On a Behavior of Trajectories of a Certain Family of Cubic Dynamic Systems in a Poincare Circle

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Abstract. We present the results of the investigation of a family of dynamic systems on a real plane containing reciprocal cubic and square polynomials in their right parts. A Poincare method of serial mappings has been used. All possible for the systems under consideration topologically different types of phase portraits in a Poincare circle have been described and constructed. The total amount of such portraits appears to be over 200. Coefficient criteria of their realization have been outlined.

1. Introduction
Dynamic systems are interesting for investigators in the numeric fields of science being mathematical models of a phenomenon (a process), where ”statistical events” (i.e. fluctuations) we may ignore and disregard. Global characteristics of an arbitrary dynamic system appear to be the follows: its initial state and a law of transforming it into a different state. A totality of all admissible states of a given dynamic system is called a phase space of it.

It’s naturally to distinguish dynamic systems with the discrete (for one hand) and with the continuous time (for another hand). Dynamic systems of the first (discrete) kind are called cascades, and their behavior can be described with a sequence of states of a system. Dynamic systems of the second kind, with continuous time, are called flows. For them a state of a system is defined on a real (or on an imaginary) axis for every given moment of time. Both abovementioned kinds of dynamic systems appear to be the main subject of study in the fields of the topological and the symbolic dynamics. Usually both kinds of dynamic systems, the cascades as well as the flows, can be described with an autonomous system of differential equations. Such a differential system is defined in a certain domain and satisfy the Cauchy theorem conditions (of existence and uniqueness of solutions of differential equations) in it.

Periodical solutions of differential equations correspond to closed phase curves of dynamic systems, and the same time singular points of differential equations correspond to equilibrium positions of dynamic systems. A mostly important task of the theory of dynamic systems is to study curves, which are defined by differential equations [3, 4]. For this purpose it is necessary to split a phase space of a system under consideration into trajectories and study a limit behavior of them: to find out and categorize equilibrium positions, to analyze repulsive and attracting manifolds (sources and sinks).

Notions of the basic importance in the theory of dynamic systems are the follows:
1) a notion of equilibrium states’ stability, i.e. an ability of a system to remain near an equilibrium state (or on a given manifold) for an arbitrary long time period under considerably small initial data changes, as well as

2) a notion of a system’s roughness (i.e. saving of properties under relatively small changes in a very model itself). A rough system qualitatively preserves its motion character despite of satisfactory small changes of its parameters.

Up to date some types of normal autonomous second-order differential systems with polynomial right parts were successfully studied. Among them, for example, are quadratic dynamic systems [2]; systems, which polynomial right parts contain nonzero linear terms; homogeneous cubic systems; dynamic systems with nonlinear homogeneous terms of the odd degrees, such as 3, 5, 7 [3], having a center or a focus in a singular point (0,0) [4]; some additional types of dynamic systems.

We consider a large family of systems on a real plane \( x, y \) of their phase variables

\[
\frac{dx}{dt} = X(x, y), \quad \frac{dy}{dt} = Y(x, y),
\]

for which \( X(x, y), Y(x, y) \) be reciprocal forms of \( x \) and \( y \), \( X \) be a cubic, and \( Y \) be a square form, \( X(0,1) > 0, Y(0,1) > 0 \). A purpose of the present work is to find out and construct in a Poincare circle all topologically admissible for the Eq.(1) – systems kinds of phase portraits, as well as to indicate criteria of portraits’ appearance. Authors successfully use a Henry Poincare’s method of consecutive mappings. At a first step we use a central mapping (from a center \((0, 0, 1)\) of a sphere \( \Sigma \) (augmented with a line at infinity) onto a sphere \( \Sigma : X^2 + Y^2 + Z^2 = 1 \) with identified diametrically opposite points; at the second step the orthogonal mapping of a lower enclosed semi sphere of a sphere \( \Sigma \) onto a circle \( \Omega : x^2 + y^2 \leq 1 \) with identified diametrically opposite points of its boundary \( \Gamma \) is used.

The sphere \( \Sigma \) is called in the scientific literature the Poincare sphere and the circle \( \Omega \) is called the Poincare circle [1].

2. Basic notation and definitions

\( \varphi(t,p), p = (x,y) \) – a fixed point := a solution (a motion) of an Eq.(1) – system with initial data \((0, p)\).

\( L_p : \varphi = \varphi(t,p), t \in I_{\text{max}}, \) – a trajectory of a motion \( \varphi(t,p) \).

\( L_p^{(-)} := (+(-)) \) – a semi trajectory of a trajectory \( L_p \).

\( O \) – curve of a system := its semi trajectory \( L_{p_0}(p \neq O, s \in \{+,-\}) \), adjoining to a point \( O \) under a condition that \( st \to +\infty \).

\( O^{(+)} \) – curve of a system := its \( O \) – curve \( L_p^{+(-)} \).

\( O^{(+)} \) – curve of a system := its \( O \) – curve, adjoining to a point \( O \) from a domain \( x > 0 \) \((x < 0)\).

\( TO \) – curve of a system := its \( O \) – curve, which, being supplemented by a point \( O \), touches some ray in it.

A nodal bundle of \( NO \)-curves of a system := an open continuous family of its \( TO \)-curves \( L_{p_0} \), where \( s \in \{+,-\} \) is a fixed index, \( p \in \Lambda \), \( \Lambda \) – a simple open arc, \( L_{p_0} = \{p\} \).

A saddle bundle of \( SO \)-curves of a system, a separatrix of the point \( O \) := a fixed \( TO \) – curve, which is not included into some bundle of \( NO \)-curves of a system.

\( E, H, P \) – \( O \)-sectors of a system: an elliptical, a hyperbolic, a parabolic ones.

A topological type \((T\text{-type})\) of a singular point \( O \) := a word \( A_0 \) which includes the letters \( N, S \) (a word \( B_0 \) which includes the letters \( E, H, P \) and describes a circular order of bundles \( N, S \) of its \( O \)-curves (of its \( O \)-sectors \( E, H, P \)) when traversing the point \( O \) in the “\(+\)”-direction (counterclockwise), starting with some of bundles.

\[
P(u) := X(1,u) \equiv p_0 + p_1 u + p_2 u^2 + p_3 u^3,
\]
\[ Q(u) := Y(1, u) \equiv a + bu + cu^2. \]

**Note 1.** ∀ Eq.(1) – system:
1) the polynomial’s \( Q(u) \) (polynomial’s \( P(u) \)) real roots appear to be actually angular coefficients of isoclines of a zero (isoclines of the infinity);
2) real roots of polynomials \( P(u), Q(u) \) we always number in an ascending order;
3) it’s important to know the both forms of a \( T \)-type of a singular point \( 0 \), and its form \( B_0 \) is easy to obtain using the form \( A_0 \) (and backwords, see the Corollary 1).

3. **Topological type \( (T – type) \) of a singular point \( O(0,0) \)**
A method of exceptional directions of a system in the point \( O \) is used for finding of all \( O \)-curves and splitting of their totality into bundles \( N, S \) [1]. As one can determine from this source, the equation of exceptional directions for the point \( O \) of the Eq.(1) – system is:

\[ xY(x, y) = x(ax^2 + bxy + cy^2) = 0. \]

This equation defines:
1) if \( d \equiv b^2 - 4ac > 0 \), then simple straight lines \( x = 0 \) and \( y = q_i x, i = 1, 2, q_1 < q_2 \),
2) if \( d = 0 \), the straight line \( x = 0 \) and the double straight line \( y = qx, q = -b/2c, 3) \) if \( d \geq 0 \) only a straight line \( x = 0 \).

**Theorem 1** A topological type of a singular point \( O(0,0) \) of the Eq.(1) – system is defined by the words \( A_0 \) and \( B_0 \), and they:
1) if \( d > 0 \) (depending on signs of values \( P(q_i), i = 1, 2 \)), have forms, indicated in a Table 1,
2) if \( d = 0 \) (depending on signs of values \( q \) and \( P(q) \)) – have forms, indicated in a Table 2,
3) if \( d \leq 0 \) : have the forms \( A_0 = \text{SOS}^0 \), \( B_0 = HH \) [5].

**Table 1.** A singular point \( O(0,0) \) topological types in the case of \( d > 0 \) \( (r = \overline{1, 6}) \).

| \( r \) | \( P(q_1) \) | \( P(q_2) \) | \( A_0 \) | \( B_0 \) |
|-------|------------|------------|-------|-------|
| 1, 4  | +          | +          | \( S_0S^1_+N^2_+S^0N^1S^2_+ = S_0S^1_+NS^2_+ \) | \( PH^2 \) |
| 2     | -          | -          | \( S_0N^2_+S^0S^1_+N^2 = NS^2_+S^0S^2_+ \) | \( PH^2 \) |
| 3, 6  | -          | +          | \( S_0N^1_+N^2_+S^0S^1_+S^2_+ \) | \( PEPH^3 \) |
| 5     | +          | -          | \( S_0S^0_+S^2_+S^0N^1N^2 = H^3PEP \) |

**Note 2.** New symbols appeared in the Theorem 1.
A symbol \( S_0(S^0) \) stands for a bundle \( S \), which is adjoining to the point \( O(0,0) \) from the domain \( x > 0 \) along a semi axis \( x = 0 \), \( y < 0, t \to +\infty \) (along a semi axis \( x = 0 \), \( y > 0, t \to -\infty \)).
A lower sign index "+" or "-" of every bundle \( N \) or \( S \), different from \( S_0 \) and \( S_0 \), shows wheather a bundle consists of \( O_+ \)-curves or of \( O_- \)-curves. Upper index 1 or 2 shows wheather its \( O \)-curves adjoin to the point \( O \) along a straight line \( y = q_1x \) or a line \( y = q_2x \). In the Table 2 (lines 5, 6) a bundle \( N \) includes both \( O_+ \) – curves and \( O_- \) – curves simultaneously, thus it doesn’t need any lower index.

**Corollary 1.** Theorem 1 allows us to infer: Eq.(1) – systems do not have limit cycles on the \( R^2_{xy} \) plane.
Table 2. A singular point $O(0,0)$ topological types in the case of $d = 0$.

| $q$ | $P(q)$ | $A_0$ | $B_0$ |
|-----|--------|-------|-------|
| +   | +      | $S_0S_+N_+S_-^0$ | $H^2P$ |
| -   | -      | $S_0N_+S_-^0$ | $PH^2$ |
| +   | -      | $S_0S_+S_-^0$ | $H^2P$ |
| -   | +      | $S_0S_-^0N_-$ | $PH^2$ |
| 0   | +      | $S_0S_-^0$ | $H^2P$ |
| 0   | -      | $NS_+N_+S_-^0N_-$ | $PH^2$ |

Proof A Poincare index of the singular point $O$ surrounded by a limit cycle must be equal to 1 [1]. The same time the Bendixon’s formula for an index of an isolated singular point of a smooth dynamic system is:

$$I(O) = 1 + \frac{e - h}{2},$$

where $e(h)$ stands for a number of elliptical (hyperbolic) $O$-sectors of the system. This formula and Theorem 1 mean: for the singular point $O(0,0)$ of every Eq.(1) – system Poincare index $I(O) = 0$, so this singular point isn’t surrounded with a limit cycle [1, 2].

**Corollary 2.** 11 different topological types are possible for a singular point $O(0,0)$ of an Eq.(1) – system, and for every Eq.(1) – system the singular point $O(0,0)$ has less than 5 separatrices (actually 2, 3 or 4 separatrices may appear).

4. Infinitely remote singular points (IR-points)

This item is devoted to a behavior of trajectories of the Eq.(1) – systems in a neighborhood of infinity. We use a method of Poincare consecutive transformations (serial mappings) [1].

The first Poincare transformation

$$x = \frac{1}{z}, \quad y = \frac{u}{z} \quad (u = \frac{y}{x}, \quad z = \frac{1}{x})$$

transforms Eq. (1)-system into another system; after a time change $dt = -z^2d\tau$ in the Poincare coordinates $u, z$ it has the form

$$\frac{du}{d\tau} = P(u)u - Q(u)z, \quad \frac{dz}{d\tau} = P(u)z,$$

$P(u) :\equiv X(1,u)$ and $Q(u) :\equiv Y(1,u)$ are reciprocal polynomials.

For the new system $z = 0$ is the invariant axis (it consists of system’s trajectories). On this axis singular points $O_i(u_i, 0), \ i = 0, m$, are situated, where $u_i, i = 1, m$ are all real roots of the polynomial $P(u)$, and $u_0 = 0$; the same time may exist $i_0 \in \{1, \ldots, m\} : u_{i_0} = 0$. Let’s call such points IR-points of the 1-st kind for the Eq.(1) – system.

The second Poincare transformation

$$x = \frac{v}{z}, \quad y = \frac{1}{z} \quad (v = \frac{x}{y}, \quad z = \frac{1}{y})$$
also an ambiguously maps a phase plane $R^2_{x,y}$ onto a Poincare sphere $\Sigma$ with the diametrically opposite points identified, which is considered without its equator, and every Eq. (1) – system transforms into a system, which in the coordinates $\tau, v, z$ has a form:

$$\frac{dv}{d\tau} = -X(\nu, 1) + Y(\nu, 1)\nu z, \quad \frac{dz}{d\tau} = Y(\nu, 1)z^2.$$ 

It is determined on the whole sphere $\Sigma$, and on the whole $(\nu, z)$ – plane $\hat{a}$, tangent to a sphere $\Sigma$ at a point $D = (0, 1, 0)$ [1].

**Theorem 2** If $u = 0$ is a multiplicity $k \in \{0, \ldots, 3\}$ root of a polynomial $P(u)$, then topological types of $IR$-points $O^\pm_0(0, 0)$, described with words $A^\pm_0$, have forms, indicated in the Table 3 (here $a$ and $pk$ are coefficients of the Eq. (1) – system).

| $k$ | $a_p k$ | $A^+_0$ | $A^-_0$ |
|-----|---------|---------|---------|
| 0, 2 | +(-) | $N_+(N_-)$ | $N_-(N_+)$ |
| 1, 3 | +(-) | $N_-N_+(\emptyset)$ | $\emptyset(N_-N_+)$ |

**Corollary 4.** Infinitely remote singular points $O^\pm_0$ of any Eq. (1) – system have no separatrices.

**Theorem 3** If $u_i(\neq 0)$ is a multiplicity $k_i \in \{1, 2, 3\}$ root of a polynomial $P(u)$, then for such an Eq. (1) – system a value $g_i = P(k_i)(u_i)Q(u_i) \neq 0$ and words $A^\pm_i$, which determine $T$-types of infinitely remote points $O^\pm_i(u_i, 0)$ of the system (depending on signs of numbers $u_i$ and $g_i$, and a value of $k_i$) have shown in the Table 4 forms [5].

| $u_i$ | $k_i$ | $g_i$ | $A^+_i$ | $A^-_i$ |
|-------|-------|-------|---------|---------|
| +(-)  | 1, 3  | +     | $N_+(N_-)$ | $-(S_+)$ |
| +(-)  | 1, 3  | -     | $S_-(S_+)$ | $N_+(N_-)$ |
| +(-)  | 2     | +     | $S_-N_+(\emptyset)$ | $\emptyset(N_-S_+)$ |
| +(-)  | 2     | -     | $\emptyset(SN - S_+)$ | $S_-N_+(\emptyset)$ |

**Corollary 5.** Theorems 2 and 3 show: for infinitely remote singular points of Eq. (1) – systems only a finite number – thirteen – of different topological types are possible; their further study allows to see: infinitely remote points of each Eq. (1) – system have only $m$ separatrices: one separatrice per each singular point $O_i(u_i, 0)$, $1, m$.

**Note 3.** For tables 3 and 4 a lower sign index “+” or “-” of every bundle $N$ or $S$ shows whether a bundle adjusts to a point $O^+_i(O^-_i)$ from the side $u > u_i$ or from the side $u < u_i$ of the isocline $u = u_i$. In the Table 3, line 1, a bundle $N$ doesn’t need a lower sign index, because it includes $O^+_i$-curves ($O^-_i$-curves) in every domain $-u_i < 0$ [5].
5. Systems containing different amounts of independent multipliers in their right parts

Further investigation includes alternate studies of those Eq.(1) – systems, which right parts have decompositions of polynomials \( X(x, y), Y(x, y) \) into real forms of lower degrees containing different amounts of independent multipliers: 3 and 2 multipliers; different situations with 2 and 2 multipliers; 3 and 1 multipliers; 2 and 1 etc. Specific investigation methods were especially developed for this work.

We describe the situation for one of these cases below as an example.

6. Systems with various combinations of 2 different multipliers in the both right parts of them

\[
X(x, y) = p(y - u_1 x)^{k_1}(y - u_2 x)^{k_2}, \quad Y(x, y) = q(y - q_1 x)(y - q_2 x),
\]

where \( p, q, u_1, u_2, q_1, q_2 \in R, \, p > 0, q > 0, u_1 < u_2, q_1 < 2, u(i) \neq q_j \) for each \( i, j \in \{1, 2\} \), \( k_1, k_2 \in N, \, k_1 + k_2 = 3 \).

We’ll consider the two classes of Eq.(2)-systems. Let the \( A \)-class to include systems with \( k_1 = 1, \, k_2 = 2 \), and the \( B \)-class to include systems with \( k_1 = 2, \, k_2 = 1 \).

Thus the \( A \)-class dynamic system looks like:

\[
\frac{dx}{dt} = p(y - u_1 x)^2(y - u_2 x)^2, \quad \frac{dy}{dt} = q(y - q_1 x)(y - q_2 x).
\]

For an arbitrary Eq.(3) – system we introduce the follows notions and some new concepts.

\[
P(u) := (X(1, u) = p(u - u_1)(u - u_2)^2, \quad Q(u) := Y(1, u) = q(u - q_1)(u - q_2),
\]

and \( RSP(RSQ) \) let be an ascending sequence of all real roots of polynomials \( P(u)(Q(u)) \); then \( RSPQ \) be an ascending sequence of all real roots of both polynomials \( P(u) \) and \( Q(u) \). We have six possible different variants of \( RSPQ \): \( C_2^4 = \frac{4!}{2!2!} = 6 \), and number them from 1 to 6 (in some order).

One useful notion: an \( r \)-family be a totality of Eq.(3) – dynamic systems with the RSPQ number \( r \) from the list of six available variants. A research of each fixed family of the Eq.(3) – systems needs the follows steps.

1) For every singular point of a given system we introduce notions of \( S \) (saddle) and \( N \) (node) bundles of semi trajectories, adjacent to a chosen singular point; a notion of its separatrix and a notion of its topodynamical (TD) type.

2) We divide an \( r \)-family into \( s \)-subfamilies, \( s \in \{1, \ldots, 5\} \). Then we find out TD-types of singular points and separatrices of singular points \( \forall s = \frac{1}{5} \).

3) Per each of 5 subfamilies we investigate behavior of separatrices and study a question of uniqueness of their global continuations from a small neighborhood of a singular point to all their lengths in the Poincare circle \( \Omega \), together with a question of their mutual arrangement in \( \Omega \).

The last is invariable in the case when for a given \( s \) a global continuation of each separatrix of any singular point of the \( s \)-subfamily is unique. That means: all systems of a chosen \( s \)-subfamily have one common phase portrait in a Poincare circle.

Oppositely, if for a fixed \( s \)-subfamily, for example, exist \( m \) separatrices with not unique global continuations, this subfamily must be divided into \( m \) additional subsubfamilies of the next order.

Their further study shows: for every subsubfamily a global continuation of each separatrix is unique, and their mutual arrangement in \( \Omega \) is invariable. So a topological type of a phase portrait of all systems belonging to a given subsubfamily in the \( \Omega \) Poincare circle is determined.
4) We describe (and draw) phase portraits in $\Omega$ for the Eq.(3)-systems, $r = 1, 6$, both in a table and in a graphic form, and outline per each one close to coefficient criteria of its existence [5, 6, 10].

A conclusion is: Eq.(3) – systems of the 1-family have in $\Omega$ 13 different topological types of phase portraits,

Eq.(3) – systems of the 2-family have 7 types,

the 3-family shows 10 types,

the 4, 5 and 6-families have 5 different types of phase portraits per each taken family number.

Totally all family of Eq.(3) – dynamic systems of the A-class shows us 45 different topological types of phase portraits in a Poincare circle [6, 7, 10].

7. Conclusions

The article is devoted to the original study; its main task is to find out and describe all topologically different phase portraits in a Poincare circle for the dynamic systems of some extended family with its numerical subfamilies. All phase portraits were constructed in two forms (a descriptive, or table, and in a graphic form). Every table includes 5 or 6 lines. Each line describes one invariant cell of the phase portrait -its boundary, as well as a source and a sink of its phase flow. Such a table we call a descriptive phase portrait.

Also in a process of study several new effective investigation methodics were developed and successfully applied [8, 9, 10].

It is a theoretical work, but due to above-mentioned new methods it may be useful for applied research also.

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