CANCELLATION OF INFRARED DIVERGENCES IN QED AT NONZERO TEMPERATURE

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Talk presented at the 3rd Workshop on Thermal Field Theories and their Applications, Banff, Alberta (August, 1993)

ABSTRACT

The radiation produced by a classical charged current coupled to a quantized $A_\mu$ is first computed. To each order in $\alpha$, all infrared divergences cancel between the virtual $\gamma$’s and the real $\gamma$’s that are absorbed from or emitted into the plasma. When all orders of perturbation theory are summed, the finite answer predicts a suppression of radiation with $\omega < \alpha T$. The analysis of QED then consists of two steps. First, a general probability at $T \neq 0$ is organized so that all the virtual $e^\pm, \gamma$ are in the amplitudes and all the real $e^\pm, \gamma$ are in the phase space integrations. Next, the cancellations of IR divergences between virtual and real are demonstrated.

1. Introduction

Field theories containing massless particles contain infrared divergences, which may be classified into three types as follows.

Soft or infrared divergences: When an on-shell electron ($p^2 = m^2$) emits an on-shell photon ($k^2 = 0$) the resulting electron is off-shell with a propagator

$$\frac{-1}{(p-k)^2 - m^2} = \frac{1}{2p \cdot k} = \frac{1}{2(E - |\vec{p}| \cos \theta)|k|}$$

Since this emission amplitude is proportional to $1/k$, the emission rate is

$$\int \frac{d^3k}{2k} \frac{|1/2|^2}{k |[1 + n_B(k)]} \quad n_B(k) \equiv \frac{1}{\exp(k/T) - 1}$$

At $T = 0$ this gives a logarithmic divergence at small $k$ but the divergence cancels in physical quantities. At $T \neq 0$ the divergence is linear at small $k$ because of the Bose-Einstein function $n_B$. This paper will sketch how the linear and logarithmic divergences all cancel in physical quantities.
Mass or collinear divergences: If the electron mass were zero in Eq. (1), then $E = p$ and there would be a singularity in the amplitude at $\theta = 0$ for any value of $k$. These collinear singularities beset nonabelian theories but are not present in massive QED, which is the subject here.

Coulomb divergences: The total cross section for charged particle scattering is divergent

$$\sigma_{\text{Tot}} \sim \int \frac{\sin \theta}{\sin^4 \theta} \sim \int \frac{dt}{t^2} \sim \infty$$

At $T = 0$ this is not too troublesome. At $T \neq 0$ it arises in self-energy discontinuities at two loops as can be seen even in the classical expression $\Gamma = n v \sigma_{\text{Tot}}$ for the collision rate. At $T \neq 0$ this divergence is reduced to logarithmic by Braaten-Pisarski resummation\(^2\) but it is not eliminated.

2. IR Cancellation for Semiclassical Bremsstrahlung

2.1. Bremsstrahlung to First Order in $\alpha$

The first step is to examine the radiation produced by a charged particle that scatters while passing through a fixed-temperature plasma. To first order in $\alpha$ the probability of radiating energy $\omega = |\vec{k}|$ is

$$2 \omega \frac{dP_1}{d^3k} = \sum_{\text{pol}} |\mathcal{M}(k)|^2 \left[ 1 + n_B(\omega) \right]/(2\pi)^3$$

When $\omega$ is small, the matrix element is

$$\mathcal{M}(k) = J^\mu(k) e_\mu = ie \left( \frac{p' \cdot \epsilon}{p' \cdot k} - \frac{p \cdot \epsilon}{p \cdot k} \right) e^{-k/2\Lambda}$$

regardless of the spin of the charged particle. Here $\Lambda$ is a momentum cutoff that is necessary later. The radiation is mostly parallel to $\vec{p}$ or $\vec{p}'$. When integrated over angles the result is

$$\frac{dP_1}{d\omega} = \frac{A}{\omega} \left[ 1 + n_B(\omega) \right] e^{-\omega/2\Lambda} \quad A(p \cdot p') \equiv \frac{\alpha}{\pi} \left[ \frac{1}{v} \ln \left( \frac{1 + v}{1 - v} \right) - 2 \right]$$

where $v$ is defined by $p \cdot p' = m^2 (1 - v^2)^{-1/2}$. Except for the statistical factor $n_B$, Eq. (6) is a classical formula. When $\omega \ll T$ it predicts $dP_1/d\omega \approx AT/\omega^2$. This is totally unphysical because the energy radiated in low energy modes, below some $E_{\text{max}}$, would be infinite: $\int_0^{E_{\text{max}}} d\omega dP_1/d\omega = \infty$.

2.2 Bremsstrahlung to All Orders in $\alpha$

To improve on the first-order result, we couple the classical current in Eq. (5) to the quantized radiation field $A_\mu$. The generating function for all multi-photon
amplitudes can be obtained by a functional integration over $A_{\mu}$. From this one obtains the multi-photon amplitudes $\mathcal{M}$. The probability that any number $n$ of real photons in the plasma will radiate a net energy $\omega$ is

$$\frac{dP}{d\omega} = \sum_{n=1}^{\infty} \int d\Phi_1...d\Phi_n \delta(k_1^0 + ... + k_n^0 - \omega) \frac{1}{n!} \sum_{\text{pol}} |\mathcal{M}(k_1, ... k_n)|^2$$  \hfill (7)

This weights photon emission ($k_0^0 > 0$) with the statistical factor $1+n_B$; and photon absorption ($k_0^0 < 0$) with the factor $n_B$.

Each amplitude $\mathcal{M}$ is infrared divergent from closed loops of virtual photons. Each integration $d\Phi$ over the real photons is also infrared divergent. The virtual and real contributions exponentiate to give

$$\frac{dP}{d\omega} = \int_{-\infty}^{\infty} \frac{dz}{2\pi} e^{-i\omega z} \exp[\mathcal{R}(z)]$$  \hfill (9)

$$\mathcal{R}(z) \equiv \int \frac{d^3k}{2k(2\pi)^3} J_{\mu}(k) J^\mu(k) \left([1 + n_B] e^{ikz} + n_B e^{-ikz} - [1 + 2n_B]\right)$$  \hfill (10)

Real emission, real absorption, and virtual photons are automatically weighted by $1 + n_B$, $n_B$, and $1 + 2n_B$, respectively. Even though $J_\mu \sim 1/k$ and $n_B \sim 1/k$ at small $k$, Eq. (10) is completely finite in the region $k \to 0$. It is possible to compute $\mathcal{R}(z)$ and to compute the Fourier transform $dP/d\omega$. The final result is

$$\frac{dP}{d\omega} = |\Gamma(\frac{A}{2} + i \frac{\omega}{2\pi T})|^2 e^{\omega^2/2T} e^{-|\omega|/A} \left(\frac{2\pi T}{A}\right)^A$$  \hfill (11)

where $A$ is given in Eq. (6). The most interesting feature of Eq. (11) is the appearance of two dimensionless scales: $A \ll 1$ and $\omega/T$. If $A\pi \ll \omega/T$ then

$$\frac{dP}{d\omega} \approx \frac{A}{\omega} [1 + n_B(\omega)] e^{-\omega/\Lambda} \quad (A\pi T \ll \omega)$$  \hfill (12)

which coincides with the first-order result Eq. (6). However this does not apply at $\omega \ll T$. At small energy,

$$\frac{dP}{d\omega} \approx \frac{AT}{\omega^2 + (A\pi T)^2} \quad (\omega \ll T)$$  \hfill (13)
Naturally Eq. (12) and (13) agree in the region of overlap. At very small energies, $\omega \ll A\pi T$, $dP/d\omega$ is constant rather than increasing like $1/\omega^2$. The interpretation of the small $\omega$ suppression is that the quantity $2A\pi T = \Gamma_r$ is a damping rate produced by the radiation reaction that is required by unitarity\textsuperscript{3}.

3. IR Cancellation for Thermal QED

3.1 Separation of Real and Virtual Particles

We now set aside the semiclassical approximation and turn to the full quantum field theory. Each species of particle has four types of propagator in thermal field theory: $S_{ab}$ for electrons and $D_{\mu\nu}^{ab}$ for photons with $a, b = 1$ or 2. The problem is that $D_{12} = D_{21} \sim \delta(k^2)$ and $S_{12} = -S_{21} \sim \delta(p^2 - m^2)$ actually represent real, on-shell particles but this is obscured by the finite-temperature Feynman rules. Before looking for infrared cancellations, it is very convenient to reorganize thermal probabilities as squares of amplitudes that contain only $S_{11}$ and $D_{\mu\nu}^{11}$, integrated over physical phase space.

For definiteness, consider the process $e^- (p_1) + \text{plasma} \rightarrow \text{anything}$, with rate

$$R(p_1) = \sum_{F,I} |< F|C|I >|^2 \quad C \equiv [S, b^\dagger(p_1)]$$

(14)

Using completeness and thermofield dynamics\textsuperscript{4}, one can write this as

$$R(p_1) = < 0(\beta)|C^\dagger C|0(\beta) >= \sum_F |< F(\beta)|C|0(\beta) >|^2$$

(15)

where $|F(\beta) >$ is a complete set of thermal states in the Foch space built out of Bogoliubov-transformed creation operators\textsuperscript{4}. For example, the contribution of $a^\dagger_\beta|0(\beta) >$ to $R$ is

$$\int \frac{d^3k}{2k(2\pi)^3}[1 + n_B(k)] |< 0(\beta)||a,C||0(\beta) >|^2$$

(16)

The contribution of $\tilde{a}^\dagger_\beta|0(\beta) >$ to $R$ is

$$\int \frac{d^3k}{2k(2\pi)^3}n_B(k)|< 0(\beta)||\tilde{a}^\dagger, C||0(\beta) >|^2$$

(17)

The complete rate is

$$R(p_1) = \sum_{\ell=2}^\infty \int d\Psi_2...d\Psi_\ell \sum_{n=0}^\infty \int d\Phi_1...d\Phi_n \frac{1}{n!} |M_{\ell,n}(p_1, ...p_\ell; k_1, ...k_n)|^2$$

(18)
where \(d\Phi\) is the photon phase space from Eq. (8), \(d\Psi\) is the fermion phase space, and \(\mathcal{M}_{\ell n}\) is the amplitude for \(\ell\) real leptons (including the observed \(p_1\)) and \(n\) real photons (either initial or final). The amplitude is

\[
M_{\ell,n} = \langle 0| [a^\#(\vec{k}_n), [a^\#(\vec{k}_{n-1}), \ldots [b^\#(\vec{p}_2), [b^\dagger(\vec{p}_1), S]] \ldots] |0(\beta) \rangle
\]  

(19)

In the interaction picture, the averaging required for these amplitudes is determined by the operator \(S\exp(-\beta H)\) rather than by \(\exp(-\beta H)\). Consequently these amplitudes contain only the propagators \(D_{11}^{\mu\nu}\) and \(S_{11}\) and none of the cut propagators \(D_{12}^{\mu\nu}\) and \(S_{12}\). Eq. (18) therefore provides a complete separation between real particles (contained in the phase space \(d\Psi d\Phi\)) and virtual particles (represented by the propagators \(D_{11}\) and \(S_{11}\)). Ref. 5 and 6 present diagramatic arguments for this result.

### 3.2 IR Cancellation to All Orders in \(\alpha\)

To analyze the rate in Eq. (18) one can repeat the analysis of Yennie, Frautschi, and Suura with some modifications. For fixed fermion momenta, define

\[
R_\ell(p_1, \ldots p_\ell) = \sum_{n=0}^\infty \int d\Phi_1 \ldots d\Phi_n \frac{1}{n!} |M_{\ell,n}|^2
\]  

(20)

Virtual photons inside the amplitude \(\mathcal{M}_{\ell n}\) can only cause an IR divergence when they are on-shell and attached to “external” fermions \(p_1, \ldots p_\ell\). Each end of a photon line attached to an “external” fermion gives a multiplicative factor \(e2p^\mu/2p \cdot k\) plus non-IR terms. At each order of \(\alpha\), \(\mathcal{M}_{\ell n}\) has a maximum IR divergence \(\alpha^p\) plus non-leading divergences \(\alpha^{p-1}, \alpha^{p-2}, \ldots\). When summed to all orders, the leading and non-leading divergences from virtual photons all exponentiate.

Next we examine infrared divergences produced by real photon emission and absorption. These arise from the integrations \(d\Phi\) in Eq. (20) whenever \(\mathcal{M}_{\ell n} \sim 1/k\). The diagrams in which \(n\) real photon lines are attached to the “external” fermions in all possible ways, give multiplicative factors \(e2p^\mu/2p \cdot k\) plus non-IR terms. The contribution to Eq. (20) of \(n\) real photons has an IR divergence \(\alpha^n\) plus non-leading divergences \(\alpha^{n-1}, \alpha^{n-2}, \ldots 1\). When summed over \(n\), the leading and non-leading divergences from real photons also exponentiate. The final result is

\[
R_\ell(p_1, \ldots p_\ell) = \sum_{m=0}^\infty \int_{-\infty}^{\infty} d\omega \frac{dP}{d\omega} \int d\Phi_1 \ldots d\Phi_m \beta_{\ell,m}(k_1, \ldots k_m)/m!
\]  

(21)
with all IR divergences contained in the quantity

$$\frac{dP}{d\omega} = \int_{-\infty}^{\infty} \frac{dz}{\pi} e^{-i\omega z} \exp[\mathcal{R}(z)]$$  \hspace{1cm} (22)

$$\mathcal{R}(z) = \int \frac{d^3k}{2k(2\pi)^3} \sum_{a,b=1}^{\ell} e_a e_b (p_a \cdot p_b) (1 + n_B) e^{ikz} + n_B e^{-ikz} - [1 + 2n_B] e^{-k/\Lambda}$$  \hspace{1cm} (23)

This integration is IR finite because of the delicate cancellation between real emission $(1 + n_B)$, real absorption $(n_B)$, and virtual photons $(1 + 2n_B)$. The integrations are the same as in Sec. 2 with the result

$$\frac{dP}{d\omega} = \frac{|\Gamma(A/2 + i\omega/2\pi)|^2}{2\pi^2 T^2} \frac{e^{\omega/2T} e^{-|\omega|/\Lambda}}{(2\pi T)^A}$$  \hspace{1cm} (24)

but now $A$ depends on all the “external” fermion momenta:

$$A(p_1, ... p_\ell) \equiv - \sum_{a,b=1}^{\ell} \frac{e_a e_b}{8\pi^2} \frac{1}{v_{ab}} \ln \left( \frac{1 + v_{ab}}{1 - v_{ab}} \right) \geq 0$$  \hspace{1cm} (25)

and $v_{ab}$, defined by $p_a \cdot p_b/m^2 = (1 - v_{ab}^2)^{-1/2}$, is relative velocity of charge $a$ in the rest frame of $b$.

Eq. (24) is infrared finite and has the behavior $dP/d\omega \to \text{constant}$ as $\omega \to 0$. Therefore, for fixed momenta of the “external” fermions, the rate in Eq. (21) contains no infrared divergences.

4. Conclusions

Although the infrared divergences have all been eliminated, the analysis is not quite complete. The full rate requires integration over the thermalized fermions:

$$R(p_1) = \sum_{\ell=2}^{\infty} \int d\Psi_2 ... d\Psi_\ell R_\ell(p_1, ... p_\ell)$$  \hspace{1cm} (26)

The fermion integrations do not affect the infrared finiteness of $R(p_1)$. However, there will still be Coulomb divergences that arise when any one of the momentum transfers vanishes: $(p_a - p_b)^2 \to 0$. It is known that the usual zero-temperature Coulomb divergence $\int \sin \theta d\theta/\theta^4$ is reduced to logarithmic, $\int \sin \theta d\theta/\theta^2$ at $T \neq$
0 due to Braaten-Pisarski resummation. These logarithmic divergences are not eliminated.

For definiteness, the process $e^{-}(\vec{p}_1) + \text{plasma} \rightarrow \text{anything}$ was treated specifically. One can analyze a general process $\{A\} + \text{plasma} \rightarrow \{B\} + \text{anything}$, where $\{A\}$ and $\{B\}$ are any sets of $e^{\pm}, \gamma$ by using the same amplitudes $M_{\ell,n}$. The rate corresponding to Eq. (18) would not be integrated over the lepton and photon momenta that enter and leave the plasma. These fixed external momenta cause no complication, but make the notation a bit more cumbersome. The same arguments apply and show that all infrared divergences cancel in the generalized rates. However, the logarithmic Coulomb divergence remain.

5. Acknowledgements

It is a pleasure to thank R. Kobes and G. Kunstatter for organizing an excellent workshop. This work was supported in part by the U.S. National Science Foundation under grant PHY-9213734.

6. References

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