On interrelations between strongly, weakly and chord separated set-systems (a geometric approach)

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Abstract. We consider three types of set-systems that have interesting applications in algebraic combinatorics and representation theory: maximal collections of the so-called strongly separated, weakly separated, and chord separated subsets of a set \([n] = \{1, 2, \ldots, n\}\). These collections are known to admit nice geometric interpretations; namely, they are bijective, respectively, to rhombus tilings on the zonogon \(Z(n, 2)\), combined tilings on \(Z(n, 2)\), and fine zonotopal tilings (or “cubillages”) on the 3-dimensional zonotope \(Z(n, 3)\). We describe interrelations between these three types of set-systems in \(2^n\), by studying interrelations between their geometric models. In particular, we completely characterize the sets of rhombus and combined tilings properly embeddable in a fixed cubillage, explain that they form distributive lattices, give efficient methods of extending a given rhombus or combined tiling to a cubillage, and etc.

Keywords: strongly separated sets, weakly separated sets, chord separated sets, rhombus tiling, cubillage, higher Bruhat order

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1 Introduction

For a positive integer \(n\), the set \(\{1, 2, \ldots, n\}\) with the usual order is denoted by \([n]\). For a set \(X \subseteq [n]\) of elements \(x_1 < x_2 < \ldots < x_k\), we may write \(x_1 x_2 \ldots x_k\) for \(X\), \(\min(X)\) for \(x_1\), and \(\max(X)\) for \(x_k\) (where \(\min(X) = \max(X) := 0\) if \(X = \emptyset\)). We use three binary relations on the set \(2^n\) of all subsets of \([n]\). Namely, for subsets \(A, B \subseteq [n]\), we write:

\[(1.1) \quad (i) \ A < B \text{ if } \max(A) < \min(B) \text{ (global dominating);} \]
\[(ii) \quad A \prec B \text{ if } (A - B) < (B - A), \text{ where } A' - B' \text{ denotes } \{i' : A' \ni i' \notin B'\} \text{ (global dominating after cancelations);} \]

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(iii) \( A \succ B \) if \( A - B \neq \emptyset \), and \( B - A \) can be expressed as a union of nonempty subsets \( B', B'' \) so that \( B' < (A - B) < B'' \) (splitting).

We also say that \( A \) surrounds \( B \) if there are no elements \( i < j < k \) of \([n]\) such that \( i, k \in B - A \) and \( j \in A - B \) (equivalently, if either \( A = B \) or \( A < B \) or \( B < A \) or \( B \succ A \)). The above relations are used in the following notions.

**Definitions.** Following Leclerc and Zelevinsky [7], sets \( A, B \subseteq [n] \) are called strongly separated (from each other) if \( A < B \) or \( B < A \) or \( A = B \), and called weakly separated if either \(|A| \leq |B|\) and \( A \) surrounds \( B \), or \(|B| \leq |A|\) and \( B \) surrounds \( A \) (or both). Following terminology of Galashin [5], \( A, B \subseteq [n] \) are called chord separated if one of \( A, B \) surrounds the other (equivalently, if there are no elements \( i < j < k < l \) of \([n]\) such that \( i, k \) belong to one, and \( j, \ell \) to the other set among \( A - B \) and \( B - A \)).

(The third notion for \( A, B \) is justified by the observation that if \( n \) points labeled 1, 2, \ldots, \( n \) are disposed on a circumference \( O \), in this order cyclically, then there exists a chord to \( O \) separating \( A - B \) from \( B - A \).)

Accordingly, a collection (set-system) \( F \subseteq 2^{[n]} \) is called strongly (resp. weakly, chord) separated if any two of its members are such. For brevity, we refer to strongly, weakly, and chord separated collections as \( s-, w-, \) and \( c\)-collections, respectively. In the hierarchy of these collections, any \( s\)-collection is a \( w\)-collection, and any \( w\)-collection is a \( c\)-collection, but the converse need not hold. Such collections are encountered in interesting applications (in particular, \( w\)-collections appeared in [7] in connection with the problem of quasi-commuting flag minors of a quantum matrix). Also they admit impressive geometric-combinatorial representations (which will be discussed later).

An important fact is that these three sorts of collections possess the property of purity. More precisely, we say that a domain \( D \subseteq 2^{[n]} \) is \( s\)-pure (\( w\)-pure, \( c\)-pure) if all inclusion-wise maximal \( s\)-collections (resp. \( w\)-collections, \( c\)-collections) in \( D \) have the same cardinality, which in this case is called the \( s\)-rank (resp. \( w\)-rank, \( c\)-rank) of \( D \). We will rely on the following results on the full domain \( 2^{[n]} \).

1. (1.2) \[ 2^{[n]} \text{ is } s\text{-pure and its } s\text{-rank is equal to } \binom{n}{2} + \binom{n}{1} + \binom{n}{0} = \frac{1}{2}n(n+1)+1. \]
2. (1.3) \[ 2^{[n]} \text{ is } w\text{-pure and its } w\text{-rank is equal to } \binom{n}{2} + \binom{n}{1} + \binom{n}{0}. \]
3. (1.4) \[ 2^{[n]} \text{ is } c\text{-pure and its } c\text{-rank is equal to } \binom{n}{3} + \binom{n}{2} + \binom{n}{1} + \binom{n}{0}. \]

(The phenomenon of \( w\)-purity has also been established for some other important domains, see [9, 8, 10]; however, those results are beyond the main stream of our paper.)

As is seen from (1.2)–(1.4), the \( c\)-rank of \( 2^{[n]} \) is at \( O(n) \) times larger that its \( s\)- and \( w\)-ranks (which are equal), and we address the following issue: given a maximal \( c\)-collection \( C \subset 2^{[n]} \), what can one say about the sets \( S(C) \) and \( W(C) \) of inclusion-wise maximal \( s\)-collections and \( w\)-collections, respectively, contained in \( C \)?

It turns out that a domain \( C \) of this sort need not be \( s\)-pure or \( w\)-pure in general, as we show by an example in Sect. 3.1. Nevertheless, the sets of \( s\)-collections and \( w\)-collections contained in \( C \) and having the maximal size (equal to \( \frac{1}{2}n(n+1)+1 \),
denoted as $S^*(C)$ and $W^*(C)$, respectively, have nice structural properties, and to present them is just the main purpose of this paper.

On this way, we are based on the following known geometric-combinatorial constructions for s-, w-, and c-collections. As it follows from results in [1], each maximal s-collection in $2^{[n]}$ corresponds to the vertex set of a rhombus tiling on the n-zonogon in the plane, and vice versa. A somewhat more sophisticated planar structure, namely, the so-called combined tilings, or combies, on the n-zonogon are shown to represent the maximal w-collections in $2^{[n]}$, see [3]. As to the maximal c-collections, Galashin [5] recently showed that they are bijective to subdivisions of the 3-dimensional zonotope $Z(n,3)$ into parallelotopes. For brevity, we liberally refer to such subdivisions as cubillages, and its elements (parallelotopes) as cubes.

In this paper, we first discuss interrelations between strongly and chord separated set-systems. A brief outline is as follows.

(a) For a maximal c-collection $C \subset 2^{[n]}$, let $Q = Q(C)$ be its associated cubillage (where the elements of $C$ correspond to the 0-dimensional cells, or vertices, of $Q$ regarded as a complex). Then for each $S \in S^*(C)$, its associated rhombus tiling $T(S)$ is viewed (up to a piecewise linear deformation) as a 2-dimensional subcomplex of $Q$, called an s-membrane in it. Furthermore, these membranes (and therefore the members of $S^*(C)$) form a distributive lattice with the minimal and maximal elements to be the “front side” $Z_{fr}$ and “rear side” $Z_{re}$ of the boundary subcomplex of $Z(n,3)$, respectively. This lattice is “dense”, in the sense that any two s-collections whose s-membranes are neighboring in the lattice are obtained from each other by a standard flip, or mutation (which involves a hexagon, or, in terminology of Leclerc and Zelevinsky [7], is performed “in the presence of six witnesses”).

(b) It is natural to raise a “converse” issue: given a maximal s-collection $S \subset 2^{[n]}$, what can one say about the set $C(S)$ of maximal c-collections containing $S$? One can efficiently construct an instance of such c-collections, by embedding the tiling $T(S)$ (as an s-membrane) into the “empty” zonotope $Z(n,3)$ and then by growing, step by step (or cube by cube), a required cubillage containing $T(S)$. In fact, the set of cubillages for $C(S)$ looks like a “direct product” of two sets $Q^{-}$ and $Q^{+}$, where the former (latter) is formed by partial cubillages consisting of “cubes” filling the volume of $Z(n,3)$ between the surfaces $Z_{fr}$ and $T(S)$ (resp. between $T(S)$ and $Z_{re}$).

A somewhat similar programme is fulfilled for w-collections, and on this way, we obtain main results of this paper. We consider a maximal c-collection $C \subset 2^{[n]}$ and cut each cube of the cubillage $Q$ associated with $C$ into two tetrahedra and one octahedron, forming a subdivision of $Z(n,3)$ into smaller pieces, denoted as $Q^{2}$ and called the fragmentation of $Q$. We show that each combi $K(W)$ associated with a maximal by size w-collection $W \subset W^*(C)$ is related to a set of 2-dimensional subcomplexes of $Q^{2}(C)$, called w-membranes. Like s-membranes, the set of all w-membranes of $Q^{2}$ are shown to form a distributive lattice with the minimal element $Z_{fr}$ and the maximal element $Z_{re}$, and any two neighboring w-membranes in the lattice are linked by either a tetrahedral flip or an octahedral flip (the latter corresponds, for a w-collection, to a “mutation in the presence of four witnesses”, in terminology of [7]). As to the “converse
direction”, we consider a fixed maximal w-collection \( W \subset 2^{[n]} \) and develop an efficient geometric method to construct a cubillage containing the combi \( K(W) \). Also additional results on interrelations between s- and w-collections from one side, and c-collections from the other side are presented.

This paper is organized as follows. Section 2 recalls definitions of rhom bus tilings and combined tilings on a zonogon and fine zonotopal tilings (“cubillages”) on a 3-dimensional zonotope, and reviews their relations to maximal s-, w-, and c-collections in \( 2^{[n]} \). Section 3 starts with an example of a maximal c-collection in \( Z(n, 3) \) that is neither s-pure nor w-pure. Then it introduces s-membranes in a cubillage, discusses their relation to rhombus tilings, and describes transformations of cubillages on \( \mathbb{Z}(n, 3) \) to ones on \( Z(n-1, 3) \) and back, that are needed for further purposes. Section 4 studies the structure of the set of s-membranes in a fixed cubillage and, as a consequence, describes the lattice \( S^*(C) \). Section 5 discusses the task of constructing a cubillage containing one or two prescribed rhombus tilings. Then we start studying interrelations between maximal w- and c-collections. In Section 6 we introduce w-membranes in the fragmentation \( Q^\otimes \) of a fixed cubillage \( Q \), explain that they form a lattice, demonstrate a relationship to combined tilings, and more. The concluding Section 7 is devoted to the task of extending a given combi to a cubillage, which results in an efficient algorithm of finding a maximal c-collection in \( 2^{[n]} \) containing a given maximal w-collection.

2 Backgrounds

In this section we recall the geometric representations for s-, w-, and c-collections that we are going to use. For disjoint subsets \( A \) and \{\( a, \ldots, b \)\} of \([n]\), we use the abbreviated notation \( A a \ldots b \) for \( A \cup \{a, \ldots, b\} \), and write \( A - c \) for \( A - \{c\} \) when \( c \in A \).

2.1 Rhombus tilings

Let \( \Xi = \{\xi_1, \ldots, \xi_n\} \) be a system of \( n \) non-colinear vectors in the upper hyperplane \( \mathbb{R} \times \mathbb{R}_{\geq 0} \) that follow in this order clockwise around \((0, 0)\). The zonogon generated by \( \Xi \) is the \( 2n \)-gone that is the Minkowski sum of segments \([0, \xi_i], i = 1, \ldots, n\), i.e., the set

\[
Z = Z_{\Xi} := \{\lambda_1 \xi_1 + \ldots + \lambda_n \xi_n : \lambda_i \in \mathbb{R}, 0 \leq \lambda_i \leq 1, i = 1, \ldots, n\},
\]

also denoted as \( Z(n, 2) \). A tiling that we deal with is a subdivision \( T \) of \( Z \) into tiles, each being a parallelogram of the form \( \sum_{k \in X} \xi_k + \{\lambda \xi_i + \lambda' \xi_j : 0 \leq \lambda, \lambda' \leq 1\} \) for some \( i < j \) and some subset \( X \subseteq [n] - \{i, j\} \). In other words, the tiles are not overlapping (have no common interior points) and their union is \( Z \). A tile determined by \( X, i, j \) as above is called an \( ij \)-tile and denoted as \( \rho(X|ij) \).

We identify each subset \( X \subseteq [n] \) with the point \( \sum_{i \in X} \xi_i \) in \( Z \) (assuming that the generators \( \xi_i \) are \( \mathbb{Z} \)-independent). Depending on the context, we may think of \( T \) as a 2-dimensional complex and associate to it the planar directed graph \((V_T, E_T)\) in which each vertex (0-dimensional cell) is labeled by the corresponding subset of \([n]\) and each edge (1-dimensional cell) that is a parallel translate of \( \xi_i \) for some \( i \) is called an \( i \)-edge,
or an edge of type (or color) $i$. In particular, the left boundary of the zonogon is the directed path $(v_0, e_1, v_1, \ldots, e_n, v_n)$ in which each vertex $v_i$ is the set $[i]$ (and $e_i$ is an i-edge), whereas the right boundary of $Z$ is the directed path $(v'_0, e'_1, v'_1, \ldots, e'_n, v'_n)$ with $v'_i = [n] - [n - i]$ (and $e'_i$ being an $(n - i + 1)$-edge).

We call the vertex set $V_T$ (regarded as a set-system in $2^{[n]}$) the spectrum of $T$. In fact, the graphic structure of $T$ (and therefore its spectrum) does not depend on the choice of generating vectors $\xi_i$ (by keeping their ordering clockwise). In the literature one often takes vectors of the same euclidean length, in which case each tile becomes a rhombus and $T$ is called a rhombus tiling. In what follows we will liberally use this term whatever generators $\xi_i$ are chosen.

One easily shows that for any $1 \leq i < j \leq n$, there exists a unique $ij$-tile, or $ij$-rhombus, in $T$. The central property of rhombus tilings is as follows.

**Theorem 2.1** [7] The correspondence $T \mapsto V_T$ gives a bijection between the set $RT_n$ of rhombus tilings on $Z(n, 2)$ and the set $S_n$ of maximal s-collections in $2^{[n]}$.

In particular, each maximal s-collection $S$ determines a unique rhombus tiling $T$ with $V_T = S$, and this $T$ is constructed easily: each pair of vertices of the form $X, Xi$ is connected by (straight line) edge from $X$ to $Xi$; then the resulting graph is planar and all its faces are rhombi, giving $T$. Two rhombus tilings play an especial role. The spectrum of one, called the standard tiling and denoted as $T_{st}^n$, is formed by all intervals in $[n]$, i.e., the sets $I_{ij} := \{i, i + 1, \ldots, j\}$ for $1 \leq i \leq j \leq n$, plus the “empty interval” $\emptyset$. The other one, called the anti-standard tiling and denoted as $T_{ant}^n$, has the spectrum consisting of all co-intervals, the sets of the form $[n] - I_{ij}$. These two tilings for $n = 4$ are illustrated on the picture.

Next, as it follows from results in [7], $RT_n$ is endowed with a poset structure. In this poset, $T_{st}^n$ and $T_{ant}^n$ are the unique minimal and maximal elements, respectively, and a tiling $T$ immediately precedes a tiling $T'$ if $T'$ is obtained from $T$ by one strong (or hexagonal) raising flip (and accordingly $T$ is obtained from $T'$ by one strong lowering flip). This means that

\[(2.1) \text{there exist } i < j < k \text{ and } X \subseteq [n] - \{i, j, k\} \text{ such that: } T \text{ contains the vertices } X, Xi, Xj, Xk, Xij, Xjk, \text{ and the set } V_{T'} \text{ is obtained from } V_T \text{ by replacing } Xj \text{ by } Xik.\]

(This transformation is called in [7] a “mutation in the presence of six witnesses”, namely, $X, Xi, Xk, Xij, Xjk, Xijk$.) See the picture.
We denote the corresponding hexagon in $T$ as $H = H(X|ijk)$ and say that $H$ has \textit{Y-configuration} (\textit{\Lambda-configuration}) if the three rhombi spanning $H$ are as illustrated in the left (resp. right) fragment of the above picture.

\subsection*{2.2 Combined tilings}

For tilings of this sort, the system $\Xi$ generating the zonogon is required to satisfy the additional condition of \textit{strict convexity}, namely: for any $1 \leq i < j < k \leq n$,

$$\xi_j = \lambda \xi_i + X' \xi_k, \quad \text{where } \lambda, X' \in \mathbb{R}_{>0} \text{ and } \lambda + X' > 1.$$  \hfill (2.2)

Besides, we use vectors $\epsilon_{ij} := \xi_j - \xi_i$ for $1 \leq i < j \leq n$. A \textit{combined tiling}, or simply a \textit{combi}, is a subdivision $K$ of $\mathbb{Z}_2$ into certain polygons specified below. Like the case of rhombus tilings, a combi $K$ may be regarded as a complex and we associate to it a planar directed graph $(V_K, E_K)$ in which each vertex corresponds to some subset of $\{k\}$ and each edge is a parallel translate of either $\xi_i$ or $\epsilon_{ij}$ for some $i, j$. In the later case we say that the edge has \textit{type} $ij$. We call $V_K$ the \textit{spectrum} of $K$.

There are three sorts of tiles in $T$: $\Delta$-tiles, $\nabla$-tiles, and lenses. A $\Delta$-tile ($\nabla$-tile) is a triangle with vertices $A, B, C \subseteq \{n\}$ and edges $(B, A), (C, A), (B, C)$ (resp. $(A, C), (A, B), (B, C)$) of types $i, j$ and $ij$, respectively, where $i < j$. For purposes of Sect. 6 we denote this tile as $\Delta(A|ij)$ (resp. $\nabla(A|ij)$), call $(B, C)$ its \textit{base} edge and call $A$ its \textit{top} (resp. \textit{bottom}) vertex. See the left and middle fragments of the picture.

In a \textit{lens} $\lambda$, the boundary is formed by two directed paths $U_\lambda$ and $L_\lambda$, with at least two edges in each, having the same beginning vertex $\ell_\lambda$ and the same end vertex $r_\lambda$; see the right fragment of the above picture. The upper boundary $U_\lambda = (v_0, e_1, v_1, \ldots, e_p, v_p)$ is such that $v_0 = \ell_\lambda$, $v_p = r_\lambda$, and $v_k = X_i k$ for $k = 0, \ldots, p$, where $p \geq 2$, $X \subseteq \{n\}$ and $0 < i_0 < i_1 < \cdots < i_p$ (so $k$-th edge $e_k$ is of type $i_{k-1} i_k$). And the lower boundary $L_\lambda = (u_0, e'_1, u_1, \ldots, e'_q, u_q)$ is such that $u_0 = \ell_\lambda$, $u_q = r_\lambda$, and $u_m = Y - j_m$ for $m = 0, \ldots, q$, where $q \geq 2$, $Y \subseteq \{n\}$ and $j_0 > j_1 > \cdots > j_q$ (so $m$-th edge $e'_m$ is of type $j_m, j_{m-1}$). Then $Y = X i_0 j_0 = X i_p j_q$, implying $i_0 = j_q$ and $i_p = j_0$, and we say that the lens $\lambda$ has \textit{type} $i_0 j_0$. Note that $X$ as well as $Y$ need not be a vertex in $K$; we refer to
and $Y$ as the *lower* and *upper root* of $\lambda$, respectively. Due to condition (2.2), each lens $\lambda$ is a convex polygon of which vertices are exactly the vertices of $U_\lambda \cup L_\lambda$.

**Remark 1.** In the definition of a combi introduced in [3], the generators $\xi_i$ are assumed to have the same euclidean length. However, taking arbitrary (cyclically ordered) generators subject to (2.2) does not affect, in essence, the structure of the combi and its spectrum, and in what follows we will vary the choice of generators when needed. Next, to simplify visualizations, it will be convenient to think of edges of type $i$ as “almost vertical”, and of edges of type $ij$ as “almost horizontal”; following terminology of [3], we refer to the former edges as $V$-edges, and to the latter ones as $H$-edges. Note that any rhombus tiling turns into a combi without lenses in a natural way: each rhombus is subdivided into two “semi-rhombi” $\Delta$ and $\nabla$ by drawing the “almost horizontal” diagonal in it.

The picture below illustrates a combi $K$ having one lens $\lambda$ for $n = 4$; here the $V$-edges and $H$-edges are drawn by thick and thin lines, respectively.

![Diagram of a combi

We will rely on the following central result on combies.

**Theorem 2.2** [3] The correspondence $K \mapsto V_K$ gives a bijection between the set $K_n$ of combined tilings on $\mathbb{Z}(n,2)$ and the set $W_n$ of maximal $w$-collections in $2^n$.

In particular, each maximal $w$-collection $W$ determines a unique combi $K$ with $V_K = W$, and [3] explains how to construct this $K$ efficiently.

By results in [1, 2, 3], the set $K_n$ forms a poset in which $T_{\text{st}}$ and $T_{\text{ant}}$ are the unique minimal and maximal elements, respectively, and a combi $K$ immediately precedes a combi $K'$ if $K'$ is obtained from $K$ by one *weak raising flip* (and accordingly $K$ is obtained from $K'$ by one *weak lowering flip*). This means that

\[(2.3) \text{there exist } i < j < k \text{ and } X \subseteq [n] - \{i, j, k\} \text{ such that: } K \text{ contains the vertices } X_i, X_j, X_k, X_{ij}, X_{jk}, \text{ and the set } V_{K'} \text{ is obtained from } V_K \text{ by replacing } X_j \text{ by } X_{ik}.\]

(Using terminology of [7], one says that $V_K$ and $V_{K'}$ are linked by a “mutation in the presence of four witnesses”, namely, $X_i, X_k, X_{ij}, X_{jk}$.)
2.3 Cubillages

Now we deal with the zonotope generated by a special cyclic configuration \( \Theta \) of vectors in the space \( \mathbb{R}^3 \) with coordinates \((x, y, z)\). It consists of \( n \) vectors \( \theta_i = (x_i, y_i, 1) \), \( i = 1, \ldots, n \), with the following strict convexity condition:

\[
(2.4) \quad x_1 < x_2 < \cdots < x_n, \quad \text{and each } (x_i, y_i) \text{ is a vertex of the convex hull } H \text{ of the points } (x_1, y_1), \ldots, (x_n, y_n) \text{ in the plane } z = 1.
\]

An example with \( n = 5 \) is illustrated in the picture (where \( y_i = x_i^2 \) and \( x_i = -x_{6-i} \)).

The zonotope \( Z \) generated by \( \Theta \), also denoted as \( Z(n, 3) \), is the sum of segments \([0, \theta_i], i = 1, \ldots, n\). A fine zonotopal tiling of \( Z \) is a subdivision \( Q \) of \( Z \) into parallelotopes of which any two can intersect only by a common face and any face of the boundary of \( Z \) is a one of some parallelotope. This is possible only if each parallelotope is of the form \( \sum_{i \in X} \theta_i + \{\lambda \theta_i + \lambda' \theta_j + \lambda'' \theta_k : 0 \leq \lambda, \lambda', \lambda'' \leq 1\} \) for some \( i < j < k \) and \( X \subseteq [n] - \{i, j, k\} \). (For more aspects of fine zonotopal tilings on zonotopes generated by cyclic configurations, see, e.g., [6].) For brevity, we liberally refer to parallelotopes as cubes, and to \( Q \) as a cubillage.

Depending on the context, we also may think of a cubillage \( Q \) as a polyhedral complex or as the corresponding set of cubes. In particular, (in the former case) by a vertex, edge, rhombus in \( Q \) we mean, respectively, (the closure of) a 0-, 1-, 2-dimensional cell of this complex, and (in the latter case) when writing \( \zeta \in Q \), we mean that \( \zeta \) is a cube of \( Q \).

Like the case of zonogons and rhombus tilings, each subset \( X \subseteq [n] \) is identified with the point \( \sum_{i \in X} \theta_i \) in \( Z \) (assuming that generators \( \theta_i \) are \( \mathbb{Z} \)-independent), and we apply to an edge, rhombus, and cube in \( Q \) terms an \( i \)-edge, \( ij \)-rhombus, and \( ijk \)-cube in a due way (where \( i < j < k \)). Also we say that such a rhombus (cube) is of type \( ij \) (resp. \( ijk \)). The edges are directed according to the generating vectors. An \( ij \)-rhombus (\( ijk \)-cube) with the bottom vertex \( X \) is denoted as \( \rho(X|ij) \) (resp. \( \zeta(X|ijk) \)). As a specialization to \( d = 3 \) of a well-known fact about fine zonotopal tilings on zonotopes \( Z(n, d) \) generated by cyclic vector configurations in \( \mathbb{R}^d \) with an arbitrary dimension \( d \), the following is true:

\[
(2.5) \quad \text{for any } 1 \leq i < j < k \leq n, \text{ a cubillage } Q \text{ has exactly one } ijk \text{-cube.}
\]

The directed graph formed by the vertices and edges occurring in \( Q \) is denoted by \( G_Q = (V_Q, E_Q) \) and we call the vertex set \( V_Q \) regarded as a set-system in \( 2^{[n]} \) the spectrum of \( Q \). The following property is of most importance for us.
Theorem 2.3 [5] The correspondence \( Q \mapsto V_Q \) gives a bijection between the set \( Q_n \) of cubillages on \( Z(n,3) \) and the set \( C_n \) of maximal \( c \)-collections in \( 2^n \).

Next, in our study of interrelations of \( s \)- and \( c \)-collections, we will use the projection \( \pi : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \) along the second coordinate vector, i.e., given by \( \pi(x,y,z) := (x,z) \). Then \( \pi(Z) \) is the zonogon generated by the vectors \( \pi(\theta_1), \ldots, \pi(\theta_n) \) (which lie in the “upper half-plane” and are numbered clockwise, in view of (2.4)). Let us represent the boundary \( \text{bd}(Z) \) of \( Z \) as the union \( Z_{\text{fr}} \cup Z_{\text{re}} \) of its front and rear sides, i.e., \( Z_{\text{fr}} \) (\( Z_{\text{re}} \)) is formed by the points \( (x,y,z) \in Z \) with \( y \) minimal (resp. maximal) in \( \pi^{-1}(x,z) \). Then \( Z_{\text{rim}} := Z_{\text{fr}} \cap Z_{\text{re}} \) is the closed piecewise linear curve being the union of two directed paths connecting the vertices \( \emptyset \) and \( [n] \) in \( G_Q \). We call \( Z_{\text{rim}} \) the rim of \( Z \).

Condition (2.4) provides that (2.6) the maximal affine sets in \( Z_{\text{fr}} \) and \( Z_{\text{re}} \) are rhombi which are projected by \( \pi \) to the standard and antistandard tilings in \( \pi(Z) \), respectively (defined in Sect. 2.1).

We identify \( Z_{\text{fr}} \) and \( Z_{\text{re}} \) with the corresponding polyhedral complexes.

Finally, for \( h = 0, 1, \ldots, n \), the intersection of \( Z \) with the horizontal plane \( z = h \) is denoted as \( \Sigma_h \) and called \( h \)-th section of \( Z \); the definition of \( \Theta \) implies that \( \Sigma_h \) contains all vertices \( X \) of size \( |X| = h \) in \( Q \).

3 S-membranes

This section starts with an example of cubillages whose spectra are neither \( s \)-pure nor \( w \)-pure. Then we consider a fixed cubillage \( Q \) on \( Z(n,3) \), introduce a class of 2-dimensional subcomplexes in it, called \( s \)-membranes, explain that each of them is isomorphic to a rhombus tiling \( T \) on \( Z(n,2) \) such that \( V_T \subset V_Q \), and vice versa (thus obtaining a geometric description of \( S^*(V_Q) \)), and demonstrate some other structural properties.

3.1 An example

Consider the zonotope \( Z = Z(4,3) \). The vertices of its boundary \( \text{bd}(Z) \) are the intervals and co-intervals on the set \([4]\) (cf. [2.6]), and there are exactly two subsets of \([4]\) that are neither intervals nor co-intervals, namely, 13 and 24. So 13 and 24 are just those “points” from \( 2^{[4]} \) that are contained in the interior of \( Z \). Since they are not chord separated, there are exactly two cubillages on \( Z \): one containing 13 and the other containing 24 (taking into account that the vertices of \( \text{bd}(Z) \) belong to any cubillage and that each cubillage is determined by its spectrum, by Theorem [2.3]).

Lemma 3.1 For the cubillage \( Q \) on \( Z(4,3) \) that contains 13, the set \( V_Q \) is neither \( s \)-pure nor \( w \)-pure.

Proof Let \( R, V_1, V_2 \) be the vertices in the rim, front side, and rear side of \( Z(4,3) \), respectively (for definitions, see the end of Sect. [2.3]). Then \( R \) consists of the eight
intervals of the form \([i] \) or \([4] \) \((0 \leq i \leq 4)\); \(V_1\) is \(R\) plus the intervals \(2, 3, 23\); and \(V_2\) is \(R\) plus the co-intervals \(14, 124, 134\). Note that the vertices (intervals) of the rim of any zonotope \(Z(n, 3)\) are strongly separated from any subset of \([n]\).

Consider the set \(S := R \cup \{2, 124\}\). It is a subset of 
\[ V_Q = V_1 \cup V_2 \cup \{13\} = R \cup \{2, 3, 23, 14, 124, 134, 13\}. \]

Observe that \(S\) is an \(s\)-collection (since \(2 \subset 124\)) but not an \(s\)-collection of maximum size in \(2^4\) (since \(|S| = 10\) but \(|V_1| = 11\)). We have \(3, 23 \sqsubset 124\) but \(|3|, |23| < |124|\), and \(2 \triangleright 14, 134, 13\) but \(|2| < |14|, |134|, |13|\). Thus, \(S\) is a maximal \(s\)-collection and a maximal \(w\)-collection in \(V_Q\), yielding the result.

In fact, using results on \(s\) - and \(w\)-membranes given later, one can strengthen the above lemma by showing that for \(n \geq 4\), the spectrum \(V_Q\) of any cubillage \(Q\) on \(Z(n, 3)\) is neither \(s\)-pure nor \(w\)-pure (we omit a proof here).

### 3.2 S-membranes

Like the definitions of \(Z^\text{fr}\) and \(Z^\text{re}\) from Sect. 2.3 for \(S \subseteq Z = Z(n, 3)\), let \(S^\text{fr} (S^\text{re})\) denote the set of points \((x, y, z) \in S\) with \(y\) minimum (resp. maximum) for each \(\pi^{-1}(x, z)\), called the front (resp. rear) part of \(S\). (In other words, \(S^\text{fr}\) and \(S^\text{re}\) are what is seen in \(S\) in the directions \((0, 1, 0)\) and \((0, -1, 0)\), respectively.) This is extended in a natural way when we deal with a subcomplex of a cubillage in \(Z\).

**Example.** In view of (2.4), for a cube \(\zeta = \zeta(X|ijk)\) (where \(i < j < k\)), \(\zeta^\text{fr}\) is formed by the rhombi \(\rho(X|i j), \rho(X|j i), \rho(X|j k)\), while \(\zeta^\text{re}\) is formed by \(\rho(X|ik), \rho(X|i j), \rho(X|j k)\). See the picture.

**Definition.** A 2-dimensional subcomplex \(M\) of a cubillage \(Q\) is called an \(s\)-membrane if \(\pi\) is injective on \(M\) and sends it to a rhombus tiling on the zonogon \(Z(n, 2)\). In other words, \(M\) is a disk (i.e., a shape homeomorphic to a circle) whose boundary coincides with \(Z^\text{rim}\) and such that \(M = M^\text{fr}\).

In particular, both \(Z^\text{fr}\) and \(Z^\text{re}\) are \(s\)-membranes. Therefore, up to a piecewise linear deformation, we may think of \(M\) as a rhombus tiling whose spectrum is contained in \(V_Q\). So the vertex set \(V_M\) of \(M\) belongs to \(S^*(V_Q)\). Moreover, a sharper property takes place (which can be deduced from general results on higher Bruhat orders and related aspects in \([8, 11, 12]\); yet we prefer to give a direct and shorter proof).

**Theorem 3.2** The correspondence \(M \mapsto V_M\) gives a bijection between the \(s\)-membranes in a cubillage \(Q\) on \(Z(n, 3)\) and the set \(S^*(V_Q)\) of maximum by size \(s\)-collections contained in \(V_Q\).
In light of explanations above, it suffices to prove the following

**Proposition 3.3** For any rhombus tiling $T$ on $Z(n, 2)$ with $V_T \subset V_Q$, there exists an $s$-membrane $M$ in $Q$ isomorphic to $T$.

This proposition will be proved in Sect. 3.4, based on a more detailed study of structural features of cubillages and operations on them given in the next subsection.

### 3.3 Pies in a cubillage

Given a cubillage $Q$ on $Z = Z(n, 3)$, let $\Pi_i = \Pi_i(Q)$ be the part of $Z$ covered by cubes of $Q$ having edges of color $i$, or, let us say, $i$-cubes. When it is not confusing, we also think of $\Pi_i$ as the set of $i$-cubes or as the corresponding subcomplex of $Q$. We refer to $\Pi_i$ as the $i$-th pie of $Q$. When $i = n$ or 1, the pie structure becomes rather transparent, which will enable us to apply some useful reductions.

To clarify the structure of $\Pi_n$, we first consider the set $U$ of $n$-edges lying in $\text{bd}(Z)$. Since the tilings on the sides $Z_{fr}$ and $Z_{re}$ of $Z$ are isomorphic to $T^{st}_n$ and $T^{ant}_n$, respectively (cf. (2.6)), one can see that

\[ (3.1) \text{ the beginning vertices of edges of } U \text{ are precisely those contained in the cycle } C = P' \cup P'', \text{ where } P' \text{ is the subpath of left path of } Z^{\text{rim}} \text{ from the bottom vertex } \emptyset \text{ to } [n-1], \text{ and } P'' \text{ is the path in } Z^{fr} \text{ passing the vertices } [n-1] - [i] \text{ for } i = n-1, n-2, \ldots, 0; \text{ in other words, } C \text{ is the rim of the zonotope } Z(n-1, 3) \text{ generated by } \theta_1, \ldots, \theta_{n-1}. \]

Accordingly, the end vertices of edges of $U$ lie on the cycle $C' := C + \theta_n$; this $C'$ is viewed as the rim of the zonotope $Z(n-1, 3)$ shifted by $\theta_n$. The area of $\text{bd}(Z)$ between $C$ and $C'$ is subdivided into $2(n-2)$ rhombi of types $\ast n$ (where $\ast$ means an element of $[n-1]$); we call this subdivision the belt of $\Pi_n$. See the picture with $n = 5$.

Now fix an $n$-edge $e = (X, Xn)$ not on $\text{bd}(Z)$ and consider the set $S$ of cubes in $\Pi_n$ containing $e$. Each cube $\zeta \in S$ is viewed as the (Minkowski) sum of some rhombus $\rho = \rho(X|ij)$ and the segment $[0, \theta_n]$, and an important fact is that $\rho$ belongs to the front side of $\zeta$ (in view of (2.4) and $n > i, j$). Gluing together such rhombi $\rho$, we obtain a disk lying on the front side of the shape $\hat{S} := \cup(\zeta \in S)$ and containing $X$ as an interior point, and $\hat{S}$ is just the sum of this disk and $[0, \theta_n]$. Based on this local behavior, one can conclude that
(3.2) $\Pi_n$ is the sum of a disk $D$ and the segment $[0, \theta_n]$; this disk lies in $\Pi_n^\text{fr}$ and its boundary is formed by the cycle $C$ as in (3.1).

Then $D' := D + \theta_i$ is a disk in $\Pi_n^\text{re}$ whose boundary is the cycle $C'$ as above.

The facts that $D^\text{fr} = D$ and $C = \tilde{Z}^{\text{im}}(n - 1, 3)$ imply that $D$ is subdivided into rhombi which (being projected by $\pi$) form a rhombus tiling on $Z(n - 1, 2)$. And similarly for $D'$.

In what follows we write $\Pi_n^-$ for $D$, $\Pi_n^+$ for $D'$, $Z_n^-(Z_n^+)$ for the (closed) subset of $Z$ between $Z^\text{fr}$ and $\Pi_n^- (\text{resp.} \Pi_n^+)$, and $Q_n^-(Q_n^+)$ for the portion (partial cubillage) of $Q$ lying in $Z_n^-$ (resp. $Z_n^+$). One can see that

(3.3) the edges of $G_Q$ connecting $Z_n^-$ and $Z_n^+$ are directed from the former to the latter and are exactly the $n$-edges of $Q$; as a consequence, each vertex of $Q_n^-$ is in $[n - 1]$ and each vertex of $Q_n^+$ is of the form $Xn$, where $X \subseteq [n - 1]$.

The following operation is of importance.

$n$-Contraction. Delete the interior of $\Pi_n$ and shift $Z_n^+$ together with the cubillage $Q_n^+$ filling it by the vector $-\theta_n$. As a result, the disks $\Pi_n^-$ and $\Pi_n^+$ merge and we obtain a cubillage on the zonotope $Z(n - 1, 3)$; it is denoted by $Q_n^\text{con}$ and called the contraction of $Q$ by (the color) $n$.

Note that $\Pi_n$ becomes an s-membrane of $Q_n^\text{con}$. Also the following is obvious:

(3.4) each cube $\zeta = \zeta(X|ijk)$ of $Q$ with $k < n$ (i.e. not contained in $\Pi_n$) one-to-one corresponds to a cube $\zeta'$ of $Q_n^\text{con}$; this $\zeta'$ is of the form $\zeta(X|ijk)$ if $\zeta \in Q_n^-$, and $\zeta(X - n|ijk)$ if $\zeta \in Q_n^+$.

Next we introduce a converse operation.

$n$-Expansion. Let $M$ be an s-membrane in a cubillage $Q'$ on the zonogon $Z' = Z(n - 1, 3)$. Define $Z^-(M) (Z^+(M))$ to be the part of $Z'$ between $(Z')^\text{fr}$ and $M$ (resp. between $M$ and $(Z')^\text{re}$), and define $Q^-(M) (Q^+(M))$ to be the subcubillage of $Q'$ contained in $Z^-(M)$ (resp. $Z^+(M)$). The $n$-expansion operation for $(Q, M)$ consists in shifting $Z^+(M)$ together with $Q^+(M)$ by $\theta_n$ and filling the space “between” $M$ and $M + \theta_n$ by the corresponding set of $n$-cubes, denoted as $Q^n_n(M)$. More precisely, each rhombus $\rho(X|ij)$ in $M$ (where $i < j < n$ and $X \subseteq [n - 1] - \{i, j\}$) generates the cube $\zeta(X|ijn)$ of $Q^n_n(M)$. A fragment of the operation is illustrated in the picture.

\begin{center}
\begin{tikzpicture}
  \node at (0,0) {in $M$:
  \begin{itemize}
    \item a cube $\zeta \subseteq \Pi_n^-$
  \end{itemize}};
  \node at (3,0) {in $Q^n_n(M)$:
  \begin{itemize}
    \item $\zeta(X|ijn)$
  \end{itemize}};
\end{tikzpicture}
\end{center}

Using the facts that $M^\text{fr} = M$ and that the boundary cycle of $M$ is the rim of $Z'$, one can see that
(3.5) taken together, the sets of cubes in $Q^-(M)$, $Q^0_n(M)$ and \{$\zeta + \theta_n : \zeta \in Q^+(M)$\} form a cubillage in $Z = Z(n, 3)$.

We denote this cubillage as $Q(M) = Q_n(Q', M)$ and call it the \textit{n-expansion} of $Q'$ using $M$. There is a natural relation between the $n$-contraction and $n$-expansion operations, as follows. (A proof is straightforward and left to the reader as an exercise.)

\textbf{Proposition 3.4} The correspondence $(Q', M) \mapsto Q(M)$, where $Q'$ is a cubillage on $Z(n - 1, 3)$, $M$ is an s-membrane in $Q'$, and $Q(M)$ is the $n$-expansion of $Q'$ using $M$, gives a bijection between the set of such pairs $(Q', M)$ in $Z(n - 1, 3)$ and the set of cubillages in $Z(n, 3)$. Under this correspondence, $Q'$ is the $n$-contraction of $Q = Q(M)$ and $M$ is the image of the $n$-pie in $Q$ under the $n$-contraction operation.

We will also take advantages from handling the 1-pie of a cubillage $Q$ on $Z(n, 3)$ and applying the corresponding 1-contraction and 1-expansion operations, which are symmetric to those concerning the color $n$ as above. More precisely, if we make a mirror reflection of $\Theta$ by replacing each generator $\theta_i = (x_i, y_i, 1)$ by $(-x_i, y_i, 1)$, denoted as $\theta'_n = -1\theta_n$, then the 1-edges of $Q$ turn into $n$-edges of the corresponding cubillage $Q'$ on $Z(\theta'_1, \ldots, \theta'_n)$, and the 1-pie of $Q$ turns into the $n$-pie of $Q'$. This leads to the corresponding counterparts of (3.1)–(3.5) and Proposition 3.4.

\textbf{Remark 2.} The usage of $j$-pies and operations on them is less advantageous when $1 < j < n$. The trouble is that if a cube $\zeta$ containing a $j$-edge is of the form $\zeta(X|ijk)$, where $i < j < k$, then $\zeta$ is the sum of the rhombus $\rho = \rho(X|ik)$ and segment $[0, \theta_j]$, but $\rho$ lies on the rear side of $\zeta$. For this reason, a relation between $j$-pies and rhombus tilings becomes less visualized. However, we will not use $j$-pies with $1 < j < n$ in this paper.

\section{Applications of the contraction and expansion operations}

We start with the following assertion, using terminology and notation as above.

\textbf{Proposition 3.5} Let $Q$ be a cubillage on $Z = Z(n, 3)$.

(i) If $Q$ contains vertices $X$ and $X_i$, then it does the edge $(X, X_i)$.

(ii) If $Q$ contains vertices $X, X_i, X_j, X_{ij}$ ($i < j$), then it does the rhombus $\rho(X|ij)$.

(iii) If $Q$ contains a set $S$ of eight vertices $X, X_i, X_j, X_k, X_{ij}, X_{ik}, X_{jk}, X_{ijk}$ ($i < j < k$), then it does the cube $\zeta(X|ijk)$.

\textbf{Proof} We use induction on $n$. Let us prove (iii), denoting by $Q'$ the cubillage on $Z(n - 1, 3)$ that is the $n$-contraction of $Q$. Three cases are possible.

(a) Suppose that $k < n$ and $n \notin X$. Then $S$ belongs to the vertex set of the subcubillage $Q^-_n$ (cf. (3.3)) and, therefore, to the vertex set of $Q'$. By induction, $Q'$ contains the cube on $S$, namely, $\zeta = \zeta(X|ijk)$. From (3.5) and Proposition 3.4 it follows that under the $n$-expansion operation for $Q'$ using the s-membrane $M := \Pi_n$,
\( \zeta \) becomes a cube in \( Q \), as required. (Recall that \( \Pi^-_n \) is the corresponding disk in \( \Pi^{fr}_n \), defined in the paragraph before (3.3.).)

(b) Suppose that \( n \in X \). Then \( k < n \) and \( S \) belongs to the vertex set of \( Q^+_n \). Therefore, \( S' := \{ Y - n : Y \in S \} \) is contained in \( V_{Q'} \) and, moreover, in the vertex set of the subcubillage \( Q^+(M) \) of \( Q' \) (where \( M \) is as in (a)). So, by induction, \( Q^+(M) \) contains the cube \( \zeta' = \zeta(X - n|ijk) \). The \( n \)-contraction operation for \( Q' \) using \( M \) transfers \( \zeta' \) to the desired cube \( \zeta(X|ijk) \) in \( Q \).

(c) Now let \( n \notin X \) and \( k = n \). Then the set \( S^- := \{ X, Xi, Xj, Xi\} \) belongs to \( \Pi^-_n \) (and \( Q^-_n \)), and the set \( S^+ := \{ Xn, Xin, Xjn, Xijn \} \) to \( \Pi^+_n \) (and \( Q^+_n \)). The \( n \)-expansion operation shifts \( S^+ \) by \( -\theta_n \) and merges it with \( S^- \) (which lies in \( M \)). By induction, \( Q' \) contains the rhombus \( \rho = \rho(X|ij) \). The \( n \)-expansion operation for \( Q' \) using \( M \) transforms \( \rho \) into the cube \( \zeta(Z|ijk) \) in \( Q^{fr}_n(M) \), and therefore, in \( Q \) (cf. (3.5)), as required.

Assertions in (i) and (ii) are shown in a similar fashion (even easier).

Based on this proposition, we now prove Proposition 3.3

Let \( Q \) be a cubillage on \( Z(n, 3) \), and \( T \) a rhombus tiling on \( Z(n, 2) \) with \( V_T \subset V_Q \) (regarding vertices as subsets of \([n]\)). For each rhombus \( \rho = \rho(X|ij) \) in \( T \), the vertices of the form \( X, Xi, Xj, Xi\) are contained in \( Q \) as well, and by (ii) in Proposition 3.5, \( Q \) contains a rhombus \( \rho' \) on these vertices. Then \( \rho = \pi(\rho') \). Combining such rhombi \( \rho' \) in \( Q \) determined by the rhombi \( \rho \) on \( T \), we obtain a 2-dimensional subcomplex \( M \) in \( Q \) which is bijectively mapped by \( \pi \) onto \( T \). Hence \( M \) is an s-membrane in \( Q \) isomorphic to \( T \), yielding Proposition 3.3 and Theorem 3.2.

4 The lattice of s-membranes

As mentioned in the Introduction, the set \( S^*(C) \) of maximal by size strongly separated collections \( S \subset 2^{[n]} \) that are contained in a fixed maximal chord separated collection \( C \subset 2^{[n]} \) has nice structural properties. Due to Theorems 2.1, 2.3, 3.2, it is more enlightening to deal with equivalent geometric objects, by considering a cubillage \( Q \) on the zonotope \( Z = Z(n, 3) \) and the set \( \mathcal{M}(Q) \) of s-membranes in \( Q \).

Using notation as in Sect. 3.3 for an s-membrane \( M \in \mathcal{M}(Q) \), we write \( Z^-(M) \) (\( Z^+(M) \)) for the (closed) region of \( Z \) bounded by the front side \( Z^{fr} \) of \( Z \) and \( M \) (resp. by \( M \) and the rear side \( Z^{re} \)) and write \( Q^-(M) \) (\( Q^+(M) \)) for the set of cubes of \( Q \) contained in \( Z^-(M) \) (resp. \( Z^+(M) \)). The sets \( Q^-(M) \) and \( Q^+(M) \) are important in our analysis and we call them the front heap and the rear heap of \( M \), respectively.

Consider two s-membranes \( M, M' \in \mathcal{M}(Q) \) and form the sets \( N := (M \cup M')^{fr} \) and \( N' := (M \cup M')^{re} \). Then both \( N, N' \) are bijective to \( Z' \) by \( \pi \). Also one can see that for any rhombus \( \rho \) in \( M \), if some interior point of \( \rho \) belongs to \( N \) (\( N' \)), then the entire \( \rho \) lies in \( N \) (resp. \( N' \)), and similarly for \( M' \). These observations imply that:

(4.1) (i) both \( N \) and \( N' \) are s-membranes in \( Q \);

(ii) the front heap \( Q^-(N) \) of \( N \) is equal to \( Q^-(M) \cap Q^-(M') \), and the front heap \( Q^-(N') \) of \( N' \) is equal to \( Q^-(M) \cup Q^-(M') \).
(Accordingly, the rear heaps of $N$ and $N'$ are $Q^+(N) = Q^+(M)\cup Q^+(M')$ and $Q^+(N') = Q^+(M)\cap Q^+(M')$.) By [11], the front heaps of s-membranes constitute a distributive lattice, which gives rise to a similar property for the s-membranes themselves.

Proposition 4.1 The set $\mathcal{M}(Q)$ of s-membranes in $Q$ is endowed with the structure of distributive lattice in which the meet and join operations for $M, M' \in \mathcal{M}(Q)$ produce the s-membranes $M \land M'$ and $M \lor M'$ such that $Q^-(M \land M') = Q^-(M) \cap Q^-(M')$ and $Q^-(M \lor M') = Q^-(M) \cup Q^-(M')$. ~

It is useful to give an alternative description for this lattice, which reveals an intrinsic structure and a connection with flips in rhombus tilings. It is based on a natural partial order on $Q$ defined below. Recall that for a cube $\zeta$, the front side $\zeta^{fr}$ and the rear side $\zeta^{re}$ are formed by the rhombi as indicated in the Example in Sect. 3.2.

Definition. For $\zeta, \zeta' \in Q$, we say that $\zeta$ immediately precedes $\zeta'$ if $\zeta^{re} \cap (\zeta')^{fr}$ contains a rhombus.

Lemma 4.2 The directed graph $\Gamma_Q$ whose vertices are the cubes of $Q$ and whose edges are the pairs $(\zeta, \zeta')$ such that $\zeta$ immediately precedes $\zeta'$ is acyclic.

Proof Consider a directed path $P = (\zeta_0, e_1, \zeta_1, \ldots, e_p, \zeta_p)$ in $\Gamma_Q$ (where $e_r$ is the edge $(\zeta_{r-1}, \zeta_r)$). We show that $P$ is not a cycle (i.e., $\zeta_0 \neq \zeta_p$ when $p > 0$) by using induction on $n$. This is trivial if $n = 3$.

We know that for any $n$-cube $\zeta = \zeta(X|ijn)$ of $Q$, its front rhombus $\rho(X|ij)$ belongs to the front side $\Pi_{n}^{fr}$ of $\Pi_n$, its rear rhombus $\rho(X|ijn)$ belongs to the rear side $\Pi_{n}^{re}$, and the other rhombi of $\zeta$, namely, $\rho(X|in), \rho(X|jn), \rho(X|in)$, lie in the interior or belt of $\Pi_n$ (for definitions, see Sect. 3.3). This implies that if for some $r$, the cubes $\zeta_{r-1}$ and $\zeta_r$ belong to different sets among $Q_n^-, Q_n^+, \Pi_n$, then the edge $e_r$ goes either from $Q_n^-$ to $\Pi_n$, or from $\Pi_n$ to $Q_n^+$. Therefore, $P$ crosses each of the disks $\Pi_{n}^{fr}$ and $\Pi_{n}^{re}$ at most once, implying that $\zeta_0 = \zeta_p$ would be possible only if the vertices of $P$ are entirely contained in exactly one of $Q_n^-, Q_n^+, \Pi_n$.

Let $Q'$ be the $n$-contraction of $Q$. We assume by induction that $\Gamma_{Q'}$ is acyclic. Then the cases $\zeta_r \in Q_n^-$ and $\zeta_r \in Q_n^+$ are impossible (subject to $\zeta_0 = \zeta_p$), taking into account that $Q'$ is obtained by combining $Q_n^-$ and the cubes of $Q_n^+$ shifted by $-\theta_n$.

It remains to show that the subgraph $\Gamma'$ of $\Gamma_Q$ induced by the cubes of $\Pi_n$ is acyclic. To see this, observe that for an $n$-cube $\zeta = \zeta(X|ijn)$ of $Q$, its rear rhombus lying in the interior or belt of $\Pi_n$ are $\rho_1 := \rho(X|in)$ and $\rho_2 := \rho(X|ijn)$. So if $\zeta$ and another $n$-rhombus $\zeta'$ are connected by edge $(\zeta, \zeta')$ in $\Gamma'$, then $\zeta'$ shares with $\zeta$ either $\rho_1$ or $\rho_2$. Let us associate with $\zeta, \zeta'$ the corresponding rhombi $\rho, \rho'$ on $\Pi_n^{fr}$, respectively. Then $\rho = \rho(X|ij)$ and $\rho' = (X'|i'j')$ for some $X'$ and $i' < j'$. These rhombi have an edge in common; namely, the edge $e = (X, X'i)$ if $\zeta, \zeta'$ share the rhombus $\rho_1$, and the edge $e' = (Xi, X'ij)$ if $\zeta, \zeta'$ share $\rho_2$. Note that (under the projection by $\pi$) both $e, e'$ belong to the left boundary of $\rho$, in view of $i < j$.

These observations show that the subgraph $\Gamma'$ is isomorphic to the graph $\Gamma''$ whose vertices are the rhombi in $\Pi_n^{fr}$ and whose edges are the pairs $(\rho, \rho')$ such that $\rho, \rho'$ share
an edge lying in the left boundary of \( \rho \) (and in the right boundary of \( \rho' \)). This \( \Gamma'' \) is acyclic. (Indeed, consider the rhombus tiling \( T := \pi(\Pi_n^b) \) on \( Z(n-1,2) \). Then any directed path from \( \emptyset \) to \([n-1]\) in \( T \) may be crossed by an edge of \( \Gamma'' \) only from right to left, not back, whence \( \Gamma'' \) is acyclic.)

**Corollary 4.3** The graph \( \Gamma_Q \) induces a partial order \( \prec \) on the cubes of \( Q \). Moreover, the ideals of \((Q, \prec)\) (i.e., the subsets \( Q' \subseteq Q \) satisfying \((\zeta \in Q', \zeta' \prec \zeta \implies \zeta' \in Q')\) are exactly the front heaps \( Q^{-}(M) \) of s-membranes \( M \in \mathcal{M}(Q) \).

Here the second assertion can be concluded from the fact that the ideals of \((Q, \prec)\) are the sets of cubes \( Q' \subseteq Q \) such that \( \Gamma_Q \) has no edge going from \( Q - Q' \) to \( Q' \).

Using Corollary 4.3, we now explain a relation to strong flips in rhombus tilings. For convenience we identify an s-membrane \( M \in \mathcal{M}(Q) \) with the corresponding rhombus tiling \( \pi(M) \) on \( Z(n,2) \). In particular, the minimal s-membrane \( Z^{fr} \) is identified with the standard tiling \( T_n^{st} \), and the maximal s-membrane \( Z^{re} \) with the antistandard tiling \( T_n^{ant} \).

Let \( M \in \mathcal{M}(Q) \) be different from \( T_n^{st} \). Then the heap \( J := Q^{-}(M) \) is nonempty. Since \( \Gamma_Q \) is acyclic, \( J \) has a maximal element \( \zeta = \zeta(X|ijk) \) (i.e., there is no \( \zeta' \in J \) with \( \zeta < \zeta' \)). Then \( M \) contains all rear rhombi of \( \zeta \), namely, \( \rho(X|ij), \rho(X|jk), \rho(X|ki) \). They span the hexagon \( H(X|ijk) \) having \( \Lambda \)-configuration and we observe that

\[
(4.2) \quad \text{for } M, J, \zeta \text{ as above, the set } J' := J - \{\zeta\} \text{ is an ideal of } (Q, \prec) \text{ as well, and the s-membrane (rhombus tiling) } M' \text{ with } Q^{-}(M') = J' \text{ is obtained from } M \text{ by replacing the rhombi of } \zeta^{re} \text{ by the rhombi forming } \zeta^{fr} \text{ (namely, } \rho(X|ij), \rho(X|jk), \rho(X|ki) \text{), or, in other words, by the lowering flip involving the hexagon } H(X|ijk) \text{ (see the picture in the end of Sect. 2.1).}
\]

(Of especial interest are principal ideals of \((Q, \prec)\); each of them is determined by a cube \( \zeta \in Q \) and consists of all \( \zeta' \in Q \) from which \( \zeta \) is reachable by a directed path in \( \Gamma_Q \). The s-membrane corresponding to such an ideal admits only one lowering flip within \( Q \), namely, that determined by the rhombi of \( \zeta \). Symmetrically: considering \( M \in \mathcal{M}(Q) \) different from \( T_n^{ant} \) and its rear heap \( R := Q^{+}(M) \), and choosing an element \( \zeta \in R \) that admits no \( \zeta' \in R \) with \( \zeta' < \zeta \), we can make the raising flip by replacing in \( M \) the rhombi of \( \zeta^{fr} \) by the ones of \( \zeta^{re} \). When \( R \) is formed by some \( \zeta \in Q \) and all \( \zeta' \in Q \) reachable from \( \zeta \) by a directed path in \( \Gamma_Q \), then \( M \) admits only one raising flip within \( Q \), namely, that determined by the rhombi of \( \zeta \).)

In terms of maximal s-collections, (4.2) together with Proposition 4.1 implies the following

**Corollary 4.4** Let \( C \) be a maximal chord separated collection in \( 2^{[n]} \). The set \( S^*(C) \) of maximal by size strongly separated collections in \( C \) is a distributive lattice with the minimal element \( \mathcal{I}_n \) and the maximal element \( \text{co-} \mathcal{I}_n \) (being the set of intervals and the set of co-intervals in \([n]\), respectively) in which \( S \in S^*(C) \) immediately precedes \( S' \in S^*(C) \) if and only if \( S' \) is obtained from \( S \) by one raising flip (“in the presence of six witnesses”).

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Remark 3. Note that the set of all maximal $s$-collections in $2^{[n]}$ is a poset (with the unique minimal element $\mathcal{I}_n$ and the unique maximal element co-$\mathcal{I}_n$ and with the immediate preceding relation given by strong flips as well); however, in contrast to $\mathbb{S}^*(C)$, this poset is not a lattice already for $n = 6$, as is shown in Ziegler [12].

Note also that a triple $\tau$ of rhombi in an $s$-membrane $M \in \mathcal{M}(Q)$ that spans a hexagon need not belong to one cube of $Q$ (in contrast to (iii) in Proposition 3.5 where $Q$ contains a cube if all eight vertices of this cube belong to $V_Q$). In this case, $(M, \tau)$ determines a flip in the variety of all rhombus tilings on $Z(n, 2)$ but not within $\mathcal{M}(Q)$.

5 Embedding rhombus tilings in cubillages

In this section we study cubillages on $Z(n, 3)$ containing one or more fixed $s$-membranes.

5.1 Extending an $s$-membrane to a cubillage

We start with the following issue. Given a maximal strongly separated collection $S \subset 2^{[n]}$, let $\mathcal{C}(S)$ be the set of maximal chord separated collections containing $S$. How to construct explicitly one instance of such $c$-collections? A naive method consists in choosing, step by step, a new subset $X$ of $[n]$ and adding it to the current collection including $S$ whenever $X$ is chord separated from all its members. However, this method is expensive as it may take exponentially many (w.r.t. $n$) steps.

An efficient approach involving geometric interpretations and using flips in $s$-membranes consists in the following. We build in the “empty” zonotope $Z = Z(n, 3)$ the abstract $s$-membrane $M = M(S)$ with $V_M = S$, by embedding $S$ (as the corresponding set of points) in $Z$ and forming the rhombus $\rho(X|ij)$ for each quadruple of the form $\{X, Xi, Xj, Xij\}$ in $S$. Then we construct a cubillage $Q$ containing this $s$-membrane (thus obtaining $S \subset V_Q \in \mathcal{C}(S)$, as required).

This is performed in two stages. At the first stage, assuming that $M$ is different from $Z^{fr}$ (equivalently, $\pi(M) \neq T_{n}^{st}$), we grow, step by step, a partial cubillage $Q'$ filling the region $Z^{-}(M)$ between $Z^{fr}$ and $M$, starting with $Q' := \emptyset$. At each step, the current $Q'$ is such that $(Q')^{re} = M$ and $(Q')^{fr}$ forms an $s$-membrane $M'$. If $M' = Z^{fr}$, we are done. Otherwise $\pi(M') \neq T_{n}^{st}$ implies that $M'$ contains at least one triple of rhombi spanning a hexagon having $\Lambda$-configuration (see the end of Sect. 2.1). We choose one hexagon $H = H(X|ijk)$ of this sort, add to $Q'$ the cube $\zeta = \zeta(X|ijk)$ induced by $H$, and update $M'$ accordingly by replacing the rhombi of $H$ by the other three rhombi in $\zeta$ (which form $\zeta^{fr}$); we say that the updated $M'$ is obtained from the previous one by the lowering flip using $\zeta$. And so on until we reach $Z^{fr}$.

At the second stage, acting in a similar way, we construct a partial cubillage $Q''$ filling the region $Z^{+}(M)$ between $M$ and $Z^{re}$. Namely, a current $Q''$ is such that $(Q'')^{fr} = M$, and $(Q'')^{re}$ forms an $s$-membrane $M''$. Unless $M'' = Z^{re}$, we choose in $M''$ a hexagon $H$ having $Y$-configuration, add to $Q''$ the cube $\zeta$ induced by $H$ and update $M''$ accordingly, thus making the raising flip using $\zeta$. And so on.

Eventually, $Q := Q' \cup Q''$ becomes a complete cubillage in $Z$ containing $M$, as
required. Since the partial cubillages \( Q', Q'' \) are constructed independently,

\[(5.1) \text{the set } Q(M) \text{ of cubillages on } Z = Z(n,3) \text{ containing a fixed s-membrane } M \text{ is represented as the "direct union" of the sets } Q^-(M) \text{ and } Q^+(M) \text{ of partial cubillages filling } Z^-(M) \text{ and } Z^+(M), \text{ respectively, i.e., } Q(M) = \{ Q' \cup Q'': Q' \in Q^-(M), \ Q'' \in Q^+(M) \}. \]

**Remark 4.** When \( M = Z^r \) (\( M = Z^e \)), \( Q^+(M) \) (resp. \( Q^-(M) \)) becomes the set \( Q_n \) of all cubillages on \( Z(n,3) \). The latter set is connected by 3-flips (defined in the Introduction); moreover, a similar property is valid for fine tilings on zonotopes of any dimension, as a consequence of results in [8].

It light of this, one can consider an arbitrary s-membrane \( M \) and ask about the connectedness of the set \( Q(M) \) of cubillages w.r.t. 3-flips that preserve \( M \). This is equivalent to asking whether or not for any two partial cubillages \( Q, Q' \in Q^-(M) \), there exists a sequence \( Q_0, Q_1, \ldots, Q_p \in Q^-(M) \) such that \( Q_0 = Q, \ Q_p = Q' \), and each \( Q_r \) \( (r = 1, \ldots, p) \) is obtained by a 3-flip from \( Q_{r-1} \); and similarly for \( Q^+(M) \). The answer to this question is affirmative (a proof is left to a forthcoming paper).

### 5.2 Cubillages for two s-membranes

One can address the following issue. Suppose we are given two abstract s-membranes \( M, M' \) properly embedded in \( Z = Z(n,3) \). When there exists a cubillage \( Q \) on \( Z \) containing both \( M \) and \( M' \)? The answer is clear: if and only if the set \( V_M \cup V_{M'} \) is chord separated. However, one can ask: how to construct such a \( Q \) efficiently?

For simplicity, consider the case of “non-crossing” s-membranes, assuming that \( M \) is situated in \( Z \) before \( M' \), i.e., \( M \) lies in \( Z^-(M') \).

A partial cubillage \( Q' \) filling \( Z^-(M) \) and a partial cubillage \( Q'' \) filling \( Z^+(M') \) always exist and can be constructed by the method as in Sect. 5.1. So the only problem is to construct a partial cubillage \( \tilde{Q} \) filling the space between \( M \) and \( M' \), i.e., \( Z(M, M') := Z^+(M) \cap Z^-(M') \); then \( \tilde{Q} := Q' \cup \tilde{Q} \cup Q'' \) is as required. Conditions when a required \( \tilde{Q} \) does exist are exposed in the proposition below.

We need some definitions. Consider a rhombus tiling \( T \) on the zonogon \( Z' = Z(n,2) \) and a color \( i \in [n] \). For each \( i \text{-edge } e \text{ in } T \), let \( m(e) \) be the middle point on \( e \), and for each \( j \in [n] - \{i\} \), let \( c(\rho) \) be the central point of the \( \{i, j\} \)-rhombus \( \rho \) in \( T \) (where \( \rho \) is the \( ij \)-rhombus when \( i < j \), and the \( ji \)-rhombus when \( j < i \)). For such a \( \rho \) and the \( i \)-edges in it, say, \( e \) and \( e' \), connect \( c(\rho) \) by straight-line segments with each of \( m(e) \) and \( m(e') \). One easily shows that the union of these segments over all \( j \) produces a non-self-intersecting piecewise linear curve connecting the middle points of the two \( i \)-edges on the left and right boundaries of \( Z' \), denoted as \( D_i \) and called \( i \)-th (undirected) dual path for \( T \). (The set \( \{D_1, \ldots, D_n\} \) matches a pseudo-line arrangement, in a sense.)

**Definitions.** Let \( 1 \leq i < j < k \leq n \) and let \( \rho \) be the \( ik \)-rhombus in \( T \). The triple \( ijk \) is called normal if \( \rho \) lies above \( D_j \), and an inversion for \( T \) if \( \rho \) lies below \( D_j \). The set of inversions for \( T \) is denoted by \( \text{Inv}(T) \). Also we say that a triple \( ijk \) in \( T \) is elementary if the rhombi of types \( ij, ik \) and \( jk \) in it span a hexagon (which has \( Y \)-configuration if \( ijk \) is normal, and \( \Lambda \)-configuration if \( ijk \) is an inversion).
See the picture where a normal triple (an inversion) $ijk$ is illustrated in the left (resp. right) fragment and the corresponding dual paths are drawn by dotted lines.

**Proposition 5.1** Let $M, M'$ be two s-membranes in $Z = Z(n, 3)$ such that $M \subset Z^- (M')$. Then a partial cubillage $\tilde{Q}$ filling $Z(M, M')$ (and therefore a cubillage on $Z$ containing both $M, M'$) exists if and only if $\text{Inv}(M) \subseteq \text{Inv}(M')$. Such a $\tilde{Q}$ consists of $|\text{Inv}(M')| - |\text{Inv}(M)|$ cubes and can be constructed efficiently.

One direction in this proposition is rather easy. Indeed, suppose that a partial cubillage $\tilde{Q}$ filling $Z(M, M')$ does exist. Take a minimal (w.r.t. the order $\prec$ as in Sect. 4) cube $\zeta = \zeta(X|ijk)$ in $\tilde{Q}$. Then the rhombi of $\zeta^r$ belong to $M$ and span the hexagon $H = H(X|ijk)$ having $Y$-configuration. Hence the triple $ijk$ in $M$ is normal and elementary (using terminology for $M$ as that for the tiling $\pi(M)$). By making the flip in $M$ using $\zeta$, we obtain an s-membrane in which $ijk$ becomes an inversion, and the fact that $ijk$ is elementary implies that no other triple $i'j'k'$ changes its status. Also the new s-membrane becomes closer to $M'$. Applying the procedure $|\tilde{Q}|$ times, we reach $M'$. This shows “only if” part in Proposition 5.1.

As to “if” part, its proof is less trivial and relies on a result by Felsner and Weil. Answering an open question by Ziegler [12], they proved the following assertion (stated in [4] in equivalent terms of pseudo-line arrangements).

**Theorem 5.2** [4] Let $T, T'$ be rhombus tilings on $Z(n, 2)$ and let $\text{Inv}(T) \subset \text{Inv}(T')$. Then $T$ has an elementary triple contained in $\text{Inv}(T') - \text{Inv}(T)$.

(This is a 2-dimensional analog of the well-known fact that for two permutations $\sigma, \sigma' \in S_n$ with $\text{Inv}(\sigma) \subset \text{Inv}(\sigma')$, $\sigma$ has a transposition in $\text{Inv}(\sigma') - \text{Inv}(\sigma)$. Ziegler [12] showed that the corresponding assertion in dimension 3 or more is false.)

Now Theorem 5.2 implies that if $M, M'$ are s-membranes with $\text{Inv}(M) \subset \text{Inv}(M')$, then there exists a cube $\zeta = \zeta(X|ijk)$ such that $\zeta^r \subset M$ and $ijk \in \text{Inv}(M') - \text{Inv}(M)$. The flip in $M$ using $\zeta$ increases the set of inversions by $ijk$. This enables us to recursively construct a partial cubillage filling $Z(M, M')$ starting with $\zeta$, and “if” part of Proposition 5.1 follows.

6 W-membranes and quasi-combies

In this section we deal with a maximal c-collection $C$ in $2^{[n]}$ and its associated cubillage $Q$ on the zonotope $Z = Z(n, 3)$ (i.e., with $V_Q = C$), and consider the class $\mathbf{W}^*(C)$ of
 maximal by size weakly separated collections contained in \( C \). (Recall that \( C \) need not be w-pure, by Lemma 3.1.) Since each \( W \in W^*(C) \) is the spectrum of a combi on the zonogon \( Z' = Z(n, 2) \) (cf. Theorem 2.2), a reasonable question is how a combi \( K \) with \( V_K \subset V_Q \) (regarding vertices as subsets of \([n]\)) relates to the structure of \( Q \). We have seen that maximal by size \( s \)-collections in \( C \) and their associated rhombus tilings on \( Z' \) are represented by \( s \)-membranes, that are special 2-dimensional subcomplexes in \( Q \). In case of weak separation, we will represent combies via \( w \)-membranes, that are subcomplexes of a certain subdivision, or fragmentation, of \( Q \). Also, along with a combi \( K \) with \( V_K \subset V_Q \), we will be forced to deal with the set of so-called quasi-combies accompanying \( K \), which were introduced in [3] and have a nice geometric interpretation in terms of \( Q \) as well.

6.1 Fragmentation of a cubillage and quasi-combies

The fragmentation \( Q^\frak{f} \) of a cubillage \( Q \) on \( Z = Z(n, 3) \) is the complex obtained by cutting \( Q \) by the horizontal planes through the vertices of \( Q \), i.e., the planes \( z = h \) for \( h = 0, \ldots , n \). This subdivides each cube \( \zeta = \zeta(X|ijk) \) into three pieces: the lower tetrahedron \( \zeta^\triangledown \), the middle octahedron \( \zeta^\square \), and the upper tetrahedron \( \zeta^\triangle \), called the \( \triangledown \)-, \( \square \)-, and \( \triangle \)-fragments of \( \zeta \), respectively. Depending on the context, we also may think of \( Q^\frak{f} \) as the set of such fragments over all cubes. We say that a fragment has height \( h + \frac{1}{2} \) if it lies between the planes \( z = h \) and \( z = h + 1 \).

It is convenient to visualize faces of \( Q^\frak{f} \) as though looking at them from the front and slightly from below, i.e., along a vector \((0, 1, \epsilon)\), and accordingly use the projection \( \pi_\epsilon : \mathbb{R}^3 \to \mathbb{R}^2 \) defined by \( \pi_\epsilon(x, y, z) = (x, z - \epsilon y) \) for a sufficiently small \( \epsilon > 0 \). One can see that \( \pi_\epsilon \) transforms the generators \( \theta_1, \ldots , \theta_n \) for \( Z' = Z(n, 2) \) as in (2.1) into generators for \( Z' = Z(n, 2) \) which are adapted for combies, i.e., satisfy the strict convexity condition (2.2).

For \( S \subset Z \), let \( S^\frak{fr} \) (\( S^\frak{re} \)) denote the set of points of \( S \) seen from the front (from the rear) in the direction related to \( \pi_\epsilon \), i.e., the points \((x, y, z) \in S \cap \pi_\epsilon^{-1}(\alpha, \beta) \) with \( y \) minimum (resp. maximum), for all \((\alpha, \beta) \in \mathbb{R}^2 \). In particular, when replacing the previous projection \( \pi \) by \( \pi_\epsilon \), all facets (triangles) of the fragments of a cube become fully seen from the front or rear; see the picture.

Thus, all 2-dimensional faces in \( Q^\frak{f} \) are triangles, and we conditionally refer to those of them that lie in horizontal sections \( z = h \) as horizontal triangles, and to the other ones (halves of rhombi in \( Q \)) as vertical ones. Horizontal triangles \( \tau \) are divided into two groups. Namely, \( \tau \) is called upper (lower) if it has vertices of the form \( Xi, Xj, Xk \) (resp. \( Y - k, Y - j, Y - i \)) for \( i < j < k \), and therefore its “obtuse” vertex \( Xj \) (resp.
$Y - j$ is situated above the edge $(Xi, Xk)$ (resp. below the edge $(Y - k, Y - i)$),
called the longest edge of $\tau$ (which is justified when $\varepsilon$ is small). Equivalently, an upper
(lower) horizontal $\tau$ belongs to an $\nabla$-fragment (resp. $\Delta$-fragment).

Accordingly, we refer to the edges in horizontal sections as horizontal ones, or $H$-
edges, and to the other edges as vertical ones, or $V$-edges (adapting terminology for
combies from Sect. 2).

For $h \in [n]$, let $Q^h$ denote the section of $Q$ at height $h$ (consisting of horizon-
tal triangles). The triangulation $Q^h$ partitioned into upper and lower triangles will be
of use in what follows. (A nice property of $Q^h$ pointed out in [5] is that its
spectrum (the set of vertices regarded as subsets of $[n]$) constitutes a maximal w-
collection in $\binom{n}{h}$.) For example, if $Q$ is the cubillage on $Z(4, 3)$ for four cubes
$\zeta(1\{123\}), \zeta(\emptyset\{134\}), \zeta(1\{234\}), \zeta(3\{124\})$, then the triangulations $Q^1, Q^2, Q^3$ are as illus-
trated in the picture, where the sections of these cubes are labeled by $a, b, c, d,
respectively.

\begin{center}
\begin{tikzpicture}
\node (A) at (0,0) {$Q^1$};
\node (B) at (1,0) {$Q^2$};
\node (C) at (2,0) {$Q^3$};
\node (1) at (-1,0) {1};
\node (2) at (-0.5,0) {2};
\node (3) at (0.5,0) {3};
\node (4) at (1,0) {4};
\node (12) at (0.5,0) {\small 12};
\node (13) at (1,0) {\small 13};
\node (14) at (1.5,0) {\small 14};
\node (23) at (1,0) {\small 23};
\node (24) at (2,0) {\small 24};
\node (34) at (1.5,0) {\small 34};
\node (123) at (1,0) {\small 123};
\node (134) at (1.5,0) {\small 134};
\draw (A) -- (2);
\draw (A) -- (3);
\draw (A) -- (4);
\draw (B) -- (12);
\draw (B) -- (13);
\draw (B) -- (14);
\draw (B) -- (23);
\draw (B) -- (24);
\draw (B) -- (34);
\draw (C) -- (123);
\draw (C) -- (134);
\end{tikzpicture}
\end{center}

6.2 W-membranes

**Definition.** A 2-dimensional subcomplex $M$ of the fragmentation $Q^Z$ is called a w-
membrane if $M$ is a disk which is bijectively projected by $\pi_\varepsilon$ on $Z(n, 2)$; equivalently,
the boundary of the disk $M$ is the rim $Z^{\text{rim}}$ of $Z = Z(n, 3)$ and $M = M^\text{fr}$.

Arguing as in Sect. 3 for s-membranes, one shows that the set $\mathcal{M}(Q^Z)$ of w-
membranes in $Q^Z$ constitutes a distributive lattice.

More precisely, associate with a w-membrane $M$: (a) the part $Z^-(M)$ ($Z^+(M)$) of
$Z$ between $Z^h$ and $M$ (resp. between $M$ and $Z^{re}$); and (b) the subcomplex $Q^\ast_\varepsilon (M)$
($Q^\ast_- (M)$) of $Q^Z$ contained in $Z^-(M)$ (resp. $Z^+(M)$), called the front heap (resp. rear
heap) when it is regarded as the corresponding set of $\nabla$-, $\square$-, and $\Delta$-fragments.

Then (similar to (1.1)) for two w-membranes $M, M' \in \mathcal{M}(Q^Z)$, we have:

(6.1) (i) both $N := (M \cup M')^\varepsilon$ and $N' := (M \cup M')^{\text{re}}$ are w-membranes;

(ii) $Q^\varepsilon_N = Q^\varepsilon_M \cap Q^\varepsilon_{M'}$ and $Q^{\text{re}}_N = Q^{\text{re}}_M \cup Q^{\text{re}}_{M'}$.

**Proposition 6.1.** $\mathcal{M}(Q^Z)$ is a distributive lattice in which operations $\wedge$ and $\vee$
are applied to $M, M' \in \mathcal{M}(Q^Z)$ produce w-membranes $M \wedge M'$ and $M \vee M'$ such that $Q^\ast_\varepsilon (M \wedge
M') = Q^\ast_- (M) \cap Q^\ast_\varepsilon (M')$ and $Q^\ast_- (M \vee M') = Q^\ast_- (M) \cup Q^\ast_- (M')$.

Next, for fragments $\tau, \tau'$ in $Q^Z$, we say that $\tau$ immediately precedes $\tau'$ if $\tau^{\text{re}} \cap (\tau')^\varepsilon$
consists of a (vertical or horizontal) triangle. Accordingly, we define the directed graph
$\Gamma_{Q^Z}$ whose vertices are the fragments in $Q^Z$ and whose edges are the pairs $(\tau, \tau')$ such
that $\tau$ immediately precedes $\tau'$. 
Lemma 6.2  The graph $\Gamma_{Q^\Box}$ is acyclic.

Proof  Consider a directed path $P = (\tau_0, e_1, \tau_1, \ldots, e_p, \tau_p)$ in $\Gamma_{Q^\Box}$. We show that $P$ is not a cycle as follows.

If consecutive fragments $\tau = \tau_{i-1}$ and $\tau' = \tau_i$ share a horizontal triangle $\sigma$ of height $h$ (i.e., lying in the plane $z = h$), then the construction of $\pi$, together with the equality $\sigma = \tau_{\Box} \cap (\tau'_\Box)$ implies that $\tau$ lies below and $\tau'$ lies above the plane $z = h$. On the other hand, if $\tau$ and $\tau'$ share a vertical triangle, then both $\tau, \tau'$ have the same height.

Thus, it suffices to show that if all fragments $\tau_i$ in $P$ have the same height, then $P$ is not a cycle. This assertion follows from Lemma 4.2 and the observation that if fragments $\tau, \tau'$ of $Q^\Box$ share a vertical triangle $\sigma$, and $\tau$ immediately precedes $\tau'$, then the cubes $\zeta, \zeta'$ of $Q$ containing these fragments (respectively) share the rhombus $\rho$ including $\sigma$ and such that $\rho = \zeta_{\Box} \cap (\zeta'_{\Box})_{\Box}$.

Corollary 6.3  The graph $\Gamma_{Q^\Box}$ induces a partial order $\prec$ on the fragments of $Q^\Box$. The ideals of $(Q^\Box, \prec)$ are exactly the front heaps $Q^{\bullet-}(M)$ of w-membranes $M \in \mathcal{M}(Q^\Box)$.

When a w-membrane $M$ is different from the minimal w-membrane $Z_{\Box}$, the ideal $F := Q^{\bullet-}(M)$ has at least one maximal element, i.e., a fragment $\tau \in F$ such that there is no $\tau' \in F - \{\tau\}$ with $\tau \prec \tau'$. Equivalently, the rear side $\tau_{\Box}$ is entirely contained in $M$. The lowering flip in $M$ using $\tau$ replaces the triangles of $\tau_{\Box}$ by the ones of $\tau_{\Box}$, producing a w-membrane $M'$ closer to $Z_{\Box}$, namely, such that $Q^{\bullet-}(M') = F - \{\tau\}$. Note that this flip preserves the set of vertices (i.e., $V_{M'} = V_M$) if $\tau$ is a $\triangle$- or $\Delta$-fragment, in which case we refer to this as a tetrahedral (lowering) flip. See the picture.

In contrast, if $\tau$ is a $\Box$-fragment, then the set of vertices does change, namely, $V_{M'} = (V_M - \{X_{ik}\}) \cup \{X_j\}$, where $\tau$ belongs to the cube $\zeta(X|ijk)$; we refer to such a flip as octahedral or essential. See the picture.

Symmetrically, when $M \neq Z_{\Box}$, its rear heap $R := Q^{\bullet+}(M)$ has at least one minimal fragment $\tau$, i.e., such that there is no $\tau' \in R - \{\tau\}$ with $\tau' \prec \tau$. Equivalently, $\tau_{\Box}$ is entirely contained in $M$. The raising flip in $M$ using $\tau$ produces a w-membrane $M'$
closer to $Z^{re}$. Such flips, referred to as tetrahedral and octahedral (or essential) as before, are illustrated on the above two pictures as well.

Making all possible lowering or raising *tetrahedral* flips starting with a given w-membrane $M$, we obtain a set of w-membranes with the same spectrum $V_M$, denoted as $E(M)$ and called the *escort* of $M$. Of especial interest is a w-membrane $L \in E(M)$ that has the maximum number of V-edges. Such an $L$ admits neither a $\nabla$-fragment $\tau$ with $\tau_{re} \subset L$, nor a $\Delta$-fragment $\tau'$ with $(\tau')_{fr} \subset L$, since a lowering flip in the former case and a raising flip in the latter case would increase the number of V-edges. We call $L$ a *fine* w-membrane.

We shall see later that the w-membranes correspond to the so-called *non-expensive* quasi-combies, and the fine w-membranes to the combies, which are compatible with $Q^\otimes$. The following auxiliary statement will be of use.

(6.2) (i) Let $Q^\otimes$ contain a vertical $\Delta$-triangle $\Delta$ and a lower horizontal triangle $\sigma$ sharing an edge $e$ that is the longest edge of $\sigma$ (and the base edge of $\Delta$). Then $\Delta$ and $\sigma$ belong to the same $\Delta$-fragment $\tau$ of $Q^\otimes$ (thus forming $\tau_{fr}$).

(ii) Symmetrically, if a vertical $\nabla$-triangle $\nabla$ and an upper horizontal triangle $\sigma$ share an edge that is the longest edge of $\sigma$, then $\nabla \cup \sigma = \tau_{re}$ for some $\nabla$-fragment $\tau$ of $Q^\otimes$.

Indeed, let $\rho$ be the rhombus in $Q$ containing the triangle $\Delta$ as in (i). This $\rho$ is a facet of one or two cubes of $Q$ and $\sigma$ lies in the section of one of them, $\zeta$ say, by the horizontal plane containing $e$. Since $\sigma$ is lower, the only possible case is when $\Delta$ and $\sigma$ form the front side of the $\Delta$-fragment of $\zeta$, as required. The case (ii) is symmetric.

A useful consequence of (6.2) is:

(6.3) for any horizontal triangle $\sigma$ of a fine w-membrane $L$, the longest edge of $\sigma$ belongs to one more (lower or upper) horizontal triangle of $L$.

Indeed, if $\sigma$ is lower, then its longest edge belongs to neither a vertical $\nabla$-triangle (since $\pi_{re}$ is injective on $L$), nor a vertical $\Delta$-triangle (otherwise $\sigma \cup \Delta$ would be as in (6.2)(i) and one could make a lowering flip increasing the number of V-edges). When $\sigma$ is upper, the argument is similar (using (6.2)(ii)).

### 6.3 Quasi-combies and w-membranes

We assume that the zonogon $Z' := Z(n,2)$ is generated by the vectors $\xi_i = \pi_{re}(\theta_i)$, $i = 1, \ldots, n$, where the $\theta_i$ are as in (2.4); then the $\xi_i$ satisfy (2.2). Speaking of combies and etc., we use terminology and notation as in Sect 2.2.

A *quasi-combi* on $Z'$ is defined in the same way as a combi, with the only difference that the requirement that for any lens $\lambda$, the lower boundary $L_{\lambda}$, as well as the upper boundary $U_{\lambda}$, has at least two edges is now withdrawn; so one of $L_{\lambda}$ and $U_{\lambda}$ is allowed to have only one edge. When all vertices of $\lambda$ are contained in $L_{\lambda}$, and therefore $U_{\lambda}$ has a unique edge, namely, $(\ell_{\lambda}, r_{\lambda})$, we say that $\lambda$ is a *lower semi-lens*. Symmetrically,
when the set $V_\lambda$ of vertices of $\lambda$ belongs to $U_\lambda$, $\lambda$ is called an upper semi-lens. An important special case of a semi-lens $\lambda$ is a (lower of upper) triangle.

We refer to the $\Delta$- and $\nabla$-tiles of a quasi-combi $K$ as vertical ones, and to the lenses and semi-lenses in it as horizontal ones. This is justified by the fact that all vertices $A$ of a horizontal tile have the same size, or, let us say, lie in the same level $h = |A|$, whereas a vertical tile has vertices in two levels.

A quasi-combi is called fully triangulated if all its tiles are triangles. An immediate observation is that

\[(6.4)\] $\pi_r$ maps any w-membrane $M$ of $Q^\otimes$ to a fully triangulated quasi-combi (regarding $M$ as a 2-dimensional complex).

In what follows we will liberally identify $M$ with $\pi_r(M)$ and speak of a w-membrane as a quasi-combi. A property converse to (6.4), in a sense, is valid in a more general situation. Before stating it, we introduce four simple operations on a quasi-combi $K$.

(S) Splitting a horizontal tile. For chosen a lens $\lambda$ of $K$ and non-adjacent vertices $u, v$ in $L_\lambda$ or in $U_\lambda$, the operation cuts $\lambda$ into two pieces (either one lens and one semi-lens or two semi-lenses) by connecting $u, v$ by the line-segment $[u, v]$. When $\lambda$ is a lower (upper) semi-lens and $u, v$ is a pair of non-adjacent vertices in $L_\lambda$ (resp. $U_\lambda$), the operation acts similarly.

(M) Merging two horizontal tiles. Suppose that $\lambda'$ and $\lambda''$, which are either two semi-lenses or one lens and one semi-lens, have a common edge $e$ that is the longest edge of at least one of them, $\lambda'$ say, i.e., $e = (\ell_{\lambda'}, r_{\lambda'})$. The operation merges $\lambda', \lambda''$ into one piece $\lambda := \lambda' \cup \lambda''$.

One can see that both operations result in correct quasi-combies. Two examples are illustrated in the picture.

\[
\begin{array}{c}
\text{(S)} \\
\text{(M)} \\
\end{array}
\]

The next two operations involve semi-lenses and vertical triangles and resemble, to some extent, tetrahedral flips in w-membranes. Here by a lower (upper) fan in a quasi-combi $K$ we mean a sequence of $\nabla$-tiles $\nabla_r = \nabla(X|i_{r-1}i_r)$ (resp. $\Delta$-tiles $\Delta_r = \Delta(Y|i_{r-1}i_r)$, $r = 1, \ldots, p$, where $i_0 < \cdots < i_p$ (resp. $i_0 > \cdots > i_p$); i.e., these triangles have the same bottom vertex $X$ (resp. the same top vertex $Y$) and two consecutive triangles share a vertical edge.

(E) Eliminating a semi-lens. Suppose that the longest edge $e = (\ell_\lambda, r_\lambda)$ of a lower semi-lens $\lambda$ belongs to a $\Delta$-tile $\Delta = \Delta(Y|ji)$ ($j > i$). Then $e$ is the base edge $(Y - j, Y - i)$ of $\Delta$, and $\lambda$ has type $ij$ and the upper root just at $Y$. The operation of eliminating $\lambda$ replaces $\lambda$ and $\Delta$ by the corresponding upper fan ($\Delta_r$: $r = 1, \ldots, p$), where each $\Delta_r$ has the top vertex $Y$ and its base edge is $r$-th edge in $L_\lambda$. Symmetrically, if an upper semi-lens $\lambda$ and a $\nabla$-tile $\nabla$ share an edge $e$ (which is the longest edge of
λ and the base edge of ∇), then the operation replaces λ and ∇ by the corresponding lower fan (∇_r: r = 1, . . . , p), where the base edge of ∇_r is r-th edge in U_λ.

(C) Creating a semi-lens. This operation is converse to (E). It deals with a lower or upper fan of vertical triangles and replaces them by the corresponding pair consisting of either an upper semi-lens and a ∇-tile, or a lower semi-lens and a ∆-tile.

Again, it is easy to check that (E) and (C) result in correct quasi-combies. These operations are illustrated in the picture (where p = 3).

Now, given a quasi-combi K, we consider the set Ω(K) of all quasi-combies K’ on Z’ with the same spectrum V_K, called the escort of K (note that when K matches a w-membrane, Ω(K) can be larger than E(M)). We observe the following

Lemma 6.4 (i) Ω(K) contains exactly one combi. (ii) Ω(K) is the set of quasi-combies that can be obtained from K by use of operations (S),(M),(E),(C). In particular, V_K is a maximal w-collection in 2^n.

Proof Choosing an arbitrary quasi-combi K’ ∈ Ω(K) and applying to K’ a series of operations (M) and (E), one can produce K* having no semi-lenses at all (since each application of (M) or (E) decreases the number of semi-lenses). Therefore, K* is a combi with V_{K*} = V_K =: S. Moreover, K* is the unique combi with the given spectrum S, by Theorem 2.2 (see also [3, Th. 3.5]). This gives (i). Now (i) implies (ii) (since any K’ ∈ Ω(K) can be obtained from K* using (S) and (C), which are converse to (M) and (E)).

As a consequence of (6.4) and Lemma 6.4, we obtain

Corollary 6.5 The spectrum of any w-membrane is a maximal w-collection in 2^n.

Definition. A quasi-combi K is called compatible with a cubillage Q if each edge of K is (the image by π_e of) an edge of Q^Q. (In particular, V_K ⊂ V_Q.)

Proposition 6.6 Let K be a quasi-combi on Z’ = Z(n,2) compatible with a cubillage Q on Z(n,3). Then the horizontal tiles (lenses and semi-lenses) of K can be triangulated so as to turn K into (the image by π_e of) a w-membrane in Q^Q.

Proof Let τ be a ∆- or ∇-tile in K; then the edges of τ belong to Q^Q. Arguing as in the proof of Proposition 3.5 (using induction on n and considering the n-contraction of Q and its fragmentation), one can show that τ is a face (a vertical triangle) of Q^Q. Now consider a lens or semi-lens λ of K lying in level h, say. Since all edges of λ belong
to $Q^2$, the polygon $\lambda$ must be subdivided into a set of triangles in the section of $Q^2$ by the plane $z = h$. Combining such sets and vertical triangles $\tau$ as above, we obtain a disk bijective to $Z'$ by $\pi$, yielding a w-membrane $M$ in $Q^2$ with $V_K \subset V_M$. Now the fact that both $V_M$ and $V_K$ are maximal w-collections implies $V_K = V_M$, and the result follows.

Let us say that a quasi-combi $K$ is non-expensive if all semi-lenses in it are triangles and there is no semi-lens $\lambda$ whose longest edge $(\ell_\lambda, r_\lambda)$ is simultaneously either an edge of a lens or the longest edge of another semi-lens. In particular, any combi is non-expensive.

Note that (6.4) and Proposition 6.6 imply that each w-membrane $M$ one-to-one corresponds (via $\pi$) to a fully triangulated quasi-combi compatible with $Q$ and having the same spectrum $V_M$. One more correspondence following from Proposition 6.6 concerns non-expensive quasi-combies.

**Corollary 6.7** Each w-membrane $M$ one-to-one corresponds to a non-expensive quasi-combi $K$ compatible with $Q$ and such that $V_K = V_M$. Non-expensive quasi-combies with the same escort have the same set of lenses.

Indeed, for a non-expensive quasi-combi $K$, the corresponding w-membrane $M$ is obtained by subdividing each lens of $K$ into triangles of $Q^2$. We also use the fact that each application of (E) matches a tetrahedral flip in the corresponding w-membrane (since each semi-lens is a triangle), and a series of such operations results in a combi with the same set of lenses.

A sharper version of above results is stated by weakening the requirement of compatibility.

**Theorem 6.8** For each maximal by size w-collection $W$ contained in the spectrum $V_Q$ of a cubillage $Q$, there exists a w-membrane $M$ in $Q^2$ with $V_M = W$.

**Proof** Let $K$ be the combi with $V_K = W$. In light of reasonings in the proof of Proposition 6.6, it suffices to show that

(6.5) each vertical triangle $\tau$ of $K$ is extended to a rhombus of $Q$ (and therefore $\tau$ is a face of $Q^2$).

To see this, we rely on the following fact (which is interesting in its own right).

**Claim.** Let a set $Y \subset [n]$ be chord separated from each of $X, X1, Xn$ for some $X \subset [n] - \{1, n\}$. Then $Y$ is chord separated from $X1n$ as well.

**Proof of the Claim.** Let $1, \ldots, n$ be disposed in this order on a circumference $O$. Let $Y' := Y - X$ and $X' := X - Y$. One may assume that $1, n \notin Y'$ (otherwise the chord separation of $Y$ and $X1n$ immediately follows from that of $Y, X, X1, Xn$).

If $Y$ and $X1n$ are not chord separated, then there are elements $x, x' \in X1n$ and $y, y' \in Y'$ such that the corresponding chords $e = [x, x']$ and $e' = [y, y']$ “cross” each
other. Then \( \{x, x'\} \neq \{1, n\} \) (since 1, \( n \) are neighboring in \( O \)). So one may assume that \( x \in X' \) (and \( x' \in X'1n \)). But in each possible case \((x' \in X', x' = 1 \text{ or } x' = n)\), the chord \( e \) crossing \( e' \) connects two elements of either \( X' \) or \( X'1 \) or \( X'n \); a contradiction.

Now consider a \( \nabla \)-tile \( \nabla = \nabla(X|ij) \) of \( K \) (having the vertices \( X, Xi, Xj \) with \( i < j \)). If \( \{i, j\} = \{1, n\} \), then, by the Claim (and Theorem \ref{thm:3.3}), \( Xij \) is chord separated from all vertices of \( Q \), and the maximality of \( V_Q \) implies that \( Xij \) is a vertex of \( Q \) as well. Hence, by Proposition \ref{prop:3.3}(ii), \( Q \) contains the rhombus \( \rho(X|ij) \), as required.

So we may assume that at least one of \( j < n \) and \( 1 < i \) takes place. Assuming the former, we use induction on \( n \) and argue as follows.

Let \( Q' \) be the \( n \)-contraction of \( Q \), and \( M \) the \( s \)-membrane in \( Q' \) that is the image of the \( n \)-pie in \( Q \) (for definitions, see Sect. \ref{sect:3.3}). Besides \( Q' \), we need to consider the reduced set \( W' := \{A \subseteq [n-1]: A \text{ or } An \text{ or both belong to } W\} \). Then \( W' \) is a maximal \( w \)-collection in \( 2^{[n-1]} \), and as is shown in \ref{thm:3.3},

\[
\text{(6.6) if } \tau \text{ is a vertical triangle of } K \text{ having type } ij \text{ with } j < n, \text{ if } A, B, C \text{ are the vertices of } \tau, \text{ and if } K' \text{ is the combi on } Z(n-1,2) \text{ with } V_{K'} = W', \text{ then } K' \text{ has a vertical triangle with the vertices } A - n, B - n, C - n.
\]

Now consider two: \( n \notin X \) and \( n \in X \).

If \( n \notin X \), then \( X, Xi, Xj \) are vertices of \( Q' \) and simultaneously vertices of the reduced combi \( K' \). By \ref{thm:3.3}, \( K' \) has the tile \( \nabla' = \nabla(X|ij) \). By induction, the vertices of \( \nabla' \) are extended to a rhombus \( \rho' \) of \( Q' \). This \( \rho' \) is lifted to \( Q \), as required.

If \( n \in X \), then \( Q' \) and \( K' \) have vertices \( X', Xi, Xj \) for \( X' := X - n \). \( K' \) has the triangle \( \nabla' = \nabla(X'|ij) \) (by \ref{thm:3.3}), the vertices of \( \nabla' \) are extended to a rhombus \( \rho' \) of \( Q' \), and \( \rho' \) is lifted to the desired rhombus \( \rho(X|ij) \) in \( Q \).

The case of a \( \Delta \)-tile \( \Delta = \Delta(Y|ji) \) of \( K \) with \( i < j < n \) is symmetric.

Finally, if \( 1 < i < j = n \), we act in a similar fashion, but applying to \( Q \) the \( 1 \)-contraction operation, rather than the \( n \)-contraction one (this is just the place where we use the \( 1 \)-contraction mentioned in Sect. \ref{sect:3.3}; the details are left to the reader.

This completes the proof of the theorem.

7 Extending a combi to a cubillage

The purpose of this section is to explain how to efficiently extend a fixed maximal \( w \)-collection in \( 2^{[n]} \) to a maximal \( c \)-collection, working with their geometric interpretations: combies and cubillages. Our construction will imply the following

**Theorem 7.1** Given a maximal weakly separated collection \( W \subset 2^{[n]} \), one can find, in polynomial time, a maximal chord separated collection \( C \subset 2^{[n]} \) including \( W \).

**Proof** It is convenient to work with an arbitrary fully triangulated quasi-combi \( K \) with \( V_K = W \). The goal is to construct a cubillage \( Q \) on \( Z = Z(n, 3) \) whose fragmentation
$Q^\otimes$ contains $K$ as a w-membrane. (Note that it is routine to construct the (unique) combi with the spectrum $W$ (see [3] for details), and to form $K$, we subdivide each lens of the combi into the pair of upper and lower semi-lenses and then triangulate them arbitrarily. The resulting cubillage $Q$ will depend on the choice of such triangulations.)

We start with properly embedding $K$ into the “empty” zonotope $Z$, and our method consists of two phases. At the first (second) phase, we construct a partial fragmentation $F^-$ (resp. $F^+$) consisting of $\nabla^-$, $\square^-$, and $\Delta$-fragments of some cubes $\zeta(X|ijk)$ (where, as usual, $i < j < k$ and $X \subseteq [n] - \{i, j, k\}$) filling the region $Z^-(K)$ of $Z$ between $Z^\ell$ and $K$ (resp. the region $Z^+(K)$ between $K$ and $Z^\tau$). (For definitions, see Sect. 6.1.) Then $F := F^+ \cup F^-$ is a subdivision of $Z$ into such fragments, and it is not difficult to realize that $F$ is just the fragmentation $Q^\otimes$ of some cubillage $Q$; so $Q^\otimes$ is as required for the given $K$.

Next we describe the first phase. At each step in it, we deal with one more w-membrane $M$ such that

(*) $M$ lies entirely in $Z^-(K)$, and there is a partial fragmentation $F'$ filling the region $Z(M, K)$ between $M$ and $K$ (i.e., $F'$ is a subdivision of $Z(M, K)$ into $\nabla^-$, $\square^-$, and $\Delta$-fragments).

If $M$ (regarded as a fully triangulated quasi-combi) has no horizontal triangle (semi-lens), then $M$ is, in essence, a rhombus tiling in which each rhombus $\rho(X|ij)$ is cut into two vertical triangles, namely, $\nabla(X|ij)$ and $\Delta(Xij|ji)$. So $M$ can be identified with the corresponding s-membrane, and we can construct a partial cubillage $Q^\prime$ filling the region $Z^-(M)$ (between $Z^\ell$ and $M$) by acting as in Sect. 5.1. Combining $Q'$ and $F'$, we obtain the desired fragmentation $F^-$ filling $Z^-(K)$

Now assume that $M$ has at least one semi-lens, and let $h$ be minimum so that the set $\Lambda$ of semi-lenses in the level $z = h$ is nonempty. Choose $\lambda \in \Lambda$ such that no edge in its lower boundary $L_\lambda$ belongs to another semi-lens. (The existence of such a $\lambda$ is provided by the acyclicity of the directed graph whose vertices are the elements of $\Lambda$ and whose edges are the pairs $(\lambda, \lambda')$ such that $U_\lambda$ and $L_{\lambda'}$ share an edge, which follows, e.g., from Lemma 4.2.) Two cases are possible.

Case 1: $\lambda$ is an upper triangle, i.e., $L_\lambda$ consists of a single edge, namely, $e = (\ell_\lambda, r_\lambda)$. Let $U_\lambda$ have vertices $Xi = \ell_\lambda$, $Xj$ and $Xk = r_\lambda$ ($i < j < k$). Then $e$ belongs to a $\nabla$-tile in $M$, namely, $\nabla = \nabla(X|ik)$. Form the $\nabla$-fragment $\tau = \zeta\nabla(X|ijk)$ (the lower tetrahedron with the vertices $X, Xi, Xj, Xk$). We add $\tau$ to $F'$ and accordingly make the lowering flip in $M$ using $\tau$ (which replaces the triangles $\lambda, \nabla$ forming $\tau_{\nabla}^\ell$ by $\nabla(X|ij)$ and $\nabla(X|jk)$ forming $\tau_{\nabla}^\ell$; see Sect. 6.2). The new $M$ is a correct fully triangulated quasi-combi (embedded as a w-membrane in $Z$), which is closer to $Z^\ell$.

Case 2: $\lambda$ is a lower triangle. Then $L_\lambda$ consists of two edges: $e = (\ell_\lambda = Y - k; Y - j)$ and $e' = (Y - j, Y - i = r_\lambda)$, where $i < j < k$. Also by the choice of $h$, the edges $e, e'$ belong to $\nabla$-tiles of $M$, namely, those of the form $\nabla = \nabla(X|jk)$ and $\nabla' = \nabla(X'|ij)$, respectively, where $X := Y - \{j, k\}$ and $X' := Y - \{i, j\}$. See the left fragment of the picture.
Let $A := Y - j = (Xk = X'i)$. Note that the “sector” between the edges $(X, A)$ and $(X', A)$ is filled by an upper fan $(\Delta_1, \ldots, \Delta_p)$, where $\Delta_r = \Delta(A|i_r-1i_r)$ and $k = i_0 > i_1 \cdots > i_p = i$ (cf. Sect 6.2). Consider two possibilities.

**Subcase 2a:** $p = 1$, i.e., the fan consists of only one tile, namely, $\Delta = \Delta(A|ki)$. Observe that the vertices $X, X', X_j, X_j'$, $A$ belong to an octahedron, namely, $\tau = \zeta'(\bar{X}|ijk)$, where $\bar{X}$ denotes $X - i = X' - k$. Moreover, the triangles $\lambda, \Delta, \nabla, \nabla'$ form the rear side of $\tau$. We add $\tau$ to $F'$ and accordingly make the octahedral flip in $M$ using $\tau$, which replaces $\tau'_{re}$ by the front side $\tau'_{fr}$ formed by four triangles shared the new vertex $A' := \bar{X}j$. (See the middle fragment of the above picture where the new triangles are indicated by solid lines.) The new $M$ is again a correct $w$-membrane closer to $Z^B$. Note that under the flip, the semi-lens $\lambda$ is replaced by an upper semi-lens $\lambda'$ in level $h - 1$ (this $\lambda'$ has the longest edge $(X, X')$ and the top $A'$).

**Subcase 2b:** $p > 1$. Then $X$ and $X'$ are connected in $M$ by the path $P$ that passes the vertices $X = A - i_0, A - i_1, \ldots, A - i_p = X'$. We make two transformations. First we connect $X$ and $X'$ by line-segment $\bar{c}$. Since $\bar{c}$ lies in the region $Z^-(M)$ (in view of (2.2)), so does the entire truncated polyhedral cone $\Sigma$ with the top vertex $A$ and the base polygon $B$ bounded by $P \cup \bar{c}$. See the right fragment of the above picture (where $p = 3$). We subdivide $B$ into $p - 1$ triangles $\sigma_1, \ldots, \sigma_{p-1}$ (having vertices on $P$) and extend each $\sigma_i$ to tetrahedron $\tau_i$ with the top $A$. These $\tau_1, \ldots, \tau_{p-1}$ subdivide $\Sigma$ into $\Delta$-fragments (each being of the form $\zeta^A(A|i_\alpha i_\beta i_\gamma)$ for some $0 \leq \alpha < \beta < \gamma \leq p$). Observe that the rear side $\Sigma'_{re}$ of $\Sigma$ is formed by the fan $(\Delta_1, \ldots, \Delta_p)$, whereas $\Sigma'_{fr}$ consists of the lower horizontal triangles $\sigma_1, \ldots, \sigma_{p-1}$ plus the vertical triangle with the top $A$ and the base $\bar{c}$, denoted as $\bar{\Delta}$.

We add the fragments $\tau_1, \ldots, \tau_{p-1}$ to $F'$ and accordingly update $M$ by replacing the triangles of $\Sigma'_{re}$ by the ones of $\Sigma'_{fr}$ (as though making $p - 1$ lowering tetrahedral flips). The new $w$-membrane has the upper fan at $A$ consisting of a unique $\Delta$-tile, namely, $\bar{\Delta}$, and now we make the second transformation, by applying the octahedral flip as in Subcase 2a (involving the triangles $\lambda, \bar{\Delta}, \nabla, \nabla'$ on the same vertices $X, X', X_j, X_j', A$).

Doing so, we eventually get rid of semi-lenses in the current $M$; so $M$ becomes an $s$-membrane in essence, which enables us to extend the current $F'$ to the desired fragmentation $F^-$ filling $Z^-(K)$ (by acting as in Sect. 5.1).

At the second phase, we act “symmetrically”, starting with $M := K$ and moving toward $Z'^{re}$, in order to obtain a fragmentation $F^+$ filling $Z^+(K)$. Then $F^- \cup F^+$ is as required, and the theorem follows.
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