On mysteriously missing T-duals, H-flux and the T-duality group

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A general formula for the topology and H-flux of the T-duals of type II string theories with H-flux on toroidal compactifications is presented here. It is known that toroidal compactifications with H-flux do not necessarily have T-duals which are themselves toroidal compactifications. A big puzzle has been to explain these mysterious “missing T-duals”, and our paper presents a solution to this problem using noncommutative topology. We also analyze the T-duality group and its action, and illustrate these concepts with examples.

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T-duality is a symmetry of type II string theories that involves exchanging a theory compactified on a torus with a theory compactified on the dual torus. The T-dual of a type II string theory compactified on a circle, in the presence of a topologically nontrivial NS 3-form H-flux, was analyzed in special cases in \cite{15}. There it was observed that T-duality changes not only the H-flux, but also the spacetime topology. A general formalism for dealing with T-duality for compactifications arising from a free circle action was developed in \cite{5}. This formalism was shown to be compatible with two physical constraints: (1) it respects the local Buscher rules \cite{1}, and (2) it yields an isomorphism with twisted K-theory, in which the Ramond-Ramond charges and fields take their values \cite{11, 12, 13}. It was shown in \cite{5} that T-duality exchanges the first Chern class with the fiberwise integral of the H-flux, thus giving a formula for the T-dual spacetime topology. In this note we will present an account for the T-dual need not be another toroidal compactification. A similar phenomenon was noticed in \cite{15} in the special case of the trivial \textsuperscript{2}T bundle over \textsuperscript{2}T with non-trivial H-flux. We also show that the generalized T-duality group acts to generate the complete list of T-dual pairs related to a given toroidal compactification with H-flux. We will explain these results by providing examples and applications.

In this letter we will consider type II string theories on target \(d\)-dimensional manifolds \(X\), which are assumed to admit free, rank \(n\) torus actions. While for most physical applications one wants \(d = 10\), we do not need to assume this, and in fact \(X\) could represent a partial reduction of the original 10-dimensional spacetime after preliminary compactification in \(10-d\) dimensions. The space of orbits of the torus action on \(X\) is given by a \((d-n)\)-dimensional manifold, which we call \(Z\). The freeness of the action implies that each orbit is a torus and that none of these tori degenerate. As a result \(X\) is a principal torus bundle over the base \(Z\), and so its topology is entirely determined by the topology of the base \(Z\) together with the first Chern class \(c\) of the bundle \(X \overset{p}{\rightarrow} Z\) in \(H^2(Z, \mathbb{Z}^n)\). This viewpoint is useful in that it automatically identifies some gauge equivalent configurations, excludes configurations not satisfying some equations of motion and imposes the Dirac quantization conditions. The Chern class \(c\) is represented by a vector valued closed 2-form with integral periods, the curvature \(F\). We will discuss conditions under which the pair \((X \overset{p}{\rightarrow} Z, H)\) has a T-dual, either another pair \((X \overset{p}{\rightarrow} Z, H^\#)\) with the same base \(Z\) (the “classical” case) or a more general non-commutative object (the “nonclassical” case). In both cases, there should be a sense in which string theory on the original space \(X\) (with H-flux \(H\)) is equivalent to a theory on the T-dual.

Basic setup: Let \(p: X \rightarrow Z\) be a principal \(T\)-bundle as above, where \(T = (S^1)^n = \text{torus}^n\) is a rank \(n\) torus. Let \(H \in H^3(X, \mathbb{Z})\) be an H-flux on \(X\) satisfying \(\iota^* H = 0\), \(\iota^*: H^3(X, \mathbb{Z}) \rightarrow H^3(T, \mathbb{Z})\), where \(\iota: T \hookrightarrow X\) is the inclusion of a fiber. (This condition is automatically satisfied when \(n \leq 2\).)

The simplest case when the condition \(\iota^*(H) = 0\) does not apply is \(X = \text{torus}^3\), when considered as a rank 3, principal torus bundle over a point, with H-flux a non-zero integer multiple of the volume 3-form on \(\text{torus}^3\). When \(\iota^*(H) \neq 0\), there is no T-dual in the sense we are considering, even in what we call the “nonclassical” sense.

It turns out that nontrivial bundles are always T-dual to trivial bundles with non-zero H-flux. Therefore we will need to include the fluxes \(H\) and \(H^\#\) in our toroidal compactifications, which are then topologically determined by the triples \((Z, c, H)\) and \((Z, c^\# , H^\#)\), where \(H\) and \(H^\#\) are closed three-forms on the total spaces \(X\) and \(X^\#\) respectively.

Our result on classical T-duals: Suppose that we are in the basic setup as above. Choose a basis \(\{ T^k_j \}_{j=1}^k, k = \binom{n}{2}\) for \(H_2(T, \mathbb{Z})\) consisting of 2-tori, and push this forward into \(H_2(X, \mathbb{Z})\) via \(\iota_*\). We
can consider the cohomology classes
\[ \int_{T^2_j} H = H \cap \iota_*(\mathbb{T}^2_j) \in H^1(X, \mathbb{Z}). \]

These classes restrict to 0 on the fibers, since \( \iota^*(H) = 0 \). Using the following exact sequence, derived from the spectral sequence of the torus bundle,
\[ 0 \to H^1(Z, \mathbb{Z}) \xrightarrow{\nu} H^1(X, \mathbb{Z}) \xrightarrow{\zeta} H^1(T, \mathbb{Z}) \to \cdots, \]
we see that the classes \( \int_{T^2_j} H = H \cap \iota_*(\mathbb{T}^2_j) \in H^1(X, \mathbb{Z}) \) come from unique classes \( \{ \beta_j \}_{j=1}^k \in H^1(Z, \mathbb{Z}) \). Set
\[ p_1(H) = (\beta_1, \ldots, \beta_k) \in H^1(Z, \mathbb{Z}^k). \]

If \( p_1(H) = 0 \in H^1(Z, \mathbb{Z}^k) \), and in particular if \( Z \) is simply connected, then there is a classical T-dual to \( (p, H) \), consisting of \( p^\# : X^\# \to Z \), which is another principal \( T \)-bundle over \( Z \), and \( H^\# \in H^3(X^\#, \mathbb{Z}) \), the T-dual H-flux on \( X^\# \). One obtains a commuting diagram of the form
\[
\begin{array}{ccc}
X \times_{\mathbb{Z}^k} X^\# & \xrightarrow{(p^\#) \times (p^\#)} & X^\# \\
\downarrow{p^\#} & \searrow{(p^\#)^*} & \searrow{(p^\#)^*} \\
X & \xrightarrow{p} & X^\# \\
\end{array}
\]

In this case, the compactifications topologically specified by \((Z, c, H)\) and \((Z, c^\#, H^\#)\) are T-dual if \( c, c^\# \in H^2(Z, \mathbb{Z}^n) \) are related as follows:

Let \( c_j, j = 1, \ldots, n \), be the components of \( c \). Let \( X_j \xrightarrow{\pi_j} Z \) be the principal \( T^{n-1} \)-bundle subbundle of \( X \) obtained by deleting \( c_j \), i.e. the Chern class of \( X_j \) is
\[ c(\pi_j) = (c_1, \ldots, \hat{c}_j, \ldots, c_n). \]

Then \( X \xrightarrow{p_j} X_j \) is a principal \( S^1 \)-bundle whose Chern class is equal to \( \pi_j^*(c_j) \). Define \( X_j \xrightarrow{\pi^\#_j} Z \), \( X^\# \xrightarrow{\nu^\#_j} X^\# \) etc. similarly. Then we have
\[ (\pi_j)^*(c^\#_j) = (p_j)_!(H) \quad \text{and} \quad (\pi^\#_j)^*(c_j) = (p^\#_j)_!(H^\#). \]

Here the correspondence space \( X \times_{\mathbb{Z}} X^\# \) is the submanifold of \( X \times X^\# \) consisting of pairs of points \((x, y)\) such that \( p(x) = p^\#(y) \), and has the property that it implements the T-duality between \((p, H)\) and \((p^\#, H^\#)\). It also turns out that \( p^\#_j(H^\#) = 0 \in H^1(Z, \mathbb{Z}^k) \) and that the T-dual of \((p^\#, H^\#)\) is \((p, H)\). So in this case, T-duality exchanges the integral of the H-flux (over a basis of circles in the fibers) with the first Chern class. The condition in the result above determines, at the level of cohomology, the curvatures \( F \) and \( F^\# \). However the NS field strengths are only determined up to the addition of a three-form on the base \( Z \), because the integral of such a form over a basis of circles in the fibers vanishes. This settles a conjecture in [8], and was also considered by [9].

The simplest higher rank example is \( X = S^2 \times \mathbb{T}^2 \), considered as the trivial \( \mathbb{T}^2 \)-bundle over \( Z = S^2 \), with H-flux equal to \( H = k_1a \wedge b_1 + k_2a \wedge b_2 \), where we use the Künneth theorem to identify \( H^3(S^2, \mathbb{Z}) \otimes H^1(\mathbb{T}^2, \mathbb{Z}) \), and \( a \) is the generator of \( H^2(S^2, \mathbb{Z}) \cong \mathbb{Z} \). \( b_1, b_2 \) are the generators of \( H^1(\mathbb{T}^2, \mathbb{Z}) \cong \mathbb{Z}^2 \) and \( k_1, k_2 \in \mathbb{Z} \). Since \( S^2 \) is simply connected, \( p_1(H) = 0 \) and the T-dual of \((S^2 \times \mathbb{T}^2, H)\) is the nontrivial rank 2 torus bundle \( P \) over \( S^2 \) with Chern class \( c_1(P) = (k_1a, k_2a) \in H^2(S^2, \mathbb{Z}) \oplus H^2(S^2, \mathbb{Z}) = H^2(S^2, \mathbb{Z}^2) \), and with H-flux equal to zero. This example generalizes easily by taking the Cartesian product with a manifold \( M \), and pulling back the \( H \)-flux to the product and arguing as before, we see that the T-dual of \((M \times S^2 \times \mathbb{T}^2, H)\) is \((M \times P, 0)\).

**Our result on nonclassical T-duals:** Suppose that we are in the basic setup as above. If \( p_1(H) \neq 0 \in H^1(Z, \mathbb{Z}^k) \), then there is no classical T-dual to \((p, H)\); however, there is a nonclassical T-dual consisting of a continuous field of (stabilized) noncommutative tori \( A_f \) over \( Z \), where the fiber over the point \( z \in Z \) is equal to the rank \( n \) noncommutative torus \( A_{f(z)} \) (see Figure 1 below).

**FIG. 1:** In the diagram, the fiber over \( z \in Z \) is the noncommutative torus \( A_{f(z)} \), which is represented by a foliated torus, with foliation angle equal to \( f(z) \).

This suggests an unexpected link between classical string theories and the “noncommutative” ones, obtained by “compactifying” matrix theory on tori, as in [4] (cf. also [19], §6–7]). We now recall the definition of the rank \( n \) noncommutative torus \( A_{\theta} \), cf. [13]. This algebra (stabilized by tensoring with the compact operators \( K \)) occurs geometrically as the foliation algebra associated to Kru- necker foliations on the torus [3]. In [4], the same algebra...
occurs naturally from studying the field equations of the IKKT (Ishibashi-Kawai-Kitazawa-Tsuchiya) model compactified on n-tori, or from the study of BPS states of the BFSS (Banks-Fisher-Shenker-Susskind) model. (The IKKT and BFSS models are both large-N matrix models in which Poisson brackets in the Lagrangian are replaced by matrix commutators.) For each \( \theta \in \mathbb{T}^k \), identified with a hermitian matrix \( \theta = (\theta_{ij}) \), \( i, j = 1, \ldots, n \), \( \theta_{ij} \in \mathbb{S}^1 \) with 1’s down the diagonal, the noncommutative torus \( A_\theta \) is defined abstractly as the \( C^* \)-algebra generated by \( n \) unitaries \( U_j, j = 1, \ldots, n \) in an infinite dimensional Hilbert space satisfying the commutation relation \( U_i U_j = \theta_{ij} U_j U_i \), \( i, j = 1, \ldots, n \). Elements in \( A_\theta \) can be represented by infinite power series

\[
 f = \sum_{m \in \mathbb{Z}^n} a_m U^m, \tag{4}
\]

where \( a_m \in \mathbb{C} \) and \( U^m = U_1^{m_1} \cdots U_n^{m_n} \), for all \( m = (m_1, \ldots, m_n) \in \mathbb{Z}^n \).

A famous example of a principal torus bundle with non-T-dualizable H-flux is provided by \( \mathbb{T}^3 \), considered as the trivial \( \mathbb{T}^2 \)-bundle over \( \mathbb{T} \), with \( H \) given by \( k \) times the volume form on \( \mathbb{T}^3 \). \( H \) is non T-dualizable in the classical sense since \( p_!(H) \neq 0 \in H^1(\mathbb{T}, \mathbb{Z}) \). Alternatively, there are no non-trivial principal \( \mathbb{T}^2 \)-bundles over \( \mathbb{T} \), since \( H^2(\mathbb{T}, \mathbb{Z}) = 0 \), that is, there is no way to dualize the H-flux by a (principal) torus bundle over \( \mathbb{T} \), cf. [4]. This is an example of a mysteriously missing T-dual. This example is covered by our result on nonclassical T-duals above. The T-dual is realized by a field of stabilized noncommutative tori fibered over \( \mathbb{T} \). Let \( \mathcal{H} = L^2(\mathbb{T}) \) and consider the the projective unitary representation \( \rho_\theta : \mathbb{Z}^2 \to \text{PU}(\mathcal{H}) \) in which the generator of the first \( Z \) factor acts by multiplication by \( z^k \) (where \( \mathbb{T} \) is thought of as the unit circle in \( \mathbb{C} \)) and the generator of the second \( Z \) factor acts by translation by \( \theta \in \mathbb{T} \). Then the Mackey obstruction of \( \rho_\theta \) is \( \theta^k \in \mathbb{T} \cong H^2(\mathbb{T}^2, \mathbb{T}) \). Let \( K(\mathcal{H}) \) denote the algebra of compact operators on \( \mathcal{H} \) and define an action \( \alpha \) of \( \mathbb{Z}^2 \) on continuous functions on the circle with values in compact operators, \( C(\mathbb{T}, K(\mathcal{H})) \), given at the point \( \theta \) by \( \rho_\theta \). Define the \( C^* \)-algebra \( B \), which is obtained by inducing the \( \mathbb{Z}^2 \) action to an action of \( \mathbb{R}^2 \) on \( B = \text{Ind}_{\mathbb{Z}^2}^{\mathbb{R}^2} (C(\mathbb{T}, K(\mathcal{H})), \alpha) \), i.e.

\[
 B = \{ f : \mathbb{R}^2 \to C(\mathbb{T}, K(\mathcal{H})): f(t + g) = \alpha(g)(f(t)), t \in \mathbb{R}^2, g \in \mathbb{Z}^2 \}. \tag{5}
\]

Then \( B \) is a continuous-trace \( C^* \)-algebra having spectrum \( \mathbb{T}^3 \) and Dixmier-Douady invariant \( H \). \( B \) also has an action of \( \mathbb{R}^2 \) whose induced action on the spectrum of \( B \) is the trivial bundle \( \mathbb{T}^3 \to \mathbb{T} \). Then our noncommutative T-dual is the crossed product algebra \( B \rtimes \mathbb{R}^2 \cong C(\mathbb{T}, K(\mathcal{H})) \times \mathbb{Z}^2 = A_\theta \), which has fiber over \( \theta \in \mathbb{T} \) given by \( K(\mathcal{H}) \times_{\rho_\theta} \mathbb{Z}^2 \cong A_\theta \otimes K(\mathcal{H}) \), where \( A_\theta \) is the noncommutative 2-torus. In fact, the crossed product \( B \rtimes \mathbb{R}^2 \) is isomorphic to the (stabilized) group \( C^* \)-algebra \( C^*(\mathcal{H}_Z) \otimes K \), where \( \mathcal{H}_Z \) is the integer Heisenberg-type group,

\[
 H_Z = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{Z} \right\}. \tag{5}
\]

In summary, the nonclassical T-dual of \( (\mathbb{T}^3, H = k) \) is \( A_\theta = C^*(\mathcal{H}_Z) \otimes K \). As required in order to match up RR charges, the \( K \)-theory of this algebra is the same as the \( K \)-theory of \( \mathbb{T}^3 \) with twist given by our H-flux, or \( k \) times the volume form.

This example generalizes easily by taking the Cartan product with a manifold \( M \). Pulling back the H-flux to the product and arguing as before, we see that \( (M \times \mathbb{T}^3, H = k) \) is T-dual to \( C(M) \otimes C^*(\mathcal{H}_Z) \otimes K \). For instance, if the dimension of \( M \) is seven, then \( M \times \mathbb{T}^3 \) is ten dimensional, yielding examples of spacetime manifolds that are relevant to type II string theory.

**Our results on the T-duality group:**

It is important to realize that a fixed space \( X \) sometimes be given the structure of a principal torus bundle over \( Z \) in many different ways. For example, given a free action of a torus \( T = \mathbb{T}^n \) on \( X \), with quotient space \( Z = X/T \), we can for every element \( g \in \text{Aut}(\mathbb{T}^n) = \text{GL}(n, \mathbb{Z}) \) define a new free action of \( T \) on \( X \), twisted by \( g \), by the formula \( x \cdot g t = x \cdot g(t) \). (Here \( t \in T \) is the original free right action of \( T \) on \( X \), and \( \cdot g \) is the new twisted action.) If \( c \in H^2(\mathbb{T}, \mathbb{Z}^m) \) was the Chern class of the original bundle, the Chern class of the \( g \)-twisted bundle is \( g \cdot c \), with \( g \) acting via the action of \( \text{GL}(n, \mathbb{Z}) \) on \( \mathbb{Z}^m \).

The group \( \text{GL}(n, \mathbb{Z}) \) embeds in \( O(n, n; \mathbb{Z}) \), the subgroup of \( \text{GL}(2n, \mathbb{Z}) \) preserving the quadratic form defined by \( \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \), via \( a \mapsto \begin{pmatrix} a & 0 \\ 0 & (a^t)^{-1} \end{pmatrix} \) (see [11, §2.4]). This larger group \( O(n, n; \mathbb{Z}) \) is often called the T-duality group. In fact we will consider the still larger generalized T-duality group \( G(n, n; \mathbb{Z}) = O(n, n; \mathbb{Z}) \times (\mathbb{Z}/2) \) of matrices in \( \text{GL}(2n, \mathbb{Z}) \) preserving the form \( \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \) up to sign. Good references for the T-duality group include [10] (for the state of the theory up to 1994) and [13] for more current developments.

Suppose that we are in the basic setup as above, with \( Z \) simply connected, so that one is always guaranteed to have a classical T-dual. Then the generalized T-duality group \( G(n, n; \mathbb{Z}) \) acts on the set of T-dual pairs \((p, H)\) and \((p^#, H^#)\) to generate all related T-dual pairs. **All of these pairs are physically equivalent.** The restriction of the action to \( G(n, n; \mathbb{Z}) \) (as embedded above) corresponds to twisting of the action on the same underlying space as above.

When \( Z \) is not simply connected and \( p_!(H) \neq 0 \), it is not clear that one has an action of the full T-duality group. But the action of \( G(n, n; \mathbb{Z}) \) always sends the pair consisting of \((p, H)\) and its nonclassical T-dual to another nonclassical pair, involv-
ing continuous fields of (stabilized) noncommutative tori over $\mathbb{Z}$.

We illustrate the action of the generalized T-duality group in the simplest case of circle bundles with H-flux, in which case the generalized T-duality group reduces to $GO(1,1;\mathbb{Z})$, a dihedral group of order 8.

Consider the example of the 3 dimensional lens space $L(1,p) = S^3/Z_p$, with H-flux $H = q$ times the volume form, cf. [17]. Here $p, q \in \mathbb{Z}$, and initially we take $p \neq 0$. Then $L(1,p)$ is a circle bundle over the 2-dimensional sphere $S^2$ and has first Chern class equal to $p$ times the volume form of $S^2$. Then, as shown in [8, §4.3], $(L(1,p), H = q)$ and $(L(1,q), H = p)$ are T-dual to each other, and the element \[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\] of $O(1,1;\mathbb{Z})$ interchanges them. The element \[
\begin{pmatrix}
-1 & 0 \\
0 & -1
\end{pmatrix}
\] of the T-duality group $O(1,1;\mathbb{Z})$ lies in the subgroup $GL(1,\mathbb{Z})$, embedded as above, and acts by twisting the $S^1$ action on $L(1,p)$. This twisted action makes $L(1,p)$ into a circle bundle over $S^2$ having first Chern class equal to $-p$ times the volume form of $S^2$. This bundle is denoted $L(-1, -p)$, and its total space is diffeomorphic to $L(1,p)$, though by an orientation-reversing diffeomorphism. Therefore the action of \[
\begin{pmatrix}
-1 & 0 \\
0 & -1
\end{pmatrix}
\] on the pair $(L(1,p), H = q)$ and $(L(1,q), H = p)$ gives rise to a new T-dual pair $(L(-1,-p), H = -q)$ and $(L(1,-q), H = -p)$. The group $GO(1,1;\mathbb{Z})$ is generated by the two elements of $O(1,1;\mathbb{Z})$ just discussed and by \[
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix},
\] which replaces the original T-dual pair by the pair consisting of $(L(1,p), H = -q)$ and $(L(1,-q), H = p)$. Here we have tacitly assumed $p, q \geq 2$: we can extend things to other values of $p$ and $q$ by making the convention that $L(1,1) = S^3$ and $L(1,0) = S^2 \times S^1$. This defines the T-duality in [8, §4.3]. Thus in general there are 8 different (bundle, H-flux) pairs with equivalent physics, corresponding to $(\pm p, \pm q)$ and $(\pm q, \pm p)$.

This example generalizes easily by taking the Cartesian product with a manifold $M$. For instance, if the dimension of $M$ is seven, then we obtain 8 different (bundle, H-flux) pairs in the same $GO(1,1;\mathbb{Z})$-orbit as $M \times L(1,p)$. All of these are ten-dimensional spacetime manifolds relevant to type II string theory.

We end with some open problems. A critical verification of any proposed duality is that the anomalies should match on both sides. This was checked for T-duality involving circle bundles with H-flux in [8], but remains to be analyzed in the general torus bundle case with H-flux. It also remains to be determined whether or not the group $GO(n,n;\mathbb{Z})$ also operates in the nonclassical case. Another problem is to extend our results to non-free torus actions [20], in which case it could be relevant to mirror symmetry.

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