New Regularity Criteria for Weak Solutions to the MHD Equations in Terms of an Associated Pressure

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Communicated by G.P.Galdi

Abstract. We assume that Ω is either a smooth bounded domain in $\mathbb{R}^3$ or $\Omega = \mathbb{R}^3$, and $\Omega'$ is a sub-domain of $\Omega$. We prove that if $0 \leq T_1 < T_2 \leq T \leq \infty$, $(u, b, p)$ is a suitable weak solution of the initial–boundary value problem for the MHD equations in $\Omega \times (0, T)$ and either $F_\gamma(p-)$ $\in L^\infty(T_1, T_2; L^{3/2}(\Omega'))$ or $F_\gamma(B+)$ $\in L^\infty(T_1, T_2; L^{3/2}(\Omega'))$ for some $\gamma > 0$, where $F_\gamma(s) = s \ln(1 + s)^{1+\gamma}$, $B = p + \frac{1}{2} |u|^2 + \frac{1}{2} |b|^2$ and the subscripts “−” and “+” denote the negative and the nonnegative part, respectively, then the solution $(u, b, p)$ has no singular points in $\Omega' \times (T_1, T_2)$. If $b \equiv 0$ then our result generalizes some previous known results from the theory of the Navier–Stokes equations.

Mathematics subject classification (2010). Primary 35B65, 35Q30, 35Q35; Secondary 76W05, 76D05.

Keywords. MHD equations, Navier–Stokes equations, Pressure, Regularity.

1. Introduction and the Main Result

1.1. The MHD Initial—Boundary Value Problem

The motion of a viscous incompressible electrically conductive fluid in a domain $\Omega \subset \mathbb{R}^3$ in the time interval $(0, T)$ (where $0 < T \leq \infty$), in the absence of an external specific body force and an external magnetic induction, is described by the system of magneto-hydro-dynamical equations (which is abbreviated to MHD equations), whose dimensionless form is

\[ \partial_t u + u \cdot \nabla u - \nabla p + \frac{1}{R_f} \Delta u, \]

\[ \partial_t b - \nabla (u \times b) = \frac{1}{R_m} \Delta b, \]

\[ \text{div } u = \text{div } b = 0. \]

The unknowns are the velocity $u \equiv (u_1, u_2, u_3)$ of the fluid, the magnetic field $b \equiv (b_1, b_2, b_3)$ and the pressure $p$. The positive parameters $R_f$ and $R_m$ represent the fluid Reynolds number and the magnetic Reynolds number. These numbers are defined by the formulas $R_f = UL/\nu$ and $R_m = UL/\lambda$, where $U$ is the characteristic velocity of the flow, $L$ is the characteristic length, $\nu$ is the kinematic viscosity and $\lambda$ is the magnetic diffusivity. The positive parameter $\kappa$ equals $\sigma B^2 L/(\rho U \mu)$, where $B$ is the characteristic unit of the magnetic field, $\sigma$ is the electric conductivity, $\rho$ is the density and $\mu$ is the magnetic permeability of the fluid. Note that $\lambda$, $\mu$ and $\sigma$ are related through the formula $\mu \sigma = \lambda^{-1}$. Choosing appropriately the characteristic unit $B$, we may further assume, without loss of generality, that $\kappa = 1$.

Note that the Eqs. (1.1) and (1.2) are often written in the modified form

\[ \partial_t u + u \cdot \nabla u - b \cdot \nabla b = -\nabla(p + \frac{1}{2} |b|^2) + \frac{1}{R_f} \Delta u, \]
\[ \partial_t b + u \cdot \nabla b - b \cdot \nabla u = \frac{1}{\mathcal{R}_m} \Delta b. \] (1.2)

By analogy with \( p + \frac{1}{2} |u|^2 \), which is called the Bernoulli pressure in the fluid mechanics, it is logical to call \( p + \frac{1}{2} |b|^2 \) the magnetic Bernoulli pressure. Eqs. (1.1), (1.2) and (1.3) are completed by the initial conditions

\[ u|_{t=0} = u_0, \quad b|_{t=0} = b_0 \quad \text{in } \Omega \] (1.4)

and also by appropriate boundary conditions if \( \Omega \neq \mathbb{R}^3 \).

\( \ast \) \( \Omega \) is either the whole space \( \mathbb{R}^3 \) or a bounded domain in \( \mathbb{R}^3 \) with the boundary of the class \( C^{2+\alpha} \) for some \( \alpha > 0 \) throughout the whole paper, and if \( \Omega \neq \mathbb{R}^3 \) then we consider the so called Navier–type boundary conditions

\[
\begin{align*}
a) & \quad u \cdot n = b \cdot n = 0, \\
b) & \quad \text{curl } u \times n = \text{curl } b \times n = 0 \quad \text{on } \partial \Omega \times (0, T)
\end{align*}
\]
(1.5)

for both the velocity \( u \) and the magnetic field \( b \). (We denote by \( n \) the outer normal vector field on \( \partial \Omega \).) Note that \( b \) satisfies (1.5) in the case of a perfectly conductive wall.

### 1.2. The Choice of the Pressure and the Main Result

Let \( \gamma \) be a positive parameter. Define

\[ \mathcal{F}_\gamma (s) := s \left[ \ln (1 + s) \right]^{1+\gamma} \]

for \( s \geq 0 \). Notice that the function \( \mathcal{F}_\gamma \) is increasing and strictly convex on \([0, \infty)\).

The next theorem deals with a suitable weak solution \((u, b, p)\) of the problem (1.1), (1.2), (1.3), (1.4) and (1.5). If \( \Omega = \mathbb{R}^3 \) then the function \( \rho \) is automatically essentially bounded in some neighborhood of each regular point \((x_0, t_0)\) of the solution \((u, b, p)\), see Sect. 1.6. However, this need not be true if \( \Omega \) is bounded, because \( \rho \) is unique up to an arbitrary additive function \( \zeta = \zeta(t) \in L^{5/3}(0, T) \). Nevertheless, among all “admissible” pressures, which differ from one another at most by an additive function \( \zeta \in L^{5/3}(0, T) \), there exists a class \( \mathcal{P} \) of pressures, that are essentially bounded in some neighborhood of each regular point. (The reasons are explained in Sect. 1.6.) From now on, we assume that \( \rho \in \mathcal{P} \).

The definitions of a weak and a suitable weak solution and a singular or a regular point of the solution are summarized in Sect. 1.5.

Our main result is formulated in this theorem:

**Theorem 1.** Let \( \Omega \) be a domain in \( \mathbb{R}^3 \), satisfying the condition (\( \ast \)). Let \((u, b, p)\) be a suitable weak solution of the MHD initial–boundary value problem (1.1), (1.2), (1.3), (1.4) and (1.5) in \( \Omega \times (0, T) \), \( \Omega' \) be a subdomain of \( \Omega \) and \( 0 \leq T_1 < T_2 \leq T \). Suppose that there exists \( \gamma > 0 \), such that at least one of the conditions

\[
\begin{align*}
a) & \quad \mathcal{F}_\gamma (p_-) \in L^\infty (T_1, T_2; L^{3/2}(\Omega')), \\
b) & \quad \mathcal{F}_\gamma (B_+) \in L^\infty (T_1, T_2; L^{3/2}(\Omega')), \quad \text{where } B := p + \frac{1}{2} |u|^2 + \frac{1}{2} |b|^2
\end{align*}
\]

holds. Then the set \( \mathcal{S}((T_1, T_2) \times \Omega') \) of singular points of the solution \((u, b, p)\) in \( \Omega' \times (T_1, T_2) \) is empty. Consequently, the functions \( u \) and \( b \) are Hölder continuous in \( \Omega' \times (T_1, T_2) \).

The subscripts “–”, respectively “+”, denote the negative, respectively nonnegative part. As the negative part is taken “positively”, one has \( p = p_+ - p_- \) and \( B = B_+ - B_- \).

Note that condition a) is stronger than the condition \( p_- \in L^\infty (T_1, T_2; L^{3/2}(\Omega')) \) and weaker than the condition “there exists \( q > \frac{3}{2} \) such that \( p_- \in L^\infty (T_1, T_2; L^q(\Omega')) \).” The same remark also holds on condition b).

At this point, we also consider it necessary to note that the conclusions of Theorem 1 concern only the interior regularity of the solution \((u, b, p)\) in \( \Omega' \times (T_1, T_2) \) and they say nothing on possible singular points on \( \partial \Omega' \times (T_1, T_2) \), or, if \( \partial \Omega \cap \partial \Omega' \neq \emptyset \), also on \( \partial \Omega \times (T_1, T_2) \). The treatment of \((u, b, p)\) in the neighborhood of \( \partial \Omega \) would require a new approach and it would go beyond the scope of this paper. However, as our
result denies the existence of singular points just in the interior of \( \Omega' \) and not on the boundary of \( \Omega' \) or on the boundary of \( \Omega \), there arises a natural question about the role of the boundary conditions (1.5). This is explained in Sect. 1.6.

1.3. Comparison with Previous Related Results

There have already appeared relatively many regularity criteria for weak solutions to the MHD equations in literature, that involve a posteriori conditions on the pressure – let us at first mention the papers see [2,4,5,10,15,18,20,21]. While additional a posteriori conditions are imposed on both \( p \) and \( b \) in [21], all other named papers in fact impose additional conditions on the magnetic Bernoulli pressure \( p + \frac{1}{2}|b|^2 \). Moreover, all the aforementioned papers (with the exception of [15]) study the regularity in \( \mathbb{R}^3 \times (0,T) \). The reason is not only formal: the methods applied in [2,4,5,10,18,20] and [21] use the integration by parts over \( \mathbb{R}^3 \). If one applies the same procedure to a domain \( \Omega \), that differs from \( \mathbb{R}^3 \), then the boundary integrals do not seem to be controllable. (The reason lies in the presence of the magnetic field \( b \) and the boundary conditions, satisfied by \( b \) on \( \partial \Omega \times (0,T) \). The same problem does not appear in the theory of the Navier–Stokes equations, where \( b = 0. \)) In [15], the authors estimate the boundary integrals, assuming that domain \( \Omega \) is convex.

All the papers [2,4,5,10,15,18,20] and [21] use the condition \( R_f = R_m \), which enables one to transform the MHD equations to the so called symmetric form, where the new unknowns functions are \( w^+ := u + b \) and \( w^- := u - b \). This possibility is important, otherwise the used methods fail. Some authors write that one can assume that \( R_f = R_m \) “without loss of generality”. This is, however, not correct, because \( R_f \) and \( R_m \) are independent dimensionless quantities. On the contrary, the condition \( R_f = R_m \) is from the physical point of view very restrictive.

Compared to the aforementioned papers, we do not need the identity \( R_f = R_m \) and we impose an additional assumption only on the negative part of \( p \), respectively the nonnegative part of \( B \). Moreover, we consider a flow in \( \Omega \times (0,T) \), where \( \Omega \) need not be \( \mathbb{R}^3 \) and the sub-domain \( \Omega' \subset \Omega \), where we derive a statement on regularity, need not coincide with \( \Omega \).

It should be noted that results, where additional conditions have been imposed only on the negative part of \( p \) or the nonnegative part of the Bernoulli pressure \( p + \frac{1}{2}|u|^2 \) are known from the theory of the Navier–Stokes equations, see e.g. the papers [12,14] and [16]. In the last cited paper, G. Seregin and V. Šverák consider a suitable weak solution \( u, p \) to the Navier–Stokes equations in \( \mathbb{R}^3 \times (0,\infty) \). The authors say that a scalar function \( g: \mathbb{R}^3 \times (0,\infty) \rightarrow [0,\infty) \) satisfies condition (C) if to any \( t_0 > 0 \) there exists \( R_0 > 0 \) such that

\[
A(t_0) := \sup_{x_0 \in \mathbb{R}^3} \sup_{t_0 - R_0^2 \leq t \leq t_0} \int_{|x - x_0| < R_0} \frac{g(x,t)}{|x - x_0|} \, dx < \infty
\]

and for each fixed \( x_0 \in \mathbb{R}^3 \) and each fixed \( R \in (0,R_0] \), the function

\[
t \mapsto \int_{|x - x_0| < R} \frac{g(x,t)}{|x - x_0|} \, dx
\]

is continuous at \( t_0 \) from the left. The main result of [16] says that if there exists a function \( g \), satisfying condition (C), so that the normalized pressure

\[
p(x,t) := \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x - y|} \, \text{div} \, \text{div} \left[ u(y,t) \otimes u(y,t) \right] \, dy
\]

satisfies at least one of the inequalities

\[
a) \quad |u(x,t)|^2 + 2p(x,t) \leq g(x,t), \quad b) \quad p(x,t) \geq -g(x,t)
\]

for all \( x \in \mathbb{R}^3 \) and \( 0 < t < \infty \), then \( u \) is Hölder-continuous in \( \mathbb{R}^3 \times (0,\infty) \), i.e. \( u \) is regular.

The method from [16] has been extended by K. Kang and J. Lee to the MHD Eqs. (1.1), (1.2) and (1.3). (See Theorem 1.3 in [9].) The authors deal with the MHD equations in \( \mathbb{R}^3 \times (0,\infty) \), assume that
\( \mathcal{R}_f = \mathcal{R}_m = 1 \) and their first sufficient condition for regularity of a suitable weak solution \( u, b, p \) of the MHD system (1.1), (1.2) and (1.3) coincides with (1.7b). The second condition is analogous to (1.7a): it requires
\[
|u(x, t)|^2 + |b(x, t)|^2 + 2p(x, t) \leq g(x, t) \tag{1.8}
\]
for all \( x \in \mathbb{R}^3 \) and \( 0 < t < \infty \).

It should be also noted that the methods, applied in the last cited papers [9] and [16], strongly use that fact that the considered flow fills in the whole space \( \mathbb{R}^3 \). (Otherwise \( p \) need not satisfy (1.6) or an analogous formula, used in [9].) Thus, our Theorem 1 generalizes Theorem 1.3 from paper [9] in these points: our conditions a) and b) are weaker than the conditions (1.7b) and (1.8) used in [9], our domain \( \Omega \) need not be the whole space \( \mathbb{R}^3 \) and we do not assume that \( \mathcal{R}_f = \mathcal{R}_m = 1 \). If one considers \( b \equiv 0 \) then our Theorem 1 represents a generalization of the results from [16].

### 1.4. On the Structure of This Paper

We assume that \( t_0 \in (T_1, T_2) \) is the so called \( \Omega' \)-epoch of irregularity of the solution \( (u, b, p) \), which means that the solution is smooth in \( \Omega' \times (t_0 - \xi, t_0) \) for some \( \xi > 0 \) and blows up for \( t \to t_0 \) in a neighborhood of some point in \( \Omega' \). Our aim is to show, by contradiction, that this cannot happen if \( (u, b, p) \) satisfies the assumptions of Theorem 1.

We assume that \( \mathcal{B} \subset \mathcal{B} \subset \Omega' \) is an arbitrarily chosen ball inside \( \Omega' \). The MHD Eqs. (1.1), (1.2) and (1.3) are “localized” to a neighborhood of \( \mathcal{B} \) in Sect. 2. (Then the new unknown functions \( \hat{u}, \hat{b} \) and \( \hat{p} \) are supported just in the neighborhood of \( \mathcal{B} \).) Sect. 3 contains some auxiliary results. We show that \( (u, b, p) \) does not blow up in \( \mathcal{B} \) for \( t \to t_0 \) in Sects. 4 and 5. Important formulas (4.14), used in Sects. 4 and 5, are derived in Sect. 6. The derivation is postponed to a separate section, because it is just technical and its presentation inside Sects. 4 or 5 would disrupt the logical sequence of the text.

### 1.5. Notation and Some Definitions

We denote vector functions and spaces of vector functions by boldface letters. \( C_{0, \sigma}(\Omega) \) denotes the linear space of all infinitely differentiable divergence–free vector functions in \( \Omega \), with a compact support, and \( L^2_{\sigma}(\Omega) \) is the closure of \( C_{0, \sigma}(\Omega) \) in \( L^2(\Omega) \). Finally, \( W^{1,2}_{\sigma}(\Omega) := W^{1,2}(\Omega) \cap L^2_{\sigma}(\Omega) \).

Recall the definitions of a weak solution, associated pressure and a suitable weak solution:

Given \( u_0, b_0 \in L^2_{\sigma}(\Omega) \), a pair of functions \( (u, b) \) is said to be a weak solution to the problem (1.1), (1.2), (1.3), (1.4) and (1.5), if \( u, b \in L^\infty(0, T; L^2_{\sigma}(\Omega)) \cap L^2(0, T; W^{1,2}_{\sigma}(\Omega)) \) and the integral equations
\[
\int_0^T \int_\Omega \left[ -u \cdot \partial_t \phi + \frac{1}{\mathcal{R}_f} \text{curl} \ u \cdot \text{curl} \ \phi + u \cdot \nabla \phi \cdot u - b \cdot \nabla \phi \cdot b \right] \, dx \, dt
= \int_\Omega u_0 \cdot \phi(x, 0) \, dx,
\]
\[
\int_0^T \int_\Omega \left[ -b \cdot \partial_t \phi + \frac{1}{\mathcal{R}_m} \text{curl} \ b \cdot \text{curl} \ \phi - \text{curl}(u \times b) \cdot \phi \right] \, dx \, dt
= \int_\Omega b_0 \cdot \phi(x, 0) \, dx
\]
are satisfied for all \( \phi \in C^\infty_c((0, T); W^{1,2}_{\sigma}(\Omega)) \). A distribution \( p \) in \( QT := \Omega \times (0, T) \) is called an associated pressure if \( u, b \) and \( p \) satisfy the Eq. (1.1) in the sense of distributions in \( QT \). The triplet \( (u, b, p) \) is
called a suitable weak solution to the problem (1.1), (1.2), (1.3), (1.4) and (1.5), if the associated pressure \( p \) is a function from \( L^{5/3}(Q_T) \) and \( u, b, p \) satisfy the so called localized energy inequality

\[
\int_{Q_T} 2 \left( \frac{1}{R_f} |\nabla u|^2 + \frac{1}{R_m} |\nabla b|^2 \right) \partial_t \theta \, dx \, dt \\
\leq \int_{Q_T} \left[ |u|^2 \left( \partial_t \theta + \frac{1}{R_f} \Delta \theta \right) + |b|^2 \left( \partial_t \theta + \frac{1}{R_m} \Delta \theta \right) \\
+ (|u|^2 + |b|^2 + 2p) (u \cdot \nabla \theta) - 2(u \cdot b) (b \cdot \nabla \theta) \right] \, dx \, dt
\]

for every nonnegative infinitely differentiable scalar function \( \theta \), compactly supported in \( Q_T \).

Note that the existence of a weak solution to the problem (1.1), (1.2), (1.3), (1.4) and (1.5) can be established, applying the same technique as for the Navier–Stokes equations. (This technique is described in a series of papers and books. Particularly, the Navier–type boundary conditions for the velocity are considered e.g. in [19, Theorem 6.3].) A suitable weak solution can be again constructed in the same way as for the Navier–Stokes equations, following e.g. the method described in [1].

A space–time point \((x, t) \in Q_T\) is said to be a regular point of the solution \((u, b, p)\) if there exists a neighborhood \(U(x, t) \subset Q_T\) of this point such that both \(u\) and \(b\) are essentially bounded in \(U(x, t)\). Other points in \(Q_T\) are called singular points. (The functions \(u\) and \(b\) are in fact Hölder–continuous in some neighborhood of each regular point, see [11].) Let us denote by \(\mathcal{S}(Q_T)\) the set of all singular points of the solution \((u, b, p)\) in \(Q_T\). Particularly, if \(I \subset (0, T)\) and \(\Omega_I \subset \Omega\), then \(\mathcal{S}(\Omega_I \times I)\) denotes the set of all singular points in \(\Omega_I \times I\). (It should be noted that the question whether \(\mathcal{S}(Q_T)\) is nonempty is open.)

If \(t \in (0, T)\) then we also denote by \(\mathcal{S}_t(\Omega)\) the set of \(x \in \Omega\), such that \((x, t)\) is a singular point of the solution \((u, b, p)\). Obviously, the set \(\mathcal{S}_t(\Omega_I)\) is closed in \(\Omega_I \times I\) and the set \(\mathcal{S}(\Omega_I)\) is closed in \(\Omega_I\).

Paper [8] contains a series of criteria for regularity of the solution \((u, b, p)\) at a given point \((x_0, t_0)\) in \(Q_T\), from which one can deduce that the 1–dimensional Hausdorff measure of \(\mathcal{S}(Q_T)\) is zero. The results hold for domains \(\Omega\) of several types, including domains satisfying condition (⋆). Although the authors state in the introduction that they consider the homogeneous boundary conditions for \(u\) and \(b\), their definition of a suitable weak solution is independent of boundary conditions and as their main results are based just on local interior estimates of the velocity and the magnetic field, they are also valid for suitable weak solutions with the boundary conditions (1.5). The same information on the 1D Hausdorff measure of \(\mathcal{S}(Q_T)\) also directly follows from [3].

### 1.6. Regularity of the Pressure in the Neighborhood of Regular Points and the role of the Boundary Conditions for Velocity

If \(\Omega = \mathbb{R}^3\) and \(b = 0\) (which means that \((u, p)\) is a suitable weak solution to the Navier–Stokes equations) then \(p\) and all its spatial derivatives are essentially bounded in some neighborhood of each regular point. This follows from Theorem 4 in [17]. If \(b \neq 0\) then the same property of \(p\) can be proven by means of exactly the same arguments as in [17], using the representation formula

\[
p(x, t) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x - y|} \text{div} \left[ u(y, t) \otimes u(y, t) - b(y, t) \otimes b(y, t) \right] \, dy.
\]

If \(\Omega\) is a smooth bounded domain in \(\mathbb{R}^3\) then it follows from [15, Sect. 4 and Sect. 6.3] that \(\nabla p\), together with all its spatial derivatives, is essentially bounded in an neighborhood of each regular point. Let us show that \(p\) can be modified by an additive function \(\zeta = \zeta(t) \in L^{5/3}(0, T)\) so that \(p + \zeta\) is essentially bounded in some neighborhood of each regular point.

Thus, let \(x_0\) be a point in \(\Omega\), such that \((x_0, t)\) is a regular point of the solution \((u, b, p)\) for all \(t \in (0, T)\). (Such a point exists, because the set \(\mathcal{S}(Q_T)\) is closed in \(Q_T\) and the 1D Hausdorff measure of \(\mathcal{S}(Q_T)\) is zero.) Then to any \(\delta > 0\) there exists a spatial neighborhood \(U(x_0) \subset \Omega\), such that \(\nabla p\) (together with
all its spatial derivatives) is essentially bounded in $U(x_0) \times [\delta, T - \delta]$. Denote by $\overline{p}(t)$ the mean value of $p(\cdot, t)$ in $U(x_0)$. Obviously, $\overline{p} \in L^{5/3}(0, T)$ and $p - \overline{p}$ is in $L^{\infty}(U(x_0) \times (\delta, T - \delta))$.

Let $(x_1, t_1)$ be another regular point of the solution $(u, b, p)$. Choose $\delta > 0$ so small that $2\delta < t_1 < T - 2\delta$. There exists a simple smooth curve $C$ of a finite length in $\Omega$, connecting the points $x_0$ and $x_1$, such that each point of $C \times \{t_1\}$ is a regular point of $(u, b, p)$. (This follows again from the facts that $\mathcal{T}(Q_T)$ is closed in $Q_T$ and the 1D Hausdorff measure of $\mathcal{T}(Q_T)$ is zero.) As $C$ is compact, there exists a space–time neighborhood $U(C)$, such that $\nabla p$ is essentially bounded in $U(C)$. Consequently, there exist a spatial neighborhood $U(x_1) \subset \Omega$ and $\xi > 0$, such that $p - \overline{p}$ is essentially bounded in $U(x_1) \times (t_1 - \xi, t_1 + \xi)$.

Put $\zeta(t) := -\overline{p}(t)$. We have proven that $p + \zeta$ is essentially bounded in some space–time neighborhood of any regular point. Obviously, $(u, b, p + \zeta)$ is a suitable weak solution of the problem (1.1), (1.2), (1.3), (1.4) and (1.5).

Recall that we denote by $\mathcal{P}$ the family of all pressures, associated with the weak solution $(u, b)$, that are essentially bounded in some neighborhood of each regular point, and we assume that the pressure $p$ we deal with belongs to $\mathcal{P}$. (See Sect. 1.2.) The essential boundedness of $p$ in the neighborhood of each regular point enables us to deduce that the function $f$, defined by formula (2.5), is in $L^\infty(T_1 + 2\delta, T_2 - 2\delta; L^\infty(\mathbb{R}^3))$ at the end of Sect. 2.1. This implies that the function $p_2$, defined by (5.4), is in $L^\infty(T_1 + 2\delta, T_2 - 2\delta; L^\infty(\mathbb{R}^3))$ (see Sect. 5.1), and consequently the function $\hat{p}_1$, defined by (5.3), satisfies the inclusion (5.5). This inclusion further plays an important role in the proof of Theorem 1.

Note that in order to show that the pressure $p$ can be chosen so that $p \in \mathcal{P}$, we have strongly used the fact that $\nabla p$ is essentially bounded in some space–time neighborhood of any regular point of the solution $(u, b, p)$. This property of $p$ follows from [15, Sects. 4 and 6.3] in the case that $\Omega = \mathbb{R}^3$ or $\Omega$ is a smooth bounded domain and $u$ satisfies the Navier–type boundary conditions (1.5). If $u$ satisfies Dirichlet’s or Navier’s boundary conditions then the arguments used above fail, because if $(x, t)$ is a regular point of $(u, b, p)$ then we only know that there exist $U(x)$ and $\xi > 0$ such that $\nabla p$ is in $L^\mu(t - \xi, t + \xi; L^\infty(U(x)))$ for any $\mu \in (1, 2)$ (in the case of homogeneous Dirichlet’s boundary condition for $u$) or $\nabla p \in L^4(t - \xi, t + \xi; L^\infty(U(x)))$ (in the case of Navier’s slip boundary conditions for $u$, see [15]).

The dependence on the boundary conditions is not surprising, because the pressure is a global quantity and its local behavior anywhere in $\Omega$ is affected, among other things, by boundary conditions satisfied by the velocity.

2. Localization of the Initial–Boundary Value Problem (1.1), (1.2), (1.3), (1.4) and (1.5)

Let $(u, b, p)$, respectively $\Omega'$, respectively $(T_1, T_2)$, be the suitable weak solution, respectively the domain, respectively the interval, from Theorem 1. As the solution $(u, b, p)$ is suitable, one can apply the same arguments as in the theory of the Navier–Stokes equations (see the so called “Theorem on Structure” in [6]) and show that the time interval $(0, T)$ can be expressed as a union $\bigcup_{\gamma \in \Gamma'}(a_\gamma, b_\gamma) \cup T$, where set $\Gamma$ is at most countable, the Lebesgue measure of $T$ is zero, the intervals $(a_\gamma, b_\gamma)$ are mutually disjoint and the functions $u$ and $b$ are of the class $C^2$ on each set $\Omega \times (a_\gamma, b_\gamma)$ for $\gamma \in \Gamma$. This implies that $(T_1, T_2) = \bigcup_{\gamma \in \Gamma'}(a'_\gamma, b'_\gamma) \cup T'$, where $\Gamma'$ is at most countable, $T' \subset T$, the intervals $(a'_\gamma, b'_\gamma)$ are non-overlapping and the functions $u$ and $b$ are of the class $C^2$ on each set $\Omega' \times (a'_\gamma, b'_\gamma)$ for $\gamma \in \Gamma'$. In accordance with Sect. 1.4, we assume (by contradiction) that $t_0 \in (T_1, T_2)$ is an $\Omega'$-epoch of irregularity. It means that $t_0$ coincides with one of the numbers $b'_\gamma$ (for $\gamma \in \Gamma'$) and the solution blows up for $t \to t_0$—in a neighborhood of some point in $\Omega'$.

2.1. Localization to the Neighborhood of $\mathcal{B}$ and the Definition of the Functions $\hat{u}, \hat{b}, \hat{p}$

Recall that $\mathcal{B}$ is an arbitrary ball inside $\Omega'$, which means that $\mathcal{B} \subset \overline{\mathcal{B}} \subset \Omega'$. For $0 < \rho_1 < \rho_2$, we denote by $U_{\rho_1}(\mathcal{B})$ the neighborhood $\{x \in \mathbb{R}^3; \text{dist}(x, \mathcal{B}) < \rho_1\}$ and by $A_{\rho_1, \rho_2}(\mathcal{B})$ the annulus $\{x \in \mathbb{R}^3; \rho_1 < \text{dist}(x, \mathcal{B}) < \rho_2\}$.
There exist positive numbers $\rho_1$ and $\rho_2$ such that $0 < \rho_1 < \rho_2$, $U_{\rho_2}(B) \subset \Omega'$ and $\frac{A_{\rho_1,\rho_2}(B)}{2} \times (T_1, T_2)$ contains no singular points of the solution $(u, b, p)$. This follows (by means of the same arguments as in [14] or [13]) from the fact that the 1-dimensional Hausdorff measure of the set $\mathcal{H}(\Omega' \times (T_1, T_2))$ is zero. Thus, $u$ and $b$ are essentially bounded in $A_{\rho_1,\rho_2}(B) \times (T_1 + \delta, T_2 - \delta)$ for each $\delta > 0$ sufficiently small. Applying the results of [11], one can deduce that $u$, $b$, $\partial_t b$ and all their spatial derivatives (of all orders) are bounded and Hölder-continuous in $A_{\rho_3,\rho_4}(B) \times (T_1 + 2\delta, T_2 - 2\delta)$ for all $\rho_3$ and $\rho_4$ such that $\rho_1 < \rho_3 < \rho_4 < \rho_2$. Moreover, due to Sect. 1.6), $p$ and all its spatial derivatives, and consequently also $\partial_t u$ together with all its spatial derivatives, are essentially bounded in $A_{\rho_3,\rho_4}(B) \times (T_1 + 2\delta, T_2 - 2\delta)$.

We further assume that $\delta > 0$, satisfying $T_1 + 2\delta < t_0 < T_2 - 2\delta$, is fixed.

Note that the assumption ($\star$) on domain $\Omega$ enables us to apply the results from [15].

Let $\rho_5$ and $\rho_6$ be positive numbers, satisfying $\rho_3 < \rho_5 < \rho_6 < \rho_4$. Let $\eta$ be an infinitely differentiable cut-off function in $\mathbb{R}^3$ with values in $[0, 1]$, such that

$$\eta = \begin{cases} 
1 & \text{in } U_{\rho_5}(B), \\
0 & \text{in } \mathbb{R}^3 \setminus U_{\rho_6}(B).
\end{cases}$$

Obviously, $\text{div} (\eta u) = \nabla \eta \cdot u$ and $\text{div} (\eta b) = \nabla \eta \cdot b$. In order to obtain divergence-free functions, we put

$$\hat{u} := \eta u - u_{\text{corr}} \quad \text{and} \quad \hat{b} := \eta b - b_{\text{corr}},$$

where the corrections $u_{\text{corr}}$ and $b_{\text{corr}}$ satisfy $\text{div} u_{\text{corr}} = \nabla \eta \cdot u$ and $\text{div} b_{\text{corr}} = \nabla \eta \cdot b$. The existence of appropriate $u_{\text{corr}}$ and $b_{\text{corr}}$ follows e.g. from [7, Theorem III.3.3]. Due to this theorem, to given $m \in \{0\} \cup \mathbb{N}$, there exists a linear mapping $\mathfrak{B}_m$ from $W^{m,2}_0(A_{\rho_3,\rho_4}(B)) \to W^{m+1,2}_0(A_{\rho_3,\rho_4}(B))$, such that for all $f \in W^{m,2}_0(A_{\rho_3,\rho_4}(B))$, satisfying $\int_{A_{\rho_3,\rho_4}(B)} f \, dx = 0$, one has

1. $\text{div} \mathfrak{B}_m f = f$ a.e. in $A_{\rho_3,\rho_4}(B)$,
2. $\|\nabla^{m+1} \mathfrak{B}_m f\|_2(A_{\rho_3,\rho_4}(B)) \leq c \|\nabla^m f\|_2(A_{\rho_3,\rho_4}(B))$,

where $c$ is independent of $f$. (Mapping $\mathfrak{B}_m$ is the so called Bogovskij–Pileckas operator.) Obviously, $\partial A_{\rho_3,\rho_4}(B) = \partial U_{\rho_3}(B) \cup \partial U_{\rho_4}(B)$. Since $\eta = 1$ on $\partial U_{\rho_3}(B)$, $\eta = 0$ on $\partial U_{\rho_4}(B)$ and $\text{div} u = 0$ in $A_{\rho_3,\rho_4}(B)$, we have

$$\int_{A_{\rho_3,\rho_4}(B)} \nabla \eta \cdot u \, dx = \int_{\partial U_{\rho_3}(B)} \eta u \cdot n \, dS + \int_{\partial U_{\rho_4}(B)} \eta u \cdot n \, dS - \int_{A_{\rho_3,\rho_4}(B)} \eta \text{div} u \, dx = \int_{\partial U_{\rho_3}(B)} u \cdot n \, dS - \int_{\partial U_{\rho_4}(B)} u \cdot n \, dS - \int_{A_{\rho_3,\rho_4}(B)} \text{div} u \, dx = 0.$$

The function $\nabla \eta \cdot b$ satisfies the same identities. Thus, we may put

$$u_{\text{corr}}(., t) := \mathfrak{B}_3[\nabla \eta \cdot u(., t)] \quad \text{and} \quad b_{\text{corr}}(., t) := \mathfrak{B}_3[\nabla \eta \cdot b(., t)].$$

As $\nabla \eta \cdot u \in L^\infty(T_1 + 2\delta, T_2 - 2\delta; W^{3,2}_0(A_{\rho_3,\rho_4}(B)))$, the function $u_{\text{corr}}$ belongs to $L^\infty(T_1 + 2\delta, T_2 - 2\delta; W^{4,2}_0(A_{\rho_3,\rho_4}(B)))$. Applying the same arguments, we obtain that $b_{\text{corr}} \in L^\infty(T_1 + 2\delta, T_2 - 2\delta; W^{5,2}_0(A_{\rho_3,\rho_4}(B)))$, too.

Extending $u_{\text{corr}}$ and $b_{\text{corr}}$ by zero outside $A_{\rho_3,\rho_4}(B)$, and extending also $\eta u$ and $\eta b$ by zero outside $\Omega$, we observe that the functions $\hat{u}$ and $\hat{b}$, defined by (2.1), are divergence-free in $\mathbb{R}^3 \setminus (T_1 + 2\delta, T_2 - 2\delta)$, they coincide with $u$ and $b$, respectively, in $U_{\rho_2}(B) \times (T_1 + 2\delta, T_2 - 2\delta)$, they are equal to zero in $(\mathbb{R}^3 \setminus U_{\rho_4}(B)) \times (T_1 + 2\delta, T_2 - 2\delta)$ and all their spatial derivatives are essentially bounded in $A_{\rho_3,\rho_4}(B) \times (T_1 + 2\delta, T_2 - 2\delta)$.

Put $\tilde{p}(., t) := \eta p(., t)$ in $\Omega$ and $\tilde{p}(., t) := 0$ in $\mathbb{R}^3 \setminus \Omega$. Multiplying the Eqs. (1.1), (1.2) by $\eta$ and using the formulas

$$\eta \nabla (p + \frac{1}{2}|b|^2) = \nabla (\eta p + \frac{1}{2} \eta |b|^2) - p \nabla \eta - \frac{1}{2} |b|^2 \nabla \eta$$

and

$$\eta \text{div} (\eta u) = \nabla \eta \cdot u$$

we arrive at

$$\int_{A_{\rho_3,\rho_4}(B)} \eta \text{div} \tilde{p} \, dx = \int_{A_{\rho_3,\rho_4}(B)} \tilde{p} \, dx = \int_{A_{\rho_3,\rho_4}(B)} \eta \text{div} \tilde{p} \, dx.$$
\begin{equation}
= \nabla (\eta \rho + \frac{1}{2} \eta^2 |b|^2 + \frac{1}{2} \eta (1 - \eta) |b|^2) - \rho \nabla \eta - \frac{1}{2} |b|^2 \nabla \eta
\end{equation}
\begin{equation}
= \nabla (\tilde{\rho} + \frac{1}{2} |\tilde{b}|^2) - \nabla (\tilde{b} \cdot \tilde{c}_\text{corr}) - \frac{1}{2} \nabla |\tilde{c}_\text{corr}|^2 + \frac{1}{2} \nabla (\eta (1 - \eta) |b|^2]
\end{equation}

one can deduce that \((\tilde{u}, \tilde{b}, \tilde{p})\) is a suitable weak solution to the system of equations

\begin{equation}
\partial_t \tilde{u} + \tilde{u} \cdot \nabla \tilde{u} - \tilde{b} \cdot \nabla \tilde{b} = -\nabla (\tilde{\rho} + \frac{1}{2} |\tilde{b}|^2) + \frac{1}{R_f} \Delta \tilde{u} + f, \tag{2.2}
\end{equation}
\begin{equation}
\partial_t \tilde{b} + \tilde{u} \cdot \nabla \tilde{b} - \tilde{b} \cdot \nabla \tilde{u} = \frac{1}{R_m} \Delta \tilde{b} + g, \tag{2.3}
\end{equation}
\begin{equation}
\text{div} \tilde{u} = \text{div} \tilde{b} = 0 \tag{2.4}
\end{equation}

in \(\mathbb{R}^3 \times (T_1 + 2\delta, T_2 - 2\delta)\), where

\begin{equation}
f = -\partial_t u_{\text{corr}} - \eta (1 - \eta) u \cdot \nabla u + \eta (1 - \eta) b \cdot \nabla b + \eta (u \cdot \nabla \eta) u - \eta (b \cdot \nabla \eta) b
\end{equation}
\begin{equation}
- u_{\text{corr}} \cdot \nabla (\eta u) - \eta u \cdot \nabla u_{\text{corr}} + u_{\text{corr}} \cdot \nabla u_{\text{corr}} + \nabla (\eta b) + \eta b \cdot \nabla b_{\text{corr}}
\end{equation}
\begin{equation}
- b_{\text{corr}} \cdot \nabla b_{\text{corr}} + \nabla [(\eta b + b_{\text{corr}}) \cdot b_{\text{corr}}] + \frac{1}{2} \nabla |b_{\text{corr}}|^2 - \frac{3}{2} \nabla (\eta (1 - \eta) |b|^2]
\end{equation}
\begin{equation}
+ \frac{1}{2} |b|^2 \nabla \eta - \frac{2}{R_f} \nabla \eta \cdot \nabla u - \frac{1}{R_f} (\Delta \eta) u + \rho \nabla \eta, \tag{2.5}
\end{equation}
\begin{equation}
g = -\partial_t b_{\text{corr}} - \eta (1 - \eta) u \cdot \nabla b + \eta (1 - \eta) b \cdot \nabla u + \eta (u \cdot \nabla \eta) u - \eta (b \cdot \nabla \eta) b
\end{equation}
\begin{equation}
- u_{\text{corr}} \cdot \nabla (\eta b) - \eta u \cdot \nabla b_{\text{corr}} + u_{\text{corr}} \cdot \nabla b_{\text{corr}} + \nabla (\eta u) + \eta b \cdot \nabla u_{\text{corr}}
\end{equation}
\begin{equation}
- b_{\text{corr}} \cdot \nabla b_{\text{corr}} + \frac{2}{R_m} \nabla \eta \cdot \nabla b - \frac{1}{R_m} (\Delta \eta) b. \tag{2.6}
\end{equation}

The functions \(f(.,t)\) and \(g(.,t)\) are (for each fixed \(t \in (T_1 + 2\delta, T_2 - 2\delta)\)) supported in \(A_{\rho_3,\rho_4} (\mathbb{B})\) and \(f, g \in L^\infty (T_1 + 2\delta, T_2 - 2\delta; L^\infty (\mathbb{R}^3))\). The same statements also hold on the spatial derivatives of \(f\) and \(g\).

By analogy with \(\mathcal{I} (\Omega_1 \times I)\) and \(\mathcal{I}_I (\Omega_1)\), we denote by \(\tilde{\mathcal{I}} (\Omega_1 \times I)\) the set of singular points of the solution \((\tilde{u}, \tilde{b}, \tilde{p})\) in \(\Omega_1 \times I\) and by and \(\tilde{\mathcal{I}}_I (\Omega_1)\) the set of \(x \in \Omega_1\), such that \((x, t)\) is a singular point of \((\tilde{u}, \tilde{b}, \tilde{p})\).

### 3. Several Auxiliary Results

The purpose of this section is to present and prove two lemmas and one corollary, which will later clarify the reasons for the use of the function \(\mathcal{I}_I \) in conditions a) and b) of Theorem 1.

**Lemma 1.** Let \(f\) be a nonnegative measurable function in \(\mathbb{R}^3\). If \(\mathcal{I}_I (\rho) \in L^{3/2} (\mathbb{R}^3)\) then there exist \(R_0 \in (0,1]\) and \(c_1 > 0\), depending only on \(\gamma\), such that

\begin{equation}
\int_{B_r(x_0)} f(x) \, dx \leq c_1 \frac{r}{(\ln r^{-1})^{1+\gamma}} \tag{3.1}
\end{equation}

for all \(0 < r < R_0\) and \(x_0 \in \mathbb{R}^3\).

**Proof.** We split the integral into two parts and for \(0 < r < 1\) denote

\begin{equation}
J_1(x_0, r) := \int_{B_r(x_0) \cap M_1} f(x) \, dx,
\end{equation}
\begin{equation}
J_2(x_0, r) := \int_{B_r(x_0) \cap M_2} f(x) \, dx, \tag{3.2}
\end{equation}

where $M_1 := \{ x \in \mathbb{R}^3; f(x) \leq 1 \}$ and $M_2 := \{ x \in \mathbb{R}^3; f(x) > 1 \}$. It is easy to see that
\[
J_1(x_0, r) \leq \int_{B_r(x_0)} dx = \frac{4\pi}{3} r^3 = \frac{4\pi}{3} r^2 (\ln r^{-1})^{1+\gamma} \frac{r}{(\ln r^{-1})^{1+\gamma}}.
\]
Since an elementary computation shows that
\[
\lim_{r \to 0^+} r^2 (\ln r^{-1})^{1+\gamma} = 0,
\]
there is a positive number $R_1$, depending only on $\gamma$, such that for all $0 < r < R_1$ and $x_0 \in \mathbb{R}^3$,
\[
J_1(x_0, r) \leq \frac{4\pi}{3} \frac{r}{(\ln r^{-1})^{1+\gamma}}.
\]
(3.3)

We write the second integral as
\[
J_2(x_0, r) = \int_{B_r(x_0)} g(x) \, dx,
\]
where $g(x) := f(x) \chi_{M_2}(x)$. ($\chi_{M_2}$ is the characteristic function of set $M_2$.) For $s \geq 0$, define
\[
\Phi_\gamma(s) := \mathcal{F}_\gamma^{3/2}(s) = s^{3/2} \left[ \ln (1+s) \right]^{-\gamma/2}.
\]
(3.4)

As the function $\mathcal{F}_\gamma$ is increasing and strictly convex on $[0, \infty)$, the function $\Phi_\gamma$ has the same properties. Now, we apply Jensen’s inequality to obtain that
\[
\Phi_\gamma \left( \frac{3}{4\pi r^3} \int_{B_r(x_0)} g(x) \, dx \right) \leq \frac{3}{4\pi r^3} \int_{B_r(x_0)} \Phi_\gamma(g(x)) \, dx.
\]
(3.5)

Since $\mathcal{F}_\gamma(f) \in L^{3/2}(\mathbb{R}^3)$, we have
\[
c_2 := \frac{3}{4\pi} \int_{\mathbb{R}^3} \Phi_\gamma(g(x)) \, dx < \infty.
\]

As $\Phi_\gamma$ is bijective and its inverse function $\Phi_\gamma^{-1}$ is also increasing, we obtain from (3.5) that
\[
J_2(x_0, r) \equiv \int_{B_r(x_0)} g(x) \, dx \leq \frac{4\pi r^3}{3} \Phi_\gamma^{-1} \left( \frac{3}{4\pi r^3} \int_{B_r(x_0)} \Phi_\gamma(g(x)) \, dx \right)
\]
\[
= \frac{4\pi r^3}{3} \Phi_\gamma^{-1}(r^{-3}c_2) = \frac{4\pi r^3}{3} A_\gamma(r),
\]
(3.6)

where $A_\gamma(r) := \Phi_\gamma^{-1}(r^{-3}c_2)$. Since $\Phi_\gamma(A_\gamma(r)) = \mathcal{F}_\gamma^{3/2}(A_\gamma(r)) = r^{-3}c_2$, we have
\[
r^2 \mathcal{F}_\gamma(A_\gamma) = r^2 A_\gamma(r) [\ln (1 + A_\gamma(r))]^{1+\gamma} = c_2^{2/3}.
\]

Thus, we obtain the identities
\[
r^3 A_\gamma(r) = r^2 A_\gamma(r) [\ln (1 + A_\gamma(r))]^{1+\gamma} \left( \frac{\ln r^{-1}}{\ln (1 + A_\gamma(r))} \right)^{1+\gamma} \frac{r}{(\ln r^{-1})^{1+\gamma}}
\]
\[
= c_2^{2/3} \left( \frac{\ln r^{-1}}{\ln (1 + A_\gamma(r))} \right)^{1+\gamma} \frac{r}{(\ln r^{-1})^{1+\gamma}}.
\]
(3.7)

Combining (3.6) and (3.7) we get
\[
J_2(x_0, r) \leq \frac{4\pi}{3} c_2^{2/3} \left( \frac{\ln r^{-1}}{\ln (1 + A_\gamma(r))} \right)^{1+\gamma} \frac{r}{(\ln r^{-1})^{1+\gamma}}.
\]
(3.8)

Finally, we show that there is a positive number $R_2$, depending only on $\gamma$, such that
\[
\frac{\ln r^{-1}}{\ln (1 + A_\gamma(r))} \leq 1
\]
(3.9)
for all $0 < r < R_2$. Then we can take $R_0 = \min\{R_1, R_2\}$ and, from (3.3), (3.8), and (3.9), we deduce that (3.1) holds with

$$c_1 := \frac{4\pi}{3} + \frac{4\pi}{3} \frac{2^{3/2}}{c_2}.$$  

Inequality (3.9) is equivalent to $r^{-1} - 1 \leq A_\gamma(r)$. Since $\Phi_\gamma$ is increasing and $\Phi_\gamma(A_\gamma(r)) = r^{-3}c_2$, it is also equivalent to $r^3 \Phi_\gamma(r^{-1} - 1) < r^3 \Phi_\gamma(A_\gamma(r)) = c_2$. Thus, if we prove that

$$\lim_{r \to 0^+} r^3 \Phi_\gamma(r^{-1} - 1) = 0,$$

(3.10)

then there is a positive number $R_2$, depending only on $\gamma$, such that for all $0 < r < R_2$, (3.9) holds true. Indeed, we have $r^3 \Phi_\gamma(r^{-1} - 1) = (r^2 \mathcal{F}_\gamma(r^{-1} - 1))^{3/2}$ and

$$\lim_{r \to 0^+} r^2 \mathcal{F}_\gamma(r^{-1} - 1) = \lim_{r \to 0^+} r(1 - r) (\ln r^{-1})^{1+\gamma} = 0,$$

which yields (3.10). This completes the proof. \(\square\)

**Lemma 2.** Let $f$ be a nonnegative measurable function in $\mathbb{R}^3$. If there exists $c_3 > 0$ and $R_0 \in (0, 1]$ such that

$$\int_{B_r(x_0)} f(x) \, dx \leq c_3 \frac{r}{(\ln r^{-1})^{1+\gamma}}$$

(3.11)

for all $0 < r < R_0$ and $x_0 \in \mathbb{R}^3$, then

$$\lim_{r \to 0^+} \sup_{x_0 \in \mathbb{R}^3} \int_{B_r(x_0)} \frac{f(x)}{|x - x_0|} \, dx = 0.$$  

(3.12)

**Proof.** Define the measure $\mu$ on $\mathbb{R}^3$ by the formula $d\mu = f(x) \, dx$. Then we can rewrite the integral in (3.12) as

$$\int_{B_r(x_0)} \frac{f(x)}{|x - x_0|} \, dx = \int_{B_r(x_0)} \frac{d\mu}{|x - x_0|} = \int_0^\infty \mu\{x \in B_r(x_0); |x - x_0|^{-1} > \xi\} \, d\xi = \int_0^\infty \mu\{x \in B_r(x_0); |x - x_0| < \zeta\} \frac{d\zeta}{\xi^2},$$

where the last equality comes from the substitution $\xi = \zeta^{-1}$. In order to estimate the last integral, we split it into two parts and use the condition (3.11). We obtain

$$\int_0^\infty \mu\{x \in B_r(x_0); |x - x_0| < \zeta\} \frac{d\zeta}{\xi^2}$$

$$= \int_0^r \mu(B_\zeta(x_0)) \frac{d\zeta}{\xi^2} + \int_r^\infty \mu(B_r(x_0)) \frac{d\zeta}{\xi^2}$$

$$= \int_0^r \left( \int_{B_\zeta(x_0)} f(x) \, dx \right) \frac{d\zeta}{\xi^2} + \frac{1}{r} \int_{B_r(x_0)} f(x) \, dx$$

$$\leq \int_0^r \frac{c_3 \zeta}{(\ln \zeta^{-1})^{1+\gamma}} \frac{d\zeta}{\xi^2} + \frac{c_3}{(\ln r^{-1})^{1+\gamma}}.$$  

(3.13)

It is easy to see that

$$\lim_{r \to 0^+} \frac{c_3}{(\ln r^{-1})^{1+\gamma}} = 0.$$

Finally, an elementary computation shows that the last integral in (3.13) becomes

$$\int_0^r \frac{c_3 \zeta}{(\ln \zeta^{-1})^{1+\gamma}} \frac{d\zeta}{\xi^2} = \int_0^r \frac{c_3}{\zeta^{(\ln \zeta)^{1+\gamma}}} d\zeta = c_3 \int_{-\infty}^{ln r} \frac{d\eta}{(-\eta)^{1+\gamma}}$$

$$= c_3 \int_{-\ln r}^0 \frac{d\eta}{\eta^{1+\gamma}} = \frac{c_3}{\gamma (\ln r^{-1})^\gamma},$$
which also goes to 0 as \( r \to 0^+ \). This completes the proof. \( \square \)

**Corollary 1.** Let \( f \) be a nonnegative measurable function in \( \mathbb{R}^3 \times I \), where \( I \subset \mathbb{R} \). If there exists \( c_4 > 0 \) such that \( \| \mathcal{F}_\gamma(f(\cdot, t)) \|_{3/2; \mathbb{R}^3} \leq c_4 \) for all \( t \in I \) then

\[
\lim_{r \to 0^+} \sup_{x_0 \in \mathbb{R}^3} \int_{B_r(x_0)} \frac{f(x,t)}{|x - x_0|} \, dx = 0
\]  

(3.14)

uniformly with respect to \( t \in I \).

**Proof.** The validity of (3.14) follows from Lemmas 1 and 2. The uniformity of the limit follows from the fact that constant \( c_1 \) in (3.1) (where we now consider \( f(x,t) \) instead of just \( f(x) \)) can be chosen to be independent \( x_0 \) and \( t \), due to the boundedness of \( \| \mathcal{F}_\gamma(f(\cdot, t)) \|_{3/2; \mathbb{R}^3} \) for \( t \in I \). \( \square \)

### 4. Proof of Theorem 1 Under Condition a).

Recall that \((\hat{u}, \hat{b}, \hat{p})\) is a suitable weak solution to the system of MHD Eqs. (2.2), (2.3) and (2.4) in \( \mathbb{R}^3 \times (T_1 + 2\delta, T_2 - 2\delta) \), satisfying the identities \( \hat{u} = u \), \( \hat{b} = b \), \( \hat{p} = p \) in \( U_{\rho_3}(\mathcal{B}) \times (T_1 + 2\delta, T_2 - 2\delta) \) and \( \hat{u} = 0 \), \( \hat{b} = 0 \), \( \hat{p} = 0 \) in \( [\mathbb{R}^3 \setminus U_{\rho_4}(\mathcal{B})] \times (T_1 + 2\delta, T_2 - 2\delta) \). As \( p \) is essentially bounded in \( A_{\rho_3, \rho_4}(\mathcal{B}) \times (T_1 + 2\delta, T_2 - 2\delta) \), condition a) of Theorem 1 yields

\[
\mathcal{F}_\gamma(\hat{p}_-) \in L^\infty(T_1 + 2\delta, T_2 - 2\delta; L^{3/2}(\mathbb{R}^3)).
\]  

(4.1)

Recall that the \( \Omega' \)-epoch of irregularity \( t_0 \) coincides with \( b'_\gamma \). In addition to the inequalities \( T_1 + 2\delta < t_0 < T_2 - 2\delta \), we may assume without loss of generality that \( a'_\gamma \) is so close to \( b'_\gamma \) and \( \delta > 0 \) is so small that \( T_1 + 2\delta \leq a'_\gamma < b'_\gamma \equiv t_0 < T_2 - 2\delta \). It follows from the construction of \( \hat{u} \) and \( \hat{b} \) and from the smoothness of \( u \) and \( b \) on \( \Omega' \times (a'_\gamma, b'_\gamma) \) that \( \hat{u} \) and \( \hat{b} \) are of the class \( C^2 \) on \( \Omega' \times (a'_\gamma, b'_\gamma) \).

#### 4.1. Splitting of \( \hat{p} \) to the Sum \( \hat{p}_1 + \hat{p}_2 \)

Applying the operator \( \text{div} \) to Eq. (2.2), we obtain

\[
\Delta [\hat{p}(\cdot, t) + \frac{1}{2} \hat{b}(\cdot, t)^2] = -\text{div}[\hat{u}(\cdot, t) \cdot \nabla \hat{u}(\cdot, t) - \hat{b}(\cdot, t) \cdot \nabla \hat{b}(\cdot, t)] + \text{div} f
\]

in \( \mathbb{R}^3 \) for all \( t \in (a'_\gamma, b'_\gamma) \). This yields, on the other hand, that

\[
\hat{p} = \hat{p}_1 + \hat{p}_2,
\]  

(4.2)

where

\[
\hat{p}_1(x,t) := -\frac{\hat{b}(x,t)^2}{2} + \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x - y|} \, \text{div}[\hat{u}(y,t) \cdot \nabla \hat{u}(y,t)] \, dy
\]

\[
-\hat{b}(y,t) \cdot \nabla \hat{b}(y,t)
\]

(4.3)

\[
\hat{p}_2(x,t) := -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x - y|} \, \text{div} f(y,t) \, dy.
\]  

(4.4)

It follows from the properties of the function \( f \) (see the end of Sect. 2.1) that \( \hat{p}_2 \in L^\infty(T_1 + 2\delta, T_2 - 2\delta; L^\infty(\mathbb{R}^3)) \). Hence \( \Phi_\gamma(\hat{p}_2) \in L^\infty(T_1 + 2\delta, T_2 - 2\delta; L^\infty(\mathbb{R}^3)) \). (Recall that the function \( \Phi_\gamma \) is defined by formula (3.4).) Obviously, \( \hat{p}_1 = (\hat{p} - \hat{p}_2)_- \leq \hat{p}_- + \hat{p}_2 \). Furthermore, due to Jensen’s inequality and the inequality \( \Phi_\gamma(2s) \leq 2^{3+\frac{5}{2}\gamma} \Phi_\gamma(s) \), we have \( \Phi_\gamma((\hat{p}_- + \hat{p}_2)_-) \leq \frac{1}{2} \Phi_\gamma(2\hat{p}_-) + \frac{1}{2} \Phi_\gamma(2\hat{p}_2) \leq 2^{3+\frac{5}{2}\gamma} \Phi_\gamma(\hat{p}_-) + 2^{3+\frac{5}{2}\gamma} \Phi_\gamma(\hat{p}_2) \). These inequalities and the inclusion (4.1) imply that \( \Phi_\gamma(\hat{p}_-) \in L^\infty(T_1 + 2\delta, T_2 - 2\delta; L^{3/2}(\mathbb{R}^3)) \), which means that

\[
\mathcal{F}_\gamma(\hat{p}_-) \in L^\infty(T_1 + 2\delta, T_2 - 2\delta; L^{3/2}(\mathbb{R}^3)).
\]  

(4.5)
4.2. More on the Formula (4.3)

The integral on the right hand side of (4.3) has a sense at all points \((x, t) \in \mathbb{R}^3 \times (a_\gamma, t_0)\). Let us show, that it also has a sense at all regular points \((x, t_0)\) of the solution \((\hat{u}, \hat{b}, \hat{p})\). Thus, let \(d > 0\) and \(x\) be a point in \(\mathbb{R}^3\), whose distance from \(\mathcal{S}_{t_0}(\mathbb{R}^3)\) (the set of all singular points of \((\hat{u}, \hat{b}, \hat{p})\)) in \(\mathbb{R}^3\) on the time level \(t_0\) is greater than or equal to \(2d\). Considering \(t = t_0\) and splitting the integral to the sum of the integral over \(B_d(x)\) and the integral over \(\mathbb{R}^3 \setminus B_d(x)\), and applying twice the integration by parts to the integral over \(\mathbb{R}^3 \setminus B_d(x)\), we obtain

\[
\int_{\mathbb{R}^3} \frac{1}{|x-y|} \ \text{div} \left[ \hat{u}(y, t_0) \cdot \nabla \hat{u}(y, t_0) - \hat{b}(y, t_0) \cdot \nabla \hat{b}(y, t_0) \right] \ dy = \int_{\mathbb{R}^3} \frac{1}{|x-y|} \ \frac{\partial^2}{\partial y_i \partial y_j} \left[ \hat{u}_i(y, t_0) \hat{u}_j(y, t_0) - \hat{b}_i(y, t_0) \hat{b}_j(y, t_0) \right] \ dy
\]

\[
= \int_{\mathbb{R}^3} \left[ \frac{1}{|x-y|} \left( \frac{\partial}{\partial y_i} \left( \frac{1}{|x-y|} \right) \right) n^x_j \left[ \hat{u}_i(y, t) \hat{u}_j(y, t) \right] - \hat{b}_i(y, t) \hat{b}_j(y, t) \right] \ dy + I_d^{(1)}(x, t_0) + I_d^{(2)}(x, t_0), \tag{4.6}
\]

where \(\hat{u}_i\) and \(\hat{b}_i\) \((i = 1, 2, 3)\) are the components of \(\hat{u}\) and \(\hat{b}\), respectively,

\[
I_d^{(1)}(x, t) = \int_{B_d(x)} \frac{1}{|x-y|} \ \frac{\partial^2}{\partial y_i \partial y_j} \left[ \hat{u}_i(y, t) \hat{u}_j(y, t) - \hat{b}_i(y, t) \hat{b}_j(y, t) \right] \ dy
\]

\[
+ \int_{S_d(x)} \frac{n^x_i}{|x-y|} \ \frac{\partial}{\partial y_j} \left[ \hat{u}_i(y, t) \hat{u}_j(y, t) - \hat{b}_i(y, t) \hat{b}_j(y, t) \right] \ dy S
\]

\[
- \int_{S_d(x)} \frac{\partial}{\partial y_i} \left( \frac{1}{|x-y|} \right) n^x_j \left[ \hat{u}_i(y, t) \hat{u}_j(y, t) \right] - \hat{b}_i(y, t) \hat{b}_j(y, t) \right] \ dy S, \tag{4.7}
\]

\[
I_d^{(2)}(x, t) = \int_{\mathbb{R}^3 \setminus B_d(x)} \frac{1}{|x-y|} \ \frac{\partial^2}{\partial y_i \partial y_j} \left( \frac{1}{|y-x|} \right) \left[ \hat{u}_i(y, t) \hat{u}_j(y, t) - \hat{b}_i(y, t) \hat{b}_j(y, t) \right] \ dy
\]

\[
= \int_{\mathbb{R}^3 \setminus B_d(x)} \left[ \mathbb{K}(y-x) : \left[ \hat{u}(y, t) \otimes \hat{u}(y, t) \right. \right] - \left. \hat{b}(y, t) \otimes \hat{b}(y, t) \right] \ dy. \tag{4.8}
\]

Here, \(S_d(x)\) denotes the sphere \(\{y \in \mathbb{R}^3: |y-x| = d\}\), \(n^x = (n^x_1, n^x_2, n^x_3)\) is the normal vector field on \(S_d(x)\), oriented to the interior of \(S_d(x)\) (which means that \(n^x(y) = (x-y)/d\) for \(y \in S_d(x)\)) and \(\mathbb{K}(y-x) := \nabla^2 |y-x|^{-1}\) is the second order tensor with the entries

\[
k_{ij}(y-x) = \frac{\partial^2}{\partial y_i \partial y_j} \left( \frac{1}{|y-x|} \right) = -\frac{\partial}{\partial y_i} \frac{y_j - x_j}{|y-x|^3} = 3 \frac{(y_i - x_i)(y_j - x_j)}{|y-x|^3} \frac{(y_i - x_i)}{|y-x|^3}
\]

for \(i, j = 1, 2, 3\). As all integrals in \(I_d^{(1)}(x, t_0)\) and \(I_d^{(2)}(x, t_0)\) converge, the right hand side of (4.3) (where \(t = t_0\)) makes sense. Thus, since \(d > 0\) can be chosen arbitrarily small , the function \(\hat{p}_1\) is defined by formula (4.3) not only in \(\mathbb{R}^3 \times (a_\gamma, t_0)\), but also at all regular points of the solution \((\hat{u}, \hat{b}, \hat{p})\) on the time level \(t = t_0\).

4.3. Important Integral Identities

Let \(x_0 \in \mathbb{R}^3, t \in (a_\gamma, t_0), R > 0\) and \(\alpha \in [0, 1]\). By (4.3), we have

\[
\int_{B_R(x_0)} |x_0 - y|^{-\alpha} \left[ \hat{p}_1(y, t) + \frac{1}{2} |\hat{b}(y, t)|^2 \right] \ dy
\]
Particularly, choosing \( \alpha = 1 \) and \( \alpha = 0 \), we get

\[
\int_{B_R(x_0)} \frac{1}{|x - x_0|} \left( 2\hat{p}_1 + |\hat{u}_p^{x_0}|^2 + |\hat{b}_p^{x_0}|^2 \right) \, dx
\]
\[ \int_{B_R(x_0)} \frac{1}{R} \left( 3\tilde{p}_1 + |\tilde{u}|^2 + \frac{1}{2} |\tilde{b}|^2 \right) \, dx \]
\[ = \int_{\mathbb{R}^3 \setminus B_R(x_0)} \frac{R^2}{|x - x_0|^3} \left[ 2 |\tilde{u}^{x_0}|^2 - |\tilde{u}^{x_0}_p|^2 - 2 |\tilde{b}^{x_0}|^2 + |\tilde{b}^{x_0}_p|^2 \right] \, dx. \quad (4.14) \]

4.4. The Continuity of \( \hat{u} \) and \( \hat{b} \) From \( (a'_\gamma , t_0] \) to \( L^2(\mathbb{R}^3) \)

As the solution \( (\tilde{u}, \tilde{b}, \tilde{p}) \) has no singular points in \( \mathbb{R}^3 \times (a'_\gamma , t_0) \), the norms \( \|\tilde{u}(., t)\|_{2; \mathbb{R}^3} \) and \( \|\tilde{b}(., t)\|_{2; \mathbb{R}^3} \) depend continuously on \( t \) for \( t \in (a'_\gamma , t_0) \). Our next aim in this subsection is to prove that
\[ \lim_{t \to t_0^-} \left( \|\tilde{u}(., t) - \tilde{u}(., t_0)\|_{2; \mathbb{R}^3} + \|\tilde{b}(., t) - \tilde{b}(., t_0)\|_{2; \mathbb{R}^3} \right) = 0. \quad (4.15) \]
We shall use the next lemma:

**Lemma 3.** Let \( \delta \in (0, t_0 - a'_\gamma) \). Then these two implications hold:
\[ \sup_{R > 0, \ x_0 \in U_{\rho_4}(\mathbb{R})} \frac{1}{R} \ \text{ess sup}_{t_0 - \delta < t < t_0} \|\tilde{u}(., t)\|_{2; B_R(x_0)} < \infty \]
\[ \implies \lim_{t \to t_0^-} \|\tilde{u}(., t) - \tilde{u}(., t_0)\|_{2; U_{\rho_4}(\mathbb{R})} = 0, \quad (4.16) \]
\[ \sup_{R > 0, \ x_0 \in U_{\rho_4}(\mathbb{R})} \frac{1}{R} \ \text{ess sup}_{t_0 - \delta < t < t_0} \|\tilde{b}(., t)\|_{2; B_R(x_0)} < \infty \]
\[ \implies \lim_{t \to t_0^-} \|\tilde{b}(., t) - \tilde{b}(., t_0)\|_{2; U_{\rho_4}(\mathbb{R})} = 0. \quad (4.17) \]

**Proof.** As the function \( u \) is weakly continuous from \( (t_0 - \delta, t_0] \) to \( L^2(\mathbb{R}^3) \), it is also weakly continuous from \( (t_0 - \delta, t_0] \) to \( L^2(B_R(x_0)) \). Hence, due to the lower semi-continuity of the norm of \( \tilde{u} \) in \( L^2(B_R(x_0)) \) in dependence of \( t \), we have
\[ \text{ess sup}_{t_0 - \delta < t < t_0} \|\tilde{u}(., t)\|_{2; B_R(x_0)}^2 = \sup_{t_0 - \delta < t \leq t_0} \|\tilde{u}(., t)\|_{2; B_R(x_0)}^2. \quad (4.18) \]
Hence one can consider just sup instead of ess sup on the left hand side of the implication (4.16). Then the validity of the implication (4.16) follows from Lemma 3.2 in [16]. The validity of the implication (4.17) can be confirmed in the same way. \( \square \)

In order to prove (4.15), let us at first show that the premises in the implications (4.16) and (4.17) in Lemma 3 are satisfied.

Due to (4.5), there exists set \( T \subset (a'_\gamma , t_0) \) of 1D Lebesgue measure zero such that the norm \( \|\tilde{\mathcal{F}}_{\gamma}(\tilde{p}_{1-}(., t))\|_{3/2; \mathbb{R}^3} \) is uniformly bounded for \( t \in (a'_\gamma , t_0) \setminus T \). Then, due to Corollary 1, there exists \( R_0 > 0 \) independent of \( x_0 \), such that
\[ \int_{\mathbb{R}^3} \frac{\tilde{p}_{1-}(x, t)}{|x - x_0|} \leq 1 \quad (4.19) \]
for all \( R \in (0, R_0) \), \( x_0 \in U_{\rho_4}(\mathbb{R}) \) and \( t \in (a'_\gamma , t_0) \setminus T \).

Let \( x_0 \in U_{\rho_4}(\mathbb{R}) \) and \( R \in (0, R_0) \). It follows from the second identity in (4.14) that at each time \( t \in (a'_\gamma , t_0) \setminus T \), we have
\[ \frac{1}{R} \int_{B_R(x_0)} \left( |\tilde{u}|^2 + \frac{1}{2} |\tilde{b}|^2 \right) \, dx \]
\[ \leq \frac{1}{R} \int_{B_R(x_0)} \left( |\tilde{u}|^2 + \frac{1}{2} |\tilde{b}|^2 + 3 [\tilde{p}_{1} + \tilde{p}_{1-}] \right) \, dx \]
respectively. Then the first term on the right hand side can be estimated as follows:

\[ \int_{B_R(x_0)} \frac{1}{|x - x_0|} \left( 2\hat{p}_1 + |\hat{u}_{p_0}^D|^2 + |\hat{b}_r^D|^2 \right) \, dx + \frac{3}{R} \int_{B_R(x_0)} \hat{p}_1 - \, dx \]

\[ \leq \int_{B_{R_0}(x_0)} \frac{1}{|x - x_0|} \left( 2\hat{p}_1 + |\hat{u}_{p_0}^D|^2 + |\hat{b}_r^D|^2 \right) \, dx + \frac{3}{R} \int_{B_{R_0}(x_0)} \hat{p}_1 - \, dx \]

\[ \leq \int_{B_{R_0}(x_0)} \frac{1}{|x - x_0|} \left( 2\hat{p}_1 + |\hat{u}_{p_0}^D|^2 + |\hat{b}_r^D|^2 \right) \, dx + \frac{3}{R} \int_{B_{R_0}(x_0)} \hat{p}_1 - \, dx \]

\[ \leq \int_{R_3 \setminus B_{R_0}(x_0)} \left( 2|\hat{u}_{r_0}^D|^2 - |\hat{u}_{p_0}^D|^2 - 2|\hat{b}_r^D|^2 - |\hat{b}_p^D|^2 \right) \, dx + \frac{3}{R} \int_{B_{R_0}(x_0)} \hat{p}_1 - \, dx \]

\[ \leq \frac{c}{R_0} \int_{R_3 \setminus B_{R_0}(x_0)} \left( \|\hat{u}(.t)\|^2 + \|\hat{b}(.t)\|^2 \right) \, dx + \frac{3}{R_0} \int_{B_{R_0}(x_0)} \hat{p}_1 - \, dx \]

where \( A_{R,R_0}(x_0) := \{ x \in R^3; R < |x - x_0| < R_0 \} \) and \( c \) is independent of \( x_0, t, R \) and \( R_0 \).

We have shown that the terms \( R^{-1} \|u(.,t)\|_{2:B_R(x_0)} \) and \( R^{-1} \|b(.,t)\|_{2:B_R(x_0)} \) are bounded above and the bound is independent of \( x_0, t, R \) for \( x_0 \in U_{\rho_d}(\mathcal{B}), t \in (a_\gamma, t_0) \times \mathcal{T} \) and \( R \in (0, R_0) \). If \( R > R_0 \), then, obviously, \( R^{-1} \|u(.,t)\|_{2:B_R(x_0)} \leq R_0^{-1} \|u(.,t)\|_{2:R^3} \leq c/R_0 \), where \( c \) is independent of \( x_0, t, R \) and \( R_0 \). (The same inequalities also hold for function \( b \).) This shows that the premises in the implications (4.16) and (4.17) are true. Thus, the statements of the implications are also true. Taking into account that \( \hat{u}(.,t) = \hat{b}(.,t) = 0 \) in \( R^3 \setminus U_{\rho_d}(\mathcal{B}) \), we obtain (4.15).

### 4.5. The Continuity of \( I^{(2)}_d \) in \( R^3 \times (a'_\gamma, t_0] \)

Let \( d > 0 \). Recall that the function \( I^{(2)}_d \) is defined by (4.8). Assume that \( \{ (x_n, t_n) \} \) is a sequence of points in \( R^3 \times (a'_\gamma, t_0] \), converging to a point \( (x_s, t_s) \in R^3 \times (a'_\gamma, t_0] \) for \( n \to \infty \). Obviously,

\[
|I^{(2)}_d(x_n, t_n) - I^{(2)}_d(x_s, t_s)| \\
\leq |I^{(2)}_d(x_n, t_n) - I^{(2)}_d(x_n, t_s)| + |I^{(2)}_d(x_n, t_s) - I^{(2)}_d(x_s, t_s)|.
\]

(4.20)

Denote, for simplicity, the expressions \( \hat{u}(y, t) \otimes \hat{u}(y, t) \) and \( \hat{b}(y, t) \otimes \hat{b}(y, t) \) by \( \hat{U}(y, t) \) and \( \hat{B}(y, t) \), respectively. Then the first term on the right hand side can be estimated as follows:

\[
|I^{(2)}_d(x_n, t_n) - I^{(2)}_d(x_n, t_s)|
\]
4.6. The Continuity of \( \hat{p}_1 \) and \( \hat{p}_{1-} \) in \( (\mathbb{R}^3 \times (a'_\gamma, t_0)) \setminus \widehat{\mathcal{F}}(\mathbb{R}^3 \times (a'_\gamma, t_0)) \)

Recall that points of \( \widehat{\mathcal{F}}(\mathbb{R}^3 \times (a'_\gamma, t_0)) \) may appear in \( \mathbb{R}^3 \times (0, t_0) \) only on the time level \( t = t_0 \). The function \( \hat{p}_1 \) satisfies

\[
\hat{p}_1(x, t) = -\frac{\|\hat{b}(x, t)\|^2}{2} + \frac{1}{4\pi} \left[ I^{(1)}_d(x, t) + I^{(2)}_d(x, t) \right],
\]

where

\[
I^{(2)}_d(x, t) - I^{(2)}_d(x, t) = \int_{\mathbb{R}^3 \setminus B_d(x_n)} K(y - x_n) : \left[ \hat{\nabla}(y, t) - \hat{\nabla}(y, t) \right] \, dy
\]

Since \( t_0 \in (0, t_0) \), the right hand side tends to zero for \( n \to \infty \) due to the continuity of \( \|\hat{u}(., t)\|_{2; \mathbb{R}^3} \) and \( \|\hat{b}(., t)\|_{2; \mathbb{R}^3} \) for \( t \in (a'_\gamma, t_0) \). The second term on the right hand side of (4.20) can be estimated in this way:

\[
|I^{(2)}_d(x, t) - I^{(2)}_d(x, t)|
\]

If \( n \) is so large that \( |x_n - x| < d \) then the first modulus on the right hand side is less than or equal to

\[
\frac{c}{d^3} \left( \int_{B_d(x_n) \setminus B_d(x_n)} + \int_{B_d(x_n) \setminus B_d(x_n)} \right) \frac{\partial^2}{\partial y_i \partial y_j} \left( \frac{1}{|y - x_n|} - \frac{1}{|y - x_n|} \right) \leq c \frac{|x_n - x|}{d^4}
\]

where \( c \) is independent of \( n \). This tends to zero for \( n \to \infty \), because both \( \hat{\nabla}(., t) \) and \( \hat{\nabla}(., t) \) are in \( L^1(\mathbb{R}^3) \times \mathbb{R}^3 \times \mathbb{R}^3 \) and the measures of \( B_d(x_n) \setminus B_d(x_n) \) and \( B_d(x_n) \setminus B_d(x_n) \) tend to zero as \( n \to \infty \). In order to show that the second modulus on the right hand side of (4.22) also tends to zero as \( n \to \infty \), consider \( n \) so large that \( |x_n - x| < \frac{1}{2} d \). Then \( |y - x_n| > \frac{1}{2} d \) for \( y \in \mathbb{R}^3 \setminus B_d(x_n) \).

Obviously, for these \( y \), the inequality \( |y - x| \geq d \) also holds true. Hence

\[
|k_{ij}(y - x_n) - k_{ij}(y - x_n)| = \left| \frac{\partial^2}{\partial y_i \partial y_j} \left( \frac{1}{|y - x_n|} - \frac{1}{|y - x_n|} \right) \right| \leq c \frac{|x_n - x|}{d^4},
\]

where \( c \) is independent of \( d \), \( x_n \) and \( x_n \). Thus, the second modulus on the right hand side of (4.22) is bounded above by

\[
c \frac{|x_n - x|}{d^4} \int_{\mathbb{R}^3} \left| \hat{\nabla}(y, t) - \hat{\nabla}(y, t) \right| \, dy,
\]

which tends to zero as \( n \to \infty \). Thus, we have shown that for each \( d > 0 \), the function \( I^{(2)}_d \) is continuous on \( \mathbb{R}^3 \times (a'_\gamma, t_0) \).
where the functions $I_{d}^{(1)}$ and $I_{d}^{(2)}$ are defined by (4.7) and (4.8). Also recall that we have already proven the continuity of $I_{d}^{(2)}$ in $\mathbb{R}^3 \times (a'_c, t_0]$ for any $d > 0$ in Sect. 4.5. We still need to show that $I_{d}^{(1)}$ is continuous in $(\mathbb{R}^3 \times (a'_c, t_0)) \setminus \mathcal{I}(\mathbb{R}^3 \times (a'_c, t_0))$.

The function $I_{d}^{(1)}$ is continuous at each point $(x, t) \in \mathbb{R}^3 \times (0, t_0]$, whose distance from $\mathcal{I}(\mathbb{R}^3 \times (a'_c, t_0))$ is greater than or equal to $2d$, due to the Hölder continuity of $\hat{u}$ and $\hat{b}$ in $U_{2d}(x, t)$ (the $2d$–neighborhood of $(x, t)$). Hence the same statement on continuity can also be made on $I_{d}^{(1)} + I_{d}^{(2)}$. However, as the sum $I_{d}^{(1)} + I_{d}^{(2)}$ is independent of $d$, because it equals $4\pi \hat{p}_1 + 2\pi |\hat{b}|^2$, it is a continuous function on the whole set $(\mathbb{R}^3 \times (a'_c, t_0)) \setminus \mathcal{I}(\mathbb{R}^3 \times (a'_c, t_0))$.

Consequently, $\hat{p}_1$ and $\hat{p}_1^-$ are continuous functions in $(\mathbb{R}^3 \times (a'_c, t_0)) \setminus \mathcal{I}(\mathbb{R}^3 \times (a'_c, t_0))$, too.

4.7. The Boundedness of $\|\mathcal{F}_\gamma(\hat{p}_1^-(\cdot, t))\|_{3/2; \mathbb{R}^3}$ up to the Epoch of Irregularity $t_0$

Let $\Omega_1$ be a bounded domain in $\mathbb{R}^3$ and $d > 0$. Since $\mathcal{F}_{t_0}(\mathbb{R}^3)$ is a compact subset of $U_{\rho_2}(B)$, we have

$$\int_{\Omega_1} \mathcal{F}_{3/2}^{\gamma}(\hat{p}_1^- (y, t_0)) \, dy = \lim_{d \to 0+} \int_{\Omega_1 \setminus U_d(\mathcal{F}_{t_0}(\mathbb{R}^3))} \mathcal{F}_{3/2}^{\gamma}(\hat{p}_1^- (y, t_0)) \, dy,$$

which does not exclude that both sides are equal to $\infty$. As $\hat{p}_1^-$ is continuous on $M_1 := (\mathbb{R}^3 \times (a'_c, t_0)) \setminus \mathcal{I}(\mathbb{R}^3 \times (a'_c, t_0))$ and $M_2 := [\Omega_1 \setminus U_d(\mathcal{F}_{t_0}(\mathbb{R}^3))] \times [t_1, t_0]$ (where $t_1 := (a'_c + t_0)/2$) is a compact subset of $M_1$, the function $\mathcal{F}_\gamma(\hat{p}_1^-)$ is uniformly continuous on $M_2$. Hence

$$\int_{\Omega_1 \setminus U_d(\mathcal{F}_{t_0}(\mathbb{R}^3))} \mathcal{F}_{3/2}^{\gamma}(\hat{p}_1^- (y, t_0)) \, dy = \lim_{t \to t_0^-} \int_{\Omega_1 \setminus U_d(\mathcal{F}_{t_0}(\mathbb{R}^3))} \mathcal{F}_{3/2}^{\gamma}(\hat{p}_1^- (y, t)) \, dy$$

$$\leq \sup_{t \in [t_1, t_0]} \|\mathcal{F}_\gamma(\hat{p}_1^-(\cdot, t))\|_{3/2; \mathbb{R}^3}^{3/2} \leq \text{ess sup}_{0 < t < t_0} \|\mathcal{F}_\gamma(\hat{p}_1^-(\cdot, t))\|_{3/2; \mathbb{R}^3}^{3/2} =: c_5.$$

This shows that the integral in the limit on the right hand side of (4.23) is finite and less than or equal to $c_5$. As $c_5$ is independent of $d$, the integral on the left hand side of (4.23) is less than or equal to $c_5$, too. Since $c_5$ is also independent of $\Omega_1$, the integral $\int_{\mathbb{R}^3} \mathcal{F}_{3/2}^{\gamma}(\hat{p}_1^- (y, t_0)) \, dy$ is less than or equal to $c_5$ as well.

4.8. Completion of the Proof of Theorem 1 Under Condition $a$)

In order to deny the existence of a singular point of the suitable weak solution $(\hat{u}, \hat{b}, \hat{p})$ on the time level $t = t_0$, we shall use the next lemma, which follows from Theorem 1.1 in [9]:

**Lemma 4.** Let $x_0 \in \mathbb{R}^3$. There exists $\epsilon_\ast > 0$, independent of $x_0$, such that if

$$\sup_{0 < R < R_\ast} \sup_{t_0 - R^2 \leq t \leq t_0} \frac{1}{R} \left( \|\hat{u}(\cdot, t)\|_{2; B_R(x_0)}^2 + \|\hat{b}(\cdot, t)\|_{2; B_R(x_0)}^2 \right) < \epsilon_\ast$$

(4.24)

for some $R_\ast > 0$ then $(x_0, t_0)$ is a regular point of the solution $(\hat{u}, \hat{b}, \hat{p})$.

Let $x_0 \in \mathbb{R}^3$. Our aim is to show that there exists $R_\ast > 0$ such that (4.24) holds. Note that due to Corollary 1 and the results of Sect. 4.7, we have

$$\lim_{r \to 0+} \int_{B_r(x_0)} \frac{\hat{p}_1^-(x, t)}{|x - x_0|} \, dx = 0$$

(4.25)
for all \( t \in (a'_\gamma, t_0) \). Since the norm \( \| \mathcal{S}_\gamma(\tilde{p}_1(\cdot, t)) \|_{3/2, 3} \) is bounded as a function of \( t \) on \( (a'_\gamma, t_0) \), the limit in (4.25) is uniform with respect to \( t \in (0, t_0) \). Moreover, at time \( t_0 \), we also have

\[
\int_{B_R(x_0)} \frac{1}{|x - x_0|} \left[ |\tilde{u}_p^{x_0}(x, t_0)|^2 + |\tilde{b}_r^{x_0}(x, t_0)|^2 + 2\tilde{p}_1(x, t_0) \right] \, dx
\]

\[
= \int_{B_R(x_0)} \frac{1}{|x - x_0|} \left[ |\tilde{u}_p^{x_0}(x, t_0)|^2 + |\tilde{b}_r^{x_0}(x, t_0)|^2 \right] \, dx + 2\tilde{p}_1(x, t_0) \, dx + \int_{B_R(x_0)} \left[ \frac{2\tilde{p}_1(x, t_0)}{|x - x_0|} \right] \, dy
\]

\[
= \int_{\mathbb{R}^3 \setminus B_R(x_0)} \frac{R^2}{|x - x_0|^3} \left[ 2|\tilde{u}_p^{x_0}(x, t_0)|^2 - |\tilde{u}_p^{x_0}(x, t_0)|^2 \right] \, dx
\]

\[
- 2|\tilde{b}_r^{x_0}(x, t_0)|^2 + |\tilde{b}_r^{x_0}(x, t_0)|^2 \right] \, dx
\]

\[
\leq \int_{B_R(x_0)} \left[ |\tilde{u}_p^{x_0}(x, t_0)|^2 + |\tilde{b}_r^{x_0}(x, t_0)|^2 \right] \, dx + 2\tilde{p}_1(x, t_0) \, dx \quad \text{(by (4.14))}
\]

\[
\leq \int_{B_R(x_0)} \left[ |\tilde{u}_p^{x_0}(x, t_0)|^2 + |\tilde{b}_r^{x_0}(x, t_0)|^2 \right] \, dx + 2\tilde{p}_1(x, t_0) \, dx < \frac{\epsilon_*}{4}.
\]

Applying (4.15), we deduce that there exists \( \theta > 0 \) so small that the inequality

\[
\int_{\mathbb{R}^3 \setminus B_R(x_0)} \frac{R^2}{|x - x_0|^3} \left[ 2|\tilde{u}_p^{x_0}(x, t)|^2 - |\tilde{u}_p^{x_0}(x, t)|^2 \right] \, dx
\]

\[
+|\tilde{b}_r^{x_0}(x, t)|^2 \right] \, dx < \frac{\epsilon_*}{2}
\]

holds for \( t \in (t_0 - \theta, t_0) \). Then, due to (4.14) and (4.27), we also have for all \( R \in (0, R_*) \) and on each time level \( t \in (t_0 - \theta, t_0) \):

\[
\frac{1}{R} \int_{B_R(x_0)} \left( |\tilde{u}|^2 + \frac{1}{2} |\tilde{b}|^2 \right) \, dx \leq \frac{1}{R} \int_{B_R(x_0)} \left( |\tilde{u}|^2 + \frac{1}{2} |\tilde{b}|^2 + 3\tilde{p}_1 \right) \, dx
\]
As this holds independently of $R$ and $t$ for $R \in (0, R_s]$ and $t \in (t_0 - \delta, t_0)$, we observe that (4.24) is satisfied. Thus, due to Lemma 4, $(x_0, t_0)$ is a regular point of the solution $(\hat{u}, \hat{b}, \hat{p})$. As point $x_0$ was chosen arbitrarily in $\mathbb{R}^3$, the solution has no singular points on the time level $t_0$. As the solution $(u, b, p)$ coincides with $(\hat{u}, \hat{b}, \hat{p})$ in $\mathcal{B} \times (a'_+, t_0)$, $(u, b, p)$ has no singular points in $\mathcal{B}$ at the time $t_0$. Due to the freedom in the choice of the ball $\mathcal{B}$ inside $\Omega'$, $t_0$ cannot be an $\Omega'$-epoch of irregularity of the solution $(u, b, p)$. Consequently, the solution $(u, b, p)$ has no singular points in $\Omega' \times (T_1, T_2)$. Using the results of [11], we can state that $u$ and $b$ are Hölder-continuous in $\Omega' \times (T_1, T_2)$. The proof of Theorem 1 (under condition a)) is completed.

### 5. Proof of Theorem 1 Under Condition b)

The contents of Sects. 4.1–4.3 can be copied without any changes. In Sect. 4.4 (on the left continuity of $\hat{u}$ and $\hat{b}$ as elements of $L^2(\mathbb{R}^3)$ in dependence on time in $(a'_+, t_0)$), we used the inclusion (4.5), following from condition a) of Theorem 1. We show in the next subsection that the same conclusion (formulated by means of (4.15)) can also be proven if we consider the inclusion

$$\mathcal{F}_\gamma(\hat{B}_+) \in L^\infty(T_1 + 2\delta, T_2 - 2\delta; L^{3/2}(\mathbb{R}^3)), \quad (5.1)$$

where $\hat{B} := \frac{1}{2} |\hat{u}|^2 + \frac{1}{2} |\hat{b}|^2 + \hat{p}_1$, instead of (4.5). Note that (5.1) follows from condition b) of Theorem 1 by means of the same arguments as in the case of (4.5).
5.1. The Left Continuity of \( \hat{u} \) and \( \hat{b} \) in the \( L^2 \)-norm at the Time \( t_0 \)

As in Sect. 4.4, we deduce that there exists a set \( T \subset (a', t_0) \) of the 1D Lebesgue measure zero and \( R_0 > 0 \) such that

\[
\int_{B_R(x_0)} \frac{1}{|x - x_0|} \left( 2\tilde{p}_1 + |\hat{u}_{x_0}|^2 + |\hat{b}_x|^2 \right) \, dx \leq 1 \tag{5.2}
\]

for all \( R \in (0, R_0) \) and \( t \in (a', t_0) \setminus T \). Let \( t \in (a', t_0) \setminus T \), \( x_0 \in \mathbb{R}^3 \) and \( R_0 > 0 \). We will use the identities (4.14) in the form

\[
\int_{B_R(x_0)} \frac{1}{|x - x_0|} \left( 2\tilde{p}_1 + |\hat{u}_{p}|^2 + |\hat{b}_x|^2 \right) \, dy \\
= -\frac{1}{2R} \int_{B_R(x_0)} \left( |\hat{u}|^2 + |\hat{b}|^2 \right) \, dx + \frac{3}{2R} \int_{B_R(x_0)} \left( 2\tilde{p}_1 + |\hat{u}|^2 + |\hat{b}|^2 \right) \, dy \\
= \int_{\mathbb{R}^3 \setminus B_R(x_0)} \frac{R^2}{|x - x_0|^3} \left[ 2|\hat{u}_{x_0}|^2 - |\hat{u}_p|^2 - 2|\hat{b}_x|^2 + |\hat{b}_p|^2 \right] \, dx. \tag{5.3}
\]

Then, for \( 0 < R \leq R_0 \) and each time \( t \in (a', t_0) \), we have

\[
\frac{1}{2R} \int_{B_R(x_0)} \left( |\hat{u}|^2 + |\hat{b}|^2 \right) \, dx = \frac{3}{2R} \int_{B_R(x_0)} \left( |\hat{u}|^2 + |\hat{b}|^2 + 2\tilde{p}_1 \right) \, dx \\
- \int_{B_R(x_0)} \frac{1}{|x - x_0|} \left( |\hat{u}_{x_0}|^2 + |\hat{b}_x|^2 + 2\tilde{p}_1 \right) \, dx \\
\leq \frac{3}{2R} \int_{B_R(x_0)} \left| \hat{B}_+ \right| \, dx + \int_{B_R(x_0)} \frac{1}{|x - x_0|} \left[ 2\hat{B}_+ - \left( |\hat{u}_{x_0}|^2 + |\hat{b}_x|^2 + 2\tilde{p}_1 \right) \right] \, dx \\
- \int_{B_R(x_0)} \frac{2\hat{B}_+}{|x - x_0|} \, dx \\
\leq \int_{B_R_0(x_0)} \frac{\hat{B}_+}{|x - x_0|} \, dx + \int_{B_R_0(x_0)} \frac{1}{|x - x_0|} \left[ 2\hat{B}_+ - \left( |\hat{u}_{x_0}|^2 + |\hat{b}_x|^2 + 2\tilde{p}_1 \right) \right] \, dx \\
- \int_{\mathbb{R}^3 \setminus B_R_0(x_0)} \frac{R_0^2}{|x - x_0|^3} \left[ 2|\hat{u}_{x_0}|^2 - |\hat{u}_p|^2 \right. \\
- \left. 2|\hat{b}_x|^2 + |\hat{b}_p|^2 \right] \, dx. \tag{5.4}
\]

Obviously,

\[
\left| \int_{\mathbb{R}^3 \setminus B_R_0(x_0)} \frac{R_0^2}{|x - x_0|^3} \left[ 2|\hat{u}_{x_0}|^2 - |\hat{u}_p|^2 \right. \right. \\
- \left. \left. 2|\hat{b}_x|^2 + |\hat{b}_p|^2 \right] \, dx \right| \leq \frac{c}{R_0},
\]

where \( c \) is independent of \( x_0 \), \( t \), \( R \) and \( R_0 \). The boundedness of the first term on the right hand side of (5.4), independent of \( t \) for \( t \in (0, t_0) \setminus T \), can now be justified by means of the same arguments as the boundedness of the analogous integral in Sect. 4.4. The validity of the premises in the implications (4.16) and (4.17) can now be also confirmed in the same way as at the end of Sect. 4.4. The statements of these implications imply that (4.15) holds.
Further on, the contents of Sect. 4.5 can be copied without any change and the contents of Sects. 4.6 and 4.7 can be copied with the only change that we replace $\hat{p}_{1-}$ by $\hat{B}_+$ and we also use the Hölder–continuity of $\hat{u}$ and $\hat{b}$ in the neighborhood of regular points.

5.2. Completion of the Proof of Theorem 1 Under Condition b)

Let $x_0 \in \mathbb{R}^3$. We will show that there exists $R_0 > 0$ such that (4.24) holds.

Let $R > 0$ and $t \in (a'_-, t_0]$. Using the identity between the first two lines in (5.3), and at the end also the identity between the first and the third lines, we get

$$\frac{1}{2R} \int_{B_R(x_0)} (|\hat{u}|^2 + 2|\hat{b}|^2) \, dx$$

$$= \frac{3}{2R} \int_{B_R(x_0)} (|\hat{u}|^2 + |\hat{b}|^2 + 2\hat{p}_1) \, dx$$

$$- \int_{B_R(x_0)} \frac{1}{|x - x_0|} (|\hat{u}^x|_0^2 + |\hat{b}^x|_0^2 + 2\hat{p}_1) \, dx$$

$$\leq \frac{3}{R} \int_{B_R(x_0)} \hat{B}_+ \, dx - \int_{B_R(x_0)} \frac{2\hat{B}_+}{|x - x_0|} \, dx$$

$$+ \int_{B_R(x_0)} \frac{1}{|x - x_0|} [2\hat{B}_+ - (|\hat{u}^x|_0^2 + |\hat{b}^x|_0^2 + 2\hat{p}_1)] \, dx$$

$$\leq \int_{B_R(x_0)} \frac{\hat{B}_+}{|x - x_0|} \, dx + \int_{B_R(x_0)} \frac{1}{|x - x_0|} (|\hat{u}^x|_0^2 + |\hat{b}^x|_0^2) \, dx$$

$$+ \int_{B_R(x_0)} \frac{1}{|x - x_0|} [2\hat{B}_+ - (|\hat{u}|^2 + |\hat{b}|^2 + 2\hat{p}_1)] \, dx. \tag{5.5}$$

The right hand side is finite, because it equals

$$3 \int_{B_R(x_0)} \frac{\hat{B}_+}{|x - x_0|} \, dx + \int_{B_R(x_0)} \frac{1}{|x - x_0|} (|\hat{u}^x|_0^2 + |\hat{b}^x|_0^2) \, dx$$

$$- \int_{B_R(x_0)} \frac{1}{|x - x_0|} (|\hat{u}|^2 + |\hat{b}|^2 + 2\hat{p}_1) \, dx$$

$$= 3 \int_{B_R(x_0)} \frac{\hat{B}_+}{|x - x_0|} \, dx - \int_{B_R(x_0)} \frac{1}{|x - x_0|} (|\hat{u}^x|_0^2 + |\hat{b}^x|_0^2 + 2\hat{p}_1) \, dx$$

$$= 3 \int_{B_R(x_0)} \frac{\hat{B}_+}{|x - x_0|} \, dx$$

$$- \int_{\mathbb{R}^3 \setminus B_R(x_0)} \frac{R^2}{|x - x_0|^{3\alpha}} [2|\hat{u}^x|_0^2 - |\hat{u}^x|_0^2 - 2|\hat{b}^x|_0^2 + |\hat{b}^x|_0^2] \, dx$$

$$\leq 3 \int_{B_R(x_0)} \frac{\hat{B}_+}{|x - x_0|} \, dx + \frac{c}{R}, \tag{5.6}$$

where $c$ is independent of $x_0$, $t$ and $R$.

Let $\epsilon_*$ be the number from Lemma 4. By analogy with (4.25), we have

$$\lim_{r \to 0^+} \int_{B_r(x_0)} \frac{\hat{B}_+(x, t)}{|x - x_0|} \, dx = 0 \tag{5.7}$$
uniformly with respect to $t \in (0, t_0]$. Choose $R_* > 0$ so small that
\begin{equation}
3 \int_{B_{R_*}(x_0)} \frac{\hat{B}_+(x, t)}{|x - x_0|} \, dx < \frac{\epsilon_*}{4}
\end{equation}
for all $t \in (0, t_0]$ and
\begin{align*}
&\int_{B_{R_*}(x_0)} \frac{1}{|x - x_0|} \left( (|\tilde{u}_r^x(x, t_0)|^2 + |\tilde{b}_p^x(x, t)|^2) \right) \, dx \\
&\quad + \int_{B_{R_*}(x_0)} \frac{1}{|x - x_0|} \left( (|\tilde{u}(x, t_0)|^2 + |\tilde{b}(x, t_0)|^2 + 2 \tilde{p}_1(x, t_0)) - \right) \, dx \\
&\quad < \frac{\epsilon_*}{4}.
\end{align*}
This choice of $\epsilon_*$ is possible, because of (5.7) and due to the fact that the second integral in (5.9) equals the right hand side of (5.5) (with $R = R_*$ and $t = t_0$), which is finite and consists of three integrals over $B_{R_*}(x_0)$ with nonnegative integrands. Then, by analogy with (4.29) and from the comparison of (5.6) with (5.7), we obtain
\begin{align*}
- \int_{\mathbb{R}^3 \setminus B_{R}(x_0)} \frac{R^2}{|x - x_0|} \left[ 2|\tilde{u}_r^x| - |\tilde{u}_p^x| - 2|\tilde{b}_r^x| + |\tilde{b}_p^x| \right] \, dx &\quad \bigg|_{t = t_0} \\
&= - \int_{B_{R}(x_0)} \frac{\hat{B}_+}{|x - x_0|} \, dx + \int_{B_{R}(x_0)} \frac{1}{|x - x_0|} \left( |\tilde{u}_r^x|^2 + |\tilde{b}_p^x|^2 \right) \, dx \\
&\quad + \int_{B_{R}(x_0)} \frac{1}{|x - x_0|} \left( |\tilde{u}|^2 + |\tilde{b}|^2 + 2 \tilde{p}_1 \right) \, dx \\
&\quad \bigg|_{t = t_0} \leq \frac{\epsilon_*}{4}.
\end{align*}
Due to (4.15), there exists $\vartheta > 0$ so small that
\begin{equation}
- \int_{\mathbb{R}^3 \setminus B_{R}(x_0)} \frac{R^2}{|x - x_0|^3} \left[ 2|\tilde{u}_r^x|^2 - |\tilde{u}_p^x|^2 - 2|\tilde{b}_r^x|^2 + |\tilde{b}_p^x|^2 \right] \, dx < \frac{\epsilon_*}{2}
\end{equation}
at all times $t \in (t_0 - \vartheta, t_0]$. Applying this inequality and inequality (5.8), and using also the fact that the left hand side of (5.5), i.e. $(2R)^{-1} \int_{B_{R}(x_0)} (|\tilde{u}|^2 + 2|\tilde{b}|^2) \, dx$, is equal to the expression on the line (5.6), we observe that the inequality
\begin{equation}
\frac{1}{2R} \int_{B_{R}(x_0)} (|\tilde{u}|^2 + 2|\tilde{b}|^2) \, dx < \frac{\epsilon_*}{4} + \frac{\epsilon_*}{2} = \frac{3\epsilon_*}{4}
\end{equation}
holds for all $t \in (t_0 - \vartheta, t_0]$ and $R \in (0, R_*)$. The proof can now be completed in Sect. 5.

6. Derivation of Formulas (4.10) and (4.11)

Recall that $x, x_0 \in \mathbb{R}^3$, $R > 0$ and $\alpha \in [0, 1]$ in (4.10) and (4.11). Obviously,
\begin{equation}
\int_{B_R(x_0)} \frac{|x_0 - y|^{-\alpha}}{|x - y|} \, dy = \int_0^R r^{-\alpha} \left( \int_{S_r(x_0)} \frac{dy}{|x - y|} \right) \, dr.
\end{equation}
The inside integral over $S_r(x_0)$ depends on $x$ only through $|x_0 - x|$. Thus, we may assume, without loss of generality, that $x_0 = 0$ and $x = (0, 0, r\alpha)$, where $a = |x|/r$. We use the transformation to the spherical coordinates: $y = (r \cos \varphi \sin \vartheta, r \sin \varphi \sin \vartheta, r \cos \vartheta)$. Then
\begin{equation*}
|x - y|^2 = \left( r \cos \varphi \sin \vartheta, r \sin \varphi \sin \vartheta, r \cos \vartheta - r\alpha \right)^2 = r^2 \left[ 1 + a^2 - 2a \cos \vartheta \right],
\end{equation*}
and therefore, as the Jacobian of the transformation equals \( r^2 \sin \vartheta \) and using also the change of variables 
\( 1 + a^2 - 2a \cos \vartheta = z \), we obtain
\[
\int_{S_r(0)} \frac{d_yS}{|x - y|} = 2\pi \int_0^r \frac{r^2 \sin \vartheta}{r \sqrt{1 + a^2 - 2a \cos \vartheta}} \, dr = \pi r^2 \int_{1 + a^2 - 2a}^1 \frac{dz}{\sqrt{z}} = 2\pi r^2 \left[ \sqrt{z} \right]_{(1+a)^2} = 2\pi r^2 \left[ (1 + a) \mp (1 - a) \right],
\]
where the sign “−” holds if \( 1 - a \geq 0 \) (which means that \( |x| \leq r \)) and “+” holds if \( 1 - a < 0 \) (which means that \( |x| > r \)). Thus, returning to a general point \( x_0 \) instead of the special case \( x_0 = 0 \), we get
\[
\int_{S_r(x_0)} \frac{d_yS}{|x - y|} = \begin{cases} 
4\pi r & \text{if } |x - x_0| \leq r, \\
4\pi r^2 |x - x_0|^{-1} & \text{if } |x - x_0| > r.
\end{cases} \tag{6.2}
\]

Let us at first assume that \(|x - x_0| < R\). Then, by (6.1), we have
\[
\int_{B_R(x_0)} \frac{|x_0 - y|}{|x - y|} \, dy = \int_0^R r^{-\alpha} \int_{S_r(x_0)} \frac{d_yS}{|x - y|} \, dr = \int_0^{|x_0 - x|} 4\pi r^{2-\alpha} \, dr + \int_{|x - x_0|}^R 4\pi r^{1-\alpha} \, dr
\]
\[
= \frac{4\pi |x - x_0|^{2-\alpha}}{3 - \alpha} + \frac{4\pi R^{2-\alpha}}{2 - \alpha} - \frac{4\pi |x - x_0|^{2-\alpha}}{2 - \alpha} = \frac{4\pi R^{2-\alpha}}{2 - \alpha}.
\]

This yields
\[
\nabla_x^2 \int_{B_R(x_0)} \frac{|x_0 - y|}{|x - y|} \, dy = -\frac{4\pi}{(3 - \alpha)(2 - \alpha)} \nabla_x^2 |x - x_0|^{2-\alpha}. \tag{6.3}
\]

Since
\[
\nabla_x^2 |x - x_0|^\beta = \beta(\beta - 2) |x - x_0|^\beta - 4 (x - x_0) \otimes (x - x_0) + \beta |x - x_0|^\beta - 2, \tag{6.4}
\]
equality (6.3) (where we use (6.4) with \( \beta = \alpha - 2 \)) yields (4.10).

Suppose now that \(|x - x_0| > R\). Then (6.1) and (6.2) imply that
\[
\int_{B_R(x_0)} \frac{|x_0 - y|}{|x - y|} \, dy = \int_0^R r^{-\alpha} \int_{S_r(x_0)} \frac{d_yS}{|x - y|} \, dr = \int_0^R 4\pi r^{2-\alpha} \, dr
\]
\[
= \frac{4\pi R^{3-\alpha}}{3 - \alpha}.
\]

This, together with (6.4) (which we use with \( \beta = -1 \)), yield formula (4.11).

Acknowledgements. The first author has been supported by the Academy of Sciences of the Czech Republic (RVO 67985840) and by the Grant Agency of the Czech Republic, grant No. GA19-04243S. The second author acknowledges the support of the National Research Foundation of Korea No. 2021R1A2C4002840 and No. 2015R1A5A1009350.

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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(accepted: May 25, 2021; published online: June 17, 2021)