On Regular Locally Scalar Representations of Graph $\tilde{D}_4$ in Hilbert Spaces

S. A. Kruglyak †, L. A. Nazarova, A. V. Roiter.

† Institute of Mathematics of National Academy of Sciences of Ukraine, Tereshchenkovska str., 3, Kiev, Ukraine, ind. 01601
E-mail: red@imath.kiev.ua

Representations of quivers corresponding to extended Dynkin graphs are described up to equivalence in [1]. Locally scalar representations of graphs in the category of Hilbert spaces were introduced in [2], and such representations are naturally classified up to unitary equivalence.

Representations of $*$-algebras generated by linearly related orthogonal projections are studied in [3–10] and others. The connection between locally scalar representations of several graphs (trees, which include also Dynkin graphs) and representations of such $*$-algebras is stated in [11], and we further use this connection.

The present paper is dedicated to the classification of indecomposable regular (see [12]) locally scalar representations of the graph $\tilde{D}_4$ (for $\tilde{D}_4$ those are indecomposable locally scalar representations in the dimension $(2; 1, 1, 1, 1)$). The answer obtained for the corresponding $*$-algebra in [5, 6], in our opinion, cannot be satisfactory and definitive. We will obtain explicit formulas, expressing matrix elements of a representation by a character (see [2]) of a locally scalar representation and two “free” real parameters.

1. Let $\mathcal{H}$ be the category of finite-dimensional Hilbert spaces, and $\tilde{D}_4$ be an extended Dynkin graph

![Graph $\tilde{D}_4$](image)

Remind (see [2]) that a representation $\Pi$ of the graph $\tilde{D}_4$ associates a space $H_i \in \mathcal{H}$ to each vertex $i$ ($i = \overline{0, 4}$), and a pair of interadjoint linear operators $\Pi(\gamma_i) = \{\Gamma_{0i}; \Gamma_{0i}^*\}$ to each edge $\gamma_i$, where $\Gamma_{0i} : H_i \to H_0$, $\Gamma_{0i}^* = \Gamma_{0i}^\ast$.

A morphism $C : \Pi \to \tilde{\Pi}$ is a family $C = \{C_i\}_{i=0,4}$ of operators $C_i : \Pi(i) \to \tilde{\Pi}(i)$ such that
the diagrams

\[
\begin{array}{c}
H_i \xrightarrow{\Gamma_{ji}} H_j \\
\downarrow C_i \quad \downarrow C_j \\
\tilde{H}_i \xrightarrow{\tilde{\Gamma}_{ji}} \tilde{H}_j
\end{array}
\]

are commutative, i.e. \( C_j \Gamma_{ji} = \tilde{\Gamma}_{ji} C_i \).

Let \( M_i \) be the set of vertices connected with a vertex \( i \) by an edge, \( A_i = \sum_{j \in M_i} \Gamma_{ij} \Gamma_{ji} \).

Representation \( \Pi \) is called locally scalar [2] if all operators \( A_i \) are scalar; \( A_i = \alpha_i I_{H_i} \), where \( I_{H_i} \) is identity operator in a space \( H_i \). Since \( A_i \) is a positive operator, \( \alpha_i \geq 0 \). A vector \( \{ \dim \Pi(i) \} \) is a dimension of a finite-dimensional representation \( \Pi_i \); if \( A_i = f(i) I_{H_i} \) then \( \{ f(i) \} \) is called a character of locally scalar representation \( \Pi \).

Further we will denote as \( \text{Rep}(\tilde{D}_4, f) \) the category of finite-dimensional locally scalar representations of the graph \( \tilde{D}_4 \) in \( H \) with given character \( f \).

We will assume that \( \alpha_i = f(i) > 0, \ i = 0,4 \) and the character is normalized: \( f(0) = \alpha_0 = 1 \).

On the other hand, consider the following *-algebra over the field \( \mathbb{C} \):

\[
\mathcal{P}_{4,f} = \mathbb{C}\langle p_1, p_2, p_3, p_4 \mid p_i = p_i^*, p_i^2 = \sum_{i=1}^{4} \alpha_i p_i = e \rangle,
\]

where \( \alpha_i = f(i) \), \( e \) is the identity of the algebra, and the category \( \text{Rep} \mathcal{P}_{4,f} \) of finite-dimensional *-representation of the algebra \( \mathcal{P}_{4,f} \).

Let \( \Pi \in \text{Rep}(\tilde{D}_4, f) \), \( \Pi(i) = H_i \), \( \pi(\gamma_i) = \{ \Gamma_{i,0}; \Gamma_{0,i} \} \). Let us construct a representation \( \pi \) of the algebra \( \mathcal{P}_{4,f} \) by the following way: \( \pi(p_i) = \frac{1}{\alpha_i} \Gamma_{0,i} \cdot \Gamma_{i,0} = P_i \). If \( C : \Pi \to \tilde{\Pi} \) is a morphism in \( \text{Rep}(\tilde{D}_4, f) \) then \( C_0 : \pi \to \tilde{\pi} \) is a morphism in the category \( \text{Rep} \mathcal{P}_{4,f} \) (\( C_0 \) is the operator interlacing representations \( \pi \) and \( \tilde{\pi} \)). Define a functor

\[
\Phi : \text{Rep}(\tilde{D}_4, f) \to \text{Rep} \mathcal{P}_{4,f}
\]

putting \( \Phi(\Pi) = \pi \), \( \Phi(C) = C_0 \). Clearly, the functor \( \Phi \) is the equivalence of categories.

Let \( \pi \in \text{Rep} \mathcal{P}_{4,f} \) be a representation in the space \( H_0 \). Set \( H_i = \text{Im} P_i \), \( i = 1,4 \); \( \Gamma_{0,i} : H_i \to H_0 \) is the natural injection of the space \( H_i \) into \( H_0 \), then, putting \( \Pi(\gamma_i) = \{ \Gamma_{0,i}; \Gamma_{i,0}^* \} \), we obtain a representation from \( \text{Rep}(\tilde{D}_4, f) \). If \( C_0 : \pi \to \tilde{\pi} \) set \( C_i = C_0 \mid_{H_i} \). If \( \Phi^{-1}(\pi) = \Pi \), \( \Phi^{-1}(C_0) = \{ C_i \}_{i=0,4} \) then \( \Phi \Phi^{-1} \sim I_{\text{Rep} \mathcal{P}_{4,f}} \), \( \Phi^{-1} \Phi \sim I_{\text{Rep}(\tilde{D}_4, f)} \).

2. Consider representations of the *-algebra \( \mathcal{P}_{4,f} \) for

\[
0 < \alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \alpha_4 < 1, \quad \sum_{i=1}^{4} \alpha_i = 2.
\]

(in the other cases representations of the *-algebras \( \mathcal{P}_{4,f} \) are reduced to the simplest representations by the Coxeter functors [13]).
Let us make the substitution of generators in the algebra $\mathcal{P}_{4,f}$ [5]:

\[
\begin{align*}
x &= \alpha_2 p_2 + \alpha_3 p_3 - \frac{1}{2} \beta_1 e, & \beta_1 &= (2 - \alpha_1 + \alpha_2 + \alpha_3 - \alpha_4)/2, \\
y &= \alpha_1 p_1 + \alpha_3 p_3 - \frac{1}{2} \beta_2 e, & \beta_2 &= (2 + \alpha_1 - \alpha_2 + \alpha_3 - \alpha_4)/2, \\
z &= \alpha_1 p_1 + \alpha_2 p_2 - \frac{1}{2} \beta_3 e, & \beta_3 &= (2 + \alpha_1 + \alpha_2 - \alpha_3 - \alpha_4)/2.
\end{align*}
\]

Denote also

\[
\begin{align*}
\gamma_1 &= (\alpha_1^2 - \alpha_2^2 - \alpha_3^2 + \alpha_4^2)/4, \\
\gamma_2 &= (-\alpha_1^2 + \alpha_2^2 - \alpha_3^2 + \alpha_4^2)/4, \\
\gamma_3 &= (-\alpha_1^2 - \alpha_2^2 + \alpha_3^2 + \alpha_4^2)/4.
\end{align*}
\]

It is easy to check that $\gamma_1 \leq \gamma_2 \leq \gamma_3$ and $0 \leq \gamma_2$.

The new generators $x, y, z$ satisfy the system of relations

\[
\begin{align*}
\{y, z\} &= \gamma_1 e, \\
\{z, x\} &= \gamma_2 e, \\
\{x, y\} &= \gamma_3 e, \\
(x + y + z)^2 &= \alpha_4^2 e.
\end{align*}
\]

The equalities (1) imply

\[
\begin{align*}
p_1 &= \frac{-x + y + z}{2\alpha_1} + \frac{1}{2} e, \\
p_2 &= \frac{x - y + z}{2\alpha_2} + \frac{1}{2} e, \\
p_3 &= \frac{x + y - z}{2\alpha_3} + \frac{1}{2} e, \\
p_4 &= \frac{-x - y - z}{2\alpha_4} + \frac{1}{2} e.
\end{align*}
\]

3. Let $\gamma_3 = 0$, then $0 \leq \gamma_2 \leq \gamma_3$ implies $\gamma_2 = 0$, hence $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \frac{1}{2}$ (this case is considered in [4]), and so $\gamma_1 = 0$.

In this case the system of relation (2) has a form

\[
\begin{align*}
\{y, z\} &= 0, \\
\{z, x\} &= 0, \\
\{x, y\} &= 0, \\
x^2 + y^2 + z^2 &= \frac{1}{4} e.
\end{align*}
\]

Let $\pi$ be an indecomposable two-dimensional representation of the algebra $\mathcal{P}_{4,f}$ and $\pi(x) = X$, $\pi(y) = Y$, $\pi(z) = Z$.

a) Let $Z = 0$. The matrix $X$ can be diagonalized as a matrix of self-adjoint operator. The relations of anticommutation imply that the triple $X, Y, Z = 0$ is indecomposable only in the case when the diagonalized matrix $X$ equals

\[
\begin{bmatrix}
-\lambda & 0 \\ 0 & \lambda
\end{bmatrix}
\]. Then $Y = \begin{bmatrix} 0 & y_{12} \\ y_{12} & 0 \end{bmatrix}$, $y_{12} \neq 0,$
and the element $y_{12}$ can be made positive by the admissible transformations. Therefore, we obtain the case

$$X = \lambda \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad Y = \mu \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}; \quad \lambda > 0, \mu > 0.$$

b) Let $Z \neq 0$ and $X = 0$. Then the matrix $Y$ can be diagonalized: $Y = \begin{bmatrix} -\mu & 0 \\ 0 & \mu \end{bmatrix}$, $\mu > 0$ (in the other cases the triple of matrices turn out to be decomposable). Then from $\{y, z\} = 0$ we obtain $Z = \begin{bmatrix} 0 & z_{12} \\ z_{12} & 0 \end{bmatrix}$ and one can reduce the matrix $Z$ (does not changing the $Y$) by the admissible transformations to the form $Z = \begin{bmatrix} 0 & i\nu \\ i\nu & 0 \end{bmatrix}$, $\nu > 0$. The triple of matrices $X, Y, Z$ is reduced by means of a unitary matrix $U = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$ by the transformation $UXU^*$, $UYU^*$, $UZU^*$ to the form

$$X = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad Y = \mu \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Z = \nu \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}; \quad \mu > 0, \nu > 0.$$

c) Let $X \neq 0$, $Z \neq 0$, $Y = 0$. In this case the matrices $X, Y, Z$ can be reduced to the form

$$X = \lambda \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad Z = \nu \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}; \quad \lambda > 0, \nu > 0.$$

d) $X \neq 0$, $Y \neq 0$, $Z \neq 0$. In this case the matrices $X, Y, Z$ can be reduced to the form

$$X = \lambda \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad Y = \mu \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Z = \nu \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}; \quad \lambda > 0, \mu > 0, \nu \in \mathbb{R}, \nu \neq 0.$$

Thus, unifying these cases, we may consider that

$$X = \lambda \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad Y = \mu \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Z = \nu \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix},$$

where $\lambda^2 + \mu^2 + \nu^2 = \frac{1}{4}$ (follows from (4)), and either $\lambda > 0$, $\mu > 0$, $\nu \in \mathbb{R}$; either $\lambda = 0$, $\mu > 0$, $\nu > 0$; or $\lambda > 0$, $\mu = 0$, $\nu > 0$.

Formulas (3) imply

$$P_1 = \begin{bmatrix} \frac{1}{2} - \lambda & \mu + \nu i \\ \mu - \nu i & \frac{1}{2} + \lambda \end{bmatrix}, \quad P_2 = \begin{bmatrix} \frac{1}{2} + \lambda & -\mu + \nu i \\ -\mu - \nu i & \frac{1}{2} - \lambda \end{bmatrix},$$

$$P_3 = \begin{bmatrix} \frac{1}{2} + \lambda & \mu - \nu i \\ \mu + \nu i & \frac{1}{2} - \lambda \end{bmatrix}, \quad P_4 = \begin{bmatrix} \frac{1}{2} - \lambda & -\mu - \nu i \\ -\mu + \nu i & \frac{1}{2} + \lambda \end{bmatrix}.$$
If \( \mu + \nu i = \sqrt{\mu^2 + \nu^2} e^{i\varphi} = \sqrt{\frac{1}{4} - \lambda^2} e^{i\varphi} \) then, passing on to the unitary equivalent representation by means of the matrix \( U = \begin{bmatrix} e^{-i\varphi} & 0 \\ 0 & 1 \end{bmatrix} \), we may consider that

\[
P_1 = \begin{bmatrix} \frac{1}{2} - \lambda & \sqrt{\frac{1}{4} - \lambda^2} \\ \sqrt{\frac{1}{4} - \lambda^2} & \frac{1}{2} + \lambda \end{bmatrix}, \quad P_2 = \begin{bmatrix} \frac{1}{2} + \lambda & e^{ix} \sqrt{\frac{1}{4} - \lambda^2} \\ -e^{-ix} \sqrt{\frac{1}{4} - \lambda^2} & \frac{1}{2} - \lambda \end{bmatrix},
\]

\[
P_3 = \begin{bmatrix} \frac{1}{2} + \lambda & -e^{-ix} \sqrt{\frac{1}{4} - \lambda^2} \\ -e^{ix} \sqrt{\frac{1}{4} - \lambda^2} & \frac{1}{2} - \lambda \end{bmatrix}, \quad P_4 = \begin{bmatrix} \frac{1}{2} - \lambda & -\sqrt{\frac{1}{4} - \lambda^2} \\ \sqrt{\frac{1}{4} - \lambda^2} & \frac{1}{2} + \lambda \end{bmatrix} \tag{5}
\]

\[
0 \leq \lambda < 1/2, \quad 0 < \chi < \pi/2 \text{ when } \lambda = 0,
\]

\[-\pi/2 < \chi \leq \pi/2 \text{ when } 0 < \lambda < 1/2.\]

4. Let \( \gamma_3 \neq 0 \), then \( \{x, y\} = \gamma_3 e \) implies \( X \neq 0 \) and \( Y \neq 0 \). Moreover, \( X \) and \( Y \) has no zero eigenvalues. Indeed, let matrix \( X \) has the form after digonalization: \( X = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \). In the matrices \( Y \) and \( Z \) either \( y_{12} \neq 0 \) or \( z_{12} \neq 0 \) (or else the triple of matrices is decomposable). \( \{x, y\} = \gamma_3 e \), \( \{x, z\} = \gamma_2 e \) imply \( (\lambda_1 + \lambda_2)z_{12} = 0 \) and \( (\lambda_1 + \lambda_2)y_{12} = 0 \). Therefore we can conclude that \( -\lambda_1 = \lambda_2 = \lambda \neq 0, \lambda > 0 \). The same reasoning is useful also for \( Y \).

Let \( X = \begin{bmatrix} -\lambda & 0 \\ 0 & \lambda \end{bmatrix} \). \( \{x, y\} = \gamma_3 e \) implies \( y_{11} = -\frac{\gamma_3}{2\lambda}, \ y_{22} = \frac{\gamma_3}{2\lambda}; \ \{x, z\} = \gamma_2 e \) implies \( z_{11} = -\frac{\gamma_2}{2\lambda}, \ z_{22} = \frac{\gamma_2}{2\lambda} \).

\[
X = \lambda \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad Y = \frac{1}{2\lambda} \begin{bmatrix} -\gamma_3 & y_{12} \\ y_{12} & \gamma_3 \end{bmatrix}, \quad Z = \frac{1}{2\lambda} \begin{bmatrix} -\gamma_2 & z_{12} \\ z_{12} & \gamma_2 \end{bmatrix}.
\]

\( \{y, z\} = \gamma_1 e \) implies

\[
\frac{1}{4\lambda^2} \begin{bmatrix} 2\gamma_2\gamma_3 + y_{12}z_{12} + \overline{y}_{12}z_{12} & 0 \\ 0 & 2\gamma_2\gamma_3 + y_{12}\overline{z}_{12} + \overline{y}_{12}z_{12} \end{bmatrix} = \gamma_1 I,
\]

hence

\[
y_{12}\overline{z}_{12} + \overline{y}_{12}z_{12} = 4\gamma_1 \lambda^2 - 2\gamma_2\gamma_3. \tag{6}
\]

Let us turn to the unitary equivalent representation by means of the unitary matrix \( U = \begin{bmatrix} e^{i\varphi} & 0 \\ 0 & 1 \end{bmatrix} \) so that \( y_{12} + z_{12} = r_1 \) would be real positive; at that \(-y_{12} + z_{12}\) remains to be complex in general, \(-y_{12} + z_{12} = r_2 e^{ix}\).

Then \( (2) \) implies

\[
(X + Y + Z)^2 = \frac{1}{4\lambda^2} \begin{bmatrix} (2\lambda^2 + \gamma_2 + \gamma_3)^2 + (y_{12} + z_{12})^2 & 0 \\ 0 & (2\lambda^2 + \gamma_2 + \gamma_3)^2 + (y_{12} + z_{12})^2 \end{bmatrix} = \alpha_4^2 I
\]

\[
\lambda = \begin{bmatrix} \alpha_4 \\ \alpha_4 \end{bmatrix}.
\]
and
\[ r_1^2 = (y_{12} + z_{12})^2 = 4\alpha_4^2\lambda^2 - (2\lambda^2 + \gamma_2 + \gamma_3)^2, \]

from which it easy to obtain
\[
\begin{align*}
    r_1 &= \sqrt{-4\lambda^4 + 2(\alpha_2^2 + \alpha_4^2)\lambda^2 - \frac{1}{4}(\alpha_3^2 - \alpha_2^2)^2} \\
    &= \sqrt{-4 \left( \lambda^2 - \frac{(\alpha_3^2 - \alpha_2^2)}{4} \right) \left( \lambda^2 - \frac{(\alpha_4^2 + \alpha_2^2)}{4} \right)},
\end{align*}
\]

\[ \frac{\alpha_4 - \alpha_1}{2} \leq \lambda \leq \frac{\alpha_4 + \alpha_1}{2}. \]

\[
\begin{cases}
    y_{12} + z_{12} = r_1, \\
    -y_{12} + z_{12} = r_2 e^{i\chi},
\end{cases}
\]

\[
y_{12} z_{12} = \frac{(r_1^2 - r_2^2)}{2} = 4\gamma_1 \lambda^2 - 2\gamma_2 \gamma_3 \] (the last equality follows from (8)).

Thus,
\[
r_2^2 = r_1^2 + 4\gamma_2 \gamma_3 - 8\gamma_1 \lambda^2 = -4\lambda^4 + 2(\alpha_2^2 + \alpha_3^2)\lambda^2 - \frac{1}{4}(\alpha_3^2 - \alpha_2^2)^2 \geq 0
\]

and
\[
    -y_{12} + z_{12} = r_2 e^{i\chi} = e^{i\chi} \sqrt{-4\lambda^4 + 2(\alpha_2^2 + \alpha_3^2)\lambda^2 - \frac{1}{4}(\alpha_3^2 - \alpha_2^2)^2} =
\]

\[
    = e^{i\chi} \sqrt{-4 \left( \lambda^2 - \frac{(\alpha_3^2 - \alpha_2^2)}{4} \right) \left( \lambda^2 - \frac{(\alpha_4^2 + \alpha_2^2)}{4} \right),}
\]

\[ \frac{\alpha_3 - \alpha_2}{2} \leq \lambda \leq \frac{\alpha_3 + \alpha_2}{2}. \]

Now we can directly pass on to the determining of the operators of the representation
\[ P_1, P_2, P_3, P_4: \]

\[
P_1 = \frac{-X + Y + Z}{2\alpha_1} + \frac{1}{2} I =
\]

\[
= \frac{1}{4\alpha_1 \lambda} \left[ \begin{array}{c}
2\lambda^2 + 2\alpha_1 \lambda - \frac{1}{2}(\alpha_3^2 - \alpha_2^2) \\
\sqrt{-4\lambda^4 + 2(\alpha_2^2 + \alpha_4^2)\lambda^2 - \frac{1}{4}(\alpha_3^2 - \alpha_2^2)^2}
\end{array} \right]
\]

\[
\times \left[ \begin{array}{c}
-4\lambda^4 + 2(\alpha_2^2 + \alpha_4^2)\lambda^2 - \frac{1}{4}(\alpha_3^2 - \alpha_2^2)^2 \\
-2\lambda^2 + 2\alpha_1 \lambda + \frac{1}{4}(\alpha_3^2 - \alpha_2^2)
\end{array} \right],
\]

\[
= \frac{1}{4\alpha_1 \lambda} \left[ \begin{array}{c}
2 \left( \lambda - \frac{1}{2}(\alpha_4 - \alpha_1) \right) \left( \lambda + \frac{1}{2}(\alpha_4 + \alpha_1) \right) \\
\sqrt{-4 \left( \lambda^2 - \frac{1}{4}(\alpha_4 - \alpha_1)^2 \right) \left( \lambda^2 - \frac{1}{4}(\alpha_4 + \alpha_1)^2 \right)}
\end{array} \right]
\]

\[
\times \left[ \begin{array}{c}
-4 \left( \lambda^2 - \frac{1}{4}(\alpha_4 - \alpha_1)^2 \right) \left( \lambda^2 - \frac{1}{4}(\alpha_4 + \alpha_1)^2 \right) \\
-2 \left( \lambda + \frac{1}{2}(\alpha_4 - \alpha_1) \right) \left( \lambda - \frac{1}{2}(\alpha_4 + \alpha_1) \right)
\end{array} \right],
\]

\[
P_2 = \frac{X - Y + Z}{2\alpha_2} + \frac{1}{2} I =
\]

\[
= \frac{1}{4\alpha_2 \lambda} \left[ \begin{array}{c}
-2\lambda^2 + 2\alpha_2 \lambda + \frac{1}{2}(\alpha_3^2 - \alpha_2^2) \\
e^{i\chi} \sqrt{-4\lambda^4 + 2(\alpha_2^2 + \alpha_3^2)\lambda^2 - \frac{1}{4}(\alpha_3^2 - \alpha_2^2)^2}
\end{array} \right]
\]

\[
\times \left[ \begin{array}{c}
2\lambda^2 + 2\alpha_2 \lambda - \frac{1}{2}(\alpha_3^2 - \alpha_2^2) \\
e^{i\chi} \sqrt{-4 \left( \lambda^2 - \frac{1}{4}(\alpha_3 + \alpha_2)^2 \right) \left( \lambda^2 - \frac{1}{4}(\alpha_3 - \alpha_2)^2 \right)}
\end{array} \right],
\]

\[
= \frac{1}{4\alpha_2 \lambda} \left[ \begin{array}{c}
-2 \left( \lambda - \frac{1}{2}(\alpha_3 + \alpha_2) \right) \left( \lambda + \frac{1}{2}(\alpha_3 - \alpha_2) \right) \\
e^{i\chi} \sqrt{-4 \left( \lambda^2 - \frac{1}{4}(\alpha_3 + \alpha_2)^2 \right) \left( \lambda^2 - \frac{1}{4}(\alpha_3 - \alpha_2)^2 \right)}
\end{array} \right]
\]

\[
\times \left[ \begin{array}{c}
2 \left( \lambda + \frac{1}{2}(\alpha_3 + \alpha_2) \right) \left( \lambda - \frac{1}{2}(\alpha_3 - \alpha_2) \right)
\end{array} \right].
\]
Thus, indecomposable regular (not necessarily normalized) locally scalar representations of the graph $\widetilde{D}_4$ depend on 6 real parameters (on the 4 of 5 parameters $\alpha'_1$, $\alpha'_2$, $\alpha'_3$, $\alpha'_4$, $\alpha'_0$, connected by the relation $\alpha'_1 + \alpha'_2 + \alpha'_3 + \alpha'_4 = 2\alpha'_0$, $\alpha_i = \frac{\alpha'_i}{\alpha'_0}$, and parameters $\lambda$ and $\chi$).

References

1. L. A. Nazarova Representations of quivers of infinite type. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 37 (1973), 752–791.
2. S. A. Kruglyak, A. V. Roiter Locally scalar representations of graphs in the category of Hilbert spaces. Funct. Anal. Appl. 39 (2005), no. 2, 91–105

3. Yu. N. Bespalov. Sets of orthoprojectors connected by relations. Ukrainian Math. J. 44 (1992), no. 3, 269–277.

4. V. Ostrovskyi and Yu. Samoilenko Introduction to the theory of representations of finitely presented *-algebras. I. Representations by bounded operators. Reviews in Mathematics and Mathematical Physics, 11, pt.1. Harwood Academic Publishers, Amsterdam, 1999. iv+261 pp. ISBN: 90-5823-042-2

5. D. V. Galinsky, S. A. Kruglyak Representations of *-algebras generated by linearly related orthogonal projections. (Ukrainian.) Visnyk Kyiv. derzh. Univ. No. 2, 24–31, 1999.

6. D. V. Galinsky. Representations of *-algebras generated by orthogonal projections satisfying a linear relation. Methods Funct. Anal. Topology 4 (1998), no. 3, 27–32.

7. D. V. Galinsky, M. A. Muratov On representations of algebras generated by sets of three and four orthoprojections. Spectral and evolutionary problems, Vol. 8 (Sevastopol, 1997), 15–22, Tavria Publ., Simferopol, 1998.

8. S. A. Kruglyak, V. I. Rabanovich, Yu. S. Samoilenko On sums of projections. Funct. Anal. Appl. 36 (2002), no. 3, 182–195

9. S. Kruglyak, A. Kyrychenko. On four orthogonal projections that satisfy the linear relation $\alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 + \alpha_4 P_4 = I$, $\alpha_i > 0$. Symmetry in nonlinear mathematical physics, Part 1, 2 (Kyiv, 2001), 461–465, Pr. Inst. Mat. Nats. Akad. Nauk Ukr. Mat. Zastos., 43, Part 1, 2, Natsional. Akad. Nauk Ukrainy, Inst. Mat., Kiev, 2002.

10. A. Strelets. Description of the representations of the algebras generated by four linearly related idempotents. J. Algebra Appl. 4 (2005), no. 6, 671–681.

11. S. Kruglyak, S. Popovich, Yu. Samoilenko. The spectral problem and *-representations of algebras associated with Dynkin graphs. J. Algebra Appl. 4 (2005), no. 6, 761–776.

12. I. K. Redchuk, A. V. Roiter. Singular locally scalar representations of quivers in Hilbert spaces, and separating functions. Ukrainian Math. J. 56 (2004), no. 6, 947–963.

13. S. A. Kruglyak. Coxeter functors for a certain class of *-quivers and *-algebras. Methods Funct. Anal. Topology 8 (2002), no. 4, 49–57.