ORTHOPROJECTORS ON PERTURBATIONS OF SPLINES SPACES

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Abstract. We show that $L^\infty$-norms of orthoprojectors on certain types of perturbations of spline spaces are bounded independently of the knot sequence. Explicit applications of this result are given, one of them being orthoprojectors onto Chebyshevian spline spaces.

1. Introduction

In this paper, we extend Shadrin’s theorem [13] on the boundedness of the polynomial spline orthoprojector on $L^\infty$ by a constant that does not depend on the underlying univariate grid to certain perturbations of spline spaces.

One of the main reasons to consider this extension of spline orthoprojectors is that in recent years, it turned out that in many cases (see e.g. [13, 11, 7, 6, 3, 10, 9]), sequences of orthogonal projections onto classical spline spaces corresponding to arbitrary grid sequences behave like sequences of conditional expectations (or, more generally, like martingales) and we want to extend martingale type results to an even larger class of orthogonal projections.

In order to explain those martingale type results, we have to introduce a little bit of terminology: Let $k$ be a positive integer, $(\mathcal{F}_n)$ an increasing sequence of interval $\sigma$-algebras of sets in $[0,1]$, where we say that a $\sigma$-algebra is an interval $\sigma$-algebra if it is generated by a finite partition of $[0,1]$ into intervals of positive length. Moreover, let

$$S_n^{(k)} = \{ f \in C^{k-2}[0,1] : f \text{ is a polynomial of order } k \text{ on each atom of } \mathcal{F}_n \}$$

be the spline space of order $k$ corresponding to $\mathcal{F}$ and define $P_n^{(k)}$ as the orthogonal projection operator onto $S_n^{(k)}$ with respect to the $L^2$ inner product on $[0,1]$ with Lebesgue measure $| \cdot |$. The space $S_n^{(1)}$ (interpreting $C^{-1}[0,1]$ as the space of all real-valued functions on $[0,1]$) consists of piecewise constant functions and $P_n^{(1)}$ is the conditional expectation operator with respect to the $\sigma$-algebra $\mathcal{F}_n$. Similarly to the definition of martingales, we introduce the following notion pertaining to spline spaces: let $(f_n)_{n\geq0}$ be a sequence of integrable functions, we call this sequence a $k$-martingale spline sequence (adapted to $(\mathcal{F}_n)$), if

$$P_n^{(k)} f_{n+1} = f_n, \quad n \geq 0.$$  

Classical martingale theorems such as Doob’s inequality, the martingale convergence theorem or Burkholder’s inequality in fact carry over to $k$-martingale spline sequences corresponding to arbitrary filtrations $(\mathcal{F}_n)$ of the above type. Indeed, we have for any positive integer $k$

(i) (Shadrin’s theorem) there exists a constant $c_k$ depending only on $k$ such that

$$\sup_n \| P_n^{(k)} : L^1 \to L^1 \| \leq c_k,$$
(ii) there exists a constant $c_k$ depending only on $k$ such that for any $k$-martingale spline sequence $(f_n)$ and any $\lambda > 0$,
\[ \left| \left\{ \sup_n |f_n| > \lambda \right\} \right| \leq c_k \frac{\sup_n \|f_n\|_{L^1}}{\lambda}. \]

(iii) for all $p \in (1, \infty]$ there exists a constant $c_{p,k}$ depending only on $p$ and $k$ such that for all $k$-martingale spline sequences $(f_n)$,
\[ \left\| \sup_n |f_n| \right\|_{L^p} \leq c_{p,k} \sup_n \|f_n\|_{L^p}, \]

(iv) if $(f_n)$ is an $L^1$-bounded $k$-martingale spline sequence, then $(f_n)$ converges almost surely to some $L^1$-function.

(v) for all $p \in (1, \infty)$, scalar-valued $k$-spline-differences converge unconditionally in $L^p$, i.e. for all $f \in L^p$,
\[ \left\| \sum_n \pm (P_n^{(k)} - P_{n-1}^{(k)}) f \right\|_{L^p} \leq c_{p,k} \|f\|_{L^p}, \]

for some constant $c_{p,k}$ depending only on $p$ and $k$.

Item (i) is proved in [13], for a considerably shorter proof we refer to [4]. Banach space valued versions of (iii)–(v) are proved in [11, 6] and (v) is proved in [7]. For periodic spline spaces some of those properties are proved in [8, 5]. The basic starting point in proving the results (ii)–(v) independently of the filtration $(\mathcal{F}_n)$ is Shadrin’s theorem (i) and in this paper we prove its analogue for certain perturbations of spline spaces.

An important tool in the analysis of the operators $P_n^{(k)}$ as well as in the formulation of our perturbation result in Section 2 are special localized bases of the spaces $S_n^{(k)}$ and perturbations thereof. In the case of the space $S_n^{(k)}$ this is the so called B-spline basis $(M_i)$, normalized in $L^1$. By definition, the spaces $(S_n^{(k)})_n$ are nested, i.e., $S_n^{(k)} \subset S_{n+1}^{(k)}$ for all $n$. Moreover, the B-splines $(M_i)$ have the following properties:

(a) $\text{supp } M_i \cap \text{supp } M_j = \emptyset$ for $|i - j| \geq k$,
(b) each function $M_j$ only depends on the local form of $\mathcal{F}_n$, i.e., on $\mathcal{F}_n \cap \text{supp } M_j$,
(c) $\text{supp } M_i$ is a union of atoms of $\mathcal{F}_n$.

Note that the uniform (in $n$) $L^1$-boundedness of $P_n^{(k)}$ stated in (i) and their uniform $L^\infty$-boundedness are equivalent, as $P_n^{(k)}$ is self-adjoint with respect to the inner product $\langle f, g \rangle = \int_0^1 f(x)g(x) \, dx$ since it is an orthogonal projection. Defining the renormalized B-spline function $N_i = (|\text{supp } M_i|/k)M_i$, the uniform boundedness of $\|P_n^{(k)}\|_{L^\infty} = \|P_n^{(k)} : L^\infty \to L^\infty\|$ can be rephrased in terms of the Gram matrix $G = (\langle M_i, N_j \rangle)$. In fact, the uniform boundedness of $\|P_n^{(k)}\|_{L^\infty}$ is equivalent (see [11] or [13]) to the uniform estimate
\[ (1.1) \quad \|G^{-1}\|_{\infty} \leq C_k, \]

where $C_k$ is some constant depending only on the spline order $k$.

2. Projectors onto perturbed spline spaces

In this section we define what we mean by perturbations of spline spaces and prove the corresponding theorem about the uniform boundedness on $L^\infty$ of the associated orthoprojectors.

Let $k$ be an arbitrary positive integer, $\mu$ be a non-atomic probability measure on $[0, 1]$ and $\theta : [0, \infty) \to [0, \infty)$ be an increasing function with $\lim_{t \to 0} \theta(t) = 0$. For any interval $\sigma$-algebra $\mathcal{F}$ on $[0, 1]$, let $S_{\mathcal{F}}^{(k)}$ be the spline space of order $k$ corresponding to $\mathcal{F}$ and
let $S_{F,p}^{(k)} \subset L^2(\mu)$ be a finite dimensional linear space. As above, we denote by $(M_i)$ the B-spline basis of $S_{F,k}^{(k)}$ and additionally, we use the notation $\langle f, g \rangle_\mu = \int_0^1 f(x)g(x)d\mu(x)$ and $|F|_\mu = \max_A \mu(A)$, where max is taken over all atoms $A$ of $F$. We also define the $F$-support $\text{supp}_F f$ of a function $f : [0, 1] \to \mathbb{R}$ to be the smallest subset of $[0, 1]$ that is a union of atoms of $F$ and contains the support $\text{supp} f$ of $f$.

We say that the collection $(S_{F,p}^{(k)})_F$ is a $(\mu, \theta)$-perturbation of the spline spaces $(S_{F,k}^{(k)})_F$ with constant $C$ if, for any $F$, $S_{F,p}^{(k)}$ admits a basis $(M_i^p)$ so that for any indices $i,j$, we have

1. $|\mu(\text{supp}_F M_i^p)(M_i^p, M_j^p)_\mu - |\text{supp}_F M_j|(M_i, M_j)| \leq \theta(|F|_\mu)$,
2. $\|M_i^p \cap \text{supp}_F M_j^p = \emptyset$ for $|i - j| \geq C$,
3. $\|M_i^p\|_{L^\infty(\mu)} \cdot \mu(\text{supp}_F M_i^p) \leq C$.

If $(S_{F,p}^{(k)})_F$ is a $(\mu, \theta)$-perturbation of $(S_{F,k}^{(k)})_F$, we say that the spaces $(S_{F,p}^{(k)})_F$ are compatible if for each $G \subset F$, the following conditions are satisfied:

4. Nestedness: $S_{G,p}^{(k)} \subset S_{F,p}^{(k)}$,
5. Local structure of the basis: for each set $I \subset [0, 1]$ so that the trace $\sigma$-algebras $F \cap I$ and $G \cap I$ coincide and each basis function $M_i^p$ of $S_{F,p}^{(k)}$ with $\text{supp}_F M_i^p \subset I$, we also have $M_i^p \in S_{G,p}^{(k)}$.

If $(S_{F,p}^{(k)})_F$ is a $(\mu, \theta)$-perturbation of $(S_{F,k}^{(k)})_F$, denote by $P_{F,\mu} : L^2(\mu) \to L^2(\mu)$ the orthogonal projection operator onto the space $S_{F,p}^{(k)}$ with respect to the inner product $\langle \cdot, \cdot \rangle_\mu$.

**Remark 2.1.** For B-spline functions $(M_i)$ that form a basis of some spline space $S_{F,k}^{(k)}$, the notions of support and $F$-support coincide by property [4] on page 2, i.e. we have $\text{supp} M_i = \text{supp}_F M_i$ for any index $i$.

The collection of spline spaces $(S_{F,k}^{(k)})_F$ is a compatible $(|\cdot|, 0)$-perturbation of itself.

The last condition [5] means that the basis function $M_i^p$ is, in some sense, determined only by the local structure of $F$. Observe that by their very definition and property [3] on page 2 of the B-spline functions $(M_i)$, the spline spaces $(S_{F,k}^{(k)})_F$ are compatible.

If $\mu$ is an arbitrary non-atomic finite measure on $[0, 1]$, we can define the probability measure $\overline{\mu} = \mu/m$ with $m = \mu[0,1]$. Additionally the functions $\overline{M_i^p} = m M_i^p$ and $\overline{\theta}(t) = \theta(mt)$ satisfy conditions [1]-[3] as well as $P_{F,\overline{\mu}} = P_{F,\mu}$. Therefore, there is no loss of generality in assuming $\mu$ to be a probability measure.

**Theorem 2.2.** Let $k$ be a positive integer, $\mu$ be a non-atomic probability measure on $[0, 1]$, $\theta : [0, \infty) \to [0, \infty)$ be an increasing function with $\lim_{t \to 0} \theta(t) = 0$ and $C$ be a positive constant. Assume that $(S_{F,p}^{(k)})_F$ is a $(\mu, \theta)$-perturbation of $(S_{F,k}^{(k)})_F$ with constant $C$.

Then, there exists a constant $K_1$ depending only on $C$ and $k$ so that

$$\sup_F \|P_{F,\mu} : L^\infty(\mu) \to L^\infty(\mu)\| \leq K_1,$$

where $\sup$ is taken over all interval $\sigma$-algebras $F$ with $|F|_\mu \leq \varepsilon$ for $\varepsilon > 0$ taken so that $\theta(\varepsilon) \leq k/(4CkC)$ with the constant $C_k$ from [1].

Additionally, if the spaces $(S_{F,p}^{(k)})_F$ are compatible we have

$$\sup_F \|P_{F,\mu} : L^\infty(\mu) \to L^\infty(\mu)\| \leq K_2,$$
where sup is taken over all interval $\sigma$-algebras $\mathcal{F}$ and $K_2$ is a constant depending only on $C$, $\theta$ and $k$.

Proof. Let $\mathcal{F}$ be an arbitrary interval $\sigma$-algebra with $|\mathcal{F}|_\mu \leq \varepsilon$ and $(M_i)$, $(M_i^p)$ be the bases corresponding to the spaces $S_{\mathcal{F},\mu}^{(k)}$ and $S_{\mathcal{F},p}^{(k)}$, respectively, satisfying conditions (1)–(3). Let $G = \langle \langle M_i, N_j \rangle \rangle_{ij}$ and $G_p = \langle \langle M_i^p, N_j^p \rangle \rangle_{ij}/\mu$, where $N_j = (|\text{supp } M_j|/k) M_j$. Let $\theta(q)$, $\mu(q)$ be an arbitrary interval $\sigma$-algebra with $|\mathcal{F}|_\mu \leq \varepsilon$, with $\varepsilon$ so that $\theta(\varepsilon) \leq k/(4C_kC)$, we obtain

$$
\|X\|_\infty \leq \|G^{-1}\|_\infty \|G_p - G\|_\infty \leq 1/2.
$$

Thus we have $(I - X)^{-1} = \sum_{k=0}^\infty X^k$ and

$$
\|(I - X)^{-1}\|_\infty \leq \sum_{k=0}^\infty \|X\|^k_\infty \leq 2.
$$

Notice that $(G_p)^{-1} = (G + (G_p - G))^{-1} = (I + G^{-1}(G_p - G))^{-1} = (I - X)^{-1} G^{-1}$. Using estimate (2.3) together with (2.2) we obtain

$$
\|(G_p)^{-1}\|_\infty \leq 2\|G^{-1}\|_\infty \leq 2C_k.
$$

Additionally, we observe that we have a similar bound also for the norm $\|G_p\|_\infty$ of the banded matrix $G_p$ by properties (2) and (3):

$$
\|G_p\|_\infty = \max_i \sum_j \langle M_i^p, N_j^p \rangle \mu
$$

\leq \max_i \sum_{j: \text{supp } M_i^p \cap \text{supp } M_j^p \neq \emptyset \mu} \|M_i^p\|_{L^1(\mu)} \|N_j^p\|_{L^\infty(\mu)}

\leq 2C \max_{i,j} \|M_i^p\|_{L^\infty(\mu)} \mu(\text{supp } M_i^p) \cdot \frac{\mu(\text{supp } M_j^p)}{k} \leq 2C^3/k.

Next, we apply Denko’s theorem [2] that states in particular— that the boundedness of $\|G_p\|_\infty$ and $\|(G_p)^{-1}\|_\infty$ of a banded matrix $G_p$ is sufficient to deduce the geometric decay of the matrix $(a_{ij})_{ij} = G_p^{-1}$, i.e. we have for any $i, j$ the estimate $|a_{ij}| \leq c q^{|i-j|}$, where $c$ and $q \in (0, 1)$ are two constants depending only on $\|G_p\|_\infty$ and $\|(G_p)^{-1}\|_\infty$.

In our case this means that $c$ and $q$ only depend on $C$ and $k$. Using this matrix, the projection operator $P_{\mathcal{F},\mu}$ onto $S_{\mathcal{F},p}$ is given by the formula

$$
P_{\mathcal{F},\mu} f(t) = \sum_{i,j} a_{ij} \langle f, M_j^p \rangle \mu N_i^p(t).
$$

Hence, by (3) and the geometric decay estimate for $a_{ij}$, we have for $\mu$-almost-every $t \in [0, 1]$ the inequality

$$
|P f(t)| \leq \sum_{i,j: t \in \text{supp } N_i^p} |a_{ij}| \|f\|_{L^\infty(\mu)} \|M_j^p\|_{L^1(\mu)} \|N_i^p\|_{L^\infty(\mu)}
$$
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where (\( S \))

\[ \sum_{i,j \in \text{supp} N_i^p} q^{i-j} \| f \|_{L^\infty(\mu)}. \]

Now we use again property \( [2] \) to deduce

\[ \| Pf \|_{L^\infty(\mu)} \leq K_1 \| f \|_{L^\infty(\mu)} \]

with \( K_1 = 4cC^3 \sum_{j=0}^{\infty} q^j / k \) depending only on \( C \) and \( k \). This concludes the proof of the first assertion of the theorem.

Next, we assume that the spaces \( (S_{\mathcal{F},p})_{\mathcal{F}} \) are compatible as well. Let \( \mathcal{G} \) be an interval \( \sigma \)-algebra with \( |G|_\mu \geq \varepsilon \). Let \( U \subset [0, 1] \) be the point set consisting of all atoms \( A \) of \( \mathcal{G} \) with \( \mu(A) \leq \varepsilon \). Next, we let \( \mathcal{F} \) be an interval \( \sigma \)-algebra with \( |\mathcal{F}|_\mu \leq \varepsilon \) that is a refinement of \( \mathcal{G} \) and coincides with \( \mathcal{G} \) on \( U \) and has the property that for any atom \( A \subset U^c \) of \( \mathcal{F} \), we have \( \mu(A) \geq \varepsilon / 2 \). This is possible since \( \mu \) is assumed to be nonatomic. Let \( N_i^p \) be a basis function from \( S_{\mathcal{F},p}^{(k)} \), given by \( [2.1] \), with \( \text{supp}_\mathcal{F} N_i^p \subset U \).

Then, since \( S_{\mathcal{F},p}^{(k)} \) and \( S_{\mathcal{G},p}^{(k)} \) are compatible, the function \( N_i^p \) is also contained in \( S_{\mathcal{G},p}^{(k)} \). Since the operators \( P_{\mathcal{G},\mu} \) and \( P_{\mathcal{F},\mu} \) are both orthogonal projections, we get that both \( P_{\mathcal{G},\mu} f - f \) and \( P_{\mathcal{F},\mu} f - f \) are orthogonal to the function \( N_i^p \) in \( L^2(\mu) \). This implies

\[ \langle (P_{\mathcal{F},\mu} - P_{\mathcal{G},\mu}) f, N_i^p \rangle_\mu = 0, \quad \text{supp}_\mathcal{F} N_i^p \subset U. \]

Therefore, since \( S_{\mathcal{G},p}^{(k)} \subset S_{\mathcal{F},p}^{(k)} \), we expand

\[ (P_{\mathcal{F},\mu} - P_{\mathcal{G},\mu}) f = \sum_{i : \text{supp}_\mathcal{F} N_i^p \not\subset U} d_i N_i^{p,*}, \]

where \( (N_j^{p,*}) \) denotes the basis of \( S_{\mathcal{F},p}^{(k)} \) that is dual to the basis \( (N_j^p) \) w.r.t the inner product in \( L^2(\mu) \). Moreover, the coefficients \( d_i \) and the functions \( N_i^{p,*} \) are given by the formulas

\[ d_i = \langle (P_{\mathcal{F},\mu} - P_{\mathcal{G},\mu}) f, N_i^p \rangle_\mu, \quad N_i^{p,*} = \sum_j a_{ij} N_j^p = \sum_j b_{ij} N_j^{p,*}, \]

where the matrix \( (a_{ij}) \), as above, denotes the inverse \( G_p^{-1} \) of the Gram matrix \( G_p = \langle (M_i^p, N_j^p) \rangle_\mu \) and the matrix \( (b_{ij}) \) denotes the inverse of the matrix \( \langle (N_i^p, N_j^p) \rangle_\mu \). Since \( N_i^p = a_i M_i^p \) with \( a_i := \mu(\text{supp}_\mathcal{F} N_i^p) / k \), the coefficients \( a_{ij} \) and \( b_{ij} \) are related by the equation \( b_{ij} = a_{ij} / a_j \). Since the matrix \( (b_{ij}) \) is symmetric, we also have \( b_{ij} = b_{ji} = a_{ji} / a_i \). The geometric decay of the matrix \( (a_{ij}) \) and (3) imply the pointwise estimate

\[ |N_i^{p,*}(t)| \leq \sum_j |b_{ij}| |N_j^p(t)| \leq \frac{cC}{a_i} \sum_{j : \text{supp}_\mathcal{F} N_j^p} q^{|i-j|}, \quad \text{for } \mu\text{-a.e. } t. \]

Denoting by \( j(t) \) any fixed index with \( t \in \text{supp}_\mathcal{F} N_j^{p,(t)} \), we obtain the estimate

\[ |N_i^{p,*}(t)| \leq \frac{c_1}{\mu(\text{supp}_\mathcal{F} N_j^p)} q^{|i-j(t)|}, \]

with \( c_1 := 4kcC^2 q^{-C} \sum_{t=0}^{\infty} q^t \). Next, we estimate the coefficients \( d_i \) from equation (2.5). Since \( P_{\mathcal{F},\mu} \) and \( P_{\mathcal{G},\mu} \) are both orthogonal projections, they have an \( L^2(\mu) \)-norm of 1. Thus, we estimate \( d_i \) as

\[ |d_i| \leq 2 \| f \|_{L^2(\mu)} \| N_i^p \|_{L^2(\mu)} \leq \frac{2C}{k} \| f \|_{L^\infty(\mu)} \mu(\text{supp}_\mathcal{F} N_i^p)^{1/2}, \]

where in the last step we used property \( [3] \).
Now, insert estimates (2.6) and (2.7) into (2.5) and observe that for \( i \) such that \( \text{supp}_F N_i \not\subset U \), we have \( \mu(\text{supp}_F N_i) \geq \varepsilon/2 \) by definition of \( F \). Therefore, for \( t \)-almost-every \( t \in [0, 1] \),

\[
|(P_{F, \mu} - P_{G, \mu})f(t)| = \left| \sum_{i: \text{supp}_F N_i \not\subset U} d_i N_i^{p, \ast}(t) \right| \leq 4c_1 \frac{C}{k \varepsilon^{1/2}} \sum_{i: \text{supp}_F N_i \not\subset U} \|f\|_{L^\infty} q^{1-j(t)}(t) \leq c_3 \|f\|_{L^\infty} \]

with \( c_3 := 8c_1 C \sum_{j=0}^{\infty} q^j / \varepsilon^{1/2} \). Taking the supremum over \( t \in [0, 1] \), we obtain that the operator \( P_{F, \mu} - P_{G, \mu} : L^\infty(\mu) \to L^\infty(\mu) \) is bounded by \( c_3 \). Therefore this and the first part of the Theorem 2.2 imply

\[
\|P_{G, \mu}\|_{L^\infty(\mu)} \leq \|P_{G, \mu} - P_{F, \mu}\|_{L^\infty(\mu)} = \|P_{F, \mu}\|_{L^\infty(\mu)} \leq c_3 + K_1
\]

and thus we get the second assertion of the theorem with \( K_2 := c_3 + K_1 \). \( \square \)

3. Applications

In this section we investigate concrete applications of Theorem 2.2. The first example considers projection operators on classical spline spaces corresponding to different measures and the second example considers orthoprojectors onto Chebyshevian spline spaces.

3.1. Weighted spline spaces. We consider the setting of standard B-splines \((M_j)\) on an interval \(\sigma\)-algebra \(F\), \(S_T = \text{span}(M_j)\). Moreover, consider the measure \(\mu\) with \(d\mu = w \, dx, w : [0, 1] \to (0, \infty)\) being a continuous function satisfying the inequalities \(M^{-1} \leq w \leq M\) for some constant \(M\). Set \(P_i^p = M_i/w(c_i)\) with the center \(c_i\) of \(\text{supp} M_i\).

We now show properties (1)–(3) on page 3 in this setting. First consider property (1): we see that

\[
|\mu(\text{supp} M_i^p, M_j^p)\mu - |\text{supp} M_j|(M_i, M_j)|
\]

\[
\leq \int_{\text{supp} M_i \cap \text{supp} M_j} M_i(x) M_j(x) \, |\text{supp} M_j| \left| \frac{\mu(\text{supp} M_j)}{|\text{supp} M_j|} \frac{w(x)}{w(c_i)w(c_j)} - 1 \right| \, dx.
\]

Note that \(\mu(\text{supp} M_j)/|\text{supp} M_j| = w(\xi_j)\) for some \(\xi_j \in \text{supp} M_j\). Moreover, for \(x \in \text{supp} M_i \cap \text{supp} M_j\),

\[
\left| \frac{w(\xi_j)w(x)}{w(c_i)w(c_j)} - 1 \right| \leq M^2 (w(\xi_j)|w(x) - w(c_j)| + w(c_j)|w(\xi_j) - w(c_j)|)
\]

\[
\leq 2M^3 \omega(w, k|F|),
\]

where \(\omega(f, \delta) := \sup_{|x-y|<\delta} |f(x) - f(y)|\) denotes the modulus of continuity of \(f\). As

\[
\int_{\text{supp} M_i \cap \text{supp} M_j} M_i(x) M_j(x) \, |\text{supp} M_j| \, dx \leq (|\text{supp} M_i|)(\|M_j\|_\infty|\supp M_j|) \leq k^2
\]

and \(\omega(w, k|F|) \to 0\) as \(|F|/M \leq |F|\mu \to 0\) by the continuity of the function \(w\), we obtain property (1). Property (2) follows from the corresponding properties of B-splines and property (3) follows from the inequality

\[
|\|M_i^p\|_{L^\infty(\mu)} \cdot \mu(\text{supp} M_i^p) \leq M^2 |\|M_i\|_{L^\infty} |\text{supp} M_i| \leq kM^2.
\]

Thus, an application of Theorem 2.2 yields the following.
Corollary 3.1. Suppose that \( d\mu = w \, dx \) for some continuous function \( w \) on \([0,1]\) satisfying \( M^{-1} \leq w \leq M \) for some constant \( M > 0 \). For an interval \( \sigma \)-algebra \( \mathcal{F} \) and a non-negative integer \( k \), let \( S_{\mathcal{F}} \) be the corresponding spline space of order \( k \) and \( P_{\mathcal{F},\mu} \) the orthogonal projection operator onto \( S_{\mathcal{F}} \) with respect to the inner product \( \langle \cdot, \cdot \rangle_{\mu} \).

Then, there exists a constant \( C \), depending only on \( k \), \( M \) and the modulus of continuity of \( w \), so that

\[
\sup_{\mathcal{F}} \|P_{\mathcal{F},\mu} : L^\infty(\mu) \to L^\infty(\mu)\| \leq C,
\]

where \( \sup \) is taken over all interval \( \sigma \)-algebras \( \mathcal{F} \).

3.2. Chebyshevian spline spaces. Here, we only give the necessary definitions and results pertaining to Chebyshevian spline spaces used to apply Theorem 2.2. As a basic reference and for more information about Chebyshevian splines, we refer to the book [12], in particular Chapter 9. Suppose that for a positive integer \( k \), \( w = (w_1, \ldots, w_k) \) is a vector consisting of \( k \) positive functions (weights) on \([0,1]\) with \( w_i \in C^{k-i+1}[0,1] \) for all \( i \in \{1, \ldots, k\} \). Then define the vector \( u = (u_1, \ldots, u_k) \) of functions by

\[
u_1 = w_1 \quad \text{and} \quad u_i(x) = w_1(x) \int_0^x w_2(s_2) \cdots \int_0^{s_i-1} w_i(s) \, ds \cdots \, ds_2, \quad i = 2, \ldots, k.
\]

Let \( \mathcal{F} \) be an interval \( \sigma \)-algebra and define the Chebyshevian spline space \( S_{\mathcal{F},w} \) as

\[
S_{\mathcal{F},w} = \{ f \in C^{k-2}[0,1] : f \in \text{span}\{u_1, \ldots, u_k\} \text{ on each atom of } \mathcal{F} \}.
\]

If we choose the constant weight functions \( w_1 = \cdots = w_k = \text{const} \), we get the classical spline space \( S_{\mathcal{F}} \) of order \( k \). Given the weights \( w = (w_1, \ldots, w_k) \) and the corresponding system \( u = (u_1, \ldots, u_k) \), we define its dual system \( u^* = (u_1^*, \ldots, u_k^*, u_{k+1}^*) \) by

\[
u_i^*(x) = u_0^*(x) = \int_0^x w_k(s_k) \int_0^{s_k} w_{k-1}(s) \cdots \int_0^{s_{k-i+3}} w_{k-i+2}(s) \, ds \cdots \, ds_k
\]

for \( i = 2, \ldots, k+1 \). Moreover, we define for \( j = 0, \ldots, k \) the functions \( u_{j+1}^* \) by

\[
u_j^*(x) = \int_0^x w_{k-j}(s_k) \int_0^{s_k} w_{k-1}(s) \cdots \int_0^{s_{k-i+3}} w_{k-i+2}(s) \, ds \cdots \, ds_k
\]

for \( i = 2, \ldots, k+1 - j \).

Next, define \( h_i^w(x, y) := u_1(x) \) and

\[
h_j^w(x, y) := u_1(x) \int_y^x w_2(s_2) \int_y^{s_2} \cdots \int_y^{s_j} w_j(s) \, ds \cdots \, ds_2, \quad j = 2, \ldots, k.
\]

Then, the functions \( g_j^w := \mathbb{1}_{x \geq y}(x, y) h_j^w(x, y), j = 1, \ldots, k, \) are the analogues of the truncated power functions \( (x-y)^k_{+1} \) for polynomials, which we get by choosing the weight functions \( w_i = 1 \).

Let \( s_1 \leq s_2 \leq \cdots \leq s_k \) be an increasing sequence of real numbers. We set \( d_i := \max\{0 \leq j \leq k-1 : s_i = \cdots = s_{i-j}\} \) and define the expression

\[
D \left( \begin{array}{c} s_1, \ldots, s_k \\ u_1, \ldots, u_k \end{array} \right) := \det \left( D^{d_i} u_j(t_i) \right)_{i,j=1}^k.
\]

where the letter \( D \) on the right hand side denotes the ordinary differential operator. Now define the sequence \( (t_i)_{i=0}^n \) of grid points of the interval \( \sigma \)-algebra \( \mathcal{F} \) to be the increasingly ordered sequence of boundary points of atoms of \( \mathcal{F} \), where the points 0 and 1 each appear \( k \) times and every other point appears once in the sequence \( (t_i)_{i=0}^n \).
Then, the Chebyshevian B-spline function $M_i^w$ for the weights $w = (w_1, \ldots, w_k)$ and for $x \in [0, 1]$ is defined by

\begin{equation}
M_i^w(x) = (-1)^k \frac{D\left(\frac{t_i, \ldots, t_{i+k}}{u_1^*, \ldots, u_k^*, g_i^w(x, \cdot)}\right)}{D\left(\frac{t_i, \ldots, t_{i+k}}{u_1^*, \ldots, u_{k+1}^*}\right)}, \quad i = 0, \ldots, n - k.
\end{equation}

The system of functions $(M_i^w)_{i=0}^{n-k}$ forms an algebraic basis of the Chebyshevian spline space $S_{F,w}$ and each function $M_i^w$ has the properties

\begin{equation}
M_i^w > 0 \text{ on } (t_i, t_{i+k}), \quad M_i^w = 0 \text{ on } (t_i, t_{i+k})^c, \quad \int_0^1 M_i^w(x) \, dx = 1.
\end{equation}

If we choose the weights $w_1 = \cdots = w_k = 1$, the above definition yields the corresponding polynomial B-spline functions $M_i$, normalized in $L^1$ for which we have the pointwise estimate $M_i(x) \leq k/(t_{i+k} - t_i)$.

In order to show properties (1)–(3) on page 3 for the Chebyshevian spline spaces $(S_{F,w})_F$ and the corresponding B-spline functions $(M_i^w)$, we need the following result about the difference between $M_i^w$ and the classical B-spline function $M_i$:

**Proposition 3.2.** Let $M > 0$ be a constant such that $1/M \leq w_i \leq M$ for $i = 1, \ldots, k$.

Then there exists a constant $C > 0$ depending only on $M$ and $k$ so that the following pointwise estimates are true:

\[
|M_i^w(x) - M_i(x)| \leq \frac{C}{|\text{supp } M_i|} \max_{j=1,\ldots,k} \omega(w_j, |\text{supp } M_i|),
\]

\[
|M_i^w(x)| \leq \frac{C}{|\text{supp } M_i|},
\]

where $\omega(f, \delta) = \sup_{|x-y|<\delta} |f(x) - f(y)|$, as before, denotes the modulus of continuity of $f$.

**Proof.** Clearly the second estimate is a consequence of the first estimate and the inequality $M_i(x) \leq k/(t_{i+k} - t_i)$.

In order to estimate the difference between the Chebyshevian B-spline function $M_i^w$ and the classical B-spline function $M_i$, we separately estimate numerator and denominator in (3.4). First we perform determinant rules to $D\left(\frac{t_i, \ldots, t_{i+k}}{u_1^*, \ldots, u_{k+1}^*}\right)$; as the first column in this matrix consists entirely of 1-entries, we replace, for $i = 2, \ldots, k + 1$, the $i$th row by the difference between the $i$th and the $(i-1)$st row. Recalling the corresponding definitions and factoring out $w_k(s_1) \cdots w_k(s_k)$ by multilinearity of the determinant, we obtain

\[
D\left(\frac{t_i, \ldots, t_{i+k}}{u_1^*, \ldots, u_{k+1}^*}\right) = \int_{t_i}^{t_{i+1}} \cdots \int_{t_{i+k-1}}^{t_{i+k}} w_k(s_1) \cdots w_k(s_k) D\left(\frac{s_1, \ldots, s_k}{u_{1,1}^*, \ldots, u_{1,k}^*}\right) \, ds_k \cdots \, ds_1.
\]

Denote by $p_i^*(t) = t^{i-1}/(i-1)!$ the function $u_i^*$ corresponding to the choice of weight functions $w_1 = \cdots = w_k = 1$. By induction on $k$ we infer the estimate

\begin{equation}
\prod_{j=1}^k \left(\min_{t \in [t_i, t_{i+k}]} w_j(t)\right)^j \frac{D\left(\frac{t_i, \ldots, t_{i+k}}{u_1^*, \ldots, u_{k+1}^*}\right)}{D\left(\frac{t_i, \ldots, t_{i+k}}{p_1^*, \ldots, p_{k+1}^*}\right)} \leq \prod_{j=1}^k \left(\max_{t \in [t_i, t_{i+k}]} w_j(t)\right)^j.
\end{equation}
Note that \( D \left( \begin{array}{c} t_{i}, \ldots, t_{i+k} \\ u_1^*, \ldots, u_{k+1}^* \end{array} \right) \) is a constant multiple of the Vandermonde determinant, i.e. \( D \left( \begin{array}{c} t_{i}, \ldots, t_{i+k} \\ u_1^*, \ldots, u_{k+1}^* \end{array} \right) = c \prod_{i \leq r < s \leq i+k} (t_s - t_r) \), where \( c \) depends only on \( k \). Since \( M^{-1} \leq w_i \leq M \) for \( i = 1, \ldots, k \), by (3.6) there exists a constant \( C \) depending only on \( M \) and \( k \) so that

\[
(3.7) \quad C^{-1} \prod_{i \leq r < s \leq i+k} (t_s - t_r) \leq D \left( \begin{array}{c} t_{i}, \ldots, t_{i+k} \\ u_1^*, \ldots, u_{k+1}^* \end{array} \right) \leq C \prod_{i \leq r < s \leq i+k} (t_s - t_r),
\]

Denote

\[
q_k = D \left( \begin{array}{c} t_{i}, \ldots, t_{i+k} \\ u_1^*, \ldots, u_{k+1}^* \end{array} \right), \quad \varepsilon_k = D \left( \begin{array}{c} t_{i}, \ldots, t_{i+k} \\ u_1^*, \ldots, u_{k+1}^* \end{array} \right) - q_k,
\]

where \( \overline{w}_1^*, \ldots, \overline{w}_{k+1}^* \) correspond to the choice of constant weights \( \overline{w}_j = \min_{t \in [t_i, t_{i+k}]} w_j(t) \). Thus, (3.6) implies

\[
(3.8) \quad 0 \leq \varepsilon_k \leq q_k \prod_{j=1}^k \left( 1 + \frac{\omega(w_j, t_{i+k} - t_i)}{\min_{t \in [t_i, t_{i+k}]} w_j(t)} \right)^j - 1 \leq C q_k \max_{j=1,\ldots,k} \omega(w_j, t_{i+k} - t_i),
\]

for some constant \( C \) depending only on \( M \) and \( k \).

Similarly, we estimate the numerator \( D \left( \begin{array}{c} t_{i}, \ldots, t_{i+k} \\ u_1^*, \ldots, u_{k}^*, g_k(x, \cdot) \end{array} \right) \) in (3.4). To this end, observe that the definition (3.2) of \( g_j^w \) yields

\[
\frac{\partial}{\partial y} g_j^w(x, y) = -w_j(y)g_{j-1}^w(x,y), \quad j \geq 2, x \neq y.
\]

Therefore, performing the same determinant rules as above, we write

\[
D \left( \begin{array}{c} t_{i}, \ldots, t_{i+k} \\ u_1^*, \ldots, u_{k}^*, g_k(x, \cdot) \end{array} \right) = - \int_{t_i}^{t_{i+1}} \cdots \int_{t_{i+k-1}}^{t_{i+k}} w_k(s_1) \cdots w_k(s_k) D \left( \begin{array}{c} s_1, \ldots, s_k \\ u_1^*, \ldots, u_{k}^*, g_k(x, \cdot) \end{array} \right) ds_1 \cdots ds_k.
\]

Using this formula, induction on \( k \) yields (for \( x \in (t_i, t_{i+k}) \))

\[
(3.9) \quad \prod_{j=1}^k \min_{t \in [t_i, t_{i+k}]} w_j(t)^j \leq \frac{D \left( \begin{array}{c} t_{i}, \ldots, t_{i+k} \\ u_1^*, \ldots, u_{k}^*, g_k(x, \cdot) \end{array} \right)}{D \left( \begin{array}{c} t_{i}, \ldots, t_{i+k} \\ p_1^*, \ldots, p_{k}^*, \overline{g}_k(x, \cdot) \end{array} \right)} \leq \prod_{j=1}^k \max_{t \in [t_i, t_{i+k}]} w_j(t)^j, \quad
\]

where \( g_k(x, \cdot) \) denotes the function \( g_k^w(x, \cdot) \) corresponding to the choice of the weight functions \( w_i = 1 \). Define

\[
r_k(x) = D \left( \begin{array}{c} t_{i}, \ldots, t_{i+k} \\ p_1^*, \ldots, p_{k}^*, \overline{g}_k(x, \cdot) \end{array} \right), \quad \delta_k(x) = D \left( \begin{array}{c} t_{i}, \ldots, t_{i+k} \\ u_1^*, \ldots, u_{k}^*, g_k(x, \cdot) \end{array} \right) - r_k(x),
\]

where \( \overline{g}_k(x, \cdot) \) corresponds to the choice of constant weights \( \overline{w}_j = \min_{t \in [t_i, t_{i+k}]} w_j(t) \). Thus, using (3.9) implies

\[
(3.10) \quad |\delta_k(x)| \leq (-1)^k r_k(x) \cdot \left( \prod_{j=1}^k \left( 1 + \frac{\omega(w_j, t_{i+k} - t_i)}{\min_{t \in [t_i, t_{i+k}]} w_j(t)} \right)^j - 1 \right) \leq C (-1)^k r_k(x) \max_{j=1,\ldots,k} \omega(w_j, t_{i+k} - t_i),
\]

where the constant \( C \) depends only on \( M \) and \( k \).
Note that the ordinary B-spline $M_i(x)$ satisfies
\[
\frac{r_k(x)}{q_k} = \frac{D\left(\frac{t_i, \ldots, t_{i+k}}{w_i, \ldots, w_{k+1}}\right)}{D\left(\frac{t_i, \ldots, t_{i+k}}{w_i, \ldots, w_{k+1}}\right)} = \frac{D\left(\frac{t_i, \ldots, t_{i+k}}{1, \ldots, (x-t)^{k-1}}\right)}{D\left(\frac{t_i, \ldots, t_{i+k}}{1, \ldots, (x-t)^{k-1}}\right)} = (-1)^k M_i(x).
\]
From the pointwise estimate $M_i(x) \leq k/(t_{i+k} - t_i)$ we therefore get
\[
|r_k(x)| = (-1)^k r_k(x) \leq k(t_{i+k} - t_i)^{-1} q_k.
\]
Hence, summarizing the above estimates (3.10) and (3.11),
\[
|\sum_{i=1}^{k} r_k(x) - M_i(x)| = |\sum_{i=1}^{k} r_k(x)| - |\sum_{i=1}^{k} M_i(x)| \leq C(t_{i+k} - t_i)^{-1} \max_{j=1, \ldots, k} \omega(w_j, t_{i+k} - t_i),
\]
which concludes the proof of Proposition 3.2. 

Now we show properties (1)–(3) on page 3 with the functions $M_i^p = M_i^w$ and Lebesgue measure $\mu$. It follows from (3.5) that $\sup \mathcal{F} M_i^p = \sup \mathcal{F} M_i^w$. By Proposition 3.2 we have the following estimate
\[
|\mu(\sup \mathcal{F} M_i^p) - \mu(\sup \mathcal{F} M_i^w)| \leq C \max_{i=1, \ldots, k} \omega(w_i, k),
\]
where the constant $C$ depends only on $M$ and $k$. This confirms property (1). Moreover, property (2) is satisfied by definition and (3) is a consequence of Proposition 3.2. Moreover the spaces $(\mathcal{F}, \mathcal{F})$ are compatible by the definition of the Chebyshevian B-spline functions $M_i^w$. Therefore, an application of Theorem 2.2 yields

**Corollary 3.3.** Let $k$ be a positive integer and suppose that $w = (w_1, \ldots, w_k)$ is a vector of functions on $[0, 1]$ with $w_i \in C^{k+1}([0, 1])$ satisfying the inequalities $M^{-1} \leq w_i \leq M$ for any $i = 1, \ldots, k$. For an interval $\sigma$-algebra $\mathcal{F}$, let $S_{\mathcal{F}, w}$ be the corresponding space of Chebyshevian splines and $P_{\mathcal{F}, w}$ the orthogonal projection operator onto $S_{\mathcal{F}, w}$ with respect to the inner product $\langle \cdot, \cdot \rangle$. Then, there exists a constant $C$, depending only on $k, M$ and the moduli of continuity of the weight functions $w_1, \ldots, w_k$, so that
\[
\sup_{\mathcal{F}} \|P_{\mathcal{F}, w} : \mathcal{F} \rightarrow L^\infty\| \leq C,
\]
where $\sup$ is taken over all interval $\sigma$-algebras $\mathcal{F}$.

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