Hilbert Norms For Graded Algebras

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Abstract

This paper presents a solution to a problem from superanalysis about the existence of Hilbert-Banach superalgebras. Two main results are derived:
1) There exist Hilbert norms on some graded algebras (infinite-dimensional superalgebras included) with respect to which the multiplication is continuous.
2) Such norms cannot be chosen to be submultiplicative and equal to one on the unit of the algebra.

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1 Introduction

The type of norms investigated in this article are generalizations of norms used for the symmetric tensor algebra in the white noise analysis [7][11] or in the Malliavin calculus [20]. But now more general algebras are included, especially the algebra of antisymmetric tensors (Grassmann algebra) and \( \mathbb{Z}_2 \)-graded algebras (superalgebras) related to supersymmetry and to quantum probability [15].

A locally convex commutative superalgebra is a \( \mathbb{Z}_2 \)-graded locally convex space \( \mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1 \) equipped with an associative continuous multiplication having the following property: for any \( a, b \in \mathcal{E}_0 \cup \mathcal{E}_1 \), \( ab \neq 0 \) the product satisfies \( ab = (-1)^{p(a)p(b)}ba \) with the parity function \( p \), which is defined on \( (\mathcal{E}_0 \cup \mathcal{E}_1) \setminus \{0\} \) with \( p(\mathcal{E}_0 \setminus \{0\}) = 0 \), \( p(\mathcal{E}_1 \setminus \{0\}) = 1 \), and \( p(ab) = |p(a) - p(b)| \). Typical examples are Grassmann algebras with finite or countable sets of generators. In superanalysis one considers modules over (commutative) superalgebras [16][8][19][7][18][10][14]. It is quite easy to define an infinite-dimensional Grassmann algebra with a non-Hilbertian norm [10]. But for a long time it was unknown whether the topology of a locally convex superalgebra - including the Grassmann algebra - can be defined with a Hilbert norm, and moreover, whether this norm can be chosen to be simultaneously submultiplicative and equal to one at the unit of the algebra. The paper gives a complete solution to these problems. Our theorems imply a positive answer to the first question and a negative answer to the second question.

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3In the pioneering works of Martin [14] and of Berezin [13] the Grassmann algebra itself has been used instead of these modules.
2 General considerations

Let \( \mathcal{A} \) be an algebra over the field \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \) with unit \( e_0 \). The product is denoted by \( a, b \in \mathcal{A} \rightarrow ab \in \mathcal{A} \). We assume that \( \mathcal{A} \) is provided with a positive definite inner product \( a, b \in \mathcal{A} \rightarrow (a \mid b) \in \mathbb{K} \). The corresponding Hilbert norm \( \|a\| = \sqrt{(a \mid a)} \geq 0 \) is normalized at the unit \( \|e_0\| = 1 \). We are interested in such norms which allow a uniform estimate for the product of the algebra

\[
\|ab\| \leq \gamma \|a\| \|b\| \tag{1}
\]

with a constant \( \gamma \geq 1 \). In this section we prove under rather general conditions that this constant has the lower limit \( \gamma \geq \sqrt{\frac{4}{3}} \).

**Theorem 1** Let \( \mathcal{A} \) be an algebra over the field \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \) with dimension \( \dim \mathcal{A} \geq 2 \). If this algebra satisfies the properties

i) \( \mathcal{A} \) is provided with a Hilbert inner product \((. \mid .)\) normalized at the unit \( e_0 \), \( \|e_0\|^2 = (e_0 \mid e_0) = 1 \),

ii) there exists at least one element \( f \in \mathcal{A}, f \neq 0 \), such that \( e_0, f \) and \( f^2 = ff \) satisfy \( (e_0 \mid f) = (f \mid f^2) = 0 \) and \( (e_0 \mid f^2) \geq 0 \),

then the norm estimate \( \|ab\| \leq \gamma \|a\| \|b\| \) is not valid for some \( a, b \in \mathcal{A} \), if \( \gamma < \sqrt{\frac{3}{4}} \).

**Proof** Since \( f \neq 0 \) we can normalize this element and assume \( \|f\| = 1 \). Take \( a = e_0 + \lambda f \) with \( \lambda \in \mathbb{R} \). Then \( a^2 = e_0 + 2\lambda f + \lambda^2 f^2 \) and

\[
\|a^2\|^2 = 1 + 2\lambda^2 (e_0 \mid f^2) + 4\lambda^2 + \lambda^4 \|f^2\|^2 \geq 1 + 4\lambda^2.
\]

On the other hand \( \|a\|^2 = 1 + \lambda^2 \), and \( \|a^2\|^2 \leq \gamma^2 \|a\|^4 \) implies \( 1 + 4\lambda^2 \leq \gamma^2 (1 + \lambda^2)^2 \). But this inequality is true for all \( \lambda \geq 0 \) only if \( \gamma^2 \geq \sup_{\lambda \geq 0}(1 + 4\lambda^2)(1 + \lambda^2)^{-2} = \frac{4}{3} \). \( \square \)

This Theorem obviously applies to the tensor algebra \( T = \oplus_{n=0}^{\infty} T_n \), where \( T_n \) is the subspace of tensors of degree \( n \), and the norm is defined in the standard way as

\[
\|f\|^2 = \sum_{n=0}^{\infty} w_n \|f_n\|^2 \quad \text{if} \quad f = \sum_{n=0}^{\infty} f_n, \quad f_n \in T_n
\]

with arbitrary positive weights \( w_n > 0, n \in \mathbb{N} \) and \( w_0 = 1 \). In that case we can simply choose an element \( f \in T_1, f \neq 0 \), to satisfy the assumptions with \((e_0 \mid f \otimes f) = 0 \).

Theorem 1 can also be applied to a large class of algebras \( \mathcal{A} \) which can be derived from the tensor algebra \( T \) by the following modifications of the product.

1. The product is generated by \( f, g \in \mathcal{A}_1 = T_1 \rightarrow f \circ g := f \otimes g + (-1)^{\chi} g \otimes f \) where \( \chi = 0, 1 \mod 2 \) is a parity factor.

2. The product is generated by \( f, g \in \mathcal{A}_1 = T_1 \rightarrow f \circ g := f \otimes g + (-1)^{\chi} g \otimes f + \omega(f, g)e_0 \). Here \( \chi \) is again a parity factor and \( \omega(., .) : \mathcal{A}_1 \times \mathcal{A}_1 \rightarrow \mathbb{K} \) is a bilinear pairing.

The first class of algebras includes the algebra of symmetric tensors, the algebra of antisymmetric tensors (Grassmann algebra), and tensor products of these algebras, including...
the $\mathbb{Z}_2$-graded algebras (superalgebras) used in quantum field theory. The assumptions of the Theorem 1 are satisfied for any non-vanishing element $f \in A_1 = T_1$.

The second class includes the Clifford product, the (symmetric) Wiener product, the antisymmetric Wiener product (with antisymmetric $\omega$) and Le Jan’s supersymmetric Wiener-Grassmann product [4][13][15]. In these cases the assumptions of Theorem 1 are satisfied if there exists a non-vanishing $f \in A_1$ with $\omega(f, f) \geq 0$. Such a vector can always be found

if the algebra is complex, or

if the algebra is real and $\omega$ is not negative definite.

The last constraint is satisfied for the symmetric Wiener product on real spaces, and for the real Clifford system in quantum field theory [2]. In both cases the form $\omega$ is positive definite.

Moreover Theorem 1 is obviously true for any unital algebra $A$, which has a nilpotent element $f$ that is orthogonal to the unit element. If we only know that $A$ has at least one nilpotent element, we can derive the weaker

**Corollary 1** Let $A$ be an algebra which satisfies condition i) of Theorem 1. If this algebra has a nilpotent element $f$, then the norm estimate $\|ab\| \leq \|a\| \|b\|$ is not valid for some $a, b \in A$.

**Proof** We assume again $\|f\| = 1$. Then $a = e_0 + \lambda f$ with $\lambda \in \mathbb{R}$ and $a^2 = (e_0 + \lambda f)^2 = e_0 + 2\lambda Re(e_0, f) + \lambda^2$ and $\|a^2\|^2 = 1 + 4\lambda Re(e_0, f) + 4\lambda^2$. If $Re(e_0, f) = 0$ we can apply the arguments given in the proof for Theorem 1. If $Re(e_0, f) = \gamma \neq 0$, then we chose $\lambda = -2\gamma$, and $\|a^2\|^2 = 1 + 8\gamma^2 \leq 1 = \|a\|^4$ is a contradiction. $\Box$

### 3 Norm estimates for $\mathbb{Z}$-graded algebras

In this section we present Hilbert norm estimates for rather general $\mathbb{Z}$-graded algebras $A$ over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. We assume the following structure of $A$.

1. The algebra is the direct sum $A = \bigoplus_{n=0}^{\infty} A_n$ of orthogonal spaces $A_n$. Thereby $A_0$ is the one dimensional space $\mathbb{K}$ spanned by the unit $e_0$ of the algebra. The product $a \circ b$ maps $A_p \times A_q$ into $A_{p+q}$ for all $p, q \in \{0, 1, \ldots\}$.

2. The spaces $A_n$ are provided with Hilbert norms $\|\cdot\|_n$, $n = 0, 1, \ldots$. The unit has norm $\|e_0\|_0 = 1$. The product of two homogeneous elements $a_p \in A_p$ and $b_q \in A_q$ satisfies

$$\|a_p \circ b_q\|_{p+q} \leq \|a_p\|_p \|b_q\|_q$$

if $a_p \in A_p$ and $b_q \in A_q$.

3. The algebra is provided with a family of Hilbert norms

$$\|a\|_{(\sigma)}^2 = \sum_{n=0}^{\infty} w_n(\sigma) \|a_n\|^2_n \text{ if } a = \sum_{n=0}^{\infty} a_n, \ a_n \in A_n$$

(4)
with $\sigma \in \mathbb{R}$. The factors $w_n(\sigma), n = 0, 1, \ldots$, are positive weights with the normalization $w_0(\sigma) = 1$ for all $\sigma \in \mathbb{R}$. The weights satisfy the inequalities $w_n(\sigma) \leq w_n(\tau)$ for all $n \in \mathbb{N}$ if $\sigma \leq \tau$.

An immediate consequence of these assumptions is $\|a\|_{(\sigma)} \leq \|a\|_{(\tau)}$ for all $a \in \mathcal{A}$ if $\sigma \leq \tau$. A simple example of such an algebra $\mathcal{A}$ is the tensor algebra $\mathcal{T}$. Its standard norm satisfies (3) with weights $w_n = 1$ for all $n = 0, 1, \ldots$. More interesting examples are the algebras of symmetric tensors or of antisymmetric tensors. With the notation $f \circ g$ for both the symmetric and the antisymmetric tensor product the estimate (3) is satisfied by the norms

$$\|f_1 \circ f_2 \circ \cdots \circ f_n\|_n^2 = \begin{cases} (n!)^{-1} \text{per}(f_\mu | f_\nu) & \text{for symmetric tensors}, \\ (n!)^{-1} \text{det}(f_\mu | f_\nu) & \text{for antisymmetric tensors}, \end{cases}$$

but it is violated if the factor $(n!)^{-1}$ is omitted. The standard norm (3) is defined without the factor $(n!)^{-1}$. In the notations used here it corresponds therefore to a norm (4) with a weight function $w_n = n!$.

**Theorem 2** If there exists a constant $\delta(\sigma, \tau, \rho) > 0$ such that the weight functions satisfy the inequalities

$$(p + q - 1)w_{p+q}(\rho) \leq \delta(\sigma, \tau; \rho)w_p(\sigma)w_q(\tau)$$

for values of $\sigma, \tau$ and $\rho$ with $\sigma \leq \rho$ and $\tau \leq \rho$, then the product of $\mathcal{A}$ is estimated by

$$\|a \circ b\|_{(\rho)} \leq \gamma \cdot \|a\|_{(\sigma)} \cdot \|b\|_{(\tau)}$$

where the constant $\gamma$ is $\gamma = \sqrt{3} \max(1, \delta(\sigma, \tau, \rho))$.

**Proof** For $a = a_0 + a_+$ and $b = b_0 + b_+$ with $a_0, b_0 \in \mathcal{A}_0 = \mathbb{K}$ and $a_+ = \sum_{n=1}^{\infty} a_n$, $b_+ = \sum_{n=1}^{\infty} b_n$ with $a_n, b_n \in \mathcal{A}_n, n \in \mathbb{N}$ the norm of $a \circ b$ is calculated by

$$\|a \circ b\|_{(\rho)}^2 = \|a_0b_0 + a_0b_+ + a_+b_0 + a_+ \circ b_+\|_{(\rho)}^2$$

$$\leq |a_0b_0|^2 + 3 \left(|a_0|^2 \|b_+\|_{(\rho)}^2 + \|a_+\|_{(\rho)}^2 |b_0|^2 + \|a_+ \circ b_+\|_{(\rho)}^2 \right)$$

$$\leq |a_0b_0|^2 + 3 \left(|a_0|^2 \|b_+\|_{(\rho)}^2 + \|a_+\|_{(\rho)}^2 |b_0|^2 + \sum_{n \geq 1} w_n(\rho) \|\sum_{p+q=n} a_p \circ b_q\|_{n}^2 \right)$$

The symbol $\sum'$ means summation with the constraint $p \geq 1, q \geq 1$. The sum $\sum'_{p+q=n, p \geq 1, q \geq 1, \ldots} = \sum'_{p+q=n} \ldots$ has $n-1$ terms, hence

$$\left| \sum'_{p+q=n, p \geq 1, q \geq 1} a_p \circ b_q \right|_{n}^2 \leq (n - 1) \sum_{p+q=n} |a_p \circ b_q|_{n}^2 \leq (n - 1) \sum_{p+q=n} \|a_p\|_{p}^2 \|b_q\|_{q}^2.$$  

If $w_n(\rho)$ is chosen such that (3) is satisfied we obtain

$$\sum_{n \geq 1} w_n(\rho) \left| \sum'_{p+q=n} a_p \circ b_q \right|_{n}^2 \leq \delta \cdot \left( \sum_{p \geq 1} w_p(\sigma) \|a_p\|_{p}^2 \right) \cdot \left( \sum_{q \geq 1} w_q(\tau) \|b_q\|_{q}^2 \right).$$

The “standard” inner product of the symmetric/antisymmetric tensor algebra is characterized by the following property. Let $\mathcal{F}_i, i = 1, 2$, be two orthogonal subspaces of the space $\mathcal{A}_1$. Denote by $\mathcal{A}(\mathcal{F}_i)$ the subalgebra generated by elements $f \in \mathcal{F}_i$. Then $(a_1 \circ a_2 | b_1 \circ b_2) = (a_1 | b_1)(a_2 | b_2)$ holds for all $a_i \in \mathcal{A}(\mathcal{F}_i), i = 1, 2.$
\[ \leq \delta \|a_+\|_{(\sigma)}^2 \|b_+\|_{(\tau)}^2 \]. For \( \rho \leq \sigma, \tau \) we have in addition the inequalities
\[ \|a_+\|_{(\rho)} \leq \|a_+\|_{(\sigma)}^2 \text{ and } \|b_+\|_{(\rho)} \leq \|b_+\|_{(\tau)}^2 \] such that finally
\[
\|a \circ b\|_{(\rho)}^2 \leq |a_0b_0|^2 + 3 \left( |a_0|^2 \|b_+\|_{(\tau)}^2 + \|a_+\|_{(\sigma)} \|b_0\|^2 + \delta \|a_+\|_{(\sigma)}^2 \|b_+\|_{(\tau)}^2 \right) \\
\leq 3\gamma \|a\|_{(\sigma)} \|b\|_{(\tau)}^2.
\]
where \( \gamma = \max(1, \delta) \).

As the first application of Theorem 2 we derive norms with respect to which the product of the algebra is continuous. In that case the inequality (3) has to be satisfied for identical weights \( \wp(\sigma) = \wp(\tau) = \wp(\rho) = \wp, \ p \geq 1. \) If we fix \( q = 1 \) then (3) implies \( p \cdot \wp_{p+1} \leq \delta \cdot \wp_p \cdot \wp_1 \) for \( p \in \mathbb{N}. \) As a consequence we obtain \( \wp_p \leq \delta^{p-1} ((p - 1)!)^{-1} \wp_1, p \geq 1. \) The slowest decrease of the weights which might be possible according to our estimates is therefore \( \wp_p \sim ((p - 1)!)^{-1}. \) The proof that such a solution actually exists follows from
the simple estimate
\[
\left( \frac{m + n}{m} \right) \geq \frac{(m + n)!}{m!} \geq m + n \text{ if } m, n \geq 0. \text{ Hence } (p + q - 1) \frac{1}{(p + q)!} = \\
\frac{1}{(p+q-2)!} \leq \frac{1}{(p-1)!} \frac{1}{(q-1)!} \text{ is valid for all } p, q \geq 1. \text{ Since }
\frac{2^{m+n}}{2} \geq \left( \frac{m + n}{m} \right) = \frac{(m + n)!}{m!} \geq m + n \text{ if } m, n \geq 1,
\]
also \( (p + q - 1) \frac{1}{(p+q)!} \leq \frac{1}{(p+q-1)!} \leq \frac{1}{p! q!} \) follows for all \( p, q \geq 1. \) We have therefore derived

**Corollary 2** If the norm is defined with the weights \( w_0 = 1, w_n = ((n - 1)!)^{-1}, n \geq 1, \) or with \( w_0 = 1, w_n = (n!)^{-1}, n \geq 1, \) the product of the algebra is continuous with the uniform norm estimate
\[
\|a \circ b\| \leq \sqrt{3} \|a\| \|b\|. \tag{9}
\]

As a more general class of norms we choose weights
\[
w_0 = 1, \quad w_n(\sigma, \rho, s) = (n!)^{\sigma/2}2^{\rho m}(1 + n)^{s} \text{ if } n \geq 1, \tag{10}
\]
with real parameters \( \sigma, \rho, s. \) These weights satisfy the inequalities
\[
w_n(\sigma_1, \rho_1, s_1) \leq w_n(\sigma_2, \rho_2, s_2) \text{ if } \sigma_1 \leq \sigma_2, \rho_1 \leq \rho_2, s_1 \leq s_2. \text{ We denote by } \|a\|_{(\sigma, \rho, s)} \text{ the norm (4)} \text{ defined with the weights } w_n(\sigma, \rho, s). \text{ The estimate (8) and the bounds (9)} \text{ and the bounds } \frac{(m + n)!}{m!n!} \geq \frac{2^{m!}}{(m!^2)} \geq const \cdot 2^{m} m^{-1/2} \text{ if } n \geq m \geq 1 \text{ and } 1 \leq \frac{(1+m)(1+n)}{1+m+n} \leq 1 + \min(m, n) \text{ yield inequalities of the type (3)} \text{ also for these norms. We obtain }
\]
\[
(p + q - 1)w_{p+q}(\sigma, \rho, s) \leq \delta w_p(\sigma', \rho', s')w_q(\sigma', \rho', s') \text{ if } p, q \geq 1 \tag{11}
\]
with a constant \( \delta \geq 1 \) if \( \sigma = \sigma' = -1 \) with \( \rho = \rho' \in \mathbb{R} \) and \( s = s' \leq 0, \) or if \( \sigma = \sigma' < -1 \) with \( \rho = \rho' \in \mathbb{R} \) and \( s = s' \in \mathbb{R}. \)

The generalizations of (3) are therefore
\[
\|a \circ b\|_{(-1, \rho, s)} \leq \sqrt{3} \|a\|_{(-1, \rho, s)} \cdot \|b\|_{(-1, \rho, s)} \text{ if } \rho \in \mathbb{R}, s \leq 0, \tag{12}
\]
and
\[
\|a \circ b\|_{(\sigma, \rho, s)} \leq \gamma \|a\|_{(\sigma, \rho, s)} \cdot \|b\|_{(\sigma, \rho, s)} \text{ if } \sigma < -1, \rho \in \mathbb{R}, s \in \mathbb{R}. \tag{13}
\]
Here $\gamma$ takes some value $\gamma \geq \sqrt{3}$ depending on the choice of the parameters $\sigma$ and $s$.

Moreover, the inequalities (11) are valid for $(\sigma, \rho, s) \neq (\sigma', \rho', s')$ if $\sigma < \sigma'$ or if $\sigma = \sigma'$ and $\rho < \rho'$. The corresponding estimates for the norms are

$$\|a \circ b\|_{(\sigma, \rho, s)} \leq \gamma \|a\|_{(\sigma', \rho', s')} \cdot \|b\|_{(\sigma', \rho', s')} \quad \text{if} \quad \sigma < \sigma' \quad \text{for all} \quad \rho, \rho', s, s' \in \mathbb{R},$$

and

$$\|a \circ b\|_{(\sigma, \rho, s)} \leq \gamma \|a\|_{(\sigma, \rho', s')} \cdot \|b\|_{(\sigma, \rho', s')} \quad \text{if} \quad \rho < \rho' \quad \text{for all} \quad \sigma, s, s' \in \mathbb{R}. \quad (15)$$

The value of $\gamma \geq \sqrt{3}$ depends on the choice of the parameters.

For the tensor algebra and for algebras of symmetrized tensors\footnote{This class of algebras does not only include the algebra of symmetric tensors and the algebra of antisymmetric tensors (Grassmann algebra), but also the $\mathbb{Z}_2$-graded algebras (superalgebras) used in supersymmetric quantum field theory.} the Hilbert space $A_1 = \mathcal{H}$ generates the whole algebra. Given a (self-adjoint/positive) operator $A$ on $\mathcal{H}$, the mapping $\Gamma(A)e_0 = e_0$ and $\Gamma(A)(f_1 \circ f_2 \circ \ldots \circ f_n) := (Af_1) \circ (Af_2) \circ \ldots \circ (Af_n)$ for $f_\mu \in \mathcal{H}$, $\mu = 1, \ldots, n$, and $n \in \mathbb{N}$, defines a unique (self-adjoint/positive) operator $\Gamma(A)$ on the algebra $\mathcal{A}$, which satisfies the relation

$$\Gamma(A)(a \circ b) = (\Gamma(A)a) \circ (\Gamma(A)b). \quad (16)$$

The norms (10) with the weights (10) are then easily generalized to

$$\|a\|^2_{(\sigma, \rho, s)} = \sum_{n=0}^{\infty} (n!)^\sigma \|\Gamma(A)^\rho a_n\|^2_n (1 + n)^s \quad \text{if} \quad a = \sum_{n=0}^{\infty} a_n, \quad a_n \in \mathcal{A}_n. \quad (17)$$

If $A$ is an invertible positive operator with lower bound $A \geq 2 \cdot id$, then $\Gamma(A)$ satisfies $\|(\Gamma(A))^{-\rho} a\|_n \leq 2^{-\rho n} \|a\|_n$ for $a \in \mathcal{A}_n$ if $\rho \geq 0$. This bound and the relation (16) imply that the estimates (12), (13) and (15) are also valid for the norms (17), moreover (14) holds if $\rho \leq \rho'$.

If $A^{-1}$ is a Hilbert-Schmidt operator then a family of norms (17) can be used to define a nuclear topology on the algebra $\mathcal{A}$. For the symmetric tensor algebra that has been done in the white noise calculus and in the Malliavin calculus, see e.g. \footnote{This class of algebras does not only include the algebra of symmetric tensors and the algebra of antisymmetric tensors (Grassmann algebra), but also the $\mathbb{Z}_2$-graded algebras (superalgebras) used in supersymmetric quantum field theory.} \footnote{This class of algebras does not only include the algebra of symmetric tensors and the algebra of antisymmetric tensors (Grassmann algebra), but also the $\mathbb{Z}_2$-graded algebras (superalgebras) used in supersymmetric quantum field theory.} \footnote{This class of algebras does not only include the algebra of symmetric tensors and the algebra of antisymmetric tensors (Grassmann algebra), but also the $\mathbb{Z}_2$-graded algebras (superalgebras) used in supersymmetric quantum field theory.} \footnote{This class of algebras does not only include the algebra of symmetric tensors and the algebra of antisymmetric tensors (Grassmann algebra), but also the $\mathbb{Z}_2$-graded algebras (superalgebras) used in supersymmetric quantum field theory.} For the algebra of antisymmetric tensors and for the superalgebras such nuclear topologies can be found in \footnote{This class of algebras does not only include the algebra of symmetric tensors and the algebra of antisymmetric tensors (Grassmann algebra), but also the $\mathbb{Z}_2$-graded algebras (superalgebras) used in supersymmetric quantum field theory.} \footnote{This class of algebras does not only include the algebra of symmetric tensors and the algebra of antisymmetric tensors (Grassmann algebra), but also the $\mathbb{Z}_2$-graded algebras (superalgebras) used in supersymmetric quantum field theory.} \footnote{This class of algebras does not only include the algebra of symmetric tensors and the algebra of antisymmetric tensors (Grassmann algebra), but also the $\mathbb{Z}_2$-graded algebras (superalgebras) used in supersymmetric quantum field theory.} But the estimates of these references are not strong enough to derive the results with a single Hilbert norm as presented in Corollary \footnote{This class of algebras does not only include the algebra of symmetric tensors and the algebra of antisymmetric tensors (Grassmann algebra), but also the $\mathbb{Z}_2$-graded algebras (superalgebras) used in supersymmetric quantum field theory.} and in eqs. \footnote{This class of algebras does not only include the algebra of symmetric tensors and the algebra of antisymmetric tensors (Grassmann algebra), but also the $\mathbb{Z}_2$-graded algebras (superalgebras) used in supersymmetric quantum field theory.} \footnote{This class of algebras does not only include the algebra of symmetric tensors and the algebra of antisymmetric tensors (Grassmann algebra), but also the $\mathbb{Z}_2$-graded algebras (superalgebras) used in supersymmetric quantum field theory.} \footnote{This class of algebras does not only include the algebra of symmetric tensors and the algebra of antisymmetric tensors (Grassmann algebra), but also the $\mathbb{Z}_2$-graded algebras (superalgebras) used in supersymmetric quantum field theory.}.

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