On directed homotopy equivalences and a notion of directed topological complexity

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Abstract

This short note introduces a notion of directed homotopy equivalence (or dihomotopy equivalence) and of “directed” topological complexity (which elaborates on the notion that can be found in e.g. [9]) which have a number of desirable joint properties. In particular, being dihomotopically equivalent implies having bisimilar natural homologies (defined in [5]). Also, under mild conditions, directed topological complexity is an invariant of our directed homotopy equivalence and having a directed topological complexity equal to one is (under these conditions) equivalent to being dihomotopy equivalent to a point (i.e., to being “dicontractible”, as in the undirected case). It still remains to compare this notion with the notion introduced in [8], which has lots of good properties as well. For now, it seems that for reasonable spaces, this new proposal of directed homotopy equivalence identifies more spaces than the one of [8].

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1 Introduction

The aim of this note is to introduce another notion of directed homotopy equivalence than the one of [8], hoping to get other insights on directed topological spaces. The view we are taking here is that of topological complexity, as defined in [9], adapted to directed topological spaces.

Let us briefly motivate the interest of this “directed” topological complexity notion. In the very nice work of M. Farber, it is observed that the very important planification problem in robotics boils down to, mathematically speaking, finding a section to the path space fibration \( \chi : PX = X^I \to X \times X \) with \( \chi(p) = (p(0), p(1)) \). If this section is continuous, then the complexity is the lowest possible (equal to one), otherwise, the minimal number of discontinuities that would encode such a section would be what is called the topological complexity of \( X \). This topological complexity is both understandable algorithmically, and topologically, e.g. as \( s \) having a continuous section is equivalent to \( X \) being contractible. More generally speaking, the topological complexity is defined as the Schwartz genus of the path space fibration, i.e. is the minimal cardinal of partitions of \( X \times X \) into “nice” subspaces \( F_i \) such that \( s_{F_i} : F_i \to PX \) is continuous.

This definition perfectly fits the planification problem in robotics where there are no constraints on the actual control that can be applied to the physical apparatus that is supposed to be moved from point \( a \) to point \( b \). In many applications, a physical apparatus may have dynamics that can be described as an ordinary differential equation in the state variables \( x \in \mathbb{R}^n \) and in time \( t \), parameterized by control parameters \( u \in \mathbb{R}^p \), \( \dot{x}(t) = f(t, x(t)) \). These parameters are generally bounded within some set \( U \), and, not knowing the precise control law (i.e. parameters \( u \) as a function of time \( t \)) to be applied, the way the controlled system can evolve is as one of the solutions of the differential inclusion \( \dot{x}(t) \in F(t, x(t)) \) where \( F(t, x(t)) \) is the set of all \( f(t, x(t), u) \) with \( u \in U \). Under some classical conditions, this differential inclusion can be proven to have solutions on at least a small interval of time, but we will not discuss this further here. Under the same conditions, the set of solutions of this differential inclusion naturally generates a d-space (a very general structure of directed space, where a preferred subset of paths is singled out, called directed paths, see e.g. [13]). Now, the planification problem in the presence of control constraints equates to finding sections to the analogues to the path space fibrations taking a dipath to its end points. This is developed in next section, and we introduce a notion of directed homotopy equivalence that has precisely, and in a certain non technical sense, minimally, the right properties with respect to this directed version of topological complexity.

The development of the notion of directed topological complexity and of its properties, together with applications to optimal control is joint work with coauthors, and will be published separately.

Mathematical context : The context is that of d-spaces [13].

Definition 1 ([13]). A directed topological space, or d-space \( X = (X, dX) \) is a topological space equipped with a set \( dX \) of continuous maps \( p : I \to X \) (where \( I = [0, 1] \) is the unit segment with the usual topology inherited from \( \mathbb{R} \) ), called directed paths or d-paths, satisfying three axioms :

- every constant map \( I \to X \) is directed
- \( dX \) is closed under composition with non-decreasing maps from \( I \) to \( I \)
- \( dX \) is closed under concatenation

For \( X \) a d-space, let us note by \( PX \) (resp. \( TX \)) the topological space, with compact open topology, of dipaths (resp. the trace space, i.e. \( PX \) modulo increasing homeomorphisms of the unit directed interval) in \( X \). \( PX(a, b) \) (resp. \( TX(a, b) \)) is the sub-space of \( PX \) (resp. of \( TX \)) containing only dipaths (resp. traces) from point \( a \in X \) to point \( b \in X \). We write \( * \) for the concatenation map from \( PX(a, b) \times PX(b, c) \) to \( PX(a, c) \) (resp. on trace spaces), which is continuous.

A dmap \( f \) from d-space \( X \) to d-space \( Y \) is a continuous map from \( X \) to \( Y \) that also maps elements from \( dX \) to elements of \( dY \) (i.e. they preserve directed paths).

In what follows, we will be particularly concerned with the following map :\footnote{That would most probably not qualify for being called a fibration in the directed setting.}
Definition 2. We define the dipath space map $\chi: PX \to X \times X$ of $X$ by $\chi(p) = (p(0), p(1))$ for $p \in PX$.

Because $PX$ only contains directed paths, the image of $\chi$ is just a subset of $X \times X$, called $\Gamma_X = \{(x, y) \mid \exists p \in PX, \ p(0) = x, \ p(1) = y\}$. On the classical case, we do not need to force the restriction to the image of the path space fibration, since the notions of contractibility and path-connectedness are simple enough to be defined separately. In the directed setting, dicontractibility, and “directed connectedness” are not simple notions and will be defined here through the study of the dipath space map.

In order to study this map, in particular when looking at conditions under which there exists “nice” sections to it, we need a few concepts from directed topology.

2 Some useful directed topological constructs

Let $X$ be a d-space. We define as in [6], $\preceq$ the preorder on $X$, $x \preceq y$ iff there exists a dipath from $x$ to $y$. We define the category $\mathcal{P}X$ whose:

- objects are pairs of points $(x, y)$ of $X$ such that $x \preceq y$ (i.e. objects are elements of $\Gamma_X$)
- morphisms (called extensions) from $(x, y)$ to $(x', y')$ are pairs $(\alpha, \beta)$ of dipaths of $X$ with $\alpha$ going from $x'$ to $x$ and $\beta$ going from $y$ to $y'$

We now define, for each $X$ d-space, the functor $\mathcal{P}X$ from $\mathcal{P}X$ to $\text{Top}$ with :

- $\mathcal{P}X(x, y) = PX(x, y)$
- $\mathcal{P}X(\alpha, \beta)(a) = \alpha \ast a \ast \beta$, where $(\alpha, \beta)$ is a morphism from $(x, y)$ to $(x', y')$ and $a$ is a trace from $x$ to $y$ (i.e. an element of $TX(x, y)$).

Remark : There is an obvious link to profunctors and to enriched category theory that we will not be contemplating here. Similar ideas from enriched category theory in directed algebraic topology have already been used in [15, 16] and [8].

A homotopy is a continuous function $H: I \times X \to Y$. We say that two maps $f, g: X \to Y$ are homotopic if there is a homotopy $H$ such that $H(0, \_)$ $= f$ and $H(1, \_)$ $= g$. This is an equivalence relation, compatible with composition.

A d-homotopy equivalence is a dmap $f: X \to Y$ which is invertible up to homotopy, i.e., such that there is a dmap $g: Y \to X$ with $f \circ g$ and $g \circ f$ homotopic to identities. We say that two dspaces are d-homotopy equivalent if there is a d-homotopy equivalence between them.

In such a case, $f$ and $g$ being dmaps induce

$$ Pf : PX \to PY $$

resp.

$$ Pg : PY \to PX $$

which are continuously bigraded maps in the sense that $P f _ { a , b } : PX(a, b) \to PY(f(a), f(b))$ and this bigrading is continuous in $a, b$ (resp. $P g _ { c , d } : PY(c, d) \to PX(g(c), g(d))$, continuous in $c, d$).

We write $PY^f$ for the sub topological space of $PY$ of dipaths from $f(a)$ to $f(b)$ in $Y$ for some $(a, b) \in \Gamma_X$ (resp. $PX^g$ for the sub topological space of $PX$ of dipaths in $X$ from $g(c)$ to $g(d)$ for some $(c, d) \in \Gamma_Y$).

2 By analogy with the classical path space fibration - but fibration may be a bad term in that case in directed algebraic topology.
3 Dihomotopy equivalences

3.1 Dihomotopy equivalence and dcontractibility

Definition 3. Let $X$ and $Y$ be two $d$-spaces. A dihomotopy equivalence between $X$ and $Y$ is given by:

- A $d$-homotopy equivalence between $X$ and $Y$, $f : X \to Y$ and $g : Y \to X$.
- A map $F : PY \to PX$ continuously bigraded as $F_{a,b} : PY(f(a), f(b)) \to PX(a, b)$ such that $(Pf_{a,b}, F_{a,b})$ is a homotopy equivalence between $PX(a, b)$ and $PY(f(a), f(b))$.
- A map $G : PX \to PY$, continuously bigraded as $G_{c,d} : PX(g(c), g(d)) \to PY(c, d)$ such that $(Pg_{c,d}, G_{c,d})$ is a homotopy equivalence between $PX(g(c), g(d))$ and $PY(c, d)$.
- These homotopy equivalences are natural in the following sense:
  - For the two diagrams below (separately), for all $(\alpha, \beta) \in PX$, there exists $(\gamma, \delta) \in PY$ (with domains and codomains induced by the diagrams below) such that they commute$^3$ up to homotopy

\[
\begin{array}{ccc}
PX(a, b) & \xrightarrow{Pf_{a,b}} & PY(f(a), f(b)) \\
\downarrow{F_{a,b}} & & \downarrow{G_{c,d}} \\
PX(\alpha, \beta) & & PX(\gamma, \delta) \\
\uparrow{Pf'_{a',b'}} & & \uparrow{Pf'_{a',b'}} \\
PX(a', b') & \xrightarrow{Pf'_{a',b'}} & PY(f'(a'), f'(b'))
\end{array}
\]

with $g(c') = u'$ and $g(d') = v'$.

- For the two diagrams below (separately), for all $(\gamma, \delta) \in PY$ there exists $(\alpha, \beta) \in PX$ such that they commute up to homotopy

\[
\begin{array}{ccc}
PX(a, b) & \xrightarrow{Pf_{a,b}} & PY(f(a), f(b)) \\
\downarrow{F_{a,b}} & & \downarrow{G_{c,d}} \\
PX(\alpha, \beta) & & PX(\gamma, \delta) \\
\uparrow{Pf'_{a',b'}} & & \uparrow{Pf'_{a',b'}} \\
PX(a', b') & \xrightarrow{Pf'_{a',b'}} & PY(f'(a'), f'(b'))
\end{array}
\]

with $f(a') = u'$ and $f(b') = v'$.

We sometimes write $(f, g, F, G)$ for the full data associated to the dihomotopy equivalence $f : X \to Y$. Note that in the definition above, we always have the following diagrams that commute on the nose, so that the conditions above only consists of 6 commutative diagrams up to homotopy:

\[
\begin{array}{ccc}
PX(a, b) & \xrightarrow{Pf_{a,b}} & PY(f(a), f(b)) \\
\downarrow{Pf'_{a,b}} & & \downarrow{F_{a,b}} \\
PX(\alpha, \beta) & & PX(\gamma, \delta) \\
\uparrow{Pf'_{a',b'}} & & \uparrow{Pf'_{a',b'}} \\
PX(a', b') & \xrightarrow{Pf'_{a',b'}} & PY(f'(a'), f'(b'))
\end{array}
\]

\[
\begin{array}{ccc}
PX(a, b) & \xrightarrow{Pf_{a,b}} & PY(f(a), f(b)) \\
\downarrow{Pf'_{a,b}} & & \downarrow{F_{a,b}} \\
PX(\alpha, \beta) & & PX(\gamma, \delta) \\
\uparrow{Pf'_{a',b'}} & & \uparrow{Pf'_{a',b'}} \\
PX(a', b') & \xrightarrow{Pf'_{a',b'}} & PY(f'(a'), f'(b'))
\end{array}
\]

\[
\begin{array}{ccc}
PX(a, b) & \xrightarrow{Pf_{a,b}} & PY(f(a), f(b)) \\
\downarrow{Pf'_{a,b}} & & \downarrow{F_{a,b}} \\
PX(\alpha, \beta) & & PX(\gamma, \delta) \\
\uparrow{Pf'_{a',b'}} & & \uparrow{Pf'_{a',b'}} \\
PX(a', b') & \xrightarrow{Pf'_{a',b'}} & PY(f'(a'), f'(b'))
\end{array}
\]

\[
\begin{array}{ccc}
PX(a, b) & \xrightarrow{Pf_{a,b}} & PY(f(a), f(b)) \\
\downarrow{Pf'_{a,b}} & & \downarrow{F_{a,b}} \\
PX(\alpha, \beta) & & PX(\gamma, \delta) \\
\uparrow{Pf'_{a',b'}} & & \uparrow{Pf'_{a',b'}} \\
PX(a', b') & \xrightarrow{Pf'_{a',b'}} & PY(f'(a'), f'(b'))
\end{array}
\]

\[
\begin{array}{ccc}
PX(a, b) & \xrightarrow{Pf_{a,b}} & PY(f(a), f(b)) \\
\downarrow{Pf'_{a,b}} & & \downarrow{F_{a,b}} \\
PX(\alpha, \beta) & & PX(\gamma, \delta) \\
\uparrow{Pf'_{a',b'}} & & \uparrow{Pf'_{a',b'}} \\
PX(a', b') & \xrightarrow{Pf'_{a',b'}} & PY(f'(a'), f'(b'))
\end{array}
\]

\[3\text{Meaning both squares, one with } Pf_{a,b} \text{ and } Pf'_{a',b'} \text{ and the other with } F_{a,b} \text{ and } F'_{a',b'} ; \text{ and similarly for the diagram on the right hand side.} \]
Remark: This definition clearly bears a lot of similarities with Dwyer-Kan weak equivalences in simplicial categories (see e.g. [2]). The main ingredient of Dwyer-Kan weak equivalences being exactly that $Pf$ induces a homotopy equivalence. But our definition adds continuity and “extension” or “bisimulation-like” conditions to it, which are instrumental to our theorems and to the classification of the underlying directed geometry.

Remark: There is an obvious notion of deformation diretract of $X \subseteq Y$, which is a dihomotopy equivalence $r : Y \to X$ such that the inclusion map from $X$ to $Y$ is the left homotopy inverse of $r$ and $r \circ i = Id$. But there seems to be no reason that in general, two dihomotopically equivalent spaces are diretracts of a third one (the mapping cylinder in the classical case). It may be true with a zigzag of diretracts though, as in [8].

A dicontractible directed space is a space for which there exists a directed deformation retract onto one of its points. Particularizing once again the definition above, we get:

Definition 4. Let $X$ be a d-space. $X$ is dicontractible if there is a continuous map $R : \{s\} \to PX$, continuously bigraded, such that $R_{c,d}$ are homotopy equivalences (hence in particular, all path spaces of $X$ are contractible).

3.2 Strong dihomotopy equivalence

Algebraically speaking, there is a simple condition that enforces the bisimulation condition above (Lemma[1], that we call strong dihomotopy equivalence (see below Definition[5]). We do not know if it is equivalent to dihomotopy equivalence for a large class of directed spaces.

Definition 5. Let $X$ and $Y$ be two d-spaces. A strong dihomotopy equivalence between $X$ and $Y$ is given by:

- A d-homotopy equivalence between $X$ and $Y$, $f : X \to Y$ and $g : Y \to X$.
- A map $F : PY^f \to PX$ continuously bigraded as $F_{a,b} : PY(f(a), f(b)) \to PX(a, b)$ such that $(Pf_{a,b}, F_{a,b})$ is a homotopy equivalence between $PX(a, b)$ and $PY(f(a), f(b))$.
- A map $G : PX^g \to PY$, continuously bigraded as $G_{c,d} : PX(g(c), g(d)) \to PY(c, d)$ such that $(Pg_{c,d}, G_{c,d})$ is a homotopy equivalence between $PY(c, d)$ and $PX(g(c), g(d))$.
- (a) For all $\alpha \in PX(a', a), \nu \in PY(f(a), f(b))$ and $\beta \in PX(b, b')$, $F_{a', b'}(Ff_{a', a}(\alpha) * \nu * Pf_{b, b'}(\beta)) \sim \alpha * F_{a, b}(\nu) * \beta$.
- (b) For all $\gamma \in PY(c', c), \nu \in PX(g(c), g(d))$ and $\delta \in PY(d, d')$, $G_{c', c'}(Pg_{c', c}(\gamma) * \nu * Pg_{d, d'}(\delta)) \sim \gamma * G_{c, c}(\nu) * \delta$.
- (c) For all $\gamma \in PY^f(f'(a)), \nu \in PY(f(a), f(b))$ and $\delta \in PY^f(f(b), f')$, there exists $(a', b') \in PX$ such that $f(a') = u'$, $f(b') = v'$ and $F_{a', b'}(\gamma * \nu * \delta) \sim F_{a', b}(\gamma) * F_{a, b}(\nu) * F_{b, b'}(\delta)$.
- (d) For all $\alpha \in PX^g(g', g), \nu \in PX(g(c), g(d))$ and $\beta \in PX^g(g(d), g')$, there exists $(c', d') \in PY$ such that $g(c') = u'$, $g(d') = v'$ and $G_{c', c'}(\gamma * \nu * \beta) \sim G_{c, c}(\gamma) * G_{c, d}(\nu) * G_{d, d'}(\delta)$.

Lemma 1. Strong dihomotopy equivalences are dihomotopy equivalences.

Proof. Let $f : X \to Y$ be a strong dihomotopy equivalence; it comes with $g : Y \to X$, $G : PX \to PY$ and $F : PY \to PX$. Let $(\alpha, \beta) \in PX$, with $\alpha \in PX(a', a)$ and $\beta \in PX(b, b')$. In order to prove that $f$ is a dihomotopy equivalence, we must find $(\gamma, \delta)$ such that the diagram below involving $F_{a, b}$ and $F_{a', b'}$ commutes up to homotopy (the other diagram is commutative, for free)
This commutes indeed with $\gamma = P f_{a',a}(\alpha)$ and $\delta = P f_{b,b}(\beta)$ because of property (a) of strong dihomotopy equivalence $f$.

Now, consider the diagram below, involving $G_{c,d}$ and $G_{c',d'}$

with $g(c') = u'$ and $g(d') = v'$. Let $\gamma = G_{c',d'}(\alpha)$ and $\delta = G_{d,d'}(\beta)$. Property (d) implies that the diagram above commutes up to homotopy.

Similarly, let $(\gamma, \delta) \in PY$, property (c) of $f$ implies that the following diagram commutes up to homotopy by taking $\alpha = F_{a',a}(\gamma)$ and $\beta = F_{b,b}(\delta)$

with $f(a') = u'$ and $f(b') = v'$.

And finally, property (b) implies that the following diagram commutes up to homotopy, by taking $\alpha = P g_{c',d'}(\gamma)$ and $\beta = P g_{d,d'}(\delta)$

\[ \begin{align*}
PX(a,b) & \xrightarrow{P f_{a,b}} PY(f(a),f(b)) \\
PX(a,b) & \xrightarrow{F_{a,b}} PX(\gamma,\delta) \\
PX(a',b') & \xrightarrow{P f_{a',b'}} PY(f(a'),f(b')) \\
PX(a',b') & \xrightarrow{F_{a',b'}} PX(\gamma,\delta)
\end{align*} \]

\[ \begin{align*}
PX(g(c),g(d)) & \xrightarrow{G_{c,d}} PY(c,d) \\
PX(g(c),g(d)) & \xrightarrow{P g_{c,d}} PX(\gamma,\delta) \\
PX(g(c'),g(d')) & \xrightarrow{G_{c',d'}} PY(c',d') \\
PX(g(c'),g(d')) & \xrightarrow{P g_{c',d'}} PX(\gamma,\delta)
\end{align*} \]

3.3 Simple properties and examples of directed homotopy equivalences

The first obvious (but important) observation is that directed homotopy equivalence refines ordinary homotopy equivalence. Also, directed homotopy equivalence is an invariant of dihomeomorphic dspaces :

**Lemma 2.** Let $X, Y$ be two directed spaces. Suppose there exists $f : X \to Y$ a dmap, which has an inverse, also a dmap. Then $X$ and $Y$ are directed homotopy equivalent.
Proof. Take \( g = f^{-1} \), \( F = Pg \) and \( G = Pf \). This data forms a directed homotopy equivalence. \( \square \)

Now, natural homology \([3]\) is going to be an invariant of dihomotopy equivalence, as it should be:

**Lemma 3.** Let \( X, Y \) be two directed spaces. Suppose \( X \) and \( Y \) are directed homotopy equivalent. Then \( X \) and \( Y \) have bisimilar natural homotopy and homology (in the sense of \([3]\)).

**Proof.** Suppose \( f : X \to Y \) and \( g : Y \to X \) the underlying dmaps, forming the homotopy equivalence which is a directed homotopy equivalence.

The bisimulation relation we are looking for is the relation:

\[
\{(a, b), Pf_{a,b}, (f(a), f(b)) \mid (a, b) \in \Gamma_X \} \cup \{(g(c), g(d)), Pg_{c,d}, (c, d) \mid (c, d) \in \Gamma_Y \}
\]

The diagrams defining the directed homotopy equivalence imply that \( R \) is hereditary with respect to extension maps. \( \square \)

Unfortunately, unlike Dwyer-Kan equivalences, or classical homotopy equivalences, our dihomotopy equivalences do not have the 2-out-of-3 property. We only have preservation by composition, as shown in next Lemma, but also, for surjective dihomotopy equivalences, two thirds of the 2-out-of-3 property, as shown in Proposition 1.

**Lemma 4.** Compositions of dihomotopy equivalences are dihomotopy equivalences.

**Proof.** Suppose \( f_1 : X \to Y \) and \( f_2 : Y \to Z \) are dihomotopy equivalences. We have quadruples \( (f_1, g_1, F_1, G_1), (f_2, g_2, F_2, G_2) \) as in Definition \([3]\). Now, it is obvious to see that its composite \( (f_2 \circ f_1, g_1 \circ g_2, F_2 \circ F_1, G_2 \circ G_1) \) is a dihomotopy equivalence from \( X \) to \( Z \). \( \square \)

The problem for getting a general 2-out-of-3 property on dihomotopy equivalences can be exemplified as follows. Suppose that \( f^1 \) is a dihomotopy equivalence and that \( f^2 \circ f^1 \) is a dihomotopy equivalence. In particular, by 2-out-of-3 on classical homotopy equivalence, we know that \( f^2 \) is a homotopy equivalence, with homotopy inverse \( g^1 \). Consider now the following composites, for all \((a, b) \in PX\)

\[
\begin{align*}
P_X(a,b) & \xrightarrow{Pf^1_{a,b}} PY(f^1(a),f^1(b)) \xrightarrow{Pf^2_{f^1(a),f^1(b)}} PZ(f^2 \circ f^1(a),f^2 \circ f^1(b)) \\
& \xrightarrow{Pf^2 \circ f^1_{a,b}} P_Y(a,b)
\end{align*}
\]

Because of 2-out-of-3 for classical homotopy equivalences, and as \( Pf^2 \circ f^1 \) and \( Pf^1 \) in the diagram above are homotopy equivalences, \( Pf^2_{f^1(a),f^1(b)} \) is a homotopy equivalence. But we need it to be a homotopy equivalence for all \((c, d) \in PY\), not only the ones in the image of \( f^1 \). Similarly to Dwyer-Kan equivalences (see e.g. \([2]\)), it is reasonable to add in the definition of dihomotopy equivalences the assumption that \( f \) is surjective on points. Then we have

**Proposition 1.** If \( f^1 : X \to Y \) and \( f^2 : Y \to Z \) are such that \( f^1 \) and \( f^2 \circ f^1 \) are surjective dihomotopy equivalences, then \( f^2 \) is a surjective dihomotopy equivalence.

**Proof.** In that case, we get that \( Pf^2 \) induces homotopy equivalences from all \( PY(c, d) \) (call the homotopy inverse \( F^2 \)). Now, notice that, denoting by \( F_{a,b}^{21} \) the homotopy inverse of \( Pf^2 \circ f^1_{a,b} = Pf^2 \circ Pf_{a,b}^{1} \), and by \( F_{a,b}^{1} \) the homotopy inverse of \( Pf_{a,b}^{1} \). \( F^{21} = Pf_{a,b}^{1} \circ F_{a,b}^{21} \) is a right homotopy inverse to \( Pf^2_{f^1(a),f^1(b)} \), because

\[
\begin{align*}
Pf^2_{f^1(a),f^1(b)} \circ (Pf_{a,b}^{1} \circ F_{a,b}^{21}) &= (Pf^2_{f^1(a),f^1(b)} \circ Pf_{a,b}^{1}) \circ F_{a,b}^{21} \\
&= (P(f^2 \circ f^1)_{a,b}) \circ F_{a,b}^{21} \\
&\sim Id_{PZ(f^2, f^1(a),f^2, f^1(b))}
\end{align*}
\]
We therefore have a right homotopy inverse \( F^2_{c,d} \) and a homotopy inverse \( F^2_{c,d} \) of \( P^2_{c,d} \). So,

\[
F^2_{c,d} \sim (F^2_{c,d} \circ P^2_{c,d} \circ F^2_{c,d}) = F^2_{c,d} \circ (P^2_{c,d} \circ F^2_{c,d}) \sim F^2_{c,d}
\]

and

\[
F^2_{c,d} \circ P^2_{c,d} \sim F^2_{c,d} \circ P^2_{c,d} \sim \text{Id}_{PY(c,d)}
\]

therefore \( F^2_{c,d} \) is a homotopy inverse of \( P^2_{c,d} \). Moreover, as a composition of maps \( P^1_{c,d} \) with \( F^2_{c,d} \) which are continuous in \((c, d) \in PY\), it is a continuously bigraded map on \((c, d)\).

We also have to look at the map induced by \( g^2 \). In the following diagram

\[
\begin{array}{ccc}
P^X(g^2(e), g^2(f)) & \overset{P^Y(g^2(e), g^2(f))}{\longrightarrow} & P^Z(e, f) \\
\bigcirc & & \bigcirc
\end{array}
\]

by 2-out-of-3 for classical homotopy equivalences, without any other assumptions, we get that \( P^1_{e,f} \) is a homotopy equivalence (with homotopy inverse \( G^1_{e,f} \)) for all path spaces \( P^Z(e, f) \). We note as above that, denoting by \( G^2_{e,f} \) the homotopy inverse of \( P^1_{e,f} \), \( G^2_{e,f} = G^2_{e,f} \circ P^1_{g^2(e), g^2(f)} \) is a left homotopy inverse of \( P^2_{e,f} \). As before, we easily get that \( G^2_{e,f} \sim G^2_{e,f} \) and \( G^2_{e,f} \) is a homotopy inverse of \( P^1_{e,f} \) which forms a continuously bigraded map.

Now we have to check the extension diagrams of Definition 3. Let \((\alpha, \beta) \in PY\) with \( \alpha \in PY(c', c), \beta \in PY(d', d)\). As \( f^1 \) is surjective, \( c = f(a) \) and \( d = f(b) \) for some \((a, b) \in PX\). As \( f^1 \) is a dihomotopy equivalence, we have the existence of \((\alpha', \beta') \in PX\) as in the diagram below.

Now, we use the fact that \( f^2 \circ f^1 \) is a dihomotopy equivalence, and we get a map \((\alpha, \beta) \in PX\) such that the following diagram commutes up to homotopy

\[
\begin{array}{ccc}
PX(a, b) & \overset{P^1_{a,b}}{\longrightarrow} & PY(c, d) \\
F^2_{c,d} & \overset{F^1_{c,d}}{\longrightarrow} & F^1_{c,d} \\
PX(a', b') & \overset{P^1_{a', b'}}{\longrightarrow} & PY(c', d') \\
PX(a', b') & \overset{P^1_{a', b'}}{\longrightarrow} & PY(c', d') \\
F^2_{c', d'} & \overset{F^1_{c', d'}}{\longrightarrow} & F^1_{c', d'}
\end{array}
\]

In the diagram above, \( F^2_{c,d} \) (represented as a dashed arrow) is actually the composite \( P^1_{a,b} \circ F^2_{c,d} \) as shown before; similarly, \( F^2_{c', d'} \) is the composite \( P^1_{a', b'} \circ F^2_{c', d'} \). Therefore we have the extension property needed for \( F^2_{c,d} \). The five other diagrams can be proven in a similar manner, by pulling back or pushing forward the existence of maps using the diagrams for \( P^1, P^1, F^1, G^1 \) (resp. \( P^1 \circ f^1, P^1 \circ g^2, F^1, G^1 \).

\[\square\]

**Remark:** Similarly, if we have a surjective \( f^2 : Y \to Z \) that is a dihomotopy equivalence, and \( f^1 : X \to Y \) such that \( f^2 \circ f^1 \) is a surjective dihomotopy equivalence, first, there is no reason why \( f^1 \) should be surjective. Similarly as with \( g^2 \) before, we can prove that \( P^1_{a,b} \) forms a continuous bigraded family of homotopy equivalences with a continuous bigraded family of homotopy inverses. The problem is with \( P^1_{a,b} \) which we can only prove to have a continuous family of homotopy inverses on spaces \( PY(g^2(e), g^2(f)) \). The only result we can have in general is dual.
to the one of Proposition 1. If \( f^2 : X \to Y \) and \( f^2 : Y \to Z \) are such that \( f^2 \) and \( f^2 \circ f^1 \) are dihomotopy equivalences with surjective homotopy inverses, then \( f^2 \) is a dihomotopy equivalence with surjective homotopy inverse.

**Example 1.** The unit segment is dicontractible. The wedge of two segments is dicontractible (which makes shows the version of dicontractibility discussed here to be notably different from that used in the framework of [8]). Note that in view of applications to directed topological complexity, this is coherent with the fact that directed topological complexity should be invariant under directed homotopy equivalence. For any two pair of points in \( \Gamma \) of directed segment or of a wedge of two segments, there is indeed a continuous map depending on this pair of points to the unique dipath going from one to the other.

**Example 2.** The Swiss flag is not directed homotopy equivalent to the hollow square. This can be seen already using natural homology, that distinguishes the two, see e.g. [5].

More precisely, we consider the following d-spaces (SF on the left, HS on the right), coming from PV processes, which are subspaces of \( \mathbb{R}^2 \) and whose points are within the white part in the square (the grey part represents the forbidden states of the program) and whose dipaths are non decreasing paths for the componentwise ordering on \( \mathbb{R}^2 \). They are homotopy equivalent using the two maps (and even dmaps), \( f \) from SF to HS and \( g \) from HS to SF, depicted below (\( f \) is the map on the left):

![Diagram](image.png)

The points in light grey are the points which do not belong to the image of those maps. The problem is that those two programs are quite different: SF has a dead-lock in \( \alpha \) and inaccessible states, while HC does not. Topologically, they do not have the same (directed) components in the sense of [11].

It is easy to see that although \( Pf \) and \( Pg \) are homotopy equivalences as well, \( (f,g) \) does not induce a dihomotopy equivalence in our sense. Take a point \( \alpha \) in the lower convexity of the Swiss flag, and consider the constant dipath on \( f(\alpha) \) in HS. \( Pf \) maps the constant path on \( \alpha \) onto the constant path on \( f(\alpha) \) but we can extend in \( P(HS)^f \) this constant path to paths \( v \) from \( f(\alpha) \) to the image by \( f \) of the upper right point, which is again the upper right point in HS. This extension makes the corresponding path space within HS homotopy equivalent to two points, whereas there are no path from \( \alpha \) to the right upper point in SF.

In natural homology, SF and HS do not have bisimilar natural homologies since, considering the pair of points \((\alpha,\alpha)\) in \( SF \times SF \), all extensions of this pair of points will give 0th homology of there corresponding path space equal to \( \mathbb{Z} \), whereas there are extensions of any pair of points \((\beta,\beta)\) in HS which give 0th homology of there corresponding path space equivalent to \( \mathbb{Z}^2 \).

![Diagram](image.png)

Figure 1: Naive equivalence between the Fahrenberg’s matchbox \( M \) and its upper face \( T \)
The constant map 
\[ g \]
\[ 0 \] in our sense. As a matter of fact, consider points \( T \) of 
\[ \gamma, \delta \]
\[ 4 \] Dicontractibility and the dipath space map

Definition 6. Let \( X \) be a d-space. \( X \) is said to be weakly dicontractible if its natural homotopy (equivalently, natural homology \[ 5 \], by directed Hurewicz, all up to bisimulation) is the natural system - which we will denote by \( \Box \) - on \( 1 \) (the final object in \( \text{Cat} \)) with value \( \mathbb{Z} \) on the object, in dimension 0 and 0 in higher dimensions.

The following is a direct consequence of Lemma 3 but we give below a simple and direct proof of it:

Lemma 5. Let \( X \) be a dicontractible d-space. Then \( X \) is weakly dicontractible in the sense of Definition 6.

Proof. Suppose \( X \) is dicontractible. Therefore, by Definition 4 we have a continuous map \( R : \{ * \} \to PX \), continuously bigraded, which are are homotopy equivalences. All extension maps \( \gamma, \delta \) induce identities modulo homotopy, trivially. Now, these diagrams of spaces induce, in homology, a diagram which has \( \mathbb{Z} \) as value on objects for dimension 0, and 0 for higher dimensions.

It is a simple exercise to see that such diagrams are bisimilar to the one point diagram which has only \( \mathbb{Z} \) as value in dimension 0, and 0 in higher dimension. This shows weak dicontractibility.

Theorem 1. Suppose \( X \) is a contractible d-space. Then, the dipath space map has a continuous section if and only if \( X \) is dicontractible.

Proof. As \( X \) is contractible, we have \( f : X \to \{ a_0 \} \) (the constant map) and \( g : \{ a_0 \} \to X \) (the inclusion) which form a (classical) homotopy equivalence. Trivially, \( f \) and \( g \) are dmaps.

Suppose that we have a continuous section \( s \) of \( \chi \). There is an obvious inclusion map \( i : \{ s(a,b) \} \to PX(a,b) \), which is continuously bigraded in \( a \) and \( b \). Define \( R \) to be that map. Now the constant map \( r : PX(a,b) \to \{ s(a,b) \} \) is a retraction map for \( i \). We define

\[
H : PX \times [0,1] \to PX
\]

\[
(u,t) \to v \text{ s.t. } \begin{cases} v(x) = u(x) & \text{if } 0 \leq x \leq \frac{t}{2} \\ v(x) = s \left( u \left( \frac{t}{2} \right), u \left( 1 - \frac{t}{2} \right) \right) & \text{if } \frac{t}{2} \leq x \leq 1 - \frac{t}{2} \\ v(x) = u(x) & \text{if } 1 - \frac{t}{2} \leq x \leq 1 
\end{cases}
\]

\( H(u,t) \) is extended by continuity for \( t = 1 \) as being equal to \( u \).

As concatenation and evaluation are continuous and as \( s \) is continuous in both arguments \( H \) is continuous in \( u \in PX \) and in \( t \). \( H \) induces families \( H_{a,b} : PX(a,b) \times [0,1] \to PX(a,b) \), and because \( H \) is continuous in \( u \) in the compact-open topology, this family \( H_{a,b} \) is continuous in \( a \) and \( b \) in \( X \). Finally, we note that \( H(u,1) = u \) and \( H(u,0) = s(u(0), u(1)) = i \circ r(u) \). Hence \( r \) is a deformation retraction and \( PX(a,b) \) is homotopy equivalent to \( \{ s(a,b) \} \) and has the homotopy type we expect (is contractible for all \( a \) and \( b \)), meaning that \( R \) is a continuously bigraded homotopy equivalence.

The homotopy \( H_{a,b} \) shows also that any extension map from \( PX(a',b') \) to \( PX(a,b) \) is homotopic to the identity \( \Box \). Therefore, \( X \) is dicontractible.

4 Conversely, suppose \( X \) is dicontractible. We have in particular a continuous family (in \( a, b \) in \( X \)) of maps \( R_{a,b} : \{ * \} \to PX(a,b) \). Define \( s(a,b) = R_{a,b}(*) \), this is a continuous section of \( \chi \).

\[ \Box \]Note the link with the fact that \( s(a,a') \) and \( s(b',b) \) are Yoneda invertible.
Remark: Sometimes, we do not know right away, in the theorem above, that \( X \) is contractible. But instead, there is an initial state in \( X \), i.e., a state \( a_0 \) from which every point of \( X \) is reachable. Suppose then that, as in the Theorem above, \( \chi \) has a continuous section \( s : \Gamma_X \to PX \). Consider \( s'(a,b) = s^{-1}(a_0,a) \ast s(a_0,b) \) the concatenation of the inverse dipath, going from \( a \) to \( a_0 \), with the dipath going from \( a_0 \) to \( b \) : this is a continuous path from \( a \) to \( b \) for all \( a, b \) in \( X \). Now, \( s' \) is obviously continuous since concatenation, and \( s \), are. By a classical theorem \([9]\), this implies that \( X \) is contractible and the rest of the theorem holds.

5 Directed topological complexity

Definition 7. The directed topological complexity \( \overline{TC}(X) \) of a \( d \)-space \( X \) (which is also an Euclidean Neighborhood retract, or ENR) is the minimum number \( n \) (or \( \infty \) if no such \( n \) exists) such that \( \Gamma_X \) can be partitioned into \( n \) ENRs\(^5\) \( F_1, \ldots, F_n \) such that there exists a map \( s : \Gamma_X \to PX \) (not necessarily continuous, of course!) with :

- \( \chi \circ s = \text{Id} \) (\( s \) is a, non-necessarily continuous, section of \( \chi \))
- \( s|_{F_i} : F_i \to PX \) is continuous

Example 4. Consider the PV program \( \text{PaVa} | \text{PaVa} \) (a is a mutex), the component category (see e.g. [11]) has 4 regions \( C_1, C_2, C_3 \), and \( C_4 \), with a unique morphism from \( C_1 \) to \( C_2 \), \( C_1 \) to \( C_3 \), \( C_2 \) to \( C_4 \), and \( C_3 \) to \( C_4 \) (but two from \( C_1 \) to \( C_4 \)). We have \( \overline{TC}(X) = 2 \) since \( \Gamma_X \) can be partitioned into \( \{(x,y) \mid x \in C_i, y \in C_j, (i,j) \neq (1,4)\} \) and \( C_1 \times C_4 \). More generally speaking, the lifting property of components, Proposition 7 of [10], in the case when we have spaces \( X \) with components categories (such as with the cubical complexes of [8]) implies that we can examine the dihomotopy type of \( X \) through the dihomotopy equivalent space, quotient of \( X \) by its components.

Example 5. Consider now \( X \) to be a cube minus an inner cube, seen as a partially ordered space with componentwise ordering, and as such, a \( d \)-space. Note that it is dihomotopy equivalent to a hollow cube, hence we will examine its directed topological complexity through this di retract. Up to homotopy, this is \( S^2 \) (seen as a sphere with unit radius centered in 0, in \( \mathbb{R}^3 \)), for which we know (e.g. [12]) that \( TC(X) = 3 \). A simple partition which shows that it is at most 3 is

- \( F_1 = \{(x,y) \mid x \neq -y\} \)
- \( F_2 = \{(x,-x) \mid x \neq x_0\} \)
- \( F_3 = \{(x_0, -x_0)\} \)

where \( x_0 \) is some point of \( S^2 \) that we can choose, for which we have a smooth vector field \( v \) on \( S^2 \), which is non zero everywhere except at \( x_0 \) (such a point must exist by general theorems, and we can find a vector field which will only be zero at one point). On \( F_1 \), we take as section to the classical path space fibration, the Euclidean geodesic path from \( x \) to \( y \). On \( F_2 \), we take the path from \( x \) to \( -x \) which follows the vector field \( v \). Finally, we take any path from \( x_0 \) to \( -x_0 \).

This partition can be used to find a upper bound for the directed topological complexity of the hollow cube (or equivalently, the cube minus an inner cube). Note that in general, the two notions are incomparable. Because if \( X \times X = \bigcup_{i=1}^{k} F_i \) (\( k \) is \( TC(X) \)) with \( s|_{F_i} \), continuous section of the path space fibration, then we do not know if on each \( F_i \) we can find a continuous section of the dipath space map. Conversely, if \( \Gamma_X = \bigcup_{i=1}^{l} F_i \) (\( l \) is \( \overline{TC}(X) \)), each \( s|_{F_i} \) provides us with a continuous section, locally to \( F_i \), of the dipath space map and of the path space fibration, but unfortunately, we can only cover that way \( \Gamma_X \subseteq X \times X \). These two notions are only trivially comparable when the set of directed paths is equal to the set of all continuous paths, in which case the directed topological complexity is equal to the classical topological complexity.

\(^5\)This could be replaced as in the classical case, by asking for a covering by open sets, or by a covering by closed sets.
Still, in some cases, such as in the case of the hollow cube, we can carefully examine the partition given by the classical topological complexity, to, in general, find an upper bound to the directed topological complexity. Here, we can choose $x_0$ to be the final point of $X$ (the hollow cube), and strip down $F_1$, $F_2$ and $F_3$ to be the part of the previous partition, intersected with $X$. Then on each of these three sets we have a trivial section to the dipath space map. Hence $\overrightarrow{TC}(X) \leq 3$.

Martin Raussen observed\footnote{Private communication during the Hausdorff Institute “Applied Computational Algebraic Topology” semester, on the 14th September 2017.} that in fact, $\overrightarrow{TC}(X) = 2$ which shows an essential difference to the classical case.

Remark: Consider the universal covering of an NPC cubical complex (as in \cite{1,2}). It is CAT(0). We conjecture that $\overrightarrow{TC}(X)$ is the number of maximal configurations of the corresponding prime event structure (see \cite{1,5,17,1}, or equivalently, the number of levels in the universal covering (see e.g. \cite{7}), or the number of maximal dipaths starting in the initial point, up to dihomotopy).

**Lemma 6.** Let $X$ and $Y$ be two dihomotopy equivalent spaces. Then $\overrightarrow{TC}(X) = \overrightarrow{TC}(Y)$.

**Proof.** As $X$ and $Y$ are dihomotopy equivalent, we have $f : X \rightarrow Y$ and $g : Y \rightarrow X$ dmaps, which form a homotopy equivalence between $X$ and $Y$. We also get $G_{c,d} : PX(g(c),g(d)) \rightarrow PY(c,d)$ which is inverse modulo homotopy to $F_{g_{c,d}}$ and varies continuously according to $c,d$; and $F_{a,b} : PX(a,b) \rightarrow PY(f(a),f(b))$ which is inverse modulo homotopy to $P_{f_{a,b}}$, varying continuously according to $a$ and $b$. We note that $f$ and $g$ induce continuous maps $f^* : \Gamma_X \rightarrow \Gamma_Y$ and $g^* : \Gamma_Y \rightarrow \Gamma_X \ (\Gamma_X \text{ and } \Gamma_Y \text{ inherit the product topology of } X \text{, resp. } Y)$.

Suppose first $k = \overrightarrow{TC}(X)$. Thus we can write $\Gamma_X = F_1^X \cup \ldots \cup F_k^X$ such that we have a map $s : \Gamma_X \rightarrow PX$ with $\chi \circ s = Id$ and $s|_{F_k^X}$ is continuous.

Define $F_i^Y = \{ u \in \Gamma_Y \mid g^*(u) \in F_i^X \}$ (which is an ENR - or open if we choose the alternate definition - as $F_i^X$ is ENR and $g^*$ is continuous) and define $t_{F_i^Y}(u) = G_u \circ s|_{F_i^X} \circ g^*(u) \in PY(u)$ for all $u \in F_i^Y \subseteq \Gamma_Y$. This is a continuous map in $u$ since $s|_{F_i^X}$ is continuous, $g^*$ is continuous, and $G$ is continuously bigraded. Therefore $\overrightarrow{TC}(Y) \leq \overrightarrow{TC}(X)$.

Conversely, suppose $t : \overrightarrow{TC}(Y)$, $\Gamma_Y = F_1^Y \cup \ldots \cup F_k^Y$ such that we have a map $t : \Gamma_Y \rightarrow PY$ with $\chi \circ t = Id$ and $t|_{F_i^Y}$ is continuous. Now define $F_i^X = \{ u \in \Gamma_X \mid f^*(u) \in F_i^Y \}$ (which is an ENR - or open set if we choose the alternate definition - as $F_i^Y$ is ENR and $f^*$ is continuous) and define $s|_{F_i^X}(u) = F_u \circ t|_{F_i^Y} \circ f^*(u) \in PX(u)$ for all $u \in F_i^X \subseteq \Gamma_X$. This is a continuous map in $u$ since $t|_{F_i^Y}$ is continuous, $f^*$ is continuous, and $F$ is continuously bigraded. Therefore $\overrightarrow{TC}(X) \leq \overrightarrow{TC}(Y)$. Hence we conclude that $\overrightarrow{TC}(X) = \overrightarrow{TC}(Y)$ and directed topological complexity is an invariant of dihomotopy equivalence. \qed

Remark: The proof above is enlightening in that it uses all homotopy equivalences generated by the data that $f$ is a dihomotopy equivalence. We would have had only a Dwyer-Kan type of equivalence, we would not have had that directed topological equivalence is a dihomotopy invariant.

6 Conclusion

There are numerous developments to this, in studying directed topological complexity with a view to control theory, but also on the more fundamental level of the structure of dihomotopy equivalences.

There is for instance an interesting notion of weak-equivalence, coming out of our dihomotopy equivalence (same conditions, but inducing isomorphisms of the fundamental groups of the different path spaces, with similar extension conditions). This weak-equivalence should have good properties.
with respect to our dihomotopy equivalence, have the 2-out-of-6 property, and a refined form of
natural homology should come out as a derived functor in that framework. This is left for another
venue.

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