New Family of Exact Soliton Solutions for the Nonlinear Three-Wave Interaction Equations

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Abstract: Problem statement: To obtain new exact traveling wave soliton solutions for the three-wave interaction equation in a dispersive medium and a non zero phase mismatch. Approach: The tanh method is usually used to find a traveling wave analytic soliton solutions for one nonlinear wave and evolution partial differential equation. Here, we generalize this method to solve a system of nonlinear evolution partial differential equations, then we use this generalization to find new family of exact traveling wave soliton solutions for the nonlinear three-wave interaction equation. Results: We were able to generalize the tanh method and apply this generalization to the (TWI) system of (PDE’s). We derive a system of algebraic Eq. 28-32 and introduced some interested sets of solutions for this system, these sets of solutions leads us to write explicit analytic new family of soliton solutions for the three-wave interaction equation. Conclusion: The generalization of the tanh method is proved its efficiency in obtaining exact solutions for nonlinear evolution partial differential equations. This method also can be used similarly to obtain exact solutions for another interested nonlinear evolution system of partial differential equations.

Key words: Soliton solutions, tanh method, three-wave interaction, non zero phase mismatch

INTRODUCTION

Obtaining an exact solution for a nonlinear equation is considered an interesting problem for mathematicians, so, what if we have a system of nonlinear equations? As an example on those systems is the nonlinear Three-Wave Interaction (TWI) system of Partial Differential Equations (PDE’s), which represents a mathematical model for three interacting optics waves? This system describes many physical phenomena, such as, the resonant quadratic nonlinear interaction of three optics waves (Ibragimov and Struthers, 1997), the second harmonic generation process which produces the first coherent or laser light source (Rushehitskii, 1996; Kumar et al., 2008) and the study of the model in x^2 materials (Chen et al., 2004). Many analytic solutions for the (TWI) system were found, such as, the solution of this system when it includes the phase mismatch (\Delta k) (Ibragimov et al., 2001), the solution of this system when it includes the second order dispersion (Werner and Drummond, 1994; Menyuk et al., 1994; Tahar, 2007), the solution of this system when it doesn’t include the second order dispersion by the Inverse Scattering Transform method (IST) (Ibragimov et al., 1998; Batiha, 2007) and the exact soliton solution found by some ansatz introduced by (Huang, 2000).

In this study we introduce a direct generalization of the tanh method (Wazwaz, 2004) to solve a nonlinear evolution system of (PDE’s), then we apply this generalization and get new families of soliton solutions for the (TWI) system of (PDE’s) which includes nonzero quadratic dispersion coefficients \{g_1, g_2, g_3\} and a nonzero phase mismatch (\Delta k) in Eq. 14. In all of the obtained solutions we mentioned that replacing the tanh function with the coth function will keep our solutions valid.

MATERIALS AND METHODS

This method is used to solve many nonlinear wave and evolution equations, such as, the soliton solutions found for many forms of the fifth order Kdv equation (Wazwaz, 2007). In using this method, we are looking for a traveling wave solution \( u(x, y, z, t) \) to a given nonlinear (PDE) in the form:

\[ H(u, u_x, u_y, u_z, u_{xx}, u_{yy}, u_{zz}, ... ) = 0 \]  

(1)

If we assume that:

\[ \theta = c_1 x + c_2 y + c_3 z - \lambda t \]  

(2)
where, \(c_i, \lambda\) are unknown constants to be found, then Eq. 1 will be transformed to an Ordinary Differential Equation (ODE) in the form:

\[
F(U, U', U'', \ldots) = 0
\]  

(3)

where, \(U(0) = u(x, y, z, t)\) and the derivatives appear in (3) are with respect to \(\theta\). If we introduce the variable \(Y(\theta) = \tanh(\theta)\) then the derivatives appear in (3) become:

\[
\frac{dU}{d\theta} = (1 - Y^2)U(\theta)
\]

\[
\frac{d^2U}{d\theta^2} = (-1 + Y^2)(2YU(\theta) + (-1 + Y^2)U'(\theta))
\]

(4)

where similar formula for higher derivatives can be obtained as well. The tanh method now admits solutions of the form:

\[
U(\theta) = u(x, y, z, t) = S(Y) + \sum_{i=0}^{n} a_i Y(\theta)^i
\]

(5)

where, \(n\) is an unknown positive integer, which can be determined from the resulted equation by balancing the linear and nonlinear terms of the highest orders. After determining \(n\) we substitute Eq. 5 in Eq. 3. Doing this, will give us an algebraic coefficients of some powers of \(Y(\theta)\), making these coefficients zeros, will produce a system of algebraic equations in \(a_i\) and \(c_i\), then we solve this system and substitute the result in (2) and (5), to get a solution \(u(x, y, z, t)\) for (1).

**Generalization of the tanh method:** This is a direct generalization for the above tanh method. Here, we suppose that we have a system of nonlinear evolution (PDE’s) in the form:

\[
\tilde{H}(t, \bar{X}; u_r(t, \bar{X}), u_{rr}(t, \bar{X}), u_{r,s_i} (t, \bar{X}),\ldots) = 0
\]

(6)

where, \(\bar{X} = [x_1, x_2, \ldots, x_n]\), \(u_r(t, \bar{X})\), \(1 \leq r \leq n\), are the required solutions and:

\[
u_{r_{s_i} \ldots s_m}(t, \bar{X}) = \frac{\partial^n u_r(t, \bar{X})}{\partial x_{s_i} \partial x_{s_j} \ldots \partial x_m}
\]

(7)

If we assume that:

\[
\theta = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n - \lambda t
\]

(8)

where, \(c_i, \lambda\) are unknown constants to be found, then the partial derivatives in (7) becomes:

\[
\frac{\partial^n u_r(t, \bar{X})}{\partial x_{s_i} \partial x_{s_j} \ldots \partial x_m}
\]

(9)

if we substitute Eq. 9 in Eq. 6, then Eq. 6 will be transformed to a system of (ODE’s) in the form:

\[
\tilde{F}(u_r(\theta), u_{r}'(\theta), u_{r}''(\theta), \ldots) = 0
\]

(10)

where the derivatives appear in Eq. 10 are with respect to \(\theta\). If we assume that:

\[
Y(\theta) = \tanh(\theta)
\]

(11)

then the derivatives appear in (10) are given by:

\[
\frac{dU}{d\theta} = (1 - Y^2)u_r'(\theta)
\]

\[
\frac{d^2U}{d\theta^2} = (-1 + Y^2)(2Yu_r'(\theta) + (-1 + Y^2)u_r''(\theta))
\]

(12)

where, several formula for \(\frac{d^3u_r}{d\theta^3}, \frac{d^4u_r}{d\theta^4}, \ldots\) can be easily obtained as well. The generalized tanh method now admits solutions of the form:

\[
u_r(\theta) = u_r(t, x_1, x_2, \ldots, x_n) = S_r(Y) + \sum_{i=0}^{n} A_{ri} Y(\theta)^i
\]

(13)

Where:

\(n_r = \) Positive integers depend on \(r\)

\(A_{ri} = \) Constant coefficients (usually \(A_{ri}\) are complex numbers) yet to be determined

From the resulted system of (ODE) we determine \(n_r\) by balancing the highest nonlinear terms with the highest orders of the given system. After determining \(n_r\) we substitute Eq. 13 in Eq. 10, to get a system of equations, where in each individual equation in this system, there will be some algebraic coefficients contains \(A_{ri}\) and \(c_i\), multiplied by some powers of \(Y(\theta)\), setting these coefficients with zero, will give us a system of algebraic equations, doing the same for the rest of the equations and collecting all these algebraic equations, we will get a universal system of algebraic equations, doing the same for this universal system, if we are able to find such a solution we substitute it in Eq. 8 and 13, then the result is an explicit analytic formula for the required solution \(u_r(\theta) = u_r(t, x_1, x_2, \ldots, x_n)\) for the given system of (PDE’s) in (6).
**Basic equations:** We will apply the generalized tanh method to the following nonlinear evolution (TWI) system of (PDE's) given by (Ibragimov et al., 1998; Faiedh et al., 2006):

\[
i(\partial_z A_1 + \frac{1}{v_1} \partial_t A_1) - g_1 \partial_{zz} A_1 = -\sigma A_1 A_1^* e^{i k z} e^{i \omega t},
\]

\[
i(\partial_z A_2 + \frac{1}{v_2} \partial_t A_2) - g_2 \partial_{zz} A_2 = -\sigma A_2 A_2^* e^{i k z} e^{i \omega t},
\]

\[
i(\partial_z A_3 + \frac{1}{v_3} \partial_t A_3) - g_3 \partial_{zz} A_3 = -\sigma A_3 A_3^* e^{i k z} e^{i \omega t},
\]

(14)

The system in (14) describes three interacting plane waves traveling in the positive z direction in a nonlinear dispersive medium, with the following associated electric fields \( E_j(z, t) \) given by:

\[
E_j(z, t) = A_j(z, t) e^{i q_j z} e^{i \omega_j t}, \quad j = 1, 2, 3
\]

(15)

Where:

- \( A_j(z, t) = \) The slowly varying complex-amplitude envelopes of the three waves
- \( \omega_j = \) The center frequencies
- \( K_j = \) The wave numbers given by \( K_j = \frac{n_j \omega_j}{c} \)
- \( c = \) The speed of light
- \( n_j = \) The refractive indexes
- \( v_j = \) The group velocities of the three waves which are in general different from each others
- \( g_j = \) The second-order non zero dispersion coefficients
- \( \sigma = \) The nonlinear coupling constant given by \( \sigma = \frac{2 \pi n_j \omega_j}{k_0 c^2} \)
- \( \chi_{nl} = \) The nonlinear dielectric susceptibility
- \( \Delta k = \) The phase velocity mismatch given by \( \Delta k = k_3 - k_2 - k_1 \)
- \( * = \) For the complex conjugate, \( i^2 = -1 \)

If we assume that:

\[
A_{1,2} \rightarrow Q_{1,2}, \quad A_3 \rightarrow -Q_3, \quad g_{1,2} \rightarrow g_3, \quad g_3 \rightarrow -G_3,
\]

\[
\sigma \rightarrow -p_1, \quad \sigma \rightarrow -p_2, \quad \sigma \rightarrow -p_3
\]

(16)

then the system in (14) will be transformed to the following symmetric form:

\[
i(\partial_{Q_1} + \frac{1}{v_1} \partial_t Q_1) - G_1 \partial_{zz} Q_1 = -p_1 Q_1 Q_1^* e^{i k z} e^{i \omega t},
\]

\[
i(\partial_{Q_2} + \frac{1}{v_2} \partial_t Q_2) - G_2 \partial_{zz} Q_2 = -p_2 Q_2 Q_2^* e^{i k z} e^{i \omega t},
\]

\[
i(\partial_{Q_3} + \frac{1}{v_3} \partial_t Q_3) - G_3 \partial_{zz} Q_3 = -p_3 Q_3 Q_3^* e^{i k z} e^{i \omega t},
\]

(17)

if we also let (Ibragimov et al., 1998):

\[
S = (t - \frac{1}{v_1} z), \quad \eta = z, \quad Q_r = Q_{1,2} q_r(S, \eta), \quad r = 1, 2, 3
\]

(18)

Where:

\[
Q_{1,2} = \frac{-i}{\sqrt{-p_1 p_3}}, \quad Q_2 = \frac{i}{\sqrt{-p_1 p_3}}, \quad Q_3 = \frac{1}{\sqrt{-p_1 p_2}}
\]

(19)

then the system in (17) will be rescaled to become:

\[
i(\partial_{q_1} - \gamma_1 \frac{\partial q_1}{\partial S} + \sigma_1 \frac{\partial^2 q_1}{\partial S^2}) = q_1^* e^{i \omega_1 t}, \quad r = 1, 2, 3
\]

(20)

where, \( a_{1,2} = -g_{1,2}, \quad a_3 = g_3, \quad \gamma_1 = 0 \), is added for purposes of obtaining symmetric solutions later, while \( \gamma_2 \) and \( \gamma_3 \) which are called the temporal walk-off parameters are given by the formula:

\[
\gamma_r = \left( \frac{1}{v_1} - \frac{1}{v_r} \right), \quad r = 2, 3
\]

(21)

**Applying the generalized tanh method:** To apply the generalized tanh method, Eq. 6 is now our system given in (20). To transform the system of (PDE’s) in (20) to a system of (ODE’s) as in equation (10), we use the following assumptions suggested in (Huang, 2000):

\[
qu_r(S, \eta) = u_r(\Theta) e^{i \Theta}, \quad \Theta = \Omega S - K \eta, \quad \Omega(S, \eta) = K, \eta - \Omega, S, \quad r = 1, 2, 3
\]

(22)

where, \( \Omega, K, K, \Omega \) are unknown real constants to be determined. Using Eq. 21 and 22, the system in (20) will be transformed to the following system of (ODE’s):
\[ u_i(0) + u_i'(0)u_i(0) + iX_iu_i(0) - \tau_iu_i(0) = 0 \quad (23) \]

Where:
\[ \Phi_i = (K_i + \Omega_i(\gamma_i + \alpha_i\Omega_i)), \quad X_i = (K_i + \Omega_i(\gamma_i + 2\alpha_i\Omega_i)) \]
\[ \tau_i = \alpha_i\Omega_i^2, \quad i = 1, 2, 3, \quad K_j = -(K_i + K_j - \Delta k), \quad \Omega_j = -\Omega_i + \Omega_j \]

Notice that from Eq. 24, if later we able to find \( \Phi_i \) and \( X_i \), this will give us 6 real values for them, then we can easily use these values to find \( \Omega_1, \Omega_2, K_1, K_2, K, \Omega \).

The maximum powers appear in Eq. 25 are 2n and \( n + 2 \), those powers come from the terms \( S_i(Y)S_j(Y) \) and \( YS_i(Y), YS_j(Y) \). If we make these powers equal, then we get \( n_1 = n_2 = n_3 = 2 \), so Eq. 13 becomes:
\[ u_i(0)A_i + B_i \tanh(\theta) + C_i \tanh^2(\theta), r = 1, 2, 3 \quad (26) \]

where, \( A_i, B_i, C_i \) are generally complex numbers yet to be determined. If we substitute Eq. 26 in 23 and use the identity:
\[ \tanh^*(\theta) = \tanh^{*2}(\theta)(1 - \text{sech}^2(\theta)) \quad (27) \]

then we get a system of equations consists of the following algebraic coefficients of some powers of \( Y(\theta) \), these coefficients are:

**The constant coefficients:**
\[ (A_i + C_i)\Phi_i + B_i'\Phi_i + (A_j + C_j)(A_i + C_i) = 0 \quad (28) \]

**The coefficients of \( \tanh(\theta) \):**
\[ B_i\Phi_i + B_i'(A_i + C_i) + B_i''(A_i + C_i) = 0 \quad (29) \]

**The coefficients of \( \text{sech}^4(\theta) \):**
\[ (4\tau_i - \Phi_i)C_i + iB_iX_i - B_i'\Phi_i - A_iC_j - (A_j + 2C_j)C_i = 0 \quad (30) \]

**The coefficients of \( \text{sech}^2(\theta)\tanh(\theta) \):**
\[ 2B_i\tau_i + 2iC_iX_i - B_i'\Phi_i - B_i'\Phi_i = 0 \quad (31) \]

**The coefficients of \( \text{sech}^4(\theta) \):**
\[ -6C_i\tau_i + C_jC_i = 0 \quad (32) \]

where, \( \{i, j, k\} \) take the values \{1, 2, 3\}, \{2, 1, 3\} and \{3, 1, 2\}. The system (28-32) is a nonlinear system of 15 equations in 15 unknowns, namely \( A_i, B_i, C_i, \Phi_i \) and \( X_i \).

**RESULTS**

We were able to find the following interested sets of solutions for the system in (28-32), however, finding another interested sets of solutions for this system is still an open problem. In all of these founded sets some of the unknowns were arbitrary constants, which mean that the system has infinitely many similar solutions.

**Set 1:** Let \( r = 1, 2 \) and 3, choose arbitrary real constants \( \Phi_r \neq 0 \) and \( X_r \), choose \( \{\delta_1, \delta_2, \delta_3\} \), such that \( \delta_1 + \delta_2 + \delta_3 = 2n\pi \), \( n \) is an integer and:
\[ B_i = C_i = 0, \quad A_i = -\sqrt{\Phi_i}\Phi_i e^{i\delta_i} \]
\[ A_j = -\sqrt{\Phi_i}\Phi_i e^{i\delta_j}, \quad A_j = -\sqrt{\Phi_i}\Phi_i e^{i\delta_j} \quad (33) \]

**Set 2:** Let \( r = 1, 2 \) and 3, \( \{i, j, k\} \) have the values \{1, 2, 3\}, \{2, 1, 3\} and \{3, 2, 1\}, \( \tau_r \) are arbitrary real numbers, choose \( \{\delta_1, \delta_2, \delta_3\} \) such that \( \delta_1 + \delta_2 + \delta_3 = 2n\pi \), \( n \) is an integer and:
\[ B_i = 0, \quad X_i = 0, C_i = 6\sqrt{\tau_i}\Phi_i e^{i\delta_i}, \quad A_i = -C_i, \Phi_i = 4\tau_i \quad (34) \]

Notice that, since \( \tau_r = \alpha_r\Omega_r^2 \) are arbitrary real numbers, this means, \( \Omega_i \) is actually chosen to be arbitrary real number.

**Set 3:** Let \( r = 1, 2 \) and 3, \( \{i, j, k\} \) have the values \{1, 2, 3\}, \{2, 1, 3\} and \{3, 2, 1\}, \( \tau_r \) are arbitrary real numbers, choose \( \{\delta_1, \delta_2, \delta_3\} \), such that \( \delta_1 + \delta_2 + \delta_3 = 2n\pi \), \( n \) is an integer and:
\[ B_r = 0, X_r = 0, C_r = 6\left(\tau_r e^{15}\right), A_r = -\frac{1}{3} C_r, \Phi_r = -4\tau_r \] \hspace{1cm} (35)

**Set 4:** Let \( r = 1, 2 \) and \( 3 \), \( \{i, j, k\} \) have the values \( \{1, 2, 3\} \), \( \{2, 1, 3\} \) or \( \{3, 2, 1\} \), choose arbitrary non zero real numbers \( b_r, A_1, A_2 \) and:

\[ C_r = \tau_r = 0, \] \hspace{1cm} (36)

\[ A_3 = \frac{-A_1 A_2 b_1 + b_1 b_2}{A_1 b_1 + A_2 b_2} \]

\[ \Phi_1 = \frac{(A_1^2 + b_1^2)b_1}{A_1 b_1 + A_2 b_2} \]

\[ \Phi_3 = \frac{A_3 b_1 + A_1 b_2 b_3}{b_3} \]

**DISCUSSION**

To find an explicit formula for \( Q_r(z, t) \) in (17), we do the following steps:

- Substitute the values of \( \Phi_r, X_r \) in (24) and solve the resulted equations to get \( \Omega, K, K_r, \Omega_r \).
- Substitute the values of \( \Omega, K, K_r, \Omega_r \) in (22) to get \( \theta(S, \eta), \theta_r(S, \eta) \).
- Substitute the values of \( A_r, B_n, C_r \) in (26) to get \( u_r(\theta) \).
- Substitute the formula for \( u_r(\theta), \theta(S, \eta), \theta_r(S, \eta) \), in (21), to get \( q_r(S, \eta) \).
- Use (19) and \( q_r(S, \eta) \) to write an explicit formula for \( Q_r(z, t) \) in (18).

Applying the above steps on Set 1, we get the following solution for (17):

\[ Q_1(z, t) = \frac{1}{P_1 \sqrt{P_3}} e^{i(K_1 x + z \Omega_1)} \]

(37)

\[ Q_2(z, t) = \frac{1}{P_1 \sqrt{P_3}} e^{i(K_2 x + z \Omega_2)} \]

\[ Q_3(z, t) = \frac{1}{P_1 \sqrt{P_3}} e^{i(K_3 x + z \Omega_3)} \]

Applying the above steps on Set 2, we get the following solution for (17):

\[ Q_1(z, t) = \frac{1}{P_1 \sqrt{P_3}} e^{i(K_1 x + z \Omega_1)} \]

\[ Q_2(z, t) = \frac{1}{P_1 \sqrt{P_3}} e^{i(K_2 x + z \Omega_2)} \]

\[ Q_3(z, t) = \frac{1}{P_1 \sqrt{P_3}} e^{i(K_3 x + z \Omega_3)} \]

Applying the above steps on Set 3, we get the following solution for (17):

\[ Q_1(z, t) = \frac{1}{P_1 \sqrt{P_3}} e^{i(K_1 x + z \Omega_1)} \]

\[ Q_2(z, t) = \frac{1}{P_1 \sqrt{P_3}} e^{i(K_2 x + z \Omega_2)} \]

\[ Q_3(z, t) = \frac{1}{P_1 \sqrt{P_3}} e^{i(K_3 x + z \Omega_3)} \]

Applying the above steps on Set 4, we get the following solution for (17):

\[ Q_1(z, t) = \frac{1}{P_1 \sqrt{P_3}} e^{i(K_1 x + z \Omega_1)} \]

\[ Q_2(z, t) = \frac{1}{P_1 \sqrt{P_3}} e^{i(K_2 x + z \Omega_2)} \]

\[ Q_3(z, t) = \frac{1}{P_1 \sqrt{P_3}} e^{i(K_3 x + z \Omega_3)} \]

Notice that if instead of the assumption in (26), we assume that:

\[ u_r(\theta) = A_1 + B_1 \coth(\theta) + C_1 \coth^r(\theta), \quad r = 1, 2, 3 \]

then we will obtain the same system as in (28-32), so in the solutions (37-40), we can replace \( \tanh \) with \( \coth \) and still get a solution.

**CONCLUSION**

We were able to generalize the \( \tanh \) method, then apply this generalization on the Three-Wave-Interaction
(TWI) system of (PDE’s). We were able to construct an algebraic nonlinear system of 15 complex equations and found 4 different sets of solutions for this system, then construct 4 new families of analytic traveling wave soliton solutions for the (TWI) system. We are very positive that there are more sets of solutions for our algebraic system, which means that there are more similar family of analytic solutions for the (TWI) system which can be constructed as well.

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