Leray-Schauder degree for the resonant $Q$-curvature problem in even dimensions

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Abstract

In this paper, using the theory of critical points at infinity of Bahri\cite{5}, we derive an exact bubbling rate formula for the resonant prescribed $Q$-curvature equation on closed even-dimensional Riemannian manifolds. Using this, we derive new existence results for the resonant prescribed $Q$-curvature problem under a positive mass type assumption. Moreover, we derive a compactness theorem for conformal metrics with prescribed $Q$-curvature under a non-degeneracy assumption. Furthermore, combining the bubbling rate formula with the construction of some blowing-up solutions, we compute the Leray-Schauder degree of the resonant prescribed $Q$-curvature equation under a non-degeneracy and Morse type assumption.

Key Words: GJMS operator, $Q$-curvature, Blow-up analysis, Critical points at infinity, Pseudo-gradient, Topological degree.

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1 Introduction and statement of the results

One of the most recurrent question in conformal geometry is the problem of finding conformal metrics for which a certain curvature quantity is equal to a prescribed function, e.g. constant. As a model of such problems, we have the problem of existence of conformal metrics with prescribed Gauss curvature on closed Riemannian surfaces, namely the Nirenberg problem and the more general Kazdan-Warner problem.

There exists also analogues of the Gauss curvature in high even dimensions which enjoy similar properties which are relevant to conformal geometry. To better introduce those curvatures, we recall some facts about the theory of closed Riemannian surfaces. It is a well known fact that the Laplace-Beltrami operator on closed Riemannian surfaces $(\Sigma, g)$ is conformally covariant of bidegree $(0, 2)$, and governs the transformation laws of the Gauss curvature under conformal changes of the background metric $g$. In fact, under the conformal change of metric $g_u = e^{2u}g$, we have

\begin{equation}
\Delta_{g_u} = e^{-2u} \Delta_g; \quad -\Delta_g u + K_g = K_{g_u} e^{2u},
\end{equation}

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where $\Delta_g$ and $K_g$ (resp. $\Delta_{g_u}$ and $K_{g_u}$) are the Laplace-Beltrami operator and the Gauss curvature of $(\Sigma, g)$ (resp. of $(\Sigma, g_u)$). Moreover, we have the Gauss-Bonnet formula which relates $\int_{\Sigma} K_g dV_g$ and the topology of $\Sigma$

$$\int_{\Sigma} K_g dV_g = 2\pi \chi(\Sigma),$$

where $\chi(\Sigma)$ is the Euler characteristic of $\Sigma$ and $dV_g$ is the volume form on $\Sigma$ with respect to $g$. From this we have that $\int_{\Sigma} K_g dV_g$ is a topological invariant, hence a conformal invariant too. We point out that the conformal invariance of $\int_{\Sigma} K_g dV_g$ can also be seen just by integrating equation (1) which is of divergence structure.

In 1983, Paneitz\cite{Pan83} has discovered a conformally covariant differential operator $P_g$ on four-dimensional closed Riemannian manifolds $(M, g)$ (known now as Paneitz operator). To the Paneitz operator, Branson\cite{Br86} has associated a natural curvature invariant $Q_g$ called $Q$-curvature. They are defined in terms of the Ricci tensor $Ric_g$ and the scalar curvature $R_g$ of the Riemannian manifold $(M, g)$ as follows

$$P_g = \Delta_g^2 + div_g \left( \frac{2}{3} R_g g - 2 Ric_g \nabla_g \right), \quad Q_g = -\frac{1}{12} (\Delta_g R_g - R_g^2 + 3 |Ric_g|^2),$$

where $div_g$ is the negative divergence and $\nabla_g$ is the gradient with respect to $g$. As the Laplace-Beltrami operator is conformally covariant of bidegree $(0, 2)$, and governs the transformation laws of the Gauss curvature under conformal changes, we have also that the Paneitz operator is conformally covariant of bidegree $(0, 4)$, and governs the transformation laws of the $Q$-curvature under conformal changes of the background metric. Indeed under a conformal change of metric $g_u = e^{2u} g$, we have

$$P_{g_u} = e^{-4u} P_g, \quad P_{g_u} u + 2 Q_g = 2 Q_{g_u} e^{4u}.$$  

Apart from this analogy, we have also an extension of the Gauss-Bonnet identity which is the Chern-Gauss-Bonnet formula

$$\int_M (Q_g + \frac{|W_g|^2}{8}) dV_g = 4\pi^2 \chi(M),$$

where $W_g$ denotes the Weyl tensor of $(M, g)$ and $\chi(M)$ is the Euler characteristic of $M$. Hence from the pointwise conformal invariance of $|W_g|^2 dV_g$, it follows that $\kappa_P := \int_M Q_g dV_g$ is also conformally invariant. Moreover, the conformal invariance of $\int_M Q_g dV_g$ can also be seen by just integrating the second equation of (3) which is also of divergence structure, like in the case of Gauss curvature for closed Riemannian surfaces.

On the other hand, there are high order analogues of the Laplace-Beltrami operator and the Paneitz operator for high even-dimensional closed Riemannian manifolds and also to the associated curvature invariants. More precisely, given a closed $n$-dimensional Riemannian manifold $(M, g)$ with $n \geq 2$ and even, in 31, it was introduced a family of conformally covariant differential operators $P^n_g$ whose leading term is $(-\Delta_g)^n$. Moreover, in 12, some curvature invariants $Q^n_g$ was defined, naturally associated to $P^n_g$. In low dimensions, we have the following relations

$$P^n_g = -\Delta_g, \quad Q^n_g = K_g, \quad P^n_g = P_g, \quad \text{and} \quad Q^n_g = 2 Q_g.$$  

It turns out that $P^n_g$ is self-adjoint and annihilates constants. Furthermore, as for the Laplace-Beltrami operator on closed Riemannian surfaces and the Paneitz operator on closed four-dimensional Riemannian manifolds, for every closed $n$-dimensional Riemannian manifold $(M, g)$ with $n \geq 2$ and even, we have that after a conformal change of metric $g_u = e^{2u} g$

$$P^n_{g_u} = e^{-nu} P^n_g, \quad P^n_{g_u} u + Q^n_{g_u} = Q^n_{g_u} e^{nu}.$$  

Thus, as in the 2-dimensional and 4-dimensional cases, by integrating the second equation of (4) and using the fact that $P^n_g$ is self-adjoint and annihilates constants, it follows that $\kappa^n_P := \int_M Q^n_g dV_g$ is also conformally invariant. Furthermore, there exists also an analogue of the Chern-Gauss-Bonnet formula, which is a consequence of a formula of Alexakis\cite{A} (see also 2), and reads as follows

$$\int_M (Q^n_g + |\tilde{W}_g|) dV_g = \frac{(n-1)!}{2} \omega_n \chi(M),$$
As for the Kazdan-Warner problem for closed Riemannian surfaces, given a closed $\omega$ with mass $(\omega)$, a smooth solution of (7) can be found by looking at critical points of the following geometric functional:

$$J(u) := (P_g^n u, u) + 2 \int_M Q_g^n u dV_g - \frac{2}{n} \kappa_g^n \log \left( \int_M Ke^{nu} dV_g \right), \quad u \in W^{2,2}(M),$$

where $W^{2,2}(M)$ is the space of functions on $M$ which are of class $W^{2,2}$ in each coordinate system.

The asymptotic behaviour of sequences of solutions

$$P_g^n u_l + t_l Q_g^n = t_l \kappa_g^n Ke^{nu_l} \quad \text{in} \quad M,$$

with $\ker P_g^n \simeq \mathbb{R}$, $K : M \to \mathbb{R}$ smooth and positive, and $t_l \to 1$ as $l \to +\infty$ plays an important role in the Variational Analysis of $J$ in the resonant case, i.e. when $\kappa_g^n \in (n-1)!\omega_n\mathbb{N}$. In this paper, we are interested in the exact bubbling rate formula for (8) in the resonant case and its applications to existence, compactness and topological degree-computation.

In order to state our results clearly, we first fix some notation and make some definitions. For $m \in \mathbb{N}^*$ such that

$$\kappa_g^n = (n-1)!m\omega_n,$$

we define $\mathcal{F}_K : M^m \setminus F_m(M) \to \mathbb{R}$ as follows

$$\mathcal{F}_K(a_1, \ldots, a_m) := \sum_{i=1}^m \left( H(a_i, a_i) + \sum_{j \neq i} G(a_i, a_j) + \frac{2}{n} \log(K(a_i)) \right),$$

where $F(M^m)$ denotes the fat Diagonal of $M^m$, namely $F(M^m) := \{ A := (a_1, \ldots, a_m) \in M^m : \text{there exists } i \neq j \text{ with } a_i = a_j \}$, $G$ is the Green’s function of $P_g^n(\cdot) + \frac{1}{m} Q_g^n$ with mass $(n-1)!\omega_n$ satisfying the normalization $\int_M Q_g^n(x) G(\cdot, x) dV_g(x) = 0$, and $H$ is its regular part, see Section 2 for more information. Furthermore, we define

$$\text{Crit}(\mathcal{F}_K) := \{ A \in M^m \setminus F_m(M) : A \text{ critical point of } \mathcal{F}_K \}.$$  

Moreover, for $A = (a_1, \ldots, a_m) \in M^m \setminus F_m(M)$, we set

$$\mathcal{F}_t^A(x) := e^{\kappa(H(a_i, x) + \sum_{j \neq i} G(a_i, a_j)) + \frac{1}{n} \log(K(a_i))},$$

and define

$$\mathcal{L}_K(A) := -\sum_{i=1}^m (\mathcal{F}_t^A)^{\frac{m-1}{m}}(a_i) L_g((\mathcal{F}_t^A)^{\frac{m-1}{m}})(a_i),$$

where

$$L_g := -\Delta_g + \frac{(n-2)}{4(n-1)} R_g.$$
is the conformal Laplacian associated to $g$. We set also

$$(13) \quad F_\infty := \{ A \in \text{Crit}(F_K) : \mathcal{L}_K(A) < 0 \},$$

and

$$(14) \quad i_\infty(A) := (n+1)m - 1 - \text{Morse}(A, F_K),$$

where $\text{Morse}(F_K, A)$ denotes the Morse index of $F_K$ at $A$. Finally, we say

$$(15) \quad (ND)_0 \text{ holds if for every } A \in \text{Crit}(F_K), \mathcal{L}_K(A) \neq 0,$$

$$(16) \quad (ND)_- \text{ holds if for every } A \in \text{Crit}(F_K), \mathcal{L}_K(A) < 0,$$

$$(17) \quad (ND)_+ \text{ holds if for every } A \in \text{Crit}(F_K), \mathcal{L}_K(A) > 0,$$

and

$$(18) \quad (ND) \text{ holds if } (ND)_0 \text{ holds and } F_K \text{ is a Morse function.}$$

Now, we are ready to state our results and we start with the exact bubbling rate formula for sequences of blowing up solutions to $\mathbb{S}$ under the assumption $\ker P_g \simeq \mathbb{R}$.

**Theorem 1.1.** Let $(M, g)$ be a closed $n$-dimensional Riemannian manifold with $n \geq 4$ even such that $\ker P_g \simeq \mathbb{R}$ and $\kappa^m_g = (n-1)! m \omega_n$ with $m \in \mathbb{N}^*$. Assuming that $K$ is a smooth positive function on $M$ and $u_l$ is a sequence of blowing up solutions to $\mathbb{S}$ with $t_l \to 1$ as $l \to +\infty$, then up to a subsequence, we have that for $l$ large enough, there holds

$$t_l - 1 = \frac{\epsilon_{n,m}^K(A) e^{-2 \max_{i=1}^l u_i}}{(F_1^A(a_i))^{\frac{2}{m}}} [\mathcal{L}_K(A) + o_l(1)],$$

with $A = (a_1, \ldots, a_m) \in \text{Crit}(F_K)$ and $\epsilon_{n,m}^K(A)$ is a positive constant depending only on $K, A, n$ and $m$.

Theorem 1.1 and our work in the nonresonant case (i.e $\kappa^m_g \notin (n-1)! \omega \mathbb{N}^*$) imply the following existence result for conformal metrics with prescribed $Q$-curvature.

**Corollary 1.2.** Let $(M, g)$ be a closed $n$-dimensional Riemannian manifold with $n \geq 4$ even such that $\ker P_g \simeq \mathbb{R}$ and $\kappa^m_g = (n-1)! m \omega_n$ with $m \in \mathbb{N}^*$. Assuming that $K$ is a smooth positive function on $M$ such that $(ND)_-$ or $(ND)_+$ holds, then $K$ is the $Q$-curvature of a Riemannian metric conformally related to $g$.

In the critical case, i.e $\kappa^m_g = (n-1)! \omega_n$, Theorem 1.1 implies the following existence of minimizer of the functional $J$.

**Corollary 1.3.** Let $(M, g)$ be a closed $n$-dimensional Riemannian manifold with $n \geq 4$ even such that $\ker P_g \simeq \mathbb{R}$, $P_g^m \geq 0$ and $\kappa^m_g = (n-1)! \omega_n$. Assuming that $K$ is a smooth positive function on $M$ such that $(ND)_+$ holds, then $K$ is the $Q$-curvature of a Riemannian metric $g_u = e^{2u} g$ with $u$ a minimizer of $J$ on $H^2(M)$.

Theorem 1.1 implies also the following compactness theorem for conformal metrics with prescribed $Q$-curvature.
Corollary 1.4. Let \((M, g)\) be a closed \(n\)-dimensional Riemannian manifold with \(n \geq 4\) even such that \(\ker P^n_g \simeq \mathbb{R}\) and \(\kappa^n_g = (n-1)!m\omega_n\) with \(m \in \mathbb{N}^*\). Assuming that \(K\) is a smooth positive function on \(M\) such that \((ND)_{0}\) holds, then for every \(k \in \mathbb{N}\), there exists a large positive constant \(C_k\) such that for every \(u\) solution of \((7)\),
\[
\|u\|_{C^k(M)} \leq C_k.
\]
Corollary 1.4 implies that the Leray-Schauder degree of equation \((7)\) is well-defined. Indeed, we have

Theorem 1.5. Let \((M, g)\) be a closed \(n\)-dimensional Riemannian manifold with \(n \geq 4\) even such that \(\ker P^n_g \simeq \mathbb{R}\) and \(\kappa^n_g = (n-1)!m\omega_n\) with \(m \in \mathbb{N}^*\). Assuming that \(K\) is a smooth positive function on \(M\) such that \((ND)_{0}\) holds, then the Leray-Schauder degree \(d_m\) of equation \((7)\) is well-defined. Furthermore, if \((ND)\) holds, then there exists \(L_0 > 0\) such that
\[
d_m = \begin{cases} 
(-1)^m \left(1 - \sum_{A \in \mathcal{F}_\infty} (-1)^{i(A)}(1)\right) = \chi(L^*, J - L) - \sum_{A \in \mathcal{F}_\infty} (-1)^{i(A)} & \text{if } m = 1, \\
(-1)^m \left(\frac{1}{m!} \prod_{i=1}^{n-1} (1 - \chi(M)) - \frac{1}{m!} \sum_{A \in \mathcal{F}_\infty} (-1)^{i(A)}\right) & \text{if } m \geq 2.
\end{cases}
\]
for all \(L \geq L_0\).

2 Notations and Preliminaries

In this brief section, we fix our notations and give some preliminaries. First of all, we recall that \((M, g)\) and \(K\) are respectively the given underlying closed \(n\)-dimensional Riemannian manifold with \(n \geq 4\) even, and the prescribed function with the following properties (until otherwise said)
\[
\ker P^n_g = \mathbb{R}, \quad \kappa^n_g = (n-1)!m\omega_n \quad \text{for some } m \in \mathbb{N}^* \quad \text{and } K \text{ is a smooth positive function on } M.
\]
Furthermore, \(u_l\) and \(t_l\) are respectively sequence of smooth functions and real numbers, satisfying
\[
P^n_g u_l + t_l Q^n_g = t_l \kappa^n_g K e^{nu_l} \quad \text{in } M, \quad \text{and } t_l \to 1 \quad \text{as } l \to +\infty.
\]

We are going to discuss the asymptotics near the singularity of the Green’s function \(G\) of the operator \(P^n_g(\cdot) + \frac{1}{m!} Q^n_g\) with mass \((n-1)!\omega_n\) satisfying the normalization \(\int_M G(\cdot, y) Q^n_g(y) dV_g(y) = 0\) and make some related definitions.

In the following, for a Riemannian metric \(\tilde{g}\) on \(M\), we will use the notation \(B^\tilde{g}(r)\) to denote the geodesic ball with respect to \(\tilde{g}\) of radius \(r\) and center \(p\). We also denote by \(d_{\tilde{g}}(x, y)\) the geodesic distance with respect to \(\tilde{g}\) between two points \(x\) and \(y\) of \(M\), \(\exp_{\tilde{g}}\) the exponential map with respect to \(\tilde{g}\) at \(x\). \(\text{inj}_{\tilde{g}}(M)\) stands for the injectivity radius of \((M, \tilde{g})\), \(dV_{\tilde{g}}\) denotes the Riemannian measure associated to the metric \(\tilde{g}\). Furthermore, we recall that \(\nabla_{\tilde{g}}, \Delta_{\tilde{g}}, R_{\tilde{g}},\) and \(\text{Ric}_{\tilde{g}}\) will denote respectively the gradient, the Laplace-Beltrami operator, the scalar curvature and Ricci curvature with respect to \(\tilde{g}\). For simplicity, we will use the notation \(B_p(r)\) to denote \(B^\tilde{g}(r)\), namely \(B_p(r) = B^\tilde{g}(r)\). \(M^2\) stands for the cartesian product \(M \times M\), while \(\text{Diag}(M)\) is the diagonal of \(M^2\).

For \(1 \leq p \leq \infty\) and \(k \in \mathbb{N}, \theta \in [0, 1]\), \(L^p(M), W^{k,p}(M), C^k(M),\) and \(C^{k,\theta}(M)\) stand respectively for the standard Lebesgue space, Sobolev space, \(k\)-continuously differentiable space and \(k\)-continuously differential space of Hölder exponent \(\beta\), all with respect \(g\) (if the definition needs a metric structure, and for precise definitions and properties, see [4] or [30]). Given a function \(u \in L^1(M)\), \(\bar{u}\) and \(\overline{\pi}_{Q^n}\) denote respectively its average on \(M\) with respect to \(g\) and the sign measure \(Q^n_g dV_g\), that is
\[
\bar{u} = \frac{\int_M u(x) dV_g(x)}{Vol_g(M)},
\]
with \(Vol_g(M) = \int_M dV_g\) and
\[
\overline{\pi}_{Q^n} = \frac{1}{(n-1)!\omega_n m} \int_M Q^n_g(x) u(x) dV_g(x).
\]
For $\varepsilon > 0$ and small, $\lambda \in \mathbb{R}_+$, $\lambda \geq \frac{1}{r}$, and $a \in M$, $O_{\lambda,\varepsilon}(1)$ stands for quantities bounded uniformly in $\lambda$, and $\varepsilon$, and $O_{a,\varepsilon}(1)$ stands for quantities bounded uniformly in $a$ and $\varepsilon$. For $l \in \mathbb{N}^+$, $O_l(1)$ stands for quantities bounded uniformly in $l$ and $o_l(1)$ stands for quantities which tends to $0$ as $l \to +\infty$. For $\varepsilon$ positive and small, $a \in M$ and $\lambda \in \mathbb{R}_+$ large, $\lambda \geq \frac{1}{r}$, $O_{a,\lambda,\varepsilon}(1)$ stands for quantities bounded uniformly in $a$, $\lambda$, and $\varepsilon$. For $\varepsilon$ positive and small, $p \in \mathbb{N}^+$, $\lambda := (\lambda_1, \cdots, \lambda_p) \in (\mathbb{R}_+)^p$, $\lambda_i \geq \frac{1}{r}$ for $i = 1, \cdots, p$, and $A := (a_1, \cdots, a_p) \in M^p$ (where $(\mathbb{R}_+)^p$ and $M^p$ denotes respectively the cartesian product of $p$ copies of $\mathbb{R}_+$ and $M$), $O_{A,\lambda,\varepsilon}(1)$ stands for quantities bounded uniformly in $A$, $\lambda$, and $\varepsilon$. Similarly for $\varepsilon$ positive and small, $p \in \mathbb{N}^+$, $\lambda := (\lambda_1, \cdots, \lambda_p) \in (\mathbb{R}_+)^p$, $\lambda_i \geq \frac{1}{r}$ for $i = 1, \cdots, p$, and $A := (a_1, \cdots, a_p) \in M^p$ (where $\mathbb{R}^p$ denotes the cartesian product of $p$ copies of $\mathbb{R}$, $O_{A,\lambda,\varepsilon}(1)$ will mean quantities bounded from above and below independent of $\bar{\alpha}, A, \lambda$, and $\varepsilon$. For $x \in \mathbb{R}$, we will use the notation $O(x)$ to mean $|x|O(1)$ where $O(1)$ will be specified in all the contexts where it is used. Large positive constants are usually denoted by $C$ and the value of $C$ is allowed to vary from formula to formula and also within the same line. Similarly small positive constants are also denoted by $c$ and their value may varies from formula to formula and also within the same line.

We call $\bar{m}$ the number of negative eigenvalues (counted with multiplicity) of $P_{\bar{g}}^n$. We point out that $\bar{m}$ can be zero, but it is always finite. If $\bar{m} \geq 1$, then we will denote by $E_- \subset W_{\frac{Q}{2}}^{1,2}(M)$ the direct sum of the eigenspaces corresponding to the negative eigenvalues of $P_{\bar{g}}^n$. The dimension of $E_-$ is of course $\bar{m}$. On the other hand, we have the existence of an $L^2$-orthonormal basis of eigenfunctions $v_1, \cdots, v_{\bar{m}}$ of $E_-$ satisfying
\begin{equation}
P_{\bar{g}}^n v_i = \mu_r v_r \quad \forall \ r = 1 \cdots \bar{m},
\end{equation}
\begin{equation}
\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{\bar{m}} < 0 \leq \mu_{\bar{m}+1} \leq \cdots
\end{equation}
where $\mu_r$’s are the eigenvalues of $P_{\bar{g}}^n$ counted with multiplicity. We define also the pseudo-differential operator $P_{\bar{g}}^{n,+}$ as follows
\begin{equation}
P_{\bar{g}}^{n,+} u = P_{\bar{g}}^n u - 2 \sum_{r=1}^{\bar{m}} \mu_r (\int_M u v_r dV_{\bar{g}}) v_r.
\end{equation}

Basically $P_{\bar{g}}^{n,+}$ is obtained from $P_{\bar{g}}^n$ by reversing the sign of the negative eigenvalues and we extend the latter definition to $\bar{m} = 0$ for uniformity in the analysis and recall that in that case $P_{\bar{g}}^{n,+} = P_{\bar{g}}^n$. Now, for $t > 0$ we set
\begin{equation}
J_t(u) := \langle P_{\bar{g}}^{n,+} u, u \rangle + 2t \int_M Q_{\bar{g}}^n u dV_{\bar{g}} - \frac{2(n-1)!\omega_n m}{n} \log \int_M K e^{\alpha u} dV_{\bar{g}}, \ u \in W_{\frac{Q}{2}}^{1,2}(M),
\end{equation}
and hence $J = J_1$.

We will use the notation $\langle \cdot, \cdot \rangle$ to denote the $L^2$ scalar product and $\langle \cdot, \cdot \rangle_{W_{\frac{Q}{2}}^{1,2}}$ for the $W_{\frac{Q}{2}}^{1,2}$-scalar product. On the other hand, it is easy to see that
\begin{equation}
\langle u, v \rangle_{p^n} := \langle P_{\bar{g}}^{n,+} u, v \rangle, \ u, v \in \{ w \in W_{\frac{Q}{2}}^{1,2}(M) : \pi_{Q^n} w = 0 \}
\end{equation}
defines a inner product on $\{ u \in W_{\frac{Q}{2}}^{1,2}(M) : \ u_{Q^n} = 0 \}$ which induces a norm equivalent to the standard norm $|| \cdot || := \sqrt{\langle \cdot, \cdot \rangle_{W_{\frac{Q}{2}}^{1,2}}}$ of $W_{\frac{Q}{2}}^{1,2}(M)$ (on $\{ u \in W_{\frac{Q}{2}}^{1,2}(M) : \ u_{Q^n} = 0 \}$) and denoted by
\begin{equation}
|| u ||_{p^n} := \sqrt{\langle u, u \rangle_{p^n}}, \ u \in \{ w \in W_{\frac{Q}{2}}^{1,2}(M) : \pi_{Q^n} w = 0 \}.
\end{equation}

$B_1^m$ will stand for the closed ball of center 0 and radius $r$ in $\mathbb{R}^m$. $S^{m-1}$ will denote the boundary of $B_1^m$. Given a set $X$, we define $X \times \tilde{B}_1^m$ to be the cartesian product $X \times \tilde{B}_1^m$ where the tilde means that $X \times \partial B_1^m$ is identified with $\partial B_1^m$. For $m \geq 2$, we denote by $B_{m-1}(M)$ the set of formal barycenters of $M$ of order $m - 1$, namely
\begin{equation}
B_{m-1}(M) := \{ \sum_{i=1}^{m-1} \alpha_i \delta_{a_i} : a_i \in M, \alpha_i \geq 0, i = 1, \cdots, m - 1, \sum_{i=1}^{m-1} \alpha_i = 1 \},
\end{equation}
Finally, we set

\[ A_{m-1,\bar{m}} := B_{m-1}(M) \times \bar{B}^\bar{m}_1. \]

In the sequel also, \( J^c \) with \( c \in \mathbb{R} \) will stand for \( J^c := \{ u \in W^{2,2}_\#(M) : J(u) \leq c \} \). For \( X \) a topological space, \( \chi(X) \) denotes the Euler characteristic of \( X \) with \( \mathbb{Z}_2 \) coefficients and for \( (X, Y) \) a topological pair, \( \chi(X, Y) \) denotes the Euler characteristic of \( X \) with \( \mathbb{Z}_2 \) coefficients.

As above, in the general case, namely \( \bar{m} \geq 0 \), for \( \epsilon \) small and positive, \( \bar{\beta} := (\beta_1, \ldots, \beta_{\bar{m}}) \in \mathbb{R}^{\bar{m}} \) with \( \beta_i \) close to \( 0 \), \( i = 1, \ldots, \bar{m} \) (where \( \mathbb{R}^{\bar{m}} \) is the empty set when \( \bar{m} = 0 \)), \( \bar{\lambda} := (\lambda_1, \ldots, \lambda_{\bar{m}}) \in (\mathbb{R}_+)^{\bar{m}} \), \( \lambda_i \geq \frac{1}{\epsilon} \) for \( i = 1, \ldots, \bar{m} \), \( \bar{\alpha} := (\alpha_1, \ldots, \alpha_{\bar{m}}) \in \mathbb{R}^{\bar{m}} \), \( \alpha_i \) close to \( 1 \) for \( i = 1, \ldots, \bar{m} \), and \( \lambda := (\lambda_1, \ldots, \lambda_{\bar{m}}) \in \mathbb{R}^{\bar{m}} \), \( p \in \mathbb{N}^*, \quad w \in W^{2,2}_\# \) with \( \| w \| \) small, \( O_{\bar{a}, \bar{\lambda}, \bar{\beta}, w, \epsilon}(1) \) will stand quantities bounded independent of \( \bar{\alpha}, \lambda, \bar{\lambda}, \bar{\beta}, w \) and \( \epsilon \).

As in [56] and [53], given a point \( b \in \mathbb{R}^n \) and a positive real number, we define \( \delta_{b, \lambda} \) to be the standard bubble, namely

\[ \delta_{b, \lambda}(y) := \log \left( \frac{2\lambda}{1 + \lambda^2|y - b|^2} \right), \quad y \in \mathbb{R}^n. \]

The functions \( \delta_{b, \lambda} \) verify the following equation

\[ (-\Delta_{\mathbb{R}^n})^2 \delta_{b, \lambda} = (n - 1)! e^{-n\delta_{b, \lambda}} \text{ in } \mathbb{R}^n. \]

Geometrically, equation (32) means that the metric \( g = e^{2\lambda_{b, \lambda}} dx^2 \) (after pull-back by stereographic projection) has constant \( Q \)-curvature equal to \( (n - 1)! \), where \( dx^2 \) is the standard metric on \( \mathbb{R}^n \).

Using the existence of conformal normal coordinates (see [17] or [33]), we have that, for \( a \in M \) there exists a function \( u_a \in C^\infty(M) \) such that

\[ g_a = e^{2u_a} g \text{ verifies } \text{det} g_a(x) = 1 \text{ for } x \in B^\lambda_{\epsilon}(a). \]

with \( 0 < \epsilon_0 < \epsilon_a < \frac{\text{inj}_a(M)}{10} \) for some small positive \( \epsilon_0 \) satisfying \( \epsilon_0 < \frac{\text{inj}_a(M)}{10} \).

Now, for \( 0 < \rho < \min\{\frac{\text{inj}_a(M)}{4}, \frac{\epsilon_0}{2}\} \), we define a smooth cut-off function \( \chi_\rho : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) satisfying the following properties:

\[ \begin{cases} 
\chi_\rho(t) = t & \text{for } t \in [0, \rho], \\
\chi_\rho(t) = 2\rho & \text{for } t \geq 2\rho, \\
\chi_\rho(t) \in [\rho, 2\rho] & \text{for } t \in [\rho, 2\rho].
\end{cases} \]

Using the cut-off function \( \chi_\rho \), we define for \( a \in M \) and \( \lambda \in \mathbb{R}_+ \) the function \( \hat{\delta}_{a, \lambda} \) as follows

\[ \hat{\delta}_{a, \lambda}(x) := \log \left( \frac{2\lambda}{1 + \lambda^2\chi_\rho^2(d_{g_a}(x, a))} \right). \]

For every \( a \in M \) and \( \lambda \in \mathbb{R}_+ \), we define \( \varphi_{a, \lambda} \) to be the unique solution of the following projection problem

\[ \begin{cases} 
P^n_a \varphi_{a, \lambda} + \frac{1}{m} Q^n = (n - 1)! \omega_n \frac{e^{-n\delta_{a, \lambda} + u_a}}{\int_M e^{-n\delta_{a, \lambda} + u_a} dV_g} & \text{in } M, \\
\int_M Q^n_a(x) \varphi_{a, \lambda}(x) dV_g(x) = 0.
\end{cases} \]

Now, we recall that \( G \) is the unique solution of the following PDE

\[ \begin{cases} 
P^n G(a, \cdot) + \frac{1}{m} Q^n G(\cdot) = (n - 1)! \omega_n \delta_a(\cdot), \\
\int_M Q^n G(a, x) dV_g(x) = 0.
\end{cases} \]

Using (37), it is easy to see that the following integral representation formula holds

\[ u(x) - \mathbb{P}Q^n = \frac{1}{(n - 1)!\omega_n} \int_M G(x, y) P^n u(y), \quad u \in C^n(M), \quad x \in M, \]
where \( u_{Q^n} \) is defined as in section 2. It is a well know fact that \( G \) has a logarithmic singularity. In fact \( G \) decomposes as follows

\[
(39) \quad G(a,x) = \log \left( \frac{1}{\chi^2_g(d_g(a,x))} \right) + H(a,x).
\]

where \( H \) is the regular par of \( G \). Furthermore, it is also a well-know fact that

\[
(40) \quad G \in C^\infty(M^2 \setminus Diag(M)), \quad \text{and} \quad H \in C^{3,\beta}(M^2) \ \forall \beta \in (0,1).
\]

Now, using (39) and (41) combined with the symmetry of \( H \), it is easy to see that

\[
(41) \quad \frac{\partial F_K(a_1, \ldots, a_m)}{\partial a_i} = \frac{2}{n} \frac{\nabla_g F_i^A(a_i)}{F_i^A(a_i)}, \quad i = 1, \ldots, m.
\]

Next, we set

\[
(42) \quad l_K(A) := \sum_{i=1}^m \left( \frac{\Delta_g A_i^A(a_i)}{(F_i^A(a_i))^{\frac{2}{n}}} - \frac{n}{2(n-1)} R_g(a_i)(F_i^A(a_i))^2 \right),
\]

and have

\[
(43) \quad l_K(A) = \frac{2n}{n-2} \mathcal{L}_K(A), \quad \forall A \in Crit(\mathcal{F}_K).
\]

### 3 An expansion of \( \nabla J_{t_i} \) at infinity

In this section, we present a useful expansion of \( \nabla J_{t_i} \) at infinity. In order to do that, we first fix \( A \) to a large positive constant. Next, like in [54], and [56] (see also [53]), for \( \epsilon \) and \( \eta \) small positive real numbers, we first denote by \( V(m, \epsilon, \eta) \) the \((m, \epsilon, \eta)\)-neighborhood of potential critical points at infinity, namely

\[
V(m, \epsilon, \eta) := \{ u \in W^{1,2}(M) : a_1, \ldots, a_m \in M, \lambda_1, \ldots, \lambda_m > 0, \ ||u - \pi_{Q^n} - \sum_{i=1}^m \varphi_{a_i, \lambda_i}|| + \}
\]

\[
(44) \quad ||\nabla W^{1,2} J(u)|| = O \left( \sum_{i=1}^m \frac{1}{\lambda_i} \right) \lambda_i \geq \frac{1}{\epsilon}, \quad \frac{2}{\Lambda} \leq \frac{\lambda_i}{\lambda_j} \leq \frac{\Lambda}{2}, \ \text{and} \ d_g(a_i, a_j) \geq 4C\eta \ \text{for} \ i \neq j,
\]

where \( C \) is a large positive constant, \( \nabla W^{1,2} J \) is the gradient of \( J \) with respect to the \( W^{1,2} \)-topology, \( O(1) := O_{\Lambda, \lambda, u, \epsilon} \) meaning bounded uniformly in \( \lambda := (\lambda_1, \ldots, \lambda_n) \), \( A := (a_1, \ldots, m) \), \( u, \epsilon \). Next, as in [56] and [53], following the ideas of Bahri-Coron[9], we have that for \( \eta \) a small positive real number with \( 0 < 2\eta < \theta \), there exists \( \epsilon_0 = \epsilon_0(\eta) > 0 \) such that

\[
(45) \quad \forall \ 0 < \epsilon \leq \epsilon_0, \ \forall u \in V(m, \epsilon, \eta), \ \text{the minimization problem} \ \min_{B_{\epsilon, \eta}} ||u - \pi_{Q^n} - \sum_{i=1}^m \alpha_i \varphi_{a_i, \lambda_i} - \sum_{r=1}^m \beta_r (v_r - (v_r)_{Q^n})||_{p^n}
\]

has a unique solution, up to permutations, where \( B_{\epsilon, \eta} \) is defined as follows

\[
(46) \quad B_{\epsilon, \eta} := \{ (\alpha, A, \lambda, \beta) \in \mathbb{R}^m \times M^m \times (0, +\infty)^m \times \mathbb{R}^m : |\alpha_i - 1| \leq \epsilon, \lambda_i \geq \frac{1}{\epsilon}, i = 1, \ldots, m, \ \ d_g(a_i, a_j) \geq 4C\eta, i \neq j, |\beta_r| \leq R, r = 1, \ldots, m\}. \]

Moreover, using the solution of (15), we have that every \( u \in V(m, \epsilon, \eta) \) can be written as

\[
(47) \quad u - \pi_{Q^n} = \sum_{i=1}^m \alpha_i \varphi_{a_i, \lambda_i} + \sum_{r=1}^m \beta_r (v_r - (v_r)_{Q^n}) + w,
\]

where \( w \) verifies the following orthogonality conditions

\[
(48) \quad \langle Q^n_g, w \rangle_{p^n} = \langle \varphi_{a_i, \lambda_i}, w \rangle_{p^n} = \langle \frac{\partial \varphi_{a_i, \lambda_i}}{\partial \lambda_i}, w \rangle_{p^n} = \langle \frac{\partial \varphi_{a_i, \lambda_i}}{\partial a_i}, w \rangle_{p^n} = \langle v_r, w \rangle = 0, i = 1, \ldots, m, \quad \text{if} \ \ r = 1, \ldots, m.
\]
and the estimate

$$||w|| = O \left( \sum_{i=1}^{m} \frac{1}{\lambda_i} \right),$$

where here $O(1) := O_{\alpha_A, \lambda, \beta, w, \epsilon}(1)$, and for the meaning of $O_{\alpha_A, \lambda, \beta, w, \epsilon}(1)$, see Section 2. Furthermore, the concentration points $a_i$, the masses $\alpha_i$, the concentrating parameters $\lambda_i$ and the negativity parameter $\beta_r$ in (47) verify also

$$d_g(a_i, a_j) \geq 4C \eta, \ i \neq j = 1, \cdots, m, \ \frac{1}{\Lambda} \leq \frac{\lambda_i}{\lambda_j} \leq \Lambda, \ i, j = 1, \cdots, m, \ \lambda_i \geq \frac{1}{\epsilon},$$

and

$$\sum_{r=1}^{\bar{m}} |\beta_r| + \sum_{i=1}^{m} |\alpha_i - 1| \sqrt{\log \lambda_i} = O \left( \sum_{i=1}^{m} \frac{1}{\lambda_i} \right)$$

with still $O(1)$ as in (49). Using the neighborhood of potential critical points at infinity, we have the following Lemma. For its proof see Lemma 3.1 in [51] and Proposition 3.3 in [50].

**Lemma 3.1.** Let $\epsilon$ and $\eta$ be small positive real numbers with $0 < 2\eta < \varrho$ where $\varrho$ is as in [34]. Assuming that $u_l$ is a sequence of blowing up critical point of $J_l$ with $|u_l|_{Q, \gamma} = 0, l \in \mathbb{N}$ and $t_l \to 1$ as $l \to +\infty$, then there exists $l_{\epsilon, \eta}$ a large positive integer such that for every $l \geq l_{\epsilon, \eta}$, we have $u_l \in V(m, \epsilon, \eta)$.

Now, we present some gradient estimates for $J_t$. We start with the following one.

**Lemma 3.2.** Assuming that $\eta$ is a small positive real number with $0 < 2\eta < \varrho$ where $\varrho$ is as in [34], and $\epsilon \leq \epsilon_0$ where $\epsilon_0$ is as in [35], then for $a_i \in M$ concentration points, $\alpha_i$ masses, $\lambda_i$ concentration parameters ($i = 1, \cdots, m$) and $\beta_r$ negativity parameters ($r = 1, \cdots, \bar{m}$) satisfying (50), we have that for $l$ large enough and for every $j = 1, \cdots, m$, there holds

$$\left\langle \nabla J_{l_i} \left( \sum_{i=1}^{m} \alpha_i \varphi_{a_i, \lambda_i} + \sum_{r=1}^{\bar{m}} \beta_r (\nu_r - (\nu_r)_{Q^\gamma}) \right), \lambda_j \frac{\partial \varphi_{a_j, \lambda_j}}{\partial \lambda_j} \right\rangle = 2(n-1)! \omega_n \alpha_j \tau_j$$

$$- \frac{c_2^2 (n-1)! \omega_n}{n \lambda_j^2} \left( \frac{\Delta_{g_{a_j}} F^A_j(a_j)}{F^A_j(a_j)} - \frac{n}{2(n-1)} R_g(a_j) \right) - \frac{2(n-1)! \omega_n}{(n-2) \lambda_j^2} \sum_{i=1, i \neq j}^{\bar{m}} \tau_i \Delta_{g_{a_j}} G(a_j, a_i) + \frac{c_2^2 (n-1)! \omega_n}{n \lambda_j^2} \tau_j \left( \frac{\Delta_{g_{a_j}} F^A_j(a_j)}{F^A_j(a_j)} - \frac{n}{2(n-1)} R_g(a_j) \right)$$

$$+ O \left( \sum_{i=1}^{\bar{m}} |\alpha_i - 1|^2 + \sum_{i=1}^{\bar{m}} |\beta_r|^3 + \sum_{i=1}^{m} \frac{1}{\lambda_i^3} \right),$$

where $A := (a_1, \cdots, a_m)$, $c_2^2$ is a positive real number depending only on $n$, and for $i = 1, \cdots, m$,

$$\tau_i := 1 - t_i \frac{\gamma_i}{D}, \quad D := \int_{M} K(x) e^{n \left( \sum_{i=1}^{m} \alpha_i \varphi_{a_i, \lambda_i}(x) + \sum_{r=1}^{\bar{m}} \beta_r \nu_r(x) \right)} dV_g(x),$$

with

$$\gamma_i := c_n^2 \lambda_i^2 n^{2n \alpha_i - n} F^A_1(a_i) G_i(a_i),$$

where

$$c_n^2 := \int_{\mathbb{R}^n} \frac{1}{(1 + |y|^2)^n} dy$$

$$G_i(a_i) := e^{n((\alpha_i - 1)H(a_i, a_i) + \sum_{j=1, j \neq i}^{m} \alpha_j - 1)G(a_j, a_i)) - e^{n \frac{1}{(n-2)} \sum_{j=1, j \neq i}^{m} \alpha_j \Delta_{g_{a_i}} G(a_j, a_i)} e^{\frac{n}{2(n-2)} \sum_{j=1, j \neq i}^{m} \alpha_j \Delta_{g_{a_i}} H(a_j, a_i)}$$

$$\times e^{n \sum_{r=1}^{\bar{m}} \beta_r \nu_r(a_i)},$$

c_2^2$ is a positive real number depending only on $n$ and for the meaning of $O_{\alpha_A, \lambda, \beta, w, \epsilon}(1)$, see Section 2.
Furthermore, we have

\[
\left< \nabla J_{t_i} \left( \sum_{i=1}^{m} \alpha_i \varphi_{a_i, \lambda_i} + \sum_{r=1}^{m} \beta_r (v_r - (v_r)_Q^n) \right), \varphi_{a_j, \lambda_j} \right> = \\
- \sum_{i=1}^{m} c_n^2 (n-1)! \gamma_n \left( \frac{\Delta_{g_n} F^A_i(a_i)}{F^A_i(a_i)} - \frac{n}{2(n-1)} R_g(a_i) \right) + \bar{c}_n (1 - t_i) m \\
+ O \left( \sum_{i=1}^{m} |\alpha_i - 1|^2 + \sum_{r=1}^{m} |\beta_r|^3 + \sum_{i=1}^{m} \tau_i^3 + \sum_{i=1}^{m} \frac{1}{\lambda_i^3} \right),
\]

where \( A, O(1), c_n^2, \) and \( \tau_i (i = 1, \ldots, m) \) are as above and \( \bar{c}_n \) is positive constant depending only on \( n \).

**Proof.** The proof follows the same strategy as in Lemma 5.1 and Corollary 5.2 in [50]. \( \blacksquare \)

**Remark 3.3.** We would like to remark that the \( \tau_i \)'s depends on \( t_i \), but for the seek of simplicity in notations, we have decided to omit this dependency.

Next, we present a gradient estimate for \( \nabla J_{t_i} \) in the direction of the \( \alpha_i \)'s. Indeed, we have:

**Lemma 3.4.** Assuming that \( \eta \) is a small positive real number with \( 0 < 2\eta < \varphi \) where \( \varphi \) is as in [34], and \( 0 < \varepsilon \leq \varepsilon_0 \) where \( \varepsilon_0 \) is as in [35], then for \( \alpha_i \in \mathcal{M} \) concentration points, \( \alpha_i \) masses, \( \lambda_i \) concentration parameters (\( i = 1, \ldots, m \)), and \( \beta_r \) negativity parameters (\( r = 1, \ldots, \bar{m} \)) satisfying (50), we have that for \( l \) large enough and for every \( j = 1, \ldots, m \), there holds

\[
\left< \nabla J_{t_i} \left( \sum_{i=1}^{m} \alpha_i \varphi_{a_i, \lambda_i} + \sum_{j=1}^{m} \beta_r (v_r - (v_r)_Q^n) \right), \varphi_{a_j, \lambda_j} \right> = \\
(2 \log \lambda_j + H(a_j, a_j) - C_2^a) \frac{1}{\alpha_j} \left< \nabla J_{t_i} \left( \sum_{i=1}^{m} \alpha_i \varphi_{a_i, \lambda_i} + \sum_{r=1}^{m} \beta_r (v_r - (v_r)_Q^n) \right), \lambda_j \frac{\partial \varphi_{a_j, \lambda_j}}{\partial \lambda_j} \right> \\
+ \sum_{i=1, i \neq j}^{m} G(a_j, a_i) \left< \nabla J_{t_i} \left( \sum_{i=1}^{m} \alpha_i \varphi_{a_i, \lambda_i} + \sum_{r=1}^{m} \beta_r (v_r - (v_r)_Q^n) \right), \lambda_i \frac{\partial \varphi_{a_j, \lambda_j}}{\partial \lambda_i} \right> \\
+ 4(n-1)! \omega_n (\alpha_j - 1) \log \lambda_j + O \left( \log \lambda_j \left( \sum_{i=1}^{m} \frac{|\alpha_i - 1|}{\log \lambda_i} + \sum_{r=1}^{m} |\beta_r| \left( \frac{1}{\log \lambda_i} + \sum_{i=1}^{m} \frac{1}{\lambda_i^3} \right) \right) \right),
\]

where \( O(1) \) is as in Lemma 3.2 and \( C_2^a \) is a constant depending only on \( n \).

**Proof.** It follows from the same arguments as in the proof of Lemma 5.3 in [50]. \( \blacksquare \)

Now, we derive a gradient estimate for \( \nabla J_{t_i} \) with respect to the \( \alpha_i \)'s. Precisely, we have:

**Lemma 3.5.** Assuming that \( \eta \) is a small positive real number with \( 0 < 2\eta < \varphi \) where \( \varphi \) is as in [34], and \( 0 < \varepsilon \leq \varepsilon_0 \) where \( \varepsilon_0 \) is as in [35], then for \( \alpha_i \in \mathcal{M} \) concentration points, \( \alpha_i \) masses, \( \lambda_i \) concentration parameters (\( i = 1, \ldots, m \)), and \( \beta_r \) negativity parameters (\( r = 1, \ldots, \bar{m} \)) satisfying (50), we have that for \( l \) large enough and for every \( j = 1, \ldots, m \), there holds

\[
\left< \nabla J_{t_i} \left( \sum_{i=1}^{m} \alpha_i \varphi_{a_i, \lambda_i} + \sum_{r=1}^{m} \beta_r (v_r - (v_r)_Q^n) \right), \frac{1}{\lambda_j} \frac{\partial \varphi_{a_j, \lambda_j}}{\partial a_j} \right> = \\
- \frac{4c_n^2 (n-1)! \omega_n}{n \lambda_j} \frac{\nabla_g F^A_i(a_j)}{F^A_i(a_j)} \\
+ O \left( \sum_{i=1}^{m} \frac{|\alpha_i - 1|^2}{\lambda_j} \right) \\
+ O \left( \sum_{i=1}^{m} \frac{1}{\lambda_j^3} + \sum_{r=1}^{m} \frac{|\beta_r|^2}{\lambda_j^3} + \sum_{i=1}^{m} \tau_i^3 \right),
\]

where \( A := (a_1, \ldots, a_m) \), \( O(1) \) is as in Lemma 3.2 and for \( i = 1, \ldots, m \), \( \tau_i \) is as in Lemma 3.2.
PROOF. It follows from the same arguments as in the proof of Lemma 5.4 in [50].

Finally, we have the following estimate for \( \nabla J_t \) in the direction of the \( \beta_r \)'s.

**Lemma 3.6.** Assuming that \( \eta \) is a small positive real number with \( 0 < 2\eta < \varrho \) where \( \varrho \) is as in (34), and \( 0 < \epsilon \leq \epsilon_0 \) where \( \epsilon_0 \) is as in (15), then for \( a_i \in M \) concentration points, \( \alpha_i \), masses, \( \lambda_i \) concentration parameters \( (i = 1, \ldots, m) \), and \( \beta_r \) negativity parameters \( (r = 1, \ldots, \bar{m}) \) satisfying (30), we have that for \( l \) large enough and for every \( s = 1, \ldots, \bar{m} \), there holds here holds

\[
\left\langle \nabla J_t \left( \sum_{i=1}^{m} \alpha_i \varphi_{a_i, \lambda_i} + \sum_{r=1}^{\bar{m}} \beta_r (v_r - (v_r)_Q) \right), v_i - (v_i)_Q \right\rangle = 2\mu_i \beta_i + O \left( \sum_{i=1}^{m} |\alpha_i| + 1 + \sum_{i=1}^{m} |\tau_i| + \sum_{i=1}^{m} \frac{1}{\lambda_i^2} \right),
\]

where \( O(1) \) is as in Lemma 3.2 and for \( i = 1, \ldots, m \), \( \tau_i \) is as in Lemma 3.2.

**Proof.** It follows from the same arguments as in the proof of Lemma 5.5 in [50].

**Remark 3.7.** We would like to point out that the gradient estimates in (51)-(55) holds in \( C^1 \) as function of the variables \( (\bar{a}, A, \bar{\lambda}, \bar{\beta}, \bar{\tau}) \) with \( \bar{a} = (a_1, \ldots, a_m), A = (a_1, \ldots, a_m), \bar{\lambda} = (\lambda_1, \ldots, \lambda_m), \bar{\beta} = (\beta_1, \ldots, \beta_\bar{m}), \bar{\tau} = (\tau_1, \ldots, \tau_m) \) where for \( B, C, D \) some \( C^1 \)-functions of the variables \( (\bar{a}, A, \bar{\lambda}, \bar{\beta}, \bar{\tau}) \), \( B = C + O(D) \) in \( C^1 \) in the variables \( (\bar{a}, A, \bar{\lambda}, \bar{\beta}, \bar{\tau}) \) means \( B = C + O(D) \) and \( \nabla B = \nabla C + O(D) \) with \( \nabla \) denoting the gradient with respect to \( (\bar{a}, A, \bar{\lambda}, \bar{\beta}, \bar{\tau}) \).

4 Refined location of \( u_t \)

In this section, we improve the location of \( u_t \) given by Lemma 3.1. In order to do that, we divide this section into two subsections. In the first one, we derive a finite-dimensional parametrization of \( u_t \). In the second one, we present the improvement we talked about above, by using the later parametrization of \( u_t \).

4.1 Finite-dimensional parametrization of \( u_t \)

As already mentioned above, in this subsection, we give a finite-dimensional parametrization of \( u_t \). For this end, we start with the following Lemma.

**Lemma 4.1.** Assuming that \( \eta \) is a small positive real number with \( 0 < 2\eta < \varrho \) where \( \varrho \) is as in (34), and \( 0 < \epsilon \leq \epsilon_0 \) where \( \epsilon_0 \) is as in (15) and \( u = \overline{u}_Q + \sum_{i=1}^{m} \alpha_i \varphi_{a_i, \lambda_i} + \sum_{r=1}^{\bar{m}} \beta_r (v_r - (v_r)_Q) + w \in V(m, \epsilon, \eta) \) with \( w \), the concentration points \( a_i \), the masses \( \alpha_i \), the concentrating parameters \( \lambda_i \) \((i = 1, \ldots, m)\), and the negativity parameters \( \beta_r \) \((r = 1, \ldots, \bar{m})\) verifying (18)-(50), then we have

\[
J_t(u) = J_t \left( \sum_{i=1}^{m} \alpha_i \varphi_{a_i, \lambda_i} + \sum_{r=1}^{\bar{m}} \beta_r (v_r - (v_r)_Q) \right) - f_t(w) + Q_t(w) + o(||w||^2),
\]

where

\[
f_t(w) := 2(n-1)! \omega_n t^n \frac{\int_M K e^{n \sum_{i=1}^{m} \alpha_i \varphi_{a_i, \lambda_i} + n \sum_{r=1}^{\bar{m}} \beta_r v_r w dV_g}}{\int_M K e^{n \sum_{i=1}^{m} \alpha_i \varphi_{a_i, \lambda_i} + n \sum_{r=1}^{\bar{m}} \beta_r v_r dV_g}},
\]

and

\[
Q_t(w) := ||w||^2_{L^n} - \eta \omega_n t^n \frac{\int_M K e^{n \sum_{i=1}^{m} \alpha_i \varphi_{a_i, \lambda_i} + n \sum_{r=1}^{\bar{m}} \beta_r v_r w^2 dV_g}}{\int_M K e^{n \sum_{i=1}^{m} \alpha_i \varphi_{a_i, \lambda_i} + n \sum_{r=1}^{\bar{m}} \beta_r v_r dV_g}}.
\]

Moreover, setting

\[
E_{a_i, \lambda_i} := \{ w \in W^{1,2}(M) : \langle \varphi_{a_i, \lambda_i}, w \rangle p_n = \langle \frac{\partial \varphi_{a_i, \lambda_i}}{\partial \lambda_i}, w \rangle p_n = \langle \frac{\partial \varphi_{a_i, \lambda_i}}{\partial a_i}, w \rangle p_n = 0, \}
\]

\[
\langle w, Q^p_g \rangle = (v_r, w) = 0, r = 1, \ldots, \bar{m}, \text{ and } ||w|| = O \left( \sum_{i=1}^{m} \frac{1}{\lambda_i} \right),
\]

(59)
and

\[ A := (a_1, \cdots, a_m), \quad \bar{\lambda} = (\lambda_1, \cdots, \lambda_m), \quad E_{A, \bar{\lambda}} := \cap_{i=1}^m E_{a_i, \lambda_i}, \]

we have that, the quadratic form \( Q \) is positive definite in \( E_{A, \bar{\lambda}} \). Furthermore, the linear part \( f \) verifies that, for every \( w \in E_{A, \bar{\lambda}} \), there holds

\[
 f_l(w) = O \left[ ||w|| \left( \sum_{i=1}^m \left| \frac{\nabla F_i^A(a_i)}{\lambda_i} \right| + \sum_{i=1}^m |\alpha_i - 1| \log \lambda_i + \sum_{r=1}^m |\beta_r| + \sum_{i=1}^m \frac{\log \lambda_i}{\lambda_i^4} \right) \right],
\]

where here \( o(1) = o_l, a, \bar{\lambda}, \bar{\nu}, w, \epsilon(1) \) and \( O(1) := O_l, a, \bar{\lambda}, \bar{\nu}, w, \epsilon(1) \) and for their meanings see section [2].

**Proof.** The proof is the same as the one Proposition 6.1 in [50].

Like in [50] and for the same reasons, we have that Lemma [41] implies the following direct corollary.

**Corollary 4.2.** Assuming that \( \eta \) is a small positive real number with \( 0 < 2\eta < \varrho \) where \( \varrho \) is as in [34], \( 0 < \epsilon \leq \epsilon_0 \) with \( \epsilon_0 \) is as in [54], and \( u := \sum_{i=1}^m \alpha_i \varphi_{a_i, \lambda_i} + \sum_{r=1}^m \beta_r (v_r - (v_r)_{Q^n}) \) with the concentration points \( a_i \), the masses \( \alpha_i \), the concentrating parameters \( \lambda_i \) \( (i = 1, \cdots , m) \) and the negativity parameters \( \beta_r \) \( (r = 1, \cdots , m) \) satisfying [50], then for \( l \) large enough, there exists a unique \( \bar{w}_l(\bar{\alpha}, A, \bar{\lambda}, \bar{\beta}) \in E_{A, \bar{\lambda}} \) such that

\[
 J_l(u + \bar{w}_l(\bar{\alpha}, A, \bar{\lambda}, \bar{\beta})) = \min_{w \in E_{A, \bar{\lambda}}, u + w \in V(m, \epsilon, \eta)} J_l(u + w),
\]

where \( \bar{\alpha} := (\alpha_1, \cdots, \alpha_m), \quad A := (a_1, \cdots, a_m), \quad \bar{\lambda} := (\lambda_1, \cdots, \lambda_m) \) and \( \bar{\beta} := (\beta_1, \cdots, \beta_m) \).

Furthermore, for \( l \) large enough, the map \( (\bar{\alpha}, A, \bar{\lambda}, \bar{\beta}) \mapsto \bar{w}_l(\bar{\alpha}, A, \bar{\lambda}, \bar{\beta}) \in C^1 \) and satisfies the following estimate

\[
 \frac{1}{C} ||\bar{w}(\bar{\alpha}, A, \bar{\lambda}, \bar{\beta})||^2 \leq ||f_l(\bar{w}(\bar{\alpha}, A, \bar{\lambda}, \bar{\beta}))|| \leq C ||\bar{w}(\bar{\alpha}, A, \bar{\lambda}, \bar{\beta})||^2,
\]

for some large positive constant \( C \) independent of \( l, \bar{\alpha}, \bar{\lambda}, \) and \( \bar{\beta} \), hence

\[
 ||\bar{w}(\bar{\alpha}, A, \bar{\lambda}, \bar{\beta})|| = O \left( \sum_{i=1}^m \left| \frac{\nabla F_i^A(a_i)}{\lambda_i} \right| + \sum_{i=1}^m |\alpha_i - 1| \log \lambda_i + \sum_{r=1}^m |\beta_r| + \sum_{i=1}^m \frac{\log \lambda_i}{\lambda_i^4} \right),
\]

where \( O(1) := O_l, a, \bar{\lambda}, \bar{\nu}, w, \epsilon(1) \) and for its meaning see section [2]. Moreover, assuming that \( u_0 := \sum_{i=1}^m \alpha_i^0 \varphi_{a_i^0, \lambda_i^0} + \sum_{r=1}^m \beta_r^0 (v_r - (v_r)_{Q^n}) \) with the concentration points \( a_i^0 \), the masses \( \alpha_i^0 \), the concentrating parameters \( \lambda_i^0 \) \( (i = 1, \cdots , m) \) and the negativity parameters \( \beta_r^0 \) \( (r = 1, \cdots , m) \) satisfying [50], then for \( l \) large enough, there exists an open neighborhood \( U_l^0 \) of \( (\bar{\alpha}^0, A^0, \bar{\lambda}^0, \bar{\beta}^0) \) (with \( \bar{\alpha}^0 := (\alpha_1^0, \cdots, \alpha_m^0), \quad A^0 := (a_1^0, \cdots, a_m^0), \quad \bar{\lambda}^0 := (\lambda_1^0, \cdots, \lambda_m^0) \) and \( \bar{\beta}^0 := (\beta_1^0, \cdots, \beta_m^0) \) such that for every \( (\bar{\alpha}, A, \bar{\lambda}, \bar{\beta}) \in U_l^0 \) with \( \bar{\alpha} := (\alpha_1, \cdots, \alpha_m), \quad A := (a_1, \cdots, a_m), \quad \bar{\lambda} := (\lambda_1, \cdots, \lambda_m), \quad \bar{\beta} := (\beta_1, \cdots, \beta_m) \), and the \( a_i \), the \( \alpha_i \), the \( \lambda_i \) \( (i = 1, \cdots , m) \) and the \( \beta_r \) \( (r = 1, \cdots , m) \) satisfying [50], and \( w \) satisfying [44] with \( \sum_{i=1}^m \alpha_i \varphi_{a_i, \lambda_i} + \sum_{r=1}^m \beta_r (v_r - (v_r)_{Q^n}) + w \in V(m, \epsilon, \eta) \), we have the existence of a change of variable

\[
 w \rightarrow V_l
\]

from a neighborhood of \( \bar{w}(\bar{\alpha}, A, \bar{\lambda}, \bar{\beta}) \) to a neighborhood of \( 0 \) such that

\[
 J_l \left( \sum_{i=1}^m \alpha_i \varphi_{a_i, \lambda_i} + \sum_{r=1}^m \beta_r (v_r - (v_r)_{Q^n}) + w \right)
\]

\[
 J_l \left( \sum_{i=1}^m \alpha_i \varphi_{a_i, \lambda_i} + \sum_{r=1}^m \beta_r (v_r - (v_r)_{Q^n}) + \bar{w}(\bar{\alpha}, A, \bar{\lambda}, \bar{\beta}) \right)
\]

\[
 + \frac{1}{2} \theta J_l \left( \sum_{i=1}^m \alpha_i \varphi_{a_i, \lambda_i} + \sum_{r=1}^m \beta_r (v_r - (v_r)_{Q^n}) + \bar{w}(\bar{\alpha}, A, \bar{\lambda}, \bar{\beta}) \right) (V_l, V_l).
\]

Thus, with this new variable, in \( V(m, \epsilon, \eta) \) we have a splitting of the variables \( (\bar{\alpha}, A, \bar{\lambda}, \bar{\beta}) \) and \( V_l \), and \(-V_l\) is a pseudogradient of \( J_{l+1} \) in the direction of \( V_l \). Using this fact, we have the following Proposition which was the goal of this subsection.
Proposition 4.3. Assuming that $u_l$ is a sequence of blowing up solutions to (53), then for $l$ large enough there holds 

$$u_l - \bar{u} = \sum_{i=1}^m \alpha_i \varphi_{a_i, \lambda_i} + \sum_{r=1}^m \beta_r (v_r - (v_r)_Q) + \tilde{w}_l (\bar{a}_i, A_i, \bar{\lambda}_i, \tilde{\beta}_i),$$

with $\alpha_i := (\alpha_1', \ldots, \alpha_m')$, $A_i := (a_1', \ldots, a_m')$, $\lambda_i := (\lambda_1', \ldots, \lambda_m')$, and $\tilde{\beta}_i := (\beta_1', \ldots, \beta_m')$. Furthermore, for $l$ large enough, the concentration points $a_i$, the masses $\alpha_i$, the concentrating parameters $\lambda_i$ ($i = 1, \ldots, m$) and the negativity parameters $\beta_r$ ($r = 1, \ldots, \bar{m}$) satisfy (50).

Proof. It follows from the fact that $u_l$ is a solution to (53) implies $\nabla J_l(u_l) = 0$, the fact that $J_l$ is invariant by translation by constants combined with (45), (47)–(50), Lemma 3.1 Corollary 12 and the discussion right after it. \hfill \blacksquare

4.2 Refined estimates for the finite-dimensional parameters of $u_l$

As already mentioned at the beginning of this section, in this subsection we derive refined estimates for the finite-dimensional parameters of $u_l$ in the formula (67). In order to do that, we start by constructing a pseudo-gradient for $J_l(\bar{a}, A, \bar{\lambda}, \tilde{\beta})$, where $J_l(\bar{a}, A, \bar{\lambda}, \tilde{\beta}) := J_l(\sum_{i=1}^m \alpha_i \varphi_{a_i, \lambda_i} + \sum_{r=1}^m \beta_r (v_r - (v_r)_Q) + \tilde{w}_l (\bar{a}, A, \bar{\lambda}, \tilde{\beta}))$ in a suitable subset of $V(m, \epsilon, \eta)$. Indeed, setting

$$V_{\text{deep}}^l (m, \epsilon, \eta) := \{ u \in V(m, \epsilon, \eta) : \sum_{i=1}^m |\nabla gF^A(a_i)|_{\lambda_i} + \sum_{i=1}^m |\alpha_i - 1| + \sum_{i=1}^m |\tau_i| + \sum_{r=1}^m 1_{X_r}, \lambda_i \} \leq C_0 \sum_{i=1}^m 1_{X_r},$$

with $C_0$ a large positive constant, we have the following Proposition.

Proposition 4.4. Assuming that $\eta$ is a small positive real number with $0 < 2\eta < \varrho$ where $\varrho$ is as in (43), and $0 < \epsilon \leq \epsilon_0$ where $\epsilon_0$ is as in (45), then for $l$ large enough, we have that there exists a pseudogradient $W := W(l)$ of $J_l(\bar{a}, A, \bar{\lambda}, \tilde{\beta})$ in $V(m, \epsilon, \eta) \setminus V_{\text{deep}}^l (m, \epsilon, \eta)$ such that for every $u := \sum_{i=1}^m \alpha_i \varphi_{a_i, \lambda_i} + \sum_{r=1}^m \beta_r (v_r - (v_r)_Q) \in V(m, \epsilon, \eta) \setminus V_{\text{deep}}^l (m, \epsilon, \eta)$ with the concentration points $a_i$, the masses $\alpha_i$, the concentrating parameters $\lambda_i$ ($i = 1, \ldots, m$) and the negativity parameters $\beta_r$ ($r = 1, \ldots, \bar{m}$) satisfying (50), there holds

$$< -\nabla J_l(u), W > \geq c \left( \sum_{i=1}^m 1_{X_r} \right) + \sum_{i=1}^m |\nabla gF^A(a_i)|_{\lambda_i} + \sum_{i=1}^m |\alpha_i - 1| + \sum_{i=1}^m |\tau_i| + \sum_{r=1}^m |\beta_r|,$$

where $c$ is a small positive constant independent of $l$, $A := (a_1', \ldots, a_m')$, $\bar{a} = (\alpha_1', \ldots, \alpha_m')$, $\bar{\lambda} = (\lambda_1', \ldots, \lambda_m')$, $\tilde{\beta} = (\beta_1', \ldots, \beta_m')$ and $\epsilon$. Furthermore, for every $u := \sum_{i=1}^m \alpha_i \varphi_{a_i, \lambda_i} + \sum_{r=1}^m \beta_r (v_r - (v_r)_Q) \in V(m, \epsilon, \eta) \setminus V_{\text{deep}}^l (m, \epsilon, \eta)$ with the concentration points $a_i$, the masses $\alpha_i$, the concentrating parameters $\lambda_i$ ($i = 1, \ldots, m$) and the negativity parameters $\beta_r$ ($r = 1, \ldots, \bar{m}$) satisfying (50), and $\tilde{w}_l (\bar{a}, A, \bar{\lambda}, \tilde{\beta})$ is as in (62), there holds

$$< -\nabla J_l(u), W > + \frac{\partial \tilde{w}_l(W)}{\partial(\bar{a}, A, \bar{\lambda}, \tilde{\beta})} \geq c \left( \sum_{i=1}^m 1_{X_r} \right) + \sum_{i=1}^m |\nabla gF^A(a_i)|_{\lambda_i} + \sum_{i=1}^m |\alpha_i - 1| + \sum_{i=1}^m |\tau_i| + \sum_{r=1}^m |\beta_r|,$$

where $c$ is still a small positive constant independent of $l$, $A := (a_1', \ldots, a_m')$, $\bar{a} = (\alpha_1', \ldots, \alpha_m')$, $\bar{\lambda} = (\lambda_1', \ldots, \lambda_m')$, $\tilde{\beta} = (\beta_1', \ldots, \beta_m')$ and $\epsilon$.

Proof. The argument is the same as the one of Proposition in (56). \hfill \blacksquare

Finally, we are going to achieve the goal of this section, by establishing a refined location of $u_l$, by exploiting its criticality for $J_l$, its finite-dimensional parametrization given by the previous subsection and Proposition 4.3. Precisely, we have:

Lemma 4.5. Let $\eta$ be a small positive real number with $0 < 2\eta < \varrho$ where $\varrho$ is as in (43) and $0 < \epsilon \leq \epsilon_0$ where $\epsilon_0$ is as in (45). Assuming that $u_l$ is a sequence of blowing-up solutions to (21), then for $l$ large enough, we have

$$u_l \in V_{\text{deep}}^l (m, \epsilon, \eta).$$
Proof. It follows from the fact that \( u_l \) is a solution to (5) implies \( \nabla J_l(u_l) = 0 \) combined with Proposition 4.3 and Proposition 4.4. ■

5 Proof of the results

In this section, we present the proof of Theorem 1.1 Corollary 1.4. We start with the one of Theorem 1.1 in order to do that, we are going to show the following result from which Theorem 1.1 follows directly, thanks to the formula (14) and to the used scaling blowing up formula which is given by point a) of Lemma 2.3 in [54].

Theorem 5.1. Let \((M, g)\) be a closed four-dimensional Riemannian manifold such that \( \ker P_g \simeq \mathbb{R} \), and \( \kappa^n = (n-1)!m\omega_n \) with \( m \in \mathbb{N}^+ \). Assuming that \( K \) is a smooth positive function on \( M \), \( \epsilon \) and \( \eta \) be small positive real numbers with \( 0 < 2\eta < \alpha \) where \( \alpha \) is as in (34) and \( u_l \) is a sequence of blowing up solutions to (21), then for \( l \) large enough, we have that \( u_l \in V^t_{\text{deep}}(m, \epsilon, \eta) \)

\[
u_l - (u_l)_{Q^n} = \sum_{i=1}^{m} \alpha^i_l \phi^i_l \chi^i_l + \sum_{r=1}^{m} \beta^r_l (v_r - (v_r)_{Q^n}) + + \bar{\omega} (\bar{\alpha}, A_l, \bar{\lambda}_l, \bar{\beta}_l),
\]

and

\[
t_l - 1 = \frac{\tilde{e}^K_{n,m}(A^i)}{(F^A(a_i))^{\frac{n-m}{2}}(\lambda^i_l)^2} \left[ l_K(A^i) + O \left( \frac{1}{\lambda^i_l} \right) \right], \quad i = 1, \ldots, m
\]

with \( V^t_{\text{deep}}(m, \epsilon, \eta) \) defined by (38), \( \bar{\alpha}_l := (\alpha^1_l, \ldots, \alpha^m_l) \), \( A_l := (a^1_l, \ldots, a^m_l) \), \( \bar{\lambda}_l := (\lambda^1_l, \ldots, \lambda^m_l) \), \( \bar{\beta}_l = (\beta^1_l, \ldots, \beta^m_l) \), the concentration points \( z^i_l \), the masses \( a^i_l \), the concentrating parameters \( \lambda^i_l \) \( (i = 1, \ldots, m) \), and the negativity parameters \( \beta^r_l \) \( (r = 1, \ldots, \bar{m}) \) satisfy (50), \( A^i \rightarrow A \in \text{Crit}(\mathcal{F}_K) \) as \( l \rightarrow +\infty \), \( \tilde{e}^K_{n,m}(A^i) \rightarrow \tilde{e}^n_{m}(A) > 0 \), and \( l_K(\cdot) \) is defined by (42).

Proof of Theorem 5.1

The proof is the same as the one of Theorem 5.1 in [55]. ■

Proof of Corollary 1.2

Case 1: \((ND)_-\) holds

In this case, the result follows from our work [49] in the nonresonant case by considering the functional \( J_{1+\varepsilon} \) with \( \varepsilon \) positive and small combined with Theorem 1.1.

Case 2: \((ND)_+\) holds

In this case, the same argument as above works by considering \( J_{1-\varepsilon} \) with \( \varepsilon \) positive and small. We add that for \( m = 1 \) and \( \bar{m} = 0 \), by the existence of minimizers in the subcritical case, we can take the solution to be be a minimizer of \( J \). On the other hand when \( m \geq 2 \), the result follows also from the characterization of the "true" critical points at infinity of \( J \), and the topology of very high and very negative sublevels of \( J \) established in our work [51]. ■

Proof of Corollary 1.4

Clearly Theorem 1.1 combined with standard elliptic regularity theory imply the result. ■

6 Proof of Theorem 1.5

In this section, we prove Theorem 1.5. We start by showing a deep local characterization at infinity around critical points of \( \mathcal{F}_K \) for blowing-up solution of the type considered in Lemma 4.5. Indeed setting

\[
V^t_{\text{deep}}(m, \epsilon, \eta)(A^0) := \{ u \in V^t_{\text{deep}}(m, \epsilon, \eta) : \text{dist}(a_i, a^0_i) \leq \tilde{C}_0 \frac{1}{\lambda^i_l}, \quad i = 1, \ldots, m \}
\]

for \( A^0 = (a^0_1, \ldots, a^0_m) \in \text{Crit}(\mathcal{F}_K) \) with \( \tilde{C}_0 \) a large positive constant, we have:
Proposition 6.1. Let \( \eta \) be a small positive real number with \( 0 < 2 \eta < \varrho \) where \( \varrho \) is as in \((\text{54})\) and \( 0 < \epsilon \leq \epsilon_0 \) where \( \epsilon_0 \) is as in \((\text{45})\). Assuming that \( u_t \) is a sequence of blowing-up solutions to \((\text{21})\), then for \( l \) large enough, we have

\[
u_t \in \mathcal{V}_{\text{deep}}^l(m, \epsilon, \eta)(A),
\]

for some \( A \in \text{Crit}(\mathcal{F}_K) \).

**Proof.** Using Theorem \((\text{5.1})\) we have \( u_t \in \mathcal{V}_{\text{deep}}^l(m, \epsilon, \eta) \) and

\[
u_t - (\nu_t)_{Q^n} = \sum_{i=1}^m \alpha_i^t \varphi_i^t \chi_i^t + \sum_{r=1}^m \beta_r^t (v_r - (v_r)_{Q^n}) + \nu_t(\alpha^t, A^t, \chi^t, \beta^t)
\]

with \( ||\nu_t(\alpha^t, A^t, \chi^t, \beta^t)|| = O\left( \sum_{i=1}^m |\nabla \mathcal{F}_A^l(\alpha_i^t)|_{\chi_i^t} + \sum_{i=1}^m |\alpha_i^t - 1| \log \lambda_i^t + \sum_{r=1}^m |\beta_r^t| + \sum_{i=1}^m \log \lambda_i^t \right) \).

Thus using the definition of \( \mathcal{V}_{\text{deep}}^l(m, \epsilon, \eta) \), we have

\[
\frac{|\nabla \mathcal{F}_A^l(\alpha_i^t)|}{\chi_i^t} \leq C_0 \frac{1}{(\lambda_i^t)^2}
\]

This implies \( A^t = (a_i^t) \to A \in \text{Crit}(\mathcal{F}_K) \), thank to \((\text{11})\). Moreover, the non-degeneracy of \( \mathcal{F}_K \) implies

\[
d_{\mathcal{L}}(a_i^t, a_i) \leq C_0 \frac{1}{\lambda_i^t} \quad i = 1, \ldots, m,
\]

for some large \( C_0 > 0 \). Hence \( u_t \in \mathcal{V}_{\text{deep}}^l(m, \epsilon, \eta)(A) \). \( \blacksquare \)

**Remark 6.2.** The bubbling rate formula in Theorem \((\text{4.4})\) implies that for \( u_t \in \mathcal{V}_{\text{deep}}^l(m, \epsilon, \eta)(A) \), we have \( u_t \in \mathcal{V}_{\text{deep}}^l(m, \epsilon, \eta)(A) \), where \( \mathcal{V}_{\text{deep}}^l(m, \epsilon, \eta)(A) \) is defined as in \((\text{68})\) with \( t_1 \) replaced by \( 1 \) and \( A^0 \) replaced by \( A \).

In the next proposition, we show that for any \( A \in \text{Crit}(\mathcal{F}_K) \) with \( \mathcal{L}_K(A) < 0 \), there exists \( u_t \in \mathcal{V}_{\text{deep}}^l(m, \epsilon, \eta)(A) \) for \( t \approx 1^{-} \) when \( \mathcal{F}_K \) is a Morse function. The set \( \mathcal{V}_{\text{deep}}^l(m, \epsilon, \eta)(A) \) is defined as in \((\text{68})\) with \( t_1 \) replaced by \( t \) and \( A^0 \) replaced by \( A \).

**Proposition 6.3.** Let \( (M, g) \) be a closed \( n \)-dimensional Riemannian manifold with \( n \geq 4 \) even such that \( \ker \mathcal{F}_g^n \simeq \mathbb{R} \) and \( \kappa_g^n = (n - 1)! n \omega_n \). Assuming that \( K \) is a smooth positive function on \( M \) such that \( \mathcal{F}_K \) is a Morse function and \( A \in \text{Crit}(\mathcal{F}_K) \) with \( \mathcal{L}_K(A) < 0 \), then for \( t \approx 1^{-} \), there exist \( u_t \) verifying

\[
P_g^m u_t + tQ_g^n = t \kappa_g^n K e^{nu_t} \text{ in } M
\]

such that

\[
u_t \in \mathcal{V}_{\text{deep}}^l(m, \epsilon, \eta)(A).
\]

**Proof.** By Theorem \((\text{5.1})\) and Proposition \((\text{6.1})\), we must look for a solution \( u_t \in \mathcal{V}_{\text{deep}}^l(m, \epsilon, \eta)(A) \) for \( t \approx 1^{-} \) verifying

\[
u_t - (\nu_t)_{Q^n} = \sum_{i=1}^m \alpha_i^t \varphi_i^t \chi_i^t + \sum_{r=1}^m \beta_r^t (v_r - (v_r)_{Q^n}) + \nu_t(\alpha^t, A^t, \chi^t, \beta^t),
\]

with \( \alpha^t = (\alpha_1^t, \ldots, \alpha_m^t) \), \( A^t = (a_1^t, \ldots, a_m^t) \), \( \chi^t = (\chi_1^t, \ldots, \chi_m^t) \), \( \beta^t = (\beta_1^t, \ldots, \beta_m^t) \), and \( \nu_t(\alpha^t, A^t, \chi^t, \beta^t) \) verifies

\[
J_t(\sum_{i=1}^m \alpha_i^t \varphi_i^t \chi_i^t + \sum_{r=1}^m \beta_r^t (v_r - (v_r)_{Q^n}) + \nu_t(\alpha^t, A^t, \chi^t, \beta^t)) = \min_{w \in E_{A^t, \chi^t, \beta^t}} \left\{ \sum_{i=1}^m \alpha_i^t \varphi_i^t \chi_i^t + \sum_{r=1}^m \beta_r^t (v_r - (v_r)_{Q^n}) + w \right\}
\]
with
\[ \dot{z}^t = \sum_{i=1}^{m} \alpha_i^t \varphi_{a_i^t} \lambda_i^t + \sum_{r=1}^{\bar{m}} \beta_r^t (v_r - \langle v_r \rangle_{Q^m}) \]
and
\[ \| \dot{w}_t(\hat{\alpha}^t, A^t, \hat{\lambda}^t, \hat{\beta}^t) \| = O \left( \sum_{i=1}^{m} \left| \nabla_g F_i^A(a_i^t) \right| / \lambda_i^t + \sum_{i=1}^{m} |\alpha_i^t - 1| \log \lambda_i^t + \sum_{r=1}^{\bar{m}} |\beta_r^t| + \sum_{i=1}^{m} \log \left( \lambda_i^t / (\lambda_i^t)^2 \right) \right), \]
where \( E_{A^t, \hat{\lambda}^t} \) is as in (70) with \( (A, \hat{\lambda}) \) replaced by \( (A^t, \hat{\lambda}^t) \). Thus, we have
\[
\langle \nabla J_t(\sum_{i=1}^{m} \alpha_i^t \varphi_{a_i^t} \lambda_i^t + \sum_{r=1}^{\bar{m}} \beta_r^t (v_r - \langle v_r \rangle_{Q^m}) + \dot{w}_t(\hat{\alpha}^t, A^t, \hat{\lambda}^t, \hat{\beta}^t), \dot{w} \rangle = 0, \quad \forall \dot{w} \in E_{A^t, \hat{\lambda}^t}.
\]
Hence, setting
\[ z^t = \sum_{i=1}^{m} \alpha_i^t \varphi_{a_i^t} \lambda_i^t + \sum_{r=1}^{\bar{m}} \beta_r^t (v_r - \langle v_r \rangle_{Q^m}) + \dot{w}_t(\hat{\alpha}^t, A^t, \hat{\lambda}^t, \hat{\beta}^t), \]
we have
\[ \nabla J_t(z^t) = 0 \]
is equivalent to
\[ \nabla J_t(z^t), \lambda_j^t \frac{\partial \varphi_{a_j^t} \lambda_j^t}{\partial a_j^t} = \left\langle \nabla J_t(z^t), \frac{1}{\lambda_j^t} \frac{\partial \varphi_{a_j^t} \lambda_j^t}{\partial \alpha_j^t} \right\rangle + \left\langle \nabla J_t(z^t), \varphi_{a_j^t} \lambda_j^t \right\rangle = \left\langle \nabla J_t(z^t), v_r - \langle v_r \rangle_{Q^m} \right\rangle = 0, \quad j = 1, \ldots, m, \ r = 1, \ldots, \bar{m}.
\]
Using Lemma 3.4 and Corollary 3.2 with \( t_l \) replaced by \( t \) combined with (75) and recalling that we are looking for \( z^t \in V^t_{\text{deep}}(m, \epsilon, \eta)(A) \), we have
\[ \left\langle \nabla J_t(z^t), v_r - \langle v_r \rangle_{Q^m} \right\rangle = \left\langle \nabla J_t(\sum_{i=1}^{m} \alpha_i^t \varphi_{a_i^t} \lambda_i^t + \sum_{r=1}^{\bar{m}} \beta_r^t (v_r - \langle v_r \rangle_{Q^m}), v_r - \langle v_r \rangle_{Q^m} \right\rangle \]
\[ + O \left( \sum_{i=1}^{m} \left| \nabla_g F_i^A(a_i^t) \right| / \lambda_i^t \right) + \sum_{i=1}^{m} |\alpha_i^t - 1| + \sum_{r=1}^{\bar{m}} |\beta_r^t| + \sum_{i=1}^{m} \log \left( \lambda_i^t / (\lambda_i^t)^2 \right) \right), \]
As in Remark 3.7, (79) holds in \( C^1 \) of the variables \( (\hat{\alpha}^t, A^t, \hat{\lambda}^t, \hat{\beta}^t, \tau^t) \) with \( \tau^t = (\tau_1^t, \ldots, \tau_m^t) \) with \( \tau_i^t \) as in Remark 3.2 in with \( t_l \) replaced by \( t \). Thus, using Lemma 3.4, Remark 3.7 and (79), we have
\[ \left\langle \nabla J_t(z^t), v_r - \langle v_r \rangle_{Q^m} \right\rangle = 0 \]
is equivalent to
\[ 2 \mu_r \beta_r^t + O \left( \sum_{i=1}^{m} \left| \nabla_g F_i^A(a_i^t) \right| / \lambda_i^t \right) + \sum_{i=1}^{m} |\alpha_i^t - 1| + \sum_{i=1}^{m} |\tau_i^t| + \sum_{r=1}^{\bar{m}} |\beta_r^t| + \sum_{i=1}^{m} \log \left( \lambda_i^t / (\lambda_i^t)^2 \right) \right) = 0, \]
with (81) holding in \( C^1 \) of the variables \( (\hat{\alpha}^t, A^t, \hat{\lambda}^t, \hat{\beta}^t, \tau^t) \) in the sense defined in Remark 3.7. Thus by implicit function theorem we have a unique \( \hat{\beta}_r^t = \hat{\beta}_r^t(A^t, \hat{\alpha}^t, \hat{\lambda}^t, \tau^t) \) solving (51) with \( z^t \in V^t_{\text{deep}}(m, \epsilon, \eta)(A) \) for \( t \approx 1^- \). Similarly, using Lemma 3.5 and Corollary 3.2 with \( t_l \) replaced by \( t \) combined with (75), we have
\[ \left\langle \nabla J_t(z^t), \frac{1}{\lambda_j^t} \frac{\partial \varphi_{a_j^t} \lambda_j^t}{\partial a_j^t} \right\rangle = \left\langle \nabla J_t(\sum_{i=1}^{m} \alpha_i^t \varphi_{a_i^t} \lambda_i^t + \sum_{r=1}^{\bar{m}} \beta_r^t (v_r - \langle v_r \rangle_{Q^m}), \frac{1}{\lambda_j^t} \frac{\partial \varphi_{a_j^t} \lambda_j^t}{\partial \alpha_j^t} \right\rangle \]
\[ + O \left( \sum_{i=1}^{m} \left| \nabla_g F_i^A(a_i^t) \right| / \lambda_i^t \right) + \sum_{i=1}^{m} |\alpha_i^t - 1| + \sum_{r=1}^{\bar{m}} |\beta_r^t| + \sum_{i=1}^{m} \log \left( \lambda_i^t / (\lambda_i^t)^2 \right) \right), \]
with (82) holding in $C^1$ of the variables $(\tilde{\alpha}^t, A^t, \tilde{\lambda}^t, \tau^t)$. Thus, using Lemma 3.5, Remark 3.7, and (82), we have

$$\left\langle \nabla J_t(z^t), \frac{1}{\lambda_j^t} \frac{\partial \varphi_{a^t_j, \lambda_j^t}}{\partial a_j^t} \right\rangle = 0$$

is equivalent to

$$(84)$$

$$-4e^{2}(n+1-1)\omega_n \frac{\partial \varphi}{\partial \lambda_j^t} + O \left( \sum_{i=1}^{m} \left[ \frac{\partial \varphi}{\partial \lambda_j^t} \right]^{2} + \sum_{i=1}^{m} |\lambda_j^t| - 1 + \sum_{i=1}^{m} |\lambda_j^t| + \sum_{i=1}^{m} |\tau_j^t|^{2} \right) = 0,$$

with (84) holding in $C^1$ of the variables $(\tilde{\alpha}^t, A^t, \tilde{\lambda}^t, \tau^t)$. Thus as above, using the implicit function theorem we have a unique $A^t = A^t(\tilde{\alpha}^t, \tilde{\lambda}^t, \tau^t)$ solving (84) with $z^t \in V^t_{\text{deep}}(m, \epsilon, \eta)(A)$ for $t \approx 1^{-}$. Again as above, using Lemma 4.1 and Corollary 4.2 with $t_1$ replaced by $t$ combined with (75), we have

$$\left\langle \nabla J_t(z^t), \varphi_{a^t_j, \lambda_j^t} \right\rangle = \left\langle \nabla J_t(\sum_{i=1}^{m} \alpha_i^t \varphi_{a^t_i, \lambda_j^t} + \sum_{r=1}^{\tilde{m}} \beta_r^t (\nu_r - (\nu_r)_{Q^t})), \varphi_{a^t_j, \lambda_j^t} \right\rangle$$

$$+ O \left( \sum_{i=1}^{m} \left[ \frac{\partial \varphi}{\partial \lambda_j^t} \right]^{2} + \sum_{i=1}^{m} |\lambda_j^t| - 1 + \sum_{i=1}^{m} |\lambda_j^t| + \sum_{i=1}^{m} |\tau_j^t|^{2} \right)$$

with (85) holding in $C^1$ of the variables $(\tilde{\alpha}^t, \tilde{\lambda}^t, \tau^t)$. Thus, using Lemma 3.2 and Lemma 3.4, Remark 3.7, and (85), we have

$$\left\langle \nabla J_t(z^t), \varphi_{a^t_j, \lambda_j^t} \right\rangle = 0$$

is equivalent to

$$(86)$$

$$4(n+1-1)\omega_n (\alpha_j^t - 1) \log \lambda_j^t + O \left( \sum_{i=1}^{m} \left[ \frac{\partial \varphi}{\partial \lambda_j^t} \right]^{2} + \sum_{i=1}^{m} |\lambda_j^t| - 1 + \sum_{i=1}^{m} |\lambda_j^t| + \sum_{i=1}^{m} |\tau_j^t|^{2} \right) = 0,$$

with (87) holding in $C^1$ of the variables $(\tilde{\alpha}^t, \tilde{\lambda}^t, \tau^t)$. Thus as above using the implicit function theorem we have a unique $\tilde{\lambda}^t = \tilde{\alpha}^t(\tilde{\lambda}^t, \tau^t)$ solving (87) with $z^t \in V^t_{\text{deep}}(m, \epsilon, \eta)(A)$ for $t \approx 1^{-}$. Again as above, using Lemma 4.1 and Corollary 4.2 with $t_1$ replaced by $t$ combined with (75), we have

$$\left\langle \nabla J_t(z^t), \lambda_j^t \frac{\partial \varphi_{a^t_j, \lambda_j^t}}{\partial \lambda_j^t} \right\rangle = \left\langle \nabla J_t(\sum_{i=1}^{m} \alpha_i^t \varphi_{a^t_i, \lambda_j^t} + \sum_{r=1}^{\tilde{m}} \beta_r^t (\nu_r - (\nu_r)_{Q^t})), \lambda_j^t \frac{\partial \varphi_{a^t_j, \lambda_j^t}}{\partial \lambda_j^t} \right\rangle$$

$$+ O \left( \sum_{i=1}^{m} \left[ \frac{\partial \varphi}{\partial \lambda_j^t} \right]^{2} + \sum_{i=1}^{m} |\lambda_j^t| - 1 + \sum_{i=1}^{m} |\lambda_j^t| + \sum_{i=1}^{m} |\tau_j^t|^{2} \right),$$

with (88) holding in $C^1$ of the variables $(\tilde{\lambda}^t, \tau^t)$. Thus, using Lemma 3.2, Remark 3.7, and (88), we have

$$\left\langle \nabla J_t(z^t), \lambda_j^t \frac{\partial \varphi_{a^t_j, \lambda_j^t}}{\partial \lambda_j^t} \right\rangle = 0$$

is equivalent to

$$(89)$$

$$2(n+1-1)\omega_n (\alpha_j^t) \tau_j^t + O \left( \sum_{i=1}^{m} \left[ \frac{\partial \varphi}{\partial \lambda_j^t} \right]^{2} + \sum_{i=1}^{m} |\lambda_j^t| - 1 + \sum_{i=1}^{m} |\lambda_j^t| + \sum_{i=1}^{m} |\tau_j^t|^{2} \right)$$

with (90) holding in $C^1$ of the variables $(\tilde{\lambda}^t, \tau^t)$. Thus as above, using the implicit function theorem we have a unique $\tau^t = \tau^t(\lambda^t)$ solving (90) with $z^t \in V^t_{\text{deep}}(m, \epsilon, \eta)(A)$ for $t \approx 1^{-}$. Finally for $t \approx 1^{-1}$, we have $\forall i = 1, \cdots, m$ there exists a unique $\lambda_i^t$ such that

$$(t - 1)(F^t(a_i))^{n-2} = \frac{2m \epsilon(K(A), L(A))}{n - 2(\lambda_i^t)^2},$$
with \( e_{n,m}^K(A) \) as in Theorem 6.1. Hence

\[
z^t = \sum_{i=1}^{m} \tilde{z}_i^t \varphi_{\tilde{z}_i^t, \tilde{z}_i^t} + \sum_{r=1}^{m} \tilde{z}_r^t (v_r - (v_r)_Q) + \tilde{w}_t (\tilde{z}_r^t, \tilde{z}_r^t, \tilde{z}_r^t)
\]

with \( \tilde{z}_r^t = \tilde{x}_r^t, \tilde{z}_r^t = \tilde{x}_r^t (\tilde{x}_r^t, \tilde{x}_r^t) \), \( \tilde{z}_r^t = \tilde{x}_r^t (\tilde{x}_r^t, \tilde{x}_r^t) \), and \( \tilde{z}_r^t = \tilde{x}_r^t (\tilde{x}_r^t, \tilde{x}_r^t) \) verifies

\[
\nabla J_t (z^t) = 0.
\]

Thus \( u^t = z^t - \log \int_M e^{4z} dV_g \) satisfies (73) and (74) thereby ending the proof of the proposition. □

Our work [49] in the non-resonant case, Corollary 1.4 and Theorem 6.1 imply the following proposition important in the calculation of \( d_m \) as carried below.

**Proposition 6.4.** Let \((M, g)\) be a closed \( n \)-dimensional Riemannian manifold with \( n \geq 4 \) even such that \( \ker P^n_g \simeq \mathbb{R} \) and \( \kappa^n = (n - 1)! \omega_n \). Assuming that \( K \) is a smooth positive function on \( \mathcal{M} \) such that \((ND)\) holds. There exist \( \epsilon_{m,n} > 0, C_m > 0, C_t > C_{n,m} (t \simeq 1) \) with \( C_{n,m} \) and \( \epsilon_{m,n} \) depending only on \( m \) and \( n \), \( C_t \) continuous in \( t \) and \( \lim_{t \to 1} C_t = +\infty \) such that for every \( u \) solution of

\[
P^n_g u + tQ^n_g = tKe^{nu},
\]

with \( |(n - 1)\omega_n (t - 1)| < \epsilon_{m,n} \), we have

1) \( \|u\| < C_t \) \( \forall t \neq 1 \).

2) \( \|u\| \leq C_{m,n} \) \( for \) \( t = 1 \).

3) If \( t \neq 1 \), then we have

i) Either \( \|u\| < C_{m,n} \),

ii) Or \( \|u\| \geq C_{m,n} \) and \( u \in V_{\text{deep}}^t(M, \epsilon, \eta) := \bigcup_{A \in \text{Crit}(\kappa)} V_{\text{deep}}^t(M, \epsilon, \eta)(A) \).

**Proof.** 1) follows from the compactness result in [49] (see also [26, 41]), while 2) follows from Corollary 1.4 and 3) follows from Theorem 6.1. □

**Proof of Theorem 6.5.**
Let \( X := \{ u \in W^T(M) : \int_M K e^{nu} dV_h = 1 \} \) and \( T : X \to X \) be defined by

\[
T(u) = (P^n_g)^{-1} (\kappa^n_g K e^{nu} - Q^n_g), \quad u \in X.
\]

Then \( u \) is a solution of (77) is equivalent to \((I - T)u = 0\). On the other hand, Corollary 1.4 and Theorem 6.4 imply the Leray-Schauder degree of (77) \( d_m = d_m (K) \) is well-defined and verifies

\[
d_m = \text{deg} (I - T, B_{C_{n,m}}, 0).
\]

From the work of Malchiodi [43] in the non-resonant case, there exists \( L_0 > 0 \) such that for all \( L \geq L_0 \),

\[
1 - \chi (A_{m-1, m}) = \chi (J^L_t, J^{-L}_t), \quad \forall t \in (0, 1).
\]

From our work [54], up to taking \( L_0 \) larger we have

\[
\chi (J^L_t, J^{-L}_t) = \chi (J^L_t, J^{-L}_t), \quad \forall t \in (0, 1).
\]

Similarly to (77) and (91), for \( t \in (0, 1) \) we consider the equation

\[
P^n_g u + tQ^n_g = t\kappa^n_g K e^{nu},
\]

and the operator

\[
T_t (u) = (P^n_g)^{-1} (t\kappa^n_g K e^{nu} - tQ^n_g), \quad u \in X.
\]
Then Proposition 6.4 implies the Leray-Schauder degree of \( d_m \) is well-defined and is given by
\[
d_m = \deg(I - T, B_{C_1}, 0), \quad \forall t \in (0, 1).
\]
Furthermore, the work of Malchodi\[53\] implies
\[
d_m = 1 - \chi(A_{m-1}, \bar{m}), \quad \forall t \in (0, 1)
\]
Let us define
\[
d_m^- = \lim_{t \to 1^-} d_m^t.
\]
Then Theorem 1.1, Remark 6.2, and Theorem 6.4 imply
\[
d_m^- = \deg(I - T, B_{C_{n,m}}, 0) + \deg(I - T, V_{\text{deep}}^1(m, \epsilon, \eta), 0),
\]
where
\[
V_{\text{deep}}^1(m, \epsilon, \eta) = \bigcup_{A \in \mathcal{F}_\infty} V_{\text{deep}}^1(m, \epsilon, \eta)(A)
\]
On the other hand, our Morse lemma at infinity in \[54\] and Poncare-Hopf theorem imply
\[
\deg(I - T, V_{\text{deep}}^1(m, \epsilon, \eta), 0) = \frac{1}{m!} \sum_{A \in \mathcal{F}_\infty} (-1)^{i_\infty(A) + \bar{m}}
\]
Thus, we get
\[
d_m^- = d_m + \frac{1}{m!} \sum_{A \in \mathcal{F}_\infty} (-1)^{i_\infty(A) + \bar{m}}
\]
So, we obtain
\[
1 - \chi(A_{m-1}, \bar{m}) = \chi(J^L, J^{-L}) = d_m + \frac{1}{m!} \sum_{A \in \mathcal{F}_\infty} (-1)^{i_\infty(A)}
\]
this implies
\[
d_m = \chi(J^L, J^{-L}) - \frac{1}{m!} \sum_{A \in \mathcal{F}_\infty} (-1)^{i_\infty(A) + \bar{m}} = 1 - \chi(A_{m-1}, \bar{m}) - \frac{1}{m!} \sum_{A \in \mathcal{F}_\infty} (-1)^{i_\infty(A) + \bar{m}}
\]
Hence, recalling
\[
1 - \chi(A_{m-1}, \bar{m}) = (-1)^\bar{m} \text{ for } m = 1
\]
and
\[
1 - \chi(A_{m-1}, \bar{m}) = (-1)^\bar{m} \frac{1}{(m-1)!} \prod_{i=1}^{m-1} (i - \chi(M)), \text{ for } m \geq 2
\]
we have the result follows.

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