A < 0 Quantum Gravity in 2+1 Dimensions I: 
Quantum States and Stringy S-Matrix

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Abstract

We consider the theory of pure gravity in 2+1 dimensions, with negative cosmological constant. The theory contains simple matter in the form of point particles; the later are classically described as lines of conical singularities. We propose a formalism in which quantum amplitudes for process involving black holes and point particles are obtained as Liouville field theory (LFT) correlation functions on Riemann surfaces $X$. Point particles are described by LFT vertex operators, black holes (asymptotic regions) are in correspondence with boundaries of $X$. We analyze two examples: the amplitude for emission of a particle by the BTZ black hole, and the amplitude of black hole creation by two point particles. We then define an inner product between quantum states. The value of this inner product can be interpreted as the amplitude for one set of point particles to go into another set producing black holes. The full particle S-matrix is then given by the sum of all such amplitudes. This S-matrix is that of a non-critical string theory, with the world-sheet CFT being essentially the Liouville theory. $\Lambda < 0$ quantum gravity in 2+1 dimensions is thus a string theory.

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1 Introduction

The present paper develops a quantum theory of gravity in (2+1)-dimensions, with negative cosmological constant. We consider pure gravity, that is a theory of only the metric. The only type of matter we allow is a point particle, which is classically described as a line of conical singularities. This is a purely geometrical description, with no extra fields describing particles present. This should be contrasted to by now standard setting of AdS$_3$/CFT correspondence of string theory, where the 3d theory that appears contains, in addition to the metric, a large number of other fields.

The main motivation for studying this seemingly much too simple theory is that, despite its deceiving simplicity, it contains all interesting phenomena one can expect of a gravity theory. The theory is rich enough in that it contains both black holes [1] and matter in the form of point particles [2]. The later can collide and form the former, as was first demonstrated by Matschull in [3], see also [4]. At the same time the theory is simple in that it does not have local degrees of freedom. Thus, effectively, one has to deal with only a finite number of the degrees of freedom, which makes the usual quantum mechanics applicable. This gives hope for an explicit construction of the quantum theory. If constructed, the quantum theory can be expected to be a non-trivial playground for testing various ideas about quantum gravity. Thus, one can expect to give definite answers to such important conceptual questions as the geometrical origin of the black hole entropy and information loss in black hole evaporation.

A great deal is known about the classical theory, and important advances have been made in its quantization. We will not be able to give all references. Let us only highlight the main developments. The work [5] of Brown and Henneaux showed that the algebra of asymptotic symmetries of asymptotically AdS (2+1) gravity is the infinite dimensional Virasoro algebra of central charge $c = 3l/2G$. The quantum theory should therefore contain the same Virasoro algebra among its symmetries, and is thus a conformal field theory of this central charge. Work [6] used the Chern-Simons (CS) formulation of the theory to show that asymptotic degrees of freedom can be encoded in a single scalar field on the boundary of spacetime. The effective theory that appears on the boundary is Liouville field theory (LFT). Subsequent works [2, 3, 4, 10] also used the CS formulation and clarified the question of what are the degrees of freedom (DOF) in asymptotically AdS (2+1) gravity. Works [11, 12] were important in understanding the DOF in the geometrodynamics formulation. The physical picture that can be extracted from all these works, and also works on AdS$_3$/CFT correspondence in string theory, is that the quantum theory can be formulated holographically as a CFT living on the boundary of the spacetime, and this CFT is expected [3] to be the quantum Liouville theory. However, as we explain in section 7 the actual boundary theory is not Liouville theory, although it is closely related to LFT. Most of the constructions in the present paper are general enough so that it does not matter what CFT is used. We shall thus refer to this CFT as Liouville theory, although one should keep in mind the remarks of section 7.

In the present paper we further develop this “holographic” picture. From the technical point of view, the main novelty will be the usage of quantum LFT on Riemann surfaces for calculations in the Lorentzian signature quantum gravity. Thus, the quantum theory is developed using a suitable analytic continuation procedure.
To motivate our prescription we, following Maldacena [13], consider Hartle-Hawking (HH) states. A HH state [14] is a wave-functional $\Psi^{(2)g}$ that depends on the metric $(2)g$ on a spacelike slice $X$. This wave-functional is obtained as the path integral over Euclidean signature metrics $(3)g$ on a manifold $M$ whose boundary $\partial M = X$:

$$
\Psi^{(2)g} = \int \mathcal{D}^{(3)g} e^{-S_{E}[^{(3)g}]},
$$

The integral should be taken over metrics $(3)g$ whose restriction on the boundary $X$ is $(2)g$. As we shall explain in detail below, the spacelike slices $X$ of a typical spacetime are Riemann surfaces. Thus one should just take “the interior” of $X$ to be $M$. In other words, $M$ is the “solid” $X$. Such $M$ is not unique. Let us, however, at this stage be schematic. We shall return to the issue of interpretation of different $M$’s below. It is not hard to evaluate the path integral. As the AdS/CFT correspondence suggests, and as was argued in [15], the Euclidean path integral over $M$ is equal to some CFT partition function on $X$. The CFT here is expected [6] to be the quantum Liouville theory, or a close relative thereof, see below. Thus, the HH state is essentially given by the CFT partition function on the spatial slice $X$. In the bulk of the paper we give details of this construction and explain what Riemann surfaces arise as $X$. A special care is necessary in case of a rotating spacetime, for then there is no natural spatial slice to consider. Our construction of HH states thus generalizes and clarifies the proposal of [13].

An important new step is to introduce point particles. It has been proposed in [16] that point particles should be described by LFT vertex operators. These are operators of the form $e^{\eta \phi / b}$, where $\phi$ is the Liouville field, and $b$ is the Liouville coupling constant. As is well-known, for $\eta$ real such an operator can be thought of as making a conical singularity of the deficit angle $4\pi \eta$ in the surface. In other words, at least semi-classically, the LFT partition function on $X$ with such an operator inserted equals to the partition function on the same $X$ but with a conical singularity at the point of insertion. Since point particles are essentially conical singularities it is natural to describe them by these vertex operators. Generalizing the construction of HH states to the case when point particles are present, we have that the states are given by correlation functions on a Riemann surface $X$.

One can now interpret HH states as quantum amplitudes. As we discuss in detail in the bulk of the paper, simplest HH states describe amplitudes of, for example, black hole creation out of two point particles, or emission of a particle by a black hole. Taking the amplitude norm squared one gets the probability for these processes, which is of direct physical interest.

We then go onto constructing the S-matrix for point particles. First we define a natural scalar product between HH states. The main idea of this product is as follows. A HH state is the CFT partition function on $X$. The surface $X$ is a Riemann surface with boundary, one boundary component for every asymptotic region. There is then a natural inner product between two HH states for $X_1, X_2$ that have the same number of asymptotic regions. One just glues two surfaces with boundary to obtain a closed Riemann surface. The CFT partition function on this closed surface is the value of the inner product between the two states. The scalar product so defined gives transition, or topology changing amplitudes, see more on this interpretation below.
We use this proposal to define scattering amplitudes for point particles. These amplitudes are given by the CFT partition function on a closed Riemann surface with insertion of a number of vertex operators describing particles. Thus, the S-matrix for particles is given by a CFT partition function with insertions. Intermediate states in this scattering can be interpreted as black holes produced by particles. The genus of the Riemann surface on which CFT correlator is calculated is related to the number of asymptotic regions and the internal topology of the black hole.

The last step in our construction is to form the full S-matrix for particle scattering. It is obtained by summing over all possible ways that an initial configuration of particles can go to the final one. In particular, one has to sum over all black hole intermediate states that can be produced by particles. The resulting S-matrix is given by the same prescription that is used in string theory. Namely, it is given by the sum over genera integral over the moduli of a CFT correlator. The corresponding formula is given in the main body of the paper.

Thus, somewhat surprisingly, we obtain a stringy S-matrix for point particles. We should emphasize that this S-matrix is only similar to, but does not coincide with the S-matrix of any of the critical string theories. In particular, what we get is a non-critical CFT with the non-zero Brown-Henneaux value of $c$. We must note that, our CFT being the Liouville theory, the non-critical string theory we get is essentially 2d gravity.

The paper is organized as follows. In the next section we review the classical description of black hole and point particle spacetimes. An important technical and conceptual tool is that of an analytic continuation, which we describe in Section 3. We then define HH states in Section 4. Section 5 uses these HH states to obtain probabilities of simple processes involving point particles. We discuss the probability of a black hole creation by point particles and the Hawking emission process. The S-matrix for point particles is defined in Section 6. We conclude with a discussion of open problems and future directions.

## 2 Black hole and point particle spacetimes

In this section we review how the black hole and point particle spacetimes are constructed. Our main references here are [17, 18], see also [19] and the companion paper [20]. For more information on point particles see [21] and references therein.

Since there are no local DOF in 3d gravity, all spacetimes look locally like the maximally symmetric one, that is AdS$_3$. Let us thus start by reminding the reader some very basic facts about the Lorentzian AdS$_3$, out of which more complicated spaces will be obtained by identifications of points. The spacetime is best viewed as the interior of an infinite cylinder. The cylinder itself is the conformal boundary $\mathcal{I}$ of the spacetime. It is timelike, unlike the null conformal boundary of an asymptotically flat spacetime. All light rays propagating inside AdS start and end on $\mathcal{I}$. In this picture the constant time slices are copies of the Poincare (unit) disc. The unit disc is isometric to the upper half plane $\mathbb{U}$; we shall use both models. The isometry group of the Lorentzian signature AdS$_3$ is $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$. The spacetimes itself can be viewed as the (universal cover of the) $\text{SL}(2, \mathbb{R})$ group manifold, and the isometry group acts by the left and right multiplication. For
Figure 1: BTZ black hole: geometry of the time symmetry surface.

more details on AdS$_3$ see any of the references [17, 18, 19].

Let us now turn to the black hole spacetimes. We start with non-rotating spacetimes. The description of the non-rotating black holes is greatly facilitated by the fact that there is a surface $t = 0$ of time symmetry. This surface is preserved by the discrete group $\Gamma$ one uses to identify points. Thus, $\Gamma$ is actually a subgroup of the group $\text{SL}(2, \mathbb{R}) \subset \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$ that fixes the $t = 0$ plane. This “diagonal” $\text{SL}(2, \mathbb{R})$ consists of transformations of the form $x \rightarrow gxg^T$, $g \in \text{SL}(2, \mathbb{R})$, where we imply the $\text{SL}(2, \mathbb{R})$ group manifold model of AdS$_3$. Note that this is not the usual diagonal $\text{SL}(2, \mathbb{R})$ consisting of transformations $x \rightarrow gxg^{-1}$, $g \in \text{SL}(2, \mathbb{R})$, which fixes the origin of AdS$_3$. Transformations fixing the $t = 0$ plane act on it by isometries. Thus, the geometry of the surface $t = 0$ is that of the quotient of the unit disc by the action of $\Gamma \subset \text{SL}(2, \mathbb{R})$. Such 2-manifolds are nothing but Riemann surfaces. Once the geometry of the $t = 0$ plane is understood one just “evolves” the identifications in time to obtain a spacetime, see [17].

Let us see how this works on examples. Consider first the case of the non-rotating BTZ BH. In this case the discrete group is generated by a single hyperbolic element. Its action on the $t = 0$ plane can be understood by finding the fundamental region. In the case of $\Gamma$ generated by a single element $A$ the fundamental region is that between two geodesics on $U$ mapped into one another by $A$, see Fig. 1(a). It is clear that the quotient space has the topology of the $S^1 \times \mathbb{R}$ wormhole with two asymptotic regions, each having the topology of $S^1$, see Fig. 1(b). The BTZ angular coordinate runs from one geodesic to the other. The distance between the two geodesics measured along their common normal is precisely the horizon circumference. It can also be determined from the trace of the generator:

$$\frac{1}{2} \text{Tr} A = \cosh \pi r_+. \quad (2.1)$$

The black hole mass is then

$$M/\pi = 1 + r_+^2. \quad (2.2)$$

Here we work in the units $8\pi G = l = 1$, where $l = 1/\sqrt{-\Lambda}$. We take the empty AdS$_3$ to have zero mass. The “zero mass”, or, more correctly, zero size black hole then corresponds to $M = \pi$. Having understood the time symmetry surface geometry, one can obtain the spacetime geometry by “evolving” in time the $t = 0$ slice, see [17]. One finds that the resulting spacetime is indeed a black hole, in the sense that there is a region causally disconnected from the asymptotic infinity.
Let us consider more complicated initial slice geometries. We now consider the group \( \Gamma \) to be generated by two hyperbolic elements. For example, let the fundamental region be the part of the unit disc between four geodesics, as in Fig. 2(a). Let us identify these geodesics cross-wise. It is straightforward to show that the resulting geometry has only one asymptotic region, consisting of all four parts of the infinity of the fundamental region. With little more effort one can convince oneself that the resulting geometry is one asymptotic region “glued” to a torus, see Fig. 2(b). The spacetime obtained by evolving this geometry is a single asymptotic region black hole, but the topology inside the event horizon is now that of a torus. See [17] for more details on this spacetime.

A group generated by two elements can also be used to obtain a three asymptotic region black hole [17]. The fundamental region on the \( t = 0 \) plane is again the region bounded by four geodesics. They are, however, now identified side-wise, see Fig. 3(a). One can clearly see that the initial slice geometry has three asymptotic regions, with the black hole separated from the asymptotic regions by three horizons, see in Fig. 3(b). Evolving this, one gets a spacetime with three asymptotic regions and corresponding event horizons. See [17] for more details.

Taking the group \( \Gamma \) to be more complicated one constructs a large class of spacetimes. In particular, one can have a single asymptotic region black hole with an arbitrary Riemann surface inside the horizon. More generally, one can have a black hole with any number of asymptotic regions, and with any number of handles hidden behind the horizon(s).

To get point particle spacetimes one has to take some of the generators to be elliptic. Let us again consider the case of a non-rotating spacetime with a time symmetry plane \( t = 0 \). This
describes point particles that are at rest at $t = 0$. The spacetime describing a single point particle is obtained by taking the group $\Gamma \in \text{SL}(2, \mathbb{R})$ generated by a single elliptic element $A$. The elliptic $A$ has a fixed point inside the unit disc. After identifications are carried out this point becomes a conical singularity. The deficit angle is determined by the holonomy of $\text{Tr} A$. We have:

$$\frac{1}{2} \text{Tr} A = \cos \pi \alpha,$$

and the deficit angle is $4\pi \eta = 2\pi(1 - \alpha)$. This describes a point particle of mass

$$M/\pi = 1 - \alpha^2.$$  

Particle’s mass is then bounded from above by $M = \pi$. A particle of this maximal mass corresponds to a parabolic generator and can be viewed as the “zero mass” black hole. The geometry of surface $U/\Gamma$ is that of a cone over a circle. The tip of the cone is a singularity. It should be thought of as the point where the worldline of the particle intersects the $t = 0$ plane.

Spacetimes that contain both black holes and point particles are obtained by using a group $\Gamma$ generated by several elements of different type. One has to use Klein combination theorems, see e.g. [22], to select individual generators in such a way that the resulting time symmetry surface has the desired topology. In particular, one can have any of the black holes discussed above with one (or several) point particles crossing the time symmetry surface. The crossings can occur both behind the horizon and in the asymptotic regions. For example, one can have a BTZ black hole with a point particle located outside of the horizon, in one of the asymptotic regions. This configuration will be used in our description of the Hawking emission process.

Let us now turn to the rotating case, that is to the case with no plane of time symmetry in the spacetime. To describe and classify these spacetimes to the extent we understand non-rotating ones it turns out to be convenient to perform an analytic continuation [19]. We describe this in the next section.

### 3 Analytic continuation

The material presented in this section is from [19].

The idea is to analytically continue the discrete group $\Gamma \subset \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$ to some discrete group $\Sigma$ of $\text{SL}(2, \mathbb{C})$, which is the isometry group of the Euclidean AdS$_3$, same as the hyperbolic space $\mathbb{H}^3$. One then uses $\Sigma$ to obtain a quotient $M = \mathbb{H}^3/\Sigma$. Hyperbolic 3-manifolds such as $M$ are well understood. One uses this knowledge to understand and classify the rotating spacetimes. More precisely, it is best to concentrate on the conformal boundary $\tilde{X}$ of the resulting 3-manifold $M$. This is a Riemann surface, and its topology and the moduli are in one-to-one correspondence with the geometry of the corresponding black hole spacetime.

Let us first cast the non-rotating case in this language. The group $\Gamma$ is then a subgroup of $\text{SL}(2, \mathbb{R})$. The analytic continuation prescription in this case is to take the same group $\Gamma$ but think of it as a subgroup of $\text{SL}(2, \mathbb{C})$. There is then a simple relation between the geometry of the time symmetry plane and the conformal boundary $\tilde{X}$ of $M = \mathbb{H}^3/\Gamma$. It turns out that $\tilde{X}$ is simply
the Schottky double of the geometry of the time symmetry plane $X$. Let us remind the reader that the Schottky double exist for any Riemann surface. For a compact surface $X$, the double $\tilde{X}$ is simply two disconnected copies of $X$, with all moduli replaced by their complex conjugates in the second copy. For a Riemann surface with boundary, as is the case for the time symmetry surfaces of our black holes (boundaries are asymptotic regions), the double is obtained by taking two copies of $X$, taking complex conjugates of all the moduli in the second copy, and gluing them along the boundary to obtain a closed connected Riemann surface $\tilde{X}$. The Schottky double is obtained exactly by uniformizing the surface $X$ by a Fuchsian group $\Gamma$, and then acting by $\Gamma$ on the whole complex plane instead of the upper half-plane. The double $\tilde{X}$ is simply the quotient $\mathbb{C}/\Gamma$, where $\mathbb{C}$ is the complement of the set of fixed points of action of $\Gamma$ on $\mathbb{C}$. As an example, let us give a picture of the fundamental domain for the action on $\mathbb{C}$ of the group $\Gamma$ generated by two hyperbolic generators. The upper half of the Fig. 4 is simply the fundamental domain on the unit disc, Fig. 2(a) or Fig. 3(a), mapped into the upper half-plane. Considering the action on the full complex plane, and taking the quotient, one gets a closed Riemann surface, in this case of genus two – the Schottky double of $X$.

Having explained why the boundary of the Euclidean space is the Schottky double of the initial slice geometry, let us use this to give a simple relation between the number of asymptotic regions $K$, the number of handles $G$ behind the horizon, and the genus $g$ of the Euclidean boundary $\tilde{X}$. As is not hard to see:

$$g = 2G + K - 1.$$  \hspace{1cm} (3.1)

We see that several different time symmetry surface geometries can correspond to the same genus of the double $\tilde{X}$. This is because Riemann surface $\tilde{X}$ can be cut into two “equal” pieces in several different ways. This is somewhat reminiscent of string theory, where a single Riemann surface corresponds to many different field theory Feynman graphs, depending on how one decomposes the surface into pants. We shall see more parallels with string theory in what follows.

Let us note that curves along which one cuts $\tilde{X}$ to obtain two copies of $X$ correspond to asymptotic regions on $X$. In particular, the holonomy along such a curve $\alpha$ (that is the trace of an element of $\Gamma$ whose axis projects on $\alpha$) determines, see (2.1) the size (and the mass) of the horizon separating this asymptotic region from the rest of spacetime. Thus, to distinguish different spacetimes that give rise to the same Euclidean boundary $\tilde{X}$ one should mark $\tilde{X}$ with a set of homotopy non-trivial non-intersecting curves that divide $\tilde{X}$ into two pieces of equal topology and
correspond to asymptotic regions.

Let us discuss what kind of 3-manifold $M$ one gets as $\mathbb{H}^3/\Gamma$. Obtained by identification of points in $\mathbb{H}^3$, $M$ is a constant negative curvature 3-manifold. Its conformal boundary is precisely the surface $\tilde{X}$. Let us note, however, that given a Riemann surface $\tilde{X}$ there are many different 3-manifolds $M$ whose boundary is $\tilde{X}$. To get a specific manifold one has to select on $\tilde{X}$ a basis of $\pi_1(\tilde{X})$ consisting of curves $\alpha_i, \beta_i$. Such surface $\tilde{X}$ is called marked. There is then a single 3-manifold $M$ whose boundary is $\tilde{X}$ and such that curves $\alpha_i$ are all contractible inside $M$. This manifold is obtained by the Schottky uniformization of $\tilde{X}$. Namely, given a marked Riemann surface $\tilde{X}$ there is a unique Schottky group $\Sigma$, that is a group freely generated by $g$ elements $L_1, \ldots, L_g$, such that $\tilde{X}$ is given by the quotient of the complex plane by $\Sigma$ and all generators $L_i$ project to the curves $\beta_i$.

One can then use this Schottky group to obtain a hyperbolic 3-manifold $M = \mathbb{H}^3/\Sigma$. The curves $\beta_i$ are non-contractible inside $M$, whereas $\alpha_i$ are contractible.

In our case the manifold $M$ is a specific one, given by $M = \mathbb{H}^3/\Gamma$. In practice $\Gamma$ comes with a set of generators chosen, in other words it is marked, and the axes of these generators project to curves on $\tilde{X}$ that are non-contractible inside $M$. In particular, the curves on $\tilde{X}$ that correspond to asymptotic regions, that is divide $\tilde{X}$ into two parts $X$, are all either projections of the axes of generators of $\Gamma$, or are obtained as commutators of the generators. In any case they are non-contractible inside $M$ due to the presence of horizons. Thus, $M$ can be pictured as the “exterior” of $\tilde{X}$ in $S^3$, not as the interior.

Let us now generalize this picture to rotating spacetimes. The main idea is to take the group $\Gamma$ of a non-rotating spacetime and deform it into a subgroup of $\text{SL}(2, \mathbb{C})$, this deformation taking into account the rotation. As we showed in [19], it is natural to consider the Fenchel-Nielsen deformation. The result of the deformation is a quasi-Fuchsian group $\Gamma^\tau$. The Euclidean boundary $\tilde{X}$ is then the quotient of the complex plane with respect to the transformations from $\Gamma^\tau$. Geometrically, turning on the rotating in one of the asymptotic regions amounts to performing the Fenchel-Nielsen twist on the Riemann surface $\tilde{X}$ along the geodesic $\alpha$ that corresponds to this asymptotic region. The angular velocity of the asymptotic region is determined by the (imaginary part of the) trace of an element of $\Gamma^\tau$ whose axes projects on $\alpha$. This brief description will be sufficient for our purposes. See [19] for more detail.

Let us summarize. A non-rotating black hole is described in the Euclidean signature by the Schottky double $\tilde{X}$ of the time symmetry plane $X$. The closed Riemann surface $\tilde{X}$ is the conformal boundary of a hyperbolic 3-manifold $M$—analytic continuation of the black hole spacetime. A set of geodesics on $\tilde{X}$ that separates it into two parts is in one-to-one correspondence with asymptotic regions of $X$. One turns on the rotation by performing the Fenchel-Nielsen twist along these geodesics. The holonomy along these geodesics determine the size and rotation of the corresponding horizons. This analytic continuation construction gives an explicit description and classification of the rotating spacetimes. Whereas the non-rotating spacetimes are classified by their time symmetry plane $X$ geometry, rotating black holes are classified by the geometry of the double $\tilde{X}$.

Although we have only described the case of hyperbolic generators, that is no point particles, the above discussion generalizes to the case when particles are present. When the spacetime is non-
rotating, and the particles are in rest on the time symmetry surface, one has a group $\Gamma \subset \text{SL}(2, \mathbb{R})$ that contains elliptic (and/or parabolic) elements. One can again consider the action of $\Gamma$ on the whole complex plane and obtain the double $\tilde{X}$. It is a closed surface with conical singularities, whose number is twice the number of singularities on the time symmetry surface. The position of singularities on $\tilde{X}$ is obtained from the position on $X$ by taking the mirror image with respect to the real axes. The case of rotating spacetimes, or case of particles with momentum is more complicated. One can still form a double $\tilde{X}$ that has the same topology (and the same number of insertions) as in the non-rotating case. However, the position of insertions on $\tilde{X}$ is more complicated to determine. It depends both on the rotation of the spacetime and on the momentum of particles. In the present paper, being rather schematic, we will not need a precise relation. It is given in the companion paper [20].

4 Hartle-Hawking states

Having reviewed how the spacetimes in question are described, we are ready to consider quantum theory. As we have explained in the introduction, the idea is to consider HH states. In the context of negative cosmological constant gravity in (2+1) dimensions such states were considered in [13]. The author considered only simplest of such states. In this section we generalize his considerations, in particular to spacetimes containing point particles and to the rotating case. The next section considers some simple physical processes that can be analyzed with the help of the amplitudes we define here. Section 6 defines an inner product for HH states.

Let us consider the non-rotating case first. In this case there is a plane $t = 0$ of time symmetry. It is natural to define HH states using the plane $X$. Thus, a state $\Psi$ will be a quantum state of our spacetime at time $t = 0$. It is defined as the path integral over metrics of Euclidean signature on a 3-manifold $M$ whose boundary is $X$. There are different possible choices of such $M$. Let us for now be schematic, leaving a discussion on the choice of $M$ till later. At this stage one should think of $M$ as being the “interior” of the Riemann surface $X$. We will now utilize the result of [15] which states that the Euclidean path integral over a 3-manifold $M$ whose boundary is $X$ is equal to the CFT partition function on $X$:

$$\int_M D^{(3)}g \ e^{-S_{E^{(3)}g}} = Z_{\text{CFT}}[X].$$

This result is, of course, in the general spirit of AdS$_3$/CFT correspondence, specialized to the case when no fields except for the metric is present. The relation (4.1) was obtained in [15] by using the CS formulation of $\Lambda < 0$ Euclidean 3D gravity. It was shown that the gravity partition function holomorphically factorizes, and thus is a partition function of some CFT. There are reasons to believe [4] that this CFT is related to the quantum Liouville theory. For now we shall refer to this CFT as simply Liouville field theory (LFT), but we note that the relevant CFT is rather a certain close relative of LFT that incorporates the point particle states, see section 7 for a discussion on this point.

We must note that the result (4.1) was obtained in [13] for the case of a closed Riemann surface $X$, while spatial slices that are relevant for our purposes have boundaries (asymptotic regions).
Some extra thought is required to establish an analog of (4.1) for surfaces with boundaries. Let us for the moment analyze the case of a closed $X$. Such spacetimes are simple cosmological models, and were considered in this context in, e.g., [23]. Thus, for the case of a non-rotating spacetime with a closed time symmetry plane $X$ the HH state is given by the Liouville partition function on $X$. Note that $Z_{\text{CFT}}[X]$ only depends non-trivially on the conformal structure of $X$, that is, it is a function on the Teichmuller space $T_X$ of $X$, not on the space of metrics $(2)g$ on $X$.

One can interpret the fact that $\Psi[(2)g]$ only depends on the conformal structure, not on the full metric $(2)g$ as follows. Recall that in the canonical approach to quantum gravity a quantum state $\Psi$ should satisfy the Wheeler-DeWitt equation. In other words, a physical state must be annihilated by the constraints. A way to get such states is to first find the reduced phase space at the classical level, and then consider functions on the reduced configuration space. It is known that the reduced phase space of 2+1 gravity is the cotangent bundle over the Teichmuller space of $X$. In the context of zero cosmological constant this is a well-known result due to Moncrief [24]. In our negative cosmological case one can understand this result as follows. Let us use the CS formulation. Then $\Lambda < 0$ Lorentzian signature gravity is $\text{SL}(2,\mathbb{R}) \times \text{SL}(2,\mathbb{R})$ CS theory. The reduced phase space of CS theory on a manifold of topology $X \times \mathbb{R}$ is the space of homomorphisms $\pi_1(X) \rightarrow \text{SL}(2,\mathbb{R}) \times \text{SL}(2,\mathbb{R})$. This space is parametrized by holonomies of the two connections along the loops generating $\pi_1(X)$. One can introduce complex coordinates on this space by performing the analytic continuation of $\text{SL}(2,\mathbb{R}) \times \text{SL}(2,\mathbb{R})$ to $\text{SL}(2,\mathbb{C})$, in the way we explained in the previous section, and that is described in more detail in [19]. Under this continuation the CS symplectic structure goes to the CS symplectic structure on the space of homomorphisms $\pi_1(X) \rightarrow \text{SL}(2,\mathbb{C})$. This later space is known to be the same as the space of projective structures on $X$, which is the same as the cotangent bundle the Teichmuller space $T_X$, see e.g. [25]. The reduced phase space being $T^*T_X$ explains why it is natural for the quantum states $\Psi$ to be functions on $T_X$.

Having said this, there are two cautionary remarks in order. First, because the partition function is modular invariant it is a function on the moduli (Riemann) space $\mathcal{M}_X$ rather than on $T_X$. Second, the CFT partition function is not really a function, but rather a section of a line bundle over $\mathcal{M}_X$. Indeed, due to the presence of the conformal anomaly $Z_{\text{CFT}}$ depends not only on the moduli, but also on a representative of the metric $(2)g$ on $X$. Under a change of this representative by a conformal rescaling $(2)g \rightarrow e^{\varphi}(2)g$ the partition function gets multiplied by the exponent of the classical Liouville action for $\varphi$. Thus, to make $Z_{\text{CFT}}$ a function on $\mathcal{M}_X$ we have to select a representative $(2)g$ for every point in $\mathcal{M}_X$. A natural representative is the canonical (Poincare) metric of constant negative curvature that exists (and is unique) on every $g > 1$ Riemann surface. When we talk about the partition function as a function on $\mathcal{M}_X$ we will always imply this choice.

It seems to be the right time to discuss the issue of the choice of $M$ that needs to be made to compute the path integral in (4.1). For the case of a closed $X$ that we are presently discussing, there are different possible choices that can be distinguished by specifying which set of loops on $X$ is contractible inside $M$. One needs to specify $g$ such homotopy non-trivial loops on $X$, where $g$ is the genus of $X$. Any two such sets of loops are related by a modular transformation on $X$. On the other hand, the right hand side of (4.1), that is, the CFT partition function, is modular
invariant. This means that the left hand side should be interpreted as containing not only the path integral over metrics on $M$ of a fixed topology, but also a sum over different 3-manifolds $M$ that have the same conformal structure on $X = \partial M$. Such a sum over topologies is by now standard in the setting of AdS/CFT correspondence, see [26] for the first discussion of this issue. Thus, for a closed $X$, a choice of $M$ in (4.1) is not important, one should sum over all possibilities.

Let us now return to the case relevant for our purposes, that is, when Riemann surface $X$ has a boundary. There is one circular boundary component for every asymptotic region. Motivated by the result (4.1) valid for a closed surface $X$, we define an open $X$ HH state to be the CFT partition function on $X$ as well. From the discussion that follows it will become clear why this definition is natural. We shall see, however, that there are some important subtle differences from the closed case.

To evaluate the CFT partition function on $X$ with boundary one has to specify boundary conditions. In the case of rational CFT there is a special subset of boundary conditions, the one considered in the original work by Cardy [27]. These Cardy boundary conditions are in one-to-one correspondence with primaries of the CFT. For a non-compact CFT, as is Liouville theory, one has similar Cardy boundary states, see [28, 29, 30] for a recent discussion of boundary LFT. The LFT boundary states are labelled by a single real number, for which we shall employ lower case Latin letters. Specifying this boundary label for every asymptotic region one gets Riemann surface with labels. The HH state is defined as the Liouville partition function on such $X$. Schematically,

$$\Psi_{\text{HH}}[a, b, c] = Z_{\text{CFT}} \left[ \begin{array}{c} \text{a} \\ \text{b} \\ \text{c} \end{array} \right],$$

(4.2)

where we have used the geometry of the three asymptotic region black hole for illustration. Boundaries of $X$ correspond to asymptotic regions. One should think of the number labelling each boundary as in a certain sense dual to the size of the corresponding horizon, see more on this below.

One can also consider the time symmetry surface geometries containing point particles. The HH states are still given by the CFT partition function on the corresponding Riemann surface. The point particles are incorporated by inserting LFT vertex operators. For example, the HH state of the three asymptotic region black hole with three point particles on the time symmetry slice is given by:

$$\Psi_{\text{HH}}[\xi_1, \xi_2, \xi_3; a, b, c] = Z_{\text{CFT}} \left[ \begin{array}{c} \text{a} \\ \text{b} \\ \text{c} \end{array} \right].$$

(4.3)

The right hand side can either be interpreted as the LFT partition function on a 3-holed sphere with three vertex operators inserted, or as the partition function of the 3-holed sphere with three conical singularities. The parameters $\xi_i$ are those of the LFT vertex operators $e^{\xi_i \phi}$. In the semi-classical limit $b \to 0$, where $b$ is the LFT coupling constant, the relevant parameter is $\eta = \xi b$. The deficit angle created is equal to $4\pi \eta = 2\pi (1 - \alpha)$, where the parameter $\alpha$ is the one introduced for point particles in (2.3).
Actually, one can similarly think of the holes as being made by an insertion into the sphere of non-local LFT vertex operators. While vertex operators creating conical singularities are given by $e^{\xi \phi}$, with $\xi$ real, non-local vertex operators that create holes correspond to $\xi = Q/2 + iP$, where $Q = b + 1/b$, and $P$ is proportional to the size of the hole it makes. Thus, one can obtain the boundary states (4.2) as a linear combination of the correlation functions with non-local operators inserted. This is done by introducing the Ishibashi states. Schematically,

$$\langle a \mid = \int_0^\infty \frac{dP}{\pi} U(P) \langle P \mid,$$

where $\langle P \mid$ is the Ishibashi state (sum over the descendants) constructed from the primary state created by $e^{\xi \phi}$ with $\xi = Q/2 + iP$, and $U(P)$ is the one point function in the boundary LFT, see [28]. The boundary condition $a$ is thus dual (in the above sense) to the size $P$ of the hole.

Let us now introduce another, equivalent definition of the open $X$ HH states. It is one of the defining properties of the boundary CFT, see, e.g., [31], that the full CFT partition function on a surface with boundary equals to the chiral CFT partition function on the Schottky double:

$$Z_{\text{CFT}}[X] = Z_{\text{chiral}}[\tilde{X}].$$

The partition function on the right hand side is evaluated in the presence of holonomies. There is one holonomy for every boundary component. Each holonomy is labelled by the corresponding boundary condition. As we have seen in the previous section, the Schottky double $\tilde{X}$ of the time symmetry plane is the boundary of the Euclidean 3-manifold that is the analytic continuation of the black hole spacetime. Thus, our HH states also have the interpretation of the chiral CFT partition function on the Euclidean boundary. Schematically,

$$\Psi_{\text{HH}}[a, b, c] = Z_{\text{chiral}}.$$ 

As is shown in [13], this chiral CFT is nothing else but the SL(2, C) CS theory for one of the two complex conjugate connections that describe gravity in the Euclidean AdS. The chiral CFT partition function, or, in other words, a conformal block, is a particular quantum state of the CS theory. Thus, the chiral definition (4.6) of HH states makes it clear that they are indeed quantum states in that they can be obtained as the path integral of exponential of $i$ times the SL(2, C) CS action.

The definition (4.6) HH states as chiral CFT partition function on the double $\tilde{X}$ allows us to be more precise on the issue which manifold $M$ to take when evaluating the Euclidean 3d gravity path integral. As we have discussed above, for a closed surface $X$ the result is the full CFT partition function on $X$, which is modular invariant. One sums over all possible $M$’s in the path integral to produce a modular invariant answer. For an open surface $X$ the situation is more complicated. We recall that the chiral CFT partition function, in other words a conformal block, or a state of CS theory, has complicated transformation properties with respect to modular transformations. It can be thought of as a section of a very non-trivial holomorphic fiber bundle over the moduli space $M_{\tilde{X}}$. It is best, however, to think of it as a section of a (trivial) holomorphic fiber bundle
over the Teichmuller space $T_{\tilde{X}}$. Note that it is important where the holonomies are located on $\tilde{X}$. Choosing a different set of holonomies for boundary conditions, or a different labelling of them changes the HH state (4.6). Thus, HH states depend explicitly on a marking of $\tilde{X}$ with a set of curves that correspond to asymptotic regions. On the other hand (4.2) is invariant under the modular transformations that leave the boundary of $X$ (that is, asymptotic regions) invariant.

Having defined $\Psi_{\text{HH}}$ as the chiral CFT partition function on the double $\tilde{X}$, we have $\Psi_{\text{HH}}$ equal to the SL(2, C) CS theory partition function on a specific 3D manifold $\tilde{M}$, whose boundary is $\tilde{X}$. The manifold $\tilde{M}$ is just the analytic continuation of the black hole spacetime, that is the space $H^3/\Gamma$. Since the Fuchsian group $\Gamma$ is marked, this is a specific 3D space, with generators of $\Gamma$ corresponding to asymptotic regions non-contractible inside $\tilde{M}$.

Another useful interpretation of HH states that results from (4.6) is as follows. As is shown in [15], the phase space of each of the two SL(2, C) CS theories is the holomorphic (or anti-holomorphic) part of the full phase space, which is the cotangent bundle over the Teichmuller space of $\tilde{X}$. This holomorphic part of $T^*T_{\tilde{X}}$ is itself naturally isomorphic to $T_{\tilde{X}}$. Thus, our HH states can also be thought as particular states in the Hilbert space $\mathcal{H}$ obtained by quantizing the Teichmuller space of $\tilde{X}$. In other words, $\Psi_{\text{HH}}$ are particular holomorphic functions of the moduli of $\tilde{X}$. Different choice of boundary conditions (or marking of $\tilde{X}$) give different states in the same Hilbert space $\mathcal{H}$. Thus, HH states of the Lorentzian signature theory are particular states in the Hilbert space obtained by quantizing the Teichmuller space of the Euclidean boundary $\tilde{X}$. This makes the theory of quantum Teichmuller spaces developed in [32, 33] directly relevant to quantum gravity in (2+1) dimensions.

This last interpretation allows for an immediate generalization of HH states to the rotating case. On the first sight this seems to be impossible, for there is no time symmetry plane in this case. However, as we saw above, the non-rotating HH states can also be interpreted as states in the Hilbert space $\mathcal{H}$ of the quantum Teichmuller space of the Schottky double $\tilde{X}$. Recall now that the rotation can be incorporated as the Fenchel-Nielsen twist on $\tilde{X}$. One gets a new Riemann surface $\tilde{X}^\tau$ of the same genus, but with different values of the moduli. Thus, the same Hilbert space $\mathcal{H}$ obtained by quantizing the Teichmuller space $T_{\tilde{X}}$ also contains states that correspond to rotating spacetimes. They are just different holomorphic functions of the moduli of $\tilde{X}$. One should be able to obtain these states by evaluating the chiral CFT partition function (4.5) in the presence of more general holonomies. It can be expected that the general holonomy (and thus a boundary condition) in Liouville CFT is characterized by a single complex number. Such more general boundary condition is dual to a more general vertex operator $e^{a\phi}$ with $a$ not necessarily of the form $a = Q/2 + iP$. The conformal dimension for states created by such operators is complex. The imaginary part of this conformal dimension is the angular momentum of the corresponding asymptotic region. We will not need details on the rotating states in the discussion that follows. It suffices to know that they exist.
5 Quantum amplitudes for processes involving point particles

Having understood how the HH states are constructed, we can use them to study simple physical processes. There are two main processes of interest: (i) emission (absorption) of a point particle by the BTZ black hole, and (ii) creation of a black hole by two point particles. In this section we explain how the quantum probabilities for these processes can be obtained. We will be rather schematic and give no detailed calculations. Details on the BH creation process are given in the companion paper [20].

5.1 Emission of a point particle by the BTZ BH

Black holes emit particles. Having point particles in our theory, we expect that there is an emission process in which a point particle gets created in the vicinity of the horizon and propagates away from the hole. Only massless particles can reach infinity in AdS. Thus, there is a strict version of the Hawking process only for lightlike particles. Massive particles are also expected to be created, but they will eventually fall back into the black hole. There is a question how to describe these processes, for there is a conceptual difference from the original Hawking [34] setup for black hole evaporation. Indeed, there are no extra fields present to carry away radiation. Thus, one cannot neither quantize fields in the black hole background, which is the original Hawking derivation [34], nor consider the hole as a quantum mechanical system in external fields, see, e.g., Bekenstein [35]. A derivation that is applicable to our case is that of [36], which treats the emission process as quantum tunneling. In this approach the analysis is performed in the geometric optics approximation, in which radiation travels along (null) geodesics. This is directly applicable to our case with quanta traveling along geodesics being point particles. For the AdS black holes the analysis was performed in [37], see also [38] for a derivation for the case of BTZ BH. These results tell us that our black holes are indeed expected to radiate point particles.

To describe this emission qualitatively we need to find a HH wave function whose amplitude squared will give the probability for the process. According to our prescription, a HH state is the LFT partition function on the relevant spatial slice. For the case of emission, this slice has the geometry of the infinite throat, with a point particle in one of the asymptotic regions. The relevant HH state is thus the LFT partition function on a two-holed sphere with a single vertex operator inserted. Each of the two boundaries corresponding to the asymptotic regions must be assigned a boundary condition. The two boundary conditions need not be the same. Thus, we get a state:

$$\Psi_{\text{HH}}[a, b; \eta] = Z_{\text{CFT}} \left[ \begin{array}{c} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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taking the absolute value square of the amplitudes (5.1) and summing over the boundary conditions. The sum over boundary conditions glues two copies of the cylinder together to produce a torus. The full emission probability is then given by the CFT partition function on the torus with two insertions:

$$\mathcal{P} = \sum_{a,b} Z_{\text{CFT}} \left[ \begin{array}{c} s \cr \eta \cr s \end{array} \right] \overline{Z_{\text{CFT}} \left[ \begin{array}{c} s \cr \eta \cr s \end{array} \right]} = Z_{\text{CFT}} \left[ \begin{array}{c} \eta \cr \eta \cr \eta \end{array} \right].$$  \hfill (5.2)

The probability $\mathcal{P}$ is a function of the modulus of the torus, which codes the parameters of the black hole, of the mass parameter $\eta$, and of the relative position of the two insertion points, which codes particle’s momentum.

To see that this is indeed the “right” answer for the emission/absorption probability we recall some facts about the absorption of a scalar field by the BTZ black hole. The point is that in the limit of large frequency, this absorption probability is given by essentially the same two-point function on the torus. In this limit the geometric optics approximation is valid, which means that absorption for waves is the same as absorption for particles. The fact that the absorption probability is given by the two-point function means that our answer (5.2) is essentially correct.

Absorption probability for scalar (and other) fields propagating in the background of the BTZ BH can be found exactly. This is usually discussed for massless field, when the field can reach the infinity, see for results on the minimally coupled massless scalar $[39]$. For massive particles the absorption cross-section can be read off from the AdS/CFT prediction for the emission rate, see $[40]$. Importantly, in all the cases the absorption probability can be expressed as a certain integral of the thermal two point function in the CFT. More precisely, as was demonstrated by $[41, 42]$ the absorption probability for a wave with momentum along the boundary equal to $p = (\omega, p)$ is proportional to

$$\int d^2 x \ e^{ip \cdot x} G(t - i\epsilon, x),$$ \hfill (5.3)

where $p \cdot x = wt - px$ and the integral is taken over the location of the insertion of one of the operators in the two-point function. The thermal two-point function is given by

$$G(t, x) = \langle \mathcal{O}(t, x) \mathcal{O}(0, 0) \rangle_{T_H} = C \left( \frac{\pi T_+}{\sinh \pi T_+ x_+} \right)^{2h_+} \left( \frac{\pi T_-}{\sinh \pi T_- x_-} \right)^{2h_-}.$$ \hfill (5.4)

Here $C$ is a normalization constant, unimportant for us, and $x_\pm = t \pm x$ are the usual null coordinates. The left and right temperatures are related to $T_H$ as:

$$\frac{2}{T_H} = \frac{1}{T_+} + \frac{1}{T_-}.$$ \hfill (5.5)

The conformal dimension is, for a massive field of mass $m$

$$h_+ = \frac{1}{2} (1 + \sqrt{1 + m^2}).$$ \hfill (5.6)

The integral (5.3) gives

$$\sinh (\beta \cdot p) \frac{(2\pi T_+)^{2h_+ - 1} (2\pi T_-)^{2h_- - 1}}{\Gamma(2h_+)} \left| \Gamma \left( h_+ + i \frac{p_+ \beta_+}{4\pi} \right) \Gamma \left( h_+ + i \frac{p_- \beta_-}{4\pi} \right) \right|^2.$$ \hfill (5.7)
Here $\beta_{\pm} = 1/T_{\pm}$, and $p_{\pm} = \omega \pm p$. This matches the absorption probability calculated from the wave equation in the black hole background. For large frequencies (large $p$), the integral (5.3) is dominated by $x_{\pm} \sim 1/p_{\pm}$. Thus, the large frequency absorption probability is given by just the two point function. Thus, our answer (5.2) is qualitatively correct, at least for large frequencies.

5.2 Black hole creation by two point particles

Let us now discuss the black hole creation process. We will be schematic. A detailed analysis is presented in the companion paper [20].

The classical process of black hole creation by two point particles was described by Matschull [3]. The case of non-zero impact parameter is analyzed in [4]. The case of massive particles is analyzed in [43].

Let us consider the simplest case of a head-on collision of two massive particles of equal mass $M/\pi = 1 - \alpha^2$. The elliptic generators describing particles can be taken, for instance, to be:

$$A_1 = e^{s(\gamma_0 + \kappa\gamma_2)}, \quad A_2 = e^{s(\gamma_0 - \kappa\gamma_2)}. \quad (5.8)$$

Here $\kappa < 1$ is the boost parameter and $s$ is related to the identification angle via:

$$\frac{1}{2}\text{Tr}(A_{1,2}) = \cos(s\sqrt{1 - \kappa^2}) = \cos \pi \alpha. \quad (5.9)$$

The trace of the product of these two generators plays the role of the order parameter for the black hole creation. It is given by:

$$\frac{1}{2}\text{Tr}(A_1A_2) = 1 - 2\frac{\sin^2(\alpha)}{1 - \kappa^2}. \quad (5.10)$$

Massless particles correspond to the limit $\kappa \to 1$. In this limit the order parameter (5.10) becomes $1 - 2s^2$. A black hole is created when $s \geq 1$, see [3]. This is consistent with one’s intuition. Indeed, in order to create a black hole out of massless particles the particles must be boosted to some minimal momentum. At this threshold a zero mass black hole is created.

Another important case to consider is that of two particles whose combined mass is enough to create a black hole for any value of the boost parameter. As is not hard to see from (5.10), this corresponds to $\alpha = \pi/2$. Then there is a black hole created for arbitrary small values of the boost $\kappa$.

Let us now indicate how all this can be described in the quantum theory. As in the previous subsection, we would like to find a HH state whose norm squared would give the probability for the process. The classical process described above corresponds to creation of a single asymptotic region black hole. Thus, the relevant spatial slice geometry is that of a disc with two operator insertions. The HH state is then, schematically

$$\Psi_{\text{HH}}[a; \eta, \eta] = Z_{\text{CFT}} \left[ \begin{array}{c} \eta \rightarrow \alpha \rightarrow \eta \end{array} \right]. \quad (5.11)$$
This wave-function depends on the boundary condition $a$, on the rest mass of the particles $\eta$, and on the relative position of the insertion points, which encodes the particle’s relative momentum. A precise relation between the position and momentum is given in the companion paper [20].

To obtain the BH creation probability let us take $|\Psi_{HH}|^2$ and sum over all possible boundary conditions. Summing over boundary conditions is equivalent to “erasing” them, and the result is the 4-point function on the sphere:

$$\sum_a |\Psi[a; \eta, \eta]|^2 = Z_{LFT} \begin{bmatrix} \eta & \eta & \eta & \eta \end{bmatrix}.$$ (5.12)

This quantity has the interpretation of the probability of two point particles colliding and forming a BH. To get the probability of creation of a particular size BH we have to project the 4-point function on some intermediate state. We have, schematically,

$$Z_{LFT} \begin{bmatrix} \eta & \eta & \eta & \eta \end{bmatrix} = \sum_P \eta \cdots P \cdots \eta.$$ (5.13)

The sum here is actually an integral, see [20]. Each term in the sum (5.13) has the interpretation of the probability of creation of a BH of a particular size determined by the label $P$.

As is explained in [20], in the semi-classical limit of small AdS curvatures, the 4-point function is dominated by the exponential of the classical Liouville action. Then, for a large BH created, the Liouville action is proportional to the BH size. One thus obtains an exponentially small answer for the probability. As is explained in the companion paper [20] the total probability of creating a horizon is given by a sum of probabilities of all possible topologies inside the horizon, the process (5.11) giving the simplest topology. It is argued in [20] that the total probability is close to unity.

## 6 Inner product and the S-matrix

In this section we introduce an inner product in the space of HH states, and define the point particle S-matrix. This leads to a stringy interpretation of our formalism.

The state (4.2) can be interpreted as the amplitude for a black hole with three asymptotic regions of size $a, b, c$. The state (4.3) can be similarly interpreted as the amplitude for three asymptotic region black hole with three point particles. One can imagine a process in which one configuration goes into the other. The amplitude for such a process should be given by some inner product. There is a natural inner product that can be constructed for HH states under consideration. One should simply glue two Riemann surfaces with boundary to form a closed surface, and consider the full CFT partition function on it. To illustrate this, let us consider two states. The first one $\Psi_1$ is the state of three asymptotic region black hole with three point particles. The second state $\Psi_2$ is that of the three asymptotic region black hole with a handle inside and two point particles:

$$\Psi_1 = Z_{CFT} \begin{bmatrix} \eta & b & c \\ a & \eta \\ b & c \end{bmatrix}, \quad \Psi_2 = Z_{CFT} \begin{bmatrix} \eta & b & c \\ a & \eta \\ b & c \end{bmatrix}.$$ (6.1)
Their inner product is defined as
\[ \langle \Psi_1 | \Psi_2 \rangle := Z_{\text{CFT}} \begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix}. \] (6.2)

It should be interpreted as the amplitude for the handle inside the horizons plus two particles be converted into three point particles. This process happens at fixed boundary conditions for all three asymptotic regions. Of course, this particular amplitude may be zero, but (6.2) illustrates the general principle.

Let us now consider a different problem. Assume that one is interested in the total amplitude for this process, for all possible values of the boundary conditions at infinity. This more general amplitude can be obtained by summing over the boundary conditions. Such a sum over a complete set is equivalent to having no boundary conditions at all. Thus, we have:
\[ \sum_{a,b,c} Z_{\text{CFT}} \begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix} = Z_{\text{CFT}} \begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix}. \] (6.3)

This is the total amplitude for two particles to collide with a handle and produce three particles. In this amplitude one only fixes the mass of the particles and moduli of the surface. One can consider an even more general amplitude where only the masses are fixed. This is obtained by integrating over the insertion positions, and over the surface moduli:
\[ \langle \alpha_1, \ldots, \alpha_k | \alpha_{k+1}, \ldots, \alpha_n \rangle_g = \int dmXd\bar{m}_X Z_{\text{CFT}} \begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix}. \] (6.4)

Here \( \alpha_1, \ldots, \alpha_k \) and \( \alpha_{k+1}, \ldots, \alpha_n \) are parameters of the incoming and outgoing particles, \( V_\alpha \) are the corresponding vertex operators. This amplitude only depends on the genus of \( X \) and on the particle masses. To form the full S-matrix for the particles one should in addition sum over the genus. Different genera can be differently weighted by the “string coupling constant” \( g_s \). The full S-matrix for particles is then given by the following expression:
\[ \langle \alpha_1, \ldots, \alpha_k | \alpha_{k+1}, \ldots, \alpha_n \rangle = \frac{1}{g_s^2} \sum g_s^2 \int dmXd\bar{m}_X Z_{\text{CFT}} \begin{bmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{bmatrix}. \] (6.5)

This is the usual string theory expression for the S-matrix. An important difference, however, is that the CFT we are working with is non-critical. Our asymptotic states describing point particles are also different from the usual asymptotic states of string theory.

## 7 Remarks on unitarity and BH entropy

Up to this point we have assumed that the CFT in question is Liouville theory. As we shall now see, this is not quite right: the “correct” CFT is a certain close relative of LFT that incorporates the point particle states.

\footnote{More precisely, one sums over all possible intermediate states, thus gluing two Riemann surfaces together. This is what we mean by a sum over boundary conditions.}
We start by reminding the reader that the Liouville field theory is not capable of reproducing the
entropy of BTZ black holes. This important point was emphasized, e.g., in \cite{45,46}. This is because
the effective central charge of Liouville field theory is equal to one, and, in Strominger’s derivation
\cite{17} of BTZ BH entropy, it is the effective central charge that has to be used in Cardy’s formula
for the density of states, see \cite{13} for a discussion on this. This raised doubts, see \cite{16}, on whether
the quantum theory of pure gravity in 2+1 dimensions is capable of reproducing the BH entropy
microscopically. The string theory on AdS$_3$, on the other hand, does not suffer from this problem,
because the relevant CFT, which is the one coming from the D1/D5 system, contains many fields,
and its effective central charge is large and equal to the Brown-Henneaux value. This seeming
incapability of pure gravity in 2+1 dimensions to account for the BH entropy microscopically
lead \cite{16} to the conclusion that “gravity is thermodynamics” and one does not get anywhere by
attempting to quantize it: the correct microscopic theory is string theory. This is a viewpoint
currently shared by most of the string community.

A related point is as follows. It can sometimes be heard that LFT is non-unitary. This statement
is not quite right as it stands, what is actually meant by this is that LFT cannot contain non-
normalizable (conical singularity) states if one wants to have a unitary theory. This conclusion
follows from the well-known fact that CFT correlators contain only the normalizable states from the
continuous spectrum when decomposed into intermediate states. This is true even for correlation
functions of vertex operators creating the non-normalizable (conical singularity) states. Thus,
allowing for the non-normalizable point particle states one gets a non-unitary theory. Indeed, it
is a theory in which one has both normalizable and non-normalizable states as external, but only
normalizable states appear as the internal states. The unitary LFT is a consistent truncation that
only allows the normalizable states: both as external and internal. The effective central charge
of this unitary theory is equal to one; it thus does not have enough states to account for the BH
entropy microscopically.

Thus, the only consistent theory seems to be the one with no particle (non-normalizable) states.
There is something unsatisfactory with this picture, however. Indeed, LFT gives definite answers
even for correlators of the non-normalizable states. These correlators definitely make sense, and
are of great interest, for example, for the problem of accessory parameters for uniformization, and
also for understanding of the symplectic geometry of the moduli space of punctured surfaces, see
\cite{48}. It is thus quite unfortunate if it is not possible to make sense of LFT with non-normalizable
states.

We have many reasons to believe that this is not so. In other words, there are reasons to believe
that there exists another CFT, very closely related to the usual Liouville theory, which consistently
incorporates the LFT non-normalizable states. Semi-classically, this other CFT is related to the
classical geometry of Riemann surfaces, as is the usual LFT. However, the correlation functions
of this theory, when decomposed into intermediate states, contain not only the states from the
continuum spectrum, but also the point particle (non-normalizable) states. For example, the 4-
point function of point particles vertex operators contains, in addition to the usual continuous part
What we have here is a product of two 3-point functions, and the sum is taken over a set of intermediate particle states. There are reasons to believe that the set of states that appears is discrete. The quantization of the particle spectrum comes as the requirement of rationality of the deficit angle. It is expected that states with rational deficit angle can be made normalizable by choosing the inner product appropriately. We shall refer to the point particle states that appear in (7.1) as states of the discrete spectrum.

The reasons to believe the picture outlined are few. We shall mention just a couple of them. First, it is known that the usual LFT is intimately related to (essentially gives) the quantization of the moduli space of compact Riemann surfaces. One can expect a similar quantum theory to exist for the moduli spaces of surfaces with conical singularities. The quantization of this more general moduli space will give conformal blocks of the CFT in question. A reason to expect an appearance of the discrete part in the spectrum is an analogy with the representation theory of SL(2, R). As is well-known, there are unitary irreducible representations of two main types: the continuous and discrete series. One can similarly expect the appearance of these two series of representations in the CFT. In fact, this is much more than an analogy, for the theory of quantum Teichmüller spaces, see [32, 33], is based on the representation theory of the quantum group SL_{q}(2, R). The usual Liouville theory is based on a certain continuous series of representations of SL_{q}(2, R), see [49]. One can expect that there is another CFT based on both the continuous and discrete series (continuous representations possibly of a different series).

We have reasons to believe that the CFT whose structure we outlined is a certain theory based on SL(2, C) WZW model. This theory was discussed in [15]. It was shown in this reference that it is this theory, not the usual LFT, that arises from the Euclidean path integral in Λ < 0 gravity in 3 dimensions. To explain what that theory is, and what is its difference with the usual LFT, we recall that there are two natural types of structures that can be given a Riemann surface. These are: (i) a conformal structure; (ii) a projective structure. A conformal structure can be parametrized by prescribing a map from the Riemann surface X in question to the upper half-plane U. Such a map is exactly what is used to obtain the so-called operator formalism for LFT, see, e.g., [51]. The Liouville theory can be thought of as arising by considering the path integral over such maps. A projective structure, on the other hand, arises by giving a set of charts of complex coordinates on X, with transition functions given by fractional linear transformations. In other words, a projective structure is a map from X to the complex plane. One can consider the path integral over such maps (projective structures). This defines a theory that is different from, but is closely related to the usual LFT. We have reasons to believe that this theory is essentially the LFT with point particle states added to the spectrum. An attempt to demonstrate this here would take us too far. We hope to return to this issue in the future.

Thus, in this and in the companion paper we have to ask the reader to assume that there exists a CFT that is essentially LFT with the point particle states added to the spectrum. This theory
would solve the above-mentioned problem with BH entropy. Indeed, this CFT would have zero conformal dimension of the lowest lying state. Thus, its effective central charge would be equal to the LFT value \( c = 1 + 6Q^2 \). Therefore, at least naively, by adding point particles to the theory one does get enough degrees of freedom to account for the entropy.

As is clear, for example, from (7.1), the role of point particle states is to bring disconnected world-sheets into the game. Thus, our proposal for obtaining a large central charge CFT can be rephrased by saying that the central charge comes via the disconnected world-sheets. We note that this is essentially the same mechanism for obtaining a large central charge as the one advocated in [50]. It was argued in this reference that in the string theory on \( \text{AdS}_3 \times S^3 \times T^4 \) a large central charge comes by considering the second quantized theory, which includes disconnected world-sheets. This is very similar to the mechanism proposed in this paper. This analogy at the very least suggests a natural name for the CFT conjectured. We propose to call it the second quantized Liouville field theory.

Let us summarize. The usual LFT does not account for the BH entropy microscopically. However, one can conjecture the existence of another, closely related CFT, which contains the point particle (non-normalizable) states in the spectrum, and which thus must be capable of reproducing the entropy. In the picture suggested the point particle states are necessary to explain entropy microscopically. This is consistent with the fact that black holes can be made out of point particles, and can evaporate into them. Thus, point particles seem to be the constituents that black holes are made of.

The CFT arising as the path integral over the projective structures remains to be studied better, in order to demonstrate that it is indeed LFT with the point particle states added to the spectrum. However, there are many physically interesting situations in which the particle states only appear as external. Then the usual Liouville theory must be sufficient for calculations, at least in the semi-classical regime when correlators are dominated by the classical Liouville action. An example of a calculation along these lines is given in the companion paper [20].

8 Conclusions

We have proposed a formalism in which amplitudes for physical processes involving point particles can be computed as the quantum LFT partition function on Riemann surfaces. We have sketched how this formalism can be applied to simple physical processes such as a particle emission by the BTZ black hole, and the black hole creation process. A much more detailed analysis of the BH production process is presented in the companion paper [20].

Our proposal to use Riemann surfaces to do calculations in Lorentzian signature quantum theory can be understood as a version of analytic continuation. Indeed, it is only in the non-rotating case that one can use the time symmetry surface to define HH states. When there is no time symmetry plane, as is the case of spacetimes with particles of non-zero momentum, or rotating spacetimes, one defines HH states using the double \( \tilde{X} \). The surface \( \tilde{X} \) naturally arises as the boundary of the Euclidean 3-manifold that is the analytic continuation of the spacetime.
Our usage of Riemann surfaces leads us to the string theory interpretation of the formalism. We have argued that transition (scattering) amplitudes for point particles are given by CFT correlators on closed Riemann surfaces. Sum over all such amplitudes gives the S-matrix for particles. This S-matrix is that of a non-critical string theory, whose world-sheet CFT is essentially the Liouville field theory, see, however, remarks in the previous section. Thus, this string theory is essentially 2D gravity. Thus, somewhat surprisingly, we obtain that the hologram of the gravity theory in 2+1 dimensions is also gravity, but in one dimension less. This is somewhat reminiscent of brane world ideas. An important difference, however, is that in the brane worlds scenarios the theory on the brane is always more than pure gravity: one has, in particular, gauge fields in addition to the metric.

Our construction have implications not just for the pure gravity considered here, but also for other gravitational theories that include AdS$_3$ as part of the background. Indeed, any such theory will have non-trivial topology black holes and point particles as its classical solutions. Since topologically non-trivial black holes seem to force one to consider higher genus Riemann surfaces, this leads to holographic theory being a string theory. These ideas applied to the usual string theory AdS$_3$/CFT setting lead to a puzzle, for they seem to suggest that one string theory in bulk is dual to another string theory on the boundary. It would be of interest to find whether this interpretation is correct, and, in case it is, obtain a relation between the two string theories.

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