OPTIMAL CONVERGENCE RATES FOR THE
THREE-DIMENSIONAL TURBULENT FLOW EQUATIONS

DONGFEN BIAN AND BOLING GUO

Abstract. In this paper we are concerned with the convergence rate of solutions to the three-dimensional turbulent flow equations. By combining the $L^p$-$L^q$ estimates for the linearized equations and an elaborate energy method, the convergence rates are obtained in various norms for the solution to the equilibrium state in the whole space, when the initial perturbation of the equilibrium state is small in $H^3$-framework. More precisely, the optimal convergence rates of the solutions and its first order derivatives in $L^2$-norm are obtained when the $L^p$-norm of the perturbation is bounded for some $p \in [1, \frac{6}{5})$.

AMS Subject Classification 2000: 35Q35, 65M12, 76F60, 93D20.

Key words and phrases: Turbulent flow equations; $k$-$\varepsilon$ model; optimal convergence rate; energy estimates.

1. Introduction

We consider in this work the turbulent flow equations for compressible flows on $\mathbb{R}^3$, 
\[
\begin{align*}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) - \Delta u - \nabla p &= -\frac{2\pi}{3} \nabla (\rho k), \\
(\rho h)_t + \text{div}(\rho u h) - \Delta h &= \frac{\partial p}{\partial t} + S_k, \\
(\rho k)_t + \text{div}(\rho u k) - \Delta k &= G - \rho \varepsilon, \\
(\rho \varepsilon)_t + \text{div}(\rho u \varepsilon) - \Delta \varepsilon &= C_1 G - C_2 \rho \varepsilon^2.
\end{align*}
\] (1.1)

with 
\[S_k = [\mu (\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}) - 2 \delta_{ij} \mu (\frac{\partial u_i'}{\partial x_k} \frac{\partial u_k'}{\partial x_j} + \frac{\partial u_j'}{\partial x_k} \frac{\partial u_k'}{\partial x_i}) + \frac{\mu_t}{\rho^2} \frac{\partial p}{\partial x_j} \frac{\partial p}{\partial x_i}], \]
\[G = \frac{\partial u_i'}{\partial x_j} \mu e (\frac{\partial u_j'}{\partial x_j} + \frac{\partial u_i'}{\partial x_i}) - \frac{2\pi}{3} \delta_{ij} (\rho k + \mu (\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}))],
\]
where $\delta_{ij}$ is given by $\delta_{ij} = 0$ if $i \neq j$, $\delta_{ij} = 1$ if $i = j$, dynamic viscosity $\mu$ and the eddy viscosity $\mu_t$ are positive constants satisfying $\mu + \mu_t = \mu_e$, and $C_1, C_2$ are also two adjustable positive constants.

Here $\rho, u, h, k,$ and $\varepsilon$ denote the density, velocity, total enthalpy, turbulent kinetic energy and rate of viscous dissipation, respectively. The pressure $p$ is a smooth function of $\rho$. In this paper, without loss of generality, we have renormalized some constants to be 1. The system (1.1) is formed by combining effect of turbulence on time-averaged Navier-Stokes equations with the $k$-$\varepsilon$ model equations.

All flows encountered in engineering practice, both simple ones such as two-dimensional jets, wakes, pipe flows and flat plate boundary layers and more complicated three-dimensional ones, become unstable above a certain Reynolds number. At low Reynolds numbers flows are laminar. Flows in the laminar regime are described by the continuity and Navier-Stokes equations which have been studied by
At high Reynolds numbers flows are observed to become turbulent. A chaotic and random state of motion develops in which the velocity and pressure change continuously with time within substantial regions of flow. More precisely, at values of the Reynolds number above $Re_{crit}$ a complicated series of events takes place which eventually leads to a radical change of the flow character. In the final state the flow behavior is random and chaotic. The motion becomes intrinsically unsteady even with constant imposed boundary conditions. The velocity and all other flow properties vary in a random and chaotic way. Turbulence stands out as a prototype of multi-scale phenomenon that occurs in nature. It involves wide ranges of spatial and temporal scales which makes it very difficult to study analytically and prohibitively expensive to simulate computationally. Many, if not most, flows of engineering significance are turbulent, so the turbulent flow regime is not just of theoretical interest. Up to now, although many physicists and mathematicians studied turbulent flows, there are not any general theory suitable for them. Fluid engineers need access to viable tools capable of representing the effects of turbulence.

This paper is devoted to study decay rates for the system (1.1) and proves the optimal convergence rates of its solutions under suitable assumptions. Bian-Guo [3] has obtained the global existence of smooth solutions to the system (1.1) under the condition that the initial data are close to the equilibrium state in $H^3$-framework. More precisely, this result is expressed in the following.

**Proposition 1.1.** Assume that initial data are close enough to the constant state $(\bar{\rho}, 0, 0, \bar{k}, 0)$, i.e. there exists a constant $\delta_0$ such that if

$$\| (\rho_0 - \bar{\rho}, u_0, h_0, k_0 - \bar{k}, \varepsilon_0) \|_{H^3(\mathbb{R}^3)} \leq \delta_0,$$

then the system (1.1) admits a unique smooth solution $(\rho, u, h, k, \varepsilon)$ such that for any $t \in [0, \infty)$,

$$\| (\rho - \bar{\rho}, u, h, k - \bar{k}, \varepsilon) \|_{H^3}^2 + \int_0^t \| \nabla \rho \|_{H^2}^2 + \| (\nabla u, \nabla h, \nabla k, \nabla \varepsilon) \|_{H^3}^2 \, ds \leq C \| (\rho_0 - \bar{\rho}, u_0, h_0, k_0 - \bar{k}, \varepsilon_0) \|_{H^3}^2,$$

where $C$ is a positive constant.

Based on this stability result, the main purpose in this paper is to investigate the optimal convergence rates in time to the stationary solution. We remark that the convergence rate is an important topic in the study of the fluid dynamics for the purpose of the computation [13] [15]. The main idea in this paper is to combine the $L^p-L^q$ estimates for the linearized equations and an improved energy method which includes the estimation on the higher power of $L^2$-norm of solutions. By doing this, the optimal convergence rates for the solutions to the nonlinear problem (1.1) in various norms can be obtained and are stated in the following theorem.

**Theorem 1.2.** Let $\delta_0$ be the constant defined in Proposition 1.1. There exist constants $\delta_1 \in (0, \delta_0)$ and $C > 0$ such that the following holds. For any $\delta \leq \delta_1$, if

$$\| (\rho_0 - \bar{\rho}, u_0, h_0, k_0 - \bar{k}, \varepsilon_0) \|_{H^3(\mathbb{R}^3)} \leq \delta,$$

and for some $p \in [1, \frac{5}{3})$,

$$\rho_0 - \bar{\rho}, u_0, h_0, k_0 - \bar{k}, \varepsilon_0 \in L^p(\mathbb{R}^3),$$

then...
then the smooth solution \((\rho, u, h, k, \varepsilon)\) in Proposition 1.2 enjoys the estimates for \(t \in [0, \infty)\),
\[
\|\rho - \bar{\rho}, u, h, k - \bar{k}, \varepsilon\|_q \leq C(1 + t)^{-\sigma(p,q,0)}, \quad 2 \leq q \leq 6,
\]
\[
\|\rho - \bar{\rho}, u, h, k - \bar{k}, \varepsilon\|_\infty \leq C(1 + t)^{-\sigma(p,2,1)},
\]
\[
\|\nabla (\rho - \bar{\rho}, u, h, k - \bar{k}, \varepsilon)\|_{H^2} \leq C(1 + t)^{-\sigma(p,2,1)},
\]
\[
\|\rho_t, u_t, h_t, k_t, \varepsilon_t\|_2 \leq C(1 + t)^{-\sigma(p,2,1)},
\]
where \(\sigma(p,q,l)\) are defined by
\[
\sigma(p,q,l) = \frac{3}{2} \left( \frac{1}{p} - \frac{1}{q} \right) + \frac{l}{2}.
\]

Remark 1.3. Equation (1.5) shows that the perturbation of initial data around the constant state \((\bar{\rho}, 0, 0, \bar{k}, 0)\) is bounded in \(L^p\)-norm, for some \(p \in \left[1, \frac{6}{5}\right]\), which need not be small.

Remark 1.4. The linearized equations of (1.1) around the constant state \((\bar{\rho}, 0, 0, \bar{k}, 0)\) take the following form:
\[
\begin{cases}
\alpha_t + \gamma \text{div} v = 0, \\
v_t + \gamma \nabla u - \lambda \Delta v - \lambda \nabla \text{div} v = 0, \\
h_t - \lambda \Delta h = 0, \\
m_t - \lambda \Delta m = 0, \\
\varepsilon_t - \lambda \Delta \varepsilon = 0,
\end{cases}
\]
where \(\gamma, \lambda\) are positive constants which will be given precisely in Section 2. Compared to the decay estimates of the solutions to the above linearized equations by using Fourier analysis [22] stated in Lemma 2.1 in the next section, Theorem 1.2 gives the optimal decay rates for the solution in \(L^q\)-norm, for any \(2 \leq q \leq 6\), and its first order estimates in \(L^2\)-norm. Note that the convergence rates of the derivatives of higher order in \(L^2\)-norm and the solution in \(L^\infty\)-norm are not the same as those for linearized equations.

Remark 1.5. We mainly use the method in [12] to prove Theorem 1.2. But our problem is much more difficult because of the strong coupling between velocity, total enthalpy, turbulent kinetic energy and rate of viscous dissipation. Moreover, our result is better than that in [12]. Here we assume \(L^p\)-norm of initial data \(\rho_0 - \bar{\rho}, u_0, h_0, k_0 - \bar{k}, \varepsilon_0\) is bounded, for some \(p \in \left[1, \frac{6}{5}\right]\), instead of \(\|(\rho_0 - \bar{\rho}, u_0, h_0, k_0 - \bar{k}, \varepsilon_0)\|_{L^1} < \infty\).

**Notation:** Throughout the paper, \(C\) stands for a general constant, and may change from line to line. The norm \(\|(A, B)\|_X\) is equivalent to \(\|A\|_X + \|B\|_X\) and \(\|A\|_{X \cap Y} = \|A\|_X + \|A\|_Y\). The norms in the Sobolev Spaces \(H^m(\mathbb{R}^3)\) and \(W^{m,q}(\mathbb{R}^3)\) are denoted respectively by \(\|\cdot\|_{H^m}\) and \(\|\cdot\|_{W^{m,q}}\) for \(m \geq 0, q \geq 1\). In particular, for \(m = 0\), we will simply use \(\|\cdot\|_p\). Moreover, \(\langle \cdot, \cdot \rangle\) denotes the inner-product in \(L^2(\mathbb{R}^3)\). Finally,
\[
\nabla = (\partial_1, \partial_2, \partial_3), \quad \partial_i = \partial_{x_i}, \quad i = 1, 2, 3,
\]
and for any integer \(l \geq 0\), \(\nabla^l f\) denotes all derivatives up to \(l\)-order of the function \(f\). And for multi-indices \(\alpha, \beta\) and \(\xi\)
\[
\alpha = (\alpha_1, \alpha_2, \alpha_3), \quad \beta = (\beta_1, \beta_2, \beta_3), \quad \xi = (\xi_1, \xi_2, \xi_3),
\]
we use
\[ \frac{\partial \alpha}{\partial x} = \frac{\partial \alpha_1}{\partial x_1} + \frac{\partial \alpha_2}{\partial x_2} + \frac{\partial \alpha_3}{\partial x_3}, \quad |\alpha| = \sum_{i=1}^{3} \alpha_i, \]
and \[ C^\beta_\alpha = \frac{\alpha!}{\beta!(\alpha - \beta)!} \] when \( \beta \leq \alpha \).

2. Preliminaries

We will reformulate the problem (1.1) as follows. Set
\[ \gamma = \sqrt{p'(\bar{\rho}) + \bar{k}}, \quad \lambda = \frac{1}{\bar{\rho}}. \]
Introducing new variables by
\[ a = \rho - \bar{\rho}, \quad v = \frac{1}{\gamma \lambda}, \quad h = h, \quad m = k - \bar{k}, \quad \varepsilon = \varepsilon, \]
the initial value problem (1.1) is reformulated as
\[
\begin{aligned}
& a_t + \gamma \text{div} v = F_1, \\
& v_t + \gamma \nabla a - \lambda \Delta v - \lambda \nabla \text{div} v = F_2, \\
& h_t - \lambda \Delta h = F_3, \\
& m_t - \lambda \Delta m = F_4, \\
& \varepsilon_t - \lambda \Delta \varepsilon = F_5,
\end{aligned}
\]
with \( S^1_k \) and \( G^1 \) the corresponding \( S_k \) and \( G \) in the variables of \((a, v, h, m, \varepsilon)\).

Let \( U = (a, v), \quad U_0 = (a_0, v_0), \quad F = (F_1, F_2), \]
\[ A = \begin{pmatrix} 0 & \gamma \text{div} \\ \gamma \nabla & \lambda \Delta + \lambda \nabla \text{div} \end{pmatrix} \]
and \( E(t) \) be the semigroup generated by the linear operator \( A \), then we can rewrite the solution for the first two equations of the nonlinear problem (1.1) as
\[
U(t) = E(t)U_0 + \int_0^t E(t-s)F ds.
\]

The semigroup \( E(t) \) has the following properties on the decay in time, which can be found in [21, 22] and will be applied to the integral formula (2.3).
Lemma 2.1. Let \( t \geq 0 \) be an integer and \( 1 \leq p < q < \infty \). Then for any \( t \geq 0 \), it holds that
\[
\| \nabla^l E(t) U_0 \|_q \leq C(1 + t)^{-\sigma(p,q,l)} \| U_0 \|_{L^p \cap H^l},
\]
with \( \sigma(p,q,l) \) defined by (1.10).

To treat the last three equations, we introduce the semigroup \( S(t) \) generated by \( \lambda \Delta \), then (2.1) become
\[
\begin{align*}
\dot{h}(t) &= S(t)h_0 + \int_0^t S(t-s)F_3 ds, \\
\dot{m}(t) &= S(t)m_0 + \int_0^t S(t-s)F_4 ds, \\
\dot{\varepsilon}(t) &= S(t)\varepsilon_0 + \int_0^t S(t-s)F_5 ds.
\end{align*}
\]

We state the large-time behavior of solutions to the last three equations of the system (2.1) as the following lemma which can be obtained by direct calculation or can refer to [29].

Lemma 2.2. For the solution \((h,m,\varepsilon)\) of the last three equations of the system (2.1) with Cauchy data \( h(x,0) = h_0, m(x,0) = m_0, \varepsilon(x,0) = \varepsilon_0 \), there exists a constant \( C \) such that
\[
\begin{align*}
\| (\nabla^l h, \nabla^l m, \nabla^l \varepsilon) \|_q &\leq C(1 + t)^{-\sigma(p,q,l)} \| (h_0, m_0, \varepsilon_0) \|_p \\
&+ C \int_0^t (1 + t-s)^{-\sigma(p,q,l)} \| (F_3, F_4, F_5) \|_p, \quad l = 0, 1,
\end{align*}
\]
for any \( t \geq 0 \), \( 1 \leq p, q \leq +\infty \), as well as \( \sigma \) is defined by (1.10).

For later use we list some Sobolev inequalities as follows, cf. [2111].

Lemma 2.3. Let \( \Omega \subset \mathbb{R}^3 \) be the whole space \( \mathbb{R}^3 \), or half space \( \mathbb{R}_+ \), or the exterior domain of a bounded region with smooth boundary. Then

(i) \( \| f \|_{L^p(\Omega)} \leq C \| \nabla f \|_{L^2(\Omega)}, \) for \( f \in H^1(\Omega) \).

(ii) \( \| f \|_{L^p(\Omega)} \leq C \| f \|_{H^1(\Omega)}, \) for \( 2 \leq p \leq 6 \).

(iii) \( \| f \|_{C^0(\Omega)} \leq C \| f \|_{W^{1,6}(\Omega)} \leq C \| \nabla f \|_{H^1(\Omega)}, \) for \( f \in H^2(\Omega) \).

(iv) \( \int_{\Omega} f \cdot g \cdot h dx \leq \varepsilon \| \nabla f \|_{L^2}^2 + \frac{\varepsilon}{\varepsilon} \| g \|_{H^1}^2 \| h \|_{L^2}^2, \) for \( \varepsilon > 0, f, g \in H^1(\Omega), h \in L^2(\Omega) \).

(v) \( \int_{\Omega} f \cdot g \cdot h dx \leq \varepsilon \| g \|_{L^2}^2 + \frac{\varepsilon}{\varepsilon} \| \nabla f \|_{H^1}^2 \| h \|_{L^2}^2, \) for \( \varepsilon > 0, f \in H^2(\Omega), g, h \in L^2(\Omega) \).

Finally, the following elementary inequality [12] will also be used.

Lemma 2.4. If \( r_1 > 1 \) and \( r_2 \in [0, r_1] \), then it holds that
\[
\int_0^t (1 + t-s)^{-r_1} (1 + s)^{-r_2} ds \leq C_1(r_1, r_2) (1 + t)^{-r_2},
\]
where \( C_1(r_1, r_2) \) is defined by
\[
C_1(r_1, r_2) = \frac{2^{r_2+1}}{r_1 - 1}.
\]
3. Basic estimates

In this section we shall establish two basic inequalities for the proof of the optimal convergence rates in section 4. One inequality is the decay rate of the first order derivatives, while the other is the energy estimate.

Lemma 3.1. Let \( W = (U, h, m, \varepsilon) \) be the solution to the problem (2.1), then under the assumptions of Theorem 1.2, we have

\[
\|\nabla W\|_2 \leq CE_0 (1 + t)^{-\sigma(p, 2; 1)} + C \int_0^t (1 + t - s)^{-\sigma(p, 2; 1)} \|\nabla W\|_{L^2} ds,
\]

with \( E_0 = \|W(0)\|_{L^p \cap H^1} = \|(U_0, h_0, m_0, \varepsilon_0)\|_{L^p \cap H^1} = \|(a_0, v_0, h_0, m_0, \varepsilon_0)\|_{L^p \cap H^1} \) and \( 1 \leq p < \frac{2}{\sigma} \).

Proof. Let \( l = 1 \) in (2.1), we then have from (2.1)

\[
\|\nabla U(t)\|_2 \leq CE_0 (1 + t)^{-\sigma(p, 2; 1)} + C \int_0^t (1 + t - s)^{-\sigma(p, 2; 1)} \|(F_1, F_2)\|_{L^p \cap H^1} \, ds,
\]

which together with (2.1) implies that

\[
\|\nabla W(t)\|_2 \leq CE_0 (1 + t)^{-\sigma(p, 2; 1)} + C \int_0^t (1 + t - s)^{-\sigma(p, 2; 1)} \|(F_1, F_2)\|_{L^p \cap H^1} \, ds
\]

+ \|(F_3, F_4, F_5)\|_{L^2} \, ds.

(3.2)

For \( \|(F_1, F_2)\|_{L^p \cap H^1} \), we estimate as follows. For \( 1 \leq p < \frac{2}{\sigma} \), the term in \( F_1 \) can be estimated as

\[
- \gamma \lambda \text{div}(av) \|_p \leq C(\|\partial_t av\|_p + \|a \partial_t v\|_p)
\]

\[
\leq C\|v\|_{2;1} \|\nabla a\|_2 + C\|\nabla v\|_2 \|a\|_{2;1}
\]

\[
\leq C\delta \|(\nabla a, \nabla v)\|_2.
\]

With the help of Hölder inequality and Lemma 2.3 it holds that

\[
- \gamma \lambda \text{div}(av) \|_{H^1} \leq C(\|\partial_t av\|_2 + \|a \partial_t v\|_2 + \|\nabla \text{div}(av)\|_2)
\]

\[
\leq C\delta \|(\nabla a, \nabla v)\|_{H^1}.
\]

Similarly, the terms in \( F_2 \) can be estimated as

\[
\left\| \frac{1}{\rho} - \frac{1}{\rho}(\Delta v + \nabla \text{div} v) \right\|_p \leq C\|\nabla^2 v\|_2 \|a\|_{2;1} \leq C\delta \|\nabla^2 v\|_2,
\]

\[
\left\| \frac{1}{\rho} - \frac{1}{\rho}(\Delta v + \nabla \text{div} v) \right\|_{H^1}
\]

\[
\leq C\left\| \left( \frac{1}{\rho} - \frac{1}{\rho} \right)(\Delta v + \nabla \text{div} v) \right\|_2 + \|\nabla(\frac{1}{\rho} - \frac{1}{\rho})(\Delta v + \nabla \text{div} v)\|_2
\]

\[
\leq C(\|a\|_\infty \|\nabla^2 v\|_2 + \|a\|_\infty \|\nabla^3 v\|_2 + \|\nabla a\|_\infty \|\nabla^2 v\|_2)
\]

\[
\leq C\delta \|\nabla v\|_{H^2},
\]

\[
\left\| - \frac{1}{\gamma \lambda} \left( \frac{p'(a + \rho)}{a + \rho} - \frac{p' \rho}{\rho} + \frac{2(m + \tilde{k})}{3(a + \rho)} + \frac{2k}{3\rho} \right) \nabla a \right\|_p
\]

\[
\leq C(\|\nabla a\|_2 \|a\|_{2;1} + \|\nabla a\|_2 \|m\|_{2;1})
\]

\[
\leq C\delta \|\nabla a\|_2,
\]
which will be estimated as follows.

where \( J \)

\[
\begin{align*}
\| - \frac{1}{\gamma \lambda} \left( \frac{p'(a + \bar{\rho})}{a + \bar{\rho}} - \frac{p'(\bar{\rho})}{\bar{\rho}} + \frac{2(m + \bar{k})}{3(a + \bar{\rho})} - \frac{2\bar{k}}{3\bar{\rho}} \right) \nabla a \|_{H^1} \\
\leq C \left( \| \left( \frac{p'(a + \bar{\rho})}{a + \bar{\rho}} - \frac{p'(\bar{\rho})}{\bar{\rho}} + \frac{2(m + \bar{k})}{3(a + \bar{\rho})} - \frac{2\bar{k}}{3\bar{\rho}} \right) \nabla a \|_2 \\
+ \| \nabla \left[ \left( \frac{p'(a + \bar{\rho})}{a + \bar{\rho}} - \frac{p'(...)
\leq C \| a\nabla a + m\nabla a \|_2 + \| a\nabla^2 a + m\nabla^2 a + (\nabla a)^2 + \nabla a\nabla m \|_2 \\
\leq C\delta \| \nabla a \|_{H^2},
\end{align*}
\]

Next, we estimate for \( \| (F_3, F_4, F_5) \|_p \). Almost as same as the estimation of \( \| (F_1, F_2) \|_{L^p \cap H^1} \), we get

\[
\| (F_3, F_4, F_5) \|_p \leq C\delta \| (\nabla a, \nabla v, \nabla m, \nabla \varepsilon) \|_{H^2}. \tag{3.4}
\]

Inserting (3.3) and (3.4) into (3.2), we complete the proof of Lemma 3.1.

\[\square\]

**Lemma 3.2.** Let \( W = (U, h, m, \varepsilon) \) be the solution to the problem (2.1), then under the assumptions of Theorem 1.1 if \( \delta > 0 \) is sufficiently small then it holds that

\[
\frac{dM(t)}{dt} + \| \nabla^2 a \|_{H^1}^2 + \| \nabla^2 (v, h, m, \varepsilon) \|_{H^2}^2 \leq C\delta \| \nabla (a, v, h, m, \varepsilon) \|_{H^2}, \tag{3.5}
\]

where \( M(t) \) is equivalent to \( \| \nabla (a, v, h, m, \varepsilon) \|_{H^2}^2 \), i.e., there exists a positive constant \( C_2 \) such that

\[
C_2^{-1} \| \nabla (a, v, h, m, \varepsilon) \|_{H^2}^2 \leq M(t) \leq C_2 \| \nabla (a, v, h, m, \varepsilon) \|_{H^2}^2 \tag{3.6}
\]

**Proof.** Let \( \alpha \) be any multi-index with \( 1 \leq |\alpha| \leq 3 \). Applying the operator \( \partial^\alpha_x \) to (2.1) and then taking inner product with \( \partial^\alpha_x a, \partial^\alpha_x v, \partial^\alpha_x h, \partial^\alpha_x m \) and \( \partial^\alpha_x \varepsilon \), one gets

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \| \partial^\alpha_x (a, v, h, m, \varepsilon) \|_2^2 + \lambda \| \nabla \partial^\alpha_x (a, v, h, m, \varepsilon) \|_2^2 + \lambda \| \text{div} \partial^\alpha_x (a, v, h, m, \varepsilon) \|_2^2 \\
= \langle \partial^\alpha_x a, \partial^\alpha_x F_1 \rangle + \langle \partial^\alpha_x v, \partial^\alpha_x F_2 \rangle + \langle \partial^\alpha_x h, \partial^\alpha_x F_3 \rangle + \langle \partial^\alpha_x m, \partial^\alpha_x F_4 \rangle \\
+ \langle \partial^\alpha_x \varepsilon, \partial^\alpha_x F_5 \rangle \\
=: J_1^\alpha(t) + J_2^\alpha(t) + J_3^\alpha(t) + J_4^\alpha(t) + J_5^\alpha(t),
\end{align*}
\]

where \( J_i^\alpha(t), i = 1, 2, 3, 4, 5 \), are the corresponding terms in the above equation which will be estimated as follows.
Now let’s estimate for $J_1^a(t)$. It follows from Lemma 2.3 that

$$J_1^a \sim \langle \partial_j \partial_i (a v^i), \partial_j a \rangle = \langle \partial_j \partial_i (a v^i + \partial_i a \partial_j v^i + \partial_j a \partial_i v^i + \partial_j \partial_i v^i a, \partial_j a) \rangle \leq C \| \nabla^2 a \|_2 \| \nabla a \|_2 \| v \|_\infty + \| \nabla a \|_\infty \| \nabla^2 a \|_2 + \| \nabla^2 a \|_2 \| a \|_\infty \leq \delta \| \nabla a \|^2_2 + \frac{C}{\delta} \| \nabla^2 a \|^2_2 \| v \|_{H^2} + \frac{C}{\delta} \| \nabla^2 a \|^2_2 \| a \|_{H^2}.$$  

$$J_2^a \sim \langle \partial_j^2 \partial_i (a v^i), \partial_j^2 \partial^2 \xi a \rangle = \langle \nabla^2 a v + \nabla a \nabla v + \nabla^2 v a, \nabla^2 a \rangle \leq \delta \| \nabla^3 a \|^2_2 + \frac{C}{\delta} \left( \| \nabla^2 a \|^2_2 \| v \|_{H^2} + \| \nabla v \|^2_2 \| \nabla a \|_\infty + \| \nabla^2 v \|^2_2 \| a \|_{H^2} \right) \leq \delta \| \nabla^3 a \|^2_2 + \frac{C}{\delta} \left( \| \nabla^2 a \|^2_2 \| v \|_{H^2} + \| \nabla v \|^2_2 \| \nabla a \|^2_2 \| a \|_{H^2} \right).$$

Hence

$$J_1^a \leq C \delta \sum_{1 \leq |\alpha| \leq 3} \| \partial^2 a \|_2^2 + C \delta \sum_{1 \leq |\alpha| \leq 4} \| \partial^2 v \|_2^2. \quad (3.7)$$

Similarly, by Lemma 2.3 and Hölder inequality, $J_2^a$ can be estimated as

$$J_2^a \sim \langle \partial_j \left( \frac{1}{\rho} - \frac{1}{\bar{\rho}} \right) (\Delta v + \nabla \text{div} v) - \frac{1}{\gamma \lambda} \left( \lambda \left( \frac{p'(a + \bar{\rho})}{a + \bar{\rho}} - \frac{p'(\bar{\rho})}{\bar{\rho}} + \frac{2(m + \bar{k})}{3(a + \bar{\rho})} - \frac{2k}{3\rho} \right) \nabla a \right) \nabla m, \partial_j v \rangle \leq C \left( \| \nabla a \|^3_3 \| \nabla v \|_a \| \nabla^2 v \|_2 + \| \nabla^3 v \|_2 \| \nabla v \|_2 \| a \|_{H^2} + \| \nabla(a, m) \|^3_3 \| \nabla \|_2 \| v \|_2 \right) \leq \delta \left( \| \nabla^2 v \|_2^2 + \| \nabla^3 v \|_2^2 + \| \nabla^2 (a, m) \|_2^2 + \frac{C}{\delta} \left( \| \nabla v \|^2_2 \| (a, m) \|^2_2 + \| \nabla v \|^2_2 \| \nabla m \|_2^2 \right) \right) \leq C \delta \| \nabla^2 (a, m) \|^2_2 + C \delta \sum_{1 \leq |\alpha| \leq 3} \| \partial^2 v \|_2^2.$$
Adding (3.10) and (3.11) together implies
\[ J_2^* \sim \langle \partial^\beta [\frac{1}{\rho} - \frac{1}{\tilde{\rho}}/(\Delta v + \nabla \text{div} v) - \frac{1}{\gamma\lambda}(p'(a + \tilde{\rho}) - \frac{p'(\tilde{\rho})}{\tilde{\rho}}) + \frac{2(m + \tilde{k})}{3(a + \tilde{\rho})} - \frac{2\tilde{k}}{3\tilde{\rho}} \rangle a \]
\[-\frac{2}{3\gamma\lambda} \nabla m], \partial_x^{\alpha+\xi} v \]
\[ \sim \langle \nabla a(\Delta v + \nabla \text{div} v) + a \nabla^3 v + \nabla a \nabla m + a \nabla^2 a + m \nabla^2 a + \nabla^2 m, \nabla^3 v \rangle \]
\[ \leq C(\|\nabla a\|_3\|\nabla^2 v\|_6\|\nabla^3 v\|_2 + \|\nabla^3 v\|_2\|a\|_\infty + \|\nabla a\|_3\|\nabla a\|_6\|\nabla^3 v\|_2 \]
\[ + \|\nabla a\|_6\|\nabla^3 v\|_2\|\nabla m\|_3 + \|\nabla^2 a\|_2\|\nabla^3 v\|_2\|a, m\|_\infty + \|\nabla^2 m\|_3\|\nabla^3 v\|_2\|m + \tilde{k}\|_6 \]
\[ \leq C\delta \sum_{1 \leq |\alpha| \leq 3} \|\partial_x^\alpha a\|_2^2 + C\delta \sum_{1 \leq |\alpha| \leq 3} \|\partial_x^\alpha (v, m)\|_2^2, \text{ with } \alpha = \beta + \xi, |\beta| = |\xi| = 1, \]

\[ J_3^* \sim \langle \partial^\beta [\frac{1}{\rho} - \frac{1}{\tilde{\rho}}/(\Delta v + \nabla \text{div} v) - \frac{1}{\gamma\lambda}(p'(a + \tilde{\rho}) - \frac{p'(\tilde{\rho})}{\tilde{\rho}}) + \frac{2(m + \tilde{k})}{3(a + \tilde{\rho})} - \frac{2\tilde{k}}{3\tilde{\rho}} \rangle a \]
\[-\frac{2}{3\gamma\lambda} \nabla m], \partial_x^{\alpha+\xi} v \]
\[ \sim \langle a \nabla^4 v + \nabla a \nabla^3 v + \nabla^2 a \nabla^2 v + a \nabla^3 a + \nabla a \nabla^2 a + \nabla m \nabla^2 a + \nabla^2 m \nabla a \]
\[ + m \nabla^3 a + \nabla^3 m, \nabla^4 v \rangle \]
\[ \leq C\delta \|\nabla^4 v\|_2^2 + \frac{C}{\delta} (\|\nabla a\|_3^2 \|\nabla^3 v\|_2^2 + \|\nabla^2 v\|_2^2 \|\nabla^2 a\|_6^2 + \|\nabla^3 a\|_2^2 \|a\|_\infty^2 \]
\[ + \|\nabla^2 a\|_2^2 \|\nabla (a, m)\|_2^2 + \|\nabla^2 m\|_2^2 \|\nabla a\|_\infty^2 + \|\nabla^3 a\|_2^2 \|m\|_\infty^2 + \|\nabla^3 m\|_3 \|\nabla m\|_2 \]
\[ \leq C\delta \sum_{1 \leq |\alpha| \leq 3} \|\partial_x^\alpha a\|_2^2 + C\delta \sum_{1 \leq |\alpha| \leq 4} \|\partial_x^\alpha (v, m)\|_2^2, \text{ with } \alpha = \beta + \xi, |\beta| = 2, |\xi| = 1. \]

Incorporating the above estimates, it holds that
\[ J_2^*(t) \leq C\delta \sum_{1 \leq |\alpha| \leq 3} \|\partial_x^\alpha a\|_2^2 + C\delta \sum_{1 \leq |\alpha| \leq 4} \|\partial_x^\alpha (v, m)\|_2^2. \tag{3.8} \]

Moreover, \( J_3^*(t), J_4^*(t) \) and \( J_5^*(t) \) can be estimated similarly by using Lemma 2.3 and Hölder inequality,
\[ |J_3^*(t), J_4^*(t), J_5^*(t)| \leq C\delta \sum_{1 \leq |\alpha| \leq 3} \|\partial_x^\alpha a\|_2^2 + C\delta \sum_{1 \leq |\alpha| \leq 4} \|\partial_x^\alpha (v, h, m, \varepsilon)\|_2^2. \tag{3.9} \]

On the other hand, we apply \( \partial_x^\alpha v \) to (2.1) with \( 1 \leq |\alpha| \leq 2 \), and then take inner product with \( \nabla \partial_x^\alpha a \) to yield
\[ \langle \partial_x^\alpha v, \nabla \partial_x^\alpha a \rangle + \gamma \|\nabla \partial_x^\alpha a\|_2^2 = \lambda \langle \partial_x^\alpha \Delta v + \partial_x^\alpha \nabla \text{div} v, \partial_x^\alpha \nabla a \rangle + \langle \partial_x^\alpha F_2, \partial_x^\alpha \nabla a \rangle, \tag{3.10} \]
and similarly from (2.1),
\[ \langle \partial_x^\alpha v, \nabla \partial_x^\alpha a \rangle = -\gamma \langle \partial_x^\alpha v, \nabla \partial_x^\alpha \text{div} v \rangle + \langle \partial_x^\alpha v, \partial_x^\alpha F_1 \rangle. \tag{3.11} \]

Adding (3.10) and (3.11) together implies
\[ \frac{d}{dt} \langle \partial_x^\alpha v, \nabla \partial_x^\alpha a \rangle + \gamma \|\nabla \partial_x^\alpha a\|_2^2 = \lambda \langle \partial_x^\alpha \Delta v + \partial_x^\alpha \nabla \text{div} v, \partial_x^\alpha \nabla a \rangle \]
\[ + \langle \partial_x^\alpha F_2, \partial_x^\alpha \nabla a \rangle - \gamma \langle \partial_x^\alpha v, \nabla \partial_x^\alpha \text{div} v \rangle + \langle \partial_x^\alpha v, \partial_x^\alpha F_1 \rangle. \tag{3.12} \]
By Hölder inequality and similar to the estimation of $J_2^n(t)$, the right hand side can be bounded by
\[
\frac{7}{2} \| \nabla \partial_x^a a \|_2^2 + C \sum_{1 \leq |\alpha| \leq 2} \| \partial_x^a \nabla v \|_H^4 + C \delta \sum_{1 \leq |\alpha| \leq 3} \| \partial_x^a a \|_2^2 + C \delta \sum_{1 \leq |\alpha| \leq 4} \| \partial_x^a (v, h, m, \varepsilon) \|_2^2.
\]
(3.13)

Therefore, if we define
\[
M(t) = C_1 \sum_{1 \leq |\alpha| \leq 2} \| \partial_x^a (a, v, h, m, \varepsilon) \|_2^2 + \sum_{1 \leq |\alpha| \leq 2} \langle \partial_x^a v, \nabla \partial_x^a a \rangle,
\]
and choosing $\delta$ sufficiently small, then (3.11) and (3.13) imply that
\[
\frac{dM(t)}{dt} + C_1 (\| \nabla^2 a \|_H^4 + \| \nabla^2 (v, h, m, \varepsilon) \|_H^4) \leq C \delta \| \nabla (a, v, h, m, \varepsilon) \|_2^2,
\]
where $C_1$ is a positive constant independent of $\delta$. Thus we arrive at the proof of the lemma.

4. Optimal convergence rates

The optimal convergence rates can be proved by first improving the estimates given in Lemma 3.1 and Lemma 3.2 to the estimates on the $L^2$-norms of solutions to higher power and then letting the power tend to infinity. By the inequalities \(3.1\) and \(5.3\), we have the following lemmas.

Lemma 4.1. Let $W = (U, h, m, \varepsilon)$ be the solution to the problem (2.1), then under the assumptions of Theorem 1.2, if $\delta > 0$ is sufficiently small, then for any integer $n \geq 1$, and for some $p \in [1, \frac{3}{2})$, it holds that
\[
\int_0^t (1 + s)^l \| \nabla W(s) \|_2^{2n} ds \leq (CE_0)^{2n} + (C\delta)^{2n} \int_0^t (1 + s)^l \| \nabla^2 W(s) \|_H^{2n} ds,
\]
where $l = 0, 1, \cdots, N = \lfloor 2n \left( \frac{3}{2p} - \frac{1}{4} \right) \rfloor - 2$, the constant $E_0$ is given in Lemma 3.1.

Lemma 4.2. Let $W = (U, h, m, \varepsilon)$ be the solution to the problem (2.1), then under the assumptions of Theorem 1.2, if $\delta > 0$ is sufficiently small, then for any integer $n \geq 1$, and for some $p \in [1, \frac{3}{2})$, it holds that
\[
(1 + t)^l M(t)^n + n \int_0^t (1 + s)^l M(s)^{n-1} \| \nabla^2 W(s) \|_H^2 ds \leq 2M(0)^n + (C_3E_0)^{2n} + 8C_2n \left( \frac{3}{2p} - \frac{1}{4} \right) \int_0^t (1 + s)^{l-1} M(s)^{n-1} \| \nabla^2 W(s) \|_H^2 ds,
\]
where $l = 0, 1, \cdots, N = \lfloor 2n \left( \frac{3}{2p} - \frac{1}{4} \right) \rfloor - 2$, the constant $C_2$ is given in Lemma 3.3, $C_3$ is independent of $\delta$.

Remark 4.3. Lemma 4.1 and Lemma 4.2 are similar to that in [12]. But Lemma 4.1 and Lemma 4.2 in this work hold for general $p$ with $p \in [1, \frac{3}{2})$. Note that lemmas hold only for $p = 1$ in [12]. For completeness, we state the proofs of Lemma 4.1 and Lemma 4.2 as follows.
Proof of Lemma 4.1. Fix any integer \( n \geq 1 \). By taking (3.1) to power \( 2n \) and multiplying it by \((1 + t)^l, l = 0, 1, \cdots, N, \) integrating the resulting inequality over \([0, t]\) gives that

\[
\int_0^t (1 + \tau)^l \| \nabla W(\tau) \|^2_{2n} d\tau \leq (CE_0)^{2n} \int_0^t (1 + \tau)^{-2n(\frac{3}{2p} - \frac{1}{4}) + l} d\tau \\
+ (C\delta)^{2n} \int_0^t (1 + \tau)^l \left[ \int_0^\tau (1 + \tau - s)^{-\frac{3}{2p}} \| \nabla W(s) \|_{H^2}^2 ds \right]^{2n} d\tau.
\]

(4.3)

It follows from the Hölder inequality that

\[
\left[ \int_0^\tau (1 + \tau - s)^{-\frac{3}{2p}} \| \nabla W(s) \|_{H^2} ds \right]^{2n} \\
\leq \left[ \int_0^\tau (1 + \tau - s)^{-r_1}(1 + s)^{-r_2} ds \right]^{2n-1} \\
\times \int_0^\tau (1 + \tau - s)^{-\frac{4}{2p}}(1 + s)^l \| \nabla W(s) \|_{H^n}^{2n} ds,
\]

where

\[ r_1 = \left( \frac{3}{2p} - \frac{1}{4} - \frac{2}{3n} \right) \frac{2n}{2n - 1} \]

and

\[ r_2 = \frac{l}{2n - 1}. \]

Notice that \( \frac{3}{2p} - \frac{1}{4} \leq r_1 \leq \frac{3}{2p} - \frac{1}{4} \) and \( r_2 \in [0, r_1] \) for \( n \geq 1 \) and \( 0 \leq l \leq N = [2n(\frac{3}{2p} - \frac{1}{4}) - 2] \), from Lemma 2.4 one deduces that

\[
\int_0^\tau (1 + \tau - s)^{-r_1}(1 + s)^{-r_2} ds \leq C_1(r_1, r_2)(1 + \tau)^{-r_2} \leq C(1 + \tau)^{-r_2}, \]

(4.5)

where \( C_1(r_1, r_2) \) given by (2.8) is bounded uniformly for \( n \geq 1 \). Hence, (4.3) together with (4.4) and (4.5) leads to

\[
\int_0^t (1 + \tau)^l \| \nabla W(\tau) \|^2_{2n} d\tau \leq (CE_0)^{2n} \frac{1}{2n(\frac{3}{2p} - \frac{1}{4}) + l - 1} \\
+ (C\delta)^{2n} \int_0^t (1 + s)^l \| \nabla W(s) \|_{H^n}^{2n} \int_s^t (1 + \tau - s)^{-\frac{4}{2p}} d\tau ds \\
\leq (CE_0)^{2n} + (C\delta)^{2n} \int_0^t (1 + s)^l \| \nabla W(s) \|_{H^2}^{2n} ds \\
\leq (CE_0)^{2n} + (C\delta)^{2n} \int_0^t (1 + s)^l \| \nabla W(s) \|_2^{2n} + \| \nabla^2 W(s) \|_{H^1}^{2n} ds.
\]

(4.6)

Here we have used the fact

\[
2n(\frac{3}{2p} - \frac{1}{4}) - l - 1 \geq 2n(\frac{3}{2p} - \frac{1}{4}) - 2n(\frac{3}{2p} - \frac{1}{4}) + 2 - 1 = 1.
\]

Thus if \( \delta > 0 \) is sufficiently small such that \((C\delta)^{2n} \leq \frac{1}{2}\) in the final inequality of (4.6), then (4.6) implies (4.1). We finish the proof of Lemma 4.1. \( \Box \)
Proof of Lemma 4.2. Multiplying (3.3) by \((1 + t)^l M(t)^{n-1}\) for \(l = 0, 1, \ldots, N\) and integrating it over \([0, t]\) give that

\[
(1 + t)^l M(t)^n + n \int_0^t (1 + s)^l M(s)^{n-1} \|\nabla^2 W(s)\|_{H^1}^2 ds \\
\leq M(0)^n + C\delta n \int_0^t (1 + s)^l M(s)^{n-1} \|\nabla W(s)\|_2^2 ds + l \int_0^t (1 + s)^{l-1} M(s)^n ds. 
\]  

(4.7)

For the second term on the right hand side of (4.7), from the Young inequality, (3.6) and Lemma 4.1, it holds that for any \(\eta > 0\),

\[
\delta n \int_0^t (1 + s)^l M(s)^{n-1} \|\nabla W(s)\|_2^2 ds \\
\leq \delta n \int_0^t (1 + s)^l [\frac{n-1}{n} \eta M(s)^n + \frac{1}{n \eta^{n-1}} \|\nabla W(s)\|_{2^n}^2] ds \\
\leq \delta n C_2 \eta \int_0^t (1 + s)^l M(s)^{n-1} (\|\nabla W(s)\|_2^2 + \|\nabla^2 W(s)\|_{H^1}^2) ds \\
+ \delta \eta^{1-n} \int_0^t (1 + s)^l \|\nabla W(s)\|_{2^n}^2 ds \\
\leq \delta n C_2 \eta \int_0^t (1 + s)^l M(s)^{n-1} \|\nabla W(s)\|_2^2 ds \\
+ \delta n C_2 \eta \int_0^t (1 + s)^l M(s)^{n-1} \|\nabla^2 W(s)\|_{H^1}^2 ds \\
+ \delta \eta^{1-n} [\gamma \|\nabla W(s)\|_2^2 + \eta^{n-1} ] \int_0^t (1 + s)^l M(s)^{n-1} \|\nabla W(s)\|_2^2 ds \\
+ \delta n C_2 \eta \int_0^t (1 + s)^l M(s)^{n-1} \|\nabla^2 W(s)\|_{H^1}^2 ds \\
+ \delta \eta^{1-n} C \eta^{n-1} \int_0^t (1 + s)^l M(s)^{n-1} \|\nabla W(s)\|_2^2 ds.
\]  

(4.8)

Choosing \(\eta = \frac{1}{2C_2} \) in (1.8), it holds that

\[
\delta n \int_0^t (1 + s)^l M(s)^{n-1} \|\nabla W(s)\|_2^2 ds \\
\leq 2\delta (2C_2)^{n-1} \|\nabla W(s)\|_2^2 + \delta n [1 + \gamma (\delta)^{2n} (2C_2)^{n-1} ] \int_0^t (1 + s)^l M(s)^{n-1} \|\nabla^2 W(s)\|_{H^1}^2 ds \\
\leq \delta \|\nabla W(s)\|_2^2 + \delta n [1 + \gamma (\delta)^{2n} ] \int_0^t (1 + s)^l M(s)^{n-1} \|\nabla^2 W(s)\|_{H^1}^2 ds.
\]  

(4.9)
Thus if $\delta > 0$ is sufficiently small such that $C\delta \leq 1$ in (4.9), then $(C\delta)^{2n} \leq 1$ for any $n \geq 1$. And from (4.9), it is easy to show that

\[
\delta n \int_0^t (1 + s)^l M(s)^{n-1} \|\nabla W(s)\|_H^2 \, ds \\
\leq (CE_0)^{2n + 2\delta n} \int_0^t (1 + s)^l M(s)^{n-1} \|\nabla^2 W(s)\|_H^2 \, ds.
\]

(4.10)

Similarly, we can estimate for the third term on the right hand side of (4.7) as

\[
l \int_0^t (1 + s)^l M(s)^n \, ds \\
\leq (CE_0)^{2n + \delta n(C\delta)^{2n-1}} \int_0^t (1 + s)^l M(s)^{n-1} \|\nabla^2 W(s)\|_H^2 \, ds \\
+ 2lC_2 \int_0^t (1 + s)^{l-1} M(s)^{n-1} \|\nabla^2 W(s)\|_H^2 \, ds,
\]

which together with (4.7) and (4.10) arrives at

\[
(1 + t)^l M(t)^n + n \int_0^t (1 + s)^l M(s)^{n-1} \|\nabla^2 W(s)\|_H^2 \, ds \\
\leq M(0)^n + (CE_0)^{2n + \delta n[C + (C\delta)^{2n-1}]} \int_0^t (1 + s)^l M(s)^{n-1} \|\nabla^2 W(s)\|_H^2 \, ds \\
+ 2lC_2 \int_0^t (1 + s)^{l-1} M(s)^{n-1} \|\nabla^2 W(s)\|_H^2 \, ds.
\]

(4.12)

Choose $\delta > 0$ sufficiently small such that for any $n \geq 1$, it follows that

\[\delta[C + C\delta]^{2n-1} \leq \frac{1}{2},\]

then it follows from (4.12) that

\[
(1 + t)^l M(t)^n + n \int_0^t (1 + s)^l M(s)^{n-1} \|\nabla^2 W(s)\|_H^2 \, ds \\
\leq 2M(0)^n + (C_3E_0)^{2n} + 4lC_2 \int_0^t (1 + s)^{l-1} M(s)^{n-1} \|\nabla^2 W(s)\|_H^2 \, ds,
\]

which gives (4.12) because $l \leq N \leq 2n(\frac{3}{2p} - \frac{1}{4})$. Thus we complete the proof of Lemma 4.2.

\[
\text{Proof of Theorem 1.2.} \quad \text{Let } \delta > 0 \text{ be small enough such that Lemma 4.2 holds for any } n \geq 2. \text{ For any fixed integer } n \geq 2, \text{ from Lemma 4.2, we get that the inequality (4.2) holds for any } l = 0, 1, \ldots, N. \text{ When } l = 1, \text{ (4.2) reads}
\]

\[
(1 + t)M(t)^n + n \int_0^t (1 + s)M(s)^{n-1} \|\nabla^2 W(s)\|_H^2 \, ds \\
\leq 2M(0)^n + (C_3E_0)^{2n} + 8C_2n(\frac{3}{2p} - \frac{1}{4}) \int_0^t M(s)^{n-1} \|\nabla^2 W(s)\|_H^2 \, ds.
\]

(4.13)
It follows from (1.3) in Proposition 1.1 that
\[
\int_0^t M(s)^{n-1} \|\nabla^2 W(s)\|_H^2 \, ds \leq \left[ \sup_{s \geq 0} M(s) \right]^{n-1} \int_0^t \|\nabla^2 W(s)\|_H^2 \, ds \\
\leq (C_2 C_0 \delta^2)^{n-1} C_0 \delta^2 \leq (C_2 C_0 \delta^2)^n.
\]
which together with (4.13) shows that
\[
(1 + t) M(t)^n + n \int_0^t (1 + s) M(s)^{n-1} \|\nabla^2 W(s)\|_H^2 \, ds \\
\leq 2M(0)^n + (C_3 E_0)^{2n} + 8C_2n\left(\frac{3}{2p} - \frac{1}{4}\right)(C_2 C_0 \delta^2)^n.
\]

For $1 \leq l \leq N$, by induction one can arrive at
\[
(1 + t)^l M(t)^n + n \int_0^t (1 + s)^l M(s)^{n-1} \|\nabla^2 W(s)\|_H^2 \, ds \\
\leq [2M(0)^n + (C_3 E_0)^{2n}] \sum_{i=1}^l [8C_2\left(\frac{3}{2p} - \frac{1}{4}\right)]^i + n[8C_2\left(\frac{3}{2p} - \frac{1}{4}\right)]^l(C_2 C_0 \delta^2)^n.
\]
In fact, suppose that (4.15) holds for $1 \leq l \leq N - 1$. Then from (4.12), it holds that
\[
(1 + t)^{l+1} M(t)^n + n \int_0^t (1 + s)^{l+1} M(s)^{n-1} \|\nabla^2 W(s)\|_H^2 \, ds \\
\leq 2M(0)^n + (C_3 E_0)^{2n} + 8C_2n\left(\frac{3}{2p} - \frac{1}{4}\right) \int_0^t (1 + s)^l M(s)^{n-1} \|\nabla^2 W(s)\|_H^2 \, ds \\
\leq [2M(0)^n + (C_3 E_0)^{2n}] + 8C_2\left(\frac{3}{2p} - \frac{1}{4}\right) \sum_{i=1}^l [8C_2\left(\frac{3}{2p} - \frac{1}{4}\right)]^{i-1} \\
+ n[8C_2\left(\frac{3}{2p} - \frac{1}{4}\right)]^l(C_2 C_0 \delta^2)^n
\]
\[
\leq [2M(0)^n + (C_3 E_0)^{2n}] \sum_{i=1}^{l+1} [8C_2\left(\frac{3}{2p} - \frac{1}{4}\right)]^{i-1} + n[8C_2\left(\frac{3}{2p} - \frac{1}{4}\right)]^{l+1}(C_2 C_0 \delta^2)^n,
\]
which combining with (4.13) gives that (4.15) holds for any $1 \leq l \leq N$.

Specially,
\[
(1 + t)^N M(t)^n \leq [2M(0)^n + (C_3 E_0)^{2n}] \frac{\left[8C_2\left(\frac{3}{2p} - \frac{1}{4}\right)\right]^N - 1}{8C_2\left(\frac{3}{2p} - \frac{1}{4}\right) - 1} + n\left[8C_2\left(\frac{3}{2p} - \frac{1}{4}\right)\right]^N(C_2 C_0 \delta^2)^n.
\]
Note that
\[
2n\left(\frac{3}{2p} - \frac{1}{4}\right) - 3 \leq N = [2n\left(\frac{3}{2p} - \frac{1}{4}\right) - 2] \leq 2n\left(\frac{3}{2p} - \frac{1}{4}\right) - 1.
\]
It is not difficult to prove that
\[
(1 + t)^{2n(\frac{3}{2p} - \frac{1}{4}) - 3} \leq C^{2n(\frac{3}{2p} - \frac{1}{4})}[M(0)^n + E_0^{2n} + \delta^{2n}],
\]
which gives that
\[
M(t)^{\frac{3}{2p} - \frac{1}{4}} \leq C[M(0)^n + E_0^{2n} + \delta^{2n}]^{\frac{3}{2p} - \frac{1}{4}} (1 + t)^{-\left(\frac{3}{2p} - \frac{1}{4}\right) + \frac{3}{2p} - \frac{1}{4}}.
\]
Since $M(0)$, $E_0$ and $\delta$ are independent of $n$, one gets
\[
[M(0)^n + E_0^{2n} + \delta^{2n}]^{\frac{1}{n}} \to \max\{\sqrt{M(0)}, E_0, \delta\}, \text{ as } n \to \infty.
\]
The above relation implies that
\[
M(t)^{\frac{1}{2}} \leq C \max\{\sqrt{M(0)}, E_0, \delta\}(1 + t)^{-(\frac{d}{2} - \frac{1}{4})},
\]
that is,
\[
\|W(t)\|_{H^2} \leq C \max\{\sqrt{M(0)}, E_0, \delta\}(1 + t)^{-\sigma(p,2;1)},
\]
which together with Lemma 2.3 implies (1.7) and (1.8).

Now, estimate for (1.6). For this purpose, applying (1.8), Lemma 2.1 and Lemma 2.2 leads to
\[
\|W(t)\|_2 \leq C E_0(1 + t)^{-\sigma(p,2;0)} + C \int_0^t (1 + t - s)^{-\sigma(p,2;0)}(\|F(W)\|_p + \|F(W)\|_2)ds
\]
\[
\leq C E_0(1 + t)^{-\sigma(p,2;0)} + C \delta \int_0^t (1 + t - s)^{-\sigma(p,2;0)}\|\nabla W(s)\|_{H^1}ds
\]
\[
\leq C E_0(1 + t)^{-\sigma(p,2;0)} + C \delta \int_0^t (1 + t - s)^{-\sigma(p,2;0)}(1 + s)^{-\sigma(p,2;1)}ds
\]
\[
\leq C(1 + t)^{-\sigma(p,2;1)}.
\]
By interpolation, we have that (1.6) holds for any $2 \leq q \leq 6$.

For (1.9), from (2.1), we get
\[
\|\partial_t W(t)\|_2 \leq \|\gamma \text{div} v\|_2 + \|(F_1, F_2, F_3, F_4, F_5)\|_2 + \| - \gamma \nabla a + \lambda \Delta v + \lambda \text{div} v\|_2
\]
\[
+ \|\lambda \Delta h\|_2 + \|\lambda \Delta m\|_2 + \|\lambda \Delta \varepsilon\|_2
\]
\[
\leq C(\|\nabla (a, v, h, m, \varepsilon)\|_2 + \|\nabla^2 (v, h, m, \varepsilon)\|_2)
\]
\[
\leq C E_0(1 + t)^{-\sigma(p,2;1)}.
\]
Thus, (1.9) is proved. The proof of Theorem 1.2 is complete. \qed

References

[1] H. Abidi, Équation de Navier-Stokes avec densité et viscosité variables dans l’espace critique, Rev. Mat. Iberoam. 23(2)(2007) 537–586.
[2] R. Adams, Sobolev spaces. New York, Academic Press, 1985.
[3] D. F. Bian and B. L. Guo, Global existence of smooth solutions to three-dimensional turbulent flow equations, arXiv:1203.5566v2 [math.AP] 9 Apr 2012.
[4] M. Cannone, Ondelettes, Paraproduits et Navier-Stokes equation, Nouveaux essais, Diderot éditeurs, Paris, 1995.
[5] M. Cannone, Harmonic analysis tools for solving the incompressible Navier-Stokes equations, Handbook of Mathematical fluid Dynamics, vol. III, North-Holland, Amsterdam, 2004.
[6] M. Cannone, F. Planchon, M. Schonbek, Strong solutions to the incompressible Navier-Stokes equations in the half-space, Comm. Partial Differential Equations 25(5-6)(2000) 903–924.
[7] Q. L. Chen, C. X. Miao, Z. F. Zhang, Well-posedness in critical spaces for compressible Navier-Stokes equations with density dependent viscosities, Rev. Mat. Iberoam. 26(2010) 915–946.
[8] R. Danchin, Local theory in critical spaces for compressible viscous and heat-conductive Gases, Comm. Partial Differential Equations 26(2001) 1183–1233.
[9] R. Danchin, Well-posedness in critical spaces for barotropic viscous fluids with truly not constant density, Comm. Partial Differential Equations 32(2007) 1373–1397.
[10] R. Danchin, P. B. Mucha, A critical functional framework for the inhomogeneous Navier-Stokes equations in the half-space, J. Funct. Anal. 256(3)(2009) 881–927.
[11] K. Deckelnick, Decay estimates for the compressible Navier-Stokes equations in unbounded domains, Math. Z. 209(1992), 115-130.
[12] R. J. Duan, S. Ukai, T. Yang and H. J. Zhao, Optimal convergence rates for the compressible Navier-Stokes equations with potential force, Math. Models Methods Appl. Sci. 17(5)(2007) 737-758.
[13] I. A. Ene and J. Paulin, Homogenization and two-scale convergence for a Stokes or Navier-Stokes flow in an elastic thin porous medium, Math. Models Methods Appl. Sci. 6(7)(1996) 941-955.
[14] H. Fujita and T. Kato, On Navier-Stokes initial value problem, I. Arch. Rational Mech. Anal. 16(1964) 269–315.
[15] Y. Giga, Solutions for semilinear parabolic equations in $L^p$ and regularity of weak solutions of the Navier-Stokes system, J. Differential Equations 62(2)(1986) 186–212.
[16] Y. Giga, T. Miyakawa, Solutions in $L^r$ of the Navier-Stokes initial value problem, Arch. Ration. Mech. Anal. 89(1985) 267–281.
[17] Y. Giga, H. Sohr, Abstract $L^p$ estimates for the cauchy problem with applications to the Navier-Stokes equations in exterior domains, J. Funct. Anal. 102(1991) 72–94.
[18] O. Goubet, Behavior of small finite element structures for the Navier-Stokes equations, Math. Models Methods Appl. Sci. 6(1)(1996) 1-32.
[19] H. Iwashita, $L^q$-$L^r$ estimates for solutions of the nonstationary Stokes equations in an exterior domain and the Navier-Stokes initial value problems in $L^q$ spaces, Math. Ann. 285(2)(1989) 265–288.
[20] T. Kato, Strong $L^p$-solutions of Navier-Stokes equations in $\mathbb{R}^N$ with applications to weak solutions, Math. Z 187(1984) 471–480.
[21] T. Kobayashi, Some estimates of solutions for the equations of motion of compressible viscous fluid in an exterior domain in $\mathbb{R}^3$, J. Differential Equations 184(2002) 587–619.
[22] T. Kobayashi and Y. Shibata, Decay estimates of solutions for the equations of motion of compressible viscous and heat-conductive gases in an exterior domain in $\mathbb{R}^3$, Commun. Math. Phys. 200(1999) 621–659.
[23] H. Kozono, Global $L^N$-solution and its decay property for the Navier-Stokes equations in half-space $\mathbb{R}^N_+$, J. Differential Equations 79(1)(1989) 79–88.
[24] O. Ladyzhenskaya, V. Solonnikov, The unique solvability of an initial-boundary value problem for viscous incompressible inhomogeneous fluids, J. Soviet Math. 9(1978) 697–749.
[25] P. L. Lion, Mathematics Topics in Fluid Mechanics, vol. 1, Incompressible Models, Clarendon Press, Oxford, 1996.
[26] A. Matsumura and T. Nishita, The initial value problem for the equations of motion of viscous and heat conductive gases, J. Math. Kyoto Univ., 20(1)(1980) 67-104.
[27] A. Meyer, Wavelets, Paraproducts and Navier-Stokes equations, Current developments in mathematics, International press, 1996.
[28] Z. P. Xin, Blow up of smooth solutions to the compressible Navier-Stokes equation with compact density, Comm. Pure Appl. Math. 51(1998) 229–240.
[29] E. Zuazua, Time asymptotics for heat and dissipative wave equations, June 2003, Available at: http://www.uam.es/enrique.zuazua.

The Graduate School of China Academy of Engineering Physics, P. O. Box 2101, Beijing 100088, PR China,

E-mail address: biandongfen@mail.com

Institute of Applied Physics and Computational Mathematics, P. O. Box 8009, Beijing 100088, PR China,

E-mail address: gbl@iapcm.ac.cn