A note on the use of the word 'likelihood' in statistics and meteorology

Stephen Jewson, Anders Brix and Christine Ziehmann *
Risk Management Solutions, London, United Kingdom

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Abstract

We highlight the different uses of the word likelihood that have arisen in statistics and meteorology, and make the recommendation that one of these uses should be dropped to prevent confusion and misunderstanding.

1 Introduction

We discuss the different meanings of the word likelihood as used in classical statistics and meteorology. In section 2 we describe how the word is used in classical statistics, and in section 3 we describe how it has been used in meteorology. In section 4 we discuss the differences and express the opinion that one of the two uses should be dropped. In section 5 we summarize.

2 Classical statistics definition of likelihood

Imagine that we have two datasets \( x_i \) and \( y_i \), for a range of values of \( i \). We might attempt to build a statistical model for \( x_i \) in terms of \( y_i \), or \( y_i \) in terms of \( x_i \).

As an example we will consider linear regression, and will consider building a model for \( y_i \) in terms of \( x_i \). The model we will consider can be written in the following equivalent ways:

\[
y_i = \alpha + \beta x_i + \sigma e_i
\]

where

\[
e_i \sim N(0,1)
\]

or

\[
y_i \sim N(\alpha + \beta x_i, \sigma^2)
\]

or

\[
p(y_i|x_i) = N(\alpha + \beta x_i, \sigma^2)
\]

To fit the parameters \((\alpha, \beta, \sigma)\) of this model given the data \( x_i \) and \( y_i \) one would typically consider finding those parameters that maximise the likelihood \( L(\alpha, \beta, \sigma) \), where \( L \) is defined as:

\[
L(\alpha, \beta, \sigma) = p(y|x, \alpha, \beta, \sigma)
\]

In the case of linear regression, this happens to be equivalent to finding parameters that minimise the sum of squared errors.

We note that the definition of \( L \) as \( L(\alpha, \beta, \sigma) = p(y|x, \alpha, \beta, \sigma) \) arises because we are using \( x_i \) to model \( y_i \), i.e. \( x \) is the 'input', 'independent variable', 'predictor', 'regressor', 'explanatory variable' or 'covariate', while \( y \) is the 'output', 'dependent variable', 'regressand', 'response variable' or 'predictand'.

If we had set out to use \( y_i \) to model \( x_i \) (using linear regression, or any other model) then the likelihood would have been defined as \( L(\theta) = p(x|y, \theta) \) where \( \theta \) represents the parameters of this new model. We

*Correspondence address: RMS, 10 Eastcheap, London, EC3M 1AJ, UK. Email: x@stephenjewson.com
see that the definition of likelihood is thus entirely dependent on what is being used to model what, and what model is being used.

In general, once we have decided on a model then the likelihood is the probability density (or, for discrete rather than continuous models, just the probability) of the predictand given the predictors, as a function of the parameters of the model.

Use of the word likelihood in this context comes from the original works of Fisher in the 1920s, such as Fisher (1922). A recent overview of the use of the likelihood in statistical inference is given by Casella and Berger (2002), and likelihood-based fitting of parameters is also discussed in Press et al. (1992).

2.1 Application to making probabilistic forecasts

We now consider a meteorological application: making probabilistic forecasts. Probabilistic forecasts are made by taking the inherently non-probabilistic output from numerical models (consisting of, for example, single integrations, ensemble members, or ensemble means and spreads) and fitting a statistical model to them to generate probabilities. In other words, given a non-probabilistic model forecast \( f \) we make a probabilistic forecast of the observations \( o \). Our forecast can be written as \( p(o|f) \) i.e. a probability distribution of different possible observations, given the forecast we have available. Note that the notation \( p(o|f) \) does not yet specify what model is used to convert \( f \) to \( o \).

For temperature, a reasonable way to make a probabilistic forecast is to use linear regression, with the input \( f_i \) being either a single forecast or an ensemble mean. This then means we can write:

\[
p(o_i|f_i) = N(\alpha + \beta f_i, \sigma^2)
\]

If we wish to use information from the mean \( m_i \) and spread \( s_i \) of an ensemble forecast, then we can use the spread regression model of Jewson et al. (2003):

\[
p(o_i|f_i) = N(\alpha + \beta m_i, (\gamma + \delta s_i)^2)
\]

The usual way to fit the parameters of either of these models would be to find those parameters that maximise the likelihood, defined as \( L = L(\alpha, \beta, \sigma) = p(o|f, \alpha, \beta, \sigma) \) (for the regression model) or \( L = L(\alpha, \beta, \gamma, \delta) = p(o|f, \alpha, \beta, \gamma, \delta) \) (for the spread regression model). The likelihood is defined as \( p(o|f) \) simply because we are trying to predict the observations \( o \) from the forecast \( f \). If, for some reason, we wanted to predict the forecast from the observations (it is not immediately obvious why one would want to do this, but there may be reasons), then we would define the likelihood as \( p(f|o) \).

3 Murphy and Winkler definition of likelihood

Murphy and Winkler (1987) (henceforth MW) discuss ways in which one can validate probabilistic forecasts, and in particular introduce the following definitions:

- \( p(f|o) \) is the likelihood
- \( p(o) \) is the base rate
- \( p(o|f) \) is the calibration
- \( p(f) \) is the refinement

Following this paper, a number of other meteorologists (such as Jolliffe and Stephenson (2003) and Wilks (2001)) have used the word likelihood to refer to \( p(f|o) \) and the word calibration to refer to \( p(o|f) \).

4 Discussion

We see that the MW definition of the word likelihood is subtly different from the original definition as used in classical statistics. In particular, MW define likelihood once and for all as \( p(f|o) \) irrespective of whether \( o \) is being modelled in terms of \( f \), or \( f \) is being modelled in terms of \( o \). The classical statistics definition of likelihood, on the other hand, depends on what is being used to model what.

This creates some confusion, especially when one tries to apply classical statistical methods to forecast calibration as described in section 2.1. Because of this, we advocate that the MW definition should not be used. Our reasons for taking this position are:
• The classical statistical definition of the word likelihood is the original definition.
• It is used, and understood, by many thousands of applied mathematicians.
• It has been in use for over 80 years.
• The MW definition is a restriction of the original definition.
• It is only used, and understood, by a very small number of meteorologists involved in the field of probabilistic forecast verification.
• It is very recent.

5 Summary

Statisticians have used the phrase likelihood for over 80 years, with a particular meaning, following Fisher (1922). A relatively recent paper by Murphy and Winkler (1987) attempts to redefine this word when applied to meteorological forecasts and observations. This undermines the original meaning, creates confusion, and is not very helpful in building connections between statistics and meteorology. We therefore strongly advise that meteorologists working in forecast verification should not use the definition of Murphy and Winkler (1987), and should stick to the original definition of Fisher (1922).

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