INNER TABLEAU TRANSLATION PROPERTY OF THE WEAK ORDER AND RELATED RESULTS

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Abstract. Let SYT_n be the set of all standard Young tableaux with n cells and \( \leq_{\text{weak}} \) be Melnikov’s the weak order on \( \text{SYT}_n \). The aim of this paper is to introduce a conjecture on the weak order, named the property of inner tableau translation, and discuss its significance. We will also prove the conjecture for some special cases.

1. Introduction

The weak order first introduced by Melnikov and well studied in [12, 13, 14] due its strong connections to Kazhdan-Lusztig and geometric order on standard Young tableau, where the latter are induced from the representation theory of special linear algebra and symmetric group. We have the following inclusion among all of these three orders:

\[ \text{weak order} \subseteq \text{Kazhdan-Lusztig (KL) order} \subseteq \text{geometric order}. \]

The fact that its definition uses just the combinatorics of tableaux such as Knuth relations and the weak order on symmetric group, gives the weak order an important place among these orders. On the other hand the only justification for its well-definedness is induced from above inclusion, in other words there is no self contained proof of this basic fact.

Our aim here is to bring the attention to the following conjecture on the weak order (which is first asked in [21]), called the property of inner tableau translation. This property is known to be satisfied by Kazhdan-Lusztig and geometric orders and its importance on the weak order relies on the fact that it provides a self contained proof for the well-definedness of this order.

Conjecture 1.1. Given two tableau \( S <_{\text{weak}} T \) having the same inner tableau \( R \), replacing \( R \) with another same shape tableau \( R' \) in \( S \) and \( T \) still preserves the weak order.

In the following we first provide the definitions and related background for the weak order. In the third section, by assuming the conjecture we will provide a self contained proof for the well definedness the weak order and we close this section with the discussion on how this conjecture plays specific role in studies of Poirier and Reutenauer Hopf algebra on standard Young tableaux. In the last section, we prove the conjecture for the case when the inner tableau \( R \) has hook shape or a shape which consists of two rows or two columns.

2. Related background

2.1. Definition of the weak order. The definition of the weak order uses well known Robinson-Schensted (RSK) correspondence which bijectively assigns to every permutation \( w \in S_n \) a pair of same shape tableaux \( (I(v), R(w)) \in \text{SYT}_n \times \text{SYT}_n \), where \( I(w) \) and \( R(w) \) are called the insertion and recording tableau of \( w \) respectively. On the other hand an equivalence relation \( u \sim_K w \) due to Knuth [8] plays a crucial role in this correspondence. Namely:

\[ u \sim_K w \iff I(u) = I(w). \]

We will denote the corresponding equivalence classes in \( S_n \) by \( \{Y_T\}_{T \in \text{SYT}_n} \).

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Let us explain these algorithms briefly. Denote by \((I_{i-1}, R_{i-1})\) the same shape tableaux obtained by insertion and recording algorithms on the first \(i-1\) indices of \(w = w_1 \ldots w_n\). In order to get \(I_i\), if \(w_i\) is greater then the last number on the first row of \(I_{i-1}\), it is concatenated to the right side of the first row of \(I_{i-1}\), otherwise, \(w_i\) replaces the smallest number, say \(a\) among all numbers in the first row greater than \(w_i\) and this time insertion algorithm is applied to \(a\) on the next row. Observe that after finitely many steps the insertion algorithm terminates with a new added cell. The resulting tableau is then \(I_i\) and recording tableau \(R_i\) is found by filling this new cell in \(R_{i-1}\) with the number \(i\). We illustrate these algorithms with the following example

Example 2.1. Let \(w = 52413\). Then,

\[
\begin{align*}
I_1 &= 5 & I_2 &= 2 \quad \Rightarrow I_3 &= 2 \quad \Rightarrow I_4 &= 1 \quad \Rightarrow I_5 &= 1 \\
R_1 &= 1 & R_2 &= 1 \quad \Rightarrow R_3 &= 1 \\
& & & & \Rightarrow R_4 &= 1 \\
& & & & \Rightarrow R_5 &= 1 \\
& & & & \Rightarrow (I(w)) \\
& & & & \Rightarrow (R(w))
\end{align*}
\]

Definition 2.2. We say \(u, w \in S_n\) differ by one Knuth relation, written \(u \overset{K}{\sim} w\), if

\[
\begin{align*}
either w &= x_1 \ldots yxz \ldots x_n \text{ and } u &= x_1 \ldots yzx \ldots x_n \\
or \quad w &= x_1 \ldots xzy \ldots x_n \text{ and } u &= x_1 \ldots zxy \ldots x_n
\end{align*}
\]

for some \(x < y < z\). Two permutations are called Knuth equivalent, written \(u \overset{K}{\sim} w\), if there is a sequence of permutations such that

\[
u = u_1 \overset{K}{\sim} u_2 \ldots \overset{K}{\sim} u_k = w.
\]

Schützenberger’s jeu de taquin slides [20] are one of the combinatorial operations on tableaux that we apply often in the following sections.

Definition 2.3. Let \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)\) and \(\mu = (\mu_1, \mu_2, \ldots, \mu_l)\) be two Ferrers diagrams such that \(\mu \subset \lambda\). Then the corresponding skew diagram is defined to be the set of cells

\[
\lambda/\mu = \{c : c \in \lambda, c \notin \mu\}.
\]

A skew diagram is called normal if \(\mu = \emptyset\). A partial skew tableau of shape \(\lambda/\mu\) is an array of distinct integers elements whose rows and columns increase. A standard skew tableau of shape \(\lambda/\mu\) is partial skew tableau whose elements are \(\{1, 2, \ldots, n\}\).

We next illustrate the forward and backward slides of Schützenberger’s jeu de taquin [20] without the definition.

Example 2.4. Let \(P = \begin{array}{ccc}
4 & 2 & 4 \\
2 & 5 & 3 \\
1 & 3 & 1
\end{array}\) and \(Q = \begin{array}{ccc}
2 & 4 & 4 \\
3 & 5 & 1 \\
1 & 3 & 1
\end{array}\). Below we illustrate a forward and backward slide on \(P\) and \(Q\) through the cells indicated by dots.

\[
\begin{array}{cccccc}
\cdot & 4 & 2 & 4 & 2 & 4 \\
2 & 5 & \cdot & 5 & \cdot & 3 & 5 \\
1 & 3 & 1 & 3 & 1 & \cdot & 1
\end{array}
\]

\[
\begin{array}{cccccc}
\cdot & 4 & 2 & 4 & \cdot & 4 \\
3 & 5 & \cdot & 5 & \cdot & 2 & 5 \\
1 & \cdot & 1 & 3 & 1 & 3 & 1 & 3
\end{array}
\]

\[
\begin{array}{cccccc}
\cdot & 4 & 2 & 4 & \cdot & 4 \\
3 & 5 & \cdot & 5 & \cdot & 2 & 5 \\
1 & \cdot & 1 & 3 & 1 & 3 & 1 & 3
\end{array}
\]
The other main ingredient of the weak order is the \( \text{(right) weak Bruhat order} \), \( \leq_{\text{weak}} \), on \( S_n \) which obtained by taking the transitive closure of the following relation:

\[
u \leq_{\text{weak}} w \text{ if } w = u \cdot s_i \text{ and } \text{length}(w) = \text{length}(u) + 1\]

where \( s_i \) denotes the adjacent transposition \( (i, i+1) \) and \( \text{length}(w) \) measures the size of a reduced word of \( w \). The weak order has an alternative characterization \( [9] \text{ Prop. 3.1} \) in terms of \( \text{(left) inversion sets} \) namely

\[
u \leq_{\text{weak}} w \text{ if and only if } \text{Inv}_L(u) \subset \text{Inv}_L(w)
\]

where \( \text{Inv}_L(u) := \{(i, j) : 1 \leq i < j \leq n \text{ and } u^{-1}(i) > u^{-1}(j)\} \).

**Definition 2.5.** The weak order \( (\text{SYT}_n, \leq_{\text{weak}}) \), first introduced by Melnikov \([12]\) under the name \textit{induced Duflo order}, is the partial order induced by taking transitive closure of the following relation:

\[
S \leq_{\text{weak}} T \text{ if there exist } \sigma \in \mathcal{Y}_S, \tau \in \mathcal{Y}_T \text{ such that } \sigma \leq_{\text{weak}} \tau.
\]

The necessity of taking the transitive closure in the definition of the weak order is illustrated by the following example (cf. Melnikov \([12] \text{ Example 4.3.1}\)).

**Example 2.6.** Let \( R = \begin{array}{ccc}
1 & 2 & 5 \\
3 & 4 & 4 \\
\end{array} 
\), \( S = \begin{array}{ccc}
1 & 4 & 5 \\
3 & 3 & 4 \\
\end{array} 
\), \( T = \begin{array}{ccc}
1 & 4 & 5 \\
3 & 3 & 5 \\
\end{array} 
\) with

\[
\mathcal{Y}_R = \{31425, 34125, 34152, 34152, 34512\}, \quad \mathcal{Y}_S = \{32145, 32415, 32451, 34215, 34251, 34521\}, \quad \mathcal{Y}_T = \{32154, 32514, 35214, 32541, 35241\}.
\]

Here \( R <_{\text{weak}} S \) since 34125 <\( \text{weak} \) 34215, and \( S <_{\text{weak}} T \) since 32145 <\( \text{weak} \) 32154. Therefore \( R <_{\text{weak}} T \).

On the other hand, for every \( \rho \in \mathcal{Y}_R \) and for every \( \tau \in \mathcal{Y}_T \) we have \( (2, 4) \in \text{Inv}_L(\rho) \) but \( (2, 4) \notin \text{Inv}_L(\tau) \).

**2.2. Some basic properties of the weak order.** For \( u \in S_n \) and \( 1 \leq i < j \leq n \), let \( u_{[i,j]} \) be the word obtained by restricting \( u \) to the segments \([i, j]\) and \( \text{std}(u_{[i,j]}) \in S_{j-i+1} \) be the permutation obtained from \( u_{[i,j]} \) by subtracting \( i-1 \) from each letter.

Similarly for \( S \in \text{SYT}_n \) and \( 1 \leq i < j \leq n \), let \( S_{[i,j]} \) be the normal shape tableau obtained by restricting \( S \) to the segments \([i, j]\) and by applying Schützenberger’s back word jeu-de-taquin slides. Then \( \text{std}(S_{[i,j]}) \in \text{SYT}_{j-i+1} \) be the tableau obtained from \( S_{[i,j]} \) by subtracting \( i-1 \) from each letter.

In fact \( \text{Inv}_L(u) \subset \text{Inv}_L(w) \) gives \( \text{Inv}_L(u_{[i,j]}) \subset \text{Inv}_L(w_{[i,j]}) \) for all \( 1 \leq i < j \leq n \) and hence

\[
u \leq_{\text{weak}} w \text{ implies } \text{std}(u_{[i,j]}) \leq_{\text{weak}} \text{std}(w_{[i,j]}) \text{ for all } 1 \leq i < j \leq n.
\]

The following basic fact about \( RSK \), Knuth equivalence, and jeu-de-taquin are essentially due to Knuth and Schützenberger; see Knuth \([7] \text{ Section 5.1.4}\) for detailed explanations.

**Lemma 2.7.** Given \( u \in S_n \), let \( I(u) \) be the insertion tableau of \( u \). Then for \( 1 \leq i < j \leq n \),

\[
\text{std}(I(u_{[i,j]})) = I(\text{std}(u_{[i,j]})).
\]

Therefore we have following:

**Lemma 2.8.** The weak order restricts to segments, i.e.,

\[
S \leq T \text{ implies } \text{std}(S_{[i,j]}) \leq \text{std}(T_{[i,j]}) \text{ for all } 1 \leq i < j \leq n.
\]

**Remark 2.9.** Melnikov shows in \([12] \text{ Page 45}\) that the geometric order also restricts to segments. On the other hand the same fact about Kazhdan-Lusztig order was first shown by Barbash and Vogan \([1]\) for arbitrary finite Weyl groups (see also work by Lusztig \([10]\)) whereas the generalization to Coxeter groups is due to Geck \([4] \text{ Corollary 3.4}\).

Now recall that \( \text{(left) descent set} \) of a permutation \( \tau \) is defined by

\[
\text{Des}_L(\tau) := \{i : 1 \leq i < n-1 \text{ and } \tau^{-1}(i) > \tau^{-1}(i+1)\}.
\]
Figure 1. The weak on $\text{SYT}_n$ for $n = 2, 3, 4, 5$.

On the other hand the descent set of the standard Young tableau $T$ is described intrinsically by

$$\text{Des}(T) := \{(i, i + 1) : 1 \leq i \leq n - 1 \text{ and } i + 1 \text{ appears in a row below } i \text{ in } T\}.$$  

As a consequence of a well-known properties of RSK we have the following basic fact:

**Lemma 2.10.** For any $\tau \in \mathcal{Y}_T$ we have

$$\text{Des}_L(\tau) = \text{Des}(T)$$

i.e., the left descent set is constant on Knuth classes.

We let $(2^{[n-1]}, \subseteq)$ be the Boolean algebra of all subsets of $[n-1]$ ordered by inclusion.

**Lemma 2.11.** Let $\leq$ be any order on $\text{SYT}_n$ which is stronger than the weak order and restricts to segments. Then the map

$$(\text{SYT}_n, \leq) \mapsto (2^{[n-1]}, \subseteq)$$

sending any tableau $T$ to its descent set $\text{Des}(T)$ is order preserving.

We denote by $(\text{Par}_n, \leq_{\text{dom}})$ the set of all partitions of the number $n$ ordered by the opposite (or dual) dominance order, that is, $\lambda \leq_{\text{dom}} \mu$ if

$$\lambda_1 + \cdots + \lambda_k \geq \mu_1 + \cdots + \mu_k$$

for all $k$.

The following can be easily deduced from Greene’s theorem [5].
Lemma 2.12. $S \leq_{\text{weak}} T$ implies $\text{shape}(T) \leq_{\text{dom}} \text{shape}(S)$

Recall that for a standard Young tableau $T$, $T^\dagger$ denotes the transpose of $T$ whereas $T^{\text{evac}}$ denotes the tableau found by applying the Schützenberger’s [19] evacuation map on $T$. For any $\tau = \tau_1 \tau_2 \ldots \tau_n \in \mathcal{Y}_T$ we have

$$\tau^\dagger = \tau_{n} \tau_{n-1} \ldots \tau_1 \in \mathcal{Y}_T^\dagger$$
$$\tau^{\text{evac}} = (n+1-\tau_n)(n+1-\tau_{n-1})\ldots(n+1-\tau_1) \in \mathcal{Y}_T^{\text{evac}}$$

Proposition 2.13. Suppose $S \leq_{\text{weak}} T$ in SYT$_n$. Then

1. $S^{\text{evac}} \leq_{\text{weak}} T^{\text{evac}}$.
2. $T^\dagger \leq_{\text{weak}} S^\dagger$.

Proof. Let $w_0$ be the longest element in $S_n$. Then the maps

$$w \mapsto w_0 w \text{ and } w \mapsto ww_0$$

are clearly anti-automorphisms and hence $w \mapsto w_0 w w_0$ is a automorphism of $(S_n, \leq_{\text{weak}})$. On the other hand $I(w w_0)$ is just the transpose tableau of $I(w)$ [16] whereas $I(w_0 w w_0)$ is nothing but the evacuation of $I(w)$ [19].

2.3. Inner tableau translation property. The dual Knuth relations $\sim_{K}$ on $S_n$ plays the main role in the definition of inner tableau translation property. In its most basic form this relation is defined through the Knuth relations applied on the inverse of permutations. Namely,

$$\sigma \sim_{K^*} \tau \text{ in $S_n$ if and only if } \sigma^{-1} \sim_{K} \tau^{-1}.$$  

An equivalent definition can be given by taking the transitive closure of the following: We say $\sigma$ and $\tau$ differs by a single dual Knuth relation determined by the triple $\{i, i+1, i+2\}$ if

either $\sigma = \ldots i + 1 \ldots i + 2 \ldots$ and $\tau = \ldots i + 2 \ldots i + 1 \ldots$

or $\sigma = \ldots i + 1 \ldots i + 2 \ldots i \ldots$ and $\tau = \ldots i \ldots i + 2 \ldots i + 1 \ldots$

Since left descent sets are all equal for the permutations lying in the same Knuth class, the dual Knuth relation defines an action on the standard Young tableaux. In order to present this action let us give the following definition.

Definition 2.14. For $S \in \text{SYT}_n$ let $A = \{(i, j)\}$ be a cell lying in $\text{shape}(S)$, where $i$ denotes the row number counted from the top and $j$ denotes the column number counted from the left. Then

$$(S, A, \text{ne}) := \{(k, l) \mid k < i \text{ and } l \geq j\}$$

$$(S, A, \text{sw}) := \{(k, l) \mid k \geq i \text{ and } l < j\}$$

Suppose that $\sigma \in \mathcal{Y}_S$ has $i \in \text{Des}(\sigma)$ but $i \not\in \text{Des}(\sigma)$. Therefore $\sigma$ has one of the following form

$$\sigma = \ldots i + 1 \ldots i + i + 2 \ldots \text{ or } \sigma = \ldots i + 1 \ldots i + 2 \ldots$$

Now denote by $C_i, C_{i+1}$ and $C_{i+2}$ the cells labeled by $i, i + 1$ and $i + 2$ in $S$, respectively. Then

either $C_i \in (S, C_{i+1}, \text{ne}) \cap (S, C_{i+2}, \text{sw})$ or $C_{i+2} \in (S, C_{i+1}, \text{ne}) \cap (S, C_{i}, \text{sw})$

and the action of a single dual Knuth relation determined by the triple $\{i, i+1, i+2\}$ on $S$ interchanges the places of $\sigma$ in the first case and it interchanges the places of $i$ in the second case.

The following theorem (see [17, Proposition 3.8.1]) provides an important characterization of the dual Knuth relation.

Proposition 2.15. Let $S, T \in \text{SYT}_n$. Then $S \sim_{K^*} T$ if and only if $\text{shape}(S) = \text{shape}(T)$.

Definition 2.16. Let $\{\alpha, \beta\} = \{i, i+1\}$ and $\text{SYT}_n^{[\alpha, \beta]} := \{T \in \text{SYT}_n \mid \alpha \in \text{Des}(T), \beta \not\in \text{Des}(T)\}$. Then we have inner translation map

$$\mathcal{V}_{[\alpha, \beta]} : \text{SYT}_n^{[\alpha, \beta]} \mapsto \text{SYT}_n^{[\beta, \alpha]}$$

which send every tableau $T \in \text{SYT}_n^{[\alpha, \beta]}$ to a tableau obtained as a result of the action of the single dual Knuth relation determined by the triple $\{i, i+1, i+2\}$.
The inner translation map is first introduced by Vogan in [22] where he also shows that Kazhdan-Lusztig order is preserved under this map. For geometric order this result is due to Melnikov [15, Proposition 6.6]. On the other hand the example given below shows that the weak order does not satisfy this property.

**Example 2.17.**

\[
\begin{array}{ccc}
1 & 2 & 4 \\
3 & 5 & 6 \\
\end{array} \leq_{\text{weak}} \begin{array}{ccc}
1 & 2 & 4 \\
3 & 6 & 5 \\
\end{array} \quad \text{but} \quad \begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & 6 \\
\end{array} \not\leq_{\text{weak}} \begin{array}{ccc}
1 & 2 & 5 \\
3 & 6 & 4 \\
\end{array}
\]

where the latter pair is obtained from the former by applying a single dual Knuth relation on the triple \{3, 4, 5\}.

A weaker version of the inner translation property can be defined in the following manner:

**Definition 2.18.** For \(1 \leq k < n\) and \(R \in \text{SYT}_k\), let \(\text{SYT}_k^R := \{ T \in \text{SYT}_n \mid T_{[1,k]} = R \} \). Then for \(R, R' \in \text{SYT}_k\), having the same shape, we have inner tableau translation map

\[ V_{[R, R']} : \text{SYT}_n^R \rightarrow \text{SYT}_n^{R'} \]

which send every \(T \in \text{SYT}_n^R\) to the tableau \(T'\) obtained by replacing \(R\) with \(R'\).

As a consequence of Proposition 2.15, one can generate \(T'\) by a sequence of dual Knuth relations applied on the subtableau \(R\) of \(T\). Therefore if a partial order is preserved under inner translation map then it is also preserved under inner tableau translation map. Hence Kazhdan-Lusztig and geometric orders have this property. On the other hand it is still reasonable to ask whether the weak order is preserved under the inner tableau translation property.

**Conjecture 2.19.** Let \(S \preceq_{\text{weak}} T\) be a covering relation in \(\text{SYT}_n^R\) and \(R'\) be a tableau obtained by applying to \(R\) a single dual Knuth relation. Then

\[ V_{[R, R']} (S) \preceq_{\text{weak}} V_{[R, R']} (T) \text{ in } \text{SYT}_n^{R'}. \]

In other words the weak order on standard Young tableau is preserved under the inner tableau translation map.

**Remark 2.20.** Recall that any tableau \(R' \in \text{SYT}_r\) with the same shape as \(R\), can be obtained by applying to \(R\) a sequence of dual Knuth relation by Proposition 2.15. Therefore one can generalize the conjecture for any tableau \(R\) and \(R'\) having the same shape.

As it is stated earlier this conjecture is checked by computer programing up to \(n = 9\). In the last section we also show that for a specific case the conjecture is true.

### 3. Applications of the Conjecture

#### 3.1. Well-definedness of the weak order

By assuming Conjecture 2.19 we first prove the following result.

**Theorem 3.1.** The weak order on \(\text{SYT}_n\) is well defined.

**Proof.** It is enough to show that if \(S \preceq_{\text{weak}} T\) and \(S \neq T\) then \(T \not\preceq_{\text{weak}} S\). By Lemma 2.11 we know that \(S \preceq_{\text{weak}} T\) implies \(\text{Des}(S) \subseteq \text{Des}(T)\) and if \(\text{Des}(T) \setminus \text{Des}(S) \neq \emptyset\) then clearly \(T \not\preceq_{\text{weak}} S\). Now we suppose that \(\text{Des}(T) = \text{Des}(S)\). Let \(k\) be the smallest integer satisfying

\[ S_{[1,k]} = T_{[1,k]}. \]

So \(k < n\) and \(S_{[1,k+1]}\) and \(T_{[1,k+1]}\) differ only by the position of the corner cells labeled by \(k + 1\). On the other hand by Lemma 2.8 have

\[ S_{[1,k+1]} \preceq_{\text{weak}} T_{[1,k+1]} \]

and Lemma 2.12 together with the fact that \(\text{shape}(T_{[1,k]}) = \text{shape}(S_{[1,k]})\) gives

\[ \text{shape}(T_{[1,k+1]}) \preceq_{\text{dom shape}} \text{shape}(S_{[1,k+1]}). \]
The last argument shows that $T' \not\preceq_{\text{weak}} S'$ and therefore by Conjecture 2.19, $T \not\preceq_{\text{weak}} S$. □

3.2. Poirier-Reutenauer Hopf algebra on $\mathbb{Z}SYT = \oplus_{n \geq 0} \mathbb{Z}SYT_n$. Following the work of Malvenuto and Reutenauer on permutations [11], Poirier-Reutenauer construct two graded Hopf algebra structures on $\mathbb{Z}$ module of all plactic classes $\{PC_T\}_{T \in SYT}$, where $PC_T = \sum_{P(u)=T} u$. The product structure of the one that concerns us here is given by

$$PC_T \ast PC_{T'} = \sum_{P(u)=T} \text{shf}(u, \overline{w})$$

where $\overline{w}$ is obtained by increasing the indices of $w$ by the length of $u$ and $\text{shf}$ denotes the shuffle product. Then the bijection sending each plactic class to its defining tableau gives us a Hopf algebra structure on the $\mathbb{Z}$ module of all standard Young tableaux, $\mathbb{Z}SYT = \oplus_{n \geq 0} \mathbb{Z}SYT_n$.

In [10] Poirier and Reutenauer explain this product using jeu de taquin slides. Following an analogous result of Loday and Ronco [9] Thm. 4.1 on permutations, the author shows the following result in [21]: For $S \in SYT_k$, $T \in SYT_l$ where $k + l = n$, let $T$ the tableau which is obtained by increasing the indices of $T$ by $k$. Denote by $S/T$ the tableau whose columns are obtained by concatenating the columns of $T$ over $S$ below and by $S \setminus T$ the tableau whose rows are obtained by concatenating the rows of $T$ over $S$ from the right. Then by [21] Thm. 4.2

$$S \setminus T = \sum_{R \in SYT_n : S \setminus T \preceq_{\text{weak}} R \preceq_{\text{weak}} S/T} R$$

Namely the product structure can be read on the weak order poset of standard Young tableaux.

Example 3.2. Let $S = \begin{array}{ccc} 1 & 2 \\ 3 & \end{array}$ and $T = \begin{array}{c} 1 \\ 2 \end{array}$. Then $S \setminus T = \begin{array}{ccc} 1 & 2 \\ 3 & 5 & 4 \end{array}$, $S/T = \begin{array}{c} 3 \\ 4 \end{array}$. Then

$$PC_{\begin{array}{ccc} 1 & 2 \\ 3 & \end{array}} \ast PC_{\begin{array}{c} 1 \\ 2 \end{array}} = \text{shf}(312, 54) + \text{shf}(132, 54)$$

$$= PC_{\begin{array}{ccc} 1 & 2 & 4 \\ 3 & 5 & \end{array}} + PC_{\begin{array}{ccc} 1 & 2 & 4 \\ 3 & 4 & \end{array}} + PC_{\begin{array}{ccc} 3 & 4 & 2 \\ 5 & \end{array}}.$$ 

On the other hand one can check from Figure 1 that the product $S \ast T$ is equal to the sum of all tableaux in the interval $[S \setminus T, S/T]$.

By using the facts that $(S \setminus T)^{\text{evac}} = T^{\text{evac}} \setminus S^{\text{evac}}$ and $(S/T)^{\text{evac}} = T^{\text{evac}} / S^{\text{evac}}$ and Proposition 2.13 one can easily deduce the following corollary to Conjecture 2.19.

Corollary 3.3. Let $S, S', T, T'$ be standard Young tableaux satisfying

$$\text{shape}(S) = \text{shape}(S') \text{ and } \text{shape}(T) = \text{shape}(T').$$
Then the intervals of the weak order \([S,T, S/T]\) and \([S', T', S'/T']\) are isomorphic. Equivalently, the shuffle product \(S \ast T\) is determined by the shapes of the tableaux rather than the tableaux itself.

### 4. The cases where the conjecture holds

**Lemma 4.1.** Suppose that \(S \leq_{\text{weak}} T\) is a covering relation in \(\text{SYT}_n^R\) and \(T, T' \in \text{SYT}_k\) has the same shape. If \(\text{std}(S_{k+1,n}) \leq_{\text{weak}} \text{std}(T_{k+1,n})\) then \(V_{[R,R']}(S) \leq_{\text{weak}} V_{[R,R']}(T)\).

**Proof.** It is enough to consider the case when \(R\) and \(R'\) differ by only one dual Knuth relation determined by the triples \(\{i, i+1, i+2\}\) for some \(i \leq k - 2\). Since \(S \leq_{\text{weak}} T\) is a covering relation, there exist \(\sigma, \tau \in \mathcal{Y}_T\) such that for some \(i < n\),

\[
\sigma = a_1 \ldots a_j a_{j+1} \ldots a_n \leq a_1 \ldots a_{j+1} a_j \ldots a_n = \tau, \text{ where } a_j < a_{j+1}
\]

i.e., \(\sigma < \tau\) is also covering relation the right weak order on \(S_n\). On the other hand by Lemma 2.8 we have

\[
I(\sigma_{k+1,n}) = S_{k+1,n} \text{ and } T_{k+1,n} = I(\tau_{k+1,n})
\]

Furthermore the assumption \(S_{k+1,n} \leq_{\text{weak}} T_{k+1,n}\) yields that \(\sigma_{k+1,n} \leq \tau_{k+1,n}\) and \(\sigma_{[1,k]} = \tau_{[1,k]}\).

Now applying the dual Knuth relation determined by the triple \(\{i, i+1, i+2\}\) on \(\sigma\) and \(\tau\) gives two new permutations say \(\sigma'\) and \(\tau'\) such that \(\sigma' \prec \tau'\) in the right weak Bruhat order and therefore

\[
V_{[R,R']}(S) = I(\sigma') \leq_{\text{weak}} I(\tau') = V_{[R,R']}(T).
\]

Now if there exist a tableau \(Q \in \text{SYT}_n^{R'}\) satisfying \(V_{[R,R']}(S) \leq_{\text{weak}} Q \leq_{\text{weak}} V_{[R,R']}(T)\) then we have

\[
S \leq_{\text{weak}} V_{[R', R]}(Q) \leq_{\text{weak}} T
\]

which is clearly a contradiction. Hence \(V_{[R,R']}(S) \leq_{\text{weak}} V_{[R,R']}(T)\).

**Lemma 4.2.** Suppose

\[
R = \begin{array}{ccc}
1 & 2 & 3 \\
2 & & \\
3 & &
\end{array} \quad \text{and} \quad R' = \begin{array}{ccc}
1 & 2 & \\
3 & & \\
& &
\end{array}
\]

Then \(S \leq_{\text{weak}} T\) in \(\text{SYT}_n^R\) if and only if \(V_{[R,R']}(S) \leq_{\text{weak}} V_{[R,R']}(T)\) in \(\text{SYT}_n^{R'}\).

**Proof.** Since \(S \leq_{\text{weak}} T\) there exist \(\sigma \in \mathcal{Y}_S\) and \(\tau \in \mathcal{Y}_T\) such that for some \(1 \leq j < n\) we have

\[
\sigma = a_1 \ldots a_j a_{j+1} \ldots a_n \quad \text{and} \quad \tau = a_1 \ldots a_{j+1} a_j \ldots a_n, \text{ where } a_j < a_{j+1}.
\]

Observe that since \(S\) and \(T\) have the same inner tableau \(R\), we have \(\{\sigma_{[1,3]}, \tau_{[1,3]}\} \subset \mathcal{Y}_R = \{213, 321\}\).

If \(\{a_j, a_{j+1}\} \neq \{1, 3\}\) then applying the dual Knuth relation determined by \(\{1, 2, 3\}\) on \(\sigma\) and \(\tau\) yields two permutations \(\sigma' \prec \tau'\) in \(\mathcal{Y}_R\) such that \(\sigma' \prec \tau'\) in the right weak Bruhat order. Therefore

\[
V_{[R,R']}(S) \leq_{\text{weak}} V_{[R', R']}(T)
\]

and it must be a covering relation.

Suppose \(\{a_j, a_{j+1}\} = \{1, 3\}\). Since \(\mathcal{Y}_R = \{213, 321\}\), the insertion tableau \(I(a_1 \ldots a_{j-1})\) must have the number 2 its first row left most position. Let for some \(i \leq j - 1\)

\[
b_1 \ldots b_{i-1} 2 b_i \ldots b_j - 1
\]

be the row word of \(I(a_1 \ldots a_{j-1})\) obtained by reading numbers in each row of \(I(a_1 \ldots a_{j-1})\) from left to right, starting from last row. Therefore the sequence \(2 b_{i+1} \ldots b_{j-1}\) labels the first row and moreover

\[
b_1 \ldots b_{i-1} 2 b_{i+1} b_{i+2} \ldots b_{j-1} 1 3 a_{j+2} \ldots a_n \in \mathcal{Y}_S
\]

\[
b_1 \ldots b_{i-1} 2 b_{i+1} b_{i+2} \ldots b_{j-1} 3 1 a_{j+2} \ldots a_n \in \mathcal{Y}_T.
\]

On the other hand since \(2 < b_{i+1} < \ldots < b_{j-1}\) we have

\[
2 b_{i+1} b_{i+2} \ldots b_{j-1} 3 \sim_k 2 b_{i+1} b_{i+2} \ldots b_{j-1}
\]

\[
2 b_{i+1} b_{i+2} \ldots b_{j-1} 3 \sim_k b_{i+1} b_{i+2} 3 1 2 \ldots b_{j-1}
\]

and moreover

\[
b_1 \ldots b_{i-1} 2 b_{i+1} 3 b_{i+2} \ldots b_{j-1} a_{j+2} \ldots a_n \in \mathcal{Y}_S
\]

\[
b_1 \ldots b_{i-1} b_{i+2} 1 3 b_{i+2} \ldots b_{j-1} a_{j+2} \ldots a_n \in \mathcal{Y}_T.
\]
On the other hand applying dual Knuth relation determined by \{1,2,3\} we get
\[ b_1 \ldots b_{i-1} 3 b_{i+1} 2 b_{i+2} \ldots b_{j-1} a_{j+2} \ldots a_n \in \mathcal{Y}_{(R',R)}(S) \text{ and } b_1 \ldots b_{i-1} b_{i+1} 3 1 2 b_{i+2} \ldots b_{j-1} a_{j+2} \ldots a_n \in \mathcal{Y}_{(R',R)}(T) \]
which are clearly the generator of \( \mathcal{V}_{(R',R)}(S) \trianglelefteq \mathcal{V}_{(R',R)}(T) \).

Suppose now \( S \trianglelefteq \mathcal{T} \) is a covering relation in \( SYT_n^{R'} \). Since \( R' = R \), by Proposition 2.13 we have
\[ T^t \trianglelefteq S^t \text{ in } SYT_n^R \text{ by Proposition 2.13} \]
Now by the previous result we have \( \mathcal{V}_{(R,R)}(T^t) \trianglelefteq \mathcal{V}_{(R,R)}(S^t) \) and therefore
\[ \mathcal{V}_{(R,R)}(S) = (\mathcal{V}_{(R,R)}(S^t))^t \trianglelefteq \mathcal{V}_{(R,R)}(T^t))^t = \mathcal{V}_{(R,R)}(T). \]

\[ \square \]

**Proposition 4.3.** Suppose that \( S \trianglelefteq \mathcal{T} \) in \( SYT_n^R \) where \( R \in SYT_k \) has exactly two rows. If \( R' = S \) is another tableau in \( SYT_k \) having the same shape with \( R \) then \( \mathcal{V}_{(R,R)}(S) \trianglelefteq \mathcal{V}_{(R,R)}(T) \) in \( SYT_n^{R'} \).

**Proof.** Suppose that \( S \trianglelefteq \mathcal{T} \) is a covering relation in \( SYT_n^R \) and \( R \in SYT_k \) has two rows. When \( k < 3 \) there is nothing to prove. For \( k = 3 \) the only case that needs to be explored is when \( R \) has non vertical or non horizontal shape, hence Lemma 1.2 gives the desired result.

So we suppose the statement is true for \( k-1 \) and let \( R \in SYT_k \). It is enough to consider the case when \( R \) and \( R' \) differ by only one dual Knuth relation determined by the triple \( \{i, i+1, i+2\} \) where \( i + 2 = k \). If \( i + 2 < k \) then \( S_{i+2} = T_{i+2} \) of \( R \) has still two rows and induction gives the desired result. If \( i + 2 = k \) then we have the following classes of possibilities for the tableau \( R \):

\[
\begin{array}{cccc}
\ast & k-2 & k \\
\ast & k-1 & k \\
\ast & k-2 & k-1 \\
\ast & k-1 & k \\
\ast & k-2 & k-1 \\
\ast & k-1 & k \\
\end{array}
\]

(1.1) (a) (b) (c) (d)

Observe that in the last two classes the dual Knuth relation determined by \{k-2, k-1, k\} interchanges the places of \( k-1 \) and \( k-2 \) and so they refer to the cases with smaller inner tableau \( S_{i+2} = T_{i+2} \) of \( R \) and the induction argument gives the required result.

For the first two classes we have the following analysis: Since \( S \trianglelefteq \mathcal{T} \) there exist \( \sigma \in \mathcal{Y}_S \) and \( \tau \in \mathcal{Y}_T \) such that \( \sigma < \tau \) is also a covering relation the right weak order on \( S_n \) i.e., for some \( 1 \leq j < n \) we have
\[ \sigma = a_1 \ldots a_j a_{j+1} \ldots a_n \text{ and } \tau = a_1 \ldots a_{j+1} a_j \ldots a_n, \text{ where } a_j < a_{j+1}. \]

If \( \{a_j, a_{j+1}\} \neq \{k, k-2\} \) then applying the dual Knuth relation determined by \( \{k, k-1, k-2\} \) on \( \sigma \) and \( \tau \) yields two permutations \( \sigma' \in \mathcal{V}_{(R,R)}(S) \text{ and } \tau' \in \mathcal{V}_{(R,R)}(T) \) which still have \( \sigma' < \tau' \) in the right weak order. Therefore \( \mathcal{V}_{(R,R)}(S) \trianglelefteq \mathcal{V}_{(R,R)}(T) \) and it must be a covering relation.

Now let \( \{a_j, a_{j+1}\} = \{k, k-2\} \), i.e.,
\[ \sigma = a_1 \ldots a_{j-1} (k-2) k a_{j+2} \ldots a_n \text{ and } \tau = a_1 \ldots a_{j-1} k (k-2) a_{j+1} \ldots a_n \]

**Case 1.** We first consider the case illustrated in (1.1)-(a), where \( k-1 \) comes before \( k \) in every permutations in the Knuth classes of \( S \) and \( T \). Therefore the tableau \( I(a_1 \ldots a_{j-1}) \) must have the number \( k-1 \) and it must be located in the first row, since otherwise the number \( k \) drops to the second row of \( T \) at the end of the insertion of \( \tau \) and that is clearly a contradiction. Let for some \( r \leq j-1 \)
\[ b_1 \ldots b_{r-1} (k-1) b_{r+1} b_{r+2} \ldots b_{j-1} \]
be the row word of \( I(a_1 \ldots a_{j-1}) \) obtained by reading numbers in each row of \( I(a_1 \ldots a_{j-1}) \) from left to right, starting from last row. Therefore \( (k-1) b_{r+1} b_{r+2} \ldots b_{j-1} \) lies on the first row and so \( k-1 < b_{r+1} < b_{r+2} < \ldots < b_{j-1}. \)

Now it is easy to see that
\[ b_1 \ldots b_{r-1}(k-1)b_{r+1}b_{r+2} \ldots b_{j-1}(k-2)k a_{j+2} \ldots a_n \sim b_1 \ldots b_{r-1}(k-1)b_{r+1}(k-2)k \ldots b_{j-1}a_{j+2} \ldots a_n \]
lies in the Knuth class \( S \) where as
\[ b_1 \ldots b_{r-1}(k-1)b_{r+1}b_{r+2} \ldots b_{j-1}(k-2)a_{j+2} \ldots a_n \sim b_1 \ldots b_{r-1}(k-1)(k-2)k \ldots b_{j-1}a_{j+2} \ldots a_n \]
lies in the Knuth class of $T$. Moreover applying dual Knuth relation determined by $\{k, k - 1, k - 2\}$ on the latter permutations we get

\[
b_1 \ldots b_{r-1} k b_{r+1}(k-2) (k-1) \ldots b_{j-1} a_{j+2} \ldots a_n \in \mathcal{V}_{[R, R']}(S)
\]

\[
b_1 \ldots b_{r-1} b_{r+1} k (k-2) (k-1) \ldots b_{j-1} a_{j+2} \ldots a_n \in \mathcal{V}_{[R, R']}(T)
\]

and therefore $\mathcal{V}_{[R, R']}(S) \leq_{\text{weak}} \mathcal{V}_{[R, R']}(T)$.

**Case 2.** For the case illustrated in (4.1)-(b), we have

\[
\text{Suppose first that the left most cell in its first row is labeled by a number, say } x, \text{ which is smaller then } k - 1. \text{ This implies that insertion of the sequence } (k-2)k \text{ in to } I(a_n \ldots (k-1) \ldots a_{j+2}) \text{ places the sequence } (k-2)k \text{ to the right of } x \text{ but this contradicts to the fact that the inner tableau } R^t \text{ has at most two columns. Therefore we have } x = k - 1. \text{ Now let for some } r \leq n - j - 1
\]

\[
b_1 \ldots b_{r-1} (k-1) b_{r+1} \ldots b_{n-j-1}
\]

be the row word of $I(a_n \ldots (k-1) \ldots a_{j+2})$. Therefore $(k-1) b_{r+1} b_{r+2} \ldots b_{n-j-1}$ lies on the first row and so $k - 1 < b_{r+1} b_{r+2} \ldots b_{n-j-1}$ which yields

\[
b_1 \ldots b_{r-1} (k-1) b_{r+1} \ldots b_{n-j-1} k (k-2) a_{j+1} \ldots a_1 \sim b_1 \ldots b_{r-1} b_{r+1} (k-1) (k-2) k \ldots b_{n-j-1} a_{j+1} \ldots a_1
\]

lies in $\mathcal{V}_{[R, R']}$ whereas

\[
b_1 \ldots b_{r-1} (k-1) b_{r+1} \ldots b_{n-j-1} (k-2) k a_{j+1} \ldots a_1 \sim b_1 \ldots b_{r-1} b_{r+1} (k-1) (k-2) k \ldots b_{n-j-1} a_{j+1} \ldots a_1
\]

lies in $\mathcal{V}_{T^t}$. Now reversing and then applying dual Knuth relation determined by $\{k, k - 1, k - 2\}$ on the latter permutations we get

\[
a_1 \ldots a_{j-1} b_{n-j-1} \ldots (k-1)(k-2) k b_{r+1} b_{r+2} \ldots b_{n-j-1} \in \mathcal{V}_{[R, R']}(S)
\]

\[
a_1 \ldots a_{j-1} b_{n-j-1} \ldots (k-1)(k-2) b_{r+1} k b_{r+2} \ldots b_{n-j-1} \in \mathcal{V}_{[R, R']}(T)
\]

and therefore $\mathcal{V}_{[R, R']}(S) \leq_{\text{weak}} \mathcal{V}_{[R, R']}(T)$.

Lastly the fact that resulting relations are in fact covering relations follows directly.

\[\square\]

**Corollary 4.4.** Suppose that $S \leq_{\text{weak}} T$ is a covering relation in $\text{SYT}_n^R$ where $R \in \text{SYT}_k$ has exactly two columns. If $R'$ is another tableau in $\text{SYT}_k$ having the same shape with $R$ then $\mathcal{V}_{[R, R']}(S) \leq_{\text{weak}} \mathcal{V}_{[R, R']}(T)$ is also a covering relation in $\text{SYT}_n^{R'}$.

**Proof.** By Proposition 2.13 we have $T^t \leq_{\text{weak}} S^t$ in $\text{SYT}_n^{R^t}$ where $R^t$ has exactly two rows. Now $(R^t)'$ has the same shape with $R^t$ and by previous theorem $\mathcal{V}_{[R', (R')']}(T^t) \leq_{\text{weak}} \mathcal{V}_{[R', (R')']}(S^t)$. Therefore

\[
\mathcal{V}_{[R, R']}(S) = (\mathcal{V}_{[R', (R')']}(S^t))^t \leq_{\text{weak}} (\mathcal{V}_{[R', (R')']}(T^t))^t = \mathcal{V}_{[R, R']}(T).
\]

\[\square\]

**Definition 4.5.** For $T \in \text{SYT}_n$ and $A$ is a corner cell of $T$, denote by

\[
T^{\uparrow A} \text{ and } \eta(T^{\uparrow A})
\]

the tableau obtained by applying reverse insertion algorithm to $T$ through the corner cell $A$ and respectively the number which leaves the tableau at the end.

The following result on the hook shape tableaux is easy to deduce by using reverse RSK algorithm.
**Lemma 4.6.** Let $R \in \text{SYT}_k$ be a tableau of hook shape with more than two rows and two columns and suppose that the only two corner cells of $R$, say $A$ and $B$ are labeled by $k$ or $k-1$. Then

$$\eta(R^T) \neq \eta(R^B)$$

and if $a_1 \ldots a_k$ and $b_1 \ldots b_k$ be two permutations in the Knuth class of $R$ with $a_k = b_k$ then

$$I(a_1 \ldots a_{k-1}) = I(b_1 \ldots b_{k-1}).$$

**Proof.** Since the tableaux required to have more than two rows and two columns it is enough consider the following tableaux together with their transposes, where $k$ labels the cell $A$ and $k-1$ labels the cell $B$.

\[
\begin{array}{c}
* & k-2 & k \\
* & k-1 & k-1 \\
\end{array}
\]

Clearly $\eta(R^T) \neq \eta(R^B)$ and this shows that if $a_k = b_k$ then they must leave the tableau at the end of a reverse insertion applied on the same corner cell, say $A$. Therefore $I(a_1 \ldots a_{k-1}) = R^T = I(b_1 \ldots b_{k-1})$. 

**Proposition 4.7.** Suppose that $S \leq_{\text{weak}} T$ is a covering relation in $\text{SYT}_n^R$ where $R \in \text{SYT}_k$ has hook shape. If $Q$ is another tableau in $\text{SYT}_k$ having the same shape with $R$ then $V_{[R,Q]}(S) \leq_{\text{weak}} V_{[R,Q]}(T)$ is also a covering relation in $\text{SYT}_n^R$.

**Proof.** Here we must deal with the case when $R$ is not a horizontal or a vertical tableau. On the other hand for $k \leq 4$ the only non horizontal or vertical hook shape tableaux have either two rows or two columns and Proposition 4.3 gives the required result. Therefore in the rest we assume that $n > k > 4$ and $R$ has more than two rows and two columns.

We may assume that $R$ and $Q$ differ by only one dual Knuth relation determined by the triple $\{i, i+1, i+2\}$. If $i+2 < k$ then the subtableau $S[i,i+2] = T[i,i+2]$ of $R$ has still hook shape and induction gives the desired result. So let $R$ and $R'$ differ by a single dual Knuth relation determined by $\{k-2, k-1, k\}$. Since $R$ has a hook shape this implies the dual Knuth relation interchanges the places of $k$ and $k-1$ i.e., the only two corner cells of $R$ are occupied by $k$ and $k-1$.

Now since $S \leq_{\text{weak}} T$ is a covering relation, there exist $\sigma \in \mathcal{Y}_S$ and $\tau \in \mathcal{Y}_T$ such that $\sigma < \tau$ and $\sigma$ is also a covering relation the right weak order on $S_n$, i.e., for some $1 \leq j < n$ we have

$$\sigma = a_1 \ldots a_j a_{j+1} \ldots a_n$$

and

$$\tau = a_1 \ldots a_{j+1} a_j \ldots a_n,$$

where $a_j < a_{j+1}$.

If $\{a_j, a_{j+1}\} \neq \{k, k-2\}$ the result follows as discussed in the proof of Proposition 4.3. So in the rest we assume that $\{a_j, a_{j+1}\} = \{k, k-2\}$.

Observe that we have either $\sigma_i = a_1 = \tau_1$ or $\sigma_n = a_n = \tau_n$. WLOG assume $\sigma_n = a_n = \tau_n$ (the first one can be dealt with the same method on the transposes of the tableaux). Therefore there exist some corner cells say $A_S$ and $A_T$ of $S$ and $T$ respectively such that

$$\eta(S^T A_S) = a_n = \eta(T^T A_T)$$

and therefore

$$V_{[R,R']} \left( S' \right) \leq_{\text{weak}} V_{[R,R']} \left( T' \right)$$

and hence

$$V_{[R,R']} \left( S' \right) = V_{[R,R']} \left( S' \right) \leq_{\text{weak}} V_{[R,R']} \left( T' \right) \leq_{\text{weak}} V_{[R,R']} \left( T \right)$$

**Case 1.** If $a_n > k$ then $S'$ and $T'$ have still the same inner tableau $R$. Moreover since $S' \leq_{\text{weak}} T'$, we have by induction

$$V_{[R,R']} (S') \leq_{\text{weak}} V_{[R,R']} (T')$$

and therefore

$$V_{[R,R']} (S) = V_{[R,R']} (S') \leq_{\text{weak}} V_{[R,R']} (T') \leq_{\text{weak}} V_{[R,R']} (T)$$

**Case 2.** If $a_n \leq k$ then the number $a_n$ leaves the tableaux through the sub-tableau $R$ in both reverse insertion $S' = S^T A_S$ and $T' = T^T A_T$. Recall that the only two corners of $R$ is labeled by $k$ and $k-1$. This result by Lemma 4.6 that $a_n$ leaves $R$ as a result of the reverse insertion algorithm applied on the same corner cell say $C$, i.e.,

$$a_n = \eta(R^T C)$$
Recall that $R$ has more than two rows and two columns which leaves us with the following possibilities:

\begin{equation}
\begin{array}{ccc}
* & k-2 & k \\
* & k-2 & * \\
k-1 & k & k \\
\end{array}
\end{equation}

In the first two cases of (4.4), one can observe easily that either $a_n = k$ or $a_n < k - 2$, but $a_n = k$ contradicts to the assumption that \{ \(a_j, a_{j+1}\) = \{k, k - 2\}. Therefore $a_n < k - 2$ and the application of dual Knuth relation determined by \(k - 2, k - 1, k\) to the inner tableau $R' = R^{TC}$ gives $Q' = Q^{TC}$. Now by induction we have $\mathcal{V}_{\mathcal{R}', \mathcal{Q}'}(S') \leq_{\text{weak}} \mathcal{V}_{\mathcal{R}', \mathcal{Q}'}(T')$ and therefore

$$\mathcal{V}_{\mathcal{R}, \mathcal{Q}}(S) = \mathcal{V}_{\mathcal{R}', \mathcal{Q}'}(S') \leq_{\text{weak}} \mathcal{V}_{\mathcal{R}', \mathcal{Q}'}(T') \leq_{\text{weak}} \mathcal{V}_{\mathcal{R}, \mathcal{Q}}(T).$$

In the last two cases (4.3), we have either either $a_n = k - 1$ or $a_n < k - 2$. If $a_n < k - 2$ then the required result follows as above. On the other hand if $a_n = k - 1$ then for the tableau $R' = R^{TC}$ we have the following possibilities

\begin{equation}
\begin{array}{ccc}
* & k-2 & * \\
* & k-2 & * \\
k & k & k \\
\end{array}
\end{equation}

where in both cases every permutation in the Knuth class have the subsequence $k(k - 2)$. This shows that any two permutations in the Knuth class of $R$ that ends with $k - 1$ must have the subsequence $k(k - 2)$ and this again contradicts to the fact that \{ \(a_j, a_{j+1}\) = \{k, k - 2\}.

\[\square\]

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