Reasonable multi-item mechanisms are not much better than item pricing

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Abstract

Multi-item mechanisms can be very complex offering many different bundles to the buyer that could even be randomized. Such complexity is thought to be necessary as the revenue gaps between randomized and deterministic mechanisms, or deterministic and simple mechanisms are huge even for additive valuations.

We challenge this conventional belief by showing that these large gaps can only happen in unrealistic situations. These are situations where the mechanism overcharges a buyer for a bundle while selling individual items at much lower prices. Arguably this is impractical as the buyer can break his order into smaller pieces paying a much lower price overall. Our main result is that if the buyer is allowed to purchase as many (randomized) bundles as he pleases, the revenue of any multi-item mechanism is at most $O(\log n)$ times the revenue achievable by item pricing, where $n$ is the number of items. This holds in the most general setting possible, with an arbitrarily correlated distribution of buyer types and arbitrary valuations.

We also show that this result is tight in a very strong sense. Any family of mechanisms of subexponential description complexity cannot achieve better than logarithmic approximation even against the best deterministic mechanism and even for additive valuations. In contrast, item pricing that has linear description complexity matches this bound against randomized mechanisms.

1 Introduction

It is well known that revenue optimal mechanisms can be complicated: when a seller has more than one item to sell to a buyer, the seller may price bundles of items rather than just the individual items; better still the seller may offer random subsets of items, also called lotteries; even more troublingly, the seller may offer an infinitely large menu of such options. A primary line of enquiry within algorithmic mechanism design aims to establish simplicity versus optimality tradeoffs: is it possible to obtain some fraction of the optimal revenue via simple mechanisms, i.e. mechanisms that can be described easily, understood easily, and are prevalent in practice?

To quantify this tradeoff let us introduce some standard notation. Let $\text{Rev}$ denote the optimal revenue obtained by using the most general kind of mechanism—a menu of lottery pricings. Let $\text{DRev}$ denote the optimal revenue achieved by a deterministic mechanism—a menu of bundle pricings. Two “simple” mechanisms have been studied extensively in literature: item pricings, where each item is assigned a price and the buyer can buy any subset at the sum of the constituent prices; and bundle pricings, where every bundle is sold at a single constant price. The optimal
revenues achievable by item pricings and bundle pricings are denoted SRev and BRev respectively. The goal is then to bound the ratios REV/DREV, or REV/\text{max}(SRev, BRev), etc.

Briest et al. [5] were the first to show that revenue optimal mechanism design exhibits a particularly diabolical curse of dimensionality: whereas REV/SRev is equal to 1 when the seller has only one item to sell, the ratio becomes infinite with three items even if the buyer is unit demand. Hart and Nisan [15] improved this to show that the ratio is infinite already with just two items. In fact, if the buyer has additive values over two items, meaning that the value of the bundle of two items is the sum over the individual item values, the ratio of REV/DREV can also be infinite. Even focusing on just deterministic mechanisms, the situation is not much better: the ratio of DREV to SRev can be as large as \Omega(2^n/n) with n items and additive values. Daskalakis in his 2014 survey [11] summarized the situation as such: “Multi-item auctions defy intuition.”

Further work along these lines suggested that the existence of a good tradeoff depends on properties of the buyer’s valuation – both the structure of the value function as well as the distribution from which it is drawn. For example, two parallel lines of work investigating unit demand buyers [8, 10] and additive valuation buyers [14, 16, 1] respectively established that when the buyer’s values for different items are drawn independently the larger of SRev and BRev is within a constant factor of REV. These results extend to more general settings with subadditive valuations [18, 7] as well as multiple buyers [9, 20, 7, 6], but continue to require some degree of independence across individual item values.2

Our work presents an alternate view of the simplicity versus optimality tradeoff. Our thesis is that if mechanisms are required to be “reasonable”, the curse of dimensionality nearly disappears. What do we mean by reasonable mechanisms? Let us illustrate through an example adapted from Hart and Nisan [15]. Consider a seller with n items and a single buyer with additive values. Order all of the 2^n - 1 subsets of the items in weakly increasing order of size, and let S_i denote the i-th set in this ordering. The buyer’s value function is picked randomly from a set of 2^n - 1 different types. The i-th type is realized with probability proportional to n^{-i} and values every item in the set S_i at n^i/|S_i| and every item not in S_i at 0. It is now straightforward to see that a mechanism \mathcal{M} that offers the set S_i at a price of n^{i-1} extracts as revenue a 1/n fraction of the buyer’s total expected value, or \Omega(2^n/n)—for every i, the type i buyer extracts the most utility by purchasing the set S_i. On the other hand, the buyer’s expected value is distributed according to the equal revenue distribution, and so any bundle pricing extracts at most O(1) revenue and any item pricing extracts at most O(n). The implication is that optimal deterministic mechanisms can obtain exponentially larger revenue than simple ones.

We argue that \mathcal{M} is not a realistic mechanism. The single-buyer mechanism design problem is a convenient abstraction for settings with unlimited supply and multiple i. i. d. buyers. Upon finding a (near-)optimal mechanism for the single-buyer setting, we can apply the mechanism as-is in the latter setting, once for each buyer. However, in that context, mechanisms such as \mathcal{M} offer buyers opportunities for arbitrage. Consider, in particular, a set S_i of size 2. \mathcal{M} sells both of the items in S_i individually at a price no more than a 1/n fraction of the price of S_i. A buyer of type i can then participate in the mechanism twice, purchasing the constituents of S_i individually and paying far less to the mechanism than before. In effect, \mathcal{M} is not Sybil-proof.

1Note that for unit demand buyers DREV=SREV.

2[10], [18], and [4] allow item values to be correlated by being defined as linear functions over a common set of random variables, but the latter is required to be independent. Psomas et al. [17] perform an investigation of smoothed complexity for revenue optimal mechanism design over correlated value distributions, but their results are largely negative.
Single-buyer mechanisms can without loss of generality be described as menus of options where each option is a random allocation (a.k.a. lottery) paired with a price. We consider a mechanism unreasonable if the buyer is disallowed from purchasing more than one menu option. We say that a mechanism is Sybil-proof if the buyer is allowed to purchase any multi-set of menu options (of arbitrary size).\textsuperscript{3} For deterministic menus, Sybil-proofness simply means that the prices assigned to different menu options are subadditive over subsets of items. As a corollary, item pricing and bundle pricing are already Sybil-proof. For general mechanisms, on the other hand, imposing the constraint of Sybil-proofness greatly limits the extent to which mechanisms can price discriminate between buyers of different types and, in particular, disallows the gap examples of Briest et al. and Hart and Nisan.

We can now again ask whether arbitrary, complicated Sybil-proof mechanisms can obtain unboundedly larger revenue (or, even exponentially or polynomially larger revenue, with respect to the dimension \( n \)) relative to simple mechanisms. Our main result is that they cannot:

**Theorem 1.1.** For any arbitrary distribution over arbitrary valuation functions,

\[
\frac{\text{SybilProofRev}}{\text{SRev}} = O(\log n).
\]

Here \( \text{SybilProofRev} \) denotes the optimal revenue achievable through (potentially randomized) Sybil-proof mechanisms. Briest et al. [5] previously studied the gap between Sybil-proof mechanisms\textsuperscript{4} and item pricings for the special case of unit-demand buyers and proved the same upper bound. Our main contribution is to extend this result to arbitrary valuation functions and distributions over valuations. Indeed we make no assumptions whatsoever on the buyer’s valuations other than that they are monotone non-decreasing in the set of items allocated—allocating extra items to the buyer never lowers his value.

Can we do even better? We show that the above result is tight in a very strong sense:

**Theorem 1.2.** There exists a distribution over additive values for which no mechanism with a sub-exponential description complexity can obtain a \( o(\log n) \) fraction of the optimal deterministic Sybil-proof revenue.

Observe that item pricings can be described using \( O(n \log R) \) bits when values lie in the range \([1, R]\). We construct a distribution over additive values with \( R < n \) and a deterministic subadditive pricing such that no mechanism that can be represented using \( 2^{o(n^{1/6})} \) bits can obtain a \( o(\log n) \) fraction of the revenue of the subadditive pricing. We further show that for single-minded buyers, the gap cannot be improved in general even if the pricing we are comparing against is a submodular function. Briest et al. previously showed a weaker result for unit-demand buyers, namely that the ratio \( \frac{\text{SybilProofRev}}{\text{SRev}} \) can be \( \Omega(\log n) \).

**Menu size.** Our results also have implications for the menu size complexity of optimal auctions. The menu size of an auction, defined as the number of different outcomes the seller offers to the buyer, has been studied extensively in literature as a measure of complexity for single-buyer

\textsuperscript{3}When the menu contains lotteries, there is a slight distinction between whether the buyer can select a multi-set of options adaptively depending on outcomes of previous lotteries, or non-adaptively. Our results apply to both settings.

\textsuperscript{4}Briest et al. used the term “buy-many setting” for what we call Sybil-proof mechanisms.
mechanisms (see, e.g., [15, 12, 2, 13]). One criticism of this notion of menu size is that some mechanisms can be described much more succinctly than indicated by their menu size, as is the case for item pricing. Hart and Nisan [15] introduced the alternate concept of “additive menu size”, namely the number of “basic” options a buy-many mechanism offers, and showed that even mechanisms with small additive menu size cannot capture a good fraction of the optimal revenue. Our results show that allowing the buyer to purchase multiple options doesn’t just allow a more succinct description of the mechanism, it also fundamentally changes the set of mechanisms available to the seller. In particular, buy-many mechanisms with even infinite additive menu size cannot capture any finite fraction of the overall optimal revenue $\text{Rev}$, as their revenue is bounded by $\text{SybilProofRev}$.

Our work calls for a new investigation of additive menu-size complexity of Sybil-proof mechanisms. A natural question, for example, is whether one can always obtain a constant fraction of the optimal Sybil-proof revenue via mechanisms with finite additive menu size. If so, what menu size is necessary to obtain a $1 - \epsilon$ fraction of the optimal Sybil-proof revenue? Does it matter whether the buyer can select multiple options adaptively or non-adaptively? We leave these questions to future work.

**Demand distributions and a reformulation.** Our lower bounds are constructed essentially by viewing the problem of approximating the revenue of one pricing function using another in the following manner. Let $f : 2^n \rightarrow \mathbb{R}$ be the subadditive function we are trying to approximate. Let $\Pi$ denote the “demand distribution”, or the distribution over sets that are bought when the function $f$ is offered as a price menu to the buyer. Then, the expected revenue of $f$ can be written as

$$E_{S \sim \Pi}[f(S)]$$

(1)

Let $g : 2^n \rightarrow \mathbb{R}$ be another pricing function that we are trying to construct. Then, without any further information about the buyer’s value distribution (and ignoring incentives for a moment), we may conclude that for any set $S$ with $g(S) \leq f(S)$, $g$ obtains a revenue of $g(S)$ from buyers that purchase $S$ under $f$, and for any set $S$ with $g(S) > f(S)$, $g$ obtains a revenue of 0. Therefore, a conservative estimate for the revenue obtained by $g$ is:

$$E_{S \sim \Pi}[g(S)1_{g(S) \leq f(S)}]$$

(2)

In fact, there is a distribution over single-minded buyers for which the expected revenues of $f$ and $g$ are exactly equal to the expressions above: we pick a set $S$ from the distribution $\Pi$ and assign a value of $f(S)$ to every superset of $S$, including itself, and a value of 0 everywhere else.

Now we may ask, for a given function $f$, does there exist a simple $g$ for which the expression in (2) approximates the expression in (1)? As we discussed above, the answer is yes with a gap of $O(\log n)$, but we cannot hope for a smaller gap in general.

Can we obtain a better bound in special cases? We show for single-minded buyers that if the demand distribution $\Pi$ is a product distribution then the better of item pricing and bundle pricing obtains a constant factor approximation to the revenue of $f$. Our approach for proving this result is very similar to that of Rubinstein and Weinberg [18] who show a constant upper bound on the ratio of $\text{Rev}$ to $\max(\text{SRev}, \text{BRev})$ when the buyer’s value function is subadditive.

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5Single-minded buyers are interested in buying a specific subset $S$ of items at some value.
over independent item values. Despite the similarity in techniques, our setting is quite different from that of Rubinstein and Weinberg. In our setting, the buyer’s value function is single-minded (and in particular, not subadditive), the optimal pricing we compare against is subadditive (not arbitrary), and the independence is across the items demanded by the buyer (and not the item values). Further exploring the simplicity versus optimality tradeoff in terms of demand distributions is another fruitful direction for future work.

2 Notation and definitions

We study the following single-buyer mechanism design problem. A seller has \( n \) heterogeneous items to sell to a single buyer. The buyer’s type is given by a valuation function \( v \) that assigns non-negative values to every set of items: \( v : 2^{[n]} \rightarrow \mathbb{R}^+ \). Values are monotone, meaning that for any \( S \) and \( T \) with \( S \subset T \subseteq [n] \), \( v(S) \leq v(T) \). The buyer’s type is drawn from an arbitrary known distribution \( D \) over the set of all possible valuation functions.

Any selling mechanism can be described as a menu over options, each of which assigns a price to a random allocation or lottery. Let \( \Delta = \Delta(2^{[n]}) \) denote the set of all probability distributions over sets of items and \( \lambda \in \Delta \) denote a “lottery” or random allocation. We describe a mechanism using a pricing function \( p : \Delta \rightarrow \mathbb{R}^+ \); the price assigned to a lottery \( \lambda \in \Delta \) is then given by \( p(\lambda) \).

If a buyer with valuation \( v \) buys a lottery \( \lambda \in \Delta \) at price \( p(\lambda) \), her utility from the purchase is given by

\[
u(v, p, \lambda) := E_{S \sim \lambda} [v(S)] - p(\lambda).
\]

If a buyer with valuation \( v \) purchases a multiset \( \Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_k\} \) of lotteries with price function \( p \), her utility from the purchase is given by

\[
u(v, p, \Lambda) := E_{S_i \sim \lambda_i, \forall i \in [k]} \left[ v \left( \bigcup_i S_i \right) \right] - \sum_{i \in [k]} p(\lambda_i).
\]

We sometimes use \( v(\lambda) \) as shorthand for \( E_{S \sim \lambda} [v(S)] \) and \( v(\Lambda) \) likewise for the value assigned to the union of sets drawn from a multiset \( \Lambda \) of lotteries.

Sybil-proofness and optimal revenue. We say that a lottery \( \lambda \) dominates another lottery (or a multiset of lotteries) \( \lambda' \) if there exists a coupling between a random draw \( S \) from \( \lambda \) and a random draw (or union of draws) \( S' \) from \( \lambda' \) such that \( S \) is a superset of \( S' \). We are now ready to define Sybil-proofness.

Definition 2.1. A mechanism or pricing \( p : \Delta \rightarrow \mathbb{R}^+ \) is Sybil-proof if for every multiset \( \Lambda \) of lotteries there exists a single lottery \( \lambda \) dominating it that is cheaper: \( p(\lambda) \leq \sum_{\lambda' \in \Lambda} p(\lambda') \).

Observe that if a lottery \( \lambda \) dominates another lottery (or a multiset of lotteries) \( \lambda' \), then any buyer with a monotone valuation function obtains higher expected value from \( \lambda \) than from \( \lambda' \). We therefore get the following observation:

\[\text{Since we are not investigating menu size, we will assume that the pricing assigns a price to every lottery. It is easy to extend a partial pricing to a complete one: we assign to every lottery the price of the cheapest option that dominates it, or a price of infinity if no such option exists.}\]
Fact 2.1. Given any Sybil-proof pricing $p$, for any buyer type $v$ it is optimal for the buyer to purchase a single lottery $\lambda = \arg \max_{\lambda \in \Delta} (v(\lambda) - p(\lambda))$.

Given a Sybil-proof pricing $p$, we use $\lambda_p(v) := \arg \max_{\lambda \in \Delta} (v(\lambda) - p(\lambda))$ to denote the lottery purchased by a buyer with value function $v$, and $u(v, p)$ as the corresponding utility achieved. For convenience, we overload notation and use $p(v) := p(\lambda_p(v))$ to denote the price paid by the buyer. Given a distribution $D$ over buyer types, we write $\text{REV}_D(p) := \mathbb{E}_{v \sim D}[p(v)]$ as the revenue of the pricing $p$. The optimal Sybil-proof revenue for distribution $D$ is given as follows; we drop the subscript $D$ when it is clear from the context.

$$\text{SYBIL-ProofRev}_D := \max_{\text{Sybil-proof pricings } p} \text{REV}_D(p).$$

Adaptive Sybil-proofness. Our definition of Sybil-proofness guards against buyers that purchase multiple options from the given menu and receive a random allocation for each. The buyer can perform even better if these menu options are selected sequentially and adaptively—that is, if the buyer observes the instantiation of each random allocation before deciding whether and what to purchase next. Guarding against such an adaptive buyer places a further restriction on the prices the mechanism can charge. Let $\mathcal{A}$ denote a buying strategy, that is, an adaptive sequence of lotteries. Let $\Lambda_\mathcal{A}$ denote the (random) sequence of lotteries bought in $\mathcal{A}$. As before, we say that $\mathcal{A}$ is dominated by a lottery $\lambda$ if there exists a coupling between a random draw $S$ from $\lambda$ and a random union of draws $S'$ from $\Lambda_\mathcal{A}$ such that $S$ is a superset of $S'$. We can now define a more restrictive definition of Sybil-proofness:

Definition 2.2. A mechanism or pricing $p : \Delta \rightarrow \mathbb{R}_0^+$ is Adaptive Sybil-proof if for every adaptive buying strategy $\mathcal{A}$ there exists a single lottery $\lambda$ dominating it that is cheaper: $p(\lambda) \leq \mathbb{E}_{\Lambda_\mathcal{A}} \left[ \sum_{\lambda' \in \Lambda_\mathcal{A}} p(\lambda') \right]$.

Of course there is no way for the seller to enforce whether buyers can make purchasing decisions adaptively or non-adaptively. So we believe that realistic mechanisms should satisfy adaptive Sybil-proofness. Nevertheless, all of our positive results apply for the more restrictive notion of (non-adaptive) Sybil-proofness.

Deterministic mechanisms. Deterministic mechanisms price only deterministic sets (a.k.a. bundles) of items—$p : 2^{[n]} \rightarrow \mathbb{R}_0^+$. A deterministic pricing $p$ is Sybil-proof if and only if it is monotone and subadditive, that is, for any $S_1 \subseteq S_2 \subseteq [n]$, $p(S_1) \leq p(S_2)$, and for any set of bundles, $\mathcal{T} \subseteq 2^{[n]}$, the prices of the union of bundles is no more than the sum of individual bundle prices: $\sum_{S \in \mathcal{T}} p(S) \geq p(\cup_{S \in \mathcal{T}} S)$. The optimal deterministic Sybil-proof revenue for distribution $D$ is given as follows.

$$\text{SubadditiveRev}_D := \max_{\text{Deterministic monotone subadditive pricings } p} \text{REV}_D(p).$$

Simple pricings. An item pricing is a deterministic additive pricing: $p(S) = \sum_{i \in S} p\{i\}$ for all $S \subseteq [n]$. A bundle pricing is a constant pricing that assigns the same price to every set: $p(S) = p([n])$ for all $S \subseteq [n]$. Observe that item priceings and bundle priceings are always Sybil-proof. We use $\text{SRev}$ and $\text{BRev}$ to denote the optimal revenue achievable using item priceings and bundle priceings respectively (over an implicit distribution $D$).
3 Approximation via item pricing

In this section we present our main upper bound, namely that the ratio between $\text{SYBILPROOFREV}_D$ and $\text{SREV}_D$ is bounded by $O(\log n)$ for any value distribution $D$. This result is based on two main ideas. Both ideas are present implicitly in the work of Bries et al. [5] for unit-demand buyers. We formalize and extend these ideas to arbitrary valuations.

The first idea is that given any Sybil-proof pricing $p$, it is possible to define an item pricing $q$ that (essentially) point-wise approximates $p$ within a factor of poly$(n)$. That is, for every set $S$, $p(S)$ is within a factor of poly$(n)$ of $q(S)$.

Ignoring incentives for the moment, suppose we just want to maximize the quantity $\mathbf{E}_{S \sim D} \left[ \frac{g(S)}{BD} \right]$ for some function $g$. Then, it suffices to set $g$ to be an appropriate scaling of $q$; there are $O(\log n)$ scales of interest, and therefore there exists one that provides an $O(\log n)$ approximation.

Our second idea handles incentives. Consider a suite of pricings $\{g_\alpha\}$ defined over a range of scale factors $\alpha \in [\ell, h]$, where $g_\alpha := \alpha q$. We want to argue that this suite of pricings collectively obtains good revenue from every buyer type. Consider starting at $\alpha = \ell$ and gradually increasing the scaling factor and, correspondingly, prices. As the prices increase, the buyer’s utility from purchasing his favorite bundle (which may change with $\alpha$) weakly decreases. This decrease in the buyer’s utility is captured in the form of the seller’s revenue. In effect, the total revenue obtained by the pricings in $\{g_\alpha\}$ is proportional to the difference in utility the buyer obtains at $g_\ell$ and $g_h$ respectively. In order to relate this to the revenue of the pricing $p$, we set $\ell$ low enough so that the buyer’s utility is comparable to the price he pays under $p$ and set $h$ high enough so that the buyer’s utility is tiny no matter what bundle he purchases. Observe that in carrying out this argument we cannot make the assumption that the buyer’s value is bounded. Indeed the range of the buyer’s values can be super-exponential in $n$ or even unbounded.

We first illustrate these ideas over deterministic pricings and then extend them to arbitrary Sybil-proof pricings in Section 3.2.

3.1 Upper bound for deterministic monotone subadditive pricings

**Theorem 3.1.** For any distribution $D$, we have $\text{SUBADDITIVEREV}_D \leq O(\log n)\text{SREV}_D$.

**Proof.** Let $p$ be the subadditive pricing that achieves revenue $\text{SUBADDITIVEREV}_D$. Let $q$ be an item pricing defined by $q(\{i\}) = p(\{i\})$ for all $i \in [n]$ and $q(S) = \sum_{i \in S} q(\{i\})$ for all $S \subseteq [n]$. Observe that for all $S \subseteq [n]$, the subadditivity of $p$ implies

$$\frac{1}{n} q(S) \leq p(S) \leq q(S).$$

For $\alpha \in [1/2n, 1]$, let $g_\alpha := \alpha q$. Consider a buyer with value function $v$. Let $r_v(\alpha)$ denote the utility the buyer derives when offered the pricing $g_\alpha$, and $S_\alpha$ denote the corresponding set purchased:

$$r_v(\alpha) = \max_{S \subseteq [n]} (v(S) - \alpha q(S))$$

$$S_\alpha = \arg \max_{S \subseteq [n]} (v(S) - \alpha q(S)).$$

$p(S)$ is straightforward to define for deterministic mechanisms. For randomized mechanisms, we can bound it by determining the cheapest way of obtaining all of the elements in $S$ with near certainty.
By the envelope theorem it holds that \( r_v'(\alpha) = -q(S_\alpha) \). Therefore, the seller’s revenue from this buyer is given by

\[
a q(S_\alpha) = -\alpha r_v'(\alpha).
\]

Now, consider picking \( \alpha \) from the equal revenue distribution over \([1/2n, 1]\) with density function \( 1/(\alpha \log(2n)) \), and offering the buyer the pricing \( g_\alpha \). Then, the expected revenue from this buyer is:

\[
\int_{1/2n}^{1} a q(S_\alpha) \cdot \frac{1}{\alpha \log(2n)} d\alpha = -\frac{1}{\log(2n)} \int_{1/2n}^{1} r_v'(\alpha) d\alpha = \frac{r_v(1/2n) - r_v(1)}{\log(2n)}.
\]

Finally, observe that

\[
r_v(1/2n) = \max_{S \subseteq [n]} \left( v(S) - \frac{1}{2n} q(S) \right) \geq \max_{S \subseteq [n]} \left( v(S) - \frac{1}{2} p(S) \right) = u(v, p) + \frac{1}{2} p(v).
\]

On the other hand,

\[
r_v(1) = \max_{S \subseteq [n]} (v(S) - q(S)) \leq \max_{S \subseteq [n]} (v(S) - p(S)) = u(v, p).
\]

We therefore obtain a revenue of at least \( \frac{1}{2 \log(2n)} p(v) \) from the buyer with value function \( v \). The result now follows by taking expectations over \( v \) and recalling that \( g_\alpha \) is an item pricing.

### 3.2 Upper bound for randomized Sybil-proof pricings

We will now prove our main theorem:

**Theorem 3.2.** For any distribution \( \mathcal{D} \), we have \( \text{SYBILPROOFREV}_\mathcal{D} \leq O(\log n)\text{SREV}_\mathcal{D} \).

**A proof for adaptively Sybil-proof pricings.** Our proof proceeds in a manner quite similar to that of Theorem 3.1. Let \( p \) be the given Sybil-proof pricing, and let us assume at first that \( p \) is adaptively Sybil-proof. We begin by defining base prices for each of the items. Informally, these are the minimum prices the buyer needs to pay in expectation to obtain item \( i \) with certainty:

\[
q(\{i\}) = \min_{\lambda \in \Delta} \frac{p(\lambda)}{\Pr[i \in \lambda]}
\]

where we write \( \Pr[i \in \lambda] \) as shorthand for \( \Pr_{S \sim \lambda}[i \in S] \). Let \( \lambda_i \) denote the lottery that defines the price for item \( i \): \( \lambda_i = \arg \max_{\lambda \in \Delta} \frac{p(\lambda)}{\Pr[i \in \lambda]} \). Let \( q(S) = \sum_{i \in S} q(\{i\}) \) for all \( S \subseteq [n] \).

Observe that for any lottery \( \lambda \in \Delta \), the adaptive buyer can draw a set \( T \) from \( \lambda \) and purchase this set by iterating until he obtains the item. The expected price paid by the buyer in this strategy is precisely \( \mathbf{E}_{T \sim \lambda}[q(T)] = \sum_{i \in [n]} (q(\{i\}) \Pr[i \in \lambda]) \). Let us call this latter sum \( q(\lambda) \). Since \( p \) is adaptively Sybil-proof, we have \( p(\lambda) \leq q(\lambda) \).

On the other hand, by the definition of \( q \), \( p(\lambda) \geq q(\{i\}) \Pr[i \in \lambda] \) and therefore, summing over all \( i \in [n] \) and dividing by \( n \), we have \( p(\lambda) \geq \frac{1}{n} q(\lambda) \).

We will now define our suite of item pricings. As before, for \( \alpha \in [1/2n, 1] \), define \( g_\alpha := a q, S_\alpha = \arg \max_{S \subseteq [n]} (v(S) - a q(S)) \), and \( r_v(\alpha) = v(S_\alpha) - a q(S_\alpha) \). Also, as before, the expected revenue obtained by \( g_\alpha \) when \( \alpha \) is picked from the equal revenue distribution over \([1/2n, 1]\) with density function \( 1/(\alpha \log(2n)) \) is given by \( (r_v(1/2n) - r_v(1))/\log(2n) \). The equations (3) and (4) now easily follow from our observation that \( \frac{1}{n} q(\lambda) \leq p(\lambda) \leq q(\lambda) \).
The general case. Applying this argument to non-adaptive buyers requires some modification because it is no longer necessarily true that \( p(\lambda) \leq q(\lambda) \). Let us first reiterate the parts of the argument that work as before. We define the pricing \( q \) as:

\[
q(\{i\}) = \min_{\lambda \in \Delta} \frac{p(\lambda)}{\Pr[i \in \lambda]} \quad \forall i \in [n] \quad \text{and} \quad q(S) = \sum_{i \in S} q(\{i\}) \quad \forall S \subseteq [n].
\]

Let \( \alpha \) be picked from the range \([1/2n, 4n]\) with density function \(1/(\alpha \log(8n^2))\). Define \( g_\alpha, S_\alpha, r_\nu(\alpha) \), and \( \lambda_i \) as before. We then have:

\[
p(\lambda) \geq \frac{1}{n} q(\lambda)
\]

\[
E_\alpha[g_\alpha(v)] = \frac{r_\nu(1/2n) - r_\nu(4n)}{\log(8n^2)}
\]

and,

\[
r_\nu(1/2n) = \max_{S \subseteq [n]} \left( v(S) - \frac{1}{2n} q(S) \right) \geq v(\lambda_\nu(v)) - \frac{1}{2n} q(\lambda_\nu(v))
\]

\[
\geq v(\lambda_\nu(v)) - \frac{1}{2} p(\lambda_\nu(v)) = u(v, p) + \frac{1}{2} p(v).
\]

where \( \lambda_\nu(v) \) is the lottery bought by the buyer in pricing \( p \).

Define \( h = 4n \). It remains to bound the buyer’s utility in \( g_h \), namely \( r_\nu(h) \). Let \( T_h \) be the set purchased by the buyer when offered the pricing \( g_h \). Then \( r_\nu(h) = v(T_h) - hq(T_h) \). We want to bound this utility in terms of the utility the buyer gets under the pricing \( p \), so let us consider how much it would cost the buyer to acquire \( T_h \) under \( p \). In particular, fix some number \( m \) and suppose that the buyer purchases a multiset \( \Lambda_m \) that contains \( m_i = \lceil \frac{m}{\Pr[i \in \lambda_i]} \rceil \) copies of \( \lambda_i \) for all \( i \in T_h \). Then, the probability that some \( i \in T_h \) does not belong to the random allocation drawn from this multiset is at most \( (1 - \Pr[i \in \lambda_i])^m < 2^{-m} \). Accordingly, the probability that \( T_h \) is not a subset of the random allocation drawn from \( \Lambda_m \) is at most \( n2^{-m} \). The total price of \( \Lambda_m \) is \( \sum_{i \in T_h} \lceil \frac{m}{\Pr[i \in \lambda_i]} \rceil p(\lambda_i) \leq (m + 1) \sum_{i \in T_h} q(\{i\}) = (m + 1)q(T_h) \). We therefore have:

\[
u(v, p) \geq u(v, p, \Lambda_m) \geq (1 - n2^{-m})v(T_h) - (m + 1)q(T_h) \quad \text{for all } m.
\]

Now, let \( k \) be defined such that \( n2^{-k}v(T_h) = q(T_h) \), that is, \( k = \log(nv(T_h)/q(T_h)) \). Then we get:

\[
u(v, p) \geq v(T_h) - (k + 2)q(T_h)
\]

and,

\[
r_\nu(h) \leq u(v, p) + (k + 2 - h)q(T_h).
\]

Now, if \( k + 2 \leq h \), then we are already done. So, for the remainder of the proof, assume that \( k > h - 2 \geq 3n \). We will construct a different item pricing to recover the quantity \( (k+2-h)q(T_h) < kq(T_h) \) as revenue from the buyer with value \( v \).

For any \( T \subseteq [n] \) and positive integer \( a \), the (uniform) item pricing \( \tilde{g}_{T,a} \) is defined as follows:

\[
\tilde{g}_{T,a}(S) = 2^{a+n}q(T)|S| \quad \text{for all } S \subseteq [n].
\]

Now consider a buyer with value function \( v \) and with \( k = \log(nv(T_h)/q(T_h)) \). When offered the pricing \( \tilde{g}_{T_h,a} \) with \( a < k - n - 2\log n \), this buyer obtains
a utility of at least \( v(T_h) - \tilde{g}_{T_h,a}(T_h) \geq v(T_h) - 2^{a+n} q(T_h) \geq v(T_h) - \frac{1}{n} 2^k q(T_h) = 0 \). Therefore, the buyer buys at least one item under this pricing and pays at least \( 2^{a+n} q(T_h) \). In other words,
\[
\tilde{g}_{T_h,a}(v) \geq 2^{a+n} q(T_h) \quad \text{for } a < k - n - 2 \log n.
\]

Let \( T \) be drawn from the uniform distribution over \( 2^{[n]} \) and \( a \) be drawn from the geometric distribution with mean 2, that is, \( \Pr[a = x] = 2^{-x-1} \) for every \( x \in \mathbb{N} \). Then, we get:
\[
E_{T,a}[\tilde{g}_{T,a}(v)] \geq \frac{1}{2n} \sum_{a=0}^{k-n-2\log n-1} \frac{1}{2a+1} 2^{a+n} q(T_h)
\]
\[
= \frac{1}{2}(k - n - 2 \log n - 1)q(T_h)
\]
\[
\geq \frac{1}{4} kq(T_h) \quad \text{for } k \geq h - 2. \quad (8)
\]

Putting together Equations (5), (6), (7), and (8), we get that for any buyer valuation \( v \):
\[
\log(8n^2)E_{\alpha}[g_{\alpha}(v)] + 4E_{T,a}[\tilde{g}_{T,a}(v)] \geq \frac{1}{2} p(v).
\] (9)

The theorem now follows by taking expectations over the valuation function \( v \).

## 4 Lower Bound

In this section we present our main lower bound, namely that the \( O(\log n) \) approximation achieved in the previous section is tight in a very strong sense. We show that there exists a buyer such that no mechanism with sub-exponential description complexity can \( o(\log n) \)-approximate the revenue from optimal deterministic Sybil-proof pricing. The main idea behind this lower bound is that there exist (additive) valuation functions and subadditive pricings that extract almost the entire value, that are parameterized by exponentially many independent parameters. Obtaining a good approximation would essentially amount to achieving a good quality compression of these parameters into sub-exponentially many bits, an impossible task. We begin by describing our construction for single-minded valuation functions and then extend it to additive values. It is worth noting that our result gives a lower bound on \textsc{SubadditiveRev}/\textsc{SRev}, and not just \textsc{SybilProofRev}/\textsc{SRev}. Briest et al. [5] provided a matching lower bound on \textsc{SybilProofRev}/\textsc{DRev}.

### 4.1 The basic construction and a lower bound for single-minded buyers

Let \( f \) and \( g \) be any two functions defined over the subsets of \([n]\). Let \( \Pi \) be a distribution over sets, \( \Pi \in \Delta(2^{[n]}) \). We will say that \( g \) \( e \)-approximates \( f \) from below over \( \Pi \) if the following holds:
\[
E_{S \sim \Pi}[g(S) 1_{g(S) \leq f(S)}] \geq \frac{1}{c} E_{S \sim \Pi}[f(S)].
\]

Our argument will proceed in two parts. First, we will show that for any “small” class of functions \( g \), there exists a subadditive (in fact, submodular) \( f \) and a distribution \( \Pi \) such that no \( g \) in the class can \( o(\log n) \)-approximate \( f \) from below over \( \Pi \). Then we will show that for any \( f \) and \( \Pi \) of the form constructed in the first step, there exists a distribution over valuation functions such that the optimal subadditive revenue is a constant fraction of \( E_{S \sim \Pi}[f(S)] \), while the revenue of any other function \( g \) is no more than \( E_{S \sim \Pi}[g(S) 1_{g(S) \leq f(S)}] \). Together this will imply the following theorem.
Theorem 4.1. For any large enough \( n \), there exists a distribution \( \mathcal{D} \) over single-minded valuation functions over \( n \) items and a deterministic submodular pricing function \( p \) such that \( \text{Rev}_p(p) \) is a factor of \( \Omega(\log n) \) larger than the revenue of any pricing that can be described using \( 2^{o(n^{1/6})} \) bits.

Let us begin by describing the class of subadditive functions \( f \) that we will use in our argument. Let \( \mathcal{S} = \{S_1, S_2, \ldots, S_N\} \) be a collection of \( N \) subsets of \( [n] \). Let \( \beta = (b_1, \ldots, b_N) \) be a vector of integers of size \( N \), where each coordinate is picked from the range \([b_{\min}, b_{\max}]\). We define a partial function \( f_{\mathcal{S}, \beta} \) as follows: for all \( S_i \in \mathcal{S} \), set \( f_{\mathcal{S}, \beta}(S_i) = b_i \). The following lemma follows from the work of Balcan and Harvey [3] and shows that we can pick both \( N \) and \( b_{\max}/b_{\min} \) to be sufficiently large while ensuring that \( f \) is submodular.\(^8\) See the appendix for a proof.

Lemma 4.2. Let \( N = 2^{n^{1/6}/8} \), \( b_{\min} = n^{1/6} \), and \( b_{\max} = n^{1/3} \). Then, there exists a collection of sets \( \mathcal{S} = \{S_1, S_2, \ldots, S_N\} \), such that for each \( i, |S_i| = n^{1/3} \); for each \( i \neq j, |S_i \cap S_j| \leq n^{1/6} \); and for any integral vector \( \beta \in [b_{\min}, b_{\max}]^N \), the partial function \( f_{\mathcal{S}, \beta} \) can be completed to a matroid rank function.

Let \( \Pi \) be the uniform distribution over the collection \( \mathcal{S} \). Our next lemma argues that for any small class of functions \( \mathcal{G} \), there exists a vector \( \beta \), such that no function in \( \mathcal{G} \) can \( c \)-approximate \( f_{\mathcal{S}, \beta} \) from below over \( \Pi \). Observe that because \( \Pi \) only places non-zero mass over sets in \( \mathcal{S} \), we do not need to specify a completion of \( f_{\mathcal{S}, \beta} \) for this lemma.

Lemma 4.3. Let \( m = \log(b_{\max}/b_{\min}), \ c \leq m/8 \), and \( \mathcal{G} \) be an arbitrary class of functions defined over the subsets of \( [n] \) with \( |\mathcal{G}| \leq 2^{o(N/4^m)} \). Then there exists an integral vector \( \beta \in [b_{\min}, b_{\max}]^N \) such that no function \( g \in \mathcal{G} \) can \( c \)-approximate \( f_{\mathcal{S}, \beta} \) from below over \( \Pi \).

Proof. Fix a function \( g \in \mathcal{G} \). We will pick \( \beta \) from a distribution and show that the probability that \( g \) \( c \)-approximates the corresponding function \( f_{\mathcal{S}, \beta} \) is small. For each \( i \in [N] \), draw \( b_i \) independently according to the following truncated geometric distribution: \( \Pr[b_i = 2^k b_{\min}] = \frac{2^{-k}}{1 - 2^{-m}} \) for \( 1 \leq k \leq m \). Let \( h_i = g(S_i) \mathbb{1}_{g(S_i) \leq f_{\mathcal{S}, \beta}(S_i)} \) be a random variable that depends on \( \beta \). Then the statement that \( g \) \( c \)-approximates \( f_{\mathcal{S}, \beta} \) from below over \( \Pi \) is equivalent to the statement that

\[
\sum_{i \in [N]} h_i \geq \frac{1}{c} \sum_{i \in [N]} f_{\mathcal{S}, \beta}(S_i). \tag{10}
\]

Observe that over the randomness in \( \beta \), the variables \( h_i \) are independent and bounded. For all \( i \), \( h_i \leq f_{\mathcal{S}, \beta}(S_i) \leq 2^m b_{\min} \). Furthermore,

\[
\mathbb{E}[h_i] = g(S_i) \Pr[f_{\mathcal{S}, \beta}(S_i) \geq g(S_i)] \leq g(S_i) \frac{2b_{\min}}{g(S_i)} = 2b_{\min}.
\]

On the other hand,

\[
\mathbb{E}[f_{\mathcal{S}, \beta}(S_i)] = \mathbb{E}[h_i] = \frac{m}{1 - 2^{-m}} b_{\min}.
\]

---

\(^8\)Matroid rank functions are a subclass of monotone submodular functions, which in turn are a subclass of all monotone subadditive functions.
We can now bound the probability that (10) holds by applying concentration to the sums of $h_i$ and $f_{S,\beta}(S_i)$ respectively.

$$
\Pr_\beta \left[ \sum_i h_i \geq \frac{1}{c} \sum_i f_{S,\beta}(S_i) \right] \\
\leq \Pr \left[ \sum_i h_i \geq \frac{1}{2c} \mathbb{E} \left[ \sum_i f_{S,\beta}(S_i) \right] \right] + \Pr \left[ \sum_i f_{S,\beta}(S_i) < \frac{1}{2} \mathbb{E} \left[ \sum_i f_{S,\beta}(S_i) \right] \right] \\
= \Pr \left[ \sum_i h_i \geq \frac{m}{2c(1 - 2^{-m})} N b_{\min} \right] + \Pr \left[ \sum_i f_{S,\beta}(S_i) < \frac{m}{2(1 - 2^{-m})} N b_{\min} \right] \\
\leq \exp \left( -\frac{2(\frac{m}{2c(1 - 2^{-m})} N b_{\min} - 2 N b_{\min})^2}{N(2^m b_{\min})^2} \right) + \exp \left( -\frac{2(\frac{m}{2(1 - 2^{-m})} N b_{\min})^2}{N(2^m b_{\min})^2} \right) \\
= O(2^{-N4^{-m}}).
$$

Here the third line uses Hoeffding’s inequality by observing that $0 \leq h_i, f_{S,\beta}(S_i) \leq 2^m b_{\min}$; the last line follows using $c \leq m/8$. The lemma now follows by taking the union bound over all $g \in \mathcal{G}$. □

We are now ready to prove Theorem 4.1.

**Proof of Theorem 4.1.** Let $b_{\min}, b_{\max}, N$, and $\mathcal{S}$ be as given in Lemma 4.2. Let $\Pi$ be the uniform distribution over $\mathcal{S}$. Let $m = \log(b_{\max}/b_{\min})$, $c = m/8$, and observe that the class $\mathcal{G}$ of all pricing functions that can be described using $2^{n(1/6)}$ bits has size $2^{n(1/6)}$. So we can apply Lemma 4.3 to obtain a vector $\beta$.

Now we will define a distribution $\mathcal{D}$ over valuation functions as follows. For $i \in [N]$, let $v_i$ be the function that takes on value $b_i$ over any superset of $S_i$ and value 0 otherwise. Function $v_i$ is instantiated with probability $1/N$. Let $f$ be the completion of $f_{S,\beta}$ as given by Lemma 4.2. Consider a pricing function that assigns a price of $\frac{1}{2}f(S)$ to any set $S \subseteq [n]$. Since this function is subadditive and a buyer with value $v_i$ is single-minded and can afford to buy the set $S_i$, the mechanism obtains a revenue of $\frac{1}{2}f(S_i)$ from this buyer. The mechanism’s expected revenue over the distribution $\mathcal{D}$ is then $\frac{1}{2} \mathbb{E}_{S \sim \Pi} [f_{S,\beta}(S)]$.

On the other hand, for any monotone pricing function $g$, a buyer with value function $v_i$ purchases the set $S_i$ if and only if $g(S_i) \leq v_i(S_i) = b_i = f_{S,\beta}(S_i)$. Therefore, the revenue of such a function $g$ over $\mathcal{D}$ is at most $\mathbb{E}_{S \sim \Pi} \left[ g(S) \mathbbm{1}_{g(S) \leq f(S)} \right]$.

The theorem now follows by applying Lemma 4.3. □

### 4.2 A lower bound for additive buyers

We now extend our lower bound to additive value buyers. Our construction is similar to that for Theorem 4.1 but there are some subtle differences. As before, let $\mathcal{S} = \{S_1, S_2, \ldots, S_N\}$ and $\beta = (b_1, \ldots, b_N) \in [b_{\min}, b_{\max}]^N$. We now state and prove our lower bound for additive buyers.

**Theorem 4.4.** For any large enough $n$, there exists a distribution $\mathcal{D}$ over additive valuation functions over $n$ items and a deterministic monotone subadditive pricing function $p$ such that $\text{REV}_\mathcal{D}(p)$ is a factor of $\Omega(\log n)$ larger than the revenue of any mechanism that can be described using $2^{o(n^{1/6})}$ bits.
Proof. Let $N$ and $S$ be as defined in Lemma 4.2. Let $b_{\min} = 2n^{1/6}$, $b_{\max} = n^{1/3}$, and fix an integral vector $\beta \in [b_{\min}, b_{\max}]^N$. Consider the following distribution $D_{\beta}$ over value functions. For $i \in [N]$, $v_{i,\beta}$ is an additive function that takes on the value $b_i/|S_i|$ over all items in $S_i$, and 0 on items not in $S_i$. Observe that $v_{i,\beta}$ is a uniform additive valuation and $v_{i,\beta}(S_i) = b_i$. Function $v_{i,\beta}$ is instantiated with probability $1/N$.

Now consider the pricing function $f_{S,\beta}$ defined over the sets in $S$ as $f_{S,\beta}(S_i) = \frac{1}{2}b_i$ for all $S_i \in S$, and extended to arbitrary sets in the natural way: $f_{S,\beta}(S) = \min_{A \subseteq [N]: \cup_{i \in A} S_i \supseteq S} \sum_{i \in A} f_{S,\beta}(S_i)$. Observe that $f_{S,\beta}(S_i) < v_{i,\beta}(S_i)$. We claim that $f_{S,\beta}$ is subadditive and therefore extracts revenue $\frac{1}{2}b_i$ from the buyer with type $v_{i,\beta}$. To see this, recall that the buyer obtains positive utility from the set $S_i$. Suppose the buyer instead decides to buy the collection $S_{i_1}, S_{i_2}, \cdots, S_{i_k}$. Since $|S_{i_j} \cap S_{i_k}| \leq n^{1/6}$, $v_{i,\beta}(S_{i_1} \cup S_{i_2} \cup \cdots \cup S_{i_k}) \leq kn^{-1/6}b_i$. Thus his utility from buying these sets is at most

$$kn^{-1/6}b_i - \sum_{j=1}^{k} \frac{1}{2}b_{i_j} \leq kn^{-1/6}b_{\max} - \frac{1}{2}kb_{\min} = 0.$$ 

Therefore, we have $f_{S,\beta}(v_{i,\beta}) = f_{S,\beta}(S_i)$.

Now let $G$ be the class of mechanisms/pricings in the statement of the theorem and fix any $g \in G$. We will again define a distribution over instances by defining a distribution over the vectors $\beta$, as in the proof of Lemma 4.3. Let $m = \log(b_{\max}/b_{\min})$. For all $i \in [N]$, draw $b_i$ independently from the following truncated geometric distribution: $\Pr[b_i = 2^kb_{\min}] = \frac{2^{-k}}{1 - e^{-m}}$ for $1 \leq k \leq m$.

Let $h_i = g(v_{i,\beta})$ be the revenue $g$ obtained from the buyer with value function $v_{i,\beta}$. Observe that $h_i$ is a random variable that depends on $b_i$. As before, the variables $h_i$ are independent and bounded by $2^m b_{\min}$. Furthermore, we can bound the expectation of $h_i$ by observing that a buyer with valuation $v_{i,\beta}$ is an additive buyer with uniform values over items in $S_i$. Selling a subset of items to this buyer is equivalent to selling fractional amounts of a single item to a single-parameter buyer. The optimal revenue from this buyer, over the randomness in $b_i$, is bounded by the revenue of single posted price. Therefore, $E[h_i] \leq 2b_{\min}$. We now apply the same concentration argument as in the proof of Lemma 4.3 to obtain

$$\Pr_{\beta} \left[ \sum_i h_i \geq \frac{1}{c} \sum_i f_{S,\beta}(S_i) \right] \leq O(2^{-NA^4-m}).$$ 

The theorem now follows by taking the union bound over $g \in G$.  

\[\square\]

5 A constant upper bound for special demand distributions

In the previous section we proved that we cannot obtain an $o(\log n)$-approximation to deterministic subadditive pricing via simple mechanisms for arbitrary distributions over valuation functions. However, when the distributions satisfy a certain property simple mechanisms, in particular item or bundle pricings, are able to do better.

Definition 5.1. Given a deterministic Sybil-proof pricing $p$ and a distribution $D$ over valuations, the demand distribution $\Pi_{p,D}$ is the distribution over sets of items that specifies the random set of items bought by a buyer with value drawn from $D$ under pricing $p$: for all $S \subseteq [n]$, $\Pi_{p,D}(S) = \Pr_{v \sim D}[S = \arg\max_{T \subseteq [n]}(v(T) - p(T))]$.\footnote{Generally speaking, specifying the demand distribution requires specifying a tie-breaking rule between multiple sets of equal utility. We will focus here on single-minded buyers, so the possibility of tie-breaking will not arise.}
Observe that for any pricing \( p \) and value distribution \( D \), the revenue of the pricing is precisely \( \mathbf{E}_{v \sim D}[p(v)] = \mathbf{E}_{S \sim \Pi \times D}[p(S)] \). We will now show that if the buyer is single-minded and the demand distribution is a product distribution over items, then the revenue of \( p \) can be approximated by a simple pricing.

\[ \text{Theorem 5.1.} \quad \text{Given a deterministic monotone subadditive pricing function} \ p \ \text{and value distribution} \ D \ \text{over single-minded values, suppose that the demand distribution} \ \Pi \times D \ \text{is a product distribution over items, then} \ \operatorname{REV}_D(p) < 22.67 \max(\operatorname{SREV}_D, \operatorname{BREV}_D). \]

\[ \text{Proof.} \quad \text{As in the work of [18], we will break up set of items and correspondingly the revenue obtained by} \ p \ \text{into two components: over one of these components, a.k.a. the core, the price will concentrate around its expectation and can be approximated using a bundle pricing; over the other, a.k.a. the tail, a significant fraction of the revenue will be contributed by singleton items and can be recovered using an item pricing.} \]

We write \( \Pi \) as shorthand for \( \Pi\times\mathcal{D} \). Number the items in decreasing order of their individual prices: \( p(\{1\}) \geq p(\{2\}) \geq \cdots \geq p(\{n\}) \). For items \( i \in [n] \) let \( \pi_i \) denote the marginal probability that \( i \) is purchased, that is, \( \pi_i = \Pr_{S \sim \Pi}[i \in S] \). Find the index \( k \in [n] \) such that \( \pi_1 + \pi_2 + \cdots + \pi_k < \frac{1}{2} \) and \( \pi_1 + \pi_2 + \cdots + \pi_{k+1} \geq \frac{1}{2} \). The high value, low probability items \( \{1, 2, \cdots , k\} \) will form the core, and the remaining items will form the tail. For any subset \( S \subseteq [n] \), define \( S_{\text{TAIL}} = S \cap \{1, 2, \cdots , k\} \) and \( S_{\text{CORE}} = S \cap \{k+1, \cdots , n\} \). By the subadditivity of \( p \) we have \( p(S) \leq p(S_{\text{TAIL}}) + p(S_{\text{CORE}}) \). Therefore, \( \mathbf{E}_{S \sim \Pi}[p(S_{\text{TAL}})] + \mathbf{E}_{S \sim \Pi}[p(S_{\text{CORE}})] \) is an upper bound of the revenue of \( p \).

We will first bound the contribution of the core, \( \mathbf{E}_{S \sim \Pi}[p(S_{\text{CORE}})] \) using the following concentration lemma for subadditive functions from [19]. To apply the lemma, we observe that \( p(S_{\text{CORE}}) \) is Lipschitz with a Lipschitz constant of \( \max_{i > k} p(\{i\}) = p(\{k+1\}) \) because \( S_{\text{CORE}} \) contains only items with index larger than \( k \).

\[ \text{Theorem 5.2.} \quad \text{(Corollary 12 from [19]) Suppose that} \ f(X) \ \text{is a non-negative} \ c \ \text{-Lipschitz subadditive function, where} \ X \in \{0, 1\}^n \ \text{is drawn from a product distribution} \ D. \ \text{If} \ a \ \text{is the median of} \ f, \ \text{then for any} \ k > 0, \ \Pr_{X \sim D}[f(X) \geq 3a + k] \leq 2^{-k/c}. \]

Setting \( a = \text{median}_{S \sim \Pi}(p(S_{\text{CORE}})) \) and \( c = p(\{k+1\}) \), we obtain

\[ \mathbf{E}_{S \sim \Pi}[p(S_{\text{CORE}})] = \int_{t \geq 0} \Pr[p(S_{\text{CORE}}) > t] dt \leq 3a + \int_{k \geq 0} 2^{-k/c} dk = 3a + \frac{4}{\ln 2}c. \]

We now observe that we can recover both of the terms above using a bundle pricing. In particular, by setting a constant bundle price of \( a \), we obtain a revenue of \( a \Pr_{v \sim D}[v(\{i\}) > a] \geq a \Pr_{S \sim \Pi}[p(S) > a] \geq a \Pr_{S \sim \Pi}[p(S_{\text{CORE}}) > a] = a/2 \). On the other hand, by setting a constant bundle price of \( c \), we obtain a revenue of \( c \Pr_{v \sim D}[v(\{i\}) > c] \geq c \Pr_{S \sim \Pi}[p(S) > c] \geq c \Pr_{S \sim \Pi}[\exists i \leq k + 1 : i \in S] \). The latter probability can be bounded from below as:

\[ \Pr_{S \sim \Pi}[\exists i \leq k + 1, i \in S] = 1 - (1 - \pi_1)(1 - \pi_2) \cdots (1 - \pi_{k+1}) \geq 1 - \left( \frac{1 - \sum_{i \leq k+1} \pi_i}{k+1} \right)^{k+1} \geq 1 - e^{-1/2}, \]

where the first inequality follows by applying Jensen’s inequality, and the second follows by using \( \sum_{i \leq k+1} \pi_i \geq 1/2 \).

Thus we can bound \( \mathbf{E}[p(S_{\text{CORE}})] \) as follows:

\[ \mathbf{E}[p(S_{\text{CORE}})] \leq 3a + \frac{4}{\ln 2}c \leq 6 \operatorname{BREV} + \frac{4}{(1 - e^{-1/2}) \ln 2} \operatorname{BREV} \leq 20.67 \operatorname{BREV}. \]
Now we bound $E[p(S_{\text{TAIL}})]$. By subadditivity,

$$E[p(S_{\text{TAIL}})] \leq \sum_{i \leq k} p(S_{\text{TAIL}} \cap \{i\}) = \sum_{i \leq k} \pi_i p(\{i\}).$$

Recall that $\pi_1 + \pi_2 + \cdots + \pi_k \leq \frac{1}{2}$. Consider the additive pricing that sells items in $\{1, 2, \cdots, k\}$ at item prices $q_i = p(\{i\})$, and allocates items $i > k$ for free.

Consider a buyer that purchases a set $S$ under pricing $p$ with $S_{\text{TAIL}} = \{i\}$. In the pricing $q$, the buyer continues to afford the set $S$ at a lower price of $q_i = p(\{i\}) < p(S)$. Therefore, the revenue of the pricing $q$ from this buyer is at least $q_i = p(\{i\})$. Since this buyer is instantiated with probability at least $\Pr[S_{\text{TAIL}} = \{i\}] \geq \frac{1}{2} \pi_i$, we get that $E[p(S_{\text{TAIL}})] \leq 2 \text{Rev}_D(q) \leq 2 \text{SRev}$. Then combining the above cases we get $\text{Rev}_D(p) \leq 22.67 \max(\text{SRev}_D, \text{BRev}_D)$.

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A Deferred proofs

A.1 Proof of Lemma 4.2

Lemma 4.2. Let \( N = 2^{n^{1/6}/8} \), \( b_{\text{min}} = n^{1/6} \), and \( b_{\text{max}} = n^{1/3} \). Then, there exists a collection of sets \( S = \{S_1, S_2, \ldots, S_N\} \), such that for each \( i \) \( |S_i| = n^{1/3} \); for each \( i \neq j \), \( |S_i \cap S_j| \leq n^{1/6} \); and for any integral vector \( \beta \in [b_{\text{min}}, b_{\text{max}}]^N \), the partial function \( f_{S, \beta} \) can be completed to a matroid rank function.

Proof. We need the following lemmas from [3].
Lemma A.1. (Theorem 9 of [3]) Let μ and τ be non-negative integers. \( f : 2^{|N|} \to \mathbb{R} \) is called \((μ, τ)\)-large if \( f(J) \geq 0, \forall |J| < τ; f(J) \geq μ, \forall τ \leq |J| \leq 2τ − 2 \). Then for any sets \( S_1, \ldots, S_N \subseteq [n] \), \( \mathcal{I} = \{I : |I| \leq μ \wedge |I \cap S(J)| \leq f(J), \forall J \subseteq [N], |J| < τ\} \) is the family of independent sets of a matroid, here \( S(J) = \bigcup_{j \in J} S_j \).

Lemma A.2. (Theorem 13 of [3]) Let \( G(U \cup V, E) \) be bipartite graph. G is called a \( d, L, \epsilon \)-lossless expander if \( Γ(u) = d, \forall u \in U; Γ(J) \geq (1 − \epsilon)d|J|, \forall J \subseteq U, |J| \leq L \). Then if \( |U| = N, |V| = n, d \geq 2 \log N/\epsilon, n \geq 6Ld/\epsilon, \) a \( (d, L, \epsilon)\)-lossless expander exists.

We prove that for \( N = 2n^{1/6}/8 \), \( d = b_{\text{max}} = \mu = n^{1/3}, b_{\text{min}} = n^{1/6}, \epsilon = b_{\text{min}}/4μ = \frac{1}{4n^{1/6}}, \tau = \frac{2b_{\text{min}}}{n^{1/6}} = 2n^{1/6}, L = 2τ = 4n^{1/6} \), there exists \( S_1, \ldots, S_N \) such that for any \( b_1, \ldots, b_N \in [b_{\text{min}}, b_{\text{max}}] \), \( f(S_j) = b_i \) is a matroid rank function.

Let \( S_j = Γ(\{j\}) \) in the lossless expander (can check feasible under parameters above). Then \(|S_j| = d = n^{1/3}, |S_i \cap S_j| = |S_i| \wedge |S_j| = |Γ(\{i, j\})| \leq 2(n^{1/3} - 1) < n^{1/6} \). Let \( h(J) = \sum_{j \in J} b_j - (μ|J| - |S(J)|) \).

Lemma A.3. \( h \) is \((μ, τ)\)-large.

Proof. Notice that \( h(J) \geq b_{\text{min}}|J| - \epsilon|J|μ = 3ε|J|. \) When \( |J| \leq τ \), \( h(J) \geq 0; \) when \( |J| > τ \), \( h(J) \geq 3ετμ = 1.5μ > μ. \)

By Theorem A.1, \( \mathcal{I} = \{|I| \leq μ \wedge |I \cap S(J)| \leq h(J), \forall |J| < τ\} \) is the family of independent sets of a matroid. For any \( S_i \), pick \( B_i \subseteq S_i \) such that \( |B_i| = b_i \). Now we show that \( B_i \) is a maximum independent subset of \( S_i \). We need to verify the following properties:

- \( \text{rank}(S_i) \leq b_i \). This is true since for any \( S \subseteq S_i \), if \( S \) is independent, then \( |S \cap S_i| \leq h(\{i\}) = b_i \).

- \( B_i \) is independent. Only need to show that for any \( J, |B_i \cap S(J)| \leq h(J) \). When \( i \notin J, |J| \geq 2 \), \( h(J) = b_i + \sum_{j \in J, j \neq i} b_j - \epsilon|J|μ \geq b_i + (|J| - 1)b_{\text{min}} - |J|εμ \geq b_i \) since \( εμ = \frac{1}{4}b_{\text{min}} \). When \( i \notin J, |B_i \cap S(J)| \leq \sum_{j \in J} |B_i \cap S_j| \leq \sum_{j \in J} |S_i \cap S_j| \leq |J| \cdot 2eμ < 3εμ|J| \leq h(J) \). Thus \( B_i \in \mathcal{I} \), then \( B_i \) is independent.

Thus for any \( b \in [b_{\text{min}}, b_{\text{max}}]^N \), \( f(S_i) = b_i \) can be extended to a feasible matroid rank function. \( \square \)