ON STOCHASTIC MULTI-GROUP LOTKA-VOLTERRA ECOSYSTEMS WITH REGIME SWITCHING

RUI WANG
School of Mathematics and Statistics, Northeast Normal University
Changchun, Jilin 130024, China

XIAOYUE LI∗
School of Mathematics and Statistics, Northeast Normal University
Changchun, Jilin 130024, China

DENIS S. MUKAMA
School of Mathematics and Statistics, Northeast Normal University
Changchun, Jilin 130024, China

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ABSTRACT. Focusing on stochastic dynamics involving continuous states as well as discrete events, this paper investigates dynamical behaviors of stochastic multi-group Lotka-Volterra model with regime switching. The contributions of the paper lie on: (a) giving the sufficient conditions of stochastic permanence for generic stochastic multi-group Lotka-Volterra model, which are much weaker than the existing results in the literature; (b) obtaining the stochastic strong permanence and ergodic property for the mutualistic systems; (c) establishing the almost surely asymptotic estimate of solutions. These can specify some realistic recurring phenomena and reveal the fact that regime switching can suppress the impermanence. A couple of examples and numerical simulations are given to illustrate our results.

1. Introduction. The Lotka-Volterra model proposed by Lotka [16] and Volterra [24] describes the interaction of n-species growth by
\[ \dot{x}(t) = \text{diag}(x_1(t), \ldots, x_n(t)) \left[ b + Ax(t) \right], \]
where \( x(t) = (x_1(t), \ldots, x_n(t))^T \in \mathbb{R}^n \), each \( x_i(t) \) represents the concentration size of \( i \)-th species, \( b = (b_1, \ldots, b_n)^T \in \mathbb{R}^n \) is the intrinsic growth rate vector, \( A = (a_{ij})_{n \times n} \in \mathbb{R}^{n \times n} \) is the community matrix in which \( a_{ij} \) measures the action of species \( j \) on the growth rate of species \( i \). Generally, the intra-specific (if \( i = j \)) interaction is competitive as individuals of the same species compete for resources, food, or habitat, thus \( a_{ii} \leq 0 \). If \( a_{ij} \leq 0 \) (\( 1 \leq i, j \leq n, i \neq j \)), (1) is termed competitive system while if \( a_{ij} \geq 0 \) (\( 1 \leq i, j \leq n, i \neq j \)), it is referred to as facultative mutualism system [5].

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∗ Corresponding author: Xiao Yue Li.
In fact, ecosystems are always subject to environmental noises. It is necessary to reveal how the noise affects the population ecosystems. Stochastic population ecosystems have been drawn great attention by many researchers depicting more realistic ecosystems. Lotka-Volterra models perturbed by white noise have been investigated in [18] [20] while the authors in [19] [23] [27] go further by considering the delay effects on the stochastic population ecosystems. The population habitats are distinguished by circumstance such as nutrition or rainfall [3] [22] that at random instants the population growth is subjected to a palpable change. Such random changes can’t be described precisely by the white noises induced by Brownian motion. In fact, the model involving both continuous dynamics and discrete events is more pertinent, in which the discrete events are modeled by a continuous-time Markov chain; see [19] [23] [27]. Our aim is to investigate the stochastic multi-group Lotka-Volterra ecosystem with environmental fluctuations described by

\[ dx_i(t) = x_i(t) \left( b_i(\gamma(t)) + \sum_{j=1}^{n} a_{ij}(\gamma(t))x_j(t) \right) dt + x_i(t) \sum_{j=1}^{d} \sigma_{ij}(\gamma(t))dB_j(t), \]

or equivalently, in the matrix form,

\[ dx(t) = \text{diag}(x_1(t), \cdots, x_n(t)) \left( [b(\gamma(t)) + A(\gamma(t))x(t)] dt + \sigma(\gamma(t))dB(t) \right), \]

with initial value \( x(0) = x_0 \in \mathbb{R}^+_n \), \( \gamma(0) = \gamma_0 \in \mathbb{S} := \{1, 2, \cdots, m\} \). \( \gamma(\cdot) \) is a continuous-time Markov chain with finite-state space \( \mathbb{S} \) and the generator \( Q = (q_{ij})_{m \times m} \in \mathbb{R}^{m \times m} \) satisfying \( q_{ij} \geq 0 \) for \( i \neq j \); \( \sum_{j \in \mathbb{S}} q_{ij} = 0 \) for each \( i \in \mathbb{S} \). \( B(\cdot) \) is a standard \( d \)-dimensional Brownian motion.

The dynamics of ecosystems with regime switching are interesting and amazing. Considering a two-dimensional predator-prey Lotka-Volterra ecosystem switching between two individual systems described by ordinary differential equations (ODEs), Takeuchi et al. in [23] revealed that both individual systems develop periodically but switching between them makes them become neither permanent nor dissipative. Various large-time behaviors of stochastic Lotka-Volterra ecosystems with regime switching have attracted more and more attention recently. For instance, criteria of extinction and permanence on sample paths to the predator-prey systems are yielded in [11]; The principle of coexistence and exclusion for the two-species competitive system are proved in [21]; More asymptotic properties for competitive systems and generic systems are referred to [3] [13] [17] [28] [29]. As far as we know there are few results on mutualism dynamics except for the ergodicity in [14], which is one of our motivations.

Taking into account the dynamics of ecosystem switching between different habitat, we always ask ourselves: Whether the species will be permanent when their individual habitat is impermanent? To illustrate this, consider a stochastic predator-prey system switching randomly between two individual environments described by

\[
\begin{cases}
  dx_1(t) = x_1(t) \left( 2 - x_2(t) \right) dt + x_1(t)(dB_1(t) + dB_2(t)), \\
  dx_2(t) = x_2(t) \left( 2 \right) dt + x_2(t)(\sqrt{2}dB_1(t) + dB_2(t)),
\end{cases}
\]

and

\[
\begin{cases}
  dx_1(t) = x_1(t) \left( 1 - \frac{1}{2}x_1(t) - 2x_2(t) \right) dt + x_1(t)(\sqrt{2}dB_1(t) + 2dB_2(t)), \\
  dx_2(t) = x_2(t) \left( 2 + x_1(t) - \frac{1}{2}x_2(t) \right) dt + x_2(t)(dB_1(t) + 2dB_2(t)),
\end{cases}
\]
modulated by a Markov chain $\gamma(t)$ with generator $Q = \begin{pmatrix} -9 & 9 \\ 1 & -1 \end{pmatrix}$. Note that the sum of community matrix of (4) and its transpose has eigenvalue 0. The existing results in the literatures such as [11, 14, 20, 28] cannot deal with this case. Figure 1 pictures two sample trajectories (resp., colored in black and blue) associated to (4) and (5). We see that the black (resp. blue) trajectory increases to infinity (tends to zero). So the species in both environments are impermanent. What will happen to the switched system? This inspires us to formulate this paper.

Figure 1. Sample paths of $|x(t)|$ of (4) (left) and (5) (right).

In this paper, we investigate permanence and asymptotic estimate of the sample paths for generic stochastic ecosystems with regime switching, then make further efforts on the mutualism dynamics involving strong permanence and existence of the invariant measure. Our contributions are as follows:

- We reveal an interesting feature that the regime switching has average effect on the dynamics and can suppress the impermanence. For instance, the switched ecosystem is stochastically permanent by Theorem 3.8 although both (4) and (5) are impermanent; see Figure 2.
- Compared with the results on the asymptotic estimates of moments in the literature [11, 28] and [19, pp.157-163], we relax the restriction on the coefficients required in [28], and the restriction on the generator of the Markov chain $\gamma(t)$ which is often required by utilizing the M-Matrix theory in [11] and [19, pp.157-163].
- Combining stochastic Lyapunov analysis and asymptotic analysis, we obtain the criterion on stochastic strong permanence of the mutualism ecosystem; see Theorem 5.4. Moreover, we analyze the almost surely asymptotic properties including the growth rate and sample paths in time average from both the above and below; see Theorems 4.1-4.3.
- Compared with the results on the ergodicity of the mutualism ecosystem in the literature [14, 20], we relax the restrictions on the coefficients and the generator of the Markov chain $\gamma(t)$ required in [14] and cover the main result on the ergodicity of the mutualism ecosystem without regime switching in [20].

The above discussions rely on the value of switching rate (generator) which is constant. Hutchinson [7] suggested that sufficiently frequent switching can reverse the trend of competitive exclusion for the competing species. It inspires a large amount of work on competitive dynamics with temporal fluctuations of the environment. Recently, Benaim and Lobry [2] investigated a two-species competitive
Lotka-Volterra system switching between two different environments (described by ODEs) that are both favorable to the same species and showed rigorously that the sufficient large switching rate depending on time $t$ can lead to extinction of this species or coexistence of the two competing species as $t$ tends to infinity. Comparing with the results in [2], we focus on the dynamics of the generic stochastic ecosystem involving several interspecific relationships switched with a constant switching frequency due to the complexity and nonlinearity of stochastic differential equations (SDEs), then go further to more precise large-time behaviors of the mutualism system. We shed light on the average effect of environmental switching that switching can suppress the impermanence.

The main methods of this paper are stochastic Lyapunov analysis and asymptotic analysis. Yin and Zhu in [27] began to utilize Fredholm alternative result to construct the Lyapunov functions depending on states and yielded stability and instability of linearized switching SDEs [27] pp. 203-207 and linearized switching ODEs [27] pp. 227-230. Note that (3) is a $n$-dimensional nonlinear system, especially the interaction among the species brings about more complexity and requires more computational effort. We borrow the Yin and Zhu’s idea in [27], construct several Lyapunov functions basing on the special structure, and yield our required dynamics of (3) by combining stochastic Lyapunov analysis with the asymptotic analysis. Moreover, we think this method is an attractive option to investigate more dynamical properties for nonlinear switching systems.

The rest of this paper is organized as follows. We introduce some preliminaries in Section 2. Section 3 is devoted to stochastic permanence of generic Lotka-Volterra ecosystem. Section 4 analyzes the almost surely asymptotic properties of the solutions including the growth rate and the time average. Section 5 begins to investigate mutualistic Lotka-Volterra ecosystem and yields the stochastic strong permanence. Section 6 goes a step further to examine ergodic property and positive recurrence, which accounts for some recurring events of the population ecosystem. Section 7 discusses some examples and carries out numerical simulations to illustrate our main results. Finally, Section 8 is the concluding remarks.

2. Preliminaries. Suppose that both $\gamma(\cdot)$ and $B(\cdot) = (B_1(\cdot), \ldots, B_d(\cdot))^T$ are defined independently in a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). For each $k \in S$, $b(k) = (b_1(k), \ldots, b_n(k))^T \in \mathbb{R}_+^n$, $A(k) = (a_{ij}(k))_{n \times n} \in \mathbb{R}^{n \times n}$, $\sigma(k) = (\sigma_{ij}(k))_{n \times d} \in \mathbb{R}^{n \times d}$ which is the noise intensity matrix.
A generator $Q$ or its corresponding Markov chain $\gamma(t)$ is called \textit{irreducible} if the following linear equations

$$\pi Q = 0, \quad \sum_{i=1}^{m} \pi_i = 1,$$

have a unique solution $\pi = (\pi_1, \cdots, \pi_m) \in \mathbb{R}^{1 \times m}$ (a row vector) satisfying $\pi_i > 0$ for each $i \in S$. Such a solution is termed a stationary distribution \cite{27}. Throughout this paper, as a standing assumption, we assume $\pi U$ above for each $\pi U$ have a unique solution for different appearances. For any sequence $1 \leq i \leq n$ for example, $\pi U$ and $\pi u$ to denote the corresponding matrix multiplications with compatible dimensions. That is, for example, $U \in \mathbb{R}^{m \times m}$ and $u \in \mathbb{R}^{m} := \mathbb{R}^{m \times 1}$, respectively.

Denote the positive cone by $\mathbb{R}^n_+ = \{ x \in \mathbb{R}^n : x_i > 0 \text{ for all } 1 \leq i \leq n \}$ and the nonnegative positive cone by $\mathbb{R}^n_0 = \{ x \in \mathbb{R}^n : x_i \geq 0 \text{ for all } 1 \leq i \leq n \}$. If $A$ is a vector or matrix, denote its transpose by $A^T$ and its trace norm by $|A| = \sqrt{\text{trace}(A^T A)}$. If $A = (a_{ij})_{n \times n}$ is a symmetric $n \times n$ matrix, its largest eigenvalue is denoted by $\lambda_{\text{max}}(A)$, and define $\lambda_{\text{max}}^+(A) = \sup_{x \in \mathbb{R}^n_+, |x|=1} x^T A x$. Indeed, $\lambda_{\text{max}}^+(A) \leq \lambda_{\text{max}}(A)$. We state a useful lemma which can be found in \cite{19} pp. 58-62.

**Lemma 2.1.** \cite{19} pp. 58-62 $\lambda_{\text{max}}^+(A) \leq \max_{1 \leq i \leq n} \left( a_{ii} + \sum_{j \neq i} [a_{ij}]^+ \right)$, where $[a]^+ := a \lor 0$ for any $a \in \mathbb{R}$.

Let $(y(t), \gamma(t))$ be the diffusion process described by

$$dy(t) = f(y(t), \gamma(t))dt + g(y(t), \gamma(t))dB(t),$$

$$y(0) = y_0, \quad \gamma(0) = \gamma_0,$$

where

$$f : \mathbb{R}^n \times S \rightarrow \mathbb{R}^n, \quad g : \mathbb{R}^n \times S \rightarrow \mathbb{R}^{n \times d}.$$ 

For any $k \in S$ and any $V(y, t, k)$ defined on $\mathbb{R}^n \times \mathbb{R}_+$ which is continuously twice differentiable in $y$ and once in $t$, define $\mathcal{L}_y V$ by

$$\mathcal{L}_y V(y, t, k) = V_t(y, t, k) + V_y(y, t, k)f(y, k) + \frac{1}{2} \text{trace} \left[ g^T(y, k)V_{yy}(y, t, k)g(y, k) \right]$$

$$+ \sum_{i=1}^{m} q_{ki} V(y, t, l),$$

where

$$V_t(y, t, k) = \frac{\partial V(y, t, k)}{\partial t}, \quad V_y(y, t, k) = \left( \frac{\partial V(y, t, k)}{\partial y_1}, \cdots, \frac{\partial V(y, t, k)}{\partial y_n} \right),$$

and

$$V_{yy}(y, t, k) = \left( \frac{\partial^2 V(y, t, k)}{\partial y_j \partial y_l} \right)_{n \times n}.$$ 

For convenience, denote by $K$ a generic positive constant whose value may be different for different appearances. For any sequence $\{c_i(k)\}$ ($i = 1, \cdots, n; k = 1, \cdots, m$), define

$$\hat{c}(k) = \min_{1 \leq i \leq n} c_i(k), \quad \hat{c}_i = \min_{1 \leq k \leq m} c_i(k), \quad \hat{c} = \min_{1 \leq i \leq n, 1 \leq k \leq m} c_i(k),$$

$$\check{c}(k) = \max_{1 \leq i \leq n} c_i(k), \quad \check{c}_i = \max_{1 \leq k \leq m} c_i(k), \quad \check{c} = \max_{1 \leq i \leq n, 1 \leq k \leq m} c_i(k).$$
For any sequence \( \{ c_i \} \) \((i = 1, \cdots, n)\) or \( \{ c(k) \} \) \((k = 1, \cdots, m)\), we define \( \hat{c} \) and \( \check{c} \) similarly.

3. **Stochastic permanence.** Stochastic differential equation (SDE) (3) models the population growth, each component of its solution representing the concentration size of the species. They should not only be nonnegative but also not explode to infinity at any finite time. Although the classical theory of SDEs (see e.g. [4, 19]) is not applicable directly to SDE (3), the existence of the global positive solution has been obtained by the standard technique in [11]. In this paper we cite this result and its corresponding assumption.

**Assumption 1.** For each \( k \in S \), there exist positive numbers \( c_1(k), \cdots, c_n(k) \), such that

\[
\lambda_k := \lambda_{\text{max}}(\bar{C}(k)A(k) + A^T(k)\bar{C}(k)) \leq 0,
\]

where \( \bar{C}(k) := \text{diag}(c_1(k), \cdots, c_n(k)) \).

**Lemma 3.1.** [11] Under Assumption 1, for any initial value \( x_0 \in \mathbb{R}^n_+ \), \( \gamma_0 \in S \), there is a unique solution \( x(t) \) to the SDE (3) and the solution will remain in \( \mathbb{R}^n_+ \) for all \( t \geq 0 \) almost surely.

In this lemma, \( \lambda_k \leq 0 \) for each \( k \in S \) is necessary for the global existence of solutions because even for \( n = 1 \), if some \( \lambda_k > 0 \), solutions may explode at finite time [12]. The above result provides us with a great opportunity to analyze the stochastic permanence. For clarity, we cite the definitions of stochastic permanence and its relatives in [11].

**Definition 3.2.** [11] The solutions of SDE (3) are called stochastically ultimately bounded from the above (resp. the below), if for any \( \epsilon \in (0, 1) \), there is a positive constant \( \chi(= \chi(\epsilon)) \) such that for any initial value \( x_0 \in \mathbb{R}^n_+ \), \( \gamma_0 \in S \), the solution of SDE (3) has the property that

\[
\limsup_{t \to +\infty} P \{|x(t)| > \chi\} < \epsilon \quad \text{and} \quad \limsup_{t \to +\infty} P \{|x(t)| < \chi\} < \epsilon.
\]

**Definition 3.3.** [11] The SDE (3) is called stochastically permanent if its solutions are stochastically ultimately bounded from both the above and below.

We begin with a criterion on asymptotic upper boundedness of the moment, and make use of it to obtain the stochastically ultimate upper boundedness of SDE (3).

**Theorem 3.4.** Under Assumptions 1 and \( \pi \lambda < 0 \), for \( p > 0 \) sufficiently small the solution \( x(t) \) of SDE (3) with any initial value \( x_0 \in \mathbb{R}^n_+ \), \( \gamma_0 \in S \) has the property that

\[
\limsup_{t \to \infty} E (\log^p |x(t)|) \leq K,
\]

where \( \lambda = (\lambda_1, \cdots, \lambda_m)^T \).

**Proof.** Since the proof is rather technical we divided it into 3 steps.

**Step 1.** In order for the required assertion (9), we begin to construct an appropriate Lyapunov function depending on the states in \( S \). Under Assumption 1 define \( C_k := (c_1(k), \cdots, c_n(k)) \) for each \( k \in S \). Note that

\[
\pi [C - (\pi C) I_m] = 0,
\]
where $C := \left(\frac{\lambda_1}{2|\pi C_1|^2}, \ldots, \frac{\lambda_m}{2|\pi C_m|^2}\right)^T$, $\|m := (1, \ldots, 1)^T$. Then it follows from the Fredholm Alternative result (c.f. [27, pp.362-366]) that

$$Q\alpha = C - (\pi C) \|m$$

has a solution $\alpha = (\alpha_1, \ldots, \alpha_m)^T \in \mathbb{R}^m$. Thus

$$\frac{\lambda_k}{2|C_k|^2} - \sum_{l=1}^{m} q_{kl}\alpha_l = \pi C \leq \frac{\pi \lambda}{2 \min_{1 \leq k \leq m} \{|C_k|^2\}} < 0, \ \forall k \in S. \ (10)$$

Choose a small constant $0 < p_1 \leq 1$, such that for any $0 < p \leq p_1$,

$$1 - \alpha_k p > 0, \ k = 1, \ldots, m.$$

Define $V_1 : \mathbb{R}^n_+ \times S \to \mathbb{R}_+$ by

$$V_1(x, k) = (1 - \alpha_k p)[1 + \log(C_k x + 1)]^p.$$

The direct calculation implies

$$V_{1,x}(x, k) = (1 - \alpha_k p)p[1 + \log(C_k x + 1)]^{p-1} \frac{C_k}{C_k x + 1},$$

$$V_{1,xx}(x, k) = (1 - \alpha_k p)p(p-1)[1 + \log(C_k x + 1)]^{p-2} \frac{C_k^T C_k}{(C_k x + 1)^2}$$

$$- (1 - \alpha_k p)p[1 + \log(C_k x + 1)]^{p-1} \frac{C_k^T C_k}{(C_k x + 1)^2}$$

$$= (1 - \alpha_k p)p[1 + \log(C_k x + 1)]^{p-2}[p - 2 - \log(C_k x + 1)] \frac{C_k^T C_k}{(C_k x + 1)^2}.$$

Then it follows from the inequality $a > \log a$ for any $a > 0$ and the non-positivity of $\lambda_k$ that

$$\mathcal{L}_x V_1(x, k) =
\begin{align*}
p(1 - \alpha_k p) [1 + \log(C_k x + 1)]^{p-1} & \left( \frac{x^T C(k)b(k)}{C_k x + 1} + \frac{x^T C(k)A(k)x}{C_k x + 1} \right) \\
& + \frac{1}{2} p(1 - \alpha_k p) [1 + \log(C_k x + 1)]^{p-2} [p - 2 - \log(C_k x + 1)] \left| \frac{x^T C(k)\sigma(k)}{C_k x + 1} \right|^2 \\
& + \sum_{l=1}^{m} q_{kl}(1 - \alpha_l p) [1 + \log(C_l x + 1)]^p
\end{align*}
$$

$$\leq
\begin{align*}
p(1 - \alpha_k p) [1 + \log(C_k x + 1)]^{p-1} & \left[ b(k) + \frac{\lambda_k |x|^2}{2(C_k x + 1)} \right] \\
& + \sum_{l=1}^{m} q_{kl}(1 - \alpha_l p) [1 + \log(C_l x + 1)]^p
\end{align*}
$$

$$\leq
\begin{align*}
p(1 - \alpha_k p) [1 + \log(C_k x + 1)]^{p-1} & \left[ b(k) + \frac{\lambda_k (C_k x + 1 - 1)^2}{2|C_k|^2(C_k x + 1)} \right] \\
& + \sum_{l=1}^{m} q_{kl}(1 - \alpha_l p) [1 + \log(C_l x + 1)]^p
\end{align*}$$
\[
\begin{align*}
&\leq p(1 - \alpha_k p) \left[ 1 + \log(C_k x + 1) \right]^{p-1} \left[ \hat{b}(k) + \frac{\lambda_k (C_k x + 1)}{2|C_k|^2} - \frac{\lambda_k}{|C_k|^2} \right] \\
&\quad + \sum_{l=1}^{m} q_{kl}(1 - \alpha_p) \left[ 1 + \log(C_l x + 1) \right]^{p} \\
&\leq p(1 - \alpha_k p) \left[ 1 + \log(C_k x + 1) \right]^{p-1} \left( \frac{\lambda_k}{2|C_k|^2} \log(C_k x + 1) + \hat{b}(k) - \frac{\lambda_k}{|C_k|^2} \right) \\
&\quad + \sum_{l=1}^{m} q_{kl}(1 - \alpha_p) \left[ 1 + \log(C_l x + 1) \right]^{p} \\
&= p V_1(x, k) \varphi(x, k),
\end{align*}
\]
where
\[
\varphi(x, k) := \frac{\lambda_k}{2|C_k|^2} + \left( \frac{\hat{b}(k) - \frac{3\lambda_k}{2|C_k|^2}}{1 + \log(C_k x + 1)} \right) - \frac{1}{p(1 - \alpha_k p)} \sum_{l=1}^{m} q_{kl}(1 - \alpha_p).
\]

**Step 2.** In order to obtain the upper bound of \( L_x V_1 \), we compute the limits of \( V_1 \) and \( \varphi \) as \( |x| \to 0^+ \) and \( |x| \to +\infty \). For each \( k \in \mathbb{S} \),
\[
\lim_{|x| \to 0^+} V_1(x, k) = 1 - \alpha_k p, \quad \lim_{|x| \to +\infty} V_1(x, k) = +\infty,
\]
and
\[
\lim_{|x| \to 0^+} \varphi(x, k) = \hat{b}(k) - \frac{\lambda_k}{|C_k|^2} + \frac{1}{p(1 - \alpha_k p)} \sum_{l=1}^{m} q_{kl}(1 - \alpha_p).
\]
For any \( l, k \in \mathbb{S} \) using
\[
\frac{1 + \log(\hat{c}_l \sum_{i=1}^{n} x_i + 1)}{1 + \log(\hat{c}_k \sum_{i=1}^{n} x_i + 1)} < \frac{1 + \log(C_l x + 1)}{1 + \log(C_k x + 1)} < \frac{1 + \log(\hat{c}_l \sum_{i=1}^{n} x_i + 1)}{1 + \log(\hat{c}_k \sum_{i=1}^{n} x_i + 1)}
\]
and
\[
\lim_{|x| \to +\infty} \frac{1 + \log(\hat{c}_l \sum_{i=1}^{n} x_i + 1)}{1 + \log(\hat{c}_k \sum_{i=1}^{n} x_i + 1)} = 1, \quad \lim_{|x| \to +\infty} \frac{1 + \log(\hat{c}_l \sum_{i=1}^{n} x_i + 1)}{1 + \log(\hat{c}_k \sum_{i=1}^{n} x_i + 1)} = 1,
\]
yields
\[
\lim_{|x| \to +\infty} \frac{1 + \log(C_l x + 1)}{1 + \log(C_k x + 1)} = 1.
\]
Recalling the property of the generator that \( \sum_{l=1}^{m} q_{kl} = 0 \), we have
\[
\frac{1}{p(1 - \alpha_k p)} \sum_{l=1}^{m} q_{kl}(1 - \alpha_p) = -\frac{1}{1 - \alpha_k p} \sum_{l=1}^{m} q_{kl} \alpha_l \\
= -\left( \sum_{l=1}^{m} q_{kl} \alpha_l + \frac{\alpha_k p}{1 - \alpha_k p} \sum_{l=1}^{m} q_{kl} \alpha_l \right).
\]
Choose a small constant \( 0 < p_0 \leq p_1 \), such that for any \( 0 < p \leq p_0 \),
\[
\omega_k = \pi C - \frac{\alpha_k p}{1 - \alpha_k p} \sum_{l=1}^{m} q_{kl} \alpha_l < 0, \quad k = 1, \ldots, m.
\]
Step 3. Now, choose a sufficiently small positive constant $\zeta = \zeta(p)$ such that
\[ \varpi_k + \frac{\zeta}{p} < 0, \ k = 1, \ldots, m. \] (20)

Hence it follows from (15), (16), (19) and (20), we have
\[ \lim_{|x| \to 0^+} V_1(x, k)(p\varphi(x, k) + \zeta) \leq (1 - \alpha_kp) \left( b(k)p - \frac{\lambda_k}{|C_k|^2} + \zeta \right) - p \sum_{l=1}^m q_{kl}\alpha_l; \]

and
\[ \lim_{|x| \to +\infty} V_1(x, k)(p\varphi(x, k) + \zeta) = \lim_{|x| \to +\infty} pV_1(x, k)(\varpi_k + \frac{\zeta}{p}) = -\infty. \]

Thus,
\[ \sup_{x \in \mathbb{R}_+^N, k \in \mathbb{S}} V_1(x, k)(p\varphi(x, k) + \zeta) \leq K < +\infty. \]

Define $V_2 : \mathbb{R}_+^n \times \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}_+$ by
\[ V_2(x, t, k) = e^{\xi t}V_1(x, k). \]

Using the generalized Itô formula, we have
\[ V_2(x(t), t, \gamma(t)) = V_2(x_0, 0, \gamma_0) + \int_0^t \mathcal{L}_xV_2(x(s), s, \gamma(s))ds + M_{V_2}(t), \] (21)

where
\[ \mathcal{L}_xV_2(x, t, k) = \zeta e^{\xi t}V_1(x, k) + e^{\xi t}\mathcal{L}_xV_1(x, k) \leq e^{\xi t}V_1(x, k)(p\varphi(x, k) + \zeta) \leq Ke^{\xi t}. \] (22)

In view of the generalized Itô formula,
\[ M_{V_2}(t) := V_2(x(t), t, \gamma(t)) - V_2(x_0, 0, \gamma_0) - \int_0^t \mathcal{L}_xV_2(x(s), s, \gamma(s))ds \]
is a local martingale. Define
\[ \tau_R = \inf \{ t \geq 0 : x(t) \in \mathbb{R}_+^n, x_i(t) > R, \ i = 1, \ldots, n. \} . \]

Then $\{\tau_R\}$ is monotonically increasing. By Lemma 3.1, $\lim_{t \to \infty} \tau_R = \infty$ a.s. For any $t \geq 0$, $t \wedge \tau_R$ is a bounded stopping time and $t \wedge \tau_R \to t$ a.s. The local martingale property implies that $\mathbb{E}[M_{V_2}(t \wedge \tau_R)] = 0$ and it therefore follows from (21) and (22) that
\[ \mathbb{E} [V_2(x(t) \wedge \tau_R), t \wedge \tau_R, \gamma(t \wedge \tau_R)] \]
\[ = \mathbb{E} [V_2(x_0, 0, \gamma_0)] + \mathbb{E} \int_0^{t \wedge \tau_R} \mathcal{L}_xV_2(x(s), s, \gamma(s))ds. \] (23)
Letting $R \to \infty$, by virtue of the monotonic convergence theorem and the dominated convergence theorem (c.f. [19, pp. 8, 9])

$$\mathbb{E}[V_2(x(t), t, \gamma(t))] = \mathbb{E}[V_2(x_0, 0, \gamma_0)] + \mathbb{E} \int_0^t \mathcal{L}_x V_2(x(s), s, \gamma(s))ds$$

$$\leq \mathbb{E}[V_2(x_0, 0, \gamma_0)] + Ke^{\zeta t}.$$ (24)

Then we have

$$(1 - \alpha_p)\mathbb{E}\{e^{\zeta t} [1 + \log (e^{\zeta t}|x(t)| + 1)]^p\} \leq (1 - \alpha_p)\mathbb{E}[1 + \log (C(\gamma_0)x_0 + 1)]^p + Ke^{\zeta t}.$$

So

$$\mathbb{E}[1 + \log (e^{\zeta t}|x(t)| + 1)]^p \leq \frac{1 - \alpha_p}{1 - \alpha_p} e^{-\zeta t}\mathbb{E}[1 + \log (C(\gamma_0)x_0 + 1)]^p + K.$$

Letting $t \to \infty$,

$$\limsup_{t \to \infty} \mathbb{E}[1 + \log (e^{\zeta t}|x(t)| + 1)]^p \leq K,$$

then the required assertion follows. \hfill \Box

**Remark 1.** Since the proof of Theorem 3.4 is rather technical, now we give an outline to help the reader. In Step 1 basing on the Fredholm alternative result we construct the Lyapunov function $V_1(x, k)$ depending on the states in $\mathbb{S}$ which can reveal the effect of switching randomly between $m$-environment states more precisely. We go a step further to compute the limits of $\mathcal{L}_x V_1(x, k)$ as $|x| \to 0^+$ and $|x| \to +\infty$. Finally, we choose an appropriate constant $\zeta$, then estimate $\mathcal{L}_x(e^{\zeta t}V_1(x, k))$ and analyze the asymptotical properties of $\mathbb{E}(e^{\zeta t}V_1(x(t), \gamma(t)))$. Thus the required result follows.

**Remark 2.** From Theorem 3.4 the sign of $\pi \lambda$ is a key for asymptotic upper boundedness of the moment. In fact, $\pi \lambda$ can be regarded as weighted arithmetic mean of $\lambda_1, \ldots, \lambda_m$ with positive weights $\pi_1, \ldots, \pi_m$. Therefore some element $\lambda_k$ with a large weight $\pi_k$ contributes more to the weighted mean than how does the element with a low weight. By virtue of the law of large number, roughly, the stationary distribution $\pi_k$ determines the fraction of the time spent by Markov chain $\gamma(t)$ in state $k$. The parameters of the $k$-th subsystem determine $\lambda_k$ representing some character of $k$-th environment. Therefore this theorem reveals that the asymptotic upper boundedness of moment results from the perform of all environments and fluctuation randomly between them.

**Theorem 3.5.** Solutions of SDE (3) are stochastically ultimately bounded from the above under the conditions of Theorem 3.4.

**Proof.** Fix $p > 0$ sufficiently small such that (9) holds. For any $\varepsilon \in (0, 1)$, let $\chi = e^{(\frac{K}{\varepsilon})^p}$, where $K$ come from (9). Thus, by virtue of Chebyshev’s inequality,

$$\limsup_{t \to +\infty} \mathbb{P}\{|x(t)| > \chi\} = \limsup_{t \to +\infty} \mathbb{P}\{\log |x(t)| > \log \chi\} \leq \limsup_{t \to +\infty} \frac{\mathbb{E}(\log^p |x(t)|)}{\log^p \chi} \leq \varepsilon.$$

The desired assertion follows. \hfill \Box

In Assumption 1 we set $C(k) = I$ $(n \times n$ identity matrix). By virtue of Lemma 2.1, we obtain the following corollary directly.
Corollary 1. If for each \( k \in \mathbb{S} \), \( a^n(k) \leq 0 \) and \( \pi a^u < 0 \), where \( a^u := (a^u(1), \cdots, a^u(m))^T \) and \( a^n(k) := \max_{1 \leq i \leq n} \left( 2a_{ii}(k) + \sum_{j \neq i} a_{ij}(k) + a_{ji}(k) \right) \), then the conclusions of Theorems 3.6 and 3.5 still hold.

Remark 3. Zhu and Yin in [28] claimed that if for each \( k \in \mathbb{S} \) and \( i, j = 1, \cdots n \) with \( i \neq j \), \( a_{ii}(k) < 0 \), \( a_{ij}(k) \leq 0 \), solutions of (3) are stochastically ultimately bounded from the above. Obviously, Corollary 1 generalizes this result.

The positivity of \( x(t) \) gives hints to the use of reciprocal transformation. Thus we estimate the asymptotic upper bound of the moment of \( 1/|x(t)| \) in order to obtain the stochastic ultimate boundedness from the below. For convenience, define

\[
\tilde{\sigma}_j(k) := \max_{1 \leq i \leq n} \{\sigma_{ij}(k)\}, \quad \tilde{\beta}(k) := \tilde{b}(k) - \frac{1}{2} \sum_{j=1}^d \tilde{\sigma}_j^2(k), \quad \tilde{\beta} := (\tilde{\beta}(1), \cdots, \tilde{\beta}(m))^T.
\]

(25)

Theorem 3.6. Under Assumption 1 and \( \pi \tilde{\beta} > 0 \), for \( \theta > 0 \) sufficiently small the solution \( x(t) \) of (3) with any initial value \( x_0 \in \mathbb{R}^n_+ \), \( \gamma_0 \in \mathbb{S} \) has the property that

\[
\lim_{t \to \infty} \sup \mathbb{E} \left[ |x(t)|^{-\theta} \right] \leq K.
\]

Proof. In order to obtain the upper boundary of the moment of \( 1/|x(t)| \) with some order, we define another Lyapunov function \( V_3(x, k) \in \mathbb{R}^n_+ \times \mathbb{S} \). From the Fredholm Alternative result (c.f. [27], pp. 362-366)

\[
Qe = -\tilde{\beta} + \left( \frac{\pi \tilde{\beta}}{} \right) 1_m
\]

has a solution \( e = (e_1, \cdots, e_m)^T \in \mathbb{R}^m \). Choose a constant \( 0 < \theta_1 < 1 \) such that for each \( 0 < \theta \leq \theta_1 \), we have

\[1 - e_k \theta > 0, \quad k = 1, \cdots, m.\]

Define \( V_3 : \mathbb{R}^n_+ \times \mathbb{S} \to \mathbb{R}_+ \) by

\[
V_3(x, k) = (1 - e_k \theta) \left( 1 + \frac{1}{\sum_{j=1}^n x_j} \right)^\theta.
\]

Next, similar to the proof of Lemma 3.4, we can obtain the required assertion. To avoid duplication, we omit the remaining part. \( \square \)

Using the method which is similar to that used in Theorem 3.5 we yield the following theorem directly.

Theorem 3.7. Solutions of SDE (3) are stochastically ultimately bounded from the below under the conditions of Theorem 3.6.

The result on permanence follows directly from Theorems 3.5 and 3.7.

Theorem 3.8. Under Assumption 1 \( \pi \lambda < 0 \) and \( \pi \tilde{\beta} > 0 \), the SDE (3) is stochastically permanent.

Remark 4. Here the permanence is about \( |x(t)| \), namely, the sum of all concentrations of the species. Thus it means that the population as a whole can survive in the future with large probability. Under Assumption 1 \( \pi \lambda < 0 \) can prevent the population growth wild by Theorem 3.5 while \( \pi \tilde{\beta} > 0 \) can keep the population survival not extinctive by Theorem 3.7.
Theorem 3.7 generalizes the main result in [11]. To complete this section, let us compare our results with that in [11]. We first cite the aforementioned result in the reference which can be considered as a corollary of ours.

Corollary 2. [11] The SDE (3) is stochastically permanent under the following conditions

(A.1) The generator \( Q = (q_{ij})_{m \times m} \) satisfies that for some \( u \in S \), \( q_{iu} > 0 \) for all \( i \neq u \).

(A.2) Assumption \( I \) holds, moreover, each \( \lambda_k < 0 \), \( k = 1, \ldots, m \).

(A.3) \( \pi \beta > 0 \).

Remark 5. We highlight our new results in this remark as follows.

1. Note that (A.1) can be regarded as a sufficient condition for irreducibility. In [11], (A.1) was used to facilitate the application of M-matrix theory. In this paper, we only assume irreducibility rather than (A.1) and irreducibility. It is readily seen that the condition we used are much weaker than requiring some entire column of the generator to be positive. In fact, from [8], we can see that we do not need each entry of an entire column of the generator but only need each entry of a column after some finite number of iterations to be positive.

2. Note that (A.2) has been improved in the current paper. Assumption 1 implies \( \lambda_k \leq 0 \) for each \( k \in S \). Thus \( \pi \lambda < 0 \) holds even if some \( \lambda_k < 0 \).

4. Asymptotic properties. In this section we investigate the almost surely asymptotic properties of the solutions. We begin with the estimation of the growth rate. For convenience, we give some notations and an assumption as follows.

\[
\bar{\sigma}_j(k) := \min_{1 \leq i \leq n} |\sigma_{ij}(k)|, \quad \bar{\beta}(k) := \bar{b}(k) - \frac{1}{2} \sum_{j=1}^{d} \bar{\sigma}_j^2(k), \quad \text{and} \quad \bar{\beta} := (\bar{\beta}(1), \ldots, \bar{\beta}(m))^T.
\]

(26)

Assumption 2. There exist positive numbers \( c_1, \ldots, c_n \), such that

\[
\tilde{\lambda} := \max_{k \in S} \{\lambda_k^+ (C A(k) + A^T(k) \tilde{C})\} \leq 0,
\]

(27)

where \( \tilde{C} = \text{diag}(c_1, \ldots, c_n) \).

We shall establish the estimate for

\[
\limsup_{t \to +\infty} \frac{\log |x(t)|}{t}
\]

almost surely, which is called the sample Lyapunov exponent or simply Lyapunov exponent.

Theorem 4.1. Under Assumption 2, the solution \( x(t) \) of (3) satisfies

\[
\limsup_{t \to +\infty} \frac{\log |x(t)|}{t} \leq \pi \bar{\beta} \text{ a.s.}
\]

(28)

Particularly, if \( \pi \bar{\beta} < 0 \), then \( \lim_{t \to +\infty} |x(t)| = 0 \text{ a.s.} \)

Proof. Under Assumption 2 define \( C := (c_1, \ldots, c_n) \). It follows from the Itô formula that

\[
\log(C x(t)) - \log(C x_0)
\]
Particularly, if \( \pi \) from (29) and Assumption 3 that proof. Assumption 1 guarantees the global existence of the solutions. Then it follows

\[ \text{noise in tensity can make all species extinct.} \]

\[ \text{en vironmental noise can suppress exponential growth of the population and large} \]

\[ \text{there exist positive numbers} \]

\[ \text{Assumption 3.} \]

\[ \text{Remark 6.} \]

\[ \text{By the definition} \]

\[ \text{which implies} \]

\[ \text{Note that the quadratic variation of} \]

\[ \langle M_j(t), M_j(t) \rangle = \int_0^t \frac{1}{(Cx(s))^2} \left( \sum_{i=1}^n c_i x_i(s) \sigma_{ij}(\gamma(s)) \right)^2 ds \leq Kt. \]

\[ \text{By the law of large numbers for local martingales (c.f. [13]), we obtain} \]

\[ \frac{1}{t} \sum_{j=1}^d M_j(t) = \frac{1}{t} M_j(t) \to 0, \text{ as } t \to \infty \text{ a.s.} \]

\[ \text{which implies} \]

\[ \limsup_{t \to +\infty} \frac{1}{t} \log(Cx(t)) \leq \limsup_{t \to +\infty} \frac{1}{t} \int_0^t \tilde{\beta}(\gamma(s)) ds = \pi \tilde{\beta}. \]

Therefore the desired assertion follows from \( \hat{\epsilon} \leq \epsilon \sum_{i=1}^n x_i \leq Cx. \)

**Remark 6.** By the definition \( \tilde{\beta} \) of (26), Theorem 4.1 reveals the fact that the environmental noise can suppress exponential growth of the population and large noise intensity can make all species extinct.

On the other hand we look for the lower bound of the growth rate. For convenience we impose the following assumption.

**Assumption 3.** There exist positive numbers \( c_1, \cdots, c_n \), such that

\[ \hat{\lambda} := \min_{k \in \mathbb{S}} \{ \lambda_{\min}(\tilde{C}A(k) + A^T(k)\tilde{C}) \} \geq 0 \]

where \( \tilde{C} = \text{diag}(c_1, \cdots, c_n). \)

**Theorem 4.2.** Under Assumptions 4 and 3 the solution \( x(t) \) of (5) satisfies

\[ \liminf_{t \to +\infty} \frac{\log |x(t)|}{t} \geq \pi \tilde{\beta} \text{ a.s.} \]

Particularly, if \( \pi \tilde{\beta} > 0 \), then \( \lim_{t \to +\infty} |x(t)| = \infty \text{ a.s.} \)

**Proof.** Assumption 1 guarantees the global existence of the solutions. Then it follows from (29) and Assumption 3 that

\[ \log(Cx(t)) - \log(Cx_0) \]
where SDE (3) is called persistence in mean. Using the inequality \( \log(\cdot) \), Under Assumption 2 and Theorem 4.3.

\[ \lim_{t \to +\infty} \frac{\lambda(\gamma(s)) + \hat{\lambda} |x(s)|^2}{2C} - \frac{1}{2} \sum_{j=1}^{d} \hat{\sigma}_{j}^2(\gamma(s)) \leq -\sum_{j=1}^{d} M_{j}(t) \]

\[ \geq \int_{0}^{t} \hat{\beta}(\gamma(s))ds + \sum_{j=1}^{d} M_{j}(t). \]

Using a similar argument we get the desired assertion. \( \square \)

**Corollary 3.** If each \( A(k) \) (\( k \in \mathbb{S} \)) is an antisymmetric matrix, then the conclusion of Theorem 4.2 still holds.

To end this section we establish the estimate of sample paths in time average, more precisely, the estimates of

\[ \lim_{t \to +\infty} \frac{1}{t} \int_{0}^{t} \omega(x(s))ds \quad \text{and} \quad \limsup_{t \to +\infty} \frac{1}{t} \int_{0}^{t} |x(s)|ds. \]

If there exist positive constants \( K_{1} \) and \( K_{2} \) such that

\[ K_{1} \leq \lim_{t \to +\infty} \frac{1}{t} \int_{0}^{t} |x(s)|ds \leq \limsup_{t \to +\infty} \frac{1}{t} \int_{0}^{t} |x(s)|ds \leq K_{2}, \]

SDE (3) is called persistence in mean in some references.

**Theorem 4.3.** Under Assumption 2 and \( \hat{\lambda} < 0 \), the solution \( x(t) \) of (3) satisfies

\[ -\frac{2\hat{\gamma}}{\lambda}[\pi\hat{\beta} \vee 0] \leq \lim_{t \to +\infty} \frac{1}{t} \int_{0}^{t} |x(s)|ds \leq \limsup_{t \to +\infty} \frac{1}{t} \int_{0}^{t} |x(s)|ds \leq \frac{2|C|}{\lambda}[\pi\hat{\beta} \vee 0] \ a.s., \]

where \( \hat{\lambda} \) and \( \hat{\beta} \) are defined by (27) and (32) respectively.

**Proof.** By (30), we have

\[ \log(Cx(t)) \leq \log(Cx_{0}) + \int_{0}^{t} \hat{\lambda}(\gamma(s))ds + \frac{\hat{\lambda}}{2|C|} \int_{0}^{t} |x(s)|ds + \sum_{j=1}^{d} M_{j}(t) \]

\[ := F_{1}(t) + \frac{\hat{\lambda}}{2|C|} \int_{0}^{t} |x(s)|ds \ a.s. \]

(31) implies \( \lim_{t \to +\infty} \frac{F_{1}(t)}{t} = \pi\hat{\beta} \ a.s. \). Then there exists a set \( \Omega_{1} \in \mathcal{F}, P(\Omega_{1}) = 1 \) such that for any fixed \( \varepsilon_{1} > 0, \omega \in \Omega_{1} \), there exists a constant \( T_{1} = T_{1}(\varepsilon_{1}, \omega) > 0 \) such that

\[ \frac{F_{1}(t)}{t} \leq \pi\hat{\beta} + \varepsilon_{1}, \forall t > T_{1}. \]

Using the inequality \( \log(Cx) \geq \log \hat{c} + \log |x| \),

\[ \log |x(t)| \leq (\pi\hat{\beta} + \varepsilon_{1}) t - \log \hat{c} + \frac{\hat{\lambda}}{2|C|} \int_{0}^{t} |x(s)|ds, \forall t \geq T_{1}. \]

(34)

Define

\[ g(t) := \int_{0}^{t} |x(s)|ds, \forall t \geq 0. \]

(35)

Inequality (34) implies that

\[ e^{-\frac{\hat{\lambda}}{\pi\hat{\beta} + \varepsilon_{1}} g(t)} \frac{dg(t)}{dt} \leq \frac{e^{\hat{\lambda}(\pi\hat{\beta} + \varepsilon_{1}) t}}{\hat{c}}, \forall t \geq T_{1}. \]
Integrating both sides from $T_1$ to $t$ gives

\[
-\frac{2|C|}{\lambda} \left( e^{-\frac{1}{2\sigma^2} g(t)} - e^{-\frac{1}{2\sigma^2} g(T_1)} \right) \leq \frac{1}{c(\pi \beta + \epsilon_1)} \left( e^{(\pi \beta + \epsilon_1) t} - e^{(\pi \beta + \epsilon_1) T_1} \right), \quad \forall t \geq T_1.
\]

So,

\[
e^{-\frac{1}{2\sigma^2} g(t)} \leq e^{-\frac{1}{2\sigma^2} g(T_1)} - \frac{\hat{\lambda}}{2c(\pi \beta + \epsilon_1)|C|} \left( e^{(\pi \beta + \epsilon_1) t} - e^{(\pi \beta + \epsilon_1) T_1} \right), \quad \forall t \geq T_1.
\]

Taking the logarithm of both sides leads to

\[
g(t) \leq -\frac{2|C|}{\lambda} \log \left[ e^{-\frac{1}{2\sigma^2} g(T_1)} - \frac{\hat{\lambda}}{2c(\pi \beta + \epsilon_1)|C|} \left( e^{(\pi \beta + \epsilon_1) t} - e^{(\pi \beta + \epsilon_1) T_1} \right) \right], \quad \forall t \geq T_1.
\]

Dividing both sides by $t$ and then taking the upper limit gives

\[
\limsup_{t \to +\infty} \frac{1}{t} \int_0^t |x(s)| ds \leq -\frac{2|C|}{\lambda} \left( \pi \hat{\beta} \right) a.s.
\]

The arbitrariness of $\epsilon_1$ implies

\[
\limsup_{t \to +\infty} \frac{1}{t} \int_0^t |x(s)| ds \leq -\frac{2|C|}{\lambda} [\pi \hat{\beta} \vee 0] a.s.
\] (36)

On the other hand, it follows from (29) that

\[
\log(Cx(t)) \geq \log(Cx_0) + \int_0^t \left( \hat{b}(\gamma(s)) - \frac{1}{2} \sum_{j=1}^d \hat{\sigma}_j^2(\gamma(s)) + \frac{\hat{\lambda}|x(s)|^2}{2Cx(s)} \right) ds + \sum_{j=1}^d M_j(t)
\]

\[
\geq \log(Cx_0) + \int_0^t \hat{\beta}(\gamma(s)) ds + \int_0^t \frac{\hat{\lambda}}{2c}|x(s)| ds + \sum_{j=1}^d M_j(t)
\]

\[
:= F_2(t) + \int_0^t \frac{\hat{\lambda}}{2c}|x(s)| ds \quad a.s.
\] (37)

Note that \(\lim_{t \to +\infty} \frac{F_2(t)}{t} = \pi \hat{\beta}\) a.s. There exists a set \(\Omega_2 \in \mathcal{F}, \mathbb{P}(\Omega_2) = 1\) such that for any fixed \(0 < \epsilon_2 < \pi \hat{\beta}, \omega \in \Omega_2\), there exists a constant \(T_2 = T_2(\epsilon_2, \omega) > 0\) such that

\[
\frac{F_2(t)}{t} > \pi \hat{\beta} - \epsilon_2, \quad \forall t > T_2,
\]

Substituting this into (37) and using the inequality \(\log(Cx) \leq \log |C| + \log |x|\),

\[
\log |x(t)| \geq \left( \pi \hat{\beta} - \epsilon_2 \right) t - \log |C| + \int_0^t \frac{\hat{\lambda}}{2c}|x(s)| ds, \quad \forall t > T_2.
\]

By (35),

\[
e^{-\frac{1}{2\sigma^2} g(t)} \frac{dg(t)}{dt} \geq \frac{e^{(\pi \beta - \epsilon_2) t}}{|C|}, \quad \forall t > T_2.
\]
Similar to the estimate of the superior limit we can obtain
\[
\liminf_{t \to +\infty} \frac{1}{t} \int_0^t |x(s)| ds \geq -\frac{2\hat{c}}{\lambda} [\pi\hat{\beta} \lor 0] \text{ a.s.} \tag{38}
\]
The desired assertion follows from (36) and (38). \[\square\]

**Remark 7.** Li et al. [11] claimed that the SDE (3) with \(d = 1\) satisfies (33) under the conditions (A.1) and (A.3) of Corollary 2 besides the conditions of Theorem 4.3. Here we remove the conditions (A.1) and (A.3) the conclusion still holds.

5. **Stochastic strong permanence.** In this section we begin to focus on mutualistic population system. The parameters of mutualistic systems are as follows.

**Assumption 4.** For each \(k \in \mathbb{S}\) and \(i, j = 1, 2, \cdots, n\) with \(i \neq j\), \(a_{ii}(k) \leq 0\), \(a_{ij}(k) \geq 0\).

Although the stochastic permanence implies that \(|x(t)|\) can keep away from 0 with large probability in long-time scale, it doesn’t guarantee that each component of \(x(t)\) keeps away from 0. In this section we concentrate on the survival of each species, which is termed *stochastic strong permanence*. For clarity we give its definition.

**Definition 5.1.** The SDE (3) is called stochastically strongly permanent if it is stochastically ultimately bounded from the above and below for each component.

Next we investigate the upper boundedness of the moment of each component \(1/x_i(t)\).

**Theorem 5.2.** Under Assumption 7 and 4, if for some \(1 \leq i \leq n\), \(\pi\beta_i > 0\), then for any \(\eta_i > 0\) sufficiently small the \(i\)-th component \(x_i(t)\) of the solution of SDE (3) has the property that
\[
\limsup_{t \to \infty} \mathbb{E} \left[ x_i^{-\eta_i}(t) \right] \leq K, \tag{39}
\]
where \(\beta_i(k) := b_i(k) - \frac{1}{2} \sum_{j=1}^d \sigma_{ij}^2(k)\) for each \(k \in \mathbb{S}\) and \(\beta := (\beta_i(1), \cdots, \beta_i(m))^T\).

**Proof.** Since the proof is rather technical we divided it into 3 steps.

**Step 1.** In order for the required assertion on \(x_i(t)\) we begin to construct an appropriate Lyapunov function depending on the states in \(\mathbb{S}\). It follows from the Fredholm Alternative result (c.f. [27] pp.362-366) the equation
\[
Qv_i = -\beta_i + (\pi\beta_i) I_m \tag{40}
\]
has a solution \(v_i = (v_{i1}, \cdots, v_{im})^T \in \mathbb{R}^m\). Hence
\[
\beta_i(k) + \sum_{l=1}^m q_{kl}v_{il} = \pi\beta_i > 0, \quad k = 1, \cdots, m. \tag{41}
\]
Choose two constants \(0 < \eta_i^{(1)} \leq \frac{1}{2}\) such that for each \(0 < \eta_i \leq \eta_i^{(1)}\),
\[
1 - v_{ik}\eta_i \geq \frac{1}{2}, \quad k = 1, \cdots, m.
\]
Define \(W_i : \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}_+\) by
\[
W_i(x, k) = (1 - v_{ik}\eta_i) \left( \frac{1}{x} + 1 \right)^{\eta_i}. \tag{42}
\]
Then the direct calculation shows

\[ W_{ix}(x, k) = -(1 - v_i \eta_i) \eta_i \left( \frac{1}{x} + 1 \right)^{\eta_i - 1} \frac{1}{x^2}, \]

\[ W_{ixx}(x, k) = (1 - v_i \eta_i) \eta_i \left( \frac{1}{x} + 1 \right)^{\eta_i - 2} \left[ 2 \left( \frac{1}{x} + 1 \right) \frac{1}{x^3} + (\eta_i - 1) \frac{1}{x^4} \right]. \]

Hence it follows from (2) and the property of the generator that

\[
\begin{align*}
\mathcal{L}_x W_i(x_i, k) &= (1 - v_i \eta_i) \eta_i \left( \frac{1}{x_i} + 1 \right)^{\eta_i - 1} \left[ -\frac{1}{x_i} \left( b_i(k) + \sum_{j=1}^{n} a_{ij}(k)x_j \right) \right] \\
&\quad + \frac{1}{2} (1 - v_i \eta_i) \eta_i \left( \frac{1}{x_i} + 1 \right)^{\eta_i - 2} \left[ \frac{2}{x_i} + (\eta_i - 1) \frac{1}{x_i^2} \left( \sum_{j=1}^{d} \sigma^2_{ij}(k) \right) \right] \\
&\quad + \sum_{l=1}^{m} q_{kl}(1 - v_i \eta_i) \left( \frac{1}{x_i} + 1 \right)^{\eta_i} \leq (1 - v_i \eta_i) \eta_i \left( \frac{1}{x_i} + 1 \right)^{\eta_i - 1} \left[ -\frac{1}{x_i} (b_i(k) + a_{ii}(k)x_i) \right] \\
&\quad + \frac{1}{2} (1 - v_i \eta_i) \eta_i \left( \frac{1}{x_i} + 1 \right)^{\eta_i - 2} \left[ \frac{2}{x_i} + (\eta_i - 1) \frac{1}{x_i^2} \left( \sum_{j=1}^{d} \sigma^2_{ij}(k) \right) \right] \\
&\quad + \sum_{l=1}^{m} q_{kl}(1 - v_i \eta_i) \left( \frac{1}{x_i} + 1 \right)^{\eta_i}, \quad (43)
\end{align*}
\]

where

\[ \psi_i(x_i, k) := \left( -b_i(k) + \sum_{j=1}^{d} \sigma^2_{ij}(k) + a_{ii}(k) \right) \frac{1}{x_i + 1} + \frac{\eta_i - 1}{2} \sum_{j=1}^{d} \sigma^2_{ij}(k) \left( \frac{1}{x_i + 1} \right)^2 \]

\[ - a_{ii}(k) - \sum_{l=1}^{m} q_{kl}v_{li} - \frac{v_i \eta_i}{1 - v_i \eta_i} \sum_{l=1}^{m} q_{kl}v_{li}. \]

**Step 2.** In order to obtain the upper bound of \( \mathcal{L}W_i \), we analyze the limits of \( W_i \) and \( \psi_i \) as \( x_i \to 0^+ \) and \( x_i \to +\infty \). Obviously, for each \( k \in S \)

\[
\lim_{x_i \to 0^+} W_i(x_i, k) = +\infty, \quad \lim_{x_i \to +\infty} W_i(x_i, k) = 1 - v_i \eta_i. \quad (45)
\]

Now choose a small constant \( 0 < \eta_i^{(0)} \leq \eta_i^{(1)} \), such that for each \( 0 < \eta_i \leq \eta_i^{(0)} \)

\[ s_{ik} := \pi \beta_i - \eta_i \frac{d}{2} \sum_{j=1}^{d} \sigma^2_{ij}(k) + \frac{v_i \eta_i}{1 - v_i \eta_i} \sum_{l=1}^{m} q_{kl}v_{li} > 0, \quad \forall k \in S. \]

Hence it follows from the definition of \( \beta_i(k) \) and (44) that for each \( k \in S \)

\[
\lim_{x_i \to 0^+} \psi_i(x_i, k) = -b_i(k) + \frac{\eta_i + 1}{2} \sum_{j=1}^{d} \sigma^2_{ij}(k) - \sum_{l=1}^{m} q_{kl}v_{li} - \frac{v_i \eta_i}{1 - v_i \eta_i} \sum_{l=1}^{m} q_{kl}v_{li}.
\]
Theorem 5.4. \[ \text{strong permanence follows directly from Theorem 3.5 and Theorem 5.3.} \]

\[
= -\pi \beta_i + \frac{\eta_i}{2} \sigma_i^2(k) - \frac{v_{ik} \eta_i}{1 - v_{ik} \eta_i} \sum_{i=1}^{m} q_{ki} v_{ii} = -\zeta_k, \tag{46}
\]

and

\[
\lim_{x_i \to +\infty} \psi(x_i, k) = -a_i(k) - \sum_{i=1}^{m} q_{ki} v_{ii} - \frac{v_{ik} \eta_i}{1 - v_{ik} \eta_i} \sum_{i=1}^{m} q_{ki} v_{ii}. \tag{47}
\]

**Step 3.** Choose a positive constant \( \kappa_i = \kappa_i(\eta_i) \) sufficiently small such that \( \eta_i \zeta_k - \kappa_i > 0, \ k = 1, \ldots, m. \)

It then follows from (45)-(47) that

\[
\lim_{x_i \to 0^+} W_i(x_i(k)) (\eta_i \psi_i(x_i, k) + \kappa_i) = -\infty,
\]

and

\[
\lim_{x_i \to +\infty} W_i(x_i, k) (\eta_i \psi_i(x_i, k) + \kappa_i)
= - (1 - v_{ik} \eta_i) \eta_i \left( a_i(k) + \sum_{i=1}^{m} q_{ki} v_{ii} + \frac{v_{ik} \eta_i}{1 - v_{ik} \eta_i} \sum_{i=1}^{m} q_{ki} v_{ii} - \frac{\kappa_i}{\eta_i} \right).
\]

Thus, for each \( i, \)

\[
\sup_{x_i \in \mathbb{R}_+, k \in \mathbb{S}} W_i(x_i, k) (\eta_i \psi_i(x_i, k) + \kappa_i) < +\infty,
\]

which implies

\[
\mathcal{L}_x \left[ e^{\kappa t} W_i(x_i, k) \right] = \kappa_i e^{\kappa t} W_i(x_i, k) + e^{\kappa t} \mathcal{L}_x W_i(x_i, k)
= e^{\kappa t} W_i(x_i, k) (\eta_i \psi_i(x_i, k) + \kappa_i) \leq K e^{\kappa t}.
\]

Using the generalized Itô formula (c.f. [27], p. 29, eq. (2.7)), we get

\[
\mathbb{E} \left[ e^{\kappa t} W_i(x_i(t), \gamma(t)) \right] = \mathbb{E} \left[ W_i(x_i(0), \gamma(0)) \right] + \mathbb{E} \int_0^t \mathcal{L}_x \left[ e^{\kappa \tau} W_i(x_i(s), \gamma(s)) \right] ds
\leq \mathbb{E} \left[ W_i(x_i(0), \gamma(0)) \right] + K e^{\kappa t}.
\]

Recalling the definition of \( W_i \) we have

\[
(1 - \hat{v}_i \eta_i) \mathbb{E} \left[ e^{\kappa \tau} \left( \frac{1}{x_i(t)} + 1 \right)^{\eta_i} \right] \leq (1 - \hat{v}_i \eta_i) \mathbb{E} \left[ \left( \frac{1}{x_i(0)} + 1 \right)^{\eta_i} \right] + K e^{\kappa t},
\]

which implies

\[
\mathbb{E} \left[ \left( \frac{1}{x_i(t)} + 1 \right)^{\eta_i} \right] \leq \frac{1 - \hat{v}_i \eta_i}{1 - \hat{v}_i \eta_i} \mathbb{E} \left[ \left( \frac{1}{x_i(0)} + 1 \right)^{\eta_i} \right] e^{-\kappa \tau} + K. \tag{48}
\]

Therefore

\[
\limsup_{t \to \infty} \mathbb{E} \left[ x_i^{-\eta_i}(t) \right] \leq \limsup_{t \to \infty} \mathbb{E} \left[ \left( \frac{1}{x_i(t)} + 1 \right)^{\eta_i} \right] \leq K.
\]

**Theorem 5.3.** Under the conditions of Theorem 5.3 the \( i \)-th component \( x_i(t) \) of the solution of SDE (3) is stochastically ultimately bounded from the below.

The proof is rather standard and hence is omitted. The result on stochastic strong permanence follows directly from Theorem 4.5 and Theorem 5.3.

**Theorem 5.4.** Under Assumptions 4 and 5, if \( \pi \lambda < 0 \) and each \( \pi \beta_i > 0 \) for \( i = 1, \ldots, n \), the SDE (3) is stochastically strongly permanent.
Basing on Corollary 1 and Theorem 5.3, we get the following corollary.

**Corollary 4.** Under Assumption 4, if for each $k \in S$, $a_u(k) \leq 0$, $\pi a_u \leq 0$ and each $\pi \beta_i > 0$ for $i = 1, \ldots, n$, where $a_u(k)$ and $a_u$ are defined in Corollary 1, the SDE (3) is stochastically strongly permanent.

We complete this section by estimating the lower bound of the growth rate on each species.

**Theorem 5.5.** Under the conditions of Theorem 5.2, for any $\eta_i > 0$ sufficiently small, the solution $x(t)$ of (3) satisfies

$$\liminf_{t \to +\infty} \frac{\log x_i(t)}{\log t} \geq -\frac{1}{\eta_i} \text{ a.s.} \quad (49)$$

**Proof.** Define $U_i : \mathbb{R}_+ \to \mathbb{R}_+$ by $U_i(x) = \left(\frac{1}{x} + 1\right)^{\eta_i}$.

Similar to the proof of Theorem 5.2, we can obtain

$$\mathcal{L}_x U_i(x_i) \leq \eta_i \left(\frac{1}{x_i} + 1\right)^{\eta_i} \left[-b_i(k) + a_{ii}(k) + \sum_{j=1}^{d} \sigma^2_{ij}(k) \frac{1}{1 + x_i} \right. + \left. \frac{\eta_i - 1}{2} \sum_{j=1}^{d} \sigma^2_{ij}(k) \left(\frac{1}{x_i + 1}\right)^2 - a_{ii}(k) \right] \leq K U_i(x_i). \quad (50)$$

Then by the generalized Itô formula,

$$dU_i(x_i(t)) \leq K U_i(x_i(t))dt - \eta_i \left(\frac{1}{x_i(t)} + 1\right)^{\eta_i - 1} \frac{1}{x_i(t)} \sum_{j=1}^{d} \sigma_{ij}(\gamma(t)) dB_j(t).$$

Inequality (48) in the proof of Theorem 5.2 implies

$$\mathbb{E}[U_i(x_i(t))] \leq K, \forall t \geq 0.$$ 

Choose $\Delta$ sufficiently small such that

$$K \Delta + 3 \eta_i \max_{k \in S} \left\{ \left( \sum_{j=1}^{d} \sigma^2_{ij}(k) \right)^{\frac{1}{2}} \right\} \Delta^{\frac{1}{2}} < \frac{1}{2}. \quad (51)$$

For any positive integer $r$, (50) implies that

$$\mathbb{E} \left[ \sup_{(r-1)\Delta \leq t \leq r\Delta} U_i(x_i(t)) \right]$$

$$\leq \mathbb{E}[U_i(x_i((r-1)\Delta))] + \mathbb{E} \left[ \sup_{(r-1)\Delta \leq t \leq r\Delta} \left| \int_{(r-1)\Delta}^{t} K U_i(x_i(s))ds \right| \right]$$

$$+ \mathbb{E} \left[ \sup_{(r-1)\Delta \leq t \leq r\Delta} \left| \int_{(r-1)\Delta}^{t} \eta_i \left(\frac{1}{x_i(s)} + 1\right)^{\eta_i - 1} \frac{1}{x_i(s)} \sum_{j=1}^{d} \sigma_{ij}(\gamma(s)) dB_j(s) \right| \right]. \quad (52)$$
Direct calculation yields
\[
E \left[ \sup_{(r-1)\Delta \leq t \leq r\Delta} \left| \int_{(r-1)\Delta}^{t} KU_i(x_i(s))ds \right| \right] \leq K\Delta E \left[ \sup_{(r-1)\Delta \leq t \leq r\Delta} U_i(x_i(t)) \right].
\]  
(53)

Using the Burkholder-Davis-Gundy inequality (c.f. [18, Theorem 7.3, pp.40]),
\[
E \left[ \sup_{(r-1)\Delta \leq t \leq r\Delta} \left| \int_{(r-1)\Delta}^{t} \eta_i \left( \frac{1}{x_i(s)} + 1 \right)^{\eta_i-1} \frac{1}{x_i(s)} \left( \sum_{j=1}^{d} \sigma_{ij}(\gamma(s))dB_j(s) \right) ds \right| \right] \leq 3E \left\{ \int_{(r-1)\Delta}^{r\Delta} \eta_i^2 \left( \frac{1}{x_i(s)} + 1 \right)^{2\eta_i} \left( \sum_{j=1}^{d} \sigma_{ij}^2(\gamma(s)) \right) ds \right\}^{\frac{1}{2}} \leq 3\eta_i \max_{k \in S} \left\{ \left( \sum_{j=1}^{d} \sigma_{ij}^2(k) \right)^{\frac{1}{2}} \right\} \Delta^{\frac{1}{2}} E \left[ \sup_{(r-1)\Delta \leq t \leq r\Delta} U_i(x_i(t)) \right].
\]  
(54)

Substituting (53) and (54) into (52) gives
\[
E \left[ \sup_{(r-1)\Delta \leq t \leq r\Delta} U_i(x_i(t)) \right] \leq E \left[ U_i((r-1)\Delta) \right] + K\Delta + 3\eta_i \max_{k \in S} \left\{ \left( \sum_{j=1}^{d} \sigma_{ij}^2(k) \right)^{\frac{1}{2}} \right\} \Delta^{\frac{1}{2}} E \left[ \sup_{(r-1)\Delta \leq t \leq r\Delta} U_i(x_i(t)) \right],
\]
which together with (51) implies
\[
E \left[ \sup_{(r-1)\Delta \leq t \leq r\Delta} U_i(x_i(t)) \right] \leq K.
\]  
(55)

For any \( \varepsilon > 0 \), by the Chebyshev inequality,
\[
P \left\{ \omega : \sup_{(r-1)\Delta \leq t \leq r\Delta} U_i(x_i(t)) > (r\Delta)^{1+\varepsilon} \right\} \leq \frac{K}{(r\Delta)^{1+\varepsilon}}, \quad r = 1, 2, \ldots.
\]  
(56)

Note that
\[
\sum_{r=1}^{\infty} \frac{K}{(r\Delta)^{1+\varepsilon}} < \infty.
\]

Thus, by the virtue of the Borel-Cantelli lemma, there exists a set \( \Omega_0 \in F \), with \( P(\Omega_0) = 1 \) and an integer-valued random variable \( r_0(\omega) > \frac{1}{\Delta} + 2 \), such that for almost all \( \omega \in \Omega_0,
\[
\sup_{(r-1)\Delta \leq t \leq r\Delta} U_i(x_i(t)) \leq (r\Delta)^{1+\varepsilon}, \quad \text{whenever } r > r_0.
\]  
(57)

Then \( \forall t : (r-1)\Delta \leq t \leq r\Delta, \)
\[
\frac{\log U_i(x_i(t))}{\log t} \leq \frac{(1+\varepsilon) \log(r\Delta)}{\log((r-1)\Delta)}.
\]

Therefore
\[
\limsup_{t \to +\infty} \frac{\log U_i(x_i(t))}{\log t} \leq 1 + \varepsilon \quad \text{a.s.}
\]
The arbitrariness of $\varepsilon$ implies
\[
\limsup_{t \to +\infty} \frac{\log U_i(x_i(t))}{\log t} \leq 1 \quad \text{a.s.}
\]
Therefore the desired assertion follows from the definition of $U_i$. \hfill $\Box$

6. **Ergodicity.** For deterministic population systems, the stability of positive equilibrium point is among of the interesting topics, which implies all species coexist steadily. However, many stochastic systems don’t posses a determinstic positive equilibrium state. Recently, for stochastic population systems, the stability of the “stochastic positive equilibrium state”-the existence of the stationary distribution (see e.g. [23]) has drawn increasing attention [11, 12, 14, 20, 21, 26]. This section is devoted to obtaining the existence of a unique stationary distribution for the mutualistic Lotka-Volterra ecosystem (3). The positive recurrence of Markovian process $(x(t), \gamma(t))$ is often required for its ergodicity in $\mathbb{R}^n_+ \times \mathbb{S}$, see [3, 27]. Now we introduce a useful lemma on the positive recurrence as stated in [27].

**Lemma 6.1.** [27, p.77, p.87] The process $(y(t), \gamma(t))$ given by (7) is positive recurrent in $\mathbb{R}^n_+ \times \mathbb{S}$ if the following hypotheses hold.

(1) For each $k \in \mathbb{S}$,
\[
\xi^T g(y, k)g^T(y, k)\xi \geq \kappa_1|\xi|^2, \forall \xi \in \mathbb{R}^n,
\]
with some constant $\kappa_1 \in (0, 1]$ for all $y \in \mathbb{R}^n$;

(2) There exists a nonempty open set $D \subset \mathbb{R}^n$ with compact closure and a non-negative function $V(\cdot, k) : D^c \to \mathbb{R}$ for each $k \in \mathbb{S}$ such that $V(\cdot, k)$ is twice continuously differentiable and that for some $\alpha > 0$,
\[
L_y V(y, k) \leq -\alpha, \forall y \in D^c.
\]

**Assumption 5.** For each $k \in \mathbb{S}$, $\text{Rank}(\sigma(k)) = n$.

**Theorem 6.2.** Under Assumption 5 and the conditions of Theorem 5.4, the process $(x(t), \gamma(t))$ given by (7) is positive recurrent in $\mathbb{R}^n_+ \times \mathbb{S}$.

**Proof.** For each $1 \leq i \leq n$, define $z_i(t) := \log x_i(t)$ for $t \geq 0$, and $z(t) := (z_1(t), \cdots, z_n(t))^T$. By the generalized Itô formula, for each $i$,
\[
dz_i(t) = \left( b_i(\gamma(t)) + \sum_{j=1}^n a_{ij}(\gamma(t))e^{z_j(t)} - \frac{1}{2} \sum_{j=1}^d \sigma_i^2(\gamma(t)) \right) dt + \sum_{j=1}^d \sigma_{ij}(\gamma(t)) dB_j(t).
\]

(58)

Obviously, the positive recurrence of $(x(t), \gamma(t))$ in $\mathbb{R}^n_+ \times \mathbb{S}$ is equivalent to that of $(z(t), \gamma(t))$ in $\mathbb{R}^n \times \mathbb{S}$. Thus it suffices to prove the process $(z(t), \gamma(t))$ given by (58) satisfies the conditions of Lemma 6.1. Note that $\sigma(k)\sigma^T(k)$ is positive-definite under Assumption 5, thus condition (1) holds. We now proceed to prove that condition (2) holds. Choose $p, \eta_i > 0$ satisfying Theorem 3.4 and Theorem 5.2. Define $H : \mathbb{R}^n \times \mathbb{S} \to \mathbb{R}$ by
\[
H(z, k) := (1 - \alpha_k p) \left[ 1 + \log \left( \sum_{i=1}^n c_i(k)e^{z_i} + 1 \right) \right]^p + \sum_{i=1}^n (1 - \nu_i \eta_i) (e^{-z_i} + 1)^\eta_i
\]
\[
:= F(z, k) + \sum_{i=1}^n G_i(z_i, k),
\]

(59)
where $\alpha = (\alpha_1, \ldots, \alpha_m)^T$ and $v_i = (v_{i1}, \ldots, v_{im})^T$ are given by (10) in Theorem 3.4 and (40) in Theorem 5.2 respectively. Then direct calculations imply

$$L_zF(z, k) = p(1 - \alpha_k p) \left[ 1 + \log \left( \sum_{i=1}^n c_i(k)e^{z_i} + 1 \right) \right]^{p-1} \sum_{i=1}^n c_i(k)b_i(k)e^{z_i} + \sum_{i,j=1}^n c_i(k)a_{ij}(k)e^{z_i}e^{z_j} \sum_{i=1}^n c_i(k)e^{z_i} + 1$$

$$+ \frac{1}{2} p(1 - \alpha_k p) \left[ 1 + \log \left( \sum_{i=1}^n c_i(k)e^{z_i} + 1 \right) \right]^{p-2} \sum_{j=1}^d \left( \sum_{i=1}^n c_i(k)e^{z_i} \sigma_{ij}(k) \right)^2 \sum_{i=1}^n c_i(k)e^{z_i} + 1$$

$$+ \sum_{l=1}^m q_{kl} (1 - \alpha_l p) \left[ 1 + \left( \sum_{i=1}^n c_i(l)e^{z_i} + 1 \right) \right]^p,$$

(60)

and

$$L_zG_i(z_i, k) = -(1 - v_{i1}\eta_i)\eta_i (e^{-z_i} + 1)^{n_i-1}e^{-z_i}\left( b_i(k) + \sum_{j=1}^n a_{ij}(k)e^{z_i} \right)$$

$$+ \frac{1}{2} (1 - v_{i1}\eta_i)\eta_i (e^{-z_i} + 1)^{n_i-2} \left[ (\eta_i + 1)e^{-2z_i} + 2e^{-z_i} \right] \left( \sum_{j=1}^d \sigma_{ij}(k) \right)^2$$

$$+ \sum_{l=1}^m q_{kl} (1 - v_{il}\eta_k) (e^{-z_i} + 1)^{n_i}.$$

It follows from (12) and (13)

$$L_zF(z, k) = L_zV_1((e^{z_1}, \ldots, e^{z_n})^T, k), \quad L_zG_i(z_i, k) = L_zW_i(e^{z_i}, k).$$

(61)

Then the first equation in (61) together with (13), (13), (16) and (19) in the proof of Theorem 3.4 leads to

$$\limsup_{z_i \to -\infty, \forall i=1, \ldots, n} L_zF(z, k) = \limsup_{|x_i| \to 0^+} L_zV_1(x, k) \leq p(1 - \alpha_k p) \left( b(k) - \frac{\lambda_k}{C_k^2} \right) - p \sum_{l=1}^m q_{kl} \alpha_l,$$

and for any $1 \leq i \leq n$,

$$\lim_{z_i \to -\infty} L_zF(z, k) = \lim_{|x_i| \to \infty} L_zV_1(x, k) = -\infty.$$

(62)

On the other hand, the second equation in (61) together with (44) and (45)-(47) also leads to

$$\lim_{z_i \to -\infty} L_zG_i(z_i, k) = \lim_{x_i \to 0^+} L_zW_i(x_i, k) = -\infty,$$

(63)
and
\[
\limsup_{z_i \to +\infty} L_z G(z, k) = \limsup_{x_i \to +\infty} L_z W_i(x_i, k) \leq -\eta_i (1 - v_i \eta_i) a_{ii}(k) - \eta_i \sum_{l=1}^{m} q_{kl} v_i.
\]

Thus,
\[
\sup_{(x, k) \in \mathbb{R}_+^n \times S} \left( L_z V_1(x, k) + \sum_{i=1}^{n} L_z W_i(x_i, k) \right) < \infty,
\]
which implies
\[
L_z H(z, k) \leq K, \forall (z, k) \in \mathbb{R}_+^n \times S.
\]

By the virtue of (62) and (63), choose a large positive constant \(N\), such that for each \(1 \leq i \leq n\),
\[
L_z F(z) \leq -K - 1, \text{ whenever } z_i \geq N,
\]
and
\[
L_z G_i(z_i, k) \leq -K - 1, \text{ whenever } z_i \leq -N.
\]

Define
\[
D_N := \{ z \in \mathbb{R}_+^n : -N < z_i < N, i = 1, \cdots, n \}.
\]

Note that
\[
L_z H(z, k) \leq -1, \forall (z, k) \in D_N^c \times S.
\]

Therefore, the desired assertion follows. \(\square\)

Furthermore, using Theorem 4.3 in [27, p.114], we yield the following theorem.

**Theorem 6.3.** Under the conditions of Theorem 6.2, the process \((x(t), \gamma(t))\) of (3) has a unique stationary distribution in \(\mathbb{R}_+^n \times S\).

**Corollary 5.** Under Assumption [5] and the conditions of Corollary 4, the process \((x(t), \gamma(t))\) of (3) has a unique stationary distribution in \(\mathbb{R}_+^n \times S\).

Theorem 6.3 generalizes the main results in [14, 20]. To complete this section, we compare our results with those in [14, 20]. We first cite the following results. To put it another way, the aforementioned results in the references can be considered as special cases or corollaries of ours.

**Corollary 6.** [14] The process \((x(t), \gamma(t))\) of (3) has a unique stationary distribution in \(\mathbb{R}_+^n \times S\) under Assumption [5] and conditions (A.1)-(A.3) of Corollary 2.

**Remark 8.** (A.1)-(A.3) have been improved completely in the current paper. The highlights with regard to (A.1) and (A.2) are illustrated in Remark 3. From the definition of \(\beta_i(k)\) in Theorem 5.2 (A.3) may be false even if each \(\pi \beta_i < 0\). Thus each \(\pi \beta_i < 0\) is weaker than (A.3). Thus (A.3) is improved.

**Remark 9.** For the special case \(m = 1\), Mao in [20] gave the existence of the unique stationary distribution but needed \(-A(1)\) is a non-singular M-matrix besides the conditions of Theorem 6.3. Thus Theorem 6.3 generalizes the result of [20].
7. Numerical examples. Theorem 3.8 and Theorem 5.4 concern the permanence while Theorem 4.1 and Theorem 4.2 concentrate on the impermanence including the extinction and exponential growth. For $S = \{k\}$, as a special case, the Lotka-Volterra system

$$dx(t) = \text{diag}(x_1(t), \ldots, x_n(t)) \left[ (b(k) + A(k)x(t)) \, dt + \sigma(k) dB(t) \right]$$

(64)
called the subsystem of the switching system $[3]$ is permanent or not depends its parameters. Precisely, if $\lambda_k < 0$, $\beta(k) > 0$, it is permanent while it is impermanent if $\lambda \leq 0$, $\beta(k) < 0$ or $\lambda_k < 0$, $\lambda \geq 0$, $\beta(k) > 0$. Obviously, each $\lambda_k < 0$, $\beta(k) > 0$ for $k = 1, \ldots, m$ implies $\pi \lambda = \sum_{k=1}^m \pi_k \lambda_k < 0$ and $\pi \hat{\beta} = \sum_{k=1}^m \pi_k \hat{\beta}_k > 0$, by virtue of Theorem 3.8 the switching system $[3]$ is stochastically permanent. Other cases that $\lambda \leq 0$, $\beta(k) < 0$ or $\lambda_k < 0$, $\lambda \geq 0$, $\beta(k) > 0$ are similar. Therefore, if all subsystems $[64]$ are permanent (impermanent) the switching system $[3]$ keeps their properties. While if some subsystems are permanent and some are not, the overall behavior of SDE $[3]$ may be permanent or extinctive depending on both environments represented by the parameters and the environmental switching represented by $\pi$. However, a striking example was given by Lawley et al. in $[10]$ that a process switching between two linear ordinary differential equations, where the individual parameter matrix and the coefficient average of two matrices are all Hurwitz (all eigenvalues have strictly negative real part) but at least one individual matrix is not normal, goes to infinity at large time for certain values of the switching rate. It indicates for the non-normal case that a certain value of switching rate can bring delicately about a very different large-time behavior from both the individual systems and the average system. On the other hand, they showed also that the switching process tends to zero at large time when each individual matrix is normal and Hurwitz. So the strange dynamical phenomenon doesn’t appear for the normal case. Although $[3]$ is nonlinear and stochastic, the parameter matrices of the linear parts of its individual drift terms are all normal. Thus our results support the opinion on the normal case of $[10]$ in some sense. Next we give another example to demonstrate the effectiveness and applicability of this normal case results.

Example 7.1. Consider the three-species competitive SDE $[3]$ with $\gamma(t)$ taking values in $S = \{1, 2\}$,

$$b(1) = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad A(1) = \begin{pmatrix} 0 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix}, \quad \sigma(1) = \begin{pmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{3} & 1 & 2 \\ 1 & 1 & \sqrt{3} \end{pmatrix};$$

$$b(2) = \begin{pmatrix} 4 \\ 3 \\ 3 \end{pmatrix}, \quad A(2) = \begin{pmatrix} -1 & 0 & -1 \\ -1 & -1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & -1 \end{pmatrix}, \quad \sigma(2) = \begin{pmatrix} 1 & 1 & 1 \\ \frac{\sqrt{3}}{2} & 1 & 1 \end{pmatrix}.$$ 

Let $C(k) = I$ for any $k \in S$, and compute

$$\lambda_1 = \sup_{x \in \mathbb{R}^3, \|x\| = 1} x^T (A(1) + A(1)^T) x = 0, \quad \lambda_2 \leq \lambda_{\text{max}}(A(2) + A(2)^T) = -1 < 0.$$ 

Thus Assumption $[1]$ holds. Moreover,

$$\hat{\beta}(1) = \frac{7}{2}, \quad \hat{\beta}(1) = \frac{1}{2} < 0; \quad \hat{\beta}(2) = 1 > 0, \quad \hat{\beta}(2) = \frac{11}{4}.$$ 

Therefore, by Theorem 3.8 and 4.1 the subsystem $[64]$ of state-1 is extinctive, while the subsystem of state-2 is stochastically permanen, see Figure 3.
Case 1. Let the generator of the Markov chain $\gamma(t)$ be

$$Q = \begin{pmatrix} -4 & 4 \\ 1 & -1 \end{pmatrix}.$$

By solving the linear equation (6) we obtain the unique stationary distribution

$$\pi = (\pi_1, \pi_2) = \left(\frac{1}{5}, \frac{4}{5}\right).$$

Then

$$\pi \lambda = \sum_{k=1}^{2} \pi_k \lambda_k < 0; \quad \pi \hat{\beta} = \sum_{k=1}^{2} \pi_k \hat{\beta}(k) = \frac{1}{10} > 0.$$

By Theorems 3.8 the switching system is stochastically permanent, see Figure 4.

Case 2. Let the generator of the Markov chain $\gamma(t)$ be

$$Q = \begin{pmatrix} -1 & 1 \\ 6 & -6 \end{pmatrix}.$$

By solving the linear equation (6) we obtain the unique stationary distribution

$$\pi = (\pi_1, \pi_2) = \left(\frac{6}{7}, \frac{1}{7}\right).$$

Then

$$\pi \hat{\beta} = \sum_{k=1}^{2} \pi_k \hat{\beta}(k) = -\frac{1}{28} < 0.$$

By Theorems 4.1 the switching system is extinctive, see Figure 5.
Figure 5. Case 2. A sample path of $|x(t)|$ of the switching system in Example 7.1.

Clearly, the main results in [11, 14, 20, 28] are infeasible for the case that there exists some $a_{ii}(k) = 0$. However, our results can deal with this case. Therefore, our results have wider range of applications. For mutualistic systems, we give more precise results including the stochastic strong permanence and the existence of the unique stationary distribution. We complete this section by giving another example on a mutual site system to illustrate the rest part results.

Example 7.2. Consider the two-species mutualistic SDE (3) with $\gamma(t)$ taking values in $S = \{1, 2, 3\}$ and

\[
\begin{align*}
    b(1) &= \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \quad A(1) = \begin{pmatrix} -1 & 0 \\ 1 & -2 \end{pmatrix}, \quad \sigma(1) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}; \\
    b(2) &= \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad A(2) = \begin{pmatrix} -4 & 1 \\ 2 & -2 \end{pmatrix}, \quad \sigma(2) = \begin{pmatrix} 2 \\ 2 \\ \sqrt{2} \end{pmatrix}; \\
    b(3) &= \begin{pmatrix} 5 \\ 6 \end{pmatrix}, \quad A(3) = \begin{pmatrix} -3 & 1 \\ 1 & -1 \end{pmatrix}, \quad \sigma(3) = \begin{pmatrix} 2 \\ 1 \\ \sqrt{2} \end{pmatrix}.
\end{align*}
\]

Let $\bar{C}(k) = I$ for any $k \in S$, and compute

\[
\begin{align*}
    \lambda^+_{\min}(A(1) + A^T(1)) &= -4, \quad \lambda_1 = -3 + \sqrt{2}; \\
    \lambda^+_{\min}(A(2) + A^T(2)) &= -8, \quad \lambda_2 = -6 + \sqrt{13}; \\
    \lambda^+_{\min}(A(3) + A^T(3)) &= -6, \quad \lambda_3 = -4 + 2\sqrt{2}.
\end{align*}
\]

and

\[
\hat{\lambda} = -8, \quad \bar{\lambda} = -3 + \sqrt{2}.
\]

Moreover,

\[
\begin{align*}
    \hat{\beta}(1) &= -\frac{1}{2}, \quad \tilde{\beta}(1) = 2, \quad \beta_1(1) = \frac{1}{2}, \quad \beta_2(1) = 1; \\
    \hat{\beta}(2) &= -\frac{11}{2}, \quad \tilde{\beta}(2) = -1, \quad \beta_1(2) = -2, \quad \beta_2(2) = -\frac{9}{2}; \\
    \hat{\beta}(3) &= 2, \quad \tilde{\beta}(3) = 5, \quad \beta_1(3) = \frac{5}{2}, \quad \beta_2(3) = \frac{9}{2}.
\end{align*}
\]

Therefore, by the virtues of Theorem 5.4, 6.3 and 4.1, the subsystems of state-1 and state-3 are stochastically strongly permanent and have a unique stationary density; while the subsystem of state-2 is extinctive; see Figure 6, 7, 8.
Figure 6. A sample path of $x_1(t)$ and $x_2(t)$ of state-1, state-2 and state-3 in Example 7.2

Figure 7. Stationary distribution and scatter plot of a sample path of state-1 in Example 7.2

Figure 8. Stationary distribution and scatter plot of a sample path of state-3 in Example 7.2

Case 1. Let the generator of the Markov chain $\gamma(t)$ be

$$Q = \begin{pmatrix} -6 & 6 & 0 \\ 0 & -6 & 6 \\ \frac{3}{2} & 0 & -\frac{3}{2} \end{pmatrix},$$

where each column has a non-diagonal element 0. The main results of [11, 14] can't deal with such case. Solving the linear equation we obtain the stationary
distribution of $\gamma(t)$,
\[ \pi = (\pi_1, \pi_2, \pi_3) = \left( \frac{1}{6}, \frac{1}{6}, \frac{2}{3} \right). \]
Therefore we have
\[ \pi_{\hat{\beta}} = \frac{1}{3}, \pi_{\check{\beta}} = \frac{7}{2}, \pi_{\beta_1} = \frac{17}{12} > 0, \pi_{\beta_2} = \frac{29}{12} > 0. \]
By Theorems 5.4 and 6.2 the switching system is stochastically strongly permanent and has a unique stationary distribution, see Figure 9, 10. Moreover, by Theorems 4.3 the following inequality holds.
\[ \frac{1}{12} \leq \liminf_{t \to +\infty} \frac{1}{t} \int_0^t |x(s)| ds \leq \limsup_{t \to +\infty} \frac{1}{t} \int_0^t |x(s)| ds \leq 3\sqrt{2} + 2 \text{ a.s.} \]
see Figure 11.
Case 2. Let the generator of the Markov chain $\gamma(t)$ be

$$Q = \begin{pmatrix} -8 & 8 & 0 \\ 0 & -1 & 1 \\ 8 & 0 & -8 \end{pmatrix}.$$  

By solving the linear equation (6) we obtain the unique stationary distribution

$$\pi = (\pi_1, \pi_2, \pi_3) = \left(\frac{1}{10}, \frac{4}{5}, \frac{1}{10}\right).$$

Therefore we have

$$\pi \beta = -\frac{1}{10} < 0.$$  

By Theorems 4.1 the switching system is extinctive, see Figure 12.

![Figure 12. Case 2. A sample path of $x_1(t)$ and $x_2(t)$ of the switching system in Example 7.2.](image)

8. **Concluding remarks.** This paper investigated stochastic permanence and several asymptotic properties of stochastic Lotka-Volterra model with regime switching. For the mutualistic system we give more precise results including the stochastic strong permanence and the existence of the stationary distribution. We relaxed the restrictions (A.1)-(A.3) of both coefficients and the generator required by the previous references. An interesting fact is revealed that the regime switching can suppress the impermanence. We demonstrated our results through numerical examples as illustrated in Section 7.

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E-mail address: wangr706@nenu.edu.cn
E-mail address: lixy209@nenu.edu.cn
E-mail address: dmukama86@gmail.com