GENERALIZED DISCRETE $q$-HERMITE II POLYNOMIALS AND $q$-DEFORMED OSCILLATOR

KAMEL MEZLINI

Abstract. In this paper, we present an explicit realization of $q$-deformed Calogero-Vasiliev algebra whose generators are first-order $q$-difference operators related to the generalized discrete $q$-Hermite II polynomials recently introduced in [13]. Furthermore, we construct the wave functions and we determine the $q$-coherent states.

1. Introduction

The $q$-deformed harmonic oscillator algebras [10, 11, 15, 16] have been intensively studied in recent years due to their crucial role in diverse areas of mathematic and physics. The basic interest in $q$-deformed algebras resides in the generalization of the fundamental symmetry concept of classical Lie algebras. Many algebraic constructions have been proposed to describe various generalization of the quantum harmonic oscillator in the literature. The difficulty for most of them is to realize an explicit form of the associated Hamiltonian eigenfunctions. It is well known that the Hermite polynomials are connected to the realization of the classical harmonic oscillator algebra. It is natural then, that generalizations of quantum harmonic oscillator lead to generalizations of the Hermite polynomials. An explicit realization of the $q$-harmonic oscillator has has been explored by many authors see for example Atakishiev [2, 3], Borzov [6], also Kulish and Damaskinsky [15], where the eigenfunctions of the corresponding Hamiltonian are given explicitly in terms of the $q$-deformed Hermite polynomials. The generators of the corresponding algebra are realized in terms of first-order difference operators.

In particular, as pointed out by Macfarlane in [16, 17], the Calogero-Vasiliev oscillator generalizes the parabose oscillator and its $q$-deformation describes the $q$-analogue of the parabose oscillator [11]. In one dimensional case, Rosenblum in [19] studied the generalized Hermite polynomials associated with the Dunkl operator and used them to construct the eigenfunctions of the parabose oscillator Hamiltonian. This oscillator, as it has been shown in [17], is linked to two-particle Calogero model [7].

The purpose of this paper is to explore the generalized discrete $q$-Hermite II polynomials $\tilde{h}_{n,\alpha}(x;q)$, recently introduced in [13] to construct the Hamiltonian eigenfunctions for the $q$-deformed Calogero-Vasiliev oscillator. This allows to find an explicit form of the generators of the corresponding algebra in terms of $q$-difference operators.

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This paper is organized as follows: in Section 2, we recall some notations and useful results from [13] about the generalized discrete $q$-Hermite II polynomials $\tilde{h}_{n,\alpha}(x;q)$. In Section 3, we review briefly the Fock space description of the Calogero-Vasiliev oscillator and its $q$-deformation as developed by Macfarlane in [16, 17]. In Section 4, we introduce an explicit form of the eigenfunctions of the $q$-deformed Calogero-Vasiliev Hamiltonian oscillator. This directly leads to the dynamical symmetry algebra $su_q(1,1)$, whose generators are explicitly constructed in terms of the $q$-difference operators, we construct the family of coherent states of this oscillator. Finally, we investigate the limiting case of the $q$-deformed Calogero-Vasiliev oscillator.

2. Notations and Preliminary

For the convenience of the reader, we provide in this section a summary of the mathematical notations and definitions used in this paper. We refer to the general references [12] and [13] for the definitions and notations. Throughout this paper, we assume that $0 < q < 1$ and we write $\mathbb{R}_q = \{ \pm q^n, n \in \mathbb{Z}\}$.

2.1. Basic symbols. For a complex number $a$, the $q$-shifted factorials are defined by:

$$(a;q)_0 = 1; \quad (a;q)_n = \prod_{k=0}^{n-1} (1 - aq^k), n = 1, 2, \ldots; \quad (a;q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k).$$

The $q$-numbers and the $q$-factorials are defined as follows:

$$[x]_q = \frac{1 - q^x}{1 - q}, \quad x \in \mathbb{C} \quad \text{and} \quad n!_q = [1]_q[2]_q\ldots[n]_q, \quad [0]_q = 1, \quad n \in \mathbb{N}.$$ 

For $\alpha \in \mathbb{R}$, we define the generalized $q$-integers and the generalized $q$-factorials by

$$(2n)_{q,\alpha} = [2n]_q, \quad [2n + 1]_{q,\alpha} = [2n + 2\alpha + 2]_q; \quad n!_{q,\alpha} = [1]_{q,\alpha}[2]_{q,\alpha}\ldots[n]_{q,\alpha}$$

and the generalized $q$-shifted factorials by

$$(q;q)_{n,\alpha} := (1 - q)^n n!_{q,\alpha}.$$}

Remark that we can rewrite (2.2) as

$$(q;q)_{2n,\alpha} = (q^2;q^2)_n (q^{2\alpha+2};q^2)_n, \quad (q;q)_{2n+1,\alpha} = (q^2;q^2)_n (q^{2\alpha+2};q^2)_{n+1}.$$ 

We may express the generalized $q$-factorials as

$$(2n)!_{q,\alpha} = \frac{(1 + q)^{2n} \Gamma_{q^2}(\alpha + n + 1) \Gamma_{q^2}(n + 1)}{\Gamma_{q^2}(\alpha + 1)},$$

$$(2n + 1)!_{q,\alpha} = \frac{(1 + q)^{2n+1} \Gamma_{q^2}(\alpha + n + 2) \Gamma_{q^2}(n + 1)}{\Gamma_{q^2}(\alpha + 1)},$$

where $\Gamma_q$ is the $q$-Gamma function given by (see [12])

$$\Gamma_q(z) = \frac{(q;q)_\infty}{(q^2;q)_\infty} (1 - q)^{1-z}, \quad z \neq 0, -1, -2, \ldots$$
and tends to $\Gamma(z)$ when $q$ tends to $1^-$. In particular, we have the limits

\begin{align}
\lim_{q \to 1^-} (2n)_q! &= \frac{2^{2n} n! \Gamma(\alpha + n + 1)}{\Gamma(\alpha + 1)} = \gamma_{\alpha + \frac{1}{2}}(2n), \\
\lim_{q \to 1^-} (2n + 1)_q! &= \frac{2^{2n+1} n! \Gamma(\alpha + n + 2)}{\Gamma(\alpha + 1)} = \gamma_{\alpha + \frac{1}{2}}(2n + 1),
\end{align}

where $\gamma_\nu$ is the Rosenblum’s generalized factorials (see [19]).

**Remark 2.1.** If $\alpha = -\frac{1}{2}$, then we get $(q; q)_n,\alpha = (q; q)_n$ and $n!_q,\alpha = n!_q$.

2.2. The **generalized $q$-exponential functions**. The two Euler’s $q$-analogues of the exponential function are given by (see [12])

\begin{align}
E_q(z) &= \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2} z^k}{(q; q)_k} = (-z; q)_\infty, \\
e_q(z) &= \sum_{k=0}^{\infty} \frac{z^k}{(q; q)_k} = \frac{1}{(z; q)_\infty}, \quad |z| < 1.
\end{align}

For $z \in \mathbb{C}$, the generalized $q$-exponential functions are defined by (see [13])

\begin{align}
E_{q,\alpha}(z) &= \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2} z^k}{(q; q)_k,\alpha}, \\
e_{q,\alpha}(z) &= \sum_{k=0}^{\infty} \frac{z^k}{(q; q)_k,\alpha}, \quad |z| < 1
\end{align}

and

\begin{align}
\psi_\alpha^\lambda(z) &= \sum_{n=0}^{\infty} b_{n,\alpha} (i \lambda z; q^2), \quad \lambda \in \mathbb{C},
\end{align}

where

\begin{align}
b_{n,\alpha}(z; q^2) &= q^{\frac{[n]}{2}} \frac{(q^2)_{[\frac{1}{2}]+1}}{n!_q,\alpha} z^n
\end{align}

and $[x]$ denoting the integer part of $x \in \mathbb{R}$. Note that $\psi_\alpha^\lambda(z)$ is the $q$-Dunkl kernel defined in [4].

A particular case, where $\alpha = -\frac{1}{2}$, by Remark [21] it follows that $E_{q,\alpha}(z) = E_q(z)$ and $e_{q,\alpha}(z) = e_q(z)$. 


The generalized $q$-derivatives. The Jackson’s $q$-derivative $D_q$ (see [12] [14]) is defined by:

$$D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z}.$$  

We also need a variant $D^+_q$, called forward $q$-derivative of the (backward) $q$-derivative $D^-_q = D_q$ as defined in (2.10):

$$D^+_q f(z) = \frac{f(q^{-1}z) - f(z)}{(1-q)z}.$$  

Note that $\lim_{q \to 1^-} D_q f(z) = \lim_{q \to 1^-} D^+_q f(z) = f'(z)$ whenever $f$ is differentiable at $z$.

The generalized backward and forward $q$-derivative operators $D_{q,\alpha}$ and $D^+_{q,\alpha}$ are defined as (see [13])

$$D_{q,\alpha} f(z) = \frac{f(z) - q^{2\alpha+1}f(qz)}{(1-q)z},$$  

$$D^+_{q,\alpha} f(z) = \frac{f(q^{-1}z) - q^{2\alpha+1}f(z)}{(1-q)z}.$$  

The generalized $q$-derivatives operators are given by

$$\Delta_{\alpha,q} f = D_q f_e + D_{q,\alpha} f_o,$$  

$$\Delta^+_{\alpha,q} f = D^+_q f_e + D^+_{q,\alpha} f_o,$$  

where $f_e$ and $f_o$ are respectively the even and the odd parts of $f$.

For $\alpha = -\frac{1}{2}$, we have $D_{q,\alpha} = D_q$, $D^+_{q,\alpha} = D^+_q$, $\Delta_{q,\alpha} = D_q$ and $\Delta^+_{q,\alpha} = D^+_q$.

We can rewrite the $q$-Dunkl operator introduced in [4] by means of the generalized $q$-derivative operators as

$$\Lambda_{\alpha,q} f = \Delta^+_{\alpha,q} f_e + \Delta_{\alpha,q} f_o.$$  

It is noteworthy that for a differentiable function $f$, we have

$$\lim_{q \to 1^-} \Delta_{\alpha,q} f = \lim_{q \to 1^-} \Delta^+_{\alpha,q} f = \Lambda_{\alpha+\frac{1}{2}} f,$$  

where $\Lambda_\nu$ is the classical Dunkl operator defined by

$$\Lambda_\nu f(x) = f'(x) + \frac{\nu}{x} [f(x) - f(-x)].$$
2.4. The $q$-Dunkl transform. We shall need the Jackson $q$-integral defined by (see [12, 14]).

$$\int_{-\infty}^{\infty} f(x) d_q x = (1 - q) \sum_{n=0}^{\infty} q^n f(q^n) + (1 - q) \sum_{n=-\infty}^{\infty} q^n f(-q^n).$$

For $p \geq 1$, we denote by $L^p_{\alpha,q}(\mathbb{R}_q)$ the space of complex-valued functions $f$ on $\mathbb{R}_q$ such that

$$\|f\|_{q,p} = \left( \int_{-\infty}^{\infty} |f(x)|^p |x|^{2\alpha+1} d_q x \right)^{\frac{1}{p}} < \infty.$$ 

The generalized $q$-exponential function $\psi_{\lambda}^{\alpha,q}(x)$ defined in (2.8) gives rise to a $q$-integral transform, called the $q$-Dunkl transform on the real line, which was introduced in [4] as

$$F_{D}^{\alpha,q}(f)(\lambda) = K_{\alpha} \int_{-\infty}^{+\infty} f(x) \psi_{-\lambda}^{\alpha,q}(x)|x|^{2\alpha+1} d_q x, \quad f \in L^1_{\alpha,q}(\mathbb{R}_q),$$

where

$$K_{\alpha} = \frac{(1 - q)^{\alpha} (q^{2\alpha+2}; q^2)_\infty}{2 (q^2; q^2)_\infty}.$$ 

It satisfies the following Plancheral theorem:

**Theorem 2.1.** $F_{D}^{\alpha,q}$ is an isometric isomorphism of $L^2_{\alpha,q}(\mathbb{R}_q)$ and for $f \in L^2_{\alpha,q}(\mathbb{R}_q)$, we have

(2.19) $$\|F_{D}^{\alpha,q}(f)\|_{q,2} = \|f\|_{q,2}$$

and the following inversion formula

(2.20) $$f(x) = K_{\alpha} \int_{-\infty}^{+\infty} F_{D}^{\alpha,q} f(\lambda) \psi_{\lambda}^{\alpha,q}(\lambda)|\lambda|^{2\alpha+1} d_q \lambda, \quad \forall x \in \mathbb{R}_q.$$ 

2.5. The generalized discrete $q$-Hermite II polynomials. The generalized discrete $q$-Hermite II polynomials $\{\tilde{h}_{n,\alpha}(x; q)\}_{n=0}^{\infty}$ are defined by (see [13])

(2.21) $$\tilde{h}_{n,\alpha}(x; q) := (q; q)_n \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k q^{-2nk} q^{(2k+1)\alpha} x^{n-2k}.$$ 

They have the following properties:

- **Generating function:**

(2.22) $$e_{q^2}(-z^2) E_{q,\alpha}(xz) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}}}{(q; q)_n} \tilde{h}_{n,\alpha}(x; q) z^n.$$ 

- **Inversion formula:**

(2.23) $$x^n = (q; q)_n \sum_{k=0}^{\left[\frac{n}{2}\right]} q^{-2nk+3k^2} \tilde{h}_{n-2k,\alpha}(x; q) (q^2; q^2)_k (q; q)_{n-2k}.$$
• **Forward shift operator:**

\[
\tilde{h}_{n,\alpha}(q^{-1}x; q) - q^{(2\alpha+1)\theta_{n+1}}\tilde{h}_{n,\alpha}(x; q) = q^{-n}(1 - q^n)x\tilde{h}_{n-1,\alpha}(x; q),
\]

where \(\theta_n\) is defined to be 0 if \(n\) is odd and 1 if \(n\) is even.

• **Backward shift operator:**

\[
\tilde{h}_{n,\alpha}(x; q) - q^{(2\alpha+1)\theta_{n+1}}(1 + q^{-2\alpha-1}x^2)\tilde{h}_{n,\alpha}(q; q) = -q^n\frac{1 - q^{n-1-(2\alpha+1)\theta_n}}{1 - q^{-n-1}}x\tilde{h}_{n+1,\alpha}(x; q).
\]

• **Orthogonality relation:**

\[
\int_{-\infty}^{\infty} \tilde{h}_{n,\alpha}(x; q)\tilde{h}_{m,\alpha}(x; q)\omega_{\alpha}(x; q)|x|^{2\alpha+1}d_qx = d_{n,\alpha}^2\delta_{n,m},
\]

where

\[
\omega_{\alpha}(x; q) = e_{q^2}(-q^{-2\alpha-1}x^2)
\]

and

\[
d_{n,\alpha} = c_{\alpha}q^{\frac{2\alpha+1}{2}}(q; q)_{n,\alpha}^{\frac{1}{2}}, \quad c_{\alpha} = \sqrt{\frac{(-q^{-2\alpha-1}, -q^{2\alpha+3}, q^{2\alpha+2}; q^2)_{\infty}}{2(1-q)(-q, -q^2; q^2)_{\infty}}}.
\]

3. **The Calogero-Vasiliev Oscillator and q-deformation**

3.1. **The Calogero-Vasiliev Oscillator.** The Calogero-Vasiliev oscillator algebra \([16, 17]\) (also called the deformed Heisenberg algebra with reflection \([8]\)) is generated by the operators \(\{I, a, a^+, N, K\}\) subject to the Hermiticity conditions

\[
(a^+)^* = a, \quad N^* = N, \quad K^* = K^{-1}
\]

and it satisfies the relations

\[
[a, a^+] = I + 2\nu K, \quad \nu \in \mathbb{R}, \quad 2\nu + 1 > 0, \quad K^2 = I,
\]

\[
[N, a] = -a, \quad [N, a^+] = a^+, \quad [N, K] = 0,
\]

where \([A, B] = AB - BA\). The operators \(a^-, a^+\) and \(N\) generalize the annihilation, creation and number operators related to the classical harmonic oscillator.

This oscillator, as it has been shown by Macfarlane in \([16]\), also describes a parabose oscillator of order \(p = 2\nu + 1\). In particular, it is linked to two-particle Calogero model \([17]\) and Bose-like oscillator \([19]\). This algebra has a basic one-dimensional explicit realization in terms of difference-differential operators

\[
A_{\nu} = \frac{1}{\sqrt{2}}(\Lambda_{\nu} + xI), \quad A_{\nu}^+ = \frac{1}{\sqrt{2}}(\Lambda_{\nu} - xI),
\]
where $I$ is the identity mapping and $A_\nu$ is the Dunkl operator defined by (2.18). The Hamiltonian is expressed as

$$H = \frac{1}{2} \{ A_\nu, A_\nu^+ \} = \frac{1}{2} (-\Lambda_\nu^2 + x^2 I),$$

where $\{ A, B \} = AB + BA$. The eigenvalues of $H$ are $n + \frac{1}{2} + \nu$ and the corresponding eigenvectors $\phi_\nu^\nu(x)$, which are the generalized Hermite functions introduced by Rosenblum in [19] as

$$(3.5) \quad \phi_\nu^\nu(x) = \left( \frac{\gamma_\nu(n)}{\Gamma(\nu + \frac{1}{2})} \right)^{\frac{1}{2}} e^{\frac{-x^2}{2}} H_\nu(x)\left(\begin{array}{c} n \\ \nu \end{array}\right)!,$$

where $\gamma_\nu$ is the generalized factorial

$$\gamma_\nu(n) = 2^n \left[ \frac{n}{2} \right] ! \Gamma \left( \nu + \left[ \frac{n+1}{2} \right] + \frac{1}{2} \right) \Gamma \left( \nu + \frac{1}{2} \right)$$

and $H_\nu(x)$ is the generalized Hermite polynomials.

$\{ \phi_\nu^\nu(x) \}_{n \in \mathbb{N}}$ is a complete orthonormal set in the Hilbert space $L_\nu^2(\mathbb{R})$ of Lebesgue measurable functions $f$ on $\mathbb{R}$ with

$$||f||_\nu := \left( \int_{-\infty}^{\infty} |f(x)|^2 |x|^{2\nu} dx \right)^{\frac{1}{2}} < \infty,$$

on which the conjugation relations (3.1) are satisfied. Let $\mathcal{S}_\nu$ be the space spanned by the generalized Hermite functions $\{ \phi_\nu^\nu(x) \}_{n=0}^\infty$. The operators $A_\nu$, $A_\nu^+$ and $N$ act on $\mathcal{S}_\nu$ as follows

$$(3.6) \quad A_\nu^+ \phi_\nu^{\nu 2n}(x) = \sqrt{2n + 2\nu + 1} \phi_\nu^{\nu 2n+1}(x),$$
$$A_\nu^+ \phi_\nu^{\nu 2n+1}(x) = \sqrt{2n + 2\nu} \phi_\nu^{\nu 2n+2}(x),$$
$$N \phi_\nu^{\nu n}(x) = n \phi_\nu^{\nu n+1}(x).$$

The number operator $N$ is given explicitly in terms of the creation and annihilation operators by

$$N = \frac{1}{2} \{ A_\nu, A_\nu^+ \} - \frac{2\nu + 1}{2}.$$

$K$ is realized in terms of the $N$ operator $K = (-1)^N$. Obviously, the operators $A_\nu$, $A_\nu^+$, $N$ and $K$ satisfy the commutation relations (3.2) and (3.3) on $\mathcal{S}_\nu$.

It is well known that in one dimension the two-particle Calogero system realizes an irreducible representations of Lie algebra $su(1, 1)$ [18]. Then one can easily verify that the operators

$$K_+ = \frac{1}{2}(A_\nu^+)^2, \quad K_- = \frac{1}{2} A_\nu^2, \quad \text{and} \quad K_0 = \{ A_\nu, A_\nu^+ \}/4.$$
satisfy the commutation relations
\[ [K_+, K_-] = -2K_0, \quad [K_0, K_\pm] = \pm K_\pm, \quad \text{on } \mathfrak{S}_\nu. \]
Thus, \( K_0, K_+ \) and \( K_- \) are the generators of Lie algebra \( su(1, 1) \). The representations are characterized by eigenvalues of the Casimir operator given by
\[ C = K_0^2 + \{ K_+, K_- \}/2, \]
which commutes with \( K_0 \) and \( K_\pm \). It follows from (3.6) that \( C \) takes the value
\[ -3/16 + \nu(\nu \pm 1) \]
throughout the even and odd subspaces of \( \mathfrak{S}_\nu \). Thus \( \mathfrak{S}_\nu^\pm \) carry out unitary irreducible representations of \( su(1, 1) \) with distinct eigenvalues of the Casimir operator \( C \).

3.2. The \( q \)-deformed Calogero-Vasiliev oscillator. The \( q \)-deformed Calogero-Vasiliev oscillator algebra is defined as the associative initial algebra generated by the operators \( \{ b, b^+, N \} \), which satisfy the relations

\[
\begin{align*}
[N, b] &= -b, \quad [N, b^+] = b^+, \quad (b^+)^* = b, \quad N^* = N, \\
bb^+ - q^{\pm(1+2\nu K)}b^+b &= [1 + 2\nu K]_q q^{\mp(N+\nu - \nu K)},
\end{align*}
\]
where \([x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}\) is an alternative definition of \( q \)-numbers and \( K = (-1)^N \).

The Fock representation of this \( q \)-oscillator algebra is constructed on a Hilbert space \( \mathcal{H} \) with the orthonormal basis \( \{ e_n^+ \}_n^\infty \). The operators \( b, b^+ \) and \( N \) act on the subspace \( \mathfrak{S}_{q\nu} \) spanned by the basis vectors \( e_n \) according to the formulas (see [16, 17, 20])

\[
\begin{align*}
b^+e_{2n} &= \sqrt{2n + 2\nu + 1}_q e_{2n+1}, \quad n = 0, 1, 2, ..., \\
b^+e_{2n-1} &= \sqrt{2n}_q e_{2n}, \quad n = 1, 2, ..., \\
Ne_n &= ne_n, \quad n = 0, 1, 2, ....
\end{align*}
\]
It follows from (3.9) that we have the following equalities
\[
bb^+ = [N + 1 + \nu(1 + K)]_q, \quad b^+b = [N + \nu(1 - K)]_q \quad \text{on } \mathfrak{S}_{q\nu}.
\]
The operators \( b, b^+ \) and \( N \) directly lead to the realisation of the quantum algebra \( su_q(1, 1) \) with the generators (see [15, 16, 17])
\[ K_+ = \beta(b^+)^2, \quad K_- = \beta b^2, \quad K_0 = \frac{1}{2}(N + \nu + \frac{1}{2}), \quad \beta = ([2]_q)^{-1}. \]
They satisfy the commutation relations
\[ [K_0, K_\pm] = \pm K_\pm, \quad [K_-, K_+] = [2K_0]_q^2 \quad \text{on } \mathfrak{S}_{q\nu} \]
and the conjugation relations
\[ (K_0)^* = K_0, \quad (K_+)^* = K_. \]
The Casimir operator $C$, which by definition commutes with the generators $K_\pm$ and $K_0$ is

$$C = \left[ K_0 - \frac{1}{2} \right]^2 q^2 - K_+ K_-.$$

The action of the operator $C$ on the vectors $e_n$ is given by the formulas

$$Ce_{2n} = \left[ \frac{2\nu - 1}{4} \right]^2 q^2 e_{2n}, \quad Ce_{2n+1} = \left[ \frac{2\nu + 1}{4} \right]^2 q^2 e_{2n+1}.$$

In the space $\mathcal{G}_{q\nu}$ the Casimir operator $C$ has two eigenvalues $\left[ \frac{2\nu \pm 1}{4} \right]^2 q^2$, with eigenvectors in the subspaces $\mathcal{G}_{q\nu}^\pm$ formed by the even and odd basis vectors $e_n$, respectively. Thus $\mathcal{G}_{q\nu}$ splits into the direct sum of two $su_q(1,1)$-irreducible subspaces $\mathcal{G}_{q\nu}^+$ and $\mathcal{G}_{q\nu}^-$. In particular Macfarlane in [17] has explored the links between the $q$-Deformed Calogero-Vasiliev Oscillator and the $q$-analogue of the parabose oscillator of order $p = 2\nu + 1$ studied in [11].

4. Realization of the $q$-deformed Calogero-Vasiliev oscillator

In this section we discuss an explicit realization of one-dimensional $q$-deformed Calogero-Vasiliev oscillator algebra. We give an explicit expression of the representation operators $b$, $b^+$ and $N$ defined in the previous subsection in terms of $q$-difference operators. It is known that such representation can be realized on a Hilbert space, on which all these operators are supposed to be well defined and the conjugation relations in (3.7) hold. For this purpose we take, as Hilbert space, the space $L^2_{q,\alpha}(\mathbb{R}_q)$, equipped with the scalar product

$$(\psi_1, \psi_2) = \int_{-\infty}^{\infty} \psi_1(x) \overline{\psi_2(x)} |x|^{2\alpha+1} d_q x.$$

We, now, construct a convenient orthonormal basis of $L^2_{q,\alpha}(\mathbb{R}_q)$ consisting of the $(q,\alpha)$-deformed Hermite functions defined by

$$\phi_n^\alpha(x; q) = d_{n,\alpha} \sqrt{\omega_{\alpha}(x; q)} \tilde{h}_{n,\alpha}(x; q),$$

where $\tilde{h}_{n,\alpha}(x; q)$, $\omega_{\alpha}(x; q)$ and $d_{n,\alpha}$ are given by (2.21), (2.27) and (2.28), respectively.

**Proposition 4.1.** $\{\phi_n^\alpha(x; q)\}_{n=0}^\infty$ is a complete orthonormal set in $L^2_{q,\alpha}(\mathbb{R}_q)$.

**Proof:**

The (discrete) orthogonality relation (2.26) for $\tilde{h}_{n,\alpha}(x; q)$ can be written as

$$\int_{-\infty}^{\infty} \phi_n^\alpha(x; q) \phi_m^\alpha(x; q) |x|^{2\alpha+1} d_q x = \delta_{n,m}.$$

Thus $\{\phi_n^\alpha(x; q)\}_{n=0}^\infty$ is an orthonormal set in $L^2_{q,\alpha}(\mathbb{R}_q)$. Let us prove that it is complete. Suppose that there exists $f \in L^2_{q,\alpha}(\mathbb{R}_q)$ orthogonal to all $\phi_n^\alpha(x; q)$, that is

$$\int_{-\infty}^{\infty} \phi_n^\alpha(x; q) f(x) |x|^{2\alpha+1} d_q x = 0, \quad \text{for all } n \in \mathbb{N}.$$
By using the inverse formula (2.23), we obtain
\[
\int_{-\infty}^{\infty} \sqrt{\omega_o(x; q) x^n f(x)} |x|^{2\alpha+1} d_q x = 0, \quad \text{for all} \ n \in \mathbb{N}.
\]

\[
F_D^{\alpha,q}(\sqrt{\omega_o(\cdot; q)f})(\lambda) = K_\alpha \int_{-\infty}^{+\infty} \sqrt{\omega_o(x; q)f(x)} \phi_o^{\alpha,q}(x) |x|^{2\alpha+1} d_q x,
\]

So,
\[
= K_\alpha \sum_{n=0}^{\infty} b_n (-i\lambda; q^2) \int_{-\infty}^{\infty} \sqrt{\omega_o(x; q)x^n f(x)} |x|^{2\alpha+1} d_q x
\]
\[
= 0.
\]

But, since \( f \in L^2_{q,a}(\mathbb{R}_q) \) and \( \omega_o(\cdot; q) \) is bounded on \( \mathbb{R}_q \), we deduce that \( \sqrt{\omega_o(\cdot; q)f} \in L^2_{q,a}(\mathbb{R}_q) \) and from the Plancherel theorem, we get \( f = 0 \). \( \square \)

We denote by \( \delta_q \) the \( q \)-dilatation operator in the variable \( x \), defined by \( \delta_q f(x) = f(qx) \), and the operator of multiplication by a function \( g \) will be denoted also by \( g \).

Let \( \mathfrak{S}_{q\alpha} \) be the finite linear span of \((q, \alpha)\)-deformed Hermite functions \( \phi_n^\alpha(x; q) \). From the forward and backward shift operators (2.24) and (2.25), we define the operators \( a \) and \( a^+ \) on \( \mathfrak{S}_{q\alpha} \) in a \( 2 \times 2 \) matrix form by

\[
a f = \frac{q^{\frac{\alpha}{2}}}{\sqrt{1-q x}} \begin{pmatrix} \delta_q^{-1} \sqrt{1+q^{-2\alpha-1}x^2} - 1 & 0 \\ 0 & \delta_q^{-1} \sqrt{1+q^{2\alpha-1}x^2} - q^{2\alpha+1} \end{pmatrix} \begin{pmatrix} f_e \\ f_o \end{pmatrix}
\]

\[
a^+ f = \frac{q^{\frac{2\alpha+1}{2}}}{\sqrt{1-q x}} \begin{pmatrix} \sqrt{1+q^{-2\alpha-1}x^2} \delta_q - 1 & 0 \\ 0 & \sqrt{1+q^{2\alpha-1}x^2} \delta_q - q^{-2\alpha-1} \end{pmatrix} \begin{pmatrix} f_e \\ f_o \end{pmatrix}
\]

where \( f_e \) and \( f_o \) are respectively the even and the odd parts of \( f \in \mathfrak{S}_{q\alpha} \).

The reader may verify that these operators are indeed mutually adjoint in the Hilbert space \( L^2_{q,a}(\mathbb{R}_q) \).

The action of the operators \( a \) and \( a^+ \) on the basis \( \{\phi_n^\alpha(x; q)\}_{n=0}^{\infty} \) of \( L^2_{q,a}(\mathbb{R}_q) \) leads to the explicit results:

**Proposition 4.2.**

\[
a \phi_0^\alpha(x; q) = 0,
\]

\[
a \phi_n^\alpha(x; q) = \sqrt{[n]_{q,a}} \phi_{n-1}^\alpha(x; q), \quad n \geq 1,
\]

\[
a^+ \phi_n^\alpha(x; q) = \sqrt{[n+1]_{q,a}} \phi_{n+1}^\alpha(x; q),
\]

\[
\phi_n^\alpha(x; q) = (n!_{q,a})^{-\frac{1}{2}} a^+ a^n \phi_0^\alpha(x; q),
\]

where \( [n]_{q,a} \) is defined by (2.10).
Proof: (4.4) is an immediate consequence of the definition (4.1). (4.5) and (4.6) follow from the forward and backward shift operators (2.24) and (2.25) and from the fact that
\[ d_{n,\alpha} = q^{n-\frac{1}{2}} \sqrt{[n]_{q,\alpha}} d_{n-1,\alpha}. \]
(4.7) is a consequence of (4.6). □

From (4.5) and (4.6), one deduces that
(4.8) \[ a^+ a^\alpha_n(x; q) = [n]_{q,\alpha} a^\alpha_n(x; q) \]
and
(4.9) \[ a a^+ a^\alpha_n(x; q) = [n+1]_{q,\alpha} a^\alpha_n(x; q). \]

The number operator \( N \) is defined in this case by the relations
(4.10) \[ a^+ a = [N]_{q,\alpha}, \quad a a^+ = [N+1]_{q,\alpha} \quad \text{on} \quad \mathfrak{S}_{q\alpha}. \]
The formulas (4.10) can be inverted to determine an explicit expression of the operator \( N \) as follows
(4.11) \[ N := \frac{1}{2 \log q} \log [1 - (1-q)aa^+] + \frac{1}{2 \log q} \log [1 - (1-q)a^+a] - \alpha - 1. \]
From (4.8), (4.9) and (4.11), we obtain
(4.12) \[ N a^\alpha_n(x; q) = n a^\alpha_n(x; q) \]
and
(4.13) \[ [N, a] = -a, \quad [N, a^+] = a^+ \quad \text{on} \quad \mathfrak{S}_{q\alpha}. \]

Now, we shall construct explicitly the generators \( b \) and \( b^+ \) of the \( q^{\frac{1}{2}} \)-deformed Calogero-Vasiliev algebra defined in the previous subsection by means of the operators \( a \) and \( a^+ \) in the following way
\[ b = q^{-\frac{N+(K+1)(\alpha+1)}{4}} a, \quad b^+ = a^+ q^{-\frac{N+(K+1)(\alpha+1)}{4}}, \quad K = (-1)^N. \]
Using the relation
\[ [x]_{q^{\frac{1}{2}}} = q^{-\frac{x+1}{2}} [x]_q, \]
one easily verifies that the actions of operators \( b \) and \( b^+ \) on the basis \( \{\phi^\alpha_n(x; q)\}_{n=0}^{\infty} \) are given by
(4.14) \[
\begin{align*}
    b\phi^\alpha_0(x; q) & = 0, \\
    b\phi^\alpha_n(x; q) & = \sqrt{[n]_{q^{\frac{1}{2}},\alpha}} \phi^\alpha_{n-1}(x; q), \quad n \geq 1, \\
    b^+ \phi^\alpha_n(x; q) & = \sqrt{[n+1]_{q^{\frac{1}{2}},\alpha}} \phi^\alpha_{n+1}(x; q),
\end{align*}
\]
where
\[ [2n]_{\frac{1}{2},\alpha} = [2n]_{\frac{1}{2}} \quad \text{and} \quad [2n + 1]_{\frac{1}{2},\alpha} = [2n + 2\alpha + 2]_{\frac{1}{2}}. \]

From (4.14), the basis vectors \( \phi_n^\alpha(x; q) \) may also be expressed in terms of the operator \( b^+ \) and \( \phi_0^\alpha(x; q) \) as follows
\[
\phi_n^\alpha(x; q) = \frac{1}{\sqrt{[n]_{\frac{1}{2},\alpha}!}} (b^+)^n \phi_0^\alpha(x; q),
\]
where
\[ [n]_{\frac{1}{2},\alpha} = [1]_{\frac{1}{2},\alpha} [2]_{\frac{1}{2},\alpha} \ldots [n]_{\frac{1}{2},\alpha}. \]

From the above facts, we may check that equation (3.7) holds and
\[
(4.15) \quad b^+b = [N + 1 + \nu(1 + K)]_{\frac{1}{2}}^{-1} \quad \text{on} \quad S_{q\alpha}.
\]

We deduce from (4.15) that the operators \( b \) and \( b^+ \) satisfy the relations
\[
(4.16) \quad b^+b = [1 + 2\nu K]_{\frac{1}{2}}^{-1} q^{1+2\nu K} \quad \text{on} \quad S_{q\alpha}.
\]

This leads to an explicit expressions for the generators \( \{b, b^+, N\} \) of the \( q \)-deformed Calogero-Vasiliev Oscillator algebra. The corresponding Hamiltonian is defined from \( b \) and \( b^+ \) according to
\[
(4.17) \quad H = \frac{1}{2} \{b, b^+\}.
\]

The functions \( \phi_n^\alpha(x; q) \) are eigenfunctions of this Hamiltonian with eigenvalues
\[
E_{q\alpha}(n) = \frac{1}{2} \left( [n]_{\frac{1}{2},\alpha} + [n + 1]_{\frac{1}{2},\alpha} \right).
\]

Thus, we recover in the limit \( q \to 1 \) the eigenvalues of the Hamiltonian of the Calogero-Vasiliev oscillator.

In the same manner, as in the case of \( su(1, 1) \), by virtue of the results of the previous subsection, we construct an explicit realization of the operators \( B_-, B_+ \) and \( B_0 \) generators of the quantum algebra \( su_{\frac{1}{2}}(1, 1) \) in terms of the oscillation operators \( b, b^+ \) and \( N \) by setting
\[
B_+ = \gamma (b^+)^2, \quad B_- = \gamma b^2, \quad B_0 = \frac{1}{2} (N + \alpha + 1), \quad \gamma = ([2]_{\frac{1}{2}})^{-1}.
\]

From (4.14), we derive the actions of these operators on the basis \( \{\phi_n^\alpha(x; q)\}_{n=0}^\infty \)
\[
(4.18) \quad B_0\phi_n^\alpha(x; q) = \frac{1}{2} (n + \alpha + 1)\phi_n^\alpha(x; q),
\]
\[
B_+\phi_n^\alpha(x; q) = \gamma \sqrt{[n + 2]_{\frac{1}{2},\alpha} [n + 1]_{\frac{1}{2},\alpha}} \phi_{n+2}^\alpha(x; q),
\]
\[
B_-\phi_n^\alpha(x; q) = \gamma \sqrt{[n]_{\frac{1}{2},\alpha} [n - 1]_{\frac{1}{2},\alpha}} \phi_{n-2}^\alpha(x; q), \quad n \geq 1.
\]
It follows that
\begin{align}
B_- B_+ \phi^\alpha_{2n}(x; q) &= \gamma^2 [2n + 2]_{q^2} [2n + 2\alpha + 2]_{q^2} \phi^\alpha_{2n}(x; q), \\
B_- B_+ \phi^\alpha_{2n+1}(x; q) &= \gamma^2 [2n + 2]_{q^2} [2n + 2\alpha + 4]_{q^2} \phi^\alpha_{2n+1}(x; q), \\
B_+ B_- \phi^\alpha_{2n}(x; q) &= \gamma^2 [2n]_{q^2} [2n + 2\alpha]_{q^2} \phi^\alpha_{2n}(x; q), \\
B_+ B_- \phi^\alpha_{2n+1}(x; q) &= \gamma^2 [2n]_{q^2} [2n + 2\alpha + 2]_{q^2} \phi^\alpha_{2n+1}(x; q).
\end{align}

(4.19)

Using the following identity (see [5] p.58)

\begin{equation}
[x]_q [y - z]_q + [z]_q [y - x]_q + [z]_q [x - y]_q = 0,
\end{equation}

(4.20)

with \(x = 2n + 2\), \(y = -2n - 2\alpha\), \(z = 2\) and with \(x = 2n + 2\), \(y = -2n - 2\alpha - 2\), \(z = 2\) respectively, we obtain

\begin{align}
[2n + 2]_{q^2} [2n + 2\alpha + 2]_{q^2} - [2n]_{q^2} [2n + 2\alpha]_{q^2} &= [2]_{q^2} [4n + 2\alpha + 2]_{q^2}, \\
[2n + 2]_{q^2} [2n + 2\alpha + 4]_{q^2} - [2n]_{q^2} [2n + 2\alpha + 2]_{q^2} &= [2]_{q^2} [4n + 2\alpha + 4]_{q^2}.
\end{align}

By the identity \([2x]_{q^2} = [2]_{q^2} [x]_q\), we obtain

\begin{align}
[4n + 2\alpha + 2]_{q^2} &= [2]_{q^2} [2n + \alpha + 1]_q, \\
[4n + 2\alpha + 4]_{q^2} &= [2]_{q^2} [2n + \alpha + 2]_q,
\end{align}

from which follows the following commutation relations

\[ [B_0, \pm B] = \pm B, \quad [B_-, B_+] = [2B_0]_q \quad \text{on} \quad \mathcal{S}_{q\alpha} \]

and the conjugation relations

\[ B^*_0 = B_0, \quad B^*_+ = B_- \quad \text{on} \quad \mathcal{S}_{q\alpha}. \]

We conclude an explicit realization of generators \(B_0, B_-\) and \(B_+\) of the quantum algebra \(su_{q^2}(1, 1)\).

To analyze the irreducible representations of \(su_{q^2}(1, 1)\) algebra, we need the invariant Casimir operator \(C\), which in this case has the form:

\[ C = \left[ B_0 - \frac{1}{2} \right]_q^2 + B_+ B_. \]

From (4.18) and (4.19) we obtain the action of this operator on the basis \(\{ \phi^\alpha_n(x; q) \}_{n=0}^\infty\)

\[ C \phi^\alpha_{2n}(x; q) = \left( \left[ n + \frac{\alpha}{2} \right]_q^2 - [n]_q [n + \alpha]_q \right) \phi^\alpha_{2n}(x; q), \]

\[ C \phi^\alpha_{2n+1}(x; q) = \left( \left[ n + \frac{\alpha + 1}{2} \right]_q^2 - [n]_q [n + \alpha + 1]_q \right) \phi^\alpha_{2n+1}(x; q). \]

Using (4.20) with \(x = n + \frac{\alpha}{2}, \ y = n, \ z = -\frac{\alpha}{2}\) and with \(x = n + \frac{\alpha + 1}{2}, \ y = n, \ z = -\frac{\alpha + 1}{2}\) respectively, we deduce

\[ \left[ n + \frac{\alpha}{2} \right]_q^2 - [n]_q [n + \alpha]_q = \left[ \frac{\alpha}{2} \right]_q^2. \]
\[
\left[ n + \frac{\alpha + 1}{2} \right]_q^2 - [n]_q [n + \alpha + 1]_q = \left[ \frac{\alpha + 1}{2} \right]_q^2.
\]

Then, the Casimir operator \( C \) has two eigenvalues \( \left[ \frac{2\alpha + 1}{4} \right]_q^2 \) in the subspaces \( \mathfrak{S}^\pm_{q \alpha} \) formed by the even and odd basis vectors \( \{ \phi^\alpha_n(x; q) \}_{n=0}^\infty \), respectively. Thus \( \mathfrak{S}_{q \alpha} \) splits into the direct sum of two \( su_q(1, 1) \)-irreducible subspaces \( \mathfrak{S}^+_{q \alpha} \) and \( \mathfrak{S}^-_{q \alpha} \).

In particular Macfarlane in [17] showed that this oscillator realises the \( q \)-deformed parabose oscillator of order \( p = 2\nu + 1 \) studied in [11]. Hence we derive an explicit realizations of the annihilation and creation operators of \( q \)-deformed parabose oscillator in terms of \( q \)-difference operators.

4.1. The \( q \)-coherent states. The normalized \( q \)-coherent state \( \varphi_\zeta(x; q) \) related to the \( q \)-deformed Calogero-Vasiliev oscillator is defined as the eigenfunction of the annihilation operator \( a \) with eigenvalue \( \zeta \in \mathbb{C} \),

\[
a \varphi_\zeta(x; q) = \zeta \varphi_\zeta(x; q) \text{ on } \mathfrak{S}_{q \alpha}.
\]

Theorem 4.1. The \( q \)-coherent states are of the form

\[
\varphi_\zeta(x; q) = \frac{c_\alpha \sqrt{\omega_\alpha(x; q)}}{\sqrt{e_{q,\alpha}(-1 - q)\zeta^2}} e_{q^\frac{1}{2}}(-q(1 - q)\zeta^2) E_{q,\alpha}(q^\frac{1}{2a} (1 - q)^\frac{1}{2} x \zeta),
\]

where \( c_\alpha \) is given in (2.28).

Proof: By expressing \( \varphi_\zeta(x; q) \) in terms of the wave functions \( \phi^\alpha_n(x; q) \), we get

\[
\varphi_\zeta(x; q) = \sum_{n=0}^{+\infty} f_{n,\alpha}(q) \phi^\alpha_n(x; q).
\]

From the eigenvalue equations (4.4) and (4.5), we can write

\[
a \varphi_\zeta(x; q) = \sum_{n=0}^{+\infty} f_{n,\alpha}(q) \sqrt{[n]_{q,\alpha}} \phi^\alpha_{n-1}(x; q).
\]

Replace \( \varphi_\zeta(x; q) \) by the series (4.23) in (4.21) and equate the coefficients of \( \phi^\alpha_n(x; q) \) on both sides to get

\[
f_{n+1,\alpha}(q) \sqrt{[n + 1]_{q,\alpha}} = \zeta f_{n,\alpha}(q).
\]

By iterating the last relation, we get since \( f_{0,\alpha}(q) = C_0 = C_0(\zeta) \), the relations

\[
f_{1,\alpha}(q) = \frac{C_0 \zeta}{\sqrt{[1]_{q,\alpha}}}, \quad f_{2,\alpha}(q) = \frac{C_0 \zeta^2}{\sqrt{2[1]_{q,\alpha}}}, \quad \cdots, f_{n,\alpha}(q) = \frac{C_0 \zeta^n}{\sqrt{n[1]_{q,\alpha}}},
\]

which, inserted into the expansion (4.23), give

\[
\varphi_\zeta(x; q) = C_0(\zeta) \sum_{n=0}^{+\infty} \frac{\zeta^n}{\sqrt{n[1]_{q,\alpha}}} \phi^\alpha_n(x; q).
\]
Now, for $\zeta, \zeta' \in \mathbb{C}$, we have the scalar product
\[
\int_{-\infty}^{+\infty} \varphi_\zeta(x;q) \varphi_{\zeta'}(x;q) |x|^{2\alpha+1} d_q x
= C_0(\zeta) C_0(\zeta') \sum_{n,k=0}^{+\infty} \frac{\zeta^n \zeta'^k}{\sqrt{n!_{q,\alpha}} \sqrt{k!_{q,\alpha}}} \int_{-\infty}^{+\infty} \phi_n^{\alpha}(x;q) \phi_k^{\alpha}(x;q) |x|^{2\alpha+1} d_q x.
\]
But, the orthogonality relation (2.26) implies that
\[
\int_{-\infty}^{+\infty} \varphi_\zeta(x;q) \varphi_{\zeta'}(x;q) |x|^{2\alpha+1} d_q x = C_0(\zeta) C_0(\zeta') \sum_{n=0}^{+\infty} \zeta^n \zeta'^n.
\]
By the relation (2.7), we get
\[
\int_{-\infty}^{+\infty} \varphi_\zeta(x;q) \varphi_{\zeta'}(x;q) |x|^{2\alpha+1} d_q x = C_0(\zeta) C_0(\zeta') e_{q,\alpha}(- (1-q) \zeta \zeta').
\]
The normalized condition requires to choose $C_0(\zeta) = \frac{1}{\sqrt{e_{q,\alpha}(- (1-q) \zeta^2)}}$
So, we can write
\[
\varphi_\zeta(x;q) = \frac{1}{\sqrt{e_{q,\alpha}(- (1-q) \zeta^2)}} \sum_{n=0}^{+\infty} \frac{\zeta^n}{\sqrt{n!_{q,\alpha}}} \phi_n^{\alpha}(x;q).
\]
From the relations (4.1) and (2.28), we obtain
\[
\varphi_\zeta(x;q) = \frac{\sqrt{\omega_\alpha(x;q)}}{\sqrt{e_{q,\alpha}(- (1-q) \zeta^2)}} \sum_{n=0}^{+\infty} \frac{\zeta^n}{\sqrt{n!_{q,\alpha}}} c_\alpha q^\frac{1}{2} \frac{(q;q)_n}{(q;q)_n} \tilde{h}_{n,\alpha}(x;q),
\]
which can be rewritten as
\[
\varphi_\zeta(x;q) = \frac{c_\alpha \sqrt{\omega_\alpha(x;q)}}{\sqrt{e_{q,\alpha}(- (1-q) \zeta^2)}} \sum_{n=0}^{+\infty} q^{\frac{n(n-1)}{2}} \left( q^\frac{1}{2} (1-q)^{\frac{1}{2}} \right)^n (q;q)_n \tilde{h}_{n,\alpha}(x;q).
\]
So, from the generating function (2.22) for the polynomials $\tilde{h}_{n,\alpha}(x;q)$, we get the explicit form of the normalized $q$-coherent state (4.22).

4.2. Limit to the Calogero oscillator.

Lemma 4.1.
\[
\lim_{q \to 1^-} (1-q^2)^{\alpha+\frac{1}{2}} \phi_n^{\alpha}(\sqrt{1-q^2} x; q) = \phi_n^{\alpha+\frac{1}{2}}(x),
\]
where $\phi_n^{\mu}$ is the Rosenblum’s Hermite function defined by formula (3.3).
To show this, one first verifies easily that
\[ \lim_{q \to 1^-} \frac{\tilde{h}_{n,a}(\sqrt{1-q^2}; q)}{(1-q^2)^{\frac{a}{2}}} = \frac{H_n^{\alpha + \frac{1}{2}}(x)}{2^n}, \]
where \( H_n^{\alpha + \frac{1}{2}}(x) \) is the Rosenblum’s Hermite polynomials.

\[ (4.26) \quad \lim_{q \to 1^-} \omega_{\alpha}(\sqrt{1-q^2}; q) = \exp(-x^2). \]

We have the limits (see [1] Theorem 10.2.4)
\[ (4.27) \quad \lim_{q \to 1^-} d_{n,a}(1-q^2)^{\frac{a}{2}} = \lim_{q \to 1^-} c_{\alpha} \lim_{q \to 1^-} q^{\frac{a}{2}} \frac{(q;q)_n^{\frac{1}{2}}}{(q;q)_n}(1-q^2)^{\frac{a}{2}} = 2^{\frac{a}{2}} \sqrt{\frac{\gamma_{\alpha + \frac{1}{2}}(n)}{n!}} \lim_{q \to 1^-} c_{\alpha}. \]

We have the limits (see [1] Theorem 10.2.4)
\[
\begin{align*}
\lim_{q \to 1^-} (-q^{-2\alpha-1}; q^2)_\infty & = 2^{\alpha+1} \\
\lim_{q \to 1^-} (-q^{-2\alpha+3}; q^2)_\infty & = 2^{-\alpha-1} \\
\lim_{q \to 1^-} (q^{2\alpha+2}; q^2)_\infty (1-q^2)^{\alpha} & = \lim_{q \to 1^-} \frac{1}{\Gamma_q(\alpha + 1)} = \frac{1}{\Gamma(\alpha + 1)}. 
\end{align*}
\]

Then,
\[ \lim_{q \to 1^-} (1-q^2)^{\frac{a+1}{2}} c_{\alpha} = \frac{1}{\sqrt{\Gamma(\alpha + 1)}} \]

\[ (4.27) \quad \lim_{q \to 1^-} (1-q^2)^{\frac{a+1}{2}} d_{n,a} \tilde{h}_{n,a}(\sqrt{1-q^2}; q) = \left( \frac{\gamma_{\alpha + \frac{1}{2}}(n)}{\Gamma(\alpha + 1)} \right)^{\frac{1}{2}} \frac{H_n^{\alpha + \frac{1}{2}}(x)}{2^n n!} \]
\[ \lim_{q \to 1^-} (1-q^2)^{\frac{a+1}{2}} \phi_n^{\alpha}(\sqrt{1-q^2}; q) = \left( \frac{\gamma_{\alpha + \frac{1}{2}}(n)}{\Gamma(\alpha + 1)} \right)^{\frac{1}{2}} \exp(-x^2) \frac{H_n^{\alpha + \frac{1}{2}}(x)}{2^n n!} \]
\[ = \phi_n^{\alpha + \frac{1}{2}}(x). \]

In the limit as \( q \to 1^- \) the \( q \)-Calogero-Vasiliev oscillator reduces to the Calogero oscillator. To show this, one first verifies easily that
\[ a\phi_n^{\alpha}(x; q) = \sqrt{q(1-q)} \omega_{\alpha}(x; q) \Delta_{n,q}^+(d_{n,a} \tilde{h}_{n,a}(.; q))(x), \]
where \( \Delta_{n,q}^+ \) is given by (2.15). One rescales \( x \to \sqrt{1-q^2}x \), we get
\[ a\phi_n^{\alpha}(\sqrt{1-q^2}x; q) = \frac{\sqrt{q(1-q)} \omega_{\alpha}(\sqrt{1-q^2}; q)}{\sqrt{1+q}} \Delta_{n,q}^+(d_{n,a} \tilde{h}_{n,a}(\sqrt{1-q^2}; q)). \]
Using the limits (4.26), (4.27) and (2.17), we find that
\[
\lim_{q \to 1^-} (1 - q^2)^{\frac{\alpha+1}{2}} a\phi_n^\alpha(\sqrt{1 - q^2 x}; q) = \frac{\exp(-\frac{x^2}{2})}{\sqrt{2}} \Lambda_{\alpha+\frac{1}{2}} \left( \frac{\gamma_{\alpha+\frac{1}{2}}(n)}{\Gamma_{\frac{1}{2}}(\alpha + 1)2^\frac{\alpha}{2}n!} \right) H_n^{\alpha+\frac{1}{2}}(x).
\]

By definition of the Rosenblum’s Hermite function \( \phi_n^\alpha \) (3.5) and the properties of the Dunkl operator \( \Lambda_\alpha \), we have
\[
\lim_{q \to 1^-} (1 - q^2)^{\frac{\alpha+1}{2}} a\phi_n^\alpha(\sqrt{1 - q^2 x}; q) = \frac{1}{\sqrt{2}} (\Lambda_{\alpha+\frac{1}{2}} + xI) \phi_n^{\alpha+\frac{1}{2}}(x),
\]
where \( I \) is the identity operator. In the same way, we can write
\[
a^+ \phi_n^\alpha(x; q) = \sqrt{q} \omega_n(\sqrt{1 - q^2 x}; q) d_{n,\alpha} \left(-H_{\alpha,q} \Delta_{\alpha,q} + \frac{x}{1 - q} \delta_q\right) \tilde{h}_{n,\alpha}(x; q),
\]
where \( \Delta_{\alpha,q} \) is the operator (2.14) and
\[
H_{\alpha,q} : f = f_e + f_o \longmapsto f_e + q^{2\alpha+1} f_o.
\]

Hence, we get
\[
a^+ \phi_n^\alpha(\sqrt{1 - q^2 x}; q) = \sqrt{q} \omega_n(\sqrt{1 - q^2 x}; q) d_{n,\alpha} \left(-\frac{1}{\sqrt{1 + q}} H_{\alpha,q} \Delta_{\alpha,q} + \sqrt{1 + q} x \delta_q\right) \tilde{h}_{n,\alpha}(\sqrt{1 - q^2 x}; q).
\]

By (4.26), (4.27) and (2.17), we obtain
\[
\lim_{q \to 1^-} (1 - q^2)^{\frac{\alpha+1}{2}} a^+ \phi_n^\alpha(\sqrt{1 - q^2 x}; q) = \exp(-\frac{x^2}{2}) \left[-\frac{1}{\sqrt{2}} \Lambda_{\alpha+\frac{1}{2}} + \sqrt{2} x I\right] \left( \frac{\gamma_{\alpha+\frac{1}{2}}(n)}{\Gamma_{\frac{1}{2}}(\alpha + 1)2^\frac{\alpha}{2}n!} \right) H_n^{\alpha+\frac{1}{2}}(x)
= \frac{1}{\sqrt{2}} (-\Lambda_{\alpha+\frac{1}{2}} + xI) \phi_n^{\alpha+\frac{1}{2}}(x).
\]

Note that if we replace \( \alpha + \frac{1}{2} \) by \( \nu \) we obtain the annihilation and creation operators of one-dimensional two-particle Calogero oscillator given by (3.4).

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KAMEL MEZLINI. University of Carthage, High Institute of Applied Sciences and Technologies of Mateur, 7030, Tunisia.

E-mail address: kamel.mezlini@lamsin.rnu.tn
E-mail address: kamel.mezlini@yahoo.fr