An Infinite Series of Perfect Quadratic Forms and Big Delaunay Simplexes in $\mathbb{Z}^n$.

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Abstract

George Voronoi (1908-09) introduced two important reduction methods for positive quadratic forms: the reduction with perfect forms, and the reduction with $L$-type domains.

A form is perfect if can be reconstructed from all representations of its arithmetic minimum. Two forms have the same $L$-type if Delaunay tilings of their lattices are affinely equivalent. Delaunay (1937-38) asked about possible relative volumes of lattice Delaunay simplexes. We construct an infinite series of Delaunay simplexes of relative volume $n-3$, the best known as of now. This series gives rise to a new infinite series of perfect forms $TF_n$ with remarkable properties: e.g. $TF_5 = D_5$, $TF_6 = E_6^*$, $TF_7 = \varphi_{15}^*$; for all $n$ the domain of $TF_n$ is adjacent to the domain of the 2-nd perfect form $D_n$. Perfect form $TF_n$ is a direct $n$-dimensional generalization of Korkine and Zolotareff’s 3-rd perfect form $\phi 2^5$ in 5 variables. It is likely that this form is equivalent to Anzin’s (1991) form $h_n$.

Keywords: $Sym_n(\mathbb{R})$, Quadratic Form, Perfect Form, Point Lattices, Voronoi Reduction of the 1st and 2nd types, Delaunay Tiling ($L$-partition), $L$-type, Repartitioning Complex, $E_6$, $E_6^* = E_6^3$, $D_n$, Gosset Polytope $2_{21}$, Dual Systems of Integral Vectors

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1 Introduction and main result

Positive quadratic forms (referred to as PQFs) in $n$ indeterminate form a closed cone $\mathfrak{P}(n)$ of dimension $N = \frac{n(n+1)}{2}$ in $\mathbb{R}^N$, and this cone is the main object of study in our paper. The interior of $\mathfrak{P}(n)$ consists of positive definite forms of rank $n$. We abbreviate positive definite forms as PDQFs. PDQFs serve as algebraic representations of point lattices. There is a one-to-one correspondence between isometry classes of
n-lattices and integral equivalence classes (i.e. with respect to \(GL(n, \mathbb{Z})\)-conjugation) of PDQFs in \(n\) indeterminates. For basic results of the theory of lattices and PQFs and their applications see Ryshkov and Baranovskii (1978), Gruber and Lekkerkerker (1987), Erdős, Gruber and Hammer (1989), Conway and Sloane (1999).

\(GL(n, \mathbb{Z})\) acts pointwise on the space of quadratic forms \(Sym(n, \mathbb{R}) \cong \mathbb{R}^N\). A polyhedral reduction partition \(R\) of \(\mathcal{P}(n)\) is a partition of this cone into open convex polyhedral cones such that:

**Definition 1.1**

1. it is invariant with respect to \(GL(n, \mathbb{Z})\);
2. there are finitely many inequivalent cones in this partition;
3. for each cone \(C\) of \(R\) and any PQF \(f\) in \(n\) indeterminates, \(f\) can be \(GL(n, \mathbb{Z})\)-equivalent to at most finitely many forms lying in \(C\).

The partition into perfect cones and the \(L\)-type partition (also referred to as the Voronoi partition of the 2nd kind, or the partition into Voronoi reduction domains) are important polyhedral reduction partitions of \(\mathcal{P}(n)\). (Our usage of term *domain* is lax; it should be clear from the context whether we mean the whole arithmetic class, or just one element of this class.) These partitions have been intensively studied in geometry of numbers since times of Korkin, Zolotareff (1873) and Voronoi (1908-1909), and more recently in combinatorics (e.g. Deza et al. (1997)), and algebraic geometry (e.g. Alexeev (1999a,b)). In most previous works (e.g. Voronoi (1908,1909), Ryshkov, Baranovskii (1976), Dickson (1972)) the \(L\)-type partition of \(P(n)\), or sub-cones of \(P(n)\), was constructed by refining the perfect partition. It is not an exaggeration to say that in almost any systematic study, except for Engels’ computational investigations, \(L\)-types were approached via perfect forms. For example, Voronoi started classifying 4-dimensional \(L\)-types by analyzing the Delaunay \((L\)-tilings of forms lying in the 1st \((A_n)\) and 2nd \((D_n)\) perfect domains. The same route was followed by Ryshkov and Baranovskii (1976). It was widely believed that the \(L\)-type partition is the refinement of the perfect partition, i.e. each convex cone of the perfect partition is the union of finitely many convex cones from the \(L\)-type partition. This conjecture is implicit in Voronoi’s memoirs (1908-1909), and explicit in Dickson (1972), where he showed that the first perfect domain is the only perfect domain which coincided with an \(L\)-type domain. More information on this refinement conjecture and its failure for \(\mathcal{P}(n) \subset Sym(6, \mathbb{R})\) can be found in our note ”Voronoi-Dickson hypothesis on perfect forms ands \(L\)-types” published in this volume.

In our paper we continue the above-mentioned line of research on the relashionship between these two reduction partitions. However, we go in the opposite direction. We construct an arithmetic class of perfect forms from an arithmetic class of \(L\)-types.

**Remark 1.2** Since there is a lot of overlap in references between this paper and our paper on the Voronoi-Dickson hypothesis, some of the references for this paper should be found in the bibliography for the other paper. Such references are quoted in italics, e.g. Baranovski (1991).
2 Perfect and $L$-type partitions

2.1 $L$-types

**Definition 2.1** Let $L$ be a lattice in $\mathbb{R}^n$. A convex polyhedron $P$ in $\mathbb{R}^n$ is called a Delaunay cell of $L$ with respect to a positive quadratic form $f(x,x)$ if:

1. for each face $F$ of $P$ we have $\text{conv}(L \cap F) = F$;

2. there is a quadric circumscribed about $P$, called the empty ellipsoid of $P$ when $f(x,x)$ is positive definite, whose quadratic form is $f(x,x)$ (in case rank $f < n$, this quadric is an elliptic cilinder);

3. no points of $L$ lie inside the quadric circumscribed about $P$.

When $f = \sum_{i=1}^n x_i^2$, our definition coincides with the classical definition of Delaunay cell in $\mathbb{R}^n$. Delaunay cells form a convex face-to-face tiling of $L$ that is uniquely defined by $L$ (Delaunay, 1937). Two Delaunay cells are called homologous is they can be mapped to each other with a composition of a lattice translation and a central inversion with respect to a lattice point.

**Definition 2.2** PQFs $f_1$ and $f_2$ belong to the same convex $L$-domain if the Delaunay tilings of $\mathbb{Z}^n$ with respect to $f_1$ and $f_2$ are identical. $f_1$ and $f_2$ belong to the same $L$-type if these tilings are equivalent with respect to $GL(n,\mathbb{Z})$.

The following proposition establishes the equivalence between the Delaunay’s definition of $L$-equivalence for lattices and the notion of $L$-equivalence for arbitrary PQFs, which is introduced above.

**Proposition 2.3** Positive definite forms $f$ and $g$ belong to the same $L$-type if the corresponding lattices belong to the same $L$-type with respect to the form $\sum_{i=1}^n x_i^2$.

**Theorem 2.4** (Voronoi) The partition of $\mathcal{P}(n)$ into $L$-types is a reduction partition. Moreover, it is face-to-face.

The notions of Delauny tiling and $L$-type are important in the study of extremal and group-theoretic properties of lattices (see also ”Voronoi-Dickson Hypothesis...”). Barnes and Dickson (1967, 1968) and, later, in a geometric form, Delaunay et al. (1969, 1970) proved the following

**Theorem 2.5** The closure of any $N$-dimensional convex $L$-domain contains at most one local minimum of the sphere covering density. The group of $GL(n,\mathbb{Z})$-automorphisms of the domain maps this form to itself.
Using this approach, Delaunay, Ryshkov and Baranovskii (1963, 1976) found the best lattice coverings in $\mathbb{E}^4$ and $\mathbb{E}^5$. The theory of $L$-types also has numerous connections to combinatorics and, in particular, to cuts, hypermetrics, and regular graphs (see Deza et al. (1997)). Recently, V. Alexeev (1999a,b) found exciting connections between compactifications of moduli spaces of principally polarized abelian varieties and $L$-types.

2.2 Perfect cones

The $L$-type partition of $\mathfrak{P}(n)$ is closely related to the theory of perfect forms originated by Korkine and Zolotareff (1873). Let $f(x, x)$ be a PDQF. The arithmetic minimum of $f(x, x)$ is the minimum of this form on $\mathbb{Z}^n$. The integral vectors on which this minimum is attained are called the representations of the minimum, or the minimal vectors of $f(x, x)$: these vectors have the minimal length among all vectors of $\mathbb{Z}^n$ when $f(x, x)$ is used as the metrical form. Form $f(x, x)$ is called perfect if it can be reconstructed up to scale from all representations of its arithmetic minimum. In other words, a form $f(x, x)$ with the arithmetic minimum $m$ and the set of minimal vectors $\{v_k | k = 1, ..., 2s\}$ is perfect if the system

$$\sum_{i,j=1}^{n} a_{ij} v_k^i v_k^j = m,$$

where $k = 1, ..., 2s$, has a unique solution $(a_{ij})$ in $\text{Sym}(n, \mathbb{R}) \cong \mathbb{R}^N$ (indeed, uniqueness requires at least $n(n + 1)$ minimal vectors).

**Definition 2.6** PQFs $f_1$ and $f_2$ belong to the same cone of the perfect partition if they both can be written as strictly positive linear combinations of some subset of minimal vectors of a perfect form $\phi$. $f_1$ and $f_2$ belong to the same perfect type if there is $f'_1$, equivalent to $f_1$, such that $f'_1$ and $f_2$ belong to the same cone of the perfect partition.

**Theorem 2.7** (Voronoi) The partition of $\mathfrak{P}(n)$ into perfect domains is a reduction partition. Moreover, it is face-to-face. Each 1-dimensional cone of this partition lies on $\partial \mathfrak{P}(n)$.

Perfect forms play an important role in lattice sphere packings. Voronoi’s theorem (1908) says that if a form is extreme—i.e., a maximum of the packing density—it must also be perfect (see Coxeter (1951), Conway, Sloane (1988) for the proof). The notion of eutactic form arises in the study of the dense lattice sphere packings and is directly related to the notion of perfect form. The reciprocal of $f(x, x)$ is a form whose Gramm matrix is the inverse of the Gramm matrix of $f(x, x)$. The dual form is normally denoted by $f^*(x, x)$. A form $f(x, x)$ is called eutactic if the dual form $f^*(x, x)$ can be written as $\sum_{k=1}^{s} \alpha_k (v_k \cdot x)^2$, where $\{v_k | k = 1, ..., s\}$ is the set of mutually non-collinear minimal vectors of $f(x, x)$, and $\alpha_k > 0$.

**Theorem 2.8** (Voronoi) A form $f(x, x)$ is a maximum of the sphere packing density if and only if $f(x, x)$ is perfect and eutactic.
Voronoi gave an algorithm finding all perfect domains for given \( n \). This algorithm is known as Voronoi’s reduction with perfect forms. For the computational analysis of his algorithm and its improvements see Martinet (1996). The perfect forms and the incidence graphs of perfect partitions of \( \mathfrak{P}(n) \) have have been completely described for \( n \leq 7 \).

2.3 Relationship between perfect domains and L-types: interpretable and non-interpretable perfect walls.

Voronoi (1908-09) proved that for \( n = 2, 3 \) the \( L \)-partition and the perfect partition of \( \mathfrak{P}(n) \) coincide. The perfect facet \( D_4 \) (the 2nd perfect form in 4 variables) exemplifies a new pattern in the relation of these partitions. Namely, the facet \( D_4 \) is decomposed into a number of simplicial \( L \)-type domains like a pie: this decomposition consists of the cones with apex at the affine center of this facet over the \((N - 2)\)-faces. These simplexes are \( L \)-type domains of two arithmetic types: type I is adjacent to the the perfect/\( L \)-type domain of \( A_4 \), type II is adjacent to an arithmetically equivalent \( L \)-type domain (also type II, indeed) from the \( L \)-subdivision of the adjacent \( D_4 \) domain (for details see Delaunay et al. (1963, 1968)).

Following the lead of Voronoi, Delaunay et al. proved that for \( n = 4 \) the tiling of \( \mathfrak{P}(n) \) with \( L \)-type domains refines the partition of this cone into perfect domains. Ryshkov and Baranovskii (1975) proved the refinement hypothesis for \( n = 5 \). See another our paper from this volume to learn why this hypothesis fails for \( n = 6 \). In cases where \( L \)-type domains refine perfect ones, the \( L \)-type is changing on each perfect wall. We call such perfect walls interpretable. It is not yet clear why some perfect walls are interpretable, while other, like the wall between domains of types \( E_6 \) and \( E_6^* \), are not.

Below we construct an infinite series of Delaunay polytopes \( R_n \) on \( n + 2 \) vertices in \( \mathbb{Z}^n \) (Theorem 3.2). We prove that they are repartitioning complexes. One of the two triangulations of this polytope has a Delaunay simplex of relative volume \( n - 3 \). This triangulation defines an \( N \)-dimensional \( L \)-type domain which is a subcone of a perfect domain \( TF_n \), that we describe in Theorem 5.1. Domain of type \( TF_n \) shares a wall with domain of type \( D_n \). All forms lying on this wall have the repartitioning complex \( R \) in its Delaunay tiling.

3 Fat Symplexes

It is well known that the Delaunay tiling of lattice \( E_6 \) consists of Gosset polytopes \( G \) (e.g. see Baranovskii 1991). For properties of \( G \)-tope see Coxeter (1973, 1995). All faces of \( G \)-tope are regular polytopes. Let \( S_4 \) be a 4-face of \( G \)-tope which is a common facet of two cross-polytopal facets. Since any pair of vertices of \( G \)-tope is either a diagonal or an edge of a cross-polytopal facet, there are only two vertices
of the polytope which do not have common edges with vertices of $S_4$. The volume of the convex hull of $S_4$ and these two “distant” vertices is 3 times the volume of a fundamental simplex of $E_6$. Using a computer program we checked that there are no other simplexes of volume 3, Delaunay or not, in $G$-tope (below we prove that all simplexes inscribed into $G$-tope are Delaunay). There are exactly 216 4-faces that serve as common facets of pairs of cross-polytopal facets and all of them are equivalent with respect to the group of the $G$-tope. Therefore, there are exactly 216 Delaunay simplexes of relative volume 3 (we will often omit the word *relative*) in $G$-tope. They are all equivalent with respect to the isometry group of $G$-tope. According to Ryshkov and Baranovskii (1998) there is only one arithmetic type of triple Delaunay simplexes in 6-lattices, and there are no Delaunay simplexes of volume greater than 3 in 6-dimensional lattices.

In an appropriate coordinate system the vertices (here the column-vectors) of this simplex have the following form.

$$
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -3
\end{pmatrix}
$$

There is a repartitioning complex $R_6$ one of whose triangulations includes the above simplex. The vertices of this repartitioning complex are the vertices of the above simplex plus the vertex $(0, 0, 0, 0, 0, 0, 1)^T$.

Delaunay (1937) asked about possible volumes of Delaunay simplexes. Ryshkov (1973) showed that in every dimension $2r + 1$ there is a lattice with a Delaunay simplex of relative volume $r$. Namely, Ryshkov proved that lattice $A_n^k$ for $n \geq 2k + 1$ has a Delaunay simplex of rel. volume $k$. Ryshkov also noticed that in the case of $A_n^k$ the existence of big Delaunay simplexes is closely related to another interesting phenomenon: for $n \geq 9$ perfect lattice $A_n^k$ is not generated by its shortest vectors. Earlier, Coxeter (1951) made a similar observation about the relevance of these two phenomena in case of $A_n^k$, but he did not know for sure if $A_n^k$ had such big simplexes.

We generalized the construction of the above simplex to the following series of simplexes of volume $n - 3$. Although, to our knowledge, this is the best infinite series of big Delaunay simplexes, in Leech lattice $\Lambda_{24}$ all Delaunay simplexes are non-fundamental, and the biggest of them has volume 20480. Haase and Ziegler (2000) showed that for $n > 3$ there are empty lattice simplexes of arbitrary large volume (not Delaunay, indeed). A trivial upper bound on the rel. volume of a Delaunay simplex is $\frac{n!}{2}$.
The corresponding repartitioning complex is obtained by adjoining vertex $(0, \ldots, 0, 1)^T$. In this paper we use a short-hand notation for $n$-vectors that have few distinct integral coordinates and for families of such vectors obtained from some $n$-vector by all circular permutations of selected subsets of its coordinates. Here are the rules:

1. $m^k$ stands for $k$ consecutive positions filled with $m$’s.

2. Square brackets $[a_1 \ldots a_n]$ are used to denote a vector (or type of vector) that can be obtained from this vectors by circular permutations in sequences of coordinates that are separated by commas and bordered on the sides by semicolons and/or brackets.

3. A family of vectors that are obtained from vector $[a_1 \ldots a_n] = (a_1 \ldots a_n)^T$ by all admissible (see 2) permutations is denoted by $[a_1 \ldots a_n]^{#}$, where $#$ is the number of such vectors; if $# = 1$, we omit $#$.

Example: $[1^{n-3}, 0^2, 3]^{(n-1)}$ stands for all vectors with 3 at the last entry, two 0’s and $n - 3$ 1’s among the first $n - 1$ coordinates.

To prove that the above simplex is a lattice Delaunay simplex we need the theory of $(0, 1)$-dual systems developed by Erdahl and Ryshkov (1990, 1991 a, b). Let $S$ be a set of integer vectors in $\mathbb{Z}^n$. The $(0,1)$-dual of $S$ is the set of all integer vectors in $\mathbb{Z}^n$ that have the scalar product of 0 or 1 with all vectors of $S$. We denote the $(0,1)$-dual of $S$ by $S^0$. Erdahl and Ryshkov (1990) showed that if the double dual of an integral simplex has only $n + 2$ points, then this simplex is a Delaunay simplex for some PDQFs.

**Theorem 3.1** (Erdahl, Ryshkov) Let $S$ be a set of vectors in $\mathbb{Z}^n$. If $(S^0)^0 \backslash [0^n]$ consists of $n + 1$ linearly independent vectors, then there is an $N$-dimensional cone of PDQFs for which $S$ is a Delaunay simplex in $\mathbb{Z}^n$.

Using the Erdahl-Ryshkov theorem we verify in the following proposition that our series is indeed a series of lattice Delaunay simplexes.

**Theorem 3.2** For any $n > 3$ there are lattices with Delaunay simplex of volume $n - 3$. 
Proof. Let $S_n$ be a simplex in $\mathbb{Z}^n$ whose vertices are the columns of the following matrix:

$$S_n = \begin{pmatrix} I_{n-1} & 1_{n-1} \\ 0_{n-1}^T & -(n-3) \end{pmatrix}.$$

Here $I_{n-1}$ is the $(n-1) \times (n-1)$ identity matrix, $1_{n-1}$ is the column $(n-1)$-vector of ones, and $0_{n-1}^T$ is the row $(n-1)$-vector of zeros. $S_n^0$ cannot have vectors with negative numbers in positions 1 through $(n-1)$, for $S_n$ contains an identity submatrix $I_{n-1}$ (we use $S_n$ to refer to both the simplex and its matrix). Since $S_n$ has a vector with $-(n-3)$ at the last coordinate, $S_n^0$ does not have vectors with the absolute value of the last coordinate different from 0 or 1; meanwhile, the last coordinate cannot be negative, since it would imply that one of the first $(n-1)$ coordinates is negative. Thus, 0 and 1 are the only choices for the last coordinate of a vector of $S_n^0$. If a vector of $S_n^0$ has 0 at the last position, it can have at most one 1 among the other coordinates. Evidently, if a vector of $S_n^0$ has 1 at the last position, it can have either $(n-2)$ or $(n-1)$ ones among the other coordinates. Therefore, the dual of $S_n$ consists of the following $(0,1)$-vectors: (1) all vectors with 0 at the last position and one 1 among the first $(n-1)$ coordinates, i.e. $[1^n, 0^{n-2}; 0^{n-1}]$; (2) all vectors with 1 at the last position and $(n-2)$ 1’s among the first $(n-1)$ coordinates, i.e. $[1^{n-2}, 0; 1^{n-1}]$; (3) all vectors with 1 at the last position and $(n-3)$ 1’s among the first $(n-1)$ coordinates, i.e. $[1^{n-3}, 0^2; 1^{n-1}]$; (4) the zero vector $[0^n]$. It is easy to see that $(S_n^0)^0 = S_n \cup [0^{n-1}; 1]$, i.e., the double dual of $S_n$ is obtained by adding a vector with zero coordinates, except for only 1 at the very last position. Denote by $D$ the matrix whose columns are the elements of $S_n^0 \setminus [0^n]$. The images of vectors of $S_n^0 \setminus [0^n]$ under the Voronoi mapping are linearly independent in $\mathbb{R}^N$, since there is an $N \times (N-1)$ matrix $M$ such that $D^T M = I_{n-1}$. We omit details here; however we will provide them in a more consecutive paper. Thus, rank one forms corresponding to the vectors of $S_n^0$ span a cone of co-dimension 1 in $\mathcal{P}(n)$. This is the cone of all PDQFs for which $\text{conv} \ (S_n^0)^0$ is a Delaunay cell in $\mathbb{Z}^n$; it is interesting that for $n = 6$ this cone coincides with the wall between the domains of $\phi_6^0 \ (\phi_6^0 \sim D_6)$ and $E_6^*$.

By the above theorem $S_n$ is a Delaunay simplex in $\mathbb{Z}^n$ for some PDQFs. ■

4 Tame Wall and Interpretability

There are many instances of coincidence of $L$-walls and perfect walls. In particular, for $n \leq 6$ all perfect walls are also $L$-walls, although already for $n = 4$ there are $L$-walls which are not perfect walls (see Ryshkov, Baranovskii 1976).

Definition 4.1 We call a wall between two $N$-dimensional $L$-type domains $P$ interpretable if it is also a wall between two perfect domains. Conversely a perfect wall is called $L$-interpretable if is also a wall between two $L$-type domains.
Let \( R_n \) be the repartitioning complex obtained by adding \([0^{n-1}; 1]\) to \( S_n \), i.e. \( R_n = S_n \cup [0^{n-1}; 1] \). As shown above, the Voronoi images of the vectors of \( S_n^0 \) span a cone of co-dimension 1 in \( \mathbb{R}^N \). It follows from the definition of \((0, 1)\)-dual system that the interior of this cone consists of all PDQFs that have \( R_n \) among their Delaunay cells. Voronoi showed that any such cone must be an L-wall. We call this wall \( TW(n) \). Let us prove that for any \( n > 4 \) \( TW(n) \) is a wall between the second perfect form \( D_n \) and a new perfect form. This new form exhibits a very interesting geometric behavior in all dimensions, but this will be the subject of another paper.

5 Perfect Wall Tamed by Big Simplex

Theorem 5.1 For any \( n > 4 \) the cone \( TW(n) \) is a common wall of the perfect domain of type \( D_n \) and the domain of perfect form \( TF_n \), where \( TF_n = (a_{ij})_n \) is defined as follows. For even \( n \):

\[
a_{ii} = 1 \text{ if } 1 \leq i \leq n - 1; \quad a_{nn} = \frac{1}{2} n^2 - \frac{7}{2} n + 7; \quad a_{ij} = \frac{n - 4}{2(n - 2)} \text{ for } i \neq j, j \neq n
\]

\[
a_{in} = -\frac{1}{2} n^2 - 6n + 10 \quad \text{for } i < n
\]

For odd \( n \):

\[
a_{ii} = 1 \text{ if } 1 \leq i \leq n - 1; \quad a_{nn} = \frac{n^3 - 8n^2 + 23n - 20}{2(n - 1)};
\]

\[
a_{ij} = \frac{n - 3}{2(n - 1)} \quad \text{for } i \neq j, j < n; \quad a_{in} = (-1) \frac{n^2 - 5n + 8}{2(n - 1)} \quad \text{for } i < n
\]

In lower dimensions: \( TF_5 \sim \phi_2^5 \) (III-d perfect form of Korkine and Zolotareff), \( TF_6 \sim E_6^* \).

Proof. To prove that \( TW(n) \) is a wall of a perfect domain we have to complement the vectors of \( V(R_n^+) \) to a set \( P \) of at least \( \frac{n(n+1)}{2} \) primitive integral vectors such that the Voronoi image of these vectors defines a hyperplane in \( \mathbb{R}^N \). We call elements of \( P \setminus R_n^+ \) complimentary vectors. Below we give two ways to complement \( V(R_n^+) \) to the set Voronoi images of minimal vectors for a perfect form.

A) Second Perfect Form.

The complimentary vectors are of type \([1, -1, 0^{n-3}; 0]\). Thus they lie in one of the half-spaces defined by \( TW(n) \). Let \((f_{ij})\) be a quadratic form defined by the following formulae: \( f_{ii} = 1 \) for \( i < n - 1 \), \( f_{nn} = 1 + \binom{n-2}{2} \), \( f_{ij} = \frac{1}{2} \) for \( i \neq j \) and \( i < n \), and

\[
\]
\[ f_m = -\frac{(n-2)}{2}. \] The minimum of \( f \) is 1 and it is attained on all vectors of \( R_n^+ \) and all \( \binom{n-1}{2} \) vectors of type \([1, -1, 0^{n-3}; 0]\). Denote by \( \text{CompD}_n \) the set of all vectors whose coordinates are obtained by circular permutations of the first \( n-1 \) positions of \([1, -1, 0^{n-3}; 0]\), except for \([-1; 0^{n-3}; 1; 0]\). Let us show that \((f_{ij})\) is integrally equivalent to \( D_n \). With respect to the scalar product defined by \((f_{ij})\), the following integral vectors forms a Coxeter diagram for \( D_n : \text{CompD}_n, [0^{n-1}; 1; 0], [1^{n-2}; 0; 1] \). In this diagram \([0^{n-3}; 1; -1; 0]\) is the vertex of valence 3, and \([0^{n-1}; 1; 0], [1^{n-2}; 0; 1] \) are the vertices of the two leaves of the diagram which are adjacent to \([0^{n-3}; 1, -1; 0]\). Vectors \([0^{n-1}; 1; 0], [1^{n-2}; 0; 1] \) and vectors of \( \text{CompD}_n \) obviously form a basis of \( \mathbb{Z}^n \). Therefore \((f_{ij})\) is integrally equivalent to \( D_n \). Notice that \(|\{R_n^+ \cup [1, -1, 0^{n-3}; 0]\}_{n-1}^{\mathbb{Z}}| = n(n-1)\), which is half the number of minimal vectors of \( D_n \). Therefore \( R_n^+ \cup [1, -1, 0^{n-3}; 0]\) and their inverses are all of the minimal vectors of \((f_{ij}) \sim D_n\).

B) Generalization of the Third Perfect Form.

Notice that \( TF_n \) is 1 for all \( v \in R_n^+ \). Set \( w := \lfloor \frac{n}{2} \rfloor \). The choice of complimentary vectors depends on the parity of the dimension.

1. When \( n \) is even and \( n > 4 \) the complimentary vectors for \( TF_n \) are: \([(w - 2)^{n-1}; w - 1], [w - 1, (w - 2)^n - 2; w - 1]^{n-1}, [(w - 1)^{n-1}; w] \). The total number of minimal vectors of is \( n(n + 3) \)

2. When \( n \) is odd the complimentary vector for \( TF_n \) is: \([(w - 1)^{n-1}; w] \). The total number of minimal vectors is \( n(n + 1) \)

With respect to the standard scalar product \( TW(n) \) is defined by the equation \( \mathbf{n} \cdot \mathbf{x} = 0 \), where \( \mathbf{n} \) is given by formulae \( n_{ij} = 0 \) for \( i < n \); \( n_{in} = \frac{n - 2}{2(n - 4)} \) for \( i < n \); \( n_{ij} = 1 \) for \( i \neq j \), \( j < n \). \( n_{nn} = \frac{n^3 - 9n^2 + 24n - 19}{2(n - 3)} \). For any \( n > 3 \) \( TF_n \) is a unique hyperplane in \( \mathbb{R}^N \) containing both \( R_n^+ \) and the complementary vectors, because for both even and odd \( n \) the complementary vectors form non-zero scalar products with \( \mathbf{n} \) and therefore do not belong to the hyperplane containing the cone \( TW(n) \). To prove that \( TF_n \) is positive definite it is enough to show that \( \det TF_n > 0 \), since all other main minors of \( TF_n \) correspond to bases formed by vectors of length 1 with angles \( \arccos a_{ij} \) between them (here we use the correspondence between bases and quadratic form). If we can construct a basis whose Gramm matrix is \( (a_{ij}) \), then \( \det TF_n > 0 \). Evidently, there exist \( n - 1 \) vectors \( \mathbf{v}_1, ... \mathbf{v}_{n-1} \) in \( \mathbb{R}^n \) so that \( \mathbf{v}_i \cdot \mathbf{v}_j = a_{ij} \), for \( i, j < n \). Now, the norm of \( \mathbf{v}_n \) must be \( a_{nn} \). If the cos of the (acute) angle between, say, \( \mathbf{v}_1 \) and \( \mathbf{d} := \sum_{j=1}^{n-1} \mathbf{v}_j \) (\( \mathbf{d} \) is the diagonal of the parallelogram based on basis \( \mathbf{v}_1, ... \mathbf{v}_{n-1} \)) is greater than \( |\arccos \frac{a_{mn}}{\sqrt{a_{im}a_{mn}}}| \), where \( i < n \), then one can construct vector \( \mathbf{v}_n \) of norm \( a_{nn} \) such that \( \mathbf{v}_i \cdot \mathbf{v}_n = a_{in} \).
For odd $n$ this is equivalent to showing that:

$$\frac{a_{nn}^2}{a_{nm}a_{ii}} = \frac{2(n^2 - 6n - 10)^2(n - 1)}{4(n - 2)^2(n^3 - 8n^2 + 23n - 20)} < \frac{(2 + \frac{(n-2)(n-3)}{n-1})^2}{2(n^2 - 5n + 8)} = \frac{(v_1 \cdot d)^2}{(v_1 \cdot v_1)(d \cdot d)}$$

For even $n$ this is equivalent to showing that:

$$\frac{a_{nn}^2}{a_{nm}a_{ii}} = \frac{(-10 + n^2 - 6n)^2}{2(n^2 - 7n + 14)(n - 2)^2} < \frac{2(n - 2)}{(n - 1)} = \frac{(v_1 \cdot d)^2}{(v_1 \cdot v_1)(d \cdot d)}$$

Using elementary algebra and calculus we have checked that both these inequalities hold for all $n > 4$.

The arithmetic minimum of $TF_n$ is 1 and the number of minimal vectors is $n(n+3)$ for even $n$, and $n(n + 1)$ for odd $n$. For example, this can be shown by the method of projective inequalities orginated by Korkin and Zolotareff (see Anzin (1991)). Unfortunately the proof is too tedious and we have to leave it out. We will publish the proof in another, more technical paper.

For all $n$ the number of minimal vectors of $TF_n$ is equal to that of $h_n$ of Anzin (1991). It is not difficult to show (Anzin, private communication) that for $n = 5, 6, 7$ $TF_n$ is equivalent to his form $h_n$. We refer to $TW(n)$ as a tame perfect wall because it admits an interesting $L$-interpretation described above. For $n = 6$ $TF_6 \sim E_6^*$ (proved in "Voronoi-Dickson Hypothesis...", this volume) and $TW(n)$ is one of the three (up to $GL(n, \mathbb{Z})$-equivalence) walls of the domain of $E_6^*$ (see Barnes (1957) ) and . The other two walls, called $W_2(24)$ and $W_3(21)$ by Barnes, are wild, as there is no change of $L$-type at almost all interior points of these perfect walls. The proof that $W_2(24)$ is not interpretable can be found in the other paper by us from this volume. We plan to publish the proof that $W_3(21)$ is not interpretable later. For $n = 7$ $TF_7 \sim \phi_{15}^7$ from Stacey’s (1973, 1975) list (see Anzin (1991) and Martinet (1996) for $\phi_{15}^7$). After this paper had been submitted for publication Maxim Anzin noticed that in lower dimensions ($n = 5, 6, 7$) our series coincides with Anzin’s (1991) series $h_n$. Maxim informed us that he is about to prove that $TF_n \sim h_n$ for all $n$.

6 The case of $n=6$: $L$-partition of $E_6$

In this subsection we discuss Delaunay tilings of lattices lying in a small neigbourhood of $E_6$ in the space of parameters. More specifically, we look at the $L$-partition of $\mathfrak{P}(n)$ near the ray corresponding of $E_6$. The Delaunay tiling of lattice $E_6$ is formed by congruent copies of the Gosset polytope $(2_{21}$ in Coxeter’s notation), which is the convex hull of a unique two-distance spherical set in $\mathbb{E}^6$. We refer to the Gosset polytope as the $G_6$-tope. The $G_6$-topes of the Delaunay tiling of $E_6$ fall into two translation classes. The star of a lattice point is formed by 54 $G_6$-topes, 27 in each translation class.
The $G_6$-tope is quite remarkable. It has 27 vertices, 216 edges, 72 regular simplicial facets, and 36 regular cross-polytopal facets (e.g. Coxeter (1995)). Thus, the vertices of the $G_6$-tope form a spherical two distance set. Polytopes whose vertices form a spherical two distance sets are interesting combinatorial objects (see Deza and Laurent (1997), Deza, Grishukhin, Laurent (1992)). In the case of $G_6$-tope the two distance structure is realized so that for each vertex $v$ of the $G_6$-tope there is a vector $p_v$ such that the vertex set of $G_6$-tope can be represented as $v \cup V_1 \cup V_2$, where $V_1 = \{u \in S \mid (u - v) \cdot p = 1\}$, and $V_2 = \{u \in S \mid (u - v) \cdot p = 2\}$. For a detailed description of geometric and group theoretic properties of the $G_6$-tope see (Coxeter (1973, 1995)).

Below, we show that for every subset of vertices of a Delaunay cell of $E_6$, $E_6$ can be perturbed so that this subset becomes a Delaunay cell for the perturbed lattice. In particular, this implies that there are perturbations of $E_6$ having a Delaunay simplex of volume 3, the maximal relative volume of a Delaunay lattice simplex in $\mathbb{E}^6$.

**Proposition 6.1** For every convex polytope $D$ whose vertex set is a subset of the vertex set of the $G_6$-tope there is a perturbation of $E_6$ making $D$ a Delaunay polytope for the perturbed lattice.

**Proof.** Denote by $\phi_{E_6}(x)$ an inhomogenous quadratic function whose quadratic part is $E_6$, and such that $\phi_{E_6}(x) = 0$ is an ellipsoid circumscribing the $G$-tope. For $\alpha > 0$ consider quadratic function

$$\phi(x) = \phi_{E_6}(x) + \alpha \sum_{v \notin D} (p_v \cdot x - 1)(p_v \cdot x - 2).$$

When $\alpha$ is sufficiently small the quadratic part of $\phi_{E_6}(x)$ is close to $E_6$ in the space of parameters. The ellipsoid $\phi_{E_6} = 0$ circumscribes $D$, since forms $\alpha (p_v \cdot x - 1)(p_v \cdot x - 2)$, $v \notin D$ guarantee that all vertices of the $G$-tope that are not in $D$ lie outside of $\phi_{E_6} = 0$. $lacksquare$

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