Estimates of parabolic cylinder functions on the real axis

A.A. POKROVSKI
Laboratory of Quantum Networks, Institute for Physics, St-Petersburg State University, St.Petersburg 198504, Ulyanovskaya 1.
E-mail: alexis.pokrovski@mail.ru

We estimate the expressions $F(\pm x, \lambda)$ and $F(\pm ix, -\lambda)$, where

$$F(x, \lambda) = \frac{\partial}{\partial x} U(-\frac{1}{2}, x \sqrt{2} + U(-\frac{1}{2}, x \sqrt{2}) \sqrt{x^2 - \lambda}$$

and $U$ is the standard solution of the parabolic cylinder equation, satisfying $U(a, x) \sim x^{-a-1/2} e^{-x^2}$ as $x \to +\infty$. The estimates are valid in rather complicated domains and refine there the classical result of Olver. The estimates with real $x$ are important for the spectral analysis of non-analytically perturbed quantum harmonic oscillator.

We determine the part of the real $x$-axis, which is within the domains of the estimates. This requires a detailed study of the image of the real $x$-axis in the standard quasiclassical variable.

Keywords: Parabolic cylinder functions; Quasiclassical estimates

2000 Mathematics Subject Classification: 34M60, 33C15, 34E20, 81Q20.

1 Introduction

Consider the solution $\psi$ of the equation

$$-y''(x, \lambda) + x^2 y(x, \lambda) = \lambda y(x, \lambda), \quad y' = \frac{\partial y}{\partial x},$$

defined by $\psi(x, \lambda) = U(-\frac{1}{2}, x \sqrt{2})$, where $U$ is the standard parabolic cylinder function $U(a, x) = D_{a-\frac{1}{2}}(x)$ (see [1], [2]). Our main result is the estimates of the expressions $F(\pm x, \lambda)$ and $F(\pm ix, -\lambda)$, where $F(x, \lambda) = y'(x, \lambda) + \psi(x, \lambda) \sqrt{x^2 - \lambda}$. Each of the four $F$’s is evidently related to one of the four solutions $\psi(\pm x, \lambda), \psi(\pm ix, -\lambda)$ of (1.1). The estimates and their (rather complicated) domains are given in Theorem 5.1. In particular, for $F(x, \lambda)$ the estimate has the form

$$|F(x, \lambda)| \leq C |\phi(\lambda)| \rho(x, \lambda) e^{-\lambda \xi(x)}$$

where $C$ is an absolute constant, $\xi(t) = t \int_{1}^{t} \sqrt{s^2 - 1} \, ds$ and $\phi(\lambda) = 2^{\frac{3}{2}} \sqrt{\pi} \left(\frac{1}{2\pi}\right)^{\frac{3}{2}}$ are positive on $(1, \infty)$ and defined on $\mathbb{C} \setminus (-\infty, 1]$ and $\mathbb{C} \setminus \mathbb{R}^-$, respectively. The estimates of $F(-x, \lambda)$ and $F(\pm ix, -\lambda)$ have similar form, with modifications taking into account branching of $\xi$, $\phi$ and $\sqrt{x^2 - \lambda}$. 

1
We take special care of the case of real \( x \); to this end we study the image of the real axis in the relevant quasiclassical variable (see Section 4). Then we localize the image within the domains of the estimates. By Theorem 3.1 for each real \( x \) and each ray \( \arg \lambda = \text{const} \) the estimates for at least two of the four expressions \( F(\pm x, \lambda) \), \( F(\pm ix, -\lambda) \) are fulfilled and the two related solutions \( \psi \) are linearly independent. The last issue is important for applications that we discuss below.

Our result is a refinement of the following estimates due to Olver [3]:

\[
|\psi(x, \lambda)| \leq C \frac{|\phi(\lambda)|}{\rho_0(x, \lambda)} e^{-\frac{1}{2} \sqrt{x \lambda}}, \quad |\psi'(x, \lambda)| \leq C|\phi(\lambda)\rho_0(x, \lambda)e^{-\frac{1}{2} \sqrt{x \lambda}|, \quad (1.3)
\]

where \( \rho_0(x, \lambda) = 1 + |\lambda| e^{-\frac{1}{2} |x^2 - \lambda|^{\frac{1}{3}}} \). These estimates are valid for all values of \( x \) and \( \lambda \) (for special choice of branches of \( \xi, \phi \) and \( \sqrt{x^2 - \lambda} \), discussed in [3]), whereas the domain of our estimates is smaller. For example, for \( \lambda > 0 \) the asymptotics (1.2) holds true only for \( \sqrt{\lambda} \leq x \).

The estimates, obtained in the present paper, give a tool for improvement of the error bound in (1.2). We plan to make this in a separate paper.

In the proof we follow the classical scheme used in [5]. In Section 2 we list the necessary properties of the parabolic cylinder functions. In Section 3 we introduce the quasiclassical variable \( z = z_\lambda(x) \); the equation (1.1) becomes a perturbed Airy equation (3.14) with the perturbation \( V_0 \) decaying in both \( z \) and \( \lambda \). In Section 4 we study the properties of the family of curves \( \Gamma_\lambda = z_\lambda(\mathbb{R}^+) \), taking special care of its positioning relatively to the sectors of decay of Airy functions. Since \( \Gamma_\lambda \) for small \( \lambda > 0 \) is not within one such sector, we split the curve by the image \( z_* = z_\lambda(x_*) \) of a suitably defined turning point \( x_* \in \mathbb{R}^+ \).

In Section 5 we formulate the main result. We prove it first in \( z \)-variable. We fix four solutions \( A_0, A_\pm, A_* \) of Eq. (3.14), asymptotically close to one of the Airy functions. For each solution we write integral equation. Then we modify these integral equations by separating the exponential multiplier, writing \( A_0(z, \lambda) = e^{-\frac{z^2}{2} - \frac{x}{2}} a_0(z, \lambda) \) and similarly introducing \( a_\pm, a_* \). Analyzing the modified integral equations, we prove the main estimates in terms of \( z \)-derivatives of \( a_0, a_\pm \) and \( a_* \) (Theorem 5.4). The domains of these estimates turn out to be rather complicated. Comparing the asymptotics as \( x \to \infty \), we identify each of \( A_\psi(z_\lambda(x), \lambda)/\sqrt{z_\lambda'(x)} \) with one of \( \psi(\pm x, \lambda), \psi(\pm ix, -\lambda) \). This yields Theorem 5.4. We also give the connection formulas and calculate the Wronskians for \( A_\psi \).

In Appendix A we list the properties of the auxiliary family of curves, that are used in the proof of Theorem 5.4. In Appendix B we accomplish the study of properties of \( \Gamma_\lambda = z_\lambda(\mathbb{R}^+) \) by estimating the integrals of \( \left[ \frac{1}{(1 + |z|)^n} \right] \) and \( \left[ \frac{e^{-\frac{z^2}{2}}}{(1 + |z|)^n} \right] \) along the family of curves \( \Gamma_\lambda(z) \). For the sake of completeness some technical results from [4] are reproduced in Section 4 and Appendix B.

**Notations.**

- We set \( \mathbb{C}_+ = \{ z \in \mathbb{C} : \text{Im} z > 0 \} \), \( \overline{\mathbb{C}}_+ = \{ z \in \mathbb{C} : \text{Im} z \geq 0 \} \).
For the spectral parameter $\lambda \in \mathbb{C} \setminus \{0\}$ we set $\lambda = |\lambda| e^{2i\vartheta}, \vartheta \in [0, \pi/2$.

The functions $\log z$ and $z^\alpha = e^{\alpha \log z}$ for $\alpha \in \mathbb{C}$ take their principal values on $\mathbb{C} \setminus \mathbb{R}_-$.

We denote by $S[\alpha, \beta]$ the sector $\{ z \in \mathbb{C} : \arg z \in [\alpha, \beta]\}$; similarly we define $S(\alpha, \beta), S[\alpha, \beta)$ and $S(\alpha, \beta]$. By $S[-\pi, \pi]$ we denote the complex plane cut along $(-\infty, 0]$, where the upper and the lower sides of the cut are included, but not identified.

We also set $S_R[\alpha, \beta] = \{ z \in S[\alpha, \beta] : |z| \geq R\}$.

## 2 Elementary properties of parabolic cylinder functions

In this section we list the necessary properties of standard parabolic cylinder functions. We use the solution $\psi$ of Eq. (1.1), given by $\psi(x, \lambda) = U(-\frac{\lambda}{2}, x\sqrt{2})$, where $U$ is the standard parabolic cylinder function (see [1]). In the notation of Whittaker [2] for the parabolic cylinder function $D_n(z)$ and the confluent hypergeometric function $W_{k,m}(x)$ we have $\psi(x, \lambda) = D_{\frac{\lambda}{2} - 1}(x\sqrt{2}) = \frac{2^{\frac{\lambda}{2} - 1}}{\sqrt{x}} W_{\frac{\lambda}{2} - 1}(x^2)$. The solution $\psi$ is uniquely defined by its asymptotics

$$\psi(x, \lambda) \sim (x\sqrt{2})^{\frac{\lambda}{2} - 1} e^{-\frac{x^2}{2}} \quad \text{as} \quad |x| \to \infty, \quad |\arg x| \leq \frac{3\pi}{4} - \epsilon \quad \text{for any} \quad \epsilon > 0. \quad (2.5)$$

It is an entire function of $x$ and an entire function of $\lambda$. Other important solutions of (1.1) are $\psi(-x, \lambda)$ and $\psi(\pm ix, -\lambda)$. The connection formulas are

$$\psi(\pm ix, -\lambda) = \frac{\Gamma(\frac{1-\lambda}{2})}{\sqrt{2\pi}} \left( e^{i\frac{\pi}{4}(\lambda+1)} \psi(\pm x, \lambda) + e^{-i\frac{\pi}{4}(\lambda+1)} \psi(\mp x, \lambda) \right) \quad (2.6)$$

$$\psi(\pm x, \lambda) = \frac{\Gamma(\frac{1+\lambda}{2})}{\sqrt{2\pi}} \left( e^{i\frac{\pi}{4}(\lambda-1)} \psi(\pm ix, -\lambda) + e^{-i\frac{\pi}{4}(\lambda-1)} \psi(\mp ix, -\lambda) \right) \quad (2.7)$$

## 3 The changes of variables

In the variable $t = \frac{x}{\sqrt{\lambda}}$ the equation (1.1) becomes

$$w''(t) = \lambda^2 (t^2 - 1) w(t), \quad w(t) = y(t\sqrt{\lambda}). \quad (3.8)$$

Introduce the function $\xi(t) = \int_1^t \sqrt{s^2 - 1} \, ds$, such that $\xi > 0$ for $t > 0$, defined on $\mathbb{C} \setminus (-\infty, 1]$. We have

$$\xi(t) = \frac{1}{2} \left( t \sqrt{t^2 - 1} - \log(t + \sqrt{t^2 - 1}) \right) = \frac{t^2}{2} - \frac{1}{2} \log 2t - \frac{1}{4} + O(t^{-2}), \quad t \to \infty. \quad (3.9)$$

$\xi$ has finite branch points $t = +1$ and $t = -1$, which we denote by $A$ and $E$, respectively.

**Definition of $\eta(t)$.** Introduce the function

$$\eta(t) = \left( \frac{2}{3} \xi(t) \right)^{\frac{3}{2}}, \quad t \in \mathbb{C} \setminus (-\infty, -1]. \quad (3.10)$$
Unlike $\xi$, it is analytic on $(-1,1)$ and $t=1$ is its regular point. Define the points $\eta_0 = \eta(0) = -(3\pi \sqrt{2})^2$ and $\eta_E = \eta(-1) = -(3\pi \sqrt{2})^2$. The mapping $\eta : DT \mapsto \mathbb{C} \setminus (-\infty, \eta_E]$ is an analytic isomorphism, where $DT = (-1,1) \cup \{t : |\arg \xi(t)| < \frac{3\pi}{2}\}$. We denote by $t(\eta)$ the function inverse to $\eta(t)$ and defined on $\mathbb{C} \setminus (-\infty, \eta_E]$.

The boundary of $DT$ is $\gamma_\pm \cup \gamma_+$, where $\gamma_\pm = \{t \in \mathbb{C}_\pm : |\arg \xi(t)| = \pm \frac{3\pi}{2}, |\xi(t)| \geq \frac{\pi}{2}\}$. The curves $\gamma_\pm$ and $\gamma_{E+}$ emanate from $E$ and have tangential inclinations to the positive real axis of $\pm \frac{3\pi}{2}$, respectively. By (3.9), the curves $\gamma_{\pm}$ are asymptotic to the rays $\arg t = \pm \frac{3\pi}{2}$, respectively. The domain $DT$ and the curves $\gamma_{\pm}$ are schematically presented on Fig 1a.

For any fixed $\lambda \in \mathbb{S}[-\pi, \pi] \setminus \{0\}$ transition from $x$-variable to $z$-variable is given by the function

$$z_\lambda(x) = \lambda^{\frac{2}{3}} \eta \left( \frac{x}{\sqrt{\lambda}} \right) \equiv \lambda^{\frac{2}{3}} \left( \frac{2}{3} \xi \left( \frac{x}{\sqrt{\lambda}} \right) \right)^{\frac{2}{3}}, \quad x \in D_X(\lambda) \overset{\text{def}}{=} \{x \in \mathbb{C} : \frac{x}{\sqrt{\lambda}} \in DT\}. \tag{3.11}$$

For a fixed $\lambda \in \mathbb{S}[-\pi, \pi] \setminus \{0\}$ each mapping

$$z_\lambda(\cdot) : D_X(\lambda) \mapsto D_Z(\lambda) \overset{\text{def}}{=} \mathbb{C} \setminus \{z : \arg z = \arg z_E(\lambda), |z| \geq |z_E(\lambda)|\} \tag{3.12}$$

is an analytic isomorphism. We denote by $x_\lambda(\cdot)$ the function inverse to $z_\lambda(\cdot)$ and defined on $D_Z(\lambda)$. Evidently $z_\lambda(\sqrt{\lambda}) = 0$; the images of $x = 0$ and $x = -\sqrt{\lambda}$ in $z$-variable are

$$z_0 = z_\lambda(0) = \lambda^{\frac{2}{3}} \eta_0 = -\lambda^{\frac{2}{3}} \left( \frac{3\pi \sqrt{2}}{\sqrt{\lambda}} \right)^{\frac{2}{3}}, \quad z_E(\lambda) = z_\lambda(-\sqrt{\lambda}) = \lambda^{\frac{2}{3}} \eta_E = -\lambda^{\frac{2}{3}} \left( \frac{3\pi \sqrt{2}}{\sqrt{\lambda}} \right)^{\frac{2}{3}}. \tag{3.13}$$

The parabolic cylinder equation in $z$-variable. $y$ is a solution of (1.1) in $D_X(\lambda)$ if and only if $u(z, \lambda) = \frac{y(x_\lambda(z), \lambda)}{\sqrt{\partial_x x_\lambda(z)}}$ solves the equation

$$\partial_z^2 u(z, \lambda) - zu(z, \lambda) = V_0(z, \lambda) u(z, \lambda), \quad z \in D_Z(\lambda), \tag{3.14}$$

where we use the notation $\partial_z u = \frac{\partial u}{\partial z}$ and where

$$V_0(z, \lambda) = v \left( z \lambda^{-\frac{2}{3}} \right) \lambda^{-\frac{4}{3}}, \quad v(\eta) = \sqrt{t(\eta)} \frac{d^2}{d\eta^2} \frac{1}{\sqrt{t(\eta)}}. \tag{3.15}$$

and $t(\cdot)$ denotes the function, inverse to $\eta(\cdot)$, given by (3.10). The function $v(\eta)$ is analytic in $\mathbb{C} \setminus (-\infty, \eta_E]$, hence $V_0$ is analytic in both $\lambda$ and $z$ for $(\lambda, z) \in (\mathbb{C} \setminus \mathbb{R}_-) \times D_Z(\lambda)$. Since $v(\eta)$ has the uniform asymptotics $v(\eta) \sim \frac{7}{6} \eta^{-2}$ as $|\eta| \to \infty$, it is unbounded only at $\eta = \eta_E$. Thus for any $\epsilon > 0$ there exists a constant $C$ such that

$$|v(\eta)| \leq \frac{C}{(1+|\eta|)^2}, \quad \text{for } |\eta - \eta_E| \geq \epsilon, \quad \eta \notin (-\infty, \eta_E]. \tag{3.16}$$

The curve $\Gamma_{\lambda} = z_\lambda(\mathbb{R}_+)$. Each mapping $z_\lambda(\cdot) : \mathbb{R}_+ \mapsto \Gamma_{\lambda} = z_\lambda(\mathbb{R}_+)$ is a real analytic isomorphism. If $\lambda > 0$, then $\Gamma_{\lambda} = [z_0, \infty) \subset \mathbb{R}$ is a half-line ($z_0$ is given by (3.13)). The curve is schematically presented on Fig 1b. For $z_1, z_2 \in \Gamma_{\lambda}$ such that $x_\lambda(z_1) \leq x_\lambda(z_2)$ define the curves, that play the role of an interval, by

$$\Gamma_{\lambda}(z_1, z_2) = \{z : x_\lambda(z) \in [x_\lambda(z_1), x_\lambda(z_2)]\}, \quad \Gamma_{\lambda}(z_1, \infty) = \{z : x_\lambda(z) \geq x_\lambda(z_1)\}. \tag{3.17}$$

Now we generalize the notion of the turning point for non-positive $\lambda$. 


Definition 3.1. For each \( \lambda \in S[\pi, -\pi] \setminus \{0\} \) define \( x_+ = x_+(\lambda) \in \mathbb{R}_+ \) and \( x_- = x_- (\lambda) \in \Gamma_\lambda \) by \( |z_+(x_-)| = \min_{x \in \mathbb{R}_+} |z_+(x)| \) and \( z_* = z_*(x_*) \). We also define \( t_* \) and \( r_* \) by \( t_* = \frac{z_*}{|z_*|} \) and \( r_* = |t_*| \).

By Lemma 4.3 the point \( z_* \) is defined correctly (is unique). Throughout the paper we use \( z_* \) according to this definition, omitting dependence from \( \lambda \) for brevity. We set

\[
\Gamma^- = \Gamma_\lambda(z_0, z_*), \quad \Gamma^+ = \Gamma_\lambda(z_*, \infty), \quad \Gamma_\lambda = \Gamma^- \cup \Gamma^+.
\]

(3.17)

Note that for \( \lambda > 0 \) we have \( x_* = \sqrt{\lambda} \).

Separating small and large \( \arg \lambda \). Further analysis of Eq. (3.14) employs different technique for small and large arguments of \( \lambda \). We formalize these two cases by introducing \( \delta \) and considering separately \( |\arg \lambda| \leq \delta \) and \( \delta < |\arg \lambda| \leq \pi \). Here and below we fix

\[
\delta \in (0, \frac{\pi}{3}).
\]

(3.18)

Throughout the paper the constant \( C \) in the estimates depends on \( \delta \); for brevity we indicate this dependence explicitly only in Theorem 5.1.

For each \( \lambda \) define the domain \( D^\delta_+(\lambda) \) in \( z \)-plane as follows. Let \( B_\varepsilon(\lambda) \) denote the disk of radius \( |z_E(\lambda)| \sin \varepsilon \), centered at \( z_E(\lambda) \). Define the points \( w_{\pm\delta}(\lambda) \) where the curves \( \text{Im}(ze^{i(\frac{\pi}{3} + \frac{\delta}{3})})^\frac{3}{2} \) \( = \text{const} \) are tangential to \( D^\delta(\lambda) \): \( \text{Im}(w_{\pm\delta}(\lambda)e^{i(\frac{\pi}{3} + \frac{\delta}{3})})^\frac{3}{2} = \sup_{z \in B_\varepsilon(\lambda)} \text{Im}(ze^{i(\frac{\pi}{3} + \frac{\delta}{3})})^\frac{3}{2} \). \( w_\delta(\lambda) \) is defined for \( \lambda \in S_{1/2}^{3}[\delta, \pi] \), \( w_{-\delta}(\lambda) \) is defined for \( \lambda \in S_{1/2}^{3}[0, \pi - \delta] \). Fix \( \varepsilon \in (0, \frac{\delta}{3}) \) sufficiently small to ensure that for any \( \lambda \in S^{3}_{1/2}[\delta, \pi] \) holds \( \text{Im}(w_\delta(\lambda)e^{i(\frac{\pi}{3} + \frac{\delta}{3})})^\frac{3}{2} \leq \text{Im}(z_0(\lambda)e^{i(\frac{\pi}{3} + \frac{\delta}{3})})^\frac{3}{2} \). The domain’s complement is the disk \( B_\varepsilon(\lambda) \) plus the disk’s shadow from the point light source at the origin,

\[
D^\delta_-(\lambda) = \mathbb{C} \setminus (B_\varepsilon(\lambda) \cup \{ z : |\arg z - \arg z_E(\lambda)| \leq \varepsilon, |z| \geq |z_E(\lambda)| \cos \varepsilon \}).
\]

(3.19)

The complement to \( D^\delta_+(\lambda) \) is schematically presented on Fig[1]b) by the dashed region.

Here is the motivation of this definition and of the following one. We prove the main estimate in \( \lambda \)-variable using integral equation, equivalent to (3.14). The kernel of the integral equation is a product of Airy functions and of the effective potential \( V_0 \). By (3.16), we have the estimate

\[
|V_0(z, \lambda)| \leq \frac{C}{|\lambda|^\frac{3}{2} + |z|^2} \quad \text{for} \quad z \in D^\delta_+(\lambda).
\]

(3.20)

For brevity, we indicate this dependence explicitly only in Theorem 5.1.

Airy functions allow convenient estimates on the family of curves \( \text{Im}(ze^{i\varphi})^\frac{3}{2} = \text{const} \), \( |\varphi| \leq \frac{\pi}{3} - \frac{\delta}{3} \) (the curves are discussed in details in Appendix A). Thus the estimates for the whole kernel hold in a subdomain of \( D^\delta_+(\lambda) \), whose points are attainable along a curve from this family. Next we define an important part of this subdomain.

Definition 3.2. \( H_{\pm\delta}(\lambda) \) is the part of the sector \( S[-\pi \pm \frac{\delta}{3}, -\frac{\pi}{3} \pm \frac{\delta}{3}] \), that lies above the curve \( \text{Im}(ze^{i(\frac{\pi}{3} + \frac{\delta}{3})})^\frac{3}{2} = \text{const} \), tangential to the boundary of \( D^\delta_+(\lambda) \),

\[
H_{\pm\delta}(\lambda) = \{ z \in S[-\pi \pm \frac{\delta}{3}, -\frac{\pi}{3} \pm \frac{\delta}{3}] : \text{Im}(ze^{i(\frac{\pi}{3} + \frac{\delta}{3})})^\frac{3}{2} \geq \text{Im}(w_{\pm\delta}(\lambda)e^{i(\frac{\pi}{3} + \frac{\delta}{3})})^\frac{3}{2} \}.
\]

\( H_{+\delta}(\lambda) \) is defined for \( \lambda \in S_{1/2}^{3}[0, \pi - \delta] \), \( H_{-\delta}(\lambda) \) is defined for \( \lambda \in S_{1/2}^{3}[\delta, \pi] \).

\( ^1 \)This choice of \( \varepsilon \) is used in Section 5 when we show that \( \Gamma_\lambda \) (or its parts \( \Gamma_\lambda^{\pm} \)) is within the range of the estimates of Theorem 5.1.
Figure 1: a) The ray $e^{-i\vartheta} \mathbb{R}_+$ for $\vartheta \in [0, \frac{\pi}{2}]$ is in the domain $D_T$ in $t$-plane. b) The point $z_*$ divides the curve $\Gamma_\lambda = \Gamma^-_\lambda \cup \Gamma^+_\lambda$. The curve $\text{Im}(ze^{i(\frac{\pi}{3} - \delta)})^{\frac{3}{2}} = \text{const}$ is tangential to the boundary of $D^\delta_Z(\lambda)$ and is separated away from $\Gamma_\lambda$. The complement to $D^\delta_Z(\lambda)$ is dashed. Here $2\vartheta = \arg \lambda$.

4 Properties of $\Gamma_\lambda$

In this section we find out the properties of the curve $\Gamma_\lambda = z_\lambda(\mathbb{R}_+)$. Since $\xi$ is symmetric with respect to the real axis, we consider only the case $\text{Im} \lambda \geq 0$. In this case we set $\xi(t) = \xi(t - i0)$ for $t \in (-\infty, 1]$ and follow this agreement throughout the paper.

The relation
\[
\frac{2}{3} z_\lambda(x)^{\frac{3}{2}} = \lambda \xi(t), \quad t = \frac{x}{\sqrt{\lambda}} = re^{-i\vartheta} \in S[-\frac{\pi}{2}, 0], \quad \lambda = |\lambda| e^{2i\vartheta}
\]
(4.21)
reduces the study of the family of curves $\Gamma_\lambda = z_\lambda(\mathbb{R}_+), \lambda \in \mathbb{C}_+$ to the study of the family $e^{2i\vartheta} \xi(e^{-i\vartheta} \mathbb{R}_+)$ for $\vartheta \in [0, \frac{\pi}{2}]$. We use the representation of $t \in e^{-i\vartheta} \mathbb{R}_+$ in the form
\[
t = re^{-i\vartheta} = 1 + \eta e^{-i\varphi}, \quad \varphi \in [0, \pi], \quad r, \eta \geq 0.
\]
(4.22)
Using this representation and (3.9), we have
\[
\xi(t) = e^{-i\frac{3\varphi}{2}} \int_0^\eta \sqrt{2 + se^{-i\varphi}} \cdot s^{\frac{3}{2}} ds, \quad \partial_r \xi(t) = e^{-i\vartheta} \sqrt{t^2 - 1} = \sqrt{\eta e^{-i(\vartheta + \varphi)}} \sqrt{1 + re^{-i\vartheta}}.
\]
(4.23)
Lemma 4.1. Fix $\vartheta \in [0, \frac{\pi}{2}]$ and let $t \in e^{-i\vartheta} \mathbb{R}_+$. Write $t$ in the form (4.22). Then

1. $\frac{2}{3} \sin^{\frac{3}{2}} \vartheta \leq |\xi(t)|$; if $|t - 1| \leq 1$, then $\frac{2}{3}|t - 1|^\frac{3}{2} \leq |\xi(t)| \leq 2|t - 1|^\frac{3}{2},$
2. $\arg \xi(t) \in \left[ -\frac{3\varphi}{2} - \vartheta, -\frac{3\varphi}{2} - \vartheta \right], -\varphi \in \left[ \frac{2}{3} \arg \xi(t), \frac{2}{3} \arg \xi(t) + \frac{\vartheta}{3} \right],$
3. $\arg \partial_r \xi(t) \in \left[ -\frac{\vartheta}{2} - \vartheta, -2\vartheta \right] \cap \left( -\frac{\vartheta}{2} - \frac{3\varphi}{2}, -\frac{\vartheta}{2} - \vartheta \right],$
4. if $\vartheta \in (0, \frac{\pi}{2}]$, then $\arg \xi(e^{-i\vartheta} \mathbb{R}_+) = \left[ -\frac{3\varphi}{2} - \vartheta, -2\vartheta \right]$; $\arg \xi(e^{-i\vartheta} \mathbb{R}_+) = \left\{ -\frac{3\varphi}{2}, 0 \right\}$.
5. if \(-\pi - \vartheta \leq \arg \xi(t)\), then \(\arg \left(e^{2i\partial} \partial_t \xi(t)\right) \in \left[-\frac{\pi}{2} + \frac{\vartheta}{2}, \frac{\vartheta}{2}\right]\).

6. if \(-\frac{\pi}{2} - 2\vartheta \leq \arg \xi(t)\), then \(\arg \left(e^{2i\partial} \partial_t \xi(t)\right) \in \left[-\frac{\pi}{6} - \frac{\vartheta}{6}, \frac{\vartheta}{2}\right]\).

Proof. Assume \(|t - 1| \leq 1\) and consider the integrand in (4.23). For \(s \in [0, \eta]\) we have \(1 \leq \Re \sqrt{2 + se^{-i\varphi}}\) and \(|\sqrt{2 + se^{-i\varphi}}| < 3\). Substituting these estimates in (4.23) yields \(\frac{2}{3} \eta^\frac{3}{2} - \frac{2}{3} \leq |\xi(t)| < 3 \cdot \frac{2}{3} \eta^\frac{3}{2}, \eta = |t - 1|\). The relation \(\arg \frac{\eta}{\eta - 1} = \sin \vartheta\) for \(\vartheta \in [0, \frac{\pi}{2}]\) finishes the proof.

Consider the integrand in (4.23). For \(s \in [0, \eta]\) we have \(\sqrt{2 + se^{-i\varphi}} \in \left[-\frac{\vartheta}{2}, 0\right]\), hence \(\arg \xi(t) + 3\frac{\vartheta}{2} = \arg \left(\int_0^s \sqrt{2 + se^{-i\varphi}} \cdot s^\frac{3}{2} ds\right) \in \left[-\frac{\vartheta}{2}, 0\right]\).

In the second identity in (4.23) we have \((t^2 - 1) \in \left(-\pi, -2\vartheta\right),\) so that \(\frac{\sqrt{t^2 - 1}}{2} \in \left[-\frac{\pi}{2}, -\vartheta\right]\) and \(\arg \partial_t \xi(t) \in \left[-\frac{\pi}{2} - \vartheta, -2\vartheta\right]\). Using \(\arg 1 + re^{-i\varphi} \in \left(-\frac{\vartheta}{2}, 0\right]\) in the same formula, we obtain \(\arg \partial_t \xi(t) \in \left(-\frac{\vartheta}{2} - \frac{3\vartheta}{2}, -\frac{\vartheta}{2} - \vartheta\right]\).

Fix \(\vartheta \in (0, \frac{\pi}{2})\). By the principle of boundary correspondence for conformal mappings, \(\xi(S[-\frac{\vartheta}{2}, 0]) \subset \xi(S[-\frac{\vartheta}{2}, 0]),\) and \(\arg \partial_t \xi(t) \in \left[-\frac{\pi}{2} - \vartheta, -2\vartheta\right]\). Using \(\arg 1 + re^{-i\varphi} \in \left(-\frac{\vartheta}{2}, 0\right]\) in the same formula, we obtain \(\arg \partial_t \xi(t) \in \left(-\frac{\vartheta}{2} - \frac{3\vartheta}{2}, -\frac{\vartheta}{2} - \vartheta\right]\). Therefore, \(\arg \xi(t) \in \left[-\frac{\pi}{2} - 2\vartheta\right]\). It remains to prove that \(\arg \left(e^{2i\partial}\xi(t)\right) < 0\).

By Lemma 4.2 for a fixed \(\vartheta\) the function \(\Im \left(e^{2i\partial}\xi(re^{i\varphi})\right)\) strictly decreases in \(r\). Therefore, using \(\xi(S[-\frac{\vartheta}{2}, 0]) \subset \xi(S[-\frac{\vartheta}{2}, 0])\) and \(\arg (0) = \frac{\vartheta}{2}\), we conclude that the curve \(e^{2i\partial}\xi(e^{i\varphi})\) crosses only the negative half of the imaginary axis. Again using monotone decrease of \(\Im \left(e^{2i\partial}\xi(re^{i\varphi})\right)\) in \(r\), we obtain \(\arg \left(e^{2i\partial}\xi(t)\right) < 0\), which comletes the proof.

Using the second identity in (4.23) and \(\sqrt{1 + re^{-i\varphi}} \in \left[-\frac{\vartheta}{2}, 0\right]\), we obtain
\[
\arg \left(e^{2i\partial}\partial_t \xi(t)\right) \in \left[-\frac{\vartheta}{2} + \frac{\vartheta}{2}, -\frac{\vartheta}{2} + \vartheta\right].
\] (4.24)

By hypothesis and \(\arg \xi(t) \in \left[-\pi - \vartheta, -2\vartheta\right]\). Thus using \(\eta\) we obtain \(-\varphi \in \left[-\frac{\vartheta}{2}(\pi + \vartheta), -\vartheta\right]\). Substituting this into (4.24) proves 5.

By \(\eta\) \(-\frac{\pi}{2} - \frac{\vartheta}{2} \leq \arg \xi(t) \leq -\frac{\vartheta}{2}\). Using \(\arg \sqrt{1 + re^{-i\varphi}} + 1 \in \left[-\frac{\vartheta}{2}, 0\right]\), we obtain \(\arg \left(e^{2i\partial}\partial_t \xi(t)\right) = \arg \left(e^{2i\partial}\sqrt{t^2 - 1}\right) \in \left[-\frac{\pi}{2} - \frac{\vartheta}{2}, \frac{\vartheta}{2}\right].\)

Lemma 4.2. Let \(t = re^{-i\varphi}\), \(r \geq 0\) for a fixed \(\vartheta \in \left(0, \frac{\pi}{2}\right]\). Then

1. if \(\vartheta \in \left(0, \frac{\pi}{2}\right]\) and \(r \geq 0\) then \(\Im \left(e^{2i\partial}\partial_t \xi(t)\right) < 0\) and \(\Re \left(e^{2i\partial}\partial_t \xi(t)\right) > 0\),

2. if \(\vartheta = 0\) and \(r \in \left[0, 1\right]\), then \(\Im \left(e^{2i\partial}\partial_t \xi(t)\right) < 0\) and \(\Re \xi(t) = 0\),

3. if \(\vartheta = 0\) and \(r \in \left(1, \infty\right]\), then \(\Im \xi(t) = 0\) and \(\Re \left(e^{2i\partial}\partial_t \xi(t)\right) > 0\),

4. if \(\arg \left(e^{2i\partial}\xi(t)\right) \in \left(-\pi, -\frac{\pi}{2} + \vartheta\right]\), then \(\partial_t \arg \xi(t) > 0\),

5. if \(\arg \left(e^{2i\partial}\xi(t)\right) \in \left[-\frac{3\vartheta}{2}, -2\vartheta\right]\), then \(\partial_t \arg \xi(t) \geq 0\),

6. if \(\arg \left(e^{2i\partial}\xi(t)\right) \in \left(-\frac{3\vartheta}{2}, -\pi + \vartheta\right]\), then \(\partial_t |\xi(t)| \leq 0\),

7. if \(\arg \left(e^{2i\partial}\xi(t)\right) \in \left(-\frac{3\vartheta}{2}, -\pi\right]\), then \(\partial_t |\xi(t)| > 0\).

Proof. For \(\vartheta \in \left(0, \frac{\pi}{2}\right]\), the result follows from Lemma 4.1.3. For \(\vartheta = 0\), the result follows from (3.3) by direct calculation.

Direct calculation yields \(\partial_t \arg \xi(t) = \frac{\partial}{\partial t} \Im \ln \xi(re^{-i\varphi}) = \frac{|\xi(t)|}{\xi(t)} \sin \{\arg \partial_t \xi(t) - \arg \xi(t)\}\). By Lemma 4.1.3, \(\partial_t \arg \xi(t)\) is strictly positive for \(\arg \xi(t) \in \left(-\pi - 2\vartheta, -\frac{\pi}{2} - \vartheta\right)\).
We have \( \frac{\partial}{\partial r} \arg \xi(r e^{-i\theta}) = \text{Im} \left( \frac{\partial_r \xi(t)}{\xi(t)} \right) \). By Lemma 4.1.8, \( \arg \partial_r \xi(t) \in \left[ -\frac{\pi}{2} - \frac{5\theta}{6}, -\frac{\pi}{2} - \theta \right] \), so using Lemma 4.1.2 yields \( \arg \partial_r \xi(t) = \left[ \frac{1}{2} \arg \xi(t) - \frac{3\theta}{2}, \frac{1}{3} \arg \xi(t) - \frac{5\theta}{6} \right] \). Hence, \( \arg \left( \frac{\partial_r \xi(t)}{\xi(t)} \right) \in \left[ -\frac{2}{3} \arg \xi(t) - \frac{3\theta}{2}, -\frac{2}{3} \arg \xi(t) - \frac{5\theta}{6} \right] \). By hypothesis and Lemma 4.1.11, \( \arg \xi(t) \in \left[ -\frac{3\pi}{2}, -2\theta - \frac{\theta}{6} \right] \), which implies \( \arg \left( \frac{\partial_r \xi(t)}{\xi(t)} \right) \in [0, \pi] \), as required.

We have \( \partial_r \arg \xi(t)^2 = 2|\xi(t)||\xi(t)| \cos \{ \arg \partial_r \xi(t) - \arg \xi(t) \} \). By Lemma 4.1.8, \( \partial_r \arg \xi(t) \) is positive for \( \arg \xi(t) \in (-\frac{3\pi}{2} - 2\theta, -\theta - \pi - \theta) \).

**Lemma 4.3.** For each \( \lambda = |\lambda| e^{2i\vartheta} \in \mathbb{C}_+ \setminus \{0\} \) there exists a unique \( x_\vartheta \geq 0 \) such that \( |z_\lambda(x_\vartheta)| = \min_{x \geq 0} |z_\lambda(x)| \). Define \( t_* \in e^{-i\theta} \mathbb{R}_+ \) and \( z_* \in \Gamma_\lambda \) by \( t_* = \frac{\pi}{\sqrt{\lambda}} \) and \( z_* = z_\lambda(x_*) \). Then

1. \( |z_\lambda(t)| \) is strictly decreasing on \( [0, x_\vartheta) \) and strictly increasing on \( (x_\vartheta, \infty) \).

2. for \( \lambda \) fixed \( \text{Re} z_\lambda(t)^2 \) is non-decreasing on \( \mathbb{R}_+ \). If \( \lambda > 0 \) and \( x \in [x_\vartheta, \infty) \), or if \( 0 < \lambda \leq \pi \) and \( x \in \mathbb{R}_+ \), then it is strictly increasing,

3. if \( \vartheta = 0 \), then \( x_* = 0 \) and \( x_* = \sqrt{\lambda} \).

4. if \( \vartheta \in (0, \frac{\pi}{2}) \), then \( \arg z_* \in \left[ -\frac{\pi}{2} - \frac{\vartheta}{2}, -\frac{\pi}{2} + \frac{5\vartheta}{6} \right] \), \( -\pi + \frac{5\vartheta}{6} \leq \arg \left( e^{2i\vartheta} \xi(t_*) \right) \).

5. \( |t_*| \leq \sqrt{2} \).

**Proof.** First we prove existence and uniqueness of \( x_* \). By (1.21), it is sufficient to prove that there exist a unique minimum of \( |\xi(t)| \) on \( e^{-i\theta} \mathbb{R}_+ \). For \( \arg \lambda = 0 \) the result is evident. Fix \( \vartheta \in (0, \frac{\pi}{2}) \) and let \( t = re^{-i\vartheta} \). Direct calculation yields \( \partial_r |\xi(t)|^2 \big|_{r=0} = -\left( \pi / \sqrt{2} \right) \cos \vartheta < 0 \). By Lemma 4.1.10 and Lemma 4.1.11, we have \( \partial_r |\xi(t)| > 0 \) as \( r \to \infty \). Therefore there exists at least one point \( t \in e^{-i\theta} \mathbb{R}_+ \) such that \( \partial_r |\xi(t)| = 0 \). This point is unique if

\[
\frac{1}{2} \frac{\partial^2}{\partial t^2} |\xi(t)|^2 = |\xi'(t)|^2 + \text{Re} \left( e^{-2i\vartheta} \xi''(t) \xi(t) \right) > 0 \quad \text{for} \quad t \in S[-\frac{\pi}{2}, 0].
\]

(4.25)

By \( \xi'(t) = \sqrt{t^2 - 1} \) and \( \xi''(t) = \frac{t}{\sqrt{t^2 - 1}} \), it is sufficient to show that

\[
\left| \frac{\xi(t)}{w(t)} \right| < 1 \quad \text{for} \quad \text{Re} t \geq 0,
\]

(4.26)

where \( w(t) = \frac{(\xi'(t))^2}{\xi'(t)} = \frac{(\xi(1))^2}{t} \) on \( \mathbb{C} \setminus (-\infty, 1] \) and \( w(t) > 0 \) for \( t > 1 \). Note that \( \frac{\xi(1)}{w(1)} \) is analytic in the half-plane \( \text{Re} t > -1 \), since both \( w \) and \( \xi \) change sign when crossing the cut \( (-1, 1] \). Thus we prove (1.26) applying the principle of maximum for the expanding half-disks \( D_R = \{ t : \text{Re} t \geq 0, |t| \leq R \} \). For \( t \to \infty \) we have \( w(t) \sim t^2 \) and \( \xi(t) \sim \frac{t}{2} \) uniformly in \( |\arg t| \leq \frac{3\pi}{4} \). Hence \( \left| \frac{\xi(t)}{w(t)} \right| \to \frac{1}{2} \) uniformly on the arcs \( |t| = R, |\arg t| \leq \frac{\pi}{2} \) as \( R \to \infty \). For \( \text{Re} t = 0 \) deformation of the integration path in (5.9) gives \( |\xi(t)|^2 \leq (\frac{1}{2})^2 + |t|^2 (1 + |t|/2)^2 \), so that \( |\xi(t)| \leq \frac{7}{12} w(t) \). Thus the inequality \( |\xi| < \frac{7}{12} \) holds on the boundary of \( D_R \) for sufficiently large \( R \). By the maximum principle, this yields (4.26), (4.25) and uniqueness of the point \( t_* \in e^{-i\theta} \mathbb{R}_+ \) satisfying \( \partial_r |\xi(t_*)| = 0 \). By (4.21), \( x_* = \frac{t_*}{\sqrt{\lambda}} \) is the unique minimum of \( |z_\lambda(t)| \) on \( \mathbb{R}_+ \), as required.
Therefore, the result follows from Lemma 4.2.1.

By (4.21), for \( x = e^{-i\vartheta} \) the sign of \( \partial_x \Re \xi(x) \) is the same as that of \( \partial_x \Re (e^{2i\vartheta} \xi(re^{-i\vartheta})) \). Thus, the result follows from Lemma 4.2.1.

For \( \arg \lambda = 0 \) the result is evident; fix \( \vartheta \in (0, \frac{\pi}{2}) \) and let \( t = re^{-i\vartheta} \), \( r \geq 0 \). We have \( \partial_r |\xi(t)|^2 = 2 \Re \left( \xi(t) \partial_r \xi(t) \right) \). By Lemma 4.1.3, for \( t = 1 + \eta e^{-i\vartheta} \in S[-\frac{\pi}{2}, 0] \) we have \( \arg \xi(t) \in [-\frac{3\pi}{2} - \frac{\vartheta}{2}, -\frac{\vartheta}{2}] \); by Lemma 4.1.5, we have \( \arg \partial_r \xi(t) \in [\frac{\pi}{2} + \vartheta, \frac{\pi}{2} + \frac{3\vartheta}{2}] \). Thus \( \arg \left( \xi(t) \partial_r \xi(t) \right) \in [-\varphi + \frac{\vartheta}{2}, -\varphi + \frac{3\vartheta}{2}] \) and we have \( \partial_r |\xi(t)|^2 < 0 \) for \( -\varphi \in \left(-\frac{3\pi}{2} - \frac{\vartheta}{2}, -\frac{\vartheta}{2} - \frac{3\vartheta}{2}\right) \) and \( \partial_r |\xi(t)|^2 > 0 \) for \( -\varphi \in \left(-\frac{\pi}{2} - \frac{\vartheta}{2}, -\frac{\pi}{2} - \frac{3\vartheta}{2}\right) \). Therefore, the point \( t_* = r_*e^{-i\vartheta} = 1 + \eta_*e^{-i\vartheta} \) satisfies

\[
-\varphi_* \in \left[-\frac{\pi}{2} - \frac{3\vartheta}{2}, -\varphi - \frac{\vartheta}{2}\right].
\]

Using (4.27) and Lemma 4.1.2 we obtain

\[
\arg \xi(t_*) \in \left[-\frac{3\pi}{4} - \frac{11\vartheta}{4}, -\frac{3\pi}{4} - \frac{3\vartheta}{4}\right].
\]

By (4.21), this proves \( z_* \in \left[-\frac{\pi}{2} - \frac{\vartheta}{2}, -\frac{\pi}{2} + \frac{5\vartheta}{6}\right] \). By (4.28) for \( 0 \leq \vartheta \leq \frac{3\pi}{11} \) and Lemma 4.1.10 for \( \frac{11\vartheta}{6} \leq \vartheta \leq \frac{\pi}{2} \), we have \( -\pi + \frac{\vartheta}{2} \leq \arg(e^{2i\vartheta} \xi(t_*)) \), as required.

By (4.21), for \( t = re^{-i\vartheta} = 1 + \eta e^{-i\vartheta} \) and \( r < |t_*| \) we have \( -\pi \leq -\varphi \leq -\frac{\pi}{2} - \frac{\vartheta}{2} \). Using Lemma 4.1.3 we obtain \( -\frac{\pi}{2} + \vartheta \leq \arg(e^{2i\vartheta} \partial_r \xi(t)) \leq \vartheta - \frac{\vartheta}{2} \leq -\frac{\pi}{4} + \frac{3\vartheta}{4} \).

Geometric considerations show that \( r = \frac{\sin \varphi}{\sin(\varphi - \vartheta)} \). By (4.27), for \( t_* = r_*e^{-i\vartheta} = 1 + \eta_*e^{-i\vartheta} \) we have \( -\frac{\pi}{2} - \frac{\vartheta}{2} \leq \varphi \leq -\frac{\pi}{2} + \frac{\vartheta}{6} \), so that \( r_* = \frac{\sin \varphi_*}{\sin(\varphi_* - \vartheta)} \leq \frac{1}{\sin \frac{\pi}{2}} = \sqrt{2} \).

In the next Lemma we analyse the curves \( \Gamma_\lambda^\pm \), defined in (3.17).

**Lemma 4.4.** Let \( \lambda = |\lambda|e^{2i\vartheta} \in \mathbb{C}_+ \backslash \{0\} \). Then

1. if \( \vartheta = 0 \), then \( \Gamma^-_\lambda = \left[-\frac{3\pi}{8}|\lambda|, 0\right] \), \( \Gamma^+_\lambda = [0, +\infty) \), \( \Gamma^+_{-\lambda} \subset S[-\pi + \frac{3\pi}{3}, 0] \), \( \Gamma^-_{-\lambda} \subset S[-\pi + \frac{3\pi}{3}, -\frac{\vartheta}{2} + \frac{5\vartheta}{6}] \), \( \Gamma^+_{-\lambda} \subset S[-\frac{\pi}{2} - \frac{\vartheta}{2}, 0] \),

2. if \( 0 < \arg \lambda \leq \delta \), then \( \Gamma^-_\lambda \subset S[-\pi + 0, -\frac{\vartheta}{2} + \frac{5\vartheta}{6}] \), \( \Gamma^-_{-\lambda} \subset S[-\frac{\pi}{2} - \frac{\vartheta}{2}, 0] \),

3. if \( \vartheta = 0 \), then \( \inf_{|z| \in \Gamma_\lambda^\pm} |z| \geq |\lambda|^\frac{3}{2} \sin \vartheta \),

4. \( \Gamma^-_\lambda \subset \{z : |z| \leq (\frac{3\pi}{8}|\lambda|)^\frac{1}{2}\} \), the length of \( \Gamma^-_\lambda \) satisfies \( |\Gamma^-_\lambda| \leq C|\lambda|^{\frac{3}{2}} \).

**Proof**

For \( \vartheta = 0 \) the result is evident. For \( \vartheta \in (0, \frac{\pi}{2}) \) the assertion on \( \Gamma^-_\lambda \) follows from Lemma 4.1.4 and relation (4.21).

Now prove the assertion on \( \Gamma^+_\lambda \). By Lemma 4.1.3, we have only show that \( z\lambda(x) \) is non-decreasing at \( x = x_* \). Writing Lemma 4.1.3 in terms of \( t_* = r_*e^{-i\vartheta} = 1 + \eta_*e^{-i\vartheta} \) yields (4.28). Hence the hypothesis of Lemma 4.2.1 is satisfied for \( t_* \), so that \( \partial_x \arg \xi(t_*) \geq 0 \). By (4.21), \( \partial_x \arg z\lambda(x_*) \geq 0 \). This completes the proof.

By (4.21) of this Lemma, we have \( \Gamma^-_\lambda \subset S[-\pi, -\frac{\pi}{2} + \frac{5\vartheta}{6}] \) and \( \Gamma^+_{-\lambda} \subset S[-\frac{\pi}{2} - \frac{\vartheta}{2}, 0] \). By definition (3.18), \( \delta < \frac{\pi}{4} \), which implies the result.

The result follows from Lemma 4.1.10 and relation (4.21).

The inclusion follows from Lemma 4.3.1 and (3.13). Using the definition (3.11), we obtain \( |\Gamma^-_\lambda| = |\lambda|^{\frac{3}{2}} \int_{0}^{t_*} |\eta'(re^{-i\vartheta})|dr \). The integral is bounded uniformly in \( \lambda \), since \( \eta(t) \) is analytic for \( \Re t > -1 \) and \( |t_*| \leq \sqrt{2} \) (see Lemma 4.3.3). This proves the estimate for \( |\Gamma^-_\lambda| \).
5 The main estimates

In this section we prove the main estimates (5.30) (5.33). We consider only the case $\text{Im} \lambda \geq 0$, since for $\text{Im} \lambda \leq 0$ the results are analogous. In order to formulate the result we recall the necessary definitions and notations.

By (3.11), we have $z_\lambda(x) = \left(\frac{2}{\pi} \lambda \xi(\frac{x}{\lambda})\right)^{\frac{1}{2}}$, $\xi(t) = \int_1^t \sqrt{s^2 - 1} ds$, where $\xi(t) > 0$ for $t > 1$. By Definition 3.1 $|x_\lambda(\cdot)|$ on $\mathbb{R}_+$ has the unique minimum in $x_* \equiv x_*(\lambda)$; for $\lambda > 0$ $x_* = \sqrt{\lambda}$ is the turning point. By (1.2), $\phi(\lambda) = 2^\frac{3}{2} \sqrt{\pi} \left(\frac{\lambda}{2\pi}\right)^{\frac{1}{2}}$ takes its principal value on $\mathbb{C} \setminus \mathbb{R}_-$. According to (1.2),

$$\rho(x, \lambda) = \frac{1 + |\lambda|^\frac{1}{2} + |x^2 - \lambda|^\frac{1}{2}}{1 + |\lambda|^{\frac{1}{2}} + |x^2 - \lambda|^{\frac{1}{2}}}.$$ 

For $\arg \lambda > 0$ we define by $\sqrt{x^2 - \lambda}$ the branch, analytic on $\mathbb{R}_+$, such that $\arg \sqrt{x^2 - \lambda} \to 0$ as $x \to +\infty$. For $\arg \lambda = 0$ we set $\sqrt{x^2 - \lambda} = \sqrt{x^2 - (\lambda + i0)}$.

Using Definition 3.2 for $H_{-\delta}(\lambda)$ and (3.19) for $D_{-\delta}(\lambda)$, introduce the domains

$$D_{\delta}(\lambda) \overset{\text{def}}{=} D_{\delta}^2(\lambda) \cap \{ z : |\arg z| \leq \pi - \frac{\delta}{3} \}, \quad \lambda \in S_{1/2}[0, \pi],$$

$$D_{+\delta}(\lambda) \overset{\text{def}}{=} D_{\delta}^2(\lambda) \cap \left( S\left(\frac{\pi}{3} + \frac{4\delta}{3}, \frac{\pi}{3} - \frac{\delta}{3}\right) \cup H_{-\delta}(\lambda) \cup S[-\pi + \frac{4\delta}{3}, \frac{\pi}{3} - \frac{\delta}{3}] \right), \quad \lambda \in S_{1/2}[0, \pi - \delta],$$

$$D_{-\delta}(\lambda) \overset{\text{def}}{=} D_{\delta}^2(\lambda) \cap \{ z : \left| \frac{\pi}{3} - \arg z \right| \geq \frac{\delta}{3} \}, \quad \lambda \in S_{1/2}[0, \pi],$$

$$D_{\delta}(\lambda) \overset{\text{def}}{=} D_{\delta}^2(\lambda) \cap \left( S\left(\frac{\pi}{3} + \frac{4\delta}{3}, \frac{\pi}{3} + \frac{4\delta}{3}\right) \cup H_{\delta}(\lambda) \cup S[-\frac{\pi}{3} + \frac{4\delta}{3}, -\frac{\pi}{3} + \frac{4\delta}{3}] \right), \quad \lambda \in S_{1/2}[\delta, \pi],$$

which are schematically presented of Fig.2.

We are interested in the range of $\arg \lambda$ such that $\Gamma_{\lambda}^\pm$, given by (3.17), are within the estimate’s validity domain. By Lemma 4.13, $\text{Im}(z_\lambda(x)e^{i(\frac{x^2}{2} - \lambda)})^{\frac{1}{2}}$ is non-decreasing in $x$ for $\delta \leq \arg \lambda \leq \pi$. Taking into account the definition (3.19), we conclude that for $\delta \leq \arg \lambda \leq \pi$ the curve $\text{Im}(ze^{i(\frac{x^2}{2} - \lambda)})^{\frac{1}{2}} = \text{const}$, tangential to the boundary of $D_{\delta}^2(\lambda)$, does not intersect $\Gamma_{\lambda}$,

$$\Gamma_{\lambda} \cap \{ z \in S[-\frac{\pi}{3} + \frac{4\delta}{3}, -\frac{\pi}{3} + \frac{4\delta}{3}] : \text{Im}(ze^{i(\frac{x^2}{2} - \lambda)})^{\frac{1}{2}} = \text{Im}(w_{\delta}(\lambda)e^{i(\frac{\pi}{2} - \lambda)})^{\frac{1}{2}} \} = \emptyset, \quad \lambda \in S_{1/2}[\delta, \pi]. \quad (5.29)$$

**Theorem 5.1.** Let $0 < \delta < \frac{\pi}{3}$ and let $\psi$ be the solution of Eq. (1.1), satisfying (2.3). Then there exist a positive number $C_{\delta}$, independent of $x$ and $\lambda$, such that

For $0 \leq \arg \lambda \leq \pi$ and $z_\lambda(x) \in D_{\delta}^2(\lambda)$ we have

$$|\psi'(x, \lambda) + \psi(x, \lambda)\sqrt{x^2 - \lambda}| \leq C_{\delta}|\phi(\lambda)\rho(x, \lambda)e^{-\lambda \xi(\frac{x}{\sqrt{\lambda}})}|. \quad (5.30)$$

In particular, the estimate holds for $\delta \leq \arg \lambda \leq \pi$, $x \in \mathbb{R}_+$ and $0 \leq \arg \lambda \leq \delta$, $x \in [x_*(\lambda), \infty]$.

For $0 \leq \arg \lambda \leq \pi - \delta$ and $z_\lambda(x) \in D_{\delta}^2(\lambda)$ we have

$$|\psi'(ix, -\lambda) + i\psi(ix, -\lambda)\sqrt{x^2 - \lambda}| \leq C_{\delta}|\phi(-\lambda)e^{-i\frac{\pi}{2}\lambda}\rho(x, \lambda)e^{\lambda \xi(\frac{x}{\sqrt{\lambda}})}|. \quad (5.31)$$

In particular, the estimate holds for $0 \leq \arg \lambda \leq \pi - \delta$ and $x \in \mathbb{R}_+$.

For $0 \leq \arg \lambda \leq \pi$ and $z_\lambda(x) \in D_{\delta}^2(\lambda)$ we have

$$|\psi'(-ix, -\lambda) + i\psi(-ix, -\lambda)\sqrt{x^2 - \lambda}| \leq C_{\delta}|\phi(-\lambda)\rho(x, \lambda)e^{-\lambda \xi(\frac{x}{\sqrt{\lambda}})}|. \quad (5.32)$$
In particular, the estimate holds for $0 \leq \arg \lambda \leq \delta$ and $x \in [0, x_*(\lambda)]$.

For $\delta \leq \arg \lambda \leq \pi$ and $z_\lambda(x) \in D_\delta^\delta(\lambda)$ we have

$$
|\psi'(-x, \lambda) + \psi(-x, \lambda) \sqrt{x^2 - \lambda}| \leq C_\delta |\phi(\lambda)e^{-\frac{i\pi}{2}\lambda}\rho(x, \lambda)e^{\frac{x}{\Delta\lambda}}|.
$$

(5.33)

In particular, the estimate holds for $\frac{\pi}{2} - \frac{\delta}{2} \leq \arg \lambda \leq \pi$ and $x \in \mathbb{R}_+$. 

The plan of the proof is as follows. In Definition 5.2 we introduce the solutions $A_0$, $A_\pm$ and $A_*$ of the parabolic cylinder equation in $z$-coordinate (3.14). Each solution is asymptotically close to one of the Airy functions $\text{Ai}(z)$, $\text{Ai}(ze^{\pm i\frac{\pi}{3}})$ in some sector. It is convenient to separate explicitly the exponential multipliers of the solutions and proceed in terms of the modified ones, which we denote by $a_0$, $a_\pm$ and $a_*$ (see Definition 5.3). In Theorem 5.4 we estimate the $z$-derivative of the modified solutions. For each modified solution we specify the domain where the estimate is uniform.
in both $z$ and $\lambda$; we find the range of $\arg \lambda$ such that $\Gamma^+_{\lambda}$ or $\Gamma^-_{\lambda}$ (or both) is within the domain. Finally we transform the estimates of $\partial_z a_0$, $\partial_z a_\pm$ and $\partial_z a_*$ into the estimates of Theorem 5.1 using the relation of $A_\nu(z, \lambda)$ to the parabolic cylinder functions $\psi(\pm x, \lambda)$, $\psi(\pm ix, -\lambda)$. We also present connection formulas and calculate Wronskians for solutions $A_0$, $A_\pm$ and $A_*$.

**Definition 5.2.** $A$, $A_\pm$ and $A_*$ are the solutions of Eq. (3.14), satisfying the following asymptotics as $|z| \to \infty$:

$$A_0(z, \lambda) = \text{Ai}(z)(1 + O(z^{-\frac{2}{3}})), \quad \partial_z A_0(z, \lambda) = \text{Ai}'(z)(1 + O(z^{-\frac{2}{3}}))$$ (5.34)

for $\lambda \in S_{1/2}[0, \pi]$ and $z \in S[-\pi + \frac{4\theta}{3} + \varepsilon, \pi - \frac{\delta}{3}]$;

$$A_+(z, \lambda) = \text{Ai}(z\omega)(1 + O(z^{-\frac{2}{3}})), \quad \partial_z A_+(z, \lambda) = \omega \text{Ai}'(z\omega)(1 + O(z^{-\frac{2}{3}}))$$ (5.35)

for $\lambda \in S_{1/2}[0, \pi - \delta]$ and $z \in S[-\pi + \frac{4\theta}{3} + \varepsilon, \frac{\pi}{3} - \frac{\delta}{3}]$;

$$A_-(z, \lambda) = \text{Ai}(z\omega)(1 + O(z^{-\frac{2}{3}})), \quad \partial_z A_-(z, \lambda) = \overline{\omega} \text{Ai}'(z\omega)(1 + O(z^{-\frac{2}{3}}))$$ (5.36)

for $\lambda \in S_{1/2}[0, \pi]$ and $z \in S[-\frac{\pi}{3} + \frac{\delta}{3}, \frac{\pi}{3} + \frac{4\theta}{3} - \varepsilon]$;

$$A_*(z, \lambda) = \text{Ai}(z\omega)(1 + O(z^{-\frac{2}{3}})), \quad \partial_z A_*(z, \lambda) = \omega \text{Ai}'(z\omega)(1 + O(z^{-\frac{2}{3}}))$$ (5.37)

for $\lambda \in S_{1/2}[\delta, \pi]$ and $z \in S[\frac{\pi}{3} + \frac{\delta}{3}, \frac{4\theta}{3} - \varepsilon]$.

Here $\delta$ and $\varepsilon$ be given by (3.18) and (3.19), respectively.

The sectors from the definition are schematically presented on Fig. 3.

![Figure 3](image_url)

**Figure 3:** The sectors on $z$-plane, where $A_0$, $A_\pm$ and $A_*$ are asymptotically close to Airy functions (here $2\theta = \arg \lambda \in [0, \pi]$). Arrows indicate the sectors where the solutions decay.

The functions $A_0$ and $A_-$ are asymptotic to $\text{Ai}(z)$ and $\text{Ai}(z\omega)$, respectively. The sector of exponential decay of $\text{Ai}(z\omega)$ is divided in two subsectors by the ray $\arg z = -\pi + \frac{4\theta}{3}$ (see (3.12)). The solution $A_+$ is asymptotic to $\text{Ai}(z\omega)$ in one subsector, $A_*$ — in another one.
Further analysis of the perturbed Airy equation (3.14) use the following connection and asymptotic formulas for the Airy functions (see [1]):

\[
\{\text{Ai}(z), \text{Bi}(z)\} = 1, \quad \text{Bi}(z) = i \left(2e^{-i\pi/3}\text{Ai}(\omega z) - \text{Ai}(z)\right), \quad \omega = e^{i2\pi/3}, \quad (5.38)
\]

\[
\text{Ai}(z) = e^{-i\pi/4}\text{Ai}(z\omega) + e^{i\pi/4}\text{Ai}(z\overline{\omega}), \quad \text{Bi}(z) = ie^{-i\pi/4}\text{Ai}(z\omega) - ie^{i\pi/4}\text{Ai}(z\overline{\omega}). \quad (5.39)
\]

\[
\text{Ai}(z) = \frac{e^{-\frac{2}{3}z^{3/2}}}{2\sqrt{\pi z^{3/2}}}(1 + O(z^{-\frac{3}{2}})), \quad |z| \to \infty, \quad |\arg z| < \pi - \epsilon, \quad \forall \epsilon > 0. \quad (5.40)
\]

Next we introduce the modified solutions \(a_\nu\), where the exponential multipliers are separated explicitly. The basic estimates are formulated in terms of these modified solutions.

**Definition 5.3.** For \(|\lambda| > 1/2\) and \(z \in D^\delta_2(\lambda)\)

\[
a_0(z, \lambda) = e^{\frac{2}{3}z^{3/2}}A(z, \lambda), \quad \text{for} \quad \lambda \in S(-\pi, \pi), \quad z \not\in \mathbb{R}_-, \quad (5.41)
\]

\[
a_+(z, \lambda) = e^{\frac{2}{3}(z\omega)^{3/2}}A_+(z, \lambda), \quad \text{for} \quad \lambda \in S(-\pi, \pi), \quad z \not\in e^{i\frac{\pi}{3}}\mathbb{R}_+, \quad (5.42)
\]

\[
a_-(z, \lambda) = e^{\frac{2}{3}(z\overline{\omega})^{3/2}}A_-(z, \lambda), \quad \text{for} \quad \lambda \in S(-\pi, \pi), \quad z \not\in e^{-i\frac{\pi}{3}}\mathbb{R}_+, \quad (5.43)
\]

\[
a_*(z, \lambda) = e^{\frac{2}{3}(z\omega)^{3/2}}A_*(z, \lambda), \quad \text{for} \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \quad z \not\in e^{i\frac{\pi}{3}}\mathbb{R}_+. \quad (5.44)
\]

The next theorem is essentially Theorem 5.1 in \(z\)-variable. It gives the estimates of the \(z\)-derivatives of \(a_\nu\), uniform in both \(z\) and \(\lambda\). It is proved using an equivalent integral equation. The domains of the estimates are defined in the beginning of this section. Their complicated structure is the cost of transition from asymptotic formulas to uniform estimates. \(^2\)

**Theorem 5.4.** The solutions \(A_0, A_\pm, A_*\) exist and are defined uniquely. The solutions are analytic in both \(\lambda\) and \(z\) for \(\lambda \in S_{1/2}(\pi, \pi)\) and \(z \in D_2(\lambda)\). The asymptotics (5.34–5.37) are uniform in both \(z\) and \(\lambda\), and the error term can be replaced by \(O(z^{-\frac{3}{2}}\lambda^{-\frac{3}{2}})\).

Fix \(\delta \in (0, \frac{\pi}{3})\) and let \(D^\delta_2(\lambda)\) be given by (5.19). Then the estimate

\[
|\partial_za_\nu(z, \lambda)| \leq \frac{C_\delta}{(1 + |z|)^{\frac{5}{4}}}, \quad (5.45)
\]

where \(C_\delta\) is independent of \(z\) and \(\lambda\), is fulfilled in the following cases:

\[\nu = 0\] and \(z \in D^0_\delta(\lambda), \lambda \in S_{1/2}[0, \pi].\] In particular, the estimate is valid for \(\lambda \in S_{1/2}[\delta, \pi], z \in \Gamma_{\lambda}\) and for \(\lambda \in S_{1/2}[0, \delta], z \in \Gamma^+_{\lambda}\).

\[\nu = +\] and \(z \in D^+_\delta(\lambda), \lambda \in S_{1/2}[0, \pi - \delta].\] In particular, the estimate is valid for \(\lambda \in S_{1/2}[0, \pi - \delta], z \in \Gamma_\lambda\).

\[\nu = -\] and \(z \in D^-_\delta(\lambda), \lambda \in S_{1/2}[0, \pi].\] In particular, the estimate is valid for \(\lambda \in S_{1/2}[0, \delta], z \in \Gamma^-_{\lambda}\).

\(^2\) Derivation of the classical estimates [130] in [3] also involves detailed discussion of domains.
\[ \nu = \ast \text{ and } z \in D^0_\ast(\lambda), \ \lambda \in S_{1/2}[\delta, \pi]. \] In particular, the estimate is valid for \( \lambda \in S_{1/2}[\pi/2 - \delta, \pi], \)

\[ z \in \Gamma_\lambda. \]

**Proof of Theorem 5.4.** We present the proof only for \( A_0 \) when \( \lambda \in S_{1/2}[0, \pi] \) and \( z \in S(-\pi, \frac{\pi}{3}) \cap D^0_\ast(\lambda) \); for other cases the proof is analogous. Note that existence of a solution of \( 3.14 \) implies its analyticity in \( D^0_\ast(\lambda) \) (by Fuchs theorem, see Ch.5 §3 in [6]).

The scheme of the proof is as follows. First we replace \( 3.14 \) by an equivalent integral equation. Since its integrand is analytic, we choose the integration path to be a curve \( \Upsilon_\varphi(z) \), given by Definition 6.1 (see Appendix A). Using the estimate of the integrand on these curves, standard iteration scheme yields the required estimate of \( z \)-derivative of the solution.

For any \( z \in S(-\pi, \frac{\pi}{3}) \cap D^0_\ast(\lambda) \) there exist a contour \( \Upsilon_\varphi(z) \) with \( |\varphi| \leq \frac{\pi}{3} - \frac{\delta}{3} \), which lies within the same domain: \( \Upsilon_\varphi(z) \subset S(-\pi, \frac{\pi}{3}) \cap D^0_\ast(\lambda) \). Since \( V_0 \) is analytic (see its definition \( 3.15 \)), \( u \) solves \( 3.14 \) if it is a solution of the integral equation

\[ u(z, \lambda) = \text{Ai}(z) + \int_{\Upsilon_\varphi(z)} J_0(z, s)V_0(s, \lambda)u(s, \lambda) \, ds, \quad z \in S(-\pi, \frac{\pi}{3}) \cap D^0_\ast(\lambda), \tag{5.46} \]

where \( J_0(z, s) = \text{Ai}(s)B_1(z) - \text{Ai}(z)B_1(s) \). Here \( \int_{\Upsilon_\varphi(z)} f(s) ds \) denotes the complex line integral of \( f \) along the infinite curve \( \Upsilon_\varphi(z) \). We have to treat \( 5.46 \) as a formal equation; it is justified below using standard iteration technique. We rewrite the last equation in terms of \( v = e^{\frac{2\pi i}{3}} u \):

\[ v(z, \lambda) = a(z) + \int_{\Upsilon_\varphi(z)} J(z, s)V_0(s, \lambda)v(s, \lambda) \, ds, \quad a(z) \equiv \text{Ai}(z)e^{\frac{2\pi i}{3}}, \tag{5.47} \]

where \( J(z, s) = J_0(z, s)e^{\frac{2\pi i}{3}(s^\frac{3}{2} - z^\frac{3}{2})} \). To provide continuity as \( \text{arg} \, \lambda \downarrow 0 \), everywhere below we require that \( z^\frac{3}{2} \) takes its values on the lower side of the cut \( (-\infty, 0] \) for \( z < 0 \). (Since for \( \text{Im} \, \lambda > 0 \) the curve \( \Gamma_\lambda \) lies in the lower half-plane \( \text{Im} \, z < 0 \).) By Definition 5.3 and (5.38), we have

\[ J(z, s) = -2\pi e^{-\frac{2\pi i}{3}} \left( a(z)a(s) - e^{\frac{4\pi i}{3}(s^{-\frac{3}{2}} - z^{-\frac{3}{2}})} a(z)a(s) \right), \quad \omega = e^{\frac{2\pi i}{3}}. \tag{5.48} \]

Set \( v_0(z) = a(z) \) and consider the iterations

\[ v_n(w, \lambda) = \int_{\Upsilon_\varphi(w)} J(w, s)V_0(s, \lambda)v_{n-1}(s, \lambda) \, ds \quad \text{for} \quad w \in \Upsilon_\varphi(z). \]

We estimate \( v_n \) in terms of the majorizing functions \( b_n(w, \lambda) = \sup_{s \in \Upsilon_\varphi(w)} (1 + |s|)^\frac{3}{2} |v_n(s, \lambda)| \), defined on \( \Upsilon_\varphi(z) \). By Lemma 6.11, \( b_n(w, \lambda) \) is non-decreasing on \( \Upsilon_\varphi(z) \). By (5.48),

\[ |a(z)| \leq \frac{C}{(1 + |z|)^\frac{3}{2}}, \quad |a'(z)| \leq \frac{C}{(1 + |z|)^\frac{3}{2}}, \quad |\text{arg} \, z| \leq \pi - \varepsilon, \quad \forall \varepsilon > 0. \tag{5.49} \]

Therefore, using (3.20) and Lemma 6.22, we obtain

\[ |J(w, s)V_0(s, \lambda)| \leq \frac{C}{(1 + |w|)^\frac{3}{2}(|\lambda|^{\frac{3}{2}} + |s|^2)(1 + |s|)^\frac{3}{2}} \quad \text{for} \quad s \in \Upsilon_\varphi(w) \]
uniformly in \( w \in S(-\pi, \frac{\pi}{3}) \cap D^2_Z(\lambda) \). Thus

\[
 b_{n+1}(w) \leq C \int_{\gamma(w)} \frac{b_n(s)|ds|}{(|s|^{\frac{4}{3}} + |s|^2)(1 + |s|)^{\frac{7}{3}}}, \quad w \in \gamma(z),
\]

where we denote by \( \int_C f(s)|ds| \) the line integral of \( f \) along \( G \) with respect to the arc length \( |ds| = \sqrt{(dx)^2 + (dy)^2} \). By (5.49), we have \( b_0 \leq C \), so the integrals converge absolutely. Using the induction principle, we obtain

\[
(1 + |w|)^{\frac{4}{3}} |v_n(w, \lambda)| \leq b_n(w, \lambda) \leq \frac{1}{n!} \left( \frac{C}{|\lambda|^{\frac{4}{3}} (1 + |s|)^{\frac{7}{3}}} \right)^n, \quad w \in \gamma(z). \tag{5.50}
\]

By Lemma 6.2.4, the integral in parenthesis is bounded. Hence, the series \( \sum_{n=0}^{\infty} v_n \) converges uniformly and absolutely and its sum \( v = \sum_{n=0}^{\infty} v_n \) solves \( 5.47 \). Thus \( u = e^{-\frac{\lambda}{3}z^3} \) solves \( 3.14 \). Moreover, (5.50) implies that for \( \lambda \in S_{1/2}[0, \pi] \) and \( z \in S(-\pi, \frac{\pi}{3}) \cap D^2_Z(\lambda) \)

\[
|v(z, \lambda)| \leq \frac{1}{(1 + |z|)^{\frac{4}{3}}} \sum_{n=0}^{\infty} b_n(z, \lambda) \leq \frac{C}{(1 + |z|)^{\frac{4}{3}}}. \tag{5.51}
\]

Now show that \( u \) has the asymptotics (5.34). We write \( v \) as

\[
v(z, \lambda) = a(z) + a(z)I_p - a(z \omega) e^{\frac{\lambda}{3}z^3} I_e, \tag{5.52}
\]

where

\[
I_p = -2\pi i e^{-\frac{i\pi}{3}} \int_{\gamma(z)} V_0(s, \lambda) a(s \omega) v(s, \lambda) ds, \quad I_e = 2\pi i e^{-\frac{i\pi}{3}} \int_{\gamma(z)} e^{-\frac{\lambda}{3}z^3} V_0(s, \lambda) a(s) v(s, \lambda) ds.
\]

By (3.20), (5.49), (5.51) and Lemma 6.2.4,

\[
|I_e| \leq \frac{C}{|\lambda|^{\frac{4}{3}} (1 + |z|)^{\frac{7}{3}}} \quad \text{for} \quad z \in S[-\pi + \frac{\pi}{3}, \frac{\pi}{3}] \cap D^2_Z(\lambda). \tag{5.53}
\]

In order to estimate \( I_p \) we observe that it is given by an integral with an analytic integrand. So for \( z \in S(-\pi + \frac{4\pi}{3} + \varepsilon, \frac{\pi}{3}) \) we deform the integration path to the ray \( \{s : \arg s = \arg z, |s| \geq |z|\} \), which lies within \( S(-\pi, \frac{\pi}{3}) \cap D^2_Z(\lambda) \). Thus using (5.49), (3.20) and (5.51), we obtain

\[
|I_p| \leq \frac{C}{(1 + |z|)^{\frac{4}{3}}}, \quad |I_p| \leq \frac{C}{|\lambda|^{\frac{4}{3}} (1 + |z|)^{\frac{7}{3}}} \quad \text{for} \quad z \in S[-\pi + \frac{4\pi}{3} + \varepsilon, \frac{\pi}{3}]. \tag{5.54}
\]

Using \( a(z) = Ai(z) e^{\frac{\lambda}{3}z^3} \), (5.52), (5.53) and (5.54), we obtain for \( u = e^{-\frac{\lambda}{3}z^3} v \) the estimates

\[
|u(z, \lambda) - Ai(z)| \leq C \frac{|e^{-\frac{\lambda}{3}z^3}|}{(1 + |z|)^{\frac{4}{3} + \frac{7}{3}}}, \quad |u(z, \lambda) - Ai(z)| \leq C \frac{|e^{-\frac{\lambda}{3}z^3}|}{|\lambda|^{\frac{4}{3}} (1 + |z|)^{\frac{7}{3} + \frac{4}{3}}}
\]

15
uniformly in \( \lambda \in S_{1/2}[0, \pi] \) and \( z \in S[-\pi + \frac{4\theta}{3} + \varepsilon, \frac{7\pi}{3}] \). By the first estimate and (5.30), \( u \) satisfies the first asymptotics in (5.31). By the second estimate, the error term in (5.31) can be replaced by \( O(\lambda^{-\frac{3}{2}} |z|^{-\frac{1}{2}}) \).

Taking z-derivative of (5.32) and using (5.33) and (5.34), we obtain for \( \lambda \in S_{1/2}[0, \pi] \) and \( z \in S[-\pi + \frac{4\theta}{3} + \varepsilon, \frac{7\pi}{3}] \) the estimates

\[
|\partial_z u(z, \lambda) - A i'(z)| \leq C \frac{|e^{-\frac{2}{\lambda}z|\lambda|}|}{(1 + |z|)^{\frac{3}{2} - \frac{1}{4}}}, \quad |\partial_z u(z, \lambda) - A i'(z)| \leq C \frac{|e^{-\frac{2}{\lambda}z|\lambda|}|}{|\lambda|^{\frac{5}{2}} (1 + |z|)^{\frac{3}{2} - \frac{1}{4}}},
\]

so that \( u \) satisfies the second asymptotics in (5.31) and its error term can be replaced by \( O(\lambda^{-\frac{3}{2}} |z|^{-\frac{1}{2}}) \).

We demonstrated that \( u \) has the asymptotics (5.33), hence \( A_0 = u = e^{-\frac{2}{\lambda}z|\lambda|} v \) and \( a_0 = v \). Now prove (5.44) for \( A_0 \). By (5.49), (5.50), (5.51) and Lemma 6.2, we conclude that \( I_p \) is uniformly bounded for \( \lambda \in S_{1/2}[0, \pi] \) and \( z \in S[-\pi + \frac{4\theta}{3} + \varepsilon, \frac{7\pi}{3}] \cap D_2(\lambda) \). Taking z-derivative of (5.32) and estimating \( |\partial_z u(z, \lambda) - A_i'(z)| \) using (5.49), (5.53) and boundedness of \( I_p \), we obtain (5.43) for \( \nu = 0 \).

Next we relate the solutions \( A_\nu \) to \( \psi(\pm x, \lambda) \) and \( \psi(\pm ix, -\lambda) \). The next Corollary follows from the asymptotics (5.34) (5.37) for \( A_\nu \), (2.25) for \( \psi \) and \( z_\lambda(x) = \left( \frac{3}{4} \right) \sqrt{x} \left( 1 + O(x^{-1}) \right) \) for \( |\arg x-\vartheta'| < \pi \) (see [3.9, 3.11]). Note that multiplication by \( \phi \) (or \( \phi^{-1} \)), which has different values on the upper and the lower sides of the cut \( \mathbb{R} \), annihilates with the similar behavior of \( A_\nu / \sqrt{z_\lambda'} \).

**Corollary 5.5.** Let \( |\lambda| \geq \frac{1}{2}, \lambda \in \mathbb{C} \setminus \mathbb{R} \) and \( z_\lambda(x) \in D_2(\lambda) \). Then

\[
A_0) \quad \psi(x, \lambda) = \phi(\lambda) A_0(z_\lambda(x), \lambda) \frac{A_0(z_\lambda(x), \lambda)}{\sqrt{z_\lambda'(x)}} \quad \text{for } \lambda \in S_{1/2}(-\pi, \pi), \quad (5.55)
\]

\[
A_-) \quad \psi(-ix, -\lambda) = \frac{2^\frac{3}{2} \pi}{\phi(\lambda)} e^{i \frac{\pi}{4} e^{-\frac{2}{\lambda}z|\lambda|}} A_{-}(z_\lambda(x), \lambda) \sqrt{z_\lambda'(x)} \quad \text{for } \lambda \in S_{1/2}[\pi + \delta, \pi], \quad (5.56)
\]

\[
A_+) \quad \psi(ix, -\lambda) = \frac{2^\frac{3}{2} \pi}{\phi(\lambda)} e^{-i \frac{\pi}{4} e^{-\frac{2}{\lambda}z|\lambda|}} A_{+}(z_\lambda(x), \lambda) \sqrt{z_\lambda'(x)} \quad \text{for } \lambda \in S_{1/2}[\pi, \pi - \delta], \quad (5.57)
\]

\[
A_*) \quad \psi(-x, \lambda) = e^{\pm i \frac{\pi}{6} e^{\pi i / 4}} \phi(\lambda) A_{*}(z_\lambda(x), \lambda) \frac{A_*}(z_\lambda(x), \lambda)}{\sqrt{z_\lambda'(x)}} \quad \text{for } \pm \arg \lambda \in [\delta, \pi], \frac{1}{2}, \quad (5.58)
\]

where \( \phi(\lambda) \) is given by (1.2).

**Proof of Theorem 5.1** We give the proof only for \( \psi(x, \lambda) \) and \( \text{Im} \lambda \geq 0 \); for other cases the proof is analogous. By (5.55), we have \( \phi(\lambda) a_0(z_\lambda(x), \lambda) = \psi(x, \lambda) \sqrt{z_\lambda(x)} \exp \left( \frac{2}{3} z_\lambda(x) \frac{x}{\lambda} \right) \). Taking x-derivative of this identity and using (3.11), we obtain

\[
\psi'(x, \lambda) + \psi(x, \lambda) \sqrt{x^2 - \lambda} = \phi(\lambda) e^{-\frac{\lambda x}{\sqrt{\lambda}} \sqrt{\lambda}} \left( \sqrt{z_\lambda(x)} \frac{\partial}{\partial z} a_0(z_\lambda(x), \lambda) - \frac{2}{3} \frac{z_\lambda''(x)}{z_\lambda'(x)^2} a_0(z_\lambda(x), \lambda) \right),
\]

where we write \( z_\lambda \) in place of \( z_\lambda(x) \) for brevity. Using the estimate (5.43) of Theorem 5.4 we have

\[
|\psi'(x, \lambda) + \psi(x, \lambda) \sqrt{x^2 - \lambda}| \leq C_\delta |\phi(\lambda) e^{-\frac{\lambda x}{\sqrt{\lambda}} \sqrt{\lambda}}| \left( \frac{\sqrt{z_\lambda'(x)}}{1 + |z_\lambda|^\frac{3}{2}} + \frac{z_\lambda''(x)}{z_\lambda'(x)^2} \frac{1}{1 + |z_\lambda|^\frac{3}{2}} \right) \quad (5.59)
\]
for $z_\lambda(x) \in D_0(\lambda) \subset D_Z(\lambda)$. Using (3.9 3.11), we conclude that for $x \in D_X(\lambda)$ (that is, $z_\lambda(x) \in D_Z(\lambda)$)

$$|z_\lambda(x)| \leq C \frac{|x^2 - \lambda|}{|\lambda|^{\frac{1}{2}} + |x^2 - \lambda|^{\frac{1}{2}}}, \quad |z'_\lambda(x)| \leq C(|\lambda|^{\frac{1}{2}} + |x^2 - \lambda|^{\frac{1}{2}}), \quad \left| \frac{z''_\lambda}{z'_\lambda} \right| \leq C \frac{|z'_\lambda|}{1 + |z_\lambda|}. \quad (5.60)$$

Now (5.59) and (5.60) yield (5.30), as required. ■

**Corollary 5.6 (Symmetries).** Let $z \in D_Z(\lambda)$. Then

$$A_0(z, \lambda) = A_0(z, \bar{\lambda}), \quad A_\iota(z, \lambda) = A_\iota(z, \bar{\lambda}), \quad A_\pm(z, \lambda) = A_\mp(z, \bar{\lambda}). \quad (5.61)$$

Next we use (5.55) and the identity $\phi(-\lambda)\phi(\lambda) = 2^{\frac{4}{3}}\pi e^{\pm i\frac{2\lambda}{3}}$ for $\pm \Im \lambda > 0$ to rewrite the connection formulas (2.6 2.7) in terms of $A_\iota$.

**Corollary 5.7 (Connection formulas).** Let $A_\alpha$ denote $A_\alpha(z, \lambda)$ for $\alpha = 0, +, -, \ast$, where $z \in D_Z(\lambda)$. Then

$$A_0 = \frac{2\sqrt{\pi}}{\wp'(\lambda)} \Gamma(\frac{\lambda+1}{2}) \left[ e^{-i\frac{2}{3}} A_+ + e^{i\frac{2}{3}} A_- \right], \quad \text{for} \quad -\pi < \arg \lambda < \pi, \quad (5.62)$$

$$A_\pm = \frac{e^{\mp i\frac{2}{3}\lambda}}{2 \cos \frac{\lambda}{2}} \frac{1}{\wp'(\lambda)} \Gamma(\frac{\lambda+1}{2}) \left[ e^{\pm i\lambda} e^{\pm i\frac{2}{3}} A_0 + A_0 \right], \quad \text{for} \quad 0 < \pm \arg \lambda < \pi, \quad (5.63)$$

$$A_\pm = \frac{e^{\mp i\frac{2}{3}\lambda}}{2 \cos \frac{\lambda}{2}} \frac{1}{\wp'(\lambda)} \Gamma(\frac{\lambda+1}{2}) \left[ e^{\pm i\lambda} A_\iota + e^{\pm i\frac{2}{3} A_0} \right], \quad \text{for} \quad 0 < \pm \arg \lambda < \pi, \quad (5.64)$$

$$A_\iota = \frac{2\sqrt{\pi}}{\wp'(\lambda)} \Gamma(\frac{\lambda+1}{2}) \left[ e^{\pm i\lambda} e^{\mp i\frac{2}{3} A_\iota} A_\mp + A_\mp \right], \quad \text{for} \quad 0 < \pm \arg \lambda < \pi. \quad (5.65)$$

Next we find the Wronskians $W\{f, g\} = fg' - f'g$ for the solutions $A_\iota$, using the asymptotics (5.34 5.37) and the connection formulas (5.62 5.66).

**Corollary 5.8.** For $0 < \arg \lambda < \pi$ we have

$$W\{A_0, A_\pm\} = \frac{e^{i\pi\frac{2}{3}}}{2\pi}, \quad W\{A_\iota, A_+\} = -\frac{e^{i\pi\frac{2}{3}}}{2\pi} e^{i\pi\lambda}, \quad W\{A_\iota, A_-\} = \frac{e^{i\pi\frac{2}{3}}}{2\pi}, \quad (5.66)$$

$$W\{A_0, A_\iota\} = \frac{e^{-i\pi\frac{2}{3}}}{\pi} e^{i\frac{2}{3}\lambda} \cos(\frac{\pi\lambda}{2}) \frac{2\sqrt{\pi}}{\wp'(\lambda)} \Gamma(\frac{\lambda+1}{2}), \quad W\{A_-, A_+\} = \frac{e^{-i\pi\frac{2}{3}}}{2\pi} \frac{1}{\wp'(\lambda)} \Gamma(\frac{\lambda+1}{2}). \quad (5.67)$$

6 Appendix A

We define the family of curves $\Upsilon_\varphi(z)$ and study its properties. We use it in the proof of Theorem 5.7.

**Definition 6.1.** For a complex point $z \in S(-\pi, \pi)$ and an angle $\varphi \in [-\frac{\pi}{3}, \frac{\pi}{3}]$, satisfying $|\arg z - \varphi| \leq \frac{2\pi}{3}$, we set

$$\Upsilon_\varphi(z) = \left\{ s \in S ||\arg z, \varphi|| : \Im(se^{-i\varphi})^\frac{3}{2} = \Im(ze^{-i\varphi})^\frac{3}{2}, \Re(se^{-i\varphi})^\frac{3}{2} \geq \Re(ze^{-i\varphi})^\frac{3}{2} \right\}, \quad (A.1)$$

where $S ||\arg z, \varphi||$ denotes the sector $S[\arg z, \varphi]$ if $\arg z \leq \varphi$ and $S[\varphi, \arg z]$ otherwise.
The curve $\Upsilon_\varphi(z)$ is asymptotic to the ray $\arg s = \varphi$. If $\arg z = \varphi$, then $\Upsilon_\varphi(z)$ degenerates into the ray $e^{i\varphi}[|z|, \infty)$. If $|\arg z - \varphi| = \frac{2\pi}{3}$, then $\Upsilon_\varphi(z)$ degenerates into the sum of the interval $e^{i\arg z}[0, |z|]$ and the ray $e^{i\varphi}\mathbb{R}_+$. The curves $\arg(ze^{i\varphi})^{3/2} = \text{const}$ are schematically presented on Fig. 4a).

**Lemma 6.2.** 1. If $w \in \Upsilon_\varphi(z)$, then $\Upsilon_\varphi(w) \subset \Upsilon_\varphi(z)$.

2. If $s \in \Upsilon_\varphi(z) \setminus \{z\}$ and $|\varphi| < \frac{\pi}{3}$, then $\Re(s^{\frac{3}{2}} - z^{\frac{3}{2}}) = \cos \frac{3\varphi}{2} \cdot \Re((s e^{-i\varphi})^{\frac{3}{2}} - (z e^{-i\varphi})^{\frac{3}{2}}) > 0$.

3. Let $\alpha \in \mathbb{R}$ and $\delta > 0$. Then for $|\arg z| \leq \pi - \frac{2\delta}{3}$ there exists $\varphi$ such that $|\varphi| \leq \frac{\pi}{3} - \frac{\delta}{3}$, $|\arg z - \varphi| \leq \frac{2\pi}{3} - \frac{\delta}{3}$ and

$$\int_{\Upsilon_\varphi(z)} \frac{|e^{-\frac{3}{2}s^\frac{3}{2}}|}{(1 + |s|)^{\alpha + \frac{1}{2}}} |ds| \leq C_\delta \frac{|e^{-\frac{3}{2}z^\frac{3}{2}}|}{(1 + |z|)^{\alpha + \frac{1}{2}}}.$$  \hspace{1cm} (A.2)

where $C_\delta$ is independent of $z$ and $\varphi$.

4. Let $\alpha > 1$. Then the integral $\int_{\Upsilon_\varphi(z)} \frac{|ds|}{(1 + |s|)^{\alpha}}$ is bounded uniformly in $z$.

**Proof.** 1 is evident. To prove 2 observe that $s^{\frac{3}{2}} = e^{\frac{3\varphi}{2}}(se^{-i\varphi})^{\frac{3}{2}}$, where we take the principal value of non-integer powers on $\mathbb{C} \setminus \mathbb{R}_-$. Hence, $\Re s^{\frac{3}{2}} = \cos \frac{3\varphi}{2} \cdot \Re(se^{-i\varphi})^{\frac{3}{2}} + \sin \frac{3\varphi}{2} \cdot \Im(se^{-i\varphi})^{\frac{3}{2}}$, where the last term is constant for all $s \in \Upsilon_\varphi(z)$. Subtracting the same formula with $s = z$ yields the result.

Now prove 3. Let $x = \Re(ze^{-i\varphi})^{\frac{3}{2}}$, $y = \Im(ze^{-i\varphi})^{\frac{3}{2}}$. Parametrize the curve $\Upsilon_\varphi(z)$ by $s(t) = e^{i\varphi}(t + iy)^{\frac{3}{2}}$, $t \in [x, \infty)$. By 2 $\Re s^{\frac{3}{2}} = t \cos \frac{3\varphi}{2} + y \sin \frac{3\varphi}{2}$; using $|ds| = \frac{3}{2} \frac{dt}{|t + iy|^{\frac{3}{2}}}$, we obtain

$$\int_{\Upsilon_\varphi(z)} \frac{|e^{-\frac{3}{2}s^\frac{3}{2}}|}{(1 + |s|)^{\alpha}} |ds| \leq C e^{-\frac{3}{4}y^2} \sin \frac{3\varphi}{2} I, \quad I = \int_x^{\infty} \frac{e^{-x\frac{3}{4} \cos \frac{3\varphi}{2}} dt}{(1 + |t + iy|)^{\frac{3}{2}}} \frac{2}{|t + iy|^{\frac{3}{2}}},$$  \hspace{1cm} (A.3)

where we used $(1 + |s|)^{-\alpha} \leq C(1 + |t + iy|)^{-\frac{3\varphi}{2}}$. It remains to prove that

$$I \leq \frac{C \cdot e^{-ex}}{(1 + |w|)^{\frac{3}{2}(\alpha + \frac{1}{2})}}, \quad \text{where} \quad w = x + iy, |\arg w| \leq \pi - \frac{\delta}{2} \quad \text{and} \quad \varepsilon = \frac{\pi}{3} \cos \frac{3\varphi}{2}. \hspace{1cm} (A.4)$$

For $|w| \leq 1$ this is evident. For $|w| > 1$ and $|\arg w| \leq \frac{\pi}{2}$ we use $|t + iy| \geq |w|$ to obtain

$$I \leq \frac{C}{(1 + |w|)^{\frac{3}{2}(\alpha + \frac{1}{2})}} \int_x^{\infty} e^{-xt} dt. \quad \text{Since} \quad \varepsilon \geq \frac{\pi}{2} \sin \frac{3\varphi}{2}, \quad \text{this yields} \quad (A.4).$$

It remains to consider $|w| > 1$ and $\frac{\pi}{2} \leq |\arg w| \leq \pi - \frac{\delta}{2}$. Using $\sin \frac{\delta}{2} \leq |w| \sin \frac{\delta}{2} \leq |y|$, we have

$$I \leq \frac{1}{|y|^{\frac{3}{2}(\alpha + \frac{1}{2})}} \int_x^{\infty} e^{-xt} dt \leq \frac{3}{4(\sin \frac{\delta}{2})^{\frac{3}{2}(\alpha + \frac{1}{2})} + 1} \frac{e^{-ex}}{|w|^{\frac{3}{2}(\alpha + \frac{1}{2})}} \leq \frac{e^{-ex}}{(1 + |w|)^{\frac{3}{2}(\alpha + \frac{1}{2})}},$$

which also implies (A.4). Substituting (A.4) into (A.3) completes the proof.

To prove 4 we use the same parametrization $s(t)$ as in the proof of 3. This gives

$$\int_{\Upsilon_\varphi(z)} \frac{|ds|}{(1 + |s|)^{\alpha}} |ds| \leq \int_x^{\infty} \frac{2 \frac{dt}{|t + iy|^{\frac{3}{2}}} |t + iy|^{\frac{1}{2}}}{(1 + |t + iy|^{\frac{3}{2}})^{\alpha + \frac{1}{2}}} \leq \frac{2}{\alpha - 1}. \hspace{1cm} (A.5)$$
7 Appendix B: Integration along $\Gamma_\lambda$

In this section we estimate the integrals of $\frac{1}{(1+|z|)^\alpha}$ and $\frac{|e^{\frac{z}{3}\bar{z}}|}{(1+|z|)^\alpha}$. We show that for $\delta \leq |\arg \lambda| \leq \pi$ the integrals along the family of curves $\Gamma_\lambda(z)$ allow the same estimates as the integrals along $\mathbb{R}_+$. For $|\arg \lambda| \leq \delta$ this is true only for $z \in \Gamma_\lambda^+$. The integrals along $\Gamma_\lambda^+$ for $|\arg \lambda| \leq \delta$ allow the same estimate as those along $[-|\lambda|^\frac{2}{3}, 0]$.

For any continuous function $f$ on a smooth curve $G$ we denote the usual complex line integral by $\int_G f(s) \, ds$. We denote by $\int_G f(s) \, |ds|$ the line integral of $f$ along $G$ with respect to the arc length $|ds| = \sqrt{(dx)^2 + (dy)^2}$. For integration along the infinite curve $G$ we use the same notation $\int_G f(s) \, ds = \lim_{R \to \infty} \int_{|s| \leq R} f(s) \, ds$ for absolutely converging integrals.

Now we formulate the main result of this section.

**Theorem 7.1.** Let $|\lambda| \geq 1/2$ and $\alpha \in \mathbb{R}$. Fix $\delta \in (0, \frac{\pi}{3})$ and assume that either a) $z \in \Gamma_\lambda$, $\delta \leq |\arg \lambda| \leq \pi$ or b) $z \in \Gamma_\lambda^+$. Then the following estimates are fulfilled:

\[
\begin{align*}
\int_{\Gamma_\lambda(z)} \frac{|e^{-\frac{4}{3}z\bar{z}}|}{(1+|s|)^\alpha} \, |ds| &\leq C \frac{|e^{\frac{4}{3}z\bar{z}}|}{(1+|z|)^{\alpha+\frac{2}{3}}}, \\
\int_{\Gamma_\lambda(w,z)} \frac{|e^{\frac{4}{3}z\bar{z}}|}{(1+|s|)^\alpha} \, |ds| &\leq C \frac{|e^{\frac{4}{3}z\bar{z}}|}{(1+|z|)^{\alpha+\frac{2}{3}}}, \quad \text{where} \quad \begin{cases} 
  w = z_0 & \text{for} \quad \delta \leq |\arg \lambda| \leq \pi, \\
  w = z_* & \text{for} \quad |\arg \lambda| \leq \delta,
\end{cases} \\
\int_{\Gamma_\lambda(z)} \frac{|ds|}{(1+|s|)^\alpha} &\leq C \frac{1}{(1+|z|)^{\alpha-1}}, \quad \alpha > 1, \\
\int_{\Gamma_\lambda(z)} \frac{|ds|}{(1+|s|)||\lambda|^{-\frac{2}{3}}\alpha} &\leq C \frac{\alpha^{-1} + \ln(1 + 2|\lambda|)}{(1+|z|)||\lambda|^{-\frac{2}{3}}\alpha}, \quad \alpha > 0,
\end{align*}
\]

where $C$ is independent of $\lambda$ and $z$.

**Theorem 7.2.** Let $|\lambda| \geq 1/2$. Then the following estimates are fulfilled:

\[
\begin{align*}
\int_{\Gamma_\lambda} \frac{|ds|}{(1+|s|)^\alpha} &\leq \begin{cases} 
  C(1-\alpha)^{-1}|\lambda|^{-\frac{2}{3}(1-\alpha)} & \text{for} \quad 0 \leq \alpha < 1, \\
  C\log(1+2|\lambda|) & \text{for} \quad \alpha = 1, \\
  C(\alpha-1)^{-1} & \text{for} \quad \alpha > 1,
\end{cases} \\
\int_{\Gamma_\lambda} \frac{|ds|}{|\lambda|^{\frac{2}{3}} + |s|^2} &\leq \frac{C}{|\lambda|^{\frac{2}{3}}},
\end{align*}
\]

where $C$ is independent of $\lambda$ and $z$.

We consider only the case $\text{Im} \lambda \geq 0$; for $\text{Im} \lambda \leq 0$ the proof is analogous. As a prerequisite for the proof we estimate $\int_{\Gamma_\lambda(w,z)} |f(s)| |ds|$ for the cases a) and b) of the hypothesis of Theorem 7.1. Similarly we estimate $\int_{\Gamma_\lambda} |f(s)||ds|$ for $0 \leq \arg \lambda \leq \delta$.

Introduce a convenient parametrization of $\Gamma_\lambda$ for $\delta < |\arg \lambda| \leq \pi$ and of $\Gamma_\lambda^+$ for $0 \leq |\arg \lambda| \leq \delta$. By definition of $\Gamma_\lambda = z_\lambda(\mathbb{R}_+)$, the mapping $z_\lambda(\cdot)$ already gives the parametrization by $x \in \mathbb{R}_+$. Define the new parameter $\zeta$ as a function of $x$ by $\zeta = \text{Re}(e^{2i\varphi}(\frac{x}{\sqrt{\lambda}}))$. This is a smooth one-to-one
mapping, since by Lemma 4.2.1, $\text{Re}(e^{2i\vartheta}\xi(\frac{e^{-i\vartheta}}{\sqrt{\lambda}}))$ is strictly increasing in $x$. We also introduce
the function $v_{\vartheta}$, which maps the real part of $e^{2i\vartheta}\xi(re^{-i\vartheta})$ ($r \geq 0$) to its imaginary part: for $\varpi = \text{Re}(e^{2i\vartheta}\xi(re^{-i\vartheta}))$, we set $v_{\vartheta}(\varpi) = \text{Im}(e^{2i\vartheta}\xi(re^{-i\vartheta}))$. The curve $e^{2i\vartheta}\xi(e^{-i\vartheta}\mathbb{R}^+)$ is schematically presented on Fig 4(b). The parametrization of $\Gamma_\lambda$ in terms of $\varpi$ is given by

$$s_\lambda(\varpi) = |\lambda|^{1/2} \left[ \frac{2}{3}(\varpi + iv_{\vartheta}(\varpi)) \right]^{3/2}, \quad \text{so that} \quad |ds| = \sqrt{1 + \left( \frac{dv_{\vartheta}(\varpi)}{d\varpi} \right)^2 \frac{|\lambda|d\varpi}{|s_\lambda(\varpi)|}}. \quad (B.7)$$

By (B.10), (B.11) and (B.7), for a point $z_\lambda(x) = s_\lambda(\varpi)$ on $\Gamma_\lambda$ we have

$$\frac{2}{3} \frac{z_{\lambda}^{1/3}(x)}{|\lambda|} = e^{2i\vartheta}\xi(t) = \varpi + iv_{\vartheta}(\varpi) = \frac{2}{3} s_\lambda(\varpi), \quad \text{where} \quad t = \frac{x}{\sqrt{\lambda}} = re^{-i\vartheta}, \quad r \geq 0. \quad (B.8)$$

Let us show that

$$\left| \frac{dv_{\vartheta}(\varpi)}{d\varpi} \right| \leq C \quad \text{in cases} \quad \begin{array}{ll}
\text{a)} & s_\lambda(\varpi) \in \Gamma_\lambda, \quad \delta < \arg \lambda \leq \pi, \\
\text{b)} & s_\lambda(\varpi) \in \Gamma_\lambda^+, \quad 0 \leq \arg \lambda \leq \delta.
\end{array} \quad (B.9)$$

For $\varpi = \text{Re}(e^{2i\vartheta}\xi(t))$, where $t = re^{-i\vartheta}$ and $r \geq 0$, we have

$$\frac{dv_{\vartheta}(\varpi)}{d\varpi} = \frac{d}{dt} \text{Im}(e^{2i\vartheta}\xi(re^{-i\vartheta})) = \frac{d}{d\varpi} \text{Re}(e^{2i\vartheta}\xi(re^{-i\vartheta})) = \tan \arg(e^{2i\vartheta}\partial_\varpi(\xi(t))) = \tan \arg(e^{2i\vartheta}(t^2 - 1)). \quad (B.10)$$

In case a) by Lemma 4.13, we have $\arg\partial_{\varpi}(\xi(t)) \in [-\frac{\pi}{2} - \vartheta, -2\vartheta]$. Hence $\arg(e^{2i\vartheta}\partial_{\varpi}(\xi(t)) \in [-\frac{\pi}{2} + \delta, 0$ and $\left| \frac{dv_{\vartheta}(\varpi)}{d\varpi} \right| \leq \cot \frac{\delta}{2}$ uniformly for $0 \leq \arg \lambda \leq \pi$.

In case b) by definition (B.18), we have $0 \leq \vartheta < \frac{n}{10}$, so that (B.13) (equivalent to Lemma 4.13) implies $-\pi - \vartheta \leq \arg \xi(t)$. Therefore by Lemma 4.16, $\arg(e^{2i\vartheta}\partial_{\varpi}(\xi(t)) \in [-\frac{\pi}{3} + \frac{\delta}{3}, \frac{\delta}{3}]$. Hence $\left| \frac{dv_{\vartheta}(\varpi)}{d\varpi} \right| \leq \tan \frac{\delta}{3}$, as required.

Thus for $\Gamma_\lambda(z_1, z_2) \subset \Gamma_\lambda, \delta < \arg \lambda \leq \pi$ and for $\Gamma_\lambda(z_1, z_2) \subset \Gamma_\lambda^+, 0 \leq \arg \lambda \leq \delta$ the estimate (B.9) implies

$$\int_{\Gamma_\lambda(z_1, z_2)} |f(s)||ds| \leq C\lambda| \int_{\varpi_1}^{\varpi_2} \left| \frac{f(s_\lambda(\varpi))}{\sqrt{|s_\lambda(\varpi)|}} \right| d\varpi, \quad \text{where} \quad \varpi_{1,2} = \text{Re} \left| \frac{2}{3} \frac{z_{\lambda}^{1/2}}{|\lambda|} \right|. \quad (B.11)$$

By Lemma 4.16, the last inequality in Lemma 4.13 and (B.8), we have

$$-\pi + \frac{\pi}{22} \leq \arg(\varpi + iv_{\vartheta}(\varpi)) \quad \text{for} \quad s_\lambda(\varpi) \in \Gamma_\lambda^+, \quad 0 \leq \arg \lambda \leq \pi. \quad (B.12)$$

By (B.10), (B.8), Lemma 4.10 and (B.18), we have

$$\left| \frac{dv_{\vartheta}(\varpi)}{d\varpi} \right| \leq \tan(\frac{\pi}{4} - \frac{\pi}{22}) < 1 \quad \text{if} \quad \varpi \geq 0, \quad 0 \leq \arg \lambda \leq \delta. \quad (B.13)$$

Now we estimate $\int_{\Gamma_\lambda^-} |f(s)||ds|$ for $0 \leq \arg \lambda \leq \delta$. We introduce the parametrization of $\Gamma_\lambda^-$, symmetric in a sense to (B.7): now the imaginary part of $\frac{z_{\lambda}^{1/2}}{|\lambda|}$ becomes the parameter. Let $x$
Let us show that $\frac{du}{dx}$ of $x$ we obtain $-\left[\text{Im}(\psi(x))\right]$.

By Lemma 4.3.4 and (3.18), arg $\lambda$ gives $\chi$. Let $v_\phi(x)$, which maps the imaginary part of $e^{2i\theta} \xi(r e^{-i\theta})$ ($r \geq 0$) to its real part: for $\chi = \text{Im}(e^{2i\theta} \xi(r e^{-i\theta}))$ we set $u_\phi(\chi) = \text{Re}(e^{2i\theta} \xi(r e^{-i\theta}))$. The parametrization of $\Gamma_\lambda$ in terms of $\chi$ is

$$w_\lambda(\chi) = |\lambda|^\frac{3}{2} \left[\frac{3}{2}(u_\phi(\chi) + i\chi)^\frac{3}{2}\right], \quad |dw| = \sqrt{1 + \left(\frac{du_\phi(\chi)}{d\chi}\right)^2} \frac{|\lambda|}{|w_\lambda(\chi)|}.$$  \hspace{1cm} (B.14)

Let us show that $\frac{du_\phi}{d\chi}$ is uniformly bounded. For $\chi = \text{Im}(e^{2i\theta} \xi(t))$, where $t = re^{-i\theta}$ and $r \geq 0$, we have

$$\frac{du_\phi(\chi)}{d\chi} = \frac{\partial}{\partial \chi} \text{Re}(e^{2i\theta} \xi(r e^{-i\theta})) = \cot(2i\theta \partial_t \xi(t)).$$

By Lemma 4.3.4 and (3.18), $\arg(e^{2i\theta} \partial_t \xi(t))$ is uniformly bounded and for $0 \leq \arg \lambda \leq \delta$ we have

$$\int_{\Gamma_\lambda} |f(s)||ds| \leq C|\lambda| \int_{\chi_*} \frac{|f(w_\lambda(\chi))|}{\sqrt{|w_\lambda(\chi)|}} d\chi, \quad \text{where} \quad \chi_* = \text{Im} \left[\frac{2z_0^\frac{3}{2}}{3|\lambda|}\right].$$  \hspace{1cm} (B.15)

and $\chi_0 = \frac{\pi}{4} \cos 2\theta$.

**Lemma 7.3.** Let $\lambda \in S_{1/2}[0, \pi]$. For each $\lambda$ define $\lambda_0$, $\lambda_*$ and $\lambda_1 > 0$ by $s_\lambda(\lambda_0) = z_0$, and $v_\theta(\lambda_1) = -\lambda_1$. Then $\lambda_* \in [\lambda_0, \lambda_1]$ and there exist a positive number $C$, independent of $\lambda$ and $z$, such that

1. $|s_\lambda(\lambda)| \leq C|\lambda|$ for $\delta \leq \arg \lambda \leq \pi$ and $\lambda \in [\lambda_0, \lambda_1]$.
2. $|s_\lambda(\lambda)|^\frac{3}{2} \leq C \text{Im} z_0^\frac{3}{2}$ for $0 \leq \arg \lambda \leq \delta$ and $\lambda \in [\lambda_*, \lambda_1]$.
3. $|s_\lambda(\lambda)|^\frac{3}{2} \leq \frac{3}{2\sqrt{2}} |\lambda| |\lambda| \text{ for } 0 \leq \arg \lambda \leq \pi \text{ and } \lambda \in [\lambda_1, \infty]$.

**Proof.** Our main instrument is the relation (B.8). Together with (B.13) it implies uniqueness of $\lambda_*$. Using Lemma 4.2.3 and Lemma 4.3.3, we conclude that $\lambda_* \in [\lambda_0, \lambda_1]$. By (B.8) and Lemma 4.2.1, $v_\theta(\lambda)$ is non-increasing. The points $\lambda_{0,1}$ are schematically presented on Fig (4b).

1. Write $t$ such that $\lambda^* \tilde{\eta}(t) = s_\lambda(\lambda)$ in the form (4.22): $t = re^{-i\theta} = 1 + \eta e^{-i\varphi}$, $\varphi \in [0, \pi]$, $\eta \geq 0$. By Lemma 4.2.3 and the definition of $z_1$, we have $\arg(\tilde{\xi}(t)) \leq -\frac{\pi}{4} - 2\theta$. Now using Lemma 4.1.2 we obtain $-\frac{\pi}{4} \leq \vartheta - \varphi \leq -\frac{\pi}{6}$, so that the identity $r = \frac{\sin \varphi}{\sin(\varphi - \theta)}$ yields $|t| \leq \frac{1}{\sin \frac{\pi}{6}}$. Applying (B.8), we conclude that $\frac{2}{3}s_\lambda(\lambda^*)^\frac{3}{2} \subseteq \lambda \xi \{t : |t| \sin \frac{\pi}{6} \leq 1\}$ for $\lambda \in [\lambda_0, \lambda_1]$, hence $|s_\lambda(\lambda)| \leq C|\lambda|^\frac{3}{2}$. Lemma 4.3.3 gives $|\lambda|^\frac{3}{2} \sin \frac{\theta}{2} \leq |z_1|$. This yields $|s_\lambda(\lambda)| \leq C|\lambda|$, as required.

2. It suffice to consider $\arg \lambda \neq 0$. The estimates (B.9), (B.12) implies $|v_\phi(0)| \leq C|v_\phi(\lambda_*)|$ and (B.13) implies $|v_\phi(\lambda_1)| \leq C|v_\phi(\lambda_*)|$, so that $|v_\phi(\lambda_1)| \leq C|v_\phi(\lambda_*)|$. By (B.8), (B.18), Lemma 4.3.3 in the form (4.28) and definition of $\lambda_1$, we have $-\frac{2\pi}{4} - \frac{\pi}{8} \leq \arg(\psi + iv_\phi(\lambda)) \leq -\frac{\pi}{4}$ for $\lambda \in [\lambda_*, \lambda_1]$. Taking into account non-increasing of $v_\phi(\lambda)$, we conclude that $|\lambda| \leq |v_\phi(\lambda)| \cot \frac{\theta}{2}$ and
Figure 4: a) the curves \( \arg(ze^{i\varphi})^{\frac{\lambda}{\bar{\lambda}}} = \text{const} \) in the sector \( S[-\varphi - \frac{2\pi}{3}, -\varphi + \frac{2\pi}{3}] \); b) the curve \( e^{2i\theta} \xi(e^{-i\theta} \Re) \).

therefore \( |x + iv\theta(x)| \leq C|v\theta(x_*)| \) for \( x \in [x_*, x_1] \). By (B.8), this is equivalent to \( |s_\lambda(x)|^{\frac{\lambda}{\bar{\lambda}}} \leq C|\text{Im} z_\lambda^{\frac{\lambda}{\bar{\lambda}}}| \), as required.

By definition of \( x_1 \), (B.8), and Lemma 4.2.3, we have \( s_\lambda(x)^{\frac{\lambda}{\bar{\lambda}}} \in S[-\frac{\pi}{4}, 0] \) for \( x \in [x_1, \infty) \), as required. ■ Note that \( \lambda_0 = \text{Re} \frac{2}{3} \frac{\lambda}{|\bar{\lambda}|} = -\frac{\pi}{4} \sin 2\vartheta \).

Proof of Theorem 7.1. First we prove (B.1). Consider the case \( 0 \leq \arg \lambda \leq \pi \), \( z \in \Gamma^+ \). By Lemma 4.3.1, for \( s_\lambda(x) \in \Gamma^+ \) the function \( |s_\lambda(x)| \) is non-decreasing. Thus using (B.7), (B.11), \( |\lambda x|^{\frac{\lambda}{\bar{\lambda}}} \leq \sqrt{|s_\lambda(x)|} \) for \( |z| \leq 1 \) and \( \sqrt{|z|} \leq \sqrt{|s_\lambda(x)|} \) for \( |z| > 1 \), we obtain

\[
\int_{\Gamma_\lambda(z)} \frac{|e^{-\frac{4s^2}{3\lambda}}|}{(1 + |s|)^\alpha} |ds| \leq C|\lambda| \int_{\text{Re} \frac{2}{3} \frac{\lambda}{|\lambda|}}^\infty \frac{e^{-2|\lambda|\varphi}}{\sqrt{|s_\lambda(x)|}} \leq C \frac{|e^{-\frac{4s^2}{3\lambda}}|}{(1 + |z|)^{\alpha + \frac{1}{2}}}
\]

Consider the case \( \delta \leq \arg \lambda \leq \pi \), \( z \in \Gamma^- \). By 4 and 5 of Lemma 4.1, \( |z_*| \) is bounded away from zero and \( |z| \leq C|z_*| \). So using (B.7), (B.11) and the definition of \( z_* \) (3.1) we obtain

\[
\int_{\Gamma_\lambda(z)} \frac{|e^{-\frac{4s^2}{3\lambda}}|}{(1 + |s|)^\alpha} |ds| \leq C|\lambda| \int_{\text{Re} \frac{2}{3} \frac{\lambda}{|\lambda|}}^\infty \frac{e^{-2|\lambda|\varphi}}{\sqrt{|s_\lambda(x)|}} \leq C \frac{|e^{-\frac{4s^2}{3\lambda}}|}{(1 + |z|)^{\alpha + \frac{1}{2}}}
\]

as required. This completes the proof of (B.1).
Consider (B.2). Introduce the notations \( a = \text{Re} \frac{2 \zeta^2}{3 |\alpha|} \), \( b = \text{Re} \frac{2 w^2}{3 |\alpha|} \) (here and below we omit dependence of \( \lambda \)). Applying (B.11) we obtain

\[
\int_{\Gamma_{\lambda}(w,z)} \frac{|e^{\frac{4}{3}z^2}|}{(1 + |s|)^{\alpha}} |ds| \leq C I(b, a), \quad \text{where} \quad I(b, a) = \int_{b}^{a} \frac{e^{2|\lambda|\varepsilon} |\lambda| |d\varepsilon|}{(1 + |s_{\lambda}(\varepsilon)|)^{\alpha} \sqrt{|s_{\lambda}(\varepsilon)|}}. \tag{B.16}
\]

Let \( \delta < \text{arg} \lambda \leq \pi \). Suppose that \( z \in \Gamma_{\lambda}(z_0, z_1) \), where \( z_0 \) is given by (B.13) and \( z_1 = s_{\lambda}(\varepsilon_1) \) for \( \varepsilon_1 \) is defined in Lemma 7.3. By 3 and 4 of Lemma 4.4, \( |z_1| \) is bounded away from zero and \( |z| \leq C |z_1| \). Therefore,

\[
I(b, a) \leq I(z_0, a) \leq \frac{1}{(1 + |z_1|^{\alpha} |z_1|^{\frac{1}{2}})} \int_{z_0}^{a} e^{2|\lambda|\varepsilon} |\lambda| |d\varepsilon| \leq C \frac{e^{2|\lambda|\varepsilon}}{(1 + |z|^{\alpha} + \frac{1}{2})} \leq C \frac{|e^{\frac{4}{3}z^2}|}{(1 + |z|)^{\alpha + \frac{1}{2}}},
\]

By (B.16), the last estimate proves (B.2) for \( z \in \Gamma_{\lambda}(z_0, z_1) \).

Now suppose that \( z \in \Gamma_{\lambda}(z_1, \infty) \). By Lemma 7.3, \( a > 0 \) and \( |z| \leq C |\lambda|a^{\frac{3}{4}} \). Therefore,

\[
I(z_0, a/2) \leq \int_{z_0}^{a/2} e^{2|\lambda|\varepsilon} |\lambda| |d\varepsilon| \leq e^{\lambda a} \leq C \frac{e^{2|\lambda|\varepsilon}}{(1 + (|\lambda|a)^{\frac{3}{4}})^{\alpha + \frac{1}{2}}} \leq C \frac{|e^{\frac{4}{3}z^2}|}{(1 + |z|)^{\alpha + \frac{1}{2}}},
\]

where we used \( e^{-t} \leq \frac{C}{(1 + t)^{\alpha}} \) for \( t \geq 0 \). Using Lemma 4.3, Lemma 7.3 and \( a > 0 \), we obtain

\[
I(a/2, a) \leq C \frac{e^{2|\lambda|\varepsilon}}{(|\lambda|a)^{\frac{3}{4} (\alpha + \frac{1}{2})}} \leq C \frac{|e^{\frac{4}{3}z^2}|}{(1 + |z|)^{\alpha + \frac{1}{2}}},
\]

By (B.16), the two last displayed formulas prove (B.2) for \( z \in \Gamma_{\lambda}(z_1, \infty) \).

Now let \( 0 < \text{arg} \lambda < \delta \) (for \( \lambda > 0 \) the proof is by direct calculation). For \( |z| \leq 1 \) the result is evident, so we consider the case \( |z| > 1 \). Suppose that \( z \in \Gamma_{\lambda}(z_*, z_1) \). By Lemma 4.3 and Lemma 7.3, \( 0 < |v_\varphi(z_*)\lambda|^{\frac{3}{4}} \leq |z_*| \leq |z| \leq C |v_\varphi(z_*)\lambda|^{\frac{3}{4}} \). Hence,

\[
I(z_*, a) \leq C \int_{z_*}^{a} e^{2|\lambda|\varepsilon} |\lambda| |d\varepsilon| \leq C \frac{|e^{\frac{4}{3}z^2}|}{|z^{\alpha + \frac{1}{2}}|} \leq C \frac{|e^{\frac{4}{3}z^2}|}{(1 + |z|)^{\alpha + \frac{1}{2}}},
\]

By (B.16), this proves (B.2) for \( z \in \Gamma_{\lambda}(z_*, z_1) \).

It remains to consider \( z \in \Gamma_{\lambda}(z_1, \infty) \). By definition of \( z_1 \), \( a > 0 \). Using the substitution \( t = |\varepsilon\lambda|^{\frac{3}{4}} \) and Lemma 7.3, we have

\[
I(z_*, a) \leq \int_{-\infty}^{a} \frac{C \cdot e^{2|\lambda|\varepsilon} |\lambda| |d\varepsilon|}{(1 + |\varepsilon\lambda|^{\frac{3}{2}} |\lambda| |\varepsilon|^{\frac{1}{2}})} \leq \int_{-\infty}^{C \cdot e^{2|\lambda|\varepsilon}} \frac{C \cdot e^{2|\lambda|\varepsilon}}{(1 + a |\lambda|^{\frac{3}{4}})^{\alpha + \frac{1}{2}}} \leq C \frac{|e^{\frac{4}{3}z^2}|}{(1 + |z|)^{\alpha + \frac{1}{2}}},
\]

By (B.16), this proves (B.2).

Next we prove (B.3). Consider the case \( 0 \leq \text{arg} \lambda \leq \pi \), \( z \in \Gamma_+ \). By Lemma 12.4, \( v_\varphi(z) \) is non-increasing. By (B.8) and the last estimate in Lemma 4.3, \( v_\varphi(z) \leq 0 \). Thus using (B.7), (B.11) and the substitution \( t = |\varepsilon\lambda|^{\frac{3}{4}} \) we obtain

\[
\int_{\Gamma_{\lambda}(\varepsilon)} \frac{|ds|}{(1 + |s|)^{\alpha}} \leq \frac{C \cdot I}{|\lambda|^{\frac{3}{4}(\alpha - 1)}}, \quad I = \int_{a}^{\infty} \frac{2 \beta |\varepsilon|^{\frac{3}{4}} |\lambda| |d\varepsilon|}{(|\lambda| + |v|^{\frac{3}{4}} + \varepsilon)^{\alpha}} = \int_{|a|^{\frac{3}{4}} \text{sign} a}^{\infty} \frac{dt}{(|t| + |v|^{\frac{3}{4}} + \varepsilon)^{\alpha}}, \tag{B.17}
\]
where \( a = \text{Re} \left( \frac{2}{3} \frac{3}{|\lambda|} \right), \ v = \text{Im} \left( \frac{2}{3} \frac{3}{|\lambda|} \right), \ \varepsilon = \left( \frac{3}{2} |\lambda| \right)^{-\frac{2}{3}}. \) For \( a \geq 0 \) direct calculation yields \( I \leq \frac{C}{\alpha - 1} (|a|^\frac{2}{3} + \varepsilon v^2)^{\frac{2}{3}} \). By (1.77), this gives (1.3). For \( a < 0 \) we have

\[
I \leq 2 \int_0^\infty \frac{dt}{(t + |v|^{\frac{2}{3}} + \varepsilon)^{\alpha - 1}} \leq \frac{2/(\alpha - 1)}{(|v|^{\frac{2}{3}} + \varepsilon)^{\alpha - 1}} \leq \frac{C/(\alpha - 1)}{(|v|^{\frac{2}{3}} + \varepsilon)^{\alpha - 1}},
\]

where the last estimate follows from (1.22). Now (1.18) and (1.17) again give (1.3).

It remains to consider the case \( \delta < \text{arg} \lambda \leq \pi, \ z \in \Gamma^{-}. \) By (1.8) and Lemma 1.4.3, we have \( (\frac{3}{2})^2 (\sin \frac{\theta}{2})^3 \leq \varepsilon^2 + v^2(\varepsilon). \) Thus using (1.71), (1.11) and Lemma 1.4.4, we have

\[
\int_{\Gamma(z)} \left| \frac{ds}{|s|^{\alpha - 1}} \right| \leq C \int_{|\lambda|^{-\frac{2}{3}}}^{\infty} \frac{d\varepsilon}{\varepsilon^{\frac{2}{3}} + v^{2}(\varepsilon)} \leq C \int_{|a|^{\frac{2}{3}}}^{\infty} \frac{dt}{(\varepsilon + |v|^{\frac{2}{3}} + |t|)^{\alpha}},
\]

as required.

Now prove (1.4). The proof is similar to that of (1.3). Consider the case \( 0 \leq \text{arg} \lambda < \pi, \ z \in \Gamma^{+}. \) By Lemma 1.2.1, \( v(\varepsilon) \) is non-increasing; by (1.8) and the last estimate in Lemma 1.3.3, we have \( v(\varepsilon) \leq 0. \) Thus using (1.71), (1.11) and the substitution \( t = \varepsilon^2 \) we obtain

\[
\int_{\Gamma(z)} \left| \frac{ds}{|s|^{\alpha - 1}} \right| \leq C J, \quad J = \int_{|a|^{\frac{2}{3}}}^{\infty} \frac{dt}{(\varepsilon + |v|^{\frac{2}{3}} + |t|)^{\alpha}},
\]

where \( a = \text{Re} \left( \frac{2}{3} \frac{3}{|\lambda|} \right), \ v = \text{Im} \left( \frac{2}{3} \frac{3}{|\lambda|} \right), \ \varepsilon = \left( \frac{3}{2} |\lambda| \right)^{-\frac{2}{3}}. \) For \( a \geq 0 \) direct estimate of the last integral gives (1.4). For \( a < 0 \) we expand the integration range to \((-\infty, \infty)\) and use symmetry of the integrand. This gives

\[
J \leq 2 \int_{|v|^{\frac{2}{3}}}^{\infty} \frac{dt}{(\varepsilon + |t|)^{\alpha}} \leq 2 \alpha^{-1} + \ln(1 + \varepsilon^{-1}) + \frac{1}{\alpha - 1}.
\]

Now we use (1.12) to deduce (1.4) from the last inequality.

It remains to estimate the integral over \( \Gamma(z, z_0) \) for \( z \in \Gamma^{-}, \ \delta < \text{arg} \lambda < \pi. \) By (1.8) and Lemma 1.4.4, \( \zeta_0 \) and \( z\lambda^{-\frac{2}{3}} \) are bounded. So we use (1.71), (1.11) and the substitution \( t = |\lambda|^{-\frac{2}{3}} \) to obtain

\[
\int_{\Gamma(z)} \frac{(1 + |s|)|ds|}{(1 + |s||\lambda|^{-\frac{2}{3}})^{\alpha}} \leq C \int_0^{\|\lambda\|^{-\frac{2}{3}}} \frac{dt}{\varepsilon + t} \leq C \ln(1 + 2|\lambda|) \leq C \frac{\alpha^{-1} + \ln(1 + 2|\lambda|)}{(1 + |z||\lambda|^{-\frac{2}{3}})^{\alpha}}.
\]

Combining the last estimate with the result for \( z \in \Gamma^{+} \) completes the proof. ■

**Proof of Lemma 7.2.** It is sufficient to consider \( \lambda \in S_{1/2}[0, \pi]. \) First prove (1.5); by (1.15), we have

\[
\int_{\Gamma(z)} \frac{|ds|}{(1 + |s|)^{\alpha}} \leq C I = \frac{3}{2} \int_{|\lambda|^{-\frac{2}{3}}}^{\chi_0} \frac{d\lambda}{|\lambda|^{\frac{2}{3}(\alpha - 1)}}^\alpha,
\]

where \( \varepsilon = \left( \frac{3}{2} |\lambda| \right)^{-\frac{2}{3}}, \ \chi_0 = \text{Im}(e^{2i\vartheta}(\varepsilon(0))) = \frac{\pi}{4} \cos 2\vartheta, \ \chi_* = \text{Im}(e^{2i\vartheta}(\varepsilon(t_*))). \) By Lemma 1.3.5, \( |\chi_*| \) is uniformly bounded; by the last inequality in Lemma 1.3.3, \( \chi_* \leq 0. \) The change of variable \( t = |\lambda|^{-\frac{2}{3}} \) gives in \( I \) and further direct estimate give (1.5).
Now prove (B.6). For $\delta \leq \arg \lambda \leq \pi$ we use the parametrization of $\Gamma_\lambda$, given by (B.7) and the estimate (B.11). Similarly for $0 \leq \arg \lambda \leq \delta$ we use (B.7) and (B.11) on $\Gamma^+\lambda$ and (B.14), (B.15) on $\Gamma^-\lambda$. In the both cases we obtain

$$\int_{\Gamma_\lambda} \frac{|ds|}{|\lambda|^{\frac{4}{3}} + |s|^2} \leq C \int_{-\infty}^{\infty} \frac{dt}{|\lambda|^\frac{2}{3} |t|^{\frac{4}{3}} (1 + |t|^\frac{2}{3})^2} \leq C.$$  \hspace{1cm} (B.21)

8 Acknowledgements

The author is thankful to E.Korotyaev, A.Badanin and D.Chelkak for useful discussions.

References

[1] Abramowitz, M. and Stegun, A. (Eds.), (1970), *Handbook of mathematical functions* (N.Y.: Dover Publications Inc.).

[2] Whittaker, E.T. and Watson, G.N., 1927, *A course of modern analysis. V.2* (Cambrodge Univ. Press).

[3] Olver, F.W.J., 1960, Two inequalities for parabolic cylinder functions. *Cambs.Philos.*, 57, No.4, 811–822.

[4] Klein, M., Korotyaev, E. and Pokrovski, A., 2005, Spectral Asymptotics of the Harmonic Oscillator Perturbed by Bounded Potentials. *Annales Henri Poincare*, 6, 747 - 789. (arxiv.org/math-ph/031206)

[5] Olver, F.W.J., 1959, Uniform asymptotic expansionmns for Weber parabolic cylinder functions of large orders. *J.Res.Nat.Bur.Stand.*, 63B, 131–169.

[6] Olver, F.W.J., 1997, *Asymptotics and special functions* (Mass.: A.K.Peters, Wellesley).