Bayesian Variable Selection for Linear Regression with the $\kappa$-$G$ Priors

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Abstract

In this article we develop a new methodology for Bayesian variable selection in multiple linear regression that is independent of the standard indicator vector method. Serving as an extension of Zellner’s $g$-prior, we extend the original scalar $g$ to a diagonal matrix $G$ that controls the stability of the prior on the coefficients $\beta$, and each of the elements $g_j$ controls the stability of its corresponding dimension. From the Metropolis-within-Gibbs sampling method, the posterior values of $G$ are sampled and those promising variables tend to have a $g_j$’s that are close to 0. Thus the promising variables are chosen based on the posterior of $g_j$. As each of the $g_j$’s is a stabilizer of its own dimension, the $1 - g_j$ values imitates the posterior inclusion probability (PIP) as in the standard methodology.

Keywords:

1. Introduction

Consider a normal multiple linear regression (MLR) model having the form

$$y = X\beta + \varepsilon, \varepsilon \sim N(0, \sigma^2 I)$$

(1)

where $y$ is an $n \times 1$ response vector, $X$ an $n \times p$ data matrix, $\beta$ a $p \times 1$ coefficient vector and $\varepsilon$ the random error. A very interesting and important question to answer is which predictors belong to the true model.

O’Hara and Sillanpää (2011) gives a thorough review of the approaches in variable selection from the Bayesian perspective. A typical approach is the indicator method, where a vector $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_p)$ of indicators is assigned as

$$\gamma_j = \begin{cases} 
1 & \text{if } x_j \text{ is in the model} \\
0 & \text{if } x_j \text{ is not in the model}
\end{cases}$$

(2)

for $j = 1, 2, \ldots, p$. Given some prior information about $p(\beta)$, $p(\sigma^2)$, and $p(\gamma)$, the goal is to find the posterior $p(\gamma \mid y)$ and $p(y \mid \gamma)$ from which the variable selection can be inferred with the help of Bayes factor (See Fernández et al. (2001) and Barbieri and Berger (2004)). Multiple difficulties arise from this method. First, such method requires an examination through the model space, which is of size $2^p$. When the dimensionality $p$ is large, such examination is not feasible. A second difficulty is that the posterior $p(y \mid \gamma)$ is often not of analytical form, resulting in the Bayes factor not having a closed form either.

To avoid the first difficulty, George and McCulloch (1993) and George and McCulloch (1997) proposed a method of stochastic search variable selection (SSVS). Starting from a formulation of Bayesian hierarchical model, the promising subsets are identified with higher posterior probability.
The computational burden is alleviated by using Gibbs sampling indirectly searching the possible subsets.

Zellner (1986) proposed a “reference informative prior” for the coefficients $\beta$ that takes on the form of

$$
\beta_{\gamma} \sim N\left(0, g\sigma^2(X_{\gamma}^T X_{\gamma})\right).
$$

(3)

The idea of Zellner’s $g$-prior is rather simple, and yet it yields a neat closed form for $p(y | \gamma)$ and the corresponding Bayes factor. It thus remains a conventional choice of prior in terms of Bayesian variable selection. Multiple works have been done to extend the original Zellner’s $g$-prior. Specifically, Liang et al. (2008) proposed a study on mixtures of $g$-priors which provides a family of hyperpriors on $g$ while still preserves the tractability on the marginal likelihood. Bové and Held (2011) developed an extension of the classical Zellner’s $g$-prior to generalized linear models, given a large family of hyperprior on $g$. Maruyama and George (2011) introduced a fully Bayes formulation with an orthogonal decomposition on the matrix $X_{\gamma}^T X_{\gamma}$ the method resolves the problem in which a dataset is of the form $p > n$.

Our work focuses on a specific formulation of the prior on $\beta$. Motivated by Agliari and Carvi Parisetti (1988), and George and McCulloch (1993), we inject a diagonal matrix $G = \text{diag}(g_1, g_2, \ldots, g_p)$ to the variance of $\beta$ such that the promising variables are selected based on the posterior values of $g_i$’s, with some stochastic method, like the Gibbs sampling. Unlike Agliari and Carvi Parisetti (1988), in which the authors injected a diagonal matrix directly to the linear model, we still keep the linear model of the form in (1). Further, instead of keeping the indicator $\gamma$ in the formulation as in George and McCulloch (1993), our work is completely independent of such indicator methodology. A prior is given to each of the $g_i$’s in $G$, and variable selection is achieved through the sampling of $G$. Also, since the matrix $G$ exists in the variance of $\beta$, it serves as a parametric analogy to Tipping (2001), in which each of the $\beta_j$’s follows a Gaussian prior of $N(0, \alpha_j^{-1})$. A large value of $\alpha_j$ is equal to a high precision, and thus an insignificant $\beta_j$.

Section 2 provides the details of the $\kappa$-$G$ priors formulation and the joint posterior of the parameters. It also discusses the posterior and conditional distributions of each parameter, with details of some of the limiting properties. Section 3 provides the computational details. An algorithm involving Metropolis-within-Gibbs sampling is introduced in this section. The methodology is tested with multiple examples and the results are provided in Section 4 with some discussion. We draw a conclusion of this work in Section 5.

## 2. The $\kappa$-$G$ Priors Formulation

Given an MLR model having the form of (1), we introduce a prior distribution on $\beta$ that has the form

$$
\beta \sim N_p\left(\beta_0, \kappa \sigma^2 \left(GX^T X G\right)^{-1}\right).
$$

(4)

Intuitively the mean $\beta_0$ is usually assigned as 0. For the components of the variance, $\kappa$ is a continuous variable that controls the total scale of the variance, and is further assigned an Inverse Gamma prior distribution by conjugacy. $G = \text{diag}(g_1, g_2, \ldots, g_p)$ plays a key role as the stabilizer in this formulation, where each of the $g_i$’s controls its corresponding dimension, and all of the $p$ elements follow an i.i.d. Beta distribution. Finally, we still keep the Jeffreys’ prior for $\sigma^2$, that is
\[ p(\sigma^2) \propto (\sigma^2)^{-1}. \] Thus, the \( \kappa \)-G method is formulated as
\[
\begin{align*}
    y \mid \beta, \sigma^2 &\sim N_n \left( y \mid X\beta, \sigma^2 I \right) \\
    \beta \mid G, \kappa, \sigma^2 &\sim N_p \left( \beta \mid \beta_0, \kappa \sigma^2 \left( G X^T X G \right)^{-1} \right) \\
    G &\sim \prod_{j=1}^p \text{Beta} \left( g_j \mid a, b \right) \\
    \kappa &\sim \text{IG} \left( \alpha, \theta \right) \\
    p(\sigma^2) &\propto (\sigma^2)^{-1}
\end{align*}
\] (5)

For the prior distribution of \( G \), the hyper-parameters \( a \) and \( b \) are preferably equal to each other and both less than 1, indicating that the \( g_j \)'s come from a symmetric distribution that has support between 0 and 1, with higher density close to the boundaries. Some of the properties of \( g_j \)'s and its utility in variable selection will be further discussed in the following content, but intuitively, serving as a direct analogy to Tipping (2001), \( \beta_j \) would have a large precision with a center at \( \beta_0 \) if the corresponding \( g_j \) is large. The choice of inverse Gamma distribution for \( \kappa \) is due to conjugacy, and the parameters \( \alpha \) and \( \theta \) do not have a significant impact to variable selection in practice, although improper values would result in relatively intense computation.

Following the formulation in (5), the joint posterior distribution is given by
\[
\begin{align*}
p \left( \beta, \kappa, G, \sigma^2 \mid y \right) &\propto p(y \mid \beta, \sigma^2) p(\beta \mid G, \kappa, \sigma^2) p(\kappa) p(G) p(\sigma^2) \\
&\propto N_n \left( y \mid X\beta, \sigma^2 I \right) \times N_p \left( \beta \mid \beta_0, \kappa \sigma^2 \left( G X^T X G \right)^{-1} \right) \\
&\times \text{IG} \left( \kappa \mid \alpha, \theta \right) \times \prod_{j=1}^p \text{Beta} \left( g_j \mid a, b \right) \times (\sigma^2)^{-1} \\
&\propto \frac{1}{|\sigma^2 I|^{1/2}} \exp \left\{ -\frac{1}{2\sigma^2} (y - X\beta)^T (y - X\beta) \right\} \times \\
&\quad \frac{1}{|\kappa \sigma^2 (G X^T X G)^{-1}|^{1/2}} \exp \left\{ -\frac{1}{2\kappa \sigma^2} (\beta - \beta_0)^T G X^T X G (\beta - \beta_0) \right\} \\
&\quad \times \kappa^{-\alpha - 1} \exp \left( -\frac{\theta}{\kappa} \right) \times \left( \prod_{j=1}^p g_j a^{-1} (1 - g_j) b^{-1} \right) \times (\sigma^2)^{-1}
\end{align*}
\] (6)

2.1. Posterior properties of \( \beta \)

Directly following the formulation in (6), we readily obtain the posterior distribution of \( \beta \) as
\[
\begin{align*}
p \left( \beta \mid \kappa, G, \sigma^2, y \right) &\propto \exp \left\{ -\frac{1}{2\sigma^2} (y - X\beta)^T (y - X\beta) \right\} \times \\
&\quad \exp \left\{ -\frac{1}{2\kappa \sigma^2} (\beta - \beta_0)^T G X^T X G (\beta - \beta_0) \right\} \\
&\sim N_p \left( \tilde{\beta}, \tilde{\sigma}_\beta^2 \right)
\end{align*}
\] (7)
where \( \tilde{\mu}_\beta \) and \( \tilde{\sigma}_\beta^2 \) are the mean and variance of the posterior and take on the form of

\[
\tilde{\mu}_\beta = \left( X^T X + \frac{1}{\kappa} G X^T X G \right)^{-1} \left( X^T y + \frac{1}{\kappa} G X^T X G \beta_0 \right)
\]
\[
\tilde{\sigma}_\beta^2 = \sigma^2 \left( X^T X + \frac{1}{\kappa} G X^T X G \right)^{-1}
\]

(8)

Consider the following two limiting cases. First, if \( G \) is a null matrix, that is, \( g_j \to 0, \forall i \), the posterior mean and variance are

\[
\tilde{\mu}_\beta = \left( X^T X \right)^{-1} X^T y = \hat{\beta}^{(\text{OLS})}
\]
\[
\tilde{\sigma}_\beta^2 = \sigma^2 \left( X^T X \right)^{-1} = \text{Var} \left( \hat{\beta}^{(\text{OLS})} \right)
\]

(9)

It indicates that in the extreme condition where all \( g_j \)'s approach 0, this formulation would degenerate to ordinary least square (OLS). In fact, this is reasonable since with \( G \) being a null matrix, the prior information of \( \beta \) has infinite variance.

The second limiting case is that \( G = I_p \), i.e., a \( p \)-dimensional identity matrix, and \( \beta_0 = 0 \), which is a fairly reasonable assumption. Under this condition,

\[
\tilde{\mu}_\beta = \left( X^T X + \frac{1}{\kappa} X^T X G \right)^{-1} X^T y
\]
\[
= \frac{\kappa}{\kappa + 1} \hat{\beta}^{(\text{OLS})}
\]

(10)

This is exactly the same as the original Zellner’s g-prior with \( \kappa = g \), and with \( \kappa \to \infty \), \( \tilde{\mu}_\beta \) converges to \( \hat{\beta}^{(\text{OLS})} \). In terms of \( \kappa \), the convergence from \( \tilde{\mu}_\beta \) to \( \hat{\beta}^{(\text{OLS})} \) does not require a specific \( G \), as long as \( \kappa \to \infty \).

2.2. Posterior properties of \( G \)

We then derive the posterior distribution of \( G \) given \( y, \sigma^2 \) and \( \kappa \) by integrating out \( \beta \):

\[
p \left( G \mid y, \sigma^2, \kappa \right) = \int G \cdot p(G) p \left( y \mid \beta, \sigma^2 \right) p(\beta) d\beta
\]
\[
= \prod_{j=1}^p \text{Beta}(g_j \mid a, b) \int \mathcal{N}_p \left( y \mid X \beta, \sigma^2 I \right) \times \mathcal{N}_p \left( \beta \mid 0, \kappa \sigma^2 (G X^T X G) \right) d\beta
\]
\[
\alpha |G|^{a-1} |I_p - G|^{b-1} \times \frac{1}{|\kappa \sigma^2 (G X^T X G)|^{-1/2}} \times
\]
\[
\int \exp \left\{ -\frac{1}{2\sigma^2} (y - X \beta)^T (y - X \beta) \right\} \times \exp \left\{ -\frac{1}{2\kappa \sigma^2} \beta^T G X^T X G \beta \right\} d\beta
\]
\[
\alpha |G|^a |I_p - G|^{b-1} \sigma^2 \left( X^T X + \frac{1}{\kappa} G X^T X G \right)^{-1} \times
\]
\[
\exp \left\{ \frac{1}{2} y^T X \left( X^T X + \frac{1}{\kappa} G X^T X G \right)^{-1} X^T y \right\} \times
\]
\[
\int_{\beta} N_p \left( \beta \mid \left( X^T X + \frac{1}{\kappa} G X^T X G \right)^{-1} X^T y, \sigma^2 \left( X^T X + \frac{1}{\kappa} G X^T X G \right)^{-1} \right) d\beta
\]

\[
\propto |G|^a |I_p - G|^{b-1} \left( x^T X + \frac{1}{\kappa} G X^T X G \right)^{-1/2} \times
\exp \left\{ \frac{1}{2} y^T X \left( X^T X + \frac{1}{\kappa} G X^T X G \right)^{-1} X^T y \right\}
\]

A question emerges naturally: how does \( G \) connect with variable selection? Such question can be broken into two parts: is “\( X_j \) is not a true predictor” a necessary condition of \( g_j \to 1 \), and, is \( g_j \to 1 \) a sufficient condition of “\( X_j \) is not a true predictor”. We provide a 2 dimensional informal proof here to demonstrate some aspects of these two questions.

Consider the case where \( p = 2 \) and \( \kappa = \sigma^2 = 1 \). Let \( A = X^T X + G X^T X G \), and

\[
A = \begin{pmatrix}
(1 + g_1^2) \sum x_i^2 & (1 + g_1 g_2) \sum x_i x_2 \\
(1 + g_1 g_2) \sum x_i x_2 & (1 + g_2^2) \sum x_i^2
\end{pmatrix}.
\]

Combining (11) and (12), we have an expression for \( p(G \mid y, \sigma^2, \kappa) \) in 2 dimension:

\[
p(G \mid y, \sigma^2, \kappa) \propto \prod_{i=1}^2 (a_1 a_3 - a_2^2)^{-1/2} \times
\exp \left\{ \frac{1}{2(a_1 a_3 - a_2^2)} \left( \sum x_i y_i \right) \left( a_3 \sum x_1 x_1 y_i - a_2 \sum x_2 x_1 y_i + \right) \right.
\left. \left( \sum x_2 x_2 y_i \right) \left( a_1 \sum x_1 x_2 y_i - a_2 \sum x_2 x_1 y_i \right) \right\}
\]

where \( a_1 \) and \( a_3 \) are the diagonal elements in \( A \) and \( a_2 \) the off-diagonal element in \( A \).

We further assume that \( X_1 \) is a true predictor of \( y \) and \( X_2 \) is not, and there is no significant correlation between the two variables. Figure 1 provides a contour of \( p(G \mid y, \sigma^2, \kappa) \) with regard to \( g_1 \) and \( g_2 \). It is obvious from the contour plot that within the region \( \{ (g_1, g_2) \mid g_1 \in (0, 1), g_2 \in (0, 1) \} \) the posterior is maximized as \( g_1 \to 0 \) and \( g_2 \to 1 \). Thus, to say \( X_j \) is not a predictor is the
same as to say $g_j \to 1$, and vice versa. This is very similar to the result in Tipping (2001), where $\alpha \to \infty$ would indicate that the corresponding vector is not relevant.

2.3. Conditional distribution of $\kappa$, and $\sigma^2$

From the formulation in (6), we also derive the conditional distributions for $\kappa$, $\sigma^2$ and $G$. The scale parameter $\kappa$ has a closed-form conditional distribution by conjugacy

$$p \left( \kappa \mid \beta, \sigma^2, G, y \right) \propto \kappa^{-p/2 - \alpha - 1} \exp \left\{ -\frac{\theta}{\kappa} \frac{1}{2\kappa \sigma^2} (\beta - \beta_0)^T G X^T X G (\beta - \beta_0) \right\}$$

$$\sim IG \left( \tilde{\alpha}, \tilde{\theta} \right) \tag{14}$$

where

$$\tilde{\alpha} = \frac{p}{2} + \alpha$$

$$\tilde{\theta} = \frac{1}{2\sigma^2} (\beta - \beta_0)^T G X^T X G (\beta - \beta_0) + \theta$$

The parameter $\sigma^2$ also has a closed-form conditional distribution by conjugacy

$$p \left( \sigma^2 \mid \beta, \kappa, G, y \right) \propto (\sigma^2)^{-n + p/2 - 1} \exp \left\{ -\frac{1}{2\sigma^2} (y - X \hat{\beta})^T (y - X \hat{\beta}) \right\} \times$$

$$\exp \left\{ -\frac{1}{2\kappa \sigma^2} (\beta - \beta_0)^T G X^T X G (\beta - \beta_0) \right\}$$

$$\sim IG \left( n + p/2, \frac{1}{2} s^2 + \frac{1}{2} (\beta - \hat{\beta})^T X^T X (\beta - \hat{\beta}) + \frac{1}{2\kappa} (\beta - \beta_0)^T G X^T X G (\beta - \beta_0) \right) \tag{15}$$

where

$$s^2 = (y - X \hat{\beta})^T (y - X \hat{\beta})$$

$$\hat{\beta} = \left( X^T X \right)^{-1} X^T y$$

3. Aspects of Computation

If we are to use the MCMC method to obtain the sample path of $\beta$, $\kappa$, $\sigma^2$, and $G$, the conditional distributions of such are required to have closed-form expressions. However, as stated in the previous section, the posterior of $G$ does not have such expression. Thus, in terms of computation, we use Gibbs sampling to update $\beta$, $\kappa$ and $\sigma^2$, and Metropolis-Hastings algorithm within Gibbs to update $G$. For each draw of $G$, the acceptance ratio in the Metropolis algorithm is

$$r = \frac{p(G^{(t)} \mid y, \sigma^2, \kappa) / J(G^{(t)} \mid G^{(t-1)})}{p(G^{(t-1)} \mid y, \sigma^2, \kappa) / J(G^{(t-1)} \mid G^{(t)})} \tag{16}$$
where \( p(G | \cdots) \) is given by Equation (11) and \( J(\cdot) \) is the proposal distribution and is defined as

\[
J(G^{(s)} | G^{(t-1)}) \sim \prod_{j=1}^{p} \text{Beta}(g_j^{(s)} | \cdot, \cdot).
\]

(17)

The parameters in the proposal distribution may differ in different cases. As usual, \( G^{(s)} \) is accepted with probability \( \min(1, r) \). The whole algorithm of the \( \kappa \)-G formulation of Bayesian variable selection is as shown in Algorithm 1.

**Algorithm 1: The \( \kappa \)-G formulation**

**Input:** data matrix \( X \), response \( y \), initial values \( \beta^{(0)}, (\sigma^2)^{(0)}, \kappa^{(0)} \) and \( g^{(0)}_i \).

1. for \( t = 1 \) to \( T \) do
2.   Update \( \beta^{(t)} \) based on \( p(\beta^{(t)} | \kappa^{(t-1)}, (\sigma^2)^{(t-1)}, G^{(t-1)}, y) \) as in Equation (7);
3.   Update \( \kappa^{(t)} \) based on \( p(\kappa^{(t)} | \beta^{(t)}, (\sigma^2)^{(t-1)}, G^{(t-1)}, y) \) as in Equation (14);
4.   Update \( (\sigma^2)^{(t)} \) based on \( p((\sigma^2)^{(t)} | \beta^{(t)}, \kappa^{(t)}, G^{(t-1)}, y) \) as in Equation (15);
5.   Accept \( G^{(t)} = G^{(s)} \) with probability \( \min(1, r) \) as in Equation (16).
6. end

After drawing samples of \( g_j^{(t)} \), the estimated posterior mean of \( g_i \) is given by

\[
\mathbb{E}(g_j | y) = \hat{g}^{(Bayes)}_j = \frac{1}{T} \sum_{t=1}^{T} g_j^{(t)}.
\]

(18)

As stated above, true predictors would have a \( g_j \) value that is close to 0, and other \( g_j \)'s would have values that are close to 1.

4. **Numerical Examples**

In this section we demonstrate our methodology with two simulated examples. The first one is a continuation of Section 2.2, where a 2 dimensional case is considered. A second example of \( p = 4 \) is also given in this section.

4.1. **The 2-d simulation**

Consider again when \( p = 2 \). \( X_1 \) and \( X_2 \) both have 30 observations and come from an i.i.d. \( \mathcal{N}(0, 1) \), and the true model is given by

\[
y_i = 2x_{i1} + \mathcal{N}(0, 1).
\]

In Algorithm 1, we have \( a = b = 0.5, \alpha = \theta = 1, \) and \( T = 10000 \). For the Metropolis step, we use an independent uniform proposal

\[
J(G^{(s)} | G^{(t-1)}) \prod_{j=1}^{p} \text{Beta}(g_j^{(s)} | 1, 1).
\]

Figure 2 gives a histogram of the sample path of \( g_j \). It is not surprising from the informal proof
in Section 2.2 that $g_1$ is dense close to 0 whereas $g_2$ close to 1. In fact, if we compute posterior mean, we have $\hat{g}_1^{\text{(Bayes)}} = 0.08$ and $\hat{g}_2^{\text{(Bayes)}} = 0.80$. We can treat small values of $g$ as an indication of “relevant” vector or variables. We can also treat $1 - g$ as an imitation of the posterior inclusion probability (PIP) in the indicator method of Bayesian variable selection.

4.2. The 4-d simulation

This simulated example is a mild extension of the previous one. The data contains 30 observations and 4 variables that come from an i.i.d. $N(0, 1)$. The true model is given by

$$y_i = 2(x_{i1} + x_{i3}) + N(0, 1).$$

Most of the setup in the MCMC method is the same as the previous example, except that the prior parameters of $g_j$ are $a = b = 0.3$ instead of 0.5. By doing so, we would expect that the prior on $g_j$ has higher density towards either 0 or 1, instead of varying around 0.5. Figure 3 provides a comparison of histograms of $g_j$'s. Again, those $g_j$'s that are close to 0 indicate that the corresponding variables are promising. Table 1 summarizes the posterior mean of the $g_j$'s and the $1 - g_j$ values. From the values of $1 - g_j$, we can readily see that they are indeed mimicking the PIP.
Table 1: Summary of the posterior mean of $g_j$

|       | $\hat{g}_{\text{(Bayes)}}$ | $1 - \hat{g}_{\text{(Bayes)}}$ |
|-------|-----------------------------|----------------------------------|
| $X_1$ | 0.095                       | 0.905                            |
| $X_2$ | 0.804                       | 0.196                            |
| $X_3$ | 0.096                       | 0.904                            |
| $X_4$ | 0.810                       | 0.190                            |

5. Conclusion

In this article we have demonstrated a new methodology for Bayesian variable selection. We have proved that the formulation of the prior on $\beta$ is actually a generalization of Zellner’s classical $g$-prior, and shown the link between such formulation and the relevance vector machine. It is crucial to notice that the whole formulation depends only on the parameters $\kappa$ and $G$, and is completely independent to the standard Bayesian variable selection method, which is to use the indicator vector $\gamma$ and search through the model space with size $2^p$. Based on this $\kappa$-$G$ formulation, the promising variables tend to have $g_j$ values that are close to 0, while the $g_j$’s of the other variables are close to 1. The small $g_j$’s also coincides to a large PIP in the indicator methodology. From the aspect of computation, although we lose closed-form expression from the posterior of $G$, the intractability can be compensated by the Metropolis algorithm inside a Gibbs sampler.

Some future works include a systematic proof of the limiting properties on each dimension. A main difficult issue in such study is to seek a general form for $(X^T X + G X^T X G)^{-1}$. The extension of this methodology to categorical predictors or responses, and the resolution of the problem where the datasets have a form of $p > n$ are also some of the topics that are interesting to study.

References

Agliari, A. and C. Carpi Parisetti (1988). A-g Reference Informative Prior: A Note on Zellner’s $g$-Prior. *Journal of the Royal Statistical Society, Series D.*

Barbieri, M. and J. Berger (2004). Optimal Predictive Model Selection. *The Annals of Statistics.*

Bové, D. and L. Held (2011). Hyper-g Priors for Generalized Linear Models. *Bayesian Analysis.*

Fernandéz, C., E. Ley, and M. Steel (2001). Benchmark Priors for Bayesian Model Averaging, *Journal of Econometrics.*

George, E. and R. McCulloch (1993). Variable Selection via Gibbs Sampling. *Journal of American Statistical Association.*

George, E. and R. McCulloch (1997). Approaches for Bayesian Variable Selection. *Statistica Sinica.*

Harrison, D. and D. Rubinfeld (1978). Hedonic House Prices and the Demand for Clean Air. *Journal of Environmental Economics and Management.*

Liang, F., R. Paulo, G. Molina, M. Clyde, and J. Berger (2008). Mixtures of g-Priors for Bayesian Variable Selection. *Journal of American Statistical Association.*

Maruyama, Y. and E. George (2011). Fully Bayes Factors with a Generalized g-Prior. *The Annals of Statistics.*
O’Hara, R. and M. Sillanpää (2011). A Review of Bayesian Variable Selection: What, How and Which. *Bayesian Analysis*.

Tipping, M. (2001). Sparse Bayesian Learning and the Relevance Vector Machine. *Journal of Machine Learning Research*.

Zellner, A. On Assessing Prior Distributions and Bayesian Regression Analysis with g-Prior Distributions. In *Bayesian Inference and Decision Techniques: Essays in Honor of Bruno de Finetti*. 