplify \(\text{Cov}\{B_{[a]}(r), C_{[a]} (r')\}\) as

\[
\text{Cov}\{B_{[a]}(r), C_{[a]} (r')\} = \frac{\pi_{[a]} (r, r') - \pi_{[a]} (r) \pi_{[a]} (r')}{n_{[a]} \pi_{[a]} (r')} \bar{Y}_{[a]} (r') \cdot (n_{[a]} - 1) \sum_{i: A_i = a} \bar{Y}_i (r) = 0 = 0.
\]

(3) When \(a = a'\) and \(r = r'\),

\[
\text{Cov}\{B_{[a]}(r), C_{[a]} (r)\} = \text{Cov} \left[\frac{n_{[a]} \pi_{[a]}(r)}{n_{[a]} \pi_{[a]}(r')} \sum_{i: A_i = a} I(R_i = r) \bar{Y}_i(r), \frac{n_{[a]} \pi_{[a]}(r)}{n_{[a]} \pi_{[a]}(r')} \sum_{j: A_j = a} I(R_j = r) \bar{Y}_j(r)\right]^{-1} \sum_{i: A_i = a} I(R_i = r) \bar{Y}_i(r) \sum_{j: A_j = a} I(R_j = r) \bar{Y}_j(r)
\]

\[
+ \frac{n_{[a]} \pi_{[a]}(r) \pi_{[a]} (r')}{n_{[a]} \pi_{[a]} (r')} \bar{Y}_{[a]} (r') \sum_{i: A_i = a} \bar{Y}_i (r) \sum_{j: A_j = a} \bar{Y}_j (r) \text{Cov}\{I(R_i = r), I(R_j = r)\}
\]

\[
+ \frac{n_{[a]} \pi_{[a]}(r) \pi_{[a]} (r')}{n_{[a]} \pi_{[a]} (r')} \bar{Y}_{[a]} (r') \sum_{i: A_i = a} \bar{Y}_i (r) \sum_{j: A_j = a} \bar{Y}_j (r) \text{Cov}\{I(R_i = r), I(R_j = r)\}
\]

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\[
= \frac{\pi_{[a][a]}(r, r) - \pi_{[a]}(r) \pi_{[a]}(r)}{n_{[a]}^2 \pi_{[a]}(r) \pi_{[a]}(r)} \sum_{i \neq j; A_i = a, A_j = a} \bar{Y}_i(r) \bar{Y}_j(r)
+ \frac{\pi_{[a]}(r) - \pi_{[a]}(r) \pi_{[a]}(r)}{n_{[a]}^2 \pi_{[a]}(r) \pi_{[a]}(r)} \sum_{i; A_i = a} \bar{Y}_i(r) \bar{Y}_j(r),
\]

where the last equality follows from (A7). We can further simplify \(\text{Cov}\{B_{[a]}(r), C_{[a]}(r)\}\) as

\[
\text{Cov}\{B_{[a]}(r), C_{[a]}(r)\} = \frac{\pi_{[a][a]}(r, r) - \pi_{[a]}(r) \pi_{[a]}(r)}{n_{[a]}^2 \pi_{[a]}(r) \pi_{[a]}(r)} \sum_{i; A_i = a} \bar{Y}_i(r) \times (n_{[a]} - 1) \bar{Y}_j(r) - 0 = 0.
\]

\[\square\]

**Proof of Lemma A3.** Similarly to the proofs of Lemmas A1 and A2, we discuss three cases separately, and use (A7) to calculate the covariance between two treatment indicators. (1) When \(a \neq a'\),

\[
\text{Cov}\{C_{[a]}(r), C_{[a']}\}(r') = \text{Cov}\left[\{n_{[a]} \pi_{[a]}(r)\}^{-1} \sum_{i; A_i = a} I(R_i = r) \bar{Y}_i([a](r)), \{n_{[a']} \pi_{[a']}\}(r')^{-1} \sum_{j; A_j = a'} I(R_j = r') \bar{Y}_j([a'](r'))\right]
\[
= \{n_{[a]}n_{[a']} \pi_{[a]}(r) \pi_{[a']}\}(r')^{-1} \bar{Y}_i([a](r)) \bar{Y}_j([a'](r')) \sum_{i; A_i = a} \sum_{j; A_j = a'} \text{Cov}\{I(R_i = r), I(R_j = r')\}
\[
= \frac{\pi_{[a][a']} (r, r') - \pi_{[a]}(r) \pi_{[a']}\}(r')}{{n_{[a]}n_{[a']} \pi_{[a]}(r) \pi_{[a']}\}(r')} \bar{Y}_i([a](r)) \bar{Y}_j([a'](r')) n_{[a]}n_{[a']}
\[
= \frac{\pi_{[a][a']} (r, r') - \pi_{[a]}(r) \pi_{[a']}\}(r')}{{n_{[a]}n_{[a']} \pi_{[a]}(r) \pi_{[a']}\}(r')} \bar{Y}_i([a](r)) \bar{Y}_j([a'](r')),
\]

where the last equality follows from (A7). Based on the definitions of \(d_{[a][a']} (r, r')\) and \(c_{[a][a']} (r, r')\) in (7) and (8), we can further simplify \(\text{Cov}\{C_{[a]}(r), C_{[a']}\}(r')\) as

\[
\text{Cov}\{C_{[a]}(r), C_{[a']}\}(r') = \left(\frac{\pi_{[a][a']} (r, r')}{{\pi_{[a]}(r) \pi_{[a']}\}(r')} - 1\right) \bar{Y}_i([a](r)) \bar{Y}_j([a'](r')) = \frac{d_{[a][a']} (r, r')}{\left(n_{[a]} n_{[a']}\right)^{1/2}} \bar{Y}_i([a](r)) \bar{Y}_j([a'](r'))
\[
= \frac{c_{[a][a']} (r, r')}{\left(n_{[a]} n_{[a']}\right)^{1/2}} \bar{Y}_i([a](r)) \bar{Y}_j([a'](r')).
\]

(2) When \(a = a'\) and \(r \neq r'\),

\[
\text{Cov}\{C_{[a]}(r), C_{[a]}\}(r') = \text{Cov}\left[\{n_{[a]} \pi_{[a]}(r)\}^{-1} \sum_{i; A_i = a} I(R_i = r) \bar{Y}_i([a](r)), \{n_{[a]} \pi_{[a]}\}(r')^{-1} \sum_{j; A_j = a} I(R_j = r') \bar{Y}_j([a](r'))\right]
\[
= \{n_{[a]}^2 \pi_{[a]}(r) \pi_{[a]}\}(r')^{-1} \bar{Y}_i([a](r)) \bar{Y}_j([a](r')) \sum_{i \neq j; A_i = a, A_j = a} \text{Cov}\{I(R_i = r), I(R_j = r')\}
\[
+ \{n_{[a]}^2 \pi_{[a]}(r) \pi_{[a]}\}(r')^{-1} \bar{Y}_i([a](r)) \bar{Y}_j([a](r')) \sum_{i; A_i = a} \text{Cov}\{I(R_i = r), I(R_i = r')\}
\]

\[12\]
= \frac{\pi_{[a][a]}(r, r') - \pi_{[a][a]}(r) \pi_{[a][a]}(r')}{n_{[a]}^2 \pi_{[a]}(r) \pi_{[a]}(r')} \bar{Y}_{[a]}(r) \bar{Y}_{[a]}(r') n_{[a]} (n_{[a]} - 1)
- \frac{\pi_{[a][a]}(r) \pi_{[a][a]}(r')}{n_{[a]}^2 \pi_{[a]}(r) \pi_{[a]}(r')} \bar{Y}_{[a]}(r) \bar{Y}_{[a]}(r') n_{[a]},

where the last equality follows from (A7). Based on the definitions of \(d_{[a][a]}(r, r')\) and \(c_{[a][a]}(r, r')\) in (7) and (8), we can further simplify \(\text{Cov}\{C_{[a]}(r), C_{[a]}(r')\}\) as

\[
\text{Cov}\{C_{[a]}(r), C_{[a]}(r')\} = \left\{ \left(1 - n_{[a]}^{-1}\right) \left(\frac{\pi_{[a][a]}(r, r')}{\pi_{[a]}(r) \pi_{[a]}(r')} - 1\right) - n_{[a]}^{-1}\right\} \bar{Y}_{[a]}(r) \bar{Y}_{[a]}(r')
= \left\{ \left(1 - n_{[a]}^{-1}\right) n_{[a]}^{-1}d_{[a][a]}(r, r') - n_{[a]}^{-1}\right\} \bar{Y}_{[a]}(r) \bar{Y}_{[a]}(r')
= \left(1 - n_{[a]}^{-1}\right) \frac{d_{[a][a]}(r, r') - 1}{n_{[a]}} \bar{Y}_{[a]}(r) \bar{Y}_{[a]}(r')
= \frac{c_{[a][a]}(r, r')}{n_{[a]}} \bar{Y}_{[a]}(r) \bar{Y}_{[a]}(r').
\]

(3) When \(a = a'\) and \(r = r'\),

\[
\text{Var}\{C_{[a]}(r)\} = \text{Cov} \left[ \{n_{[a]} \pi_{[a]}(r)\}^{-1} \sum_{i:A_i=a} I(R_i = r) \bar{Y}_{[a]}(r), \{n_{[a']} \pi_{[a']} (r')\}^{-1} \sum_{j:A_j=a'} I(R_j = r') \bar{Y}_{[a']} (r') \right]
= \left\{ n_{[a]}^2 \pi_{[a]}^2 (r) \right\}^{-1} \bar{Y}_{[a]}^2 (r) \sum_{i:A_i=a} \text{Cov} \{I(R_i = r), I(R_i = r)\}
+ \left\{ n_{[a]}^2 \pi_{[a]}^2 (r) \right\}^{-1} \bar{Y}_{[a]}^2 (r) \sum_{\ell:A_{\ell} = a} \text{Cov} \{I(R_i = r), I(R_i = r)\}
= \frac{\pi_{[a][a]}(r, r) - \pi_{[a][a]}^2 (r)}{n_{[a]}^2 \pi_{[a]}^2 (r)} \bar{Y}_{[a]}^2 (r) n_{[a]} (n_{[a]} - 1) + \frac{\pi_{[a][a]}(r) - \pi_{[a][a]}^2 (r)}{n_{[a]}^2 \pi_{[a]}^2 (r)} \bar{Y}_{[a]}^2 (r) n_{[a]},
\]

where the last equality follows from (A7). Based on the definitions of \(d_{[a][a]}(r, r)\) and \(c_{[a][a]}(r, r)\) in (7) and (8), we can further simplify \(\text{Var}\{C_{[a]}(r)\}\) as

\[
\text{Var}\{C_{[a]}(r)\} = \left\{ \left(1 - n_{[a]}^{-1}\right) \left(\frac{\pi_{[a][a]}(r, r)}{\pi_{[a]}^2 (r)} - 1\right) + n_{[a]}^{-1} \pi_{[a]}^{-1} (r) - n_{[a]}^{-1}\right\} \bar{Y}_{[a]}^2 (r)
= \left\{ \left(1 - n_{[a]}^{-1}\right) n_{[a]}^{-1}d_{[a][a]}(r, r) + n_{[a]}^{-1} \pi_{[a]}^{-1} (r) - n_{[a]}^{-1}\right\} \bar{Y}_{[a]}^2 (r)
= \left(1 - n_{[a]}^{-1}\right) \frac{d_{[a][a]}(r, r) + \pi_{[a]}^{-1} (r) - 1}{n_{[a]}} \bar{Y}_{[a]}^2 (r) = \frac{c_{[a][a]}(r, r)}{n_{[a]}} \bar{Y}_{[a]}^2 (r).
\]

\[
\Box
\]
Proof of Lemma A4. For $1 \leq a \leq H$ and $r \neq r' \in \mathcal{R}$, by definition, we have

$$S_a^2(r) + S_a^2(r') - S_a^2(r-r') = (n_a - 1)^{-1} \left[ \sum_{i: A_i = a} \bar{Y}_i^2(r) + \sum_{i: A_i = a} \bar{Y}_i^2(r') - \sum_{i: A_i = a} \left\{ \bar{Y}_i(r) - \bar{Y}_i(r') \right\}^2 \right]$$

$$= (n_a - 1)^{-1} \times 2 \sum_{i: A_i = a} \bar{Y}_i(r) \bar{Y}_i(r') = 2S_a^2(r, r'),$$

and

$$\overline{Y_a}(r)Y_a(r') + (2n_a)^{-1} \left\{ S_a^2(r) + S_a^2(r') - S_a^2(r-r') \right\}$$

$$= \{n_a(n_a - 1)\}^{-1} \sum_{i \neq f:A_i = A_j = a} Y_i(r)Y_j(r') + \{n_a(n_a - 1)\}^{-1} \left\{ \sum_{i: A_i = a} Y_i(r)Y_i(r') - n_a \bar{Y}_a(r)\bar{Y}_a(r') \right\}$$

$$= \{n_a(n_a - 1)\}^{-1} \left\{ \sum_{i: A_i = a} \sum_{j: A_j = a} Y_i(r)Y_j(r') - n_a \bar{Y}_a(r)\bar{Y}_a(r') \right\}$$

$$= \{n_a(n_a - 1)\}^{-1} \left\{ n_a^2 \bar{Y}_a(r)\bar{Y}_a(r') - n_a \bar{Y}_a(r)\bar{Y}_a(r') \right\} = \bar{Y}_a(r)\bar{Y}_a(r').$$

\[ \square \]

A2.3. Proofs of the theorems for general assignment mechanism

Proof of Theorem 1. First, we calculate the sampling variance of estimated subgroup average peer effect. From (A6) and Lemmas A1–A3, the covariances, Cov \{B_a(r), C_a(r)\}, Cov \{B_a(r), C_a(r')\}, Cov \{B_a(r'), C_a(r)\} and Cov \{B_a(r'), C_a(r')\}, are all zero for $r \neq r'$. Therefore, the sampling variance of subgroup average peer effect estimator is

$$\text{Var} \left\{ \hat{t}_a(r, r') \right\}$$

$$= \text{Var} \left\{ B_a(r) + C_a(r) - B_a(r') - C_a(r') \right\}$$

$$= \text{Var} \left\{ B_a(r) \right\} + \text{Var} \left\{ B_a(r') \right\} - 2Cov \left\{ B_a(r), B_a(r') \right\} + \text{Var} \left\{ C_a(r) \right\} + \text{Var} \left\{ C_a(r') \right\} - 2Cov \left\{ C_a(r), C_a(r') \right\}$$

$$= \left( n_a - 1 \right) \frac{\pi_a(r) - \pi_a(a)(r, r)}{n_a^2 \pi_a^2(a)} S_a^2(r) + \left( n_a - 1 \right) \frac{\pi_a(r') - \pi_a(a)(r', r')}{{n_a}^2 \pi_a^2(a)} S_a^2(r')$$

$$+ 2 \left( n_a - 1 \right) \frac{\pi_a(a)(r, r')}{n_a^2 \pi_a(r) \pi_a(a)} S_a^2(r, r') + n_a^{-1} \left\{ c_{a[a]}(r, r) \bar{Y}_a^2(r) + c_{a[a]}(r', r') \bar{Y}_a^2(r') - 2c_{a[a]}(r, r') \bar{Y}_a(r)\bar{Y}_a(r') \right\}$$

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Replacing $2S_{[a]}(r, r')$ and $\bar{Y}_{[a]}(r)\bar{Y}_{[a]}(r')$ by their expressions in Lemma A4, we can rewrite the sampling variance of $\hat{\tau}_{[a]}(r, r')$ as

$$\text{Var} \left\{ \hat{\tau}_{[a]}(r, r') \right\} = (n_{[a]} - 1) \frac{\pi_{[a]}(r) - \pi_{[a]}(r', r)}{n_{[a]}^2 \pi_{[a]}^2(r)} S_{[a]}^2(r) + \left( n_{[a]} - 1 \right) \frac{\pi_{[a]}(r') - \pi_{[a]}(r', r')}{n_{[a]}^2 \pi_{[a]}^2(r')} S_{[a]}^2(r')$$

$$+ \frac{(n_{[a]} - 1) \pi_{[a]}(r, r')}{n_{[a]}^2 \pi_{[a]}(r) \pi_{[a]}(r')} \left\{ S_{[a]}^2(r) + S_{[a]}^2(r') - S_{[a]}^2(r - r') \right\}$$

$$+ n_{[a]}^{-1} \left\{ c_{[a][a]}(r, r) Y_{[a]}^2(r) + c_{[a][a]}(r', r') Y_{[a]}^2(r') - 2 c_{[a][a]}(r, r') Y_{[a]}(r) Y_{[a]}(r') \right\}$$

$$- \frac{2 c_{[a][a]}(r, r') S_{[a]}^2(r) + S_{[a]}^2(r') - S_{[a]}^2(r - r')}{n_{[a]}} \frac{2}{2 n_{[a]}}.$$ (A8)

We then combine the terms and calculate the coefficients of $S_{[a]}^2(r), S_{[a]}^2(r')$ and $S_{[a]}^2(r - r')$ in (A8), separately. From the definitions of $c_{[a][a]}(r, r')$ and $b_{[a]}(r)$, we can simplify the coefficient of $S_{[a]}^2(r)$ as

$$n_{[a]}^{-1} \left\{ 1 - n_{[a]}^{-1} \right\} \left( \pi_{[a]}^{-1}(r) - \frac{\pi_{[a]}(r, r)}{\pi_{[a]}^2(r)} + \frac{\pi_{[a]}(r, r')}{\pi_{[a]}(r) \pi_{[a]}(r')} \right) - n_{[a]}^{-1} c_{[a][a]}(r, r') \right\} = n_{[a]}^{-1} b_{[a]}(r),$$

and similarly simplify the coefficient of $S_{[a]}^2(r')$ as $n_{[a]}^{-1} b_{[a]}(r')$. We can also simplify the coefficient of $S_{[a]}^2(r - r')$ as

$$- \frac{(n_{[a]} - 1) \pi_{[a]}(r, r')}{n_{[a]}^2 \pi_{[a]}(r) \pi_{[a]}(r')} + \frac{c_{[a][a]}(r, r')}{n_{[a]}^2} = -n_{[a]}^{-1}.$$

Therefore, (A8) reduces to

$$\text{Var} \left\{ \hat{\tau}_{[a]}(r, r') \right\} = n_{[a]}^{-1} \left\{ b_{[a]}(r) S_{[a]}^2(r) + b_{[a]}(r') S_{[a]}^2(r') - S_{[a]}^2(r - r') \right\}$$

$$+ n_{[a]}^{-1} \left\{ c_{[a][a]}(r, r) Y_{[a]}^2(r) + c_{[a][a]}(r', r') Y_{[a]}^2(r') - 2 c_{[a][a]}(r, r') Y_{[a]}(r) Y_{[a]}(r') \right\}.$$

Second, we calculate the covariance between two estimated subgroup average peer effects. For $a \neq a'$ and $r \neq r' \in \mathcal{R}$, according to Lemmas A1–A3, the covariances, $\text{Cov} \{ B_{[a]}(r), B_{[a']}(r') \}$, $\text{Cov} \{ B_{[a]}(r), C_{[a']}(r') \}$, $\text{Cov} \{ B_{[a]}(r'), B_{[a']}(r) \}$, $\text{Cov} \{ B_{[a]}(r'), C_{[a']} (r) \}$, $\text{Cov} \{ B_{[a]}(r), C_{[a']} (r') \}$, $\text{Cov} \{ B_{[a]}(r'), C_{[a']} (r) \}$ and $\text{Cov} \{ B_{[a]}(r'), C_{[a']} (r') \}$, are all zero. Therefore the sampling covariance between $\hat{\tau}_{[a]}(r, r')$ and $\hat{\tau}_{[a']} (r, r')$ is

$$\text{Cov} \left\{ \hat{\tau}_{[a]}(r, r'), \hat{\tau}_{[a']} (r, r') \right\} = \text{Cov} \left\{ B_{[a]}(r) + C_{[a]}(r) - B_{[a]}(r') - C_{[a]}(r'), B_{[a']} (r) + C_{[a']} (r) - B_{[a']} (r') - C_{[a']} (r') \right\}$$

$$= \text{Cov} \left\{ C_{[a]}(r), C_{[a']} (r) \right\} + \text{Cov} \left\{ C_{[a]}(r'), C_{[a']} (r') \right\} - \text{Cov} \left\{ C_{[a]} (r), C_{[a']} (r') \right\} - \text{Cov} \left\{ C_{[a]} (r'), C_{[a']} (r) \right\}$$

$$= (n_{[a]} n_{[a']}^{-1})^{1/2} \left\{ c_{[a][a']} (r, r) \bar{Y}_{[a]} (r) \bar{Y}_{[a']} (r) + c_{[a][a']} (r', r') \bar{Y}_{[a]} (r') \bar{Y}_{[a']} (r') \right\}.$$

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Proof of Theorem 2. First, we prove that $\hat{s}_a^2(r)$ defined in Section 3.3 is unbiased for the finite population variance $S_a^2(r).$ Note that

\[
E \left\{ \hat{Y}_a^2(r) \right\} = E \left\{ \frac{n_a \pi_a}{\pi_a^2} \right\}^{-1} \sum_{i : A_i = a} I(R_i = r) Y_i(r) \times \{n_a \pi_a \}^{-1} \sum_{j : A_j = a} I(R_j = r) Y_j(r)
\]

\[
= \frac{n_a^2 \pi_a^2}{n_a \pi_a^2} \sum_{i : A_i = a} Y_i(r) Y_j(r) \Pr(R_i = r, R_j = r)
\]

\[
+ \frac{n_a^2 \pi_a^2}{n_a \pi_a^2} \sum_{i : A_i = a} Y_i(r) Y_j(r) \frac{\pi_a}{\pi_a^2} \sum_{i : A_i = a} Y_i^2(r),
\]

where the last equality follows from the definitions of $\pi_a(r)$ and $\pi_a[a](r,r).$ We can then simplify $E \{ \hat{Y}_a^2(r) \}$ as

\[
E \left\{ \hat{Y}_a^2(r) \right\} = \frac{\pi_a[a](r,r)}{n_a^2 \pi_a^2} \sum_{i : A_i = a} Y_i(r) Y_j(r) + \frac{\pi_a(r) - \pi_a[a](r)}{n_a^2 \pi_a^2} Y_i^2(r)
\]

\[
= \frac{\pi_a(a)(r,r)}{n_a^2 \pi_a^2} \sum_{i : A_i = a} Y_i^2(r).
\]
The mean of $s^2_{[a]}(r)$ is

$$E \left\{ s^2_{[a]}(r) \right\} = \frac{n_{[a]} \pi^2_{[a]}(r)}{(n_{[a]} - 1) \pi_{[a]}(r, r)} \left[ \frac{n_{[a]} + c_{[a]}(r, r)}{n_{[a]}^2 \pi_{[a]}(r)} \sum_{i : A_i = a} E \left\{ I(R_i = r) \right\} Y_i^2(r) - E \left\{ \hat{Y}_{[a]}^2(r) \right\} \right]$$

$$= \frac{n_{[a]} \pi^2_{[a]}(r)}{(n_{[a]} - 1) \pi_{[a]}(r, r)} \left[ \frac{n_{[a]} + c_{[a]}(r, r)}{n_{[a]}^2 \pi_{[a]}(r)} \sum_{i : A_i = a} Y_i^2(r) - E \left\{ \hat{Y}_{[a]}^2(r) \right\} \right]$$

$$= \frac{n_{[a]} \pi^2_{[a]}(r)}{(n_{[a]} - 1) \pi_{[a]}(r, r)} \left[ \frac{(n_{[a]} - 1) \pi_{[a]}(r, r) + \pi_{[a]}(r)}{n_{[a]}^2 \pi_{[a]}(r)} \sum_{i : A_i = a} Y_i^2(r) - E \left\{ \hat{Y}_{[a]}^2(r) \right\} \right],$$

where the last equality follows from the definition of $c_{[a]}(r, r)$ in (8). Using (A9), we can further simplify $E \left\{ s^2_{[a]}(r) \right\}$ as

$$E \left\{ s^2_{[a]}(r) \right\} = \frac{n_{[a]} \pi^2_{[a]}(r)}{(n_{[a]} - 1) \pi_{[a]}(r, r)} \left\{ \frac{(n_{[a]} - 1) \pi_{[a]}(r, r) + \pi_{[a]}(r)}{n_{[a]}^2 \pi_{[a]}(r)} \sum_{i : A_i = a} Y_i^2(r) - \frac{\pi_{[a]}(r)}{\pi_{[a]}^2(r)} \sum_{i : A_i = a} Y_i^2(r) - \frac{\pi_{[a]}(r)}{\pi_{[a]}^2(r)} \sum_{i : A_i = a} \hat{Y}_i^2(r) \right\}$$

$$= \frac{n_{[a]} \pi^2_{[a]}(r)}{(n_{[a]} - 1) \pi_{[a]}(r, r)} \left\{ \frac{\pi_{[a]}(r)}{\pi_{[a]}^2(r)} \sum_{i : A_i = a} Y_i^2(r) - \frac{\pi_{[a]}(r)}{\pi_{[a]}^2(r)} \sum_{i : A_i = a} \hat{Y}_i^2(r) \right\}$$

$$= (n_{[a]} - 1)^{-1} \left\{ \sum_{i : A_i = a} Y_i^2(r) - n_{[a]} \hat{Y}_{[a]}^2(r) \right\} = S^2_{[a]}(r).$$

Second, we prove the unbiasedness of the estimator for $\hat{Y}_{[a]}^2(r)$. For $1 \leq a \leq H$ and $r \in \mathcal{R}$, according to (A9) and the unbiasedness of $s^2_{[a]}(r)$ for $S^2_{[a]}(r)$,

$$E \left[ \frac{n_{[a]} \hat{Y}_{[a]}^2(r) - \{b_{[a]}(r) - 1\} s_{[a]}^2(r)}{n_{[a]} + c_{[a]}(r, r)} \right] = n_{[a]} E \left\{ \hat{Y}_{[a]}^2(r) \right\} - \{b_{[a]}(r) - 1\} E \left\{ s_{[a]}^2(r) \right\}$$

$$= \frac{n_{[a]} \pi_{[a]}(r, r)}{n_{[a]} + c_{[a]}(r, r)} \left\{ \frac{\pi_{[a]}(r)}{\pi_{[a]}^2(r)} \sum_{i : A_i = a} Y_i^2(r) + \frac{\pi_{[a]}(r) - \pi_{[a]}(r)}{n_{[a]}^2 \pi_{[a]}^2(r)} \sum_{i : A_i = a} Y_i^2(r) \right\} - \frac{b_{[a]}(r) - 1}{n_{[a]} + c_{[a]}(r, r)} S^2_{[a]}(r),$$

which, based on the definitions of $c_{[a]}(r, r)$ and $b_{[a]}(r)$, further reduces to

$$\frac{n_{[a]} \pi_{[a]}(r, r)}{(n_{[a]} - 1) \pi_{[a]}(r, r) + \pi_{[a]}(r)} \left\{ \frac{\pi_{[a]}(r)}{\pi_{[a]}^2(r)} \sum_{i : A_i = a} Y_i^2(r) + \frac{\pi_{[a]}(r) - \pi_{[a]}(r)}{n_{[a]}^2 \pi_{[a]}^2(r)} \sum_{i : A_i = a} Y_i^2(r) \right\}$$

$$- \frac{\{\pi_{[a]}(r) - \pi_{[a]}(r)\} \sum_{i : A_i = a} Y_i^2(r) - n_{[a]} \hat{Y}_{[a]}^2(r)}{n_{[a]} \{(n_{[a]} - 1) \pi_{[a]}(r, r) + \pi_{[a]}(r)\}} \left\{ \sum_{i : A_i = a} Y_i^2(r) - n_{[a]} \hat{Y}_{[a]}^2(r) \right\}$$

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Third, we prove the unbiasedness of the estimator for \( \bar{Y}_{[a]}(r) \). For 1 \( \leq a \leq H \) and \( r \neq r' \in \mathcal{R} \),

\[
E \left\{ \frac{n_{[a]}}{n_{[a]} - 1} \frac{\pi_{[a]}(r) \pi_{[a]}(r')}{\pi_{[a]}(r,r')} \hat{Y}_{[a]}(r) \hat{Y}_{[a]}(r') \right\} = \frac{n_{[a]}}{n_{[a]} - 1} \frac{\pi_{[a]}(r) \pi_{[a]}(r')}{\pi_{[a]}(r,r')} \sum_{i:A_i = a} I(R_i = r) Y_i(r) + \sum_{j:A_j = a} I(R_j = r') Y_j(r') \]

where the second last equality holds because \( \text{pr}(R_i = r, R_j = r') = 0 \) for \( r \neq r' \in \mathcal{R} \).

Fourth, we prove that, for \( a \neq a' \) and \( r, r' \in \mathcal{R} \), \( \pi_{[a]}(r) \pi_{[a']}\pi_{[a]'}(r)/\pi_{[a]}\pi_{[a']}\pi_{[a]'}(r,r') \cdot \hat{Y}_{[a]}(r) \hat{Y}_{[a']}\pi_{a'}(r') \) is unbiased for \( \bar{Y}_{[a]}(r) \bar{Y}_{[a']}\pi_{a'}(r') \). For \( a \neq a' \) and \( r, r' \in \mathcal{R} \), any units \((i,j)\) such that \( A_i = a \) and \( A_j = a' \) must satisfy \( i \neq j \), and therefore

\[
E \left\{ \frac{\pi_{[a]}(r) \pi_{[a']}\pi_{[a]'}(r')}{\pi_{[a]}\pi_{[a']}\pi_{[a]'}(r,r')} \hat{Y}_{[a]}(r) \hat{Y}_{[a']}\pi_{a'}(r') \right\} = \frac{\pi_{[a]}(r) \pi_{[a']}\pi_{[a]'}(r')}{\pi_{[a]}\pi_{[a']}\pi_{[a]'}(r,r')} \sum_{i:A_i = a} \pi^{-1}_{[a]}(r) I(R_i = r) Y_i(r) \times n^{-1}_{[a]} \sum_{j:A_j = a'} \pi^{-1}_{a'}(r') I(R_j = r') Y_j(r') \]

A3. More technical details about complete randomization

Proof of Proposition 2. We first show that the numerical implementation in Section 2.4.2 generates treatment assignments under complete randomization. For any treatment assignment \( z \) with \( L(z) = z \), by definition, there are \( l_t \) groups with group attribute set \( g_t \) for 1 \( \leq t \leq T \). Thus, there
are \( \prod_{t=1}^{T} l_t! \) ways to arrange these \( m \) groups such that the first \( l_1 \) groups have group attribute set \( g_1 \), the next \( l_2 \) groups have group attribute \( g_2 \), ..., and the last \( l_T \) groups have group attribute \( g_T \). Each of the \( \prod_{t=1}^{T} l_t! \) arrangements is a group assignment from the numerical implementation, and all group assignments from the numerical implementation have the same probability. Therefore, under the numerical implementation, any assignment \( z \) with \( L(z) = z \) corresponds to \( \prod_{t=1}^{T} l_t! \) realizations and will have the same probability.

We then prove Proposition 2. From the above discussion, it is equivalent to consider the distribution of \( (R_1, \ldots, R_n) \) under the group assignment generated from the numerical implementation. Under the numerical implementation, for each \( 1 \leq a \leq H \) and the \( n_{[a]} \) units with attribute \( a \), the \( l_1 \times g_1(a) \) units in the first \( l_1 \) groups must receive treatment \( g_1 \setminus \{a\} \), the next \( l_2 \times g_2(a) \) units in the next \( l_2 \) groups must receive treatment \( g_2 \setminus \{a\} \), ..., and the last \( l_T \times g_T(a) \) units in the last \( l_T \) groups must receive treatment \( g_T \setminus \{a\} \), and the assignments of the \( n_{[a]} \) units into these \( m \) groups have the same probability. From the relationship between \( n_{[a]} \) and \( L_t(z)g_t(a) \) in (4), the first conclusion (1) in Proposition 2 holds. The second conclusion (2) in Proposition 2 follows directly from the independence among the group assignments for units with different attributes under the numerical implementation.

As a direct consequence of Proposition 2, we have the following results characterizing the probability law of complete randomization, and we will use them in later proofs.

**Proposition A2.** Under Assumptions 1 and 2, and under complete randomization defined in Section 2.4.2, for \( 1 \leq a, a' \leq H \) and \( r, r' \in \mathcal{R} \), we have \( \tau_{[a]}(r) = n_{[a]} \) if \( a = n_{[a]}/n_{[a]} \), \( b_{[a]}(r) = n_{[a]}/n_{[a]} \), \( c_{[a][a']}(r, r') = 0 \), and

\[
\pi_{[a][a']} (r, r') = \begin{cases}
\frac{n_{[a]} n_{[a']}}{n_{[a]} n_{[a']}} & \text{if } a = a'; \\
\frac{n_{[a]} n_{[a']}}{n_{[a]} - (n_{[a]} - 1)} & \text{if } a = a', r \neq r'; \\
\frac{n_{[a]} n_{[a']}}{n_{[a]} - (n_{[a]} - 1)} & \text{if } a = a', r = r'.
\end{cases}
\]

The formulas of \( \tau_{[a]}(r) \) and \( \pi_{[a][a']}(r, r') \) are standard in completely randomized experiments with multiple treatments, the formula of \( \pi_{[a][a']}(r, r') \) with \( a \neq a' \) follows from the independence between treatments of units with different attributes, and the formulas of \( d_{[a][a']}(r, r'), c_{[a][a']}(r, r') \) and \( b_{[a]}(r) \) follow from their definitions in (7)–(9).

**Proof of Theorem 3.** We first consider the point and interval estimator for the subgroup average peer effect. Under complete randomization, for the \( n_{[a]} \) units with attribute \( a \), we are essentially conducting a complete randomized experiments with \( n_{[a]} \) units receiving treatment \( r \). Based on Lemma A4 and the regularity condition (ii) in Condition 1, the finite population covariance between potential outcomes \( S_{[a]}(r, r') \) has a limit. From Li and Ding (2017, Theorem 5), \( \hat{\tau}_{[a]}(r, r') \) is asymptotically Normal:

\[
\sqrt{n_{[a]}} \left\{ \hat{\tau}_{[a]}(r, r') - \tau_{[a]}(r, r') \right\} \xrightarrow{d} \mathcal{N} \left( 0, \lim_{n \to \infty} n_{[a]} \text{Var} \{\tau_{[a]}(r, r')\} \right),
\]
where \( \lim_{n \to \infty} n_{[a]} \text{Var}\{ \hat{\tau}_{[a]}(r, r') \} \) exists due to the convergence of proportions of units receiving different treatments \( n_{[a]r}/n_{[a]} \) and the finite population variances of potential outcomes and individual peer effects \( S_{[a]}^2(r) \) and \( S_{[a]}^2(R) \).

Moreover, according to Li and Ding (2017, Proposition 3), the sample variance of observed outcomes in the subgroup consisting of units with attribute \( a \) receiving treatment \( r \), \( s_{[a]}^2(r) \), is consistent for the population analogue \( S_{[a]}^2(r) \), in the sense that \( s_{[a]}^2(r) - S_{[a]}^2(r) \overset{p}{\to} 0 \). Thus, the variance estimator \( \hat{V}_{[a]}(r, r') \) satisfies

\[
\frac{n_{[a]} \hat{V}_{[a]}(r, r') - n_{[a]} \text{Var}\{ \hat{\tau}_{[a]}(r, r') \}}{s_{[a]}^2(r)} = \frac{n_{[a]}}{n_{[a]r}} \left\{ s_{[a]}^2(r) - S_{[a]}^2(r) \right\} + \frac{n_{[a]}}{n_{[a]r'}} \left\{ s_{[a]}^2(r') - S_{[a]}^2(r') \right\} \overset{p}{\to} 0.
\]

Therefore, the Wald-type confidence interval for \( \tau_{[a]}(r, r') \) is asymptotically conservative.

Second, we consider the confidence interval for the average peer effect \( \tau(r, r') \). Based on Slutsky’s theorem, \( \hat{\tau}(r, r') \) is asymptotically Normal:

\[
\sqrt{n} \left\{ \hat{\tau}(r, r') - \tau(r, r') \right\} = \sum_{a=1}^{H} \sqrt{w_{[a]}} \sqrt{\frac{n_{[a]}}{n_{[a]r}}} \left\{ \hat{\tau}_{[a]}(r, r') - \tau_{[a]}(r, r') \right\} \overset{d}{\to} \mathcal{N} \left( 0, \lim_{n \to \infty} n \text{Var}\{ \hat{\tau}(r, r') \} \right).
\]

Moreover, the variance estimator \( \hat{V}(r, r') \) satisfies that

\[
n \hat{V}(r, r') - n \text{Var}\{ \hat{\tau}(r, r') \} = \sum_{a=1}^{H} w_{[a]} \left\{ n_{[a]} \hat{V}_{[a]}(r, r') - n_{[a]} \text{Var}\{ \hat{\tau}_{[a]}(r, r') \} - S_{[a]}^2(r-r') \right\} \overset{p}{\to} 0.
\]

Therefore, the Wald-type confidence interval for \( \tau(r, r') \) is asymptotically conservative.

**Proof of Theorem 4.** We prove the three conclusions in Theorem 4 as follows.

First, let \( |R| \) dimensional column vectors

\[
Y_{i}(R) = (Y_{i}(r_{1}), \ldots, Y_{i}(r_{|R|}))^{\top}, \quad \bar{Y}_{[a]}(R) = (\bar{Y}_{[a]}(r_{1}), \ldots, \bar{Y}_{[a]}(r_{|R|}))^{\top}, \quad \hat{Y}_{[a]}(R) = (\hat{Y}_{[a]}(r_{1}), \ldots, \hat{Y}_{[a]}(r_{|R|}))^{\top}
\]

consist of unit \( i \)'s all potential outcomes, all subgroup average potential outcomes, and all subgroup average potential outcome estimators, respectively. Then

\[
\theta_{i}(R) = \Gamma Y_{i}(R), \quad \theta_{[a]}(R) = \Gamma \bar{Y}_{[a]}(R), \quad \hat{\theta}_{[a]}(R) = \Gamma \hat{Y}_{[a]}(R).
\]

Based on the equivalence relationship in Proposition 2 and the variance formula in Li and Ding (2017, Theorem 3), the sampling covariance matrix of \( \hat{Y}_{[a]}(R) \) under complete randomization is

\[
\text{Cov}\{ \hat{Y}_{[a]}(R) \} = \text{diag} \left\{ \frac{S_{[a]}^2(r_1)}{n_{[a]r_1}}, \ldots, \frac{S_{[a]}^2(r_{|R|})}{n_{[a]r_{|R|}}} \right\} - \frac{1}{n_{[a]}(n_{[a]} - 1)} \sum_{i:A_i=a} \{Y_{i}(R) - \bar{Y}_{[a]}(R)\} \{Y_{i}(R) - \bar{Y}_{[a]}(R)\}^{\top},
\]

which implies the sampling covariance of \( \hat{\theta}_{[a]}(R) = \Gamma \hat{Y}_{[a]}(R) \).

Second, the regularity conditions of Theorem 4 and Li and Ding (2017, Theorem 5) imme-
diately imply the asymptotic Normality of \( \sqrt{n_a} \{ \hat{Y}_{[a]}(\mathcal{R}) - \tilde{Y}_{[a]}(\mathcal{R}) \} \), which further implies the asymptotic Normality of \( \sqrt{n_a} \{ \hat{\theta}_{[a]}(\mathcal{R}) - \theta_{[a]}(\mathcal{R}) \} \).

Third, according to Li and Ding (2017, Proposition 3), \( s^2_{[a]}(r) \) is consistent for \( S^2_{[a]}(r) \). Moreover, the second term in the covariance formula of \( \hat{\theta}_{[a]}(\mathcal{R}) \) is a positive semi-definite matrix. Therefore, the Wald-type confidence set using variance estimator (17) is asymptotically conservative.

Note that the second term in the covariance formula of \( \hat{\theta}_{[a]}(\mathcal{R}) \) is actually the finite population covariance matrix of \((\theta_i(r_1), \theta_i(r_2), \ldots, \theta_i(r_{|\mathcal{R}|}))^T\) for units with attribute \( a \) scaled by \( n_{[a]}^{-1} \), and these centered individual potential outcomes \( \theta_i(r) \)'s can be represented as linear functions of the individual peer effects \( \tau_i(r, r') \)'s. Thus, when the individual peer effects for units with the same attribute are additive, the finite population covariance matrix of \((\theta_i(r_1), \theta_i(r_2), \ldots, \theta_i(r_{|\mathcal{R}|}))^T\) for units with attribute \( a \) are zero, and the Wald-type confidence sets for \( \theta_{[a]}(\mathcal{R}) \) become asymptotically exact. □

**A4. More on random partitioning**

In this section, we discuss details of random partitioning. In particular, we give the formulas for \( \pi_{[a]}(r) \) and \( \pi_{[a][a']}(r, r') \), based on which we can get the formulas for \( d_{[a][a']}(r, r'), c_{[a][a']}(r, r') \) and \( b_{[a]}(r) \), the unbiased point estimators for peer effects, the sampling variances of peer effects estimators, and the corresponding variance estimators.

Each \( r \in \mathcal{R} \) is a set containing \( K \) unordered but replicable elements from \( \{1, 2, \ldots, H\} \). Let \( r(a) \) be the number of elements in set \( r \) that are equal to \( a \). If \( a \) itself belongs to \( r \), let \( r \setminus \{a\} \) be the set containing the remaining \( K - 1 \) elements, by deleting an element \( a \) from the set \( r \).

**Theorem A1.** Under random partitioning, for \( 1 \leq a, a' \leq H \) and \( r, r' \in \mathcal{R} \), the probability that a unit \( i \) with attribute \( A_i = a \) receives treatment \( r \) is

\[
\pi_{[a]}(r) = \Pr(R_i = r) = \frac{\binom{n_{[a]} - 1}{r(a)} \prod_{1 \leq q \leq H, q \neq a} \binom{n_{[q]} - 1}{r(q)}}{\binom{n - 1}{K}}, \quad (A10)
\]

and the probability that two different units \((i \neq j)\) with attributes \( A_i = a \) and \( A_j = a' \) receive treatments \( r \) and \( r' \) is

\[
\pi_{[a][a']}(r, r') = \Pr(R_i = r, R_j = r') = \frac{K}{n - 1} \cdot \psi_{[a][a']}(r, r') + \frac{n - K - 1}{n - 1} \cdot \phi_{[a][a']}(r, r'), \quad (A11)
\]

where

\[
\psi_{[a][a']}(r, r') = \begin{cases} \frac{\binom{n_{[a]} - 1}{r(a)} \binom{n_{[a']} - 1}{r(a')}}{\binom{n - 1}{K}}, & \text{if } a' \in r, a' \in r', r \setminus \{a\} = r' \setminus \{a\}, a \neq a', \\ \frac{\binom{n_{[a]} - 1}{r(a)}}{\binom{n - 1}{K}}, & \text{if } a' \in r, a \in r', r \setminus \{a\} = r' \setminus \{a\}, a = a', \quad (A12) \\ 0, & \text{otherwise}, \end{cases}
\]
and

\[
\phi_{[a][a']} (r, r') = \begin{cases} 
\binom{n[a][a'] - 1}{r[a]} \binom{n[a][a'] - 1 - r(a')}{r'[a']} \prod_{1 \leq q \leq K, q \neq a} \binom{n[a][q]}{r[q]}, & \text{if } a \neq a', \\
\binom{n[a][a'] - 2}{r[a]} \binom{n[a][a'] - 2 - r(a')}{r'[a']} \prod_{1 \leq q \leq K, q \neq a} \binom{n[a][q]}{r[q]}, & \text{if } a = a'.
\end{cases}
\]

We give some intuition to explain the formulas of \( \pi_{[a]} (r) \) and \( \pi_{[a][a']} (r, r') \) under random partitioning. First, in (A10), the denominator \( \binom{n-1}{K} \) denotes the total number of possible peers for unit \( i \), and the numerator denotes the number of possible peers such that unit \( i \) receives treatment \( r \). Second, for any two different units \( i \) and \( j \) with attributes \( a \) and \( a' \), we consider two cases according to whether units \( i \) and \( j \) are in the same group or not. The coefficients \( K/(n-1) \) and \( (n-K-1)/(n-1) \) in (A11) are the probabilities that units \( i \) and \( j \) are in the same group and not in the same group, respectively. Correspondingly, \( \psi_{[a][a']} (r, r') \) and \( \phi_{[a][a']} (r, r') \) represent the conditional probabilities that units \( i \) and \( j \) receive treatments \( r \) and \( r' \) given that \( i \) and \( j \) are and are not in the same group.

When units \( i \) and \( j \) are in the same group, they have \( K-1 \) common peers, and therefore, the treatment \( R_i \) of unit \( i \) consists of unit \( j \)'s attribute and the \( K-1 \) common peers' attributes, and the treatment \( R_j \) consists of unit \( i \)'s attribute and the \( K-1 \) common peers' attributes. Therefore, \( \psi_{[a][a']} (r, r') \) is positive if and only if \( a' \in r, a \in r' \) and \( r \setminus \{a'\} = r' \setminus \{a\} \). In (A12), when \( \psi_{[a][a']} (r, r') \neq 0 \), the denominator \( \binom{n-2}{K-1} \) counts the number of possible \( K-1 \) units in the same group as units \( i \) and \( j \), and the numerator counts the number of possible \( K-1 \) units in the same group as units \( i \) and \( j \) such that units \( i \) and \( j \) receive treatments \( r \) and \( r' \). In (A13), the denominator \( \binom{n-2}{K} \binom{n-2-K}{K} \) counts the number of possible peers for units \( i \) and \( j \), and the numerator counts the number of possible peers for units \( i \) and \( j \) such that units \( i \) and \( j \) receive treatments \( r \) and \( r' \).

Proof of Theorem A1. First, we calculate \( \pi_{[a]} (r) \). Assume that unit \( i \) has attribute \( A_i = a \). The total number of possible peers of unit \( i \) is \( \binom{n-1}{K} \), and the total number of possible peers of unit \( i \) such that unit \( i \) receives treatment \( r \) is

\[
\binom{n[a] - 1}{r(a)} \prod_{1 \leq q \leq K, q \neq a} \binom{n[q]}{r(q)},
\]

where \( \binom{n[a] - 1}{r(a)} \) counts the number of possible choice of the peers of unit \( i \) with attribute \( a \), and \( \binom{n[q]}{r(q)} \) counts the number of possible choice of the peers of unit \( i \) with attribute \( q \neq a \). Under random partitioning, any other \( K \) units have the same probability to be in the same group as unit \( i \). Therefore, (A10) holds.

Second, we calculate \( \pi_{[a][a']} (r, r') \). Assume that \( i \neq j \) are two units with attributes \( A_i = a \) and \( A_j = a' \). Under random partitioning, we can decompose the probability \( \text{pr}(R_i = r, R_j = r') \) into two parts according to whether units \( i \) and \( j \) are in the same group:

\[
\pi_{[a][a']} (r, r') = \text{pr}(R_i = r, R_j = r') = \text{pr}(j \in Z_i, R_i = r, R_j = r') + \text{pr}(j \notin Z_i, R_i = r, R_j = r')
\]
Because any other \( K \) units have the same probability to be the peers of unit \( i \), by symmetry, 
\[
\Pr(j \in Z_i) \Pr(R_i = r, R_j = r' \mid j \in Z_i) + \Pr(j \notin Z_i) \Pr(R_i = r, R_j = r' \mid j \notin Z_i).
\]

(A14)

We then consider the two conditional probabilities \( \psi_{[a][a']} (r, r') \equiv \Pr(R_i = r, R_j = r' \mid j \in Z_i) \) and \( \phi_{[a][a']} (r, r') \equiv \Pr(R_i = r, R_j = r' \mid j \notin Z_i) \).

When units \( i \) and \( j \) are in the same group, units \( i \) and \( j \) are peers of each other and they have \( K - 1 \) common peers. Therefore, they have positive probability to receive treatments \( r \) and \( r' \) if and only if \( r \) and \( r' \) satisfy \( a' \in r, a \in r' \), and \( r \setminus \{a'\} = r' \setminus \{a\} \). When \( a' \in r, a \in r' \), and \( r \setminus \{a'\} = r' \setminus \{a\} \), the total number of possible peers of units \( i \) and \( j \) such that units \( i \) and \( j \) receive treatments \( r \) and \( r' \) is

\[
\begin{cases}
\frac{(n_{[a]} - 1)_2}{(r(a) - 1)_2} \prod_{1 \leq q \leq H \neq a, a'} \left( \frac{n_{[a]} - q}{r(q)} \right), & \text{if } a \neq a', \\
\frac{(n_{[a]} - 2)_2}{(r(a) - 1)_3} \prod_{1 \leq q \leq H \neq a} \left( \frac{n_{[a]} - q}{r(q)} \right), & \text{if } a = a'.
\end{cases}
\]

Note that the total number of possible peers of units \( i \) and \( j \) is \( \binom{n - 2}{K - 1} \). Because any possible peers of units \( i \) and \( j \) have the same probability, by symmetry,

\[
\psi_{[a][a']} (r, r') = \Pr(R_i = r, R_j = r' \mid j \in Z_i)
\]

\[
= \begin{cases}
\frac{(n_{[a]} - 1)_2}{(r(a) - 1)_2} \prod_{1 \leq q \leq H \neq a, a'} \left( \frac{n_{[a]} - q}{r(q)} \right), & \text{if } a \neq a', \\
\frac{(n_{[a]} - 2)_2}{(r(a) - 1)_3} \prod_{1 \leq q \leq H \neq a} \left( \frac{n_{[a]} - q}{r(q)} \right), & \text{if } a = a'.
\end{cases}
\]

When units \( i \) and \( j \) are not in the same group, the total number of their possible peers is \( \binom{n - 2}{K} \), and the total number of their possible peers such that units \( i \) and \( j \) receive treatments \( r \) and \( r' \) is

\[
\begin{cases}
\frac{(n_{[a]} - 1)_2}{(r(a) - 1)_2} \prod_{1 \leq q \leq H \neq a, a'} \left( \frac{n_{[a]} - q}{r(q)} \right), & \text{if } a \neq a', \\
\frac{(n_{[a]} - 2)_2}{(r(a) - 1)_3} \prod_{1 \leq q \leq H \neq a} \left( \frac{n_{[a]} - q}{r(q)} \right), & \text{if } a = a'.
\end{cases}
\]

Because any possible peers of units \( i \) and \( j \) have the same probability, by symmetry,

\[
\phi_{[a][a']} (r, r') = \Pr(R_i = r, R_j = r' \mid j \notin Z_i)
\]

\[
= \begin{cases}
\frac{(n_{[a]} - 1)_2}{(r(a) - 1)_2} \prod_{1 \leq q \leq H \neq a, a'} \left( \frac{n_{[a]} - q}{r(q)} \right), & \text{if } a \neq a', \\
\frac{(n_{[a]} - 2)_2}{(r(a) - 1)_3} \prod_{1 \leq q \leq H \neq a} \left( \frac{n_{[a]} - q}{r(q)} \right), & \text{if } a = a'.
\end{cases}
\]

We have computed the four terms in (A14), and Theorem A1 follows directly. \( \square \)