The antiferromagnetic \( \phi^4 \) Model, II. The one-loop renormalization

Vincenzo Branchina

Laboratory of Theoretical Physics, Louis Pasteur University
3 rue de l’Université 67087 Strasbourg, Cedex, France

Hervè Mohrbach

Laboratory of Theoretical Physics, Louis Pasteur University
3 rue de l’Université 67087 Strasbourg, Cedex, France
and
LPLI-Institut de Physique, F-57070 Metz, France

Janos Polonyi

Laboratory of Theoretical Physics, Louis Pasteur University
3 rue de l’Université 67087 Strasbourg, Cedex, France
and
Department of Atomic Physics, L. Eőtvős University
Puskin u. 5-7 1088 Budapest, Hungary
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Abstract

It is shown that the four dimensional antiferromagnetic lattice \( \phi^4 \) model has the usual non-asymptotically free scaling law in the UV regime around the chiral symmetrical critical point. The theory describes a scalar and a pseudoscalar particle. A continuum effective theory is derived for low energies. A possibility of constructing a model with a single chiral boson is mentioned.

I. INTRODUCTION

This is the second of two papers where we study the impact of higher derivative terms in field theories. In ref. [1] we have pointed out that the presence of these terms in a self-interacting single component scalar field theory produces tree level effects which may drive

\*branchina@crnvax.in2p3.fr
\†mohrbach@crnvax.in2p3.fr
\‡polonyi@fresnel.u-strasbg.fr
the formation of new vacua which is not accounted for by the decoupling theorem \[2\]. Three phases have been studied in the $\Phi^4$ theory by means of the mean-field approximation, the paramagnetic, $<\Phi(x)>_0$, the ferromagnetic, $<\Phi(x)>_1 \neq 0$ and the antiferromagnetic where $<\Phi(x)>$ is an oscillating function. $2^d$ bands have been found in the dispersion relation for the elementary excitations above these vacua in a certain range of the coupling constants of the d-dimensional theory. A reduced Brillouin zone was introduced for each band. Two zones describe particle like excitations and the others decouple in the mean-field continuum limit.

For a special choice of the coefficients of the higher order derivatives the theory possesses a formal chiral symmetry which allows us to decouple the two particles. The two decoupled modes correspond to the sublattices consisting of the even and the odd lattice sites. The theory which has nonvanishing field variables on one of the sublattices only is local and describes a chiral boson. In fact the space inversion exchanges the two sublattices and there is no space inversion partner of the particle in such a model.

We extend the analysis \[1\] in the upper critical dimension, $d = 4$, for theories in the vicinity of the chiral invariant critical point to the one-loop order and show that the beta functions of the lattice regulated theory with our $O(\Box^2)$ term in the lagrangian are those of an ordinary $\Phi^4$ model and give a renormalized lagrangian in terms of the continuum field variables. The one-loop renormalizability turns out to be a nontrivial consequence of the perturbative renormalizability around the critical point of the usual $\Phi^4$ model without higher order derivative terms because we have to render the dynamics for the two particles finite by fine tuning the set of the parameters of the bare lagrangian with a single quantum field.

There is a formal similarity between the tricritical point of the $\phi^6$ model and the chiral symmetrical theory. The mean field solution of the model with the potential

$$V(\phi) = \frac{g_2}{2}\phi^2 + \frac{g_4}{4!}\phi^4 + \frac{g_6}{6!}\phi^6$$

shows a tricritical point at $g_4 = 0$ which separates the second and the first order phase transition lines with different scaling laws \[3\]. In our case the dispersion relation

$$G^{-1}(p) = m^2 + p^2 - c_2 a^2 p^4 + c_4 a^4 p^6$$

produces a tricritical point when $c_2$ is sufficiently large to give an absolute minimum at nonvanishing values of the momentum. When $G^{-1}$ at the minimum is negative an inhomogeneous condensate is formed. The chiral symmetrical point where this happens is the Lifshitz point. This was introduced in \[4\] where the $\epsilon$-expansion was used to find out the scaling laws. A scalar model where the dispersion relation has a single minimum at nonvanishing momentum was considered in \[3\] and \[4\]. The phase transition towards the inhomogeneous vacuum was identified in the mean field level and the quantum fluctuations were taken into account in \[4\]. We will be working at $d = 4$ and extend the loop computation into the phase with inhomogeneous condensate which generates a "dangerous irrelevant variable" \[5\]. The dispersion relation of our model has several minima hence it contains several particle modes simultaneously. The condensate formation mechanism selects one of these particle sectors in a manner reminiscent of the spontaneous symmetry breaking.

There is a technical problem to solve in achieving this goal because more than one particle corresponds to the same quantum field. The formal problem is that higher order derivative
terms in the kinetic energy imply the presence of states with negative metric [8] and may render the effective action complex. But we argue that far below the momentum scale of the condensate we find only two particles, both with positive metric. Their dispersion relations can be replaced in the continuum limit by the usual quadratic expressions coming from a manifestly hermitean free lagrangian [9]. The situation turns out to be somehow similar to the species doubling of the lattice fermions where one finds several particle modes in the dispersion relation of a single bispinor field. We introduce a $2^d$-component field variable, $\Phi_\alpha(x)$, $\alpha = 1, \cdots, 2^d$, for the computation of the one-loop generator functional for the 1PI functions of the different excitation bands and show that it can be made finite by an appropriate fine tuning of the coupling constants of the original lagrangian.

The organization of the paper is the following. The basic tools of the perturbation expansion are developed in Section 2. The computation of the effective potential is presented in Section 3. The elimination of the divergences is shown and the finite renormalized coupling constants are obtained in Section 4. Section 5 is devoted to a simple effective theory which reproduces our model at low energy. A brief conclusion is in Section 6.

II. THE PERTURBATION EXPANSION

We develop the basic formula for the perturbation expansion in the scalar $\phi^4$ model in $d = 4$ with higher order derivatives by keeping the original field variable, $\Phi(x)$. As in [1], the theory is regularized on the lattice.

A. The lagrangian

The model considered contains a one component field variable, $\Phi(x)$, and is defined by the bare, cut-off lagrangian,

$$\mathcal{L} = \frac{1}{2} \partial_\mu \Phi(x) \mathcal{K}\left(\frac{(2\pi)^2}{\Lambda^2} \Box\right) \partial_\mu \Phi(x) + \frac{m^2_B}{2} \Phi^2(x) + \frac{\lambda}{4} \Phi^4(x),$$

where

$$\mathcal{K}(z) = 1 + c_2 z.$$  

We write this lagrangian as

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$$

with

$$\mathcal{L}_1 = \frac{1}{2} \partial_\mu \Phi \mathcal{K}\left(\frac{(2\pi)^2}{\Lambda^2} \Box\right) \partial_\mu \Phi + \frac{m^2_R}{2} \Phi^2 + \frac{\lambda_R}{4} \Phi^4,$$

and

$$\mathcal{L}_2 = \frac{\delta Z}{2} \partial_\mu \Phi \mathcal{K}\left(\frac{(2\pi)^2}{\Lambda^2} \Box\right) \partial_\mu \Phi + \frac{\delta m^2}{2} \Phi^2 + \frac{\delta \lambda}{4} \Phi^4.$$
We will use $\mathcal{L}_1$ non-perturbatively in the selection of the saddle point and $\mathcal{L}_2$ perturbatively in removing the UV divergences of the loop-corrections. The bare parameters are defined as $m_B^2 = m_R^2 + \delta m^2$ and $\lambda_B = \lambda_R + \delta \lambda$. We have no counter terms for the coupling constants $c_j$ because their leading order renormalization is at the tree-level.

We employ lattice regularization where one introduces the dimensionless variables $x^\mu$, $\varphi = a^{d/2-1} \Phi$, $m_L^2 = m^2 a^2$ and the unit vectors $(e_\mu)^\nu = \delta_{\mu \nu}$ and writes the action as

$$S_1[\varphi] = \sum_x \mathcal{L}_1(x) = \sum_x \left\{ K[\varphi; x] + V(\varphi(x)) \right\},$$

where

$$K[\varphi; x] = -\frac{1}{2} \varphi(x) \left[ A \varphi(x) + \sum_\mu \left( B(\varphi(x + e_\mu) + \varphi(x - e_\mu)) \right. \right.$$

$$+ C(\varphi(x + 2e_\mu) + \varphi(x - 2e_\mu))$$

$$\left. \left. \left. + \sum_{\mu \neq \nu} \left( E(\varphi(x + e_\mu + e_\nu) + 2\varphi(x + e_\mu - e_\nu) + \varphi(x - e_\mu - e_\nu) \right) \right] \right\}$$

and

$$V(\varphi(x)) = \frac{m_{LR}^2 \varphi^2(x)}{2} + \frac{\lambda_R}{4} \varphi^4(x),$$

The coefficients of the kinetic energy are

$$A = -2d + (4d^2 + 2d)c_2,$$

$$B = 1 - 4dc_2,$$

$$C = c_2,$$

$$E = c_2.$$  (11)

The field variable, $\varphi = \varphi_{vac} + \phi$, is the sum of the tree-level vacuum,

$$\varphi_{vac}(x) = \varphi_1 + \varphi_{2d}\chi(x),$$

where

$$\chi(x) = (-1)^{d-1} \sum_{\mu=1}^d x^\mu,$$  (13)

and the quantum fluctuations, $\phi(x)$. We will study the theory in the para- ($\varphi_1 = \varphi_{2d} = 0$), ferro- ($\varphi_1 \neq 0$, $\varphi_{2d} = 0$) and the (1,2) antiferromagnetic ($\varphi_1 = 0$, $\varphi_{2d} \neq 0$) phases in $d = 4$. The lagrangian for the quantum fluctuations is

$$\mathcal{L}_{1P} = \frac{1}{2} \partial_\mu \phi(x) K \left( \frac{(2\pi)^2}{\Lambda^2} \Box \right) \partial_\mu \phi(x) + \frac{m_{LR}^2}{2} \phi^2(x) + \frac{\lambda_R}{4} \phi^4(x),$$

$$\mathcal{L}_{1F} = \frac{1}{2} \partial_\mu \phi(x) K \left( \frac{(2\pi)^2}{\Lambda^2} \Box \right) \partial_\mu \phi(x) + \frac{1}{2} \left( m_{LR}^2 + 3\lambda_R \varphi_1^2 \right) \phi^2(x)$$

$$+ \lambda_R \varphi_1 \phi^3(x) + \frac{\lambda_R}{4} \phi^4(x),$$

$$\mathcal{L}_{1AF} = \frac{1}{2} \partial_\mu \phi(x) K \left( \frac{(2\pi)^2}{\Lambda^2} \Box \right) \partial_\mu \phi(x) + \frac{1}{2} \left( m_{LR}^2 + 3\lambda_R \varphi_{2d}^2 \right) \phi^2(x)$$

$$+ \lambda_R \varphi_{2d} \chi(x) \phi^3(x) + \frac{\lambda_R}{4} \phi^4(x).$$  (14)
\[ \mathcal{L}_{2P} = \frac{\delta Z}{2} \partial_\mu \phi(x) \mathcal{K} \left( \frac{(2\pi)^2}{\Lambda^2} \Box \right) \partial_\mu \phi(x) + \frac{\delta m_L^2}{2} \phi^2(x) + \frac{\delta \lambda}{4} \phi^4(x), \]
\[ \mathcal{L}_{2F} = \frac{\delta Z}{2} \partial_\mu \phi(x) \mathcal{K} \left( \frac{(2\pi)^2}{\Lambda^2} \Box \right) \partial_\mu \phi(x) + \varphi_1 (\delta m_L^2 + \delta \lambda \varphi_1^2) \phi(x) \]
\[ = \frac{1}{2} (\delta m_L^2 + 3 \delta \lambda \varphi_1^2) \phi^2(x) + \delta \lambda \varphi_1 \phi^3(x) + \frac{\delta \lambda}{4} \phi^4(x), \]
\[ \mathcal{L}_{2AF} = \frac{\delta Z}{2} \partial_\mu \phi(x) \mathcal{K} \left( \frac{(2\pi)^2}{\Lambda^2} \Box \right) \partial_\mu \phi(x) + \varphi_{2d} \chi(x) (\delta m_L^2 + \delta \lambda \varphi_{2d}^2) \phi(x) \]
\[ = \frac{1}{2} (\delta m_L^2 + 3 \delta \lambda \varphi_{2d}^2) \phi^2(x) + \delta \lambda \varphi_{2d} \phi^3(x) + \frac{\delta \lambda}{4} \phi^4(x), \] (15)

where the tree-level vacuum is given by

\[ P : \quad \varphi_{P1} = 0 \quad \varphi_{P2d} = 0, \]
\[ F : \quad \varphi_{F1} = -\frac{m^2_{LR}}{\lambda} \varphi_{F2d} = 0, \]
\[ AF : \quad \varphi_{AF1} = 0 \quad \varphi_{AF2d} = -\frac{m^2_{LR} + M^2_L}{\lambda}. \] (16)

Here \( M^2_L \) stands for the eigenvalue of the kinetic energy on the antiferromagnetic vacuum,

\[ \mathcal{K} \left( \frac{(2\pi)^2}{\Lambda^2} \Box \right) \chi = M^2_L (d, c_2) \chi, \] (17)

with

\[ M^2_L (d, c_2) = 4d \mathcal{K}(-4d) = 4d(1 - 4dc_2). \] (18)

The tree-level conditions for the three phases shown in Fig. 1 are

\[ P : \quad m^2_{LR} \geq 0 \quad m^2_{LR} + M^2_L \geq 0, \]
\[ F : \quad m^2_{LR} \leq 0 \quad M^2_L \geq 0, \]
\[ AF : \quad m^2_{LR} + M^2_L \leq 0M^2_L \leq 0. \] (19)

**B. The free propagator**

The free propagator,

\[ < \phi(x) \phi(y) > = \int_{p \leq \pi} \frac{d^dp}{(2\pi)^d} e^{-ipx} G(p), \] (20)

is given by

\[ G^{-1}(p) = \tilde{m}^2_{LR} + \hat{p}_\mu \hat{p}^\mu \mathcal{K}(-\hat{p}_\mu \hat{p}^\mu), \] (21)

where mass parameter with the tilde includes the shift due to the condensate.
\[ \tilde{m}_{LR}^2 = \begin{cases} m_{LR}^2, & \text{P}, \\ -2m_{LR}^2, & \text{F}, \\ -2m_{LR}^2 - 3M_L^2(d, c_2), & \text{AF}, \end{cases} \]  

in the different phases and

\[ \hat{p}_\mu = 2 \sin \frac{p_\mu}{2}. \]  

We further write

\[ G^{-1}(p) = P^2(p) - c_2P^4(p) + \tilde{m}_{LR}^2, \]  

with the help of

\[ P^2(p) = 4 \sum_\mu \sin^2 \frac{p_\mu}{2}. \]  

It is advantageous to divide the Brillouin zone,

\[ \mathcal{B} = \left\{ \mathbf{k}_\mu, \ |k_\mu| \leq \pi \right\}, \]  

into \(2^d\) restricted zones,

\[ \mathcal{B}_\alpha = \left\{ |k_\mu - P_\mu(\alpha)| \leq \frac{\pi}{2} \right\}, \]  

whose centers are at

\[ P_\mu(\alpha) = \pi n_\mu(\alpha), \]  

where \(n_\mu(\alpha) = 0, 1\) and the index \(1 \leq \alpha \leq 2^d\) is given by

\[ \alpha = 1 + \sum_{\mu=1}^{d} \alpha_\mu 2^{\mu-1}. \]  

The propagator for the zone \(\mathcal{B}_\alpha\) is

\[ G_\alpha(p) = G(P(\alpha) + p). \]  

It turns out that only the Brillouin zones \(\alpha = 1\) and \(2^d\) contain particle like excitations and the corresponding propagators are

\[ G^{-1}_\alpha(p) = \tilde{m}_{LR}^2(\alpha) + Z(\alpha)p^2 + O(p^4), \]  

where the mass and the wave function renormalization constant are given in Table 1. Note that \(\tilde{m}_{LR}^2(1) = m_{LR}^2\).

The fact that the vacuum is a single Fourier mode offers the possibility of recovering the energy-momentum conservations in the anti-ferromagnetic phase. The possible translations which keep the vacuum invariant consist of an even number of shift of the integer lattice coordinates. The corresponding spectrum of the momentum operator is

\[ p_{AF\mu} = p_\mu \ (\text{mod} \pi). \]  

In fact, the function mod\(\pi\) substracts the part of the momentum which can be exchanged with the antiferromagnetic vacuum and the resulting value is conserved. In this manner the momentum non-conservation is traded for the exchange of the particle type, the "flavor dynamics".
C. Chiral symmetry

The chiral transformation

\[ \chi : \phi(x) \longrightarrow \chi(x)\phi(x), \]

which appears as the shift

\[ p_\mu \rightarrow p_\mu + P_\mu(2^d) \]

in the Fourier space is a symmetry of the lagrangian when

\[ c_2 = \frac{1}{4d}, \quad c_4 = 0. \]

The two particle species are degenerate in the chiral invariant theory.

The operator \( P_\pm = \frac{1}{2}(1 \pm \chi) \) identifies the fields which belong to the even or odd sublattices,

\[ P_\pm \phi_\pm = \phi_\pm. \]

The chiral transformation is represented by

\[ \phi_\pm \rightarrow \pm \phi_\pm, \]

so the chiral fields \( \phi_+ \) and \( \phi_- \) decouple in the chiral invariant theory. The inversion of odd number of coordinates exchanges the chiral fields. The low energy excitations in \( B_1 \) and \( B_{16} \) correspond to

\[ \tilde{\phi}_\pm = \phi_+ \pm \phi_- , \]

where the fields \( \phi_\pm \) are slowly varying. Thus the low energy excitations of the zones \( B_1 \) and \( B_{16} \) have space inversion parity +1 and −1, respectively.

III. THE EFFECTIVE POTENTIAL

The renormalization of the theory will be performed in the para-ferro and the \((1,2)\) antiferromagnetic phase in the one-loop order by making the effective potential cut-off independent. It is easy to verify that this latter is enough, i.e. the wavefunction renormalization constant is finite at the one-loop order, \( \delta Z = 0. \)

A. A one-loop diagram

In order to develop the appropriate notation we consider first a simple example, the contribution of the second graph of Fig. 2 in the most complicated case, the \((1,2)\) antiferromagnetic phase,
\[
\Sigma(k) = \frac{1}{2} \lambda_R^2 \varphi^2 \int_{p \leq \pi} \frac{d^d p}{(2\pi)^d} G(k + p) G(p + P(2^d)). \tag{39}
\]

The lattice cut-off of the loop integrals, \( p < \pi \), should always be understood as the constraint \( |p_\mu| < \pi \), for \( \mu = 1, \ldots, d \) imposed on the torus \( D \) unless it is stated otherwise. An integral like this can be written in a simpler form by the help of the following matrix notation. The loop integration is split into the sum over the \( 2^d \) restricted Brillouin zones,

\[
\int_{p \leq \pi} d^d p f(p) = \sum_{\alpha=1}^{2^d} \int_{p \leq \pi/2} d^d p f(P(\alpha) + p), \tag{40}
\]

in particular,

\[
\int_{p \leq \pi} d^d p G(p) = \sum_{\alpha=1}^{2^d} \int_{p \leq \pi/2} d^d p f(P(\alpha) + p) = \sum_{\alpha=1}^{2^d} \int_{p \leq \pi/2} d^d p G_\alpha(p). \tag{41}
\]

Returning to our one-loop integral (39) we find

\[
\Sigma_c(k) = \frac{1}{2} \lambda_R^2 \varphi^2 \sum_{\alpha=1}^{2^d} \int_{p \leq \pi/2} \frac{d^d p}{(2\pi)^d} G_\alpha(k + p) G_\alpha(p), \tag{42}
\]

where we have introduced the region complementer to \( \alpha \),

\[
\bar{\alpha} = 2^d + 1 - \alpha. \tag{43}
\]

To simplify further the latter expression we now promote \( \alpha \) to be an internal index distinguishing different kind of fluctuations and define the propagator,

\[
G_{\alpha,\beta}(p) = \delta_{\alpha,\beta} G(P(\alpha) + p), \tag{44}
\]

which is diagonal in this new internal space. The contribution considered to the self energy is then written in matrix notation,

\[
\Sigma(k) = \frac{1}{2} \lambda_R^2 \varphi^2 \int_{p \leq \pi/2} \frac{d^d p}{(2\pi)^d} \text{tr}[G(k + p) \gamma^{2^d} G(p)] \gamma^{2^d}], \tag{45}
\]

by the help of the matrix

\[
\gamma^{2^d}_{\alpha,\beta} = \delta_{\alpha+\beta,2^d+1}, \tag{46}
\]

which describes the change of the type of particle after scattering off the vacuum.

**B. The one-loop effective potential**

Let us denote the usual 1PI functions by \( \Gamma^{(n)}(p_1, \ldots, p_n) \). The 1PI function for the excitations of the type \( \alpha_1, \ldots, \alpha_n \) is given as

\[
\Gamma^{(n)}(P(\alpha_1) + p_1, \ldots, P(\alpha_n) + p_n). \tag{47}
\]
The generator function for the zero momentum excitations, the effective potential, is defined as

\[ V_{\text{eff}}(\Phi) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\alpha_1, \ldots, \alpha_n} \Phi_{\alpha_1} \cdots \Phi_{\alpha_n} \Gamma^{(n)}(P(\alpha_1), \ldots, P(\alpha_n)). \] (48)

The matrix \( \gamma^{2d} \) in (45) reflects a modification of the Feynman rules. Whenever a propagator \( G_{\alpha}(p) \) is inserted in a graph it contains the momentum \( P_\mu(\alpha) + p \). We keep track of the first term of this sum by introducing a \( 2^d \)-component field, \( \Phi_{\alpha} \), in such a manner that the \( \alpha \)-th component will be responsible of the excitations in \( B_\alpha \). Thus the Feynman rules are those of a \( 2^d \)-component field with the propagator (44) and each external line with \( p = 0 \) is represented by the insertion of the matrix

\[ \bar{\Phi} = \sum_{\alpha=1}^{2^d} \gamma^{\alpha} \Phi_{\alpha}, \] (49)

where

\[ \gamma_{\rho,\sigma}^{\alpha} = \prod_{\mu=1}^{d} \delta_{\sigma,\mu + \alpha_{\mu} (\text{mod} 2),0} \] (50)

takes care of the change of the particle type at each vertex due to the momentum flowing from the external leg. We will use either the index \( \alpha \) or its vector representative, \( n_\mu(\alpha) \), in the formulae.

Taking advantage of the matrix formalism introduced above we obtain

\[
V_{\text{eff}}(\Phi) = \frac{1}{2} \int_{p \leq \pi/2} \frac{d^d p}{(2\pi)^d} \text{tr} \ln[\mathcal{P}^2(P + p)\mathcal{K}(-\mathcal{P}^2(P + p))] \\
+ \bar{m}^2_{LR} + 6\lambda_R \bar{\Phi} \Phi + 3\lambda_R \Phi^2, \\
= \frac{1}{2} \int_{p \leq \pi/2} \frac{d^d p}{(2\pi)^d} \text{tr} \ln[\mathcal{P}^2(P + p)\mathcal{K}(-\mathcal{P}^2(P + p))] \\
+ \bar{m}^2_{LR} + 3\lambda_R (\bar{\Phi} + \Phi)^2, \] (51)

where the matrix \( P \) is given by

\[ P_{\alpha,\beta} = \delta_{\alpha,\beta} P(\alpha) \] (52)

and the vacuum field is

\[ \Phi = \varphi_1 \gamma^1 + \varphi_2 \gamma^{2d}. \] (53)

The complete one-loop effective potential \( V^{(0)}(\Phi) + V_{\text{eff}}^{(1)}(\Phi) \) for the background field

\[ \bar{\Phi} = \Phi_1 \gamma^1 + \Phi_2 \gamma^{2d}, \] (54)

is obtained in (A3) and (A11).
\[ V^{P(0)}(\Phi) = \frac{1}{2} \left( P^2(P(1))K(-P^2(P(1))) + m^2_{LR} + \delta m^2_L \right) \Phi^2_1 \\
+ \frac{1}{2} \left( P^2(P(2^d))K(-P^2(P(2^d))) + m^2_{LR} + \delta m^2_L \right) \Phi^2_{2d} \\
+ \frac{\lambda + \delta \lambda}{4} (\Phi^4_1 + \Phi^4_{2d} + 6\Phi^2_1\Phi^2_{2d}), \]

\[ V^{F(0)}(\Phi) = V^{P(0)}(\Phi + \varphi_F), \]

\[ V^{AF(0)}(\Phi) = V^{P(0)}(\Phi + \varphi_{AF}). \] (55)

\[ V^{P(1)}_{\text{eff}}(\Phi) = \frac{1}{2} \int_{p \leq \pi/2} \frac{d^d p}{(2\pi)^d} \sum_{\alpha=1}^{2^d-1} \ln \left\{ \right. \\
\left. \times \left[ P^2(P(\alpha) + p)K(-P^2(P(\alpha) + p)) \\
+ m^2_{LR} + 3\lambda_R(\Phi^2_{2d} + \Phi^2_1) \right] \\
\left[ P^2(P(\bar{\alpha}) + p)K(-P^2(P(\bar{\alpha}) + p)) \\
+ m^2_{LR} + 3\lambda_R(\Phi^2_{2d} + \Phi^2_1) \right] - 36\lambda^2_R\Phi^2_1\Phi^2_{2d} \right\} \]

\[ V^{F(1)}_{\text{eff}}(\Phi) = V^{P(1)}(\Phi + \varphi_F), \]

\[ V^{AF(1)}_{\text{eff}}(\Phi) = V^{P(1)}(\Phi + \varphi_{AF}). \] (56)

The mass parameter of the effective potential in the ferro- and the antiferromagnetic phase after the shift \( \Phi \rightarrow \Phi + \varphi \) is given by (22).

IV. THE RENORMALIZATION IN \( D = 4 \)

The divergences arising in the one-loop integral for the effective potential are isolated by expanding the logarithm in the integrand. We reintroduce the lattice spacing and use dimensional quantities in the rest of this paper. One finds three divergent integrals,

\[ \mathcal{D}_1 = \sum_{\alpha=1}^{16} \int_{p \leq \pi/2} \frac{d^d p}{(2\pi)^d} G_{\alpha}(p) \]

\[ \mathcal{D}_2 = \sum_{\alpha=1}^{16} \int_{p \leq \pi/2} \frac{d^d p}{(2\pi)^d} G_{\alpha}(p)^2 \]

\[ \mathcal{D}_2 = \sum_{\alpha=1}^{8} \int_{p \leq \pi/2^5} \frac{d^d p}{(2\pi)^d} G_{\alpha}(p)G_{\bar{\alpha}}(p), \] (57)

and the divergent part of \( V^{AF(1)}(\Phi) \) turns out to be
\[ V_{\text{div}}^{AF(1)}(\Phi) = \frac{C}{2} D_1 - \frac{C^2}{4} D_2 - \frac{B^2}{2} \bar{D}_2 \]
\[ = \frac{3}{2} \lambda_R D_1 [(\Phi_{16} + \varphi_{16})^2 + \Phi_1^2] \]
\[ - \frac{1}{8} \lambda_R^2 36 \bar{D}_2 [(\Phi_{16} + \varphi_{16})^4 + \Phi_1^4 + 6(\Phi_{16} + \varphi_{16})^2 \Phi_1^2] \]
\[ - \frac{9}{4} \lambda_R^2 \Delta D_2 [(\Phi_{16} + \varphi_{16})^4 + \Phi_1^4 - \frac{4}{3} (\Phi_{16} + \varphi_{16})^2 \Phi_1^2], \]

where
\[ \Delta D_2 = D_2 - 2 \bar{D}_2, \] (59)

and
\[ B = 6 \lambda_R \Phi_1 (\phi_{16} + \Phi_{16}), \]
\[ C = 3 \lambda_R (2 \varphi_{16} \Phi_{16} + \Phi_{16}^2 + \Phi_1^2). \] (60)

The corresponding expression for the ferromagnetic phase can be obtained by the exchange \( 1 \leftrightarrow 16 \) of the internal index. The condensate has to be set to zero, \( \varphi_\alpha = 0 \), in the expressions of the paramagnetic phase.

The choice of the mass counterterm,
\[ \delta m_L^2 = -3 \lambda_R D_1 - 9 \lambda_R^2 \varphi_{16}^2 \bar{D}_2, \] (61)
is straightforward after comparing (58) with (55). But there is a problem with the counterterm \( \delta \lambda \) because it can not eliminate the divergences for both particles in the same time when \( \Delta D_2 \neq 0 \).

**A. Renormalization with chiral symmetry**

The remedy of the problem of the divergences \( O(\Phi^4) \) comes from the observation that the chiral symmetry protects against the unwanted divergences. In fact, the chiral transformation, (33), acts as
\[ \Phi_\alpha \to \Phi_\bar{\alpha}, \]
\[ G_\alpha(p) \to G_\bar{\alpha}(p) \] (62)
on the variables of the effective potential and the propagator and the chiral symmetry requires
\[ G_\alpha(p) = G_\bar{\alpha}(p) \] (63)
which reduces the number of divergences since we gain the relation
\[ \Delta D_2 = D_2 - 2 \bar{D}_2 = 0. \] (64)
The divergent part of the effective potential is now written as
\[ V_{\text{div}}^{AF(1)}(\Phi) = 3\lambda R \mathcal{D}_1[(\Phi_{16} + \varphi_{16})^2 + \Phi_1^2] \\
- 9\lambda^2 R \mathcal{D}_2[(\Phi_{16} + \varphi_{16})^4 + \Phi_1^4 + 6(\Phi_{16} + \varphi_{16})^2\Phi_1^2]. \] (65)

Comparing it with (55) we arrive at the choice
\[ \delta \lambda = 18\lambda R \mathcal{D}_2. \] (66)

Thus one can eliminate the divergences of the chiral symmetrical theory in either of the phases by the help of the appropriate fine tuning of the parameters \( m_B^2 \) and \( \lambda_B \) of the original lagrangian. The chiral invariant theory is invariant under the exchange of the two degenerate particles. Using the chiral fields, \( \Phi_\pm = \Phi_1 \pm \Phi_{16} \), one can decouple the two particle modes. Let’s consider for example the case of the paramagnetic phase. Replacing in (55) the appropriate values of \( \delta m^2_L \) and \( \delta \lambda \), as given respectively in (61) and (66), from (55) and (56) we get for the effective potential along the chiral line \( \chi_P \) (see Fig.1),
\[ V_{\text{eff}}^P(\Phi_1, \Phi_{16}) = V_{\text{eff}}^{ch}(\Phi_+) + V_{\text{eff}}^{ch}(\Phi_-), \] (67)

where
\[ V_{\text{eff}}^{ch}(\Psi) = \frac{1}{2} m_R^2 \Psi^2 + \frac{1}{4} \lambda_R \Psi^4 + \frac{1}{2} \sum_{\alpha=1}^{2^{d-1}} \int_{p \leq \frac{\pi}{a\alpha}} \frac{d^4p}{(2\pi)^4} \]
\[ \times \ln \left[ (P(\alpha) + p)^2 K \left( \frac{(2\pi)^2}{\Lambda^2} (P(\alpha) + p)^2 \right) + m_R^2 + 6\lambda_R \Psi^2 \right] \\
- 3\lambda_R \Psi^2 \int_{p \leq \frac{\pi}{a\alpha}} \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 + m_R^2} \\
+ 9\lambda_R^2 \Psi^2 \int_{p \leq \frac{\pi}{a\alpha}} \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2 + m_R^2)^2}. \] (68)

The same is true along the chiral lines in the other phases. This decoupling arises because in either of the phases at the chiral line the lattice decouples into two different sublattices as explained in Ref. [1].

**B. Renormalization around the symmetrical point**

In order to remove the symmetry with respect to the exchange of the two particles we consider the four dimensional theory with the tree-level cut-off dependence
\[ m_{LR}^2 = m_R^2 a^2, \quad c_2 = \frac{1}{16} \left[ 1 + \sigma(\mu)^{2+\kappa} \right], \] (69)

where \( \sigma = \pm 1 \) and \( \mu \) is a mass parameter to characterize the split of the degeneracy in the spectrum,
\[ \mathcal{M}_L^2 = -16\sigma(\mu)^{2+\kappa}. \] (70)

The quantities referring to the symmetrical theory, \( \mu = 0 \), will be labelled with a star. Since there are several possibilities in reaching the continuum limits as shown in Fig. 1b we collect
the corresponding conditions for \( m^2_R(2^d) = m^2_R(1) \) in Table 2. We will find that the one-loop corrections do not change qualitatively the tree-level spectrum. The degeneracy, \( m^2_R(2^d) = m^2_R(1) \), is achieved analytically at the chiral line indicating that the chiral symmetry is not broken dynamically. We find \( m^2_R(1) < m^2_R(2^d) \) in the phase \( F \) and in the region \( P_F \) (\( P_F \) is the region of the paramagnetic phase on the left side of the chiral line \( \chi_F \), see Fig.1b). On the contrary, \( m^2_R(2^d) < m^2_R(1) \) in the regions \( AF \) and \( P_AF \) (on the right side of the chiral line \( \chi_F \) in Fig.1b). We found no singularities in the effective potential due to the discontinuity in the momentum of the condensate when the chiral line is reached from the phases \( F \) or \( AF \).

The complication we face is that there will be finite \( \mu \)-dependent corrections from the counterterms in the vicinity of the symmetrical theory, \( D_2 \neq 2\bar{D}_2 \). The detailed study of the \( \mu \) dependence in the limit \( a \to 0 \) is presented in Appendix B. One finds that the \( \mu \)-dependence drops out from the finite part of the effective potential and is finite for \( D_2 \) and \( \bar{D}_2 \). By introducing

\[
D_2 = D^*_2 + \delta D_2, \\
\bar{D}_2 = \bar{D}^*_2 + \delta \bar{D}_2,
\]

with \( D^*_2 = 2\bar{D}^*_2 \) one finds the finite expressions

\[
\delta D_2 = -\frac{1}{16\pi^2} \ln \frac{\tilde{m}^2_R(1)}{\tilde{m}^2_R(16)}, \\
\delta \bar{D}_2 = -\frac{1}{16\pi^2} \frac{\tilde{m}^2_R(16)}{\tilde{m}^2_R(16) - \tilde{m}^2_R(1)} \ln \frac{\tilde{m}^2_R(16)}{\tilde{m}^2_R(1)},
\]

(72)

with

\[
\tilde{m}^2_R(1) = \begin{cases} 
  m^2_R & \text{P} \\
  -2m^2_R & \text{F} \\
  -2m^2_R + 48\mu^2(a\mu)^{\kappa} & \text{AF},
\end{cases}
\]

(73)

and

\[
\tilde{m}^2_R(16) = \begin{cases} 
  m^2_R - 16\sigma\mu^2(a\mu)^{\kappa} & \text{P}, \\
  -2m^2_R + 16\mu^2(a\mu)^{\kappa} & \text{F}, \\
  -2m^2_R + 32\mu^2(a\mu)^{\kappa} & \text{AF}.
\end{cases}
\]

(74)

These expressions lead to the counterterms

\[
\delta m^2 = -3\lambda_R D_1 - 9\lambda^2_{R\varphi^2_{16}} \varphi_D^2, \\
\delta \lambda = 18\lambda^2_R \varphi_D^2.
\]

(75)

It is well known that the spontaneous symmetry breaking in a ferromagnetic theory changes the counterterms by a cut-off independent finite piece and influences the renormalization group flow at finite energies only. One could, in principle, encounter a different situation in the antiferromagnetic phase because the condensate is formed at the cut-off scale. Furthermore one band of the elementary excitations, in \( B_{16} \), belongs to the staggered modes which show fast oscillation at the cut-off scale. It is the fine tuning of the value of the minimum of the dispersion relation in the zone \( B_{16} \) which eliminates the divergent phase dependence in the counterterms and restricts the effects of the phase transitions in the infrared region.
C. Mass spectrum

We are now in the position to follow the renormalization in the vicinity of the critical system. The effective potential is written as the sum of the finite and divergent part,

\[ V_{\text{eff}}^{(1)} = V_{\text{fin}}^{(1)} + V_{\text{div}}^{(1)}, \]

where the second term in the right hand side is defined by (58). One should bear in mind that starting with a single mass parameter in the bare lagrangian we have already introduced different masses for the propagators in the zones \( B_1 \) and \( B_{16} \). The physical masses which contain the radiative corrections are given by the derivative of the effective potential.

The Brillouin zone \( B_1 \): The mass square of the excitations is given by

\[ \partial^2_{\Phi_1} V_{\text{eff}}^{AF}(\Phi) \bigg|_{\Phi=0} = \tilde{m}_R^2 + \delta m^2 + 3(\lambda_R + \delta \lambda) \varphi_{16}^2 + 3\lambda_R \Delta D_1 - 9\lambda_R^2 \varphi_{16}^2 \Delta D_2 - 36\lambda_R^2 \varphi_{16}^2 \bar{D}_2 + \partial^2_{\Phi_1} V_{\text{fin}}^{AF(1)}(\Phi) \bigg|_{\Phi=0}, \]

\[ = m_{\text{ph}}^2(1) \]  

By the help of the counterterms (75) we find

\[ m_{\text{ph}}^2(1) = \tilde{m}_R^2 + 3\lambda_R \varphi_{16}^2 (1 - 3\lambda_R \delta \Delta D_2 - 12\lambda_R \delta \bar{D}_2) + \partial^2_{\Phi_1} V_{\text{fin}}^{AF(1)}(\Phi) \bigg|_{\Phi=0}, \]

where

\[ \delta \Delta D_2 = -\frac{1}{16\pi^2} \ln \frac{-\tilde{m}_R^2 + 16\mu^2(a\mu)^\kappa}{-\tilde{m}_R^2 + 24\mu^2(a\mu)^\kappa}, \]

\[ \delta \bar{D}_2 = \frac{1}{16\pi^2} \left( 1 + \frac{-\tilde{m}_R^2 + 16\mu^2(a\mu)^\kappa}{-8\sigma\mu^2(a\mu)^\kappa} \ln \frac{-\tilde{m}_R^2 + 16\mu^2(a\mu)^\kappa}{-\tilde{m}_R^2 + 24\mu^2(a\mu)^\kappa} \right). \]

The computation of the finite part of the effective potential in Appendix C yields vanishing result for the second derivatives with respect either field variables in all phases. So we arrive at

\[ m_{\text{ph}}^2(1) = -2\tilde{m}_R^2 + 48\mu^2(\kappa) - 18\lambda_R^2 \varphi_{16}^2 \Delta D_2(\kappa), \]

in the continuum limit with

\[ \mu^2(\kappa) = \lim_{a \to 0} \mu^2(a\mu)^\kappa = \begin{cases} 0 & \kappa > 0, \\ \mu^2 & \kappa = 0, \\ \infty & \kappa < 0, \end{cases} \]

and

\[ \Delta D_2(\kappa) = \lim_{a \to 0} [\Delta D_2(\mu) - 2\delta \bar{D}_2(\mu)] \]

\[ = \lim_{a \to 0} [\delta \Delta D_2 - 2\delta \bar{D}_2] \]

\[ = \begin{cases} \frac{\tilde{m}_R^2(1)}{\tilde{m}_R^2(1) - \tilde{m}_R^2(16)} \ln \frac{\tilde{m}_R^2(1)}{\tilde{m}_R^2(16)} & \kappa = 0, \\ 0 & \kappa > 0. \end{cases} \]
The Brillouin zone $B_{16}$: One finds
\[
\partial^2_{\Phi_{16}} V_{\text{eff}} (\Phi) \bigg|_{\Phi=0} = G^{-1}_{16}(0) + \delta m^2 + 3\delta \lambda \varphi^2_{16} + 3\lambda R D_1 \\
-36\lambda_R^2 \varphi^2_{16} D_2 + \partial^2_{\Phi_{16}} V_{\text{fin}}^{AF(1)} (\Phi) \bigg|_{\Phi=0} \\
= -2\bar{m}^2_R + 32\mu^2 (\mu\kappa) + 27\lambda_R^2 \varphi^2_{16} (2\delta D_2 - \delta D_2) \\
= m^2_{ph}(16),
\]
which results
\[
m^2_{ph}(16) = -2\bar{m}^2_R + 32\mu^2 (\kappa) - 27\lambda_R^2 \varphi^2_{16} \Delta \bar{D}_2 (\kappa).
\]

Our conclusion is that for $\kappa > 0$ $\delta \bar{D}_2 = \delta D_2 = 0$ so the two particles become degenerate and the chiral symmetry is restored in the continuum limit. For $\kappa = 0$ the mass spectrum stays non-degenerate. Finally the masses diverge as expected when $\kappa < 0$.

The ferromagnetic phase: We have, in a similar manner
\[
m^2_{ph}(1) = -2m^2_R - 18\lambda_R^2 \varphi^2_1 \Delta D_2 (\kappa),
\]
\[
m^2_{ph}(16) = -2m^2_R + 16\mu^2 (\kappa) - 27\lambda_R^2 \varphi^2_1 \Delta \bar{D}_2 (\kappa).
\]

The paramagnetic phase: The renormalized masses for the line $\chi_p$ are
\[
m^2_{ph}(1) = m^2_R,
\]
\[
m^2_{ph}(16) = m^2_R + 16\mu^2 (\kappa).
\]
showing the presence of the chiral symmetry. In the remaining part of the paramagnetic phase we find a non-degenerate spectrum,
\[
m^2_{ph}(1) = m^2_R,
\]
\[
m^2_{ph}(16) = m^2_R + 16\mu^2 (\kappa).
\]
The $P - AF$ transition line corresponds to the spectrum
\[
m^2_{ph}(1) = m^2_R,
\]
\[
m^2_{ph}(16) = 16\mu^2 \lim_{a \to 0} (\mu a)^\kappa.
\]

D. Coupling constant renormalization

The Brillouin zone $B_1$: The definition of the renormalized coupling constant is
\[
\partial^4_{\Phi_1} V_{\text{eff}} (\Phi) \bigg|_{\Phi=0} = 6(\lambda_R + \delta \lambda) - 54\lambda_R^2 D_2 + \partial^4_{\Phi_1} V_{\text{fin}}^{AF(1)} (\Phi) \bigg|_{\Phi=0} \\
= 6\lambda_{ph}(1),
\]
giving
\[
\lambda_{ph}(1) = (\lambda_R + \delta \lambda) - 9\lambda_R^2(D_2^* + \delta D_2) + \frac{1}{6} \partial_{\Phi_1}^4 V_{fin}^{AF(1)}(\Phi)|_{\Phi=0}. \tag{91}
\]

With our choice of the counterterms we have
\[
\lambda_{ph}(1) = \lambda_R - 9\lambda_R^2 \lim_{a \to 0} \delta D_2 + \frac{1}{6} \lim_{a \to 0} \partial_{\Phi_1}^4 V_{fin}^{AF(1)}(\Phi)|_{\Phi=0} \tag{92}
\]
in the continuum limit.

The Brillouin zone \( \mathcal{B}_{16} \): The self-coupling constant for the field \( \Phi_{16} \) is
\[
\lambda_{ph}(16) = \lambda_R - 9\lambda_R^2 \lim_{a \to 0} \delta D_2 + \frac{1}{6} \lim_{a \to 0} \partial_{\Phi_{16}}^4 V_{fin}^{AF(1)}(\Phi)|_{\Phi=0}. \tag{93}
\]

For the coupling constant which mixes the two fields we have
\[
\partial_{\Phi_1}^2 \partial_{\Phi_{16}}^2 V_{eff}^{AF}(\Phi)|_{\Phi=0} = 6(\lambda_R + \delta \lambda) - 18\lambda_R^2(D_2^* + \delta D_2) - 72\lambda_R^2(D_2^* + \delta D_2) + \partial_{\Phi_1}^2 \partial_{\Phi_{16}}^2 V_{fin}^{AF(1)}(\Phi)|_{\Phi=0} \tag{94}
\]
In the continuum limit it is
\[
\lambda_{ph}(1, 16) = \lambda_R - 3\lambda_R^2 \lim_{a \to 0} (\delta D_2 - 4\delta D_2) + \frac{1}{6} \lim_{a \to 0} \partial_{\Phi_1}^2 \partial_{\Phi_{16}}^2 V_{fin}^{AF(1)}(\Phi)|_{\Phi=0}. \tag{95}
\]
The finite part of the effective potential, \( V_{fin}^{AF(1)}(\Phi) \), is computed in Appendix C. The corresponding expressions in the ferro- and the paramagnetic phases are formally the same.

V. A LOW ENERGY EFFECTIVE THEORY

Our theory with a single quantum field contains two particles and its antiferromagnetic vacuum is in the ultraviolet regime. So it is not obvious that the evolution of the coupling constants for the two particle like excitations obeys the renormalization group equations which hold for the usual para- or ferromagnetic theories. In order to obtain the renormalization group equation for the potential of the model we introduce the running cut-off, \( k \), implemented in each restricted Brillouin zone in a spherical symmetric manner,
\[
\mathcal{D}_\alpha(k) = \left\{ \left( p - \frac{\Lambda}{2} n(\alpha) \right)^2 \leq k^2 \right\}, \tag{96}
\]
where \( \Lambda = 2\pi/a \) and the contributions coming from the edges of the toroidal Brillouin zones are left out. We approximate the dispersion relation in \( \mathcal{D}_1(k) \) and \( \mathcal{D}_{16}(k) \) with an \( O(4) \) invariant parabola and neglect the non-particle like excitations. These approximations involve irrelevant operators of the perturbative continuum limit which should not influence the finite energy behavior. Thus the renormalization group equation \cite{10} for the potential is
\[ k \partial_k V_k^P(\Phi_1, \Phi_2) = \frac{1}{2} \Omega_d k^d \ln \left\{ \left[ k^2 + \tilde{m}_R^2(1) + \partial_\Phi^2, V_k^P(\Phi) \right] \right. \]
\[ \left. \left[ k^2 + \tilde{m}_R^2(2^d) + \partial_{\Phi_2}^2 V_k^P(\Phi) \right] - \partial_{\Phi_2} \partial_\Phi V_k^P(\Phi) \right\} \]
\[ V_k^F(\Phi) = V_k^P(\Phi + \varphi_F), \]
\[ V_k^{AF}(\Phi) = V_k^P(\Phi + \varphi_{AF}), \] (97)

in the leading order of the gradient expansion where \( \Omega_d \) stands for the solid angle in \( d \) dimensions. The coefficients of the higher order terms in \( k^2 \) in the logarithm are kept fixed in our approximation.

Consider now the following renormalizable continuum lagrangian for a scalar and a pseudoscalar field, \( \tilde{\phi}_+(x), \tilde{\phi}_-(x) \), respectively, with momentum space cut-off,
\[ L = \frac{1}{2} (\partial_\mu \tilde{\phi}_+)^2 + \frac{1}{2} (\partial_\mu \tilde{\phi}_-)^2 + V(\tilde{\phi}_+, \tilde{\phi}_-), \] (98)

whose renormalization group equation in the leading order of the gradient expansion is
\[ k \partial_k V_k(\tilde{\phi}_+, \tilde{\phi}_-) = \frac{1}{2} \Omega_d k^d \ln \left\{ \left[ k^2 + \partial_\tilde{\phi}_+^2 V_k(\tilde{\phi}) \right] \left[ k^2 + \partial_\tilde{\phi}_-^2 V_k(\tilde{\phi}) \right] - \partial_\tilde{\phi}_- \partial_\tilde{\phi}_+ V_k(\tilde{\phi}) \right\}. \] (99)

The renormalization group flow of this model agrees with our higher derivative theory at low energies when the initial condition
\[ V_\Lambda(\tilde{\phi}_+, \tilde{\phi}_-) = \frac{m_B^2}{2} \tilde{\phi}_+^2 + \frac{m_B^2}{2} \tilde{\phi}_-^2 + \lambda_B (\tilde{\phi}_+^4 + \tilde{\phi}_-^4 + 6 \tilde{\phi}_+^2 \tilde{\phi}_-^2) \] (100)
is chosen. In other words, the model (98), (100) is equivalent with (3) at low energy when the continuum limit is taken. The correspondence between the phases is the following,
\[ P \iff < \tilde{\phi}_+ > = 0, < \tilde{\phi}_- > = 0, \]
\[ F \iff < \tilde{\phi}_+ > \neq 0, < \tilde{\phi}_- > = 0, \]
\[ AF \iff < \tilde{\phi}_+ > = 0, < \tilde{\phi}_- > \neq 0. \] (101)

The conserved momentum of the antiferromagnetic phase is (32) and the exchange of the momentum \( \pi n(16)\mu/2 \) on the lattice with the vacuum corresponds to the exchange of the scalar and the pseudoscalar particle. Due to the vertex \( \tilde{\phi}_+^2 \tilde{\phi}_-^2 \) in the lagrangian a pseudoscalar particle can decay into two scalar ones in the antiferromagnetic phase and the parity is not conserved.

The one-loop scaling laws of our theory agree with a usual two component \( \phi^A \) model up to irrelevant terms. Thus one may suppose that our theory is not asymptotically free and consequently becomes trivial in the continuum limit. In this case when the cut-off can not be eliminated from the interacting theory the irrelevant terms which were neglected in the comparison might be important and generate different physical content.
VI. CONCLUSIONS

The one-loop vacuum polarization effects were studied in the para-, ferro- and (1, 2) antiferromagnetic phases of the four dimensional $\phi^4$ model around the chiral invariant critical point. One can identify two particle like excitations in each phase. The one-loop divergences were eliminated by an appropriate fine tuning of the parameters of the bare lagrangian and the resulting theory was found to be equivalent at low energies with a usual renormalizable model made by a scalar and a pseudoscalar field. In this continuum limit where the length scale of antiferromagnetic vacuum or the pseudoscalar staggered particle mode tends to zero the well known problems about the unitarity disappear.

One should emphasize that even though the cut-off can be removed and the continuum limit can be taken at the one-loop level the theory can be defined by relying heavily on the regulator. The renormalized continuum theory exists only when the regulator is taken into account both at the tree- and loop-levels in a systematical manner.

The possibility of removing the divergences in the presence of an apparently non-renormalizable term in the lagrangian is in principle a serious threat for the usual strategy of Particle Physics where the universality is used to limit our investigations to the class of renormalizable theories. But the result that our model reproduces the infrared structure of a conventional renormalizable one is reassuring because it indicates that there is no new universality class encountered.

The antiferromagnetic phase is certainly different compared to the usual $\phi^4$ model with $c_j = 0$. But even the para- and the ferromagnetic phases in our higher derivative model become unusual, as well, in the vicinity of the chiral invariant critical point. This is because the dispersion relation develops in all of these phases a second minimum which can be fine tuned around this critical point in such a manner that another particle like excitation, the analogue of the rotons of superfluids, appears. This particle has staggered excitation modes which allow us to introduce the chiral fields which are exchanged between each other under the space-time inversions. These chiral fields decouple in the chiral symmetrical theory. By considering field variables only in the sublattice of the even lattice one can construct models with a single chiral boson. Such a decoupling of the modes is reminiscent of the fermion doubling on the lattice and the resulting model with a single chiral boson is local and satisfies the reflection positivity.

Our computation was made at the one-loop level only. There is no conceptual problem in extending our work to higher loop orders though the treatment of the overlapping divergences with unconventional dispersion relation represents a challenging problem. It remains to be seen if the perturbative elimination of the divergences can be achieved beyond the one-loop order. If a theory in the antiferromagnetic phase turns out to be renormalizable then its vacuum appears homogeneous in physical measurements. It is the structure of the excitations only which betrays the non-trivial structure of the vacuum of such a theory. There are numerical indications of the continuum limit in the antiferromagnetic phase for other models with antiferromagnetic vacuum.

One should mention that there are other possible continuum limits in our model away from the chiral invariant critical region when the mass parameter is kept at a cut-off independent value. This parameter plays a role analogous of the $\kappa$-parameter of the Wilson fermions. In fact, the excitations of the restricted zone $B_1$ decouple when $m^2_{LR} = O(a^0) \neq 0$. 

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The only left over excitations in $B_{16}$ become critical along the P-AF transition line. Thus the approach of the critical line with a fixed mass parameter results in a theory which contains a single pseudoscalar particle. In a similar manner certain regions of the $c_4 \neq 0$ part of the phase diagram may become critical and offer a continuum limit. This is because the renormalizability is a rather straightforward issue when only one particle is left in our model.

Finally we mention the problem of triviality. It is a frustrating experience that the simplest models such as the $\phi^4$ and QED which are used in the textbooks to demonstrate the renormalization of Quantum Field Theories might well be non-renormalizable if they are found to be trivial. In this case the study of their ultraviolet scaling behaviour serves phenomenological interest and a real ultraviolet fixed point can be achieved by asymptotically free models only. The one-loop ultraviolet structure of our theory turned out to be similar to the conventional $\phi^4$ model. This leads us to assume that our theory is not asymptotically free and perhaps trivial because its coupling constant which is marginal at the tree-level becomes irrelevant due to the one-loop contributions. This suggests the extension of the investigation of the antiferromagnetic vacuum to other, more involved asymptotically free models which may preserve their renormalizability and offer a more consistent example of a non-homogeneous vacuum which actually appears homogeneous in the experiments.
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FIG. 1. The phase boundary between the paramagnetic (P), ferromagnetic (F) and the antiferromagnetic (AF) phase for $c_4 = 0$. The two particles are degenerate along the chiral symmetric lines $\chi_P$, $\chi_F$, $\chi_{AF}$. (a): The plane $(m^2_{LR}, M_L^2)$; (b): The plane $(c_2, m^2_{LR})$. The arrows show the different continuum limits at the critical point CR.

FIG. 2. The one-loop self-energy graphs.
TABLE I. The parameters of the propagator for $B_1$ and $B_{16}$.

| phase | $\tilde{m}^2_{LR}(1)$ | $\tilde{m}^2_{LR}(16)$ | $Z(1)$ | $Z(16)$ |
|-------|---------------------|---------------------|--------|--------|
| P     | $m^2_{LR}$          | $m^2_{LR} + M^2_L$  | 1      | $-1 + 32c_2$ |
| F     | $-2m^2_{LR}$        | $-2m^2_{LR} + M^2_L$| 1      | $-1 + 32c_2$ |
| AF    | $-2m^2_{LR} - 3M^2_L$| $-2m^2_{LR} - 2M^2_L$| 1      | $-1 + 32c_2$ |

TABLE II. The different ways of approaching the critical point of Fig.1b. $P_F$ and $P_{AF}$ are the regions in the paramagnetic phase respectively on the left and on the right side of the chiral line $\chi_P$.

| phase | $\kappa$ | $\sigma$ | $m^2_R$ | $\tilde{m}^2_R(16)$ | $\tilde{m}^2_R(1) - \tilde{m}^2_R(16)$ |
|-------|----------|----------|--------|---------------------|---------------------------------|
| $P_F$ | 0        | -1       | > 0    | $m^2_R + 16\mu^2(\kappa)$ | $-16\mu^2(\kappa)$ |
| $\chi_P$ | > 0     | ±1       | > 0    | $m^2_R$               | 0                               |
| $P_{AF}$ | 0      | +1       | > $16\mu^2$ | $m^2_R - 16\mu^2(\kappa)$ | $16\mu^2(\kappa)$ |
| $P - AF$ | 0      | +1       | = $16\mu^2$ | 0                     | $16\mu^2(\kappa)$ |
| $AF$   | 0        | +1       | < $16\mu^2$ | $-2m^2_R + 32\mu^2(\kappa)$ | $16\mu^2(\kappa)$ |
| $\chi_{AF}$ | > 0     | +1       | < 0    | $-2m^2_R$               | 0                               |
| $\chi_F$ | > 0     | -1       | < 0    | $-2m^2_R$               | 0                               |
| $F$    | 0        | -1       | < 0    | $-2m^2_R + 16\mu^2(\kappa)$ | $-16\mu^2(\kappa)$ |
ACKNOWLEDGMENTS

We thank Jan Stern for interesting discussions.

APPENDIX A: COMPUTATION OF THE EFFECTIVE POTENTIAL

The tree-level: The tree-level effective potential for the background field

\[ \Phi = \Phi_1 \gamma^1 + \Phi_2 \gamma^2, \tag{A1} \]

in the antiferromagnetic phase is the sum of the renormalized potential and the counterterms,

\[ V_{AF}^{(0)}(\Phi) = \frac{1}{2} \left[ m_{LR}^2 + \delta m_L^2 + 3(\lambda_R + \delta \lambda) \varphi_{2d}^2 \right] \Phi_1^2 \]

\[ + \frac{1}{2} \left[ P^2(2d) K(-P^2(2d)) + m_{LR}^2 + \delta m_L^2 + 3(\lambda_R + \delta \lambda) \varphi_{2d}^2 \right] \Phi_2^2 \]

\[ + (\delta m_L^2 + \delta \lambda \varphi_{2d}^2) \varphi_{2d} \Phi_{2d} + \varphi_{2d} (\lambda_R + \delta \lambda) (\Phi_{2d}^3 + 3\Phi_{2d}^2 \Phi_{2d}) \]

\[ + \frac{\lambda_R + \delta \lambda}{4} (\Phi_1^4 + \Phi_{2d}^4 + 6\Phi_1^2 \Phi_{2d}^2), \tag{A2} \]

what can be written up to a constant as

\[ V_{AF}^{(0)}(\Phi) = \frac{1}{2} \left[ P^2(1) K(-P^2(1)) + m_{LR}^2 + \delta m_L^2 \right] \Phi_1^2 \]

\[ + \frac{1}{2} \left[ P^2(2d) K(-P^2(2d)) + m_{LR}^2 + \delta m_L^2 \right] (\Phi_{2d}^2 + \varphi_{2d}^2)^2 \]

\[ + \frac{\lambda_R + \delta \lambda}{4} [\Phi_1^4 + (\Phi_{2d} + \varphi_{2d})^4] + \frac{3}{2} (\lambda_R + \delta \lambda) \Phi_1^2 (\Phi_{2d} + \varphi_{2d})^2. \tag{A3} \]

The one-loop level: The next step is to obtain the one-loop contribution,

\[ V_{AF}^{(1)}(\Phi) = \frac{1}{2} \int_{p \leq \pi/2} \frac{d^d p}{(2\pi)^d} \text{tr} \ln \left[ (P + p)^2 K(-(P + p)^2) \right. \]

\[ + m_{LR}^2 + 6\lambda_R (\Phi_1 + \gamma^2 \Phi_{2d}) \varphi_{2d} \gamma^2 \]

\[ + 3\lambda_R (\Phi_1 + \gamma^2 \Phi_{2d})^2 \]. \tag{A4} \]

Since \((\gamma^2)^2 = \gamma^1 = 1\) we can write this integral as

\[ \int_{p \leq \pi/2} \frac{d^d p}{(2\pi)^d} \ln \left| A(p) + B \gamma^1 \right|, \tag{A5} \]

where
\begin{equation}
A(p) + B \gamma^{2d} = \begin{pmatrix}
A_1(p) & 0 & \cdots & 0 & 0 \\
0 & A_2(p) & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & A_{2d-1}(p) & 0 \\
0 & 0 & \cdots & 0 & A_{2d}(p)
\end{pmatrix}
+ \begin{pmatrix}
0 & 0 & \cdots & 0 & B \\
0 & 0 & \cdots & B & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & B & \cdots & 0 & 0 \\
B & 0 & \cdots & 0 & 0
\end{pmatrix}
\end{equation}

(A6)

and

\begin{align*}
A_\alpha(p) &= G^{-1}_\alpha(p) + C, \\
B &= 6 \lambda_R \Phi_1(\varphi_2^d + \Phi_{2d}), \\
C &= 3 \lambda_R (2 \varphi_{2d} \Phi_{2d}^2 + \Phi_1^2). 
\end{align*}

(A7)

The determinant in question is

\begin{equation}
\det[A(p) + B \gamma^{2d}] = \prod_{\alpha=1}^{2d-1} (A_\alpha A_{\bar{\alpha}} - B^2). 
\end{equation}

(A8)

In this manner we obtain

\begin{equation}
V_{eff}^{AF(1)}(\Phi) = \frac{1}{2} \int_{\vec{p} \leq \pi/2} \frac{d^d p}{(2\pi)^d} \ln \det[A(p) + B \gamma^{2d}]
= \frac{1}{2} \int_{\vec{p} \leq \pi/2} \frac{d^d p}{(2\pi)^d} \sum_{\alpha=1}^{2d-1} \ln[A_\alpha(p) A_{\bar{\alpha}}(p) - B^2].
\end{equation}

(A9)

In order to isolate the UV divergences it is advantageous to write

\begin{align*}
V_{eff}^{AF(1)}(\Phi) &= \frac{1}{2} \int_{\vec{p} \leq \pi/2} \frac{d^d p}{(2\pi)^d} \sum_{\alpha=1}^{2d-1} \ln[(G_\alpha^{-1} + C)(G_{\bar{\alpha}}^{-1} + C) - B^2]
= \int_{\vec{p} \leq \pi/2} \frac{d^d p}{(2\pi)^d} \sum_{\alpha=1}^{2d-1} \ln[1 + CG_\alpha + CG_{\bar{\alpha}} + (C^2 - B^2)G_\alpha G_{\bar{\alpha}}]
+ \int_{\vec{p} \leq \pi/2} \frac{d^d p}{(2\pi)^d} \sum_{\alpha=1}^{2d-1} \ln[G_\alpha^{-1} G_{\bar{\alpha}}^{-1}]. 
\end{align*}

(A10)

The detailed form of (A9) reads as

\begin{align*}
V_{eff}^{AF(1)}(\Phi) &= \frac{1}{2} \int_{\vec{p} \leq \pi/2} \frac{d^d p}{(2\pi)^d} \sum_{\alpha=1}^{2d-1} \ln \left\{ \left[ (P(\alpha) + p)^2 K(- (P(\alpha) + p)^2) + m^2_{LR} \Phi_{2d}^2 + \Phi_1^2 \right] \times \right\}
\end{align*}
\[
\left[ (P(\bar{\alpha}) + p)^2 K(-(P(\bar{\alpha}) + p)^2) \\
+ m_{LR}^2 + 3\lambda_R (2\varphi_2 \Phi_2 + \Phi_2^2 + \Phi_2^0) \right] \\
\left[ (P(\bar{\alpha}) + p)^2 K(-(P(\bar{\alpha}) + p)^2) \\
+ m_{LR}^2 + 3\lambda_R (2\varphi_1 \Phi_1 + \Phi_1^2 + \Phi_1^0) \right]
\] \\
\left\{ (P(\bar{\alpha}) + p)^2 K(-(P(\bar{\alpha}) + p)^2) \\
+ m_{LR}^2 + 3\lambda_R (2\varphi_2 \Phi_2 + \Phi_2^2 + \Phi_2^0) \right\}.
\] (A11)

The corresponding expressions for the ferromagnetic phase are obtained by the exchange \(1 \leftrightarrow 2d\),

\[
V_{eff}^{(1)}(\Phi) = \frac{1}{2} \int_{p \leq \pi/2} \frac{d^d p}{(2\pi)^d} \sum_{\alpha=1}^{2d-1} \ln \left\{ \left[ (P(\alpha) + p)^2 K(-(P(\alpha) + p)^2) \\
+ m_{LR}^2 + 3\lambda_R (2\varphi_1 \Phi_1 + \Phi_1^2 + \Phi_1^0) \right] \right. \\
\left. \left[ (P(\bar{\alpha}) + p)^2 K(-(P(\bar{\alpha}) + p)^2) \\
+ m_{LR}^2 + 3\lambda_R (2\varphi_2 \Phi_2 + \Phi_2^2 + \Phi_2^0) \right] \right\}.
\] (A12)

In the paramagnetic phase one sets \(\varphi_\alpha = 0\),

\[
V_{eff}^{P(1)}(\Phi) = \frac{1}{2} \int_{p \leq \pi/2} \frac{d^d p}{(2\pi)^d} \sum_{\alpha=1}^{2d-1} \ln \left\{ \left[ (P(\alpha) + p)^2 K(-(P(\alpha) + p)^2) \\
+ m_{LR}^2 + 3\lambda_R (\Phi_1^2 + \Phi_1^0) \right] \right. \\
\left. \left[ (P(\bar{\alpha}) + p)^2 K(-(P(\bar{\alpha}) + p)^2) \\
+ m_{LR}^2 + 3\lambda_R (\Phi_2^2 + \Phi_2^0) \right] \right\}.
\] (A13)

**APPENDIX B: EXPANSION AROUND THE SYMMETRICAL THEORY**

We study in this Appendix the dependence of the loop integrals on the scale parameter \(\mu\) what controls the strength of the explicit breaking of the chiral symmetry. For this end we need the continuum limit for the integrals of the type

\[
\lim_{a \to 0} I_\alpha = \lim_{a \to 0} \int_{p \leq \pi/a} d^d p C_\alpha^m(p).
\] (B1)

Close to the symmetrical point the propagator has two maxima, namely in the regions \(\alpha = 1, 16\). In case of a convergent integral one expects that most of the contribution comes from the regions around these maxima. But due to the presence of divergencies the problem has to be considered more carefully.
The Brillouin zone $B_1$: The $\mu$-dependence shows up in the antiferromagnetic phase only. One expands around $p = 0$,

$$G_1^n(p) = \left(p^2 + \tilde{m}_R^2(1) + ba^2p^4 + ca^4p^6 + O(a^6p^8)\right)^n$$  \hspace{1cm} \text{(B2)}

where

$$b = b_0 + b_1(\mu a)^{2+\kappa}$$
$$c = c_0 + c_1(\mu a)^{2+\kappa}$$ \hspace{1cm} \text{(B3)}

are dimensionless functions. One finds

$$\int d^4p G_1^n(p) = \int d^4p \frac{1}{(p^2 + \tilde{m}_R^2(1))^n} \times \left[ 1 - n \frac{ba^2p^4 + ca^4p^6 + O(a^6p^8)}{p^2 + \tilde{m}_R^2(1)} \right.$$ \hspace{1cm} \text{(B4)}

$$+ \frac{n(n+1)}{2} \left( \frac{ba^2p^4 + ca^4p^6 + O(a^6p^8)}{p^2 + \tilde{m}_R^2(1)} \right)^2 + \cdots \right].$$

The corrections to the usual first term are of the form

$$\int \frac{a^k p^\ell}{(p^2 + \tilde{m}_R^2(1))^{n+m}}$$ \hspace{1cm} \text{(B5)}

where $\ell - k - 2m = 0$ in order to keep the dimension of each contribution. The one-loop integral is finite for $4 - 2(n+m) + \ell - k < 0$ i.e. for $n \geq 3$. Since $k \geq 0$ the finiteness implies vanishing corrections and $\mu$-independence,

$$\lim_{a \to 0} \int \frac{a^k p^\ell}{(p^2 + \tilde{m}_R^2(1))^{n+m}} = \int_{-\infty}^{\infty} d^4p(p^2 + \tilde{m}_R^2(1))^{-n},$$ \hspace{1cm} \text{(B6)}

for $n \geq 3$. The non-vanishing corrections arise for $n = 1$ and 2.

The mass renormalization can be carried out without difficulties for $\mu \neq 0$. In order to follow the renormalization of the coupling constant we need the $\mu$-dependence for $n = 2$,

$$\lim_{a \to 0} \int \frac{d^4p}{(2\pi)^4} G_1^2(p) = \int_{-\infty}^{\infty} \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2 + \tilde{m}_R^2(1))^2} + I_{\text{fin}}^*,$$

$$= \frac{1}{16\pi^2} \left( \frac{\Lambda^2}{\tilde{m}_R^2(1)} - 1 \right) + I_{\text{fin}}^*$$ \hspace{1cm} \text{(B7)}

where $\Lambda = 2\pi/a$ and $I_{\text{fin}}$ is a finite function of $\mu$. Since $I_{\text{fin}}$ depends on $\mu$ through the combination $(\mu a)^{2+\kappa}$ and the UV divergence is logarithmic only the finite part of (B7) becomes $\mu$-independent in the continuum limit.

The Brillouin zone $B_{16}$: The mass is $\mu$-dependent in each phase and we write $\tilde{m}_R^2(16) = \tilde{m}_R^2(1) + \tilde{m}_R^{2'}$ where

$$\tilde{m}_R^{2'} = -16\sigma \mu^2(\mu a)^\kappa.$$ \hspace{1cm} \text{(B8)}
The expansion is made around $p = P(16)$,

$$G_{16}^{-n}(p) = \left(p^2 + \tilde{m}_R^2(1) + \tilde{m}_R^2 + ba^2 p^4 + ca^4 p^6 + O(a^6 p^8)\right)^n,$$

(B9)

$$\int d^4 p G_{16}^n(p) = \int d^4 p \frac{1}{(p^2 + \tilde{m}_R^2(1))^n} \times \left[1 - n \frac{\tilde{m}_R^2 + ba^2 p^4 + ca^4 p^6 + O(a^6 p^8)}{p^2 + \tilde{m}_R^2(1)} + \frac{n(n + 1)}{2} \left(\frac{\tilde{m}_R^2 + ba^2 p^4 + ca^4 p^6 + O(a^6 p^8)}{p^2 + \tilde{m}_R^2(1)}\right)^2 + \cdots \right].$$

(B10)

The repetition of the argument followed in the previous case yields to the $\mu$-independent finite result,

$$\lim_{\alpha \to 0} \int_{-\frac{\pi}{2a}}^{\frac{\pi}{2a}} d^4 p G_{16}^n(p) = \int_{-\infty}^{\infty} \frac{d^4 p}{(p^2 + \tilde{m}_R^2(1) + \tilde{m}_R^2)^n},$$

(B11)

for $n \geq 3$. For $n = 2$ one finds

$$\lim_{\alpha \to 0} \int_{-\frac{\pi}{2a}}^{\frac{\pi}{2a}} d^4 p G_{16}^2(p) = \int_{-\infty}^{\infty} \frac{d^4 p}{(p^2 + \tilde{m}_R^2(1) + \tilde{m}_R^2)^2} + I_{fin}^* = \frac{1}{16 \pi^2} \left(\ln \frac{\Lambda^2}{\tilde{m}_R^2(1) + \tilde{m}_R^2} - 1\right) + I_{fin}^*$$

(B12)

$$= \frac{1}{16 \pi^2} \left(\ln \frac{\Lambda^2}{\tilde{m}_R^2(1)} - \ln \frac{\tilde{m}_R^2(16)}{\tilde{m}_R^2(1)} - 1\right) + I_{fin}^*.$$

There is a similar result for the mixed product,

$$\lim_{\alpha \to 0} \int_{-\frac{\pi}{2a}}^{\frac{\pi}{2a}} d^4 p G_1(p) G_{16}(p) = \int_{-\infty}^{\infty} \frac{d^4 p}{(2\pi)^4 (p^2 + \tilde{m}_R^2(1)) (p^2 + \tilde{m}_R^2(1) + \tilde{m}_R^2)}$$

$$+ I_{fin}^* = \frac{1}{16 \pi^2} \left(\ln \frac{\Lambda^2}{\tilde{m}_R^2(1) + \tilde{m}_R^2} - \ln \frac{\tilde{m}_R^2(16)}{\tilde{m}_R^2(1)}\right) + I_{fin}^*.$$  

(B13)

The regions $B_\alpha$, $\alpha = 2, \cdots, 15$: We can find a real number, $\gamma$, such that

$$G_\alpha(p) \leq \gamma a^2,$$

(B14)

or

$$\int_{-\frac{\pi}{2a}}^{\frac{\pi}{2a}} d^4 p G_\alpha^n(p) \leq \gamma^n a^{2n} \int_{-\frac{\pi}{2a}}^{\frac{\pi}{2a}} d^4 p.$$  

(B15)

This integral is vanishing in the continuum limit for $n \geq 3$. For $n = 2$ one has
\[ G_\alpha(p)G_\beta(p) = G_\alpha^*(p)G_\alpha^*(p) \left[ 1 + \sigma \mu^\kappa a^{\kappa-2} \left( \sin^4(p + P(\alpha))G_\beta^*(p) + \ldots \right) \right]. \quad \text{(B16)} \]

From
\[ \mu^n a^{n-2} \sin^4(p + P(\alpha)) \leq \mu a^{n-2} \quad \text{(B17)} \]
and
\[ G_\beta^*(p) \leq \gamma a^2 \quad \text{(B18)} \]
one obtains
\[ \mu^n a^{n-2} \sin^4(p + P(\alpha)) G_\beta^3(p) \leq \mu \gamma^3 a^{n+4} \quad \text{(B19)} \]
which yields the equation
\[ \lim_{a \to 0} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} d^4p G_\alpha(p)G_\beta(p) = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} d^4p G_\alpha^*(p)G_\beta^*(p) \quad \text{(B20)} \]
in the continuum limit.

So one finds the divergent part
\[ D_2 = D_2^* + \delta D_2, \]
\[ D_2 = D_2^* + \delta D_2, \quad \text{(B21)} \]
with \( D_2^* = 2D_2^* \) and
\[ \delta D_2 = -\frac{1}{16\pi^2} \ln \frac{\tilde{m}_R^2(16)}{\tilde{m}_R^2(1)} \]
\[ \delta \bar{D}_2 = \frac{1}{16\pi^2} \left( 1 - \frac{\tilde{m}_R^2(1)}{\tilde{m}_R^2(16)} \ln \frac{\tilde{m}_R^2(16)}{\tilde{m}_R^2(1)} \right). \quad \text{(B22)} \]

**APPENDIX C: THE FINITE PART OF THE EFFECTIVE POTENTIAL**

To compute the finite part of the effective potential we start from equation (A10), separate the finite contributions and cancel the divergences by the counterterms. Actually the finite part of the potential in the continuum limit depends only the terms containing \( G_1 \) et \( G_{16} \).

**Non-degenerate masses, \( (\kappa = 0) \):** We seek
\[ V_{fin}^{AF(1)}(\Phi) = \frac{1}{2} \int_{p \leq \Lambda} \frac{d^4p}{(2\pi)^4} \ln[(p^2 + \tilde{m}_R^2(1) + C)(p^2 + \tilde{m}_R^2(16) + C) - B^2] \]
\[ - \frac{C}{2} \int_{p \leq \Lambda} \frac{d^4p}{(2\pi)^4} \left( \frac{1}{p^2 + \tilde{m}_R^2(1)} + \frac{1}{p^2 + \tilde{m}_R^2(16)} \right) \]
\[ + \frac{C^2}{4} \int_{p \leq \Lambda} \frac{d^4p}{(2\pi)^4} \left( \frac{1}{(p^2 + \tilde{m}_R^2(1))^2} + \frac{1}{(p^2 + \tilde{m}_R^2(16))^2} \right) \]
\[ + \frac{B^2}{2} \int_{p \leq \Lambda} \frac{d^4p}{(2\pi)^4} \left( \frac{1}{(p^2 + \tilde{m}_R^2(1))(p^2 + \tilde{m}_R^2(16))} \right) \quad \text{(C1)} \]

\[ (\text{B16}) \]

From
\[ \mu^n a^{n-2} \sin^4(p + P(\alpha)) \leq \mu a^{n-2} \quad \text{(B17)} \]
and
\[ G_\beta^*(p) \leq \gamma a^2 \quad \text{(B18)} \]
one obtains
\[ \mu^n a^{n-2} \sin^4(p + P(\alpha)) G_\beta^3(p) \leq \mu \gamma^3 a^{n+4} \quad \text{(B19)} \]
which yields the equation
\[ \lim_{a \to 0} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} d^4p G_\alpha(p)G_\beta(p) = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} d^4p G_\alpha^*(p)G_\beta^*(p) \quad \text{(B20)} \]
in the continuum limit.

So one finds the divergent part
\[ D_2 = D_2^* + \delta D_2, \]
\[ D_2 = D_2^* + \delta D_2, \quad \text{(B21)} \]
with \( D_2^* = 2D_2^* \) and
\[ \delta D_2 = -\frac{1}{16\pi^2} \ln \frac{\tilde{m}_R^2(16)}{\tilde{m}_R^2(1)} \]
\[ \delta \bar{D}_2 = \frac{1}{16\pi^2} \left( 1 - \frac{\tilde{m}_R^2(1)}{\tilde{m}_R^2(16)} \ln \frac{\tilde{m}_R^2(16)}{\tilde{m}_R^2(1)} \right). \quad \text{(B22)} \]

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**Non-degenerate masses, \( (\kappa = 0) \):** We seek
\[ V_{fin}^{AF(1)}(\Phi) = \frac{1}{2} \int_{p \leq \Lambda} \frac{d^4p}{(2\pi)^4} \ln[(p^2 + \tilde{m}_R^2(1) + C)(p^2 + \tilde{m}_R^2(16) + C) - B^2] \]
\[ - \frac{C}{2} \int_{p \leq \Lambda} \frac{d^4p}{(2\pi)^4} \left( \frac{1}{p^2 + \tilde{m}_R^2(1)} + \frac{1}{p^2 + \tilde{m}_R^2(16)} \right) \]
\[ + \frac{C^2}{4} \int_{p \leq \Lambda} \frac{d^4p}{(2\pi)^4} \left( \frac{1}{(p^2 + \tilde{m}_R^2(1))^2} + \frac{1}{(p^2 + \tilde{m}_R^2(16))^2} \right) \]
\[ + \frac{B^2}{2} \int_{p \leq \Lambda} \frac{d^4p}{(2\pi)^4} \left( \frac{1}{(p^2 + \tilde{m}_R^2(1))(p^2 + \tilde{m}_R^2(16))} \right) \quad \text{(C1)} \]
which gives after integration

\[
V_{fin}^{AF(1)}(\Phi) = \frac{1}{128\pi^2} \left\{ (a_{nd}^2 - b_{nd}^2) \ln \frac{\tilde{m}_R^2(16)}{\tilde{m}_R^2(1)} + (a_{nd}^2 + b_{nd}^2 + d_{nd}) \ln \frac{2a_{nd}b_{nd} - d_{nd}}{2\tilde{m}_R^2(16)} \right. \\
+ d_{nd} \frac{\tilde{m}_R^2(16) + \tilde{m}_R^2(1)}{\tilde{m}_R^2(1) - \tilde{m}_R^2(16)} \ln \frac{\tilde{m}_R^2(16)}{\tilde{m}_R^2(1)} - 2c_{nd}(\tilde{m}_R^2(16) + \tilde{m}_R^2(1)) \\
- 6c_{nd}^2 - d_{nd} - (a_{nd} + b_{nd}) \sqrt{(a_{nd} - b_{nd})^2 + 2d_{nd}} \right. \\
\times \ln \frac{a_{nd} + b_{nd} - \sqrt{(a_{nd} - b_{nd})^2 + 2d_{nd}}}{a_{nd} + b_{nd} - \sqrt{(a_{nd} - b_{nd})^2 + 2d_{nd}}} \right\},
\]

(C2)

with

\[
a_{nd} = \tilde{m}_R^2(1) + 3\lambda_R(2\varphi_{16}\Phi_{16} + \Phi_{16}^2 + \Phi_1^2),
\]

\[
b_{nd} = \tilde{m}_R^2(16) + 3\lambda_R(2\varphi_{16}\Phi_{16} + \Phi_{16}^2 + \Phi_1^2),
\]

\[
c_{nd} = 3\lambda_R(2\varphi_{16}\Phi_{16} + \Phi_{16}^2 + \Phi_1^2),
\]

\[
d_{nd} = 72\lambda_R^2\Phi_1^2(\varphi_{16} + \Phi_{16})^2,
\]

\[
\varphi_{16}^2 = \frac{1}{\lambda_R}(-m_R^2 + 16\mu^2).
\]

(C3)

We find

\[
\partial_{\Phi_1}^2 V_{fin}^{AF(1)}(\Phi) \bigg|_{\Phi=0} = \delta^2_{\Phi_1} V_{fin}^{AF(1)}(\Phi) \bigg|_{\Phi=0} = 0,
\]

\[
\partial_{\Phi_1}^4 V_{fin}^{AF(1)}(\Phi) \bigg|_{\Phi=0} = \frac{\lambda_R^2\varphi_{16}^2}{128\pi^2} \left[ \frac{124416\lambda_R\varphi_{16}^2}{\tilde{m}_R^2(1) - \tilde{m}_R^2(16)} \right] \\
+ \frac{10368}{\tilde{m}_R^2(1) - \tilde{m}_R^2(16)} \ln \frac{\tilde{m}_R^2(16)}{\tilde{m}_R^2(1)} \\
+ 62208\lambda_R^2\varphi_{16}^2 \frac{\tilde{m}_R^2(16) + \tilde{m}_R^2(1)}{(\tilde{m}_R^2(1) - \tilde{m}_R^2(16))^2} \ln \frac{\tilde{m}_R^2(16)}{\tilde{m}_R^2(1)} \right\},
\]

(C4)

\[
\partial_{\Phi_1}^4 V_{fin}^{AF(1)}(\Phi) \bigg|_{\Phi=0} = \frac{5184\lambda_R^2\varphi_{16}^2}{128\pi^2} \left[ \frac{1}{\tilde{m}_R^2(1) + \tilde{m}_R^2(16)} - \frac{\lambda_R\varphi_{16}^2}{\tilde{m}_R^2(1) + \tilde{m}_R^2(16)} \right],
\]

Degenerate masses, \((\kappa > 0)\): If the two masses are degenerate then the effective potential is
\[ V_{\text{fin}}^{\text{AF}(1)}(\Phi) = \frac{1}{64\pi^2} \left[ (a_d^2 + b_d^2) \ln \frac{a_d^2 - b_d^2}{\tilde{m}^4_R} - 3b_d^2 - 2\tilde{m}^2_R c_d - 3c_d^2 \right. \\
\left. - 2a_d b_d \ln \frac{a_d - b_d}{a_d + b_d} \right], \]  
(C5)

with

\[ \tilde{m}_R^2 = -2m_R^2 \]
\[ a_d = \tilde{m}_R^2 + 3\lambda_R (2\varphi_{16} \Phi_{16} + \Phi_{16}^2 + \Phi_1^2), \]
\[ b_d = 6\lambda_R \Phi_1 (\varphi_{16} + \Phi_{16}), \]
\[ c_d = 3\lambda_R (2\varphi_{16} \Phi_{16} + \Phi_{16}^2 + \Phi_1^2), \]
\[ \varphi_{16} = -\frac{m_R^2}{\lambda_R}. \]  
(C6)

In this case
\[ \partial_{\Phi_1}^4 V_{\text{fin}}^{\text{AF}(1)} \bigg|_{\Phi=0} = \partial_{\Phi_{16}}^4 V_{\text{fin}}^{\text{AF}(1)} \bigg|_{\Phi=0} = \partial_{\Phi_1}^2 \partial_{\Phi_{16}}^2 V_{\text{fin}}^{\text{AF}(1)} \bigg|_{\Phi=0} = 648\lambda_R^2. \]  
(C7)

The effective potential and its derivatives of the ferromagnetic phase can be obtained from the corresponding formulae of the antiferromagnetic phase by exchanging the index 1 \(\leftrightarrow\) 16. In the paramagnetic phase one finds
\[ \partial_{\Phi_1}^2 V_{\text{fin}}^{\text{P}(1)} \bigg|_{\Phi=0} = \partial_{\Phi_{16}}^2 V_{\text{fin}}^{\text{P}(1)} \bigg|_{\Phi=0} = 0. \]  
(C8)

Observe that the particle of the zone \(B_{16}\) remains massless along the P-AF transition line.