ON LINEAR SHIFTS OF FINITE TYPE AND THEIR ENDOMORPHISMS

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Abstract. Let $G$ be a group and let $A$ be a finite-dimensional vector space over an arbitrary field $K$. We study finiteness properties of linear subshifts $\Sigma \subset A^G$ and the dynamical behavior of linear cellular automata $\tau : \Sigma \to \Sigma$. We say that $G$ is of $K$-linear Markov type if, for every finite-dimensional vector space $A$ over $K$, all linear subshifts $\Sigma \subset A^G$ are of finite type. We show that $G$ is of $K$-linear Markov type if and only if the group algebra $K[G]$ is one-sided Noetherian. We prove that a linear cellular automaton $\tau$ is nilpotent if and only if its limit set, i.e., the intersection of the images of its iterates, reduces to the zero configuration. If $G$ is infinite, finitely generated, and $\Sigma$ is topologically mixing, we show that $\tau$ is nilpotent if and only if its limit set is finite-dimensional. A new characterization of the limit set of $\tau$ in terms of pre-injectivity is also obtained.

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1. Introduction

Let $G$ be a group and let $A$ be a set, called the alphabet. The set $A^G := \{x : G \to A\}$, consisting of all maps from $G$ to $A$, is called the set of configurations over the group $G$ and the alphabet $A$. We equip $A^G = \prod_{g \in G} A$ with its prodiscrete uniform structure, i.e., the product uniform structure obtained by taking the discrete uniform structure on each factor $A$ of $A^G$. Thus, two configurations are “close” if they coincide on a “large” subset of $G$. Note that $A^G$ is a totally disconnected Hausdorff space and that $A^G$ is compact if and only if $A$ is finite. The shift action of the group $G$ on $A^G$ is the action defined by $(g, x) \mapsto gx$, where $gx(h) := x(g^{-1}h)$ for all $g, h \in G$ and $x \in A^G$. This action is uniformly continuous with respect to the prodiscrete uniform structure.

A closed $G$-invariant subset $\Sigma \subset A^G$ is called a subshift of $A^G$.

Given subsets $D \subset G$ and $P \subset A^D$, the set

$$
\Sigma(D, P) = \Sigma(A^G; D, P) := \{x \in A^G : (g^{-1}x)|_D \in P \text{ for all } g \in G\}
$$

is a $G$-invariant subset of $A^G$ (here $(g^{-1}x)|_D \in A^D$ denotes the restriction of the configuration $g^{-1}x$ to $D$). When $D$ is finite, $\Sigma(D, P)$ is also closed in $A^G$, and therefore is a subshift. One then says that $\Sigma(D, P)$ is the subshift of finite type, briefly SFT, associated with $(D, P)$ and that $D$...
(resp. $P$) is a defining memory set (resp. a defining set of admissible patterns) for $Σ$. Note that a defining set of admissible patterns for an SFT is not necessarily finite.

Let $B$ be another alphabet set. A map $τ: B^G → A^G$ is called a cellular automaton, briefly a CA, if there exist a finite subset $M ⊂ G$ and a map $µ: B^M → A$ such that

\[ τ(x)(g) = µ((g^{-1}x)|_M) \quad \text{for all } x ∈ B^G \text{ and } g ∈ G. \]

Such a set $M$ is then called a memory set and $µ$ is called a local defining map for $τ$. It is immediate from the above definition that every CA $τ: B^G → A^G$ is uniformly continuous and $G$-equivariant (cf. [6] Theorem 1.1), see also [8] Theorem 1.9.1).

More generally, if $Σ_1 ⊂ B^G$ and $Σ_2 ⊂ A^G$ are subshifts, a map $τ: Σ_1 → Σ_2$ is called a CA if it can be extended to a CA $\tilde{τ}: B^G → A^G$.

Suppose now that $A$ and $B$ are vector spaces over a field $K$. Then $A^G$ and $B^G$ inherit a natural $K$-vector space structure. A subshift $Σ ⊂ A^G$ which is also a vector subspace of $A^G$ is called a linear subshift. A $K$-linear CA $τ: B^G → A^G$ is called a linear CA. Note that a CA $τ: B^G → A^G$ with memory set $M ⊂ G$ is linear if and only if the associated local defining map $µ: B^M → A$ is $K$-linear (see [8] Section 8.1]).

More generally, given linear subshifts $Σ_1 ⊂ B^G$ and $Σ_2 ⊂ A^G$, a map $τ: Σ_1 → Σ_2$ is called a linear CA if it is the restriction of some linear CA $\tilde{τ}: B^G → A^G$.

A linear subshift $Σ ⊂ A^G$ is called a linear-sofic subshift provided there exists a vector space $B$, a linear SFT $Σ' ⊂ B^G$, and a linear CA $τ: B^G → A^G$ such that $τ(Σ') = Σ$.

In our recent papers [12, 20] we introduced the notion of an algebraic sofic subshift $Σ ⊂ A^G$, where $A$ is the set of $K$-points of an algebraic variety over an algebraically closed field $K$, and studied cellular automata $τ: Σ → Σ$ whose local defining maps are induced by algebraic morphisms. When referring to these notions, we shall refer to the “algebraic setting”. When the field $K$ is algebraically closed, linear-sofic subshifts and linear cellular automata are algebraic sofic subshifts and algebraic cellular automata, respectively. Therefore, several results in [12, 20] hold true in the present setting, even when the field $K$ is not algebraically closed, by a direct adaptation of the proofs given therein. However the proofs in the algebraic setting are much more technical and involved, and one of the purposes of this paper is to present simpler and more direct proofs of these results in the linear setting. We also obtain several new results and consequences as indicated below.

Our first result is a linear version of the well known characterization of SFT with finite alphabets by the descending chain condition (see [12] Theorem 10.1] and [20] Proposition 6.1] for a similar result in the algebraic setting, and [20] Proposition 9.17] for the more general admissible group shifts, [20] Definition 9.11].)

**Theorem 1.1.** Let $G$ be a countable group and let $A$ be a finite-dimensional vector space over a field $K$. Let $Σ ⊂ A^G$ be a linear subshift. Then the following conditions are equivalent:

(a) $Σ$ is a SFT;
(b) every decreasing sequence of linear subshifts of $A^G$

\[ Σ_0 ⊃ Σ_1 ⊃ ⋯ ⊃ Σ_n ⊃ Σ_{n+1} ⊃ ⋯ \]

such that $Σ = \bigcap_{n∈\mathbb{N}} Σ_n$, eventually stabilizes (that is, there exists $n_0 ∈ \mathbb{N}$ such that $Σ_{n_0} = Σ_n$ for all $n ≥ n_0$).

**Corollary 1.2.** Let $G$ be a countable group and let $A$ be a finite-dimensional vector space over a field $K$. Then the following conditions are equivalent:

(a) every linear subshift $Σ ⊂ A^G$ is an SFT;
(b) $A^G$ satisfies the descending chain condition for linear subshifts, that is, every decreasing sequence of linear subshifts of $A^G$

\[ Σ_0 ⊃ Σ_1 ⊃ ⋯ ⊃ Σ_n ⊃ Σ_{n+1} ⊃ ⋯ \]

eventually stabilizes.

Given a field $K$, we say that a group $G$ is of $K$-linear Markov type provided that the equivalent conditions in Corollary 1.2 hold for every finite-dimensional vector space $A$ over $K$. 
Let $G$ be a group and let $K$ be a field. Given $\alpha \in K[G]$ we define $\alpha^* \in K[G]$ by setting $\alpha^*(g) := \alpha(g^{-1})$ for all $g \in G$. It is straightforward to check that the map $\alpha \mapsto \alpha^*$ yields a $K$-algebra isomorphism of the group algebra $K[G]$ onto the opposite algebra $K[G]^{opp}$. As a consequence, $K[G]$ is left-Noetherian if and only if it is right-Noetherian and, if this is the case, we simply say that $K[G]$ is one-sided Noetherian. In the proofs, however, in order to use a working definition at hand, we shall always refer to left Noetherianity.

We have the following characterization of countable groups of $K$-linear Markov type.

**Theorem 1.3.** Let $G$ be a countable group and let $K$ be a field. Then the group algebra $K[G]$ is one-sided Noetherian if and only if $G$ is of $K$-linear Markov type.

Recall that a group $G$ is said to be *polycyclic* if it admits a subnormal series with cyclic factors, that is, a finite sequence $G = G_0 \supset G_1 \supset \cdots \supset G_n = \{1_G\}$ of subgroups such that $G_{i+1}$ is normal in $G_i$ and $G_i/G_{i+1}$ is (possibly infinite) cyclic group, for $i = 0, 1, \ldots, n - 1$. More generally, $G$ is said to be *polycyclic-by-finite* if it admits a polycyclic subgroup of finite index.

The following result is the linear version of a famous result by Klaus Schmidt [30, Theorem 4.2] (see also [20]). The third-named author considered the notion of an *admissible Artinian group structure* (cf. [26, Definition 9.1]): this includes, for instance, a group structure of finite Morley rank, e.g., an algebraic group, an Artinian group, or an Artinian module. Since a finite-dimensional vector space over a field $K$ is an Artinian $K$-module and therefore it is naturally equipped with an admissible Artinian group structure (see [26, Example 9.12]), Corollary [14] also constitutes a simpler case of the more general results [26, Theorem 9.13] or [26, Theorem 1.11], where the alphabet set can be taken as an admissible Artinian group structure. Also note that the proof of [30, Theorem 4.2] heavily relies on the fact that the alphabet set therein is a *compact* Lie group, so that the configuration space (equipped with the product topology) is itself compact. We deduce Corollary [14] below from Theorem [13] and a result of Philip Hall ([18], see also [24, Corollary 2.8]) extending Hilbert’s basis theorem on Noetherian rings. An alternative self-contained proof of Corollary [14] using only linear symbolic dynamics, is presented in Remark [6.0].

**Corollary 1.4.** Let $K$ be a field. Then all polycyclic-by-finite groups (e.g., the free abelian groups $\mathbb{Z}^d$, $d \geq 1$) are of $K$-linear Markov type.

The free group $F_2$ (and, more generally, any group which contains a subgroup which is not finitely generated) is not of $K$-linear Markov type (see Section [9]). Groups of $K$-linear Markov type, or, more generally, *monoids of $K$-linear Markov type*, satisfy interesting topological properties. For example, it is shown in [27] that the natural action of every finitely generated abelian monoid of linear CA on any linear subshift satisfies the shadowing property.

Let now $f: X \to X$ be a selfmap of a set $X$.

One has $X \supset f(X) \supset f^2(X) \supset \cdots \supset f^n(X) \supset f^{n+1}(X) \supset \cdots$ and the set $\Omega(f) := \bigcap_{n\geq 1} f^n(X) \subset X$ is called the *limit set* of $f$. This is the set of points of $X$ that occur after iterating $f$ arbitrarily many times. The notion of a limit set was introduced in the framework of cellular automata by Wolfram [31] and was subsequently investigated for instance in [13], [17], [19], [22], and [12].

Observe that $f(\Omega(f)) \subset \Omega(f)$. The inclusion may be strict (cf. [12, Proposition A.2.(iii)] and Example (3) in Section [7.3] and equality holds if and only if every $x \in \Omega(f)$ admits a *backward orbit*, i.e., a sequence $(x_i)_{i\geq 0}$ of points of $X$ such that $x_0 = x$ and $f(x_{i+1}) = x_i$ for all $i \geq 0$. Clearly, $f$ is surjective if and only if $\Omega(f) = X$. Note also that $\text{Per}(f) := \bigcup_{n \geq 1} \{x \in X : f^n(x) = x\}$, the set of *$f$-periodic points*, is contained in $\Omega(f)$ and that $\Omega(f^n) = \Omega(f)$ for every $n \geq 1$. One says that the map $f$ is *stable* if $f^{n+1}(X) = f^n(X)$ for some $n \geq 1$.

Assume that $X$ is a topological space and $f: X \to X$ is a continuous map. One says that $x \in X$ is a *recurrent* (resp. *non-wandering*) point of $f$ if for every neighborhood $U$ of $x$, there exists $n \geq 1$ such that $f^n(x) \in U$ (resp. $f^n(U)$ meets $U$). Let $R(f)$ (resp. NW($f$)) denote the set of recurrent (resp. non-wandering) points of $f$. It is immediate that $\text{Per}(f) \subset R(f) \subset \text{NW}(f)$ and that NW($f$) is a closed subset of $X$.
Suppose now that $X$ is a uniform space and let $f : X \to X$ be a uniformly continuous map. One says that a point $x \in X$ is chain-recurrent if for every entourage $E$ of $X$ there exist an integer $n \geq 1$ and a sequence of points $x_0, x_1, \ldots, x_n \in X$ such that $x = x_0 = x_n$ and $(f(x_i), x_{i+1}) \in E$ for all $0 \leq i \leq n - 1$. We shall denote by $\text{CR}(f)$ the set of chain-recurrent points of $f$. Observe that $\text{CR}(f)$ is always closed in $X$.

We shall establish the following result (compare with \cite[Theorem 1.3]{12} in the algebraic setting).

**Theorem 1.5.** Let $G$ be a finitely generated group and let $A$ be a finite-dimensional vector space over a field $K$. Let $\Sigma \subset A^G$ be a linear subshift and let $\tau : \Sigma \to \Sigma$ be a linear CA. Then the following hold:

(i) $\Omega(\tau)$ is a linear subshift of $A^G$;

(ii) $\tau(\Omega(\tau)) = \Omega(\tau)$;

(iii) $\text{Per}(\tau) \subset \text{R}(\tau) \subset \text{NW}(\tau) \subset \text{CR}(\tau) \subset \Omega(\tau)$;

(iv) if $\Omega(\tau)$ is of finite type then $\tau$ is stable;

(v) if $\Omega(\tau)$ is finite-dimensional then $\tau$ is stable.

In the above theorem, we may relax the condition on $G$ being finitely generated provided we assume in addition that the linear subshift $\Sigma \subset A^G$ is linear-sofic. We thus have the following.

**Corollary 1.6.** Let $G$ be a group and let $A$ be a finite-dimensional vector space over a field $K$. Let $\Sigma \subset A^G$ be a linear-sofic subshift (e.g., a linear SFT) and let $\tau : \Sigma \to \Sigma$ be a linear CA. Then properties (i) – (v) in Theorem 1.5 hold.

As in \cite{12}, the proof relies on the analysis of the so called space-time inverse system associated with a CA (cf. Section 3.2).

Let $G$ be a group and let $A$ be a set. A CA $\tau : \Sigma_1 \to \Sigma_2$ between subshifts of $A^G$ is called pre-injective if whenever $x, y \in A^G$ are two configurations that coincide outside of a finite subset of $G$ and satisfy $\tau(x) = \tau(y)$, then one has $x = y$. When $A$ is a vector space over a field $K$ and $\Sigma_1, \Sigma_2 \subset A^G$ are linear subshifts, a linear CA $\tau : \Sigma_1 \to \Sigma_2$ is pre-injective if and only if the restriction of $\tau$ to the vector subspace of configurations in $\Sigma_1$ with finite support is injective. A subshift $\Sigma \subset A^G$ is called strongly irreducible if there exists a finite subset $\Delta \subset G$ such that for all $x, y \in A^G$ and for all finite subsets $E, F \subset G$ such that $E \cap F \Delta = \emptyset$, then there exists $z \in \Sigma$ such that $z|_E = x|_E$ and $z|_F = y|_F$.

We obtain the following characterization of limit sets of linear cellular automata in terms of pre-injectivity.

**Corollary 1.7.** Let $G$ be a polycyclic-by-finite group and let $A$ be a finite-dimensional vector space. Let $\Sigma \subset A^G$ be a strongly irreducible linear subshift and let $\tau : \Sigma \to \Sigma$ be a linear CA. Then $\Omega(\tau)$ is the largest strongly irreducible linear subshift $\Lambda \subset A^G$ contained in $\Sigma$ such that $\tau(\Lambda) \subset \Lambda$ and $\tau|_\Lambda$ is pre-injective.

Given a set $X$, one says that a map $f : X \to X$ is nilpotent if there exist a constant map $c : X \to X$ and an integer $n_0 \geq 1$ such that $f^{n_0} = c$. This implies $f^n = c$ for all $n \geq n_0$. Such a constant map $c$ is then unique and we say that the unique point $x_0 \in X$ such that $c(x) = x_0$ for all $x \in X$ is the terminal point of $f$. The terminal point of a nilpotent map is its unique fixed point. Observe that if $f : X \to X$ is nilpotent with terminal point $x_0$ then $\Omega(f) = \{x_0\}$ is a singleton. The converse is true in general (cf. \cite[Proposition A.2.(ii)]{12} and Example (1) in Section 7.3).

For linear cellular automata we establish the following characterization of nilpotency.

**Theorem 1.8.** Let $G$ be a group and let $A$ be a finite-dimensional vector space over a field $K$. Let $\Sigma \subset A^G$ be a linear-sofic subshift (e.g., a linear SFT) and let $\tau : \Sigma \to \Sigma$ be a linear CA. Then the following conditions are equivalent:

(a) $\tau$ is nilpotent;

(b) $\Omega(\tau) = \{0\}$.

The analog of Theorem 1.8 for classical cellular automata follows from \cite[Theorem 3.5]{14}. In the algebraic setting, this corresponds to \cite[Theorem 1.4]{12}. 
Given a set $X$, one says that a map $f: X \to X$ is pointwise nilpotent if there exist a point $x_0 \in X$ such that for each $x \in X$ there exists an integer $n_x \geq 1$ such that $f^{n_x}(x) = x_0$ for all $n \geq n_x$. Such a point $x_0$ is clearly unique and it is called the terminal point of $f$. If $f$ is nilpotent then it is also pointwise nilpotent and the terminal points relative to the two notions of nilpotency coincide.

Let $G$ act on a Hausdorff topological space $X$. One says that the action is topologically mixing provided that given two nonempty open subsets $U, V \subset X$ there exists a finite subset $F \subset G$ such that $U \cap gV \neq \emptyset$ for all $g \in G \setminus F$. If $A$ is a set, a subshift $\Sigma \subset A^G$ is said to be topologically mixing if the restriction to $\Sigma$ of the $G$-shift is topologically mixing.

We obtain the following characterization of nilpotency for linear cellular automata over infinite groups.

**Theorem 1.9.** Let $G$ be a finitely generated infinite group and let $A$ be a finite-dimensional vector space over a field $K$. Let $\Sigma \subset A^G$ be a topologically mixing linear subshift (e.g., $\Sigma = A^G$) and let $\tau: \Sigma \to \Sigma$ be a linear CA. Then the following conditions are equivalent:

(a) $\tau$ is nilpotent;
(b) $\tau$ is pointwise nilpotent;
(c) there exists $n_0 \in \mathbb{N}$ such that $\tau^{n_0}(\Sigma)$ is finite-dimensional;
(d) $\Omega(\tau)$ is finite-dimensional;
(e) $\Omega(\tau) = \{0\}$.

In the above theorem, we may relax the condition of being finitely generated on the infinite group $G$ provided we assume that the subshift $\Sigma \subset A^G$ is, in addition, linear-sofic. We thus have the following.

**Corollary 1.10.** Let $G$ be an infinite group and let $A$ be a finite-dimensional vector space over a field $K$. Let $\Sigma \subset A^G$ be a topologically mixing linear-sofic subshift (e.g., $\Sigma = A^G$) and let $\tau: \Sigma \to \Sigma$ be a linear CA. Then conditions (a) - (e) in Theorem 1.9 are all equivalent.

The analog of Theorem 1.9 for classical cellular automata follows from [12, Theorem 3.5] (see also [17, Corollary 4]). In the algebraic setting, this corresponds to [12, Theorem 1.5], but the equivalence of (a), (b), (c), and (e) with the point (d) is a new result.

The paper is organized as follows. In Section 2 we fix notation and establish some preliminary results. In particular, we study linear SFTs and show that every finite-dimensional linear subshift $\Sigma \subset A^G$ is of finite type if the group $G$ is finitely generated (Proposition 2.4). We introduce the notion of a memory set for a linear-sofic subshift $\Sigma$ and, given a cellular automaton $\tau: \Sigma \to \Sigma$, we review the properties of the restriction cellular automaton $\tau_H: \Sigma_H \to \Sigma_H$, for any subgroup $H$ containing both a memory set for $\tau$ and a memory set for $\Sigma$. As an application, we establish a relation between the limit sets $\Omega(\tau)$ and $\Omega(\tau_H)$ of $\tau$ and $\tau_H$, respectively, and deduce that $\tau$ is nilpotent if and only if $\tau_H$ is nilpotent (Lemma 2.7). In Section 3 we review from [12] the notion of space-time inverse system, together with its inverse limit, associated with a cellular automaton $\tau: \Sigma \to \Sigma$, where $\Sigma$ is a linear-sofic subshift in $A^G$, with $G$ a countable group and $A$ a finite-dimensional vector space over a field $K$. As an application, in the subsequent section we prove the closed image property for linear cellular automata: we show that essentially under the above assumptions, $\tau(\Sigma)$ is closed in the prodiscrete topology in $A^G$ (Theorem 4.1). In Section 5 we present the proofs of all results stated in the Introduction. In Section 6 we further investigate the class of groups of $K$-linear Markov type. We show that this class is closed under the operation of taking subgroups, quotients, and extensions by finite and cyclic groups, and that it is contained in the class of Noetherian groups (the latter are the groups satisfying the maximal condition on subgroups). Finally, in the last section we present some examples/counterexamples and discuss some further remarks. In particular, in Subsection 7.2 we present an example of a linear cellular automaton $\tau: A^G \to A^G$, where $G$ is any non-periodic group (e.g., $G = \mathbb{Z}$) and $A$ is any infinite-dimensional vector space, which does not satisfy the closed image property. At last, in Subsection 7.3 we study nilpotency and pointwise nilpotency for linear cellular automata over infinite-dimensional vector spaces and present some examples of the associated limit sets. As
a byproduct, we show that the conclusions of Theorem 1.9 may fail to hold, in general, if the finite-dimensionality of the alphabet set $A$ is dropped from the assumptions.

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2. Preliminaries

2.1. Notation. We use the symbols $\mathbb{Z}$ for the integers and $\mathbb{N}$ for the non-negative integers.

We write $A^B$ for the set consisting of all maps from a set $B$ into a set $A$. Let $C \subset B$. If $x \in A^B$, we denote by $x|_C$ the restriction of $x$ to $C$, that is, the map $x|_C : C \to A$ given by $x|_C(c) = x(c)$ for all $c \in C$. If $X \subset A^B$, we set $X_C := \{x|_C : x \in X\} \subset A^C$.

Let $E, F$ be subsets of a group $G$. We define $EF := \{gh : g \in E, h \in F\}$ and define inductively $E^n$ for all $n \in \mathbb{N}$ by $E^0 := \{1_G\}$ and $E^{n+1} := E^nE$.

Let $A$ be a set and let $E$ be a subset of a group $G$. Given $x \in A^E$, we define $gx \in A^E$ by $(gx)(h) := x(g^{-1}h)$ for all $h \in gE$.

2.2. Topologically transitive linear subshifts. An action of a group $G$ on a Hausdorff topological space $X$ is called topologically transitive provided that given any two nonempty open subsets $U, V \subset X$ there exists $g \in G$ such that $U \cap gV \neq \emptyset$. Moreover, if $A$ is a set, a subshift $\Sigma \subset A^G$ is said to be topologically transitive if the restriction to $\Sigma$ of the $G$-shift is topologically transitive. It is straightforward that if the acting group $G$ is infinite, then every topologically mixing $G$-action (e.g., any topologically mixing subshift $\Sigma \subset A^G$) is topologically transitive.

The following result, which we shall use in the proof of Theorem 1.9, has some interest on its own. We thank the referee for pointing out a gap in our original argument and providing us with an outline of the present proof.

Proposition 2.1. Let $G$ be a group and let $A$ be a vector space over a field $K$. Let $\Sigma \subset A^G$ be a linear subshift and suppose that $\Sigma$ is topologically transitive and finite dimensional. Then $\Sigma = \{0\}$.

Proof. Suppose, by contradiction, that $\Sigma$ is nontrivial. Let $x_0 \in \Sigma \setminus \{0\}$ and let $g_0 \in G$ such that $a := x_0(g_0) \neq 0_A$. Consider the open subsets $U_1 := \{x \in \Sigma : x(g_0) = 0_A\}$ and $V_1 := \{x \in \Sigma : x(g_0) = a\}$. Note that $U_1 \neq \emptyset$ since $0 \in U_1$ and $V_1 \neq \emptyset$ since $x_0 \in V_1$. By topological transitivity, we can find $h_1 \in G$ such that $U_1 \cap h_1V_1 \neq \emptyset$. Thus, if $z_1 \in U_1 \cap h_1V_1$ and $g_1 := h_1g_0$, we have $z_1(g_0) = 0_A$ and $z_1(g_1) = a$. Consider now the open subsets $U_2 := \{x \in \Sigma : x(g_0) = x(g_1) = 0_A\}$ and $V_2 := \{x \in \Sigma : x(g_0) = 0 \text{ and } x(g_1) = a\}$. Note that $U_2 \neq \emptyset$ since $0 \in U_2$ and $V_2 \neq \emptyset$ since $x_1 \in V_2$. By topological transitivity, we can find $h_2 \in G$ such that $U_2 \cap h_2V_2 \neq \emptyset$. Thus, if $z_2 \in U_2 \cap h_2V_2$ and $g_2 := h_2g_1$, we have $z_2(g_0) = z_2(g_1) = 0_A$ and $z_2(g_2) = a$. Continuing this way, we inductively find $z_1, z_2, \ldots \in \Sigma$ and group elements $g_1, g_2, \ldots \in G$ such that $z_n(g_0) = z_n(g_1) = \cdots = z_n(g_{n-1}) = 0_A$ and $z_n(g_n) = a$ for all $n \geq 1$. As $a \neq 0_A$, it is straightforward that the configurations $z_1, z_2, \ldots$ are linearly independent, contradicting the assumption that $\Sigma$ is finite dimensional. We deduce that $\Sigma = \{0\}$. □

2.3. Linear subshifts of finite type. We begin with two simple useful facts.

Lemma 2.2. Let $A$ be a set and let $G$ be a group. Let $D \subset G$ and let $P \subset A^D$. Let $\Sigma := \Sigma(D, P) \subset A^G$. Let $E \subset G$ such that $D \subset E$. Then one has $\Sigma = \Sigma(E, \Sigma_E)$. In particular, $\Sigma = \Sigma(D, \Sigma_D)$.

Proof. Let $x \in \Sigma$ and let $g \in G$. Then $(g^{-1}x)|_E \in \Sigma_E$. Thus $\Sigma \subset \Sigma(E, \Sigma_E)$. Conversely, let $x \in \Sigma(E, \Sigma_E)$. Then, for every $g \in G$, we have $(g^{-1}x)|_D = ((g^{-1}x)|_E)|_D \in (\Sigma_E)_D = \Sigma_D \subset P$, since $D \subset E$. Therefore, $x \in \Sigma(D, P) = \Sigma$, and the conclusion follows. □

The following lemma states that every linear SFT admits a defining set of admissible patterns which is a vector space.
Lemma 2.3. Let $G$ be a group and let $A$ be a vector space over a field $K$. Let $Σ ⊂ A^G$ be a linear SFT and let $D ⊂ G$ be a memory set for $Σ$. Then there exists a vector subspace $W ⊂ A^D$ such that $Σ = Σ(D,W)$.

Proof. The set $W := Σ_D = \{x|_D : x ∈ Σ\} ⊂ A^D$ is a vector subspace of $A^D$ and we have $Σ = Σ(D,W)$ by Lemma 2.2.

Proposition 2.4. Let $G$ be a finitely generated group and let $A$ be a vector space over a field $K$. Then every finite-dimensional linear subshift $Σ ⊂ A^G$ is of finite type.

Proof. Let $S ⊂ G$ be a finite generating subset of $G$. After replacing $S$ by $S ∪ S^{-1} ∪ \{1_G\}$, we can assume that $S = S^{-1}$ and $1_G ∈ S$. Then, given any element $g ∈ G$, there exist $n ∈ \mathbb{N}$ and $s_1,s_2,\ldots,s_n ∈ S$ such that $g = s_1s_2\cdots s_n$. The minimal $n ∈ \mathbb{N}$ in such an expression of $g$ is the $S$-length of $g$, denoted by $ℓ_S(g)$. For every $n ∈ \mathbb{N}$ we set $B_n := \{g ∈ G : ℓ_S(g) ≤ n\}$.

Let $Σ ⊂ A^G$ be a finite-dimensional linear subshift. For every $n ∈ \mathbb{N}$ denote by $π_n : Σ → Σ_{B_n}$ the restriction map. Note that $π_n$ is linear and that setting $Σ_n := ker π_n$ we have that $(Σ_n)_{n∈\mathbb{N}}$ is a decreasing sequence of vector subspaces of $Σ$. Now, on the one hand, since $Σ$ is a decreasing sequence of vector subspaces of $Σ$, then there exist $n_0 ∈ \mathbb{N}$ such that $Σ_n = Σ_{n_0}$ for all $n ≥ n_0$. We deduce that $Σ_n = \{0\}$. Thus, setting $Δ := Σ_{n_0}$, the restriction map $Σ → Σ_{Δ}$ is injective (in fact bijective).

Set $D := SΔ ⊂ G$ and $W := Σ_D ⊂ A^D$, and let us show that $Σ = Σ(D,W)$.

Let $x ∈ Σ$. Then for every $g ∈ G$ we have $g^{-1}x ∈ Σ$ so that $(g^{-1}x)|_D ∈ Σ_D = W$. This shows that $x ∈ Σ(D,W)$, and the inclusion $Σ ⊂ Σ(D,W)$ follows.

Conversely, suppose that $x ∈ Σ(D,W)$. By definition of $Σ(D,W)$, for every $g ∈ G$, there exists $x_g ∈ Σ$ such that $(g^{-1}x)|_D = (x_g)|_D$. Observe that, given $g ∈ G$, such an $x_g$ is unique since $Δ ⊂ D$. Let us show, by induction on the $S$-length of $g$, that

\begin{equation}
 x_g = g^{-1}x_1g
\end{equation}

for all $g ∈ G$. If $ℓ_S(g) = 0$, then $g = 1_G$ and there is nothing to prove. Suppose now that (2.1) is satisfied for all $g ∈ G$ such that $ℓ_S(g) = n$ and let $h ∈ G$ such that $ℓ_S(h) = n + 1$. Then there exist $g ∈ G$ with $ℓ_S(g) = n$ and $s ∈ S$ such that $h = gs$. For all $d ∈ Δ$, we have

\begin{align*}
x_h(d) &= (h^{-1}x)(d) \quad (\text{since } Δ ⊂ D) \\
&= (g^{-1}x)(sd) \quad (\text{since } h = gs) \\
&= x_g(sd) \quad (\text{since } SΔ ⊂ D) \\
&= (g^{-1}x_1g)(sd) \quad (\text{by our induction hypothesis}) \\
&= (h^{-1}x_1g)(d) \quad (\text{since } h = gs).
\end{align*}

Thus $x_h$ and $h^{-1}x_1g$ coincide on $Δ$. As $x_h,h^{-1}x_1g ∈ Σ$, this implies that $x_h = h^{-1}x_1g$. By induction, we conclude that (2.1) holds for all $g ∈ G$. Since $1_G ∈ D$, we deduce that

\begin{equation}
 x(g) = (g^{-1}x)(1G) = x_g(1G) = (g^{-1}x_1g)(1G) = x_1g(g)
\end{equation}

for all $g ∈ G$. This shows that $x = x_1g ∈ Σ$, and the inclusion $Σ(D,W) ⊂ Σ$ follows.

In conclusion, $Σ = Σ(D,W)$ is a shift of finite type.

The condition that $G$ is finitely generated cannot be removed from the assumptions in Proposition 2.4. In fact we have the following (cf. [29, Lemma 1]; see also Section 6).

Corollary 2.5. Let $G$ be a group and $A$ be a nontrivial finite-dimensional vector space over a field $K$. Consider the subshift $Σ ⊂ A^G$ consisting of all constant configurations. Then $Σ$ is a SFT if and only if $G$ is finitely generated.

Proof. As the linear subshift $Σ$ satisfies $dim_K(Σ) = dim_K(A) < ∞$, it is a SFT whenever $G$ is finitely generated by Proposition 2.3.

Conversely, suppose that $Σ$ is an SFT. Thus there exists a finite subset $D ⊂ G$ and $P ⊂ A^D$ such that $Σ = Σ(D,P)$. Consider the subgroup $H ⊂ G$ generated by $D$ and let $a ∈ A$ such that $a ≠ 0$. Then the configuration $x ∈ A^G$ such that $x(g) = 0$ if $g ∈ H$ and $x(g) = a$ otherwise belongs
to $\Sigma(D, P)$ since, for each $g \in G$, either $gD \subset H$ or $gD \subset G \setminus H$. As $\Sigma(D, P) = \Sigma$ and every configuration in $\Sigma$ is constant, we conclude that $H = G$. Therefore $G$ is finitely generated. $\square$

2.4. Restriction of linear-sofic subshifts and of linear CAs. Let $G$ be a group and let $A$ be a vector space over a field $K$. Recall that a linear subshift $\Sigma \subset A^G$ is said to be a linear-sofic subshift if there exists a vector space $B$ over $K$, an SFT $\Sigma' \subset B^G$, and a linear cellular automaton $\tau : B^G \to A^G$ such that $\Sigma = \tau(\Sigma')$. We shall refer to a finite subset $M \subset G$ containing both a memory set for $\Sigma'$ as well as a memory set for $\tau$ as to a memory set for the linear-sofic subshift $\Sigma$.

Let $\Sigma \subset A^G$ be a linear-sofic subshift. Let $H \subset G$ be a subgroup of $G$ containing a memory set for $\Sigma$. Denote by $G/H := \{gH : g \in G\}$ the set of all right cosets of $H$ in $G$. As the right cosets of $H$ in $G$ form a partition of $G$, we have a natural factorization

$$A^G = \prod_{c \in G/H} A^c$$

in which each $x \in A^G$ is identified with $(x|_c)_{c \in G/H} \in \prod_{c \in G/H} A^c$. The above factorization of $A^G$ induces a factorization (cf. [12, Lemma 2.8])

$$\Sigma = \prod_{c \in G/H} \Sigma_c,$$

where $\Sigma_c = \{x|_c : x \in \Sigma\}$ is a vector subspace of $A^c$ for all $c \in G/H$.

Let $T \subset G$ be a complete set of representatives for the right cosets of $H$ in $G$ such that $1_G \in T$. Then, for each $c \in G/H$, we have a linear uniform homeomorphism $\phi_c : \Sigma_c \to \Sigma_H$ given by $\phi_c(y)(h) = y(gh)$ for all $y \in \Sigma_c$, where $g \in T$ represents $c$.

Now suppose in addition that $\tau : \Sigma \to \Sigma$ is a linear CA which admits a memory set contained in $H$. Then we have $\tau = \prod_{c \in G/H} \tau_c$, where $\tau_c : \Sigma_c \to \Sigma_c$ is the linear map defined by setting $\tau_c(y) := \tau(x)|_c$ for all $y \in \Sigma_c$, where $x \in \Sigma$ is any configuration extending $y$. Note that for each $c \in G/H$, the linear maps $\tau_c$ and $\tau_H$ are conjugated by $\phi_c$, i.e., we have $\tau_c = \phi_c^{-1} \circ \tau_H \circ \phi_c$. This allows us to identify the action of $\tau_c$ on $\Sigma_c$ with that of the restriction cellular automaton $\tau_H$ on $\Sigma_H$.

The following extends [9, Theorem 2.1] (cf. [12, Lemma 2.10]).

**Lemma 2.6.** Let $G$ be a group, let $A$ be a vector space over a field $K$, and let $\Sigma \subset A^G$ be a linear-sofic subshift. Let $\tau : A^G \to A^G$ be a cellular automaton. Let $H \subset G$ be a subgroup containing memory sets for both $\Sigma$ and $\tau$. Then $\tau(\Sigma)$ is closed in $A^G$ if and only if $\tau_H(\Sigma_H)$ is closed in $A^H$.

**Proof.** With the above notation, we have $\tau(\Sigma) = \prod_{c \in G/H} \tau_c(\Sigma_c)$. It is immediate that $\tau_H(\Sigma_H)$ is closed in $A^H$ if $\tau(\Sigma)$ is closed in $A^G$. Conversely, if $\tau_H(\Sigma_H)$ is closed in $A^H$, then so are $\tau_c(\Sigma_c) = \phi_c^{-1}(\tau_H(\Sigma_H))$ in $A^c$ for all $c \in G/H$, since the $\phi_c : A^c \to A^H$ are uniform homeomorphisms. Consequently, $\tau(\Sigma)$ is closed in $A^G$ whenever $\tau_H(\Sigma_H)$ is closed in $A^H$ since the product of closed subspaces is closed in the product topology. $\square$

2.5. Nilpotent linear cellular automata. Let $G$ be a group, let $A$ be a vector space over a field $K$, and let $\tau : \Sigma \to \Sigma$ be a linear cellular automaton, where $\Sigma \subset A^G$ is a linear subshift. By linearity, $\tau$ is nilpotent if and only if there exists an integer $n_0 \geq 1$ such that $\tau^{n_0} = 0$. Moreover, $\tau^n(\Sigma)$, $n \in \mathbb{N}$, and therefore $\Omega(\tau)$ are vector subspaces of $\Sigma$.

The following is the linear version of [12, Lemma 2.9].

**Lemma 2.7.** Let $G$ be a group, let $A$ be a vector space over a field $K$, and let $\Sigma \subset A^G$ be a linear-sofic subshift. Let $\tau : \Sigma \to \Sigma$ be a linear cellular automaton. Let $H \subset G$ be a subgroup containing memory sets of both $\Sigma$ and $\tau$. Then the following hold:

(i) $\Omega(\tau) = \prod_{c \in G/H} \Omega(\tau_c)$;
(ii) $\Omega(\tau)$ is linearly uniformly homeomorphic to $\Omega(\tau_H)^{G/H}$;
(iii) $\tau$ is nilpotent if and only if $\tau_H : \Sigma_H \to \Sigma_H$ is nilpotent.
Lemma 3.1. \[26, \text{Lemma 9.15}\] for more details.

Of the affine schemes, the general treatment by Bourbaki \[2, \text{Theorem 1, TG II. Section 5}\], or

\[\begin{align*}
\text{consisting of all } (\varphi) & \in G^n, \\
\text{every pair of elements admits an upper bound. An } & \\
\text{inverse system } & \\
\text{the following data: (1) a set } Z & \\
\text{such that } & \\
\varphi_{ij} & \text{for all } i, j, k \in I \text{ such that } i \preceq j \preceq k. \\
\text{One then speaks of the inverse system } (Z_i, \varphi_{ij}) & \\
\text{or simply } (Z_i) & \text{if the index set and the transition} \\
\text{maps are clear from the context. One says that an inverse } & \\
\text{system } (Z_i, \varphi_{ij}) & \text{satisfies the Mittag-Leffler condition provided that for each } i \in I \text{ there exists } j \in I \text{ with } i \preceq j \text{ such that } \varphi_{ik}(X_k) = \\
\text{the inverse limit of an inverse system } & \\
\text{consisting of all } (z_i)_{i \in I} & \text{such that } \varphi_{ij}(z_j) = z_i \text{ for all } i \preceq j. \\
\text{The following useful lemma is an application of the classical Mittag-Leffler lemma to affine } & \\
\text{inverse systems (see, e.g. } [16] \text{ Section I.3], where Grothendieck used it in his study of the cohomology of affine schemes, the general treatment by Bourbaki } [2] \text{ Theorem 1, TG II. Section 5], or } \\
\text{Lemma 3.1 for a self-contained proof in the countable case. Cf. also } [25] \text{ Proposition 4.2} & \\
\text{and } [26] \text{ Lemma 9.15} \text{ for more details).} \\
\text{Lemma 3.1. Let } K & \text{ be a field. Let } (X_i, f_{ij}) \text{ be an inverse system indexed by an index set } I, \text{ where each } X_i \text{ is a nonempty finite-dimensional } K\text{-affine space and each transition map } f_{ij} : X_j \to X_i \text{ is a } K\text{-affine map for all } i \preceq j. \text{ Then } \lim_{i \in I} X_i \neq \emptyset. \\
\text{Proof. We have } & \\
\tau^n(\Sigma) & = \prod_{c \in G/H} \tau^n_c(\Sigma) \text{ for all } n \in \mathbb{N}, \text{ so that } \\
\Omega(\tau) & = \bigcap_{n \in \mathbb{N}} \prod_{c \in G/H} \tau^n_c(\Sigma) = \bigoplus_{c \in G/H} \Omega(\tau_c). \\
\text{This proves (i). It is then clear that } & \\
\phi := \prod_{c \in G/H} \phi_c : A^G & \to (A^H)^{G/H}, \text{ yields, by restriction, a } \\
\text{linear uniform homeomorphism } & \\
\Omega(\tau) & = \prod_{c \in G/H} \Omega(\tau_c) \to \Omega(\tau_H^{G/H}). \text{ This proves (ii).} \\
\text{We have that } & \\
\tau & \text{ is nilpotent if and only if there exists an integer } n_0 \geq 1 \text{ such that } \tau^{n_0}(\Sigma) = \{0\}. \\
\text{By the above discussion, this is equivalent to } & \\
\tau_H^{n_0}(\Sigma_H) & = \{0\}, \text{ that is, to } \tau_H \text{ being nilpotent.} \quad \square \\
\text{Proposition 2.8. Let } G & \text{ be a group and let } A \text{ be a finite-dimensional vector space over a field } K. \\
\text{Suppose that } & \\
\dim_K(A) & = d \text{ and that } A \text{ is equipped with a basis } B. \text{ Let } \tau : A^G \to A^G & \text{ be a linear } \\
\text{CA. Then the following conditions are equivalent: } & \\
\text{(a) } & \\
\tau & \text{ is nilpotent; } \\
\text{(b) the matrix } & \\
M_B(\tau) & \in \text{Mat}_d(K[G]) \text{ is nilpotent.} \\
\text{Proof. The proof follows immediately from the fact that } & \\
\text{LCA}(G, A) & \text{ and } \text{Mat}_d(K[G]) \text{ are isomorphic as } K\text{-algebras, and that nilpotency is preserved under } K\text{-algebra homomorphisms.} \quad \square \\
\text{3. Space-time inverse systems of linear cellular automata} \\
\text{3.1. Inverse limits of sets. Let } (I, \preceq) & \text{ be a directed set, i.e., a partially ordered set in which every pair of elements admits an upper bound. An inverse system of sets indexed by } I \text{ consists of the following data: (1) a set } Z_i & \text{ for each } i \in I; \text{ (2) a transition map } \varphi_{ij} : Z_j \to Z_i \text{ for all } i, j \in I \text{ such that } i \preceq j. \text{ Furthermore, the transition maps must satisfy the following conditions: } \\
\varphi_{ii} & = \text{Id}_{Z_i} \text{ (the identity map on } Z_i \text{) for all } i \in I, \\
\varphi_{ij} \circ \varphi_{jk} & = \varphi_{ik} \text{ for all } i, j, k \in I \text{ such that } i \preceq j \preceq k. \\
\text{One then speaks of the inverse system } (Z_i, \varphi_{ij}), \text{ or simply } (Z_i) & \text{ if the index set and the transition maps are clear from the context. One says that an inverse system } (Z_i, \varphi_{ij}) \text{ satisfies the Mittag-Leffler condition provided that for each } i \in I \text{ there exists } j \in I \text{ with } i \preceq j \text{ such that } \varphi_{ik}(X_k) = \\
\varphi_{ij}(X_j) & \text{ for all } j \preceq k. \\
\text{The inverse limit of an inverse system } (Z_i, \varphi_{ij}) & \text{ is the subset } \\
\lim_{i \in I} (Z_i, \varphi_{ij}) & = \lim_{i \in I} Z_i \subset \prod_{i \in I} Z_i \\
\text{consisting of all } (z_i)_{i \in I} & \text{such that } \varphi_{ij}(z_j) = z_i \text{ for all } i \preceq j. \\
\text{The following useful lemma is an application of the classical Mittag-Leffler lemma to affine } & \\
\text{inverse systems (see, e.g. } [16] \text{ Section I.3], where Grothendieck used it in his study of the cohomology of affine schemes, the general treatment by Bourbaki } [2] \text{ Theorem 1, TG II. Section 5], or } \\
\text{Lemma 3.1 for a self-contained proof in the countable case. Cf. also } [25] \text{ Proposition 4.2} & \\
\text{and } [26] \text{ Lemma 9.15} \text{ for more details).} \\
\text{Lemma 3.1. Let } K & \text{ be a field. Let } (X_i, f_{ij}) \text{ be an inverse system indexed by an index set } I, \text{ where each } X_i \text{ is a nonempty finite-dimensional } K\text{-affine space and each transition map } f_{ij} : X_j \to X_i \text{ is a } K\text{-affine map for all } i \preceq j. \text{ Then } \lim_{i \in I} X_i \neq \emptyset.
3.2. Space-time inverse systems. Let \( G \) be a group and let \( A \) be a vector space over a field \( K \). Let \( \Sigma \subset A^G \) be a linear subshift and let \( \tau: \Sigma \to \Sigma \) be a linear CA. Let \( \bar{\tau}: A^G \to A^G \) be a linear CA extending \( \tau \) and let \( M \subset G \) be a memory set for \( \bar{\tau} \). Since every finite subset of \( G \) containing a memory set for \( \bar{\tau} \) is itself a memory set for \( \bar{\tau} \), we can choose \( M \) such that \( 1_G \in M \) and \( M = M^{-1} \).

Let \( \mathcal{P}^*(G) \) denote the set of all finite subsets of \( G \) containing \( 1_G \) equipped with the ordering given by inclusion. Also equip \( N \) with the natural ordering. Equip \( I := \mathcal{P}^*(G) \times N \) with the product ordering \( \preceq \). Thus, given \( \Omega, \Omega' \in \mathcal{P}^*(G) \) and \( n, n' \in N \), we have \( (\Omega, n) \preceq (\Omega', n') \) if and only if \( \Omega \subset \Omega' \) and \( n \leq n' \). It is clear that \( (I, \preceq) \) is directed.

We construct an inverse system \( (\Sigma_{\Omega,n})(\Omega,n) \in I \) indexed by \( I \) as follows.

Firstly, given \( (\Omega, n) \in I \), we set \( \Sigma_{\Omega,n} := \Sigma_{OM^n} = \{ x|_{\Omega M^n} : x \in \Sigma \} \subset A^{OM^n} \).

To define the transition maps \( \Sigma_{\Omega,n'} \to \Sigma_{\Omega,n} ((\Omega, n) \preceq (\Omega', n')) \) of the inverse system \( (\Sigma_{\Omega,n})(\Omega,n) \in I \), it is clearly enough to define, for all \( \Omega, \Omega' \in \mathcal{P}^*(G) \) and \( n, n' \in N \), with \( \Omega \subset \Omega' \) and \( n \leq n' \), the horizontal transition map \( p_{\Omega,\Omega',n}: \Sigma_{\Omega,n'} \to \Sigma_{\Omega,n} \), the vertical transition map \( q_{\Omega,n,n'}: \Sigma_{\Omega,n} \to \Sigma_{\Omega,n'} \), and verify that the diagram

\[
\begin{array}{ccc}
\Sigma_{\Omega,n'} & \xrightarrow{p_{\Omega,\Omega',n}} & \Sigma_{\Omega,n} \\
\downarrow{q_{\Omega,n,n'}} & & \downarrow{q_{\Omega,n',n}} \\
\Sigma_{\Omega,n} & \xleftarrow{p_{\Omega,n,n'}} & \Sigma_{\Omega,n'}
\end{array}
\]

is commutative, i.e.,

\[
q_{\Omega,n,n'} \circ p_{\Omega,\Omega',n} = p_{\Omega,\Omega',n'} \circ q_{\Omega',n,n'}
\]

for all \( \Omega, \Omega' \in \mathcal{P}^*(G) \) and \( n, n' \in N \), with \( \Omega \subset \Omega' \) and \( n \leq n' \).

We define \( p_{\Omega,\Omega',n} \) as being the linear map obtained by restriction to \( \Omega M^n \subset \Omega' M^n \). Thus, for all \( \sigma \in \Sigma_{\Omega',n} = \Sigma_{\Omega'M^n} \), we have

\[
p_{\Omega,\Omega',n}(\sigma) = \sigma|_{\Omega M^n}.
\]

We now define \( q_{\Omega,n,n'} \). If \( n = n' \), then \( q_{\Omega,n,n'} = q_{\Omega,n,n} := \text{Id}_{\Omega M^n}: \Omega M^n \to \Omega M^n \), is the identity map. Suppose now that \( n + 1 \leq n' \). We first observe that, given \( x \in \Sigma \) and \( g \in G \), it follows from \( \text{[1.2]} \) applied to \( \bar{\tau} \) that \( \tau^{n''-n}(x)(g) \) only depends on the restriction of \( x \) to \( gM^{n''-n} \). As \( gM^{n''-n} \subset \Omega M^n \) \( \Omega M^{n''-n} = \Omega M^n \) for all \( g \in \Omega M^n \), we deduce from this observation that, given \( \sigma \in \Sigma_{\Omega,n,n} = \Sigma_{\Omega M^n} \) and \( x \in \Sigma \) extending \( \sigma \), the formula

\[
q_{\Omega,n,n'}(\sigma) := \tau^{n''-n}(x)|_{\Omega M^n}
\]

yields a well-defined element \( q_{\Omega,n,n'}(\sigma) \in \Sigma_{\Omega M^n} = \Sigma_{\Omega,n} \), and hence a linear map \( q_{\Omega,n,n'}: \Sigma_{\Omega,n} \to \Sigma_{\Omega,n} \).

**Definition 3.2.** The inverse system \( (\Sigma_{\Omega,n})(\Omega,n) \in I \) is called the space-time inverse system associated with the triple \( (\Sigma, \tau, M) \).

When \( G \) is countable, we can simplify the above construction by slightly modifying the definitions therein. Since \( G \) is countable, we can find a sequence \( (M_n)_{n \in \mathbb{N}} \) of finite subsets of \( G \) such that

\[
\text{(M-1)} \quad M_0 = \{1_G\} \quad \text{and} \quad M_1 = M \quad \text{(the memory set for} \ \bar{\tau}),
\]

\[
\text{(M-2)} \quad M_i M_j \subset M_{i+j} \quad \text{for all} \ i, j \in \mathbb{N},
\]

\[
\text{(M-3)} \quad \bigcup_{n \in \mathbb{N}} M_n = G.
\]

For instance, if \( G \) is finitely generated and \( M \) in addition generates \( G \), then one may take \( M_n := M^n \) for all \( n \in \mathbb{N} \). We equip \( \mathbb{N}^2 \) with the product ordering \( \preceq \), that is, given \( i, j, k, l \in \mathbb{N} \), we have \( (i, j) \preceq (k, l) \) if and only if \( i \leq k \) and \( j \leq l \). We then construct an inverse system \( (\Sigma_{ij})_{i,j \in \mathbb{N}} \) indexed by the directed set \( (\mathbb{N}^2, \preceq) \) by setting

\[
\Sigma_{ij} := \Sigma_{M_{i+j}} = \{ x|M_{i+j} : x \in \Sigma \} \subset A^{M_{i+j}}
\]

and defining, for all \( i, j \in \mathbb{N} \), the unit-horizontal transition map \( p_{ij}: \Sigma_{i+1,j} \to \Sigma_{ij} \) as being the linear map obtained by restriction to \( M_{i+j} \subset M_{i+j+1} \), and the unit-vertical transition map
with the triple \( \{ B_i \} \) of subspaces. It is clear from the constructions of the inverse system (Theorem 4.1) (see also [8, Theorem 8.8.1]).

**Definition 3.3.** The inverse system \((\Sigma_{ij})_{i,j \in \mathbb{N}}\) is called the space-time inverse system associated with the triple \((\Sigma, \tau, (M_n)_{n \in \mathbb{N}})\).

3.3. **Space-time-systems and limit sets.** We keep the assumptions and notation from the above subsection. Let us fix \( n \in \mathbb{N} \). Then, in our space-time inverse system we get an horizontal inverse system \((\Sigma_{\Omega,n})_{\Omega \in \mathcal{P}^*(G)}\) indexed by \(\mathcal{P}^*(G)\) whose transition maps are the restriction maps \(p_{\Omega,\Omega';n}: \Sigma_{\Omega',M^n} \to \Sigma_{\Omega M^n}\), \(\Omega, \Omega' \in \mathcal{P}^*(G)\) such that \(\Omega \subset \Omega'\). Note that the horizontal inverse system satisfies the Mittag-Leffler condition and that in fact, as it immediately follows from the closedness of \(\Sigma\) in \(A^G\) and the fact that \(G M^n = G\), one has that the limit

\[
\Sigma_n \coloneqq \lim_{\Omega \in \mathcal{P}^*(G)} \Sigma_{\Omega,n}
\]

can be identified with \(\Sigma\) in a canonical way.

Moreover, the linear maps \(q_{\Omega,n,n'}: \Sigma_{\Omega,n'} \to \Sigma_{\Omega,n}\), for \(\Omega \in \mathcal{P}^*(G)\), define an inverse system linear morphism from the inverse system \((\Sigma_{\Omega,n'})_{\Omega \in \mathcal{P}^*(G)}\) to the inverse system \((\Sigma_{\Omega,n})_{\Omega \in \mathcal{P}^*(G)}\). This yields a linear limit map \(\tau_{n,n'}: \Sigma_n \to \Sigma_{n'}\). Using the identifications \(\Sigma_n = \Sigma_{n'} = \Sigma\), we have \(\tau_{n,n'} = \tau^{n'-n}\) for all \(n, n' \in \mathbb{N}\) such that \(n \leq n'\). We deduce that the limit

\[
\lim_{\Omega \in \mathcal{P}^*(G)} \Sigma_{\Omega,n} = \lim_{n \in \mathbb{N}} \Sigma_n
\]

is the set of backward orbits of \(\tau\), that is, the set consisting of all sequences \((x_n)_{n \in \mathbb{N}}\) such that \(x_n \in \Sigma\) and \(x_n = \tau(x_{n+1})\) for all \(n \in \mathbb{N}\). Each such a sequence satisfies that \(x_n = \tau^n(x_0)\) for all \(n \in \mathbb{N}\), and hence \(x_0 \in \Omega(\tau)\). This determines a canonical linear map

\[
\Phi: \lim_{\Omega \in \mathcal{P}^*(G)} \Sigma_{\Omega,n} \to \Omega(\tau).
\]

**Proposition 3.4.** Let \(G\) be a group and let \(A\) be a finite dimensional vector space over a field \(K\). Let \(\Sigma \subset A^G\) be a linear subshift and let \(\tau: \Sigma \to \Sigma\) be a linear CA. Let \(M \subset G\) be a memory set for \(\tau\) which is symmetric and contains \(1_G\), and consider the space-time inverse system associated with the triple \((\Sigma, \tau, M)\). Then the canonical map \(\Phi\) is surjective.

**Proof.** Let \(y_0 \in \Omega(\tau) \subset \Sigma\). For every \(\Omega \in \mathcal{P}^*(G)\) and \(n \in \mathbb{N}\), define a finite dimensional affine subspace \(B_{\Omega,n} \subset A^{\Omega M^n}\) by setting

\[
B_{\Omega,n} := \left( q_{\Omega,0,1} \circ q_{\Omega,1,2} \circ \cdots \circ q_{\Omega,n-1,n} \right)^{-1}(y_0|\Omega) \subset \Sigma_{\Omega M^n}.
\]

By definition of \(\Omega(\tau)\), for every \(n \in \mathbb{N}\) there exists an element \(y_n \in \Sigma\) such that \(\tau^n(y_n) = y_0\). Hence, it follows from the definition of the transition maps \(q_{\Omega,j-1,j}\) and of \(\Omega_j, j \in \mathbb{N}\), that \(y_n|\Omega M^n \in B_{\Omega,n}\). In particular, \(B_{\Omega,n} \neq \emptyset\) for every \(\Omega \in \mathcal{P}^*(G)\) and \(n \in \mathbb{N}\). By restricting the transition maps of the space-time inverse system \((\Sigma_{\Omega,n})_{(\Omega,n) \in I}\) to the sets \(B_{\Omega,n}\), we obtain a well-defined inverse subsystem \((B_{\Omega,n})_{(\Omega,n) \in I}\) of finite dimensional affine spaces with affine transition maps. By Lemma 3.1 we can find

\[
x \in \lim_{(\Omega,n) \in I} B_{\Omega,n} \subset \lim_{(\Omega,n) \in I} \Sigma_{\Omega,n}.
\]

It is clear from the constructions of the inverse system \((B_{\Omega,n})_{(\Omega,n) \in I}\) and of the map \(\Phi\) that \(\Phi(x) = y_0\). This shows that \(\Phi\) is surjective. \(\square\)

4. **The closed-image-property for linear cellular automata**

Using the space-time inverse system, we give a short proof of the following result extending [9, Theorem 1.4] (see also [8, Theorem 8.8.1]).

**Theorem 4.1.** Let \(G\) be a group and let \(A\) be a finite-dimensional vector space over a field \(K\). Let \(\Sigma \subset A^G\) be a linear subshift and let \(\tau: A^G \to A^G\) be a linear CA. Then \(\tau(\Sigma)\) is a linear subshift of \(A^G\).
Proof. Since the cellular automaton $\tau$ is linear and $G$-equivariant, its image $\tau(\Sigma)$ is a $G$-invariant vector subspace of $A^G$. We thus only need to show that $\tau(\Sigma)$ is closed in $A^G$. Let $M \subset G$ be a memory set for $\tau$ which is symmetric and contains $1_G$, and consider the space-time inverse system associated with the triple $(\Sigma, \tau, M)$ as in Section 3.2.

Suppose that $x \in \Sigma$ belongs to the closure of $\tau(\Sigma)$. We must show that $x \in \tau(\Sigma)$.

For every $\Omega \in \mathcal{P}^*(G)$, define an affine subspace $Z_\Omega \subset A^{BM}$ by setting

$$Z_\Omega := (q_{\Omega,0,1})^{-1}(x|_\Omega) \cap \Sigma_{\Omega,1}.$$ 

Since $x$ belongs to the closure of $\tau(\Sigma)$, it follows that $Z_\Omega \neq \emptyset$ for all $\Omega \in \mathcal{P}^*(G)$. By restricting the projections $p_{\Omega,\Omega'} : A^{BM} \to A^{BM}$ (cf. (3.2)), $\Omega, \Omega' \in \mathcal{P}^*(G)$, with $\Omega \subset \Omega'$ to the $Z_\Omega$'s, we obtain affine maps $\pi_{\Omega,\Omega'} : Z_{\Omega'} \to Z_\Omega$ of the inverse system $(Z_\Omega)_{\Omega \in \mathcal{P}^*(G)}$. It then follows from Lemma 3.1 that $\lim_{\Omega \in \mathcal{P}^*(G)} Z_\Omega = \lim_{\Omega \in \mathcal{P}^*(G)} \Sigma_{\Omega,1} = \Sigma$ (cf. (3.4)) have $\tau(c) = x$. This shows that $\tau(\Sigma)$ is closed.

\[\Box\]

Remark 4.2. We observe that the hypothesis of finite-dimensionality of the vector space $A$ in Theorem 1.1 cannot be dropped, as the example in Section 7.2 below shows.

5. Proofs

Proof of Theorem 1.1. Suppose $\Sigma$ is of finite type. Hence $\Sigma = \Sigma(D, W) \subset A^D$, where $D \subset G$ is finite and $W \subset A^D$ is a vector subspace (cf. Lemma 2.3). Let $\Sigma_0 \supset \Sigma_1 \supset \cdots$ be a decreasing sequence of linear subshifts of $A^G$ such that $\bigcap_{n \geq 1} \Sigma_n = \Sigma$. Let $(M_n)_{n \in \mathbb{N}}$ be a sequence of linear subshifts of $A^G$ satisfying conditions (M-1)-(M-3) and such that $D \subset M_1$. Consider the inverse system $(X_{ij})_{i,j \in \mathbb{N}}$ defined by setting $X_{ij} := (\Sigma_j)_{M_i} \subset A^{M_i}$. Observe that $X_{i+1,j} \subset X_{ij}$ since $\Sigma_{i+1} \subset \Sigma_j$ for all $i, j \in \mathbb{N}$. Also, we define the transition maps $p_{ij} : X_{i+1,j} \to X_{ij}$ by setting $p_{ij}(x) := x|_{M_i}$, for all $x \in X_{i+1,j} = (\Sigma_j)_{M_{i+1}}$ and $d_{ij} : X_{i+1,j} \to X_{ij}$ as the inclusion maps.

The decreasing sequence $(X_{ij})_{i,j \in \mathbb{N}}$ of finite-dimensional vector spaces eventually stabilizes so that there exists $j_0 \geq 1$ such that $X_{1,j} = X_{1,j_0}$ for all $j \geq j_0$. Set $W' := X_{1,j_0}$ and let us show that $\Sigma$ equals the linear SFT $\Sigma' := \Sigma(M_1, W')$. First note that $\Sigma_{j_0} \subset \Sigma'$ so that $\Sigma \subset \Sigma'$. Conversely, let $w \in W'$. We construct an inverse subsystem $(Z_{ij})_{i,j \geq j_0}$ of $(X_{ij})_{i,j \geq j_0}$ as follows. For $i \geq 1$ and $j \geq 0$, consider the affine subspace of $X_{ij}$:

$$Z_{ij} := \{x \in X_{ij} : x|_{M_i} = w\} \subset X_{ij}.$$ 

The transition maps of $(Z_{ij})_{i,j \geq j_0}$ are well-defined as the restrictions of the transition maps of the system $(X_{ij})_{i,j \geq j_0}$.

By our construction, each $Z_{ij}$ is clearly nonempty. Hence, Lemma 4.1 implies that there exists $x = (x_{ij})_{i,j \geq j_0} \in \lim Z_{ij}$. Let $y \in A^G$ be defined by $y(g) = x_{ij}(g)$ for every $g \in G$ and any large enough $i \geq 1$ such that $g \in M_i$. Observe that $x_{ij} = x_{ik}$ for every $i \geq 1$ and $0 \leq k \leq j$ since the vertical transition maps $X_{ik} \to X_{ij}$ are simply inclusions. Consequently, for every $n \in \mathbb{N}$, we have $y \in \Sigma_n$ by (3.4). Hence $y \in \Sigma$. By construction, $y|_{M_i} = w$. Since $w$ was arbitrary, this shows that $W' \subset \Sigma_{j_0}$. Hence, $\Sigma' = \Sigma(M_1, W') \subset \Sigma(M_1, \Sigma_{j_0}) = \Sigma$. The last equality follows from Lemma 2.2 as $D \subset M_1$. Therefore, $\Sigma' = \Sigma$ and $\Sigma_n = \Sigma$ for all $n \geq j_0$. This proves the implication (a) $\implies$ (b).

Suppose now that $\Sigma \subset A^G$ is a linear subshift which is not of finite type. Let $(M_n)_{n \in \mathbb{N}}$ be a sequence of finite subshifts of $G$ satisfying conditions (M-2)-(M-3). For every $n \in \mathbb{N}$, set $W_n := \Sigma_{M_n}$ (as in Section 3.2). Then, $W_n$ is a vector subspace of $A^{M_n}$. For every $n \in \mathbb{N}$ we consider the linear SFT $\Sigma_n := (M_n, W_n)$. As $(\Sigma_{M_n})_{M_n} = \Sigma_{M_n}$, it is clear that $\Sigma \subset \Sigma_{n+1} \subset \Sigma_n$ for all $n \in \mathbb{N}$. We claim that $\Sigma = \bigcap_{n \in \mathbb{N}} \Sigma_n$. We only need to prove that $\bigcap_{n \in \mathbb{N}} \Sigma_n \subset \Sigma$. Let $x \in \bigcap_{n \in \mathbb{N}} \Sigma_n$. Then by definition of $\Sigma_n$, we find that $x|_{M_n} \in W_n = \Sigma_{M_n}$ for every $n \in \mathbb{N}$. Thus, since $\Sigma$ is closed, $x \in \lim_{n \in \mathbb{N}} \Sigma_n = \Sigma$ (cf. (3.3)) and hence $\bigcap_{n \in \mathbb{N}} \Sigma_n \subset \Sigma$. However, the decreasing sequence $(\Sigma_n)_{n \in \mathbb{N}}$ cannot stabilize since, otherwise, the subshift $\Sigma$ would be of finite type. This shows that (b) $\implies$ (a). The proof of Theorem 1.1 is complete. $\Box$
Proof of Corollary 1.2. Suppose first that $A^G$ satisfies condition (b) and let $\Sigma \subset A^G$ be a linear subshift. Let $(D_n)_{n \in \mathbb{N}}$ be an increasing sequence of finite subsets of $G$ such that $\bigcup_{n \in \mathbb{N}} D_n = G$. For every $n \in \mathbb{N}$ let $W_n := \Sigma_{D_n} \subset A^{D_n}$. Then $\Sigma_n := \Sigma(D_n, W_n) \subset A^G$ is a linear SFT and $\Sigma_n \supset \Sigma_1 \supset \cdots \supset \Sigma_n \supset \Sigma_{n+1} \cdots$ for all $n \in \mathbb{N}$. We claim that $\bigcap_{n \in \mathbb{N}} \Sigma_n = \Sigma$. Since $\Sigma_n \supset \Sigma$ for all $n \in \mathbb{N}$, we only need to show that $\bigcap_{n \in \mathbb{N}} \Sigma_n \subset \Sigma$. Let $x \in \bigcap_{n \in \mathbb{N}} \Sigma_n$. This means that for each $n \in \mathbb{N}$ there exists $x_n \in \Sigma$ such that $x|_{D_n} = x_n|_{D_n}$. Since the sequence $(D_n)_{n \in \mathbb{N}}$ is exhausting and $\Sigma$ is closed in the prodiscrete topology, we deduce that $x \in \Sigma$. This proves the claim. Since $A^G$ satisfies condition (b), there exists $n_0 \in \mathbb{N}$ such that $\Sigma_n = \Sigma_{n_0}$ for all $n \geq n_0$. We deduce that $\Sigma = \Sigma_{n_0}$ is of finite type.

Conversely, suppose that every linear subshift $\Sigma \subset A^G$ is of finite type and let $(\Sigma_n)_{n \in \mathbb{N}}$ be a decreasing sequence of linear subshifts. Set $\Sigma := \bigcap_{n \in \mathbb{N}} \Sigma_n \subset A^G$. Then $\Sigma$ is a linear subshift and, by our assumptions, it is of finite type. It follows from Theorem 1.1 that the sequence $(\Sigma_n)_{n \in \mathbb{N}}$ eventually stabilizes. The proof of Corollary 1.2 is complete. □

Proof of Theorem 1.3. We first observe that if $G$ is uncountable then, on the one hand $G$ is not finitely generated and thus is not Noetherian (that is, it does not satisfy the maximal condition on subgroups) and therefore the group algebra $K[G]$ is not one-sided Noetherian (cf. [24, Lemma 2.2, Chapter 10]), and, on the other hand, $G$ is not of $K$-linear Markov type, since the linear subshift consisting of all constant configurations in $K^G$ is not of finite type (cf. Corollary 2.5).

Thus, in order to prove Theorem 1.3, it is not restrictive to assume that $G$ is countable.

Recall that $\text{LCA}(G, A)$ denotes the $K$-algebra of all linear cellular automata $\tau: A^G \to A^G$ (cf. [8, Section 8.1]).

The evaluation map $(\tau, x) \mapsto \tau(x)$, where $\tau \in \text{LCA}(G, A)$ and $x \in A^G$, yields a $K$-bilinear map $\text{LCA}(G, A) \times A^G \to A^G$.

Given a subset $\Gamma$ in $\text{LCA}(G, A)$, set
\[
\Gamma^\perp := \bigcap_{\tau \in \Gamma} \ker(\tau) \subset A^G.
\]

Since every map $\tau \in \text{LCA}(G, A)$ is linear, continuous, and $G$-equivariant, we deduce immediately that its kernel $\ker(\tau)$ is a linear subshift of $A^G$. Moreover, since the set of all linear subshifts in $A^G$ is closed under intersections, we have that $\Gamma^\perp$ is a linear subshift of $A^G$.

Given a subset $\Sigma \subset A^G$, set
\[
\Sigma^\perp := \{ \tau \in \text{LCA}(G, A) : \Sigma \subset \ker(\tau) \} \subset \text{LCA}(G, A).
\]

We claim that $\Sigma^\perp$ is a left ideal in $\text{LCA}(G, A)$. First of all, we clearly have $0 \in \Sigma^\perp$, since $\Sigma \subset A^G = \ker(0)$. Suppose that $\tau_1, \tau_2 \in \Sigma^\perp$. Then $(\tau_1 - \tau_2)(x) = \tau_1(x) - \tau_2(x) = 0 - 0 = 0$ for all $x \in \Sigma$, showing that $\tau_1 - \tau_2 \in \Sigma^\perp$. Finally, if $\tau \in \text{LCA}(G, A)$, we have $(\tau \circ \tau_1)(x) = \tau(\tau_1(x)) = \tau(0) = 0$ for all $x \in \Sigma$, showing that $\tau \circ \tau_1 \in \Sigma^\perp$. This proves the claim.

We note also that if $\Sigma_1, \Sigma_2 \subset A^G$, then
\[
\Sigma_1 \subset \Sigma_2 \implies \Sigma_1^\perp \subset \Sigma_2^\perp.
\]

We have the following key lemma:

**Lemma 5.1.** Let $G$ be a group, let $A$ be a vector space over a field $K$, and let $\Sigma \subset A^G$ be a linear subshift. Then
\[
(\Sigma^\perp)^\perp = \Sigma.
\]

**Proof.** It trivially follows from the definitions that $\Sigma \subset (\Sigma^\perp)^\perp$. In order to show the other inclusion, let $x \in A^G \setminus \Sigma$ and let us show that $x \notin (\Sigma^\perp)^\perp$. Since $\Sigma$ is closed, by the definition of prodiscrete topology we can find a finite subset $\Omega \subset G$ such that $x|_{\Omega} \notin \Sigma_{\Omega}$. It is a classical and easy argument in Linear Algebra that there exists a linear map $\mu: A^\Omega \to A$ such that $\mu(x|_{\Omega}) \equiv 0$, that is, $\Sigma_{\Omega} \subset \ker(\mu)$, and $\mu(x|_{\Omega}) \neq 0$. It is then clear that the linear CA $\tau$ with memory set $\Omega$ and local defining map $\mu$ satisfies that $\Sigma \subset \ker(\tau)$, that is, $\tau \in \Sigma^\perp$, but $\tau(x) \neq 0$. Thus $x \notin (\Sigma^\perp)^\perp$. □

In the proof of the following lemma, we explicitly use the $K$-algebra isomorphism $\text{Mat}_d(K[G]) \cong \text{LCA}(G, K^d)$ we alluded to above (cf. [8, Corollary 8.7.8]) for $d = 1$. This is given by associating
with each $\alpha \in K[G]$ the linear cellular automaton $\tau_\alpha : K^G \to K^G$ with memory set $M_\alpha := \{ g \in G : \alpha(g) \neq 0 \}$, the support of $\alpha$, and local defining map $\mu_\alpha : K^{M_\alpha} \to K$ defined by setting

$$\mu_\alpha(y) := \sum_{h \in M_\alpha} \alpha(h)y(h)$$

for all $y \in K^{M_\alpha}$.

We shall also make use of the following notation. Given $\alpha \in K[G]$, for every finite subset $E \subset G$ such that $M_\alpha \subset E$ we define the linear map $\mu_{\alpha,E} : K^E \to K$ by setting $\mu_{\alpha,E} := \mu_\alpha \circ \pi_{M_\alpha,E}$, where $\pi_{M_\alpha,E} : K^E \to K^{M_\alpha}$ is the projection map induced by the inclusion $M_\alpha \subset E$. Note that $\mu_{\alpha,E}$ is the local defining map of $\tau_\alpha$ associated with the memory set $E$.

**Lemma 5.2.** Let $G$ be a countable group and let $K$ be a field. Let $\Gamma \subset K[G]$ be a left ideal. Suppose that $\Gamma^\perp \subset K^G$ is a linear SFT. Then $\Gamma$ is a finitely generated left ideal.

**Proof.** Since $G$ is countable, we can find an increasing sequence $(E_n)_{n \in \mathbb{N}}$ of finite subsets of $G$ such that $G = \bigcup_{n \in \mathbb{N}} E_n$. For every $n \in \mathbb{N}$, let $\Gamma_n \subset \Gamma$ be the ideal of $K[G]$ generated by the elements of $\Gamma$ whose supports are contained in $E_n$. Then $\Gamma_n \subset \Gamma_{n+1}$ for all $n \in \mathbb{N}$ and $\Gamma = \bigcup_{n \in \mathbb{N}} \Gamma_n$. We thus obtain a decreasing sequence $(\Gamma_n^\perp)_{n \in \mathbb{N}}$ of linear subshifts of $K^G$.

Remark that we can write

$$\Gamma^\perp = \bigcap_{\alpha \in \Gamma} \ker(\tau_\alpha) = \bigcap_{n \in \mathbb{N}} \bigcap_{\alpha \in \Gamma_n} \ker(\tau_\alpha) = \bigcap_{n \in \mathbb{N}} \Gamma_n^\perp.$$

Since, by hypothesis, the linear subshift $\Gamma^\perp \subset K^G$ is of finite type, we deduce from Theorem 1.1 that there exists $n_0 \in \mathbb{N}$ such that $\Gamma_n^\perp = \Gamma_n^\perp\!\!\!\perp$ for every $n \geq n_0$, equivalently, $\Gamma^\perp = \Gamma_{n_0}^\perp$.

**Claim.** $\Gamma = \Gamma_{n_0}$.

**Proof of the claim.** Set $J := \Gamma_{n_0}$ and suppose by contradiction that there exists $\alpha \in \Gamma \setminus J$. Let $m_0 \in \mathbb{N}$ be such that $E_{m_0}$ contains the support $M_\alpha \subset G$ of $\alpha$.

For every $m \in \mathbb{N}$, we set $V_m := K^{E_m}$ and denote by $V_m^\ast$ the dual $K$-vector space of $V_m$. Given a vector subspace $W_m \subset V_m$ (resp. $J_m \subset V_m^\ast$) we set $W_m^\perp := \{ v^\ast \in V_m^\ast : W_m \subset \ker(v^\ast) \} \subset V_m^\ast$ (resp. $J_m^\perp := \bigcap_{v^\ast \in J_m} \ker(v^\ast) \subset V_m^\ast$. Since $V_m$ is finite-dimensional, we have $(J_m^\perp)^\perp = J_m$.

We then denote by $J_m \subset J$ the subset containing all elements of $J$ whose supports are contained in $E_m$. Observe that $J_m \subset J_{m+1}$ and $J = \bigcup_{m \geq m_0} J_m$. We regard $J_m$ as a linear subspace of $V_m^\ast$ via the map $\beta \mapsto \mu_{\beta,E_m}$. This way, setting $W_m := \bigcap_{\beta \in J_m} \ker(\mu_{\beta,E_m}) \subset V_m$, we have $W_m = J_m^\perp$ and therefore

$$\{ v^\ast \in V_m^\ast : W_m \subset \ker(v^\ast) \} = W_m^\perp = (J_m^\perp)^\perp = J_m.$$

From this we deduce that for every $m \geq m_0$

$$U_m := W_m \setminus \ker(\mu_{\alpha,E_m}) \neq \emptyset.$$

Indeed, otherwise, we would have $W_m \subset \ker(\mu_{\alpha,E_m})$ so that, by (5.5), $\alpha \in J_m \subset J$, a contradiction since $\alpha \notin J$.

For every $m \geq m_0$, let $\pi_{nm} : K^{E_m} \to K^{E_n}$ be the projection map induced by the inclusion $E_n \subset E_m$. It is clear that $\pi_{nm}(U_m) \subset U_n$ since $\ker(\mu_{\alpha,E_m}) = \ker(\mu_{\alpha}) \times K^{E_m \setminus M_\alpha} \subset K^{E_m}$ and $\pi_{nm}(W_m) \subset W_n$ for all $m \geq n \geq m_0$. Therefore, we obtain an inverse system $(U_m)_{m \geq m_0}$ of nonempty sets with transition maps $\varphi_{nm} := \pi_{nm}|U_m : U_m \to U_n$ for $m \geq n \geq m_0$.

As in Lemma 4.1, an immediate application of the Mittag-Leffler condition to the inverse system $(U_m)_{m \geq m_0}$ shows that there exists a configuration $c \in \lim_{m \geq m_0} U_m \subset \lim_{m \geq m_0} W_m$. Let us show that $c \in J^\perp = \bigcap_{\beta \in J} \ker(\tau_\beta) \subset K^G$. Let $\beta \in J$ and let $g \in G$. Since $J$ is an ideal of $K[G]$ and $J = \bigcup_{m \geq m_0} J_m$, there exists $m \geq m_0$ such that $g^\beta \in J_m$. Since $c|_{E_m} \in W_m$, it follows from the definition of $W_m$ that

$$\tau_\beta(c)(g) = \mu_{\beta,E_m}(g^{-1}c|_{E_m}) = \mu_{g\beta,E_m}(c|_{E_m}) = 0.$$

Since $g \in G$ was arbitrary, this shows that $\tau_\beta(c) = 0$. Since $\beta \in J$ was arbitrary, this shows that $c \in J^\perp$. On the other hand, by construction, we have that $\mu_\alpha(c|_{M_\alpha}) \neq 0$ so that $\tau_\alpha(c) \neq 0$. Since $\alpha \in J$, we deduce that $c \notin J^\perp$, a contradiction. The claim is proved.
We are now in a position to show that $\Gamma$ is finitely generated as a left ideal. With the above notation, $J_{n_0}$, the subset consisting of all elements in $J = \Gamma$ whose supports are contained in $E_{n_0}$ is a vector subspace of $V_{n_0} = K^{E_{n_0}}$, and therefore is finite dimensional. It is then clear that any vector basis of $J_{n_0}$ also generates $\Gamma_{n_0} = \Gamma$ as a left ideal. We conclude that $\Gamma$ is a finitely generated left ideal of $K[G]$.

We are now in a position to prove Theorem 1.3.

Recall that we assume that $G$ is countable. Suppose first that the group algebra $K[G]$ is one-sided Noetherian. Let $A$ be a finite-dimensional vector space over $K$ and let $d = \dim_K(A)$. We then observe that since $K[G]$ is one-sided Noetherian, so is the finitely generated left $K[G]$-module $\text{Mat}_d(K[G])$, the $K$-algebra of $d \times d$ matrices with coefficients in the group ring $K[G]$. Since every left ideal in $\text{Mat}_d(K[G])$ is trivially a left $K[G]$-module, we deduce that $\text{Mat}_d(K[G])$ is one-sided Noetherian as well as a ring. As mentioned above (cf. [8, Corollary 8.7.8]), once fixed a vector basis for $A$, there exists a canonical $K$-algebra isomorphism of LCA($G, A$) onto $\text{Mat}_d(K[G])$. We deduce that LCA($G, A$) is one-sided Noetherian.

In order to show that $G$ is of $K$-linear Markov type, let $(\Sigma_n)_{n \in \mathbb{N}}$ be a decreasing sequence of linear subshifts in $A^G$ and let us show that it stabilizes. Setting $\Gamma_n := \Sigma_n^+$ for all $n \in \mathbb{N}$, we get an increasing sequence $(\Gamma_n)_{n \in \mathbb{N}}$ of left ideals in LCA($G, A$). Since the latter is left-Noetherian, such a sequence stabilizes, that is, there exists $n_0 \in \mathbb{N}$ such that $\Gamma_n = \Gamma_{n_0}$ for all $n \geq n_0$. It then follows from Lemma 5.1 that $\Sigma_n = \Gamma_n^+ = \Gamma_{n_0}^+ = \Sigma_{n_0}$ for all $n \geq n_0$, that is, $(\Sigma_n)_{n \in \mathbb{N}}$ stabilizes. This shows that $G$ is of $K$-linear Markov type.

Conversely, suppose that $G$ is of $K$-linear Markov type and let $\Gamma \subset K[G]$ be a left ideal. Then the linear subshift $\Gamma^+ \subset K[G]$ is of finite type. Lemma 6.2 implies that $\Gamma$ is finitely generated. This shows that the group algebra $K[G]$ is one-sided Noetherian.

The proof of Theorem 1.3 is complete. □

Proof of Corollary 1.4. Let $G$ be a polycyclic-by-finite group and let $K$ be a field. It follows from a famous result of P. Hall [15] (see also [24, Corollary 10.2.5]) that $K[G]$ is one-sided Noetherian. We then deduce from Theorem 1.3 that $G$ is of $K$-linear Markov type. □

Remark. (1) At our knowledge, it is not known whether or not there exist groups $G$, other than the polycyclic-by-finite groups, whose group algebra $K[G]$ is one-sided Noetherian. See Section 6 for more on this.

(2) An alternative and self-contained proof of Corollary 1.4 is obtained from Lemma 6.2 and Lemma 6.3 below combined with an easy induction argument. For the details see Remark 6.6.

Proof of Theorem 1.5. (i) It follows from Theorem 4.1 that $\tau^n(\Sigma)$ is a linear subshift in $A^G$ for all $n \in \mathbb{N}$. Since the intersection of any family of linear subshifts is itself a linear subshift, we deduce that $\Omega(\tau) = \bigcap_{n \in \mathbb{N}} \tau^n(\Sigma)$ is a linear subshift.

(ii) Let $x \in \Omega(\tau)$, that is, $x \in \tau^n(\Sigma)$ for every $n \geq 0$. Thus $\tau(x) \in \tau^{n+1}(\Sigma)$ for every $n \geq 0$ and it follows that $\tau(x) \in \Omega(\tau)$. Therefore, $\tau(\Omega(\tau)) \subset \Omega(\tau)$. For the converse inclusion, let $y \in \Omega(\tau)$. Let also $M \subset G$ be a memory set for $\tau$ such that $I_G \subset M$ and $M = M^{-1}$. Then, by Proposition 3.2, there exists $x \in \lim_{\leftarrow} (\tau(\Omega_n)) \subset \Omega(\tau)$ such that $\Phi(x) = y$. On the other hand, (3.3) tells us that $\Phi^{-1}(y) \subset \lim_{\leftarrow} (\tau(\Omega_n)) \subset \Omega(\tau)$ is the set of backward orbits of $y$ under $\tau$. Hence, we can find $z \in \Omega(\tau)$ such that $\tau(z) = y$. Thus, $\Omega(\tau) \subset \tau(\Omega(\tau))$ and equality follows.

(iii) As already mentioned in the Introduction, the inclusions $\text{Per}(\tau) \subset R(\tau) \subset \text{NW}(\tau)$ are immediate from the definitions. In [12, Proposition 2.2] it is shown that if $X$ is a uniform space and $f: X \to X$ is a continuous map, then $\text{NW}(f) \subset \text{CR}(f)$. Since every cellular automaton is continuous, we deduce that $\text{NW}(\tau) \subset \text{CR}(\tau)$. In [12, Proposition 2.3] it is shown that if $X$ is a Hausdorff uniform space and $f: X \to X$ is a uniformly continuous map such that $f^n(X)$ is closed in $X$ for all $n \in \mathbb{N}$, then $\text{CR}(f) \subset \Omega(f)$. In our setting, uniform continuity of $\tau$ is a general property of cellular automata already mentioned in the Introduction. Moreover, $\tau^n(\Sigma)$ is closed in $\Sigma$ for all $n \in \mathbb{N}$ by Theorem 4.1. We deduce the last inclusion, namely $\text{CR}(\tau) \subset \Omega(\tau)$. □
(iv) Suppose that $\Omega(\tau)$ is of finite type. It follows from Theorem 1.1 that the sequence $(\tau^n(\Sigma))_{n\in\mathbb{N}}$ eventually stabilizes, that is, there exists $n_0 \geq 1$ such that $\tau^n(\Sigma) = \tau^{n_0}(\Sigma)$ for all $n \geq n_0$. This shows that $\tau$ is stable.

(v) Suppose that $\Omega(\tau)$ is finite-dimensional. It follows from Proposition 2.4 that $\Omega(\tau)$ is of finite type. Using (iv) we deduce that $\tau$ is stable.

This ends the proof of Theorem 1.5.

\[\square\]

**Proof of Corollary 1.6.** We only need to prove the statements for $G$ not finitely generated. Let $M \subset G$ be a finite subset serving as a memory set for both \(\Sigma\) and \(\tau\), and denote by $H \subset G$ the subgroup generated by $M$.

The proof of Theorem 1.5(ii) did not use any finite generation assumption on $G$ and therefore holds true in the present setting as well.

(i) It follows from Theorem 1.5 (i) applied to the restriction cellular automaton $\tau_H : \Sigma_H \rightarrow \Sigma_H$ that $\Omega(\tau_H)$ is a linear subshift. As a consequence, $\tau(\tau_c)$ are closed in $A^\tau$ for all $c \in G/H$. As products of closed subspaces are closed in the product topology, we deduce from Lemma 2.7(i) that $\Omega(\tau) = \prod_{c \in G/H} \Omega(\tau_c)$ is also closed in $A^G$. Since $\Omega(\tau)$ is a $K$-linear and $G$-invariant subset of $A^G$, we conclude that it is a linear subshift of $A^G$.

(ii) It follows from Theorem 1.5 (ii) applied to the restriction cellular automaton $\tau_H : \Sigma_H \rightarrow \Sigma_H$ that $\tau_H(\Omega(\tau_H)) = \Omega(\tau_H)$. As a consequence, $\tau(\Omega(\tau_c)) = \Omega(\tau_c)$ for all $c \in G/H$. We deduce from Lemma 2.7(i) that $\tau(\Omega(\tau)) = \prod_{c \in G/H} \tau(\Omega(\tau_c)) = \prod_{c \in G/H} \Omega(\tau_c) = \Omega(\tau)$.

(iii) Just note that, by virtue of Theorem 1.1, $\tau_H^0(\Sigma_H)$ is closed in $A^H$ for all $n \in \mathbb{N}$. Hence, by Lemma 2.6, $\tau^n(\Sigma)$ is closed in $A^G$ for all $n \in \mathbb{N}$, and the proof of Theorem 1.5(iii) applies verbatim.

(iv) Up to enlarging $M \subset G$, if necessary, we may suppose that $M$ also serves as a memory set for the SFT $\Omega(\tau)$, say $\Omega(\tau) = \Sigma(M, W) \subset A^G$ for some $W \subset A^M$. We have that $\Omega(\tau)_H = \Omega(\tau_H) = \Sigma(M, W) \subset A^H$ is of finite type as well. It then follows from Theorem 1.5(iv) applied to the restriction cellular automaton $\tau_H$, that $\tau_H$ is stable. Since stability is invariant under the operation of restriction, we deduce that $\tau$ is itself stable.

(v) If $\Omega(\tau)$ is finite-dimensional, so is $\Omega(\tau_H) = \Omega(\tau)_H$. It then follows from Theorem 1.5(v) applied to the restriction cellular automaton $\tau_H$, that $\tau_H$ is stable. Thus $\tau$ is itself stable.

This proves Corollary 1.6.

\[\square\]

**Proof of Corollary 1.7.** We first observe that every polycyclic-by-finite group is amenable (see, for instance, [8 Chapter 4]). Let $\Lambda \subset A^G$ be a strongly irreducible subshift such that $\Lambda \subset \Sigma$, $\tau(\Lambda) \subset \Lambda$, and such that the restriction linear CA $\tau|_\Lambda : \Lambda \rightarrow \Lambda$ is pre-injective. Since $G$ is polycyclic-by-finite, Corollary 1.4 ensures that $\Lambda$ is a linear subshift of finite type. Since $G$ is amenable, the implication pre-injectivity $\implies$ surjectivity in the Garden of Eden theorem [10] (Theorem 1.2) yields the equality $\tau(\Lambda) = \Lambda$. It follows immediately that $\Lambda \subset \Omega(\tau)$.

Theorem 1.5(i) and Corollary 1.4 imply that $\Omega(\tau)$ is a linear subshift of finite type. Thus, by Theorem 1.5(iv), $\tau$ is stable and therefore there exists an integer $n \geq 1$ such that $\tau^n(\Sigma) = \Omega(\tau)$. Since the image of a strongly irreducible subshift under a CA is also strongly irreducible, it follows that $\Omega(\tau)$ is a strongly irreducible linear SFT. By Theorem 1.5(ii), $\tau(\Omega(\tau)) \subset \Omega(\tau)$ and the restriction linear CA $\tau|_{\Omega(\tau)} : \Omega(\tau) \rightarrow \Omega(\tau)$ is surjective. We can thus conclude from the implication surjectivity $\implies$ pre-injectivity in the Garden of Eden theorem [10] (Theorem 1.2) that $\tau|_{\Omega(\tau)}$ is pre-injective.

The proof of Corollary 1.7 is complete.

\[\square\]

**Proof of Theorem 1.8.** Let $H \subset G$ be a finitely generated subgroup containing both a memory set for $\Sigma$ and a memory set for $\tau$. By virtue of Lemma 2.7, we have, on the one hand that $\tau$ is nilpotent if and only if $\tau_H$ is, and, one the other hand that $\Omega(\tau) = \{0\}$ if and only if $\Omega(\tau_H) = \{0\}$. Thus, it is not restrictive to suppose that $G = H$ is finitely generated.

Suppose that $\tau$ is nilpotent. Then there exists $n_0 \geq 1$ such that $\tau^{n_0}(\Sigma) = \{0\}$. It then follows that $\Omega(\tau) = \{0\}$. 

Conversely, suppose (b). Then $\Omega(\tau)$ is of finite type. By the characterization of linear SFT in Theorem 1.1, the sequence $(\tau^n(\Sigma))_{n \in \mathbb{N}}$ eventually stabilizes, that is, there exists $n_0 \geq 1$ such that $\tau^{n_0}(\Sigma) = \Omega(\tau) = \{0\}$. This shows that $\tau$ is nilpotent.

This completes the proof of Theorem 1.8.

**Proof of Theorem 1.9** We shall prove the implications

$$
(a) \iff (b) \quad \text{and} \quad (a) \implies (c) \implies (d) \implies (e) \implies (a).
$$

The implication $(a) \implies (b)$ is trivial.

Suppose that $\tau$ is pointwise nilpotent, so that for every $x \in \Sigma$, there exists an integer $n_\tau \geq 1$ such that $\tau^n(x) = 0$ for all $n \geq n_\tau$. Since $G$ is finitely generated, it is countable. Then, the configuration space $A^G$, being a countable product of discrete (and therefore completely metrizable) spaces, it admits a complete metric compatible with its topology, and hence is a Baire space. Since $\Sigma$ is closed in $A^G$, it is a Baire space as well. For each integer $n \geq 1$, the set

$$
X_n := (\tau^n)^{-1}(0) = \{x \in \Sigma : \tau^n(x) = 0\}
$$

is a linear subshift of $A^G$ contained in $\Sigma$. We have $\Sigma = \bigcup_{n \geq 1} X_n$ by our hypothesis on $\tau$. By the Baire category theorem, there is an integer $n_0 \geq 1$ such that $X_{n_0}$ has a nonempty interior. Since $\Sigma$ is topologically mixing and $G$ is infinite, $\Sigma$ is topologically transitive. It follows from a standard fact (cf. [12, Lemma A.3]) that $X_{n_0} = \Sigma$, equivalently, $\tau^{n_0}(\Sigma) = \{0\}$. The latter is equivalent to $\tau$ being nilpotent, and the implication (b) $\implies$ (a) follows. From the first implication we deduce that in fact $(a) \iff (b)$.

The implication (a) $\implies$ (c) is obvious.

The implication (c) $\implies$ (d) is clear since $\Omega(\tau)$ is a vector subspace of $\tau^{n_0}(\Sigma)$.

The implication (d) $\implies$ (e) follows from Proposition 2.1 since any topologically mixing action of an infinite group is topologically transitive.

Finally, suppose (e). As $\Omega(\tau)$ is of finite type, we deduce from Theorem 1.1 that the sequence $(\tau^n(\Sigma))_{n \in \mathbb{N}}$ eventually stabilizes, that is, there exists an $n_0 \in \mathbb{N}$ such that $\tau^{n_0}(\Sigma) = \Omega(\tau) = \{0\}$. Thus $\tau$ is nilpotent. This shows the outstanding implication (e) $\implies$ (a), and the proof of Theorem 1.9 is complete.

**Proof of Corollary 1.10** We only need to prove the equivalences for $G$ not finitely generated. Let $H \subset G$ be a finitely generated subgroup containing both a memory set for $\Sigma$ and a memory set for $\tau$. Observe that $[G : H] = \infty$, since $G$ is not finitely generated. Denote by $\tau_H : \Sigma_H \to \Sigma_H$ the corresponding restriction cellular automaton.

The implication (a) $\implies$ (b) is trivial.

Suppose (b). It is straightforward that $\tau_H$ is also pointwise nilpotent. It then follows from the finitely generated case (i.e., from the implication (b) $\implies$ (a) in Theorem 1.9) that $\tau_H$ is nilpotent. We then deduce from Lemma 2.7(iii) that $\tau$ is itself nilpotent. This shows the implication (b) $\implies$ (a). Combined with the previous implication, this gives the equivalence (a) $\iff$ (b).

The implications (a) $\implies$ (c) $\implies$ (d) are trivial.

Suppose (d). Recalling that $H$ has infinite index in $G$, we deduce from Lemma 2.7(i) that $\Omega(\tau_H) = \{0\}$ and $\Omega(\tau) = \{0\}$. This shows the implication (d) $\implies$ (c).

The final implication (e) $\implies$ (a) follows from Theorem 1.8.

The proof of Corollary 1.10 is complete.

6. Groups of $K$-linear Markov type

We have seen in Corollary 2.3 that the condition that $G$ be finitely generated cannot be removed from the assumptions in Proposition 2.4. More generally, if a finitely generated group $G$ admits a subgroup $H$ which is not finitely generated (for instance, if $G$ contains a subgroup $K$ isomorphic to $F_2$, the free group of rank 2, and $H = [K, K] \subset K$ its commutator subgroup) then the subshift consisting of all configurations $x \in A^G$ which are constant on each left coset of $H$ in $G$ is not of finite type (note that $H$ has necessarily infinite index in $G$).

Recall that a group $G$ satisfies the maximal condition on subgroups if any ascending sequence $G_0 \subset G_1 \subset \cdots \subset G_n \subset G_{n+1} \subset \cdots \subset G$ of subgroups eventually stabilizes, that is, there exists
Suppose that \( n_0 \geq 1 \) such that \( G_n = G_{n_0} \) for all \( n \geq n_0 \). It is immediately verified that a group \( G \) satisfies the maximal condition on subgroups if and only if all of its subgroups are finitely generated. A group satisfying the maximal condition on subgroups is also called a Noetherian group. From the above discussion we immediately deduce the following.

**Corollary 6.1.** Let \( G \) be a group of \( K \)-linear Markov type for some field \( K \). Then \( G \) is Noetherian. In particular, \( G \) is finitely generated.

As remarked above, we don’t know whether or not the class of polycyclic-by-finite groups coincides with the class of groups of \( K \)-linear Markov type. We remark that there exist Noetherian groups constructed by A.Y. Olshanskii \[23\], for which the group algebra is not known to be one-sided Noetherian, equivalently (cf. Theorem 1.3), it is not known whether or not they are of \( K \)-linear Markov type.

On the other hand, it follows from the work of L. Bartholdi \[1\] and P. Kropholler and K. Lorensen \[21\], and Theorem 1.3, that if \( G \) is of \( K \)-linear Markov type, then \( G \) is necessarily amenable.

We refer to Mathoverflow \[13\] for other interesting information.

In the next two lemmas we show that the class of groups of \( K \)-linear Markov type is closed under finite and cyclic extensions.

**Lemma 6.2.** Let \( G \) be a countable group and let \( H \subset G \) be a normal subgroup of finite index. Suppose that \( H \) is of linear Markov type. Then also \( G \) is of linear Markov type.

*Proof.* Let \( A \) be a finite-dimensional vector space and let \( \Sigma \subset A_G \) be a linear subshift. Let \( T \subset G \) be a complete set of representatives for the cosets of \( H \) in \( G \), so that \( G = HT \). Then \( B := A_T \) is a finite-dimensional vector space and the map \( \varphi : A^G \to B^H \) defined by

\[
(\varphi(x)(h))(t) = x(ht)
\]

for all \( x \in A^G \), \( h \in H \), and \( t \in T \), is a linear isomorphisms and uniform homeomorphism. Moreover,

\[
(\varphi(kx)(h))(t) = (kx)(ht) = x(k^{-1}ht) = (\varphi(x)(k^{-1}h))(t) = (k\varphi(x)(h))(t)
\]

for all \( x \in A^G \), \( k \in K \), \( h \in H \), and \( t \in T \), showing that \( \varphi \) is \( H \)-equivariant.

Then \( \Sigma' := \varphi(\Sigma) \) is a linear subshift in \( B^H \). Since \( H \) is of linear Markov type, and \( B \) is finite-dimensional, there exists a finite subset \( D_H \subset H \) and a subspace \( P_H \subset B^{D_H} \) such that \( \Sigma' = \Sigma(B^H; D_H, P_H) \). Set \( D := D_H T \subset G \) and consider the map \( \psi : A^D \to B^{D_H} \) defined by

\[
(\psi(y)(h))(t) = y(ht)
\]

for all \( y \in A^D \), \( h \in D_H \), and \( t \in T \). Then \( \psi \) is a linear isomorphism and a uniform homeomorphism. Let us set \( P := \psi^{-1}(P_H) \subset A^D \). Note that if \( x \in A^G \), \( h \in D_H \), and \( t \in T \), we have

\[
(\varphi(x)|_{D_H}(h))(t) = (\varphi(x)(h))(t) = x(ht) = x|_D(ht) = (\psi(x|_D)(h))(t)
\]

so that

\[
\varphi(x)|_{D_H} = \psi(x|_D).
\]

It follows that

\[
\begin{align*}
x \in \Sigma & \iff tx \in \Sigma, \text{ for all } t \in T & \text{(by } G \text{-invariance of } \Sigma) \\
& \iff \varphi(tx) \in \Sigma', \text{ for all } t \in T & \text{(by definition of } \Sigma' \text{ and } \varphi \text{ being 1-1)} \\
& \iff (h\varphi(tx))|_{D_H} \in P_H, \text{ for all } h \in H \text{ and } t \in T & \text{(since } \Sigma' = \Sigma(B^H; D_H, P_H)) \\
& \iff (\varphi(htx))|_{D_H} \in P_H, \text{ for all } h \in H \text{ and } t \in T & \text{(by } H \text{-equivariance of } \varphi) \\
& \iff (\varphi(gx))|_{D_H} \in P_H, \text{ for all } g \in G & \text{(since } G = HT) \\
& \iff \psi^{-1}((\varphi(gx))|_{D_H}) \in P, \text{ for all } g \in G & \text{(by definition of } P \text{ and } \psi \text{ being 1-1)} \\
& \iff (gx)|_{D} \in P, \text{ for all } g \in G & \text{(by } 6.3) \\
& \iff x \in \Sigma(A^G; D, P).
\end{align*}
\]

This shows that \( \Sigma = \Sigma(A^G; D, P) \) is of finite type. We deduce that \( G \) is of linear Markov type. \( \Box \)
The following is the linear (and therefore simpler) version of the more general result [26 Theorem 7.2].

**Lemma 6.3.** Let $G$ be a countable group and let $H \subset G$ be a normal subgroup such that $G/H$ is infinite cyclic. Suppose that $H$ is of linear Markov type. Then also $G$ is of linear Markov type.

**Proof.** Let $A$ be a finite-dimensional vector space and let $\Sigma \subset A^G$ be a linear subshift. Let $a \in G$ such that $aH$ generates $G/H \cong \mathbb{Z}$ and set $T' := \{a^n : n \in \mathbb{Z}\}$. Then $T'$ is a complete set of representatives for the cosets of $H$ in $G$ so that $G = HT'$. Since $H$ is also countable, we can find an increasing sequence $(F_m)_{m \in \mathbb{N}}$ of finite subsets $F_m \subset H$ such that $1_H \in F_0$ and $H = \bigcup_{m \in \mathbb{N}} F_m$.

For $i,j \in \mathbb{Z} \cup \{-\infty, +\infty\}$ and $i \leq j$ let us set $T_i^j := \{a^i, a^{i+1}, \ldots, a^j\} \subset T'$.

**Claim 1.** For every $n \geq 1$ the set

$$X_n := \{x|_H : x \in \Sigma \text{ such that } x(g) = 0_A \text{ for all } g \in HT_{-n}^{-1}\} \subset A^H$$

is a linear subshift in $A^H$.

**Proof of the Claim.** Let $n \geq 1$. The fact that $X_n$ is a vector subspace of $A^H$ is clear. Let now $x \in X_n$. Then, there exists $h \in \Sigma$ such that $x = y|_H$ and $y(h) = 0_A$ for all $g \in HT_{-n}^{-1}$. Given $h \in H$, we have $hy \in \Sigma$, because $\Sigma$ is a subshift in $A^G$, and $(hy)(g) = y(h^{-1}g) = 0_A$ for all $g \in HT_{-n}^{-1}$, since $h^{-1}g \in hHT_{-n}^{-1} = HT_{-n}^{-1}$. It follows that $hx = (hy)|_H \in X_n$, and this shows that $X_n$ is $H$-invariant. We are only left to show that $X_n$ is closed with respect to the prodiscrete topology in $A^H$. For $k \geq n$ let us set

$$X_{n,k} := \{x|_{F_kT_{-k}^{-1}} : x \in \Sigma \text{ such that } x(g) = 0_A \text{ for all } g \in HT_{-n}^{-1}\} \subset \Sigma_{F_kT_{-k}^{-1}}.$$

Note that $X_{n,k}$ is a finite-dimensional vector space. For $m \geq k \geq n$, let $\pi_{k,m} : A^{F_mT_{-m}^{-1}} \to A^{F_kT_{-k}^{-1}}$ denote the projection map. Note that if $x \in A^{F_mT_{-m}^{-1}}$ satisfies that $x(g) = 0_A$ for all $g \in F_mT_{-m}^{-1}$, then $\pi_{k,m}(x)(g') = 0_A$ for all $g' \in F_kT_{-k}^{-1}$. Hence, setting $p_{k,m} := \pi_{k,m}|_{X_{n,m}}$ we have $p_{k,m} : X_{n,m} \to X_{n,k}$, and $(X_{n,k}, p_{k,m})_{m \geq k \geq n}$ is an inverse system of finite-dimensional vector spaces.

Let $z \in A^H$ be a configuration belonging to the closure of $X_n$ in $A^H$. We must show that $z \in X_n$. By definition, for each $k \geq n$ there exists a configuration $x_k \in \Sigma$ such that

$$x_k|_{F_k} = z|_{F_k} \quad \text{and} \quad x_k(g) = 0_A \text{ for all } g \in HT_{-n}^{-1}.$$

Let us set

$$X_{n,k}(z) := \{x|_{F_kT_{-k}^{-1}} : x \in \Sigma \text{ such that } x|_{F_k} = z|_{F_k} \text{ and } x(g) = 0_A \text{ for all } g \in HT_{-n}^{-1}\} \subset X_{n,k}.$$

Note that $X_{n,k}(z)$ is an affine subset in $A^{F_kT_{-k}^{-1}}$. Moreover, for $i \leq j$ we have that $p_{i,j}(X_{n,j}(z)) \subset X_{n,i}(z)$, showing that $(X_{n,k}(z))$ is an inverse system (in fact, an inverse subsystem of $(\Sigma_{F_kT_{-k}^{-1}})$).

By Lemma 8.1 there exists $x \in \lim_{\leftarrow k \geq n} X_{n,k}(z) \subset \lim_{\leftarrow k \geq n} \Sigma_{F_kT_{-k}^{-1}} = \Sigma$. By construction, we have

$$x(g) = 0_A \text{ for all } g \in F_kT_{-k}^{-1} \quad \text{and} \quad x|_{F_k} = z|_{F_k}, \text{ for all } k \geq n,$$

so that, letting $k \to \infty$,

$$x(g) = 0_A \text{ for all } g \in HT_{-n}^{-1} \quad \text{and} \quad x|_H = z.$$

This shows that $z = x|_H \in X_n$. The claim follows. \hfill \Box

It is clear that $X_n \supset X_{n+1}$ for all $n \geq 1$. Thus, as a consequence of Claim 1, $(X_n)_{n \in \mathbb{N}}$ is a decreasing sequence of linear subshifts of $A^H$. Since $A$ is finite-dimensional and $H$ is of linear Markov type, by Corollary 1.2 the above sequence must stabilize: there exists $n_0 \in \mathbb{N}$ such that $X_n = X_{n_0}$ for all $n \geq n_0$. Thus, setting

$$X := \bigcap_{n \in \mathbb{N}} X_n = \{x|_H : x \in \Sigma \text{ such that } x(g) = 0_A \text{ for all } g \in HT_{-n_0}^{-1}\} \subset A^H$$

we have that $X$ is a linear subshift in $A^H$ and, moreover,

$$X = X_{n_0} = \{x|_H : x \in \Sigma \text{ such that } x(g) = 0_A \text{ for all } g \in HT_{-n_0}^{-1}\}.$$
Consider the finite set $T := T_{-n_0} \subset T'$ and set $\Omega := HT \subset G$. The action of $H$ on $\Omega$ by left multiplication induces an action of $H$ on $A^\Omega$: this is given by setting $(hx)(kt) := x(h^{-1}kt)$ for all $h, k \in H, x \in A^\Omega$, and $t \in T$.

**Claim 2.** The subset $\Sigma_\Omega \subset A^\Omega$ is $H$-invariant and closed with respect to the prodiscrete topology on $A^\Omega$.

**Proof of the Claim.** Let $z \in \Sigma_\Omega$. Then there exists $x \in \Sigma$ such that $z = x|_\Omega$. Given $h \in H$, we have, for all $k \in H$ and $t \in T$,

$$(hx)(kt) = z(h^{-1}kt) = x(h^{-1}kt) = (hx)(kt) = (hx)|_\Omega(kt).$$

Since $hx \in \Sigma$, we deduce that $hx = (hx)|_\Omega \in \Sigma_\Omega$. This shows that $\Sigma_\Omega$ is $H$-invariant.

Since $G$ is countable, we can find an increasing sequence $(E_n)_{n \in \mathbb{N}}$ of finite subsets of $G$ such that $G = \bigcup_{n \in \mathbb{N}} E_n$. Setting $F_n := E_n \cap \Omega$ for all $n \in \mathbb{N}$, we obtain an increasing sequence $(F_n)_{n \in \mathbb{N}}$ of finite subsets of $\Omega$ such that $\Omega = \bigcup_{n \in \mathbb{N}} F_n$. Let $d \in A^\Omega$ and suppose it belongs to the closure of $\Sigma_\Omega$ in $A^\Omega$. We must show that $d \in \Sigma_\Omega$. For each $n \in \mathbb{N}$ there exists $y \in \Sigma_\Omega$ such that $d|_{F_n} = y|_{F_n}$. Since $y \in \Sigma_\Omega$, there exists $x \in \Sigma$ such that $y = x|_\Omega$. Setting $z := x|_{E_n} \in \Sigma_{E_n}$, we have $z|_{F_n} = x|_{F_n} = (x|_\Omega)|_{F_n} = y|_{F_n} = d|_{F_n}$, so that the finite-dimensional affine set

$$Z_n := \{z \in \Sigma_{E_n} : z|_{F_n} = d|_{F_n}\} \subset \Sigma_{E_n}$$

is nonempty. It is clear that for $m, n \in \mathbb{N}$ with $m \geq n$ the restriction map $\pi_{n,m} : A^{E_m} \to A^{E_n}$ induces, by restriction, a well defined linear map $p_{nm} : Z_m \to Z_n$. Hence, by applying Lemma 3.1 to the inverse system $(Z_m, p_{nm})$, there exists $x \in A_{-\infty} \subset Z_n \subset \lim_{n \to \infty} \Sigma_{E_n} = \Sigma$. By definition, we have $x|_{F_n} = d|_{F_n}$ for every $n \in \mathbb{N}$, so that $x|_\Omega = d$. This shows that $d = x|_\Omega \in \Sigma_\Omega$. We deduce that $\Sigma_\Omega$ is closed, and the claim follows. □

**Claim 3.** $\Sigma = \Sigma(A^G, \Omega, \Sigma_\Omega)$. 

**Proof of the Claim.** Let us set $\Sigma = \Sigma(A^G, \Omega, \Sigma_\Omega) \subset A^G$. It is clear that $\Sigma \subset \Sigma$. To prove the converse inclusion, let $y \in \Sigma$. Then, there exists $z_0 \in \Sigma$ such that $z_0|_\Omega = y|_\Omega$. Since also $a^{-1}y \in \Sigma$, there exists $y_0 \in \Sigma$ such that $y_0|_\Omega = (a^{-1}y)|_\Omega$. As a consequence, setting $z_1 := a y_0 \in \Sigma$, one has $z_1(a\omega) = (a^{-1}z_1)(\omega) = y_0(a\omega) = (a^{-1}y)(\omega) = y(a\omega)$ for all $\omega \in \Omega$, equivalently, $z_1|_\Omega = y|_\Omega$. Note that $a\Omega = aHT = HaT = HaT_{-n_0} = T_{-n_0+1}H$ so that $\Omega \cap (a\Omega) = HT_{-n_0+1} \cap T_{-n_0+1}$. Thus, for the configuration $z := z_0 - z_1 \in \Sigma$ we have $z(g) = 0_A$ for all $g \in HT_{-n_0}$. Moreover, if $g' \in HT_{-n_0}$, then $g := a^{n_0+1}g' \in HT_{-n_0}$ and therefore $(a^{-n_0}z)(g') = z(a^{n_0}g') = z(g) = 0_A$. As a consequence, the configuration $v := (a^{-n_0-1}z)|_H \in A^H$ is in $X$ (cf. (6.6)). Set

$$L(v) := \{x \in \Sigma : x(g) = 0_A \text{ for all } g \in HT_{-\infty} \text{ and } x|_H = v\} \subset A^G.$$

Clearly, $L(v)$ is a nonempty affine subspace: keeping in mind (6.6), there exists $z' \in \Sigma$ with $z'|_H = (a^{-n_0-1}z)|_H = v$ and $z'(g) = 0_A$ for all $g \in HT_{-\infty}$, so that $z' \in L(v)$. Let $c \in L(v)$ and consider the configuration $x := z_0 - a^{n_0+1}c \in \Sigma$. Let $h \in H$ and set $h' := a^{-n_0-1}ha_{n_0+1} \in H$. Then, for $1 \leq n \leq n_0$ we have

$$(6.6) \quad x(ha^n) = z_0(ha^n) - (a^{n_0+1}c)(ha^n) = z_0(ha^n) - c(h'a^{n_0-1}) = z_0(ha^n) = y(ha^n),$$

where we used the fact that, on the one hand $-n_0 \leq n - n_0 - 1 \leq -1$ and $c|_{HT_{-n_0}} = 0$, and, on the other hand, $HT_{-n_0} \subset HT_{-n_0} \subset \Omega$ and $z_0|_\Omega = y|_\Omega$. Moreover,

$$x(ha^{n_0+1}) = x(a^{n_0+1}h') = z_0(a^{n_0+1}h') - (a^{n_0+1}c)(a^{n_0+1}h')$$

$= z_0(a^{n_0+1}h') - c(h') = z_0(a^{n_0+1}h') - (a^{-n_0-1}z)(h') \quad \text{(since } c \in L(v)\text{)}$

$= z_0(a^{n_0+1}h') - z(a^{n_0+1}h') = z_1(a^{n_0+1}h') \quad \text{(since } z = z_0 - z_1\text{)}$

$= z_1(a^{n_0+1}h') = y(a^{n_0+1}h') \quad \text{(since } z_1|_\Omega = y|_\Omega\text{)}$

$= y(a^{n_0+1}h') = y(ha^{n+1}).$
Keeping in mind (6.6), this shows that \( x|_{HT^n_{i_0+1}} = y|_{HT^n_{i_0+1}} \). An immediate induction on \( m \geq 1 \) yields a sequence \((x_m)_{m \geq 1}\) in \( \Sigma \) such that

\[(6.7) \quad x_m|_{HT^n_i} = y|_{HT^n_i}\]

for all \( m \geq 1 \).

Let now \( F \subset G \) be a finite subset. Then we can find \( i, j \in \mathbb{Z} \), with \( i \leq j \), such that \( F \subset HT^j_i \). Setting \( m := j - i + 1 \), it follows that \( a^{-i+1} F \subset HT^m_i \). Consider the configuration \( y' := a^{-i+1} y \in \Sigma' \). Then by using (6.7) applied to \( y' \), we can find \( x'_m \in \Sigma \) such that \( x'_m|_{HT^m_i} = y'|_{HT^m_i} \). Then setting \( x_m := a^{-i}_m \) \( x'_m \in \Sigma \), we obtain \( x_m|_{HT^m_i} = y|_{HT^m_i} \) so that, in particular, \( x_m|_F = y|_F \). Since \( \Sigma \) is closed and \( F \) was arbitrary, this shows that \( y \in \Sigma \). This proves \( \Sigma \subset \Sigma \), and the claim follows.

The remaining of the proof of the lemma follows step by step the end of the proof of Lemma 6.2 with \( G \) replaced by \( \Omega \) and \( \Sigma' \) replaced by \( \varphi(\Omega) \). We thus set \( B := A^T \), so that \( B \) is a finite-dimensional vector space and the map \( \varphi: A^D \to B^H \) defined by (6.1) is an \( H \)-equivariant linear isomorphism and uniform homeomorphism. By virtue of Claim 2, we have that \( \Sigma' := \varphi(\Omega) \subset B^H \) is a subshift. Since \( H \) is of linear Markov type, and \( B \) is finite-dimensional, there exists a finite subset \( D_H \subset H \) and a subspace \( P_H \subset B^{D_H} \) such that \( \Sigma' = (\Sigma(B^H; D_H, P_H)) \). Then, setting \( D := DT \subset G \) and \( P := \psi^{-1}(P_H) \subset A^D \), where \( \psi: A^D \to B^{D_H} \) is as in (6.1), we have that \( \Sigma = \Sigma(A^G; D, P) \) is of finite type.

**Proposition 6.4.** Let \( K \) be a field. Then the class of \( K \)-linear Markov groups is closed under the operations of taking subgroups, quotients, and extensions by finite or cyclic groups.

**Proof.** Let \( G \) be a group, let \( H \subset G \) be a subgroup, and let \( A \) be a finite-dimensional vector space over a field \( K \). Given a subshift \( \Sigma \subset A^H \) we set

\[ \Sigma^{(G)} := \{ x \in A^G : (gx)|_H \in \Sigma \text{ for all } g \in G \} \subset A^G. \]

Roughly speaking, \( \Sigma^{(G)} \) is the set of all configurations in \( A^G \) whose restriction to each left coset \( c \in G/H \) yields – modulo the bijection \( h \mapsto gh \), induced by an element \( g \in c \), which identifies \( H \) and \( c \) – an element in \( \Sigma \).

It is easy to see that \( \Sigma^{(G)} \subset A^G \) is a linear subshift and that it is of finite type if and only if \( \Sigma \) is of linear Markov type, so that it is of finite type if and only if \( \Sigma \) is normal in \( G \).

Suppose now that \( H \) is normal in \( G \) and denote by \( \pi: G \to K := G/H \) the canonical quotient homomorphism. Given a subshift \( \Sigma \subset A^K \) we denote by

\[ \Sigma(G) := \{ x \circ \pi : x \in \Sigma \} \subset A^G. \]

Roughly speaking, \( \Sigma(G) \) is the set of all configurations \( x \in A^G \) which are constant on each left coset \( c \in G/H \) and such that, if \( T \subset G \) is a complete set of representatives of the cosets of \( H \) in \( G \), then the restriction \( x|_T \) yields – modulo the bijection \( \pi|_T: T \to K \) – an element in \( \Sigma \). Assume that \( G \) is of linear Markov type. Once again, it is easy to see that \( \Sigma(G) \subset A^G \) is a linear subshift and that it is of finite type if and only if \( \Sigma \) is. We deduce that \( K \) is of linear Markov type as well.

The fact that the class of groups of linear Markov type is closed under finite or cyclic extensions follows from Lemma 6.2 and Lemma 6.3, respectively.

It is a well known fact (see, e.g., [28] Theorem 5.4.12) that a solvable group is polycyclic if and only if it is Noetherian. Similarly, one has that a virtually solvable group is polycyclic-by-finite if and only if it Noetherian (cf. [29] Lemma 6). From Corollary 1.4 and Corollary 6.1 we deduce the following (cf. [29] Theorem 5):

**Corollary 6.5.** Let \( G \) be a virtually solvable group and let \( K \) be a field. Then the following conditions are equivalent:

(a) \( G \) is of \( K \)-linear Markov type;
(b) \( G \) is Noetherian;
(c) \( G \) is polycyclic-by-finite.
Remark 6.6. As mentioned above, we can directly deduce Corollary [1.4] from Lemma [6.2] and Lemma [6.3] thus without using P. Hall’s theorem. For the sake of completeness, we produce here the alternative proof, by induction. Thus, suppose that $G$ is a polycyclic-by-finite group. Then $G$ admits a subnormal series $G = G_n \supset G_{n-1} \supset \cdots \supset G_1 \supset G_0 = \{1_G\}$ whose factors are finite or cyclic groups. We first observe that if $G$ is a trivial group, then it is of $K$-linear Markov type. Indeed, let $A$ be a finite-dimensional vector space over a field $K$. Then, setting $D := \{1_G\}$ and identifying $A$ with $A^D$ and $A^G$, we have that the (identity) map

$$B \mapsto \Sigma(A^G, D, B)$$

yields a bijection between subspaces $B \subset A$ and subshifts $\Sigma \subset A^G$. Since every descending sequence of vector subspaces of a finite-dimensional vector space eventually stabilizes, it follows from Corollary [1.2] that all subshifts $A$ are of finite type. This proves the base of induction. A recursive application of Lemma [6.2] or Lemma [6.3] then shows that $G_n, G_{n-1}, \ldots, G_1, G_0 = G$ are all of $K$-linear Markov type.

7. Examples and final remarks

7.1. The descending chain condition. Let $G$ be a group and let $A$ be an infinite-dimensional vector space over a field $K$. Then $A$ admits a strictly decreasing sequence $(A_n)_{n \in \mathbb{N}}$ of vector subspaces and the sequence $(\Sigma_n)_{n \in \mathbb{N}}$, where $\Sigma_n := A^G_n \subset A^G$, is a strictly decreasing sequence of linear subshifts of $A^G$. Thus $A^G$ does not satisfy the descending chain condition for linear subshifts.

7.2. The closed image property. In [9, Section 5] it is shown that if $A$ is an infinite-dimensional vector space and $G$ is any nonperiodic group, then there exists a linear cellular automaton $\tau: A^G \to A^G$ whose image $\tau(A^G)$ is not closed in $A^G$. This shows that Theorem [4.1] fails to hold in general if the finite-dimensionality of the alphabet $A$ is dropped.

Explicitly, the linear cellular automaton $\tau: A^G \to A^G$ we alluded to above can be defined as follows. Since $A$ is infinite-dimensional, we can find a sequence $(a_i)_{i \in \mathbb{N}}$ of linearly independent vectors in $A$. Let $E$ denote the vector subspace spanned by the $a_i$’s and let $F$ be a vector subspace such that $A = E \oplus F$. Let $\psi: A \to A$ denote the linear map defined by setting $\psi(a_i) = a_{i+1}$ for all $i \in \mathbb{N}$ and $\psi|_E = 0$. Since $G$ is nonperiodic, there exists an element $g \in G$ of infinite order. Then the cellular automaton $\tau: A^G \to A^G$ with memory set $M = \{1_G, g\} \subset G$ and local defining map $\mu: A^M \to A$ given by

$$\mu(y) := y(g) - \psi(y(1_G)),$$

for all $y \in A^M$, satisfies that $\tau(A^G)$ is not closed in $A^G$ (cf. [9, Lemma 5.2]; see also [8, Example 8.8.3]).

7.3. Nilpotency for linear cellular automata. Let $G$ be a group and let $A$ be a vector space over a field $K$. Given a linear map $f: A \to A$, we denote by $\tau_f: A^G \to A^G$ the LCA with memory set $M := \{1_G\}$ and associated local defining map $\mu_f := f: A = A^M \to A$. In other words, $\tau_f = \prod_{g \in G} f$ so that, in particular, $\tau_f^n(A^G) = \prod_{g \in G} f^n(A)$ for all $n \in \mathbb{N}$. As a consequence,

$$(7.1) \quad \Omega(\tau_f) = \bigcap_{n \in \mathbb{N}} \tau_f^n(A^G) = \bigcap_{n \in \mathbb{N}} \prod_{g \in G} f^n(A) = \prod_{g \in G} \bigcap_{n \in \mathbb{N}} f^n(A) = \bigcap_{g \in G} \Omega(f) = \Omega(f)^G.$$

Note that $f$ is nilpotent (resp. pointwise nilpotent) if and only if $\tau_f$ is nilpotent (resp. pointwise nilpotent).

Suppose that $A$ is infinite-dimensional. Let $\{e_n : n \in \mathbb{N}\} \subset A$ be an independent subset and set $A_1 := \text{span}_K \{e_n : n \in \mathbb{N}\}$ and $A_2 := A \ominus A_1$.

(1) Consider the linear map $f: A \to A$ defined by setting $f(e_n) = e_{n+1}$ for all $n \in \mathbb{N}$ and $f(a) = 0$ for all $a \in A_2$. It is then clear that $\Omega(f) = \{0\}$ so that, by (7.1), $\Omega(\tau_f) = \{0\}$. However, $\tau_f$ is not pointwise nilpotent (and therefore not nilpotent either).

(2) Consider the linear map $f: A \to A$ defined by setting $f(e_0) = 0, f(e_n) = e_{n-1}$ for all $n \geq 1$, and $f(a) = 0$ for all $a \in A_2$. Then $f$ and therefore $\tau_f$ are surjective so that $\tau_f$ is not nilpotent, $\Omega(\tau_f) = A^G$. However, $f$ and therefore $\tau_f$ are pointwise nilpotent.
Consider, for each $n \geq 1$, the set $I_n := \{0, 1, \ldots, n\}$ and the map $g_n: I_n \to I_n$ given by $g_n(k) := k - 1$ if $k \geq 1$ and $g_n(0) = 0$. Let $X$ be the set obtained by taking disjoint copies of the sets $I_n$, $n \geq 1$, and identifying all copies of 0 in a single point $y_0$ and all copies of 1 in a single point $y_1 \neq y_0$. Then the maps $g_n$ induce a well-defined quotient map $g: X \to X$. Clearly $\Omega(g) = \{y_0, y_1\}$ and $g(\Omega(g)) = \{y_0\}$. Since $X$ is countable, we can find a bijection $\varphi: \mathbb{N} \to X$ such that $\varphi(0) = y_0$ and $\varphi(1) = y_1$. Setting $h: \varphi^{-1} \circ g \circ \varphi: \mathbb{N} \to \mathbb{N}$ we thus have $\Omega(h) = \{0, 1\}$ and $h(\Omega(h)) = \{0\}$. Consider the linear map $f: A \to A$ defined by setting $f(e_{a}) := e_{h(a)}$ for all $a \in \mathbb{N}$ and $f(a) = 0$ for all $a \in A_2$. Then $\Omega(f) = \text{span}_K \{e_0, e_1\} = K e_0 + K e_1$ while $f(\Omega(f)) = \text{span}_K \{e_0\} = K e_0$. As a consequence, $\tau_f(\Omega(f)) = (Ke_0)^G \subset (Ke_0 + Ke_1)^G = \Omega(\tau_f)$.

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