Comparative production of the scalar and tensor mesons in 
\[ \gamma \gamma^* (Q^2) \rightarrow \eta \pi^0 \] reaction

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Abstract

The prediction of the cross section \( \sigma(\gamma \gamma^* (Q^2) \rightarrow \eta \pi^0) \) based on the simultaneous description of the Belle data on the \( \gamma \gamma \rightarrow \eta \pi^0 \) reaction and the KLOE data on the \( \phi \rightarrow \eta \pi^0 \gamma \) decay is presented.

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I. INTRODUCTION

Study of the nature of light scalars $a_0(980)$ and $f_0(980)$, well-established part of the proposed light scalar mesons nonet [1], is one of the central problems of nonperturbative QCD, it is important for understanding the way chiral symmetry is realized in the low energy region and, consequently, for the understanding of confinement. Naively one might think that the scalar $a_0(980)$ and $f_0(980)$ mesons are also the $q\bar{q}$ P-wave states with the same quark structure as $a_2(1320)$ and $f_2(1270)$, respectively. But now there are many indications that the above scalars are four quark states [2–27].

One of these indications is the suppression of the $a_0(980)$ and $f_0(980)$ resonances in the $\gamma\gamma \rightarrow \eta\pi^0$ and $\gamma\gamma \rightarrow \pi\pi$ reactions, respectively, predicted in 1982 [1], $\Gamma_{a_0\gamma\gamma} \approx \Gamma_{f_0\gamma\gamma} \approx 0.27$ keV, and confirmed by experiment [1]. The elucidation of the mechanisms of the $\sigma(600)$, $f_0(980)$, and $a_0(980)$ resonance production in the $\gamma\gamma$ collisions confirmed their four quark structure [8, 26, 27]. Light scalar mesons are produced in $\gamma\gamma$ collisions mainly via rescatterings, that is, via the four quark transitions. As for the $a_2(1320)$ and $f_2(1270)$, the well-known $q\bar{q}$ states, they are produced mainly via the two quark transitions (direct couplings to $\gamma\gamma$).

Another argument in favor of the four-quark nature of $a_0(980)$ and $f_0(980)$ is the fact that the $\phi(1020) \rightarrow a_0\gamma$ and $\phi(1020) \rightarrow f_0\gamma$ decays go through the kaon loop: $\phi \rightarrow K^+K^- \rightarrow a_0\gamma$, $\phi \rightarrow K^+K^- \rightarrow f_0\gamma$, i.e. via the four quark transition [10, 13–16, 19, 20, 22–25]. The kaon loop model was suggested in Ref. [10] and confirmed by the experiment ten years later [17, 18, 21].

Recently it was proposed in Ref. [28] to study the production of scalar and tensor mesons comparatively to investigate the nature of light scalar mesons in the reactions $e^+e^- \rightarrow \gamma^* \rightarrow (a_0 + a_2)\gamma \rightarrow \gamma\pi^0\eta$ and $e^+e^- \rightarrow \gamma^* \rightarrow (f_0 + f_2)\gamma \rightarrow \gamma\pi^0\pi^0$ (i.e. in the timelike region of $\gamma^*$).

In the given paper we consider the reaction $\gamma^*\gamma \rightarrow \pi^0\eta$ in the spacelike region of $\gamma^*$. In 2009 Belle Collaboration published high-statistical data on the $\gamma\gamma \rightarrow \eta\pi^0$ reaction [29]. These data revealed the specific feature of the $\gamma\gamma \rightarrow \eta\pi^0$ cross section: it turned out sizable in the region between the $a_0(980)$ and $a_2(1320)$ resonances, that certainly indicates the presence of additional contributions. The experimenters took into account the putative heavy isovector scalar $a'_0$ with mass about 1.3 GeV (they called it $a(Y)$) along with $a_0(980)$ and $a_2(1320)$ together with the polynomial coherent background [29].
In the theoretical works Ref. [26, 27] $a_0(980)$ is produced mainly by loops, and the Born contribution to $\gamma \gamma \rightarrow \eta \pi^0$ plays role of coherent background, see Figs. [1] [2]. The kaon formfactor $G_{K^+}(t, u)$ in loop diagrams $\gamma \gamma \rightarrow K^+K^- \rightarrow a_0$ was introduced in those papers.

The given paper is the development of the Refs. [26, 27]. Basing on their theoretical results, we perform new fitting of the Belle data simultaneously with the KLOE data on the $\phi \rightarrow \eta \pi^0 \gamma$ decay. We show that in the kaon loop model of the $\gamma \gamma \rightarrow K^+K^- \rightarrow a_0$ transition the kaon formfactor $G_{K^+}(t, u)$ is not required for good data description, as well as in the $\phi \rightarrow K^+K^- \rightarrow a_0 \gamma \rightarrow \eta \pi^0 \gamma$ decay and $\phi \rightarrow K^+K^- \rightarrow f_0 \gamma \rightarrow \pi^0 \pi^0 \gamma$ [10, 13–25].

The obtained $a_0(980)$ coupling constants are in good agreement with the four-quark model prediction [10].

Using results on $\gamma \gamma \rightarrow \eta \pi^0$ we predict the $\sigma(\gamma^* \gamma \rightarrow \eta \pi^0, s, Q^2)$, where $s$ is the $\gamma^* \gamma$ invariant mass and $Q^2$ is the $\gamma^*$ virtuality.

The measurement of the cross-section $\sigma(\gamma^*(Q^2) \gamma \rightarrow \eta \pi^0)$ would allow additional check of the models of the $a_0(980)$ structure and our understanding of the mechanism of the reactions $\gamma^* \gamma \rightarrow \eta \pi^0$ as well as $\gamma \gamma \rightarrow \eta \pi^0$.

The theoretical description of the $\gamma^* \gamma \rightarrow \eta \pi^0$ reaction may be found in Sec. [II]. The results on the $\gamma \gamma \rightarrow \eta \pi^0$ data description are presented in Sec. [III]. The prediction of the $\sigma(\gamma^* \gamma \rightarrow \eta \pi^0, s, Q)$ is in Sec. [IV]. It is found that the role of vector excitations ($\rho’, \omega’, \rho, \ldots$) is crucial for the $a_2$ production. The conclusion is in Sec. [V].

Note the noticed misprints in Refs. [26, 27] are mentioned in Ref. [30].

II. THEORETICAL DESCRIPTION OF THE $\gamma^* \gamma \rightarrow \eta \pi^0$ REACTION

All formulas for the $\gamma \gamma \rightarrow \eta \pi^0$ reaction were derived in the Refs. [26, 27], these results are used to fit the experimental data. In this section we derive formulas for the $\gamma^*(q) \gamma(k) \rightarrow \eta(q_1)\pi^0(q_2)$ reaction, $Q^2 = -q^2 > 0$. The results of the Refs. [26, 27] are reached in the limit $Q^2 \rightarrow 0$.

According to the Refs. [26, 27], we use a model for the helicity amplitudes $M_\lambda$ ($\lambda$ is the difference between photon helicities), taking into account electromagnetic Born contributions from $\rho, \omega, K^*, K$ exchanges and strong elastic and inelastic final-state interactions in $\pi^0\eta$, $\pi^0\eta’, K^+K^-$, and $K^0\bar{K}^0$ channels, as well as the contributions due to the direct interaction of the resonances with photons.
FIG. 1: Diagrammatical representation for the helicity amplitudes $\gamma^*\gamma \to \pi^0\eta$.

FIG. 2: The Born $\rho, \omega, K^*, K$ exchange diagrams for $\gamma^*\gamma \to \pi^0\eta$, $\gamma^*\gamma \to \pi^0\eta'$, and $\gamma^*\gamma \to K\bar{K}$.
\[
M_0(\gamma^*\gamma \rightarrow \pi^0\eta; s, Q, \theta) = M_0^{\text{Born}}(\gamma^*\gamma \rightarrow \pi^0\eta; s, Q, \theta) + \tilde{I}_{\pi^0\eta}^V(s, Q) T_{\pi^0\eta \rightarrow \pi^0\eta}(s) \\
+ \tilde{I}_{\pi^0\eta'}^V(s, Q) T_{\pi^0\eta' \rightarrow \pi^0\eta}(s) + \left( \tilde{I}_{K^+K^-}^{K^+} (s, Q) - \tilde{I}_{K^0\bar{K}^0}^{K^-} (s, Q) \right) T_{K^+K^- \rightarrow \pi^0\eta}(s) \\
+ M_{\text{res}}(s, Q) + M_0(\gamma^*\gamma \rightarrow a_2(1320) \rightarrow \pi^0\eta; s, Q),
\]
(1)
\[
M_1(\gamma^*\gamma \rightarrow \pi^0\eta; s, Q, \theta) = M_1^{\text{Born}}(\gamma^*\gamma \rightarrow \pi^0\eta; s, Q, \theta) \\
+ M_1(\gamma^*\gamma \rightarrow a_2(1320) \rightarrow \pi^0\eta; s, Q),
\]
(2)
\[
M_2(\gamma^*\gamma \rightarrow \pi^0\eta; s, Q, \theta) = M_2^{\text{Born}}(\gamma^*\gamma \rightarrow \pi^0\eta; s, Q, \theta) \\
+ M_2(\gamma^*\gamma \rightarrow a_2(1320) \rightarrow \pi^0\eta; s, Q),
\]
(3)

the diagrams corresponding to these amplitudes are shown in Figs. 1, 2. Here \(\theta\) is the angle between \(\pi^0\) and \(\gamma\) (or \(\eta\) and \(\gamma^*\)) momenta in the \(\gamma^*\gamma\) center-of-mass system.

The cross-sections \(\sigma_\lambda \equiv \sigma_\lambda(\gamma^*\gamma \rightarrow \eta\pi^0, s, Q)\) are related to the amplitudes \(M_\lambda\) as
\[
\sigma_\lambda = \frac{\rho_{\pi\eta}(s)}{64\pi s \rho_{\gamma^*\gamma}} \int_{0.8}^{0.8} |M_\lambda|^2 d\cos\theta, \quad \rho_{\gamma^*\gamma} = 1 + Q^2/s,
\]
(4)
\[
\bar{\sigma}_\lambda = \frac{1}{2a} \int_{\sqrt{s}+a}^{\sqrt{s}-a} \sigma_\lambda(s')d\sqrt{s'},
\]
(5)
and the full cross-section \(\sigma(\gamma^*\gamma \rightarrow \eta\pi^0, s, Q) = \sigma_0 + \sigma_1 + \sigma_2\). The limits \(|\cos\theta| \leq 0.8\) and bin size \(2a = 20\text{ MeV}\) are in experiment Ref. [29], where the data on \(\bar{\sigma}_0 + \bar{\sigma}_2\) at \(Q = 0\) are presented for different values of \(\sqrt{s}\) \((\sigma_1 = \bar{\sigma}_1 = 0\text{ at } Q = 0)\).

The Born term \(M_0^{\text{Born}}\) is caused by equal contributions of the \(\rho\) and \(\omega\) exchange mechanisms [9, 11]:
\[
M_0^{\text{Born}}(\gamma^*\gamma \rightarrow \pi^0\eta; s, Q, \theta) = 2g_{\omega\pi\gamma}g_{\omega\pi\eta}F_{\eta\pi^0}^{\text{Born}}(Q) \left[ \frac{A_0(s, t, Q, m_\eta, m_\pi)G_\omega(s, t)}{t - m_\omega^2} \right. \\
+ \left. \frac{A_0(s, u, Q, m_\pi, m_\eta)G_\omega(s, u)}{u - m_\omega^2} \right],
\]
(6)
\[
M_1^{\text{Born}}(\gamma^*\gamma \rightarrow \pi^0\eta; s, Q, \theta) = 2g_{\omega\pi\gamma}g_{\omega\pi\eta}F_{\eta\pi^0}^{\text{Born}}(Q) \left[ \frac{A_1(s, t, Q, m_\eta, m_\pi)G_\omega(s, t)}{t - m_\omega^2} \right. \\
+ \left. \frac{A_1(s, u, Q, m_\pi, m_\eta)G_\omega(s, u)}{u - m_\omega^2} \right],
\]
(7)
\[
M_2^{\text{Born}}(\gamma^*\gamma \rightarrow \pi^0\eta; s, Q, \theta) = \frac{2g_{\omega\pi\gamma}g_{\omega\pi\eta}F_{\eta\pi^0}^{\text{Born}}(Q)}{4} \left( m_\pi^2m_\omega^2 - tu \right)
\]
+ \frac{Q^2}{s+Q^2} \left( t - m_\pi^2 \right) \left[ \frac{G_\omega(s,t)}{t-m_\omega^2} + \frac{G_\omega(s,u)}{u-m_\omega^2} \right],

(8)

where

\begin{align*}
A_0(s, t, Q, m_1, m_2) &= \frac{t(s + Q^2)}{4} + \frac{Q^2(t - m_2^2)^2}{4(s + Q^2)}, \\
A_1(s, t, Q, m_1, m_2) &= Q \sqrt{-t + \frac{(m_2^2 - m_1^2 - Q^2)^2}{4s}},
\end{align*}

t and u are the Mandelstam variables for the reaction \( \gamma^* \gamma \to \eta \pi^0 \):

\begin{align*}
t &= m_\pi^2 - \frac{1 + Q^2/s}{2} \left( s + m_\pi^2 - m_\eta^2 - s \rho_{\eta\pi\eta} \cos \theta \right), \\
u &= m_\eta^2 - \frac{1 + Q^2/s}{2} \left( s + m_\eta^2 - m_\pi^2 + s \rho_{\eta\pi\eta} \cos \theta \right),
\end{align*}

here and below \( \rho_{ab}(s) = 2p_{ab}(s)/\sqrt{s} = (1 - m_a^2/s)(1 - m_b^2/s), m_\pm = m_a \pm m_b \) \((ab = \eta\pi^0, K^+K^-, K^0\bar{K}^0, \eta'\pi^0)\), where \( p_{ab} \) is the module of the momentum of \( a \) (or \( b \)) particles in the s.c.m., \( g_{\omega\pi\pi}^2 = 12\pi \Gamma_{\omega \to \pi\pi} [(m_\omega^2 - m_\pi^2)/(2m_\omega)]^{-3} \approx 0.519 \text{ GeV}^{-2} \), \( g_{\omega\eta\gamma}^2 = 12\pi \Gamma_{\omega \to \eta\gamma} [(m_\omega^2 - m_\eta^2)/(2m_\omega)]^{-3} \approx 1.86 \times 10^{-2} \text{ GeV}^{-2} \). According to Vector Dominance Model the factor \( F_{\eta\pi^0}^{\text{Born}}(Q) \) dealing with resonance excitation in the \( \gamma^* \) leg is

\begin{equation}
F_{\eta\pi^0}^{\text{Born}}(Q) = \frac{1}{2} \left( \frac{1}{1 + Q^2/m_\rho^2} + \frac{1}{1 + Q^2/m_\omega^2} \right) \tag{9}
\end{equation}

In the corresponding Born amplitudes for \( \gamma^* \gamma \to \pi^0 \eta' \) the constant \( g_{\omega\eta'\gamma}^2 = 4\pi \Gamma_{\omega \to \eta'\gamma} [(m_{\eta'}^2 - m_\omega^2)/(2m_\omega)]^{-3} \approx 1.86 \times 10^{-2} \text{ GeV}^{-2} \) \[1\], \( F_{\eta'\pi^0}^{\text{Born}}(Q) = F_{\eta\pi^0}^{\text{Born}}(Q) \). As in Refs. \[26,27\], we take the formfactor \( G_\omega(s, t) = G_\rho(s, t) \) of forms

\begin{equation}
G_\omega(s, t) = \exp[(t - m_\omega^2)b_\omega(s)], \tag{10}
\end{equation}

\begin{equation}
b_\omega(s) = b_\omega^0 + (\alpha'_\omega/4) \ln[1 + (s/s_0)^4], \tag{11}
\end{equation}

where \( b_\omega^0 = 0, \ \alpha'_\omega = 0.8 \text{ GeV}^{-2} \), and \( s_0 = 1 \text{ GeV}^2 \). Form factors for the \( K^* \) exchange result from the above by substitution of \( m_{K^*} \) for \( m_\omega \), other parameters are the same.
The function $\tilde{I}^\nu_{s\eta}(s, Q)$ reads

$$\tilde{I}^\nu_{s\eta}(s, Q) = \frac{8}{\pi} \int_{(m_{\eta} + m_\omega)^2}^{\infty} ds' \rho_{\eta\eta}(s') \frac{M_{00}^{\text{Born}}(\gamma^*\gamma \rightarrow \pi^0\eta; s, Q)}{s'(s' - s - i\varepsilon)}$$

(12)

$$M_{00}^{\text{Born}}(\gamma^*\gamma \rightarrow \pi^0\eta; s, Q) = \frac{1}{2} \int_{-1}^{1} M_0^{\text{Born}}(\gamma^*\gamma \rightarrow \pi^0\eta; s, Q, \theta) \, d\cos \theta$$

(13)

The $\tilde{I}^\nu_{s\eta}(s, Q), \tilde{I}^{K^++}_{K^+K^-}(s, Q)$ and $\tilde{I}^{K^-o}_{K^0K^0}(s, Q)$ are built analogously. The amplitudes $M_{0}^{\text{Born}K^+}(\gamma^*\gamma \rightarrow K\bar{K}; s, \theta)$ for the $K^*$ exchanges result from Eqs. (6), (7), (8) with the use of substitutions $m_\omega \rightarrow m_{K^*}, G_\omega \rightarrow G_{K^*}, m_\pi \rightarrow m_K, m_\eta \rightarrow m_K$, and $2g_{\omega\pi\gamma}g_{\omega\eta\gamma} \rightarrow g_{K^*K\gamma}^2$, where $g_{K^+K^-K^+}^{2} \approx 0.064$ GeV$^{-2}$ and $g_{K^0K^0K^0}^{2} \approx 0.151$ GeV$^{-2}$. The factors $F_{KK}^{\text{Born}K^+}(Q)$ and $F_{KK}^{\text{Born}K^-}(Q)$ may be taken in our case as

$$F_{KK}^{\text{Born}K^+}(Q) = \frac{3/2}{1 + Q^2/m_\rho^2} + \frac{1/2}{1 + Q^2/m_\omega^2} - \frac{1}{1 + Q^2/m_\phi^2};$$

(14)

$$F_{KK}^{\text{Born}K^-}(Q) = \frac{3/4}{1 + Q^2/m_\rho^2} - \frac{1/4}{1 + Q^2/m_\omega^2} + \frac{1/2}{1 + Q^2/m_\phi^2};$$

(15)

The $\tilde{I}^{K^+}_{K^+K^-}(s, Q)$ is

$$\tilde{I}^{K^+}_{K^+K^-}(s, Q) = -F_K(Q)\alpha\left\{1 + \frac{Q^2}{s + Q^2} \left( \rho_{K^+K^-}(s) \left( \ln \frac{1 + \rho_{K^+K^-}(s)}{1 - \rho_{K^+K^-}(s)} - i\pi \right) - \frac{m_{K^+}^2}{Q^2} \left( -\ln^2 \frac{1 + \rho_{K^+K^-}(s)}{1 - \rho_{K^+K^-}(s)} - \ln \rho_{K^+K^-}(s) \right) \right) \right\}$$

(16)

$$F_K(Q) = \frac{1/2}{1 + Q^2/m_\rho^2} + \frac{1/6}{1 + Q^2/m_\omega^2} + \frac{1/3}{1 + Q^2/m_\phi^2};$$

(17)

The amplitudes of the pseudoscalar pairs rescattering are

$$T_{s\eta \rightarrow s\eta}(s) = T_0^1(s) = \frac{\eta_0(s)e^{i\delta_0(s)} - 1}{2i\rho_{s\eta}(s)} =$$

(18)
\[ T^{bg}_{\pi\eta}(s) + e^{2i\delta^{bg}_{\pi\eta}(s)}T^{res}_{\pi^0\eta^0}(s), \]  
\[ T^{\pi^0\eta^0\rightarrow\pi^0\eta^0}(s) = T^{res}_{\pi^0\eta^0\rightarrow\pi^0\eta^0}(s) e^{i[\delta^{bg}_{\pi^0\eta^0}(s)+\delta^{bg}_{\pi^0\eta^0}(s)]}, \]  
\[ T_K^{K^+K^0\rightarrow\pi^0\eta^0}(s) = T^{res}_{K^+K^0\rightarrow\pi^0\eta^0}(s) e^{i[\delta^{bg}_{KK}(s)+\delta^{bg}_{K\bar{K}}(s)]}, \]

where \( T^{bg}_{\pi\eta}(s) = (e^{2i\delta^{bg}_{\pi\eta}(s)} - 1)/(2i\rho_{\pi\eta}(s)), \) \( T^{res}_{\pi^0\eta^0\rightarrow\pi^0\eta^0}(s) = (\eta_0^{1}(s)e^{2i\delta^{res}_{\pi^0\eta^0}(s)} - 1)/(2i\rho_{\pi^0\eta^0}(s)), \) \( \delta^{01}_{\pi\eta}(s) = \delta^{bg}_{\pi\eta}(s) + \delta^{res}_{\pi^0\eta^0}(s), \delta^{bg}_{\pi\eta}(s), \delta^{bg}_{\pi^0\eta^0}(s), \) and \( \delta^{bg}_{KK}(s), \delta^{bg}_{K\bar{K}}(s) \) are the phase shifts of the elastic background contributions in the channels \( \pi\eta, \pi\eta', \) and \( KK \) with isospin \( I = 1, \) respectively.

When \( a'_0 \) is taken into account the resonant amplitudes of the processes \( ab \rightarrow \eta^0\pi^0 \) are

\[ T^{res}_{ab\rightarrow\eta^0\pi^0}(s) = \sum_{R,R'} \frac{g_{Rab}G^{-1}_{RRe}g_{R'e\eta}m^0}{16\pi}, \]

where \( R, R' = a_0, a'_0 \) and pair \( ab = \gamma\gamma, \eta^0\pi, K^+K, \eta^0\pi. \) In case of \( ab = \gamma\gamma \) the constants \( g_{Rab} \equiv g_{Rab}^{(0)} \) are related to the width as

\[ \Gamma^{(0)}_{R\rightarrow\gamma\gamma} = \frac{|m^2_R g^{(0)}_{(0)}|^2}{16\pi m_R} \]  

The matrix of the inverse propagators is

\[ G_{RR'} \equiv G_{RR'}(m) = \begin{pmatrix} D_{a'_0}(m) & -\Pi_{a'_0a}(m) \\ -\Pi_{a'_0a}(m) & D_{a}(m) \end{pmatrix}, \]

\[ \Pi_{a'_0a}(m) = \sum_{a,b} \frac{g_{a'b}}{g_{a'a}} \Pi_{a}(m) + C_{a'0a}, \]

where \( m \) is the invariant mass of the \( \eta^0\pi^0 \) system, \( m^2 = s, \) the constant \( C_{a'_0a} \) incorporates the subtraction constant for the transition \( a_0(980) \rightarrow (0^-0^-) \rightarrow a'_0 \) and effectively takes into account contribution of multi-particle intermediate states to \( a_0 \leftrightarrow a'_0 \) transition, see Ref. 13. The inverse propagator of the R scalar meson is presented also in Refs. 3, 5, 10, 13:

\[ D_R(m) = m^2_R - m^2 + \sum_{ab}[Re\Pi^{ab}_{R}(m^2_R) - \Pi^{ab}_{R}(m^2)], \]

where \( \sum_{ab}[Re\Pi^{ab}_{R}(m^2_R) - \Pi^{ab}_{R}(m^2)] = Re\Pi_{R}(m^2_R) - \Pi_{R}(m^2) \) takes into account the finite width corrections of the resonance which are the one loop contribution to the self-energy of the \( R \) resonance from the two-particle intermediate \( ab \) states.
For pseudoscalar $a, b$ mesons and $m_a \geq m_b$, $m \geq m_+$ one has:

$$
\Pi^{ab}_R(m^2) = \frac{g^2_{Rab}}{16\pi} \left[ \frac{m_+ m_-}{\pi m^2} \ln \frac{m_b}{m_a} + \right.
$$

$$
+ \rho_{ab} \left( i + \frac{1}{\pi} \ln \frac{\sqrt{m^2 - m_-^2} - \sqrt{m^2 - m_+^2}}{\sqrt{m^2 - m_-^2} + \sqrt{m^2 - m_+^2}} \right) \right]
$$

(25)

$m_- \leq m < m_+$

$$
\Pi^{ab}_R(m^2) = \frac{g^2_{Rab}}{16\pi} \left[ \frac{m_+ m_-}{\pi m^2} \ln \frac{m_b}{m_a} - |\rho_{ab}(m)| + 
$$

$$
+ \frac{2}{\pi} |\rho_{ab}(m)| \arctan \frac{\sqrt{m_+^2 - m_-^2}}{\sqrt{m_-^2}} \right].
$$

(26)

$m < m_-$

$$
\Pi^{ab}_R(m^2) = \frac{g^2_{Rab}}{16\pi} \left[ \frac{m_+ m_-}{\pi m^2} \ln \frac{m_b}{m_a} - \right.
$$

$$
- \frac{1}{\pi} \rho_{ab}(m) \ln \frac{\sqrt{m_+^2 - m_-^2} - \sqrt{m_-^2 - m_+^2}}{\sqrt{m_+^2 - m_-^2} + \sqrt{m_-^2 - m_+^2}} \right].
$$

(27)

The constants $g_{Rab}$ are related to the width

$$
\Gamma_R(m) = \sum_{ab} \Gamma(R \rightarrow ab, m) = \sum_{ab} \frac{g^2_{Rab}}{16\pi m} \rho_{ab}(m).
$$

(28)

Note that we take into account intermediate states $\eta\pi^0, K\bar{K}, \eta'\pi^0$ in the $a_0(980)$ and $a'_0$ propagators:

$$
\Pi_{a_0} = \Pi_{a_0}^{\eta\pi^0} + \Pi_{a_0}^{K^+K^-} + \Pi_{a_0}^{K^0\bar{K}^0} + \Pi_{a_0}^{\eta'\pi^0},
$$

(29)

and the same for $a'_0$. Note that $g_{a_0K^+K^-} = -g_{a_0K^0\bar{K}^0}$ and $g_{a'_0K^+K^-} = -g_{a'_0K^0\bar{K}^0}$.

For the background phase shifts we use the parametrizations from Refs. 26, 27:

$$
e^{2i\delta_{ab}^{\eta\pi}(s)} = \frac{1 + iF_{ab}(s)}{1 - iF_{ab}(s)},
$$

(30)

where

$$
F_{\eta\pi}(s) = \frac{\sqrt{1 - (m_\eta + m_\pi)^2/s} (c_0 + c_1 (s - (m_\eta + m_\pi)^2))}{1 + c_2 (s - (m_\eta + m_\pi)^2)^2},
$$

(31)

$$
F_{K\bar{K}}(s) = f_{K\bar{K}} \sqrt{s - 4m_{K^+}^2},
$$

(32)

$$
F_{\eta'\pi'}(s) = f_{\eta'\pi'} \sqrt{s - (m_{\eta'} + m_\pi)^2}.
$$

(33)
Note that analytical continuation of the phases under the thresholds changes modules of corresponding amplitudes. The parameterization of $F_{K\bar{K}}(s)$ slightly differs from Refs. \[26, 27\].

The amplitude

$$M_{\text{res}}^{\text{direct}}(s, Q) = 16\pi s T_{\gamma\gamma\rightarrow\eta\pi}(s) F_{\gamma\gamma\rightarrow\eta\pi}(s) e^{i\delta_{\eta\pi}(s)}$$

(34)

$$F_{\text{direct}}(s, Q) = \frac{1}{2} \left( \frac{1}{1 + Q^2/m_{\rho}^2} + \frac{1}{1 + Q^2/m_{\omega}^2} \right) (1 + Q^2/s)$$

(35)

describes the $\gamma^*\gamma \rightarrow \pi^0\eta$ transition caused by the direct coupling constants of the $a_0$ and $a_0'$ resonances to photons $g_{a_0\gamma\gamma}^{(0)}$ and $g_{a_0'\gamma\gamma}^{(0)}$: the factor $s(1 + Q^2/s)$ appears due to the gauge invariance (the effective Lagrangian $\sim F_{\mu\nu} F^{\mu\nu} a_0$, $M_{\text{res}}^{\text{direct}}(s, Q) \sim (kq)$).

It is known that in the reaction $\gamma^*\gamma \rightarrow a_2 \rightarrow \eta\pi$ tensor mesons are produced mainly by the photons with the opposite helicity states. The effective Lagrangian in this case is

$$L = g_{a_2\gamma\gamma} T_{\mu\nu} F_{\mu\sigma} F_{\nu\sigma},$$

(36)

$$F_{\mu\sigma} = \partial_{\mu} A_{\sigma} - \partial_{\sigma} A_{\mu}$$

where $A_{\mu}$ is a photon field and $T_{\mu\nu}$ is a tensor $a_2$ field. So in the frame of Vector Dominance Model we assume that the effective Lagrangian of the reaction $a_2 \rightarrow V(1)V(2)$ is \[31\]:

$$L = g_{a_2V(1)V(2)} T_{\mu\nu} F_{\mu\sigma}^{V(1)} F_{\nu\sigma}^{V(2)};$$

(37)

$$F_{\mu\sigma}^{V(i)} = \partial_{\mu} V(i)_{\sigma} - \partial_{\sigma} V(i)_{\mu}; \ i = 1, 2.$$ 

Matrix elements for $a_2(1320)$ contribution are Ref. \[31\]:

$$M_2(\gamma^*\gamma \rightarrow a_2(1320) \rightarrow \pi^0\eta; s, Q, \theta) = A(s, Q) \sin^2 \theta$$

(38)

$$M_1(\gamma^*\gamma \rightarrow a_2(1320) \rightarrow \pi^0\eta; s, Q, \theta) = -\sqrt{2} A(s, Q) \sqrt{\frac{Q^2}{s}} \sin \theta \cos \theta$$

(39)

$$M_0(\gamma^*\gamma \rightarrow a_2(1320) \rightarrow \pi^0\eta; s, Q, \theta) = -A(s, Q) \frac{Q^2}{3s}$$

(40)
A(s, Q) = 20\pi F_{a_2}(Q) \sqrt{\frac{6s\Gamma_{a_2\rightarrow\gamma\gamma}(s)\Gamma_{a_2\rightarrow\eta\pi^0}(s)}{\rho_{\eta\pi^0}(s)}} \frac{1}{D_{a_2}(s)} \left(1 + \frac{Q^2}{s}\right) \quad (41)

The $F_{a_2}(Q)$ is defined and discussed in Sec. IV. The $F_{a_2}(0) = 1$, and the $F_{a_2}$ dependence on $Q$ does not influence on the $\gamma\gamma \rightarrow \eta\pi^0$ process. Note that Eq. (41) differs from Eq. (3) in Ref. 31 because of normalization of the amplitude and the sign of $g_{a_2\gamma\gamma}$, which is taken negative. The inverse sign does not lead to essential consequences for fitting the data and analytical continuation to non-zero $Q^2$.

According to Refs. 26, 27,

$$D_{a_2}(m^2) = m_{a_2}^2 - m^2 - im\Gamma_{a_2}(m),$$

where

$$\Gamma_{a_2}(m) = \Gamma^{tot}_{a_2} \frac{p^2_{\eta\pi}(m)}{m^2} \frac{D_2(r_{a_2\eta\pi}(m_{a_2}))}{D_2(r_{a_2\eta\pi}(m))},$$

and

$$\Gamma_{a_2\rightarrow\eta\pi^0}(m) = Br(a_2 \rightarrow \eta\pi^0)\Gamma_{a_2}(m)$$

Here $D_2(x) = 9 + 3x^2 + x^4$ [32], and we take $m_{a_2} = 1318.3$ MeV, $\Gamma^{tot}_{a_2} = 105$ MeV, and $Br(a_2 \rightarrow \eta\pi^0) = 0.145$ from Ref. 1. $\Gamma_{a_2\rightarrow\gamma\gamma}(s) = (\sqrt{s}/m_{a_2})^3\Gamma_{a_2\rightarrow\gamma\gamma}(m_{a_2})$, $Br(a_2 \rightarrow \gamma\gamma) = 9.4 \times 10^{-6}$ [1]. We also take $r_{a_2} = 1.9$ GeV$^{-1}$ from Refs. 26, 27.

The theoretical description of the KLOE data Ref. 21 on the mass spectrum $dBr(\phi \rightarrow \gamma\pi^0\eta, m)/dm$ is the same as in Ref. 22, with obvious change

$$\frac{g_{a_0K^+K^-a_0\eta\pi^0}}{D_{a_0}(m)} \rightarrow \sum_{R,R'} g_{R^0K^+K^-R'R} g_{R'R\eta\pi^0},$$

so the amplitude of the signal process $\phi(p) \rightarrow \gamma(a_0 + a'_0) \rightarrow \gamma(q)\pi^0(k_1)\eta(k_2)$ is

$$M_{sig} = e^{i\delta_B} g(m) \sum_{R,R'} g_{R^0K^+K^-R'R} g_{R'R\eta\pi^0} \left(\phi\epsilon - \frac{(\phi q)(\epsilon p)}{(pq)}\right),$$

where $m^2 = (k_1 + k_2)^2$, $\phi_\alpha$ and $\epsilon_\mu$ are the polarization vectors of $\phi$ meson and photon, the function $g(m)$ is given below. The $\delta_B = \delta_{\eta\pi^0}^bg + \delta_{KK}^bg$.

The matrix element of the background process $\phi(p) \rightarrow \pi^0\rho^0 \rightarrow \gamma(q)\pi^0(k_1)\eta(k_2)$ is
The mass spectrum is

\[
\frac{d\Gamma(\phi \to \gamma\pi^0\eta, m)}{dm} = \frac{d\Gamma_{\text{sig}}(m)}{dm} + \frac{d\Gamma_{\text{back}}(m)}{dm} + \frac{d\Gamma_{\text{int}}(m)}{dm},
\]

where the mass spectrum for the signal is

\[
\frac{d\Gamma_{\text{sig}}(m)}{dm} = \frac{2|g(m)|^2 p_{\eta\pi}(m^2_\phi - m^2)}{3(4\pi)^3 m^3_\phi} \left| \sum_{R,R'} g_{R+K} G_{R'\gamma\eta\phi} g_{R'\eta\phi} \right|^2.
\]

The mass spectrum for the background process \(\phi \to \pi^0\rho \to \gamma\pi^0\eta\) is [19]:

\[
\frac{d\Gamma_{\text{back}}(m)}{dm} = \frac{(m^2_\phi - m^2)p_{\pi\eta}}{128\pi^3 m^3_\phi} \int_{-1}^1 dx A_{\text{back}}(m, x),
\]

where

\[
A_{\text{back}}(m, x) = \frac{1}{3} \sum |M_B|^2 = \frac{1}{24} (m_\eta^4 m_\pi^4 + 2m^2 m_\eta^2 m_\pi^2 m_\rho^2 - 2m_\eta^4 m_\pi^2 m_\rho^2 - 2m_\eta^2 m_\pi^2 m_\rho^2 + 2m^4 m_\rho^4 - 2m^2 m_\eta^2 m_\rho^4 + m^4 m_\rho^4 - 2m^2 m_\eta^2 m_\rho^4 + 4m_\eta^2 m_\pi^2 m_\rho^4 + 4m^4 m_\rho^4 + 2m^2 m_\eta^6 - 2m_\eta^2 m_\pi^6 - 2m_\pi^2 m_\rho^6 + m_\rho^8 - 2m_\eta^4 m_\pi^2 m_\rho^2 - 2m^2 m_\eta^2 m_\rho^2 + 2m_\eta^2 m_\pi^2 m_\rho^2 - 2m_\eta^2 m_\pi^2 m_\rho^4 + 2m^2 m_\eta^2 m_\rho^4 - 2m^2 m_\pi^2 m_\rho^6 + m^4 m_\rho^4 + m^4 m_\rho^4) \times \frac{|g_{\phi\pi\pi} g_{\rho\gamma}|^2}{|D_{\rho}(m_\rho)|^2},
\]

and

\[
m_\rho^2 = m^2_\eta + \frac{(m^2 + m^2_\eta - m^2_\pi)(m^2_\phi - m^2)}{2m^2} - \frac{(m^2_\phi - m^2)x}{m} p_{\pi\eta} \quad \text{and} \quad p_{\pi\eta} = \sqrt{(m^2 - (m_\eta - m_\pi)^2)(m^2 - (m_\eta + m_\pi)^2)} / 2m.
\]

The term of the interference between the signal and the background processes is written in the following way:

\[
\frac{d\Gamma_{\text{int}}(m)}{dm} = \frac{(m^2_\phi - m^2)p_{\pi\eta}}{128\pi^3 m^3_\phi} \int_{-1}^1 dx A_{\text{int}}(m, x),
\]
where

\[ A_{\text{int}}(m, x) = \frac{2}{3} Re \sum M_{\text{sig}} M_B^* = \frac{1}{3} \left( (m^2 - m_{\phi}^2)\bar{m}_\rho^2 + \frac{m_{\phi}^2(m_{\rho}^2 - m_{\eta}^2)^2}{m_{\phi}^2 - m^2} \right) \times \]
\[ Re\left\{ e^{i\delta} g(m) \left( \sum_{R, R'} g_{R \rightarrow K^+ - K^-}^{-1} Re g_{R' \rightarrow K^-} g_{\phi \rightarrow \rho \phi} \right) \right\} D_{\rho}^*(\bar{m}_\rho) \]. \tag{53} \]

The \( \delta \) is additional relative phase between \( M_{\text{sig}} \) and \( M_B \), does not mentioned in Eqs. (45) and (46). It is assumed to be constant and takes into account, for example, \( \rho \pi \) rescattering effects, see [33].

In the \( \phi \rightarrow K^+ K^- \rightarrow a_0 \gamma \) loop model \( g(m) \) has the following forms:

For \( m < 2m_{K^+} \)

\[ g(m) = \frac{e}{2(2\pi)^2} g_{\phi \rightarrow K^+ K^-} \left\{ 1 + \frac{1 - \rho_{K^+ K^-}^2(m^2)}{\rho_{K^+ K^-}^2(m_{\phi}^2) - \rho_{K^+ K^-}^2(m^2)} \times \right. \]
\[ \left[ 2|\rho_{K^+ K^-}(m^2)| \arctan \frac{1}{|\rho_{K^+ K^-}(m_{\phi}^2)|} - \rho_{K^+ K^-}(m_{\phi}^2) \lambda(m_{\phi}^2) + i\pi \rho_{K^+ K^-}(m_{\phi}^2) - \right. \]
\[ \left. -(1 - \rho_{K^+ K^-}(m_{\phi}^2)) \left( \frac{1}{4}(\pi + i\lambda(m_{\phi}^2))^2 - \right. \right. \]
\[ \left. \left. \left( \arctan \frac{1}{|\rho_{K^+ K^-}(m_{\phi}^2)|} \right)^2 \right\} \right\}, \tag{54} \]

where

\[ \lambda(m^2) = \ln \frac{1 + \rho_{K^+ K^-}(m^2)}{1 - \rho_{K^+ K^-}(m^2)} \quad ; \quad \frac{e^2}{4\pi} = \alpha = \frac{1}{137}. \tag{55} \]

For \( m \geq 2m_{K^+} \)

\[ g(m) = \frac{e}{2(2\pi)^2} g_{\phi \rightarrow K^+ K^-} \left\{ 1 + \frac{1 - \rho_{K^+ K^-}^2(m^2)}{\rho_{K^+ K^-}^2(m_{\phi}^2) - \rho_{K^+ K^-}^2(m^2)} \times \right. \]
\[ \times \left[ \rho_{K^+ K^-}(m^2)(\lambda(m^2) - i\pi) - \rho_{K^+ K^-}(m_{\phi}^2)(\lambda(m_{\phi}^2) - i\pi) - \right. \]
\[ \left. \frac{1}{4}(1 - \rho_{K^+ K^-}^2(m_{\phi}^2)) \left( (\pi + i\lambda(m_{\phi}^2))^2 - (\pi + i\lambda(m_{\phi}^2))^2 \right) \right\}. \tag{56} \]

Note that \( g(m) \rightarrow -\tilde{I}_{K^+ K^-}(s, Q)/16\pi e, \) see Eq. (16), for \( m^2 \rightarrow s, m_{\phi}^2 \rightarrow -Q^2. \)

### III. RESULTS OF THE DATA DESCRIPTION

Using the above theoretical framework we fit the data Ref. [29] and Ref. [21] and obtain results shown in Table I and Figs. 3, 4, and 5.
FIG. 3: The $\gamma \gamma \rightarrow \eta \pi^0$ cross section, $|\cos \theta| < 0.8$. The curves correspond to Fit 1, points are the Belle data [29]. The curve on the a) figure represents cross section as is, while the curve on the b) figure represents averaged cross section: each point of the curve is the cross section averaged over the $\pm 10$ MeV neighborhood.

The function to minimize, $\chi^2$ function, in the present paper consists of three terms:

$$
\chi^2 = \chi^2_{\gamma\gamma} + \chi^2_{\text{spec}} + \chi^2_{\text{add}}.
$$

(57)

The $\chi^2_{\gamma\gamma}$ and $\chi^2_{\text{spec}}$ are usual $\chi^2$ functions for the $\sigma(\gamma \gamma \rightarrow \eta \pi^0, s)$ and $\eta \pi^0$ spectrum of the $\phi \rightarrow \eta \pi^0 \gamma$ reaction.

The $\chi^2_{\text{add}}$ represents additional terms to reach some desired aims: lower $f_{K\bar{K}}$ to reduce influence of the analytical continuation of the $e^{i\delta_{bg}}_{K\bar{K}}$ under the $K\bar{K}$ threshold on the $|M_{sig}|$, see Eq. (45); coupling constants close to the relations Eq. (58); not large $g_{a_0\gamma\gamma}$ and not very large $f_{\pi\eta'}$. The $\chi^2_{\text{add}}$ was used to obtain Fits 1, 3, and 4, for the Fit 2 $\chi^2_{\text{add}} = 0$.

In the Fit 1 the $m_{a_0} = 1400$ MeV is in the middle of agreed corridor $\approx 1300 - 1500$ MeV [1], the coupling constant relations are close to the naive four quark model predictions [10]:

$$
g_{a_0\eta\pi^0} = \sqrt{2}\sin(\theta_p + \theta_q)g_{a_0K^+K^-} = (0.85 \div 0.98)g_{a_0K^+K^-}
$$

$$
g_{a_0\pi'\pi^0} = -\sqrt{2}\cos(\theta_p + \theta_q)g_{a_0K^+K^-} = -(1.13 \div 1.02)g_{a_0K^+K^-}
$$

(58)

In brackets the first values correspond to $\theta_p = -18^\circ$ and the second ones to $\theta_p = -11^\circ$. The $\theta_q = 54.74^\circ$. 

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FIG. 4: Plot of the Fit 1 curve and the KLOE data (points) on the $\phi \rightarrow \eta\pi^0\gamma$ decay. Cross points are omitted in fitting. The box point (0.999 GeV) was omitted in Fits 3 and 4 of Ref. [22], but we keep it.

FIG. 5: The comparison of the KLOE data on $\phi \rightarrow \eta\pi^0\gamma$ decay and Fit 1. Histograms show Fit 1 curve averaged over each bin for a) $\phi \rightarrow \eta\pi^0\gamma$, $\eta \rightarrow \gamma\gamma$ and b) $\phi \rightarrow \eta\pi^0\gamma$, $\eta \rightarrow \pi^+\pi^-\pi^0$ samples, see details in Ref. [22].

In Fit 1 direct couplings of $a_0$ and $a_0'$ with the $\gamma\gamma$ channel are close to values in Refs. [26, 27].

One can see that the quality of experimental data description is good, see also Figs. 3, 4 and 5. This means that the data agrees with the four quark model scenario.

The Fit 2 is the result of minimization of the classical $\chi^2$ function, $\chi^2_{cl} = \chi^2_{\gamma\gamma} + \chi^2_{spec}$, without additional terms. Only $m_{a_0'}$ was fixed, $m_{a_0'} = 1400$ MeV, all the other parameters were free. One can see that the coupling constants go not too far from the four quark model prediction, and kaon loop contribution gives main contribution to $\Gamma(a_0 \rightarrow \gamma\gamma, m)$. 

---

Table I. Properties of the resonances and main characteristics

| Fit | 1     | 2     | 3     | 4     |
|-----|-------|-------|-------|-------|
| $m_{a_0}$, MeV | 993.86 | 994.50 | 995.34 | 988.69 |
| $g_{a_0K^+K^-}$, GeV | 2.7457 | 2.4311 | 3.8748 | 3.7148 |
| $g_{a_0K^+K^-}/4\pi$, GeV$^2$ | 0.60 | 0.47 | 1.19 | 1.10 |
| $g_{a_0\eta\pi}$, GeV | 2.7417 | 3.0929 | 3.6936 | 3.6084 |
| $g_{a_0\eta\pi}/4\pi$, GeV$^2$ | 0.60 | 0.76 | 1.09 | 1.04 |
| $g_{a_0\eta\pi}^{(0)}$, GeV | -2.8596 | -4.6179 | -4.1549 | -3.9537 |
| $g_{a_0\eta\pi}^{(0)}/4\pi$, GeV$^2$ | 0.65 | 1.70 | 1.37 | 1.24 |
| $g_{a_0\gamma\gamma}^{(0)}$, $10^{-3}$ GeV$^{-1}$ | 1.8 | 2.7810 | 4.5919 | 3.2520 |
| $\Gamma_{a_0\rightarrow\eta\pi}$, keV | 0.063 | 0.151 | 0.414 | 0.203 |
| $\langle \Gamma_{a_0\rightarrow\gamma\gamma}^{direct} >_{\eta\pi} \rangle$, keV | 0.019 | 0.031 | 0.030 | 0.024 |
| $\langle \Gamma_{a_0\rightarrow(KK+\eta\eta^0+\eta'\eta^0)\rightarrow\gamma\gamma} >_{\eta\pi} \rangle$, keV | 0.126 | 0.113 | 0.141 | 0.129 |
| $\langle \Gamma_{a_0\rightarrow(KK+\eta\eta^0+\eta'\eta^0+direct)\rightarrow\gamma\gamma} >_{\eta\pi} \rangle$, keV | 0.225 | 0.236 | 0.273 | 0.243 |
| $\Gamma_{a_0}(m_{a_0})$, MeV | 116.76 | 140.81 | 218.65 | 186.75 |
| $\Gamma_{a_0}^{eff}$, MeV | 34.6 | 57.24 | 38.76 | 44.76 |
| $m_{a_0}$, MeV | 1400 | 1400 | 1300 | 1500 |
| $g_{a_0'K^+K^-}$, GeV | 1.6304 | 0.2551 | 0.9280 | 2.3595 |
| $g_{a_0'K^+K^-}/4\pi$, GeV$^2$ | 0.21 | 0.005 | 0.07 | 0.44 |
| $g_{a_0'\eta\pi}$, GeV | -3.1161 | -1.4861 | -2.5542 | -2.8173 |
| $g_{a_0'\eta\pi}/4\pi$, GeV$^2$ | 0.77 | 0.18 | 0.52 | 0.63 |
| $g_{a_0'\eta'\pi}$, GeV | -4.7541 | -7.1931 | -5.1977 | -5.7646 |
| $g_{a_0'\eta'\pi}/4\pi$, GeV$^2$ | 1.80 | 4.12 | 2.15 | 2.64 |
| $g_{a_0'\gamma\gamma}$, $10^{-3}$ GeV$^{-1}$ | 5.5 | 9.927 | 8.2322 | 8.8103 |
| $\Gamma_{a_0'}^{(0)}(m_{a_0'})$, keV | 1.65 | 5.38 | 2.96 | 5.21 |
| $\Gamma_{a_0'}(m_{a_0'})$, MeV | 330.91 | 399.4 | 271.0 | 453.40 |
| $C_{a_0a_0'}$, GeV$^2$ | 0.02130 | 0.30218 | -0.02131 | -0.03393 |
In Fits 3 and 4 the $m_{a_0}'$ is set to 1300 MeV and 1500 MeV correspondingly. Fits 3 and 4 show that the experimental data under consideration allow large range of $m_{a_0}'$ values. Note that it is possible to obtain good fits with $m_{a_0}' = 1200$ MeV and $m_{a_0}' = 1700$ MeV also.

Note that both the data on $\gamma\gamma \rightarrow \eta\pi^0$ and the data on $\phi \rightarrow \eta\pi^0\gamma$ decay can be described without $a_0'$ contribution. But good $\gamma\gamma \rightarrow \eta\pi^0$ description (we achieved $\chi^2/n.d.f. = 15.1/27$) requires large width of the $a_0(980)$ ($\Gamma_{a_0}(m_{a_0}) \approx 650$ MeV, $\Gamma_{a_0}' \approx 100$ MeV), which contradicts the data on the $\phi \rightarrow \eta\pi^0\gamma$ decay (see also Ref. 22). So it is not possible to describe both data simultaneously without $a_0'$.

The $g_{a_0\gamma\gamma}^{(0)}$ is caused by the $q\bar{q}$ component of the $a_0(980)$, $a_{2q}^0 = (u\bar{u} - d\bar{d})/\sqrt{2}$, while in the four quark model $2$ $a_{2q}^0 = (u\bar{u}s\bar{s} - d\bar{d}s\bar{u})/\sqrt{2}$ decays to $\gamma\gamma$ only via loops. The obtained values of $g_{a_0\gamma\gamma}^{(0)}$ mean that the point-like contribution $\phi a_0\gamma$, $g_{\phi a_0\gamma}$, is negligible: the $g_{a_0\omega\gamma} \approx g_{a_0\gamma\gamma}/10$, and $g_{a_0\phi\gamma}$ should be less than $g_{a_0\omega\gamma}$ at $\sim 20$ times because of $\phi - \omega$ mixing. Special fit with point-like contributions $g_{\phi a_0\gamma}$ and $g_{\phi a_0'\gamma}$ confirmed that it is not possible to derive these constants from the current data.

Remind that there should be no confusion due to relatively large $a_0$ width in Table I. The invariant mass spectrum of $\eta\pi^0$ in $a_0 \rightarrow \eta\pi^0$ is given by the relation

$$
\frac{dN_{\eta\pi^0}}{dm} \sim \frac{2m^2}{\pi} \frac{\Gamma(a_0 \rightarrow \eta\pi^0, m)}{|D_{a_0}(m)|^2}.
$$

(59)

The effective (visible) width of this distribution $\Gamma_{a_0}^{eff}$ is much less then the nominal width $\Gamma(a_0 \rightarrow \eta\pi^0, m_{a_0})$ due to strong $a_0K\bar{K}$ coupling, usually $\Gamma_{a_0}^{eff}$ is $50 - 70$ MeV, see Table I.
and Ref. [22].

The close story is with \( \Gamma(a_0 \to \gamma\gamma) \). Following [8, 9, 34, 35], one can determine the width of the \( a_0(980) \to \gamma\gamma \) decay averaged over the resonance mass distribution in the \( \eta\pi^0 \) channel:

\[
\langle \Gamma_{a_0 \to \gamma\gamma} \rangle_{\eta\pi^0} = \int_{0.9 \text{ GeV}}^{1.1 \text{ GeV}} \frac{s}{4\pi^2} \sigma_{0}^{\text{res}}(\gamma\gamma \to \pi^0\eta; s) d\sqrt{s}
\]

(60)

where \( \langle \Gamma_{a_0 \to \gamma\gamma} \rangle_{\eta\pi^0} \equiv< \Gamma_{a_0 \to (K\bar{K} + \eta\pi^0 + \eta'\pi^0 + \text{direct}) \to \gamma\gamma > \eta\pi^0 \), the integral is taken over the region occupied by the \( a_0(980) \) resonance, and the \( \sigma_{0}^{\text{res}} \) is determined by the matrix element that contains only the resonance contributions from the rescatterings and direct transitions in Eq. (1), i.e. all contributions mentioned in Eq. (1) at \( Q = 0 \) except the Born one:

\[
M_{0}^{\text{res}}(s) = \bar{I}_{\pi^0\eta}(s,0) T_{\pi^0\eta\to\pi^0\eta}(s) + \bar{I}_{\pi^0\eta'}(s,0) T_{\pi^0\eta'\to\pi^0\eta}(s) + \left( \bar{I}_{K^+K^-}(s,0) - \bar{I}_{K^0\bar{K}^0}(s,0) + \bar{I}_{K^+K^-}(s,0) \right) T_{K+K^-\to\eta\pi^0}(s) + M_{\text{direct}}^{\text{res}}(s,0)
\]

(61)

This quantity is an adequate characteristic of the coupling of the \( a_0(980) \) resonance with a \( \gamma\gamma \) pair. One can also consider particular contributions to \( \langle \Gamma_{a_0 \to \gamma\gamma} \rangle_{\eta\pi^0} \). The obtained results for \(< \Gamma_{a_0 \to \gamma\gamma} > \eta\pi^0 > \), \(< \Gamma_{a_0 \to (K\bar{K} + \eta\pi^0 + \eta'\pi^0 + \text{direct}) \to \gamma\gamma > \eta\pi^0 \), and \(< \Gamma_{a_0 \to (K\bar{K} + \eta\pi^0 + \eta'\pi^0 + \text{direct}) \to \gamma\gamma > \eta\pi^0 \) are shown in Table I. One can see that the averaged values are much less than the values at \( m = m_{a_0} \), for example, \(< \Gamma_{a_0 \to \gamma\gamma} > \eta\pi^0 > \) is at 3-10 times less than \( \Gamma_{0}^{(0)} a_0 \to \gamma\gamma \) depending on Fit.

As one can see in Table I, the experimental data allow rather large deviation of the \( a_0 \) and \( a_0' \) features.

IV. NON-ZERO Q

At \( Q \to \infty \) the matrix elements of Born diagrams in Eqs. (1), (2), and (3) with vector mesons in t-channel fall exponentially because of factor Eq. (10), the kaon loop contribution \( \bar{I}_{K^+K^-}(s, Q) T_{K+K^-\to\pi^0\eta}(s) \) is \( \sim \ln^2(Q^2/m^2_{K^+})/Q^2 \) [36], the direct \( a_0 \to \gamma^*\gamma \) interaction Eq. (34) does not depend on \( Q \) at \( Q \to \infty \). The obtained asymptotics of the direct \( \gamma^*\gamma \to a_0 \) transition is similar to the QCD based asymptotics Ref. [37, 38] for the \( q\bar{q} \) meson and the obtained asymptotics for the kaon loop contribution \( \sim 1/Q^2 \) is similar to the QCD asymptotics for the four quark meson. This fact emphasizes that the direct transition goes through the \( q\bar{q} \) component of the \( a_0 \), which is small in the four-quark model, and the
FIG. 6: The $\sigma_0(\gamma\gamma^*(Q^2) \to \eta\pi^0, s) + \sigma_2(\gamma\gamma^*(Q^2) \to \eta\pi^0, s)$, $|\cos \theta| < 0.8$, for Fit 1, $a = -0.082$ and $b = -0.15$. a) Solid line $Q^2 = 0$, dashed line $Q^2 = 0.25$ GeV$^2$, long-dashed line $Q^2 = 1$ GeV$^2$; b) Solid line $Q^2 = 2.25$ GeV$^2$, dashed line $Q^2 = 4$ GeV$^2$, long-dashed line $Q^2 = 6.25$ GeV$^2$.

FIG. 7: The average $\gamma\gamma^*(Q^2) \to \eta\pi^0$ cross section, $|\cos \theta| < 0.8$, for the same parameters as in Fig. 6. Each point of the curve is the cross section averaged over the $\pm 10$ MeV neighborhood. a) Solid line $Q^2 = 0$, dashed line $Q^2 = 0.25$ GeV$^2$, long-dashed line $Q^2 = 1$ GeV$^2$; b) Solid line $Q^2 = 2.25$ GeV$^2$, dashed line $Q^2 = 4$ GeV$^2$, long-dashed line $Q^2 = 6.25$ GeV$^2$.

rescattering contributions domination is the manifestation of the four-quark nature of the $a_0$.

For the $\gamma^*\gamma \to a_2$ transition the obtained asymptotics are

$$M_0(\gamma^*\gamma \to a_2) \sim F_{a_2}(Q)Q^4$$  \hspace{1cm} (62)

$$M_1(\gamma^*\gamma \to a_2) \sim F_{a_2}(Q)Q^3$$  \hspace{1cm} (63)
FIG. 8: The $\sigma_1(\gamma\gamma^*(Q^2) \to \eta\pi^0, s), |\cos \theta| < 0.8$, for the same parameters as in Fig. 6. Solid line $Q^2 = 0.25$ GeV$^2$, dashed line $Q^2 = 1$ GeV$^2$, long-dashed line $Q^2 = 6.25$ GeV$^2$.

$M_2(\gamma^*\gamma \to a_2) \sim F_{a_2}(Q)Q^2$  \hspace{1cm} (64)

The hierarchy $M_0 : M_1 : M_2 : Q : 1$ is the consequence of the Lagrangian Eq. (37) and agrees with the QCD based prediction $M_\lambda(\gamma^*\gamma \to a_2) \sim Q^{-\lambda}$ [37, 38]. To reach this asymptotics one have to conclude that $F_{a_2}(Q) \sim 1/Q^4$, and this is possible when vector excitations $\rho', \rho'', \omega',$ and $\omega''$ are taken into account. Basing on Ref. [28], let’s take

$F_{a_2}(Q) = \tilde{F}_{a_2}(Q)/\tilde{F}_{a_2}(0)$, where

$\tilde{F}_{a_2}(Q) = \frac{g_{a_2\rho\omega}}{f_\rho f_\omega} \left( \frac{1}{1 + Q^2/m_\rho^2} + \frac{1}{1 + Q^2/m_\omega^2} \right) + \frac{g_{a_2\rho'\omega'}}{f_{\rho'} f_{\omega'}} \left( \frac{1}{1 + Q^2/m_{\rho'}^2} + \frac{1}{1 + Q^2/m_{\omega'}^2} \right) + \frac{g_{a_2\rho''\omega''}}{f_{\rho''} f_{\omega''}} \left( \frac{1}{1 + Q^2/m_{\rho''}^2} + \frac{1}{1 + Q^2/m_{\omega''}^2} \right)$

Here $a = g_{a_2\rho'\omega'} f_{\rho'} f_{\omega'}/g_{a_2\rho\omega} f_\rho f_\omega$ and $b = g_{a_2\rho''\omega''} f_{\rho''} f_{\omega''}/g_{a_2\rho\omega} f_\rho f_\omega$. The requirement

$m_\rho^2 + m_\omega^2 + a(m_{\rho'}^2 + m_{\omega'}^2) + b(m_{\rho''}^2 + m_{\omega''}^2) = 0$  \hspace{1cm} (66)

leads to elimination of the contribution proportional to $1/Q^2$, so $F_{a_2}(Q) \sim 1/Q^4$ satisfying the QCD based asymptotics. This asymptotics is a manifestation of the vector excitation contribution.
FIG. 9: The $\sigma_0(\gamma\gamma^\ast(Q^2) \rightarrow \eta\pi^0, s) + \sigma_2(\gamma\gamma^\ast(Q^2) \rightarrow \eta\pi^0, s)$, $|\cos\theta| < 0.8$, for Fit 1, $a = 3.835$ and $b = -3$.  

a) Solid line $Q^2 = 0$, dashed line $Q^2 = 0.25$ GeV$^2$, long-dashed line $Q^2 = 1$ GeV$^2$;  
b) Solid line $Q^2 = 2.25$ GeV$^2$, dashed line $Q^2 = 4$ GeV$^2$, long-dashed line $Q^2 = 6.25$ GeV$^2$.

FIG. 10: The $\sigma_1(\gamma\gamma^\ast(Q^2) \rightarrow \eta\pi^0, s)$, $|\cos\theta| < 0.8$, for the same parameters as in Fig. 9.  

Solid line $Q^2 = 0.25$ GeV$^2$, dashed line $Q^2 = 1$ GeV$^2$, long-dashed line $Q^2 = 6.25$ GeV$^2$.

The requirement Eq. (66) do not fix both $a$ and $b$. The ratio $a/b$ is the question for the experiment.

Let’s take in Eq. (65) $m_{\rho'} = m_{\omega'} = 1450$ MeV and $m_{\rho''} = m_{\omega''} = 1700$ MeV in agreement with Ref. [1].

If one takes $a = -0.082, b = -0.15$ together with the parameters of Fit 1, the $a_0$ and $a_2$ peaks fall synchronously with $Q$ increase, see Figs. 6 and 7 where the sums $\sigma_0(s, Q) + \sigma_2(s, Q)$ and $\bar{\sigma}_0(s, Q) + \bar{\sigma}_2(s, Q)$ are shown for different values of $Q$. The $a_2$ peak starts
FIG. 11: The $\sigma_0(\gamma\gamma^*(Q^2) \rightarrow \eta\pi^0, s) + \sigma_2(\gamma\gamma^*(Q^2) \rightarrow \eta\pi^0, s)$, $|\cos \theta| < 0.8$, for Fit 1, $a = -4.413$ and $b = 3$. a) Solid line $Q^2 = 0$, dashed line $Q^2 = 0.25$ GeV$^2$, long-dashed line $Q^2 = 1$ GeV$^2$; b) Solid line $Q^2 = 2.25$ GeV$^2$, dashed line $Q^2 = 4$ GeV$^2$, long-dashed line $Q^2 = 6.25$ GeV$^2$.

FIG. 12: The $\sigma_1(\gamma\gamma^*(Q^2) \rightarrow \eta\pi^0, s)$, $|\cos \theta| < 0.8$, for the same parameters as in Fig. 11. Solid line $Q^2 = 0.25$ GeV$^2$, dashed line $Q^2 = 1$ GeV$^2$, long-dashed line $Q^2 = 6.25$ GeV$^2$.

donating only at $Q \sim 5$ GeV. The $\sigma_1(s, Q)$ is shown in Fig. 11. In this case vector excitations contribution is relatively small at $Q = 0$ as one could expect from general considerations.

The Eqs. (65), (66) also contain cases when vector excitations contribution exceeds the $\rho$ and $\omega$ one even at $Q = 0$. Let’s consider two of them.

In case $a = 3.835$ and $b = -3$ the $a_0$ and $a_2$ peaks in $\sigma_0(s, Q) + \sigma_2(s, Q)$ fall synchronously for $Q < 1$ GeV, then the $a_2$ peak starts dominating, see Fig. 9. The $\sigma_1(s, Q)$ is shown in
The choice $a = -4.413$ and $b = 3$ leads to Fig. 11: the $a_2$ peak grows at low $Q$ and falls under the value at $Q = 0$ only at $Q > 1$ GeV. Starting from $Q \approx 0.5$ GeV the $a_2$ peak dominates the $a_0$ peak, see Fig. 11. The $\sigma_1(s, Q)$ for this case is shown in Fig. 12.

We don’t consider scenarios that seem doubtful. For example, for vast region of $a$ and $b$ the $F_{a_2}(Q) = 0$ at $Q < 1$ GeV.

New Belle experiment could clarify what scenario is realized.

How vector excitations influence on contributions to the $\gamma\gamma \rightarrow \eta\pi^0$ amplitude, not involving the $a_2$ meson, is the question for separate investigation. The kaon loop contribution probably does not change dramatically because the couplings of the vector excitations to kaon channel, obtained in [40], are small. Other contributions are not so large. The direct transition $M_{\text{res}}^{\text{direct}}$ is small in agreement with the four quark model scenario. The Born diagrams with vector excitations in t-channel are suppressed due to large masses of these particles.

V. CONCLUSION

The experimental data on the $\gamma\gamma \rightarrow \eta\pi^0$ evidence in favor of the four quark model of the $a_0(980)$. The data is well described with the scenario based on four quark model: relations Eqs. (58) and small $g^{(0)}_{a_0\gamma\gamma}$. The obtained values of $g^{(0)}_{a_0\gamma\gamma}$ mean that the point-like coupling $a_0\phi\gamma$ gives negligible contribution to $\phi \rightarrow \eta\pi^0\gamma$ process. The production (and decay) of the $a_0(980)$ via rescatterings, i.e. via the four quark transitions, is the main qualitative argument in favour of the four quark nature of the $a_0(980)$.

We think the predicted $Q^2$ behaviour of the cross section, shown in Figs. 6, 7 and 8 is reliable for $Q < 1 \div 2$ GeV [31, 41]. At higher $Q$ it may be treated as a guide. Note that at very high $Q$ the QCD should be used.

The strong influence of vector excitations at $Q \sim 1$ GeV is considered also, see Figs. 9, 11, 12.

We don’t use the kaon formfactor $G_{K^+}(t, u)$, introduced in Refs. [26, 27], since the data can be explained without it, and the processes $\phi \rightarrow \eta\pi^0\gamma$ as well as $\phi \rightarrow \pi^0\pi^0\gamma$ are described without this factor also.

As for comparative production of $a_0$ and $a_2$ in $\sigma_0 + \sigma_2$, at high $Q \gtrsim 5$ GeV the $a_2$ contribution dominates in all variants. In the intermediate region $Q \sim 1$ GeV the $a_0$ and
$a_2$ peaks fall synchronously if vector excitations contribution is relatively small at $Q = 0$ as one could expect from general considerations. But in principle it is not excluded that the $a_2$ peak dominates in the intermediate region too, see Figs. 9, 11.

The perspectives of study of the $f_0(980)$ and $f_2(1270)$ comparative production in $\gamma^*(Q^2)\gamma \to \pi^0\pi^0$ reaction are poor because the $f_0(980)$ peak is considerably less than the $f_2(1270)$ peak in $\gamma\gamma \to \pi^0\pi^0$ already [8, 27]. As for the $f_2(1270)$ production, it is similar to the $a_2(1320)$ production. Emphasize that the best process to study the $a_2$ production is the $\gamma^*(Q^2)\gamma \to a_2 \to \rho\pi$ reaction because $Br(a_2 \to \rho\pi) = 70\%$ [1] and the background is expected to be small.

VI. ACKNOWLEDGEMENTS

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[30] The following typos are noticed in Ref. [26, 27]:

1. The curve on Fig. 19b in Ref. [27] is drawn with $g_{\rho\gamma\gamma}^{(0)} = 0.536 \times 10^{-3}$ GeV$^{-1}$ and $g_{f_0\gamma\gamma}^{(0)} = 0.652 \times 10^{-3}$ GeV$^{-1}$ (the multiplier $10^{-3}$ was missed in the text).

2. In Table I of Ref. [26] and Appendix II of Ref. [27] ($g_{a_0\gamma\gamma}^{(0)}; g_{a_0'\gamma\gamma}^{(0)}$) = (1.83; −5.9) $\times$10$^{-3}$ GeV$^{-1}$ instead of (1.77; −11.5) $\times$10$^{-3}$ GeV$^{-1}$.

3. The $c_0$ in Ref. [27] is −0.603, not −0.603.

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