On Geometric Circulant Matrices Whose Entries are Bi-Periodic Fibonacci and Bi-Periodic Lucas Numbers

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Abstract

In this study, we obtain upper and lower bounds for the spectral norms of the geometric circulant matrices with the bi–periodic Fibonacci numbers and bi–periodic Lucas numbers, respectively. Then we give some bounds for the spectral norms of Kronecker and Hadamard products of these matrices.

1. Introduction

The well-known Fibonacci and Lucas sequences are given by the following recursive equations: for \( n \geq 0 \),

\[
F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n,
\]

and

\[
L_0 = 2, \quad L_1 = 1, \quad L_{n+2} = L_{n+1} + L_n,
\]

respectively.

Many researchers gave various generalizations of the Fibonacci sequence in past fifty years. An interesting one, called bi–periodic Fibonacci sequence, was introduced by Edson and Yayenie in [5] as follows:

\[
q_0 = 0, \quad q_1 = 1, \quad \text{and} \quad q_n = \begin{cases} 
aq_{n-1} + q_{n-2}, & \text{if } n \text{ is even}; \\
bq_{n-1} + q_{n-2}, & \text{if } n \text{ is odd}.
\end{cases} \quad (n \geq 2),
\]

where \( a \) and \( b \) are nonzero real numbers. They obtained many identities for the sequence \( \{q_n\}_{n=0}^{\infty} \). For instance, they gave the following extended Binet formula

\[
q_n = \left( \frac{1-\xi(n)}{ab} \right) \left( \frac{\alpha^n - \beta^n}{\alpha - \beta} \right), \quad (n \geq 0)
\]

where \( \alpha = \left( ab + \sqrt{a^2b^2 + 4ab} \right) / 2 \) and \( \beta = \left( ab - \sqrt{a^2b^2 + 4ab} \right) / 2 \). Here, \( \xi(n) = n - 2 \left\lfloor \frac{n}{2} \right\rfloor \) is the parity function.

In [4], Bilgici gave a general form of the Lucas sequence similar to the generalized Fibonacci sequence \( \{q_n\}_{n=0}^{\infty} \) as follows:

\[
l_0 = 2, \quad l_1 = a, \quad \text{and} \quad l_n = \begin{cases} 
al_{n-1} + l_{n-2}, & \text{if } n \text{ is odd}; \\
b_{n-1} + l_{n-2}, & \text{if } n \text{ is even}.
\end{cases} \quad (n \geq 2),
\]
where \(a\) and \(b\) are nonzero real numbers. He also derived many identities for the sequence \(\{l_n\}_{n=0}^{\infty}\). For example, he gave the following extended Binet formula

\[
l_n = \left(\frac{a^{l(n)}}{(ab)^{l(n)/2}}\right)(\alpha^n + \beta^n), \quad (n \geq 0).
\]

The \(n \times n\) \(r\)-circulant matrix, \(C_r\), associated with the numbers \(c_0, c_1, \ldots, c_{n-1}\) is of the form

\[
c_{ij} = \begin{cases} 
c_{j-i}, & j \geq i \\
rc_{n+j-i}, & j < i \end{cases}
\]

that is

\[
C_r = \begin{pmatrix} 
c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\
c_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\
c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
c_2 & c_3 & c_4 & \cdots & c_0 & c_1 \\
c_1 & c_2 & c_3 & \cdots & c_{n-1} & c_0
\end{pmatrix}.
\]

For \(r = 1\), the \(r\)-circulant matrix \(C_r\) reduces to circulant matrix \(C\), i.e.,

\[
C = \begin{pmatrix} 
c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\
c_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\
c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
c_2 & c_3 & c_4 & \cdots & c_0 & c_1 \\
c_1 & c_2 & c_3 & \cdots & c_{n-1} & c_0
\end{pmatrix}.
\]

Circulant matrices, \(r\)-circulant matrices, and their versions have been studied in many papers. For example, in [20], Solak found some bounds for the spectral norms of circulant matrices with the Fibonacci and Lucas number entries. Afterwards, Shen and Cen [16] developed Solak’s results. Later, many researchers studied different types of these matrices. For more details, we refer the interested reader to [1–3, 6, 8, 9, 12, 15, 17–19, 21–23, 25].

In [10], Kızılates and Tuğlu defined the \(n \times n\) geometric circulant matrix, \(C_r\), associated with the numbers \(c_0, c_1, \ldots, c_{n-1}\) as

\[
C_r = \begin{pmatrix} 
c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\
s_{n-2}c_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\
s_{n-3}c_{n-2} & s_{n-1}c_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
s_2c_2 & s_1c_3 & s_0c_4 & \cdots & c_0 & c_1 \\
s_1c_1 & s_0c_2 & s_1c_3 & \cdots & s_{n-1}c_{n-1} & c_0
\end{pmatrix}.
\]

They calculated bounds for the spectral norms of geometric circulant matrices with the generalized Fibonacci numbers and hyperharmonic Fibonacci numbers. Same authors [11] also found the norms of geometric and symmetric circulant matrices with the Tribonacci numbers. In [13], Köme and Yazlık presented some bounds for the spectral norms of the \(r\)-circulant matrices with the bi–periodic Fibonacci and Lucas numbers.

The purpose of this paper is to find some new upper and lower bounds for the spectral norms of the geometric circulant matrices with the bi-periodic Fibonacci numbers and hyperharmonic Lucas numbers. In [13], Köme and Yazlık presented some bounds for the spectral norms of the \(r\)-circulant matrices with the bi–periodic Fibonacci and Lucas numbers.

Now we need the following definitions and lemmas to derive new bounds.

The Euclidean (Frobenius) norm of matrix \(A\) (\(A = (a_{ij})\) be any \(m \times n\) matrix) is defined as

\[
\|A\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2}.
\]

Let \(A^H\) is conjugate transpose of the matrix \(A\). Then the spectral norm of matrix \(A\) is defined as

\[
\|A\|_2 = \sqrt{\max_{1 \leq i \leq n} \lambda_i(A^HA)}.
\]

The following inequality [7] holds:

\[
\frac{1}{\sqrt{n}} \|A\|_F \leq \|A\|_2 \leq \|A\|_F.
\]

**Lemma 1.1** ([7]). Let \(A = (a_{ij})\) and \(B = (b_{ij})\) be any \(m \times n\) matrices and let \(A \circ B\) is the Hadamard product of \(A\) and \(B\). Then

\[
\|A \circ B\|_2 \leq \|A\|_2 \|B\|_2.
\]
Lemma 1.2 ([14]). Let $A = (a_{ij})$ and $B = (b_{ij})$ be any $m \times n$ matrices. Then

$$\|A \otimes B\|_2 \leq r_1(A) c_1(B)$$

where

$$r_1(A) = \max_{1 \leq i \leq m} \sqrt{\sum_{j=1}^{n} |a_{ij}|^2} \quad \text{and} \quad c_1(B) = \max_{1 \leq j \leq n} \sqrt{\sum_{i=1}^{m} |b_{ij}|^2}.$$  

Lemma 1.3 ([7]). Let $A = (a_{ij})$ and $B = (b_{ij})$ be any $m \times n$ matrices and let $A \otimes B$ be the Kronecker product of $A$ and $B$. Then

$$\|A \otimes B\|_2 = \|A\|_2 \|B\|_2.$$  

2. Main Results

Theorem 2.1. Let $F = C_a \left( \left( \frac{a}{b} \right)^{\frac{q_0}{q}} q_1, \left( \frac{a}{b} \right)^{\frac{q_1}{q}} q_1, \ldots, \left( \frac{a}{b} \right)^{\frac{q_{n-1}}{q}} q_{n-1} \right)$ be an $n \times n$ geometric circulant matrix where $a$ and $b$ are nonzero positive real numbers and $s \in \mathbb{C}$. Then

(i) If $|s| > 1$, we have

$$\frac{q_{n-1}q_n}{b} \leq \|F\|_2 \leq \sqrt{\frac{|s|^{2n-2} - 1}{|s|^2 - 1}} \frac{q_{n-1}q_n}{b}.$$  

(ii) If $|s| < 1$, we have

$$\frac{|s|^2b}{\sqrt{\sqrt{ab} + 4}} \frac{2|s|^2 - 1}{|s|^2 + 1} |s|^2 + 2 - 2 \left( \frac{|s|^{2n} - (-1)^n}{|s|^2 + 1} \right) \|F\|_2 \leq \sqrt{\frac{(n-1)q_{n-1}q_n}{b}}.$$  

Proof. If we consider the definition of $F = C_a \left( \left( \frac{a}{b} \right)^{\frac{q_0}{q}} q_0, \left( \frac{a}{b} \right)^{\frac{q_1}{q}} q_1, \ldots, \left( \frac{a}{b} \right)^{\frac{q_{n-1}}{q}} q_{n-1} \right)$, then we have the following matrix:

$$F = \begin{pmatrix} (\frac{a}{b})^{\frac{q_0}{q}} q_0 & (\frac{a}{b})^{\frac{q_1}{q}} q_1 & (\frac{a}{b})^{\frac{q_2}{q}} q_2 & \cdots & (\frac{a}{b})^{\frac{q_{n-1}}{q}} q_{n-1} \\ s(\frac{a}{b})^{\frac{q_0}{q}} q_n & (\frac{a}{b})^{\frac{q_1}{q}} q_n & (\frac{a}{b})^{\frac{q_2}{q}} q_n & \cdots & (\frac{a}{b})^{\frac{q_{n-1}}{q}} q_n \\ s^2(\frac{a}{b})^{\frac{q_0}{q}} q_2 & s(\frac{a}{b})^{\frac{q_1}{q}} q_2 & (\frac{a}{b})^{\frac{q_2}{q}} q_2 & \cdots & (\frac{a}{b})^{\frac{q_{n-1}}{q}} q_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s^{n-1}(\frac{a}{b})^{\frac{q_0}{q}} q_1 & s^{n-2}(\frac{a}{b})^{\frac{q_1}{q}} q_1 & s^{n-3}(\frac{a}{b})^{\frac{q_2}{q}} q_1 & \cdots & (\frac{a}{b})^{\frac{q_{n-1}}{q}} q_1 \\ \end{pmatrix}.$$  

Thus we get the Euclidean norm of the matrix $F$ as

$$\|F\|_2^2 = \sum_{k=0}^{n-1} (n-k) \left( \frac{a}{b} \right)^{\xi(k)} \xi_k^2 + \sum_{k=1}^{n-1} k \left( \frac{a}{b} \right)^{\xi(k)} \xi_k^2.$$  

(i) If $|s| > 1$, from [24, Theorem 2.3], we obtain

$$\|F\|_2^2 \geq \sum_{k=0}^{n-1} (n-k) \left( \frac{a}{b} \right)^{\xi(k)} \xi_k^2 + \sum_{k=1}^{n-1} k \left( \frac{a}{b} \right)^{\xi(k)} \xi_k^2 = n \sum_{k=0}^{n-1} \left( \frac{a}{b} \right)^{\xi(k)} \xi_k^2.$$

So we have

$$\frac{1}{\sqrt{n}} \|F\|_E \geq \sqrt{\frac{q_{n-1}q_n}{b}}.$$  

that is

$$\sqrt{\frac{q_{n-1}q_n}{b}} \leq \|F\|_2.$$
Now, let us choose the matrices

\[ A = \begin{pmatrix}
q_0 & 1 & 1 & \cdots & 1 & 1 \\
s & q_0 & 1 & \cdots & 1 & 1 \\
s^2 & s & q_0 & \cdots & 1 & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
s^{n-1} & s^{n-2} & s^{n-3} & \cdots & s & q_0
\end{pmatrix} \tag{2.1} \]

and

\[ B = \begin{pmatrix}
\left( \frac{\xi_0}{b} \right)^2 q_0 & \left( \frac{\xi_1}{b} \right)^2 q_1 & \left( \frac{\xi_2}{b} \right)^2 q_2 & \cdots & \left( \frac{\xi_{n-1}}{b} \right)^2 q_{n-1} \\
\left( \frac{\xi_2}{b} \right)^2 q_{n-1} & \left( \frac{\xi_0}{b} \right)^2 q_0 & \left( \frac{\xi_1}{b} \right)^2 q_1 & \cdots & \left( \frac{\xi_{n-2}}{b} \right)^2 q_{n-2} \\
\left( \frac{\xi_1}{b} \right)^2 q_2 & \left( \frac{\xi_2}{b} \right)^2 q_{n-1} & \left( \frac{\xi_0}{b} \right)^2 q_0 & \cdots & \left( \frac{\xi_{n-3}}{b} \right)^2 q_{n-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\left( \frac{\xi_{n-1}}{b} \right)^2 q_1 & \left( \frac{\xi_2}{b} \right)^2 q_2 & \left( \frac{\xi_1}{b} \right)^2 q_3 & \cdots & \left( \frac{\xi_0}{b} \right)^2 q_0
\end{pmatrix} \tag{2.2} \]

such that \( F = A \circ B \). Therefore we obtain

\[ r_1(A) = \max_{1 \leq s \leq n} \left[ \sum_{i=1}^{n} (a_{ij})^2 \right] = \sqrt{q_0^2 + \sum_{k=1}^{n-1} |s|^{2k}} = |s| \sqrt{\frac{|s|^{2n-2} - 1}{|s|^2 - 1}} \]

and

\[ c_1(B) = \max_{1 \leq j \leq n} \left[ \sum_{i=1}^{n} (b_{ij})^2 \right] = \sqrt{\sum_{k=0}^{n-1} \left( \frac{\xi(k)}{b} \right)^2} = \sqrt{\frac{q_{n-1}q_n}{b}} \]

By Lemma 1.2, we get

\[ \sqrt{\frac{q_{n-1}q_n}{b}} \leq \| F \|_2 \leq |s| \sqrt{\frac{\left( \frac{|s|^{2n-2} - 1}{|s|^2 - 1} \right) q_{n-1}q_n}{b}}. \]

(ii) If \( |s| < 1 \), then we have

\[ \| F \|_2^2 \geq \sum_{k=0}^{n-1} (n-k) |s|^{n-k} \left( \frac{\alpha}{b} \right)^2 \xi(k)^2 q_k^2 + \sum_{k=1}^{n-1} k |s|^{n-k} \left( \frac{\beta}{|s|^2 ab} \right)^2 \xi(k)^2 q_k^2 \]

\[ = n |s|^{2n} \sum_{k=0}^{n-1} \left( \frac{\xi_k}{|s|^2} \right)^2 \]

\[ = an |s|^{2n} \sum_{k=0}^{n-1} \left( \frac{\alpha^2}{|s|^2 ab} \right)^k + \left( \frac{\beta^2}{|s|^2 ab} \right)^k - \sum_{k=0}^{n-1} \left( \frac{\xi_k}{|s|^2} \right)^2 \]

\[ = an |s|^{2n} \sum_{k=0}^{n-1} \left( \frac{2|s|^{2n+2} - |s|^{2n}(ab+2) - |s|^2 l_{2n+2} - 2 \left( \frac{|s|^{2n} - (-1)^n}{|s|^2 + 1} \right)}{|s|^4 - |s|^2 (ab+2) + 1} \right) \]

Therefore we obtain the following lower bound:

\[ \frac{|s| \sqrt{ab}}{b \sqrt{ab} + 4} \sqrt{\frac{2|s|^{2n+2} - |s|^{2n}(ab+2) - |s|^2 l_{2n+2} - 2 \left( \frac{|s|^{2n} - (-1)^n}{|s|^2 + 1} \right)}{|s|^4 - |s|^2 (ab+2) + 1}} \leq \| F \|_2. \]

In the meantime, let the matrices \( A \) and \( B \) be given in (2.1) and (2.2) such that \( F = A \circ B \). Then we have

\[ r_1(A) = \max_{1 \leq i \leq n} \left[ \sum_{j=1}^{n} (a_{ij})^2 \right] = \sqrt{n - 1} \]

and

\[ c_1(B) = \max_{1 \leq j \leq n} \left[ \sum_{i=1}^{n} (b_{ij})^2 \right] = \sqrt{\sum_{k=0}^{n-1} \left( \frac{\xi(k)}{b} \right)^2} = \sqrt{\frac{q_{n-1}q_n}{b}}. \]

Combining the above equations, we get the following inequality:

\[ \frac{|s| \sqrt{ab}}{b \sqrt{ab} + 4} \sqrt{\frac{2|s|^{2n+2} - |s|^{2n}(ab+2) - |s|^2 l_{2n+2} - 2 \left( \frac{|s|^{2n} - (-1)^n}{|s|^2 + 1} \right)}{|s|^4 - |s|^2 (ab+2) + 1}} \leq \| F \|_2 \leq \frac{\sqrt{n - 1} q_{n-1} q_n}{b}. \]
Theorem 2.2. Let $L = C_f\left(\left(\frac{a}{b}\right)^{\frac{k}{n}} l_0, \left(\frac{a}{b}\right)^{\frac{k}{n}} l_1, \ldots, \left(\frac{a}{b}\right)^{\frac{k}{n}} l_{n-1}\right)$ be an $n \times n$ geometric circulant matrix where $a$ and $b$ are nonzero positive real numbers and $s \in \mathbb{C}$. Then

(i) If $|s| > 1$, we have

$$\sqrt{\frac{l_{n-1} l_n + 2a}{b}} \leq ||L||_2 \leq \sqrt{\left(\frac{|s|^{2n} - 1}{|s|^2 - 1}\right) l_{n-1} l_n + 2a}$$

(ii) If $|s| < 1$, we have

$$\frac{|s| \sqrt{ab}}{b} \sqrt{\frac{2|s|^{2n+2} - |s|^{2n} (ab + 2) - |s|^2 l_{2n} + l_{2n-2}}{|s|^2 (ab + 2) + 1} + 2 \left(\frac{|s|^{2n} - (-1)^n}{|s|^2 + 1}\right)} \leq ||L||_2 \leq \sqrt{\frac{n}{b} (l_{n-1} l_n + 2a)}$$

Proof. Firstly, we have the following matrix:

$$L = \begin{pmatrix}
\left(\frac{a}{b}\right)^{\frac{k}{n}} l_0 & \left(\frac{a}{b}\right)^{\frac{k}{n}} l_1 & \left(\frac{a}{b}\right)^{\frac{k}{n}} l_2 & \cdots & \left(\frac{a}{b}\right)^{\frac{k}{n}} l_{n-1} \\
\left(\frac{a}{b}\right)^{\frac{k}{n}} l_{n-1} & \left(\frac{a}{b}\right)^{\frac{k}{n}} l_0 & \left(\frac{a}{b}\right)^{\frac{k}{n}} l_1 & \cdots & \left(\frac{a}{b}\right)^{\frac{k}{n}} l_{n-2} \\
\left(\frac{a}{b}\right)^{\frac{k}{n}} l_{n-2} & \left(\frac{a}{b}\right)^{\frac{k}{n}} l_{n-1} & \left(\frac{a}{b}\right)^{\frac{k}{n}} l_0 & \cdots & \left(\frac{a}{b}\right)^{\frac{k}{n}} l_{n-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\left(\frac{a}{b}\right)^{\frac{k}{n}} l_1 & s^{-1} \left(\frac{a}{b}\right)^{\frac{k}{n}} l_2 & s^{-2} \left(\frac{a}{b}\right)^{\frac{k}{n}} l_3 & \cdots & s^{-n+1} \left(\frac{a}{b}\right)^{\frac{k}{n}} l_0
\end{pmatrix}$$

Thus we get the Euclidean norm of the matrix $L$ as

$$||L||_2^2 = \sum_{k=0}^{n-1} \left|\left(\frac{a}{b}\right)^{\frac{k}{n}} l_k\right|^2 + \sum_{k=0}^{n-1} k \left|\left(\frac{a}{b}\right)^{\frac{k}{n}} l_k\right|^2.$$ 

(i) If $|s| > 1$, from [13, Theorem 2.1], we get

$$||L||_2^2 \geq \sum_{k=0}^{n-1} \left(\frac{a}{b}\right)^{\frac{k}{n}} l_k^2 + n \sum_{k=0}^{n-1} k \left(\frac{a}{b}\right)^{\frac{k}{n}} l_k^2$$

$$= n \sum_{k=0}^{n-1} \left(\frac{a}{b}\right)^{\frac{k}{n}} l_k^2$$

$$= \frac{n}{b} (l_{n-1} l_n + 2a).$$

So we obtain

$$\frac{1}{\sqrt{n}} ||L||_2 \geq \sqrt{\frac{l_{n-1} l_n + 2a}{b}}$$

that is

$$\sqrt{\frac{l_{n-1} l_n + 2a}{b}} \leq ||L||_2.$$
such that \( L = C \circ D \). Therefore we have

\[
r_1(C) = \max_{1 \leq i \leq n} \sqrt{\sum_{j=1}^{n} |c_{ij}|^2} = \sqrt{\sum_{k=0}^{n-1} s^{2k}} = \sqrt{\frac{|s|^{2n} - 1}{|s|^2 - 1}}
\]

and

\[
c_1(D) = \max_{1 \leq i \leq n} \sqrt{\sum_{j=1}^{n} |d_{ij}|^2} = \sqrt{\sum_{k=0}^{n-1} \left( \frac{a}{b} \right)^{k+1} t_k^2} = \sqrt{\frac{l_n - 1 + 2a}{b}}.
\]

By Lemma 1.2, we obtain

\[
\sqrt{\frac{l_n - 1 + 2a}{b}} \leq ||L||_2 \leq \sqrt{\frac{|s|^{2n} - 1}{|s|^2 - 1}} \left( \frac{l_n - 1 + 2a}{b} \right).
\]

(ii) If \( |s| < 1 \), then we get

\[
||L||_2^2 \geq \sum_{k=0}^{n-1} (n - k) |s|^{n-k}^2 \left( \frac{a}{b} \right)^{k+1} t_k^2 + \sum_{k=0}^{n-1} k |s|^{n-k}^2 \left( \frac{a}{b} \right)^{k+1} t_k^2
\]

\[
= n |s|^{2n} \sum_{k=0}^{n-1} \left( \frac{a^2}{|s|^2} \right)^k + \sum_{k=0}^{n-1} k |s|^{n-k} \left( \frac{\beta^2}{|s|^2} \right)^k + 2 \sum_{k=0}^{n-1} \left( \frac{1}{|s|^2} \right)^k
\]

\[
= n |s|^{2n} \left( 2 |s|^{2n-2} - |s|^{2n} (ab + 2) - |s|^2 t_2a + t_2a - 2 + 2 |s|^{2n} - (-1)^n \right).
\]

Thus we get the following inequality:

\[
|s| \sqrt{ab} \sqrt{\frac{2 |s|^{2n-2} - |s|^{2n} (ab + 2) - |s|^2 t_2a + t_2a - 2 + 2 |s|^{2n} - (-1)^n}{|s|^2 - |s|^2 (ab + 2) + 1}} \leq ||L||_2.
\]

In the meantime, let the matrices \( C \) and \( D \) be given in (2.3) and (2.4) such that \( L = C \circ D \). Then we have

\[
r_1(C) = \max_{1 \leq i \leq n} \sqrt{\sum_{j=1}^{n} |c_{ij}|^2} = \sqrt{n}
\]

and

\[
c_1(D) = \max_{1 \leq i \leq n} \sqrt{\sum_{j=1}^{n} |d_{ij}|^2} = \sqrt{\sum_{k=0}^{n-1} \left( \frac{a}{b} \right)^{k+1} t_k^2} = \sqrt{\frac{l_n - 1 + 2a}{b}}.
\]

Combining the above equations, we have the following inequality:

\[
|s| \sqrt{ab} \sqrt{\frac{2 |s|^{2n-2} - |s|^{2n} (ab + 2) - |s|^2 t_2a + t_2a - 2 + 2 |s|^{2n} - (-1)^n}{|s|^2 - |s|^2 (ab + 2) + 1}} \leq ||L||_2 \leq \sqrt{n \frac{l_n - 1 + 2a}{b}}.
\]

Corollary 2.3. Let \( a \) and \( b \) be nonzero positive real numbers and the matrices \( F \) and \( L \) be as in Theorem 2.1 and Theorem 2.2, respectively.

(i) If \( |s| > 1 \), then we have

\[
\|F \circ L\|_2 \leq \frac{|s|}{b \left| s^2 - 1 \right|} \sqrt{\left( |s|^{2n-2} - 1 \right) \left( |s|^{2n} - 1 \right) q_{n-1} q_n (l_{n-1} l_n + 2a)}.
\]

(ii) If \( |s| < 1 \), then we have

\[
\|F \circ L\|_2 \leq \frac{1}{b} \sqrt{n (n-1) q_{n-1} q_n (l_{n-1} l_n + 2a)}.
\]

Proof. The proof follows from Lemma 1.1, Theorem 2.1 and Theorem 2.2.

Corollary 2.4. Let \( a \) and \( b \) be nonzero positive real numbers and the matrices \( F \) and \( L \) be as in Theorem 2.1 and Theorem 2.2, respectively.
(i) If $|s| > 1$, then we have
\[
\|F \otimes L\|_2 \geq \frac{1}{b} \sqrt{q_{n-1}q_n (l_{n-1} l_n + 2a)}
\]
and
\[
\|F \otimes L\|_2 \leq \frac{|s|}{b \left( |s|^2 - 1 \right)} \sqrt{\left( |s|^{2n-2} - 1 \right) \left( |s|^{2n} - 1 \right) q_{n-1}q_n (l_{n-1} l_n + 2a)}.
\]

(ii) If $|s| < 1$, then we have
\[
\|F \otimes L\|_2 \geq \frac{a |s|^2}{b \sqrt{|ab| + 4}} \sqrt{\left( 2 |s|^{2n} - |s|^{2n} (ab + 2) - |s|^2 l_{2n} + l_{2n-2} \right)^2 - 4 \left( |s|^{2n} - (-1)^n \right)^2}
\]
and
\[
\|F \otimes L\|_2 \leq \frac{1}{b} \sqrt{n(n-1) q_{n-1}q_n (l_{n-1} l_n + 2a)}.
\]

Proof. The proof follows from Lemma 1.3, Theorem 2.1 and Theorem 2.2.

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