Large-Maximal submodules

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Abstract. The goal of this research is to introduce the concept of Large-maximal submodule, also we will consider some properties of it, such that a proper submodule \( N \) of an \( R \)-module \( M \) is said to be Large-maximal (L-maximal) submodule of \( M \) if there exists a submodule \( K \) of \( M \) such that \( N < K \leq M \), then \( K \) is essential submodule of \( M \) \((K \leq_e M)\).

Keywords: maximal submodules, L-maximal submodules.

1. Introduction

Throughout this paper \( R \) represents a commutative ring with identity. It is well known that a proper submodule \( N \) of an \( R \)-module \( M \) is called maximal, if whenever \( K \) is a submodule of \( M \) with \( N < K \leq M \) implies that \( K = M \). Inaam and Riyadh in [2] introduced the concept of almost maximal submodules, where a proper submodule \( N \) of an \( R \)-module \( M \) is called almost maximal, if whenever \( K \) is an essential submodule of \( M \) with \( N < K \leq M \) implies that \( K = M \), where a submodule \( K \) of \( M \) is said to be essential \((K \leq_e M)\) if for every submodule \( N \) of \( M \) with \( K \cap N = (0) \) implies that \( N = (0) \). Muna and Shiren in [5] introduced the concept of S-maximal submodules, where a proper submodule \( N \) of an \( R \)-module \( M \) is called S-maximal, if whenever \( K \) is a semi essential submodule of \( M \) with \( N < K \leq M \) implies that \( K = M \), where a submodule \( K \) of \( M \) is called semi essential if \( K \cap N \neq (0) \), for every nonzero prime submodule \( N \) of \( M \) [9]. In this paper, we introduce the concept of Large-maximal submodule as a generalization of maximal submodule, where a proper submodule \( N \) of an \( R \)-module \( M \) is said to be Large-maximal (L-maximal) submodule of \( M \) if there exists a submodule \( K \) of \( M \) such that \( N < K \leq M \), then \( K \) is essential submodule of \( M \) \((K \leq_e M)\).

2. Large-maximal submodule

In this section we introduce the concept of Large-maximal submodule and many of its properties, also we give the concept of Large-Local module.

Definition (2.1): Let \( M \) be an \( R \)-module, a proper submodule \( N \) of \( M \), is called Large-maximal (L-maximal) submodule of \( M \) if there exists a submodule \( K \) of \( M \) such that \( N < K \leq M \), then \( K \) is essential submodule of \( M \) \((K \leq_e M)\). An ideal \( I \) is called L-maximal ideal if there exists an ideal \( J \) of \( R \) such that \( I < J \leq R \), then \( J \) is essential ideal of \( R \) \((J \leq_e R)\).

Remarks and Examples (2.2):
1- Every maximal submodule is L-maximal submodule. Thus in $Z_4$ as Z-module, $\{0, 2\}$ is maximal and L-maximal submodule since $\{0, 2\} < Z_4 \leq Z_4$ and $Z_4 \leq e \leq Z_4$.

2- The converse of (1) is not true, as the following example: In $Z$ as Z-module, $4Z$ is L-maximal submodule since $4Z < 2Z \leq Z$ and $2Z \leq e \leq Z$ but $4Z$ is not maximal submodule since $2Z \neq Z$.

3- A submodule of L-maximal submodule need not be L-maximal submodule, as the following example: In $Z_{36}$ as Z-module, $42Z_{36}$ is L-maximal since $4Z_{36} < 2Z_{36} \leq Z_{36}$ and $2Z_{36} \leq e \leq Z_{36}$ but $12Z_{36}$ not L-maximal since $12Z_{36} < 4Z_{36} \leq Z_{36}$ and $4Z_{36}$ is not essential in $Z_{36}$.

4- If $M$ is a uniform module, then every submodule of $M$ is L-maximal. For examples: the Z-modules $Q^\infty$ and $Z^{\infty}$.

5- The converse of (4) is not true, as the following example: In $Z_6$ as Z-module, $2Z_6$ and $3Z_6$ are maximal hence L-maximal submodule by (1), but $Z_6$ is not uniform module.

6- Every essential submodule of $M$ is L-maximal submodule of $M$.

Proof: Let $N$ be a proper submodule of $M$ such that $N \leq e \leq M$ and let $N < K \leq M$, then we have $N \leq e \leq K \leq e \leq M$ by [1], so $K \leq e \leq M$ hence $N$ is L-maximal submodule of $M$.

7- If $\frac{M}{N}$ is uniform module, then any submodule of $M$ is L-maximal.

Proof: Let $N$ be a proper submodule of $M$ such that $N < K \leq M$, thus $\frac{K}{N} \leq \frac{M}{N}$ and since $\frac{M}{N}$ is uniform then $\frac{K}{N} \leq e \leq \frac{M}{N}$ and hence $K \leq e \leq M$, so $N$ is L-maximal submodule of $M$.

8- If $\frac{M}{N}$ is simple, then $N$ is maximal submodule by [6] and hence $N$ is L-maximal submodule by (1).

9- Every non zero F-regular module has a maximal submodule [3], and hence L-maximal submodule by (1), where an R-module $M$ is called F-regular if every submodule of $M$ is pure [3].

10- Let $N$ and $K$ are proper submodules of $M$ such that $N < K$, if $N$ is L-maximal submodule in $K$ and $K$ is L-maximal submodule in $M$ then $N$ is not necessary L-maximal submodule in $M$, as the following example: Let $M = Z_{42}$, $K = 2Z_{42}$ and $N = 6Z_{42}$, $N$ is L-maximal in $K$ since $6Z_{42} < 2Z_{42} \leq 2Z_{42}$ and $2Z_{42} \leq e \leq 2Z_{42}$, also $K$ is L-maximal in $M$ since $2Z_{42} < Z_{42} \leq Z_{42}$ and $Z_{42} \leq e \leq Z_{42}$ but $N$ is not L-maximal in $M$ since $6Z_{42} < 3Z_{42} \leq 2Z_{42}$ and $3Z_{42} \leq e \leq Z_{42}$.

11- Let $M$ be a semisimple module and $N$ be a proper submodule of $M$ then $N$ is maximal submodule of $M$ if and only if, $N$ is L-maximal submodule of $M$.

12- Every chained module is uniform module, and then every submodule is L-maximal submodule by (4), where an R-module $M$ is called chained if for each submodules $U$ and $V$ of $M$, then either $U \leq V$ or $V \leq U$ [7].

13- Every integral domain is uniform module, and then every submodule is L-maximal submodule by (4).

14- Every simple module is uniform module, and then every submodule is L-maximal submodule by (4).

Proposition (2.3): Let $N$ and $K$ are proper submodules of $M$ such that $N \leq K$, if $N$ is L-maximal submodule of $M$ then $K$ is L-maximal submodule of $M$. 

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Proof: Let $K < W \leq M$. Since $N \leq K$ and $N$ is $L$-maximal submodule of $M$ then $N < W \leq M$ and $W \leq M$, so $K$ is $L$-maximal submodule of $M$.

Proposition (2.4): Let $N$ and $K$ are proper submodules of $M$, if $N \cap K$ is $L$-maximal submodule of $M$ then both $N$ and $K$ are $L$-maximal submodules of $M$.

Proof: Since $N \cap K \leq N$ and $N \cap K$ is $L$-maximal submodule of $M$ then by proposition (2.3), we have $N$ is $L$-maximal submodule of $M$, similarity $K$ is $L$-maximal submodule of $M$.

Remark (2.5): The converse of proposition (2.4) is not true, by (2.2).

Proposition (2.6): Let $N$ and $K$ are proper submodules of $M$, if $N$ and $K$ are $L$-maximal submodules of $M$ then $N + K$ is $L$-maximal submodule of $M$.

Proof: Since $N \leq N + K$ and $N$ is $L$-maximal submodule of $M$ then by proposition (2.3), we have $N + K$ is $L$-maximal submodule of $M$.

Remark (2.7): The converse of proposition (2.6) is not true, by (2.2).

Proposition (2.8): Let $f: M_1 \rightarrow M_2$ be an epimorphism where $M_1$ and $M_2$ be an $R$-modules such that, if $N$ is $L$-maximal submodule of $M_2$ then $f^{-1}(N)$ is $L$-maximal submodule of $M_1$.

Proof: Let $f^{-1}(N) < K \leq M_1$, so $f(f^{-1}(N)) < f(K) \leq M_2$ and hence $N < f(K) \leq M_2$, since $N$ is $L$-maximal submodule of $M_2$ then $f(K) \leq M_2$ and hence $f^{-1}(f(K)) = K \leq M_1$ [1], so $f^{-1}(N)$ is $L$-maximal submodule of $M_1$.

Proposition (2.9): Let $N$ is $L$-maximal submodule of $M$ and $I$ be an ideal of $R$, if $(N:_M I)$ is a proper submodule of $M$, then $(N:_M I)$ is $L$-maximal submodule of $M$.

Proof: Since $N < (N:_M I)$ and $N$ is $L$-maximal submodule of $M$, then by proposition (2.3), we have $(N:_M I)$ is $L$-maximal submodule of $M$.

Sometimes $(N:_M I) = M$, for example: If $M$ is multiplication module, then any submodule $N$ of $M$ can be written as $N = IM$ and hence $(N:_M I) = M$.

Remark (2.10): The converse of proposition (2.9) is not true, as the following example: In $M = Z_{12}$ as $Z$-module and the ideal $I = 2Z$ of $Z$, so $(N:_M I) = \{0, 2, 4, 6, 8, 10\}$ is $L$-maximal submodule of $M$ since $(N:_M I) = \{0, 2, 4, 6, 8, 10\} < Z_{12} \leq Z_{12}$ and $Z_{12} \leq Z_{12}$ but $N = \{0, 6\}$ is not $L$-maximal submodule of $M$ since $N = \{0, 6\} < \{0, 3, 6, 9\} \leq Z_{12}$ and $\{0, 3, 6, 9\}$ is not essential submodule in $Z_{12}$.

Remark (2.11): Every multiplication module contains an $L$-maximal submodule.

Proof: Since every multiplication module has a maximal submodule, hence it has $L$-maximal submodule by (2.2).

Corollary (2.12): Every cyclic $R$-module has $L$-maximal submodule.

Proof: Since every cyclic module is multiplication module, so we get the result by Remark (2.11).

Theorem (2.13): Let $M$ be a faithful, finitely generated and Multiplication $R$-module and $N$ be a submodule of $M$, then the following are equivalent:

1- $N$ is $L$-maximal submodule of $M$

2- $(N:_R M)$ is $L$-maximal ideal of $R$
$N = IM$ for some L-maximal ideal $I$ of $R$

Proof: (1) $\Rightarrow$ (2) Let $(N; R) < J \leq R$ where $J$ be an ideal of $R$, since $M$ is Multiplication, then $N = (N;R)M \leq JM \leq RM = M$ by [4], so $N \leq JM \leq M$ and since $N$ is L-maximal submodule of $M$ by (1), then $JM \leq eM = RM$ so $JM \leq eRM$ and hence $J \leq eR$ and $(N; R)$ is L-maximal ideal of $R$.

(2) $\Rightarrow$ (3) Since $M$ is Multiplication, then $N = (N;R)M$ by [4] and by (2), $(N; R)$ is L-maximal ideal of $R$, hence $N = IM$ for some L-maximal ideal $I$ of $R$.

(3) $\Rightarrow$ (1) Let $N < K \leq M$ so by (3), $N = IM$ for some L-maximal ideal $I$ of $R$ and $K = JM$ for some L-maximal ideal $J$ of $R$, since $M$ is Multiplication, then $N = IM < JM \leq RM = M$ and since $M$ is faithful, finitely generated and Multiplication then, $I < J \leq R$ by [8], and by (3) $I$ is L-maximal ideal of $R$, hence $J \leq eR$ then $JM \leq eM$ by [4], so $K \leq eM$ hence $N$ is L-maximal submodule of $M$.

Remark (2.14): For R-module $M$, if $N_1$ and $N_2$ are L-maximal submodules of $M$, then not necessary that $N_1 \oplus N_2$ is L-maximal submodule of $M$, as the following example: In $Z_6$ as Z-module it is clear that $N_1 = 2Z_6$ and $N_2 = 3Z_6$ are L-maximal submodules, but $2Z_6 \oplus 3Z_6 = Z_6$ is not L-maximal since $Z_6$ is not proper submodule of $Z_6$.

Now, we give the following definition.

Definition (2.15): A nonzero R-module $M$ is called Large-Local (L-Local) Module, if $M$ has only one L-maximal submodule which contains all proper submodule of $M$. A ring $R$ is called Large-Local (L-Local) Ring, if $R$ is an L-Local R-module.

Example (2.16): In $Z_6$ as Z-module, the Z-module $(\bar{2})$ is an L-Local Module since it has only one L-maximal submodule which is $(\bar{4})$.

Proposition (2.17): Let $M$ be a nonzero Multiplication and L-Local R-Module and let $N$ is an L-maximal submodule of $M$, if $N \neq (0)$ then $N$ is an essential submodule of $M$.

Proof: Let $K$ be a submodule of $M$ such that $N \cap K = (0)$, since $M$ is a nonzero Multiplication then by Remark(2.11), $M$ has L-maximal submodule $N$, also since $M$ is L-Local Module then $M$ has only one L-maximal submodule which is $N$ and contain all proper submodule $K$, so $K \leq N$ and hence $K = (0)$ and $N$ is an essential submodule of $M$.

Corollary (2.18): Let $R$ be an L-Local and let $I$ is an L-maximal ideal of $R$, if $I \neq (0)$ then $I$ is an essential ideal of $R$.

Theorem (2.19): Let $M$ be an R-module and $N$ be a proper submodule of $M$, consider the following:

1- $N$ is maximal submodule
2- $N$ is almost maximal submodule
3- $N$ is L-maximal submodule

If $M$ is chained module then (1) $\iff$ (2) and if $M$ is semisimple module then (2) $\iff$ (3) $\iff$ (1).

Proof: (1) $\iff$ (2) by [2].

(2) $\Rightarrow$ (3) Let $N < K \leq M$ since $N$ is almost maximal submodule then $K = M$, where $K$ be essential submodule of $M$, hence $N$ is L-maximal submodule.

(3) $\Rightarrow$ (2) Since $N$ is L-maximal submodule then $K \leq eM$, but $M$ is semisimple then $K = M$, hence $N$ is almost maximal submodule.
(3) \Leftrightarrow (1) \text{ Clear by (2.2).}

3. References

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