When the Schur functor induces a triangle-equivalence between Gorenstein defect categories

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Abstract Let \( R \) be an Artin algebra and \( e \) be an idempotent of \( R \). Assume that \( \text{Tor}^i_{eRe}(Re, G) = 0 \) for any \( G \in \text{Gproj}_eRe \) and \( i \) sufficiently large. Necessary and sufficient conditions are given for the Schur functor \( S_e \) to induce a triangle-equivalence \( D_{\text{def}}(R) \cong D_{\text{def}}(eRe) \). Combining this with a result of Psaroudakis et al. (2014), we provide necessary and sufficient conditions for the singular equivalence \( D_{\text{sg}}(R) \cong D_{\text{sg}}(eRe) \) to restrict to a triangle-equivalence \( \text{Gproj}_R \cong \text{Gproj}_{eRe} \). Applying these to the triangular matrix algebra \( T = \begin{pmatrix} A & B \\ 0 & R \end{pmatrix} \), corresponding results between candidate categories of \( T \) and \( A \) (resp. \( B \)) are obtained. As a consequence, we infer Gorensteinness and CM (Cohen-Macaulay)-freeness of \( T \) from those of \( A \) (resp. \( B \)). Some concrete examples are given to indicate that one can realize the Gorenstein defect category of a triangular matrix algebra as the singularity category of one of its corner algebras.

Keywords Schur functors, triangle-equivalences, singularity categories, Gorenstein defect categories, triangular matrix algebras

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1 Introduction

Let \( R \) be an Artin algebra. In the study of the stable homological algebra and Tate cohomology, Buchweitz [9] introduced the singularity category \( D_{\text{sg}}(R) \) of \( R \). This category is a certain Verdier quotient of the bounded derived category \( D^b(\text{mod } R) \) of \( \text{mod } R \) modulo the bounded homotopy category of \( \text{proj } R \), where \( \text{mod } R \) is the category of finitely generated left \( R \)-modules and \( \text{proj } R \) is its subcategory consisting of projective modules. Later on, this category was reconsidered by Orlov [27] in the setting of algebraic geometry and turned out to have a closed relation with the “homological mirror symmetry conjecture”. The singularity category of \( R \) measures the “regularity” of \( R \) in the sense that \( D_{\text{sg}}(R) = 0 \) if and only if \( R \) has the finite global dimension.

By the fundamental result in [9], the stable category \( \text{Gproj } R \) of finitely generated Gorenstein projective modules might be regarded as a triangulated subcategory of \( D_{\text{sg}}(R) \) via a triangulated embedding functor...
F. Besides, $F : \text{Gproj } R \to \mathbb{D}_{sg}(R)$ is a triangle-equivalence provided that $R$ is Gorenstein (see [9, 19]). Inspired by this, Bergh et al. [8] considered the Verdier quotient $\mathbb{D}_{\text{def}}(R) \equiv \mathbb{D}_{sg}(R)/\text{Im } F$, and they called it the Gorenstein defect category of $R$. This category measures how far the algebra $R$ is from being Gorenstein. More precisely, $R$ is Gorenstein if and only if $\mathbb{D}_{\text{def}}(R)$ is trivial. Nowadays, singularity categories and related topics have been studied by many authors (see, for example, [10, 11, 15, 24, 25, 30, 33]).

It attracts much attention when two algebras share the same singularity category up to a triangle-equivalence (see [10, 13, 14, 29]). In this case, we call these two algebras singular equivalent and such an equivalence is called a singular equivalence. Let $R$ be an Artin algebra and $e \in R$ be an idempotent. The Schur functor associative with $e$ is defined to be $S_e = eR \otimes_R \cdot : \text{mod } R \to \text{mod } eRe$ (see [18]), and we denote by $T_e = Re \otimes_{eRe} \cdot : \text{mod } eRe \to \text{mod } R$ its left adjoint. Recall that $e$ is said to be singularly-complete (see [10]) if the projective dimension of any $R/ReR$-module is finite as an $R$-module. Chen [10] showed that if $e$ is singularly-complete and the projective dimension $\text{pd}_{eRe} eR < \infty$, then the Schur functor induces a singular equivalence

$$\mathbb{D}_{sg}(R) \xrightarrow{\mathbb{D}_{sg}(S_e)} \mathbb{D}_{sg}(eRe).$$

Later on, Psaroudakis et al. [29, Corollary 5.4] generalized this result and showed that the converse also holds true; as a consequence, they obtained that if $R$ is Gorenstein and the Schur functor $S_e$ is an eventually homological isomorphism, then there exists a triangle-equivalence $\text{Gproj } R \simeq \text{Gproj } eRe$ between their stable categories of Gorenstein projective modules (see [29, Corollary 5.6]). We note that under the assumptions of [29, Corollary 5.6], both $R$ and $eRe$ are Gorenstein (see [29, Corollary 4.7]), so the triangle-equivalence $\text{Gproj } R \simeq \text{Gproj } eRe$ is trivial—it coincides with the singular equivalence

$$\mathbb{D}_{sg}(R) \xrightarrow{\mathbb{D}_{sg}(S_e)} \mathbb{D}_{sg}(eRe).$$

We wonder whether or not a triangle-equivalence $\text{Gproj } R \simeq \text{Gproj } eRe$ could be got from the singular equivalence

$$\mathbb{D}_{sg}(R) \xrightarrow{\mathbb{D}_{sg}(S_e)} \mathbb{D}_{sg}(eRe)$$

even if $R$ or $eRe$ is not Gorenstein, and thus the following question is natural.

**Question 1.** When does the singular equivalence

$$\mathbb{D}_{sg}(R) \xrightarrow{\mathbb{D}_{sg}(S_e)} \mathbb{D}_{sg}(eRe)$$

restrict to a triangle-equivalence $\text{Gproj } R \simeq \text{Gproj } eRe$?

Recall that the Gorenstein defect category is a Verdier quotient of the singularity category modulo the isomorphic image of the stable category of Gorenstein projective modules. It is not hard to see the above question is equivalent to when the singular equivalence

$$\mathbb{D}_{sg}(R) \xrightarrow{\mathbb{D}_{sg}(S_e)} \mathbb{D}_{sg}(eRe)$$

induces a triangle-equivalence $\mathbb{D}_{\text{def}}(R) \simeq \mathbb{D}_{\text{def}}(eRe)$ between their Gorenstein defect categories. However, due to Kong and Zhang’s results (see [23, Section 6]), the Gorenstein defect category could also be regarded as a Verdier quotient of the bounded derived category. Hence there is a great possibility that the triangle-equivalence $\mathbb{D}_{\text{def}}(R) \simeq \mathbb{D}_{\text{def}}(eRe)$ is not induced by the singular equivalence

$$\mathbb{D}_{sg}(R) \xrightarrow{\mathbb{D}_{sg}(S_e)} \mathbb{D}_{sg}(eRe).$$

This means that the Schur functor might induce the triangle-equivalence $\mathbb{D}_{\text{def}}(R) \simeq \mathbb{D}_{\text{def}}(eRe)$ directly even though it does not induce a singular equivalence $\mathbb{D}_{sg}(R) \simeq \mathbb{D}_{sg}(eRe)$. From this viewpoint, we consider the following more general question.
Question 2. When does the Schur functor induce a triangle-equivalence $\mathcal{D}_{\text{def}}(R) \simeq \mathcal{D}_{\text{def}}(eRe)$ between Gorenstein defect categories?

To state our results precisely, let us first introduce some terminology and notations. Let $R$ be an Artin algebra. An $R$-module $M$ is called Gorenstein projective (see [5, 17]) if there is an exact sequence

$$P^\bullet = \cdots \to P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} P^1 \to \cdots$$

of $\text{proj } R$ with $M \cong \ker d^0$ such that $\text{Hom}_R(P^\bullet, Q)$ is exact for every $Q \in \text{proj } R$. Denote by $\text{Gproj } R$ the subcategory of $\text{mod } R$ consisting of Gorenstein projective modules. Given a module $M \in \text{mod } R$, the Gorenstein projective dimension $\text{Gpd}_R M$ of $M$ is defined to be

$$\text{Gpd}_R M = \inf \{ n | \text{there exists an exact sequence } 0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0, \text{ where each } G_i \in \text{Gproj } R \}.$$

An idempotent $e \in R$ is called Gorenstein singularly-complete if the Gorenstein projective dimension of any $R/ReR$-module is finite as an $R$-module.

Theorem 1.1. Let $R$ be an Artin algebra and $e$ be an idempotent of $R$. Assume that $\text{Tor}_i^{eRe}(Re, G) = 0$ for any $G \in \text{Gproj } eRe$ and $i$ sufficiently large. Then the Schur functor $S_e$ induces a triangle-equivalence of Gorenstein defect categories $\mathcal{D}_{\text{def}}(R) \simeq \mathcal{D}_{\text{def}}(eRe)$ if and only if the following conditions are satisfied:

(C1) $\text{Gpd}_{eRe} S_e(F) < \infty$ for any $F \in \text{Gproj } R$.

(C2) $\text{Gpd}_RT_e(G) < \infty$ for any $G \in \text{Gproj } eRe$.

(C3) The idempotent $e$ is Gorenstein singularly-complete.

In this case, $R$ is Gorenstein if and only if $eRe$ is Gorenstein.

Note that Theorem 1.1 answers Question 2 properly. Moreover, it also provides a reduction technique to test Gorensteinness of $R$ from that of $eRe$. To prove this theorem, for a given algebra $R$, we describe the Gorenstein defect category as the Verdier quotient $\mathcal{D}^b(\text{mod } R)/\mathcal{D}^b(\text{mod } R)_{\text{fgp}}$ (comparing with [23, 34]), where $\mathcal{D}^b(\text{mod } R)_{\text{fgp}}$ is the subcategory of $\mathcal{D}^b(\text{mod } R)$ consisting of complexes with finite Gorenstein projective dimension. The advantage is that this not only provides a concise proof of the converse of Buchweitz's theorem (see [9, Theorem 4.4.1]), but also makes the isomorphic image of $\text{Gproj } R$ into $\mathcal{D}_{\text{sg}}(R)$ controllable by the finiteness of Gorenstein projective dimension of homological bounded complexes (see Section 2). Somehow, our proof is then ascribed to considering when the subcategories consisting of complexes with finite Gorenstein projective dimension are preserved under the derived functors of $S_e$ and $T_e$.

Combining Theorem 1.1 with [29, Corollary 5.4], we then get the answer to Question 1.

Corollary 1.2. Let $R$ be an Artin algebra and $e$ be an idempotent of $R$. Assume that $\text{Tor}_i^{eRe}(Re, G) = 0$ for any $G \in \text{Gproj } eRe$ and $i$ sufficiently large. Then the Schur functor $S_e$ induces the following exact commutative diagram:

$$
\begin{array}{ccc}
0 & \longrightarrow & \text{Gproj } R \\
\downarrow & & \downarrow \\
\text{Gproj } eRe & \longrightarrow & \mathcal{D}_{\text{sg}}(eRe) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{D}_{\text{def}}(eRe) \\
\end{array}
$$

with all the vertical functors triangle-equivalences if and only if the following conditions are satisfied:

(C1) $\text{Gpd}_{eRe} S_e(F) < \infty$ for any $F \in \text{Gproj } R$.

(C2) $\text{Gpd}_RT_e(G) < \infty$ for any $G \in \text{Gproj } eRe$.

(C3) The idempotent $e$ is singularly-complete and $\text{pd}_{eRe} eR < \infty$.

We note that the assumptions in Theorem 1.1 and Corollary 1.2 are mild in the setting of triangular matrix algebras (see Section 4). Consequently, we obtain equivalent characterizations for the existences of triangle-equivalences of Gorenstein defect categories (resp. stable categories of Gorenstein projective modules, singularity categories) between a triangular matrix algebra and one of its corner algebras (see Section 4).
Theorems 4.4 and 4.6). Also some concrete examples are given to indicate that one can realize the Gorenstein defect category of a triangular matrix algebra as the singularity category of one of its corner algebras (see Examples 4.5 and 4.7).

The rest of this paper is organized as follows. In Section 2, we give descriptions of $\text{Gproj } R$ and $\mathbb{D}_{\text{def}}(R)$ by using the subcategory $\text{D}^b(\text{mod } R)_{\text{sg}}$ and then prove the converse of Buchweitz’s theorem. In Section 3, we provide proofs of the above-mentioned theorem and corollary. In Section 4, some applications in the setting of triangular matrix algebras are given. As a consequence, we infer Gorensteinness and CM-freeness of a triangular matrix algebra from one of its corner algebras.

2 Gorenstein defect categories

Throughout, all the algebras are Artin algebras over some commutative Artinian ring and all the modules are finitely generated. For a given algebra $R$, denote by $\text{mod } R$ the category of left $R$-modules; right $R$-modules are viewed as left $R^{\text{op}}$-modules, where $R^{\text{op}}$ is the opposite algebra of $R$. We use $\text{proj } R$ to define the subcategory of $\text{mod } R$ consisting of projective modules. The $*$-bounded derived category of $\text{mod } R$ and the homotopy category of $\text{proj } R$ are denoted by $\mathbb{D}^*(\text{mod } R)$ and $\mathbb{K}^*(\text{proj } R)$, respectively, where $* \in \{\text{blank}, +, -, b\}$.

Usually, we use $R M$ (resp. $M R$) to define a left (resp. right) $R$-module $M$, and $\text{pd}_R M$ (resp. $\text{pd} M R$) to define the projective dimension of $R M$ (resp. $M R$). For a subclass $\mathcal{X}$ of $\text{mod } R$, denote by $\mathcal{X}^\perp$ (resp. $\perp \mathcal{X}$) the subcategory consisting of modules $M \in \text{mod } R$ such that $\text{Ext}_R^n(X, M) = 0$ (resp. $\text{Ext}^n_R(M, X) = 0$) for any $X \in \mathcal{X}$.

Let
$$X^\bullet = \cdots \to X^{-1} \xrightarrow{d^{-1}} X^0 \xrightarrow{d^0} X^1 \to \cdots$$
be a complex in $\text{mod } R$. For any integer $n$, we set $Z^n(X^\bullet) = \text{Ker } d^n$ and $B^n(X^\bullet) = \text{Im } d^{n-1}$; and denote by $H^n(X^\bullet) = Z^n(X^\bullet)/B^n(X^\bullet)$ the $n$-th homology of $X^\bullet$.

It is well known that the subcategory $\text{Gproj } R$ of $\text{mod } R$ consisting of Gorenstein projective modules is a Frobenius category, and hence its stable category $\text{Gproj } R$ is a triangulated category (see [20]). Consider the composition of the embedding functor $\text{Gproj } R \hookrightarrow \mathbb{D}^b(\text{mod } R)$ and the localization functor $\mathbb{D}^b(\text{mod } R) \to \mathbb{D}_{\text{sg}}(R)$. It induces a functor $F : \text{Gproj } R \to \mathbb{D}_{\text{sg}}(R)$, which sends every Gorenstein projective module to the stalk complex concentrated in degree zero.

**Lemma 2.1** (Buchweitz’s theorem [9, Theorem 4.4.1]). Keep the above notations. The canonical functor $F : \text{Gproj } R \to \mathbb{D}_{\text{sg}}(R)$ is an embedding triangle-functor. Furthermore, $F$ is a triangle-equivalence provided that $R$ is Gorenstein (i.e., the left and right self-injective dimensions of $R$ are finite).

According to Buchweitz’s theorem, $\text{Im } F$ is a triangulated subcategory of $\mathbb{D}_{\text{sg}}(R)$.

**Definition 2.2** (See [8]). We call the Verdier quotient $\mathbb{D}_{\text{def}}(R) := \mathbb{D}_{\text{sg}}(R)/\text{Im } F$ the Gorenstein defect category of $R$.

Recall from [12] that $R$ is called Cohen-Macaulay free, or simply, CM-free if $\text{Gproj } R = \text{proj } R$, while $R$ is called Cohen-Macaulay finite, or simply, CM-finite (see [7]) if $\text{Gproj } R$ is of finite representation type. Let $R$ be CM-finite and $\{G_1, G_2, \ldots, G_n\}$ be the set of all the pairwise non-isomorphic indecomposable Gorenstein projective modules. Recall that the opposite
$$\text{Aus}(R) := \text{End}(\bigoplus_{1 \leq i \leq n} G_i)^{\text{op}}$$
of the endomorphism algebra $\text{End}(\bigoplus_{1 \leq i \leq n} G_i)$ is called the relative Auslander algebra (see [23]) or Cohen-Macaulay Auslander algebra (see [7]) of $R$.

**Remark 2.3.**

1. It is not hard to see that $R$ is CM-free if and only if $\mathbb{D}_{\text{def}}(R) = \mathbb{D}_{\text{sg}}(R)$.

2. Following [23, Corollary 6.10], if $R$ is CM-finite, then there is a triangle-equivalence $\mathbb{D}_{\text{def}}(R) \simeq \mathbb{D}_{\text{sg}}(\text{Aus}(R))$. 

Recently, the Gorenstein defect category \( D_{\mathrm{def}}(R) \) was reconsidered by Kong and Zhang \([23]\). What’s more, they found that \( D_{\mathrm{def}}(R) \) is triangle-equivalent to \( D^b(\mathrm{mod} \, R) / (\text{Gproj} \, R) \), where \( \text{Gproj} \, R \) is the triangulated subcategory of \( D^b(\mathrm{mod} \, R) \) generated by \( \text{Gproj} \, R \). We wonder what is the exact form of objects in \( \text{Gproj} \, R \). We introduce the following definition.

**Definition 2.4.** A complex \( X^\bullet \in D^b(\mathrm{mod} \, R) \) is said to have finite Gorenstein projective dimension if \( X^\bullet \) is isomorphic to some bounded complex consisting of Gorenstein projective modules in \( D^b(\mathrm{mod} \, R) \).

We note that the finiteness of Gorenstein projective dimension for a complex in Definition 2.4 coincides with that of \([31]\) (see, for example, \([31]\, \text{Construction 5.5}\)). Denote by \( D^b(\mathrm{mod} \, R)_{\mathrm{fgp}} \) the subcategory of \( D^b(\mathrm{mod} \, R) \) consisting of complexes with finite Gorenstein projective dimension. We give the following equivalent characterizations when an object of \( D^b(\mathrm{mod} \, R) \) lies in \( D^b(\mathrm{mod} \, R)_{\mathrm{fgp}} \). It should be noticed that some of these assertions could also be obtained by Kong and Zhang’s terminology in \([23]\, \text{Subsection 6.2}\).

**Lemma 2.5.** Let \( X^\bullet \in D^b(\mathrm{mod} \, R) \). Then the following statements are equivalent:

1. \( X^\bullet \in D^b(\mathrm{mod} \, R)_{\mathrm{fgp}} \).
2. For any quasi-isomorphism \( P^\bullet \to X^\bullet \) with \( P^\bullet \in K^{-b}(\mathrm{proj} \, R) \), one has \( Z^i(P^\bullet) \in \text{Gproj} \, R \) for \( i \leq 0 \), where \( K^{-b}(\mathrm{proj} \, R) \) is the full subcategory of \( D^b(\mathrm{mod} \, R) \) consisting of complexes with finite non-zero homology.
3. There exists a quasi-isomorphism \( P^\bullet \to X^\bullet \) with \( P^\bullet \in K^{-b}(\mathrm{proj} \, R) \) such that \( Z^i(P^\bullet) \in \text{Gproj} \, R \) for \( i \leq 0 \).

**Proof.** (1) \( \Rightarrow \) (2). Let \( P^\bullet \to X^\bullet \) be a quasi-isomorphism with \( P^\bullet \in K^{-b}(\mathrm{proj} \, R) \). Since \( X^\bullet \in D^b(\mathrm{mod} \, R)_{\mathrm{fgp}} \), we get \( X^\bullet \cong G^\bullet \) in \( D^b(\mathrm{mod} \, R) \) with \( G^\bullet \) a bounded complex consisting of Gorenstein projective modules. It follows that \( P^\bullet \cong G^\bullet \) in \( D^b(\mathrm{mod} \, R) \) and then there is a quasi-isomorphism \( f : P^\bullet \to G^\bullet \) by \([4, \text{1.4.1}]\). Hence the mapping cone

\[
\text{Con}(f) = \cdots \to P^{n-1} \to P^n \to P^{n+1} \oplus G^n \to \cdots
\]

is acyclic. Note that \( \text{Con}(f) \) is bounded above with each degree in \( \text{Gproj} \, R \) and \( \text{Gproj} \, R \) is closed under kernels of epimorphisms (see \([21, \text{Theorem 2.5}]\)). We conclude that \( Z^i(P^\bullet) \cong Z^{i-1}(\text{Con}(f)) \in \text{Gproj} \, R \) for \( i \leq 0 \).

(2) \( \Rightarrow \) (3) is trivial.

(3) \( \Rightarrow \) (1). Let \( P^\bullet \to X^\bullet \) be a quasi-isomorphism with \( P^\bullet \in K^{-b}(\mathrm{proj} \, R) \) and \( Z^i(P^\bullet) \in \text{Gproj} \, R \) for \( i \leq 0 \). Since \( P^\bullet \in K^{-b}(\mathrm{proj} \, R) \), \( P^\bullet \) is isomorphic to the following complex in \( D^b(\mathrm{mod} \, R) \):

\[
G^\bullet := 0 \to Z^i(P^\bullet) \to P^t \to \cdots \to P^{s+1} \to P^s \to 0,
\]

where \( s \) is the supremum of index \( i \in \mathbb{Z} \) such that \( P^i \neq 0 \) and \( t \) is the index such that \( Z^i(P^\bullet) \in \text{Gproj} \, R \) and \( H^j(P^\bullet) = 0 \) for any \( i \leq t \). Hence \( X^\bullet \cong G^\bullet \) in \( D^b(\mathrm{mod} \, R) \) with \( G^\bullet \) a bounded complex consisting of Gorenstein projective modules and then \( X^\bullet \in D^b(\mathrm{mod} \, R)_{\mathrm{fgp}} \).

**Remark 2.6.** It follows from Lemma 2.5 and \([5, \text{Theorem 3.1}]\) that an \( R \)-module \( M \) viewed as a stalk complex concentrated in degree zero lies in \( D^b(\mathrm{mod} \, R)_{\mathrm{fgp}} \) if and only if \( \text{Gpd}_R \, M < \infty \).

Let \( X^\bullet \) be a complex of \( R \)-modules. The length \( l(X^\bullet) \) of \( X^\bullet \) is defined to be the cardinal of the set \( \{ X^i \neq 0 \mid i \in \mathbb{Z} \} \). Let \( n \in \mathbb{Z} \). Denote by \( X^\bullet_{\geq n} \) the complex with the \( i \)-th component equaling \( X^i \) whenever \( i \geq n \) and \( 0 \) elsewhere.

**Theorem 2.7.** \( D^b(\mathrm{mod} \, R)_{\mathrm{fgp}} \) is a thick subcategory of \( D^b(\mathrm{mod} \, R) \). Furthermore, \( D^b(\mathrm{mod} \, R)_{\mathrm{fgp}} = (\text{Gproj} \, R) \).

**Proof.** It follows from \([34, \text{Proposition 3.2}]\) that \( D^b(\mathrm{mod} \, R)_{\mathrm{fgp}} \) is a triangulated subcategory of \( D^b(\mathrm{mod} \, R) \). To get the first assertion, it suffices to show that \( D^b(\mathrm{mod} \, R)_{\mathrm{fgp}} \) is closed under direct summands. In fact, let \( X^\bullet_1 \oplus X^\bullet_2 \in D^b(\mathrm{mod} \, R)_{\mathrm{fgp}} \) with \( X^\bullet_1, X^\bullet_2 \in D^b(\mathrm{mod} \, R) \). Choose quasi-isomorphisms \( P^\bullet_1 \to X^\bullet_1 \) and \( P^\bullet_2 \to X^\bullet_2 \) with \( P^\bullet_1, P^\bullet_2 \in K^{-b}(\mathrm{proj} \, R) \). It follows that \( P^\bullet_1 \oplus P^\bullet_2 \to X^\bullet_1 \oplus X^\bullet_2 \) is a quasi-isomorphism. Notice that \( X^\bullet_1, X^\bullet_2 \in D^b(\mathrm{mod} \, R)_{\mathrm{fgp}} \) and \( P^\bullet_1, P^\bullet_2 \in K^{-b}(\mathrm{proj} \, R) \). It follows from Lemma 2.5 that \( Z^i(P^\bullet_1 \oplus P^\bullet_2) \in \text{Gproj} \, R \) for \( i \leq 0 \). Since \( Z^i(P^\bullet_1 \oplus P^\bullet_2) \cong Z^i(P^\bullet_1) \oplus Z^i(P^\bullet_2) \), we get \( Z^i(P^\bullet_1), Z^i(P^\bullet_2) \in \text{Gproj} \, R \) for \( i \leq 0 \). Then by Lemma 2.5, we get \( X^\bullet_1, X^\bullet_2 \in D^b(\mathrm{mod} \, R)_{\mathrm{fgp}} \).
Note that every Gorenstein projective module viewed as a stalk complex concentrated in degree zero has finite Gorenstein projective dimension. Thus $\text{Gproj} R \subseteq \mathbb{D}^b(\text{mod} R)_{\text{fgp}}$ and then $(\text{Gproj} R) \subseteq \mathbb{D}^b(\text{mod} R)_{\text{fgp}}$. On the other hand, let $G^\bullet$ be a bounded complex consisting of Gorenstein projective $R$-modules. We show $G^\bullet \in (\text{Gproj} R)$ to complete the proof. We proceed by induction on the length $l(G^\bullet)$ of $G^\bullet$. If $l(G^\bullet) = 1$, it is trivial to verify $G^\bullet \in (\text{Gproj} R)$. Now let $l(G^\bullet) = n \geq 2$. We may suppose $G^m \neq 0$ and $G^i = 0$ for $i < m$. Then we have the following triangle in $\mathbb{D}^b(\text{mod} R)$:

$$G^m[-m-1] \to G^\bullet_{m+1} \to G^\bullet \to G^m[-m].$$

By the induction hypothesis, we have that both $G^m[-m-1]$ and $G^\bullet_{m+1}$ lie in $(\text{Gproj} R)$. Therefore, $G^\bullet \in (\text{Gproj} R)$. \hfill $\square$

**Proposition 2.8.** Let $X^\bullet \in \mathbb{D}^b(\text{mod} R)$ be a bounded complex. If each $X^i$ is of finite Gorenstein projective dimension as an $R$-module, then $X^\bullet \in \mathbb{D}^b(\text{mod} R)_{\text{fgp}}$. Furthermore, $\mathbb{D}^b(\text{mod} R)_{\text{fgp}} = \mathbb{D}^b(\text{mod} R)$ if and only if $R$ is Gorenstein.

**Proof.** We proceed by induction on the length $l(X^\bullet)$ of $X^\bullet$. If $l(X^\bullet) = 1$, it is trivial to verify $X^\bullet \in \mathbb{D}^b(\text{mod} R)_{\text{fgp}}$. Now let $l(X^\bullet) = n \geq 2$. We may suppose $X^m \neq 0$ and $X^i = 0$ for $i < m$. Then we have the following triangle in $\mathbb{D}^b(\text{mod} R)$:

$$X^m[-m-1] \to X^\bullet_{m+1} \to X^\bullet \to X^m[-m].$$

By the induction hypothesis, we have that both $X^m[-m-1]$ and $X^\bullet_{m+1}$ lie in $\mathbb{D}^b(\text{mod} R)_{\text{fgp}}$. Therefore, $X^\bullet \in \mathbb{D}^b(\text{mod} R)_{\text{fgp}}$.

Note that $R$ is Gorenstein if and only if every module in $\text{mod} R$ has finite Gorenstein projective dimension by [22, Theorem]. Thus $\mathbb{D}^b(\text{mod} R)_{\text{fgp}} = \mathbb{D}^b(\text{mod} R)$ if and only if $R$ is Gorenstein. \hfill $\square$

Since $\mathbb{D}^b(\text{mod} R)_{\text{fgp}}$ is a thick subcategory of $\mathbb{D}^b(\text{mod} R)$, one gets $\mathbb{D}^b(\text{mod} R)_{\text{fgp}}/\mathbb{K}^b(\text{proj} R)$ is a thick subcategory of $\mathbb{D}_{sg}(R)$. Let

$$\iota : \mathbb{D}^b(\text{mod} R)_{\text{fgp}}/\mathbb{K}^b(\text{proj} R) \to \mathbb{D}_{sg}(R)$$

be the canonical embedding functor. Notice that

$$\mathbb{D}_{sg}(R)/(\mathbb{D}^b(\text{mod} R)_{\text{fgp}}/\mathbb{K}^b(\text{proj} R)) \simeq \mathbb{D}^b(\text{mod} R)/\mathbb{D}^b(\text{mod} R)_{\text{fgp}}$$

is a triangle-equivalence (see [32, Corollary 4.3]). There exists a canonical quotient functor

$$\pi : \mathbb{D}_{sg}(R) \to \mathbb{D}^b(\text{mod} R)/\mathbb{D}^b(\text{mod} R)_{\text{fgp}}.$$

We have both $\iota$ and $\pi$ are triangle-functors.

The following result seems clear (see, e.g., [34, Theorem 3.4]), and we provide a proof here.

**Theorem 2.9.** We have the following exact commutative diagram:

$$
\begin{array}{cccccc}
0 & \to & \text{Gproj} R & \overset{F}{\to} & \mathbb{D}_{sg}(R) & \to & \mathbb{D}_{\text{def}}(R) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathbb{D}^b(\text{mod} R)_{\text{fgp}}/\mathbb{K}^b(\text{proj} R) & \overset{\iota}{\to} & \mathbb{D}_{sg}(R) & \overset{\pi}{\to} & \mathbb{D}^b(\text{mod} R)/\mathbb{D}^b(\text{mod} R)_{\text{fgp}} & \to & 0
\end{array}
$$

with all the vertical functors triangle-equivalences.

**Proof.** In view of Buchweitz’s theorem, it suffices to show that $\text{Im} F = \mathbb{D}^b(\text{mod} R)_{\text{fgp}}/\mathbb{K}^b(\text{proj} R)$.

Note that every Gorenstein projective module viewed as a stalk complex concentrated in degree zero has finite Gorenstein projective dimension. Thus

$$\text{Im} F \subseteq \mathbb{D}^b(\text{mod} R)_{\text{fgp}}/\mathbb{K}^b(\text{proj} R).$$
Now let $X^\bullet \in \mathbb{D}^b(\text{mod} \ R)_{\text{fgp}}/\mathbb{K}^b(\text{proj} \ R)$. It follows from Lemma 2.5 that there exists a quasi-isomorphism $P^\bullet \to X^\bullet$ with $P^\bullet \in \mathbb{K}^{-b}(\text{proj} \ R)$ such that $Z^i(P^\bullet) \in \text{Gproj} \ R$ for $i < 0$. Hence $P^\bullet$ is isomorphic to the following complex in $\mathbb{D}^b(\text{mod} \ R)$:

$$G^\bullet := 0 \to Z^i(P^\bullet) \to P^i \to \cdots \to P^{s-1} \to P^s \to 0,$$

where $s$ is the supremum of index $i \in \mathbb{Z}$ such that $P^i \neq 0$ and $t$ is the index such that $Z^i(P^\bullet) \in \text{Gproj} \ R$ and $H^i(P^\bullet) = 0$ for any $i \leq t$. Consider the following triangle in $\mathbb{D}_{\text{def}}(\text{mod} \ R)$:

$$Z^i(P^\bullet)[{-t}] \to P_{\geq t}^\bullet \to G^\bullet \to Z^i(P^\bullet)[{-t + 1}].$$

Since $P_{\geq t}^\bullet \in \mathbb{K}^b(\text{proj} \ R)$, $G^\bullet \cong Z^i(P^\bullet)[{-t + 1}]$ and then $X^\bullet \cong Z^i(P^\bullet)[{-t + 1}]$ in $\mathbb{D}_{\text{def}}(\text{mod} \ R)$. Hence $X^*[t - 1] \cong Z^i(P^\bullet)$, i.e., $X^*[t - 1] \in \text{Im} \ F$. Since $\text{Im} \ F$ is a triangulated subcategory, $X^\bullet \in \text{Im} \ F$ and then $\mathbb{D}^b(\text{mod} \ R)_{\text{fgp}}/\mathbb{K}^b(\text{proj} \ R) \subseteq \text{Im} \ F$. \hfill $\Box$

As a consequence, we obtain the following corollary, which implies that the converse of Buchweitz’s theorem also holds true. Note that this has also been proved by many other authors (see, e.g., Beligiannis [6], Bergh et al. [8], Kong and Zhang [23] and Zhu [35]).

**Corollary 2.10.** The following statements are equivalent:

1. $F: \text{Gproj} \ R \to \mathbb{D}_{\text{def}}(\text{R})$ is a triangle-equivalence.
2. $\mathbb{D}_{\text{def}}(\text{R}) = 0$.
3. $R$ is Gorenstein.

**Proof.** (1) $\iff$ (2) is trivial, and (2) $\iff$ (3) follows from Proposition 2.8 and Theorem 2.9. \hfill $\Box$

### 3 Proofs of the main results

In this section, let $R$ be an Artin algebra and $e$ an idempotent of $R$. Recall from [18, Chapter 6] that the Schur functor associative with $e$ is defined to be

$$S_e = eR \otimes_R - : \text{mod} \ R \to \text{mod} \ eRe,$$

and it was also called the restriction functor in [2, Subsection I.6]. Clearly, $S_e$ admits a fully faithful left adjoint

$$T_e = eRe \otimes_{eRe} - : \text{mod} \ eRe \to \text{mod} \ R$$

and a fully faithful right adjoint

$$L_e = \text{Hom}_{eRe}(eR, -) : \text{mod} \ eRe \to \text{mod} \ R.$$

Recall from [28, 29] that a *recollement* between abelian categories $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ is a diagram

$$\begin{array}{ccc}
\mathcal{A} & \xrightarrow{i_\ast} & \mathcal{B} \\
i^\ast & & j_\ast \\
\downarrow & & \downarrow \\
_i & & j
\end{array}$$

satisfying the following conditions:

1. $(i^\ast, i_\ast, j^\ast)$ and $(j, j^\ast, j_\ast)$ are adjoint triples.
2. The functors $i_\ast$, $j_\ast$ and $j_\ast$ are fully faithful.
3. $\text{Im} \ i_\ast = \text{Ker} \ j^\ast$.

We need the following fact.

**Lemma 3.1** (See [28, 29]). We have the following recollement between module categories:

$$\begin{array}{cccc}
\text{mod} \ R/ReR & \xrightarrow{\text{inc}} & \text{mod} \ R/ReR \\
\text{Im} \ R/ReR & \xrightarrow{\text{inc}} & \text{mod} \ R \\
\text{Hom}_{R}(R/ReR, -) & \xrightarrow{j_\ast} & \text{Hom}_{eRe}(eR, -) \\
\text{Hom}_{R}(R/ReR, -) & \xrightarrow{j_\ast} & \text{mod} \ eRe.
\end{array}$$
where inc : mod R/ReR → mod R denotes the inclusion functor induced by the canonical ring homomorphism R → R/ReR. In the following, the image of the functor inc : mod R/ReR → mod R is identified with mod R/ReR for simplicity.

Since $S_e$ is exact, it lifts to the $*$-bounded derived functors $\mathbb{D}^*(S_e) : \mathbb{D}^*(mod R) \to \mathbb{D}^*(mod eRe)$ via $\mathbb{D}^*(S_e)(X^*) = S_e(X^*)$ for each complex $X^* \in \mathbb{D}^*(mod R)$, where $* \in \{blank, +, -, b\}$. Meanwhile, it is not hard to see that $T_e$ preserves projective modules. So $T_e$ lifts to a derived functor $\mathbb{D}^-(T_e) : \mathbb{D}^-(mod eRe) \to \mathbb{D}^-(mod R)$ (comparing with [16, Proposition 2.1]).

**Lemma 3.2.** The following statements hold true:

1. The $*$-bounded derived functor $\mathbb{D}^*(S_e) : \mathbb{D}^*(mod R) \to \mathbb{D}^*(mod eRe)$ restricts to

   $$\mathbb{D}^b(S_e)_{fgp} : \mathbb{D}^b(mod R)_{fgp} \to \mathbb{D}^b(mod eRe)_{fgp}$$

   if and only if $\text{Gpd}_{eRe} S_e(F) < \infty$ for any $F \in \text{Gproj} R$.

2. Assume that $\text{Gpd}_R T_e(G) < \infty$ for any $G \in \text{Gproj} eRe$. Then the derived functor $\mathbb{D}^-(T_e) : \mathbb{D}^-(mod eRe) \to \mathbb{D}^-(mod R)$ restricts to

   $$\mathbb{D}^b(T_e)_{fgp} : \mathbb{D}^b(mod eRe)_{fgp} \to \mathbb{D}^b(mod R)_{fgp}$$

   if and only if $\text{Tor}^e_{Re}(R, G) = 0$ for any $G \in \text{Gproj} eRe$ and $i$ sufficiently large.

**Proof.** (1) Assume that $\mathbb{D}^*(S_e) : \mathbb{D}^*(mod R) \to \mathbb{D}^*(mod eRe)$ restricts to

   $$\mathbb{D}^b(S_e)_{fgp} : \mathbb{D}^b(mod R)_{fgp} \to \mathbb{D}^b(mod eRe)_{fgp}.$$  

   For any $F \in \text{Gproj} R$, we have that $\mathbb{D}^b(S_e)(F) = S_e(F)$ lies in $\mathbb{D}^b(mod eRe)_{fgp}$. By Remark 2.6, we get $\text{Gpd}_{eRe} S_e(F) < \infty$. Conversely, assume that $\text{Gpd}_{eRe} S_e(F) < \infty$ for any $F \in \text{Gproj} R$. Let $X^*$ be a bounded complex of Gorenstein projective $R$-modules. It follows that $\mathbb{D}^*(S_e)(X^*) = S_e(X^*)$, and it is a bounded complex with each degree of finite Gorenstein projective dimension. Hence $\mathbb{D}^*(S_e)(X^*) \in \mathbb{D}^b(mod eRe)_{fgp}$ by Proposition 2.8. Thus $\mathbb{D}^*(S_e) : \mathbb{D}^*(mod R) \to \mathbb{D}^*(mod eRe)$ restricts to $\mathbb{D}^b(S_e)_{fgp} : \mathbb{D}^b(mod R)_{fgp} \to \mathbb{D}^b(mod eRe)_{fgp}$.

(2) For the “if” part, we show $\mathbb{D}^-(T_e)(\mathbb{D}^b(mod eRe)_{fgp}) \subseteq \mathbb{D}^b(mod R)_{fgp}$. To do this, it suffices to show $\mathbb{D}^-(T_e)(Y^*) \in \mathbb{D}^b(mod R)_{fgp}$ for any bounded complex $Y^*$ of Gorenstein projective $eRe$-modules. We proceed by induction on the length $l(Y^*)$ of $Y^*$.

If $l(Y^*) = 1$, we may suppose that $Y^* = Y$ is the stalk complex concentrated in degree 0. Take a projective resolution

$$\cdots \to P^{-n} \to \cdots \to P^{-1} \to P^0 \to Y \to 0$$

of $Y$, and set

$$P^* = \cdots \to P^{-n} \to \cdots \to P^{-1} \to P^0 \to 0.$$  

It follows that $\mathbb{D}^-(T_e)(Y) \cong T_e(P^*) = Re \otimes eRe P^*$, and it is a complex of projective $R$-modules. Since $Y \in \text{Gproj} eRe$, one has each cycle $Z^i(P^*) \in \text{Gproj} eRe$. Then by assumption, we obtain $\text{Gpd}_R T_e(Z^i(P^*)) < \infty$ for every integer $i$. Note that $\text{Tor}^e_{Re}(R, G) = 0$ for any $G \in \text{Gproj} eRe$ and $i$ sufficiently large. It follows that $T_e(P^*)$ is exact in degree $i$ and hence $Z^i(T_e(P^*)) \cong T_e(Z^i(P^*))$ whenever $i \ll 0$. Thus there exists some integer $n_0 \gg 0$ such that $T_e(P^*)$ is isomorphic to its truncation complex

$$G^* := 0 \to T_e(Z^{-n_0}(P^*)) \to T_e(P^{-n_0}) \to \cdots \to T_e(P^{-1}) \to T_e(P^0) \to 0.$$  

Note that $T_e(P^i) \in \text{proj} R$ for every integer $i$ and $\text{Gpd}_R T_e(Z^{-n_0}(P^*)) < \infty$. Therefore, by Proposition 2.8 we have $\mathbb{D}^-(T_e)(Y) \in \mathbb{D}^b(mod R)_{fgp}$.

Now suppose that $l(Y^*) = n \geq 2$ and the claim holds true for any integer less than $n$. Then $Y^*$ must be of the following form:

$$Y^* = 0 \to Y^{m+1} \to Y^{m+2} \to \cdots \to Y^{m+n} \to 0.$$
It induces a triangle
\[ Y^{m+1}[-m-2] \to Y\mathbf{\gamma}_{m+2} \to Y^\bullet \to Y^{m+1}[-m-1] \]
in \( \mathcal{D}^- (\text{mod} \, R) \). Then we have the following triangle in \( \mathcal{D}^- (\text{mod} \, R) \):
\[ \mathcal{D}^- (T_c)(Y^{m+1})[-m-2] \to \mathcal{D}^- (T_c)(Y\mathbf{\gamma}_{m+2}) \to \mathcal{D}^- (T_c)(Y^\bullet) \to \mathcal{D}^- (T_c)(Y^{m+1})[-m-1]. \]
By the induction hypothesis, we see that both \( \mathcal{D}^- (T_c)(Y^{m+1})[-m-2] \) and \( \mathcal{D}^- (T_c)(Y\mathbf{\gamma}_{m+2}) \) lie in \( \mathbb{D}^b(\text{mod} \, R) \). Thus \( \mathcal{D}^- (T_c)(Y^\bullet) \in \mathbb{D}^b(\text{mod} \, R) \).

For the “only if” part, assume that \( \mathcal{D}^- (T_c) : \mathcal{D}^- (\text{mod} \, R) \to \mathcal{D}^- (\text{mod} \, R) \) restricts to
\[ \mathbb{D}^b(T_c) \mathbb{D}^b(\text{mod} \, eRe) \to \mathbb{D}^b(\text{mod} \, R). \]
Let \( G \in \text{Gproj} \, eRe \). Take a projective resolution \( P^\bullet \to G \) of \( G \). It follows that \( \mathcal{D}^- (T_c)(G) \cong T_c(P^\bullet \to G) \), and this should be a complex in \( \mathbb{D}^b(\text{mod} \, R) \). Thus \( T_c(P^\bullet \to G) \) has finite cohomology and then \( \text{Tor}_{i}^{eRe}(Re, G) = 0 \) for \( i \) sufficiently large.

Denote by \( \mathbb{D}^b(\text{mod} \, R)_{\text{mod} \, R/ReR} \) the subcategory of \( \mathbb{D}^b(\text{mod} \, R) \) consisting of complexes with cohomology in \( \text{mod} \, R/ReR \). It is not hard to see that \( \mathbb{D}^b(\text{mod} \, R)_{\text{mod} \, R/ReR} \) is a thick subcategory of \( \mathbb{D}^b(\text{mod} \, R) \) generated by \( \text{mod} \, R/ReR \).

Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. For the “if” part, since \( 0 \to \text{mod} \, R/ReR \to \text{mod} \, R \to \text{mod} \, eRe \to 0 \) is an exact sequence of module categories by Lemma 3.1, it follows from [26, Theorem 3.2] that \( \mathbb{D}^b(S_c) : \mathbb{D}^b(\text{mod} \, R) \to \mathbb{D}^b(\text{mod} \, eRe) \) induces a triangle-equivalence
\[ \mathbb{D}^b(S_c) : \mathbb{D}^b(\text{mod} \, R) / \mathbb{D}^b(\text{mod} \, R)_{\text{mod} \, R/ReR} \to \mathbb{D}^b(\text{mod} \, eRe). \]
Notice that \( e \) is Gorenstein singularly-complete, and thus any \( R/ReR \)-module has finite Gorenstein projective dimension as an \( R \)-module. As \( \mathbb{D}^b(\text{mod} \, R)_{\text{mod} \, R/ReR} \) is generated by \( \text{mod} \, R/ReR \), one has \( \mathbb{D}^b(\text{mod} \, R)_{\text{mod} \, R/ReR} \subseteq \mathbb{D}^b(\text{mod} \, R)_{\text{mod} \, R/ReR} \). Then by Theorem 2.9, we obtain
\[ \mathbb{D} \mathbb{D}^b(\text{mod} \, eRe)(\mathbb{D}^b(\text{mod} \, R)_{\text{mod} \, R/ReR}) = \mathbb{D}^b(\text{mod} \, eRe)(\mathbb{D}^b(\text{mod} \, R)_{\text{mod} \, R/ReR}). \]
Since \( \text{Gpd}_{\text{eRe}} S_c(F) < \infty \) for any \( F \in \text{Gproj} \, R \), by Lemma 3.2 we have \( \mathbb{D}^b(S_c)(\mathbb{D}^b(\text{mod} \, R)_{\text{mod} \, R/ReR}) \subseteq \mathbb{D}^b(\text{mod} \, eRe)(\mathbb{D}^b(\text{mod} \, R)_{\text{mod} \, R/ReR}) \subseteq \mathbb{D}^b(\text{mod} \, eRe)(\mathbb{D}^b(\text{mod} \, R)_{\text{mod} \, R/ReR}) \). This implies the following exact commutative diagram:
\[
\begin{array}{cccccc}
0 & \to & \mathbb{D}^b(\text{mod} \, R)_{\text{mod} \, R/ReR} & \to & \mathbb{D}^b(\text{mod} \, R)_{\text{mod} \, R/ReR} & \to & \mathbb{D}^b(\text{mod} \, eRe) & \to & 0 \\
& & \mathbb{D}^b(\text{mod} \, eRe) & \to & \mathbb{D}^b(\text{mod} \, eRe) & \to & \mathbb{D} \mathbb{D}^b(\text{mod} \, eRe) & \to & 0 \\
& & \mathbb{D} \mathbb{D}^b(\text{mod} \, eRe) & \to & \mathbb{D} \mathbb{D}^b(\text{mod} \, eRe) & \to & \mathbb{D} \mathbb{D}^b(\text{mod} \, eRe) & \to & 0 \\
\end{array}
\]
where \( \mathbb{D}^b(S_c)_{\text{mod} \, R/ReR} \) is the restriction of \( \mathbb{D}^b(S_c) \). Since \( \mathbb{D}^b(S_c) \) is fully faithful, so is \( \mathbb{D}^b(S_c)_{\text{mod} \, R/ReR} \).

Next, we show that \( \mathbb{D}^b(S_c)_{\text{mod} \, R/ReR} : \mathbb{D}^b(\text{mod} \, R)_{\text{mod} \, R/ReR} \to \mathbb{D}^b(\text{mod} \, eRe)(\mathbb{D}^b(\text{mod} \, R)_{\text{mod} \, R/ReR}) \) is essentially surjective (or dense). By [16, Proposition 2.1] we have the following diagram:
\[
\begin{array}{cccccc}
\mathcal{D}^- (T_c) & \to & \mathcal{D}^- (T_c) & \to & \mathcal{D}^- (T_c) & \to & 0 \\
\mathcal{D}^- (R) & \to & \mathcal{D}^- (R) & \to & \mathcal{D}^- (R) & \to & 0 \\
\mathcal{D}^- (S_c) & \to & \mathcal{D}^- (S_c) & \to & \mathcal{D}^- (S_c) & \to & 0 \\
\end{array}
\]
such that \( \left( \mathcal{D}^- (T_c), \mathcal{D}^- (S_c) \right) \) is an adjoint pair with \( \mathcal{D}^- (T_c) \) fully faithful. In view of Lemma 3.2, this diagram restricts to the following diagram:
\[
\begin{array}{cccccc}
\mathbb{D}^b(T_c)_{\text{mod} \, R/ReR} & \to & \mathbb{D}^b(T_c)_{\text{mod} \, R/ReR} & \to & \mathbb{D}^b(T_c)_{\text{mod} \, R/ReR} & \to & 0 \\
\mathbb{D}^b(S_c)_{\text{mod} \, R/ReR} & \to & \mathbb{D}^b(S_c)_{\text{mod} \, R/ReR} & \to & \mathbb{D}^b(S_c)_{\text{mod} \, R/ReR} & \to & 0 \\
\end{array}
\]
such that $(\mathcal{D}^b(T_e)_{fgp}, \mathcal{D}^b(S_e)_{fgp})$ is an adjoint pair with $\mathcal{D}^b(T_e)_{fgp}$ fully faithful. Then for any $Y^\bullet \in \mathcal{D}^b(\mod \, eRe)_{fgp}$, set $X^\bullet = \mathcal{D}^b(T_e)_{fgp}(Y^\bullet)$. It follows that

$$Y^\bullet \cong \mathcal{D}^b(S_e)_{fgp} \mathcal{D}^b(T_e)_{fgp}(Y^\bullet) \cong \mathcal{D}^b(S_e)_{fgp}(X^\bullet).$$

Then $\mathcal{D}^b(S_e)_{fgp} : \mathcal{D}^b(\mod \, R)_{fgp} \to \mathcal{D}^b(\mod \, eRe)_{fgp}$ is dense and hence

$$\mathcal{D}^b(S_e)_{fgp} \to \mathcal{D}^b(\mod \, eRe)_{fgp}$$

is dense. To conclude, $\mathcal{D}^b(S_e)_{fgp}$ is a triangle-equivalence. Therefore, we infer that $\mathcal{D}_{\def}(S_e) : \mathcal{D}_{\def}(R) \to \mathcal{D}_{\def}(eRe)$ is a triangle-equivalence from the above exact commutative diagram.

Conversely, let $F \in \Gproj R$. It follows from Proposition 2.8 that $F$ is zero in $\mathcal{D}_{\def}(R)$. Since $\mathcal{D}_{\def}(S_e) : \mathcal{D}_{\def}(R) \to \mathcal{D}_{\def}(eRe)$ is a triangle-equivalence, we get that $\mathcal{D}_{\def}(S_e)(F) = S_e(F)$ is zero in $\mathcal{D}_{\def}(\mod \, eRe)$ and hence $S_e(F) \in \mathcal{D}^b(\mod \, eRe)_{fgp}$. Therefore, from Remark 2.6 we have $\Gproj_{eRe} S_e(F) < \infty$ and (C1) follows. To get (C2) and (C3), for any $G \in \Gproj eRe$ and $M \in \mod \, R/ReR$, we show that both $M$ and $T_e(G)$ have finite Gorenstein projective dimension as $R$-modules. Following Lemma 3.1, we obtain that $S_e(M) = 0$ and $G \cong S_eT_e(G)$ and hence both $S_e(M)$ and $S_eT_e(G)$ are zero in $\mathcal{D}_{\def}(eRe)$. Notice that $\mathcal{D}_{\def}(S_e) : \mathcal{D}_{\def}(R) \to \mathcal{D}_{\def}(eRe)$ is an equivalence. We obtain that both $M$ and $T_e(G)$ are zero in $\mathcal{D}_{\def}(R)$. Therefore by the foregoing proof, we conclude that both $M$ and $T_e(G)$ have finite Gorenstein projective dimension as $R$-modules as desired.

Combining Theorem 1.1 with [29, Corollary 5.4], we get the proof of Corollary 1.2.

**Proof of Corollary 1.2.** The “only if” part follows directly from Theorem 1.1 and [29, Corollary 5.4], and we prove the “if” part. Combining [29, Corollary 5.4] with Theorem 1.1, the Schur functor induces a singular equivalence $\mathcal{D}_{sg}(S_e) : \mathcal{D}_{sg}(R) \to \mathcal{D}_{sg}(eRe)$ and a triangle-equivalence of Gorenstein defect categories $\mathcal{D}_{\def}(S_e) : \mathcal{D}_{\def}(R) \to \mathcal{D}_{\def}(eRe)$. Then we have the following exact commutative diagram:

$$
\begin{array}{c}
0 \longrightarrow \Gproj R \longrightarrow \mathcal{D}_{sg}(R) \longrightarrow \mathcal{D}_{\def}(R) \longrightarrow 0 \\
\mathcal{D}_{sg}(S_e) \downarrow \mathcal{D}_{\def}(S_e) \\
0 \longrightarrow \Gproj eRe \longrightarrow \mathcal{D}_{sg}(eRe) \longrightarrow \mathcal{D}_{\def}(eRe) \longrightarrow 0.
\end{array}
$$

Hence it is not hard to see that $\mathcal{D}_{sg}(S_e) : \mathcal{D}_{sg}(R) \to \mathcal{D}_{sg}(eRe)$ restricts to a triangle-equivalence $\Gproj eRe \simeq \Gproj R$ and then we get the desired result.

### 4 Applications in triangular matrix algebras

In this section, we deal with the triangular matrix algebra $T = \left( \begin{array}{c} A & M \\ 0 & B \end{array} \right)$, where the corner algebras $A$ and $B$ are Artin algebras and $A \cdot M \cdot B$ is an $A\cdot B$-bimodule.

Recall that a left $T$-module is identified with a triple $(X, Y, \phi)$, where $X \in \mod \, A$, $Y \in \mod \, B$ and $\phi : M \otimes_B Y \to X$ is an $A$-morphism. If there is no possible confusion, we omit the morphism $\phi$ and write $(X, Y)$ for short. For example, we write $(M \otimes_B Y, Y)$ for the $T$-module $(M \otimes_B Y, Y, \id)$. A $T$-morphism $(X, Y, \phi) \to (X', Y', \phi')$ is identified with a pair $(f, g)$, where $f \in \Hom_A(X, X')$ and $g \in \Hom_B(Y, Y')$ such that the following diagram:

$$
\begin{array}{c}
M \otimes_B Y \xrightarrow{\phi} X \\
\downarrow 1 \otimes g \\
M \otimes_B Y' \xrightarrow{\phi'} X'
\end{array}
$$

is commutative.

A sequence

$$0 \to (X_1, Y_1, \phi_1) \xrightarrow{(f_1, g_1)} (X_2, Y_2, \phi_2) \xrightarrow{(f_2, g_2)} (X_3, Y_3, \phi_3) \to 0$$

is commutative.
in mod $T$ is exact if and only if $0 \to X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \to 0$ and $0 \to Y_1 \xrightarrow{g_1} Y_2 \xrightarrow{g_2} Y_3 \to 0$ are exact in mod $A$ and mod $B$, respectively. Indecomposable projective $T$-modules are exactly $(P,0)$ and $(M \otimes Q,Q)$, where $P$ runs over indecomposable projective $A$-modules, and $Q$ runs over indecomposable projective $B$-modules. We refer the reader to [1,3,18] for more details.

Recall from [33] that $AM_B$ is compatible if $M \otimes_B -$ carries every acyclic complex of projective $B$-modules to acyclic $A$-complex and $M \in (Gproj A)$. If $AM_B$ is compatible, it is not hard to see $\text{Tor}_i^B(M,G) = 0$ for any $G \in Gproj B$ and $i \geq 1$.

**Lemma 4.1** (See [33, Theorem 1.4]). Let $T = (A M_B)$ be a triangular matrix algebra with $AM_B$ compatible. Then $(X,Y,\phi) \in Gproj T$ if and only if $Y \in Gproj B$ and $\phi : M \otimes_B Y \to X$ is an injective $A$-morphism with $\text{Coker} \phi \in Gproj A$.

As a consequence, we have the following corollary.

**Corollary 4.2.** Let $T = (A M_B)$ be a triangular matrix algebra with $AM_B$ compatible. The following statements hold true:

1. $Gpd_T(X,0) = Gpd_A X$.
2. Assume that $Gpd_A M \otimes_B G < \infty$ for any $G \in Gproj B$. Then $Gpd_T(0,Y) < \infty$ if and only if $Gpd_B Y < \infty$.

**Proof.** (1) is clear and we only prove (2). For the “only if” part, let $Gpd_T(0,Y) = n$ for some integer $n \geq 0$. Then there exists an exact sequence

$$0 \to (P_n,Q_n,\phi_n) \to \cdots \to (P_1,Q_1,\phi_1) \to (P_0,Q_0,\phi_0) \to (0,Y) \to 0$$

in mod $T$ with each $(P_i, Q_i, \phi_i) \in Gproj T$. It follows that

$$0 \to Q_n \to \cdots \to Q_1 \to Q_0 \to Y \to 0$$

is an exact sequence in mod $B$. Since each $(P_i, Q_i, \phi_i) \in Gproj T$, it follows from Lemma 4.1 that each $Q_i \in Gproj B$. Therefore, $Gpd_B Y \leq n$.

For the “if” part, we first claim $Gpd_T(0,G) < \infty$ for any $G \in Gproj B$. To get this, let $G \in Gproj B$. Consider the following exact sequence of $T$-modules:

$$0 \to (M \otimes_B G,0) \to (M \otimes_B G,G) \to (0,G) \to 0.$$ 

By assumption, $Gpd_A M \otimes_B G < \infty$. As a consequence of (1), we get $Gpd_T(M \otimes_B G,0) < \infty$. Notice that $(M \otimes_B G,G) \in Gproj T$. We obtain that $Gpd_T(0,G) < \infty$ and the claim follows.

Now assume $Gpd_B Y = m$ for some integer $m \geq 0$. Take a Gorenstein projective resolution $0 \to G_m \to \cdots \to G_1 \to G_0 \to Y \to 0$ of $Y$. We have $0 \to (0,G_m) \to \cdots \to (0,G_1) \to (0,G_0) \to (0,Y) \to 0$ is exact in mod $T$. By the claim, one has $Gpd_T(0,G_i) < \infty$ for every $i$. Therefore, we conclude that $Gpd_T(0,Y) < \infty$.

Let $e_A = (1 0 \ 0)$ and $e_B = (0 1 \ 0)$ be the idempotents of $T$. It is known that $A \cong e_A T e_A \cong T / T e_B T$ and $B \cong e_B T e_B \cong T / T e_A T$ as algebras. Denote by $S_{e_A}$ and $S_{e_B}$ the Schur functors associative to $e_A$ and $e_B$, respectively.

We have the following observation.

**Lemma 4.3.** The following statements hold true:

1. $\text{Tor}_i^A(T e_A, F) = 0$ for any $F \in Gproj A$ and $i \geq 1$.
2. If $AM_B$ is compatible, then $\text{Tor}_i^B(T e_B, G) = 0$ for any $G \in Gproj B$ and $i \geq 1$.

**Proof.** (1) Since $T e_A \cong A$ as right $A$-modules, this assertion is trivial.

(2) Note that $T e_B \cong M_B \oplus B_B$ is an isomorphism of right $B$-modules. Since $AM_B$ is compatible, $\text{Tor}_i^B(M,G) = 0$ for any $G \in Gproj B$ and $i \geq 1$. Hence $\text{Tor}_i^B(T e_B, G) = 0$ for any $G \in Gproj B$ and $i \geq 1$.

To take $R = T$ and $e = e_A$ as in Theorem 1.1 and Corollary 1.2, we get the following theorem.
**Theorem 4.4.** Let $T = (\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix})$ be a triangular matrix algebra with $AM_B$ compatible. The following statements hold true:

1. The Schur functor $S_{e_A}$ induces a triangle-equivalence $\mathcal{D}_{\text{def}}(T) \simeq \mathcal{D}_{\text{def}}(A)$ if and only if $B$ is Gorenstein and $\text{Gpd}_A M \otimes_B Y < \infty$ for any $Y \in \text{Gproj} B$. In this case, we have a singular equivalence $\mathcal{D}_{\text{sg}}(\text{Aus}(T)) \simeq \mathcal{D}_{\text{sg}}(\text{Aus}(A))$ between the relative Auslander algebras $\text{Aus}(T)$ and $\text{Aus}(A)$ provided that $T$ is CM-finite.

2. (Comparing with [10, Theorem 3.3]) The Schur functor $S_{e_A}$ induces the following exact commutative diagram:

$$
\begin{array}{cccccc}
0 & \longrightarrow & \text{Gproj} T & \longrightarrow & \mathcal{D}_{\text{sg}}(T) & \longrightarrow & \mathcal{D}_{\text{def}}(T) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \text{Gproj} A & \longrightarrow & \mathcal{D}_{\text{sg}}(A) & \longrightarrow & \mathcal{D}_{\text{def}}(A) & \longrightarrow & 0
\end{array}
$$

with all the vertical functors triangle-equivalences if and only if $B$ has finite global dimension, $\text{pd}_A M < \infty$ and $\text{Gpd}_A M \otimes_B Y < \infty$ for any $Y \in \text{Gproj} B$. In this case, $T$ is Gorenstein (resp. CM-free) if and only if so is $A$.

**Proof.** Since $e_A T e_A \cong A$, we prove by replacing $R$ with $T$ and $e$ with $e_A$ in Theorem 1.1 and Corollary 1.2, respectively. By Lemma 4.3(1), $\text{Tor}_i^A(T e_A, F) = 0$ for any $F \in \text{Gproj} A$ and $i \geq 1$.

(1) For the “only if” part, assume that $S_{e_A}$ induces a triangle-equivalence $\mathcal{D}_{\text{def}}(T) \simeq \mathcal{D}_{\text{def}}(A)$. Then the conditions (C1)–(C3) in Theorem 1.1 are satisfied. For any $Y \in \text{Gproj} B$, it follows from Lemma 4.1 that $(M \otimes_B Y, Y) \in \text{Gproj} T$. Since $S_{e_A}(M \otimes_B Y, Y) \simeq M \otimes_B Y$, we infer $\text{Gpd}_A M \otimes_B Y < \infty$ from (C1). Now let $Z \in \text{mod} B$. Note that $Z$ viewed as a $T$-module is isomorphic to $(0, Z)$. We infer that $\text{Gpd}_T(0, Z) < \infty$ from (C3). In view of Corollary 4.2(2), we obtain $\text{Gpd}_B Z < \infty$. Then it follows from [22, Theorem] that $B$ is Gorenstein.

For the “if” part, assume that $B$ is Gorenstein and $\text{Gpd}_A M \otimes_B Y < \infty$ for any $Y \in \text{Gproj} B$. In view of Theorem 1.1, it suffices to show that the conditions (C1)–(C3) in Theorem 1.1 are satisfied. Let $(X, Y, \phi) \in \text{Gproj} T$. It follows from Lemma 4.1 that $Y \in \text{Gproj} B$ and $\phi : M \otimes_B Y \to X$ is an injective $A$-morphism with $\text{Coker} \phi \in \text{Gproj} A$. Now consider the following exact sequence of $A$-modules:

$$
0 \to M \otimes_B Y \xrightarrow{\phi} X \to \text{Coker} \phi \to 0.
$$

Because $\text{Coker} \phi \in \text{Gproj} A$ and $\text{Gpd}_A M \otimes_B Y < \infty$, we have that $\text{Gpd}_A X < \infty$. Noticing that $X \cong S_{e_A}(X, Y, \phi)$, we have that (C1) follows. Now for any $F \in \text{Gproj} A$, we have $\text{Hom}_T(T_{e_A}(F), (X', Y', \phi')) \cong \text{Hom}_A(F, S_{e_A}(X', Y', \phi')) \cong \text{Hom}_A(F, X') \cong \text{Hom}_T((F, 0), (X', Y', \phi'))$ for any $(X', Y', \phi') \in \text{mod} T$. By the Yoneda lemma, we have $T_{e_A}(F) \cong (F, 0)$. Following Lemma 4.1, $T_{e_A}(F) \in \text{Gproj} T$ and then the condition (C2) follows. For (C3), let $Z \in \text{mod} B$. Notice that $Z$ viewed as a $T$-module is isomorphic to $(0, Z)$. We show $\text{Gpd}_T(0, Z) < \infty$. Since $B$ is Gorenstein, one gets $\text{Gpd}_B Z < \infty$. Hence it follows from Corollary 4.2(2) that $\text{Gpd}_T(0, Z) < \infty$.

For any $X \in \text{Gproj} A$, we have $(X, 0) \in \text{Gproj} T$ by Lemma 4.1. Thus the CM-finiteness of $T$ implies that of $A$. By Remark 2.3, we infer a singular equivalence $\mathcal{D}_{\text{sg}}(\text{Aus}(T)) \simeq \mathcal{D}_{\text{sg}}(\text{Aus}(A))$ from the triangle-equivalence $\mathcal{D}_{\text{def}}(A) \simeq \mathcal{D}_{\text{def}}(T)$.

(2) For the “only if” part, assume that $S_{e_A}$ induces such an exact commutative diagram. It follows from Corollary 1.2 that $e_A$ is singularly-complete and $\text{pd}_A e_A T < \infty$. Notice that $e_A T \cong A \oplus M$ is an isomorphism of $A$-modules. We obtain $\text{pd}_A M < \infty$. Now for any $Z \in \text{mod} B$, since $e_A$ is singularly-complete, it follows that $\text{pd}_T(0, Z) < \infty$. We may assume $\text{pd}_T(0, Z) = n$ for some integer $n \geq 0$. Take a projective resolution

$$
0 \to (P_n, Q_n, \phi_n) \to \cdots \to (P_1, Q_1, \phi_1) \to (P_0, Q_0, \phi_0) \to (0, Z) \to 0
$$

of $(0, Z)$. It follows that

$$
0 \to Q_n \to \cdots \to Q_1 \to Q_0 \to Z \to 0
$$
is a projective resolution of $Z$ and then $\text{pd}_B Z < \infty$. This implies that $B$ has the finite global dimension. Note that $S_{e_A}$ induces a triangle-equivalence $D_{\text{def}}(T) \simeq D_{\text{def}}(A)$. We have that $\text{Gpd}_A M \otimes_B Y < \infty$ for any $Y \in \text{Gproj} B$ as a consequence of (1).

For the “if” part, assume that $B$ has the finite global dimension, $\text{pd}_A M < \infty$ and $\text{Gpd}_A M \otimes_B Y < \infty$ for any $Y \in \text{Gproj} B$. By the foregoing proof, we know that the conditions (C1) and (C2) in Corollary 1.2 are satisfied. To get this assertion, in view of Corollary 1.2, it suffices to show that $e_A$ is singularly-complete and $\text{pd}_A e_A T < \infty$. Since $\text{pd}_A T < \infty$, we infer $\text{pd}_A e_A T < \infty$ from the isomorphism $e_A T \cong A \oplus M$. Now let $Z \in \text{mod} B$. Because $B$ has the finite global dimension, one has $\text{pd}_B Z < \infty$. If $Z \in \text{proj} B$, we get that $\text{pd}_A M \otimes_B Z < \infty$, since $\text{pd}_A M < \infty$. It follows from [10, Lemma 3.1(1)] that $\text{pd}_T(M \otimes_B Z, 0) < \infty$. Notice that $(M \otimes_B Z, Z)$ is projective, and hence we infer $\text{pd}_T(0, Z) < \infty$ from the short exact sequence

$$0 \to (M \otimes_B Z, 0) \to (M \otimes_B Z, Z) \to (0, Z) \to 0.$$  

Now assume $\text{pd}_B Z = n$ for some integer $n > 0$. Take a projective resolution

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to Z \to 0$$

of $Z$. We have that

$$0 \to (0, P_n) \to \cdots \to (0, P_1) \to (0, P_0) \to (0, Z) \to 0$$

is exact in $\text{mod} T$. Note that $\text{pd}_T(0, P_i) < \infty$ for every $i$. We conclude that $\text{pd}_T(0, Z) < \infty$ and hence $e_A$ is singularly-complete.

Note that $T$ is Gorenstein if and only if $D_{\text{def}}(T) = 0$ (see Corollary 2.10); while $T$ is CM-free if and only $\text{Gproj} T = 0$. Then we infer that $T$ is Gorenstein (resp. CM-free) if and only if so is $A$ from such exact commutative diagram.

\begin{example}
Let $k$ be a field and $Q$ the following quiver:

$$1 \xrightarrow{\alpha} 2 \xleftarrow{\gamma} 3 \xrightarrow{\delta} 4.$$

Consider the $k$-algebra $T = kQ/I$, where $I$ is generated by $\beta \gamma$, $\gamma \beta$ and $\delta \beta$. Let $e_i$ be the idempotent corresponding to the vertex $i$ and put $e = e_2 + e_3 + e_4$. Define $A = eTe$ and $B = e_1Te_1$. Then $T = (\frac{A}{B})$ with $M = eTe_1$. Clearly $B = k$, and hence every $B$-module (left or right) is projective. It is easy to verify that $AM$ is projective and then $\text{Gpd}_A M \otimes_B Y < \infty$ for any $Y \in \text{mod} B$. Following Theorem 4.4, we get the following exact commutative diagram:

$$0 \to \text{Gproj} T \to D_{\text{sg}}(T) \to D_{\text{def}}(T) \to 0$$

$$0 \to \text{Gproj} A \to D_{\text{sg}}(A) \to D_{\text{def}}(A) \to 0$$

with all the vertical functors triangle-equivalences. Note that $A$ is of radical square zero but not self-injective. Following [12], $A$ is CM-free, and then so is $T$. Thus $\text{Gproj} T$ and $\text{Gproj} A$ are trivial. Hence we get triangle-equivalences $D_{\text{def}}(T) = D_{\text{sg}}(T) \simeq D_{\text{sg}}(A)(= D_{\text{def}}(A))$.

To take $R = T$ and $e = e_B$ as in Theorem 1.1 and Corollary 1.2, we get the following theorem.

\begin{theorem}
Let $T = (\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix})$ be a triangular matrix algebra with $AM_B$ compatible. The following statements hold true:

(1) The Schur functor $S_{e_A}$ induces a triangle-equivalence $D_{\text{def}}(T) \simeq D_{\text{def}}(B)$ if and only if $A$ is Gorenstein. In this case, we have a singular equivalence $D_{\text{sg}}(\text{Aus}(T)) \simeq D_{\text{sg}}(\text{Aus}(B))$ between the relative Auslander algebras $\text{Aus}(T)$ and $\text{Aus}(B)$ provided that $T$ is CM-finite.

\end{theorem}
(2) (Compare [33, Theorem 2.2]) The Schur functor $S_{e_B}$ induces the following exact commutative diagram:

$$
\begin{array}{cccc}
0 & \rightarrow & \text{Gproj } T & \rightarrow & \mathbb{D}_{sg}(T) & \rightarrow & \mathbb{D}_{def}(T) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \text{Gproj } B & \rightarrow & \mathbb{D}_{sg}(B) & \rightarrow & \mathbb{D}_{def}(B) & \rightarrow & 0 \\
\end{array}
$$

with all the vertical functors triangle-equivalences if and only if $A$ has the finite global dimension. In this case, $T$ is Gorenstein (resp. CM-free) if and only if so is $B$.

**Proof.** Notice that $e_B T e_B \cong B$. We prove by replacing $R$ with $T$ and $e$ with $e_B$ in Theorem 1.1 and Corollary 1.2, respectively. Since $AM_B$ is compatible, by Lemma 4.3, we get $\text{Tor}^B_1(T e_B, G) = 0$ for any $G \in \text{Gproj } B$ and $i \geq 1$.

(1) Let $(X, Y, \phi) \in \text{Gproj } T$. It follows that $S_{e_B}(X, Y, \phi) \cong Y$. Following Lemma 4.1, we have $Y \in \text{Gproj } B$ and then the condition (C1) in Theorem 1.1 follows. Now for any $G \in \text{Gproj } B$, we have

$$
\text{Hom}_T(T e_B(G), (X', Y', \phi')) \cong \text{Hom}_B(G, S_{e_B}(X', Y', \phi'))
\cong \text{Hom}_B(G, Y') \cong \text{Hom}_T((M \otimes_B G, G), (X', Y', \phi'))
$$

for every $(X', Y', \phi') \in \text{mod } T$. By the Yoneda lemma, we have $T e_B(G) \cong (M \otimes_B G, G)$. Then $T e_B(G)$ is a Gorenstein projective $T$-module by Lemma 4.1 and hence the condition (C2) in Theorem 1.1 follows. Thus to get the desired assertion, in view of Theorem 1.1, it suffices to show that $A$ is Gorenstein if and only if $e_B$ is Gorenstein singularly-complete. Note that every $A$-module $W \in \text{mod } A$ viewed as a $T$-module is isomorphic to $(W, 0)$. Combining Corollary 4.2(1) with [22, Theorem], we conclude that $A$ is Gorenstein if and only if every $A$-module has finite Gorenstein projective dimension as a $T$-module if and only if $e_B$ is Gorenstein singularly-complete.

In this case, assume that $T$ is CM-finite. Notice that $(M \otimes_B Y, Y) \in \text{Gproj } T$ for any $Y \in \text{Gproj } B$. One gets that $B$ is also CM-finite. By Remark 2.3, we infer a singular equivalence $\mathbb{D}_{sg}(\text{Aus}(T)) \simeq \mathbb{D}_{sg}(\text{Aus}(B))$ from the triangle-equivalence $\mathbb{D}_{def}(T) \simeq \mathbb{D}_{def}(B)$.

(2) By the foregoing proof, we get the conditions (C1) and (C2) in Corollary 1.2 are always satisfied. Notice that there exists an isomorphism of $B$-modules $e_B T \cong B$. We get that $e_B T$ is projective. In view of Corollary 1.2, it suffices to show that $A$ has the finite global dimension if and only if $e_B$ is Gorenstein singularly-complete. Note that every $A$-module $W \in \text{mod } A$ viewed as a $T$-module is isomorphic to $(W, 0)$. It follows from [10, Lemma 3.1(1)] that $\text{pd}_T(W, 0) = \text{pd}_A W$. Therefore, $A$ has the finite global dimension if and only if $\text{pd}_T(W, 0) < \infty$ if and only if $e_B$ is singularly-complete.

In this case, by a similar argument to that in the proof of Theorem 4.4(2), we conclude that $T$ is Gorenstein (resp. CM-free) if and only if so is $B$. \qed

**Example 4.7.** (1) Let $k$ be a field and $Q$ be the following quiver:

$$
\begin{array}{c}
1 \rightarrow 2 \\
\gamma \downarrow \beta \\
3 \rightarrow 4 \\
\beta' \downarrow \theta \\
\rightarrow 5
\end{array}
$$

Consider the $k$-algebra $T = kQ/I$, where $I$ is generated by $\beta' \beta$, $\beta \beta'$, $\theta \beta$ and $\alpha \gamma - \delta \beta$. Let $e_i$ be the idempotent corresponding to the vertex $i$ and put $e = e_3 + e_4 + e_5$. Define $A = (1 - e)T(1 - e)$ and $B = eT e$. Then $T = (A M B)$ with $M = (1 - e)T e$. It is easy to see that the global dimension of $A$ is $1$ and $M_B$ is projective. Then $A M_B$ is compatible. Following Theorem 4.6, we get the following exact
commutative diagram:

\[
\begin{array}{cccccc}
0 & \longrightarrow & \text{Gproj } T & \longrightarrow & \mathbb{D}_{\text{sg}}(T) & \longrightarrow & \mathbb{D}_{\text{def}}(T) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \text{Gproj } B & \longrightarrow & \mathbb{D}_{\text{sg}}(B) & \longrightarrow & \mathbb{D}_{\text{def}}(B) & \longrightarrow & 0
\end{array}
\]

with all the vertical functors triangle-equivalences. Notice that \( B \) is CM-free, and then so is \( T \). Thus \( \text{Gproj } T \) and \( \text{Gproj } B \) are trivial. Hence we get triangle-equivalences \( \mathbb{D}_{\text{def}}(T) \cong \mathbb{D}_{\text{sg}}(T) \cong \mathbb{D}_{\text{sg}}(B) \) (= \( \mathbb{D}_{\text{def}}(B) \)).

(2) Let \( k \) be a field and \( Q \) the following quiver:

\[
\begin{array}{c}
1 \\
\gamma \\
3 \\
\beta \\
5
\end{array}
\begin{array}{c}
\alpha \\
\alpha' \\
\beta' \\
\beta' \\
\delta
\end{array}
\]

Consider the \( k \)-algebra \( T = kQ/I \), where \( I \) is generated by \( \alpha'\alpha, \alpha\alpha', \beta'\beta, \beta\beta' \). Let \( e_1 \) be the idempotent corresponding to the vertex \( i \) and put \( e = e_3 + e_4 + e_5 \). \( A = (1 - e)T(1 - e) \) and \( B = eTe \). Then \( T = (A, M) \) with \( M = (1 - e)T(1 - e) \). Clearly, \( A \) is self-injective and \( B \) is CM-free. It is easy to see that \( A \) and \( B \) are projective and then \( eA \) is compatible. In view of Theorem 4.6(1), we get a triangle-equivalence \( \mathbb{D}_{\text{def}}(T) \cong \mathbb{D}_{\text{def}}(B) = \mathbb{D}_{\text{sg}}(B) \). However, since the global dimension of \( A \) is infinite, the Schur functor \( S_\gamma \) does not induce an exact commutative diagram as that in Theorem 4.6(2). Thus we conclude that \( S_\gamma \) induces neither a singular equivalence \( \mathbb{D}_{\text{sg}}(T) \cong \mathbb{D}_{\text{sg}}(B) \) nor a triangle-equivalence \( \text{Gproj } T \cong \text{Gproj } B \).

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