Eventually dendric subshifts
Francesco Dolce and Dominique Perrin
July 16, 2018

Contents
1 Introduction 1
2 Eventually dendric subshifts 3
3 Complexity of subshifts 4
4 Asymptotic equivalence 8
5 Simple trees 12
6 Conjugacy 13
7 A property of eventually dendric shifts 16
8 Generalized extension graphs 17
9 Complete bifix decoding 21

Abstract
We introduce a class of subshifts called eventually dendric. This class generalizes the class of dendric subshifts studied in previous papers as the class of subshifts whose language is a tree set and defined by a property of the extensions of each word. Our main result is that the class of eventually dendric subshifts is closed under conjugacy.

1 Introduction

Minimal subshifts have been extensively studied. They appear naturally as the subshifts generated by primitive morphisms as well as the natural coding of transformations such as rotations of the circle or interval exchange transformations. Several books are devoted to the study of such subshifts (see [10] or [11] for example).

Among minimal subshifts, some special classes have received attention. In particular Sturmian subshifts have been introduced and described by Arnoux
and Rauzy (and are often called Arnoux-Rauzy subshifts). The class of dendric subshifts has been introduced in [3] (under the name of tree sets given to their language). A dendric subshift \( X \) is defined by introducing the extension graph of a word in the language \( L(X) \) of \( X \) and by requiring that this graph is a tree for every word in \( L(X) \). The class of dendric subshifts generalizes both the class of Sturmian subshifts and of interval exchange subshifts. It has many interesting properties which involve free groups. In particular, in a dendric subshift \( X \) on the alphabet \( A \), the group generated by the set of return words to some word in \( L(X) \) is the free group on the alphabet and, in particular, has \( \text{Card}(A) \) elements. This generalizes a property known for Sturmian subshifts whose link with automorphisms of the free group was noted by Arnoux and Rauzy.

However, the class of dendric subshifts is not closed under conjugacy, and thus is not a natural class from the dynamical point of view. We prove here, following a suggestion of Fabien Durand, that one obtains a class which is closed under conjugacy by weakening as follows the definition. The class of eventually dendric subshifts, introduced in this paper, is defined by the property that the extension graph of every word \( w \) in the language of the subshift is a tree for every long enough word \( w \). Our main result is that the class of eventually dendric subshifts is closed under conjugacy (Theorem 6.1).

The paper is organized as follows. In the first section, we introduce the definition of the extension graph and of an eventually dendric subshift. In Section 3, we recall some mostly known properties on the complexity of a subshift and of special words. We prove a result which characterizes eventually dendric subshifts by the extension properties of special words (Proposition 3.5). In Section 4, we use the classical notion of asymptotic equivalence to give a second characterization of eventually dendric subshifts (Proposition 4.5). In Section 5, we introduce the notion of a simple tree and we prove in a minimal eventually dendric subshift, the extension graph of every long enough word is a simple tree (Proposition 5.1), a property which holds trivially for every word in a Sturmian subshift. In Section 6, we prove our main result (Theorem 6.1) using the previous results. In the next sections (Section 7 to 9), we prove additional properties of eventually dendric subshifts. We first prove that eventually dendric subshifts are minimal as soon as they are irreducible (Theorem 7.1), a property already known for dendric subshifts [7]. Next we introduce generalized extension graphs in which extension by words of of fixed length replace extension by letters. We prove that one obtain an equivalent definition of eventually dendric subshifts using these generalized extension graphs (Theorem 8.1). Finally, we prove that the class of eventually dendric subshifts is closed under complete bifix decoding, a result already known for dendric subshifts.

Acknowledgements We thank Valérie Berthé, Paulina Cecchi, Fabien Durand and Samuel Petite for useful conversations on this subject and especially Fabien Durand which suggested to us the statement of the main result.
2 Eventually dendric subshifts

Let $X$ be a subshift on the alphabet $A$. We denote by $L(X)$ the language of $X$, which is the set of finite factors of the elements of $X$. A language $L$ on the alphabet $A$ is the language of a subshift if and only if it is factorial (that is contains the factors of its elements) and extendable (that is for any $w \in L$ there are letters $a, b \in A$ such that $awb \in L$).

For $n \geq 0$ we denote

$$L_n(X) = L(X) \cap A^n \quad L_{\geq n}(X) = \cup_{m \geq n} L_m(X).$$

For $w \in L(X)$ and $n \geq 1$, we denote

$$G_n(w, X) = \{ u \in L_n(X) \mid uw \in L(X) \} \quad D_n(w, X) = \{ v \in L_n(X) \mid vw \in L(X) \} \quad E_n(w, X) = \{ (u, v) \in G_n(w, X) \times D_n(w, X) \mid uuv \in L(X) \}.$$

The extension graph of order $n$ of $w$, denoted $E_n(w, X)$, is the undirected graph with set of vertices the disjoint union of $G_n(w, X)$ and $D_n(w, X)$ and with edges the elements of $E_n(w, X)$.

When the context is clear, we denote $G_n(w), D_n(w), E_n(w)$ instead of $G_n(w, X), D_n(w, X), E_n(w, X)$ and $E_n(w, X)$.

A path in an undirected graph is reduced if it does not contain successive equal edges. A graph is a tree if and only if there is a unique reduced path between any pair of vertices. For any $w \in L(X)$, since any vertex of $G_n(w)$ is connected to at least one vertex of $D_n(w)$, the graph $E_n(w)$ is a tree if and only if there is a unique reduced path between every pair of vertices of $G_n(w)$ (resp. $D_n(w)$).

The subshift $X$ is said to be eventually dendric with threshold $m \geq 0$ if $E_1(w)$ is a tree for every word $w \in L_{\geq n}(X)$. It is said to be dendric if we can choose $m = 0$.

The languages of dendric subshifts were introduced in [3] under the name of tree sets. An important example of dendric subshifts is formed by Sturmian subshifts (also called Arnoux-Rauzy subshifts), which are by definition such that $L(X)$ is closed by reversal and such that for every $n$ there exists a unique $w_n \in L_n(X)$ such that Card$(D_1(w_n)) = Card(A)$ and such that for every $w \in L_n(X) \setminus \{w_n\}$ one has Card$(D_1(w)) = 1$ (see [3]).

Example 2.1 Let $X$ be the Fibonacci subshift, which is generated by the morphism $a \mapsto ab, b \mapsto a$. It is a Sturmian subshift (see [3]). The graph $E_1(a)$ is shown in Figure 1 on the left. The graph $E_3(a)$ is shown on the right.

The tree sets of characteristic $c \geq 1$ introduced in [2, 7] give an example of eventually dendric subshifts.
Example 2.2 Let $X$ be the subshift generated by the morphism $a \mapsto ab, b \mapsto cda, c \mapsto cd, d \mapsto abc$. Its language is a tree set of characteristic 2 ([2, Example 4.2]) and it is actually a specular set. The extension graph $E_1(\varepsilon)$ is shown in Figure [2].

Since the extension graphs of all nonempty words are trees, the subshift is eventually dendric with threshold 1.

3 Complexity of subshifts

Let $X$ be a subshift. For a word $w \in L(X)$, we denote

$$g(w) = \text{Card}(G_1(w)), \ e(w) = \text{Card}(E_1(w)), \ d(w) = \text{Card}(D_1(w)).$$

For any $w \in L(X)$, we have $1 \leq g(w), d(w) \leq e(w)$. The word $w$ is left-special if $g(w) > 1$, right-special if $d(w) > 1$ and bispecial if it is both left-special and right-special.

We define the multiplicity of $w$ as

$$m(w) = e(w) - g(w) - d(w) + 1.$$

We say that $w$ is strong if $m(w) \geq 0$, weak if $m(w) \leq 0$ and neutral if $m(w) = 0$.

It is clear that

1. if $E_1(w)$ is acyclic, then $w$ is weak,
2. if $E_1(w)$ is connected, then $w$ is strong,
3. if $E_1(w)$ is a tree, then $w$ is neutral.

Proposition 3.1 Let $X$ be a subshift and let $w \in L(X)$. If $w$ is neutral, then

$$g(w) - 1 = \sum_{b \in D_1(w)} (g(wb) - 1) \quad (3.1)$$
Proof. Since $w$ is neutral, we have $e(w) = g(w) + d(w) - 1$. Thus
\[
\sum_{b \in D_1(w)} (g(wb) - 1) = e(w) - d(w) = g(w) - 1.
\]

Set further
\[
p_n(X) = \text{Card}(L_n(X)),
\]
\[
s_n(X) = p_{n+1}(X) - p_n(X),
\]
\[
b_n(X) = s_{n+1}(X) - s_n(X).
\]
The sequence $p_n(X)$ is called the complexity of the subshift $X$.

The following result is from [5] (see also [3, Lemma 2.12]). We include a proof for convenience.

**Proposition 3.2** We have for all $n \geq 0$,
\[
s_n(X) = \sum_{w \in L_n(X)} (g(w) - 1) = \sum_{w \in L_n(X)} (d(w) - 1) \tag{3.2}
\]
and
\[
b_n(X) = \sum_{w \in L_n(X)} m(w). \tag{3.3}
\]
In particular, the number of left-special (resp. right-special) words of length $n$ is bounded by $s_n(X)$.

Proof. We have
\[
\sum_{w \in L_n(X)} (g(w) - 1) = \sum_{w \in L_n(X)} g(w) - \text{Card}(L_n(X)) = \text{Card}(L_{n+1}(X)) - \text{Card}(L_n(X)) = p_{n+1} - p_n = s_n(X)
\]
with the same result for $\sum_{w \in L_n(X)} (d(w) - 1)$. Next
\[
\sum_{w \in L_n(X)} m(w) = \sum_{w \in L_n(X)} (e(w) - g(w) - r(w) + 1) = p_{n+2}(X) - 2p_{n+1}(X) + p_n(X) = s_{n+1}(X) - s_n(X) = b_n(X).
\]

We will use the following easy consequence of Proposition 3.2.
Proposition 3.3 Let $X$ be a subshift. If $X$ is eventually dendric, then the sequence $s_m(X)$ is eventually constant.

Proof. Let $n \geq 1$ be such that the extension graph of every word in $L_{\geq n}(X)$ is a tree. Then $b_m(X) = 0$ for every $m \geq n$. Thus $s_m(X) = s_{m+1}(X)$ for every $m \geq n$, whence our conclusion. 

The converse of Proposition 3.3 is not true, as shown by the following example.

Example 3.4 Let $X$ be the Chacon ternary subshift, which is the substitutive subshift generated by the morphism $\varphi : a \mapsto aabc, b \mapsto bc, c \mapsto abc$. It is well known that the complexity of $X$ is $p_n(X) = 2n + 1$ and thus that $s_n = 2$ for all $n \geq 0$ (see [10, Section 5.5.2]). The extension graphs of $abc$ and $bca$ are shown in Figure 3.

Thus $m(abc) = 1$ and $m(bca) = -1$. Let now $\alpha$ be the map on words defined by $\alpha(x) = aabc\varphi(x)$. Let us verify that if the extension graph of $x$ is the graph of Figure 3 on the left, the same holds for the extension graph of $y = \alpha(x)$. Indeed, since $axa \in L(X)$, the word $\varphi(axa) = aabc\varphi(x)aabc = ayaabc$ is also in $L(X)$ and thus $(a, a) \in E_1(y)$. Since $cxa \in L(X)$ and since a letter $c$ is always preceded by a letter $b$, we have $bcxa \in L(X)$. Thus $\varphi(bcxa) = bcyabc \in L(X)$ and thus $(c, a) \in E_1(y)$. The proof of the other cases is similar. The same property holds for a word $x$ with the extension graph on the right of Figure 3. This shows that there is an infinity of words whose extension graph is not a tree and thus the Chacon set is not eventually dendric.

Let $X$ be a subshift. We define $SG_n(X)$ (resp. $SG_{\geq n}(X)$) as the set of left-special words of $L(X)$ of length $n$ (resp. at least $n$).

The following result expresses the fact that eventually dendric subshifts are characterized by an asymptotic property of left-special words which is a local version of the property defining Sturmian subshifts.

Proposition 3.5 A subshift $X$ is eventually dendric if and only if there is an integer $n \geq 0$ such that any word $w$ of $SG_{\geq n}(X)$ has exactly one right extension $wb \in SG_{\geq n+1}(X)$ with $b \in A$ which is moreover such that $g(wb) = g(w)$.

Proof. Assume first that $X$ is eventually dendric with threshold $m$. Then any word $w$ in $SG_{\geq m}(X)$ has at least one right extension in $SG(X)$. Indeed, since $G_1(w)$ has at least two elements and since the graph $E_1(w)$ is connected, there is at least one element of $D_1(w)$ which is connected by an edge to more than one element of $G_1(w)$.
Next, Equation (3.1) shows that for any \( w \in SG_{\geq m}(X) \) which has more than one right extension in \( SG(X) \), one has \( g(wb) < g(w) \) for each such extension. Thus the number of words prefix of one another which have more than one right extension is bounded by \( \text{Card}(A) \). This proves that there exists an \( n \geq m \) such that for any \( w \in L_{\geq n}(X) \) there is exactly one \( b \in A \) such that \( wb \in SG(X) \). Moreover, one has then \( g(wb) = g(w) \) by Equation (3.1).

Conversely, assume that the condition is satisfied for some integer \( n \). For any word \( w \) in \( L_{\geq n}(X) \), the graph \( E_1(w) \) is acyclic since all vertices in \( D_1(w) \) except at most one have degree 1. Thus \( w \) is weak. Let \( N \) be the length of \( w \). Then for every word \( u \) of length \( N \) and every \( b \in A \), one has \( g(ub) = 0 \) except for one letter \( b \) such that \( g(ub) = g(u) \). Thus, by Proposition 3.2

\[
s_N(X) = \sum_{u \in L_N(X)} (g(u) - 1) = \sum_{v \in L_{N+1}(X)} (g(v) - 1) = s_{N+1}(X).
\]

This shows that \( b_N = 0 \) for every \( N \geq n \) and thus, by Proposition 3.2 again, all words in \( L_{\geq n}(X) \) are neutral. Since all graphs \( E_1(w) \) are moreover acyclic, this forces that these graphs are trees and thus that \( X \) is eventually dendric with threshold \( n \).

We give below an example of a subshift which is shown to be eventually dendric using Proposition 3.3.

**Example 3.6** Let \( X \) be the Tribonacci subshift, which is the Sturmian subshift generated by the substitution \( \varphi : a \mapsto ab, b \mapsto ac, c \mapsto a \) and let \( \alpha \) be the morphism \( \alpha : a \mapsto a, b \mapsto a, c \mapsto c \). Let \( \varphi^\omega(a) \) be the right infinite word having all \( \varphi^n(a) \) for \( n \geq 1 \) as prefixes. The left-special words for \( X \) are the prefixes of \( \varphi^\omega(a) \). Indeed, it is easy to verify that if \( w \) is left-special, then \( \varphi(w) \) is also left-special.

Note that the set \( L(X) \cap c\{a, b\}^*c \) is

\[
\{cabac, cabaabc, cababc\}.
\]

Since these three words are of distinct lengths, it follows that the restriction of \( \alpha \) to the set \( L(X) \cap c\{a, b\}^*c \) is injective.

Next we claim that the left-special words for \( \alpha(X) \) containing a letter \( c \) are the prefixes of \( \alpha(\varphi^\omega(a)) \) or \( aao(\varphi^\omega(a)) \) containing a letter \( c \). Indeed, if \( w \) is a prefix of \( \varphi^\omega(a) \), we have \( G_1(w, X) = \{a, b, c\} \) and thus \( G_1(\alpha(w), \alpha(X)) = \{a, c\} \) showing that \( \alpha(w) \) is left-special. Next, \( G_3(w, X) = \{aba, bac, cab\} \) and thus \( G_1(aao(w), \alpha(X)) = \{a, c\} \) showing that \( aao(w) \) is left-special. Conversely, assume that \( u \) is left-special for \( \alpha(X) \) and contains a \( c \). Since \( u \) is a prefix of a word ending with \( c \), we may assume that \( u \) ends with \( c \). Set \( u = a^jcv \) with \( j \geq 0 \). By a previous remark, there is a unique word \( s \in L(X) \) such that \( csc \in L(X) \) and \( \alpha(csc) = cvc \). Since every word in \( L(X) \) of length at least 7 contains a \( c \), we have \( j \leq 6 \). It is easy to verify by inspection of the possible left extensions of \( c \) in \( L(X) \) that \( u \) is left-special only when \( j = 3 \) or \( j = 5 \) (see Figure 4).

7
If \( j = 3 \), then \( u = \alpha(w) \) where \( w = abacsc \) is left-special in \( L(X) \) and thus is a prefix of \( \varphi^x(a) \). If \( j = 5 \), then \( u \) is the common image by \( \alpha \) of \( ababacsc \) and \( baabacsc \). Then \( w = abacsc \) is left-special in \( X \) and thus is a prefix of \( \varphi^x(a) \). Since \( u = aac(w) \), the claim is proved.

It follows from the claim that the subshift \( \alpha(X) \) satisfies the condition of Proposition 3.5 with \( n = 4 \). Thus we conclude that \( \alpha(X) \) is dendric with threshold at most 4. The threshold is actually 4 since \( a^3 \) has multiplicity 1 in \( \alpha(X) \).

### 4 Asymptotic equivalence

We denote by \( \sigma \) the shift transformation of \( A^\mathbb{Z} \) defined by \( y = \sigma(x) \) if \( y_n = x_{n+1} \) for every \( n \in \mathbb{Z} \). We denote by the same symbol the shift transformation on the set \( A^\mathbb{N} \) of right-infinite words defined by the same formula for every \( n \in \mathbb{N} \). One must however use this abuse of notation carefully because the shift transformation is a bijection on \( A^\mathbb{Z} \) but not on \( A^\mathbb{N} \). Thus \( \sigma^{-1} \) is well defined on \( A^\mathbb{Z} \) but not on \( A^\mathbb{N} \).

The orbit of \( x \in A^\mathbb{Z} \) is the class of equivalence of \( x \) under the action of the shift transformation. Thus \( y \) is in the orbit of \( x \) if there is an \( n \in \mathbb{Z} \) such that \( x = \sigma^n(y) \). We say that \( x \) is a shift of \( y \) if they belong to the same orbit.

For \( x \in A^\mathbb{Z} \), denote \( x^- = \cdots x_{-2}x_{-1} \) and \( x^+ = x_0x_1 \cdots \) and \( x = x^- \cdot x^+ \). When \( X \) is a subshift, we denote \( X^+ \) the set of right infinite words \( u \) such that \( u = x^+ \) for some \( x \in X \).

A right infinite word \( u \in A^\mathbb{N} \) is a tail of the two-sided infinite word \( x \in A^\mathbb{Z} \) if \( u = y^+ \) for some shift \( y \) of \( x \), that is \( u = x_nx_{n+1} \cdots \) for some \( n \in \mathbb{Z} \).

Let \( X \) be a subshift on the alphabet \( A \). The right asymptotic equivalence is the equivalence on \( X \) defined as follows. Two elements \( x, y \) of \( X \) are asymptotically equivalent if there exists two shifts \( x', y' \) of \( x, y \) such that \( x'^+ = y'^+ \).

In other words, \( x, y \) are right asymptotic equivalent if they have a common tail (see Figure 5).

\[ \text{Figure 5: Two right asymptotic sequences } x, y. \]
The classes of the asymptotic equivalence not reduced to one orbit are called right asymptotic classes (they are called in [8] asymptotic components).

**Example 4.1** The Fibonacci subshift $X$ has one right asymptotic class formed of the shifts of the two elements $x, y \in X$ such that $x^+ = y^+ = \varphi_\omega(a)$ where $\varphi_\omega(a)$ is the Fibonacci word, that is the right infinite word having all $\varphi^n(a)$ for $n \geq 1$ as prefixes. Indeed, let $x, y \in X$ be such that $x^+ = y^+$ with $x \neq y$. Then all finite prefixes of $x^+ = y^+$ are left-special and thus are prefixes of $\varphi_\omega(a)$. Thus $x^+ = y^+ = \varphi_\omega(a)$.

If $C$ is an asymptotic class, it is, by the definition of asymptotic equivalence, a union of orbits. The following result is proved in [8, Lemma 3.2] under a weaker hypothesis that we shall not need here. We give a proof for the sake of completeness.

**Proposition 4.2** Let $X$ be a subshift such that the sequence $s_n(X)$ is bounded by $k$. Then the number of asymptotic classes is finite and at most equal to $k$.

**Proof.** Let $(x_1, y_1), \ldots, (x_k, y_k)$ be $k$ pairs of distinct elements of $X$ such that $x_i^+ = y_i^+$ for $1 \leq i \leq k$ belonging to asymptotic classes $C_1, \ldots, C_k$. For $n$ large enough the prefixes of length $n$ of the $x_i^+$ are $k$ distinct left-special words and thus $k \leq s_n(X)$ since by Proposition 3.2 the number of left-special words is bounded by $s_n(X)$. This shows that the number of asymptotic classes is finite by Proposition 3.3.

Let $X$ be a subshift. For an asymptotic class $C$ of $X$, we denote $\omega(C) = \text{Card}(o(C)) - 1$ where $o(C)$ is the set of orbits contained in $C$. For a right infinite word $u \in X^+$, let

$$g_C(u) = \text{Card}\{a \in A \mid x^+ = au \text{ for some } x \in C\}.$$ We denote by $S(C)$ the set of right infinite words $u$ such that $g_C(u) \geq 2$.

A nonempty subshift is minimal if it does not contain properly another nonempty subshift. As well known, $X$ is minimal if and only if it is uniformly recurrent, that is for any $w \in L(X)$ there exists an $n \geq 0$ such that $w$ is a factor of any word in $L_n(X)$. If $X$ is minimal and infinite, then there exists for every $w \in L(X)$ an integer $n \geq 1$ such that $w^n \notin L(X)$. Indeed, otherwise, $L(X)$ contains the periodic word with period $w$ and thus $X$ is equal to the finite subshift formed by the shifts of $\cdots ww \cdot ww \cdots$.

The following statement can be seen as an infinite counterpart of Proposition 3.2.

**Proposition 4.3** Let $X$ be a minimal subshift and let $C$ be a right asymptotic class. Then

$$\omega(C) = \sum_{u \in S(C)} (g_C(u) - 1)$$ (4.1)

where both sides are simultaneously finite.
Proof. We may assume that $X$ is infinite since otherwise, there is only one asymptotic class formed of one orbit.

We first claim that if $u, v \in S(C)$, there exist $n, m \geq 0$ such that $\sigma^n(v) = \sigma^m(u)$. Indeed, let $x, y \in C$ be such that $x^+ = u$ and $y^+ = v$. Since $x, y$ are right asymptotic, there exist $n', m' \in \mathbb{Z}$ such that $\sigma^{n'}(x)^+ = \sigma^{m'}(y)^+$. Applying $\sigma^p$ with $p \geq 0$ large enough on both sides, we obtain $\sigma^n(x)^+ = \sigma^m(y)^+$ for $n = n' + p \geq 0$ and $m = m' + p \geq 0$. Then $\sigma^n(u) = \sigma^m(v)$ and the claim is proved.

Assume first that $\omega(C)$ is finite. We build a finite tree $T(C)$ having $o(C)$ as set of leaves as follows. The set of nodes of $T(C)$ is $o(C) \cup S(C)$. For every $x \in C$, there is at least one $y \in X$ with $x ney$ such that $y^+ = x^+$. Let $n \geq 0$ be the minimal integer such that $x_{n-1} \neq y_{n-1}$. Then $\sigma^{-n}(x)^+$ is in $S(C)$ and depends only on the orbit of $x$. We define it as the father of the orbit of $x$ in the tree $T(C)$. For $u \in S(C)$, the father of $u$ is $\sigma^n(u)$ where $n \geq 0$ is the minimal integer if it exists such that $\sigma^n(u) \in S(C)$. Since $X$ is minimal, we have then $\sigma^n(u) \neq u$.

There is only one $u \in S(C)$ which does not have a father. Indeed, let $u, v \in S(C)$ be distinct and without a father in $T(C)$. There are by the claim made above integers $n, m \geq 0$ such that $\sigma^n(u) = \sigma^m(v)$. Assume $n \leq m$ and choose $n$ minimal. If $n = 0$, then $v$ is an ancestor of $u$ in $T(C)$ and thus $u$ has a father, a contradiction. Otherwise, we have $u_{n+1} \neq v_{m-1}$ and thus $\sigma^n(u) = \sigma^m(v)$ is in $S(C)$ and is a common ancestor of $u$ and $v$, a contradiction again.

To show that $T(C)$ is a tree, there remains to show that for any $u \in o(C) \cup S(C)$ there exists a path from $u$ to the root. No element of $S(C)$ can be an ancestor of itself since otherwise it would imply that $\sigma^n(u) = u$ for some $n > 0$ which forces $u$ to be periodic, a contradiction with the hypothesis that $X$ is minimal and infinite. Since $\omega(C)$ is finite, the sequence $(u_1, u_2, \ldots)$ where $u_{i+1}$ is the father of $u_i$ ends when $u_i$ is the root of $T(C)$.

Formula (4.1) now follows from the fact that in any finite tree with $n$ leaves and and a set $V$ of internal vertices, one has $n - 1 = \sum_{v \in V} (d(v) - 1)$ where $d(v)$ is the number of sons of vertex $v$.

Assume now that the right hand side of Equation (4.1) is finite. Then the set $S(C)$ is finite and thus $T(C)$ is again a tree with a finite number of interior nodes. Since the degree of each node is finite, it implies that it has also a finite number of leaves. Thus $\omega(C)$ is finite and Equation (4.1) also holds.

Example 4.4 Consider again the image $\alpha(X)$ of the Tribonacci subshift by the morphism $\alpha : a \mapsto a, b \mapsto a, c \mapsto c$ (Example 3.6).

There is only one asymptotic class $C$ made of three orbits represented in Figure 6 on the left. The class is formed of the orbits of $x, y, z$ where $x^+ = \alpha(\varphi^n(a))$ and $y^+ = z^+ = aax^+$. The tree $T(C)$ is shown on the right.

Let us now deduce from Proposition 3.3 a characterization of eventually
dendric subshifts in terms of asymptotic classes. For a subshift \( X \), denote

\[ \omega(X) = \sum \omega(C) \]

where the sum is over the asymptotic classes \( C \) of \( X \).

**Proposition 4.5** A minimal subshift \( X \) is eventually dendric if and only if:

1. The sequence \( s_n(X) \) is eventually constant, and
2. We have \( \lim s_n(X) = \omega(X) \).

**Proof.** Assume first that \( X \) is eventually dendric. Then assertion 1 holds by Proposition 3.3. To prove assertion 2, consider an integer \( n \) large enough so that the condition of Proposition 3.5 holds (it implies that \( s_m(X) \) is constant for \( m \geq n \)). Let us consider an asymptotic class \( C \).

Let \( \pi \) be the map assigning to \( u \in A^\mathbb{N} \) its prefix of length \( n \). Then \( \pi \) maps \( S(C) \) into \( SG_n(X) \). The map \( \pi \) is injective since otherwise some word in \( SG_{\geq n}(X) \) would have more than one extension on the right, contrary to Proposition 3.5. Next the sets \( \pi(S(C)) \) form a partition of \( SG_n(X) \).

Thus, by Equation (3.2),

\[ s_n(X) = \sum_{w \in SG_n(X)} (g(w) - 1) = \sum_C \sum_{w \in S(C)} (g_C(u) - 1) = \sum_C \omega(C) \]

where the last equality follows from Equation (4.1).

Conversely, if the two conditions are satisfied, let \( n \) be large enough so that \( s_m(X) = s_n(X) \) for all \( m \geq n \). The number of left-special words of length \( m \) is bounded by \( s_m(X) \) and thus it is finite. We may thus also assume that \( n \) is large enough so that the prefixes of length \( n \) of the words of \( S(C) \) for every asymptotic class \( C \) are distinct.

Then, every word \( w \) of \( SG_n(X) \) has exactly one right extension \( wb \) in \( SG_{n+1}(X) \). It is moreover such that \( g(w) = g(wb) \) and thus \( X \) is eventually dendric by Proposition 3.5.
5 Simple trees

The diameter of a tree is the maximal length of simple paths. A tree is simple if its diameter is at most 3. Note that if the tree is the extension graph $E_n(w)$ in some subshift $X$ of a bispecial word $w$, then the diameter of $E_n(G)$ is at least 3 and this happens if and only if two vertices of $E_n(w)$ on the same side (that is, both in $G_n(x)$ or both in $D_n(w)$) are connected to a common vertex on the opposite side.

For example, if $X$ is the Fibonacci subshift, then $E_1(a)$ is simple while $E_3(a)$ is not (see Example 2.1).

We prove the following additional property of the graphs $E_k(w)$.

**Proposition 5.1** Let $X$ be a minimal and eventually dendric subshift. For any $k \geq 1$ there exists an $n \geq 1$ such that $E_k(w)$ is a simple tree for every $w \in L \geq n(X)$.

We first prove the following lemma.

**Lemma 5.2** Let $X$ be a minimal and eventually dendric subshift. For every $k \geq 1$ there is an $n \geq 1$ such that if $p, w \in L(X)$ with $|p| \leq k$ and $|w| \geq n$ are such that $pw, w \in SG(X)$, then $pw, w$ have a unique right extension in $SG(X)$ for some letter $b \in A$ which is moreover such that $g(pwb) = g(pw)$ and $g(wb) = g(w)$.

**Proof.** Consider two asymptotic classes $C, D$ and let $u \in S(C)$, $v \in S(D)$. If $C, D$ are distinct, we cannot have $pu = v$ for some word $p$. Thus there is an integer $n$ such that if $w$ is the prefix of length $n$ of $u$, then $pw$ is not a prefix of $v$. Since there is a finite number of words $p$ of length at most $k$, a finite number of asymptotic classes (by Proposition 4.2) and since for each such class the set $S(C)$ is finite, we infer that for every $k$ there exists an $n$ such that for every pair of asymptotic classes $C, D$ and any $u \in S(C), v \in S(D)$, if $w$ is a prefix of $u$ and $pw$ a prefix of $v$, with $|p| \leq k$ and $|w| = n$, then $C = D$.

Next, assume that $w$ is a prefix of $u$ and $pw$ a prefix of $v$ with $u, v \in S(C)$ for some asymptotic class $C$. If $v \neq pu$, then there is a right extension $u'$ of $w$ such that $pu'$ is not a prefix of $v$. By contraposition, if $n$ is large enough, we have $v = pu$.

We thus choose $n$ large enough so that:

1. All elements of $S(C)$ for all asymptotic components $C$ have distinct prefixes of length $n$;
2. For every pair of asymptotic classes $C, D$ and any $u \in S(C), v \in S(D)$, if $w$ is prefix of $u$ and $pw$ is prefix of $v$ with $|p| \leq k$ and $|w| = n$ then $C = D$ and $pu = v$.

We moreover assume that $n$ is large enough so that the condition of Proposition 3.5 holds. Note that Proposition 3.5 also holds since it is the case where $p$ is empty.
Consider $p, w$ with $|p| = k$ and $|w| = n$ such that $pw, w$ are left-special. By condition 1, there are asymptotic components $C, D$ and elements $u \in S(C)$ and $v \in S(D)$ such that $w$ is a prefix of $u$ and $pw$ a prefix of $v$. Because of condition 2, we must have $g^k(v) = u$ (in particular $C = D$). Thus there is a unique letter $b \in A$ such that $wb, pw \in SG(X)$ which is moreover such that $g(wb) = g(w)$ and $g(pwb) = g(pw)$ by Proposition 3.5.

\textbf{Proof of Proposition 5.1} We choose $n$ such that Proposition 3.5 and Lemma 5.2 hold.

We prove by induction on $\ell$ with $1 \leq \ell \leq k$ that for any $p, q \in G_\ell(w)$ there is an $r \in D_k(w)$ such that $pwr, qwr \in L(X)$.

The property is true for $\ell = 1$. Indeed, set $p = a$ and $q = b$. Apply iteratively Proposition 3.5 to obtain letters $c_1, \ldots, c_k$ such that $g(wc_1 \cdots c_i) = g(wc_1 \cdots c_{i+1})$ and set $r = c_1 \cdots c_k$. Then $awr, bwr \in L(X)$.

Assume next that the property is true for $\ell - 1$ and consider $ap, bq \in G_\ell(w)$ with $a, b \in A$. Replacing if necessary $w$ by some longer word, we may assume that $p, q$ end with different letters and thus that $w$ is left-special. By the induction hypothesis, there is a word $r \in D_k(w)$ such that $pur, qwr \in L(X)$. By Lemma 5.2 the first letter of $r$ is the unique letter $c$ such that $g(pwc) = g(qwc) = g(wc)$. Thus $apwc, bqwc \in L(X)$. Applying Lemma 5.2 iteratively in this way, we obtain that $apwr, bqwr \in L(X)$.

\section{Conjugacy}

The following result shows that the property of being eventually dendric is a dynamical property.

\textbf{Theorem 6.1} The class of eventually dendric subshifts is closed under conjugacy.

We first treat the following particular case of conjugacy. Let $X$ be a subshift on the alphabet $A$ and let $k \geq 1$. Let $f : L_k(X) \to A_k$ be a bijection from the set $L_k(X)$ of blocks of length $k$ of $X$ onto an alphabet $A_k$. The map $\gamma_k : X \to A_k^\mathbb{Z}$ defined for $x \in X$ by $y = \gamma_k(x)$ if for every $n \in \mathbb{Z}$

$$y_n = f(x_n \cdots x_{n+k-1})$$

is the $k$-th higher block code on $X$. The subshift $X^{(k)} = \gamma_k(X)$ is called the $k$-th higher block subshift of $X$. It is well known that the $k$-th higher block code is a conjugacy.

We extend the bijection $f : L_k(X) \to A_k$ to a map still denoted $f$ from $L_{\geq k}(X)$ to $L_{\geq 1}(X^{(k)})$ by $f(a_1a_2 \cdots a_n) = f(a_1 \cdots a_k) \cdots f(a_{n-k+1} \cdots a_n)$. Note that all nonempty elements of $L(X^{(k)})$ are image by $f$ on an element of $L(X)$, i.e., $L(X^{(k)}) = \{f(w) \mid w \in L_{\geq k}(X)\} \cup \{\varepsilon\}$. 

\begin{center}
\begin{tabular}{c}
\end{tabular}
\end{center}

13
Example 6.2 Let $X$ be the Fibonacci subshift. We show that the 2-block extension $X^{(2)}$ of $X$ is eventually dendric with threshold 1. Set $A_2 = \{ u, v, w \}$ with $f : aa \mapsto u, ab \mapsto v, ba \mapsto w$. Since $X$ is dendric, the graph $E_1(w)$ is a tree for every word $w \in L(X^{(2)})$ of length at least 1 (but not for $w = \varepsilon$). Thus $X^{(2)}$ is eventually dendric. It is actually a tree subshift of characteristic 2 since the graph $E_1(\varepsilon)$ is the union of two trees (see Figure 7).

![Figure 7: The extension graphs $E(\varepsilon)$ and $E_1(vw)$.](image)

Lemma 6.3 For every $k \geq 1$, the $k$-th higher block subshift $X^{(k)}$ is eventually dendric if and only if $X$ is eventually dendric.

Proof. We define for every $w \in L_{\geq k}(X)$ a map from $E_1(w)$ to $E_1(f(w))$ as follows.

To every $a \in G_1(w)$, we associate the first letter $\lambda(a)$ of $f(aw)$ and to every $b \in D_1(w)$, we associate the last letter $\rho(b)$ of $f(wb)$. Then, since $f(awb) = \lambda(a)f(w)\rho(b)$, the pair $(a, b)$ is in $E_1(w)$ if and only if $(\lambda(a), \rho(b))$ is in $E_1(f(w))$. Thus, the maps $\lambda, \rho$ define an isomorphism from $E_1(w)$ onto $E_1(f(w))$.

Thus we conclude that $X$ is eventually dendric with threshold $m$ if and only if $X^{(k)}$ is eventually dendric with threshold $M$ with $0 \leq M \leq \sup(1, m - k + 1)$.

Example 6.4 Let $X$ be the Fibonacci subshift. For all $k \geq 2$, $X^{(k)}$ is an eventually dendric subshift with threshold 1.

Example 6.5 Let $X$ be the subshift associated to the two-sided infinite word $\cdots abab \cdot abab \cdots$. $X$ is an eventually dendric subshift with threshold 1 (the empty word has 2 connected components). For every $k \geq 1$, the subshift $X^{(k)}$ is eventually dendric with threshold 1.

A morphism $\alpha : A^* \to B^*$ is called alphabetic if $\alpha(A) \subset B$.

Lemma 6.6 Let $X$ be an eventually dendric subshift on the alphabet $A$ and let $\alpha : A^* \to B^*$ be an alphabetic morphism which induces a conjugacy from $X$ onto a subshift space $Y$. Then $Y$ is eventually dendric.

Proof. Since $\alpha$ is invertible, there exists an integer $r \geq 0$ and a map $f : B^{2r+1} \to A$ such that for $x = (x_k)_{k \in \mathbb{Z}}$ and $y = (y_k)_{k \in \mathbb{Z}}$, one has $y = \alpha(x)$ if and only if for every $k \in \mathbb{Z}$, one has (see Figure 8)

$$x_k = f(y_{k-r} \cdots y_{k-1} y_k y_{k+1} \cdots y_{k+r}).$$
For \( s, t \in L_r(Y) \) and \( u \in L(Y) \) we denote by \( f_{s,t}(v) \) the word of \( L(X) \) obtained by applying \( f \) to \( v \) with context \( s, t \) (see Figure 9). Formally, if \( svt = b_{1-r} \cdots b_{n+r} \), we have \( f_{s,t}(v) = a_1 \cdots a_n \) where \( a_i = f(b_{i-r} \cdots b_i \cdots b_{i+r}) \). Set \( u = f_{s,t}(v) \). Note again that we have in this case \( v = \alpha(u) \).

Let \( n \) be the integer given by Proposition 5.1 for \( k = r + 1 \). We claim that every graph \( \mathcal{E}_1(w) \) for \( |w| \geq n + 2r \) is a tree. Let indeed \( s, t \in L_r(Y) \) and \( v \in L_{≥n}(Y) \) be such that \( w = svt \). Let \( u = f_{s,t}(v) \).

Let \( E'_k(u) = \{(p, q) \in G_k(u) \times D_k(u) \mid \alpha(puq) \in BwB\} \)

and let \( G'_k(u) \) (resp. \( D'_k(u) \)) be the set of \( p \in G_k(u) \) (resp. \( q \in D_k(u) \)) which are connected to \( D_k(u) \) (resp. \( G_k(u) \)) by an edge in \( E'_k(u) \). Let \( \mathcal{E}'_k(u) \) be the subgraph of \( \mathcal{E}_k(u) \) obtained by restriction to the set of vertices which is the disjoint union of \( G'_k(u) \) and \( D'_k(u) \).

Claim 1. The graph \( \mathcal{E}'_k(u) \) is a simple tree. Indeed, by Proposition 5.1, the graph \( \mathcal{E}_k(u) \) is a simple tree. We may assume that \( u \) is bispecial (otherwise, the property is obviously true). Let \( (p, q) \) be an edge of \( \mathcal{E}'_k(u) \). Then \( (p, q) \) is an edge of \( \mathcal{E}_k(u) \) and since the latter is a simple tree either \( p \) is the unique vertex in \( G_k(u) \) such that \( pu \) is right-special or \( q \) is the unique vertex in \( D_k(u) \) such that \( uq \) is left-special (both cases can occur simultaneously). Assume the first case, the other being proved in a symmetric way. If \( (p', q') \) is another edge of \( \mathcal{E}'(u) \), then \( (p, q') \) is an edge of \( \mathcal{E}_k(u) \). Since \( \alpha(p) \in Bs \) and \( \alpha(q) \in tB \), we have actually \( (p, q') \in \mathcal{E}'_k(u) \). Thus \( \mathcal{E}'_k(u) \) contains the two vertices of \( \mathcal{E}_k(u) \) connected to at most one other vertex and this implies that \( \mathcal{E}'_k(u) \) is a simple tree.

For \( p \in G'_k(u) \), let \( \lambda(p) \) be the first letter of \( \alpha(p) \) and for \( q \in D'_k(u) \), let \( \rho(q) \) be the last letter of \( \alpha(q) \).

Claim 2. The graph \( \mathcal{E}_1(w) \) is the image by the maps \( \lambda, \rho \) of the graph \( \mathcal{E}'_k(u) \). Indeed, one has \( (a, b) \in E_1(w) \) if and only if there exist \( (p, q) \in \mathcal{E}'_k(u) \) such that \( \lambda(p) = a \) and \( \rho(q) = b \).

It is easy to verify that the image of a simple tree by a graph morphism is a again a simple tree. Thus \( \mathcal{E}_1(w) \) is a simple tree, which concludes the proof.
We are now ready to prove the theorem.

**Proof of Theorem 6.1** Every conjugacy is a composition of a higher block code and an alphabetic morphism. Thus Theorem 6.1 is a direct consequence of Lemmas 6.3 and 6.6. 

**Example 6.7** We have seen in Example 3.6 that the image of the Tribonacci subshift by the morphism $\alpha : a \mapsto a, b \mapsto a, c \mapsto c$ is eventually dendric. This is actually a consequence of Theorem 6.1 since $\alpha$ is a conjugacy, as we have seen in Example 3.6. The images of a Sturmian subshift by a non trivial alphabetic morphism have been investigated in [12].

7 A property of eventually dendric shifts

A subshift $X$ is **irreducible** if for any $u, v \in L(X)$ there is a word $w$ such that $uwv \in L(X)$ (equivalently $L(X)$ is called recurrent). A minimal subshift is irreducible but the converse is false.

We want to prove the following curious property of eventually dendric subshifts.

**Theorem 7.1** An eventually dendric subshift is minimal if and only if it is irreducible.

For $w \in L(X)$, set $\pi(w) = g(w) - 1$ and for a set $W \subset L(X)$, set $\pi(W) = \sum_{w \in W} \pi(w)$. By Proposition 3.1 the map $\pi$ satisfies for every neutral word $w$ the additivity rule satisfied by a right probability. Thus the following statement expresses a property which reduces to a familiar property of prefix codes with respect to a right probability.

**Proposition 7.2** Let $X$ be an eventually dendric subshift with threshold $m$. Then for any prefix code $U \subset L(X)$ we have

$$\pi(U) \leq \pi(L_{\leq m}(X)).\quad (7.1)$$

**Proof.** Let $U \subset L(X)$ be a prefix code. We first assume that $U$ is finite.

For any proper prefix $w$ of $U$ which is neutral, we have by Proposition 3.1

$$\sum_{a \in D_1(w)} \pi(wa) = \pi(w).\quad (7.2)$$

If for some $u \in U$ we have $u = wa$ with $a \in A$ and $w$ neutral, then we have by Equation (7.2), $\pi(U) = \pi(U')$ where $U'$ is the prefix code $U' = (U \setminus wA) \cup w$. Thus, applying iteratively this argument, we obtain $\pi(U) = \pi(V)$ where $V$ is a prefix code such that for any $v \in V$, one has $v = wa$ with $a \in A$ and $w$ not neutral. Since every word of length $m$ is neutral, we have $V \subset L_{\leq m}(X)$, and thus Equation (7.1) follows.
Finally, if $U$ is infinite, let $U_n$ be the set of words in $U$ of length at most $n$. Then $U_n$ is finite and thus $\pi(U_n) \leq \pi(L_{\leq m}(X))$. Since $\pi(U)$ is the limit of the $\pi(U_n)$, the result follows.

Let $X$ be a subshift. The set of complete return words to a word $w \in L(X)$ is the set $\mathcal{CR}(w)$ of words having exactly two factors equal to $w$, one as a proper prefix and the other one as a proper suffix. It is clear that $X$ is minimal if and only if it is irreducible and if for every word $w$ the set of complete return words to $w$ is finite.

**Proof of Theorem 7.1.** Assume that $X$ is an irreducible subshift is eventually dendric with threshold $m$. Consider a word $w \in L(X)$ and let $S$ be the set of proper suffixes of $\mathcal{CR}(w)$. For $s \in S$, denote $\alpha(s) = \text{Card}\{a \in A \mid as \in S \cup \mathcal{CR}(w)\} - 1$. Then $\mathcal{CR}(w)$ is finite if and only if $S$ is finite. Moreover in this case, since $\mathcal{CR}(w)$ is a suffix code, we have by a well known property of trees

$$\text{Card}(\mathcal{CR}(w)) = \alpha(S) + 1.$$  \hspace{1cm} \text{(7.3)}

Let $Y$ be the set of words in $S$ which are not proper suffixes of $w$. We claim that $Y$ is a prefix code. Indeed, if $u, uv \in Y$, then $w$ is a proper suffix of $u$ and thus is an internal factor of $uv$, a contradiction unless $v = \varepsilon$. Thus $Y$ is prefix.

We have $\alpha(s) = 0$ for any proper suffix $s$ of $w$ since any word in $\mathcal{CR}(w)$ has $w$ as a proper suffix. Next we have $\alpha(s) = \pi(s)$ for any $s \in Y$. Indeed, if $au \in L(X)$ for $u \in S$ and $a \in A$, then $au \in \mathcal{CR}(w) \cup S$ since $L(X)$ is recurrent. Thus we have $\alpha(Y) = \alpha(S) = \pi(S)$. But, by Proposition 7.2 we have $\pi(Y) \leq \pi(L_{\leq m}(X))$. Thus we have

$$\alpha(S) \leq \pi(L_{\leq m}(X)).$$

By Equation (7.3), this implies that $\mathcal{CR}(w)$ is finite. Thus $X$ is minimal.

Note that the proof shows that Theorem 7.1 holds for the more general class of subshifts which are eventually neutral, in the sense that there is an integer $m$ such that every word of length at least $m$ is neutral. This class includes the subshifts $X$ such that $L(X)$ is neutral with characteristic $c$ introduced in [7] and for which Theorem 7.1 is proved in [7] with a similar proof.

Note also that the proof of Theorem 7.1 shows that in a minimal eventually dendric subshift the cardinality of complete return words is bounded. There exist minimal subshifts which do not have this property (see [9, Example 3.17]).

8 Generalized extension graphs

We will now see how the conditions on extension graphs can be generalized to graphs expressing the extension by words having different length.

We will need the following notions. Let $X$ be a subshift. A prefix code (resp. suffix code) $U \subset L(X)$ is $X$-maximal if it is not properly contained in any prefix code (resp. suffix code) $V \subset L(X)$.
A set $U \subset L(X)$ is said to be right $X$-complete (resp. left $X$-complete) if any long enough word of $L(X)$ has a prefix (resp. suffix) in $U$.

It is not difficult to show that a prefix code (resp. a suffix code) $U \subset L(X)$ is $X$-maximal if and only if it is right $X$-complete (resp. left $X$-complete) (see Proposition 3.3.1).

For $U, V$ and $w \in L(X)$, let $G_U(w) = \{ u \in U \mid uw \in L(X) \}$ and $D_V(w) = \{ v \in V \mid vw \in L(X) \}$.

Let $U \subset L(X)$ (resp. $V \subset L(X)$) be an $X$-maximal suffix code (resp. prefix code). The generalized extension graph of $w$ relative to $U, V$ is the following undirected graph $E_{U,V}(w)$. The set of vertices is made of two disjoint copies of $G_U(w)$ and $D_V(w)$. The edges are the pairs $\langle u, v \rangle \in G_U(w) \times D_V(w)$ such that $uwv \in L(X)$. In particular $E_n(w) = E_{L_n(X), L_n(X)}(w)$.

**Proposition 8.1** For every $n \geq 1$ and $m \geq 0$, the graph $E_n(w)$ is a tree for all $w \in L_{\geq m}(X)$ if and only if $E_{n+1}(w)$ is a tree for all words $w \in L_{\geq m}(X)$.

The proof uses the following statement. The only if part is Lemmas 3.8 and 3.10).

**Lemma 8.2** Let $X$ be a subshift and let $w \in L(X)$. Let $U \subset L(X)$ be a finite $X$-maximal suffix code and let $V \subset L(X)$ be finite $X$-maximal prefix code. Let $\ell \in L(X)$ be such that $A\ell \cap L(X) \subset U$ and such that $E_{A\ell}(\ell w)$ is a tree. Set $U' = (U \setminus A\ell) \cup \ell$. The graph $E_{U', V}(w)$ is a tree if and only if the graph $E_{U, V}(w)$ is a tree.

**Proof.** We need only to prove the if part.

First, note that the hypothesis that $E_{A\ell}(\ell w)$ is a tree guarantees that the left vertices $A\ell$ in $E_{U', V}(w)$ are clustered: for any pair of vertices $a\ell, b\ell$ there exists a unique reduced path from $a\ell$ to $b\ell$ in $E_{U', V}(w)$ using as left vertices only elements of $A\ell$. Indeed, such a path exists since the subgraph $E_{A\ell}(\ell w)$ of $E_{U', V}(w)$ is isomorphic to $E_{A\ell}(\ell w)$ that is connected. Since $E_{U', V}(w)$ is a tree, this path is unique.

Let $v, v' \in D_V(w)$ be two distinct vertices and let $\pi$ be the unique reduced path from $v$ to $v'$ in $E_{U', V}(w)$. We show that we can find a unique reduced path $\pi'$ from $v$ to $v'$ in $E_{U', V}(w)$.

If $\pi$ does not pass by $A\ell$, we can simply define $\pi'$ as a path passing by the same vertices than $\pi$. Otherwise, we can decompose $\pi$ in a unique way as a concatenation of a path $\pi_1$ from $v$ to a vertex in $A\ell$ not passing by $A\ell$ before, followed by a path from $A\ell$ to $A\ell$ (using on the left only vertices from $A\ell$) and a path $\pi_2$ from $A\ell$ to $v'$ without passing in $A\ell$ again. We consider in $E_{U', V}(w)$ the unique path $\pi'_1$ from $v$ to $\ell$ obtained by replacing the last vertex of $\pi_1$ by $\ell$ and the unique reduced path $\pi'_2$ from $\ell$ to $v'$ obtained by replacing the first vertex of $\pi_2$ by $\ell$. In this case we define $\pi'$ as the concatenation of $\pi'_1$ and $\pi'_2$.

The reduced path $\pi'$ is unique. Indeed, let us suppose that we have a different path $\pi^*$ from $v$ to $v'$ in $E_{U', V}(w)$. If $\pi^*$ does not pass (on the left) by $\ell$ then we would find a path having the same vertices in $E_{U', V}(w)$ which is impossible since
the graph is acyclic. Let us suppose that both $\pi'$ and $\pi^*$ passes by $\ell$. Without loss of generality let us suppose that we have a cycle in $\mathcal{E}_{U',V}(w)$ passing by $\ell$ and $v$ (the case with $v'$ being symmetric). Let us define by $\pi_0'$ and $\pi_0^*$ the two distinct subpaths of $\pi'$ and $\pi^*$ respectively going from $v$ to $\ell$. Since $L(X)$ is biextendable, we can find $a\ell$, $b\ell \in U$, with $a,b \in A$ not necessarily distinct, and two reduced paths $\pi_1$ from $v$ to $a\ell$ and and $\pi_2$ from $v$ to $b\ell$ in $\mathcal{E}_{U',V}(w)$ obtained from $\pi_0'$ and $\pi_0^*$ by replacing the vertex $\ell$ by $a\ell$ and $b\ell$ respectively. From the remark at the beginning of the proof we know that we can find a reduced path in $\mathcal{E}_{U,V}(w)$ from $a\ell$ to $b\ell$. Thus we can find a nontrivial cycle in $\mathcal{E}_{U,V}(w)$, which contradicts the acyclicity of the graph.

A symmetric statement holds for $r \in L(X)$ such that $rA \cap L(X) \subset V$ and $\mathcal{E}_{U,A}(wr)$ is a tree, with $V' = (V \setminus rA) \cup r$: the graph $\mathcal{E}_{U,V}(w)$ is a tree if and only if $\mathcal{E}_{U',V}(w)$ is a tree.

Lemma 8.3 Let $n \geq 1$, let $m \geq 0$ and let $V$ be a finite X-maximal prefix code. If $\mathcal{E}_{L_n(X),V}(w)$ is a tree for every $w \in L_{\geq m}(X)$ then for each word $\ell \in L_{\geq n-1}(X)$, the graph $\mathcal{E}_{A,V}(\ell w)$ is a tree.

Proof. The graph $\mathcal{E}_{A,V}(\ell w)$ is obtained from $\mathcal{E}_{L_n(X),V}(\ell w)$ by identifying the vertices of $G_n(\ell w)$ ending with the same letter. Since $\mathcal{E}_{L_n(X),V}(\ell w)$ is connected, $\mathcal{E}_{A,V}(\ell w)$ is also connected.

Set $\ell = \ell'\ell''$ with $|\ell'| = n-1$. The graph $\mathcal{E}_{A,V}(\ell w)$ is isomorphic to $\mathcal{E}_{A',V}(\ell''w)$ which is a subgraph of $\mathcal{E}_{A}(\ell''w)$ and thus it is acyclic.

Thus $\mathcal{E}_{A,V}(\ell w)$ is a tree.

A symmetric statement holds for $n \geq 1$ and $U$ a finite X-maximal suffix code: If $\mathcal{E}_{U,L_n(X)}(w)$ is a tree for every $w \in L_{\geq m}(X)$ if and only if $\mathcal{E}_{U,A}(wr)$ is a tree for every $r \in L_{\geq n-1}(X)$ and $w \in L_{\geq m}(X)$.

Proof of Proposition [X.7] We proceed in several steps.

Step 1. Assume first that $\mathcal{E}_{n}(w)$ is tree for every word $w \in L_{\geq m}(X)$. We fix some $w \in L_{\geq m}(X)$.

Step 1.1 We claim that for any finite X-maximal suffix code $U$ formed of words of length $n$ or $n+1$, the graph $\mathcal{E}_{U,L_n(X)}(w)$ is a tree by induction on $\gamma_{n+1}(U) = \text{Card}(G_U(w) \cap A^{n+1})$.

The property is true for $\gamma_{n+1}(U) = 0$, since then $\mathcal{E}_{U,L_n(X)}(w) = \mathcal{E}_{n}(w)$. Assume now that $\gamma_{n+1}(U) > 0$. Let $a\ell$ with $a \in A$ be a word of length $n+1$ in $G_U(w)$. Since $U$ is an X-maximal suffix code with words of length $n$ or $n+1$, we have $A\ell \cap L(X) \subset U$. Let us consider $U' = (U \setminus A\ell) \cup \ell$. Since $\gamma_{n+1}(U') < \gamma_{n+1}(U)$, by induction hypothesis the graph $\mathcal{E}_{U',L_n(X)}(w)$ is a tree. Moreover, by Lemma [X.8] the graph $\mathcal{E}_{U,L_n(X)}(\ell w)$ is a tree.

Thus, by assertion 1 of Lemma [X.9] the graph $\mathcal{E}_{U,L_n(X)}(w)$ is a tree. This proves the claim.

Step 1.2 We now claim that for any finite X-maximal prefix code $V$ formed of words of length $n$ or $n+1$, the graph $\mathcal{E}_{L_{n+1}(X),V}(w)$ is a tree by induction on $\delta_{n+1}(V) = \text{Card}(D_V(w) \cap A^{n+1})$.
The property is true for $\delta_n + 1 (V) = 0$, since the graph $E_{n+1} (V) = E_{n+1} (X, L_n (X), V (w))$, is a tree by Step 1.1. Assume now that $\delta_n + 1 (V) > 0$. Let $r a$ with $a \in A$ be a word of length $n + 1$ in $D_V (w)$. Since $V$ is an $X$-maximal prefix code with words of length $n$ or $n + 1$, we have $r A \cap L(X) \subset U$. Let us consider $V' = (V \setminus r A) \cup r$. Since $\delta_n + 1 (V') < \delta_n + 1 (V)$, by induction hypothesis the graph $E_{n+1} (X, V') (w)$ is a tree. Moreover, by the symmetric version of Lemma 8.3, the graph $E_{n+1} (X, A) (w)$ is a tree. This proves the claim.

Since $E_n + 1 (w) = E_{n+1} (X, L_n (X), V (w))$, we conclude that $E_n + 1 (w)$ is a tree. Step 2 Assume now that $E_n + 1 (w)$ is a tree for every $w \in L_{\geq m} (X)$. Fix some $w \in L_{\geq m} (X)$.

Step 2.1 We first claim that $E_{U, n+1} (X) (w)$ is a tree for every $X$-maximal suffix code $U$ formed of words of length $n$ or $n + 1$ by induction on $\gamma (U) = Card(G_U (w) \cap A^n)$.

The property is true if $\gamma_n (U) = 0$, since then $E_{U, n+1} (X) (w) = E_{n+1} (w)$.

Assume next that $\gamma_n (U) > 0$. Let $\ell \in G_U (w) \cap A^n$. Set $W = (U \setminus \ell) \cup A \ell$ or equivalently $U = (W \setminus A \ell) \cup \ell$. Then $\delta_n (W) < \delta_n (U)$ and consequently $E_{W, n+1} (X) (w)$ is a tree by induction hypothesis. On the other hand, by Lemma 8.3, the graph $E_{n+1} (X, A) (\ell w)$ is also a tree. By Assertion 2 of Lemma 8.2 the graph $E_{U, n+1} (X, A) (w)$ is a tree and thus the claim is proved.

Step 2.2 We now claim that $E_{n} (w) = E_{U, V} (w)$ for $U = V = L_n (X)$, it follows from the claim that $E_n (w)$ is a tree.

The following result shows that in the definition of eventually dendric subshifts, one can replace the graphs $E_1 (w)$ by $E_n (w)$ with the same threshold.

**Theorem 8.4** Let $X$ be a subshift. For every $m \geq 1$, the following conditions are equivalent.

(i) $X$ is eventually dendric with threshold $m$,

(ii) the graph $E_n (w)$ is a tree for every $n \geq 1$ and every word $w \in L_{\geq m}$,

(iii) there is an integer $n \geq 1$ such that $E_n (w)$ is a tree for every word $w \in L_{\geq m} (X)$.

**Proof.** (i) $\Rightarrow$ (ii). It is proved by ascending induction on $n$ using iteratively Proposition 8.1.

(ii) $\Rightarrow$ (iii). It is obvious.

20
(iii) ⇒ (i). It is proved by descending induction on \( n \) using Proposition 8.1.

9 Complete bifix decoding

Let \( X \) be a subshift on the alphabet \( A \). A subset of \( L(X) \) is two-sided \( X \)-complete if it is both left and right \( X \)-complete.

A bifix code is both a prefix code and a suffix code. A bifix code \( U \subset L(X) \) is \( X \)-maximal if it is not properly contained in a bifix code \( V \subset L(X) \). If a bifix code \( U \subset X \) is right \( X \)-complete (resp. left \( X \)-complete), it is an \( X \)-maximal bifix code since it is already an \( X \)-maximal prefix code (resp. suffix code). It can be proved conversely that if \( X \) is irreducible, a finite bifix code is \( X \)-maximal if and only if it is two-sided \( X \)-complete (see [1, Theorem 4.2.2]). This is not true in general, as shown by the following example.

**Example 9.1** Let \( X \) be the subshift such that \( L(X) = a^*b^* \). The set \( U = \{aa, b\} \) is an \( X \)-maximal bifix code. Indeed, it is a bifix code and it is left \( X \)-complete as one may verify. However it is not right \( X \)-complete since no word in \( ab^* \) has a prefix in \( U \).

Let \( X \) be a subshift and let \( U \) be a two-sided \( X \)-complete finite bifix code. Let \( \varphi : B \to U \) be a coding morphism for \( U \), that is, a bijection from an alphabet \( B \) onto \( U \) extended to a morphism from \( B^* \) into \( A^* \). Then \( \varphi^{-1}(L(X)) \) is factorial and, since \( U \) is two-sided complete, it is extendable. Thus it is the language of a subshift called the complete bifix decoding of \( X \) with respect to \( U \).

For example, for any \( n \geq 1 \), the set \( L_n(X) \) is a two-sided complete bifix code and the corresponding complete bifix decoding is the decoding of \( X \) by non-overlapping \( n \)-blocks. It can be identified with the dynamical system \((X, \sigma^n)\).

In [3, Theorem 3.13] it is proved that the maximal bifix decoding of an irreducible dendric subshift is a dendric subshift. Actually, the hypothesis that \( X \) is irreducible is only used to guarantee that the \( X \)-maximal bifix code used for the decoding is also an \( X \)-maximal prefix code and an \( X \)-maximal suffix code. In the definitions used here of a maximal bifix decoding, we do not need this hypothesis.

**Theorem 9.2** Any complete bifix decoding of an eventually dendric subshift is an eventually dendric subshift having the same threshold.

Note that any \( X \)-maximal suffix code \( U \) one has \( \text{Card}(U) \geq \text{Card}(X \cap A) \). Indeed, every \( a \in A \) appears as a suffix of (at least) an element of \( X \).

**Lemma 9.3** Let \( X \) be an eventually dendric subshift with threshold \( n \). For any \( w \in L_{\geq n}(X) \), any \( X \)-maximal suffix code \( U \) and any \( X \)-maximal prefix code \( V \), the graph \( E_{U,V}(w) \) is a tree.
Proof. We use an induction on the sum of the lengths of the words in $U, V$. The property is true if the sum is equal to $2\text{Card}(X \cap A)$. Indeed, for every $w \in L_{\geq n}(X)$ one has $U = G(w)$ and $V = D(w)$ and thus $\mathcal{E}_{U,V}(w) = \mathcal{E}_1(w)$ is a tree. Otherwise, we may assume that $U$ contains words of length at least 2 (the case with $V$ being symmetrical). Let $u \in U$ be of maximal length. Set $u = a\ell$ with $a \in A$. Since $U$ is an $X$-maximal suffix code, we have $A\ell \cap L(X) \subset U$. Set $U' = (U \setminus A\ell) \cup \ell$. By induction hypothesis, the graphs $\mathcal{E}_{U',V}(w)$ and $\mathcal{E}_{A,V}(\ell w)$ are trees. Thus, by Lemma 8.2, $\mathcal{E}_{U,V}(w)$ is also a tree. 

Proof of Theorem 9.2. Assume that $X$ is eventually dendric with threshold $n$. Let $\varphi : B \to U$ be a coding morphism for $U$ and let $Y$ be the decoding of $X$ corresponding to $U$. Consider a word $w$ of $L(Y)$ of length at least $n$. By Lemma 9.3 and since $|\varphi(w)| \geq n$, the graph $\mathcal{E}_{U,U}(\varphi(w))$ is a tree. But for $b, c \in B$, one has $bwc \in L(Y)$ if and only if $\varphi(bwc) \in L(X)$, that is, if and only if $(\varphi(b), \varphi(c)) \in \mathcal{E}_1(\varphi(w))$. Thus $\mathcal{E}_1(w)$ is isomorphic to $\mathcal{E}_{U,U}(\varphi(w))$ and thus $\mathcal{E}_1(w)$ is a tree. This shows that $Y$ is eventually dendric with threshold $n$. 

Example 9.4 Let $X$ be the Fibonacci subshift. Then $U = \{aa, aba, b\}$ a is an $X$-maximal bifix code. Let $\varphi : \{u, v, w\} \to U$ be the coding morphism for $U$ defined by $\varphi : u \mapsto aa, v \mapsto aba, w \mapsto b$. The complete bifix decoding of $X$ with respect to $U$ is an eventually dendric subshift with threshold 0. It is actually the natural coding of an interval exchange transformation on three intervals (see [1]). The extension graphs $\mathcal{E}_1(\varepsilon, Y)$ and $\mathcal{E}_1(v, Y)$ are shown in Figure 1.

![Figure 10: The graphs $\mathcal{E}_1(\varepsilon, Y)$ and $\mathcal{E}_1(v, Y)$.](image)

A particular case of complete bifix decoding is related to a notion which is well-known in topological dynamics, namely the skew product of two dynamical systems (see [1]). Indeed, assume that when we start with a permutation group $G$ on a set $Q$ and a morphism $f : A^* \to G$. We denote $q \mapsto q \cdot w$ the result of the action of the permutation $f(w)$ on the point $q \in Q$. Fix a point $i \in Q$. The set of words $w$ such that $i \cdot w = i$ is a submonoid generated by a bifix code $U$ which is two-sided complete (this follows from [2] Theorem 4.2.11). The corresponding decoding is a shift space which is related to the skew product of $(X, \sigma)$ and $(G, Q)$. It is the shift space $Y$ on the alphabet $A \times Q$ formed by the labels of the two-sided infinite paths on the graph with vertices $Q$ and edges $(p, q)$ labeled $(a, p)$ for $a \in A$ such that $p \cdot f(a) = q$. The decoding of $X$ corresponding to $U$ is the dynamical system induced by $Y$ on the set of $y \in Y$ such that $y_0 = (a, i)$ for some $a \in A$. 

22
Example 9.5 Let $X$ be the Fibonacci shift, let $Q = \{1, 2\}$ and $G = \mathbb{Z}/2\mathbb{Z}$. Let $f : A^* \to G$ be the morphism $a \mapsto (12), b \mapsto (1)$. Choosing $i = 1$, the bifix code $U$ build as above is $U = \{aa, aba, b\}$ as in Example 9.4.

References

[1] Jean Berstel, Clelia De Felice, Dominique Perrin, Christophe Reutenauer, and Giuseppina Rindone. Bifix codes and Sturmian words. *J. Algebra*, 369:146–202, 2012.

[2] Valérie Berthé, Clelia De Felice, Vincent Delecroix, Francesco Dolce, Julien Leroy, Dominique Perrin, Christophe Reutenauer, and Giuseppina Rindone. Specular sets. *Theoret. Comput. Sci.*, 684:3–28, 2017.

[3] Valérie Berthé, Clelia De Felice, Francesco Dolce, Julien Leroy, Dominique Perrin, Christophe Reutenauer, and Giuseppina Rindone. Acyclic, connected and tree sets. *Monatsh. Math.*, 176(4):521–550, 2015.

[4] Valérie Berthé, Clelia De Felice, Francesco Dolce, Julien Leroy, Dominique Perrin, Christophe Reutenauer, and Giuseppina Rindone. Bifix codes and interval exchanges. *J. Pure Appl. Algebra*, 219(7):2781–2798, 2015. (http://dx.doi.org/10.1016/j.jpaa.2014.09.028).

[5] Julien Cassaigne. Complexité et facteurs spéciaux. *Bull. Belg. Math. Soc. Simon Stevin*, 4(1):67–88, 1997. Journées Montoises (Mons, 1994).

[6] Isaac P. Cornfeld, Sergei V. Fomin, and Yakov G. Sinai. *Ergodic theory*, volume 245 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1982. Translated from the Russian by A. B. Sosinskii.

[7] Francesco Dolce and Dominique Perrin. Neutral and tree sets of arbitrary characteristic. *Theoret. Comput. Sci.*, 658(part A):159–174, 2017.

[8] Sebastian Donoso, Fabien Durand, Alejandro Maass, and Samuel Petite. On automorphism groups of low complexity subshifts. *Ergod. Th. Dynam. Sys.*, 36:64–95, 2016.

[9] Fabien Durand, Julien Leroy, and Gwenaël Richomme. Do the properties of an $S$-adic representation determine factor complexity? *J. Integer Seq.*, 16(2):Article 13.2.6, 30, 2013.

[10] N. Pytheas Fogg. *Substitutions in dynamics, arithmetics and combinatorics*, volume 1794 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2002. Edited by V. Berthé, S. Ferenczi, C. Mauduit and A. Siegel.

[11] Martine Queffélec. *Substitution dynamical systems—spectral analysis*, volume 1294 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, second edition, 2010.
[12] Vojtech Vesely. Properties of morphic images of $S$-adic sequences. Master’s thesis, Czech Technical University, 2018.