PRODUCT OF STATISTICAL MANIFOLDS WITH A NON-DIAGONAL METRIC

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Abstract. In this paper, we generalize the dualistic structures on warped product manifolds to the dualistic structures on generalized warped product manifolds. We develop an expression of curvature for the connection of the generalized warped product in relation to those corresponding analogues of its base and fiber and warping functions. We show that the dualistic structures on the base $M_1$ and the fiber $M_2$ induce a dualistic structure on the generalized warped product $M_1 \times M_2$ and conversely, moreover, $(M_1 \times M_1, G_{\tilde f_1 f_2})$ or $(M_1 \times M_1, \tilde g_{f_1 f_2})$ is statistical manifold if and only if $(M_1, g_1)$ and $(M_1, g_1)$ are. Finally, some interesting consequences are also given.

1. Introduction. The warped product provides a way to construct new pseudo-Riemannian manifolds from the given ones, see [8],[4] and [3]. This construction has useful applications in general relativity, in the study of cosmological models and black holes. It generalizes the direct product in the class of pseudo-Riemannian manifolds and is defined as follows. Let $(M_1, g_1)$ and $(M_2, g_2)$ be two pseudo-Riemannian manifolds and let $f_1 : M_1 \rightarrow \mathbb{R}^+$ be a positive smooth function on $M_1$, the warped product of $(M_1, g_1)$ and $(M_2, g_2)$ is the product manifold $M_1 \times M_2$ equipped with the metric tensor $g_{f_1} := \pi_1^* g_1 + (f_1 \circ \pi_1)^2 \pi_2^* g_2$, where $\pi_1$ and $\pi_2$ are the projections of $M_1 \times M_2$ onto $M_1$ and $M_2$ respectively. The manifold $M_1$ is called the base of $(M_1 \times M_2, g_{f_1})$ and $M_2$ is called the fiber. The function $f_1$ is called the warping function.

The doubly warped product is constructed in the class of pseudo-Riemannian manifolds generalizing the warped product and the direct product, it is obtained by homothetically distorting the geometry of each base $M_i \times \{q\}$ and each fiber $\{p\} \times M_2$ to get a new "doubly warped" metric tensor on the product manifold and defined as follows. For $i \in \{1, 2\}$, let $M_i$ be a pseudo-Riemannian manifold equipped with metric $g_i$, and $f_i : M_i \rightarrow \mathbb{R}^+$ be a positive smooth function on $M_i$. The well-known notion of doubly warped product manifold $M_i \times_{f_1 f_2} M_j$ is defined as the product manifold $M = M_i \times M_j$ equipped with pseudo-Riemannian metric which is denoted by $g_{f_1 f_2}$, given by

$$g_{f_1 f_2} = (f_2 \circ \pi_2)^2 \pi_1^* g_1 + (f_1 \circ \pi_1)^2 \pi_2^* g_2.$$

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The generalized warped product is defined as follows. Let $c$ be an arbitrary real number and let $g_i$, $(i = 1, 2)$ be a Riemannian metric tensors on $M_i$. Given a smooth positive function $f_i$ on $M_i$, the generalized warped product of $(M_1, g_1)$ and $(M_2, g_2)$ is the product manifold $M_1 \times M_2$ equipped with the metric tensor $G_{f_1,f_2}$ (see [6]), explicitly, given by

$$G_{f_1,f_2}(X,Y) = (f_2^i)^2 g_1^{\pi_1}(d\pi_1(X), d\pi_1(Y)) + (f_1^j)^2 g_2^{\pi_2}(d\pi_2(X), d\pi_2(Y))$$

$$+ c f_1^h f_2^v (X(f_1^h)Y(f_2^v) + X(f_2^v)Y(f_1^h)).$$

For all $X, Y \in \Gamma(TM_1 \times M_2)$. When the warping functions $f_1 = 1$ or $f_2 = 1$ or $c = 0$ we obtain a warped product or direct product.

Dualistic structures are closely related to statistical mathematics. They consist of pairs of affine connections on statistical manifolds, compatible with a pseudo-Riemannian metric [1]. Their importance in statistical physics was underlined by many authors: [5],[2] etc.

Let $M$ be a pseudo-Riemannian manifold equipped with a pseudo-Riemannian metric $g$ and let $\nabla$, $\nabla^*$ be the affine connections on $M$. We say that a pair of affine connections $\nabla$ and $\nabla^*$ are compatible (or conjugate ) with respect to $g$ if

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X^* Z) \text{ for all } X, Y, Z \in \Gamma(TM),$$

(1)

where $\Gamma(TM)$ is the set of all tangent vector fields on $M$. Then the triplet $(g, \nabla, \nabla^*)$ is called the dualistic structure on $M$.

We note that the notion of "conjugate connection " has been attributed to A.P. Norden in affine differential geometry literature (Simon, 2000) and was independently introduced by (Nagaoka and Amari, 1982) in information geometry, where it was called "dual connection" (Lauritzen, 1987). The triplet $(M, \nabla, g)$ is called a statistical manifold if it admits another torsion-free connection $\nabla^*$ satisfying the equation (1). We call $\nabla$ and $\nabla^*$ duals of each other with respect to $g$.

In the notions of terms on statistical manifolds, for a torsion-free affine connection $\nabla$ and a pseudo-Riemannian metric $g$ on a manifold $M$, the triple $(M, \nabla, g)$ is called a statistical manifold if $\nabla g$ is symmetric. If the curvature tensor $R$ of $\nabla$ vanishes, $(M, \nabla, g)$ is said to be flat.

This paper extends the study of dualistic structures on warped product manifolds, [9], to dualistic structures on generalized warped products in pseudo-Riemannian manifolds. We develop an expression of curvature for the connection of the generalized warped product in relation to those corresponding analogues of its base and fiber and warping functions.

The paper is organized as follows. In section 2, we collect the basic material about Levi-Civita connection, the notion of conjugate, horizontal and vertical lifts and the generalized warped products.

In section 3, we show that the projection of a dualistic structure defined on a generalized warped product space $(M_1 \times M_2, G_{f_1,f_2})$ induces dualistic structures on the base $(M_1, g_1)$ and the fiber $(M_2, g_2)$. Conversely, there exists a dualistic structure on the generalized warped product space induced by its base and fiber.

In section 4, we show that the projection of a dualistic structure defined on a generalized warped product space $(M_1 \times M_2, \tilde{g}_{f_1,f_2})$ induces dualistic structures on the
base \((M_1, g_1)\) and the fiber \((M_2, g_2)\). Conversely, there exists a dualistic structure on the generalized warped product space induced by its base and fiber and finally, some interesting consequences are also given.

2. Preliminaries.

2.1. Statistical manifolds. We recall some standard facts about Levi-Civita connections and the dual statistical manifold. Many fundamental definitions and results about dualistic structure can be found in Amari's monograph ([1],[2]).

Let \((M,g)\) be a pseudo-Riemannian manifold. The metric \(g\) defines the musical isomorphisms
\[
\sharp_g : T^*M \rightarrow TM \\
\alpha \mapsto \sharp_g(\alpha)
\]
such that \(g(\sharp_g(\alpha), Y) = \alpha(Y)\), and its inverse \(\flat_g\). We can thus define the cometric \(\tilde{g}\) of the metric \(g\) by:
\[
\tilde{g}(\alpha, \beta) = g(\sharp_g(\alpha), \sharp_g(\beta)).
\]
(2)

A fundamental theorem of pseudo-Riemannian geometry states that given a pseudo-Riemannian metric \(g\) on the tangent bundle \(TM\), there is a unique connection (among the class of torsion-free connection) that "preserves" the metric; as long as the following condition is satisfied:
\[
X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \text{ for } X, Y, Z \in \Gamma(TM)
\]
(3)

Such a connection, denoted as \(\hat{\nabla}\), is known as the Levi-Civita connection. Its component forms, called Christoffel symbols, are determined by the components of pseudo-metric tensor as ("Christoffel symbols of the second Kink")
\[
\hat{\Gamma}^k_{ij} = \sum_l \frac{1}{2} g^{kl} \left( \frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{ij}}{\partial x^l} - \frac{\partial g_{ij}}{\partial x^l} \right)
\]
and ("Christoffel symbols of the first Kink")
\[
\hat{\Gamma}_{ij,k} = \frac{1}{2} \left( \frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k} \right).
\]
The Levi-Civita connection is compatible with the pseudo metric, in the sense that it treats tangent vectors of the shortest curves on a manifold as being parallel.

It turns out that one can define a kind of "Compatibility" relation more generally than expressed by the (3), by introducing the notion of "Conjugate" (denoted by *) between two affine connections.

Let \((M, g)\) be a pseudo-Riemannian manifold and let \(\nabla, \nabla^*\) be an affine connections on \(M\). A connection \(\nabla^*\) is said to be "conjugate" to \(\nabla\) with respect to \(g\) if
\[
X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X^* Z) \text{ for } X, Y, Z \in \Gamma(TM)
\]
(4)

Clearly,
\[(\nabla^*)^* = \nabla^*\]
Otherwise, \(\nabla^*\), which satisfies the (3), is special in the sense that it is self-conjugate
\[(\nabla^*)^* = \nabla^*\]
Because pseudo-metric tensor \(g\) provides a one-to-one mapping between vectors in the tangent space and co-vectors in the cotangent space, the equation (1) can also
be seen as characterizing how co-vector fields are to be parallel-transported in order to preserve their dual pairing \(<.,.>\) with vector fields. Writing out the equation explicitly,

\[
\frac{\partial g_{ij}}{\partial x^k} = \Gamma_{ki,j} + \Gamma^*_{kj,i},
\]

where

\[
\nabla^* \partial_i \partial_j = \sum_l \Gamma^*_{ij,l} \partial_l
\]

so that

\[
\Gamma^*_{kj,i} = g(\nabla^*_\partial_j \partial_k, \partial_i) = \sum_l g_{il} \Gamma^*_{kj,l}.
\]

In the following, a manifold \(M\) with a pseudo-metric \(g\) and a pair of conjugate connections \(\nabla, \nabla^*\) with respect to \(g\) is called a ”pseudo-Riemannian manifold with dualistic structure” and denoted by \((M,g,\nabla, \nabla^*)\).

Obviously, \(\nabla\) and \(\nabla^*\) (or equivalently, \(\Gamma\) and \(\Gamma^*\)) satisfy the relation

\[
\hat{\nabla} = \frac{1}{2}(\nabla + \nabla^*)\] (or equivalently, \(\hat{\Gamma} = \frac{1}{2}(\Gamma + \Gamma^*)\)).

Thus an affine connection \(\nabla\) on \((M,g)\) is metric if and only if \(\nabla^* = \nabla\) (that it is self-conjugate).

For a torsion-free affine connection \(\nabla\) and a pseudo-Riemannian metric \(g\) on a manifold \(M\), the triplet \((M,\nabla, g)\) is called a statistical manifold if \(\nabla g\) is symmetric. If the curvature tensor \(R\) of \(\nabla\) vanishes, \((M,\nabla, g)\) is said to be flat.

For a statistical manifold \((M,\nabla, g)\), the conjugate connection \(\nabla^*\) with respect to \(g\) is torsion-free and \(\nabla^* g\) symmetric. Then the triplet \((M,\nabla^*, g)\) is called the dual statistical manifold of \((M,\nabla, g)\) and \((\nabla, \nabla^*, g)\) the dualistic structure on \(M\). The curvature tensor of \(\nabla\) vanishes if and only if that of \(\nabla^*\) does and in such a case, \((\nabla, \nabla^*, g)\) is called the dually flat structure \([2]\).

More generally, in information geometry, a one-parameter family of affine connections \(\nabla^{(\lambda)}\) indexed by \(\lambda\) (\(\lambda \in \mathbb{R}\)), called \(\lambda\)-connections, is introduced by Amari and Nagaoka in \([1],[2]\).

\[
\nabla^{(\lambda)} = \frac{1+\lambda}{2} \nabla + \frac{1-\lambda}{2} \nabla^*\] (or equivalently, \(\Gamma^{(\lambda)} = \frac{1+\lambda}{2} \Gamma + \frac{1-\lambda}{2} \Gamma^*\)).

(6)

Obviously, \(\nabla^{(0)} = \nabla\).

It can be shown that for a pair of conjugate connections \(\nabla, \nabla^*\), their curvature tensors \(R, R^*\) satisfy

\[
g(R(X,Y)Z,W) + g(Z,R^*(X,Y)W) = 0,\]

(7)

and more generally

\[
g(R^{(\lambda)}(X,Y)Z,W) + g(Z,R^{*\lambda}(X,Y)W) = 0.
\]

(8)

If the curvature tensor \(R\) of \(\nabla\) vanishes, \(\nabla\) is said to be flat.

So, \(\nabla\) is flat if and only if \(\nabla^*\) is flat. In this case, \((M,g,\nabla, \nabla^*)\) is said to be dually flat.

When \(\nabla, \nabla^*\) is dually flat, then \(\nabla^{(\lambda)}\) is called \(\lambda\)-transitively flat \([?]\). In such case, \((M,g,\nabla^{(\lambda)}, \nabla^{*\lambda})\) is called an ”\(\lambda\)-Hessian manifold”, or a manifold with \(\lambda\)-Hessian structure.
2.2. Horizontal and vertical lifts. Throughout this paper $M_1$ and $M_2$ will be respectively $m_1$ and $m_2$ dimensional manifolds, $M_1 \times M_2$ the product manifold with the natural product coordinate system and $\pi_1 : M_1 \times M_2 \rightarrow M_1$ and $\pi_2 : M_1 \times M_2 \rightarrow M_2$ the usual projection maps.

We recall briefly how the calculus on the product manifold $M_1 \times M_2$ derives from that of $M_1$ and $M_2$ separately. For details see [8].

Let $\phi_1$ in $C^\infty(M_1)$. The horizontal lift of $\phi_1$ to $M_1 \times M_2$ is $\phi^h_1 = \phi_1 \circ \pi_1$. One can define the horizontal lifts of tangent vectors as follows. Let $p_1 \in M_1$ and let $X_{p_1} \in T_{p_1}M_1$. For any $p_2 \in M_2$ the horizontal lift of $X_{p_1}$ to $(p_1, p_2)$ is the unique tangent vector $X^h_{(p_1, p_2)}$ in $T_{(p_1, p_2)}(M_1 \times M_2)$ such that $d_{(p_1, p_2)} \pi_1(X^h_{(p_1, p_2)}) = X_{p_1}$ and $d_{(p_1, p_2)} \pi_2(X^h_{(p_1, p_2)}) = 0$.

We can also define the horizontal lifts of vector fields as follows. Let $X_1 \in \Gamma(TM_1)$. The horizontal lift of $X_1$ to $M_1 \times M_2$ is the vector field $X^h_1 \in \Gamma(T(M_1 \times M_2))$ whose value at each $(p_1, p_2)$ is the horizontal lift of the tangent vector $(X_1)_{p_1}$ to $(p_1, p_2)$.

For $(p_1, p_2) \in M_1 \times M_2$, we will denote the set of the horizontal lifts to $(p_1, p_2)$ of all the tangent vectors of $M_1$ at $p_1$ by $\mathcal{L}(p_1, p_2)(M_1)$. We will denote the set of the horizontal lifts of all vector fields on $M_1$ by $\mathcal{L}(M_1)$.

The vertical lift $\phi^v_2$ of a function $\phi_2 \in C^\infty(M_2)$ to $M_1 \times M_2$ and the vertical lift $X^v_2$ of a vector field $X_2 \in \Gamma(TM_2)$ to $M_1 \times M_2$ are defined in the same way using the projection $\pi_2$. Note that the spaces $\mathcal{L}(M_1)$ of the horizontal lifts and $\mathcal{L}(M_2)$ of the vertical lifts are vector subspaces of $\Gamma(T(M_1 \times M_2))$ but neither is invariant under multiplication by arbitrary functions $\phi \in C^\infty(M_1 \times M_2)$.

Observe that if $\{ \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{m_1}} \}$ is the local basis of the vector fields (resp. $\{ dx_1, \ldots, dx_{m_1} \}$ is the local basis of 1-forms) relative to a chart $(U, \Phi)$ of $M_1$ and $\{ \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_{m_2}} \}$ is the local basis of the vector fields (resp. $\{ dy_1, \ldots, dy_{m_2} \}$ the local basis of the 1-forms) relative to a chart $(V, \Psi)$ of $M_2$, then $\{ (\frac{\partial}{\partial x_1})^h, \ldots, (\frac{\partial}{\partial x_{m_1}})^h \}$, $\{ (\frac{\partial}{\partial y_1})^v, \ldots, (\frac{\partial}{\partial y_{m_2}})^v \}$ is the local basis of the vector fields (resp. $\{ dx_1, \ldots, dx_{m_1} \}$, $(\frac{\partial}{\partial y_1})^v, \ldots, (\frac{\partial}{\partial y_{m_2}})^v \}$ is the local basis of the 1-forms) relative to the chart $(U \times V, \Phi \times \Psi)$ of $M_1 \times M_2$.

The following lemma will be useful later for our computations.

**Lemma 2.1.**

1. Let $\phi_i \in C^\infty(M_i)$, $X_i, Y_i \in \Gamma(TM_i)$ and $\alpha_i \in \Gamma(T^*(M_i))$, $i = 1, 2$. Let $\phi = \phi^h_1 + \phi^v_2$, $X = X^h_1 + X^v_2$ and $\alpha, \beta \in \Gamma(T^*(M_1 \times M_2))$. Then
   
   i/ For all $(i, I) \in \{(1, h), (2, v)\}$, we have
   
   $X_i^I(\phi) = X_i(\phi_i)^I$, $[X_i, Y_i] = [X_i, Y_i]^I$ and $\alpha_i^I(X) = \alpha(X_i)^I$.

   ii/ If for all $(i, I) \in \{(1, h), (2, v)\}$ we have $\alpha(X_i^I) = \beta(X_i^I)$, then $\alpha = \beta$.

2. Let $\omega_i$ and $\eta_i$ be $r$-forms on $M_i$, $i = 1, 2$. Let $\omega = \omega^h_1 + \omega^v_2$ and $\eta = \eta^h_1 + \eta^v_2$. We have
   
   $d\omega = (d\omega_1)^h + (d\omega_2)^v$ and $\omega \wedge \eta = (\omega_1 \wedge \eta_1)^h + (\omega_2 \wedge \eta_2)^v$.

**Proof.** See [7].

**Remark 1.** Let $X$ be a vector field on $M_1 \times M_2$, such that $d\pi_1(X) = \phi(X_1 \circ \pi_1)$ and $d\pi_2(X) = \phi(X_2 \circ \pi_2)$, then $X = \phi X^h_1 + \phi X^v_2$.\[\square\]
2.3. The generalized warped product. Let $\psi: M \to N$ be a smooth map between smooth manifolds and let $g$ be a metric on $k$-vector bundle $(F,P_F)$ over $N$. The metric $g^\psi: \Gamma(\psi^{-1}F) \times \Gamma(\psi^{-1}F) \to C^\infty(M)$ on the pull-back $(\psi^{-1}F,P_{\psi^{-1}F})$ over $M$ is defined by

$$g^\psi(U,V)(p) = g_p((U_p,V_p)), \quad \forall U,V \in \Gamma(\psi^{-1}F), \ p \in M.$$ 

Given a linear connection $\nabla^N$ on $k$-vector bundle $(F,P_F)$ over $N$, the pull-back connection $\tilde{\nabla}$ is the unique linear connection on the pull-back $(\psi^{-1}F,P_{\psi^{-1}F})$ over $M$ such that

$$\tilde{\nabla}_X(W \circ \psi) = \nabla^N_{\varphi(W)}W, \quad \forall W \in \Gamma(F), \ \forall X \in \Gamma(TM).$$  

Further, let $U \in \psi^{-1}F$ and let $p \in M$, $X \in \Gamma(TM)$. Then

$$\tilde{\nabla}_X(U)(p) = (\nabla^N_{\varphi(U)}(\psi(p))),$$

where $\tilde{U} \in \Gamma(F)$ with $\tilde{U} \circ \psi = U$.

Now, let $\pi_i$, $i=1,2$, be the usual projection of $M_1 \times M_2$ onto $M_i$, given a linear connection $\tilde{\nabla}$ on vector bundle $TM_i$, the pull-back connection $\tilde{\nabla}$ is the unique linear connection on the pull-back $M_1 \times M_2 \rightarrow \pi^{-1}_i(TM_i)$ such that for each $Y_i \in \Gamma(TM_i)$, $X \in \Gamma(TM_1 \times M_2)$

$$\tilde{\nabla}_X(Y_i \circ \pi_i) = \tilde{\nabla}_{\pi_i(X)}Y_i.$$  

Further, let $(p_1,p_2) \in M_1 \times M_2$, $U \in \psi^{-1}_i(TM)$ and $X \in \Gamma(TM_1 \times M_2)$. Then

$$\tilde{\nabla}_X(U)(p_1,p_2) = (\tilde{\nabla}_{\pi_i(X)}(\varphi(p_1,p_2))\tilde{U})(p_1).$$

Now, let $c$ be an arbitrary real number and let $g_i$, $(i = 1, 2)$ be a Riemannian metric tensors on $M_i$. Given a smooth positive function $f_i$ on $M_i$, the generalized warped product of $(M_1,g_1)$ and $(M_2,g_2)$ is the product manifold $M_1 \times M_2$ equipped with the metric tensor (see [6])

$$G_{f_1,f_2} = (f_2)^2 \pi_1^* g_1 + (f_1)^2 \pi_2^* g_2 + cf_1 f_2 df_1^h \otimes df_2^h,$$

Where $\pi_i$, $(i = 1, 2)$ is the projection of $M_1 \times M_2$ onto $M_i$ and

$$df_i^h = df_i^h \otimes df_i^h + df_i^h \otimes df_i^h.$$

For all $X,Y \in \Gamma(TM_1 \times M_2)$, we have

$$G_{f_1,f_2}(X,Y) = (f_2)^2 g_1^{\pi_1}(d\pi_1(X),d\pi_1(Y)) + (f_1)^2 g_2^{\pi_2}(d\pi_2(X),d\pi_2(Y)) + cf_1 f_2 (X(f_1^h)Y(f_2^h) + X(f_2^h)Y(f_1^h)).$$

It is the unique tensor fields such that for any $X_i,Y_i \in \Gamma(TM_i)$, $(i = 1, 2)$

$$g_{f_1,f_2}(X_i,Y_k^I) = \begin{cases} (f_2^{I-1})^2 g_i(X_i,Y_i), & \text{if } (i,I) = (k,K) \\ cf_1 f_k^K X_i(f_i)^I Y_k(f_k)^K, & \text{otherwise} \end{cases}$$

If either $f_1 \equiv 1$ or $f_2 \equiv 1$ but not both, then we obtain a singly warped product. If both $f_1 \equiv 1$ and $f_2 \equiv 1$, then we have a product manifold. If neither $f_1$ nor $f_2$ is constant and $c = 0$, then we have a nontrivial doubly warped product. If neither $f_1$ nor $f_2$ is constant and $c \neq 0$, then we have a nontrivial generalized warped product.

Now, Let us assume that $(M_i,g_i)$, $(i = 1, 2)$ is a smooth connected Riemannian manifold. The following proposition provides a necessary and sufficient condition
for a symmetric tensor field $G_{f_1, f_2}$ of type $0, 2$ of two Riemannian metrics to be a Riemannian metric.

**Proposition 1.** [6] Let $(M_i, g_i), (i = 1, 2)$ be a Riemannian manifold and let $f_i$ be a positive smooth function on $M_i$ and $c$ be an arbitrary real number. Then the symmetric tensor field $G_{f_1, f_2}$ is Riemannian metric on $M_1 \times M_2$ if and only if

$$0 \leq c^2 g_1(\text{grad}f_1, \text{grad}f_1)h g_2(\text{grad}f_2, \text{grad}f_2)v < 1.$$  

(14)

**Corollary 1.** [6] If the symmetric tensor field $G_{f_1, f_2}$ of type $(0, 2)$ on $M_1 \times M_2$ is degenerate, then for any $i \in \{1, 2\}$, $g_i(\text{grad}f_i, \text{grad}f_i)$ is positive constant $k_i$ with

$$k_i = \frac{1}{c^2k_{3-i}}. $$

In all what follows, we suppose that $f_1$ and $f_2$ satisfies the inequality (14).

**Lemma 2.2.** [6] Let $X$ be an arbitrary vector field of $M_1 \times M_2$, if there exist $\varphi_i, \psi_i \in C^{\infty}(M_i)$ and $X_i, Y_i \in \Gamma(TM_i), (i = 1, 2)$ such that

$$
\begin{cases}
G_{f_1, f_2}(X, Z_i^h) = G_{f_1, f_2}(\varphi_i X_i^h + \varphi_i X_i^h, Z_1^h), & \forall Z_i \in \Gamma(TM_i), \\
G_{f_1, f_2}(X, Z_2^h) = h^h G_{f_1, f_2}(\psi_i Y_i^h + \psi_i Y_i^h, Z_2^h), & \forall Z_i \in \Gamma(TM_i), \\
\end{cases}
$$

Then we have,

$$
X = \varphi_i X_i^h + \psi_i Y_i^h + cf_i f_2^v \{\psi_i Y_i^h(f_i^h) - \varphi_i X_i^h(f_i^h)\} \text{grad}(f_2^v)
$$

$$
- cf_i f_2^v \{\psi_i Y_i^h(f_2^v) - \varphi_i X_i^h(f_2^v)\} \text{grad}(f_1^h)
$$

(15)

**3. Dualistic structure with respect to $G_{f_1, f_2}$.**

**Proposition 2.** Let $(g_{f_1, f_2}, \nabla, \nabla^*)$ be a dualistic structure on $M_i \times M_2$. Then there exists an affine connections $\nabla, \nabla^*$ on $M_i$, such that $(g_i, \nabla, \nabla^*)$ is a dualistic structure on $M_i$ $(i = 1, 2)$.

**Proof.** Taking the affine connections on $M_i, (i = 1, 2)$.

$$
\begin{cases}
(\nabla X_i^L Y_i^L, Z_i^L) = G_{f_1, f_2}(\nabla X_i^L Y_i^L, Z_i^L), & \forall X_i, Y_i, Z_i \in \Gamma(TM_i) \\
(\nabla X_i^L Y_i^L) = G_{f_1, f_2}(\nabla X_i^L Y_i^L, Z_i^L), & \forall X_i, Y_i, Z_i \in \Gamma(TM_i) \\
(\nabla X_i^L Y_i^L) = G_{f_1, f_2}(\nabla X_i^L Y_i^L, Z_i^L), & \forall X_i, Y_i, Z_i \in \Gamma(TM_i) \\
(\nabla X_i^L Y_i^L) = G_{f_1, f_2}(\nabla X_i^L Y_i^L, Z_i^L), & \forall X_i, Y_i, Z_i \in \Gamma(TM_i)
\end{cases}
$$

(16)

Since, $d\pi(L_i^L) = 0, Z_i^L(f_{3-i}^L) = 0$ and for any $X \in \Gamma(TM_1 \times M_2), g_{f_1, f_2}(X, Z_i^L) = (f_{3-i}^L)^2 g_{f_i, f_i}(\text{grad}(X), Z_i) + cf_i f_2^v X(f_{3-i}^L) Z_i(f_i)^L, \text{then the equation (23)}$ is equivalent to

$$
(f_{3-i}^L)^2(X_i(g_i(Y_i, Z_i)))^L = (f_{3-i}^L)^2 \{g_i(\nabla X_i Y_i, Z_i) + g_i(Y_i, \nabla X_i Z_i)\}^L.
$$

Where $(i, I), (3-i, J) \in \{(1, h), (2, v)\}$. Hence, the pair of affine connections $\nabla$ and $\nabla^*$ are conjugate with respect to $g_i$.  

$\square$
Proposition 3. Let \((g_1, \tilde{\nabla}, \tilde{\nabla}')\) be a dualistic structure on \(M_i\) \((i = 1, 2)\). Then there exists a dualistic structure on \(M_1 \times M_2\) with respect to \(G_{f_1, f_2}\).

Proof. Let \(\nabla\) and \(\nabla^*\) be the connections on \(M_1 \times M_2\) given by

\[
\begin{align*}
d\pi_1(\nabla_X Y) &= \nabla_X d\pi_1(Y) + Y(\ln f_1^i) d\pi_1(X) + X(\ln f_1^i) d\pi_1(Y) \\
&+ \frac{1}{f_1^i f_2^j(1 - f_2^j f_2^j)} \left\{ \left( \frac{f_2^j}{f_2^j} \right)^2 B_{ij}^*(X, Y) - c b_i^1 f_1^j B_{ij}^*(X, Y) \right\} \{ \text{grad} f_i \} \circ \pi_1, \\
&- c f_i^1 (1 - c b_i^j) [X(f_i^1)Y(f_i^j) + X(f_i^j)Y(f_i^1)] \{ \text{grad} f_i \} \circ \pi_1, \\
d\pi_2(\nabla_X Y) &= \nabla_X d\pi_2(Y) + Y(\ln f_1^i) d\pi_2(X) + X(\ln f_1^i) d\pi_2(Y) \\
&+ \frac{1}{f_1^i f_2^j(1 - f_2^j f_2^j)} \left\{ \left( \frac{f_2^j}{f_2^j} \right)^2 B_{ij}^*(X, Y) - c b_i^1 f_1^j B_{ij}^*(X, Y) \right\} \{ \text{grad} f_i \} \circ \pi_2, \\
&- c f_i^2 (1 - c b_i^j) [X(f_i^1)Y(f_i^2) + X(f_i^2)Y(f_i^1)] \{ \text{grad} f_i \} \circ \pi_2,
\end{align*}
\]

(17)

for any \(X, Y \in \Gamma(TM_1 \times M_2)\). Where \(B_{ij}^*\) and \(B_{ij}^*\) \((i, j = 1, 2)\) the \((0, 2)\) tensors fields of \(f_1^i\) given respectively by

\[
\begin{align*}
B_{ij}^*(X, Y) &= \frac{1}{c f_i^j} \left\{ X(Y(f_i^j)) - g_i^j \left( \tilde{\nabla}_X d\pi_1(Y), (\text{grad} f_i) \circ \pi_1 \right) \right\} \\
&+ c X(f_i^1)Y(f_i^j) - \frac{1}{f_1^j} g_i^j (d\pi_1(X), d\pi_1(Y)),
\end{align*}
\]

and

\[
\begin{align*}
B_{ij}^*(X, Y) &= \frac{1}{c f_i^j} \left\{ X(Y(f_i^j)) - g_i^j \left( \tilde{\nabla}_X d\pi_1(Y), (\text{grad} f_i) \circ \pi_1 \right) \right\} \\
&+ c X(f_i^1)Y(f_i^j) - \frac{1}{f_1^j} g_i^j (d\pi_1(X), d\pi_1(Y)),
\end{align*}
\]

\(j = i - 3\) and \((i, I), (j, J) \in \{(1, h), (2, v)\}\).

Or, for any \(X_i, Y_i \in \Gamma(TM_i)\) \((i = 1, 2)\)

\[
\begin{align*}
\nabla_{X_1}Y_1^h &= \left( \tilde{\nabla}_{X_1}Y_1^h \right) + f_2^j B_{ij}^*(X_1, Y_1^h) \text{grad}(f_2^j); \\
\nabla_{X_2}Y_2^v &= \left( \tilde{\nabla}_{X_2}Y_2^v \right) + f_2^j B_{ij}^*(X_2, Y_2^v) \text{grad}(f_2^j); \\
\nabla_{X_1}Y_1^h &= \left( \tilde{\nabla}_{X_1}Y_1^h \right) + f_2^j B_{ij}^*(X_1, Y_1^h) \text{grad}(f_2^j); \\
\nabla_{X_2}Y_2^v &= \left( \tilde{\nabla}_{X_2}Y_2^v \right) + f_2^j B_{ij}^*(X_2, Y_2^v) \text{grad}(f_2^j); \\
\nabla_{X_1}Y_2^v &= \nabla_{X_1}Y_2^v = -c X_1(f_1^h)Y_2^v(f_2^v) \left\{ f_2^j \text{grad}(f_2^j) + f_2^j \text{grad}(f_2^j) \right\} \\
&+ \left( Y_2(f_2^v) \right)^{f_2^v} X_1^{f_1^h} + (X_1(f_2^v))^{f_2^v} Y_2^{f_2^v} \\
\nabla_{X_2}Y_1^h &= \nabla_{X_2}Y_1^h = \nabla_{X_2}Y_2^v.
\end{align*}
\]

(18)
Where $B_i$ and $B_i^*$ ($i = 1, 2$) the $(0, 2)$ tensors fields of $f_i$ given respectively by

$$B_i(X_i, Y_i) = c f_i \left\{ X_i(Y_i(f_i)) - \hat{\nabla}_X Y_i(f_i) \right\} + c X_i(f_i) Y_i(f_i) - g_i(X_i, Y_i),$$

and

$$B_i^*(X_i, Y_i) = c f_i \left\{ X_i(Y_i(f_i)) - \hat{\nabla}_X Y_i(f_i) \right\} + c X_i(f_i) Y_i(f_i) - g_i(X_i, Y_i),$$

Let us assume that $(g_i, \hat{\nabla}, \hat{\nabla}^*)$ is a dualistic structures on $M_i$, $i = 1, 2$. Let $A$ be a tensor field of type $(0, 3)$ defined for any $X, Y, Z \in \Gamma(TM_1 \times M_2)$ by

$$A(X, Y, Z) = X(G_{f_1 f_2}(Y, Z)) - G_{f_1 f_2}(\nabla X Y, Z) - G_{f_1 f_2}(Y, \nabla^*_X Z),$$

if $X_i, Y_i, Z_i \in \Gamma(TM_i)$, $i = 1, 2$, then we have

$$X_i^j(G_{f_1 f_2}(Y_i^j, Z_i^j)) = X_i^j((f_{3-i})^2 g_i(X_i, Y_i)^j).$$

Since $d\pi_{3-i}(X_i^j) = 0$, it follows that $d\pi_{3-i}(X_i^j)(f_{3-i}) = X_i^j(f_{3-i}) = 0$, and hence

$$X_i^j(G_{f_1 f_2}(Y_i^j, Z_i^j)) = (f_{3-i})^2(X(g_i(Y_i, Z_i))^j),$$

as $(g_i, \hat{\nabla}, \hat{\nabla}^*)$ is dualistic structure, we have thus

$$X_i^j(G_{f_1 f_2}(Y_i^j, Z_i^j)) = (f_{3-i})^2\{g_i(\hat{\nabla}_X Y_i, Z_i)^j + g_i(Y, \hat{\nabla}_X Z_i)^j\},$$

from Equations (13), (18), then it’s easily seen that the following equation holds

$$A(X_i^j, Y_i^j, Z_i^j) = 0$$

In the different lifts ($i \neq j$), we have

$$X_i^j(G_{f_1 f_2}(Y_i^j, Z_j^j)) = c f_j^j(Z_j(f_j))^j X_i^j((f_i(Y_i(f_i))))^j,$$

$$G_{f_1 f_2}(\nabla X_i^j Y_i^j, Z_j^j) = f_j^j\{c f_i X_i(Y_i(f_i)) + c X_i(f_i) Y_i(f_i) - g_i(X_i, Y_i)^j\} Z_j(f_j)^j,$$

and

$$G_{f_1 f_2}(\nabla X_i^j Z_j^j, Y_i^j) = f_j^j g_i(X_i, Y_i)^j Z_j(f_j)^j.$$

We add these equations and obtain

$$A(X_i^j, Y_i^j, Z_i^j) = 0$$

Hence the same applies for $A(X_j^j, Y_j^j, Z_i^j) = A(X_i^j, Y_j^j, Z_i^j) = 0$.

This proves that $\nabla^*$ is conjugate to $\nabla$ with respect to $G_{f_1 f_2}$.

We recall that the connection $\nabla$ on $M_1 \times M_2$ induced by $\hat{\nabla}$ and $\hat{\nabla}^*$ on $M_i$ and $M_2$ respectively, is given by Equation (18).

**Proposition 4.** $(M_1, \hat{\nabla}, g_1)$ and $(M_2, \hat{\nabla}, g_2)$ are statistical manifolds if and only if $(M_1 \times M_2, G_{f_1 f_2}, \nabla)$ is a statistical manifold.

**Proof.** Let us assume that $(M_i, \hat{\nabla}, g_i)$ $(i = 1, 2)$ is statistical manifold.

Firstly, we show that $\nabla$ is torsion-free. Indeed; by Equation (17), we have for any $X, Y \in \Gamma(TM_1 \times M_2)$

$$d\pi_i(T(X, Y)) = \hat{\nabla}_X d\pi_i(Y) - \hat{\nabla}_Y d\pi_i(X) - d\pi_i([X, Y])$$

where $\hat{\nabla}_X$ and $\hat{\nabla}_Y$ are connections on $M_i$ induced by $\nabla$.
Since for \( i = 1, 2 \), \( \nabla \) is torsion-free, then

\[
\nabla_X d\pi_i(Y) - \nabla_Y d\pi_i(X) = d\pi_i([X, Y])
\]

Therefore, from Remark 1, the connection \( \nabla \) is torsion-free.

Secondly, we show that \( \nabla G_{f_1, f_2} \) is symmetric. In fact; for \( i = 1, 2 \),

\[
(\nabla G_{f_1, f_2})(X_i^f, Y_i^f, Z_i^f) = X_i^f(G_{f_1, f_2}(Y_i^f, Z_i^f)) - G_{f_1, f_2}(\nabla X_i^f Y_i^f, Z_i^f) - G_{f_1, f_2}(Y_i^f, \nabla X_i^f Z_i^f)
\]

by Equations (13), (18) and since \( (\nabla g_i) \), \( i = 1, 2 \), is symmetric, we have

\[
(\nabla G_{f_1, f_2})(X_i^f, Y_i^f, Z_i^f) = (f_{ij}^l)^2((\nabla g_i)(X_i, Y_i, Z_i))^l
\]

\[
= (f_{ij}^l)^2((\nabla g_i)(Y_i, X_i, Z_i))^h
\]

\[
= (\nabla G_{f_1, f_2})(Y_i^f, X_i^f, Z_i^f).
\]

In the different lifts, we have

\[
(\nabla G_{f_1, f_2})(X_i^f, Y_i^f, Z_i^f) = (\nabla G_{f_1, f_2})(X_{i-1}^f, Y_i^f, Z_i^f) = (\nabla G_{f_1, f_2})(X_i^f, Y_{i-1}^f, Z_i^f) = 0,
\]

Therefore, \( (\nabla G_{f_1, f_2}) \) is symmetric. Thus \( (M_1 \times M_2, g_{f_1, f_2}, \nabla) \) is a statistical manifold.

Conversely, if \( (M_1 \times M_2, G_{f_1, f_2}, \nabla) \) is a statistical manifold, then \( (\nabla G_{f_1, f_2}) \) is symmetric and \( \nabla \) is torsion-free, particularly, when \( X_i, Y_i, Z_i \in \Gamma(TM_i) \), we have

\[
\left\{
\begin{array}{l}
(\nabla G_{f_1, f_2})(X_i^f, Y_i^f, Z_i^f) = (\nabla G_{f_1, f_2})(Y_i^f, X_i^f, Z_i^f), \\
T(X_i^f, Y_i^f) = 0.
\end{array}
\right.
\]

Then, by Equations (13) and (18), we obtained, for \( i = 1, 2 \), \( \nabla g_i \), is symmetric and \( \nabla \), is torsion-free. Therefore, \( (M_i, \nabla, g_i), i = 1, 2 \), is statistical manifold. \( \square \)

4. Dualistic structure with respect to \( \tilde{g}_{f_1, f_2} \). Let \( c \) be an arbitrary real number and let \( g_i \) (\( i = 1, 2 \)) be a Riemannian metric tensors on \( M_i \). Given a smooth positive function \( f_i \) on \( M_i \), we define a metric tensor field on \( M_1 \times M_2 \) by

\[
\tilde{g}_{f_1, f_2} = \pi_i^* g_i + (f_i^h)^2 \pi_2^* g_2 + \frac{c^2}{2} (f_i^h)^2 df_1^h \odot df_1^h.
\]

(19)

Where \( \pi_i \) (\( i = 1, 2 \)) is the projection of \( M_1 \times M_2 \) onto \( M_i \) (see [6]).

For all \( X, Y \in \Gamma(TM_1 \times M_2) \), we have

\[
\tilde{g}_{f_1, f_2}(X, Y) = g_i^\pi_i(d\pi_1(X), d\pi_1(Y)) + (f_i^h)^2 g_2^\pi_2(d\pi_2(X), d\pi_2(Y)) + (cf_i^h)^2 X(f_i^h)Y(f_i^h).
\]

It is the unique tensor fields such that for any \( X_i, Y_i \in \Gamma(TM_i) \), \( i = 1, 2 \)

\[
\left\{
\begin{array}{l}
\tilde{g}_{f_1, f_2}(X_i^h, Y_i^h) = g_i(X_i, Y_i)^h + (cf_i^h)^2 X_1(f_i)Y_1(f_i)^h, \\
\tilde{g}_{f_1, f_2}(X_i^h, Y_i^* ) = \tilde{g}_{f_1, f_2}(Y_i^*, X_i^h) = 0, \\
\tilde{g}_{f_1, f_2}(X_i^*, Y_i^* ) = (f_i^h)^2 g_2(X_2, Y_2)^e.
\end{array}
\right.
\]

(20)

Proposition 5. Let \( (\tilde{g}_{f_1, f_2}, \nabla, \nabla^*) \) be a dualistic structure on \( M_1 \times M_2 \). Then there exists an affine connections \( \nabla, \nabla^* \) on \( M_i \), such that \( (\tilde{g}_i, \nabla, \nabla^*) \) is a dualistic structure on \( M_i \) (\( i = 1, 2 \)).
Proof. Taking the affine connections on \( M_i \), \( i = 1, 2 \).

\[
\begin{cases}
\hat{\nabla}_X Y_1 \circ \pi_1 = d\pi_1(\hat{\nabla}_X Y_1^h) + (cf_2^\flat)^2 H^{f_1}_1(X^h, Y_1^h)(\text{grad}^1 f_1) \circ \pi_1, \\
\hat{\nabla}_X Y_1 \circ \pi_2 = d\pi_1(\hat{\nabla}_X Y_1^h) + (cf_2^\flat)^2 H^{f_1}_2(X^h, Y_1^h)(\text{grad}^1 f_1) \circ \pi_2,
\end{cases}
\]

\[
\begin{cases}
\hat{\nabla}_X Y_2 \circ \pi_1 = \frac{1}{(f'_1)^2} d\pi_2(\hat{\nabla}_X Y_2^h) \\
\hat{\nabla}_X Y_2 \circ \pi_2 = \frac{1}{(f'_2)^2} d\pi_2(\hat{\nabla}_X Y_2^h).
\end{cases}
\]

Therefore, we have for all \( X_1, Y_1, Z_i \in \Gamma(TM_i) \) \( i = 1, 2 \).

\[
X_i^f(\hat{g}_{f_1 f_2}(Y_i^1, Z_1^f)) = \hat{g}_{f_1 f_2}(\hat{\nabla}_{X^f_i} Y_i^1, Z_1^f) + \hat{g}_{f_1 f_2}(Y_i^1, \hat{\nabla}_{X^f_i} Z_1^f).
\]

Since, \( d\pi_i(Z_i^f) = 0 \), \( X_i^f(f^i_{j-1}) = 0 \) and for any \( X \in \Gamma(TM_1 \times M_2) \),

\[
\hat{g}_{f_1 f_2}(X, Z_i^f) = \begin{cases}
g_i^{\hat{h}_1}(d\pi_1(X), Z_1 \circ \pi_1) + (cf_2^\flat)^2 X(f_1^h)Z_1(\text{grad}^1 f_1), & \text{if } (i, I) = (1, h) \\
(f_1^h)^2 g_2^{\hat{h}_2}(d\pi_2(X), Z_2 \circ \pi_2), & \text{if } (i, I) = (2, v)
\end{cases}
\]

Substituting from Equations (21) and (22) into Formula (23) gives

\[
\begin{align*}
(X_1(g_1(Y_1, Z_1)))^h & = g_i^{\hat{h}_1}(\hat{\nabla}_{X^f_1} Y_1^1, Z_1 \circ \pi_1) + g_i^{\hat{h}_1}(\hat{\nabla}_{X^f_1} Y_1^1, Y_1 \circ \pi_1), \\
(f_1^h)^2 (X_2(g_2(Y_2, Z_2)))^v & = (f_1^h)^2 \left\{ g_2^{\hat{h}_2}(\hat{\nabla}_{X^f_2} Y_2^1, Z_2 \circ \pi_2) + g_2^{\hat{h}_2}(\hat{\nabla}_{X^f_2} Y_2^1, Y_2 \circ \pi_2) \right\}.
\end{align*}
\]

Hence, the pair of affine connections \( \hat{\nabla} \) and \( \hat{\nabla}^* \) are conjugate with respect to \( g_i \).

\[\square\]

**Proposition 6.** Let \((g_i, \hat{\nabla}, \hat{\nabla}^*)\) be a dualistic structure on \( M_i \) \( i = 1, 2 \). Then there exists a dualistic structure on \( M_1 \times M_2 \) with respect to \( \hat{g}_{f_1 f_2} \).

**Proof.** Let \( \nabla \) and \( \nabla^* \) be the connections on \( M_1 \times M_2 \) given by

\[
\begin{cases}
\nabla_{X^1_1} Y_1^h = (\hat{\nabla}_{X^1_1} Y_1^1)^h + \frac{(cf_2^\flat)^2 H^{f_1}_1(X^h_1, Y_1^h)}{1+(cf_2^\flat)^2 b_1^h}(\text{grad}^1 f_1)^h \\
- c^2 f_2^\flat(X_1(\ln f_1)Y_1(\ln f_1))^h(\text{grad}^2 f_2)^v,
\end{cases}
\]

\[
\begin{cases}
\nabla_{X^2_1} Y_2^h = (\hat{\nabla}_{X^2_1} Y_2^1)^h - \frac{H^{f_1}_2(X_2, Y_2)^v}{1+(cf_2^\flat)^2 b_1^h}(\text{grad}^1 f_1)^h, \\
\nabla_{X^1_1} Y_1^v = (\hat{\nabla}_{X^1_1} Y_1^1)^v + \frac{(cf_2^\flat)^2 H^{f_1}_1(X^h_1, Y_1^h)}{1+(cf_2^\flat)^2 b_1^h}(\text{grad}^1 f_1)^v \\
- c^2 f_2^\flat(X_1(\ln f_1)Y_1(\ln f_1))^v(\text{grad}^2 f_2)^h,
\end{cases}
\]

\[
\begin{cases}
\nabla_{X^2_1} Y_2^v = (\hat{\nabla}_{X^2_1} Y_2^1)^v - \frac{H^{f_1}_2(X_2, Y_2)^h}{1+(cf_2^\flat)^2 b_1^h}(\text{grad}^1 f_1)^h, \\
\nabla_{X^1_1} Y_2^h = \nabla_{X^1_1}^* Y_2^h = \nabla_{X^1_1}^* Y_2^v = \nabla_{X^1_1}^* X^1_1 = \nabla_{X^1_1}^* Y_2^v,
\end{cases}
\]

for any \( X_i, Y_i \in \Gamma(TM_i) \) \( i = 1, 2 \). Where \( H^{f_1} \) and \( H^{f_1}^* \) are the Hessian of \( f_1 \) with respect to \( \hat{\nabla} \) and \( \hat{\nabla}^\ast \) respectively.

Let us assume that \((g_i, \hat{\nabla}, \hat{\nabla}^* )\) is a dualistic structures on \( M_i \), \( i = 1, 2 \). Let \( A \) be a tensor field of type \((0,3)\) defined for any \( X, Y, Z \in \Gamma(TM_1 \times M_2) \) by

\[
A(X, Y, Z) = X(\hat{g}_{f_1 f_2}(Y, Z)) - \hat{g}_{f_1 f_2}(\hat{\nabla}_X Y, Z) - \hat{g}_{f_1 f_2}(Y, \hat{\nabla}_X Z),
\]
Since \( d\pi_{3-i}(X^I_i) = 0 \), it follows that 
\[
X^I_i(f_{3-i}) = d\pi_{3-i}(X^I_i)(f_{3-i}) = 0, \quad \forall (i, I) \in \{(i, h), (2, v)\},
\]
and hence, for all \( X_i, Y_i, Z_i \in \Gamma(TM_i) \) \((i = 1, 2)\), we have 
\[
\begin{align*}
X^i_1 \left( \tilde{g}_{f_1, f_2}(Y^h_i, Z^h_i) \right) &= \left( X_1(g_1(Y_1, Z_1)) \right)^h + \left( c f^i_2 \right)^2 \left( Y_1(f_1)X_1(Z_1(f_1)) + Z_1(f_1)X_1(Y_1(f_1)) \right)^h, \\
X^i_2 \left( \tilde{g}_{f_1, f_2}(Y^h_i, Z^h_i) \right) &= \left( c f^i_2 \right)^2 \left( X_2(g_2(Y_2, Z_2)) \right)^h.
\end{align*}
\]
as \((g_i, \tilde{\nabla}, \tilde{\nabla}^*)\) is dualistic structure and from Equations \((20), (24)\), then it's easily seen that the following equation holds 
\[
A(X^I_i, Y^I_i, Z^I_i) = 0, \quad \forall (i, I), (j, J) \in \{(i, h), (2, v)\}.
\]
In the different lifts \((i \neq j)\), we have 
\[
X^I_i(\tilde{g}_{t_1, t_2}(Y^I_i, Z^I_j)) = 0,
\]
\[
\begin{align*}
\tilde{g}_{t_1, t_2}(\tilde{\nabla}_X^I Y^I_i, Z^I_j) &= -c f^2_{t_2} X_1(f_1)^h Y_1(f_1)^h Z_2(f_2)^v, \\
\tilde{g}_{t_1, t_2}(\tilde{\nabla}_Y^I Y^I_i, Z^I_j) &= -f^1_{t_2} g_2(X_2, Y_2)^v Z_1(f_1)^h,
\end{align*}
\]
and 
\[
\begin{align*}
\tilde{g}_{t_1, t_2}(Y^I_i, \tilde{\nabla}_X^I Z^I_j) &= -c f^2_{t_2} X_1(f_1)^h Y_1(f_1)^h Z_2(f_2)^v, \\
\tilde{g}_{t_1, t_2}(Y^I_i, \tilde{\nabla}_Y^I Z^I_j) &= f^1_{t_2} g_2(X_2, Y_2)^v Z_1(f_1)^h,
\end{align*}
\]
We add these equations and obtain 
\[
A(X^I_i, Y^I_i, Z^I_j) = 0, \quad \forall (i, I), (j, J) \in \{(i, h), (2, v)\}.
\]
Hence the same applies for \((X^I_j, Y^I_j, Z^I_i) = A(X^I_i, Y^I_i, Z^I_i) = 0\).
This proves that \(\tilde{\nabla}^*\) is conjugate to \(\tilde{\nabla}\) with respect to \(\tilde{g}_{t_1, t_2}\). \(\Box\)

We recall that the connection \(\tilde{\nabla}\) on \(M_1 \times M_2\) induced by \(\tilde{\nabla}^1\) and \(\tilde{\nabla}^2\) on \(M_1\) and \(M_2\) respectively, is given by Equation \((24)\).

**Proposition 7.** \((M_1, \tilde{\nabla}, g_1)\) and \((M_2, \tilde{\nabla}, g_2)\) are statistical manifolds if and only if \((M_1 \times M_2, \tilde{g}_{t_1, t_2}, \tilde{\nabla})\) is a statistical manifold.

**Proof.** Let us assume that \((M_i, \tilde{\nabla}, g_i) \quad (i = 1, 2)\) is statistical manifold. Firstly, we show that \(\tilde{\nabla}\) is torsion-free. Indeed; by Equation \((24)\), we have for any \(X, Y \in \Gamma(TM_1 \times M_2)\)
\[
d\pi_i(T(X, Y)) = \tilde{\nabla}_X d\pi_i(Y) - \tilde{\nabla}_Y d\pi_i(X) - d\pi_i([X, Y])
\]
Since for \(i = 1, 2\), \(\tilde{\nabla}^i\) is torsion-free, then 
\[
\tilde{\nabla}_X d\pi_i(Y) - \tilde{\nabla}_Y d\pi_i(X) = d\pi_i([X, Y])
\]
Therefore, from Remark 1, the connection \(\tilde{\nabla}\) is torsion-free.
Secondly, we show that \(\tilde{\nabla} G_{f_1, f_2}\) is symmetric. In fact; for \((i, I) \in \{(i, h), (2, v)\}\), 
\[
(\tilde{\nabla} \tilde{g}_{f_1, f_2})(X^I_i, Y^I_i, Z^I_j) = X^I_i(\tilde{g}_{f_1, f_2}(Y^I_i, Z^I_j)) - \tilde{g}_{f_1, f_2}(X^I_i, Y^I_i, Z^I_j) - \tilde{g}_{f_1, f_2}(Y^I_i, X^I_i, Z^I_j)
\]
by Equations \((20), (24)\) and since \((\tilde{\nabla} g_i), i = 1, 2\), is symmetric, we have 
\[
(\tilde{\nabla} \tilde{g}_{f_1, f_2})(X^I_i, Y^I_i, Z^I_j) = (\tilde{\nabla} \tilde{g}_{f_1, f_2})(Y^I_i, X^I_i, Z^I_j).
\]
In the different lifts, for all \((i, I), (j, J) \in \{(i, h), (2, v)\}\), we have 
\[
(\tilde{\nabla} \tilde{g}_{f_1, f_2})(X^I_i, Y^I_i, Z^I_j) = (\tilde{\nabla} \tilde{g}_{f_1, f_2})(X^I_{3-i}, Y^I_{i}, Z^I_{j}) = (\tilde{\nabla} \tilde{g}_{f_1, f_2})(X^I_{3-i}, Y^I_{j}, Z^I_{i}) = (\tilde{\nabla} \tilde{g}_{f_1, f_2})(X^I_{3-i}, Y^I_{j}, Z^I_{i}) = 0.
\]
Therefore, \((\nabla \tilde{g}_{1_{i_2}})\) is symmetric. Thus \((M_1 \times M_2, \tilde{g}_{1_{i_2}}, \nabla)\) is a statistical manifold.

Conversely, if \((M_1 \times M_2, \tilde{g}_{1_{i_2}}, \nabla)\) is statistical manifold, then \((\nabla \tilde{g}_{1_{i_2}})\) is symmetric and \(\nabla\) is torsion-free, particularly, when \(X_i, Y_i, Z_i \in \Gamma(TM_i)\), we have

\[
\begin{align*}
(\nabla \tilde{g}_{1_{i_2}})(X^I_i, Y^I_i, Z^I_i) &= (\nabla \tilde{g}_{1_{i_2}})(Y^I_i, X^I_i, Z^I_i), & \forall i = 1, 2, \\
T(X^I_i, Y^I_i) &= 0.
\end{align*}
\]

Then, by Equations (20) and (24), we obtained, for \(i = 1, 2\), \(\tilde{g}_{1_i}\) is symmetric and \(\nabla\) is torsion-free. Therefore, \((M_i, \tilde{\nabla}, g_i), i = 1, 2\), is statistical manifold. \(\Box\)

At first, note that \((M_1 \times M_2, \tilde{g}_{1_{i_2}}, \nabla)\) is the statistical manifold induced from \((M_1, g_1, \tilde{\nabla})\) and \((M_2, g_2, \tilde{\nabla})\).

Now, let \((M_i, \tilde{\nabla}, g_i)\) and \((M_j, \tilde{\nabla}, g_j)\) be two statistical manifolds and let \(\tilde{\mathcal{R}}, \tilde{\mathcal{R}}\) and \(\mathcal{R}\) be the curvature tensors with respect to \(\tilde{\nabla}, \tilde{\nabla}\) and \(\nabla\) respectively.

**Proposition 8.** Let \((M_i, \tilde{\nabla}, \tilde{\nabla}, g_i), (i = 1, 2)\) be a connected statistical manifold.

Assume that the gradient of \(f_i\) is parallel with respect to \(\tilde{\nabla}\) and \(\tilde{\nabla}^i (i = 1, 2)\). Then for any \(X_i, Y_i, Z_i \in \Gamma(TM_i)\) \((i = 1, 2)\) we have

1. \(\mathcal{R}(X^h_i, Y^h_i)Z^h_1 = (\mathcal{R}^i(X_1, Y_1)Z_1)^h,\)
2. \(\mathcal{R}(X^v_2, Y^v_2)Z^v_2 = (\mathcal{R}^2(X_2, Y_2)Z_2)^v - \frac{b_2}{1 + (c f_2)^{b_1}} \{(X_2 \wedge g_2 Y_2)Z_2\}^v + \frac{c^2 f_2^h f_2^v b_1}{(1 + (c f_2)^{b_1})} \{(X_2 \wedge g_2 Y_2)Z_2\}^v (\text{grad} f_2)^v,\)
3. \(\mathcal{R}(X^h_i, Y^h_i)Z^v_2 = 0,\)
4. \(\mathcal{R}(X^h_1, Y^v_2)Z^h_1 = \frac{c^2 f_1^h (\ln f_1)^{b_1} f_1 (\ln f_1)^{b_1} Y_2 f_2^v}{1 + (c f_2)^{b_1}} \{(X_2 \wedge g_2 Y_2)Z_2\}^v (\text{grad} f_2)^v,\)

where the wedge product \((X_2 \wedge g_2 Y_2)Z_2 = g_2(Y_2, Z_2)X_2 - g_2(X_2, Z_2)Y_2.\)

**Proof.** Long but straightforward calculations as in proof of the proposal (2), where it uses the fact that connections are compatible with the metric. We obtained the same results as in (2), knowing we use only the connections are symmetrical. \(\Box\)

**Corollary 2.** Let \((M_i, \tilde{\nabla}, \tilde{\nabla}, g_i), (i = 1, 2)\) be a connected statistical manifold. Assume that \(f_1\) is a non-constant positive function and \(c \neq 0.\)

If \((\tilde{\nabla}, \tilde{\nabla}, \tilde{\nabla}, g_{i_2})\) is a dually flat structure then \((\tilde{\nabla}, \tilde{\nabla}, g_1)\) is also dually flat and \((\tilde{\nabla}, \tilde{\nabla}, g_2)\) has a constant sectional curvature.

**Proof.** Let \((\tilde{\nabla}, \tilde{\nabla}, \tilde{\nabla}, g_{i_2})\) be a dually flat structure.

By 1. of Proposition 8, for any \(X_1, Y_1, Z_1 \in \Gamma(TM_1)\), we have

\[
\mathcal{R}(X_1, Y_1)Z_1 = 0.
\]

From Equation (7), Since \((M_i, \tilde{\nabla}, g_i)\) \((i = 1, 2)\) is a statistical manifold, we have

\[
\mathcal{R}(X_1, Y_1)Z_1 = 0.
\]
Hence \((M_1, \nabla, \nabla^*, g_1)\) is dually flat.

By 4. of Proposition 8, for any \(X_1, Z_1 \in \Gamma(TM_1)\) and \(Y_2 \in \Gamma(TM_2)\), we have
\[
\frac{c^2 X_1 (\ln f_1)^h Z_1 (\ln f_1)^h Y_2 (f_2)^\nu}{1 + (cf_2^v)^2 b_1} (\nabla f_2) = 0.
\]

So \(f_2\) is a constant function since \(f_1\) is non-constant function and \(M_2\) is assumed to be connected.

Moreover, By 2. of Proposition 8, for any \(X_2, Y_2, Z_2 \in \Gamma(TM_2)\), we have
\[
\mathcal{R}(X_2, Y_2)Z_2 = \frac{b_1}{1 + (cf_2^v)^2 b_1} \{ (X_2 \wedge g_2 Y_2) Z_2 \}^\nu,
\]

Since \(b_1\) and \(f_2\) are constants, it follows from the previous equality that \((\nabla, \nabla^*, g_2)\) has a constant sectional curvature \(\frac{b_1}{1 + (cf_2^v)^2 b_1}\).

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