MEAN-FIELD LIMITS OF PHASE OSCILLATOR NETWORKS AND THEIR SYMMETRIES

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ABSTRACT. Rather than considering a large collection of interacting dynamical nodes on a graph, one typically considers a continuous limits to understand the dynamics. Depending on the type of network under consideration, different continuum descriptions may be appropriate: Sometimes, it is sufficient to track the evolution of a distribution of states, sometimes, one wants to track the state of a particular node. Here, we explicitly analyze the symmetry properties of infinite-dimensional continuum descriptions of phase oscillator networks. While symmetries and their consequences have been widely studied for finite network dynamical systems, much less is known about their continuous limits. We come across some phenomena that have no finite-dimensional counterpart, such as noninvertible symmetries. Our insights allow us to identify different continuous limits as invariant subspaces within a single dynamical framework. While we focus on phase oscillator networks, our results also apply to other dynamical systems on graph limits.

1. INTRODUCTION

Network dynamical systems—dynamical nodes that interact along network connections—arise in various contexts in science and technology, ranging from interacting neural units to power grid networks [28, 31]. If there are finitely many dynamical nodes, a graph $G$ is a natural abstraction of the network structure: Two nodes interact if there is an edge connecting the nodes. An essential challenge is to understand how the properties of the graph affect the collective network dynamics, i.e., the joint dynamics of all nodes. The structure of $G$ has important consequences for the network dynamics, especially if the nodes are (almost) identical: Automorphisms of $G$ yield symmetries of the network dynamical system, which give rise to invariant sets in phase space or determines what bifurcations can happen generically. While symmetries of dynamical systems is a classical topic [10, 11], there has recently been renewed interest in the structure of network dynamical systems, also due to the interest in symmetry breaking phenomena [3]. From a practical perspective, one can calculate symmetries to understand the emergence of “clustered dynamics” where nodes synchronize partially [27]. From a theoretical perspective, one can use symmetries and their generalizations to elucidate the phase space structure [25].

Often, network dynamical systems of interest have a large number of nodes. For example, the human brain is a network of billions of cells. Rather than considering a high-dimensional
dynamical system with finitely many nodes, it is often easier to understand the dynamics of a continuum with infinitely many nodes. For the sake of simplicity, we restrict our discussion here to networks of phase oscillators where the state of each node is given by a phase variable on the circle $T := \mathbb{R}/2\pi\mathbb{Z}$. For example, the phase of oscillator $k \in \{1, \ldots, N\}$ in the Kuramoto model [17] with identical oscillators on a graph evolves according to

$$\frac{d}{dt} \theta_k = \omega + \frac{1}{N} \sum_{j=1}^{N} A_{jk} \sin(\theta_j - \theta_k),$$

where $A_{jk}$ are the coefficients of the adjacency matrix of $G$; see also [29]. Depending on $G$ and the perspective one wants to take, several approaches are feasible to describe the limit of large networks.

First, if $G$ is the complete graph on $N$ vertices—as in the traditional Kuramoto model—then the networks interactions are solely through the mean field $Z = \frac{1}{N} \sum_{j=1}^{N} \exp(i\theta_j)$ with $i := \sqrt{-1}$. In this case, in the limit $N \to \infty$ one typically considers the corresponding mean-field system which describes the evolution of a probability measure $\mu(t)$ on $T$ that describes the probability to find an oscillator at time $t$ at phase $\varnothing \in T$ [32, 23]. For the Kuramoto model of identical oscillators, that is (1) with $A_{jk} = 1$, the evolution of an initial measure $\mu(0) = \mu_0$ is determined by the transport equation

$$\mu(t) = \Phi\# \mu_0$$

where $\#$ denotes the push forward of a measure and $\Phi$ is the flow induced by

$$\frac{d}{dt} \varnothing = \omega + \int_{T} \sin(\varnothing - \theta) d\mu(t)(\theta).$$

If the measures are sufficiently regular, there is a corresponding partial differential equation to describe the evolution of probability densities. These equations give explicit insights into bifurcation structure [9] and the Ott–Antonsen [26] and Watanabe–Strogatz [36] reductions have been instrumental in elucidating the dynamics (see also [5] and references therein). The mean-field transport equations can be generalized to networks that consist of multiple populations with interaction beyond a sine function, such as higher harmonics and nonpairwise (“higher-order”) interactions [4].

Second, if $G_N$ is a family of dense graphs on $N$ vertices that gives rise to a graphon $W$ as the limiting object as $N \to \infty$ [20], then one can consider dynamics of the corresponding graphon system [21]: Nodes are indexed by $x \in I := [0,1]$ so the state of the system is determined by an integrable function $\theta(x, t)$ that describes the phase $\theta \in T$ of oscillator $x \in I$ at time $t$. For the Kuramoto model (1), the phase $\theta(x, t)$ evolves according to

$$\frac{d}{dt} \theta(x, t) = \omega + \int_{I} W(x, y) \sin(\theta(y, t) - \theta(x, t)) dy$$

The stability of specific solutions can be understood within the context of (3) [22].
Third, one can consider mean-field equations on graph limits [7]; we refer to this as the mean-field graphon system. Here vertices are indexed by $I$ and the state at $x \in I$ at time $t$ is determined by a probability distribution $\mu(x, t)$. The measures are coupled through network interaction and an initial family of measures $\mu_0 = \mu(x, 0)$ for the Kuramoto model evolves according to the transport equation

$$\mu(x, t) = \Phi \# \mu_0$$

where $\Phi$ is now the flow induced by

$$\frac{d}{dt} \theta(x, t) = \omega + \int_I W(x, y) \int_T \sin(\vartheta - \theta(x, t)) \, d\mu(y, t)(\vartheta) \, dy.$$  

Within this framework, one can analyze stability and bifurcations of particular solutions of interest [7, 8].

How do symmetries organize the phase space structure in the limit of large networks? And how do these approaches relate to each other? As mentioned above, the consequences of symmetries for finite network dynamical systems are well understood. At the same time, graphon automorphisms arise naturally in the theory of graph limits [20]. Therefore, one can expect that automorphisms of the underlying network lead to symmetries for the network evolution in the limit of large networks.
The main contribution of this paper is twofold. First, we will analyze the symmetries of the mean-field graphon system. This will elucidate the overall structure of the (infinite-dimensional) phase space. Second, we will use these insights to show that the mean-field system and the graphon system are contained in the mean-field graphon system. Figure 1 illustrates the embeddings. We now introduce the main results informally and outline the structure of the paper.

After giving a more formal introduction to the topics discussed above in Section 2, the main results of Section 3 elucidate the symmetries and corresponding fixed point sets of the mean-field graphon system (4), shown in the middle of the diagram in Figure 1. We will see that the graphon automorphisms are symmetries of the dynamical system, but that—in contrast to finite networks—general symmetries need not to be invertible. We will specifically discuss graphons that are invariant under rotations, homogeneous coupling in clusters, and phase-shift symmetries that arise through the network interactions.

In Section 4 we consider the mean-field system and a close relative, the multi-population mean-field system; the former is trivially contained in the latter as a one-population system (A in Figure 1). The main result of the section is that for homogeneous networks (given by $W = 1$), the mean-field dynamics are contained in the mean-field graphon system as dynamics on a fixed point set of the graphon automorphisms (B). In Section 5 we concentrate on the evolution of Dirac measures for the mean-field graphon system. This allows to recover the graphon system as a subsystem of the mean-field graphon system (C). Moreover, finite networks are a special case of these Dirac measures if the oscillator indices are discretized (D). The finite system can also be seen as a multi-population mean-field system where the measure of each population is a Dirac measure (E). Finally, for $W = 1$ the graphon system can be projected onto the mean-field system by stripping away the oscillator labels and considering their distribution instead (F).

2. Preliminaries

Our recipe contains three ingredients: graphons, symmetries of dynamical system, and ergodic theory. In this section we prepare the first two ingredients. Starting from finite graphs, we will introduce graphons and their automorphisms. We will review the general theory of symmetries of dynamical systems and apply it to finite phase oscillator networks. This example serves as an appetizer for the main courses of the article. Ergodic transformations will be introduced in Section 3.

2.1. Graphs, Graphons, and Their Automorphisms

2.1. Definition. A graph is given by a set of vertices, and a set of unordered pairs of vertices called edges. If the vertices are finite in number, we denote them by $1, \ldots, N$ and call the graph finite. A finite graph is completely described by its adjacency matrix $A = (A_{jk})$ by
setting $A_{jk} = 1$ if $jk$ is an edge and $A_{jk} = 0$ otherwise. A graph is sometimes called a network.

Let $I = [0, 1]$ be endowed with the Lebesgue measure $\lambda$. Subsets of $I$ of measure zero will be systematically neglected, two functions over $I$ will be identified if they are equal up to a set of measure zero.

2.2. Definition. A graphon is a symmetric measurable function $W : I \times I \to [0, 1]$. Two graphons are identified if they are equal up to a set of measure zero.

A graphon can be thought of as a weighted graph on the vertex set $I$, where $xy$ is an edge if and only if $W(x, y) > 0$. Every finite graph $G$ can be embedded in a graphon in a natural way: Divide $I$ into $N$ intervals $I_1, \ldots, I_N$ and set $W = 1$ or $W = 0$ over $I_j \times I_k$ depending on whether $jk$ is an edge of $G$ or not. We call

$$W_G(x, y) = \begin{cases} 1, & \text{if } \lceil Nx \rceil \lceil Ny \rceil \text{ is an edge of } G \\ 0, & \text{otherwise} \end{cases}$$

(5)

the canonical embedding of $G$. This provides a geometrical way of representing the adjacency matrix of $G$. Graphons were introduced as limit objects of graph sequences [20]: One can imagine that as $N$ goes to infinity, the canonical embeddings give pictures with higher and higher resolution which converge to a measurable function; see also [15, Lemma 3.3].

2.3. Definition. An automorphism of a graph is a bijection $\gamma$ of the set of vertices preserving the set of edges, that is such that $\gamma(j)\gamma(k)$ is an edge if and only if $jk$ is an edge.

In terms of adjacency matrix this is equivalent to $A_{\gamma(j)\gamma(k)} = A_{jk}$. In order to adapt this definition to graphons, bijections must be replaced by measure-preserving transformations.

2.4. Definition. Given two measurable spaces $X$ and $Y$, a measure $\mu$ on $X$ and a measurable map $\gamma : X \to Y$, we call push-forward the measure $\gamma \# \mu$ on $Y$ is determined by $\gamma \# \mu(A) = \mu(\gamma^{-1}A)$ for any measurable set $A$. 
2.5. Definition. Let $X$ be a measurable space and $\mu$ a measure on $X$. A measurable map $\gamma : X \to X$ is called measure-preserving if $\gamma \# \mu = \mu$.

Measure-preserving transformations, rather than bijections, are better suited for graphons. For example, while bijections preserve the quantity of edges in a finite graph, measure-preserving transformations preserve the quantity of edges in a graphon. Note that bijections are in general not measure-preserving and measure-preserving transformations are in general not bijective; however the inverse of an invertible measure-preserving transformation is measure-preserving.

2.6. Definition. An automorphism of a graphon $W$ is an invertible measure-preserving transformation $\gamma : I \to I$ such that $W(\gamma(x), \gamma(y)) = W(x, y)$ for almost every $x, y \in I$.

As graphs can be embedded in graphons via the canonical embedding, graph automorphisms can be realized as graphon automorphisms via interval exchange transformations, which are known to be measure-preserving [34].

2.2. Symmetries of dynamical systems

2.7. Definition. A flow on a set $X$ is an action of the additive group of real numbers $\mathbb{R}$ on the set $X$, that is, a map $\Phi : X \times \mathbb{R} \to X$ satisfying
\[
\Phi_0(x) = x, \quad \Phi_s(\Phi_t(x)) = \Phi_{s+t}(x)
\]
for every $x \in X$ and every $s, t \in \mathbb{R}$. A set together with a flow is called continuous-time dynamical system.

Since we only consider continuous-time dynamical systems, we will omit this specification. Usually dynamical systems are given by differential equations, but those we will study are not, so we will need the following general definitions.

2.8. Definition. Consider a dynamical system on $X$ determined by a flow $\Phi$. For every point $x \in X$ the function $t \mapsto \Phi_t(x)$ is called the trajectory of $x$. If this map is constant then $x$ is stationary or an equilibrium. A subset $A \subseteq X$ is dynamically invariant if $\Phi_t(A) \subseteq A$ for every $t \in \mathbb{R}$.

2.9. Definition. A symmetry of a dynamical system $X, \Phi$ is a map $\gamma : X \to X$ satisfying
\[
\Phi_t(\gamma(x)) = \gamma(\Phi_t(x))
\]
for every $x \in X$ and every $t \in \mathbb{R}$. A point $x \in X$ is fixed by the symmetry $\gamma$ if $\gamma(x) = x$. A set $A \subseteq X$ is called $\gamma$-invariant if $\gamma(A) \subseteq A$.

A map $\gamma : X \to X$ is a symmetry if the trajectory of $\gamma(x)$ is equal to the image with respect to $\gamma$ of the trajectory of $x$. In simpler terms, applying $\gamma$ and letting the time pass or letting the time pass first and then applying $\gamma$ gives the same result. The following lemma is one of the most important consequences.
2.10. Lemma. Let \( \Phi \) be a flow on \( X \). Let \( \gamma \) be a symmetry of the dynamical system. Then the fixed points of \( \gamma \) form a dynamically invariant set and the stationary points form a set which is \( \gamma \)-invariant.

Proof. Both the assertions follows from (6). Suppose that \( \gamma(x) = x \), then \( \gamma(\Phi_t(x)) = \Phi_t(\gamma(x)) = \Phi_t(x) \). Suppose that \( \Phi_t(x) = x \), then \( \gamma(\Phi_t(x)) = \gamma(\gamma(x)) = \gamma(x) \). \( \square \)

It is often useful to consider more than one symmetry at a time. Let \( \Gamma \) be a set of symmetries. In this case \( x \) is called fixed by \( \Gamma \) if \( \gamma(x) = x \) for every \( \gamma \in \Gamma \) and \( A \) is called \( \Gamma \)-invariant if \( \gamma(A) \subseteq A \) for every \( \gamma \in \Gamma \). The \( \Gamma \)-fixed point set is dynamically invariant and the the set of stationary points of the flow is \( \Gamma \)-invariant. In most applications \( \Gamma \) is a group, usually a compact Lie group. Note that for every dynamical system \( \Gamma = \mathbb{R} \) is a group of symmetries, but of course this gives no information.

We illustrate these notions in the context of finite phase oscillator networks, see [2] for an in-depth discussion. This motivates the study of symmetries in mean-field limits. Recall that \( T = \mathbb{R}/2\pi\mathbb{Z} \).

2.11. Definition. Let \( A \) be the adjacency matrix of a graph, let \( D : T \to \mathbb{R} \) be Lipschitz continuous. The finite system is the dynamical system in \( T^N \) induced by the differential equations

\[
\frac{d}{dt}\theta_k(t) = \frac{1}{N} \sum_{j=1}^{N} A_{jk} D(\theta_j(t) - \theta_k(t))
\]

for \( k = 1, \ldots, N \). A solution is a function \( \theta = (\theta_1, \ldots, \theta_N) : \mathbb{R} \to T^N \) satisfying (7) for every \( t \in \mathbb{R} \).

The solutions are the trajectories of the dynamical system. The functions \( \theta_k \) describe the evolution of the phase of the oscillator \( k \). Let \( \Gamma(G) \) be the automorphism group of \( G \). Let \( \theta \) be a solution and \( \gamma \in \Gamma(G) \). We have

\[
\frac{d}{dt}\theta_{\gamma(k)}(t) = \frac{1}{N} \sum_{j=1}^{N} A_{j\gamma(k)} D(\theta_j(t) - \theta_{\gamma(k)}(t)) = \frac{1}{N} \sum_{j=1}^{N} A_{\gamma(j)\gamma(k)} D(\theta_{\gamma(j)}(t) - \theta_{\gamma(k)}(t)) = \frac{1}{N} \sum_{j=1}^{N} A_{jk} D(\theta_{\gamma(j)}(t) - \theta_{\gamma(k)}(t))
\]

\(^1\)In contrast to (1) we have now set \( \omega = 0 \) (and will do for the remainder of the article). We can do this without loss of generality because oscillators are coupled through phase differences.
and so \((\theta_{\gamma(1)} \ldots \theta_{\gamma(N)})\) is also a solution. Therefore \(\Gamma(G)\) is a group of symmetries, the action on the phase space is given by exchanging the coordinates. Some authors prefer the action of \(\gamma^{-1}\), we do not since we will not distinguish between left and right actions.

Suppose that the transposition exchanging two vertices \(j\) and \(k\) is a graph automorphism, then the phases \(\theta_j\) and \(\theta_k\) never cross each other: Either \(\theta_j(t) = \theta_k(t)\) for all \(t\) or for no \(t\). We call this the no-crossing property. The no-crossing property depends heavily on the topology of the graph: Two oscillators have the no-crossing property if and only if they have the same neighbors. If \(G\) is complete than any pair satisfy the no-crossing property and the dynamics can be restricted to the canonical invariant region \(\theta_1 < \ldots < \theta_N\). If \(G\) is disconnected then the oscillators can be partitioned into two subsets evolving independently.

3. The mean-field graphon system and its symmetries

The mean-field graphon system describes mean-field dynamics on a graphon. To set the stage, we first will follow [15] and briefly review the mean-field graphon system and discuss existence and uniqueness of its solutions. Then we will elucidate the symmetries of the system and how they constrain the dynamics. Applications to graphons invariant under rotations and to the analysis of the cluster states will be presented. The structure imposed by the symmetry on the dynamics will be a crucial ingredient to show that the mean-field graphon system contains a range of simpler systems that have been analyzed independent of one another, including the mean-field system and the graphon system.

3.1. Existence and uniqueness of solutions

3.1. Definition. Let \(\mathcal{M}\) be the set of Borel probabilities on \(T\) endowed with the Wasserstein metric

\[
d_{\mathcal{M}}(\mu, \nu) = \sup_f \left| \int_T f \, d\mu - \int_T f \, d\nu \right|
\]

where \(f\) varies among the 1-Lipschitz continuous functions \(T \to \mathbb{R}\). Let \(\mathcal{M}^I\) be set of maps \(I \to \mathcal{M}\) such that the preimage of an open set is measurable. In \(\mathcal{M}^I\) we define the metric

\[
d_{\mathcal{M}^I}(\mu, \nu) = \int_I d_{\mathcal{M}}(\mu(x), \nu(x)) \, dx.
\]

As always throughout this article we identify two functions over \(I\) if they are equal up to a set of measure zero.

Dynamics will take place on \(\mathcal{M}^I\). Thus, trajectories will be \(\mathcal{M}^I\)-valued maps \(\mu\) depending on time, often written as \(\mathcal{M}\)-valued functions \(\mu(x, t)\) depending on \(x\) and \(t\). We will use the symbol \(\bullet\) to indicate a missing argument, for example \(\mu(\bullet, t)\) denotes the map \(x \mapsto \mu(x, t)\).
3.2. Lemma. Let $W$ be a graphon and let $D : T \to \mathbb{R}$ be 1-Lipschitz continuous. Fix a continuous $\mu : \mathbb{R} \to M^I$ and $x \in I$. For every $t_0 \in \mathbb{R}$ and $u_0 \in T$ there exist a unique solution $u : \mathbb{R} \to T$ to the following initial value problem

\begin{align}
\frac{d}{dt} u(t) &= \int_I W(x, y) \int_T D(v - u(t)) \, d\mu(y, t)(v) \, dy \\
u(t_0) &= u_0
\end{align}

(9a) (9b)

Proof. The right hand side of the differential equation is uniformly Lipschitz continuous in $u$ and continuous in $t$, see [15, Lemma 2.2]. □

Equations (9) induce a flow $\Phi(\mu, x, t, \bullet) : T \to T$, where $\Phi(\mu, x, t, u_0) = u(t)$ is the solution starting at $u(0) = u_0$. Note that $\mu$ and $x$ are fixed, we have a flow for each choice.

From here on we will assume that $D : T \to \mathbb{R}$ is 1-Lipschitz continuous and that $W$ is a graphon satisfying the following equation for every $x \in I$ we have

\[ \lim_{\delta \to 0} \int_I |W(x + \delta, y) - W(x, y)| \, dy = 0. \]

(10)

3.3. Definition. The mean-field graphon system is given by

\begin{align}
\mu(x, t) &= \Phi(\mu, x, t, \bullet) \# \mu(x, 0) \\
\mu(x, 0) &= \mu_0(x)
\end{align}

(11a) (11b)

where $\Phi(\mu, x, t, \bullet) : T \to T$ is the flow induced by (9) and where $\mu_0 \in M^I$. These equations must hold for every $t$ and almost every $x$.

In [15] the solution theory for (11) is developed locally, that is, with time restricted to some finite interval $[0, T]$. We will show that existence and uniqueness can be extended to the whole real line (but we claim nothing about the continuous dependence on the initial data, which was also studied in [15]). Instead of starting from scratch, we will obtain global existence and uniqueness formally from the local result.

3.4. Theorem. [15, Theorem 2.4] Assume that $D : T \to \mathbb{R}$ is 1-Lipschitz continuous and $W$ satisfies (10). Fix an arbitrary $T > 0$. Then for every $\mu_0 \in M^I$ there is a unique $\mu : [0, T] \to M^I$ satisfying (11).

Since $T > 0$ is arbitrary, we have existence and uniqueness in the future. For a general dynamical system existence and uniqueness in the future do not imply existence and uniqueness in the past. Our main argument is that inverting time can be taken care of by changing the sign of $D$, which clearly preserve Lipschitz continuity.

3.5. Theorem. Assume that $D : T \to \mathbb{R}$ is 1-Lipschitz continuous and $W$ satisfies (10). Then for every $\mu_0 \in M^I$ there is a unique $\mu : \mathbb{R} \to M^I$ satisfying (11).
Proof. Consider an auxiliary system where \( D \) has been replaced by \( -D \). Since \( D \) and \( -D \) are 1-Lipschitz continuous, both the systems have existence and uniqueness in the future; let \( \Psi_D \) and \( \Psi_{-D} \) the corresponding semiflows on \( \mathcal{M}^I \). We will prove that for every \( t > 0 \) the maps \( \Psi_{D,t} \) and \( \Psi_{-D,t} \) are inverses of each other, so that both the semiflows are flows and existence and uniqueness hold globally.

Fix \( T > 0 \) and let \( \mu : [0, T] \to \mathcal{M}^I \) be a solution of (11). If \( u(t) \) solves (9) then \( u(T - t) \) solves (9) with \( t \) replaced by \( T - t \) and \( D \) replaced by \( -D \). Therefore

\[
\Phi_D(\mu, x, T - t, \bullet) = \Phi_{-D}(\nu, x, t, \Phi_D(\mu, x, T, \bullet))
\]

where \( \nu(x, t) = \mu(x, T - t) \). It follows that

\[
\nu(x, t) = \Phi_D(\mu, x, T - t, \bullet) \# \mu(x, 0) = \Phi_{-D}(\nu, x, t, \bullet) \# \Phi_D(\mu, x, T, \bullet) \# \mu(x, 0) = \Phi_{-D}(\nu, x, t, \bullet) \# \nu(x, 0).
\]

In particular for \( t = T \) we get \( \mu(x, 0) = \Psi_{-D,T}(\mu(x, T)) \). We conclude that \( \Psi_{D,T} \) and \( \Psi_{-D,T} \) are inverses of each other.

\[\square\]

3.2. Symmetries of the mean-field graphon system

In this section we will introduce symmetries induced by graphon automorphisms, they extend graph automorphisms which are symmetries in the finite system. Let \( \gamma : I \to I \), then \( \gamma \) acts naturally by right-composition on the set of \( \mathcal{M} \)-valued functions: If \( \mu : I \to \mathcal{M} \) then \( \mu \circ \gamma : I \to \mathcal{M} \). By \( W^\gamma \) we will denote the function \( W(\gamma(x), \gamma(y)) \).

3.6. Proposition. Let \( \gamma : I \to I \) be a measure-preserving transformation satisfying \( W^\gamma = W \). Then \( \gamma \) acts on \( \mathcal{M}^I \) by \( \mu \mapsto \mu \circ \gamma \). This action is a symmetry of the mean-field graphon system.

Proof. Recall that \( \mathcal{M}^I \) is defined so that the preimages of open sets are measurable. Given \( \mu \in \mathcal{M}^I \) and a measurable set \( A \subseteq \mathcal{M} \), we have \( (\mu \circ \gamma)^{-1} A = \gamma^{-1} \mu^{-1} A \). This set is measurable since \( \mu^{-1} A \) is a measurable set and \( \gamma : I \to I \) is a measurable function.

We now prove that every such \( \gamma \) is a symmetry of the dynamical system. Let \( \mu(x, t) \) be a solution of (11) with initial value \( \mu_0(x) \). We need to show that \( \mu(\gamma(x), t) \) is a solution with initial value \( \mu_0(\gamma(x)) \). The initial value condition is trivial. It remains to prove that (11a) is satisfied if \( \mu(x, t), \mu \) and \( \mu(x, 0) \) are replaced by \( \mu(\gamma(x), t), \mu \circ \gamma \) and \( \mu(\gamma(x), 0) \) respectively. By definition

\[
\mu(x, t) = \Phi(\mu, x, t, \bullet) \# \mu(x, 0)
\]

holds for almost every \( x \). Since \( \gamma \) is measure-preserving then

\[
\mu(\gamma(x), t) = \Phi(\mu, \gamma(x), t, \bullet) \# \mu(\gamma(x), 0)
\]
holds for almost every \( x \) as well. Since our goal is
\[
\mu(\gamma(x), t) = \Phi(\mu \circ \gamma, x, t, \bullet) \# \mu(\gamma(x), 0),
\]
it remains to prove that \( \Phi(\mu, \gamma(x), t, \bullet) \) is equal to \( \Phi(\mu \circ \gamma, x, t, \bullet) \). We compare the equations defining these flows. Note that \( \Phi(\mu \circ \gamma, x, t, \bullet) \) is given by
\[
\frac{d}{dt} u(t) = \int_I W(x, y) \int_T D(v - u(t)) \, d\mu(\gamma(y), t)(v) \, dy
\]
while \( \Phi(\mu, \gamma(x), t, \bullet) \) is given by
\[
\frac{d}{dt} u(t) = \int_I W(\gamma(x), y) \int_T D(v - u(t)) \, d\mu(y, t)(v) \, dy.
\]
The latter can be obtained from the former by replacing \( W(x, y) \) with \( W(\gamma(x), \gamma(y)) \) and then then \( \gamma(y) \) with \( y \). Since \( W^\gamma = W \) and \( \gamma \) is measure-preserving, then none of these operations change the right hand side. \( \square \)

Proposition \([3.6]\) does not require \( \gamma \) to be invertible. If the extra hypothesis of being invertible is required, then one obtains exactly the definition of graphon automorphism.

**3.7. Theorem.** The automorphism group of the graphon \( \text{Aut}(W) \) is a group of symmetries of the mean-field graphon system.

**Proof.** Every \( \gamma \in \text{Aut}(W) \) satisfies the hypothesis of Proposition \([3.6]\). \( \square \)

We will see concrete examples of graphons and related groups of symmetries, some are listed in Figure \([3.2]\). We will rarely consider the whole automorphism group \( \text{Aut}(W) \), rather we will focus on a subgroup of \( \text{Aut}(W) \). To avoid misunderstandings, we call any subgroup of \( \text{Aut}(W) \) a group of automorphisms, while \( \text{Aut}(W) \) is instead referred to as the automorphism group.
3.3. Graphons invariant under rotations

As a first example we consider graphons invariant under rotations. Let us start with a graph $G$, let $A$ be its adjacency matrix. If $G$ is the cycle with ordered vertices $1, \ldots, N$ then an obvious group of symmetries is the cyclic group $\mathbb{Z}/N\mathbb{Z}$. Cycles are not the only graphs with this property: We can connect the $k$ nearest neighbors to each vertex and still maintain the same symmetries. In general $G$ is invariant under $\mathbb{Z}/N\mathbb{Z}$ if and only if $A_{jk}$ depends on $j - k \mod N$ only. Something similar holds for graphons.

The orthogonal group $SO(2)$ is the group of linear rotations of the euclidean plane. In this section $SO(2)$ acts on the vertex set $I$ as follows: First identify $SO(2)$ with $\mathbb{R}/\mathbb{Z}$, then $\rho \in SO(2)$ maps $x \mapsto x + \rho \mod 1$. It is easy to see that these maps are measure-preserving. For which graphons are they automorphism?

3.8. Proposition. $SO(2) \subseteq \text{Aut}(W)$ if and only if $W(x, y)$ depends on $x - y \mod 1$ only.

Proof. In order to lighten the notation, we will omit $\mod 1$ during this proof. If $W(x, y)$ is a function of $x - y$, then trivially $W(x, y) = W(x + \rho, y + \rho)$. Conversely, suppose $SO(2) \subseteq \text{Aut}(W)$. Take an irrational rotation $\rho$. Fix any $y \in I$. Then $f(x) = W(x, y + x)$ is a square integrable function satisfying $f \circ \rho = f$. By uniqueness of the Fourier expansion

$$f(x) = \sum f_n e^{i2\pi nx}$$

for every $n$ we must have $f_n e^{i2\pi n\rho} = f_n$ for every $n$. Since $\rho$ is irrational this implies we have $f_n = 0$ for every $n \neq 0$. Therefore $f$ is constant.

The proof shows that if a graphon invariant under an irrational rotation then it depends on $x - y$ only and thus it is invariant under any rotation, see Figure 4. On the other hand a rational rotation generates a finite cyclic group, leading to graphons invariant under $\mathbb{Z}/n\mathbb{Z}$, which is a strictly weaker requirement.

Assume that $W(x, y)$ depends on $x - y \mod 1$ only. In this case $SO(2)$ acts transitively on $I$ as a group of symmetries. This implies that there is a form of homogeneity among the
solutions of the dynamical system: A solution always gives rise to a continuum of solutions that are related to each other by the SO(2)-symmetry action.

### 3.4. Non-invertible symmetries

Invertibility is not required in Proposition 3.6. There are many measure-preserving transformations of $I$ which are not invertible. We give an explicit example. For simplicity, we will suppose $W = 1$ so that $W^\gamma = W$ is trivially satisfied. Consider the map $\gamma(x) = 2x \mod 1$. For every integrable function $f$ on $I$ we have

$$
\int_I f(\gamma(x)) \, dx = 2 \int_{[0,1/2]} f(2x) \, dx = \int_I f(x) \, dx
$$

and so $\gamma \# \lambda = \lambda$, showing that $\gamma$ is measure-preserving and thus a symmetry. The map $\gamma$ is clearly not invertible, but despite this it effectively acts as a relabelling of the indexes, preserving the dynamical properties of the system. Intuitively speaking, the reason is that after $\gamma$ is applied, the same quantity of $x$ have the same value $\mu(x)$, which is what determines the dynamics if $W = 1$.

Note that the bijections (permutations) of $\{1, \ldots, N\}$ are exactly the measure-preserving transformations of the set $\{1, \ldots, N\}$ endowed with discrete uniform probability. On the other hand the bijections $\gamma$ of $I$ in general do not map $\mu \in M^I$ to $\mu \circ \gamma \in M^I$. We conclude that preserving the measure of the vertex set, rather than invertibility, is the fundamental property of symmetries in phase oscillator networks.

There is connection between permutations and measure-preserving transformations: The transformations preserving the Lebesgue measure can be approximated by permutations [16, 18]. For example $\gamma(x) = 2x \mod 1$ can be approximated by (invertible) permutations despite being non-invertible, and there is no good reason to exclude it as a symmetry of the continuous limit.

### 3.5. Cluster states

A point in $\mu \in M^I$ is a cluster state if different nodes take the same value, or—more precisely—if there is a measurable set $A \subseteq I$ such that $\mu(x) = \mu(y)$ for all $x, y \in A$. Some cluster states are dynamically invariant, which ones depends on the network structure. For finite oscillator networks cluster states can be determined from symmetries. Here we given an analogous description for the mean-field graphon system.

Let $A \subseteq I$ be an interval, let $\mu \in M^I$. Our first goal is detecting whether $\mu : I \to M$ is constant on $A$ using measure-preserving transformations. To this end we recall the notion of ergodic transformation.
3.11. Theorem. Let $A$ be a measurable set with finite measure $\lambda$. A measure-preserving transformation $\gamma : X \to X$ is ergodic if for every measurable set $A \subseteq X$ we have $\gamma^{-1}(A) = A$ only if $\lambda(A) = 0$ or $\lambda(X \setminus A) = 0$.

We have already encountered two examples ergodic transformations: The map $x \mapsto 2x \mod 1$ and the irrational rotations of the interval. Ergodic transformations are characterized by the following property:

3.10. Theorem. Let $\gamma : X \to X$ be ergodic. Let $f : X \to \mathbb{R}$ be a measurable function such that $f \circ \gamma = f$. Then $f$ is constant on $X$ up to a set of measure zero.

We refer the reader to [35, Theorem 1.6] for a proof. Since our functions are $\mathcal{M}$-valued, rather than $\mathbb{R}$-valued, a generalization of this result is needed. Unfortunately going from $\mathbb{R}$ to $\mathcal{M}$ makes the proof quite technical. If $A \subseteq I$ is an interval, we identify the measure-preserving transformations of $A$ with the measure-preserving transformations of $I$ leaving $I \setminus A$ fixed.

3.11. Theorem. Let $A \subseteq I$ be an interval and $\gamma$ an ergodic transformation of $A$. Let $\mu \in \mathcal{M}^I$ such that $\mu \circ \gamma = \mu$. Then $\mu$ is constant on $A$.

Proof. If $A$ is empty or contains just one point, there is nothing to prove. Otherwise $A$ and $I$ are isomorphic, from a measure-theoretic perspective, up to a normalization of the measures. It is enough to prove the case $A = I$, the general one follows by considering the restriction $\mu|_A$ and applying the isomorphism.

The metric space $\mathcal{M}$ is separable [6] and therefore Lindelöf [30]. This means that every open cover of $\mathcal{M}$ contains a countable subcover. In particular for every $r > 0$ we can cover $\mathcal{M}$ with countably many open balls of radius $r$.

Let $\gamma$ and $\mu \in \mathcal{M}^I$ be as in the hypothesis. Take a countable cover $\{B_n^r\}_n$ of $\mathcal{M}$ where each $B_n^r$ is an open ball of radius $r > 0$. Let $\mu^{-1}(B_n^r) \subseteq I$ be the preimage of $B_n^r$ with respect to $\gamma : I \to \mathcal{M}$. Since $B_n^r$ is open then $\mu^{-1}(B_n^r)$ is measurable. Since $\{B_n^r\}_n$ covers $\mathcal{M}$, then $\{\mu^{-1}(B_n^r)\}_n$ covers $I$. Since $\mu$ is $\gamma$-invariant, then the sets $\mu^{-1}(B_n^r)$ are $\gamma$-invariant. Ergodicity implies that the measure of each $\mu^{-1}(B_n^r)$ is either 0 or 1. Since their countable union has measure 1, one of them has measure 1.

We showed that for every $r > 0$ there is an open ball $B^r$ of radius $r$ such that $\mu^{-1}(B^r)$ has measure 1. We now take $r = 1/m$ where $m$ vary among the positive integers and consider the intersection $\bigcap_m B^{1/m}$. Since each $\mu^{-1}(B^{1/m})$ has measure 1 and the intersection is countable then $\mu^{-1}\left(\bigcap_m B^{1/m}\right)$ has also measure 1.

Since $\mathcal{M}$ is complete then $\bigcap_n B^{1/m}$ is either empty or it contains just one point. Since its preimage has positive measure, it must contain one point. We conclude that $\bigcap_n B^{1/m} = \{\mu_0\}$ for some $\mu_0 \in \mathcal{M}$ and $\mu(x) = \mu_0$ for almost every $x$. \hfill \Box

Theorem 3.11 tells us that if $\mu \circ \gamma = \mu$ for an ergodic transformation of $A$ then $\mu$ is constant on $A$. Of course, the converse is trivially true. We thus reached our goal of
characterizing constancy in terms of measure-preserving transformations. The next step is understanding when an ergodic transformation of A is a graph automorphism.

### 3.12. Definition

Let $A \subseteq I$ be an interval. We say that $W$ is homogeneous with respect to $A$ if for every $x \in A$ and every $y \in I$ the value $W(x, y)$ is independent on $x \in A$.

By definition $W$ is homogeneous with respect to $A$ if and only if the restriction $W|_{A \times I}$ is constant with respect to the first variable. Note that $W|_{A \times I}$ is constant with respect to the first variable if and only if $W|_{I \times A}$ is constant with respect to the second variable, in particular $W|_{A \times A}$ must be constant.

Homogeneity has a natural interpretation in terms of networks. Recall that $W(x, y)$ denotes the link strength between $x$ and $y$. If $W|_{A \times I}(x, y)$ is independent on $x \in A$ then every node $x \in A$ is connected to the same neighbors with the same strength. Two special cases are hubs, which are fully connected to the network, and isolated vertices. For hubs and isolated vertices $W|_{A \times I}(x, y)$ is independent on both $x \in A$ and $y \in I$. Examples are represented in Figure 3.5.

### 3.13. Lemma

The following are equivalent:

(i) $W$ is homogeneous with respect to $A$;

(ii) $W^\gamma = W$ for every measure-preserving transformation $\gamma$ of $A$.

**Proof.** Suppose that $W$ is homogeneous with respect to $A$ and let $\gamma$ be a measure-preserving transformation of $I$ leaving $I \setminus A$ fixed. It is easy to see that $W^\gamma = W$.

Conversely, suppose that $W^\gamma = W$ for every measure-preserving transformation $\gamma$ of $A$. We will prove that $x \mapsto W(x, y)$ is independent on $x \in A$.

First we prove that $x \mapsto W(x, y)$ is independent on $x \in A$ if $y \in I \setminus A$. So let $y \in I \setminus A$ and let $\gamma$ be an ergodic transformations of $A$. By definition we have $\gamma(y) = y$. Then the function $f(x) = x \mapsto W(x, y)$ satisfies $f \circ \gamma = f$ and by Theorem 3.10, it constant on $A$.

It remains to prove that $x \mapsto W(x, y)$ is independent on $x \in A$ if $y \in A$. For this, choose an ergodic transformation $\gamma$ of $A$ so that $\gamma \times \gamma$ is an ergodic transformation of $A \times A$, this
is possible by [35, Corollary 1.10.1]. Since $W|_{A \times A}$ is $\gamma \times \gamma$-invariant and $\gamma \times \gamma$ is ergodic, then $W|_{A \times A}$ is constant, concluding the proof. □

First, we characterized the cluster state in terms of symmetries. Second, we characterized which networks allows these symmetries. By combining these results we get the following theorem.

3.14. Theorem. Suppose that $W$ is homogeneous with respect to an interval $A$. Then the maps $\mu \in M^I$ constant over $A$ form a dynamically invariant set.

Proof. By Lemma 3.13 we have $W^\gamma = W$ for every measure-preserving transformation of $A$. By Proposition 3.6 these transformations are symmetries of the mean-field graphon system. By Theorem 3.11 the fixed point set is given by the maps $\mu \in M^I$ that are constant over $A$. Being the fixed point set of a set of symmetries, it is dynamically invariant. □

3.6. Phase-shift symmetries

Graphon automorphism are the most important symmetries and are the main focus of this work. However, the system (11) has a phase-shift symmetry that comes from the fact that the coupling function $D$ depends only on the phase differences; for finite systems this is considered in [2]. We include a discussion here for completeness.

While graphon automorphisms acts on $M^I$ through $I$, here we consider an action on $M^I$ through $M$. Let $SO(2)$ act on $T$ (rather than on $I$ as above) as the group of rotations of the circle as follows: First identify $SO(2)$ with $R/2\pi Z$ so that $\rho \in SO(2)$ acts by $u \mapsto u + \rho \mod 2\pi Z$, then $\mu \mapsto \rho \# \mu$ induces an action on measures. Note that if $\mu$ is a Dirac measure at $u$, then $\rho \# \mu$ is a Dirac measure at $u + \rho$, showing that the action on the set of measures extends the action on the circle.

3.15. Theorem. The group $SO(2)$ acts on $M^I$ via $\mu \mapsto \rho \# \mu$, resulting in a group of symmetries of the mean-field graphon system.

Proof. First, we check that $SO(2)$ actually acts on $M^I$. Recall that $M^I$ is defined so that the preimages of open sets are measurable. Given $\rho \in SO(2)$ and $\mu \in M^I$ and a measurable set $A \subseteq M$, we have $(\rho \# \mu)^{-1} A = \mu^{-1} (\rho^{-1} \# A)$. The set $\rho^{-1} \# A$ is open since the metric $d_M$ defining the topology of $M$ is invariant under rotation. Thus $\mu^{-1} (\rho^{-1} \# A)$ is open by definition of $M^I$ and $\rho \# \mu \in M^I$.

Second, we prove that every $\rho \in SO(2)$ is a symmetry of the mean-field graphon system. Let $\mu(x, t)$ be a solution of (11a) with initial value $\mu_0(x)$. We want to show that $\rho \# \mu(x, t)$ is a solution with initial value $\rho \# \mu_0(x)$. The initial value condition is trivial. It remains to prove that (11a) is satisfied when $\mu(x, t)$, $\mu$ and $\mu_0(x)$ are replaced by $\rho \# \mu(x, t)$, $\rho \# \mu$ and $\rho \# \mu(x, 0)$ respectively. By definition

$$\mu(x, t) = \Phi(\mu, x, t, \bullet) \# \mu(x, 0)$$
which implies
\[ \rho \# \mu(x, t) = \rho \# \Phi(\mu, x, t, \bullet) \# \mu(x, 0). \]

Our goal is to show that
\[ \rho \# \mu(x, t) = \Phi(\rho \# \mu, x, t, \bullet) \# \rho \# \mu(x, 0). \]

It remains to check that \( \rho + \Phi(\mu, x, t, u_0) \) is equal to \( \Phi(\rho \# \mu, x, t, \rho + u_0) \). Since \( u(t) = \Phi(\mu, x, t, u_0) \) satisfies \( u(0) = u_0 \) and
\[
\frac{d}{dt} u(t) = \int_I W(x, y) \int_T D(v - u(t)) \, d\mu(y, t)(v) \, dy
\]
then \( w(t) = \rho + \Phi(\mu, x, t, u_0) \) satisfies \( w(0) = \rho + u_0 \) and
\[
\frac{d}{dt} w(t) = \int_I W(x, y) \int_T D(v + \rho - w(t)) \, d\mu(y, t)(v) \, dy
\]
\[
= \int_I W(x, y) \int_T D(v - w(t)) \, d\rho \# \mu(y, t)(v) \, dy
\]
from which \( w(t) = \Phi(\rho \# \mu, x, t, \rho + u_0) \), concluding the proof.

We realized \( \text{SO}(2) \) as a group of symmetries, what is the corresponding invariant space? The phase shift can be seen as an action of the 1-dimensional torus onto itself. By a well-known theorem due to Haar, locally compact Abelian groups has a unique regular probability measure which is invariant under the group operation \([13]\). In the case of \( T \) this measure is \( \lambda/2\pi \). The map \( \mu \in \mathcal{M} \) which is constant \( \mu(x) = \lambda/2\pi \) for every \( x \) is indeed fixed by \( \text{SO}(2) \) and stationary with respect to dynamics.

### 3.7. Disconnected networks

Let \( A, B \) two disjoint non-trivial intervals whose union is \( I \). If \( W \) is zero on \( A \times B \) then \( A \) and \( B \) are disconnected. Connectivity in graphons is defined in \([14]\). If \( A \) and \( B \) are disconnected we expect \( \mu|_A \) and \( \mu|_B \) to evolve independently: By this we mean that \( t \mapsto \mu|_A(\bullet, t) \) does not depend on \( \mu|_B(\bullet, 0) \) and \( t \mapsto \mu|_B(\bullet, t) \) does not depend on \( \mu|_A(\bullet, 0) \). If this is the case, we can rotate two copies of \( T \) independently as
\[
(12) \quad \mu|_A \times \mu|_B \mapsto \rho \# \mu|_A \times \sigma \# \mu|_B
\]
making \( \text{SO}(2) \times \text{SO}(2) \) a group of symmetries.

### 3.16. Proposition. Let \( A, B \) two disjoint non-trivial intervals whose union is \( I \). Assume that \( D \) is not constant. Then the following are equivalent:

(i) \( A \) and \( B \) are disconnected,
(ii) \( \mu_A \) and \( \mu_B \) evolve independently,
(iii) \( \text{SO}(2) \times \text{SO}(2) \) is a group of symmetries via the action \([12]\).
Figure 6. A disconnected graphon.

Proof. (i) \(\Rightarrow\) (ii) Let \(x \in A\). The evolution of \(\mu_A(x)\) is given by \(\mu_A(x, t) = \Phi(\mu, x, t, \bullet)\#\mu(x, 0)\) where \(\Phi(\mu, x, t, \bullet)\) is given by

\[
\frac{d}{dt} u(t) = \int_I W(x, y) \int_T D(v - u(t)) \, d\mu(y, t)(v) \, dy = \int_A W(x, y) \int_T D(v - u(t)) \, d\mu_A(y, t)(v) \, dy
\]

which is independent on \(\mu_B\). By symmetry \(\mu_B\) does not depend on \(\mu_A\).

(ii) \(\Rightarrow\) (iii) Trivial.

(iii) \(\Rightarrow\) (i) For every \(\rho \in SO(2)\) the transformation \(\mu_A, \mu_B \mapsto \mu_A, \rho\#\mu_B\) is a symmetry. Let \(x \in A\). We must have

\[
\int_B W(x, y) \int_T D(v - u(t)) \, d\mu(y, t)(v) \, dy = \int_B W(x, y) \int_T D(v + \rho - u(t)) \, d\mu(y, t)(v) \, dy
\]

for every \(t\). Take \(t = 0\) and \(u(0) = 0\). Since a symmetry transforms the whole space \(M^I\), in particular we can choose \(\mu(x, 0) = \delta_v\) for every \(x \in I\), where \(\delta_v\) is the Dirac measure at some point \(v \in T\). We get

\[
D(v) \int_B W(x, y)dy = D(v + \rho) \int_B W(x, y) \, dy.
\]

Finally, this has to hold for every \(\rho \in SO(2)\). Therefore either \(D\) is constant or for every \(x \in A\) the integral \(\int_B W(x, y)dy\) is zero, which implies \(W_{A \times B} = 0\) since \(W \geq 0\).

3.17. Remark. In Proposition 3.16 we excluded the case that \(D\) is constant. If \(D\) is constant we have

\[
\int_I W(x, y) \int_T D(v - u(t)) \, d\mu(y, t)(v) \, dy = D \int_I W(x, y) \, dy
\]

and so for every \(x\) the measure \(\mu(x, 0)\) rotates rigidly around \(T\) with constant speed proportional to \(\int_I W(x, y) \, dy\). Therefore, although \(W\) may be non trivial, the system actually behaves like an uncoupled system with different intrinsic frequencies. From a dynamical
point of view this is not very interesting, but it can be interesting from a network science perspective since \( \int_I W(x, y) \, dy \) can be interpreted as the degree of node \( x \), suggesting that the structure of \( W \) may be reconstructed from the dynamics. Something similar is done for finite networks \[33\]. Instead, there may be cases where \( D \) is not constant everywhere on \( T \) but only on specific intervals; cf. \[1\].

4. **Mean-field dynamics**

In the previous sections we considered mean-field dynamics on structured networks determined by graphons. Here we will now focus on the case \( W = 1 \), that is, the network is fully connected with equal weights, or \( W \) with block structure (we make this precise below). The main result of this section is to show that traditional mean-field dynamics (2) as well as multi-population mean-field dynamics (defined below) arise on dynamically invariant subspaces of the mean-field graphon system.

4.1. **The mean-field system**

Assume that \( W = 1 \). We first make the notion of the mean-field system, introduced for the Kuramoto model with identical oscillators in (2), precise.

4.1. **Definition.** The mean-field system is given by

\[
\begin{align*}
\mu(t) &= \Phi(\mu, t, \bullet) \# \mu(0) \\
\mu(0) &= \mu_0
\end{align*}
\]

(13a)

(13b)

where \( \mu_0, \mu(t) \in \mathcal{M} \) and \( \Phi(\mu, t, \bullet) : T \to T \) is the flow induced by

\[
\begin{cases}
\frac{d}{dt} u(t) = \int_T D(v - u(t)) \, d\mu(t)(v), \\
u(0) = u_0.
\end{cases}
\]

(14)

Essentially, this is obtained from the mean-field graphon system by forgetting the dependency on \( x \). Let us identify \( \mathcal{M} \) inside \( \mathcal{M}^I \) as the set of constant measure-valued functions. We will prove that \( \mathcal{M} \) is dynamically invariant with respect to the mean-field graphon system and that the restricted dynamics is given by (13).

Let \( A \subseteq I \) be an interval. By \( \Gamma_A \) we denote the group of invertible measure-preserving transformations of \( A \), which are identified with the measure preserving transformations of \( I \) whose restriction on \( I \setminus A \) is the identity. Note that for every graphon \( W \) the automorphism group \( \text{Aut}(W) \) is a subgroup of \( \Gamma_I \). By Theorem 3.10 we have \( \text{Aut}(W) = \Gamma_I \) if and only if \( W = 1 \).
4.2. Theorem. Consider the mean-field graphon system, given by (11). Suppose that \( W = 1 \). Then every measure-preserving transformation of \( I \) is a symmetry of the dynamical system. Moreover \( M \subseteq \mathcal{M} \), the set of constant functions, is dynamically invariant. The mean-field graphon dynamics restricted to \( M \) coincides with the mean-field dynamics (13).

Proof. Since \( W = 1 \) every measure-preserving transformation of \( I \) satisfies \( W^\gamma = W \) and so, by Proposition 3.6, it is a symmetry. Let \( \mu \in \mathcal{M} \) such that \( \mu^\gamma = \mu \) for every symmetry \( \gamma \). Let \( \gamma \) be ergodic. By Theorem 3.11 it follows that \( \mu : I \to \mathcal{M} \) is constant. Therefore \( \mathcal{M} \) is the fixed point set of a set of symmetries and therefore it is dynamically invariant. It remains to show that the restricted dynamics is the mean-field system.

Let \( t \mapsto \mu(t) \) be a solution of the mean-field graphon system evolving in \( \mathcal{M} \). Then \( \mu(t) = \Phi(\mu, x, t, \bullet) \# \mu(0) \) where \( \Phi(\mu, x, t, \bullet) : T \to T \) is given by

\[
\frac{d}{dt}u(t) = \int_I W(x, y) \int_T D(v - u(t)) \, d\mu(t)(v) \, dy = \int_T D(v - u(t)) \, d\mu(t)(v)
\]

since \( W = 1 \). Therefore \( \Phi(\mu, x, t, \bullet) \) does not depend on \( x \) and is equal to \( \Phi(\mu, t, \bullet) \), the flow of the mean-field system.

Being fixed by any measure-preserving transformation, all the structural symmetries are lost in the mean-field system. Of course, one still has the phase-shifts considered in Theorem 3.15.

4.2. Multi-population mean-field systems

We can consider a system where a finite number of populations, each given by mean-field equations, interact with each other. Let \( G \) be a graph with \( N \) vertices. Let \( A \) be the adjacency matrix of \( G \) and let \( W \) be the canonical embedding of \( A \). If \( W \) can be obtained from a finite graph in this way, we say that it has block structure.

4.3. Definition. The multi-population mean-field system is given by

\[
\begin{align}
\mu(j, t) &= \Phi(\mu, j, t, \bullet) \# \mu(j, 0) \\
\mu(j, 0) &= \mu_0(j)
\end{align}
\]

for \( j = 1, \ldots, N \) where \( \Phi(\mu, j, t, \bullet) : T \to T \) is the flow induced by

\[
\begin{cases}
\frac{d}{dt}u(t) = \frac{1}{N} \sum_{k=1}^N A_{jk} \int_T D(v - u(t)) \, d\mu(k, t)(v) \\
u(0) = u_0
\end{cases}
\]
This is a special case of the dynamics considered in [4], where “higher-order” interactions are allowed. The proof of the following result is similar to that of Theorem 4.2 and will be omitted.

4.4. Theorem. Consider the mean-field graphon system, given by (11). Suppose that \( W \) is the canonical embedding of a graph \( G \) on \( N \) vertices. Then

\[
\Gamma_{I_1} \times \ldots \times \Gamma_{I_N}
\]

is a group of symmetries, where \( I_k = [k - 1/N, k/N] \). The fixed point set is given by the \( \mu \in M^I \) that are constant on each \( I_k \). Let \( \mu|_{I_k} = \mu(k) \). Then the dynamics on the fixed point set coincides with the multi-population mean-field system, given by (15).

Note that the automorphism group of a canonical embedding can be larger than the group considered in Theorem 4.4; this implies that in contrast to the mean-field system the multi-population mean-field system can still have interesting structural symmetries, see Remark 5.4.

5. Graphon dynamics

The graphon system, introduced for the Kuramoto model as (3), describes the evolution of individual phases in a continuum of oscillators, whose interaction is determined by a graphon \( W \); cf. [21]. The main result of this section allows us to see the graphon system as the dynamics on a dynamically invariant subspace of the mean-field graphon system. As a consequence, the graphon system will inherit the symmetry properties of the mean-field graphon system in Section 3. We will further relate this system to finite oscillator networks and their symmetries and the mean-field system.

5.1. The graphon system

The graphon system describes the evolution of phases \( \theta \) on the circle \( T \).

5.1. Definition. The graphon system is given by

\[
\begin{cases}
\frac{d}{dt} \theta(x, t) = \int_{I} W(x, y) D(\theta(y, t) - \theta(x, t)) \, dy \\
\theta(x, 0) = \theta_0(x)
\end{cases}
\]

(17)

where \( \theta(\cdot, t) \) and \( \theta_0 \) are measurable functions \( I \to T \). These equations must hold for every \( t \) and almost every \( x \).

Let us denote by \( \delta_\theta \) the Dirac measure with support \( \{\theta\} \). Let \( D \subseteq M \) be the set of Dirac measures. Note that the Wasserstein metric of \( M \) restricts to \( D \) to the usual metric of the circle, so \( T \) and \( D \) are isomorphic as metric spaces. Recall that \( M^I \) is defined so that the preimages of open sets are measurable, thus the functions \( I \to D \) belonging to \( M^I \) are the
same as the measurable functions $I \rightarrow T$. Let $T^I$ denote the set of these functions. We identify $T$ with $\mathcal{D}$ as just described, so that $T \subseteq \mathcal{M}$ and $T^I \subseteq \mathcal{M}^I$.

**5.2. Theorem.** Consider the mean-field graphon system, given by (11). Then the set $T^I \subseteq \mathcal{M}^I$ is dynamically invariant. The mean-field graphon dynamics restricted to $T^I$ coincides with the graphon dynamics, given by (17).

*Proof.* Let $\mu : \mathbb{R} \rightarrow \mathcal{M}^I$ be a solution of the mean-field graphon system such that $\mu(0) \in T^I$. For every $x$ and $t$ the measure $\mu(x, t)$ is the push-forward of the Dirac measure $\mu(x, 0)$ and so it is a Dirac measure. It is easy to compute it: Let $\mu(x, t) = \delta_{\theta(x, t)}$, then

$$\delta_{\theta(x,t)} = \Phi(\mu, x, t, \bullet) \# \delta_{\theta(x,0)} = \delta_{\Phi(\mu, x, t, \theta(x,0))}.$$ 

Therefore $\theta(x, t) = \Phi(\mu, x, t, \theta(x, 0))$ and $\theta(x, t)$ satisfies the differential equation

$$\frac{d}{dt}\theta(x, t) = \int_I W(x, y) \int_T D(v - \theta(x, t)) \, d\delta_{\theta(y, t)}(v) \, dy = \int_I W(x, y) \, D(\theta(y, t) - \theta(x, t)) \, dy,$$

concluding the proof. □

As opposed to the mean-field system, the graphon system inherits all the symmetries of the mean-field graphon system. We will see two applications.

### 5.2. Finite networks within the graphon system

The graphon system closely resembles the finite system, although with an (uncountably) infinite number of oscillators.

**5.3. Theorem.** Consider the graphon system, given by (17). Suppose that $W$ is the canonical embedding of a graph $G$ on $N$ vertices. Then

$$\Gamma_{I_1} \times \ldots \times \Gamma_{I_N}$$

is a group of symmetries, where $I_k = [k - 1/N, k/N]$. Its fixed point set is given by the functions $\theta : I \rightarrow T$ that are constant on every $I_k$. The dynamics on this set can be identified with the finite system, given by (7).

*Proof.* First note that $\Gamma_{I_1} \times \ldots \times \Gamma_{I_N}$ is a group of automorphisms of $W$. By Proposition 3.6 it follows that it is a group of symmetries. By Theorem 3.11 the fixed point set is given by the functions $\theta : I \rightarrow T$ that are constant on each $I_k$. Being the fixed-point set of a group of symmetries, it is dynamically invariant. It remains to show that the induced dynamics can be identified with the finite system.
Let \( \theta_k(t) \) be the value of \( \theta(x, t) \) on the interval \( I_k \). For every \( k \) and every \( x \in I_k \) we have

\[
\frac{d}{dt} \theta_k(t) = \int_I W(x, y) D(\theta(y, t) - \theta_k(t)) dy = \sum_{j=1}^N \int_{I_j} A_{jk} D(\theta_j(t) - \theta_k(t)) dy = \frac{1}{N} \sum_{j=1}^N A_{jk} D(\theta_j(t) - \theta_k(t))
\]

concluding the proof. \( \square \)

Note that the finite system coincides with the set-theoretic intersection of the multi-population mean-field system and the graphon system.

5.4. Remark. Suppose that \( W \) has a block structure, let \( G \) be the associated graph. We know that \( \Gamma_{I_1} \times \ldots \times \Gamma_{I_N} \subseteq \text{Aut}(W) \), but \( \text{Aut}(W) \) can actually be larger. Indeed we have a natural embedding of \( \text{Aut}(G) \) into \( \text{Aut}(W) \), where permutations are realized as interval-exchange transformations. In this way we obtain a semi-direct product

\[(\Gamma_{I_1} \times \ldots \times \Gamma_{I_N}) \rtimes \text{Aut}(G).\]

In particular \( \text{Aut}(G) \) acts on the fixed point set of \( \Gamma_{I_1} \times \ldots \times \Gamma_{I_N} \), which in the case of the graphon system is the finite system. This shows how the symmetries of the finite system are carried into the graphon system via the canonical embedding. The same holds for the multi-population mean-field system.

5.3. NO CROSSING IN THE GRAPHON SYSTEM

In Section 2.2 we discussed the no-crossing property in the context of finite systems: If two oscillators have the same neighbors then they can be interchanged by a symmetry, resulting in their phases not being allowed to cross each other. A similar property holds for the graphon system. Let \( A \subseteq I \) be an interval, vertices with the same neighbors corresponds to \( W \) being homogeneous with respect to \( A \) and by Theorem 3.14 we know that the functions \( \theta : I \to T \) constant on \( A \) form a dynamically invariant set. Therefore Theorem 3.14 can be interpreted as the no-crossing property for graphon systems.

Having to deal with sets instead of single points may be unsatisfactory in some contexts, but it convenient when applying the machinery of measure theory, thus convenient in developing our theory of symmetries. If we abandon, for a moment, our habit of identifying functions up to a set of measure zero, then the value of a function at a single point becomes relevant again. In this case one can obtain a pointwise version of the no-crossing property, not by using symmetries but by a direct argument, which we include here for completeness.
5.5. Theorem. Let $x, y \in I$ such that $W(x, z) = W(y, z)$ holds for every $z \in I$. Then the state $\theta(x) = \theta(y)$ is dynamically invariant.

Proof. Since $W(x, z) = W(y, z)$ for every $z \in I$ we have

$$\frac{d}{dt} \theta(y, t) - \frac{d}{dt} \theta(x, t) = \int_I W(x, z) \left( D(\theta(z, t) - \theta(x, t)) - D(\theta(z, t) - \theta(y, t)) \right) \, dz.$$ 

Since $|W| \leq 1$ and $D$ is 1-Lipschitz continuous the right hand side is bounded above by $|\theta(y, t) - \theta(x, t)|$. It follows that the function $h(t) = \theta(y, t) - \theta(x, t)$ satisfies

$$\left| \frac{d}{dt} h(t) \right| \leq |h(t)|$$

for every $t \in \mathbb{R}$.

We claim that such a function either has no zeros or is constantly zero. Indeed, suppose that $h(0) > 0$ but $h(t) = 0$ for some $t > 0$. Without loss of generality we can suppose $h(s) > 0$ for $s \in [0, t)$. For every $s \in [0, t)$ we have

$$\frac{d}{ds} h(t - s) \leq h(t - s)$$

and by the Grönwall’s inequality $h(t - s) \leq h(t)e^{s}$, at $s = 0$ we find $h(t) \geq h(0)e^{-t}$, a contradiction. \qed

5.4. CLOSING THE LOOP: PROJECTING THE GRAPHON TO THE MEAN-FIELD SYSTEM

In this section we will show how the mean-field system arises from the graphon system when we “forget” the oscillator labels. In the graphon system a measurable map $\theta : I \to T$ evolves over time, while in the mean-field system a measure on $T$ evolves over time. For each $t$ the push-forward $\theta(\cdot, t) \# \lambda$ is a measure on $T$ and it is natural to ask its relation to the mean-field system. Intuitively, this means that we forget the oscillators indexes and focus on their distribution instead. We will prove that if $W = 1$ then $\theta(\cdot, t) \# \lambda$ evolves according to the mean-field system.

5.6. Proposition. Suppose that $W = 1$ and let $\theta$ be a solution of the graphon system (17). Then $\mu(t) = \theta(\cdot, t) \# \lambda$ is solution of the mean-field system (13).

Proof. We need to check that $\mu(t) = \theta(\cdot, t) \# \lambda$ satisfies $\mu(t) = \Phi(\mu, t, \cdot) \# \mu(0)$, where $\Phi(\mu, t, \cdot)$ is given by (14). By definition $\mu(t) = \Phi(\mu, t, \cdot) \# \mu(0)$ is the same as

$$\theta(\cdot, t) \# \lambda = \Phi(\mu, t, \cdot) \# \theta(\cdot, 0) \# \lambda$$

and therefore it suffices to prove $\theta(x, t) = \Phi(\mu, t, \theta(x, 0))$. Since $W = 1$ we have

$$\frac{d}{dt} \theta(x, t) = \int_I D(\theta(y, t) - \theta(x, t)) \, dy = \int_T D(v - \theta(x, t)) \, d\mu(t)(v)$$
concluding the proof. □

5.7. Remark. In is interesting to note that Proposition 5.6 together with the no-crossing property of Theorem 5.5 guarantees the existence, in the mean-field system, of solutions admitting a density. We sketch the argument. Let \( h(t) = \theta(y, t) - \theta(x, t) \) as in the proof of Theorem 5.5. Suppose that \( h > 0 \). We know that \( h(t) \geq h(0) e^{-t} \) and similarly one can obtain \( h(0) e^{-t} \leq h(t) \leq h(0) e^t \). It follows that for every \( t \) we have

\[
\inf_{x,y \in \mathbf{I}} \frac{\left| \theta(y, 0) - \theta(x, 0) \right|}{|y - x|} e^{-t} \leq \inf_{x,y \in \mathbf{I}} \frac{\left| \theta(y, t) - \theta(x, t) \right|}{|y - x|} \leq \inf_{x,y \in \mathbf{I}} \frac{\left| \theta(y, 0) - \theta(x, 0) \right|}{|y - x|} e^t.
\]

Therefore the Lipschitz continuity of \( \theta \) and its inverse is propagated over time. This guarantees the existence of a density.

6. Discussion

Symmetries and the resulting invariant subspaces provide a framework that allows for a unified perspective on different continuous limits of oscillator networks. For the mean-field graphon system, the automorphism of the underlying graph limit provide an important class of symmetries. But the set of symmetries include more exotic elements such as noninvertible symmetries. Including these symmetries, we obtain a symmetry semigroup. These insights allowed to relate the mean-field graphon model to other continuum limits of network dynamical systems; we review these relationships that were informally introduced in the introduction in more detail. For specific graphons, mean-field limits arise as subspaces that are fixed point subspaces of subgroups of the symmetry group. For \( W = 1 \), the diagonal \( \{ \mu(x) = \mu(y) : x, y \in \mathbf{I} \} \subset \mathcal{M}^{\mathbf{I}} \) is fixed under any graphon automorphism; the dynamics on the diagonal correspond to the mean-field system (Theorem 4.2). If \( W \) has block structure, then the dynamics on the invariant subspace that is fixed by automorphisms that preserve each block correspond to the multi-population mean-field system (Theorem 4.4). For arbitrary \( W \), the graphon system describes the evolution of a continuum of phases at each node \( x \in \mathbf{I} \). These dynamics correspond to the invariant subspace \( \mu(x) = \delta_x \) (Theorem 5.2). The finite system is contained in the graphon system by considering the canonical embedding of the finite graph as a graphon and the subspace where \( \delta_x = \delta_y \) if \( x, y \in \mathbf{I} \) are in the same block (Theorem 5.3). Finally, one can project the graphon system to the corresponding mean-field system by forgetting the oscillator label and just considering the evolution of the oscillator distribution: As above, if \( W = 1 \) all oscillators are identically coupled and one can project to the mean-field system (Theorem 5.6), and if \( W \) has block structure one just has to remember the index of the block to obtain the multi-population mean-field system.

\(^2\)Noninvertible symmetries have—in a different context—also attracted attention in high-energy physics recently; cf. [12, 24].
The embeddings of the mean-field system and the graphon systems into the mean-field graphon system correspond to two distinct notions of network synchrony. First, we can have \textit{spatial synchrony}, that is, synchrony with respect to \( x \in I \): This happens when different \( \mu(x) \) have the same value. This first type of synchrony is related to dynamically invariant cluster states, the mean-field system and the multi-population mean-field system. Second, we have \textit{phase synchrony} at \( x \in I \), that is, synchrony in the sense of the distribution \( \mu(x) \) on \( T \): This happens when some points of \( T \) have positive \( \mu(x) \)-measure; in the case of full phase-synchrony \( \mu(x) \) is a Dirac measure. Both of these are interesting in the mean-field graphon system. The two types of synchrony may seem unrelated, but when \( W = 1 \) and the graphon system is projected to the mean-field—see Proposition 5.6—then non-trivial intervals where \( \theta \) is constant correspond to points of positive \( \theta\#\lambda \)-measure; in particular \( \theta \) is fully spatially-synchronized if and only if \( \theta\#\lambda \) is fully phase-synchronized. Note that if \( W = 1 \) studying synchrony in the graphon system, or in the mean-field system, is essentially the same.

Many of our results focus on an interval \( A \subseteq I \), but they can be applied to much more general cases. Indeed, it is enough to require \( \gamma A \) to be an interval for some \( \gamma \in \Gamma_I \). We sketch the argument in the case of cluster states. Suppose that \( A \) is not an interval but \( \gamma A \) is. Suppose that \( W \) is \( \Gamma_A \)-invariant, we would like to conclude that the solutions constant on \( A \) form a dynamically invariant set. Note that Lemma 3.13 and Theorem 3.14 do not directly apply to \( W \) and \( A \), since \( A \) is not an interval, but they apply to \( W^{\gamma^{-1}} \) and \( \gamma A \). As a consequence, in the mean-field graphon system with graphon \( W^{\gamma^{-1}} \) the solutions constant on \( \gamma A \) form a dynamically invariant set. It follows that, in the \( W \) system, the solutions constant on \( A \) form a dynamically invariant set.

For a given network structure, identifying its symmetries gives insights into the network dynamics as clustered solutions can correspond to dynamically invariant sets forced by symmetry. Recently, there has been an interest in numerically identifying network symmetries [27] using efficient algorithms [19]. Note that the symmetries of finite networks can be distinct from those of their continuous limit discussed here. For example, a network dynamical system whose network connectivity is sampled from a finite Erdős–Renyi (ER) random graph typically has very few symmetries. By contrast, ER random graphs limit to a constant graphon and the corresponding (mean-field) graphon system has maximal symmetry. Our results give insights about how to identify symmetries (and thus clustered dynamics): Symmetries can arise by having subsets of vertices \( A \) homogeneously coupled in the graphon, see Lemma [3.13]. If \( W \) is differentiable this homogeneity conditions turns into \( \partial_x W|_{A \times I} = 0 \), suggesting ways to computationally identify symmetries through the graphon. Moreover, symmetries restrict the linearization [10, 11] and thus identifying symmetries not only gives existence of clustered dynamics but also yields insights into their stability.
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