Casimir energy and a cosmological bounce

Carlos A R Herdeiro and Marco Sampaio

Departamento de Física e Centro de Física do Porto, Faculdade de Ciências da Universidade do Porto, Rua do Campo Alegre, 687, 4169-007 Porto, Portugal

Received 13 October 2005, in final form 17 November 2005
Published 28 December 2005
Online at stacks.iop.org/CQG/23/473

Abstract

We review different computation methods for the renormalized energy–momentum tensor of a quantized scalar field in an Einstein static universe. For the extensively studied conformally coupled case, we check their equivalence; for different couplings, we discuss violation of different energy conditions. In particular, there is a family of masses and couplings which violate the weak and strong energy conditions but do not lead to spacelike propagation. Amongst these cases is that of a minimally coupled massless scalar field with no potential. We also point out a particular coupling for which a massless scalar field has vanishing renormalized energy–momentum tensor. We discuss the backreaction problem and in particular the possibility that this Casimir energy could both source a short inflationary epoch and avoid the big bang singularity through a bounce.

PACS numbers: 04.62.+v, 98.80.—k

(Some figures in this article are in colour only in the electronic version)

1. Introduction

The Casimir effect [1] is a manifestation of the vacuum fluctuations of a quantum field. It was first considered in systems with boundaries, and it is known that the Casimir force is highly sensitive to the size, geometry and topology of such boundaries. In particular, it may change from attractive to repulsive depending on such shape [2]. But the Casimir force is also present in systems without boundaries and a compact topology, since the latter imposes periodicity conditions which resemble boundary conditions.

If our universe is either open or flat, with non-trivial topology, or closed, every quantum field living on it should generate a Casimir-type force, which has led many authors to study the Casimir effect in FRW models (see [2] and references therein for a review). In the case of a spherical universe, most computations of the Casimir energy, or more generically, of the renormalized stress–energy tensor, have focused on conformally coupled scalar fields. For instance, a massless conformally coupled scalar field, the electromagnetic field and the...
massless Dirac field on an Einstein static universe (ESU) have been considered in [12–14]; their Casimir energies have been shown to be of the form \( \alpha/R^4 \), with \( \alpha \), respectively, \( 1/(480\pi^2) \), \( 11/(240\pi^2) \) and \( 17/(1920\pi^2) \). Note that all of these are positive. Since these are all conformally coupled fields, they have equation of state \( p = \rho/3 \) and obey the strong energy condition. This means the Casimir force is attractive. But this is not always so in the cosmological context. Zel’dovich and Starobinskii [3] have indeed verified long ago that the Casimir energy of a scalar field could drive inflation in a flat universe with toroidal topology.

The purpose of this paper is to exhibit a family of quantum scalar fields which originate a repulsive Casimir force in a closed universe, since they violate the strong energy condition. Interestingly, this family includes the simplest case one could consider: a minimally coupled massless scalar field with no potential. Our computation will be performed in the Einstein static universe (ESU), which can be faced as an approximation to a dynamical FRW model in sufficiently small time intervals, and avoids having to deal with the complexities of quantum field theory in a time-dependent spacetime, like particle creation\(^1\).

2. Quantum scalar field with arbitrary coupling in the ESU

We consider the ESU, which is a well-known solution of the Einstein equations sourced by a perfect fluid with positive energy density \( \rho > 0 \) and zero pressure \( p = 0 \) together with a positive cosmological constant \( \Lambda > 0 \): \( G_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu} \), with \( T_{\mu\nu} = \rho u_{\mu} u_{\nu} \). The metric is

\[
ds^2 = -dt^2 + R^2 \left[ \frac{1}{\sigma_1 R^2} + \frac{1}{\sigma_2 R^2} + \frac{1}{\sigma_3 R^2} \right].
\]

We have written the metric on the unit 3-sphere, \( d\Omega^3 \), in terms of right forms on \( SU(2) \). In order to achieve a static solution, the cosmological constant and the energy density are related by \( \Lambda = \rho/2 = 1/R^2 \). Another viewpoint is that the ESU is supported by a perfect fluid with \( p = -\rho/3 = -1/R^2 \).

It is well known that this universe is unstable against small radial perturbations as was first argued by de Sitter. The reason is that the energy density of the perfect fluid increases/decreases with decreasing/increasing radius, whereas that of the cosmological constant is kept constant. Since the former gives an attractive contribution and the latter a repulsive one, any displacement from the original equilibrium position will grow, rendering such an original position unstable. But even without such classical perturbations this universe is unstable due to quantum mechanics. These are the instabilities we will focus on, in the spirit discussed at the end of the introduction.

Let us consider a free (i.e., with no potential) scalar field \( \Phi \), with mass \( \mu \) and with a coupling (not necessarily conformal) to the Ricci scalar of the background \( \mathcal{R} = 6/R^2 \), governed by

\[
(\Box - \xi \mathcal{R}) \Phi = \mu^2 \Phi.
\]

Conformal coupling is obtained in four spacetime dimensions by taking the coefficient \( \xi = 1/6 \) (and the theory is then conformal if \( \mu = 0 \)), whereas minimal coupling corresponds to \( \xi = 0 \). The compactness of the spatial sections of the background guarantees a discrete mode spectrum which can be easily obtained using elementary group theory. We take the d’Alembertian in the form

\[
\Box = -\frac{\partial^2}{\partial t^2} + \frac{4}{R^2} \left( (k_1^2)^2 + (k_2^2)^2 + (k_3^2)^2 \right) = -\frac{\partial^2}{\partial t^2} + \frac{4}{R^2} k^2,
\]

\(^1\) Recent works on the Casimir effect on the ESU are, e.g., [4–6] whereas works on the regularization of the vacuum energy in spherical cavities are, e.g., [7–9].
where $k^R$ are the right vector fields dual to $\sigma^R_i$ and $k^2$ is one of the two Casimirs in $SO(4)$. Note that the eigenfunctions of the Klein–Gordon operator $(\Box - \xi R - \mu^2)$ may be taken in the form
\[ \Phi_n = e^{-i\omega_n t} D_j^{m_L, m_R}, \tag{1} \]
where the index $n$ represents all quantum numbers $j, m_L, m_R$ and $D_j^{m_L, m_R}$ represents a Wigner $D$-function \cite{10}. Such a function may be thought of as a spherical harmonic on the 3-sphere or as a matrix element of the rotation operator $\langle j, m_L | \hat{R}(\alpha, \beta, \gamma) | j, m_R \rangle$, where $| j, m \rangle$ is the basis of a representation of $SU(2)$ and $(\alpha, \beta, \gamma)$ are Euler angles. It follows straightforwardly that the dispersion relation becomes
\[ \omega_j^2 = \frac{(2j + 1)^2 + 6\xi - 1}{R^2} + \mu^2, \quad j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots \tag{2} \]
with the degeneracy of each frequency being $d_j = (2j + 1)^2$, in agreement with the spectrum found in \cite{12}. For the case of a minimally coupled field, this spectrum was first found by Schrödinger \cite{11}. Note that there are no unstable modes for $\xi \in \mathbb{R}^+_0$, which includes minimal and conformal coupling. This is the range of couplings which we will analyse in the following.

Canonical quantization of the scalar field can be performed unambiguously. One finds the mode expansion
\[ \Phi = \sum_n \hat{a}_n^+ \Psi_n + \hat{a}_n \Psi_n^*, \quad \Psi_n = \sqrt{\frac{2j + 1}{2\omega_j} V} \Phi_n, \]
with $V = 2\pi^2 R^3$ being the volume of the constant $t$ hypersurfaces and with the operators $\hat{a}_n, \hat{a}_n^+$ obeying the usual commutation relation $[\hat{a}_n, \hat{a}_m^+] = \delta_{nm}$.

The classical energy–momentum tensor of the scalar field is more conveniently written in the natural tetrad basis $e^i = \{ dt, \frac{R\sigma_1}{R}, \frac{R\sigma_2}{R}, \frac{R\sigma_3}{R} \}$.

\[ T_{ab} = k_a \Phi k_b \Phi - \frac{g_{ab}}{2} k_c \Phi k^c \Phi - \frac{\mu^2}{2} g_{ab} \Phi^2 + \xi (G_{ab} - \nabla_a k_b + g_{ab} \Box) \Phi^2, \tag{3} \]

where we have denoted $k_a = \{ \partial/\partial t, k^R \}$. The conformal case ($\xi = 1/6, \mu = 0$) has zero trace; quantum mechanically, however, the renormalized energy–momentum tensor for a conformally coupled, free massless scalar field generically develops a trace anomaly, which can be written solely in terms of geometric quantities of the background \cite{15, 16}. In four spacetime dimensions such an anomaly takes the form
\[ \langle T^a_\mu \rangle_{\text{ren}} = \frac{1}{120(4\pi)^2} \left\{ C_{abcd} C^{abcd} - \frac{1}{3} (R_{abcd} R^{abcd} - 4 R_{ab} R^{cd} + 4 R^2) \right\} + \chi \nabla^2 R. \tag{4} \]

The coefficient $\chi$ is renormalization scheme dependent. This trace is zero for the ESU: the Ricci scalar is constant, the Weyl tensor $C_{abcd}$ vanishes since the geometry is conformally flat and the second Euler density also vanishes since the Euler characteristic of any odd dimensional sphere is zero.

3. Renormalization

Denoting the non-vanishing components of $\langle T^a_\mu \rangle$ as $\langle T^0_0 \rangle \equiv -\rho$ and $\langle T^1_1 \rangle = \langle T^2_2 \rangle = \langle T^3_3 \rangle \equiv p$, we find the unrenormalized quantities
\[ \rho_0 = \frac{1}{V} \sum_{n=1}^{\infty} n^2 \omega_n^2, \quad p_0 = \frac{1}{V} \sum_{n=1}^{\infty} n^2 \omega_n^2 - \frac{\mu^2}{6\omega_n}. \tag{5} \]
with
\[ \omega_n = \frac{\sqrt{n^2 + \mu^2 R^2 + 6\xi} - 1}{R}, \quad n \in \mathbb{N}. \] (6)

In [13], the particular case with \( \xi = 1/6 \) was studied. These infinite quantities were renormalized by introducing a damping factor \( e^{-\beta n} \) in the sums, subtracting the flat space contribution and taking the parameter \( \beta \) to zero. The result is
\[ \rho_{\text{ren}} = \frac{1}{4\pi^2 R^4} \left\{ \sum_{n=1}^{N-1} n^3 \left[ 1 + \left( \frac{\mu R}{n} \right)^2 \right] - n^3 - \frac{(\mu R)^2}{2n} + \frac{(\mu R)^4}{8n} \right\} + \sum_{n=N}^{\infty} \sum_{k=3}^{\infty} b_k \frac{(\mu R)^{2k}}{n^{2k-3}} \]

\[ + \frac{1}{120} - \frac{(\mu R)^2}{24} - \frac{(\mu R)^4}{32} \left( 4 \ln \frac{\mu R}{2} + 1 + 4\gamma \right), \]

(7)

\[ p_{\text{ren}} = \frac{1}{12\pi^2 R^4} \left\{ \sum_{n=1}^{N-1} n^3 \left[ \frac{n^2}{\sqrt{1 + \left( \frac{\mu R}{n} \right)^2}} \right] - n^3 - \frac{(\mu R)^2}{2n} + \frac{3(\mu R)^4}{8n} \right\} + \sum_{n=N}^{\infty} \sum_{k=3}^{\infty} c_k \frac{(\mu R)^{2k}}{n^{2k-3}} \]

\[ + \frac{1}{120} + \frac{(\mu R)^2}{24} + \frac{(\mu R)^4}{32} \left( 12 \ln \frac{\mu R}{2} + 7 + 12\gamma \right), \]

where \( \gamma \) is Euler’s constant, \( N \) is an integer such that \( N > \mu R \) and the explicit formulae for the coefficients \( b_k, c_k \) are
\[ b_k = \frac{(-1)^{k-1}(2k - 3)!!}{2^k k!}, \quad c_k = \frac{(-1)^{k}(2k - 1)!!}{2^k k!}. \]

These expressions are not very enlightening; therefore, we plot the renormalized energy density and pressure, for fixed radius, in terms of the mass in figure 1. The point we would like to emphasize is that they are always positive. Therefore, we conclude that a conformally coupled scalar field with arbitrary mass does not violate the strong energy condition and hence produces solely an attractive effect in the universe.

At this point, let us make a comment which also works as a consistency check. The first law of thermodynamics yields \( p = -\partial E/\partial V \), where the total energy \( E = \rho V = \rho 2\pi^2 R^3 \). We can easily check that this holds both for the infinite unrenormalized quantities (5) and the finite renormalized ones (7)

\[ p_0 = -\frac{1}{3R^2} \frac{\partial (R^3 \rho_0)}{\partial R}, \quad p_{\text{ren}} = -\frac{1}{3R^2} \frac{\partial (R^3 \rho_{\text{ren}})}{\partial R}. \]

Thus, we could have only computed the energy density and have deduced the pressure via the first law of thermodynamics, which obviously only states the conservation of energy \( T^{\mu\nu}_{\mu\nu} = 0 \). This is exactly what shall be done in the cases to follow.

As a check on the result (7) let us consider another renormalization method, namely, the zeta function. We can write the regularized expression as
\[ \tilde{\rho}(s) = \frac{\mu_0^s}{4\pi^2 R^{3-s}} \sum_{n=1}^{\infty} \frac{n^2}{(n^2 + a^2)^{s/2}}, \]

(8)
where \( \mu_0 \) has dimension of mass and
\[ a^2 = \mu^2 R^2 + 6\xi - 1, \]
which, for \( a^2 \geq 0 \), has the appropriate form to be written in terms of Epstein–Hurwitz zeta functions

\[
\bar{\rho}(s) = \frac{\mu_{a+1}^2}{4\pi^2 R^{3-s}} (\zeta_{EH}(s/2 - 1, a^2) - a^2 \zeta_{EH}(s/2, a^2)). \tag{9}
\]

In order to obtain the renormalized energy density, we should now consider the limit of this expression when \( s \to -1 \).

The Epstein–Hurwitz zeta function is only defined by the infinite sum \( \sum_{n=1}^{\infty} (n^2 + a^2)^{-s} \) for \( \text{Re}(s) > 1/2 \). Its analytic continuation to other values of \( s \), which converges for all \( s \) with the exception of an infinite number of poles, is given by (see, e.g., [18])

\[
\zeta_{EH}(s, a^2) = \frac{1}{2(a^2)^s} + \frac{\sqrt{\pi}}{2} \frac{\Gamma(s - 1/2)}{\Gamma(s)} (a^2)^{1/2-s} + \frac{2\pi^4 (a^2)^{1/4-s/2}}{\Gamma(s)} \sum_{n=1}^{\infty} n^{s-1/2} K_{s-1/2}(2\pi n \sqrt{a^2}),(10)
\]

where \( K_v \) are modified Bessel functions and \( a^2 \) is required to be positive. It is simple to see that the infinite series converges for all \( s \), since the modified Bessel functions have asymptotic behaviour

\[ K_v(z) \sim \frac{\sqrt{\pi}}{2z} e^{-z}, \quad |z| \to \infty. \]

The poles of \( \zeta_{EH}(s, a^2) \) arise at the poles of \( \Gamma(s - 1/2) \), that is at \( s = 1/2 - n, n \in \mathbb{N}_0 \).

Since \( a^2 \) must be positive this technique does not apply in the case of a massless minimally coupled scalar field. Let us focus on the case with \( a^2 > 0 \). Applying (10) to (9), we find

\[
\bar{\rho}(s) = \frac{\mu_{a+1}^2 R^4}{2V} \left( \frac{\sqrt{\pi}}{4} \frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s}{2} + \frac{1}{2}\right)} \sqrt{a^2}^{1-s} + \frac{2\pi^4}{\sqrt{\pi}} \frac{\Gamma\left(\frac{s}{2} - \frac{1}{2}\right)}{\Gamma\left(\frac{s}{2} - 1\right)} \sum_{n=1}^{\infty} \left( \frac{2\pi n}{\sqrt{a^2}} \right)^{s-\frac{1}{2}} K_{s-\frac{3}{2}}(2\pi n \sqrt{a^2}) \right) + \frac{a^2}{2-s} \left( \frac{2\pi n}{\sqrt{a^2}} \right)^{s-\frac{1}{2}} K_{s-\frac{3}{2}}(2\pi n \sqrt{a^2}) \right). \tag{11}
\]
The point $s = -1$ is exactly at one of the poles of the analytic continuation of the Epstein–Hurwitz zeta function. That is, the right-hand side of the last formula is an infinite quantity, and it might seem that the zeta function method fails. With the correct interpretation this is not so. Let us consider separately two cases:

- **Conformal coupling** ($\xi = 1/6$) and arbitrary mass. This case is obtained by substituting $a^2 = \mu^2 R^2$ in (11). The infinite contribution, i.e., the term with $\Gamma(-2)$ in the $s = -1$ limit, is exactly the $R$-independent term. Thus, despite its (infinite) contribution to the Casimir energy it will not contribute to the Casimir force. This argument would be enough to neglect it, and to suspect this is the flat space contribution. To confirm this is indeed the case compute the flat space result by taking the infinite radius limit of (8):

$$\lim_{R \to +\infty} \tilde{\rho}(s) = \frac{\mu^2 R^2}{4\pi^2} \int_0^{+\infty} dk \, k^2 (k^2 + \mu^2)^{-s/2} = \frac{\mu^2 R^2}{4\pi^2} \frac{\Gamma\left(\frac{s-3}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} (\mu)^{3-s}.$$ 

This is exactly the infinite term obtained after zeta function regularization.

- **Nonconformal coupling.** In the generic case, the divergent term cannot be identified with the flat space contribution, since it does depend on $R$. However, it turns out that the correct result is still obtained by simply dropping out the divergent terms, as will be shown below. Doing so, the final answer for the renormalized vacuum energy density of a scalar field with generic coupling is

$$\rho_{\text{ren}} = \frac{1}{4\pi^2 R^4} \sum_{n=1}^{+\infty} \left( \frac{3(\mu^2 R^2 + 6\xi - 1)}{2\pi^2 n^2} K_{-\frac{3}{2}}(2\pi n \sqrt{\mu^2 R^2 + 6\xi - 1}) \right.$$

$$+ \left. \frac{(\mu^2 R^2 + 6\xi - 1)^{3/2}}{\pi n} K_{-1}(2\pi n \sqrt{\mu^2 R^2 + 6\xi - 1}) \right). \quad (12)$$

An expression equivalent to (12) was first derived, for the particular case of $\xi = 1/6$, in [19], using a different renormalization technique. It was argued therein that such a renormalized energy–momentum tensor could lead to a self-consistent ESU solution of the semiclassical Einstein equations. We shall come back to this point in section 4.

The advertised check on our calculation of the Casimir energy for a massive, conformally coupled, scalar in the ESU can be performed graphically; plotting the result (12) for $\xi = 1/6$, we have checked that it perfectly coincides with the previous one (7), plotted in figure 1.

The zeta function method confirmed the damping factor method used originally in [13]. But it is unfortunately inapplicable to the most general situation when $a^2 < 0$ which includes the interesting case of a massless minimally coupled scalar field. We will now study this case by a variation of the damping factor method. The essential ingredient will be the Abel–Plana formula which we use in the form [20]

$$\sum_{m=b}^{+\infty} G(m) = \int_{b}^{+\infty} G(t) \, dt + \frac{G(b)}{2} + i \int_{0}^{+\infty} \frac{G(it + b) - G(-it + b)}{\exp(2\pi t) - 1} \, dt. \quad (13)$$

We regularize the vacuum energy density in the following way:

$$\tilde{\rho}(\beta) = \frac{1}{4\pi^2} \sum_{n=1}^{+\infty} \frac{n^2}{R^4} \sqrt{\frac{n^2}{R^2} + \frac{\mu^2 R^2 + 6\xi - 1}{R^2}} e^{-\beta \Omega(n/R)}, \quad (14)$$

where $\Omega$ is a positive function of its argument that grows sufficiently fast with $n$ so as to make the sum converge for any $\beta > 0$. A convenient choice obeys

$$\frac{d \Omega}{d \left(\frac{n}{R}\right)} = \frac{n^2}{R^2} \sqrt{\frac{n^2}{R^2} + \frac{\mu^2 R^2 + 6\xi - 1}{R^2}};$$
it can be written explicitly as

\[ \Omega \left( \frac{n}{R} \right) = \frac{1}{8} \left\{ \frac{n^2}{R^2} + \frac{\mu^2 R^2}{R^2} + 6 \xi - 1 \right\}^{3/2} - \frac{\mu^2 R^2}{R^2} + 6 \xi - 1 \left( \frac{n^2}{R^2} + \frac{\mu^2 R^2 + 6 \xi - 1}{R^2} \right)^{1/2} \]

\[ - \left( \frac{\mu^2 R^2 + 6 \xi - 1}{R^2} \right)^2 \ln \left( \frac{n^2}{R^2} + \left( \frac{\mu^2 R^2 + 6 \xi - 1}{R^2} \right)^{1/2} \right) \].

The integration constant has been chosen such that \( \Omega \to 0 \) as the frequency vanishes, which means that the damping factor is not altering the long wavelength modes. This seems to be the most physical choice, since these long wavelength modes are the ones responsible for the Casimir energy. We divide the situation with a stable spectrum into two cases which we analyse separately:

- \( \mu^2 R^2 + 6 \xi - 1 > 0 \) (includes conformal coupling with arbitrary mass). Applying the Abel–Plana formula (13) with \( b = 0 \) to the regularized expression (14), we find

\[ \tilde{\rho} (\beta) = \frac{1}{4 \pi^2} \int_0^{+\infty} d\Omega \, e^{-\beta \Omega} \]

\[ - \frac{i}{4 \pi^2 R^4} \int_0^{+\infty} t^2 \left[ \frac{\mu^2 R^2 + 6 \xi - 1}{R^2} (\bar{\Omega}(\beta / R, x) - \bar{\Omega}(\beta / R, x)) e^{-\beta \bar{\Omega}(\beta / R)} \right] \]

The first integral became exactly the contribution of flat space (which justifies our choice of \( \Omega \)). Indeed, from (14)

\[ \lim_{R \to +\infty} \tilde{\rho} (\beta) = \frac{1}{4 \pi^2} \int_0^{+\infty} dx \, x^2 \sqrt{x^2 + \mu^2} e^{-\beta \bar{\Omega}(x)} = \frac{1}{4 \pi^2} \int_0^{+\infty} d\Omega \, e^{-\beta \Omega}, \]

where

\[ \bar{\Omega}(x) = \lim_{R \to +\infty} \Omega(x). \]

The second integral, where \( a = \sqrt{\mu^2 R^2 + 6 \xi - 1} \), converges in the \( \beta \to 0 \) limit. Thus,

\[ \rho_{\text{ren}} = - \frac{i}{4 \pi^2 R^4} \int_0^{+\infty} t^2 \left[ \frac{\mu^2 R^2 + 6 \xi - 1}{R^2} (\bar{\Omega}(\beta / R, x) - \bar{\Omega}(\beta / R, x)) \right] \]

The square root representation in the complex plane that has been used has a branch cut for \( t \in [-a, a] \). Splitting the integral into \( \int_{-a}^{a} + \int_{a}^{+\infty} \), one can check that only the second one contributes. The final result is

\[ \rho_{\text{ren}} = \frac{1}{2 \pi^2 R^4} \int_{\sqrt{\mu^2 R^2 + 6 \xi - 1}}^{+\infty} t^2 \sqrt{t^2 - (\mu^2 R^2 + 6 \xi - 1)} \exp(2 \pi t) - 1 \]

This expression was first obtained for the special case of conformal coupling \( \xi = 1/6 \) in [17] (see also [2]). Specializing for such coupling, the plot of this function exactly coincides with the ones in figure 1, again checking the result. It is quite striking that expressions apparently as distinct as (7), (12) and (16) are actually different representations of the same function.

For a more general coupling with \( \mu^2 R^2 + 6 \xi - 1 > 0 \), we can also check that the result (16) coincides with the one computed with the zeta function method (12), where we dropped the divergent term, even though it could not be clearly interpreted as the flat space contribution.
Figure 2. Renormalized total vacuum energy $2\pi^2 \rho_{\text{ren}}$ and pressure $2\pi^2 p_{\text{ren}}$ for $R = 1$, as a function of $\mu \in [0, 1.5]$, for six different couplings $\xi \in [0, 1/6]$. As $\xi$ increases the colour of the line in the corresponding graph becomes darker. $\xi = 0$ corresponds to the most negative curve in both graphs.

Such a term should renormalize the gravitational action in the way described in [21]. Since we are mostly interested in the renormalized energy–momentum tensor, which will be determined by the finite part of the result, we will not dwell any longer on this point.

- $-1 \leq \mu^2 R^2 + 6\xi - 1 < 0$ (includes minimal coupling with zero mass). Applying the Abel–Plana formula (13) with $b = 1$ to the regularized expression (14), we find

$$\bar{\rho}(\beta) = \int_{\Omega(\frac{1}{R})}^{+\infty} \frac{d\Omega}{4\pi^2} e^{-\beta \Omega} + \frac{\sqrt{\mu^2 R^2 + 6\xi}}{8\pi^2 R^4} e^{-\beta \Omega(\frac{1}{2})} + \frac{i}{4\pi^2 R^4} \int_{0}^{+\infty} \frac{f_\beta(1 + i t) - f_\beta(1 - i t)}{\exp(2\pi t) - 1} dt,$$

where

$$f_\beta(x) \equiv x^2 \sqrt{x^2 + \mu^2 R^2 + 6\xi} - 1 e^{-\beta \Omega(\frac{1}{2})}.$$

The flat space contribution is still (15), when $\mu^2 > 0$. For the tachyonic case, the flat space contribution is more subtle, so we will restrict our analysis to $-1 \leq \mu^2 R^2 + 6\xi - 1 < 0$ and $\mu^2 > 0$. Subtracting the flat space quantity and removing the regulator $\beta$, we find the renormalized energy density

$$\rho_{\text{ren}} = -\frac{1}{32\pi^2 R^4} \left[ (\mu^2 R^2 + 6\xi)^2 - 3\sqrt{\mu^2 R^2 + 6\xi} - (\mu^2 R^2 + 6\xi - 1)^2 \right] \times \ln \left( \frac{1 + \sqrt{\mu^2 R^2 + 6\xi}}{\sqrt{|1 - 6\xi - \mu^2 R^2|}} \right) + \frac{i}{4\pi^2 R^4} \int_{0}^{+\infty} \frac{f_0(1 + i t) - f_0(1 - i t)}{\exp(2\pi t) - 1} dt. \quad (17)$$

Despite the apparent unfriendly expression, this is a real quantity which can be plotted without great difficulty. Using the last formula and (16) we have plotted several cases, with different $\xi$s, in figure 2. The most noticeable feature is that both the renormalized energy density and pressure may become negative for a range of values. Clearly, this leads to violations of the strong energy condition, as will be discussed in more detail in the following section. Let us close by remarking that for zero mass, the renormalized energy density and pressure can be written, for arbitrary coupling, in the form

$$\rho_{\text{ren}} = \frac{\alpha}{R^5}, \quad p_{\text{ren}} = \frac{\alpha}{3R^5}. \quad (18)$$
Casimir energy and a cosmological bounce

\[ \alpha \equiv -\frac{1}{32\pi^2} \left[ (6\xi)^2 - 3\sqrt{6\xi} - (6\xi - 1)^2 \ln \left( \frac{1 + \sqrt{6\xi}}{\sqrt{1 - 6\xi}} \right) \right] + \frac{i}{4\pi^2} \int_0^{\infty} \frac{f_0(1 + it) - f_0(1 - it)}{\exp(2\pi t) - 1} \, dt. \] (19)

The quantity \( \alpha \) is plotted as a function of \( \xi \) in figure 3. In particular, the special value of the coupling where \( \alpha \) changes sign corresponds to a theory where the renormalized energy–momentum tensor vanishes in the ESU. Its numerical value is \( \xi_c \simeq 0.05391 \). In figure 4, some energy conditions are displayed.
4. Discussion

The main purpose of this paper was to point out that the Casimir effect in the ESU becomes repulsive for a family of scalar fields with various couplings to the Ricci scalar and masses. This can lead to an inflationary era in the early universe, which generically seems to be too short to solve the usual big bang model problems. More interestingly, it can lead to a cosmological bounce. To understand how, let us briefly consider the backreaction problem by taking the semiclassical Einstein equations,

\[ G_{\mu\nu} + \Lambda g_{\mu\nu} = T^{(\text{matter})}_{\mu\nu} + \langle T_{\mu\nu}^\phi \rangle. \]

Analysing the simple massless case (18), for \( \alpha > 0 \), we conclude that the quantum fluid can support a self-consistent Einstein static universe with some radius, a fact first noticed in [19]. Considering the Friedmann equation and Raychaudhuri equations (reinserting Newton’s constant)

\[ \ddot{R}^2 + k = \frac{8\pi G}{3} \rho R^2, \quad \ddot{R} = -\frac{4\pi G}{3} (\rho + 3p)R, \]

and taking the quantum fluid and a positive cosmological constant

\[ \rho = \Lambda + \frac{\alpha}{R^4}, \quad p = -\Lambda + \frac{\alpha}{3R^4}, \]

a self-consistent solution is obtained with

\[ k = +1, \quad \Lambda = \frac{\alpha}{R^4}, \quad R = \sqrt{\frac{16\pi G \alpha}{3}}. \]  

(20)

Clearly, this solution suffers from fine tuning and is a universe of the order of the Planck size. A stable ESU was recently obtained in the context of loop quantum cosmology in [22]. For the case with \( \alpha < 0 \), the fluid can, with the help of dust and a positive cosmological constant, produce an inflationary era followed by a decelerating phase and the present accelerating era. Indeed, taking

\[ \rho = \Lambda - \frac{|\alpha|}{R^4} + \frac{\eta}{R^3}, \quad p = -\Lambda - \frac{|\alpha|}{3R^4}, \]

\[ \dot{R}^2 + k = \frac{8\pi G}{3} \rho R^2, \quad \ddot{R} = -\frac{4\pi G}{3} (\rho + 3p)R, \]

Figure 5. Left: scale factor for a universe with a cosmological constant, dust and the quantum fluid of a massless scalar field; right: detail near \( t = 0 \) clearly showing the bounce structure.
where $\eta > 0$ characterizes the dust, one finds the generic behaviour displayed in figure 5. The universe has a minimal radius $R = R_c$ where it bounces from a collapsing to an expanding epoch, thus avoiding the big bang singularity. For $R > R_c$ there is a short accelerating phase, which is followed by a matter era and then cosmological constant domination$^2$.

What is most interesting on having this Casimir-energy-induced bounce is that it does not rely on any particular form of a potential, and the scalar field need not have any classical effects whatsoever. It is its very existence that originates the effect. Of course, this is by no means a complete cosmological model of our universe. One obvious problem is how to include the radiation era in the picture, which has the same $1/R^4$ dependence for the energy density as the Casimir energy–momentum tensor. Indeed, nucleosynthesis data constrain the influence that the quantum fluid could have over that period and therefore, since they vary equally with $R$, over any period where they both exist. A way around this problem could be to assume that radiation is only created after the quantum fluid domination epoch. Such a mechanism could be similar to the usual reheating at the end of inflation.

In any case, this simple model illustrates the point that the Casimir energy could both source an early inflationary epoch and avoid the big bang singularity in a closed universe. Note that despite having zero mass $\mu$, the non-minimal coupling $\xi$ works in the Einstein static universe as an effective mass of order $1/R^2$. Thus, this massless case can have its classical dynamics frozen during inflation due to such an effective mass.

It would certainly be interesting to further generalize this analysis to the case of a non-static FRW model where the dynamical Casimir effect takes place.

Acknowledgments

We are very grateful to Pedro Avelino, Filipe Paccetti Correia, Malcolm Perry and Gary Gibbons for discussions and suggestions. We would especially like to thank J S Dowker for reading the manuscript. CH is supported by FCT through the grant SFRH/BPD/5544/2001. MS was supported by the Marie Curie research grant MERG-CT-2004-511309. This work was also supported by Fundação Calouste Gulbenkian through Programa de Estímulo à Investigação and by the FCT grants POCTI/FNU/38004/2001 and POCTI/FNU/50161/2003. Centro de Física do Porto is partially funded by FCT through POCTI programme.

References

[1] Casimir H B G 1948 On the attraction between two perfectly conducting plates Proc. Kon. Nederl. Akad. Wet. 51 793
[2] Bordag M, Mohideen U and Mostepanenko V M 2001 New developments in the Casimir effect Phys. Rep. 353 1 (Preprint quant-ph/0106045)
[3] Zeldovich Y B and Starobinsky A A 1984 Quantum creation of a universe in a nontrivial topology Sov. Astron. Lett. 10 135
[4] Elizalde E and Tort A C 2004 A note on the Casimir energy of a massive scalar field in positive curvature space Mod. Phys. Lett. A 19 111 (Preprint hep-th/0306049)
[5] Brevik I, Milton K A and Odintsov S D 2002 Entropy bounds in spherical space Preprint hep-th/0210286
[6] Altaie M B and Setare M R 2003 Finite-temperature scalar fields and the cosmological constant in an Einstein universe Phys. Rev. D 67 044018 (Preprint gr-qc/0301009)
[7] Cognola G, Vanzo L and Zerbini S 1992 Vacuum energy in arbitrarily shaped cavities J. Math. Phys. 33 222
[8] Bordag M, Elizalde E, Kirsten K and Leseduarte S 1997 Casimir energies for massive fields in the bag Phys. Rev. D 56 4896 (Preprint hep-th/9608071)

$^2$ Recent examples of bouncing universes in string-inspired gravity are [23, 24].
[9] Cognola G, Elizalde E and Kirsten K 2001 Casimir energies for spherically symmetric cavities J. Phys. A: Math. Gen. 34 7311 (Preprint hep-th/9906228)

[10] Edmonds A R 1974 Angular Momentum in Quantum Mechanics 3rd edn (Princeton, NJ: Princeton University Press) chapter 4

[11] Schrödinger E 1938 Comment. Pont. Acad. Sci. 2 321

[12] Ford L H 1975 Quantum vacuum energy in general relativity Phys. Rev. D 11 3370

[13] Ford L H 1976 Quantum vacuum energy in a closed universe Phys. Rev. D 14 3304

[14] Birrell N D and Davies P C W 1982 Quantum Fields in Curved Space (Cambridge: Cambridge University Press) p 186

[15] Brown L S 1977 Stress tensor trace anomaly in a gravitational metric: scalar fields Phys. Rev. D 15 1469

[16] Wald R M 1978 Trace anomaly of a conformally invariant quantum field in curved space-time Phys. Rev. D 17 1477

[17] Mamayev S G, Mostepanenko V M and Starobinsky A A 1976 Sov. Phys.—JETP 43 823

[18] Nesterenko V V and Pirozhenko I G 1997 Justification of the zeta function renormalization in rigid string model J. Math. Phys. 38 6265 (Preprint hep-th/9703097)

[19] Dowker J S and Critchley R 1977 Vacuum stress tensor in an Einstein universe. Finite temperature effects Phys. Rev. D 15 1484

[20] Olver F W J 1974 Asymptotics and Special Functions (New York: Academic)

[21] Streeruwitz E 1975 Vacuum fluctuations of a quantized scalar field in a Robertson–Walker universe Phys. Rev. D 11 3378

[22] Mulryne D J, Tavakol R, Lidsey J E and Ellis G F R 2005 An emergent universe from a loop Phys. Rev. D 71 123512 (Preprint astro-ph/0502589)

[23] Lu H, Vazquez-Poritz J F and Wang J E 2004 De Sitter bounces Class. Quantum Grav. 21 4963 (Preprint hep-th/0406028)

[24] Biswas T, Mazumdar A and Siegel W 2005 Bouncing universes in string-inspired gravity Preprint hep-th/0508194