Charged fermion in two-dimensional curved spaces of constant Gaussian curvature with constant magnetic flux

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Abstract. We investigate the behavior of spin-1/2 particles (electron and positron) confined to the Gaussian curvature surfaces. For the non-negative Gaussian curvatures, we present the preliminary results in cylindrical and spherical cases. To be specific we use the deformed hyperbolic solutions to obtain eigenvalues of the Dirac equation in the presence of an axial gauge field. Our results demonstrate the quantized energy and eigenstates of fermion. The quantization of energy depends on the spin-orbit coupling and the Landau quantization. The imaginary energy is obtained from the negative Gaussian curvatures. It is interpreted as the quasi normal mode (QNM). The angular momentum of fermion is shifted by addition of the constant magnetic flux. The fermion behaves like boson when the flux is half-integer.

1. Introduction
Costa [1,2] was the first to investigate the non-relativistic quantum mechanics in the presence of extrinsic and intrinsic curvature. For the relativistic property of the half-spin particle, Dirac equation reveals interesting physical consequences of the particle constraints in (2+1)-dimensional spacetime system. A quantum effect of particle is determined by the curvature of the space.

When charged particles are subjected to a magnetic field perpendicular to a surface, the orbits will be quantized [3]. The Aharonov-Bohm(AB) effect is a quantum mechanical phenomenon. It gives a phase shift in which a charged particle is affected by a gauge potential. When a particle travelling in the region with zero magnetic fields, it can still acquire a phase shift [4]. For example, the charged particles are confined to carbon nanotubes in the presence magnetic flux [5]. The Quantum Hall effects [6, 7] are notably phenomena in the constrained charged fermions with gauge fields. In the quantum system, strain is equivalent to an effective gauge field, e.g. electrons in deformed nanotube and graphene experience deformed potential generated from the strain tensors [8–10], a particle in metal sheet induced by strain or electromagnetic field, such as the investigation of the fermions interaction on a curved surface [11–15]. Recently, many people are studying the phenomenon of the following systems; The electronic properties of graphene and its multilayers [16–19], and applications in curved graphene [20–27].

Possibility to study relevant gravitational effects related to the quantum character of the quasiparticles in the condensed matter system. The charged fermion in two-dimensional spherical space was studied by Garcia et al. [28]. They used the Dirac equation to study the massless
changed fermion in the C$_{60}$ molecule is placed in the presence of an AB flux, and the appearance of an analogue of the Aharonov-Carmi phase in a rotating C$_{60}$ molecule [29]. Gonzalez et al. [30] studied the electronic structure of the junctions between a graphene layer and carbon nanotubes. Recently, this group is considering the graphene wormholes where the nanotube bridge attached to two graphene sheets [31]. In 2018, Cariglia et al. [32] considered non-relativistic Lévy-Leblond fermion an essentially smooth simplified spacetime, namely a Bronnikov-Ellis wormhole. A. Iorio and G. Lambiase [33] investigated the surfaces of constant Gaussian curvature is realized by a two-dimensional axially-symmetric curved space. They usually have Hilbert horizons, and this phenomenon is consistently to the BTZ black hole [34].

In this work, we investigate physical properties of the charged fermions confined on the surface of constant Gaussian curvature which are subjected to an external magnetic field along the direction of the local flatness axis $u$. In Section 2, basic geometric and gauge setup are established. In Section 3, we show a simple interpretation of the results of a cylindrical tube in terms of the angle between the spin and orbital angular momentum of the fermion confined on the surface. In Section 4, the Dirac equation in two-dimensional spherical space is used to analyze the (1+2)-dimensional stationary state of the charged fermion surrounded the two-dimensional spherical surface. Here, we associate a doublet of spinors interacting with a characteristic curvature of space, the presence of an axial gauge field. To compare the hyperbolic pseudosphere (what we called two-dimensional wormhole) to general case could be analyzed in Section 5 to identify the crucial role of surface with negative Gaussian curvature. In Section 6, the special cases of constant Gaussian curvature are considered. Finally, we summarize and discuss our results in Section 7.

2. Geometric and gauge setup of curved spaces

The position vector on the axially-symmetric curved space is

$$\mathbf{r}(u, \theta) = R(u) \cos \theta \, \hat{i} + R(u) \sin \theta \, \hat{j} \pm \int du \sqrt{1 - (R'(u))^2} \, \hat{k}. \quad (1)$$

The constraint on $z$ follows from the relation $d|\mathbf{r}|^2 = dx^2 + dy^2 + dz^2 = du^2 + R^2 d\theta^2$. It gives the Hilbert horizons at $R'(u_H) = \pm 1$.

Embedding (1+2)-dimensional curved space into (1+3)-dimensional space generates effective gravity or effective curvature to the reduced spacetime. Any particle or quasiparticle living on the reduced spacetime will experience the spacetime curvature.

The transformation matrix between the two coordinates is then

$$\frac{\partial x'_{\mu'}}{\partial x^\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & R'(u) \cos \theta & R(u) \sin \theta & \sqrt{1 - (R'(u))^2} \\ 0 & -R(u) \sin \theta & R(u) \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (2)$$

Where $dx'_{\mu'} = \{cdt, dx, dy, dz\}$ as the (1+3)-dimensional Minkowski spacetime coordinates, and $dx^{\mu} = \{cdt, du, d\theta\}$ as the (1+2)-dimensional curved spacetime coordinates. In the (1+2)-dimensional curved spacetime, the line element is in the following form

$$ds^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} = -c^2 dt^2 + d|\mathbf{r}|^2, \quad (3)$$

where the (1 + 2)-dimensional metric is $g_{\mu\nu}$. The zweibein $e^a_\mu$ is then defined as

$$e^a_\mu = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & R(u) \end{pmatrix}. \quad (4)$$
where $g_{\mu\nu} \equiv e_a^\mu e_b^\nu \eta_{ab}$, and $\eta_{ab} = \text{diag}(-1, 1, 1)$ in $(1+2)$-dimensions and $a, b \in \{0, 1, 2\}$. We consider an electron (or charged fermion) in Minkowski space subject to the curved spacetimes embedding constraints. The fermion will experience the effective curvature that can be addressed by considering the Dirac equation in curved $(1+2)$-dimensional spacetime

$$\left[ \gamma^a e_a^\mu \left( -\hbar \nabla_\mu + i \frac{e}{c} A_\mu \right) - M c \right] \Psi = 0. \quad (5)$$

$\Psi = \Psi(t, u, \theta)$ represents the Dirac spinor field and $M$ represents the rest mass of the particle, $c$ is the speed of light in the curved spacetimes, $e$ is electric charge, and $A_\mu$ is the electromagnetic four-potential. The $\gamma^a$ are the Dirac matrices given by

$$\gamma^0 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & i\sigma^k \\ -i\sigma^k & 0 \end{pmatrix}, \quad (6)$$

where $\sigma^k$ are the Pauli matrices.

The covariant derivative of the spinor interaction with gauge field in the curved space is given by

$$\nabla_\mu \equiv \partial_\mu - \Gamma_\mu$$

where the spin connection $\Gamma_\mu$ [35] is

$$\Gamma_\mu = -\frac{1}{4} \gamma^a \gamma^b e_a^\nu \left[ \partial_\mu \left( g_{\nu\beta} e_b^\beta \right) - e_{\beta b}^\beta \Gamma_{\beta\mu\nu} \right], \quad (7)$$

where $\beta, \mu, \nu \in \{t, u, \theta\}$ and the Christoffel symbols $\Gamma_{\beta\mu\nu}$ given by

$$-\Gamma_{u\theta\theta} = \Gamma_{\theta u\theta} = \Gamma_{\theta\theta u} = \frac{1}{2} \partial_u R^2 = RR', \quad (8)$$

and zero otherwise. The spin connections become

$$\Gamma_t = 0, \quad \Gamma_u = 0, \quad \Gamma_\theta = \frac{1}{2} \gamma^1 \gamma^2 R'. \quad (9)$$

In this work, we will apply an external magnetic field is uniform with respect to the plane $(x, y)$ in the magnetic flux through the circular area enclosed at a fixed $z$ is constant, namely $B_z \sim 1/R^2$. Due to the axial symmetry, the electromagnetic four-potential can be expressed in the axial gauge as

$$A'_\mu(t, x, y, z) = (0, -\frac{1}{2} B y, \frac{1}{2} B x, 0), \quad (10)$$

and in the $(1 + 2)$-dimensional spacetime coordinates as

$$A_\mu(t, u, \theta) = \frac{\partial x'_{\nu'}}{\partial x^\mu} A'_{\nu'}(t, x, y, z) = \left( 0, 0, \frac{1}{2} B R^2 \right). \quad (11)$$

The magnetic field is then given by

$$\vec{B} = \left( -\frac{x}{2} \partial_x B, -\frac{y}{2} \partial_z B, B \right). \quad (12)$$

The Dirac equation equation (5) can be written in the form

$$\left( \frac{M c}{\hbar} + i \partial_t c \right) \begin{pmatrix} \hbar \nabla_\mu + i \frac{e}{c} A_\mu \\ -i T \end{pmatrix} \Psi = 0, \quad (13)$$

where $T = \frac{M c}{\hbar} - i \partial_t c$. 

where $D$ is a differential operator

$$D \equiv \sigma^1 \left( \partial_u + \frac{R'}{2R} \right) + \sigma^2 \left( \frac{\partial_\theta - i \phi}{\phi_0} \right),$$

where $\phi = \int \vec{B} \cdot d\vec{a} = \pi R^2 B$ and the magnetic flux quantum is defined as $\phi_0 \equiv \hbar c/e$.

The first term is equivalent to the Dirac equation with the pseudo gauge potential $A_u(u) \equiv \frac{i \hbar c R'}{2eR}$ in the $u$ direction, is generated by the curvature along the $\theta$ direction, $\Gamma_\theta$. In this sense, the intrinsic gravity connection can be interpreted as the effective (imaginary) gauge connection (in the locally perpendicular direction) that leads to the complexity of the energy and the emergence of the QNMs and unstable modes on a surface with the negative Gaussian curvature (see in Section 5-6). The second term is similar to a spin-orbit-curvature coupling potential [36].

In the presence of external magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}$ along the $z$ direction, the charged fermion moving in $\theta$ direction is expected to form a stationary state with quantized angular momentum and energy, i.e. the Landau levels in the curved space with hole. To show this, we need to solve for the stationary states of the system.

Consider a stationary state of the Dirac spinor $\Psi(t, u, \theta)$ in the form

$$\Psi(t, u, \theta) = e^{-\frac{i}{\hbar} Et \sigma^3} \chi(u) \varphi(u),$$

where $\chi(u), \varphi(u)$ are two-component spinors, for stationary states, the wave function needs to be single-valued at every point in spacetime. $\Psi(t, u, \theta)$ must be a periodic function in $\theta \in [0, 2\pi]$, the orbital angular momentum quantum number $m = 0, \pm 1, \pm 2, ...$. Equation (13) can be rewritten in the form of the second derivative equation

$$0 = \varphi''(u) + \frac{R'}{R} \varphi'(u) + \left[ \frac{R''}{2R} + \frac{\hbar m \sigma R' - \hat{m}^2 - (R'/2)^2}{R^2} + k^2 \right] \varphi(u),$$

where the new orbital angular momentum in the presence of magnetic flux $\hat{m} = m - \frac{\phi}{\phi_0}$ [37]. We have used the momentum parameter $k^2 \equiv (E^2 - M^2 c^4)/\hbar^2 c^2$ and $\sigma$ is a spin-state index corresponding to spin up ($\sigma = +1$) or down ($\sigma = -1$) of the fermion for each eigenvalue of $\sigma^3$.

In these calculation, you can see in reference [38]. In this paper, we will demand the wave function to be regular in the constant Gaussian curved space and will not specify the boundary condition at the Hilbert horizons.

3. Cylindrical Geometry

To understand essential physics of the magnetized charged fermion in the curved spaces, consider a simple case when $R(u)$ is constant, i.e. a cylindrical tube. In this case, the intrinsic (Gaussian) curvature is zero, so we can identify which effects are induced by the “gravity”. Equation (16) become

$$0 = \varphi''(u) + \left[ k^2 - \left( \frac{\hat{m}}{R} \right)^2 \right] \varphi(u).$$

Assuming the solution in the form $\varphi_{(tube)}(u) \sim \exp(i k_u u)$ to obtain

$$E^2_{m,n} = M^2 c^4 + \left( \frac{\hbar c}{R} \right)^2 \left[ \left( \frac{n\pi R}{2r} \right)^2 + \left( m - \frac{\phi}{\phi_0} \right)^2 \right].$$

Setting $2r$ as the length of the cylinder with the boundary conditions $\varphi(u = 0) = \varphi(u = 2r) = 0$, we get $k_u = n\pi / 2r$ where $n = 0, 1, 2, ...$. The energy is purely real and only normal modes exist.
4. Spherical Geometry
In this section, we focus on a surface of positive constant Gaussian curvature when \( R(u) = d \cos(u/r) \). Equation (16) become

\[
0 = (1 - X^2)\varphi''(X) - 2X\varphi'(X) + \left[ k^2r^2 - \frac{1}{4} + \frac{\hat{m}\sigma X}{d} - \left( \frac{\hat{m}}{d} \right)^2 - \frac{1}{4} \right] \varphi(X),
\]

when we perform the coordinate transformation \( X(u) \equiv rR'(u)/d = -\sin \phi \), and the zenith angle \( \phi = u/r \). Now we define solution \( \varphi_{(sphere)}(X) = (1 - X)^{\alpha}(1 + X)^{\beta}P(X) \), the equation of motion (19) can be rewritten as

\[
0 = (1 - X^2)P''(X) + 2\left[ \beta - \alpha - X(\beta + \alpha + 1) \right] P'(X) + \left[ k^2r^2 - \left( \beta + \alpha + \frac{1}{2} \right)^2 \right] P(X),
\]

where we assume

\[
\alpha = \pm \left( \frac{1}{4} - \frac{\hat{m}\sigma r}{2d} \right), \quad \beta = \pm \left( \frac{1}{4} + \frac{\hat{m}\sigma r}{2d} \right),
\]

Equation (20) is Jacobi Differential Equation, the energy levels become

\[
E_{m,n}^2 = M^2 c^4 + \hbar^2 c^2 k_{m,n}^2 = M^2 c^4 + \frac{\hbar^2 c^2}{r^2} \left( n + \frac{1}{2} + \alpha + \beta \right)^2.
\]

Depending on the sign choices of \((\alpha, \beta)\), the resulting equation of motion and the corresponding energy levels will be dependent or independent of the spin-orbit coupling term \( \sim \sigma mr/d \). The general solution can be expressed in the form

\[
\Psi(t, \phi, \theta) = e^{-\frac{t}{\hbar}E_{m,n}t}e^{im\theta} \left( \frac{-ihcD\varphi(\phi)}{E_{m,n} + M \omega^2} \right),
\]

where \( \varphi(\phi) = \varphi_{m,n}(1 + \sin \phi)^{\alpha}(1 - \sin \phi)^{\beta}P_n^{(2\alpha,2\beta)}(-\sin \phi) \).

5. Wormhole Geometry
The surface of the wormhole is realized by a two-dimensional axially-symmetric curved space of a constant negative Gaussian curvature. To be specific, we will choose a deformed hyperbolic wormhole described by \( R(u) = d \cosh_q(u/r) \) where \( d \) is the radius of wormhole at the midpoint between the two edges and \( r \) is the radius of curvature of the wormhole surface along \( u \) direction. In this case, a surface has a constant negative Gaussian curvature. \( R(u) \) is based on a \( q \)-deformation of the usual hyperbolic functions [39, 40] which are defined by

\[
cosh_q(x) \equiv \frac{e^x + qe^{-x}}{2}, \quad \sinh_q(x) \equiv \frac{e^x - qe^{-x}}{2}, \quad \tanh_q(x) = \frac{\sinh_q(x)}{\cosh_q(x)}.
\]

Definitions same to else hyperbolic functions but note that almost all relations known from the usual hyperbolic functions have been modified, for example

\[
cosh_q^2(x) - \sinh_q^2(x) = q, \quad \frac{d}{dx} \sinh_q(x) = \cosh_q(x), \quad \frac{d}{dx} \tanh_q(x) = \frac{q}{\cosh_q^2(x)}.
\]
Figure 1. Geometric structure of the curved surface where \( d \) is a radius at minimum radius function \( R(u = 0) \). And \( r \) is the scale parameter of the curved surfaces along \( u \) direction. (a) Hyperbolic pseudosphere (wormhole) surface where \( R(u) = d \cosh(q(u/r)) \). The Hilbert horizons of the wormhole are at \( u_H = r \ln\left(\frac{r}{d} \pm \frac{\sqrt{r^2/d^2 + 1}}{}\right) \) where \( q = 1 \). (b) Beltrami pseudosphere surface. The Hilbert horizon of Beltrami surface is at \( u_H = r \ln\left(\frac{2r}{d}\right) \) where \( q = 0 \). (c) Elliptic pseudosphere surface, gives the Hilbert horizons at \( u_H = r \ln\left(\frac{r}{d} \pm \frac{\sqrt{r^2/d^2 - 1}}{}\right) \) where \( q = -1 \).

Now performing the similar transformations using \( X(u) = rR'(u)/d = \sinh_q(u/r) \) and \( \varphi(q)(X) = (\sqrt{q} + iX)^\dot{\alpha}(\sqrt{q} - iX)^\dot{\beta}\Phi(X) \). The equation of motion (16) now takes the form

\[
0 = (1 - Y^2)\Phi''(Y) + 2 \left[ (\dot{\alpha} - \dot{\beta}) - (\dot{\beta} + \dot{\alpha} + 1) Y \right] \Phi'(Y) - \left[ k^2r^2 + (\dot{\beta} + \dot{\alpha} + \frac{1}{2})^2 \right] \Phi(Y),
\]

where we define \( X \equiv -i\sqrt{q}Y \), and \( \dot{\alpha}, \dot{\beta} \) now becomes

\[
\dot{\alpha} = \pm \left( \frac{1}{4} + \frac{i\sigma \cdot \hat{m} \cdot r}{2d} \right), \quad \dot{\beta} = \pm \left( \frac{1}{4} - \frac{i\sigma \cdot \hat{m} \cdot r}{\sqrt{q}2d} \right),
\]

the energy levels become

\[
E_{m,n}^2 = M^2c^4 + \hbar^2c^2k^2_{m,n} = M^2c^4 - \frac{\hbar^2c^2}{r^2} \left( n + \frac{1}{2} + \dot{\alpha} + \dot{\beta} \right)^2.
\]

For the sign choice of \((\dot{\alpha}, \dot{\beta})\) as \((+, +)\) and \((-,-)\), the energy is independent of the magnetic flux and spin-orbit term. On the other hand, for another sign choice \((+, -)\) and \((-,+),\) the energy levels depend on the spin-orbit term, are a complex quantity which can be interpreted as the quasi-normal modes (QNMs). For the QNMs with negative imaginary parts, the curvature effects leak the energy of the fermion away from the wormhole as long as the angle \( \theta = \arccos R' \) between the \( \sigma^3 \)-spin component and orbital angular momentum is not \( \pi/2 \). For the choice \((+, -)\), the positive-energy solution is unstable with positive imaginary part for \( \sigma \hat{m} < 0 \) and
for the choice \((-,+).\) For these states, the fermion will either slowly decay away or spin off the wormhole due to the curvature effect.

It is challenging to give physical interpretation to the states from the choice \((+,-)\) and \((-,-)\). They have negative momentum square along \(u\) direction, they do not feel the magnetic field and do not have the angular momenta. It is most natural to identify them with diffusive modes (due to imaginary momentum along the wormhole direction \(u\)) with \(m = 0\). However, the energy of these modes can be either real or purely imaginary depending on the quantum number \(n\).

Finally, the solutions of equation (15) is

\[
\varphi(q)(t,u,\theta) = \Phi_0 e^{-\frac{i}{\hbar}E_m u \theta} (1 + Y)^\delta (1 - Y)^\beta P_n^{(2\beta,2\delta)}(Y),
\]

where \(Y(u) = \frac{i}{\sqrt{q}} \sinh_q(u/r)\). These solutions, we checked also NDSolve in Wolfram Mathematica.

6. Pseudosphere Geometry

There are special cases when the deformation parameter \(q = 0, -1\), i.e. the Beltrami and Elliptic pseudosphere that are not captured in the general analysis, we address them in this section.

6.1. Beltrami Geometry

For \(q = 0\) to obtain \(R(u) = \frac{d}{2} e^{u/r}\), the equation of motion (16) becomes

\[
0 = \varphi''(u) + \frac{1}{r} \varphi'(u) + \left[ k^2 + \frac{1}{4r^2} + 2 \frac{\hat{m} \sigma}{dr} e^{-u/r} + \left( \frac{2 \hat{m}}{d} \right)^2 e^{-2u/r} \right] \varphi(u),
\]

The general solution can be expressed in the form

\[
\varphi_{(q=0)}(u) = Z^{\frac{n}{2}} e^{-Z/2} \left[ C_{11} F_1 \left( \kappa - \frac{\sigma}{2}, \kappa, Z \right) + C_2 U \left( \kappa - \frac{\sigma}{2}, \kappa, Z \right) \right],
\]

where we define \(\kappa \equiv (1 - 2i \kappa r)/2\) and \(Z(u) = \frac{4\hat{m} \sigma}{d} e^{-u/r}\) that takes the value \(Z \in [4\hat{m} \sigma/d, 0]\) for \(u \in [0, \infty)\) respectively. \(1 F_1, U\) is the confluent hypergeometric function of the First kind and second kind.

Regularity at \(Z = 4\hat{m} \sigma/d > 1\) demands that the series of the hypergeometric function truncates at finite power of \(Z\) giving the energy quantization

\[
E_n^2 = M^2 c^4 + \hbar^2 c^2 \kappa^2 = M^2 c^4 - \frac{\hbar^2 c^2}{r^2} \left( n + \frac{1 - \sigma}{2} \right)^2.
\]

Remarkably, the energies do not depend on the magnetic field and \(m\) at all, only the wave functions have \(\hat{m}\) dependence. All \(\hat{m}\) states degenerate in each energy level \(E_n\). They have the negative momentum square along \(u\) direction again.

6.2. Elliptic Geometry

For elliptic pseudosphere all formulae of the wormhole cases can be used. Notably since \(q = -1\), the parameters \(\hat{\alpha}, \hat{\beta}\) become purely real and we can simply make replacement \(\sqrt{q} = i\) in all the results of the wormhole cases. The spin-orbit and all magnetic induced coupling terms in equation (28) become real. For choice the sign of parameters \((\hat{\alpha}, \hat{\beta})\) as \((+, -)\) and \((-+, +)\), the QNMs only occur for highly excited states where the coupling terms are greater than the rest mass energy \(\sim Mc^2\). In this case, the energies are purely imaginary for when QNMs appear.
For (+,+) and (−,−) modes are not affected by the wormhole geometry, they are diffusive modes for highly excited states. Topologically, the elliptic surface is distinctively different from the hyperbolic(wormhole) and Beltrami ones. They haven’t QNMs for low n states in contrast to the hyperbolic and Beltrami cases because the space starts at \( R(u = 0) = 0 \), so the modes cannot leak out through the hole.

7. Conclusions
We consider spin-1/2 charged fermions confined to curved surfaces of constant Gaussian curvature with the constant magnetic flux through the area enclosed by a circle at a fixed \( z \).

The curvature connection of curved spaces generates effective gauge connection resulting in the induced spin-orbit coupling of the fermion on the surface. The coupling is genuinely gravitational since it exists even in the absence of the magnetic field. Adding external magnetic fields in the tangent direction to the surface, the new interaction is the combined effect of gravity and gauge field on the charged fermions, appearing in the terms of the angular momentum of the fermion and the magnetic field.

A simple picture to help understanding these results is the following. When a fermion is confined to the 2-dimensional curved space its \( \sigma^3 \)-spin component is perpendicular to the surface (since the zweibein is locally defined in the tangent space of the surface) while the orbital angular momentum is pointing along the \( z \)-direction. The spin-orbit coupling \( \sim \vec{\sigma} \cdot \vec{m} \hat{z} \) is thus generated for generic curved spaces with curvature. For cylindrical tube, the spin-orbit coupling disappears together with the Landau coupling between spin and the magnetic field. The only remaining interaction is the orbital-magnetic Landau coupling. Note that the \( \sigma^3 \)-spin component is pointing along the direction of the normal vector of the curvature of surface since the zweibein \( e^a_\mu \) is defined on the tangent space of the curved surface. Also, because \( R'(u) = \cos \delta \) where \( \delta \) is the angle between the \( \sigma^3 \)-spin component and the \( z \) axis, the spin-orbit coupling term can thus be rewritten as \( \sim \vec{\sigma} \cdot \vec{m} \hat{z} = \sigma m \cos \delta = \sigma m R' \). The spin-orbit coupling vanishes when \( R' = \cos \delta = 0 \) or \( \delta = \pi/2 \), i.e. when the normal vector of the surface is perpendicular to \( \hat{z} \) (see at figure 1). In the spherical, the energy levels contain the interaction spin-orbit coupling term, depending on choice of \( (\alpha, \beta) \). However, the energy still depends on the quantum number \( n \).

For every choice of \( \alpha \) and \( \beta \), sufficiently highly excited states with large \( n \) or strong magnetic flux \( \phi \) will always give QNMs. The energy naturally leaks out of the curved space of negative Gaussian curvature when the fermion is sufficiently excited. This is consistent with the existence of Hilbert horizons at finite \( \alpha_H \equiv r \ln \left( \frac{r}{d} \pm \sqrt{\left( \frac{r}{d} \right)^2 + q} \right) \). The negative curvature QNMs have much greater oscillation frequency under the damping rate (when \( \text{Im}[E] < 0 \)) [41]. Regardless of the magnetic field, the imaginary parts in the energy expression have the gravitational origin. The QNMs will vanish when the curvature is zero as we can see again in Section (3.4).

It is possible to alter the statistics of boson-like fermions when the flux-flux quantum ratio is half-integer. It’s a good riddle to the investigation of this novel in real experimental. This quite likely lead to several profound electronic properties and future applications.

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