Bounds on the Capacity of the Blockwise Noncoherent APSK-AWGN Channels

Daniel C. Cunha and Jaime Portugheis
Department of Communications, School of Electrical and Computer Engineering
State University of Campinas
C.P. 6101, 13083-852, Campinas-SP, Brazil
Emails: dcunha@decom.fee.unicamp.br, jaime@decom.fee.unicamp.br

Abstract—Capacity of M-ary Amplitude and Phase-Shift Keying (M-APSK) over an Additive White Gaussian Noise (AWGN) channel that also introduces an unknown carrier phase rotation is considered. The phase remains constant over a block of L symbols and it is independent from block to block. Aiming to design codes with equally probable symbols, uniformly distributed channel inputs are assumed. Based on results of Peleg and Shamai for M-ary Phase Shift Keying (M-PSK) modulation, easily computable upper and lower bounds on the effective M-APSK capacity are derived. For moderate M and L and a broad range of Signal-to-Noise Ratios (SNR's), the bounds come close together. As in the case of M-PSK modulation, for large L the coherent capacity is approached.

I. INTRODUCTION

Coherent reception is not possible for many bandpass transmission systems. In these systems, it is commonly assumed that the unknown carrier phase rotation is constant over a block of L symbols and independent from block to block. One approach adopted to solve the problem of detection of information transmitted over these systems is Multiple Symbol Differential Detection (MSDD) [1]. The system modulation is usually M-ary Phase Shift Keying (M-PSK), but in the case of high spectral efficiencies, M-ary Amplitude and Phase-Shift Keying (M-APSK) with independent phase and amplitude modulations is preferable [2], [3].

The capacity of a noncoherent AWGN channel in the case of input symbols drawn from an M-PSK modulation has been investigated by Peleg and Shamai [4]. It was shown that capacity can be achieved by uniformly, independently and identically distributed (u.i.i.d) symbols. For the case of detection with an overlapping of one symbol, upper and lower bounds on capacity were given. Aiming to design codes with equally probable symbols, u.i.i.d channel inputs are here assumed. In this case, the capacity is denominated effective. Extending the results for M-PSK modulation, easily computable upper and lower bounds on the effective M-APSK capacity are derived.

The outline of the paper is as follows. In Section II we compare the effective capacity of some APSK constellations in the case of coherent reception. In Section III we define the noncoherent channel model. Section IV describes the derivation of the upper and lower bounds on the capacity of the noncoherent APSK channels. Section V concludes the paper presenting numerical results.

II. M-APSK SIGNAL CONSTELLATIONS

We consider APSK constellation diagrams which consist of N different amplitude rings, each one with P phase values. The amplitude values of the rings differ by a constant factor r denominated ring ratio. Such constellations will be denoted by M-APSK (N, P), with M = NP. Fig. 1 shows two examples of constellations for N = 2 with P = 4 and P = 8.

Since it is expected that the noncoherent capacity approaches that of a coherent channel for large values of block length, we are interested in calculating this capacity. For fixed SNR, the capacity of an APSK alphabet depends on the ring ratio. For a uniform input distribution, the capacity of an APSK alphabet, \( C^* \), can be efficiently evaluated by Monte Carlo methods [5]. By doing so, we could obtain the values of r that maximize \( C^* \) for each SNR. Fig. 2 shows results for three constellations: 8-APSK(2, 4), 16-APSK(2, 8) and 16-APSK(4, 4). \( E_s \) is the average constellation energy and \( N_0 \) is the one-sided noise spectral density. The results show that 16-APSK(2, 8) has a greater capacity when compared to 16-APSK(4, 4).

It was observed that the optimal value of r does not change significantly for SNR’s greater than 2 dB. This observation led us to choose constellations with fixed r in order to compute bounds on the capacity of noncoherent channels. For 8-APSK(2, 4) and 16-APSK(2, 8), \( r = 2.42 \) and \( r = 2.0 \) were chosen, respectively. This last value was also suggested in [3].
The input of the channel is a vector of length \( S = [s_0, s_1, ..., s_{L-1}] \), whose components \( s_l = a_l \exp (j\phi_l) \) represents APSK-modulated symbols. Their average energy is \( E_a \). The amplitudes \( a_l \) can assume one of \( N \) possible discrete values and \( \phi_l \) can assume one of \( P \) discrete phases, so the signal \( s_l \) belongs to a \( M \)-APSK \((N, P)\) constellation. The output is also a vector of length \( L, R = [r_0, r_1, ..., r_{L-1}] \), whose components may be expressed as

\[
r_l = s_l \exp (j\theta) + n_l \quad , \quad l = 0, 1, ..., L - 1
\]

where \( \theta \) is a phase shift introduced by the channel uniformly distributed over the interval \([0, 2\pi] \) and \( n_l \) are independent circularly symmetric Gaussian noise variables, whose real and imaginary parts are each zero mean with variance \( \sigma^2 \). The SNR is then defined as \( E_a / N_0 \). In the following we will use the vectors \( A = [a_0, a_1, ..., a_{L-1}] \) and \( \Phi = [\phi_0, \phi_1, ..., \phi_{L-1}] \) that can be defined by using the components \( s_l \) of \( S \).

Since an input distribution for \( S \) is assumed, we would like to obtain the \textit{Average Mutual Information} (AMI), \( I_{nc} \), of the channel described above using the formula:

\[
I_{nc} = I(S; R) = E_{S,R} \log_2 \left( \frac{P(R|S)}{P(R)} \right)
\]

where \( E_{S,R} \) denotes the statistical expectation taken with respect to variables \( S \) and \( R \). The transition probability densities \( P(R|S) \) are given by [1]:

\[
P(R|S) = \frac{1}{(2\pi\sigma^2)^L} \exp \left[ -\frac{1}{2\sigma^2} \sum_{l=0}^{L-1} \left( |r_l|^2 + |s_l|^2 \right) \right] 
\]

where \( I_0(\cdot) \) is the modified Bessel function of the first kind of order zero. The probability density \( P(R) \) can be obtained by the following equation:

\[
P(R) = \sum_S P(R|S)P(S),
\]

where \( P(S) \) is the distribution of the channel input \( S \).

The computing of \( I_{nc} \) is rather complicated for large \( L \) and \( M (= NP) \). It is then appropriate to resort to bounds. As in [4], we consider the case where there exists overlapping of one symbol between consecutive blocks. Therefore the following normalization for the capacity (in bits per modulation symbol) is used throughout

\[
C_{nc} = \frac{I_{nc}}{L-1} \quad .
\]

**IV. Bounds**

The steps to derive the bounds are similar to those done in [4] for MPSK signals. The phase rotation \( \theta \) is viewed as an additional channel input with AMI \( I_v = I(\theta; S; R) \). Then the chain rule for mutual information [6] is applied to \( I_v \) resulting in

\[
I_{nc} = I_v - I(R; \theta|S)
\]

and hence, \( I_{nc} = I(S; R|\theta) - I(\theta; R|S) + I(\theta; R) \). Equivalent to MPSK signals, the first term is the AMI over the APSK-AWGN coherent channel while the term \( [I(\theta; R|S) - I(\theta; R)] \) represents the degradation due to unknown \( \theta \).

An upper bound on \( I_{nc} \) is derived by computing (1) for \( \theta \) discretely and uniformly distributed over the same number of input phases, i.e., \( \theta \) has the same distribution of \( \phi_l \). Therefore, we have

\[
I(S; R|\theta) \leq (L - 1) C_c \quad ,\]

where \( C_c \) is the APSK-AWGN coherent channel capacity. The coefficient \((L - 1)\) is used in (8) because of overlapping of one symbol in detection.

For evaluating \( I(\theta; R) \), we will consider the channel model shown in Fig. 3. This is a Single Input Multiple Output (SIMO) channel [7], with \( \theta \) as the single input. Define \( \phi'_l = (\theta \oplus \phi_l), \quad l = 1, 2, ..., L - 1 \), where \( \oplus \) is sum modulus \( 2\pi \). Since the \( \phi_l \) are u.i.i.d. variables, the \( \phi'_l \) are independent of \( \theta \) implying that \( p(r_l|\theta), \quad l = 1, 2, ..., L - 1 \), are also independent of \( \theta \). Consequently, we have

\[
I(\theta; r_l) = 0, \quad l = 1, 2, ..., L - 1
\]
we have any of the information. Due to the fact that phase rotations do not change the mutual AWGN channel. For example, considering 8-APSK average of capacities of PSK modulations over a coherent or
\[ I(\theta; R) = \frac{1}{2} C_{c-4PSK(A)} + \frac{1}{2} C_{c-4PSK(rA)} , \]
where \( C_{c-4PSK(A)} \) and \( C_{c-4PSK(rA)} \) are the 4-PSK channel capacities for two amplitudes, A and \( rA \), respectively.
Finally, \( I(\theta; R|S) \) is given by the following equation [8]:
\[ I(\theta; R|S) = \sum_{\alpha} P_S(\alpha) I(\theta; R|S = \alpha) . \]

By using again the concept of a SIMO channel, \( I(\theta; R|S) \) is obtained by computing \( I(\theta; r_l|s_l) \), the AMI of the \( l \)-th component, with SNR increased by a factor of \( L \). As above, we have
\[ I(\theta; r_l|s_l) = I(\theta; r_l|a_l) = \sum_k P_{a_l}(k)I(\theta; r_l|a_l = k) . \]

Therefore, \( I(\theta; R) \) is evaluated using the same reasoning that was applied to computation of \( I(\theta; R|S) \).

The lower bound is also obtained starting with \( I(\theta; r_0) \), but knowing that the unknown phase \( \theta \) is a continuous uniformly distributed variable. Following [4], we incorporate the inequality
\[ I(\theta; R) \geq I(\theta; r_0) \]
to \( I(\theta; R|S) \) yielding:
\[ I_{nc} \geq I(S; R|\theta) - I(\theta; R|S) + I(\theta; r_0) . \]

The first term of the right hand side of \( I_{nc} \) is identical to the first term of the upper bound. The third term, \( I(\theta; r_0) \), is also given by \( I(\theta; r_0) \) but with a continuous \( \theta \). Each AMI \( I(\theta; r_0|a_0 = k) \) equals the capacity of a coherent continuous input phase modulated channel [9]. The second term of \( I_{nc} \) is also equivalent to the second term of the upper bound, except that we have to calculate capacities for a channel with a single continuous input. All these capacities were evaluated efficiently using Monte Carlo methods.

V. NUMERICAL RESULTS

Figs. 3 and 4 illustrate the results for 8-APSK(2, 4) and 16-APSK(2, 8) constellations, respectively. Solid lines represent results for the upper bounds while dashed lines represent them for the lower bounds. For 8-APSK(2, 4) and \( L = 2 \), the bounds come close together with SNR’s less than 0 dB whereas for 16-APSK(2, 8) and \( L = 2 \) this happens with SNR’s less than 6 dB. It can be seen that as \( L \) increases, the bounds become close to coherent channel capacity. Moreover, for \( L = 8, 16, 32 \), the bounds come close together over a broad range of SNR’s (the difference between the upper and lower bounds is less than 0.1 bit/symbol). Therefore, we can conclude that the coherent capacity is approached.

Fig. 3. Channel model for evaluating \( I(\theta; R) \).

Fig. 4. Bounds on the capacity of the noncoherent 8-APSK(2, 4)-AWGN channel. ∗: \( L = 8 \), □: \( L = 16 \), △: \( L = 32 \), —: 8-APSK(2, 4)-AWGN coherent channel capacity.
Fig. 5. Bounds on the capacity of the noncoherent 16-APSK(2, 8)-AWGN channel. ○: \( L = 2 \), □: \( L = 8 \), △: \( L = 16 \), *: \( L = 32 \), —: 16-APSK(2, 8)-AWGN coherent channel capacity.

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