On the third moment of $L\left(\frac{1}{2}, \chi_d\right)$: the rational function field case

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Abstract

In this note, we prove the existence of a secondary term in the asymptotic formula of the cubic moment of quadratic Dirichlet $L$-functions

$$\sum_{d_0-\text{monic \& sq. free}} \sum_{\deg d_i = D} L\left(\frac{1}{2}, \chi_{d_0}\right)^3$$

over rational function fields on the order of $q^{\frac{1}{4}D}$. This term is in perfect analogy with the $x^{\frac{1}{4}}$-term indicated in our joint work [18] for the corresponding asymptotic formula over the rationals.

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1 Introduction

Statement of the main results. Let \( \mathbb{F} \) be a finite field with \( q \) elements. For simplicity, we will assume throughout that \( q \equiv 1 \pmod{4} \). For monic polynomials \( d_0, m \in \mathbb{F}[x] \) with \( d_0 \) square-free, let \( \chi_{d_0}(m) = (d_0/m) \) be the usual quadratic residue symbol, and consider the generating series of the cubic moments of the central values of quadratic Dirichlet \( L \)-functions

\[
W(\xi) = \sum_{D \geq 0} \left( \sum_{d_0 - \text{monic & sq. free}} \sum_{\deg d_0 = D} L\left( \frac{1}{2}, \chi_{d_0} \right)^3 \right) \xi^D.
\]

This series is absolutely convergent for complex \( \xi \) with sufficiently small (depending upon the size \( q \) of \( \mathbb{F} \)) absolute value.

The main result of this note is the following

**Theorem 1.1.** — The function \( W(\xi) \) has meromorphic continuation to the open disk \( |\xi| < q^{-1/3} \). It is analytic in this region, except for poles of order seven at \( \xi = \pm q^{-1} \) and simple poles at \( \xi = \pm q^{-3/4}, \pm iq^{-3/4} \), and the principal part at each of these poles is explicitly computable.

The principal parts of \( W(\xi) \) at the poles \( \xi = \pm q^{-1} \) can be computed as in [18, Section 3.2], and accordingly will not be discussed any further here. The residues at the remaining (simple) poles will be computed in Section 7, see (30).

Let \( \zeta(s) = \zeta_{\mathbb{F}(x)}(s) \) denote the zeta function of the field \( \mathbb{F}(x) \). As a consequence of Theorem 1.1, we have the following asymptotic formula for the cubic moments of the central values of quadratic Dirichlet \( L \)-functions.

**Theorem 1.2.** — For every small \( \delta > 0 \) and \( D \in \mathbb{N} \), we have

\[
\sum_{d_0 - \text{monic & sq. free}} \sum_{\deg d_0 = D} L\left( \frac{1}{2}, \chi_{d_0} \right)^3 = \frac{q^D}{\zeta(2)} Q(D, q) + q^{2D} R(D, q) + O_D\left( q^{D(\frac{2}{3} + \delta)} \right)
\]

for explicitly computable \( Q(D, q) \) and \( R(D, q) \).

An expression for \( Q(D, q) \) can be easily obtained from the principal parts of \( W(\xi) \) at \( \xi = \pm q^{-1} \). We will not pursue this calculation here, as there are alternative ways to compute \( Q(D, q) \) (see [28, Section 8 (a)] and [2, Section 5.3]). The computation of \( R(D, q) \) will be given in Section 8. However, for the convenience of the reader, we give here the expression of \( R(D, q) \):

\[
P(x) := (1-x)^3(1+x)(1+4x+11x^2+10x^3-11x^4+11x^6-4x^7-x^8) = 1-14x^3-x^4+78x^5+\ldots
\]

is the polynomial appearing in [34], then

\[
R(D, q) = \frac{1}{4}(1 + q^{1/4} + 10q^{1/2} + 7q^{3/4} + 20q + 7q^{3/4} + 10q^{1/4} + q^2) \zeta\left( \frac{1}{2} \right)^3 \prod_{p} P\left( \frac{1}{\sqrt{|p|}} \right)
\]

\[
+ \frac{(-1)^D}{4}(1 - q^{1/4} + 10q^{1/2} - 7q^{3/4} + 20q - 7q^{3/4} + 10q^{1/2} - q^2) \zeta\left( \frac{1}{2} \right)^3 \prod_{p} P\left( \frac{1}{\sqrt{|p|}} \right)
\]

\[
+ \frac{1}{2} R\left( i^D (1 - iq^{1/4} - 4q^{1/2} + 7iq^{3/4} + 6q - 7iq^{3/4} - 4q^{1/2} + iq^{1/4} + q^2) \right) L\left( \frac{1}{2}, \chi_{0} \right)^3 \prod_{p} P\left( (-1)^{\deg p} / \sqrt{|p|} \right)
\]

the products in the right-hand side being over all monic irreducibles of \( \mathbb{F}[x] \), and where

\[
\zeta\left( \frac{1}{2} \right) = \frac{1}{1-\sqrt{q}} \quad \text{and} \quad L\left( \frac{1}{2}, \chi_{0} \right) = \frac{1}{1+\sqrt{q}}.
\]

Similar results over the rationals will appear in the forthcoming manuscript [20].
Relation to previous work. Understanding the moments of various families of $L$-functions at the center of symmetry is a very important problem in analytic number theory. A classical example is the family of quadratic Dirichlet $L$-series whose moments attracted considerable attention over the years. Heuristics to determine the main terms in the asymptotic formula for the moments of this family were given in [17] and [18]. Besides the conjectural main terms, this asymptotic formula should also have a finer part consisting generically of infinitely many lower order terms. The first\(^1\) instance when such a lower order term occurs is the cubic moment, and to justify this assertion is the subject of this note and [20]; so far, the evidence supporting the existence of this additional term was limited to the conditional result in [34], and the extensive computations and experiments in [1]. This particular moment is the highest over the rationals for which an asymptotic formula has been established, see [29], [18] and [33]. In the rational function field case, the corresponding asymptotics for the third and fourth moments have been established in [22] and [23]\(^2\), respectively. It is by no means a coincidence that the error terms in both [33] and [22] are of size comparable to the size of the corresponding secondary terms asserted.

The approach we take is based on Weyl group multiple Dirichlet series. These are series associated to root systems over global fields (containing sufficient roots of unity) of the form

$$Z(s;m,\Psi) = \sum_{n=(n_1,\ldots,n_r)} \frac{H(n;m)\Psi(n)}{\prod |n_i|^{s_i}}$$

$m = (m_1,\ldots,m_r)$ being a twisting parameter, satisfying (Weyl) groups of functional equations; see for details [11], [6] and [16]. If $m_i = 1$ for all $i$, the series is said to be untwisted. The most important part of $Z(s;m,\Psi)$ is the function $H$, giving the structure of the multiple Dirichlet series. Via a twisted multiplicativity (see, for example, [6] and [16]), this function is determined by its values on prime powers. Equivalently, the multiple Dirichlet series is determined by its $p$-parts\(^3\), i.e.,

$$\sum_{k_1,\ldots,k_r \geq 0} H(p^{k_1},\ldots,p^{k_r};p^{l_1},\ldots,p^{l_r}) |p|^{-k_1 \cdots -k_r l_r} \quad \text{(with } p^l \parallel m_i, \text{ } p \text{ prime)}.$$  

There are several different methods of representing the $p$-parts of multiple Dirichlet series, namely,

- Definition by the “averaging method”, sometimes known as the Chinta-Gunnells method, see [15] and [16].
- Definition as spherical $p$-adic Whittaker functions, see [7] and [8].
- Definition as sums over crystal bases, see [7] and [25].
- Definition as partition functions of statistical-mechanical lattice models, see [4] and [5].

The equivalence of the Chinta-Gunnells method with the Whittaker definition was established by McNamara [26].

The Chinta-Gunnells method and the Whittaker definition were recently extended to infinite root systems in [24] and [27], respectively, and in [19] the author, in joint work with Paşol, applied Deligne’s theory of weights in the context of moduli spaces of admissible double covers to express the coefficients of the $p$-parts of untwisted multiple Dirichlet series associated to arbitrary moments of quadratic Dirichlet $L$-series in terms of $q$-Weil numbers, where $q = |p|$. The axiomatic approach introduced in [19] has also been applied in [31] and [32] to construct untwisted Weyl group multiple Dirichlet series associated to affine root systems.

Overview of the argument. The main ideas involved in the proof of Theorem 1.1 can be summarized as follows. As in [18], we first write

$$\mathcal{W}(q^{-w}) = \sum_{h \text{-monic}} \mu(h) \underbrace{Z\left(\frac{1}{2},\frac{1}{2},w,1;h\right)}_{\text{Möbius function on } \mathbb{F}^*} \quad \text{(for } \Re(w) > 1)$$

1The lower order terms we are referring to are all of magnitude larger than the threshold $s^{\frac{1}{2}}$. The additional term noticed in [21], besides being certainly of different origin, it is a special feature of the first moment.

2In this case, the asymptotic formula proved contains only the leading three terms.

3In [14, Corollary 5.8] it is shown that the $p$-parts of untwisted Weyl group multiple Dirichlet series constructed from quadratic characters are uniquely determined. This implies the remarkable fact that untwisted quadratic Weyl group multiple Dirichlet series over rational function fields coincide, after a simple change of variables, with their own $p$-parts.
where \( Z \) is a multiple Dirichlet series with a certain congruence condition. For every monic and square-free polynomial \( h \), this function will be expressed in terms of twisted (in the sense of [18]) multiple Dirichlet series. Unlike [18], the formula we use (see Section 5) is a finite sum of terms of the form

\[
|h|^{-2w} \chi_{c_2}^e (c_1) Z^{(i)} \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; w; \chi_{c_2}, \chi_{c_1} \right) \prod_{p \mid c_1} F (|p|^{-w}, \ldots, |p|^{-w}; |p|) |p|^{-w} \cdot \prod_{p \mid c_2} G^{(e_p)} (|p|^{-w}, \ldots, |p|^{-w}; |p|)
\]

with \( c_i \), monic, \( h = c_1 c_2 c_3 \), and for each monic irreducible \( p \mid c_2 c_3 \), the quantity \( \varepsilon_p = 0 \) or 1 according as \( p \) divides \( c_3 \) or \( p \) divides \( c_2 \). The functions \( F \) and \( G^{(e_p)} \) represent a (normalized) partition of the local \( p \)-part of the \( D_4 \)-untwisted Weyl group multiple Dirichlet series (associated to the cubic moment) corresponding to odd and even weighted monomials, and with negative degree terms in \( |p|^{-w} \) removed.

We will prove that the above series representation of \( W(q^{-w}) \) converges absolutely and uniformly on every compact subset of the half-plane \( \Re (w) > 2/3 \), away from the points \( w \in \mathbb{C} \) for which \( q^{-w} = \pm q^{-1} \), or \( q^{-w} = \pm q^{-3/4} \), \( \pm i q^{-3/4} \). To show this, we will exhibit additional decay of the function \( Z \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; w; 1; h \right) \) in \( |h| \). This will be done in two steps:

* We first show that, for \( \Re (w) \geq \frac{5}{4} \), the functions \( F \) and \( G^{(e)} \) are bounded, independent of \( w, |p| \), and

\[
G^{(1)} (|p|^{-w}, \ldots, |p|^{-w} ; |p|) \ll |p|^{-\frac{3}{2}}
\]

see Lemma 6.1. These estimates provide sufficient decay in the parameters \( c_1 \) and \( c_2 \).

* To obtain the required decay in the remaining parameter, we use again the properties of the \( p \)-parts of the \( D_4 \)-untwisted Weyl group multiple Dirichlet series combined with an inductive argument, to improve upon the convexity bound (19) of \( Z^{(i)} \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; w; \chi_{c_2}, \chi_{c_1} \right) \) in the \( c_3 \)-aspect, see Proposition 6.3.

The reader will no doubt have noticed the special role played by the \( p \)-parts in the argument. In [19] it is shown that the coefficients of these generating series can be expressed in terms of the eigenvalues of Frobenius acting on the \( \ell \)-adic étale cohomology of moduli of admissible double covers of genus zero stable curves with marked points, hence in terms of Weil algebraic integers. Thus, the most conceptual reason behind the asymptotics and estimates discussed in Section 6 is precisely the dominance condition (see [19]) satisfied by the \( p \)-parts of the untwisted multiple Dirichlet series associated to any (not just cubic) moment of quadratic Dirichlet \( L \)-functions. However, in the present context we take advantage of the completely explicit nature of the \( D_4 \)-Weyl group multiple Dirichlet series (see Appendix B) to deduce the relevant facts about its \( p \)-parts.

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## 2 Notation and preliminaries

Let \( \mathbb{F} \) be a finite field with \( q \equiv 1 \pmod{4} \) elements. For a non-zero \( m \in \mathbb{F}[x] \), we define its norm by \( |m| = q^{\deg m} \). For polynomials \( d, m \in \mathbb{F}[x] \), with \( m \) monic, let \( (d/m) \) denote the Kronecker symbol, defined as a completely multiplicative function of \( m \), for every fixed \( d \), and if \( m = p \) is irreducible then \( (d/p) = 0 \) if \( p \mid d \) and \( (d/p) = \pm 1 \) if \( p \nmid d \), the + or − sign being determined according to whether \( d \) is congruent to a square modulo \( p \) or not; we take \( (d/1) = 1 \). The symbol \( (d/m) \) is also completely multiplicative as a function of \( d \), for every \( m \). Since we are assuming that \( q \equiv 1 \pmod{4} \), we have the simpler quadratic reciprocity law:

\[
\left( \frac{d}{m} \right) = \left( \frac{m}{d} \right) \quad \text{(for coprime non-constant monic polynomials \( d, m \in \mathbb{F}[x] \)).}
\]

In addition, if \( b \in \mathbb{F}^\times \) then \( \left( \frac{b}{m} \right) = \text{sgn}(b)^{\deg m} \) for all non-constant \( m \in \mathbb{F}[x] \), where, for \( d(x) = b_0 x^d + b_1 x^{d-1} + \cdots + b_d \in \mathbb{F}[x] \) \( (b_0 \neq 0) \), we define \( \text{sgn}(d) = 1 \) if \( b_0 \in \left( \mathbb{F}^\times \right)^2 \) and \( \text{sgn}(d) = -1 \) if \( b_0 \notin \left( \mathbb{F}^\times \right)^2 \).
For \( d = d_0 \) square-free, let \( \chi_{d_0}(m) = (d_0/m) \). The \( L \)-series attached to the character \( \chi_{d_0} \) is defined by
\[
L(s, \chi_{d_0}) = \sum_{m\in\mathbb{Z}[x]} \chi_{d_0}(m)|m|^{-s} = \prod_{p-\text{monic \& irred.}} (1 - \chi_{d_0}(p)|p|^{-s})^{-1} \quad \text{(for complex } s \text{ with } \Re(s) > 1).\]

It is well-known that \( L(s, \chi_{d_0}) \) is a polynomial in \( q^{-s} \) of degree \( d_0 - 1 \) when \( d_0 \) is non-constant; if \( d_0 \in \mathbb{F}^\times \) then
\[
L(s, \chi_{d_0}) = \zeta(s) = \frac{1}{1 - q^{1-s}} \quad \text{(when } \sgn(d_0) = 1) \quad \text{and} \quad L(s, \chi_{d_0}) = \frac{1}{1 + q^{1-s}} \quad \text{(when } \sgn(d_0) = -1).\]

Moreover, if we define \( \gamma_d(s, d) \) by
\[
\gamma_d(s, d) := q^{\frac{1}{2}(\deg d)(\deg d - \frac{1}{2})} (1 - \sgn(d)q^{-s})(1 + \deg d)^{\frac{1}{2}} (1 - \sgn(d)q^{s-1})(1 - \deg d)^{\frac{1}{2}}
\]
then the function \( L(s, \chi_{d_0}) \) satisfies the functional equation
\[
L(s, \chi_{d_0}) = \gamma_d(s, d_0) |d_0|^\frac{1}{2} L(1 - s, \chi_{d_0}).
\]

### 2.1 The Chinta-Gunnells action

We shall now recall an important technique developed by Chinta and Gunnells [15] to produce certain rational functions associated to classical root systems, which they subsequently used as building blocks to construct Weyl group multiple Dirichlet series (over any global field) twisted by quadratic characters. Strictly speaking, we shall apply this construction only when the underlying root system is \( D_4 \), and therefore, the material included in Appendix B suffices for our purposes. However, we feel that the approach taken here is applicable to similar problems in other contexts, and for this reason, we opted to present this background material in some generality.

Let \( \Phi \) be a rank \( r \) irreducible simply-laced root system, and let \( W = W(\Phi) \) denote the Weyl group of \( \Phi \). Fix an ordering of the roots and decompose \( \Phi = \Phi^+ \cup \Phi^- \) into positive and negative roots. Let \( \alpha_1, \alpha_2, \ldots, \alpha_r \) be the simple roots and let \( \sigma_i \in W \) be the simple reflection through the hyperplane perpendicular to \( \alpha_i \). The simple reflections generate the Weyl group and satisfy the relations \( (\sigma_i, \sigma_j)^{r_{ij}} = 1 \) with \( r_{ij} = 1 \) for all \( i \) and \( r_{ij} \in \{2, 3\} \) if \( i \neq j \). The indices \( i \) and \( j \) are said to be adjacent if \( i \neq j \) and \( r_{ij} = 3 \). The action of the simple reflections on the roots is given by
\[
\sigma_i \alpha_j = \begin{cases} 
\alpha_i + \alpha_j & \text{if } i \text{ and } j \text{ are adjacent} \\
-\alpha_j & \text{if } i = j \\
\alpha_j & \text{otherwise}.
\end{cases}
\]

Let \( \Lambda(\Phi) \) denote the root lattice of \( \Phi \). Every element \( \lambda \) of the root lattice has a unique representation as an integral linear combination of the simple roots
\[
\lambda = \sum_{i=1}^r k_i \alpha_i.
\]

Let \( d(\lambda) := \sum_{i=1}^r k_i \) be the height function on \( \Lambda(\Phi) \).

In this setting, Chinta and Gunnells [15] introduced a Weyl group action on the field of rational functions \( \mathbb{C}(z_1, \ldots, z_r) \) in \( r \) variables and used it to construct multiple Dirichlet series over global fields having analytic continuation to \( \mathbb{C}^r \) and satisfying a group of functional equations isomorphic to \( W \).

To define this group action, denote \( \mathfrak{z} = (z_1, \ldots, z_r) \), and for \( \lambda \in \Lambda(\Phi) \), set \( \mathfrak{z}^\lambda := z_1^{k_1} \cdots z_r^{k_r} \) with \( k_i \) determined by \( \lambda \) as above. Following [15], define \( \mathfrak{z}' = \mathfrak{z}^\lambda \), where
\[
\mathfrak{z}'_j = \begin{cases} 
-z_j & \text{if } i \text{ and } j \text{ are adjacent} \\
z_j & \text{otherwise}
\end{cases}
\]

\(^4\)Very often in this work, a monic polynomial \( d \) will be expressed as \( d = d_0 d_i^2 \) with \( d_0 \) monic and square-free, which justifies the notation.
and \( \sigma_i z = z' \), where

\[
z'_j = \begin{cases} \sqrt{q}z_i z_j & \text{if } i \text{ and } j \text{ are adjacent} \\ \frac{1}{(q z_j)} & \text{if } i = j \\ z_j & \text{otherwise.} \end{cases}
\]

Here \( q \geq 1 \) is a fixed parameter. One checks easily that \( \frac{z_i}{z_j} = z, \frac{z_i}{z_j} = \frac{z_i}{z_j} \), and that

\[
\sigma_i \tau_i z = \begin{cases} \tau_i \sigma_i z & \text{if } i \text{ and } j \text{ are adjacent} \\ \sigma_i z & \text{otherwise.} \end{cases}
\]

Letting \( f_i(z) = (f(z) + f(z^i))/2 \), for \( f \in \mathbb{C}(z) \), one defines the action of a simple reflection \( \sigma_i \), on \( \mathbb{C}(z) \) by

\[
(f|\sigma_i)(z) = -\frac{1 - qz_i}{qz_i(1 - z_i)} f_i + \frac{1}{\sqrt{q}z_i} f_i(\sigma_i z).
\]

In [15, Lemma 3.2] it has been verified that this action extends to a \( W \)-action on \( \mathbb{C}(z) \).

Using this Weyl group action, one can construct a \( W \)-invariant rational function \( f \in \mathbb{C}(z) \) such that \( f(0, \ldots, 0; q) = 1 \), and satisfying the following limiting condition:

\[
\text{for each } i = 1, \ldots, r, \text{ the function } (1 - z_i) \cdot f(z; q)|_{z_j = 0} \text{ for all } j \text{ adjacent to } i \text{ is independent of } z_i. \tag{3}
\]

The rational function satisfying these conditions is unique. When the root system is \( D_4 \), the uniqueness of this function follows easily from [10, Theorem 3.7] by a simple specialization, and in the general case, it follows similarly from the results in [31] and [32]. To construct this function, let \( \Delta(z) \) be defined by

\[
\Delta(z) = \prod_{\alpha \in \Phi^+} (1 - q^{\ell(\alpha)}z^{2\alpha})
\]

and, for \( \sigma \in W \), put

\[
j(\sigma, z) = \frac{\Delta(z)}{\Delta(\sigma z)}.
\]

Note that \( j(\sigma_i, z) = -qz_i^2 \) for each simple reflection \( \sigma_i \), and that this function satisfies the one-cocycle relation

\[
j(\sigma' \sigma, z) = j(\sigma', \sigma z) j(\sigma, z) \quad \text{for all } \sigma, \sigma' \in W.
\]

Finally, we define the rational function \( f(z; q) \) by

\[
f(z; q) = \Delta(z)^{-1} \cdot \sum_{\sigma \in W} j(\sigma, z)(1|\sigma)(z). \tag{4}
\]

The fact that this function satisfies the required conditions is established in [15, Theorem 3.4].

The rational function (4) corresponding to the root system \( D_4 \) is further discussed in Appendix B, and will be used in the next section to construct a family of multiple Dirichlet series over rational function fields satisfying the usual analytic properties.

### 3 Multiple Dirichlet series

Consider the rational function \( f \) defined in Appendix B, Eqs. (31), (32), and expand it in a power series

\[
f(z_1, \ldots, z_r, z_{r+1}; q) = \sum_{k_1, \ldots, k_r, l \geq 0} a(k_1, \ldots, k_r, l; q) z_1^{k_1} \cdots z_r^{k_r} z_{r+1}^l \quad \text{(with } r = 3\text{)}.
\]

\[\text{We shall assume throughout that } r = 3. \text{ However, since most of the functions (and other quantities) involved can be defined for other values of } r \text{ as well, we prefer (in such instances) to denote this value by } r - \text{ rather than taking it to be } 3.\]
We now use the coefficients of $f$ to construct the relevant family of multiple Dirichlet series.

Let $c \in \mathbb{F}[x]$ be monic and square-free, and fix a factorization $c = c_1 c_2 c_3$ (with $c_i \in \mathbb{F}[x]$ monic). Choose a $\Theta_0 \in \mathbb{F}^* \setminus \left(\mathbb{F}^*\right)^2$, and let $a_1, a_2 \in \{1, \Theta_0\}$. For $s = (s_1, \ldots, s_{r+1})$ with $\Re(s_i)$ sufficiently large, we define the multiple Dirichlet series

$$Z^{(c)}(s; \chi_{a_2 c_2}, \chi_{a_1 c_1}) := \sum_{m_1, \ldots, m_r, \text{monic}} \frac{\chi_{a_1 c_1 d_1}(m_1) \cdots \chi_{a_1 c_1 d_0}(m_r) \chi_{a_2 c_2}(d_0)}{|m_1|^{s_1} \cdots |m_r|^{s_r} |d_{r+1}|} \cdot A(m_1, \ldots, m_r, d)$$

(5)

where $m_i$ $(i = 1, \ldots, r)$ denotes the part of $m_i$ coprime to $d_0$, and the coefficients $A(m_1, m_2, m_3, \ldots, m_r, d)$ are defined as follows:

(i) If $p \in \mathbb{F}[x]$ is a monic irreducible, put

$$A(p^{k_1}, \ldots, p^{k_r}, p^l) = a(k_1, \ldots, k_r, l; \chi_{a_1 c_1}^{\deg p})$$

(ii) For monic $m_1, \ldots, m_r$ with $(m_1, \ldots, m_r, d, c) = 1$, we have

$$A(m_1, \ldots, m_r, d) = \prod_{p^{k_i} \| m_i} A(p^{k_1}, \ldots, p^{k_r}, p)$$

the product being taken over monic irreducibles $p \in \mathbb{F}[x]$.

The series (5) has two alternative expressions allowing us to show that $Z^{(c)}(s; \chi_{a_2 c_2}, \chi_{a_1 c_1})$ admits meromorphic continuation and satisfies a group of functional equations. Indeed, for every monic polynomial $d = d_0 d_1^2$ coprime to $c$, we can express

$$\sum_{m_1, \ldots, m_r, \text{monic}} m_{1, \ldots, m_r, c} \cdot A(m_1, \ldots, m_r, d) = \prod_{p^{k_i} \| d} \left( \sum_{l - \text{odd}} \frac{A(p^{k_1}, \ldots, p^{k_r}, p^l)}{|p|^{k_1 + \cdots + k_r}} \prod_{l' \| d} \left( \sum_{l' - \text{even}} \frac{\chi_{a_1 c_1 d_0}(p) A(p^{k_1}, \ldots, p^{k_r}, 1)}{|p|^{k_1 + \cdots + k_r}} \right) \right).$$

If $l = 0$, we have

$$A(p^{k_1}, \ldots, p^{k_r}, 1) = a(k_1, \ldots, k_r, 0; \chi_{a_1 c_1}^{\deg p}) = 1$$

(see Appendix B, (33)) and thus

$$\prod_{l \equiv 1 \pmod{2}} P_{l} \left( |p|^{-s_1}, \ldots, |p|^{-s_r}; q^{\deg p} \right) \prod_{l \equiv 0 \pmod{2}} P_{l} \left( |p|^{-s_1}, \ldots, |p|^{-s_r}; q^{\deg p} \right).$$

The remaining part ($l \neq 0$) of the two products can be expressed as

$$\prod_{l \equiv 1 \pmod{2}} P_{l} \left( |p|^{-s_1}, \ldots, |p|^{-s_r}; q^{\deg p} \right) \prod_{l \equiv 0 \pmod{2}} P_{l} \left( |p|^{-s_1}, \ldots, |p|^{-s_r}; q^{\deg p} \right).$$
Thus, if we define the Dirichlet polynomial
\[
P_r(s_1, \ldots, s_r ; \chi_{a_1 \in d_0}) = \prod_{p^s \| d} P_t(|p|^{-s_1}, \ldots, |p|^{-s_r}; q^{\deg p})
\]
\[
\cdot \prod_{p^s \| d} P_0(\chi_{a_1 \in d_0}(p)|p|^{-s_1}, \ldots; \chi_{a_1 \in d_0}(p)|p|^{-s_r}; q^{\deg p})
\]
(6)
then we can write
\[
Z(c)(s; \chi_{a_2 \in 2}, \chi_{a_1 \in 1}) = \sum_{(d, c) = 1}^{\infty} \frac{\sum_{d = d_0 d_1^r} \chi_{a_1 \in d_0}(\tilde{m}) \cdot \chi_{a_1 \in 1}(d_0)}{|d|^{\nu + 1}} A(m_1, \ldots, m_r, d)
\]
\[
\cdot \prod_{p^s \| m_1} \left( \sum_{l = 1}^{\infty} \frac{A(p^{k_1}, \ldots, p^{k_r}, p^l)}{|p|^{\nu + 1}} \right)
\]
\[
\cdot \prod_{p^s \| m_2} \left( \sum_{l = 1}^{\infty} \frac{\chi_{a_2 \in 2}(p) A(p^{k_1}, \ldots, p^{k_r}, p^l)}{|p|^{\nu + 1}} + \frac{A(p^{k_1}, \ldots, p^{k_r}, p^l)}{|p|^{(\nu + 1)\nu + 1}} \right)
\]
(7)
Now fix monics \( m_1, \ldots, m_r \) coprime to \( c \), and write \( m_1 \cdots m_r = n_0 n_1^2 \) with \( n_0 \) square-free. As
\[
A(p^{k_1}, \ldots, p^{k_r}, p^l) = a(k_1, \ldots, k_r, l; q^{\deg p}) = 0
\]
if \( \sum k_i \equiv l \pmod{1} \) (see Appendix B, (34)), we have
\[
\sum_{(d, c) = 1}^{\infty} \frac{\chi_{a_1 \in d_0}(\tilde{m}) \cdot \chi_{a_1 \in 1}(d_0)}{|d|^{\nu + 1}} A(m_1, \ldots, m_r, d)
\]
\[
= \chi_{a_1 \in d_0}(n_0) \cdot \prod_{p^s \| m_1} \left( \sum_{l = 0}^{\infty} \frac{A(p^{k_1}, \ldots, p^{k_r}, p^l)}{|p|^{\nu + 1}} \right)
\]
\[
\cdot \prod_{p^s \| m_2} \left( \sum_{l = 1}^{\infty} \frac{\chi_{a_2 \in 2}(p) A(p^{k_1}, \ldots, p^{k_r}, p^l)}{|p|^{\nu + 1}} + \frac{A(p^{k_1}, \ldots, p^{k_r}, p^l)}{|p|^{(\nu + 1)\nu + 1}} \right)
\]
Since
\[
A(1, \ldots, 1, p^l) = a(0, \ldots, 0, l; q^{\deg p}) = 1
\]
(see Appendix B, (33)) we can again compute the part corresponding to \( k_1 = \cdots = k_r = 0 \) as
\[
\prod_{p^s \| n_1} \left( \sum_{l = 0}^{\infty} \frac{\chi_{a_2 \in 2}(p^l)}{|p|^{\nu + 1}} \right) = \prod_{p^s \| n_2} \left( 1 - \chi_{a_2 \in 2}(p)|p|^{-\nu + 1} \right)^{-1} = L^{(c = n_1)}(s_1 + 1; \chi_{a_2 \in 2}).
\]
The remaining part of the expression is
\[
\chi_{a_1 \in 1}(n_0) \cdot \prod_{p^s \| m_1} Q_2(|p|^{-\nu + 1}; q^{\deg p}) \cdot \prod_{p^s \| m_2} \left( 1 - \chi_{a_2 \in 2}(p)|p|^{-\nu + 1} \right)^{-1} Q_2(\chi_{a_2 \in 2}(p)|p|^{-\nu + 1}; q^{\deg p}).
\]
As before, for \( m = (m_1, \ldots, m_r) \), we define the Dirichlet polynomial
\[
Q_m(s_1 + 1; \chi_{a_2 \in 2}) = \prod_{p^s \| m_1} Q_2(|p|^{-\nu + 1}; q^{\deg p}) \cdot \prod_{p^s \| m_2} Q_2(\chi_{a_2 \in 2}(p)|p|^{-\nu + 1}; q^{\deg p}).
\]
The polynomials

We now apply \( (7) \) and compute the residues at some of its poles.

3.1 Functional equations and analytic continuation of multiple Dirichlet series

The polynomials \( P_d(s_1, \ldots, s_r; \chi_{a_1 c_1 d_0}) \) are symmetric in \( s_1, \ldots, s_r \), and by (35) we have

\[
P_d(s_1, \ldots, s_r; \chi_{a_1 c_1 d_0}) = |d_1|^{1-2s_1} P_d(1-s_1, \ldots, s_r; \chi_{a_1 c_1 d_0}).
\]

(10)

The polynomials \( Q_m(s_1; \chi_{a_2 c_2 n_0}) \) satisfy the functional equation

\[
Q_m(s_1+1; \chi_{a_2 c_2 n_0}) = |m_1|^{1-2s_1} Q_m(1-s_1+1; \chi_{a_2 c_2 n_0})
\]

(11)

where, for \( m = (m_1, \ldots, m_r) \), we write \( m_1 \cdots m_r = n_0 n_1^2 \) with \( n_0 \) square-free.

We now apply (2), (10) and (11) to describe the functional equations of \( Z^{(c)}(s; \chi_{a_2 c_2}, \chi_{a_1 c_1}) \). We shall follow here [18]; see also [15].

First assume that \( \deg c_2 \) is even. We split the sum in (9) into two parts according as \( \deg n_0 \) is even or odd. By applying (10), (11), and (2) in the form

\[
L^{(c_1 c_2)}(s_1+1; \chi_{a_2 c_2 n_0}) = \gamma_d(s_1+1, a_2 c_2 n_0) |c_2 n_0|^{1-s_1} L^{(c_1 c_2)}(1-s_1+1; \chi_{a_2 c_2 n_0})
\]

we find that

\[
Z^{(c)}(s; \chi_{a_2 c_2}, \chi_{a_1 c_1})
= \gamma_d(s_1+1, a_2) \sum_{(m_1 \cdots m_r, c_1) = 1 \atop m_1 \cdots m_r = n_0 n_1^2 \atop \deg n_0 \text{ even}} L^{(c_1 c_2)}(1-s_1+1; \chi_{a_2 c_2 n_0})
\]

\[
\times \frac{L^{(c_1 c_2)}(1-s_1+1; \chi_{a_2 c_2 n_0}) L^{(c_1 c_2)}(s_1+1; \chi_{a_2 c_2 n_0})}{|m_1|^{1-s_1+1} |m_r|^{1-s_1+1}}
\]

\[
+ \gamma_d(s_1+1) \sum_{(m_1 \cdots m_r, c_1) = 1 \atop m_1 \cdots m_r = n_0 n_1^2 \atop \deg n_0 \text{ odd}} L^{(c_1 c_2)}(1-s_1+1; \chi_{a_2 c_2 n_0})
\]

\[
\times \frac{L^{(c_1 c_2)}(s_1+1; \chi_{a_2 c_2 n_0}) L^{(c_1 c_2)}(s_1+1; \chi_{a_2 c_2 n_0})}{|m_1|^{1-s_1+1} |m_r|^{1-s_1+1}}
\]

where we put

\[
\gamma_d(s_1+1, a_2) = \frac{q^{2s_1+1} - 1}{q^{2s_1+1} - 1 - \text{sgn}(a_2) q^{-s_1+1}} \quad \text{and} \quad \gamma_d(s_1+1) = q^{s_1+1} \frac{d_1}{a_2}.
\]

Notice that in the first sum we have also used the fact that \( \chi_{a_1 c_1}(n_0) = \chi_{c_1}(n_0) \) when \( n_0 \) has even degree.

To simplify this expression, multiply \( Z^{(c)}(s; \chi_{a_2 c_2}, \chi_{a_1 c_1}) \) by the product \( \prod_{p \mid c_1 c_2} (1 - |p|^{2s_1+2}) \). If we define \( U_m(s_{r+1}) = 1 \) for \( m \in \mathbb{F}^+ \), and

\[
U_m(s_{r+1}) = \prod_{p \mid m} \frac{|p|^{s_{r+1}+1} (1 - |p|^{1-s_{r+1}})}{1 - |p|^{1-s_{r+1}}}
\]
for m square-free of positive degree (the product being over the monic irreducible divisors of m), we can express:

\[
\frac{L_{c_1c_3}(1 - s_{r+1} \chi_{d_2c_2m_0})}{L_{c_1c_3}(s_{r+1}; \chi_{d_2c_2m_0})} 
\prod_{p \mid c_1c_3} (1 - |p|^{2s_{r+1} - 2}) = \prod_{p \mid c_1c_3} (1 + \chi_{d_2c_2m_0}(p)|p|^{s_{r+1} - 1}) \left(1 - \chi_{d_2c_2m_0}(p)|p|^{-s_{r+1}}\right)
\]

\[
= \prod_{p \mid c_1c_3} (1 - |p|^{-1}) \left(1 + \chi_{d_2c_2m_0}(p)U_{p}(s_{r+1})\right) = \frac{\phi(c_1c_3)}{|c_1c_3|} \sum_{m \mid c_1c_3} \chi_{d_2c_2m_0}(m)U_{m}(s_{r+1}).
\]

Here \(\phi\) is Euler’s totient function over \(\mathbb{F}[x]\). Letting

\[
\sigma_{i} s := (s_{1}, \ldots, s_{i}, \ldots, s_{r}, s_{t} + s_{r+1} - \frac{1}{2}) \text{ for } i \leq r, \text{ and } \sigma_{r+1} s := (s_{1} + s_{r+1} - \frac{1}{2}, \ldots, s_{r} + s_{r+1} - \frac{1}{2}, 1 - s_{r+1})
\]

it follows that

\[
\prod_{p \mid c_1c_3} (1 - |p|^{2s_{r+1} - 2}) \cdot Z^{(c)}(s; \chi_{d_2c_2}, \chi_{d_1c_1})
\]

\[
= \frac{1}{2} \gamma_{\ell}^{c}(s_{r+1}; a_{2}) \frac{\phi(c_1c_3)}{|c_1c_3|} \sum_{m \mid c_1c_3} \chi_{d_2c_2m}(m)U_{m}(s_{r+1}) \left\{Z^{(c)}(\sigma_{r+1}s; \chi_{d_2c_2}, \chi_{d_1c_1/m}) + Z^{(c)}(\sigma_{r+1}s; \chi_{d_2c_2}, \chi_{d_1c_1/m}/e)\right\}
\]

\[
+ \frac{1}{2} \gamma_{\ell}^{c}(s_{r+1}) \frac{\phi(c_1c_3)}{|c_1c_3|} \sum_{m \mid c_1c_3} \chi_{d_2c_2m}(m)U_{m}(s_{r+1}) \left\{Z^{(c)}(\sigma_{r+1}s; \chi_{d_2c_2}, \chi_{d_1c_1/m/e}) - Z^{(c)}(\sigma_{r+1}s; \chi_{d_2c_2}, \chi_{d_1c_1/m/e})\right\}.
\]

In this formula, the two sums are over all divisors \(m\) of \(c_1c_3\), and \(e\) denotes the gcd of \(m\) and \(c_1\).

When \(deg c_2\) is odd, the functional equation is obtained by just switching the factors \(\gamma_{\ell}^{c}(s_{r+1}; a_{2})\) and \(\gamma_{\ell}^{c}(s_{r+1})\).

We can combine the two cases, and write this functional equation as

\[
Z^{(c)}(s; \chi_{d_2c_2}, \chi_{d_1c_1}) = \frac{1}{2} |c_1|^{1/2-s_{r+1}} \frac{\phi(c_1c_3)}{|c_1c_3|} \prod_{p \mid c_1c_3} (1 - |p|^{2s_{r+1} - 2})^{-1}
\]

\[
\cdot \sum_{g \in (1, b_0)} \chi_{d_1g}(c_1) \left\{\gamma_{\ell}^{c}(s_{r+1}; a_{2}) + sgn(a_{2} \theta) \gamma_{\ell}^{c}(s_{r+1})\right\} \sum_{m \mid c_1c_3} \chi_{d_2c_2m}(m)U_{m}(s_{r+1})Z^{(c)}(\sigma_{r+1}s; \chi_{d_2c_2}, \chi_{d_1c_1/m/e}).
\]

Similarly, we can use the expression (7) of \(Z^{(c)}(s; \chi_{d_2c_2}, \chi_{d_1c_1})\), and the functional equations (10) and

\[
L^{(c_2c_3)}(s_1, \chi_{d_2c_2}, \chi_{d_1c_1}) = \gamma_{\ell}(s_1, a_1c_1) \mid c_1 \mid^{s_{r+1}} L^{(c_2c_3)}(1 - s_1, \chi_{d_1c_1}) \frac{L_{c_2c_3}(1 - s_{r+1} - s_1; \chi_{d_1c_1})}{L_{c_2c_3}(s_{r+1}; \chi_{d_1c_1})}
\]

to get:

\[
Z^{(c)}(s; \chi_{d_2c_2}, \chi_{d_1c_1}) = \frac{1}{2} |c_1|^{1/2-s_{r+1}} \frac{\phi(c_1c_3)}{|c_1c_3|} \prod_{p \mid c_1c_3} (1 - |p|^{2s_{r+1} - 2})^{-1}
\]

\[
\cdot \sum_{g \in (1, b_0)} \chi_{d_2g}(c_1) \left\{\gamma_{\ell}^{c}(s_1; a_{1}) + sgn(a_{2} \theta) \gamma_{\ell}^{c}(s_1)\right\} \sum_{\ell \mid c_1c_3} \chi_{d_1\ell}(\ell)U_{\ell}(s_1)Z^{(c)}(\sigma_{r+1}s; \chi_{d_2c_2}, \chi_{d_1\ell}).
\]

Of course, by symmetry, we have similar functional equations in the variables \(s_2, \ldots, s_{r}\). Writing explicitly now \(r = 3\) and taking \(s_1 = s_2 = s_3 = s\), we can express the functional equation \(\sigma_{2}, \sigma_{3}, \sigma_{3}\) as

\[
Z^{(c)}(s; \chi_{d_2c_2}, \chi_{d_1c_1}) = \frac{1}{2} |c_1|^{1/2-s_{r+1}} \frac{\phi(c_1c_3)}{|c_1c_3|} \prod_{p \mid c_1c_3} (1 - |p|^{2s_{r+1} - 2})^{-3}
\]

\[
\cdot \sum_{g \in (1, b_0)} \chi_{d_2g}(c_1) \left\{\gamma_{\ell}^{c}(s; a_{1}) + sgn(a_{2} \theta) \gamma_{\ell}^{c}(s)\right\} \sum_{\ell \mid c_1c_3} \chi_{d_1\ell}(\ell)W_{\ell}(s)Z^{(c)}(\sigma_{r+1}s; \chi_{d_2c_2}, \chi_{d_1\ell}/\ell^{2}).
\]
where $V_i(s) = W_i(s) = 1$ for $\ell \in \mathbb{F}^\times$, and
\[
V_i(s) = \prod_{p | \ell} \left( 1 + 3 U_p^2(s) \right) \quad \text{and} \quad W_i(s) = \prod_{p | \ell} \frac{U_p(s) \left( 3 + U_p^2(s) \right)}{1 + 3 U_p^2(s)}
\]
for $\ell$ square-free of positive degree. This functional equation will be used in the next section.

As in [15] and [18], by applying the above functional equations and Bochner’s theorem [3], it follows that $Z^{(c)}(s; \chi_{a_2c_2}, \chi_{a_1c_1})$ can be meromorphically continued to $\mathbb{C}^{r+1}$ ($r = 3$). Moreover, as in [18, Proposition 4.11], the function
\[
(1 - q^{3-4w})(1 - q^{2-2w})^7 Z^{(c)}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; w; \chi_{a_2c_2}, \chi_{a_1c_1})
\]
is entire and has order one. This function is, in fact, a polynomial in $q^w$, but we shall not need this piece of information.

### 3.2 Convexity bound

We shall now obtain a convexity bound for the function (15) analogous to that proved in [18, Proposition 4.12] over the rationals.

To obtain this estimate, we first note that by Proposition B.1 and (6) we have
\[
|P_i(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; \chi_{a_1c_1}d_0)| \leq \left( \frac{843}{1 - 5^{2n}} \right)^{\text{mon}(d_1)} |d_1|^{\frac{1}{2} + \eta}
\]
for every small positive $\eta$. Here $\text{mon}(d_1)$ denotes the number of distinct monic irreducible factors of $d_1$. Choosing, for example, $\eta = 1/5$, we find easily that
\[
|Z^{(c)}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, w; \chi_{a_2c_2}, \chi_{a_1c_1})| \leq \left(\frac{1}{2}\right)^{\text{mon}(c_2c_3)} \sum_{(d_0, c_1) = 1} L(\frac{1}{2}, \chi_{a_1c_1}d_0)^3 \sum_{d_1 - \text{monic and square free}} \frac{(1776)^{\text{mon}(d_1)}}{|d_1|^{3/10}}
\]
\[
< \left(\frac{1}{2}\right)^{\text{mon}(c_2c_3)} \zeta(\frac{11}{10})^{1776} \sum_{(d_0, c_1) = 1} L(\frac{1}{2}, \chi_{a_1c_1}d_0)^3 \sum_{d_1 - \text{monic and square free}} \frac{1}{|d_1|^{3/10}} \ll \delta_q |c_1|^{\delta} \frac{1}{1 - q^{1 + \delta - w}}
\]
for all $w \in \mathbb{C}$ with $\Re(w) > 1$. By Theorem A.1, the last series is convergent. Moreover, for $w > 1$ and small $0 < \delta < w - 1$, we have
\[
\sum_{(d_0, c_1) = 1} \sum_{d_1 - \text{monic and square free}} \frac{L(\frac{1}{2}, \chi_{a_1c_1}d_0)^3}{|d_1|^{w - \delta}} \ll \delta_q |c_1|^{\delta} \sum_{d_1 - \text{monic}} \frac{1}{|d_1|^{w - \delta}} \ll \delta_q |c_1|^{\delta} \frac{1}{1 - q^{1 + \delta - \Re(w)}}
\]
The implied constant can be taken to be $64 q^{30/\delta}$. It follows that
\[
|Z^{(c)}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, w; \chi_{a_2c_2}, \chi_{a_1c_1})| \leq 64 \zeta(\frac{11}{10})^{1776} \left(\frac{1}{2}\right)^{\text{mon}(c_2c_3)} \frac{q^{30/\delta} |c_1|^{\delta}}{1 - q^{1 + \delta - \Re(w)}}.
\]
We shall now establish a similar estimate when $w \in \mathbb{C}$ with $\Re(w) = -\delta$, for small positive $\delta$. To do so, we shall apply the functional equation corresponding to the Weyl group element $\tau := \sigma_3 \sigma_7 \sigma_7 \sigma_7 \sigma_7 \sigma_7 \sigma_3 \sigma_7 \sigma_7$, relating the values of $Z^{(c)}(s; \chi_{a_2c_2}, \chi_{a_1c_1})$ to the values of a linear combination of similar multiple Dirichlet series at $\tau s$, and then make use of (16). Note that
\[
\tau(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, w) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1 - w).
\]
Following [13] and [18], we write the functional equations (12), (13) in the matrix notation. If we denote by \( Z(c)(s) \) the column vector whose entries are the multiple Dirichlet series \( Z(c)(s; \chi, \lambda_1 \lambda_2) \), then there are matrices \( X_c(s_4) \) and \( Y_c(s) \) such that

\[
\vec{Z}(c)(s) = X_c(s_4) \cdot \vec{Z}(c)(\sigma_4 s) \quad \text{and} \quad \vec{Z}(c)(s) = Y_c(s_4) \cdot \vec{Z}(c)(\sigma_4 s) \quad \text{for } i = 1, 2, 3.
\]

Taking \( s = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, w) \) and applying successively the functional equations corresponding to \( \sigma_1, \sigma_2, \ldots, \sigma_4 \), we obtain:

\[
\vec{Z}(c)(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, w) = M(w) \cdot \vec{Z}(c)(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1 - w)
\]

where the matrix \( M(w) \) is given by

\[
M(w) = X_c(w) Y_c(w)^{-1} X_c(2w - \frac{1}{2}) Y_c(w)^{-1} X_c(w).
\]

We shall now estimate the entries of the matrix \( M(w) \). Let \( s_4 \in \mathbb{C} \) with \( \Re(s_4) < 0 \). Since

\[
|\gamma_c(s_4)| + |\gamma_c(s_4)| < 4 \quad |U_n(s_4)| \leq 3^{o(m)} |m|^{-\Re(s_4)}
\]

and

\[
\prod_{\ell \mid c_1 c_2} \left( 1 - |p|^{2\Re(s_4) - 2} \right)^{-1} \leq \left( \frac{2\delta}{\delta^2} \right)^{o(m)}
\]

we have by (12) that

\[
|Z(c)(s; \chi, \lambda_1 \lambda_2)| \leq 4^{o(m)} (c_1 c_2 + 1) \sum_{m \mid c_1 c_2} |m|^{-\Re(s_4)} \cdot \frac{1}{2} \sum_{\ell \mid c_1 c_2} \left| \frac{1}{2} \sum_{\ell \mid c_1 c_2} \left| Z(c)(\sigma_4 s; \chi, \lambda_1 \lambda_2) \right| \right|
\]

(17)

Note that for every divisor \( m \) of \( c_1 c_2 \), the monic polynomial \( c_1 m / l^2 \) is also a divisor of \( c_1 c_2 \). Conversely, to every pair \((l_1, l_2')\) with \( l_1 \mid c_1 \) and \( l_2' \mid c_2 \) there corresponds \( m := (c_1 l_1) l_2' \).

Similarly, for \( s \in \mathbb{C} \) with \( \Re(s) < 0 \), we have

\[
|Z(c)(s; \chi, \lambda_1 \lambda_2)| \leq 4^{o(m)} (c_1 c_2 + 1) \sum_{m \mid c_1 c_2} |m|^{-\Re(s_4)} \cdot \frac{1}{2} \sum_{\ell \mid c_1 c_2} \left| \frac{1}{2} \sum_{\ell \mid c_1 c_2} \left| Z(c)(\sigma_4 s; \chi, \lambda_1 \lambda_2) \right| \right|
\]

(18)

To estimate the entries of the matrix \( M(w) \) when \( \Re(w) = -\delta \), we need to estimate an expression \( E \) of the form

\[
E = |c_1|^\delta |c_2|^{\frac{3}{2}} |b_2|^{\frac{1}{2}} \left( |b_1| |m|^{\frac{1}{2}} \right) |b_3|^\delta |b_4|^\delta
\]

where \( m, b_2, \ldots, b_5 \in \mathbb{F}[x] \) are (monic) divisors of \( c \) such that \( (c_2, b_2) = (b_1, m) = (b_2, b_{i+1}) = 1 \), and \( b_4 = b_2 m \) modulo squares. Let \( p \) be a monic irreducible divisor of \( c \). If \( p \mid c_1 \) then \( p \mid b_2 \), and the power of \( p \) dividing \( b_2 m b_4 \) cannot exceed 4; it is 4 if and only if \( p \mid m \), and hence \( p \mid b_4 \). If \( p \mid c_2 \) then the power of \( p \) dividing \( b_2 m b_4 \) cannot exceed 6; it is 6 if and only if \( p \mid b_2 \) and \( p \mid b_4 \) (hence \( p \mid m + n \)). Thus

\[
E \leq |c_1|^\delta |c_2|^{\delta + 106} |c_2|^{3 + 108}.
\]

Since the dimension of the matrices \( X_c \) and \( Y_c \) is \( 4 \cdot 3^{o(c)} \), it follows from (16) that, for \( w \in \mathbb{C} \) with \( \Re(w) = -\delta \), we have

\[
|Z(c)(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, w; \chi, \lambda_1 \lambda_2)| \ll \delta \quad 18^{o(c)} |c_1|^\delta |c_2|^{\delta + 106}.
\]

Thus by applying the Phragmen-Lindelöf principle, for every \( \delta > 0 \), we have the estimate

\[
\left| (1 - q^{3 - 4w})(1 - q^{2 - 2w}) Z(c)(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, w; \chi, \lambda_1 \lambda_2) \right| \ll \delta \quad 20^{o(c)} |c_1|^\delta |c_2|^{\delta + 106} |c_2|^{3 + 108}.
\]

for all \( w \) with \( 0 \leq \Re(w) \leq 1 \).

As noted in the introduction, one of the main ingredients in the proof of Theorem 1.1 is an improvement of (19) in the \( c_3 \)-aspect. This will be established in Proposition 6.3.
4 Poles of multiple Dirichlet series and their residues

Throughout this section we are assuming that \( r = 3 \). By \( (9) \), the multiple Dirichlet series \( Z^{(c)}(s; \chi_{02c}, \chi_{01c}) \) has a (simple) pole at \( s \rho = 4 \) only if \( a_2 = c_2 = 1 \), and the part of \( Z^{(c)}(s; \chi_{01c}) \) that contributes to this pole is

\[
\zeta^{(c)}(s) \sum_{(m_1, \ldots, m_r, c_1 c_2 c_3) = 1} \frac{Q_m(s_1+1)}{|m_1|^{a_1} \cdots |m_r|^{a_r}} = \zeta^{(c)}(s_1+1) \prod_{p \equiv 1 \pmod{2}} \sum_{s \equiv 0 \pmod{2}} \frac{Q_p(p^{s}+1; |p|)}{|p|^{a_1 s_1 + \cdots + a_r s_r}}. 
\]

From the definition of the polynomials \( Q_m(t_1 + q) \) (see Appendix B) it is straightforward to check that

\[
\frac{Z^{(c)}}{\zeta(s)} \sum_{s \equiv 1} = \zeta^{(c)}(2s_1 + 2s_2 + 2s_3 - 1) \prod_{i=1}^{3} \zeta^{(c)}(2s_i) \prod_{1 \leq i \leq j \leq 3} \zeta^{(c)}(s_i + s_j). \tag{20}
\]

This “modified” residue of \( Z^{(c)}(s; \chi_{01c}) \) (to which we shall refer as residue) is more convenient to work with in our context. We also have that

\[
\frac{Z^{(c)}}{L(s, \chi_{0c})} \bigg|_{q^{-s}=q^{-1}} = \frac{Z^{(c)}}{\zeta(s)} \bigg|_{s_1 = 1} \tag{21}
\]

For our purposes, it will suffice to compute the residues at the remaining poles of \( Z^{(c)}(s; \chi_{02c}, \chi_{01c}) \) only when \( s_1 = s_2 = s_3 = \frac{1}{2} \) and \( a_1 = 1 \).\(^6\) Fix \( \vartheta' \in \{1, \theta_0\} \), and let \( \rho(\vartheta') \) be such that \( \rho(\vartheta') \in \{ \pm \vartheta \} \) if \( \vartheta' = \theta_0 \). We define

\[
\Gamma(a_2, \vartheta'; \rho(\vartheta')) = \sum_{\vartheta \in \{1, \theta_0\}} \left\{ \gamma^{(c)}(s_1; a_2) + \text{sgn}(\vartheta) \gamma^{(c)}(s_2) \right\} \left\{ \gamma^{(c)}(s_3; \vartheta) + \text{sgn}(a_2 \vartheta') \gamma^{(c)}(s_4) \right\} \bigg|_{q^{-s}=\rho(\vartheta')q^{-\frac{1}{2}}}. \tag{22}
\]

For the reader’s convenience, we give the explicit values of \( \Gamma(a_2, \vartheta'; \rho(\vartheta')) \) in the following table.

| \( a_2 \) | \( \rho(\vartheta') \) | \( \Gamma(a_2, \vartheta'; \rho(\vartheta')) \) |
|---|---|---|
| 1 | 1 | 2 \( (1 + q^{1/6} + 10q^{1/2} + 7q^{3/6} + 20q^{1/6} + 7q^{3/6} + 10q^{1/2} + q^{3/6} + q^{1/2}) \) |
| \( \theta_0 \) | -1 | 2 \( (1 + q^{1/6} + 10q^{1/2} + 7q^{3/6} + 20q^{1/6} + 7q^{3/6} + 10q^{1/2} + q^{3/6} + q^{1/2}) \) |
| 1 | -1 | 2 \( (1 - q^{1/6} + 10q^{1/2} - 7q^{3/6} + 20q - 7q^{3/6} + 10q^{1/2} - q^{3/6} + q^{1/2}) \) |
| 1 | i | 2 \( (1 - iq^{1/6} - 4q^{1/2} + 7iq^{3/6} + 6q - 7iq^{3/6} - 4q^{1/2} + iq^{3/6} + q^{1/2}) \) |
| \( \theta_0 \) | -i | 2 \( (1 - iq^{1/6} - 4q^{1/2} + 7iq^{3/6} + 6q - 7iq^{3/6} - 4q^{1/2} + iq^{3/6} + q^{1/2}) \) |
| 1 | -i | 2 \( (1 + iq^{1/6} - 4q^{1/2} - 7iq^{3/6} + 6q + 7iq^{3/6} - 4q^{1/2} - iq^{3/6} + q^{1/2}) \) |
| \( \theta_0 \) | i | 2 \( (1 + iq^{1/6} - 4q^{1/2} - 7iq^{3/6} + 6q + 7iq^{3/6} - 4q^{1/2} - iq^{3/6} + q^{1/2}) \) |

Note that there are four distinct values of \( \Gamma(a_2, \vartheta'; \rho(\vartheta')) \) in total, indicated with four different colors.

**Proposition 4.1.** — Let \( c \in \mathbb{F}[x] \) be monic and square-free, and let \( a_2, \vartheta' \in \{1, \theta_0\} \). Suppose that \( c = c_1 c_2 c_3 \) with \( c_i \in \mathbb{F}[x] \)

\(^6\) As in \( [18] \), the only possible poles of the function \( Z^{(c)}(s; \chi_{02c}, \chi_{01c}) \) (with \( s_1 = s_2 = s_3 = 1/2 \)) may occur when \( q^{-s} = \pm 1/q \) of order at most seven, and \( q^{-s} = \pm i q^{-3/4}, \pm i q^{-5/4} \) of order at most one.
monic for all \(i\). Then, for \(p(\vartheta')\) as above, we have

\[
\left(1 - p(\vartheta') q^{3 - s_4}\right) Z^{(c)} \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, s_4; \chi_{\Delta_0 \mathbb{C}_2}, \chi_{\mathbb{C}_1}\right)_{q^{-s_4} = \rho(\vartheta') q^{-\frac{1}{2}}}
= \frac{\chi_{\Delta_0 \mathbb{C}_2}(c_1)}{8} \Gamma(a_2, \vartheta'; p(\vartheta')) L \left(\frac{1}{2}, \chi_{\mathbb{C}_0}\right)^7
\cdot p(\vartheta')^{1/2} \prod_{p | c_1} \left(1 - \chi_{\mathbb{C}_0}(p) |p|^{-1/2}\right)^2 \left(1 + \chi_{\mathbb{C}_0}(p) |p|^{-1/2}\right) \left(1 + 6 \chi_{\mathbb{C}_0}(p) |p|^{-1/2} + |p|^{-1}\right)
\cdot \prod_{p | c_2} \left(1 - \chi_{\mathbb{C}_0}(p) |p|^{-1/2}\right)^2 \left(1 + \chi_{\mathbb{C}_0}(p) |p|^{-1/2}\right) \left(3 + 7 \chi_{\mathbb{C}_0}(p) |p|^{-1/2} + 3 |p|^{-1}\right)
\cdot \prod_{p | c_3} \left(1 - \chi_{\mathbb{C}_0}(p) |p|^{-1/2}\right)^2 \left(1 + \chi_{\mathbb{C}_0}(p) |p|^{-1/2}\right) \left(1 + 7 \chi_{\mathbb{C}_0}(p) |p|^{-1/2} + 13 |p|^{-1} + 7 \chi_{\mathbb{C}_0}(p) |p|^{-3/2} + |p|^{-2}\right).
\]

**Proof.** We first apply the functional equation (12) (recall that \(r = 3\), and write

\[
Z^{(c)}(s; \chi_{\Delta_0 \mathbb{C}_2}, \chi_{\mathbb{C}_1}) = \frac{1}{2} |c_2|^{1 - s_4} \frac{\varphi(c_1 c_2)}{|c_1 c_2|} \prod_{p | c_1} \left(1 - |p|^{2s_4 - 2}\right)^{-1}
\cdot \sum_{\vartheta \in \{1, \vartheta_0\}} \chi_{\Delta_0}(c_2) \left\{ \gamma'_c \left(s_4; a_2, \vartheta; s_1\right) + \operatorname{sgn}(\vartheta) \gamma'_c \left(s_4; a_2, \vartheta; s_1\right) \right\}
\prod_{\ell | c_2 \ell} \left(1 - |\ell|^{2s_4 - 2}\right)^{-1}
\cdot \sum_{\ell | c_2 \ell} \chi_{\mathbb{C}_1}(\ell) W(s + s_4 + \ell) Z^{(c)}(s; \chi_{\Delta_0 \mathbb{C}_2}, \chi_{\mathbb{C}_1})
\]

Letting \(s_1 = s_2 = s_3 = s\), we have by (14) that

\[
Z^{(c)}(s; \chi_{\Delta_0 \mathbb{C}_2}, \chi_{\mathbb{C}_1}) = \frac{1}{2} \left|c_1 m\right|^{1 - s_4} \frac{\varphi(c_1 c_2)}{|c_1 c_2|} \prod_{p | c_1 m} \left(1 - |p|^{2s_4 - 2}\right)^{-1}
\cdot \sum_{\vartheta \in \{1, \vartheta_0\}} \chi_{\Delta_0}(c_2) \chi_{\Delta_0}(c_1) \left\{ \gamma'_c \left(s_4; a_2, \vartheta; s_1\right) + \operatorname{sgn}(\vartheta) \gamma'_c \left(s_4; a_2, \vartheta; s_1\right) \right\}
\prod_{\ell | c_2 \ell} \left(1 - |\ell|^{2s_4 - 2}\right)^{-1}
\cdot \sum_{\ell | c_2 \ell} \chi_{\mathbb{C}_1}(\ell) W(s + s_4 + \ell) Z^{(c)}(s; \chi_{\Delta_0 \mathbb{C}_2}, \chi_{\mathbb{C}_1})}
\]

and thus we can write

\[
Z^{(c)}(s; \chi_{\Delta_0 \mathbb{C}_2}, \chi_{\mathbb{C}_1}) = \frac{\chi_{\Delta_0}(c_1)}{4} |c_2|^{1 - s_4} \frac{\varphi(c_1 c_2)}{|c_1 c_2|} \prod_{p | c_1} \left(1 - |p|^{2s_4 - 2}\right)^{-1}
\cdot \sum_{\vartheta \in \{1, \vartheta_0\}} \chi_{\Delta_0}(c_2) \chi_{\Delta_0}(c_1) \left\{ \gamma'_c \left(s_4; a_2, \vartheta; s_1\right) + \operatorname{sgn}(\vartheta) \gamma'_c \left(s_4; a_2, \vartheta; s_1\right) \right\}
\prod_{\ell | c_2 \ell} \left(1 - |\ell|^{2s_4 - 2}\right)^{-1}
\cdot \sum_{\ell | c_2 \ell} \chi_{\mathbb{C}_1}(\ell) W(s + s_4 + \ell) Z^{(c)}(s; \chi_{\Delta_0 \mathbb{C}_2}, \chi_{\mathbb{C}_1})}
\]
The multiple Dirichlet series \( Z^{(c)}(\theta; m, c; \chi_{a_{2}c_{2}}) \) that contribute to the residue we are interested in occur whenever \( c_{2} \ell = b^{2} \), i.e., \( \ell = b = c_{2} \). Thus, for \( s \in \mathbb{C} \) and \( \theta' \in \{1, \theta_0\} \), we have

\[
(1 - \rho(\theta'))^{q^{3-3s-2}}\theta^{c_{2}}Z^{(c)}(s; \chi_{a_{2}c_{2}}, \chi_{c_{1}}) |_{q^{-s} = \rho(\theta')q^{3/2}}
\]

\[
= \frac{\chi_{a_{2}c_{2}}(c_{1})}{8c_{2}^{3}s} \prod_{\ell | c_{2}} (\ell - s - \frac{1}{2}) \frac{\zeta(c_{2})}{\zeta(c_{1})} \prod_{\ell | c_{1}} (1 - \ell)^{s-1} \cdot \sum_{\theta \in \{1, \theta_0\}} \{ \gamma_{s}(s + s + 4) \}
\]

\[
\cdot \sum_{m | c_{1}c_{2}} \frac{\chi_{c_{1}}(m)}{\chi_{a_{2}c_{2}}(c_{1})} U_{c_{1}}(s) |_{c_{1} | c_{3}} \frac{c_{1}^3}{c_{2}^{3}(s + s + 4) \prod_{\ell | c_{2}} (1 - \ell)^{s-1}}
\]

where \( \rho(\theta') \in \{1\} \) if \( \theta' = 1 \) or \( \rho(\theta') \in \{1 \} \) if \( \theta' = \theta_0 \). Here \( s \) is such that \( q^{-s} = \rho(\theta')q^{3/2} \). Letting \( S \) temporarily denote the inner sum over \( m \), we can write

\[
S = \sum_{\ell | c_{1}} \sum_{\ell | c_{3}} \frac{\chi_{c_{1}}(m)}{\chi_{a_{2}c_{2}}(c_{1})} \frac{c_{1}^3}{c_{2}^{3}(s + s + 4) \prod_{\ell | c_{2}} (1 - \ell)^{s-1}}
\]

\[
= |c_{2}|^{3(s + s + 2)} \frac{\zeta(c_{2})^{3}}{\zeta(c_{3})} \frac{c_{2}^{3}}{c_{1}^{3}(s + s + 4) \prod_{\ell | c_{2}} (1 - \ell)^{s-1}}
\]

\[
\cdot \prod_{\ell | c_{1}} \left\{ \sum_{\ell | c_{2}} (1 + \chi_{c_{1}}(m) |_{c_{1} | c_{3}} U_{c_{1}}(s)) |_{c_{1} | c_{3}} \frac{c_{1}^3}{c_{2}^{3}(s + s + 4) \prod_{\ell | c_{2}} (1 - \ell)^{s-1}} \right\}
\]

the products being over monic irreducibles. Taking \( s = \frac{1}{2} \) and \( q^{-s} = \rho(\theta')q^{3/2} \), it follows easily from the definitions of \( U_{c}(s), V_{c}(s) \) and \( W_{c}(s) \) that

\[
(1 - \rho(\theta'))^{q^{3-3s}}Z^{(c)}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, s; \chi_{a_{2}c_{2}}, \chi_{c_{1}}) |_{q^{-s} = \rho(\theta')q^{3/2}}
\]

\[
= \frac{\chi_{a_{2}c_{2}}(c_{1})}{8} \Gamma(a_{2}; \theta); \rho(\theta') \cdot L(1, c_{1})
\]

\[
\cdot \prod_{\ell | c_{1}} (1 - \chi_{c_{1}}(m) |_{c_{1} | c_{3}} U_{c_{1}}(s)) |_{c_{1} | c_{3}} \frac{c_{1}^3}{c_{2}^{3}(s + s + 4) \prod_{\ell | c_{2}} (1 - \ell)^{s-1}}
\]

as asserted.

In particular, if \( c = a_{2} = 1 \), we have

\[
(1 - \rho(\theta'))^{q^{3-2s}}Z^{(c)}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, s; 1, 1) |_{q^{-s} = \rho(\theta')q^{3/2}} = \frac{1}{8} \Gamma(1, \theta'; \rho(\theta')) L(1, c_{1})
\]

equality which can also be verified directly from the explicit expression of \( Z(s; 1, 1) \) given in the second appendix.
5 Sieving

For $h \in \mathbb{F}[x]$ square-free monic and $a_2 \in \{1, \theta_b\}$, put

$$Z(s; \chi_{a_2}; h) = \sum_{m_1, \ldots, m_r, d_{\text{monic}}, \text{sq. free}} \frac{\chi_{a_0}(\widehat{m}_1) \cdots \chi_{a_0}(\widehat{m}_r) \chi_{a_2}(d_0)}{|m_1|^{s_1} \cdots |m_r|^{s_r} |d|^{s_{r+1}}} \cdot A(m_1, \ldots, m_r, d_0).$$

The series in the right-hand side is absolutely convergent if $s_1, \ldots, s_{r+1}$ are complex numbers with sufficiently large real parts. Let $\mu(h)$ denote the Möbius function defined for non-zero polynomials over $\mathbb{F}$ by $\mu(h) = (-1)^{\omega(h)}$ if $h$ is square-free, and $h$ is a constant times a product of $\omega(h)$ distinct monic irreducibles, and $\mu(h) = 0$ if $h$ is not square-free; it is understood that $\mu(h) = 1$ if $h \in \mathbb{F}^\times$. We have the usual property of Möbius functions:

$$\sum_{h \mid d \text{ h-monic}} \mu(h) = \begin{cases} 1 & \text{if } \deg d = 0 \\ 0 & \text{if } \deg d \geq 1. \end{cases}$$

We have the following simple lemma:

**Lemma 5.1.** — For $a_2 \in \{1, \theta_b\}$ and $s = (s_1, \ldots, s_{r+1}) \in \mathbb{C}^{r+1}$ with $\Re(s_i)$ sufficiently large, define

$$Z_0(s; \chi_{a_2}) = \sum_{d_{\text{monic}} \& \text{sq. free}} L(s_1; \chi_{a_0}) \cdots L(s_r; \chi_{a_0}) \chi_{a_2}(d_0) |d_0|^{-s_{r+1}}.$$

Then we have the equality

$$Z_0(s; \chi_{a_2}) = \sum_{h \text{-monic}} \mu(h) Z(s; \chi_{a_2}; h). \quad (23)$$

**Proof:** The right-hand side of the equality is

$$\sum_{m_1, \ldots, m_r, d_{\text{monic}}, \text{sq. free}} \frac{\chi_{a_0}(\widehat{m}_1) \cdots \chi_{a_0}(\widehat{m}_r) \chi_{a_2}(d_0)}{|m_1|^{s_1} \cdots |m_r|^{s_r} |d_0|^{s_{r+1}}} \cdot A(m_1, \ldots, m_r, d_0)$$

where, as before, $\widehat{m}_i$ is the part of $m_i$ coprime to $d_0$. Recall that the coefficients $A(m_1, \ldots, m_r, d_0)$ are multiplicative and that, for every monic irreducible $p$, $A(p^{k_1}, \ldots, p^{k_r}, p) = 0$, unless $k_1 = \cdots = k_r = 0$ in which case $A(1, \ldots, 1, p) = 1$. It follows that the above sum equals

$$\sum_{m_1, \ldots, m_r, d_{\text{monic}}, \text{sq. free}} \frac{\chi_{a_0}(m_1) \cdots \chi_{a_0}(m_r) \chi_{a_2}(d_0)}{|m_1|^{s_1} \cdots |m_r|^{s_r} |d_0|^{s_{r+1}}} \cdot A(m_1, \ldots, m_r, 1)$$

and our assertion follows from the fact that $A(m_1, \ldots, m_r, 1) = 1$. \hfill \Box

We can express the function $Z(s; \chi_{a_2}; h)$ in terms of the multiple Dirichlet series $Z^{(c)}(s; \chi_{a_2'}, \chi_{a_1(c)})$, discussed in the previous sections, as follows. Let $c$ be a monic divisor of $h$, and write $h = cc'$. Decompose

$$c = p_{i}^{r_i} \text{ and } c' = p'_{i+1}^{r_{i+1}} \cdots p'_{i+1}^{r_i}$$

into monic irreducibles. Consider

$$\frac{\chi_{cd_0}(\widehat{m}_1) \cdots \chi_{cd_0}(\widehat{m}_r) \chi_{a_2}(cd_0)}{|m_1|^{s_1} \cdots |m_r|^{s_r} |cd_0|^{s_{r+1}}} \cdot A(m_1, \ldots, m_r, cd_0, d_0^2) \quad (\text{with } d_0 \text{ square-free coprime to } h) \quad (24)$$

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representing a term of $Z(s, \chi_{\alpha_2}, h)$. From this expression, we can factor out a piece corresponding to $h$ (i.e., to $p_1, \ldots, p'_l$). Let $p_i^{c_i}, (p'_i)^{d_i} \parallel m_i$ (for $1 \leq i \leq l$, $l+1 \leq j \leq n$ and $1 \leq k \leq r$), and $p_i^{b_i}, (p'_i)^{b'_i} \parallel c d_i^2$ with $b_i \geq 3$ odd and $b'_i \geq 2$ even. Since $p_j \mid c d_h$ (hence $p_j \not\mid \tilde{m}_k$), we can factor out from (24) the product
\[
\prod_{i=1}^l A(p_{i_1}^{a_{i_1}}, \ldots, p_{i_r}^{a_{i_r}}, p_i^{b_i}) \prod_{j=l+1}^n \frac{\chi_c(p'_j)^{b'_j} A((p'_j)^{b'_j}, (p'_j)^{b'_j})(p'_j)^{b'_j}}{|p'_j|^{a'_j + \cdots + a'_j + b'_j}} \quad \text{(with } b'_j \geq 2 \text{ even}).
\]
To isolate the remaining irreducibles, we note that $(p'_i)^{d_i} \parallel \tilde{m}_k$ for all $j$ and $k$. Thus we can also factor out from (24) the product
\[
\prod_{j=l+1}^n \frac{\chi_c(p'_j)^{a'_j + \cdots + a'_j} A((p'_j)^{a'_j}, \ldots, (p'_j)^{a'_j}, (p'_j)^{b'_j})}{|p'_j|^{a'_j + \cdots + a'_j + b'_j}} \quad \text{(with } b'_j \geq 2 \text{ even}).
\]
Consequently, we can write the expression (24) as
\[
\chi_c(\tilde{h}_1) \cdots \chi_c(\tilde{h}_r) \chi_{\alpha_2}(d_0) A(n_1, \ldots, n_r, d_0, d_1^2) \prod_{j=l+1}^n \frac{\chi_c(p'_j)^{a'_j + \cdots + a'_j} A((p'_j)^{a'_j}, \ldots, (p'_j)^{a'_j}, (p'_j)^{b'_j})}{|p'_j|^{a'_j + \cdots + a'_j + b'_j}} \prod_{i=1}^l A(p_{i_1}^{a_{i_1}}, \ldots, p_{i_r}^{a_{i_r}}, p_i^{b_i}) \prod_{j=l+1}^n \frac{\chi_c(p'_j)^{a'_j + \cdots + a'_j} A((p'_j)^{a'_j}, \ldots, (p'_j)^{a'_j}, (p'_j)^{b'_j})}{|p'_j|^{a'_j + \cdots + a'_j + b'_j}}.
\]
Here $n_1, \ldots, n_r, d_0, d_1$ are coprime to $h$. Let $\varepsilon = (\varepsilon_j)_{1 \leq j \leq s_a}$ with $\varepsilon_j \in \{0, 1\}$ be defined by
\[
\alpha'_j + \cdots + \alpha'_j \equiv \varepsilon_j \pmod{2}.
\]
If we put $c'_\varepsilon = \prod_{i=1}^l \varepsilon_i (p'_i)^{b'_i}$, we have (by the quadratic reciprocity law) that
\[
\prod_{j=l+1}^n \frac{\chi_c(p'_j)^{a'_j + \cdots + a'_j}}{|p'_j|^{a'_j + \cdots + a'_j}} = \chi_c^\varepsilon(d_0).
\]
Accordingly, if we let
\[
F(z_1, \ldots, z_r; q) := z_r^{s+1} f_{\text{odd}}(z_1, \ldots, z_r; q) - z_r^{-s+1}
\]
and, for $a \in \{0, 1\}$,
\[
G^a(z_1, \ldots, z_r; q) := \frac{1}{2} \left( f_{\text{even}}(z_1, \ldots, z_r, z_r; q) - \prod_{k=1}^{r} (1 - z_k)^{-1} \right) z_r^{-s+1}
\]
\[
+ \frac{(-1)^{r+2}}{2} \left( f_{\text{even}}(-z_1, \ldots, -z_r, z_r; q) - \prod_{k=1}^{r} (1 + z_k)^{-1} \right) z_r^{-s+1}
\]
with $f_{\text{odd}}$ and $f_{\text{even}}$ defined in Appendix B, we obtain the key equality:
\[
Z(s, \chi_{\alpha_2}, h) = |h|^{-2s+1} \sum_{h \equiv c \varepsilon \pmod{e}} \sum_{\varepsilon = (\varepsilon_j)_{p'_j} c'_\varepsilon} Z^h(s, \chi_{\alpha_2}, \chi_{c}) \prod_{p \mid c} F(|p|^{-2}, \ldots, |p|^{-2}; |p|) |p|^{-2s+1}
\]
\[
\times \chi_{c'_\varepsilon}(c) \prod_{p' \mid c'} G^a(p'_1) |p'|^{-2}, \ldots, |p'|^{-2}; |p'|). \quad \text{(26)}
\]
Notice that the right-hand side gives the analytic continuation of $Z(s, \chi_{\alpha_2}, h)$. Our main goal is to show that, for $s = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, w)$, the series obtained by substituting (26) into (23) converges absolutely and uniformly on every compact subset of the half-plane $\Re(w) > 2/3$, away from the points $w \in \mathbb{C}$ for which $q^{-w} = \pm q^{-1}$, or $q^{-w} = \pm q^{-3/4}, \pm i q^{-3/4}$. 

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6 Estimates

To prove Theorem 1.1, we will use (26) in conjunction with the estimates of the local parts of the untwisted multiple Dirichlet series $Z(s; 1, 1)$ provided by the following elementary lemmas.

**Lemma 6.1.** — For $|z| \leq q^{-\frac{1}{2}}$, we have the asymptotics

$$F(q^{\frac{1}{2}}, q^{-\frac{1}{2}}, q^{\frac{1}{2}}, z; q) = 14 + qz^2 + O(z^3)$$

and the estimate

$$G^{(a)}(q^{\frac{1}{2}}, q^{-\frac{1}{2}}, q^{\frac{1}{2}}, z; q) = 14 + qz^2 + O(q^{-1})$$

the implied constants in the $O$-symbols being independent on $z$ and $q$.

**Proof.** Using the formulas in the Appendix B and the definitions of $F$ and $G^{(a)} (a \in \{0, 1\})$, one finds that

$$G^{(a)}(q^{\frac{1}{2}}, q^{-\frac{1}{2}}, q^{\frac{1}{2}}, z; q) = \frac{1}{2}(1 - q^{-\frac{1}{2}})^{-3} \cdot \frac{1 + (7 - 14q^{\frac{1}{2}} + 6q^{-1} - q^{\frac{1}{2}})z^2 + 7(1 - 4q^{\frac{1}{2}} + 4q^{-1} - q^{\frac{1}{2}})z^4 + (1 - 6q^{\frac{1}{2}} + 14q^{-1} - 7q^{\frac{1}{2}})z^6 - q^{\frac{1}{2}}z^8}{(1 - z^2)^2(1 - qz^2)}$$

and

$$G^{(1)}(q^{\frac{1}{2}}, q^{-\frac{1}{2}}, q^{\frac{1}{2}}, z; q) = \frac{1}{2}(1 - q^{-\frac{1}{2}})^{-3} \cdot \frac{1 + (7 - 14q^{\frac{1}{2}} + 6q^{-1} - q^{\frac{1}{2}})z^2 + 7(1 - 4q^{\frac{1}{2}} + 4q^{-1} - q^{\frac{1}{2}})z^4 + (1 - 6q^{\frac{1}{2}} + 14q^{-1} - 7q^{\frac{1}{2}})z^6 - q^{\frac{1}{2}}z^8}{(1 - z^2)^2(1 - qz^2)}$$

From this explicit formulas we see easily that

$$|F(q^{\frac{1}{2}}, q^{-\frac{1}{2}}, q^{\frac{1}{2}}, z; q) - 14 - qz^2| \leq \frac{15q^7 + 119q^6 + 412q^5 + 812q^4 + 994q^3 + 770q^2 + 363q^{-1} + 99}{(1 - q^{-1})^8} |z|^2$$

$$|G^{(a)}(q^{\frac{1}{2}}, q^{-\frac{1}{2}}, q^{\frac{1}{2}}, z; q) - 14 - qz^2| \leq \frac{(1 + q^{-1})(15q^7 + 120q^6 + 420q^5 + 843q^4 + 1064q^3 + 866q^2 + 427q^{-1} + 153)}{q(1 - q^{-1})^{11}}$$

and

$$|G^{(1)}(q^{\frac{1}{2}}, q^{-\frac{1}{2}}, q^{\frac{1}{2}}, z; q)| \leq \frac{q^{-7} + 10q^{-6} + 36q^{-5} + 65q^{-4} + 121q^{-3} + 134q^{-2} + 70q^{-1} + 31}{(1 - q^{-1})^{10}} \frac{1}{\sqrt{q}}$$

from which the lemma follows.
The estimates in the above lemma show that there is additional decay in (26) in the conductors of the characters \( \chi_c \) and \( \chi_{\omega(\chi')} \). However, this is not sufficient, as we would need enough decay in \(|h|\).

To this end, define

\[
 f^\pm_{\text{even}}(z_1, \ldots, z_{r+1}; q) = (f_{\text{even}}(z_1, \ldots, z_{r+1}; q) \pm f_{\text{even}}(-z_1, \ldots, -z_r, z_{r+1}; q))/2 \quad \text{(with } r = 3) \]

where \( f_{\text{even}} \) is as defined in Appendix B. We will use the next lemma and an inductive argument to improve the convex bound (19), precisely in the \( c_r \)-aspect.

**Lemma 6.2.** — For every real \( q \geq 5 \) and \(|z| \leq q^{-\frac{1}{2}}\), we have the estimates

\[
 |f_{\text{odd}}(q^{-\frac{1}{2}}, q^{\frac{1}{2}}, q^{\frac{1}{2}}, z; q)| < 17|z|
\]

\[
 |f_{\text{even}}^-(q^{-\frac{1}{2}}, q^{\frac{1}{2}}, q^{\frac{1}{2}}, z; q)| < 58q^{-\frac{1}{2}}
\]

and if \( q \equiv 1 \pmod{4} \) is a prime power, we have the inequality

\[
 \frac{1}{|f_{\text{even}}^+(q^{-\frac{1}{2}}, q^{\frac{1}{2}}, q^{\frac{1}{2}}, z; q)|} < 20.
\]

**Proof.** We have the explicit expressions:

\[
 f_{\text{odd}}(q^{-\frac{1}{2}}, q^{\frac{1}{2}}, q^{\frac{1}{2}}, z; q) = \frac{z(1 + 7z^2 + 7z^4 + z^6)}{(1 - z)^3 (1 - qz^2)}
\]

\[
 f_{\text{even}}^-(q^{-\frac{1}{2}}, q^{\frac{1}{2}}, q^{\frac{1}{2}}, z; q) = \frac{3 + q^{-1} + (10 - 17q^{-2} + 3q^{-3})z^2 + (3 - 17q^{-1} + 10q^{-2})z^4 + (q^{-1} + 3q^{-2})z^6}{\sqrt{q}(1 - q^{-1})^3 (1 - z)^3 (1 - qz^2)}
\]

and

\[
 \frac{1}{|f_{\text{even}}^+(q^{-\frac{1}{2}}, q^{\frac{1}{2}}, q^{\frac{1}{2}}, z; q)|} = \frac{(1 - q^{-1})^3 (1 - z)^3 (1 - qz^2)}{1 + 3q^{-1} + (7 - 15q^{-1} + q^{-2} - q^{-3})z^2 + (7 - 35q^{-1} + 55q^{-2} - 7q^{-3})z^4 + (1 - q^{-1} + 15q^{-2} - 7q^{-3})z^6 - (3q^{-2} + q^{-3})z^8}.
\]

It follows that

\[
 |f_{\text{odd}}(q^{-\frac{1}{2}}, q^{\frac{1}{2}}, q^{\frac{1}{2}}, z; q)| \leq \frac{1 + 7|z|^2 + 7|z|^4 + |z|^6}{(1 - |z|^2)^3 (1 - q|z|^2)} |z| \leq \frac{1 + 7q^{-1} + 7q^{-2} + q^{-3}}{(1 - q^{-1})^3} |z|.
\]

The expression

\[
 \frac{1 + 7q^{-1} + 7q^{-2} + q^{-3}}{(1 - q^{-1})^3} \quad \text{(for } q \geq 5) \]

is increasing as a function of \( q^{-1} \), and its value when \( q = 5 \) is 16.0217... < 17. We have similarly

\[
 |f_{\text{even}}^-(q^{-\frac{1}{2}}, q^{\frac{1}{2}}, q^{\frac{1}{2}}, z; q)| \leq \frac{3 + 11q^{-1} + 20q^{-2} + 20q^{-3} + 11q^{-4} + 3q^{-5} - q^{-6}}{(1 - q^{-1})^3} \cdot q^{-\frac{1}{2}} < 58q^{-\frac{1}{2}}
\]

as we had asserted.

Now the numerator of \( 1/|f_{\text{even}}^+(q^{-\frac{1}{2}}, q^{\frac{1}{2}}, q^{\frac{1}{2}}, z; q)| \) is

\[
 (1 - q^{-1})^3 (1 - z^2)(1 - qz^2) < (1 + q^{-1})^8 \leq (6/5)^8.
\]
To obtain a lower bound for the denominator, we first assume that \( q \geq 9 \). In this case we have
\[
|1 + 3q^{-1} + (7 - 15q^{-1} + q^{-2} - q^{-3})\zeta^2 + (7 - 35q^{-1} + 35q^{-2} - 7q^{-3})\zeta^4 + (1 - q^{-1} + 15q^{-2} - 7q^{-3})\zeta^6 - (3q^{-2} + q^{-3})\zeta^8| \\
\geq 1 + 3q^{-1} + (7 - 15q^{-1} + q^{-2} - q^{-3})\zeta^2 + (7 - 35q^{-1} + 35q^{-2} - 7q^{-3})\zeta^4 + (1 - q^{-1} + 15q^{-2} - 7q^{-3})\zeta^6 - (3q^{-2} + q^{-3})\zeta^8 \\
\geq 1 - 4q^{-1} - 22q^{-2} - 37q^{-3} - 37q^{-4} - 22q^{-5} - 10q^{-6} - q^{-7} > 2/9.
\]

When \( q = 5 \) we have
\[
\left| \frac{8}{5} - \frac{8z^5}{125}(2z^6 - 21z^4 - 21z^2 - 63) \right| \geq \frac{8}{5} \frac{8|z|^2}{125}|2z^6 - 21z^4 - 21z^2 - 63| \geq \frac{8}{5} \frac{8}{625} \left( 63 + \frac{21}{5} + \frac{21}{25} + \frac{2}{125} \right) > \frac{2}{9}
\]
The last assertion follows from these inequalities.

For ease of notation, we let \( Z(c)(w; \chi_{a_2c_2}, \chi_{a_1c_1}) = Z(c)\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, w; \chi_{a_2c_2}, \chi_{a_1c_1}\right) \) with \( a_1, a_2 \in \{1, \theta_0\} \).

**Proposition 6.3.** — Let \( c_1, c_2 \) and \( c_3 \) be monic polynomials such that \( c = c_1c_2c_3 \) is square-free, and let \( \omega(c) \) denote the number of irreducible factors of \( c \). If we define
\[
\tilde{Z}(c)(w; \chi_{a_2c_2}, \chi_{a_1c_1}) = \left(1 - q^{-3/4}\right)\left(1 - q^{-2/4}\right)^2 Z(c)(w; \chi_{a_2c_2}, \chi_{a_1c_1}) \quad a_1, a_2 \in \{1, \theta_0\}
\]
then, for every \( \delta > 0 \), we have the estimate
\[
\tilde{Z}(c)(w; \chi_{a_2c_2}, \chi_{a_1c_1}) \ll_{\delta} A_0 \omega(c_2) A_1 \omega(c_3) |c_1|^{\delta} |c_2|^{\delta} |c_3|^{\delta} \max\left(3^{-4\delta}, 2, \frac{20\delta}{3} \right) \left( \frac{1}{\delta} \right)
\]
with \( A_0 = 20^9 \) and \( A_1 = 20 + 1500 \cdot 20^9 \), for all \( w \) with \( \frac{1}{2} \leq \Re(w) \leq \frac{3}{4} \).

**Proof.** We proceed by induction on \( \omega(c_1) \). If \( c_1 = 1 \), our estimate was established in (19); in other words, for every \( \delta > 0 \), \( c_1, c_2 \) monics such that \( c_1c_2 \) is square-free, and \( w \) with \( \frac{1}{2} \leq \Re(w) \leq \frac{3}{4} \), we have
\[
\tilde{Z}(c_1,c_2)(w; \chi_{a_2c_2}, \chi_{a_1c_1}) \leq B(\delta,q) 20^{\delta} |c_2|^{\delta} |c_3|^{\delta} \max(3^{-4\delta}, 2, \frac{20\delta}{3}) \left( \frac{1}{\delta} \right)
\]
for some positive constant \( B(\delta,q) \).

Let \( c_1, c_2 \) and \( c_3 \) be monic polynomials with \( c_1c_2c_3 \) square-free. For \( s = (s_1, \ldots, s_4) \in \mathbb{C}^4 \) with \( \Re(s) \) sufficiently large, consider the multiple Dirichlet series
\[
Z(c_1c_2c_3)(s; \chi_{a_2c_2}, \chi_{a_1c_1}) = \sum_{m_1, m_2, m_3, a \text{ d-monic}} \frac{\chi_{a_1c_1}(d_0) \chi_{a_2c_2}(\overline{d_0}) \chi_{a_1c_1}(\overline{d_0}) \chi_{a_2c_2}(d_0) \chi_{a_3c_3}(d_0)}{|m_1|^{|s_1|} |m_2|^{|s_2|} |m_3|^{|s_3|} |d|^4} \cdot A(m_1, m_2, m_3, a, d).
\]

If \( p \in \mathbb{P}[x] \) is a monic irreducible, \( p \nmid c_1c_2c_3 \), we can write (as before):
\[
Z(c_1c_2c_3)(s; \chi_{a_2c_2}, \chi_{a_1c_1}) = \chi_{a_2c_2}(p) Z(c_1c_2\cdot p)(s; \chi_{a_2c_2}, \chi_{a_1c_1}) f_{\text{odd}}(t^{deg p}, t_{4}^{deg p}, q^{deg p}) \\
+ \frac{\chi_{a_1c_1}(p)}{2} Z(c_1c_2\cdot p)(s; \chi_{a_2c_2}, \chi_{a_1c_1}) \left( f_{\text{even}}(t^{deg p}, t_{4}^{deg p}, q^{deg p}) - f_{\text{even}}(-t^{deg p}, t_{4}^{deg p}, q^{deg p}) \right) \\
+ \frac{1}{2} Z(c_1c_2\cdot p)(s; \chi_{a_2c_2}, \chi_{a_1c_1}) \left( f_{\text{even}}(t^{deg p}, t_{4}^{deg p}, q^{deg p}) + f_{\text{even}}(-t^{deg p}, t_{4}^{deg p}, q^{deg p}) \right)
\]
where we set \( t_{i} = q^{-s_{i}} \), and \( \pm t^{deg p} \) stands for \( (\pm t_{1}^{deg p}, \pm t_{2}^{deg p}, \pm t_{3}^{deg p}) \). Setting \( s_{i} = \frac{1}{2} \) for \( i = 1, 2, 3 \) and \( s_{4} = w \), we have by analytic continuation that
\[
\tilde{Z}(c_1c_2c_3)(w; \chi_{a_2c_2}, \chi_{a_1c_1}) = \chi_{a_2c_2}(p) \tilde{Z}(c_1c_2\cdot p)(w; \chi_{a_2c_2}, \chi_{a_1c_1}) f_{p}(w) + \chi_{a_1c_1}(p) \tilde{Z}(c_1c_2\cdot p)(w; \chi_{a_2c_2}, \chi_{a_1c_1}) f_{p}(w) \\
+ \tilde{Z}(c_1c_2\cdot p)(w; \chi_{a_2c_2}, \chi_{a_1c_1}) f_{p}^{+}(w).
\]
Here
\[ f_p(w) := f_{\alpha \beta}(|p|^{-\frac{1}{2}}, |p|^{\frac{1}{2}}, |p|^{\frac{1}{2}}, |p|^{\frac{1}{2}}, |p|^{\frac{1}{2}}; |p|) \quad \text{and} \quad f_p^\delta(w) := f_{\alpha \beta}^\delta(|p|^{-\frac{1}{2}}, |p|^{\frac{1}{2}}, |p|^{\frac{1}{2}}, |p|^{\frac{1}{2}}, |p|^{\frac{1}{2}}; |p|). \]

Applying the inequalities in Lemma 6.2 to \( f_p(w) \) and \( f_p^\delta(w) \), it follows that, for \( \Re(w) \geq \frac{1}{2} \),
\[ \left| \mathcal{Z}(c_1c_2c_3p, \mathcal{X}_{a_2c_2}, \mathcal{X}_{a_1c_1}) \right| < 20 \left| \mathcal{Z}(c_1c_2c_3) \left( w; \mathcal{X}_{a_2c_2}, \mathcal{X}_{a_1c_1} \right) \right| + 17 \cdot 20 \left| \mathcal{Z}(c_1p c_2c_3) \left( w; \mathcal{X}_{a_2c_2}, \mathcal{X}_{a_1c_1}, p \right) \right| |p|^{-\Re(w)} \]
\[ + 58 \cdot 20 \left| \mathcal{Z}(\bar{c}c_2c_3p) \left( w; \mathcal{X}_{a_2c_2}, \mathcal{X}_{a_1c_1} \right) \right| |p|^{-\frac{1}{2}}. \]

Let \( K(c_1, c_2, c_3, w, \delta, q) \) denote the right-hand side of (27), i.e.,
\[ K(c_1, c_2, c_3, w, \delta, q) = B(\delta, q) A_0^{c_0(c_2c_3)} A_1^{c_1} |c_1|^{3(1-\Re(w)) + \delta} |c_2|^{c_2} |c_3|^{\max\{3-4\Re(w), 2-\Re(w)\} + \delta}. \]

Taking \( w \) such that \( \frac{1}{2} \leq \Re(w) \leq \frac{4}{5} \), we have by the induction hypothesis
\[ \left| \mathcal{Z}(c_1p) \left( w; \mathcal{X}_{a_2c_2}, \mathcal{X}_{a_1c_1} \right) \right| < K(c_1, c_2, c_3, w, \delta, q) \quad \left( 20 + 340 A_0 |p|^{3-4\Re(w) + \delta} + 1600 A_0 |p|^{2-\Re(w) + \delta} \right) \]
and the proposition follows.

Using the last proposition, we can now estimate the function \( \mathcal{Z}(w; \mathcal{X}_{a_2}; h) := Z \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, w; \mathcal{X}_{a_2}; h \right) \).

**Theorem 6.4.** — For \( h \in \mathbb{F}[x] \) square-free monic and \( a_2 \in \{ 1, \theta_h \} \), put
\[ \mathcal{Z}(w; \mathcal{X}_{a_2}; h) = \left( 1 - q^{3-4w} \right) \left( 1 - q^{2-2w} \right) \mathcal{Z}(w; \mathcal{X}_{a_2}; h). \]

Then, for every \( \delta > 0 \), we have
\[ \mathcal{Z}(w; \mathcal{X}_{a_2}; h) \ll_{\delta,q} A^{\delta(h)} |h|^{2-\frac{3\Re(w)}{2} + \delta} \quad (28) \]
on the strip \( \frac{7}{5} \leq \Re(w) \leq \frac{4}{5} \), and
\[ |h|^{2w} \mathcal{Z}(w; \mathcal{X}_{a_2}; h) \ll_{\delta,q} A^{\delta(h)} \quad (29) \]
on the strip \( \frac{1}{2} \leq \Re(w) \leq 1 + \delta \), where \( A \) is an explicitly computable constant.

**Proof.** By (26) we have
\[ \left| \mathcal{Z}(w; \mathcal{X}_{a_2}; h) \right| \leq |h|^{-3\Re(w)} \sum_{h' \in c_1'} \sum_{\epsilon \in \left( \mathbb{P} \right)_{p_1'}} \left| \mathcal{Z}(h) \left( w; \mathcal{X}_{a_2h'}, \mathcal{X}_{h'} \right) \right| \prod_{p \in c_1'} |F(|p|^{-\frac{1}{2}}, \ldots, |p|^{-w}; |p|)| |p|^{-\Re(w)} \]
\[ \cdot \prod_{p' \in c_1'} \left| G\left( \epsilon, p_1 \right) \left( |p'|^{-\frac{1}{2}}, \ldots, |p'|^{-w}; |p'| \right) \right|. \]

It follows from Proposition 6.3 and Lemma 6.1 that, for every \( \delta > 0 \) and \( w \in \mathbb{C} \) with \( \frac{7}{5} \leq \Re(w) \leq \frac{4}{5} \),
\[ \mathcal{Z}(w; \mathcal{X}_{a_2}; h) \ll_{\delta,q} B^{\delta(h)} \quad |h|^{2-\frac{3\Re(w)}{2} + \delta} \sum_{h' \in c_1'} \sum_{\epsilon \in \left( \mathbb{P} \right)_{p_1'}} 1 \ll_{\delta,q} (3B)^{\delta(h)} |h|^{2-\frac{3\Re(w)}{2} + \delta} \]
for some explicitly computable constant \( B \). In particular, if \( \Re(w) = \frac{4}{5} \), we have
\[ |h|^{2w} \mathcal{Z}(w; \mathcal{X}_{a_2}; h) \ll_{\delta,q} (3B)^{\delta(h)} |h|^\delta. \]
On the other hand, if $\Re(w) = 1 + \delta$ we have by (16) that

$$|h|^{2w} \tilde{z}(w, \chi_{a_2}; h) \ll \delta, q \left(11B\right)^{\omega(h)} |h|^{\delta}. $$

The function $|h|^{2w} \tilde{z}(w, \chi_{a_2}; h)$ is holomorphic on an open neighborhood of the strip $\frac{4}{3} \leq \Re(w) \leq 1 + \delta$. This function is also of finite order on the strip and thus, the second estimate follows from the Phragmen-Lindelöf principle.

This completes the proof of the theorem.

To establish the analytic continuation of $Z_0(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, w, \chi_{a_2})$ to the half-plane $\Re(w) > \frac{2}{3}$, we shall need the following elementary lemma:

**Lemma 6.5.** — For any $A > 1$, the Dirichlet series

$$D_A(s) := \sum_{h \text{ monic} \& \text{ sq. free}} A^{\omega(h)} |h|^{-s}$$

is absolutely convergent in the half-plane $\Re(s) > 1$.

**Proof.** First the series is absolutely convergent for $\Re(s)$ sufficiently large. To see this, choose $n \geq 1$ such that $A < q^n$. Since $\omega(h) \leq \deg h$ for any square-free polynomial, we have $A^{\omega(h)} \leq |h|^n$. Thus, for $\Re(s) = \sigma > n + 1$,

$$\sum_{h \text{ monic} \& \text{ sq. free}} A^{\omega(h)} |h|^{-\sigma} < \sum_{h \text{ monic}} |h|^{-\sigma},$$

the last series being obviously convergent.

Now $D_A(s)$ has the Euler product expression

$$D_A(s) = \prod_{m=1}^{\infty} \left(1 + A q^{-m}\right)^{\text{Irr}_q(m)} \quad \text{ (for } \Re(s) > n + 1)$$

where $\text{Irr}_q(m)$ is the number of (monic) irreducible polynomials of degree $m$ over $\mathbb{F}$. From the well-known formula $\sum_{d|m} d \text{Irr}_q(d) = q^m$ for $m \geq 1$, we have $\text{Irr}_q(m) \leq q^m/m$. By using this estimate and the familiar inequality $\log(1 + y) < y$ for $y > 0$, we have, for $s = \sigma > n + 1$,

$$\log D_A(\sigma) = \sum_{m=1}^{\infty} \text{Irr}_q(m) \log(1 + A q^{-m}) < A \sum_{m=1}^{\infty} \frac{q^m(1 - \sigma)}{m}. $$

Thus the Euler product expression of $D_A(\sigma)$ converges when $\sigma > 1$, from which the lemma follows.

---

**7 Proof of Theorem 1.1**

The function $\tilde{z}_0(w, \chi_{a_2}) := (1 - q^{-1 - 4w})(1 - q^{-2 - 2w})Z_0(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, w, \chi_{a_2})$ is holomorphic in the half-plane $\Re(w) > 1$, and in this region, we have (by Lemma 5.1 and analytic continuation) that

$$\tilde{z}_0(w, \chi_{a_2}) = \sum_{h \text{ monic}} \mu(h) \tilde{z}(w, \chi_{a_2}; h). \quad (29)$$

By Theorem 6.4 and Lemma 6.5, the series in the right-hand side converges uniformly on every compact subset of the half-plane $\Re(w) > 2/3$, and the meromorphic continuation of $Z_0(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, w, \chi_{a_2})$ now follows from Weierstrass Theorem. The values $w \in \mathbb{C}$ for which $q^{-w} = \pm q^{-1}$, or $q^{-w} = \pm q^{-3/2}, \pm i q^{-3/2}$, are the only possible poles of this function. The principal parts of $Z_0(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, w, \chi_{a_2})$ at $q^{-w} = \pm q^{-1}$ can be computed following the arguments in [18, Section 3.2].

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To compute the residues at the remaining poles, fix \( \tilde{\theta}' \in \{1, \theta_0\} \), and let \( \rho(\tilde{\theta}') \) be such that \( \rho(\tilde{\theta}') \in \{ \pm 1 \} \) if \( \tilde{\theta}' = 1 \) or \( \rho(\tilde{\theta}') \in \{ \pm i \} \) if \( \tilde{\theta}' = \theta_0 \). Letting

\[
P(x) = (1-x)^5 (1+x) (1+4x+11x^2+10x^3-11x^4+11x^6-4x^7-x^8)
\]

\[
= 1-14x^3-x^4+78x^5+\ldots
\]

we have that

\[
\left( 1 - \rho(\tilde{\theta}') q^{\frac{3}{2}} \right) Z_\theta \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, w, \chi_{\theta_2} \right) \bigg|_{q^{-\theta} = \rho(\tilde{\theta}') q^{\frac{3}{2}}} = \frac{1}{8} \Gamma(a_2, \tilde{\theta}'; \rho(\tilde{\theta}')) L\left( \frac{1}{2}, \chi_{\theta'} \right)^3 \prod_p P\left( \chi_{\theta'}(p) \sqrt{|p|} \right).
\]

(30)

the product in the right-hand side being over all monic irreducibles in \( \mathbb{F}[x] \). Note that \( P(x) \) is precisely the polynomial appearing in the analogous calculation of Zhang [34] in the context of the cubic moment of quadratic Dirichlet \( L\)-series over the rationals.

To justify (30), we first apply Proposition 4.1 and (26). Indeed, let

\[
P_1(c) = \rho(\tilde{\theta}')^{\deg c} |c|^{-1/2} \prod_{p|c} \left( 1 - \chi_{\theta'}(p) |p|^{-1/2} \right)^3 \left( 1 + \chi_{\theta'}(p) |p|-1/2 \right)^2 \left( 1 + 6 \chi_{\theta'}(p) |p|^{-1/2} + |p|^{-1} \right)
\]

\[
P_2(c, c') = |c'|^{-1/2} \prod_{p|c'} \left( 1 - \chi_{\theta'}(p) |p|^{-1/2} \right)^3 \left( 1 + \chi_{\theta'}(p) |p|^{-1/2} \right) \left( 3 + 7 \chi_{\theta'}(p) |p|^{-1/2} + 3 |p|^{-1} \right)
\]

\[
P_3 \left( \frac{c'}{c} \right) = \prod_{p|c} \left( 1 - \chi_{\theta'}(p) |p|^{-1/2} \right)^3 \left( 1 + \chi_{\theta'}(p) |p|^{-1/2} \right) \left( 1 + 7 \chi_{\theta'}(p) |p|^{-1/2} + 13 |p|^{-1} + 7 \chi_{\theta'}(p) |p|^{-3/2} + |p|^{-2} \right).
\]

In (26) set \( s_1 = s_2 = s_3 = \frac{1}{2} \) and \( s_4 = w \). Multiplying the resulting equality by \( 1 - \rho(\tilde{\theta}') q^{\frac{3}{2}} \) and then taking the value \( q^{-w} = \overline{\rho(\tilde{\theta}')} q^{\frac{3}{2}} \), it follows from Proposition 4.1 that

\[
\left( 1 - \rho(\tilde{\theta}') q^{\frac{3}{2}} \right) Z_\theta \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, w, \chi_{\theta_2}; h \right) \bigg|_{q^{-w} = \overline{\rho(\tilde{\theta}')} q^{\frac{3}{2}}} = \frac{1}{8} \Gamma(a_2, \tilde{\theta}'; \rho(\tilde{\theta}')) L\left( \frac{1}{2}, \chi_{\theta'} \right)^3
\]

\[
\cdot \chi_{\theta'}(h) |h|^{-3/2} \sum_{h = c'} P_1(c) \prod_{p|c} F\left( |p|^{-1/2}, |p|^{-1/2}, |p|^{-1/2}, \overline{\rho(\tilde{\theta}')} \deg p |p|^{-3/4}; |p| \right) \overline{\rho(\tilde{\theta}')} \deg p |p|^{-3/4}
\]

\[
\sum_{e = (e_p, p') \in c'} P_2(c', c') P_3 \left( \frac{c'}{c} \right) \prod_{p' \in c'} G_e^{(\chi_{\theta'})} \left( |p'|^{-1/2}, |p'|^{-1/2}, |p'|^{-1/2}, \overline{\rho(\tilde{\theta}')} \deg p' |p'|^{-3/4}, |p'| \right).
\]

Recalling the explicit expressions of \( F \) and \( G^{(\chi_{\theta'})} \) (see the proof of Lemma 6.1), the equality (30) follows now from (29) and a routine computation. This completes the proof of the theorem.

8 Proof of Theorem 1.2

The proof is a standard application of the residue theorem. First replace \( q^{-w} \) in \( Z_\theta \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, w, 1 \right) \) by \( \xi \), and denote the resulting function by \( \mathcal{W}(\xi) \). Thus

\[
\mathcal{W}(\xi) = \sum_{D \geq 0} \sum_{d_0 \text{monic } \& \text{ sq. free}} \sum_{\text{deg } d_0 = D} L\left( \frac{1}{2}, \chi_{\theta_0} \right)^3 \xi^{d_0}.
\]
By Theorem 1.1, this function is meromorphic in the open disk \(|\xi < q^{-2/3}\). For small positive \(\delta\), let \(A_\delta = \{\xi \in \mathbb{C} : q^{-2} \leq |\xi| \leq q^{-2/3-\delta}\}\), and for \(D \geq 0\), consider the contour integral

\[
I(D) = \frac{1}{2\pi i} \int_{\partial A_\delta} \frac{\mathcal{W}(\xi)}{\xi^{D+1}} d\xi.
\]

We have

\[
\sum_{d_0-\text{monic} \& \text{ sq. free}} \frac{L\left(\frac{1}{2}, \chi_{d_0}\right)^3}{\xi^{D+1}} = \frac{1}{2\pi i} \int_{|\xi| = q^{-2}} \frac{\mathcal{W}(\xi)}{\xi^{D+1}} d\xi
\]

and by applying (29) and (28),

\[
\int_{|\xi| = q^{-2/3-\delta}} \frac{\mathcal{W}(\xi)}{\xi^{D+1}} d\xi \ll_{\delta, q} q^{D(\frac{1}{2} + \delta)}
\]

giving the error term in the asymptotic formula. By the residue theorem, \(I(D)\) is the sum of the residues at the poles of the function \(\mathcal{W}(\xi)/\xi^{D+1}\) in the annulus \(A_\delta\), i.e., \(\xi = \pm q^{-1}\) and \(\xi = \pm q^{-3/4}, \pm i q^{-3/4}\). The sum corresponding to the poles at \(\xi = \pm q^{-1}\) gives the main contribution to the asymptotic formula, and can be computed as in [18]; see also [28, Section 8 (a)] and [2, Section 5.3]. Now, from the proof of Theorem 1.1, the sum of the residues at \(\xi = \pm q^{-3/4}, \pm i q^{-3/4}\) of the integrand is given by

\[
-\frac{1}{2}(1 + q^{1/4} + 10q^{1/4} + 7q^{1/4} + 20q + 7q^{3/4} + 10q^{3/4} + q^2)q^{\frac{3D}{2}} \zeta\left(\frac{1}{2}\right)^7 \cdot \prod_p \left(1/\sqrt{|p|}\right)
\]

\[
-\frac{(-1)^D}{4}(1 - q^{1/4} + 10q^{1/4} - 7q^{3/4} + 20q - q^{3/4} + 10q^{3/4} + q^2)q^{\frac{3D}{2}} \zeta\left(\frac{1}{2}\right)^7 \cdot \prod_p \left(1/\sqrt{|p|}\right)
\]

\[
-\frac{1}{2} \Re\left(i^D(1 - iq^{1/4} - 4iq^{1/4} + 7iq^{3/4} + 6q - 7iq^{3/4} - 4q^{3/4} + iq^{3/4} + q^2)q^{\frac{3D}{2}} L\left(\frac{1}{2}, \chi_{d_0}\right)^3 \cdot \prod_p \left((-1)^{\deg p}/\sqrt{|p|}\right)
\]

Thus, letting \(R(D, q)\) denote the last expression times \(-q^{\frac{3D}{2}}\), we have that

\[
\sum_{d_0-\text{monic} \& \text{ sq. free}} L\left(\frac{1}{2}, \chi_{d_0}\right)^3 = \frac{q^{\frac{3D}{2}}}{\zeta(2)} Q(D, q) + q^{\frac{3D}{2}} R(D, q) + O_{b, q}\left(q^{D(\frac{1}{2} + \delta)}\right)
\]

which completes the proof.

A Appendix

To obtain the estimate (16), we have used the Lindelöf-type bound established in the following

**Theorem A.1.** — Let \(E_q\) be a finite field of odd characteristic, and let \(d\) be a square-free polynomial over \(E_q\) of degree \(D \geq 3\). Then, for any \(t \in \mathbb{R}\), we have

\[
|L\left(\frac{1}{2} + it, \chi_d\right)| < 4|d|^{10/\log^* 4}.
\]

**Proof.** We shall follow closely the argument in the proof of [9, Theorem 5.1]. Let \(C_d\) denote the (elliptic/hyperelliptic) curve corresponding to \(d\), and consider the numerator \(P_d(u)\) of the zeta function of \(C_d\). Then

\[
L(s, \chi_d) = (1 \pm q^{-s})^{\nu(D)} P_d(q^{-s})
\]

with \(\nu(D) = (1 + (-1)^D)/2\) and the \(+\) or \(-\) sign is determined according to whether the leading coefficient of \(d\) is a square in \(E_q^*\) or not. We estimate the factor \((1 \pm q^{-s})^{\nu(D)}\) (for \(s = \frac{1}{2} + it\)) trivially:

\[
|1 \pm q^{-s}|^{\nu(D)} \leq 1 + q^{-1/2} \leq 1 + \frac{1}{\sqrt{3}}.
\]

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It is well-known (see [30]) that

\[ P_d(u) = \prod_{m=1}^{2g} (1 - \sqrt[q]{\theta} e^{i\omega_m} u) \in \mathbb{Z}[u] \]

with \( \omega_m \in \mathbb{R} \) for all \( m \); the genus \( g \) of the curve \( C_d \) is obtained from the degree \( D \) of the polynomial \( d \): \( 2g = D - 1 \) if \( D \) is odd, and \( 2g = D - 2 \) if \( D \) is even.

Now, by [12, Theorem 8.1], for every non-negative integer \( N \) and every monic polynomial

\[ F(z) = \prod_{m=1}^{M} (z - \alpha_m) \quad (\alpha_1, \ldots, \alpha_M \in \mathbb{C} \text{ with } |\alpha_m| \leq 1 \text{ for all } 1 \leq m \leq M) \]

we have the estimate

\[ \sup_{|z| \leq 1} \log |F(z)| \leq M (N + 1)^{-1} \log 2 + \sum_{n=1}^{N} n^{-1} \left| \sum_{m=1}^{M} \alpha_m^n \right| \]

Fix an algebraic closure \( \overline{\mathbb{F}_q} \) of \( \mathbb{F}_q \). Let \( \chi_n \) (\( n \geq 1 \)) denote the non-trivial real character of \( \mathbb{F}_q^* \), extended to \( \overline{\mathbb{F}_q} \) by setting \( \chi_n(0) := 0 \). Applying this bound to \( P_d(u) \), we have

\[ \log |P_d(u)| < D (N + 1)^{-1} \log 2 + \sum_{n=1}^{N} n^{-1} \left| \sum_{m=1}^{2g} \chi_{n|m} \right| \]

for every \( u \in \mathbb{C} \) with \( |u| = 1/\sqrt{q} \). Recalling that, for a prime \( \ell \) different from the characteristic of \( \mathbb{F}_q \) and \( n \geq 1 \),

\[ q^{n/2} \sum_{m=1}^{2g} \chi_{n|m} = \text{Tr}(F^*| H^1_r(C_d, \mathbb{Q}_l)) = \sum_{\theta \in \mathbb{F}_q^*} \chi_n(d(\theta)) \]

where \( \overline{C_d} := C_d \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q} \) and \( F^* \) is the endomorphism of the \( \ell \)-adic étale cohomology induced by the Frobenius morphism \( F: \overline{C_d} \to \overline{C_d} \), we have trivially

\[ \left| \sum_{m=1}^{2g} \chi_{n|m} \right| \leq q^{-n/2} + q^{n/2}. \]

Consequently, if \( N \geq 1 \) we have

\[ \sum_{n=1}^{N} n^{-1} \left| \sum_{m=1}^{2g} \chi_{n|m} \right| \leq \sum_{n=1}^{N} n^{-1} \left( q^{-n/2} + q^{n/2} \right) < \sum_{n=1}^{\infty} n^{-1} q^{-n/2} + 2 \sum_{N/2 \leq n \leq N} n^{-1} q^{n/2} < \log (1 - 3^{-1/2})^{-1} + 4 (1 - 3^{-1/2})^{-1} \cdot \frac{q^{N/2}}{N}. \]

Thus, for all \( N \geq 1 \), we obtain the estimate

\[ \log |P_d(u)| < D (N + 1)^{-1} \log 2 + \log (1 - 3^{-1/2})^{-1} + 8 (1 - 3^{-1/2})^{-1} q^{N/2} (N + 1)^{-1}. \]

Choosing \( N = \left\lfloor \frac{2 \log D}{2 \log q} \right\rfloor \), we see that

\[ |P_d(u)| < \left( 1 - 3^{-1/2} \right)^{-1} \cdot \left| d \right|^{\frac{\log q}{2 \log q} + \frac{1}{2} (1 - 3^{-1/2})^{-1} \cdot \frac{1}{2 \log q}} \quad \text{(if } D \geq \sqrt{q} \text{)} \]

and

\[ |P_d(u)| < \left| d \right|^{\frac{\log q}{2 \log q}} \quad \text{(if } D < \sqrt{q} \text{)} \]

from which the theorem follows.
B Appendix

Let $F_q$ be a finite field with $q$ elements of odd characteristic. In [10] we have constructed a multiple Dirichlet series associated to the fourth moment of quadratic Dirichlet $L$-series over $F_q(\chi)$. By setting one of the first four variables of this multiple Dirichlet series to zero and by applying the recurrence relations in the proof of Theorem 3.7 of loc. cit., one obtains the explicit expression of the multiple Dirichlet series associated to the cubic moment. Explicitly, this series is, in fact, a rational function

$$Z(z_1, z_2, z_3; q) = \frac{N(z_1, z_2, z_3; q)}{D(z_1, z_2, z_3; q)}$$

with numerator given by

$$N(z_1, z_2, z_3; q) = 1 - q^3 z_1 z_2 - q^2 z_1 z_3 - q^2 z_1 z_4 + q^2 z_2 z_3 - q^2 z_2 z_4 + q^2 z_3 z_4 - q^3 z_1 z_3 - q^3 z_1 z_4 + q^3 z_2 z_4 + q^3 z_3 z_4 - q^4 z_1 z_4 + q^4 z_2 z_4 + q^4 z_3 z_4 - q^5 z_1 z_4 + q^5 z_2 z_4 + q^5 z_3 z_4 - q^6 z_1 z_4 + q^6 z_2 z_4 + q^6 z_3 z_4$$

and denominator

$$D(z_1, z_2, z_3; q) = (1 - q z_1)(1 - q z_2)(1 - q z_3)(1 - q z_4) - q^6 z_1 z_2 z_3 z_4 - q^5 z_1 z_2 z_3 - q^5 z_1 z_2 z_4 - q^5 z_1 z_3 z_4 - q^5 z_2 z_3 z_4 - q^4 z_1 z_2 z_3 z_4 - q^4 z_1 z_2 z_3 - q^4 z_1 z_2 z_4 - q^4 z_1 z_3 z_4 - q^4 z_2 z_3 z_4$$

In other words, with notation as in Section 3, Eq. (7), we have that

$$Z(q, q^{-1}, q^{-3}, q^{-4}; q) = Z^{(1)}(s; 1, 1) = \sum_{d=d_0 d_1^2} \frac{\prod_{d=1}^{\infty} L(s, \chi_{d_0}) \cdot P_{d}(s, z_2, s; \chi_{d_0})}{|d|^{s/2}}$$

Then the function

$$f(z_1, z_2, z_3; q) = f_{d_1}(z_1, z_2, z_3, z_4; q) := Z(q z_1, q z_2, q z_3, q z_4; 1/q)$$

is precisely the rational function obtained by considering the Chinta-Gunnells average (4) for the root system $D_4$ with central node corresponding to $z_1$. This fact can be checked either by a direct computation of the Chinta-Gunnells average, or by simply verifying that the rational function $f$ is $W$-invariant with respect to the Weyl group action defined in 2.1. $f(0, \ldots, 0; q) = 1$, and that it satisfies the condition (3). Expanding $f$ in a power series

$$f(z_1, z_2, z_3; q) = \sum_{k_1, k_2, k_3, l \geq 0} a(k_1, k_2, k_3, l; q) z_1^{k_1} z_2^{k_2} z_3^{k_3} z_4^l$$
we see easily that
\[ a(k_1, k_2, k_3, 0; q) = a(0, 0, 0, l; q) = 1 \quad \text{(for all } k_1, k_2, k_3, l \geq 0). \quad (33) \]

When \( l = 1 \) these coefficients vanish, unless \( k_1 = k_2 = k_3 = 0 \). Moreover, if \( \sum k_i \equiv l \equiv 1 \pmod{2} \) then
\[ a(k_1, k_2, k_3, l; q) = 0. \quad (34) \]

Now define
\[ f_{\text{odd}}(z_1, z_2, z_3; q) = (f(z_1, z_2, z_3, z_4; q) - f(z_1, z_2, z_3, -z_4; q)) / 2 \]
and
\[ f_{\text{even}}(z_1, z_2, z_3; q) = (f(z_1, z_2, z_3, z_4; q) + f(z_1, z_2, z_3, -z_4; q)) / 2. \]

One checks that the numerator of \( f_{\text{odd}}(z_1, z_2, z_3; z_4; q) \) is divisible by \((1 - z_4)(1 - z_2)(1 - z_3)\), and thus we can write
\[ f(z_1, z_2, z_3, z_4; q) = f_{\text{even}}(z_1, z_2, z_3, z_4; q) + f_{\text{odd}}(z_1, z_2, z_3, z_4; q) = (1 - z_4)^{-1}(1 - z_2)^{-1}(1 - z_3)^{-1} \sum_{l, \text{even}} P_l(z_1, z_2, z_3, z_4; q) z_4^l + \sum_{l, \text{odd}} P_l(z_1, z_2, z_3, z_4; q) z_4^l \]
for \(|z_4|\) sufficiently small (depending on the other variables). The symmetric polynomials \( P_l(z_1, z_2, z_3, z_4; q) \) defined by this expression of \( f \) were used in Section 3 to define the Dirichlet polynomial (6). Similarly, the polynomials \( Q_l(z_4; q) \) are defined by the expansion
\[ f(z_1, z_2, z_3, z_4; q) = (1 - z_4)^{-1} \sum_{l, \text{even}} Q_l(z_4; q) z_4^l + \sum_{l, \text{odd}} Q_l(z_4; q) z_4^l. \]

From the \( W \)-invariance of \( f \) we deduce the functional equations:
\[ P_l(z_1, z_2, z_3; q) = (\sqrt{q} z_l)^{-a_l} P_l \left( \frac{1}{\sqrt{q} z_l}; z_1, z_2, z_3; q \right) \quad \text{and} \quad Q_l(z_4; q) = (\sqrt{q} z_l)^{a_l} Q_l \left( \frac{1}{\sqrt{q} z_4}; q \right) \quad (35) \]
with \( a_n = 0 \) or 1 according as \( n \) is even or odd.

For the reader’s convenience, we include here the explicit expressions of some specializations of the rational functions introduced in this appendix. We have
\[ f_{\text{odd}}(q^{-rac{1}{2}}, q^{-rac{3}{4}}, q^{-rac{1}{2}}, z; q) = \frac{z(1 + 7z^2 + 7z^4 + z^6)}{(1 - z^2)^2(1 - qz^2)} \]
\[ f_{\text{even}}(q^{-rac{1}{2}}, q^{-rac{3}{4}}, q^{-rac{1}{2}}, z; q) = (1 - q^{-rac{1}{2}})^{-1} \cdot \frac{1 + (7 - 14q^{-rac{1}{2}} + 6q^{-1} - q^{-rac{3}{2}})z^2 + 7(1 - 4q^{-rac{1}{2}} + 4q^{-1} - q^{-rac{3}{2}})z^4 + (1 - 6q^{-rac{1}{2}} + 14q^{-1} - 7q^{-rac{3}{2}})z^6 - q^{-rac{3}{2}}z^8}{(1 - z^2)^2(1 - qz^2)} \]
and
\[ f_{\text{even}}(-q^{-rac{1}{2}}, -q^{-rac{3}{4}}, -q^{-rac{1}{2}}, z; q) = (1 + q^{-rac{1}{2}})^{-1} \cdot \frac{1 + (7 + 14q^{-rac{1}{2}} + 6q^{-1} + q^{-rac{3}{2}})z^2 + 7(1 + 4q^{-rac{1}{2}} + 4q^{-1} + q^{-rac{3}{2}})z^4 + (1 + 6q^{-rac{1}{2}} + 14q^{-1} + 7q^{-rac{3}{2}})z^6 + q^{-rac{3}{2}}z^8}{(1 - z^2)^2(1 - qz^2)} \]

One can use these formulas to estimate \( P_l(\pm q^{-rac{1}{2}}, \pm q^{-rac{3}{4}}, \pm q^{-rac{1}{2}}, q^{-rac{1}{2}}; q) \). Indeed, taking \(|z| = q^{-\frac{1}{4} - n}\) for small \( \eta > 0 \), we have the inequalities
\[ |f_{\text{odd}}(q^{-rac{1}{2}}, q^{-rac{3}{4}}, q^{-rac{1}{2}}, z; q) z^{-1}| \leq \frac{1 + 7|z|^2 + 7|z|^4 + |z|^6}{(1 - |z|^2)^2(1 - q|z|^2)} < \left( \frac{1 + |z|^2}{1 - |z|^2} \right)^7 \frac{1}{1 - q|z|^4} < \left( \frac{\sqrt{q} + 1}{\sqrt{q} - 1} \right)^7 \frac{1}{1 - q^{-4n}} \]

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The same bound holds for \((1 + q^{-\frac{1}{2}})^3|f_{even}(\pm q^{-\frac{1}{2}}, \pm q^{-\frac{1}{2}}, \pm q^{-\frac{1}{2}}, z; q)|\), since by the maximum principle we have
\[
(1 + q^{-\frac{1}{2}})^3|f_{even}(\pm q^{-\frac{1}{2}}, \pm q^{-\frac{1}{2}}, \pm q^{-\frac{1}{2}}, z; q)| < (1 + q^{-\frac{1}{2}})^3 \cdot \max_{|u| = q^{-1/4}} |(1 - qu^{4})f_{even}(\pm q^{-\frac{1}{2}}, \pm q^{-\frac{1}{2}}, \pm q^{-\frac{1}{2}}, u; q)|
\]
\[\leq \left(\frac{\sqrt{q} + 1}{\sqrt{q} - 1}\right)^7.
\]
If \(q \geq 5\), we have
\[
\left(\frac{\sqrt{q} + 1}{\sqrt{q} - 1}\right)^7 \cdot \frac{1}{1 - q^{-\frac{\eta}{4}}} < \frac{843}{1 - 5^{-\frac{\eta}{4}}}.
\]
Then by applying Cauchy’s inequality we obtain:

**Proposition B.1.** — For every small positive \(\eta\), \(q \geq 5\) and \(l \geq 1\) we have
\[
|P_{l}(\pm q^{-\frac{1}{2}}, \pm q^{-\frac{1}{2}}, \pm q^{-\frac{1}{2}}, q)| < \frac{843}{1 - 5^{-\frac{\eta}{4}}} q^{(l-a_{l})(\frac{\eta}{4})}
\]

where \(a_{l} = 0\) or 1 according as \(l\) is even or odd.

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