We present a prescription for obtaining Bell’s inequalities for $N > 2$ observers involving more than two alternative measurement settings. We give examples of some families of such inequalities. The inequalities are violated by certain classes of states for which all standard Bell’s inequalities with two measurement settings per observer are satisfied.

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Which quantum states do not allow a local realistic description \[22\]? Despite the considerable research efforts, this question remains open. One of the reasons for this is that our present tools to test local realism are not optimal. Most of Bell’s inequalities are for the case in which only two measurement settings can be chosen by each observer. One can call such inequalities "standard" ones. The usual Bell inequalities for bipartite two-dimensional systems \[1\, 2\] including the Clauser-Horne-Shimony-Holt (CHSH) inequality \[3\] and inequalities for bipartite higher-dimensional systems \[4\] are standard ones. Similarly, multipartite Bell’s inequalities like those of Mermin and Klyshko \[5\] and the recently found inequalities for correlation functions \[6\, 7\] also belong to the class.

One can expect that the full set of Bell inequalities for the case when the observers can choose between more than two observables should give a more stringent constraints on local realistic description of quantum predictions than the standard inequalities.

The derivation of such constraints is important for several reasons: First, the new inequalities can shed new light on the subtle relation between non-separability (impossibility to decompose quantum state of a composite system as a convex sum of product states) and violation of local realism. They may extend the class of non-separable states which cannot be described by local realistic models. There are non-separable mixed states that admit local realistic description \[8\], but the recent results show increasing subtlety in this relation, as the number of systems grows. Some multi-particle pure entangled states satisfy all standard Bell correlation function inequalities \[9\]. For example, the generalized GHZ states

$$|\psi\rangle = \cos \alpha |111\rangle + \sin \alpha |000\rangle,$$

satisfy all standard inequalities in the range $\alpha \in [0, \pi/12]$ \[9\, 10\]. Second, the violation of Bell’s inequalities is an important primitive for building quantum information protocols that decrease the communication complexity \[11\, 12\] and is a criterion for the efficient extraction of secure key in quantum key distribution protocols \[13\]. On the basis of Bell’s inequalities involving many measurement settings one can expect to build new quantum communication complexity protocols and strengthen the criteria for secure quantum key distribution.

Thus far we still lack a general and efficient method for deriving Bell inequalities involving more than two measurement settings per observer. One possible approach is to compute Bell’s inequalities that define the facets of the full correlation polytope \[14\]. Using this method a multipartite Bell inequality was found with 3 measurement settings per party, which can be violated even when the CHSH inequality is satisfied \[15\]. Yet this approach is limited by its connection to the computationally hard NP-problems \[16\] and thus applicable only for small numbers of parties, measurement settings and outcomes \[30\].

Various approaches combining numerical and analytical methods were proposed to derive Bell’s inequalities with more than two settings. In \[17\, 18\] Bell’s inequalities for two parties and many measurement settings were derived. Recently Wu and Zong \[19\] derived an inequality for three parties, which involves 4 local settings for two observers and 2 settings for the third one. This inequality can be violated by the states \[11\] for all values of $\alpha$ and thus is a more stringent condition on local realism than the all standard Bell inequalities \[6\, 7\]. In Ref. \[20\] a multiparty “functional” Bell inequality for a continuous range of settings of the apparatus at each site was derived. It is stronger than the standard Bell inequalities for certain classes of states, such as a family of bound entangled states \[21\], and the GHZ states \[20\].

Here we give an analytical method for deriving Bell’s inequalities for $N > 2$ parties and many measurement settings. The inequalities involving many measurement settings are generalizations of the standard inequalities. That is, by reducing the number of settings to two for each observer one recovers the full set of the standard Bell inequalities \[6\, 7\]. The inequalities reveal violations of local realism for wide classes of states, for which the standard Bell’s inequalities fail in this task. A derivation of the necessary and sufficient condition for the violation of the inequalities will be presented in a separate publication \[22\].

Before we give the method for many measurement settings, it is worthwhile to recall the method of obtaining the full set of correlation function Bell’s inequalities with two measurement settings per observer. Consider $N$ observers and allow each of them to choose between two dichotomic observables, determined by some local param-
eters denoted by $\tilde{n}_1$ and $\tilde{n}_2$. We choose such a notation for brevity; of course, each observer can choose independently two arbitrary directions. The assumption of local realism implies existence of two numbers $A^1_j$ and $A^2_j$, each taking values +1 or -1, which describe the predetermined result of a measurement by the $j$-th observer of the observable defined by $\tilde{n}_1$ and $\tilde{n}_2$, respectively. The following algebraic identity holds for the predetermined results for a single run of the experiment \[3\]:

$$\sum_{s_1, \ldots, s_N = \pm 1} S(s_1, \ldots, s_N) \prod_{j=1}^N [A^1_j + s_j A^2_j] = \pm 2^N,$$  \hspace{0.5cm} (2)

where $S(s_1, \ldots, s_N)$ stands for an arbitrary “sign” function of the summation indices $s_1, \ldots, s_N = \pm 1$, such that its values are only $\pm 1$, i.e. $S(s_1, \ldots, s_N) = \pm 1$. For a specific run of the experiment one can introduce the product of the local results $\prod_{j=1}^N A^j_{k_j}$, with $k_j = 1, 2$. The correlation function is the average over many runs of the experiment $E_{k_1, \ldots, k_N} = \langle \prod_{j=1}^N A^j_{k_j}\rangle_{\text{avg}}$. After averaging the expression \[3\] over the ensemble of the runs of the experiment one obtains the following set of Bell inequalities for local realistic correlation functions \[4\]:

$$\sum_{s_1, \ldots, s_N = \pm 1} S(s_1, \ldots, s_N) \sum_{k_1, \ldots, k_N = 1, 2} s_1^{k_1-1} \cdots s_N^{k_N-1} E_{k_1, \ldots, k_N} \leq 2^N.$$

Since there are $2^N$ different functions $S$, the above inequality represents a set of $2^N$ Bell inequalities \[5\].

Following the above ideas, and generalizing the trick introduced in \[11\], we show how to obtain Bell’s inequalities involving many measurement settings. We explain the method for the case of three observers, and then show how to generalize it to an arbitrary number of observers.

Suppose that the first two observers are allowed to choose between four settings $\{1, 2, 3, 4\}$, and the third one between two settings $\{1, 2\}$. We denote the family of inequalities that will be obtained as $4 \times 4 \times 2$ (this family contains the inequality of Wu and Zong \[1\]). To avoid too many indices we introduce a new notation for the local realistic values: $A_1, A_2, A_3$ and $A_4$ stand for the predetermined results for the first observer under the local setting 1, 2, 3 and 4, respectively, $B_1, B_2, B_3$ and $B_4$ are the similar values for the second observer, and $C_1$ and $C_2$ are the predetermined values for the third observer (for the given run).

The local realistic results for the pair of settings 1 and 2 of Alice and Bob satisfy the identity \[2\], i.e.

$$A_{12,12,S} \equiv \sum_{s_1, s_2 = \pm 1} S(s_1, s_2)(A_1 + s_1 A_2)(B_1 + s_2 B_2) = \pm 4,$$  \hspace{0.5cm} (3)

where 1 and 2 are chosen from a larger set of four measurement settings $\{1, 2, 3, 4\}$. Similarly one defines $A_{34,34,S'}$ by replacing $A_1, A_2, B_1, B_2$ by $A_3, A_4, B_3, B_4$ respectively and $S$ by $S'$. Depending on the value of $s = \pm 1$ one has $(A_{12,12,S} + s A_{34,34,S'}) = \pm 8$, or 0. Therefore, the following algebraic identity holds:

$$\sum_{s_1', s_2' = \pm 1} S''(s_1', s_2')(A_{12,12,S} + s_1' A_{34,34,S'})(C_1 + s_2'C_2) = \pm 16.$$  \hspace{0.5cm} (4)

With the use of \[3\], after averaging over the runs, we can generate a family of $(2^3)^3 = 2^{12}$ new Bell’s inequalities \[6\]. They form a necessary condition for local realistic description to hold. Note that inequalities involving three settings for a given observer can also be obtained by, e.g., choosing the settings 2 and 3 identical.

Let us give an example. The Wu-Zong inequality is obtained if one chooses $S''(1, 1) = S''(1, -1) = S''(-1, 1) = -S''(-1, -1)$. In such a case one obtains

$$\sum_{s_1, s_2 = \pm 1} S(s_1, s_2)(A_1 + s_1 A_2)(B_1 + s_2 B_2)(C_1 + C_2)$$

$$+ \sum_{s_1, s_2 = \pm 1} S'(s_1, s_2)(A_3 + s_1 A_4)(B_3 + s_2 B_4)(C_1 - C_2) = \pm 8.$$  \hspace{0.5cm} (5)

If one now puts $S' = \pm S''$ and $S = \pm S''$, and averages the resulting algebraic identity over the runs, one obtains

$$| \sum_{g=1,2} (E_{11g} + E_{21g} + E_{12g} - E_{22g}) |$$

$$+ | \sum_{g=1,2} (-1)^g(E_{33g} + E_{43g} + E_{34g} - E_{44g}) | \leq 4,$$

which is equivalent to the inequality \[7\].

For $N = 3$ Eq. \[4\] leads to the three families of non-trivial Bell inequalities. One can generate the full set of members of a family by permuting the local settings in an inequality belonging to the given family. In the table we give representative inequalities of the families, and their maximal quantum values.

| Typical inequality | Max. quantum value |
|--------------------|--------------------|
| $| - E_{111} - E_{331} + E_{112} - E_{332} | \leq 2$ | $2\sqrt{2}$ |
| $| - 2E_{111} - E_{331} - E_{431} - E_{341} + 2E_{112} - E_{332} - E_{432} - E_{342} + E_{442} | \leq 4$ | $4\sqrt{3}$ |
| $| - E_{111} - E_{331} - E_{431} + E_{221} - E_{332} - E_{432} - E_{441} + E_{441}$
  $+ E_{112} + E_{212} - E_{122} - E_{332} - E_{432} - E_{342} + E_{442} | \leq 4$ | $8$ |
The first family of Bell’s inequalities involves two settings for each observer (we shall denote this property by $2 \times 2 \times 2$) and thus belongs to the standard inequalities. Yet, the second and third family are new, of the type $3 \times 3 \times 2$ and $4 \times 4 \times 2$, respectively.

The results for some classes of states are summarized below. The new Bell inequalities often give more stringent conditions on local realism than the standard inequalities. All numerical results are obtained using the “amoeba” procedure.

(1) The maximal violations occur for the GHZ states: $|\psi\rangle = 1/\sqrt{2}(|000\rangle + |111\rangle)$ for all families.

(2) For $\rho = (1 - f)|\psi\rangle\langle\psi| + f |0\rangle\langle0|/8$, where $|\psi\rangle$ is the GHZ state, and $\rho /8$ is the completely mixed state (noise), the highest possible fraction of noise such that the state still does not allow a local realistic description is $f < \frac{1}{2}$ for both the standard and new Bell inequalities.

(3) Both the second and third family are violated by the generalized GHZ states $|\psi\rangle$ for the whole range of $\alpha$. For $0 \leq \alpha \leq \frac{\pi}{2}$, that is, the range of the parameter, for which none of the standard Bell inequalities $\|A\|\leq 1$ is violated, the third family is violated by the factor $\sqrt{1 + \sin^2 2\alpha}$. This shows that in this case the family is a stronger entanglement witness than the full set of (256) standard Bell’s inequalities $\|A\|\leq 1$.

(4) The $W$ state: $|W\rangle = 1/\sqrt{3}(|001\rangle + |010\rangle + |100\rangle)$ violates the inequalities of the third family by the factor of 1.7449, whereas for the standard correlation function Bell inequalities the factor is only 1.523. The highest possible fraction of the white noise that can be admixed to the $W$ state such that the resulting state still violates the inequality is $f < 0.4269$ for the new inequalities, whereas in the case of standard inequalities $f < 0.3434$.

(5) One way to show impossibility of local realistic description of mixed entangled states is to distill from them pure entanglement, that can violate Bell’s inequality. Yet this is not possible for bound entangled states. They are usually tested via various Bell’s inequalities. We have tested a three-qubit bound entangled state that has tripartite but no bipartite entanglement, i.e. the entanglement across any split into two parties is zero. The state is $\rho = 1/4(1 - \sum_{j=1}^{4} |\psi_j\rangle\langle\psi_j|)$, where $|\psi_1\rangle = |01\rangle$, $|\psi_2\rangle = |10\rangle$, $|\psi_3\rangle = |+\rangle$ and $|\psi_4\rangle = |-\rangle$ with $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$. However, both the standard and new Bell’s inequalities are satisfied.

The method just applied can be generalized in various ways to different numbers of measurement settings and to arbitrary number of observers. The following examples illustrate the strength of the method.

(a) $3 \times 3 \times 2$-type inequalities: The inequalities involving 3 settings for the first two observers and 2 settings for the last one can be obtained by, e.g., choosing the settings 1 and 2 identical for the two observers. If we put $A_2 = A_1$ and $B_2 = B_1$ in Eq. 10, after averaging over the runs, we obtain the following inequality:

$$4| \sum_{k=1,2} E_{11k} | + \sum_{s_1, s_2 = \pm 1} \sum_{i, j = 3, 4} \sum_{k = 1, 2} s_1^{i-1}s_2^{j-1}(-1)^{k-1}E_{ij} | \leq 8.$$ 

It is violated by the states $|\psi\rangle$ for the full range of $\alpha$. Again in the range of $0 < \alpha < \pi/2$, the violation is by the factor $\sqrt{1 + \sin^2 2\alpha}$. Thus, in this case, the $3 \times 3 \times 2$ inequalities are just as efficient as the $4 \times 4 \times 2$ ones.

(b) $4 \times 4 \times \ldots \times 4 \times 2$-type of inequality: The multipartite inequalities involving 4 measurement settings for the first $N - 1$ observers and 2 settings for the last observer can be derived as follows. Analogously as in the case of two observers in Eq. 8, for the local realistic predetermined values for the $N - 1$ observers one can introduce

$$A_{12, \ldots, 12, s} = \sum_{s_{1, \ldots, s_{N-1} = \pm 1}} S(s_1, \ldots, s_{N-1}) \prod_{j=1}^{N-1} (A_j^s + s_j A_j^0)$$

where 1 and 2 are two local settings chosen from a larger set of four settings, $\{1, 2, 3, 4\}$, for the $j$-th observer. By Eq. 2 one has $A_{12, \ldots, 12, s} = \pm 2^{N-1}$. Similarly, the local realistic results for the pairs of settings 3 and 4, and a different sign function $S'$, satisfy the identity $A_{34, \ldots, 34, s'} = \pm 2^{N-1}$. One has $[A_{12, \ldots, 12, s} + s_{34, \ldots, 34, s'}] = \pm 2^N$ or 0, depending on the value of $s = \pm 1$. By including the $N$-th observer, who can choose between 2 measurement settings $\{1, 2\}$ one obtains

$$\sum_{s_1, s_2 = \pm 1} S(s_1, s_2) (A_{12, \ldots, 12, s} + s_1 A_{34, \ldots, 34, s'}) (A_1^3 + s_2 A_3^0) = \pm 2^N.$$ 

One can use this expression for generating new Bell inequalities for $N$ observers in the same way as it was previously done in Eq. 11 for three observers.

(c) $2^{N-1} \times 2^{N-1} \times 2^{N-2} \times 2^{N-3} \times \ldots \times 2$-type of inequality: The method can also be applied to obtain Bell’s inequalities involving exponential (in $N$) number of measurement settings. The starting point is the identity $1$ of the type $4 \times 4 \times 2$ as derived above. One can introduce a similar identity for the settings 5, 6, 7, 8, for the first two observers, and 3, 4, for the third one. One can allow the forth observer to choose between two settings. Applying the same method as the one described in (b), one obtains an algebraic identity which generates Bell’s inequalities of the $8 \times 8 \times 4 \times 2$ type. One may apply the method iteratively, increasing the number of observers by one, to obtain inequalities of the type given above.

It is clear that the method can be extended to various combinations of the numbers of measurement settings, and observers. In Ref. 27 one can find another application of the method: a family of Bell inequalities for $N = 5$ qubits, which involves 8 settings for first two observers and 4 settings, for the other three.
In summary, we present new Bell’s inequalities involving many measurement settings and prove that they give more stringent conditions on the possibility of a local realistic description of quantum states, than the standard Bell’s inequalities for two settings per observer.

Let us remark on some practical implications of our results for communication complexity problems (problems of computing a function if its inputs are distributed among separated parties [28]). In Ref. [11] it was proven that for every Bell’s inequality there exists a communication complexity problem, for which the protocol assisted by states which violate the inequality is more efficient than the standard classical description of quantum states, than the standard protocol utilizing them cannot have any advantage over classical ones, as long as, the local inputs of the function have only two possible values. However, a byproduct of the analysis given above is that if one considers 3 or more values for the local inputs, a quantum communication complexity protocols involving generalized GHZ states can be more efficient than any classical one (since the criterion of their superiority is violation of the related Bell inequality with 3 or more settings).

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[28] A. C.-C. Yao, in Proceedings of the 11th Annual ACM Symposium on Theory of Computing, 200-213 (1979).
[29] Realism supposes that measurement results are predetermined by the properties the particles carry prior to and independent of observations. Locality supposes that these results are independent of any action at space-like separations.
[30] Because of the explosive growth of the number of inequalities, with growing number of particles, even the task of printing of the inequalities is a NP-problem!
[31] It is well known that any stochastic local realistic theory can be formulated with an underlying deterministic one. Thus, here, we consider only the deterministic theories.
[32] This set of inequalities is a sufficient and necessary condition for the correlation functions entering them to have a local realistic model.
[33] One has $2^4$ different functions $S$ (and also $S'$ and $S''$), thus the full multiplicity is $2^{12}$. 