Relativistic charged spheres: II. Regularity and stability

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Abstract. We present new results concerning the existence of static, electrically charged, perfect fluid spheres that have a regular interior and are arbitrarily close to a maximally charged black-hole state. These configurations are described by exact solutions of Einstein’s field equations. A family of these solutions had already been found (de Felice et al 1995 Mon. Not. R. Astron. Soc. 277 L17) but here we generalize that result to cases with different charge distributions within the spheres and show, in an appropriate parameter space, that the set of such physically reasonable solutions has a non-zero measure. We also perform a perturbation analysis and identify the solutions which are stable against adiabatic radial perturbations. We then suggest that the stable configurations can be considered as classic models of charged particles. Finally, our results are used to show that a conjecture of Kristiansson et al (1998 Gen. Rel. Grav. 30 275) is incorrect.

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1. Introduction

It is well known that static, spherically symmetric, uncharged perfect fluids cannot be held in equilibrium below a certain radius without developing singularities inside. This radius is $9M/4$ for an incompressible fluid sphere of total mass–energy $M$; it can be even larger than that for more realistic equations of state (Buchdahl 1959). The possibility of holding a non-singular object in stable equilibrium but compact enough to be close (in fact, arbitrarily close) to a black-hole state, is of great interest not only in order to judge the state of matter in this quasi-critical condition, that is about to turn into a black hole, but also to yield a classic model of charged massive particles which might have astrophysical and cosmological implications. Although one can reach this goal with non-perfect fluids, a perfect fluid solution of the type mentioned was found recently (de Felice et al 1995) but with the presence of an electric charge. Electric charges inhibit the growth of spacetime curvature and therefore they are an efficient means of avoiding singularities inside matter and enhance stability even for quasi-critical configurations.

In this paper we extend the above-mentioned analysis (de Felice et al 1995) to a more general case and, in section 2, show the existence of regular solutions very close to a black-hole state as points in a parameter space (figures 1–3). In section 3 we study the stability of these solutions against adiabatic radial perturbations and show the domain of stable solutions in the
same parameter space (figures 1 and 2). Having solutions which describe very compact sources, we are able to verify the correctness of a conjecture (Kristiansson et al. 1998) according to which the limit of regular embedding in the Euclidean space of conformally reduced spacelike sections of the Reissner–Nordström metric, coincides with the limit of regularity of the internal solutions. In section 4 we show that this is not true.

Throughout the paper we use geometrized units \( (c = 1 = G) \) and metric signature +2.

### 2. Existence of regular quasi-critical spheres

In Schwarzschild coordinates the line element of a spherically symmetric spacetime reads

\[
ds^2 = -e^{\eta} \, dt^2 + e^\lambda \, dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right)
\]

where \( \eta = \eta(r, t) \) and \( \lambda = \lambda(r, t) \). The energy–momentum tensor of a perfect, electrically charged fluid takes the form (de Felice and Clarke 1990):

\[
T^{\mu \nu} = (\rho + p)u^\mu u^\nu + pg^{\mu \nu} + \frac{1}{4\pi} \left( F^{\mu \alpha} F_\alpha^\nu - \frac{1}{4} g^{\mu \nu} F_{\alpha \beta} F^{\alpha \beta} \right)
\]

where \( F^{\alpha \beta} \) satisfies Maxwell’s equations:

\[
F_{[\alpha \beta \gamma]} = 0, \quad F^{\mu \nu, \nu} = 4\pi j^\mu.
\]

Here \( j^\mu \) is the current 4-vector, \( \rho \) and \( p \) are the energy density and the isotropic pressure of matter measured in its rest frame, respectively; \( \rho \) and \( p \) are expressed in (geometrized) units of length\(^{-2}\).

In the spherically symmetric case the only non-vanishing component of Maxwell’s tensor is (Bekenstein 1971)

\[
F^{tr} = e^{-(\lambda + \eta)/2} \frac{Q(r, t)}{r^2}
\]

where

\[
Q(r, t) = \int_0^r e^{(\lambda + \eta)/2} 4\pi r^2 j^t \, dr
\]

is the total electric charge within a sphere of radius \( r \) at time \( t \); it then follows that

\[
F^{\alpha \beta} F_{\alpha \beta} = -\frac{2Q^2}{r^4}
\]

where charge is expressed in units of length. If we require that the configuration be static, then \( u^\mu = e^{-\eta/2} \delta^\mu_t \), hence Einstein’s equations which one needs to solve for the interior metric, take the form

\[
e^{-\lambda} \left( \frac{\lambda'}{r} - \frac{1}{r^2} \right) + \frac{1}{r^2} = \frac{Q^2}{r^4} + 8\pi \rho, \quad (2)
\]

\[
e^{-\eta} \left( \frac{\eta'}{r} + \frac{1}{r^2} \right) + \frac{1}{r^2} = \frac{Q^2}{r^4} - 8\pi p \quad (3)
\]

where a prime denotes a derivative with respect to \( r \).

Let the sphere be incompressible with a constant matter–energy density \( \rho_m \) and the charge distribution within the sphere be given by

\[
Q(r) = Q_0 \left( \frac{r}{R} \right)^n
\]

where
where \( n \) is a constant parameter, \( R \) is the coordinate radius of the boundary and \( Q_0 \) is the total charge. Integrating equation (2) with respect to coordinate radius \( r \), one obtains

\[
e^{-\lambda} = 1 - \frac{2}{3} \pi n_0 r^2 - \frac{Q_0^2}{(2n-1)R^{2n}}.
\]

Combining this result with (1) and (4), we see that the charge density is given by:

\[
e^{\eta/2} = \frac{Q_0 n r^n}{\sqrt{4 \pi R^n e^{\lambda/2}}} = \frac{Q_0 n r^n}{\sqrt{4 \pi R^n e^{\lambda/2}}},
\]

then we have \( n \geq 3 \) which prevents the divergence of charge density at the centre.

Introducing scale parameters such as

\[
p_0 = \frac{1}{3} \rho_m, \quad r_0 = \left( \frac{3}{4 \pi \rho_m} \right)^{1/2}
\]

and setting

\[
Y \equiv \frac{p}{p_0}, \quad \tilde{\xi} \equiv \frac{r}{r_0}
\]

the Oppenheimer–Wolkoff equation and equation (5) for such a sphere read, respectively, (de Felice et al 1995):

\[
\frac{dY}{d\tilde{\xi}} = n\tilde{\xi}^{2n-3} - (Y + 3) \frac{(Y + 1)\tilde{\xi} - [(n - 1)/(2n - 1)]\alpha \tilde{\xi}^{2n-3}}{\tilde{\xi}^2 - [\alpha/(2n - 1)]\tilde{\xi}^{2(2n-1)}},
\]

\[
e^{-\lambda} = 1 - 2\tilde{\xi}^2 - \frac{\alpha}{(2n - 1)}\tilde{\xi}^{2(2n-1)},
\]

where

\[
\alpha = \left( \frac{Q_0 n^{n-1}}{R^n} \right)^2.
\]

An obvious condition is \( \alpha > 0 \). Now we see that the set of equations (3), (8) and (9) is complete.

The boundary of the physical configuration is identified by the conditions

\[
Y(\tilde{\xi}_b) = 0, \quad e^\eta(\tilde{\xi}_b) = e^{-\lambda}(\tilde{\xi}_b) = 1 - 2\tilde{\xi}_b^2 - \frac{\alpha}{2n - 1}\tilde{\xi}_b^{2(2n-1)},
\]

where \( \tilde{\xi}_b = R/r_0 \). Then, for a given set of \( (\alpha, \tilde{\xi}_b, n) \), one can solve equations (3), (8) and (9) and obtain the structure of the corresponding configuration. (Note that when one integrates equation (8) numerically from its boundary inward, pressure \( Y \) may diverge before \( \tilde{\xi} \) reaches zero for some values of \( (\alpha, \tilde{\xi}_b, n) \). Since there is a singularity in such a configuration, it is physically meaningless. However, we still use the parameter \( (\alpha, \tilde{\xi}_b, n) \) to represent such solutions.) However, not all of the solutions are acceptable, two basic requirements of a regular static configuration are:

(a) that there be no singularity inside the sphere; and
(b) that the radius of the boundary be larger than the external horizon size of the black hole with the same mass and charge.

For a given \( n \), each point in the \((\tilde{\xi}_b, \alpha)\)-plane represents a solution. In figures 1–3 we plot the functions which allow one to single out the points which correspond to physical configurations.
Let $M$ be the total mass–energy of the solution, then the relations between $\bar{\xi}_b$ and the other parameters are

$$\frac{R}{M} = \frac{1}{\bar{\xi}_b^2 \left[ 1 + \bar{\xi}_b^{2n-2} \left( \frac{n\alpha}{2n-1} \right) \right]}, \quad (11)$$

$$\frac{Q_0}{M} = \sqrt{\alpha} \frac{1}{\bar{\xi}_b^{3-n} \left[ 1 + \bar{\xi}_b^{2n-2} \left( \frac{n\alpha}{2n-1} \right) \right]}, \quad (12)$$

One can show that $R/M$ changes monotonically along the curve $Q_0/M = \text{constant}$. Let $Q_0 = M$, then (12) gives the curve $b$ where $Q_0 = M$. Inside (figure 1, for $n > 3$) or below (figure 2, for $n = 3$) this curve, $Q_0 > M$. 

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**Figure 1.** Configurations in the parameter space for $n = 5$.

**Figure 2.** Configurations in the parameter space for $n = 3$. 

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| a | R=R | b | Q=0 | c | Y(0)=inf | d | γ=4 | e | γ=5/3 |
|---|-----|---|-----|---|---------|---|-----|---|-------|
| n=5 | n=3 | n=3 | n=3 | n=3 | n=3 | n=3 | n=3 | n=3 | n=3 |
The horizon sizes for the charged black hole are defined as the solutions of the equation $e^\lambda(R) = 0$ and the results are $R_\pm = M \pm \sqrt{M^2 - Q_0^2}$ for a given $M$ and $Q_0$. Let $e^\lambda(\bar{\xi}_b) = 0$, equation (9) gives curve $a$ where $R = R_+$. Above this curve, we have $R < R_+$, so the solutions are unacceptable. We notice that this curve is tangential to the curve $Q_0 = M$ at the point $\bar{\xi}_b^2 = (n - 1)/(2n - 1)$, $\alpha = [(n - 1)/(2n - 1)]^{1-n}$. On the left-hand side of this point, curve $a$ represents configurations with their boundary at the external horizon. Consequently, physical configurations must lie below this curve.

For given $\alpha$ and $\bar{\xi}_b^2$, one can solve equation (8) numerically to obtain $Y(\bar{\xi})$. We find that the central pressure $Y(0)$ will diverge as $\bar{\xi}_b^2$ increases to some value for fixed $\alpha < [(n - 1)/(2n - 1)]^{1-n}$, then we obtain the curve $c$, where $Y(0) = \inf$, which approaches the tangential point of curves $a$ and $b$. Above this curve, there are pressure singularities inside the configurations. So the regular static configurations must lie below this curve. (Curves $d$ and $e$ are for later use.)

We see from the figures that, for these special models, no regular solution exists which describes a compact charged sphere having $Q_0 < M$ and radius $R$ arbitrarily close to $R_+$ since any such solution would lie above curve $c$ where a pressure singularity emerges inside the sphere. A more general result is assured by the following:

**Theorem.** If the total electric charge of a static perfect fluid ball is smaller than its total mass, then there is no regular static configuration having a radius arbitrarily close to the size of the external horizon (Yu et al 1999).

As stated in de Felice et al (1995), the curves in figures 1–3 show that a finite region in the parameter space exists, which is the region between curves $b$ and $c$, where regular solutions describing static charged balls of matter may be found with a compactness arbitrarily close to the critical one for transition to a black-hole state, namely with $R \to R_+$ and $Q_0 \to M$.

It is of interest to study the behaviour of the pressure inside some of these configurations as a function of parameter $\alpha$. For a fixed $n$, this parameter gives a measure of the total charge...
Figure 4. The pressure of the configurations for \( n = 3 \).

within the body, so we show in figure 4 plots of the internal pressure as a function of radius \( \xi \), for \( n = 3 \), \( \xi_b^2 = 0.36 \) and increasing values of \( \alpha \) (the corresponding configurations are indicated by ‘+’ in figure 2). It is interesting to see how charge contributes to a decrease of the central pressure, a property which justifies the very existence of regular solutions describing compact balls of matter.

Regular solutions also exist inside (for \( n > 3 \)) or below (for \( n = 3 \)) curve \( b \) in figures 1 and 2, where \( Q_0 > M \). However, as appears from figure 4, the central pressure is negative in these solutions.

In what follows we shall discuss the stability of these solutions against adiabatic radial perturbations.

3. Stability

We will consider small radial perturbations for the charged, perfect fluid balls, then any fluid element which was at \( r \) in the unperturbed configuration is displaced to \( r + \xi(r, t) \) in the perturbed one where \( \xi \ll r \). In what follows we shall use the subscript ‘\( i \)’ to denote the unperturbed configuration, then we have (Misner et al 1973)

\[
\begin{align*}
\eta(r, t) &= \eta_i(r) + \delta\eta(r, t) \\
\lambda(r, t) &= \lambda_i(r) + \delta\lambda(r, t) \\
p(r, t) &= p_i(r) + \delta p(r, t) \\
\rho(r, t) &= \rho_i(r) + \delta\rho(r, t) \\
\bar{n}(r, t) &= \bar{n}_i(r) + \delta\bar{n}(r, t)
\end{align*}
\]

(13)

where \( \bar{n} \) is the baryon number density; the variations are small and such that \( \delta a/a_i \sim \xi/r \), \( a \) being any of the above parameters. Furthermore, we assume that the charge distribution in the vibrating configuration remains unchanged, namely \( Q(r + \xi(r, t)) = Q_i(r) \), meaning that
there are no electric currents for the comoving observer. To first order in the variations, the Lagrangian perturbations read:

\[
\begin{align*}
\Delta p(r, t) &= p(r + \xi, t) - p_i(r) \\
&= \delta p(r, t) + p'_i \xi \\
\Delta \rho(r, t) &= \delta \rho(r, t) + \rho'_i \xi \\
\Delta \bar{n}(r, t) &= \delta \bar{n}(r, t) + \bar{n}'_i \xi \\
\Delta Q(r, t) &= 0.
\end{align*}
\]

(14)

From baryon conservation we have

\[
\frac{d}{d\tau} \bar{n} = \bar{n}' - \bar{n} \bar{u}^\mu \bar{u}_\mu.
\]

(15)

Since \(u'/u' = \ddot{\xi}\), the dot denoting differentiation with respect to the coordinate time \(t\) and \(u^\mu u_\mu = -1\), in the perturbations we have to first order:

\[
\begin{align*}
u' &= e^{-\lambda/2}\left(1 - \frac{1}{2} \delta \eta\right) \\
u' &= e^{-\lambda/2} \ddot{\xi}.
\end{align*}
\]

(16)

Using the relation

\[
u^\mu \nu^\nu = \left(-g\right)^{-1/2} \left[\left(-g\right)^{1/2} u^\mu\right]_\mu
\]

and integrating equation (15) with respect to coordinate time (first-order analysis), we deduce

\[
\Delta \bar{n} = -\bar{n}_i \left[r^{-2} e^{-\lambda/2} \left(r^2 e^{\lambda/2} \bar{\xi}' + \frac{1}{2} \delta \lambda\right) - \bar{\xi} p'_i\right].
\]

(17)

For adiabatic variations we have

\[
\frac{\Delta p}{\Delta \bar{n}} = \frac{p_i}{\bar{n}_i} \gamma
\]

where \(\gamma\) is the adiabatic index, then from (17) and (14) we obtain

\[
\delta p = -\gamma p_i \left[r^{-2} e^{-\lambda/2} \left(r^2 e^{\lambda/2} \bar{\xi}' + \frac{1}{2} \delta \lambda\right) - \bar{\xi} p'_i\right].
\]

(18)

From Einstein’s equations, we obtain

\[
\begin{align*}
e^{-\lambda/2} \frac{r' \eta'_i + 1}{r^2} &+ \delta \lambda + e^{-\lambda/2} \left(\delta \eta\right)' = \frac{2 Q_i' Q_i^*}{r^4} \ddot{\xi} + 8\pi \delta p \\
e^{-\lambda/2} \left(\delta \lambda\right)' &= -8\pi \left(p_i + p_i\right) \ddot{\xi} = -\frac{e^{-\lambda/2}}{r} \left(\lambda' + \eta'\right) \ddot{\xi}.
\end{align*}
\]

(19)

(20)

Thus, integrating (20) with respect to coordinate time and choosing the integration constant so that \(\delta \lambda = 0\) when \(\bar{\xi} = 0\), we obtain

\[
\delta \lambda = -e^{\lambda/2} 8\pi r (p_i + p_i) \bar{\xi} = -(\lambda' + \eta') \bar{\xi}.
\]

(21)

Using (21), (18) and (3) to eliminate, respectively, \(\delta \lambda\), \(\delta p\) and \(\eta'\) in (19), we have

\[
\left(\delta \eta\right)'' = e^{\lambda/2} \left(\frac{2 Q_i' Q_i^*}{r^4} \frac{2}{\dot{\xi}^2} + 8\pi \delta p\right) + \frac{r' \eta_i' + 1}{r} \delta \lambda
\]

\[
= -8\pi \gamma p_i \frac{1}{2} e^{\lambda/2} \bar{\xi}' \left(r^2 e^{-\lambda/2} \frac{\dot{\xi}'}{\bar{\xi}'}\right) + 8\pi \left(p_i' r - (p_i + p_i)\right) e^{\lambda/2} \bar{\xi} = -\frac{2 Q_i' Q_i^*}{r^3} \bar{\xi} e^{\lambda/2}.
\]

(22)
If we project the identity
\[ T_{\nu ; \mu} \equiv 0 \]
(23)
parallel to the matter flow, namely
\[ u^\mu T_{\nu ; \mu} = 0, \]
we obtain, from the chosen forms of \( u^\mu \) and \( T_{\mu \nu} \):
\[ \frac{d \rho}{d \tau} + \left( \rho + p \right) \frac{d \bar{n}}{d \tau} - \frac{Q}{4 \pi r^2} \frac{d Q}{d \tau} = 0. \]
(24)
Integrating (24) with respect to time and using (14) and (17) to remove \( \Delta Q \), \( \Delta \rho \) and \( \Delta \bar{n} \) we deduce after some algebra:
\[ \delta \rho = -\left( \rho_i + p_i \right) \left( -d \rho + \frac{p}{r} \right) - \frac{Q}{4 \pi r^2} \frac{d Q}{d \tau}. \]
(25)
If we now project (23) transversely to the matter flow, namely
\[ h^{\mu \nu} T_{\sigma ; \mu} \equiv 0, \]
where
\[ h^{\mu \nu} = \delta^{\mu \nu} + u^\mu u^\nu, \]
we obtain
\[ \left( \rho + p \right) u^\mu u^\nu \frac{d \rho}{d \tau} = -p, \]
(26)
The \( r \)-component of (26) yields
\[ \left( \rho_i + p_i \right) c^k \eta_i \xi = \left( \delta p \right)' - \left( \delta \rho + \delta p \right) \frac{1}{2} \eta_i' \]
(27)
while the \( t \)-component is trivial and gives
\[ p_i' = \frac{Q_4 Q_i'}{4 \pi r^4} - \left( \rho_i + p_i \right) \frac{1}{2} \eta_i'. \]
By using the initial value equations (18), (25) and (22) to re-express \( \delta p \), \( \delta \rho \) and \( \delta \eta \)' in terms of \( \xi \) in equation (27), we obtain
\[ \left( \rho_i + p_i \right) c^k \eta_i \xi = \left[ \left( \rho_i + p_i \right) \left( 1 + \gamma \right) - \frac{Q_4 Q_i'}{4 \pi r^4} \right] \frac{1}{2} \eta_i' \]
\[ + \left( \rho_i + p_i \right) \left( \frac{4 \pi \gamma p_i}{r^3} c^k \eta_i \xi + 4 \left( \rho_i + p_i \right) c^k \eta_i \xi \right) \]
\[ - \frac{4 \pi p_i r^3 c^k \eta_i \xi}{r^3} + \frac{Q_4 Q_i'}{4 \pi r^4} \frac{1}{2} \eta_i' \]
\[ = \left[ \left( \rho_i + p_i \right) \left( 1 + \gamma \right) - \frac{Q_4 Q_i'}{4 \pi r^4} \right] \frac{1}{2} \eta_i' \]
(28)
where \( \eta_i = r^2 c^k \eta_i \xi \) is a renormalized displacement function. From the above, the physically acceptable solutions of the dynamic equation must satisfy the following boundary conditions:
\[ \frac{\xi}{r} \text{ finite or zero as } r \to 0 \]
(29)
\[ \Delta p = -\gamma \rho_i r^2 c^k / (r^2 c^k) \xi \to 0 \text{ as } r \to R \]
(30)
where \( R \) is the radius of the ball.
Assume that $\zeta(r, t)$ has a sinusoidal time dependence:

$$\zeta(r, t) = \zeta(r) e^{-i\omega t}$$

(31)

hence the dynamic equation and the boundary conditions reduce to an eigenvalue problem for the angular frequency $\omega$ and amplitude $\zeta(r)$:

$$(F \zeta')' + (H + \omega^2 W) \zeta = 0$$

(32)

so that $\zeta / r^3$ is finite or zero as $r \to 0$ and $\gamma p_i r^{-2} e^{\eta/2} \zeta'(r) \to 0$ as $r \to R$, where

$$F = \gamma p_i r^{-2} e^{\lambda_i/2 + \eta/2}$$

(33)

$$H = e^{\lambda_i/2 + \eta/2} \left[ \frac{(p_i' - (Q_i r')/(4\pi r^3))^2}{r^2 (\rho_i + p_i)} + \frac{Q_i Q_i'}{4\pi r^6} - p_i' \right] - \frac{8\pi p_i (\rho_i + p_i)}{\pi r^7} \left[ \frac{Q_i Q_i''}{4\pi r^6} - \frac{Q_i Q_i'}{\pi r^7} \right]$$

(34)

$$W = (\rho_i + p_i) r^{-2} e^{\lambda_i/2 + \eta/2}.$$  

(35)

By using the variational principle (Mathews and Walker 1970), one obtains

$$\omega^2 = \text{extreme of} \int_0^R (F \zeta'^2 - H \zeta^2) \, dr \int_0^R W \zeta^2 \, dr$$

(36)

so, the configuration is stable against adiabatic radial perturbations only if

$$\int_0^R (F \zeta'^2 - H \zeta^2) \, dr > 0$$

(37)

for any $\zeta$ which satisfies the boundary conditions (29) and (30).

Using the simplest trial function $\zeta \propto r^3$ (Chandrasekhar 1964) and assuming that the adiabatic index $\gamma$ was constant throughout the spheres, we calculated the stability of such configurations. The domain of stable solutions in the parameter space is shown in figures 1 and 2: curve $d$ for $\gamma = 4$, the configurations are unstable above this curve, curve $e$ for $\gamma = \frac{5}{3}$, the configurations are unstable above this curve, and the stability curves approach curve $c$ as the adiabatic index approaches infinity. Evidently this limit corresponds to perfect rigidity.

With the inclusion of these curves, figures 1 and 2 allow us to identify a region on the parameter space which corresponds to solutions which are physically significant, being regular and stable against adiabatic radial perturbations. Since these very compact configurations are electrically charged, they are probably not describing astrophysical objects but, perhaps, classic models of massive elementary particles. In this case it would be desirable to extend our analysis to a more detailed modelling of the interior, since the adiabatic index depends critically on the equation of state (Akmal et al 1998).

4. Regular solutions and regular embeddings

Given a Riemannian manifold $M$ with Lorentzian metric $g$, it may be heuristically helpful to find embedding diagrams of any subset of $M$ into Euclidean space. More specifically, one has to find a surface in a 3-Euclidean space which is isometric to a given 2-surface of $M$ (Bini and Jantzen 1999). This problem does not always have a solution. It has been argued (Kristiansson et al 1998) that the limiting condition for regular embedding of spacelike sections
of the vacuum Schwarzschild and Reissner–Nordström solutions, conformally reduced by the factor $\Omega = g_{00}$, does coincide with the limiting condition for their regular, static, spherically symmetric internal solutions. We shall show here that this is not true.

Consider the Reissner–Nordström spacetime solution:

$$\text{d}t^2 = -\left(1 - \frac{2m}{r} + \frac{Q_0^2}{r^2}\right) \text{d}r^2 + \left(1 - \frac{2m}{r} + \frac{Q_0^2}{r^2}\right)^{-1} \text{d}r^2 + r^2 (\text{d}\theta^2 + \sin^2 \theta \text{d}\phi^2)$$

(38)

then consider the conformal $t = \text{constant}$ spacelike subspace:

$$\text{d}\Sigma^2 = \left(1 - \frac{2m}{r} + \frac{Q_0^2}{r^2}\right)^{-2} \text{d}r^2 + \left(1 - \frac{2m}{r} + \frac{Q_0^2}{r^2}\right)^{-1} r^2 (\text{d}\theta^2 + \sin^2 \theta \text{d}\phi^2).$$

(39)

The two-dimensional surface $\theta = \pi/2$, namely

$$\text{d}\sigma^2 = \left(1 - \frac{2m}{r} + \frac{Q_0^2}{r^2}\right)^{-2} \text{d}r^2 + \left(1 - \frac{2m}{r} + \frac{Q_0^2}{r^2}\right)^{-1} r^2 \text{d}\phi^2$$

(40)

is matched to a surface $z = z(R)$ in the three-dimensional Euclidean space:

$$\text{d}\sigma_2^2 = \text{d}R^2 + R^2 \text{d}\phi^2 + \text{d}z^2$$

$$= \left[1 + \left(\frac{\text{d}z}{\text{d}R}\right)^2\right] \text{d}R^2 + R^2 \text{d}\phi^2$$

(41)

where $R, \phi, z$ are cylindrical coordinates. Comparing (41) with (40) we obtain the condition for regular embedding as

$$\left(\frac{\text{d}z}{\text{d}R}\right)^2 = \left(1 - \frac{2m}{r} + \frac{Q_0^2}{r^2}\right)\left(1 - \frac{3m}{r} + \frac{2Q_0^2}{r^2}\right)^{-2} - 1 \geq 0.$$ 

(42)

From this we deduce, for each $M$ and $Q_0$, the condition for regular embedding as $r \geq \tilde{r}$ where $\tilde{r}$ is the solution of the equation

$$1 - \frac{2m}{\tilde{r}} + \frac{Q_0^2}{\tilde{r}^2} = \left(1 - \frac{3m}{\tilde{r}} + \frac{2Q_0^2}{\tilde{r}^2}\right)^2.$$ 

(43)

If $Q_0 = 0$, we easily deduce $\tilde{r} = 9M/4$, a well known limit for regular internally static, electrically neutral, spherically symmetric and perfect fluid solutions (Buchdahl 1959). If $Q_0 \neq 0$, there are several internal solutions which match the external Reissner–Nordström one. Here, of the various solutions we shall consider those with $n = 3$ and $n = \infty$. The plots of the limit of regularity for these solutions, namely where the central pressure diverges, are compared in the $(r-Q_0)$-plane and shown in figure 5 where $r$ is the coordinate radius of the sphere’s boundary. Although the two curves coincide at $r = 9M/4$ when $Q_0 = 0$ and $r = M$ when $Q_0 = M$, as expected, they differ from each other everywhere else, curve $n = \infty$ being at larger radii than curve $n = 3$ for each value of $Q_0$. Now it happens that the condition for infinite central pressure in the $n = \infty$ case (a perfect conductor) can be expressed analytically through solving the Oppenheimer–Wolkoff equation for conductive incompressible perfect fluids and reads

$$\sqrt{1 - \frac{2m}{\tilde{r}} + \frac{Q_0^2}{\tilde{r}^2}} = \frac{\tilde{r}m - Q_0^2}{3\tilde{r}m - 2Q_0^2}.$$ 

(44)

This coincides with condition (43) for regular embedding. It is then clear that regular solutions for $n = 3$ exist which have radii smaller than the corresponding limit of regular embedding; this is sufficient to reject the conjecture.
5. Conclusions

The use of plots in figures 1–5 highlights the result that regular solutions of Einstein’s equations exist describing charged spheres arbitrarily close to a black hole. These solutions fill a region in a parameter space which is thin but finite so it is not a set of measure zero. However, not all these solutions are stable against adiabatic radial perturbations. We have made a perturbation analysis and have established a stability criterion given by relation (37). We have deduced the stability curves for configurations with adiabatic indices $\gamma = \frac{5}{3}$ and $\gamma = 4$ (respectively, curves e and d in figures 1 and 2), assuming that $\gamma$ was constant throughout the sphere. We see that, for moderate values of $\gamma$, the limits of stability prevent stable compact charges from being arbitrarily close to a black-hole state but also from having a total charge $Q_0$ arbitrarily close to their mass. These situations would be possible if the adiabatic index diverged; in this case, in fact, the stability curve approaches curve c which, we recall, is the limit of regularity for our solutions. Clearly, a diverging $\gamma$ corresponds to a complete rigidity while, in a less extreme case, $\gamma$ would critically depend on the equation of state (Akmal et al 1998).

Although our solutions can be thought of as being classic (non-quantum) models of charged particles, it would be interesting to find solutions of Einstein’s equations which describe charged spheres with a more detailed physical description of their interior.

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References

Akmal A, Pandharipande V R and Ravenhall D G 1998 Phys. Rev. C 58 1804
Bekenstein J D 1971 Phys. Rev. D4 2185
Bini D and Jantzen R T 1999 Circular orbits in Kerr space times: embedding diagrams Int. Rep. CNR-Napoli
Buchdahl H A 1959 Phys. Rev. 116 1027
Chandrasekhar S 1964 Phys. Rev. Lett. 12 114
de Felice F and Clarke C J S 1990 Relativity on Curved Manifolds (Cambridge: Cambridge University Press)
de Felice F, Yu Yunqiang and Fang Jing 1995 Mon. Not. R. Astron. Soc. 277 L17
Kristiansson S, Sonego S and Abramowicz M 1998 Gen. Rel. Grav. 30 275
Mathews J and Walker R L 1970 Mathematical Methods of Physics (Menlo Park, CA: Benjamin-Cummings)
Misner C W, Thorne K S and Wheeler J A 1973 Gravitation (San Francisco, CA: Freeman)
Yu Yunqiang and Liu Siming 1999 Relativistic charged balls Commun. Theor. Phys. at press