Linear harmonic oscillator in spaces with degenerate metrics

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Abstract

With the help of contraction method we study the harmonic oscillator in spaces with degenerate metrics, namely, on Galilei plane and in the flat 3D Cayley-Klein spaces $R_3(j_2,j_3)$. It is shown that the inner degrees of freedom are appeared which physical dimensions are different from the dimension of the space.

PACS: 02.40.Dr; 45.05.+x; 45.20.Jj

Keywords: Harmonic oscillator, Degenerate metrics, Fiber space

1 Introduction

Recently a family of classical superintegrable systems was defined on the spaces of constant curvature (or Cayley-Klein spaces) with nondegenerate metrics from classical groups [1,2], as well as from quantum groups [3,4]. But among Cayley-Klein spaces there are spaces with degenerate metrics, which are not considered in the these papers. Meanwhile, it is quite reasonable take one of the superintegrable systems, say harmonic oscillator, on the spherical space and study the chain of its contractions to Euclid space and then to Galilei space with degenerate metrics. Since contractions from the spherical space to the Euclid space for different classical superintegrable systems are discussed in [1,4], in this paper we consider contraction of harmonic oscillator from Euclid to Galilei plane and its behavior in the flat 3D Cayley-Klein spaces $R_3(j_2,j_3)$. 

2 Geometrical properties of Galilei plane

The metrics of three flat Cayley-Klein planes in Cartesian coordinates can be described in unified manner \[6\] as

\[
ds^2 = dx^2 + j^2 dy^2,
\]

where parameter \(j = 1, \iota, i\). Euclid plane with signature \((+, +)\) is obtained for \(j = 1\), Minkowski plane with pseudoeuclidean metrics of signature \((+, -)\) is obtained for \(j = i\) and nilpotent value of the parameter \(j = \iota, \iota^2 = 0, \iota/\iota = 1\) correspond to Galilei plane with degenerate metrics of signature \((+, 0)\). Galilei plane is the simplest fiber space with 1D base \(\{x\}\) and 1D fiber \(\{y\}\), therefore has two independent metrics: first in the base and second in the fiber

\[
ds_{b}^2 = dx^2, \quad ds_{f}^2 = \frac{1}{j^2}ds^2|_{dx=0} = \frac{1}{j^2} \left(dx^2 + j^2 dy^2\right)|_{dx=0} = dy^2.
\]

A bundle of lines through a point on three Cayley-Klein planes has different properties relative to the plane automorphism \[5\]. On Euclid plane, any two lines of the bundle are transformed to each other by rotation around the point. On Galilei plane, there is one isolated line that do not superposed with any other line of the bundle by Galilei boost. On Minkowski plane, there are two isolated lines that are invariant with respect to Lorentz transformations.

![Figure 1: A bundle of lines on Euclid E₂, Galilei G₂ and Minkowski M₂ planes.](image)

If one interpret these planes in some physical context, then on Euclid plane all lines must have the same physical dimension \([x] = [y]\). On Galilei
plane, there are infinite many lines with physical dimension identical with dimension of the base \([x]\) and one isolated line in the fiber with some different physical dimension \([y] \neq [x]\). On Minkowski plane, there are three types of lines, namely, \(x\)-like, \(y\)-like and zero-like \((x = \pm y)\) that can be used for modelling of three physically different quantities. The most known but not unique interpretations of Galilei and Minkowski planes are kinematical one, when \(x\)-axis is interpreted as time and \(y\)-axis is interpreted as space of \((1+1)\) kinematics.

The flat 3D Cayley-Klein spaces \(R_3(j_2, j_3)\) are defined by the metrics
\[
ds^2 = dx^2 + j_2^2 dy^2 + j_2^2 j_3^2 dz^2,
\]
where parameters \(j_2 = 1, \tau_2, i, j_3 = 1, \tau_3, i, \tau_2^2 = \tau_3^2 = 0, \tau_2 \tau_3 = \tau_3 \tau_2 \neq 0, \tau_k/\tau_k = 1, k = 2, 3\) [6]. These spaces provide more possibilities for unification of different physical quantities within the bounds of one geometry. In particular, for \(j_2 = \tau_2, j_3 = \tau_3\) doubly fiber space \(R_3(\tau_2, \tau_3)\) with two projections is obtained. The first projection has 1D base \(\{x\}\) and 2D fiber \(\{y, z\}\), the second projection acts in 2D fiber \(\{y, z\}\) and has 1D base \(\{y\}\) and 1D fiber \(\{z\}\). There are three independent metrics
\[
ds^2_b = dx^2, \quad ds^2_1 = \frac{1}{j_2^2} ds^2|_{dx=0} = dy^2, \quad ds^2_2 = \frac{1}{j_2^2 j_3^2} ds^2|_{dx=dy=0} = dz^2,
\]
therefore three different physical quantities \([x] \neq [y] \neq [z]\) can be modelled by the space \(R_3(\tau_2, \tau_3)\).

Unlike euclidean and pseudoeuclidean geometries, where only one and three physical quantities can be modelled, respectively, fiber geometries enable to unify arbitrary many different physical quantities under appropriate dimensions and fibers.

3 Harmonic oscillator in Galilei plane

The action for the linear harmonic oscillator in Euclid plane reads as
\[
S^* = \int_{t_1^*}^{t_2^*} \left\{ \frac{m^*}{2} \left( \dot{x}^* \dot{x}^* + \dot{y}^* \dot{y}^* \right) - \gamma^* \left( x^* x^2 + y^* y^2 \right) \right\} dt^*,
\]
where by star * are marked the initial euclidean variables.

To obtain the action for Galilei plane we use the method of unified description of Cayley-Klein spaces, groups, algebras etc. [6]. The main idea
of this method is that construction suitable for all Cayley-Klein cases can be obtained from an analogous construction for spherical space, orthogonal group, orthogonal algebra etc. by an appropriate transformation with the help of contraction parameters.

Let us transform Cartesian coordinates as follows: \( x^* = x, \ y^* = jy, \) then with \( t^* = t, \ m^* = m, \ \gamma^* = \gamma \) we obtain the action for the harmonic oscillator in the form

\[
S = S^*(\rightarrow) = \int_{t_1}^{t_2} \left\{ \frac{m}{2} \left( x^2 + j^2 y^2 \right) - \gamma \left( x^2 + j^2 y^2 \right) \right\} \, dt, \tag{6}
\]

where the arrow \((\rightarrow)\) means that transformed variables are substituted instead of initial ones. For \( j^2 = i^2 = 0, \) that is for Galilei plane, the one dimensional linear oscillator along the base \( \{x\} \) is described by the action

\[
S_b = \int_{t_1}^{t_2} \left\{ \frac{m}{2} x^2 - \gamma x^2 \right\} \, dt. \tag{7}
\]

In standard classical mechanics the time \( t \) is the real continuous parameter associated with nondegenerate metrics. There are two independent metrics on Galilei plane, so it looks quite natural to introduce two real continuous parameters. One of them associated with the metrics in the base as the time \( t \) and another one associated with the metrics in the fiber as the “time” \( \tilde{t} \).

To obtain the action for the fiber let us transform in addition to Cartesian coordinates the “time” \( t^* = j\tilde{t} \) and as before \( m^* = m, \ \gamma^* = \gamma, \) then

\[
S = \frac{1}{j} S^*(\rightarrow) = \int_{\tilde{t}_1}^{\tilde{t}_2} \left\{ \frac{m}{2} \left( \frac{1}{j^2} \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 \right) - \gamma \left( x^2 + j^2 y^2 \right) \right\} \, d\tilde{t}. \tag{8}
\]

As it is easy to see, the same action \( \boxed{\text{Eq. (8)}} \) can be obtained by the mass renormalization \( m = j^2 m^* \) with untouched “time” \( t^* = \tilde{t}, \ S = S^*(\rightarrow) \) and \( L = L^*(\rightarrow), \) where \( L \) is Lagrangian. Therefore this approach can be used when instead of an action a Lagrangian is regarded.

For nilpotent value of the parameter \( j = i \) this mass renormalization is similar to the mass renormalization in \( \phi^4 \) quantum field theory (compare with Eq. (9.36) in \( \boxed{\text{[7]}} \))

\[
m^2 = m_1^2 \left( 1 + \frac{g}{16\pi^2\epsilon} \right) \xrightarrow{\epsilon \to 0} m^2 = \frac{m_1^2}{\epsilon^4}, \tag{9}
\]

where \( m_1 \) is physical mass, \( m \) is unobserved mass and \( \epsilon' \approx i^4. \)
To avoid an indefinite expressions in the action (8) it is necessary to put
\( dx = 0 \), which define the fiber \( x = x_0 = \text{const} \), then \( dx/d\tilde{t} = 0 \). After that
the action
\[
S_f = \int_{\tilde{t}_1}^{\tilde{t}_2} \left\{ \frac{m}{2} \left( \frac{dy}{d\tilde{t}} \right)^2 - \gamma x_0^2 \right\} d\tilde{t}
\] (10)
describe the free "'motion'' \( y = u_0 \tilde{t} + y_0 \) in this fiber. Here \( u_0, y_0 \) are integration constants. So instead of one parameter harmonic oscillator trajectory in Euclid plane we have two parameter family of trajectories, that fill up the band \( 2A \times R \) in Galilei \( G_2 \) plane (see Fig.2).

![Figure 2: Two parameter family of harmonic oscillator trajectories in Galilei \( G_2 \) plane.](image)

If one interpret the base as the space axis \([x] = \text{[space]}\), then the fiber must have some different physical dimension \([y] \neq \text{[space]}\) and can be regarded as some inner degree of freedom.

4 Harmonic oscillator in spaces \( R_3(j_2, j_3) \)

The harmonic oscillator in three dimensional Euclid space \( R_3 \) is described by the action
\[
S^* = \int_{t_1}^{t_2} \left\{ \frac{m^*}{2} \left( \dot{x}^* + y^* + z^* \right)^2 - \gamma^* \left( x^* + y^2 + z^* \right) \right\} dt^*.
\] (11)
The transition from \( R_3 \) to the space \( R_3(j_2, j_3) \) is given by the following transformation of Cartesian coordinates
\[
x^* = x, \quad y^* = j_2 y, \quad z^* = j_2 j_3 z
\] (12)
and the action is given by

\[ S = S^*(\to) = \int_{t_1}^{t_2} \left\{ \frac{m}{2} \left( \dot{x}^2 + j_2^2 \dot{y}^2 + j_3^2 \dot{z}^2 \right) - \gamma \left( x^2 + j_2^2 y^2 + j_3^2 \dot{z}^2 \right) \right\} dt, \]  

(13)

where \( t^* = t, \ m^* = m, \ \gamma^* = \gamma. \)

Let \( j_2 = 1, \ j_3 = \iota_3, \) that is the space \( \mathbb{R}^3(1, \iota_3) \) has 2D base \( \{ x, y \} \) and 1D fiber \( \{ z \}. \) The two dimensional linear oscillator in the base is described by the action

\[ S_b = \int_{t_1}^{t_2} \left\{ \frac{m}{2} \left( \dot{x}^2 + \dot{y}^2 \right) - \gamma \left( x^2 + y^2 \right) \right\} dt, \]  

(14)

which is identical with (5) and its trajectories are ellipses. With \( t^* = j_3 \tilde{t} \) the action (11) is transformed to

\[ S = \frac{1}{j_3} S^*(\to) = \int_{\tilde{t}_1}^{\tilde{t}_2} \left\{ \frac{m}{2} \left( \frac{dz}{d\tilde{t}} \right)^2 - \gamma \left( x^2 + y^2 + j_3^2 \dot{z}^2 \right) \right\} d\tilde{t}. \]  

(15)

If we put \( dx = dy = 0, \) which define the fiber \( x = x_0, y = y_0, \) then \( dx/d\tilde{t} = dy/d\tilde{t} = 0 \) and the action

\[ S_f = \int_{\tilde{t}_1}^{\tilde{t}_2} \left\{ \frac{m}{2} \left( \frac{dz}{d\tilde{t}} \right)^2 - \gamma \left( x_0^2 + y_0^2 \right) \right\} d\tilde{t} \]  

(16)

describe the free "motion" \( z = w_0 \tilde{t} + z_0 \) in this fiber. So we obtain two parameter family of trajectories, that fill up the elliptic cylinder in \( \mathbb{R}^3(1, \iota_3) \) (see Fig.3).

In the case of doubly fiber space \( \mathbb{R}^3(\iota_2, \iota_3) \) according with (11) the one dimensional linear oscillator along the main base \( \{ x \} \) is described by the action (11), the action for the base \( \{ y \} \) of the second projection is given by (10) and the action for the last fiber \( \{ z \} \) is obtained with the help of time transformation \( t^* = j_2 j_3 \tilde{t} \) in the form

\[ S = \frac{1}{j_2 j_3} S^*(\to)|_{dx=dy=0} = \int_{\tilde{t}_1}^{\tilde{t}_2} \left\{ \frac{m}{2} \left( \frac{dz}{d\tilde{t}} \right)^2 - \gamma x_0^2 \right\} d\tilde{t}. \]  

(17)

As a result, we have three parameter family of trajectories, that fill up the region \( 2A \times R \times R \) in the space \( \mathbb{R}^3(\iota_2, \iota_3). \) If one interpret the main base as the
space axis $[x] = \text{[space]}$, then the base $\{y\}$ in the first fiber and the second fiber $\{z\}$ all must have some different physical dimension $[z] \neq [y] \neq [x] = \text{[space]}$ and can be regarded as two inner degree of freedom of the system under consideration.

5 Conclusion

We have considered the harmonic oscillator in fiber Cayley-Klein spaces which have several independent metrics in each base and in each fiber. Correspondingly, several actions and several real continuous parameters (or several “times”) are appeared. The action in the main base is obtained by the coordinate transformations. The action in the fiber is obtained by the time transformation coupled with coordinate ones. In the case of Lagrangian the mass renormalization can be used instead of time transformation. The fiber coordinates can be interpreted as some inner degrees of freedom of the system which physical dimensions are different from the dimension of the base coordinates. In general, the geometries with degenerate metrics are suitable tools for unification of arbitrary many different physical quantities under appropriate dimensions and fibers.

Acknowledgements

The author would like to thank J.F. Cariñena, F.J. Herranz and V.V. Kura-tov for helpful discussions.
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