A CUBIC NONCONVENTIONAL ERGODIC AVERAGE WITH MULTIPLICATIVE OR MANGOLDT WEIGHTS

E. H. EL ABDALAOUI AND XIANGDONG YE

Abstract. We show that the cubic nonconventional ergodic averages of any order with a bounded multiplicative function weight converge almost surely to zero provided that the multiplicative function satisfies a strong Daboussi-Delange condition. We further obtain that the Cesàro mean of the self-correlations and some moving average of the self-correlations of such multiplicative functions converge to zero. Our proof gives, for any $N \geq 2$,

$$\frac{1}{N} \sum_{m=1}^{N} \left| \frac{1}{N} \sum_{n=1}^{N} \nu(n)\nu(n+m) \right| \leq \frac{C}{\log(N)^\epsilon},$$

and

$$\frac{1}{N^2} \sum_{n,p=1}^{N} \left| \frac{1}{N} \sum_{m=1}^{N} \nu(m)\nu(n+m)\nu(m+p)\nu(n+m+p) \right| \leq \frac{C}{\log(N)^\epsilon},$$

where $C, \epsilon$ are some positive constants and $\nu$ is a bounded multiplicative function satisfying a Daboussi-Delange condition with logarithmic speed. We further establish that the cubic nonconventional ergodic averages of any order with Mangoldt weight converge almost surely provided that all the systems are nilsystems.

1. Introduction.

The purpose of this note is motivated, on one hand, by the recent great interest on the Möbius function from the dynamical point view, and on the other hand, by the problem of the multiple recurrence which goes back to the seminal work of Furstenberg [28]. This later problem has nowadays a long history.

The dynamical study of Möbius function was initiated recently by Sarnak in [51]. There, Sarnak made a conjecture that the Möbius function is orthogonal to any deterministic dynamical sequence. Notice that this precise the definition of a reasonable sequence in the Möbius randomness law mentioned by Iwaniec-Kowalski in [39, p.338]. Sarnak further mentioned that Bourgain’s approach allows to prove that for almost all point $x$ in any measurable dynamical system $(X, \mathcal{A}, T, \mathbb{P})$, the Möbius function is orthogonal to any dynamical sequence $f(T^n x)$. In [2], using a spectral theorem combined with Davenport’s estimation and Etmaidi’s trick, the authors gave a simple proof of this fact. Subsequently, Cuny and Weber gave a
proof in which they showed that there is a rate in this almost sure convergence [16]. They further used Bourgain’s method to prove that the almost sure convergence holds for other arithmetical functions, like the divisor function, the theta function and the generalized Euler totient function. Very recently, using Green-Tao estimation [31] combined with the method in [2], Eisner in [22] proved that almost surely the dynamical sequence $f(T^n \cdot x)$, where $p$ is an integer polynomial, is orthogonal to the Möbius function. She further proved that the Möbius function is a good weight (with limit 0) for the multiple polynomial mean ergodic theorem by using Qing Chu’s result [15]. Subsequently, in a very recent preprint [25], Host and Frantzikinakis established that any multiplicative function with mean value along any arithmetic sequence is a good weight for the multiple polynomial mean ergodic theorem (with limit 0 for aperiodic multiplicative functions).

Here, we are interested in the pointwise convergence of cubic nonconventional ergodic averages with multiplicative functions satisfying a strong Daboussi-Delange condition.

The convergence of cubic nonconventional ergodic averages was initiated by Bergelson in [8], where convergence in $L^2$ was shown for order 2 and under the extra assumption that all the transformations are equal. Under the same assumption, Bergelson’s result was extended by Host and Kra for cubic averages of order 3 in [35], and for arbitrary order in [36]. Assani proved that pointwise convergence of cubic nonconventional ergodic averages of order 3 holds for not necessarily commuting maps in [4, 5], and further established the pointwise convergence for cubic averages of arbitrary order when all the maps are equal. In [17], Chu and Frantzikinakis completed the study and established the pointwise convergence for the cubic averages of arbitrary order. Very recently, Huang-Shao and Ye [38] gave a topological-like proof of the pointwise convergence of the cubic nonconventional ergodic average when all the maps are equal. They further applied their method to obtain the pointwise convergence of nonconventional ergodic averages for a distal system.

Here, we establish that the cubic averages of any order with the multiplicative function weight satisfying a strong Daboussi-Delange condition converge to zero almost surely. The proof depends heavily on the double recurrence Bourgain’s theorem (DRBT for short) [11]. As a consequence, we obtain that the cubic averages of any order with Möbius or Liouville weights converge to zero almost surely. We further obtain an estimation of the speed of the convergence of the Césaro mean of the self-correlation of the multiplicative function satisfying a Daboussi-Delange condition with logarithmic speed. Those Césaro means of the self-correlation converge obviously to zero. Our proofs yields that some moving averages of the self-correlation of the multiplicative function satisfying a Daboussi-Delange condition with logarithmic speed are summable along any divergent geometric sequence.

We further obtain with the help of Davenport estimation that some moving averages of the self-correlation of Möbius function or Liouville function converge to zero, and they are summable along any divergent geometric sequence. Moreover,
We establish that the cubic nonconventional ergodic averages weighted with von-Mangoldt function converge provided that all the systems are nilsystems. This result is obtained as a consequence of the very recent result of Ford-Green-Konyagin-Tao [24] combined with some ingredients from [4].

The paper is organized as follows. In section 2, we recall the main ingredients needed for the proof. In section 3, we state our main results and its consequences, and prove our first main result. In section 4, we give proofs of our other main results.

When this paper was under preparation, we learned that Matomäki, Radzwill and Tao [47] proved that for any natural number \(k\), and for any \(10 \leq H \leq X\), we have
\[
\sum_{1 \leq h_1, h_2, \ldots, h_k \leq H} \left| \sum_{1 \leq n \leq X} \lambda(n+h_1) \cdots \lambda(n+h_k) \right| \ll k \left( \frac{\log \log H}{\log H} + \frac{1}{\log \frac{\log H}{1300} X} \right) H^{k-1} X.
\]
In the case \(k = 2\), this gives
\[
\sum_{1 \leq h \leq X} \left| \sum_{1 \leq n \leq X} \lambda(n) \lambda(n+h) \right| \ll k \left( \frac{\log \log H}{\log H} + \frac{1}{\log \frac{\log H}{1300} X} \right) H X.
\]
This last estimation is largely bigger than our estimation when \(H = X\).

We remind that besides, some estimation of limsup and liminf of some correlations of Liouville were obtained by several authors: Graham and Hansely, Harman-Pintz and Wolke, and Cassaigne-Ferenczi-Mauduit-Rivat and Sárközy. We refer to [13] for more details and for the references on the subject. We further refer to the recent work of Matomäki, Radzwill [48] in which the authors proved that for any \(h \geq 1\), there exits \(\delta(h) > 0\) such that
\[
\frac{1}{X} \left| \sum_{j=1}^{X} \lambda(j) \lambda(j+1) \right| < 1 - \delta(h),
\]
for all large enough \(X > 1\).

2. Basic definitions and tools.

Recall that the Liouville function \(\lambda : \mathbb{N}^* \rightarrow \{-1, 1\}\) is defined by
\[
\lambda(n) = (-1)\Omega(n),
\]
where \(\Omega(n)\) is the number of prime factors of \(n\) counted with multiplicities with \(\Omega(1) = 1\). Obviously, \(\lambda\) is completely mutiplicative, that is, \(\lambda(nm) = \lambda(n)\lambda(m)\), for any \(n, m \in \mathbb{N}^*\). The integer \(n\) is said to be not square-free if there is a prime number \(p\) such that \(n\) is in the class of 0 mod \(p^2\). The Môbius function \(\mu : \mathbb{N} \rightarrow \{-1, 0, 1\}\) is define as follows
\[
\mu(n) = \begin{cases} 
\lambda(n), & \text{if } n \text{ is square-free} \\
1, & \text{if } n = 1 \\
0, & \text{otherwise}.
\end{cases}
\]
This definition of Môbius function establishes that the the restriction of Liouville function and Môbius function to set of square free numbers coincide. Nevertheless, the Môbius function is only mutiplicative, that is, \(\mu(mn) = \mu(m)\mu(n)\)
whenever $n$ and $m$ are coprime.

We further remind that the von Mangoldt function is given by

$$\Lambda(n) = \begin{cases} \log(p), & \text{if } n = p^\alpha, \text{ for some prime } p \text{ and } \alpha \geq 1; \\ 0, & \text{otherwise.} \end{cases}$$

We shall need the following crucial result due to Davenport [20]

**Proposition 2.1.** For any $\epsilon > 0$, for any $N \geq 2$, we have

$$\sum_{n=1}^{N} \mu(n)e^{2\pi int} = O\left(\frac{N}{\ln(N)^e}\right),$$

uniformly in $t$.

By Batman-Chowla’s argument in [7] we have

**Proposition 2.2.** For any $\epsilon > 0$, for any $N \geq 2$, we have

$$\sum_{n=1}^{N} \Lambda(n)e^{2\pi int} = O\left(\frac{N}{\ln(N)^e}\right),$$

uniformly in $t$.

The previous estimation seems to be the best known result. However, if we restrict our self to the subset of the circle and allowed the estimation non to be uniform, Murty and Sankaranarayanan [49] proved the following

**Proposition 2.3.** For any $\epsilon > 0$, for any $N \geq 2$, we have

$$\sum_{n=1}^{N} \Lambda(n)e^{2\pi int} = O\left(N^{\frac{3}{5}+\epsilon}\right),$$

for all $t$ of type 1.

We remind that the irrational number $t$ is said to be of type $\eta$ if $\eta$ is the supremum of all $\gamma$ for which $\lim \inf_{q \to +\infty} q^\gamma \|qt\|_Z = 0$, where, as is customary, $\|x\|_Z$ denotes the distance to the nearest integer.

The authors in [49] pointed out that the following estimation is implicit in [20]. That is, for all $t$ of type 1,

$$\sum_{n=1}^{N} \Lambda(n)e^{2\pi int} = O\left(N^{\frac{1}{2}+\epsilon}\right),$$

Here, obviously the constant in the estimation depend on $\alpha$. At this point, one may ask if Proposition 2.3 can be improved by establishing that the estimation is uniform. If the answer is yes then our result (Theorem 3) can be much more improved. But, following the state of art in the subject, we can define the following class of mutiplicative functions.

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1See the proof of Lemma 1 in [7].
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The multiplicative function is said to satisfy the strong Daboussi-Delange condition with logarithmic speed if for any \( N \geq 2 \), we have

\[
\left| \frac{1}{N} \sum_{n=1}^{N} \nu(n)e^{2\pi i n \alpha} \right| \leq \frac{c}{\log(N)^{\kappa}}, \quad \text{for all } \alpha,
\]

with \( c, \kappa \) are positive constants, \( \kappa < 1 \).

By Propositions 2.1, 2.2, the M"obius and Liouville functions satisfy the Daboussi-Delange condition with logarithmic speed. We further say that the multiplicative function \( \nu \) satisfies the strong Daboussi-Delange condition if

\[
\sup_{\alpha \in \mathbb{R}} \left| \frac{1}{N} \sum_{n=1}^{N} \nu(n)e^{2\pi i n \alpha} \right| \rightarrow 0 \quad \text{as } N \rightarrow +\infty.
\]

The M"obius and Liouville functions are connected to the Riemann zeta function by the following

\[
\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \quad \text{and} \quad \frac{\zeta(2s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} \quad \text{for any } s \in \mathbb{C} \text{ with } \Re(s) > 1.
\]

Let us recall that Chowla made a conjecture in [14] on the multiple self-correlation of \( \mu \) which can be stated as follows:

**Conjecture (of Chowla).** For each choice of \( 1 \leq a_1 < \cdots < a_r \), \( r \geq 0 \), with \( i_s \in \{1, 2\} \), not all equal to 2, we have

\[
\sum_{n \leq N} \mu^{i_0}(n) \cdot \mu^{i_1}(n + a_1) \cdot \cdots \cdot \mu^{i_r}(n + a_r) = o(N).
\]

In [51], Sarnak noticed that Chowla conjecture is a notorious conjecture in number theory, and formulated the following conjecture:

**Conjecture (of Sarnak).** For any dynamical system \((X,T)\), where \( X \) is a compact metric space and \( T \) is a homeomorphism of zero topological entropy, for any \( f \in C(X) \) and any \( x \in X \), we have

\[
\sum_{n \leq N} f(T^n x) \mu(n) = o(N).
\]

He further announced that Chowla conjecture implies Sarnak conjecture, and wrote “we persist in maintaining Conjecture \( \text{(5)} \) as the central one even though it is much weaker than Conjecture \( \text{(5)} \). The point is that Conjecture \( \text{(6)} \) refers only to correlations of \( \mu \) with deterministic sequences and avoids the difficulties associated with self-correlations.” For the ergodic proof of the fact that Chowla conjecture implies Sarnak conjecture we refer the reader to [2] and the references therein.

At this point, let us mention that in many cases Sarnak’s conjecture holds as a consequence of the following criterion.

**Proposition 2.4** (Katai-Bourgain-Sarnak-Ziegler criterion). Let \((X, \mathcal{A}, \mu)\) be a Lebesgue probability space and \( T \) be an invertible measure preserving transformation. Let \( \nu \) be a multiplicative, \( f \in L^\infty \) with \( \|f\|_\infty \leq 1 \) and \( \varepsilon > 0 \). Assume that
for almost all point \( x \in X \) and for all different prime numbers \( p \) and \( q \) less than \( \exp(1/\varepsilon) \), we have

\[
\limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N} f(T^{pn}x)f(T^{qn}x) \right| < \varepsilon, \tag{7}
\]

then, for almost all \( x \in X \), we have

\[
\limsup_{N \to \infty} \left| \frac{1}{N} \sum_{n=1}^{N} \nu(n)f(T^{nx}) \right| < 2\sqrt{\varepsilon \log 1/\varepsilon}. \tag{8}
\]

Let us notice that we state Katai-Bourgain-Sarnak-Ziegler criterion in the form that we will use. This form can be derived from the original one by putting \( u(n) = f(T^{pn}x) \) for \( x \in X' \) with \( \mu(X') = 1 \). Moreover, a slight modification of the proof yields that the Wiener-Wintner version of it holds. But, we do not need such generalization.

Given an arithmetical function \( A : \mathbb{N} \to \mathbb{C} \), and a positive integer \( N \in \mathbb{N} \), for \( n \in \{1, \ldots, N\} \), we define a self-correlation coefficient \( c_{n,N}(A) \) by

\[
c_{n,N}(A) = \frac{1}{N} \sum_{m=1}^{N} A(m)A(m+n). \]

According to Wiener [53], if the limit exists for each \( n \), then this defines the so-called spectral measure of \( A \).

**Cubic averages and related topics.** Let \((X, \mathcal{B}, \mathbb{P})\) be a Lebesgue probability space and given three measure preserving transformations \( T_1, T_2, T_3 \) on \( X \). Let \( f_1, f_2, f_3 \in L^\infty(X) \). The cubic nonconventional ergodic averages of order 2 with weight \( A \) are defined by

\[
\frac{1}{N^2} \sum_{n,m=1}^{N} A(n)A(m)A(n+m)f_1(T_1^{n}x)f_2(T_2^{m}x)f_3(T_3^{n+m}x).
\]

This nonconventional ergodic average can be seen as a classical one as follows

\[
\frac{1}{N^2} \sum_{n,m=1}^{N} \tilde{f}_1(T_1^n(A,x))\tilde{f}_2(T_2^m(A,x))\tilde{f}_3(T_3^{n+m}(A,x)),
\]

where \( \tilde{f}_i = \pi_0 \otimes f_i, \tilde{T}_i = (S \otimes T_i), \) \( i = 1, 2, 3 \) and \( \pi_0 \) is define by \( x = (x_n) \mapsto x_0 \) on the space \( Y = \mathbb{C}^N \) equipped with some probability measure.

The study of the cubic averages is closely and strongly related to the notion of seminorms introduced in [29] and [30]. They are nowadays called Gowers-Host-Kra’s seminorms.

Assume that \( T \) is an ergodic measure preserving transformation on \( X \). Then, for any \( k \geq 1 \), the Gowers-Host-Kra’s seminorms on \( L^\infty(X) \) are defined inductively as follows

\[
\|f\|_1 = \left| \int f d\mu \right|;
\]

\[
\|f\|_{k+1}^2 = \lim_{H \to \infty} \frac{1}{H} \sum_{l=1}^{H} \|f \circ T^l\|_k^2.
\]
For each $k \geq 1$, the seminorm $|||.||_k$ is well defined. For details, we refer the reader to \cite{36} and \cite{34}. Notice that the definitions of Gowers-Host-Kra’s seminorms can be easily extended to non-ergodic maps as it was mentioned by Chu and Frantzikinakis in \cite{17}.

The importance of the Gowers-Host-Kra’s seminorms in the study of the non-conventional multiple ergodic averages is due to the existence of a $T$-invariant sub-$\sigma$-algebra $Z_{k-1}$ of $X$ that satisfies

$$\mathbb{E}(f|Z_{k-1}) = 0 \iff |||f|||_k = 0.$$  

This was proved by Host and Kra in \cite{36}. The existence of the factors $Z_k$ was also shown by Ziegler in \cite{55}. We further notice that Host and Kra established a connection between the $Z_k$ factors and the nilsystems in \cite{36}.

**Nilsystems and nilsequences.** The nilsystems are defined in the setting of homogeneous space\footnote{For a nice account of the theory of the homogeneous space we refer the reader to \cite{19,42} pp.815-919}. Let $G$ be a Lie group, and $\Gamma$ a discrete cocompact subgroup (Lattice, uniform subgroup) of $G$. The homogeneous space is given by $X = G/\Gamma$ equipped with the Haar measure $h_X$ and the canonical complete $\sigma$-algebra $\mathcal{B}_c$. The action of $G$ on $X$ is by the left translation, that is, for any $g \in G$, we have $T_g(x\Gamma) = g.x\Gamma = (gx)\Gamma$. If further $G$ is a nilpotent Lie group of order $k$, $X$ is said to be a $k$-step nilmanifold. For any fixed $g \in G$, the dynamical system $(X, \mathcal{B}_c, h_X, T_g)$ is called a $k$-step nilsystem. The basic $k$-step nilsequences on $X$ are defined by $f(g^n x\Gamma) = (f \circ T_g^n)(x\Gamma)$, where $f$ is a continuous function of $X$. Thus, $(f(g^n x\Gamma))_{n \in \mathbb{Z}}$ is any element of $\ell^\infty(\mathbb{Z})$, the space of bounded sequences, equipped with uniform norm $||(a_n)||_\infty = \sup_{n \in \mathbb{Z}} |a_n|$. A $k$-step nilsequence, is a uniform limit of basic $k$-step nilsequences. For more details on the nilsequences we refer the reader to \cite{37} and \cite{9}.

Recall that the sequence of subgroups $(G_n)$ of $G$ is a filtration if $G_1 = G$, $G_{n+1} \subset G_n$, and $[G_n, G_p] \subset G_{n+p}$, where $[G_n, G_p]$ denotes the subgroup of $G$ generated by the commutators $[x, y] = x^{-1}y^{-1}x y$ with $x \in G_n$ and $y \in G_p$. The lower central filtration is given by $G_1 = G$ and $G_{n+1} = [G, G_n]$. It is well known that the lower central filtration allows to construct a Lie algebra $\text{gr}(G)$ over the ring $\mathbb{Z}$ of integers. $\text{gr}(G)$ is called a graded Lie algebra associated to $G$ \cite{10} p.38. The filtration is said to be of degree or length $l$ if $G_{l+1} = \{e\}$, where $e$ is the identity of $G$. We denote by $G^e$ the identity component of $G$. Since $X = G/\Gamma$ is compact, we can assume that $G/G^e$ is finitely generated \cite{44}.

If $G$ is connected and simply-connected with Lie algebra $\mathfrak{g}$\footnote{The term ‘nilsequence’ was coined by Bergelson-Host and Kra in 2005 \cite{9}.}, then $\exp : G \rightarrow \mathfrak{g}$ is a diffeomorphism, where $\exp$ denotes the Lie group exponential map. We further have, by Mal’cev’s criterion, that $\mathfrak{g}$ admits a basis $X = \{X_1, \cdots, X_m\}$ with rational
structure constants $[g, h]$, that is,

$$[X_i, X_j] = \sum_{n=1}^{m} c_{ijn} X_n,$$

for all $1 \leq i, j \leq k,$

where the constants $c_{ijn}$ are all rational.

Let $X = \{X_1, \cdots, X_m\}$ be a Mal’cev basis of $g$, then any element $g \in G$ can be uniquely written in the form $g = \exp \left( t_1 X_1 + t_2 X_2 + \cdots + t_m X_m \right)$, $t_i \in \mathbb{R}$, since the map $\exp$ is a diffeomorphism. The numbers $(t_1, t_2, \cdots, t_k)$ are called the Mal’cev coordinates of the first kind of $g$. In the same manner, $g$ can be uniquely written in the form $g = \exp(s_1 X_1) \cdot \exp(s_2 X_2) \cdots \exp(s_m X_m)$, $s_i \in \mathbb{R}$. The numbers $(s_1, s_2, \cdots, s_k)$ are called the Mal’cev coordinates of the second kind of $g$. Applying Baker-Campbell-Hausdorff formula, it can be shown that the multiplication law in $G$ can be expressed by a polynomial mapping $\mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ [50, p.55], [30]. This gives that any polynomial sequence $g$ in $G$ can be written as follows

$$g(n) = \gamma_1^{p_1(n)}, \cdots, \gamma_m^{p_m(n)},$$

where $\gamma_1, \cdots, \gamma_m \in G$, $p_i : \mathbb{N} \to \mathbb{N}$ are polynomials [30]. Given $n, h \in \mathbb{Z}$, we put

$$\partial_h g(n) = g(n + h)g(n)^{-1}.$$

This can be interpreted as a discrete derivative on $G$. Given a filtration $(G_n)$ on $G$, a sequence of polynomial $g(n)$ is said to be adapted to $(G_n)$ if $\partial_{h_1} \cdots \partial_{h_l} g$ takes values in $G_i$ for all positive integers $i$ and for all choices of $h_1, \cdots, h_l \in \mathbb{Z}$. The set of all polynomial sequences adapted to $(G_n)$ is denoted by poly($\mathbb{Z}$, $(G_n)$).

Furthermore, given a Mal’cev’s basis $X$ one can induce a right-invariant metric $d_X$ on $X$ [30]. We remind that for a real-valued function $\phi$ on $X$, the Lipschitz norm is defined by

$$\|\phi\|_L = \|\phi\|_\infty + \sup_{x \neq y} \frac{|\phi(x) - \phi(y)|}{d_X(x, y)}.$$  

The set $L(X, d_X)$ of all Lipschitz functions is a normed vector space, and for any $\phi$ and $\psi$ in $L(X, d_X)$, $\phi \psi \in L(X, d_X)$ and $\|\phi \psi\|_L \leq \|\phi\|_L \|\psi\|_L$. We thus get, by Stone-Weierstrass theorem, that the subsalgebra $L(X, d_X)$ is dense in the space of continuous functions $C(X)$ equipped with uniform norm $\|\cdot\|_\infty$. It turns out that for a Lipschitz function, the extension from an arbitrary subset is possible without increasing the Lipschitz norm, thanks to Kirszbraun-Mcshane extension theorem [21, p.146].

In this setting, we remind the following fundamental Green-Tao’s theorem on the strong orthogonality of the Möbius function to any $m$-step nilsequence, $m \geq 1$.

**Proposition 2.5.** [31, Theorem 1.1]. Let $G/\Gamma$ be a $m$-step nilmanifold for some $m \geq 1$. Let $(G_p)$ be a filtration of $G$ of degree $l \geq 1$. Suppose that $G/\Gamma$ has a $Q$-rational Mal’cev basis $X$ for some $Q \geq 2$, defining a metric $d_X$ on $G/\Gamma$. Suppose that $F : G/\Gamma \to [-1, 1]$ is a Lipschitz function. Then, for any $\Lambda > 0$, we have the bound,

$$\sup_{g \in \text{poly}(\mathbb{Z}, (G_p))} \left| \frac{1}{N} \sum_{n=1}^{N} \mu(n) F(g(n) \Gamma) \right| \leq C \left( 1 + \frac{\|F\|_L}{\log \Lambda N} \right).$$
where the constant $C$ depends on $m, l, A, Q, N \geq 2$.

We further need the following decomposition theorem due to Chu-Frantzikinakis and Host from [18, Proposition 3.1].

**Proposition 2.6** (NSZE-decomposition theorem [18]). Let $(X, \mathcal{A}, \mu, T)$ be a dynamical system, $f \in L^\infty(X)$, and $k \in \mathbb{N}$. Then for every $\varepsilon > 0$, there exist measurable functions $f_{ns}, f_z, f_e$, such that

(a) $\|f_{\kappa}\|_\infty \leq 2\|f\|_\infty$ with $\kappa \in \{ns, z, e\}$.
(b) $f = f_{ns} + f_z + f_e$ with $\|f_z\|_{k+1} = 0; \|f_e\|_1 < \varepsilon$; and
(c) for $\mu$ almost every $x \in X$, the sequence $(f_{ns}(T^nx))_{n \in \mathbb{N}}$ is a $k$-step nilsequence.

3. Main results and the proof of Theorem 1.

3.1. The first main result and its proof. We state our first main result as follows.

**Theorem 1.** The cubic nonconventional ergodic averages of any order with a bounded multiplicative function weight converge almost surely to zero provided that the multiplicative function satisfies a strong Daboussi-Delange condition, that is, for any $k \geq 1$, for any $f_i \in L^\infty(X)$, $i = 1, \ldots, k$, for almost all $x$, we have

\[
\frac{1}{N^k} \sum_{n_1, \ldots, n_k, e_1, \ldots, e_k \in C^*} \prod_{e \in \{0, 1\}^{k+1}} \nu(n,e)f_e(T^{n,e}x) \xrightarrow{N \to +\infty} 0,
\]

where $n = (n_1, \ldots, n_k), e = (e_1, \ldots, e_k), C^* = \{0,1\}^k \setminus \{(0, \ldots, 0)\}$, $n.e$ is the usual inner product, and $\nu$ is the multiplicative function satisfying a strong Daboussi-Delange condition.

The proof of our first main result (Theorem 1) for $k \geq 2$ is essentially based on the inverse Gowers norms theorem due to Green, Tao and Ziegler [32] combined with the recent result of Zorin-Kranich on the extension of Wiener-Wintner’s version of Bourgain double recurrence theorem to the nilsequence case [56] (WWBDRT for short). Precisely, we will need the following results.

**Proposition 3.1** (Discrete inverse theorem for Gowers norms [32]). Let $N \geq 1$ and $s \geq 1$ be integers, and let $\delta > 0$. Suppose $f : \mathbb{Z} \to [-1, 1]$ is a function supported on $\{1, \ldots, N\}$ such that

\[
\frac{1}{N^{s+2}} \sum_{n,m \in [1,N]^{s+1}} \prod_{e \in \{0,1\}^{s+1}} f(n + m.e) \geq \delta.
\]

Then there exists a filtered nilmanifold $G/\Gamma$ of degree $\leq s$ and complexity $O_{s,\delta}(1)$, a polynomial sequence $g : \mathbb{Z} \to G$, and a Lipschitz function $F : G/\Gamma \to \mathbb{R}$ of Lipschitz constant $O_{s,\delta}(1)$ such that

\[
\frac{1}{N} \sum_{n=1}^{N} f(n)F(g(n)\Gamma) \gg_{s,\delta} 1.
\]

For the definition of complexity, we refer to [32]. The second result that we will need is the following
Proposition 3.2 (Nilsequence’s version of WWBDRT [56]). Let \((X, \mu, T)\) be an ergodic dynamical system and \(a, b\) distinct non-zero integers. Then, for any \(f, g \in L^\infty\), there exists a measurable set \(X'\) of full measure such that for any filtration \((G_p)\) of length \(l \geq 2\) and for any function \(F\) in the Sobolev space \(W^{r, 2^l}\) for any \(x \in X'\), we have
\[
\sup_{g \in \text{poly}(\mathbb{Z}, (G_p))} \left| \frac{1}{N} \sum_{n=0}^{N-1} f_1(T^{an}x) f_2(T^{bn}x) F(g(n)\Gamma) \right| \leq C \min\{ \|f_1\|_{U^{2+1}}^{2^l+1}, \|f_2\|_{U^{2+1}}^{2^l+1} \},
\]
where \(r = \sum_{m=1}^{l} (d_m - d_{m+1}) \binom{l}{m-1}\) with \(d_i\) is the dimension of \(G_i\) and \(C\) is a constant which depends only on the nilmanifold \(G/\Gamma\).

At this point, let us give a proof of our main result.

Proof of Theorem 1. Notice that the strong Daboussi-Delange condition implies that \(\nu\) is aperiodic, that is,
\[
\frac{1}{N} \sum_{n=1}^{N} \nu(an + b) \xrightarrow{N \to +\infty} 0, \text{ for all } a, b \in \mathbb{N}.
\]
Therefore, by Theorem 2.2 from [26], for any nilsequence \((u_n)\), we have
\[
\frac{1}{N} \sum_{n=1}^{N} \nu(n) u_n \xrightarrow{N \to +\infty} 0.
\]

Now, for the case \(k = 1\), we refer to [2, Section 3]. Let us assume from now that \(k \geq 2\).

We proceed by contradiction. Assume that (9) does not hold. Then, there exist \(\delta > 0\), a functions \(f_e, e \in V_k\) and \(\mu(A) > 0\) such that for each \(x \in A\)
\[
\limsup_{N \to +\infty} \frac{1}{N} \sum_{n \in \mathbb{N}^k} \prod e \in C^* \nu(n) e (T^n e x) \geq \delta,
\]
where \(\nu\) is a bounded multiplicative function which satisfies a strong Daboussi-Delange condition. Whence, by [17, Proposition 3.2], we have
\[
\|\nu(n) f_e \circ T^n_e(x)\|_{U^{k+1}} \geq \delta,
\]
for some \(e \in V_k\) and for any \(x \in A\).

This combined with Proposition [5,1] yields that there exist a nilmanifold \(G/\Gamma\), a filtration \((G_p)\), a polynomial sequence \(g\) and a Lipschitz function \(F\) such that
\[
\frac{1}{N} \sum_{n=1}^{N} \nu(n) f_e (T^n_e x) F(g(n)\Gamma) \gg_{k, \delta} 1.
\]

Applying the decomposition theorem (Proposition [2,6]), we may write \(f_e = f_{e, ns} + f_{e,z} + f_{e,e}\) and may assume that we have \(\|f_{e,z}(T^n_e(x))\|_{U^{k+1}} = 0\). Notice that we further have
\[
\frac{1}{N} \sum_{n=1}^{N} \nu(n) f_{e, ns} (T^n_e x) F(g(n)\Gamma) \xrightarrow{N \to +\infty} 0,
\]
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since the pointwise product of two nilsequences is a nilsequence and $\nu$ satisfies the strong Daboussi-Delange condition.

We thus get, up to some small error, that

$$
\frac{1}{N} \sum_{n=1}^{N} \nu(n) f_{\nu}(T_{e}^{n}x) F(g(n)\Gamma) \sim \frac{1}{N} \sum_{n=1}^{N} \nu(n) f_{\nu}(T_{e}^{n}x) F(g(n)\Gamma).
$$

Applying Katai-Bourgain-Sarnak-Ziegler criterion, we get that the re exist distinct prime numbers $p, q$ such that

$$
\limsup \left| \frac{1}{N} \sum_{n=1}^{N} f_{\nu}(T_{e}^{pn}(x)) f_{\nu}(T_{e}^{qn}(x)) F(g(np)\Gamma) F(g(nq)\Gamma) \right| \gg_{k, \delta} 1
$$

for a subset $B$ with positive measure. This contradicts Proposition 3.2 since $\mu(B) > 0$ and $\|f_{\nu}(T_{e}^{n}(x))\|_{U^{k+1}} = 0$, and the proof of the theorem is complete. ■

3.2. Other main results. Our second main result can be stated as follows

**Theorem 2.** Assume that for some $i \in \{1, 2, 3\}$ (resp. $i \in \{1, 2, 3, 4, 5, 6, 7\}$), $T_{i}$ has a discrete spectrum. Then, the cubic ergodic averages of order 2 (resp. of order 3) with multiplicative function weight which satisfies the strong Daboussi-Delange condition converge almost surely to 0.

Obviously, we have the following corollary.

**Corollary 1.** Assume that for some $i \in \{1, 2, 3\}$ (resp. $i \in \{1, 2, 3, 4, 5, 6, 7\}$), $T_{i}$ is a nilsystem. Then, the cubic ergodic averages of order 2 (resp. of order 3) with the Möbius or Liouville function weight converge almost surely to 0.

The proof of Corollary 1 can be obtained independently and we shall give such proof in section 4.

The proof of Theorem 2 is divided into subsections in section 4. In subsection 4.1, we prove that for any $N \geq 2$,

$$
\frac{1}{N} \sum_{m=1}^{N} \nu(n) \nu(n + m) \leq \frac{C}{\log(N)^{\varepsilon}},
$$

and

$$
\frac{1}{N^{2}} \sum_{n,p=1}^{N} \left| \frac{1}{N} \sum_{m=1}^{N} \nu(m) \nu(n + m) \nu(m + p) \nu(n + m + p) \right| \leq \frac{C}{\log(N)^{\varepsilon}},
$$

where $C, \varepsilon$ are some positive constants and $\nu$ is a bounded multiplicative function which satisfies a weak Daboussi-Delange condition, that is,

$$
\left| \frac{1}{N} \sum_{n=1}^{N} \nu(n)e^{2i\pi n\alpha} \right| \leq \frac{c}{\log(N)^{\varepsilon}}, \text{ for all } \alpha,
$$

with $c$ is a positive constant.

In subsection 4.3, we provide a simpler proof of Corollary 1 based on the proof of our second main result.

Notice that our result involves the self-correlation of $\mu$. Furthermore, as far as we know the problem of the self-correlation of $\mu$ is still open. This problem is known as Elliott’s conjecture and it can be stated as follows.
Conjecture (of Elliott). \[23\]

\[
\lim_{{N \to +\infty}} c_{h,N}(\mu) = \begin{cases} 
0 & \text{if } h \neq 0 \\
\frac{6}{\pi^2} & \text{if not}
\end{cases}
\]

P.D.T.A. Elliott wrote in his 1994’s AMS Memoirs that “even the simple particular cases of the correlation (when \(h = 1\) in (10)) are not well understood. Almost surely the M" obius function satisfies (10) in this case, but at the moment we are unable to prove it.”

Recently, el Abdalaoui and Disertori in \[1\] established that the \(L^1\)-flatness of trigonometric polynomials with M" obius coefficients implies Elliott’s conjecture. The \(L^1\)-flatness of trigonometric polynomials with coefficients in \(\{0, 1, -1\}\) is an open problem in harmonic analysis and spectral theory of dynamical systems (see \[3\] and the references therein).

Nevertheless, as a consequence of Theorem 1, we have the following result

**Corollary 2.** Let \(\nu\) be the M"obius function or the Liouville function. Then

\[
\frac{1}{N^k} \sum_{{n \in [1,N]^k}} \prod_{{e \in C^*}} \nu(n.e) \xrightarrow[\ N \to +\infty\] 0,
\]

where \(n = (n_1, \ldots, n_k), \ e = (e_1, \ldots, e_k), \ C^* = \{0, 1\}^k \setminus \{(0, \ldots, 0)\}, \ n.e\ is the usual inner product.

**Proof.** Take \(f_e = 1\) in Theorem 1

Our third main result can be stated as follows.

**Theorem 3.** Let \(\nu\) be a multiplicative function satisfying the Daboussi-Delange condition with logarithmic speed. Then, for any \(N \geq 2\),

\[
\left| \frac{1}{N} \sum_{{m=1}}^{N} \left| \frac{1}{N} \sum_{{n=1}}^{N} \nu(n)\nu(n+m) \right| \right| \leq \frac{C}{\log(N)^\kappa},
\]

and

\[
\left| \frac{1}{N^2} \sum_{{n,p=1}}^{N} \left| \frac{1}{N} \sum_{{m=1}}^{N} \nu(m)\nu(n+m)\nu(m+p)\nu(n+m+p) \right| \right| \leq \frac{C}{\log(N)^\kappa},
\]

where \(C\) is some positive constant.

It is easy to see that from Davenport estimation \[4\] that we have the following

**Corollary 3.** Let \(\Lambda\) be the von Mangoldt function. Then, for any \(N \geq 2\), for almost all \(t,\)

\[
\left| \frac{1}{N} \sum_{{n=1}}^{N} \Lambda(n)e^{2\pi nt} \right| \xrightarrow[\ n \to +\infty\] 0.
\]

We further have, by the proof of Theorem 3, the following corollary,
Corollary 4. For any $\rho > 1$ there exist two sequences of integer $m_1, n_1$ such that
\[
\frac{1}{\left\lceil \rho m_1 \right\rceil} \sum_{k=1}^{\left\lceil \rho m_1 \right\rceil} \mu(k) \mu(k + n_1) \to 0,
\]
and
\[
\frac{1}{\left\lceil \rho m_1 \right\rceil} \sum_{k=1}^{\left\lceil \rho m_1 \right\rceil} \lambda(k) \lambda(k + n_1) \to 0.
\]
We further have, for any integer $N \geq 2$, for any $\epsilon > 0$,
\[
\frac{1}{N} \sum_{n=1}^{N} \left\lvert \frac{1}{N} \sum_{m=1}^{N} \mu(m) \mu(n + m) \right\rvert \leq \frac{C}{\log(N)^c},
\]
and
\[
\frac{1}{N} \sum_{n=1}^{N} \left\lvert \frac{1}{N} \sum_{m=1}^{N} \lambda(m) \lambda(n + m) \right\rvert \leq \frac{C}{\log(N)^c},
\]
where $C$ is a constant which depends only on $\epsilon < 1$.

This gives
\[
\frac{1}{N} \sum_{n=1}^{N} \left\lvert \frac{1}{N} \sum_{m=1}^{N} \mu(m) \mu(n + m) \right\rvert \to 0,\]
and
\[
\frac{1}{N} \sum_{n=1}^{N} \left\lvert \frac{1}{N} \sum_{m=1}^{N} \lambda(m) \lambda(n + m) \right\rvert \to 0.
\]

Before starting to prove our main results, let us point out that it suffices to establish our results for a dense set of functions. Indeed, assume that the convergence holds for some $f_T, T = (1, 0, \cdots, 0)$, and let $g$ be such that $\|f_T - g\|_1 < \epsilon$, for a given $\epsilon > 0$. Put
\[
\psi_N(f_T) = \frac{1}{N} \sum_{n_1=1}^{N} \nu(n_1) f_T(T_1^{n_1}(x)) \frac{1}{N^{k-1}} \sum_{m \in \{1, N^{k-1} \}} \prod_{n \in \{n_1, m\}} \nu(n, e) f_e(T_e^{n-e}(x)).
\]

Then,
\[
\left\lvert \psi_N(f_T) - \psi_N(g) \right\rvert \leq \frac{1}{N} \sum_{n_1=1}^{N} \left\lvert f_T(T_1^{n_1} x) - g(T_1^{n_1} x) \right\rvert \frac{1}{N^{k-1}} \sum_{m \in \{1, N^{k-1} \}} \prod_{n \in \{n_1, m\}} \nu(n, e) f_e(T_e^{n-e}(x)) \right\rvert
\]
\[
\leq c \left( \prod_{e \in C_e \setminus \{T \}} \|f_e\|_\infty \right) \frac{1}{N} \sum_{n_1=1}^{N} |f_1(T_1^{n_1} x) - g(T_1^{n_1} x)|.
\]
Letting $N \to \infty$, it follows
\[
\limsup |\psi_N(f_T) - \psi_N(g)| \leq \|f_T - g\|_1 \leq \epsilon,
\]
by Birkhoff ergodic theorem combined with our assumption. Notice that the maps $T_e$ are supposed to be ergodic. From now on, without loss of generality, we will assume that $T_e$ are ergodic.
4. The proof of Theorems 2 and 3

In this subsection we will show Theorems 2 and 3. In the process to do we also provide the proof of Corollary 4.

4.1. The case $k = 2$. In this subsection, we focus our study on the case $k = 2$ and at least one of the maps $T_i$, $i = 1, 2, 3$ has a discrete spectrum.

Proof of Theorem 2 Let assume that $T_1$ has a discrete spectrum and let $f_1$ be an eigenfunction with eigenvalue $\lambda$. Then, for almost all $x \in X$, we can write

$$\left| \frac{1}{N^2} \sum_{n,m=1}^{N} \nu(n)\nu(m)\nu(n+m)f_1(T^n_1 x)f_2(T^m_2 x)f_3(T^{n+m}_3 x) \right|$$

$$= \left| \frac{1}{N} \sum_{m=1}^{N} \nu(m)f_2(T^m_2 x)\frac{1}{N} \sum_{n=1}^{N} \nu(n)\nu(n+m)\lambda^n f_3(T^{n+m}_3 x) \right|$$

$$= \left| \frac{1}{N} \sum_{m=1}^{N} \nu(m)f_2(T^m_2 x)\frac{1}{N} \sum_{n=1}^{N} \nu(n)\nu(n+m)\lambda^{n+m} f_3(T^{n+m}_3 x) \right|$$

$$\leq \frac{1}{N} \sum_{m=1}^{N} |f_2(T^m_2 x)| \left| \frac{1}{N} \sum_{n=1}^{N} \nu(n)\nu(n+m)\lambda^{n+m} f_3(T^{n+m}_3 x) \right|$$

$$\leq \|f_2\|_{\infty} \frac{1}{N} \sum_{m=1}^{N} \left| \frac{1}{N} \sum_{n=1}^{N} \nu(n)\nu(n+m)\lambda^{n+m} f_3(T^{n+m}_3 x) \right|$$

Applying Cauchy-Schwarz inequality we can rewrite (12) as

$$\left| \frac{1}{N^2} \sum_{n,m=1}^{N} \nu(n)\nu(m)\nu(n+m)f_1(T^n_1 x)f_2(T^m_2 x)f_3(T^{n+m}_3 x) \right|$$

$$\leq \|f_2\|_{\infty} \left( \frac{1}{N} \sum_{m=1}^{N} \left| \frac{1}{N} \sum_{n=1}^{N} \nu(n)\nu(n+m)\lambda^{n+m} f_3(T^{n+m}_3 x) \right|^2 \right)^{\frac{1}{2}}$$

Furthermore, by Bourgain’s observation [11] equations (2.5) and (2.7), we have

$$\sum_{m=1}^{N} \left| \frac{1}{N} \sum_{n=1}^{N} \nu(n)\nu(n+m)\lambda^{n+m} f_3(T^{n+m}_3 x) \right|^2$$

$$= \sum_{m=1}^{N} \int_{T} \left( \frac{1}{N} \sum_{n=1}^{N} \nu(n)z^{-n} \right) \left( \sum_{p=1}^{2N} \nu(p)f_3(T^p_3 x)(\lambda z)^p \right)^2 z^{-m} dz$$

$$\leq \int_{T} \left| \frac{1}{N} \sum_{n=1}^{N} \nu(n)z^{-n} \right|^2 \left( \sum_{p=1}^{2N} \nu(p)f_3(T^p_3 x)(\lambda z)^p \right)^2 dz.$$
Then, for any $m \in \mathbb{Z}$,

$$
\hat{\Phi}_N(m) = \int_T \left( \frac{1}{N} \sum_{n=1}^{N} \nu(n)z^{-n} \right) \left( \sum_{p=1}^{2N} \nu(p)f_3(T_3^p x) (\lambda z)^p \right) z^{-m} dz.
$$

Whence

$$
\sum_{m=1}^{N} | \int_T \left( \frac{1}{N} \sum_{n=1}^{N} \nu(n)z^{-n} \right) \left( \sum_{p=1}^{2N} \nu(p)f_3(T_3^p x) (\lambda z)^p \right) z^{-m} dz |^2
\leq \sum_{m=1}^{N} \left| \hat{\Phi}_N(m) \right|^2
\leq \int_T |\Phi_N(z)|^2 dz.
$$

Now, combining (13) with (14) we can assert that

$$
\left| \frac{1}{N^2} \sum_{n,m=1}^{N} \nu(n)\nu(m)\nu(n+m)f_1(T_1^n x)f_2(T_2^m x)f_3(T_3^{n+m} x) \right|
= \left| \frac{1}{N} \sum_{m=1}^{N} \nu(m)f_2(T_2^m x) \frac{1}{N} \sum_{n=1}^{N} \nu(n)\lambda^n\nu(n+m)f_3(T_3^{n+m} x) \right|
\leq \|f_2\|_\infty \left( \frac{1}{N} \sup_{z \in T} \left| \frac{1}{N} \sum_{n=1}^{N} \nu(n)z^{-n} \right|^2 \int_T \left| \sum_{p=1}^{2N} \nu(p)f_3(T_3^p x)(\lambda z)^p \right|^2 dz \right)^{\frac{1}{2}},
$$

We thus get

$$
\left| \frac{1}{N^2} \sum_{n,m=1}^{N} \nu(n)\nu(m)\nu(n+m)f_1(T_1^n x)f_2(T_2^m x)f_3(T_3^{n+m} x) \right|
\leq \|f_2\|_\infty \sup_{z \in T} \left| \frac{1}{N} \sum_{n=1}^{N} \nu(n)z^{-n} \right| \left( \frac{1}{N} \sum_{p=1}^{2N} |\nu(p)|^2 \right)^{\frac{1}{2}} \|f_3\|_\infty,
$$

since the map $z \mapsto \lambda z$ is a measure-preserving transformation, and we have

$$
\int_T \left| \sum_{p=1}^{2N} \nu(p)f_3(T_3^p x) z^p \right|^2 dz = \sum_{p=1}^{2N} |\nu(p)|^2 |f_3(T_3^p x)|^2 \leq \sum_{p=1}^{2N} |\nu(p)|^2 \|f_3\|_\infty^2.
$$

It follows from our assumption on the multiplicative function $\nu$ that we have

$$
\left| \frac{1}{N^2} \sum_{n,m=1}^{N} \nu(n)\nu(m)\nu(n+m)f_1(T_1^n x)f_2(T_2^m x)f_3(T_3^{n+m} x) \right|
\leq \sqrt{2}\|f_2\|_\infty \|f_3\|_\infty \frac{C}{\log(N)\varepsilon},
$$

where $C$ and $\varepsilon$ are positive constants, and $\varepsilon < 1$. Letting $N \to \infty$, we conclude that the almost sure convergence holds. The proof of the theorem for the case $k = 2$ is complete. □
4.2. **On the self-correlation of Möbius and Liouville functions.** Before the proof of Theorem 2 for $k = 3$, we discuss the self-correlation of Möbius and Liouville functions first. Notice that we have actually proved

**Factor 1.** Let $\nu$ be the Möbius or the Liouville function. Then, for any $N \geq 2$,

$$ \frac{1}{N} \sum_{n=1}^{N} \nu(n) \nu(n + m) \leq \frac{C}{\log(N)^\varepsilon}, $$

where $C$ and $\varepsilon$ are positive constants, $\varepsilon < 1$.

**Proof.** This follows by combining (12), (14), (15), (3) with (4) and by taking $\lambda = 1$ and $f_3 = 1$. □

At this point, the proof of the second part of Corollary 4 follows.

Let $\rho > 1$, then for $N = [\rho^m]$ with some $m \geq 1$, (18) takes the form

$$ \frac{1}{[\rho^m]} \sum_{n \leq [\rho^m]} \left| \sum_{n \leq [\rho^m]} \nu(n) \nu(n + m) \right| < \infty. $$

This gives

$$ \sum_{m \geq 1} \frac{1}{[\rho^m]} \sum_{n \leq [\rho^m]} \left| c_{n, [\rho^m]} \right| < \infty. $$

Let $(\delta_l)$ be a sequence of positive numbers such that $\delta_l \to 0$. Then, for any $l \geq 1$ there exists a positive integer $m_l$ such that

$$ \sum_{m \geq m_l} \frac{1}{[\rho^m]} \sum_{n \leq [\rho^m]} \left| c_{n, [\rho^m]} \right| < \delta_l. $$

This gives, for any $m \geq m_l$,

$$ \frac{1}{[\rho^m]} \sum_{n \leq [\rho^m]} \left| c_{n, [\rho^m]} \right| < \delta_l. $$

Hence, there exists $n_l \leq [\rho^m]$ such that

$$ \left| c_{n_l, [\rho^m]} \right| < \delta_l. $$

By letting $l$ go to $\infty$, we get $c_{n_l, [\rho^m]} \to 0$. This prove the first part of Corollary 4. □

Note that we have proved more, namely,

**Corollary 5.** For any $\rho > 1$,

$$ \sum_{m \geq 1} \frac{1}{[\rho^m]} \sum_{n=1}^{[\rho^m]} \sum_{k=1}^{[\rho^m]} \mu(k) \mu(k + n) < \infty. $$

**Remark 1.** It is shown in [1] that if Sarnak’s conjecture holds with some technical assumption then the self-corrections of $\mu$ satisfy

$$ \frac{1}{2N} \sum_{n=-N}^{N} \mu(n) \mu(n + k) \to 0, $$

as $N \to \infty$. 


for any $k < 0$.

4.3. The case $k = 3$. In this subsection we give the proof of Theorem for $k = 3$. For simplicity, we shall prove the following.

\[
\left| \frac{1}{N^3} \sum_{n,p,m=1}^N \nu(n)\nu(m)\nu(p)\nu(n+m)\nu(n+p)\nu(m+p)\nu(n+m+p) \right| \leq \frac{C}{\log(N)},
\]

where $C$ and $\epsilon > 0$ are positive constants, and $\nu$ is a bounded multiplicative function which satisfies the weak Daboussi-Delange condition.

Notice that by the triangle inequality, we have

\[
\left| \frac{1}{N^3} \sum_{n,p,m=1}^N \nu(n)\nu(m)\nu(p)\nu(n+m)\nu(n+p)\nu(m+p)\nu(n+m+p) \right| 
\leq \frac{1}{N^2} \sum_{n,p=1}^N \left| \frac{1}{N} \sum_{m=1}^N \nu(n)\nu(m)\nu(p)\nu(n+m)\nu(n+p)\nu(m+p)\nu(n+m+p) \right|.
\]

But, since $\nu$ is bounded (say by one), we can write

\[
\frac{1}{N^2} \sum_{n,p=1}^N \left| \frac{1}{N} \sum_{m=1}^N \nu(n)\nu(m)\nu(p)\nu(n+m)\nu(n+p)\nu(m+p)\nu(n+m+p) \right| 
\leq \frac{1}{N^2} \sum_{n,p=1}^N \left| \frac{1}{N} \sum_{m=1}^N \nu(m)\nu(n+m)\nu(m+p)\nu(n+m+p) \right|.
\]

Let $p$ be fixed and apply Cauchy-Schwarz inequality combined with Bourgain observation to get

\[
\frac{1}{N} \sum_{n=1}^N \left| \frac{1}{N} \sum_{m=1}^N \nu(m)\nu(n+m)\nu(m+p)\nu(n+m+p) \right| 
\leq \left( \frac{1}{N} \sum_{n=1}^N \left| \frac{1}{N} \sum_{m=1}^N \nu(m)\nu(n+m)\nu(m+p)\nu(n+m+p) \right|^2 \right)^{1/2}
\leq \left( \frac{1}{N} \sum_{n=1}^N \left| \hat{\Psi}_{N,p}(n) \right|^2 \right)^{1/2}
\]

where

\[
\Psi_{N,p}(z) = \left( \frac{1}{N} \sum_{m=1}^N \nu(m)z^{-m} \right) \left( \sum_{m'=1}^N \nu(m')\nu(m'+p)z^{m'} \right).
\]

As before, applying Bessel-Parseval inequality, we can write

\[
\frac{1}{N} \sum_{n=1}^N \left| \frac{1}{N} \sum_{m=1}^N \nu(m)\nu(n+m)\nu(m+p)\nu(n+m+p) \right| 
\leq \left( \frac{1}{N} \int \left| \Psi_{N,p}(z) \right|^2 dz \right)^{1/2}.
\]

\[5\]We extend the definition of $\mu$ to the negative integer in usual fashion.
Now, by our assumption on $\nu$, we have
\[ |\Psi_{N,p}(z)| \leq C \frac{\nu(m')\nu(m' + p)z^{m'}}{\log(N)^\epsilon}. \]

Therefore,
\[ \frac{1}{N} \int |\Psi_{N,p}(z)|^2 dz \leq C^2 \left( \frac{1}{N} \sum_{m'=1}^{N} \nu^2(m')\nu^2(m' + p) \right). \]

It turns out that for any $p$, one can estimate the quantity
\[ \frac{1}{N} \sum_{m'=1}^{N} \nu^2(m')\nu^2(m' + p). \]

for $\nu \in \{\mu, \lambda\}$, thanks to Mirsky Theorem. But we don’t need such estimation. Here, we observe only that $\nu^2$ is bounded by one. Whence, for any $p$,
\[ \left( \frac{1}{N} \int |\Psi_{N,p}(z)|^2 dz \right)^{\frac{1}{2}} \leq C \frac{1}{\log(N)^\epsilon}. \]

This proves Corollary 4. A careful application of our previous machinery allows us to prove Theorem 3 and the detailed verification is left to the reader. ■

Proof of Theorem 3 It follows from subsections 4.1 and 4.3.

4.4. On the nilsystem case. In this subsection, we present an alternative proof of our second main result when $k = 2$ or $3$, $\nu \in \{\mu, \lambda\}$ and at least one of the dynamical systems is a nilsystem.

Proof of Corollary 7 Let us assume that $T_1$ is an elementary nilsystem of order $s$, that is, $T_1$ is an ergodic $s$-step nilsystem on $X = G/\Gamma$, where $G$ is a nilpotent Lie group of dimension $s$ and $\Gamma$ is a discrete subgroup. By the density argument, it suffices to prove the theorem for a nilsequence $(f_1(T^n x))$, $x \in X$, $f_1$ is a continuous function on $X$. Now, by Leibman’s observation [44], we can embed $G$ into a connected and simply-connected nilpotent Lie group $\hat{G}$ with a cocompact subgroup $\hat{\Gamma}$ such that $X = G/\Gamma$ is isomorphic to a sub-nilmanifold of $\hat{X} = \hat{G}/\hat{\Gamma}$, with all translations from $G$ represented in $\hat{G}$. Furthermore, by Tietze-Uryshon extension theorem [21, p.48], we can extend $f_1$ to $\hat{X}$. Hence, we are reduced to prove our main result for the nilsystems on $\hat{X}$.

Analyzing the proof given in subsection 4.1 we need to estimate
\[ \sup_{z \in \mathbb{T}} \left| \frac{1}{N} \sum_{n=1}^{N} \mu(n)f_1(T^n x)z^{-n} \right|. \]

Again, by the density argument, we may assume that $f_1$ is Lipschitz. We further notice that the sequence $(\alpha_n z^{-n})_n$ can viewed as a nilsequence on $Y = \hat{G}/\hat{\Gamma} \times \mathbb{R}/\mathbb{T}$. But the group $\hat{G} \times \mathbb{R}$ is connected and simply-connected, and the function $F_1(x, z) =
Let \( f_1(x)z^{-1} \) be Lipschitz. Then, we can apply Green-Tao’s Theorem (Theorem 1.1 in [31]) for a given filtration \((H_n)\) of \( \hat{G} \times \mathbb{R} \) of length \( m \geq 1 \). This gives,

\[
\sup_{z \in T} \left| \frac{1}{N} \sum_{n=1}^{N} \mu(n)f_1(T^n x)z^{-n} \right| \leq C \frac{1 + \|f_1\|_L}{\log^A(N)},
\]

for any \( A > 0 \), uniformly on \( x \) and \( z \). Letting \( N \) go to infinity, we get

\[
\sup_{z \in T} \left| \frac{1}{N} \sum_{n=1}^{N} \mu(n)f_1(T^n x)z^{-n} \right| \xrightarrow{N \to +\infty} 0.
\]

Whence,

\[
\left| \frac{1}{N^2} \sum_{n,m=1}^{N} \mu(n)\mu(m)\mu(n+m)f_1(T^n x)f_2(T^m x)f_3(T^{n+m} x) \right| \xrightarrow{N \to +\infty} 0.
\]

by (16), which ends the proof of the theorem for the case \( k = 2 \), the rest of the proof is left to the reader. \( \square \)

Note that by (19), we have proved the following popular and well-known result.

**Proposition 4.1.** Sarnak’s conjecture holds for any nilsystem.

**Question.** A natural problem suggested by our result is the following: Assume that \( T \) satisfies Sarnak’s conjecture, do we have for any continuous function, for all \( x \in X \),

\[
\sup_{t} \left| \frac{1}{N} \sum_{n=1}^{N} \mu(n)f(T^n x)e^{2\pi int} \right| \xrightarrow{N \to +\infty} 0? \quad (WWS)
\]

Notice that the topological entropy of the cartesian product of two dynamical flow on compact set is the sum of their topological entropy [33].

### 4.5. On the cubic average with Mangoldt weight.

The study of the correlations of von Mangoldt function is of great importance in number theory, since it is related to the famous old conjecture of the twin numbers and more generally to Hardy-Littlewood k-tuple conjecture. It is also related to Riemann hypothesis and the Goldbach conjectures. Here we will establish the following.

**Theorem 4.** The cubic nonconventional ergodic averages of any order with von Mangoldt function weight converge almost surely provided that the systems are nilsystems.

The proof of Theorem 4 is largely inspired from Ford-Green-Konyagin-Tao’s proof of the following theorem [24]. It is used also some elementary fact on Gowers uniformity semi-norms.

**Proposition 4.2 (Ford-Green-Konyagin-Tao [24]).** The Gowers norm of von Mangoldt function is positive. Precisely, for any \( d \geq 1 \),

\[
\frac{1}{N^d} \sum_{\vec{n} \in [1,N]^d} \prod_{e \in C^*} \Lambda(\vec{n},e) \xrightarrow{N \to +\infty} \prod_p \beta_p,
\]

where

\[
\beta_p = \frac{1}{p^d} \sum_{\vec{n} \in (\mathbb{Z}/p\mathbb{Z})^d} \prod_{e \in C^*} \Lambda_{\mathbb{Z}/p\mathbb{Z}}(\vec{n},e).
\]
The function $\Lambda_{Z/pZ}$ is the local von Mangoldt function, that is, the $p$-periodic function defined setting $\Lambda_{Z/pZ}(b) = \frac{1}{p-1}$ when $b$ is coprime to $p$ and $\Lambda_{Z/pZ}(b) = 0$ otherwise.

As it is mentioned in [24], the fundamental ingredients in the proof of Proposition 4.2 are the Möbius orthogonality to the nilsequences (Proposition 4.1) combined with the inverse Gowers theorem (Proposition 3.1) and the “W-trick”. We will need also the following lemma from [27].

**Lemma 1.** Let $(a_n)$ be a bounded sequence of complex numbers. Then, we have

$$\left| \frac{1}{\pi(N)} \sum_{\text{prime } p \leq N} a_p - \frac{1}{N} \sum_{n=1}^{N} \Lambda'(n)a_n \right| \rightarrow 0.$$  

**Proof of Theorem 4.** For the case $k = 1$. The result holds for any dynamical system. Indeed, the result follows from Lemma 1 combined with the main result from [54]. Let us assume from now that $k \geq 2$. Then, as in Ford-Green-Konyagin-Tao’s proof, it suffices to see that the following holds

$$\frac{1}{N^k} \sum_{\vec{n} \in [1,N]^k} \left( \prod_{e \in C^*} \Lambda'_{b_i,W}(\vec{n},e) - 1 \right) \prod_{e \in C^*} f_e(T^{b_i,W}e_x),$$  

(NCSM)

where $b_i \in [1,W]$, $i = 1, \cdots, k$ coprime to $W$.

But $\Lambda'_{b,W} - 1$ is orthogonal to the nilsequences by Green-Tao result. Therefore the limit of the quantity (CNSM) is zero. □

**Question.** According to our result Theorem 4, we ask if is it true that the cubic nonconventional ergodic averages of any order with von Mangoldt function weight converge almost surely?

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