Cosmological constant, matter, cosmic inflation and coincidence

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We present a possible understanding to the issues of cosmological constant, inflation, matter and coincidence problems based only on the Einstein equation and Hawking particle production. The inflation appears and results agree to observations. The CMB large-scale anomaly can be explained and the dark-matter acoustic wave is speculated. The entropy and reheating are discussed. The cosmological term $\Omega_\Lambda$ tracks down the matter $\Omega_M$ until the radiation-matter equilibrium, then slowly varies, thus the cosmic coincidence problem can be avoided. The relation between $\Omega_\Lambda$ and $\Omega_M$ is shown and can be examined at large redshifts.

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Introduction. In the standard model of modern cosmology ($\Lambda$CDM), the cosmological constant, inflation, dark matter and coincidence problem have been long standing issues since decades, though many models and efforts have been made to approach these issues, and readers are referred to review articles and professional books, for example, see Refs. [1–11]. We present here the possible scenario based only on the Einstein equation, in which the cosmological term generates (couples to) the matter via the Hawking pair production of particles and antiparticles. As an effective field theory of the Einstein gravity, two physically relevant area operators of the Ricci scalar $R$ and cosmological term $\Lambda$ can be possibly realized in the scaling domain [12],

$$\mathcal{A}_{EC}^\text{eff} = \int \frac{d^4x}{16\pi G} \det(-g)(R - 2\Lambda),$$

and high-dimensional operators are suppressed. The gravitation constant $G \sim \ell_{\text{pl}}^2 = M_{\text{pl}}^{-2}$ is the smallest area at the Planck cutoff. Whereas the cosmological constant represents the intrinsic scale $\Lambda \propto \xi^{-2}$, the scaling invariant correlation length square $\xi^2$ is the largest area at the Universe horizon [13]. We will further show such a dynamics of cosmological constant $\Lambda$ and its function in Universe evolution.

The effective action [1] yields the Einstein equation for the spacetime of Einstein tensor $G^{ab}$
coupling to the matter of energy-momentum tensor $T_{\mu\nu}^M$,

$$\mathcal{G}^{ab} = -8\pi G T_{\mu\nu}^M; \quad \mathcal{G}^{ab} = R^{ab} - (1/2)g^{ab}R - \Lambda g^{ab}. \quad (2)$$

Its covariant differentiation and the Bianchi identity are

$$\mathcal{G}^{ab}_{;\ b} = -8\pi [G T_{\mu\nu}^M]_{;\ b}, \quad [R^{ab} - (1/2)\delta^{ab}R]_{;\ b} \equiv 0, \quad (3)$$

which lead to the conservation law,

$$(\Lambda)_{;\ b} g^{ab} = 8\pi (G)_{;\ b} T_{\mu\nu}^M + 8\pi G (T_{\mu\nu}^M)_{;\ b}, \quad (4)$$

with varying cosmological term $\Lambda_{;\ b} = (\Lambda)_{,\ b}$ and coupling $(G)_{;\ b} = (G)_{,\ b}$. Despite its essence of spacetime origin, the cosmological $\Lambda$-term in $\mathcal{G}_{ab}$ can be moved to the RHS of Eq. (2) and formally expressed as $T_{\mu\nu}^\Lambda$, analogously to the $T_{\mu\nu}^M$ of a perfect fluid,

$$T_{\mu\nu}^\Lambda = p_{\mu\nu}^\Lambda + (\rho_{\mu\nu}^\Lambda + \rho_{\mu\nu})U_a U^a, \quad (5)$$

by implementing a negative mass density $\rho_{\Lambda} = \Lambda/(8\pi G) \equiv -p_{\Lambda}$. Equation (4) is equivalent to the conservation law $T_{\mu\nu}^{ab}_{;\ b} \equiv (T_{\mu\nu}^\Lambda + T_{\mu\nu}^M)_{;\ b} = 0$. The energy density $\rho_{\mu\nu}^\Lambda$ and pressure $p_{\mu\nu}^\Lambda$ are in the comoving frame of four velocity $U^a = (1, 0, 0, 0)$.

In the Robertson-Walker spacetime $ds^2 = a^2(t)dx^2$ of zero curvature, Eqs. (2) and (4) become,

$$h^2 = g(\Omega_M + \Omega_{\Lambda}), \quad h = H/H_0, \quad \Omega_{M,\Lambda} = \rho_{M,\Lambda}/\rho_c^0, \quad (6)$$

$$\frac{dh^2}{dx} + 2h^2 = g \left(2\Omega_{\Lambda} - (1 + 3\omega_M)\Omega_M \right), \quad (7)$$

$$\frac{d}{dx} \left[g(\Omega_{\Lambda} + \Omega_M) \right] = -3g(1 + \omega_M)\Omega_M, \quad (8)$$

where $g = G/M_{\text{pl}}^{-2}$, $H = \dot{a}/a$, $\omega_M = p_M/\rho_M$, $x = \ln(a/a_0)$, and $d(\cdots)/dt = Hd(\cdots)/dx$ [14]. The characteristic scales $H_0, a_0$, and $\rho_c^0 = 3H_0^2/(8\pi M_{\text{pl}}^{-2})$ depend on the Universe evolution epoch: inflation, reheating, radiation and matter dominated epochs.

Here we consider only the constant $G = M_{\text{pl}}^{-2}$ [15] and set the reduced Planck scale $8\pi G = m_{\text{pl}}^{-2} = 1$, unless otherwise stated. Equations (6-7) and (8) are recasted into two independent equations,

$$h^2 = (\Omega_M + \Omega_{\Lambda}), \quad (9)$$

$$\frac{d}{dx} (\Omega_{\Lambda} + \Omega_M) = -3(1 + \omega_M)\Omega_M, \quad (10)$$

which reduces to the Friedmann equations for the constant cosmological term $\Omega_{\Lambda}$. Equations (8) and (10) show the interaction between the cosmological term and matter. Moreover, the matter
term $\Omega_m(h)$ is generated by the spontaneous pair production of particles and antiparticles from the spacetime horizon $h$, as will be shown in next section. In turn, $\Omega_m(h) \neq 0$ dynamically leads to $h^2$ and $\Omega_\Lambda$ decrease via Eq. (10), it changes via Eq. (9). This completely determines the variations of $h^2(x)$ and $\Omega_\Lambda(h)$ and $\Omega_m(h)$ scaling in the Universe evolution.

**Pair production.** To calculate $\Omega_m(h)$, we consider the pair production of spin-1/2 particles and antiparticles in the exact De Sitter spacetime of the constant $H$ and scaling factor $a(t) = e^{Ht}$. The averaged number density of pairs produced from $t_o = 0$ to $t \approx 2\pi H^{-1}$ is [16] [17]

$$n_M = \frac{H^3}{2\pi^2} \int_0^\infty dz z^2 |\beta_k^{(n)}(t)|^2$$

$$= \frac{H^3 e^{\pi \mu}}{16\pi} \int_0^\infty dz \frac{z^3}{\sqrt{z^2 + \mu^2}} F_{\nu}^{(n)}(z, \mu),$$

(11)

where $z \equiv kH^{-1}e^{-Ht}$, the particle mass $\mu = m/H$ and momentum $k$, the Bogolubov coefficient up to the $n$-th adiabatic order $|\beta_k^{(n)}(t)|_{k \to \infty} \sim \mathcal{O}(1/k^{n+2})$ in the ultraviolet (UV) limit. Due to the exact De Sitter symmetry ($H = \text{const}$), the energy-momentum tensor of produced pairs $T_{\mu \nu}^M \propto g^{\mu \nu}$ [16] [18]. Since the back reaction of pair production leads to a slowly decreasing $H$ and breaks the exact symmetry, we assume $T_{\mu \nu}^M$ to be spatially homogenous and in the form [5]

$$\rho_M = 2 \frac{H^3}{2\pi^2} \int_0^\infty dz z^2 \epsilon_k |\beta_k^{(n)}(t)|^2$$

$$= 2 \frac{H^4 e^{\pi \mu}}{16\pi} \int_0^\infty dz z^3 F_{\nu}^{(n)}(z, \mu),$$

(12)

$$p_M = 2 \frac{H^3}{2\pi^2} \int_0^\infty dz z^2 \frac{(k/a)^2}{3\epsilon_k} |\beta_k^{(n)}(t)|^2$$

$$= \frac{\rho_M}{3} - 2 \frac{H^4 e^{\pi \mu}}{3 \times 16\pi} \int_0^\infty dz \frac{z^3}{\sqrt{z^2 + \mu^2}} F_{\nu}^{(n)}(z, \mu),$$

(13)

where the spectrum of created particles $\epsilon_k = a^{-1}[(k/a)^2 + m^2]^{1/2}$. To ensure the UV finiteness of Eqs. (11), (12) and (13), the appropriate adiabatic order $n$ is considered,

$$F_{\nu}^{(n)}(z, \mu) = \left| f_1^{(n)}(z) H_{\nu-1}^{(1)}(z) - i f_2^{(n)}(z) H_\nu^{(1)}(z) \right|^2,$$

(14)

where $\sigma_{\pm} \equiv [(z^2 + \mu^2)^{1/2} \pm \mu]^{1/2}$, $\nu = 1/2 - i\mu$, $H_\nu^{(1)}(z)$ is the Hankel function of the first kind, and $f_{1,2}^{(n)} = 1 + \sum_{i=1}^n F_{1,2}^{(i)}$ [16].

The spacetime of the horizon $H$ produces particles and antiparticles of different masses $m \gtrsim H$ and degeneracies $g_d$. We simply introduce the unique mass scale “$m$” to effectively describe the total contribution of pairs to Eqs. (11), (12) and (13), and its value is determined by observations. These particles and antiparticles can be both dark matter and usual matter particles. It should be also noted that the pair productions of bosonic particles and antiparticles are not considered
FIG. 1: The inflation appears, as \( h \) and \( \Omega_\Lambda(h) \), \( \omega_M = p_M/\rho_M \) slowly decrease in the e-folding number. Here \( m = H_o = 1 \). Note that the superscript or subscript \( "o" \) indicates the quantities at the inflation beginning, not to be confused with \( "0" \) standing for the present time.

here, since their number density \( n_B^M \) goes to zero for \( m/H \gg 1 \) and has a spurious divergence for \( m/H \ll 1 \) [16]. Their quantitative contributions to the energy density and pressure of matter content are postponed for future studies.

Cosmic inflation. Starting from the initial conditions \( \Omega_\Lambda^o = h^2_o \) at the reduced Planck scale \( \Lambda_o = 3H^2_o \approx m^2_{\text{pl}}, \Omega_\Lambda(h) \gg \Omega_M(h), \) and \( \Omega_\Lambda(h) \) governs the varying spacetime horizon \( h \) in the inflation epoch. Here we select the initial scale \( H_o = m_{\text{pl}} < M_{\text{pl}} \) so that the details of quantum gravity and/or Planck transition could possibly be ignored and Eqs. (9) and (10) approximately hold. Numerically integrating Eqs. (9,10) and (12,13) with the initial condition \( h^2_o = 1 \) and \( h^2_o \gtrsim \Omega_\Lambda^o \gg \Omega_M^o, \) we find that the cosmic inflation of very slowly decreasing \( h^2 \) and \( \Omega_\Lambda(h) \) is indeed a solution, as illustrated in Fig. 1. The reason is that the pair production (11) is not so rapid that \( h^2 \) decreases slowly, see Eq. (10), as a function of e-folding numbers \( \ln(a/a_o) \). This in turn justifies our approximate calculations (12) and (13) by using formulas for a constancy \( H \). As a result, we obtain the solution to the cosmological “constant”, slowly varying as an “area” law \( \Lambda = 3H^2_o \Omega_\Lambda(h) \approx 3H^2 \) or \( \Omega_\Lambda(h) \approx h^2 \).

Due to the continuous pair productions, \( \Omega_M/\Omega_\Lambda \) increasing, \( H \) and \( \Omega_\Lambda \) decreasing, the inflation ends at \( a = a_e \), which can be estimated by the expansion rate \( H_e \) being smaller than the averaged pair-production rate \( \Gamma_M \approx dN/(2\pi dt) \approx (H/2\pi)dN/dx \) and the number of particles \( N = n_M H^{-3}/2 \). However, \( H_e < \Gamma_M(H_e) \) occurs in \( m \gg H \), where it is difficult to perform numerical calculations of Hankel functions [19].

To explicitly show the inflation physics in \( m \gg H \), we explore asymptotic expressions:

\[
\begin{align*}
n_M &\approx \chi_m H^2, \quad \Gamma_M \approx -(\chi m/4\pi)(H^{-1}dH/dx), \\
\rho_M &\approx 2\chi m^2 H^2(1 + s), \quad p_M \approx (s/3)\rho_M,
\end{align*}
\]
\[ \omega_M \approx s/3, \text{ where } \chi \approx 1.85 \times 10^{-3} \text{ and } s \approx 1/2(H/m)^2 \] The leading order of both \( n_M \) and \( \rho_M \) follows the area law \( \propto H^2 \), rather than the volume law \( \text{[11]} \). The physical picture is the large number (or degeneracies \( g_d \)) \( N \approx H^{-1}/m^{-1} \gg 1 \) of pairs produced mainly in the thin layer of the width \( 1/m \) on the horizon surface area \( H^{-2} \).

Consequently, Eq. \( \text{(10)} \) becomes
\[ dH^2/dx \approx -2\chi m^2H^2(1 + \omega_M)(1 + s), \] yielding \( H \approx H_s \exp -\chi m^2 x = H_s(a/a_s)^{-\chi m^2} \), slowly decreasing for \( \chi m^2 = \chi (m/m_{pl})^2 \ll 1 \). This solution shows the features of the inflation epoch. The initial scale \( H_s \) corresponds to the interested mode of the pivot scale \( k_s \) crossed the horizon \( (c_s k_s = H_s a_s) \) for CMB observations. At this pivot scale, one calculates the scalar, tensor power spectra and their ratio
\[ \Delta^2_{\chi} = \frac{1}{8\pi^2} \frac{H_s^2}{m^2_{pl}} \epsilon c_s; \Delta^2_{h} = \frac{2}{\pi^2} \frac{H_s^2}{m^2_{pl}} \; r \equiv \frac{\Delta^2_{h}}{\Delta^2_{\chi}} = 16 \epsilon c_s, \] where \( c_s < 1 \) due to the Lorentz symmetry broken by the time dependence of the background \( \text{[3]} \).

Their deviations from the scale invariance \( \Delta(R) \equiv d^n \ln \Delta_{\chi}(k)/d(\ln k)^n \approx d^n \ln \Delta_{\chi}(k_s)/dx^n \):
\[ n_s - 1 = \Delta(1) \approx -2\epsilon - \eta - \kappa, \; \alpha_s = \Delta(2) \approx n_s', \]
\[ n_t = \Delta(1) \approx -2\epsilon, \; \tilde{n}_t = \Delta(2) \approx n_t', \]
and \( \tilde{\alpha}_s = \Delta(3) \approx \alpha_s' \), and we calculate
\[ \epsilon \equiv -H'/H|_{k_s} \approx \chi m^2 (1 + s), \]
\[ \eta \equiv \epsilon'/\epsilon|_{k_s} \approx -3\chi m^2 s \approx -3 s \epsilon, \]
\( \kappa = c_s'/c_s \) and their derivatives \( \eta' = d\eta/dx \approx -3\eta^2, \eta'' \approx \eta^2 - 3\eta^3, \eta'' \approx 9\eta^4 - 6\eta^2 \epsilon^2 \).

Based on two observational values at \( k_s = 0.05 \text{ Mpc}^{-1} \) \( \text{[20]} \): (i) \( n_s \approx 0.965 \), we estimate \( m \lesssim 3.08 m_{pl} \) by \( 2\epsilon \approx 2\chi m^2 \lesssim 1 - n_s \approx 0.035 \) for \( \epsilon > \eta \) and assuming \( 2\epsilon < \kappa \); (ii) \( \Delta^2_R = A_s \approx 2.1 \times 10^{-9} \), Eq. \( \text{(18)} \) gives the inflation scale \( H_s = 3.15 \times 10^{-5} (r/0.1)^{1/2} m_{pl} \), and Eq. \( \text{(15)} \) gives the pair-production rate \( \Gamma_M = \chi m/4\pi \epsilon = 7.9 \times 10^{-6} m_{pl} \). The inflation ends at \( \Gamma_M > H_e \), i.e., \( (\chi m/4\pi) \epsilon > H_s \exp - (\epsilon N_e) \),
\[ N_e = \ln \left( \frac{a_e}{a_s} \right) > \frac{2}{1 - n_s} \ln \left[ \frac{7.91 \times 10^{-4} (r/0.1)^{1/2}}{(1 - n_s) \chi (m/m_{pl})} \right], \]
yielding the results \( r < 0.037, 0.052 \) for \( N_e = 50, 60 \), in agreement with observations \( \text{[20]} \). Replacing the unique mass parameter \( m \) by the observed quantity of spectral index \( n_s \); \( 2\chi (m/m_{pl})^2 \approx (1 - n_s) \).
FIG. 2: On the Figure 28 of the Planck 2018 results [20] for constraints on the tensor-to-scalar ratio $r$, we plot the parameter-free $(n_s - r)$ relation [23] that shows in the observed $n_s$-range, two QFC curves respectively representing $N_{end} = 60$ and $N_{end} = 50$ are consistently inside the blue zone constrained by several observational data sets. The real values of $r$ ratio should be below the curves due to the nature of inequality (23). As a short notation, the abbreviation QFC stands for the name “quantum field cosmology” [14, 25].

As a result, being independent of any free parameter, Equation (22) yields a definite $(n_s - r)$-relationship between the spectral index $n_s$ and the scalar-tensor-ratio $r$,

$$(r/0.1) < 7.97 \times 10^5 \chi (1 - n_s)^3 e^{(1-n_s)N_{end}},$$

for a given $N_{end}$ value of inflation $\epsilon$-folding number, see Fig. 2. Moreover, $n_s' < e^2 \approx (1 - n_s)^2/4$, $n_s'' < e^3 \approx (1 - n_s)^3/8$, and we need to know $\kappa$ for further parameter constrains. In this inflation epoch $H_s > H > H_c$, $\Omega_\Lambda = (H/H_s)^2 - \Omega_M$ dominates over $\Omega_M \approx (\chi m^2/3)(H/H_s)^2$, and the cosmological “constant” $\Lambda = 3H_s^2\Omega_\Lambda \propto H^2$.

Using Eqs. (9) and (10), we recast Eqs. (18) and (21) as

$$\Delta^2(k) = \frac{1}{12\pi^2} \frac{H^2 R_M^{-1}}{m_{pl}^2 (1 + \omega_M)} c_s, \quad \epsilon = \frac{3}{2} (1 + \omega_M) R_M,$$

where $R_M = \Omega_M/\Omega_{\Lambda + \Omega_M}$). In the “pre-inflation” epoch $H_0 > H > H_s$, see Fig. 1. $\omega_M$ varies from $\sim 1/3$ ($H/m \sim 1$) to 0 ($H/m \ll 1$), while $H$ and $\Omega_{\Lambda, M}$ slowly vary a few percent only, implying that $\Delta^2(k)$ [24] decreases $3/4$ at most. This probably explains the large-scale anomaly of the low amplitude of the CMB power spectrum at low-$\ell$ multipole, and implies that $n_s$ decreases, $\epsilon$ and $r$ increase as $k_s$ goes to large scales. Moreover, there could be the acoustic wave of dark-matter density perturbation $\delta \rho_M / \rho_M$ in the “pre-inflation” epoch, described by the sound velocity
c_s^M = \omega_s^{1/2} \neq 0. Analogously to baryon acoustic oscillations, these dark-matter sound waves should probably have imprinted in the both CMB and matter power spectra at large scales of \( k_s \sim 10^{-3} \text{Mpc}^{-1} \).

**Entropy and reheating.** The pair production from the spacetime is an entropically favorable process, and pairs can in turn annihilate back to the spacetime. Considering the processes as thermal emissions and absorptions, we discuss their entropy and temperatures associating with Eq. (5), using the first law of thermal dynamics in the volume \( V = \frac{4\pi}{3} H^{-3} \),

\[
dQ_M = T_M dS_M = d(\rho_M V) + p_M dV - \mu_M dN, \quad (25)
\]

\[
dQ_\Lambda = T_\Lambda dS_\Lambda = d(\rho_\Lambda V) + p_\Lambda dV = d(\rho_\Lambda)V, \quad (26)
\]

where the entropy \( S_\Lambda \) is related to the horizon entropy as shown below. Since the chemical potentials of fermions \( F \) and anti-fermions \( \bar{F} \) are equal and opposite \( \mu_M = -\bar{\mu}_M \) in the pair production \( dN = d\bar{N} \), the chemical potential \( \mu_M = \mu_M + \bar{\mu}_M = 0 \) in Eq. (25). From the particle number conservation \( (n_M U^a)_{,b} = 0 \) and the total energy conservation along the fluid flowing line \( U_a(T_M^{ab} + T_\Lambda^{ab})_{,b} = 0 \), we obtain

\[
T_M dS_M + T_\Lambda dS_\Lambda = 0. \quad (27)
\]

This relates to the total entropy conservation, implying the adiabatic evolution of the Universe composed by matter and spacetime. In fact, Eq. (27) is equivalent to the conservation law (10), provided with Eqs. (25) and (26). However, the matter entropy \( S_M \) renders the physical sense of the spacetime entropy \( S_\Lambda \), via the back and forth processes of spacetime producing pairs, which annihilate to the spacetime.

For no pair production \( \Omega_M = 0, \Lambda = 3H^2 \) and \( H = \text{const.} \), Eq. (26) gives

\[
S_\Lambda = \frac{\rho_\Lambda V}{T_\Lambda} = \frac{3H^2}{4G} V = \frac{\pi M_{pl}^2}{H^2} = \frac{1}{4} A = S_H, \quad (28)
\]

provided \( T_\Lambda = T_H = H/2\pi \) Hawking temperature, where \( S_H \) is the entropy of an De Sitter spacetime [21]. The entropy (28) relates to the total number of spacetime quantum states \( (q,p) \) on the horizon area \( A = 4\pi H^{-2}/\ell_{pl}^2 \), fluctuating \( \delta q \cdot \delta p \approx 1 \), \( \delta q \approx \ell_{pl} \) and \( \delta p \approx M_{pl} \) at the Planck scale. The characteristic state is \( \delta q \approx 2\pi H^{-1} \) and \( \delta p \approx T_H \).

In the inflation epoch \( H > \Gamma_M \) and \( \Omega_\Lambda \gg \Omega_M \neq 0 \), \( H \) and \( \Omega_\Lambda \) slowly decrease, due to pair production. However the rate of pairs annihilating back to the spacetime is smaller than the inflation rate \( H \), i.e., \( \Gamma_\text{anni}^M = \Gamma_M < H \), so that the pairs are far from reaching an equilibrium or equipartition with the inflating spacetime. Equations (26) and (27) give

\[
T_M dS_M = -d(\rho_\Lambda)V \approx -(3H^2/8\pi G)V = 2\epsilon T_H S_H dx, \quad (29)
\]
where \( T_\Lambda \approx T_H \gg T_M \).

After the inflation epoch, \( \Gamma_M > H \) implies that pairs have large density and rate to annihilate back to the spacetime. This epoch should be studied by integrating Eqs. (9) and (10) with the rate equation \[22\]
\[
\frac{dn_M}{dt} + 3Hn_M = \Gamma_M \left(n^{T_\Lambda}_M - n_M\right),
\]
(30)
where \( n^{T_\Lambda}_M \) is the thermal density of pairs in an equilibrium with the spacetime at the temperature \( T_\Lambda \). If pairs reach a thermal equilibrium with the spacetime, namely \( T_\Lambda = T_M = T_H \) and \( n^{T_\Lambda}_M = n_M \) in Eq. (30), \( \frac{dn_M}{dx} + 3n_M = 0 \) and \( n_M \propto a^{-3} \). The total entropy conservation \( dS_M = -dS_\Lambda \) (27) indicates that the spacetime entropy is converted to the matter entropy, as \( \Omega_\Lambda \) decreases and \( \Omega_M \) increases,
\[
S_M(\bar{H}) - S_M(H_e) = S_\Lambda(H_e) - S_\Lambda(\bar{H}),
\]
(31)
from the inflation ending \( H_e \) to the reheating end \( \bar{H} \). We may consider the approximations: (i) \( S_M(H_e) \approx 0 \) as \( \Omega_\Lambda(H_e) \ll \Omega_\Lambda(H_e) \); (ii) \( S_\Lambda(H_e) \approx S_\Lambda(H_e) \) of Eq. (28) as \( \rho_\Lambda(H_e) \approx 3H^2_e \); (iii) the reheating ends up with \( S_\Lambda(\bar{H}) \approx 0 \) and \( \bar{H} \approx 0 \), so that the spacetime entropy converted to the matter entropy is maximal \( S_M(\bar{H}) \approx S_\Lambda(H_e) \). Actually, at a certain point the pairs decay to light particles rather than annihilate to the spacetime, thus are out of thermal equilibrium, \( n^{T_\Lambda}_M \) exponentially decreases and \( T_M > T_\Lambda \). The Universe stops acceleration and starts deceleration for \( 2\Omega_\Lambda \leq (1 + 3\omega_M)\Omega_M \).

The enormous matter entropy (temperature) is generated (increased) by the decay of massive pairs to light particles, when the decay rate \( \Gamma^\text{decay}_M \propto g_Y^2 m > \Gamma_M > H \), where \( g_Y \) is the Yukawa coupling between the massive pairs and light particles. The term \( \Gamma^\text{decay}_M n_M \) should be added to the RHS of Eq. (30), and the particle number conservation law changes to \( (n_M U^a)_b - \Gamma^\text{decay}_M n_M \).

Postponing detailed studies of the complex reheating epoch, we postulate the reheating epoch ends at \( \tilde{a}, \tilde{t}, \tilde{\rho}_e = 3\bar{H}^2, \) and \( \bar{T} \), when an enormous amount of light particles decouples from the \( \Omega_\Lambda \), and approximately follow their own conservation law \( (\tilde{x} = \ln a/\tilde{a}) \),
\[
d\Omega_M/d\tilde{x} \approx -3(1 + \omega_M)\Omega_M, \quad \Omega_M(\tilde{a}) = \bar{\Omega}_M \gg \bar{\Omega}_\Lambda.
\]
(32)
We henceforth use \( \Omega_M \) and \( \omega_M \) to represent the “usual” matter that had been produced by the end of the reheating, governing the Universe evolution later on.

**Cosmic coincidence.** In the standard cosmology epoch, we separately consider two matter contributions: (i) the “coupled” matter \( \Omega_M^\Lambda(h) \) (\( \Omega_M^\Lambda \ll \Omega_\Lambda \)) and \( \omega_M^\Lambda \), representing the particle-antiparticle pairs produced after the reheating end, computed by Eqs. (11-13) since \( \tilde{t} = 0 \); (i)
the usual matter $\Omega_M \approx \tilde{\Omega}_M \exp(-3(1 + \omega_M)) x$ of Eq. (32), neglecting $\Omega_M$-annihilation to $\Omega_\Lambda$ and $\Omega_M^\Lambda$-decay to $\Omega_M$. In this approximation, the conservation law (10) decouples into Eq. (32) and

$$\frac{d}{dx} \left( \Omega_\Lambda + \Omega_M^\Lambda \right) \approx -3\left( 1 + \omega_M + \omega_M^{\text{decay}} \right) \Omega_M^\Lambda,$$

where we incorporate $(n_M U^a)_b = -\Gamma_M^{\text{decay}} n_M$ and introduce $\omega_M^{\text{decay}} \equiv \Gamma_M^{\text{decay}} / H$ for particle-antiparticle pairs decay to relativistic/non-relativistic particles in the radiation/matter dominate epoch. The $\omega_M^{\text{decay}}$ value depends on the final states and phase space of particles that they subsequently decay. As a result, Eqs. (9) and (10) are recast as

$$h^2 = (\Omega_M + \Omega_\Lambda), \quad \frac{d\Omega_\Lambda}{dx} \approx -3(1 + \omega_M^{\text{decay}}) \Omega_M^\Lambda (h),$$

where we rewrite $(\Omega_M^\Lambda + \Omega_\Lambda)$ as a new $\Omega_\Lambda$, since it overall represents “dark energy” in observations. It shows $\Omega_\Lambda$ indirectly interacting with $\Omega_M$ through $h$. Note $\omega_M^\Lambda \approx 0$ for $H/m \ll 1$, see Eq. (16).

To calculate $\Omega_M^\Lambda (h)$, we introduce another mass scale $\tilde{m}$ in Eq. (12). In the physical regime $\tilde{m} \gg H$, using $\Omega_M^\Lambda \approx 2\chi \tilde{m}^2 h^2/3$ in Eq. (34), analogously to the asymptotic expressions (15,16) for $\chi m^2 \to \chi \tilde{m}^2 \ll 1$, we obtain

$$\frac{d\Omega_\Lambda}{dx} + \tau \Omega_\Lambda = -\tau \Omega_M, \quad \tau \equiv 2 \chi \tilde{m}^2 (1 + \omega_M^{\text{decay}}).$$

In the radiation dominate epoch starting from the reheating end (32), the solution ($x = \ln a/\tilde{a}$ and $\omega_M = 1/3$) is

$$\Omega_\Lambda = \frac{\tau_R \tilde{\Omega}_M}{4 - \tau_R} e^{-4x} + e^{-\tau_R x} \tilde{C} = \frac{\tau_R}{4 - \tau_R} \Omega_M \propto h^2,$$

where $\tau_R \approx 2 \chi \tilde{m}^2 [1 + \omega_M^{\text{decay}}]$. Here we choose the initial condition $\tilde{C} = 0$ at $a = \tilde{a}$, i.e., $\tilde{\Omega}_\Lambda = \tau_R \tilde{\Omega}_M / (4 - \tau_R)$ at the reheating end (32), for the reason that the transitions from the reheating end to the standard cosmology start are radiation dominate and continuous, they have the same $\omega_M$ and $\omega_M^{\text{decay}}$ values. Solution (36) shows that in a long dark epoch, $\Omega_\Lambda \ll \Omega_M$ and $\Omega_\Lambda$ tracks $23$ down $\Omega_M$ until the Universe reaches the radiation-matter equilibrium ($a_{\text{eq}}/\tilde{a} = (\tilde{T}/T_{\text{eq}}) \sim 10^{15}\text{GeV} / 10\text{eV} \sim 10^{23}$,

$$\Omega_{\Lambda}^{\text{eq}} \approx (\tau_R / 4) \Omega_M^{\text{eq}} \ll 1, \quad \Omega_M^{\text{eq}} = \Omega_M (a_{\text{eq}}) \lesssim 1,$$

in unit of the density $\rho_{\text{eq}}^{\text{eq}} = 3H_{\text{eq}}^2$. Equations (9) and (36) give $\Omega_\Lambda \approx (\tau_R / 4) h^2$ in this dark epoch.

In the matter dominate epoch starting from $a_{\text{eq}}$ (37) to the present epoch $a \approx a_0$ and $(a/a_{\text{eq}}) \approx (1 + z) \sim 10^4$, the solution to Eq. (35) ($x = \ln a/a_{\text{eq}}$ and $\omega_M = 0$) is

$$\Omega_\Lambda = \frac{\tau_M}{3 - \tau_M} \Omega_M + e^{-\tau_M x} C^\text{eq} \approx \frac{\tau_M}{3} h^2 + C^\text{eq},$$

(38)
where \( \tau_M \approx 2 \chi \tilde{m}^2 [1 + \omega_{M,M}^{\text{decay}}] \) and the initial condition \( C^{\text{eq}} \) is fixed by Eq. (37)

\[
C^{\text{eq}} = 2 \chi \tilde{m}^2 \Delta \omega_M^{\text{decay}} \Omega_M^{\text{eq}}.
\]

The \( \Delta \omega_M^{\text{decay}} \) represents the effective variation from \( \omega_{M,R}^{\text{decay}} \) to \( \omega_M^{\text{decay}} \), and \( \Delta \omega_M^{\text{decay}} > 0 \) for a larger and recursively generated phase space of final states of particles and their subsequent decays [24]. Actually, the \( C^{\text{eq}} \) (39) is the integration over discontinuous transitions from the radiation dominate epoch to the matter dominate one.

In this light epoch, Eq. (38) shows that the first term decreases as \( \Omega_M \approx \Omega_M^{\text{eq}} (1 + z)^{-3} \) and \( \Omega_\Lambda \) fails to track down \( \Omega_M \), approaching to a slowly varying “constant” \( e^{-\tau_M x} C^{\text{eq}} \), which recalls its value (37) at the radiation-matter equilibrium. As a result, we obtain the ratio

\[
\frac{\Omega_\Lambda}{\Omega_M} \approx (\tau_M/3) + 2 \chi \tilde{m}^2 \Delta \omega_M^{\text{decay}} (1 + z)^3, \quad (1 + z) = (a/a^{\text{eq}})^{-1}
\]

Using current observations \( \Omega_\Lambda^0 \approx 0.7 \) and \( \Omega_M^0 \approx 0.3 \), correspondingly \( z \sim 10^4 \), we obtain

\[
2 \chi \tilde{m}^2 \Delta \omega_M^{\text{decay}} \approx (1 + z)^{-3} \Omega_\Lambda/\Omega_M \approx 2.3 \times 10^{-12}.
\]

If \( \Delta \omega_M^{\text{decay}} \sim \mathcal{O}(1) \), \( \tau_M \approx \tau_R \sim \mathcal{O}(10^{-12}) \) and the mass scale \( \tilde{m} \sim 10^{14} \) GeV coincides with the reheating temperature \( \tilde{T} \).

These results give us an insight into the issue of the cosmic coincidence at the present time. The \( \Omega_\Lambda \) and \( \Omega_M \) relation shows that the cosmic coincidence of \( \Omega_\Lambda \) and \( \Omega_M \) values appears naturally without any extremely fine tuning, since the matter dominated epoch of \( z \sim 10^{3-4} \) is much shorter than the radiation dominated epoch of \( (a^{\text{eq}}/a) \sim 10^{23} \), when the \( \Omega_\Lambda \) tracks down \( \Omega_M \) and the ratio \( \Omega_\Lambda/\Omega_M \) is constant. Otherwise we would have the cosmic coincidence problem of an incredibly fine tuning the values \( \tilde{\Omega}_\Lambda \) and \( \tilde{\Omega}_M \) at the reheating end at the order \( \sim (10^{-23})^4 \times (10^{-4})^3 \sim 10^{-104} \), so as to reach their present observational values of the same order of magnitude. To describe this scenario, we use the ratio \( \Omega_\Lambda/\Omega_M \), which is independent of the different units used in different epochs. In Fig. 3, we plot the ratio \( \Omega_\Lambda/\Omega_M \) from the radiation dominated epoch (36) to the matter dominated epoch (40) for an explicit illustration.

The \( \Omega_\Lambda - \Omega_M \) relation (38) can be rewritten in units of the critical density \( \rho_c = 3H_0^2 \) today,

\[
\Omega_\Lambda \approx (\tau_M/3) \Omega_M^0 (1 + z)^3 + \Omega_\Lambda^0 (1 + z)^7_M,
\]

and \( \Omega_\Lambda^0 \approx \Omega_\Lambda^{\text{eq}} \sim 10^{-12} \Omega_M^{\text{eq}} \). This can possibly be examined with observations [25]. In particular, how to examine the \( \Omega_\Lambda \)-transition from the present “constant” \( \sim (1 + z)^7_M \) back to the track-down evolution \( \sim (1 + z)^3 \) at the large redshift \( z \sim 10^{3-4} \). We speculate that such \( \Omega_\Lambda \)-transition should induce the peculiar fluctuations of gravitational field that imprint on the CMB spectrum, analogously to the integrated Sachs-Wolfe effect.
FIG. 3: The ratio $\frac{\Omega_{\Lambda}}{\Omega_{M}}$ is plotted as a function of $\ln(a/a_{\text{eq}})$, where the scaling factor $a$ runs from the reheating end $\tilde{a}$, through the radiation-matter equilibrium $a_{\text{eq}}$ to the present time $a_{0}$, $\tilde{a} < a_{\text{eq}} < a_{0}$. It shows that (i) the tracking-down behavior: the ratio is a small constant $\sim 10^{-12}$ for $\ln(a/a_{\text{eq}}) < 0$; (ii) the tracking-down failure occurs around the radiation-matter equilibrium $\ln(a/a_{\text{eq}}) = 0$; (iii) $\Omega_{\Lambda} \approx \text{const.}$ and $\Omega_{M} \sim (a/a_{\text{eq}})^{-3}$, the ratio $\Omega_{\Lambda}/\Omega_{M}$ increases to $O(1)$ at the present time $(a/a_{\text{eq}}) \sim 10^{4}$ and $\ln(a/a_{\text{eq}}) \approx 9.2$. When $\Omega_{\Lambda}/\Omega_{M} = 1/2$, the Universe turns from the deceleration phase to the acceleration phase. The cosmological term $\Omega_{\Lambda}$ will dominate over the matter term $\Omega_{M}$ in future.

Some remarks. We emphasize that the area law (15) and (16) are crucial for obtaining the law $\Omega_{\Lambda} \propto h^{2}$ in the cosmic inflation and $\Omega_{\Lambda} \sim \Omega_{M}$ coincidence in the present time. The initial value $\Omega_{\Lambda}^{\circ} \propto H_{0}^{2}$ at the Planck scale should be attributed to the spacetime quantum fluctuation at the Planck scale [28]. Oppositely to the matter and its negative gravitational potential, the $\Omega_{\Lambda}$ physically represents a negative mass, whose positive potential not only leads to the pair production, but also to the Universe acceleration. In turn, these pairs “screen” the positive potential, increase $\Omega_{M}$ and deepen the negative potential. The present value $\Omega_{\Lambda}^{0} \propto H_{0}^{2}$ [29] is the consequence of $\Omega_{\Lambda}$ creating and interacting with $\Omega_{M}$ in the Universe evolution. The positivity of total energy is expected as long as Eqs. (9) and (10) hold. Full numerical approach to this problem is very inviting. The lengthy article in details can be found in Ref. [26].

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