Some Characterizations of Exponential Distribution Based on Order Statistics

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Abstract

In this paper some new characterizing theorems of exponential distribution based on order statistics are presented. Some existing results are generalized and the open conjecture by Arnold and Villasenor is solved.

keywords: Characterization, order statistics, exponential distribution, Stirling numbers of the second kind

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1 Introduction

There is an abundance of characterizations of exponential distribution and among them a considerable part is based on properties of order statistics. Most of them could be found in \([1, 3, 7]\) and \([10]\). Recently Arnold and Villasenor \([4]\) proposed a series of characterizations based on the sample of size two and stated some conjectures on their generalization. They also proposed a new method of proof which can be used when the density in question is analytic. Later Yanev and Chakraborty \([12]\) proved by this method two characterization theorems concerning maximum of sample of size three as well as the characterization based on consecutive maxima \([6]\).

In this paper we extend the generalizations to arbitrary order statistics. We consider the case of consecutive order statistics via convolution with independent random variable from the same distribution. Similar problems have been studied in \([11]\) and \([5]\). Their formulations are slightly different in terms that the convolution in question includes a random variable with fixed distribution.

The other case we consider is the characterization based on representation of \(k\) th order statistic of sample of size \(n\) as a weighted sum of \(k\) sample members. This problem has a history. Alsamullah and Rahman \([2]\) proved the theorem when the representation is valid for all \(k\). Later Huang \([9]\) showed that the condition in general cannot be relaxed to just one value of \(k\). We prove that under the assumption of analyticity of the density function the theorem is valid even under this relaxed condition. As a corollary we solve the conjecture of Arnold and Villasenor regarding representation of sample maximum stated in \([4]\).

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2 Auxiliary results

In this section we present four combinatorial identities that will be used in the proofs of characterization theorems. In all of them appear Stirling numbers of the second kind. A Stirling number of second kind, denoted \( \{ a \, b \} \), represents the number of ways to partition a set of \( a \) objects into \( b \) non-empty subsets. In proofs of our lemmas we use the following well-known identities (see e.g. [8]).

\[
\begin{align*}
\{ a \, b \} &= \{ a - 1 \, b - 1 \} + b \{ a - 1 \, b \}, \\
\{ a + 1 \, b + 1 \} &= \sum_{l=0}^{a} \binom{a}{l} \{ 1 \, b \}, \\
\{ a + b + 1 \, b \} &= \sum_{l=0}^{b} l \{ a + l \, l \}, \\
\{ a \, b \} &= \sum_{l=0}^{b} \binom{b}{l} a(a - 1) \cdots (a - l + 1).
\end{align*}
\]

We proceed with the lemmas necessary for the proofs of the characterization theorems.

**Lemma 2.1** For integers \( k, n, r \) such that \( 1 < k \leq n \) and \( r \geq 0 \) it holds

\[
\begin{align*}
&\sum_{j=k-2}^{k+r-1} \sum_{i=0}^{j-k+2} \binom{n-k}{i}(i+k-2)! \{ j+1 \, i+k-1 \}(k-1)n^{k+r-1-j} \\
&= \sum_{i=0}^{r+1} \binom{n-k}{i}(i+k-1)! \{ k+r+1 \, i+k \}.
\end{align*}
\]

**Proof.** We prove the lemma by induction on \( r \). For \( r = 0 \) the equality \((5)\) simplifies to

\[
(k-1)! \left( n + \binom{k}{k-1} + (k-1)(n-k) \right) = (k-1)! \left( \binom{k+1}{k} + k(n-k) \right),
\]

which is true because of \((1)\). Thus the statement of the lemma holds for \( r = 0 \) for all \( 1 < k \leq n \).

Let us now suppose that \((5)\) is satisfied for \( r-1 \) for all \( 1 < k \leq n \). The left hand side of \((5)\) can be split as

\[
\begin{align*}
&\sum_{j=k-2}^{k+r-2} \sum_{i=0}^{j-k+2} \binom{n-k}{i}(i+k-2)! \{ j+1 \, i+k-1 \}(k-1)n^{k+r-1-j} \\
+ &\sum_{i=0}^{r+1} \binom{n-k}{i}(i+k-2)! \{ k+r \, i+k-1 \}(k-1).
\end{align*}
\]
Using the induction hypothesis on the first summand we have that the expression above is equal to

\[ \sum_{i=0}^{r} \binom{n-k}{i} (i + k - 1)! \binom{k + r}{i + k} n + \sum_{i=0}^{r+1} \binom{n-k}{i} (i + k - 2)! \binom{k + r}{i + k - 1} (k - 1). \]

(6)

It remains to prove that (6) is equal to

\[ \sum_{i=0}^{r+1} \binom{n-k}{i} (i + k - 1)! \binom{k + r + 1}{i + k}. \]

which can be written as

\[ \sum_{i=0}^{r+1} \binom{n-k}{i} (i + k - 1)! \binom{k + r}{i + k} + \sum_{i=0}^{r} \binom{n-k}{i} (i + k)! \binom{k + r}{i + k}. \]

(7)

Grouping the corresponding summands from (6) and (7) we get

\[ \sum_{i=0}^{r} \binom{n-k}{i} (i + k - 1)! \binom{k + r}{i + k} (n - k - i) = \sum_{i=0}^{r+1} \binom{n-k}{i} (i + k - 2)! \binom{k + r}{i + k - 1} i. \]

The last equality is easily shown putting \( j = i + 1 \) in the first sum.

**Lemma 2.2** For integers \( k, n, r \) such that \( 1 < k \leq n \) and \( r \geq 0 \) it holds

\[ \sum_{j=k-2}^{k+r-1} \sum_{i=0}^{j} \binom{n-k+1}{i} (i + k - 2)! \binom{j+1}{i + k - 1} (k - 1)(n - k + 1)^{k+r-1-j} = \sum_{i=0}^{r+1} \binom{n-k}{i} (i + k - 1)! \binom{k + r + 1}{i + k}. \]

(8)

The proof of this lemma is analogous to the proof of lemma 2.1 so we omit it here.

**Lemma 2.3** For integers \( k, n, r \) such that \( 1 < k \leq n \) and \( r \geq 0 \) it holds

\[ \sum_{i=0}^{r+1} \binom{n-k}{i} (i + k - 1)! \binom{k + r + 1}{i + k}. \]

(9)

**Proof.** We prove the lemma by induction on \( n \). For any \( r \) and \( k \) and \( n = k \) the expression (9) simplifies to identity (3).

Suppose now that the equality (9) is true for any \( k \), any \( r \) and \( n - 1 \). We need to prove that it is also true for \( n \). Transforming the left hand side of (9) we get

\[ \sum_{i=0}^{r+1} \binom{n-k}{i} (i + k - 1)! \binom{k + r + 1}{i + k}. \]
Lemma 2.4 For integers \( k, n, r \) such that \( 1 < k \leq n \) and \( r \geq 0 \) it holds

\[
\sum_{i=0}^{r+1} \binom{n-k}{i} \left( n-k-i \right)^{k} \sum_{s=1}^{r} \frac{(i+s-1)!}{(s-1)!} \left\{ s+r \right\} + \sum_{s=1}^{k} \frac{(i+s)!}{(s-1)!} \left\{ s+r \right\}
\]

\[
\sum_{i=0}^{r+1} \binom{n-k}{i} \left( n-k-i \right)^{k} \sum_{s=1}^{r} \frac{(i+s-1)!}{(s-1)!} \left\{ s+r \right\} + \sum_{i=0}^{r+1} \frac{(n-k-1)!}{(s-1)!} \left\{ s+r \right\}
\]

\[
\sum_{i=0}^{r+1} \binom{n-k}{i} \left( n-k-i \right)^{k} \sum_{s=1}^{r+1} \frac{(i+s-1)!}{(s-1)!} \left\{ s+r \right\}
\]

\[
\sum_{i=0}^{r+1} \binom{n-k}{i} \left( n-k-i \right)^{k} \sum_{s=1}^{r+1} \frac{(i+s-1)!}{(s-1)!} \left\{ s+r \right\} + \sum_{s=1}^{r} \frac{(n-k-1)!}{(s-1)!} \left\{ s+r \right\}
\]

Applying the identity (11), shifting the index \( s \) in the last inner sum and separating the term for \( s = k + 1 \), the expression above becomes

\[
\sum_{i=0}^{r+1} \binom{n-k}{i} \left( n-k-i \right)^{k} \sum_{s=1}^{r+1} \frac{(i+s-1)!}{(s-1)!} \left\{ s+r \right\}
\]

\[
\sum_{i=0}^{r+1} \binom{n-k}{i} \left( n-k-i \right)^{k} \sum_{s=1}^{r+1} \frac{(i+s-1)!}{(s-1)!} \left\{ s+r \right\}
\]

\[
\sum_{i=0}^{r+1} \binom{n-k}{i} \left( n-k-i \right)^{k} \sum_{s=1}^{r+1} \frac{(i+s-1)!}{(s-1)!} \left\{ s+r \right\}
\]

Grouping the first two summands together and applying the induction hypothesis to the result we get the right hand side of (10). □

**Lemma 2.4** For integers \( k, n, r \) such that \( 1 < k \leq n \) and \( r \geq 0 \) it holds

\[
\sum_{i=0}^{r+1} \binom{n-k}{i} \left( n-k-i \right)^{k} \sum_{s=1}^{r} \frac{(i+s-1)!}{(s-1)!} \left\{ s+r \right\} = \sum_{i=0}^{r+1} \binom{n-k}{i} \left( i+k \right) \sum_{s=1}^{r+1} \frac{(i+s)!}{(s-1)!} \left\{ i+r+1 \right\}.
\]

**Proof.** The proof is done using the strong induction on \( r \). For any \( k \) and \( n \) and \( r = 0 \) we have

\[
n + (n-1) + \cdots + (n-k+1) = \binom{k+1}{k} + (n-k)k,
\]

which is obviously true since \( \binom{k+1}{k} = \frac{k(k+1)}{2} \). Suppose now that (10) is satisfied up to \( r - 1 \). Then it remains to prove that it is satisfied for \( r \).
Applying lemma 2.1 to the two inner sums and grouping the summands we get

\[ \sum_{j_1, \ldots, j_k \geq 0} n^{j_1} (n-1)^{j_2} \cdots (n-k+1)^{j_k} = \sum_{j_1, \ldots, j_k \geq 0} (n-1)^{j_2} \cdots (n-k+1)^{j_k} \]

\[ + \sum_{j_1=1}^{r+1} n^{j_1} \sum_{j_2, \ldots, j_k \geq 0} (n-1)^{j_2} \cdots (n-k+1)^{j_k}. \quad (11) \]

Remark 2.5 The statements of lemmas 2.3 and 2.4 are also true for \( k = 1 \).

They could be easily proven analogously.

\[ \sum_{j_1, \ldots, j_k \geq 0} n^{j_1} (n-1)^{j_2} \cdots (n-k+1)^{j_k} = (n-k+1)^{r+1} \]

\[ + \sum_{i=1}^{k-1} \sum_{j_1=1}^{r+1} (n-l+1)^j \sum_{i=0}^{r+1-j} \binom{n-k}{i} \frac{(i+k-l-1)!}{(k-l-1)!} \{ k-l+r+1-j \}. \]

Substituting the index \( j \) with \( m = k+r-1-l-j \) and, subsequently, the index \( l \) with \( s = k-l+1 \), as well as applying the identity \( (s) \) to \((n-k+1)^{r+1}, (12)\) becomes

\[ \sum_{s=0}^{r} \sum_{i=0}^{r-s-2} \frac{(n-k+s)!}{(s-1)!} \frac{(m+1)}{(i+s-1)!} \{ m+1 \} \]

\[ + \sum_{i=0}^{r+1} \frac{(n-k+1)!}{(n-k+1-i)!} \{ i \} \]

\[ = \sum_{s=0}^{r} \sum_{i=0}^{r-s-2} \frac{(n-k+s)!}{(s-1)!} \frac{(m+1)}{(i+s-1)!} \{ m+1 \} \]

\[ + (n-k+1) \sum_{i=1}^{r+1} \frac{(n-k)}{(i-1)!} \{ r+1 \}. \]

Applying lemma 2.1 to the two inner sums and grouping the summands we get

\[ \sum_{s=1}^{r} (n-k+s) \frac{(n-k)}{i} \frac{(i+s-1)!}{(s-1)!} \{ r+s \}. \]

Applying now lemma 2.4 we obtain the right hand side of \((10)\). Hence the proof is completed. \( \square \)

Remark 2.5 The statements of lemmas 2.3 and 2.4 are also true for \( k = 1 \).

They could be easily proven analogously.
3 Main results

In the beginning we state and prove two lemmas that will play an important role in the proofs of the theorems. They are similar to those from [6].

Let \( F \) be a class of continuous distribution functions \( F \) such that \( F(0) = 0 \) and whose density function \( f \) allows expansion in Maclaurin series for all \( x > 0 \).

**Lemma 3.1** Let \( F \) be a distribution function that belongs to \( F \). If for all natural \( q \) holds

\[
f^{(q)}(0) = (-1)^q f^{q+1}(0),
\]

then \( f(x) = \lambda e^{-\lambda x} \) for some \( \lambda > 0 \).

**Proof.** Expanding the function \( f \) in Maclaurin series for positive values of \( x \) we get

\[
f(x) = \sum_{q=0}^{\infty} f^{(q)}(0) \frac{x^q}{q!} = \sum_{q=0}^{\infty} (-1)^q f^{q+1}(0) \frac{x^q}{q!} = f(0)e^{-f(0)x}.
\]

For \( f(0) > 0 \) this is the density of exponential distribution with \( \lambda = f(0) \).

**Lemma 3.2** Let \( F \) be a distribution function that belong to the class \( F \). Denote \( A_m(x) = F^m(x)f(x) \). If the condition (13) is satisfied for all \( 0 \leq k \leq r - m, r > m \), then

\[
A^{(r)}_m(0) = (-1)^{r-m} f^{r+1}(0) \binom{r+1}{m+1} m!.
\]

**Remark 3.3** For \( r \leq m \) the statement of the lemma is valid without any condition imposed on derivatives of \( f \).

**Proof.** The \( r \) th derivative of \( A_m(x) \) is

\[
A^{(r)}_m(x) = \sum_{j_1, \ldots, j_{m+1} \geq 0 \atop j_1 + \cdots + j_{m+1} = r} \binom{r}{j_1, \ldots, j_{m+1}} F^{(j_1)}(x) \cdots F^{(j_m)}(x) f^{(j_{m+1})}(x).
\]

Using the fact that \( F(0) = 0 \) we get

\[
A^{(r)}_m(0) = \sum_{j_1, \ldots, j_{m+1} \geq 0 \atop j_1 + \cdots + j_{m+1} = r} \binom{r}{j_1, \ldots, j_{m+1}} f^{(j_1-1)}(0) \cdots f^{(j_{m-1})}(0)f^{(j_{m+1})}(0).
\]
Since all derivatives are of orders smaller or equal to \( r - m \) using (13) we obtain

\[
A_m^r(0) = \sum_{j_1, \ldots, j_m \geq 0} \binom{r}{j_1, \ldots, j_m} (-1)^{r-m} f^{r+1}(0)
\]

\[
= (-1)^{r-m} f^{r+1}(0) \sum_{j_1, \ldots, j_m \geq 1, j_{m+1} = 0} \binom{r}{j_1, \ldots, j_m}
\]

\[
+ (-1)^{r-m} f^{r+1}(0) \sum_{j_1, \ldots, j_m \geq 1, j_{m+1} = r} \binom{r}{j_1, \ldots, j_m}
\]

\[
= (-1)^{r-m} f^{r+1}(0) \left( \binom{r}{m+1} (m+1)! + \binom{r}{m} m! \right)
\]

\[
= (-1)^{r-m} f^{r+1}(0) m! \left[ \binom{r+1}{m+1} \right].
\]

In the last line we used the identity (1). □

Let \( X_{(k,n)} \) be the \( k \) th order statistics from the sample of size \( n \). We now state the characterization theorems.

**Theorem 3.4** Let \( X_1, \ldots, X_n \) be a random sample from the distribution \( F \) that belongs to \( \mathcal{F} \). Let \( k \) be a fixed number such that \( 1 < k \leq n \). If

\[
X_{(k-1,n-1)} + \frac{1}{n} X_n \overset{d}{=} X_{(k,n)}
\]

then \( X \sim \mathcal{E}(\lambda), \lambda > 0 \).

**Proof.** Equalizing the densities from (17) we get

\[
\int_0^x \frac{(n-1)!}{(k-2)!(n-k)!} f^{k-2}(x-y)(1-F(x-y))^{n-k} f(x-y)n f(ny)dy
\]

\[
= \frac{n!}{(k-1)!(n-k)!} f^{k-1}(x)(1-F(x))^{n-k} f(x),
\]

or

\[
(k-1) \sum_{i=0}^{n-k} (-1)^i \binom{n-k}{i} \int_0^x A_{i+k-2}(x-y)f(ny)dy
\]

\[
= f(x) \sum_{i=0}^{n-k} (-1)^i \binom{n-k}{i} \int_0^x A_{i+k-2}(y)dy.
\]

Using induction we prove that (13) holds for every natural \( q \) which by lemma 3.1 implies that \( f(x) \) is exponential density.

Differentiating integral equation (18) \( k \) times we get
eliminating zero terms we get

\[ (n-k)! f^{(k-1)}(0) + f(0) A^{(k-1)}_{k-2}(0) = (n-k)! f^{k+1}(0), \]

from where we get \( f'(0) = -f^2(0) \), which means that (15) holds for \( q = 1 \).

Suppose now that (15) is satisfied for all \( k \leq r \). We shall prove that it holds for \( q = r+1 \).

Differentiating the integral equation (15) \( k + r \) times, letting \( x = 0 \) and eliminating zero terms we get

\[
(k - 1) \sum_{i=0}^{n-k} (-1)^i \binom{n-k}{i} \sum_{j=0}^{k-1} n^{k-1-j} f^{(k-1-j)}(0) A^{(j)}_{i+k-2}(0) + \int_0^x A^{(k)}_{i+k-2}(x-y) f(ny)dy
\]

\[ = \sum_{i=0}^{n-k} (-1)^i (i+k-1) \binom{n-k}{i} \sum_{j=0}^{k-1} \binom{k}{j} f^{(k-j)}(x) A^{(j)-1}_{i+k-2}(0) + f^{(k)}(x) \int_0^x A^{(k)}_{i+k-2}(y)dy. \]

Letting \( x = 0 \) and eliminating zero terms we get

\[
(k - 1) (n f'(0)(k-2)! f^{k-1}(0) + f(0) A^{(k-1)}_{k-2}(0) - (n-k)(k-1)! f^{k+1}(0)) =
\]

\[
(k - 1) f(0) A^{(k-1)}_{k-2}(0) + (k-1)! f'(0)(k-2)! f^{k-1}(0) k(n-k)(k-1)! f^{k+1}(0),
\]

The terms for \( i = 0 \) and \( j = k + r - 1 \) are equal and hence they cancel out.

Splitting the summation into two parts for \( i = 0 \) and \( i > 0 \) we get

\[
(k - 1) \left( n^{r+1} f^{(r+1)}(0) A^{(k-2)}_{k-2}(0) + \sum_{j=0}^{k+r-2} n^{k+r-1-j} f^{(k+r-1-j)}(0) A^{(j)}_{i+k-2}(0) \right)
\]

\[ + (k - 1) \sum_{i=1}^{r+1} \sum_{j=i+k-2}^{k+r-1} (-1)^i \binom{n-k}{i} n^{k+r-1-j} f^{(k+r-1-j)}(0) A^{(j)}_{i+k-2}(0)
\]

\[ = (k - 1) \left( \binom{k+r}{k-1} f^{(r+1)}(0) A^{(k-2)}_{k-2}(0) \right)
\]

\[ + (k - 1) \sum_{j=1}^{k+r-2} \binom{k+r}{j+1} f^{(k+r-1-j)}(0) A^{(j)}_{k-2}(0)
\]

\[ + \sum_{i=1}^{r+1} \sum_{j=i+k-2}^{k+r-1} (-1)^i \binom{n-k}{i} (i+k-1) \binom{k+r}{j+1} f^{(k+r-1-j)}(0) A^{(j)}_{i+k-2}(0). \]
Applying the induction hypothesis to the derivatives of functions \( f \) and, consequently, via lemma 3.2 to \( A_{i+k-2} \) and grouping the summands we obtain

\[
f^{(r+1)}(0) = (-1)^{r+1} f^{k+r+1}(0) (k-1)! \left[ n^{r+1} - \binom{k+r}{k-1} \right]
\]

\[
= (-1)^{r+1} f^{k+r+1}(0) (k-1)! \left[ \sum_{j=k-1}^{k-2} \left( \binom{k+r}{j+1} - n^{k+r-1-j} \right) \binom{j+1}{k-1} \right]
\]

\[
+ \sum_{i=1}^{r+1} \sum_{j=i+k-2}^{k+r-1} \binom{n-k}{i} (i+k-1) \binom{k+r}{j+1}
\]

\[
- (k-1)n^{k+r-1-j}(i+k-2)! \binom{j+1}{i+k-1}.
\]

To prove the induction step it remains to show that

\[
(k-1)! \left[ n^{r+1} - \binom{k+r}{k-1} + \sum_{j=k-1}^{k-2} \left( n^{k+r-1-j} - \binom{k+r}{j+1} \right) \binom{j+1}{k-1} \right]
\]

\[
= \sum_{i=0}^{r+1} \sum_{j=i+k-2}^{k+r-1} \binom{n-k}{i} (i+k-1) \binom{k+r}{j+1}
\]

\[
- (k-1)n^{k+r-1-j}(i+k-2)! \binom{j+1}{i+k-1}.
\]

Joining the summation for \( i = 0 \) and \( i > 0 \) back together we get

\[
\sum_{i=0}^{r+1} \sum_{j=i+k-2}^{k+r-1} \binom{n-k}{i} (i+k-1) n^{k+r-1-j}(i+k-2)! \binom{j+1}{i+k-1}
\]

\[
= \sum_{i=0}^{r+1} \binom{n-k}{i} (i+k-1)! \sum_{j=i+k-2}^{k+r-1} \binom{k+r}{j+1} \binom{j+1}{i+k-1}.
\]

Using identity (2) and lemma 2.1 we complete the proof. \qed

**Theorem 3.5** Let \( X_1, \ldots, X_n \) be a random sample from the distribution \( \mathcal{F} \) that belongs to \( \mathcal{F} \) and let \( X_0 \) be a random variable independent of the sample that follows the same distribution. Let \( k \) be a fixed number such that \( 1 < k \leq n \). If

\[
X_{(k-1:n)} + \frac{1}{n-k+1} X_0 \overset{d}{=} X_{(k:n)}
\]

then \( X \sim \mathcal{E}(\lambda), \lambda > 0 \).

We omit the proof since it follows completely analogous procedure to the proof of theorem 3.3 with the application of lemma 2.2 in the last step.

**Theorem 3.6** Let \( X_1, \ldots, X_n \) be a random sample from the distribution \( \mathcal{F} \) that belongs to \( \mathcal{F} \). Let \( k \) be a fixed number such that \( 1 \leq k \leq n \). If

\[
\frac{1}{n} X_1 + \frac{1}{n-1} X_2 + \cdots + \frac{1}{n-k+1} X_k \overset{d}{=} X_{(k:n)}
\]

then \( X \sim \mathcal{E}(\lambda), \lambda > 0 \).
Proof. Let \( k \geq 2 \). Equalizing the respective densities as in the previous proof we get

\[
\int_0^x f(n(x-y_2)) \cdots \int_0^{y_{k-1}} f((n-k+2)(y_{k-1}-y_k)) f((n-k+1)y_k) dy_2 \cdots dy_k
= \frac{1}{(k-1)!} f(x) \sum_{i=0}^{n-k} (-1)^i \binom{n-k}{i} \int_0^\infty A_{k-2+i}(x) dx.
\]  

(21)

Denote the left hand side of (21) with \( J_{k,n}(x) \). Obviously, it can be expressed as

\[
J_{k,n}(x) = \int_0^x f(n(x-y_2)) J_{k-1,n-1}(y_2) dy_2,
\]

\[
J_{1,1}(x) = f((n-k+1)x).
\]

The \((k+r)\) th derivative of \( J_{k,n} \) is

\[
J_{k,n}^{(k+r)}(x) = \sum_{j=0}^{k+r-1} n^j f^{(j)}(0) J_{k-1,n-1}^{(k+r-j-1)}(x)
+ \int_0^x f^{(k+1)}(n(x-y_2)) n^{k+r} J_{k-1,n-1}^{(r+1)}(y_2) dy_2.
\]

Letting \( x = 0 \) we get

\[
J_{k,n}^{(k+r)}(0) = \sum_{j=0}^{k+r-1} n^j f^{(j)}(0) J_{k-1,n-1}^{(k+r-j-1)}(0),
\]

\[
J_{1,1}^{(s)}(0) = (n-k+1)^s f^{(s)}(0), \text{ for every } s \geq 0.
\]  

(22)

Applying the recurrence relation (22) \( k-1 \) times we obtain

\[
J_{k,n}^{(k+1)}(0) = \sum_{j_1=0}^{k+r-1} n^{j_1} f^{(j_1)}(0) \sum_{j_2=0}^{k+r-2-j_1} (n-1)^{j_2} f^{(j_2)}(0) \cdots
\]

\[
\cdots \sum_{j_k=0}^{r+1-k-\sum_{l=1}^{k-2} j_l} (n-k+2)^{j_k-1} f^{(j_k-1)}(n-k+1) \int_0^\infty f^{(r+1-k-\sum_{l=1}^{k-2} j_l)}(0) dx.
\]

Then the \((k+r)\) th derivative of the left hand side of (21) becomes

\[
\sum_{j_1+\cdots+j_k=r+1} n^{j_1}(n-1)^{j_2} \cdots (n-k+1)^{j_k} f^{(j_1)}(0) f^{(j_2)}(0) \cdots f^{(j_k)}(0).
\]
As before we shall prove by induction that (13) holds for every \( q \). For \( r = 0 \), the \( k \) th derivative of (21) at \( x = 0 \) is

\[
(n + n - 1 + \cdots + n - k + 1)f'(0)f^{k-1}(0) = \frac{1}{(k-2)!}f(0)A_{k-2}^{(k-1)}(0) + f'(0)f^{k-1}(0)k - k(n - k)f^{k+1}(0).
\]

(23)

From (16) we can get that

\[
A_{k-2}^{(k-1)}(0) = f^{k-2}(0)f'(0)(k - 1)! + (k - 2)f'(0)f^{k-2}(0)\left(\frac{k - 1}{2}\right).
\]

Inserting this in (23) we get \( f'(0) = -f^2(0) \) which means (13) holds for \( q = 1 \). Suppose now that (13) is satisfied for all \( q \leq r \). We shall prove that it holds for \( q = r + 1 \). The \( (k + r) \) th derivative of (21) at \( x = 0 \) is

\[
\sum_{j_1, \ldots, j_k \geq 0} n^{j_1} (n - 1)^{j_2} \cdots (n - k + 1)^{j_k} f^{(j_1)}(0)f^{(j_2)}(0) \cdots f^{(j_k)}(0)
\]

(24)

Applying the induction hypothesis the left hand side of (24) becomes

\[
f^{k-1}(0)f^{(r+1)}(n^{r+1} + \cdots + (n - k + 1)^{r+1}) + \sum_{0 \leq j_1, \ldots, j_k < r+1} n^{j_1} (n - 1)^{j_2} \cdots (n - k + 1)^{j_k} (-1)^{r+1} f^{r+1+k}(0),
\]

while the right hand side of (24) can be expressed as

\[
\sum_{i=1}^{r+1} \left( \begin{array}{c} n - k \\ i \end{array} \right) \sum_{j_i + \cdots + j_k = k-2} ^{k+r-1} \left( \begin{array}{c} k + r \\ j_i + 1 \end{array} \right) (k - 1)!(i + k - 1)! f^{k+r+1}(0)(i + k - 1)! (k - 1)!
\]

\[
+ \sum_{j_i + \cdots + j_k = k-1} ^{k+r-2} f^{k+r+1}(0)(i + k - 2)! (k - 2)!
\]

\[
+ \left( \begin{array}{c} k + r \\ k - 1 \end{array} \right) f^{(r+1)}(0)(i + k - 2)! (k - 2)! + \frac{1}{(k - 2)!}f(0)A_{k-2}^{(k+r-1)}(0).
\]

The term \( A_{k-2}^{(k+r-1)}(0) \) can be evaluated using (13) and (16) as

\[
A_{k-2}^{(k+r-1)}(0) = \frac{(k + r - 1)!}{(r + 1)!}f^{k-2}(0)f^{(r+1)}(0)
\]

\[
+ \frac{(k + r - 1)!}{(r + 2)!}(k - 2)f^{k-2}(0)f^{(r+1)}(0)
\]

\[
+ \sum_{1 \leq j_1, \ldots, j_k < r+2} ^{k} \sum_{0 \leq j_1, \ldots, j_k < k+r-1} (-1)^{r+1} f^{r+k}(0)\frac{(k + r - 1)!}{j_1! \cdots j_k!}.
\]
After transformations given above and grouping the summands \((2\) becomes 

\[
f^{(r+1)}(0)
\left(n^{r+1} + \cdots + (n - k + 1)^{r+1} - \binom{k + r}{k - 1} - \binom{k + r - 1}{k - 2} - \binom{k + r - 1}{k - 3}\right)
\]

\[
= (-1)^{r+1} f^{r+2}(0) \sum_{j=\frac{k-1}{2}}^{k+r-2} \binom{k + r}{j + 1} \binom{j + 1}{k - 1}
\]

\[
+ \sum_{i=1}^{r+1} \binom{n - k}{i} \sum_{j=1}^{k-r-2} \binom{k + r}{j + 1} \frac{(i + k - 1)!}{(k - 1)!} \frac{j + 1}{i + k - 1}
\]

\[
+ \frac{1}{(k - 2)!} \sum_{1 \leq j_1, \ldots, j_k < r+2} \frac{(k + r - 1)!}{j_1! \cdots j_{k-1}!} - \sum_{0 \leq j_1, \ldots, j_k < r+1} \sum_{j_1 + \cdots + j_k = k+r+1} n^{j_1} (n - 1)^{j_2} \cdots (n - k + 1)^{j_k}.
\]

To prove the induction step it remains to show that the expressions in the brackets on both sides are equal. Joining the summands back together and applying the identity \(2\) we obtain

\[
\sum_{j_1, \ldots, j_k \geq 0} n^{j_1} (n - 1)^{j_2} \cdots (n - k + 1)^{j_k} = \sum_{i=0}^{r+1} \binom{n - k}{i} \frac{(i + k - 1)!}{(k - 1)!} \frac{j + 1}{i + k - 1},
\]

which follows from lemma \(2\). Hence, the proof for \(k \geq 2\) is completed.

In case of \(k = 1\) the proof is done in an analogous way, but it is much simpler, so we omit it here. \(\Box\)

The following corollary, which follows directly from theorem \(3.6\), is a conjecture stated in \(4\). 

Corollary 3.7 Let \(X_1, \ldots, X_n\) be a random sample from the distribution \(F\) that belongs to \(F\). If

\[
X_1 + \frac{1}{2} X_2 + \cdots + \frac{1}{n} X_n \overset{d}{=} X_{(n,n)},
\]

then \(X \sim \mathcal{E}(\lambda), \lambda > 0\). 

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