Criterion of quantum synchronization and controllable quantum synchronization based on an optomechanical system

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Abstract

We propose a quantitative criterion to determine whether the coupled quantum systems can achieve complete synchronization or phase synchronization in the process of analyzing quantum synchronization. Adopting the criterion, we discuss the quantum synchronization effects between optomechanical systems and find that the error between the systems and the fluctuation of error is sensitive to coupling intensity by calculating the largest Lyapunov exponent of the model and quantum fluctuation, respectively. By taking the appropriate coupling intensity, we can control quantum synchronization even under different logical relationships between switches. Finally, we simulate the dynamical evolution of the system to verify the quantum synchronization criterion and to show the ability of synchronization control.

Keywords: quantum synchronization, optomechanical system, Lyapunov exponent

(Some figures may appear in colour only in the online journal)

1. Introduction

In recent years, synchronization effects of two or more interconnected classical systems have aroused comprehensive attention because synchronization phenomena are found widely in nature. For example, Huygens found that two clocks with different swings at an initial time will appear synchronous with time evolvement; it was also observed that fireflies glow synchronously and the oscillation of heart cells in humans or animals will keep in step with each other. At the same time, synchronization effects exhibit inimitable application potential in many fields, such as the synchronous transmission of information in the Internet, the synchronous transmission and amplification of signals between coupled lasers, the encryption and decryption of signals using chaotic synchronization technology, and so on. To date, the synchronization of classical system has gradually become the focus of investigation in numerous fields. The groundbreaking work on theoretical exploration of classical synchronization was made by Yamada and Fujisaka who put forward a criterion to judge synchronization behaviors by calculating the Lyapunov exponent of the coupling system and obtained synchronization conditions [1]. After this, Pecora and Carroll found synchronization phenomena in the electronic circuit and designed a circuit scheme of encrypted communications using synchronization techniques, which demonstrates the attractive application prospect of synchronization effects and arouses the intense research interest for synchronization theory and application [2]. Recently, many effective synchronization techniques have been proposed in order to achieve complete or phase synchronization of classical systems [3–7].

Naturally, it is expected that a similar synchronization phenomenon in quantum systems will be found in order to realize the synchronous transmission of quantum information or states due to the unique advantages of synchronization effects. However, it is difficult to define precisely some concepts that describe synchronization in quantum systems, like ‘tracks’ and ‘errors’. Even some concepts used in classical dynamics are completely unsuitable to be used in quantum dynamics because of remarkable differences between two kinds of systems. Therefore, the relative research of quantum synchronization once was unfeasible. The optomechanical system, as a
representative of mesoscopic systems, has attracted widespread attention and systematic discussion recently [8–11, 25]. Mesoscopic systems exhibit simultaneously both properties of classical and quantum systems under certain conditions because the scale of the system is in-between the macro-system and micro-system. So some phenomena, regardless of chaos behaviors and limit cycle in classical kingdom [12–14, 22] or quantum entanglement and quantum coherent in quantum domain [15–18, 22], have been observed in optomechanical systems, which provides a reliable basis to expand synchronization theory from classical to quantum. In 2013, Mari et al extended the classical synchronization concepts to the quantum system [19] and developed a quantitative theory of synchronization for continuous variable systems evolving in the quantum regime. And in their work, two different measures quantifying the level of synchronization of coupled continuous variables are also introduced. Thereafter, some progress has been made in inter-related theories and experiments [22–24, 26].

However, in the general case, the synchronization effects are very sensitive to parameters of the systems, such as driving field, coupled intensity, and so on. Therefore, further investigation is necessary to obtain a quantitative synchronization criterion in order to determine directly whether the synchronization can be realized. Meanwhile, the synchronization criterion can also be regarded as a necessary and sufficient condition of the synchronization effect, which means that the quantum coupling systems can be adjusted and controlled to satisfy the synchronization criteria and to realize the aim of quantum synchronization. Further, the controllability and practicability of quantum synchronization can be improved.

In this work, we present a general method for discussing synchronization effects in mesoscopic quantum systems. We introduce the first order and second order measurements to describe the expectation value and the fluctuation of error, respectively, and give the necessary conditions to estimate the presence of quantum synchronization effects. Using this theory, we design a model based on an optomechanical system to realize logic control of quantum synchronization. Subsequently, we validate the criterion through the simulation.

This paper is organized as follows: In section 2, the classical synchronization theory is briefly introduced. In section 3, the processing method of quantum mesoscopic synchronization is described and the quantitative criteria for determining quantum complete synchronization and quantum phase synchronization are proposed. In section 4, a controllable quantum synchronization model based on an optomechanical system is designed and the phase synchronization effect is discussed. Finally, the summary and the prospects are given in section 5.

2. Classical synchronization theory

Considering two classical coupled systems

\[ \dot{x}_1(t) = F(x_1(t)) + U_1(x_1, x_2) \]
\[ \dot{x}_2(t) = F(x_2(t)) + U_2(x_1, x_2) \]  

(1)

where \(x_1(t)\) and \(x_2(t)\) are state variables of two systems, \(U_1\) and \(U_2\) are couplings between systems, respectively. If the error \(e_{1}(t) \equiv |x_1(t) - x_2(t)| \to 0\) when \(t \to \infty\), the complete synchronization between classical systems is realized. If the phases \(\varphi_1(t)\) and \(\varphi_2(t)\) of \(x_1(t)\) and \(x_2(t)\) meet \(\varphi_1(t) \equiv |\varphi_1(t) - \varphi_2(t)| \to 0\), the phase synchronization between the systems is obtained.

The classical synchronization criteria reported previously are mainly to analyze the stability of the error \(x_{\pm}(t)\) or phase error \(\varphi_{\pm}(t)\) and to determine whether \(x_{\pm}(t)\) or \(\varphi_{\pm}(t)\) can converge asymptotically to zero through calculating the largest Lyapunov exponent. In the next section, we will propose a criterion for quantum synchronization based on the classical synchronization theory.

3. Quantum synchronization criterion

In the Heisenberg picture, we use quadrature operators \(q_j(t)\) and \(p_j(t)\) to describe two coupled quantum systems (here, \(j = 1, 2\); \([q_j(t), p_j(t)] = \pm i\hbar\)). Hence, the error operators \(q_{\pm}(t)\) and \(p_{\pm}(t)\) between the systems can be defined as follows

\[ q_{\pm}(t) \equiv \frac{q_1(t) - q_2(t)}{\sqrt{2}} \]
\[ p_{\pm}(t) \equiv \frac{p_1(t) - p_2(t)}{\sqrt{2}} \]  

(2)

It can be seen from equation (2) that the error operators \(q_{\pm}(t)\) and \(p_{\pm}(t)\) are physical quantities that describe the differences between the conjugate mechanical quantities of two systems. However, both \(q_{\pm}(t)\) and \(p_{\pm}(t)\) cannot be very small simultaneously due to the Heisenberg uncertainty relation. Therefore, it is required to consider synthetically both values of two error operators in the quantum synchronization measurement. For this reason, Mari et al introduced the following figure of merit based on equation (2)

\[ S_c(t) = \frac{1}{2} \left( \langle q_{\pm}(t)^2 \rangle + \langle p_{\pm}(t)^2 \rangle \right)^{-1} \]  

(3)

Equation (3) is used to gauge the level of quantum complete synchronization, and its value is in the range \(0 < S_c(t) \leq 1\).

Nevertheless, it is also difficult to discuss quantum synchronization by analyzing \(S_c(t)\) directly because it is not easy to give a definite criterion for judging the synchronization or not. On the other hand, the calculation of \(S_c(t)\) is also complex in quantum systems. Fortunately, the mean value approximation is acceptable in a mesoscopic system, meaning that we can write a mesoscopic system operator \(\delta o(t)\) in the form of \(o(t) = \langle o(t) \rangle + \delta o(t)\); here, \(\langle o(t) \rangle\) is the expectation value of the operator at the moment \(t\) and it can be regarded as a description of ‘classical properties’. \(\delta o(t)\) represents the quantum fluctuation of the operator near its expectation value, and the quantum effects of the system can be embodied in \(\delta o(t)\). For a mesoscopic system, \(\delta o(t)\) is small but cannot be ignored. Then, on the basis of equation (2), the error operators...
of the systems can be rewritten as follow a
\[ q_{-}(t) = \left[ \left( \langle q_{1}(t) \rangle + \delta q_{1}(t) \right) - \left( \langle q_{2}(t) \rangle + \delta q_{2}(t) \right) \right] / \sqrt{2} \]
\[ p_{-}(t) = \left[ \left( \langle p_{1}(t) \rangle + \delta p_{1}(t) \right) - \left( \langle p_{2}(t) \rangle + \delta p_{2}(t) \right) \right] / \sqrt{2} \]
where \( \delta q_{-}(t) = [\delta q_{1}(t) - \delta q_{2}(t)]/\sqrt{2} \) and \( \delta p_{-}(t) = [\delta p_{1}(t) - \delta p_{2}(t)]/\sqrt{2} \). As discussed earlier, the quantum effects of the systems are embodied in quantum fluctuations near their expectation value; that is, \( \delta \phi(t) \) and \( \delta q_{-}(t) \) in the discussion of the synchronization deviation influenced by quantum effects, and equation (3) can be rewritten as
\[ S'_{c}(t) = \left( \delta q_{-}(t)^{2} + \delta p_{-}(t)^{2} \right) \]

It should be noted that \( S'_{c}(t) \) in equation (5) is a measurement of the error operator’s quantum fluctuations near its expectation value. \( S'_{c}(t) \) will equal \( S_{c}(t) \) and it is a synchronization measurement only if \( \langle q_{-}(t) \rangle \rightarrow 0 \) and \( \langle p_{-}(t) \rangle \rightarrow 0 \). Therefore, we define \( S'_{c}(t) \) as a second order measurement to reflect the differences between systems generated by the quantum noise even though the classical synchronization conditions are reached. Correspondingly, \( \langle q_{-}(t) \rangle \rightarrow 0 \) and \( \langle p_{-}(t) \rangle \rightarrow 0 \) can be defined as a first order measurement of quantum synchronization. Now that \( \langle q_{-}(t) \rangle \) and \( \langle p_{-}(t) \rangle \) satisfy the classical properties, the stability analysis method that calculates the largest Lyapunov exponent of the system can be used to determine whether \( \langle q_{-}(t) \rangle \) and \( \langle p_{-}(t) \rangle \) tend stably to zero. Thus, it is only necessary to control the stability of \( \langle q(t) \rangle \) and \( \langle p(t) \rangle \); that is, we can control mesoscopic quantum synchronization instead of analyzing \( S_{c}(t) \) directly.

Similarly, in the discussion on quantum phase synchronization, the phase error operator can be defined as follows
\[ \phi_{c}(t) = \left[ \phi_{1}(t) - \phi_{2}(t) \right] / \sqrt{2} \]

In summary, the synchronization effects in mesoscopic quantum systems can be discussed through the following steps:

a. Write the operator equations of a system’s conjugate mechanical quantities in the Heisenberg picture, define the error operators, and take them as the form of fluctuations near their expectation value; that is, \( \delta \phi(t) \) and \( \delta q(t) \).

b. Perform a stability analysis for \( \langle \phi(t) \rangle \) and calculate the largest Lyapunov exponent of the error equations. If the largest Lyapunov exponent is less than zero, the evolution of \( \langle \phi(t) \rangle \) can tend to zero stably after a certain time; however, it may be ruleless oscillation.

c. If the largest Lyapunov exponent is less than zero, it is necessary to discuss the magnitude of the noise \( \delta \phi(t) \) and to calculate \( S'_{c}(t) \) and \( S''_{c}(t) \) based on equations (5) and (8), respectively. Conversely, if \( S'_{c}(t) \) and \( S''_{c}(t) \) keep a constant but not zero, the synchronization between the quantum systems is achieved.

4. Design of the controlled quantum synchronization model and quantum phase synchronization

A controlled quantum synchronization model is designed based on the quantum optomechanical system in order to check the validity of the previously mentioned quantitative criteria. In this model, we can realize quantum synchronization control through different logical relationships of the switches, shown in figure 1.

Two coupled optomechanical systems are driven by a laser and interact mutually through a phonon tunnel and a fiber, which can be controlled by the opening or closing of the switches \( K_{1} \) and \( K_{2} \). The Hamiltonian of the system can be given directly after a rotating approximation
Here, \( a_j^+ \) and \( a_j \) are the optical creation and annihilation operators for the system \( j \), and \( b_j^+ \) and \( b_j \) are the mechanical creation and annihilation operators. \( \Delta_j \) and \( \omega_j \) are the optical detunings and the mechanical frequencies, respectively. \( g \) is the optomechanical coupling constant and \( E \) is the laser intensity that drives the optical cavities. \( \mu \) is the intensity of the phonon tunnel and \( \lambda \) is the coupling constant of the fiber. The switches \( K_1 \) and \( K_2 \) can change \( \mu \) and \( \lambda \) values from zero to a positive constant by being turned on and off. After considering the dissipative effects, the following quantum Langevin equations can be written in the Heisenberg picture through the input-output properties

\[
\partial_t a_1 = \left[ -\kappa + i\Delta_1 + \text{ig} \left( b_1^+ + b_1 \right) \right] a_1 + E - i\lambda a_2 + \sqrt{2\kappa} a_1^{\text{in}} \\
\partial_t a_2 = \left[ -\kappa + i\Delta_2 + \text{ig} \left( b_2^+ + b_2 \right) \right] a_2 + E - i\lambda a_1 + \sqrt{2\kappa} a_2^{\text{in}} \\
\partial_t b_1 = \left[ -\gamma - i\omega_1 \right] b_1 + \text{ig} a_1^+ a_1 + \text{i} \rho b_2 + \sqrt{2\gamma} b_1^{\text{in}} \\
\partial_t b_2 = \left[ -\gamma - i\omega_2 \right] b_2 + \text{ig} a_2^+ a_2 + \text{i} \rho b_1 + \sqrt{2\gamma} b_2^{\text{in}}
\]

(9)

where \( \kappa \) and \( \gamma \) are the optical and mechanical damping rates. \( a_j^{\text{in}} \) and \( b_j^{\text{in}} \) are the input bath operators, which satisfy \( \langle a_j^{\text{m}}(t) a_j^{\text{m}}(t') + a_j^{\text{m}}(t') a_j^{\text{m}}(t) \rangle = \delta_{jj} \delta(t-t') \) and \( \langle b_j^{\text{m}}(t) b_j^{\text{m}}(t') + b_j^{\text{m}}(t') b_j^{\text{m}}(t) \rangle = (2\lambda_{bb} + 1) \delta_{jj} \delta(t-t') \), where \( \lambda_{bb} = \exp \left( \frac{-i u T}{2m} \right) - 1 \)^{-1}.

Using the expectation value and quantum fluctuation to replace operators from equation (10), we can get the following two equations.

\[
\begin{pmatrix}
-\kappa & -\Delta_1 - 2g \text{ Re} \left[ B_1 \right] & 0 \\
\Delta_1 + 2g \text{ Re} \left[ B_1 \right] & -\kappa & -\lambda \\
0 & -\lambda & -\kappa \\
0 & -\kappa & -\Delta_2 - 2g \text{ Re} \left[ B_2 \right] \\
2g \text{ Re} \left[ A_1 \right] & 2g \text{ Im} \left[ A_1 \right] & 0 \\
0 & 0 & 2g \text{ Re} \left[ A_2 \right]
\end{pmatrix}
\begin{pmatrix}
\lambda \mu \\
\end{pmatrix}
= \begin{pmatrix}
-2g \text{ Im} \left[ A_1 \right] & 0 & 0 & 0 \\
0 & 2g \text{ Re} \left[ A_1 \right] & 0 & 0 \\
0 & 0 & -2g \text{ Im} \left[ A_2 \right] & 0 \\
0 & 0 & 0 & 2g \text{ Re} \left[ A_2 \right]
\end{pmatrix}
\begin{pmatrix}
\lambda \mu \\
\end{pmatrix}
\]

(10)

where \( A_j = \langle a_j \rangle \) and \( B_j = \langle b_j \rangle \).

The parts of the quantum fluctuation are

\[
\partial_t \delta a_1 = \left[ -\kappa + i\Delta_1 + \text{ig} \left( B_1^+ + B_1 \right) \right] \delta a_1 + \text{ig} A_1 \left( \delta b_1^+ + \delta b_1 \right) - \text{i} \rho \delta a_2 + \sqrt{2\lambda} \delta a_1^{\text{in}} \\
\partial_t \delta a_2 = \left[ -\kappa + i\Delta_2 + \text{ig} \left( B_2^+ + B_2 \right) \right] \delta a_2 + \text{ig} A_2 \left( \delta b_1^+ + \delta b_2 \right) - \text{i} \rho \delta a_1 + \sqrt{2\lambda} \delta a_2^{\text{in}} \\
\partial_t \delta b_1 = \left[ -\gamma - i\omega_1 \right] \delta b_1 + \text{ig} A_1^+ \delta a_1 + \rho \delta b_2 + \sqrt{2\gamma} \delta b_1^{\text{in}} \\
\partial_t \delta b_2 = \left[ -\gamma - i\omega_2 \right] \delta b_2 + \text{ig} A_2^+ \delta a_2 + \rho \delta b_1 + \sqrt{2\gamma} \delta b_2^{\text{in}}
\]

(11)

The dynamic properties of the cavity and oscillator can be described by their own conjugate mechanical quantities, i.e.,

\[
x_j = (a_j^+ + a_j)/\sqrt{2} \\
y_j = (a_j^+ - a_j)/\sqrt{2} \\
q_j = (b_j^+ + b_j)/\sqrt{2} \\
p_j = (b_j^+ - b_j)/\sqrt{2}
\]

(12)

Substituting equation (13) into equation (12), equation (12) can be expressed in the matrix form

\[
\partial_t u = Su + \xi
\]

(14)

where \( u \) is a vector \((\delta x_1, \delta y_1, \delta x_2, \delta y_2, \delta q_1, \delta p_1, \delta q_2, \delta p_2)^T\) and \( \xi \) means an input vector \((\delta x_1^{\text{in}}, \delta y_1^{\text{in}}, \delta x_2^{\text{in}}, \delta y_2^{\text{in}}, \delta q_1^{\text{in}}, \delta p_1^{\text{in}}, \delta q_2^{\text{in}}, \delta p_2^{\text{in}})^T\). \( S \) is an \( 8 \times 8 \) time-dependent matrix.

\[
\begin{pmatrix}
-\kappa & -\Delta_1 - 2g \text{ Re} \left[ B_1 \right] & 0 \\
\Delta_1 + 2g \text{ Re} \left[ B_1 \right] & -\kappa & -\lambda \\
0 & -\lambda & -\kappa \\
0 & -\kappa & -\Delta_2 - 2g \text{ Re} \left[ B_2 \right] \\
2g \text{ Re} \left[ A_1 \right] & 2g \text{ Im} \left[ A_1 \right] & 0 \\
0 & 0 & 2g \text{ Re} \left[ A_2 \right]
\end{pmatrix}
\begin{pmatrix}
\lambda \mu \\
\end{pmatrix}
= \begin{pmatrix}
-2g \text{ Im} \left[ A_1 \right] & 0 & 0 & 0 \\
0 & 2g \text{ Re} \left[ A_1 \right] & 0 & 0 \\
0 & 0 & -2g \text{ Im} \left[ A_2 \right] & 0 \\
0 & 0 & 0 & 2g \text{ Re} \left[ A_2 \right]
\end{pmatrix}
\begin{pmatrix}
\lambda \mu \\
\end{pmatrix}
\]

(15)
For the convenience of calculation, a covariance matrix \( C \) is defined as
\[
c_{ij}(t) = c_{ji}(t) = \frac{1}{2} \{ u_i(t) u_j(t) + u_j(t) u_i(t) \}
\]
and the evolution of matrix \( C \) can be determined by equation (17):
\[
\partial_t C = SC + CS^\top + N
\]
where \( N \) is a diagonal noise correlation matrix defined by:
\[
N_{ij}(t-t') = \frac{1}{2} \{ \xi_i(t) \bar{\xi}_j(t') + \bar{\xi}_j(t') \xi_i(t) \}
\]
(18)

At this point, the dynamical analysis of the model we used has already been finished and the system will evolve according to equations (11) and (14). Subsequently, we are going to discuss the synchronization effects between the systems by using those equations.

We will discuss the phase synchronization between the oscillators of the systems. First, define the 'classical' part (expectation value) of the phase error on the basis of equation (6):
\[
\langle \phi_i(t) \rangle = \left[ \langle \phi_1(t) \rangle - \langle \phi_2(t) \rangle \right] / \sqrt{2} = \theta(t) / \sqrt{2}
\]
where
\[
\langle \phi_1(t) \rangle = \text{arg}[B1(t)]
\]
\[
\langle \phi_2(t) \rangle = \text{arg}[B2(t)]
\]  
(19)

The evolution of \( \theta(t) \) can be simulated numerically with equations (11), (19), and (20) simultaneously under certain initial conditions, and the largest Lyapunov exponent of the error can be calculated by the following equation
\[
L_\gamma = \lim_{t \to \infty} - \ln \left| \frac{\delta \theta(t)}{\delta \theta(0)} \right|
\]
(21)

\( \delta \theta(0) \) and \( \delta \theta(t) \) in equation (21) mean the disturbances of the phase errors when \( t = 0 \) and \( t = t \), respectively, and \( L_\gamma \) can be seen as the eigenvalue of the Jacobian matrix corresponding to \( \delta \theta \). Because the phase synchronization is controlled by the intensity of the phonon tunnel and the coupling constant of the fiber, we calculate the largest Lyapunov exponent of phase error with the variation of \( \mu \) and \( \lambda \). In calculation, the damping rates and the intensity of the driving field are assumed to be equal in both systems, but there are differences in frequencies and initial conditions. The values of \( \omega_1, g, \kappa \) and \( \gamma \) are the same as in Mari’s work so that the conclusion is more easily verified by the experiment. Moreover, we properly reduce the intensity of the driving field in order to highlight the coupling function in synchronization. Otherwise, a driving field that is too strong will dilute the coupling effect, which leads the system to the ‘forced’ synchronous effect. It can be known from equations (9) and (11) that the phonon coupling can directly influence mechanical oscillators; however, the photon coupling can only influence them indirectly by changing the light field in the cavities. In order for two switches to have similar ability to control synchronization, we reduce the intensity of phonon channel (\( \mu \)) and increase the coupling constant of the fiber. Therefore, we calculate the Lyapunov exponent in the region of \( \lambda \in [0, 0.2], \mu \in [0, 0.01] \) and the calculation result is shown in figure 2.

Instead of analyzing the largest Lyapunov exponent concretely, as discussed earlier, we can determine whether the system is in the synchronization state only by comparing the largest Lyapunov exponent with zero. Therefore, figure 2 is redrawn in this form: some parameter regions where the largest Lyapunov exponent is greater than zero are projected and marked in red; conversely, the regions where the largest Lyapunov exponent is less than zero are marked in blue. Moreover, we also plot a curve of the Lyapunov exponent with a fixed \( \lambda \) in order to display it more clearly, shown in figure 3.

If \( \mu \) and \( \lambda \) are in the blue regions of figure 3(a), the largest Lyapunov exponent is less than zero, indicating the evolution of the phase error tends to zero stably after a certain time and the systems reach synchronization. By contrast, the largest Lyapunov exponent is greater than zero, while \( \mu \) and \( \lambda \) are in the red regions, and the systems are not synchronous because the phase error tends to random oscillations. Therefore, it can be seen from figure 3(a) that the systems will not synchronize when they are not coupled (\( \mu = 0 \) and \( \lambda = 0 \)). Once there is coupling between systems, however, the red regions will be replaced gradually by blue area with the increase of \( \mu \) and \( \lambda \). We expect to control the synchronization by the switches \( K_1 \) and \( K_2 \) together, i.e., it will happen only if two switches meet the logic 'AND'. Then we choose the parameters according to the following principles: the largest Lyapunov exponent is greater than zero when \( \mu = 0 \) and \( \lambda \neq 0 \) as well as \( \mu \neq 0 \) and \( \lambda = 0 \), but it must be less than zero as \( \mu \neq 0 \) and \( \lambda \neq 0 \) at the
same time. According to this, the appropriate parameters can be found in $\lambda \in [0.14, 0.2]$ and $\mu \in [0.004, 0.007]$ based on figure 3, and within the range the systems will achieve phase synchronization only if switches $K_1$ and $K_2$ are closed synchronously. (It is worth noting that a lone point with a different color from the surrounding ones can be ignored as an error.) In other words, we can control the synchronization effect with switches $K_1$ and $K_2$ by aforementioned characteristics.

In the previous discussion, we give a range of parameters $\mu$ and $\lambda$ but not exact values. This is because the ‘classical’ error can tend to zero for all parameters in that range. For quantum synchronization, not only the expectation value of errors tending to zero is necessary, but also the error fluctuations need to be as small as possible. For the sake of reaching a perfect synchronization, we need to select appropriate values of $\mu$ and $\lambda$ and to ensure minimal quantum fluctuations. It is the reason why we calculate the second order measurement $S'_p(t)$ of the quantum phase synchronization.

To calculate $S'_p(t)$, the matrix $C$ defined in equation (16) is transformed first as

$$C'(t) = U(t)C(t)U(t)^\dagger$$

(22)

here $U(t) = \text{diag}[e^{-i\phi_{a_1}}, e^{i\phi_{a_1}}, \ldots]$ and $\phi_{a_1} = \arg(a_1(t))$, $\phi_{a_2} = \arg(a_2(t))$, ..., $C'(t)$ can be obtained by substituting the matrix $C'$ into equation (17) and $S'_p(t)$ can be expressed as

$$S'_p(t) = \frac{1}{2} \left\langle \frac{\delta \phi(t)^2}{\delta t} \right\rangle^{-1}$$

$$= \frac{1}{2} \left\langle \frac{1}{2} \left( \delta p_1^2 + \delta p_2^2 - 2\delta p'_1 \delta p'_2 \right) \right\rangle^{-1}$$

$$= \frac{1}{2} \frac{1}{2} \left( C'_{66} + C'_{68} - 2C'_{68} \right)^{-1}. \quad (23)$$

Time-averaged $S'_p(t)$ is further calculated in order to show directly the size of quantum fluctuation under the different parameters.

$$\overline{S'_p} = \lim_{T \to \infty} \frac{1}{T} \int_0^T S'_p(t) dt$$

(24)

and the calculation result is shown in figure 4.

Although the phase expectation values of every system tend to be equal, the quantum fluctuations between the systems under different parameters still influence the perfection of quantum phase synchronization, as shown in figure 4. The fluctuation of system error will be reduced to the minimum extent while taking $\mu = 0.004$ and $\lambda = 0.16$, which draws the conclusion that the best effect of synchronization has been reached.

The dynamical evolution of the system is simulated here to test the validity of our criterion. Before the simulation, we let $\mu = 0.004$ and $\lambda = 0.16$ and assume that the initial phase error between the systems is $\theta(0) = \frac{\pi}{2}$. The remaining parameters are the same as the ones used in figure 2. The

Figure 3. (a): Comparison between the largest Lyapunov exponent and zero. Blue areas mean the largest Lyapunov exponent is less than zero, and red areas mean the largest Lyapunov exponent is greater than zero. (b): Evolution of the largest Lyapunov exponent with varied $\mu$.

Figure 4. $S'_p$ with varied $\mu$ and $\lambda$. In this calculation, we let $T = 2000$ and other parameters are same as figure 2.
Simulation results are illustrated in figure 5 in which the unit of ordinate is $\pi$. It can be known from figures 5(a)–(c) that the synchronization between systems will not be achieved as long as any switch of $K_1$ and $K_2$ is opened. Upon further inspection, we notice that two different systems will never achieve phase synchronization in the other parameters ($\omega$, $g$, $\kappa$ and $\gamma$) but will have the same $E$ when couples disappear. Only when two switches are both closed can the systems realize synchronization, shown in figure 5(d). This result is identical with our analysis, and it also verifies the quantum synchronous criterion proposed in our work. By the way, the logical relationship ‘AND’ of two switches is taken as an example in figure 3; however, the logical relationships ‘OR’ or ‘exclusive-OR’ between the switches can also be selected to realize the quantum synchronization of the systems by adjusting appropriate parameters of $\mu$ and $\lambda$.

5. Conclusions

In this paper, we investigate quantum synchronization effects and present the quantitative criteria of complete synchronization and phase synchronization between quantum systems. Further, we realize the quantum phase synchronization between coupled optomechanical systems by using our criteria. By calculating the largest Lyapunov exponent, we find that the systems will not reach synchronization unless switches $K_1$ and $K_2$ are closed synchronously (satisfying the logic relation ‘AND’) when the parameter values are taken at the ranges $\lambda \in [0.14, 0.2]$ and $\mu \in [0.004, 0.007]$. At the same time, we obtain the information that the fluctuation of system error will reduce to a minimum while $\mu = 0.004$ and $\lambda = 0.16$ by calculating the second order measurement $S^\prime_p(t)$ of the quantum phase synchronization. Finally, the dynamical evolution of the system is simulated in order to test the validity of our criterion under the aforementioned parameters. Since the concrete quantum synchronization criteria have been proposed and the control theory of quantum synchronization effects is simple and efficient in the work, other designers can set different synchronization conditions to satisfy their aims. We believe that our work can bring certain application values in quantum communication, quantum control, and quantum logical gates.

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Figure 5. Evolution of the phase error between the systems. (a): $K_1$ and $K_2$ are both opened ($\mu = 0$, $\lambda = 0$); (b): $K_1$ is opened and $K_2$ is closed ($\mu = 0$, $\lambda = 0.16$); (c): $K_1$ is closed and $K_2$ is opened ($\mu = 0.004$, $\lambda = 0$) and (d): $K_1$ and $K_2$ are both closed ($\mu = 0.004$, $\lambda = 0.16$).
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