Why Extension-Based Proofs Fail

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Abstract

We prove that a class of fundamental shared memory tasks are not amenable to certain standard proof techniques in the field. We formally define a class of extension-based proofs, which contains impossibility arguments like the valency proof by Fisher, Lynch and Paterson of the impossibility of wait-free consensus in an asynchronous system. We introduce a framework which models such proofs as an interaction between a prover and an adversarial protocol. Our main contribution is showing that extension-based proofs are inherently limited in power: for instance, they cannot establish the impossibility of solving \((n - 1)\)-set agreement among \(n \geq 3\) processes in a wait-free manner. This impossibility result does have proofs based on combinatorial topology. However, it was unknown whether proofs based on simpler techniques were possible.
1 Introduction

One of the most well-known results in the theory of distributed computing, due to Fischer, Lynch, and Paterson [FLP85], is that there is no deterministic, wait-free protocol solving consensus among \( n \geq 2 \) processes in an asynchronous message passing system. In fact, they showed that, even if at most one process may crash, it is possible to construct an infinite execution in which no process terminates. Their result has been extended to asynchronous shared memory systems where processes communicate by reading from and writing to shared registers [Abr88, CIL87, Her91, LAA87].

Chaudhuri [Cha93] conjectured that these impossibility results could be generalized to the \( k \)-set agreement problem: given \( 1 \leq k < n \), each of the \( n \) processes begins with an input in \( \{0,1,\ldots,k\} \). Each process that does not crash must output a value that is the input of some process (validity) and, collectively, at most \( k \) different values may be output (agreement). In particular, consensus is just 1-set agreement. Chaudhuri’s conjecture was eventually proved in three concurrent papers by Borowsky and Gafni [BG93], Herlihy and Shavit [HS99], and Saks and Zaharoglou [SZ00]. These papers all relied on sophisticated machinery from combinatorial topology, which they used to model the space of all reachable configurations of the system. Later on, Attiya and Castañeda [AC11] and Attiya and Paz [AP12] showed how to obtain the same results using purely combinatorial techniques, without explicitly using topology. A common feature of these impossibility proofs is that they assume the existence of a deterministic, wait-free protocol, and first argue that it has only finitely many executions, and then show that at least \( k + 1 \) different values are output in one of the executions. This implies that any deterministic protocol for \( k \)-set agreement among \( n > k \) processes in an asynchronous system has an execution in which some process takes infinitely many steps, without returning a value. However, these proofs do not construct such an infinite execution.

In contrast, impossibility proofs for deterministic, wait-free consensus in asynchronous systems explicitly construct an infinite execution by repeatedly extending a finite execution by the steps of some processes. Fischer, Lynch, and Paterson introduced valency arguments to show that such extensions are possible in the case of consensus. A natural question arises: is there a “constructive” proof of the impossibility of \( k \)-set agreement that explicitly constructs an infinite execution by repeated extensions?

Our contributions. In this paper, we formally define the class of extension-based proofs. We also prove that there is no extension-based proof of the impossibility of a deterministic, wait-free protocol solving \( k \)-set agreement among \( n > k \geq 2 \) processes in an asynchronous system using only registers.

We view a proof of the impossibility of solving a task as an interaction between a prover and any protocol that claims to solve the task. The prover has to refute this claim. To do so, it repeatedly queries the protocol about the states of processes in configurations that can be reached in a small number of steps from configurations it already knows about. The goal of the prover is to construct a bad execution, i.e. an execution in which some processes take infinitely many steps without terminating, or output values that do not satisfy the specifications of the task. The definition of extension-based proofs is presented in Section 3. An extension-based proof of the impossibility of deterministic, wait-free solutions to consensus among 2 processes in an asynchronous shared memory system appears in Section 3.

A key observation is that, from the results of its queries, many protocols are indistinguishable to the prover. It must construct a single execution that is bad for all these protocols. To prove that no prover can construct a bad execution, in Section 4, we show how an adversary can adaptively define
a protocol in response to any specific prover’s queries. In this adversarial protocol, all processes eventually terminate and output correct values in executions consistent with the results of the prover’s queries.

We use basic combinatorial topology to represent reachable configurations of a protocol. This is described in Section 2. In this view, when an extension-based prover makes queries, it is essentially performing local search on the configuration space of the protocol. Because the prover obtains incomplete information about the protocol, the adversary has some flexibility when specifying the protocol’s behaviour in configurations not yet queried by the prover.

An additional type of query is added to the definition of extension based proof in Section 5. This query makes the prover stronger and simplifies modeling valency arguments as extension-based proofs. We extend the proof in Section 4 to handle this additional type of query.

We provide a formal way of reasoning about proofs of impossibility in shared memory. This opens the door for a number of interesting directions for future work, such as generalizations of the extension-based proof model, reasoning about lower bound proofs, better understanding the exact sources of limitations of certain techniques, and exploring connections to the results in the theory of proof complexity. This is discussed in Section 7.

2 Preliminaries

2.1 NIIS Model

We consider the non-uniform iterated immediate snapshot (NIIS) model with \( n \geq 2 \) processes, introduced by Hoest and Shavit [HS06]. For wait-free computability, it is known that the NIIS model is equivalent to the standard asynchronous shared memory model, in which processes communicate by reading from and writing to shared registers. Specifically, any task that has a wait-free solution in one of these models has a wait-free solution in the other [HS06].

In the NIIS model, \( n \) processes, \( p_0, \ldots, p_{n-1} \), communicate using an infinite sequence, \( S_1, S_2, \ldots \), of shared single-writer atomic snapshot objects. Each snapshot object has \( n \) components, which are all initially \(-\), and supports two operations, \texttt{update}(v) and \texttt{scan}(). An \texttt{update}(v) operation by process \( p_i \) performed on a snapshot object updates component \( i \) of the object to have value \( v \), where \( v \) is an element of an arbitrarily large set that does not contain \(-\). A \texttt{scan}() operation returns a vector containing the value of each component of the object.

In the NIIS model, processes remember their entire history. This is called a full information protocol. Initially, \( p_i \)’s state is its input. It accesses each snapshot object at most twice, starting with \( S_1 \). The first time, it performs an \texttt{update} to set the \( i \)’th component of the object to its current state. At its next step, it performs a \texttt{scan} of the same object. The result of this \texttt{scan} is its new state. It then checks if it can output a value by consulting a map, \( \Delta \), that takes its process identifier, \( i \), and its new state as input. If \( \Delta \) returns \( \bot \), then, at its next step, \( p_i \) accesses the next snapshot object. If it returns a value \( v \neq \bot \), then \( p_i \) outputs \( v \) and terminates. A protocol in the NIIS model is completely specified by the map \( \Delta \).

A scheduler decides the order in which the processes take steps. It repeatedly selects a set of processes that are all poised to perform \texttt{updates} on the same snapshot object, schedules all of their \texttt{updates}, in order of their identifiers, and then schedules all of their next \texttt{scans}, in the same order. Note that the scheduler never selects processes that have terminated. A schedule is a sequence containing subsets of processes selected by the scheduler. For example, \((\{p_0, p_1\}, \{p_1\})\)
is a schedule in which the scheduler selects \( \{p_0, p_1\} \), followed by \( \{p_1\} \). After this schedule, \( p_1 \) has updated and scanned one more snapshot object than \( p_0 \). A protocol is wait-free if it does not have an infinite schedule.

A configuration contains the contents of each shared object and the state of each process. A process is active in a configuration if it has not terminated. A configuration is terminal if it has no active processes. An initial configuration is specified by the input of each process. Each finite schedule from an initial configuration results in a reachable configuration. The following observation is a key property of full information protocols.

**Observation 2.1.** A reachable configuration is fully specified by the set of states of all processes in the configuration (including the processes that have terminated).

Two configurations \( C \) and \( C' \) are indistinguishable to a set of processes \( P \) if every process in \( P \) has the same state in \( C \) and \( C' \). Given a schedule \( \alpha \) from a reachable configuration \( C \), we use \( C\alpha \) to denote the resulting configuration. Two finite schedules \( \alpha \) and \( \beta \) from \( C \) are indistinguishable to a set of processes \( P \) if the resulting configurations \( C\alpha \) and \( C\beta \) are indistinguishable to \( P \).

**Observation 2.2.** Suppose \( C \) and \( C' \) are two reachable configurations that are indistinguishable to \( P \), every active process in \( P \) is poised to update \( S_i \) in \( C \), each snapshot object \( S_r \) has the same contents in \( C \) and \( C' \) for all \( r \geq t \), and \( \alpha \) is a finite schedule from \( C \) containing only processes in \( P \). Then \( \alpha \) is a schedule from \( C' \) and the configurations \( C\alpha \) and \( C'\alpha \) are indistinguishable to \( P \).

Suppose \( C \) is a reachable configuration in which all active processes are poised to update the same snapshot object. For any set of processes \( P \), a \( P \)-only 1-round schedule from \( C \) is an ordered partition of the processes in \( P \) that are active in \( C \). If none of the processes in \( P \) are active in \( C \), then the empty schedule is the only 1-round schedule. A (full) 1-round schedule from \( C \) is simply a \( P \)-only 1-round schedule, where \( P \) is the set of all processes. Observe that, in the NIIS model, if \( \alpha \) is a \( P \)-only 1-round schedule from \( C \) and \( \beta \) is any full 1-round schedule from \( C \) such that \( \beta = \alpha \alpha' \), then \( \alpha \) and \( \beta \) are indistinguishable to the processes in \( P \).

For \( t > 1 \), a (full) \( t \)-round schedule from \( C \) is a schedule \( \alpha_1 \alpha_2 \cdots \alpha_t \) such that \( \alpha_1 \) is a 1-round schedule from \( C \) and, for \( 1 < i \leq t \), \( \alpha_i \) is a 1-round schedule from \( C\alpha_1 \cdots \alpha_{i-1} \). A \( P \)-only \( t \)-round schedule from \( C \) is a schedule \( \alpha_1 \alpha_2 \cdots \alpha_t \) such that there exists a (full) \( t \)-round schedule \( \beta_1 \cdots \beta_t \) from \( C \) where \( \alpha_i \) is a prefix of \( \beta_i \) for \( 1 \leq i \leq t \), \( \alpha_1 \) is a \( P \)-only 1-round schedule from \( C \), and \( \alpha_i \) is a \( P \)-only 1-round schedule from \( C\beta_1 \cdots \beta_{i-1} \) for \( 1 < i \leq t \). Notice that some processes in \( P \) may have terminated during \( \alpha_1 \cdots \alpha_{i-1} \). These processes are not included in \( \alpha_i \). If \( \alpha_1 \cdots \alpha_t \) is a \( P \)-only \( t \)-round schedule from \( C \), \( \beta_1 \cdots \beta_t \) is any full \( t \)-round schedule from \( C \), and \( \alpha_i \) is a prefix of \( \beta_i \) for all \( 1 \leq i \leq t \), then these two schedules are indistinguishable to the processes in \( P \).

Let \( P \) be the set of all active processes in some reachable configuration \( C \) and suppose that every process in \( P \) is poised to update the same snapshot object \( S_r \). For any nonempty finite schedule \( \gamma \) from \( C \), let \( P_i(\gamma) \) denote the set of processes in \( P \) that are poised to update \( S_i \) in the resulting configuration \( C\gamma \). Let \( m = \max \{ i \mid P_i(\gamma) \neq \emptyset \} \). Then there exists a schedule \( \gamma' = \gamma'_0 \cdots \gamma'_{m-1} \) from \( C \) that is indistinguishable to all processes such that \( \gamma'_i \) is a \( (\bigcup_{j=i+1}^m P_j) \)-only 1-round schedule.

### 2.2 Topological Representation of a Protocol

An (abstract) simplicial complex is a collection of sets, \( S \), that is closed under subset: for any set \( \sigma \in S \), if \( \tau \subseteq \sigma \), then \( \tau \in S \). Each set \( \sigma \in S \) is called a simplex of \( S \). If \(|\sigma| = 1\), then \( \sigma \) is called
a vertex. If $|\sigma| = 2$, then $\sigma$ is called an edge. A subcomplex of $S$ is a subset of $S$ that is also a simplicial complex.

Let $S$ be a simplicial complex and let $A$ and $B$ be non-empty subcomplexes of $S$. A path between $A$ and $B$ in $S$ of length $\ell$ is a sequence of vertices $v_0, v_1, \ldots, v_\ell$ such that $v_0 \in A$, $v_\ell \in B$, and, for $0 \leq j < \ell$, $\{v_j, v_{j+1}\}$ is an edge in $S$. Notice that a vertex may appear more than once in a path and, if $A$ and $B$ both consist of a single vertex, then we have the standard definition of a (non-simple) path in a graph. $S$ is connected if for any two vertices $u, v \in S$, there is a path between $u$ and $v$ in $S$. If $S$ is connected, then the distance between $A$ and $B$ in $S$, denoted $\text{dist}_S(A, B)$, is the minimum $\ell \geq 0$ such that there is a path between $A$ and $B$ in $S$ of length $\ell$.

In the topological view of a protocol (specified by a map $\Delta$) in the NIIS model, there is a simplicial complex $S'$ for each $t \geq 0$. $S'$ represents all configurations reachable from initial configurations by $t$-round schedules. In particular, the input complex of the protocol, $S^0$, represents all possible initial configurations.

Consider any configuration $C$ reachable from an initial configuration by a (full) $t$-round schedule. For every process $p_i$, there is a unique vertex $(i, s_i)$ in $S'$, where $s_i$ is the state of $p_i$ in $C$. The process identifier $i$ is called the colour of the vertex. It is needed because multiple processes may have the same state in $C$ (i.e. they have the same input and see the same results from all their scans). The $n$-element set $\{(i, s_i) \mid i = 0, \ldots, n - 1\}$ is a simplex in $S'$. It is called the full simplex of $S'$ corresponding to configuration $C$. By Observation 2.1, each full simplex corresponds to only one configuration. Every simplex of $S'$ is a subset of a full simplex of $S'$. Thus, all the vertices in a simplex of $S'$ have different colors.

The non-uniform chromatic subdivision operation $\chi$ maps every subcomplex $A$ of $S'$ to a subcomplex $\chi(A, \Delta)$ of $S'^{t+1}$. It has the property that $\chi(A, \Delta)$ is the union of $\chi(\sigma, \Delta)$ over all simplices $\sigma \in A$. Hoest and Shavit [HS06] proved that $S'^{t+1} = \chi(S', \Delta)$ is the non-uniform chromatic subdivision of $S'$.

Suppose $\sigma$ is a full simplex in $S'$ corresponding to a configuration $C$. Then every full simplex in $\chi(\sigma, \Delta) \in S'^{t+1}$ corresponds to a configuration reachable from $C$ by a 1-round schedule. If $\tau \subset \sigma$ and $P = \{p_i \mid i \text{ is the color of a vertex in } \sigma\}$, then each simplex in $\chi(\tau, \Delta)$ consists of the set of states of a subset of the processes in $P$ in a configuration resulting from $C$ by a $P$-only 1-round schedule. Although $\tau$ can be also be subset of a full simplex in $S'$ corresponding to a different configuration, Observation 2.2 says that the definition does not depend on the choice of this configuration.

Hoest and Shavit also showed that, if $S^0$ is connected, then, for $t \geq 0$, $S'$ is connected and, thus, the distance between any two vertices in $S'$ is well-defined. We note that the input complex of any $k$-set agreement protocol is connected.

Figure 1 contains an example of the non-uniform chromatic subdivision of a simplicial complex $S$ with 3 processes, $p_0$, $p_1$, and $p_2$. In the configuration represented by the left face of $S$, $p_0$, $p_1$, and $p_2$ have states $x$, $y$, and $z$, respectively, none of which have terminated. In the configuration represented by the right face, $p_0$ and $p_2$ have the same state, but $p_1$ has state $y'$, in which it terminates and outputs $\Delta(1, y')$. We also illustrate two subcomplexes $A$ and $B$ of $S$ and their subdivisions.

### 2.3 Non-uniform Chromatic Subdivision

In this section, we define the non-uniform chromatic subdivision operation. But, for a more comprehensive definition of the non-uniform chromatic subdivision, which is quite technical, we refer
the reader to [HS06]. We prove that the non-uniform subdivision operation has the following useful property relating the distances between subcomplexes and the distances between the nonuniform chromatic subdivisions of these subcomplexes. This lemma can also be used to fix a technical problem in one of the arguments of Hoest and Shavit [HS97, HS06]. This is discussed in Section 6.

Lemma 2.3. Let $\mathbb{A}$ and $\mathbb{B}$ be nonempty, disjoint subcomplexes of $S^t$, let $\mathbb{A}' = \chi(\mathbb{A}, \Delta)$, and let $\mathbb{B}' = \chi(\mathbb{B}, \Delta)$. Then $\text{dist}_{t+1}(\mathbb{A}', \mathbb{B}') \geq \text{dist}_t(\mathbb{A}, \mathbb{B})$. Furthermore, if every path between $\mathbb{A}$ and $\mathbb{B}$ in $S^t$ contains at least one edge between unterminated vertices, then $\text{dist}_{t+1}(\mathbb{A}', \mathbb{B}') \geq \text{dist}_t(\mathbb{A}, \mathbb{B}) + 1$.

Recall that distances are defined based on only the vertices and edges of the complex (also known as the 1-skeleton of the complex). However, the proof of this lemma relies on the higher dimensional structure of the complex $S^t$ and certain properties of non-uniform subdivision operation. These properties are listed immediately prior to its proof.

2.4 Non-uniform Chromatic Subdivision and Distance Lemma

Let $t \geq 0$ and suppose that $\Delta(v)$ is defined for all vertices $v \in S^t$.

Consider any simplex $\sigma$ in $S^t$ and suppose that $\Delta(v) = \perp$ for each vertex $v \in \sigma$. Let $I(\sigma)$ be the set of colours of the vertices in $\sigma$. The standard chromatic subdivision of $\sigma$ is the abstract simplicial complex $\chi(\sigma, \Delta)$ whose vertices are of the form $(i, \tau)$, where $\emptyset \neq \tau \subseteq \sigma$ and $i$ is the colour of some vertex in $\tau$. Intuitively, $\tau$ contains the set of process states that process $p_i$ sees in its next scan. Strictly speaking, the vertices are actually of the form $(i, \vec{\tau})$, where $\vec{\tau}_j = s$ if $(j, s) \in \tau$ and $\vec{\tau}_j = -$ if $j$ is not the colour of any vertex in $\tau$. This corresponds to the actual values seen by the processes in their next scan. A set of vertices $\{(i, \tau_i) : i \in I\}$, for some set of identifiers $I \subseteq I(\sigma)$, is a simplex in $\chi(\sigma, \Delta)$ if and only if there is an ordering $\preceq$ on $I$ such that $i \preceq j$ implies that $\tau_i \subseteq \tau_j$ and, for each $i, j \in I$, if $i$ the colour of some vertex in $\tau_j$, then $\tau_i \subseteq \tau_j$.

Now consider a simplex $\sigma$ where some vertices in $\sigma$ have been terminated. Let $\tau \subseteq \sigma$ be the set of terminated vertices in $\sigma$. The non-uniform chromatic subdivision of $\sigma$ is the abstract simplicial complex $\chi(\sigma, \Delta)$ whose vertices are the vertices in $\tau$ and the vertices in the standard chromatic subdivision $\chi(\sigma - \tau, \Delta)$ of $\sigma - \tau$. Each simplex in $\chi(\sigma - \tau, \Delta)$ is a simplex of $\chi(\sigma, \Delta)$. In addition, if $\tau' \subseteq \tau$ and $\sigma' \in \chi(\sigma - \tau, \Delta)$ is a simplex, then $\tau' \cup \sigma'$ is a simplex of $\chi(\sigma, \Delta)$.
For any subcomplex $A$ of $S^t$, the non-uniform chromatic subdivision $A(\sigma, \Delta) \subseteq S^{t+1}$ of $A$ is the union of $\chi(\sigma, \Delta)$ for all simplices $\sigma \in A$.

There is a natural geometric interpretation of an abstract simplicial complex and subdivision. Roughly speaking, a geometric simplex $\sigma$ may be represented in Euclidean space as the set of convex combinations of $|\sigma|$ affinely independent points (one per element of $\sigma$). A face of $\sigma$ is the set of convex combinations of a subset of the affinely independent points. A geometric simplicial complex $K$ is a collection of geometric simplices such that each face of $\sigma \in K$ is a simplex in $K$ and, for any two simplices $\sigma, \tau \in K$, $\sigma \cap \tau \in K$. The geometric realization of $K$ is the union of the simplices in $K$ (in Euclidean space). A geometric simplicial complex $B$ is a subdivision of $A$ if their geometric realizations are the same and each simplex in $A$ is the union of finitely many simplices in $B$.

One of the key contributions of Hoest and Shavit [HS97] is their proof that the non-uniform chromatic subdivision of a simplicial complex is in fact a subdivision (in the geometric sense). The following properties all follow from this fact:

**Proposition 2.4.** The non-uniform chromatic subdivision operation has the following properties:

- If $S^0$ is connected, then, for all $t \geq 1$, $S^t$ is connected.
- $A$ and $B$ are disjoint subcomplexes of $S^t$ if and only if $\chi(A, \Delta)$ and $\chi(B, \Delta)$ are disjoint subcomplexes of $S^{t+1}$.
- If every path between $A$ and $B$ passes through a subcomplex $C$, then every path between $\chi(A, \Delta)$ and $\chi(B, \Delta)$ passes through $\chi(C, \Delta)$.
- If $C$ contains only unterminated vertices and $C_1$ and $C_2$ are disjoint nonempty subcomplexes of $C$, then the distance between $\chi(C_1, \Delta)$ and $\chi(C_2, \Delta)$ is at least 2.

**Lemma 2.3.** Let $A$ and $B$ be nonempty, disjoint subcomplexes of $S^t$, let $A' = \chi(A, \Delta)$, and let $B' = \chi(B, \Delta)$. Then $\text{dist}_{g_{t+1}}(A', B') \geq \text{dist}_{g_t}(A, B)$. Furthermore, if every path between $A$ and $B$ in $S^t$ contains at least one edge between unterminated vertices, then $\text{dist}_{g_{t+1}}(A', B') \geq \text{dist}_{g_t}(A, B) + 1$.

**Proof.** By induction. First suppose that $\text{dist}_{g_t}(A, B) = 1$. Since $A$ and $B$ are disjoint, $A'$ and $B'$ are disjoint. Thus $\text{dist}_{g_{t+1}}(A', B') \geq 1 = \text{dist}_{g_t}(A, B)$.

So, suppose $d = \text{dist}_{g_t}(A, B) > 1$ and $\text{dist}_{g_{t+1}}(\chi(A, \Delta), \chi(B, \Delta)) \geq \text{dist}_{g_t}(\hat{A}, \hat{B})$ holds for all non-empty subcomplexes $\hat{A}$ and $\hat{B}$ of $S^t$ where $\text{dist}_{g_t}(\hat{A}, \hat{B}) < d$. Let $C$ be the largest subcomplex of $S^t$ containing only vertices $v$ such that $\text{dist}_{g_t}(v, A) = 1$ and let $C' = \chi(C, \Delta)$. Since $\text{dist}_{g_t}(A, B) > 0$, $C$ is non-empty and $\text{dist}_{g_t}(C, B) = d - 1$. Thus, by the induction hypothesis, $\text{dist}_{g_{t+1}}(A', C') \geq 1$ and $\text{dist}_{g_{t+1}}(C', B') \geq d - 1$. Since every path between $A$ and $B$ passes through $C$, every path between $A'$ and $B'$ in $S^{t+1}$ must pass through $C'$. Therefore, $\text{dist}_{g_{t+1}}(A', B') \geq \text{dist}_{g_{t+1}}(A', C') + \text{dist}_{g_{t+1}}(C', B') \geq d$ and the claim holds for $A$ and $B$.

Now suppose that every path between $A$ and $B$ in $S^t$ contains at least one edge between unterminated vertices. Let $E \neq \emptyset$ be the smallest set of edges between unterminated vertices in $S^t$ such that every path between $A$ and $B$ contains at least one edge in $E$. Viewing $S^t$ as a graph, the removal of $E$ from $S^t$ results in some number of connected components. Let $\hat{A}$ be the set of vertices in the connected components that contain at least one vertex in $A$ and let $\hat{B}$ be the set of remaining vertices in $S^t$. Let $\hat{A}$ and $\hat{B}$ be the largest subcomplexes of $S^t$ containing only vertices in $\hat{A}$ and $\hat{B}$, respectively, let $\hat{A}' = \chi(\hat{A}, \Delta)$, and let $\hat{B}' = \chi(\hat{B}, \Delta)$. Observe that $A$ is a subcomplex of
\[ A \text{ and } B \text{ is a subcomplex of } \hat{B}. \text{ Moreover, by the minimality of } E, \text{ the set of edges between } A \text{ and } \hat{B} \text{ is exactly } E. \]

Let \( C \) be the largest subcomplex of \( S' \) containing only vertices that are contained in some edge in \( E \) and let \( C' = \chi(C, \Delta) \). By definition of \( E \), every vertex in \( C \) is unterminated. Observe that every path between \( A \) and \( \hat{B} \cap C \) has to pass through \( \hat{A} \cap C \) and every path between \( A \cap C \) and \( \hat{B} \) has to pass through \( \hat{A} \cap C \). It follows that every path between \( \hat{A}' \) and \( \chi(\hat{B} \cap C, \Delta) \) has to pass through \( \chi(\hat{A} \cap C, \Delta) \) and every path between \( \chi(A \cap C, \Delta) \) and \( \hat{B}' \) has to pass through \( \chi(\hat{B} \cap C, \Delta) \). In particular, this implies that \( \text{dist}_{\text{gr}+1}(A', B') \geq \text{dist}_{\text{gr}+1}(A', \chi(\hat{A} \cap C, \Delta)) + \text{dist}_{\text{gr}+1}(\chi(\hat{A} \cap C, \Delta), \chi(\hat{B} \cap C, \Delta)) \). By the first part of the lemma, which we have already proved, \( \text{dist}_{\text{gr}+1}(A', \chi(\hat{A} \cap C, \Delta)) \geq \text{dist}_{\text{gr}}(A, \chi(\hat{A} \cap C, \Delta)) \text{ and } \text{dist}_{\text{gr}+1}(B', \chi(\hat{B} \cap C, \Delta)) \geq \text{dist}_{\text{gr}}(B, \chi(\hat{B} \cap C, \Delta)). \) Now consider \( \text{dist}_{\text{gr}+1}(\chi(\hat{A} \cap C, \Delta), \chi(\hat{B} \cap C, \Delta)) \). Since every vertex in \( C \) is unterminated, \( \chi(C, \Delta) \) is the standard chromatic subdivision of \( C \), which fully subdivides every simplex in \( C \). In particular, any vertex in \( \chi(\hat{A} \cap C, \Delta) \) is not adjacent to any vertex in \( \chi(\hat{B} \cap C, \Delta) \). It follows that \( \text{dist}_{\text{gr}+1}(\chi(\hat{A} \cap C, \Delta), \chi(\hat{B} \cap C, \Delta)) \geq 2. \) Since \( \text{dist}_{\text{gr}}(A, \chi(\hat{A} \cap C)) + \text{dist}_{\text{gr}}(B, \chi(\hat{B} \cap C)) = \text{dist}_{\text{gr}}(A, B) - 1 \), we obtain that \( \text{dist}_{\text{gr}+1}(A', B') \geq \text{dist}_{\text{gr}}(A, B) + 1. \)

### 3 Extension-Based Proofs

To prove that a task has no wait-free solution in the NIIS model, we consider a prover, which, given any protocol that supposedly solves the task, constructs a schedule from an initial configuration of the protocol such that either the schedule is finite and one of the specifications of the task is violated in the resulting configuration, or the schedule is infinite and some processes remain active. To learn information about the protocol, the prover queries the protocol about the states of processes in various reachable configurations. In the NIIS model, the only information a prover learns about the state of a process in a reachable configuration is whether that process has output a value and, if so, the value that it output. The rest of the information about its state is the same for all NIIS protocols. In other words, the protocol is entirely specified by \( \Delta \) and the prover is learning the value of the map \( \Delta \), for various process states.

More formally, an extension-based proof in the NIIS model is an interaction between a prover and any protocol defined by a map \( \Delta \). The prover starts with no knowledge about the protocol (except its initial configurations) and makes the protocol reveal information about the states of processes in various configurations by asking queries. Each query allows the prover to reach some configuration of the protocol. The interaction proceeds in phases.

In each phase \( \varphi \geq 1 \), the prover starts with a finite schedule, \( \alpha(\varphi) \), and a set, \( A(\varphi) \), of configurations that are reached by performing \( \alpha(\varphi) \) from initial configurations, which only differ from one another by the states of processes that do not appear in this schedule (as these processes might have different input values). It also maintains a set, \( A'(\varphi) \), containing the configurations it reaches from configurations in \( A(\varphi) \) during phase \( \varphi \). This set is empty at the start of phase \( \varphi \). At the start of phase 1, \( \alpha(1) \) is the empty schedule and \( A(1) \) is the set of all initial configurations of the protocol.

The prover queries the protocol by specifying a configuration \( C \in A(\varphi) \cup A'(\varphi) \) and a set of processes \( P \) that are poised to update the same snapshot object in \( C \). For each process \( p_i \in P \), let \( s_i \) denote the state of \( p_i \) in the configuration \( C' \) resulting from scheduling \( P \) from \( C \). The protocol replies to this query with \( \Delta(i, s_i) \), for each \( p_i \in P \). Notice that, by the definition of the NIIS model, this is enough for the prover to know the state of every process and the contents of every
component of every snapshot object in \( C' \). Then, the prover adds \( C' \) to \( A'(\varphi) \), and we say that the prover has reached \( C' \).

If the prover reaches a configuration in which the outputs of the processes do not satisfy the specifications of the task, it has demonstrated that the protocol is incorrect. In this case, the prover wins (and the interaction ends).

A chain of queries is a sequence of queries, \( (C_0, P_0), (C_1, P_1), \ldots \), such that, for each \( i \geq 0 \), \( C_{i+1} \) is the configuration resulting from scheduling \( P_i \) from \( C_i \). If the prover constructs an infinite chain of queries, it has demonstrated that the protocol is not wait-free. In this case, the prover also wins (and the interaction ends). In particular, the prover wins against the trivial protocol in which no process ever outputs a value, by constructing any infinite chain of queries.

After making finitely many chains of queries in phase \( \varphi \) without winning, the prover must end the phase by committing to an extension of the schedule \( \alpha(\varphi) \). More formally, at the end of phase \( \varphi \), the prover must choose a configuration \( C' \in A'(\varphi) \). Let \( \alpha' \) be a (nonempty) schedule such that \( C' \) is reached by performing \( \alpha' \) starting from some configuration \( C \in A(\varphi) \). Let \( \alpha(\varphi + 1) \) denote the schedule \( \alpha(\varphi)\alpha' \). Since \( C \in A(\varphi) \), there is an initial configuration \( I \) such that \( C \) is reached by performing \( \alpha(\varphi) \) starting from \( I \). Thus \( C' \) is reached by performing \( \alpha(\varphi + 1) \) starting from \( I \). Finally, let \( A(\varphi + 1) \) be the set of all configurations that are reached by performing \( \alpha(\varphi + 1) \) from the initial configurations that only differ from \( I \) by the states of processes that do not appear in this schedule. Then the prover begins phase \( \varphi + 1 \).

If, in every configuration in \( A(\varphi) \), every process has terminated, then \( A'(\varphi) = \emptyset \), the prover loses, and the interaction ends.

If the number of phases in the interaction is infinite, the prover has constructed an infinite schedule in which some processes remain active and, hence, the protocol is not wait-free. This is the third way that the prover can win.

To prove that a task is impossible using an extension-based proof, a prover must win against every protocol.

**An example.** We give an extension-based proof of the impossibility of solving wait-free binary consensus among 2 processes.

**Theorem 3.1.** Binary consensus among 2 processes is impossible in the NIIS model.

**Proof.** Let \( C_0 \) denote the initial configuration in which \( p_0 \) has input 0 and \( p_1 \) has input 1. Then, by validity, the solo-execution by \( p_0 \) must decide \( a_0 = 0 \) and the solo-execution by \( p_1 \) must decide \( 1 - a_0 = 1 \). The prover performs the query chain corresponding to the solo execution by \( p_0 \) from \( C_0 \). The prover wins if this does not terminate or \( p_0 \) does not output 0. Similarly, the prover performs the query chain corresponding to the solo execution by \( p_1 \) from \( C_0 \) and wins if this does not terminate or \( p_1 \) does not output 1.

The prover will either construct an infinite query chain in some phase, reach a configuration in which both 0 and 1 have been output, or inductively construct an infinite sequence of configurations \( C_1, C_2, \ldots \) and a corresponding sequence of bits \( a_1, a_2, \ldots \) such that, for all \( i \geq 1 \), \( C_i \) is reached from \( C_{i-1} \) by scheduling one set of processes (either \( \{p_0\}, \{p_1\} \), or \( \{p_0, p_1\} \)), the solo-execution by \( p_0 \) from \( C_i \) outputs \( a_i \), and the solo-execution by \( p_1 \) from \( C_i \) outputs \( 1 - a_i \). Let \( i \geq 1 \) and suppose the claim is true for \( i - 1 \).

If process \( p_0 \) is terminated (outputting value \( a_{i-1} \)) in configuration \( C_{i-1} \), then the solo execution by \( p_1 \) from \( C_{i-1} \), which outputs \( 1 - a_{i-1} \), results in a configuration in which both 0 and 1 have
been output. Similarly, if \( p_1 \) is terminated in configuration \( C_{i-1} \), then the prover has reached a configuration in which both 0 and 1 have been output. So, suppose that neither \( p_0 \) nor \( p_1 \) is terminated in \( C_{i-1} \).

From \( C_{i-1} \), the prover first performs the query chain corresponding to the schedule \( \{p_0\}, \{p_1\}, \{p_1\}, \ldots \) where \( p_0 \) is scheduled once and then \( p_1 \) is scheduled until it outputs a value \( b_i \). If that never happens, then the prover wins. If \( b_i = 1 - a_{i-1} \), then the prover ends phase \( i \), chooses \( C_i = C_{i-1}\{p_0\} \), and sets \( a_i = a_{i-1} \). Note that the solo execution by \( p_0 \) from \( C_i \) outputs \( a_i = a_{i-1} \) and the solo execution by \( p_1 \) from \( C_i \) outputs \( 1 - a_i = 1 - a_{i-1} \).

Otherwise, \( b_i = a_{i-1} \). In this case, the prover performs the query chain from \( C_{i-1} \) corresponding to the schedule \( \{p_1\}, \{p_0\}, \{p_0\}, \ldots \), where \( p_1 \) is scheduled once and then \( p_0 \) is scheduled until it outputs a value \( d_i \). If that never happens, then the prover wins. If \( d_i = a_{i-1} \), then the prover ends the round, chooses \( C_i = C_{i-1}\{p_1\} \), and sets \( a_i = a_{i-1} \). Note that the solo execution by \( p_0 \) from \( C_i \) outputs \( a_i = a_{i-1} \) and the solo execution by \( p_1 \) from \( C_i \) outputs \( 1 - a_i = 1 - a_{i-1} \).

Otherwise, \( d_i = 1 - a_{i-1} \). Then the prover performs the query \( \{p_0, p_1\} \). Note that the configurations \( C_{i-1}\{p_0, p_1\} \) and \( C_i\{p_0, p_1\} \) are indistinguishable to \( p_1 \), i.e. \( p_1 \) has the same state in both configurations. Thus, it outputs \( b_i \) in its solo execution from \( C_{i-1}\{p_0, p_1\} \). Likewise, the configurations \( C_{i-1}\{p_0, p_1\} \) and \( C_i\{p_0, p_1\} \{p_0\} \) are indistinguishable to \( p_0 \), so it outputs \( d_i \) in its solo execution from \( C_{i-1}\{p_0, p_1\} \). Then the prover ends the round, chooses \( C_i = C_{i-1}\{p_0, p_1\} \), and sets \( a_i = d_i \). Note that the solo execution by \( p_0 \) from \( C_i \) outputs \( a_i \) and the solo execution by \( p_1 \) from \( C_i \) outputs \( b_i = a_{i-1} = 1 - d_i = 1 - a_i \). Thus, the claim is true for \( i \). Hence, by induction, the claim is true for all \( i \geq 0 \).

\[ \square \]

4 What cannot be proved by extension!

In this section, we prove that no extension-based proof can show the impossibility of deterministically solving \( k \)-set agreement in a wait-free manner in the NIIS model, for \( n > k \geq 2 \) processes. Any protocol for \( n > k + 1 \) processes is also a protocol for \( k + 1 \) processes, since the remaining processes could crash before taking any steps. Therefore, it suffices to consider \( n = k + 1 \).

To show our result, we define an adversary that is able to win against every extension-based prover. The adversary maintains a partial specification of \( \Delta \) (the protocol it is adaptively constructing) and an integer \( t \geq 0 \). The integer \( t \) represents the number of non-uniform chromatic subdivisions of the input complex, \( S^0 \), that it has performed. Once the adversary has defined \( \Delta \) for each vertex in \( S^t \), then it may perform a non-uniform chromatic subdivision of \( S^t \) (or, subdivide \( S^t \)) and construct \( S^{t+1} = \chi(S^t, \Delta) \).

For each \( 0 \leq r \leq t \) and each input value \( a \in \{0, 1, \ldots, k\} \), we define \( N^r_a \) and \( T^r_a \) as the largest subcomplexes of \( S^r \) such that \( N^r_a \) contains only vertices that do not contain \( a \) and \( T^r_a \) contains only vertices \( v \) where \( \Delta(v) = a \). Notice that the definition of \( N^r_a \) does not depend on the adversary’s current specification of \( \Delta \), while the definition of \( T^r_a \) does. From these definitions, it follows that non-uniform chromatic subdivisions of these subcomplexes have simple descriptions:

**Proposition 4.1.** For \( t \geq 1 \), \( \chi(T_{a}^{t-1}, \Delta) = T_{a}^{t-1} \) and \( \chi(N_{a}^{t-1}, \Delta) = N_{a}^{t} \).

**Adversarial Strategy in Phase 1.** We define an adversarial strategy so that, after each query made by the prover in phase 1, the adversary is able to maintain the following invariants:
For each $0 \leq r < t$ and each vertex $v \in S^r$, $\Delta(v)$ is defined. If $v$ is a vertex in $S^t$, then either $\Delta(v)$ is undefined or $\Delta(v) \neq \bot$. If $s$ is the state of a process $p_i$ in a configuration that was reached by the prover, then $(i, s)$ is a vertex in $S^r$, for some $0 \leq r \leq t$, and $\Delta(i, s)$ is defined.

(2) For any input $a$, if $T_a^t$ is non-empty, then $\text{dist}_{S}^t(T_a^t, N_a^t) \geq 2$.

(3) For any two inputs $a \neq b$, if $T_a^t$ and $T_b^t$ are non-empty, then $\text{dist}_{S}^t(T_a^t, T_b^t) \geq 3$.

We note the following consequence of the invariants:

**Lemma 4.2.** For any two inputs $a \neq b$, if $T_a^t$ and $T_b^t$ are non-empty, then any path between $T_a^t$ and $T_b^t \cup N_a^t$ in $S^t$ contains at least one edge between unterminated vertices. Moreover, if the adversary defines $\Delta(v) = \bot$ for each vertex $v \in S^t$ where $\Delta(v)$ is undefined and subdivides $S^t$ to construct $S^{t+1}$, then $\text{dist}_{S}^{t+1}(T_a^{t+1}, T_b^{t+1}) \geq \text{dist}_{S}^t(T_a^t, T_b^t) + 1$ and $\text{dist}_{S}^{t+1}(T_a^{t+1}, N_a^{t+1}) \geq \text{dist}_{S}^t(T_a^t, N_a^t) + 1$.

**Proof.** Consider any path $v_0, v_1, \ldots, v_k$ between $T_a^t$ and $T_b^t \cup N_a^t$ in $S^t$. Let $v_j$ be the last vertex in $T_a^t$. Since the invariant holds after each query and $v_j \in T_b^t \cup N_a^t$, invariants (3) and (2) imply that $v_j$ and $v_k$ are at least 2. Hence, $\ell \geq j + 2$. Since $v_j$ is the last vertex in $T_a^t$, $v_{j+1}$ and $v_{j+2}$ are not in $T_a^t$. Moreover, by invariant (3), $v_{j+1}$ and $v_{j+2}$ are not in $T_c^t$ for any input $c \neq a$. Hence, $\{v_{j+1}, v_{j+2}\}$ is an edge between unterminated vertices. If the adversary defines $\Delta(v) = \bot$ for each vertex $v \in S^t$ where $\Delta(v)$ is undefined and subdivides $S^t$ to construct $S^{t+1}$, then the claim follows by Lemma 2.3 and Proposition 4.1.

Initially, the adversary sets $t = 0$ and $\Delta(v) = \bot$ for each vertex $v \in S^0$. It then subdivides $S^0$ to construct $S^t$ and increments $t$. No vertices in $S^0$ have terminated, so $T_a^0$ is empty for each input $a$. Before the first query, the prover has only reached the initial configurations. Hence, the invariants are satisfied.

Now suppose the invariants are satisfied immediately prior to a query $(C, P)$ by the prover, where $C$ is a configuration previously reached by the prover and $P$ is a set of active processes in $C$ poised to access the same snapshot object. Let $D$ be the configuration resulting from scheduling $P$ from $C$. Since each process in $P$ is poised to access the same snapshot object, by invariant (1), there exists $0 \leq r \leq t$ such that the state of each process in $P$ in configuration $C$ corresponds to a vertex in $S^r$. Since $P$ is active in $C$, $\Delta(v) = \bot$ for each such vertex $v$. Hence, by invariant (1), $r < t$. If $r < t - 1$, then invariant (1) implies that $\Delta$ is defined for each vertex corresponding to the state of a process in $D$. Hence, the adversary does not need to do anything.

So, suppose that $r = t - 1$. Let $\sigma$ denote the set of vertices in $S^t$ corresponding to the the states of $P$ in $D$. For each vertex $v \in \sigma$, if $\Delta(v)$ is undefined, the adversary defines $\Delta(v)$ as follows. If there exists an input $a$ such that the distance between $v$ and $N_a^t$ in $S^t$ is at least 3 and the distance between $v$ and $T_b^t$ in $S^t$ is at least 2, for all inputs $b \neq a$, then the adversary sets $\Delta(v) = a$. Otherwise, the adversary sets $\Delta(v) = \bot$. This ensures that invariants (2) and (3) continue to hold.

If $\Delta(v) \neq \bot$ for every vertex $v \in \sigma$, then invariant (1) holds. Otherwise, the adversary defines $\Delta(v) = \bot$ for each vertex $v \in S^t$ where $\Delta(v)$ is undefined, subdivides $S^t$ to construct $S^{t+1}$, and increments $t$. Then invariant (1) holds. By Lemma 4.2, invariants (2) and (3) continue to hold.

Therefore, the invariants hold after the prover’s query.

**No infinite chains of queries in phase 1.** Suppose that the invariants all hold after each query made by the prover in phase 1. By invariants (2) and (3), at most one value is output in any configuration reached by the prover and the value is the input of some process in the configuration.
So, the prover cannot win in phase 1 by showing that the protocol violates some task specifications. We now show that the prover also cannot win by making an infinite chain of queries in phase 1.

**Lemma 4.3.** Every chain of queries in phase 1 is finite.

*Proof.* Assume, for a contradiction, that there is an infinite chain of queries, \((C_j, P_j)\), for \(j \geq 0\). Let \(P\) be the set of processes that are scheduled infinitely often. Then, there exists \(j_0 \geq 0\) such that, for all \(j \geq j_0\), \(P_j \subseteq P\). Let \(t_0 \geq 1\) be the value of \(t\) maintained by the adversary immediately prior to query \((C_{j_0}, P_{j_0})\). By invariant (1), the state of each process in \(C_{j_0}\) corresponds to a vertex \(v \in S^r\), for some \(0 \leq r \leq t_0\). Hence, no process has accessed \(S_{t_0+1}\) in \(C_{j_0}\) and, during this chain of queries, only processes in \(P\) access \(S_t\) for \(t > t_0\). Since the processes in \(P\) eventually access \(S_{t+1}\) for all \(r \geq t_0\), and no process in \(P\) ever terminates, the adversary eventually defines \(\Delta(v) = \bot\) for each vertex \(v \in S^r\) where \(\Delta(v)\) is undefined and subdivides \(S^r\) to construct \(S^{r+1}\), for all \(r \geq t_0\).

Consider the first \(j_1 \geq j_0\) such that each process in \(P_{j_1}\) is poised to access \(S_{t_0+2}\) in \(C_{j_1}\), i.e. \(P_{j_1}\) is the first set of processes to access \(S_{t_0+2}\) in the chain of queries. By definition of \(S^{t_0+2}\), the set of states of the processes in \(P_{j_1}\) in \(C_{j+1}\) correspond to a simplex \(\sigma_1\) in \(S^{t_0+2}\). Since the adversary does not terminate any new processes, \(T_a^t = T_a^{t_0}\), for any input \(a\) and any \(t \geq t_0\).

Thus, by applying Lemma 4.2 twice, for any inputs \(a \neq b\), whenever \(T_a^{t_0}\) and \(T_b^{t_0}\) are non-empty, \(\text{dist}_{S^{t_0+2}}(T_a^{t_0+2}, T_b^{t_0+2}) \geq 5\) and \(\text{dist}_{S^{t_0+2}}(T_a^{t_0+2}, N_a^{t_0+2}) \geq 4\).

If there is a vertex \(v \in \sigma_1\) that has distance at most 2 to some vertex in \(T_a^{t_0+2}\) in \(S^{t_0+2}\), for some input \(a\), then the distance from \(v\) to \(N_a^{t_0+1}\) is at least 2 and the distance from \(v\) to non-empty \(T_b^{t_0+1}\) in \(S^{t_0+1}\) is at least 3, for all \(b \neq a\). Hence the adversary defines \(\Delta(v) \neq \bot\) after query \((C_{j_1}, P_{j_2})\), i.e. some process in \(P_{j_1} \subseteq P\) terminates. This is a contradiction.

So, each vertex in \(\sigma_1\) has distance at least 3 to \(T_a^{t_0+2}\), for all inputs \(a\) where \(T_a^{t_0+2}\) is non-empty. Consider the first \(j_2 > j_1\) such that each process in \(P_{j_2}\) is poised to access \(S_{t_0+3}\) in \(C_{j_2}\). Let \(P'\) be the set of processes that have accessed \(S_{t_0+2}\) in \(C_{j_2}\). Since each process in \(P_{j_1} \cup P_{j_2}\) has already accessed \(S_{t_0+2}\), \(P_{j_1} \cup P_{j_2} \subseteq P'\). Hence, the states of \(P'\) in \(C_{j_2}\) forms a simplex \(\sigma_2\) in \(S^{t_0+2}\) and \(\sigma_1 \subseteq \sigma_2\). It follows that each vertex in \(\sigma_2\) has distance at least 2 to \(T_a^{t_0+2}\), for all inputs \(a\) where \(T_a^{t_0+2}\) is non-empty.

Let \(a\) be the input of any process in \(P_{j_1}\). Since \(P_{j_1}\) is the first set of processes to access \(S_{t_0+2}\) and each process in \(P'\) has accessed \(S_{t_0+2}\), each vertex in \(\sigma_2\) contains \(a\) and \(\text{dist}_{S^{t_0+2}}(\sigma_2, N_a^{t_0+2}) \geq 1\). Since the distance between \(\sigma_2\) and any terminated vertex in \(S^{t_0+2}\) is at least 2, any path from a vertex \(v \in \sigma_2\) to a vertex in \(N_a^{t_0+1} \cup \bigcup_{b \neq a} T_b^{t_0+1}\) must contain at least one edge between unterminated vertices (specifically, between \(v\) and one of its neighbours). By Lemma 2.3 and Proposition 4.1, it follows that \(\text{dist}_{S^{t_0+3}}(\chi(\sigma_2, \Delta), N_a^{t_0+3}) \geq 2\) and \(\text{dist}_{S^{t_0+3}}(\chi(\sigma_2, \Delta), T_b^{t_0+3}) \geq 3\), for any input \(b \neq a\) where \(T_b^{t_0+2} = T_b^{t_0+3}\) is non-empty. The state of each process in \(P_{j_2}\) in \(C_{j_2+1}\) corresponds to a vertex \(v\) in \(\chi(\sigma_2, \Delta)\). Hence, the adversary defines \(\Delta(v) \neq \bot\) for at least one such vertex \(v\), i.e. some process in \(P_{j_2} \subseteq P\) terminates. This is a contradiction. 

**Checkmating the prover.** Since the invariants hold after every query, then the prover cannot perform an infinite chain of queries in phase 1 (Lemma 4.3). Hence, the prover does not win in phase 1 and, after performing finitely many chains of queries, the prover must choose a configuration \(C \in A'(1)\) at the end of phase 1. This determines the set of configurations \(A(2)\) it can initially query in phase 2. The adversary can update \(\Delta\) one final time so that it can answer all future queries by the prover. The prover will eventually be forced to choose a terminal configuration at the end of some phase and, consequently, will lose in the next phase.
By definition of $A'(1)$, $C$ is a configuration reached by a schedule $\alpha$ (from an initial configuration) that contains at least one set of processes. Consider the largest $r \geq 1$ such that some set of processes accesses $S_r$ in $\alpha$. Let $P$ be the first set of processes to access $S_r$ in $\alpha$ and let $a$ be the input of any process in $P$. By invariant (1), $r \leq t$. The set of states of processes in $P$ in $C$ corresponds to a simplex $\sigma$ in $S^t$.

Let $Q$ be the largest subcomplex of $S^r$ containing only vertices that are distance at most 1 from every vertex in $\sigma$. By definition, $Q$ contains every simplex corresponding to a configuration $C'$ reachable by an $r$-round schedule $\alpha'$ such that $C'$ is indistinguishable from $C$ to $P$. In particular, $P$ sees $a$ at each step. Each process in $P$ sees $S_r$ in $\alpha'$. Since $\alpha'$ is an $r$-round schedule, each active process in $C'$ has accessed $S_r$ in $\alpha'$ and sees $a$. Hence, the distance between $Q$ and $N^r_a$ in $S^r$ is at least 1. If $t > r$, then, by applying Lemma 2.3 and Proposition 4.1 $(t-r)$ times, $\delta_{\sigma^t}(\chi^{t-r}(Q, \Delta), N^r_a) \geq 1$.

By invariant (1), for each vertex $v \in S^t$, $\Delta(v)$ is either undefined or $\Delta(v) \neq \perp$. By invariant (3), each simplex in $S^t$ corresponds to a configuration in which all processes that have terminated have output the same value. For each vertex $v \in \chi^{t-r}(Q, \Delta)$ where $\Delta(v)$ is undefined, the adversary sets $\Delta(v) = a$. Then each vertex in $\chi^{t-r}(Q, \Delta)$ has terminated. Moreover, each simplex in $S^t$ corresponds to a configuration in which the processes have output at most 2 $\leq k$ different values.

In phases $\varphi \geq 2$, the prover can only query configurations reachable from some configuration in $A(2)$. Since any sufficiently long schedule from a configuration in $A(2)$ results in a configuration corresponding to a simplex in $\chi^{t-r}(Q, \Delta)$, the prover must eventually choose a configuration $D$ corresponding to a simplex in $\chi^{t-r}(Q, \Delta)$ at the end of some phase. Since each process has terminated in $D$, the prover loses in the next phase.

5 An Extension to Extension-Based Proofs

In this section, we extend the definition of an extension-based proof to include output queries, explain how the adversarial protocol can respond to these queries, and extend the proof in Section 4. Roughly speaking, output queries allow the prover to perform a valency argument [FLP85].

An output query in phase $\varphi$ is specified by a configuration $C \in A(\varphi) \cup A'(\varphi)$, a set of active processes $P$ in $C$ that are poised to access the same snapshot object, and a value $y \in \{0, 1, \ldots, k\}$. If there is a schedule from $C$ that involves only processes in $P$, i.e. a $P$-only schedule, and results in a configuration in which some process in $P$ outputs $y$, then the protocol returns some such schedule. Otherwise, the protocol returns NONE. In each phase, the prover is allowed to make finitely many output queries in addition to finitely many chains of queries.

For example, if $P$ is the set of all processes, then the sequence of output queries $(C, P, 0)$, $(C, P, 1)$, \ldots, $(C, P, k)$ enables the prover to determine which values can be output by the processes when they are scheduled starting from $C$. In particular, the prover can determine if it is only possible to output one value starting from $C$, i.e. if $C$ is univalent.

Responding to output queries in phase 1. Suppose that the invariants hold prior to an output query $(C, P, y)$ in phase 1. We show that the adversary can answer the output query so that it never conflicts with the result of any future query made in phase 1, while still maintaining the invariants. Let $r \geq 0$ be the number of times each process in $P$ has been scheduled in reaching $C$. By invariant (1), $r \leq t$. Let $\mathcal{V}$ be the largest subcomplex of $S^t$ containing only vertices representing the state of a process in $P$ in a configuration $C'$ reachable from $C$ by a $P$-only $(t-r)$-round schedule.
from \( C \). In particular, \( \mathbb{V} \) contains the possible states of each process in \( P \) after it has been scheduled \( t \) times (or it has terminated) in some configuration \( C' \) reachable from \( C \) by a \( P \)-only schedule.

If some vertex \( v \in \mathbb{V} \) has terminated with output \( y \), then the adversary returns a \( P \)-only schedule from \( C \) that corresponds to \( v \) (removing the steps of other processes in the \( P \)-only \( (t-r) \)-round schedule leading to \( v \)). If every vertex in \( \mathbb{V} \) has terminated, but none has output \( y \), then the adversary returns none. Since the adversary never changes \( \Delta(v) \) once it has been set, this does not conflict with the result of any future query. In both cases, the invariants continue to hold. So, suppose that, in \( \mathbb{V} \), at least one vertex has not terminated and no vertex has terminated with output \( y \).

Let \( U \neq \emptyset \) be the largest subcomplex of \( \mathbb{V} \) containing only unterminated vertices. For each simplex \( \tau \in U \), let \( A_{\tau} \) be the largest subcomplex of \( S' \) that only contains vertices at distance at most 1 to each vertex in \( \tau \). We consider a number of cases.

**Case 1:** For every simplex \( \tau \in U \), either \( \tau \in N^t_y \) or some vertex \( w \in A_{\tau} \) has terminated with an output \( a \neq y \). In this case, the adversary returns none. If \( \tau \in N^t_y \), then no vertex in \( \tau \) contains \( y \) and, by Proposition 4.1, no vertex in any subdivision of \( \tau \) contains \( y \). Hence, by invariant (2), the adversary never terminates any vertex in \( \tau \) (or a future subdivision of \( \tau \)) with output \( y \) as the result of a future query (not just in phase 1). If some vertex \( w \in A_{\tau} \) has terminated with output \( a \neq y \), then, since \( w \) is adjacent to each vertex in \( \tau \), \( w \) is adjacent to each vertex in a subdivision of \( \tau \) (see Section 2.4). Since invariant (3) holds, the adversary never terminates such vertices with any output other than \( a \) as the result of a future query in phase 1. In both cases, the invariants continue to hold.

**Case 2:** There is a simplex \( \tau \in U - N^t_y \) and a vertex \( w \in A_{\tau} \) such that \( w \) has terminated with output \( y \). Then the adversary defines \( \Delta(v) = \perp \) for each vertex \( v \in S' \) where \( \Delta(v) \) is undefined and subdivides \( S' \) to construct \( S'^{t+1} \). By Lemma 4.2, \( \text{dist}_{s^{t+1}}(T^{t+1}_y N^t_y) \geq 3 \) and \( \text{dist}_{s^{t+1}}(T^{t+1}_y T^{t+1}_a) \geq 4 \), for all inputs \( a \neq y \). Since \( w \) has terminated, \( \tau \) is adjacent to every vertex in \( \chi(\tau, \Delta) \subseteq S'^{t+1} \). It follows that \( \text{dist}_{s^{t+1}}(\chi(\tau, \Delta), N^t_y) \geq 2 \) and \( \text{dist}_{s^{t+1}}(\chi(\tau, \Delta), T^{t+1}_a) \geq 3 \), for all inputs \( a \neq y \). The adversary defines \( \Delta(v) = y \), for a vertex \( v \in \chi(\tau, \Delta) \), returns a \( P \)-only schedule from \( C \) corresponding to \( v \), and increments \( t \). By Lemma 4.2, the invariants are not violated by the subdivision and increment of \( t \). Since \( \Delta(v) \neq \perp \), invariant (1) continues to hold and, since \( \text{dist}_{s^{t}}(v, N^t_y) \geq 2 \) and \( \text{dist}_{s^{t}}(v, T^{t}_a) \geq 3 \), for all inputs \( a \neq y \), invariants (2) and (3) continue to hold.

**Case 3:** \( U - N^t_y \) is non-empty and, for every simplex \( \tau \in U - N^t_y \), no vertex in \( A_{\tau} \) has terminated. Let \( \tau \in U - N^t_y \) be such a simplex. Then the adversary subdivides \( S' \) twice, i.e. it defines \( \Delta(v) = \perp \) for each \( v \in S' \) where \( \Delta(v) \) is undefined, subdivides \( S' \) to construct \( S'^{t+1} \), defines \( \Delta(v) = \perp \) for each \( v \in S'^{t+1} \) where \( \Delta(v) \) is undefined, and subdivides \( S'^{t+1} \) to construct \( S'^{t+2} \). Let \( P' \subseteq P \) be the set of processes whose states appear in \( \tau \in U \). Consider the simplex \( \rho \) in \( \chi(\tau, \Delta) \subseteq S'^{t+1} \) corresponding to scheduling \( P' \) altogether from the configuration represented by \( \tau \). Since \( \tau \notin N^t_y \), some vertex in \( \tau \) contains \( y \), so each vertex in \( \rho \) contains \( y \). Hence, \( \text{dist}_{s^{t+1}}(\rho, N^{t+1}_y) \geq 1 \). Since each vertex in \( A_{\tau} \) is unterminated, \( \chi(\Delta_{\tau}, \Delta) \) is the standard chromatic subdivision of \( A_{\tau} \). It follows that every path between \( \rho \) and \( T^{t+1}_a \cup N^{t+1}_y \) in \( S'^{t+1} \), for inputs \( a \neq y \), contains at least one edge between a vertex in \( \rho \) and one of its neighbours in \( \chi(\Delta_{\tau}, \Delta) \) and each such edge is between unterminated vertices (see Section 2.4). Hence, \( \text{dist}_{s^{t+1}}(\rho, T^{t+1}_a) \geq 2 \), for all inputs \( a \). By Lemma 2.3 and Proposition 4.1, it follows that \( \text{dist}_{s^{t+2}}(\rho, N^{t+2}_y) \geq 2 \) and \( \text{dist}_{s^{t+2}}(\rho, T^{t+2}_a) \geq 3 \), for all inputs \( a \). The adversary defines \( \Delta(v) = y \), for one vertex \( v \in \chi(\rho, \Delta) \), returns a \( P \)-only schedule from \( C \) corresponding to \( v \), and sets \( t \) to \( t + 2 \). By Lemma 4.2, the invariants are not violated by the subdivisions after the value of \( t \) has been increased. Since \( \Delta(v) \neq \perp \), invariant (1) continues to
hold and, since \( \text{dist}_S(v, N_y^t) \geq 2 \) and \( \text{dist}_S(v, T^t_a) \geq 3 \), for all inputs \( a \neq y \), invariants (2) and (3) continue to hold.

**Checkmating the prover.** Since the adversary is able to maintain the invariants after each query, as in Section 4, the prover does not win in phase 1 and must choose a configuration \( C \in A'(1) \).

To ensure that the prover loses, we have to modify the adversary’s strategy at the end of phase 1 in Section 4 slightly. In particular, when the adversary defines \( \Delta(v) \) for each vertex in \( \chi^{t-r}(Q, \Delta) \), it cannot simply set \( \Delta(v) = a \), where \( a \) is an input value contained in each vertex of \( \chi^{t-r}(Q, \Delta) \).

The problem is that the adversary may have answered NONE to an output query \( (C, P, y) \) where not all vertices have terminated (in Case 1).

The modified strategy of the adversary is as follows:

1. Subdivide \( S^t \) twice to construct \( S^{t+2} \) and set \( t \) to \( t + 2 \).
2. For each terminated vertex \( w \in S^t \) and each vertex \( v \in \chi^{t-r}(Q, \Delta) \) adjacent to \( w \), if \( \Delta(v) \) is undefined, set \( \Delta(v) = \Delta(w) \).
3. For each vertex \( v \in \chi^{t-r}(Q, \Delta) \), if \( \Delta(v) \) is undefined, set \( \Delta(v) = a \).

Observe that, by applying Lemma 4.2, prior to step 2, for any two inputs \( a \neq b \) where \( T^t_b \) and \( T^t_a \) are non-empty, \( \text{dist}_S(T^t_a, T^t_b) \geq 5 \) and \( \text{dist}_S(T^t_a, N^t_a) \geq 4 \). After step 2, the distance of newly terminated vertices \( u, v \) where \( \Delta(u) \neq \Delta(v) \) is at least 3. Moreover, each newly terminated vertex \( v \) contains \( \Delta(v) \) and has distance at least 3 to \( N^t_{\Delta(v)} \). Hence, the invariants hold after step 2.

We first show that these changes to \( \Delta \) do not violate the response to any output query.

**Lemma 5.1. No output queries made in phase 1 are violated.**

**Proof.** Let \( (C, P, y) \) be an output query. Let \( t_0 \) be the value of \( t \) maintained by the adversary prior to the query and let \( V \) be the subcomplex of \( S^{t_0} \) as defined in Section 5. If some vertex \( v \in V \) has output \( y \), then the adversary cannot violate its response as it never changes \( \Delta(v) \). Similarly, if every vertex \( v \in V \) has terminated, then the adversary cannot violate its response. Now let \( U \neq \emptyset \) and \( A_\tau \), for simplices \( \tau \in U \), be as defined in Section 5. For the same reason as previously, if the adversary responds with a schedule (i.e. in Case 2 and Case 3), then it also cannot violate its response. So, suppose the adversary responds with NONE (i.e. in Case 1). Then, for every simplex \( \tau \in U \), either \( \tau \in N^t_y \) or \( A_\tau \) contains a vertex \( w \) that has terminated \( \Delta(w) \neq y \). Let \( \tau \in U \) and consider \( \tau' = \chi^{t-t_0}(\tau, \Delta) \). Suppose \( \tau \in N^t_y \). Then \( \tau' \in N^t_y \). If \( \Delta(v) \) was defined for some vertex \( v \in \tau' \) as a result of a query, then since the invariants hold after each query, by invariant (2), \( \Delta(v) \neq y \). Similarly, if \( \Delta(v) \) was defined in step 2 or 3, then \( \Delta(v) \neq y \) (by invariant (2) and definition of \( a \), respectively). Hence, the adversary does not violate its response on simplices \( \tau \in N^t_y \) and their subdivisions \( \tau' \). So, suppose \( \tau \notin N^t_y \). Then \( A_\tau \) contains a vertex \( w \) that has terminated with \( \Delta(w) \neq y \). Since \( w \) has terminated and is adjacent to every vertex in \( \tau \) in \( S^{t_0} \), it is adjacent to every vertex in \( \tau' \) in \( S^t \) (see Section 2.4). If \( \Delta(v) \) was defined for a vertex \( v \in \tau' \) as the result of a query, then since invariant (3) holds, \( \Delta(v) = \Delta(w) \neq y \). Otherwise, \( \Delta(v) \) was defined for \( v \in \tau' \) as a result of step 2. However, since the invariants hold after step 2, \( \Delta(v) = \Delta(w) \neq y \). However, this implies that if \( \Delta(v) \) is defined for any vertex in \( \tau' \), then \( \Delta(v) \neq y \). It follows that no vertex in \( \chi^{t-t_0}(U, \Delta) \) has been terminated with output \( y \). \( \square \)
Since no output query is violated, and the invariants hold prior to step 3, the rest of the argument is unchanged. In particular, the prover must commit to a terminal configuration in $\chi^{t-r}(Q, \Delta)$ at the end of some phase and, hence, loses in the next phase.

6 Distance Lemma to the Rescue

Hoest and Shavit [HS97, HS06] developed the NIIS model in order to apply topological framework to the complexity of problems, as opposed to computability. They considered $\epsilon$-agreement task, where processes must output values within $\epsilon$ of each other, and proved step complexity upper and lower bounds. Additionally, similar to consensus, a process with input 0 executed solo should return 0, while a process with input 1 executed solo should return 1.

While the lower bound of Hoest and Shavit is the more important result, the upper bound has to assign correct outputs to terminal process states. If the round complexity of the upper bound is $t$, then every vertex in $S^t$ needs to be terminal and assigned an output value.

Suppose $v$ is a vertex in $S^t$ that corresponds to a terminal state of a process with input 0 executed solo, and suppose $v'$ is a vertex in $S^t$ that corresponds to a terminal state of another process with input 1 executed solo. Notice that $v$ must be assigned output 0, while $v'$ must be assigned output 1. The condition that in any configuration, output values of all process should be within $\epsilon$ dictates that vertices with distance 1 of each other must be assigned output values that differ by at most $\epsilon$. Hence, the distance between $v$ and $v'$ in $S^t$ can’t be less than $1/\epsilon$.

Therefore, a correct upper bound argument should contain a proof for a version of a distance lemma between terminal vertices, i.e. that after subdividing $t$ times for some $t$, the distance between terminal vertices that must be assigned outputs 0 and 1 must be at least $1/\epsilon$. But close inspection of the upper bound arguments of Hoest and Shavit uncovered a technical issue, which we have confirmed in private communication. There is a problem both in the conference version [HS97] and in the updated argument that appeared in the journal version [HS06] of their paper.

The conference version [HS97] considers certain simplices separately, assigning output values to some vertices in each simplex. The output values are within $\epsilon$ of each other, as desired. However, the issue is that a vertex that is on a boundary of two simplices may be considered twice in the context of these two simplices and be assigned two different output values.

In the journal version [HS06], every process maintains the current desired output value, which initially is its input value. Each vertex corresponding to an active process state is then labeled by its desired output value, and each terminal vertex is labeled by the output value. If the desired output label associated with a vertex in $S^r$ is within $\epsilon$ of all labels of vertices at distance one in $S^r$, this vertex is terminated with the desired value as the output. This resolves the inconsistent output assignment issue of [HS97]. Otherwise, new desired outputs are assigned to active process states in round $r + 1$. A process that is scheduled first without any other process in round $r + 1$ (i.e. only sees its own update when scanning $S^{r+1}$), keeps its desired output value from the round $r$: note that this ensures that solo executions by processes return the correct value, which is the input of the process. Otherwise, if the process is not scheduled first and on its own in round $r + 1$, its new desired output label is computed as a weighted average between the maximum and the minimum of desired values of all vertices in $S^r$ that were not terminated. (The new value is either the average, $2/3$ times minimum plus $1/3$ times maximum, or $1/3$ times minimum plus $2/3$ times maximum, depending on some properties of the vertex). The vertex in $S^{r+1}$ corresponding to the state of a process after round $r + 1$ is labeled accordingly.
The issue with this argument is that after a subdivision, the output of a terminated vertex (which is also a vertex in $S^{r+1}$) may no longer be within $\epsilon$ of the desired values of vertices at distance 1. This is because the assignment of new desired values only depends on the global maximum and minimum among the desired values of active processes from round $r$. Note that we can’t fix this issue by waiting and trying to terminate all vertices in the same round, as the global minimum will remain 0, global maximum will remain 1 (as long as some process has input 0 and some process has input 1 in the initial configuration), and all labels computed in future rounds will be $1/3$, $1/2$ or $2/3$, and there will always be vertices at distance 1 with desired values differing by at least $1/6$, preventing termination for a smaller $\epsilon$.

While Lemma 2.3 shows that the distances between vertices that have to be labeled with outputs 0 and 1 will grow sufficiently after some number of rounds, a correct upper bound argument also needs to assign outputs to these vertices on intermediate paths. Let $A$ be the largest subcomplex that contains only the vertices that must be labeled with output 0, and let $B$ be the largest subcomplex that contains only the vertices that must be labeled with output 1.

Essentially, the distance lemma should be extended to show that all paths from $A$ to $B$ can be decomposed into layers of vertices (and corresponding subcomplexes) at distance 1, distance 2, etc, such that the distance only between vertices in successive layers are 1. Then, vertices in the first layer can be assigned output $\epsilon$, vertices in the second layer can be assigned output $2\epsilon$, etc. Operationally, we suspect this will be similar to existing algorithms for $\epsilon$-agreement that update desired output values based on the desired output values that they see in each round. As this is not related to our main contributions, we omit the proof of the extended version of Lemma 2.3 in this manuscript but will incorporate it in the full version after further discussion with the authors of [HS97, HS06].

7 Future Work

We developed a framework that allows us to show the limitation of valency arguments for proving impossibility results in asynchronous shared memory systems. This is done by defining a class of extension-based proofs, formalized as an interaction between a prover and a protocol. In this paper, we deliberately decided to restrict attention to the proof of impossibility of one problem in one model, to keep the discussion focused. However, our approach can be applied to other problems and other models, which we now discuss briefly.

While the NIIS model is computationally equivalent to an asynchronous shared memory model in which processes communicate by reading from and writing to shared registers, they are not equivalent in terms of space and step complexities. We have a definition for extension-based proofs in the latter model and we are working on extending our proof to show that certain complexity lower bounds cannot be obtained using extension-based proofs. In particular, we conjecture that covering arguments cannot be used to prove a lower bound on the number of registers needed for randomized wait-free solutions to $k$-set agreement among $n > k \geq 2$ processes that depends on the number of processes.

The definition of an extension-based proof can be modified to handle other termination conditions, such as obstruction-freedom. It suffices for the prover to construct a schedule that violates this condition. It might also be possible to generalize the definition of an extension-based proof to message-passing systems.

We have considered allowing the prover to perform a number of other types of queries and can
extend our adversarial protocol so that it can answer them. For example, if a prover asks the same output query \((C, P, y)\) multiple times, the protocol could be required to return different schedules each time, until it has returned all possible \(P\)-only schedules from \(C\) that output \(y\).

We cannot allow certain queries, such as asking for an upper bound on the length of any schedule. If the prover is given such an upper bound, then it can perform a finite number of chain queries to examine all reachable configurations, thereby fixing the protocol. However, we can allow the prover to use this information in a restricted way and still construct an adversarial \(k\)-set agreement protocol. For example, we might require that the prover does not use this information to decide which queries to perform or what extensions to construct, but can use this information to win when it has constructed a schedule that is longer than this upper bound.

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