A gravitational collapse singularity theorem consistent with black hole evaporation

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Abstract
The global hyperbolicity assumption present in gravitational collapse singularity theorems is in tension with the quantum mechanical phenomenon of black hole evaporation. In this work I show that the causality conditions in Penrose’s theorem can be almost completely removed. As a result, it is possible to infer the formation of spacetime singularities even in absence of predictability and hence compatibly with quantum field theory and black hole evaporation.

1 Introduction
The celebrated Penrose’s singularity theorem \[24,26\] establishes that in a globally hyperbolic spacetime admitting a non-compact Cauchy hypersurface \(H\), under the null energy condition, any trapped surface \(S\) leads to the formation of a spacetime singularity in its causal future.

If the weak cosmic censorship conjecture is correct, then the spacetime singularity will be hidden behind a horizon so leading to the formation of a black hole. These objects, particularly stationary ones, have then been investigated with the theory of quantum fields over curved spacetimes. A rather robust prediction states that the black hole will eventually evaporate in a finite, though huge, time \[14,15,30\].

It was observed by Kodama \[19\] and Wald \[29\] that the phenomenon of black hole evaporation is in tension with global hyperbolicity and hence with the possibility of predicting the evolution of the spacetime manifold by means of Einstein's equations. Recently Lesourd \[20\] has given more precise arguments which show that even causal continuity, a causality condition much weaker than global hyperbolicity, would be violated.

We see that we are in presence of a loophole argument. In order to infer the formation of singularities we need to assume global hyperbolicity. Unfortunately, one would expect the formation of a black hole and hence its very evaporation. If so the global assumption of global hyperbolicity could not have been satisfied in the first place. This kind of logical fallacy points to the possibility that
trapped surfaces do not necessarily imply the formation of singularities, at least
when quantum effects are taken into account. The situation would be clarified
and the problem solved if we could generalize Penrose’s theorem by removing
the assumption of global hyperbolicity.

Much of the discussion about the information loss paradox might have been
influenced by the fact that Penrose’s theorem requires global hyperbolicity. It is
clear that if the very classical picture of the gravitational collapse had been itself
non deterministic from the outset, then the prediction of information loss in a
quantum field theoretical context would have been perceived as less problematic
and the very feeling of a paradox would have been reduced [28]. We shall indeed
show that the very classical picture for gravitational collapse does not require
determinism.

Let us discuss whether Penrose’s theorem can really be improved. To start
with, we observe that the Einstein’s equation is not used in this theorem, so
a deterministic development of the spacetime metric is not required. Only the
null energy condition derived from that equation is used. Fortunately, weaker
averaged formulations exist that have the same effect but might be preserved
in a quantum field theoretical context [9, 11, 12, 25, 27, 31]. Alternatively, when
violations of the energy conditions are due to scalar fields one can use weaker
versions obtained by replacing the Ricci tensor with the Bakry-Emery Ricci
tensor [6, 32]. All the steps in the proof of Penrose’s theorem pass through to
the modified Ricci case, as will those of the singularity theorems that we shall
present in the following.

It must be mentioned that there already exist singularity theorems that do
not assume causality conditions. The most important is Hawking’s 1967 theo-
rem [13, 16] which establishes that, under suitable energy conditions, expanding
partial Cauchy hypersurfaces imply the existence of past singularities. Other
variants that get rid of causality assumptions have been considered by Borde [3]
and Galloway [10]. It can be observed that all these examples are cosmological
in scope. The conditions they impose on spacelike hypersurfaces have global
consequences because these hypersurfaces are themselves, so to say, global. On
the contrary, trapped surfaces may form in limited regions of spacetime and
getting rid of global causality conditions becomes more difficult.

The assumption of global hyperbolicity in Penrose’s theorem was soon re-
garded as too strong and quite undesirable already from the classical point of
view. It should be recalled that the addition of a small charge or angular mo-
momentum to a Schwarzschild black hole alters its causal structure dramatically,
for the maximally extended solutions become the Reissner-Nordström and Kerr
solutions, neither of which is globally hyperbolic.

Hawking and Ellis commented Penrose’s theorem as follows [16, p. 285]

The real weakness of the theorem is the requirement that $H$ be a
Cauchy surface. This was used in two places: first, to show that
$(M, g)$ was causally simple which implied that the generators of
$J^+(S)$ had past endpoints on $S$, [i.e. $J^+(S) = E^+(S)$] and second,
to ensure that under the [global timelike vector field flow-projection]
map every point of $\dot{J}^+(S)$ was mapped into a point of $H$.

[Penrose’s theorem] does not answer the question of whether singularities occur in physically realistic solutions. To decide this we need a theorem which does not assume the existence of Cauchy surfaces.

Hawking and Penrose’s answered these problems with the development of the singularity theorem that brings their name [17]. Unfortunately, this theorem is in several respects weaker than Penrose’s. The singularity might well be to the past of the future trapped surface so, in the context of a spacetime that had origin through a Big Bang singularity, Hawking and Penrose’s theorem does not provide any new information for what concerns the formation of a singularity through gravitational collapse. The singularity that it signals could just be the Big Bang singularity. Moreover, the genericity condition there appearing has also been criticized as not always physically justified particularly when there are inextendible causal curves imprisoned in compact sets or compact Cauchy horizons [23, Remark 6.22].

We are left with the strategy of improving Penrose’s theorem. Bardeen [1] gave an example of null geodesically complete spacetime which satisfies all the assumptions of Penrose’s theorem but global hyperbolicity [4, 16]. It was obtained through a regularization of the singularity in the Reissner-Nordström solution. From consideration of this spacetime Hawking and Ellis concluded that the global hyperbolicity condition in Penrose’s theorem is necessary [16]. Nevertheless, Borde [4] noticed that Bardeen’s example contains a compact partial Cauchy hypersurface, a fact which might indicate that rather than global hyperbolicity, what is essential for the validity of the theorem is a sort of topological condition to prevent the compact set $E^+(S)$ from wrapping around the universe (or, stated in more suggestive terms, from swallowing the whole universe). His analysis supported the hope that the global hyperbolicity condition could be weakened in Penrose’s theorem. So far this generalization had been elusive, but with this work we shall indeed show that Borde’s intuition was correct.

Since we shall have to replace the global hyperbolicity condition with some weaker conditions, let us have a look at the conformal structure that has been proposed to represent an evaporating black hole [5,15,18,29], see Fig. 1.

As the figure illustrates there are pairs of points such that $p \in J^-(q)$ but $q \notin J^+(p)$. This means that the proposed spacetime is not future reflecting though it is past reflecting [23].

In this work we first give arguments, complementing those by Lesourd [20], which show that in presence of an evaporating black hole, or more generally of an evaporating null hypersurface (to be defined), future reflectivity cannot hold. We shall then prove that a version of Penrose’s theorem holds just under past reflectivity, cf. Theorems [3.9] and [3.10].

By the previous result this condition looks as a somewhat technical notion that guarantees that we are not working inside the time dual of an evaporating black hole. There is the chance that past reflectivity holds on any reasonable
spacetime, for it might represent a sort of entropy condition. Non-time symmetric entropy conditions are imposed in PDE theory to guarantee the existence of unique solutions [8], so one is left to investigate if past reflectivity might play a similar role, see also [20,29]. The validity of past reflectivity would not exclude the possibility that information flows in from a timelike boundary, but such a flow would have to be continuous not in burst or shocks.

Also it can be noticed that we are not going to impose any causality condition save for the achronality of the trapped surface (which implies that the trapped surface does not intersect the region of chronology violation). So we do not impose the compactness of the causal diamonds, the closure of the causal relation, or the chronology of spacetime. As a consequence, we do not assume the existence of time functions, of arbitrarily small causally convex neighborhoods, or that the chronological relation distinguishes events.

In fact the condition of past-reflectivity, though conformally invariant, does not belong to the causal ladder of spacetimes, so its role is not that of preventing nasty influence from infinity nor that of getting rid of almost closed causal curves. It is rather a topological condition on the nature of the causality relation, and hence it places restrictions on how information propagates on spacetime (it belongs to the so called transverse ladder [22,23]). It is implied by causal continuity, and hence by global hyperbolicity, by the existence of a complete timelike Killing vector field [7], or by the continuity of the Lorentzian distance [2, Thm. 4.24] [23, Prop. 5.2].
2 Evaporation and violation of future reflectivity

We denote with $I$ the chronological relation, and with $J$ the causal relation. The set $E^+(p) = J^+(p) \setminus I^+(p)$ is the future horismos of $p$. The inclusion $\subset$ is reflexive.

Since in this work we shall make repeated use of the notion of future (and past) reflectivity, it is convenient to recall its many equivalent formulations, each giving different insights on its geometrical and physical meaning [23, Def. 4.6]

**Definition 2.1.** The spacetime $(M, g)$ is future reflecting if any of the following equivalent properties holds true. For every $p, q \in M$

(a) $p \in J^-(q) \Rightarrow q \in J^+(p)$,
(b) $p \in J^-(q) \Rightarrow q \in J^+(p)$,
(c) $I^-(p) \subset I^-(q) \Rightarrow I^+(q) \subset I^+(p)$,
(d) $\uparrow I^-(p) = I^+(p)$,
(e) the volume function $t^+(p) = -\mu(I^+(p))$ is continuous.

Here $\uparrow I^-(p) = \text{Int}([\bigcap_{r \in I^-(p)} I^+(r)])$ is the common future while $\mu$ is any finite spacetime measure absolutely continuous with respect to the Lebesgue measures induced by the coordinate charts.

As it is well known the black hole region is given by the set $B = M \setminus I^-(\mathcal{I}^+)$, where $\mathcal{I}^+$ is the future null infinity in Penrose’s conformal boundary [16]. The achronal set $N := \partial B = \partial I^-(\mathcal{I}^+)$ is a topologically embedded hypersurface generated by future inextendible lightlike geodesics, namely it is a $C^0$ future null hypersurface commonly referred as the horizon of the black hole.

Although the reader can keep in mind this relevant example we shall now try to make mathematical sense of what “evaporating” or “evaporated” could mean for these objects. Typically what distinguishes an evaporating black hole is the possibility, for an outside observer that has escaped the fate of falling into the black hole, of observing matter fall into the black hole up to a certain proper time. After that instant looking further does not add any new information, for the observer will not see any more matter crossing or approaching the horizon. From the point of view of the observer the gravitational collapse has come to an end. This is in stark contrast with what happens in a Schwarzschild black hole, for which an exterior observer would keep receiving new information from the matter that falls into the black hole.

In order to formalize this concept we need to understand that the observer can look at different regions of the black hole. Let $p \in N$ be a representative point of such a region and let $r \in I^-(p)$. Consider two timelike curves $\gamma$ and $\sigma$, the former curve $\gamma: [0, \infty) \rightarrow M$, $\gamma(0) = r$, $p = \gamma(a)$, $a > 0$, represents matter that leaves $r$ and crosses the horizon at $p$, while the latter curve $\sigma: [0, \infty) \rightarrow M$, $\sigma(0) = r$, $\sigma \cap I^+(N) = \emptyset$, is future inextendible and represents an observer that
looks at the infalling matter without being itself causally influenced by the horizon.

\[ p = \sigma(\tau) \]

\( \sigma \gamma N \partial \uparrow I^- (p) \)

\[ q = \sigma(\tau) \]

\( \gamma \)

\( p \)

\( N \)

Figure 2: Future reflectivity is violated by evaporating future null hypersurfaces.

We are interested in those observers \( \sigma \) that can witness the whole falling history, i.e. \( \gamma([0,a]) \subset J^- (\sigma) \). There are two possibilities, either there is \( t > 0 \) such that \( \gamma([0,a]) \subset J^- ( \sigma([0,t])) \) or there is not. Only in the former case we say that the horizon at point \( p \), or better over the whole generator passing through \( p \), has evaporated or that it is evaporating from the point of view of \( \sigma \). In fact, the observer \( \sigma \) might wait further time after the event \( \sigma(t) \) but will not receive any more information from the matter that has crossed the horizon at \( p \).

Let us redefine \( \tau = \inf t \) to be the infimum of all the parameters \( t \) for which the inclusion holds, and let \( q = \sigma(\tau) \). Then for every \( p' \in \gamma([0,a]) \) we have \( q \in J^+(p') \). Let us assume that past reflectivity holds, then \( p' \in J^- (q) \), and by the arbitrariness of \( p' \) and the fact that \( \gamma \) is timelike we get \( \gamma([0,a]) \subset J^- (q) \), namely \( \tau \) is the least value for which the inclusion holds.

Since \( \gamma([0,a]) \subset J^- (q) \), \( \gamma(a) = p \), we have \( p \in J^- (q) \). However, it cannot be \( q \in J^+(p) \) since letting \( q' = \gamma(\tau') \), \( \tau' > \tau \), we would have \( q' \gg q \), and hence \( q' \in I^+(p) \) in contradiction with the assumption that \( \gamma \) does not intersect \( J^+(N) \). We conclude that a spacetime which contains an evaporating \( C^0 \) future null hypersurface is not future reflecting if it is past reflecting.

Notice that every \( \tau' \in \gamma([0,a]) \) belongs to \( J^- (q) \), and since \( \gamma \) is timelike, to \( I^- (q) \), thus \( q \in \cap_{\tau \in I^- (p)} I^+(r) \). This is a future set whose interior is the common future \( \uparrow I^- (p) \), thus \( q \in \partial[\uparrow I^- (p) \setminus I^+(p)] \), where the set in square brackets is known to be empty under future reflectivity. An observer can see the horizon evaporate completely, not just the portion near \( p \). For that, the observer’s worldline passes through the events belonging to the set \( \partial[\cap_{\tau \in I^- (N)} I^+(r)] \). Once those events are passed the observer cannot receive any new information from the infalling matter.
3 Singularities from trapped surfaces

The reader might be familiar with the notion of future trapped set which is a non-empty set \( S \) such that \( E^+(S) \) is non-empty and compact. Under very weak causality conditions the notion of null araying set is more convenient, \[23\].

**Definition 3.1.** A future lightlike \( S \)-ray is a future inextendible causal curve which starts from \( S \) and does not intersect \( I^+(S) \). A set \( S \) is a future null araying set if there are no future lightlike \( S \)-rays.

Trapped and araying sets are related through the next result.

**Theorem 3.2.** Let \( S \) be a non-empty achronal compact set. If \( S \) is a future null araying set then it is a future trapped set.

**Proof.** By achronality \( S \subset E^+(S) \), thus \( E^+(S) \neq \emptyset \). Suppose \( E^+(S) \) is not compact then we can find a sequence \( q_n \in E^+(S) \) for which no subsequence converges to some \( q \in E^+(S) \) (this can happen because \( E^+(S) \) is non-compact or because \( E^+(S) \) is non-closed). We can find a corresponding sequence \( p_n \in S \) such that \( q_n \in J^+(p_n) \). Passing to a subsequence if necessary, we can assume \( p_n \to p \in S \). Consider the causal curves \( \sigma_n \) connecting \( p_n \) to \( q_n \). Necessarily, they do not intersect \( I^+(S) \) otherwise \( q_n \in I^+(S) \), in particular, they do not intersect \( I^+(p_n) \) and so they are achronal lightlike geodesic segments. By the limit curve theorem \[21, 22, 23\], there is a future limit causal curve \( \sigma \) starting from \( p \), to which some subsequence of \( \sigma_n \), here denoted in the same way, converges uniformly on compact subsets. The curve \( \sigma \) does not intersect \( I^+(S) \) otherwise \( q_n \in I^+(S) \), in particular, \( q_n \) converges to a point in \( E^+(S) \). But if \( \sigma \) does not connect \( p \) to \( q \) then it is future inextendible, namely an \( S \)-ray, a contradiction.

A kind of converse result holds true but it requires additional causality conditions, such as strong causality, that we are not imposing [23, Thm. 2.116].

The following technical results establishes that past reflectivity is sufficient in order to infer that \( E^+(S) \) has no edge. This result is really what makes the generalization of Penrose’s theorem possible.

**Theorem 3.3.** Let \((M, g)\) be past reflecting. If \( S \) is a compact and achronal future null araying set then \( J^+(S) = E^+(S) \) and hence \( \text{edge}(E^+(S)) = \emptyset \).

**Proof.** Suppose not then there is \( q \in J^+(S) \backslash E^+(S) \). Let \( \sigma_n \) be a sequence of timelike curves starting from \( S \) and ending at \( q_n \) with \( q_n \to q \). By achronality \( S \subset J^+(S) \). Due to the compactness of \( S \), the limit curve theorem \[21, 23\] tells us that there are either a continuous causal curve connecting some \( \tau \in S \) to \( q \), which is impossible since it would entail \( q \in J^+(S) \) and hence \( q \in E^+(S) \), a contradiction, or there is a future inextendible continuous causal curve \( \sigma^r \) with starting point \( \tau \in S \) to which some subsequence of \( \sigma_n \), here denoted in the same way, converges in a suitable parametrization. Since \( S \) is a future null araying
set, $\sigma^\tau$ cannot be a future null $S$-ray, hence it enters $I^+(S)$. Let $b \in \sigma^\tau$, such that $U \ni b$ is an open neighborhood contained in $I^+(S)$. Let $p \in I^-(b,U)$, then since $b$ is a limit point of the sequence, for sufficiently large $n$, $\sigma_n$ intersects $I^+(p,U)$, thus $q_n \in I^+(p)$ and $q \in \overline{I^-(p)}$, and by past reflectivity $p \in \overline{I^-(q)}$, thus as $p \in I^+(S)$, $q \in I^+(S)$, a contradiction.

In order to capture the idea of trapped set that swallows the whole universe, as described in [16, p. 265] and [4] we introduce the next definition. With $I_U$ we denote the chronological relation in the spacetime $U$ with the induced metric.

**Definition 3.4.** A compact and achronal set $S$ is said to have an **unavoidable** or **swallowing** future horismos if there exists an open neighborhood $U$ of $E^+(S)$ such that $I_U^-(E^+(S)) \subset \text{Int}D^-(E^+(S))$.

In fact under this condition an observer, represented by an inextendible causal curve, that were to pass through a neighborhood of the horismos $E^+(S)$ would be forced to intersect it and hence to fall into its causal influence.

**Theorem 3.5.** Let $(M, g)$ be past reflecting. Every compact and achronal future null araying set $S$ has an unavoidable future horismos $E^+(S)$. In fact this horismos is also compact and coincident with $I^+(S)$.

For an example of stably causal spacetime which is non-past reflective and which admits compact and achronal future null araying sets which have avoidable horismos, see [16, Fig. 51] or [23, Fig. 2.4]. These examples are also such that $J$ is transitive, which taking into account the transverse ladder, shows that the assumption of past reflectivity cannot be replaced by the weaker notion of transitivity of $J$.

**Proof.** By Thm. 3.2 $E^+(S)$ is non-empty and compact, moreover from Thm. 3.3 we know that $E^+(S) = I^+(S)$. First we show that there is an open set $U \ni E^+(S)$ such that $I_U^-(E^+(S)) \subset D^-(E^+(S))$ from which the inclusion of the statement follows by the openness of $I_U^-(E^+(S))$. If the inclusion $I_U^-(E^+(S)) \subset D^-(E^+(S))$ does not hold for any open neighborhood $U$ of $E^+(S)$, then by considering a sequence of nested relatively compact neighborhoods $U$ we deduce that we can find a sequence of future inextendible causal curves $\sigma_n$ starting from $p_n \in I^-(E^+(S))$ and not intersecting $E^+(S)$ such that $p_n \to p \in E^+(S)$. By the limit curve theorem there exists a future inextendible continuous causal curve $\sigma$ starting from $p$ to which a subsequence of $\sigma_n$, here denoted in the same way, converges uniformly on compact subsets. This continuous causal curve $\sigma$ enters $I^+(S)$ in fact even if it were a lightlike geodesic aligned with a generator of $E^+(S)$, as $S$ is future null araying it would have to enter $I^+(S)$. Thus it intersects $I^+(S) = I^+(E^+(S))$, and so for sufficiently large $n$ the same is true for $\sigma_n$, which then has to pass from $I^-(E^+(S))$ to $I^+(E^+(S))$ hence intersecting $\partial[I^+(E^+(S))] = \partial I^+(S) = E^+(S)$ (for clarity we have denoted the boundary with $\partial$ instead that with a dot), a contradiction since $\sigma_n$ does not intersect $E^+(S)$.

\[ \square \]
**Definition 3.6.** We say that a spacetime is *(spatially)* open if it does not contain a compact spacelike hypersurface.

The next result will clarify that Penrose’s theorem can be deduced from our Thm. 3.9.

**Proposition 3.7.** Every spacetime admitting a non-compact Cauchy hypersurface (hence globally hyperbolic) is open.

**Proof.** The argument is as in Penrose’s theorem and makes use of the flow of a global timelike vector field $v$ to project any spacelike hypersurface $\Sigma$ to the Cauchy hypersurface $H$. As $\Sigma$ is spacelike such a projection is open, but $H$ is connected (because $M$ is connected and it splits topologically as $M \simeq \mathbb{R} \times H$) thus $\Sigma$ and $H$ are homeomorphic, hence $\Sigma$ is non-compact.

**Theorem 3.8.** Let $(M, g)$ be a past reflecting and open spacetime. Then it does not admit compact achronal future null araying sets.

**Proof.** By contradiction, let $S$ be such an araying set. By Thm. 3.3 $E^+(S)$ is an achronal boundary hence a locally Lipschitz topological hypersurface, and by Thm. 3.2 it is compact. By Theorem 3.5 $\text{IntD}(E^+(S)) \neq \emptyset$, so we can find a spacelike Cauchy hypersurface for this spacetime which is homeomorphic to $E^+(S)$, hence compact. We conclude that $(M, g)$ is not spatially open, a contradiction.

A *trapped surface* is a spacelike compact codimension 2 manifold without boundary such that $\theta^+, \theta^- < 0$ all over $S$, where these quantities are the divergences of the two future lightlike geodesic congruences issued from $S$. The *null convergence condition* states that $\text{Ric}(n, n) \geq 0$, for every null vector $n$, where $\text{Ric}$ is the Ricci tensor.

We obtain the following theorem which drops the global hyperbolicity assumption from Penrose’s theorem.

**Theorem 3.9.** Let $(M, g)$ be a past reflecting spacetime which is open and satisfies the null convergence condition. Suppose that it admits an achronal future trapped surface $S$, then it is future null geodesically incomplete.

**Proof.** Let us assume future null geodesic completeness. By Thm. 3.8 $S$ admits a lightlike $S$-ray. This ray must start perpendicularly to $S$ otherwise it would be entirely contained (save for the starting point) in $I^+(S)$. Thus this ray belongs to one of the two congruences of converging lightlike geodesics issued from $S$. By the Raychaudhuri equation and null geodesic completeness, this geodesic reaches a focal point (to the surface $S$) and hence it enters $I^+(S)$, a contradiction which proves that the spacetime is future null geodesically incomplete.

Theorem 3.5 has the following corollary which shows that Borde intuition was correct.
Theorem 3.10. Let \((M, g)\) be a past reflecting spacetime which satisfies the null convergence condition. Suppose that it admits an achronal future trapped surface \(S\), then it is either future null geodesically incomplete or the horismos \(E^+(S)\) is compact, unavoidable and actually coincident with \(\tilde{I}^+(S)\).

**Proof.** Suppose that \((M, g)\) is future null geodesically complete. By arguing as in the proof of Thm. 3.9 we obtain that \(S\) is a null araying set, thus from Theorem 3.5 we get that the horismos \(E^+(S)\) is compact, unavoidable and actually coincident with \(\tilde{I}^+(S)\). 

We conclude that unless the horismos swallows the whole universe a singularity is formed in the future of the trapped surface.

4 Conclusions

In Penrose’s singularity theorem we find the following two conditions: (a) the spacetime is globally hyperbolic, and (b) there is a non-compact Cauchy hypersurface. We have shown that they can be weakened as follows: (a) the spacetime is past reflecting, and (b) the spacetime is spatially open.

By dropping the global hyperbolicity condition we have shown that determinism is not necessary in order to infer that trapped surfaces lead to spacetime singularities. The classical picture one gets of the gravitational collapse is therefore consistent with that of quantum field theory on curved backgrounds, where one would have to renounce to global hyperbolicity anyway, due to the phenomenon of black hole evaporation. Our result helps to resolve some of the tension between quantum field theory and general relativity by showing that both can accommodate coherent descriptions of black hole formation and evaporation in non-globally hyperbolic spacetimes.

We have also proved a singularity theorem in which condition (b) is replaced by a condition which states that the horismos \(E^+(S)\) of the trapped surface does not swallow the whole universe (in which case investigation through cosmological singularity theorems would have been more appropriate), so confirming a conjecture by Borde.

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