Abstracting an operational semantics to finite automata

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Abstract. There is an apparent similarity between the descriptions of small-step operational semantics of imperative programs and the semantics of finite automata, so defining an abstraction mapping from semantics to automata and proving a simulation property seems to be easy. This paper aims at identifying the reasons why simple proofs break, among them artifacts in the semantics that lead to stuttering steps in the simulation. We then present a semantics based on the zipper data structure, with a direct interpretation of evaluation as navigation in the syntax tree. The abstraction function is then defined by equivalence class construction.

Keywords: Programming language semantics; Abstraction; Finite Automata; Formal Methods; Verification

1 Introduction

Among the formalisms employed to describe the semantics of transition systems, two particularly popular choices are abstract machines and structural operational semantics (SOS). Abstract machines are widely used for modeling and verifying dynamic systems, e.g. finite automata, Büchi automata or timed automata [9,4,1]. An abstract machine can be represented as a directed graph with transition semantics between nodes. The transition semantics is defined by moving a pointer to a current node. Automata are a popular tool for modeling dynamic systems due to the simplicity of the verification of automata systems, which can be carried out in a fully automated way, something that is not generally possible for Turing-complete systems.

This kind of semantics is often extended by adding a background state composed of a set of variables with their values: this is the case of timed automata, which use background clock variables [2]. The UPPAAL model checker for timed automata extends the notion of background state even further by adding integer and Boolean variables to the state [7] which, however, do not increase the

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computational power of such timed automata but make them more convenient to use.

Another formalism for modeling transition systems is structural semantics (“small-step”, contrary to “big-step” semantics which is much easier to handle but which is inappropriate for a concurrent setting), which uses a set of reduction rules for simplifying a program expression. It has been described in detail in [16] and used, for example, for the Jinja project developing a formal model of the Java language [10]. An appropriate semantic rule for reduction is selected based on the expression pattern and on values of some variables in a state. As a result of reduction the expression and the state are updated.

\[ s' = s(v \mapsto \text{eval } \text{expr } s) \]

\[ (\text{Assign } v \text{ expr }, s) \rightarrow (\text{Unit}, s') \]  

[Assignment]

This kind of rules is intuitive; however, the proofs involving them require induction over the expression structure. A different approach to writing a structural semantics was described in [3,12] for the CMinor language. It uses a notion of continuation which represents an expression as a control stack and deals with separate parts of the control stack consecutively.

\[ (\text{Seq } e_1 \cdot e_2 \cdot \kappa, s) \rightarrow (e_1 \cdot e_2 \cdot \kappa, s) \]

\[ (\text{Empty } \cdot \kappa, s) \rightarrow (\kappa, s) \]

Here the “·” operator designates concatenation of control stacks. The semantics of continuations does not need induction over the expression, something which makes proof easier; however it requires more auxiliary steps for maintaining the control stack which do not have direct correspondance in the modeled language.

For modeling non-local transfer of control, Krebbers and Wiedijk [11] present a semantics using (non-recursive) “statement contexts”. These are combined with the above-mentioned continuation stacks. The resulting semantics is situated mid-way between [3] and the semantics proposed below.

The present paper describes an approach to translation from structural operational semantics to finite automata extended with background state. All the considered automata are an extension of Büchi automata with background state, i.e. they have a finite number of nodes and edges but can produce an infinite trace. The reason of our interest in abstracting from structural semantics to Büchi automata is our work in progress [6]. We are working on a static analysis algorithm for finding possible resource sharing conflicts in multithreaded Java programs. For this purpose we annotate Java programs with timing information and then translate them to a network of timed automata which is later model checked. The whole translation is formally verified. One of the steps of the translation procedure includes switching from structural operational semantics of a Java-like language to automata semantics. During this step we discovered some problems which we will describe in the next section. The solutions we propose extend well beyond the problem of abstracting a structured language to an automaton. It can also be used for compiler verification, which usually is cluttered up with arithmetic adress calculation that can be avoided in our approach.
The contents of the paper has been entirely formalized in the Isabelle proof assistant [14]. We have not insisted on any Isabelle-specific features, therefore this formalization can be rewritten using other proof assistants. The full Isabelle formal development can be found on the web [5].

2 Problem Statement

We have identified the following as the main problems when trying to prove the correctness of the translation between a programming language semantics and its abstraction to automata:

1. Preservation of execution context: an abstract machine always sees all the available nodes while a reduced expression loses the information about previous reductions.
2. Semantic artifacts: some reduction rules are necessary for the functionality of the semantics, but may be missing in the modeled language. Additionally, the rules can produce expressions which do not occur in the original language.

These problems occur independently of variations in the presentation of semantic rules [16] adopted in the literature, such as [10] (recursive evaluation of sub-statements) or [3,12] (continuation-style).

We will describe these two problems in detail, and later our approach to their solution, in the context of a minimalistic programming language which only manipulates Boolean values (a Null value is also added to account for errors):

```plaintext
datatype val = Bool bool | Null
```

The language can be extended in a rather straightforward way to more complex expressions. In this language, expressions are either values or variables:

```plaintext
datatype expr = Val val | Var vname
```

The statements are those of a small imperative language (similarly to [13]):

```plaintext
datatype stmt =
  Empty — no-op
  | Assign vname val — assignment: var := val
  | Seq stmt stmt — sequence: c1;c2
  | Cond expr stmt stmt — conditional: if e then c1 else c2
  | While expr stmt — loop: while e do c
```

2.1 Preservation of execution context

Problem 1 concerns the loss of an execution context through expression reductions which is a design feature of structural semantics. Let us consider a simple example.

Assume we have a structural semantics for our minimal imperative language (some rules of a traditional presentation are shown in Figure 1): we want to
translate a program written in this language into an abstract machine. Assume that the states of variable values have the same representation in the two systems; this means we only need to translate the program expression into a directed graph with different nodes corresponding to different expressions obtained by reductions of the initial program expression.

On the abstract machine level the Assign statements would be represented as two-state automata, and the Cond as a node with two outgoing edges directed to the automata for the bodies of its branches.

Consider a small program in this language Cond bexp (Assign a 5) Empty and its execution flow.

\[
\text{Cond bexp (Assign a 5) Empty} \rightarrow (\text{Assign a 5}) \xrightarrow{a=5} \text{Empty}
\]

The execution can select any of the two branches depending on the bexp value. There are two different Empty expressions appearing as results of two different reductions. The corresponding abstract machine would be a natural graph representation for a condition statement with two branches (Figure 2).

During the simple generation of an abstract machine from a program expression the two Empty statements cannot be distinguished although they should be mapped into two different nodes in the graph. We need to add more information about the context into the translation, and it can be done by different ways.

A straightforward solution would be to add some information in order to distinguish between the two Empty expressions. If we add unique identifiers to each subexpression of the program, they will allow to know exactly which subexpression we are translating (Figure 3). The advantage of this approach is its simplicity, however, it requires additional functions and proofs for identifier management.

Another solution for the problem proposed in this paper involves usage of a special data structure to keep the context of the translation. There are known examples of translations from subexpression-based semantics [10] and continuation-based semantics [12] to abstract machines. However, all these translations do not address the problem of context preservation during the translation.

\[
\begin{align*}
\text{eval bexp } s = \text{True} \quad &\Rightarrow (\text{Cond bexp } e_1 \ e_2, s) \rightarrow (e_1, s) \quad &\text{[COND]} \\
\text{eval bexp } s = \text{False} \quad &\Rightarrow (\text{Cond bexp } e_1 \ e_2, s) \rightarrow (e_2, s) \quad &\text{[CONF]}
\end{align*}
\]

Fig. 1: Semantic rules for the minimal imperative language.
Fig. 2: The execution flow and the corresponding abstract machine for the program $\text{Cond } bexp \ (\text{Assign } a \ 5) \ \text{Empty}$.

Fig. 3: The execution flow and the corresponding abstract machine for the program with subexpression identifiers $\text{Cond } n_1 \ bexp \ (\text{Assign } n_2 \ a \ 5) \ (\text{Empty } n_3)$. 
2.2 Semantic artifacts

The second problem appears because of the double functionality of the Empty expression: it is used to define an empty operator which does nothing as well as the final expression for reductions which cannot be further reduced. The typical semantic rules for a sequence of expressions look as shown on Figure 4.

\[
\begin{align*}
(e_1, s) &\rightarrow (e_1', s') \quad \text{[SEQ1]} \\
(\text{Seq } e_1 \ e_2, s) &\rightarrow (\text{Seq } e_1' \ e_2, s') \quad \text{[SEQ1]} \\
(\text{Seq Empty } e_2, s) &\rightarrow (e_2, s) \quad \text{[SEQ2]}
\end{align*}
\]

Fig. 4: Semantic rules for the sequence of two expressions.

Here the Empty expression means that the first expression in the sequence has been reduced up to the end, and we can start reducing the second expression. However, any imperative language translated to an assembly language would not have an additional operator between the two pieces of code corresponding to the first and the second expressions. The rule SEQ2 must be marked as a silent transition when translated to an automaton, or the semantic rules have to be changed.

3 Zipper-based semantics of imperative programs

3.1 The zipper data structure

Our plan is to propose an alternative technique to formalize operational semantics that will make it easier to preserve the execution context during the translation to an automata-based formalism. Our technique is built around a zipper data structure, whose purpose is to identify a location in a tree (in our case: a stmt) by the subtree below the location and the rest of the tree (in our case: of type stmt-path). In order to allow for an easy navigation, the rest of the tree is turned inside-out so that it is possible to reach the root of the tree by following the backwards pointers. The following definition is a straightforward adaptation of the zipper for binary trees discussed in [8] to the stmt data type:

```plaintext
datatype stmt-path =
  PTop |
  PSeqLeft stmt-path stmt |
  PSeqRight stmt stmt-path |
  PCondLeft expr stmt-path stmt |
  PCondRight expr stmt stmt-path |
  PWhile expr stmt-path
```

Here, PTop represents the root of the original tree, and for each constructor of stmt and each of its sub-stmts, there is a “hole” of type stmt-path where a subtree can be fitted in. A location in a tree is then a combination of a stmt and a stmt-path:
datatype stmt-location = Loc stmt stmt-path

Given a location in a tree, the function reconstruct reconstructs the original tree $\text{reconstruct} :: \text{stmt} \Rightarrow \text{stmt-path} \Rightarrow \text{stmt}$, and $\text{reconstruct-loc} (\text{Loc} \ c \ sp) =$ reconstruct c sp does the same for a location.

fun reconstruct :: stmt ⇒ stmt-path ⇒ stmt where
  reconstruct c PTop = c
  | reconstruct c (PSeqLeft sp c2) = reconstruct (Seq c c2) sp
  | reconstruct c (PSeqRight c1 sp) = reconstruct (Seq c1 c) sp
  | reconstruct c (PCondLeft e sp c2) = reconstruct (Cond e c c2) sp
  | reconstruct c (PCondRight e c1 sp) = reconstruct (Cond e c1 c) sp
  | reconstruct c (PWhile e sp) = reconstruct (While e c) sp

fun reconstruct-loc :: stmt-location ⇒ stmt where
  reconstruct-loc (Loc c sp) = reconstruct c sp

3.2 Semantics

Our semantics is a small-step operational semantics describing the effect of the execution a program on a certain program state. For each variable, the state yields Some value associated with the variable, or None if the variable is unassigned. More formally, the state is a mapping $\text{ename} \Rightarrow \text{val option}$. Defining the evaluation of an expression in a state is then standard.

Before commenting the rules of our semantics, let us discuss which kind of structure we are manipulating. The semantics essentially consists in moving around a pointer within the syntax tree. As explained in Section 3.1, a position in the syntax tree is given by a stmt-location. However, during the traversal of the syntax tree, we visit each position at least twice (and possibly several times, for example in a loop): before executing the corresponding statement, and after finishing the execution. We therefore add a Boolean flag, where True is a marker for “before” and False for “after” execution.

\[
\begin{align*}
\downarrow \text{While} & \quad \text{While} & \quad \text{While} & \quad \text{While} \\
\downarrow \text{Seq} & \quad \downarrow \text{Seq} & \quad \downarrow \text{Seq} & \quad \downarrow \text{Seq} \\
\downarrow x := T & \quad \downarrow y := F & \quad \downarrow x := T & \quad \downarrow y := F & \quad \uparrow x := T & \quad \uparrow y := F
\end{align*}
\]

Fig. 5: Example of execution of small-step semantics

As an example, consider the execution sequence depicted in Figure 5 (with assignments written in a more readable concrete syntax), consisting of the initial steps of the execution of the program $\text{While} (e, \text{Seq}(x := T, y := F))$. 

The before (resp. after) marker is indicated by a downward arrow before (resp. an upward arrow behind) the current statement. The condition of the loop is omitted because it is irrelevant here. The middle configuration would be coded as \(((\text{Loc } (x := T) \ (P\text{SeqLeft} \ (P\text{While} e \ P\text{Top}) \ (y := F))), \ True)\).

Altogether, we obtain a syntactic configuration \((\text{synt-config})\) which combines the location and the Boolean flag. The semantic configuration \((\text{sem-config})\) manipulated by the semantics adjoins the \textit{state}, as defined previously.

\begin{align*}
\text{type-synonym} & \quad \text{synt-config} = \text{stmt-location} \times \text{bool} \\
\text{type-synonym} & \quad \text{sem-config} = \text{synt-config} \times \text{state}
\end{align*}

The rules of the small-step semantics of Figure 7 fall into two categories: before execution of a statement \(s\) (of the form \(((l, \ True), s))\) and after execution (of the form \(((l, \ False), s)); there is only one rule of this latter kind: SF\_FALSE.

\begin{figure}[h]
\begin{verbatim}
fun next-loc :: stmt ⇒ stmt-path ⇒ (stmt-location × bool) where
  next-loc c PTop = (Loc c PTop, False)
  | next-loc c (PSeqLeft sp c2) = (Loc c2 (PSeqRight c sp), True)
  | next-loc c (PSeqRight c1 sp) = (Loc (Seq c1 c) sp, False)
  | next-loc c (PCondLeft e sp c2) = (Loc (Cond e c c2) sp, False)
  | next-loc c (PCondRight e c1 sp) = (Loc (Cond e c c1) sp, False)
  | next-loc c (PWhile e sp) = (Loc (While e c) sp, True)
\end{verbatim}
\end{figure}

Fig. 6: Finding the next location

Let us comment on the rules in detail:
- S\_EMPTY executes the \textit{Empty} statement just by swapping the Boolean flag.
- S\_ASSIGN is similar, but it also updates the state for the assigned variable.
- S\_SEQ moves the pointer to the substatement \(c_1\), pushing the substatement \(c_2\) as continuation to the statement path.
- S\_COND\_T and S\_COND\_F move to the \textit{then}- respectively \textit{else}- branch of the conditional, depending on the value of the condition.
- S\_WHILE\_T moves to the body of the loop.
- S\_WHILE\_F declares the execution of the loop as terminated, by setting the Boolean flag to \textit{False}.
- SF\_FALSE comes into play when execution of the current statement is finished.
  We then move to the next location, provided we have not already reached the root of the syntax tree and the whole program terminates.

The move to the next relevant location is accomplished by function \textit{next-loc} (Figure 6) which intuitively works as follows: upon conclusion of the first substatement in a sequence, we move to the second substatement. When finishing the body of a loop, we move back to the beginning of the loop. In all other cases, we move up the syntax tree, waiting for rule SF\_FALSE to relaunch the function.
((Loc Empty sp, True), s) → ((Loc Empty sp, False), s) \[\text{SEmpty}\]

((Loc (Assign vr vl) sp, True), s) → ((Loc (Assign vr vl) sp, False), s(vr → vl)) \[\text{SAssign}\]

((Loc (Seq c1 c2) sp, True), s) → ((Loc c1 (PSeqLeft sp c2), True), s) \[\text{SSeq}\]

eval e s = Bool True

((Loc (Cond e c1 c2) sp, True), s) → ((Loc c1 (PCondLeft e sp c2), True), s) \[\text{SCondT}\]

eval e s = Bool False

((Loc (Cond e c1 c2) sp, True), s) → ((Loc c2 (PCondRight e c1 sp), True), s) \[\text{SCondF}\]

eval e s = Bool True

((Loc (While e c) sp, True), s) → ((Loc c (PWhile e sp), True), s) \[\text{SWhileT}\]

eval e s = Bool False

((Loc (While e c) sp, True), s) → ((Loc (While e c) sp, False), s) \[\text{SWhileF}\]

sp ≠ PTop

((Loc c sp, False), s) → (next-loc c sp, s) \[\text{SFalse}\]

Fig. 7: Small-step operational semantics

4 Target language: Automata

4.1 Syntax

As usual, our automata are a collection of nodes and edges, with a distinguished initial state. In this general definition, we will keep the node type ‘n abstract. It will later be instantiated to synt-config. An edge connects two nodes; moving along an edge may trigger an assignment to a variable (AssAct), or have no effect at all (NoAct).

An automaton ‘n ta is a record consisting of a set of nodes, a set of edges and an initial node init-s. An edge has a source node, an action and a destination node dest. Components of a record are written between [ ] ... ].

4.2 Semantics

An automaton state is a node, together with a state as in Section 3.2.

type-synonym ‘n ta-state = ‘n * state
Executing a step of an automaton in an automaton state \((l, s)\) consists of selecting an edge starting in node \(l\), moving to the target of the edge and executing its action. Automata are non-deterministic; in this simplified model, we have no guards for selecting edges.

\[
\begin{align*}
  l &= \text{source } e \\
  l' &= \text{dest } e \\
  s' &= \text{action-effect } \text{action } e \text{ s} \\
  \text{aut} &\vdash (l, s) \rightarrow (l', s') \quad \text{[ACTION]}
\end{align*}
\]

5 Automata construction

The principle of abstracting a statement to an automaton is simple; the novelty resides in the way the automaton is generated via the zipper structure: as nodes, we choose the locations of the statements (with their Boolean flags), and as edges all possible transitions of the semantics.

To make this precise, we need some auxiliary functions. We first define a function \(\text{all-locations}\) of type \(\text{stmt} \Rightarrow \text{stmt-path} \Rightarrow \text{stmt-location list}\) which gathers all locations in a statement, and a function \(\text{nodes-of-stmt-locations}\) which adds the Boolean flags.

As for the edges, the function \(\text{synt-step-image}\) yields all possible successor configurations for a given syntactic configuration. This is of course an over-approximation of the behavior of the semantics, since some of the source tree locations may be unreachable during execution.

```plaintext
fun synt-step-image :: synt-config \Rightarrow synt-config list where
  synt-step-image (Loc Empty sp, True) = [(Loc Empty sp, False)]
| synt-step-image (Loc (Assign vr vl) sp, True) = [(Loc (Assign vr vl) sp, False)]
| synt-step-image (Loc (Seq c1 c2) sp, True) = [(Loc c1 (PSeqLeft sp c2), True)]
| synt-step-image (Loc (Cond e c1 c2) sp, True) = [(Loc c1 (PCondLeft e sp c2), True), (Loc c2 (PCondRight e c1 sp), True)]
| synt-step-image (Loc (While e c) sp, True) =
  [Loc c (PWhile e sp), True), (Loc (While e c) sp, False)]
| synt-step-image (Loc c sp, False) = (if sp = PTop then [] else [next-loc c sp])

Together with the following definitions:

fun action-of-synt-config :: synt-config \Rightarrow action where
  action-of-synt-config (Loc (Assign vn vl) sp, True) = AssAct vn vl
| action-of-synt-config (Loc c sp, b) = NoAct

definition edge-of-synt-config :: synt-config \Rightarrow synt-config edge list where
  edge-of-synt-config s =
  map(\lambda t. (\text{source } = s, \text{action } = \text{action-of-synt-config } s, \text{dest } = t)) (synt-step-image s)

definition edges-of-nodes :: synt-config list \Rightarrow synt-config edge list where
  edges-of-nodes nds = concat (map edge-of-synt-config nds)

we can define the translation function from statements to automata:

...
fun stmt-to-ta :: stmt ⇒ synt-config ta where
stmt-to-ta c =
(let nds = nodes-of-stmt-locations (all-locations c PTop) in
  (nodes = nds, edges = edges-of-nodes nds, init-s = ((Loc c PTop), True) ))

6 Simulation Property

We recall that the nodes of the automaton generated by stmt-to-ta are labeled by configurations (location, Boolean flag) of the syntax tree. The simulation lemma (Lemma 1) holds for automata with appropriate closure properties: a successor configuration wrt. a transition of the semantics is also a label of the automaton (nodes-closed), and analogously for edges (edges-closed) or both nodes and edges (synt-step-image-closed).

The simulation statement is a typical commuting-diagram property: a step of the program semantics can be simulated by a step of the automaton semantics, for corresponding program and automata states. For this correspondence, we use the notation \( \approx \), even though it is just plain syntactic equality in our case.

Lemma 1 (Simulation property).
Assume that synt-step-image-closed aut and \( (((lc, b), s) \approx ((\text{lca}, \text{ba}), \text{sa})) \). If \( ((lc, b), s) \rightarrow ((lc', b'), s') \), then there exist \( \text{lca}' \), \( \text{ba}' \), \( \text{sa}' \) such that \( \text{lca}' \), \( \text{ba}' \) \( \in \) set (nodes aut) and the automaton performs the same transition: aut \( \vdash ((\text{lca}, \text{ba}), \text{sa}) \rightarrow ((\text{lca}', \text{ba}'), \text{sa}') \) and \( ((lc', b'), s') \approx ((\text{lca}', \text{ba}'), \text{sa}') \).

The proof is a simple induction over the transition relation of the program semantics and is almost fully automatic in the Isabelle proof assistant.

We now want to get rid of the precondition synt-step-image-closed aut in Lemma 1. The first subcase (edge closure), is easy to prove. Node closure is more difficult and requires the following key lemma:

Lemma 2.
If \( lc \in \text{set (all-locations c PTop)} \) then \( \text{set (map fst (synt-step-image (lc, b)))} \subseteq \text{set (all-locations c PTop)} \).

With this, we obtain the desired

Lemma 3 (Closure of automaton). synt-step-image-closed (stmt-to-ta c)

For the proofs, see [5].

Let us combine the previous results and write them more succinctly, by using the notation \( \rightarrow^* \) for the reflexive-transitive closure for the transition relations of the small-step semantics and the automaton. Whenever a state is reachable by executing a program \( c \) in its initial configuration, then a corresponding \( (\approx) \) state is reachable by running the automaton generated with function stmt-to-ta:

Theorem 1.
If \( ((\text{Loc c PTop}, \text{True}), s) \rightarrow^* (\text{cf}', s') \) then \( \exists \text{cf}a' \text{ sa}'. \text{stmt-to-ta} c \vdash (\text{init-s (stmt-to-ta c)}, s) \rightarrow^* (\text{cf}a', \text{sa}') \wedge (\text{cf}', s') \approx (\text{cf}a', \text{sa}') \).
Obviously, the initial configuration of the semantics and the automaton are in the simulation relation $\equiv$, and for the inductive step, we use Lemma 1.

7 Removal of silent transitions

Our technique for converting the operational semantics of a program to a finite automaton generally results in automata containing a large number of silent transitions. Although harmless, such transitions are only a technical device resulting from the structured nature of operational semantics: thus, they lack any usefulness in the context of an automaton.

Rather than producing immediately an automaton free of silent transitions, it is possible (and also quite convenient) to remove them as a final operation. This is obtained by means of a $\tau$-closure algorithm, where $\tau$ is the label for silent transitions generally used in the literature (in our case, $\tau = \text{NoAct}$).

$\tau$-closure amounts to computing, for each node in the automaton, the set of those nodes which can be reached from it by taking any finite number of silent transitions. The following $\text{tauclose-step}$ computes the set of the nodes of an automaton $M$ that can be reached from a node $s$ after taking one silent transition. The argument $x$ is used as an accumulator when iterating the operation several times, and should be $\emptyset$ initially:

\[
\text{definition } \text{tauclose-step} :: \ 'n \text{ ta} \Rightarrow \ 'n \Rightarrow \ 'n \text{ set} \Rightarrow \ 'n \text{ set} \text{ where} \\
\text{tauclose-step } M \ s \ x = \{s\} \cup x \cup \{ n \in \text{set (nodes } M) . \exists e \in \text{set (edges } M) \text{. source } e \in x \land \text{action } e = \text{NoAct} \land \text{dest } e = n \}\]

The proof that $\text{tauclose-step}$ is monotonically increasing ($\text{tauclose-step } M \ s \ x \subseteq \text{tauclose-step } M \ s \ y$ for all $x, y$ such that $x \subseteq y$) is trivial.

\[
\text{lemma } \text{mono-tauclose-step} : \ \text{mono (tauclose-step } M \ s) 
\]

Then, the operation $\text{tauclose}$ is defined as the least fixpoint of the monotonic operator:

\[
\text{definition } \text{tauclose} :: \ 'n \text{ ta} \Rightarrow \ 'n \Rightarrow \ 'n \text{ set} \text{ where} \\
\text{tauclose } M \ n = \text{lfp (tauclose-step } M \ n) 
\]

To obtain a $\tau$-closed automaton, we simply map the nodes of the input automaton to their $\tau$-closed counterpart (and similarly for the initial node). To compute the set of edges, we consider the rationale behind the definition of the $\tau$-closure of an automaton. Informally, being in a certain node or in any other node reachable from it only by means of silent transitions, is equivalent. When we compute the $\tau$-closure of a certain node, we are essentially identifying all the nodes in it: thus the edges with source $\text{tauclose } M \ s1$ should be those that leave any of the nodes in the $\tau$-closure. To make things more formal, let us introduce the notation $x \xrightarrow{\alpha} y$ for edges going from node $x$ to node $y$ labeled with action $\alpha$: using this notation, the edges of the $\tau$-closed automaton are taken to be those in the form $\text{tauclose } M \ s1 \xrightarrow{\alpha} \text{tauclose } M \ s2$, such that for some $s \in \text{tauclose } M \ s1$, $s \xrightarrow{\alpha} s2$ is a non-silent transition in the input automaton.
The automaton obtained by $\tau$-closure (see example in Figure 8) has no silent edges any more: when a silent transition is taken in the input automaton, the corresponding operation in its $\tau$-closure is to stay in the same node; when a non-silent transition $s \xrightarrow{\alpha} s'$ is taken in the input automaton, a transition with the same label and target is taken in its $\tau$-closure: however the source of this transition does not have to be $\text{tauclose } s$, but can be the $\tau$-closure of any node from which $s$ can be reached by taking silent transitions.

This correspondence between an automaton and its $\tau$-closure, is expressed by the following simulation:

definition tau-sim :: 'n1 ta ⇒ 'n2 ta ⇒ bool where
tau-sim M1 M2 =
(∃R. R (init-s M1) (init-s M2) ∧}
∀ s1 s2. R s1 s2 →
(∀ s1′ a. (source = s1, action = a, dest = s1′) ∈ set (edges M1) →
(a = NoAct ∧ R s1′ s2) ∨
(∃ s2′.(source = s2, action = a, dest = s2′) ∈ set (edges M2) ∧ R s1′ s2′)))

In our case, we shall instantiate the type parameter 'n2 with 'n1 set and take the relation R to be such that R s s′ ⟺ (s ∈ set (nodes M) ∧ s′ ∈ set (nodes (tauclose-ta M)) ∧ s ∈ s′).

We are able to prove the simulation for all well formed automata. An automaton is well formed (regular-ta) when its initial nodes and the sources and targets of all its edges are in the set of its nodes.

definition regular-ta :: 'n ta ⇒ bool where
  regular-ta M =
  (init-s M ∈ set (nodes M) ∧
  (∀ e ∈ set (edges M). source e ∈ set (nodes M) ∧ dest e ∈ set (nodes M)))

Theorem 2 (simulation of τ-closure).
If regular-ta M then tau-sim M (tauclose-ta M).

The proof follows from the definitions, proceeding by cases on the possible actions.

As a final remark, it is worth noting that the definition of taulose is not entirely satisfying, given that there exists no general method to compute a fixpoint in a finite amount of time. In our case, however, the fixpoint can be computed by iterating the taulose-step function, since it is monotonically increasing with a finite upper bound, namely the set of nodes of the input automaton. Thus, we can define the following “computational” version of the τ-closure operation:

function taulose-comp-aux :: 'n ta ⇒ 'n ⇒ 'n set ⇒ 'n set ⇒ 'n set
  taulose-step M s x = x =⇒
  taulose-comp-aux M s x = x
| taulose-step M s x = x =⇒
  taulose-comp-aux M s x = taulose-comp-aux M s (taulose-step M s x)
by (atomize-elim, auto)

(* termination proof omitted *)

termination proof
(relation measure (λ(M,s,x).length (filter (λv.(v /∈ x)) (s # nodes M))),
  simp, unfold measure-def)
fix M s x
assume hneq:taulose-step M s x /∈ x
from hneq mono-taulose-step have ∃ c.(c ∈ taulose-step M s x ∧ c /∈ x)
  by (unfold mono-def taulose-step-def,auto)
from this obtain c where hcin:c ∈ taulose-step M s x and hcnout:c /∈ x by blast
have hmagic:
  length [v← s # nodes M . v /∈ taulose-step M s x]
  < length [v← s # nodes M . v /∈ x] =⇒
  ((M, s, taulose-step M s x), M, s, x)
\[ \in \text{inv-image less-than} \ (\lambda(M, s, x). \ length [v \leftarrow s \ # \ nodes M . v \notin x]) \]
by (simp)
from \text{hneq} have \( x \subseteq \tauclose-step M s x \)
by (unfold \tauclose-step-def,auto)
moreover from \text{hcin \ hnotin} have \( c \in \text{set} (s \ # \ nodes M) \)
by (unfold \tauclose-step-def,auto)
moreover note \text{hcin \ hnotin}
ultimately have
\[ \text{length} [v \leftarrow s \ # \ nodes M . v \notin x] \]
by (rule-tac filter-subset,auto)
from this \text{hmagic} show
\((M, s, \tauclose-step M s x), M, s, x) \in \text{inv-image less-than} \ (\lambda(M, s, x). \ length [v \leftarrow s \ # \ nodes M . v \notin x]) \]
by (rule_tac substituteठ)
qed

\textbf{definition} \( \text{tauclose-comp} :: \ 'n \Rightarrow \ 'n \Rightarrow \ 'n \Rightarrow \ 'n \) \text{set where}
\( \text{tauclose-comp} M s = \text{tauclose-comp-aux} M s \) \{\}

The function \( \text{tauclose-comp-aux} \) cannot be proved to be total automatically; we provide such a proof based on the finite upper bound argument we have just mentioned. As expected, we can show that \( \text{tauclose} \) and \( \text{tauclose-comp} \) compute the same function.

\textbf{lemma} \( \text{tauclose-comp-aux-sound} : \)
\textbf{assumes} \( x \subseteq \text{tauclose} M s \)
\textbf{shows} \( \text{tauclose-comp-aux} M s x = \text{tauclose} M s \)
\textbf{using} \( \text{assms} \)
\textbf{proof} \((\text{induct} M s x \ \text{rule:} \text{tauclose-comp-aux-induct}, \text{unfold} \text{tauclose-def},\text{simp})\)
fix \( Ma \ sa \ xa \)
assume \( \text{tauclose-step} Ma sa xa = xa xa \subseteq \text{lfp} (\text{tauclose-step} Ma sa) \)
from \text{this show} \( xa = \text{lfp} (\text{tauclose-step} Ma sa) \) by (\text{unfold} \text{lfp-def},auto)
next
fix \( Ma \ sa \ xa \)
assume \( \text{tauclose-step} Ma sa xa \neq xa \)
and \( \text{th:} \text{tauclose-step} Ma sa xa \subseteq \text{lfp} (\text{tauclose-step} Ma sa) \implies \)
\( \text{tauclose-comp-aux} Ma sa (\text{tauclose-step} Ma sa xa) = \text{lfp} (\text{tauclose-step} Ma sa) \)
and \( xa \subseteq \text{lfp} (\text{tauclose-step} Ma sa) \)
from \text{this show} \( \text{tauclose-comp-aux} Ma sa xa = \text{lfp} (\text{tauclose-step} Ma sa) \)
\textbf{proof} \((\text{simp, \text{rule-tac \ th, \text{simp}}})\)
assume \( xa \subseteq \text{lfp} (\text{tauclose-step} Ma sa) \)
from \text{this show} \( \text{tauclose-step} Ma sa xa \subseteq \text{lfp} (\text{tauclose-step} Ma sa) \)
by (\text{subst \text{lfp-unfold}, \text{unfold} \text{tauclose-step-def}, \text{auto} \ simp \ add: \text{mono-tauclose-step}})
qed
qed

\textbf{lemma} \( \text{tauclose-comp-sound} : \)
\textbf{shows} \( \text{tauclose-comp} M s = \text{tauclose} M s \)
by (\text{unfold} \text{tauclose-comp-def}, \text{auto} \ simp \ add: \text{tauclose-comp-aux-sound})

Theorem 3.
\[\text{tauclose-comp } M \ s = \text{tauclose } M \ s\]

The proof is by functional induction on \(\text{tauclose-comp-aux}\).

8 Conclusions

This paper has presented a new kind of small-step semantics for imperative programming languages, based on the zipper data structure. Our primary aim is to show that this semantics has decisive advantages for abstracting programming language semantics to automata. Even if the generated automata have a great number of silent transitions, these can be removed.

The playground of our formalizations is proof assistants, in which SOS has become a well-established technique for presenting semantics of programming languages. In principle, our technique could be adapted to other formalization tools like rewriting-based ones [15].

We are currently in the process of adopting this semantics in a larger formalization from Java to Timed Automata [6]. As most constructs (zipper data structure, mapping to automata) are generic, we think that this kind of semantics could prove useful for similar formalizations with other source languages. The proofs (here carried out with the Isabelle proof assistant) have a pleasingly high degree of automation that are in sharp contrast with the index calculations that are usually required when naming automata states with numbers.

Renaming nodes from source tree locations to numbers is nevertheless easy to carry out, see the code snippet provided on the web page [5] of this paper. For these reasons, we think that the underlying ideas could also be useful in the context of compiler verification, when converting a structured source program to a flow graph with basic blocks, but before committing to numeric values of jump targets.

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