Primordial black holes in linear and non-linear regimes

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Abstract. We revisit the formation of primordial black holes (PBHs) in the radiation-dominated era for both linear and non-linear regimes, elaborating on the concept of an apparent horizon. Contrary to the expectation from vacuum models, we argue that in a cosmological setting a density fluctuation with a high density does not always collapse to a black hole. To this end, we first elaborate on the perturbation theory for spherically symmetric space times in the linear regime. Thereby, we introduce two gauges. This allows to introduce a well defined gauge-invariant quantity for the expansion of null geodesics. Using this quantity, we argue that PBHs do not form in the linear regime irrespective of the density of the background. Finally, we consider the formation of PBHs in non-linear regimes, adopting the spherical collapse picture. In this picture, over-densities are modeled by closed FRW models in the radiation-dominated era. The difference of our approach is that we start by finding an exact solution for a closed radiation-dominated universe. This yields exact results for turn-around time and radius. It is important that we take the initial conditions from the linear perturbation theory. Additionally, instead of using uniform Hubble gauge condition, both density and velocity perturbations are admitted in this approach. Thereby, the matching condition will impose an important constraint on the initial velocity perturbations $\delta^h_0 = -\delta_0/2$. This can be extended to higher orders. Using this constraint, we find that the apparent horizon of a PBH forms when $\delta > 3$ at turn-around time. The corrections also appear from the third order. Moreover, a PBH forms when its apparent horizon is outside the sound horizon at the re-entry time. Applying this condition, we infer that the threshold value of the density perturbations at horizon re-entry should be larger than $\delta_{th} > 0.7$. 

Keywords: cosmological perturbation theory, primordial black holes

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1 Introduction

In the standard model of cosmology, astrophysical objects are originated from the early universe quantum fluctuations which became classical as they were stretched to super-horizon scales in an exponentially expanding period. If the density perturbations exceed some threshold value, primordial black holes (PBHs) might form. Many numerical investigations are conducted to study the threshold value for the density perturbations (see refs. [1–3, 5, 6]). Moreover, the threshold value in terms of metric perturbations is investigated in ref. [4]. Recent numerical studies can be also found in refs. [7–9]. Many studies are focused on these black holes, since these black holes are the most important candidate for the Hawking radiation from black holes [10, 11]. Chiefly, the dark nature of these black holes makes them one of the nominated objects for some fraction of the dark matter [12]. Located in an expanding background, PBHs are dubbed as cosmological black holes. Thus, it is important to apply concepts of dynamical black holes to study PBHs.

The evolving nature of the universe implies that PBHs should be classified as dynamical black holes [13]. These black holes belong to a larger class named cosmological black holes. These black holes evolve in the FRW background. In contrast to the stationary black holes, we need a more subtle approach to define mass [14], horizon [15–18] and Hawking radiation scenario [19–22] for these black holes. Located in the cosmological expanding background, the cosmological black holes have special properties at late times [23]. One important feature of the black holes is their horizon. Specifically, the event horizon is often attributed as the black hole boundary. However, the global nature of the event horizon is not appropriate for evolving black holes which are studied in the numerical relativity. Alternatively, the apparent horizon is used to distinguish the black hole boundary. The advantage is that since the apparent horizon is quantified by the expansion of null geodesics, it is not a global quantity.

The earlier work by Carr et al. [1] puts a lower and an upper threshold on the density contrast. However, the later work by Kopp et al. [24] argues that the maximum value $\delta_{\text{max}}$ is not directly related to the separate universe but to the geometry of the over-dense region. A full discussion of this subject is presented in ref. [25]. Moreover, even when the density
perturbation is greater than the threshold value, this does not guarantee that PBHs form. The necessary and essential condition is the formation of the apparent horizon. The apparent horizon formation yields a trapped surface.

In the early universe, when a density fluctuation forms, its density is of the order of the background. Given that this density can be very high, one might assume that this density collapses to a black hole. This is not always true because the over-density is located in a cosmological background. The fate of a density fluctuation depends on the formation of its apparent horizon. We will elaborate on the notion of the apparent horizon for the PBHs’ formation in both linear and non-linear regimes. Thereby, we find that linear super-horizon perturbations do not form separate universes. This is consistent with the results in ref. [25].

In the non-linear regime, we use a closed FRW model for over densities. However, we take a different approach from ref. [25]. First, we derive an exact solution for the evolution of an over-density in the radiation-dominated era. This allows to find the exact expressions for turn-around radius and time. We then provide this exact solution with the initial conditions from the linear perturbation theory. This imposes an important constraint between initial density and Hubble’s parameter at linear order. We highlight that our approach can be extended to any order. Instead of using uniform Hubble gauge as in ref. [25], we use this constraint to show that at turn around time, the over-density is \( \delta(t_m) > 3 \). This also implies that the corrections to this result start from the third order.

The outline of paper is as follows. In the linear regime we first revisit the PBH formation with emphasis on gauge-invariance of its apparent horizon. In this framework, we first develop the linear perturbation theory for spherically symmetric perturbations to study PBHs in section II. Eventually, we use this gauge-invariant quantity to argue that PBHs do not form in the linear regime. Section III is devoted to PBH formation in the non-linear regime with emphasis on initial conditions. We employ a spherical collapse model to find the threshold value of PBH formation. We obtain exact expressions for an over-density in the radiation-dominated era. Our model initial condition is set by the perturbation theory at early times. We show that when a black-hole forms, the density contrast \( \delta \) has the universal value 3. We then estimate the threshold value of the density contrast at horizon re-entry necessary for the PBH formation. Finally, we conclude in section IV.

2 PbH formation in linear regime

In the standard model of cosmology, it is known that all structures form from the early universe density perturbations. The seeds for these perturbations are set on super-horizon scales during the inflation period. These perturbations also leave their imprints on the CMB as fluctuations on the average temperature. In this framework, the perturbations are first outside the Hubble horizon. When they enter the horizon, they might start growing and eventually collapse into non-linear structures. In this section, we start with developing the perturbation theory for spherically symmetric perturbations. We apply this to obtain a gauge-invariant expression for the apparent horizon.

2.1 Perturbation theory with a symmetry

In order to study PBHs, we need to do simulations in numerical relativity. This is complicated except for simplified models. Nevertheless, analytical calculations may provide physical insights into the very nature of the formation of black holes. A black hole with a spheri-
cality is one of the most common assumptions that allows us to define analytical expressions for quantities such as mass, horizon and light cone dynamics [15–18].

We start with a detailed study of spherically symmetric perturbations. Our goal is to find gauge-invariant expressions for the apparent horizon. Thus, we consider a background that is perturbed and the perturbations also have a spherical symmetry. This implies that the Lie derivative of the perturbed metric vanishes with respect to its killing vectors. To the first order, we obtain

$$\mathcal{L}_{\Delta x} \varepsilon = 0$$

where $\varepsilon$ is the killing vector. If we make a gauge transformation $x^\mu \rightarrow x^\mu + \Delta x^\mu$, the perturbed metric will transform as

$$\tilde{\delta} g = \delta g - \mathcal{L}_{\Delta x} \tilde{\varepsilon}.$$ 

Similarly, the killing vectors will also transform as

$$\tilde{\delta} \varepsilon = \delta \varepsilon - \mathcal{L}_{\Delta x} \tilde{\varepsilon}.$$ 

To keep the spherical symmetry we require that

$$\mathcal{L}_{\Delta x} \varepsilon + \mathcal{L}_{\tilde{\varepsilon}} \mathcal{L}_{\Delta x} \tilde{\varepsilon} = 0.$$ 

Up to the first order this yields

$$\mathcal{L}_{\Delta x} \varepsilon + \mathcal{L}_{\tilde{\varepsilon}} \mathcal{L}_{\Delta x} \tilde{\varepsilon} \tilde{g} = 0.$$ 

We further suppose that $\mathcal{L}_{\Delta x} \tilde{\varepsilon} = 0$. This implies that the killing vectors are left invariant by the transformations. Moreover, we should also have $\mathcal{L}_{\tilde{\varepsilon}} \mathcal{L}_{\Delta x} \tilde{g} = 0$. That is because the tensor $\mathcal{L}_{\Delta x} \tilde{g}$ is spherically symmetric. Equivalently, we may start by studying the gauge transformations on the metric and use this to find the general form of the transformations.

We then require that the transformed metric keeps the same form after the transformation of the form $x^\mu \rightarrow x^\mu + \epsilon^\mu$. Up to the first order, we infer that the most general transformation is simply given as

$$\epsilon = (\epsilon^0(t, r), \epsilon^r(t, r), K(S^2)),$$

where $K(S^2)$ are the killing vectors on the sphere.

### 2.2 Transformation of perturbations

Before deriving a gauge-invariant expression for the apparent horizon, we derive how spherical perturbations change under the transformations given by eq. (2.4). The perturbed metric that is also spherically symmetric is

$$ds^2 = (-a^2 + \delta g_{00}(t, r))dt^2 + (a^2 + \delta g_{rr}(t, r))dr^2 + 2g_{0r}(t, r)drdt + a^2 r^2 (1 - 2E(t, r))d\Omega^2.$$ 

Expanding eq. (2.5) yields the transformations of the perturbed quantities as

$$\delta \tilde{g}_{00} = -2a^2 \tilde{\psi} = \delta g_{00} + 2a^2 \epsilon^0_0 + 2\mathcal{H}a^2 \epsilon^0,$$

$$\delta \tilde{g}_{0r} = a^2 \tilde{\psi} = \delta g_{0r} + 2a^2 \epsilon^0 r + a^2 \epsilon^0 r,$$

$$\delta \tilde{g}_{rr} = -2a^2 \tilde{\psi} = \delta g_{rr} + 2a^2 \epsilon^r r + 2\mathcal{H}a^2 \epsilon^0,$$

$$\delta \tilde{g}_{\theta\theta} = -2a^2 \tilde{\psi} = \left(-2E + 2\mathcal{H}a^2 \epsilon^0 + 2a^2 \epsilon^r r \right) \tilde{g}_{\theta\theta},$$
where the quantities $\varphi, B, \psi$ and $E$ are introduced for convenience and $\mathcal{H}$ is the conformal Hubble’s parameter. The transformation in terms of $\varphi, B, \psi$ and $E$ is also given by

$$\begin{align*}
\delta \varphi &= -(\epsilon^0_0 + \mathcal{H} \epsilon^0), \\
\delta \psi &= -\epsilon^r_r - \mathcal{H} \epsilon^0, \\
\delta B &= -\epsilon^r_t + \epsilon^0_t, \\
\delta E &= -\frac{\epsilon^r}{r} - \mathcal{H} \epsilon^0.
\end{align*}$$

(2.7)

Note that any scalar $S$ such as $\delta \rho$ transforms as

$$\delta S = - (\epsilon^0 \partial_t \bar{S} + \epsilon^r \partial_r \bar{S}).$$

Similarly, expansion for null geodesics (A.4) transforms as

$$\delta \Theta = - (\epsilon^0 \partial_t \bar{\Theta} + \epsilon^r \partial_r \bar{\Theta}),$$

(2.11)

where $\Theta$ depends on time and space in the background. This implies that the expansion is not gauge-invariant.

### 2.3 Gauge fixing

Whenever we perturb quantities with respect to an averaged background in general relativity, we have to define the surface on which this average is taken. As we change the coordinates, the value of these perturbations will change. This is because not all degrees of freedom are physical in perturbation theory. Hence, we face the notion of gauge fixing. One method to tackle this problem is to define quantities which are invariant under coordinate transformations. These gauge-invariant quantities are not unique. Some are defined using only metric perturbations like Bardeen potentials, whereas some may be constructed using metric and matter perturbations simultaneously. In this section we denote matter density and its perturbation by $\rho$ and $\delta \rho$ respectively. Moreover, the scalar perturbation which is a scalar quantity transforms as $\delta \tilde{\rho} = \delta \rho - L \epsilon \tilde{\rho}$.

We define two new gauge-invariant quantities as

$$\begin{align*}
\Phi &= \varphi - \frac{1}{a} \frac{\partial}{\partial t} \left( \frac{a \delta \rho}{\bar{\rho}} \right), \\
\Psi &= \psi - \mathcal{H} \delta \rho - \epsilon \bar{\rho} \left( -E + \mathcal{H} \frac{\delta \rho}{\bar{\rho}} \right).
\end{align*}$$

(2.12)

(2.13)

where $E$, $\varphi$ and $\psi$ are introduced in eq. (2.6). Note that because we constructed these gauge-invariant quantities using matter and the metric perturbations, they do not correspond to Newtonian gauge potentials.

- **Uniform density gauge**: in this gauge we set $\delta \rho = E = 0$. By setting $\delta \rho = 0$, we fixed the temporal gauge freedom $\varphi^0$. Using eq. (2.10) we find that spatial gauge freedom can now be fixed by setting $E = 0$. The gauge is completely fixed in this case. The expansion in this gauge is

$$\Theta = \frac{\sqrt{2}}{r \alpha} \left( r (1 - \varphi) \mathcal{H} + (1 + \psi - B) \right).$$

(2.14)
• **Newtonian like gauge:** the other gauge is given by setting $E = B = 0$. If we set $E = 0$, this yields $\epsilon' = -r \dot{H} c^0$. When we set $B = 0$, this yields

$$\dot{\epsilon}(H c_0^0) + \frac{\epsilon_0}{r} \frac{\partial}{\partial r} = 0. \quad (2.15)$$

Let us introduce the new variables $\mathcal{H}\dot{\epsilon} = \dot{\eta}$ and $\frac{\partial}{\partial r} = \partial_R$ to solve this equation. We find that the solution is given by

$$A = H\epsilon_0^0 = f(\eta - R), \quad (2.16)$$

where $f(\eta - R)$ is any well behaved function of $(\eta - R)$. The gauge is partially fixed as we have the remaining gauge freedom by eq. (2.16). The expansion in this gauge is

$$\Theta = \sqrt{2} \frac{a}{r} \left( \mathcal{H}r(1 - \dot{\varphi}) + 1 + \dot{\psi} \right). \quad (2.17)$$

### 2.4 Gauge invariant definition for expansion

Using eq. (2.11), we define a gauge-invariant expansion as

$$\tilde{\Theta} = \Theta - \partial_t \tilde{\Theta} \frac{\delta \rho}{\rho} + \partial_\nu \tilde{\Theta} \left( -r E + r \mathcal{H} \frac{\delta \rho}{\rho} \right). \quad (2.18)$$

This gauge-invariant quantity reduces to the expansion in the uniform density gauge given by eq. (2.14) where we set $\delta \rho = E = 0$. The expansion in this gauge has a physical meaning because its value is invariant. Note that the uniform-density gauge has the advantage that all perturbations are captured in the metric. The expansion defined in this gauge is also a pure geometric quantity.

Moreover, the expansion for outgoing null geodesics in the flat FRW background is

$$\Theta = \sqrt{2} \frac{a}{r} \left( \mathcal{H}r + 1 \right). \quad (2.19)$$

Note that this quantity never vanishes for out-going null geodesics in the background. Although the expansion vanishes for in-going null geodesics at $r_{\cos} = \frac{1}{\mathcal{H}}$, this surface is the cosmological horizon. A black hole is characterized by vanishing outgoing expansion. In other words, a homogeneous and isotropic universe never collapses to a black hole regardless of its density.

Eventually, eqs. (2.18), (2.14) and (2.17) imply that the out-going expansion is nonzero as long as we have $|g_{ab}| \gg |\delta g_{ab}|$. This is to say that black holes do not form inside the cosmological horizon in the linear regime. This is in contrast to the intuition that in early universe when density is high, small perturbations may lead to black holes.

### 2.5 PBH formation in the long wavelength limit

One interesting case is that the perturbations collapse to black holes when they are already outside the horizon. We argue that this can not happen. To study this situation, the linear perturbation theory that we developed in the last section can be used. However, when super-horizon perturbations are linear, the super-horizon limit can be studied using a different approach. The super-horizon limit has the advantage that the full non-linear equations can be simplified without assuming that potential perturbations are small. Hence, we need the
long-wavelength limit of full non-linear equations. The method, named gradient expansion, has been used in studying the early universe models [31]. In this approach, each quantity is expanded in powers of $\epsilon = \frac{k}{H}$. This approach has been extended to PBHs to set their initial conditions outside the horizon [4]. We just review the main points here.

Before taking the long-wave length limit, the metric is written in a conformally decomposed form as

$$ds^2 = -(\alpha^2 - \psi^4 \beta^2 r^2) dt^2 + 2\psi^4 a^2 \beta r dr dt + \psi^4 a^2 (dr^2 + r^2 d\Omega^2),$$

(2.20)

where $a$ is the scale factor. The extrinsic curvature is also decomposed to trace and traceless parts as $K_{ij} = \psi^4 a^2 \tilde{A}_{ij} + \frac{1}{3} \psi^4 a^2 \tilde{\gamma}_{ij} K$ where $\tilde{\gamma}_{ij}$ is the spatial flat metric in spherical coordinates. Expansion in this metric is given by

$$\Theta = 2 \frac{\dot{a}}{a} + A + \frac{1}{\psi^2 a} \left( \frac{2}{r} + 4 \frac{\partial_r \psi}{\psi} \right),$$

(2.21)

and $A = \tilde{A}_r$. In order to know the value of the expansion on super-horizon scales, we need to estimate the order of magnitude for $\psi$ and its first derivative $\partial_r \psi$. In the gradient expansion, it is assumed that $\delta \rho = O(\epsilon^2)$ and the gradient of any variable is $\partial \psi = \psi \times O(\epsilon)$. The long-wave length limit of other variables is found using the $3 + 1$ formalism equations. This assumption leads to $\psi - 1 \sim O(\epsilon^2)$, $\partial_r \psi \sim O(\epsilon^1)$ [4, 27]. It is important to note that $\psi$ is not necessary small. Because the first term is positive and the third term in negligible, a black hole can not form as long as the perturbations are outside the Hubble horizon. Note that $A$ vanishes in the background. Let us point that because $A$ is very small $\sim O(\epsilon^2)$, it does not affect the formation of the apparent horizon [27].

3 PBH formation in non-linear regime: spherical collapse in the radiation era

After considering the linear regime and super-horizon scales, we move to the non-linear regimes. Although the non-linear study of black-hole formation in the radiation era can be done in numerical relativity, analytical models illuminate our physical intuition. In order to study black hole formation in the non-linear regimes, we consider a toy model. In this model, we use the spherical collapse of an over-dense sphere in a flat FRW radiation-filled background. This collapsing region is modeled by a closed FRW metric. Additionally, we also include a velocity perturbation in our model in terms of different Hubble’s expansion rates. The exact expressions for evolution of this over-density are also presented in this section. We show that the over-density of the collapsing region is non-linear when the apparent horizon forms, provided that initial density perturbations are small. Our method can be extended to non-linear order. We implement the initial conditions provided by the perturbation theory. Thereby, this provides a constraint on the value of $\delta_0$ which is the initial Hubble’s rate perturbation. Eventually, we find the threshold value for the density fluctuation by requiring that the sound horizon is located inside the cosmological horizon. Thereby, we will have $\delta_0 = 0.7$. Since the value of $\delta_0$ is related to the primordial power spectrum set by the inflation, the inflationary perturbations should have a substantial power.

Adopting the spherical collapse model for the PBH formation, we assign a spherically averaged over-density $\delta(R, t)$ to every over-dense patch. The smoothed over-density can be found via applying an appropriate window function. Generally, the window function picks
up only long modes with wavelength larger than the smoothing scale. This can be justified as the (spatial) oscillations of the short modes \( q \gtrsim 1/R \) are averaged out on the smoothing scale \( R \). Hence, we assume the following uniform over-density \( \delta_{sc} \) for the fluctuations

\[
\delta_{sc} = \int d^3q \, \delta(q) |W(qR)|^2. \tag{3.1}
\]

This is roughly equal to \( \delta_{sc} \approx \sqrt{\mathcal{P}(k)} = \sqrt{\frac{k^3}{2\pi^2} \mathcal{P}(k)} \), where \( \mathcal{P}(k) \) is the dimensionless variance. That is the amplitude of a fluctuation is related to the square root of the primordial power spectrum evaluated at the scale \( R \).

According to the Birkhoff theorem, the geometry of a spherically symmetric homogeneous over-dense region is well described by a FRW metric

\[
ds^2 = -dt^2 + R(t)^2(d\chi^2 + \sin^2(\chi)d\Omega^2), \tag{3.2}
\]

with a density \( \rho = \bar{\rho}(1 + \delta_0) \) where \( \delta_0 \) is the initial density contrast and \( K \) is the curvature constant. The Friedmann equation for this region is given by

\[
\tilde{H}^2 = \frac{D}{R^4} - \frac{K}{R^2}, \tag{3.3}
\]

where \( D = 8\pi G \bar{\rho}(1 + \delta_0)R_0^4/3 \) quantifies the total initial energy of the over-dense region. Using the apparent horizon definition in appendix A, we find that the apparent horizon is located at

\[
\tilde{R} = \cot(\chi). \tag{3.4}
\]

In the case that the black hole boundary is the \( \chi = \frac{\pi}{2} \) surface, the apparent horizon or the black hole boundary will be located at the turnaround point, where we have \( \tilde{R} = 0 \).

Using eq. (3.3), we obtain an exact solution for the scale factor of the collapsing sphere \( R(t) \) as

\[
R(t) = \sqrt{R_0^2 + 4D(4K - 1)(t - t_0) - K(t - t_0)^2}. \tag{3.5}
\]

The over-dense region first expands till the time of maximum extension \( t_m \) and collapses afterwards. The maximum expansion point, also named turnaround point \( R_m \), is

\[
R_m = \sqrt{\frac{D}{K}}. \tag{3.6}
\]

The turnaround time is also given as

\[
t_m = \frac{\sqrt{D - KR_0^2}}{K} + t_0. \tag{3.7}
\]

The curvature of the over dense region is arbitrary.

Let us assume, without any reference to the linear theory, the following initial condition for the radial velocity of the over-dense sphere

\[
\frac{\dot{R}(t_0)}{R(t_0)} = H_0 = H_0(1 + \delta_0^h), \tag{3.8}
\]
where $H_0$ denotes the Hubble parameter of the background metric and $\delta_h^0$ is the initial Hubble parameter perturbation. Using eq. (3.3), this initial condition totally determines the curvature constant of the interior metric as

$$K = R_0^2(D/R_0^4 - \tilde{H}_0^2) = R_0^2H_0^2\left[(1 + \delta_0) - (1 + \delta_h^0)^2\right].$$

(3.9)

Using this, we find that the over-density at turnaround is

$$\delta(t_m) = \rho_{sc} - \bar{\rho} \bigg|_{t_m} = 3 + \delta_o^2 - 4\delta_h^2 + O(\delta^3).$$

(3.10)

This implies that the over-density at turnaround has the universal value of 3 up to corrections up to the third order of the initial perturbations. Thereby, we can extend the previous results in refs. [25] and [28] to higher orders. This result shows that at turnaround $t = t_m$, when the apparent horizon forms and collapse starts, the over-density is already non-linear.

The next step is to specify the initial condition of the spherical collapse. At early times well before the curvature term dominates and the dynamics of the over-dense region leads to a re-collapse, the evolution of the smoothed patch should coincide with the evolution from linear perturbation theory. More specifically, the prediction of the linear theory is

$$\delta(t) \sim a^2(t).$$

(3.11)

Expanding (3.5) for early times to linear order of the perturbations and noting that $a(t) = \sqrt[4]{4\lambda} t^{\frac{1}{2}}$, over-density of the collapsing region is derived as

$$\delta = (\delta_0 - 2\delta_h^0)H_0 t + \frac{\delta_0 + 2\delta_h^0}{4H_0^3 t}.$$ 

(3.12)

Later during the evolution, the decaying solution is negligible. Thereby, we obtain

$$\delta = (\delta_0/2 - \delta_h^0) t/t_0.$$ 

(3.13)

Note that in the radiation-dominated universe $H(t) = 1/2t$. Matching this expression to the linear theory prediction imposes a very important constraint on the velocity perturbation. This constraint is given as

$$\delta_h^0 = -\delta_0/2.$$ 

(3.14)

This implies that if the over-density starts as a linear perturbation, its velocity perturbation is minus half of its density perturbation.

Let us now set the initial conditions when the central region enters the background horizon $a_0 = \frac{2}{\pi H}$ and $t_0 = \frac{1}{2H_0}$ and use the initial condition for $\delta_h^0$. The final state of a medium to collapse under its own gravity is determined by a length scale, named the Jeans length, where pressure gradients tend to oppose the collapse. The Jeans length for a fluid with the sound speed $c_s$ is given by $l_j = c_s \sqrt{\frac{\pi}{2\rho}}$. However, this scale is derived using Newtonian considerations in the perturbation theory. Since we study black holes in the radiation-dominated era, we need a relativistic definition. We assume that the sound horizon scale could represent the Jeans length [26]. This means that the apparent horizon should be located outside the sound horizon. In this framework, the sound horizon is defined by

$$L_j = \frac{1}{\sqrt{3}a(t_m)} \int_{t_0}^{t_m} \frac{dt'}{a(t')}.$$ 

(3.15)
Figure 1. The physical sound horizon $L_j$ and $R_{\text{max}} = \frac{\pi}{2} R_m$. Allowed region is where sound horizon is inside the apparent horizon. The dashed line and the thick line represent the sound horizon and the apparent horizon respectively.

Figure 1 shows the physical sound horizon and physical turnaround radius for different $\delta_0$. We can infer from figure 1 that the threshold value for the density perturbations at horizon-entry should be larger than 0.7 to overcome the pressure gradients. This also indicates that the initial perturbations have enough power at the entry scale. Simple models of inflation do not satisfy this condition. Note that matching the initial conditions to the perturbation theory solution is different than taking uniform Hubble slicing as in [26]. By matching, we are mildly taking into account the pressure gradients initially. Hence, we have larger threshold.

4 Conclusion

In this paper our aim is twofold. We apply the concept of an apparent horizon for dynamical black holes to revisit the formation of PBHs in the early universe for both linear and non-linear regimes. Given that the event horizon is limited by the global nature of spacetime evolution, it is advantageous to define a black hole by its apparent horizon. The apparent horizon is quantified by the expansion of null geodesics which distinguishes the trapped surface of the black hole.

First, we develop the perturbation theory in a spherically symmetric spacetime and then we introduce two gauges. We also define a gauge-invariant quantity for the expansion of null geodesics. It is important to note that a high density fluctuation in the early universe does not always collapse. That is because black holes in a FRW universe are dynamical black holes. By applying the concept of the apparent horizon, we conclude that trapped surfaces do not form in the linear and super-horizon regimes irrespective of the density of the background. Hence, PBHs cannot form in these situations (this has been seen in [4] numerically).

In the non-linear regime, we implemented a closed FRW patch to model the collapse of PBHs in the radiation dominated era. In our model the boundary of the collapsing region first forms at $\chi = \pi/2$. The turnaround point coincides with the black hole apparent horizon. Specifically, we derive an exact expression for the evolution of a closed FRW universe in radiation-dominated era. Our approach allows both density and velocity perturbations. Moreover, we take the initial condition from the perturbation theory as we require our model to be linear initially. This will set an important constraint between the initial density and velocity perturbations. Additionally, we also find that at turn around time the overdensity is
\( \delta > 3 \). It is important to note that second order corrections to this result vanish. Furthermore, a black hole should overcome pressure forces to collapse. Taking the sound horizon as the Jeans length, we require the apparent horizon to be outside the sound horizon. Thereby, we infer that the threshold value of density perturbations at horizon re-entry must be larger than \( \delta_{th} > 0.7 \).

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A Light cone dynamics

Let \( S \) be a closed and orientable two-surface which is (smoothly) embedded in a four-dimensional time-oriented spacetime \( (M, g_{ab}) \) which has a metric compatible covariant derivative \( \nabla_a \). There are just two future-directed null directions normal to \( S \). Let \( \ell^a \) and \( n^a \) be null vector fields pointing in these directions; in situations where this is meaningful we will always take \( \ell^a \) and \( n^a \) as outgoing and ingoing respectively. If we further suppose that \( \ell \cdot n = -1 \) then there is only one remaining degree of rescaling freedom in the definition of these vector fields.

The intrinsic geometry of \( S \) is defined by the induced metric. The definition of these quantities is independent of the choice of null vectors made above. Nevertheless, for our purposes it is most useful to express them in terms of these vectors. Hence, the induced metric on \( S \) can be written as

\[
q_{ab} = g_{ab} + \ell_a n_b + \ell_b n_a. \tag{A.1}
\]

The covariant derivative operator \( d_a = q^b_a \nabla_b \) and (two-dimensional) Ricci scalar \( \hat{R} \) are defined on this two-surface. The extrinsic geometry which shows how \( S \) is embedded in \( M \). The extrinsic curvature is defined by how the a normal vectors change over \( S \) as in the usual way. The extrinsic curvatures are

\[
k^{(\ell)}_{ab} = q^c_a q^d_b \nabla_c \ell_d \quad \text{and} \quad k^{(n)}_{ab} = q^c_a q^d_b \nabla_c n_d. \tag{A.2}
\]

We can decompose the extrinsic curvatures as

\[
k^{(\ell)}_{ab} = \frac{1}{2} \Theta (\ell) q_{ab} + \sigma^{(\ell)}_{ab} \quad \text{and} \quad k^{(n)}_{ab} = \frac{1}{2} \Theta (n) q_{ab} + \sigma^{(n)}_{ab}. \tag{A.3}
\]

where

\[
\Theta(\ell) = q^{ab} \nabla_a \ell_b \quad \text{and} \quad \Theta(n) = q^{ab} \nabla_a n_b, \tag{A.4}
\]

are the expansions which are the traces of the extrinsic curvatures and the shears

\[
\sigma^{(\ell)}_{ab} \equiv \left( q^c_{(a} q^d_{b)} - \frac{1}{2} q_{ab} q^{cd} \right) \nabla_c \ell_d \quad \text{and} \quad \sigma^{(n)}_{ab} = \left( q^c_{(a} q^d_{b)} - \frac{1}{2} q_{ab} q^{cd} \right) \nabla_c n_d, \tag{A.5}
\]

are the trace-free parts. The rotation tensors are

\[
w^{(\ell)}_{ab} \equiv q^c_{[a} q^d_{b]} \nabla_c \ell_d \quad \text{and} \quad w^{(n)}_{ab} \equiv q^c_{[a} q^d_{b]} \nabla_c n_d. \tag{A.6}
\]
Here () and [ ] denote symmetrization and antisymmetrization of indexes. For hypersurface orthogonal null foliation \[30\] the rotation tensor is zero. If we define
\[
\kappa_X = -n_a X^b \nabla_b \ell^a, \tag{A.7}
\]
the Raychaudhuri equation will be
\[
\mathcal{L}_\ell \Theta(\ell) = \kappa_\ell \Theta(\ell) - (1/2) \Theta^2(\ell) - \sigma^{(\ell)}_{ab} \sigma(\ell)^{ab} - G_{ab} \ell^a \ell^b. \tag{A.8}
\]
In the case of the affine parameter for \(\ell\), we find that the \(\kappa_\ell = 0\). Similar equation for null geodesic is
\[
\mathcal{L}_n \Theta(\ell) = -\frac{R}{2} + w_a w^a + d_a w^a + G_{ab} \ell^a n^b, \tag{A.9}
\]
where is \(w_a = -q_b n_c \nabla_b \ell^c\). In the spherically symmetric case
\[
ds^2 = -e^\nu(t,r) dt^2 + e^\psi(t,r) dr^2 + R(t,r)^2 d\Omega^2, \tag{A.10}
\]
and shear free foliation, these two equations reduce to
\[
\mathcal{L}_\ell \Theta(\ell) = \kappa_\ell \Theta(\ell) - (1/2) \Theta^2(\ell) - G_{ab} \ell^a \ell^b,
\mathcal{L}_n \Theta(\ell) = -\frac{1}{R^2} + \Theta_\ell \Theta_n + G_{ab} \ell^a n^b. \tag{A.11}
\]
We can define the marginally trapped surface \(\bar{H}\) as \(\Theta_\ell = 0\) and this surface is foliated by spacelike two spheres. We can always write a tangent vector to \(\bar{H}\) as
\[
V^a = \ell^a - C n^a \tag{A.12}
\]
Since on the \(\bar{H}\) we have \(\mathcal{L}_V \Theta(\ell) = 0\) we find that
\[
C = \frac{\mathcal{L}_\ell \Theta(\ell)}{\mathcal{L}_n \Theta(\ell)} \bigg|_{\bar{H}} = \frac{G_{ab} \ell^a \ell^b}{\frac{1}{R^2} - G_{ab} \ell^a n^b}. \tag{A.13}
\]

**Black hole definition.** A smooth, three-dimensional, space-like sub-manifold (possibly with boundary) \(\bar{H}\) of spacetime is said to be a trapping horizon if it can be foliated by a family of closed 2-manifolds such that on each leaf \(S\) the expansion \(\Theta(\ell)\) of one null normal \(\ell^\mu\) vanishes; and the expansion \(\Theta(n)\) of the other null normal \(n^\mu\) is negative. This surface separates the trapped surface, \(\Theta(n), \Theta(\ell) < 0\), from untrapped one \(\Theta(n) < 0, \Theta(\ell) > 0\).

There are diverse definition for the black hole boundary in the dynamical cases. One definition is the dynamical horizon as the black hole boundary \[13\] which is a spacelike trapping horizon. A foliation independent definition for the apparent horizon comes from trapping boundary which is boundary of all trapped surfaces.

Similarly, the cosmological horizon is defined as the closed 2-manifolds such that on each leaf \(S\) the expansion \(\Theta(n) = 0\) of one null normal \(n^\mu\) vanishes; and the expansion \(\Theta(\ell) > 0\) of the other null normal \(\ell^\mu\) is positive.

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