SYMPLECTIC CACTI, VIRTUALIZATION AND BERENSTEIN-KIRILLOV GROUPS

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Abstract. We explicitly realize an internal action of the symplectic cactus group, originally defined by Halacheva for any finite dimensional complex reductive Lie algebra, on crystals of Kashiwara-Nakashima tableaux. Our methods include a symplectic version of jeu de taquin due to Sheats and Lecouvey, symplectic reversal, and virtualization due to Baker. As an application, we define and study a symplectic version of the Berenstein-Kirillov group and show that it is a quotient of the symplectic cactus group.

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1. Introduction

The cactus group was originally defined by Henriques–Kamnitzer [HeKa06] in the context of coboundary categories defined by Drinfeld [Dr90]. Coboundary categories are monoidal categories equipped with a commutor, that is, a collection of natural isomorphisms $\sigma_{A,B} : A \otimes B \rightarrow B \otimes A$ satisfying certain properties. The idea of studying the cactus group was originally due to A. Berenstein and was taken up by Henriques–Kamnitzer in [HeKa06], who defined it and further showed that it can be realized as the fundamental group of the moduli space of marked real genus zero stable curves. The original idea of Berenstein was to construct a commutor in the category of crystals of a complex, finite-dimensional Lie algebra, by first defining an involution $\xi_B : B \rightarrow B$ for each crystal $B$, which flips the crystal by exchanging highest weight elements with lowest weight elements. In the case of $\mathfrak{gl}(n,\mathbb{C})$ with the tableau model for the highest weight crystal $B(\lambda)$ it was known that $\xi_B(\lambda)$ coincides with the Schützenberger involution on semi-standard Young tableaux of shape $\lambda$ [BerZel96]. See [BuSc17, Sections 4.3, 14.3.3] and the references therein.

Important examples of coboundary categories are the categories of $\mathfrak{gl}(n,\mathbb{C})$-crystals and the category of hives. The latter turned out to be in fact isomorphic [HeKa06] by connecting a modified octahedron recurrence [RR86, KTW04, Sp07], the jeu de taquin and Schützenberger involution procedures on Young tableaux. The axioms of the coboundary category are in fact related to the octahedron recurrence of Knutson, Tao and Woodward, as is explained in [KTW04], and Speyer [Sp07].

The cactus group $J_n$ is generated by the interval reversing maps on multiple tensor products of objects in a coboundary category

$$\sigma_{p,q} : V_1 \otimes \cdots \otimes V_n \rightarrow V_1 \otimes \cdots \otimes V_{p-1} \otimes V_q \otimes \cdots \otimes V_{p+1} \otimes \cdots \otimes V_n$$

for $1 \leq p < q \leq n$ and subject to certain relations.

Recently [Ha20, Ha16], Halacheva has defined the cactus group $J_{\theta}$ for any finite-dimensional complex reductive Lie algebra $\mathfrak{g}$. Let $\mathfrak{g}$ be a complex, reductive Lie algebra with Dynkin diagram $X$. There is a Dynkin diagram automorphism $\theta : X \rightarrow X$
defined by $\alpha_{\theta(i)} = -w_0\alpha_i$, where $w_0$ is the longest element of the Weyl group of $\mathfrak{g}$. The cactus group $J_{\mathfrak{g}}$ is the group generated by $\sigma_I$, where $I$ runs over all connected sub-Dynkin diagrams of $X$, subject to the following relations:

\begin{align}
\sigma_I^2 &= 1, \\
\sigma_I\sigma_J &= \sigma_J\sigma_I \text{ if } J \subseteq X, J \cup I \text{ is disconnected} \\
\sigma_I\sigma_J &= \sigma_{\theta_I(J)}\sigma_I \text{ if } J \subset I
\end{align}

where $\theta_I$ is the automorphism on $I$ defined by the longest element of the parabolic group $W_I$. Then Halacheva has defined an internal action of the cactus group $J_{\mathfrak{g}}$ on a normal $\mathfrak{g}$-crystal by partial Schützenberger–Lusztig involutions $\xi_I$. From this action we know that partial Schützenberger–Lusztig involutions satisfy the cactus group $J_{\mathfrak{g}}$ relations [HaKaRyWe20]. Halacheva [Ha20] initiated a combinatorial study of the cactus group for $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ by comparing the action of $J_n = J_{\mathfrak{gl}(n, \mathbb{C})}$ on a normal $\mathfrak{gl}(n, \mathbb{C})$-crystal with that of the Berenstein-Kirillov group on Gelfand-Tsetlin patterns (or semi-standard Young tableaux) [BerKir95]. Using a different approach, Chmutov, Glick and Pylyavskyy [CGP16] have also found relationships between those two groups.

Our results compose a combinatorial study of the cactus group for the symplectic Lie algebra $\mathfrak{sp}(2n, \mathbb{C})$. There are many models for $C_n$ crystals, De Concini tableaux [DeCo79], King tableaux [Ki75], and Lakshmibai-Seshadri [LakSes91] and Littelmann paths [Lit95, Lit97], but we work with Kashiwara–Nakashima tableaux, for which a rich combinatorial structure exists [KasNak91, HonKan02, Lec02, Lec07, Lec07]. We review the basics in Sections 3 and 4. For each connected sub-Dynkin diagram $I$ of $X$, we define the explicit action of $\xi_I$ on a given Kashiwara–Nakashima tableau. The algorithmic procedure for that action is given by virtualization. In the case when $I$ forms a Dynkin diagram of type $C_{n-k}$, it is also given by the $I$-partial symplectic reversal, a symplectic analogue of partial reversal on $A_{n-1}$ tableaux. Thereby we provide a combinatorial action of the cactus generators $\sigma_I$ on the set of Kashiwara-Nakashima tableaux on the alphabet $C_n$. This is addressed in Section 8. The case of $I = X$ has already been developed by Santos in [Sa21a], where he defines an operation on straight shaped Kashiwara-Nakashima tableaux which is a symplectic analogue of the Schützenberger involution operation, also known as evacuation, on straight shaped $A_{n-1}$ semi-standard Young tableaux. This procedure includes the symplectic jeu-de-taquin defined by Sheats in [Sh99], and further developed by Lecouvey [Lec02] using crystal isomorphisms. This is the content of Section 7.

For $I \subseteq X$ such that $I$ forms a Dynkin diagram of type $C_{n-k}$, we define an algorithm for $I$-partial symplectic reversal which generalizes Santos’ algorithm in the sense that, when $I = X$, our algorithm is exactly the same. The $C_{n-k}$ symplectic reversal extends symplectic $C_{n-k}$ evacuation to arbitrary semi-standard skew tableaux on the alphabet $C_n$. The $C_{n-k}$ reversal of a such semi-standard skew tableau $P$ on the alphabet $C_n$, is characterized to be the unique skew tableau coplactic equivalent to $P$ and plactic equivalent to the $C_{n-k}$ evacuation of the symplectic rectification of $P$. 
An important inspiration behind our generalization is the operation of tableau-switching \cite{BSS96} of Benkart, Sottile and Stroomer on $A_{n-1}$ semi-standard tableaux. Given an admissible tableau on the alphabet $C_n$, we start off by freezing the entries corresponding to nodes not appearing in $I$, creating at the same time a new Young tableau $U$ with Young shape defined by the positive frozen entries as well as a skew tableau $P$ consisting of the non-frozen entries. The tableau pair $(U,P)$, sharing a common border, pass through each other via symplectic jeu de taquin (SJDT for short). After performing this procedure, a new pair $(R,V)$ with $R$ the symplectic rectification of $P$ and $V$ consisting of the entries of $U$ as well as some new, colored letters. Each color records a precise instance of the symplectic rectification of $P$. Our symplectic colourful tableau switching is reversible since SJDT is reversible. It reduces to the $A_{n-1}$ tableau switching on tableaux in the alphabet $[n]$. This work is carried out in detail in subsections 9.2 and 9.3 of this paper and illustrated in Section 9.4.

For the general case we use the virtualization map defined by Baker \cite{Ba00}, that is, an injective map

$$E : KN(\lambda, n) \to SSYT(\lambda^A, n, \bar{n})$$

which assigns to the type $C_n$-crystal $KN(\lambda, n)$ a subset of the type $A_{2n-1}$-crystal $SSYT(\lambda, n, \bar{n})$ in a reversible way. This is discussed in Section 5. We show that one may apply the map $E$ then perform a certain partial Schützenberger–Lusztig involution in the type $A_{2n-1}$-crystal without leaving the image of $E$ reverse the virtualization map $E$ and obtain our desired result. Additionally, in Definition 3 (Section 6) we define the virtual symplectic cactus group $J_{2n}$ and show that it is a subgroup of $J_{2n}$ isomorphic to the symplectic cactus group $J_{sp(2n, C)}$. In Theorem 3, Section 8, an action of the virtual symplectic cactus group on the set $SSYT(\lambda^A, n, \bar{n})$ is defined. The subset $E(KN(\lambda, n))$ is preserved as shown in subsections 9.5.1 and 9.6. In particular, in Subsection 9.6, we realize such action of the virtual symplectic cactus group on the virtual images of Kashiwara-Nakashima tableaux and show that it virtualizes the action of the symplectic cactus group on Kashiwara-Nakashima tableaux. This work is illustrated in Example 9.7.

As an application, (Section 10), we define symplectic Bender–Knuth involutions combinatorially (Definition 7). We define the symplectic Berenstein–Kirillov group $BK_{C_n}$ as the free group generated by the partial symplectic Schützenberger–Lusztig involutions with respect to connected subdiagrams of the type $C_n$ Dynkin diagram of the form $I = [n]$ modulo the relations they satisfy on Kashiwara-Nakashima tableaux of any straight shape in the alphabet $C_n$. These generators of $BK_{C_n}$ satisfy the relations of the symplectic cactus group. It is shown that symplectic Bender–Knuth involutions are also generators of $BK_{C_n}$.

We study relations for $BK_{C_n}$ under the virtualization map $E$. More precisely, we consider the relations satisfied by the embedding of generators of $BK_{C_n}$ in $E(KN(\lambda, n)) \subseteq SSYT(\lambda, n, \bar{n})$; we call this group (Definition 8) the virtual symplectic Berenstein–Kirillov group $BK_{2n}$, a subgroup of $BK_{2n}$ satisfying, in particular, the relations of the virtual cactus group $J_{2n}$. Proposition 12 gives the virtual symplectic Bender–Knuth
The free group generated by \( B \) and \( r \) and the hyperoctahedral group \( B_g \) weight lattice associated to the root system of \( g \). We will often abuse notation and write a Dynkin diagram \( I \) interval \( [\theta] \) which leaves the diagram invariant, \( \theta \) has the original notation \( W \) subgroup \( \alpha \subseteq J \) \( i \in I \) \( w_0 \in W \) the longest Weyl group element. We will use the numbering of the vertices of the diagram associated to the root system of \( g \) and partially supported by the grant MAESTRO NCN UMO-2019/34/A/ST1/00263. J.T. was also supported by the grant SONATA NCN UMO-2021/43/D/ST1/02290 and partially supported by the grant MAESTRO NCN UMO-2019/34/A/ST1/00263.

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3. Basics

Let \( \mathfrak{g} \) be a finite dimensional, complex, semisimple Lie algebra. Let \( I \) be the Dynkin diagram associated to the root system of \( \mathfrak{g} \) \( \Delta = \{\alpha_i : i \in I\} \) the set of simple roots, \( W \) its Weyl group, generated by the simple reflections \( \{r_i : i \in I\} \), and \( w_0 \in W \) the longest Weyl group element. We will use the numbering of the vertices of \( I \) given by \([\text{Bo VI}]\). The Dynkin diagram has an automorphism, a permutation of its nodes which leaves the diagram invariant, \( \theta : I \rightarrow I \) defined by \( \alpha_{\theta(i)} = -w_0\alpha_i \), for any node \( i \in I \), where \( w_0 \) is the longest element of \( W \). We will also denote by \( \Lambda \) the integral weight lattice associated to the root system of \( \mathfrak{g} \). For a connected sub-diagram of \( I \), \( J \subseteq I \), denote by \( \theta_J : J \rightarrow J \) the Dynkin diagram automorphism of \( J \), which satisfies \( \alpha_{\theta_J(j)} = -w'_0\alpha_j \), for any node \( j \in J \), where \( w'_0 \) is the longest element of the parabolic subgroup \( W^J \subseteq W \) (the Weyl group for \( \mathfrak{g} \) restricted to \( J \) \([\text{BjBr05}]\)). When \( J = I \) one has the original notation \( \theta_I = \theta \). We focus on the cases where \( \mathfrak{g} = \mathfrak{g}(n, \mathbb{C}), \mathfrak{sp}(2n, \mathbb{C}) \). We will often abuse notation and write a Dynkin diagram \( I \) with \( n \) nodes as the interval \( [n] \). The corresponding Weyl groups are the symmetric group \( \mathfrak{S}_n \) on \( n \) letters and the hyperoctahedral group \( B_n \) respectively, where \( B_n \) is the free group generated by \( r_1, \ldots, r_{n-1}, r_n \) subject to the relations

\[
\begin{align*}
r_i^2 &= 1, \quad 1 \leq i \leq n, \quad (5) \\
(r_ir_j)^2 &= 1, \quad 1 \leq i < j \leq n, \quad |i - j| > 1, \quad (6) \\
(r_ir_{i+1})^3 &= 1, \quad 1 \leq i \leq n - 2, \quad (7) \\
(r_{n-1}r_n)^4 &= 1. \quad (8)
\end{align*}
\]

The free group generated by \( r_1, \ldots, r_{n-1}, r_n \), subject to the relations above, for \( 1 \leq i, j < n \), is \( \mathfrak{S}_n \) realized by the simple transpositions \( r_i = (i, i + 1) \) on the set \( [n] \). The group \( B_n \) has \( 2^n n! \) elements and is realized by the signed transpositions \( r_i = (\bar{i}, \bar{i} + 1) \bar{(i, i + 1)}, \bar{i} = 1, \ldots, n - 1, \) and \( r_n = (n, \bar{n}) \) on the set \( \{1 < \cdots < n < \bar{n} < \cdots < 2 < 1\} \). That is, we may see \( B_n \) embedded in \( \mathfrak{S}_{2n} \) by folding \( \{1 < \cdots < n < \bar{n} < \cdots < 2 < 1\} \) through
a central symmetry. The long element of $B_n$ has length $n^2$, while the long element of $G_n$ has length $n(n-1)/2$. For instance, $r_1r_2r_1r_2 = r_2r_1r_2r_1$ is the long element of $B_2$, and $r_3r_2r_1r_3r_2r_1r_3r_2r_1$ is the long element of $B_3$ [Br05].

Occasionally, for the sake of clarity, we write $w_0^A$ and $w_0^C$ for the corresponding longest elements of $G_n$ and $B_n$ respectively, or simply $w_0$ when there is no room for confusion. Given a vector $v \in \mathbb{Z}^n$, we have that $r_i$, with $i \in [n-1]$, acts on $v$, $r_iv$, swapping the $i$-th and the $(i+1)$-th entries, and $r_n$ acts on $v$, $r_nv$, changing the sign of the last entry. Henceforth, $w_0^A$ reverses $v$, $w_0^A(v_1, \ldots, v_n) = (v_n, \ldots, v_1)$, and $w_0^C$ changes the sign of the entries of $v$, $w_0^Cv = -v$.

Recall the $gl(n, \mathbb{C})$ simple roots $\alpha_i = e_i - e_{i+1}$, $i \in [n-1]$, and the $sp(2n, \mathbb{C})$ simple roots $\alpha_i = e_i - e_{i+1}$, $i \in [n-1]$ and $\alpha_n = 2e_n$, where $e_i$, $i \in [n]$, is the $\mathbb{R}^n$ standard basis. The $A_{n-1}$ Dynkin diagram automorphisms above, since $-w_0\alpha_i = -(-\alpha_{n-i}) = \alpha_{n-i}$, is given by $\theta(i) = n - i$, with $i \in I = [n-1]$. For instance,

$$
n = 5 \quad A_4
$$

$$
n = 6 \quad A_5
$$

The $C_n$ Dynkin diagram automorphisms above, since for $w_0 \in B_n$, $-w_0\alpha_i = -(-\alpha_i) = \alpha_i$, is given by $\theta(i) = i$, with $i \in I = [n]$. The weight lattices are $\Lambda = \mathbb{Z}^n$ for both Lie algebra $gl(n, \mathbb{C})$ and $sp(2n, \mathbb{C})$.

3.1. Levi sub-algebras. Let $I$ be a finite Dynkin diagram. A sub-diagram $J$ of $I$ obtained by deleting from $I$ a subset of its nodes is the Dynkin diagram of a semisimple Lie algebra $g_J \subset g_I$ known as a Levi sub-algebra which is the Levi component of the reductive Lie sub-algebra of $g_I$ generated by the Chevalley generators associated to the nodes of $J$.

**Example 1.** If we remove the last node (the one labelled by $n$) from the Dynkin diagram of type $C_n$, we obtain a Dynkin diagram of type $A_{n-1}$ which corresponds to the Levi sub-algebra $sl_n$ of $sp(2n, \mathbb{C})$.

$$
C_n \quad 1 \quad 2 \quad 3 \quad \cdots \quad n-1 \quad n
$$

$$
A_{n-1} \quad 1 \quad 2 \quad 3 \quad \cdots \quad n-1
$$

**Example 2.** The semisimple Lie algebra $sl(3, \mathbb{C}) \times sp(4, \mathbb{C})$ is a Levi sub-algebra of $sp(12, \mathbb{C})$. Note that the semisimple Lie algebra $sl_3 \times sl_2$ is not a Levi sub-algebra of $sp(2n, \mathbb{C})$, as its Dynkin diagram of type $A_{n-1} \times A_1$ cannot be obtained from the type $C_n$ diagram by deleting some of its vertices.
4. Normal $\mathfrak{gl}(n, \mathbb{C})$, $\mathfrak{sp}(2n, \mathbb{C})$-crystals and Levi restrictions

Crystals corresponding to finite-dimensional (quantum group) $U_q(\mathfrak{g})$-representations belong to a family of crystals called normal crystals $[BuSc17, HaKaRyWe20]$. In classical types, these crystals may be realized by a tableau model $[KasNak91]$ and have nice combinatorial properties. Normal crystals arise as the crystals associated to the finite-dimensional representations of a quantum group $U_q(\mathfrak{g})$ for some Lie algebra $\mathfrak{g}$ $[BuSc17]$. These crystals decompose into connected components, one for each irreducible component to the representation at hand. The Levi restriction of a normal crystal is still a normal crystal, and the union of some connected components of a normal crystal is also a normal crystal $[BuSc17, HaKaRyWe20]$. The crystals that we deal with are tableau crystals for finite-dimensional representations of $U_q(\mathfrak{gl}(n, \mathbb{C}))$ and $U_q(\mathfrak{sp}(2n, \mathbb{C}))$.

A $U_q(\mathfrak{sp}(2n, \mathbb{C}))$-crystal is a finite set $\mathfrak{B}$ along with maps

$$\text{wt} : \mathfrak{B} \to \mathbb{Z}^n, \quad e_i, f_i : \mathfrak{B} \to \mathfrak{B} \cup \{0\}, \quad \varepsilon_i, \varphi_i : \mathfrak{B} \to \mathbb{Z},$$

obeying the following axioms for any $b, b' \in \mathfrak{B}$ and $i \in I$,

- $b' = e_i(b)$ if and only if $b = f_i(b)$,
- if $f_i(b) \neq 0$ then $\text{wt}(f_i(b)) = \text{wt}(b) - \alpha_i$;
- if $e_i(b) \neq 0$, then $\text{wt}(e_i(b)) = \text{wt}(b) + \alpha_i$, and
- $\varepsilon_i(b) = \max\{a \in \mathbb{Z}_{\geq 0} : e_i^a(b) \neq 0\}$ and $\varphi_i(b) = \max\{a \in \mathbb{Z}_{\geq 0} : f_i^a(b) \neq 0\}$.

where $\alpha_i = \frac{2\alpha_i}{\langle \alpha_i, \alpha_i \rangle}$ are the coroots.

**Remark 1.** Our abstract $U_q(\mathfrak{g})$-crystals are defined with the additional condition that they are seminormal $[BuSc17]$.

The crystal graph of $\mathfrak{B}$ is the directed graph with vertices in $\mathfrak{B}$ and edges labelled by $i \in I$. If $f_i(b) = b'$ for $b, b' \in \mathfrak{B}$, then we draw an edge $b \to b'$. See Example 4. Given an arbitrary subset $J \subseteq I$, $\mathfrak{B}_J$ is defined to be the crystal $\mathfrak{B}$ restricted to the sub-diagram $J$ of $I$, the Levi branched crystal. The crystal graph of $\mathfrak{B}_J$ has the same vertices as $\mathfrak{B}$ but the arrows are only those labelled in $J$; that is, we forget the maps $e_i, f_i, \varphi_i$, and $\varepsilon_i$, for $i \notin J$ $[BuSc17]$. The weight map is $\mathfrak{B} \xrightarrow{\text{wt}} \Lambda \to \Lambda_J$, where $\text{wt}$ is the weight map of $\mathfrak{B} \Lambda$ is the weight lattice of $\mathfrak{B}$ and $\Lambda_J$ is the weight lattice of $\mathfrak{g}_J$. If we restrict to $J = [n - 1]$, then we obtain a $U_q(\mathfrak{gl}(n, \mathbb{C}))$-crystal. For instance, if we restrict a $U_q(\mathfrak{sp}(2n, \mathbb{C}))$-crystal to $J = [n - 1]$, then we obtain a $U_q(\mathfrak{gl}(n, \mathbb{C}))$-crystal.

Given $b \in \mathfrak{B}$, $\mathfrak{B}(b)$ denotes the unique connected component of $\mathfrak{B}$ containing $b$.

A $U_q(\mathfrak{g})$-crystal is normal if it is isomorphic to a disjoint union of the crystals $\mathfrak{B}(\lambda)$ where $\mathfrak{B}(\lambda)$ is the crystal associated to an irreducible, finite-dimensional $U_q(\mathfrak{sp}(2n, \mathbb{C}))$-representation of highest weight $\lambda$, where $\lambda \in \Lambda$ is a dominant weight. In this work, dominant weights in $\mathbb{Z}^n$ correspond precisely to partitions, that is, weakly decreasing vectors in $\mathbb{Z}^n$ with positive entries. An important property of normal crystals $\mathfrak{B}$ is the existence of a unique highest weight vertex for each connected component of $\mathfrak{B}$ that is, an element which is a source in the corresponding crystal graph, whose weight is dominant. In $\mathfrak{B}(\lambda)$, the highest weight vertex $x$ has weight $\text{wt}(x) = \lambda$. Note that we work solely with highest weight crystals, namely, crystals $\mathfrak{B}$ wherein there exists a
highest weight element $u \in \mathbb{B}$ such that for each $b \in \mathbb{B}$ there exists a finite sequence $a_1, a_2, \ldots, a_l \in I$ such that $b = f_{a_l} \cdots f_{a_2}f_{a_1}(u)$.

4.1. Kashiwara-Nakashima tableaux. Let $\mathbb{B}(\lambda)$ be the irreducible $U_q(\mathfrak{sp}(2n, \mathbb{C}))$-crystal with highest weight a partition $\lambda$ of at most $n$ parts. We realize $\mathbb{B}(\lambda)$ as the crystal $KN(\lambda, n)$ of $C_n$ tableaux, that is, Kashiwara-Nakashima tableaux \cite{KasNak91} of shape $\lambda$ on the alphabet $C_n = \{1 < \cdots < n < \bar{n} < \cdots < 1\}$.

The irreducible $U_q(\mathfrak{gl}(n, \mathbb{C}))$-crystal with highest weight a partition $\lambda$ of at most $n$ parts is realized as the crystal $SSYT(\lambda, n)$ of semi-standard tableaux of shape $\lambda$ on the alphabet $[n]$. We also will refer to these tableaux as the $A_{n-1}$ tableaux of shape $\lambda$. The crystal $SSYT(\lambda, n)$ is a connected sub-crystal of $KN(\lambda, n)$.

Kashiwara-Nakashima tableaux (KN for short) are semi-standard Young tableaux in the alphabet $C_n$ which satisfy some extra conditions. They are a variation of De Concini symplectic tableaux \cite{DeCo79}. A semi-standard Young tableau of any shape (skew or straight) with entries in $C_n$ is KN if and only if the following two conditions hold.

- Each one of its columns is admissible.
- Its splitting is a semi-standard Young tableau.

**Definition 1.** Let $C$ be a semi-standard column in the alphabet $C_n$ of length at most $n$. Let $Z = \{z_1 > \ldots > z_m\}$ be the set of non-barred letters $z$ in $C_n$ such that both $z$ and $\bar{z}$ both appear in $C$. We say that the column $C$ is admissible if there exists a set $T = \{t_1 > \ldots > t_m\}$ of unbarred letters $t$ that satisfies:
  - $t, \bar{t} \not\in C$;
  - $t_1 < z_1$ and is maximal with this property;
  - $t_i < \min(t_{i-1}, z_i)$ and is maximal with this property.

The split of a column is the two-column tableau $lC rC$ where $lC$ is the column obtained from $C$ by replacing $z_i$ by $t_i$ and possibly re-ordering, and $rC$ is obtained from $C$ by replacing $\bar{z}_i$ by $\bar{t}_i$ and possibly re-ordering. The splitting of a tableau consisting of admissible columns is the concatenation of the splits of its columns.

Given $\mu \subseteq \lambda$ partitions with at most $n$ parts, $KN(\lambda/\mu, n)$ denotes the normal $C_n$ crystal of KN tableaux of skew shape $\lambda/\mu$ on the alphabet $C_n$ \cite[Lemma 6.1.3, Corollary 6.3.9]{Lec02}.

**Example 3.** Let $n = 2$. The column \begin{array}{c} 2 \end{array} is admissible, however, \begin{array}{c} 1 \end{array} is not. Notice that although each one of its columns is admissible, the tableau \begin{array}{c} 2 \ 2 \\ 2 \end{array} is not KN, because its split,

\begin{array}{c|c|c|c} 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \end{array}

is not semi-standard.
We will mostly use the notation and definitions from [Lec02, Lec07]. We also refer the reader to the references therein.

**Example 4.** The $U_q(\mathfrak{sp}(4, \mathbb{C}))$ crystal $KN(\lambda, 2)$ for $\lambda = (2, 1)$. Each node in the graph represents an element of the crystal. There is a blue, respectively red, arrow connecting an element $a$ to an element $b$ whenever $f_1(a) = b$, respectively $f_2(a) = b$.

**Remark 2.** [Lec02, Remark 2.2.2] The maximal height of an admissible column is $n$. Moreover, a column $C$ is admissible if and only if, for any $m \in [n]$, the number $N(m)$ of letters $x$ in $C$ such that either $x \leq m$ or $x \geq \bar{m}$ satisfies $N(m) \leq m$. Moreover, if there exists in $C$ a letter $m \leq n$ such that $N(m) > m$, then $C$ contains a pair $(z, \bar{z})$ satisfying $N(z) > z$.

**Remark 3.** In [Lec02], coadmissible columns are defined as well (see [Lec02, p.301]). We will not delve into details, here, however, we remark that there exists a bijection between admissible and coadmissible columns given by filling in the shape of the given admissible column $C$ with the unbarred letters of $lC$ from top to bottom in increasing
order, followed by the barred letters of \( rC \) in the same fashion. We will denote this bijection by \( \Phi \) and use it in \( \mathbb{Z}/2 \).

4.1.1. Levi branching of \( KN \) tableau crystals. For \( J \subseteq I \), \( KN_J(\lambda, n) \) is the restriction of \( KN(\lambda, n) \) to the sub-diagram \( J \) of \( I \); as a crystal graph it has the same set of vertices \( KN(\lambda, n) \) but only the arrows labelled by \( J \), and it is also a normal crystal. The highest weight elements of \( KN_J(\lambda, n) \) are those \( C_n \) tableaux in \( KN(\lambda, n) \) where the only incoming edges are colored in \( [n] \setminus J \). See the LHS of Example 5.

**Example 5.** We have the Levi-branched crystals \( KN(\lambda, 2)\{2\} \) and \( KN(\lambda, 2)\{1\} \) respectively from left to right for \( \lambda = (2, 1) \).

If \( J = [p, q], 1 \leq p < q \leq n \), the crystal graph \( KN_J(\lambda, n) \) consists of the \( KN \) tableaux of \( KN(\lambda, n) \) with arrows colored in \( J \). Recall the \( C_n \) signature rule [KasNak91, Lec02, BuSc17] to compute the action of the crystal operators on a word in the alphabet \( C_n \).
If \( q < n \), the Levi branched crystal \( \mathbf{KN}_{[p,q]}(\lambda, n) \) is a type \( A_{q-p+1} \) normal crystal. The Weyl group is \( W^J = \mathcal{S}_{[p,q+1]} \), the symmetric group on the letters \( \{ p, \ldots, q + 1 \} \) and generators \( r_j = (j,j+1)(j,j+1) \), \( j \in J \). We say that the entries outside of \( [\pm p, q + 1] = \{ p < \cdots < q + 1 \} \) are frozen, which amounts to saying that the KN tableaux of the set \( \mathbf{KN}(\lambda, n) \) in the same connected component of \( \mathbf{KN}_{[p,q]}(\lambda, n) \) are stable in the entries over \( [p,q+1] \setminus [\pm p, q + 1] \) under the action of the Kashiwara operators \( f_i, e_i, i \in [p,q] \). That is, if \( q < n \), in the same connected component of \( \mathbf{KN}_{[p,q]}(\lambda, n) \), the subtableaux consisting of the letters \( \{ 1 < \cdots < p - 1 \} \), \( \{ p - 1 < \cdots < q + 2 \} \) are the same. In Example 5, since \( \mathfrak{sl}(2, \mathbb{C}) = \mathfrak{sp}(2, \mathbb{C}) \), we get two crystals of types \( A_1 = C_2 \).

5. Virtualization

In this section we closely follow Baker [Ba00a, Section 2] and adopt the notation used there. In Example 9.7, we present a detailed example of the content in this section. We include it later rather than earlier because it includes some more information which is not yet presented up to the end of this section.

5.1. Baker embedding and Baker recording tableau. Let

\[
\lambda = \lambda_1 \omega_1 + \cdots + \lambda_n \omega_n \in \mathbb{Z}^n
\]

with \( \omega_j = \sum_{i=1}^{j} e_i \in \mathbb{Z}^n \) the fundamental weights of type \( C_n \). Let

\[
\omega_j^A = \sum_{i=1}^{j} e_i \in \mathbb{Z}^{2n} \text{ for } 1 \leq j \leq n \tag{9}
\]

\[
\omega_j^A = \omega_{2n-j+1}^A = \sum_{i=1}^{2n-j+1} e_i \in \mathbb{Z}^{2n} \text{ for } 1 < j \leq n \tag{10}
\]

be the \( A_{2n-1} \) fundamental weights, and the \( \mathbb{Z}^{2n} \) partition

\[
\lambda^A = 2\lambda_n \omega_n^A + \sum_{i=1}^{n-1} \lambda_i (\omega_i^A + \omega_{i+1}^A).
\]

Let \( \text{SSYT}(\lambda^A, n, \bar{n}) \) be the type \( A_{2n-1} \) crystal of semi-standard Young tableaux in the alphabet \( \mathcal{C}_{n,\bar{n}} \) of shape \( \lambda^A \). We will denote the corresponding crystal operators by \( f_i^A \) for \( i \in \mathcal{C}_{n,\bar{n}} \) and consider, for \( 1 \leq i \leq n \), the operators \( f_i^E = f_i^A f_{i+1}^A \), \( i < n \), and \( f_n^E = (f_n^A)^2 \). Let \( \mathcal{Z} \) denote the virtualization map defined on type \( C_n \) Kashiwara-Nakashima tableaux defined by Baker [Ba00a, Proposition 2.2, Proposition 2.3]. More precisely, \( \mathcal{Z} \) is an injective map.
such that \( \mathcal{E}(f_i(T)) = f_i^E \mathcal{E}(T) \) for \( T \in \mathcal{KN}(\lambda, n) \), \( 1 \leq i \leq n \). We will denote by \( E^{-1} \) the restriction of any left inverse of \( \mathcal{E} \) to the image of \( \mathcal{KN}(\lambda, n) \) under \( \mathcal{E} \).

Given an admissible column \( C \) of shape \( \omega_i \), \( 1 \leq i \leq n \), denote by \( \psi(C) \) its Baker virtual split \([Ba00a, \text{Proposition 2.2}]\), a two column type \( A_{2n-1} \) tableau of shape \( \omega_i^A + \omega_{2n-i}^A \). The map \( \psi \) is injective and embeds admissible columns of length \( i \) into \( \text{SSYT}(\omega_i^A + \omega_{2n-i}^A), 1 \leq i \leq n \). We define \( \psi^{-1} \) analogously to \( E^{-1} \). From [Ba00a, \text{Proposition 2.3}] we know that, if we write \( \psi^{-1} \) as a concatenation of its columns, that is, \( T = C_k \cdots C_1 \), then

\[
\mathcal{E}(T) = [\emptyset \leftarrow w(\psi(C_1)) \leftarrow \cdots \leftarrow w(\psi(C_k))]
\]

where the word \( w(\psi(C)) \) of a two-column \( \psi(C) \) is given by the Japanese reading of its two columns (from top to bottom and right to left), and \( T \leftarrow w \) is the column insertion of a word \( w \) into a semi-standard Young tableau \( T \) \([Fu97]\).

Let \( T_\lambda \in \mathcal{KN}(\lambda, n) \) be the highest weight element; that is, \( T_\lambda \) is the Yamanouchi tableau of shape and weight \( \lambda \) on the alphabet \( [n] \). Then \( \mathcal{E}(T_\lambda) = T_\lambda^A \) is the highest weight element of \( \text{SSYT}(\lambda^A, n, \bar{n}) \) that is, the \( A_{2n-1} \) Yamanouchi tableau of shape and weight \( \lambda^A \) in the alphabet \( C_n^\bar{A} \). The image of \( \mathcal{KN}(\lambda, n) \) by \( \mathcal{E} \) in \( \text{SSYT}(\lambda^A, n, \bar{n}) \) is the crystal generated by acting with the lowering operators \( f_i^E \) on the highest weight element \( T_\lambda^A \) of \( \text{SSYT}(\lambda^A, n, \bar{n}) \). For \( T \in \mathcal{KN}(\lambda, n) \) a tableau, where \( T = C_k \cdots C_1 \), we have

\[
w_T = w(\psi(C_1)) \cdots w(\psi(C_k)).
\]

Then \( w_T \) is a word in \( C_n^\bar{A} \), the monoid of words in the alphabet \( C_n^\bar{A} \) and \( \mathcal{E}(T) = \emptyset \leftarrow w_T \).

We will call the recording tableau of the column insertion of \( w_T \), \( Q(w_T) \), the Baker recording tableau associated to \( T \).

**Proposition 1.** For \( T \in \mathcal{KN}(\lambda, n) \), the Baker recording tableau \( Q(w_T) \) depends only on \( \lambda \). From now on, we will denote by \( Q_\lambda \) the Baker recording tableau associated to \( \lambda \).

**Proof.** By abuse of notation, we will denote by the same symbols the type \( A_{2n-1} \) crystal operators on \( C_n^\bar{A} \) and those on semi-standard Young tableaux in the same alphabet. Now, we know that there exists a sequence \( 1 \leq i_1, \ldots, i_k \leq n \) such that \( f_{i_k} \cdots f_{i_1}(T_\lambda) = T \). Therefore \( f_{i_k}^E \cdots f_{i_1}^E(\mathcal{E}(T_\lambda)) = \mathcal{E}(T) \), where \( \mathcal{E}(T_\lambda) = T_\lambda^A \), the highest weight element of \( \text{SSYT}(\lambda^A, n, \bar{n}) \) and so

\[
f_{i_k}^E \cdots f_{i_1}^E(w_{T_\lambda}) = w_T
\]

(recall that \( f_n^E = (f_n^A)^2 \)) because the connected components of the crystal \( C_n^\bar{A} \) of words of type \( A_{2n-1} \) with highest weight elements \( w_{T_\lambda} \) and \( w(\mathcal{E}(T_\lambda)) = w(T_\lambda) \) have the same weight \( \lambda^A \) and are hence isomorphic. In particular, both \( w_T \) and \( w_{T_\lambda} \) belong to the same connected component of the crystal \( C_n^\bar{A} \) of words of type \( A_{2n-1} \), namely, the connected component containing the Yamanouchi word \( w_{T_\lambda} \) of weight \( \lambda^A \) (recall that all words \( w_T \) have the same rectification shape \( \lambda^A \) and that all \( A_{2n-1} \) crystal operators
commute with *jeu de taquin*). Now, we consider a version of the RSK correspondence \[^{Fu97}\] which is a bijection
\[
\begin{array}{c}
\mathbb{C}^n \leftrightarrow \bigcup_{\ell(\mu) \leq 2n} \text{SSYT}(\mu, n, \bar{n}) \times \text{SYT}(\mu)
\end{array}
\]  \tag{12}

where SYT(\mu) is the set of standard Young tableaux of shape \(\mu\), \(P(w) = \emptyset \leftarrow w\) and \(Q(w)\) is the corresponding recording tableau which encodes the sequence of shapes produced by the column insertion of \(w_T\). In particular, for each standard Young tableau \(Q\) of shape \(\mu\) the pre-image RSK\(^{-1}(\text{SSYT}(\mu, n, \bar{n}) \times \{Q\})\) is a crystal isomorphic to \(\text{SSYT}(\mu, n, \bar{n})\), and all of these pre-images are disjoint and cover \(\mathbb{C}^n\). In particular this means that all the words \(w_T\) for \(T \in \text{KN}(\lambda, n)\) are contained in the same connected component of \(\mathbb{C}^n\).

Thereby, \(Q(w_T) = Q(w_{T_\lambda})\) for all \(T \in \text{KN}(\lambda, n)\). \(\square\)

**Corollary 1.** Let \(\lambda = \omega_{m_1} + \cdots + \omega_{m_k}, 1 \leq m_1 \leq \cdots \leq m_k \leq n\), and let \(\lambda^A = \omega^A_{2n-m_1} + \omega^A_{2n-m_1} + \cdots + \omega^A_{2n-m_k} + \omega^A_{m_k} \in \mathbb{Z}^{2n}\).

Then \(Q_\lambda\) can be written out of the shape \(\lambda^A\) as a sequence of shapes by adding successively the columns \(\omega_{m_1}^A, \omega_{2n-m_1}^A, \ldots, \omega_{m_k}^A, \omega_{2n-m_k}^A\), whose boxes are filled along columns, top to bottom with consecutive numbers from 1 to \(|\lambda^A|\):

\[
\emptyset \subset \omega_{m_1}^A \subset \omega_{2n-m_1}^A + \omega_{m_1}^A \subset \omega_{2n-m_1}^A + \omega_{m_1}^A + \omega_{m_2}^A
\subset \omega_{2n-m_2}^A + \omega_{2n-m_1}^A + \omega_{m_1}^A
\subset \cdots \subset \omega_{m_k}^A + \cdots + \omega_{2n-m_k}^A + \omega_{m_k}^A + \omega_{m_{k-1}}^A
\subset \omega_{2n-m_k}^A + \omega_{m_k}^A + \cdots + \omega_{2n-m_1}^A + \omega_{m_1}^A = \lambda^A.
\]

Given a partition \(\lambda\) with at most \(n\) parts, and \(T = C_k \cdots C_1 \in \text{KN}(\lambda, n)\), let \(\Psi(T) = (w(\psi(C_1)), \ldots, w(\psi(C_k))) \in \mathbb{C}^n\) (here the word is presented as a \(k\)-tuple) and \(\Psi^{-1} = (\psi^{-1}, \ldots, \psi^{-1})\). Then \(E^{-1}(T), Q_\lambda = \text{RSK}\Psi(T) = (P(w_T), Q_\lambda)\) and

\[
E^{-1} = \Psi^{-1}\text{RSK}^{-1}(\mathbb{C}^n) \times \{Q_\lambda\}
\]

where RSK\(^{-1}(\mathbb{C}^n) \times \{Q_\lambda\}\) denotes the inverse of RSK restricted to \(\mathbb{C}^n \times \{Q_\lambda\}\).

**Remark 4.** Let \(T \in \text{KN}(\lambda, n)\) and \(E(T) \in \text{SSYT}(\lambda^A, n, \bar{n})\). Then

\[
\text{wt}(E(T)) = \text{wt}(w_T) = (\alpha_1, \ldots, \alpha_n, \alpha_n, \ldots, \alpha_1) \in \mathbb{Z}^{2n}
\]

is such that

\[
2\text{wt}(T) = (\alpha_1 - \alpha_1, \ldots, \alpha_n - \alpha_n) \in \mathbb{Z}^{2n}.
\]  \tag{14}
5.2. The Levi branched crystal and virtualization. Recall that a Levi branched crystal $\mathcal{B}_J$, $J \subseteq I$, $I$ a Dynkin diagram, is obtained by ignoring the maps $f_i, e_i, \varphi_i, \varepsilon_i$, for $i \notin J$. Let $I$ be the $A_{2n-1}$ Dynkin diagram with nodes $\{1, \ldots, n, \bar{n}, \ldots, \bar{2}\}$.

![Dynkin diagram](image)

For each connected sub-diagram $J = [p, q]$ or $[k, n]$ with $1 \leq p \leq q < n$ and $k \leq n$, of $[n]$, let $\bar{J} = [k, k+1]$ or $[n, k+1]$, if $k < n$, be the corresponding connected sub-diagram of $[\bar{n}, \bar{2}]$, and $\bar{J} = \emptyset$ if $k = n$.

Each connected component of the Levi branched crystal $\text{SSYT}_{J \cup \bar{J}}(\lambda, n)$ with $J = [p, q]$, $[k, n]$, $1 \leq p \leq q < n$, $k \leq n$, is embedded via $\mathcal{L}$ into a connected component of the Levi branched crystal $\text{SSYT}_{J \cup \bar{J}}(\lambda, n)$ such that $J \cup \bar{J}$ is a disconnected diagram of $[1, \ldots, n, \bar{n}, \ldots, \bar{2}]$ if $q < n$, and otherwise, $J \cup \bar{J} = [k, k+1]$ with $n+1 = n$. Consider the Levi branching of the type $C_n$ crystal $\text{KN}(\lambda, n)$ to $A_{q-p+1}$, $1 \leq p \leq q < n$, and $C_{n-k+1}$, $k \leq n$. The Levi type $A_{q-p+1}$ Dynkin diagram is obtained via folding from the Levi subtype $A_{q-p+1} \times A_{q-p+1}$ of $A_{2n-1}$ which is obtained by removing the nodes $1, \ldots, p-1, q+1, \ldots, n, \bar{n}, \ldots, q+2, p+2, \ldots, \bar{2}$ from the $A_{2n-1}$ Dynkin diagram. The Levi type $C_{n-k+1}$, $k \leq n$, is obtained via folding from the Levi subtype $A_{2n-2k+1}$ of $A_{2n-1}$ obtained by removing the nodes $1, \ldots, k-1, \bar{k}, \ldots, \bar{2}$ from the $A_{2n-1}$ Dynkin diagram $[\text{BuSc}17]$.

In [Ba00a Proposition 2.3 (ii)], it is shown that given $b \in \text{KN}(\lambda, n)$, the $C_n$ crystal length functions $\varepsilon_i^c$, $\varphi_i^c$, $1 \leq i \leq n$, on $b$, and the $A_{2n-1}$ crystal length functions $\varepsilon_i^A$, $\varphi_i^A$, $1 \leq i < n$, $\varepsilon_n^A$, and $\varphi_n^A$, $1 \leq i < n$, $\varphi_n^C$, on $\mathcal{L}(b)$ are nicely related:

$$
\varepsilon_i^c(b) = \varepsilon_i^A(\mathcal{L}(b)) = \varepsilon_{i+1}^A(\mathcal{L}(b)), \quad 1 \leq i < n, \quad \text{and} \quad \varepsilon_n^c(b) = 1/2 \varepsilon_n^A(\mathcal{L}(b)),
$$

and similarly for $\varphi_i^c(b)$, $1 \leq i \leq n$, where $\varepsilon_i(b) = \max\{k \in \mathbb{Z}_{\geq 0} : \varepsilon_i^A(b) \neq 0\}$ and $\varphi_i(b) = \max\{k \in \mathbb{Z}_{\geq 0} : f_i^A(b) \neq 0\}$. This means that $b$ is the highest (lowest) weight element of a connected component $U$ of $\text{KN}(\lambda, n)$ if and only if, for all $i \in J$,

$$
\varepsilon_i^A(\mathcal{L}(b)) = \varepsilon_i^A(\mathcal{L}(b)) = \varepsilon_i^C(b) = 0, \quad \text{for all} \ i \in J \setminus \{n\}
$$

and

$$
\varepsilon_n^A(\mathcal{L}(b)) = \varphi_n^C(b) = 0, \quad \text{if} \ n \in J.
$$

Henceforth,

$$
\varepsilon_i^C(b) = 0, \ i \in J \Leftrightarrow \varepsilon_i^A(\mathcal{L}(b)) = 0, \ i \in J \cup \bar{J}
$$

and

$$
\varphi_i^C(b) = 0, \ i \in J \Leftrightarrow \varphi_i^A(\mathcal{L}(b)) = 0, \ i \in J \cup \bar{J}.
$$

In other words, because our crystals are seminormal, $\mathcal{L}(b)$ is the highest weight element of the connected component $V$ of $\text{SSYT}_{J \cup \bar{J}}(\lambda, n)$ containing $\mathcal{L}(b)$ and $\mathcal{L}(U)$. It is therefore unique. A similar statement holds for the lowest weight element. The next proposition now easily follows.
**Remark 6.** Note that when \( T \in \text{SSYT}(\mu, n, \bar{n}) \), with \( \mu \) a partition with at most \( 2n \) parts, \( T \) may be decomposed into two disjoint semi-standard tableaux \( T^+ \) and \( T^- \), where \( T^+ \) is the semi-standard tableau of shape \( \mu/\mu_+ \) on the alphabet \([n]\) defined by the entries of \( T \) in \([n]\), that is, \( T^+ \in \text{SSYT}(\mu, n, \bar{n}) \), called the positive part of \( T \), and \( T^- \) is the semi-standard tableau of shape \( \mu/\mu_- \) on the alphabet \([\bar{n}, 1]\) defined by the entries of \( T \) in \([\bar{n}, 1]\), that is, \( T^- \in \text{SSYT}(\mu, n, \bar{n}) \), called the negative part of \( T \). Provided that we only apply \( f_i^A \), \( e_i^A \) with \( i \in J \cup J' \) disconnected such that \( J \subseteq [n-1] \) and \( J' \subseteq [\bar{n}, 2] \), respectively, this shape decomposition is preserved. Those crystal operators perform the shape decomposition above because, according to the type \( A_{2n-1} \) signature rule, they only change positive (resp. negative) letters into positive (resp. negative) letters.

For \( J \cup J' \) disconnected, \( f_j^j f_j^A = f_j^j f_j^A \), with \( j \in J, j' \in J' \). We then write, for \( \{j_1, \ldots, j_r\} \subseteq J \) and \( \{j'_1, \ldots, j'_m\} \subseteq J' \),

\[
f_{j_1}^A \cdots f_{j_r}^A \cdots f_{j'_1}^A \cdots f_{j'_m}^A(T) = f_{j_r}^A \cdots f_{j_1}^A(T^+) \sqcup f_{j'_m}^A \cdots f_{j'_1}^A(T^-). \tag{15}
\]

6. **The cactus group and its virtualization**

Halacheva [Ha16, HaKaRyWe20] has defined the cactus group \( J_n \) associated to a finite-dimensional reductive Lie algebra \( \mathfrak{g} \) over \( \mathbb{C} \), a generalization of the cactus group \( J_n \) corresponding to \( \mathfrak{gl}(n, \mathbb{C}) \) [HeKa06], in terms of generators and relations.

**Definition 2.** Let \( \mathfrak{g} \) be a finite-dimensional, reductive Lie algebra with Dynkin diagram \( I \). The cactus group \( J_n \) has generators \( s_J \) where \( J \) runs over the connected sub-diagrams of the Dynkin diagram \( I \) of \( \mathfrak{g} \) and relations:

\[
\begin{align*}
& (a) \quad s_J^2 = 1, \quad \text{for all } J \subseteq I, \\
& (b) \quad s_J s_{J'} = s_{J'J} s_J, \quad \text{for all } J, J' \subseteq I \text{ such that } J \cup J' \text{ is disconnected,} \\
& (c) \quad s_{J'} s_J = s_{J(J')} s_J, \quad \text{for all } J' \subseteq J \subseteq I.
\end{align*}
\]

**Remark 6.** Note that when \( J' \subseteq J \), \( c_{J'} \) says that \( s_J \) commutes with \( s_{J'} \) by reversing \( J' \) with respect to \( J \). We also have a group epimorphism \( J_n \to W \) taking \( s_J \) to \( w_0^J \) ([HaKaRyWe20], [Ha16, Remark 10.0.1]). Together with \( \theta_J \), this implies the relations \( w_0^J w_0^J w_0 = w_0^\theta(J) \) and \( w_0^J w_0^J w_0^J = w_0^\theta(J') \).

If \( I \) is the \( A_{n-1} \) Dynkin diagram, \( \theta_J \) acts on \( J \) by reversing the connected interval of nodes \( J \), whereas in the \( C_n \) type it depends on whether \( J \) contains the node with label \( n \) or not.

**Lemma 1.** The cactus group \( J_{\mathfrak{g}(n, \mathbb{C})} = J_n \) is the group with generators \( s_J \), where \( J \) runs over all connected sub-diagrams of \( I = [n-1] \), the \( A_{n-1} \) Dynkin diagram, subject to the relations.
1A. $s^2_j = 1, J \subseteq [n-1]$,
2A. $s_{J} s_{J'} = s_{J'} s_{J}$, for all $J, J' \subseteq [n-1]$ such that $J \cup J'$ is disconnected.
3A. $s_{[p,q]} s_{[k,l]} = s_{[p,q-\ell,p+q-k]} s_{[p,q]}$ for $[k,l] \subset [p,q] \subseteq [n-1]$.

**Proof.** Relations 1 and 2 translate directly to 1A. and 2A. Consider two nested intervals $[k,l] \subset [p,q] \subseteq [n-1]$. The Weyl group $W^{[p,q]}$ is the quotient $W/\text{Stab}_W([1,p-1] \cup [q+1,n])$ and $w_{0}^{J}(\alpha_j) = -\alpha_{p+q-j}$, $j \in J$. Then $\theta_{[p,q]}(d) = p + q - d$ for $d \in [p,q]$. After this observation one sees that Relation 3 translates directly into 1A above.

**Remark 7.** The first and third relations ensure that the $n - 1$ elements of the form $s_{[i,j]}$, $1 \leq k \leq n - 1$, generate $J_n$ since any $s_{[i,j]}$ may be written as

$$s_{[i,j]} = s_{[1,j]}^{i-1} s_{[1,j-1]} s_{[1,j]}.$$

By conjugation with $s_{[i,n-1]}$, the elements $s_{[i,n-1]}$, $1 \leq i \leq n - 1$, also form a set of generators.

**Lemma 2.** The cactus group $J_{\text{sp}(2n,\mathbb{C})}$ is the group with generators $s_{J}$, where $J$ runs over all connected sub-diagrams of $I = [n]$, the $C_n$ Dynkin diagram $I$, subject to the relations

1C. $s^2_j = 1, J \subseteq [n]$,
2C. $s_{J} s_{J'} = s_{J'} s_{J}$, for all $J, J' \subseteq [n-1]$ such that $J \cup J'$ is disconnected,
3C. (i) $s_{[p,n]} s_{[k,l]} = s_{[k,l]} s_{[p,n]}$, $[k,l] \subseteq [p,n] \subseteq [n]$,
(ii) $s_{[p,q]} s_{[k,l]} = s_{[p,q-\ell,p+q-k]} s_{[p,q]}$, $[k,l] \subseteq [p,q] \subseteq [n-1]$.

**Proof.** Relations 1 and 2 translate directly to 1A. and 2A. Consider two nested intervals $[k,l] \subset [p,q]$. If $[p,q] \subset [n-1]$, we are in type $A$, hence 3C.(ii) holds, which is just relation 3A. If $q = n$, then we are in type $C$. The Weyl group $W^{[p,n]}$ is the restriction of the hyperoctahedral group $B_n$ to the generators $r_p, \ldots, r_n$, (as a group of signed permutations, it is the restriction to the set $[\pm p,n] := \{ p < \cdots < n < \bar{n} < \cdots < \bar{p} \}$), and $w_{0}^{J}(\alpha_j) = -\alpha_{j}$ for $j \in J$. Therefore $\theta_{[p,n]}(d) = d$ for $d \in [k,l]$ and Relation 3C. (i) follows directly from 3C.

**Remark 8.** Note that the elements $s_{J}$, $J \subseteq [n-1]$, subject to the relations above, generate the cactus group $J_n$. As in (17), the following are alternative $2n - 1$ generators of $J_{\text{sp}(2n,\mathbb{C})}$:

$$s_{[1,j]}, 1 \leq j \leq n - 1,$$

$$s_{[j,n]}, 1 \leq j \leq n.$$
Remark 9. We may observe that \( \mathfrak{J}_n \) is a subgroup of \( \mathfrak{J}_{\text{sp}(2n,\mathbb{C})} \) defined by the subset of generators \( s_j, J \subseteq [n - 1] \), indexed by connected sub-diagrams of the \( A_{n-1} \) connected sub-diagram \([n - 1]\) of the \( C_n \) Dynkin diagram \( I = [n] \), subject to the relations above 1.C, 2.C and 3.C, (ii).

Proposition 3. If \( g \) is a finite-dimensional reductive Lie algebra, and \( I \subset g \) is a Levi-sub-algebra, then \( J_I \) is a subgroup of \( J_g \).

Proof. Let \( I \) be the Dynkin diagram corresponding to \( g \) and \( J \subset I \) the sub-diagram corresponding to the Levi sub-algebra \( I \). Any connected sub-diagram \( K \) of \( J \) is also a connected sub-diagram of \( I \), hence one can define a map on generators by \( s^f_K \mapsto s^k_K \). Here generators of \( J_g \) are denoted by \( s^f_K \), and generators of \( J_I \) by \( s^k_K \). Remark 9 implies that this map is a morphism of groups. The map is clearly injective because the generators of \( J_g \) are all distinct.

6.1. Embedding of \( \mathfrak{J}_{\text{sp}(2n,\mathbb{C})} \) into \( J_{2n} \). We have observed that \( \mathfrak{J}_n \) is a subgroup of \( \mathfrak{J}_{\text{sp}(2n,\mathbb{C})} \). We now show that there is a group embedding of \( \mathfrak{J}_{\text{sp}(2n,\mathbb{C})} \) into \( J_{2n} \) by folding \( A_{2n-1} \) through the middle node \( n \):

Why should such an embedding exist? Let us consider the following elements of \( J_{2n} \):

\[
s'_{[p,q]} := s_{[p,q]}s_{[2n-q,2n-p]} = s_{[2n-q,2n-p]}s_{[p,q]}, \quad \text{for all } [p, q] \subseteq [n - 1].
\]

In Lemma 4 we show that these elements together with the generators \( s_{[p,2n-p]} \) for \( p \leq n \) generate a subgroup of \( J_{2n} \) isomorphic to \( \mathfrak{J}_{\text{sp}(2n,\mathbb{C})} \). Notice the similarity between this and the construction of \( \text{sp}(2n,\mathbb{C}) \) as a sub-algebra of \( \mathfrak{sl}_2 \), by folding [Kac83] Chapter 8, pp. 89 – 102. Moreover, the following lemma provides not only concrete combinatorial motivation for Lemma 4 but will also be the main ingredient in its proof.

Lemma 3. The following relations hold in \( J_{2n} \):

\[
s_{[p,2n-p]}s_{[k,2n-k]} = s_{[k,2n-k]}s_{[p,2n-p]}, \quad 1 \leq p < k < n, \quad (20)
\]

\[
s_{[p,2n-p]}s'_{[k,l]} = s'_{[k,l]}s_{[p,2n-p]}, \quad 1 \leq p < k < l < n, \quad (21)
\]

\[
s'_{[p,q]}s'_{[k,l]} = s'_{[p+q-l,p+q-k]}s'_{[p,q]}, \quad 1 \leq p < k < l < q < n. \quad (22)
\]

Proof. We have

\[
s_{[p,2n-p]}s_{[k,2n-k]} = A_3, \quad (23)
\]

\[
s_{[p,2n-p]}s_{[2n-(2n-k),2n-k]}s_{[p,2n-p]} = s_{[k,2n-k]}s_{[p,2n-p]}.
\]
for $1 \leq p < k < n$, hence relation (20) holds. Now, for $1 \leq p < k < l < n$ we have:

$$S[p,2n-p][k,l]S[2n-l,2n-k] = S[2n-l,2n-k]S[p,2n-p][2n-l,2n-k]$$

A3.

$$= S[2n-l,2n-k]S[2n-(2n-k),2n-(2n-l)]S[p,2n-p]$$

A3.

$$= S[2n-l,2n-k]S[k,l]S[p,2n-p]$$

which establishes relation (21). Finally, for $1 \leq p < k < l < q < n$ the following holds:

$$S'[p,q]S'[k,l] = S[p,q]S[2n-q,2n-p][k,l]S[2n-l,2n-k]$$

A2.

$$= S[p,q]S[k,l]S[2n-l,2n-k]$$

A3.

$$= S[p+q-l,p+q-k]S[p,q]S[2n-(p+q-k),2n-(p+q-l)]S[2n-q,2n-p]$$

A2.

$$= S[p+q-l,p+q-k][2n-(p+q-k),2n-(p+q-l)]S[p,q]S[2n-q,2n-p]$$

$$= S[p+q-l,p+q-k]S'[p,q].$$

This establishes relation (22). □

**Definition 3.** The virtual symplectic cactus group $\tilde{J}_{2n}$ is the group with generators $\tilde{s}_j$, where $J$ runs over all sub-diagrams of $I = [2n-1]$, the $A_{2n-1}$ Dynkin diagram, of the form $J = [p,2n-p]$ for all $[p,n] \subseteq [n]$, or $J = [p,q] \cup [2n-q,2n-p]$ for all $[p,q] \subseteq [n-1]$ subject to the relations

1. $\tilde{s}_j^2 = 1, J \subseteq [2n-1],$

2. $\tilde{s}_j \tilde{s}_j' = \tilde{s}_j' \tilde{s}_j$, such that $J \cup J'$ is disconnected with respect to all $[p,q] \subseteq [n],$

3. (i) $\tilde{s}_[p,2n-p][q,l] \cap [2n-l,2n-q] = \tilde{s}_[q,l] \cup [2n-l,2n-q][p,2n-p], [q,l] \subseteq [p,n] \subseteq [n],$

(ii) for $[k,l] \subseteq [p,q] \subseteq [n-1],$

$$\tilde{s}_[p,q] \cup [2n-q,2n-p][k,l] \cap [2n-l,2n-k] = \tilde{s}_[q+p-l,q+p-k] \cup [2n-p+2n-q-(2n-k),2n-p+2n-q-(2n-l)] \cap [2n-q,2n-p] = \tilde{s}_[q+p-l,q+p-k] \cup [2n-(p+q)+k,2n-(p+q)+l] \cap [2n-q,2n-p].$$

The following are $2n - 1$ alternative generators of $\tilde{J}_{2n}$

$$\tilde{s}_{[1,j]} \cup [2n-j,2n-1], 1 \leq j \leq n - 1, \quad (23)$$

$$\tilde{s}_{[j,2n-j]}, 1 \leq j \leq n. \quad (24)$$

**Proposition 4.** There is an isomorphism $\tilde{J}_{2n} \simeq J_{sp(2n,\mathbb{C})}$

**Proof.** Clearly $\tilde{J}_{2n}$ and $J_{sp(2n,\mathbb{C})}$ satisfy the same relations corresponding to all connected sub-diagrams $[p,q] \subseteq [n]$. Furthermore, the maps
We check the relations. We begin by showing that the map induced by \( \Gamma \) is indeed a group homomorphism. 

**Proof.** The following assignment defines a group injection from \( \tilde{\Gamma} \) to \( J_{2n} \):

\[
\begin{align*}
J_{\text{sp}(2n, \mathbb{C})} & \rightarrow J_{2n} \\
s_{[p,q]} & \mapsto \tilde{s}_{[p,q]}^{[2n-q,2n-p]} \\
s_{[p,n]} & \mapsto \tilde{s}_{[p,2n-p]}
\end{align*}
\]

are epimorphisms inverse to each other. This follows directly from the definitions of \( J_{2n} \) and \( J_{\text{sp}(2n, \mathbb{C})} \) (Definition 3 and Lemma 2 respectively). Therefore, \( J_{\text{sp}(2n, \mathbb{C})} \simeq J_{2n} \).

**Lemma 4.** The following assignment defines a group injection from \( J_{\text{sp}(2n, \mathbb{C})} \) to \( J_{2n} \):

\[
\begin{align*}
\Gamma : & \quad J_{\text{sp}(2n, \mathbb{C})} \rightarrow J_{2n} \\
s_{[p,q]} & \mapsto s'_{[p,q]} , \quad 1 \leq p \leq q < n, \\
s_{[p,n]} & \mapsto s'_{[p,2n-p]} , \quad 1 \leq p \leq n.
\end{align*}
\]

**Proof.** We begin by showing that the map induced by \( \Gamma \) is indeed a group homomorphism. We check the relations 1C. – 3C. from Lemma 2.

1C. We have that for \( 1 \leq p < q < n \),

\[
\Gamma(s_{[p,q]}^2) = s_{[p,q]}' \cdot s_{[p,q]}' = (s_{[p,q]}s_{[2n-q,2n-p]})^2 = s_{[p,q]}^2 s_{[2n-q,2n-p]}^2 \overset{A1}{=} 1,
\]

while for \( 1 \leq p < n \), we have \( \Gamma(s_{[p,n]}^2) = s_{[p,2n-p]}^2 = 1 \).

2C. For \( 1 \leq p < q < n \) and \( 1 \leq k < l < n \) such that \( [p, q] \cup [k, l] \) is disconnected, the sub-diagrams \( [p, q] \cup [2n-q, 2n-p] \) and \( [k, l] \cup [2n-l, 2n-k] \) of \( [2n-1] \) are disconnected, hence

\[
\Gamma(s_{[p,q]}s_{[k,l]}) \overset{A2}{=} s_{[p,q]}'s_{[k,l]}' = s_{[k,l]}'^{-1}s_{[p,q]}' = \Gamma(s_{[k,l]}s_{[p,q]}).
\]

Additionally, if \( q = n \), the sub-diagram \( [k, l] \cup [p, 2n-p] \cup [2n-l, 2n-k] \) in \( [2n-1] \) is disconnected, hence

\[
\Gamma(s_{[p,n]}s_{[k,l]}) \overset{A2}{=} s_{[p,2n-p]}s_{[k,l]}' = s_{[k,l]}'^{-1}s_{[p,2n-p]} = \Gamma(s_{[k,l]}s_{[p,n]}).
\]

3C. (i) We have that for \( 1 \leq p < k < n \) and \( 1 \leq p < k < l < n \) respectively:

\[
\Gamma(s_{[p,n]}s_{[k,n]}) = s_{[p,2n-p]}s_{[k,2n-k]} \overset{2(1)}{=} s_{[k,2n-k]}s_{[p,2n-p]} = \Gamma(s_{[k,n]}s_{[p,n]})
\]

\[
\Gamma(s_{[p,n]}s_{[k,l]}) = s_{[p,2n-p]}s_{[k,l]}s_{[2n-l,2n-k]} = s_{[2n-l,2n-k]}s_{[p,2n-p]} = \Gamma(s_{[k,l]}s_{[p,n]}).
\]

(ii) Let \( 1 \leq p < k < l < q < n \). Then

\[
\Gamma(s_{[p,q]}s_{[k,l]}) = s_{[p,q]}'s_{[k,l]}' = s_{[p+q-l,p+q-k]}'s_{[p,q]}' = \Gamma(s_{[p+q-l,p+q-k]}s_{[p,q]}).
\]
We have now finished proving that $\Gamma$ is a group morphism. To show that it is injective, one needs to show that its left inverse defined by the assignment

\[
\Gamma_{\text{left}}^{-1} : \text{im}(\Gamma) \subset J_{2n} \mapsto J_{sp(2n, C)}
\]

\[
s'_p \mapsto s[p, q], \quad 1 \leq p \leq q < n,
\]

\[
s'_{p, 2n-p} \mapsto s[p, n], \quad 1 \leq p \leq n.
\]

is also a group morphism. This however follows from the previous calculations: the generators of $\text{im}(\Gamma)$ satisfy the relations from Lemma 2, and there are no more relations between them (all possible cases have been already covered above). \[\square\]

**Proposition 5.** The group $\tilde{J}_{2n}$ is isomorphic to a subgroup of $J_{2n}$.

**Proof.** The map

\[
\tilde{J}_{2n} \mapsto J_{2n}
\]

\[
s[2n-1] \mapsto s'_p, \quad 1 \leq p \leq n
\]

is a a group injection. This follows directly after composing the maps from Proposition 4 and Lemma 4. \[\square\]

We may also think of $\tilde{J}_{2n}$ as the unfolding of $J_{sp(2n, C)}$ in $J_{2n}$.

## 7. Full Schützenberger–Lusztig involutions and algorithms

### 7.1. Full Schützenberger–Lusztig involution

Let $B(\lambda)$ be the normal crystal with highest weight $\lambda$. Let $u_\lambda$ and $u^\text{low}_\lambda$ be the highest, respectively lowest, weight elements of $B(\lambda)$. The Schützenberger–Lusztig involution $\xi$ is the unique set involution $\xi : B(\lambda) \to B(\lambda)$ such that, for all $b \in B(\lambda)$ and $i \in I$,

- $e_i \xi(b) = f_{\theta(i)} \xi(b)$
- $f_i \xi(b) = \xi f_{\theta(i)}(b)$
- $\wt(\xi(b)) = w_0 \wt(b)$

where $w_0$ is the long element of the Weyl group $W$ (see [HeKa06, BuSc17]). The involution $\xi$ acts by $w_0$ on the weights and interchanges the action of $e_i$ and $f_{\theta(i)}$. For $A_n-1$, $\xi$ acts by reversing the weight and interchanges the action of $e_i$ and $f_{n-i}$; for $C_n$, $\xi$ acts by changing the sign of the weight and interchanges the action of $e_i$ and $f_i$.

If $B$ is a normal crystal, $\mathcal{B}$ is the disjoint union of connected components, each of which is a crystal isomorphic to $B(\lambda)$ for some dominant integral weight $\lambda$. We define $\xi_B$ on $\mathcal{B}$ by applying $\xi$ to each one of its connected components. Each element of $B(\lambda)$ is generated by $u_\lambda$ (resp. $u^\text{low}_\lambda$) by applying $f_i$'s (resp. $e_i$'s). Hence the same sequence of $f_i$'s (resp. $e_i$'s) applies to the highest weight (resp. lowest weight) of any connected component of $\mathcal{B}$ isomorphic to $B(\lambda)$. 

The elements $u_\lambda$ and $u_{\lambda}^{\text{low}}$ are the unique elements of $\mathcal{B}(\lambda)$ of weight $\lambda$, respectively $w_0 \lambda$. Hence, $\wt(\xi(u_\lambda)) = w_0 \lambda$, and $\wt(\xi(u_{\lambda}^{\text{low}})) = \lambda$, and $u_{\lambda}^{\text{low}} = \xi(u_\lambda)$, $\xi(u_{\lambda}^{\text{low}}) = u_\lambda$. This implies:

$$u_{\lambda}^{\text{low}} = \xi(u_\lambda) = \xi(e_{j_r} \cdots e_{j_1}(u_{\lambda}^{\text{low}})) = f_{\theta(j_r)} \cdots f_{\theta(j_1)}(\xi(u_{\lambda}^{\text{low}})) = f_{\theta(j_r)} \cdots f_{\theta(j_1)}(u_\lambda).$$

**Corollary 2.** Let $b \in \mathcal{B}(\lambda)$ and $b = f_{j_r} \cdots f_{j_1}(u_\lambda)$. Then

$$\xi(b) = e_{\theta(j_r)} \cdots e_{\theta(j_1)} u_{\lambda}^{\text{low}}, \quad \wt(\xi(b)) = w_0 \wt(b)$$

In particular,

- in type $A_{n-1}$, $\xi(b) = e_{n-j_r} \cdots e_{n-j_1} u_{\lambda}^{\text{low}}$, and $\wt(\xi(b)) = \text{rev} \ \wt(b)$, where $\text{rev}$ is the reverse permutation (long element) of $\mathfrak{S}_n$;

- in type $C_n$, $\xi(b) = e_{j_r} \cdots e_{j_1} u_{\lambda}^{\text{low}}$, and $\wt(\xi(b)) = -\wt(b)$.

For $\mathfrak{g} = \mathfrak{gl}_n$, $\xi$ coincides with the Schützenberger involution [Len07, BerZel96] also known as evacuation (evac for short) on $\text{SSYT}(\lambda/\mu, n)$ [Fu97, St01], and as reversal on the set $\text{SSYT}(\lambda/\mu, n)$ of type $A_{n-1}$ tableaux of skew-shape $\lambda/\mu$ in the alphabet $[n]$ [BSS96].

Let $T \in \mathfrak{B} = \text{SSYT}(\lambda/\mu, n)$ and let $B(T)$ be the connected component of the crystal $\text{SSYT}(\lambda/\mu, n)$ containing $T$. Then $B(T) \simeq B(\nu)$ for some partition $\nu$ and $\text{rectification}(T) \in B(\nu)$. Thereby, $\xi(T)$ is the unique tableau in $B(T)$ such that

$$\text{rectification}(\xi(T)) = \text{evacuation}(\text{rectification}(T)),$$

where $\text{rectification}$ denotes the inverse process of rectification [BSS96, ACM19]. More precisely, the rectification (rect for short) procedure is recorded by assigning to the inner shape $\mu$ of $T$ a standard tableau $S$ to form the tableau pair $(S, T)$. The entries of $S$ govern the jeu de taquin on $T$ by sliding out all letters in the $S$ filling, from the largest to the smallest, to get a new tableau pair $(\text{rect}(T), S')$ where $S'$ is the skew standard tableau consisting of the slid letters from $S$. The anti-rectification procedure, rectification, is defined by the reverse jeu de taquin to evacuation($\text{rectification}(T)$) and is governed by the slid letters in $S'$ in the tableau pair (evacuation($\text{rectification}(T)$), $S'$) from the smallest to the largest. Eventually one obtains the tableau pair $(S_{\text{reversal}}(T))$ where

$$\text{reversal}(T) := \text{rectification}(\text{evacuation}(\text{rectification}(T))).$$

Next we will discuss $\mathfrak{g} = \mathfrak{sp}_{2n}$.

### 7.2. Lecouvey–Sheats symplectic jeu de taquin, Baker-Lecouvey insertion and Knuth equivalence.

If $T$ is a KN tableau, we consider its word $w(T) \in \mathfrak{S}_{2n}^*$ obtained by reading the columns of $T$ from rightmost to leftmost, each column read from top to bottom.
7.2.1. Lecouvey–Sheats symplectic jeu de taquin. \cite{Sh99, Lec02}

Let $T$ be a punctured KN tableau with two columns $C_1$ and $C_2$ and split form $\text{spl}(T) = \ell C_1 r C_1 \ell C_2 r C_2$, and let $C_1$ have the puncture $\ast$. Let $\alpha$ be the entry under the puncture of $r C_1$ and $\beta$ the entry to the right of the puncture of $r C_1$,

$$\text{spl}(T) = \ell C_1 r C_1 \ell C_2 r C_2 = \begin{array}{cccc}
\cdots & \cdots & \cdots & \cdots \\
\ast & \ast & \beta & \cdots \\
\cdots & \alpha & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots 
\end{array},$$

where $\alpha$ or $\beta$ may not necessarily exist. The elementary steps of the symplectic jeu de taquin, or SJDT for short, are the following:

A. If $\alpha \leq \beta$ or $\beta$ does not exist, then the puncture of $T$ will change its position with the cell beneath it. This is a vertical slide.

B. If the slide is not vertical, then it is horizontal. We then have $\alpha > \beta$ or that $\alpha$ does not exist. Let $C'_1$ and $C'_2$ be the columns obtained after the slide. We have two subcases, depending on the sign of $\beta$:

1. If $\beta$ is barred, we are moving a barred letter, $\beta$, from $\ell C_2$ to the punctured box of $r C_1$, and the puncture will occupy $\beta$’s place in $\ell C_2$. Note that $\ell C_2$ has the same barred part as $C_2$ and that $r C_1$ has the same barred part as $\Phi(C_1)$. Looking at $T$, we will have an horizontal slide of the puncture, getting $C'_2 = C_2 \setminus \{\beta\} \sqcup \{\ast\}$ and $C'_1 = \Phi^{-1}(\Phi(C_1) \setminus \ast \sqcup \{\beta\})$. In a sense, $\beta$ went from $C_2$ to $\Phi(C_1)$.

2. If $\beta$ is unbarred, the procedure is similar, but this time $\beta$ will go from $\Phi(C_2)$ to $C_1$; hence $C'_1 = C_1 \setminus \ast \sqcup \{\beta\}$ and $C'_2 = \Phi^{-1}(\Phi(C_2) \setminus \{\beta\} \sqcup \ast)$. However, in this case it may happen that $C'_1$ is no longer admissible. In this situation, if $i$ is the lowest entry such that $i, i$ appear in $C'_1$ and $N(i) > i$, we erase both $i$ and $i$ from the column and remove a cell from the bottom and from the top of the column, and place all the remaining cells orderly.

Applying elementary SJDT slides successively, eventually, the puncture will be a cell such that $\alpha$ and $\beta$ do not exist. In this case we redefine the shape to not include this cell and the jeu de taquin ends. The SJDT when applied to semi-standard tableaux in the alphabet $[n]$ reduces to the ordinary jeu de taquin.

The SJDT is reversible, meaning that we can move $\ast$, the empty cell outside of $\mu$, to the inner shape $\nu$ of a skew tableau $T$ pf shape $\mu/\nu$, simultaneously increasing both the inner and outer shapes of $T$ by one cell. The slides work similarly to the previous case: the vertical slide means that an empty cell is going up, and a horizontal slide means that an entry goes from $\Phi(C_1)$ to $C_2$ or from $C_1$ to $\Phi(C_2)$, depending on whether the slid entry is barred or not, respectively.

7.2.2. Symplectic Knuth equivalence and Baker-Lecouvey symplectic insertion. In this section we gather the necessary tools from \cite{LLT95, Lec02}. For $w \in \mathbb{C}_n^+$ let $P(w)$ be the Kashiwara-Nakashima tableau obtained by performing the Baker–Lecouvey insertion algorithm on $w$. We do not need the algorithm in this paper, but refer the reader to \cite{Ba00b, Lec02} for the original descriptions. A detailed account can also be found in \cite{Sa21b}. Given $w_1, w_2 \in \mathbb{C}_n^+$ the relation $w_1 \sim w_2 \iff P(w_1) = P(w_2)$ defines an equivalence relation on $\mathbb{C}_n^+$ known as plactic equivalence. It is the analogous relation
defined by Knuth relations in the alphabet \([n]\) \[Fu97\]. The symplectic plactic monoid is the quotient \(C^*_n/\sim\). Each plactic class is uniquely identified with a \(KN\) tableau.

The plactic monoid \(C^*_n/\sim\) can also be described as the quotient of \(C^*_n\) by the following \textit{plactic relations} (we use the notation from \[Lec02\]):

\[R1\]
\[y<x<z\] \(\sim\) \(y<z<x\) for \(x\leq y<z\) with \(z\neq \bar{x}\)

\[R2\]
\[1<x\leq y\leq \bar{x}\] \(\sim\) \[x\leq y<z\] for \(x<y<z\) with \(z\neq \bar{x}\)

\[
R3
\]
(Symplectic contraction/dilation relation) \(w\sim w\setminus\{z,\bar{z}\}\), where \(w\in C^*_n\) and \(z\in[n]\) are such that \(w\) is a non-admissible column, \(z\) is the lowest non-barred letter in \(w\) such that \(N(z) = z+1\) and any proper factor of \(w\) is an admissible column.

Remark 10. \[Sa21a\] It can be proven that given a column word \(w\in C^*_n\), any proper factor is admissible if and only if any proper prefix of \(w\) is admissible. Thus, in order to be able to apply the plactic relation \(R3\) to a non-admissible column word \(w\), we need only check that all proper prefixes of \(w\) are admissible, instead of all proper factors.

For example,

\[23\bar{4}43\sim 233\]
\[123\bar{4}43\sim 123\bar{3}\]
\[123\bar{3}\sim 12\bar{3}\]

\(27\)

When Knuth relations are applied to factors of a word, the weight is preserved while the length may not be. Knuth relations can be seen as \textit{jeu de taquin} moves on words or diagonally shaped tableaux, and each symplectic \textit{jeu de taquin} slide preserves the Knuth class of the reading word of a tableau \[Lec02\ Theorem 6.3.8\]. The words 23\bar{3}1 and 11133 are Knuth related: \(T113\sim T131\sim T13\sim 2231\sim 2321\).

7.3. Full symplectic reversal.

7.3.1. Symplectic evacuation algorithm. In \[Sa21a\], Santos introduced a symplectic evacuation algorithm on tableaux in \(\text{KN}(\lambda, n)\) denoted by \(\text{evac}^C_n\) which he proved coincides with the full Lusztig–Schützenberger involution on a given \(U_q(\mathfrak{sp}(2n, \mathbb{C}))\)-crystal \(B(\lambda)\) associated to a representation of highest weight \(\lambda\). The algorithm is defined on a given tableau \(T\in \text{KN}(\lambda, n)\) as follows. First, one complements its entries, that is, replaces all unbarred \(i\)'s by \(\bar{i}\)'s and all \(\bar{i}\)'s by \(i\)’s (this amounts to the action of \(w_0^C = -id\) on the entries of the tableau). Second, one performs a rotation by \(\pi\) to obtain a skew tableau. Finally, one performs symplectic rectification or insertion using Lecouvey–Sheats symplectic jeu de taquin \[Sh99\ Lec02\ Lec07\], or Baker–Lecouvey insertion \[Ba00b\ Lec02\ Lec07\] respectively. The resulting tableau is defined to be \(\text{evac}^C_n(T)\). We refer the reader to \[Sa21a\ Section 5\] for detailed examples of the algorithm. Santos’ evacuation mimics the Schützenberger evacuation on \(\text{SSYT}(\lambda, n)\) by replacing the action of the long element of \(S_n\) with that of the long element of \(B_n\).
7.3.2. Full symplectic reversal on KN skew tableaux. The set $KN(\lambda/\mu,m)$ is a normal $C_m$ crystal whose connected components are isomorphic to $KN(\nu,m)$ for some partition $\nu$ whose number of boxes $|\nu|$ might be less than $|\lambda| - |\mu|$. Let $n = m + j - 1$, where $1 \leq j - 1 < n$ is the number of parts of $\mu$ and $J = [j,n]$. Shifting the entries of the skew KN tableaux in $KN(\lambda/\mu,m)$ by $j - 1$, we may identify $KN(\lambda/\mu,m)$ with the (normal) full sub-crystal $B(\lambda,\mu) \subset [KN_j(\lambda,\mu)]$ consisting of the tableaux in $[KN(\lambda,\mu)]$ with entries exclusively in $1 < \cdots < j < j + 1 < \cdots < j + m < j + m < \cdots < j$ and whose sub-tableaux on the alphabet $\{1,\ldots,j-1\}$ is the fixed Yamanouchi tableau of shape $\mu$ [Lec02] Lemma 6.1.3]. $B(\lambda,\mu)$ is stable under the action of $f_i e_i, i = 1,\ldots,m$, and it decomposes into connected components of $[KN_j(\lambda,\nu)]$. That is, the crystal operators, $f_i, e_i, i = 1,\ldots,m$ do not change the skew-shape of a KN tableau on the alphabet $C_m$, and $KN(\lambda/\mu,m)$ decomposes into connected components that can be identified with the connected components of $B(\mu,\lambda)$.

In both type $A_{n-1}$ and type $C_n$, Kashiwara operators $e_i$ and $f_i$ commute with SJDT slides. Let $T \in B = KN(\lambda/\mu,m)$. An inner corner in $T$ is a box of $\mu$ such that the boxes below and to the right are not in $\mu$; an outer corner in $T$ is a box of $\lambda$ such that the boxes below and to the right are not in $\lambda$. Let $c$ be a fixed inner/outer corner of $T$. An $SJDT$ slide or a complete $SJDT$ slide to the inner corner $c$ means a slide of a box from an inner corner to an outer corner, or vice-versa. An SJDT slide to the inner/outer corner $c$ of $T$ gives a new KN skew tableau $SJDT(T,c)$, possibly with fewer/more boxes. Applying an SJDT slide to the same inner corner $c$ in all vertices of $B(T)$ defines an isomorphic crystal $B(SJDT(T,c))$ [Lec02] Theorem 6.3.8]. The images of the KN tableaux in the same connected component of $KN(\lambda/\mu,m)$ under this crystal isomorphism have the same skew shape [Lec02] Theorem 6.3.8]. Iterating the SJDT to all inner corners of $T$ rectifies $T$, producing $rect(T)$ [Sh99] Proposition 9.2], [Lec02] Theorem 6.1.9, Theorem 6.3.9].

At the end of each SJDT slide, the inner corner (outer corner) where the slide started is filled, or the column where the slide started has 2 fewer (more) boxes [Sh99] Proposition 9.2], [Lec02] Theorem 6.1.9]. The SJDT step where the tableau loses two boxes in a column has a previous step where this column is non-admissible but Knuth equivalent to the new column which is admissible. The step in reverse SJDT where the tableau gains two boxes in a column is $R_3$ Knuth equivalent to the previous one which is admissible. Therefore, in each step of SJDT we get crystals which are isomorphic. This allows, in the vein of reversal for $A_{n-1}$ skew semi-standard tableaux, the definition of symplectic reversal, $\text{reversal}^{C_n}$, on type $C_n$ skew tableaux as a coplactic extension of $\text{evacuation}^{C_n}$.

**Lemma 5.** Let $T \in B = KN(\lambda/\mu,m)$. Then $\xi^{C_n}(T)$ is the unique KN tableau in $B(T)$ that is symplectic Knuth equivalent to $\text{evac}^{C_n} rect(T)$, and

$$\text{rectification} \xi^{C_n}(T) = \text{evacuation}^{C_n}(\text{rectification}(T)).$$  \hspace{1cm} (28)

**Proof.** The crystal $B(T) \simeq B(\nu)$ for some partition $\nu$ and $\text{rectification}(T) \in B(\nu)$. The full Schützenberger-Lusztig involution on KN tableaux of straight shape satisfies $\xi^{C_n}(\text{rect}(T)) = \text{evacuation}^{C_n}(\text{rect}(T))$, and crystal operators commute with SJDT when passing from $B(T)$ to $B(\nu)$. Therefore, (28) holds. \hfill \Box
In Subsection 9.3.1 we will provide an algorithm for partial symplectic reversal on $KN_J(\lambda, n)$ with $J = [j, n]$. An algorithm for full $C_n$ reversal on $KN(\lambda/\mu, n)$ will result as a special case by considering the normal full sub-crystal $B(\mu, \lambda)$ of $KN_J(\lambda, n)$.

8. Internal cactus group action on a normal crystal

Partial Schützenberger involutions were first studied in the case $g = gl(n, \mathbb{C})$ by Berenstein and Kirillov [BerKir95] but have been defined by Halacheva in general for $g$ given $J \subseteq I$ any sub-diagram, the partial Schützenberger–Lusztig involution $\xi_J$ is defined to be the Schützenberger–Lusztig involution $\xi_B$ on the normal crystal $B_J$ [HaKaRyWe20]. The crystal $B_J$ decomposes into connected components, and we apply the Schützenberger–Lusztig involution to each connected component. Let $b \in B$ and let $u^{\text{high}}, u^{\text{low}}$ be the highest and lowest weight elements of the connected component of $B_J$ containing $b$. Let $b = f_{j_1} \cdots f_{j_r}(u^{\text{high}})$, with $j_r \cdots j_1 \in J$. Then

$$\xi_J f_j(b) = f_{\theta_{J\cup J}(j)}(\xi_J b), \quad \xi_J f_{j}(b) = e_{\theta_{J\cup J}(j)}(\xi_J b), \quad j \in J,$$

$$\xi_J b = e_{\theta_{J\cup J}(j_r)} \cdots e_{\theta_{J\cup J}(j_1)}(u^{\text{low}}), \quad \text{wt}_J(\xi_J b) = w_0^J \text{wt}_J(b).$$

**Remark 11.** If $J = K \cup K' \subseteq I$ is disconnected with $K$ and $K'$ connected sub-diagrams of $I$, we have the sub-type Dynkin diagram $K \times K'$, and the Weyl group is $W^K \times W^{K'}$ with longest elements $w_0^K$ and $w_0^{K'}$, respectively, such that $w_0^J = w_0^K w_0^{K'} = w_0^{K'} w_0^K$. Then if $\theta_K$ and $\theta_{K'}$ are the graph automorphisms defined by $w_0^K$ and $w_0^{K'}$ in $K$ and $K'$, respectively, $\theta_J = \theta_K \theta_{K'} = \theta_K \theta_{K'}$ is a graph automorphism of the Dynkin graph $K \times K'$ and hence preserves the connected sub-diagrams $K$ and $K'$ of $I$ as defined in Section 3. Thanks to [HaKaRyWe20, pages 2368, 2369], the crystal operators act componentwise on the normal crystal $B_{K \cup K'}$, and thus

$$f_k f_{k'} = f_{k'} f_k, e_k e_{k'} = e_{k'} e_k \text{ for } k \leq K, k' \leq K',$$

and $b = f_{g_1} \cdots f_{g_i} f_{g_i'} \cdots f_{g_1}(u^{\text{high}})$,

$$\xi_J b = e_{\theta_K(g_1)} \cdots e_{\theta_K(g_i)} e_{\theta_{K'}(g_i')} \cdots e_{\theta_{K'}(g_1')}(u^{\text{low}})$$

with $g_1, \cdots, g_i \in K, g_i', \ldots, g_1' \in K'$ and

$$\{g_1', \ldots, g_i', g_1, \ldots, g_i\} = \{j_1, \ldots, j_r\}.$$

This extends to a disconnected sub-diagram with more than two connected sub-diagrams. Henceforth, from [HaKaRyWe20], $\xi_K$ and $\xi_K'$ commute

$$\xi_K \xi_{K'} = \xi_{K'} \xi_K.$$

**Lemma 6.** Let $J = K \cup K' \subseteq I$ be a disconnected sub-diagram of $I$ with $K$ and $K'$ connected. Then $B_{K \cup K'}$ is a normal crystal, and the Schützenberger–Lusztig involution on $B_{K \cup K'}$, $\xi_{K \cup K'}$, satisfies

$$\xi_{K \cup K'} = \xi_K \xi_{K'} = \xi_{K'} \xi_K.$$

**Proof.** The result follows from [HaKaRyWe20], the remark above, and the fact that $\xi_K \xi_{K'} = \xi_{K'} \xi_K$ is an involution and satisfies the conditions (29), (30) above. Since
there is only one involution on the set $B_{K \cup K'}$ satisfying (29), (30), we have that $\xi_{K \cup K'} = \xi_K \xi_{K'} = \xi_{K'} \xi_K$. \hfill \qed

The partial Schützenberger–Lusztig involutions $\xi_J$ for any $J \subseteq I$ a connected Dynkin sub-diagram of $I$, satisfy the cactus relations.

**Theorem 1** (**[Haa16]**). The map $s_J \mapsto \xi_J$ for all $J \subseteq I$ connected Dynkin sub-diagrams of $I$, defines an action of the cactus group $J$ on the set $\mathcal{B}$ that is, the following is a group homomorphism

$$
\Phi_J : J \rightarrow \mathcal{B}
$$

Moreover, $\text{wt}_J(\xi_J(b)) = w''_J \text{wt}(b), b \in \mathcal{B}$

In other words, $s_J$ acts on each connected component of $\mathcal{B}$ by permuting its vertices via the evacuation exchanging highest weight and lowest weight.

**Remark 12.** (1) The action of $J$ factorizes into the braid relations of $W$. Moreover, for $G = \mathfrak{gl}(n, \mathbb{C})$, and $G = \mathfrak{sp}(2n, \mathbb{C})$, it holds that

$$
\xi_{[n-i,n-1]}^{A_n-1} = \xi_{[1,n-1]}^{A_n-1} \xi_{[1,n-1]}^{A_n-1}, \quad 1 \leq i < n,
$$

$$
\xi^{C_n} = \xi_{[1,n]}^{C_n} \xi_{[1,n]}^{C_n} \xi_{[1,n]}^{C_n}, \quad 1 \leq i \leq n,
$$

respectively.

The following corollary motivates what comes in the next section.

**Corollary 3.** (a) For the $\mathfrak{gl}(n, \mathbb{C})$-crystal $\text{SSYT}(\lambda, n)$, the map

$$
s_{[1,j]} \mapsto \xi_{[1,j]} = \text{evac}_{j+1}, \quad 1 \leq j \leq n - 1,
$$

where $\text{evac}_{j+1}$ denotes the evacuation on the sub-tableaux of straight shape obtained by restricting the entries to $\{1, \ldots, j + 1\}$ and fixing the remaining ones, defines an action of the cactus group $J_n$ on the set $\text{SSYT}(\lambda, n)$.

(b) For the $\mathfrak{sp}(2n, \mathbb{C})$-crystal $\text{KN}(\lambda, n)$ the map

$$
s_{[1,j]} \mapsto \xi_{[1,j]}^{C_n}, \quad 1 \leq j \leq n - 1,
$$

$$
s_{[j,n]} \mapsto \xi_{[j,n]}^{C_n}, \quad 1 \leq j \leq n,
$$

defines an action of $J_{\mathfrak{sp}(2n, \mathbb{C})}$ on the set $\text{KN}(\lambda, n)$, where $\xi_{[1,n]}^{C_n} = \xi_{[1,n]}^{C_n} = \text{evac}_{C_n}$, $\xi_{[1,j]}^{C_n}, 1 \leq j \leq n - 1$ is given by the Baker embedding, Theorem 7 and $\xi_{[j,n]}^{C_n}, 1 \leq j \leq n - 1$ is given either by the partial symplectic reversal in Subsection 9.3.4 or by the Baker embedding, Theorem 3.

8.1. The internal action of the virtual symplectic cactus on a $\mathfrak{gl}(2n, \mathbb{C})$-crystal. On $A_{n-1}$ semi-standard tableaux, there is a straightforward algorithm to compute the action of a partial Schützenberger–Lusztig involution $\xi_J$ with $J$ a connected $A_{n-1}$ Dynkin sub-diagram. Let $I = [n-1]$ and $J = [p,q] \subset I$, $1 \leq p < q < n$, be a connected sub-diagram. The $J$-partial reversal, $\text{reversal}_J$, is the reversal on
Let \( T \in \text{SSYT}(\lambda, n) \) then, from (25) and (26):

\[
\xi(T) = \text{glsjrev}(T)
\]

\[
:= (T_{[1,p-1]}, \text{reversal}(T_{[p+1]}), (33)
\]

\[
\text{rectification(rectification}(T_{[p+1]}))), T_{[q+2,n]}),
\]

where \( T = (T_{[1,p-1]}, T_{[p+1]}, T_{[q+2,n]}) \) is such that \( T_{[1,p-1]} \) is the tableau obtained by restricting \( T \) to the alphabet \([1,p-1]\), \( T_{[p+1]} \) is the skew tableau obtained by restricting to the alphabet \([p,q+1]\), and \( T_{[q+2,n]} \) is obtained by restricting to the alphabet \([q+2,n]\). Indeed, if \( J = [1,q] \), \( \text{reversal}_{[1,q]}(T) = \text{evac}_{q+1}(T) \). The case where \( J \) is a disconnected sub-diagram of \( I \) will be a consequence of Lemma 6.

To define an internal action of the virtual symplectic cactus group \( J_{2n} \) on a crystal \( \text{SSYT}(\mu, n, \tilde{n}) \) with \( \mu \) a partition with at most \( 2n \) parts, thanks to Lemma 6 we now explicitly characterize the partial Schützenberger-Lusztig involution on a disconnected sub-diagram \( J \cup J' \) of the \( A_{2n-1} \) Dynkin diagram such that \( J \subseteq [n-1] \) and \( J' \subseteq [\tilde{n},2] \) are connected sub-diagrams. In the case of the \( A_{2n-1} \) Dynkin diagram, we label its nodes either in \([2n-1]\) or in \([1,\ldots,n,\tilde{n},\ldots,2]\).

**Theorem 2.** Let \( J \cup J' \) be a disconnected sub-diagram of the \( A_{2n-1} \) Dynkin diagram \( I = \{1,\ldots,n,\tilde{n},\ldots,2\} \) such that \( J \subseteq [n-1] \) and \( J' \subseteq [\tilde{n},2] \) are connected sub-diagrams. Then \( \xi_{J,J'}^{A_{2n-1}} \), the Schützenberger-Lusztig involution on \( \text{SSYT}_{J,J'}(\mu, n, \tilde{n}) \), with \( \mu \) a partition with at most \( 2n \) parts, satisfies

\[
\xi_{J,J'}^{A_{2n-1}} = \xi_{J}^{A_{2n-1}} \xi_{J'}^{A_{2n-1}} = \xi_{J}^{A_{2n-1}} \xi_{J'}^{A_{2n-1}}
\]

\[
= \text{reversal}_{J}^{A_{2n-1}} \text{reversal}_{J'}^{A_{2n-1}} = \text{reversal}_{J}^{A_{2n-1}} \text{reversal}_{J'}^{A_{2n-1}}.
\]

where \( \xi_{J}^{A_{2n-1}} = \text{reversal}_{J}^{A_{2n-1}} \) and \( \xi_{J'}^{A_{2n-1}} = \text{reversal}_{J'}^{A_{2n-1}} \) are the Schützenberger-Lusztig involutions on \( \text{SSYT}_{J}(\mu, n, \tilde{n}) \) and \( \text{SSYT}_{J'}(\mu, n, \tilde{n}) \), respectively.

**Remark 13.** This statement is indeed valid for the Schützenberger-Lusztig involution on \( \text{SSYT}_{J,J'}(\mu, n) \) where \( J \cup J' \) is a disconnected sub-diagram of the \( A_{n-1} \) Dynkin diagram with \( n \) odd.

The cactus group \( J_{2n} \) acts on an \( A_{2n-1} \)-crystal of semi-standard tableaux. We now conclude that the virtual symplectic cactus \( J_{2n} \) also does.

**Theorem 3.** For the \( \mathfrak{gl}(2n, \mathbb{C}) \)-crystal of tableaux \( \text{SSYT}(\mu, 2n) \), with \( \mu \) a partition with at most \( 2n \), the map

\[
\tilde{s}_{[1,q]} \cup [2n-q,2n-1] \rightarrow \xi_{[1,q]}^{A_{2n-1}} \xi_{[2n-q,2n-1]}^{A_{2n-1}} = \text{glsjrev}_{[1,q]} \text{glsjrev}_{[2n-q,2n-1]} = \text{evac}_{q+1} \text{evac}_{2n} \text{evac}_{q+1} \text{evac}_{2n},
\]

\[
\tilde{s}_{[q,2n-q]} \rightarrow \xi_{[q,2n-q]}^{A_{2n-1}} = \text{reversal}_{[q,2n-q]}^{A_{2n-1}},
\]

\[
1 \leq q \leq n,
\]

(36)

(37)
defines an action of the virtual symplectic cactus group $\overline{J_{2n}}$ on the set $SSYT(\mu, 2n)$. That is, the following is a group homomorphism

$$\Phi_{B_{2n}} : \overline{J_{2n}} \rightarrow \xi_{J_{A_{2n-1}}}^{E},$$

where $B = SSYT(\mu, 2n)$ and $J$ as in $\text{(35)}$ or $\text{(37)}$.

Proof. Since $J_{2n}$ acts on $SSYT(\mu, 2n)$, the partial Schützenberger involutions $\xi_{J}$ with $J$ a connected sub-diagram of the $A_{2n-1}$ Dynkin diagram $I = [2n-1]$, satisfy the $J_{2n}$ cactus relations. Let $J$ run over all sub-diagrams of the $A_{2n-1}$ Dynkin diagram of the form $J = [q, 2n-q]$, $[q, n] \subseteq [n]$, and $J = [1, q] \cup [2n-q, 2n-1]$, $[1, q] \subseteq [n-1]$. Then

- $(\xi_{[1,q]∪[2n-q,2n-1]})^{2} = (\xi_{[1,q]})^{2}(\xi_{[2n-q,2n-1]})^{2} = 1$, and $(\xi_{[q,2n-q]})^{2} = 1$;
- for $J \cup J'$ disconnected with respect to all $[1, q] \subset [n]$, $(\xi_{J})^{2} = 1$;
- for all $[1, l], [1, q] \subseteq [n], \xi_{[1,l]} \cup \xi_{[1,q]} = \xi_{[1,l,q]} = \xi_{[1,l]} \cup \xi_{[1,q]}$.

Therefore, the partial Schützenberger involutions $\xi_{J}$ with $J$ any connected sub-diagram of the $A_{2n-1}$ Dynkin diagram of the form $J = [q, 2n-q]$, $[q, n] \subseteq [n]$, or $J = [1, q] \cup [2n-q, 2n-1]$, $[1, q] \subseteq [n-1]$, satisfy the virtual symplectic cactus $\overline{J_{2n}}$ relations.

9. Partial Schützenberger–Lusztig involutions and algorithms

For $J$ a connected sub-diagram in the Dynkin diagram $J = [n-1]$ of type $A_{n-1}$, the partial Schützenberger involution $\xi_{J}$ coincides with $J$-partial reversal, that is, $\text{reversal}_{J}$ $\text{(33)}$. The case wherein $J$ is a disconnected sub-diagram of $I$ has been studied in Theorem $2$ and Remark $13$.

So far, there is no known form of tableau-switching for KN tableaux. The algorithm to compute $J$-partial symplectic reversal, $\text{reversal}_{J}$ $\text{(33)}$, with $J = [p, q]$ a sub-diagram in the Dynkin diagram $I$ of type $C_n$, presented in subsection $9.3$ and summarized in $\text{(35)}$, is inspired by this problem and mimics the type $A$ partial reversal algorithm on semi-standard tableaux, summarized in $\text{(33)}$. The case $J = [p, q] \subseteq I$, $p < q < n$, is
solved by virtualization in subsection 9.5.1. In fact, all partial symplectic reversals can be virtualized as shown in subsection 9.5.1.

9.1. J has a sole node and the Weyl group action. If J has a sole node i of I, ξi := ξi(i), the Schützenberger–Lusztig involution to the i-strings (the connected components) of B(i), agrees with the Kashiwara crystal reflection operator, originally studied by Lascoux and Schützenberger in the gln case [LS81] and rediscovered by Kashiwara for any Cartan type [Kas94].

Theorem 4. [Kas94, Section 7] For i ∈ I, ξi defines an action of the Weyl group W on \( B \) \( r_i, b = ξ_i(b) \), and

1. \( r_i.wt(b) = wt(ξ_i(b)) \), for \( b \in B \),
2. \( u_0^i = w_0.u_0^i \).

The i-string of \( b \in B \) \( \varphi_i(ξ_i(b)) = ε_i(b) \), or equivalently \( ε_i(ξ_i(b)) = \varphi_i(b) \)

\[ \begin{array}{cccccccc}
ε_i(b) & b & f_i(b) & ξ_i(b) & & & & \varphi_i(b)
\end{array} \]

Proposition 6. (1) For \( U_q(\mathfrak{sp}(2n, \mathbb{C})) \): given \( i \in [n - 1] \), let \( u^- \) be a word in the alphabet \( \{i, i + 1\} \) with length \( \ell(u^-) = r \), and let \( v^+ \) be a word in the alphabet \( \{i, i + 1\} \) with length \( \ell(v^+) = s \). Then, for all \( r_i \in B_n \), \( 1 \leq i \leq n - 1 \),

\[ r_i.(u^-v^+) = ξ_i(u^-v^+) = \begin{cases} 
    u_1^i e_i^{r+s}(u_2^i) v^+, & r > s \\
    u^- v^+, & r = s \\
    u^- f_i^{r-s}(v_1^i) v_2^i, & r < s
\end{cases} \]  

(38)

such that when \( r > s \), \( u^- = u_1^i u_2^i \), with \( \ell(u_2^i) = r - s \), and when \( r < s \), \( u^- v^+ \) with \( \ell(v^+_1) = s - r \).

When \( i = n \),

\[ u^- n^- r^n = ξ_n(n^- r^n) = n^- n^r. \]  

(39)

(2) If \( b \in B_2(\lambda) \) with \( b = f_{j_1} \cdots f_{j_1}(u_\lambda) \), let \( r_k \cdots r_i r_j \) is a reduced word such that \( u_\lambda^{low} = r_k \cdots r_i r_j , u_\lambda \), then

\[ 3(b) = e_{θ(j_1)} \cdots e_{θ(j_1)}(r_k \cdots r_i r_j , u_\lambda). \]

(3) For \( U_q(\mathfrak{sp}(2n, \mathbb{C})) \): the crystal reflection operators \( ξ_i \) satisfy the relations of the Weyl group \( B_n \):

- \( ξ_i^2 = 1 \), \( 1 \leq i \leq n \),
- \( ξ_i ξ_j = ξ_j ξ_i \) \( |i - j| > 1 \), \( 1 \leq i, j \leq n \),
- \( (ξ_i ξ_{i+1})^3 = 1 \), \( 1 \leq i \leq n - 2 \),
- \( (ξ_{n-1} ξ_n)^4 = 1 \).

Example 6. From \( [KasNak91, Lec02] \), the action of \( ξ_i \) on a KN tableau is given by the signature rule on its reading word [KasNak91, Lec02]:
(1) $T = \begin{array}{cccc}
1 & 2 & 3 & 2 \\
2 & 3 & 3 & 1 \\
4 & 3 & 3 & 1 \\
3 & 3 & 1 & 1 \\
\end{array}$ \quad \rightarrow \quad \begin{array}{cccc}
+ & - & - & 3 \\
- & - & 3 & + \\
+ & - & 3 & + \\
+ & 1 & 3 & + \\
\end{array}
\rightarrow \xi_1(T) = \begin{array}{cccc}
1 & 1 & 1 & 3 \\
2 & 3 & 3 & 1 \\
4 & 3 & 3 & 1 \\
4 & 3 & 3 & 1 \\
\end{array}

\rightarrow \xi_1(T) = \begin{array}{cccc}
1 & 1 & 3 & 2 \\
2 & 3 & 3 & 1 \\
4 & 2 & 2 & 1 \\
3 & 3 & 1 & 1 \\
\end{array}
, \quad \text{wt}(\xi_1(T)) = r_1 \text{wt}(T) = r_1(-2, 1, -1, -1) = (1, -2, -1, -1)

The reading word of $T$ is $12121212$. and

$\xi_1(12121212) = -(+-) - - + (+-) \rightarrow -(+-) + + + (+-) = 12121212$

$\text{wt}(12121212) = (1, -2) = r_1 \text{wt}(12121212) = (2, 1)$

$S = \begin{array}{cccc}
2 & 3 & 3 & 3 \\
4 & 4 & 1 & 1 \\
3 & 3 & 1 & 1 \\
2 & 1 & 1 & 1 \\
\end{array}$ \quad \rightarrow \quad \begin{array}{cccc}
3 & - & 3 & 3 \\
4 & 4 & 1 & - \\
4 & 3 & + & - \\
4 & 1 & + & - \\
\end{array}
\rightarrow \xi_1(S) = \begin{array}{cccc}
2 & 3 & 3 & 2 \\
4 & 1 & 1 & 1 \\
3 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 \\
\end{array}
, \quad \text{wt}(\xi_1(S)) = r_1 \text{wt}(S) = r_1(-3, -1, 2, -1) = (-1, -3, 2, -1)

(2) $\xi_4(T) = \xi_4(12232323) = \xi_4(12232323) = \begin{array}{cccc}
1 & 2 & 2 & 3 \\
2 & 3 & 3 & 1 \\
4 & 2 & 2 & 1 \\
4 & 1 & 1 & 1 \\
\end{array}$

$\text{wt}\xi_4(T) = r_4 \text{wt}(T) = r_4(-2, 1, -1, -1) = (-2, 1, -1, 1)$.

9.2. $J$ has more than one node. The next is a follow-up of the action of the Weyl group on $i$-strings.

**Proposition 7.** Let $B(\lambda)$ be a type $C_n$ crystal, $J \subseteq I$ and $\mathcal{B}_J = B_J(\lambda)$. Let $b \in \mathcal{B}(\lambda)$.

The connected component of $\mathcal{B}_J$ containing $b$ has highest weight element $v^{b^{\text{high}}}_J$ and lowest weight element $v^{b^{\text{low}}}_J$. Then

1. $v^{b^{\text{low}}}_J = r_d \cdots r_3 b^{b^{\text{high}}}_J = \xi_d \cdots \xi_3 b^{b^{\text{high}}}_J$ where $r_d \cdots r_d$ is a short word for $w^J_0 \in W^J$ with $a, \ldots, d \in J$, and $b = f_{j_r} \cdots f_{j_1} b^{b^{\text{high}}}_J$ for some $j_r, \ldots, j_1 \in J$.

2. If $J = [p, n]$, $\mathcal{B}_{[p, n]}$ is a type $C_{n-p+1}$ crystal, then

\[ \mathcal{C}(b) = e_{j_r} \cdots e_{j_1} (r_d \cdots r_3 b^{b^{\text{high}}}_J), \quad \text{wt}_J(\mathcal{C}(b)) = -\text{wt}_J(b), \]

where $\text{wt}_J(x) \in \mathbb{Z}^{n-p+1}$, $x \in \mathcal{B}$ denotes $\text{wt}(x) \in \mathbb{Z}^n$ restricted to the entries in $[p, n]$. 

(3) If $J = [p, q]$, $1 \leq p \leq q < n$, $B_{[p,q]}$ is a type $A_{q-p+1}$ crystal, and $\xi^G(b) = e_{q-p-j_i+1} \cdots e_{q-p-1}a^b_{j,n} = (r_a \cdots r_d.b^b_{j,n})$, $wt_j(\xi^G(b)) = reverse(wt_j(b))$, where $wt_j(x) \in \mathbb{Z}^{q-p+1}$, $x \in B$ denotes $wt(x) \in \mathbb{Z}^n$ restricted to the entries in $[p, q + 1]$.

9.3. Dynkin sub-diagram $J = [j, n]$. On the set $[\text{KN}(\lambda, n)] \xi_C^C = \xi_{[1,n]}^C$, coincides with Santos’ symplectic evacuation $evac(C)$ (see 7.3.1 or [Sa21a, Section 5]). The partial Schützenberger–Lusztig involution $\xi_{[j,n]}^C$ is the Schützenberger–Lusztig involution on each connected component of $\text{KN}_{[j,n]}(\lambda, n)$.

9.3.1. The action of the Knuth operator $[R3]$ on a skew tableau. Given $1 \leq j \leq n$, the Levi branched crystal $\text{KN}_{[j,n]}(\lambda, n)$ decomposes into connected components. Let $T \in [\text{KN}(\lambda, n)]$, which belongs to some connected component of $\text{KN}_{[j,n]}(\lambda, n)$, and let $T_{[\pm j,n]}$ denote the restriction of $T$ to the alphabet $[\pm j, n]$. $T_{[\pm j,n]}$ is a KN skew tableau on the alphabet $C_n$. However, $T_{[\pm j,n]}$ might have non-admissible columns with respect to the alphabet $[\pm j, n]$. This means that by doing a shift of $-(j - 1)$ to the entries of $T_{[\pm j,n]}$, we might produce a non-admissible skew tableau on the alphabet $C_{n-j+1}$.

We show that under the action of the contractor operator $[R3]$ $T_{[\pm j,n]}$ is symplectic Knuth equivalent to a KN skew tableau on the alphabet $[\pm j, n]$. Consequently, the connected component containing $T_{[\pm j,n]}$ is symplectic Knuth equivalent to a crystal connected component of admissible skew tableaux on the alphabet $[\pm j, n]$ (of the same skew shape).

Proposition 8. [Lec02, Proposition 2.3.3] Let $C_1, \ldots, C_k$ be admissible columns on the alphabet $C_n$. Then $T = C_1C_2 \cdots C_k$ is a KN tableau on the alphabet $C_n$ if and only if $\ell(C_i) \leq r(C_{i+1})$, that is, if $\ell(C_i)r(C_{i+1})$ is a type $A_{2n-1}$ semi-standard tableau for $i = 1, \ldots, k - 1$.

Lemma 7. Let $T \in [\text{KN}(\lambda, n)]$. The restriction of $T$ to the alphabet $[\pm j, n] = [n] \setminus \{1 < \cdots < j - 1 < j - 1 < \cdots < 2 < 1\}$, $T_{[\pm j,n]}$, is a KN skew tableau on the alphabet $C_n$ where $T_{[\pm j,n]}$ might have non-admissible columns with respect to the alphabet $[\pm j, n]$.

Proof. If a cell of $T$ has a barred letter in $[\pm j - 1]$, then the cells to the southeast have barred entries in $[\pm j - 1]$, and if a cell of $T$ has a non-barred letter in $[\pm j - 1]$, then the cells to the northwest are non-barred and belong to $[j - 1]$.

The non-barred letters of $T$ in $[\pm j - 1]$ define a partition shape, say $\mu$, in $T$, and the barred letters in $[\pm j - 1]$ define a skew shape $\lambda/\nu$ where $\mu \subseteq \nu \subseteq \lambda$. Hence the cells of $T$ filled in $[\pm j, n] = [n] \setminus \{1 < \cdots < j - 1 < j - 1 < \cdots < 2 < 1\}$ define the skew shape $\nu/\mu$.

Lemma 8. Let $C_1$ and $C_2$ be two columns with entries on the alphabet $[\pm j, n]$ such that $C_1C_2$ is a skew KN tableau on the alphabet $C_n$. Assume that $C_1$ and $C_2$ have exactly $m \geq 0$ and $t \geq 0$ pairs of symmetric entries $(x, \bar{x})$, respectively, with $N(x) > x$ with respect to the alphabet $[\pm j, n]$. The columns are admissible on the alphabet $C_n$ but not necessarily on the alphabet $[\pm j, n]$ when $j > 1$. Then $C_1$ has at least $m$ boxes strictly below the row containing the last box of $C_2$, and $C_2$ has at least $t$ boxes strictly above the row containing the top box of column $C_1$.
Proof. Consider \( \text{spl}(C_1 C_2) = \ell(C_1) r(C_1) \ell(C_2) r(C_2) \) in \([\pm n]\), which is a type \( A_{2n-1} \) semi-standard tableau on the alphabet \( C \). Under the lemma’s assumptions, when \( m > 0 \), \( C_1 \) is not-admissible in \([\pm j, n]\) and has \( m > 0 \) pairs of symmetric entries \((\alpha_i, \bar{\alpha}_i)\) where the 1CC condition breaks at \( \alpha_i, i = 1, \ldots, m; \) when \( t > 0 \), \( C_2 \) is not admissible in \([\pm j, n]\) and has \( t > 0 \) pairs of symmetric entries \((\beta_i, \bar{\beta}_i)\) where the 1CC condition breaks at \( \beta_i, i = 1, \ldots, t \). Therefore, from the definition of \( \text{spl}(C_1 C_2) \), the top box of \( r(C_1) \) is filled in the interval \([\pm j, n]\), and the first \( t \) entries of \( \ell(C_2) \) are filled in \([j-1]\). Since \( r(C_1) \ell(C_2) \) is a type \( A_{2n-1} \) semi-standard tableau on the alphabet \( C \), it follows that the first \( t \) entries of column \( C_2 \) are strictly above the row containing the top box of column \( C_1 \). On the other hand, from the definition of \( \text{spl}(C_1 C_2) \), the last \( m \) boxes of \( r(C_1) \) are filled in \( \{j-1 < 2 < 1\} \), and the bottom box of \( \ell(C_2) \) is filled in \([\pm j, n]\). Similarly, since \( r(C_1) \ell(C_2) \) is a type \( A_{2n-1} \) semi-standard tableau on the alphabet \( C \), it follows that the last \( m \) entries of column \( C_1 \) are strictly below the row containing the bottom box of column \( C_2 \). \( \square \)

Let \( \mathbf{R}_3^m \) denote the iteration of the Knuth operator \( \mathbf{R}_3 \) \( m \geq 0 \) times.

**Proposition 9.** Let \( C_1, C_2 \) be two columns on the alphabet \([\pm j, n]\) such that \( C_1 C_2 \) is a skew KN tableau on the alphabet \( C \) under the conditions of the previous lemma. Let \( C_1 \equiv \mathbf{R}_3^m X \), where \( X \) is an admissible column on \([\pm j, n]\), and \( C_2 \equiv \mathbf{R}_3^{m'} Y \), where \( Y \) is an admissible column on \([\pm j, n]\). Then \( C_1 C_2 \equiv \mathbf{R}_3^{m} \mathbf{R}_3^{m'} XY \) is a skew KN tableau on \([\pm j, n]\).

**Proof.** Under our assumptions, \( C_1 \) has \( m \geq 0 \) pairs of symmetric entries \((\alpha_i, \bar{\alpha}_i)\), where \( N(\alpha_i) > \alpha_i, i = 1, \ldots, m \), with respect to the interval \([\pm j, n]\) (a column is non-admissible on \([\pm j, n]\) if \( m > 0 \)), and the \( \mathbf{R}_3 \) contraction is applied \( m \geq 0 \) times to \( C_1, C_1 \equiv \mathbf{R}_3^m X \), where \( X \) is admissible on \([\pm j, n]\). Henceforth, after applying the contraction \( \mathbf{R}_3 \) \( m \) times to \( C_1 \), the relevant \( m \) pairs of symmetric entries are deleted, the top \( m \) entries and the bottom \( m \) entries of column \( C_1 \) are made empty and the remaining entries of \( C_1 \) are put in order in the remaining \(|C_1| - 2m\) boxes of \( C_1 \) to define the admissible column \( X \) on \([\pm j, n]\). Similarly, under our assumptions about \( C_2, C_2 \) has \( t \geq 0 \) pairs of symmetric entries \((\beta_i, \bar{\beta}_i)\) such that \( N(\beta_i) > \beta_i, i = 1, \ldots, t \), with respect to the interval \([\pm j, n]\). After applying the contraction \( \mathbf{R}_3 \) \( t \) to \( C_2 \), the relevant \( t \) pairs of symmetric entries are deleted, the top \( t \) entries and the bottom \( t \) entries of column \( C_2 \) are made empty and the remaining entries of \( C_2 \) are put in order in the remaining \(|C_2| - 2t\) boxes of \( C_2 \) to define the admissible column \( Y \) on \([\pm j, n]\). Thus, from Lemma 8, the resulting pair \( XY \) of admissible columns has skew shape.

Moreover, \( XY \) is a KN skew tableau on the alphabet \([\pm j, n]\), that is, \( r(X) \ell(Y) \), with entries on the alphabet \([\pm j, n]\), is a type \( A_{2n-1} \) semi-standard tableau. By definition of \( \text{spl}(X) \text{spl}(Y) \), \( r(X) \ell(Y) \) has the same skew shape as \( XY \). Note that

\[
r(X) = r(C_1) \setminus (\{\alpha_1, \ldots, \alpha_m\} \sqcup [\pm j - 1])
\]

is obtained from \( r(C_1) \) by emptying the top \( m \) boxes and bottom \( m \) boxes of \( r(C_1) \) and by filling in order the remaining boxes of \( r(C_1) \) with \( r(C_1) \setminus (\{\alpha_1, \ldots, \alpha_m\} \sqcup [\pm j - 1]) \).
Indeed, from Lemma 8
\[ r(X) \leq \ell(C_2) \setminus [j-1], \]
and in particular, \( r(X) (\ell(C_2) \setminus [j-1]) \) is a semi-standard tableau. Recall that the top box of \( r(C_1) \) is strictly below the top \( t \) boxes of \( r(C_2) \) (exactly the ones in \( \ell(C_2) \) filled in \( [j-1] \)), and the bottom box of \( \ell(C_2) \) is strictly above the bottom \( m \) boxes of \( r(C_1) \) (exactly the ones in \( r(C_1) \) filled in \( (j-1) < \cdots < 1 \)).

Finally, note that
\[ \ell(Y) = \ell(C_2) \setminus ([j-1] \uplus \{ \beta_1, \ldots, \beta_t \}), \]
and if \( r(X) (\ell(C_2) \setminus [j-1]) \) is a semi-standard tableau, then \( r(X) (\ell(C_2) \setminus ([j-1] \uplus \{ \beta_1, \ldots, \beta_t \})) = r(X) \ell(Y) \) is also a semi-standard tableau.

9.3.2. Reduced symplectic jeu de taquin. Given \( T \in \text{KN}(\lambda/\mu, n) \) and \( j \in [n] \) such that \( T \) has all entries in \( [\pm j, n] \), the following is an algorithm to compute the reduced symplectic jeu de taquin on \( T \) on the interval \([\pm j, n]\), denoted \( \text{SJDT}_j \). The skew tableau \( T \) might not be admissible on the alphabet \([\pm j, n]\). This means that we apply the SJDT after shifting all entries in \( T \) by \(-(j-1)\) and iterating on \( T \) the contractor operator \( \text{R3} \) the needed number of times to get an admissible skew tableau on the alphabet \( C_{n-j+1} \). When \( j = 1 \), we recover the ordinary SJDT.

Definition 4. Reduced SJDT (\( \text{SJDT}_j \))
- Let \( T_j \) be the tableau obtained by replacing each non-barred entry \( c \) and barred entry \( \bar{c} \) in \( T \) by \( c-j+1 \) and \( c-j+1 \), respectively.
- If \( T_j \) is not a KN tableau in \( \text{KN}(\lambda/\mu, n-j+1) \), we have some columns containing pairs of the form \( b, \bar{b} \) such that \( b \in [n-j+1] \) is lowest in the column and \( N(b) > b \). Iteratively, we apply the Knuth contractor \( \text{R3} \) operator to \( T_j \) until we make all columns admissible. Define \( T_j \) to be the resulting tableau with all admissible columns.
- Compute SJDT on \( T_j \) as usual.
- Replace each non-barred entry \( m \) and \( \bar{m} \) in \( \text{SJDT}(T_j) \) by \( m+j-1 \) and \( m-j+1 \), respectively.

The reduced rectification to the alphabet \([\pm j, n]\), denoted \( \text{rect}_{j}, \) (\( \text{rect}_{j} \)), of \( T \) is the iteration of the \( \text{SJDT}_{j} \) to all inner corners in \( T \). Indeed, \( \text{rect}_{j}(T) \) is the shift by \( j-1 \) of all entries of \( \text{rect}(T_j) \). When \( j = 1 \) we recover the ordinary rectification.

Here is an illustrative example- first, we compute a complete SJDT slide on the interval \([\pm 1, 3]\):

\[
\begin{array}{c|c|c}
\hline
\pm & 2 & 3 \\
\hline
3 & 2 & 7 \\
\hline
3 & \frac{3}{7} & \frac{3}{7} \\
\hline
\end{array}
\quad \text{SJDT} \quad \rightarrow \quad \begin{array}{c|c|c}
\hline
1 & 1 & * \\
\hline
3 & \frac{3}{7} & \frac{3}{7} \\
\hline
\end{array}
\]

Whereas, complete the \( \text{SJDT}_{2} \) slide, the complete SJDT slide reduced to the interval \([\pm 2, 3]\), is such that:

\[
\begin{array}{c|c|c}
\hline
\pm & 2 & 3 \\
\hline
3 & 2 & 7 \\
\hline
3 & \frac{3}{7} & \frac{3}{7} \\
\hline
\end{array}
\quad \rightarrow \quad T_2 = \begin{array}{c|c|c}
\hline
\pm & 1 & 2 \\
\hline
2 & \frac{2}{7} & \frac{2}{7} \\
\hline
\end{array}
\quad \text{R3} \quad \rightarrow \quad \begin{array}{c|c|c}
\hline
\pm & 2 & 2 \\
\hline
2 & \frac{2}{7} & \frac{2}{7} \\
\hline
\end{array}
\quad \text{SJDT} \quad \rightarrow \quad \begin{array}{c|c|c}
\hline
2 & 2 & * \\
\hline
\end{array}
\]
Therefore,

\[
\begin{array}{c|c|c}
\ast & 2 & \ast \\
3 & 3 & 3 \\
\end{array}
\rightarrow
\begin{array}{c|c|c}
SJDT_2 & 3 & 3 \\
\end{array}.
\]

Another illustration: first we compute an ordinary complete SJDT slide,

\[
\begin{array}{c|c|c}
\ast & 3 & 3 \\
3 & 3 & 3 \\
\end{array}
\rightarrow
\begin{array}{c|c|c}
spl(T) = & 2 & \ast \\
& 2 & 3 \\
& 3 & 2 \\
\end{array}
\rightarrow
\begin{array}{c|c|c}
SJDT & 2 & \ast \\
& 3 & 2 \\
& 3 & 2 \\
\end{array}.
\]

On the other hand, a complete SJDT slide means:

\[
\begin{array}{c|c|c}
\ast & 3 & 3 \\
3 & 3 & 3 \\
\end{array}
\rightarrow
\begin{array}{c|c|c}
T_2 = & 2 & \ast \\
& 2 & 2 \\
& 2 & 2 \\
\end{array}
\rightarrow
\begin{array}{c|c|c}
SJDT & 1 & \ast \\
& 2 & 1 \\
& 1 & 1 \\
\end{array}
\rightarrow
\begin{array}{c|c|c}
R3 & 1 & \ast \\
& 1 & 2 \\
& 2 & 1 \\
\end{array}.
\]

Therefore,

\[
\begin{array}{c|c|c}
\ast & 3 & 3 \\
3 & 3 & 3 \\
\end{array}
\rightarrow
\begin{array}{c|c|c}
SJDT_2 & 2 & \ast \\
& 2 & 2 \\
& 3 & 2 \\
\end{array}
\rightarrow
\begin{array}{c|c|c}
SJDT_2 & 2 & \ast \\
& 3 & 2 \\
& 3 & 2 \\
\end{array}
\rightarrow
\begin{array}{c|c|c}
R3 & 2 & \ast \\
& 3 & 2 \\
& 3 & 2 \\
\end{array}
\rightarrow
\begin{array}{c|c|c}
SJDT_2 & 3 & 2 \\
& 3 & 2 \\
& 3 & 2 \\
\end{array} = \text{rect}_2(T).
\]

9.3.3. Partial symplectic reversal: colorful symplectic tableau switching. Let \( T \in \KN(\lambda, n) \) and \( j \geq 1 \). Let \( \mathcal{E} \) be the crystal connected component of \( \KN_{[j,n]}(\lambda, n) \) containing \( T \). \( \mathcal{E} \) is a highest weight crystal and all vertices of \( \mathcal{E} \) are KN tableaux on the alphabet \( \mathcal{C}_n \), with the letters in \([\pm j - 1]\) frozen, as the crystal operators in \( \mathcal{E} \) are indexed by \([j,n]\) and do not act on the entries filled in \([\pm j - 1]\).

Let \( H \) be the highest weight element of \( \mathcal{E} \) and let \( \text{wt}(H_{[j,n]}) \in \mathbb{Z}^{n-j+1} \) be its highest weight, where \( H_{[\pm j,n]} \) is the restriction of \( H \) to the alphabet \([\pm j, n]\). The restriction of \( H \) to the alphabet \([\pm j, n]\) is a skew KN tableau on the alphabet \( \mathcal{C}_n \). The entries of \( H \) in \([j-1]\) define a semi-standard tableau \( A \) of shape, say \( \mu \), and the entries in \([j-1, 1]\) define a skew semi-standard tableau \( A^- \) of shape \( \lambda/\nu \), where \( \mu \subseteq \nu \subseteq \lambda \). Hence the cells of \( H \) filled in \([\pm j, n] = [n] \setminus \{1 < \cdots < j - 1 < j - 1 < \cdots < 2 < 1\} \) define the skew shape \( \nu/\mu \), and because the crystal operators in \( \mathcal{E} \) are indexed by \([j,n]\), they do not change the skew shape \( \nu/\mu \) either. Therefore, since all the vertices of \( \mathcal{E} \) are connected to \( H \) through those crystal operators, the vertices of \( \mathcal{E} \) restricted to the alphabet \([\pm j, n]\) have the same skew shape \( \nu/\mu \) and the same semi-standard tableaux \( A \) and \( A^- \) [Lec02, Lemma 6.1.3].

Step I. The sequence of isomorphic crystals from \( T_{[\pm j, n]} \) to its reduced rectification. I.1 - The \( C_{n-j+1} \) CONNECTED CRYSTAL \( \mathcal{E}^0 \) CONTAINING \( T_{[\pm j, n]} \).

Erase in the vertices of \( \mathcal{E} \) the entries in \([\pm j - 1]\); that is, erase the semi-standard tableaux \( A \) and \( A^- \). We obtain the connected \( C_{n-j+1} \) crystal \( \mathcal{E}^0 \) of semi-standard skew tableaux of shape \( \nu/\mu \) with entries in the alphabet \([\pm j, n]\), possibly with some non-admissible columns, containing \( T_{[\pm j, n]} \). These KN skew tableaux over \( \mathcal{C}_n \) might
have non-admissible columns over \([\pm j, n]\). More precisely, \(B^0\) is the connected crystal of words on the alphabet \([\pm j, n]\), with highest element the word of \(H_{[\pm j, n]}\).

The set \(B^0\) bijects the set \(B\) with the same crystal graph structure and the same weight vertices as \(B\) Hence, \(B^0\) and \(B\) are isomorphic crystals.

**I.1.1 - The Green Inner Standard Tableau \(U_0\) for any vertex of \(B^0\).**

Define a standard tableau \(U_0\) of shape \(\mu\) filled in a completely ordered alphabet of green letters \(\{g_1 < \cdots < g_{|\mu|}\}\) where \(|\mu|\) is the number of boxes of \(\mu\). Assign the inner standard tableau \(U_0\) the inner shape of each vertex of \(B^0\). Recall \(T_{[\pm j, n]}\) is the image of \(T\) in \(B^0\); see the tableau pair \((U_0, T_{[\pm j, n]\}}\) in Figure 1.

\[
(U_0, T_{[\pm j, n]}) = (\text{Figure 1. } T_{[\pm j, n]} \text{ in the crystal } B^0 \text{ and the inner tableau } U_0).
\]

**I.2 - The \(C_{n-j+1}\) Crystal \(B^x\) of KN Skew Tableaux \([R3]\) Isomorphic to \(B^0\).**

Let \(H^0 := H_{[\pm j, n]}\) be the highest weight element of the \(C_{n-j+1}\) crystal \(B^0\). The skew tableau \(H^0\) of shape \(\nu/\mu\) may have non-admissible columns on the alphabet \([\pm j, n]\). Let \(r < s < \cdots < q < t\) be the non-admissible columns of \(H^0\). Then exactly the same columns in all vertices of \(B^0\) are non-admissible. The Knuth contraction \([R3]\) relation, Subsection 7.2.2, defines a crystal isomorphism; it commutes with the crystal operators and preserves the weight. Moreover, each time \([R3]\) is applied to a column of some vertex of \(B^0\), it is also applied to the same column in every vertex of \(B^0\) (see [Lec02, Proposition 3.2.4, Corollary 3.2.5]).

In each vertex of \(B^0\), apply the \([R3]\) contraction operation to column \(i\), for \(i = r, s, \ldots, q, t\), until column \(i\) becomes admissible. For \(i = r, s, \ldots, q, t\), each time we apply \([R3]\) to column \(i\), a pair of entries \((k, \bar{k})\) is erased (whenever \(k \in [n]\) is minimal for \(N(k) > k, k\) and \(\bar{k}\) appear in the column and all prefixes are admissible). Then the cells from the top and the bottom of the current column \(i\) are emptied; the remaining entries are placed in order in the remaining cells between those erased. We obtain a new crystal of KN skew tableaux on the alphabet \([\pm j, n]\) isomorphic to the crystal \(B^0\).

Let \(x\) be the total times \([R3]\) has to be applied to \(H^0\), from column \(r\) to column \(t\) as explained above, to get a KN skew tableau on alphabet \([\pm j, n]\). Denote the resulting KN skew tableau by \(H^x\). Note that for each column of any vertex of \(B^0\), the number of times \([R3]\) is applied is the same. We then obtain the sequence of isomorphic crystals
Define the standard tableau $U_x$ to $B^x$, with the unprimed purple letters $\{p_1, \ldots, p_x, p_{x+1}, \ldots, p_s\}$ filled with the unprimed letter at the top of the column and the primed letter at the bottom of the column. We impose the order
\[
g_1 < \cdots < g_{|\mu|} < p_1 < p_2 < \cdots < p_x < p'_x < \cdots < p'_2 < p'_1\] (40)
be a completely ordered alphabet of $|\mu| + 2x$ letters consisting of $|\mu|$ green letters and $x$ unprimed and $x$ primed purple letters.

Define the standard tableau $U_x$ of shape $\mu^x$, where $\mu \subseteq \mu^x$ and $|\mu^x| = |\mu| + x$, to be an extension of $U_0$ filled with the $|\mu|$ green letters by filling the extra $x$ cells, the total number of cells made empty at the top of each non-admissible column in a vertex of $B^0$, with the unprimed purple letters $\{p_1 < \cdots < p_x < \cdots < p_s\}$. Define the standard tableau $V_x$ of shape $\nu/\nu^x$ by filling the $x$ cells made empty at the bottom of each non-admissible column in a vertex of $B^0$ with the primed purple letters $p'_x < \cdots < p'_1$. The filling rule is as follows.

Fill successively the pair of cells made empty each time $R_3$ is applied, with one unprimed purple letter and one primed purple letter, $p_1 < p'_1, \ldots, p_x < p'_x, p_{x+1} < p'_{x+1}, \ldots, p_{x+s} < p'_{x+s}, \ldots, p_s < p'_s$, with the unprimed letter at the top of the column and the primed letter at the bottom of the column. We impose the order
\[
g_1 < \cdots < g_{|\mu|} < p_1 < \cdots < p_x < p_{x+1} < \cdots < p_{x+s} < \cdots < p_s < p'_x < \cdots < p'_1.
\]
That is, each time an unprimed purple letter and a primed purple letter are added to $U_x$ and $V_x$, respectively, the unprimed letter is strictly larger than any green letter and any unprimed purple letter already added to $U_x$, and simultaneously, the primed purple letter is strictly smaller than any primed purple letter already added to $V_x$.

By construction, the pair $(U_x, V_x)$ of inner and outer standard tableaux is the same for any vertex of $B^x$. More precisely, $U_x$ of shape $\mu^x$ is the extension of $U_0$ filled with the alphabet $\{g_1 < \cdots < g_{|\mu|} < p_1 < p_2 < \cdots < p_x\}$; $V_x$ of skew shape $\nu/\nu^x$ is filled with the alphabet of primed purple letters $p'_x < \cdots < p'_{x+s} < \cdots < p'_1$. Regarding $U_x$, extend the column $r$ of $U_0$ with the $x_r$ unprimed purple letters $p_1 < \cdots < p_{x_r}$, the column $s$ with the $x_s$ unprimed purple letters $p_{x_r+1} < \cdots < p_{x_r+s}$, and finally the column $t$ with the $x_t$ unprimed purple
letters \( p_{x+\ldots+x+q+1} < \cdots < p_{x+\ldots+x+q+1} = p_x \); regarding \( V_x \) of skew shape \( \nu/\nu_x \), start with the skew shape \( \nu/\mu_0 \), and fill the bottom \( x_r \) boxes of column \( r \) with the alphabet of primed purple letters \( p'_x < \cdots < p'_1 \), the bottom \( x_s \) of column \( s \) with the alphabet \( p'_{x_r+x_s} < \cdots < p'_{x_r+1} \), and, finally, the bottom \( x_t \) boxes of column \( t \) with the alphabet \( p'_x < \cdots < p'_{x_r+x_s+x_t} \). See the triple \((U_x,H^x,V_x)\) in Figure 2.

Figure 2. The triple \((U_x,H^x,V_x)\) with \( H^x \) in gray, \( V_x \) in purple, and \( U_0(\subseteq U_x) \) in green.

### I.3 - Rectification of the \( C_{n-j+1} \) Crystal \( B^x \) and Reduced Rectification of \( T_{[\pm j,n]} \)

Consider the triple of tableaux \((U_x,H^x,V_x)\) previously defined. Apply complete \( SJDT_j \) slides successively to the cells of \( U_x \), from the largest entry to the smallest one, to rectify \( H^x \). At the end of each complete \( SJDT_j \) slide, we get an outer cell filled with the letter where the slide started in \( U_x \). While \( H^x \) is being rectified, the cells of \( U_x \) are slid to end up as outer corners and added to the skew standard tableau \( V_x \).

The rectification of \( H^x \) does not depend on the choice of the inner corner made in each step during the rectification process \[\text{Lec}02\text{ Corollary 6.3.9}\]. Applying \( SJDT_j \) to any corner of \( U_x \) in an element of \( B^x \) (recall that for all elements of \( B^x \), \( U_x \) is the same) gives a crystal isomorphism. This observation is equivalent to the fact that the rectification does not depend on the filling of \( U_x \): \( U_x \) is a choice to keep track of the rectification of \( H^x \) (or of any other vertex of \( B^x \)). If a complete \( SJDT_j \) slide applies to an inner corner of \( H^x \), then a complete \( SJDT_j \) slide also applies to the same inner corner in every vertex of the crystal \( B^x \) and creates the same outer corner filled with the same letter.

However, if the number of boxes of \( H^x \), \(|H^x|\), exceeds the minimal number of boxes of its Knuth class, it will be necessary to apply \( SJDT_j \) more than \(|U_x| = |\mu| + x \) times to rectify \( H^x \). When \( H^x \) has the minimal number of boxes of its Knuth class, only
x unprimed purple letters and $|\mu|$ green letters will slide outwards and join the outer tableau $V_x$.

Let $2y \geq 0$ be the number of boxes of $H^x$ that exceeds the minimal number of boxes of its Knuth class, that is,

$$2y = |H^x| - \text{rectification}_j(H^x)|.$$

When $H^x$ has the minimal number of boxes of its Knuth class, $y = 0$. Necessarily $2y$ boxes of $H^x$ will be lost in the SJDT$_j$ rectification process. Henceforth, the SJDT$_j$ B.2 case will be applied $y$ times, each application creates a non-admissible column followed by the application of a contractor $\text{R3}$ operation resulting in the loss of two boxes.

**Remark 14.** Theorem 6.1.9 in Lecouvey’s paper [Lec02] says: if the B.2 case appears with the creation of a non-admissible column when applying complete SJDT to an inner corner of a KN skew tableau, it has to be at the initial column where the inner corner was originally contained.

This observation implies that each of the $y$ mentioned non-admissible columns will only occur in the columns containing the inner corners where the slide started.

The complete SJDT$_j$ slides applied successively to the entries of $U_x$, as mentioned, will transform the crystal $B^x$ into an isomorphic crystal of KN skew tableaux, as long as the SJDT$_j$ B.2 case does not create a non-admissible column. Otherwise, one has an isomorphic crystal where each vertex has a non-admissible column. In this case, we apply the contractor operator $\text{R3}$ to that column in each vertex, erasing a pair $(k, \bar{k})$ if $k \in [\pm j, n]$ is the lowest entry such that $N(k) > k$. Then, as in I.2 above, the cells from the top and the bottom of the current column are emptied and the remaining entries are placed in order. We get a new isomorphic crystal of KN skew tableaux where each vertex has a non-admissible column. As observed above, this may only happen in the $y$ columns where SJDT$_j$ was applied, specifically, those containing the inner corners where the slides started; no other boxes are deleted in the rectification process of $B^x$.

Eventually, $H^x$ is rectified to $\text{rectification}(H^x)$, as are all vertices of $B^x$, and we get the crystal $R$ of straight KN tableaux with highest weight element $\text{rectification}(H^x)$, $B^0 \simeq B^x \simeq R$.

**I.3.1 - The green-purple-red standard tableau $V$ of every vertex in the $C_{n-j+1}$ crystal $R$ containing $\text{rect}\_j(T_{[\pm j, n]})$.**

Let $B^{x,1}, B^{x,1,-}, B^{x,2}, B^{x,2,-}, \ldots, B^{x,y}, B^{x,y,-}$ be the sequence of $2y$ isomorphic crystals appearing in the rectification process from $B^x$ to $R$, tracking each complete SJDT$_j$ slide which triggers a B.2 case and the subsequent application of a contractor $\text{R3}$ operator to that non-admissible column. In particular, for $i = 1, \ldots, y$, $B^{x,i}$ is the crystal where for the $i$th time in the complete SJDT$_j$ slide, the B.2 case appeared to create a non-admissible column in the column containing the inner corner where the slide started, and $B^{x,i,-}$ is the crystal obtained by applying an $\text{R3}$ contractor operator to that non-admissible column.
For \( i = 1, \ldots, y \), let \( H^{x,i} \) and \( H^{x,i,-} \) be the pair of highest weight elements of the crystal pair \( B^{x,i} \) and \( B^{x,i,-} \), respectively. Each \( H^{x,i} \) has exactly one non-admissible column, and \( H^{x,i,-} \) has non-admissible columns.

We have to store \( 2y \) new auxiliary letters to record the \( 2y \) empty cells created by the \( y \) applications of an \( R3 \) contractor as a consequence of the creation of \( y \) non-admissible columns by the complete \( SJDT_j \) slide where the B.2 case appeared and created a non-admissible column.

Consider the triple of tableaux \( (U_x, H^x, V_x) \) corresponding to the crystal \( B^x \). Let \( (U_{x,1}, H^{x,1}, V_{x,1}) \) be the triple of tableaux obtained from \( (U_x, H^x, V_x) \) by applying complete \( SJDT_j \) slides to the entries of \( U_x \) and transforming the KN skew tableau \( H^x \) into \( H^{x,1} \), where for the first time in the complete \( SJDT_j \) slide, the B.2 case appears and creates a non-admissible column; that is, \( H^{x,1} \) has a non-admissible column, and the highest weight elements of all previous crystals obtained from \( B^x \) had all columns admissible. After the said complete \( SJDT_j \) slides to \( U_x \), \( U_{x,1} \) is the inner standard tableau of \( H^{x,1} \), and \( V_{x,1} \) is obtained from \( V_x \) by adding the sliding entries from \( U_x \) to \( V_x \). \( V_{x,1} \) is indeed a standard tableau because by construction, the entries of \( U_x \) are strictly smaller than the primed purple entries of \( V_x \),

\[ \{ g_1 < \cdots < g_{\mu_1} < p_1 < p_2 < \cdots < p_x < p'_x < \cdots < p'_1 \}. \]

The pair \( (U_{x,1}, V_{x,1}) \) of inner and outer standard tableaux is the same for every vertex of \( B^{x,1} \):

\[ U_{x,1} \subseteq U_x, \quad V_{x,1} \supseteq V_x. \]

We have to apply an \( R3 \) contractor operator to \( H^{x,1} \) (and to every vertex of \( B^{x,1} \)) to transform the non-admissible column into an admissible one: a pair of symmetric entries in each vertex of \( B^{x,1} \) will be deleted, the top and bottom cells of that column will be emptied and the remaining entries will be placed in order. Let \( B^{x,1,-} \) be the new crystal of KN skew tableaux isomorphic to \( B^{x,1} \), and let \( H^{x,1,-} \) be its highest weight element (it two has fewer boxes than \( H^{x,1} \)). Note the number of the column where \( R3 \) acts is the same for every vertex of \( B^{x,1} \). Fill the empty entries with red letters \( r_1 < r'_1 \), with \( r_1 \) on the top and \( r'_1 \) on the bottom, where in the complete \( SJDT_j \) slide, the B.2 case appears and has created a non-admissible column such that \( r_1 \) is strictly larger than any entry of \( U_{x,1} \), and \( r'_1 \) is strictly smaller than any entry of \( V_{x,1} \). \( V_{x,1} \) is filled with the entries of \( U_x \) already slid and with all primed purple letters. The cell with the red letter \( r_1 \) was the cell of \( U_x \) where the complete \( SJDT_j \) slide started and the B.2 case appeared with the creation of a non-admissible column.

Let \( U_{x,1,+} \) be the standard tableau obtained by adding the red letter \( r_1 \) to \( U_{x,1} \), and let \( V_{x,1,+} \) be the standard tableau obtained by adding the primed red letter \( r'_1 \) to \( V_{x,1} \) in the manner described,

\[ U_{x,1} \subset U_{x,1,+} \subset U_x, \quad V_{x,1,+} \supset V_{x,1} \supseteq V_x. \]

We keep applying complete \( SJDT_j \) slides to entries of \( U_{x,1,+} \), from the largest to the smallest, to rectify \( H^{x,1,-} \), so the cell \( r_1 \) will be the first to slide outwards and become an outer corner.
Let \((U_{x,2}, H^{x,2}, V_{x,2})\) be the triple of tableaux obtained from \((U_{x,1,+,}^{+}, H^{x,1,}, V_{x,1,+,}^{+})\) by applying complete \(\text{SJDT}_j\) slides to the entries of \(U_{x,1,+,}^{+}\) and transforming the KN skew tableau \(H^{x,1,}\) into \(H^{x,2}\), where for the second time in the complete \(\text{SJDT}_j\) slide, the B.2 case appears with the creation of a non-admissible column; that is, \(H^{x,2}\) has a non-admissible column, and the highest weight elements of all previous crystals obtained from \(B^{x,1,}\) had all columns admissible. After these complete \(\text{SJDT}_j\) slides to \(U_{x,1,+,}^{+},\) \(U_{x,2}\) is the inner standard tableau of \(H^{x,2}\); \(V_{x,2}\) is obtained from \(V_{x,1,+,}^{+}\) by adding the slid entries from \(U_{x,1,+,}^{+}\) to \(V_{x,1,+,}^{+}\). \(V_{x,2}\) is indeed a standard tableau because by construction the entries of \(U_{x,1,+,}^{+}\) are strictly smaller than the entries of \(V_{x,1,+,}^{+}\). At this point, the red letter \(r_1\) has already slid from \(U_{x,1,+,}^{+}\) to \(V_{x,2}\); that is, \(r_1\) is no longer in \(U_{x,2}\) and instead belongs to \(V_{x,2}\).

\[
U_{x,2} \subseteq U_{x,1} \subseteq U_{x,1,+,}^{+} \subseteq U_x, \quad V_{x,2} \supset V_{x,1,+,}^{+} \supset V_{x,1} \supset V_x
\]

Let \(B^{x,2}\) be the crystal with highest weight element \(H^{x,2}\). We have to apply the \(\text{R3}\) contractor operator to \(H^{x,2}\) (and to every vertex of \(B^{x,2}\)) to transform the non-admissible column into an admissible one: a pair of symmetric entries in each vertex of \(B^{x,2}\) will be deleted, the top and bottom cells of that column will be emptied and the remaining entries will be placed in order. Let \(B^{x,2,-}\) be the new crystal of KN skew tableaux isomorphic to \(B^{x,2}\), and let \(H^{x,2,-}\) be its highest weight element (it has two fewer boxes than \(H^{x,2}\)). Fill the empty entries with red letters \(r_2 < r'_2, r_2\) on the top and \(r'_2\) on the bottom of the column, where in the complete \(\text{SJDT}_j\) slide, the B.2 case appears and has created a non-admissible column such that \(r_2\) is strictly larger than any entry of \(U_{x,2}\), and \(r'_2\) is strictly smaller than any entry of \(U_{x,2}\) already slid. The primed letters are considered to be slid because by the time of their creation, they are outer corners.

The cell with the red letter \(r_2\) was the cell of \(U_{x,1,+,}^{+}\) where the complete \(\text{SJDT}_j\) slide started and the B.2 case appeared with the creation of a non-admissible column. Let \(U_{x,2,+}\) be the standard tableau obtained by adding the red letter \(r_2\) to \(U_{x,2}\), and let \(V_{x,2,+}\) be the standard tableau obtained by adding the primed red letter \(r'_2\) to \(V_{x,2}\). We keep applying complete \(\text{SJDT}_j\) slides to the entries of \(U_{x,2,+}\) from the biggest to the smallest to rectify \(H^{x,2,-}\).

At this point, one has the following relative ordering of the red letters, where \(r_2\) belongs to \(U_{x,2,+}\) and \(r'_2 < r_1 < r'_1\) belong to \(V_{x,2,+}\):

\[
r_2 < r'_2 < r_1 < r'_1,
\]

\[
U_{x,2} \subseteq U_{x,2,+} \subseteq U_{x,1} \subseteq U_{x,1,+,}^{+} \subseteq U_x, \quad V_{x,2,+,}^{+} \supset V_{x,2} \supset V_{x,1,+,}^{+} \supset V_{x,1} \supset V_x.
\]

Continue in this fashion. Let \(B^{x,y}\) be the crystal obtained after a complete \(\text{SJDT}_j\) slide to an entry of \(U_{x,y-1,+,}^{+}\), where the B.2 case arises and creates a non-admissible column for the \(y\)-th time. Let \(U_{x,y}\) be the standard tableau obtained from \(U_{x,y-1,+,}^{+}\) after applying the complete SJDT slides to its entries so far. We have to apply the \(\text{R3}\) contractor operator to every vertex of \(B^{x,y}\) to transform the non-admissible column into an admissible one: a pair of symmetric entries in each vertex of \(B^{x,y}\) will be deleted, the top and bottom cells of that column will be emptied and the remaining
entries will be placed in order. Let $B^{x,y,-}$ be the new crystal of KN skew tableaux isomorphic to $B^{x,y}$. Fill the empty entries with red letters $r_y < r'_y$, as before with $r_y$ on the top and $r'_y$ on the bottom of that column such that $r_y$ is strictly larger than any entry of $U_{x,y}$, and $r'_y$ is strictly smaller than any entry of $U_{x,y-1,+}$ already slid.

The cell with the red letter $r_y$ was the cell of $U_{x,y-1,+}$ where the complete $SJDT_j$ slide started and the B.2 case appeared with the creation of a non-admissible column. Let $U_{x,y,-}$ be the standard tableau obtained by adding the red letter $r_y$ to $U_{x,y}$, and let $V_{x,y,+}$ be the standard tableau obtained by adding the primed red letter $r'_y$ to $V_{x,y}$. We keep applying complete $SJDT_j$ slides to the entries of $U_{x,y,+}$ from the largest to the smallest, and eventually, we rectify $H^{x,y,-}$ without further recourse of the contractor $R3$. We reach the crystal $R$, where every vertex is rectified. The crystal $R$ is called the rectification of $B^0$.

At this point one has the following relative ordering among the $2y$ red letters:

$r_y < r'_y < \cdots < r_2 < r'_2 < \cdots < r_1 < r'_1$

and the rectification storing tableaux

$\emptyset \subset U_{x,y} \subset U_{x,y,+} \subset U_{x,y-1} \subset \cdots \subset U_{x,2} \subset U_{x,2,+} \subset U_{x,1} \subset U_{x,1,+} \subset U_x,

V \supset V_{x,y,+} \supset V_{x,y} \supset V_{x,y-1,+} \supset V_{x,y-1} \supset \cdots \supset V_{x,2,+} \supset V_{x,2} \supset V_{x,1,+} \supset V_{x,1} \supset V_x,$

where $V$ is the standard tableau obtained by adding to $V_{x,y,+}$ via sliding the letters from $U_{x,y,+}$. We have the following ordering of all colored letters, green, purple (primed and unprimed), and red (primed and unprimed) in the skew standard tableau $V$:

$g_1 < \cdots < r_y < r'_y < g_l < \cdots < r_d < r'_d < \cdots < g_{|\mu|} < \cdots < p_1 < p_2 < \cdots < r_k < r'_k < \cdots < p_i < \cdots < r_1 < r'_1 < \cdots < p_x < p'_x < \cdots < p'_1.$

(41)

We have constructed the following sequence of isomorphic crystals, stored in $V$ via the slid colorful letters:

$B^0 \xrightarrow{R3} \cdots \xrightarrow{R3} B^{x_r+x_s} \xrightarrow{R3} \cdots \xrightarrow{R3} B^{x_r+x_s+\cdots+x_1} = B^x$ \hspace{1cm} (42)

$B^x \xrightarrow{SJDT_j} \cdots \xrightarrow{SJDT_j} B^{x_{|\mu|}} \xrightarrow{R3} \cdots \xrightarrow{R3} B^{x_{|\mu|}-} \xrightarrow{SJDT_j} \cdots \xrightarrow{SJDT_j} B^{x_{|\mu|}-} \xrightarrow{R3} B^{x,y,-}$ \hspace{1cm} (43)

$B^{x,y,-} \xrightarrow{SJDT_j} \cdots \xrightarrow{SJDT_j} R.$ \hspace{1cm} (44)

**Remark 15.** In our construction, purple letters are larger than all green ones (40). However, for the red ones together with the two other colors, we just write (41).

### I.4 - The Schützenberger-Lusztig involution on the $C_{n-j+1}$ crystals $B^0$ and its rectification, the crystal $R$, and the reversal.

Let $L^0$ be the lowest weight element of the $C_{n-j+1}$ connected normal crystal $B^0$. The crystal $R$ with highest weight element $\text{rectification}_j(H^0)$ is the rectification of the crystal $B^0$ and contains $\text{rectification}_j(T_{[\pm,j,n]})$. Let $F$ be the composition of the sequence of lowering operators connecting $H^0$ to $T_{[\pm,j,n]}$ in $B^0$, $F(H^0) = T_{[\pm,j,n]}$. 

The Schützenberger-Lusztig involution $F$ in $B^0$ gives $F(T_{[\pm j,n]}) = F^{-1}(L^0)$, where $F^{-1}$ means the sequence obtained by replacing each lowering operator $f_i$ in $F$ with the corresponding raising operator $e_i$. In each crystal of the sequence (42), (43), (44) above, the same sequence $F(F^{-1})$ connects the corresponding highest (lowest) weight element to the corresponding coplactic image of $T_{[\pm j,n]}$. In particular, $F$ connects $\text{rectification}_j(H^0)$ to $\text{rectification}_j(T_{\pm j,n})$, $F(\text{rectification}_jH^0) = \text{rectification}_j(T_{[\pm j,n]})$. By Santos [S21a], the Schützenberger-Lusztig involution in $R$ guarantees that $\text{evac}^C_{n-j+1}(\text{rectification}_j(T_{[\pm j,n]})) = F^{-1}(\text{rectification}_j(L^0))$ is in $R$. Thanks to the crystal isomorphisms and Lemma 5,

$$\text{reversal}^C_{n-j+1}(T_{[\pm j,n]}) = F^{-1}(L^0) = \text{rectification}_j \text{evac}^C_{n-j+1}(\text{rectification}_j(T_{[\pm j,n]})).$$

To compute the reversal of $T_{[\pm j,n]}$ in $B^0$ without using the sequence $F$ of crystal operators and the highest/lowest weight elements $H^0$, $L^0$ of $B^0$, we use Santos’ evacuation on $\text{rect}_j(T_{[\pm j,n]})$ and the rectification sequence of crystals backwards in (42), (43), (44) stored in the standard skew tableau $V$.

**Step II. Computation of symplectic evacuation of $\text{rect}_j(T_{[\pm j,n]})$ in the $C_{n-j+1}$ crystal $R$.**

The tableau $\text{rect}_j(T_{[\pm j,n]})$ is admissible in the alphabet $[\pm j,n]$. Use Santos’ algorithm as follows: take $\pi$-rotation and change the sign of $\text{rect}_j(T_{[\pm j,n]})$; then, apply $\text{SJDT}_j$ to obtain $\text{evac}^C_{n-j+1}(\text{rectification}_j(T_{[\pm j,n]}))$ in the crystal $R$.

Replace the tableau pair $(\text{rectification}_j(T_{[\pm j,n]}), V)$ with $(\text{evac}^C_{n-j+1}(\text{rectification}_j(T_{[\pm j,n]})), V)$.

**Step III. Symplectic reversal of $T_{[\pm j,n]}$ in the $C_{n-j+1}$ crystal $B^0$.**

Consider the pair of tableaux $(\text{evac}^C_{n-j+1}(\text{rectification}_j(T_{[\pm j,n]})), V)$, where $V$ is the standard tableau consisting of all the slid letters in the rectification sequence (42), (43), (44) on the alphabet of green, purple and red letters.

Apply the reverse $\text{SJDT}_j$, $\text{RSJDT}_j$, to the entries of $V$ from the smallest to the largest to send $\text{evac}^C_{n-j+1}(\text{rect}_j(T_{[\pm j,n]}))$ to $\text{reversal}^C_{n-j+1}(T_{[\pm j,n]}) = F^{-1}(L^0)$ in the $C_{n-j+1}$ crystal $B^0$.

When the $\text{SJDT}_j$ applies to an unprimed red letter $r_i$, $i \in \{1, \ldots, y\}$, in $V$, the letter $r_i$ slides to the top of a column with the cell $r'_i$ on the bottom. At this point, we have reached the crystal $B^0$. Then we apply the dilation operator $R3$ to the column containing the pair $(r_i, r'_i)$ by erasing those entries and adding a pair of symmetric entries $(k, k)$ so that we get a non-admissible column on the alphabet $[\pm j,n]$. The $\text{SJDT}_j$ applies now to the next letter bigger than $r'_i$. In this complete reverse slide, the $\text{SJDT}_j$ B.2 case occurs.

When the reverse $\text{SJDT}_j$ slides have been applied to all non-primed purple letters, we have reached the crystal $B^0$, where the columns $r, s, \ldots, t$ have $x_i$ non-primed purple letters $p_{x_1 + \ldots + x_{i-1} + 1} < \cdots < p_{x_1 + \ldots + x_i}$ on the top and the corresponding primed letters on the bottom for $i = r, s, \ldots, t$. Then, for $i = t, \ldots, s, r$, we apply the dilation operator $R3$ to each such column $i$ $x_i$ times, and we reach the crystal $B^0$, where each vertex has non-admissible columns $r, s, \ldots, t$. In particular, we obtain $\text{reversal}^C_{n-j+1}(T_{[\pm j,n]}).$
Remark 16.

- If we put \( j = 1 \) in the colorful algorithm, we reduce to Step II of \( C_n \) evacuation.

- If \( T \in \text{SSYT}(\lambda, n) \subseteq \text{KN}(\lambda, n) \), our colorful tableau switching algorithm reduces to just the green color; that is, to the ordinary tableau switching of the tableu-pair \((U_0, T_{[j,n]})\), with \( U_0 \) a standard tableau of shape \( \mu \) and \( T_{[j,n]} \) a semi-standard skew tableau of shape \( \lambda/\mu \) on the alphabet \([j,n]\). The semi-standard skew-tableau \( T_{[j,n]} \) on the alphabet \([j,n]\) is also a \( C_{n-j+1} \) admissible tableau. 

- \( \text{R3} \) does not apply, and \( SJDT_j \) reduces to the ordinary JDT. Therefore, purple and red colors do not pop up in Step I. This means Step I returns the pair \((\text{rect}_{j}(T_{[j,n]}, V), \text{rect}_{j}(T_{[j,n]}) \text{ a semi-standard tableau in the alphabet } [j,n] \text{ with } |\lambda| - |\mu| \text{ boxes and } V \text{ completely green. Step II computes the symplectic } C_{n-j+1} \text{ evacuation of } \text{rect}_{j}(T_{[j,n]}); \text{this step produces a semi-standard tableau of the same shape on the alphabet } [\bar{n}, \bar{j}] \). Step III applies the ordinary reverse JDT to \( V \) and returns a semi-standard skew tableau of shape \( \lambda/\mu \) on the alphabet \([\bar{n}, \bar{j}]\). Step IV computes the partial reversal \( \text{reversal}_{j}^{\xi_{[j,n]}}(T), J = [j, n] \), by assigning the semi-standard tableau \( A \) of shape \( \mu \) back to the inner shape of \( \text{reversal}_{j}^{\xi_{[j,n]}}(T_{[j,n]}) \).

- The algorithm for the full \( C_m \) reversal of a KN skew tableau \( T \in \text{KN}(\lambda/\mu, m) \) results from our colorful tableau switching algorithm by considering its image \((A, \hat{T})\) in the full sub-crystal \( B(\mu, \lambda) \subseteq \text{KN}_{[j,n]}(\lambda, n) \), where \( n = m + j - 1 \). Let \( B \) be the crystal connected component of \( B(\mu, \lambda) \) containing \((A, \hat{T})\), where \( A \) is the Yamanouchi tableau of shape \( \mu \) and \( \hat{T} \) is obtained by increasing each of the entries of \( T \) by \( j - 1 \). Then, restricting \((A, \hat{T})\) to the alphabet \([\pm j, n]\), \( \hat{T} \) is an admissible \( C_{n-j+1} \) skew tableau in the \( C_{n-j+1} \) crystal \( B^0 \). Our algorithm reduces to Step I with just green and red, Step II and Step III. Finally, we subtract \( j - 1 \) from the entries of \( \text{reversal}_{j}^{\xi_{[j,n]}}(\hat{T}) \) to get \( \text{reversal}_{j}^{\xi_{[j,n]}}(T) \). However, subtraction by \( j - 1 \) replaces the last step in the reduced \( SJDT_j \), and therefore it is enough to apply \( SJDT \).

This means that the algorithm for the full \( C_m \) reversal of the KN skew tableau \( T \) results from our algorithm with \( B^0 = B(T) \) a \( C_m \) crystal, \( x = 0 \) and applying \( SJDT \) to \( U_0 \) to get \((\text{rect}(T), V)\), where \( V \) is a skew standard tableau without purple letters. \( RSJDT \) applied to \( V \) gives

\[
\text{reversal}_{j}^{\xi_{[j,n]}}(T) = \text{arectification evac}_{j}^{\xi_{[j,n]}}(\text{rectification}(T)).
\]
9.4. Examples of full and partial symplectic reversal.

Example 7. Full reversal of a skew tableau, \( J = I \). In this case, we have no purple letters, as no letters are deleted at the beginning. Let \( T = \begin{array}{ccc}
2 & 3 & 1 \\
3 & 4 & 1 \\
4 & 4 & 1 \\
1 & 1 & 1 \\
\end{array} \in KN((4, 3, 2)/(1, 3)). \) We compute \( \xi_{C^3}(T) \) as follows. First, we fill in the empty box in \( T \) with a green letter (it defines the one box standard tableau \( U_0 \), to which we perform symplectic jeu de taquin until it becomes an outer corner.

\[
(U_0, T) = \begin{array}{ccc}
2 & 2 & 1 \\
2 & 2 & 1 \\
1 & 1 & 1 \\
\end{array} \quad \xrightarrow{SJDT} \quad \begin{array}{ccc}
1 & 2 & 1 \\
2 & 1 & 1 \\
1 & 1 & 1 \\
\end{array} \quad \xrightarrow{SJDT} \quad \begin{array}{ccc}
1 & 2 & 1 \\
2 & 1 & 1 \\
1 & 1 & 1 \\
\end{array} \quad \xrightarrow{SJDT} \quad \begin{array}{ccc}
1 & 2 & 1 \\
2 & 1 & 1 \\
1 & 1 & 1 \\
\end{array} \quad \xrightarrow{SJDT} \quad \begin{array}{ccc}
1 & 2 & 1 \\
2 & 1 & 1 \\
1 & 1 & 1 \\
\end{array}
\]

We replace \( \text{rect}(T) \) with \( \text{evac}^{C^3}(\text{rect}(T)) \) in \( (\text{rect}(T), V) \) and apply reverse SJDT to \( V \) to compute \( \xi_{C^3}(T) = \text{reversal}^{C^3}(T) \):

\[
(\text{evac}^{C^3}(\text{rect}(T)), V) = \begin{array}{ccc}
1 & 1 & 2 \\
2 & 2 & 2 \\
\end{array} \quad \xrightarrow{RSJDT} \quad \begin{array}{ccc}
1 & 1 & 2 \\
2 & 2 & 2 \\
\end{array} \quad \xrightarrow{RSJDT} \quad \begin{array}{ccc}
1 & 1 & 2 \\
2 & 2 & 2 \\
\end{array} \quad \xrightarrow{RSJDT} \quad \begin{array}{ccc}
1 & 1 & 2 \\
2 & 2 & 2 \\
\end{array} = (U_0, \xi_{C^3}(T)) \Rightarrow \xi_{C^3}(T) = \begin{array}{ccc}
1 & 1 & 2 \\
2 & 2 & 2 \\
\end{array}
\]

Example 8. Let \( P = \begin{array}{ccc}
1 & 2 & 2 \\
4 & 4 & 3 \\
4 & 2 & 1 \\
\end{array} \in KN((4, 3, 3, 1), 4). \) We have wt\( (P) = (-1, 1, -2, 1). \)

To compute \( \xi_{C^3}(P) = \text{reversal}^{C^3}(P) \), we freeze the letters 1, 1 in \( P \) and consider \( P_{[\pm 2, 4]} \). \( P_{[\pm 2, 4]} \) is not an admissible \( C_3 \) tableau in the alphabet \([\pm 2, 4]\): the second column \( 24 \) is not an admissible \( C_3 \) column; \( 24 \xrightarrow{SJDT} 4 \). The column reading of \( P_{[\pm 2, 4]} \)
is $23242443 \equiv R_3 \equiv 234443$. We include this non-admissible second column in the \( SJDT_2 \) sequence to rectify $P_{[\pm 2,4]}$.

1. **Rectification of $P_{[\pm 2,4]}$**

\[
(U_0, P_{[\pm 2,4]}) = \begin{array}{c|c|c}
4 & 4 & 3 \\
3 & 2 & \\
\end{array} \xrightarrow{SJDT_2} \begin{array}{c|c|c}
4 & 4 & 3 \\
3 & p' & \\
\end{array} \xrightarrow{SJDT_2} \begin{array}{c|c|c}
4 & 4 & p \\
3 & p' & \\
\end{array} \xrightarrow{SJDT_2} \begin{array}{c|c|c}
2 & 4 & 3 \\
4 & p & \\
\end{array}
\]

\[
SJDT_2 \rightarrow \begin{array}{c|c|c}
2 & 4 & 3 \\
3 & p' & \\
\end{array} \xrightarrow{SJDT_2} \begin{array}{c|c|c}
2 & 4 & 3 \\
3 & p' & \\
\end{array}
\]

Rectification of $P_{[\pm 2,4]}$ with \( SJDT_2 \) = \( rect_2 P_{[\pm 2,4]} \).

\[
V = \begin{array}{c|c|c}
g & p & \\
r & p' & \\
r' & \\
\end{array}, \quad r < r' < g < p < p'.
\]

2. **Computation of evac\(_3 \) rect\(_2 \)(\( P_{[\pm 2,4]} \)).** Taking \( \pi \)-rotation and changing the signs of \( rect_2 (P_{[\pm 2,4]}) \), we again apply \( SJDT_2 \):

\[
SJDT_2 \rightarrow \begin{array}{c|c|c}
3 & 3 & 2 \\
3 & 4 & \\
\end{array} \xrightarrow{SJDT_2} \begin{array}{c|c|c}
3 & 3 & 2 \\
4 & 4 & \\
\end{array} \xrightarrow{SJDT_2} \begin{array}{c|c|c}
3 & 3 & 2 \\
4 & 4 & \\
\end{array} = evac_3 rect_2 P_{[\pm 2,4]}.
\]

3. **Reversal of $P_{[\pm 2,4]}$.** Replace \( rect_2 (P_{[\pm 2,4]}) \) with \( evac_3 (rect_2 (P_{[\pm 2,4]})) \) in \( (rect_2 (P_{[\pm 2,4]}), V) \) and apply \( RSJDT_2 \) to $V$.

\[
(evac_3 (rect_2 P_{[\pm 2,4]}), V) = \begin{array}{c|c|c}
3 & 3 & 7 \\
4 & g & p \\
r & p' & \\
r' & \\
\end{array} \xrightarrow{RSJDT_2} \begin{array}{c|c|c}
3 & 3 & 7 \\
4 & g & p \\
r & p' & \\
r' & \\
\end{array} \xrightarrow{RSJDT_2} \begin{array}{c|c|c}
3 & 3 & 7 \\
4 & g & p \\
r & p' & \\
r' & \\
\end{array} \xrightarrow{RSJDT_2} \begin{array}{c|c|c}
g & 2 & 7 \\
3 & 3 & p \\
4 & p' & \\
2 & \\
\end{array}
\]

\[
RSJDT_2 \rightarrow \begin{array}{c|c|c}
g & 2 & 7 \\
3 & 3 & p \\
4 & p' & \\
2 & \\
\end{array} \xrightarrow{RSJDT_2} \begin{array}{c|c|c}
g & 2 & 7 \\
3 & 3 & p \\
4 & p' & \\
2 & \\
\end{array} = (U_0, reversaf_3 (P_{[\pm 2,4]})) \Rightarrow reversaf_3 (P_{[\pm 2,4]}) = \begin{array}{c|c|c}
2 & 3 & \\
3 & 3 & 3 \\
4 & 2 & \\
2 & \\
\end{array}.
\]

4. Replace $P_{[\pm 2,4]}$ with \( reversaf_3 (P_{[\pm 2,4]}) \) in $P$ to obtain

\[
reversaf_3 (P_{[2,4]}) = \begin{array}{c|c|c|c}
1 & 2 & 3 & 1 \\
3 & 3 & 3 & \\
4 & 2 & 3 & \\
2 & \\
\end{array}, \quad wt_{[2,4]}(\xi_{[2,4]}(P)) = \begin{array}{c|c|c|c}
2 & 3 & \\
3 & 3 & 3 & \\
4 & 2 & 3 & \\
2 & \\
\end{array}
\]

\[
wt_{[2,4]}(P) = (-1, 2, -1).
\]
9.5. **Dynkin sub-diagram** $J = [1, j]$. Let $\xi_{[1,j]}^{C_n}, 1 \leq j \leq n-1$, be the Schützenberger–Lusztig involution on $\text{KN}_{[1,j]}(\lambda, n)$. $\text{KN}_{[1,j]}(\lambda, n)$ is a type $A_j$ crystal of Kashiwara-Nakashima tableaux of shape $\lambda$ on the alphabet $\pm [n]$ with lowering and raising operators $f_i$ and $e_i$, respectively, given by the type $C_n$ signature rule with $i \in [1, j]$. Notice that the unique crystal operators which change the signs of the entries are $f_n$ and $e_n$, which are forgotten.

In the next section, we give a computation of $\xi_{[1,j]}^{C_n}, 1 \leq j \leq n-1$, via virtualization $[E]$ and bring it back to $\text{KN}_{[1,j]}(\lambda, n)$ by applying $E^{-1}$. See Theorem 5 below.

9.5.1. **Embedding of a symplectic partial Schützenberger–Lusztig involution and back.** Let $U$ be a connected component of the Levi branched crystal $[\text{KN}_J(\lambda, n)]$ with $J \subseteq [n]$ and with highest and lowest weight elements $u^\text{high}$ and $u^\text{low}$, respectively. Recall from Subsection 5.2 Proposition 2 that each connected component $U$ of the Levi branched crystal $[\text{KN}_J(\lambda, n)]$ is embedded via $[E]$ into a connected component of the Levi branched crystal $\text{SSYT}_{J,J}(\lambda, n)$ with highest and lowest weight elements $[E]u^\text{high}$ and $[E]u^\text{low}$, respectively.

Note that if $J = [p, q]$, $1 \leq p \leq q < n$, then $\text{KN}_{[p,q]}(\lambda, n)$ is a type $A_{q-p+1}$ crystal. Let $T \in U$. $\text{SSYT}_{[p,q]}(\lambda^A, n, \bar{n})$ and $\text{SSYT}_{[q+1,p+1]}(\lambda^A, n, \bar{n})$ are crystal isomorphic to $\text{SSYT}_{[p,q]}(\lambda^A, n)$ and $\text{SSYT}_{[q+1,p+1]}(\lambda^A/\lambda^A_+, \bar{n})$, respectively (recall Remark 5). We may then write

$$\xi_{[p,q]}^{A_{2n-1}} \xi_{[q+1,p+1]}^{A_{2n-1}}(T) = \xi_{[q+1,p+1]}^{A_{2n-1}}(T^+) \cup \xi_{[p,q]}^{A_{2n-1}}(T^-).$$  \hspace{1cm} (47)

**Theorem 5.** Let $T \in U$, where $U$ is a connected component of the Levi branched crystal $[\text{KN}_{[p,q]}(\lambda, n)]$ of type $A_{q-p+1}, 1 \leq p \leq q < n$. Then

$$\xi_{[p,q]}^{A_{2n-1}}(T) = E^{-1} \text{reversal}_{[p,q]}^{A_{2n-1}} \xi_{[q+1,p+1]}^{A_{2n-1}}(E(T)).$$

Moreover,

$$\xi_{[p,q]}^{C_n}(T) = E^{-1} \text{reversal}_{[p,q]}^{A_{2n-1}} \xi_{[q+1,p+1]}^{A_{2n-1}}(E(T)).$$ \hspace{1cm} (48)

**Proof.** Recall Proposition 2, Remark 5 and (15). Then it follows from Theorem 2. \hspace{1cm} \square

It is now convenient to change the labeling of the $A_{2n-1}$ Dynkin diagram. Instead of $[k, k+1]$, we write $[k, 2n-k]$, and $\text{SSYT}_{[k, 2n-k]}(\lambda, n)$. This relabelling is illustrated in the picture below.

![Dynkin diagram relabelling](image)

**Theorem 6.** Let $T \in U$, where $U$ is a connected component of the Levi branched crystal $[\text{KN}_{[k,n]}(\lambda, n)]$ of type $C_{n-k+1}$ for some $1 \leq k \leq n$. Then

$$\xi_{[k,2n-k]}^{A_{2n-1}}(E(T)) = E \xi_{n[k,n]}^{C_n}(T).$$
Moreover, on \( \text{SSYT}(\lambda^A, n, \bar{n}) \)
\[
\xi^A_{2n-1}_{[k,2n-k]}(\xi^A_{2n-1}_{[k,2n-k]}) = \text{reverse}_{[k,2n-k]},
\]
and
\[
\xi^C_n_{[k,n]} = E^{-1} \text{reverse}_{[k,2n-k]} E.
\]  

(49)

Proof. Recall Corollary 2 and that, in the case of the branched crystal \( \text{SSYT}_{[k,2n-k]}(\lambda, n, \bar{n}) \), \( \theta(i) = 2n - i \), for \( i \in [k, 2n - k] \). Let
\[
T = f_{i_r} \ldots f_{i_1}(u^{\text{high}}), i_1, \ldots, i_r \in [k, n],
\]
and
\[
E(T) = f_i^A f_{2n-i}^A \ldots f_{i_1}^A f_{2n-i_1}^A(E(u^{\text{high}})), i_1, \ldots, i_r \in [k, n],
\]
and
\[
\xi^C_n_{[k,n]}(T) = e_{i_r} \ldots e_{i_1}(u^{\text{low}}).
\]

Then, from subsection 5.1,
\[
E(\xi^C_n_{[k,n]}(T)) = E(e_{i_r} \ldots e_{i_1}(u^{\text{low}})) = e_{i_r}^A e_{2n-i_r}^A \ldots e_{i_1}^A e_{2n-i_1}^A E(u^{\text{low}})
\]
and
\[
\xi^A_{[k,2n-k]}(E(T)) = e_{\theta(i_r)}^A e_{\theta(2n-i_r)}^A \ldots e_{\theta(i_1)}^A e_{\theta(2n-i_1)}^A E(\xi^C_n_{[k,n]}(T))
\]
and
\[
= e_{2n-i_r}^A e_{2n-i_r}^A \ldots e_{i_1}^A e_{2n-i_1}^A E(u^{\text{low}}) = E(\xi^C_n_{[k,n]}(T)).
\]
Finally, (49) follows (33).

Remark 17. Both \( \xi^A_{2n-1}_{[p,q]_{[2n-2n-p]}} \) and \( \xi^A_{2n-1}_{[k,2n-k]} \) act on the set \( \text{SSYT}(\lambda^A, n, \bar{n}) \) to define a permutation such that the subset \( E(\text{KN}(\lambda, n)) \) is preserved. In other words, each of these involutions defines a permutation on \( E(\text{KN}(\lambda, n)) \) when their action is restricted to this subset.

9.6. Virtualization of the action of \( J_{2n} \) on the crystal \( \text{KN}(\lambda, n) \). We have the following commutative diagram corresponding to the crystal embedding \( E \) and the partial \( C_n \) and \( A_{2n-1} \) Schützenberger–Lusztig involutions, where \( [p, q] \subseteq [n - 1] \) and \( [p, n] \subseteq [n] \) are connected subintervals of the Dynkin diagram of \( C_n \),

\[
\begin{array}{c}
\text{KN}(\lambda, n) \xrightarrow{E} \text{SSYT}(\lambda^A, n, \bar{n}) \\
\xi^C_n_{[p,n]} \downarrow \quad \xi^C_n_{[p,q]} \downarrow \quad \xi^A_{2n-1}_{[p,2n-p+1]} \downarrow \quad \xi^A_{2n-1}_{[p,q]_{[2n-2n-p+1]}} \downarrow \\
\text{KN}(\lambda, n) \quad \text{SSYT}(\lambda^A, n, \bar{n})
\end{array}
\]

\[
E \xi^C_n_{[p,n]} = \xi^A_{2n-1}_{[p,2n-p+1]} E,\quad E \xi^C_n_{[p,q]} = \xi^A_{2n-1}_{[p,q]_{[2n-2n-p+1]}} E = \xi^A_{2n-1}_{[p,q]_{[2n-2n-p+1]}} E.
\]  

(50)

Theorem 3 and Remark 17 imply that the action of \( J_{2n} \) on \( \text{SSYT}(\lambda^A, n, \bar{n}) \) preserves the subset \( E(\text{KN}(\lambda, n)) \) and thus, we have an action of \( J_{2n} \) on the set \( E(\text{KN}(\lambda, n)) \).
defined by
\[ \Phi_{\mathfrak{gl}_{2n}}^E : J_{2n} \rightarrow E \mathcal{L}[\text{KN}(\lambda, n)] \]
\[ \tilde{s}_{[p, q]|_{[q+1, p+1]}} \mapsto \xi^{A_{2n-1}}_{[p, q]|_{[q+1, p+1]}} \]
\[ \tilde{s}_{[p, p+1]} \mapsto \xi^{A_{2n-1}}_{[p, p+1]} \]
such that \( \Phi_{\mathfrak{gl}_{2n}}^E (\tilde{s}_j) = \Phi_{\mathfrak{gl}_{2n}}^E (\tilde{\iota}(s_{[1, j]} \rightarrow \tilde{s}_{[1, j]|_{[j+1, 2]}}, 1 \leq j < n, (s_{[1, j]} \rightarrow \tilde{s}_{[j+1]}), 1 \leq j < n, (s_{[j, n]} \rightarrow \tilde{s}_{[j+1]} \}, 1 \leq j < n), \text{see Proposition 3.1, and } \iota : \mathcal{L}[\text{KN}(\lambda, n)] \rightarrow \mathcal{L}[\text{KN}(\lambda, n)] \text{ the group iso-}

From (50)
\[ \tilde{\Phi}_{\mathfrak{gl}_{2n}}^E \tilde{\iota}(s_{[1, j]}) = \Phi_{\mathfrak{gl}_{2n}}^E (\tilde{s}_{[1, j]|_{[j+1, 2]}}) = \xi^{A_{2n-1}}_{[1, j]|_{[j+1, 2]}} \]
\[ = \iota \Phi_{\mathfrak{sp}(2n, \mathbb{C})}(s_{[1, j]}) = \xi^{C_{n}}_{[1, j]} E^{1} = \xi^{A_{2n-1}}_{[1, j]|_{[j+1, 2]}} \]
\[ \tilde{\Phi}_{\mathfrak{gl}_{2n}}^E \tilde{\iota}(s_{[j, n]}) = \Phi_{\mathfrak{gl}_{2n}}^E (\tilde{s}_{[j+1]}) = \xi^{A_{2n-1}}_{[j+1]} \]
\[ = \iota \Phi_{\mathfrak{sp}(2n, \mathbb{C})}(s_{[j, n]}) = \xi^{C_{n}}_{[j, n]} E^{1} = \xi^{A_{2n-1}}_{[j+1]} \]

9.7. Virtualization example.

Example 9. Consider \( n = 6, J = [1, 5] \) and the KN tableau \( T \) of shape \( \lambda = 2\omega_6 + \omega_5 + \omega_2 \):

\[
\begin{array}{cccc}
1 & 2 & 3 & \bar{3} \\
4 & 5 & \bar{5} & T \\
3 & 6 & \bar{6} & 5 \\
\bar{6} & 5 & \bar{5} & 3 \\
6 & \bar{5} & \bar{6} & 4 \\
\end{array}
\]

\( \text{wt}(T) = (-1, 2, 0, 0, -1, 1) \)

From \( \lambda \) we may immediately write \( \lambda^A = 2\omega_6^A + 2\omega_5^A + \omega_4^A + \omega_3^A + \omega_0^A + \omega_2^A \), and from \( \lambda^A \) we write the Baker recording tableau \( Q_{\lambda} \) of shape \( \lambda^A \) as a sequence of shapes where we successively fill the boxes along columns, top to bottom, from 1 to \( |\lambda^A| = 4|\omega_6^A| + |\omega_5^A| + |\omega_4^A| + |\omega_3^A| + |\omega_0^A| + |\omega_2^A| = 48 \),
\[
\omega_2 \subseteq \omega_0^A + \omega_2^A \subseteq \omega_5^A + \omega_0^A + \omega_2^A \subseteq \omega_4^A + \omega_0^A + \omega_2^A \subseteq \omega_6^A + \omega_0^A + \omega_2^A + \omega_0^A + \omega_2^A \subseteq 2\omega_6^A + \omega_4^A + \omega_0^A + \omega_2^A \subseteq 3\omega_6^A + \omega_4^A + \omega_0^A + \omega_2^A \subseteq 4\omega_6^A + \omega_4^A + \omega_0^A + \omega_2^A = \lambda^A
\]
Labelling the columns of $T$ from left to right as $C_4, C_3, C_2$ and $C_1$, we have:

$$
\begin{array}{c}
1 & 1 \\
2 & 2 \\
3 & 3 \\
4 & 5 \\
6 & 6 \\
\bar{5} & \bar{4} \\
\bar{1} & \\
\end{array}
\quad
\begin{array}{c}
1 & 2 \\
2 & 4 \\
3 & 6 \\
\bar{5} & \bar{3} \\
\bar{4} & \bar{1} \\
\end{array}
\quad
\begin{array}{c}
2 & 3 \\
4 & 5 \\
6 & 4 \\
\bar{5} & \bar{1} \\
\bar{3} & \bar{1} \\
\end{array}
\quad
\begin{array}{c}
2 & 3 \\
4 & 5 \\
6 & 5 \\
\bar{5} & \bar{3} \\
\bar{4} & \bar{1} \\
\end{array}
$$

$$
\psi(C_4) = (1, 2, 3, 4, 5, 6, \bar{5}, \bar{4}, \bar{3}, \bar{2}, \bar{1}), \psi(C_3) = (1, 2, 3, 4, 5, 6, \bar{5}, \bar{4}, \bar{3}, \bar{2}, \bar{1}), \psi(C_2) = (1, 2, 3, 4, 5, 6, \bar{5}, \bar{4}, \bar{3}, \bar{2}, \bar{1}), \psi(C_1) = (1, 2, 3, 4, 5, 6, \bar{5}, \bar{4}, \bar{3}, \bar{2}, \bar{1}).
$$

Then $[E_T]$ has shape $\lambda^A = 4 \omega_6^A + \omega_7^A + \omega_5^A + \omega_1^A + \omega_2^A$,

$$\text{wt}(E_T) = \text{wt}(w_T) = (3, 6, 4, 4, 5, 5, 3, 5, 4, 4, 2, 5),$$

and $[E_T] = [\emptyset \leftarrow w_T],

\begin{array}{c}
1 & 1 & 1 & 2 & 2 & 2 & 4 & \frac{3}{1} \\
2 & 2 & 2 & 3 & 4 & 5 & 5 & \frac{1}{1} \\
3 & 3 & 3 & 4 & 6 & 6 & \frac{3}{1} \\
4 & 5 & 6 & 6 & 6 & 5 & \frac{2}{1} \\
5 & 5 & 4 & 4 & 3 & \frac{1}{1} \\
\frac{3}{1} & \frac{2}{1} & \frac{1}{1} \\
\end{array}
$$

$[E_T] = [\emptyset \leftarrow w(\psi(C_1)) \leftarrow w(\psi(C_2)) \leftarrow w(\psi(C_3)) \leftarrow w(\psi(C_4))],

\begin{array}{c}
1 & 1 & 1 & 2 & 2 & 2 & 4 & \frac{3}{1} \\
2 & 2 & 2 & 3 & 4 & 5 & 5 & \frac{1}{1} \\
3 & 3 & 3 & 4 & 6 & 6 & \frac{3}{1} \\
4 & 5 & 6 & 6 & 6 & 5 & \frac{2}{1} \\
5 & 5 & 4 & 4 & 3 & \frac{1}{1} \\
\frac{3}{1} & \frac{2}{1} & \frac{1}{1} \\
\end{array}
$$

which has recording tableau $Q(w_T) = Q_\lambda$.

Considering the barred and unbarred parts of $[E_T]$ separately, we compute the evacuation, $\text{evac}$, of the unbarred part and the reversal, $\text{reversal}$, of the barred part, yielding:
Putting these tableaux together, one obtains

$$\xi_{[1,5]∪[6,2]}(E(T)) = \text{evac}(E(T)^+) ∪ \text{reversal}(E(T)^-).$$

Using $Q_λ$ to perform the reverse column Schensted insertion on the resulting $A_{11}$ tableau $\xi_{[1,5]∪[6,2]}(E(T))$ provides the image under $\psi$ of four KN columns $C'_1$, $C'_2$, $C'_3$, $C'_4$:

$$\psi(C'_4) = \begin{array}{cccc} 1 & 1 & 2 & 2 \\ 2 & 2 & 3 & 4 \\ 3 & 3 & 4 & 5 \\ 4 & 5 & 6 & 6 \end{array}, \quad \psi(C'_3) = \begin{array}{cccc} 1 & 1 & 2 & 2 \\ 2 & 3 & 4 & 5 \\ 3 & 5 & 6 & 5 \\ 4 & 5 & 6 & 6 \end{array}, \quad \psi(C'_2) = \begin{array}{cccc} 3 & 4 & 6 & 5 \\ 4 & 5 & 5 & 6 \\ 6 & 6 & 5 & 2 \\ 5 & 6 & 4 & 1 \end{array}, \quad \psi(C'_1) = \begin{array}{cccc} 1 & 5 & 4 & 2 \\ 5 & 5 & 6 & 6 \\ 6 & 6 & 6 & 2 \\ 4 & 1 & 2 & 1 \end{array}$$

and applying $\psi^{-1}$ to each column results in:
\[
C_4' C_3' C_2' C_1' = \begin{pmatrix}
1 & 1 & 4 & 3 \\
2 & 3 & 5 & 2 \\
3 & 5 & \text{T} & 4 \\
5 & 6 & 2 & 3 \\
6 & 5 & 1 & 2
\end{pmatrix} = \xi_{[1,5]}(T).
\]

\[
\text{wt}_{[1,5]}(\xi_C'(T)) = \text{reverse}(\text{wt}(T)) = (1, -1, 0, 0, 2, -1).
\]

This solution has been verified in SageMath.

10. The type \(C_n\) Berenstein-Kirillov group

10.1. The type \(A\) Berenstein-Kirillov group. The type \(A\) Berenstein–Kirillov group \([\text{BK}]\) (or Gelfand–Tsetlin group) \([\text{BerKir95}]\) is the free group generated by the Bender–Knuth involutions \([\text{BeKn72}]\) \(t_i\), \(i > 0\), modulo the relations they satisfy on semi-standard Young tableaux of any (straight) shape.

Definition 5. The Bender–Knuth involution \(t_i\), \(i \geq 1\), is an operation that acts on a semi-standard tableau \(T\) of any shape (skew or straight) as follows:

- pairs \((i, i+1)\) within each column of \(T\) are considered fixed, and other occurrences of \(i\)'s or \(i+1\)'s are considered free
- if a row within \(T\) has \(k\) free \(i\)'s followed by \(l\) free \(i+1\)'s, then we replace these letters by \(l\) free \(i\)'s followed by \(k\) free \(i+1\)'s.

The \(t_i\)'s have many known relations in \([\text{BK}]\) \([\text{BerKir95}]\) \([\text{CGP16}]\):

\[
t_i^2 = 1, \quad \text{for } i \geq 1 \quad [\text{BerKir95}\text{ Corollary 1.1}]
\]
\[
t_it_j = t_jt_i, \quad \text{for } |i - j| > 1 \quad [\text{BerKir95}\text{ Corollary 1.1}]
\]
\[
(t_1q_{[1,i]})^4 = 1, \quad \text{for } i > 2 \quad [\text{BerKir95}\text{ Corollary 1.1}]
\]
\[
(t_1t_2)^6 = 1, \quad [\text{BerKir95}\text{ Corollary 1.1}]
\]
\[
(t_iq_{[j,k-1]})^2 = 1, \quad \text{for } i + 1 < j < k \quad [\text{CGP16}]
\]

where

\[
q_{[1,i]} := t_1(t_2t_1)\cdots(t_{i-1}t_1), \quad \text{for } i \geq 1,
\]
\[
q_{[j,k-1]} := q_{[1,k-1]}q_{[1,k-j]}q_{[1,k-1]}, \quad \text{for } j < k.
\]

Remark 18. (1) It is not known whether the latter forms a complete set of relations.

(2) \([\text{BerKir95}\text{ Section 2}]\) On straight-shaped semi-standard Young tableaux,

\[
q_{[1,i]} = \xi_{[1,i]}, \quad i \geq 1, \quad q_{[j,k-1]} = \xi_{[j,k-1]}, \quad j < k,
\]

and \(q_{[j,j]} = q_{[1,j]}q_{[1,1]}q_{[1,j]}\) computes the crystal reflection operator \(\xi_j = \xi_{[j,j]}\), where \(q_{[1,1]} = \xi_{[1,1]} = t_1\), for \(j \geq 1\). In particular, \(q_{[1,i]} = \xi_{[1,i]} = \text{evac}_{i+1}\), the evacuation restricted to the alphabet \([i+1]\), and \(q_{[j,k-1]}\) computes the
The following are group epimorphisms from Theorem 7.

by Chmutov, Glick and Pylyavskyy via the growth diagram approach [CGP16].

$q$ satisfied by the generators $SSYT$ crystal

$i < n$

We may define the two dual sets of generators called dual Bender–Knuth involutions, and

$BK$ of $BK$

It follows from $J$ does not follow from the cactus group $BK$

$s$

$J$

$(1)$ $s_{i,j} \mapsto q_{i,j}$ [CGP16 Theorem 1.4],

$(2)$ $s_{i,j} \mapsto q_{[1,i]}$ [BerKir95 Remark 1.3], [Ha16 Section 10.2], [Ha20 Remark 3.9].

The group $BK_n$ is isomorphic to a quotient of $J_n$. The generators $q_{[1,1]}, \ldots, q_{[1,n-1]}$ of $BK_n$ (and therefore $q_{i,j}$) satisfy the relations of $J_n$.

Remark 19. It follows from [CGP16] that (53) is the only known relation which does not follow from the cactus group $J_n$ relations. It is in fact equivalent to the braid relations satisfied by the crystal reflection operators $\xi_i = \xi_{[1,n-1]}^{-1} t_i \xi_{[1,n-1]}$, $1 \leq i < n$, on a $U_q(\mathfrak{gl}(n, \mathbb{C}))$ crystal [BerKir95 Proposition 1.4], [Ro21].

Remark 20. We may define the two dual sets of generators

$t_{i-n} := q_{[1,n-1]} t_i q_{[1,n-1]}^{-1}$, $1 \leq i < n$,

called dual Bender–Knuth involutions, and

$q_{[1,i]} := q_{[1,n-1]} q_{[i,n-1]}^{-1} q_{[n-i,n-1]}$, $1 \leq i < n$,

for $BK_n$. Indeed, from Proposition 17 one has

$\tilde{t}_{n-1} = q_{[n-1,n-1]}$, $\tilde{t}_{n-i} = q_{[n-i+1,n-1]} q_{[n-i,n-1]} q_{[n-i+1,n-1]} q_{[n-i+2,n-1]}$, for $2 \leq i < n$, $q_{[n,n-1]} : = 1$, and $\text{wt}(\tilde{t}_{n-i}(T)) = r_{n-i} \cdot \text{wt}(T)$ for $T \in SSYT(\lambda, n)$ and $i < n$. 

These results are both a consequence of the combinatorial action of the cactus group $J_n$ via partial Schützenberger involutions $\xi_{[1,i]}$ on the straight-shape tableau crystal $SSYT(\lambda, n)$ as defined by Halacheva [Ha16], and the cactus group $J_n$ relations satisfied by the generators $q_{i,j} = \xi_{i,j}$ of $BK_n$ when acting on $SSYT(\lambda, n)$, as studied by Chmutov, Glick and Pylyavskyy via the growth diagram approach [CGP16].
The dual generators satisfy a list of relations similar to (51), (52), (53), (54), (55):

\[ \tilde{t}_{n-i}^2 = 1, \text{ for } i \geq 1 \]  
\[ \tilde{t}_{n-i} \tilde{t}_{n-j} = \tilde{t}_{n-j} \tilde{t}_{n-i}, \text{ for } |i - j| > 1, \]  
\[ (\tilde{t}_{n-1} \tilde{t}_{n-2})^6 = 1, \]
\[ (\tilde{t}_{n-i} \tilde{q}_{[j,k-1]})^2 = (\tilde{t}_{n-i} \tilde{q}_{[n-k+1,n-j]})^2 = 1, \text{ for } n - k < n - j < n - i - 1, \]

where

\[ \tilde{q}_{[1,n]} = \tilde{t}_{n-1}(\tilde{t}_{n-2} \tilde{t}_{n-1}) \cdots (\tilde{t}_{n-i} \tilde{t}_{n-i+1} \cdots \tilde{t}_{n-1}), \text{ for } i \geq 1, \]
\[ \tilde{q}_{[j,k-1]} := q_{[n-k+1,n-j]} = q_{[n-k+1,n-1]} [q_{[n-k+j,n-1]} q_{[n-k+1,n-1]}], \text{ for } j < k. \]

10.2. The type $C_n$ Berenstein–Kirillov group and virtualization. Symplectic Bender–Knuth involutions $\tilde{t}^{C_n}$ are not known for KN tableaux. Motivated by the fact that for $n \geq 1$, $q_{[1,1]}, \ldots, q_{[1,n-1]}$ are generators for the Berenstein-Kirillov group $\mathcal{B}K^{C_n}$ in type $A$, and that on straight shaped semi-standard tableaux, they coincide with the action of the partial Schützenberger-Lusztig involutions $\xi_{[1,i]}$, $1 \leq i < n$, we use the action of the partial Schützenberger-Lusztig involutions $\xi_{[1,i]}$, $1 \leq i \leq n - 1$, and $\xi_{[i,n]}$ $1 \leq i \leq n$, on KN tableaux of any straight shape on the alphabet $C_n$ to define the type $C_n$ Berenstein–Kirillov group $\mathcal{B}K^{C_n}$.

Definition 6. Given $n \geq 1$, the symplectic Berenstein–Kirillov group $\mathcal{B}K^{C_n}$ is the free group generated by the $2n - 1$ partial Schützenberger-Lusztig involutions

\[ q_{[1,i]}^{C_n} := \xi_{[1,i]}, 1 \leq i < n, \]

and

\[ q_{[i,n]}^{C_n} := \xi_{[i,n]}, 1 \leq i \leq n, \]

on straight shaped KN tableaux on the alphabet $C_n$ modulo the relations they satisfy on those tableaux. We also define $q_{[1,0]}^{C_n} = q_{[0,n]}^{C_n} = q_{[n+1,n]}^{C_n} := 1$ and $q_{[j,k]}^{C_n} := q_{[j+k-1]}^{C_n} q_{[1,k-j]}^{C_n} q_{[1,k-1]}^{C_n}, 1 \leq j < k \leq n$.

Thanks to Theorem 1 (31) and (32), one has that $\mathcal{B}K^{C_n}$ is a quotient of $J^{sp(2n,C)}$. The generators of $\mathcal{B}K^{C_n}$ satisfy the cactus group $J^{sp(2n,C)}$ relations.

Theorem 8. The following is a group epimorphism from $J^{sp(2n,C)}$ to $\mathcal{B}K^{C_n}$.

\[ s_{[1,j]} \mapsto q_{[1,j]}^{C_n}, 1 \leq j < n, \quad s_{[j,n]} \mapsto q_{[j,n]}^{C_n}, 1 \leq j \leq n. \]

$\mathcal{B}K^{C_n}$ is isomorphic to a quotient of $J^{sp(2n,C)}$.

We next define symplectic Bender–Knuth involutions $\tilde{t}^{C_n}$, $1 \leq i \leq 2n - 1$, on straight shaped KN tableaux that in turn generate $\mathcal{B}K^{C_n}$.

Definition 7. The $2n - 1$ symplectic Bender–Knuth involutions $\tilde{t}^{C_n}$ on KN tableaux of straight shape on the alphabet $C_n$ are defined by

\[ \tilde{t}_{1}^{C_n} := q_{[1,1]}^{C_n}, \tilde{t}_{i}^{C_n} := q_{[1,i-1]}^{C_n} q_{[1,i]}^{C_n} q_{[1,j-1]}^{C_n} q_{[1,j-2]}^{C_n}, 2 \leq i \leq n - 1, \]
As a base case, when \( i \) the statement is true for all \( k \) hypothesis implies

\[
t^i_n := q^{C_n}_{[n,n]} = \xi^C_n, \quad t^i_{n+i} := q^{C_n}_{[n-i+1,n]} q^{C_n}_{[n+i+2,n]}, \quad 2 \leq i \leq n.
\] (66)

Thanks to the \( \mathcal{J}_{\text{sp}(2n,C)} \) relations satisfied by the generators of \( \mathcal{B} \mathcal{K}^{C_n} \) \( q^{C_n}_{[j,j]} \) computes the symplectic crystal reflection operator \( \xi^C_j \), for \( 1 \leq j \leq n \), on KN tableaux (see Proposition \( \text{[6]} \)).

**Remark 21.** The symplectic Bender–Knuth involutions \( t^i_n, 1 \leq i \leq n \), act on the weights of the elements in the crystal \( \mathcal{K} \mathcal{N}(\lambda, n) \). \( \text{wt}(t^i_n(T)) = \text{wt}(\xi^C_j(T)) = r_i, \text{wt}(T), 1 \leq i \leq n \), inducing an action of the Weyl group \( B_n \) on these weights, although, as we shall see, in Subsection \( \text{[10.3]} \), they do not define an action of the hyperoctahedral group \( B_n = \langle r_1, \ldots, r_n \rangle \) on the set \( \mathcal{K} \mathcal{N}(\lambda, n) \). Let \( T \in \mathcal{K} \mathcal{N}(\lambda, n) \) and \( \text{wt}(T) = (v_1, \ldots, v_n) \in \mathbb{Z}^n \), then

\[
\text{wt}(t^i_1(T)) = r_i, \text{wt}(T), 1 \leq i < n,
\]
\[
\text{wt}(T^i_n(T)) = (v_1, \ldots, -v_n) = r_n, \text{wt}(T)
\]
\[
\text{wt}(t^i_{2n-i}(T)) = (v_1, \ldots, -v_i, \ldots, v_n) = r_n \cdots r_n-i r_n-i \cdots r_n-1 (v_1, \ldots, v_n)
= t_{n-1} \cdots t_{n-i} t_{n-i} \cdots t_{n-1} (v_1, \ldots, v_n), \quad 1 \leq i < n.
\]

**Proposition 11.** The symplectic Bender–Knuth involutions \( t^i_n, 1 \leq i \leq 2n-1 \), generate \( \mathcal{B} \mathcal{K}^{C_n} \). In particular,

1. \( q^{C_n}_{[1,i]} = p_1^{C_n} p_2^{C_n} \cdots p_i^{C_n}, 1 \leq i < n \), and
2. \( q^{C_n}_{[n,i]} = t_{n-i}^{C_n} \cdots t_{1}^{C_n}, 1 \leq i < n \),

where \( p_i^{C_n} := t_i^{C_n} \cdots t_2^{C_n} t_1^{C_n} \) is the symplectic promotion, \( 1 \leq i \leq 2n-1 \).

**Proof.** (1) We show by induction on \( i \) that \( q^{C_n}_{[1,i]} = q^{C_n}_{[1,i-1]} p_i^{C_n} \). Note that \( q^{C_n}_{[1,1]} = p_1^{C_n} = t_1^{C_n} \). Furthermore, for \( i > 1 \), \( q^{C_n}_{[1,i]} = q^{C_n}_{[1,i-1]} t_i^{C_n} \). Then, assuming that \( q^{C_n}_{[1,j]} = q^{C_n}_{[1,j]} p_i^{C_n} \) for all \( j \in [1,k-1] \), our inductive hypothesis implies

\[
q^{C_n}_{[1,k]} = q^{C_n}_{[1,k-1]} t_k^{C_n} q^{C_n}_{[1,k-2]} = q^{C_n}_{[1,k-1]} t_k^{C_n} q^{C_n}_{[1,k-2]} p_{k-1}^{C_n} = q^{C_n}_{[1,k-1]} t_k^{C_n} p_{k-1}^{C_n} = q^{C_n}_{[1,k-1]} p_{k-1}^{C_n}.
\]

(2) We proceed by induction on \( i \) in the statement \( q^{C_n}_{[n-i-1,n]} = t_{n-i}^{C_n} t_{n-i+1}^{C_n} \cdots t_n^{C_n} \). As a base case, when \( i = 1 \), we have \( t^i_n = q^{C_n}_{[n,n]} \). As an inductive step, we assume the statement is true for all \( j \in [1,k] \) for some fixed positive integer \( k \), so

\[
t_{n+k+1}^{C_n} = t_{n+k}^{C_n} = q^{C_n}_{[n,k]} q^{C_n}_{[n-(k-1),n]} \\
\Rightarrow t_{n+k}^{C_n} q^{C_n}_{[n-(k-1),n]} = q^{C_n}_{[n-k,n]} \\
\Rightarrow q^{C_n}_{[n-k,n]} = t_{n+k}^{C_n} q^{C_n}_{[n-k,n]} \cdots t_n^{C_n}.
\]

\( \square \)
Definition 8. The virtual symplectic Berenstein-Kirillov group \( \mathcal{BK}_{2n} \) is the subgroup of \( \mathcal{BK}_{2n} \) generated by the 2n - 1 involutions

\[
q_{[1,i],[2n-i,2n-1]}^{A_{2n-1}} := q_{[1,i]}^{A_{2n-1}} q_{[2n-i,2n-1]}^{A_{2n-1}}, \quad 1 \leq i < n, \quad (67)
\]

\[
q_{[2n-i,2n-1],i}^{A_{2n-1}}, \quad 1 \leq i \leq n, \quad (68)
\]

modulo the relations they satisfy when acting on semi-standard tableaux of any straight shape.

By Theorem 2, \( q_{[1,i],[2n-i,2n-1]}^{A_{2n-1}} \) coincides \( \tilde{\xi}_{A_{2n-1}}^{A_{2n-1}} \), \( 1 \leq i < n \).

Proposition 12. For \( 1 \leq i < n \), consider the Bender-Knuth involutions \( t_{i}^{A_{2n-1}} \) with their duals \( t_{i}^{A_{2n-1}} \) in \( \mathcal{BK}_{2n} \). The group \( \mathcal{BK}_{2n} \) also has the 2n - 1 generators

\[
t_{i}^{A_{2n-1}} := t_{i}^{A_{2n-1}} t_{2n-i}^{A_{2n-1}}, \quad 1 \leq i < n, \quad (69)
\]

\[
t_{n+i,1,n+i}^{A_{2n-1}} := q_{[n-i+1,n+i-1],n+i-2]}^{A_{2n-1}} q_{[n-i+2,n+i-2]}^{A_{2n-1}} = q_{[n-i+1,n+i-1]}^{A_{2n-1}}, \quad 1 \leq i \leq n, \quad (70)
\]

where \( q_{[1,1]}^{A_{2n-1}} : = 1 \). We call them the virtual symplectic Bender-Knuth involutions.

Proof. The group \( \mathcal{BK}_{2n} \) satisfies the \( J_{2n} \) relations and \( \mathcal{BK}_{2n} \subseteq \mathcal{BK}_{2n} \). Hence \( q_{[1,i]}^{A_{2n-1}} q_{[2n-i,2n-1]}^{A_{2n-1}}, \quad 1 \leq i < n \), and by (21), \( q_{[n-i+1,n+i-1],n+i-2]}^{A_{2n-1}} q_{[n-i+2,n+i-2]}^{A_{2n-1}} = q_{[n-i+1,n+i-1]}^{A_{2n-1}}, \quad 1 \leq i \leq n. \) In addition, from Remark 20 in \( \mathcal{BK}_{2n} \),

\[
t_{i}^{A_{2n-1}} t_{2n-i}^{A_{2n-1}} q_{[1,i],[2n-i,2n-1]}^{A_{2n-1}}, \quad 1 \leq i < n, \quad (71)
\]

Again by Remark 20 in \( \mathcal{BK}_{2n} \) and (71), for \( 1 \leq i < n, \)

\[
q_{[1,i],[2n-i,2n-1]}^{A_{2n-1}} = q_{[1,i]}^{A_{2n-1}} q_{[2n-i,2n-1]}^{A_{2n-1}} = p_{1}^{A_{2n-1}} \ldots p_{i}^{A_{2n-1}} q_{[1,i],2n-1]}^{A_{2n-1}}, \quad 1 \leq i \leq n, \quad (72)
\]

where \( q_{[1,i]}^{A_{2n-1}} = p_{1}^{A_{2n-1}} \ldots p_{i}^{A_{2n-1}} \) with \( p_{i}^{A_{2n-1}} := t_{i}^{A_{2n-1}} t_{2n-i}^{A_{2n-1}}, \) and
\[ p_{2n-i}^{A_{2n-1}} := q_{[2n-i, 2n-1]}^{A_{2n-1}} p_i^{A_{2n-1}} q_{[2n-i, 2n-1]}^{A_{2n-1}} = \bar{t}_{2n-i}^{A_{2n-1}} \cdots t_{2n-i-1}^{A_{2n-1}} t_{2n-i}^{A_{2n-1}} ; \ 1 \leq i < n. \]

On the other hand, for \( 1 \leq i \leq n, \)
\[ q_{[n-i+1, n+i-1]}^{A_{2n-1}} = q_n^{A_{2n-1}} (q_n^{A_{2n-1}} \cdots (q_{[n-i+1, n+i-1]}^{A_{2n-1}} \cdots q_{[n-i+1, n+i-1]}^{A_{2n-1}}) \cdots q_{[n-i+1, n+i-1]}^{A_{2n-1}} \cdots q_{[n-i+1, n+i-1]}^{A_{2n-1}}) \]
\[ = t_{n-i+1}^{A_{2n-1}} \cdots t_{n-i+1}^{A_{2n-1}} t_{n-i+1}^{A_{2n-1}} ; \]

\[ \square \]

Remark 22. If \( T \) is an \( A_{2n-1} \) semi-standard tableaux, \( \text{wt}(t_{[2n-1]}^{A_{2n-1}}(T)) = r_i r_{2n-i} \), \( \text{wt}(T) \), where \( r_i = (i+1) \) and \( r_{2n-i} = (2n-i) \), \( i \leq n \) are simple transpositions in \( \mathfrak{S}_{2n} \),
for \( 1 \leq i < n, \) and \( \text{wt}(t_{n-i+1,n+i}(T)) = (n+i+1) \text{wt}(T) \), where \( (n+i+1) \text{wt}(T) \) is the transposition of \( \mathfrak{S}_{2n} \) that swaps \( n-i+1 \) and \( n+i \), for \( 1 \leq i \leq n \).

Thanks to Theorem 9 we have that \( \mathbf{BK}_{2n} \) is a quotient of the virtual symplectic cactus \( J_{2n} \). The generators \( \mathbf{BK}_{2n} \) and \( \mathbf{BK}_{2n} \) satisfy the relations of the cactus \( J_{2n} \), or equivalently, those of the cactus \( J_{2n} \).

Theorem 9. The following is a group epimorphism from \( J_{2n} \) to \( \mathbf{BK}_{2n} \):
\[ \tilde{s}_{[i,j]}(2n-j, 2n-1) \mapsto q_{[i,j]_{[2n-j, 2n-1]}}^{A_{2n-1}}, 1 \leq j < n, \]
\[ \tilde{s}_{[j,j]}(2n-j, 2n-1) \mapsto q_{[j,j]_{[2n-j, 2n-1]}}^{A_{2n-1}}, 1 \leq j \leq n. \]

\[ \mathbf{BK}_{2n} \] is isomorphic to a quotient of \( J_{2n} \) and via the isomorphism between \( J_{2n} \) and \( J_{2n} \) that sends \( s_{[i,j]} \mapsto \tilde{s}_{[i,j]}_{[2n-j, 2n-1]}, 1 \leq j < n, \) and \( \tilde{s}_{[j,j]} \mapsto \tilde{s}_{[j,j]}_{[2n-j, 2n-1]}, 1 \leq j \leq n, \) is also isomorphic to \( J_{2n} \).

Because the action of \( J_{2n} \) on the set \( \text{SSYT}(\lambda^A, n, n) \) preserves the subset \( \mathbb{E} \text{KN}(\lambda, n) \), see Remark 14, we now relate the virtual symplectic and symplectic Bender–Knuth involutions by embedding the crystal \( \mathbf{KN}(\lambda, n) \) into the crystal \( \text{SSYT}(\lambda^A, n, n) \).

Theorem 10. The symplectic Bender–Knuth involutions \( t_i^{C_n}, 1 \leq i \leq 2n-1, \) in \( \mathbf{BK}_{2n} \) can be realized by the virtual symplectic Bender–Knuth involutions \( t_i^{A_{2n-1}} \) and \( t_i^{A_{2n-1}} \), \( 1 \leq i < n, \) and \( t_{n-i}^{A_{2n-1}} \), \( 1 \leq i \leq n, \) in \( \mathbf{BK}_{2n} \) and vice-versa.

\[ t_i^{C_n} = E^{-1} t_i^{A_{2n-i}} t_{2n-i}^{A_{2n-1}} E = E^{-1} t_i^{A_{2n-1}} t_{2n-i}^{A_{2n-1}} E, \ 1 \leq i < n, \]
\[ t_i^{C_n} = E^{-1} t_i^{A_{2n-i}} t_{n-i}^{A_{2n-1}} E, \ 1 \leq i \leq n. \]

Proof. By Theorem 5 for \( 1 \leq i < n, \)
\[ t_i^{C_n} = q_{[i, i-1]}^{C_n} q_{[i, i-1]}^{C_n} q_{[i, i-2]}^{C_n} = \]
\[ E^{-1} (t_i^{A_{2n-1}} \xi^{A_{2n-1}}) E^{-1} (q_{[i, i-1]}^{A_{2n-1}} q_{[i, i-2]}^{A_{2n-1}}) E E^{-1} (\xi^{A_{2n-1}} q_{[i, i-2]}^{A_{2n-1}}) E E^{-1} (\xi^{A_{2n-1}} q_{[i, i-2]}^{A_{2n-1}}) \]
\[ = E^{-1} (t_i^{A_{2n-1}} \xi^{A_{2n-1}}) E. \]

By Theorem 6 for \( 2 \leq i \leq n, \)
\[ t_i^{C_n} = q_{[i, i-1]}^{C_n} q_{[i, i-2]}^{C_n}, \]
and \( t_i^{C_n} = q_{[i, i-1]}^{C_n} = \xi^{C_n}. \]
10.3. Symplectic Bender–Knuth involutions, the character of the KN tableau crystal and the Weyl group action. The $C_2$ Weyl group is $B_2 = < t_1, t_2 : t_1^2 = 1, (r_1 r_2)^4 = 1 >$ with long element $r_2 r_1 r_2 r_1$, and the $C_2$ symplectic Bender–Knuth involutions are $t_1^{C_2} = \xi_1^{C_2}$, $t_2^{C_2} = \xi_2^{C_2}$, $t_3^{C_2} = \xi_3^{C_2}$, $\xi_2^{C_2} = \xi_2^{C_2}$, and one has

\[ \xi^{C_2} = t_3^{C_2} t_2^{C_2} = t_2^{C_2} t_1^{C_2} t_2^{C_2} t_1^{C_2} = t_1^{C_2} t_2^{C_2} t_3^{C_2} t_1^{C_2}. \]

From Proposition 6 and 11, $B_2$ indeed acts on the $C_2$-crystal $\text{KN}(\lambda, 2)$ via $t_1^{C_2} = \xi_1^{C_2}$ and $t_2^{C_2} := \xi_2^{C_2}$. Therefore, in this case, $t_1^{C_2}$ and $t_2^{C_2}$ define an action of Weyl group $B_2$ on the crystal $\text{KN}(\lambda, 2)$. However, the action of the Schützenberger–Lusztig involution $\xi_2^{C_2} = \xi_2^{C_2}$ on $\text{KN}(\lambda, n)$ does not coincide in general with the action of the long element of the Weyl group $B_2$, that is, $\xi_2^{C_2} = t_1^{C_2} t_2^{C_2} t_1^{C_2} t_2^{C_2} = t_2^{C_2} t_1^{C_2} t_2^{C_2} t_1^{C_2}.$

For instance, considering $T = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ in Example 11 despite

\[ \text{wt}(t_2^{C_2} t_1^{C_2} t_2^{C_2} t_1^{C_2}(T)) = (1, 0) = w_0^C(-1, 0) = \text{wt}(\xi_2^{C_2}(T)), \]

the coincidence of the actions of the Schützenberger–Lusztig involution and of the long element of the Weyl group can only be ensured when $T$ is the highest weight or lowest weight element in the crystal $\text{KN}(\lambda, n)$ as Proposition 7 ensures. But $T$ is not in that case, and in fact

\[ r_2 r_1 r_2 r_1 T = t_2^{C_2} t_1^{C_2} t_2^{C_2} t_1^{C_2} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \]

\[ = t_1^{C_2} t_2^{C_2} t_1^{C_2} t_2^{C_2} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \neq \xi_2^{C_2}(T) = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}. \]

In general, for $n \geq 3$, the symplectic Bender–Knuth involutions, $t_1^{C_n}, \ldots, t_n^{C_n}$, do not define an action of the Weyl group $B_n = < r_1, \ldots, r_n >$ on the set $\text{KN}(\lambda, n)$. On the other hand, contrary to the $A_{n-1}$ case, the Schützenberger–Lusztig involution $\xi_n^{C_n}$ is not given by the long word of $B_n$ in the first $n$ symplectic Bender–Knuth involutions. One has in fact $\xi_n^{C_n} = t_{2n-1}^{C_n} \cdots t_n^{C_n}$, as stated in Proposition 11.

To show the former claim, it is enough to recall that the first $n - 1$ generators of the Weyl group $B_n = < r_1, r_2, \ldots, r - n - 1, r_n >$ satisfy the braid relations (7). The symplectic Bender–Knuth involutions, $t_1^{C_n}, t_2^{C_n}$, in particular, do not satisfy the braid relation $(t_1^{C_n} t_2^{C_n})^3 = 1$, that is, $t_1^{C_n} t_2^{C_n} t_1^{C_n} \neq t_2^{C_n} t_1^{C_n} t_2^{C_n}$ (note however that the relation (5) is always satisfied, $(t_{n-1} t_n)^3 = (\xi_{n-1}^{C_n})^3 = 1$). By Theorem 10 to show this inequality, it is enough to consider the virtual symplectic Bender–Knuth involutions and the corresponding virtual inequality

\[ t_1^{A_{2n-1}} t_2^{A_{2n-1}} t_2^{A_{2n-1}} t_2^{A_{2n-1}} t_1^{A_{2n-1}} t_1^{A_{2n-1}} t_1^{A_{2n-1}} t_1^{A_{2n-1}} \neq t_1^{A_{2n-1}} t_2^{A_{2n-1}} t_2^{A_{2n-1}} t_1^{A_{2n-1}} t_1^{A_{2n-1}} t_1^{A_{2n-1}} t_1^{A_{2n-1}} t_1^{A_{2n-1}}. \]
We already know that \( t_i^{A_{2n-1}} \tilde{t}_{2n-i}^{A_{2n-1}} = \tilde{t}_{2n-i}^{A_{2n-1}} t_i^{A_{2n-1}} \) for any \( i \), and we now prove that
\[
\begin{align*}
\tilde{t}_1^{A_{2n-1}} \tilde{t}_{2n-2}^{A_{2n-1}} &= \tilde{t}_{2n-2}^{A_{2n-1}} t_1^{A_{2n-1}} \\
\tilde{t}_2^{A_{2n-1}} \tilde{t}_{2n-1}^{A_{2n-1}} &= \tilde{t}_{2n-1}^{A_{2n-1}} t_2^{A_{2n-1}}. 
\end{align*}
\tag{73}
\]
\[
\begin{align*}
\tilde{t}_1^{A_{2n-1}} \tilde{t}_{2n-2}^{A_{2n-1}} &= \xi_{2n-1}^{A_{2n-1}} \xi_{2n-1}^{A_{2n-1}} \xi_{2n-1}^{A_{2n-1}} \xi_{2n-1}^{A_{2n-1}} = \tilde{t}_{2n-2}^{A_{2n-1}} t_{2n-1}^{A_{2n-1}}. \\
\tilde{t}_2^{A_{2n-1}} \tilde{t}_{2n-1}^{A_{2n-1}} &= \xi_{2n-1}^{A_{2n-1}} \xi_{2n-1}^{A_{2n-1}} \xi_{2n-1}^{A_{2n-1}} \xi_{2n-1}^{A_{2n-1}} = \tilde{t}_{2n-1}^{A_{2n-1}} t_{2n-1}^{A_{2n-1}}. 
\end{align*}
\tag{74}
\]

From Proposition \[10\] and Remark \[20\] \( \tilde{t}_{2n-1}^{A_{2n-1}} = \xi_{2n-1}^{A_{2n-1}} \xi_{2n-1}^{A_{2n-1}} \xi_{2n-1}^{A_{2n-1}} \xi_{2n-1}^{A_{2n-1}} \) and \( \tilde{t}_{2n-1}^{A_{2n-1}} = \xi_{2n-1}^{A_{2n-1}} \), and indeed
\[
\tilde{t}_1^{A_{2n-1}} \tilde{t}_{2n-2}^{A_{2n-1}} = \xi_{2n-1}^{A_{2n-1}} \xi_{2n-2}^{A_{2n-1}} \xi_{2n-1}^{A_{2n-1}} \xi_{2n-1}^{A_{2n-1}} = \tilde{t}_{2n-2}^{A_{2n-1}} t_{2n-1}^{A_{2n-1}}.
\]

Similarly, \( \tilde{t}_2^{A_{2n-1}} \tilde{t}_{2n-1}^{A_{2n-1}} = \xi_{2n-1}^{A_{2n-1}} \xi_{2n-1}^{A_{2n-1}} \xi_{2n-1}^{A_{2n-1}} \xi_{2n-1}^{A_{2n-1}} = \tilde{t}_{2n-1}^{A_{2n-1}} t_{2n-1}^{A_{2n-1}}. \)

Henceforth,
\[
\begin{align*}
\tilde{t}_1^{A_{2n-1}} \tilde{t}_{2n-2}^{A_{2n-1}} &= \xi_{2n-1}^{A_{2n-1}} \xi_{2n-1}^{A_{2n-1}} \xi_{2n-1}^{A_{2n-1}} \xi_{2n-1}^{A_{2n-1}} = \tilde{t}_{2n-2}^{A_{2n-1}} t_{2n-1}^{A_{2n-1}}. \\
\tilde{t}_2^{A_{2n-1}} \tilde{t}_{2n-1}^{A_{2n-1}} &= \xi_{2n-1}^{A_{2n-1}} \xi_{2n-1}^{A_{2n-1}} \xi_{2n-1}^{A_{2n-1}} \xi_{2n-1}^{A_{2n-1}} = \tilde{t}_{2n-1}^{A_{2n-1}} t_{2n-1}^{A_{2n-1}}. 
\end{align*}
\]

and similarly,
\[
\begin{align*}
\tilde{t}_2^{A_{2n-1}} \tilde{t}_{2n-1}^{A_{2n-1}} \tilde{t}_1^{A_{2n-1}} \tilde{t}_{2n-2}^{A_{2n-1}} &= \tilde{t}_{2n-2}^{A_{2n-1}} \tilde{t}_{2n-1}^{A_{2n-1}} \tilde{t}_1^{A_{2n-1}} \tilde{t}_{2n-2}^{A_{2n-1}} = \tilde{t}_{2n-2}^{A_{2n-1}} \tilde{t}_{2n-1}^{A_{2n-1}} \tilde{t}_1^{A_{2n-1}} \tilde{t}_{2n-2}^{A_{2n-1}}.
\end{align*}
\]

If we had equality in \( (72) \), then
\[
\begin{align*}
\tilde{t}_{2n-1}^{A_{2n-1}} \tilde{t}_{2n-2}^{A_{2n-1}} \tilde{t}_{2n-1}^{A_{2n-1}} \tilde{t}_{2n-2}^{A_{2n-1}} &= \tilde{t}_{2n-2}^{A_{2n-1}} \tilde{t}_{2n-1}^{A_{2n-1}} \tilde{t}_{2n-2}^{A_{2n-1}} \tilde{t}_{2n-1}^{A_{2n-1}} \\
\iff \tilde{t}_{2n-1}^{A_{2n-1}} \tilde{t}_{2n-2}^{A_{2n-1}} \tilde{t}_{2n-1}^{A_{2n-1}} \tilde{t}_{2n-2}^{A_{2n-1}} &= \tilde{t}_{2n-1}^{A_{2n-1}} \tilde{t}_{2n-2}^{A_{2n-1}} \tilde{t}_{2n-1}^{A_{2n-1}} \tilde{t}_{2n-2}^{A_{2n-1}}.
\end{align*}
\]

Choosing the \( A_{2n-1} \) Yamanouchi tableau \( Y(\lambda^A) \) of shape \( \lambda^A \), this identity would imply
\[
\begin{align*}
\tilde{t}_{2n-1}^{A_{2n-1}} \tilde{t}_{2n-2}^{A_{2n-1}} \tilde{t}_{2n-1}^{A_{2n-1}} \tilde{t}_{2n-2}^{A_{2n-1}} (Y(\lambda^A)) &= \tilde{t}_{2n-1}^{A_{2n-1}} \tilde{t}_{2n-2}^{A_{2n-1}} \tilde{t}_{2n-1}^{A_{2n-1}} \tilde{t}_{2n-2}^{A_{2n-1}} (Y(\lambda^A)) \\
\end{align*}
\]

which is absurd unless \((\tilde{t}_1^{A_{2n-1}} \tilde{t}_2^{A_{2n-1}})^3 = 1 \) and \((\tilde{t}_{2n-1}^{A_{2n-1}} \tilde{t}_{2n-2}^{A_{2n-1}})^3 = 1 \). The latter identities do not hold either because the Bender–Knuth involutions \( t_i^{A_{2n-1}} \) and their duals \( \tilde{t}_{2n-i}^{A_{2n-1}} \), \( 1 \leq i \leq 2n \), do not satisfy the braid relations of \( \mathfrak{S}_{2n} \).

Despite the fact that the symplectic Bender Knuth involutions \( t_i^{C_n} \), \( 1 \leq i \leq n \), do not define an action of the Weyl group \( B_n \) on the set \( \text{KN}(\lambda, n) \), similarly to the type \( A_n-1 \) case, they can be used to show that the character of the crystal \( \text{KN}(\lambda, n) \) is a symmetric Laurent polynomial with respect to the action of the Weyl group \( B_n \). Let \( \mathcal{E} := \mathbb{Z}[x_1^\pm, \ldots, x_n^\pm] \) be the ring of Laurent polynomials on the variables \( x_1, \ldots, x_n \) over \( \mathbb{Z} \), and let \( \mathcal{E}^{B_n} = \{ f \in \mathcal{E} : r_i f = f, r_i \in B_n, 1 \leq i \leq n \} \) be the subring of symmetric Laurent polynomials.

The character of \( \text{KN}(\lambda, n) \) is the symplectic Schur function \( sp\lambda(x) \) in the sequence of variables \( x = (x_1, \ldots, x_n) \). Thanks to Remark \[21\] \( \text{wt}(t_i^{C_n} b) = r_i \text{wt}(b) \) for any \( b \in \text{KN}(\lambda, n) \) and \( 1 \leq i \leq n \). Therefore, since \( t_i^{C_n} \), \( 1 \leq i \leq n \), is an involution on the
set $\mathbf{KN}(\lambda, n)$ \( sp_{\lambda}(x) \) is a symmetric Laurent polynomial

\[
sp_{\lambda}(x) = \sum_{b \in \mathbf{KN}(\lambda, n)} x^{\text{wt}(b)} = \sum_{b \in \mathbf{KN}(\lambda, n)} x^{\text{wt}(t_{i}^{\alpha})}, \ 1 \leq i \leq n,
\]

\[
= \sum_{b \in \mathbf{KN}(\lambda, n)} x^{r_{i} \cdot \text{wt}(b)} = sp_{\lambda}(r_{i} \cdot x), \ 1 \leq i \leq n,
\]

where \( x^{\alpha} := x_{1}^{\alpha_{1}} \cdots x_{n}^{\alpha_{n}} \) and \( (r_{i} \cdot x)^{\alpha} = x^{r_{i} \cdot \alpha} \), for \( \alpha \in \mathbb{Z}^{n}, r_{i} \in B_{n} \).

10.4. Relations for the symplectic Berenstein–Kirillov group \( B\mathbf{K}^{C_{n}} \). Thanks to Theorem 8 and Theorem 9, we now provide the following relations. The relation \([\mathbf{5}]\) below (nontrivial for \( n \geq 3 \) and false for \( n = 2 \)) is the only one known for \( B\mathbf{K}^{C_{n}} \) which does not follow from the cactus group \( J_{\mathbf{sp}(2n, C)} \) relations, equivalently, the virtual cactus group \( J_{2n} \) relations.

**Proposition 13.** Let \( n \geq 3 \). The symplectic Bender–Knuth involutions satisfy the following relations:

1. \((t_{i}^{C_{n}})^{2} = 1, \ i = 1, \ldots, 2n - 1\).
2. \((t_{i}^{C_{n}})^{1} t_{j}^{C_{n}} = 1, \ 1 \leq i, j \leq n\).
3. \((t_{i}^{C_{n}})^{2} = 1, \ |i - j| > 1, 1 \leq i, j < n\).
4. \((t_{i}^{C_{n}})^{2} = 1, \ i < n - j\).
5. \((t_{i}^{C_{n}})^{2} = 1, \ i + 1 < j < k \leq n\).
6. \((t_{i}^{C_{n}})^{2} = 1, \ i + 1 < j \leq n\).
7. \((t_{i}^{C_{n}})^{2} = 1, \ i + 1 < j \leq n\).
8. \((t_{i}^{C_{n}})^{2} = 1, \ i < n - j\).
9. \((t_{i}^{C_{n}})^{2} = 1, \ n - i + 1 < j < k \leq n\).

The virtual symplectic Bender–Knuth involutions \( t_{i}^{A_{2n-1}} t_{2n-i}^{A_{2n-1}} = E t_{i}^{C_{n}} E^{-1}, 1 \leq i < n \), and \( t_{n-i+1,n+i}^{A_{2n-1}} = E t_{n-i+1,n+i}^{C_{n}} E^{-1}, 1 \leq i \leq n \), in \( B\mathbf{K}^{C_{n}} \) satisfy the same relations as those of \( B\mathbf{K}^{C_{n}} \) by replacing \( t_{i}^{C_{n}} \) by \( t_{i}^{A_{2n-1}} t_{2n-i}^{A_{2n-1}} \), \( 1 \leq i < n \), and \( t_{n-i+1,n+i}^{C_{n}} \) by \( t_{n-i+1,n+i}^{A_{2n-1}} \), \( 1 \leq i \leq n \).

**Proof.** Recall theorems 8 and 9.

1. \((t_{i}^{C_{n}})^{2} = (q_{i}^{C_{n}})^{2} = 1, \ (t_{i}^{C_{n}})^{2} = (q_{i}^{C_{n}})^{2} = 1. \ For 2 \leq i \leq n, \ (t_{n-i+1,n+i}^{C_{n}})^{2} = (q_{n-i+1,n+i}^{C_{n}} q_{n-i+2,n+i}^{C_{n}})^{2} = 1 \) is equivalent to the \( J_{\mathbf{sp}(2n, C)} \) relation 3C(i).

For \( 2 \leq i \leq n - 1 \),

\[
(t_{i}^{C_{n}})^{2} = q_{[1,i-1]}^{C_{n}} q_{[1,i]}^{C_{n}} q_{[1,i-1]}^{C_{n}} q_{[1,i-2]}^{C_{n}} q_{[1,i-1]}^{C_{n}} q_{[1,i-1]}^{C_{n}} q_{[1,i-2]}^{C_{n}} = 1
\]

follows from the cactus \( J_{\mathbf{sp}(2n, C)} \) relation 3C(ii) and the observations that

\[
q_{[1,i-1]}^{C_{n}} q_{[1,i-2]}^{C_{n}} = q_{[2,i-1]}^{C_{n}} q_{[1,i-1]}^{C_{n}}, \quad q_{[1,i-1]}^{C_{n}} q_{[2,i-1]}^{C_{n}} = q_{[1,i-2]}^{C_{n}} q_{[1,i-1]}^{C_{n}},
\]

\[
q_{[1,i]}^{C_{n}} q_{[2,i]}^{C_{n}} = q_{[2,i-1]}^{C_{n}} q_{[1,i]}^{C_{n}}
\]
(2) Let \( i \neq j \). From 3C(i),

\[
t_{n+i-1}^{C_n} t_{n+j-1}^{C_n} = q_{n-i+1,n}^{C_n} q_{n-i+2,n}^{C_n} q_{n-j+1,n}^{C_n} q_{n-j+2,n}^{C_n} = q_{n-i+1,n}^{C_n} q_{n-j+2,n}^{C_n} = t_{n+i-1}^{C_n} t_{n+j-1}^{C_n}.
\]

(3) Recall (51) and Remark 20. Then

\[
(t_{i}^{C_n} t_{j}^{C_n})^2 = (E^{-1} t_{i}^{A_{2n-1}} t_{j}^{A_{2n-1}} E E^{-1} t_{j}^{A_{2n-1}} t_{i}^{A_{2n-1}} E)^2 = E^{-1} (t_{i}^{A_{2n-1}} t_{j}^{A_{2n-1}})^2 (t_{j}^{A_{2n-1}} t_{i}^{A_{2n-1}})^2 E = 1, \text{ for } |i - j| > 1, 1 \leq i, j \leq n - 1.
\]

(4) For \( i < n - j \),

\[
t_{i}^{C_n} t_{n+j-1}^{C_n} = q_{1,i}^{C_n} q_{n-j+1,n}^{C_n} q_{n-j+2,n}^{C_n} = q_{1,i}^{C_n} q_{n-j+2,n}^{C_n}
\]

due to the \( \varphi(2n, C) \) relation 2C.

(5) Recall (54) and Remark 20, (51). Then

\[
(t_{i}^{C_n} t_{j}^{C_n})^6 = E^{-1} (t_{i}^{A_{2n-1}} t_{2}^{A_{2n-1}} t_{j}^{A_{2n-1}} t_{3}^{A_{2n-1}})^6 E = E^{-1} (t_{i}^{A_{2n-1}} t_{2}^{A_{2n-1}} t_{j}^{A_{2n-1}} t_{3}^{A_{2n-1}})^6 E
\]

(6) For \( i + 1 < j < k \leq n \),

\[
(t_{i}^{C_n} q_{j,k-1}^{C_n})^2 = (E^{-1} t_{i}^{A_{2n-1}} t_{2}^{A_{2n-1}} E E^{-1} t_{j}^{A_{2n-1}} t_{k}^{A_{2n-1}} E)^2 = E^{-1} (t_{i}^{A_{2n-1}} t_{2}^{A_{2n-1}} t_{j}^{A_{2n-1}} t_{2}^{A_{2n-1}})^2 E, \ 2n - k < 2n - j < 2n - i - 1
\]

(7) For \( i + 1 < j \leq n \),

\[
t_{i}^{C_n} q_{j,n}^{C_n} = E^{-1} t_{i}^{A_{2n-1}} t_{2}^{A_{2n-1}} E E^{-1} t_{j}^{A_{2n-1}} t_{n}^{A_{2n-1}} E, \ 2n - i - 1 > 2n - j \geq n
\]

(8) \((t_{n+i-1}^{C_n} q_{n+j,n}^{C_n})^2 = 1, 1 \leq i, j \leq n.\)
\[ \ell_{n+i-1}^{C_n} q_{[j,n]}^{C_n} = E^{-1} \xi_{[n-(i-1),n+i-(i-1)]}^{A_{2n-1}} \xi_{[n-(i-2),n+i-(i-2)]}^{A_{2n-1}} E \]
\[ = E^{-1} \xi_{[j,n-2j]}^{A_{2n-1}} \xi_{[n-(i-1),n+i-(i-1)]}^{A_{2n-1}} E, \text{ by Theorem } \ref{t1} \]
\[ = q_{[j,n]}^{C_n} \ell_{n+i-1}^{C_n}. \]

\[ (9) \ (\ell_{n+i-1}^{C_n} q_{[j,k-1]}^{C_n})^2 = 1, \ n - i + 1 < j < k \leq n. \]

10.5. Example: the C₂ Bender–Knuth involutions and their virtual images.

**Example 10.** We illustrate the symplectic Bender–Knuth involutions \( \ell_1^{C_2}, \ell_2^{C_2}, \ell_3^{C_2} \) in \( BK^{C_2} \) as well their virtual images in \( BK_4^{C_2} \): \( \Lambda = \mathbb{Z}^2 \) and \( B_2 = \langle r_1, r_2 \rangle \); if \( (a, b) \in \mathbb{Z}^2 \), \( r_1(a, b) = (b, a) \) and \( r_2(a, b) = (a, b) \); \( \ell_1^{C_2} = \xi_1^{C_2}, \ell_2^{C_2} = \xi_2^{C_2}, \ell_3^{C_2} = \xi_3^{C_2} \).

Using the type C₂ signature rule

\[
\begin{align*}
1, 2 & \to + \\
2, 1 & \to - \\
2 & \to + \\
\bar{2} & \to -
\end{align*}
\]

we compute \( \ell_1^{C_2} T = \xi_1^{C_2} T \), where

\[
T = \begin{bmatrix}
1 & 1 & 2 & 2 & 2 & 2 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 1 & 1
\end{bmatrix}, \quad \text{wt}(T) = (-2, 1)
\]

\[
\begin{array}{cccccccc}
\text{1112} & \text{12} & \text{2} & \text{2} & \text{2} & \text{1} & \text{2} & \text{2} & \text{1} & \text{2} & \text{2} & \text{1} & \text{2} & \text{1}
\end{array}
\]

\[
\begin{array}{cccccccc}
\text{12} & \text{12} & \text{12} & \text{12} & \text{12} & \text{12} & \text{1}
\end{array}
\]

\[ \ell_1^{C_2} T = \xi_1^{C_2} T = \begin{bmatrix}
1 & 1 & 1 & 2 & 2 & 2 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 & 1 & 1
\end{bmatrix}, \quad \text{wt}(\ell_1^{C_2} T) = (1, -2). \]

Virtualization of \( \ell_1^{C_2} \) in A₂: \( \ell_1^{C_2} = E^{-1} \xi_2^{A_1} t_1^{A_3} E = E^{-1} t_1^{A_1} \xi_2^{A_3} E \). The shape of \( T \) is \( 3\omega_1 + 5\omega_2 \), and thus the shape of \( [E] T \) and \( [E] (\ell_1^{C_2} T) \) is \( 3(\omega_1 + \omega_3) + 10\omega_2 \), where
\[ \omega_k = (1^k), \quad 1 \leq k \leq 3, \]

\[ E(T) = 1 1 1 1 1 1 1 2 2 2 2 2 2 1 1, \]

and

\[ E(t_C^1 T) = E(\xi_C^1 T) = 1 1 1 1 1 1 1 2 2 2 2 2 2 1 1. \]

Using the \( A_3 \) signature rule

\[
\begin{align*}
1 & \rightarrow + & 2 & \rightarrow + & 2 & \rightarrow - & \bar{1} & \rightarrow - \\
\bar{1} & \rightarrow + & 2 & \rightarrow - & 1 & \rightarrow -
\end{align*}
\]

we compute

\[
\xi_A^3 t_A^1 E(T) = \xi_A^2 1 1 1 1 1 1 1 1 1 1 2 2 2 2 2 2 2 1 1
\]

\[ 2 2 2 2 2 2 2 2 1 1 1 1 1 1 \]

\[ 2 1 1 \]

Therefore,

\[ t_C^1(T) = E^{-1} \xi_A^3 t_A^1 E(T). \]

Virtualization of \( t_C^2 \): \( t_C^2 = E^{-1} \xi_A^2 E \).

\[ T = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 \\ 2 & 2 & 2 & 2 & 1 & 1 \end{bmatrix}, \]

\[ t_C^2(T) = \xi_C^2(T) = 1 1 2 2 2 2 2 1 1 \]

\[ 2 2 2 1 1 \]

\[ \text{wt}(t_C^2(T)) = (-2, -1) \]

\[ E(t_C^2(T)) = \xi_A^2 E(T) \]

\[ t_C^2(T) = E^{-1} \xi_A^2 E(T) \]

\[ \xi_A^2 E(T) = \xi_A^2 1 1 1 1 1 1 1 1 1 1 2 2 2 2 2 2 2 2 1 1 \]

\[ 2 2 2 2 2 2 2 2 1 1 1 1 1 1 \]

\[ 2 1 1 \]

Therefore,

\[ t_C^2(T) = E^{-1} \xi_A^2 E(t_C^2(T)). \]
Virtualization of $t_3^{C_2}$: $t_3^{C_2} = E^{-1}\xi_2^{A_1} t_1^{A_3} (t_2^{A_3} t_1^{A_3}) (t_2^{A_3} t_2^{A_3} t_1^{A_3}) E$.

$$T = \begin{array}{ccccccc}
1 & 1 & 1 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 1 & 1 & 1
\end{array}, \quad \text{wt}(T) = (-2, 1)$$

$t_3^{C_2}(T) = \xi_2^{C_2} \xi_{[1,2]}^{C_2}(T) = \xi_2^{C_2} \xi_{[1,2]}^{C_2}(T) = \begin{array}{ccccccc}
1 & 1 & 1 & 1 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 1 & 1
\end{array}, \quad \text{wt}(t_3^{C_2}(T)) = (2, 1)$

$$E(t_3^{C_2}(T)) = E(\xi_2^{C_2} \xi_{[1,2]}^{C_2}(T)) = \xi_2^{A_1} E(\xi_{[1,2]}^{C_2}(T)) = \xi_2^{A_1} \text{evac}^A( E(T)) = \text{evac}^A \xi_2^{A_1} E(T),$$

$$t_3^{C_2}(T) = E^{-1} \xi_2^{A_1} \text{evac}^A( E(T)) = E^{-1} \xi_2^{A_1} t_1^{A_3} (t_2^{A_3} t_1^{A_3}) (t_2^{A_3} t_2^{A_3} t_1^{A_3}) E(T).$$

11. Open questions and final remarks

It remains to establish whether or not $\mathcal{BK}_n^{C_n}$ satisfies additional relations besides those of $\mathcal{BK}_n$.

Chmutov, Glick and Pylyavskyy [CGP16] have determined relationships between subsets of relations in the groups $\mathcal{BK}_n$ and $J_n$ which yield a presentation for the cactus group $J_n$ in terms of Bender–Knuth generators. Rodrigues [Ro20, Ro21] has also introduced a shifted Berenstein–Kirillov group with many parallels with the original Berenstein–Kirillov group. Following Halacheva she has defined a cactus group action of $J_n$ via partial shifted Schützenberger–Lusztig involutions (partial shifted reversal) on the Gillespie-Levibnson-Puhrboo shifted tableau crystal [GLP17]. On the other hand, with the shifted tableau switching she has defined shifted Bender–Knuth involutions, and following Chmutov, Glick and Pylyavskyy she has yield a presentation for the cactus group $J_n$ in terms of shifted Bender–Knuth generators. In the same vein, it is natural to seek precise relationships between subsets of relations in the two groups $\mathcal{BK}_n^{C_n}$ and the virtual symplectic cactus group $\bar{J}_2$. It is also natural to seek a presentation of the virtual symplectic cactus group $\bar{J}_2$ in terms of the virtual symplectic Bender–Knuth generators.

Glossary

$\mathcal{BK}_n^{C_n}$: The type $C_n$ symplectic Berenstein–Kirillov group.

$\mathcal{BK}_n$: The subgroup of $\mathcal{BK}$ generated by the first $n - 1$ Bender–Knuth involutions $t_1, \ldots, t_{n-1}$.

$\mathcal{BK}$: The Berenstein–Kirillov group (or Gelfand-Tsetlin group).

$\mathcal{KN}_I(\lambda, n)$: The Levi branched crystal of Kashiwara-Nakashima tableaux obtained by deleting in $\mathcal{KN}(\lambda, n)$ all the arrows not labelled in $J \subset I$.

$\mathcal{SSYT}(\lambda, n)$: The $U_q(\mathfrak{gl}(n, \mathbb{C}))$-crystal of semi-standard Young tableaux of shape $\lambda$ and entries in $[n]$.

$\bar{J}_2$: The virtual symplectic cactus group.
\(\widetilde{BK}_{2n}\): The virtual symplectic Berenstein-Kirillov group, a subgroup of \(BK_{2n}\) satisfying the relations of the virtual symplectic cactus group \(\widetilde{J}_{2n}\).\[1\][3][5][15][59][59]

\(E\): The virtualization map defined by Baker \[Ba00a\, Proposition 2.2, Proposition 2.3\] on type \(C\) Kashiwara-Nakashima tableaux.\[1\][11][15][26][52][63]

\(J_n\): The cactus group \(J_{gl(n,\mathbb{C})}\).\[3][15][17][26][52][63]

\(J_{sp}(2n,\mathbb{C})\): The symplectic cactus group with generators \(s_J\) for \(J\) any connected sub-diagram of the \(C\) Dynkin diagram subject to the relations in Lemma 2.\[1\][2][16][20][26][47][48][53][54][56][59][60]

\(R3\): The symplectic contraction/dilation relation in the symplectic plactic monoid \(C^*_n/\sim\).\[23][24][31][36][38][44]

\(C_n\): The monoid of words in the alphabet \(C\).\[12][13][21][23]

\(g\): Finite dimensional, complex, semisimple Lie algebra.\[2][3][5][7][15][17][20][21][25][26][29]

\(B(\lambda)\): The normal \(g\)-crystal with highest weight \(\lambda\).\[2][7][8][20][21][23][29][30][64]

\(B_J\): The Levi branched normal crystal \(B_J\), the restriction of \(B\) to the sub-diagram \(J\) of \(I\).\[7][14][25][30][64]

\(B\): A normal crystal.\[2][7][8][20][21][23][29][30][64]

\(\text{Kn}(\lambda, n)\): The \(U(sp(2n,\mathbb{C}))\) crystal of Kashiwara–Nakashima tableaux of shape \(\lambda\) in the alphabet \(C\).\[2][4][8][10][14][23][24][26][31][32][34][53]

\(\text{SSYT}(\lambda, n, A)\): The \(U_q(gl(2n,\mathbb{C}))\)-crystal of semi-standard Young tableaux of shape \(\lambda\) and entries in \(C\).\[4][11][13][47][56]

\(\text{SSYT}_J(\lambda, n)\): The Levi branched crystal, the restriction of \(\text{SSYT}(\lambda, n)\) to \(J \subseteq [n-1]\).\[27]

\(\text{reversal}^{\text{Kn}}\): J-partial symplectic reversal, the symplectic reversal \(\text{Kn}_J(\lambda, n)\) with \(J \subseteq [n]\) a connected sub-diagram containing the node \(n\).\[28][43]

\(\text{reversal}^{\text{SSYT}}\): Combinatorial procedure to compute the Schützenberger involution \(\xi\) on \(\text{Kn}(\lambda, n)\).\[24]

\(\text{reversal}_J\): J-partial reversal, the reversal on \(\text{SSYT}_J(\lambda, n)\) with \(J \subseteq [n-1]\).\[26][28]

\(\xi_B\): The Schützenberger–Lusztig involution on the normal crystal \(B\).\[2][20]

\(\xi_J\): The partial Schützenberger–Lusztig involution to the sub-diagram \(J \subseteq I\) is the Schützenberger–Lusztig involution \(\xi_{B_J}\) on the normal crystal \[57][25][28][30][31]

\(\xi\): The Schützenberger–Lusztig involution on \(B(\lambda)\).\[20][24][27][29][42]

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