Classification of pseudo-symmetric simplicial reflexive polytopes

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Abstract

Gorenstein toric Fano varieties correspond to so-called reflexive polytopes. If such a polytope contains a centrally symmetric pair of facets, we call the polytope, respectively the toric variety, pseudo-symmetric. Here we present a complete classification of pseudo-symmetric simplicial reflexive polytopes. This is a generalization of a result of Ewald on pseudo-symmetric nonsingular toric Fano varieties and recent work of Wirth. As applications we determine the maximal number of vertices, facets and lattice points, and show that the vertices can be chosen to have coordinates \(-1, 0, 1\).

Introduction

Isomorphism classes of nonsingular toric Fano varieties correspond to unimodular isomorphism classes of so-called smooth Fano polytopes; these are lattice polytopes, where the origin 0 of the lattice is contained in their interiors, and where the vertices of any facet form a basis of the lattice. Smooth Fano polytopes have been classified up to dimension four, see [WW82, Bat82, Bat99, Sat00]. However in higher dimensions classification results require more symmetries of the polytope.

Let us call a toric Fano variety centrally symmetric, if the associated Fano polytope is centrally symmetric with respect to 0. Moreover, let us denote by a del Pezzo polytope \(V_d\) the \(d\)-dimensional centrally symmetric smooth Fano polytope with \(2d + 2\) vertices \(\pm e_1, \ldots, \pm e_d, \pm (e_1 + \cdots + e_d)\), where \(e_1, \ldots, e_d\) is a basis of the lattice of even rank \(d\). We call the corresponding \(d\)-dimensional nonsingular toric Fano variety a Voskresenskij-Klyachko variety (previously called del Pezzo variety). In [VK85] Voskresenskij and Klyachko showed that any centrally symmetric nonsingular toric Fano variety is a product of projective lines and Voskresenskij-Klyachko varieties.

In [Ewa88] Ewald gave a generalization of this result by assuming that the polytope is only pseudo-symmetric. We say that a polytope \(P\) with interior point 0, respectively the associated variety, is pseudo-symmetric, if \(P\) has a centrally symmetric pair of facets. We can define a pseudo-symmetric smooth Fano
polytope \( \tilde{V}_d \) called pseudo-del Pezzo polytope as the convex hull of the \( 2d + 1 \) vertices \( \pm e_1, \ldots, \pm e_d, -e_1 - \cdots - e_d \) in above notation. We call the associated variety an Ewald variety (previously called pseudo-del Pezzo variety). Now in [Ewa88] Ewald showed that any pseudo-symmetric nonsingular toric Fano variety is a product of projective lines, Voskresenskij-Klyachko varieties and Ewald varieties. In convex-geometric language this means that any pseudo-symmetric smooth Fano polytope \( P \) splits into copies of \([-1, 1]\], del Pezzo polytopes and pseudo-del Pezzo polytopes. By a result of Casagrande in [Cas03] this holds also for any \( d \)-dimensional smooth Fano polytope \( P \) having \( d \) linearly independent vertices \( v_1, \ldots, v_d \) such that \( -v_1, \ldots, -v_d \) are also vertices in \( P \).

In the context of mirror symmetry Batyrev introduced in [Bat94] the notion of a reflexive polytope that is weaker than that of a smooth Fano polytope: A fully-dimensional lattice polytope containing the origin 0 in its interior is called reflexive, if for any facet \( F \) the unique vector in the dual vector space that evaluates \(-1\) on \( F \) is a lattice point. This implies that the dual polytope is also a reflexive polytope. Isomorphism classes of reflexive polytopes correspond to toric Fano varieties having at most Gorenstein singularities, i.e., projective toric varieties whose anticanonical divisor is an ample Cartier divisor.

Reflexive polytopes are interesting classes of lattice polytopes regarded from several different aspects and were classified up to dimension four using computer algorithms, see [KS98, KS00, KS04]. In this paper we present as a rare higher-dimensional classification result a generalization of Ewald’s theorem by regarding not only smooth Fano polytopes but simplicial reflexive polytopes, i.e., reflexive polytopes where any facet is a simplex. This is indeed a significant extension, since for instance in dimension four there are 124 isomorphism classes of smooth Fano polytopes [Bat99, Sat00] compared to 5450 isomorphism classes of simplicial reflexive polytopes in the database [KS05]. In convex-geometric language our main result reads as follows, for this recall that a crosspolytope is the combinatorial dual of a cube:

**Theorem 0.1.** Any pseudo-symmetric simplicial reflexive polytope splits up to unimodular isomorphisms uniquely into a centrally symmetric reflexive crosspolytope, del Pezzo polytopes, and pseudo-del Pezzo polytopes. The isomorphism class of any centrally symmetric reflexive crosspolytope in fixed dimension can be determined by a finite number of suitable matrix normal forms.

A more precise version of this theorem will be given in section two, divided into Theorem 2.2 and Theorem 2.5. In the case of a smooth Fano polytope our main result immediately yields the theorem of Ewald in [Ewa88]. The second part of the theorem was already formulated and proven by Wirth, a student of Ewald, in [Wir97] using the Hermite normal form theorem that we also rely on here. However our proof is independent of the results in Ewald and Wirth. While their proofs depend on explicit determinant calculations, we focus on general discrete-geometric properties of simplicial reflexive polytopes and rather deal with dual bases and the dual reflexive polytope. For this we apply observations on reflexive polytopes from [Nil05].

Some applications:

- We explicitly carry out the classification of all pseudo-symmetric simplicial reflexive polytopes up to dimension six.
In any dimension we show that there is up to isomorphism only one pseudo-symmetric simplicial reflexive polytope with the maximal number of vertices. The same statements holds for the maximal number of lattice points.

The dual polytope of a \( d \)-dimensional pseudo-symmetric simplicial reflexive polytope \( P \) has at most \( 6^{d/2} \) vertices, where the equality case is only achieved, if \( P \) splits into \( d/2 \) copies of \( V_2 \).

The last point is especially interesting, since we get a precise confirmation of the general conjecture \cite{Nil05, Conjecture 5.2} on the maximal number of vertices of a reflexive polytope.

Finally turning to a more well-known conjecture, in \cite{Ewa88} Ewald conjectured that one can always embed any \( d \)-dimensional smooth Fano polytope in \([-1, 1]^d\), as observed for pseudo-symmetric smooth Fano polytopes. This conjecture is no longer valid for general simplicial reflexive polytopes, however we show that it still holds in the pseudo-symmetric case, and moreover that their dual polytopes can be embedded into \( \lfloor \frac{d}{2} \rfloor [-1, 1]^d \).

This article is organized in the following way: In the first section the notation is fixed and basic notions are recalled. In the second section a refined version of the main result is formulated, and proven in the third section. The last section contains applications.

Remark: On advice of Batyrev we have avoided the term 'del Pezzo variety', previously used in \cite{VK85, Ewa88, Ewa96, Cas03}, since there is by now an established theory of del Pezzo varieties in the sense of Fujita.

1 Basic notions and preliminary results

The notation

Here we set up the setting of this paper:

- \( M \cong \mathbb{Z}^d \) and \( N := \text{Hom}_\mathbb{Z}(M, \mathbb{Z}) \) are dual lattices with the pairing \( \langle \cdot, \cdot \rangle \). We set \( M_\mathbb{R} := M \otimes \mathbb{Z} \cong \mathbb{R}^d \) and \( N_\mathbb{R} := N \otimes \mathbb{Z} \cong \mathbb{R}^d \).

- For a subset \( S \subseteq M_\mathbb{R} \) we denote by \( \text{conv}(S) \) the convex hull of \( S \), and by \( \text{dim}(S) \) its dimension.

- A lattice polytope in \( M_\mathbb{R} \) is the convex hull of lattice points in \( M \). Always let \( P \) be a \( d \)-dimensional lattice polytope in \( M_\mathbb{R} \) that contains the origin 0 in its interior and whose vertices are primitive lattice points. Such a \( P \) is called Fano polytope.

- The set of vertices of \( P \) is denoted by \( V(P) \), the set of facets by \( \mathcal{F}(P) \). The boundary of \( P \) is referred to as \( \partial P \). For any facet \( F \in \mathcal{F}(P) \) there is a unique inner normal \( \eta_F \in N_\mathbb{R} \) defined by \( \langle \eta_F, F \rangle = -1 \).

- Two lattice polytopes are regarded as isomorphic, if they are isomorphic under some unimodular transformation, i.e., if there is a lattice automorphism of \( M \) that maps the vertex sets mutually onto each other.
• $P$ spans a fan over its faces, denoted by $\Sigma_P$. The associated toric Fano variety is denoted by $X(M, \Sigma_P)$, briefly $X(\Sigma_P)$. This sets up a one-to-one-correspondence between isomorphism classes of Fano polytopes and isomorphism classes of toric Fano varieties, see [Bat94] or [Nil05].

**Reflexive polytopes**
The dual polytope of $P$ is defined as

$$P^* := \{ x \in \mathbb{N}^\mathbb{R} : \langle x, y \rangle \geq -1 \forall y \in P \}$$

and has as vertices precisely the inner normals of the facets of $P$.

**Definition 1.1.**

• $P$ is called smooth Fano polytope, if the vertices of any facet form a lattice basis of $M$. Equivalently, $X(\Sigma_P)$ is a nonsingular toric Fano variety.

• $P$ is called reflexive polytope, if $P^*$ is a lattice polytope. Equivalently, $X(\Sigma_P)$ is a toric Fano variety with at most Gorenstein singularities.

There is the following duality of reflexive polytopes, see [Bat94]:

$$P \subseteq M_\mathbb{R} \text{ is reflexive } \iff P^* \subseteq \mathbb{N}^\mathbb{R} \text{ is reflexive.}$$

Furthermore we need a simple but fundamental property [Nil05, Prop. 4.1]:

**Lemma 1.2.** Let $P \subseteq M_\mathbb{R}$ be a reflexive polytope, and $v, w \in \partial P \cap M$.

If $v + w \neq 0$ and there is no facet containing both $v$ and $w$, then $v + w \in \partial P \cap M$.

Another important observation is the following result, that is included in [Nil05, Lemma 5.5]:

**Lemma 1.3.** Let $P \subseteq M_\mathbb{R}$ be a reflexive polytope, $F \in \mathcal{F}(P)$, and $m \in \partial P \cap M$.

If $\langle \eta_F, m \rangle = 0$, then $m$ is contained in a facet intersecting $F$ in a codimension two face.

**Pseudo-symmetry and del Pezzo polytopes**

**Definition 1.4.**

• $P$ is called centrally symmetric, if $-P = P$.

• $P$ is called pseudo-symmetric, if there is some $F \in \mathcal{F}(P)$ with $-F \in \mathcal{F}(P)$.

For us the following smooth Fano polytopes will be especially important:

**Definition 1.5.** Let $e_1, \ldots, e_d$ be a lattice basis of $M$. Let $d$ be even.

• $V_d := \text{conv}(\pm e_1, \ldots, \pm e_d, \pm (e_1 + \cdots + e_d))$ is called a del Pezzo polytope.

• $\tilde{V}_d := \text{conv}(\pm e_1, \ldots, \pm e_d, -e_1 - \cdots - e_d)$ is called a pseudo-del Pezzo polytope.

So del Pezzo polytopes are centrally symmetric, while pseudo-del Pezzo polytopes are only pseudo-symmetric.
Hermite matrix normal forms

Let \( n \in \mathbb{N}_{\geq 1} \).

**Definition 1.6.**

- \( \operatorname{GL}_n(\mathbb{Z}) \) is the set of \( n \times n \)-matrices with integer coefficients and determinant \( \pm 1 \).

- For an arbitrary set of integers \( R \) we let \( \operatorname{Mat}_{m \times n}(R) \) be the set of \( m \times n \)-matrices with entries in \( R \). We abbreviate \( \operatorname{Mat}_n(R) := \operatorname{Mat}_{n \times n}(R) \).

- For \( n, \lambda \in \mathbb{N}_{\geq 1} \) we denote by \( \operatorname{Herm}(n, \lambda) \) the finite set of lower triangular matrices \( H \in \operatorname{Mat}_n(\mathbb{N}) \) with determinant \( \lambda \) satisfying \( h_{i,j} < h_{j,j} \) for all \( j = 1, \ldots, n-1 \) and \( i > j \).

The famous theorem of Hermite is the following (e.g., [New72] pp. 15-18):

**Theorem 1.7.** For any \( L \in \operatorname{Mat}_n(\mathbb{Z}) \) with determinant \( \lambda \neq 0 \) there exist matrices \( U \in \operatorname{GL}_n(\mathbb{Z}) \) and \( H \in \operatorname{Herm}(n, \lambda) \) such that \( UL = H \).

## 2 The main theorems

In this section Theorem 1.1 will be formulated more precisely. It is split into Theorem 2.2 and Theorem 2.5, the first one dealing with the case of the minimal number of vertices.

**Classification of centrally symmetric reflexive crosspolytopes**

Centrally symmetric reflexive crosspolytopes have been classified in [Wir97, Satz 3.3]. Here we state Wirth’s result in a somewhat strengthened form.

To simplify notation we define:

**Definition 2.1.**

- A matrix \( A \in \operatorname{Mat}_d(\mathbb{N}) \) is called Wirth matrix, if

\[
A = \begin{pmatrix}
2\text{id}_f & 0 \\
C & \text{id}_{d-f}
\end{pmatrix},
\]

where \( f \in \{0, \ldots, d-1\} \) and \( C \in \operatorname{Mat}_{(d-f) \times f}(\{0,1\}) \) such that any column of \( C \) has an odd number of 1’s. Here \( \text{id}_k \) is the \( k \times k \)-identity matrix.

- A Wirth matrix \( A \) is called 1-minimal Wirth matrix, if any row of \( C \) contains some 1. Obviously we get from a Wirth matrix a 1-minimal one called its reduction by deleting rows containing only one 1 (and the corresponding columns).

- Two matrices in \( \operatorname{Mat}_d(\mathbb{N}) \) are regarded as equivalent, if they differ only up to permutation of columns and left-multiplication by a matrix in \( \operatorname{GL}_d(\mathbb{Z}) \).
Let $P_1, P_2$ be non-zero Fano polytopes, with respect to lattices $M_1, M_2$, such that $\dim(P_1) + \dim(P_2) = d$. We say $P$ splits into factors $P_1$ and $P_2$, if $P \cong \text{conv}(P_1 \oplus \{0\}, \{0\} \oplus P_2)$ for $M \cong M_1 \oplus M_2$. Equivalently, $X(\Sigma P) \cong X(\Sigma P_1) \times X(\Sigma P_2)$; or dually, $P^* \cong P_1^* \times P_2^*$.

We say $P$ is 1-irreducible, if $P$ does not split into $[-1,1]$ and some $P_2$.

We denote by a cs-crosspolytope a centrally symmetric crosspolytope, i.e., the convex hull of a simplex (not containing 0) and its negative. Equivalently, a $d$-dimensional cs-crosspolytope is a $d$-dimensional pseudo-symmetric simplicial polytope with $2d$ vertices.

Obviously any matrix in $\text{Mat}_d(\mathbb{Z})$ with non-zero determinant defines a lattice cs-crosspolytope by taking the convex hull of its columns and their negatives. Thereby reflexive cs-crosspolytopes can be classified:

**Theorem 2.2 (Wirth, N.).**

- There is a one-to-one correspondence between equivalence classes of Wirth matrices and isomorphism classes of reflexive cs-crosspolytopes. For the inverse map take any facet of a reflexive cs-crosspolytope, then there is a lattice basis such that the coordinates of the vertices of this facet are the columns of a corresponding Wirth matrix.

- Hereby equivalence classes of 1-minimal Wirth matrices correspond to isomorphism classes of 1-irreducible reflexive cs-crosspolytopes. Moreover any reflexive cs-crosspolytope $P$ splits up to isomorphism uniquely into a 1-irreducible reflexive cs-crosspolytope $P'$ and $r$ copies of $[-1,1]$. Here an associated 1-minimal Wirth matrix of $P'$ is just the reduction of a Wirth matrix $A$ associated to $P$, and $r$ equals the number of rows of $A$ containing only one 1.

To sum up, the isomorphism class of any reflexive cs-crosspolytope $P$ is given by an up to equivalence uniquely determined 1-minimal Wirth matrix $A'$ (encoding the singular factor $P'$ of $P$) and a unique natural number $r$ (corresponding to the nonsingular factor of $P$). The determinant of $A'$ is just the index in $M$ of the lattice spanned by the vertices of $P$.

**Example 2.3.** As an illustration we give a list of all isomorphism classes of 1-irreducible reflexive cs-crosspolytopes by listing the equivalence classes of their associated 1-minimal Wirth matrices for $d \leq 6$ (this list is for $d \leq 5$ implicitly already contained in [Wirth97, 3.7,3.8]):

- $d = 2$: \[
\begin{pmatrix}
2 & 0 \\
1 & 1
\end{pmatrix}
\]

- $d = 3$: \[
\begin{pmatrix}
2 & 0 & 0 \\
0 & 2 & 0 \\
1 & 1 & 1
\end{pmatrix}
\]

- $d = 4$: \[
\begin{pmatrix}
2 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1
\end{pmatrix}, \quad
\begin{pmatrix}
2 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1
\end{pmatrix}, \quad
\begin{pmatrix}
2 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 2 & 0 \\
1 & 1 & 1 & 1
\end{pmatrix}
\]
Remark 2.4. Two matrices $A_1$ and $A_2$ in $\text{Mat}_d(\mathbb{Z})$ with the same determinant are equivalent, if for some permutation $\pi$ all entries of $A_1^\pi A_2^{-1}$ are integers, where $A_1^\pi$ is the matrix of $\pi$-permuted columns of $A_1$. For Wirth matrices this can be easily checked due to their simple structure, see [Wir97, Satz 3.9].

Classification of pseudo-symmetric simplicial reflexive polytopes

**Theorem 2.5.** Let $P \subseteq M_\mathbb{R}$ be a pseudo-symmetric simplicial reflexive polytope. Then $P$ splits up to isomorphism uniquely into a 1-irreducible reflexive cs-crosspolytope $P'$, $r$ copies of $[-1,1]$, del Pezzo polytopes, and pseudo-del Pezzo polytopes.

So $P$ splits uniquely into $P'$ (the singular factor) and a smooth Fano polytope (the nonsingular factor). We recover the original result of Ewald in [Ewa88] under milder assumptions:

**Corollary 2.6.** Let $P \subseteq M_\mathbb{R}$ be a pseudo-symmetric simplicial reflexive polytope where the vertices span the lattice $M$.

Then the corresponding toric variety $X(\Sigma_P)$ is just a product of projective lines, Voskresenskij-Klyachko varieties, and Ewald varieties. In particular $X(\Sigma_P)$ is nonsingular and $P$ a smooth Fano polytope.

**Proof.** Indeed, since by assumption the determinant of the 1-minimal Wirth matrix associated to $P'$ equals one, $P'$ has to be zero. \[ \Box \]

Now we can easily calculate that there are 1, 3, 3, 8, 8, 18 isomorphism classes of $d = 1, 2, 3, 4, 5, 6$-dimensional pseudo-symmetric smooth Fano polytopes, therefore Theorem 2.5 together with the list in Example 2.3 yields:

**Corollary 2.7.** For $d = 2, 3, 4, 5, 6$ there are exactly 4, 5, 15, 20, 50 isomorphism classes of $d$-dimensional pseudo-symmetric simplicial reflexive polytopes.
A rigorous proof of the previous result was in the centrally symmetric case up to now only known for $d \leq 3$, see [Wag95]. For $d = 4$ the centrally symmetric case of up to 10 vertices was dealt with by rather complicated and long calculations in [Wir97, Satz 5.11].

Applying [Oda88, Cor. 1.16] (and, e.g., [Nil05, Prop. 1.9]) to 2.5 yields also:

**Corollary 2.8.** Any $d$-dimensional pseudo-symmetric $\mathbb{Q}$-factorial Gorenstein toric Fano variety is the projection for the quotient of a product of projective lines, Voskresenskiy-Klyachko varieties, and Ewald varieties with respect to the action of a finite group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^f$ for $f \leq d - 1$.

The combinatorial statement sounds rather surprising:

**Corollary 2.9.** Any pseudo-symmetric simplicial reflexive polytope is combinatorially isomorphic to a pseudo-symmetric smooth Fano polytope.

**Remark 2.10.** One should not be misled by this result: Without the symmetry assumption the combinatorics of simplicial reflexive polytopes can be much more complicated than the one of smooth Fano polytopes. For instance, according to the database [KS05] the columns of the following matrix form the vertices of the only four-dimensional reflexive polytope with 7 vertices and 14 facets; it is simplicial but not a smooth Fano polytope:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & -2 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & -2 & 1 \\
0 & 0 & 1 & 0 & -1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & -1 & 1
\end{pmatrix}
\]

### 3 Proof of the main theorems

In this section Theorems 2.2 and 2.5 will be proven. We will deal directly with the general case of a pseudo-symmetric simplicial reflexive polytope, this includes therefore an independent proof of the results in [Wir97] on reflexive cs-crosspolytopes.

**The key-lemma**

As a preparation we need the following important fact (pay attention, in what follows $e_1, \ldots, e_d$ is in general not a lattice basis):

**Lemma 3.1.** Let $P$ be a simplicial reflexive polytope with facets $F$, $-F$.

Let $\mathcal{V}(F) = \{e_1, \ldots, e_d\}$, and $e_1^*, \ldots, e_d^*$ be the dual $\mathbb{R}$-basis of $N_\mathbb{R}$. For $i = 1, \ldots, d$ we denote by $F_i$ the unique facet of $P$ such that $F_i \cap F = \text{conv}(e_j : j \neq i)$. We set $u := \eta_{F} \in \mathcal{V}(P^*)$.

Let $v \in \mathcal{V}(P) \cap u^\perp$. We write $v = \sum_{i=1}^{d} q_i e_i$ as a rational linear combination. Then we have for $i = 1, \ldots, d$:

\[ q_i < 0 \iff q_i = -1 \iff v \in F_i. \]

In this case $e_i^* = \eta_{F_i} - u \in P^* \cap N$.

Moreover there are $I, J \subseteq \{1, \ldots, d\}$ with $I \cap J = \emptyset$ and $|I| = |J|$ such that

\[ v = \sum_{j \in J} e_j - \sum_{i \in I} e_i. \]
Proof. In the proof of [Nil05, Lemma 5.5] it was already readily observed that $\eta_{F_i} = u + \alpha_i e_i^*$ for $\alpha_i > 0$ and $i = 1, \ldots, d$. Hence we get $q_i = \langle e_i^*, v \rangle < 0 \iff v \in F_i$.

Now let $q_i = \langle e_i^*, v \rangle < 0$, $v \in F_i$. Therefore $F_i \cap (-F) = \emptyset$, so by duality we can apply Lemma 1.2 to the reflexive polytope $P^* \subseteq \mathbb{N}_2$ and the inner normals of $F_i$ and $-F$. This yields $\eta_{F_i} - u = \alpha_i e_i^* \in P^* \cap \mathbb{N}$. Hence $-1 \leq \langle \alpha_i e_i^*, -e_i \rangle = -\alpha_i \in \mathbb{Z}$, so $\alpha_i = 1$, and $q_i = \langle e_i^*, v \rangle = \langle \eta_{F_i} - u, v \rangle = -1$.

Obviously $v = \sum_{j=1}^d (-q_j)(-e_j)$, so applying the previous result to $-F$ yields $q_j > 0 \iff q_j = 1$, hence the remaining statement due to $\langle u, v \rangle = 0$.

The structure theorem

Proposition 3.2. Let $P$ be a simplicial reflexive polytope with facets $F_i, -F$.

Let $V(F) = \{e_1, \ldots, e_d\}$. For $i = 1, \ldots, d$ we let $v^i$ denote the unique vertex of $P$ contained in the unique facet that intersects $F$ in $\text{conv}(e_j : j \neq i)$.

- There exists a lattice basis $m_1, \ldots, m_d$ of $M$ such that in this basis the coefficient vectors of $e_1, \ldots, e_d$ are the columns of a Wirth matrix $A$.

- Any vertex of $P$ is in $\{ \pm e_1, \ldots, \pm e_d, v^1, \ldots, v^d \}$.

- There exist pairwise disjoint subsets $I_1, \ldots, I_l \subseteq \{1, \ldots, d\}$ and pairwise disjoint subsets $J_1, \ldots, J_l \subseteq \{1, \ldots, d\}$ with $I_k \cap J_k = \emptyset$ and $|I_k| = |J_k|$ for all $k = 1, \ldots, l$ such that for $i \in \{1, \ldots, d\}$ we have

$$v^i \in \eta_F^\perp \iff i \in \bigcup_{k=1}^l I_k,$$

and moreover, if $i \in I_k$ for some $k \in \{1, \ldots, l\}$, then

$$v^i = \sum_{j \in J_k} e_j - \sum_{i' \in I_k} e_i'.$$

- For $i \in I_1 \cup \cdots I_l \cup J_1 \cup \cdots \cup J_l$ the $i$th row of $A$ is of the form $(0, \ldots, 0, 1, 0, \ldots, 0)$ for $1$ at the $i$th position.

- If for $k, k' \in \{1, \ldots, l\}$ the sets $I_k$ and $J_{k'}$ intersect, then $k \neq k'$, $I_k = J_{k'}$, and $J_k = I_{k'}$.

Proof. Let $u, F_i$ defined as in Lemma 3.1. Consider the following steps for the construction of $A$ and $m_1, \ldots, m_d$:

1. Let $i \in \{1, \ldots, d\}$.
   - If $v^i \in u^\perp$, then by Lemma 3.1 $e_i^* = \eta_{F_i} - u \in \mathbb{N}$.
   - If $v^i \not\in u^\perp$, then obviously $v^i = -e_i$, so $e_i^* = \frac{\eta_{F_i} - u}{2} \in \frac{1}{2} \mathbb{N}$.
   - Hence in any case $e_1^*, \ldots, e_d^* \in \frac{1}{2} \mathbb{N}$. In particular this yields

$$2M \subseteq \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_d \subseteq M.$$
2. We define for an arbitrary lattice basis of $M$ the matrix $L \in \Mat_d(\mathbb{Z}) \cap \GL_d(\mathbb{Q})$, where the columns are the coefficient vectors of $e_1, \ldots, e_d$ in this basis. By Theorem 1.7 there exists $U \in \GL_d(\mathbb{Z})$ such that $A := UL \in \Mat_d(\mathbb{N})$ is a lower triangular matrix with $A_{i,j} \in \{0, \ldots, A_{j,j} - 1\}$ for $i > j$. Hence there is a lattice basis $m_1, \ldots, m_d$ of $A$ such that the columns of $A$ are the coefficient vectors of $e_1, \ldots, e_d$ in the lattice basis $m_1, \ldots, m_d$.

3. The first point yields that $2m_i$ is contained for $i = 1, \ldots, d$ in the column space of $A$, in particular any diagonal element of $A$ equals 1 or 2.

4. If $A_{i,j} = 1$ for $i > j$, then necessarily $A_{j,j} = 2$ and $A_{i,i} = 1$ (again one has to use that $2m_i$ is a $\mathbb{Z}$-linear combination of $e_1, \ldots, e_d$).

5. Using the previous point we assume, by possibly permutating the columns and the rows of $A$, that $A$ has a blockmatrix structure as in Definition 2.1. Since any vertex of a reflexive polytope is primitive, obviously $f \neq d$.

Now it is easy to see that $e_i^* \in N$ if and only if the $i$th row of $A$ is of the form $(0, \ldots, 0, 1, 0, \ldots, 0)$. By this observation, Lemma 3.1 (applied to $F$ and $-F$) and the first point in the proof we see that it only remains to show the last statement in the proposition:

So let $k, k' \in \{1, \ldots, l\}$ with $I_k \cap J_{k'} \neq \emptyset$. By construction $I_k \cap J_k = \emptyset$, hence $k \neq k'$.

Assume $I_{k'} \not\subseteq J_k$. Then there exists $i \in I_{k'}$, $i \not\in J_k$. Let $j \in I_k \cap J_{k'}$, in particular $j \neq i$. Now we define for the dual $\mathbb{R}$-basis $e_1^*, \ldots, e_d^*$ of $N_\mathbb{R}$ the vector $w := e_i^* - \sum_{s \neq i,j} e_s^*$. By construction it is easy to check that $w$ is an inner normal of a face of $P$ containing as vertices $e_s$ for $s = 1, \ldots, d$ with $s \neq i, j$, as well as $-e_i$ and $v^i$ and $v^j$. This is a contradiction to $P$ being simplicial.

Hence $I_{k'} \subseteq J_k$. In particular $I_{k'} \cap J_k \neq \emptyset$, so also $I_k \subseteq J_{k'}$. Since therefore $|I_{k'}| \leq |J_k| = |I_k| \leq |J_{k'}| = |I_{k'}|$, we have $I_{k'} = J_k$ and $I_k = J_{k'}$.

\begin{flushright} \Box \end{flushright}

**Proof of Theorem 2.2**

By the first point of Proposition 3.2 any reflexive cs-crosspolytope is defined by a Wirth matrix. On the other hand it was shown in [Wir97] by a simple calculation that any Wirth matrix defines a reflexive cs-crosspolytope. Furthermore if two Wirth matrices define isomorphic reflexive cs-crosspolytopes, then to see that the Wirth matrices have to be equivalent, it is enough to show that any two facets of a reflexive cs-crosspolytope are isomorphic as lattice polytopes.

For this let us assume that $P$ is a reflexive cs-crosspolytope defined by a Wirth matrix $A$ with columns $e_1, \ldots, e_d$. Now we regard a matrix where the columns are given as $e_1e_1, \ldots, e_de_d$ for $e_1, \ldots, e_d \in \{-1, 1\}$. We perform elementary row operations on this matrix by first multiplying any row that has a negative number on the diagonal by $-1$. Then we add the $i$th row to any row that has $-1$ as an entry below the diagonal in the $i$th column. This yields precisely the original matrix $A$, and finishes the proof of the first part of Theorem 2.2.
For the second part of Theorem 2.2 let $A$ be a Wirth matrix defining a reflexive cs-crosspolytope $P = \text{conv}(\pm e_1, \ldots, \pm e_d)$. Let $i \in \{1, \ldots, d\}$. Now it is easy to see that $P$ splits into $R_i := \text{conv}(\pm e_j : j \neq i)$ and $\text{conv}(\pm e_i)$ if and only if the so-called normalized volume of $R_i$ is the same as the normalized volume of $P$. Equivalently, the greatest common divisor of all $d-1$-minors of the matrix formed by the columns $e_j : j \neq i$ equals the determinant of $A$. However this happens if and only if the $i$th row of $A$ contains only one 1 (namely on the diagonal):

To see this assume $A_{i,i} = 1$ and $A_{i,j} = 1$ for $i \neq j$. Then we remove the $i$th column and the $j$th row. By Laplace developing one gets that this $d-1$-minor can never be $\det(A) = 2^f$, where $f$ is the number of 2's in $A$.

This proves that $A$ is a 1-minimal Wirth matrix if and only if $P$ is 1-irreducible. Moreover the splitting of $P$ into a 1-irreducible $P'$ and copies of $[-1,1]$ is unique up to isomorphism as lattice polytopes, since the vertices of $P'$ are uniquely determined.

### Proof of Theorem 2.5

Let $P$ be a simplicial reflexive polytope with facets $F,-F$. Proposition 3.2 immediately yields that $P$ splits in a 1-irreducible reflexive cs-crosspolytope $P'$, $r$ copies of $[-1,1]$, del Pezzo polytopes $V_{k_1}, \ldots, V_{k_s}$, and pseudo-del Pezzo polytopes $\tilde{V}_{p_1}, \ldots, \tilde{V}_{p_t}$, where the 1-irreducible reflexive cs-crosspolytope $P'$ is given by the reduction of the Wirth matrix $A$. Hence we have the splitting in Theorem 2.5.

Abbreviating, we let $P$ split into $Q$ and $S$, where $Q$ splits into $P'$ and $r$ copies of $[-1,1]$, and $S$ splits into the remaining factors. We define two vertices of $P$ to be connected, if they are contained in a minimal non-face of $P$ (a primitive collection in the language of Batyrev [Bat91]). From the description of $V_d$ and $\tilde{V}_d$ in [Cas03] we see that the connected components of $\mathcal{V}(P)$ of size $> 2$ are precisely the vertex sets of the factors in the splitting of $S$. Hence the numbers $k_1, \ldots, k_s, p_1, \ldots, p_t$ are (even combinatorial) invariants of $P$. Moreover the isomorphism type of $P$ as a lattice polytope determines the isomorphism type of $Q$, and hence $P'$ and $r$ by the second part of Theorem 2.2.

### 4 Applications

All facets are isomorphic

**Corollary 4.1.** Let $P$ be a pseudo-symmetric simplicial reflexive polytope.

Then any two facets of $P$ are isomorphic as lattice polytopes. Especially any two facets of $P$ have the same number of lattice points.

**Proof.** Since all facets of a smooth Fano polytope are isomorphic, by Theorem 2.4 we only have to regard reflexive cs-crosspolytopes. Now we use the first part of Theorem 2.2. \hfill $\square$
The maximal number of vertices

**Corollary 4.2.** Let \( P \subseteq M_\mathbb{R} \) be a pseudo-symmetric simplicial reflexive polytope.

If \( d \) is even, then

\[ |\mathcal{V}(P)| \leq 3d, \]

with equality only if \( P \) splits into \( d/2 \) copies of \( V_2 \).

If \( d \) is odd, then

\[ |\mathcal{V}(P)| \leq 3d - 1, \]

with equality only if \( P \) splits into \([-1,1]\) and \((d-1)/2\) copies of \( V_2 \).

**Proof.** By Theorem 2.5 \( P \) splits into an \( l \)-dimensional reflexive cs-crosspolytope \( Q \) and a smooth Fano polytope \( R \) splitting into del Pezzo polytopes \( V_{k_1}, \ldots, V_{k_s} \) and pseudo-del Pezzo polytopes \( \tilde{V}_{p_1}, \ldots, \tilde{V}_{p_t} \), so

\[ l + k_1 + \cdots + k_s + p_1 + \cdots + p_t = d, \]

where \( k_1, \ldots, k_s, p_1, \ldots, p_t \) are even.

Hence \( |\mathcal{V}(P)| = 2d + 2s + t \leq 2d + \dim(R) - t \leq 3d. \)

If \( |\mathcal{V}(P)| = 3d \), then \( \dim(R) = d, \) \( P \cong R, \) \( d \) is even, \( t = 0 \) and \( s = d/2. \)

So let \( d \) be odd and \( |\mathcal{V}(P)| = 3d - 1. \) Since \( P \) has dimension \( d, \) so is not even-dimensional, we cannot have \( \dim(R) = d, \) hence \( \dim(R) = d - 1. \) Therefore \( Q \) is one-dimensional, so isomorphic to \([-1,1]\). Now use the first statement for \( R \) with \( |\mathcal{V}(R)| = 3(d - 1). \)

\( \square \)

In [Cas04] the first part was recently shown to hold for arbitrary simplicial reflexive polytopes. The second part is an extension of [Nil05, Thm. 5.9].

The maximal number of facets

**Remark 4.3.** First we look at the (pseudo) del Pezzo polytopes: For this we use the notation of Definition 1.4. By setting \( e_0 := -e_1 - \cdots - e_d \) it is a straightforward calculation (see [VK85] or [Cas03]) that the facets of \( V_d \) have as vertices precisely \( \{ \pm e_j : j = 0, \ldots, d, j \neq i \} \), for fixed \( i \in \{0, \ldots, d\} \), where exactly half of the signs are equal to +1 and the others are equal to -1. Hence we get:

\[ |\mathcal{F}(V_d)| = (d + 1) \left( \frac{d}{2} \right). \]

In just the same way (see [Cas03]) we can calculate

\[ |\mathcal{F}(\tilde{V}_d)| = d \left( \frac{d - 1}{2} \right) + \sum_{i=\frac{d}{2}}^{d} \left( \frac{d}{i} \right). \]

**Corollary 4.4.** Let \( P \subseteq M_\mathbb{R} \) be a pseudo-symmetric simplicial reflexive polytope. Then

\[ |\mathcal{F}(P)| \leq 6^{d/2}, \]

where equality is only attained if \( d \) is even and \( P \) splits into \( d/2 \) copies of \( V_2 \).
Proof. By Theorem 2.5 we can assume that $P$ splits into an $l$-dimensional reflexive cs-crosspolytope $Q$, del Pezzo polytopes $V_{k_1}, \ldots, V_{k_s}$, and pseudo-del Pezzo polytopes $\tilde{V}_{p_1}, \ldots, \tilde{V}_{p_t}$, so $l + k_1 + \cdots + k_s + p_1 + \cdots + p_t = d$, where $k_1, \ldots, k_s, p_1, \ldots, p_t$ are even.

Now it is not difficult to show that

$$2n \left( \frac{2n-1}{n} \right) + \sum_{i=n}^{2n} \left( \frac{2n}{i} \right) < (2n+1) \left( \frac{2n}{n} \right) \leq 6^n$$

for $n \in \mathbb{N}_{\geq 1}$, with equality at the right only for $n = 1$.

Hence the previous remark and $|F(Q)| = 2^l \leq 6^{l/2}$ yields $|F(P)| \leq 6^{l/2}6^{k_1/2} \cdots 6^{k_s/2}6^{p_1/2} \cdots 6^{p_t/2} = 6^{d/2}$, where equality implies that $d$ is even and $P$ splits into $d/2$ copies of $V_2$.

Since $V_2 \cong V_2$, this yields for all centrally symmetric simple reflexive polytopes a confirmation of the general conjecture in [Nil05] that $V_{d/2}$ is the single reflexive polytope with the maximal number of vertices $6^{d/2}$.

### The maximal number of lattice points

In [Nil06] Thm. 6.1 it was shown that $[-1,1]^d$ solely contains the most lattice points among all $d$-dimensional centrally symmetric reflexive polytopes. Here we prove an analogous result for pseudo-symmetric simplicial reflexive polytopes.

**Definition 4.5.** Let $D_d$ be the 1-irreducible reflexive cs-crosspolytope associated to the 1-minimal Wirth matrix

$$A_d := \begin{pmatrix} 2\text{id}_{d-1} & 0 \\ 1 \cdots 1 & 1 \end{pmatrix}$$

in a lattice basis $m_1, \ldots, m_d$.

**Remark 4.6.** The dual polytope $D_d^* = \text{conv}(\pm (m_d^* - x) : x \in \sum_{i=1}^{d-1} c_i m_i^*, c_i \in \{0,1\})$ in the dual lattice basis of $m_1, \ldots, m_d$, is a reflexive polytope where the vertices are the only lattice points on the boundary.

**Corollary 4.7.** Let $P \subseteq M\mathbb{R}$ be a pseudo-symmetric simplicial reflexive polytope. Then we have:

- Any lattice point on $\partial P$ is either a vertex or the middle point of an edge.
- $|P \cap M| \leq 2d^2 + 1$. Any facet of $P$ has at most $\left( \frac{d+1}{2} \right)$ lattice points.
- The following statements are equivalent:
  1. $|P \cap M| = 2d^2 + 1$
  2. Some facet has $\left( \frac{d+1}{2} \right)$ lattice points
  3. Any facet has $\left( \frac{d+1}{2} \right)$ lattice points
  4. $P \cong D_d$
Proof. Let $F, -F \in \mathcal{F}(P)$. Applying Proposition 3.2 to $P$ and $F$ we can assume that $\mathcal{V}(F) = \{e_1, \ldots, e_d\}$ are the columns of a Wirth matrix $A$.

To the first point: Since by Corollary 4.1 any two facets are isomorphic, we may assume $m \in F \cap M, m \not\in \mathcal{V}(F)$. By looking at $A$ we get $m = (e_i + e_j)/2$ for a pair $i < j$ with $(e_i)_i = 2$. Proceeding further this shows that

$$|F \cap M| \leq \left(\frac{d}{2}\right) + d = \frac{(d+1)d}{2},$$

where equality necessarily implies $A = A_d$, and hence $P = D_d$, since by the fourth point of Proposition 3.2 we have $|\mathcal{V}(P)| = 2d$. By Corollary 4.1 this proves the bound on the number of lattice points in any facet, and (2) $\Rightarrow$ (3) $\Rightarrow$ (4).

Now we use the notation in Lemma 3.1 and Proposition 3.2, so $u := \eta_P$. Let $m \in \partial P \cap M \cap u^\perp$. If $m$ is a vertex, then $m \in \{v^1, \ldots, v^d\}$. So let $m \not\in \mathcal{V}(P)$. By Lemma 1.4 we still have $m \in F_i$ for some $i \in \{1, \ldots, d\}$. In particular $v_i \not\in u^\perp$, so $v_i = -e_i$. By the first point of this corollary we get $m = (-e_i + e_j)/2$ for some $j \neq i$.

This yields

$$|\partial P \cap M \cap u^\perp| \leq \sum_{i=1}^d |F_i \cap M \cap u^\perp| \leq d(d-1).$$

Hence, since $P$ contains no non-zero interior lattice points (e.g., [Nil05 Prop. 1.12]), we have the upper bound

$$|P \cap M| \leq 1 + 2|F \cap M| + d(d-1) \leq 2d^2 + 1,$$

where equality implies $|F \cap M|$ to be maximal. This proves the remaining implications (4) $\Rightarrow$ (1) $\Rightarrow$ (2).

The proof also shows how to easily read off the number of lattice points of a reflexive cs-crosspolytope from the associated Wirth matrix.

**Embedding in the standard lattice cube**

In [Ewa88 Conjecture 2] Ewald conjectures that, up to unimodular transformation, all vertices of a smooth Fano polytope have coordinates $1, -1, 0$ only. We say that there is an embedding into $[-1,1]^d$. This is proven for $d \leq 4$ by the existing classification. In general the conjecture does not hold for simplicial reflexive polytopes, even in dimension two. However it is true, if we assume pseudo-symmetry:

**Corollary 4.8.** (Wirth, N.) Let $P$ be a pseudo-symmetric simplicial reflexive polytope. Then $P$ can be embedded into $[-1,1]^d$.

**Proof.** Since (pseudo) del Pezzo polytopes are by definition contained in $[-1,1]^d$, by Theorem 2.2 and 2.3 we just have to show that performing row operations on a Wirth matrix $A$ we get a matrix containing only $\{-1,0,1\}$. If we assume $A_{j,j} = 2$, then there is an $i > j$ (minimal) such that $A_{i,j} = 1$. Then we just have to subtract the $i$th row from the $j$th. We proceed by induction on $j$. 

□
This result in the case of a reflexive cs-crosspolytope can be found in [Wir97, Satz 4.4]. In [Wir97, Kapitel 4] there is a discussion of the topic of embedding, where we can also find the following example that shows that we cannot drop the assumption of simpliciality:

**Remark 4.9.** Let $P \subseteq M$ be a centrally symmetric reflexive polytope. Then $P^*$ can be embedded into $[-1,1]^d$ if and only if $P$ contains a lattice basis of $M$.

This is, for instance, not true for the four-dimensional reflexive cs-crosspolytope $P$ associated to the 1-minimal Wirth matrix

\[
\begin{pmatrix}
2 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1
\end{pmatrix}
\]

since any lattice point on the boundary is $\pm$ a column.

However we can still embed $P^*$ into a small multiple of $[-1,1]^d$:

**Corollary 4.10.** Let $P$ be a pseudo-symmetric simplicial reflexive polytope. Then $P^*$ can be embedded into $[\frac{d}{2}][-1,1]^d$.

**Proof.** Since the duals of the (pseudo) del Pezzo polytopes are always contained in $[-1,1]^d$ (for this use Remark 4.3 and Cas03), by Theorem 2.5 we can assume $P = \text{conv}(\pm e_1, \ldots, \pm e_d)$. Let $m_1, \ldots, m_d$ be the lattice basis of $M$ in Proposition 3.2 such that $A = \begin{pmatrix} 2id/f & 0 \\
C & id_d-f \end{pmatrix} \in \text{Mat}_d(\mathbb{N})$. Then $A^{-1} = \begin{pmatrix}
1/2id/f & 0 \\
-C/2 & id_d-f
\end{pmatrix}$. Now the rows of $A^{-1}$ are precisely the coordinates of the dual $\mathbb{R}$-basis $e_1^*, \ldots, e_d^*$ (in the dual lattice basis $m_1^*, \ldots, m_d^*$). Furthermore for any facet $F \in \mathcal{F}(P)$ we have $\eta_F = \pm e_1^* \pm \cdots \pm e_d^* \in N$ for some signs $\pm$. Hence the vertices of $P^*$ have coordinates in $[-d/2, d/2]$ with respect to the lattice basis $m_1^*, \ldots, m_d^*$.

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