Riemann-Hilbert problem for Camassa-Holm equation with step-like initial data

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Abstract

The Cauchy problem for the Camassa–Holm equation with step-like initial conditions is reformulated as a Riemann–Hilbert problem. The the initial value problem solution is obtained then in a parametric form from the Riemann–Hilbert problem solution.

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1. Introduction

An inverse scattering approach, based on an appropriate Riemann–Hilbert problem formulation, is developed for the step-like initial value problem for the Camassa–Holm (CH) equation on the line, whose form is

\[ u_t - u_{txx} + 2\omega u_x + 3uu_x = 2u_xu_{xx} + uu_{xxx}, \quad -\infty < x < \infty, t > 0, \quad (1.1) \]

\[ u(x, 0) = u_0(x). \quad (1.2) \]

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Here \( u_0(x) \) is a step-like function, that is \( u_0(x) \to c_l \) as \( x \to -\infty \) and \( u_0(x) \to c_r \) as \( x \to +\infty \), where \( c_l, c_r \) are some real constants. We consider real-valued classical solutions \( u(x, t) \) of the CH equation (1.1), which rapidly tend to their limits as \( x \to \pm \infty \), that is for any \( T \geq 0 \)

\[
\max_{0 \leq t \leq T} \int_{-\infty}^{+\infty} (1 + |x|)^{l+1} \times \\
\times (|m(x,t) - c_l H(-x) - c_r H(x)| + |m_x(x,t)| + |m_{xx}(x,t)|) \, dx < \infty, \tag{1.3}
\]

where \( l \geq 0 \) is some integer and \( H(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0 \end{cases} \) is the Heaviside function.

The Camassa-Holm equation describes the unidirectional propagation of shallow water waves over a flat bottom (R. Camassa, D. Holm, J. Hyman, [8], [9]) as well as axially symmetric waves in a hyper-elastic rod (H. Dai [14]). Firstly it was found using the method of recursion operators as a bi-Hamiltonian equation with an infinite number of conserved functionals (A. Fokas, B. Fuchssteiner [15]).

For the case of vanishing initial data \( c_l = c_r = 0 \) the Riemann–Hilbert reformulation of CH equation and further asymptotic analysis was done by D.Shepelsky with coauthors [2], [4], [3], [7] by transforming Lax pair into suitable for asymptotical analysis form. Alternative approach based directly on the scattering theory for the underlying Sturm – Liouville operator was developed by G.Teschl and A.Kostenko with coauthors [6].

Here we derive approach based on [6], but the approach based on [2] can be developed as well. We suppose that there exists a classical solution of the Cauchy problem (1.1), (1.2), and that this solution satisfies the following condition for all values of time \( t \geq 0 \):

\[
\frac{m(x,t) + \omega}{c_r + \omega} > 0,
\]

where

\[
m(x,t) := u(x,t) - u_{xx}(x,t)
\]

is the so-called "momentum" variable. Also we suppose that \( \frac{c_l + \omega}{c_r + \omega} > 0 \).
The Camassa–Holm equation (1.1), quantity \( \frac{c_r + \omega}{c_r + \omega} \) and function \( \frac{m(x, t) + \omega}{c_r + \omega} \) are invariant (see [17], p.4) under the transformation

\[
(\omega, \ u(x, t)) \mapsto (\alpha \omega - \beta, \ v(x, t) = \alpha u(x - \beta t, \alpha t) + \beta),
\]
so it suffices to consider only the case \( c_r = 0, \ \frac{c}{\omega} \in (-1, +\infty) \). For definiteness we restrict ourselves to the case \( \frac{c}{\omega} > 0 \).

Our goal is to develop the inverse scattering approach to the CH equation with step-like initial data, in view of its further application for studying the long-time asymptotics. In section 2 we recall basic facts about Camassa–Holm equation, introduce Jost solutions, prove lemma 2.5 which guarantee boundedness of the right transmission coefficient at the edge of spectrum at the point \( \frac{i}{2} \sqrt{\frac{c}{c + \omega}} \). In sections 3 and 4 we state two Riemann–Hilbert problems, for the right and left spectral parameter, respectively. To deal with RH problem which is continuous up to conjugation contour, in section 3 we assume \( m(x, 0) \leq x \) for all \( x \). In section 4 to this end we suppose that \( l = 1 \) in the formula (1.3). In section 5 Camassa-Holm solution is reconstructed via RH problem solution. Two conservative laws for step-like solutions of CH equation are obtained ((2.8), (5.57)).

2. Lax pair for CH equation, Liouville transformation, Jost solutions and spectral functions

In this section we derive a vector Riemann – Hilbert problem directly from the scattering theory for the Sturm – Liouville operator with a step-like potential [10], [1]. We begin by recalling some required results for the Camassa – Holm equation [11], [12], [2], [6].

The starting point for our considerations is the Lax representation: the CH equation is the compatibility condition of two linear equations

\[
\frac{\omega}{m + \omega} \left( -\varphi''_{xx} + \frac{1}{4} \varphi \right) = \lambda \varphi, \quad (2.4a)
\]

\[
\varphi_t = -\left( \frac{\omega}{2\lambda} + u \right) \varphi'_x + \frac{u_x}{2} \varphi. \quad (2.4b)
\]
Assuming that $\frac{m+\omega}{\omega} \geq 0$, equation (1.1) can be equivalently written as

$$\left(\sqrt{\frac{m+\omega}{\omega}}\right)_t = - \left(u \sqrt{\frac{m+\omega}{\omega}}\right)_x. \quad (2.5)$$

Introduce Liouville transform

$$y = y(x, t) \equiv x - \int_{x}^{+\infty} \left(\sqrt{\frac{m(\bar{x}, t) + \omega}{\omega}} - 1\right) d\bar{x}, \quad (2.6)$$

$$\psi(y) = \sqrt{\frac{m + \omega}{\omega}} \varphi(x). \quad (2.7)$$

There is a conservative law

$$\rho = x \left(\sqrt{\frac{c + \omega}{\omega}} - 1\right) - c \sqrt{\frac{c + \omega}{\omega}} t + \int_{-\infty}^{x} \left(\sqrt{\frac{m(\xi, t) + \omega}{\omega}} - \sqrt{\frac{c + \omega}{\omega}}\right) d\xi + \int_{x}^{+\infty} \left(\sqrt{\frac{m(\xi, t) + \omega}{\omega}} - 1\right) d\xi \quad (2.8)$$

which can be checked directly. Due to the conservative law $\rho$, $y$ can be also expressed in the following way:

$$y = x - \int_{x}^{+\infty} \left(\sqrt{\frac{m + \omega}{\omega}} - 1\right) dr = \quad (2.9)$$

$$= -\rho + \sqrt{\frac{c + \omega}{\omega}} \left[x - ct + \int_{-\infty}^{x} \left(\sqrt{\frac{m + \omega}{c + \omega}} - 1\right) dr\right],$$

so $y \to \pm \infty$ as $x \to \pm \infty$ and vice versa.

Spectral problem (2.4a) now reads as follows:

$$-\psi_{yy}(y, t; k) + v(y, t) \psi(y, t; k) = k^2 \psi(y, t; k), \quad (2.10)$$
or, equivalently,
\[-\psi''_{yy}(y, t; k) + \left( v(y, t) + \frac{c}{4(c + \omega)} \right) \psi(y, t; k) = z^2 \psi(y, t; k). \tag{2.11} \]

Here
\[ \lambda =: k^2 + \frac{1}{4} =: z^2 + \frac{\omega}{4(c + \omega)}, \tag{2.12} \]

and
\[ v(y, t) = -\frac{m}{4(m + \omega)} + \frac{\omega}{4} \frac{m_{xx}}{(m + \omega)^2} - \frac{5\omega m_x^2}{16 (m + \omega)^3}, \tag{2.13} \]

so \( v(y, t) \to 0 \) as \( y \to +\infty \), and \( v(y, t) \to -\frac{c}{4(c + \omega)} \) as \( y \to -\infty \). From our assumption (1.3) we get that \( v(y, t) + \frac{c}{4(c + \omega)} \in L_1(\mathbb{R}_-, (1 + |y|)dy) \) and \( v(y, t) \in L_1(\mathbb{R}_+, (1 + |y|)dy) \). We will consider \( z \) as a function of \( k \),

\[ z = z(k) = \sqrt{k^2 + \frac{c}{4(c + \omega)}}, \]

where the branch of the cut across the segment \( (i \sqrt{\frac{c}{4(c + \omega)}}, -i \sqrt{\frac{c}{4(c + \omega)}}) \) is fixed by the condition \( z \sim k \) as \( k \to \infty \).

Spectral problems (2.10), (2.11) are well studied (see [13], [1], [5]), and all results known for them can be readily applied.

**Lemma 2.1.** There exist two Jost solutions \( \varphi_\pm(x, t; k) \) which solve the system of differential equations (2.4) and satisfy

\[ \lim_{x \to -\infty} e^{-i k x + \frac{\omega k x}{2 \lambda}} \varphi_+(x, t; k) = 1. \tag{2.14} \]

\[ \lim_{x \to +\infty} e^{-ikx + \frac{\omega k x}{2 \lambda}} \varphi_+(x, t; k) = 1. \tag{2.15} \]

Function \( \exp \left\{ -i \frac{t \sqrt{(c + \omega)\omega}}{2\lambda} \right\} \varphi_-(x, t; k) \) is analytic for \( \Im z(k) > 0 \) and continuous for \( \Im z(k) \geq 0 \),

function \( \exp \left\{ \frac{ikx + \omega k x}{2 \lambda} \right\} \varphi_+(x, t; k) \) is analytic for \( \Im k > 0 \) and continuous for
$\Im k \geq 0$. As $k \to \infty$, $\Im k \geq 0$, we have

$$\varphi_-(x, t; k) = \sqrt{\frac{\omega}{m + \omega}} \cdot$$

$$\cdot \exp \left\{ -iz \sqrt{\frac{c + \omega}{\omega}} \left( x + \int_{-\infty}^{x} \left( \sqrt{\frac{m + \omega}{c + \omega}} - 1 \right) d\tilde{x} - \left( c + \frac{\omega}{2\lambda} \right) t \right) \right\} \cdot$$

$$\cdot \left( 1 - \frac{1}{2ik} \int_{-\infty}^{y} \left( v(\tilde{y}, t) + \frac{c}{4(c + \omega)} \right) d\tilde{y} + O \left( k^{-2} \right) \right), \quad (2.16)$$

$$\varphi_+(x, t; k) = \sqrt{\frac{\omega}{m + \omega}} \exp \left\{ ik \left( x - \int_{x}^{+\infty} \left( \sqrt{\frac{m + \omega}{\omega}} - 1 \right) d\tilde{x} \right) - \frac{i\omega kt}{2\lambda} \right\}. \quad (2.17)$$

$$\cdot \left( 1 - \frac{1}{2ik} \int_{y}^{+\infty} v(\tilde{y}, t) d\tilde{y} + O \left( k^{-2} \right) \right), \quad (2.18)$$

as $k \to \infty$.

Moreover, the following relations are satisfied:

$$\varphi_{\pm}(x, t; k) = \varphi_{\pm}(x, t; -k),$$

$$\varphi_-(x, t; k \pm 0) = \varphi_-(x, t; k \mp 0), \quad k \in \left( i \sqrt{\frac{c}{4(c + \omega)}}, -i \sqrt{\frac{c}{4(c + \omega)}} \right);$$

$$\varphi_+(x, t; k) = \varphi_+(x, t; k), \quad k \in \left( i \sqrt{\frac{c}{4(c + \omega)}}, 0 \right);$$

$$\varphi_-(x, t; k \pm 0) = \varphi_-(x, t; k \mp 0), \quad k \in \left( i \sqrt{\frac{c}{4(c + \omega)}}, -i \sqrt{\frac{c}{4(c + \omega)}} \right).$$

Proof. This is straightforward from the corresponding results for the spectral problem (2.10) (see [10]) by virtue of the Liouville transform (2.6), (2.7). Just notice

$$\varphi_-(x, t; k) = \sqrt{\frac{\omega}{m + \omega}} \exp \left\{ -i\varphi z + \frac{i\sqrt{(c + \omega)\omega}}{2\lambda} zt \right\} \psi_-(y, t; k), \quad (2.19)$$

$$\varphi_+(x, t; k) = \sqrt{\frac{\omega}{m + \omega}} e^{-\frac{\omega k t}{2\lambda}} \psi_+(y, t; k),$$
where \( \varphi \) is a conserved quantity of CH equation (2.8) and \( \psi^\pm_K(y, t; k) \) are the Jost solutions of (2.10) determined by their asymptotics

\[
e^{+i\gamma} \psi^+_K(y, t; k) \to 1 \text{ as } y \to -\infty, \quad \Re z \geq 0
e^{-i\gamma} \psi^-_K(y, t; k) \to 1 \text{ as } y \to +\infty, \quad \Re k \geq 0.
\] (2.20)

Lemma 2.2. Jost solutions \( \varphi^\pm(x, t; k) \) determined by (2.14), (2.15) satisfy \( t \)-equation (2.4b).

Proof. We follow the well-known idea from [16], chapter 4, paragraph 2. First the statement is to be proved for real \( k \) and then can be extended by analyticity to \( \Re k \geq 0, \Re z \geq 0 \), respectively. Let \( \varphi(x, t; k) \) be one of the functions \( \varphi^\pm(x, t; k) \). It is straightforward that if \( \varphi(x, t; k) \) satisfy (1.1) and (2.4a), then

\[
\left( \varphi_t + \left( \frac{\omega}{2\lambda} + u \right) \varphi_x - \frac{u_x}{2} \varphi \right)_x = \left( \frac{1}{4} - \frac{\lambda(m + \omega)}{\omega} \right) \left( \varphi_t + \left( \frac{\omega}{2\lambda} + u \right) \varphi_x - \frac{u_x}{2} \varphi \right),
\]

i.e. \( \varphi_t + \left( \frac{\omega}{2\lambda} + u \right) \varphi_x - \frac{u_x}{2} \varphi \) satisfies the same differential equation (2.4a) as \( \varphi \). If we define by \( \hat{\varphi} \) another solution of (2.4a) such that \( \varphi \) and \( \hat{\varphi} \) are linear independent, then there exist independent on \( x \) functions \( A(t; k) \), \( B(t; k) \) such that

\[
\varphi_t(x, t; k) + \left( \frac{\omega}{2\lambda} + u(x, t) \right) \varphi_x(x, t; k) - \frac{u_x(x, t)}{2} \varphi(x, t; k) = A(t; k)\varphi(x, t; k) + B(t; k)\hat{\varphi}(x, t; k).
\]

For \( \psi(y, t; k) = \sqrt{\frac{m + \omega}{\omega}} \varphi(x, t; k), \quad \hat{\psi}(y, t; k) = \sqrt{\frac{m + \omega}{\omega}} \hat{\varphi}(x, t; k) \)
the last equation reads as

\[
\psi_t + \frac{\omega}{2\lambda} \sqrt{\frac{m + \omega}{\omega}} \psi_y = \left( \frac{um_x}{4(m + \omega)} + \frac{mt}{4(m + \omega)} + \frac{u_x}{2} + \frac{\omega m_x}{8\lambda(m + \omega)} \right) \psi = A\psi + B\hat{\psi}. \] (2.21)
a) Now let us take \( \varphi(x, t; k) = \varphi_+(x, t; k) \) and \( \hat{\varphi}(x, t; k) = \overline{\varphi_+(x, t; k)} \). Function \( \psi(x, t; k) \) defined by (2.7) has the following asymptotic behavior as \( x \to +\infty \) for \( \Im k = 0 \) :

\[
e^{-iky + \frac{2i\sqrt{c + \omega} t z(k)}{4k^2 + 1}} \psi_+(y, t; k) \to 1, \quad y \to +\infty, \quad \Im k = 0,
\]

\[
e^{-iky + \frac{2i\sqrt{c + \omega} t z(k)}{4k^2 + 1}} \psi'_+(y, t; k) \to ik, \quad y \to +\infty, \quad \Im k = 0,
\]

\[
e^{-iky + \frac{2i\sqrt{c + \omega} t z(k)}{4k^2 + 1}} \psi'_-(y, t; k) \to -i k, \quad y \to +\infty, \quad \Im k = 0.
\]

Substituting these asymptotics into (2.21) and taking the limit as \( y \to +\infty \) we get 0 in the left-hand-side, which is possible only if \( A(t; k) \equiv 0, B(t; k) \equiv 0 \).

b) Now let us take \( \varphi(x, t; k) = \varphi_-(x, t; k) \) and \( \hat{\varphi}(x, t; k) = \overline{\varphi_-(x, t; k)} \). Function \( \psi(y, t; k) \) defined (2.7) has the following asymptotic behavior as \( x \to +\infty \) for \( \Im k = 0 \) :

\[
e^{i(y + \varphi)z(k) - \frac{2i\sqrt{c + \omega} t z(k)}{4k^2 + 1}} \psi_-(y, t; k) \to 1, \quad y \to -\infty, \quad \Im z(k) = 0.
\]

\[
e^{i(y + \varphi)z(k) - \frac{2i\sqrt{c + \omega} t z(k)}{4k^2 + 1}} \psi'_-(y, t; k) \to -iz(k), \quad y \to -\infty, \quad \Im z(k) = 0.
\]

\[
e^{i(y + \varphi)z(k) - \frac{2i\sqrt{c + \omega} t z(k)}{4k^2 + 1}} \psi'_-(t, y; k) \to \frac{2i\sqrt{c + \omega} \omega z(k)}{4k^2 + 1}, \quad y \to -\infty,
\]

\[\quad \Im z(k) = 0.
\]

Substituting these asymptotics into (2.21) and taking the limit as \( y \to -\infty \) we get 0 in the left-hand-side, which is possible only if \( A(t; k) \equiv 0, B(t; k) \equiv 0 \).

\[\square\]

Next, we have the scattering relations

\[
\varphi_-(x, t; k) = a_+(k) \varphi_+(x, t; k) + b_+(k) \varphi_+(x, t; k), \quad k \in \mathbb{R} \setminus \{k = 0\}, \tag{2.22a}
\]

\[
\varphi_+(x, t; k) = a_-(k) \varphi_-(x, t; k) + b_-(k) \varphi_-(x, t; k), \quad k \in \mathbb{R}^c, \tag{2.22b}
\]

8
They possess the following properties:

Lemma 2.3. Spectral functions $a_\pm(k)$, $b_\pm(k)$ have extended domains of definition and are expressed in terms of $\varphi_\pm(x, t; k)$ as follows:

1. $W \{\varphi_-(x, t; k), \varphi_+(x, t; k)\} = 2ik a_+(k)$, $\Im z(k) \geq 0$;
2. $W \{\varphi_+(x, t; k), \varphi_-(x, t; k)\} = 2ik b_+(k)$, $z(k) \in \mathbb{R} \cap \Im k \leq 0$;
3. $W \{\varphi_-(x, t; k), \varphi_+(x, t; k)\} = 2iz(k) a_-(k)$, $\Im z(k) \geq 0$;
4. $W \{\varphi_+(x, t; k), \varphi_-(x, t; k)\} = 2iz(k) b_-(k)$, $z(k) \in \mathbb{R} \cap \Im k \geq 0$.

They possess the following properties:

I. $a_+(-k) = a_+(k)$, $\Im z(k) \geq 0$; $\overline{b_+(k)} = b_+(k)$, $z(k) \in \mathbb{R} \cap \Im k \leq 0$;
II. $a_-(k) = b_-(k)$, $k \in \mathbb{C}^*$; $\overline{b_-(k)} = b_-(k)$, $z(k) \in \mathbb{R} \cap \Im k \geq 0$;
III. $k a_+(k) = \sqrt{z(k) a_-(k)}$, $\Im z(k) \geq 0$; $\overline{k b_+(k)} = -z(k) b_-(k)$, $z(k) \in \mathbb{R} \cap \Im k \geq 0$;
IV. $b_+(k \pm 0) = a_+(k \mp 0)$, $b_-(k \pm 0) = a_-(k \mp 0)$, $b_-(k \pm 0) = a_-(k \pm 0)$, $b_+(k \mp 0) = a_+(k \pm 0)$, $k \in \left[0, \sqrt{\frac{c}{4(c + \om)}}\right]$;
V. $a_+(k) b_-(k) + a_-(k) b_+(k) = 0$, $z(k) \in \mathbb{R} \cap \Im k \geq 0$;

Lemma 2.4. Function $W(k)$ possesses the following properties:

- $W(k)$ is analytic in $\Im z(k) > 0$ and continuous in $\Im z(k) \geq 0$;
• $W(k) \neq 0$ for any $k \in \left\{ k : z(k) \in \mathbb{R} \land \Im k \geq 0 \land k \neq i\sqrt{c/4(c+\omega)} \right\}$.

• Zeros of $W(k)$ in the domain $\Im z(k) > 0$ are simple and lie in the interval $(i\sqrt{c/4(c+\omega)}, \frac{i}{2})$.

Proof. It is sufficient to prove only the third property, as first two are immediate from corresponding results from [13]. The fact that the discrete spectrum may lie only in the interval $(i\sqrt{c/4(c+\omega)}, +i\infty)$, also follows from [13]. The fact that it indeed can lie only in the interval $(i\sqrt{c/4(c+\omega)}, i2)$, can be proved as in [11, Claim 2, p. 962].

The following lemma establishes a condition on an initial data which provides $W\left(i\sqrt{c/4(c+\omega)}\right) \neq 0$.

**Lemma 2.5.** Suppose that for all $x \in \mathbb{R}$

\[
\frac{m(x,0) - c}{c + \omega} \leq 0.
\]

Then $W\left(i\sqrt{c/4(c+\omega)}\right) \neq 0$.

Proof. Define $k_0 = i\sqrt{c/4(c+\omega)}$, $\lambda_0 = \lambda(k_0)$, and suppose that $W(k_0) = 0$. In that case the Jost solutions $\varphi_-(x,t; k_0)$ and $\varphi_+(x,t; k_0)$ are linearly dependent, that is there exists a constant $C \in \mathbb{C} \setminus \{0\}$ such that $\varphi_-(x,t; k_0) = C\varphi_+(x,t; k_0)$. From (2.14), (2.15) we conclude that function $\varphi(x,t) := \exp\left\{-iz\sqrt{c/4} \cdot (c + \frac{\omega}{2x}) t\right\} \varphi_-(x,t; k_0)$ possesses the following properties (notice that $z(k_0) = 0$):

\[
\sqrt{\frac{m + \omega}{\omega}} \varphi(x,t) \to 1, \quad \varphi(x,t) \to 0, \quad \text{as} \quad x \to -\infty,
\]

\[
\varphi(x,t) \to 0, \quad \varphi_x(x,t) \to 0, \quad \text{as} \quad x \to +\infty.
\]

Equation (2.4a) can be rewritten as follows:

\[
-m_x(x,t) + \frac{1}{4}\varphi(x,t) = \lambda_0 \frac{m + \omega}{\omega} \varphi(x,t),
\]
where $\lambda_0 = k_0^2 + \frac{1}{4} = \frac{\omega}{4(c + \omega)}$. Multiply the last equation by $\psi$ and integrate from $R$ to $+\infty$:

$$
\int_R^{+\infty} -\varphi_{xx}(x, t)\psi(x, t)dx + \frac{1}{4} \int_R^{+\infty} |\varphi(x, t)|^2dx = \lambda_0 \int_R^{+\infty} \frac{m + \omega}{\omega} |\varphi(x, t)|^2dx.
$$

By integrating by parts and transferring of a summand to the right side of the equation, we get

$$
-\varphi_x \psi \bigg|_R^{+\infty} + \int_R^{+\infty} |\varphi_x(x, t)|^2dx = \int_R^{+\infty} \frac{m - c}{4(c + \omega)} |\varphi(x, t)|^2dx.
$$

Here we can take the limit as $R \to -\infty$, and so come to the equation

$$
\int_{-\infty}^{+\infty} |\varphi_x(x, t)|^2dx = \int_{-\infty}^{+\infty} \frac{m(x, t) - c}{4(c + \omega)} |\varphi(x, t)|^2dx.
$$

As the left-hand side is strictly positive, and the right-hand side is non-positive for $t = 0$, we come to contradiction with the statement of the lemma. So, $W(k_0) \neq 0$.

**Lemma 2.6.** The transmission coefficients $a_{\pm}^{-1}$ are meromorphic for $\Im z(k) > 0$ with simple poles at $i\kappa_1, ..., i\kappa_N$, $\sqrt{c/4(c + \omega)} < \kappa_N < ... < \kappa_1 < \frac{1}{2}$, and continuous for $k \in \{k : \Im z(k) \geq 0 \land \Im k \geq 0 \land z(k) \neq 0\}$. Moreover, if

1. we take $l = 0$ in (1.3) and require that $\forall x \in \mathbb{R} : c > m(x, 0)$,
   then $a_{+}^{-1}$ is continuous up to the point $z(k) = 0$;
2. we take $l = 1$ in (1.3), then $a_{-}^{-1}$ is continuous up to the point $z(k) = 0$.

Asymptotically as $k \to \infty$ we have

$$
a_{\pm}^{-1}(k) = e^{i\nu k} \left(1 + O(k^{-1})\right). \quad (2.24)
$$

The residues of $a_{+}^{-1}(k)$ and $a_{-}^{-1}(k(z))$ are given by

$$
\text{Res}_{i\kappa_j} \frac{1}{a_+(k)} = i\mu_j^2 \gamma_+^{2-j} = \text{Res}_{z(i\kappa_j)} \frac{1}{a_-(k(z))} = i\mu_j^{-1} \gamma_-^{2-j}. \quad (2.25)
$$
where
\[
\gamma_{\pm,j}^{-2} := \int_{-\infty}^{+\infty} (\varphi_{\pm}(x, t; i \kappa_j))^2 \frac{m + \omega}{\omega} dx > 0
\] (2.26)

and \(\varphi_{\pm}(x, t; i \kappa_j) = \mu_j \varphi_{\pm}(x, t; i \kappa_j)\) with quantities \(\gamma_{\pm,j}, \mu_j\) independent on \(t\).

Note that if \(a_{\pm}^K(k, t), b_{\pm}^K(k, t), \gamma_{\pm,j}^K(t), \mu_j^K(t),\)

\[W^K(t; k) \equiv W \{\psi_{\pm}^K(y, t; k), \psi_{\pm}^K(y, t; k)\} \equiv \psi_{\pm}^K \cdot \partial\psi_{\pm}^K - \psi_{\pm}^K \cdot \partial\psi_{\pm}^K\]

are the corresponding quantities for (2.10), then

\[a_{\pm}(k) = a_{\pm}^K(k, t) \exp \left\{ -i \varphi z + \frac{i \sqrt{(c + \omega)\omega}}{2\lambda} \frac{zt}{\lambda} - \frac{i \omega kt}{2\lambda} \right\},\]

\[b_{\pm}(k) = b_{\pm}^K(k, t) \exp \left\{ \pm \left( -i \varphi z + \frac{i \sqrt{(c + \omega)\omega}}{2\lambda} \frac{zt}{\lambda} + \frac{i \omega kt}{2\lambda} \right) \right\},\]

\[\gamma_{+,j} = \gamma_{+,j}^K(t) \exp \left\{ \frac{i \omega k_j t}{2\lambda_j} \right\},\]

\[\gamma_{-,j} = \gamma_{-,j}^K(t) \exp \left\{ i \varphi z_j - \frac{i \sqrt{(c + \omega)\omega}}{2\lambda_j} \frac{z_j t}{\lambda_j} \right\},\]

\[\mu_j = \mu_j^K(t) \exp \left\{ i \varphi z_j - \frac{i \sqrt{(c + \omega)\omega}}{2\lambda_j} \frac{z_j t}{\lambda_j} - \frac{i \omega k_j t}{2\lambda_j} \right\}\]

with \(\lambda_j := \lambda(k_j), z_j := z(k_j),\)

\[W(k) = \exp \left\{ -i \varphi z + \frac{i \sqrt{(c + \omega)\omega}}{2\lambda} \frac{zt}{\lambda} - \frac{i \omega kt}{2\lambda} \right\} W^K(t; k)\]

and hence all results known for (2.10) are easily applied in our situation. Particularly, as

\[\frac{i \sqrt{(c + \omega)\omega}}{2\lambda} \frac{zt}{\lambda} - \frac{i \omega kt}{2\lambda} = \frac{itc}{2 \left( z \sqrt{\frac{k}{2} + 1} + k \right)},\]

then \(a_{\pm}(k)\) are regular at the point \(k = \frac{i}{2}\).
3. Vector Riemann – Hilbert problem (1).

Suppose that the condition of lemma 2.5 is satisfied. We define a vector Riemann – Hilbert problem as follows: sectionally meromorphic function $M(x, t; k) = (M_1(x, t; k), M_2(x, t; k))$ is defined by

$$
\begin{align*}
&\left\{ \begin{array}{ll}
\sqrt{m+\omega\omega}\left(\frac{1}{a_+(k)}\varphi_-(x, t; k)e^{ig(y,t;k)}, \varphi_+(x, t; k)e^{-ig(y,t;k)}\right), & \Im z(k) > 0, \\
\sqrt{m+\omega\omega}\left(\varphi_+(x, t; k)e^{ig(y,t;k)}, \frac{1}{a_+(k)}\varphi_-(x, t; k)e^{-ig(y,t;k)}\right), & \Im z(k) < 0,
\end{array} \right.
\end{align*}
$$

(3.27)

where $g(y, t; k) = ky - \frac{2\omega kt}{4k^2 + 1}$.

We are interested in the jump relations of $M(x, t; k)$ on the contour $\Sigma = \mathbb{R} \cup \left[i\sqrt{\frac{c}{4(c+\omega)}}, -i\sqrt{\frac{c}{4(c+\omega)}}\right]$. The orientation of the contour is chosen as follows: from $-\infty$ to $+\infty$ and from $+i\sqrt{\frac{c}{4(c+\omega)}}$ to $-i\sqrt{\frac{c}{4(c+\omega)}}$. Positive side of the contour is on the left, negative is on the right. By $M_{\pm}(x, t; k)$ we denote the limit from the positive/negative side of the contour.

The scattering relations (2.22) and lemma 2.6 provides the following properties of function $M(x, t; k)$:

1. The analyticity:
   $M(x, t, \cdot)$ is meromorphic in $\mathbb{C} \setminus \Sigma$ with simple poles at $\pm i\kappa_j$ and continuous up to the boundary;
2. The jump relations: for $k \in \Sigma$
   $M_-(x, t; k) = M_+(x, t; k)J(x, t; k)$, where

   $$J(x, t; k) = \left( \begin{array}{cc}
   1 & \frac{b_+(k)}{a_+(k)} e^{-2ig(y,t;k)} \\
   \frac{b_+(k)}{a_+(k)} e^{2ig(y,t;k)} & \frac{z(k)}{k|a_+(k)|^2}
   \end{array} \right), \ k \in \mathbb{R} \setminus \{0\}; \ (3.28)$$

   $$J(x, t; k) = \left( \begin{array}{cc}
   1 & 0 \\
   \frac{z(k+0)}{ka_+(k-0)a_+(k+0)} e^{2ig(y,t;k)} & 1
   \end{array} \right), \ k \in \left(i\sqrt{\frac{c}{4(c+\omega)}}, 0\right); \ (3.29)$$
\[
J(x, t; k) = \begin{pmatrix}
1 & \frac{z(k + 0)}{k a_+(k - 0) a_+(k + 0)} e^{-2i\eta(y, t; k)} \\
0 & \frac{1}{1}
\end{pmatrix}, \quad k \in \left(0, -i \sqrt{\frac{c}{4(c + \omega)}}\right);
\]  
(3.30)

3. The pole relation: for \( j = 1, \ldots, N \)

\[
\text{Res}_{i\kappa_j} M(x, t; k) = \lim_{k \to i\kappa_j} M(x, t; k) \begin{pmatrix} 0 & 0 \\ i\gamma^2_{+j} e^{2i\eta(y, t; i\kappa_j)} & 0 \end{pmatrix}, \quad (3.31a)
\]

\[
\text{Res}_{-i\kappa_j} M(x, t; k) = \lim_{k \to -i\kappa_j} M(x, t; k) \begin{pmatrix} 0 & -i\gamma^2_{+j} e^{2i\eta(y, t; i\kappa_j)} \\ 0 & 0 \end{pmatrix}; \quad (3.31b)
\]

4. Symmetry relations:

\[
\overline{M(x, t;k)} = M(x, t; -k) = M(x, t; k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (3.32)
\]

\[
\overline{M(x, t; -k)} = M(x, t; k);
\]

5. \( M(x, t; k) \to (1 \ 1) \) as \( k \to \infty \).

**Regular Riemann–Hilbert problem.** It is useful to transform our meromorphic RH problem defined in section 3 to holomorphic RH problem. In order to achieve this, we define function \( \tilde{M}(x, t; k) \) as follows:

\[
\tilde{M}(x, t; k) = \begin{cases}
M(x, t; k) \begin{pmatrix} 1 & 0 \\ -i\gamma^2_{+j} e^{2i\eta(y, t; i\kappa_j)} \\ -i\kappa_j & 1 \end{pmatrix}, & |k - i\kappa_j| < \varepsilon, \\
M(x, t; k) \begin{pmatrix} 0 & 1 \\ i\gamma^2_{+j} e^{2i\eta(y, t; i\kappa_j)} \\ k + i\kappa_j & 1 \end{pmatrix}, & |k + i\kappa_j| < \varepsilon, \\
M(x, t; k), & \text{elsewhere},
\end{cases}
\]

where \( \varepsilon > 0 \) is sufficiently small number such that circles \( |k - i\kappa_j| \) do not intersect and lie in the domain \( \Im z(k) > 0 \).
Lemma 3.1. Vector-valued function $\hat{M}(x,t;k)$ solves the following RH problem: find a sectionally-holomorphic function $\hat{M}(x,t;k)$ which satisfies the following:

1. $\hat{M}(x,t;k)$ is holomorphic away of the contour $\Sigma$ and circuits $C_j := \{ k : |k - ik_j| = \epsilon \}$, $\overline{C}_j := \{ k : |k + ik_j| = \epsilon \}$. The orientation on $C_j$, $\overline{C}_j$ is counterclockwise, so the positive side is inside the circuits.

2. Jump condition $\hat{M}_-(x,t;k) = \hat{M}_+(x,t;k)\hat{J}(x,t;k)$ is satisfied, where

$$
\hat{J}(x,t;k) \equiv J(x,t;k), \quad k \in \Sigma,
$$

$$
\hat{J}(x,t;k) = \begin{pmatrix}
\frac{1}{k - ik_j} e^{2i\gamma(y,t;ik_j)} & 0 \\
0 & 1
\end{pmatrix} \quad k \in C_j; \quad (3.34)
$$

$$
\hat{J}(x,t;k) = \begin{pmatrix}
1 & -i\gamma^2_{+,j} e^{2i\gamma(y,t;ik_j)} \\
0 & \frac{k + ik_j}{1}
\end{pmatrix} \quad k \in \overline{C}_j; \quad (3.35)
$$

3. Symmetry relations:

$$
\overline{\hat{M}(x,t;k)} = \hat{M}(x,t;-k) = \hat{M}(x,t;k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (3.36)
$$

$$
\overline{\hat{M}(x,t;-k)} = \hat{M}(x,t;k);
$$

4. $\hat{M}(x,t;k) \to (1 \ 1)$ as $k \to \infty$.

4. Riemann–Hilbert problem (2)

Alternative Riemann-Hilbert problem can be stated for spectral parameter $z$ in assumption that $l = 1$ in (1.3). Let us define sectionally-meromorphic function

$$
\tilde{M}(x,t;z) =
$$
\[
\begin{align*}
\left\{ \begin{array}{ll}
\sqrt{\frac{m+\omega}{\omega}} \left( \frac{1}{a_-(k(z))} \varphi_+(x, t; k(z)) e^{-i\tilde{g}(y, t; z)}, \varphi_-(x, t; k(z)) e^{i\tilde{g}(y, t; z)} \right), \\
\quad \Im z > 0, \\
\sqrt{\frac{m+\omega}{\omega}} \left( \frac{\varphi_-(x, t; k(z)) e^{-i\tilde{g}(y, t; z)}}{a_-(k(z))}, \frac{1}{a_-(k(z))} \varphi_+(x, t; k(z)) e^{i\tilde{g}(y, t; z)} \right), \\
\quad \Im z < 0,
\end{array} \right.
\end{align*}
\]

where \( \tilde{g}(y, t; z) = z(y + \varphi) - \frac{2\sqrt{(c+\omega)\omega}}{4z^2 + c}\).

Scattering relations (2.22) and lemma 2.6 implies that \( \tilde{M}(x, t; z) \) satisfies the following conjugation problem:

1. \( \tilde{M}(x, t, \cdot) \) is analytic in \( \mathbb{C} \setminus \mathbb{R} \) and continuous up to the boundary;
2. \( \tilde{M}_-(x, t; z) = \tilde{M}_+(x, t; z) \tilde{J}(x, t; z) \), where

\[
\tilde{J}(x, t; z) = \begin{pmatrix}
1 & \frac{b_-(k(z))}{a_-(k(z))} e^{2i\tilde{g}(y, t; z)} \\
\frac{b_-(k(z))}{a_-(k(z))} e^{-2i\tilde{g}(y, t; z)} & \frac{b_-(k(z))}{a_-(k(z))} \end{pmatrix},
\]

\[
z \in \left( -\infty, -\sqrt{-\frac{c}{4(c+\omega)}} \right) \cup \left( \sqrt{-\frac{c}{4(c+\omega)}}, +\infty \right); \quad (4.38)
\]

\[
\tilde{J}(x, t; z) = \begin{pmatrix}
1 & \frac{b_-(k(z+ i0))}{a_-(k(z+ i0))} e^{-2i\tilde{g}(y, t; z)} \\
\frac{b_-(k(z+ i0))}{a_-(k(z+ i0))} e^{2i\tilde{g}(y, t; z)} & 0
\end{pmatrix},
\]

\[
z \in \left( -\sqrt{-\frac{c}{4(c+\omega)}}, \sqrt{-\frac{c}{4(c+\omega)}} \right); \quad (4.39)
\]
3. The pole relation: for \( j = 1, \ldots, N \)

\[
\text{Res}_{iz_j} \tilde{M}(x, t; k) = \lim_{z \to z_j} \tilde{M}(x, t; z) \begin{pmatrix} 0 & 0 \\ i\gamma_{-j}^2 e^{-2\bar{g}(y,t;z_j)} & 0 \end{pmatrix},
\]

(4.40a)

\[
\text{Res}_{-iz_j} \tilde{M}(x, t; z) = \lim_{z \to -z_j} \tilde{M}(x, t; z) \begin{pmatrix} 0 & -i\gamma_{-j}^2 e^{-2\bar{g}(y,t;z_j)} \\ 0 & 0 \end{pmatrix} ;
\]

(4.40b)

4. Symmetry relations:

\[
\tilde{M}(x, t; z) = \tilde{M}(x, t; -z) = \tilde{M}(x, t; z) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

(4.41)

5. Reconstruction of the Camassa–Holm solution from RH problem

The following lemma reads exactly as its analogue in vanishing case ([6, lemma 3.5]), but proof should be modified.

**Lemma 5.1.** Functions \( M(x, t; k) \) and \( \tilde{M}(x, t; z) \) defined in (3.27) and (4.37), respectively, satisfy the following relations:

\[
\frac{M_1(x, t; \frac{i}{2})}{M_2(x, t; \frac{i}{2})} = e^{x-y},
\]

(5.42)

\[
M_1(x, t; k)M_2(x, t; k) = \sqrt{\frac{m+\omega}{\omega}} \left( 1 + \frac{2i}{\omega} u(x, t) \left( k - \frac{i}{2} \right) + O \left( k - \frac{i}{2} \right)^2 \right),
\]

(5.43)

\[
k \to \frac{i}{2},
\]

and

\[
\frac{\tilde{M}_1(x, t; \frac{i}{2} \sqrt{\frac{\omega}{c+\omega}})}{\tilde{M}_2(x, t; \frac{i}{2} \sqrt{\frac{\omega}{c+\omega}})} = e^{\sqrt{\frac{\omega}{c+\omega}}(y+ct) - x + ct} = \exp \left\{ \int_{-\infty}^{x} \left( \sqrt{\frac{m+\omega}{c+\omega}} - 1 \right) dr \right\},
\]

17
\[
\tilde{M}_1(x, t; z)\tilde{M}_2(x, t; z) = \\
\sqrt{\frac{m + \omega}{c + \omega}} \left( 1 + \frac{2i (u(x, t) - c)}{\sqrt{\omega(c + \omega)}} \left( z - \frac{i}{2} \sqrt{\frac{\omega}{c + \omega}} \right) + O \left( z - \frac{i}{2} \sqrt{\frac{\omega}{c + \omega}} \right)^2 \right), \\
(5.45)
\]
\[
z \to \frac{i}{2} \sqrt{\frac{\omega}{c + \omega}}.
\]

**Proof** The Jost solutions \( \varphi_\pm(x, t; k) \) can be represented as follows:

\[
\varphi_+(x, t; k) = \mu_+(x, t; k) \exp \left\{ i k x - \frac{2i\omega k t}{4k^2 + 1} \right\},
\]
\[
\varphi_-(x, t; k) = \sqrt{\frac{\omega}{c + \omega}} \mu_-(x, t; k) \exp \left\{ -i \sqrt{\frac{c + \omega}{\omega}} x z + i \sqrt{\frac{c + \omega}{\omega}} t z \left( c + \frac{2\omega}{4k^2 + 1} \right) \right\},
\]
where \( \mu_\pm(x, t; k) \) are the solutions of the integral equations

\[
\mu_+(x, t; k) = 1 + \frac{k^2 + \frac{1}{4}}{2i k \omega} \int_x^{+\infty} \left( 1 - e^{-2i k (x - r)} \right) \mu_+(r, t; k) m(r, t) dr, \\
(5.46)
\]
\[
\mu_-(x, t; k) = 1 + \frac{k^2 + \frac{1}{4}}{2iz \sqrt{\omega(c + \omega)}} \int_{-\infty}^x \left( 1 - e^{2iz(x - r)\sqrt{\frac{\omega}{c + \omega}}} \right) \mu_-(r, t; k) (m(r, t) - c) dr.
(5.47)
\]

Here the values of \( \mu_\pm \) at the point \( k = \frac{i}{2} \) can be determined due to the observation that \((2.4a)\) becomes explicitly solvable at \( \lambda = 0 \).

Existence and uniqueness of solutions of \((5.47), (5.46)\) is established, for example, in [16]. Moreover, \( \mu_+ \) is analytic for \( \Im k > 0, \) \( \mu_- \) is analytic for \( \Im z(k) > 0. \)
Since \( k^2 + \frac{1}{4} = (k - \frac{i}{2})(k + \frac{i}{2}) \), we get

\[
\mu_+(x, t; k) = 1 + \frac{i}{\omega} F_+(x, t) \left( k - \frac{i}{2} \right) + O \left( k - \frac{i}{2} \right)^2, \quad k \to \frac{i}{2}, \tag{5.48}
\]

\[
\mu_-(x, t; k) = 1 - \frac{i}{\omega} F_-(x, t) \left( k - \frac{i}{2} \right) + O \left( k - \frac{i}{2} \right)^2, \quad k \to \frac{i}{2}, \tag{5.49}
\]

where

\[
F_+(x, t) = \int_{-\infty}^{+\infty} (e^{x-r} - 1) m(r, t) dr, \tag{5.50}
\]

\[
F_-(x, t) = \int_{-\infty}^{x} (e^{-x+r} - 1) (c - m(r, t)) dr. \tag{5.51}
\]

Differentiating with respect to \( x \), we get

\[
\mu'_+(x, t; k) = \frac{i}{\omega} F'_+(x, t) \left( k - \frac{i}{2} \right) + O \left( k - \frac{i}{2} \right)^2, \quad k \to \frac{i}{2}, \tag{5.52}
\]

\[
\mu'_-(x, t; k) = -\frac{i}{\omega} F'_-(x, t) \left( k - \frac{i}{2} \right) + O \left( k - \frac{i}{2} \right)^2, \quad k \to \frac{i}{2}, \tag{5.53}
\]

with

\[
F'_+(x, t) = \int_{-\infty}^{+\infty} e^{x-r} m(r, t) dr, \tag{5.54}
\]

\[
F'_-(x, t) = -\int_{-\infty}^{x} e^{-x+r} (c - m(r, t)) dr. \tag{5.55}
\]

Next, straightforward calculations show that

\[
a_+(k) = W\{\varphi_-(x, t; k), \varphi_+(x, t; k)\} = \frac{4\sqrt{\frac{\omega}{c + \omega}}}{2ik} \left( 1 + \right)
\]

19
\[ + \frac{i}{\omega} \left[ F_+ - F_- - F'_+ - F'_- - c(x + 1) + ct \left( \frac{3c}{2} + 2\omega \right) \right] \left( k - \frac{i}{2} \right) + O \left( \frac{k}{2} \right)^2 = \]

\[ = \frac{\sqrt{\omega}}{c + \omega} \left\{ 1 + \frac{i}{\omega} H_0[u] \left( k - \frac{i}{2} \right) + O \left( \frac{k}{2} \right)^2 \right\}, \quad (5.56) \]

where

\[ H_0[u] = \int_{-\infty}^{x} (c - m(r, t)) \, dr - \int_{x}^{\infty} m(r, t) \, dr - c(x + 1) + \frac{c(3c + 4\omega)t}{2} \quad (5.57) \]

is a conserved quantity of the CH equation. Substituting (5.49), (5.48) and (5.56) into (3.27), and taking into account

\[ u(x, t) = (1 - \partial_x^2)^{-1} m(x, t) = \frac{1}{2} \int e^{-|x-r|} m(r, t) \, dr, \]

we come to

\[ M_1(x, t; k) = \frac{\sqrt{m + \omega}}{\omega} e^{(x-y)/2} \left( 1 + \left[ \frac{-i}{\omega} H_0[u] - \frac{i}{\omega} F_-(x, t) + \frac{-ix}{c + \omega} + \frac{ict(3c + 4\omega)}{2\omega} \right] \left( k - \frac{i}{2} \right) + O \left( \frac{k}{2} \right)^2 \right), \]

\[ M_2(x, t; k) = \frac{\sqrt{m + \omega}}{\omega} e^{(y-x)/2} \left( 1 + \left( \frac{i}{\omega} F_+(x, t) + i(x - y) \right) \left( k - \frac{i}{2} \right) + O \left( \frac{k}{2} \right)^2 \right), \]

and thus to (5.42), (5.43). Formulas (5.44), (5.45) can be proved in analogous manner.

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