GENERALISED CLASS OF TIME FRACTIONAL BLACK SCHOLES EQUATION AND NUMERICAL ANALYSIS

RODRIGUE GNITCHOGNA BATOGNA*

Department of Mathematics and Applied Mathematics
Faculty of Natural and Agricultural Sciences, University of the Free State
P.O. Box 339 Bloemfontein 9300, South Africa

Department of Mathematics University of Namibia, Private bag 13301 Windhoek, Namibia

ABDON ATANGANA

Institute of Groundwater Studies, University of the Free State
IB 56 UFS P.O. Box 339 Bloemfontein 9300, South Africa

ABSTRACT. It is well known now, that a Time Fractional Black Scholes Equation (TFBSE) with a time derivative of real order $\alpha$ can be obtained to describe the price of an option, when for example the change in the underlying asset is assumed to follow a fractal transmission system. Fractional derivatives as they are called were introduced in option pricing in a bid to take advantage of their memory properties to capture both major jumps over small time periods and long range dependencies in markets. Recently new derivatives of Fractional Calculus with non local and/or non singular Kernel, have been introduced and have had substantial changes in modelling of some diffusion processes. Based on consistency and heuristic arguments, this work generalises previously obtained Time Fractional Black Scholes Equations to a new class of Time Fractional Black Scholes Equations. A numerical scheme solution is also derived. The stability of the numerical scheme is discussed, graphical simulations are produced to price a double barriers knock out call option.

1. Introduction. If the importance to accurately price financial derivatives is at all time reminded by volumes of trade of these financial products, the constant innovation in pricing techniques on the other hand concedes that we still don’t have to date a method we could consider absolute. From the original Fisher Black, Myron Scholes and Robert Merton models [2, 11] via Kobol processes [3], CGMY processes [4], Finite Moment Log stable Processes [6] to recent Heston models [1] the question of option pricing remains topical. Fractional Black Scholes models started receiving growing attention with noticeable work like W. Wys [13], A.cartea and D. del-castillo-Negrete[7] who established that prices of financial derivatives satisfy Fractional Partial Differential Equations, for Black-Scholes models obtained on the assumption that the dynamics of equity price, follow Jump-diffusion processes or infinite activity Levy processes. Wen Chen and Song Wang [8] considered a fractional order partial differential equation arising in the valuation of American options with the classic Riemann-Louiville, Caputo and Grunwald Letnikov fractional derivative.
of order $\alpha$. More recently Wenting Chen et al. [9] analytically priced a double barrier options based on time fractional black scholes equation in the sense of Jumarie and Caputo time fractional derivatives. H. Zhang et al [14] proposed an implicit discrete scheme for a numerical solution of a time fractional Black Scholes model, with the time fractional derivative also being defined in the sense of Guy Jumarie. As new significant results in fractional calculus has recently emerged, and fractional models continuing to be vastly applied in mathematical finance, we propose here a generalised class of time fractional Black Scholes models, together with a numerical solution of such a model with the time fractional derivative defined in the sense of the recent Caputo-Fabrizio derivative with non singular Kernel [6]. In [9] Wenting Chen et al provided a detailed derivation of a time fractional Black Scholes equation, from a bi-fractional Black Scholes equation obtained by Liang et al [14]. We briefly present the main steps of that derivation which will lead to our new equation. If we assume that both the change with time in the option price and the transmission function of the diffusion sets are fractal, if $\mathcal{Y}(S,t)$ is the total flux rate per unit time from the current time $t$ to the expiry time $T$, if the option price is denoted by $C(S,t)$ we should have the following equation

$$\int_t^T \mathcal{Y}(S,\tau)d\tau = S^{d_f-1} \int_t^T H(\tau-t)[C(S,\tau) - C(S,T)]d\tau$$

(1)

with $d_f$ being the Hausdorff dimension and $H(t) = \frac{A_\alpha}{\Gamma(1-\alpha)t^\alpha}$ where $\alpha$ is the transmission exponent, $A_\alpha$ a constant. By differentiating equation (1) and using the classic Black Scholes equation to express the flux rate $\mathcal{Y}(S,\tau)$ as a function of other terms, we achieve the following equation

$$A_\alpha S^{d_f-1} \frac{\partial^n C}{\partial t^n} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r-D)S \frac{\partial C}{\partial S} - rC = 0.$$  

(2)

The time fractional derivative is defined as

$$\frac{\partial^n C}{\partial t^n} = \frac{1}{\Gamma(n-\alpha)\alpha} \int_t^T \frac{C(S,\tau) - C(S,t)}{(\tau-t)^{\alpha+1-n}} d\tau,$$

(3)

For $\alpha = 1$,

$$\lim_{\alpha \to 1} \frac{\partial^n C}{\partial t^n} = \frac{1}{\Gamma(2-1)\alpha} \int_t^T \frac{C(S,\tau) - C(S,t)}{(\tau-t)^{1+1-2}} d\tau = \frac{\partial C}{\partial t}$$

(4)

Therefore with the values $\alpha = 1$ and $d_f = 1$ we recover the classic Black Scholes equation.

For this work, we lean on this consistency argument, to claim the regularity of the following time fractional following Black Scholes equation

$$\begin{cases}
\frac{C^\alpha_D}{\partial t^\alpha} C(S,t) + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C(S,t)}{\partial S^2} + rS \frac{\partial C(S,t)}{\partial S} - rC(S,t) = 0, \\
(S,t) \in (0, +\infty) \times (0,T),
\end{cases}$$

(5)

\[C(0,t) = C_0(t), C(\infty,t) = q(t), C(S,T) = v(S)\]

where the time fractional derivative is defined in terms of the recent Caputo-Fabrizio derivative with non singular kernel as:

$$\frac{C^\alpha_D}{\partial t^\alpha} C(S,t) = \frac{M(\alpha)}{1-\alpha} \int_t^T \frac{\partial C(S,\tau)}{\partial \tau} \exp\left[-\frac{\alpha}{1-\alpha}(t-\tau)\right] d\tau.$$  

(6)
$M(\alpha)$ is a normalisation function such that $M(0) = M(1) = 1$. However if the function does not belong to the $H^1$ space then the derivative can be reformulated as

$$C^F_\alpha D^\alpha_t C(S, t) = \frac{M(\alpha)}{1 - \alpha} \int_a^t C(S, t) - C(S, \tau) \exp\left[-\alpha \frac{t - \tau}{1 - \alpha}\right] d\tau. \quad (7)$$

We claim that the new equation (5) is regular, and in fact by taking $\alpha = 1$ we still recover the classic Black Scholes Equation.

2. Existence and uniqueness of the solution. In this section we present a shortened proof that the time fractional Black Scholes Equation (5) with Caputo-Fabrizio Derivative fractional derivative has a unique solution.

Proof. The to Caputo-Fabrizio fractional integral for a function $f$ is defined as follows:

$$I^\alpha_t (f(t)) = \frac{2(1 - \alpha)}{(2 - \alpha)M(\alpha)} f(t) + \frac{2\alpha}{(2 - \alpha)M(\alpha)} \int_0^t f(s) ds, t \geq 0. \quad (8)$$

Applying the last integral operator on equation (5) we have

$$I^\alpha_t \left( C^F_\alpha D^\alpha_t C(S, t) \right) = I^\alpha_t \left( \psi(S, t, C(S, t)) \right) \quad (9)$$

where

$$\psi(S, t, C(S, t)) = rC(S, t) - rS \frac{\partial C(S, t)}{\partial S} - \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C(S, t)}{\partial S^2}.$$

$$\Rightarrow C(S, t) - C(S, t_0) = \frac{2(1 - \alpha)}{(2 - \alpha)M(\alpha)} \psi(S, t, C(S, t)) + \frac{2\alpha}{(2 - \alpha)M(\alpha)} \int_0^t \psi(S, \tau, C(S, \tau)) d\tau.$$

Let us define the mapping $\Upsilon : H \rightarrow H$ defined as

$$\Upsilon(C(S, t)) = \frac{2(1 - \alpha)}{(2 - \alpha)M(\alpha)} \psi(S, t, C(S, t)) + \frac{2\alpha}{(2 - \alpha)M(\alpha)} \int_0^t \psi(S, \tau, C(S, \tau)) d\tau.$$

Assuming $C$ is bounded which is a fairly realistic assumption - this can also be seen as truncating the original unbounded $x$ domain into finite interval [14] resulting in $C$ being bounded - it can be shown that there is a number $k > 0$ such that

$$\|\Upsilon(C(S, t_2)) - \Upsilon(C(S, t_1))\| \leq k \|C(S, t_2) - C(S, t_1)\|. \quad (10)$$

In fact

$$\|\Upsilon(C(S, t_2)) - \Upsilon(C(S, t_1))\|$$

$$= \left\| \frac{2(1 - \alpha)}{(2 - \alpha)M(\alpha)} \psi(S, t_2, C(S, t_2)) + \frac{2\alpha}{(2 - \alpha)M(\alpha)} \int_0^{t_2} \psi(S, \tau, C(S, \tau)) d\tau - \frac{2(1 - \alpha)}{(2 - \alpha)M(\alpha)} \psi(S, t_1, C(S, t_1)) - \frac{2\alpha}{(2 - \alpha)M(\alpha)} \int_0^{t_1} \psi(S, \tau, C(S, \tau)) d\tau \right\|$$

$$\leq \frac{2(1 - \alpha)}{(2 - \alpha)M(\alpha)} k_1 \|C_1 - C_2\| + \frac{2\alpha}{(2 - \alpha)M(\alpha)} T k_1 \int_{t_1}^{t_2} \|C_1 - C_2\| d\tau$$

$$\leq \left( \frac{2k_1(1 - \alpha)}{(2 - \alpha)M(\alpha)} + \frac{2k_1\alpha T}{(2 - \alpha)} \right) \|C_1 - C_2\|.$$
by imposing
\[ k = \left( \frac{2k_1(1-\alpha)}{(2-\alpha)M(\alpha)} + \frac{2k_1\alpha T}{(2-\alpha)} \right) < 1 \]
the mapping \( \Upsilon \) is a contraction, which implies a fixed point, and thus the Time Fractional Black Scholes Equation (5) with Caputo-Fabrizio fractional derivative has a unique solution. \( \square \)

3. **Numerical solution via discretisation.** By discretising equation (5) we propose a solution to the equation. We also discuss the stability of the proposed scheme. Let us adopt the following notation

\[
\begin{align*}
t_l &= l \Delta t, \quad t_{l+1} = (l+1)\Delta t \\
\Delta t &= t_{l+1} - t_l \\
x_j &= j \Delta x, \quad x_{j+1} = (j+1)\Delta x \\
\Delta x &= x_{j+1} - x_j \\
U(x_j, t_l) &= U_j^l
\end{align*}
\]

With the notation above, let 0 = \( t_0 < t_1 < t_2 < ... < t_n = t \) the Caputo-Fabrizio time fractional derivative in equation (5) can now be written as:

\[
\frac{\text{CF}_D^\alpha C(S_l, t_n)}{\Delta t} = \frac{M(\alpha)}{1 - \alpha} \int_0^{t_n} \frac{\partial C(S_l \tau)}{\partial \tau} \exp\left[-\alpha \frac{t_n - \tau}{1 - \alpha}\right] d\tau
\]

\[
= \frac{M(\alpha)}{1 - \alpha} \sum_{j=0}^{n} \left( \int_{t_j}^{t_{j+1}} \frac{C(S_l, t_{j+1}) - C(S_l, t_j)}{\Delta t} \exp\left[-\alpha \frac{t_n - \tau}{1 - \alpha}\right] d\tau \right)
\]

\[
= \frac{M(\alpha)}{1 - \alpha} \sum_{j=0}^{n} \left( \int_{t_j}^{t_{j+1}} \frac{C(S_l, t_{j+1}) - C(S_l, t_j)}{\Delta t} \exp\left[-\alpha \frac{t_n - \tau}{1 - \alpha}\right] d\tau \right)
\]

\[
= \frac{M(\alpha)}{1 - \alpha} \sum_{j=0}^{n} \left( \frac{C_{l+1}^j - C_l^j}{\Delta t} \int_{t_j}^{t_{j+1}} \exp\left[-\alpha \frac{t_n - \tau}{1 - \alpha}\right] d\tau \right)
\]

\[
= \frac{M(\alpha)}{1 - \alpha} \sum_{j=0}^{n} \left( \frac{C_{l+1}^j - C_l^j}{\Delta t} \left( \exp\left[-\alpha \frac{t_n - \tau}{1 - \alpha}\right] \right)_{t_{j+1}}^{t_{j+1}} \right)
\]

\[
= \frac{M(\alpha)}{1 - \alpha} \sum_{j=0}^{n} \left( \frac{C_{l+1}^j - C_l^j}{\Delta t} \left( \exp\left[-\alpha \frac{t_n - t_{j+1}}{1 - \alpha}\right] \right) - \left( \exp\left[-\alpha \frac{t_n - t_j}{1 - \alpha}\right] \right) \right)
\]

\[
= \frac{M(\alpha)}{\alpha \Delta t} \left( \exp\left(-\alpha \frac{\Delta t}{1 - \alpha}\right) - 1 \right) \sum_{j=0}^{n} \left( C_{l+1}^j - C_l^j \right) \left( \exp\left(-\alpha \frac{(n-j)\Delta t}{1 - \alpha}\right) \right)
\]

If we let
\[
\delta_0 = \exp\left(-\alpha \frac{\Delta t}{1 - \alpha}\right)
\]
and

$$\delta_{n,j} = (\delta_0)^{n-j} = \exp\left(-\frac{\alpha}{1-\alpha}(n-j)\Delta t\right)$$

Moreover discretising the space derivatives in equation (5) we have

$$C_F^0 D_t^\alpha C(S_t, t_n) = \frac{M(\alpha)}{\alpha \Delta t} \left( \exp\left(\alpha \frac{\Delta t}{1-\alpha}\right) - 1 \right) \sum_{j=0}^{n} (C_t^{j+1} - C_t^j)\delta_{n,j}$$

$$= \frac{M(\alpha)}{\alpha \Delta t} (\delta_0 - 1) \sum_{j=0}^{n} (C_t^{j+1} - C_t^j)\delta_{n,j}. \quad (11)$$

For simplicity let \( a = \frac{1}{2} \sigma^2; \ h = \Delta x \); \( L_\alpha = \frac{M(\alpha)}{\alpha \Delta t} \) making use of the notation we adopted and also substituting in (11), equation (5) can be rewritten as

$$(\delta_0 - 1)L_\alpha \sum_{j=0}^{n} (C_t^{j+1} - C_t^j)\delta_{n,j} = rC_t^n - \frac{a}{h^2} S_t^n (C_t^{n+1} - 2C_t^n + C_t^{n-1})$$

$$- \frac{r}{2h} S_t (C_t^{n+1} - C_t^{n-1}), \quad (12)$$

$$(\delta_0 - 1)L_\alpha \sum_{j=0}^{n} (C_t^{j+1} - C_t^j)(\delta_0)^{n-j} = rC_t^n - \frac{a}{h^2} S_t^n (C_t^{n+1} - 2C_t^n + C_t^{n-1})$$

$$- \frac{r}{2h} S_t (C_t^{n+1} - C_t^{n-1}). \quad (13)$$

This implies

$$(\delta_0 - 1)L_\alpha (C_t^{n+1} - C_t^n) = rC_t^n - \frac{a}{h^2} S_t^n C_t^{n+1} + 2\frac{a}{h^2} S_t^n C_t^n - \frac{a}{h^2} S_t^n C_t^{n-1}$$

$$- \frac{r}{2h} S_t C_t^{n+1} + \frac{r}{2h} S_t C_t^{n-1}$$

$$- \sum_{j=0}^{n-1} (C_t^{j+1} - C_t^j)(\delta_0)^{n-j}. \quad (14)$$

Expanding and grouping likewise terms of the iteration we have

$$(\delta_0 - 1)L_\alpha C_t^{j+1} = (\delta_0 - 1)L_\alpha C_t^n + rC_t^n + 2\frac{a}{h^2} S_t^n C_t^{n+1} - \frac{a}{h^2} S_t^n C_t^{n+1} - \frac{r}{2h} S_t C_t^{n+1}$$

$$+ \sum_{j=0}^{n-1} (C_t^{j+1} - C_t^j)(\delta_0)^{n-j}. \quad (14)$$
The iterative scheme can then be fully obtained as

\[
(\delta_0 - 1) L_\alpha C_i^{n+1} = \left( (\delta_0 - 1) L_\alpha + r + 2 \frac{a}{h^2} S_l^2 \right) C_i^n - \left( \frac{a}{h^2} S_l^2 + \frac{r}{2h} S_l \right) C_{i+1}^n + \sum_{j=0}^{n-1} (C_{i+1}^{j+1} - C_i^j) (\delta_0)^{n-j}
\]

(15)

4. Stability analysis of the numerical scheme. We follow von Neumann stability analysis method to investigate the stability of the numerical scheme. We let

\[
C(S_l, t_n) = C_i^n = \tilde{C}(t_n) e^{i f S_l} = \tilde{C}_n e^{i f h} \]

\[
C_i^{n+1} = \tilde{C}_{n+1} e^{i f h}, \quad C_i^n = \tilde{C}_n e^{i f h}, \quad C_{i+1}^n = \tilde{C}_{n+1} e^{i f (l+1) h}
\]

\[
C_{i+1}^{n+1} = \tilde{C}_{n+1} e^{i f (l+1) h} = \tilde{C}_n e^{i f (l-1) h}.
\]

With the above equation (15) therefore becomes

\[
(\delta_0 - 1) L_\alpha \tilde{C}_{n+1} e^{i f h} = \left( (\delta_0 - 1) L_\alpha + r + 2 \frac{a}{h^2} S_l^2 \right) \tilde{C}_n e^{i f h} - \left( \frac{a}{h^2} S_l^2 + \frac{r}{2h} S_l \right) \tilde{C}_{n+1} e^{i f (l+1) h} + \sum_{j=0}^{n-1} (\tilde{C}_{j+1} - \tilde{C}_j) e^{i f h} (\delta_0)^{n-j},
\]

simplifying similar exponential term from both side of the equation we have:

\[
(\delta_0 - 1) L_\alpha \tilde{C}_{n+1} = \left( (\delta_0 - 1) L_\alpha + r + 2 \frac{a}{h^2} S_l^2 \right) \tilde{C}_n - \left( \frac{a}{h^2} S_l^2 + \frac{r}{2h} S_l \right) \tilde{C}_{n+1} e^{i f h} + \left( \frac{r}{2h} S_l - \frac{a}{h^2} S_l^2 \right) \tilde{C}_n e^{-i f h} - \sum_{j=0}^{n-1} (\tilde{C}_{j+1} - \tilde{C}_j) (\delta_0)^{n-j}.
\]

Factorising for the time \(n^{th}\) iteration we have

\[
(\delta_0 - 1) L_\alpha \tilde{C}_{n+1} = \left( (\delta_0 - 1) L_\alpha + r + 2 \frac{a}{h^2} S_l^2 \right) \tilde{C}_n - \left( \frac{a}{h^2} S_l^2 e^{i f h} - \frac{r}{2h} S_l e^{i f h} \right) \tilde{C}_{n+1} + \left( \frac{r}{2h} S_l e^{-i f h} - \frac{a}{h^2} S_l^2 e^{-i f h} \right) \tilde{C}_n - \sum_{j=0}^{n-1} (\tilde{C}_{j+1} - \tilde{C}_j) (\delta_0)^{n-j}.
\]
It will then follow that
\[
(\delta_0 - 1)L_\alpha \hat{C}_{n+1} = \left( (\delta_0 - 1)L_\alpha + r + \frac{2a}{h^2} S_t^2 \right) \hat{C}_n \\
- \frac{a}{h^2} S_t^2 (2 \cos (fh)) - \frac{r}{2h} S_t (2i \sin (fh)) \hat{C}_n \\
- \sum_{j=0}^{n-1} (\hat{C}_{j+1} - \hat{C}_j) (\delta_0)^{n-j}.
\]

Dividing by the coefficient, \((\delta_0 - 1)L_\alpha\), the \((n+1)\)th iteration term can be expressed as
\[
\hat{C}_{n+1} = \frac{1}{(\delta_0 - 1)L_\alpha} \left( (\delta_0 - 1)L_\alpha + r + \frac{2a}{h^2} S_t^2 \right) (1 - \cos fh) \hat{C}_n \\
- \frac{r}{2h} S_t (2i \sin (fh)) \hat{C}_n \\
- \frac{1}{(\delta_0 - 1)L_\alpha} \sum_{j=0}^{n-1} (\hat{C}_{j+1} - \hat{C}_j) (\delta_0)^{n-j}.
\]

To derive a condition for stability let us rewrite the previous equation as
\[
\hat{C}_{n+1} = \left( 1 + \frac{r}{(\delta_0 - 1)L_\alpha} + \frac{2\sigma^2}{h^2(\delta_0 - 1)L_\alpha} S_t^2 \sin^2 \left( \frac{fh}{2} \right) ight) \hat{C}_n \\
- i \frac{r}{h(\delta_0 - 1)L_\alpha} S_t \sin (fh) \hat{C}_n \\
- \frac{1}{(\delta_0 - 1)L_\alpha} \sum_{j=0}^{n-1} (\hat{C}_{j+1} - \hat{C}_j) (\delta_0)^{n-j}.
\]

We can then express the ratio \(\frac{\hat{C}_{n+1}}{\hat{C}_n}\) as:
\[
\frac{\hat{C}_{n+1}}{\hat{C}_n} = 1 + \frac{r}{(\delta_0 - 1)L_\alpha} + \frac{2\sigma^2}{h^2(\delta_0 - 1)L_\alpha} S_t^2 \sin^2 \left( \frac{fh}{2} \right) \\
- i \frac{r}{h(\delta_0 - 1)L_\alpha} S_t \sin (fh) \\
- \frac{1}{\hat{C}_n} \frac{1}{(\delta_0 - 1)L_\alpha} \sum_{j=0}^{n-1} (\hat{C}_{j+1} - \hat{C}_j) (\delta_0)^{n-j}.
\]

To stability of the scheme will be given by
\[
\left| \frac{\hat{C}_{n+1}}{\hat{C}_n} \right| < 1.
\]
This is achieved when we have
\[
\left| \frac{\tilde{C}_{n+1}}{\tilde{C}_n} \right|^2 = \left( 1 + \frac{r}{(\delta_0 - 1)L_\alpha} + \frac{2\sigma^2}{h^2(\delta_0 - 1)L_\alpha} S_t^2 \sin^2 \left( \frac{fh}{2} \right) \right)^2 + \left( \frac{r}{h(\delta_0 - 1)L_\alpha} S_t \sin(fh) \right)^2 < 1.
\]

5. **Numerical simulations.** We give here graphical solutions with different alpha values for some call and put options. We consider the time fractional Black Scholes equation (5) with no dividend.

All Figures are illustrating the price of a double barrier knock out Call option with strike price \( K = 10 \), Down-and-Out price \( DO = 3 \) and Up-and-Out price \( UO = 15 \). The volatility is \( \sigma = 0.45 \) and the risk-free rate \( r = 0.03 \).

Figure 1, shows the classic Black Scholes prices which are obtained when equation (5) is reduced to the standard case \( \alpha = 1 \).

Figure 2, shows prices for values of alpha varying from \( \alpha = 0.5, \alpha = 0.6 \) and \( \alpha = 0.67 \). From figure 2 one has the feeling that the TFBSE with Caputo-Fabrizio fractional derivative (5) seems to present a multi scale memory property capable of capturing large jumps over time interval. The price variations within less than a decile \( \alpha \)-value is remarkable, and suggests a sensitivity analysis over a range of \( \alpha \)-values needs to be conducted. Figure 3, which illustrates prices for the standard case together with \( \alpha = 0.5, \alpha = 0.6 \) and \( \alpha = 0.9 \), does reinforce that the sentiment that our solution seems to be shifting scale, or is very sensitive within specific range of \( \alpha \)-values.

This gives us the feeling that the time fractional Black Scholes equation with Caputo-Fabrizio derivative could indeed successfully capture both high jumps over small interval of times, as well as it could satisfactorily depict the long range memory property.

6. **Conclusion.** In this work we considered a new class Time Fractional Black Scholes Equation (TFBSE) with the time fractional derivative defined in the sense of Caputo-Fabrizio. We presented the existence and uniqueness of the solution to the P.D.Es, we introduced a new numerical scheme solution to the TFSBE. We discussed the stability of our solution and illustrated by pricing a double barrier option for various values of the order of differentiation \( \alpha \). We also compared to the standard case prices obtained when \( \alpha = 1 \). A noticeable feature from both figure 2 and figure 3 is that pricing with TFBSE with Caputo-Fabrizio does not exhibit the same consistent property of under pricing the option -in comparison to the standard Black Scholes [10,11] when the strike price is close to the lower barrier, and overpricing it when the strike price is closer to the upper barrier. This can only come to reinforce the already observed crossover behaviour of some fractional operator [12]. We concluded that indeed the new TFBSE has superior memory property and could satisfactorily capture both high jumps in short interval of times, as well as portray satisfactorily the long range memory property. Further investigations could focus on the sensitivity analysis of the solution to the time fractional Black Scholes equation with Caputo-Fabrizio derivative, and look in details how price variations respond to \( \alpha \)-values.
Double barrier option price solutions. Model parameters are $\sigma = 0.45, r = 0.03, T = 1, K = 10, DO = 3, UO = 15$.

Approximate solutions from equation (15) Double barrier option prices approximate solutions. Model parameters are $\sigma = 0.45, r = 0.03, T = 1, K = 10, DO = 3, UO = 15$. 
Figure 3. Approximate solutions from equation (15) Double barrier option prices approximate solutions. Model parameters are $\sigma = 0.45, r = 0.03, T = 1, K = 10, DO = 3, UO = 15$.

REFERENCES

[1] E. Alos and Y. Yang, A fractional Heston model with $H > 1/2$, Stochastics, 89 (2017), 384–399.
[2] F. Black and M. Scholes, The pricing of options and corporate liabilities, J. Polit. Econ., 81 (1973), 637–654.
[3] S. I. Boyarchenko and S. Levendorskii, Non-Gaussian Merton-Black-Scholes Theory, World Scientific, Singapore, 2002.
[4] M. Caputo and F. Fabrizio, A New Definition of Fractional Derivative without singular Kernel, Progr. Frac. differ. Appl., 1 (2015), 1–13.
[5] P. Carr, H. Geman, D. B. Madan and M. Yor, the finite structure of asset returns: An empirical investigation, The journal of Business, 75 (2002), 305–332.
[6] P. Carr and L. Wu, The finite moment log stable process and option pricing, J. Finance, 58 (2003), 753–777.
[7] A. Cartea and D. del-castillo-Negrete, Fractional diffusion models of option prices in markets with jumps, Physica A-Statistical Mechanics and its Applications, 374 (2) (2007), 749–763.
[8] W. Chen and S. Wang, A penalty method for for a fractional order parabolic variational inequality governing American put option valuation, Computers & Mathematics with Applications, 67 (2014), 77–90.
[9] W. Chen, X. Xu and S.-P. Zhu, Analytically pricing double barrier options based on a time-fractional Black Scholes equation, Computers & Mathematics with Applications, 69 (2015), 1407–1419.
[10] J.-R. Liang, J. Wang, W.-J. Zhang, W.-Y. Qiu and F.-Y. Ren, The solution to a bi-fractional Black-Scholes-Merton differential equation, Int. J. Pure Appl. Math., 58 (2010), 99–112.
[11] R. C. Merton, The theory of rational option pricing, Bell Journal of Economics and Management Science, 4 (1973), 141–183.
[12] A. A. Tateishi, H. V. Ribeiro and E. K. Lenzi, The role of fractional time-derivative operators on anomalous diffusion, Front. Phys., 5 (2017), p52.
W. Wyss, the fractional Black-Scholes equation, *Fractional Calculus and Applied Analysis Theory Applications*, 3 (2000), 51–61.

H. Zhang, F. Liu, I. Turner and Q. Yang, Numerical solution of the time fractional Black-Scholes model governing European options, *Comput. Math. Appl.*, 71 (2016), 1772–1783.

Received May 2017; revised October 2017.

E-mail address: rgbatogna@yahoo.fr
E-mail address: abdonatangana@yahoo.fr