BRANCHING PROBLEMS IN REPRODUCING KERNEL SPACES

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Abstract. For a semisimple Lie group $G$ satisfying the equal rank condition, the most basic family of unitary irreducible representations is the discrete series found by Harish-Chandra. In this paper we study some of the branching laws for these when restricted to a subgroup $H$ of the same type by combining classical results with recent work of T. Kobayashi; in particular, we prove discrete decomposability under Harish-Chandra’s condition of cusp form on the reproducing kernel.

1. Introduction

Restricting a unitary irreducible representation $\pi$ of a Lie group $G$ to a closed subgroup $H$ leads to the branching law problem, namely of finding the explicit decomposition of $\pi$ into irreducible representations of $H$. This generalizes the theory of spectral decomposition of a selfadjoint operator, and in the same way there may occur both a discrete and a continuous spectrum.

In this paper we shall consider the case of $G$ a semisimple Lie group with $\pi$ a representation in the discrete series, i.e. it occurs as a left-invariant closed subspace of $L^2(G)$; these form the celebrated discrete series of Harish-Chandra. $H$ will be a subgroup of the same type, and we shall assume that both $G$ and $H$ admit discrete series, and our aim is to find criteria ensuring that the branching law for $\pi$ gives discrete series for $H$ - perhaps even that the restriction of $\pi$ is a direct sum of such, in which case we call this discretely decomposable, and if this happens with finite multiplicities we call it $H$-admissible. We shall combine the results of Harish-Chandra on the distribution character and also the Plancherel formula for $G$ with recent results of T. Kobayashi on the admissibility of representations in order to obtain some new results on branching laws in our setting. We shall also apply the theory of reproducing kernels, using some specific models of the discrete series, this involves the spherical functions, also studied by Harish-Chandra.

Note that a special case of our situation could be with $H$ as the diagonal subgroup in $G \times G$, so that the branching problem corresponds to decomposing a tensor product; this is already a complicated problem for discrete series, and we hope our approach can lead to a deeper understanding (as an easy, yet not completely trivial consequence, we see that $\pi \otimes \pi^*$ contains continuous spectrum).

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There are (at least) two basic facts from Harish-Chandra that we use: (1) convolution by the distribution character of \( \pi \) gives the orthogonal projection onto \( \pi \), and (2) the trace of the spherical function for the lowest \( K \)-type \( W \) is the convolution (over \( K \)) of the distribution character and the character of \( W \). Our interest is only within the category of unitary representations and spectral decomposition in Hilbert spaces; thus we do not consider the smooth category and the corresponding symmetry-breaking operators of T. Kobayashi. For this note, a symmetry breaking operator is a continuous \( H \)-map from a representation of \( G \) into one of \( H \).

We now mention some of the main results in this paper:

This note is divided in eleven sections. Some of the subsections are quite technical, for example 2.1, 3.2. In subsection 3.1 we show that under certain hypotheses symmetry breaking operators agree on dense subspaces with integral operators. In all the sections, we use the fact that discrete series representations might be modeled on space of solutions to elliptic differential operators. For example, we carry out an analysis of symmetry breaking operators as differential operators (subsection 4.1, 4.2, 4.3). In particular, we show that \( H \)-admissibility implies that a certain family of symmetry breaking operators are represented by differential operators. We also show a converse statement. It is a quite different situation when we analyze projectors onto isotypic components (Examples 6.2, 6.3); under the hypothesis of admissibility, we may show projectors are represented via truncated Taylor series (section 6) or some version of infinite order differential operators. With respect to necessary and sufficient conditions for a representation to be \( H \)-discretely decomposable, we obtain criteria saying that some particular symmetry breaking operators are restrictions of differential operators (Theorem 4.3) or in terms of reproducing kernels and cusp forms of Harish-Chandra (Theorem 7.1). We also show that in the family of discrete series representations unitary discretely decomposable implies algebraic discretely decomposable. In section 8 we introduce two functions that are suitable to check the existence of discrete factors (Theorem 8.4). In two appendices we collect results on integral operators and kernel of elliptic invariant differential operators, as well as notation. Main results in this note are: Theorem 3.5, 4.3 and its corollary, Theorem 4.12, Propositions 6.6, 6.7, Theorem 7.1.

2. Preliminaries and some results

2.1. Discrete Series and Reproducing kernel. Let \( G \) be an arbitrary, matrix, connected semisimple Lie group. Henceforth we fix a maximal compact subgroup \( K \) for \( G \) and a maximal torus \( T \) for \( K \). Harish-Chandra showed that \( G \) admits square integrable irreducible representations if and only if \( T \) is a Cartan subgroup of \( G \). For this note, we always assume \( T \) is a Cartan subgroup of \( G \). Under these hypothesis, Harish-Chandra showed that the set of equivalence classes of irreducible square integrable representations is parameterized by a lattice in \( \mathfrak{h}^*_{\mathbb{R}} \). In order to state our results we need to make explicit this parametrization and set up some notation. As usual, the Lie algebra of a Lie group is denoted by the corresponding lower case German letter followed by the subindex \( \mathbb{R} \). The complexification of the Lie algebra of a Lie group is denoted by the corresponding German letter without any subscript.
\( V^* \) denotes the dual space to a vector space \( V \). Let \( \theta \) be the Cartan involution which corresponds to the subgroup \( K \), the associated Cartan decomposition is denoted by \( \mathfrak{g} = \mathfrak{k} + \mathfrak{p} \). Let \( \Phi(\mathfrak{g}, \mathfrak{t}) \) denote the root system attached to the Cartan subalgebra \( \mathfrak{t} \). Hence, \( \Phi(\mathfrak{g}, \mathfrak{t}) = \Phi_c \cup \Phi_n = \Phi_c(\mathfrak{g}, \mathfrak{t}) \cup \Phi_n(\mathfrak{g}, \mathfrak{t}) \) splits up as the union the set of compact roots and the set of noncompact roots. From now on, we fix a system of positive roots \( \Delta \) for \( \Phi_c \). As usual, the highest weight \( \lambda \) for an irreducible representation of \( K \) is denoted by \( t(\lambda) \).

From now on, we fix a system of positive roots \( \Delta \) for \( \Phi(\mathfrak{g}, \mathfrak{t}) \). A Harish-Chandra parameter for \( G \) is \( \lambda \in \mathfrak{t}_c^* \) such that \( (\lambda, \alpha) \neq 0 \), for every \( \alpha \in \Phi(\mathfrak{g}, \mathfrak{t}) \), and so that \( \lambda + \rho \) lifts to a character of \( T \). To each Harish-Chandra parameter \( \lambda \), Harish-Chandra, associates a unique irreducible square integrable representation \( (\pi^G_\lambda, V^G_\lambda) \) of \( G \) of infinitesimal character \( \lambda \). Moreover, he showed the map \( \lambda \to (\pi^G_\lambda, V^G_\lambda) \) is a bijection from the set of Harish-Chandra parameters \( \Phi(\mathfrak{g}, \mathfrak{t}) \) onto the set of equivalence classes of irreducible square integrable representations for \( G \). For short, we will refer to an irreducible square integrable representation as a discrete series representation.

Each Harish-Chandra parameter \( \lambda \) gives rise to a system of positive roots \( \Psi_\lambda = \Psi_{G,\lambda} = \{ \alpha \in \Phi(\mathfrak{g}, \mathfrak{t}) : (\lambda, \alpha) > 0 \} \).

From now on, we assume that Harish-Chandra parameter for \( G \) are dominant with respect to \( \Delta \). Whence, \( \Delta \subset \Psi_\lambda \).

Henceforth, \( \Psi \) is a system of positive roots for \( \Phi(\mathfrak{g}, \mathfrak{t}) \) containing \( \Delta \), and \( (\pi^G_\lambda, V^G_\lambda) \) a square integrable representation for \( G \) of Harish-Chandra parameter \( \lambda \) dominant with respect to \( \Psi \). \( (\tau, W) := (\pi^G_{\lambda + \rho_n}, V^G_{\lambda + \rho_n}) \) denotes the lowest \( K \)-type of \( \pi_\lambda := \pi^G_\lambda \). The highest weight of \( (\pi^G_{\lambda + \rho_n}, V^G_{\lambda + \rho_n}) \) is \( \lambda + \rho_n - \rho_c \). We recall a theorem of Vogan’s thesis which states that \( (\tau, W) \) determines \( (\pi_\lambda, V^G_\lambda) \) up to unitary equivalence. We recall the set of square integrable sections of the vector bundle determinate by the principal bundle \( K \to G \to G/K \) and the representation \( (\tau, W) \) of \( K \) is isomorphic to the space

\[
L^2(G \times \tau W) := \{ f \in L^2(G) \otimes W : f(gk) = \tau(k)^{-1}f(g), g \in G, k \in K \}.
\]

Here, the action of \( G \) is by left translation \( L_x, x \in G \). The inner product on \( L^2(G) \otimes W \) is given by

\[
(f, g)_{V_\lambda} = \int_G (f(x), g(x))_W dx
\]

where \( (\ldots, \ldots)_W \) is a \( K \)-invariant inner product on \( W \). Subsequently, \( L_D \) (resp. \( R_D \)) denotes the left infinitesimal (resp. right infinitesimal) action on functions from \( G \) of an element \( D \) in universal enveloping algebra \( U(\mathfrak{g}) \) for the Lie algebra \( \mathfrak{g} \). As usual, \( \Omega_G \) denotes the Casimir operator for \( \mathfrak{g} \). Following Hota, Enright-Wallach [OV], we realize \( V^G_\lambda := V^G_\lambda \) as the space

\[
H^2(G, \tau) = \{ f \in L^2(G) \otimes W : f(gk) = \tau(k)^{-1}f(g) \}
\]

\[
g \in G, k \in K, R_{\Omega_G}f = [(\lambda, \lambda) - (\rho, \rho)]f \}.\]
We also recall, \( R_{\Omega G} = L_{\Omega G} \) is an elliptic \( G \)-invariant operator on the vector bundle \( W \to G \times G \to G/K \) and hence, \( H^2(G, \tau) \) consists of smooth sections, moreover point evaluation \( e_x \) defined by \( H^2(G, \tau) \ni f \mapsto f(x) \in W \) is continuous for each \( x \in G \) (cf. Appendix 10.4). Therefore, the orthogonal projector \( P_\lambda \) onto \( H^2(G, \tau) \) is an integral map (integral operator) represented by the smooth matrix kernel or reproducing kernel Appendix 10.6

\[
K_\lambda : G \times G \to \text{End}_\mathbb{C}(W)
\]

which satisfies \( K_\lambda(\cdot, x)^*w \) belongs to \( H^2(G, \tau) \) for each \( x \in G, w \in W \) and

\[
(P_\lambda(f)(x), w)_W = \int_G (f(y), K_\lambda(y, x)^*w)dy, \quad x \in G, \quad w \in W, \quad f \in L^2(G \times G, W).
\]

For a close reductive subgroup \( H \), after conjugation by an inner automorphism of \( G \) we may and will assume \( L := K \cap H \) is a maximal compact subgroup for \( H \). That is, \( H \) is \( \theta \)-stable. In this note for irreducible square integrable representations \( (\pi_\lambda, V_\lambda) \) for \( G \) we would like to analyze its restriction to \( H \). In particular, we study the irreducible \( H \)-subrepresentations for \( \pi_\lambda \). A result is that any irreducible \( H \)-subrepresentation of \( V_\lambda \) is an square integrable representation for \( H \), for a proof (cf. [GW]). Thus, owing to the result of Harish-Chandra on the existence of square integrable representations it allows us that from now on we may and will assume \( H \) admits a compact Cartan subgroup. After conjugation we may assume \( U := H \cap T \) is a maximal tori in \( L = H \cap K \). Next, we consider a square integrable representation \( H^2(H, \sigma) \subset L^2(H \times \sigma, Z) \) of lowest \( L \)-type \( (\sigma, Z) \). An aim of this note is to understand the nature of the intertwining operators between the unitary \( H \)-representations \( H^2(H, \sigma) \) and \( V_\lambda^G \), the adjoint of such intertwining operators and consequences of their structure.

**Notation:** \( \mathbb{N} = \{1, 2, \ldots \} \)

3. Structure of Intertwining Maps

For this section, besides \( G, K, T, (\tau, W), (L, H^2(G, \tau)) = (\pi_\lambda^G, V_\lambda^G), H, L, U \) as in Subection 2.1. We fix \( (\nu, E) \) a finite dimensional representation for \( L \) and a continuous intertwining linear \( H \)-map \( T : L^2(H \times \nu, E) \to H^2(G, \tau) \).

**Fact 3.1.** We show \( T \) is a kernel map.

In fact, for each \( x \in G, w \in W \) the linear function \( L^2(H \times \nu, E) \ni g \mapsto (Tg(x), w)_W \) is continuous. Whence, the Riesz representation Theorem shows there exists a function

\[
K_T : H \times G \to \text{Hom}_\mathbb{C}(E, W)
\]

so that for each \( x \in G, w \in W \) the map \( h \mapsto K_T(h, x)^*w \) belongs to \( L^2(H \times \nu, E) \) and for \( g \in L^2(H \times \nu, E), w \in W \) we have the absolutely convergent integral and the equality

\[
(Tg(x), w)_W = \int_H (g(h), K_T(h, x)^*w)h dh.
\]

That is, \( T \) is the integral map

\[
Tg(x) = \int_H K_T(h, x)g(h)dh, \quad x \in G.
\]
We also have \((T(g)(x), w)_W = \int_G (T(g)(y), K_\lambda(y, x)^* w) dy\).

3.1. **Symmetry breaking operators.** Examples shows that the adjoint \(T^*\) of integral linear map \(T\) need no be an integral map, whence, we would like to know when \(T^*\) is an integral linear map. Formally, we may write \(T^* f(h) = \int_G K_T(h, x)^* f(x) dx\), where the convergence of the integral is in the weak sense. That is, for each \(g \in L^2(H \times E, E)\), \(f \in H^2(G, \tau)\), we have the absolutely convergence of the iterated integral

\[(T^* f, g)_{L^2(H \times E)} = \int_G \int_H (f(x), K_T(h, x) g(h)) dhdx.

Thus, the adjoint of an integral map is a weak integral map.

In order to study branching problems, T. Kobayashi, has introduced the concept of symmetry breaking operator. In our setting, a **symmetry breaking operator** is continuous \(H\)–map \(S : H^2(G, \tau) \to L^2(H \times E)\). For a symmetry breaking operator \(S\), the above considerations applied to \(T := S^*\) let as conclude: Under our hypothesis, a symmetry breaking operator is always a weak integral map.

In [Fo] page 45 we find an exposition on Bargmann transform \(B : L^2(\mathbb{R}^n) \to \mathcal{H}_2(C^n)\). \(B\) is an example of integral map, such that \(B^*\) may not an integral map. However, Bargmann has shown \(B^*\) restricted to certain dense subspace is an integral map. In Apendix 10.3 we provide an example of a continuous integral map into a reproducing kernel space such that its adjoint is not an integral map on the whole space. However, it is equal to an integral on certain subspace. Theorem 3.5 shows that a quite common feature for the kind of integral maps \(T\) under our consideration its adjoint \(T^*\) is an integral map on certain dense subspace of \(H^2(G, \tau)\). In order to state our results we need a few definitions.

**Definition 3.2.** A representation \((\pi_\lambda, V_\lambda)\) is discretely decomposable over \(H\), if there exists an orthogonal family of closed, \(H\)–invariant, \(H\)–irreducible subspaces of \(V_\lambda\) so that the closure of its algebraic sum is equal to \(V_\lambda\).

**Definition 3.3.** A representation \((\pi_\lambda, V_\lambda)\) is \(H\)–admissible if the representation is discretely decomposable and the multiplicity of each irreducible factor is finite.

In [KO] we find a complete list of triples \((G, H, \pi_\lambda)\) such that \((G, H)\) is a symmetric pair and \(\pi_\lambda\) is an \(H\)–admissible representation. For example, for the pair \((SO(2n, 1), SO(2k) \times SO(2n - 2k, 1))\) there is no \(\pi_\lambda\) with an admissible restriction to \(H\). Whereas, for the pair \((SU(m, n), SU(m, k) \times U(n - k))\) there are exactly \(m\) Weyl chambers \(C_1, \ldots, C_m\) in \(i\pi^*_R\) so that \(\pi_\lambda\) is \(H\)–admissible if and only \(\lambda\) belongs to \(C_1 \cup \cdots \cup C_m\).

**Definition 3.4.** A unitary representation \((\pi, V)\) is integrable, if some nonzero matrix coefficient is an integrable function.

It is a Theorem of Trombi-Varadarajan, Hecht-Schmid that: \((\pi_\lambda, V_\lambda)\) is an integrable representation if and only if \(|(\lambda, \beta)| > \sum_{\alpha \in \Phi(g, t); (\alpha, \beta) > 0} \alpha(\beta)\) for every noncompact root \(\beta\).

The next result gives more information on symmetry breaking operators.
Theorem 3.5. Let $S : H^2(G, \tau) \rightarrow L^2(H \times_v E)$ be a continuous intertwining linear $H-$map. Then,

a) If the restriction to $H$ of $(L, H^2(G, \tau))$ is discretely decomposable, then $S$ is an integral map.

b) If $(\pi_\lambda, V_\lambda)$ is an integrable representation, then $S$ restricted to the subspace of smooth vectors is an integral linear map.

In Example 3.11 we present a symmetry breaking operator so that it is equal to an integral map on the subspace of $K-$finite vectors in certain non integrable representation $H^2(G, \tau)$.

3.2. Operators from $L^2(H \times_v E)$ into $H^2(G, \tau)$. In this somewhat technical subsection we assemble properties of the matrix kernel that represent an $H-$maps as the one in the title and its adjoint. In particular, in the next Proposition, we obtain two estimates for the kernel. The second estimate yields that the adjoint map is a kernel map on the subspace of smooth vectors. To follow, we make explicit some properties of the kernel of $T$.

Proposition 3.6. Let $T : L^2(H \times_v E) \rightarrow H^2(G, \tau)$ be a continuous intertwining linear $H-$map. Then, the function $K_T$ satisfies:

a) $K_T(h, x)w = T^*(y \mapsto K_\lambda(y, x)^*w)(h)$.

b) The function $h \mapsto K_T(h, e)^*w$ is an $L-$finite vector in $L^2(H \times_v E)$.

c) $K_T$ is a smooth map. Further, $K_T(\cdot, x)^*w$ is a smooth vector.

d) There exists a constant $C$ and finitely many functions $\phi_{a,b} : G \rightarrow \mathbb{C}$ so that for every $x \in G$, $\|K_T(e, x)^*\|_{Hom_{\mathcal{G}}(W, Z_\mathcal{G})} \leq C\|T^*\| \sum_{a,b} |\phi_{a,b}(x)|$.

e) $\|K_T(\cdot, x)^*\|_{L^2(H \times_v \mathcal{G}, \omega_{Hom_{\mathcal{G}}(W, Z_\mathcal{G})})}$ is a bounded function on $G$.

f) $K_T(hh_1 s, h x k) = \tau(k^{-1}) K_T(h_1, x) \nu(s)$, $x \in G$, $s \in L$, $h, h_1 \in H$, $k \in K$.

g) If $T^*$ is a kernel map, with kernel $K_{T^*} : G \times H \rightarrow Hom(W, \mathcal{G})$ and $K_{T^*}(\cdot, h)^*z \in H^2(G, \tau)$. Then, $L_{D}^2 K_{T^*}(\cdot, \cdot) = \chi_\lambda(D) K_{T}(h, \cdot)$ for every $D$ in the center of $U(\mathcal{G})$. Here, $\chi_\lambda$ is the infinitesimal character of $\pi_\lambda$.

Note: The functions $\phi_{a,b}$ are defined as follows. We fix linear basis $\{X_{a}\}_{1 \leq a \leq N}$ (resp. $\{Y_{a}\}_{1 \leq a \leq M}$) for the space of elements in $U(\mathfrak{h})$ of degree less or equal than $\dim \mathfrak{h}$ (resp. for the space of elements in $U(\mathcal{G})$ of degree less or equal than $\dim \mathfrak{h}$). Then $\phi_{a,b}$ are defined by $Ad(x^{-1})(X_{b}) = \sum_{1 \leq a \leq M} \phi_{a,b}(x)Y_{a}$, $b = 1, \ldots, N$.

Proof of Proposition 3.6. A straightforward computation and using the invariance of Haar measure gives 1). To show b) we notice $K_T(s^{-1} h, e)^*w = K_T(h, s)^*w = (\tau(s^{-1}) K_S(h, e)^*w = K_T(h, e)^*\tau(s)w$. Since, $W$ is a finite dimensional vector space, the span of $\{L_{s}(h \mapsto K_T(h, e)^*w) : s \in L\}$ is finite dimensional.

To follow we show the a). Let $g \in L^2(H \times_v E)$ arbitrary, we have

\[
(g(\cdot), K_T(\cdot, y)^*w)(L^2(H \times_v E)) = (Tg(y), w)w \\
= (Tg(\cdot), K_\lambda(\cdot, y)^*w(L^2(G \times_v W)) \\
= (g(\cdot), T^*(t \mapsto K_\lambda(t, y)^*w(\cdot))(\cdot))L^2(H \times_v E).
\]

The first equality is justified by 3.1, the second by $P_{\lambda}(Tg) = Tg$ and (2.1), the third by definition of adjoint linear map. Since the functions
$v \mapsto K_T(v, y)^*(w), v \mapsto [T^*(t \mapsto K_{\lambda}(t, y)^*(w))](v)$ belong to $L^2(H \times \nu, E)$ and $g$ is arbitrary in $L^2(H \times \nu, E)$ we obtain a).

For further use we denote the subspace of smooth vectors in a representation $V$ by $V^\infty$, some times, to be more explicit, we write $V^{H-mfg}$ for the $H$--smooth vectors.

Next, we verify the map $K_T$ is smooth. For this we show $(h, x) \mapsto K_T(h, x)^*w$ is smooth for each $w \in W$. We first notice $K_{\lambda}(y, x) = K_{\lambda}(e, y^{-1}x)$ and $x \mapsto K_{\lambda}(x, e)^*w$ is a $K$--finite vector in $H^2(G, \tau)$. Thus, $y \mapsto K_{\lambda}(y, x)^*w = L_x(y \mapsto K_{\lambda}(y, e)^*w)$ is a smooth vector in $H^2(G, \tau)$ and hence the map $G \ni x \mapsto (G \ni y \mapsto K_{\lambda}(y, x)^*w) \in H^2(G, \tau)$ is smooth. Since $T^*$ is a continuous linear $H$--map, we have that $T^*(y \mapsto K_{\lambda}(y, x)^*w)(\cdot) \in L^2(H \times \nu, E)$. Hence, $G \ni x \mapsto T^*(y \mapsto K_{\lambda}(y, x)^*w) \in L^2(H \times \nu, E)$ is a smooth map. Next, [Wa1, section 1.6]), we endow the space of smooth vectors in a representation $(\pi, V)$ with the topology that a sequence of smooth vectors $v_n$ converges to a smooth vector $v$ iff $\pi(D)v_n$ converges to $\pi(D)v$ in norm for every $D \in U(\mathfrak{h})$. Poulsen [Po, Proposition 5.1], has shown that the space of smooth vectors $L^2(H \times \nu, E)$ in $L^2(H \times \nu, E)$ is the subspace of smooth functions $f$ so that $L_D(f)$ is square integrable for every $D \in U(\mathfrak{h})$. Further, Poulsen showed point evaluation from $L^2(H \times \nu, E)$ into $Z$ is a continuous linear map. Whence, the the following composition gives a smooth map $(h, x) \mapsto h^{-1}x \mapsto T^*(v \mapsto K_{\lambda}(v, h^{-1}x)^*w)(e)$. Now a) and the equality $b_T(e, h^{-1}x) = h_T(h, x)$ concludes the proof of c).

Subsequently we show d). We compute an upper bound for $x \mapsto K_T(e, x)^*w$. To start with we recall Sobolev’s inequality as written by Poulsen [Po, Lemma 5.1]. We fix a linear basis $\{X_b\}_{1 \leq b \leq N}$ for the elements in $U(\mathfrak{h})$ of degree less and equal to dim $\mathfrak{h}$. For a smooth vector $f \in L^2(H \times \nu, E)$ we have:

$$\|f(e)\|_Z \leq \sum_{1 \leq b \leq N} \|L_{X_b}(f)\|_{L^2(H)}.$$ 

We fix $D \in U(\mathfrak{h})$, then owing to $K_{\lambda}(\cdot, x)^*w$ is a smooth vector in $H^2(G, \tau)$ and a), we obtain $K_T(\cdot, x)^*w$ is an smooth vector and

$$L_D^{(1)}K_T(\cdot, x)^*w = L_D(T^*(K_{\lambda}(\cdot, x)^*w))(h) = T^*(L_D^{(1)}K_{\lambda}(\cdot, x)^*w)(h).$$

Thus, $\|L_D^{(1)}K_T(\cdot, x)\|_{L^2(H)} \leq \|T^*\| \|L_DK_{\lambda}(\cdot, x)^*w\|_{L^2(G)}$.

Now, for every $D \in U(\mathfrak{g})$, the equality $K_{\lambda}(y, x) = K_{\lambda}(x^{-1}y, e)$ together with the left invariance of Haar measure yields

$$\|L_DK_{\lambda}(\cdot, x)^*w\|_{L^2(G)} = \|L_{Ad(x^{-1})D}K_{\lambda}(\cdot, e)^*w\|_{L^2(G)}.$$ 

We fix a basis $\{Y_a\}_{1 \leq a \leq M}$ for the subspace of elements of $U(\mathfrak{g})$ of degree less or equal to dim $\mathfrak{h}$. We assume deg $D \leq$ dim $\mathfrak{h}$ and we write $Ad(x^{-1})D = \sum_a \phi_a(x, D)Y_a$. Whence,

$$\|L_DK_{\lambda}(\cdot, x)^*w\|_{L^2(G)} \leq \sum_a |\phi_a(x, D)||L_{Y_a}K_{\lambda}(\cdot, e)^*w|_{L^2(G)}.$$ 

For $D = X_b$ we write $\phi_{a, b}(x) := \phi_a(x, X_b)$. Let $C$ denote an upper bound for the numbers $\|L_{Y_a}K_{\lambda}(\cdot, e)^*w\|_{L^2(G)}$ for $1 \leq a \leq M$. Then, Sobolev’s inequality
and the previous inequalities gives
\[
\|K_T(e, x)^* w\|_W \leq \|T^*\| \sum_{1 \leq b \leq N} \|L_{X_b} K_\lambda(\cdot, x)^* w\|_{L^2(G)}
\]
\[
= \|T^*\| \sum_b \|L_{Ad(x^{-1}) X_b} K_\lambda(\cdot, e)^* w\|_{L^2(G)} \leq C \|T^*\| \|w\| \sum_{a,b} |\phi_{a,b}(x)|.
\]
\[
\|K_T(e, x)^*\|_{\text{Hom}(W, Z)} = \sup_{|w| \leq 1} \{\|K_T(e, x)^* w\|_W\} \leq C \|T^*\| \sum_{a,b} |\phi_{a,b}(x)|.
\]
To follow, we verify e). That is, the function \(\|K_T(\cdot, x)^*\|_{L^2(H)}\) is a bounded, for this we verify an inequality
\[
\|K_T(\cdot, x)^*\|_{L^2(H)} \leq \|K_\lambda(\cdot, e)\|_{L^2(G)} \|T\|.
\]
To begin with, for \(w \in W\), we have
\[
\|K_T(\cdot, x)^* w\|_{L^2(H)} \leq \|w\| \|K_\lambda(\cdot, e)\|_{L^2(G)} \|T\|. \quad \text{Indeed,}
\]
\[
\int_H \|K_T(h, x)^* w\|_Z^2 dh = \int_H <K_T(h, x)^* w, K_T(h, x)^* w>_E dh
\]
\[
= (T(K_T(\cdot, x)^* w)(x), w)_W
\]
and
\[
(T(K_T(\cdot, x)^* w)(x), w)_W = \int_G (T(K_T(\cdot, x)^* w)(y), K_\lambda(y, x)^* w)_W dy
\]
\[
\leq \|T((K_T(\cdot, x)^* w))_{L^2(G)}\| \left(\int_G \|K_\lambda(y, x)^* w\|_W^2 dy\right)^{1/2}
\]
\[
\leq \|w\| \|K_\lambda(\cdot, e)\|_{L^2(G)} \|T\| \|K_T(\cdot, x)^* w\|_{L^2(H)}.
\]
Therefore, after we simplify, we obtain
\[
\int_H \|K_T(h, x)^* w\|_Z^2 dh \leq \|w\| \|K_\lambda(\cdot, e)\|_{L^2(G)} \|T\| \|K_T(\cdot, x)^* w\|_{L^2(H)}.
\]
Thus,
\[
\|K_T(\cdot, x)^* w\|_{L^2(H)} \leq \|w\| \|K_\lambda(\cdot, e)\|_{L^2(G+x, x^* \otimes \text{Hom}_C(W, W))} \|T\|.
\]
To continue, we fix an orthonormal basis \(\{z_j\}\) (resp. \(\{w_i\}\)) for \(E\) (resp. for \(W\)). Then, there exists a constant \(C_1\), depends on \(E, W\), so that for each linear map \(R : W \rightarrow E\) we have,
\[
\|R\|_{\text{Hom}(W, E)} = \sup\{w : \|w\|_W \leq 1\} \{\|R(w)\|_E\} \leq C_1 \sqrt{\sum_{i,j} |\langle R w_i, z_j \rangle_E|^2}.
\]
\[
\left(\|K_T(\cdot, x)^*\|_{L^2(H+x, x^* \otimes \text{Hom}_C(W, E))}\right)^2
\]
\[
\leq C_1^2 \int_H \sum_{i,j} |(K_T(h, x)^* w_i, z_j)_E|^2 dh 
\leq C_1^2 \dim E \int_H \|K_T(h, x)^* w_i\|_E^2 dh
\leq C_1^2 \dim E \dim W \|K_\lambda(\cdot, e)\|_{L^2(G)} \|T\|^2.
\]
Whence, we obtain
\[
\|K_T(\cdot, x)^*\|_{L^2(H+x, x^* \otimes \text{Hom}_C(W, E))}
\leq C_1 \sqrt{\dim E \dim W \|K_\lambda(\cdot, e)\|_{L^2(G \times \text{Hom}(W, W))}} \|T\|.\]
3.3. Proof of Theorem 3.5.

Proof. In order to show a) we recall that Schur’s Lemma implies that for a closed $H$–irreducible subspace $N$ of $H^2(G, \tau)$ either $S(N) = \{0\}$ or $S(N)$ is a closed $H$–irreducible subspace. Thus, our hypothesis forces the image of $S$ is a sum of $H$–irreducible subspaces of $L^2(H \times_\nu E)$. Plancherel’s Theorem of Harish-Chandra, shows that closure of the sum of the $H$–irreducible subspaces in $L^2(H \times_\nu E)$, is actually, a finite sum of irreducible subspaces [HC2, Lemma 72] and Frobenius reciprocity. From now on, $L^2(H \times_\nu E)_{\text{disc}}$ denotes the sum of $H$–invariant irreducible subspaces in $L^2(H \times_\nu E)$. On the smooth vectors of an $H$–irreducible subspace $N$ the Casimir operator acts by a constant, since smooth vectors are smooth functions, the Casimir operator acts by the same constant in the whole of $N$. In [At] we find a proof that an $L^2$–eigenspace of an elliptic operator on a fiber bundle is a reproducing kernel subspace. Finally, the Casimir operator of $h$ acts as an elliptic operator on $L^2(H \times_\nu E)$. Thus, we conclude $L^2(H \times_\nu E)_{\text{disc}}$ is a finite sum of reproducing kernel subspaces and hence $L^2(H \times_\nu E)_{\text{disc}}$ is a reproducing kernel subspace. Whence, the image of $S$ is contained in a reproducing kernel subspace, which let us conclude $S$ is a kernel map. It readily follows the kernel of $S$ is equal to the kernel $K^*_S$.

Subsequently we show b) The hypothesis $S$ is a continuous $H$–map and Proposition 3.6 let us conclude that $S^*$ is an integral map. Thus, at least formally, we think $S$ as the adjoint of an integral map. For this, we formally define

$$S_0(f)(h) := \int_G K_{S^*}(h, x)^* f(x) \, dx$$

and we consider the subspace

$$\mathcal{D}_{K^*_S} := \{ f \in L^2(G \times_\tau W) : S_0(f) \in L^2(H \times_\nu E) \}.$$

It readily follows that $S_0$ restricted to $\mathcal{D}_{K^*_S}$ is an integral operator with kernel $K_{S_0}(x, h) = K_{S^*}(h, x)^*$. Besides, $S$ is an integral map when restricted to $H^2(G, \tau) \cap \mathcal{D}_{K^*_S}$. To follow, we construct a subspace $\mathcal{D}$ of $\mathcal{D}_{K^*_S}$,

$$\mathcal{D} := \{ f \in L^2(G \times_\tau W) : \int_G \|K_{S^*}(\cdot, x)^*\|_{L^2(H \times_\nu \text{Hom}(W, E))} \|f(x)\| dx < \infty \}$$

For this, we show that for $f \in \mathcal{D}$, the integral that defines the function $h \mapsto S_0(h)$ is absolutely convergent almost everywhere in $h$ and the resulting function belongs to $L^2(H \times_\nu E)$. We apply the integral version of Minkowski’s inequality for $p = 2$ and obtain

$$\left( \int_H \left( \int_G \|K_{S^*}(h, x)^* f(x)\|_E dx \right)^2 dh \right)^{1/2} \leq \int_G \left( \int_H \|K_{S^*}(h, x)^* f(x)\|_E^2 dh \right)^{1/2} dx \leq \int_G \left( \int_H \|K_{S^*}(h, x)^* \|^2_{\text{Hom}(W, E)} dh \right)^{1/2} \|f(x)\|_W dx.$$
The right hand side is a finite number because $f$ belongs to $\mathcal{D}$. Hence, \( \int_G K_{S^*}(h, x)f(x)dx \) is absolutely convergent almost everywhere in $h$ and the resulting function belongs to $L^2(H \times \nu, E)$. As a consequence we obtain that for $g \in L^2(H \times \nu, E)$ and $f \in \mathcal{D}$ the following two iterated integrals are absolutely convergent

\[
\int_G \int_H (K_{S^*}(h, x)g(h), f(x))_W dhdx = \int_H \int_G (K_{S^*}(h, x)g(h), f(x))_W dxdh.
\]

We are ready to conclude the proof of Theorem 3.5 b). Whenever $\pi_\lambda$ is an integrable discrete series representation, it follows from the work of Harish-Chandra, [HCVI, Lemma 76], that any $K$–finite vector $f$ in $H^2(G, \tau)$ is integrable with respect to Haar measure on $G$. We claim that any smooth vector in $H^2(G, \tau)$ is an integrable function. For this we recall the space of rapidly decreasing functions $\mathcal{S}(G)$ on $G$, defined by [Wa1, page 230]. Owing to [Wa1, Lemma 2.A.2.4], any rapidly decreasing function is integrable. Next, in [Wa2, Theorem 11.8.2] it is shown that the subspace $H^2(G, \tau)^{\infty}$ of smooth vectors in $H^2(G, \tau)$ is an algebraically irreducible module for $\mathcal{S}(G)$. Therefore, any smooth vector in $H^2(G, \tau)$ is the convolution of a rapidly decreasing function on $G$ times a $K$–finite vectors. Hence, any smooth vector is convolution of two integrable functions on $G$. Classical harmonic analysis yields any smooth vector in $H^2(G, \tau)$ is an integrable function. Therefore, for a smooth vector $f$ in $H^2(G, \tau)$, Proposition 3.6 forces the integral

\[
\int_G \|K_{S^*}(\cdot, x)^*\|_{L^2(H \times \nu, \otimes \nu, Hom(W, E))} \|f(x)\|_W dx
\]

is absolutely convergent. Whence, $\mathcal{D}$ contains the smooth vectors in $H^2(G, \tau)$, and we have shown b). Thus, we have shown Theorem 3.5

\[\square\]

**Example 3.7.** An application for Theorem 3.5 is: For a continuous $H$–map, we write the polar decomposition for $T = VP : L^2(H \times \sigma, Z) \to H^2(G, \tau)$ where $P = \sqrt{T^*T}$ and $V : L^2(H \times \sigma, Z) \to H^2(G, \tau)$ is a partial isometry. $V$ is usually called a generalized Bargmann transform. Then, $V$ is an integral map, and, whenever $\pi_\lambda^G$ is an integrable discrete series, the linear map $V^*$, as well as $T^*$, restricted to the subspace of smooth vectors in $H^2(G, \tau)$ is an integral map. A particular case of this is the restriction map $r : H^2(G, \tau) \to L^2(H \times_{\text{res}_\lambda(\tau)} W)$ as in Example 4.1. Here, $r^* = V\sqrt{r'^*}, \quad r = V^*r\sqrt{r'^*}$. Sometimes $r$ is injective, whence $V^*$ is injective and kernel map when $\pi_\lambda$ is integrable. In Example 3.11 we verify $V^*$ restricted to the subspace of $K$–finite vectors is an integral map despite $\pi_\lambda$ is not an integrable representation.

**Remark 3.8.** Every $f \in H^2(G, \tau)$ is a bounded function. In fact, let $K_\lambda(y, x)$ denote the matrix kernel for $H^2(G, \tau)$. Thus,

\[
f(x) = \int_G K_\lambda(y, x)f(y)dy.
\]
Then, Schwarz inequality, the equality $K_\lambda(y, x) = K_\lambda(x^{-1}y, e)$, $K_\lambda(\cdot, e)$ is square integrable and the left invariance of Haar measure, justify the inequalities presented bellow

$$\|f(x)\|_W \leq \int_G \|K_\lambda(y, x)\|_{\text{Hom}(W, W)} \|f(y)\|_W \, dg$$

$$\leq \left( \int_G \|K_\lambda(y, x)\|^2 \, dy \right)^{\frac{1}{2}} \left( \int_G \|f(y)\|^2 \, dy \right)^{\frac{1}{2}}$$

$$\leq \left( \int_G \|K_\lambda(y, e)\|^2 \, dy \right)^{\frac{1}{2}} \left( \int_G \|f(y)\|^2 \, dy \right)^{\frac{1}{2}}$$

$$= \|K_\lambda(\cdot, e)\|_{L^2(G)} \|f\|_2.$$

**Remark 3.9.** We assume $S : H^2(G, \tau) \to L^2(H \times_v E)$ is a continuous symmetry breaking operator represented by a kernel $S$. This hypothesis implies that $K_S(x, h) = K_S(h, x)^*$ and hence we may conclude:

a) $K_S$ is a smooth map.

b) $K_S(hxk, hh_1s) = \nu(s^{-1})K_S(s, h_1)\tau(k), h, h_1 \in H, x \in G, s \in L.$

c) The function $G \ni x \mapsto K_S(x, e)^*z \in W, z \in E$ belongs to $H^2(G, \tau)$ and it is $L$–finite.

**Remark 3.10.** For continuous $H$–map from $H^2(G, \tau)$ into $H^2(H, \sigma)$ or vice versa, always, they are represented by smooth kernels that enjoy the properties in Remark 3.9 or Proposition 3.6.

### 3.4. Generalized Shintani functions.

Kobayashi in [Kobs] has begun a deep analysis of functions related to $H$–intertwining linear operators $R : V_G \to V_H$ between two smooth irreducible class one representations $(\pi_G, V_G)$ $(\pi_H, V_H)$ for respectively $G, H$. For this, he fix $v_G \in V_G, v_H \in V_H$ nonzero vectors fixed by respectively $K, L$ and he defines the Shintani function $S(g) = (R(\pi_G(g)v_G), v_H)_{V_H, g} \in G$ The function $S$ satisfies: right invariant under $L$, left invariant under $K$, an eigenfunction for $R_D, D \in 3(U(g))$, and eigenfunction for $L_D : D \in 3(U(h))$. Here, $3(U(g))$ denotes the center for the universal enveloping algebra for $g$. Let $S : H^2(G, \tau) \to H^2(H, \sigma)$ a continuous intertwining linear map. Since, $H^2(H, \sigma)$ is a reproducing kernel subspace, $S$ is an integral map represented by a kernel $K_S : G \times H \to \text{Hom}_C(W, Z)$ which satisfies the statements in Remark 3.9. We now verify that properties of the function $y \mapsto K_S(y, e)$ suggest a natural generalization for the concept of Shintani function. In fact, let $\Phi(y) := K_S(y, e), y \in G.$ Thus, $\Phi$ maps $G$ into $\text{Hom}_C(W, Z).$ Among the properties of the function $\Phi$ are:

1. $\Phi$ is a smooth function.
2. $\Phi(syk) = \sigma(s)\Phi(y)\tau(k), s \in L, k \in K, y \in G.$
3. $R_D\Phi = \chi_\lambda(D)\Phi$ for $D \in 3(U(g))$.
4. $L_D\Phi = \chi_\mu(D)\Phi$ for $D \in 3(U(h))$.
5. $\Phi$ as well as its restriction to $H$ are square integrable.

The previous considerations let us to define a *generalized Shintani function*. For this we fix representations $(\tau, W)$ of $K$, $(\sigma, Z)$ of $L$ and infinitesimal characters $\chi_\lambda, \chi_\mu$ for respectively $\mathfrak{g}, \mathfrak{h}$. A generalized Shintani functions is a function $\Phi : G \to \text{Hom}_C(W, Z)$ that satisfies the five conditions enumerated in the previous paragraph. When, we dealt with Discrete Series representations, the trivial representation of a maximal compact subgroup of the
ambient group, never occurs as a $K$–type, however, it is clear that the first four stated properties are a generalization of the concept of Shintani’s function. The obvious result is:

The space of generalized Shintani functions attached to $(\tau, W), (\sigma, Z), \chi_\lambda, \chi_\mu$ is isomorphic to the space of continuous $H$–maps from $V^G_\lambda$ to $V^H_\mu$.

Example 3.11. The following example shows that statement in Theorem 3.51
b) might not be sharp. Details for some of the statements in this example are found in [DaOZ] and references therein. We set $G = SL_2(\mathbb{R})$ and $H$ equal to the subgroup of diagonal matrices in $G$. We fix as compact Cartan subgroup $T$ the subgroup of orthogonal matrices in $G$. We fix as positive root $\alpha$ one of the roots in $\Phi(\mathfrak{g}, t)$ and $\rho := \frac{1}{2}\alpha$. Then, the set of Harish-Chandra parameters for $G$ is $\{np, n \in \mathbb{Z}\{0\}\}$. The lowest $K$–type for $\pi^G_{\rho \rho}$ ($p \geq 1$) is $(\pi^{K}_{(p+1)\rho}, \mathbb{C})$. After we identify $G/K$ with the upper half plane $\mathbb{H}_+$ and we trivialize the vector bundle $G \times \pi^{K}_{(p+1)\rho}, \mathbb{C} \rightarrow G/K$ we have that $H^2(G, \pi^{K}_{(p+1)\rho})$ can be identified with the space

$$\{ f : \mathbb{H}_+ \rightarrow \mathbb{C} : f \text{ holomorphic and } \int_{\mathbb{H}_+} |f(x + iy)|^2 y^{p+1} \frac{dx dy}{y^2} < \infty \}.$$  

The bundle $H \times \pi^{K}_{(p+1)\rho} \mathbb{C} \rightarrow H/L$ is also trivial and $L^2(H \times_{\text{res}_{L}(\tau)} W)$ is identified with $L^2(\mathbb{R}_+^2, \rho \rho dt, r)$. The restriction map $r : H^2(G, \tau) \rightarrow L^2(H \times_{\text{res}_{L}(\tau)} W)$ becomes $r(f)(iy) = f(iy), y \in \mathbb{R}_+$. After we write the polar decomposition $r^* = VQ$ we have

$$V(g)(z) = \int_0^\infty e^{izt} g(it) t^{\rho} dt, \ z \in \mathbb{H}_+, \ g \in L^2(H \times_{\text{res}_{L}(\tau)} W).$$

Owing to Theorem 3.5, for an integrable discrete series, equivalently $|p| \geq 2$, the linear map $V^*$ restricted to the subspace of smooth vectors in $H^2(G, \pi^{K}_{(p+1)\rho})$ is equal to the integral linear map

$$V^*(f)(it) = \int_{\mathbb{H}_+} f(x + iy)e^{-iyt}y^{p-1} \frac{dx dy}{y^2}.$$  

(3)

For the non integrable discrete series $H^2(G, \pi^{K}_{2\rho})$ the Lebesgue integral on the right of (3) does exist for any $K$–finite vector. In fact,

$$\int_0^\infty \int_{-\infty}^\infty \left| \frac{1}{|z + i|^2} e^{-yt} \right| \frac{dx dy}{x^2 + (y + 1)^2} < +\infty \text{ for } t > 0, n \geq 0.$$  

Whence, the following integral converges absolutely,

$$\int_{\mathbb{H}_+} \left( \frac{z - i}{z + i} \right)^n \frac{1}{(z + i)^2} e^{-iyt} \frac{dx dy}{x^2 + (y + 1)^2} = c_n e^{-t}(e^{2t}(2t)^{-1} \frac{dt}{t})^n (e^{-2t}(2t)^{n+1}).$$

The inner integral is computed by means of Cauchy integral formula applied to compute the first derivative of $z \mapsto e^{-i(z-iy)t}$ at the point $-(iy + i)$. The
Sometimes we will allow the constants and different to the trivial representation. Since \( (4.1) \)
We now have that for each \( n \geq 0 \), the resulting function defined by the right hand side of \( (j) \) belongs to \( L^2(\mathbb{R}, t dt) \). Next, the set of functions \( \left( \frac{1}{z+i} \right)^n, n \geq 0 \) span the subspace of \( K \)-finite vectors. Therefore, functional analysis implies the right hand of \( (j) \) evaluated in each of this set of generators of \( (H^2(SL_2(\mathbb{R}), \pi^K_2))_{K-finite} \) is equal to \( V^* \) evaluated at each of these generators. 
This concludes the verification that \( V^* \) restricted to the subspace of \( K \)-finite vectors is an integral map.

4. INTERTWACING OPERATORS VIA DIFFERENTIAL OPERATORS

Let \( G, K, H, L, H^2(G, \tau), (\nu, E) \) be as usual. In [N],[KobPev],[Kobv] these authors have constructed \( H \)-intertwining maps between holomorphic discrete series by means of differential operators. Some of these authors also considered the case of intertwining maps between two principal series representations [KobSp]. Motivated by the fact that discrete series can be modeled as function spaces, an aim of this section is to analyze to what extent \( H \)-intertwining linear maps agree with restriction of linear differential operators. In [Kobv] is presented a general conjecture on the subject, we present a partial solution for the particular case of discrete series representations.

4.0.1. Differential Operators. For the purpose of this note a differential operator is a linear map \( S : C^\infty(G \times_\tau W) \to C^\infty(H \times_\nu E) \) so that there exists finitely many elements \( D_b \in U(g) \), finitely many elements \( \{w_c\} \) in \( W \), \( \{z_a\} \) finitely many elements \( \{z_a\} \) in \( E \), \( d_{a,b,c} \in \mathbb{C} \), and for any \( f \in C^\infty(G \times_\tau W) \) we have

\[
S(f)(h) = \sum_{a,b,c} d_{a,b,c} ([R_{D_b}f](h), w_c)_W z_a \quad \forall h \in H.
\]

Sometimes we will allow the constants \( d_{a,b,c} \) to be smooth functions on \( H \). The definition is motivated by the following result of Kobayashi, and posterior considerations that we will present. Kobayashi, in [KobPev][Theorem 2.9] constructed an isomorphism between the set of \( H \)-invariant differential operators from \( C^\infty(G \times_\tau W) \) into \( C^\infty(H \times_\nu E) \) and the space \( \text{Hom}_L(E^\vee, \text{Ind}^H_0(W^\vee)) \). As usual, \( M^\vee \) denotes the contragredient representation. The isomorphism is:

\[
\sum_{a,b,c} d_{a,b,c} z_a \otimes D_b \otimes w_c \\
\mapsto (C^\infty(G \times_\tau W) \ni f \mapsto \sum_{a,b,c} d_{a,b,c} ([R_{D_b}f](h), w_c)_W z_a)).
\]

Here, \( d_{a,b,c} \in \mathbb{C} \), \( \{z_a\} \) basis for \( E \), \( D_b \in U(g) \), \( \{w_c\} \) basis for \( W \), \( h \in H \).

WE now sketch a proof that any differential operator (according to our definition) can be replaced by a \( H \)-invariant differential operator. Indeed, since \( L \) is a compact group, any element \( D \) in \( \text{Hom}_L(E^\vee, \text{Ind}^H_0(W^\vee)) \) is a finite sum \( D = D_0 + D_1 + \cdots + D_R \). Here \( D_0 \) is \( L \)-invariant and each \( D_j, j \geq 1 \) belongs to an \( L \)-isotypic components that correspond to inequivalent representations and different to the trivial representation. Since \( D, D_0 \) maps \( C^\infty(G \times_\tau W) \)
into $C^\infty(H \times_\nu E)$ we obtain for $x \in G, s \in L \; Df(xs) = \nu(s^{-1})Df(x) = \nu(s^{-1})D_0f(x) + \nu(s^{-1})(Ad(s)D_1f)(x) + \cdots + \nu(s^{-1})(Ad(s)D_rf)(x)$. After we cancel $\nu(s^{-1})$ the orthogonality relations yields $D_jf(x) = 0$ for $j \geq 1$.

Example 4.1. Examples of differential operators are the normal derivatives as considered in [OV]. For this, we write the Cartan decomposition as $g = \mathfrak{t} + \mathfrak{p}$ and $h = 1 + \mathfrak{p}'$. We have $\mathfrak{p}' = \mathfrak{p} \cap \mathfrak{h}$. Let $(\mathfrak{p}/\mathfrak{p}')^{(n)}$ denote the $n$–th symmetric power of the orthogonal with respect to the Killing form of $\mathfrak{p}'$ in $\mathfrak{p}$. Let $\tau_n$ denote the natural representation in $Hom_\mathbb{C}((\mathfrak{p}/\mathfrak{p}')^{(n)}, W)$. Let $\lambda : S(\mathfrak{g}) \to U(\mathfrak{g})$ denote the symmetrization map. Then, for each $D \in (\mathfrak{p}/\mathfrak{p}')^{(n)}$, $f \in H^2(G, \tau), h \in H$ we compute the normal derivative of $f$ in the direction $D$ at the point $h$, $r_n(f)(D)(h) := R_{\lambda(D)}f(h)$. In [OV] it is shown that $r_n(f) \in L^2(H \times_{\tau_n} Hom_{\mathbb{C}}((\mathfrak{p}/\mathfrak{p}')^{(n)}, W))$ and the resulting map

$$r_n : H^2(G, \tau) \to L^2(H \times_{\tau_n} Hom_{\mathbb{C}}((\mathfrak{p}/\mathfrak{p}')^{(n)}, W))$$

is $H$–equivariant and continuous for $L^2$–topologies. As before, $K_\lambda$ is the matrix kernel of $P_\lambda$. The map $r_n$ is represented by the matrix kernel

$$K_{r_n} : G \times H \to Hom_{\mathbb{C}}(W; Hom_{\mathbb{C}}((\mathfrak{p}/\mathfrak{p}')^{(n)}, W))$$

given by

$$K_{r_n}(y, h)(w, D) = R_{\lambda(D)}^{(2)}(K_\lambda(y, h)w).$$

Here, the upper index (2) means we compute the derivative of $K_\lambda$ on the second variable.

4.1. Symmetry breaking via differential operators. Before we state Theorem 4.3 we derive some properties of symmetry breaking operators that are restriction of differential operators. We recall in Theorem 3.5 we have shown that when $res_H(\pi_\lambda)$ is discretely decomposable, any symmetry breaking operator is an integral map. We would like to remark that the next two results contain "automatic continuity Theorem".

Lemma 4.2. Let $S : H^2(G, \tau) \to L^2(H \times_\nu E)$ be a not necessarily continuous intertwining $H$–map such that $S$ is the restriction of a differential operator. Then, $S$ is a kernel map. That is, there exists $K_S : G \times H \to Hom_{\mathbb{C}}(W, E)$ so that $y \mapsto K_S(y, h)^*z \in H^2(G, \tau)$ for $h \in H, z \in E$ and

a) $(S(f)(h), z)_{\tau} = \int_{\mathfrak{g}}(f(y), K_S(y, h)^*z)_{\nu, E}dy$ for $f \in H^2(G, \tau), z \in E$.

b) $y \mapsto K_S(y, e)^*z$ is a $K$–finite vector for $\pi_\lambda$.

c) $K_S$ is a smooth function.

d) $S$ is continuous in $L^2$–topologies.

Conversely, if $S : H^2(G, \tau) \to L^2(H \times_\nu E)$ is an integral $H$–map so that $y \mapsto K_S(y, e)^*z$ is a $K$–finite vector for $\pi_\lambda$. Then, $S$ is continuous and $S$ is the restriction of a differential operator.

Compare b) with 3.9 c). We differ the proof of the Lemma 4.2 to until we state the following theorem.

Theorem 4.3. Let $G, K, H, L, (\tau, W), H^2(G, \tau), \pi_\lambda, (\nu, E), (\sigma, Z)$ be as in the previous paragraph. Let $S : H^2(G, \tau) \to L^2(H \times_\nu E)$ denote an intertwining linear $H$–map. The following statements holds:

If $res_H(\pi_\lambda)$ is an $H$–admissible representation and $S$ is continuous. Then, $S$ is the restriction of a linear differential operator.
For a converse statement, we have,
i) If we assume for some \( \nu \) that some nonzero linear intertwining \( H \)-map \( S : H^2(G, \tau) \to L^2(H \times_\nu E)_{\text{disc}} \) is the restriction of a linear differential operator. Then, \( \text{res}_H(\pi_\lambda) \) is discretely decomposable.

ii) If we assume for some \( \sigma \) that every nonzero linear intertwining \( H \)-map \( S : H^2(G, \tau) \to H^2(H, \sigma) \) is the restriction of a linear differential operator. Then, the multiplicity of \( (L, H^2(H, \sigma)) \) in \( \text{res}_H(\pi_\lambda) \) is finite.

iii) If we assume for every \( \sigma \) that every nonzero \( S : H^2(G, \tau) \to H^2(H, \sigma) \) is the restriction of a linear differential operator. Then, \( \text{res}_H(\pi_\lambda) \) is an \( H \)-admissible representation.

An immediate corollary to the Theorem is:

**Corollary 4.4.** If \( \pi_\lambda \) is an \( H \)-admissible representation, then, for each \( (\sigma, Z) \), every continuous linear \( H \)-map \( S : H^2(G, \tau) \to H^2(H, \sigma) \) is a differential operator as well as an integral map.

Example for maps \( S \) where the Theorem applies are the normal maps \( r_n \) defined in Example 4.1. In particular, we have \( \text{res}_H(\pi_\lambda) \) is discretely decomposable if and only if there exists an \( n \) so that the image of \( r_n \) is contained in \( L^2(H \times_{r_n} \text{Hom}_\C(\langle (p^{-1}p)(\nu) \rangle, W))_{\text{disc}} \).

Nakahama, [N][Theorem 3.6] has shown an equivalent result to our first statement in the Theorem and to the Corollary for the case of \( H \)-admissible holomorphic discrete series under the hypothesis that \( G, H \) is a symmetric pair.

**Proof of Lemma 4.2.** We fix \( \{z_j, j = 1, \ldots, \dim E\}, \{w_i, i = 1, \ldots, \dim W\} \) respective orthonormal basis for \( E, W \). Our hypothesis gives for every \( f \in H^2(G, \tau), h \in H \) the following equality holds

\[
S(f)(h) = \sum_{j,b,i} d_{j,b,i} ([R_{D_{b,i}}f](h), w_i)_W z_j.
\]

In [At] we find a proof that in the \( L^2 \)-kernel of an elliptic operator, \( L^2 \)-convergence implies uniform convergence of the sequence as well as any of its derivatives on compact sets. Since the Casimir operator acting on \( G/K \) is an elliptic operator, the result on PDE just quoted applies to \( H^2(G, \tau) \).

Hence, the equality \( (S(f)(h), z_j)_Z = \sum_{h,i} d_{j,b,i} ([R_{D_{b,i}}f](h), w_i)_W \) yields the left hand side determines a continuous linear functional on \( H^2(G, \tau) \). Thus, there exists a function \( K_S : G \times H \to \text{Hom}_\C(W, E) \) so that \( y \mapsto K_S(y, h)z_j \) belongs to \( H^2(G, \tau) \) and a) holds. The hypothesis \( S \) is an intertwining map, yields the equality \( K_S(h_1yk, h_1hs) = \nu(s^{-1})K_S(y, h)(k)h_1, h \in H, y \in G, s \in L, k \in K \). The smoothness for \( K_S \) follows from that \( K_S^* \) is equal to the map \( (y, h) \mapsto h^{-1}y \) followed by the map \( x \mapsto K_S(x, e)^* \) and that \( x \mapsto K_S(x, e)^* \) is an element of \( H^2(G, \tau) \). Next, we justify the four equalities in the computation below. The first is due to the expression of \( S \), the second is due to the identity \( L_D(f)(e) = R_Df(e) \), hence, we obtain \( (S(f)(e), z_j)_Z = \sum_{b,i,d} d_{j,b,i}([L_{D_{b,i}}f](e), w_i)_W \). The third is due to (2.1), finally, we recall for an arbitrary \( D \in U(g) \) it follows that any smooth vector \( f \in V_\infty \) is in the domain for \( L_D \), in particular \( y \mapsto K_\lambda(y, e)^*w_i \) is in the domain for \( L_D \). These four considerations justifies the following equalities for any smooth
vector $f \in H^2(G, \tau)$, we have
\[
\int_G (f(y), K_S(y, e)^* z_j)_w \, dy = (S(f)(e), z_j)
= \sum_{j,b,i} d_{j,b,i} (L_{D_b} f(e), w_i)_W
= \sum_{j,b,i} d_{j,b,i} \int_G (L_{D_b} f(y), K_\lambda(y, e)^* w_i)_W \, dy
= \sum_{j,b,i} d_{j,b,i} \int_G (f(y), L_{D_b}^{(1)} K_\lambda(y, e)^* w_i)_W \, dy.
\]

We observe, the first and last member of the above equalities defines continuous linear functionals on $H^2(G, \tau)$ and they agree on the dense subspace of smooth vectors, whence
\[
K_S(y, e)^* z_j = \sum_{b,i} d_{j,b,i} L_{D_b}^{(1)} K_\lambda(y, e)^* w_i.
\]

Since, the right hand side of the above equality is a $K$–finite vector for $\pi_\lambda$, we have shown b). To show the continuity of $S$ we notice that $S$ is defined by the Carleman kernel $K_S$ (for each $h \in H$, $K_S(\cdot, h) \in H^2(G, \tau)$) and by hypothesis the domain of the integral operator defined by $K_S$ on $L^2(G \times W)$ contains $H^2(G, \tau)$. Since a Carleman kernel determines a closed map on its maximal domain, and $H^2(G, \tau)$ is a closed subspace, we have $S : H^2(G, \tau) \to L^2(H \times \nu, E)$ is a closed linear map with domain $H^2(G, \tau)$, the closed graph Theorem leads as to the continuity of $S$.

To show the converse statement, we explicit the hypotheses on $S$: $K_S(hx, hh_1) = K_S(x, h_1), h, h_1 \in H, x \in G$; for each $Z \in E$, $y \mapsto K_S(y, e)^* z$ is a $K$–finite vector in $H^2(G, \tau)$; domain of $S$ equal to $H^2(G, \tau)$. We show $S$ is a the restriction of a differential operator. In fact, since $K_S(x, h) = K_S(h^{-1}x, e)$ and $K_S(\cdot, e)$ belongs to $H^2(G, \tau)$ we obtain $K_S(\cdot, h)$ is square integrable and hence $S$ is a Carleman map. As in the direct implication we obtain $S$ is continuous. To verify $S$ is a restriction of a differential map, we fix a nonzero vector $w \in W$. Since $H^2(G, \tau)$ is an irreducible representation, a result of Harish-Chandra shows that the underlying $(g, K)$–module for $H^2(G, \tau)$ is $U(g)$–irreducible. It readily follows that the function $K_\lambda(\cdot, e)^* w$ is nonzero (otherwise $K_\lambda$ would be the null function), therefore for each $z_j$ there exists $D_j \in U(g)$ so that $K_S(\cdot, e)^* z_j = L_{D_j} K_\lambda(\cdot, e)^* w$. For $f$ smooth vector in $H^2(G, \tau)$,
\[
S(f)(h^{-1}) = S(L_h f)(e) = \sum_{j} (SL_h f(e), z_j)_{z_j}
= \sum_{j} \int_G (L_h f(y), K_S(y, e)^* z_j)_W z_j
= \sum_{j} \int_G (L_{D_j} (L_h f)(y), K_\lambda(y, e)^* w)_W z_j
= \sum_{j} ((R_{D_j})(L_{h^{-1}} f)(e), w)_W z_j = \sum_{j} ((R_{D_j})(f)(h^{-1}), w)_W z_j.
\]
Thus, after we fix a linear basis \( \{R_i\} \) for \( U(\mathfrak{g}) \), for a smooth vector \( f \) we have
\[
S(f)(h) = \sum_{j,i} d_{i,j}([R_{R_i}(f)](h), w_j)z_j.
\]
Owing to the result on PDE quoted in the direct proof, the right hand side defines a continuous linear transformation from \( H^2(G, \tau) \) into \( C^\infty(H \times_\nu E) \). We claim, this forces \( S(f) \) to be continuous for every \( f \). In fact, each \( f \) in \( H^2(G, \tau) \) is limit of a sequence \( f_n \) of smooth vectors, whence the first and last members of the above equalities agree on each \( f_n \), if necessary going to a subsequence, the Riez-Fischer Theorem yields the left hand side pointwise converges (a.e.) to \( S(f) \). Thus, \( S(f) \) agree up to set of measure zero with a smooth function. Moreover, this argument yields \( S(f) \) is equal to the right hand side for any \( f \). Thus, we have shown \( S \) is a differential operator. This completes the proof of Lemma 4.2.

**Branching problems and rep. kernels**

For the vector spaces \( E, W \), we fix respective orthonormal basis \( \{z_j\}, \{w_i\} \). Since \( res_H(\pi_\lambda) \) is discretely decomposable, Theorem 3.5 shows \( S \) is an integral map. For each \( j \), the identity \( K_{S(l^{-1}y, e)^*}z_j = K_S(y, e)^*\nu(l)z_j \), \( l \in L, y \in G \) yields \( x \mapsto K_S(x, e)^*z_j \) is an \( L^- \)-finite vector. The hypothesis \( res_H(\pi_\lambda) \) is an admissible representation allows us to apply [Kob1, Prop. 1.6]. In this way we obtain that the subspace of \( L^- \)-finite vectors in \( H^2(G, \tau) \) is equal to the subspace of \( K^- \)-finite vectors. Whence, \( x \mapsto K_S(x, e)^*z_j \) is a \( K^- \)-finite vector. By hypothesis, \( H^2(G, \tau)_{K^-fin} \) is an irreducible representation under the action of \( U(\mathfrak{g}) \) and the function \( y \mapsto K_\lambda(y, e)^*w_i \) is nonzero and \( K^- \)-finite, hence, for each \( i, j \) there exists \( C_{j,i} \in U(\mathfrak{g}) \) so that \( [L^{(1)}_{C_{j,i}}, K_\lambda](y, e)^*w_i = K_S(y, e)^*z_j \), for all \( y \in G \). Therefore, since \( x \mapsto K_S(x, e)^*z_j \) is a smooth vector for \( G \), for \( f \in V_\lambda^\infty \) we justify as in the proof of Lemma 4.2 the fourth and sixth equality in the following computation, the fifth equality is due to (2.1),
\[
S(f)(e) = \sum_j (S(f)(e), z_j)_E z_j = \sum_j \int _G (f(y), K_S(y, e)^*z_j)_W dy z_j
\]
\[
= \sum_j \int _G (f(y), [L^{(1)}_{C_{j,i}}, K_\lambda](y, e)^*w_i)_W dy z_j
\]
\[
= \sum_j \int _G (L_{C_{j,i}}f(y), K_\lambda(y, e)^*w_i)_W dy z_j
\]
\[
= \sum_j (L_{C_{j,i}}f)(e), w_i)_W z_j = \sum_j ([R_{C_{j,i}}(f)](e), w_i)_W z_j.
\]
For \( h \in H \), we apply the previous equality to \( f := L_{h^{-1}}f \) and since \( S \) intertwines the action of \( H \) we obtain
\[
S(f)(h) = S(L_{h^{-1}}f)(e)
\]
\[
= \sum_j ([R_{C_{j,i}}(L_{h^{-1}}f)](e), w_i)_W z_j = \sum_j ([R_{C_{j,i}}(f)](h), w_i)_W z_j.
\]
After we set \( D_{j,i} := C_{j,i}^* \) and we recall definition ?? we conclude that \( S \) restricted to the subspace of smooth vectors agrees with the restriction of a differential operator. In order to show the equality for general \( f \in H^2(G, \tau) \) we argue as follows: there exists a sequence \( f_r \) of elements in \( V_\lambda^\infty \) which converges in \( L^2 \)-norm to \( f \). Owing to the Casimir operator is elliptic on
$G/K$, the sequence $f_r$ as well as any derivatives of the sequence converge uniformly on compact subsets. Moreover, owing to Harish-Chandra’s Plancherel Theorem $L^2(H \times \nu, E)_{\text{disc}}$ is a finite sum of square integrable irreducible representations. More precisely, Harish-Chandra shows $L^2(H \times \nu, E)_{\text{disc}}$ is a finite sum of eigenspaces for the Casimir operator for $\mathfrak{h}$. We know the Casimir operator acts as an elliptic differential operator on $L^2(H \times \nu, E)$, whence, we have that point evaluation is a continuous linear functional on $L^2(H \times \nu, E)_{\text{disc}}$. Finally, the hypothesis on $\text{res}_H(\pi_{\lambda})$ gives the image of $S$ is contained in $L^2(H \times \nu, E)_{\text{disc}}$. Therefore, we have justified the steps in

$$S(f)(h) = \lim_r S(f_r)(h) = \lim_r \sum_j ([R_{D_{j,i}} f_r](h), w_i)_W z_j = \sum_j ([R_{D_{j,i}} f](h), w_i)_W z_j.$$  

Whence, we have shown the first statement in Theorem 4.3.

To follow we assume for some $\sigma$ and some nonzero intertwining $H-$map $S : H^2(G, \tau) \to L^2(H \times \nu, E)_{\text{disc}}$ is the restriction of a linear differential operator, we show $\text{res}_H(\pi_{\lambda})$ is discretely decomposable.

Remark 4.5. The hypothesis on image of $S$ is quite essential. Examples and counterexamples are provided by $r, r_1, r_2, ....$

In fact, the hypothesis allows us to apply Lemma 4.2. In consequence, $y \mapsto K_S(y, e)^* z_j$ is a $K-$finite vector in $H^2(G, \tau)$. We claim, $K_S(\cdot, e)^* z_j$ is $z(U(\mathfrak{h}))-$finite. In fact, Harish-Chandra’s Plancherel Theorem shows $L^2(H \times \nu, E)_{\text{disc}}$ is equal to a finite sum of irreducible discrete series for $H$. Thus, for $f \in V_\lambda^\infty, D \in z(U(\mathfrak{h}))$ whenever the image of $S$ is contained in an irreducible subspace we have the equalities

$$\int_G (f(y), L_{D_{j,i}}(t) K_S(y, e)^* z) dy = \int_G (L_{D} f(y), K_S(y, e)^* z) dy = \chi_{\mu}(D) (S(f)(e), z)_W = \chi_{\mu}(D) \int_G (f(y), K_S(y, e)^* z) dy$$

The third equality holds because hypothesis $S(f)$ is an eigenfunction for $z(U(\mathfrak{h}))$. Therefore, the first and last members of the above equalities determine continuous linear functionals on $H^2(G, \tau)$ which agree in the dense subspace of smooth vectors. Whence, $y \mapsto K_S(y, e)^* z_j$ is an eigenfunction for $z(U(\mathfrak{h}))$. The general case readily follows from a similar computation. Thus, the hypothesis $S$ is nonzero, gives us an $z \in E$ so that $U(\mathfrak{h}) K_S(\cdot, e)^* z$ is a $z(U(\mathfrak{h}))-$finite and nonzero $U(\mathfrak{h})-$submodule of $V_{K-fin}$. We quote a result of Harish-Chandra: a $U(\mathfrak{h})-$finitely generated, $z(U(\mathfrak{h}))-$finite, $(\mathfrak{h}, L)-$module has a finite composition series. For a proof c.f. [Wa1, Corollary 3.4.7 and Theorem 4.2.1]. Therefore, the subspace $U(\mathfrak{h}) K_S(\cdot, e)^* z$ contains an irreducible $U(\mathfrak{h})-$submodule. Next, in [Kob1, Lemma 1.5] we find a proof of: If $(\mathfrak{g}, K)-$module contains an irreducible $(\mathfrak{h}, L)-$submodule, then the $(\mathfrak{g}, K)-$module is $\mathfrak{h}-$algebraically decomposable. Thus, $\text{res}_H(\pi_{\lambda})$ is algebraically discretely decomposable. The fact $V_\lambda$ is unitary yields discrete decomposable [Kob2, Theorem 4.2.6]. Whence, we have shown i).
We now assume for some $\sigma$ and every intertwining linear $H$—map $S : H^2(G, \tau) \to H^2(H, \sigma)$ is the restriction of a linear differential operator. We show, the multiplicity of $(L, H^2(H, \sigma))$ in $\text{res}_H(\pi_\lambda)$ is finite.

The first claim in Theorem 4.3 yields $\text{res}_H(\pi_\lambda)$ is discrete decomposable. Let’s assume the multiplicity of $H^2(H, \sigma)$ in $\text{res}_H(\pi_\lambda)$ is infinite. Thus, there exists $T_1, T_2, \ldots$ so that $T_j : H^2(H, \sigma) \to H^2(G, \tau)[V^H_\mu]$ are isometric immersion intertwining linear maps so that for $r \neq s$ the image of $T_r$ is orthogonal to the image of $T_s$ and the algebraic sum of the subspaces $T_r(H^2(H, \sigma)) r = 1, 2, \ldots$ is dense in $H^2(G, \tau)[V^H_\mu]$. Let $\iota : Z \to V^H[Z]$ be the equivariant immersion adjoint to evaluation at the identity. We fix a norm one vector $g_0 := \iota(z_0) \in V^H[Z]$. There are two possibilities: For some $r$ the function $K_{T_r}(\cdot, e)^*z_0$ is not a $K$—finite vector or else for every $r$ the function $K_{T_r}(\cdot, e)^*z_0$ is a $K$—finite vector. To follow, we analyze the second case, for this, we define $v_n := T_n(g_0)$ and we choose a sequence of nonzero positive real numbers $(a_n)_n$ so that $v_0 := \sum_n a_nv_n$ is not the zero vector. Due to the orthogonality for the image of the $T_r$ and the choice of the sequence, $v_0$ is not a $K$—finite vector. Since the stabilizer of $v_0$ in $H$ is equal to the stabilizer of $g_0$ on $H$, the correspondence $T : V^H_\mu \to H^2(G, \tau)$ defined by $T(h, g_0) = \frac{1}{\|v_0\|}h.v_0$ extends to an isometric immersion. We claim $S = T^*$ is not the restriction of a linear differential operator. For this, we show $S(\sum_n \frac{v_0}{\|v_0\|}) = g_0$ and $K_S(\cdot, e)^*z_0 = \sum_n \frac{a_nv_n}{\|v_0\|}$. On one hand we have the following system of equations

$$
(S(f)(e), z_0)_Z = \int_G (f(y), K_S(y, e)^*z_0)_W dy \ \forall f \in H^2(G, \tau)
$$

determine the function $K_S(\cdot, e)^*z_0$. On the other hand, for arbitrary $f \in H^2(G, \tau)$ we have

$$
\int_G (f(y), \frac{v_0(y)}{\|v_0\|})_W dy = (f, T(g_0))_{H^2(G, \tau)} = (T^*f, g_0)_{H^2(H, \sigma)} = (S(f), \iota(z_0))_{H^2(H, \sigma)} = (\iota^*(Sf), z_0)_Z = (S(f)(e), z_0)_Z.
$$

Thus, we have shown the equality $K_S(\cdot, e)^*z_0 = \sum_n \frac{a_nv_n}{\|v_0\|}$. Therefore, if $S$ were a differential operator, the fact $v_0$ is not a $K$—finite vector, yields we contradict Lemma 4.2 b). In the first case, a similar argument yields $S := T^*_r$ is not the restriction of a linear differential operator. This concludes the proof of ii) in Theorem 4.3. The statements iii) is obvious.

**Corollary 4.6.** Let $\{R_a\}$ be a linear basis for $U(g)$. Then, there exists smooth functions $g_{a,b}$ on $H$ so that

$$
K_S(y, h)^*z_j = \sum_{a,b,i} d_{j,b,i}g_{a,b}(h)(L^{(1)}_{R_a} K_\lambda)(y, h)^*w_i.
$$
In fact, let \( g_{a,b} \) be smooth functions so that \( \text{Ad}(h^{-1})D_b = \sum_a g_{a,b}(h)R_a \).

\[
K_S(y,h)^*z_j = K_S(h^{-1}y,e)^*z_j = \sum_{a,b,i} d_{j,b,i}(L_{Ra}^{(1)}K_{\lambda})(h^{-1}y,e)^*w_i \\
= \sum_{a,b,i} d_{j,b,i}(L_{Ad(h^{-1})R_a}^{(1)}K_{\lambda})(y,h)^*w_i \\
= \sum_{a,b,i} d_{j,b,i}g_{a,b}(h)(L_{Ra}^{(1)}K_{\lambda})(y,h)^*w_i.
\]

**Example 4.7.** For a real form \( H/L \) for the Hermitian symmetric space \( G/K \) and a holomorphic discrete series \( H^2(G,\tau) \) for \( G \), any nonzero intertwining linear \( H \)-map \( S : H^2(G,\tau) \to H^2(H,\sigma) \) never is the restriction of a differential operator.

The statement holds because under our hypothesis \( \text{res}_H(\pi_\lambda) \) is not discretely decomposable \([Ho]\).

4.2. Maps into \( K \)-types. A related result to Lemma 4.2, and by means of a very similar proof, is as follows. Let \((\vartheta,B)\) be an irreducible representation of \( K \). We fix a realization of \((\vartheta,B)\) as a subspace of the space of smooth sections of a bundle \( K \times_\vartheta C \to K/K_1 \). Here, \( K_1 \) is a closed subgroup of \( K \) and \((\vartheta,C)\) is a finite dimensional representation for \( K_1 \).

**Fact 4.8.** Assume \((\vartheta,B)\) is a \( K \)-type for \( H^2(G,\tau) \). Let \( S : H^2(G,\tau) \to B \) be a continuous intertwining linear map. Then, \( S \) is the restriction of a differential operator.

Indeed, we fix \( c \in C, k \in K \), then the map \( H^2(G,\tau) \ni f \mapsto (S(f)(k),c)_C \) is a continuous linear functional in \( H^2(G,\tau) \). Thus, there exists \( K_S : G \times K \to \text{Hom}_C(W,C) \) so that \( y \mapsto K_S(y,c)^*c \) belongs to \( H^2(G,\tau) \) and \((S(f)(k),c)_C = \int_G (f(y),K_S(y,k)^*c)_W dy \). Thus, \( y \mapsto K_S(y,c)^*c \) belongs to \( H^2(G,\tau)[B] \subset (V_{\lambda})_{K-\text{fin}} \). The \( U(g) \)-irreducibility for \((V_{\lambda})_{K-\text{fin}} \) implies there exists \( D \in U(g) \) so that the functions \( L_D^{(1)}K_{\lambda}(\cdot,e)w, K_S(\cdot,e)^*c \) are equal. Recalling \( K_S(y,k) = K_S(k^{-1}y,e) \) and proceeding as in the proof of Lemma 4.2 and its Corollary the Fact follows.

**Fact 4.9.** Assume \( \text{res}_H(\pi_\lambda,V_{\lambda}) \) is admissible. Then any intertwining operator from \( H^2(G,\tau) \) into \((\vartheta,B)\) presented as in Fact 4.8 is a differential operator.

In fact, the \( H \)-admissibility forces \( V_{\lambda}[B] \subset (V_{\lambda})_{K-\text{fin}} \). Now, the proof goes word by word as the one for Fact 4.8.

**Remark 4.10.** In the setting of Fact 4.8 or Fact 4.9, we further assume \( B \) is realized as the kernel of differential operators. Then, any intertwining map \( S : H^2(G,\tau) \to B \) extends to and intertwining map from the maximal globalization provided by the kernel of the Schmid operator into \( B \). The extension is a differential operator.

The proof of this remark is as the proof for Theorem 4.12.

**Remark 4.11.** Any intertwining linear map \( S : H^2(G,\tau) \to L^2(H \times_\nu E) \) restricted to the subspace \( H^2(G,\tau)_{H-\text{disc}} \) is at the same time the restriction of a kernel map and a differential operator.
4.3. Extension of an intertwining map to maximal globalization. A conjecture of Toshiyuki Kobayashi [Kolv] predicts that under certain hypothesis each continuous intertwining linear operator between two maximal globalizations of Zuckerman modules, realized via Dolbeault cohomology, are given by restriction of a holomorphic differential operator. In this subsection we show an analogous statement for the maximal globalization provided by the kernel of a Schmid operator. The symbols $G, K, (\tau, W), H^2(G, \tau), H, L, (\sigma, Z), H^2(H, \sigma)$ are as usual. Let

$$D_G : C^\infty(G \times \tau W) \to C^\infty(G \times \tau_1 W_1)$$

be the Schmid operator [Sch] [Wo]. Similarly, we have a Schmid operator $D_H : C^\infty(H \times_\sigma Z) \to C^\infty(H \times_{\sigma_1} Z_1)$. Since $D_G$ is an elliptic operator $\text{Ker}(D_G)$ is a closed subspace of the space of smooth sections. Thus, $\text{Ker}(D_G)$ becomes a smooth Frechet representation for $G$. Among the properties of the kernel of the operator $D_G$ are: $H^2(G, \tau)$ is a linear subspace of $\text{Ker}(D_G)$, the inclusion map $H^2(G, \tau)$ into $\text{Ker}(D_G)$ is continuous, the subspace of $K$-finite vectors in $\text{Ker}(D_G)$ is equal to the subspace of $K$-finite vectors for $H^2(G, \tau)$, $\text{Ker}(D_G)$ is a maximal globalization for the underlying Harish-Chandra module for $(\pi_{\lambda}, H^2(G, \tau))$. A similar statement holds for $D_H$. Now, we are ready to state the corresponding result.

**Theorem 4.12.** We assume $\text{res}_H(\pi_{\lambda})$ is an $H$-admissible representation. Then, the following two statements holds:

a) Any continuous, $H$-intertwining linear map $S : \text{Ker}(D_G) \to \text{Ker}(D_H)$ is the restriction of a differential operator.

b) Any continuous $H$-intertwining linear map $S : H^2(G, \tau) \to H^2(H, \sigma)$ extends to a continuous intertwining operator from $\text{Ker}(D_G)$ to $\text{Ker}(D_H)$.

Nakahama in [N][Theorem 3.6] has shown a similar result under the hypothesis of both $G/K, H/L$ are Hermitian symmetric spaces, the inclusion $H/L$ into $G/K$ is holomorphic, and both representations are holomorphic discrete series.

**Proof of Theorem 4.12.** We show a). Let $S$ be as in the hypothesis. Since each inclusion $H^2(G, \sigma) \subset \text{Ker}(D_G)$, $H^2(H, \sigma) \subset \text{Ker}(D_H)$ is continuous, we have for $h \in H, z \in Z$ that the linear functional $H^2(G, \tau) \ni f \mapsto (S(f)(h), z)_Z$ is continuous, whence, Riez representation theorem implies there exists an element $y \mapsto K_S(y, h)^* z$ of $H^2(G, \tau)$ so that $(S(f)(h), z)_Z = \int_G(f(y), K_S(y, h)^* z) W d y$. It readily follows that $y \mapsto K_S(y, e)^* z$ is an $L$-finite vector. Since, for discrete series, the hypothesis of $H$-admissibility implies $L$-admissibility [DV], we apply [Kolv][Proposition 1.6], hence, $y \mapsto K_S(y, e)^* z$ is $K$-finite vector. The $U(g)$-irreducibility of the subspace of $K$-finite vectors yields $K_S(\cdot, e)^* z = L^{(1)}_{D_{z, w}} \Lambda(\cdot, e)^* w$. Next, for a $K$-finite vector $f$ after a computation similar to the one in the proof of Theorem 4.3, we arrive at the equality $(S(f)(h), z)_Z = ([R_{D_{z, w}} f](h), w)_W$. The continuity of $S$ together with $\text{Ker}(D_G)$ is a maximal globalization let us conclude $S$ is the restriction of a differential operator. Thus, we have shown a). We now verify b). The hypothesis of $H$-admissibility let us to apply Theorem 4.3. Therefore, $S$ is the restriction of a differential operator. More precisely, $S(f)(h) = \sum_{a, b, i} d_{a, b, i} ([R_{D_h} f](h), w_i)_w z_a$ and $D_H(S(f)) \equiv 0$ for any
$K$–finite vector $f$ in $H^2(G, \tau)$. We extend $S$ to $\text{Ker}(D_G)$ via the previous equality. Obviously the extension is continuous in smooth topology. We claim: The image of the extension is contained in $\text{Ker}(D_H)$. Indeed, owing to the subspace of $K$–finite vectors in $H^2(G, \tau)$ is dense in $\text{Ker}(D_G)$ in smooth topology, we obtain $D_H(Sf) \equiv 0$ for every $f \in \text{Ker}(D_G)$. Whence, we have shown Theorem 4.12.

Theorem 4.12 is a step in showing the conjecture of Kobayashi. Actually, the statement in Theorem 4.12 is a solution to the conjecture of Kobayashi if we choose as maximal globalization the one's constructed via Schmid operator. In order to formulate the conjecture, we need to recall notation as well as results from Schmid thesis [Sch]. The Harish-Chandra parameter $\lambda$ gives rise to $G$–invariant complex structure on $G/T$, as well as a $K$–invariant complex structure on $K/T$ and holomorphic line bundles $L_\lambda \to G/T$, $\xi_\lambda \to K/T$ so that the representation of $K$ in $H^s(K/T, \mathcal{O}(L_\lambda))$ is equivalent to $(\tau, W)$ and the representation of $G$ on $H^s(G/T, \mathcal{O}(L_\lambda))$ is infinitesimally equivalent to $(\pi_\lambda, V^G_\lambda)$. Here $s = 1/2 \dim K/T$. Owing to the construction of the respective complex structures the inclusion map $i_K : K/T \to G/T$ is holomorphic. After we endow the space of smooth forms on $G/T$ with the smooth topology, work of Schmid, Wolf and Hon-Wei Wong shows the image of $\partial$ is closed. Thus, $H^s(G/T, \mathcal{O}(L_\lambda))$ affords a Frechet representation for $G$. Next, we describe an equivalence $F_{K,T}$ between the representations $(L, \text{Ker}(D_G))$ and $(\ell^*, H^s(G/T, \mathcal{O}(L_\lambda)))$. For this, we model $(\tau, W)$ on $(\ell^*, H^s(K/T, \mathcal{O}(L_\lambda)))$. Then, for a smooth $(0, s)$–form $\varphi$ on $G/T$ with values on $L_\lambda$ Schmid associates the function $G \ni g \mapsto F_{K,T}(\varphi)(g) := i_K^*(\ell^s_\varphi) \in H^s(K/T, \mathcal{O}(L_\lambda))$. Schmid shows that when $\varphi$ is closed we have $F_{K,T}(\varphi)$ belongs to $\text{Ker}(D_G)$ and the resulting map from $H^s(G/T, \mathcal{O}(L_\lambda))$ into $\text{Ker}(D_G)$ is bijective. We do not describe the inverse of the map $F_{K,T}$. Similarly, attached to the representation $V^H_{\mu}$, we have a holomorphic line bundle $L_\mu$ over $H/U$, and a map $F_{L,U} : H^s(H/U, \mathcal{O}(L_\mu)) \to \text{Ker}(D_H)$. The piece of the conjecture of Kobayashi that we are able to contribute is: Assume $\text{res}_H(\pi_\lambda)$ is an $H$–admissible representation. Then, for every discrete factor $H^2(H, \sigma)$ of $\text{res}_H(H^2(G, \tau))$ we have that any $H$–intertwining continuous map from $H^s(G/T, \mathcal{O}(L_\lambda))$ into $H^s(H/U, \mathcal{O}(L_\mu))$ is a holomorphic differential operator. We are able to show that an intertwining operators is a differential operators. In fact, let $S : H^s(G/T, \mathcal{O}(L_\lambda)) \to H^s(H/U, \mathcal{O}(L_\mu))$ be a continuous intertwining operator. Then $F_{L,U}SF_{K,T}^{-1}$ is a continuous intertwining linear map from $H^2(G, \tau)$ into $H^2(H, \sigma)$. Theorem 4.12 yields that this composition is a differential operator in our sense.

4.4. Comments on the relation among Hom’s. As usual, Hom$_H(\ldots, \ldots)$ denotes the space of continuous intertwining operators. We have the natural inclusions

$$\begin{align*}
\text{Hom}_{\delta,L}(H^2(G, \tau)_{K-fin}, H^2(H, \sigma)_{L-fin})
\supseteq \text{Hom}_H(H^2(G, \tau)_{H-smooth}, H^2(H, \sigma)^\infty)
\supseteq \text{Hom}_H(H^2(G, \tau), H^2(H, \sigma))
\end{align*}$$
as well as similar inclusions for $\text{Hom}'s(H^2(H, \sigma), H^2(G, \tau))$. We would like to point out that when $\text{res}_H(\pi_\lambda)$ is $H$–admissible the above inclusions are equalities. In fact, in [Kob1][Lemma 1.3, Prop. 1.6] it is shown that the above inclusions are equalities under the hypothesis of $\text{res}_H(\pi_\lambda)$ is $L$–admissible. Actually, for discrete series representations $H$–admissible is equivalent to be $L$–admissible [DV]. As a by product, Kobayashi obtained the equalities

$$H^2(G, \tau)_{L-\text{fin}} = H^2(G, \tau)_{K-\text{fin}} = \bigoplus_{M \in \{L-\text{irred}\}} H^2(G, \tau)_{L-\text{fin}}[M].$$

Here, the sum is algebraic. Thus, this work of Kobayashi together with Theorem 4.3 shows that once we know a representation $V_\lambda$ is $H$–admissible, the associated branching law problem is algebraic.

In different notes T. Kobayashi and his co-authors have done a deep study of the space of continuous intertwining linear operators between two principal series representations, their results yields estimates for dimension of such spaces as well as precise computation of such spaces. Next, we present some comments of our work for discrete series representations. For this note,

$$\text{Diff}_H(H^2(G, \tau), H^2(H, \sigma))$$

is the space of not necessarily continuous linear intertwining maps that are restriction of differential operators. We have shown in Lemma 4.2 the inclusion "automatic continuity Theorem"

$$\text{Diff}_H(H^2(G, \tau), H^2(H, \sigma)) \subset \text{Hom}_H(H^2(G, \tau), H^2(H, \sigma))$$

We have shown that a representation is $H$–admissible if and only for every $H^2(H, \sigma)$ the following equality holds,

$$\text{Diff}_H(H^2(G, \tau), H^2(H, \sigma)) = \text{Hom}_H(H^2(G, \tau), H^2(H, \sigma))$$

Besides, we have shown that if for some $H^2(H, \sigma)$ we have

$$0 < \dim \text{Diff}_H(H^2(G, \tau), H^2(H, \sigma)),$$

then $\text{res}_H(\pi_\lambda)$ is discretely decomposable. Therefore, discretely decomposable and multiplicity of $H^2(H, \sigma)$ infinite, forces some element in $\text{Hom}_H(H^2(G, \tau), H^2(H, \sigma))$ is not the restriction of a differential operator.

For a closed subgroup $L_1$ of $K$, under the hypothesis that $\pi_\lambda$ is $L_1$–admissible, after we realize each irreducible representation $(\vartheta, B)$ for $L_1$ in some space of smooth functions, we have shown the equality

$$\text{Diff}_{L_1}(H^2(G, \tau), B) = \text{Hom}_{L_1}(H^2(G, \tau), B).$$

Finally, after we realize each irreducible representation for $L_1$ in a space of smooth functions $B$ ($B$ varies with the representation), and each space is the kernel of a differential operator we have:

Each continuous $L_1$-map from $H^2(G, \tau)$ into $B$ is a differential operator and extends to a $L_1$–differential operator from $\text{Ker}(D_G)$ into $B$.

5. TWO RESULTS ON $L^2(G \times \tau W)[V^G_{\lambda_1}]$

Besides, the previous notation $(\tau, W), H^2(G, \tau)I$ we consider another square integrable irreducible representation $V^G_{\lambda_1}$ for $G$ of Harish-Chandra parameter $\lambda_1$ and lowest $K$–type $(\tau_1, W_1)$. In this section, we show that any intertwining $G$–map from $H^2(G, \tau_1)$ into $L^2(G \times \tau W)$ is an integral operator. In the
second part of this section we compute a kernel for the orthogonal projector onto the isotypic component determinate by $V^G_{\lambda_1}$.

5.1. Analysis for the elements of $\text{Hom}_G(H^2(G, \tau_1), L^2(G \times_{\tau} W))$. In this subsection, we develop a similar study to the one developed by [Kobs] on Shintani’s functions. For a Harish-Chandra parameter $\lambda_1$ we study continuous intertwining linear map $T : H^2(G, \tau_1) = V^G_{\lambda_1} \to L^2(G \times_{\tau} W)$. To begin with, we show that the natural inclusion maps

$$\text{Hom}_G(V^G_{\lambda_1}, L^2(G \times_{\tau} W))$$

extends to a continuous linear map in a Hilbert space, we have $(\text{closure}(\text{Im}S))_{K-fin} = (\text{Im}S)_{K-fin}$. Thus, closure($\text{Im}S$) is an irreducible unitary representation. In [Wn1] Lemma 3.4.11, we find a proof that $S$ extends to a continuous intertwining continuous linear map $T$ from $H^2(G, \tau_1)$ onto the closure of the image of $S$. Hence, the claim follows. Next, we show, for $T \in \text{Hom}_G(V^G_{\lambda_1}, L^2(G \times_{\tau} W))$ that

**Proposition 5.1.** $T$ is represented by a $G-$invariant smooth kernel $k_T$. We set $k(x) := K_T(x, 1)$. The function $k$ satisfies: $k : G \to \text{Hom}_C(W_1, W)$ is smooth, $k(k_1 x k_2) = \tau(k_1) k(x) \tau_1(k_2), k_1, k_2 \in K, x \in G, k$ is a solution to the equation $L_{\Omega_G} k = [\lambda_1 - (\rho, \rho)] k$, and $k$ is square integrable. Conversely, given a function $k$ which satisfies the four previous properties, then $T f(x) = \int_C k(x^{-1} y) f(y) dy$ defines a continuous intertwining continuous linear map from $H^2(G, \tau_1)$ into $L^2(G \times_{\tau} W)$.

**Proof.** For the direct affirmation we notice that since $T$ is continuous, Schur’s Lemma yields that $T^* T$ is a constant times the identity, hence, we may and will assume $T$ is an isometry into its image. Thus, $\text{Im}(T)$ is a closed irreducible left invariant subspace of $L^2(G \times_{\tau} W)$. Therefore, $\text{Im}(T)$ is included in an eigenspace of the Casimir operator, whence $\text{Im}(T)$ is contained in the kernel of an elliptic $G-$invariant operator. Therefore, $T$ is given by a kernel $K_T : G \times G \to \text{Hom}_C(W_1, W)$ so that the map $y \mapsto K_T(y, x)^* w$ belongs to $H^2(G, \tau_1)$ for each $w \in W, x \in G$. Since $T$ is an intertwining map we have the equality $K_T(y, x) = K_T(x^{-1} y, e)$. Thus, $K_T$ is a smooth function. From $K_T(ky, xk_2) = \tau(k_2^{-1}) K_T(y, x) \tau_1(k)$ we obtain $k(k_1 x k_2) = \tau(k_1) k(x) \tau_1(k_2)$. Since $y \mapsto K_T(y, x)^* w$ belongs to $H^2(G, \tau_1)$ we obtain that $L_{\Omega_G} k = [\lambda_1 - (\rho, \rho)] k$ and that $k$ is square integrable. Conversely, given $k$ that satisfies the four properties listed. Then, in [HC2] Corollary to Lemma 65 it is shown $k$ is tempered in the sense of Harish-Chandra, the hypothesis $G$ is linear let us apply [OV][Proposition 6] to $k$ and deduce $k \in L^{2-\epsilon}(G, \text{Hom}_C(Z, W))$ for $\epsilon$ small. Thus, the Kunze-Stein phenomena [C] let us conclude that the proposed formula for $T$ defines a continuous linear map. □

One way to compute the dimension of the space $\text{Hom}_G(H^2(G, \tau_1), L^2(G \times_{\tau} W))$ is via Frobenius reciprocity and Blatner’s formula. Thus, there is an explicit formula of the dimension based on the Harish-Chandra parameter for
$H^2(G, \tau_1)$, the highest weight for $W$ and a partition function associated to the noncompact roots with positive inner product with the Harish-Chandra parameter for $H^2(G, \tau_1)$. For the groups $Spin(2n, 1), U(n, 1)$ the dimension of $Hom_G(H^2(G, \tau_1), L^2(G \times \tau \ W))$ is either one or zero. It is not easy to solve the differential equation for $k$. For example, for $G = SL_2(\mathbb{R})$ the resulting equation on a maximally split Cartan subgroup is a hypergeometric equation.

5.2. **Kernel for the projector onto** $L^2(G \times \tau \ W)[V^G_{\lambda_1}]$. In this section we generalize a Theorem of [WW] shown for the isotypic component $L^2(G \times \tau \ W)[V^G_\lambda] = H^2(G, \tau)$. We also extend work of Shimeno [Shi] for the case of line bundles over $G/K$. We fix a representative $(\pi_{\lambda_1}, V^G_{\lambda_1})$ of the representation for $G$ of Harish-Chandra parameter $\lambda_1$ and we assume $(\tau, W)$ is a $K$-type for $\pi_{\lambda_1}$. We fix an orthonormal basis $\{f_j\}_{j=1...N}$ for $V^G_{\lambda_1}[W]$ and recall the spherical trace function $\phi_1(z) = d_\lambda \sum_j (\pi_{\lambda_1}^G(z)f_j, f_j)_{V^G_\lambda}$. Then,

**Proposition 5.2.** The orthogonal projector $P$ onto $L^2(G \times \tau \ W)[V^G_{\lambda_1}]$ is kernel map given the "external kernel"

$$P(f)(x) = \int_{G} \phi_1(x^{-1}y)f(y)dy \quad (\dagger)$$

and by the matrix valued kernel

$$(y, x) \mapsto \int_{K} \tau(k^{-1})\phi_1(x^{-1}yk)dk$$

**Proof.** Let $Q$ denote the orthogonal projector onto $V^G_{\lambda_1}[W]$. Then, for each nonzero intertwining $K$-map $b : V^G[W] \rightarrow W$ yields a map $f_b : V^G_{\lambda_1} \rightarrow L^2(G \times \tau \ W)$ defined by $V^G_{\lambda_1} \ni v \mapsto (G \ni x \mapsto b(Q(\pi_{\lambda_1}(x^{-1}v)))) := f_b(v, x)$. $f_b$ gives rise to an equivariant embedding of $V^G_{\lambda_1}$ into $L^2(G \times \tau \ W)[V^G_\mu]$. After we fix a linear basis for $Hom_K(V^G_\mu[W], W)$, owing to Frobenius reciprocity, the subspace $L^2(G \times \tau \ W)[V^G_{\lambda_1}]$ is equal to the linear span of the image of the functions $f_b$ when $b$ runs over the chosen linear basis for $Hom_K(V^G_{\lambda_1}[W], W)$. The spherical trace function $\phi_1$ is $K$-central, hence, it follows that for each $\Gamma$ in $L^2(G \times \tau \ W)$, the right hand side of $(\dagger)$ belongs to $\Gamma(G \times \tau \ W)$. The hypothesis $G$ is a linear group let us conclude that $\phi_1 \in L^2(G)$ for some positive $\epsilon$. Thus, the Kunze-Stein phenomena shown by M. Cowling yields the right hand side determinant a continuous linear operator on $L^2(G \times \tau \ W)$. Next, as in Appendix 10.7 we verify $\int_G \phi_1(x^{-1}y)f_b(v, y)dy = f_b(v, x)$. Harish-Chandra Plancherel Theorem yields the integral evaluated at a wave package or $f$ orthogonal to $L^2(G \times \tau \ W)[V^G_\mu] \cap (L^2(G \times \tau \ W))_{\text{disc}}$ is equal to zero. Thus, the first statement holds. The second statement readily follows. \hfill \Box

6. **Projector onto isotypic component of** $\text{res}_H(\pi_\lambda)$.  

6.1. **Projector onto isotypic components via differential operators.** In [N], it is shown that for scalar holomorphic discrete series the intertwining maps from the ambient space to the ambient space of irreducible factors are given as infinite order differential operators on the subspace of smooth vectors. The aim of this subsection is to analyze the general value of his
functions on the unit disc of $\mathbb{C}$ follow, we compute an infinite order differential operator as well as its kernel. To point out that for a differential operator $D$, to begin results. For this, we analyze ways of expressing the orthogonal projector onto an isotypic component by means of differential operators. To begin with we study an example.

**Example 6.1.** Let $G = SU(1, 1)$. We fix as $K$=diagonal matrices $(e^{i\varphi}, e^{-i\varphi})$. The characters of $K$ are $e^{in\varphi}, n \in \mathbb{Z}$. The discrete series $(\pi_\lambda, V_\lambda)$ is presented as a subspace of the space of holomorphic functions $f(z) = a_0 + a_1 z + \cdots + a_n z^n + \cdots$ on the unit disk $D = \{ z \in \mathbb{C} : |z| < 1 \}$. The action of $G$ is by homographic transformations. The action of $K$ is so that a typical isotypic subspace is $V_\lambda[V^K_\theta] = \mathbb{C}z^n$. Then, $P_{V_\lambda[V^K_\theta]}(f)(z) = c_{\lambda, \theta} z^n \frac{\partial^n f}{\partial z^n}(0)$ and $K_{V_\lambda[V^K_\theta]}(z, w) = d_{\lambda, \theta} z^n w^n$. The reproducing kernel is $K_\lambda(z, w) = \frac{k_1}{(1-\bar{z}w)c}$ and $K_{V_\lambda[V^K_\theta]}(z, w) = d_{\lambda, \theta} z^n w^n \frac{\partial^n}{\partial z^n}(K_\lambda)(0, 0)$. Since, $K_\lambda$ is a rational function and $K_{V_\lambda[V^K_\theta]}$ is a polynomial function, no derivative of $K_\lambda$ is equal to $K_{V_\lambda[V^K_\theta]}$. After we identify the space of functions on $K/K$ into $\mathbb{C}$ with $\mathbb{C}$, the intertwining from $H^2(G, \tau)$ into $H^2(K, e^{in\varphi})$ becomes the map $f \mapsto \frac{\partial^n f}{\partial z^n}(0)$. This example expresses $P_{V_\lambda[V^K_\theta]}(f)(\cdot) = F_1(\cdot)D(f)(e)$, $K_{V_\lambda[V^K_\theta]}(\cdot) = F_2(\cdot)\tilde{D}(K_\lambda)(e, e)$ where both $D, \tilde{D}$ are elements in $U(\mathfrak{sl}(2, \mathbb{R}))$ and $F_1, F_2$ smooth functions. We show this a common feature for each projector $P_{V_\lambda[V^K_\theta]}$ as well as its kernel.

**Example 6.2.** This example is a sequel to the previous one. We would like to point out that for a differential operator $D = f_0 + f_1 \delta + \cdots + f_N \partial^N + \cdots$ ($\delta = d/dz$) that maps into itself the model of $H^2(SU(1, 1), \tau)$ in holomorphic functions on the unit disc of $\mathbb{C}$, and it is equal to orthogonal projector onto $\mathbb{C}z^M$, we have $D$ is of infinite order. In fact, after a computation we obtain

$$D = \frac{Z^M}{M!} \partial^M - \frac{Z^{M+1}}{M!} \partial^{M+1} + \frac{Z^{M+2}}{2 M!} \partial^{M+2} \cdots + (-1)^p \frac{Z^{M+p}}{p! M!} \partial^{M+p} + \cdots$$

**Example 6.3.** Let $G, H, H^2(G, \tau)$ be as usual. We assume $\pi_\lambda$ is an $H-$admissible representation. We fix an isotypic component $M$ for the action of $H$. To follow, we compute an infinite order differential operator $D$ so that its restriction to the subspace of $K-$finite vectors agrees with the restriction of the orthogonal projector $P_M$ onto the isotypic component $M$. For this, we label the isotypic components as $M_1, M_2 = M, M_3, M_4, \cdots$. We fix a nonzero element $\Omega$ in the center of $U(h)$. Thus, $R := L_\Omega$ acts by a constant $c_j$ on the subspace of $H-$smooth vectors in $M_j$. We further assume $c_k \notin \{ c_1, \cdots, c_{k-1} \}$ for $k \geq 2$. The hypothesis $H-$admissible together a result of T. Kobayashi quoted in the note, imply that the subspace of $L-$finite vectors in $V_\lambda$ is equal to the subspace of $K-$finite vectors. We claim: There exists a sequence of numbers $d_0, d_1, \cdots$ such that

$$P_M = d_0(R - c_1) + d_1(R - c_1)(R - c_2) + d_2(R - c_1)(R - c_2)(R - c_3) + \cdots$$

In fact, the hypothesis let us to find $d_0$ so that $d_0(R - c_1)$ is equal to the identity in $M_2$, then we find $d_1$ so that $d_0(R - c_1) + d_1(R - c_1)(R - c_2)$ is equal to zero on $M_3$, and so on.
The proof of the next statement shows techniques that let us compute some differential operator with smooth coefficients that represents the orthogonal projector $P_M$ onto a finite dimensional $K-$invariant subspace $M$.

**Fact 6.4.** Let $(\theta, V^K_\theta)$ denote an irreducible representation for $K$. Let $P_{\lambda[V^K_\theta]}$ denotes the orthogonal projector onto $V_{\lambda[V^K_\theta]}$, let $K_{V_{\lambda[V^K_\theta]}} : G \times G \to Hom_C(W, W)$ denotes the kernel that represents $P_{\lambda[V^K_\theta]}$. We fix an orthonormal basis $\{w_i\}$ for $W$. Then, there exists a family $\{D_{i,\alpha}\} 1 \leq i \leq \text{dim} W, 1 \leq \alpha \leq \dim V_{\lambda[V^K_\theta]}$ of elements in $U(\mathfrak{g})$ and complex valued smooth functions $F_{i,\alpha}$ on $G$ so that

$$K_{V_{\lambda[V^K_\theta]}}(y, x)^*w = \sum_{i,\alpha} F_{i,\alpha}(x)(L_{D_{i,\alpha}}(1)K_{\lambda}(y, e)^*w, w_i)w_i.$$ 

Further,

$$P_{\lambda[V^K_\theta]}(f)(x) = \sum_{i,\alpha} F_{i,\alpha}(x)(L_{D_{i,\alpha}}(f)(e), w_i)w_i.$$ 

Indeed, the identity

$$(P_{\lambda[V^K_\theta]}(f)(x), w_i)_W = \int_G (f(y), K_{V_{\lambda[V^K_\theta]}}(y, x)^*w_i)_W dy \quad (d)$$

shows the function $y \mapsto K_{V_{\lambda[V^K_\theta]}}(y, x)^*w_i$ is orthogonal to any function $f$ orthogonal to $V_{\lambda[V^K_\theta]}$. Thus, $y \mapsto K_{V_{\lambda[V^K_\theta]}}(y, x)^*w_i$ belongs to $V_{\lambda[V^K_\theta]}$. The function

$$G \ni x \mapsto H_i(x) := K_{V_{\lambda[V^K_\theta]}}(\cdot, x)^*w_i \in V_{\lambda[V^K_\theta]}$$

is smooth, because $V_{\lambda[V^K_\theta]}$ is a finite dimensional vector space and the equality $(d)$ shows that evaluation of coordinate functions in our function is a smooth function. Next, let $k_w(\cdot) := K_{\lambda}(\cdot, e)^*w$. Then, $k_w$ belongs to $V_{\lambda}[W]$ and it is non zero for each nonzero $w$. We fix $i : 1 \leq i \leq \text{dim} W$. Hence, the $U(\mathfrak{g})-$irreducibility for $(V_{\lambda})_{K-fini}$ implies there exists a finite dimensional vector subspace $N_{\theta, i}$ of $U(\mathfrak{g})$ so that the linear map $R_i : N_{\theta, i} \to V_{\lambda[V^K_\theta]}$ defined by $R_i(D) := L_D(k_w_i)$ is bijective. Therefore, the composition $R_i^{-1}H_i$ is a smooth function. To follow, we fix a linear basis $D_{i,\alpha}, 1 \leq \alpha \leq \dim V_{\lambda[V^K_\theta]}$ for $N_{\theta, i}$. Hence, $R_i^{-1}H_i(x) = \sum_{\alpha} F_{i,\alpha}(x)D_{i,\alpha}$ where $F_{i,\alpha}$ are complex valued smooth functions on $G$. Thus,

$$K_{V_{\lambda[V^K_\theta]}}(y, x)^*w_i = H_i(x)(y) = R_i(R_i^{-1}(H_i(x)))(y) = \sum_{\alpha} F_{i,\alpha}(x)L_{D_{i,\alpha}}(1)K_{\lambda}(y, e)^*w_i \forall y, x \in G.$$ 

$$K_{V_{\lambda[V^K_\theta]}}(y, x)w = \sum_i (K_{V_{\lambda[V^K_\theta]}}(y, x)w, w_i)w_i$$

$$= \sum_i (w, K_{V_{\lambda[V^K_\theta]}}(y, x)^*w_i)_W w_i = \sum_{i,\alpha} (w, F_{i,\alpha}(x)L_{D_{i,\alpha}}(1)K_{\lambda}(y, e)^*w_i)_W w_i$$

$$= \sum_{i,\alpha} (F_{i,\alpha}(x)L_{D_{i,\alpha}}(1)K_{\lambda}(y, e)w, w_i)_W w_i.$$
For $f \in V_\lambda$, the following equalities holds.

$$P_{V_\lambda[V_H]}(f)(x) = \sum_i (P_{V_\lambda[V_H]}(f)(x), w_i)_W w_i = \sum_i \int_G (f(y, K_{V_\lambda[V_H]})(y, x)^*w_i)_W dyw_i$$

$$= \sum_i \int_G (f(y, \sum \alpha F_{i,\alpha}(x)L_{D_{i,\alpha}}^{(1)} \lambda(y, e)^*w_i)_W dyw_i$$

$$= \sum_i \int_G (\sum \alpha F_{i,\alpha}(x)L_{D_{i,\alpha}}^{(1)}(f)(y), \lambda(y, e)^*w_i)_W dyw_i$$

$$= \sum_i (\sum \alpha F_{i,\alpha}(x)L_{D_{i,\alpha}}^{(1)}(f)(e), w_i)_W w_i.$$

Finally, the equality of the first and last member extends to any $f \in V_\lambda$ owing that both sides agree on smooth vectors, and for a fixed $x \in G$, both sides are continuous linear functionals on $H^2(G, \tau)$. This concludes the proof of Fact 6.4.

Fact 6.5. A completely similar result to Fact 6.4 holds after we replace $V_\lambda[V_H]$ by a finite dimensional subspace $M$ of $(V_\lambda[K-f_{fin}])$. Of course, the elements $F_{i,\alpha}, D_{i,\alpha}$ depends on $M$ and are highly non unique.

Next Proposition shows a relation between discretely decomposable and projectors represented by restriction of differential operators.

**Proposition 6.6.** Assume res$_H(\pi_\lambda)$ is an $H$–admissible representation. Then for any finite dimensional subspace $M$ of $(V_\lambda[K-f_{fin}])$ the orthogonal projector onto $M$ can be expressed by means of differential operators. Conversely, if for some Harish-Chandra parameter $\mu$ for $H$, the orthogonal projector onto some nontrivial finite dimensional subspace of $(V_\lambda[V_\mu])_L-f_{fin}$ can be expressed via differential operators, then res$_H(\pi_\lambda)$ is discretely decomposable. Furthermore, if for each finite dimensional $L$–invariant subspace of $V_\lambda[V_\mu]$, its orthogonal projector is restriction of differential operator, then the multiplicity of $V_\mu$ is finite.

**Proof.** For the direct implication, the hypothesis $H$–admissible let us to apply [Kob1][Proposition 1.6]. Thus, $(V_\lambda[L-f_{fin}]) = (V_\lambda[K-f_{fin}])$. Whence, Fact 6.5 yields there exists $\{D_{i,\alpha}\}, 1 \leq i \leq \dim W, 1 \leq \alpha \leq \dim M$ elements in $U(g)$ and complex valued smooth functions $F_{i,\alpha}$ on $G$ so that for every $f \in V_\lambda$ we have

$$P_M(f)(x) = \sum_{i,\alpha} F_{i,\alpha}(x)(L_{D_{i,\alpha}}(e), w_i)_W w_i.$$

Whence, $P_M$ is computed by means of differential operators. For the converse statement, the hypothesis on $M$ gives an expression for $P_M$ as above and for $K_{P_M}$ ($w, v \in W$) we have

$$(K_{P_M}(y, x)^*w, v)_W = \sum_{i,\alpha} F_{i,\alpha}(x)(L_{D_{i,\alpha}}^{(1)} \lambda(y, e)^*w, w_i)_W (w_i, v)_W.$$

Whence, for a fixed $x \in G$, the function $y \mapsto K_{P_M}(y, x)^*w$ is $K$–finite, because the above expression for the function $y \mapsto K_{P_M}(y, x)^*w$ is as finite sum
of \( K \)-finite vectors of type \( y \mapsto LDK_\lambda(y, e)^*w \). Now, any \( H \)-smooth element in \( V_\lambda[V_\mu^H]_{\text{L-fin}} \) is \( 3(U(h)) \)-finite and, by hypothesis, \( M \subset V_\lambda[V_\mu^H]_{\text{L-fin}} \). Thus, \( KP_{M}(\cdot, x)^*w \) is \( K \)-finite and \( 3(U(h)) \)-finite. Whence, as in previous Theorem’s we conclude \( \pi_\lambda \) is algebraically decomposable. To follow we show the last statement in the Proposition. The proof goes parallel to the proof of Theorem 4.3 ii). That is, we assume \( V_\lambda[V_\mu^H] \) is not \( H \)-admissible and we built up a finite dimensional \( L \)-invariant subspace \( M \) so that \( P_\lambda \) is not a differential operator. Let \( T_j : V_\mu^H \rightarrow V_\lambda[V_\mu^H] \) be isometric immersions \( H \)-maps so that the image subspaces are pairwise orthogonal. We fix \( v_0 \in V_\mu^H[V_\mu^L] \) of norm one. There are two possibilities, either every \( T_j(v_0) \) is a \( K \)-finite vector, or at least one \( T_j(v_0) \) is not a \( K \)-finite vector. In the second case we set \( w := T_j(v_0) \) and \( M = \text{linear span } \pi_\lambda(L)w \). In the first case, we choose a sequence of positive real numbers \( (a_n) \) so that \( w := \sum a_nv_n \) is nonzero and we set \( M = \text{linear span } \pi_\lambda(L)w \). Due that \( \pi_\lambda \) is \( K \)-admissible, we have \( w \) is a not \( K \)-finite vector. As in the proof of Theorem 4.3 \( M \) is finite dimensional, a subspace of \( V_\lambda[V_\mu^H] \) and \( P_\lambda \) is not the restriction of a differential operator because \( KP_{M}(\cdot, e)^*v_0 = w \) is not a \( K \)-finite vector. \( \square \)

Next, we show that after we assume \( \text{res}_H(\pi_\lambda) \) is an \( H \)-admissible representation, then the orthogonal projector onto an isotypic component for \( H \) is equal to an infinite degree differential operator on the subspace of smooth vectors for \( G \). The result somewhat generalizes [N][Theorem 3.10] for scalar holomorphic discrete series. The set up for the next Theorem is, as usual: \( G, K, H, L, (\tau, W), H^2(G, \tau) = V_\lambda, H^2(H, \sigma) = V_\mu^H \).

**Theorem 6.7.** We assume \( \text{res}_H(\pi_\lambda) \) is an admissible representation for \( H \). We denote by \( \sigma_1, \sigma_2, \ldots \) the \( L \)-types for \( V_\mu^H \). Let \( P_{\lambda, \mu} \) denote the orthogonal projector on the isotypic component \( V_\lambda[V_\mu^H] \). Then, for \( j \in \mathbb{N}, 1 \leq i \leq \dim W, 1 \leq \alpha \leq \dim V_\lambda[V_\mu^H][V_{\sigma_j}^L] \), there exists a family \( \{D_{j, i, \alpha}\} \) of elements in \( U(\mathfrak{g}) \) and complex valued smooth functions \( F_{j, i, \alpha} \) on \( G \), such that for each smooth vector \( f \in V_\lambda \) we have

\[
P_{\lambda, \mu}(f)(x) = \sum_{j \in \mathbb{N}} \sum_{i, \alpha} F_{j, i, \alpha}(x)(LD_{j, i, \alpha}(f)(e), w_i)W w_i.
\]

The convergence is in smooth topology. Furthermore, for \( K \)-finite function \( f \) the sum on the right is finite.

**Proof.** In [DV] we find a proof that the hypothesis \( H \)-admissible implies \( V_\lambda \) is also \( L \)-admissible. The hypothesis leads that the isotypic subspaces \( V_\lambda[V_\mu^H][V_{\sigma_j}^L] \) are nonzero, finite dimensional and we have the Hilbert sum \( V_\lambda[V_\mu^H] = \oplus_j V_\lambda[V_\mu^H][V_{\sigma_j}^L] \). Further, the hypothesis of being \( H \)-admissible forces all the \( L \)-finite vectors are \( K \)-finite vectors [Kob1], hence, each subspace \( V_\lambda[V_\mu^H][V_{\sigma_j}^L] \) is contained in \( (V_\lambda)_{K-\text{fin}} \). We now apply Proposition 6.6 to obtain finitely many elements \( D_{j, i, \alpha}, i = 1, \cdots, \dim W, 1 \leq \alpha \leq \dim V_\lambda[V_\mu^H][V_{\sigma_j}^L] \) in \( U(\mathfrak{g}) \) so that

\[
P_{\lambda, \mu}(f)(x) = \sum_{i, \alpha} F_{j, i, \alpha}(x)(LD_{j, i, \alpha}(f)(e), w_i)W w_i.
\]

Next, the series \( \sum_{j} P_{\lambda, \mu}(V_\mu^H[V_{\sigma_j}^L]) \) converges pointwise to \( P_{\lambda, \mu} \). Further,[HC2] for a smooth vector \( f \) the convergence is absolute in the smooth topology. Whence, we have obtained the first statement in Theorem 6.7. The second
statement follows because \(L\)–admissible implies each isotypic component for an irreducible representation of \(K\) is contained in a finite sum of isotypic components for \(L\).

\[\square\]

6.2. **Kernel for the projector onto** \(H^2(G, \tau)[V^H_\mu]\). Let \(G, H, (\tau, W)\), \(H^2(G, \tau)\) be as usual. Let \((\pi^H_\mu, V^H_\mu)\) denote an irreducible square integrable representation for \(H\). By definition, the isotypic component, \(H^2(G, \tau)[V^H_\mu]\), for \(V^H_\mu\) is the closure of the sum of the totality of closed \(H\)–invariant subspaces in \(H^2(G, \tau)\) so that the induced action of \(H\) is equivalent to \((\pi^H_\mu, V^H_\mu)\).

Since, \(H^2(G, \tau)\) is a reproducing kernel space, we have that \(H^2(G, \tau)[V^H_\mu]\) is a reproducing kernel space. the orthogonal projector \(P_{\lambda,\mu}\) onto the isotypic component \(H^2(G, \tau)[V^H_\mu]\). Thus, the orthogonal projector onto \(H^2(G, \tau)[V^H_\mu]\) is represented by a matrix kernel \(K_{\lambda,\mu}\). In this section, under certain hypothesis, we express the matrix kernel \(K_{\lambda,\mu}\) in terms of the matrix kernel that represents the orthogonal projector onto \(H^2(G, \tau)\) and the distribution character \(\Theta_{\pi^H_\mu}\) of the representation \((\pi^H_\mu, V^H_\mu)\). We are quite convinced the formula is true under more general hypothesis. The proposed formula is in:

**Proposition 6.8.** Assume the restriction to \(H\) of \(\pi_\lambda\) is an \(H\)–admissible representation. Then, \(P_{\lambda,\mu}\) is equal to the Carleman operator given by the kernel

\[(y, x) \rightarrow d_\mu \Theta_{\pi^H_\mu}(h \mapsto K_\lambda(h^{-1}y, x)) = d_\mu \Theta_{\pi^H_\mu}(h \mapsto K_\lambda(hy, x)).\]

In order to avoid cumbersome notation, for this subsection, we write \(xv := \pi_\lambda(x)v \in G, v \in H^2(G, \tau)\). A proof of Proposition 6.8 will be given at the end of this section. For the time being, we show Proposition 6.8 under the hypothesis: \(G\) is any Lie group and \(K, H\) are compact subgroups of \(G\). Thus, \(V^H_\mu\) is a finite dimensional vector space. We fix \(N\) a reproducing kernel \(G\)–invariant subspace of \(L^2(G \times \tau, W)\). Under these hypothesis, the orthogonal projector \(P_N\) onto \(N\) is represented by different kernels. For any kernel \(K_N\) that represents the orthogonal projector \(P_N\), we want to show \(P_{N[V^H_\mu]}\) is represented by \(K_1(y, x) := \int_H d_\mu \chi_{\pi^H_\mu}(h)K_N(h^{-1}y, x)dh\). According to a classical result, the orthogonal projector onto \(L^2(G \times \tau, W)[V^H_\mu]\) is the linear operator \(\pi(d_\mu \chi_{\pi^H_\mu})\). Therefore, for \(f \in N\) and \(x \in G\) we have

\[
P_{N[V^H_\mu]}(f)(x) = \pi(d_\mu \chi_{\pi^H_\mu})(P_Nf)(x)
= \int_H d_\mu \chi_{\pi^H_\mu}(h)f(h^{-1}x)dh
= \int_H d_\mu \chi_{\pi^H_\mu}(h)\int_G K_N(y, h^{-1}x)f(y)dydh
= \int_{G} \left( \int_H d_\mu \chi_{\pi^H_\mu}(h)K_N(y, h^{-1}x)f(y)dy \right)dy
= \int_G K_1(y, x)f(y)dy.
\]

Thus, \(K_1\) is a kernel that represents the orthogonal projector onto \(N[V^H_\mu]\). This concludes the verification of Proposition 6.8 for a compact subgroup \(H\).
Our proof of Proposition 6.8 is based on an expression for the orthogonal projector onto a closed subspace \( E \) of \( H^2(G, \tau) \). For this, we fix an equivalent representation \( (\pi, V) \) of \( (L, H^2(G, \tau)) \) and we assume \( W \subset V \). Then, the map \( V \ni v \mapsto (G \ni x \mapsto f_v(x) := P_W(\pi(x^{-1})v)) \) is \( G \)-equivariant, continuous and bijective. From \( V \) onto \( H^2(G, \tau) \). We also fix an orthonormal basis \( v_i \) for \( W \) at the equality

\[
P_W(\pi(y^{-1})v) = \sum_{1 \leq j \leq \dim W} (\pi(y^{-1})v, v_j) v_j
\]

shows any element of \( H^2(G, \tau) \) is a finite sum of matrix coefficients for \( V \). We notice \((f_v, f_z)_{L^2(G)} = \frac{\dim W}{\dim W} (v, z)_V \). Hence, an unitary equivalence \( i \) from \( W \) onto \( H^2(G, \tau)[W] \) is given by \( i(v) = \frac{\dim W}{\dim W} f_v \), we have \( e_1(i(v)) = i(v)(e) = \frac{\dim W}{\dim W} v \). Hence \( f_j := i(v_j) = \frac{\dim W}{\dim W} f_v \) is an orthonormal basis for \( H^2(G, \tau)[W] \). As usual, \( \{v_j^*\} \) denotes the dual basis to the basis \( \{v_j\} \).

For \( f \in H^2(G, \tau) \) the integral below is absolutely convergent, because the product of two \( L^2 \) functions gives an integrable function. We define

\[
C_E(f)(x) := \frac{d_\lambda}{\dim W} \int_G \sum_{j,k} (P_E(\pi(y)f_j), \pi(x)f_k)_{L^2(G)} (v_k \otimes v_j^*)(f(y)) dy.
\]

The indexes in the sum run from 1 to \( \dim W \). A straightforward computation shows that for \( f \in L^2(G \times \tau W) \) the function \( C_E(f) \) belongs to \( 1(G \times \tau W) \). We want to show,

Lemma 6.9. Let \( H^2(G, \tau) \) as usual, \( E \) is a closed subspace of \( H^2(G, \tau) \). Then, \( C_E \) is equal to the orthogonal projector from \( H^2(G, \tau) \) onto \( E \).

Proof. For \( f_v \in E \) (resp. \( f_v \in E^\perp \)), we have that \( C_E(f_v) = f_v \) (resp. \( C_E(f_v) = 0 \)). In fact, \( f_v(y) = \sum_{1 \leq r \leq \dim W} (\pi(y^{-1})v, v_r) v_r \). Hence,

\[
\frac{\dim W}{d_\lambda} C_E(f_v)(x) = \sum_{k,j,r} \int_G (P_E(yf_j), xf_k)_{L^2(G)} (y^{-1}v, v_r) v^*_r v_k d y
\]

\[
= \int_G \sum_{j,k} (yf_j, P_E(xf_k))_{L^2(G)} (yv_j, v_r) v^*_r v_k d y k
\]

\[
= \frac{d_\lambda}{\dim W} \int_G \sum_{j,k} (yf_j, P_E(xf_k))_{L^2(G)} (yf_v, f_x)_{L^2(G)} d y k
\]

\[
= \frac{1}{\dim W} \sum_{j,k} (f_j, f_v)_{L^2(G)} (P_E(xf_k), f_v)_{L^2(G)} v_k
\]

\[
= \frac{1}{\dim W} \sum_{j} (f_j, f_v)_{L^2(G)} (\sum_k (P_E(xf_k), f_v)_{L^2(G)} v_k)
\]

Whence, for \( f_v \in E^\perp \) we have \( C_E(f_v) = 0 \), whereas for \( f_v \in E \), since \( P_E(f_v) = f_v \) we obtain

\[
(P_E(f_v), xf_k)_{L^2(G)} = (x^{-1}f_v, v_k)_{L^2(G)}
\]

\[
= \frac{\dim W}{\sqrt{d_\lambda}} (x^{-1}f_v, v_k)_{L^2(G)} = \frac{\sqrt{\dim W}}{\sqrt{d_\lambda}} (x^{-1}v, v_k)_V.
\]
Hence, \( \sum_k (P_E(f_v), x f_k)_{L^2(G)} v_k = \sqrt{\frac{\dim W}{d_\lambda}} f_v(x) \).

Also, \( \sum_j (f_j, v_x)_{L^2(G)} = \sum_j \frac{\sqrt{\dim W}}{\sqrt{\dim W}} (f_j, v_x)_{L^2(G)} = \frac{\sqrt{\dim W}}{\sqrt{\dim W}} \dim W. \)

Thus, \( \frac{\sqrt{\dim W}}{d_\lambda} C_E(f_v) = \frac{\dim W}{d_\lambda} f_v. \)

\( \Box \)

Note: A consequence of the Lemma is that \( C_E \) is a continuous linear operator in \( H^2(G, \tau) \). It also readily follows that \( C_E \) is a continuous linear operator on \( L^2(G \times W) \).

**Proof of Proposition 6.8.** After we recall the equality

\[ K_\lambda(y, x) = e_1 \circ P_{H^2(G, \tau)} |W| \pi^\lambda(x^{-1} y) P_{H^2(G, \tau)} |W| \circ i \]

and the previous Lemma we conclude that in order to show that the matrix kernel for the orthogonal projector onto \( H^2(G, \tau) |V^H_\mu \rangle \) is equal to the function \( (y, x) \mapsto d_{\mu} \Theta_{V^H_\mu} (h \mapsto K_\lambda(h^{-1} y, x)) \), it is equivalent to show the equality

\[ \sum_{i,j} d_{\mu} \Theta_{(\pi^H_\mu)^*} \Theta (h \mapsto (\pi^\lambda(h y) f_j, x f_i)_{L^2(G)}) v_i \otimes v^*_j \]

\[ = \sum_{i,j} (P_{\lambda,\mu}(y f_j), x f_i)_{L^2(G)} v_i \otimes v^*_j. \]

The right hand side of the above equality obviously is a well defined function, we now show the left hand side defines a function. For this, we show that for fixed \( y, x \) the function \( H \ni h \mapsto K_\lambda(h y, x) \) is tempered in the sense of Harish-Chandra. Indeed, the \( f_j \)'s are \( K \)-finite vectors, hence \( \pi^\lambda(x) f_j = L_x(f_j), \pi^\lambda(y) f_j \) are smooth vectors for \( H^2(G, \tau) \). Thus, they are tempered functions in the sense of Harish-Chandra. Since, the inner product

\[ (\pi^\lambda(h) \pi^\lambda(x) f_i, \pi^\lambda(y) f_j) \]

can be rewritten as a convolution of tempered functions evaluated in \( h \), we obtain that when we let \( h \) varies in \( G \), the matrix coefficient

\[ (\pi^\lambda(h) \pi^\lambda(x) f_i, \pi^\lambda(y) f_j) \]

is a tempered function on \( G \). In in [HIO, Proposition 2.2] it is shown that the restriction to \( H \) is a tempered function. Since the character of a discrete series representation is a tempered distribution, we obtain the left hand side defines a function of \( x, y \).

To start with, we fix \( x, y \) in \( G \) and smooth vectors \( v, w \) in \( (V^G_\lambda)^\infty \). We show:

\[ \Theta_{(\pi^H_\mu)^*} (h \mapsto (\pi^\lambda(h y) v, \pi^\lambda(x) w)_{L^2(G \times W)}) \]

\[ = (P_{\lambda,\mu}(\pi^\lambda(y) v), P_{\lambda,\mu}(\pi^\lambda(x) w))_{L^2(G, \tau)} \]

\[ = (P_{\lambda,\mu}(\pi^\lambda(y) v), \pi^\lambda(x) w)_{L^2(G,\tau)}. \]

Our hypothesis is that \( res_H(\pi^\lambda) \) is an admissible representation. Thus, there exists a subset \( Spec(res_H(\pi^\lambda)) \) of the set of Harish-Chandra parameters for \( H \) so that we have the Hilbert sum \( V^G_\lambda = \oplus_{\nu \in Spec(res_H(\pi^\lambda))} V^H_\lambda |V^H_\nu \rangle \) and \( V^H_\lambda |V^H_\nu \rangle \neq \{0\} \) if and only if \( \nu \in Spec(res_H(\pi^\lambda)) \). Moreover, the multiplicity of \( V^H_\nu \) in \( V^H_\lambda \) is finite.

To follow, we show that a \( H \)-smooth vector in \( V^H_\lambda |V^H_\nu \rangle \) is \( G \)-smooth. In fact, \( V^H_\lambda |V^H_\nu \rangle \) is a finite sum of irreducible unitary representations for \( H \). Thus, [Wa2, Theorem 11.8.2], the subspace \( (V^H_\lambda |V^H_\nu \rangle)^{H-smooth} \) is finitely generated.
over the algebra $\mathcal{S}(H)$ of rapidly decreasing functions on $H$. Hence, each element of $(V_\lambda[V^H_\nu])^{H-smooth}$ is a finite sum of functions $(\pi_\lambda)|_H^f(g)(f_1)$ where $g$ is a rapidly decreasing function on $H$ and $f_1$ is an $L-$finite element in $V_\lambda[V^H_\nu])$. Now, in [Kob1, Proposition 1.6] it is shown that the hypothesis of being $H-$admissible yields that $L-$finite elements are $K-$finite. Thus, any vector in $(V_\lambda[V^H_\nu])^{H-smooth}$ is a finite sum of the type pointed out above with $g$ rapidly decreasing on $H$ and $f_1$ a smooth vector for $G$. Further, $(V_\lambda)^\infty$ endowed with the smooth topology is a Frechet representation for $G$. Whence $(V_\lambda)^\infty$ is a Frechet representation for $H$. Therefore, since $f_1$ is a $G-$smooth vector we conclude, $(\pi_\lambda)|_H^f(g)(f_1)$ is an element of $(V_\lambda)^\infty$. Thus, every $H-$smooth vector in $(V_\lambda[V^H_\nu])^{H-smooth}$ is $G-$smooth. Next, we write for $z \in G, u \in (V_\lambda^{\mu})^\infty$

$$
\pi_\lambda(z)u = \sum_{\nu \in Spec(res_H(\pi_\lambda))} P_{\lambda,\nu}(\pi_\lambda(z)u) \quad (a).
$$

We claim the convergence of the series above is absolutely in both, $L^2-$topology and smooth topology for $(V_\lambda)^\infty$. The series converges in the topology of uniform convergence on compact sets for the function as well for any derivatives.

Our hypothesis shows that $L^2-$convergence is obvious. Since $\pi_\lambda(z)u$ is a smooth vector, we have that for every $\nu$ the vector $P_{\lambda,\nu}(\pi_\lambda(z)u)$ is $H-$smooth, the previous claim shows $P_{\lambda,\nu}(\pi_\lambda(z)u)$ is $G-$smooth. We recall a result of Harish-Chandra [HC2][Lemma 5] which asserts: The Fourier series of a smooth vector converges absolutely in smooth topology. Therefore, the Fourier series of $\pi_\lambda(z)$ as well as the Fourier series for $P_{\lambda,\nu}(\pi_\lambda(z)u)$ converges absolutely in smooth topology. Since, $H-$admissible implies $L-$admissibility, we have that the series of general term $P_{\lambda,\nu}(\pi_\lambda(z)u)$ is a rearrangement of the Fourier series for $\pi_\lambda(z)u$. Thus, the series $\sum_{\nu} P_{\lambda,\nu}(\pi_\lambda(z)u)$ converges absolutely in smooth topology. The third affirmation follows from [At].

In [HC2] it is shown the smooth vectors in $H^2(G, \tau)$ are tempered functions and convergence in smooth topology implies convergence in the space of tempered functions.

The series of functions

$$
h \mapsto (hyv, xw)_{L^2(G)} = \sum_{\nu \in Spec(res_H(\pi_\lambda))} (h \mapsto (P_{\lambda,\nu}(hyv), xw)_{L^2(G)}) \quad (b).
$$

converges in the topology for the space of tempered functions. The equality follows from the series (a) applied to $z = hyv$ and the continuity of an inner product on each variable. The convergence in tempered functions topology follows from the considerations of above.

We now verify $(P_{\lambda,\nu}(hyv), xw)_{L^2(G)} = (P_{\lambda,\nu}(hyv), P_{\lambda,\nu}xw)_{L^2(G)}$. (a) yields the equality $(P_{\lambda,\nu}(hyv), xw)_{L^2(G)} = \sum_{\nu'}(P_{\lambda,\nu}(hyv), P_{\lambda,\nu'}xw)_{L^2(G)}$. Now, we recall $P^{\lambda}_{\nu,\mu} = P^{\lambda}_{\nu,\mu}P^{\lambda}_{\nu,\nu} = P^{\lambda}_{\nu,\nu}P^{\lambda}_{\nu,\nu'} = 0$ for $\nu \neq \nu'$ and the proposed equality follows. Applying the last equality in (b) and the character of $\pi_\mu^H$ to the resulting series, we obtain

$$
\Theta_{(\pi_\mu^H),}(h \mapsto (hyv, xw)) = \sum_{\nu \in Spec(res_H(\pi_\lambda))} \Theta_{(\pi_\mu^H),}(h \mapsto (hP_{\lambda,\nu}(yv), P_{\lambda,\nu}(xw))_{L^2(G)}).
$$
Next, the function \((h \mapsto (hP_{\lambda,\nu}(yv), P_{\lambda,\nu}(xw))_{L^2(G)})\) is a matrix coefficient for \(V_\lambda[V^H]\), besides, \(P_{\lambda,\nu}(yv), P_{\lambda,\nu}(xw)\) are smooth vectors for \(H\), hence, the orthogonality relations as written in [HC2], Lemma 84 gives us

\[
\Theta_{(\pi^H)}(h \mapsto (hP_{\lambda,\nu}(yv), P_{\lambda,\nu}(xw))_{L^2(G)}) = \begin{cases} 
0 & \nu \neq \mu \\
\frac{1}{d_\mu}(P_{\lambda,\mu}(yv), P_{\lambda,\mu}(xw))_{L^2(G)} & \nu = \mu.
\end{cases}
\]

Thus,

\[
\Theta_{(\pi^H)}(h \mapsto (hyv, xw)) = \frac{1}{d_\mu}(P_{\lambda,\mu}(yv), P_{\lambda,\mu}(xw))_{L^2(G)}
\]

After we apply the above equality to \(v = f_j, w = f_i\) and adding up, we conclude a proof of Proposition 6.8 under the hypotheses of \(H\)–admissibility. 

\[\square\]

7. Criteria for discretely decomposable restriction.

As in previous sections, we keep the hypothesis and notation of Section 1. The objects are \(G, K, (\tau, W), H^2(G, \tau), H, L\). We recall the orthogonal projector \(P_\lambda\) onto \(H^2(G, \tau)\) (1.1) is given by a smooth matrix kernel \(K_\lambda(y, x) = K_\lambda(x^{-1}y, e) = \Phi_0(x^{-1}y)\), here \(\Phi_0\) is the spherical function associated to the lowest \(K\)–type \((\tau, W)\) of \(\pi^G_\lambda\). Harish-Chandra showed that \(\Phi_0\) and hence \(tr(\Phi_0)\) are tempered function for the definition of Harish-Chandra, [HC2][8.5.1]. In [HKO] we find a proof that the tempered functions on \(G\) restricted to \(H\) are tempered functions. A tempered function is called a cusp form if the integral along the unipotent radical of any proper parabolic subgroup of \(G\) of any left translate of the function is equal to zero [Wa1, 7.2.2]. Let \(r_n : H^2(G, \tau) \rightarrow L^2(H \times, r_n(p/p')^{(n)} \otimes W)\) be as in Example 4.1. The notation \(r_n(\Phi_0^*)\) means the family of functions \(r_n(K_\lambda(\cdot, y)^w) = r_n(\Phi_0^*(\cdot), w), w \in W\). The purpose of this section is to show:

**Theorem 7.1.** Let \(\pi^G_\lambda\) be a discrete series for \(G\), let \(\Phi_0\) be its lowest \(K\)–type spherical function. Then, \(r_n(\Phi_0^*)\) is a cusp form on \(H\), for every \(n = 0, 1, \ldots\), if and only if \(\pi^G_\lambda\) restricted to \(H\) is discretely decomposable. In turn, this is equivalent to: for each \(y \in G\), the restriction of \(K_\lambda(\cdot, y)\) to \(H\) is a cusp form.

**Remark 7.2.** T. Kobayashi [KO2, Theorem 2.8] has shown that for a symmetric pairs \((G, H)\), \(\pi_\lambda\) restricted to \(H\) is algebraically discretely decomposable if and only if \(res_H(\pi_\lambda)\) is \(H\)–admissible. Whence, coupling the previously quoted result of Kobayashi and Theorem 7.1, Proposition 7.3 we may state: For a symmetric pair \((G, H)\). The restriction of \(\pi_\lambda\) to \(H\) is admissible if and only if for every \(n = 0, 1, \ldots\) \(r_n(\Phi_0^*)\) is a cusp form if and only if \(\Phi_0^*\) is left \(\mathfrak{h}(U(\mathfrak{h}))\)–finite.

For symmetric pair \((G, H)\) another criteria for \(H\)–admissibility has been obtained by [HKO]. For this, they write \(H = K_0 \times H_1\), with \(K_0\) a compact subgroup and \(H_1\) a noncompact subgroup. Let \(H^{\theta}\) be the dual subgroup. Then \(H^{\theta} = K_0 \times H_2\). Let \(M_i\) denote the centralizer in \(L \cap H_1 = \ldots\)
$L \cap H_2$ of respective Cartan subspaces. Harris-He-Olafsson show that if $M_1M_2 = L \cap H_1$, then, the representation $\pi_\lambda$ restricted to $H$ is admissible and $\dim \text{Hom}_H(\pi_\mu^H, \text{res}_H(\pi_\lambda))$ is computed via a formula that involves $r_\mu$, the Harish-Chandra character for $\pi_\mu^H$ and the lowest $L$--type for $\pi_\mu^H$ and the limit of a sequence.

For any pair $(G,H)$ and $\pi_\lambda$ that satisfies Condition C, in [DV], it is shown that $\pi_\lambda$ is a $H$--admissible representation, and a "Blattner-Kostant" type formula for $\dim \text{Hom}_H(\pi_\mu^H, \text{res}_H(\pi_\lambda))$. For symmetric pairs, condition C is equivalent to $H$--admissibility.

In order to show 7.1 we first show,

**Proposition 7.3.** We let $G,H,\pi_\lambda^G, K_\lambda, \Phi_0$ be as usual. Then, $\pi_\lambda^G$ restricted to $H$ is a discretely decomposable representation for $H$ if and only if the function $y \mapsto K_\lambda(y,e)^* = \Phi_0(y)$ is left $\mathfrak{z}(U(\mathfrak{h}))$--finite.

**Proof.** For the direct implication we proceed as follows: The hypothesis $\pi_\lambda$ is discretely decomposable allows us to write $V_\lambda$ as the Hilbert sum of the $H$--isotypic components, hence, there exists a family $(P_i)_{i \in \mathbb{Z}_{\geq 0}}$ of orthogonal projectors on $V_\lambda$ which are $H$--equivariant so that we have the orthogonal direct sum decomposition $V_\lambda = \oplus_i P_i(V_\lambda)$, and for every $i$, $P_i(V_\lambda)$ is equal to the isotypic component of an irreducible $H$--module. Next, we fix $w \in W$, we recall the function $y \mapsto K_\lambda(y,e)^*(w) = k_w(y)$ is a $K$--finite element of $H^2(G,\tau)$ and $k_w$ belongs to $H^2(G,\tau)[W] \equiv W$. After we decompose $H^2(G,\tau)[W]$ as sum of irreducible $L$--submodules, we write $k_w = f_1 + \cdots + f_s$ where $f_j$ is so that the linear subspace spanned by $\pi_\lambda(L)f_j$ is an irreducible $L$--submodule of $H^2(G,\tau)[W]$. To continue, we set $f_1 := f_j$. A result of Harish-Chandra, [HC2, Lemma 70] states that an irreducible representation of $L$ is the $L$--type of at most finitely many discrete series representations for $H$. Thus, the representation of $L$ in the subspace spanned by $\pi(L)f_1$ is an $L$--type of at most finitely many discrete series representations for $H$. Therefore, $P_i(f_1) = 0$ for all but finitely many indices $i$. Let’s say $P_i(f_1) \neq 0$ for $i = 1, \ldots, N$. Since $f_1$ is a $K$--finite vector in $V_\lambda$, we have that $f_1$ is a smooth vector for $\pi_\lambda$, therefore, $P_i(f_1)$ is a $H$--smooth vectors in $P_i(V_\lambda)$. Owing to $P_i(V_\lambda)$ is an isotypic representation, we have that $\mathfrak{z}(U(\mathfrak{h}))$ applied to $P_i(f_1)$ is contained in the one dimensional vector subspace spanned by $P_i(f_1)$. Hence, $f_1$ is a finite sum of $\mathfrak{z}(U(\mathfrak{h}))$--finite vectors. Thus, $k_w$ is a $\mathfrak{z}(U(\mathfrak{h}))$--finite vector. $W$ is a finite dimensional vector space, let us conclude that the map $K_\lambda(\cdot, e)^* = \Phi_0(\cdot)^*$ is left $\mathfrak{z}(U(\mathfrak{h}))$--finite.

For the converse statement, owing to our hypothesis, for each $w \in W$ we have that $k_w$ is $\mathfrak{z}(U(\mathfrak{h}))$--finite element of $H^2(G,\tau)_{K-fin}$. A result of Harish-Chandra, [Wa1, Corollary 3.4.7 and Theorem 4.2.1], asserts that a $U(\mathfrak{h})$--finitely generated, $\mathfrak{z}(U(\mathfrak{h}))$--finite, $(\mathfrak{h},L)$--module has a finite composition series, whence we conclude that the representation of $U(\mathfrak{h})$ in $U(\mathfrak{h})k_w$ has a finite composition series. Thus, $H^2(G,\tau)_{K-fin}$ contains an irreducible sub-representation for $U(\mathfrak{h})$. Whence, [Kob1, Lemma 1.5] yields that $(H^2(G,\tau)_{\lambda})_{K-fin}$ is infinitesimally discretely decomposable as $\mathfrak{h}$--module. Finally, since $\pi_\lambda$ is unitary, in [Kob2, Theorem 4.2.6], we find a proof that
an algebraically (infinitesimally) discretely decomposable unitary representation is Hilbert discrete decomposable, hence, $\pi_\lambda$ is discretely decomposable.

**Corollary 7.4.** We assume $(\pi_\lambda, V_\lambda)$ is a Hilbert discretely decomposable representation for $H$. Then, $(\pi_\lambda, V_\lambda)$ is algebraically discretely decomposable. That is, $(V_\lambda)_{K-fn}$ can be expressed as direct sum of $U(\mathfrak{h})$—irreducible subspaces.

The Corollary follows because as in the proof of direct implication we obtain each $k_w, w \in W$ is $\mathfrak{z}(U(\mathfrak{h}))$—finite, whence, the proof for the converse statement yields that $V_\lambda$ is algebraically discretely decomposable.

Now, we are ready to show Theorem 7.1.

**Proof of Theorem 7.1.** For the direct implication, the hypothesis is $\pi_\lambda$ restricted to $H$ is discretely decomposable. Thus, Proposition 7.3 yields $K_\lambda(\cdot, e)^*w$ is $\mathfrak{z}(U(\mathfrak{h}))$—finite. Since in [OV] it is shown $r_n$ is a continuous intertwining map for $H$ and $K_\lambda(\cdot, e)^*w$ is a tempered function, a result of [HHO] previously quoted let us conclude: $r_n(K_\lambda(\cdot, e)^*w)$ is a tempered, $\mathfrak{z}(U(\mathfrak{h}))$—finite function on $H$. A result of Harish-Chandra, [HC2][Wa1, 7.2.2] implies $r_n(K_\lambda(\cdot, e)^*w)$ is a cusp form. For the converse statement, results of Harish-Chandra asserts $L^2(H \times_{\tau_n} (\mathfrak{p}/\mathfrak{p}')(\mathfrak{n}) \otimes W)_{disc}$ is a finite sum of discrete series representations for $H$ and the space of cusp forms in $L^2(H \times_{\tau_n} (\mathfrak{p}/\mathfrak{p}')(\mathfrak{n}) \otimes W)$ is contained in $L^2(H \times_{\tau_n} (\mathfrak{p}/\mathfrak{p}')(\mathfrak{n}) \otimes W)_{disc}$. Therefore, owing to the hypothesis, for every $n$, $r_n(K_\lambda(\cdot, e)^*w)$ belongs to $L^2(H \times_{\tau_n} (\mathfrak{p}/\mathfrak{p}')(\mathfrak{n}) \otimes W)_{disc}$. The $L^2$—continuity of $r_n$, yields $r_n(closure(\pi_\lambda(H)k_w))$ is contained in a finite sum of discrete representations. Whence, $\oplus_n r_n$ maps continuously the closure of $\pi_\lambda(H)k_w$ into a discrete Hilbert sum of irreducible representations. Besides, the map $\oplus_n$ is injective (the elements of $H^2(G, \tau)$ are real analytic functions). Hence, the closure of $\pi_\lambda(H)k_w$ is a discrete Hilbert sum of discrete series representations. We now proceed as in the direct proof Proposition 7.3 and obtain $k_w$ is a left $\mathfrak{z}(U(\mathfrak{h}))$—finite function. Whence, Proposition 7.3 let us conclude $res_H(\pi_\lambda)$ is Hilbert discretely decomposable. The second equivalence follows from a simple computation.

**Remark 7.5.** A simple application of Theorem 7.1 is that the tensor product representation of $G$ in $H^2(G, \tau) \boxtimes H^2(G, \tau)^*$ never is discretely decomposable, because the lowest $K$—type trace spherical function for this particular tensor product is $\phi_0(x)\overline{\phi_0(y)}$, hence, restricted to $G$ is not a cusp form.

8. Reproducing kernels and existence of discrete factors.

As usual $G, H, H^2(G, \tau) = V_\lambda, K_\lambda, H^2(H, \sigma) = V_\mu^H, K_\mu$ are as in subsection 2.1. The purpose of this section is to begin an analysis of the relation between the matrix kernel of a discrete series for $G$, the matrix kernel of a discrete series for $H$ and the existence of a nonzero $H$—intertwining linear map from one representation into the other. To begin with, since $K_\lambda$ is given by the spherical function associated to the lowest $K$—type of $\pi_\lambda^G$, the restriction of $y \mapsto tr(K_\lambda(y, x)) =: k_\lambda(y, x)$ to $H$ is a square integrable function [OV]. Let $k_\mu := trK_\mu$. We show,
Proposition 8.1. $(k_\lambda(\cdot, e), k_\mu(\cdot, e))_{L^2(H)}$ nonzero implies $V_\mu^H$ is a discrete factor for $res_H(\pi_\lambda^H)$.

The proposition is an immediate consequence of the following

Fact 8.2. Let $G, H, \pi_\lambda$ be as usual. Let $(\rho, V_1)$ an irreducible square integrable for $H$. We assume there exists smooth vector $w_1, w_2$ in $V_\lambda$ and a $L-$finite vector $z$ in $V_1$ so that

$$\int_H (\rho(h)z, z)_V L^{1}(\pi_\lambda(h)w_1, w_2)_V dh \neq 0.$$

Then, there exists a nonzero, continuous linear $H-$map from $V_1$ into $V_\lambda$.

For a proof [Va2].

Remark 8.3. The converse to Proposition 8.1 is false, as it shows the following example. We consider arbitrary $G, H$ and we assume $res_H(\pi_\lambda^H)$ is an $H-$admissible representation. Since $H^2(G, \tau)[W]$ consists of $L-$finite vectors, the subspace $H^2(G, \tau)[W]$ is contained in a finite sum of irreducible $H-$subrepresentations. Let $\pi_\mu^H$ be a representation for $H$ whose corresponding isotypic component is nonzero and contained in the orthogonal subspace spanned by the $H-$isotypic components whose intersection with $H^2(G, \tau)[W]$ is nonzero. Since, for $w \in W$, $K_\lambda(\cdot, e)^*w \in H^2(G, \tau)[W]$, we have that both $tr(K_\mu(\cdot, e)) = k_\mu(\cdot, e)$ and $tr(K_\lambda(\cdot, e)) = k_\lambda(\cdot, e)$ are linear combination of matrix coefficients of nonequivalent representations for $H$, the orthogonality relations implies $(k_\lambda(\cdot, e), k_\mu(\cdot, e))_{L^2(H)} = 0$.

One way to show a converse to Proposition 8.1 is to consider the functions

$$\Upsilon_{\mu, \lambda}(y, x) = \int_H k_\lambda(hy, x)K_\mu(h, e)^* dh$$

and

$$v_{\lambda, \mu}(y, x) := \int_H k_\mu(h_1, e)K_\lambda(h_1y, x)^* dh_1.$$

Due that both factors are square integrable in $H$, [OV], both integrals converges absolutely. Moreover, the functions $\Upsilon, v$ are continuous and defined for every $(y, x)$ because the integrand is the product of two square integrable functions for $H$, we recall the fact: convolution of two square integrable functions is defined and yields a continuous function. The two functions might be related to period integrals from Number Theory.

Under the supposition representation $\pi_\mu$ is integrable it readily follows the smoothness of both functions as well as that each function is a Carleman kernel. For this, we write $v(y, x) = res_H(L)(k_\mu(\cdot, e))(z \mapsto K_\lambda(zy, x))$, here, we think $z \mapsto K_\lambda(zy, x)$ belongs to either the Frechet space $(L^2(G) \otimes Hom_C(W, W))^\infty$ or to the Hilbert space $L^2(G) \otimes Hom_C(W, W)$, $L$ is the left regular representation on each space and $k_\mu(\cdot, e) \in L^1(H)$. Therefore, both functions $v(\cdot, x), v(y, \cdot)$ are smooth and square integrable. For $\Upsilon$ we write $\Upsilon(y, x) = res_H(L)(K_\mu(\cdot, e)^*)(z \mapsto k_\lambda(zy, x))$, here, we think $z \mapsto k_\lambda(zy, x)$ belongs to either the Frechet space $(L^2(G))^\infty$ or to the Hilbert space $L^2(G)$, $L$ is the left regular representation on each space.

We note $tr(\Upsilon(e, e)) = (k_\lambda(\cdot, e), k_\mu(\cdot, e))_{L^2(H)}$, $tr(\Upsilon(y, x)) = tr(v(y, x))$. Among properties of the functions are:
1) \( \Upsilon(s_1yk, s_2xk) = \sigma(s_1)\Upsilon_{\mu,\lambda}(y, x)\sigma(s_2^{-1}) \), for \( x \in G, s_j \in L, k \in K \).
2) The function \( \Upsilon_{\mu,\lambda} \) is real analytic.

This is because the distribution on \( G \) defined by either the function \( y \mapsto \Upsilon_{\mu,\lambda}(y, x) \) or \( x \mapsto \Upsilon_{\mu,\lambda}(y, x) \) is a real analytic \( L \)-spherical function. Indeed, both distributions are eigenfunctions of the elliptic differential operator \( R_{\Omega G} + 2R_{\Omega k} \), the regularity Theorem leads us to the real analyticity.

3) \( \Upsilon_{\mu,\lambda}(y, x) = (\Upsilon_{\mu,\lambda}(x, y))^* \).

4) Remark \ref{remark} show that some times \( \Upsilon(e, e) = 0 \).

5) \( \nu(syk_1, sxxk_2) = \tau(k_2^{-1})\nu(y, x)\tau(k_1), k_1, k_2 \in K, s \in L. \)

The main result of this subsection is:

**Theorem 8.4.** Let \( G, H, (\pi_{\lambda}, H^2(G, \tau)), (\pi_{\mu}^H, V^H_{\mu} = H^2(H, \sigma)) \) as usual. The following four conditions are equivalent:

\begin{enumerate}
  \item The isotypic component \( H^2(G, \tau)[V^H_{\mu}] \) is not zero.
  \item The function \( \Upsilon_{\mu,\lambda} \) is nonzero.
  \item The function \( tr(\Upsilon_{\mu,\lambda}) \) is nonzero.
  \item There exists \( D \in U(\mathfrak{g} \times \mathfrak{g}) \) so that \( [L_D(\Upsilon)](e, e) \neq 0. \)
  \item The function \( \nu_{\lambda,\mu} \) is nonzero.
\end{enumerate}

**Proof of Theorem 8.4.** We show \( a) \Rightarrow b) \Leftrightarrow c) \Rightarrow e) \Rightarrow a) \) and \( b) \Leftrightarrow d) \). Some of the implications are obvious. \( a) \) implies \( b) \). By hypothesis, there exists a nonzero intertwining map \( T : H^2(H, \sigma) \rightarrow H^2(G, \tau) \). Owing to Schur’s lemma \( T \) is an injective map. Let \( K_T : H \times G \rightarrow Hom_C(Z, W) \) be the kernel that represents \( T \). We have verified, for \( z \in Z \), the function \( h \mapsto K_{\mu}(h, e)^*z \) belongs to \( H^2(H, \sigma) \) and it is nonzero. Thus, \( T(K_{\mu}(\cdot, e)^*z) \) is a nonzero function for each \( z \in Z \).

Since \( k_{\lambda} \) represents \( P_{\lambda} \) we have

\[ \int_G \int_H k_{\lambda}(y, x)K_T(h, y)K_{\mu}(h, e)^*z \, dh \, dy \neq 0 \text{ for some } x \in G. \]

Now \( K_T(h, y) = K_T(e, h^{-1}y) \). Thus, after we replace \( y \) by \( hy \) and we recall Haar measure is left invariant, we obtain

\[ \int_G \int_H k_{\lambda}(hy, x)K_T(e, y)K_{\mu}(h, e)^*z \, dh \, dy \neq 0. \]

Next, using \( k_{\lambda} \) is a complex valued function, we are lead to

\[ \int_G K_T(e, y) \int_H k_{\lambda}(hy, x)K_{\mu}(h, e)^*z \, dh \, dy \neq 0. \]

Thus, \( \Upsilon \) is not the zero function. We now show \( b) \Leftrightarrow c) \) The equality (1) forces the linear span of the image of \( \Upsilon_{\mu,\lambda} \) is an \( L \times L \) invariant subspace of \( End_C(Z) \). Thus, owing to \( End_C(Z) \) is an \( L \times L \)-module irreducible we have there exists \( (y, x) \) so that \( tr(\Upsilon_{\mu,\lambda}(y, x)) \neq 0 \) unless \( \Upsilon_{\mu,\lambda} \) is equal to the zero map. \( c) \Rightarrow e) \) follows from the identity \( \tau(\nu_{\lambda,\mu}) = tr(\Upsilon_{\mu,\lambda}) \). To show \( e) \Rightarrow a) \) we notice that the hypothesis implies we may apply Fact 8.2. Therefore the isotypic component is nonzero. The two functions are real analytic, thus we conclude the proof. \( \square \)

8.1. **Some discrete factors in \( res_H(\pi_{\lambda}) \).** As before, we fix \( G, H, K, L \) and the representations \( V^G_{\lambda} = H^2(G, \tau) \). A consequence of Fact 8.2 is the following.
Fact 8.5. We assume \((\sigma, Z)\) is an \(L\)–subrepresentation of the lowest \(K\)–type \((\tau, W)\) and we further assume there exists a discrete series representation \(H^2(H, \sigma) = V^H_\mu\) with lowest \(L\)–type \((\sigma, Z)\). Then, there exists a nonzero intertwining map from \(H^2(H, \sigma)\) into \(H^2(G, \tau)\).

The proof for this fact is based on an explicit integral formulae for \(k_\mu, k_\lambda\) obtained by Flensted-Jensen, these formulae let us to apply Proposition 8.2. For details cf. [Va]. We present one application of fact 8.5 to the analysis of tensor product of two representations. For other applications cf. [Va2].

Example 8.6. Notation is as in subsection 2.1 We produce some irreducible subrepresentation of the restriction to the diagonal subgroup \(H = G_0\) of \(G := G_0 \times G_0\) (\(G_0\) a semisimple Lie group) for a tensor product \(\pi_\lambda \boxtimes \pi_\lambda\). We choose both Harish-Chandra parameters to be dominant with respect to \(\Psi\) and so that their sum is far away from the kernel of any noncompact simple root. It readily follows that \(\pi_\lambda^K\) is the Harish-Chandra parameters of a subrepresentation of the lowest \(K \times K\)–type for \(\pi_\lambda \boxtimes \pi_\lambda\). Due to our choice, the parameter \(\lambda + \lambda_1 + 2 \rho_n - \rho_c\) is far from the noncompact walls for the Weyl chamber for \(\Psi\). It follows from 2.1 that \(\lambda + \lambda_1 + 2 \rho_n - \rho_c\) is the lowest \(K\)–type of the discrete series \(\pi_{\lambda + \lambda_1 + \rho_n - \rho_c}\). Therefore, fact A, implies \(\text{res}_{G_0}(\pi_\lambda \boxtimes \pi_\lambda)\) contains the irreducible representation \(\pi_{\lambda + \lambda_1 + \rho_n - \rho_c}\).

9. Other model to realize discrete series representations.

We refer to the article [Hi] for this section. Let \(d, m\) denote a fixed \(G\)–invariant Radon measure on \(G/K\), after we normalize Haar measure on \(K\) so that \(K\) has volume one, and normalize Haar measure on \(G\), we have the equality

\[
\int_G f(x)dx = \int_{G/K} \int_K f(xk)dk \, d_m(xK).
\]

Owing to the Iwasawa decomposition for \(G = ANK\) the principal bundle \(K \to G \to G/K\) is equivalent to the trivial bundle. Therefore, for each representation \(\tau : K \to U(W)\), the vector bundle \(G \times_\tau W \to G/K\) is parallelizable. We fix a section \(\sigma\) for \(G \to G/K\) so that \(\sigma(eK) = 1\), we obtain a cocycle \(c : G \times G/K \to GL(W)\), by \(c(g, x) = \tau(\sigma(g \cdot x)^{-1}g\sigma(x))\). That is, \(c\) is a continuous map which satisfies

\[
c(gh, x) = c(g, h \cdot x)c(h, x), g, h \in G, x \in G/K, c(k, eK) = \tau(k).
\]

We now recall how the above datum gives rise a unitary representation of \(G\) equivalent to \(L^2(G \times_\tau W)\). We consider the inner product on \(W\)–valued functions on \(G/K\) defined by

\[
<F, H> := \int_{G/K} ((c(g, o)c(g, o)^*)^{-1}F(z), H(z))_W \, d_m, \, z = g \cdot eK.
\]

The group \(G\) acts unitarily on the corresponding space of square integrable functions \(L^2(G/K, W)\) by the formula

\[
g \cdot f(x) = c(g^{-1}, x)^{-1}f(g^{-1}x).
\]

A unitary equivalence of both representations is given by the map

\[
L^2(G \times_\tau W) \ni f \mapsto F \in L^2_c(G/K, W) : F(g \cdot o) := c(g, o)f(g).
\]
The inverse map is

\[ L^2(G/K, W) \ni F \mapsto f \in L^2(G, \tau) : f(g) = c(g, o)^{-1}F(g, o). \]

This unitary equivalence is due to that for \( F, H \in L^2(G/K, W) \) the following equality holds

\[
\int_G (f(g), h(g))_W dg = \int_{G/K} ([c(g, o)c(g, o)^*]^{-1}F(z), H(z))_W d_\ast m(z).
\]

For a convenient function \( K : G/K \times G/K \to \text{End}_\mathbb{C}(W) \), we consider the integral operator

\[ L^2(G/K, W) \ni F \mapsto (G/K \ni z \mapsto \int_{G/K} K(w, z)F(w)d_\ast m(w) \in W) \]

and we define for \( g, h \in G, k(g, h) := c(h, o)^{-1}K(g \cdot o, h \cdot o)c(g, o) \), then, we have the commutative diagram

\[
\begin{array}{ccc}
f & \Rightarrow & F \\
\downarrow & & \downarrow \\
\int k(x, y)f(x)dx & \Rightarrow & \int K(w, z)F(w)d_\ast m(w)
\end{array}
\]

The kernel \( k \) defines an intertwining linear operator for the left action of \( G \) on \( L^2(G \times, W) \) if and only if \( k \) is an invariant kernel, that is, \( k(ga, gb) = k(a, b), a, b, g \in G \) and of course \( k(gk, hk_k) = \tau(k_1^{-1})k(g, h)\tau(k), g, h \in G, k, k_1 \in K. \) To an invariant kernel \( k \), the corresponding kernel \( K \) satisfies

\[ K(t \cdot z, t \cdot w) = c(t, w)K(z, w)c(t, z)^{-1}, z, w \in G/K, t \in G. \]

Next we consider to kernels \( K_1, K_2 \) which defines respective intertwining linear operators on \( L^2(G/K, W) \). Then, it readily follows the equality

\[ K_1(g \cdot x, g \cdot y)K_2(g \cdot x, g \cdot y)^{-1} = c(g, x)K_1(x, y)K_2(x, y)^{-1}c(g, x)^{-1}, x, y \in G/K, g \in G. \]

In particular, whenever \( W \) is unidimensional, the kernel \( K_1 \) is equal to \( K_2 \) times a \( G \)-invariant function.

For each Iwasawa decomposition \( G = ANK \) a section of the principal bundle \( G \to G/K \) is \( \sigma(\alpha K) = an \). Therefore, from the preceding considerations we obtain a unitary equivalence between \( L^2(G \times, W) \) and \( L^2(G/K, W) \) by mean of the cocycle \( c \) associated to \( \sigma \) and \( \tau \). When \( G/K \) is a Hermitian symmetric space, it is constructed another section \( \sigma_+ \) by means of subgroups \( P_+, P_-, K_\mathbb{C} \) of the complex Lie group \( G_\mathbb{C} \). We have that \( G \subset P_+K_\mathbb{C}P_- \) and \( G/K \) is realized as a bounded domain in \( P_\mathbb{C} \). Every is uniquely written as \( g = g_+(g)\mu(g)g_-(g), \) with \( \mu_\pm \in P_\pm, \mu(g) \in K_\mathbb{C}. \) Next, we fix \( (\tau, W) \) irreducible representations of \( K \) so that it is the lowest \( K \)-type of a holomorphic discrete series \( H^2(G, \tau) \). By means of \( \sigma_+ \) and the representations \( \tau \) we define \( c_{+, \tau} = \tau(\mu(g)) \) and we have that the maps \( f \mapsto F = c_{+, \tau}(g, o)f(g) \) carries \( H^2(G, \tau) \) onto the subspace of holomorphic functions in \( L^2_{+, \tau}(G/K, W. \) Based on this model for holomorphic discrete series, many authors has contributed to the study of branching problems and harmonic analysis. Other
authors, has chosen different sections of the principal bundle $G \to G/K$. Their choice, allowed them to analyze other discrete series, branching problems, harmonic analysis. It is out of our knowledge to explicit all the work done on the subject. We would like to call the attention of work, on holomorphic discrete series, of Jacobsen-Vergne, T. Kobayashi and his colaborators. The work of G. Zhang on quaternionic discrete series [LZ].

We would like to point out that after we fix respective smooth section for the principal bundle $G \to G/K$, $H \to H/L$, we may translate the reproducing kernel $K_\lambda$ to a reproducing kernel $\tilde{K}_\lambda$ in $L^2_c(G/K, W)$, the eigenspaces of the Casimir operator to eigenspaces of the Casimir operator, as we already indicated kernel maps goes to kernel maps, it is a simple matter to verify differential operators correspond to differential operators and so on. We may say that each statement in this note has a correlative in the language of the spaces $L^2_c$. For example, a function that corresponds to an element of $H^2(G, \tau)$ growth at most as the function $\|\tau(cg, o)\|$. Differential operators intertwining $H^2(G, \tau)$ with $H^2(H, \sigma)$ corresponds to differential operators between the corresponding spaces.

9.1. An example of discrete decomposition. We consider the groups $G = U(1, n), H = U(1, n-1) \times U(1), K = U(1) \times U(n), L = U(1) \times U(n-1) \times U(1)$ and we fix an integer $\alpha$ positive and large. Then, $\tau_\alpha(k) = det(k)^\alpha$, $k \in K$ is a character of $K$. To follow, we write the decomposition of $H^2(G, \tau)$ as an $H-$representation and we compute reproducing kernels, immersions, projections, etc. Let $D_n = \{ z \in \mathbb{C}^n, |z| < 1 \}$ denote the unit ball in $\mathbb{C}^n$. The group $G$ acts means of fractional transformations, the action is transitive on $D_n$ and the isotropy subgroup at the origin of $\mathbb{C}^n$ is $K$. We fix a Lebesgue measure $dm_n$ on $\mathbb{C}^n$ and define the measure $d\mu_\alpha = (1 - |z|^2)^{-\alpha(n+1)}dm_n$.

Then, in [DOZ] we find an explicit isomorphism between the Hilbert spaces $H^2(G, \tau_\alpha)$ and $V_{n,\alpha} := O(D_n) \cap L^2(D_n, d\mu_\alpha)$. The unitary representation of $G$ on $V_{n,\alpha}$ is by means of the action:

$$\pi_\alpha(g)f(z) = \tau_\alpha(J(g^{-1}, z))^{-1}f(g^{-1}z), \quad g \in G, z \in D_n, f \in V_{n,\alpha}.$$ 

For $f \in O(D_n)$ we write the convergent power series in $D_n$,

$$f = f_0(z_1, \cdots, z_{n-1}) + f_1(z_1, \cdots, z_{n-1})z_n + f_2(z_1, \cdots, z_{n-1})z_n^2 + \cdots .$$

For an integer $m \geq 0$, we consider the linear subspace

$$\mathcal{H}_m = \{ f \in O(D_n) : \frac{\partial^p f}{\partial z_n^p}|_{z_n=0} = 0, \text{ for } 0 \leq p \leq m-1 \}.$$ 

Then, $\mathcal{H}_m := \mathcal{H}_m \cap V_{n,\alpha}$ is a closed subspace in $V_{n,\alpha}$. We denote $V_m$ for the orthogonal complement of $\mathcal{H}_{m+1}$ in $\mathcal{H}_m$. Thus, a typical element of $\mathcal{H}_m$ (resp. $V_m$) is

$$f = f_m(z_1, \cdots, z_{n-1})z_n^m + \cdots, \quad (\text{resp. } f_m(z_1, \cdots, z_{n-1})z_n^m).$$

It readily follows that the action of $H$ on the a polynomial (resp. holomorphic function) in $z_1, \cdots, z_{n-1}$ (resp. in $z_n$) is again a polynomial (holomorphic function) in the same variables. Whence, the subspaces $V_m$ are invariant for the action of $H$. Therefore, we have the orthogonal decomposition $V_{n,\alpha} = V_0 + V_1 + V_2 + V_3 + \cdots$.

The orthogonal projector $P_{a,m}$ onto $V_m$ is given by
\[ P_{\alpha,m}(f = f_0 + \cdots + f_k z_n^k + \cdots)(z) = f_m(z_1, \ldots, z_{n-1}) z_n^m \]

\[
= z_n^m \frac{\partial^m f(z_1, \ldots, z_{n-1}, 0)}{\partial z_n^m} = \int_{D_n} K_{\alpha,m}(w, z) f(w) d\mu_\alpha(w).
\]

Here, \( K_{\alpha,m} \) is the reproducing kernel for the subspace \( V_m \). Hence, \( K_\lambda = \sum_{m \geq 0} K_{\lambda,m} \). Let’s write \( z', w' \) for vectors in \( \mathbb{C}^{n-1} \). Then, up to a constant

\[
K_\lambda(w, z) = \frac{1}{(1 - (\overline{w'} z')^* z_n - \overline{w}_n z_n)^\alpha} = \sum_{m \geq 0} \left( \frac{-\alpha}{m} \right) \frac{1}{(1 - (\overline{w'} z')^* z_n)^{\alpha+m}} (\overline{w}_n z_n)^m
\]

Hence, the \( m \)-th summand is equal to \( K_{\alpha,m} \).

The representation of \( H \) on \( V_m \) is equivalent to \( H^2(H, \tau_{\alpha+m}) \). An equivariant map \( T_m \) from \( H^2(H, \tau_{\alpha+m}) \equiv V_{n-1, \alpha+m} \) onto \( V_m \) is given by

\[ g(z_1, \ldots, z_{n-1}) \mapsto g(z_1, \ldots, z_{n-1}) z_n^m. \]

A expression for \( T_m \) as integral map is

\[ T_m(g)(z', z_n) = \int_{D_{n-1}} \left( \frac{1}{(1 - (\overline{w'} z')^* z_n)^{\alpha+m}} \right) z_n^m g(w_1, \ldots, w_{n-1}) d\mu_{\alpha+m} \]

This is due to that \( \frac{1}{(1 - (\overline{w'} z')^* z_n)^{\alpha+m}} \) is the reproducing kernel for \( V_{n-1, \alpha+m} \).

Finally, an intertwining map \( S : V_{n,\alpha} \to L^2(H, \tau_{\alpha+m}) \) is

\[ S(f)(z_1, \ldots, z_{n-1}) = \frac{\partial^m f}{\partial z_n^m}(z_1, \ldots, z_{n-1}, 0). \]

10. Appendix: Recollection on elliptic PDE and basics on reproducing kernel

10.1. For two measure spaces \((Y, \mu), (X, \nu)\), a linear transformation \( T \) from \( L^2(Y) \) into \( L^2(X) \) is called an integral map, kernel map or an integral operator, if there exists a function \( K_T : Y \times X \to \mathbb{C} \) so that the function \( y \mapsto K_T(y, x) g(y) \) belongs to \( L^1(Y) \) for every \( g \) in the domain of \( T \) and the function \( Tg(x) \) is equal to \( \int_Y K_T(y, x) g(y) dy \) almost everywhere on \( x \).

In an obvious way, the definition generalizes to linear maps between vector bundles. For a measurable kernel \( k \) the domain \( D_k \) of the linear map defined by \( k \) is is the set of \( f \in L^2(Y) \) so that \( \int_Y k(y, x) f(y) d\mu(y) \) converges for almost every \( x \in X \) and the resulting function is square integrable with respect to \( \nu \). An integral map is Carleman if the function \( y \mapsto k(y, x) \) is square integrable for almost every \( x \in X \). A usual, \( F^* \) denotes the adjoint of a linear map \( F \). Formally, the adjoint of \( T \) is an integral operator with kernel \( K_T^*(x, y) := K(y, x)^* \), however, even though when \( T \) is continuous, the linear map \( T^* \) might not be equal to an integral operator on the whole dual space, as example Appendix 10.3 shows.

10.2. For an integral map \( T \) if the adjoint linear map is an integral map of kernel \( K_T^* \), then \( K_T^*(x, y) = K_T(y, x)^* \).
10.3. For an example of an integral map whose adjoint is not an integral map, we consider \( X = \mathbb{Z} \) with discrete topology and usual Haar measure and \( Y = S^1 = \mathbb{R}/2\pi i \mathbb{Z} \) with Haar measure. Point evaluation from \( \ell^2(\mathbb{Z}) \) to \( \mathbb{C} \) is continuous because \( |a_n| \leq \|(a_k)_k\|_2 \). Thus, any continuous linear map from \( L^2(S^1) \) into \( \ell^2(\mathbb{Z}) \) is a kernel map. In particular, the Fourier transform \( T : L^2(S^1) \to \ell^2(\mathbb{Z}) \) is the unitary linear map given by the kernel \( k_T(z, n) = \bar{z}^n \). Next, we verify \( T^* \) restricted to the subspace of absolutely convergent series is an integral map and globally is a weak integral map. Later on, we verify \( T^* \) is not an integral map. It readily follows that the adjoint of \( T \) is the linear map

\[
\ell^2(\mathbb{Z}) \ni (f_n)_n \mapsto \sum_n f_n z^n \in L^2(S^1).
\]

For completeness we verify that for every \( f \in \ell^2(\mathbb{Z}) \), the weak integral \( \int_{\mathbb{Z}} f_n \overline{k_T(z, n)} dn \) does exist and it is equal to the \( L^2 \)-limit of the series \( \sum_n f_n z^n \). In fact, for any \( g \in L^2(S^1) \) we have

\[
\langle \sum_n f_n z^n, g \rangle_{\ell^2(\mathbb{Z})} = \int_{S^1} (\sum_n f_n e^{inx}) \overline{g(e^{ix})} d\varphi. \quad \text{Thus,} \quad \left| \left\langle \sum_n f_n z^n, g \right\rangle_{\ell^2(\mathbb{Z})} \right| \leq \|f\|_{\ell^2(\mathbb{Z})} \|g\|_{\ell^2(\mathbb{Z})}.
\]

We now verify that \( T^* \) is not an integral map. For this, we first notice that the adjoint kernel \( k_T^*(n, z) = z^n \) defines an integral map \( (T^*)_0 : \ell^2(\mathbb{Z}) \to L^2(S^1) \) with domain the linear subspace

\[
\mathcal{D}_{k_T^*} = \{(a_n)_n : \sum_n |a_n| < \infty \}
\]

In fact, Lebesgue integral is absolutely convergent, whence \( f \) belongs to the domain of \( (T^*)_0 \) iff \( \int_K \left| k_T^*(x, y) f(x) \right| dx < \infty \forall y \in Y \), and the resulting function belongs to \( L^2(S^1) \). In our case we have to analyze the integral \( \int_{\mathbb{Z}} (f(n)K_T^*(n, z)) |dn| = \sum_n |f_n| \). Hence, if \( T^* \) where an integral operator, Appendix 10.2, would give the kernel that should represent \( T^* \) ought to be \( z^n \), hence, the domain of \( (T^*)_0 \) would be the whole \( \ell^2(\mathbb{Z}) \). A contradiction.

10.4. For the purpose of this note, a closed subspace \( V \) of \( L^2(G \times \tau W) \) is a reproducing kernel subspace, if \( V \) consists of smooth functions and evaluation at each \( x \in G \) is a continuous linear map. Hence, for each \( w \in W, x \in G \) the linear functional on \( V \)

\[
f \mapsto (e_x(f), w)
\]

is represented by a function \( k_x(\cdot)^*(w) \in V \). Thus, the function \( y \mapsto k_x(y)^*(w) \) is square integrable, smooth and the following equality hold:

\[
(f(x), w)_W = (f, k_x(\cdot)^*(w))_V, \quad x \in G, \quad w \in W, \quad f \in V.
\]

The function

\[
W \ni w \mapsto k_x(y)^*(w) \in W
\]

is linear. We define \( K_V : G \times G \to \text{End}_G(W) \) to be \( K_V(y, x)(w) = k_x(y)(w) \). Since the product of two square integrable functions gives an integrable function, we have, for \( f \in L^2(G \times \tau W) \) the integral below is absolutely convergent

\[
P_V(f)(x) := \int_G K_V(y, x) f(y) dy \quad x \in G \quad (A - 4.1)
\]

In [Hi], [OO1], [OO2] we find a proof that the map \( f \mapsto P_V(f) \) is the orthogonal projector onto \( V \). Let \( j_W : W \to W^* \) the conjugate linear map
determinate by the inner product \((...,...)_W\). For each orthonormal basis \(\{f_j, j = 1, 2, \ldots\}\) for \(V\). Then, in [At] we find a proof of

\[
K_V(y, x) = \sum_{r \geq 1} j_W(f_r(y)) \otimes f_r(x) = \sum_{r \geq 1} j_V(f_r(y)) \otimes f_r(x) \quad (A - 4.2)
\]

We analyze the convergence of the series \((A-4.2)\) in Appendix 10.5. The main examples of reproducing kernel subspaces come from the statements below, in [At] we find proofs of the stated facts.

10.5. Let \(D\) be elliptic operator which maps sections of vector bundle into sections of perhaps another vector bundle over \(G/K\). Define \(Ker_2(D)\) equal to the totality of \(L^2\)-sections which are in the kernel of \(D\) as distributions. Then, \(Ker_2(D)\) is closed in \(L^2\) and owing to the regularity theorem \(Ker_2(D)\) consists of smooth functions, moreover, \(L^2\) convergence of a sequence, implies uniform convergence on compact sets of the sequence as well as any derivative of the sequence, hence \(Ker_2(D)\) is an example of reproducing kernel Hilbert space. In [At], Prop 2.4 we find a proof of: the matrix kernel \(K_{Ker_2(D)} : G \times G \to End_C(W)\) determine by \(Ker_2(D)\) is a smooth function and the convergence of the sequence in \((A-4.2)\), as well as any derivative, is uniform on compact sets. In [At] we find a proof that \(tr(K_{Ker_2(D)}(x, x)) = \sum_j \|f_j(x)\|^2\), \(x \in G/K\). Whence, Schwarz inequality yields the convergence of \((A-4.2)\) is absolute. As a corollary, we obtain

Let \(N\) be \(L^2\)-closed subspace of \(Ker_2(D)\). Then, \(N\) is a reproducing kernel subspace and the matrix kernel for \(N\) is a smooth function. This is so, because the series that represents the matrix kernel for \(N\) is a sub-series of the absolutely convergent series \((A-4.2)\). For the same reason as before, the matrix kernel for \(N\) is a real analytic function. In particular, the matrix kernel \(K_\lambda\) for \(H^2(G, \tau)\) is a real analytic function and the matrix kernel for any closed subspace in \(H^2(G, \tau)\) is also given by a real analytic function.

10.6. The linear operator \(P_\lambda\) has two representations as an integral operator. One matrix kernel that represents \(P_\lambda\) is \(K_\lambda(y, x) = \Phi_0(x^{-1}y)\) where \(z \mapsto \Phi_0(z) := d(\pi_\lambda)P_W(\pi^G_\lambda(z)P_V)\) is the spherical function associated to the lowest \(K\)-type \(W\). Let \(\phi_0(z) = d(\pi_\lambda)tr(\Phi_0(z))\), then, the the trace kernel which represents \(P_\lambda\) is

\[
k_\lambda(y, x) := \phi_0(x^{-1}y) := d(\pi_\lambda)trP_W(\pi^G_\lambda(x^{-1}y)P_V) = d(\pi_\lambda)\sum_j(\pi^G_\lambda(x^{-1}y)f_j)_V = tr(K_\lambda(y, x)).
\]

where \(f_1, \ldots\) is an orthonormal basis for the lowest \(K\)-type \(V_\lambda[W]\). Of course, there are some identifications we have avoided to explicit. For a proof of the explicit representation of \(P_\lambda\) by \(K_\lambda, k_\lambda\) we refer to [OO2], [WW]. A sketch of proof is in next subsection. Certainly, we recover \(\Phi_0\) from \(\phi_0\) by the formula \(\Phi_0(z) = \int_K \tau(k)\phi_0(k^{-1}z)dk\).

10.7. Sketch of proof that the kernel’s \(K_\lambda, k_\lambda\) represent \(P_\lambda\) as an integral operator. The orthogonal projector onto \(H^2(G, \tau)\) is represented by the matrix
kernel \( K_\lambda(y, x) = \ldots P_W \pi_\lambda(x^{-1}y)P_W \). In fact, let \( f_v \) as in Proposition 6.8, then
\[
\int_G (P_W \pi_\lambda(x^{-1}y)P_W)f_v(y)dy = \sum_{i,s} \int_G (x^{-1}yw_i, w_s)(y^{-1}v, w_i)w_sdy
\]
\[
= \sum_{i,s} \int_G (yw_i, xw_s)(yw_i, v)w_sdy = \ldots \sum_{i,s} (w_i, w_i)(xw_s, v)w_s
\]
\[
= \ldots \sum_s (x^{-1}v, w_s)w_s = \ldots \phi(x).
\]

A trace kernel that represents the orthogonal projector onto \( H^2(G, \tau) \) is \( k_\lambda(y, x) = \ldots \phi_0(x^{-1}y) \). Indeed,
\[
\int_G tr(P_W \pi_\lambda(x^{-1}y)P_W)f_v(y)dy = \int_G \sum_{r,s} (x^{-1}yw_r, w_r)(y^{-1}v, w_i)w_idy
\]
\[
= \sum_{r,i} \int_G (yw_r, xw_r)(yw_i, v)dy = \ldots \sum_{r,i} (w_r, w_i)(xw_r, v)w_i = \ldots \phi(x).
\]

11. Notation

- \((\tau, W), (\sigma, Z), L^2(G \times \tau W), L^2(H \times \sigma Z)\) 2.1
- \(H^2(G, \tau) = V^G_\lambda = V^K_\lambda, H^2(H, \sigma) = V^H_\mu, \pi^K_\mu, \pi^K \) 2.1
- \(\pi_\lambda = \pi^K_\lambda, d_\lambda \) dimension of \( \pi_\lambda, \pi^K_\mu, P_\lambda, P_\mu, K_\lambda, K_\mu, k_\lambda, k_\mu, 2.1 \)
- \(\Phi_0, \phi_0\) Appendix 10.6
- \(M肯fin (resp. M^\infty)\) \( K \)–finite vectors in \( M \) (resp. smooth vectors in \( M \))
- \(dg, dh\) Haar measures on \( G, H \)
- A unitary representation is square integrable, equivalently a discrete series representation, (resp. integrable) if some nonzero matrix coefficient is square integrable (resp. integrable) with respect to Haar measure.
- \(\Theta_{\pi^K_\mu} (\ldots)\) Harish-Chandra character of the representation \( \pi^K_\mu \).
- For a module \( M \) (resp. a simple module \( N \), over a ring, \( M[N] \) denotes the isotypic component of \( N \) in \( M \). That is, \( M[N] \) is the sum of all irreducible submodules isomorphic to \( N \). If topology is involved, we define \( M[N] \) to be the closure of \( M[N] \).
- \(M_{disc}(M_{H-disc})\) the closure of the linear subspace spanned by the totality of irreducible submodules.
- A representation \( M \) is discretely decomposable if \( M_{disc} = (M_{H-disc}) = M \).
- A representation is \( H \)–admissible if it is discretely decomposable and each isotypic component is equal to a finite sum of irreducible representations.
- \(K_\lambda, \mu\) matrix kernel for the orthogonal projector \( P_\lambda, \mu \) onto \( H^2(G, \tau)[V^K_\mu] \).
- \(U(g)\) (resp. \( \mathfrak{u}(g)\)) universal enveloping algebra of the Lie algebra \( g\) (resp. center of universal enveloping).

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