On the integral representations of $|\Gamma(z)|^2$
and its Fourier transform

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Abstract

We derive integral representations in terms of the Macdonald functions for
the square modulus $s \mapsto |\Gamma(a + is)|^2$ of the Gamma function and its Fourier
transform when $a < 0$ and $a \neq -1, -2, \ldots$, generalizing known results in the
case $a > 0$. This representation is based on a renormalization argument using
modified Bessel functions of the second kind, and it applies to the representation
of the solutions of the Fokker-Planck equation.

Key words: Gamma function; Mellin transform; Fourier transform; Bessel functions;
Fokker-Planck equation.

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1 Introduction

The Fokker-Planck type equation

\[
\begin{aligned}
\frac{\partial U_p}{\partial t}(t, y) &= \left( y^2 \frac{\partial^2}{\partial y^2} + y \frac{\partial}{\partial y} - y^2 - p^2 \right) U_p(t, y), \quad y, t > 0, \\
U_p(0, y) &= y^p,
\end{aligned}
\]

originates from statistical physics [10]; it is also connected to the analysis of exponen-
tial functionals of Brownian motion [5], and to applications in mathematical finance,
cf. e.g. [3], [8], and references therein. The solution of (1.1) can be written using the heat kernel of the operator \( y^2 - y^2 \partial^2 / \partial y^2 - y \partial / \partial y \) (see e.g. [12]) as

\[
U_p(t, y) = \frac{2}{\pi^2} \int_0^\infty u \sinh(\pi u) K_{iu}(y) e^{-(p^2 + u^2)t} \int_0^\infty x^{p-1} K_{iu}(x) dx du.
\] (1.2)

Using the classical Mellin integral representation

\[
\Gamma(z + is) \Gamma(z - is) = 4 \int_0^\infty \left( \frac{x}{2} \right)^{2z} K_{2is}(x) \frac{dx}{x}, \quad \Re(z) > 0, \quad s \in \mathbb{R},
\] (1.3)

in the complex parameter \( z \in (0, \infty) + i \mathbb{R} \), cf. e.g. relation (26), page 331 of [6], one can show from the Fubini theorem that (1.2) can be turned for all \( p > 0 \) into the single integral representation

\[
U_p(t, y) = \frac{2p}{2\pi^2} \int_0^\infty u \sinh(\pi u) e^{-(p^2 + u^2)t} \left| \Gamma \left( \frac{p}{2} + i \frac{u}{2} \right) \right|^2 K_{iu}(y) du, \quad y, t > 0,
\]

which is more suitable for numerical implementations. Here,

\[
K_w(y) = \int_0^\infty e^{-y \cosh x} \cosh(wx) dx, \quad y > 0,
\] (1.4)

is the modified Bessel function of the second kind, or the Macdonald function, with parameter \( w \in \mathbb{C} \), cf. relation (5) page 181 of [11].

In this paper we give an extension of the Kontorovich-Lebedev transform (1.3) to all non-integer negative values of \( \Re(z) \). In particular, when \( z \in \mathbb{C} \) with \( \Re(z) \in (-1, 0) \) and \( s \in \mathbb{R} \), we show that (1.3) extends as

\[
\Gamma(z + is) \Gamma(z - is) = 4 \int_0^\infty \left( \frac{x}{2} \right)^{2z} K_{2is}(x) \left( 1 + \left( \frac{2}{x} \right)^{2z} \frac{4z}{\Gamma(1 - 2z)} K_{2z}(x) \right) \frac{dx}{x}
\] (1.5)

cf. Proposition 3.2 below for the general result, which reads

\[
\Gamma(z + is) \Gamma(z - is) = 2 \int_0^\infty K_{2is}(x) \left( 1 + \left( \frac{2}{x} \right)^{2z} \sum_{k=0}^n \frac{4(k + z)}{k! \Gamma(1 - k - 2z)} K_{2k+2z}(x) \right) \left( \frac{2}{x} \right)^{1-2z} dx
\] (1.6)

when \( z \in \mathbb{C} \) with \( \Re(z) \in (-n - 1, -n) \) and \( s \in \mathbb{R} \), for any \( n \in \mathbb{N} = \{0, 1, 2, \ldots \} \).
As an application, the representation (1.6) allows us to solve (1.1) for \( p \in (-2n-2, 0] \), \( n \in \mathbb{N} \), by the Fubini theorem, as

\[
U_p(s, y) = \frac{2}{\pi^2} \int_0^\infty u \sinh(\pi u) K_{iu}(y) e^{-(p^2+u^2)s} \int_0^\infty x^p K_{iu}(x) \frac{dx}{x} du
\]

\[
= \frac{2}{\pi^2} \int_0^\infty u \sinh(\pi u) K_{iu}(y) e^{-(p^2+u^2)s} \times \int_0^\infty x^p K_{iu}(x) \left( 1 + \sum_{k=0}^n \frac{2^{p+1}(p + 2k)}{k!\Gamma(1-p-k)} K_{-p-2k}(x) \right) \frac{dx}{x} du
\]

\[
- e^{-p^2s} \sum_{k=0}^n \frac{2^{p+1}(p + 2k)}{k!\Gamma(1-p-k)!} K_{-p-2k}(y)
\]

\[
= \frac{2^p}{2\pi^2} \int_0^\infty u \sinh(\pi u) e^{-(p^2+u^2)s} \left| \Gamma(\frac{p}{2} + i\frac{u}{2}) \right|^2 \Gamma(\frac{p+1}{2}) \right|^2 K_{iu}(y) du
\]

\[
- \sum_{k=0}^n e^{i((p+2k)^2-p^2)s} \frac{2^{p+1}(p + 2k)}{k!\Gamma(1-p-k)!} K_{-p-2k}(y),
\]

\( z \in \mathbb{R}, s > 0 \). The above expression has been originally obtained in [10] using spectral expansions, however the derivation presented here is much simpler since the argument of [10] involves severe analytical difficulties in the computation of normalization constants via the use of Meijer functions, cf. page 1641 therein.

On the other hand, Ramanujan showed in [9] that for \( a > 0 \) the Fourier transform of \( s \mapsto |\Gamma(a + is)|^2 \) satisfies the relation

\[
\int_{-\infty}^{\infty} e^{-i\xi s} |\Gamma(a + is)|^2 ds = \sqrt{\pi} \Gamma(a) \Gamma(a + 1/2)(\cosh(\xi/2))^{-2a}. \tag{1.8}
\]

This relation has been extended to all \( a \in (-1, 0) \) as an integral expression in Theorem 1.2 of [4].

As another application of (1.5), we show that it can be used to deduce an extension of (1.8) to all non-integer negative values of \( a \), using integral expressions. Namely when \( a \in (-1, 0) \) we deduce the integral representation

\[
\int_{-\infty}^{\infty} e^{-i\xi s} |\Gamma(a + is)|^2 ds \tag{1.9}
\]
\[
\frac{2\pi}{2^{2a}} \int_0^\infty x^{2a-1} \left(1 + \left(\frac{2}{x}\right)^{2a} \frac{4a}{\Gamma(1-2a)} K_{2a}(x)\right) e^{-x \cosh(\xi/2)} dx, \quad \xi \in \mathbb{R},
\]
for the Fourier transform of \( s \mapsto |\Gamma(a + is)|^2 \), which is another extension of (1.8) to \( a \in (-1,0) \), cf. Proposition 4.1 below for the general result which reads
\[
\int_{-\infty}^{\infty} e^{-i\xi s} |\Gamma(a + is)|^2 ds = \frac{2\pi}{2^{2a}} \int_0^\infty x^{2a-1} \left(1 + \left(\frac{2}{x}\right)^{2a} \sum_{k=0}^{n} \frac{4(a + k)}{k!\Gamma(1-k-2a)} K_{2a+2k}(x)\right) e^{-x \cosh(\xi/2)} dx, \quad \xi \in \mathbb{R}, a \in (-n - 1, -n), n \in \mathbb{N}.
\]

The integrability as \( x \to 0 \) in (1.5) and (1.9) is justified by the estimate
\[
x^{2a} + \frac{2^{2a+2a} a}{\Gamma(1-2a)} K_{2a}(x) = o(x^\varepsilon), \quad x \to 0,
\]
for any \( \varepsilon \in (0, 2+2a) \cap (0, -2a) \), when \( a \in (-1,0) \), cf. (2.9) and (2.12)-(2.13) below for the general case \( a \in (-n - 1, -n), n \in \mathbb{N} \).

This paper is organized as follows. In Section 2 we start by proving some asymptotic expansion and integrability results that are needed for the proof of both (1.5) and (1.9). The proof of the integral representation (1.6) and its extension to \( \Re(z) \in (-n - 1, -n) \) for all \( n \in \mathbb{N} \) are given in Section 3. Finally in Section 4 we derive the extension of the Fourier transform identity (1.9) to \( \Re(z) \in (-n - 1, -n) \) for all \( n \in \mathbb{N} \) as a consequence of Proposition 3.2.

\section{Asymptotic expansion and integrability} \label{sec:asymptotic expansion and integrability}

In this section we derive the asymptotic results needed for the proofs of (1.3)-(1.11) in Propositions 3.2 and 4.1 below. We will use the modified Bessel function of the first kind
\[
I_z(x) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+1)\Gamma(k+z+1)} \left(\frac{x}{2}\right)^{z+2k}, \quad x \in \mathbb{R}, \quad z \in \mathbb{C}.
\]
Lemma 2.1 For all $n \in \mathbb{N}$, $x \in \mathbb{R}$ and $z \in \mathbb{C}$ we have

$$
\left(\frac{2}{x}\right)^{2z} \sum_{k=0}^{n} \frac{k + z}{k! \Gamma(1 - k - 2z)} I_{2k+2z}(x) = \frac{\sin(2\pi z)}{2\pi} \tag{2.2}
$$

$$+
\sum_{l=n+1}^{\infty} \frac{1}{l!} \left(\frac{x}{2}\right)^{2l} \sum_{k=0}^{n} \binom{l}{k} \frac{k + z}{\Gamma(1 - 2z - k) \Gamma(k + l + 2z + 1)}.
$$

Proof. From (2.1) we have

$$
\left(\frac{2}{x}\right)^{2z} \sum_{k=0}^{n} \frac{k + z}{k! \Gamma(1 - k - 2z)} I_{2k+2z}(x) = \sum_{k=0}^{n} \frac{k + z}{k! \Gamma(1 - k - 2z)} \sum_{l=0}^{\infty} \frac{1}{l! \Gamma(2k + l + 2z + 1)} \left(\frac{x}{2}\right)^{2l} = \sum_{l=0}^{\infty} \left(\frac{x}{2}\right)^{2l} \sum_{k=0}^{\min(n,l)} \frac{k + z}{k! \Gamma(1 - k - 2z)(l - k)! \Gamma(k + l + 2z + 1)}. \tag{2.3}
$$

Next, using Euler’s reflection formula

$$
\frac{1}{\Gamma(k + l + 2z + 1)} = (-1)^{k+l+1} \frac{\sin(2\pi z)}{\pi} \Gamma(-2z - k - l), \quad k, l \in \mathbb{N}, \tag{2.4}
$$

cf. e.g. relation (6.1.17), page 256 of [1], we get

$$
\sum_{l=0}^{\infty} \left(\frac{x}{2}\right)^{2l} \sum_{k=0}^{l} \frac{k + z}{k! (l - k)! \Gamma(1 - k - 2z) \Gamma(k + l + 2z + 1)} = \frac{\sin(2\pi z)}{\pi} \sum_{l=0}^{\infty} \left(\frac{x}{2}\right)^{2l} (-1)^{l} \sum_{k=0}^{l} (-1)^{k} \binom{l}{k} \frac{(k + z) \Gamma(-2z - k - l)}{\Gamma(1 - 2z - k)}
$$

$$= \frac{\sin(2\pi z)}{\pi} \sum_{l=0}^{\infty} \left(\frac{x}{2}\right)^{2l} (-1)^{l} \frac{\Gamma(-2z - 2l)}{\Gamma(1 - 2z - k)} \sum_{k=0}^{l} \binom{l}{k} (k + z)(-2z - 2l)_{l-k}(2z)_{k}, \tag{2.5}
$$

where

$$
(p)_k = p(p + 1) \cdots (p + k - 1) \tag{2.6}
$$
is the shifted factorial, $p \in \mathbb{C}$, $k \geq 1$, with $(p)_0 = 1$. Next, for all $l \geq 1$ and $z \in \mathbb{C}$ we check that

$$
\sum_{k=0}^{l} \binom{l}{k} (k + z)(-2z - 2l)_{l-k}(2z)_{k}
$$
\[ z \sum_{k=0}^{l} \binom{l}{k} (-2z - 2l)_{l-k}(2z)_k - k \sum_{k=0}^{l} \binom{l}{k} (-2z - 2l)_{l-k}(2z)_k \]

\[ = z \sum_{k=0}^{l} \binom{l}{k} (-2z - 2l)_{l-k}(2z)_k - \sum_{k=1}^{l} \frac{l!}{(l-k)!(k-1)!} (-2z - 2l)_{l-k}(2z)_k \]

\[ = z \sum_{k=0}^{l} \binom{l}{k} (-2z - 2l)_{l-k}(2z)_k + 2zl \sum_{k=0}^{l-1} \binom{l-1}{k} (-2z - 2l)_{l-1-k}(2z + 1)_k \]

\[ = z(-2l)_l + 2zl(-2l + 1)_{l-1} \]

\[ = 0, \quad (2.7) \]

where we used the Pfaff-Saalschütz binomial identity

\[ \binom{p+q}{l} = \sum_{k=0}^{l} \binom{l}{k} \binom{p}{k} \binom{q}{l-k}, \quad p, q \in \mathbb{C}, \quad l \in \mathbb{N}, \]

cf. e.g. Theorem 2.2.6 and Remark 2.2.1 of [2]. As a consequence of (2.5) and (2.7) we get

\[ \sum_{l=0}^{n} \left( \frac{x^2}{2} \right)^{2l} \sum_{k=0}^{l} \frac{k + z}{k!(l-k)!\Gamma(1-k-2z)\Gamma(k+l+2z+1)} = \frac{\sin(2\pi z)}{2\pi}, \]

which allows us to rewrite (2.5) as

\[ \left( \frac{2}{x} \right)^{2z} \sum_{k=0}^{n} \frac{k + z}{k!\Gamma(1-k-2z)I_{2k+2z}(x)} \]

\[ = \frac{\sin(2\pi z)}{\pi} + \sum_{l=n+1}^{\infty} \left( \frac{x^2}{2} \right)^{2l} \sum_{k=0}^{l-1} \frac{k + z}{k!(l-k)!\Gamma(1-k-2z)\Gamma(k+2z+l+1)}, \]

and shows (2.2). \( \square \)

As a consequence of Lemma 2.1 we have the following estimates.

**Corollary 2.2** Let \( n \in \mathbb{N} \).

(i) For all \( z \in \mathbb{C} \) such that \( \Re(z) > -n - 1 \) we have

\[ \sum_{k=0}^{n} \frac{k + z}{k!\Gamma(1-k-2z)} I_{2k+2z}(x) = \frac{\sin(2\pi z)}{2\pi} \left( \frac{x^2}{2} \right)^{2z} + o(x^\varepsilon), \quad x \to 0, \quad (2.8) \]

for all \( \varepsilon \in (0, 2n + 2 + 2\Re(z)). \)
(ii) For all \( z \in \mathbb{C} \) such that \(-n - 1 < \Re(z) < -n \) we have

\[
1 + \left( \frac{2}{x} \right)^{2z} \sum_{k=0}^{n} \frac{4(z + k)}{k! \Gamma(1 - k - 2z)} K_{2k+2z}(x) = o(x^{\varepsilon - 2\Re(z)}), \quad x \to 0, \quad (2.9)
\]

for all \( \varepsilon \in (0, 2n + 2 + 2\Re(z)) \cap (0, -2n - 2\Re(z)) \).

**Proof.** (i) Relation (2.8) follows from Lemma 2.1. (ii) On the other hand, relation (2.1) with \(-2\Re(z) - 2k \geq -2\Re(z) - 2n > \varepsilon > 0\) shows that

\[
I_{-2z-2k}(x) = \sum_{l=0}^{\infty} \frac{1}{l! \Gamma(l - 2z - 2k + 1)} \left( \frac{x}{2} \right)^{-2z-2k+2l} = o(x^{\varepsilon}), \quad x \to 0, \quad (2.10)
\]

definition and the identity

\[
K_{2k+2z}(x) = \frac{\pi}{2 \sin(2\pi z)} (I_{-2z-2k}(x) - I_{2k+2z}(x)), \quad x \in \mathbb{R}, \quad (2.11)
\]

allow us to conclude to (2.9). \( \square \)

The next integrability result is a consequence of Lemma 2.1 and will be useful for the proofs of Propositions 3.1, 3.2 and 4.1 below.

**Lemma 2.3** Let \( n \in \mathbb{N} \).

(i) For all \( z \in \mathbb{C} \) such that \(-n - 1 < \Re(z) \) we have

\[
\sup_{s \in \mathbb{R}} \int_{0}^{\infty} |K_{is}(x)| \left| \frac{\sin(2\pi z)}{\pi} - 2 \left( \frac{2}{x} \right)^{2z} \sum_{k=0}^{n} \frac{k + z}{k! \Gamma(1 - k - 2z)} I_{2k+2z}(x) \right| \frac{dx}{x^{1-2z}} < \infty. \quad (2.12)
\]

(ii) For all \( z \in \mathbb{C} \) such that \(-n - 1 < \Re(z) < -n \) we have

\[
\sup_{s \in \mathbb{R}} \int_{0}^{\infty} x^{2z-1} |K_{is}(x)| \left| 1 + \left( \frac{2}{x} \right)^{2z} \sum_{k=0}^{n} \frac{4(k + z)}{k! \Gamma(1 - k - 2z)} K_{2k+2z}(x) \right| dx < \infty. \quad (2.13)
\]

**Proof.** (i) By relation (1.4), for all \( \alpha > 0 \) there exists a constant \( c_\alpha > 0 \) such that \( \cosh x > c_\alpha x^\alpha \) for all \( x > 0 \), which shows that

\[
|K_{is}(y)| \leq \int_{0}^{\infty} e^{-y \cosh x} dx \leq \int_{0}^{\infty} e^{-y c_\alpha x^\alpha} dx = \frac{\Gamma(1/\alpha)}{c_\alpha^{1/\alpha}} y^{-1/\alpha}, \quad y > 0, \quad s \in \mathbb{R}.
\]
Hence, using (2.8) we have, for all \( s \in \mathbb{R} \) and \( \alpha > 1/\varepsilon \),
\[
\int_0^1 |K_{is}(x)| \left| \frac{\sin(2\pi z)}{\pi} - \left( \frac{2}{x} \right)^{2z} \sum_{k=0}^{n} \frac{2z + 2k}{k!\Gamma(1 - k - 2z)} I_{2k+2z}(x) \right| \frac{dx}{x^{1-2z}} \leq c \Gamma \left( \frac{1}{\alpha} \right) \alpha \left( \frac{c \alpha}{c_{\alpha}} \right)^{1/\alpha} \int_0^1 \frac{dx}{x^{1-\varepsilon+1/\alpha}} < \infty, \quad s \in \mathbb{R},
\]
for some constant \( c > 0 \). Next, the bound
\[
|K_{is}(x)| \leq K_0(x), \quad x > 0, \quad s \in \mathbb{R}, \quad (2.14)
\]
that follows from relation \((3.5)\) below and the equivalences
\[
K_{is}(x) \simeq e^{-x} \sqrt{\frac{\pi}{2x}}, \quad x \to \infty, \quad s \in \mathbb{R}, \quad (2.15)
\]
and
\[
I_p(x) \simeq \frac{e^{x}}{\sqrt{2\pi x}}, \quad x \to \infty, \quad p \in \mathbb{C}, \quad (2.16)
\]
show that
\[
\sup_{s \in \mathbb{R}} \int_1^\infty |K_{is}(x)| \left| \frac{\sin(2\pi z)}{\pi} - 2 \left( \frac{2}{x} \right)^{2z} \sum_{k=0}^{n} \frac{k + z}{k!\Gamma(1 - 2z - k)} I_{2k+2z}(x) \right| \frac{dx}{x^{1-2z}} < \infty,
\]
which yields \((2.12)\) for all \( z \in \mathbb{C} \) such that \( \Re(z) > -n - 1 \).

\( (ii) \) Due to the equivalences \((2.11)\) and \((2.16)\) there also exists \( c > 0 \) such that
\[
\left| 1 + \left( \frac{2}{x} \right)^{2z} \sum_{k=0}^{n} \frac{4(k + z)}{k!\Gamma(1 - k - 2z)} K_{2k+2z}(x) \right| \leq cx^{-2z-1/2}e^x, \quad (2.17)
\]
for sufficiently large \( x > 0 \) and all \( z \in \mathbb{C} \), and this yields \((2.13)\) by replacing the use of \((2.8)\) with that of \((2.9)\) in the proof of part \( (i) \) above.

3 Analytic continuation and integral representation

In the next proposition, using analytic continuation, we prove an integral representation formula that will be applied to the proof of \((3.6)\) in Proposition \(3.2\) below.
Proposition 3.1 For all \( n \in \mathbb{N}, s \in \mathbb{R} \) and \( z \in \mathbb{C} \) with \( \Re(z) > -n - 1, \Re(z) \notin \mathbb{N} \), we have

\[
\Gamma(z + is) \Gamma(z - is) = \frac{2\pi}{\sin(2\pi z)} \sum_{k=0}^{n} \frac{k + z}{k!((z + k)^2 + s^2)\Gamma(1 - k - 2z)} (3.1)
\]

\[
+ 2^{2 - 2z} \int_{0}^{\infty} K_{2is}(x) \left( 1 - \frac{\pi}{\sin(2\pi z)} \left( \frac{2}{x} \right)^{2z} \sum_{k=0}^{n} \frac{2z + 2k}{k!\Gamma(1 - k - 2z)} I_{2k+2z}(x) \right) \frac{dx}{x^{1-2z}}.
\]

Proof. Let \( s \in \mathbb{R} \setminus \{0\} \). We will prove the equality

\[
\sin(2\pi z)\Gamma(z + is)\Gamma(z - is) = \sin(2\pi z)\Gamma(z + is)\Gamma(z - is) (3.2)
\]

\[
= \pi \sum_{k=0}^{n} \frac{2z + 2k}{k!((z + k)^2 + s^2)\Gamma(1 - k - 2z)} + \frac{4}{2^{2z}} \int_{0}^{\infty} K_{2is}(x) \left( \sin(2\pi z) - \pi \left( \frac{2}{x} \right)^{2z} \sum_{k=0}^{n} \frac{2z + 2k}{k!\Gamma(1 - k - 2z)} I_{2k+2z}(x) \right) \frac{dx}{x^{1-2z}},
\]

for all \( z = a + ib \in \mathbb{C} \) with \( a > -n - 1 \), in the following three steps.

(i) Analyticity. In (3.2), the function

\[
\sin(2\pi z)\Gamma(z + is)\Gamma(z - is)
\]

is analytic in \( \{z : z + is \notin (-N), z - is \notin (-N)\} \) and for each \( k = 0, 1, \ldots, n \) the function \((z + k)^2 + s^2)^{-1}\) is analytic in \( z \in \mathbb{C} \setminus (-N) \). On the other hand, by Lemma 2.1 we can write the integrand in (3.2) as

\[
x \mapsto \frac{K_{is}(x)}{x^{1-2z}} \left( \frac{\sin(2\pi z)}{\pi} - \left( \frac{2}{x} \right)^{2z} \sum_{k=0}^{n} \frac{2z + 2k}{k!\Gamma(1 - k - 2z)} I_{2k+2z}(x) \right)
\]

\[
= -2K_{is}(x) \sum_{k=0}^{n} \frac{x^{2z-1}}{k!\Gamma(1 - 2z - k)} \sum_{l=n+1}^{\infty} \left( \frac{x}{2} \right)^{2l} \frac{k + z}{(l-k)!\Gamma(k + 2z + l + 1)}
\]

\[
= -2K_{is}(x) \sum_{k=0}^{n} \frac{x^{2z-1}}{k!\Gamma(1 - 2z - k)\Gamma(k + 2z)} \sum_{l=n+1}^{\infty} \left( \frac{x}{2} \right)^{2l} \frac{k + z}{(l-k)!\Gamma(k + 2z + l) \cdots (k + 2z)},
\]

where for each \( k = 0, 1, \ldots, n \) the partial derivatives of

\[
z = a + ib \mapsto \sum_{l=n+1}^{\infty} \left( \frac{x}{2} \right)^{2l} \frac{k + z}{(l-k)!\Gamma(k + l + 2z) \cdots (k + 2z)}
\]
with respect to $a$ and $b$ are locally uniformly bounded by integrable functions of $x \in \mathbb{R}$ from the bounds (2.14) and (2.15), by the same arguments as in the proof of Lemma 2.3.

Hence we can exchange partial differentiation with respect to $a$ and $b$ with the integration in (3.3), showing that the Cauchy-Riemann conditions are satisfied by the integral (3.3) since all functions in the integrand are analytic in $z \in \mathbb{C}$. Consequently, all terms in (3.2) are analytic in $\{z \in \mathbb{C} : \mathcal{R}(z) > -n-1, z+is \notin (-N), z-is \notin (-N)\}$.

(ii) The equality (3.2) holds for all $s \in \mathbb{R} \setminus \{0\}$ and $z = a + ib \in (0, \infty) + i\mathbb{R}$. This follows from the integral representation (1.3) which reads

$$\Gamma(z+is) \Gamma(z-is) = 4 \int_0^\infty \left(\frac{x}{2}\right)^{2z} K_{2is}(x) \frac{dx}{x},$$

provided $a > 0$, and from the Mellin transform

$$4 \int_0^\infty K_{2is}(x) I_{2k+2z}(x) \frac{dx}{x} = \frac{1}{(z+k)^2 + s^2},$$

which is valid whenever $a + k > 0$, cf. e.g. relation (44) page 334 of [6].

(iii) By analytic continuation the relation (3.2) extends to $\{z \in \mathbb{C} : \mathcal{R}(z) > -n-1, z+is \notin (-N), z-is \notin (-N)\}$ and we conclude by dividing (3.3) by $\sin(2\pi z)$ when $z \in \mathbb{C}$ with $\mathcal{R}(z) > -n-1$ and $\mathcal{R}(z) \notin -N$.

Note that in the above proof we could also have used the unique continuation principle for real analytic functions of $a$, see e.g. Corollary 1.2.3 of [7], however real analyticity requires to check the growth rate of partial derivatives, which would have been more delicate.

Relation (1.3) can be recovered from the integral representation

$$K_{is}(y) = \frac{1}{2} \left(\frac{y}{2}\right)^{is} \int_0^\infty x^{-is-1} e^{-x-y^2/(4x)} dx, \quad s, y \in \mathbb{R},$$

cf. [11] page 183, using the Fubini theorem, as follows:

$$\Gamma(z+is) \Gamma(z-is) = \int_0^\infty x^{-2is-1} e^{-x} x^{z+is} \int_0^\infty y^{-1+z+2is} e^{-y} dy dx.$$
\[ \int_{0}^{\infty} \left( \frac{t}{2} \right)^{2z-1+2is} \int_{0}^{\infty} x^{-2is-1} e^{-x-t^2/(4x)} dx dt = 4 \int_{0}^{\infty} \left( \frac{t}{2} \right)^{2z} K_{2is}(t) \frac{dt}{t}, \quad s \in \mathbb{R}, \]

where we applied the change of variable \( y = t^2/(4x) \). However, this argument is valid only for \( \Re(z) > 0 \) due to integrability restrictions in the exchange of integrals.

We are now able to extend the above argument to all \( \Re(z) \in (-n - 1, -n), \ n \in \mathbb{N}, \) in order to prove the integral representation (3.6) which also implies (1.5).

**Proposition 3.2** For all \( n \in \mathbb{N}, \ s \in \mathbb{R} \) and \( z \in \mathbb{C} \) such that \(-n - 1 < \Re(z) < -n\) we have

\[ \Gamma(z + is) \Gamma(z - is) = 2 \int_{0}^{\infty} K_{2is}(x) \left( 1 + \left( \frac{2}{x} \right)^{2z} \sum_{k=0}^{n} \frac{4(k+z)}{k! \Gamma(1-k-2z)} K_{2k+2z}(x) \right) \left( \frac{2}{x} \right)^{1-2z} dx. \] (3.6)

**Proof.** First we note that the integrability in (3.6) follows from the bound (2.12) above. Next, from relations (2.11), (3.1), (3.4) with \(-2\Re(z) - 2k \geq -2\Re(z) - 2n > 0\), we have

\[ 2^{2z-2} \Gamma(z + is) \Gamma(z - is) \]

\[ = \frac{\pi}{\sin(2\pi z)} \int_{0}^{\infty} K_{2is}(x) \left( \frac{\sin(2\pi z)}{\pi} - \left( \frac{2}{x} \right)^{2z} \sum_{k=0}^{n} \frac{2z + 2k}{k! \Gamma(1-k-2z)} I_{2k+2z}(x) \right) \frac{dx}{x^{1-2z}} + \frac{2^{2z} \pi}{2 \sin(2\pi z)} \sum_{k=0}^{n} \frac{k+z}{k!(k+z)^2 + s^2} \Gamma(1-k-2z) \]

\[ = \frac{\pi}{\sin(2\pi z)} \int_{0}^{\infty} K_{2is}(x) \left( \frac{\sin(2\pi z)}{\pi} - \left( \frac{2}{x} \right)^{2z} \sum_{k=0}^{n} \frac{2z + 2k}{k! \Gamma(1-k-2z)} I_{2k+2z}(x) \right) \frac{dx}{x^{1-2z}} + \frac{2^{z} \pi}{\sin(2\pi z)} \sum_{k=0}^{n} \frac{2z + 2k}{k! \Gamma(1-k-2z)} \int_{0}^{\infty} K_{2is}(x) I_{2z-2k}(x) \frac{dx}{x} \]

\[ = \int_{0}^{\infty} K_{2is}(x) \left( 1 + \left( \frac{2}{x} \right)^{2z} \sum_{k=0}^{n} \frac{4(k+z)}{k! \Gamma(1-k-2z)} K_{2k+2z}(x) \right) \frac{dx}{x^{1-2z}}. \] \[ \square \]
4 Fourier transform of $|\Gamma(a + is)|^2$

We begin by proving an integral representation for the Fourier transform of

$$s \mapsto |\Gamma(a + is)|^2, \quad a \in (-n - 1, -n), \quad n \in \mathbb{N},$$

as a consequence of the integral representation (1.6) of Proposition 3.2.

**Proposition 4.1** Let $n \in \mathbb{N}$ and $a \in (-n - 1, -n)$. For all $\xi \in \mathbb{R}$ we have

$$\int_{-\infty}^{\infty} e^{-i\xi s} |\Gamma(a + is)|^2 ds = 2\pi \frac{2^a}{2^{2a}} \int_{0}^{\infty} x^{2a-1} \left( 1 + \frac{2}{x} \sum_{k=0}^{n} \frac{4(a + k)}{k!\Gamma(1 - k - 2a)} K_{2a+2k}(x) \right) e^{-x \cosh(\xi/2)} dx. \tag{4.1}$$

**Proof.** This result can be informally deduced from (3.6) in Proposition 3.2 and the Fourier-Gelfand formula

$$\int_{-\infty}^{\infty} \cos(2\pi y) e^{-i\xi s} ds = \pi (\delta(\xi/2 - y) + \delta(\xi/2 + y))$$

in distribution theory, where $\delta$ is the Dirac distribution at 0. However, with a view towards completeness, we provide a proof by approximation following the approach used in the proof of Theorem 1.1 of [4]. With the abbreviation

$$\Psi_n(x) := 1 + \frac{2}{x} \sum_{k=0}^{n} \frac{4(a + k)}{k!\Gamma(1 - k - 2a)} K_{2a+2k}(x), \quad x \in \mathbb{R}, \tag{4.2}$$

we rewrite (3.6) as

$$|\Gamma(a + is)|^2 = \frac{4}{2^{2a}} \int_{0}^{\infty} x^{2a-1} K_{2is}(x) \Psi_n(x) dx, \quad s \in \mathbb{R},$$

for $a \in (-n - 1, -n)$. Then for any $\epsilon > 0$ we have

$$\int_{-\infty}^{\infty} e^{-i\xi s} |\Gamma(a + is)|^2 ds = \int_{-\infty}^{\infty} \lim_{\epsilon \to 0} \left( e^{-2\epsilon s^2 - i\xi s} |\Gamma(a + is)|^2 \right) ds$$

$$= \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} e^{-2\epsilon s^2 - i\xi s} |\Gamma(a + is)|^2 ds$$

$$= 2^{2-2a} \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} e^{-2\epsilon s^2 - i\xi s} \int_{0}^{\infty} x^{2a-1} K_{2is}(x) \Psi_n(x) dx ds$$
\[
= \lim_{\epsilon \to 0} 2^{2-2a} \int_0^\infty x^{2a-1} \Psi_n(x) \int_{-\infty}^\infty e^{-2\epsilon s^2 - i\xi s} K_{2is}(x) ds dx, \quad (4.3)
\]
where the exchange of limit follows from the fact that \( s \to |\Gamma(a + is)|^2 \) is a rapidly decreasing function in the Schwartz class, and the last equality comes from (2.13) below which ensures the integrability required for the exchange of integrals. Next, from relation (1.4) written as

\[
K_{2is}(x) = \int_0^\infty e^{-x \cosh y} \cos(2sy) dy, \quad x > 0, \quad s \in \mathbb{R},
\]
we find

\[
\int_{-\infty}^\infty e^{-2\epsilon s^2 - i\xi s} K_{2is}(x) ds = \int_{-\infty}^\infty e^{-2\epsilon s^2 - i\xi s} \int_0^\infty e^{-x \cosh y} \cos(2sy) dy ds
\]

\[
= \int_{-\infty}^\infty e^{-x \cosh y} \int_0^\infty \cos(2sy) e^{-2\epsilon s^2 - i\xi s} ds dy
\]

\[
= \frac{1}{4} \sqrt{\frac{\pi}{2\epsilon}} \int_{-\infty}^\infty e^{-x \cosh y} (e^{-\frac{1}{2}(y-\xi/2)^2} + e^{-\frac{1}{2}(y+\xi/2)^2}) dy, \quad x > 0,
\]
hence by (1.4) we obtain

\[
\int_{-\infty}^\infty e^{-i\xi s} |\Gamma(a + is)|^2 ds
\]

\[
= 2^{2-2a} \lim_{\epsilon \to 0} \sqrt{\frac{\pi}{2\epsilon}} \int_0^\infty x^{2a-1} \Psi_n(x) \int_{-\infty}^\infty e^{-x \cosh y} (e^{-\frac{1}{2}(y-\xi/2)^2} + e^{-\frac{1}{2}(y+\xi/2)^2}) dy dx
\]

\[
= 2^{2-2a} \lim_{\epsilon \to 0} \sqrt{\frac{\pi}{2\epsilon}} \int_0^\infty x^{2a-1} \Psi_n(x) \int_{-\infty}^\infty e^{-x \cosh(y+\xi/2) - \frac{1}{2}y^2} dy dx
\]

\[
+ 2^{2-2a} \lim_{\epsilon \to 0} \sqrt{\frac{\pi}{2\epsilon}} \int_0^\infty x^{2a-1} \Psi_n(x) \int_{-\infty}^\infty e^{-x \cosh(y-\xi/2) - \frac{1}{2}y^2} dy dx
\]

\[
= 2^{2-2a} \sqrt{\frac{\pi}{2}} \lim_{\epsilon \to 0} \int_0^\infty x^{2a-1} \Psi_n(x) \int_{-\infty}^\infty e^{-x \cosh(z\sqrt{\epsilon}+\xi/2) - \frac{1}{2}z^2} dz dx
\]

\[
+ 2^{2-2a} \sqrt{\frac{\pi}{2}} \lim_{\epsilon \to 0} \int_0^\infty x^{2a-1} \Psi_n(x) \int_{-\infty}^\infty e^{-x \cosh(z\sqrt{\epsilon}-\xi/2) - \frac{1}{2}z^2} dz dx
\]

\[
= 2^{1-2a} \pi \int_0^\infty x^{2a-1} \Psi_n(x) e^{-x \cosh(\xi/2)} dx,
\]
where the required integrability follows from the bounds (2.9) and (2.17) of Section 2.

\[\square\]
In case \( a > 0 \), \( \Psi_{-1}(x) \) in (4.2) is identically equal to 1 and the proof of Proposition 4.1 also yields the Mellin transform
\[
\int_{-\infty}^{\infty} e^{-i\xi s} |\Gamma(a + is)|^2 \, ds = \frac{2\pi}{2^a} \int_{0}^{\infty} x^{2a-1}e^{-x \cosh(\xi/2)} \, dx = \frac{2\pi}{2^a} (\cosh(\xi/2))^{-2a} \Gamma(2a),
\]
which recovers (1.8), cf. also Theorem 1.1 in [4].

On the other hand, when \( a = -1/2 \) the Fourier transform of \( s \mapsto |\Gamma(-1/2 + is)|^2 \) can be explicitly computed as
\[
\int_{-\infty}^{\infty} e^{-i\xi s} |\Gamma(-1/2 + is)|^2 \, ds = 4\pi \int_{-\infty}^{\infty} \frac{e^{-i\xi s}}{(1 + 4s^2) \cosh(\pi s)} \, ds = 4\pi \log(1 + e^{-\xi}) \cosh(\xi/2) + 2\pi \xi e^{-\xi/2},
\]
cf. e.g. relation (22) page 32 of [6], whereas (4.1) yields
\[
\int_{-\infty}^{\infty} e^{-i\xi s} |\Gamma(-1/2 + is)|^2 \, ds = 4\pi \int_{0}^{\infty} \frac{1}{x} - K_{-1}(x) \right) e^{-x \cosh(\xi/2)} \frac{dx}{x},
\]
where the integrability in 0 in the above integral follows from
\[
\frac{1}{x} - K_{-1}(x) = o(x^\varepsilon), \quad x \to 0,
\]
for any \( \varepsilon \in (0, 1) \), cf. (1.11) below.

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