Random matrices theory elucidates the critical nonequilibrium phenomena

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The earlier times of evolution of a magnetic system contain more information than we can imagine. Capturing correlation matrices \( G \) of different time evolutions of a simple testbed spin system, as the Ising model, we analyzed the density of eigenvalues of \( G^T G \) for different temperatures. We observe a transition of the shape of the distribution that presents a gap of eigenvalues from critical temperature with a continuous migration to the Marchenko-Pastur law for the paramagnetic phase. We consider the analysis a promising method to be applied in other spin systems to characterize phase transitions. Our approach is different from alternatives in the literature since it uses the magnetization matrix and not the spatial matrix of spins.

Keywords: Random Matrices, Time-dependent Monte Carlo simulations, heavy atomic nucleus.

In an exciting application of random matrices, Stanley and collaborators [14, 15], using the known approach developed by Marcenko and Pastur [16, 17], showed that deviates from the bulk of spectra of random correlation matrices built with financial market assets are related to genuine correlations from Stock Market

Recently, some authors [18] interestingly investigated spectral properties of correlation matrices in near-equilibrium phase transitions. In this case, they studied correlation matrices of the \( N = L^2 \) spins of the Ising model in the two-dimensional lattice under \( \tau \) time steps of evolution to evidence the power-law spatial correlations at a phase transition display.

Similarly, [19] explored similar results in the steady-state for the correlation matrix of the asymmetric simple exclusion process. However, we believe that information about phase transitions in spin systems is still more "primitive" than we can imagine. The traces of the phase transition should reflect in properties of random matrices built from time evolutions simulated via MC simulations.
far from thermalization.

Thus, can we use alternative matrices differently from the ones considered in [18, 19], i.e., considering the critical behavior far from equilibrium? In addition, can we use the spectral properties to determine the critical parameter of the spin model studied?

Our goal in this paper is to show that it is possible. The success of our approach is to use the correct matrix that considers magnetization time series and not a matrix of the individual spins.

Thus, using this matrix that captures the collective character of the system, we show that the density of eigenvalues presents an eigenvalues gap intimately linked to the proximity of the critical system.

One performs that by first building a matrix $M$ that stores a number $N_{\text{sample}}$ of time series with $N_{\text{MC}}$ Monte Carlo (MC) steps. With this in hand, we show that the density of eigenvalues of the correlation magnetization matrix of the Ising model, built from $M$, presents a minimum strictly at its critical temperature, which corroborates the inflection point of the dispersion of eigenvalues.

In the following, we show how to define the magnetization matrix $M$ for a correct analysis of the spectra for the localization of the critical parameter of the Ising model in the earlier times of evolution. In the sequence, we present our results, followed by some summaries and our conclusions.

2-Marcenko-Pastur’s theorem and magnetization matrix: Here we define the main object for our analysis, the magnetization matrix element $m_{ij}$ that denotes the magnetization of the $j$th time series at the $i$th MC step of a system with $N = L^d$ spins. For simplicity, we used $d = 2$ (the minimal dimension to appear phase transition in the simple Ising model) in this work. Here $i = 1, \ldots, N_{\text{MC}}$, and $j = 1, \ldots, N_{\text{sample}}$. So the magnetization matrix $M$ is $N_{\text{MC}} \times N_{\text{sample}}$. In order to analyze spectral properties, an interesting alternative is to consider not $M$ but the square matrix $N_{\text{sample}} \times N_{\text{sample}}$:

$$G = \frac{1}{N_{\text{MC}}} M^T M$$

such that $G_{ij} = \frac{1}{N_{\text{MC}}} \sum_{k=1}^{N_{\text{MC}}} m_{ki}m_{kj}$. At this point, instead of working with $m_{ij}$, it is more convenient to take the Matrix $G$, defining its elements by the standard variables:

$$m_{ij}^* = \frac{m_{ij} - \langle m_j \rangle}{\sqrt{\langle m_j^2 \rangle - \langle m_j \rangle^2}}$$

where:

$$\langle m_j^2 \rangle = \frac{1}{N_{\text{MC}}} \sum_{i=1}^{N_{\text{MC}}} m_{ij}$$

Thus, using this matrix that captures the collective character of the system, we show that the density of eigenvalues presents an eigenvalues gap intimately linked to the proximity of the critical system.

We can observe that for $T < T_C$, a gap of eigenvalues characterizes the density of eigenvalues. This gap occurs until the proximity of $T_C$. For $T = 1.05T_C$ the gap almost disappears, which entirely happens for $T = 1.10T_C$. Thus we observe a migration in the density of eigenvalues as $T$ increases. The Marchenko-Pastur law (red points) fits the density of eigenvalues for large $T$ as can be observed, for example, for $T = 6.5 T_C$.

An essential computational detail is that the density of eigenvalues seems to be similar for the different number of MC steps as observed in Fig. 3 (a) for small systems (b).

Although the density of eigenvalues changes with temperature and the gap after $T_C$ disappears, we would like
to obtain a more precise parameter to localize the critical temperature of the system quantitatively. A natural choice is to compute the moments of the density of eigenvalues:

$$\langle \lambda^k \rangle = \int_{-\infty}^{\infty} \lambda^k \rho_{\text{exp}}(\lambda) d\lambda$$

where observe $\langle \lambda \rangle$ and $\text{var}(\lambda) = \langle \lambda^2 \rangle - \langle \lambda \rangle^2$ as function of $T$ which is shown in Fig. (a) and (b). We observe a notorious minimal value of $\langle \lambda \rangle$ (Fig. (a)) exactly at $T = T_C$, showing this amount captures the evolution of the density of eigenvalues and the gap of the eigenvalues that appears for $T \leq T_C$. This minimal seems to occur at $T = T_C$ independently on $N_{\text{sample}}$, keeping constant the ratio $Q = \frac{N_{\text{sample}}}{N_{\text{MC}}}$. Fig. (b) shows that at $T = T_C$ we have a corresponding inflection point.

4 - Conclusions: These results corroborate that the spectrum of correlation matrices built from the time series of magnetization of the Ising model in the earlier times of the evolution can precisely identify the critical temperature of the model. The moments of the density of eigenvalues seem to be suitable amounts to perform that.
Figure 4. (a) The minimal value of the average eigenvalue corresponds phase transition (b) The inflection point occurs strictly at the critical temperature.

The model is promising, and it deserves an exploration of other spin systems.

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