Online exploration outside a convex obstacle

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Abstract

A watchman route is a path such that a direct line of sight exists between each point in some region and some point along the path. Here, we study watchman routes outside a convex polygon, i.e., in $\mathbb{R}^2 \setminus O$, where $O$ is a convex polygon. We study the problem of a watchman route in an online setting, i.e., in a setting where the watchman is only aware of the vertices of the polygon to which it had a direct line of sight along its route. We present an algorithm guaranteeing a $\approx 89.83$ competitive ratio relative to the optimal offline path length.

1. Introduction

Exploring an unknown terrain or scanning a region of space are important tasks for autonomous robots. Many situations require exploring an unknown environment, or scanning a known environment for changes or intrusions. Some autonomous units, such as the mars rover \cite{1,2} and other space exploration vehicles are too far to control from earth, as the communication time is too long. In other cases communication is impossible due to interference or environmental conditions.

In this paper, we study the problem of scanning or exploring a region of the plane, where a convex polygonal obstacle is blocking the view and motion of the robot. We present an online algorithms guaranteeing a constant competitive ratio compared to the optimal offline path. In the offline setting the shape of the obstacle is known in advance, and an optimal path is desired, where this optimal path is the shortest path from which every point in the free space (space outside the obstacle) is viewable (a

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direct line of sight exists). For completeness, we present in the appendix a study of the offline case, and present an algorithm for finding the optimal path. Our main result is an algorithm for the online problem. In this case, the shape of the obstacle is not known in advance, and the purpose of the robot is to scan the region while studying the shape of the obstacle, where the goal is to minimize the length of the motion path. The algorithm we present guarantees a constant factor stretch of the motion path length, relative to the optimal solution of the offline problem.

2. Related work

*The watchman’s route problem* [3, 4, 5, 6, 7, 8] is a well-known optimization problem, where an algorithm for a watchman needs to be constructed so that he computes the shortest path to traverse a certain area, and from this path he must observe the entire area, we assume he has $2 \cdot \pi$ view to any distance only bounded by obstacles. In the general case it has been shown to be an NP-hard problem [9, 10, 11] (say if there are $n$ obstacles). In an offline scenario the guard is given a map of the area including its obstacles and needs to compute the shortest path. Whereas, in the online case, the watchman has to explore (i.e. discover unknown terrains in the offline case) or scan (i.e. canvas a known terrain in the online case) the area it traverses without (or with limited) prior knowledge of what lays ahead. A good example for this is when the Mars rover has no aerial aid due to interference or latency, and needs to explore the surrounding area on its own.

The watchman problem has been observed under various different constraints. Online algorithms for touring the interior of a non-convex simple polygon have been presented in [12, 13]. The traversing robot does not have a map, and is only aware of what it has explored so far. They present a $\frac{5}{4}$-competitive algorithm. This problem is similar to our online setting. However, in their case, the polygon is simply connected, whereas in our case, the outside of a polygon, which is not simply connected, is toured. Czyzowicz et al [11] give an algorithm for many obstacles (again, the obstacles may not be convex). However, even in a convex scenario, no constant competitive ratio is obtained. In the exploration algorithm of unlimited vision the complexity of the path is
\(O(P + D \cdot \sqrt{k})\) where \(P\) is the total perimeter of the terrain (including obstacle perimeter), \(D\) is the diameter of the convex hull of the terrain and \(k\) is the number of obstacles. In \([2]\) an algorithm for touring a general polygon is discussed, a competitive ratio for touring the interior of a polygon is shown to be less than 2016, and the existence for a an online competitive algorithm for touring the exterior of a general polygon is also discussed, with no details on the ratio. \([14]\) gives an algorithm for touring a polygon with holes, where the holes are polygonal and are of different colors. Their algorithm is \(\approx 600\)-competitive in the case of one hole. Here, we give an improved competitive ratio for a convex obstacle (hole) in \(\mathbb{R}^2\).

3. Problem Formulation

![Diagram](image)

Figure 1: This is an example of an optimal path around a polygon. LOS - stands for “lines of sight”. In this figure only the initial LOSs are portrayed. HP - stands for Half Planes, each edge divides the plane into 2 half planes. From the path illustrated here the robot can see every point in the plane.

A mobile point robot is placed in \(\mathbb{R}^2\) and it faces an obstacle, its mission is traversing the plane \(F\) in such a manner that its accumulated unlimited \(2 \cdot \pi\)-view (blocked by the obstacle with lines of sight) from its path covers \(F\) entirely (either by scanning or exploring) under the following constraints: An obstacle in our model is represented as an open convex polygon embedded in the plane, and denoted by \(O\).
The vertices of $O$ will be denoted $v_1, \ldots, v_n$ and the edges by $e_1, \ldots, e_n$. We define the free space as the area where the robot can move. $O$ is a sight and movement restricted area.

### 3.1. Definition and Notations

**Definition 3.1.** The free space is defined as $F = \mathbb{R}^2 \setminus O$.

This is the area where the robot can move. We call the complement of the free space - the forbidden space, notice that $O = F^c$. (It is the polygon mentioned above).

Now we define visibility in $F$.

**Definition 3.2.** For every 2 points $x, y \in F$ we say that $x, y$ are visible from each other iff $xy \cap O = \emptyset$. (where $xy$ is the straight segment between two points $x$ and $y$.)

**Definition 3.3.** Every edge $e_i \in O$, divides the plane into 2 half planes, a half plane containing the obstacle (convexity), this half plane is called a “supporting half plane”. The other half plane is contained entirely in the free space - we denote this half plane $H_i$ (relative to $e_i$).

Our polygon has a finite number $n$ of edges, and respectively $n$ half planes with arbitrary order: $H_1, \ldots, H_n$.

**Definition 3.4.** We denote $p_i \in H_i$ the (finite) sequence $p_1 \ldots p_n$ as the first points of each half plane which are visited by our robot.

**Definition 3.5.** We denote the permutation $\pi : \{H_{i \leq n}\}$ as the half planes visiting order.

**Remark:** The robot can be placed at several half planes simultaneously, however we regard to one half plane as the one it is placed in at certain time $t$ as the half plane defined by the edge it is closest to. If there are several edges that are closest, then we choose them clockwise (i.e. c.w) w.l.o.g. As to $\pi : \{H_{i \leq n}\}, k \leq n$ (visiting sequence) this is determined by the direction of advancement defined at [6.14]. Meaning if direction of motion is c.w (c.c.w), then half planes visitation is the same.

We assume our robot has unlimited vision, meaning $x$ and $y$ can be arbitrarily far. Additionally, the robot can calculate distances and angles accurately, and accumulate all the information it processes.

**Definition 3.6.** A path is a continuous function $f : [0, 1] \rightarrow F$. 


The length of a path is defined in the standard manner by a curve integral method (approximating the arc length), we refer mostly to a discrete version of the path, each segment is easily calculated with the euclidian norm. For \( A \subseteq [0, 1] \), we denote the set of points on the path \( f \) as \( P_{f(A)} = \{ f(t) | t \in A \} \) - which is the image of function \( f \).

**Definition 3.7.** Given a set of points \( S \subseteq F \), the visible set is defined as \( V(S) = \{ x \in F | \exists y \in S : \overline{xy} \cap O = \emptyset \} \).

**Definition 3.8.** A watchman route is a path such that \( V(P_{f([0, 1])}) = F \).

The optimal watchman route is a watchman route of path \( f(t) \) such that the length of \( f(t) \) is minimal among all watchman routes for this polygon.

The watchman route, can be discussed in both the floating and the fixed initial point settings, for both of the cases a polynomial time algorithms has been provided (see Appendix).

### 4. Online exploration

#### 4.1. Motivation

In competitive analysis there are many different algorithms which aim is to explore the surrounding area of an obstacle. The online algorithm is such an algorithm. It produces an exploration path, so that in the worst case when compared with the optimal algorithm (made by the adversary), comes out competitive. Construction of an online algorithm is prudent, it shows whether a task is worth while or might be too high in complexity. Furthermore, the algorithm has to have a uniform approach for all types of convex shapes, otherwise it might result in some cases, in a non-competitive ratio.

In our case in study, as in the optimal algorithm, for the robot to obtain a watchman route it also needs to visit all half planes. A logical rule of thumb will be to avoid going alongside long edges i.e. if a newly visible edge is relatively longer than the previously traversed series of segments and the alternative reasonable routes, it will not be traversed. Surprisingly, this approach does not suffice in order to get the wanted ratio because in such cases intuition dictates to make an oscillating path. However, this might be a dangerous approach, in case the algorithm is vertex dependant, it might result in divergence and thus a non-competitive ratio.
For this reason progress needs to be done cautiously. In our algorithm we imitate the approach taken in the Ski rental problem, by running an incremental spiral–search like path, which has been proven to be optimal in two dimensions 15, and also fair to say it is optimal in one dimension. We weigh the cost benefit arguments, essentially maintaining that as long as the scope ∈ OS (see definition 4.8) the robot attempts to lead to an instigation of a predetermined oscillating route, at least maintaining a temporary competitive ratio.

As soon as the scope is reduced to a CS (see definition 4.8) the traversing approach is altered according to several factors. The question is whether the robot should continue on its course in the same direction or go back (through visited regions) to explore in the opposite direction. Any decision is made by contemplating the hypothetical angles and estimated distances by the information gathered through the distance already traversed.

4.2. Problem Statement

Definition 4.1. For a given path \( f(t) \) and polygon \( O \) with vertices \( \{v_1, \ldots, v_m\} \), the seen vertices up to time \( t \) are \( \nu(t) := \{v_1, \ldots, v_m\} \cap V(P([0, t])) \)

Definition 4.2. An online watchman route algorithm is a function \( h(a, \nu(t), t) \) such that for any convex polygon \( O \) and starting point \( s \), \( f(t) := h(s, \nu(t), t) \) is a watchman route
with a starting point $s \in F$. (i.e., the decision of the algorithm at any time $t$ depends on the vertices seen up to time $t$ and thus on the scope)

**Definition 4.3.** An online watchman route algorithm, $h(s, v(t), t)$, is called $k$-competitive if for every starting point $s \in F$ and every polygon $O$, the length of $h(s, v(t), t)$ is at most $k$ times the length of $g(s, O, t)$, where $g(s, O, t)$ is the path given by the optimal offline algorithm.

### 4.3. Lower Bound

**Lemma 4.4.** Every online algorithm has a triangular obstacle, for which the competitive ratio is at least $\frac{\text{online}}{\text{OPT}} > 3 - \varepsilon$ for every $\varepsilon > 0$.

**Proof.** For any given triangle the adversary may place the robot at any starting point. A bad possible option for the best possible online algorithm occurs when the triangle has two fairly long edges $e_\ell$ and $e_\ell'$ with length $\ell$, and one very short edge denoted $e_{\varepsilon}$ with length $\varepsilon$, the robot is placed by the adversary adjacent to the middle one of the long edges of the obstacle. The OPT will produce a path of length $\frac{\ell}{2} + \varepsilon$. An online algorithm may choose to go either left or right, we assume the adversary will make it harder for any online algorithm and, therefore, the adversary knows how to place the triangle-obstacle in order to challenge the online, so that the online (any online algorithm) makes an approach to the opposite direction of what is best for it (it is oblivious). Thus, producing a path of length: $\ell + \varepsilon$. We get:

$$\frac{\text{online}}{\text{OPT}} = \frac{\ell + \frac{\ell}{2} + \varepsilon}{\frac{\ell}{2} + \varepsilon} = 3 - \varepsilon$$

□

**Corollary 4.5.** For every online algorithm and for each $\varepsilon$ there is an obstacle polygon $O$ and a starting point $s$ such that the ratio maintains $\frac{\text{online}}{\text{OPT}} > 3 - \varepsilon$ competitive ratio for any $\varepsilon > 0$.

**Remark:** This is a lower bound on the ratio, in some cases a path with a lesser ratio $c$, such that $1 < c < 3$, can be attained. However, there are several options for an online path, and the adversary has the ability to create a layout in which there is a chance in which the robot takes a path that gives a $3 - \varepsilon$ ratio.
4.4. Algorithm Specifications

**Definition 4.6.** Take any vector from the starting point \( s(t) \) towards any visible vertex, w.l.o.g \( v(t) \), denote it with \( s(t) \rightarrow v(t) \). Now take the outmost visible vertex clockwise, denote it with \( v(t) \), this is the left vertex. Take the outmost visible vertex counterclockwise, denote it with \( v(t) \), this is the right vertex. Define a **Line Of Sight**, LOS, as the line that passes through \( s(t) \) (the robot starting location) and the outmost visible vertex, whether to the left or to the right, with LOS \( (s(t), v(t)) \) and LOS \( (s(t), v(t)) \) as the lines passing through \( \{s(t), v(t)\} \) and \( \{s(t), v(t)\} \) correspondingly.

**Definition 4.7.** An **edge-chain**, is a sequence of consecutive edges of the polygon. Formally, Let \( F \) be an edge-chain made of a sequence of edges \( e_i \ldots e_j \) such that \( \forall e_i, e_i \cap e_{i+1} = v_i \) (consecutive).

Notice that an edge-chain that is visible by the robot at time \( t \) is \( e_L \ldots e_R \), where \( e_L \) and \( e_R \) are the outmost edges to the left and to the right respectively.

**Definition 4.8.** Firstly, denote the furthest visible vertices clockwise and counterclockwise as \( v_{L_{ext}}(t) \) := \( h(s(t), v(t), t) \), v_{R_{ext}}(t) := h(s(t), v(t), t) \), where \( t_L \) and \( t_R \) are the time stamps in which these vertices have been seen “extremely” (The word “extremely” in this context refers to the most extreme angles created by the locations and LOS’s visited thus far by the robot clockwise and counterclockwise.) correspondingly. Secondly, define the scope, as the angle created between LOS \( (s(t_R), v_{R_{ext}}(t_R)) \), LOS \( (s(t_L), v_{L_{ext}}(t_L)) \) counterclockwise, denoted \( \angle \text{LOS}_{R_{ext}}, \text{LOS}_{L_{ext}} \). We divide the scopes into 2 categories:

- **Opening scope**, OS, is a set of scopes defined by the triangle: LOS \( R \), LOS \( L \) and \( v_{R_{ext}}(t_R), v_{L_{ext}}(t_L) \) with the robot located inside the triangle or on its edges, i.e the unobserved area is infinite. In case there is no triangle (i.e. LOS \( R \parallel \text{LOS}_{L} \)), we also consider it to be an OS
- **Closing scope**, CS, is a set of scopes defined by the triangle: LOS \( R \), LOS \( L \) and \( v_{R_{ext}}(t_R), v_{L_{ext}}(t_L) \) with the robot located outside the triangle or on its edges, meaning the unobserved area is finite.

Notations: The segment formed between \( v_{R_{ext}}(t_R) \) and \( v_{L_{ext}}(t_L) \) the extreme vertices seen, is denoted by \( m \). The distance between the robot and \( m \) is denoted by \( \ell \).

4.5. **ONPA (The Online Path Algorithm)**

We propose an online algorithm that searches around the polygon and eventually completes an exploration path. We will show, that this algorithm is competitive, with a competitive (bound) ratio of 89.83. The algorithm consists of several stages: depending proximity the robot has different traversing options.

1. scope \( \in \text{OS} \):
• \( \text{scope} \equiv 2 \cdot \gamma \geq \frac{\pi}{2} \), spiral-search parallel to \( m_1 \), where \( m_1 \) is the line between the two extremal vertices viewed from the initial location.

• return to the starting point \( s \).

• \( \text{scope} \equiv 2 \cdot \gamma \leq \frac{\pi}{2} \), spiral-search perpendicularly to bisection line \( \hat{Q} \), where the robot moves to reduce the scope, so the angle of the triangle of the scope \( \in \text{CS} \) is \( \leq \frac{\pi}{8} \).

2. scope \( \in \text{CS} \):

Keep motion in steps of \( \ell_i \) (perpendicularly if possible) towards \( m_i \) until \( h = V(H_i|\forall i) \). If the polygon is in the way take the estimated shorter path, as long as it is not longer than the path already traversed.

4.6. Correctness

**Theorem 4.9.** The path given by the online algorithm (ONPA) has a finite length and it explores \( \mathbb{R}^2 \setminus \mathcal{O} \) entirely, i.e. it is a watchman route. Formally, \( \forall p \in F, p \in V(h(s,v(t),t)) \). (This means the robot sees all points of the free space from its path.)

**Proof.** Firstly, the robot is placed at a starting point w.l.o.g. \( s \in H_1...k \), \( k \leq n \) half planes at once. The initial oscillating iterations \( Q \) (which is the length of the projection of the first seen edge-chain) are bounded by the largest projection \( D \) which has a finite length. Thus, ensuring at a finite time \( t_m \) a CS will be found as soon as the oscillations pass the size of \( D \).

Secondly, the next part of the algorithm is bounded by approaching segments \( I_j \) to the investigated half planes, by estimation of distances and angles. The number of these segments is bounded by \( c \cdot (n-j) \) half planes, where \( c \) is some constant. At some of these segments there are points \( p_i \in I_i \) such that \( V(p_i) = H_i \).

By construction, the path \( h(t) \) is compiled of segments, that while traversed, discover additional parts of the plane. It is a brute-force approach that encourages to explore all half planes (defined by \( \mathcal{O} \)), so if needed a direct approach to the furthest unvisited region is executed, the length is bounded by the distance to the furthest vertex which is reached at time \( t_n \) (\( t_n \) is bounded).

In general:

\( \forall H_i \exists t_i \) such that for \( p_i \in H_i \ (p_i \in h(t)) \) at time \( t_i \) the robot reaches point \( p_i \), and \( V(p_i) \supseteq H_i \) (for more details see Appendix Proposition 6.5).

We get:

\[
V \left( \bigcup_{i=1}^{n} p_i \right) = \bigcup_{i=1}^{n} H_i = \mathbb{R}^2 \setminus \mathcal{O}.
\]

Resulting in a finite number of traversed segments with all half planes having been visited. Thus, the algorithm is an exploration path, making the online algorithm a watchman route, and it is bounded so the length of the path is finite. \( \square \)
Algorithm 1: ONPA (Online path algorithm)

**Data:** $LVV =$ list of visual vertices; $\forall v \in LSV$ has $x,y$ coordinates

**Result:** An online path $h(s,v(t),t)$; $h$ as described above

**Function** Check($LVV$):
- if $\exists v$ such that $v \in LVV$ twice then
  - return $||h(t)|| = \sum_{i=1}^{N} I_i$ where $I_i$ are the segments constructing the path

**Function** Finished($LVV$):
- if $\exists v$ such that $v \in LVV$ twice then
  - return True
- else
  - return False

$LSV \leftarrow LVV$;
Identify $LOS_L$ and $LOS_R$ and calculate scope;

$m_i := v_{ext}^{R}(t_R) v_{ext}^{L}(t_L)$;
Distance to $m_i := \ell_i$;
Find $\tilde{m}_i$ by the perpendicular to the bisection line;
if $Scope \in OS$ then
  - if $2 \cdot \gamma \leq \frac{\pi}{2}$ then
    - Traverse $\tilde{m}_i$ spirally, $h = h + ||\tilde{m}_i||$;
    -/* Beginning a parallel spiral-search path. */
    - Stop when $2 \cdot \gamma \leq \frac{\pi}{2}$;
  - else
    - Traverse $\min(d_R,d_L)$ spirally in ascending segments,
    - $h = h + \|2 \cdot \min(d_R,d_L)||$ (greediness), and reevaluate $Scope$;
    - Stop when finished or when $Scope \in \frac{\pi}{8}$;
    -/* $d_R$ and $d_L$ are defined as; the distances between the point on $m_i$ closest to the robot, towards the left and right vertices c.w and c.c.w correspondingly. */
  - else
    -/* A CS is found. */
    - UPDATE all;
end

while Not Finished($LVV$) do
  if $O$ is not in the way then
    - Move perpendicularly towards $m_i$;
  else
    - Move to closest point on $m_i$ on the same direction that does not intersect $O$;
end
Notice that since the robot attempts to avoid going along long edges, the cases it
does follow such edges are either when the length of the edge is no longer than what
the robot has already traversed all together, or when it has no choice but to get to the
far side of the polygon. In both cases the OPT has to make similar path decisions
(competitiveness). For this reason, greediness is a doorkeeper for wasteful traversed
sections.

4.7. Preliminaries

Lemma 4.10. Take 2 parallel segments \( \overline{AB} \parallel \overline{CD} \), such that \( |\overline{AB}| < |\overline{CD}| \) and take the
lines from A to C and from B to D (without them crossing each other between the
segments), then the lines cross on \( \overline{AB} \) side.

Theorem 4.11. • At first the scope is always an OS.

• The implementation of the spiral-search approach must result in a reduction of
the scope to a CS.

Proof. • The initial scope is the angle based at \( a = h(0) \) and its supporting lines are
the LOSs, and by definition the base of the angle is where the robot is located,
so they are placed on the same side in regard to the line defined by the extreme
vertices, inferring the algorithm always starts with an OS.

• Our obstacle is a finite convex polygon, therefore \( \exists r < \infty \) such that there is a
finite number, \( b \), such that \( \max |\langle v_i - v_j, m_1 \rangle| \leq b \cdot m_1 \) where on the l.h.s we
have the largest inter-vertex projection on the line of \( \overline{m_1} \) and the segment \( m_1 \) is
formed between \( v_{R \text{ext}}(t_R) \) and \( v_{L \text{ext}}(t_L) \) the first extreme vertices seen by the robot.
Consequentially resulting in a scope \( \in \) CS by 4.10. In our case \( \overline{CD} \) is the latest
traversed segment and \( \overline{AB} \) is the largest inter-vertex projected segment.

First priority of the online algorithm is reducing the scope from an OS to a CS,
because it indicates what type of shape the robot is dealing with (the online is oblivious
to the shape of the obstacle). We need to separate the online scenario into small
sub-scenarios. First is a scenario that competes with an OPT of type reflection (Intu-
itively the reflection scenario is generally a case in which the robot is placed far enough
from the polygon and there is a relatively acute inner angle within the obstacle that the
OPT path would choose as a pair of half planes from which to reflect and make final
approach to.). The reason for initially competing with a reflection scenario is the differ-
ence in the obtained competitive ratio in these situations. To demonstrate the extremity
of the cases, consider the naive approach - going alongside the polygon. For example,
as mentioned above, look at a triangle with two very long edges in relation with the third edge, and the robot is placed at the half plane affiliated with the small edge. The OPT will make a reflection path, while the naive online approach will go alongside the polygon so if compared: denote the small edge with \(\varepsilon\) and the long edge with \(A\), and assume the robot is fairly far from the polygon and opposite the small edge.

\[
c = \frac{\varepsilon + A}{\varepsilon + \frac{\varepsilon}{2}}
\]

Where \(c\) is the competitive ratio. It is clear that if \(\varepsilon \to 0 (\varepsilon \neq 0)\) \(c\) is non competitive.

A different approach has to be taken. In order to do so we choose a mini-ratio for oscillations along a line determined by the initial scope, in our case we choose the ratio to be 2. At each oscillation the robot multiplies its last traversed segment by a factor of 2 and then traverses that new distance towards the complete opposite direction. One dimensional spiral-search is a very popular algorithm in computational geometry. This is a pseudo filtering approach, meaning, we filter our traversed distance by blocking our next traversed distance maintaining a linear relation with the rest of this part of the path, this type of spiral-search has been shown to be 9-competitive \(^{[17]}\).

**Lemma 4.12.** \(\hat{Q}\), (the biggest projection of 2 vertices on the spiral-search line) with fringes \(A, B\) while traversed with a 1-dimension spiral-search produces a 9-competitive ratio.

**Proof.** This is a slight change from the classic approach.

1. In our case, there are 2 points \(A, B\) which are the fringes of the segment \(\hat{Q}\) which defines the CS, while utilizing the online algorithm, the robot does not have the information of the relation between its own location \(s\) and the points \(A, B\) and the relation between themselves. For this reason we discuss the worst case, w.l.o.g the robot is placed closer to point \(A\) (also viewed as the origin) the \(j-1\) segment towards \(B - \varepsilon \leq \hat{Q}\), where \(B\) is the the projection of the point \(B\) on the direction of the search. Formally:

2. Assume, for some \(j\) that \(2^{j-1} < \hat{Q} \leq 2^{j+1}\).

3. The total path length traveled is (segments are in abstract value):

\[
sp_{\text{online}} = 2 \cdot 1 + 2 \cdot 2 + 2 \cdot 2 + \ldots + 2 \cdot 2^{j-1} + 2 \cdot 2^j + \hat{Q} = 2 \cdot 2^{j+1} + \hat{Q}.
\]

Where \(sp\) stands for the spiral search. The OPT would traverse from \(A\) to \(B\) directly by \(\hat{Q}\).

\[
\frac{sp_{\text{online}}}{OPT} = \frac{2 \cdot 2^{j+1} + \hat{Q}}{\hat{Q}} \leq 1 + \frac{2 \cdot 2^{j+1}}{2^{j-1}} = 9.
\]
Thus, the iteration required gives a 9-competitive ratio.

This allows us to deal with keeping our desired competitive ratio later on without much ado. This mechanism is acceptable until the last traversed segment. As mentioned earlier, this is actually a similar approach of competitive decision making as the “Ski rental problem”. In which a person vacationing at a ski resort, rents some ski equipment each day of his vacation up until the day before the rate of renting supersedes the actual price rate of the equipment, at which point it is better to purchase the equipment, giving a 2-competitive ratio. The rationality of the consideration is made of a cost-benefit argument that is, in general, a frugal approach. As in the ski problem, our robot traverses the plane with oscillations and with each new segment ideally being the last one, it multiplies the distance with a bounded risk of squandering.

4.8. Competitiveness

In this section we prove the online algorithm (ONPA) is competitive with the OPT. To do so, we need to take the best possible OPT and compare it to the worst possible online, however, we must separate the different parts of the path into cases. For this reason we bound the OPT with $\ell_\tau$:

Definition 4.13. Define the distance to the furthest half plane, $\ell_\tau(s) := \max_i \min_f(f(s,H_i))$.

i.e. the shortest path from the starting point to the furthest half plane. Even the shortest OPT must reach the furthest half plane.

Lemma 4.14. $\ell_\tau \leq OPT \leq 3 \cdot \ell_\tau$

Proof. OPT $\geq \ell_\tau$:

By definition $\ell_\tau$ is the shortest path to the furthest half plane from the starting point. Therefore, OPT must traverse at least that in order to view the entire free space.

$OPT \leq 3 \cdot \ell_\tau$:

Lets define the sets $L = \{\min_{\ell_{i<w}} f(s,H_i)|1 \leq i \leq n\}$, $R = \{\min_{\ell_{i<w}} f(s,H_i)|1 \leq i \leq n\}$ as the half planes with minimal paths clockwise and counterclockwise correspondingly (If $\exists i : f(H_i) = f(H_{n-i})$ put $f(H_i) \in L$ or $R$ w.l.o.g.). W.l.o.g $H_\tau$ is the furthest half plane ($f(s,H_\tau) = \ell_\tau$), assume there is another unvisited half plane $H_q$, the most expensive route in terms of $\ell_\tau$, out of the paths above is going to $H_\tau$ changing direction, and coming back towards $s$, counter direction, and then going to $H_q$ which in the worst case also costs $\ell_\tau$, overall a $3 \cdot \ell_\tau$ bound (as in the lower bound section). □
Define \( \hat{m_i} \) := \( \hat{Q} \) as the projected segment from the vertices of the extreme LOSs on the second spiral-search line (bisection).

Now we need to figure out the general competitive ratio for any obstacle while implementing the online algorithm, in relation to \( \ell_\tau \) in each scenario/part of the path.

We begin with the first part of the path:

**Lemma 4.15.**

\[
\ell_\tau \geq \max \left( \min (d_L, d_R), \min (\hat{Q}_L, \hat{Q}_R) \right)
\]

where \( d_L \) and \( d_R \) are the lengths of the corresponding parts of the projection of \( m_i \) that are traversed until \( 2 \cdot \gamma \) is reduced and \( \hat{Q}_L \) and \( \hat{Q}_R \) are the lengths of the left and right parts of \( \hat{Q} \) correspondingly.

**Proof.** The algorithm always begins with \( 2 \cdot \gamma \) scope angle. We separate the cases:

1. If \( 2 \cdot \gamma > \frac{\pi}{2} \), a “parallel” spiral-search (of size \( 9 \cdot \min (d_L, d_R) \)) is utilized, we need to find the upper bound in terms of \( \ell_\tau \). There are 2 options: depending on the direction of \( \ell_\tau \).

   Define \( p_L, p_R \) to be the points of the projection of the \( m_i \) at which the online algorithm reaches an angle \( \leq \frac{\pi}{2} \) c.w and c.c.w correspondingly. Assume direction is c.w w.l.o.g:

   (a) If \( \ell_\tau \) passes through \( H_i \) such that \( p_L \in H_i \), then this means by triangle-inequality \( \ell_\tau \geq \min (d_L, d_R) \). (The spiral might not reach the minimal side first, but only in case \( d_L \leq 2 \cdot d_R \) or \( d_R \leq 2 \cdot d_L \) which is competitive.) Because \( \ell_\tau \) might attempt to reach the polygon (reaching path) and the spiral line does not.

   (b) If \( \ell_\tau \) does not pass through \( H_i \) such that \( p_L \in H_i \), this means the online goes counter direction of \( \ell_\tau \) (“greedy motivation”). By definition the path of \( \ell_\tau \) is the path from \( s \) to the furthest half plane, and \( p_L \) is either in the furthest half plane or closer (in a straight line), therefore the length from \( s \) to \( p_L = \min (d_L, d_R) \leq \ell_\tau \).

   In either case, \( \ell_\tau \geq \min (d_L, d_R) \)

2. If \( 2 \cdot \gamma \leq \frac{\pi}{2} \), a “bisection” spiral-search is utilized. First of all, need to deal with a case where the robot has a “collision” course with the polygon. This means that \( \ell_\tau \), also has to collide with the polygon (simply because if a parallel direct line is not feasible, then any short path has to approach the polygon). Moreover, the competitive ratio is the same, because both the online and \( \ell_\tau \) need to approach the polygon and the spiral acts the same (only as a broken line).

   We have 2 options:

   (a) The online and \( \ell_\tau \) face the same direction. \( \min (\hat{Q}_L, \hat{Q}_R) \) is defined by the points \( p \) (where the angle \( 2 \cdot \gamma \leq \frac{\pi}{2} \)) and the closer projection point of the extreme vertices, i.e. \( \hat{v} = \hat{Q} \cap LOS_{ext} \). \( \ell_\tau \) reaches the furthest half plane from
\(\ell \) vs. ONPA (online)

\(\ell \) may be independently small in relation to \(\widehat{Q}\). This is why we take the smaller portion \(\min(\widehat{Q}_L, \widehat{Q}_R)\). For this reason we must divide between scenarios: those with a small \(\ell_{r}(s)\) and thus non-comparable with \(\widehat{Q}\), and those with a large and comparable \(\ell_{r}(s)\). As discussed in the lemma, a non-comparable \(\ell\) is reached when the last half plane is reached "close" (also relative to the size of \(\widehat{Q}\)) to the point at which the angle \(2 \cdot \gamma\) has been reduced, this type of \(\ell\) is competitive with the online but not with \(\widehat{Q}\). So the competitive ratio in these cases is still determined in terms of \(\ell\). The second scenario indicates \(\ell\) is large and comparable with \(\widehat{Q}\), thus letting the discussion revolve only around \(\ell_{r}(s)\).

Discussing a formal scenario, the setting w.l.o.g has the robot placed at \((a, 0)\) and \((0, 0) = \overline{e}_\mu \cap \overline{e}_\sigma\) where \(\overline{e}_\mu\) and \(\overline{e}_\sigma\) are the lines of the corresponding edges \(e_\mu\) and \(e_\sigma\) that the robot makes a reflection from and last approach segment. The angle \(\alpha := \angle(\overline{e}_\mu, \overline{e}_\sigma)\) (\(\alpha\) incloses the obstacle.) from which the robot traverses \(\ell\) in the \(OPT\).

Intuition: the terms "close" and "far" refer to the relation between \(\ell\) and \(\widehat{m}_n\), where "close", is when \(\frac{\ell}{m_n} \ll 1\) and "far", is when \(\frac{\ell}{m_n} \approx 1\).

Competitive Ratio

With an elongated obstacle, the online has to compete with a longer \(\ell\), which has to stand competitive with the original \(\ell_{r}(s)\). In order to bound the path, we need to understand how to express different parts of the path in terms of \(\ell\).

**Lemma 4.16.** The ONPA algorithm has three possible stages, to bound these stages we generalize:
Proof. By the algorithm construction ONPA. (each of the parts may have length 0) □

We need to bound ONPA: We will show now the ratio of the given online algorithm: $ONPA(s) \leq (10 + 18 + 54.62) \cdot \ell_r(s) \leq (10 + 18 + 54.62) \cdot OPT$

Proof is given with the following steps:

4.8.1. Step 1:
Lemma 4.17. $I(s) \leq 9 \cdot \ell_r(s) + \ell_r (= 10 \cdot \ell_r).$ (i.e. the l.h.s of the inequality is the part of the path made by an online algorithm starting at point $s$. In the r.h.s 9 is the spiral factor, with an additional return to point $s$.)

Proof. As proven, spiral search has a 9-competitive ratio, and in the worst case the spiral reaches the furthest half plane. For simplicity in proving complexity the robot returns to the starting point and from there starts the second spiral search. Resulting in an additional payment of $\ell_r$. □

4.8.2. Steps 2, 3:
Lemma 4.18. $(II + III) \leq \ell_r \cdot \left(9 \cdot \left(\frac{\ell_r}{\sin(\frac{\pi}{8})} + \frac{3 \cdot \ell_r (\pi - \frac{\pi}{8})}{\sin(\frac{\pi}{8}) \sin(\frac{\pi}{8})}\right)\right) = \ell_r \cdot \left(\frac{36 \cdot \sqrt{2} + 21 \pi}{4 - 2 \sqrt{2}}\right) \approx 79.83$

Proof. The distance to $m_i$ is $\ell_i$. We allow (This is a restriction meant to bound the length of the path by bounding the size of the angle.) an angle (of the CS triangle) to be $\frac{\pi}{8}$.  

Resulting in the following calculations:

We have a triangle with base line spiral search which in the worst case is $3 \cdot \ell_r(s)$, denote it $A$. $O$ is located inside a triangle constructed of the second spiral search and the furthest LOSs.

The angle between the LOSs is $\frac{\pi}{8}$, we decided the angle of the bisection would be $\geq \frac{\pi}{2}$ and so, half of that angle is $\geq \frac{\pi}{4}$, thus, the complement to $\pi$ is $\leq \frac{\pi}{4}$ and from triangle external angle we get the angle $\angle (LOS, A) \leq \frac{\pi}{2}$ and the other angle is $\geq \frac{\pi}{4}$.

Denote the second spiral search with II, and the segment series after the bisection spiral search up to the point in which the algorithm is finished with III. first we express them:

$II := \frac{9 \cdot \ell_r}{\sin(\angle(LOS_{ext} A))}$ Taking $\ell_r$ to be a direct perpendicular line towards the LOS, the relation is a right angle triangle (sin). For III, we construct a circle centered at $v_{ext}$ on the opposite side of the polygon (in regard to point the direction of revolution around the polygon). Define the angle $\theta := \angle (p - v_{ext} - LOS_{ext} R \cap LOS_{ext} L)$ as noted by limiting the angle of the second spiral search $\angle (LOS_{ext} R, LOS_{ext} L) = \frac{\pi}{8}$: $\theta \geq \pi - \frac{\pi}{8}$. We denote the radius with $R$. The path is a polygonal “edge-chain”, which is convex, due to convexity of the polygon (as explained in the reaching path). $R$ is bounded within the triangle created by $A, LOS_{ext} R, LOS_{ext} L$, w.l.o.g we assume $p \in LOS_{ext} L$. The path is contained
within $R \cdot \theta$ which will bring $III \leq R \cdot \theta$, we want to bound $R$:

$$R \leq \max \left( A, LOS^{R}_{eR} \right)$$

by sine-rule we get:

$$\frac{A}{\sin(\frac{\pi}{8})} = \frac{LOS^{R}_{eR}}{\sin(\frac{\pi}{4})}$$

$$R \leq LOS^{R}_{eR} = \frac{A}{\sin(\frac{\pi}{4})}$$

$$III \leq R \cdot \theta \leq A \cdot \left( \pi - \frac{\pi}{8} \right) \sin \left( \frac{\pi}{8} \right) \leq 3 \cdot \ell \cdot \left( \frac{36 \sqrt{2} - 21 \pi}{4 - 2 \sqrt{2}} \right) \sim 79.83$$

Now we wish to minimize the lengths of the last parts of the path:

$$\text{II} := \frac{9 \ell \cdot (\pi - \frac{\pi}{4})}{\sin \left( \arctan \left( \frac{8}{\ell} \right) \right)} \geq \frac{9 \ell \cdot (\pi - \frac{\pi}{4})}{\sin \left( \arctan \left( \frac{8}{\ell} \right) \right)}$$

$$\text{III} := \frac{3 \ell \cdot (\pi - \frac{\pi}{4})}{\sin \left( \arctan \left( \frac{8}{\ell} \right) \right)} \geq \frac{3 \ell \cdot (\pi - \frac{\pi}{4})}{\sin \left( \arctan \left( \frac{8}{\ell} \right) \right)}$$

$$\min (\text{II} + \text{III}) = \ell \cdot \left( \frac{9 \ell \cdot (\pi - \frac{\pi}{4})}{\sin \left( \arctan \left( \frac{8}{\ell} \right) \right)} \right) \geq 10 + \frac{36 \sqrt{2} - 21 \pi}{4 - 2 \sqrt{2}} \sim 89.83$$

On the r.h.s the 9 comes from the spiral ratio. □

**Theorem 4.19.** The overall competitive ratio is: 10 + \frac{36 \sqrt{2} - 21 \pi}{4 - 2 \sqrt{2}}

**Proof.** Derived from the lemmas above the path $h(s,O,t)$ is bounded by $\ell_{t}$, and therefore the ratio:

$$\frac{\text{online}}{\text{OPT}} = \frac{10 + \frac{36 \sqrt{2} - 21 \pi}{4 - 2 \sqrt{2}} \cdot \ell_{t}(s)}{\ell_{t}(s)} \sim 89.83$$

□

5. Summary

We investigated the exploration task of a robot facing an unfamiliar terrain with an obstacle embedded in $\mathbb{R}^2$. The goal was to present two algorithms: one offline and the other online, proving there is a competitive relation between them. We have shown the possibilities for an optimal algorithm and calculated the competing optional online paths. We constructed the offline algorithm, as taking the minima out of all reaching and reflection paths around the obstacle. Then constructed the online algorithm, at first reducing the scope by implementing a spiral-search, which has a \(10 + \frac{36 \sqrt{2} - 21 \pi}{4 - 2 \sqrt{2}}\) relation with the shortest OPT (bounded by the path to the furthest half plane). For every polygon the ratio reduces because the offline algorithm has to invest more effort into traversing the plane. Hence we achieve a 89.83 competitive ratio as our result.

We conjecture that this ratio can be improved considerably, possibly using a variant on the 2-dimensional spiral search algorithm. We also presented a lower bound of $3 - \varepsilon$ competitive ratio.

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6. Appendix

6.1. Floating problem

6.1.1. The algorithm

The algorithm for the floating problem is based on the following observation: the minimum path always touches the polygon at least at one point, otherwise, it can be shortened by moving all segments closer to the vertex between the first and last half-plane. The path consists of a (possibly empty) motion from the first half-plane perpendicularly to one of the vertices (the furthest possible), then (possibly empty) motion along the edges of the obstacle, and finally a (possibly empty) motion from a vertex of the obstacle perpendicularly to the last half-plane. Since the length is symmetric, one can always assume that the motion is counterclockwise.

Since we go over all initial half-planes, and for each one go once over all vertices, the complexity is \(O(n^2)\) with the preprocessing step (calculating the angles of all edges) requires \(O(n)\). Thus the total complexity is \(O(n^2)\).

**Algorithm 2:** OFP (Optimal floating path)

**Data:** Vertices List \(V\)

/* Each vertex \(v \in V\) has \(x, y\) coordinates, the list is given in a counterclockwise order w.l.o.g from the starting point, and fulfills the conditions of the clockwise and counterclockwise definitions. */

**Result:** An OFP \(P\) containing a list of visited points.

Let \(\alpha_i\) be the angle between edge \(v_iv_{i+1}\) and the \(x\) axis;

\[\text{for } i := 0 \text{ to } n - 1 \text{ do}\]

\[\quad j \leftarrow i + 1;\]

\[\quad \text{while } \alpha_j - \alpha_i < \frac{\pi}{2} \mod 2\pi \text{ do}\]

\[\quad \quad j \leftarrow j + 1;\]

\[\quad \text{end}\]

Take perpendicular line between \(v_iv_{i+1}\) and \(v_j\);

\[\quad \text{while } \alpha_j - \alpha_{i-1} < \frac{\pi}{2} \mod 2\pi \text{ do}\]

\[\quad \quad \text{Take segment } v_jv_{j+1};\]

\[\quad \quad j \leftarrow j + 1;\]

\[\quad \text{end}\]

Take perpendicular line between \(v_j\) and \(v_{i-1}v_i\);

If the length of this path is smaller than the current minimum, denote it the minimum;

\[\text{end}\]
6.2. Offline exploration

6.2.1. Definitions

Definition 6.1. Given a starting point \( s \in F \), an optimal watchman route is a watchman route of path \( f(t) \) such that \( f(0) = s \) and the length of \( f(t) \) is minimal among all watchman routes starting at \( s \).

Definition 6.2. The optimal offline watchman route algorithm is a function \( g(s, O, t) \), such that for any given polygon \( O \) (defined by \( v_1 \ldots v_n \)) and starting point \( s \in F \), \( f(t) : = g(s, O, t) \) is an optimal watchman route with starting point \( s \).

The offline algorithm is a scanning algorithm (the surface is known but not seen), whereas the online algorithm is an exploration and scanning algorithm (the turf is discovered as the robot advances). The robot is placed at a non-floating point (meaning the initial point is chosen arbitrarily), from which it needs to determine what route is the optimal one so it can scan the entire free space. Before choosing a path in the offline scenario the robot calculates possible options and takes the overall minimum. First, we describe a certain approach which we call a reaching path.

Definition 6.3. Let there be points \( a, b \in F \), where \( a \) is the location point of the robot at time \( t \) and \( b \) is a target location point at time \( t + \omega \), where \( \omega \geq 0 \). Let function \( f : [0, 1] \to F \) be a path, \( f|_{t_s}^b \) is called a "reaching path" between \( a \) and \( b \) if there are \( t_s, t_\tau \in [0, 1] \) such that \( a = f(t_s), b = f(t_\tau) \) and the length of \( f_{t_s}^{t_\tau} \) is minimal.

Now we show the construction of a minimal reaching path.

Definition 6.4. A straight line \( \ell \) is called a supporting line of \( O \) iff \( \ell \) contains at least one boundary point of \( O \) and \( O \) lies entirely in one of the closed half planes bounded by \( \ell \).

A supporting half plane \( H_i \) is the closed half plane containing \( O \), \( O \subseteq H_i \).

6.2.2. Foundation Layout

Proposition 6.5. As soon as \( r_t = p_i \), where \( r_t \) is the robot location point at time \( t \) (the first point in half plane \( H_i \) that has been visited by the robot) then \( H_i \subseteq V(p_i) = \{ x \in F | x \not\in p_i \cap O = \emptyset \} \). This means \( H_i \) is completely seen as soon as one of its points has been visited by the robot. Furthermore, \( F = \bigcup_{i=1}^n H_i \).

Proof. Half plane \( H_i \) is the non-supporting half plane defined by the supporting line created from the edge \( e_i \) of the polygon. Due to polygon convexity there are no parts of the polygon in the non-supporting half plane (by definition the supporting line holds only boundary points of \( O \)), and so \( x \not\in p_i \cap O = \emptyset \), for all \( x \in H_i \). \( \square \)

Lemma 6.6. \( \exists q_i \in H_i \) such that, in order for \( q_i \) to be viewed there has to be a point \( x \in H_i \) the robot has to visit. (then we say a half plane has been visited, and not just seen)

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**Proof.** By Convexity of $O$, we get that $\forall i, H_i$ is convex, and so immediately, for each visited point $x \in H_i$ we get $q_i \in V(x)$, because $H_i \subset V(x)$ and $q_i \in H_i$. □

Scanning or exploration paths are defined so that $V(S) = F$ for some set $S \subseteq F$, the following lemma suggests $S$ be compiled from a set of half plane representative points.

**Lemma 6.7.** Seeing the entire free space is equivalent to visiting all half planes.

**Proof.** ⇒ One way is trivial, $\forall i, H_i \subseteq F$, and it is given that if $V(S) = F$, where $S$ is the set of all points $x$, so that $x \in F$. so $\forall i, H_i \subset V(S)$ where $H_i$ are the half planes defined by the polygon, so $\forall i, H_i$ have been visited, following lemma 6.6

⇐ In the other direction we need to prove that $V(p_1 \ldots p_n) = F$, where $\forall i, p_i \in H_i$. This is equivalent to proving, $V(S) \neq F \Rightarrow \exists x_i \in H_i$ for some $1 \leq i \leq n$ such that $x_i \notin V(S)$. However, following claim 6.5, if $\exists x_i \in H_i$ such that $\forall p_i \in H_i \exists x \cap O \neq \emptyset$, we get a contradiction, thus proving the lemma. □

**Lemma 6.8.** If a convex shape $A \subseteq \mathbb{R}^2$ with boundary $\partial A$ contains (envelopes) another convex shape $B \subseteq \mathbb{R}^2$ with boundary $\partial B$ then $|\partial A|$ (length of $\partial A$) is longer than $|\partial B|$. That is $B \subseteq A \Rightarrow |\partial B| \leq |\partial A|$ ([18] p. 42, Theorems 7.11 and 7.12).

**Theorem 6.9.** Let $f$ be a path (legal path which does not pass through polygon $O$), with $0 = t_0 \leq \ldots \leq t_n = 1$, times, which corresponding points on the path are $f(0) = f(t_0), \ldots, f(t_n) = f(1)$ in order. $f$ is shorter in length than any polygonal chain (legal chain) passing through those same points with that same order.

**Proof.** Assume $f : [0, 1] \to \mathbb{R}^2 \setminus O$ is a general legal path (meaning $\forall t \in [0, 1], f(t) \notin O$). Indicate $p_{\gamma} = \{f(t_\gamma)|\gamma \in [0, 1]\}$ as a set of points on the path, so that $p_s$ and $p_r$ are the first and last points respectively. A possible construction:

$$p_n = \sup \{f(t)[p_{\gamma} f(t) \cap O = \emptyset]\}$$

(Notice that the maximal $\tau$ at each step is bounded by the line of sight given by the polygon vertices, there is a finite number of vertices and therefore the number of edges in the constructed chain is finite.) where, $p_c$ and $p_n$ are the current and next points respectively. It is clear that $\sum_{i=s}^{n-1} ||(p_{i+1} p_{i})|| \leq ||f_{p_{n}}|| = \int f(x)dx$. This is an immediate result from the triangle inequality. In general, for the polygonal chain this brings:

$$\sum_{i=s}^{n-1} ||(p_{i+1} p_{i})|| \leq ||f_{p_{n}}||,$$

$f$ is a continuous function and $O$ has a finite number of edges, so the number of segments in the polygonal chain is finite. □

This construction preserves convexity (due to convexity of $O$)

**Lemma 6.10.** The shape created from the polygonal chain and the line $p_{s} p_{r}$ is convex.
Proof. Let $B$ the shape which created by the polygonal chain. Assume in contradiction $B$ is not convex, therefore, $B$ has an interior angle $\alpha > \pi$ radians. Look at the abutting points on $O_1$, $p_0, p_n \in B$, the straight segment $p_0 p_n$ is shorter than the 2 segments that enclose $\alpha$ (by triangle inequality) and it does not pass through $O_1$ because then it stands in contradiction to construction. So we have a contradiction to the construction of a minimal path (straight segments), hence $B$ is convex. □

Lemma 6.11. The boundary length $\oint_{\partial A} dt$ of any convex polygon $A$ in the supporting half plane of $p_s p_t$ is longer than $\oint_{\partial B} dt$. I.e., $\oint_{\partial B} dt \leq \oint_{\partial A} dt$

Proof. The line passing through $p_s$ and $p_t$ divides the plane and $O$ into 2 half planes and 2 convex shapes $O_1, O_2$. Note that $p_s p_t$ is a common edge of both convex hulls so it will serve as a base line of reference - we refer to it as an \textit{x-axis}. In this discussion the path $f_{\text{tri}}$ resides only in the supporting half plane of $O_1$. By our own construction according to visibility properties each segment of $B$ in the union above is a supporting line of $O_1$, (because the maximal \( t \) is bounded only by the visibility limitations - which are defined by $O$ itself.) $A$ is a convex polygon ($A$ exists because we proved a polygonal chain exists, and there is a convex polygon if a convex hull is calculated), all we need to show is that $A$ contains $B$ and by Lemma [6.10] we prove the hypothesis. Assume in contradiction that $A$ does not contain $B$. The possibilities are:

- Either there is a specific coordinate $x_m$ and a variable coordinate $y$, such that $e_i(x_m,y) - b_j(x_m,y) > 0$, with edge $e_i \in O_1$ and edge $b_j \in B$ ($O_1$ creates caves between itself and $p_s p_t$), and $a_k(x_m,y) - b_j(x_m,y) > 0, a_k \in A$ (notice $e_i(x_m,y) - a_k(x_m,y) > 0$ otherwise we have a violating path)
- Or there is a point $x_m$ such that $e_i(x_m,y) - b_j(x_m,y) < 0$ and $a_k(x_m,y) - b_j(x_m,y) < 0$ (notice $a_k(x_m,y) - e_i(x_m,y) > 0$ otherwise we have a violating path)

The differences above are obtained from the $y$ coordinate.

In the first case:
The shortest path from $p_s$ to $v_{s+1}$, which is the first approached vertex or from $v_{r-1}$ the last approached vertex, to $p_{s}$ is in a straight line, therefore if a path passes over $b_j$ there must be a vertex of $A$ such that its angle is greater than $\pi - \text{radians}$ concluding that $A$ is not convex, in contradiction to our assumptions.

In the second case:
It is easier, if $a_k$ is lower than $b_j$ then there is a point of $A$ in $O_1$ hence violating the basic restriction. So there is no such path, and $A$ contains $B$ as required. □

Theorem 6.12. A shortest polygonal path $B$ taken from $p_s$ to $p_t$ is compiled of the segments:

$$B := p_s v_1 \cup \bigcup_{j=1}^{(a-1) \mod n} e_j \cup v_q p_t,$$

or

$$B := p_s v_q \cup \bigcup_{j=q}^{(a-1) \mod n} e_j \cup v_t p_r.$$
Where \( v_i \) and \( v_q \) or \( v_q \) and \( v_i \), depending on direction, see (6.14) are the furthest seen vertices from \( p_s \) and \( p_t \) respectively, in the supporting half planes. \( q \) and \( i \) may be 0. (This is the minimal reaching path we seek)

**Proof.** To prove this theorem we need only use 6.10 and 6.11. The first applied for our theorem, shows that our polygonal path envelopes a convex polygon, and by the latter we prove that our path is shortest, and hence minimal thus proving the theorem. □

**Corollary 6.13.** The path \( B \) is the shortest path from a point \( p_s \) to a point \( p_t \) in the supporting half plane of \( O_1 \).

**Proof.** \( B \subseteq A. \) Following lemma 6.8 we get \( |\partial B| \leq |\partial A| \forall f(t). \) □

This is a discussion about an optimal “reaching path” from \( p_s \) to \( p_t \) around an obstacle. In some cases taking a reaching path is not the best approach for an optimal scanning path or optimal exploration path. In other cases a reaching path is used as a tool inside the overall path.

Another approach for scanning parts of \( F \) in an optimal manner is a “reflection”:

**Definition 6.14.** Direction: for simplicity we choose the centroid of our polygon as a point of reference for defining direction of motion. A centroid of a convex body/shape is always an interior point i.e.

\[
p_c = \frac{\sum_{i=1}^{n} M_i \cdot v_i}{\sum_{i=1}^{n} M_i},
\]

where \( p_c \in O \) is the point of the centroid and \( M_i \) is the mass of each vertex - we will assign all masses with 1 and \( v_i \) are the coordinates of each vertex in the plane (a location vector). Computation complexity of the c.o.m is \( O(n) \). Now lets define a vector \( V_{RO} := \vec{r_c} p_c \), where \( r_c \) is the robot-location and \( p_c \) is the point calculated above, this vector helps describe a direction of revolution around the obstacle. Meaning, the vector defines a line that divides the plane (and the polygon) in 2, so motion towards the left-hand-side (l.h.s) of the vector will be called clockwise and motion towards the right-hand-side (r.h.s) of the vector will be called counterclockwise.

**Definition 6.15.** Reflection is a polygonal path containing two edges such that: The hitting angle \( \alpha \) and returning angle \( \theta \) from a surface are equal, \( \alpha = \theta \).

This definition is derived from specular reflection in optics, which is an attribute of light - which by Fermat’s principle travels the shortest amount of time and distance (in the geometric description of light).

**Remark:** Note that we have a trivial scanning path (not optimal) if we discard a pair of consecutive edges of the obstacle, then the rest \( n-2 \) polygon edges are a
“scanning path” of the free space, since every half plane is visited by the edge defining it and the removed edge half plane has the vertices of its adjacent edges so it is visited as well. Meaning they can be considered as an exploration path.

Lemma 6.16. An optimal scanning path will contain at most one reflection (reflection made off $H_i$ for some $i$).

Proof. Assume by contradiction there are $n$ reflections, where $n > 1$, and that w.l.o.g motion starts clockwise (relative to $V_{RO}$). So we have to show there is a path shorter than our path containing $n - 1$ reflections, and by regressive induction we only need to show that a one-reflection-path is always shorter than a two-reflection-path (at the same scenario of course). The robot preforms a first reflection on $H_{r_1}$, which is the half plane it makes the first reflection off. After which the robot moves counterclockwise, the ordering of the half-planes is cyclic. $H_{r}$ and $H_{r_1}$ are the first and last visited half planes respectively. They are the same in every scenario, because the OPT chooses to go through the half planes that obtain the least traversed distance, and so the path will pass through all half planes, but reach them in the overall shortest segments possible. A second reflection is made off another half-plane, denoted $H_{r_2}$. Motion is now clockwise, due to convexity the robot goes through visited half planes once again. The distance traversed: $p_{r_1} \in H_{r_1}, p_{r} \in H_{r_2}, p_{r_1} \in H_{r_1}, p_{r_2} \in H_{r_2}$. (A certain point can be placed in several half planes simultaneously as explained). $f$ is a presumable optimal path, and the following lengths are the parts of the different possibilities of the path(The reflection points do not have to be the same, moreover the half planes may be different the proof only relies on the first and last half planes and points to remain the same in all cases.).

\[
\text{Length}_0 := \| f(t)_{p_1}^{p_{r_1}} \|
\]

\[
\text{Length}_1 := \| f(t)_{p_1}^{p_{r_1}} \| + \| f(t)_{p_1}^{p_{r}} \|
\]

\[
\text{Length}_2 := \| f(t)_{p_1}^{p_{r_2}} \| + \| f(t)_{p_2}^{p_{r_2}} \| + \| f(t)_{p_2}^{p_{r}} \|
\]

therefore

\[
\text{Length}_0 < \text{Length}_1 < \text{Length}_2
\]

justification: $\| f(t)_{p_1}^{p_{r_1}} \| + \| f(t)_{p_1}^{p_{r}} \| + \| f(t)_{p_2}^{p_{r_2}} \|$ the first argument is the 0-reflection path and the added arguments are the 2-reflection path extension, and therefore it is not shorter, and not optimal, except for the trivial case.

□

As a general rule of thumb an optimal route will contain an attempt to visit each half plane the least possible number of times, although a half plane might be visited more than once within an optimal algorithm and only once in other cases. Notice there are cases in which a 1-reflection is optimal and better than all 0-reflection cases in that scenario. We define 4 possible types of paths, that may take place. We must show these are the only 4 scenarios that are possible, and that one of them achieves optimality. A
combination of a reflection part and then a reaching path is basically the structure of most optimal paths. The robot always takes the overall minimum:

**Theorem 6.17.** An optimal path is constructed from one of these paths:

\[ f(t) = \min_t g(a, O, t) = \min_t \begin{cases} 
\text{A simple reaching path} \\
\text{A reflection path} \\
\text{A reflection \rightarrow reaching path} \\
\text{A reaching \rightarrow reflection \rightarrow reaching path} 
\end{cases} \]

The path is a minima over all possibilities of the paths above, where a “simple reaching path” is a reaching path from the start point and the last half plane, and in the other cases a reaching path is from the point the algorithm left off and to a certain half plane (either the last half plane or one for reflection).

Elaborately:

1. The robot goes through all half planes consecutively. The starting motion is done towards the most extreme (clockwise or counterclockwise) visible vertex in a straight line. From that point forward it goes along side the obstacle edges, until the last half plane is approached perpendicularly or straight to the \( v \in e \_ \tau \) (0 reflections).

2. The robot moves towards a certain half plane and from there it is reflected to the last half plane perpendicularly.

3. The robot begins by moving towards a certain half plane from which it reflects and continues towards a certain vertex of the polygon, next, it goes alongside the obstacle edges and if possible perpendicularly towards the last half plane.

4. The robot moves towards an extreme vertex (clockwise or counterclockwise). From there it goes alongside some of the polygon edges, then it makes its way to a half plane from which it is reflected towards an extreme vertex (back from whence it came). It traverses some edges, until it reaches the last half plane, perpendicularly if possible.

**Proof.** As shown there is only up to one reflection in the scenarios presented. By their construction, the scenarios are made of all combinations between reaching paths and reflections except for a reaching path followed by reflection, because it is not possible
in a convex scenario. Moreover, they are made by minimal part paths towards all half planes and so the minima is the optimal path overall (clockwise or counterclockwise).

If possible, the last segment of the path is done perpendicularly to the last unvisited half plane \( H_\tau \), otherwise it goes towards the closest point of \( H_\tau \) which in a convex situation is a vertex of \( e_\tau \). Any of the paths above can be a possible optimal path. The robot is situated in \( k \) non-supporting half planes at the beginning (“the waking time”) of the algorithm.

### 6.2.3. The offline algorithm

**Algorithm 3: OSP (Optimal scanning path)**

**Data:** A starting point \( s \), Vertices List \( V \)

/* Each vertex \( v \in V \) has \( x, y \) coordinates, the list is given in a counterclockwise order w.l.o.g from the starting point, and fulfills the conditions of the clockwise and counterclockwise definitions */

**Result:** An OSP \( P \) containing A list of visited points

Let \( \alpha \) be the angle between \( H_j \) and \( H_i \);

**Function Reflection\((s(t_0), H_i, H_j)\):**

\[
\text{return Segments: } I_k, I_m \text{ and Points: } s(t_0), p_r(t_1), p(t_2), t_0 \leq t_1 \leq t_2
\]

**Function Reaching\((p_s, p_\tau)\):**

Take straight line towards vertex \( v_i \);

Go along side, \( O, e_{i+1} \ldots e_{i+k}, k < n \);

Take straight line towards \( p_\tau \) as it is seen;

\[
\text{return Segments: } e_{i+1} \ldots e_{i+k} \text{ and Points: } p_s, v_i \ldots v_{i+k}, p_\tau
\]

for \( i, j := n - k \) to \( n \) do

if \( H_i, H_j \in V(p_i) \) then

Take Reflection\((s, p, H_i, H_j)\) Then compute distances of \( \text{Reaching}(p_s, p_\tau \in H_i)\);

else

Take \( \text{Reaching}(p_s, p_\tau \in H_i)\);

end

\[
f(t) = \min_i D(g(s, O, t), i);
\]

The Reflection function receives a starting point \( s \) and two half planes \( H_i \) and \( H_j \) and calculates the segments from the point \( s \) to the reflection point on the reflected half plane, \( p_\tau \in H_i \), denote this segment with \( I_k \) and the segment from that point \( p \) to the calculated half plane towards a suitable vertex of the polygon or perpendicularly
from which the robot finishes or continues with a reaching path, denoted $I_m$. These points and segments are calculated easily by taking the angle between the half planes and creating a two equation two variables system which are solved simultaneously.

6.2.4. Algorithm Complexity Proof

**Lemma 6.18.** The computational complexity of the above algorithm is $O(n^3)$.

**Proof.** The minimal of all options will be chosen as the optimal. 
Computational complexity configuration: There are $n$ vertices, $n$ calculations of angles of half planes, calculations of distances to half planes: $n - k$ half planes the robot is not placed in, and so there are $O(n^2)$ reaching paths to compute each path has $O(n)$ calculations (angles etc), therefore obtaining $O(n^3)$ time complexity.

□

It should be noted, that this algorithms aim is to find the exact optimal path and, although it might not obtain the best computational complexity. The length of the path is what we refer to as optimal (equivalently the traversing time).

**Corollary 6.19.** In an optimal scanning path if the robot is placed along side the polygon edges it will move away from it only for a reflection or perpendicular final approach.

**Proof.** Define $f_1(t_1) \rightarrow f_2(t_2)$ as a broken straight polygonal part of the path that is not along the side of the obstacle. Now assume in contradiction that such a deviation may occur in an optimal path. We know by the triangle inequality, that

$$||f_1(t_1) \rightarrow f_2(t_2)||_2 \geq ||f_1(t_1) f_2(t_2)||_2$$

because it is a straight line, in contradiction to optimality. This is true for every part of the path.

□

6.2.5. Correctness

**Theorem 6.20.** The OSP algorithm produces an optimal path $g(s, \mathcal{O}, t)$.

**Proof.** Algorithm OSP checks all possible orders of theorem [6.17] by proof of theorem [6.17] it follows that the shortest path is chosen.

□

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Figure 4: The robot starts at point S, calculates lines of sight i and h. Then takes a bisection line of the angle trapping the obstacle and then takes the perpendicular line. Determining the initial distance to traverse, it takes the projection of the extreme vertices on the perpendicular line. The created segment is marked Q. The robot starts motion towards the closer of the 2 projection points (point J) by a segment length of Q (from point S to point K). This is the beginning of the spiral-search, any new LOS is stored in memory. The robot goes towards the other side in a segment of 2·Q to point L. At this point we get an angle of size $\frac{\pi}{8} \in CS$. Now the robot starts moving perpendicularly towards the estimated half plane, until reaching point M. It continues its motion towards the line of $m_2$, perpendicularly, from M to N, and then finishes.
Figure 5: The shortest traversed path from point $S$ to point $T$. This is not an exploration path, as the robot does not visit all half planes.

Figure 6: A reflection is an incursion towards an half plane with a returning approach of an identical angle.