A Smooth Compactification of Moduli Space of Instantons and Its Application

Bohui Chen,
Dept. of Math., MIT
bchen@math.mit.edu

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1 Introduction

1.1 Compactification of Moduli of Instantons

This paper consists of two parts. The purpose of part I is to introduce a smooth compactification of moduli of instantons. Here, we say a space is smooth if it is a smooth orbifold. It is well known that moduli spaces of instantons play very important roles in the study of differential four-manifolds. The most famous example is the Donaldson theory ([6]). In general, the moduli spaces are not compact. One of the fundamental problems is to give a compactification. The most commonly used compactification is the so-called Uhlenbeck compactification ([6], [9]). The Uhlenbeck compactification works well for many purposes in the Donaldson theory. But the Uhlenbeck type compactification is less successful in tackling the problems in which studies of lower strata are involved. Examples are the Kotschick-Morgan conjecture ([16], [12]) on the wall-crossing formula and the Witten conjecture ([22], [5], [7]) on the relationship between the Donaldson invariants and the Seiberg-Witten invariants. The problem with the Uhlenbeck compactification is that the spaces are only stratified spaces and very singular in lower strata. Although the neighborhoods of lower strata can be described ([21]) and some applications are worked out ([16], [18]), they are intractable in general. For example, the Kotschick-Morgan conjecture has been around for almost ten years and a great deal of efforts has been made to solve it. So far, it remains unsolved. The author believes that the most important reason for the lack of closure is the fact that the Uhlenbeck
compactification is not good enough.

For anyone who knows algebraic geometry, it is not difficult to appreciate the importance of a good compactification, which is often the make-or-break point of a problem. A tremendous amount of literature has been devoted to construct a good compactification in various geometric problems. An important notion emerging from these studies is the notion of “stability”. Let us take the example of quantum cohomologies, which is similar to the gauge theory in many ways. A pseudo-holomorphic map from a Riemann surface (smooth or singular) has a natural automorphism given by the automorphism of Riemann surface. Such a pseudo-holomorphic map is called “stable” if the stabilizer is finite. Once the stability is achieved, the moduli space behaves as good as a smooth orbifold via so-called “virtual cycle” or “virtual integration” techniques. The moduli space of stable maps was constructed by Parker-Wolfson-Ye-Kontsevich and has been proved to be extremely important for quantum cohomologies. Indeed, the computations in the theory of quantum cohomologies has gone very far based on the moduli spaces of stable maps. Using a similar analysis, Parker-Wolfson ([19]) pointed out a similar compactification should work for moduli spaces of instantons. We refer these type compactification as the bubble tree compactification. The author was pointed out by T. Mrowka that the bubble tree compactification had been known by many people in 80’s.

However, there are instances where the stability fails when the bubble tree compactification is used for the moduli of instanton. Even in the generic situation, the corresponding moduli spaces will have singularities. As far as the author knows, there is no general method to deal with the singularities caused by the failure of stability. Unfortunately, a close examination of bubble tree compactification shows that the stability fails at “ghost strata” (see §3.2). This is very disappointed! To search a smooth and reasonable compactification, a new idea is needed!

The main purpose of this paper is to introduce a method to resolve these singularities. Technically, it is similar to the “flip” in algebraic geometry where we first perform a blow-up and then a blow-down. It is easy to do a blow-up to obtain a manifold with boundary. The key is that a blow-down can follow to obtain a closed space again. Topologically, it is equivalent to the well-known interchange of handles. Such a blow-down follows from our key observation that “ghost strata” are closely related to the Fulton-McPherson compactification ([11]) of configuration of points. This main technique is given in §4.
1.2 Wall-Crossing Formula for 4-manifolds with $b^+ = 1$

In §5, we give an affirmative answer to Kotschick-Morgan conjecture. Our new compactification is one of our main ingredients. With this smooth moduli space, we are able to apply the localization technique which is our other ingredient to solve the problem. The idea can be also applied to the moduli of $U(2)$-monopoles, which is more complicated than the case considered here ([4]): we also have a new type compactification for this moduli space; with this space, the “virtual” localization technique and the method developed in §5, this leads to an alternative proof of the Witten conjecture mentioned earlier with certain condition ([3]).

Let $X$ be a smooth, simply connected, closed compact 4-manifold. Denote by $b^+$ the maximal dimension of a maximal subspace of $H^2(X, \mathbb{R})$ on which the cup product is positive definite. Let $P$ be an $SO(3)$-bundle over $X$. It is well known that the Donaldson invariants (or Donaldson polynomials) based on the moduli space $\mathcal{M}_P(X)$, are well defined when $b^+ (X) \geq 1$. Moreover, when $b^+ \geq 2$ they are unique in the sense that the invariants are independent of metrics as long as they are generic. For $b^+ = 1$, the Donaldson invariants depend on the metric. In fact, the invariants depend on chambers in $H^2(X, \mathbb{R})$ determined by $P$. Let $\Delta_P$ be the set of chambers. Donaldson invariants are expressed as a map

$$\Psi^X_{P,c} : \Delta_P \to \text{Sym}^d(H_2(X, \mathbb{Z})),$$

where $c \in H^2(X, \mathbb{Z})$ is an integral lifting of $w_2(P)$. $\Psi^X_{P,c}(C)$ is the Donaldson polynomial in terms of the chamber $C$. The invariants vary with the chambers according to a wall-crossing formula. Let $C_1, C_{-1}$ be two chambers. Suppose $\Lambda$ is the collection of walls between them. Each wall corresponds to an element of $H^2(X, \mathbb{Z})$. So we can identify $\Lambda$ as a subset of $H^2(X, \mathbb{Z})$. The following is conjectured ([6])

**Conjecture (Kotschick-Morgan):** Suppose

$$\Psi^X_{P,c}(C_1) - \Psi^X_{P,c}(C_{-1}) = \sum_{\alpha \in \Lambda} \epsilon(c, \alpha)\delta_P(\alpha).$$

where $\epsilon(c, \alpha) = (c - \alpha)^2/2$, then

$$\delta_P(\alpha) = \sum_{i=0}^r a_i(r, d, X)q^{r-i}\alpha^{d-2r-2i}.$$
where \( q \) is the intersection form of \( X \) and the coefficients \( a_i \) depend only on \( r, d \) and homotopy type of \( X \).

Assuming the conjecture, Gottsche [12] has derived an elegant formula for \( \delta_P(\alpha) \) in terms of modular forms. So the remaining question is the conjecture itself. The core of the problem is to prove the dependence on the “homotopy type” of \( X \). For example, we show in theorem 5.12 that \( a_i \)'s depend on the Euler number \( \chi \) and signature \( \sigma \) of \( X \) which are well known homotopy invariants. One of the main ingredients is our smooth compactification. \( \delta(\alpha) \) can be computed via a localization formula of the equivariant theory for models with \( S^1 \)-action. But when we use the Uhlenbeck compactification, these models become very singular because of the structure of lower strata. In practice, computations are unavailable. It turns out that our smooth compactification fits the equivariant structure perfectly. This enables us to apply the ordinary localization theory. Beginning with this observation, we reduce the proof of the conjecture to computing \( p_1 \) of some bundle over a space, which itself is a fibration over \( X \) with fiber \( \mathcal{M}_K^b \). Fortunately, this problem can also be dealt with by using the equivariant theory again. Therefore, we prove the conjecture. Details are given in §5.3.

1.3 Acknowledgements

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Updates: This is an updated version of the author’s thesis ([4]). Recently, Feehan-Leness ([8]) announced a proof to the Witten conjecture along their cobordism approach. In the same paper, they also mentioned that their approach can also work for the Kotschick-Morgan conjecture.
2 Background

We give a quick review of some basic facts about instantons and their moduli spaces for four-manifolds. Readers are referred to [6] for most of the details.

Let \((X, g)\) be a simply connected 4-manifold with metric \(g\). There are two kinds of manifolds considered in this paper. One is manifolds with \(b_2^+ > 0\); the other is \(S^4\) with standard \(g_0\). For \(S^4\), using the stereographic projection \(s\) from the south pole to the equatorial plane, we identify \(s: S^4 \setminus \{\text{south pole}\} \rightarrow R^4\). Alternatively, 0 and \(\infty\) are used for the north and south poles of \(S^4\) respectively. Via \(s\) we pull back the standard coordinate functions on \(R^4\) to functions \(x_i, 1 \leq i \leq 4\), on \(S^4\).

Suppose that \(P\) is a principal \(G\)-bundle over \(X\), where \(G\) is either \(SO(3)\) or \(SU(2)\). When \(G = SU(2)\), let \(E\) be the associated rank 2 vector bundle of \(P\) and \(K = c_2(E)\). When \(G = SO(3)\), choose \(c \in H^2(X, \mathbb{Z})\) to be an integral lifting of \(w_2(P)\). There exists a \(U(2)\)-bundle \(E\) such that \(p_1(E) = p_1(P)\) and \(c_1(E) = c\). We say that \(E\) is associated to \((P, c)\).

Choose \(a_E\) on the determinant line bundle \(\det(E)\). For both cases define \(\mathcal{A}_E = \{A | \det(A) = a_E\}\).

\(\mathcal{G}_E\), the Gauge group of \(E\), consists of gauge transformations of \(E\) preserving \(\det(E)\). Define \(\mathcal{B}_E = \mathcal{A}_E/\mathcal{G}_E\). \(\mathcal{B}_E\) is called the configuration space. Let \(\mathcal{A}_E^* \subset \mathcal{A}_E\) be the set of non-reducible connections and \(\mathcal{B}_E^* = \mathcal{A}_E^*/\mathcal{G}_E\). All definitions can be given directly by using \(P\) instead of \(E\). A connection \(A\) is called anti-self-dual (or ASD) if

\[
F^+(A) = 0.
\]

Class \([A] \in \mathcal{B}_E\) is called an (anti-)instanton. \(\tilde{\mathcal{M}}_E(X)\) is the collection of ASD-connections of \(E\). The moduli space of instantons is

\[
\mathcal{M}_E(X) = \tilde{\mathcal{M}}_E(X)/\mathcal{G}.
\]

Let \(g_E\) be the adjoint bundle of \(E\). There is a complex

\[
0 \rightarrow \Omega^0(g_E) \xrightarrow{d_A} \Omega^1(g_E) \xrightarrow{d_A^+} \Omega^+(g_E) \rightarrow 0
\]
for any ASD-connection $A$. The complex gives the Kuranishi model for the “virtual” tangent space of $\mathcal{M}_E(X)$ at $[A]$. Let $H^i_A, i = 0, 1, 2$ be the cohomologies defined by this complex. We say that $[A]$ is regular if $H^i_A = H^3_A = 0$ and semi-regular if $H^2_A = 0$. We always assume that $H^3_A = 0$ in this paper. When $H^3_A \neq 0$, $A$ is called reducible, otherwise $A$ is called irreducible. $\mathcal{M}_E^r(X)$ is the moduli space of irreducible instantons. $[A]$ is regular iff it is irreducible. When $[A]$ is regular, $H^1_A$ is the tangent space of $[A]$ in the moduli space. When $[A]$ is semi-regular, first of all, $[A]$ is reducible: its isotropy group $\Gamma_A$ is either $S^1$ or $G$; secondly the local model of $[A]$ in the moduli space is then given by $H^1_A/\Gamma_A$. When $b_2^+(X) > 0$ and $c_2(E) \neq 0$, the transversality theorem (13, 14) says that $\mathcal{M}_E(X) = \mathcal{M}_E^r(X)$ is smooth for generic metrics $g$. For $X = S^4$ with standard metric $g_0$, instantons are regular if $c_2(E) \neq 0$. When $c_2(E) = 0$, the trivial instanton is semi-regular and its isotropy group is $SO(3)$. For general $X$, when $G$ is $SU(2)$ and $c_2(E) = 0$, $H^2$ is nontrivial for the trivial connections. We exclude this case in our paper. In general, in order to consider the trivial connection, one should introduce the “thicked moduli space” (21, 10). For generic metrics, the dimension of the moduli space is computed by the Atiyah-Singer index theory:

$$\dim \mathcal{M}_E(X) = d(p_1(E)),$$

where $d$ is defined by

$$d(n) = -2n - \frac{3}{2}(\chi + \sigma), \quad (1)$$

where $\chi$ is the Euler number of $X$ and $\sigma$ is the signature of $X$.

One would see that when $X = S^4$, the dimensions are $5(\text{mod}8)$. In fact when $X = S^4$, we are more interested in a modified moduli spaces. These are the original moduli spaces modulo a 5-dimensional group $H$. $H$ is a subgroup of the conformal group of $S^4$. It is generated by translations and dilations on $R^4$. Any conformal transformation induces an action on moduli spaces. So we can define $\mathcal{M}_E^b = \mathcal{M}_E(S^4)/H$. Alternatively we can describe this moduli space more explicitly: let $h$ be a constant less than $4\pi^2$. Define $\mathcal{M}_E^b_K$ to be the collection of ASD-connections $A$ with the properties

$$\int_{R^4} |F(A)|^2 = 0 \quad \text{and} \quad \int_{B(1)} |F(A)|^2 = h. \quad (2)$$

The first equation requires that the mass center of $A$ is 0 and the second one requires the energy in unit disk $B(1)$ to be equal to a small constant.
We call \((2)\) the **balanced condition**. An ASD-connection \(A\) is **balanced** if it satisfies \((2)\). Then \(\mathcal{M}^b_E = \mathcal{M}^b_E(S^4) / \mathcal{G}_E\), where \(\mathcal{M}^b_E(S^4)\) is the set of balanced ASD-connections.

Fix a point \(x_0 \in X\). When \(X = S^4\), we always choose \(x_0 = \infty\). Suppose \(\mathcal{G}^0_{E}\) is the subgroup of \(\mathcal{G}_E\) that consists of gauge transformations with identity at point \(x_0\). Then

\[
\tilde{M}_{E}(S^4) / \mathcal{G}_{E} = \mathcal{M}^b_E(S^4) / \mathcal{G}_E\]

is the set of balanced ASD-connections.

Fix a point \(x_0 \in X\). When \(X = S^4\), we always choose \(x_0 = \infty\). Suppose \(\mathcal{G}^0_{E}\) is the subgroup of \(\mathcal{G}_E\) that consists of gauge transformations with identity at point \(x_0\). Then

\[
u : A^* \times \mathcal{G}_E P_{x_0} \to \mathcal{B}^*_E \times X.
\]

Let \(\mathcal{M}^b_{E}(X) = u^{-1}(\mathcal{M}_E(X))\). There is another important \(SO(3)\)-bundle over \(\mathcal{B}^*_E \times X\) which plays a crucial role in the Donaldson theory. This bundle is defined by

\[
u : A^* \times \mathcal{G}_E P \to \mathcal{B}^*_E \times X.
\]

We denote this bundle by \(\mathbb{P}_{K} \) or \(\mathbb{P}_{K,c}\) depending on \(G\). In general, we write it as \(\mathbb{P}\). Similarly, \(\mathcal{M}^0_{E,X} = v^{-1}(\mathcal{M}_E(X) \times X)\). Using \(p_1(\mathbb{P})\), one defines

\[
\mu : H_*(X; \mathbb{Z}) \to H^{4-*}(\mathcal{B}^*_E; \mathbb{Q})
\]

by taking the slant product \(\mu([-]) = \frac{1}{4}p_1(\mathbb{P}) / [-]\). For the 0-class \([x_0]\), \(\mu([x_0]) = -\frac{1}{4}p_1(\mathbb{P}^0)\).

As we mentioned in the introduction, the bubble tree compactification is our initial step towards our new compactification. It is based on the following compactness theorem.

**Theorem 2.1 (Parker-Wolison).** Suppose that \(\{A_n\}_{n=1}^{\infty}\) is a sequence of instantons in \(\mathcal{M}_E(X)\). Then there exists a subsequence of \(\{A_n\}\) that converges to a bubble tree instanton \([A]\) in \(\overline{\mathcal{M}}_E(X)\).

Definitions of \(\overline{\mathcal{M}}_E(X)\) and **bubble tree instantons** are given in \(\S 3.1\). Suppose \(E, P, K\) are as above. Let \(E_l, K \geq l > 0\), be the bundles such that \(c_1(E_l) = c_1(E)\) and \(c_2(E_l) = c_2(E) - l\). Roughly speaking, a bubble tree instanton \([A]\) is a sequence of finite instantons \(([A_0], [A_1], \ldots, [A_k])\) on components: \(X\) and 4-spheres \(S^4_i, 1 \leq i \leq k\). \(X\) is called the **principal component** and \(S^4_i\)'s are called **bubbles**. We call...
[A_0]$ the background instanton of $[\mathcal{A}]$. $[A_0]$ should be in $\mathcal{M}_{E-l}(X)$ for some $l$. Then we say that $[\mathcal{A}]$ is in the $l$-level stratum. Suppose $X \neq S^4$. If $[A_0]$ is trivial, we say that $[\mathcal{A}]$ is trivial. If $[A_0]$ is reducible, we say $[\mathcal{A}]$ is reducible. Let $F_E$ be the collection of trivial bubble tree instantons. In this paper, we assume that $F_E = \emptyset$. For example, when $G = SO(3)$ and $w_2(P) \neq 0$, $F_E = \emptyset$ (13). Essentially, this is the case that we are interested in. To avoid the confusion caused by the complexity of notations, we restrict to $G = SU(2)$ and make the assumption that $F_E = \emptyset$. Also note that $E$ is determined by $K$ up to bundle isomorphisms. We replace the subscript “$E$” by $K$ in general. For example, $\mathcal{M}_K(X)$ is the same as $\mathcal{M}_E(X)$ from now on. If we consider $G = SO(3)$, then the subscript should be $K,c$.

In the next section, we will define $\overline{\mathcal{M}}_K(X)$ and its topology, then study its topological and smooth structure using the gluing theory.

We now review the problem of the wall-crossing formula. In the rest of the chapter, assume $b_2^+(X) = 1$ and $P \to X$ is an $SO(3)$ principal bundle with $w_2(P) \neq 0$. Let $c$ be an integer lift of $w_2(P)$ and $E$ be a $U(2)$-bundle over $X$ with $c_1(E) = c, p_1(E) = p_1(P)$. Set $K = c_2(E) - 0$ as before. Suppose $g_x$ is a generic metric. Let $\overline{\mathcal{M}}_K(X, g_x), \mathcal{M}_K(X, g_x)$ and $\overline{\mathcal{M}}_K(X, g_x)$ be the compactified spaces of $\mathcal{M}_K(X, g_x)$ in the sense of the Uhlenbeck compactification, the bubble tree compactification and the smooth compactification which is given later in §4. The map $\mu : H_2(X, \mathbb{Z}) \to H^2(\mathcal{M}_K(X, g_x), \mathbb{Q})$ defines 2-classes $\mu(\Sigma), \Sigma \in H_2(X)$. $\mu(\Sigma)$ can be extended to $\overline{\mu}(\Sigma)$ over $\overline{\mathcal{M}}_K(X, g_x)$ (3, 4). There are two different approaches: in 3 a line bundle $L_{\Sigma}$ over $\overline{\mathcal{M}}_K(X, g_x)$ is constructed and $\overline{\mu}(\Sigma)$ is the $c_1$ of the bundle; in 4 the proof is more algebraic-topological. Here we follow the approach of 3. Note that there are natural projections from both $\overline{\mathcal{M}}_K(X, g_x)$ and $\overline{\mathcal{M}}_K(X, g_x)$ to $\overline{\mathcal{M}}_K(X, g_x)$. So $\overline{\mu}(\Sigma)$ is also well defined over these two spaces via pull-back maps. The Donaldson invariants are defined to be the pairing of compactified moduli space with cohomology classes given by $\mu$. One can show that the Donaldson invariants are independent of compactified spaces we choose. However, when $b_2^+(X) = 1$, the pairing is no longer metric independent: let $g_{-1}, g_1$ be two generic metrics and $\lambda = g_t$ be a path of metrics connecting them. Define

$$\mathcal{M}_K(X, \lambda) := \{([A], t) | [A] \in \mathcal{M}_K(X, g_t)\}.$$ 

This forms a “cobordism” between $\mathcal{M}_K(X, g_i), i = -1, 1$. Unfortunately, it is not an actual cobordism since there may be reducible connections in the space. We can still apply compactification theories to
this space. Then $\overline{\mathcal{M}}_K^d(X,\lambda)$ forms a “cobordism” between $\overline{\mathcal{M}}_K^d(X,g_i)$, and so is true for other two different compactifications. $\bar{\mu}(\Sigma)$ are not well defined wherever reducible connections appear. Even worse, reducible connections can be at lower strata. Suppose $A$ is a reducible connection on the top stratum. Then it gives a reduction of $P$ to an $S^1$-bundle $Q_\alpha$, $P = Q_\alpha \times_{S^1} SO(3)$. Here $c_1(Q_\alpha) = \alpha, \alpha \in H^2(X,\mathbb{Z})$. We say that $A$ is reducible with respect to $\alpha$. Let $L_\alpha$ be the line bundle associated to $Q_\alpha$. Correspondingly, $E$ splits as

$$E = L^{1/2}_\alpha \otimes L^{1/2}_c \oplus L^{-1/2}_\alpha \otimes L^{1/2}_c.$$ 

If $A \in \mathcal{M}_K(X,g)$, $F_A^+ = 0$. Suppose that $\omega(g)$ is the self-dual harmonic 2-form with respect to $g$. Then $\alpha \cdot \omega(g) = 0$. In general, the reducible connection $[A]$ with respect to $\alpha$ is in $\overline{\mathcal{M}}_K(X,g)$ if and only if

$$\omega(g) \in W^\alpha := \{ x \in H^2(X,\mathbb{R}) | x^2 > 0, x \cdot \alpha = 0 \}.$$ 

where $H^2(X,\mathbb{R})$ is identified with the space of harmonic forms. We call $W^\alpha$ the wall associated to $\alpha$. $W^\alpha$ is called a wall of $P$-type if $w_2(P)$ is the reduction of $\alpha$ modulo 2 and $0 > \alpha^2 \geq p_1(P)$. The set of all walls $W^\alpha$ of $P$-type is called $P$-walls. The set of chambers $\Delta_P$ of $X$ is the set of components of the complement, in the positive cone of $H^2(X,\mathbb{R})$, of the $P$-walls. The Donaldson invariants associated to a bundle $P$ depend on the chamber where the period point of the metric is located. The invariants vary when period points of metrics pass through $P$-walls. Let $d = -p_1(P) - 3$. $\Phi_{P,c}^X : \Delta_P \to Sym^d(H_2(X,\mathbb{Z}))$ is the map such that $\Phi_{P,c}^X(C)$ defines the Donaldson invariants for the chamber $C$.

We now state the Kotschick-Morgan conjecture in terms of the Uhlenbeck compactification. Suppose $[A] \in \overline{\mathcal{M}}_K(X,\lambda)$ is a reducible connection with respect to $\alpha$. Let $r = (\alpha^2 - p_1(P))/4$. Then the family of reducible connections is $[A] \times Sym^r(X)$. Let $D(\alpha)$ denote the link of the family of reducible connections. This defines an element $\delta_P(\alpha)$ in $Sym^d(H_2(X,\mathbb{Z}))$ as follows: let $z \in Sym^d(H_2(X,\mathbb{Z}))$,

$$\delta_P(\alpha)(z) = \langle \bar{\mu}(z), D(\alpha) \rangle.$$ 

If $C_{-1}$ and $C_1$ are chambers, then

$$\Phi_{P,c}^X(C_1) - \Phi_{P,c}^X(C_{-1}) = \sum_\alpha \epsilon(c,\alpha) \delta_P(\alpha).$$ 

where $\epsilon(c,\alpha) = (-1)^{(c-\alpha)^2/2}$. 
Conjecture 2.2 ([16]). With notations as above,
\[ \delta_K(\alpha) = \sum_{i=0}^{r} a_i(r,d,X)q^{r-i}\alpha^{d-2r-2i}. \]
where \( q \) is the intersection form of \( X \) and the coefficients \( a_i \) depend only on \( r,d \) and homotopy type of \( X \).

3 Gluing Theory, The Topological and Smooth Structures of Moduli Space \( \overline{M}_K(X) \)

Before we resolve the singularities of \( \overline{M}_K(X) \), we first have to understand its smooth structure and the structure of singularities. The chapter is organized as follows: in §3.1, we define the bubble tree compactified space \( \overline{M}_K(X) \); The topological structures and the smooth structures are discussed in §3.2 and §3.3 respectively by using the gluing theory.

3.1 Bubble Trees and Strata

Let \( \mathcal{M}_K(X), \mathcal{M}_K^b \) be as before. We denote their compactified spaces by \( \overline{M}_K(X), \overline{M}_K^b \). They are stratified spaces. We begin with the description of their strata.

Each stratum is associated with a connected tree with additional requirements. Let \( T = (V,D) \) be a tree, where \( V \) is the set of vertices and \( D \) is the set of edges. Choose a vertex \( v_0 \in V \) to be the root of \( T \). For each \( v \in V \) there is a unique path connecting \( v \) and \( v_0 \). The length of the path (i.e, the cardinality of edges in the path) is called \( \text{depth}(v) \). We say that vertex \( v_i \) is an ancestor of vertex \( v_j \) if the unique path connecting \( v_i \) and \( v_j \) does not pass through \( v_0 \) and \( \text{depth}(v_i) < \text{depth}(v_j) \), and we also call \( v_j \) a descendant of \( v_i \). Moreover, if \( \text{depth}(v_j) = \text{depth}(v_i) + 1 \), \( v_i \) is called a parent of \( v_j \) and \( v_j \) is a child of \( v_i \). In other words, \( v_i \) is an ancestor of \( v_j \) and there is an edge \( e = (v_i, v_j) \) in \( D \). From now on, when we write an edge \( e = (v_i, v_j) \) we mean that \( v_i \) is the parent of \( v_j \).

For each vertex \( v \in V \), let \( V(v) \) be the set of vertices consisting of \( v \) and all descendants of \( v \). Define \( t(v) = (V(v), D(v)) \) to be the subtree of \( T \) induced by \( V(v) \), i.e, \( D(v) \) consists of all edges in \( D \) connecting vertices in \( V(v) \). We now assign each vertex \( v \) a nonnegative integer \( w(v) \). \( w(v) \) is called the weight or charge of \( v \). We call such a \( T \) to be a
A weighted tree. Given a vertex \( v \), define \( W(v) \) to be the sum of charges of all vertices in \( t(v) \). We call \( W(v) \) to be the total charge of \( v \). Let \( \text{child}(v) \) be the set of all children of the vertex \( v \).

**Definition 3.1.** A weighted tree \( T \) with a prechosen root \( v_0 \) is a bubble tree if for any vertex \( v \) we have either

- \( W(v) \neq 0 \), or

- \( |\text{child}(v)| \geq 2 \), and \( W(v_i) > 0 \) for all \( v_i \in \text{child}(v) \).

We write the bubble tree as \((T,v_0)\) or \( T \).

Note that for each vertex \( v \) in a bubble tree \( T \) the associated subtree \( t(v) \) is still a bubble tree. And \( t(v_i) \) is called a bubble tree component of \( v \) if \( v_i \in \text{child}(v) \). Given an edge \( e = (v_i, v_j) \), we say that the tree \( T' = (V', D') \) is the contraction of \( T \) at edge \( e \) if \( V' = V \setminus \{v_j\} \) and \( D' \) is the union of edges in \( D \) that do not connect \( v_j \) and new edges \( e' = (v_i, v_i), v_i \in \text{child}(v_j) \). \( W(v_i) \) is updated to \( W(v_i) + W(v_j) \). If \( T \) is a bubble tree, \( T' \) is also a bubble tree with the same root. For a bubble tree \((T,v_0)\), define the total charge \( W(T) \) of \( T \) to be \( W(v_0) \). Given an integer \( K > 0 \), let

\[ T_K = \{(T,v_0)||W(T)|| = K\} \]

We introduce a partial order on \( T_K \): \( T_1 < T_2 \) if \( T_2 \) is obtained from \( T_1 \) by a sequence of contractions. It is easy to prove

**Lemma 3.2.** \( |T_K| < \infty \).

Now we focus on \( X = S^4 \) for a moment. Suppose \((T,v_0)\) is a bubble tree and \( T = (V,D) \). We assign 4-spheres \( S^4_1 \) to each vertex \( v_i \in V \). For each edge \( e = (v_i, v_j) \) or \( v_i \in \text{child}(v_i) \), we assign a point \( d_T(e) = p_{ij} \in S^4_1 \). Moreover \( d_T(e) \neq d_T(e') \) if \( e \neq e' \). Abstractly, we define a bubble tree space \( Y \) of \( T \) to be a pair \( Y = (\prod_i S^4_1, d_T(D)) \). Geometrically, \( Y \) is realized as a quotient space \( \prod S^4_1/\sim \); for each \( e = (v_i, v_j) \), \( d_T(e) \sim \infty_j \). We denote \( Y \) by \( \prod S^4_1/d_T \). For any \( v_i \) and edge \( (v_i, v_j) \), \( d_T(v_i, v_j) \) is called a bubble point on \( S^4_1 \). For \( p = (p_1, \ldots, p_n) \in (S^4 \setminus \{\infty\})^n \) we assign charges (or energies) \( \mathbf{w} = (w_1, \ldots, w_n) \in (\mathbb{Z}^+)^n \) to it, i.e, \( w_i \) is the charge of \( p_i \). \( S_n \), the \( n \)-permutation group, acts on \( p \) and \( \mathbf{w} \) in the standard way. Define \( S_m < S_n \) to be the kernel of action on \( \mathbf{w} \) and

\[ N_{\mathbf{w}} := (S^4 \setminus \{\infty\})^n/S_m \]

The class of \( p \) in \( N_{\mathbf{w}} \) is denoted by \([p] = [p_1, \ldots, p_n]\). Note that the group action \( H \) on \( S^4 \setminus \{\infty\} \) is also well defined on \( N_{\mathbf{w}} \).
Definition 3.3. A generalized instanton on $S^4$ is an instanton $[A] \in \mathcal{M}_{k_0}(S^4)$ with $[p] = [p_1, \ldots, p_n] \in N_w$, where $w = (k_1, \ldots, k_n)$. Charges $k_i$ are called the $\delta$-mass at $p_i$. The moduli space of generalized instanton denoted by $\mathcal{M}_{k_0,w}(S^4)$ is $\mathcal{M}_{k_0}(S^4) \times N_w$. Define

$$\mathcal{M}^b_{k_0,w} = \mathcal{M}_{k_0,w}(S^4)/H.$$ 

The element $[[A],[p]]$ in $\mathcal{M}^b_{k_0,w}$ is called a balanced instanton. The representative of $[[A],[p]]$ can be chosen to satisfy the balanced condition:

1. the (generalized) mass center $m$ is the north pole, namely
   $$m := x(m([A]))k_0 + x(p_1)k_1 + \cdots + x(p_n)k_n = 0$$

2. one of the following cases holds:
   (a) the charge of $[A]$ at south hemi-sphere equals the constant $h$ and $p_i$’s are located in the open north semi-sphere,
   (b) $[A]$ is nontrivial and the charge of $[A]$ at south hemi-sphere is less than $h$. Also $p_i$’s are located in north hemi-sphere and there exists at least one $p_i$ on the equator.
   (c) $[A]$ is trivial; then
   $$x^2(p_1) + x^2(p_2) + \cdots + x^2(p_n) = \sum_{i=1}^n k_i. \quad (3)$$

The instanton is denoted by $[[[A],[p]]_w]_w$.

Example 3.4. Suppose $k_0 = 0, p = (p_1, p_2)$ and $w = (1, 1)$. Then

$$\mathcal{M}^b_{0,w} = S^3/\mathbb{Z}_2 = RP^3.$$ 

Directly from the definition, we know that

Lemma 3.5. $\mathcal{M}^b_{k_0,w}$ is a smooth manifold.

Definition 3.6. $(T,v_0)$ is a bubble tree and $T = (V,D)$. Suppose $Y$ is a bubble space of $(T,v_0)$, where $Y = (\bigsqcup_{v_i \in V} S^4)/d_T$, and instantons $[A_i] \in \mathcal{M}_{w(v_i)}(S^4)$. For each $v_i$, suppose $\text{child}(v_i) = \{v_{i1}, \ldots, v_{id_i}\}$. Let

$$p_i = (d_T(v_i, v_{i1}), \ldots, d_T(v_i, v_{id_i}))$$
Lemma 3.7. Follow the notations, one can see that \( \text{and} \)

\[ \mathfrak{w}_i = (W(v_{i_1}, \ldots , W(v_{i_d})). \]

Data \((Y, [A_i])_{v_i \in V} \) generates generalized instantons \([([A_i], [p_i])_{w_i}]) on \( S^4 \). If each element \([(A_i], [p_i])_{w_i} \) is balanced, then \((([A_i], [p_i])_{w_i})_{v_i \in V} \) is called a bubble tree instanton. The stratum, denoted by \( S_T(S^4) \), is the set of such bubble tree instantons.

Let \( \tilde{M}_T \) be the set of bubble spaces of some bubble tree instantons in \( S_T(S^4) \). Suppose \( Y \in \tilde{M}_T \). Note that \( Y \) can be represented as \((\coprod S^4, d_T) \). For each \( v \in V \), suppose \( \text{child}(v) = \{v_1, \ldots , v_n\} \) and \( \mathfrak{w}_v = (W(v_1), \ldots , W(v_n)) \). The action \( S_{w_v} \) on \((d_T(v_1), \ldots , d_T(v_n)) \) induce an action on \( d_T \) and so on \( M_T(S^4) \). Define \( M_T(S^4) = \tilde{M}_T/S_{w_v} \).

Elements of \( M_T(S^4) \) are called bubble tree manifolds.

We make a convention on representations of bubble trees and bubble tree manifolds. We denote by \( \mathcal{R}(T) \) the representation of \( T \). We define it inductively: Suppose \( v_0 \) is the root and \( \text{child}(v) = \{v_1, \ldots , v_k\} \). Their weights are \( w_i, 0 \leq i \leq k \). Then

\[ \mathcal{R}(T) = [w_0[w_1 \cdot R(t(v_1)), w_2 \cdot R(t(v_2)), \ldots , w_k \cdot R(t(v_k))]]. \]

In this equation, \( \cdot \) is only used to make the equation easy to read. Similarly, we can represent a bubble tree manifold in the same way.

Suppose \( Y \) is a bubble tree manifold of \( T \) defined by \( d_T \). To represent \( Y \) is essentially to represent \( d_T \) if \( X \) is given. We denote by \( Y(d_T) \) the representation of \( d_T \). Suppose \( p_i = d_T(v_0, v_i), 1 \leq i \leq i \), then

\[ Y(d_T) = [p_1 \cdot Y(d_{t(v_1)}), p_2 \cdot Y(d_{t(v_2)}), \ldots , p_k \cdot Y(d_{t(v_k)})]. \]

Follow the notations, one can see that

\textbf{Lemma 3.7.} \( M_T(S^4), S_T(S^4) \) are smooth manifolds.

\textbf{Proof:} We prove this by induction. If \( T \) consists of one vertex, then the lemma follows from lemma 3.5. Else, suppose

\[ \text{child}(v_0) = \{v_1, \ldots , v_n\} \text{ and } \mathfrak{w} = (W(v_1), \ldots , W(v_n)). \]

\( S_T(S^4) \) is a fiber bundle over \( \mathcal{M}^b_{w(v_0), \mathfrak{w}} \) and the fiber is a product of \( S_{t(v_i)} \). Then \( S_T(S^4) \) is smooth if \( S_{t(v_i)} \) are smooth. This can be finished inductively. The proof for the \( M_T(S^4)'s \) is the same. \( \text{q.e.d.} \)

Define

\[ \overline{M}_K = \cup_{T \in T_K} S_T(S^4). \]
Now we explain how these definitions can be modified and generalized to general $X$. Given a manifold $X$ and a bubble tree $(T, v_0)$, we define bubble tree manifolds and bubble tree instantons inductively. First assign the root $v_0$ for the manifold $X$. Then for any edge $e_i = (v_0, v_i)$ in $T$, define a bubble point $p_i := d_T(e_i) \in X$. For all vertices in the subtree $t(v_i)$, the assigned spheres are $\overline{T M_{p_i}}$, the one-point compactification of the tangent space $T M_{p_i}$. Bubble tree instantons are the modified bubble tree manifolds with instantons on each component and they satisfy the same requirements given in definition 3.6 except the one on root $X$. The instanton on $X$ is in $\mathcal{M}_{w(v_0)}(X)$. Again strata are defined to be the set of bubble tree instantons. We denote the stratum by $S_T(X)$. $M_T(X)$ is defined similarly. It is not obvious that the strata defined this way are still smooth, since the strata are parametrized by bubble points on $X$ and we define the space pointwise. However this is still true, indeed it is a fiber bundle over $X^n/S_m$ for an appropriate $w$. This can be seen easily from a better version of description given in §3.2. Define

$$\mathcal{M}_K(X) = \bigcup_{T \in \mathcal{T}_K} S_T(X).$$

We now introduce a set of terminologies concerning ghost bubbles. Suppose $T = (V, D)$ is a bubble tree and $M$ is a bubble tree manifold of $T$. By a ghost vertex $v$ we mean that $w(v) = 0$. $G_T$ is the set of ghost vertices of $T$. The corresponding component in $M$ is called a ghost bubble. If $T$ contains any ghost vertex, we say $T$ is a ghost bubble tree and $S_T(X)$ is a ghost stratum. Let $G_K \subset \mathcal{T}_K$ be the set of all ghost trees. Define

$$S_K(X) = \bigcup_{T \in G_K} S_T(X).$$

We call $S_K(X)$ the singular set of $\mathcal{M}_K(X)$. Similarly, all definitions apply to $\overline{\mathcal{M}}_K$.

### 3.2 The Gluing Theory

To patch all strata together and give $\overline{\mathcal{M}}$ a manifold (or orbifold) structure, the standard tool is the gluing theory, which has been developed intensively ([6],[21]). The basic idea is that the gluing method constructs local charts for the compactified space. In this section, we go over the gluing theory. In addition, we consider how to “patch” gluing maps for different strata. Therefore, we are enable to conclude the smoothness of the whole compactified space. This is explained in §3.3
and necessary estimates are given in this subsection. This type issue was first discussed by Ruan [20] in the quantum cohomology theory.

We are going to encounter a large collection of notations concerning fiber bundles in this paper. Suppose \( \pi : X \to Y \) be a fiber bundle with fiber \( Z \). To make the notations more suggestive, we write \( X = Y \times Z \) if no confusion is caused.

Given a stratum \( \mathcal{S}_T(X), T \in \mathcal{T}_K \), there is a so-called gluing parameter \( Gl_T \). Roughly speaking, the gluing theory is to study a gluing map
\[
\Psi_T : \mathcal{S}_T(X) \times G\ell(T) \to M_K(X).
\]
Also for any \( T < T' \) there is an associated gluing parameter \( G\ell(T,T') \subset G\ell_T \). \( \Psi_T \) maps \( \mathcal{S}_T(X) \times G\ell(T,T') \) to \( \mathcal{S}_T(X) \). In the gluing construction, a very important step is to construct approximating solutions via “splicing”. In particular, this is the case when we expect a global gluing. This step is to construct a splicing map
\[
\Psi'_T : \mathcal{S}_T(X) \times G\ell(T,T') \to B_{T'},
\]
for all \( T < T' \).

We study the base case. Suppose bubble tree \( T \) consists of only two vertices \( v_0, v \) and one edge \( e = (v_0, v) \). The weights are \( k_1 = w(v_0), k_2 = w(v) \) and let \( K = k_1 + k_2, R(T) = [k_0[k_1]] \). Assume \( k_1 k_2 \neq 0 \). Bubble tree manifolds are given by \( d_T(e) \). Suppose \( d_T(e) = p, p \in X \), then \( Y(d_T) = [p] \). So \( M_T \) is parameterized by \( X \). To be consistent with notations below, let \( X_1 = X, X_2 = S^4 \) and \( p_1 = p, p_2 = \infty \).

The stratum \( \mathcal{S}_T(X) \) is described as follows: Let \( Fr(X) \) be the frame bundle of \( X \), \( P_1 \to X \) and \( P_2 \to S^4 \) be \( SU(2) \) principal bundles with \( c_2(P_i) = k_i \). \( E_i \) are associated vector bundles of \( P_i \). Define
\[
\mathcal{M}_K(S^4, X) = Fr(X) \times_{SO(4)} \mathcal{M}_K(S^4),
\]
and
\[
\mathcal{M}_K^b(S^4, X) = Fr(X) \times_{SO(4)} \mathcal{M}_K^b.
\]
Then
\[
\mathcal{S}_T(X) = \mathcal{M}_{k_1}(X) \times \mathcal{M}_{k_2}^b(S^4, X)
\]
and it is also a bundle over \( X \). The gluing data of stratum \( \mathcal{S}_T(X) \) is a bundle
\[
gl : GL_T = \mathcal{M}_{k_1,X}^b \times Fr(X) \times_{SO(3) \times SO(4)} \mathcal{M}_{k_2}^{b,0}(S^4) \times R^+ \to \mathcal{S}_T(X),
\]
where the action $\text{SO}(3) \times \text{SO}(4)$ on elements in $\mathcal{M}_{k_2}^{b,0}(S^4)$ comes from the action of $\text{SO}(4)$ rotating $S^4$ and the action of $\text{SO}(3) = \text{SU}(2)/\{\pm 1\}$ on the base frame. Note that $M_T$ is a factor of $\mathcal{S}_T(X)$. In this case $M_T = X$. Locally for a $p \in X$ the geometric meaning of the fiber $gl^{-1}(\ast, p)$ is

$$\text{Gl}(p) := \text{Hom}_{\text{SO}(3)}(\mathfrak{g}_{E_1}(p), \mathfrak{g}_{E_2}(\infty)) \times \mathbb{R}^+.$$ 

This is isomorphic to $\text{SO}(3) \times \mathbb{R}^+ = \mathbb{R}^4 \setminus \{0\}/\mathbb{Z}_2$ = $\text{Gl}_T \setminus \{0\}$, where $\text{Gl}_T = \mathbb{R}^4/\mathbb{Z}_2$. We call the parameter $\mathbb{R}^+$ in $\text{Gl}_T$ the gluing radius. Use $\text{GL}_T(r)$ for the set in $\text{GL}_T$ with radius $\leq r$.

We say a set $U$ in a space is proper if its closure in the space is compact. The gluing theorem is now stated as

**Theorem 3.8 ([21]).** For any open proper set $U \in \mathcal{S}_T(X)$ there exists a small constant $\epsilon_0$, depending on $U$, and a gluing map $\Psi_T$

$$\Psi_T : \text{GL}_T(\epsilon_0) \cap gl^{-1}(U) \rightarrow M_K(X)$$

such that $\Psi_T$ is a diffeomorphism.

The proof consists of two parts: part 1, the local diffeomorphism of $\Psi_T$, and part 2, the injectivity of $\Psi_T$.

**Part I, local diffeomorphism of $\Psi_T$**

The idea is that $\Psi_T$ can be constructed gauge equivariantly. So we can work on $\mathcal{A}^*$ instead of $\mathcal{B}^*$. Also the local diffeomorphism is a local issue. We set up the local coordinates for gluing. Suppose that $[A_i^{b_i}], i = 1, 2$ are in $\mathcal{M}_{k_1}(X), \mathcal{M}_{k_2}^{b}; U_1, U_2^b$ are their neighborhoods; $p \in X$ and $B_p(1)$ is a ball at $p$ with radius 1. We identify the ball with the unit ball in $\mathbb{R}^4$. (Readers are referred to [21] for a careful explanation.) Then locally $\Psi_T$ is a map from

$$\Psi_T : U(\epsilon) := U_1 \times U_2^b \times B_p(1) \times \text{SO}(3) \times (0, \epsilon) \rightarrow M_K(X).$$

It is convenient to compare this gluing map with the one for a more classic model. This comparison also suggests a reasonable treatment of $B_p(1) \times \mathbb{R}^+$ in the domain of $\Psi_T$.

We modify the domain of $\Psi_T$ by: fixing a $r \in (0, \epsilon)$ and a point $p \in X$, replacing $\mathcal{M}_{k_2}^{b}$ by $\mathcal{M}_{k_2}(X_2)$. For this set-up, $X_2$ can be arbitrary. This is the case considered in [15]. But eventually, we are only interested
in $S^4$. Moreover identify $S^4 \setminus \{\infty\} = TX_p$. Define gluing data to be the bundle

$$\text{GL}'(p, r) = \mathcal{M}^0_{k_1}(X_1) \times_{SO(3)} \mathcal{M}^0_{k_2}(X_2)$$

over $\mathcal{M}_{k_1, k_2}(X_1, X_2) := \mathcal{M}_{k_1}(X_1) \times \mathcal{M}_{k_2}(X_2)$ and gluing map $\Phi(p, r) : \text{GL}'(p, r) \to \mathcal{M}_K(X)$ which is defined later.

**Proposition 3.9.** Given proper sets $U$ in $\text{GL}'(p, r)$, $\Phi(p, r)$ is a local diffeomorphism for any $r < \epsilon_0$. $\epsilon_0$ depends only on $U$.

Locally write

$$\Phi(p, r) : U_1 \times U_2 \times \text{Gl}(p) \to \mathcal{M}_K(X)$$

where $U_i \subset \mathcal{M}_{k_i}(X_i)$. We prove that this map, constructed in the proof, is a local diffeomorphism. The proof consists of four steps.

**Step 1, splicing maps and approximate solutions:** Identify $B_{p_i}^1(1) \setminus \{p_i\}$ with the cylinder $R^1 \times S^3$ by mapping points the polar coordinates $(s, \theta)$ to $(\log s, \theta)$. Let $X_i' = X_i \setminus B_{p_i}(N^{-1}r)$ for some large constant $N$. Identify the annulus $B_{p_i}(N^{-1}, Nr) := B_{p_i}(Nr) \setminus B_{p_i}(N^{-1}r)$ with the tube $[\log r - T, \log r + T] \times S^3$, where $T = \log N$. Denote the tube by $C_{p_i}[\log r - T, \log r + T]$, or $C_{p_i}$ if no confusion is caused. Define the connected sum $X_r = X'_1 \# S^4$ by gluing $X_i$ at $C_{p_i}$ with identification $(t, \theta) \sim (2 \log r - t, \theta)$. $S^4$ should be naturally treated as one-point compactification of $(TX)_p$, so we can simply assume that $X_r$ is conformal to $X$. Suppose $g_i$ are standard metrics of $X_i$. We choose a metric $g_r$ on $X_r$ with $g_r = m_i g_i$ over $X_i'$ such that $1 \leq m_i \leq 2$ on the gluing area and equal to 1 elsewhere. The Sobolev norms $L^{1,p}, L^{q}$ on $X_r$ are defined with respect to this metric. Also we fix $p, q$ such that

$$2 < p < 4 \quad \text{and} \quad q = \frac{4p}{4 - p}$$

There are several cut-off functions that are frequently used: let $i = 1, 2$, $\gamma_{i,p,r}$: equals 0 for $t < \log r - 1$ on $C_{p_i}$ and equals 1 for $t > \log r + 1$. Moreover

$$\gamma_{1,p,r} + \gamma_{2,p,r} = 1 \text{ on } X_r$$

$\beta_{i,p,r}$: equals 0 for $t < \log r - T$ on $C_{p_i}$ and equals 1 for $t > \log r - 1$.

$\eta_{i,p,r}$: equals 0 for $t < \log r + T + 1$ on $C_{p_i}$ and equals 1 for $t > \log r + 2T + 1$

Note that $\beta_{i,p,r} = 1$ over the support of $\gamma_{i,p,r}$. We drop subscripts $p, r$ from these notations unless we need to address their relations to $p, r$. 

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Convention: in this section, all constants independent of $r$ are denoted by $C, C_i$ (or $\epsilon, \epsilon_i$) if we require them bounded from above (below). They may depend on proper sets $U$.

**Lemma 3.10.** $\beta_i$ can be chosen such that $\|\nabla \beta_i\|_{L^4} \leq C (\log N)^{-3/4}$.

**Proof:** For any function $f$, $\|\nabla f\|_{L^4}$ is conformally invariant. We can construct $\beta_i$ over cylinder or $\mathbb{R}^4$. We do this on cylinder. Define

$$\tilde{\beta}(t, \theta) = \begin{cases} 
0 & t \leq 0 \\
1 & t \geq \log N \\
t(\log N)^{-1} & \text{otherwise}
\end{cases}$$

$\|\nabla \tilde{\beta}\|_{L^4}$ satisfies the requirement, however $\tilde{\beta}$ is not smooth. Let $h(x)$ be a smooth, compact supported function over $\mathbb{R}^1$ such that $\int h = 1$. Define

$$\beta(t, \theta) = \int_{\mathbb{R}^1} \tilde{\beta}(s, \theta)h(t - s)ds.$$ 

It is easy to show that $\beta$ satisfies the estimates. $\beta_i$ can be obtained from $\beta$ by shifting and reflection on $t$ coordinate. So $\nabla \beta_i$ have same estimates as $\nabla \beta$. q.e.d.

Set $\delta = C (\log N)^{-3/4}$. The following two lemmas study derivatives of our cut-off functions with respect to $r, p$.

**Lemma 3.11.** Let $f_r = \gamma_{i,r}, \beta_{i,r}$, or $\eta_{i,r}$.

$$|\frac{\partial}{\partial r} f_r| < Cr^{-1},$$

and for $f_r = \beta_{i,r}, \eta_{i,r}$

$$\|\nabla (\frac{\partial}{\partial r} f_r)\|_{L^4} < Cr^{-1}.$$ 

Here the derivatives of functions on $X_2$ is interpreted as follows: these functions are well defined over $X_r$ and hence they induce functions over $X$. In fact, they are supported in $B_p(Nr)$. The derivatives make sense when these functions are treated as functions over $X$.

**Proof:** We use $(x, \theta)$ to denote the coordinate of points on $X$ and $(s, \theta), (t, \theta)$ for points on cylinder. Suppose $f_r = \beta_{1,r}$. By definition $\beta_{1,r}(x, \theta) = \beta(\log x - \log r + T, \theta)$. Here $\beta$ is defined in lemma 3.10. A direct computation shows that

$$\frac{\partial}{\partial r} \beta_{1,r}(x, \theta) = r^{-1} \int \beta(\log x - \log r - t)h'(t)dt.$$
So $|\frac{\partial}{\partial r}\beta_{1,r}| < Cr^{-1}$. Same argument clearly works for $\gamma_1$. Since $|\nabla f|_{L^4}$ is conformally invariant, we can compute it in terms of cylinder coordinates. So

$$\nabla(\frac{\partial}{\partial r}\beta_{1,r}(s, \theta)) = r^{-1}\int \beta'(s - \log r - t)h'(t)dt.$$ 

The second inequality in the lemma for $\beta_{1,r}$ follows from this expression. Of course, the proof here works for $\eta_1$.

For $\gamma_2, \beta_2, \eta_2$ note that the derivatives are only supported in annulus $B(N^{-1}r, Nr)$. The proof essentially has no difference. q.e.d.

**Lemma 3.12.** Let $f_{p,r} = \gamma_{i,p,r}, \beta_{i,p,r}, \text{ or } \eta_{i,p,r}$. 

$$|\frac{\partial}{\partial p} f_{p,r}| < C r^{-1},$$

and for $f_r = \beta_{i,p,r}, \eta_{i,p,r}$

$$\|\nabla(\frac{\partial}{\partial p} f_r)\|_{L^4} < C r^{-1},$$

**Proof:** Note that 

$$f_{q,r}(x) = f_{p,r}(x - q).$$

We explain how this works for the estimates by describing one case. Fix a point $p$, suppose $f_{p,r} = \beta_{1,p,r}$. $\beta_{1,p,r}(x) = \beta(\log |x| - \log r + T)$. Then $\beta_{1,q,r} = \beta(\log |x - q| - \log r + T)$. Note that the derivative is only supported in annulus $B_p(N^{-1}, Nr)$. So 

$$|\frac{\partial}{\partial q}|_{q=p}\beta_{1,q,r}| = |\beta'(\log |x - q| - \log r + T)\frac{\nabla_q|x - q|}{|x - q|}|_{q=p} \leq C r^{-1}.$$ 

Other cases are similar to the computations in lemma 3.11. q.e.d.

For any $[A_i] \in U_i$, choosing representatives of the class is based on

**Lemma 3.13.** Suppose $E$ is a trivial bundle over unit ball of $R^4$. Then there exists a representative of a connection with $A_r = 0$, unique up to $A \to uAu^{-1}$ for a constant gauge transformation $u$. Moreover 

$$|A(x)| \leq |x|\sup|F_A|.$$
This is a well known result. We skip the proof. If \( A \) is chosen in this way, we say that \( A \) is a \( r \)-gauge connection at 0.

Back to the construction of approximating solutions. Suppose that \( A_i \) are \( r \)-gauge. Define \( A'_i = \eta_i A_i \). Note that \( A'_i \) are trivial on the gluing area. Then

\[
|A_i - A'_i| \leq Cr_1, \quad |F(A'_i)| \leq C, \quad (4)
\]
and

\[
\|F^+(A'_i)\|_{L^2}, \|A_1 - A'_i\|_{L^4} \leq Cr^2. \quad (5)
\]

For any \( \rho \in Gl \), one can glue bundles \( E_i \) to get a bundle \( E_\rho \) over \( X_r \) using \( \rho \). \( A'_i \) automatically give a connection \( A'_\rho \). In general, we write \( A_0 = (A_0^1, A_0^2, \rho_0) \) and let \( a = (w_1, w_2, \nu) \) be a tangent vector at \( A^0 \).

**Proposition 3.14.** \( A'_\rho \) are approximating ASD-connections,

\[
\|F^+(A'_\rho)\|_{L^p} \leq Cr^{1+4/q}.
\]

**Proof.** Note that \( F^+(A'_\rho) = F^+(A'_1) + F^+(A'_2) \).

\[
\|F^+(A'_i)\|_{L^p} \leq \|\nabla \eta_i A_i\|_{L^p} + \|\eta A_i^2\|_{L^p} =: I_1 + I_2.
\]

Note that the integral area is in \( B_{p^1}(Cr) \) and

\[
I_1 \leq \|\nabla \eta_i\|_{L^4}\|A_i\|_{L^4} \leq C\|\nabla \eta_i\|_{L^4}\|A_1\|_{L^p^1}.
\]

We can prove that \( I_1 < Cr^{1+4/q} \) by using lemma 3.10 and (4). The estimate of \( I_2 \) also follows from (4). So

\[
\|F^+(A'_i)\|_{L^p} \leq Cr^{1+4/q}.
\]

Similarly, we have estimates for \( \|F^+(A'_2)\|_{L^p} \). q.e.d.

**Proposition 3.15.** Let \( a = (w_1, w_2, e) \) be as above, then for \( z = w_i \)

\[
\|\frac{\partial}{\partial z} F^+\|_{L^p} \leq Cr^{1+4/q}\|z\|_{L^{1,p}}.
\]

**Proof.** By direct computation,

\[
\|F^+_z\| \leq C(\|\nabla z\|_{L^p} + \|z\|_{L^p})
\]

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Note that \( w_i \) are tangent vectors of a finite dimensional moduli space, they are bounded and \( C^\infty \). Moreover they are chosen so that \( w_i(p_i) = 0 \). So we can assume \( |w_i(x)| \leq C|x| \), for \( x \in B_1(p) \). Now the rest of the proof is same as the previous proposition. q.e.d.

The reason for using \( r \)-gauge connections is to keep the gluing process gauge equivariant.

**Step 2: the right inverse of \( d_{A^i}^+ \):** We assume that \( H^2_{A_i} = 0 \). This is the case that we are concerned with in this paper: when \( X = S^4 \), \( H^2_{A} = 0 \) for all ASD-connections including trivial connections; when \( X \neq S^4 \), it follows our assumption that the metric \( g \) is regular and \( F_E = \emptyset \) (cf. §2.1). Therefore there are right inverses \( P_i \)

\[
P_i: \Omega^1_{X_i}(g_{E_i}) \to \Omega^1_{X_i}(g_{E_i}),
\]
to the operators \( d_{A_i}^+ \). \( P_i \) is uniquely determined by its image. In particular, we can choose \( P_i \) by requiring that the image of \( P_i \) is perpendicular to the kernel of \( d_{A_i}^+ \) in \( L^2(\cap L^p) \). So

\[
\|P_i\xi\|_{L^q} \leq C\|\xi\|_{L^p}.
\]

Since the moduli spaces \( U_i \subset \mathcal{M}_i \) are smooth, the finite dimensional proper open sets and operators \( d_{A_i}^+, P_i \) are smoothly parameterized by the manifold \( \mathcal{M}_i \), so we have for \( z = w_i \)

\[
\left\| \frac{\partial}{\partial z} P_i \right\| \leq C\|z\|_{L^{1,p}(X_i)} \quad \text{and} \quad \left\| \frac{\partial}{\partial z} d_{A_i}^+ \right\| \leq C\|z\|_{L^{1,p}(X_i)}. \quad (6)
\]

Define \( Q_i = \beta_i P_i \gamma_i \) and let \( Q = Q_1 + Q_2 \). We have

\[
\|d_{A^i}^+ Q \xi - \xi\|_{L^p(X)} \leq C\delta\|\xi\|_{L^p(X)}. \quad (7)
\]

The proof can be found in \([6]\).

**Proposition 3.16.** For small \( r \) there exists a right inverse \( P \) to \( d_{A^i}^+ \) over \( X_r \) satisfying

\[
\|P\xi\|_{L^q} \leq C\|\xi\|_{L^p}. \quad (8)
\]

Moreover for \( w_i \in (T\mathcal{M}_i)_{A_i} \)

\[
\left\| \frac{\partial}{\partial w_i} P\xi \right\|_{L^{1,p}} \leq C\|w_i\|_{L^{1,p}(X_i)}\|\xi\|_{L^p}. \quad (9)
\]
**Proof:** By (7), we know that $d^+_A Q$ is invertible if $\delta$ is small enough. Letting $r$ small, we can choose a constant $N$ to let $\delta$ small. (see lemma 3.10). Set $P = Q(d^+_A Q)^{-1}$, (8) is obvious then. By definition of $Q_i$ and (6)

$$\| \frac{\partial}{\partial z} Q_i \xi \|_{L^{1,p}(X)} \leq C \| z \|_{L^{1,p}(X)} \| \xi \|_{L^p(X)}.$$  

Same estimate holds for $Q$. (9) follows from the following expression of the derivative of $P$ and existing estimates for $Q$:

$$P' = (Q(d^+_A Q)^{-1})' = Q'(d^+_A Q)^{-1} + P(d^+_A Q)'(d^+_A Q)^{-1}.$$  

This finishes the proof. q.e.d.

**Step 3, constructing solutions:** Here we use Taubes’ argument. It is said that all solutions are of the form $A_\rho = A'_\rho + a$, where $a = P(\xi)$ for some $\xi$. Then equation $F^+(A_\rho) = 0$ changes to

$$d^+_A a + a \wedge a = -F^+(A'_\rho).$$  

(10)

Apply $P$ to the equation,

$$a + P(a \wedge a) = P(-F^+(A'_\rho)).$$  

(11)

**Proposition 3.17.** Let $\delta_0 > 0$ be a small constant depending on $U_1 \times U_2$. For small $r \leq \delta_0$ equation (11) has unique solution with

$$\| a \|_{L^{1,p}} \leq Cr^{1+3/q}.$$  

Let $w_i$ as before

$$\| \frac{\partial}{\partial w_i} a \|_{L^{1,p}} \leq Cr^{1+3/q} \| w_i \|_{L^{1,p}},$$  

(12)

**Proof:** recall that $\| F^+(A'_\rho) \|_{L^p} \leq Cr^{1+4/q}$. By (8), the same estimate holds for $P(-F^+(A'_\rho))$. Let

$$B = \{ f \in L^{1,p}(\Omega^1_X(\mathfrak{g} E_\rho)) \| \| f \| \leq Cr^{1+3/q} \}.$$  

Define a map $H : L^{1,p} \to L^{1,p}$ by

$$H(a) = -P(a \wedge a) - P(F^+(A'_\rho)).$$
It is routine to show that $H$ maps $B$ to $B$ and it satisfies the contraction mapping principle ([3]). This proves the first statement.

Taking derivative on equation (11), we have

$$a' = -2P(a \wedge a') - P'(a \wedge a) + P'(-F^+(A'_\rho)) + P((-F^+(A'_\rho))').$$

Using the estimates for $P'(F^+(A'_\rho))'$, we have bound for $L^1,p$ norms of the last three terms. For the first term

$$\|P(a \wedge a')\|_{L^1,p} \leq C\|a \wedge a'\|_{L^p} \leq C\|a\|_{L^4}\|a'\|_{L^q} \leq \frac{1}{2}\|a'\|_{L^1,p}.$$

Absorb this term on the left side, we have (12). q.e.d.

The map $\Phi$ now can be defined as

$$\Phi(A_1, A_2, \rho) = A'_\rho + a,$$

where $a$ is solved in proposition 3.17 and it can be treated as a map defined on $U_1 \times U_2 \times Gl$. Define $\Phi'(A_1, A_2, \rho) = A'_\rho$ to be the splicing map. So $\Phi = \Phi' + a$.

**Corollary 3.18.** For any $z = w_1 \in T_Mk_i(A_i)$ or $z = \nu \in T_1(Gl)$

$$\|da(z)\|_{L^1,p} \leq Cr^{1+3/q}\|d\Phi'(z)\|_{L^1,p}$$

Namely $\|da(z)\| \ll \|d\Phi'(z)\|$.

**Proof:** By the definition of the metric $g_r$ of $X_r$, it is obvious that $C'\|w_i\| \leq \|d\Phi'(w_i)\| \leq C\|w_i\|$. So (13) is a consequence of (12). This result is still true for the tangent vector $\nu$. A simple way to see this is to replace $M_{k_2}(X_2)$ by $M^0_{k_2}(X_2)$ ($\cong Gl \times M_{k_2}$ locally) and repeat the gluing theory. Now $Gl$ is embedded in $M^0_{k_2}$ as a subspace. $\nu$ can be treated as a vector in $M^0_{k_2}$ and so by the same argument we have (12) for $z = \nu$ and so (13). q.e.d.

Since we are concerned about the property of derivatives which is a local computation, locally we can put $Gl \times U_2$ together and replace it by $U_2^0 \subset M^0_{k_2}$ from now on. This is just what we did in the proof. Note that $d\Phi' \sim Id$, so the corollary implies that $\|da\| \leq Cr^{1+3/q}$.

To study the diffeomorphism issue, it is useful to give a local coordinate chart for $M_E$. The next few pages (up to remark 3.24) is contributed to this issue.
We begin with the study of the local structure of $A$ around $\text{Im}(\Phi')$. Locally, let $W = \Phi'(U_1 \times U_2^0)$. Let $v_0 \in W$ be a fixed connection. $A_E$ is indentified as a vector space with $v_0 = 0$. Define a map

$$\mathcal{Y} : \Omega^0(g_E) \times W \times \Omega^2_+ \rightarrow A_E$$

given by

$$\mathcal{Y}(\xi, v, \eta) = \exp(\xi)(v + P_v \eta),$$

where $P_v$ is the right inverse to $d_v^+$. In the definition, $\Omega^0(g_E)$ is assigned $L^2$-$p$-norm; $\Omega^2_+$ is assigned $L^p(X_r)$-norm; $V$ and $A_E$ are assigned $L^1$-$p$-norms. We would like to conclude that $\mathcal{Y}$ is a diffeomorphism in the neighborhood of $V$.

**Proposition 3.19.** Let $V_0 = (0, v, 0)$. Let $T = D\mathcal{Y}_{V_0}$ be the tangent map.

$$T(\xi, z, \eta) = d_v \xi + z + P_v \eta.$$ 

Then $T$ is an isomorphism and

$$\|T\| \leq C, \|T^{-1}\| \leq C.$$

**Proof:** $\|T\| \leq C$ follows from the definition of norms. Conversely, we first show that

$$\|(\xi, z, \eta)\| \leq C\|\alpha\|,$$

where $\alpha = T(\xi, z, \eta)$.

We know

$$d_v \xi + z + P_v \eta = \alpha,$$

apply $d_v^+$ to the equation,

$$d_v^+ d_v \xi + d_v^+ z + \eta = d_v^+ \alpha.$$

It is easy to have

$$\|d_v^+ \alpha\| \leq C\|\alpha\|$$

$$\|d_v^+ z\| \leq Cr\|z\|$$

$$\|d_v^+ d_v \xi\| \leq Cr\|d_v \xi\|.$$

On the other hand, similar to the argument in [8], by the fact that $H^0(A) = H^0(B) = 0$ (where $v = \Phi'(A, B)$), we have

$$\|d_v \xi + z\| \geq C(\|d_v \xi\| + \|z\|).$$
Combine these inequalities together

\[ \|\eta\| = \|d_v^+ \alpha - d_v^+ d_v \xi - d_v^+ z\| \]
\[ \leq C\|\alpha\| + Cr(\|d_v \xi\| + \|z\|) \]
\[ \leq C\|\alpha\| + C'r(\|d_v \xi + z\|) \]
\[ = C\|\alpha\| + C'r(\|\alpha - \eta\|). \]

Reorganize the terms of \( \|\eta\| \), we have

\[ \|\eta\| \leq C\|\alpha\|. \]

For \( \xi, z \), the estimates follow from

\[ \|d_v \xi + z\| = \|\alpha - P_v \eta\| \leq C\|\alpha\|. \]

To see that \( T^{-1} \) exists, one can use the argument of using index theory (\[6\]). q.e.d.

In general,

**Proposition 3.20.** For any \( V = (\xi_0, v, \eta_0) \) in the \( \epsilon \)-ball of \( V_0 \), \( D\mathcal{Y}_V \) is isomorphic. Moreover,

\[ \|D\mathcal{Y}_V\| \leq C, \|D\mathcal{Y}_V^{-1}\| \leq C, \]

where \( \epsilon, C \) are independent of \( r \).

The idea is to compare the operators \( D\mathcal{Y}_V \) and \( D\mathcal{Y}_{V_0} \) by a direct computation. A similar computation is already given in \[6\]. One can find that the difference between two operators are controlled by \( \epsilon \). Hence proposition 3.19 implies this proposition.

As a consequence,

**Corollary 3.21.** \( \mathcal{Y} \) is a local diffeomorphism around \( W \). \( \mathcal{Y}(W \times \Omega^2_+) \) gives a slice for \( \mathcal{A}_E \).

By the proposition 3.17, we know that the moduli space \( \mathcal{M}_E \) near \( W \) is isomorphic to \( W \). In fact, the isomorphic map is given by

\[ \hat{a}(w) = w + a \circ (\Phi')^{-1}(w), w \in W. \]

So

**Corollary 3.22.** \( (W, \hat{a}) \) is a local coordinate chart for \( \mathcal{M}_E \).
Proof of proposition 3.9: Clearly, \( \Phi' \) is a local diffeomorphic map. Using the coordinate chart explained above, \( \Phi \) itself is \( \Phi' \). So \( \Phi \) is a local diffeomorphism. q.e.d.

Now suppose we have a smooth map \( \tilde{f} : W \to \mathcal{M}_E \subset \mathcal{A}_E \). Treating that \( W \) as a chart of \( \mathcal{M}_E \), \( \tilde{f} \) induces a map from \( f = \tilde{f}a^{-1} \) from \( \mathcal{M}_E \) to itself.

**Proposition 3.23.** Let \( \hat{f}(x) = \tilde{f}(x) - x \). Suppose

\[
\|\hat{f}\| \leq \epsilon, \|d\hat{f}\| \leq \epsilon_0,
\]

for some small constant \( \epsilon_0 > 0 \). Then \( f \) is a diffeomorphism. Here the norm used for \( \mathcal{A}_E \) is \( L^{1,p} \). Furthermore, if \( \hat{f}_i, i = 1, 2 \) are two such maps, then \( f_2 \circ f_1^{-1} \) is locally diffeomorphic.

**Proof:** Let \( v \in W \). We may assume that \( W \) is a linear vector space. Otherwise, we can identify \( W \) with the tangent space \( W_v \). Assuming \( v = 0 \) in \( W \). By \( \mathcal{Y} \), we can assume that \( \mathcal{A}_E = \Omega(g_E) \times W \times \Omega^2_+ \). Under this identification, suppose

\[
\tilde{f}(z) = (\xi(z), v(z), \eta(z)).
\]

\( f \) can be interpreted as \( f(z) = v(z) \) when \( W \) is identified with \( \mathcal{M}_E \). By proposition 3.19, we see that

\[
\|dv - I\| \leq C\|d\hat{f}\| \leq C\epsilon_0.
\]

So when \( \epsilon_0 \) is chosen small such that \( C\epsilon_0 < 0.5 \), \( dv \) is an isomorphism. Therefore \( f \) is locally diffeomorphic. q.e.d.

**Remark 3.24.** The proposition 3.23 is useful for local diffeomorphism issue. It says that if we can control difference of two maps \( \hat{f}_1 - \hat{f}_2 \) up to its derivatives in terms of the norm of Banach space(!), we can conclude the diffeomorphism on the moduli spaces. One can see that this argument works because we are taking particular norms on the Banach spaces. Roughly speaking, one of the key is that our norms are with respect to the metric of \( X_r \) instead of \( X \).

We now discuss our gluing map \( \Psi_T \). By comparing to \( \Phi \) and so following the remark 3.24, we are able to show that it is a local diffeomorphism. The main difference between \( \Psi_T \) and \( \Phi \) is that for \( \Psi_T \) we allow the gluing radius \( r \) and the bubble point \( p \in X \) to vary. In order
to make $\Psi_T$ diffeomorphic, we know just by dimension counting that we have to replace $\mathcal{M}_{k_2}(S^4)$ by $\mathcal{M}_{k_2}^b$. Locally, the map $\Psi_T$ is defined to be

$$\Psi_T(A_1, A_2, \rho, p, r) = \Phi_{p, r}(A_1, A_2, \rho).$$

This is well defined globally (21). Also, by our convention, locally we combine $(A_2, \rho)$ together to be an $A_2 \in U_2^{b,0}$. Let

$$\Psi'_T(*, p, r) = \Phi'_T(\ast).$$

Fix a point $B^0 = (A_{10}, A_{20}, r_0, p)$. Let $V_1(\epsilon_1)$ and $V_2^{b,0}(\epsilon_1)$ be $\epsilon_1$ neighborhood of $A_{10}$ in $\mathcal{M}_{k_1}(X_1)$ and $\mathcal{M}_{k_2}^{b,0}$. We construct a natural smooth map $D_{p, r_0} : V_1(\epsilon_1) \times V_2^{b,0}(\epsilon_1) \to \mathcal{M}_{k_1}(X) \times \mathcal{M}_{k_2}^{b,0} \times R^+ \times B_p(1)$. Let $h : \mathcal{M}_{k_2}^{b,0}(X_2) \to \mathcal{M}_{k_2}^{b,0}$ be the projection map and $V_2^{0} = h^{-1}(V_2^{b,0})$. For any balanced ASD-connection $A_2 \in V_2^{b,0}(\epsilon_1)$, the fiber $h^{-1}A$ is in the form of $A_2(t(\cdot - y))$, where $t \in [1 - \epsilon, 1 + \epsilon], y \in B(\epsilon)$. Define

$$D_{p, r_0}(A_1, A_2(t(\cdot - y))) = (A_1, A_2, r_0 t^{-1}, p + r_0^2 y).$$

By the geometric meanings of $t, y, r, q$, it is clear that $\Psi_T \circ D_{p, r_0} \sim \Phi_{p, r_0}$ over $V_1 \times V_2^{0}(\epsilon)$.

**Proposition 3.25.** Let $\tilde{\Psi}_T = \Psi_T \circ D_{p, r_0}$. $\tilde{\Psi}_T$ is diffeomorphic over $V_1(\epsilon) \times V_2^{0}(\epsilon)$ when $\epsilon$ is small.

**Proof:** Let $\Psi_\delta = \tilde{\Psi}_T - \Phi_{p, r_0}$. We know that $\Phi_{p, r_0}$ is diffeomorphic by proposition ?. By proposition ?. It is sufficient to show lemma ?. q.e.d.

**Lemma 3.26.** $\|d\Psi_\delta\| \leq C_{\Psi_0}^{3/4} \|d\Phi'_{p, r_0}\|$. 

**Proof:** Let $A^0 = D_{p, r_0}^{-1} B^0$. Suppose that $z$ is a tangent vector at $A^0$ and $z' = dD_{p, r_0}(z)$. Let $\Psi'_T = \Psi'_T \circ D_{p, r_0}$. The difference $\Psi_\delta$ is generated by two terms: 1) $\Psi'_T - \Phi'_{p, r_0}$ and 2) $\Psi_T - \Phi'_T$. When tangent vector $z$ is in $(TV_1)_{A_{10}} \times (TV_2^{b,0})_{A_{20}}, \Psi_\delta = 0$. So only $r$-direction vectors and $p$-direction vectors are nontrivial. To simplify the notations, we consider them separately. The case for combinations of different directions can be proved by the same arguments.

Case 1: $z' = r_0 \frac{\partial}{\partial r}$. Here with factor $r_0$, $\frac{\partial}{\partial r}$ is normalzied such that $\|z\| \sim 1$. The proof consists of two lemmas.
Lemma 3.27. \( \|d\hat{\Psi}_T'(z) - d\Phi'_{p,r_0}(z)\| \leq Cr_0^{4/q}. \)

**Proof:** Suppose \( A_r \) is the path representing \( z \) at \( A_{r_0} = A_2 \). And 
\[
A_r = A_2\left(\frac{r_0}{r}\right).
\]
By definition \( \hat{\Psi}_T'(A_1, A_r) = \Phi_{p,r}(A_1, A_2) \). On \( X \)
\[
\hat{\Psi}_T(A_1, A_r) = \eta_{1,r}(A_1) + \eta_{2,r}A_r.
\]
So
\[
\frac{\partial}{\partial r}_{|r=r_0} \hat{\Psi}_T'(A_1, A_r) = \frac{\partial}{\partial r}_{|r=r_0} \eta_{1,r}A_1 + \frac{\partial}{\partial r}_{|r=r_0} \eta_{2,r}A_2 + \eta_{2,r_0} z.
\]
Hence
\[
\|d\hat{\Psi}_T'(z) - d\Phi_{p,r_0}(z)\|_{L^1,p} \leq \|\frac{\partial}{\partial r}_{|r=r_0} \eta_{1,r}A_1\|_{L^1,p} + \|\frac{\partial}{\partial r}_{|r=r_0} \eta_{2,r}A_2\|_{L^1,p}
\]
Using lemma 3.11, the estimate is routine. q.e.d.

Similarly we also have

Lemma 3.28.
\[
\left\| \frac{\partial}{\partial z} F^+(\hat{\Psi}_T'(A_1, A_r)) \right\|_{L^p} \leq Cr_0^{4/q}
\]
\[
\left\| \frac{\partial}{\partial z} Q(\hat{\Psi}_T'(A_1, A_r)) \right\|_{L^1,p} \leq Cr_0^{-1}
\]
\[
\left\| \frac{\partial}{\partial z} P(\hat{\Psi}_T'(A_1, A_r)) \right\|_{L^1,p} \leq Cr_0^{-1}
\]
and
\[
\left\| \frac{\partial}{\partial z} \Psi_\delta(A_1, A_r) \right\|_{L^1,p} \leq Cr_0^{3/q} \tag{14}
\]

**Proof:** Most of the proofs are similar to the previous one. We explain the proof for \( \frac{\partial}{\partial z} Q \). \( Q = Q_1 + Q_2 \). We prove for \( Q_2 \). \( Q_1 \) is similar but slightly simpler. By definition \( Q_2(r) = \beta_{2,r} P_{A_r} \gamma_{2,r} \). For any \( \xi \)
\[
\| \frac{\partial}{\partial z} Q_2(r)\xi \|_{L^1,p} \leq \| \frac{\partial}{\partial r} \beta_{2,r} P_{A_r} \gamma_{2,r_0} \xi \|_{L^1,p} + \| \beta_{2,r_0} \frac{\partial}{\partial z} P_{A_r} \gamma_{2,r_0} \xi \|_{L^1,p}
\]
\[
+ \| \beta_{2,r_0} P_{A_2} \frac{\partial}{\partial z} \gamma_{2,r} \xi \|_{L^1,p} =: I_1 + I_2 + I_3.
\]
Estimate $I_1$:

\[
I_1 \leq \|\nabla (\frac{\partial}{\partial r} \beta_{2,r})\|_{L^4} \|P_{A_2} \gamma_{2,r_0} \xi\|_{L^{1,p}} + \|\frac{\partial}{\partial r} \beta_{2,r}\|_{P \gamma_{2,r_0} \xi\|_{L^{1,p}}}
\]

\[
\leq (Cr^{-1})\|\xi\|_{L^p} \leq Cr^{-1}\|\xi\|_{L^p}.
\]

Here we use lemma 3.11.

Estimate $I_2$:

\[
I_2 \leq \|\nabla \beta_{2,r_0}\|_{L^4} \frac{\partial}{\partial z} \|P_{A_r} (\gamma_{2,r_0} \xi)\|_{L^{1,p}} \leq C\|\xi\|_{L^p}.
\]

The estimate of $I_3$ is similar.

For $P$ we use the expression in proposition 3.16 and the estimates for $Q$.

(14) is a consequence of the expression of $a$ (see (11)) and all estimates we just have. q.e.d.

Combining these two lemmas, we prove the case 1.

Case 2: $z' = r_0^2 \frac{\partial}{\partial x}$.

Here $\partial/\partial x$ is a unit vector in $TM_p$. The factor $r_0^2$ normalizes the vector $z$ such that $\|z\|$ is some constant $\sim 1$. This is due to the definition of $D_{p,r_0}$.

**Lemma 3.29.** $\|d\tilde{\Psi}'_T(z) - d\Phi'_{p,r_0}(z)\| \leq Ct_0^{A/q}$.

**Proof:** Let $p(t) = p + tx$ be a path in $M$. Besides the changes of cut-off functions in the construction of gluing maps, there is another subtle change. Recall the map $\Phi'_{p,r_0}(A_1,A_2)$. We require that $A_1$ is r-gauge with respect to $p(t)$ before applying $\Phi'$ to it. Denote connections as $A_{1,p(t)}$. Set

\[
\tilde{z} = \frac{\partial}{\partial t} A_{1,p(t)}.
\]

This vector only depends on $A_1$ and $p$. So $\|\tilde{z}\|_{L^1,p} \leq C$. Let $(A_1,A_{2,t}) = D_{p,r_0}^{-1}(A_1,A_2,p(t))$. By definition,

\[
\tilde{\Psi}'_T(A_1,A_{2,r_0^2t}) = \eta_{1,p(r_0^2t),r_0} A_{1,p(r_0^2t)} + \eta_{2,p(r_0^2t),r_0} A_2(\cdot - tx).
\]

Take derivative with respect to $t$:

\[
\frac{\partial}{\partial t} \tilde{\Psi}'_T(A_1,A_{2,r_0^2t}) = r_0^2 (\frac{\partial}{\partial p} \eta_{1,p,r_0} A_1 + \eta_{1,p,r_0} \tilde{z} + \frac{\partial}{\partial p} \eta_{2,p,r_0} A_2) + \eta_{2,p,r_0} \tilde{z}
\]

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So

\[ |(d\tilde{\Psi}_T - d\Phi_{p,r_0})z| \leq r_0^2 |\partial_p \eta_{1,p,r_0} A_1 + \eta_{1,p,r_0} \tilde{z} + \partial_p \eta_{2,p,r_0} A_2| \]

Because of the factor \(r_0^2\), we indeed get better estimates than the one stated in the lemma. q.e.d.

Similarly, we also have

**Lemma 3.30.** \( \| \partial_z \Psi(A_{1,p(t)}, A_2) \|_{L^1,p} \leq Cr^{3/q} \).

Combining these two cases, we prove lemma 3.26. q.e.d.

**Part II:** The gluing map is injective.

Here for simplicity of the notation, we prove the injectivity for \(\Phi\). We state the result as

**Proposition 3.31.** \(\Phi\) is injective.

**Proof:** Suppose there exists two points in \(M_E\)

\[ B = \Phi(A_1, A_2, \gamma); B' = \Phi(A'_1, A'_2, \gamma') \]

such that \([B] = [B']\), i.e, there exists \(g \in G_E\)

\[ B = gB'. \]

Assume \(K = \Phi'(A_1, A_2, \gamma)\) and \(K' = \Phi'(A'_1, A'_2, \gamma')\). Without the loss of generality, we may assume that \(L^{1,p}\)-norms of \(A_i, A'_i\)'s are bounded. By proposition 3.17,

\[ \|K - gK'\| \leq \epsilon. \]

Now, we want to find \((A''_1, A''_2, \gamma'')\) such that for \(K'' = \Phi'(A''_1, A''_2, \gamma'')\) \([K'] = [K'']\) and

\[ \|K - K''\| \leq C\epsilon. \]

This can be done as following: since \(K, K'\)'s norm are small at the gluing area, so is \(\|dg\|\). Now fix a point \(x_0\) in the gluing area, for example, \(|x_0 - p| = r_0\). Let \(g_0 = g(x_0)\). we can choose \(g_1 = g\) over \(X \setminus B_p(r_0^{1/2})\) and be constant \(g_0\) in \(B_p(2r_0)\). Similarly, (on the \(S^4\) side), let \(g_2 = g\) over \(B_p(r_0^2)\) and be constant \(g_0\) outside \(B_p(r_0/2)\). One can show that the choice

\[ A''_1 = g_1 A'_1, A''_2 = g_2 A'_2, \gamma'' = g_0^{-1} \gamma' g_0 \]
satisfies the requirement. Let \( B'' = \Phi(A_1'', A_2'', \gamma'') \). We know that \([B] = [B'']\) and
\[
\|B - B''\| \leq C\epsilon.
\]

However, by proposition 3.21, they are located in the local diffeomorphic region. So it provides the injectivity. q.e.d.

Modulo some trivial consideration, the main idea of the proof for \( \Psi \) is identical. We skip the proof here. So we complete the proof of proposition 3.8. q.e.d.

Now suppose \( k_2 = 0 \). Then \( M_{0b}^b = \{0\}, S_T(X) = M_{k_2}^b(X) \). \( GL_T \) has a natural \( SO(3) \) action on itself. In fact, \( GL_T = M_{k_2}^b(X) \). The gluing theory does not make sense for \( \Psi_T \) for this case. The reason is that \( T \) is not a bubble tree. But a similar result to proposition 3.9 can be obtained:

**Proposition 3.9':** For any open proper set \( U \subset M_{k_1}^b(X) \) there exists a small constant \( \epsilon_0 \), depending on \( U \), and a gluing map
\[
\Phi : (GL'(p) \cap gl^{-1}(U))/SO(3) \rightarrow M_{k_1}^b(X).
\]
such that \( \Phi \) is diffeomorphic.

\( SO(3) \) is the isotropy group of the trivial connection \( A_2 \). The treatment for nontrivial isotropy group is already given as in \([4]\). This is the obstruction of constructing a smooth compactification via the bubble tree methods.

We now discuss the general cases. Suppose a bubble tree \( (T, v_0) \) is given. We describe \( S_T^b \), \( S_T(X) \) and define \( S_T^{b, 0} \). This is done inductively. The base case is that \( T \) has only one vertex \( v_0 \). Then \( S_T^b = M_{w(v_0)}^b \), \( S_T(X) = M_{w(v_0)}^b \) and \( S_T^{b, 0} := M_{w(v_0)}^{b, 0} \). We first assume that \( T \) is not a ghost tree.

**Case 1:** \( S_T^b \).

Let \( v \) be a vertex of \( T \). Let \( \text{child}(v) = \{v_1, \ldots, v_n\} \), and Suppose that \( w_i = w(v_i) \). Let \( w_{v} = (w_1, \ldots, w_n) \). Suppose that for all subtrees \((t(v_i), v_i), S_{t(v_i)}^b \) has been constructed. Let \( S_{v}^4 \) be the sphere assigned to \( v \). Then
\[
S_{t(v)} = M_{w(v), w(v)}^b \times \prod_{i=1}^{n} S_{t(v_i)}.
\]

Let \( P \) be an \( SU(2) \)-principal bundle over \( S_v^4 \) with \( c_2(P) = w(v) \). Define
bundle
\[ P_v^0 = [\tilde{M}_w^b(v) \times \mathcal{G}_P (P|_{P\cap \mathcal{M}_w^b(v)}) \times \prod_{v_i \in \text{child}(v)} P] \to \mathcal{M}_w^b(v) \times (S^4 \setminus \{\infty\})^n / S_{w^0}. \]

Note that \( \mathcal{M}_w^b(v), W(v) \) is a subspace of \( \mathcal{M}_w^b(v) \times (S^4 \setminus \{\infty\})^n / S_{w^0}. \) \( P_0^v \) defines a bundle over \( \mathcal{M}_w^b(v), W(v) \).

\[ S^{b,0}_{t(v)} = P_v^0 \times \prod_{v_i \in \text{child}(v)} SO(3) \big( \prod_{v_i \in \text{child}(v)} S^{b,0}_{t(v_i)} \times R^+ \big). \]

Drop \( P|_{\infty} \) from \( P_v^0 \) and denote the bundle by \( P_v \). Then we define
\[ \text{GL}_T = P_{v_0} \times \prod_{v_i \in \text{child}(v_0)} SO(3) \big( \prod_{v_i \in \text{child}(v_0)} (S^{b,0}_{t(v_i)} \times R^+) \big). \]

**Case 2: \( \mathcal{S}_T(X) \).**
Suppose \( \text{child}(v_0) = \{v_1, \ldots, v_n\} \) and \( w_{v_0} = (W(v_1), \ldots, W(v_n)) \). For each \( v_i \), we already have \( S^{b}_{t(v_i)} \), \( S^{b,0}_{t(v_i)} \). \( X \) is assigned to \( v_0 \) and each edge \( (v_0, v_i) \) is assigned a bubble point \( p_i \). Tuples \( (p_1, \ldots, p_n) \) are parameterized by \( B_{v_0} := (X^n \setminus \Delta) / S_{w_{v_0}} \), where \( \Delta \) is the big diagonal. Let \( P \to X \) be the \( SU(2) \) principle bundle over \( X \) with \( c_2(P) = w(v_0) \). Define
\[ P_i = Fr(X) \times SO(4) S^{b}_{t(v_i)} \to X \]
and set
\[ \mathcal{S}_T(X) = [\mathcal{M}_{w(v_0)}(X) \times (\prod_{v_i \in \text{child}(v_0)} P_i) \to (\prod_{v_i \in \text{child}(v_0)} X)] / S_{w_{v_0}}. \]

Also, define
\[ P_{v_0} = \tilde{M}_{w(v_0)} \times \mathcal{G}_P \prod_{v_i \in \text{child}(v_0)} (P \times Fr(X)) \to \mathcal{M}_{w(v_0)} \times \prod_{v_i \in \text{child}(v_0)} X. \]

The bundle of gluing data is
\[ \text{GL}_T = [P_{v_0} \times \prod_{v_i \in \text{child}(v_0)} (SO(3) \times SO(4)) \big( \prod_{v_i \in \text{child}(v_0)} (S^{b,0}_{t(v_i)} \times R^+) \big)] / S_{w_{v_0}}. \]

Define \( \text{Gl}_T \) to be the fiber of \( \text{GL}_T \). By definition of \( \text{GL}_T \),
\[ \text{Gl}_T \cong \prod_{e \in D} (R^+ \times SO(3)), \]
where $D$ is the edge set of $T$. Namely, each edge $e$ corresponds to a gluing parameter $R^+ \times SO(3)$ and we write this $SO(3)$ to be $SO(3)_e$. Treat $SO(3)$ as $SU(2)/\mathbb{Z}_2$, we get $R^+ \times SO(3) = R^4 \setminus \{0\}/\mathbb{Z}_2$. Compactify $\mathbf{GL}_T$ by adding 0-section to the bundle, it is then a $R^4/\mathbb{Z}_2$ bundle. We denote $\overline{\mathbf{GL}}_T$ for these compactified bundles. However, for simplicity, we still use $\mathbf{GL}_T$ instead of $\overline{\mathbf{GL}}_T$. Including 0-section or not can be distinguished by the contexts. Correspondingly, $R^4/\mathbb{Z}_2$ generated by the edge $e$ is written as $R^4_e/\mathbb{Z}_2$. For simplicity, at this moment, we do not want to be bothered by the stabilizer $S_w$'s. For the time being, we assume that all $S_w = 1$. Given a vertex $v$ in $T$, suppose $e_1, \ldots, e_n$ are edges connecting $v$. Define $\mathbf{GL}(v)$ to be a sub-bundle of $\mathbf{GL}_T$ with fiber isomorphic to $\prod_{i}(R^4_e/\mathbb{Z}_2)$.

Suppose $T$ is a ghost bubble tree. One can still define a bundle $\mathbf{GL}_T$ as above. But there are more structures on this bundle. Suppose $v$ is a ghost vertex. There is an $SO(3)$ action on the fiber of $\mathbf{GL}(v)$. The action is along fiber and it acts on each $SO(3)_e$ as group multiplications on suitable side. Hence, there is an $SO(3)$-action on $\mathbf{GL}_T$ given by each ghost vertex. Suppose $v_1, \ldots, v_l$ are ghost vertices of $T$. Each $v_i$ corresponds to an $SO(3)$ and denoted by $SO(3)_{v_i}$. Define the isotropy group of $T$

$$\Gamma_T = \prod_{i=1}^{l} SO(3)_{v_i}.$$ 

The gluing data now is $\mathbf{GL}_T/\Gamma_T$. For non-ghost bubble tree, define $\Gamma_T = 1$.

Suppose edge set of $T$ is $D = \{e_1, \ldots, e_n\}$. $Gl_T = \prod_{i} R^4_i/\mathbb{Z}_2$. We now define a map $\psi : Gl_T \to T_K$. Suppose $x = (x_1, \ldots, x_n) \in Gl_T$. We contract $T$ at edges $e_i$ for those $x_i \neq 0$ and set $\psi(x)$ to be the resultant tree. For any index set $I \subset \{1, \ldots, n\}$ let $x = (x_1, \ldots, x_n)$ be any point such that $x_i \neq 0$ iff $i \in I$. Let $T' = \psi(x)$. Define

$$Gl_{T,T'} = \left( \bigcap_{i \in I} (R^4_i \setminus \{0\})/\mathbb{Z}_2 \right) \times \{0\} \subset Gl_T$$

and $\mathbf{GL}_{T,T'}$ to be the sub-bundle of $\mathbf{GL}_T$ with fiber $Gl_{T,T'}$. If nontrivial $S_w$ is also considered, $\Gamma_T$ and $Gl_{T,T'}$ are still well defined. Then the gluing theory for the general case is

**Theorem 3.32 ([21]).** Let $(T, v_0) \in T_K$. For any proper set $U \subset S_T(X)$ there is a small constant $\epsilon_0$ and gluing map

$$\Psi_T : (\mathbf{GL}_T(\epsilon_0)|_U)/\Gamma_T \to \overline{\mathcal{M}}_K(X),$$

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such that \( \Psi_T \) maps \( \text{GL}_{T,T'}/\Gamma_T \) to \( S_{T'} \) diffeomorphically for any \( T' > T \).

Let

\[
D(X, K) = \{(\Psi^{-1}_T((\text{GL}_T(\epsilon_0)|_U)/\Gamma_T), \Psi^{-1}_T)|T \in T_K\}
\]

where \( U, \epsilon_0 \) are chosen as in the theorem. This defines an atlas for \( \overline{M}_K(X) \) and makes it into a topological space. Moreover

**Corollary 3.33.** \( \overline{M}_K(X) \setminus S_K(X) \) is an orbifold.

**Proof:** The only thing that needs to be checked is that the transition maps between different charts are continuous. This is proved in §7.2.8 or referred to §3.3. q.e.d.

All the constructions and results works for \( \overline{M}'_K \) parallerly.

We now have topology defined on \( \overline{M}_K(X) \) via gluing. On the other hand, we also have the Parker-Wolfson bubble tree compactification theorem. It has been known that the convergence defined in bubble tree compactification is compatible to the given topology on \( \overline{M}_K(X) \).

We now apply the Parker-Wolfson bubble tree compactification theorem to show the compactness of \( \overline{M}_K(X) \) and \( \overline{M}'_K \).

**Theorem 3.34 ([19]).** 1) For any sequence \( \{[A_n]\}_{i=1}^{\infty} \subset \mathcal{M}_K(X) \) there exists a subsequence such that it converges to a bubble tree instanton in \( \overline{M}_K(X) \) via bubble tree compactification process. 2) The bubble tree compactification is consistent with the topology defined by \( D(X, K) \). 3) For \( X = S^4 \), \( \overline{M}'_K \) is compact. Otherwise, if \( F_K = \emptyset \), \( \overline{M}_K(X) \) is compact.

**Proof:** The statement (1) is just the Parker-Wolfson theorem. The second statement can be proved by the gluing theorem. More precisely, suppose the limit \( A \in S_T \). Apply \( \Psi_T \) to \( U \times \text{Gl}_T(\epsilon) \), where \( U \) is a small neighborhood of \( A \) in \( S_T \). Since \( \Psi_T \) is a diffeomorphism map, one can show that \( [A_n] \) is in the image of \( \Psi_T \) when \( n > N \) for some \( N \). We skip the details. One can find the proof, for example, in [21]. Now we show (3). Suppose that \( \{A_n\}_{i=1}^{\infty} \) is a sequence in \( \overline{M}_K(X) \).

Take a positive sequence \( \delta_n \to 0 \). If \( A_n \) is not in \( \mathcal{M}_K(X) \), it is in some chart \( (\Psi^{-1}_T(\text{GL}_T(\epsilon_0)|_U), \Psi^{-1}_T) \). Suppose \( A_n = \Psi_T(A'_n, b_n) \), where \( A'_n \in U, b_n \in \text{Gl}_T \). Choose any \( b'_n \in \text{Gl}_T \) such that \( |b_n - b'_n| < \delta_n \) and \( \tilde{A}_n = \Psi_T(A'_n, b'_n) \in \mathcal{M}_K \). Now we have a sequence \( \tilde{A}_n \) in \( \mathcal{M}_K(X) \).

Apply the bubble tree compactification to this new sequence and get a limit \( A \in \overline{M}_K \). Clearly, this is also the limit of \( A_n \). q.e.d.

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3.3 Smoothness of $\overline{M}_K(X)$

From the previous section, we know that $\overline{M}_K(X) \setminus S_K$ is a manifold. It is natural to ask if it is smooth. The problem is to study the gluing maps given by different strata. A straightforward way is to show that the transition maps between two gluing maps have certain degrees of smoothness. In [20], a similar situation is treated and the author proved that the transition maps are $C^1$. By remark ?, a result as ? is the key. In particular, it already provides $C^\infty$-smoothness. In this subsection, we change the strategy slightly here: we “patch” the gluing maps by perturbing one of the gluing maps. Hence the transition maps are relatively simple and they are smooth automatically.

We illustrate the difficulty of the problem for the first nontrivial case. Let $T$ be the bubble tree corresponding to bubble tree manifolds

$$X \coprod S_1^4 \coprod S_2^4/p \sim \infty_1, q \sim \infty_2$$

where $p \in X, q \in S_1^4$. $Gl_T = R_1^4/Z_2 \times R_2^4/Z_2$ and the coordinate on it is denoted by $(x_1, x_2)$. For a proper open set $U \subset S_T$, let $(\psi_T^{-1}(U \times Gl_T(4\epsilon_0)), \psi_T^{-1})$ be a chart. At the mean while, there are two other charts intersecting this chart. Define the bubble trees $T_i = \psi_T(R_1^4 \setminus \{0\} \times \{0\})$. Set $V_i = B_i(\epsilon_0, 4\epsilon_0) \times \{0\} \subset R_1^4 \times \{0\}$ and $U_i = \psi_{T_i}^{-1}(U \times V_i)$. $U_i \subset S_{T_i}$ are proper in their strata. These also provide coordinate charts $(\psi^{-1}_{T_i}(U_i \times Gl_{T_i}(\epsilon_1)), \psi^{-1}_{T_i})$. The problem is whether the transition maps on the overlaps of these three charts are smooth. For example, consider charts

$$(\psi^{-1}_T(U \times Gl_T(2\epsilon_0)), \psi^{-1}_T) \text{ and } (\psi^{-1}_{T_i}(U_i \times Gl_{T_i}(\epsilon_1)), \psi^{-1}_{T_i}).$$

We prove

**Proposition 3.35.** There is a perturbed gluing map $\overline{\psi}_{T_1}$ of $\psi_{T_1}$ such that the transition map between charts $(\psi_T^{-1}(U \times Gl_T(2\epsilon_0)), \psi_T^{-1})$ and $(\overline{\psi}_{T_1}^{-1}(U_1 \times Gl_{T_i}(\epsilon_1)), \overline{\psi}_{T_1}^{-1})$ is smooth for small $\epsilon_1$, where $\epsilon_1$ depends on $\epsilon_0$.

**Proof:** We split the perturbation of $\psi_{T_1}$ into two steps.

Step 1: We know that the map $\psi_{T_1} : U \times V_1 \to S_{T_1}$ is diffeomorphic by theorem 3.32. For any proper $U$ that we consider in the gluing theory, when $\epsilon_0$, depending on $U$, is small, we may assume that $Gl_T$ is a product of two bundles $GL_T^1$ whose fiber is $R_1^4/Z_2$. Pull back $GL_T^2$
to $U \times V_1$ and denote it by $\mathbf{GL}^2_{T,1}$. $(\Psi_{T,T_1})_* \mathbf{GL}^2_{T,1}$ gives a trivialization $B_{T,T_1}$ for $\mathbf{GL}_{T_1}$ over $\Psi_{T,T_1}(U \times V_1)$ by a proper bundle map

$$B_{T,T_1} : (\Psi_{T,T_1})_* \mathbf{GL}^2_{T,1} \to \mathbf{GL}_{T_1}.$$ 

So, without the loss of generality, we can use coordinates of $\mathbf{GL}_T$ for both $\mathbf{GL}_T$ and $\mathbf{GL}_{T_1}$. Define

$$\Psi^1_T : U \times \mathbf{GL}_T(4\varepsilon_0) \to S_{T_1} \times \mathbf{GL}_{T_1}, \Psi^1_T(A, x_1, x_2) = (\Psi_T(A, x_1, 0), x_2).$$

We first construct a map $\Psi_1$ that is in the intermediate stage between $\Psi_T$ and $\Psi_T \circ \Psi^1_T$. For simplicity, we assume $U$ consists of only one point $A = (A_1, A_2, A_3)$. Recall that $\Psi_T(A, x_1, x_2)$ is constructed by: splice $A_1, A_2, A_3$ together using gluing parameter $x_1, x_2$ first, then solving for instanton. Formally write these two steps as

$$\Psi'_T(A, x_1, x_2) = A_1 \# x_1, A_2 \# x_2, A_3 = (A_1 \# x_1, A_2) \# x_2, A_3$$

and

$$\Psi_T(A, x_1, x_2) = (A_1 \# x_1, A_2) \# x_2, A_3 + a((A_1 \# x_1, A_2) \# x_2, A_3),$$

where $A_1 \# x_2, A_2$ denotes the splicing of $A_1$ and $A_2$ with respect to gluing parameter $x$. Now $\Psi_1$ is defined to be

$$\Psi_1(A, x_1, x_2) = (A_1 \# x_1, A_2 + a(A_1 \# x_1, A_2)) \# x_2, A_3 + a((A_1 \# x_1, A_2 + a(A_1 \# x_1, A_2)) \# x_2, A_3).$$

The real meaning of this expression is that we glue $A_1, A_2$ to an instanton by $x_1$ first and then glue the resultant instanton with $A_3$ by $x_2$. Our first perturbation is

**Claim 1:** $\Psi_1$ can be perturbed to a new gluing map $\overline{\Psi}_1$ defined on $U \times \mathbf{GL}_T(4\varepsilon_0)$ such that $\Psi_T = \overline{\Psi}_1$ over $U \times \mathbf{GL}_T(2\varepsilon_0)$.

*Proof:* Let $\gamma$ be a cut-off function such that $\gamma(t) = 1$, $t > 3\varepsilon_0$ and is supported in $t > 2\varepsilon_0$. Define

$$\overline{\Psi}_1(A, x_1, x_2) = [A_1 \# x_1, A_2 + \gamma(|x_1|)a(A_1 \# x_1, A_2)] \# x_2, A_3 + a((A_1 \# x_1, A_2 + \gamma(|x_1|)a(A_1 \# x_1, A_2)) \# x_2, A_3].$$

Obviously, $\overline{\Psi}_1$ satisfies our goal. We need to show that it is also a gluing map. As we did before, we compare the map with $\Psi'_T$ and study the derivative of $\overline{\Psi}_1 - \Psi'_T$. As usual, we can show that

$$||d\Psi'_T - d\overline{\Psi}_1|| \leq C(|x_1| + |x_2|)^{3/q}.$$
\[ \| \overline{\Psi}_1(A, x_1, x_2) - \Psi'_T(A, x_1, x_2) \|_{L^{1,q}} \leq C(|x_1| + |x_2|)^{1+3/q}. \]

This is sufficient to imply that \( \overline{\Psi} \) is a gluing map. q.e.d.

Let \( \Psi'_T = \overline{\Psi}_1(\Psi_T^{-1})^{-1} \). We have shown that the transition map between charts \( (\Psi_T(U \times Gl_T(2\epsilon_0)), \Psi_T^{-1}) \) and \( (\overline{\Psi}_1(U \times V_1 \times B_2(4\epsilon_0)), (\Psi'_T)^{-1}) \) is \( \Psi_T^{-1} \).

**Step 1:** One might think that \( \Psi'_T \) (or \( \Psi_1 \)) is \( \Psi_T \) intuitively, however they are different because the metrics used for two gluing maps are different. This problem is dealt with in

\textbf{Claim 2:} there exists a small constant \( \epsilon_0 > 0 \) and a gluing map \( \overline{\Psi}_T \) such that \( \overline{\Psi}_T = \Psi'_T \) over \( \Psi_T(U \times B_1(2\epsilon_0, 3\epsilon_0) \times B_2(\epsilon_1)) \) and \( \overline{\Psi}_T = \Psi_T \) over \( \Psi_T(U \times B_1(\geq 4\epsilon_0) \times B_2(\epsilon_1)) \).

**Proof:** The difference between these two maps over \( \Psi_T(U \times B_1(3\epsilon_0, 4\epsilon_0) \times B_2(\epsilon_1)) \) can be seen by what follows. We make a convention that the metric we refer to when we do comparing is the one induced by \( X^*_\sharp x, S^4 \).

Let \( B = \Psi_{T;1}(A_1, A_2, x_1, 0) \). Then

\[ \Psi'_T(\Psi_T^{-1}(A_1, A_2, A_3, x_1, x_2)) = B^*_\sharp x A_3 + a'(B^*_\sharp x A_3) \]
\[ \Psi_T(\Psi_T^{-1}(A_1, A_2, A_3, x_1, x_2)) = B^*_\sharp x A_3 + a(B^*_\sharp x A_3). \]

Here, \( a' \) and \( a \), given in proposition 3.17 by using different metrics on \( X \), are different. \( \Psi_T \) uses the standard metric while \( \Psi'_T \) uses the metric over \( X^*_\sharp x, S^4 \). We know that

\[ \| da' \| \leq Cr_2^{3/q} \| d\Psi'_T \|. \]

We also have the same estimate for \( \Psi_T \) using the standard metric on \( X \). Translate these estimates to the metric we are using now, then

\[ \| da \| \leq C(\epsilon_0)r_2^{3/q} \| d\Psi'_T \|. \]

To glue two maps together, one can do either of the following ways:
1. deform the metric on \( X \) with respect to gluing parameter \( x \) such that the metric is standard one when \( |x| > 3\epsilon_0 \);
2. deform the right inverse \( P \)'s. To be precise, let \( P, P' \) be the right inverse used to define \( a, a' \). Note that we have a simple fact: \( \lambda P + (1 - \lambda) P' \) is still a right inverse. With this, we can easily deform \( P \)'s by cut-off functions.
If we choose $\varepsilon_1$ small enough, then $r_2$ is small. By either way, $\Psi'_T$ can be deformed so that it matches $\Psi_T$ when $|x_1| > 3\varepsilon_0$. This glues $\Psi'_T$ and $\Psi_T$ together. q.e.d.

We have finished the proof of proposition 3.35. q.e.d.

**Remark 3.36.** We make a remark on a compatibility property of gluing maps $\Psi_T$ and $\Psi_T$.

$$(\Psi_{T,T_1})_*\mathrm{GL}_T^2 \xrightarrow{B_{T,T_1}} \mathrm{GL}_{T_1} \xrightarrow{\Psi_T} (\Psi_{T,T_1})_*\mathrm{GL}_T^2 \xrightarrow{B_{T,T_1}} \mathrm{GL}_{T_1} \xrightarrow{\Psi_{T,T_1}^{-1}} V_{T_1}$$

In the first diagram, the map “?” does not agree with the transition map provided by the gluing theory, while in the second one, we know the transition map is the composition of three smooth maps. Therefore, the transition map is smooth. So, roughly speaking, the patching is the patching of gluing bundles by $B_{T,T_1}$ up to $\Psi_{T,T_1}$.

We can prove the main theorem in this section.

**Theorem 3.37.** Let $K > 0$ be any integer. There is an atlas $\mathcal{D}(X,K)$ for $\overline{\mathcal{M}}_K(X)$ such that the transition maps are $C^\infty$. So $\overline{\mathcal{M}}_K(X) \setminus S_K(X)$ is a smooth orbifold.

**Proof:** The construction is based on the previous proposition. Recall that we defined a partial order on $T_K$. We introduce the charts for strata with respect to this order inductively and require that the atlas satisfies several requirements ($R1$, $R2$, $R3$)

**R1:** To each stratum $S_T$ only one chart $(\overline{\Psi}_T(U_T \times G_{\tilde{T}}(\varepsilon_0)), \overline{\Psi}_T^{-1})$ is assigned, where $U_T$ is a proper open subset of $S_T$. To simplify notations, we denote charts as $\text{chart}(V_T, \overline{\Psi}_T)$, where $V_T = \overline{\Psi}_T(U_T \times G_{\tilde{T}}(\varepsilon_0))$.

**R2:** For any two charts $\text{chart}(V_T, \overline{\Psi}_T), \text{chart}(V_{T'}, \overline{\Psi}_{T'})$, $V_T \cap V_{T'} \neq \emptyset$ iff either $T < T'$ or $T' < T$.

We start with lowest strata which are compact. Choose a $\varepsilon_0 > 0$ such that for each $T$, $(\Psi_T(S_T \times G_{\tilde{T}}(\varepsilon_0)), \Psi_T^{-1})$ gives a coordinate chart for the neighborhood of $S_T$. Set $\overline{\Psi}_T = \Psi_T$. Moreover, for small $\varepsilon_0$, $R2$ can be satisfied. Let $\mathcal{U} \subset \overline{\mathcal{M}}_K(X)$ be the union of all open sets in given charts. This set is updated as charts are created. For any stratum $T'$ whose chart has not been constructed, gluing maps of given charts induce a gluing map over $G_{\tilde{T}'}|(U \cap S_{T'})$ (we simply say “over $U \cap S_{T'}$” later).
There is a gluing map $\Psi^0_T$ over $\mathcal{U} \cap \mathcal{S}_T$ induced by defined gluing maps of lower strata. This is defined as following: for any point $(A, x)$, there exists a lower strata $T < T'$ such that $A = \Psi_{T,T'}(B,x')$. Then define the induced gluing map

$$\Psi^0_{T'}(A, x) = \Psi_T(B, x', x).$$

Note that may not be well defined since there may be more than one choice for $T$. To solve this, it is equivalent to require

**R3':** For any two charts $chart(V_{T1}, \overline{\Psi}_{T1})$ and $chart(V_{T2}, \overline{\Psi}_{T2})$, $\overline{\Psi}_{T1}$ agrees with $\overline{\Psi}_{T2}$ on the overlap of their charts.

There is no such a problem for the base case. Now suppose we are ready to give a chart for some $T \in T_K$. This means that all strata lower than $\mathcal{S}_T$ already have charts. So $\mathcal{S}_T \setminus \mathcal{U}$ is a proper set in $\mathcal{S}_T$. By induction hypothesis **R3**, we already have a gluing map $\Psi^0_T$ over $M := \mathcal{U} \cap \mathcal{S}_T$ induced from given charts. Now shrink $M$ to $M_1$ and then to $M_2$. Using $\Psi_T$ over $\mathcal{S}_T \setminus M_2$ and arguments for proposition 3.35, we glue $\Psi_T$ and $\Psi^0_T$ together and define a new gluing map $\overline{\Psi}_T$ over $GL_T(\epsilon_T) \setminus M_2$ such that it agrees with $\Psi^0_T$ over $M_1$. Here $\epsilon_T$ is a small constant depending lower charts. Now modify the lower charts such that $\mathcal{U} \cap \mathcal{S}_T$ is $M_1$ other than $M$. (This $\mathcal{U}$ does not count current chart.) This will guarantee **R3’** and so **R3**. Of course, we can make $\epsilon_T$ small enough to guarantee **R2**, **R1** is automatically true. The transition maps are smooth as explained in proposition 3.35. The new atlas is denoted by $\mathcal{D}(X, K)$. q.e.d.

**R3** is the generalization of remark 3.36. We call it the *compatibility property* of the atlas $\mathcal{D}(X, K)$. Because of this property, we can identify open set in every chart as a gluing bundle for corresponding stratum and they are patched together by bundle maps $B_{T, T'}$. This is very useful to our flip resolutions in §4.3. In fact, this was one of the main motivations of this section.

### 4 Flip Resolution and Smooth Compactification

In the previous chapter, we constructed the compactified space $\overline{\mathcal{M}}$ via bubble tree compactification. Although certain smoothness and orbifold structures are achieved, there are still singularities wherever ghost strata occur. Suppose $T \in G_K$ is a ghost tree with $g(T)$ ghost vertices.
The non-finite stabilizer is the product of $g(T)$ copies of $SO(3)$. Due to these obstructions, the advantage we can get from the bubble tree compactification, as compared to the Uhlenbeck compactification, is limited. Our task in this paper is to construct an orbifold compactification. The ingredient to solve this problem is the following key observation: $M_T$ is obtained from some manifold by $g(T)$ “blow-up”s. This structure is very similar to the Fulton-MacPherson (briefly indicated by FM) compactified space. Using these “blowing-up”s to absorb the stabilizer $SO(3)$’s, we can apply ”blowing-down”s and hence resolve singularities at $S_T$. We call this procedure the flip resolution.

4.1 Flip Resolution and Smoothing $\overline{M}_K(X)$ for $K = 2$

We start with the simplest case $K = 2$. For $K = 2$ we only need one flip resolution, so it is easy to see how the flip resolution works. For general $K$, we need to apply flip resolutions sequentially. We will describe this later.

Suppose $K = 2$. The moduli space $\overline{M}_K(X)$ contains only one ghost stratum $S_T$ where $T$ is described as follows: the vertex set $V$ of $T$ is $V = \{v_0, v_1, v_2, v_3\}$. $\text{child}(v_0) = \{v_1\}$, $\text{child}(v_1) = \{v_2, v_3\}$ and $v_1$ is a ghost vertex. $w(v_2) = w(v_3) = 1$. $R(T) = [K - 2[0[1, 1]]]$. Let $STX$ be the sphere bundle of $TX$ with the standard $\mathbb{Z}_2$ action fiber-wisely and

$$Z_T = \mathcal{M}_{K-2}(X) \times X.$$ 

Then

**Lemma 4.1.** $S_T = \mathcal{M}_{K-2}(X) \times STX/\mathbb{Z}_2$ is a bundle over $Z_T$. The gluing bundle $\Gamma_T$ is the bundle with fiber

$$Gl_T = R^1_0/\mathbb{Z}_2 \times R^1_1/\mathbb{Z}_2 \times R^1_2/\mathbb{Z}_2.$$ 

$\Gamma_T = SO(3)$. The gluing data is $\Gamma_T/\Gamma_T$.

The lemma follows from example 3.4 and the description at the end of §3.2. We skip the proof. By the construction of $\Gamma_T$, we know that $\Gamma_T$ is a pull-back bundle, say $\Gamma'_T$, over $Z_T$. In other words,

$$\Gamma_T = S_T \times_{Z_T} \Gamma'_T.$$ 

The neighborhood of the stratum is $\Gamma_T/\Gamma_T$. In particular, $SO(3)$ only acts on $\Gamma'_T$. It is worth to take a look at other strata around $S_T$.
and see how they change after flip resolutions: let \((x_0, x_1, x_2)\) denote the point in \(G\ell_T\), set \(T_0 = \Psi_T(1, 0, 0)\) and \(T_1 = \Psi_T(0, 1, 1)\). \(S_{T_0}\) is the stratum that contains two bubble points on \(X\). \(S_{T_1}\) is the stratum that contains only one bubble point on \(X\) with energy 2. For simplicity, we assume that \(\mathcal{M}_{K-2}(X)\) is a point. Then the compactification of \(S_{T_0}\) is the real blow-up of \(X^2\) along diagonal modulo \(Z_2\) action. The compactification of \(S_{T_1}\) is still singular at \(S_{T_1}\). They intersect at \(S_T\).

**Definition 4.2.** Suppose \(N\) is a compact space and \(U \to N\) is a disk bundle. Let \(T \to N\) be the sphere bundle of \(U\). The space \(\tilde{U} = [0, 1] \times S\) is called the semi-blow-up of \(U\) along \(N\). We say that \(\{0\} \times S\) is the semi-blow-up boundary of \(U\). Conversely, if \(\tilde{U} = [0, 1] \times S\) where \(S\) is a sphere bundle of some disk bundle \(U\) over \(N\), \(U\) is call the semi-blow-down of \(\tilde{U}\).

This is a rather trivial definition, however it is the case corresponding to our general model. Now let us see how we apply semi-blow-up and semi-blow-down to \(G\ell_T/\text{SO}(3)\). To simplify the notations, we ignore the orbifold structures and replace \(R^4/Z_2 \times \text{SO}(3)\) by \(R^4 \times SU(2)\). In general, it is easy to see that the construction still works for orbifold cases. The resolution consists of two steps:

**Step 1:** semi-blow up \(G\ell_T\) along \(S_T\). We call the resultant space \(\tilde{G}\ell_T\). This space is described as follows: semi-blow up \(G\ell_T'\) along \(Z_T\). The resultant bundle is

\[
(G\ell_T')^\times = Z_T \times ([0, 1] \times S^7) =: [0, 1] \times SGL_T'.
\]

So

\[
G\ell_T^\times/SU(2) = [0, 1] \times S_T \times Z_T (SGL_T'/SU(2)).
\]

**Step 2:** semi-blow down \([0, 1] \times S_T\) along \(Z_T\). Since \(S_T\) is a sphere bundle over \(Z_T\) of some vector bundle, say \(S_{T,d}\), we can apply the semi-blow down and replace \([0, 1] \times S_T\) by \(S_{T,d}\). Eventually, \(G\ell_T\) is replaced by

\[
G\ell_T = S_{T,d} \times Z_T (SGL_T/SU(2)).
\]

Clearly, none of points has non-finite stabilizer in \(G\ell_T\). The procedure here is similar to the flip in algebraic geometry. So we call such an operation *flip resolution*. The following is the formal definition.

**Definition 4.3.** Let \(G\) be a group acting on \(S^{n-1}\) freely and it induces an action on \(R^n\). Let \(SN, SV\) be sphere bundles of the vector bundles
$N, V$ over $X$, where $V$ is $n$-bundle. There is a natural diffeomorphism

$$f : SN \times_X (V \setminus X)/G \to N \setminus X \times_X SV/G$$

by

$$f(n, [v]) = (|v|n, [\frac{v}{|v|}]).$$

Replacing $SN \times_X V/G$ by $N \times_X SV/G$ via $f$ is called flip resolution. Treat $X$ as 0-section of $N$, we call $X \times_X SV/G = SV/G$ the exceptional divisor of $N \times_X SV/G$.

We denote the new space by $\mathcal{M}_K(X)$. So we have

**Lemma 4.4.** $\mathcal{M}_K(X)$ is a smooth orbifold.

To have a better understanding of this new space, we explain what the completions of $S_{T_0}$ and $S_{T_1}$ are in the new space. The exceptional divisor takes the place of the ghost stratum $S_T$. It is $\mathcal{M}_{K-2}(X) \times X \times (S^{11}/SU(2))$. One can check that the completion of $S_{T_0}$ intersects the exceptional divisor at $\mathcal{M}_{K-2}(X) \times X \times \{[1, 0, 0]\} = \mathcal{M}_{K-2}(X) \times X$. So

**Lemma 4.5.** $\mathcal{M}_{T_0}$ in $\mathcal{M}_K(X)$ is $\mathcal{M}_{K-2}(X) \times Sym^2(X)$.

One can also prove that the completion of $S_{T_1}$ is a smooth sub-orbifold in $\mathcal{M}_K(X)$ and it is

$$\mathcal{M}_{K-2}(X) \times Fr(X)_{SO(4)} \times \mathcal{M}_b^2,$$

where $\mathcal{M}_b^2$ is obtained from $\mathcal{M}_2^b$ by flip resolution in a similar way. Note that original $\mathcal{M}_2^b$ is also singular.

### 4.2 FM-Compactification and Ghost Strata

We just explained that how to apply the flip resolution to the moduli space when $K = 2$. As $K$ gets bigger, the ghost strata are very complicated and interwoven. Fortunately, they basically follow the rules of the FM-compactification of configuration space $F(X, n)$.

**Review of the FM-compactification.**

Suppose $X$ is a smooth, compact complex manifold. Let $X^{[n]} := X^n/S_n$, where $S_n$ is the permutation group. This space is also denoted by $Sym^n(X)$. (In [11], the construction is given for $X^n$.) Define

$$F(X, n) = X^{[n]} \setminus \Delta,$$
where $\Delta$ is the big diagonal in $X^{[n]}$. In general, for a tuple $I = (i_1, \ldots, i_k), 1 \leq i_1 < \cdots < i_k = n$ define

$$\Delta_I = \{(x_1, \ldots, x_n) \in X^n | x_{i_1} = \cdots = x_{i_k}\}.$$ 

Let $s : X^n \to X^{[n]}$ be the projection. $\Delta_I$ also stands for $s(\Delta_I)$, for simplicity. So $\Delta = \cup \Delta_{(a,b)}$. In [11], a compactified space $\overline{F(X,n)}$ is given. The construction can be described as follows:

1) (complex)-blow up $X^n$ along diagonal $\Delta_{(1,2,\ldots,n)}$ and denote the new manifold by $X^n_n$. Accordingly, all other $\Delta_I$ are changed after blow-up. We still use $\Delta_I$ for the updated diagonals. Note that diagonals $\Delta_I, |I| = n - 1$, are disjoint in $X^n_n$.

2) Inductively, suppose $X^n_k$ is created. If $k = 2$, we are done. Else, one can also prove that diagonals $\Delta_I, |I| = k - 1$, are disjoint in $X^n_k$ by induction. Blow up $X^n_k$ along all these diagonals and create $X^n_{k-1}$. As a result, $\Delta_I, |I| = k - 2$, are disjoint.

3) The action of $S_n$ on $X^n$ has a natural extension to $X^n_n$. Let

$$\overline{F(X,n)} = X^n_2/S_n.$$ 

There is a projection map $FM : \overline{F(X,n)} \to X^{[n]}$. It is clear that reversing the blow-up process and blowing-down $\overline{F(X,n)}$ we retrieve $X^{[n]}$. Intuitively, we introduce blow-ups wherever two or more points run into one another.

When $X$ is a real manifold, one can also construct $\overline{F(X,n)}$ using real blow-up. Here, we use “semi-blow-up” to form a “stratified” space $\overline{F(X,n)}$. Each stratum is isomorphic to some $M_T$. We explain the construction of $\overline{F(X,n)}$ for $n \leq 3$. It can be generalized to all $n$.

**Example 4.6 ($n = 2$).** The space $X^n_2$ is the semi-blow-up of $X^2$ along diagonal. And $\overline{F(X,n)} = X^n_2/S_2$. Let $(X^2)_\Delta$ be the real blow-up of $X^2$ along $\Delta$. Then

$$\overline{F(X,n)} = (X^2)_\Delta/S_2.$$ 

So when $n = 2$, this coincides with $FM$-compactification via real blow-ups. $X^n_2$ has two strata $\Omega_1$ and $\Omega_2$: $\Omega_1$ is the boundary $B$ of $\overline{F(X,n)}$ and $\Omega_2 = F(X,2)$.

**Example 4.7 ($n = 3$).** $X^n_3$ is the semi-blow-up of $X$ along the diagonal $\Delta_{(1,2,3)}$. It is a manifold with boundary $B_{123}$. $b_{123} : B_{123} \to X = \Delta_{(1,2,3)}$ is a sphere bundle over $X$. We can describe points on $b_{123}^{-1}(x)$ as follows:
suppose $z \in b_{123}^{-1}(x)$, $z$ can be written as $z = (x, [x_1, x_2, x_3]), x_i \in TX_x$ such that

$$x_1 + x_2 + x_3 = 0, x_1^2 + x_2^2 + x_3^2 = 3. \quad (15)$$

In fact, $z$ is treated as the limit of triples $(x + tx_1, x + tx_2, x + tx_3)$ in $X_3^3 \setminus \Delta$. $[x_1, x_2, x_3]$ is the equivalent class $(x_1, x_2, x_3)/\sim$, where $(x_1, x_2, x_3) \sim (x_1', x_2', x_3')$ when $x_i = \lambda x_i, \lambda > 0 \ (|FM|)$. The coordinates of neighborhood of $B_1$ can be identified as $(x, [x_1, x_2, x_3], t)$. When $t \neq 0$, $(x, [x_1, x_2, x_3], t) = (x + tx_1, x + tx_2, x + tx_3) \in X_3^3$. Usually we drop $t$ if it is equal to 0. Now we blow up $X_3^3$ along $\Delta_1, I = (1, 2), (1, 3), (2, 3)$ and get $X_3^2$. Consider $\Delta_{(1,2)}$. There is a natural normal bundle for $\Delta_{(1,2)} \setminus B_{123}$ in $X_3^3 \setminus B_{123}$: The fiber over point $(x, y)$ is identified with normal directions given by $(x + tx', x - tx', y), x \in TX_x$. This bundle can be naturally extended over $\Delta_{(1,2)} \cap B_{123}$. We describe this in the neighborhood of $B_{123}$. Given a point $(x, [x_1, x_1, x_3], t) = (x, x + tx_1, x + tx_1, x + tx_3)$ with $t > 0$, a point $y \in TX_{x+tx_1}$ in the normal direction is of the form

$$\frac{d}{ds}|_{s=0}(x + tx_1 + tsy, x + tx_2 - tsy, tx_3).$$

The path can be given in terms of our new coordinate system. One can show that it can be extended over $t = 0$. This shows the bundle converges to the normal vector of $\Delta_{(1,2)} \cap B_{123}$ in $B_{123}$. Identify the neighborhood of $\Delta_{(1,2)}$ as this normal bundle and we can apply the semi-blow-up along $\Delta_{(1,2)}$. The new boundary is denoted by $B_{12}$. This is a stratified space. We denote the coordinates for the smallest stratum $B_{12} \cap B_{123}$ in the following way:

$$(x[x_1[y, -y], x_3]) .$$

Note that $x$ is followed by a bracket $[x_1[\ldots, x_3]$ and $x_1$ is followed by $[y, -y]$ without comma. We explain the format of the notations: $x_1, x_3 \in T_xX, y, -y \in T_{x_1}(T_xX)$. The geometric meaning is that there are two points (since the bracket after $x$ contains two points) that run into $x$ and cause a blow-up at $x$ and then there are two points run into $x_1$ and cause the second blow-up at $x_1$. Moreover, the coordinates satisfy

$$2x_1 + x_3 = 0; 2x_1^2 + x_3^2 = 1; y^2 + (-y)^2 = 1 .$$

Correspondingly, blow up along $\Delta_{(2,3)}, \Delta_{(1,3)}$ and create the other two boundaries and denote them by $B_{23}, B_{12}$. The resultant space is $X_2^3$.  

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and $\overline{F(X,3)} = X^3/S_3$. Totally, there are four strata $\Omega_i, 1 \leq i \leq 4$ in the space. We describe the strata and formulate the representations of points in the strata as follows.

$$
\Omega_1 = \{(B_{12} \cup B_{23} \cup B_{13}) \cap B_{123}\}/S_3, [x, [y[z, -z], w]];$$
$$
\Omega_2 = \{B_{123} \setminus (B_{12} \cup B_{23} \cup B_{13})\}/S_3, [x[y, z, w]]$$
$$
\Omega_3 = \{(B_{12} \cup B_{23} \cup B_{13}) \setminus B_{123}\}/S_3, [x[y, -y], z];$$
$$
\Omega_4 = F(X, 3), [x, y, z]$$

We explain the notations of points: for $[x, y, z], x, y, z \in X^3 \setminus \Delta$ and $[x, y, z] = (x, y, z)/S_3$; for $[x[y, -y], z], x, z \in X^2 \setminus \Delta, y, -y \in TX_x$, this corresponds to a blow-up at $x$ and described by $x[y, -y]$; for $[x[y, z, w]]$ this means there is a blow-up at $x$ and $y, z, w \in (TX_x)^3 \setminus \Delta$; Similarly $[x[y[z, -z], w]]$ says there is a blow-up at $x$ and $y, w \in (TX_x)^2 \setminus \Delta$, and then followed by a blow-up at $y$ and $z, -z \in T(TX_x)_y$.

In general, a point in $\overline{F(X, n)}$ is denoted by

$$p = [x_0[\ldots[\ldots], x_1[\ldots], \ldots]$$

Any point $y$ included in the bracket following a point $x$ is called descendant of $x$. Two points in the same level, like $x_0, x_1$ in the example, are called siblings. By the construction, siblings should differ from each other. It is clear that the representation of points in same stratum has same format. So we also say it is the format of the stratum. There is a natural way to construct bubble trees in terms of the formats of strata.

**Example 4.8.** We continue example 4.7. We intend to build bubble trees $T_i$ such that $M_{T_i} = \Omega_i$. Suppose $T_i \in T_K$. In the construction, we do not care what the weight of the root $v_0$ is. $v_0$ is always assigned with $X$. Weights for other nontrivial vertices are all equal to 1 in the construction. Namely

$$R(T_1) = [K - 3[0[1, 0[1, 1]]]]$$
$$R(T_2) = [K - 3[0[1, 1, 1]]]$$
$$R(T_3) = [K - 3[1, 0[1, 1]]]$$
$$R(T_4) = [K - 3[1, 1, 1]]$$
and representations of bubble tree mainfolds of $T_i$ are

$$
Y(dT_1) = [x_1[x_2, x_3[x_4, x_5]]] \\
Y(dT_2) = [x_1[x_2, x_3, x_4]] \\
Y(dT_3) = [x_1, x_2[x_3, x_4]] \\
Y(dT_4) = [x_1, x_2, x_3]
$$

Following these examples, one can generalize the correspondences for $n > 3$. A general condition similar to (18) is the balanced condition (3) on ghost bubble: suppose \{x_1, \ldots, x_k\} are points on ghost bubble $S^4$, then

$$
\sum_{i=1}^{k} x_i = 0, \text{ and } \sum_{i=1}^{k} |x_i|^2 = k.
$$

We now generalize the construction to the weighted cases. Let $w = (w_1, \ldots, w_n) \in \mathbb{Z}^n, w_i > 0$ and $S_w$ be the associated permutation group. Define $X^n_w$ to be the weighted space of $X^n$. It is the collection of points $(x_1, \ldots, x_n) \in X^n$ with assigned weights $w_i$ to $x_i$. Define $F_w(X, n) = X^n_w \setminus \Delta$. Similarly, we can define

$$
F_w(X, n) = X^n_w / S_w.
$$

**Example 4.9.** $F_w(X, 3)$.

$X_{w,3}^3$: the key point is to choose a normal bundle and its sphere bundle. Define a sub-bundle $N_w \subset TX \times_X TX \times_X TX \to \Delta_{(1,2,3)} = X$. For any $x \in X$ the fiber consists of

$$
\{(x_1, x_2, x_3) \in (TX_x)^3 \mid \sum_{i=1}^{3} w_i x_i = 0\}
$$

Take sphere bundle $SN_w$ of $N_w$ by imposing

$$
w_1|x_1|^2 + w_2|x_2|^2 + w_3|x_3|^2 = w_1 + w_2 + w_3
$$

with $\sum w_i x_i = 0$. This is the balanced condition. In general if there are \{x_1, \ldots, x_k\} points with assigned weights $w = (w_1, \ldots, w_k)$, then

$$
\sum_{i=1}^{k} w_i x_i = 0 \text{ and } \sum_{i=1}^{k} w_i |x_i|^2 = \sum w_i
$$
We call this the balanced condition of weight \( w \). These definitions are made to be consistent with (3). The neighborhood of \( \Delta_{(1,2,3)} \) in \( X^3 \setminus \Delta_{(1,2,3)} \) can be identified with \( N_w \) and so with \( (0,1) \times SN_w \). The weighted blow-up is to replace the neighborhood by \( [0,1) \times S \). One can mimic example 4.7 and do (weighted)-blow-up of \( X^3 \) along \( \Delta_{(1,2)} \) and so on. The choice of bundle and its sphere bundle is made to be consistent with balanced conditions.

Weighted spaces are natural objects for our model and they are the generalization of FM-models. Given \( w = (w_1, \ldots, w_n) \), a tree \( T \) is \( w \)-type if it has only \( n \) nontrivial vertices, say \( v_1, \ldots, v_n \), with weights \( w(v_i) = w_i \) and all vertices are ghosts except leaves. Let \( G_w \) be the collection of all \( w \)-type bubble trees. One can show that

**Lemma 4.10.** \( \mathcal{F}_w(X, n) \) is a stratified space. Moreover

\[
\mathcal{F}_w(X, n) = \bigcup_{T \in G_w} M_T.
\]

Now we know that \( \mathcal{F}_w(X, n) \) is obtained by a sequence of blow-ups. And example 4.7 also shows how to identify strata of \( \mathcal{F}_w(X, n) \) with \( M_T \). If we are just looking for an orbifold compactification of \( F(X, n) \), obviously our FM-space is a not a wise choice. The easiest one, of course, is \( X^{[n]} \) itself. Note that to go from \( \mathcal{F}_w(X, n) \) to \( X^{[n]} \), we can reverse the blow-up process and do blow-downs. \( \mathcal{F}_w(X, n) \) is the right model to describe \( \overline{\mathcal{M}}_K(X) \). The redundancy of \( \mathcal{F}_w(X, n) \) comparing to \( X^{[n]} \) suggests that we might simplify \( \overline{\mathcal{M}}_K(X) \) and get a better compactification.

**Moduli spaces with marked points and Ghost Strata**

The concept of marked points is very common in the theory of \( J \)-holomorphic curves. It is natural to introduce the same concepts to our case without additional difficulties. This is not essential in our compactification, however, they make the explanation simpler.

**Definition 4.11.** Let \( w = (w_1, \ldots, w_n) \). A marked point \( w \)-instanton of \( X \) is an instanton \( A \) with an unordered tuple \( (x_1, \ldots, x_n) \), \( x_i \in X \) and \( x_i \neq x_j, i \neq j \). Moreover \( x_i \) are associated with charges \( w_i \). Denote such an element by \( [A, (x_1, \ldots, x_n)] \).

For \( w = (w_1, \ldots, w_n) \) define

\[
\mathcal{M}_{K, w} = \{ [A, (x_1, \ldots, x_k)] \text{ is a marked point} \}
\]

\[
w - \text{instanton} | A \in \mathcal{M}_{K_0}(X)
\]

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where $K_0 = K - \sum w_i$. Basically, the bubble tree compactification and gluing theory can also be generalized to this case. We denote the compactified space by $\overline{\mathcal{M}_{K,w}}(X)$. We outline how these generalizations can be done:

**Bubble tree compactification of $\overline{\mathcal{M}_{K,w}}$.** Suppose there is a sequence of marked point instantons $\{[A_n, (x_{n1}, \ldots, x_{nk})]\}_{n=1}^\infty$. Either bubble points of $\{A_n\}$ or points $x$ satisfying

$$\lim x_{ni} = \lim x_{nj} = x, i \neq j.$$

are called the bubble points. We consider the bubble point of second type. Suppose $p$ is a bubble point and $\lim x_{n1} = \lim x_{n2} = p$. Take unit disk $B_p(1)$ but treat $(A_n, x_{n1}, x_{n2})$ as generalized connections. Using the construction of bubble tree compactification [19] such as dilating, transition etc., we get a marked point instanton on sphere and so on. It is easy to see that if we fix $A_n = A$ the compactification coincides with FM compactification of weighted cases.

We can mimic §3.1 to define the compactified space which is consistent with the bubble tree compactification. Some definitions should be changed slightly:

1. **Marked point bubble trees**: Besides the general notations we mentioned in §3.1, $w_i$ are assigned to marked points. $W(v)$ is defined to be sum of $w_i$’s assigned to $v$ and $W(v_i)$ as defined in §3.1. Then definition 3.1 can be used to define *marked point bubble trees*.

2. **Marked point bubble space of a marked point bubble tree**: suppose $X$ is the bubble space assigned to the bubble tree. For each vertex $v$ suppose the assigned manifold is $X_v$. For all $w_i$ assigned to $v$ they define distinct marked points $q_i \in X_v$ which are different from the bubble points on $X_v$, and $w_i$ are weights assigned to these points $q_i$.

3. **Marked point bubble tree instanton**: treat $q_i$ as part of generalized instanton on $S^4_v$ and use the definition 3.3.

We can similarly define $\overline{\mathcal{M}_{K,w}}(X)$ and etc. The gluing theory can be extended to these spaces similarly.

We now introduce some terminologies that are related to ghost strata. Given a ghost tree $T$, we suppose $v_1, \ldots, v_l$ are all ghost vertices of $T$. If $v_i$ satisfies $W(v_i) = \min_{1 \leq j \leq l} W(v_j)$, we call $v_i$ to be an *ends* of $T$ and $W(v_i)$ to be the energy of the end. Suppose $T = (V, D)$ is ghost
tree and $v$ is an end. Let $d(v)$ be the set of vertices that are descendants of $v$ and $V' = V \setminus d(v)$. $T' = (V', D')$ is the subtree of $T$ induced by $V'$, moreover assign $W(v)$ of $T$ to $v$ to make $T'$ a marked point bubble tree. We say $T'$ is obtained by cutting $T$ at $v$.

We now consider a typical ghost tree $(T, v_0) \in T_K$: $\text{child}(v_0) = \{v_1, \ldots, v_k\}$ and $v_i, 1 \leq i \leq k$ are ghost; For each $v_i$ $\text{child}(v_i) = \{v_{i1}, \ldots, v_{il_i}\}$ and $v_{ij}$ are leaves. We now describe the stratum $S_T$. Let

$$w := (w_0, w_1, \ldots, w_k) := (w(v_0), \sum_{j=1}^{l_1} w(v_{1j}), \ldots, \sum_{j=1}^{l_k} w(v_{kj}))$$

and $K = \sum_{i=1}^{k} w_i$. It is clear that $S_T$ is a fiber bundle over $\mathcal{M}_{K,w}(X)$. The fiber is contributed by two components. One component is bubble points on ghost bubbles. This can be described as follows: let $\pi : \mathcal{M}_{K,w}(X) \to F_w(X, k)$ be the projection map on marked points. Define

$$S_T = \pi^* \prod_{i=1}^{k} SN_{w'_i},$$

$w'_i = (w(v_{i1}), \ldots, w(v_{il_i}))$. The other component comes from the moduli spaces on $v_{ij}$. Let

$$Z_T = \prod_{i=1}^{k} \prod_{j=1}^{l_i} \mathcal{M}_{w(v_{ij})}^b,$$

and $Z_T = \mathcal{M}_{K,w}(X) \times Z_T$. Then

$$S_T = S_T \times \mathcal{M}_{K,w}(X) \times Z_T.$$

Since $SN_{w'_i}$ are (almost) sphere bundles, $S_T$ are product of $k$ sphere bundles over certain base. This structure is the key to our flip resolutions. We explain how this can be done assuming that $SN_{w'_i}$ are indeed sphere bundles: first, note that The gluing bundle $GL_T$ whose fiber is $Gl_T$ is a pull back bundle $GL'_T$ over $Z_T$. In other words, it is trivial over fiber of $S_T$. The neighborhood of $S_T$ therefore is

$$U(S_T) \cong S_T \times Z_T \times GL'_T / \Gamma_T.$$

For simplicity, we assume $k = 1$. Then $\Gamma_T = SU(2)$ up to some finite group. The flip resolution on this model is exactly the same as what we did in §3.1.

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4.3 Flip Resolutions on $\overline{M}_K(X)$

We already see how a single flip resolution can be applied to resolve singularities. But when $K > 2$, ghost strata are complicated. We need a systematic process to apply flips. The arrangement is given in the proof of the following theorem. $K = 3$ is a representative case to see how this works. So along each step in the proof, the case of $K = 3$ is associated as an example. We introduce some notations for $\overline{M}_3(X)$ first: there are seven ghost strata. The trees are

\[\begin{align*}
T_1 &= [0[0[0[1, 1], 1]]]; \\
T_2 &= [0[0[1, 1]]]; \\
T_3 &= [0[0[1, 2]]]; \\
T_4 &= [0[0[1, 1]]]; \\
T_5 &= [0[1[0[1, 1]]]]; \\
T_6 &= [0[0[1, 1], 1]]; \\
T_7 &= [1[0[1, 1]]].
\end{align*}\]

Other non-ghost trees are

\[\begin{align*}
T_{01} &= [3]; \\
T_{02} &= [2[1]]; \\
T_{03} &= [1[1, 1]]; \\
T_{04} &= [1[2]]; \\
T_{05} &= [1[1[1]]]; \\
T_{1i} &= [0[T_0]], \\ &\quad 1 \leq i \leq 5; \\
T_{21} &= [0[1, 2]]; \\
T_{22} &= [0[1, 1]]; \\
T_{23} &= [0[1, 1]].
\end{align*}\]

**Theorem 4.12.** For any integer $K > 0$ by finite steps of "flips", the singularities of $\overline{M}_K(X)$ can be resolved. Denote the new spaces by $\overline{M}_K(X)$.

**Proof:** We prove this by induction for both $\overline{M}_K(X), \overline{M}_K^b$. The first nontrivial case is $K = 2$. This is proved in §4.1.

Now consider general $K$. The resolutions are done inductively on the end energies $m \leq K$ of ghost trees.

1. $m = 2$.

Pick up bubble trees in $\mathcal{T}_K$ which has at least one end with energy $m = 2$. Suppose $T$ is such a tree with $k$ such ends and $v_1, \ldots, v_k$ are these ghost vertices. Set $V = (v_1, \ldots, v_k)$. Let $T_1$ be the $k$–marked bubble tree obtained by cutting off these ends from $T$. Let $w = (K - 2k, 2, 2, \ldots, 2).$ Then $S_{T_1, w} \subset \overline{M}_K, w$. As before

\[S_T = S_T \times S_{T_1, w} Z_T.\]

Here $S_T$ is the product of sphere bundles $SN_i, 1 \leq i \leq k$, over $Z_T$, where the sphere structure of $SN_i$ is obtained from the ghost vertex $v_i$. 

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Actually, this is true only for a proper subset $U \subset S_{T_1, w}$. For simplicity we take it as $S_{T_1, w}$ right now. We write

$$Gl_T/\Gamma_T = (Gl_{T,v_1} \times \cdots \times Gl_{T,v_k}) \times Gl_{T-V} =: Gl_{T,V} \times Gl_{T-V},$$

where

$$GL_{T,v_i} = R^4_0/\mathbb{Z}_2 \times \prod_{j=1}^{k_i}(R^4_j/\mathbb{Z}_2),$$

and $Gl_{T-V}$ is defined in terms of (4.7). In this case $k_i = 2$. Correspondingly, there are bundles $GL_{T,V}$ and $GL_{T-V}$ over $Z_T$ such that

$$GL_T = \{S_T \times Z_T \quad GL_{T,V} \times Z_T \quad GL_{T-V}\}/(\prod_i SO(3)_{v_i} \times \Gamma_{T-V})$$

where $\prod_i SO(3)_{v_i} \times \Gamma_{T-V} = \Gamma_T$. We can apply $k$ flip resolutions simultaneously to $k$-pairs $(SN_i, Gl_{v_i})$. So the stabilizer for the new model is $\Gamma_{T-V}$. As we pointed out, the flip resolutions only apply to the proper set of each stratum. Namely, we apply the flip resolutions to each chart given in $D(X, K)$. We need that resolutions are compatible on different charts which intersect. This compatibility is promised by the compatibility condition $R3'$ in §3.3.

After doing flip resolutions, we update $T_K$: [1], get rid of trees whose strata appear in $S_T$. [2], distribute the exceptional divisors to those corresponding trees. We now explain how to do this. Suppose $T$ has only one end to be flipped. When the number of flip resolutions is more than one, the method is same. Suppose $v$ is the corresponding ghost vertex. Let $v_{-1}$ be the parent of $v$ and $child(v) = \{v_1, \ldots, v_k\}$. Then

$$Gl_{T,v} = R^4_0/\mathbb{Z}_2 \times (R^4_1/\mathbb{Z}_2 \times \cdots \times R^4_k/\mathbb{Z}_2).$$

Here $R^4_0$ is the basic gluing parameter assigned to the edge $(v_{-1}, v)$ and $R^4_i$ are assigned to $(v, v_i)$. The assignment of trees to points in the exceptional divisor is determined by its coordinates in $SGL_{T,v}$, the sphere in $Gl_{T,v}$, in a natural way: suppose the coordinate of the point is $[x_0, \ldots, x_k]$. Here $[\cdot]$ is the equivalence class with respects to $\mathbb{Z}_2$’s and isotropy group $SO(3)$ actions. If $x_i \neq 0, i \geq 0$, we contract $T$ at its corresponding edge. The resultant tree is assigned to the point. We now illustrate this step for $K = 3$. The ghost strata to flip when $m = 2$ are those of trees $T_1, T_5, T_6, T_7$. Among them, only $S_{T_1}$ is compact. So for the other three strata, their charts only cover some proper sets. The
resolutions for these three strata only resolve the singularities on these proper sets. The resolution of the rest is done by the resolution on $GL_{T_1}$. They would patching together well because of the compatibility condition. $T_5, T_6, T_7$ are replaced by certain exceptional divisors and they are no longer singular. The interesting one left is $T_1$. There are two components in the exceptional divisor which are still singular. One is added to $S_{T_1}$ and the other component is added to $S_{T_3}$ (or $S_{T_2}$).

Define the resulting space as $\overline{\mathcal{M}}_{K,2}(X)$.

Pick up bubble trees in $\mathcal{T}_K$ which has at least one end with energy $m$. Suppose $T$ is such a tree. For simplicity, we assume $T$ contains only one such end. For multiple ends the argument is same as what we do for $m = 2$. Suppose $v$ is the ghost vertex. Let $\text{child}(v) = \{v_1, \ldots, v_k\}, w_i = w(t(v_i))$, and $w' = (w_1, \ldots, w_k)$. We make a further assumption that $v_i$ are leaves. The reason we can make such an assumption is that by induction, $\mathcal{M}^b_{w_i}$, corresponding to $t(v_i)$, are completely resolved. So we can assume that $\mathcal{M}^b_{w_i}$ is assigned to vertex $v_i$. For instance, in our example $S_{T_2}$ can be treated as a submanifold of $S_{T_3}$ in this sense. By assumption $\sum w_i = m$. Set $w = (K - m, m)$. Let $T_1$ be the 1-marked point bubble tree obtained by cutting $T$ at $v$. We still have sphere bundles $S_T$ and $Z_T$, which it is better to denote by $Z_{T,w}$. One important fact is that $S_T$ is(!) a sphere bundle. Note that this is not true until we finish previous $(m-1)$-step flip resolutions which provide all necessary blow-downs. In fact, if not, there must exist some ghost stratum in lower level which has an end with energy less than $m$. But this is impossible. So we can continue the flip resolution to the neighborhood of $S_T$ as before. By finite steps, all ghost trees should disappear from $\mathcal{T}_K$.

As an example, let us consider $K = 3$ again. The ghost strata to resolve for $m = 3$ are $T_2, T_3, T_4$. As we said, $T_2$ and $T_3$ can be treated together to be one stratum. We still denote it by $S_{T_3}$. For $T_4$, originally $S_{T_4}$ is almost a sphere bundle but it is not a sphere bundle! To see this, it is $B_{123}$ in example 4.7 diagonals are replaced by the exceptional divisors of semi-blow-ups. Applying the flip resolutions to the step for $m = 2$, the model now corresponds to $X_3^3$. These blow-ups are killed by blowing-down. So $S_{T_4}$ is now a sphere bundle. It actually corresponds to $B_{123}$. Then we again apply flip resolutions to these strata. This completes the flip resolutions for $K = 3$. q.e.d.

We now derive some consequences from our smooth compactified
moduli spaces. First consider the $SO(3)$-bundle $\mathcal{M}^{b,0}_K$ over $\mathcal{M}^{b}_K$. After compactification, $\overline{\mathcal{M}}^{b,0}_K$ is a fibration over $\overline{\mathcal{M}}^{b}_K$, but the bundle structure fails exactly at where the ghost bubbles occur at the principal component. By exactly same constructions, the flip resolutions can be applied to both spaces. For greater convenience later on, we consider a bundle $\mathcal{R}\mathcal{M}^{b,0}_K$, which is a semi-blow-down of $\mathcal{M}^{b,0}_K \times R^+$. Basically, we just add a $R^+$ factor to the fiber $SO(3)$ and make it to be $R^4/\mathbb{Z}_2$. Note that this bundle plays a role as part of gluing parameter in $\overline{\mathcal{M}}(X)$. So by the same construction given in theorem 4.12, one can construct smooth orbifold bundle $\mathcal{R}\mathcal{M}^{b,0}_K$ over $\mathcal{M}^{b}_K$. The fiber is $R^4/\mathbb{Z}_2$. Take its sphere bundle and call it $\mathcal{M}^{b,0}_K$. It might be more natural to resolve $\mathcal{M}^{b,0}_K$ directly. In fact, this gives the same bundle. As we said, the $SO(4)$ action on $R^4$ induces an action on $\mathcal{M}^{b}_K$ and $\mathcal{M}^{b,0}_K$. It is not difficult to check that the actions can be extended to all compactification spaces. So we have

**Corollary 4.13.** $\mathcal{M}^{b,0}_K$ is an $SO(3)$-bundle over $\mathcal{M}^{b}_K$. Moreover, the bundle is $SO(4)$-equivariant.

We now consider some useful sub-orbifolds in $\overline{\mathcal{M}}_K(X)$ and their normal bundles. Let $0 < l \leq K$. We put all $l$-level strata together and denote the set by $S^l$. Each component of $S^l$, given by a partition $t$ of $l$, is a sub-orbifold and denoted by $S^l_t$. From the construction of $\overline{\mathcal{M}}_K(X)$, we have

**Corollary 4.14.** Suppose $t = (l_1, \ldots, l_k)$ is a partition of $l$. In $\overline{\mathcal{M}}_K(X)$

$$S^l_t = [\mathcal{M}_{K-l}(X) \times (Fr^k(X) \times_{SO^k(4)} \prod_{i=1}^k \mathcal{M}^{b}_{l_i})] / S^l_t.$$ 

Its normal bundle, denoted by $\mathcal{N}S^l_t$, is

$$[(\mathcal{M}_{K-l}(X) \times_{\mathcal{G}_{P-l}} \times_{Fr^k} Fr^k(X)) \times_{SO^k(4) \times SO^k(3)} \prod_{i=1}^k \mathcal{R}\mathcal{M}^{b,0}_{l_i})] / S^l_t.$$ 

In particular, when $\mathcal{M}_{K-l}$ is a point.

$$S^l_t = [Fr^k(X) \times_{SO^k(4)} \prod_{i=1}^k \mathcal{M}^{b}_{l_i}] / S^l_t. \quad (16)$$

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and
\[
\mathcal{NS}_t^i = \prod_{i=1}^k (P_i \times_X Fr(X)) \times_{SO(4) \times SO(3)} \mathcal{RM}_{t_i}^{k,0})/S_t. \tag{17}
\]

Later, for the model of the wall-crossing formula, \( \mathcal{M}_{K-t} \) is not a point. But we only take a neighborhood of a point. So the models we consider are products of \( \mathbb{C}^N \) with (19) and (20). Note that these two models are very nice. Essentially, they have product structures and so usually the computations are reduced to the case that \( t \) contains only one element.

5 Kotschick-Morgan Conjecture on Donaldson Wall-crossing Formula

In the second part of the paper we demonstrate how the new compactifications can be applied. As an example, we prove the KM-conjecture in this chapter. It turns out that the localization theory of equivariant cohomologies is very helpful to the wall-crossing formula when we working on smooth spaces. The similar situation to the Seiberg-Witten theory was studied by Cao-Zhou (\[3\]). Recall that the wall-crossing formula is expressed in terms of the link. In §5.1, we explain how links are related to the equivariant integration. Then in §5.2 & 5.3, we apply the localization formula to our new compactified moduli spaces and prove the conjecture.

5.1 Review on Equivariant Theory and Localization Formula

In this section, we simply review the equivariant theory and concentrate on the localization formula. Readers are referred to \[1,2\] and \[15\] for details.

Suppose \( G \) is a compact Lie group and \( M \) is a topological space. Set \( M_G = EG \times_G M \), where \( EG \to BG \) is the universal \( G \)-bundle. The equivariant cohomology of \( M \) is defined by
\[
H^*_G(M) := H^*(M_G).
\]

Since \( EG \times_G M \) is a bundle over \( BG \) with fiber \( M \), \( H^*_G(M) \) is a module over \( H^*(BG) \). In particular, \( H^*(BS^1) = \mathbb{C}[u], H^*(BSpin(4)) = \mathbb{C}[u, v] \)

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\(\mathbb{C}[c_R, c_L]\) (see lemma 5.10). Suppose \(\pi : E \to M\) is a \(G\)-equivariant vector bundle. The \textit{equivariant Thom class} is defined to be the Thom class of \(\pi_G : E_G \to M_G\).

If \(M\) is smooth, the equivariant de Rham theory is also available. Assume \(\dim M = 2m\). From now on, we only consider \(G = S^1\). Let \(A^*(M)\) be the graded algebra of differential forms of \(M\). Counting \(u\) as an element of degree 2, we grade polynomial ring \(A^*(M)[u]\) as well. So a degree 2\(k\) polynomial is in the form

\[
\gamma = a_0 u^k + a_1 u^{k-1} + \cdots + a_k.
\]

We call \(a_0 u^k\) is the \textit{leading term} of \(\gamma\). Define \(\gamma_{[n]} = a_n\), the 2\(n\)-form coefficient. Suppose \(V\) is the vector field on \(M\) generated by the \(S^1\) action. Define an operator \(d_{S^1} : A^*(M)[u] \to A^*(M)[u]\) by

\[
d_{S^1} \gamma = (d - i_V)\gamma, \quad \gamma \in A^*(M)[u].
\]

We say that \(\gamma\) is an \(S^1\)-equivariant form if \(\gamma\) is \(d_{S^1}\)-closed. If \(\gamma\) has degree 2\(k\) given in the form as in (21), it is equivariant iff

\[
i_V a_n = da_{n-1}, 1 \leq n \leq k.
\]

These definitions can be easily generalized to \(A^*(M)[u, u^{-1}]\) where \(u^{-1}\) has degree \(-2\).

We now explain the localization formula. From now on, we assume that all equivariant classes are represented by de Rham forms. Suppose \(F = \bigcup_i F_i\) is the union of all components of stationary sets of the action. Let \(N_i\) be the \(S^1\)-equivariant normal bundle of \(F_i\) in \(M\) and \(\Theta_i\) be its equivariant Thom classes. Then the equivariant Euler class of the bundle \(N_i\) is \(E_i = i_{F_i}^{*} \Theta_i\). We endow \(M\) with an \(S^1\)-invariant metric. Define a 1-form \(\theta\) on \(M - F\) such that \(\theta(V) = 1, \theta|_{V^\perp} = 0\). Here \(V^\perp\) is the orthogonal complement of \(V\) in the tangent space. Then \(\theta\) is a connection on the principal bundle \(M - F \to (M - F)/S^1\). Equivariant forms have some properties over \(M - F\). We list two useful lemma here.

By a direct computation, one has

**Lemma 5.1.** Suppose \(\gamma = \mu + fu\) is an equivariant two form, then \(r(\alpha) = \mu - \theta \wedge df + f d\theta\) is an \(S^1\)-invariant form over \(M - F\).

A simple result for equivariant forms with general degree is also true. We only need degree 2 forms in our application.
Lemma 5.2. If $\gamma$ is an equivariant form. Then $\gamma_{[m]}$ is exact outside $F$.

Proof: Following the definition, one can check that

$$
\gamma_{[m]} = d(\theta \wedge \gamma_{[m-1]} + \theta \wedge d\theta \wedge \gamma_{[m-2]} + \theta \wedge (d\theta)^2 \wedge \gamma_{[m-3]} + \cdots).
$$

This proves the lemma. q.e.d.

Theorem 5.3 (Atiyah-Bott). If $\gamma \in H^*_S(M)$,

$$
\int_M \gamma = \sum_i \int_{F_i} \frac{\gamma}{E_i}.
$$

(19)

Note that the degree of $E_i$ is equal to the codimension of $F_i$ and its leading term does not vanish. The right hand side of (22) makes sense (in $A^*(F_i)[u,u^{-1}]$). The same situation also holds when we restrict $\Theta_i$ over a neighborhood of $F_i$.

Proof: Let $U_i, U_i', U_i''$ be $S^1$-invariant tubular neighborhoods of $F_i$ with relations $U_i \supset U_i' \supset U_i''$. Suppose that $\Theta_i, \Theta_i'$ are the Thom classes of the normal bundles and they are supported in $U_i, U_i'$ respectively. Moreover, we require that

$$
\Theta_i |_{U_i''} = \Theta_i' |_{U_i''}.
$$

Let $\gamma_i = i_{F_i}^* \gamma$. Compare

$$
\sum_i \int_{F_i} \frac{\gamma_i}{E_i} = \sum_i \int_{U_i} \frac{\gamma_i}{\Theta_i} \wedge \Theta_i',
$$

to $\int_M \gamma$, we have

$$
\sum_i \int_{U_i} \frac{\gamma_i}{\Theta_i} \wedge \Theta_i' = \int_M \gamma = \sum_i \int_{U_i} \gamma(\frac{\Theta_i'}{\Theta_i} - 1) - \int_{M \setminus \cup_i U_i} \gamma,
$$

(20)

note that $\frac{\Theta_i'}{\Theta_i} - 1$ are supported in $M \setminus U_i''$ and are equal to $-1$ when in $M \setminus U_i'$, thus $\gamma' := \sum_i \frac{\gamma_i}{\Theta_i} \wedge \Theta_i' - \gamma$ is an well defined equivariant form which is supported away from all stationary sets. By lemma 5.2, $\int_M \gamma' = 0$. This implies that the right hand side of (23) is 0. So we prove (22). q.e.d.

The Localization theorem also has a version for manifolds with boundary ([14], [3]). Suppose $M$ is a manifold with boundary and $\partial M \cap F = \emptyset$.

We state the theorem for a special case that we need later.
**Theorem 5.4.** Suppose $\gamma = \mu + fu$ is an equivariant 2-form of $M$ and $r(\gamma)$ is the one defined in lemma 5.1. Then

$$
\int_{\partial M/S^1} r^{m-1}(\gamma) = \sum_i \int_{F_i} \frac{\gamma^{m-1}u}{E_i}.
$$

(21)

where $2m = \dim M, 2n_i = \dim F_i$.

**Proof:** Let $\tilde{\gamma} = \gamma^{m-1}u$. Consider $\int_M \tilde{\gamma}$. By the same argument used in the proof of theorem 5.3, one also has a form $\tilde{\gamma}'$ supported outside $F$. Moreover, $\tilde{\gamma}' = \tilde{\gamma}$ near $\partial M$. It is sufficient to show

$$
\int_M \tilde{\gamma}' = \int_{\partial M/S^1} r^{m-1}(\gamma).
$$

By lemma 5.2, we know that $\tilde{\gamma}' = d\eta$ for some $\eta$ over $M - F$. Note that $\eta$ is trivial over a neighborhood of $F$. And in a neighborhood of $\partial M$, one can show that

$$
\tilde{\gamma}' = \gamma = d(\theta \wedge r^{m-1}(\gamma)).
$$

Therefore by Stokes theorem

$$
\int_M \tilde{\gamma}' = \int_{\partial M} \theta \wedge r^{m-1}(\gamma).
$$

Since $r^{m-1}(\gamma)$ is horizontal, applying integration along fiber to the right side, we show (24). q.e.d.

In (?), the left hand side corresponds to the “link” in our application. Finally, we make a remark on $F$. If the stationary sets are orbifolds, these results are still true up to proper constants depending upon the orbifold structures.

### 5.2 Reducible instantons

Here we explain the local models of reducible connections in $\mathcal{M}_K(X, \lambda)$. All the constructions and techniques are well described in [6]. Here, we state most of results without giving proofs. Readers are referred to corresponding theories in [8].

First assume the reducible connection $[A]$ with respect to $\alpha$ is in the top stratum $\mathcal{M}_K(X, \lambda)$. It is known that

**Lemma 5.5.** The neighborhood of $[A]$ in $\mathcal{M}_K(X, \lambda)$ is diffeomorphic to a neighborhood $U_A/\Gamma_A$ of 0 in $\mathbb{C}^N/\Gamma_A$, where $N = -\alpha^2 - 2$ and $S^1$ action is complex multiplication.
Proof: In general, the local model at $[A]$ is given by the $\Psi^{-1}(0)$ for a $\Gamma_A$-equivariant map

$$\Psi : H^1_{[A]} \to H^2_{[A]}.$$ 

When $b^+_2 = 1$, $H^2_{[A]}$ is 1-dimensional. By choosing a generic path $\lambda$, 1-dimension parameter $t$ can be mapped onto $H^2_{[A]}$ ([17]). So the lemma follows. q.e.d.

The universal bundle $\mathbb{P}$ can not be defined over $U/S^1 \times X$. However, there is a bundle $\tilde{\mathbb{P}} \to U_A \times X$ with compatible $\Gamma_A$ action such that $\mathbb{P} = \tilde{\mathbb{P}} / \Gamma_A$ over $U - \{0\}/S^1$. Our main strategy is to apply the equivariant theory over $U_A$.

**Lemma 5.6.** For any $\Sigma \in H^2(X, \mathbb{Z})$, the restriction of $\bar{\mu}(\Sigma)$ to $U_A$ can be extended as an equivariant class $\bar{\mu}(\Sigma) + fu$ such that

$$f|\{0\} = -\frac{1}{2}\langle \alpha, \Sigma \rangle,$$

The proof is exactly the same as the one in §5.1.4 ([6]). Alternatively, one can construct $\Gamma_A$-equivariant line bundle representing $2\bar{\mu}(\Sigma)$ over $U$. $\Gamma_A$ acts weightedly on the line bundle (§5.2 in [6]). Then the equivariant Chern class of the line bundle is $2\bar{\mu}(\Sigma) - \langle \alpha, \Sigma \rangle u$.

Now consider the reducible connection $[A]$ with respect to $\alpha$ located in lower strata. Suppose $r = (\alpha^2 - p_1(P))/4$. The family of reducible connections in $\mathcal{M}^u_K(X, \lambda)$ is $[A] \times \text{Sym}^r(X)$. Let $N = -\alpha^2 - 2$ and $U$ is a neighborhood of $\{0\}$ in $\mathbb{C}^N$. When $\alpha^2 = -1$, one has to consider the obstruction bundle. The localization formula can still be applied to the so called "virtual neighborhoods" ([3]). The main result, proposition 5.9, is still true for this case. Now assume $\alpha^2 \neq -1$. The local model of $[A] \times \text{Sym}^r(X)$ can be written as $(U_A \times \mathcal{G}L_{\alpha,r})/\Gamma_A$. In general, $\mathcal{G}L_{\alpha,r}$ is rather complicated ([12], [18]). Given a $\Sigma \in H_2(X, \mathbb{Z})$, there is a line bundle $L_{\Sigma} \to X$ whose $c_1$ is $\omega$, the Poincare dual of $\Sigma$ in $H^2(X, \mathbb{Z})$. For a line bundle $L \to X$, $L^n/S_n$ is a line bundle over $\text{Sym}^n(X)$. We denote the bundle by $\text{Sym}^n(L)$. $\text{Sym}^n(\omega)$ is the $c_1$ of $\text{Sym}^n(L_{\Sigma})$. Mimicking the construction in §7 of [3], we have

**Proposition 5.7.** There is a $\Gamma_A$-equivariant line bundle $\mathcal{L}_{\alpha,r}$ over $U \times \mathcal{G}L_{\alpha,r}$ which is isomorphic to $\mathcal{L}_{\alpha,0} \otimes \text{Sym}^r(L_{\Sigma}^2)$. $\mathcal{L}_{\alpha,r}/\Gamma_A$ restricted to $(U \times \mathcal{G}L_{\alpha,r}/\Gamma_A) \setminus [A] \times \text{Sym}^r(X)$ represents $2\bar{\mu}(\Sigma)$. 

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5.3 The Proof of Kotschick-Morgan Conjecture

In this section, we apply the localization formula given in theorem 5.4 to prove the KM-conjecture.

As we mentioned, all properties in \( \overline{\mathcal{M}}_K(X, \lambda) \) can be lifted to both \( \overline{\mathcal{M}}_K(X, \lambda) \) and \( \overline{\mathcal{M}}_K(X, \lambda) \) by \( \pi_* \), \( i = 1, 2 \). Let \( U_{\alpha,r} = \pi_2^{-1}(U_A \times \mathcal{G}_{\alpha,r}) \).

To apply theorem 5.4 to \( U_{\alpha,r} \) we first study the fixed loci of \( \Gamma_A \) and how \( \bar{\mu}(\Sigma) \)'s are equivariantly extended.

**Proposition 5.8.** The fixed loci of \( \Gamma_A \) in \( U_{\alpha,r} \) are disjoint smooth components \( F_t := [A] \times \mathcal{S}'_t(X) \), where \( t = (t_1, \ldots, t_l) \) runs over all partitions of \( r \). The pull-back bundle \( \pi_*^2 \mathcal{L}_{\alpha,r} \) is a \( \Gamma_A \) equivariant line bundle. Its equivariant chern class restricted to the fixed loci \( F_t \) is \( \langle \alpha, \Sigma \rangle u + 2 \text{Sym}^r(\omega) \).

**Proof:** We explain the first statement for the case \( r = 2 \). For \( r > 2 \), the phenomenon are same.

We begin with the model in \( \overline{\mathcal{M}}_K(X, \lambda) \). We explain how \( \Gamma_A \) acts in this model and show how the situation changes after applying flip resolutions. Let \( U_1 := \pi_1^{-1}(U \times \mathcal{L}_{\alpha,r}) \). The partitions of \( r = 2 \) are \( t_1 = (1, 1) \) and \( t_2 = (2) \). The reducible connections are parameterized by two components \( F_1 := [A] \times \text{Sym}^2(X) \setminus \Delta \) and \( F_2 := [A] \times X \times \mathcal{M}_0^2 \).

Here \( F_i \) corresponds to the partition \( t_i \). Note that \( F_1 \) intersects with \( F_2 \) at ghost bubbles. We now clarify the situation around ghost bubbles. The ghost stratum can be identified as \( (U \times \text{ST}X)/\mathbb{Z}_2 \). The gluing parameter of this stratum is

\[
Gl = \prod_{i=0}^{2} R_i^4 / \mathbb{Z}_2.
\]

The gluing bundle \( GL \) is the bundle over \( U \times \text{ST}X \) with fiber isomorphic to \( Gl \). In fact, it is constructed by a pull-back bundle from \( X \)

\[
\pi^*(\mathcal{M}_K^0(X) \times \mathcal{M}_{h}^b(S^4, X))^2 \times R^+ \times R^+ \times R^+.
\]

where \( \pi : U \times \text{ST}X \to X \) is the projection map. Then the local model of the ghost stratum is \( U_1 \subset GL/(\mathbb{Z}_2 \times SO(3)) \). \( \Gamma_A \) acts only on the first factor of \( Gl \). In general, it acts only on the gluing parameters associated to bubble points on \( X \). Suppose \( (x_0, x_1, x_2) \in Gl \). The fixed loci of \( \Gamma_A \) in \( U_1 \) are those corresponding to \( (0, x_1, x_2) \) and \( (x_1, 0, 0) \). The first case follows from the action directly, the second case comes from the \( SO(3) \) quotient. They are indeed the neighborhoods of ghost stratum.
in $F_2, F_1$. This picture says that two different fixed loci $F_1, F_2$ intersect at the ghost stratum. Clearly, this awkward situation is caused by the singularity at the ghost stratum. Now we explain how the situation is modified after flip resolutions. After the resolution, the local model is

$$U = [U \times (TX/\mathbb{Z}_2) \times S^{11}/SU(2)]/\Gamma,$$

here $\Gamma$ is isomorphic to $\Gamma_A$ up to some finite group. By the definition of flip resolutions, points in $S^{11}/SU(2)$ are $[x_0, x_1, x_2]$ where $(x_0, x_1, x_2) \in S^{11}$ and the equivalent relation is given by $(x_0, x_1, x_2) \sim (c^{-1}x_0, cx_1, cx_2)$ for some $c \in SU(2)$. Now the fix loci of $\Gamma_A$ are separated. They correspond to $[0, x_1, x_2]$ and $[x_0, 0, 0]$ respectively. On the exceptional divisor after flip, $X \times [x_0, 0, 0]$ is just the natural closure of $Sym^2(X) \setminus \Delta$. These two components are in $\mathcal{S}_i, i = 1, 2$. This proves the first statement. For the second part, we use the line bundle given in proposition 5.3 and pull it back to $U_{\alpha, r}$. One can show that the weight of $\Gamma_A$-action on factor $L_\Sigma$ is trivial. Then the assertion about forms at fixed loci follows from the case of top stratum. q.e.d.

So far, we have finished the study of equivariant forms over $U_{\alpha, r}$. We take the neighborhood of each fixed locus $F_i$ and denote it by $U_{\alpha, r, i}$. We will apply theorem 5.4 to this model. Here, $\gamma$ will be the equivariant extension of $\bar{\mu}(\Sigma)$. Usually, computing the equivariant Euler classes of normal bundles at fixed loci is highly nontrivial. We first recall some facts mentioned at the end of §4.3. These can be used to reduce the computations. Recall that if $t = (t_1, \ldots, t_k)$ is a partition of $r$, the corresponding fix locus is $F_t \cong \mathcal{S}_t(X)$. Then its normal bundle is

$$N_t = U_A \times \mathcal{N}_t\mathcal{S}_t(X).$$

Since these bundles have product structures when $k > 1$, we consider $k = 1$ first. Now the situation is: let $t = (r)$, there is a bundle $N_t$ over $[A] \times \mathcal{S}_t(X) = [A] \times X \times \mathcal{M}_t^b$. The fiber is $\mathbb{C}^N \times c(SO(3))$, where $c(SO(3))$ denotes the cone of $SO(3)$. We now focus on the nontrivial part of the bundle: the subbundle $N'_t$ with fiber $c(SO(3))$. Explicitly,

$$N'_t = [P_a \times \mathcal{M}_t^{b,0}(S^4, X)]/SO(3).$$

Also

$$\mathcal{M}_t^{b,0}(S^4, X) = Fr(X) \times_{SO(4)} \mathcal{M}_t^{b,0}(S^4).$$

By theorem 5.4, we have

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Proposition 5.9. Suppose \( \theta \) is an equivariant form over \( \mathcal{U}_{\alpha,r} \). Take an equivariant neighborhood of \( F_i \) in \( N_i \), say \( U_i \). Then

\[
\int_{\partial U_i/S^1} r(\theta) = 2 \int_{F_i} \frac{1}{w^N} \frac{\theta u}{-(u - \alpha)^2 + p_1(r)},
\]

(22)

where \( p_1(r) \) is the \( p_1 \) of \( SO(3) \)-bundle \( \mathcal{M}_{\alpha,0}^b(S^4, X) \).

Proof: The trouble is to deal with \( N_i \). Without loss of generality, we assume that \( N = 0 \). Now \( N_i \) is an orbifold bundle with fiber isomorphic to \( \mathbb{R}^4/\mathbb{Z}_2 \). We first make some assumptions so that the computation can be done under the best situation: (1) in general, this orbifold bundle is not a global quotient of some \( \mathbb{R}^4 \)-bundle. Let us assume that it can be identified as a global quotient for a moment. (2) Suppose both \( SO(3) \)-bundles \( P_{\alpha, \mathcal{M}_{\alpha,0}^b(S^4, X)} \) can be lifted to \( SU(2) \)-bundles, say \( Y_1, Y_2 \). (3) Let \( Z_i = Y_i \times_{SU(2)} \mathbb{C}^2 \). As we know, \( Z_1 \) can be split as \( Z_1 = L_{\alpha}^{1/2} \oplus L_{\alpha}^{-1/2} \).

We assume that \( Z_2 \) can be also split as \( Z_2 = L_2^{1/2} \oplus L_2^{-1/2} \) where \( p_1(r) = L_2^2 \). With these assumptions, we consider the cone of \( \text{Hom}_{SU(2)}(Z_1, Z_2) \). This is actually a \( \mathbb{C}^2 \) bundle, we denote it by \( \text{Hom}(Z_1, Z_2) \). Then

\[
\text{Hom}(Z_1, Z_2) \cong (L_{\alpha}^{-1/2} \otimes L_2^{1/2}) \oplus (L_{\alpha}^{1/2} \otimes L_2^{1/2}).
\]

\( \Gamma_A = S^1 \) acts on the bundle by \( (e^{i\theta/2}, e^{-i\theta/2}) \). One can check that \( \text{Hom}(Z_1, Z_2) \) gives a double cover of \( N_i \). However, this bundle exists only when our assumptions hold. In general, we can pass to a quotient bundle of this bundle which always exists: let \( G_0 = Z_2 \times Z_2 \) act on \( \text{Hom}(Z_1, Z_2) \) by multiplication. Then the quotient bundle \( \text{Hom}(Z_1, Z_2)/G_0 \) is \( (L_{\alpha}^{-1} \otimes L_2) \oplus (L_{\alpha} \otimes L_2) \) and the \( S^1 \) action is given by \( (e^{i\theta}, e^{-i\theta}) \). There is a natural map

\[
\xi : N_i \rightarrow \text{Hom}(Z_1, Z_2)/G_0 \]

which is a double cover. Now, applying the localization formula over \( \text{Hom}(Z_1, Z_2)/G_0 \) is legal. There is a small problem to deal with: \( \theta \) is not \( G_0 \) invariant, however we can replace \( \theta \) by the average of \( g^* \theta, g \in G_0 \). This will not change the left hand side of (25). So we assume that \( \theta \) can be reduced to \( \text{Hom}(Z_1, Z_2)/G_0 \). The equivariant Euler class of \( \text{Hom}(Z_1, Z_2)/G_0 \) is then

\[
(u - \alpha + c_1(L_2))(-u + \alpha + c_1(L_2)) = -(u - \alpha)^2 + p_1(r).
\]

Counting the multiplicity of covering and apply the localization formula, we prove (25). q.e.d.
Now the main problem in the right hand side of (25) is to compute $p_1(r)$. To deal with this problem we use a trick that was essentially used in [18].

Suppose $f : X \to BSO(4)$ is the map that induces $Fr(X)$, i.e., $f^*ESO(4) = Fr(X)$. All associated bundles of $Fr(X)$ can be obtained by $f^*$. So $\mathcal{M}_b^r(S^4, X)$ and $\mathcal{M}^{b,0}_r(S^4, X)$ are the pull-backs of bundles

$$(\mathcal{M}^b_r(S^4))_{SO(4)} = ESO(4) \times_{SO(4)} \mathcal{M}^b_r(S^4)$$

and

$$(\mathcal{M}^{b,0}_r(S^4))_{SO(4)} = ESO(4) \times_{SO(4)} \mathcal{M}^{b,0}_r(S^4).$$

Moreover $(\mathcal{M}^{b,0}_r(S^4))_{SO(4)}$ is still an $SO(3)$-bundle over $(\mathcal{M}^b_r(S^4))_{SO(4)}$. We denote the $p_1$ of this bundle by $\tilde{p}_1$. Then $p_1(r) = (f^*)^* \tilde{p}_1$. By definition, $\tilde{p}_1$ is an equivariant cohomology of $M^{b}_r(S^4)$. As it is suggested in [18], we work with $Spin(4) = SU(2)_L \times SU(2)_R$ equivariant cohomology rather than with $SO(4) = SU(2) \times \mathbb{Z}_2 SU(2)$. Working with rational coefficients, no information is lost, as evident from this lemma.

**Lemma 5.10 ([13]).** Let $c_L, c_R$ be the second Chern classes of $SU(2)$-bundles

$$ESpin(4)/SU(2)_R \to BSpin(4)$$

and

$$ESpin(4)/SU(2)_L \to BSpin(4)$$

respectively, then $H^*(BSpin(4), \mathbb{Q}) = \mathbb{Q}[c_L, c_R]$.

If $s : BSpin(4) \to BSO(4)$ is the classifying map for the natural $SO(4)$ bundle over $BSpin(4)$ and $e, p_1 \in H^4(BSO(4))$ are the universal Euler class and the universal Pontrjagin class, then $s^*(p_1 + 2e) = -4c_R$ and $s^*(p_1 - 2e) = -4c_L$.

Since $\tilde{p}_1(r) \in H^4_{Spin(4)}(\mathcal{M}^b_r(S^4))$, we have

**Corollary 5.11.** $\tilde{p}_1(r) = \omega_0 + \omega_L c_L + \omega_R c_R$, where $\omega_0 \in H^4(\mathcal{M}^b_r(S^4))$ and $\omega_L, \omega_R \in H^0(\mathcal{M}^b_r(S^4))$.

Now we are able to prove our main theorem.

**Theorem 5.12.** KM-Conjecture is true.

**Proof:** Suppose $[A]$ is a reducible connection with respect to $\alpha$ and lies in level-$r$ strata, i.e., $r = (\alpha^2 - p_1(P))/4$. Using the same notations as in proposition 5.7, the neighborhood of reducible family is $\mathcal{U}_{\alpha,r}$. Let $\gamma$ be the equivariant extension of $\bar{\mu}(\Sigma)$. So $\gamma = \bar{\mu}(\Sigma) + fu$. Given a partition
of $r$, we know $\gamma$ restricted to the fixed loci $F_t$ is $\text{Sym}^r(\omega) + \frac{1}{2}\langle \alpha, \Sigma \rangle u$. Apply theorem 5.4 to this model by letting $M = \mathcal{U}_{\alpha,r}$. It is not hard to see that

$$r(\gamma) = \bar{\mu}(\Sigma) + \frac{1}{2}\langle \alpha, \Sigma \rangle d\theta.$$  

So use this information and plug $r(\gamma^d)$ into the left hand side of (25). We get exactly $\delta_P(\alpha)$. By the Stokes theorem, one can easily show that $\delta_P(\alpha)$ is the sum of $\delta_{\alpha,t}$ which is defined by

$$\delta_P(\alpha) = \sum_t \int_{\partial \mathcal{U}_t/S^1} r(\gamma) =: \sum_t \delta_{\alpha,t}.$$  

So it is sufficient to prove the following statement about $\delta_{\alpha,t}$: $\delta_{\alpha,t}$ is in the form

$$\delta_{\alpha,t} := \sum_{i=0}^{r-1} a_i(r, d, X, t) q^{r-i} x^{d-2r-2i},$$

and as it indicates $a_i$ depend on $r, d, t$ and homotopy invariants $e, \sigma$ of $X$.

First assume $t = (r)$. Therefore, the right hand side of (25) is the constant term of $u$-series

$$C(r) \frac{1}{u^{N}} \int_{\mathcal{M}_r(S^4, X)} \frac{u^{4r-2}(r \cdot \sigma + \langle \alpha, \Sigma \rangle)^{d-1}}{-(u - \alpha)^2 + p_1(r)},$$  

where $C(r)$ is some constant depending on $r$ due to the orbifold structure. Note that $\pi : \mathcal{M}_r(S^4, X) \to X$ is a bundle with fiber $\mathcal{M}_r$, we can apply the integration along fiber $\pi_*$ first. The contributions of forms in fiber direction come from $p_1(r)$. More precisely, they are terms containing $p_1^{2r-2}(r)$ and $p_1^{2r-1}(r)$. Since $\pi_*$ commutes with the pull-back $(f^*)^*$, we can apply integration along fiber at $(\mathcal{M}_r(S^4))_{\text{Spin}(4)}$ and pass down to $B\text{Spin}(4)$, then pull back by $f^*$. By corollary 5.11, the integration over fiber operation $\pi_*$ at $(\mathcal{M}_r(S^4))_{\text{Spin}(4)}$ implies

$$\pi_*(\tilde{p}_1(r))^{2(r-1)} = A, \quad \text{and} \quad \pi_*(\tilde{p}_1(r))^{2r-1} = B_{RC} + B_{LC}$$

in $H^4(B\text{Spin}(4), \mathbb{Q})$, where $A, B_R, B_L$ are universal numbers depending on $\omega_0, \omega_L, \omega_R$. By pulling back,

$$\pi_*(p_1(r))^{2(r-1)} = A \quad \text{and} \quad \pi_*(p_1(r))^{2r-1} = B_R(2e + 3\sigma) + B_L(2e - 3\sigma).$$

With this fact, applying (26) we can verify the statement for $\delta_{\alpha,t}(\alpha)$. For general $t$, the statement follows from the product property. Thus we prove the conjecture. q.e.d.
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