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Existence, stability and controllability results of fractional dynamic system on time scales with application to population dynamics

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Abstract: In this manuscript, we investigate the existence, uniqueness, Hyer-Ulam stability and controllability analysis for a fractional dynamic system on time scales. Mainly, this manuscript has three segments: In the first segment, we give the existence of solutions. The second segment is devoted to the study of stability analysis while in the last segment, we establish the controllability results. We use the Banach and nonlinear alternative Leray-Schauder–type fixed point theorem to establish these results. Also, we give some numerical examples for different time scales. Moreover, we give two applications to outline the effectiveness of these obtained results.

Keywords: controllability; existence; fractional dynamic system; stability; time scales.

AMS Subject Classification: 34N05; 34A12; 93B05; 34A08.

1 Introduction

The theory of fractional calculus began with a correspondence between Leibniz and L’Hospitalin in 1695. Nowadays, lots of literature is available on numerical as well as theoretical work on this subject. It has many applications in the fields of bioengineering [1], physics [2], viscoelasticity [3], biology [4] and mathematical networks [5, 6], etc. A few years back, many researchers and specialists have appeared incredible interest in fractional theory because of the memory character of fractional derivative, which is the generalization of integer-order derivative and can describe many phenomena of finance, biology and physics, etc. that integer-order derivative cannot clarify. For more recent work, please see [7–12]. In addition, one of the qualitative principles which could be very significant from the numerical factor, and optimization view is committed to stability analysis of the solution to dynamical systems of integer as well as fractional order. During the last few decades, the concept of stability like Mittag-Leffler function, exponential, and Lyapunov stability for different types of dynamical systems have been studied. However, an interesting kind of stability was introduced by Ulam and Hyers is known as Ulam-Hyers’s stability. This stability analysis is very useful in many applications e.g., in chemical reactions, fluid flow, semiconductors, population dynamics, heat conduction, and elasticity. There are few researchers who established the Ulam’s type of stability results for the ordinary, as well as for the fractional differential and difference equations [13–16] and references therein.

On the other hand, one investigates the continuous and discrete cases separately. The concept of time scale analysis introduced by Hilger [17] is a fairly new idea that combines the continuous and discrete analysis into one theory. Nowadays, the study of dynamic systems on time scales have got a great deal of global...
consideration and various specialists have discovered the applications of time scales in engineering, physics, economics, population dynamics, control system, and as well as in heat transfer system. For more study about time scales, we refer to the books by Bohner and Peterson [18, 19] and the papers [20–25]. In the last few years, few authors have been worked on the existence and Hyer-Ulam (HU)–type stability for the ordinary, as well as fractional order dynamic system on time scales [26–34]. In particular, Ahmadkhlanlu and Jahanshahi [29], gave the necessary and sufficient conditions for the existence of unique solution to the following fractional order dynamic system on time scales

\[ c^T D^\alpha p(t) = \Phi(t, p(t)), \quad \forall \ t \in J = [t_0, t_0 + a] \subseteq \mathbb{T}, \quad \kappa \in (0, 1], \]

where \( \Phi: J \times \mathbb{R} \rightarrow \mathbb{R} \) is a right-dense continuous bounded function and \( c^T D^\alpha \) denotes the Caputo fractional derivative on time scales.

Eventhough controllability is the principal idea in the mathematical control theory which was introduced by Kalman in 1960 and soon became an active area of research by many mathematicians. Generally speaking, controllability implies the steering the state of a dynamical control system from an arbitrary initial state to the desired final state by using an appropriate control function. Controllability results of linear and nonlinear dynamical systems have been examined by numerous mathematicians, please see [35–39]. In addition, controllability results of the dynamic system on time scales is a relatively newer field, and only few results have been reported [40–44]. Particularly, the controllability result in the finite dimensional space of a dynamic system on time scales is reported in a study by Davis et al. [41]. In [44], the authors considered the dynamic system with impulsive conditions on time scales and established the necessary and sufficient conditions for the state observability and state controllability. There is not a single manuscript which discussed the existence, HU stability and controllability results for a fractional dynamic system on time scales. Inspired by the above works, for the existence and stability results, we consider the fractional dynamic system on time scales of the form

\begin{align}
\frac{c^T}{c^T} D^\nu p(t) &= \Phi(t, p(t), q(t), r(t)), \quad t \in I = [0, T], \kappa \in (0, 1), \\
\frac{c^T}{c^T} D^\nu q(t) &= \Psi(t, p(t), q(t), r(t)), \quad t \in I, \nu \in (0, 1), \\
\frac{c^T}{c^T} D^\nu r(t) &= \Theta(t, p(t), q(t), r(t)), \quad t \in I, \omega \in (0, 1), \\
p(0) &= p_0, \quad q(0) = q_0, \quad r(0) = r_0
\end{align}

(1.1)

and for the controllability analysis, we consider the fractional dynamic system on time scales of the following form

\begin{align}
\frac{c^T}{c^T} D^\nu p(t) &= \Phi(t, p(t), q(t), r(t)) + B u(t), \quad t \in I = [0, T], \kappa \in (0, 1), \\
\frac{c^T}{c^T} D^\nu q(t) &= \Psi(t, p(t), q(t), r(t)) + C v(t), \quad t \in I, \nu \in (0, 1), \\
\frac{c^T}{c^T} D^\nu r(t) &= \Theta(t, p(t), q(t), r(t)) + D w(t), \quad t \in I, \omega \in (0, 1), \\
p(0) &= p_0, \quad q(0) = q_0, \quad r(0) = r_0
\end{align}

(1.2)

with \( 0, T \in \mathbb{T} \). \( \frac{c^T}{c^T} D^\nu, \frac{c^T}{c^T} D^\nu, \frac{c^T}{c^T} D^\nu \) are the Caputo fractional derivatives on time scales. \( B, C, D \) are the bounded operators from \( \mathbb{R} \) to \( \mathbb{R} \), \( u, v, w \in L^2(I, \mathbb{R}) \) are the control inputs. \( \Phi, \Psi, \Theta: I \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) are given functions. The results obtained in this manuscript are completely new even in the case of differential equation (when \( \mathbb{T} = \mathbb{R} \)) and in the case of difference equation (when \( \mathbb{T} = \mathbb{Z} \)).

Obviously, we can see that when \( \mathbb{T} = \mathbb{R} \), then the system (1.1) includes the well known fractional order three species prey-predator model [45], chaotic fractional order GLV model [46], toy model for three species [47], fractional-order HIV infection of CD4+ T-cells models [48, 49], etc. In the existing literature, there is no manuscript that analysed these mathematical models on time scales. We considered a more general form of the fractional dynamic system on time scales which includes these mathematical models, as well as many more models for the continuous, discrete and combination of these two. In this setting, our results are new and contribute significantly to the existing literature on the topic. The manuscript is divided into 7 sections.
In Section 2, we give some fundamental definitions, useful lemmas and preliminaries which will be used in subsequent sections. Section 3, the existence of solutions for the fractional system (1.1) is obtained. Section 4 is devoted to the investigation of the HU stability for the proposed fractional system. In Section 5, we give the controllability results for the fractional system (1.2). In Section 6, we give an illustrative example for different time scales. In the last Section 7, two applications are given to outline the effectiveness of these obtained results.

2 Preliminaries

In this segment, we define some basic notations, fundamental definitions and introduce useful lemmas. $C(I, \mathbb{R})$ forms a Banach space of all continuous functions endowed with the norm $\|p\|_C = \sup_{t \in I} |p(t)|$. Therefore, the product space $PS = C(I, \mathbb{R}) \times C(I, \mathbb{R}) \times C(I, \mathbb{R})$ is also Banach space with the norm $\|(p, q, r)\|_C = \|p\|_C + \|q\|_C + \|r\|_C$. Further, $C^1(I, \mathbb{R}) = \{p \in C(I, \mathbb{R}) : p^b \in C(I, \mathbb{R})\}$ is a Banach space with the norm $\|p\|^b = \max\{\|p\|_C, \|p^b\|_C\}$. Therefore, the product space $PS^1 = C^1(I, \mathbb{R}) \times C^1(I, \mathbb{R}) \times C^1(I, \mathbb{R})$ is also Banach space with norm $\|(p, q, r)\|^b = \|p\|^b + \|q\|^b + \|r\|^b$.

A time scale $T$ is a nonempty closed subset of real numbers. A time scale interval is defined as $[a, b]_T = \{t \in T : a \leq t \leq b\}$. In the similar way, we can define $(a, b)_T$, $[a, b)_T$, etc. We define $T^k = T \setminus \{\max T\}$ if $\max T$ exists, otherwise $T^k = T$.

**Definition 2.1.** [18] We define $\sigma : T \rightarrow T$ as a forward jump operator by $\sigma(t) = \inf\{s \in T : s > t\}$ along with $\inf \emptyset = \sup T$. Moreover, a positive function $\mu(t) = \sigma(t) - t, \forall t \in T$, is known as graininess function.

A function $\phi : T \rightarrow \mathbb{R}$ is called positive regressive (regressive) if $1 + \mu(t)\phi(t) > 0 (\neq 0), \forall t \in T$ and the collection of all positive regressive (regressive) functions are denoted by $\mathcal{R}^+ (\mathcal{R})$. A function $\phi : T \rightarrow \mathbb{R}$ is called rd-continuous on $T$, if $\phi$ is continuous at all right-dense points of $T$ and limit exist (finite) at all left dense point of $T$. The set of all rd-continuous functions from $T$ to $\mathbb{R}$ is denoted by $C_{rd}(T, \mathbb{R})$.

**Definition 2.2.** [18] The delta derivative of a function $f : T \rightarrow \mathbb{R}$ at a point $t \in T^k$ is a number $f^\Delta (t)$ (provided it exists), if there exists a neighbourhood $U$ of $t$ and $\epsilon > 0$ such that

$$\big|\big|f(\sigma(t)) - f(\tau)\big|\big| - f^\Delta (t)\big|\sigma(t) - \tau\big| \leq \epsilon|\sigma(t) - \tau|, \quad \forall \tau \in U.$$

**Definition 2.3.** [18] A function $F : [a, b]_T \rightarrow \mathbb{R}$ is called antiderivative of a function $f : [a, b]_T \rightarrow \mathbb{R}$ if $F^\Delta (t) = f(t)$ for each $t \in [a, b]_T$ is holds. Moreover, the $\Delta$-integral is defined as

$$\int_a^b f(\zeta)\Delta \zeta = F(b) - F(a).$$

**Theorem 2.4.** [18] Let $a, b \in T$ and $f \in C_{rd}(T, \mathbb{R})$, then

1. If $T = \mathbb{R}$, then

$$\int_a^b f(t)\Delta t = \int_a^b f(t)dt.$$

2. If $[a, b]$ consists of only isolated points, then

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{t \in [a,b]} \mu(t)f(t) & \text{if } a < b \\ 0 & \text{if } a = b \\ \sum_{t \in (a,b]} \mu(t)f(t) & \text{if } a > b. \end{cases}$$
3. If $T = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$, where $h > 0$, then

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{k=0}^{b-1} hf(kh) & \text{if } a < b \\ 0 & \text{if } a = b \\ -\sum_{k=1}^{a-1} hf(kh) & \text{if } a > b. \end{cases}$$

**Theorem 2.5.** [29] Let $T$ be an arbitrary time scale with $t_1, t_2 \in T$ and let $f : \mathbb{R} \to \mathbb{R}$ be a nondecreasing, continuous function. Then,

$$\int_{t_1}^{t_2} f(\zeta)\Delta \zeta \leq \int_{t_1}^{t_2} f(\zeta) d\zeta.$$  \tag{2.3}

**Definition 2.6.** [29] The delta fractional integral of an integrable function $\phi : [a, b] \to \mathbb{R}$ by

$$\delta I_a^t \phi(t) = \int_a^t \frac{(t - \zeta)^{\kappa-1}}{\Gamma(\kappa)} \phi(\zeta)\Delta \zeta,$$  \tag{2.4}

**Definition 2.7.** [29] Let $f : T \to \mathbb{R}$ be a function. Then, the Caputo fractional derivative on time scales of $f$ is defined as

$$c^T D_a^t f(t) = \int_a^t \frac{(t - \zeta)^{n-1}}{\Gamma(n-k)} f(\zeta)\Delta \zeta,$$  \tag{2.5}

where $n = \lceil k \rceil + 1$ and $f(\zeta)^{\kappa}$ denotes the $\kappa$th delta derivative of $f$.

- When $T = \bigcup_{k=0}^{\infty} [2k, 2k+1]$, then, we have

$$c^T D_a^t f(t) = \int_a^t \frac{(t - \zeta)^{n-1}}{\Gamma(n-k)} f(\zeta)\Delta \zeta = \frac{1}{\Gamma(n-k)} \sum_{k=0}^{\kappa-1} h(t - kh)^{n-k-1} f(\zeta(k), t \in T.$$

- When $T = h\mathbb{Z}$, $h > 0$, we have

$$c^T D_a^t f(t) = \int_a^t \frac{(t - \zeta)^{n-1}}{\Gamma(n-k)} f(\zeta)\Delta \zeta = \frac{1}{\Gamma(n-k)} \sum_{k=0}^{\kappa-1} h(t - kh)^{n-k-1} f(\zeta(k), t \in T.$$

- When $T = \{q^n : q > 1, n \in \mathbb{Z}\} \cup \mathbb{Z}$, then

$$c^T D_a^t f(t) = \int_a^t \frac{(t - \zeta)^{n-1}}{\Gamma(n-k)} f(\zeta)\Delta \zeta = \frac{1}{\Gamma(n-k)} \sum_{s \in \mathbb{Z}} \mu(s) (t - s)^{n-k-1} f(\zeta(s).$$

**Theorem 2.8.** [29] Let $\Phi : I \times \mathbb{R} \to \mathbb{R}$ be a function. Then, $p(t)$ is a solution of

$$c^T D_a^t p(t) = \Phi(t, p(t)), \quad t \in I, \kappa \in (0, 1), \quad p(0) = p_0,$$

iff $p(t)$ is a solution of the following equation

$$p(t) = p_0 + \frac{1}{\Gamma(\kappa)} \int_0^t (t - \zeta)^{\kappa-1} \Phi(\zeta, p(\zeta))\Delta \zeta.$$
Under the following assumptions, we prove the main results of the paper:

(A): $\Phi, \Psi, \Theta : I \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions. Also, there exist positive constants $L_{\Phi}, L_{\Psi}, L_{\Theta}, i = 1, 2, 3$, such that

$$|\Phi(t, p_1, p_2, p_3) - \Phi(t, q_1, q_2, q_3)| \leq \sum_{i=1}^{3} L_{\Phi}|p_i - q_i|, \quad \forall t \in I, p_i, q_i \in \mathbb{R}, i = 1, 2, 3,$$

$$|\Psi(t, p_1, p_2, p_3) - \Psi(t, q_1, q_2, q_3)| \leq \sum_{i=1}^{3} L_{\Psi}|p_i - q_i|, \quad \forall t \in I, p_i, q_i \in \mathbb{R}, i = 1, 2, 3,$$

$$|\Theta(t, p_1, p_2, p_3) - \Theta(t, q_1, q_2, q_3)| \leq \sum_{i=1}^{3} L_{\Theta}|p_i - q_i|, \quad \forall t \in I, p_i, q_i \in \mathbb{R}, i = 1, 2, 3.$$

(B): $\Phi, \Psi, \Theta : I \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions. Also, there exist positive constants $l_0, m_0, n_0, i = 0, 1, 2, 3$ such that

$$|\Phi(t, p, q, r)| \leq l_0 + l_1|p| + l_2|q| + l_3|r|, \quad \forall t \in I, p, q, r \in \mathbb{R},$$

$$|\Psi(t, p, q, r)| \leq m_0 + m_1|p| + m_2|q| + m_3|r|, \quad \forall t \in I, p, q, r \in \mathbb{R},$$

$$|\Theta(t, p, q, r)| \leq n_0 + n_1|p| + n_2|q| + n_3|r|, \quad \forall t \in I, p, q, r \in \mathbb{R}.$$

(W): The linear operators $^\kappa \mathsf{W}_0^\tau, ^\gamma \mathsf{W}_0^\tau, ^\omega \mathsf{W}_0^\tau : L^2(I, \mathbb{R}) \to \mathbb{R}$ defined by

$$^\kappa \mathsf{W}_0^\tau u = \frac{1}{\Gamma(\kappa)} \int_{0}^{T} (T - \zeta)^{\kappa-1} Bu(\zeta) d\zeta, \quad ^\gamma \mathsf{W}_0^\tau v = \frac{1}{\Gamma(\gamma)} \int_{0}^{T} (T - \zeta)^{\gamma-1} C v(\zeta) d\zeta,$$

$$^\omega \mathsf{W}_0^\tau w = \frac{1}{\Gamma(\omega)} \int_{0}^{T} (T - \zeta)^{\omega-1} D w(\zeta) d\zeta,$$

have the bounded invertible operators $ (^\kappa \mathsf{W}_0^\tau)^{-1}, (^\gamma \mathsf{W}_0^\tau)^{-1}$ and $ (^\omega \mathsf{W}_0^\tau)^{-1}$ which take values in $L^2(I, \mathbb{R}) \setminus \ker(^\kappa \mathsf{W}_0^\tau)$, $L^2(I, \mathbb{R}) \setminus \ker(^\gamma \mathsf{W}_0^\tau)$ and $L^2(I, \mathbb{R}) \setminus \ker(^\omega \mathsf{W}_0^\tau)$, respectively. Also, there exist positive constants $M_{\kappa,\nu}, M_{\gamma,\nu}$ and $M_{\omega,\nu}$ such that $\|(^\kappa \mathsf{W}_0^\tau)^{-1}\| \leq M_{\kappa,\nu}, \|(^\gamma \mathsf{W}_0^\tau)^{-1}\| \leq M_{\gamma,\nu}$ and $\|(^\omega \mathsf{W}_0^\tau)^{-1}\| \leq M_{\omega,\nu}$.

Also, $\mathcal{B}, \mathcal{C}, \mathcal{D}$ are continuous operator from $\mathbb{R}$ to $\mathbb{R}$ and there exist positive constants $M_{\mathcal{B}}, M_{\mathcal{C}}, M_{\mathcal{D}}$ such that $\|\mathcal{B}\| \leq M_{\mathcal{B}}, \|\mathcal{C}\| \leq M_{\mathcal{C}}, \|\mathcal{D}\| \leq M_{\mathcal{D}}$.

The equation (2.6) can be calculated for different time scales by using Theorem 2.4. For example:

- When $T = hZ, h > 0$, we have:

$$^\kappa \mathsf{W}_0^\tau u = \frac{1}{\Gamma(\kappa)} \int_{0}^{T} (T - \zeta)^{\kappa-1} Bu(\zeta) d\zeta = \frac{1}{\Gamma(\kappa)} \sum_{\nu=0}^{\kappa-1} h(T - \zeta)^{\kappa-1} Bu(\zeta).$$

- When $T = \cup_{k=0}^{n}[2k, 2k + 1]$. Let $T \in [4,5]$. Then, we have

$$^\kappa \mathsf{W}_0^\tau u = \frac{1}{\Gamma(\kappa)} \int_{0}^{T} (T - \zeta)^{\kappa-1} Bu(\zeta) d\zeta = \frac{1}{\Gamma(\kappa)} \left[ \int_{0}^{4} (T - \zeta)^{\kappa-1} Bu(\zeta) d\zeta + \int_{4}^{5} (T - \zeta)^{\kappa-1} Bu(\zeta) d\zeta + \int_{5}^{T} (T - \zeta)^{\kappa-1} Bu(\zeta) d\zeta \right].$$

- When $T = [q^n : q > 1, n \in \mathbb{Z}] \cup \mathbb{Z}$, then

$$^\kappa \mathsf{W}_0^\tau u = \frac{1}{\Gamma(\kappa)} \int_{0}^{T} (T - \zeta)^{\kappa-1} Bu(\zeta) d\zeta = \frac{1}{\Gamma(\kappa) \mu(t)} \sum_{t \in (0, T]} \mu(t) (T - t)^{\kappa-1} Bu(t).$$

Similarly, we can find $^\gamma \mathsf{W}_0^\tau v$ and $^\omega \mathsf{W}_0^\tau w$.

Throughout the manuscript, we set

$$Q_1 = \frac{T^\kappa (L_{\Phi} + L_{\Psi} + L_{\Theta})}{\Gamma(\kappa + 1)}; \quad Q_2 = \frac{T^\gamma (L_{\Phi} + L_{\Psi} + L_{\Theta})}{\Gamma(\gamma + 1)}; \quad Q_3 = \frac{T^\omega (L_{\Phi} + L_{\Psi} + L_{\Theta})}{\Gamma(\omega + 1)};$$
3 Existence of solution

Theorem 3.1. If the assumption (A) holds, then the system (1.1) has a unique solution provided

\[ \max_{1 \leq i \leq 3} |Q_i| < \frac{1}{3}, \]  

(3.7)

Proof. Consider a subset \( \Omega \subseteq PS \) such that

\[ \Omega = \{(p, q, r) \in PS : ||(p, q, r)||_C \leq \delta \}, \]

where

\[ \delta > \max \left\{ \frac{\mathcal{N}_1}{1 - 3Q_1}, \frac{\mathcal{N}_2}{1 - 3Q_2}, \frac{\mathcal{N}_3}{1 - 3Q_3} \right\}. \]

Now, we define an operator \( \Xi : \Omega \to \Omega \) as

\[ \Xi(p, q, r)(t) = \begin{pmatrix} \Xi_x(p, q, r)(t) \\ \Xi_y(p, q, r)(t) \\ \Xi_\omega(p, q, r)(t) \end{pmatrix}, t \in I, \]

(3.8)

where

\[ \Xi_x(p, q, r)(t) = p_0 + \frac{1}{\Gamma(k)} \int_0^T (t - \zeta)^{k-1} \Phi(\zeta, p(\zeta), q(\zeta), r(\zeta)) \, d\zeta, \]

\[ \Xi_y(p, q, r)(t) = q_0 + \frac{1}{\Gamma(v)} \int_0^T (t - \zeta)^{v-1} \Psi(\zeta, p(\zeta), q(\zeta), r(\zeta)) \, d\zeta, \]

\[ \Xi_\omega(p, q, r)(t) = r_0 + \frac{1}{\Gamma(\omega)} \int_0^T (t - \zeta)^{\omega-1} \Theta(\zeta, p(\zeta), q(\zeta), r(\zeta)) \, d\zeta. \]

Also, note that

\[ |\Phi(t, p, q, r)| \leq |\Phi(t, 0, 0, 0)| + |\Phi(t, 0, 0, 0)| |L\|p| + L\|q| + M\|r| + M_\phi, \]

\[ |\Psi(t, p, q, r)| \leq L\|p| + L\|q| + L\|r| + M_\psi, \]

\[ |\Theta(t, p, q, r)| \leq L\|p| + L\|q| + L\|r| + M_\theta, \]

where \( M_\phi = \sup_{t \in I} |\Phi(t, 0, 0, 0)|, M_\psi = \sup_{t \in I} |\Psi(t, 0, 0, 0)| \) and \( M_\theta = \sup_{t \in I} |\Theta(t, 0, 0, 0)|. \)

Now, we need to show that \( \Xi \) maps \( \Omega \) into itself, we have
\[
|\Xi_x(p, q, r)(t)| \leq |p_0| + \left| \frac{1}{\Gamma(\kappa)} \int_0^T (t - \zeta)^{\kappa-1} \Phi(\zeta, p(\zeta), q(\zeta), r(\zeta)) \Delta \zeta \right|
\]
\[
\leq |p_0| + \left| \frac{1}{\Gamma(\kappa)} \int_0^T (t - \zeta)^{\kappa-1} \left( L_{p_0} |p(\zeta)| + L_{q_0} |q(\zeta)| + L_{r_0} |r(\zeta)| + M_{\psi} \right) \Delta \zeta \right|
\]
\[
\leq |p_0| + \left( \frac{L_{\psi_{p_0}} + L_{\psi_{q_0}} + L_{\psi_{r_0}} + M_{\psi}}{\Gamma(\kappa)} \right) \int_0^T (t - \zeta)^{\kappa-1} \Delta \zeta.
\]
Since \((t-\zeta)^{\kappa-1}\) is an increasing function, by using Theorem 2.5 in the above equation, we get
\[
|\Xi_x(p, q, r)(t)| \leq |p_0| + \left( \frac{L_{\psi_{p_0}} + L_{\psi_{q_0}} + L_{\psi_{r_0}} + M_{\psi}}{\Gamma(\kappa)} \right) \int_0^T (t - \zeta)^{\kappa-1} \Delta \zeta
\]
\[
\leq |p_0| + \frac{T^\kappa(L_{\psi_{p_0}} + L_{\psi_{q_0}} + L_{\psi_{r_0}} + M_{\psi})}{\Gamma(\kappa + 1)} \leq N_1 + \delta Q_1.
\]
Hence,
\[
||\Xi_x(p, q, r)||_c \leq N_1 + \delta Q_1 \leq \frac{\delta}{3}, \quad (3.9)
\]
Also,
\[
|\Xi_x(p, q, r)(t)| \leq |q_0| + \left| \frac{1}{\Gamma(\nu)} \int_0^T (t - \zeta)^{\nu-1} \Psi(\zeta, p(\zeta), q(\zeta), r(\zeta)) \Delta \zeta \right|
\]
\[
\leq |q_0| + \left( \frac{L_{\psi_{p_0}} + L_{\psi_{q_0}} + L_{\psi_{r_0}} + M_{\psi}}{\Gamma(\nu + 1)} \right) \int_0^T (t - \zeta)^{\nu-1} \Delta \zeta
\]
\[
\leq N_2 + \delta Q_2.
\]
Hence,
\[
||\Xi_x(p, q, r)||_c \leq N_2 + \delta Q_2 \leq \frac{\delta}{3}, \quad (3.10)
\]
Similarly, one can find
\[
||\Xi_x(p, q, r)||_c \leq N_3 + \delta Q_3 \leq \frac{\delta}{3}, \quad (3.11)
\]
Summarize the above inequalities (3.9), (3.10) and (3.11), we get:
\[
||\Xi_x(p, q, r)||_c \leq \delta. \quad (3.12)
\]
Therefore, \(\Xi: \Omega \to \Omega\). Also, when \(t \in I\), we have
\[
|\Xi_x(p, q, r)(t) - \Xi_x(p, q, r)(t)| \leq \left| \frac{1}{\Gamma(\kappa)} \int_0^T (t - \zeta)^{\kappa-1} \left[ \Phi(\zeta, p(\zeta), q(\zeta), r(\zeta)) - \Phi(\zeta, p(\zeta), q(\zeta), r(\zeta)) \right] \Delta \zeta \right|
\]
\[
\leq \frac{1}{\Gamma(\kappa)} \int_0^T (t - \zeta)^{\kappa-1} \left( L_{p_0} |p(\zeta) - p(\zeta)| + L_{q_0} |q(\zeta) - q(\zeta)| + L_{r_0} |r(\zeta) - r(\zeta)| \right) \Delta \zeta
\]
\[
\leq \frac{T^\kappa(L_{p_0} ||p||_c + L_{q_0} ||q||_c + L_{r_0} ||r||_c)}{\Gamma(\kappa + 1)}.
\]
Thus,

$$\|\Xi_v(p, q, r) - \Xi_v(\tilde{p}, \tilde{q}, \tilde{r})\|_C \leq Q_1((|p - \tilde{p}|_C + |q - \tilde{q}|_C + |r - \tilde{r}|_C).$$  \hfill (3.13)

Also,

$$|\Xi_v(p, q, r)(t) - \Xi_v(\tilde{p}, \tilde{q}, \tilde{r})(t)| \leq \frac{1}{\Gamma(v)} \int_0^T (t - \zeta)^{v-1} |\Psi(\zeta, p(\zeta), q(\zeta), r(\zeta)) - \Psi(\zeta, \tilde{p}(\zeta), \tilde{q}(\zeta), \tilde{r}(\zeta))| \, d\zeta$$

$$\leq \frac{T^v(L_{\Phi},|p - \tilde{p}|_C + L_{\Psi},|q - \tilde{q}|_C + L_{\Psi},|r - \tilde{r}|_C)}{\Gamma(v + 1)}.$$  

Thus,

$$\|\Xi_v(p, q, r) - \Xi_v(\tilde{p}, \tilde{q}, \tilde{r})\|_C \leq Q_2((|p - \tilde{p}|_C + |q - \tilde{q}|_C + |r - \tilde{r}|_C).$$  \hfill (3.14)

Similarly, we can easily obtain

$$\|\Xi_v(p, q, r) - \Xi_v(\tilde{p}, \tilde{q}, \tilde{r})\|_C \leq Q_3((|p - \tilde{p}|_C + |q - \tilde{q}|_C + |r - \tilde{r}|_C).$$  \hfill (3.15)

After summarize the inequalities (3.13), (3.14) and (3.15), we get

$$\|\Xi(p, q, r) - \Xi(\tilde{p}, \tilde{q}, \tilde{r})\|_C \leq (Q_1 + Q_2 + Q_3)((p, q, r, (\tilde{p}, \tilde{q}, \tilde{r})).$$

Therefore, from the equation (3.7), we can conclude that $\Xi$ is a strict contraction operator. Hence, by means of Banach fixed point technique, we have a unique fixed point of $\Xi$ which is the solution of the system (1.1). $\square$

Next, by applying the Leray-Schauder alternative-type fixed point technique, existence of at least one solution for the system (1.1) under the weaker condition (B) is given.

**Theorem 3.2.** If the assumption (B) holds, then the system (1.1) has at least one solution on $I$, provided there exists a positive constant $K$ such that

$$\mathcal{N}_4 + \mathcal{N}_5 K < K.$$  \hfill (3.16)

**Proof.** Firstly, we show that the operator $\Xi$ defined by (3.8) is completely continuous. We note that continuity of functions $\Phi$, $\Psi$ and $\Theta$ implies that the operator $\Xi$ is continuous. Also, let $\Omega_1 \subseteq PS$ is bounded. Then, there exist $L_1, L_2, L_3 > 0$, such that $|\Phi(t, p, q, r)| \leq L_1, |\Psi(t, p, q, r)| \leq L_2, |\Theta(t, p, q, r)|$. Then, for any $(p, q, r) \in \Omega_1$, we have

$$|\Xi_v(p, q, r)(t)| \leq |p_0| + \frac{1}{\Gamma(k)} \int_0^T (t - \zeta)^{k-1} |\Phi(\zeta, p(\zeta), q(\zeta), r(\zeta))| \, d\zeta \leq |p_0| + \frac{L_1 T^k}{\Gamma(k + 1)}.$$  

Thus,

$$\|\Xi_v(p, q, r)\|_C \leq |p_0| + \frac{L_1 T^k}{\Gamma(k + 1)} = C_\epsilon.$$  \hfill (3.17)

Also,

$$|\Xi_v(p, q, r)(t)| \leq |q_0| + \frac{1}{\Gamma(v)} \int_0^T (t - \zeta)^{v-1} |\Psi(\zeta, p(\zeta), q(\zeta), r(\zeta))| \, d\zeta \leq |q_0| + \frac{L_1 T^v}{\Gamma(v + 1)}.$$  

Thus,

$$\|\Xi_v(p, q, r)\|_C \leq |q_0| + \frac{L_1 T^v}{\Gamma(v + 1)} = C_v.$$  \hfill (3.18)

Similarly, we have
\[ \|\Xi_\omega(p,q,r)\|_C \leq |r_0| + \frac{L_w T^w}{\Gamma(\omega + 1)} = C_\omega. \]  

(3.19)

Thus, from the above inequalities (3.17), (3.18) and (3.19), it follows that the operator \( \Xi \) is uniformly bounded.

Next, we need to show that \( \Xi \) is completely continuous. We know that any time scales can be categories into three cases namely discrete, continuous and the combination of discrete and continuous. Hence, we are considering the following three cases:

**Case 1:** When the time scales are discrete. i.e., all the points of the time scales are isolated. In this case, using Theorem 2.4, \( \Xi_\epsilon, \Xi_\nu \) and \( \Xi_\omega \) become

\[ \Xi_\epsilon(p,q,r)(t) = p_0 + \frac{1}{\Gamma(\kappa)} \sum_{\tau \in (0,1)} \mu(t)\Phi(t, p(t), q(t), r(t)), \]

\[ \Xi_\nu(p,q,r)(t) = q_0 + \frac{1}{\Gamma(\nu)} \sum_{\tau \in (0,1)} \mu(t)\Psi(t, p(t), q(t), r(t)), \]

\[ \Xi_\omega(p,q,r)(t) = r_0 + \frac{1}{\Gamma(\omega)} \sum_{\tau \in (0,1)} \mu(t)\Theta(t, p(t), q(t), r(t)), \]

which are the summation operators on a discrete finite set. Since, \( \Phi, \Psi \) and \( \Theta \) are continuous. Hence, \( \Xi_\epsilon, \Xi_\nu \) and \( \Xi_\omega \) are completely continuous. Hence \( \Xi \) is also completely continuous.

**Case 2:** When the time scales are continuous. i.e., all the points of time scales are dense. For this, let \( \tau_1, \tau_2 \in I \) such that \( \tau_1 < \tau_2 \), then

\[ |\Xi_\epsilon(p,q,r)(\tau_2) - \Xi_\epsilon(p,q,r)(\tau_1)| \leq \frac{1}{\Gamma(\kappa)} \int_{\tau_1}^{\tau_2} (\tau_2 - \zeta)^{\kappa-1} \Phi(\zeta, p(\zeta), q(\zeta), r(\zeta)) d\zeta \]

\[ - \frac{1}{\Gamma(\kappa)} \int_{\tau_1}^{\tau_2} (\tau_1 - \zeta)^{\kappa-1} \Phi(\zeta, p(\zeta), q(\zeta), r(\zeta)) d\zeta \]

\[ \leq \frac{1}{\Gamma(\kappa)} \int_{\tau_1}^{\tau_2} ((\tau_2 - \zeta)^{\kappa-1} - (\tau_1 - \zeta)^{\kappa-1}) \Phi(\zeta, p(\zeta), q(\zeta), r(\zeta)) d\zeta \]

\[ + \frac{1}{\Gamma(\kappa)} \int_{\tau_1}^{\tau_2} (\tau_2 - \zeta)^{\kappa-1} \Phi(\zeta, p(\zeta), q(\zeta), r(\zeta)) d\zeta \]  

(3.20)

Also,

\[ |\Xi_\nu(p,q,r)(\tau_2) - \Xi_\nu(p,q,r)(\tau_1)| \leq \frac{1}{\Gamma(\nu)} \int_{\tau_1}^{\tau_2} ((\tau_2 - \zeta)^{\nu-1} - (\tau_1 - \zeta)^{\nu-1}) \Psi(\zeta, p(\zeta), q(\zeta), r(\zeta)) d\zeta \]

\[ + \frac{1}{\Gamma(\nu)} \int_{\tau_1}^{\tau_2} (\tau_2 - \zeta)^{\nu-1} \Psi(\zeta, p(\zeta), q(\zeta), r(\zeta)) d\zeta \]  

(3.21)

Similarly, we can easily find

\[ |\Xi_\omega(p,q,r)(\tau_2) - \Xi_\omega(p,q,r)(\tau_1)| \leq \frac{1}{\Gamma(\omega)} \int_{\tau_1}^{\tau_2} ((\tau_2 - \zeta)^{\omega-1} - (\tau_1 - \zeta)^{\omega-1}) \Theta(\zeta, p(\zeta), q(\zeta), r(\zeta)) d\zeta \]

\[ + \frac{1}{\Gamma(\omega)} \int_{\tau_1}^{\tau_2} (\tau_2 - \zeta)^{\omega-1} \Theta(\zeta, p(\zeta), q(\zeta), r(\zeta)) d\zeta \]  

(3.22)

Clearly, one can see that the right-hand side of the above inequalities (3.20), (3.21) and (3.22) tend to zero as \( \tau_2 \to \tau_1 \). Hence, the operators \( \Xi_\epsilon, \Xi_\nu \) and \( \Xi_\omega \) are equicontinuous. Therefore, using Arzela-Ascoli theorem, one can find that the operator \( \Xi \) is completely continuous.
Case 3: When the time scales are the combination of discrete and continuous points, then, for the discrete points, using Theorem 2.4, Ξ can be written as the summation which is completely continuous (same as case 1). Also, for the continuous points, we can show that Ξ is completely continuous (same as case 2). Thus, Ξ can be written as a sum of two operators, one for the discrete points and one for the continuous points. Since, the sum of two completely continuous operators is also completely continuous. Hence, we can conclude that Ξ is completely continuous.

Thus, from the above three cases, we can say that Ξ is completely continuous.

Finally, let, for $\beta \in [0,1]$ there exists a $(p(t), q(t), r(t))$ such that $(p(t), q(t), r(t)) = \beta(\Xi(p, q, r)t)$. Then for $t \in I$, we have:

$$|p(t)| = |\beta \Xi_c (p, q, r) t| = |\beta \left( p_0 + \frac{1}{\Gamma (s)} \int_0^t (t - \zeta)^{-\lambda} \Phi (\zeta, p (\zeta), q (\zeta), r (\zeta)) d\zeta \right) |
\leq |p_0| + \frac{T^\nu (l_0 + l_1 ||p||_c + l_2 ||q||_c + l_3 ||r||_c)}{\Gamma (\nu + 1)}.
$$

Thus,

$$||p||_c \leq |p_0| + \frac{T^\nu (l_0 + l_1 ||p||_c + l_2 ||q||_c + l_3 ||r||_c)}{\Gamma (\nu + 1)}.
$$

Similarly, we get

$$||q||_c \leq |q_0| + \frac{T^\omega (m_0 + m_1 ||p||_c + m_2 ||q||_c + m_3 ||r||_c)}{\Gamma (\omega + 1)}
$$

$$||r||_c \leq |r_0| + \frac{T^\omega (n_0 + n_1 ||p||_c + n_2 ||q||_c + n_3 ||r||_c)}{\Gamma (\omega + 1)}.
$$

From the above inequalities (3.23), (3.24) and (3.25) we get

$$||p||_c + ||q||_c + ||r||_c \leq |p_0| + |q_0| + |r_0| + \frac{T^\nu (l_0 + l_1 ||p||_c + l_2 ||q||_c + l_3 ||r||_c)}{\Gamma (\nu + 1)}
+ \frac{T^\omega (m_0 + m_1 ||p||_c + m_2 ||q||_c + m_3 ||r||_c)}{\Gamma (\omega + 1)}
+ \frac{T^\omega (n_0 + n_1 ||p||_c + n_2 ||q||_c + n_3 ||r||_c)}{\Gamma (\omega + 1)}
\leq N_4 + N_5 (||p||_c + ||q||_c + ||r||_c).
$$

Thus,

$$\frac{||\langle p, q, r \rangle||_c}{N_4 + N_5 ||\langle p, q, r \rangle||_c} \leq 1.
$$

Now, the condition (3.16) gives a $K > 0$ such that $||\langle p, q, r \rangle||_c < K$. Let us suppose that $\Omega = \{ \langle p, q, r \rangle \in PS : ||\langle p, q, r \rangle||_c < K \}$. The operator $\Xi : \Omega \to PS$ is continuous and completely continuous. Therefore, from the choice of $\Omega$, there is no $\langle p, q, r \rangle \in \partial (\Omega)$ such that $(p, q, r) = \beta \Xi (p, q, r), \beta \in [0,1]$. Thus, by using the fixed point theorem due to nonlinear alternative of Leray Schauder, $\Xi$ has a fixed point which is the solution of the system (1.1).

4 Stability analysis

For $\epsilon_a > 0, \epsilon_r > 0$ and $\epsilon_\omega > 0$, consider the inequality
Lemma 4.5. \[\left\{ \begin{array}{l} c^\Gamma D^r p(t) - \Phi(t, p(t), q(t), r(t)) \leq \epsilon_p, \quad t \in I, \\ c^\Gamma D^r q(t) - \Psi(t, p(t), q(t), r(t)) \leq \epsilon_q, \quad t \in I, \\ c^\Gamma D^r r(t) - \Theta(t, p(t), q(t), r(t)) \leq \epsilon_r, \quad t \in I. \end{array} \right. \tag{4.26} \]

**Definition 4.1.** [13] System (1.1) is called HU stable if there exists a positive number $\mathcal{H}_{(i, j, l, s, k, x, y, a)}$ such that for $\epsilon > 0$ and for each solution $(p, q, r)$ of inequality (4.26), there exists a unique solution $(p, q, r)$ of system (1.1), which satisfies the following inequality
\[ |(p, q, r)(t) - (p, q, r)(t)| \leq \mathcal{H}_{(i, j, l, s, k, x, y, a)}(\epsilon), \quad \forall \ t \in I. \]

**Definition 4.2.** [13] System (1.1) is said to be generalized HU stable if there exists $\mathcal{H}_{(i, j, l, s, k, x, y, a)}(0) = 0$ such that for each solution $(p, q, r)$ of inequalities (4.26), there exists a solution $(p, q, r)$ of system (1) satisfies the following inequality
\[ |(p, q, r)(t) - (p, q, r)(t)| \leq \mathcal{H}_{(i, j, l, s, k, x, y, a)}(\epsilon), \quad \forall \ t \in I. \]

**Remark 4.3.** Definition 4.1 $\Rightarrow$ Definition 4.2.

**Remark 4.4.** A function $(p, q, r) \in PS^\Gamma$ is a solution of inequality (4.26) iff there is $f, g, h \in PS$ such that
\[
(a) |f(t)| \leq e_x, |g(t)| \leq e_y, |h(t)| \leq e_w, \forall \ t \in I.
\]
\[D^c_{-\infty} p(t) = \Phi(t, p(t), q(t), r(t)) + f(t), t \in I,\]
\[D^c_{-\infty} q(t) = \Psi(t, p(t), q(t), r(t)) + g(t), t \in I,\]
\[D^w_{-\infty} r(t) = \Theta(t, p(t), q(t), r(t)) + h(t), t \in I,\]

**Lemma 4.5.** Let $(p, q, r)$ be the solution of the inequality (4.26), then the following inequalities will be satisfied:
\[p(t) - p_0 - \frac{1}{\Gamma(k)} \int_0^t (t - \zeta)^{k-1} \Phi(\zeta, p(\zeta), q(\zeta), r(\zeta)) d\zeta \leq \frac{T_x}{\Gamma(k + 1)} e_p, \tag{4.27}\]
\[q(t) - q_0 - \frac{1}{\Gamma(\nu)} \int_0^t (t - \zeta)^{\nu-1} \Psi(\zeta, p(\zeta), q(\zeta), r(\zeta)) d\zeta \leq \frac{T_y}{\Gamma(\nu + 1)} e_q, \tag{4.28}\]
\[r(t) - r_0 - \frac{1}{\Gamma(\omega)} \int_0^t (t - \zeta)^{\omega-1} \Theta(\zeta, p(\zeta), q(\zeta), r(\zeta)) d\zeta \leq \frac{T_w}{\Gamma(\omega + 1)} e_w. \tag{4.29}\]

**Proof.** From the Remark 4.4, we have:
\[D^c_{-\infty} p(t) = \Phi(t, p(t), q(t), r(t)) + f(t), t \in I,\]
\[D^c_{-\infty} q(t) = \Psi(t, p(t), q(t), r(t)) + g(t), t \in I,\]
\[D^w_{-\infty} r(t) = \Theta(t, p(t), q(t), r(t)) + h(t), t \in I.\]

From Theorem 3.1, the solution of the above coupled system is given by
\[p(t) = p_0 + \frac{1}{\Gamma(k)} \int_0^t (t - \zeta)^{k-1} \Phi(\zeta, p(\zeta), q(\zeta), r(\zeta)) d\zeta + f(\zeta), \tag{4.30}\]
\[q(t) = q_0 + \frac{1}{\Gamma(\nu)} \int_0^t (t - \zeta)^{\nu-1} \Psi(\zeta, p(\zeta), q(\zeta), r(\zeta)) d\zeta + g(\zeta), \tag{4.31}\]
\[r(t) = r_0 + \frac{1}{\Gamma(\omega)} \int_0^t (t - \zeta)^{\omega-1} \Theta(\zeta, p(\zeta), q(\zeta), r(\zeta)) d\zeta + h(\zeta). \tag{4.32}\]
Therefore, one can easily prove the inequalities (4.27), (4.28) and (4.29).

Our next theorem gives sufficient condition under which the system (1.1) is HU stable.

\textbf{Theorem 4.6.} If the assumption (A) and the inequality (3.7) are fulfilled, then the system (1.1) is HU stable.

\textit{Proof.} Suppose that \((p, q, r)\) be a unique solution of the system (1.1) and \((p, q, r)\) be the solution of inequality (4.26). Therefore, from Theorem 3.1, we have

\[
p(t) = p_0 + \frac{1}{\Gamma(k)} \int_0^t (t-\zeta)^{k-1} \Phi(\zeta, p(\zeta), q(\zeta), r(\zeta)) d\zeta,
\]

\[
q(t) = q_0 + \frac{1}{\Gamma(v)} \int_0^t (t-\zeta)^{v-1} \Psi(\zeta, p(\zeta), q(\zeta), r(\zeta)) d\zeta,
\]

\[
r(t) = r_0 + \frac{1}{\Gamma(\omega)} \int_0^t (t-\zeta)^{\omega-1} \Theta(\zeta, p(\zeta), q(\zeta), r(\zeta)) d\zeta.
\]

For \(t \in I\), we find

\[
|p(t) - p(t)| \leq \left| p(t) - p_0 \right| + \frac{1}{\Gamma(k)} \int_0^t (t-\zeta)^{k-1} |\Phi(\zeta, p(\zeta), q(\zeta), r(\zeta))| d\zeta
\]

\[
\leq \frac{T^k}{\Gamma(k+1)} e_k + \frac{1}{\Gamma(k)} \int_0^t (t-\zeta)^{k-1} |\Phi(\zeta, p(\zeta), q(\zeta), r(\zeta))| d\zeta.
\]

Subsequently, we get

\[
\|p - p\|_c \leq \frac{T^k}{\Gamma(k+1)} e_k + \frac{T^v}{\Gamma(v+1)} e_v + \frac{T^\omega}{\Gamma(\omega+1)} e_\omega
\]

\[
\leq \frac{T^k}{\Gamma(k+1)} e_k + Q_1 (\|p - p\|_c + \|q - q\|_c + \|r - r\|_c).
\]

Similarly, we can get

\[
\|q - q\|_c \leq \frac{T^v}{\Gamma(v+1)} e_v + Q_2 (\|p - p\|_c + \|q - q\|_c + \|r - r\|_c).
\]

\[
\|r - r\|_c \leq \frac{T^\omega}{\Gamma(\omega+1)} e_\omega + Q_3 (\|p - p\|_c + \|q - q\|_c + \|r - r\|_c).
\]

From the inequalities (4.33), (4.34) and (4.35), we get

\[
\|p - p\|_c + \|q - q\|_c + \|r - r\|_c \leq \frac{T^k}{\Gamma(k+1)} e_k + \frac{T^v}{\Gamma(v+1)} e_v + \frac{T^\omega}{\Gamma(\omega+1)} e_\omega
\]

\[
+ (Q_1 + Q_2 + Q_3) (\|p - p\|_c + \|q - q\|_c + \|r - r\|_c).
\]

Hence,

\[
\| (p, q, r) - (p, q, r) \|_c \leq \left( \frac{T^k}{\Gamma(k+1)} + \frac{T^v}{\Gamma(v+1)} + \frac{T^\omega}{\Gamma(\omega+1)} \right) \frac{\epsilon}{1 - (Q_1 + Q_2 + Q_3)}
\]

\[
\leq \mathcal{H}_{(L_{p}, L_{q}, L_{r}, L_{x}, K, V, \omega)}(\epsilon),
\]

where \(\mathcal{H}_{(L_{p}, L_{q}, L_{r}, L_{x}, K, V, \omega)} = \left( \frac{T^k}{\Gamma(k+1)} + \frac{T^v}{\Gamma(v+1)} + \frac{T^\omega}{\Gamma(\omega+1)} \right) \frac{\epsilon}{1 - (Q_1 + Q_2 + Q_3)}\). Hence, the fractional system (1.1) is HU stable. Also, if we set \(\mathcal{H}_{(L_{p}, L_{q}, L_{r}, L_{x}, K, V, \omega)}(\epsilon) = \mathcal{H}_{(L_{p}, L_{q}, L_{r}, L_{x}, K, V, \omega)}(\epsilon), \mathcal{H}_{(L_{p}, L_{q}, L_{r}, L_{x}, K, V, \omega)}(0) = 0\), then system (1.1) is generalized HU stable. \(\square\)
5 Controllability results

Definition 5.1. The functions \( p, q, r \in C(I, \mathbb{R}) \) are said to be the solution of the system (1.2), if \( p(t), q(t), r(t) \) satisfies

\[
p(0) = p_0, \quad q(0) = q_0, \quad r(0) = r_0 \quad \text{and} \quad p, q, r \quad \text{is the solution of the following integral equations}
\]

\[
p(t) = p_0 + \frac{1}{\Gamma(k)} \int_0^t (t - \zeta)^{k-1} \Phi(\zeta, p(\zeta), q(\zeta), r(\zeta)) d\zeta, \quad \forall \ t \in I,
\]

\[
q(t) = q_0 + \frac{1}{\Gamma(v)} \int_0^t (t - \zeta)^{v-1} \Psi(\zeta, p(\zeta), q(\zeta), r(\zeta)) d\zeta, \quad \forall \ t \in I,
\]

\[
r(t) = r_0 + \frac{1}{\Gamma(\omega)} \int_0^t (t - \zeta)^{\omega-1} \Theta(\zeta, p(\zeta), q(\zeta), r(\zeta)) d\zeta, \quad \forall \ t \in I.
\]

Definition 5.2. System (1.2) is called controllable on \( I \), if for every \( p_0, q_0, r_0, p_T, q_T, r_T \in \mathbb{R} \), there exists rd-continuous functions \( u, v, w \in L^2(I, \mathbb{R}) \) such that the corresponding solution of system (1.2) satisfies \( p(0) = p_0, q(0) = q_0, r(0) = r_0, p(T) = p_T, q(T) = q_T \) and \( r(T) = r_T \).

We set:

\[
Q_4 = \frac{T^w(L_{q_0} + L_{q_0} + L_{q_0})}{\Gamma(k + 1)} \left( 1 + M_{q_0}M_{q_0}T^w \right),
\]

\[
Q_5 = \frac{T^w(L_{q_0} + L_{q_0} + L_{q_0})}{\Gamma(v + 1)} \left( 1 + M_{q_0}M_{q_0}T^w \right),
\]

\[
N_6 = \left( 1 + M_{q_0}M_{q_0}T^w \right) \left[ |p_0| + \frac{T^wM_{q_0}}{\Gamma(k + 1)} \right] + \frac{M_{w}M_{w}T^w}{\Gamma(k + 1)},
\]

\[
N_7 = \left( 1 + M_{q_0}M_{q_0}T^w \right) \left[ |q_0| + \frac{T^wM_{q_0}}{\Gamma(v + 1)} \right] + \frac{M_{w}M_{w}T^w}{\Gamma(v + 1)},
\]

\[
N_8 = \left( 1 + M_{q_0}M_{q_0}T^w \right) \left[ |r_0| + \frac{T^wM_{q_0}}{\Gamma(\omega + 1)} \right] + \frac{M_{w}M_{w}T^w}{\Gamma(\omega + 1)}.
\]

Lemma 5.3. Let the assumptions (A) and (W) be satisfied and \( p_T, q_T, r_T \in \mathbb{R} \) are arbitrary points. Then, the solution \( p(t), q(t), r(t) \) of the system (1.2) on \( I \) defined by (5.36), (5.37) and (5.38) respectively with the control functions

\[
u(t) = \left( \nu \right)^{\gamma(t)} p_T - p_0 - \frac{1}{\Gamma(k)} \int_0^t (t - \zeta)^{k-1} \Phi(\zeta, p(\zeta), q(\zeta), r(\zeta)) d\zeta \right) \left( t \right), \quad \forall \ t \in I,
\]

\[
u(t) = \left( \nu \right)^{\gamma(t)} q_T - q_0 - \frac{1}{\Gamma(v)} \int_0^t (t - \zeta)^{v-1} \Psi(\zeta, p(\zeta), q(\zeta), r(\zeta)) d\zeta \right) \left( t \right), \quad \forall \ t \in I,
\]

\[
u(t) = \left( \nu \right)^{\gamma(t)} r_T - r_0 - \frac{1}{\Gamma(\omega)} \int_0^t (t - \zeta)^{\omega-1} \Theta(\zeta, p(\zeta), q(\zeta), r(\zeta)) d\zeta \right) \left( t \right), \quad \forall \ t \in I.
\]

satisfies \( p(T) = p_T, q(T) = q_T, r(T) = r_T \). Also, the control functions \( u(t), v(t), w(t) \) have the estimate \( \|u(t)\| \leq M_u, \|v(t)\| \leq M_v, \|w(t)\| \leq M_w \) respectively, where
\[ M_\ell = M_\ell^\kappa \left[ |p_\ell| + |p_0| + \frac{T^k}{\Gamma(k+1)} \left( M_\phi + L_{\phi,\ell,\sup_{\ell}}|p(\ell)| + L_{\phi,\ell,\sup_{\ell}}q(\ell)| + L_{\phi,\ell,\sup_{\ell}}r(\ell)| \right) \right], \]

\[ M_\mu = M_\mu^\kappa \left[ |q_\mu| + |q_0| + \frac{T^\omega}{\Gamma(\omega+1)} \left( M_{\psi} + L_{\psi,\mu,\sup_{\mu}}|p(\mu)| + L_{\psi,\mu,\sup_{\mu}}q(\mu)| + L_{\psi,\mu,\sup_{\mu}}r(\mu)| \right) \right], \]

\[ M_\nu = M_\nu^w \left[ |r_\nu| + |r_0| + \frac{T^\omega}{\Gamma(\omega+1)} \left( M_\mu + L_{\phi,\nu,\sup_{\nu}}|p(\nu)| + L_{\phi,\nu,\sup_{\nu}}q(\nu)| + L_{\phi,\nu,\sup_{\nu}}r(\nu)| \right) \right]. \]

**Proof.** Consider the solution \( p(t) \) of the system (1.2) on \( I \) defined by (5.36). For \( t = T \) we get:

\[
p(T) = p_0 + \frac{1}{\Gamma(k)} \int_0^T (T - \zeta)^{k-1} \left[ \Phi(\zeta, p(\zeta), q(\zeta), r(\zeta)) + E u(\zeta) \right] d\zeta 
= p_0 + \frac{1}{\Gamma(k)} \int_0^T (T - \zeta)^{k-1} \Phi(\zeta, p(\zeta), q(\zeta), r(\zeta)) d\zeta 
+ \frac{1}{\Gamma(k)} \int_0^T (T - \zeta)^{k-1} \left[ p_T - p_0 - \frac{1}{\Gamma(k)} \int_0^T (T - \zeta)^{k-1} \Phi(\zeta, p(\zeta), q(\zeta), r(\zeta)) d\zeta \right] d\zeta 
= p_T.
\]

Similarly, by putting \( t = T \) in the solution (5.37) and (5.38) of the system (1.2), we get

\[ q(T) = q_T \text{ and } r(T) = r_T. \]

Also,

\[ |u(t)| \leq M_\ell^\kappa \left[ |p_\ell| + |p_0| + \frac{1}{\Gamma(k)} \int_0^T (T - r)^{k-1} |\Phi(\zeta, p(\zeta), q(\zeta), r(\zeta))| d\zeta \right] \]

\[ \leq M_\ell^\kappa \left[ |p_\ell| + |p_0| + \frac{1}{\Gamma(k)} \int_0^T (T - \zeta)^{k-1} \left[ M_\phi + L_{\phi,\ell,\sup_{\ell}}|p(\ell)| + L_{\phi,\ell,\sup_{\ell}}q(\ell)| + L_{\phi,\ell,\sup_{\ell}}r(\ell)| \right] d\zeta \right]. \]

Therefore,

\[ |u(t)| \leq M_\ell^\kappa \left[ |p_\ell| + |p_0| + \frac{T^k}{\Gamma(k+1)} \left( M_\phi + L_{\phi,\ell,\sup_{\ell}}|p(\ell)| + L_{\phi,\ell,\sup_{\ell}}q(\ell)| + L_{\phi,\ell,\sup_{\ell}}r(\ell)| \right) \right] 
= M_\ell.
\]

Similarly, we can show that \( |v(t)| \leq M_\mu \) and \( |w(t)| \leq M_\nu. \)

**Theorem 5.4.** If the assumptions (A) and (W) are satisfied, then the system (1.2) is controllable on \( I \) provided

\[ \max_{t \in [0, T]} |Q_t| < \frac{1}{3} \quad (5.42) \]

**Proof.** Let us consider a subset \( \Omega^\ast \subseteq PS \) such that

\[ \Omega^\ast = \{ (p, q, r) \in PS : \| (p, q, r) \|_c \leq \delta \}, \]

where
where
\[ \Xi''(p, q, r)(t) = \frac{1}{\Gamma(k)} \int_0^t (t - \zeta)^{k-1} \Theta(\zeta, p(\zeta), q(\zeta), r(\zeta)) + D\omega(\zeta) \Delta \zeta, \]

\[ \Xi''(p, q, r)(t) = q_0 + \frac{1}{\Gamma(V)} \int_0^t (t - \zeta)^{V-1} \Psi(\zeta, p(\zeta), q(\zeta), r(\zeta)) + CV(\zeta) \Delta \zeta, \]

\[ \Xi''(p, q, r)(t) = r_0 + \frac{1}{\Gamma(W)} \int_0^t (t - \zeta)^{W-1} \Theta(\zeta, p(\zeta), q(\zeta), r(\zeta)) + D\nu(\zeta) \Delta \zeta, \]

\[ u(t), v(t), w(t) \text{ are given by the (equations 5.39), (5.40) and (5.41) in the interval } I. \]

Now, we need to prove that \( \Xi'' \) maps \( Q\hat{\Omega}'' \) into itself, we have
\[
|\Xi''(p, q, r)(t)| \leq |p_0| + \frac{1}{\Gamma(k)} \int_0^t (t - \zeta)^{k-1} \Theta(\zeta, p(\zeta), q(\zeta), r(\zeta)) \Delta \zeta + \frac{1}{\Gamma(V)} \int_0^t (t - \zeta)^{V-1} B\omega(\zeta) \Delta \zeta
\leq |p_0| + \frac{1}{\Gamma(k)} \int_0^t (t - \zeta)^{k-1} \left( L_0|p(\zeta)| + L_0|q(\zeta)| + L_0|r(\zeta)| + M_0 \Delta \zeta + \frac{M_0M_1^T}{\Gamma(k + 1)} \right)
\leq |p_0| + \frac{L_0\delta'' + L_0\delta' + L_0\delta + M_0}{\Gamma(k)} \int_0^t (t - \zeta)^{k-1} d\zeta + \frac{M_0M_1^T}{\Gamma(k + 1)} |p_t| + |p_0|
+ \frac{T^k}{\Gamma(k + 1)} \left( L_0\delta'' + L_0\delta' + L_0\delta + M_0 \right)
\leq N_6 + \delta'' Q_4.\]

Hence,
\[ ||\Xi''(p, q, r)||_C \leq N_6 + \delta'' Q_4. \quad (5.44)\]

Similarly, one can find
\[ ||\Xi'(p, q, r)||_C \leq N_7 + \delta'' Q_3. \quad (5.45)\]
\[ ||\Xi'(p, q, r)||_C \leq N_8 + \delta'' Q_4. \quad (5.46)\]

Summarize the above inequalities (5.44), (5.45) and (5.46), we get:
\[ ||\Xi''(p, q, r)||_C \leq \delta''. \quad (5.47)\]

Therefore, \( \Xi'' : Q\hat{\Omega}'' \to Q\hat{\Omega}'' \). Also, for \( t \in I, \)
Thus, \( \| \dot{\Xi}_x (p, q, r) - \dot{\Xi}_x (\bar{p}, \bar{q}, \bar{r}) \|_c \leq \dot{\Theta}_4 \), (5.48)

Similarly, we can easily obtain
\[
\| \dot{\Xi}_x (p, q, r) - \dot{\Xi}_x (\bar{p}, \bar{q}, \bar{r}) \|_c \leq \dot{\Theta}_5 \|p - \bar{p}\|_c + \|q - \bar{q}\|_c + \|r - \bar{r}\|_c. \tag{5.49}
\]

Similarly, we can easily obtain
\[
\| \dot{\Xi}_x (p, q, r) - \dot{\Xi}_x (\bar{p}, \bar{q}, \bar{r}) \|_c \leq Q_4 \|p - \bar{p}\|_c + \|q - \bar{q}\|_c + \|r - \bar{r}\|_c. \tag{5.50}
\]

After summarizing the inequalities (5.48), (5.49) and (5.50), we get
\[
\| \dot{\Xi}_x (p, q, r) - \dot{\Xi}_x (\bar{p}, \bar{q}, \bar{r}) \|_c \leq (Q_4 + Q_5 + Q_6) \cdot \|p, q, r\| (\bar{p}, \bar{q}, \bar{r})_c.
\]

Therefore, \( \dot{\Xi}_x \) is a strict contraction operator. Thus, by means of Banach fixed point technique, we have a unique fixed point of \( \dot{\Xi}_x \) which is the solution of the system (1.2). Also, from the Lemma 5.3, we can see that \( p(t), q(t), r(t) \) satisfies \( p(T) = p_T, q(T) = q_T \) and \( r(T) = r_T \). Thus, we conclude that the system (1.2) is controllable on \( I \).

\[ \square \]

6 An illustrative example

Example 6.1. Let us consider the fractional system
\[
{^cD^T}p(t) = \frac{|p(t)|}{(t + 9)^2 (1 + |p(t)|)} + 65 (1 + q^2 (t)) + \frac{\sin (r(t))}{e^{t/4}} + \frac{1}{20 e^r}, \quad t \in I = [0, 10]^T, \\
{^cD^T}q(t) = \frac{3 + |p(t)| + |q(t)| + |r(t)|}{2e^{t/4} (1 + |p(t)| + |q(t)| + |r(t)|)} + \frac{1}{2(t^2 + 4)}, \quad t \in I, \\
{^cD^T}r(t) = \frac{\sin (p(t))}{e^{t/4}} + \frac{|q(t)|}{(t + 9)^2 (1 + |q(t)|)} + \frac{1}{65 (1 + r^2 (t))} + \frac{1}{20 e^r}, \quad t \in I,
\]

\[ p(0) = 0, \quad q(0) = 0, \quad r(0) = 0. \]

For \( t \in I \) and \( p, q, r \in \mathbb{R} \), we set
\[
\Phi(t, p(t), q(t), r(t)) = \frac{|p(t)|}{(t + 9)^2 (1 + |p(t)|)} + 65 (1 + q^2 (t)) + \frac{\sin (r(t))}{e^{t/4}} + \frac{1}{20 e^r}, \\
\Psi(t, p(t), q(t), r(t)) = \frac{3 + |p(t)| + |q(t)| + |r(t)|}{2e^{t/4} (1 + |p(t)| + |q(t)| + |r(t)|)} + \frac{1}{2(t^2 + 4)}, \\
\Theta(t, p(t), q(t), r(t)) = \frac{\sin (p(t))}{e^{t/4}} + \frac{|q(t)|}{(t + 9)^2 (1 + |q(t)|)} + \frac{1}{65 (1 + r^2 (t))} + \frac{1}{20 e^r}.
\]

Then, \( \forall t \in I, p, q, r, \bar{p}, \bar{q}, \bar{r} \in \mathbb{R} \), we have:
\[
|\Phi(t, p, q, r) - \Phi(t, \bar{p}, \bar{q}, \bar{r})| \leq \frac{1}{61} |p - \bar{p}| + \frac{1}{65} |q - \bar{q}| + \frac{1}{69} |r - \bar{r}|,
\]
\[
|\Psi(t, p, q, r) - \Psi(t, \bar{p}, \bar{q}, \bar{r})| \leq \frac{1}{61} |p - \bar{p}| + \frac{1}{65} |q - \bar{q}| + |r - \bar{r}|,
\]
\[
|\Theta(t, p, q, r) - \Theta(t, \bar{p}, \bar{q}, \bar{r})| \leq \frac{1}{61} |p - \bar{p}| + \frac{1}{65} |q - \bar{q}| + \frac{1}{69} |r - \bar{r}|.
\]
Thus, the assumption (A) is holds. Also, for \(\kappa = 1/3, \nu = 1/3, \omega = 1/4\) and \(T = 10\),
\[
Q_1 = \frac{T^\kappa (L\Phi_1 + L\Phi_2 + L\Phi_3)}{\Gamma(\kappa + 1)} = 0.1111,
\]
\[
Q_2 = \frac{T^\nu (L\Psi_1 + L\Psi_2 + L\Psi_3)}{\Gamma(\nu + 1)} = 0.0663,
\]
\[
Q_3 = \frac{T^\omega (L\Theta_1 + L\Theta_2 + L\Theta_3)}{\Gamma(\omega + 1)} = 0.0903.
\]
Therefore, all the conditions of the Theorem 3.1 are satisfied. Hence, fractional system (6.51) has a unique solution.

Now, we consider the following cases for different time scales:

**Case A.** When \(\mathbb{T} = \mathbb{R}\). We choose the final points as \(p(T) = 2, q(T) = 1, r(T) = 2\). Clearly, we can see that the trajectory of the system (6.51) shown in Figure 1 does not passes throw the desired points \(p(T) = 2, q(T) = 1\) and \(r(T) = 2\). But if we add the control functions \(u(t), v(t)\) and \(w(t)\) given by
\[
u(t) = \left(\nu T^\nu_0\right)^{-1} \left[2 - \frac{1}{\Gamma(\nu)} \int_0^T (T - \zeta)^{\nu - 1} \Psi(\zeta, p(\zeta), q(\zeta), r(\zeta)) \Delta \zeta\right] (t), t \in I,
\]
\[
u(t) = \left(\omega T^\omega_0\right)^{-1} \left[2 - \frac{1}{\Gamma(\omega)} \int_0^T (T - \zeta)^{\omega - 1} \Theta(\zeta, p(\zeta), q(\zeta), r(\zeta)) \Delta \zeta\right] (t), t \in I,
\]
where

![Figure 1: Trajectory of the system (6.51), when \(\mathbb{T} = \mathbb{R}\).](image-url)
in the system (6.51), it becomes

\[ c_T^T D^k p(t) = \frac{|p(t)|}{(t + 9)^2 (1 + |p(t)|)} + \frac{1}{65 (1 + q^2(t))} + \frac{1}{20e} \sin (r(t)) + \frac{1}{20e} + u(t), \quad t \in [0, 10]. \]

\[ c_T^T D^k q(t) = \frac{3}{2e} \sin (p(t)) + \frac{|q(t)|}{(t + 9)^2 (1 + |q(t)|)} + \frac{1}{2(t^2 + 4)} + v(t), \quad t \in I, \]

\[ c_T^T D^k r(t) = \frac{\sin (p(t))}{e^{2t}} + \frac{|p(t)|}{(t + 9)^2 (1 + |p(t)|)} + \frac{1}{65 (1 + r^2(t))} + \frac{1}{20e} + w(t), \quad t \in I. \]

Now, we can find that

\[ x^T W_0^T = 2.413, \quad y^T W_0^T = 2.413, \quad \omega^T W_0^T = 1.9619, \]

\[ Q_0 = \frac{T^n (L_0 + L_0 + L_{\nu})}{\Gamma (k + 1)} \left( 1 + \frac{M_0 M_{\nu}^n T^n}{\Gamma (k + 1)} \right) = 0.22128, \]

\[ Q_5 = \frac{T^n (L_0 + L_0 + L_{\nu})}{\Gamma (\nu + 1)} \left( 1 + \frac{M_0 M_{\nu}^n T^n}{\Gamma (\nu + 1)} \right) = 0.13257, \]

\[ Q_6 = \frac{T^n (L_0 + L_0 + L_{\nu})}{\Gamma (\omega + 1)} \left( 1 + \frac{M_0 M_{\nu}^n T^n}{\Gamma (\omega + 1)} \right) = 0.18068. \]

Therefore, all the assumptions of Theorem 5.4 are hold. Hence, system (6.52) is controllable and the controlled trajectory is shown in Figure 2.

**Case 2.** When \( T = \mathbb{Z} \). We choose the final points as \( p(T) = 1, q(T) = 2, r(T) = 1.4 \). Clearly, we can see that the trajectory of the system (6.51) shown in Figure 3 does not pass through the desired points \( p(T) = 1, q(T) = 2 \) and \( r(T) = 1.4 \). But if we add the control functions \( u(t), v(t) \) and \( w(t) \) given by

\[ u(t) = (x^T W_0^T)^{-1} \left[ 1 - \frac{1}{\Gamma (k)} \int_0^T (T - \zeta)^{k+1} \Phi (\zeta, p(\zeta), q(\zeta), r(\zeta)) d\zeta \right], \quad t \in I, \]

\[ \begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2.png}
\caption{Trajectory of the controlled system (6.52), when \( T = \mathbb{R}, p(T) = 2, q(T) = 1, r(T) = 2 \).}
\end{figure} \]
Therefore, all the assumptions of Theorem 5.4 are hold. Hence, system (6.53) is controllable and the controlled trajectory is shown in Figure 4.

\[
\begin{align*}
\nu(t) &= \left(\sqrt[\nu]{\mathcal{W}_0^\nu}\right)^{-1} \left[2 - \frac{1}{\Gamma(\nu)} \int_0^T (T - \zeta)^{\nu - 1} \Psi(\zeta, p(\zeta), q(\zeta), r(\zeta)) \Delta\zeta\right](t), t \in I, \\
w(t) &= \left(\sqrt[w]{\mathcal{W}_0^w}\right)^{-1} \left[1.4 - \frac{1}{\Gamma(\omega)} \int_0^T (T - \zeta)^{\omega - 1} \Theta(\zeta, p(\zeta), q(\zeta), r(\zeta)) \Delta\zeta\right](t), t \in I,
\end{align*}
\]

where

\[
\begin{align*}
k^\nu \mathcal{W}^\nu_0 &= \frac{1}{\Gamma(k)} \int_0^T (T - \zeta)^{k - 1} \Delta\zeta, \quad k^\nu \mathcal{W}^\nu_0 = \frac{1}{\Gamma(\nu)} \int_0^T (T - \zeta)^{-\nu - 1} \Delta\zeta, \quad \text{and} \quad k^w \mathcal{W}^w_0 = \frac{1}{\Gamma(\omega)} \int_0^T (T - \zeta)^{-w - 1} \Delta\zeta,
\end{align*}
\]

in the system (6.51), it becomes

\[
\begin{align*}
c^T \mathcal{D}^p(t) &= \frac{|p(t)|}{(t + 9)^2 (1 + |p(t)|)} + \frac{1}{65 (1 + q^2(t))} + \frac{1}{20 e^t} + u(t), \quad t \in I = [0, 10], \\
c^T \mathcal{D}^q(t) &= \frac{3 + |p(t)| + |q(t)| + |r(t)|}{2 e^{t+1}} \left(1 + |p(t)| + |q(t)| + |r(t)|\right) + \frac{1}{2(t + 4)} + v(t), \quad t \in I, \\
c^T \mathcal{D}^r(t) &= \frac{\sin(p(t))}{e^{t+1}} + \frac{|q(t)|}{(t + 9)^2 (1 + |q(t)|)} + \frac{1}{65 (1 + r^2(t))} + \frac{1}{20 e^t} + w(t), \quad t \in I,
\end{align*}
\]

\[
p(0) = 0, \quad q(0) = 0, \quad r(0) = 0.
\]

Now, we can find that

\[
\begin{align*}
k^\nu \mathcal{W}^\nu_0 &= 1.847, \quad k^\nu \mathcal{W}^\nu_0 = 1.847, \quad k^w \mathcal{W}^w_0 = 1.365, \\
Q_4 &= \frac{T^\nu (L_{0_1} + L_{0_2} + L_{0_3})}{\Gamma(\nu + 1)} \frac{1 + M_c M_w T^\nu}{\Gamma(k + 1)} = 0.2562, \\
Q_5 &= \frac{T^\nu (L_{0_1} + L_{0_2} + L_{0_3})}{\Gamma(\nu + 1)} \frac{1 + M_c M_w T^\nu}{\Gamma(\nu + 1)} = 0.1529, \\
Q_6 &= \frac{T^w (L_{0_1} + L_{0_2} + L_{0_3})}{\Gamma(\omega + 1)} \frac{1 + M_c M_w T^w}{\Gamma(\omega + 1)} = 0.2202.
\end{align*}
\]

Therefore, all the assumptions of Theorem 5.4 are hold. Hence, system (6.53) is controllable and the controlled trajectory is shown in Figure 4.

**Figure 3**: Trajectory of the system (6.51), when \( \mathbb{T} = \mathbb{Z} \).
Case 3. When $T = \frac{1}{4} \mathbb{Z}$. We choose the final points as $p(T) = 3$, $q(T) = 1.5$, $r(T) = 2$. Clearly, we can see that the trajectory of the system (6.51) shown in Figure 5 does not pass through the desired points $p(T) = 3$, $q(T) = 1.5$ and $r(T) = 2$.

But if we add the control functions $u(t)$, $v(t)$ and $w(t)$ given by

$$u(t) = \left( x \mathcal{W}_0^1 / \Gamma(k) \right) \int \left( T - \zeta \right)^{k-1} \Phi(\zeta, p(\zeta), q(\zeta), r(\zeta)) \Delta \zeta \right) \left(t, t \in I, \right.$$  

$$v(t) = \left( y \mathcal{W}_0^1 / \Gamma(v) \right) \int \left( T - \zeta \right)^{v-1} \Psi(\zeta, p(\zeta), q(\zeta), r(\zeta)) \Delta \zeta \right) \left(t, t \in I, \right.$$  

$$w(t) = \left( w \mathcal{W}_0^1 / \Gamma(w) \right) \int \left( T - \zeta \right)^{w-1} \Theta(\zeta, p(\zeta), q(\zeta), r(\zeta)) \Delta \zeta \right) \left(t, t \in I, \right.$$  

where

$$x \mathcal{W}_0^1 = \frac{1}{\Gamma(k)} \int \left( T - \zeta \right)^{k-1} \Delta \zeta, y \mathcal{W}_0^1 = \frac{1}{\Gamma(v)} \int \left( T - \zeta \right)^{v-1} \Delta \zeta \text{ and } w \mathcal{W}_0^1 = \frac{1}{\Gamma(w)} \int \left( T - \zeta \right)^{w-1} \Delta \zeta,$$

in the system (6.51), it becomes

$$c_T D^3 p(t) = \frac{|p(t)|}{(t + 9)^2 (1 + |p(t)|)} + \frac{1}{65 (1 + |q(t)|)} + \frac{\sin(r(t))}{e^{r/4}} + \frac{1}{20e^2} + u(t), \quad t \in [0, 10] \mathbb{Z},$$

$$c_T D^3 q(t) = \frac{3 + |p(t)| + |q(t)| + |r(t)|}{2e^{r/4} (1 + |p(t)| + |q(t)| + |r(t)|)} + \frac{1}{2(t^2 + 4)} + v(t), \quad t \in I,$$

$$c_T D^3 r(t) = \frac{\sin(p(t))}{e^{p/4}} + \frac{|q(t)|}{(t + 9)^2 (1 + |q(t)|)} + \frac{1}{65 (1 + r^2(t))} + \frac{1}{20e^2} + w(t), \quad t \in I,$$

$$p(0) = 0, \quad q(0) = 0, \quad r(0) = 0.$$

Now, we can find that

\[\text{Figure 4: Trajectory of the controlled system (6.53), when } T = \mathbb{Z}, p(T) = 1, q(T) = 2, r(T) = 1.4.\]
\[ \begin{align*}
\mathcal{W}_0^{\kappa} &= 1.539, \\
\mathcal{W}_0^{\nu} &= 1.539, \\
\mathcal{W}_0^{\omega} &= 1.037,
\end{align*} \]

\[ \begin{align*}
Q_4 &= \frac{T^{\kappa}(L_{\Phi_1} + L_{\Phi_2} + L_{\Phi_3})}{\Gamma(\kappa + 1)} + \frac{M_{\kappa}M_{\nu}^\omega T_{\nu}^\omega}{\Gamma(\kappa + 1)} = 0.2853, \\
Q_5 &= \frac{T^{\nu}(L_{\psi_1} + L_{\psi_2} + L_{\psi_3})}{\Gamma(\nu + 1)} + \frac{M_{\nu}M_{\psi}^\nu T_{\psi}^\nu}{\Gamma(\nu + 1)} = 0.1702, \\
Q_6 &= \frac{T^{\omega}(L_{\Theta_1} + L_{\Theta_2} + L_{\Theta_3})}{\Gamma(\omega + 1)} + \frac{M_{\psi}M_{\omega}^\psi T_{\omega}^\psi}{\Gamma(\omega + 1)} = 0.2613.
\end{align*} \]

Therefore, all the assumptions of Theorem 5.4 are hold. Hence, system (6.54) is controllable and the controlled trajectory is shown in Figure 6.

Case 4. When \([0, 6]_\mathbb{Z} \cup [7, 10]_\mathbb{R}\) (say \(D\)). We choose the final points as \(p(T) = 3, q(T) = 1, r(T) = 2\). Clearly, we can see that the trajectory of the system (6.51) shown in Figure 7 does not pass through the desired points \(p(T) = 3, q(T) = 1.5\) and \(r(T) = 2\).

But if we add the control functions \(u(t), v(t)\) and \(w(t)\) given by

\[ \begin{align*}
Q_4 &= \frac{T^{\kappa}(L_{\Phi_1} + L_{\Phi_2} + L_{\Phi_3})}{\Gamma(\kappa + 1)} + \frac{M_{\kappa}M_{\nu}^\omega T_{\nu}^\omega}{\Gamma(\kappa + 1)} = 0.2853, \\
Q_5 &= \frac{T^{\nu}(L_{\psi_1} + L_{\psi_2} + L_{\psi_3})}{\Gamma(\nu + 1)} + \frac{M_{\nu}M_{\psi}^\nu T_{\psi}^\nu}{\Gamma(\nu + 1)} = 0.1702, \\
Q_6 &= \frac{T^{\omega}(L_{\Theta_1} + L_{\Theta_2} + L_{\Theta_3})}{\Gamma(\omega + 1)} + \frac{M_{\psi}M_{\omega}^\psi T_{\omega}^\psi}{\Gamma(\omega + 1)} = 0.2613.
\end{align*} \]

Figure 5: Trajectory of the system (6.51), when \(T = \frac{1}{4}\mathbb{Z}\).

Figure 6: Trajectory of the controlled system (6.54), when \(T = \frac{1}{4}\mathbb{Z}, p(T) = 3, q(T) = 1.5, r(T) = 2\).
Now, we can find that

\[ u(t) = \left( \psi_0^\mathcal{T} \right)^{-1}\left[ 3 - \frac{1}{\Gamma(\kappa)} \int_0^T (T - \zeta)^{\kappa-1} \Phi(\zeta, p(\zeta), q(\zeta), r(\zeta)) \, d\zeta \right] (t), \quad t \in I, \]

\[ v(t) = \left( \nu_0^\mathcal{T} \right)^{-1}\left[ 1 - \frac{1}{\Gamma(\nu)} \int_0^T (T - \zeta)^{\nu-1} \Psi(\zeta, p(\zeta), q(\zeta), r(\zeta)) \, d\zeta \right] (t), \quad t \in I, \]

\[ w(t) = \left( \omega_0^\mathcal{T} \right)^{-1}\left[ 2 - \frac{1}{\Gamma(\omega)} \int_0^T (T - \zeta)^{\omega-1} \Theta(\zeta, p(\zeta), q(\zeta), r(\zeta)) \, d\zeta \right] (t), \quad t \in I, \]

where

\[ \psi_0^\mathcal{T} = \frac{1}{\Gamma(\kappa)} \int_0^T (T - \zeta)^{\kappa-1} \, d\zeta, \quad \nu_0^\mathcal{T} = \frac{1}{\Gamma(\nu)} \int_0^T (T - \zeta)^{\nu-1} \, d\zeta \]

and

\[ \omega_0^\mathcal{T} = \frac{1}{\Gamma(\omega)} \int_0^T (T - \zeta)^{\omega-1} \, d\zeta, \]

in the system (6.51), it becomes

\[ c^\mathcal{T}p(t) = \frac{|p(t)|}{(t + 9)^2 (1 + |p(t)|)} + \frac{1}{65 (1 + q^2(t))} \sin(r(t)) \frac{1}{e^{t/4}} + \frac{1}{20e^t} + u(t), \quad t \in I = [0, 10]_D, \]

\[ c^\mathcal{T}q(t) = \frac{3 + |p(t)| + |q(t)| + |r(t)|}{2e^{t/4} (1 + |p(t)| + |q(t)| + |r(t)|)} + \frac{1}{2(t^2 + 4)} + v(t), \quad t \in I, \tag{6.55} \]

\[ c^\mathcal{T}r(t) = -\frac{|q(t)|}{e^{t/4}} + \frac{1}{(t + 9)^2 (1 + |q(t)|)} + \frac{1}{65(1 + r^2(t))} + \frac{1}{20e^t} + w(t), \quad t \in I, \]

\[ p(0) = 0, \quad q(0) = 0, \quad r(0) = 0. \]

Now, we can find that

\[ \psi_0^\mathcal{T} = 0.5414, \quad \nu_0^\mathcal{T} = 0.5414, \quad \omega_0^\mathcal{T} = 0.7711, \]

\[ Q_3 = \frac{T^\mathcal{R}(L_{01} + L_{12} + L_{20})}{\Gamma(\kappa + 1)} \left( 1 + \frac{M_0 M_{01}^{\mathcal{R}}}{\Gamma(\kappa + 1)} \right) = 0.2562, \]

\[ Q_5 = \frac{T^\mathcal{R}(L_{01} + L_{02} + L_{10})}{\Gamma(\nu + 1)} \left( 1 + \frac{M_0 M_{01}^{\mathcal{R}}}{\Gamma(\nu + 1)} \right) = 0.15292, \]

\[ Q_6 = \frac{T^\mathcal{R}(L_{01} + L_{12} + L_{20})}{\Gamma(\omega + 1)} \left( 1 + \frac{M_0 M_{01}^{\mathcal{R}}}{\Gamma(\omega + 1)} \right) = 0.2270. \]
Therefore, all the assumptions of Theorem 5.4 are hold. Hence, system (6.55) is controllable and the controlled trajectory is shown in Figure 8.

7 Applications

7.1 Let us consider the three species prey-predator model of fractional order on time scales

\[ c^\alpha T D^\alpha p(t) = R p \left( 1 - \frac{p}{k} \right) - \frac{b p s}{a + p} - y p I + c_1 u(t), \alpha \in (0,1), t \in I = [0,T], \]

\[ c^\alpha T D^\alpha s(t) = \frac{c b p s}{a + p} - d s I - m s + c_2 v(t), t \in I, \]

\[ c^\alpha T D^\alpha I(t) = -n I + d s I + c y p I + c_3 w(t), t \in I, \]

with the initial conditions \( p(0) = p_0, s(0) = s_0, I(0) = I_0 \), where \( I \) is infected predator, \( s \) is susceptible predator, \( p \) is the population density of prey, \( d \) the disease transmission coefficient, \( k \) the carrying capacities of prey population, \( n \) is the death rate of infected predator, \( R \) is the growth rate of prey population, \( m \) is the death rate of susceptible predator, \( b \) is the search rate of the prey toward susceptible predator, \( a \) is saturation constant while susceptible predators attack the prey, \( c \) is the conversion rate of susceptible predator due to prey. \( u(t), v(t) \) and \( w(t) \) are some control variables with the control coefficients \( c_1, c_2, c_3 \), respectively. This model can be formulated into the fractional system (1.1) with

\[ \Phi(t,p,q,r) = R p \left( 1 - \frac{p}{k} \right) - \frac{b p q}{a + p} - y p r, \]

\[ \Psi(t,p,q,r) = \frac{c b p q}{a + p} - d q r - m q, \]

\[ \Theta(t,p,q,r) = -n r + d q r + c y p r. \]

Also, we can prove that all the conditions of the Theorems 1.1, 4.6 and 5.4 are satisfied. Hence, the system (7.56) has unique HU stable solution which is controllable.

Figure 8: Trajectory of the controlled system (6.55), when \( \overline{\mathcal{T}} = D, p(T) = 3, q(T) = 1, r(T) = 2 \).
7.2 Consider the following fractional order GLV model on time scales

\[
\begin{align*}
\frac{cd}{t}^\alpha p(t) &= p - pq + cp^2 - ap^2r + c_1u(t), \quad a \in (0, 1), \; t \in I = [0, T], \\
\frac{cd}{t}^\alpha q(t) &= -q + pq + c_2v(t), \quad t \in I, \\
\frac{cd}{t}^\alpha r(t) &= -br + ap^2r + c_3w(t), \quad t \in I,
\end{align*}
\]

with \( p(0) = p_0 \geq 0, \; q(0) = q_0 \geq 0, \; r(0) = r_0 \geq 0 \). This model can be formulated into the fractional system (1.1) with

\[
\begin{align*}
\Phi(t, p, q, r) &= p - pq + cp^2 - ap^2r, \\
\Psi(t, p, q, r) &= -q + pq, \\
\Theta(t, p, q, r) &= -br + ap^2r.
\end{align*}
\]

Also, we can prove that all the conditions of the Theorems 1.1, 4.6 and 5.4 are satisfied. Hence, the system (7.56) has unique HU stable solution which is controllable.

8 Conclusion

We have studied the existence of solutions and HU stability analysis of a fractional dynamic system (1.1) on time scales. For the existence of a unique solution, we used the Banach contraction technique and for the existence of at least one solution, we used the Leray-Schauder alternative-type fixed point theorem under the weaker condition (B). In addition, we have studied the controllability results for the fractional system (1.2). With the help of MATLAB, we have given an example with simulation for the different time scales including \( T = \mathbb{R} \) (case 1 of example 6.1), \( T = \mathbb{Z} \) (case 2 of example 6.1), \( T = \frac{1}{\pi} \mathbb{Z} \) (case 3 of example 6.1) and \( T = [0, 6] \cap \mathbb{Z} \cup [7, 10] \) \( \mathbb{R} \) (case 4 of example 6.1). Also, we have given two applications to illustrate the obtained analytical results.

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