On Finding Maximum Cardinality Subset of Vectors with a Constraint on Normalized Squared Length of Vectors Sum

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Abstract

In this paper, we consider the problem of finding a maximum cardinality subset of vectors, given a constraint on the normalized squared length of vectors sum. This problem is closely related to Problem 1 from (Eremeev, Kel’manov, Pyatkin, 2016). The main difference consists in swapping the constraint with the optimization criterion.

We prove that the problem is NP-hard even in terms of finding a feasible solution. An exact algorithm for solving this problem is proposed. The algorithm has a pseudo-polynomial time complexity in the special case of the problem, where the dimension of the space is bounded from above by a constant and the input data are integer. A computational experiment is carried out, where the proposed algorithm is compared to COINBONMIN solver, applied to a mixed integer quadratic programming formulation of the problem. The results of the experiment indicate superiority of the proposed algorithm when the dimension of Euclidean space is low, while the COINBONMIN has an advantage for larger dimensions.

Keywords: vectors sum, subset selection, Euclidean norm, NP-hardness, pseudo-polynomial time.

1 Introduction

In this paper, we study a discrete extremal problem of searching a subset of vectors with maximum cardinality, given a constraint on the normalized squared length of vectors sum. The main goal of the study is to test experimentally two different approaches to solving this problem. The first approach is based on the dynamic programming and the second
one is based on the mixed-integer mathematical programming. We also comment on the computational complexity of this problem and estimate the time complexity of a proposed algorithm based on the dynamic programming principles.

The Maximum Cardinality Subset of Vectors with a Constraint on Normalized Squared Length of Vectors Sum (MCSV) problem is formulated as follows.

Given: a set \( \mathcal{Y} = \{y_1, \ldots, y_N\} \) of points (vectors) from \( \mathbb{R}^q \) and a number \( \alpha \in (0, 1) \).

Find: a subset \( \mathcal{C} \subseteq \mathcal{Y} \) of maximum cardinality such that

\[
\frac{1}{|\mathcal{C}|} \| \sum_{y \in \mathcal{C}} y \|^2 \leq \alpha \frac{1}{|\mathcal{Y}|} \| \sum_{y \in \mathcal{Y}} y \|^2,
\]

where \( \| \cdot \| \) denotes the Euclidean norm.

If the given points of the Euclidean space correspond to people so that the coordinates of points are equal to some characteristics of these people, then the MCSV problem may be treated as a problem of finding a sufficiently balanced group of people of maximum size.

MCSV problem is closely related to Problem 1 from [5]. The main difference consists in swapping the constraint with the optimization criterion. The problems of finding a subset of vectors, analogous to the MCSV problem are typical in the Data editing and Data cleaning, where one needs to exclude some error observations from the sample (see e.g. [7, 9, 10]). A recent example of such a problem may be found in [1], where a maximum cardinality subset of vectors is sought, given a constraint that a quadratic spread of points in the subset w.r.t. its centroid is upper-bounded by a pre-specified portion of the total quadratic spread of points in the input set w.r.t. the centroid of that set.

To compare the MCSV to the problem considered in [1], we note that

\[
\frac{1}{|\mathcal{C}|} \| \sum_{y \in \mathcal{C}} y \|^2 = \sum_{y \in \mathcal{C}} \| y \|^2 - \sum_{y \in \mathcal{C}} \| y - \bar{y}(\mathcal{C}) \|^2.
\]

In the right-hand side, the first sum is the total quadratic spread of points with respect to zero, the second one is relative to the centroid \( \bar{y}(\mathcal{C}) \) of \( \mathcal{C} \). The value \( \frac{1}{|\mathcal{Y}|} \| \sum_{y \in \mathcal{Y}} y \|^2 = \sum_{y \in \mathcal{Y}} \| y \|^2 - \sum_{y \in \mathcal{Y}} \| y - \bar{y}(\mathcal{Y}) \|^2 \) characterizes the difference of analogous quadratic spreads in the initial set. Therefore the MCSV problem asks for a subset of maximum size such that, in this subset, the two mentioned above total quadratic spreads differ by not more than \( \alpha \) times from the same difference in the input set \( \mathcal{Y} \).

The MCSV problem may be also treated as a Boolean optimization problem with a quadratic constraint:

\[
\sum_{i=1}^{N} x_i \rightarrow \text{max},
\]

s.t.

\[
\sum_{j=1}^{q} \left( \sum_{i=1}^{N} y_{i(j)} \right)^2 \leq \alpha \frac{1}{N} \sum_{j=1}^{q} \left( \sum_{i=1}^{N} y_{i(j)} \right)^2 \cdot \sum_{i=1}^{N} x_i,
\]

\[
x_i \in \{0, 1\}, \quad i = 1, \ldots, N,
\]
where
\[ N \text{ is the cardinality of set } \mathcal{Y}, \]
\[ q \text{ is the dimension of the Euclidean space}, \]
\[ y_{i}^{(j)} \text{ is } j\text{-th coordinate of the } i\text{-th vector}, \]
\[ x_{i} \text{ is a Boolean variable, } x_{i} = 1 \text{ if the } i\text{-th vector is included in the solution; otherwise } x_{i} = 0 (i = 1, \ldots, N). \]

Another problem related to the MCSV is the trading hubs construction problem, emerging in electricity markets under locational marginal pricing [2, 3, 4]. A trading hub is a subset of nodes of the electricity grid that may be used to calculate a price index as an average nodal price over the hub nodes. This price index may be employed by the market participants for hedging by the means of futures contacts [2]. Assume that the set of nodes of the electricity grid which may be included into a hub is \{1, \ldots, N\} and \( c_{it} \) is the price at node \( i \), \( i = 1, \ldots, N \), at an hour \( t \), \( t = 1, \ldots, T \), where \( T \) is the length of a historic period for which the electricity prices are observed. Let \( p_{rt} \) denote the electricity price of participant \( r \), \( r = 1, \ldots, R \), at hour \( t \), and \( R \) is the number of participants. The single hub construction problem consists in minimizing the sum of squared differences of the prices of participants from the hub price, requiring that the hub contains at least \( n_{\min} \) nodes:

\[
\min_{T} \sum_{t=1}^{T} \sum_{r=1}^{R} (c_{t} - p_{rt})^2 \tag{5}
\]

s.t.
\[
c_{t} = \frac{\sum_{i=1}^{N} x_{i} c_{it}}{\sum_{i=1}^{N} x_{i}}, \quad t = 1, \ldots, T, \tag{6}
\]
\[
\sum_{i=1}^{N} x_{i} \geq n_{\min}, \tag{7}
\]
\[
x_{i} \in \{0, 1\}, \quad i = 1, \ldots, N, \quad c_{t} \geq 0, \quad t = 1, \ldots, T. \tag{8}
\]

Here the binary variables \( x_{i} \) turn into 1 whenever node \( i \) is included into the hub. The variables \( c_{t} \) define the hub price at time \( t \), \( t = 1, \ldots, T \). This problem is proved to be NP-hard in [2], where a genetic algorithm was proposed for finding approximate solutions to it. In the special case of a single market participant problem \((5)-(7)\) is equivalent to the MCSV problem, where the criterion \((5)\) and constraint \((7)\) are swapped and instead of a lower bound on the hub size (which turns into a maximization criterion in MCSV) we are given an upper bound on the sum of squared differences of the prices of participants from the hub price. Such a modification of single hub construction problem may be appropriate in situations where the required closeness of a hub price to the prices of participants may be defined, e.g. on the basis of an observation an already existing hub [8].

The paper has the following structure. In Section 2 we show that MCSV problem is NP-hard even in terms of finding a feasible solution. An exact algorithm for solving this problem using the dynamic programming approach is proposed in Section 3. The algorithm has a pseudo-polynomial time complexity in the special case of the problem, where the dimension \( q \) of the space is bounded from above by a constant and the input data are integer. Section 4 describes how the algorithm is implemented and a computational experiment is carried out. The purpose of the experiment is to analyze the algorithm and compare it with the COINBONMIN solver.
2 Problem Complexity

The following proposition shows that MCSV problem is NP-hard even in terms of finding a feasible solution.

**Theorem 1** Finding out whether MCSV has a feasible solution is NP-hard.

**Proof.** Consider an instance of the Exact Cover by 3-sets problem, i.e. the family of subsets $A_1, \ldots, A_n$ of a set $A$ where $|A_i| = 3$ for all $i$ and $|A| = 3p$ where $p$ is an integer. The question is whether there are $p$ subsets in this family whose union is $A$. This problem is known to be NP-complete [6].

Put $q = 3p$, $N = n + 1$ and for each $i = 1, \ldots, n$ let $y_i$ be a characteristic vector of the set $A_i$ (i.e. the $j$-th coordinate of $y_i$ is 1 if $j \in A_i$ and 0 otherwise). Put $y_N = (−1, \ldots, −1)$ and choose $\alpha$ in such a way that $\alpha \| \sum_{y \in Y} y \|^2 < 3$. Then the constructed instance has a feasible solution if and only if there is an exact cover by 3-sets. Indeed, if there is no such cover then for each non-empty set $C$ we have

$$\frac{1}{|C|} \| \sum_{y \in C} y \|^2 \geq \frac{3}{N} > \frac{\alpha}{|Y|} \| \sum_{y \in Y} y \|^2$$

by the choice of $\alpha$, i.e. there are no feasible solutions. The opposite implication is trivial.

3 A Pseudo-Polynomial Time Algorithm for Bounded Dimension of Space

In this section, we show that in the case of a fixed dimension $q$ of the space and integer coordinates of vectors from $Y$, the MCSV problem can be solved in a pseudo-polynomial time using the same approach as proposed in [3].

For arbitrary sets $P, Q \subseteq \mathbb{R}^q$ define their sum as

$$P + Q = \{ x \in \mathbb{R}^q \mid x = y + y', \ y \in P, \ y' \in Q \}. \tag{9}$$

For every positive integer $r$ denote by $B(r)$ the set of all vectors in $\mathbb{R}^q$ whose coordinates are integer and at most $r$ by absolute value. Then $|B(r)| \leq (2r + 1)^q$.

Let $b$ be the maximum absolute value of all coordinates of the input vectors $y_1, \ldots, y_N$. Our algorithm for the MCSV problem successively computes the subsets $S_k \subseteq B(bk), \ k = 1, \ldots, N$, where each subset $S_k$ contains all vectors that can be obtained by summing different elements of the set $\{y_1, \ldots, y_k\}$.

For $k = 1$ we assume $S_1 = \{0, y_1\}$. Then we compute

$$S_k = S_{k-1} + (\{0\} \cup \{y_k\}) \tag{10}$$

for all $k = 2, \ldots, N$, using the formula [3].

For each element $z \in S_k$ we store an integer parameter $n_z$ and a subset $C_z \subseteq Y$ such that $z = \sum_{y \in C_z} y$, where $|C_z| = n_z$ and $n_z$ is the maximum number of addends that were used to produce $z$. 
When the subset $S_N$ is computed, we find an element $z^* \in S_N$ such that $\|z^*\|^2/n_{z^*} \leq \alpha \sum_{y \in Y} y^2$ and the value $n_{z^*}$ is maximum (if such elements exist in $S_N$). The result of the algorithm is the subset $C_{z^*}$ corresponding to the found vector $z^*$ or a conclusion that the problem instance is infeasible. Let us give a formal outline of the algorithm described above.

**Initialization**

Put $C_0 := \emptyset, n_0 := 0, C_{y_1} := \{y_1\}, n_{y_1} := 1$.

Let $S_1 := \{0, y_1\}$.

**The main loop:**

For all $k = 2, \ldots, N$ do

$S_k := S_{k-1}$.

For all $z \in S_{k-1}$ do

If $S_k$ contains $z'$ such that $z' = z + y_k$ then

If $n_{z'} < n_z + 1$ then

$n_{z'} = n_z + 1$.

$C_{z'} = C_z \cup \{y_k\}$.

End if.

Else

$S_k := S_k \cup \{z + y_k\}$.

$n_{z+y_k} := n_z + 1$.

$C_{z+y_k} := C_z \cup \{y_k\}$.

End if.

End for.

End for.

Search for $z^* \in S_N$ such that $\|z^*\|^2/n_{z^*} \leq \alpha \sum_{y \in Y} y^2$ and $n_{z^*}$ is maximum.

Output $z^*$ if it exists, otherwise report the problem is infeasible.

Taking into account that computing $S_k$ takes $O(q \cdot |S_{k-1}|)$ operations, we have the following

**Theorem 2** If the coordinates of the input vectors from $Y$ are integer and each of them is at most $b$ by the absolute value then MCSV problem is solvable in $O(qN(2bN + 1)^q)$ time.

In the case of fixed dimension $q$ the running time of the algorithm is $O(N(bN)^q)$, i.e. the problem is solvable in pseudo-polynomial time in this special case.

### 4 Computational Experiments

This section contains the results of testing the dynamic programming algorithm (DP) proposed in Section 3 and the results of COINBONMIN solver (CBM). For the experiments, the DP algorithm was implemented in C++ and tested on a computer with Intel Core i7-4700 2.40GHz processor and amount of RAM 4GB. First of all, two series of instances were generated randomly. To generate these series, we fixed parameter $\alpha = 0.1$,
Table 1: CPU time comparison of the solver CBM and DP Algorithm on Series 1

| Problem | CBM value | DP value | CBM time | DP time | Problem | CBM value | DP value | CBM time | DP time |
|---------|-----------|----------|----------|---------|---------|-----------|----------|----------|---------|
| 1       | 977       | 977      | 106,8    | 40,4    | 16      | 975       | 975      | 91,1     | 32,1    |
| 2       | 971       | 971      | 131,5    | 17,0    | 17      | 984       | 984      | 132,3    | 36,3    |
| 3       | 972       | 972      | 129,3    | 65,5    | 18      | 986       | 986      | 20,7     | 18,6    |
| 4       | 986       | 986      | 17,7     | 15,2    | 19      | 971       | 971      | 180,6    | 55,1    |
| 5       | 986       | 986      | 18,7     | 17,2    | 20      | 983       | 983      | 97,5     | 63,1    |
| 6       | 981       | 981      | 91,7     | 23,5    | 21      | 978       | 978      | 36,7     | 27,4    |
| 7       | 984       | 984      | 55,5     | 19,7    | 22      | 984       | 984      | 191,5    | 39,1    |
| 8       | 965       | 965      | 232,3    | 43,2    | 23      | 977       | 977      | 64,6     | 42,5    |
| 9       | 979       | 979      | 98,6     | 57,8    | 24      | 958       | 958      | 57,6     | 31,6    |
| 10      | 968       | 968      | 99,5     | 20,7    | 25      | 986       | 986      | 20,7     | 18,6    |
| 11      | 970       | 970      | 127,7    | 29,1    | 26      | 966       | 966      | 77,3     | 40,4    |
| 12      | 990       | 990      | 27,6     | 21,5    | 27      | 965       | 965      | 324,9    | 45,3    |
| 13      | 974       | 974      | 25,6     | 23,1    | 28      | 986       | 986      | 24,9     | 22,9    |
| 14      | 964       | 964      | 255,1    | 37,5    | 29      | 965       | 965      | 65,4     | 47,4    |
| 15      | 981       | 981      | 542,1    | 46,4    | 30      | 973       | 973      | 408,7    | 53,8    |

the dimension of the space \( q = 5 \) and number of vectors \( N = 1000 \). In Series 1, the values of the vector coordinates varied from -1 to 1, in Series 2 they varied from -5 to 5 and were integers. In both series the coordinates of vectors were generated with uniform distribution.

All testing instances were solved by DP algorithm and by the package COINBONMIN, included in the GAMS package, using the quadratic programming model from Section 1 (see formulas (2) to (4)).

The results of the computational experiment for the Series 1 are presented in Table 1. Here and below, we use the bold font to emphasize the best CPU time for each of the instances. For all problems of the series, both algorithms have found optimal solutions. However in all cases, the DP algorithm found the optimal solution faster than CBN. On Series 2, in the majority of the cases DP works faster as well (The results are presented in Table 2). The Wilcoxon signed-rank test showed that the CPU times of the DP and CBN on both Series 1 and Series 2 differ with a significance level less than 5%.

We also made an experiment, with Series 3, based on the historical data on electricity prices from PJM Interconnection (USA), available at [http://www.pjm.com](http://www.pjm.com). The dimension of the space turned out to be exceedingly large for the DP algorithm to meet these challenges, while CBM algorithm was able to solve these problems. This is due to a dimension of the space \( q = 24 \). The value of the \( \alpha \) parameter was taken to be 0.1. The results of the experiment with Series 3 are shown in Table 3. It is worth noting that CBM solver could not find the optimal solution to the 4-th instance and managed to find only an approximate solution.
### Table 2: CPU time comparison of the solver CBM and DP Algorithm on Series 2

| Problem | CBM value | DP value | CBM time | DP time | Problem | CBM value | DP value | CBM time | DP time |
|---------|-----------|----------|----------|---------|---------|-----------|----------|----------|---------|
| 1       | 990       | 990      | 19.1     | 117.3   | 16      | 977       | 977      | 23.5     | 136.9   |
| 2       | 977       | 977      | 110.3    | 87.6    | 17      | 990       | 990      | 207.7    | 99.2    |
| 3       | 985       | 985      | 224.3    | 93.1    | 18      | 983       | 983      | 289.4    | 78.2    |
| 4       | 963       | 963      | 199.6    | 135.4   | 19      | 983       | 983      | 17.8     | 84.3    |
| 5       | 983       | 983      | 15.1     | 105.8   | 20      | 968       | 968      | 272.3    | 95.1    |
| 6       | 982       | 982      | 62.1     | 96.8    | 21      | 982       | 982      | 17.9     | 137.1   |
| 7       | 975       | 975      | 527.6    | 89.2    | 22      | 961       | 961      | 635.6    | 125.0   |
| 8       | 967       | 967      | 518.3    | 137.5   | 23      | 984       | 984      | 164.6    | 106.6   |
| 9       | 979       | 979      | 124.4    | 94.8    | 24      | 971       | 971      | 167.4    | 98.6    |
| 10      | 978       | 978      | 112.5    | 101.4   | 25      | 983       | 983      | 28       | 84.3    |
| 11      | 965       | 965      | 65.4     | 126.5   | 26      | 989       | 989      | 203.1    | 124.2   |
| 12      | 967       | 967      | 127.9    | 85.7    | 27      | 971       | 971      | 140.6    | 92.3    |
| 13      | 981       | 981      | 16.8     | 87.6    | 28      | 987       | 987      | 536.6    | 116.7   |
| 14      | 974       | 974      | 178.3    | 81      | 29      | 965       | 965      | 25.4     | 83.5    |
| 15      | 983       | 983      | 494.2    | 96      | 30      | 986       | 986      | 66.7     | 64.8    |

### Table 3: CPU time of the solver CBM for electricity prices “PJM Interconnection”

| Problem | CBM value | CBM time | N |
|---------|-----------|----------|---|
| 1       | 40*       | 0.311    | 43|
| 2       | 118*      | 2.293    | 152|
| 3       | 177*      | 2.503    | 199|
| 4       | 186       | 87.984   | 199|
| 5       | 223*      | 0.867    | 233|
| 6       | 397*      | 2.02     | 408|
| 7       | 630*      | 3.686    | 642|
| 8       | 625*      | 2.776    | 642|
In additional experiments, we generated three series of instances in order to investigate how the values of \( N, q \) and \( \alpha \) affect the execution time of the DP algorithm and COINBONMIN package. In Series 4, we fixed \( q = 5 \) and \( \alpha = 0.1 \) and varied \( N \) from 5 to 1000, see Fig. 1. For Series 5 we put \( N = 1000, \alpha = 0.1 \) and varied \( q \) from 1 to 7, see Fig. 2. In Series 6, we fixed \( q = 5 \) and \( N = 1000 \) and varied parameter \( \alpha \) from 0.1 to 0.9, see Fig. 3. Six problem instances were randomly generated and solved for each set of the parameters mentioned above. Average CPU times of both algorithms are presented in Figs. 1–3 where the error intervals show the standard error of the mean.

Figure 1: Average CPU time of DP and COINBONMIN as a function of \( N \)

Figure 2: Average CPU time of DP and COINBONMIN as a function of \( q \)

The results of experiments with Series 1–5 indicate that in the cases where the dimensionality of the space and the maximum value of the coordinates of input vectors are not large, the DP algorithm is the most appropriate. However, when the dimension of space increases, it is preferable to use COINBONMIN as a mixed integer quadratically constrained program solver (miqcp mode). Experiments with Series 6 show that the execution time of COINBONMIN solver is not stable w.r.t. variation of \( \alpha \), while the CPU time of the DP algorithm does not depend on this parameter (which clearly agrees with the DP algorithm description).
Conclusions

The problem of finding a maximum cardinality subset of vectors, given a constraint on the normalized squared length of vectors sum is considered for the first time. It is shown that even finding a feasible solution to this problem is NP-hard and an exact dynamic programming algorithm for solving this problem is proposed. We prove a pseudo-polynomial time complexity bound for this algorithm in the special case, where the dimension of the space is bounded from above by a constant and the input data are integer. An alternative approach to solving the problem is based on the mixed integer quadratic programming. Both approaches are compared in a computational experiment. The results of the experiment indicate that in the cases where the dimensionality of the space and the maximum value of the coordinates of input vectors are not large, the dynamic programming algorithm is the most appropriate. However, when the dimension of the space increases, it is preferable to use a mixed integer quadratically constrained program solver, like COINBONMIN.

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