Solution of Einsteins Equation for Deformation of a Magnetized Neutron Star

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Abstract. We studied the effect of very large and non-uniform magnetic field existed in the neutron star on the deformation of the neutron star. We used in our analytical calculation, multipole expansion of the tensor metric and the momentum-energy tensor in Legendre polynomial expansion up to the quadrupole order. In this way we obtain the solutions of Einstein’s equation with the correction factors due to the magnetic field are taken into account. We obtain from our numerical calculation that the degree of deformation (ellipticity) is increased when the mass is decreased.

1. Introduction
Observations of magnetic fields of pulsars are usually in the range of values from $10^8$ to $10^{12}$ G. However, new classes of pulsars such as Anomalous X-ray pulsars (AXPs) and Soft-Gamma Repeater pulsars (SGRs) have been identified produce very large magnetic fields. SGR is usually associated with remnants of a supernova, which is in fact a young neutron star [1,2]. Furthermore, measurements of periods and derivatives of periods of the corresponding pulsar provide evidence that it neutron star with very large magnetic field with strength around $\sim 10^{15}$ G. Observation results of some AXPs indicate also that AXPs surface magnetic field around of $10^{14}$ – $10^{15}$ G. The relationship between SGR observation and AXPs are not yet clear enough up to now. However, it is confirmed that a class of neutron stars that have indeed a very large magnetic field and the corresponding neutron stars is called Magnetars [3–6] If we consider a neutron star matter that influenced by very strong magnetic field, we should take care two quantum effects, namely Pauli paramagnetism (i.e., the spin interaction of Fermions with a magnetic field) and Landau’s magnetism. The other is the Lorentz force which causes deformation of stars due to magnetic field [7]. In this study, we focused on the deformation occurring in neutron stars caused by very large magnetic fields, in which the deformation of spherical stars are considered within the framework of general relativity using by Hartle and Thorne method [8,9] and using also the assumptions that non-uniform pressure as a perturbation of the total (matter and magnetic) pressures as the one already done in Mallicks work [10].
2. Formalism

The total energy density and pressure neutron star with magnetic field can be written as follows\[10\]:

\[
\varepsilon = \varepsilon_m + \frac{B^2}{8\pi} \tag{1}
\]

\[
P_{\perp} = P_m + \frac{B^2}{8\pi} \tag{2}
\]

\[
P_{\parallel} = P_m - \frac{B^2}{8\pi}, \tag{3}
\]

where \(\varepsilon_m\) and \(P_m\) is the energy density and pressure of neutron star matter, \(B^2/8\pi\) is the magnetic stress contribution. Note that for calculating \(\varepsilon_m\) and \(P_m\) we use effective field relativistic mean field (ERMF) model with BSP parameter set for nucleons coupling constants and standard SU(6) and hyperons potential energy depths to obtain hyperons (\(\Lambda, \Sigma\) and \(\Xi\)) coupling constants (see details of the corresponding model in Ref. \[11\]). For the pressure part, as it is done in Mallicks work \[10\], it can be written in isotropic form as

\[
P = P_m \pm P_B \tag{4}
\]

\[
P = P_m + \frac{B^2}{8\pi}(1 - 2\cos^2 \theta), \tag{5}
\]

\(P_B\) is the magnetic stress and \(\theta\) is the polar angle with respect to the direction of the magnetic field. Total pressure can be written also Legendre polynomial expansion

\[
P = P_m + \frac{B^2}{8\pi}\left[\frac{1}{3} - \frac{4}{3} P_2(\cos \theta)\right] \tag{6}
\]

\[
P = P_m + p_0 + p_2 P_2(\cos \theta), \tag{7}
\]

\(p_0 = B^2/24\pi\) is the monopole and \(p_2 = -4B^2/24\pi\) is the quadrupole factors due to magnetic field. The most important thing in this study is that expansion of the magnetic field correction is only for the pressure part. Since the momentum-energy tensor is multipole expansion form, if we assume neutron star is initially has a sphere form then the Schwarzschild metric should also expand by using the Hartle and Thorne method \[8, 9\]

\[
ds^2 = -e^\nu [1 + 2(h_0 + h_2 P_2(\cos \theta))] dt^2 \\
+ e^\lambda \left[1 + \frac{e^\lambda}{r} (m_0 + m_2 P_2(\cos \theta)) \right] dr^2 \\
+ r^2 [1 + 2k_2 P_2(\cos \theta)] (d\theta^2 + \sin^2 \theta d\phi^2). \tag{8}
\]

Where the metric \(\nu\) or \(\lambda\) is a function of \(r\) and they can be expressed as

\[
\nu' = -\frac{2}{(\varepsilon + P)} P' \tag{9}
\]

\[
e^\lambda = \left(1 - \frac{2Gm}{r}\right)^{-1}. \tag{10}
\]

Here \(h_0, h_2, m_0, m_2, k_2\) are also function of \(r\), respectively and they can be considered as the correction factors due to magnetic field up to second order. By solving the Einsteins equation using above metric and energy momentum tensor and also by considering the conservation law of
total momentum-energy by employing the Bianchi identity we can obtain a usual TOV equations as

\[ P' = -\frac{G (\varepsilon + P)(m + 4\pi r^3 P)}{r^2} \left(1 - 2Gm/r\right), \]

\[ m' = 4\pi r^2 \varepsilon, \]

the correction for monopole order as

\[ m'_0 = 12\pi r^2 p_0, \]

\[ h'_0 = 4\pi Gr^\lambda p_0 + \frac{r}{G} G\nu^e m_0 + \frac{1}{r^2} G e^\lambda m_0, \]

as well as for quadrupole order as

\[ h'_2 + k'_2 = h_2 \left(1 + \frac{r'}{2}\right) + \frac{1}{r} \frac{r'}{G} G m_2 \left(1 + \frac{r'}{2}\right), \]

\[ h_2 + \frac{1}{r} \frac{r'}{G} G m_2 = 0, \]

\[ h'_2 + k'_2 \left(1 + \frac{r'}{2}\right) = 4\pi Gr^\lambda p_2 + \frac{1}{r} \frac{r'}{G} G m_2 + \frac{e^\lambda}{r} G \nu^e m_2 + \frac{3}{r} e^\lambda h_2 + \frac{2}{r} e^\lambda k_2, \]

and from conservation law total momentum-energy we also obtained:

\[ p'_0 = -2\nu' p_0 - (\varepsilon + P) h'_0, \]

\[ p_2 = - (\varepsilon + P) h_2, \]

\[ p'_2 = -\frac{1}{2} \frac{r'}{G} G m_2 - \frac{e^\lambda}{r} G \nu^e m_2. \]

With some simple algebra and using equations (15), (19), and (20) we can also obtain \( h'_2 \) and \( k'_2 \) in the following form

\[ k'_2 = \frac{3}{2} p_2 \nu' + p'_2 \]

\[ h'_2 = \frac{1}{2} p_2 \nu' - p'_2. \]

Using Runge-Kutta methods, we can solve numerically Eqs. (11), (12), (13), (14), (21), and (22) with boundary condition for standard TOV equation is \( P(0) = P_c, P(R) = 0 \) and \( m(0) = 0, m(R) = M \) where \( P_c \) is pressure in the neutron star center and \( R \) is radius of neutron stars while for monopole order \( h_0(0) = 0 \) and \( m_0(0) = 0 \), for quadrupole order \( k_2(0) = h_2(0) = 0 \) and \( m_2(0) = 0 \).

In this way wee can define total mass as [10],

\[ M = M_0 + m_0, \]

where \( M_0 \) is the matter’s mass and \( m_0 \) is mass correction due to the magnetic field. In the presence of mass correction of the neutron star, the anisotropic magnetic pressure on the neutron star is affected. Following Eq.(4) where the pressure of the matter and pressure due to magnetic
Figure 1. Straight line is the radius without magnetic field correction, dash line is radius with magnetic field correction.

Figure 2. Relation between ellipticity of magnetized neutron star and the corresponding Mass.

Field are summed along the equatorial direction and reduced the polar direction, we can write the radius equatorial and polar of the deformed neutron star as[10],

\[ R_e = R + \xi_0(R) - \frac{1}{2} (\xi_2(R) + Rk_2(R)) \]  \hspace{1cm} (24)
\[ R_p = R + \xi_0(R) + (\xi_2(R) + Rk_2(R)) , \] \hspace{1cm} (25)

where \( R \) radius of spherical neutron star (without magnetic field), \( \xi_0 \) and \( \xi_2 \) define by [8–10],
\[\xi_0(r) = -h_0 \frac{r(r-2Gm(r))}{G(4\pi r^3 P_m + m(r))}\] (26)
\[\xi_2(r) = -h_2 \frac{r(r-2Gm(r))}{G(4\pi r^3 P_m + m(r))}\] (27)

Where the contribution \(\xi\) comes from the magnetic field on the magnetar surface and contribution \(k_2\) comes from integration of magnetic pressure through inside of the star [10]. To know the degree of deformation in neutron stars, we can write the ratio of ellipticity as

\[e = \sqrt{1 - \left(\frac{R_p}{R_e}\right)^2}\] (28)

3. Result and Discussion
By solving Eqs. (11), (12), (13), (14), (21), and (22), we have a relationship between the mass and the radius of the neutron stars as shown in Fig. 1. \(M\) is the total mass of stars as expressed in Eq. (23). We can see that the magnetic field can increase the mass of the neutron star.

Fig. 2 shows the relation between the mass and the corresponding ellipticity of neutron star. If the polar radius of the neutron star increases then the mass decreases and if the equatorial radius of the neutron star increases the mass increases. the maximum deformation happen in the ellipticty \(\sim 0.257\). In this case, \(R_e > R_p\). Therefore, the neutron stars are maximally in oblate form.

4. Conclusion
Using this approach, we have obtained the solution of Einsteins equation in the form of standard TOV equations and added by some monopole and quadrupole order factor equations. The latters can also lead to first order differential equations Therefore, we can solve the corresponding differential equations by utilizing boundary conditions numerically. The solution informs us that in the presence of a very large magnetic field can change the form of the neutron star from sphere into the oblate shape if the corresponding mass is decreased.

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