KdV stationary systems and their Stäckel representations

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Abstract

The notion of KdV stationary systems is introduced. Taking advantage of the Lax formalism, it is
proved that these systems have two different representations by means of the particular Stäckel systems of
Benenti type. The explicit transformation between jet coordinates and separation variables for arbitrary
number of degrees of freedom is presented. Besides, the Miura map between both representations of the
KdV stationary systems is derived.

1 Introduction

For the soliton hierarchies of PDE’s various invariant reductions lead to the Liouville integrable finite-
dimensional systems. In particular, such reductions are the stationary flows, the restricted flows, the Lax
constrained flows and others (see survey [3] and the literature therein). The KdV hierarchy is the best
known and most researched example of such a type of soliton hierarchy. Theory of its stationary flows was
developed since the early 70s. Its finite gap solutions were found by Dubrovin and Novikov [14, 12, 13] and
its Riemann theta function representation was presented by Its and Matveev [17, 18] (see the comprehensive
survey [16] and the literature therein). Then, Bogoyavlenskii and Novikov observed [10] that these flows
can be represented by the finite-dimensional Hamiltonian systems. Significant progress was made in 1987
[1], when the bi-Hamiltonian formulation for the KdV stationary flows was presented by means of the
degenerate Poisson tensors. In consequence, their Liouville integrability was proved. It was also observed
that, in fact, in the case of the KdV hierarchy there are two Hamiltonian finite-dimensional representations
of the stationary flows, connected by the Miura map [2, 21, 20].

In this article we introduce the notion of a KdV stationary system instead of an idea of a single KdV
stationary flow and prove that such systems have two different Stäckel representations from the Benenti class.
Obviously, solutions of these Stäckel systems coincide with the finite gap solutions of the related equations
from the KdV hierarchy and hence we come back to the classical results. Besides, what is interesting, the
inverse construction is also possible. Actually, starting from the particular family of Stäckel systems one
can reconstruct the related hierarchies of stationary systems and hence can reconstruct the whole KdV
hierarchy. This idea was explored for the first time in [6, 7].

The main result of this article is contained in the following theorem.

Theorem 1 The n-th KdV stationary system, which consists of first n flows from the KdV hierarchy
(2.1), together with the constraint being result of the (n + 1)-th stationary flow, i.e.
\[ u_{t_1} = K_1, \quad u_{t_2} = K_2, \quad \ldots, \quad u_{t_n} = K_n, \quad K_{n+1} = 0, \quad (1.1) \]
is equivalent to a finite-dimensional Hamiltonian system, on a 2n-dimensional phase space, represented by
a Stäckel system defined, in the first representation, by the spectral (separation) curve
\[ \lambda^{2n+1} + c\lambda^n + \sum_{k=1}^{n} H_k\lambda^{n-k} = \mu^2, \]
and in the second representation by the spectral curve
\[ \lambda^{2n} + \tilde{c}\lambda^{-1} + \sum_{k=1}^{n} H_k \lambda^{n-k} = \lambda \mu. \]

In both Stäckel representations the flows generated by Hamiltonians \( H_1, \ldots, H_n \) are in one-to-one correspondence with the flows from the stationary system \( (1.1) \), that is they are defined with respect to the same evolution parameters.

Moreover, the two Stäckel representations for each KdV stationary system \( (1.1) \) are isomorphic by a finite-dimensional Miura map on extended \((2n + 1)\)-dimensional phase space.

Only recently, in the article \[5\] the (isospectral) Lax representations for the whole Benenti class of Stäckel systems were constructed. These type of \( 2 \times 2 \) matrix Lax equations belong to the realm of the so-called Mumford systems that are associated with the separable systems with separation curves of hyperelliptic type \[19, 22\]. Taking the advantage of the Lax formalism from \[5\] we are able in this article to provide construction of explicit transformations, for arbitrary number of degrees of freedom, between jet coordinates of the KdV stationary systems and separation variables of associated Stäckel systems, which is a new result.

This article is organized as follows. In Section \[2\] we first remind the crucial facts about the KdV hierarchy, its Lax and zero-curvature representations. Next, we define the notion of KdV stationary systems and obtain their two representations together with their respective Lax equations. In Section \[3\] we collect important information on the particular class of Stäckel systems of Benenti type together with their Lax representation. Finally, in Section \[4\] we prove Theorem \[1\] first, relating the representations of the KdV stationary systems to the respective Stäckel systems and, finally, we present the explicit form of the Miura map between two different representations of the KdV stationary systems.

2 KdV hierarchy revisited

2.1 Bi-Hamiltonian structure

Let us collect some, important for further considerations, facts about the KdV hierarchy. The KdV equation
\[ u_t = \frac{1}{4} u_{xxx} + \frac{3}{2} u_{xx} \]
is a member of the bi-Hamiltonian chain of nonlinear PDE’s
\[ u_{tn} \equiv K_n = \pi_0 dH_n = \pi_1 dH_{n-1}, \quad n = 1, 2, \ldots \] (2.1)
where the two Poisson operators are
\[ \pi_0 = \partial_x, \quad \pi_1 = \frac{1}{4} \partial_x^2 + \frac{1}{2} u \partial_x + \frac{1}{2} \partial_x u. \]
The hierarchy (2.1) can be generated by the recursion operator and its adjoint
\[ N \equiv \pi_1 \pi_0^{-1} = \frac{1}{4} \partial_x^2 + u + \frac{1}{2} u_x \partial_x^{-1}, \quad N^\dagger = \frac{1}{4} \partial_x^2 + u - \frac{1}{2} \partial_x^{-1} u_x, \]
in the way such that
\[ K_{n+1} = N^n K_1, \quad \gamma_n = dH_n = (N^\dagger)^n \gamma_0, \quad n = 1, 2, \ldots. \] (2.2)
In particular, we find the following first members of the hierarchies of the KdV invariants like vector fields \( K_n \) (symmetries):
\[ K_1 = u_x, \]
\[ K_2 = \frac{1}{4} u_{xxx} + \frac{3}{2} u u_x, \]
\[ K_3 = \frac{1}{16} u_{5x} + \frac{5}{8} u u_{3x} + \frac{5}{4} u_x u_{2x} + \frac{15}{8} u^2 u_x, \]
\[ K_4 = \frac{1}{64} u_{7x} + \frac{7}{32} u u_{5x} + \frac{21}{32} u_x u_{4x} + \frac{35}{32} u_{xx} u_{3x} + \frac{35}{32} u^2 u_x + \frac{35}{32} u u_{2x} u_{2x} + \frac{35}{16} u^2 u_{3x} + \frac{35}{16} u^3 u_x, \]

\[ \vdots \]
conserved one-forms (co-symmetries) $\gamma_n$:

$\gamma_0 = 2$, \\
$\gamma_1 = u$, \\
$\gamma_2 = \frac{1}{4} u_{xx} + \frac{3}{4} u^2$, \\
$\gamma_3 = \frac{1}{16} u_{4x} + \frac{5}{3} u u_{xx} + \frac{5}{16} u_x^2 + \frac{5}{8} u^3$, \\
$\gamma_4 = \frac{1}{64} u_{6x} + \frac{7}{32} u u_{4x} + \frac{7}{16} u_x u_{3x} + \frac{21}{64} u_{2x}^2 + \frac{35}{32} u^2 u_{xx} + \frac{35}{32} u u_{2x}^2 + \frac{35}{64} u^4$, \\

and Hamiltonian densities $H_n$ of conserved functionals:

$H_0 = 2u$, \\
$H_1 = \frac{1}{2} u^2$, \\
$H_2 = -\frac{1}{8} u_x^2 + \frac{1}{4} u^3$, \\
$H_3 = \frac{1}{32} u_{2x}^2 + \frac{5}{32} u^2 u_{xx} + \frac{5}{32} u^4$, \\
$H_4 = -\frac{1}{128} u_{4x}^2 + \frac{7}{64} u u_{2x}^2 - \frac{35}{64} u^2 u_x^2 + \frac{7}{64} u^5$, \\

As $u$ belongs to the whole hierarchy (2.1) it depends on infinitely many evolution parameters $t_i$ and the spatial variable $x$: $u = u(x, t_1, t_2, t_3, ...)$). However, by the symmetry $u_{t_1} = u_x$ we can make the identification $t_1 \equiv x$, which is crucial from the point of view of stationary systems.

### 2.2 Lax representation

Alternatively, the hierarchy (2.1) can be reconstructed from the isospectral Lax equations. Actually, consider some eigenvalue problem together with time evolutions of its eigenfunctions

$$L \psi = \lambda \psi, \quad \lambda_{t_n} = 0,$$

$$A_n = B_n \psi, \quad n = 1, 2, ...,$$

where $L$ and $B_n$ are some differential operators. The compatibility conditions for (2.3) takes the form

$$L_{t_n} = [B_n, L], \quad n = 1, 2, ...,$$

known as the isospectral deformation equations, as the eigenvalues of the operator $L$ are independent of all times $t_i$ and the equations (2.3) are equivalent with the evolutionary hierarchy of PDE's (2.1). For the KdV hierarchy

$$L = \partial_x^2 + u, \quad B_n \equiv \left(L^{-\frac{1}{2}}\right)_{t_n} = \sum_{i=0}^{n-1} \left(-\frac{1}{4} \gamma_i \partial_x \right) L^{n-i-1}, \quad n = 1, 2, ...,$$

where in particular

$B_1 = \partial_x$, \\
$B_2 = \partial_x^3 + \frac{3}{2} u \partial_x + \frac{3}{4} u_x$, \\
$B_3 = \partial_x^5 + \frac{5}{2} u \partial_x^3 + \frac{15}{4} u_x^2 \partial_x^2 + \frac{5}{8} (3 u^2 + 5 u_{xx}) \partial_x + \frac{15}{16} (u_{3x} + 2 u u_x)$,
where recursively (2.9) for coefficients of \( P \) rewrite the linear problem for the KdV hierarchy (2.3), or equivalently (2.6), in the form

\[
\psi_{xx} = \lambda \psi - u \psi', \quad \psi_{tn} = P_n \psi_x - \frac{1}{2} (P_n)_x \psi, \quad n = 1, 2, \ldots, \tag{2.6}
\]

where

\[
P_n \equiv \frac{1}{2} \sum_{i=0}^{n-1} \gamma_i \lambda^{n-i-1}.
\]

Then, the compatibility conditions \( (\psi_{xx})_{tn} = (\psi_{tn})_{xx} \) of the equations (2.6) provide the hierarchy (2.4) in the form

\[
u_{tn} = 2 (P_n)_x (u - \lambda) + u_x P_n + \frac{1}{2} (P_n)_{xx} \equiv K_n, \quad n = 1, 2, \ldots. \tag{2.7}
\]

The consistency of the KdV hierarchy causes that all the \( \lambda \) terms in (2.7) mutually cancel.

The bi-Hamiltonian chain for the KdV hierarchy (2.1), on the level of co-symmetries, takes the form

\[
\pi_{\lambda} P_\lambda = 0 \quad \iff \quad (P_\lambda)_x (u - \lambda) + \frac{1}{2} u_x P_\lambda + \frac{1}{4} (P_\lambda)_{xx} = 0, \tag{2.8}
\]

where

\[
P_\lambda \equiv \sum_{i=0}^{\infty} \gamma_i \lambda^{-i-1}
\]

lies in the kernel of the Poisson pencil \( \pi_{\lambda} \equiv \pi_1 - \lambda \pi_0 \). In fact, we can integrate (2.8) to the equation

\[
- \frac{1}{2} P_\lambda (P_\lambda)_{xx} + \frac{1}{4} (P_\lambda)_x^2 - (u - \lambda) P_\lambda^2 = C(\lambda) \equiv 4\lambda^{-1}, \tag{2.9}
\]

where \( C(\lambda) \) is an arbitrary function of \( \lambda \) with coefficients being constants of integration appearing in the recursion (2.1) or, equivalently, (2.2). Here, we make the simplest possible choice \( C(\lambda) \equiv 4\lambda^{-1} \). Solving recursively (2.9) for coefficients of \( P_\lambda \) one finds that \( \gamma_0 = 2, \gamma_1 = u \) and

\[
\gamma_k = \frac{1}{16} \sum_{i=1}^{k-1} [2 \gamma_{k-i-1}(\gamma_{i} \lambda) - (\gamma_{k-i-1} \gamma_{i} \lambda) - 4 \gamma_k \cdot \gamma] + \frac{1}{4} \sum_{i=0}^{k-1} u \gamma_k \cdot \gamma_i, \quad k \geq 2. \tag{2.10}
\]

Now, the KdV flows can be obtained in the form (2.7) taking \( P_n = \frac{1}{4} [\lambda^n P_\lambda]_+ \), where \([\cdot]_+\) means, here, the projection on the polynomial part in \( \lambda \). To the best of our knowledge, the algebraic recursion formula (2.10), for the construction of co-symmetries \( \gamma_n \), was not obtained before. Let’s note that contrary to the original recursion (2.2) the formula (2.10) does not require integration.

### 2.3 Zero-curvature representation

In fact, the hierarchy (2.4) can be also reconstructed form the so-called zero-curvature equations, which are more suitable for our further considerations. Introducing the vector eigenfunction \( \Psi = (\psi, \psi_x)^T \) we can rewrite the linear problem for the KdV hierarchy (2.3), or equivalently (2.6), in the form

\[
\Psi_x = V_1 \Psi, \quad t_1 \equiv x, \tag{2.11}
\]

where

\[
V_k = \left( \begin{array}{cc}
-\frac{1}{2} (P_k)_x & P_k \\
P_k (\lambda - u) - \frac{1}{2} (P_k)_x & \frac{1}{2} (P_k)_x
\end{array} \right), \quad k = 1, 2, \ldots. \tag{2.12}
\]

In particular

\[
V_1 = \left( \begin{array}{cc}
0 & 1 \\
\lambda - u & 0
\end{array} \right), \quad V_2 = \left( \begin{array}{cc}
\frac{1}{2} u_x & -\frac{1}{4} u_x \\
-\frac{1}{4} u_x & -\frac{1}{4} u_x - \frac{1}{4} u_{xx} + \lambda + \frac{1}{4} u
\end{array} \right) \tag{2.13a}
\]

and

\[
V_3 = \left( \begin{array}{cc}
\lambda^3 - \frac{1}{2} u \lambda^2 - \frac{1}{8} (u_{xx} + 2u_x^2) & -\frac{1}{2} u_x \lambda - \frac{1}{4} (u_{xx} + 6u_{xx}) \\
-\frac{1}{2} u_x \lambda - \frac{1}{4} (u_{xx} + 6u_{xx}) & \lambda^2 + \frac{1}{4} u \lambda + \frac{1}{4} (u_{xx} + 3u_x^2)
\end{array} \right), \tag{2.13b}
\]

As a consequence of (2.5) we can represent the linear problem (2.3) by means of polynomials in the spectral variable \( \lambda \):

\[
\psi_{xx} = \lambda \psi - u \psi', \quad \psi_{tn} = P_n \psi_x - \frac{1}{2} (P_n)_x \psi, \quad n = 1, 2, \ldots, \tag{2.6}
\]
Then, the compatibility conditions \((\Psi_x)_{t_n} = (\Psi_{t_n})_x\) for (2.11) take the form of the following zero-curvature equations
\[
\frac{d}{dt} V_k = [V_k, V_1] + \frac{d}{dx} V_k, \quad k = 1, 2, \ldots, \tag{2.14}
\]
which are equivalent to the respective members of the KdV hierarchy (2.7). Here, \(\frac{d}{dx}\) and \(\frac{d}{dt}\) means the total derivatives with respect to spatial \(x\) and evolution \(t_k\) variables. The remaining zero-curvature equations coming from the conditions \((\Psi_t)_{t_n} = (\Psi_{t_n})_t\),
\[
\frac{d}{dt} V_n - \frac{d}{dt} V_m + [V_n, V_m] = 0, \quad m, n = 1, 2, \ldots, \tag{2.15}
\]
are valid as differential consequences of the commutations of the vector fields from the KdV hierarchy (2.1).

### 2.4 Stationary systems

The \((n + 1)\)-th stationary flow is determined by the following restriction on the \((n + 1)\)-th KdV symmetry:
\[
u_{t_{n+1}} = 0 \quad \text{or equivalently} \quad K_{n+1} = 0, \tag{2.16}
\]
which can be obtained by imposing on the linear problems (2.11) the constraint
\[
\Psi_{t_{n+1}} = \lambda^m \mu \Psi \tag{2.17a}
\]
or equivalently
\[
V_{n+1} \Psi = \lambda^m \mu \Psi. \tag{2.17b}
\]
The factor \(\lambda^m\) in (2.17) is a matter of later convenience. Indeed, the constraint (2.17a) and the compatibility condition \((\Psi_x)_{t_{n+1}} = (\Psi_{t_{n+1}})_x\) gives
\[
\frac{d}{dt_{n+1}} V_1 = 0, \tag{2.18}
\]
which is equivalent to (2.16), or alternatively the compatibility condition between eigenvalue problem (2.17b) and \(\Psi_x = V_1 \Psi\) yields the Lax equation
\[
\frac{d}{dx} V_{n+1} = [V_1, V_{n+1}],
\]
which combined with the zero-curvature equation (2.14) for \(k = n + 1\) gives again (2.18).

The stationary restriction (2.16) provides constraint on the infinite-dimensional (functional) manifold, on which the KdV hierarchy is defined, reducing it to the finite-dimensional submanifold. Due to the integrability the constraint is invariant with respect to all flows from the KdV hierarchy. As result the infinite hierarchy (2.1) reduces to the finite system:
\[
u_{t_1} = K_1, \quad \nu_{t_2} = K_2, \quad \ldots, \quad \nu_{t_n} = K_n, \quad K_{n+1} = 0 \tag{2.19}
\]
which further will be called the \(n\)-th KdV stationary system. The finite hierarchy of associated Lax equations is given by equations
\[
\frac{d}{dt_k} V_{n+1} = [V_k, V_{n+1}], \quad k = 1, 2, \ldots, n, \tag{2.20}
\]
completed over (2.18) and valid under the constraint (2.16). One obtains the Lax equations (2.20) simply imposing (2.18) on the zero-curvature equations (2.15) or by the compatibility conditions between eigenvalue problem (2.17b) and the respective linear problems (2.11).

After imposing the constraint (2.17b) the existence of nontrivial solutions for the respective linear problems enforces the characteristic equation
\[
\det (V_{n+1} - \lambda^m \mu I) = 0, \tag{2.21}
\]
associated with (2.16). Equation (2.21) determines the spectral curve
\[
- \frac{1}{2} P_{n+1} (P_{n+1})_{xx} + \frac{1}{4} (P_{n+1})_x^2 - (u - \lambda) P_{n+1}^2 = \lambda^2 m \mu^2, \tag{2.22}
\]
which takes the more explicit form
\[
\lambda^{2n+1} + \sum_{k=0}^{n} H_{k+m} \lambda^{n-k} = \lambda^{2m} \mu^2, \tag{2.23}
\]
where
\[
H_{k+m} = -\frac{1}{16} \sum_{i=0}^{n-k} \left[ 2\gamma_{n-i} \left( \gamma_{i+k} \right)_{xx} - \left( \gamma_{n-i} \right)_x \left( \gamma_{i+k} \right)_x + 4u \gamma_{n-i} \gamma_{i+k} \right] + \frac{1}{4} \sum_{i=1}^{n-k} \gamma_{n-i+1} \gamma_{i+k}.
\]
In particular, by (2.10)
\[
H_m = -\frac{1}{16} \sum_{i=0}^{n} \left[ 2\gamma_{n-i} \left( \gamma_{i} \right)_{xx} - \left( \gamma_{n-i} \right)_x \left( \gamma_{i} \right)_x + 4u \gamma_{n-i} \gamma_{i} \right] + \frac{1}{4} \sum_{i=1}^{n} \gamma_{n-i+1} \gamma_{i} \equiv -\gamma_{n+1} \tag{2.24a}
\]
and
\[
H_{n+m} = -\frac{1}{8} \gamma_n (\gamma_n)_{xx} + \frac{1}{16} (\gamma_n)_x^2 + \frac{1}{4} u \gamma_n^2. \tag{2.24b}
\]
In fact, the coefficients \(H_m, \ldots, H_{n+m}\) of (2.23) are constants of motion of the respective stationary system (2.19), and thus the spectral curve (2.23) describes a common level of them.

**Remark 2** In fact, the spectral curve (2.22) could be obtained without involving the above matrix Lax equations, as by (2.7) the \((n+1)\)-th stationary flow (2.16) is given by the condition
\[
2 \left( P_{n+1} \right)_x (u - \lambda) + u_x P_{n+1} + \frac{1}{2} \left( P_{n+1} \right)_3 = 0, \tag{2.25}
\]
which can be directly integrated to the spectral curve in the form:
\[
-\frac{1}{2} P_{n+1} \left( P_{n+1} \right)_x + \frac{1}{4} \left( P_{n+1} \right)^2_x - (u - \lambda) P_{n+1}^2 = C(\lambda), \tag{2.26}
\]
where \(C(\lambda)\) is an 'integral' series in \(\lambda\) with constant coefficients that must be determined or appropriately defined. Thus, differentiating the spectral curve (2.26) or (2.22) one reconstructs the stationary condition (2.25).

### 2.5 Two representations of stationary systems

Integrating separately Hamiltonian representations of the stationary flow (2.16),
\[
\mathcal{K}_{n+1} = \pi_0 \gamma_{n+1} = \pi_1 \gamma_n = 0,
\]
we obtain two representations of the \(n\)-th stationary KdV system (2.19).

Indeed, integrating the first Hamiltonian structure, \(\pi_0 \gamma_{n+1} = 0\), we find that
\[
\gamma_{n+1} + c = 0, \tag{2.27}
\]
where \(c\) is an integration constant. As result, the first representation of the \(n\)-th stationary KdV system is given by
\[
\gamma_{t_1} = \mathcal{K}_1, \quad u_{t_2} = \mathcal{K}_2, \quad \ldots, \quad u_{t_n} = \mathcal{K}_n, \quad \gamma_{n+1} + c = 0 \tag{2.28}
\]
which constitutes a system of ODE’s, with \(n\) degrees of freedom, on the \(2n\)-dimensional phase space with the jet coordinates \(u, u_x, \ldots, u_{(2n-1)x}\). Notice that, higher derivatives of \(u\) with respect to \(x\) are eliminated by the (differential) constraint coming from (2.27). The associated Lax equations are simply given by (2.20) with imposed constraint (2.27). In this case the spectral curve (2.26), choosing \(m = 0\) and taking into account (2.24a), takes the form
\[
\lambda^{2n+1} + c \lambda^n + \sum_{k=1}^{n} H_k \lambda^{n-k} = \mu^2. \tag{2.29}
\]

Integrating the second Hamiltonian structure, \(\pi_1 \gamma_n = 0\), we find another constraint
\[
\frac{1}{2} \gamma_n (\gamma_n)_{xx} - \frac{1}{4} (\gamma_n)_x^2 + \frac{1}{2} c \gamma_n^2 + 4c = 0 \tag{2.30}
\]
which gives the second representation of the \( n \)-th stationary KdV system (2.19):

\[
\begin{align*}
    u_{1} &= K_{1}, \\
    u_{2} &= K_{2}, \\
    \ldots, \\
    u_{n} &= K_{n}, \\
    \frac{1}{2} \gamma_{n}(\gamma_{n})_{xx} - \frac{1}{4} (\gamma_{n})^{2} + u\gamma_{n}^{2} + 4\hat{c} &= 0
\end{align*}
\]  

(2.31)
on the \( 2n \)-dimensional phase space with the jet coordinates \( u, u_{x}, \ldots, u_{(2n-1)x} \). Obviously, higher order derivatives are eliminated by (2.30). The Lax representation of the system (2.31) is again given by (2.20) imposing (2.30) and the spectral curve (2.23), where (2.24b), choosing \( m = 1 \), is

\[
\lambda^{2} + \hat{c}\lambda^{-1} + \sum_{k=1}^{n} H_{k}\lambda^{n-k} = \lambda\mu^{2}.
\]  

(2.32)

**Example 3** The first representation (2.25) of the stationary KdV system, given for \( n = 2 \), is constituted by the first two flows from the KdV hierarchy:

\[
\begin{align*}
    u_{1} &= u_{x} \equiv K_{1}, \\
    u_{2} &= \frac{1}{4} u_{xxx} + \frac{3}{2} uu_{x} \equiv K_{2},
\end{align*}
\]  

(2.33a)

and the constraint

\[
\frac{1}{16} u_{4x} + \frac{5}{8} uu_{xx} + \frac{5}{16} u_{x}^{2} + \frac{5}{8} u^{3} + c = \gamma_{3} + c = 0.
\]  

(2.33b)

The Lax representation of the stationary system (2.33) is given by the Lax equations

\[
\begin{align*}
    \frac{d}{dt_{1}} V_{3} &= [V_{1}, V_{3}], \\
    \frac{d}{dt_{2}} V_{3} &= [V_{2}, V_{3}],
\end{align*}
\]

where the matrices \( V_{1} \) and \( V_{2} \) are given by (2.13a) and the matrix \( V_{3} \) (2.13b) under the constraint (2.33b) takes the form

\[
V_{3} = \left( \lambda^{3} - \frac{1}{2} u\lambda^{2} - \frac{1}{8} (u_{3x} + 6uu_{x}) \lambda - \frac{1}{16} uu_{xx} - \frac{1}{32} u^{2} + \frac{1}{3} u^{3} + c \right) \lambda^{2} + \frac{1}{4} u\lambda + \frac{1}{16} (u_{3x} + 6uu_{x}).
\]  

(2.34)

The associated spectral curve (2.29) for \( n = 2 \) is given by

\[
\lambda^{5} + c\lambda^{2} + H_{1}\lambda + H_{2} = \mu^{2},
\]

where one finds the following (nontrivial) integrals of motion:

\[
\begin{align*}
    H_{1} &= \frac{1}{32} u_{x} u_{3x} - \frac{1}{64} u_{xx}^{2} + \frac{5}{32} uu_{x}^{2} + \frac{5}{64} u^{4} + \frac{1}{2} uu_{x}, \\
    H_{2} &= \frac{3}{64} uu_{x} u_{3x} + \frac{1}{256} u_{3x}^{2} - \frac{1}{128} uu_{x}^{2} + \frac{5}{64} u^{3} u_{xx} + \frac{15}{128} u^{2} u_{x}^{2} + \frac{1}{64} uu_{xx}^{2} + \frac{3}{32} u^{5} + \frac{1}{8} cu_{x} + \frac{3}{8} cu^{2}.
\end{align*}
\]  

(2.35)

**Example 4** The second representation (2.31) of the stationary KdV system, for \( n = 2 \), is given by the flows:

\[
\begin{align*}
    u_{1} &= u_{x} \equiv K_{1}, \\
    u_{2} &= \frac{1}{4} u_{xxx} + \frac{3}{2} uu_{x} \equiv K_{2},
\end{align*}
\]  

(2.36a)

and the constraint

\[
\begin{align*}
    \frac{1}{32} u_{xx} u_{4x} + \frac{3}{32} u_{x}^{2} u_{4x} - \frac{1}{64} u_{xx}^{2} - \frac{3}{16} uu_{x} u_{3x} + \frac{3}{16} uu_{x} u_{xx} + \frac{1}{4} uu_{xx}^{2} + \frac{15}{16} uu_{xx}^{2} + \frac{9}{16} u^{5} + 4\hat{c} &= 0.
\end{align*}
\]  

(2.36b)

The Lax representation of the stationary system (2.36) is given by

\[
\begin{align*}
    \frac{d}{dt_{1}} V_{3} &= [V_{1}, V_{3}], \\
    \frac{d}{dt_{2}} V_{3} &= [V_{2}, V_{3}],
\end{align*}
\]

where the matrices \( V_{1} \) and \( V_{2} \) are given by (2.13a) and the matrix \( V_{3} \) (2.13b) under the constraint (2.36b) takes the form

\[
V_{3} = \left( \lambda^{3} - \frac{1}{2} u\lambda^{2} - \frac{1}{8} (u_{3x} + 6uu_{x}) \lambda - \frac{1}{32} uu_{xx} + \frac{5}{96} u^{3} \right) \lambda^{2} + \frac{1}{4} u\lambda + \frac{1}{16} (u_{3x} + 6uu_{x}).
\]  

(2.37)
The evolution of any observable
\( \xi \)

with the integrals of motion:
\[
H_1 = -u_3^2 + 2u_x^2u_{2x} - 12uu_xu_{3x} - 4u_2u_{2x} - 20u^4u_{2x} - uu_x^2 - 30u^2u_x^2 - 24u^5 + 256\tilde{c},
\]
\[
H_2 = -u_3^2 - uu_x^2 + 2u_xu_{2x}u_{3x} + 12uu_x^2u_{2x} - 7u^2u_x^2 - 15u^4u_{2x} - 9u^6 + 256\tilde{c}u.
\]

3 Hyperelliptic separation curves and related Stäckel systems

3.1 Hamiltonian Stäckel systems

Let us consider the spectral curve in the form
\[
\sigma(\lambda) + \sum_{k=1}^{n} H_k \lambda^{-k} = \lambda^m \mu^2, \quad m \in \mathbb{Z},
\]

where \( \sigma(\lambda) \) is a (Laurent) polynomial in the variables \( \lambda \) and \( \lambda^{-1} \). The associated separable systems associated with (\ref{3.1}) belong the so-called Benenti subclass of Stäckel systems [8]. The separation relations are reconstructed by taking \( n \) copies of (\ref{3.1}) with respect to the separation (Darboux) coordinates \((\lambda, \mu)\) on a phase space \( M = T^*Q \), where \( \lambda = (\lambda_1, \ldots, \lambda_n)^T \) are local coordinates on the configuration space \( Q \) and \( \mu = (\mu_1, \ldots, \mu_n)^T \) are the (fibre) momentum coordinates. Thus, solving the linear system
\[
\sigma(\lambda_i) + \sum_{k=1}^{n} H_k \lambda_i^{-k} = \lambda_i^m \mu_i^2, \quad i = 1, \ldots, n,
\]

with respect to functions \( H_k = H_k(\lambda, \mu) \) we obtain \( n \) quadratic in momenta Hamiltonians
\[
H_k = \frac{1}{2} \mu^T K_k G_m \mu + V_k, \quad k = 1, \ldots, n,
\]

where \( G_m \) represents the contravariant metric, defined by the first Hamiltonian \( H_1 \), on the configuration space \( Q \). In fact
\[
G_m = L^m G_0, \quad G_0 = 2 \text{ diag } \left( \frac{1}{\Delta_1}, \ldots, \frac{1}{\Delta_n} \right), \quad \Delta_i = \prod_{j \neq i} (\lambda_i - \lambda_j).
\]

Here, \( K_k \) are respective Killing tensors and \( L \) is a special conformal Killing tensor [11]:
\[
K_k = (-1)^{k+1} \text{ diag } \left( \frac{\partial s_k}{\partial \lambda_1}, \ldots, \frac{\partial s_k}{\partial \lambda_n} \right), \quad L = \text{ diag } (\lambda_1, \ldots, \lambda_n),
\]

where \( s_k \) are the elementary symmetric polynomials in \( \lambda_i \). The potential functions \( V_k \) are given by
\[
V_k = (-1)^{k+1} \sum_{i=1}^{n} \frac{\partial s_k}{\partial \lambda_i} \sigma(\lambda_i)/\Delta_i.
\]

The Hamiltonians (\ref{3.2}) are in involution with respect to the canonical Poisson bracket defined in the separation variables by
\[
\{\cdot, \cdot\} = \sum_{i=1}^{n} \frac{\partial}{\partial \lambda_i} \wedge \frac{\partial}{\partial \mu_i}.
\]

The evolution of any observable \( \xi \) with respect to the Hamiltonian \( H_k \) has the form \( \xi_t = \{\xi, H_k\} \) and the Hamiltonian evolution equations are
\[
\lambda_t = \{\lambda, H_k\}, \quad \mu_t = \{\mu, H_k\}, \quad k = 1, \ldots, n.
\]
3.2 Lax representation

As it shown in [5], the Hamiltonian evolution equations (3.4) associated with the spectral curves (3.1) can be represented by (isospectral) Lax equations

\[
\frac{d}{dt} L = [U_k, L], \quad k = 1, \ldots, n, \tag{3.5}
\]

with \( L \) and \( U_k \) being \( 2 \times 2 \) quadratic (traceless) matrices depending rationally on the spectral parameter \( \lambda \).

Actually, the Lax matrix \( L \) has the form

\[
L = \begin{pmatrix} v & u \\ w & -v \end{pmatrix}, \tag{3.6}
\]

where in the separation coordinates \((\lambda, \mu)\) the coefficients are

\[
u := \prod_{k=1}^{n} (\lambda - \lambda_k) \equiv \lambda^n + \sum_{k=1}^{n} (-1)^k s_k \lambda^{n-k},
\]

\[
v := \sum_{k=1}^{n} (-1)^{k+1} \left[ \sum_{i=1}^{n} \frac{\partial s_k}{\partial \lambda_i} \frac{\lambda_i^m \mu_i}{\Delta_i} \right] \lambda^{n-k}
\]

and

\[
w := \frac{1}{u} \left[ \lambda^m \left( \frac{\sigma(\lambda)}{u} + \sum_{k=1}^{n} H_k \lambda^{n-k} \right) - v^2 \right]. \tag{3.7}
\]

In fact, \( w \) is defined so that the spectral curve (3.1) can be reconstructed from the characteristic equation

\[
\det [L - \lambda^m \mu I] = 0 \iff v^2 + uw = \lambda^{2m} \mu^2.
\]

One can show that the expression in the quadratic bracket in (3.7) factorizes so that \( w \) takes the form of a Laurent polynomial in \( \lambda \):

\[
w = \lambda^m \left[ \frac{\sigma(\lambda) - \lambda^{-m} \mu^2}{u} \right]_+.
\tag{3.8}
\]

Here, the operation \([\cdot]_+\) means the projection on the uniquely defined quotient of the division of an analytic function \( A \) over a (pure) polynomial \( u \) such that the following decomposition holds:

\[
A = \left[ \frac{A}{u} \right]_+ u + r,
\]

where the (unique) remainder \( r \) is a lower degree polynomial than the polynomial \( u \), see for details [5]. In particular when \( A \) is a Laurent polynomial we have

\[
\left[ \frac{A}{u} \right]_+ \equiv \left[ \frac{[A]_{\geq 0}}{u} \right]_{\geq 0} + \left[ \frac{[A]_{<0}}{u} \right]_{<0},
\]

where \([\cdot]_{\geq 0}\) is the projection on the part consisting of non-negative degree terms in the expansion into Laurent series at \( \infty \) and \([\cdot]_{<0}\) is the projection on the part consisting of negative degree terms in the expansion into Laurent series at \( 0 \).

The generating matrices \( U_k \) are defined by

\[
U_k := \left[ \frac{u_k L}{u} \right]_+ \equiv \left( \frac{u_k L}{u} \right)_+ - \left[ \frac{u_k L}{u} \right]_+, \quad k = 1, \ldots, n, \tag{3.9}
\]

where

\[
u_k := \left[ \frac{u}{\lambda^{n-k+1}} \right]_+ \equiv \lambda^{k-1} + \sum_{i=1}^{k-1} (-1)^{k-1} s_k \lambda^{k-1-i}.\]

In the construction of the Lax equations there is some freedom, in comparison to the general formalism from [5] we take \( g(\lambda) = \frac{1}{2} f(\lambda) = \lambda^m \).
The evolution of the Lax matrix \( (3.5) \) with respect to Hamiltonian equations \((3.4)\), and consequently the Lax equations \((3.3)\), can be directly derived from the following useful relations:

\[
\begin{align*}
\{u, H_k\} &= -2u_k v + 2u \left[ \frac{u_k v}{u} \right]_+, \\
\{v, H_k\} &= u_k w - u \left[ \frac{u_k w}{u} \right]_+, \\
\{w, H_k\} &= -2w \left[ \frac{u_k v}{u} \right]_+ + 2v \left[ \frac{u_k w}{u} \right]_+,
\end{align*}
\]

which were obtained in [5].

Moreover, considering \((3.10)\) for \( k = 1 \) and observing that \( u_1 = 1 \) and \( [\frac{w}{u}]_+ = 0 \) we can rewrite the matrices \((3.9)\) and \((3.10)\) in the form:

\[
L = \begin{pmatrix}
-\frac{1}{2} \dot{u} & u \\
-\frac{1}{2} \dot{u} + u Q & \frac{1}{2} \dot{u}
\end{pmatrix}
\]

and

\[
U_k = \begin{pmatrix}
-\frac{1}{2} \ddot{u}_k & u_k \\
-\frac{1}{2} \ddot{u}_k + u_k Q & \frac{1}{2} \ddot{u}_k
\end{pmatrix},
\]

where \( Q \equiv [\frac{w}{u}]_+ \). Here, the dot means the derivative with respect to the first Hamiltonian flow, i.e. \( \dot{\xi} \equiv \xi_{t1} \), and one can see that \( \ddot{u}_k = \left[ \frac{u_k u}{u} \right]_+ \). Now, the connection between \((3.11)\), \((3.12)\) and \((2.12)\) is apparent.

### 3.3 Viète’s coordinates

From the point of view of expressing the Hamiltonian systems \((3.4)\) in the Lax form the most practical are the so-called Viète’s (canonical) coordinates:

\[
q_i = (-1)^i s_i, \quad p_i = -\sum_{k=1}^{n} \frac{\lambda_{k}^{n-i}}{\Delta_k} \mu_{k}, \quad i = 1, \ldots, n.
\]

Let \( p = (p_1, \ldots, p_n)^T \) and \( q = (q_1, \ldots, q_n)^T \). In this coordinates the geodesic part of the Hamiltonians \( H_k \) is always polynomial function of their arguments and the potentials \( V_k \) are either polynomials or rational functions. In Viète’s coordinates the Hamiltonians \((3.2)\) take the form

\[
H_k = \frac{1}{2} p^T K_k G_m p + V_k, \quad k = 1, \ldots, n,
\]

and the respective Hamiltonian evolution equations are

\[
q_{t_k} = \{q, H_k\}, \quad p_{t_k} = \{p, H_k\},
\]

where \( \{\cdot, \cdot\} = \sum \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_j} \).

For \( \sigma(\lambda) = \sum \alpha_i \lambda^i \) the potential functions \((3.3)\) are given by \( V_k = \sum \alpha_i \psi_{k}^{(i)} \), where the so-called elementary separable potentials \( \psi_{k}^{(i)} \) can be explicitly constructed by the recursion formula [9]

\[
\psi^{(i)} = R^i \psi^{(0)}, \quad \psi^{(i)} = \left( \psi_{1}^{(i)}, \ldots, \psi_{n}^{(i)} \right)^T, \quad \psi^{(0)} = (0, \ldots, 0, -1)^T,
\]

where

\[
R = \begin{pmatrix}
-q_1 & 1 & 0 & 0 \\
\vdots & 0 & \ddots & 0 \\
\vdots & 0 & 0 & 1 \\
-q_n & 0 & 0 & 0
\end{pmatrix}, \quad R^{-1} = \begin{pmatrix}
0 & 0 & 0 & -\frac{1}{q_n} \\
1 & 0 & 0 & \vdots \\
0 & \ddots & 0 & \vdots \\
0 & 0 & 1 & -\frac{q_{n-1}}{q_n}
\end{pmatrix}.
\]

In Viète’s coordinates the metric \( G_0 \) for \( m = 0 \) has the form

\[
(G_0)^{ij} = \begin{cases}
2 & \text{if } i + j = n, \\
-2q_{i+j-n-1} & \text{if } i + j > n + 1, \\
0 & \text{otherwise},
\end{cases}
\]

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and the metrics for arbitrary \( m \) are given by \( G_m = R^n G_0 \) as in Viète’s coordinates the special conformal Killing tensor \( L \) has the matrix representation identical to \( R \) \([8]\). Moreover, the Killing tensors \( K_k \), for \( k = 1, \ldots, n \), are given by

\[
(K_k)^j_i = \begin{cases} 
q_{i-j+k-1} & \text{if } i \leq j \text{ and } k \leq j, \\
-q_{i-j+k-1} & \text{if } i > j \text{ and } k > j, \\
0 & \text{otherwise},
\end{cases}
\]

where, for convenience, we set \( q_0 = 1 \) and \( q_l = 0 \) for \( l < 0 \) or \( l > n \). Notice that \((K_1)^j_i = \delta^j_i\).

In Viète’s coordinates the \( u \) coefficient of the Lax matrix \((3.6)\) is simply given by

\[
u = \lambda n + n \sum_{k=1}^{n} q_k \lambda^{n-k},
\]

and by simple calculation, involving the change of coordinates for the metric \( G_m \), and observation that

\[
(\nu_1)^j_i = \left(\sum_{k=1}^{n} (G_m)^{kl} p_l \right) \lambda^{n-k} \equiv -\frac{1}{2} \tilde{u}.
\]

Finally \( w \) can be obtained from the formula \((3.7)\) or \((3.8)\).

4 Stäckel representations of KdV stationary systems

The first Stäckel representation of the \( n \)-th KdV stationary system is associated with the spectral curve \((2.29)\),

\[
\lambda^{2n+1} + c\lambda^n + \sum_{k=1}^{n} H_k \lambda^{n-k} = \mu^2,
\]

which is a special case of the general case \((3.1)\) with \( m = 0 \) and \( \sigma(\lambda) = \lambda^{2n+1} + c\lambda^n \). In this case, the Hamiltonians \( H_k \) are, in the Viète’s coordinates \((q, p)\), given by

\[
H_k = \frac{1}{2} p^T K_k G_0 p + \nu_k^{(2n+1)} + c\nu_k^{(n)}, \quad k = 1, \ldots, n.
\]

The second representation is associated with the spectral curve \((2.32)\),

\[
\lambda^{2n} + c\lambda^{-1} + \sum_{k=1}^{n} H_k \lambda^{-n-k} = \lambda^2 - \mu^2,
\]

which is a special case of \((3.1)\) with \( m = 1 \) and \( \sigma(\lambda) = \lambda^{2n} + c\lambda^{-1} \). In this case the Hamiltonians \( H_k \) are

\[
H_k = \frac{1}{2} p^T K_k G_1 p + \nu_k^{(2n)} + c\nu_k^{(-1)}, \quad k = 1, \ldots, n.
\]

The Lax representation \((3.5)\) of respective Hamiltonian flows \((3.14)\) are generated by the Lax operator \((3.6)\) or equivalently \((3.11)\) with the same, in both representations, \( Q \) term:

\[
Q \equiv \begin{bmatrix} w \\ u \end{bmatrix}_+ = \begin{bmatrix} \lambda^{2n+1} \\ u^2 \end{bmatrix}_{\lambda > 0} = \lambda - 2q_1.
\]

One obtains \((4.5)\) using the formula \((3.8)\) for \( w \) and observing that, in this particular cases, only the term with the highest degree in \( \sigma(\lambda) \) contributes to the form of \( Q \).

The equivalence between the appropriate Stäckel representations and the \( n \)-th stationary KdV system is apparent on the level of Lax equations \((2.20)\) and \((3.5)\) making the following identifications:

\[
L \equiv \nu_{n+1}, \quad U_k \equiv \nu_k, \quad k = 1, \ldots, n,
\]
and
\[ u \equiv P_{n+1}, \quad u_k \equiv P_k, \quad k = 1, \ldots, n. \] (4.6)

The transformation between the jet coordinates for the \( n \)-th stationary KdV system and Viète’s coordinates \((q, p)\) is given, through the KdV co-symmetries \( \gamma_i \), as

\[ q_i = \frac{1}{2} \gamma_i, \quad p_i = \frac{1}{2} \sum_{j=1}^{n} (G^{-1})_{ij} (\gamma_j)_x, \quad i = 1, \ldots, n, \quad (m = 0, 1), \] (4.7)

where we make the identification \( t_1 \equiv x \). The transformation (4.7) is a direct consequence of (4.6) and the first Hamiltonian flow (3.14) on \( q \); \( q = G_m p \). Notice that \( q_1 = \frac{1}{2} u \).

Summing up the obtained results, we get the first part of Theorem [1], the remaining part is included in Lemma [7].

**Example 5** The Stöckel representation of the first KdV stationary system for \( n = 2 \) is generated by the spectral curve

\[ \lambda^5 + c \lambda^2 + H_1 \lambda + H_2 = \mu^2. \]

The Hamiltonians (4.4) in Viète’s coordinates \((q, p)\) are

\[ H_1 = \frac{1}{2} p^T G_0 p + V_1^{(5)} + c V_1^{(2)} = 2p_1 p_2 + q_1 p_2^2 - q_1^2 - q_2^2 + 3q_1^2 q_2 + c q_1, \]
\[ H_2 = \frac{1}{2} p^T K_2 G_0 p + V_1^{(5)} + c V_1^{(2)} = p_1^2 + (q_1^2 - q_2) p_2^2 + 2q_1 p_1 p_2 - q_1^2 q_2 + 2q_1 q_2^2 + c q_2, \] (4.8)

where

\[ G_0 = 2 \begin{pmatrix} 0 & 1 \\ 1 & q_1 \end{pmatrix}, \quad K_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 1 \\ -q_2 & 1 \end{pmatrix}. \]

The related Lax operator (3.6) has the form

\[ L = \begin{pmatrix} -p_2 \lambda - p_1 - q_1 p_2 & \lambda^2 + q_1 \lambda + q_2 \\ \lambda^2 - q_1 \lambda^2 + (q_1^2 - q_2) \lambda - p_2^2 - q_1^2 + 2q_1 q_2 + c & p_2 \lambda + p_1 + q_1 p_2 \end{pmatrix}, \] (4.9)

and the generating matrices (3.3) are

\[ U_1 = \begin{pmatrix} 0 & 1 \\ \lambda - 2q_1 & 0 \end{pmatrix}, \quad U_2 = \begin{pmatrix} -p_2 & \lambda + q_1 \\ \lambda^2 - q_1 \lambda + q_1^2 - 2q_2 & p_2 \end{pmatrix}. \] (4.10)

The respective Hamiltonian flows can be now obtained directly from the Hamiltonian equations (3.14) or equivalently Lax equations (3.6). Thus, the first flow has the form

\[ \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix}_{t_1} = \begin{pmatrix} 2p_2 \\ 2p_1 + 2q_1 p_2 \\ -p_2^2 + 4q_1^2 - 6q_1 q_2 - c \\ 2q_2 - 3q_1^2 \end{pmatrix}, \] (4.11)

and the second one is given by

\[ \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix}_{t_2} = \begin{pmatrix} 2p_1 + 2q_1 p_2 \\ 2(q_1^2 - q_2) p_2 + 2q_1 p_1 \\ -2q_1 p_2^2 - 2p_1 p_2 + 3q_1^2 q_2 - 2q_1^2 \\ p_2^2 + q_1^2 - 4q_1^3 q_2 - c \end{pmatrix}. \] (4.12)

The transformation to the jet coordinates is given by (4.7), thus

\[ \begin{align*}
q_1 &= \frac{1}{2} \gamma_1 \equiv \frac{1}{2} u, \\
q_2 &= \frac{1}{2} \gamma_2 \equiv \frac{1}{8} u_{xx} + \frac{3}{8} u, \\
p_1 &= \frac{1}{4} (\gamma_2)_x - \frac{1}{4} (\gamma_1)_x \equiv \frac{1}{16} u_{3x} + \frac{1}{4} u_{ux}, \\
p_2 &= \frac{1}{4} (\gamma_1)_x \equiv \frac{1}{4} u_x.
\end{align*} \] (4.13)
Substituting (4.13) to the first (4.11) and the second flow (4.12) we obtain the equalities
\[ u_1 = (\gamma_1)_x, \quad u_2 = (\gamma_2)_x, \quad \gamma_3 + c = 0, \]
which constitute the first representation of the 2-th stationary KdV system (2.33). Substituting (4.13) to (4.9) and (4.10) one reconstructs the respective Lax matrices (2.13a) and (2.14). Substituting (4.13) to (4.8) one reconstructs the integrals of motion (2.35).

**Example 6** The St"ackel representation of the second KdV stationary system for \( n = 2 \) is generated by the spectral curve
\[ \lambda^3 + \bar{c}\lambda^{-1} + H_1\lambda + H_2 = \lambda\mu^2. \]

Then, in Vi"ete’s coordinates \((q, p)\) we find the following Hamiltonians
\[ H_1 = \frac{1}{2}p^TG_1p + V_1^{(4)} + \bar{c}V_1^{(-1)} = p_1^2 - q_2p_2^2 + q_1^2 - 2q_1q_2 + \bar{c}, \]
\[ H_2 = \frac{1}{2}p^TG_2p + V_2^{(4)} + \bar{c}V_2^{(-1)} = -2q_2p_1p_2 - q_1q_2p_2^2 + q_1^2q_2 - q_2^2 + \bar{c}q_1q_2, \]
where
\[ G_1 = 2 \begin{pmatrix} 1 & 0 \\ 0 & -q_2 \end{pmatrix} \]
and the Killing tensors are the same as in the previous example.

The related Lax operator and the generating matrices are given by
\[ L = \begin{pmatrix} \lambda^3 - q_1\lambda^2 + (q_1^2 - q_2)\lambda + (\bar{c} - q_2^2p_2)q_2^{-1} & \lambda^2 + q_1\lambda + q_2 \\ p_1\lambda - q_2p_2 \end{pmatrix} \]
and
\[ U_1 = \begin{pmatrix} 0 & 1 \\ \lambda - 2q_1 & 0 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 0 & -p_1 \\ \lambda^2 - q_1\lambda + q_1^2 - 2q_2 & p_1 \end{pmatrix}. \]

The respective Hamiltonian flows are
\[ \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix}_{t_1} = \begin{pmatrix} \frac{2p_1}{p_1^2 - 2q_2p_2} \\ \frac{-2q_2p_2}{p_1^2 - 2q_2p_2} + 3q_2^{-1} \\ \frac{-3q_2^{-1} + 2q_2}{p_1^2 + 2q_1 + \bar{c}q_2^{-2}} \end{pmatrix}, \]
\[ \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix}_{t_2} = \begin{pmatrix} -2q_2p_2 \\ -2q_1q_2p_2 - 2q_2p_1 + 3q_2^{-1} \\ 2p_1p_2 + q_1p_2^2 - q_2^2 + 2q_2 + \bar{c}q_1q_2^{-2} \end{pmatrix}. \]

The transformation to the jet coordinates (4.7) takes the form
\[ q_1 = \frac{1}{2}\gamma_1 \equiv \frac{1}{2}u, \quad q_2 = \frac{1}{2}\gamma_2 \equiv \frac{1}{8}u_{xx} + \frac{3}{8}u^2, \]
\[ p_1 = \frac{1}{4}(\gamma_1)_x \equiv \frac{1}{4}u_x, \quad p_2 = -\frac{1}{2}(\gamma_2)_x \equiv -\frac{1}{2}u_{xx} + 3u^2. \]

Substituting (4.19) to the first (4.17) and the second flow (4.18) we obtain the equalities
\[ u_{t_1} = (\gamma_1)_x, \quad u_{t_2} = (\gamma_2)_x, \quad \frac{1}{2}\gamma_2(\gamma_2)_x - \frac{1}{4}(\gamma_2)_x^2 + \bar{c} = 0, \]
which constitute the second representation of the 2-th stationary KdV system (2.36). Substituting (4.19) to (4.15), (4.16) and (4.14) one reconstructs the respective Lax matrices (2.13a), (2.37) and the integrals of motion (2.38).
We know that the Stäckel systems associated with the curves \([4.1]\) and \([4.3]\) are two representations of the same \(n\)-th stationary KdV system, thus one, in principle, supposed to be able to show equivalence between them. What is important, such an equivalence of both representations exists on an extended \((2n + 1)\)-dimensional phase spaces \(\mathcal{M} := M \otimes \mathbb{R}\).

Actually, the first representation is associated with the curve \([4.1]\) in the form
\[
\lambda^{2n+1} + \sum_{k=0}^{n} H_k \lambda^{n-k} = \mu^2,
\]
where we complete the Hamiltonians \([4.2]\), \(H_k = H_k(q, p, c)\), with an additional trivial integral of motion, \(H_0 := c\), all defined on the phase space \(\mathcal{M}\), parameterized by the extended Viète’s coordinates \((q, p, c)\).

The second case associated with the curve \([4.3]\) is now represented by
\[
\tilde{\lambda}^{2n+1} + \sum_{k=1}^{n+1} \tilde{H}_k \tilde{\lambda}^{n-k} = \tilde{\mu}^2,
\]
where the Hamiltonians \([4.24]\), \(\tilde{H}_k = \tilde{H}_k(Q, P, \tilde{c})\), are completed by an additional trivial integral of motion, \(\tilde{H}_{n+1} := \tilde{c}\), defined on the phase space \(\mathcal{M}\), parameterized by the extended Viète’s coordinates \((Q, P, \tilde{c})\).

Lemma 7 The evolution equations \([3.14]\) associated with the Stäckel systems defined by the curves \([4.20]\) and \([4.21]\) are related by the Miura map
\[
q_i = Q_i, \quad i = 1, \ldots, n
\]
\[
p_1 = -(Q_1 P_1 + Q_2 P_2 + \ldots + Q_n P_n), \quad p_i = P_{i-1}, \quad i = 2, \ldots, n,
\]
\[
c = \tilde{H}_1(Q, P, \tilde{c})
\]
and its inverse
\[
Q_i = q_i, \quad i = 1, \ldots, n
\]
\[
P_i = p_{i+1}, \quad i = 1, \ldots, n - 1, \quad P_n = -\frac{1}{q_n}(p_1 + q_1 p_2 + \ldots + q_{n-1} p_n),
\]
\[
\tilde{c} = H_n(q, p, c).
\]

Proof. It is simple to observe that the curves \([4.20]\) and \([4.21]\) are related by the relations
\[
\lambda = \tilde{\lambda}, \quad \mu = \tilde{\mu}
\]
and the following identification between Hamiltonians:
\[
c \equiv H_0(q, p, c) = \tilde{H}_1(Q, P, \tilde{c}),
\]
\[
H_i(q, p, c) = \tilde{H}_{i+1}(Q, P, \tilde{c}), \quad i = 1, \ldots, n - 1,
\]
\[
H_n(q, p, c) = \tilde{H}_{n+1}(Q, P, \tilde{c}) \equiv \tilde{c}.
\]

Notice that, the map \([4.24]\) translates immediately to the transformation between separation variables and so the Viète’s ones too, where the coordinates on the configuration space are preserved, hence \(q = Q\). Now, the connection between coordinates on the extended phase space is consequence of \([4.25]\) and the equivalence \(q = Q\). Thus, from \([3.14]\) for \(k = 1\) we have
\[
p = G_0^{-1} G_1 P \equiv R^T P \quad \iff \quad P = G_1^{-1} G_0 P \equiv (R^{-1})^T p,
\]
where \(R\), given by \([4.18]\), is the matrix representation, valid only in Viète’s coordinates, of the special conformal Killing tensor \(L\) \([8]\). Alternatively, the Miura map can be constructed through the respective changes of coordinates from the separation variables to Viète’s coordinates \([3.15]\) using the relations \([4.24]\).

The Miura map \([4.22]\), or equivalently its inverse \([4.23]\), represents the non-canonical transformation between two sets \((q, p, c)\) and \((Q, P, \tilde{c})\) of canonical coordinates on the extended phase space \(\mathcal{M}\). In consequence both Stäckel representations are bi-Hamiltonian on \(\mathcal{M}\) \([3]\) and the Miura map transforms the canonical Poisson structure of one Stäckel system into non-canonical Poisson structure of the second one.
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