A comparative review of four formulations of noncommutative quantum mechanics

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Four formulations of quantum mechanics on noncommutative Moyal phase spaces are reviewed. These are the canonical, path-integral, Weyl-Wigner and systematic formulations. Although these formulations all represent quantum mechanics on a phase space with the same deformed Heisenberg algebra, there are mathematical and conceptual differences which we discuss.

Keywords: Noncommutative geometry, noncommutative quantum mechanics, formalism, star product, Bopp's shift, Seiberg-Witten map, path-integral, Weyl-Wigner transform.

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I. INTRODUCTION

Quantum mechanics is now more than a hundred years old, starting from the Max Planck’s description in 1900 of the black body radiation problem and the paper of Albert Einstein in 1905 about quantum-based theory to explain the photoelectric effect. However quantum theory has been reformulated around 1925 with the fundamental Heisenberg commutation relations or phase space algebra and the associated uncertainty principle. One of the fundamental features is to deal with operators that do not commute, more precisely, the commutations relations between positions and momenta. The terminology noncommutative quantum mechanics is a terminology to express that the positions operators do not commute, more generally as we will review below, one can consider the situation where the commutators for coordinates and momenta are non-canonical.

One of the problems in the 1930s was how to resolve infinities in the then newly introduced quantum field theory. The idea of extending noncommutativity to the coordinates as a possible way of removing the infinite quantities appearing in field theories was first suggested by Werner Heisenberg. That is the birth of the idea of quantum spacetime, which is a generalization of the usual concept of spacetime in which some variables that ordinarily commute are assumed to be noncommuting and form a different Lie algebra.

He passed the idea to his doctoral student Rudolf Peierls in the late 1930s [1]. Peierls made use of these ideas eventually in his work related to the Landau level problem. He also noticed that electrons in a magnetic field can be regarded as moving in a quantum space. He also passed the idea to Wolfgang Pauli who then involved Robert Oppenheimer in the discussion [2]. Oppenheimer carried it to his student Hartland Snyder, who published the first concrete example in 1947 [3]. It was a period when ideas about renormalization also born and the success of the renormalization theory took over the ideas about noncommutative coordinates. Later in 1980s - 1990s Alain Connes developed noncommutative geometry [4, 5] and that has been very successfull and plausible that it has attracted the attention of particle physicists and string theorists [6-14]. The history about noncommutative spacetime can be also found in [15].

The simplest noncommutativity one can postulate is that the space-time coordinates satisfy the commutation relation

\[ [\hat{x}_i, \hat{x}_j] = i\theta_{ij}, \] (1)

where \( \theta_{ij} \) is a constant anti-symmetric tensor of dimension (length\(^2\)). Some of the consequences of the space-time coordinates noncommutativity are the following:

- the action for quantum field theory on noncommutative space is obtained from a usual quantum field theory through replacing the field products by Moyal star-product [16].
• an uncertainty relation between position measurements will a priori lead to a nonlocal theory;

• the presence of the constant $\theta_{ij}$ in equation (1) violates Lorentz invariance (if the dimension of space-time is greater than two);

• there is a persistence of the ultraviolet divergences in noncommutative quantum field theory \cite{12,14};

• the time, space noncommutativity leads to violation of unitarity and causality \cite{17}.

Despite the above (perhaps unwanted) features, there are strong motivations in studying noncommutative space-time. There is a long-held belief that in quantum theories including gravity, space-time must change its nature at distances comparable to the Planck scale. One might also study noncommutative theories as interesting analogs of theories of more direct interest, such as Yang-Mills theory. It was found that the equation (1) follows naturally as a particular low-energy limit of string theories with $\theta_{ij}$ directly related to a constant antisymmetric background field $B_{ij}$ in the presence of D-brane \cite{10,11}. Noncommutative field theory is also known to appear naturally in condensed-matter theory, for instance the theory of electrons in a magnetic field projected to the lowest Landau level (the Quantum Hall problem), which is naturally thought of as a noncommutative Chern-Simons theory \cite{18}.

In this work, however, we focus on quantum mechanics on noncommutative spaces, rather than quantum field theory. Since in noncommutative quantum mechanics with commutative time evolution, we do not encounter the problems that may occur in noncommutative quantum field theory, it is well suited as a toy model for the introduction of noncommuting coordinates. Indeed, noncommutative quantum mechanics has drawn attention of many authors. \cite{19,60}. We would like to apologize for not being able to cite all the papers relating the topic. In 2008, we have been interested in the formulation and interpretation of noncommutative quantum mechanics \cite{61} and that led us to some significant contributions in the development of noncommutative quantum mechanics \cite{62,70}.

Various formulations of noncommutative quantum mechanics are found in the literature. Are they equivalent? Do they provide different insights? Are there some problems more difficult in one formulation and more easy in another one? In this paper, we attempt to give a catalogue of formulations of noncommutative quantum mechanics. Our intention is primarily to examine the distinction between these mathematical formulations and briefly discuss then the bearing this may have on the conceptual interpretations.

II. CATALOGUE OF FORMULATIONS

A. Canonical formulation

1. New coordinates system

The noncommutative space is realized by the coordinates operators satisfying

\[ [\hat{x}_i, \hat{x}_j] = i\theta_{ij}, \]

where the operators $\hat{x}_i$, $i = 1, 2$ are the coordinate operators and $\theta_{ij}$ is real and antisymmetric, i.e., $\theta_{ij} = \theta_{ji}$, with $\epsilon_{ij}$ the completely antisymmetric tensor $\epsilon_{12} = 1$ and the spacial noncommutative parameter $\theta$ is of dimension of $(\text{length})^2$. The choice of the phase space is such that

\[ [\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}; \quad [\hat{p}_i, \hat{p}_j] = 0, \quad i, j = 1, 2, \]

or in a general setting where the momenta are also noncommutative we consider

\[ [\hat{x}_i, \hat{x}_j] = i\theta_{ij}; \quad [\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}; \quad [\hat{p}_i, \hat{p}_j] = i\eta_{ij}, \quad i, j = 1, 2, \]

where $\eta_{ij} = \eta_{ji}$ is totally antisymmetric representing the noncommutativity property of the momentum on noncommutative phase space.
The approach here is to express the set of noncommutative coordinates as linear combination of the canonical variables or vice versa. In the situation of equation (3), the most used, the Hilbert space can be consistently be taken to be exactly the same as the Hilbert space of the corresponding commutative system. The Hamiltonian has to be introduced, and that is nontrivial. Once it is done, the dynamical equation for the state $|\psi\rangle$ is the usual Schrödinger equation, $H|\psi\rangle = i\hbar \frac{\partial}{\partial t}|\psi\rangle$.

A new coordinates system is introduced as follows

$$x_i = \hat{x}_i + \frac{1}{2\hbar} \theta_{i,j} p_j, \quad p_i = \hat{p}_i,$$

where the new variables satisfy the usual canonical commutation relations

$$[x_i, x_j] = 0; \quad [x_i, p_j] = i\hbar \delta_{ij}; \quad [p_i, p_j] = 0.$$

Since the noncommutativity parameter $\theta$ is very small, the noncommutativity effects can be treated as some perturbations of the commutative counterpart up to first order $\theta$, then one can use the usual wave functions and probabilities. The difference between noncommutative and ordinary quantum mechanics consists in the choice of polarisation.

In the situation of equation (4), the representation of $\hat{x}_i$ and $\hat{p}_i$, could be the following

$$\hat{x}_i = \kappa x_i - \frac{1}{2\hbar \kappa} \theta_{ij} p_j, \quad \hat{p}_i = \kappa p_i + \frac{1}{2\hbar \kappa} \eta_{ij} x_j, \quad i, j = 1, 2,$$

where $\kappa$ is a scaling constant related to the noncommutativity of phase space.

Reference papers:

1. M. Chaichian, M.M. Sheikh-Jabbari, A. Tureanu, Hydrogen Atom Spectrum and the Lamb Shift in Noncommutative QED, Phys. Rev. Lett. vol. 86, Num.13 (2001).

2. M. Demetrian, D. Kochan, Quantum Mechanics on non-commutative plane, Acta Physica Slovaca 52, Num 1 pp: 1-9 (2002).

3. S. Bellucci, A. Nersessian, C. Sochichiu, Two phases of the noncommutative quantum mechanics, Phys. Lett. B 52: 345-349, (2001).

4. Anais Smailagic, Euro Spallucci, Isotropic representation of the noncommutative 2D harmonic oscillator, Physical Review D, Volume 65, 107701, (2002).

2. Bopp's shift and star products

The physicist Fritz Bopp was the first to consider in his paper, where he discussed some statistical implications of quantization, pseudo-differential operators obtained from a symbol by the quantization rules

$$x \rightarrow x + \frac{1}{2} i \hbar \partial_p, \quad p \rightarrow p - \frac{1}{2} i \hbar \partial_x,$$

instead of the usual correspondance $x \rightarrow x$, $p \rightarrow -\frac{i}{2} \hbar \partial_x$. In the physics literature, the operators $x \rightarrow x + \frac{1}{2} i \hbar \partial_p$ and $p \rightarrow p - \frac{i}{2} \hbar \partial_x$ are called Bopp’s shifts and this quantization procedure is called Bopp quantization. There is a connection between the Bopp’s shifts and the star product or $\star$-product, that is an associative deformation of ordinary products on phase space. The $\star$-product has been defined by Groenewold as follows

$$\star \equiv e^{\frac{i}{\hbar} \frac{1}{2} (\partial_{\rho} \theta_{\phi} - \partial_{\theta} \rho)},$$
The \(*\)-product induces Bopp’s shifts in the sense that it may be evaluated through translations of functions.

\[ f(x, p) \ast g(x, p) = f(x + \frac{i\hbar}{2} \partial_p, p - \frac{i\hbar}{2} \partial_x)g(x, p). \]  

(10)

In this approach noncommutative quantum mechanics is considered as a theory defined on a manifold where the product of functions is the Moyal one. If \( f(x) \) and \( g(x) \) are two functions, then the Moyal product is defined as

\[ (f \ast g)(x) = e^{i\theta_{ij} \partial^i \partial^j} f(x_1)g(x_2)|_{x_1 = x_2 = x}. \]  

(11)

In this sense, for instance, the time dependent Schrödinger equation

\[ i\frac{\partial \psi}{\partial t} = \left( \frac{p^2}{2m} + V(x) \right) \psi(x, t), \]  

(12)

in the noncommutative space is the same one but with the potential shifted as \( V(x - \frac{\hat{p}}{2}) \), where

\[ V(x) \ast \psi(x, t) \rightarrow V(x - \frac{\hat{p}}{2})\psi(x, t), \]  

(13)

with \( \hat{p}_i = \theta^{ij}p_j \) and \( \theta_{ij} = \theta_{\epsilon_{ij}} \), where \( \epsilon_{ij} \) is an antisymmetric tensor in two dimensions. This last fact implies that quantum mechanics in a noncommutative plane is highly nontrivial because, as the shifted potential involves in principle arbitrary powers of the momenta, we will have an arbitrary large number of derivatives in the Schrödinger equation.

**Reference papers**

5. J. Gamboa, M. Loewe and J. C. Rojas, Non-commutative Quantum Mechanics, Physical Review D, Vol. 64. Issue: 6 Article Number: 067901 (2001);

6. J. Gamboa, M. Loewe, F. Mendez and J. C. Rojas, Noncommutative Quantum Mechanics: The Two-dimensional central Field, International Journal of Modern Physics A, Vol. 17 Issue: 19 Pages: 2555-2565 (2002).

The study of noncommutative quantum mechanics can also be reduced to a variant of Bopp calculus. A variant of Bopp’s shift is the map

\[ \hat{x}_i = x_i - \frac{1}{2\hbar}\theta_{ij}p_j, \quad \hat{p}_i = p_i, \quad i, j = 1, 2, \]  

(14)

that is equivalent to equation [5], where the operators \( \hat{x}_i, \hat{p}_i, i = 1, 2 \) satisfy the algebra and the operators \( x_i, p_i, i = 1, 2 \) satisfy the algebra [6]. In the literature the equation (14) is called Bopp’s shift and this map is linked to the Moyal product. Let’s consider for instance the time independent the Schrödinger equation on noncommutative space

\[ H(\hat{x}_i, \hat{p}_i) \ast \psi = E\psi. \]  

(15)

The Moyal product can be changed into the ordinary product by using the Bopp’s shift as

\[ H(\hat{x}_i, \hat{p}_i) \ast \psi = H(x_i - \frac{1}{2\hbar}\theta_{ij}p_j, p_i)\psi = E\psi, \]  

(16)

and the noncommutative effects can be evaluated through the \( \theta \) related terms that can be treated as a perturbation in ordinary quantum mechanics, since \( \theta \ll 1 \).

**Reference papers**

7. Li, K.; Dulat, S., The Aharonov-Bohm effect in noncommutative quantum mechanics Eur. Phys. J. C Volume: 46 Issue: 3 Pages: 825-828 (2006);

8. Sahipjamal Dulat, Kang Li, Quantum Hall Effect in Noncommutative Quantum Mechanics, Eur. Phys. J.C (2009) 60: 163-168.
The Seiberg-Witten (SW) map was discovered by Nathan Seiberg and Edward Witten in the context of string theory and noncommutative geometry [10]. They argued that the ordinary gauge theory should be gauge equivalent to a noncommutative Yang-Mills field theory. They introduced a correspondence between a noncommutative gauge theory and a conventional gauge theory. Let’s briefly review the Seiberg-Witten map, the details being given in [10, 13]. Let’s consider a flat Minkowski space. On this space, we consider the coordinates $x_\mu$ as self-adjoint operators on a Hilbert space, satisfying the algebra

$$[x_\mu, x_\nu] = i\theta_{\mu\nu},$$ (17)

where $\theta_{\mu\nu}$ is real and antisymmetric. The corresponding field theory is equivalent to the usual commutative flat manifold with the product substituted by the non-local $\star$-product

$$(f \star g)(x) = \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4p}{(2\pi)^4} e^{-i(k_\mu + p_\mu) x^\mu} e^{-\frac{1}{2}i\theta_{\mu\nu} k_\mu p_\nu} f(k) \tilde{g}(p),$$ (18)

where $f$ and $g$ are functions on the manifold. The noncommutative Yang-Mills action is

$$\hat{\Sigma}_{cl} = -\frac{1}{4} \int d^4x \hat{F}_{\mu\nu} \star \hat{F}^{\mu\nu} = -\frac{1}{4} \int d^4x \hat{F}_{\mu\nu} \hat{F}^{\mu\nu},$$ (19)

where

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i\hat{A}_\mu \star \hat{A}_\nu + i\hat{A}_\nu \star \hat{A}_\mu, \quad \hat{A}_\mu = \hat{A}_\mu(A),$$ (20)

and $\hat{A}_\mu$ is a $U(1)$ gauge field. Note that $\hat{A}_\mu$ is Hermitian. The noncommutative gauge transformation is given by

$$\hat{\delta}_\lambda \hat{A}_\mu = \partial_\mu \hat{\lambda} - i\hat{A}_\mu \star \hat{\lambda} + i\hat{\lambda} \star \hat{A}_\mu \equiv \hat{D}_\mu \hat{\lambda}, \quad \hat{\lambda} = \hat{\lambda}^*, \quad \hat{D}_\mu = \partial_\mu - i\hat{A}_\mu \star.$$ (21)

with infinitesimal $\hat{\lambda} = \hat{\lambda}^*$. Seiberg and Witten have shown that an expansion in $\theta$ leads to a map between the noncommutative gauge field $\hat{A}_\mu$ and the commutative gauge field $A_\mu$ as well as their respective gauge parameters $\hat{\lambda}$ and $\lambda$, known as the Seiberg-Witten (SW) map:

$$\hat{A}_\mu(A) = A_\mu - \frac{1}{2} \theta^{\mu\sigma} A_\rho (\partial_\sigma A_\mu + F_{\sigma\mu}) + O(\theta^2)$$ (22)

$$\hat{\lambda}(\lambda, A) = \lambda - \frac{1}{2} \theta^{\rho\sigma} A_\rho \partial_\sigma \lambda + O(\theta^2),$$ (23)

where the Abelian field strength is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$ (24)

In noncommutative quantum mechanics are often called or referred to as the Seiberg-Witten (SW) maps linear (non canonical) transformations:

$$\hat{x}_i = \hat{x}_i(x_j, p_j), \quad \hat{p}_i = \hat{p}_i(x_j, p_j).$$ (25)

that relate the extended Heisenberg Algebras in equation (23) or in equation (4) to the standard Heisenberg algebra in equation (6). The Seiberg-Witten (SW) map is not unique [72]. Using one of these transformations, that is a particular SW map, it is possible to find a representation of the noncommutative observables as operators acting on the conventional Hilbert space of ordinary quantum mechanics. The states of the system are then wave functions of the ordinary Hilbert space, the Hamiltonian depend on the noncommutativity parameter in the Schroedinger equation. The Seiberg-Witten (SW) map is considered as a variant of the Bopp’s shift.

Reference papers:

9. J. Gamboa, M. Loewe, F. Mendez, J. C. Rojas, Noncommutative quantum mechanics: The two-dimensional central field, Int. J. Mod. Phys. A17 2555-2566, (2002);
B. Path integral formulation

We consider the following deformed Heisenberg algebra

\[ [\hat{q}_1, \hat{q}_2] = i\theta; \quad [\hat{q}_1, \hat{p}_1] = i\hbar; \quad [\hat{q}_2, \hat{p}_2] = i\eta; \quad [\hat{q}_2, \hat{p}_1] = 0, \quad [\hat{q}_1, \hat{p}_2] = 0. \]  \hfill (26)

The aim is to provide a path integral formulation of quantum dynamics which is consistent with equation (26). A phase space path integral is then formulated from which the commutation relations in equation (26) and the extended Heisenberg equations of motion are derived. This is formulated as follows.

In (2 + 1)-dimensional space-time, we consider the classical action

\[ S = \int_0^T dt \left( \frac{1}{2} \omega_{ij} \dot{x}_i \dot{x}_j - H(x) \right), \quad x_{1,2,3,4} = q_1, q_2, p_1, p_2, \]  \hfill (27)

where \( H \) is the hamiltonian of the system and \( \omega_{ij} = (\Theta^{-1})_{ij} \) with

\[ \Theta = \begin{pmatrix} 0 & \theta & 1 & 0 \\ -\theta & 0 & 0 & 1 \\ 0 & 0 & 0 & \eta \\ 0 & 1 & -\eta & 0 \end{pmatrix} \quad \text{and} \quad \omega = \begin{pmatrix} 0 & \eta & -1 & 0 \\ -\eta & 0 & 0 & -1 \\ 1 & 0 & 0 & \theta \\ 0 & 1 & -\theta & 0 \end{pmatrix}. \]  \hfill (28)

We assume here that \( \hbar = 1 \), and that the matrix \( \Theta \) is non singular, means \( \theta \eta \neq 1 \). The Hamiltonian equations of motion and the basic Poisson brackets are respectively

\[ \dot{x}_i = \{x_i, H\} = \Theta_{ij} \frac{\partial H}{\partial x_j}, \quad \text{and} \quad \{x_i, x_j\} = \Theta_{ij}. \]  \hfill (29)

A phase space path integral formulation of the quantum theory corresponding to action (27) is

\[ Z = \int \prod_{k=1}^4 Dx_k e^{iS} = \int \prod_{k=1}^4 Dx_k e^{\int dt(\frac{1}{2} \omega_{ij} \dot{x}_i \dot{x}_j - H(x))}. \]  \hfill (30)

- \( Z \) represents a transition amplitude between two states of a given Hilbert space,
- The time ordering of operators is enforced, as usual, by the path integral,

\[ \int DxO_1O_2e^{iS} = \langle T\{\hat{O}_1\hat{O}_2\} \rangle. \]  \hfill (31)

- Through discretization of the path integral (30) and using the time ordering of operators one derives the commutations relations

\[ [\hat{x}_i, \hat{x}_j] = i\Theta_{ij} = i(\omega^{-1})_{ij} \]  \hfill (32)

and the extended Heisenberg equations of motion

\[ \frac{d}{dt} \hat{x}_i = \Theta_{ij} \frac{\partial H}{\partial x_j} = -i[\hat{x}_i, \hat{H}]. \]  \hfill (33)
Starting with the commutations relations (26), the path integral (30) has been derived showing then the equivalence between the path integral and the operatorial formalisms. For the later formalism, a Schrödinger formulation have been performed as follows.

From the algebra (26) we have $[\hat{q}_2, \hat{p}_1] = 0$, $[\hat{q}_1, \hat{p}_2] = 0$ and that provide a basis, for instance, by the set of eivenvectors of $\hat{q}_1$ and $\hat{p}_2$, $\{|q_1, p_2\rangle\}$, or alternatively $\{|q_2, p_1\rangle\}$. For an arbitrary state $|\psi\rangle$, define the wave function (half in coordinate space, half in momentum space) $\psi(q_1, p_2, t) \equiv \langle \psi(t) | q_1, p_2 \rangle$. The action of the operators $\hat{q}_2$ and $\hat{p}_1$ on the wave function $\psi$ are respectively:

\[
\hat{q}_2 \psi = i(\partial_{p_2} - \theta \partial_{q_1})\psi; \\
\hat{p}_1 \psi = i(-\partial_{q_1} + \sigma \partial_{p_2})\psi.
\] (34) (35)

Using the above settings the transition amplitude is calculated and its path integral expression is derived.

Some comments about the path integral formulation are the following:

- The formulation has been done in (2 + 1) dimensional space-time, considerations can be easily extended to higher dimensional spaces.
- The additional quadratic couplings among phase space variables would not complicate substantially the evaluation of a noncommutative partition function, once the commutative case is under control.
- When the matrix $\Theta$ is singular, the initial two-dimensional problem reduces to the one dimensional one.

Some attempts to introduce path integrals in noncommutative quantum mechanics have been discussed in [34, 35, 43, 76–79].

Reference papers:

13. Ciprian Acatrinei, Path Integral Formulation of Noncommutative Quantum Mechanics, JHEP 0109 (2001) 007.
14. Ciprian Acatrinei, Lagrangian versus Quantization, Journal of Phys. A: Math-Gen vol: 37 issue 4 (2004).

C. Weyl-Wigner formulation (phase space)

Let’s consider in a d-dimensional space with noncommuting position and momentum variables, the extended Heisenberg algebra reads:

\[
[q_i, q_j] = i\hbar \delta_{ij}; \quad [q_i, \hat{p}_j] = i\hbar \delta_{ij}, \quad \delta_{ij} \equiv \delta_{ij}, \quad i, j = 1, \ldots, d,
\] (36)

where $\eta_{ij}$ and $\theta_{ij}$ are antisymmetric real constant ($d \times d$) matrices and $\delta_{ij}$ is the identity matrix. We assume that

\[
\Sigma_{ij} \equiv \delta_{ij} + \frac{1}{\hbar^2} \theta_{ij} \eta_{ij}
\] (37)

is equally an invertible matrix. That means that for any matrix elements $\eta$ and $\theta$, their product is considerably smaller than $\hbar^2$:

\[
\eta \theta << \hbar^2.
\] (38)

The algebra (36) is related to the standard Heisenberg algebra

\[
[q_i, q_j] = 0, \quad [q_i, p_j] = i\hbar \delta_{ij}, \quad [p_i, p_j] = 0, \quad i, j = 1, \ldots d,
\] (39)

via the Seiberg Witten (SW) map as follows

\[
\hat{q}_i = A_{ij} q_j + B_{ij} p_j, \quad \hat{p}_i = C_{ij} q_j + D_{ij} p_j,
\] (40)
where $A, B, C, D$ are real constant matrices. The transformation (40) is assumed to be invertible.

In order to present the Weyl-Wigner formulation of noncommutative quantum mechanics, let’s recall briefly the following.

The non-covariant Weyl-Wigner map is an isomorphism between the operator and the phase space representations of the ordinary quantum mechanics based on the standard Heisenberg algebra providing the simplest approach to derive most of the mathematical structure of conventional phase space quantum mechanics [74, 83, 84]. The covariant extension of this map was studied in [81] in connection with a diffeomorphism invariant formulation of Weyl-Wigner quantum mechanics.

The Weyl-Wigner formulation of noncommutative quantum mechanics relies on the covariant generalization of the Weyl-Wigner transform and the Seiberg-Witten (SW) map. The extended Weyl-Wigner map for noncommutative quantum mechanics is constructed. It is an isomorphism between the operator and phase space representations of the extended algebra [80], providing a systematic approach to derive the phase space formulation of noncommutative quantum mechanics in its most general form. It has been shown that the noncommutative Weyl-Wigner transform, and therefore the entire formulation of noncommutative quantum mechanics does not depend on the particular choice for the SW map.

- The noncommutative Wigner function is obtained by applying the extended Weyl-Wigner transform to the density matrix. It is the noncommutative counterpart of the Wigner function.
- The extended $\star$-product and an extended Moyal bracket are derived.
- The dynamical and eigenvalue equations for noncommutative quantum mechanics are given.

Details about this approach are found in the reference paper:

15. Catarina Bastos, Orfeu Bertolami, Nuno Costa Dias, Joao Nuno Prata, Weyl-Wigner Formulation of non-commutative Quantum Mechanics, J. Math. Phys. 49:072101, 2008.

Further development of the current formulation where a number of issues related to the characterization of the set of states of the theory is adressed in the following paper:

16. C. Bastos, N. C. Dias, and J. N. Prata, Wigner Measures in noncommutative quantum mechanics, Comm. Math. Phys. 299, 709-740, 2010.

D. Systematic formulation

We start by briefly introducing the noncommutative analog of field derivatives [80]. In the non-commutative approach, we consider the Hermitian operators (positions) $\hat{x}_i$, ($i = 1, 2$) satisfying the commutation relations

$$[\hat{x}_i, \hat{x}_j] = i\lambda^2 \epsilon_{ij}, \quad i, j = 1, 2,$$

where $\lambda$ is a positive constant of the dimension of length. The noncommutative analogs of field derivatives are defined as follows

$$\partial_i \hat{\varphi} = \epsilon_{ij} \frac{i}{\lambda^2} [\hat{x}_j, \hat{\varphi}] \quad i = 1, 2,$$

They satisfy the Leinix rule and reduce to the usual derivatives of the commutative limit. Using this definition (42) the momenta operators should be

$$\hat{p}_i = i\hbar \lambda^{-2} \epsilon_{ij} \text{ad}_{\hat{x}_j},$$

$\text{ad}_{\hat{x}_j} \hat{A} \equiv [\hat{x}_j, \hat{A}]$ for any operator $\hat{A}$. It is easy to verify that $[\hat{p}_1, \hat{p}_2] = 0$.

In this section we present a formalism of noncommutative quantum mechanics in complete analogy with conventional quantum mechanics as a quantum system on the Hilbert space of Hilbert Schmidt operators acting on noncommutative configuration space.

In two dimensions, the coordinates of non-commutative configuration space satisfy the commutation relation

$$[\hat{x}_i, \hat{x}_j] = i\theta \epsilon_{ij},$$
with \( \theta \) being a real positive parameter and \( \epsilon_{ij} \) the completely anti-symmetric tensor with \( \epsilon_{1,2} = 1 \).

From (44), we define creation and annihilation operators

\[
b = \frac{1}{\sqrt{2\theta}}(\hat{x} + i\hat{p}), \quad b^\dagger = \frac{1}{\sqrt{2\theta}}(\hat{x} - i\hat{p})
\]

that satisfy the Fock algebra \([b, b^\dagger] = 1\), the noncommutative configuration space is isomorphic to boson Fock space

\[
H_c = \text{span}\{|n\} = \frac{1}{\sqrt{n!}}(b^\dagger)^n|0\rangle\}_{n=0}^{\infty}
\]

where the span is taken over the field of complex numbers.

We consider now the set of Hilbert-Schmidt operators acting on the noncommutative configuration space

\[
H_q = \{\psi(\hat{x}_1, \hat{x}_2) : \psi(\hat{x}_1, \hat{x}_2) \in \mathcal{B}(H_c), \text{tr}_c(\psi(\hat{x}_1, \hat{x}_2)\dagger \psi(\hat{x}_1, \hat{x}_2)) < \infty\},
\]

where \(\text{tr}_c\) denotes the trace over non-commutative configuration space, \(\mathcal{B}(H_c)\) denotes the set of bounded operators on \(H_c\). In other words, the Hilbert space is the trace class enveloping algebra of the classical configuration space Fock algebra \([b, b^\dagger]\). As these operators are necessarily bounded, this is again a Hilbert space (recall that the set of bounded operators on a Hilbert space is again a Hilbert space) and to distinguish the classical configuration space, which is also a Hilbert space, from the quantum Hilbert space we use, respectively, \(c\) and \(q\) as subscripts. We follow the same notation to distinguish operators acting on the classical or quantum Hilbert space. Furthermore we denote states in the quantum Hilbert space by \(|\psi\rangle\) and states in the classical configuration space by \(|\psi\rangle\). The corresponding inner product is \(\langle\psi|\phi\rangle = \langle\psi, \phi\rangle = \text{tr}_c(\psi^\dagger \phi)\), which also serves to define bra states as elements of the dual space (linear functionals). Note that the trace is performed over the classical configuration space, denoted by subscript \(c\). In order to distinguish the notations for Hermitian conjugation between the two Hilbert spaces, we use the notation \(\dagger\) to denote Hermitian conjugation on noncommutative configuration space and the notation \(\hat{\dagger}\) for Hermitian conjugation on quantum Hilbert space.

The abstract Heisenberg algebra is now replaced by the non-commutative Heisenberg algebra. In two dimensions, we have

\[
[x_i, p_j] = i\hbar\delta_{ij}, \\
[x_i, x_j] = i\theta\epsilon_{ij}, \\
p_i p_j = 0.
\]

A unitary representation of this algebra in terms of operators \(\hat{X}_i\) and \(\hat{P}_i\) acting on the quantum Hilbert space \(H_q\) which is the analog of the Schrödinger representation of the Heisenberg algebra is

\[
\hat{X}_i \psi(\hat{x}_1, \hat{x}_2) = \hat{x}_i \psi(\hat{x}_1, \hat{x}_2), \\
\hat{P}_i \psi(\hat{x}_1, \hat{x}_2) = \frac{\hbar}{\theta} \epsilon_{ij} \{\hat{x}_j, \psi(\hat{x}_1, \hat{x}_2)\},
\]

the position operator acts by left multiplication and the momentum operator adjointly.

In the equation \(18\), we have considered a situation where only the coordinates are noncommutative. This was necessary to write down the representation \(19\) which requires commuting momenta to be consistent. We consider now systems in which the momenta are also noncommutative, for instance in the presence of magnetic field, where the set of commutations relations are

\[
[x_i, p_j] = i\hbar\delta_{ij}, \\
[x_i, x_j] = i\theta\epsilon_{ij}, \\
p_i p_j = i\gamma\epsilon_{ij},
\]

In order to apply our formalism, we can bring these commutations relations in equation \(50\) in the same form as \(18\) through an appropriate linear transformation on momenta and coordinates. We consider a transformation to new coordinates \(y_i\) and momenta \(\pi_i\) given by

\[
\begin{align*}
y_i &= x_i, \\
\pi_i &= \alpha p_i + \beta\epsilon_{ij} x_j
\end{align*}
\]
The new coordinates and momenta satisfy the equation \[ \alpha = \frac{\pm \hbar}{\sqrt{\hbar^2 \gamma}} \quad \beta = \frac{\hbar}{\theta} (1 - \alpha). \] (52)

The formalism can be applied now to the new coordinates, while maintaining the potential as a function of coordinates only. There is a critical value of the parameter \( \gamma = \frac{\hbar^2}{\theta} \) such that for \( \gamma < \frac{\hbar^2}{\theta} \) this representation is unitary but for \( \gamma > \frac{\hbar^2}{\theta} \) it loses its unitarity.

With the notions that we set, one can proceed with the normal quantum mechanical interpretation in the quantum Hilbert space \( \mathcal{H}_q \).

Reference papers:

17. Formulation, Interpretation and Application of non-commutative Quantum Mechanics, F. G. Scholtz, L. Gouba, A. Hafver, C. M. Rohwer, J. Phys. A 42:175303, 2009.

The present formulation has been applied succesfully to much more difficult potentials such as the spherical well. A precise meaning to piecewise constant potential in noncommutative quantum mechanics is given using this formulation. More details can be found in the following paper:

18. F. G. Scholtz, B. Chakraborty, J. Govaerts, S. Vaidya, Spectrum of the non-commutative spherical well, J. Phys. A Math-theor. 40. Issue: 48 Pages: 14581-14592, 2007.

An unambiguous formulation of the path integral representation for the propagator has been presented using this formulation. An action for a particle moving in the non commutative plane and in the presence of an arbitrary potential is derived. More details can be found in the following paper:

19. S. Gangopadhyay, F. G. Scholtz, Path-Integral Action of a Particle in the Noncommutative Plane, PRL Volume: 102. Issue: 24 Article Number: 241602, 2009.

The Gazeau-Klauder coherent states in noncommutative quantum mechanics have been studied using the this formulation. The inherent vector feature of these states has been revealed. Details are found in:

20. J. Ben Geloun and F. G. Scholtz, Coherent states in noncommutative quantum mechanics, J. Math. Phys. 50, 043505, 2009.

This formulation has been used to present a description of noncommutative quantum mechanics which may be viewed in terms of spatially extended objects. More details about this description is found in the following paper:

21. C. M. Rohwer, K. G. Zloshchastiev, L. Gouba, F. G. Scholtz, Noncommutative quantum mechanics-a perspective on structure and spatial extent, J. Phys. A Math-Theor. 43 Issue: 34, 2010.

III. CONCLUDING REMARKS

Noncommutative quantum mechanics represents a natural extension of usual quantum mechanics, in which one allows nonvanishing commutators also between the coordinates and between the momenta in general. The notion of coordinates basis and the very concept of wave functions \( \langle x, \psi \rangle \) fails. However the usual momentum space description is still valid in case the momenta commute but in the general case where the momenta do not commute it does not hold.

Various formulations of noncommutative quantum mechanics have been constructed and we attempt to list them in this paper.

For the canonical formulation that assemble the new coordinates system, the Bopp’s shift method, the star product method and the Seiberg Witten map, the procedure is to relate the extended Heisenberg algebra to the standard Heisenberg algebra by a class of linear transformations. The Bopp’s shift method is equivalent to the star product method and it is considered as an older version of the Seiberg-Witten map. The problem with the canonical formulation is that both the observables and the states beings written in terms of the Heisenberg variables do not display a simple mathematical structure.
This tends to obscure the physical meaning. Shifting the potential involves in principle arbitrary large number of derivations in the Schrödinger equation.

For the path integral formulation, the procedure is to provide a phase space path integral, starting from a classical action, that is consistent with the deformed (extended) Heisenberg algebra. The equivalence between the path integral and the operatorial formulation has been proved. Despite the claims that noncommutativity of coordinates may bar having a Lagrangian description [85], there exits such a formulation in [35], where an effective Lagrangian has been derived from a path integral approach and quasi-classical approximation.

For the Weyl-Wigner formulation, the procedure is to use the Seiberg-Witten map and the covariant generalization of the Weyl-Wigner transform to construct an isomorphism between the operator and the phase space representations of the extended Heisenberg algebra. This formulation is useful for treating general problems such as, for instance, in case where the potential is not specified.

For the systematic formulation, the procedure is to formulate noncommutative quantum mechanics, in complete analogy with commutative quantum mechanics, as a quantum system on the Hilbert space of Hilbert-Schmidt operators acting on noncommutative configuration space. This approach presents a consistent formulation and interpretational framework for non-commutative quantum mechanics, which includes an unambiguous description for position measurement.

Revising the four formulations, the path integral and the systematic approaches do not use a change of coordinates (⋆-product, Bopp's shift, SW maps). However, in a general setting where in addition the momenta do not commute we have the following situations: the path integral approach still holds in case there are some commuting phase space coordinates left so that a complete basis in the Hilbert space of the theory is provided by the set of the eigenvectors of the commuting phase space coordinates; for the systematic approach appropriate linear transformations on momenta and coordinates are needed.

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