CIR equations with multivariate Lévy noise

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Abstract

The paper is devoted to the study of the short rate equation of the form
\[ dR(t) = F(R(t))dt + \sum_{i=1}^{d} G_i(R(t-))dZ_i(t), \quad R(0) = x \geq 0, \quad t > 0, \]
with deterministic functions \( F, G_1, ..., G_d \) and a multivariate Lévy process \( Z = (Z_1, ..., Z_d) \). The equation is supposed to have a nonnegative solution which generates an affine term structure model. Two classes of noise are considered. In the first one the coordinates of \( Z \) are independent processes with regularly varying Laplace exponents. In the second class \( Z \) is a spherical processes, which means that its Lévy measure has a similar structure as that of a stable process, but with radial part of a general form. For both classes a precise form of the short rate generator is characterized. Under mild assumptions it is shown that any equation of the considered type has the same solution as the equation driven by a Lévy process with independent stable coordinates.

The paper generalizes the classical results on the Cox-Ingersoll-Ross (CIR) model, [5], as well as on its extended version from [1] and [2] where \( Z \) is a one-dimensional Lévy process.

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1 Introduction

This paper is concerned with a stochastic equation of the form

\[ dR(t) = F(R(t))dt + \sum_{i=1}^{d} G_i(R(t-))dZ_i(t), \quad R(0) = x, \quad t > 0, \]  

(1.1)

where \( F, \{G_i\}_{i=1}^{d} \) are deterministic functions, \( Z_i(t), i = 1, 2, \ldots, d \), are Lévy processes and martingales, \( x \) is a nonnegative constant. A solution \( R(t), t \geq 0 \), if nonnegative, will be identified here with the short rate process, so it defines the bank account process by

\[ B(t) := e^{\int_0^t R(s)ds}, \quad t \geq 0. \]

Related to the savings account are zero coupon bonds. Their prices form a family of stochastic processes \( P(t,T), t \in [0,T], \) parametrized by their maturity times \( T \geq 0 \). The price of a bond with maturity \( T \) at time \( T \) is equal to its nominal value, typically assumed, also here, to be 1, that is \( P(T,T) = 1 \). The family of bond prices is supposed to have the affine structure, which means that

\[ P(t,T) = e^{-A(T-t)-B(T-t)R(t)}, \quad 0 \leq t \leq T, \]

(1.2)

for some smooth deterministic functions \( A, B \). Hence, the only source of randomness in the affine model (1.2) is the short rate process \( R \) given by (1.1). As the resulting market constituted by \( (B(t), \{P(t,T)\}_{T \geq 0}) \) should exclude arbitrage, the discounted bond prices

\[ \hat{P}(t,T) := B^{-1}(t)P(t,T) = e^{-\int_0^t R(s)ds-A(T-t)-B(T-t)R(t)}, \quad 0 \leq t \leq T, \]

are supposed to be local martingales for each \( T \geq 0 \). This requirement affects in fact our starting equation. Thus the functions \( F, \{G_i\}_{i=1}^{d} \) and the noise \( Z = (Z_1, \ldots, Z_d) \) should be chosen such that (1.1) has a nonnegative solution with any \( x \geq 0 \) and such that, for some functions \( A, B \) and each \( T \geq 0 \), \( \hat{P}(t,T) \) is a local martingale on \([0,T]\). If this is the case, (1.1) will be called to generate an affine model or to be a generating equation, for short.

In the case when \( Z = W \) is a real-valued Wiener process, the only generating equation is the classical CIR equation

\[ dR(t) = (aR(t) + b)dt + C \sqrt{R(t)}dW(t), \]

(1.3)

with \( a \in \mathbb{R}, b, C \geq 0 \), due to Cox, Ingersoll, Ross, see [3]. The case with a general one-dimensional Lévy process \( Z \) was studied in [1], [2] and [3] with the following conclusion. If the variation of \( Z \) is infinite and \( G \neq 0 \), then \( Z \) must be an \( \alpha \)-stable process with index \( \alpha \in (1,2] \), with either positive or negative jumps only, and (1.1) has the form

\[ dR(t) = (aR(t) + b)dt + C \cdot R(t)^{1/\alpha}dZ(t), \]

(1.4)

with \( a \in \mathbb{R}, b \geq 0 \) and \( C \) such that it has the same sign as the jumps of \( Z \). Clearly, for \( \alpha = 2 \) equation (1.4) becomes (1.3). If \( Z \) is of finite variation then the noise enters (1.1) in the additive way, that is

\[ dR(t) = (aR(t) + b)dt + C \, dZ(t). \]

(1.5)

Here \( Z \) can be chosen as an arbitrary process with positive jumps, \( a \in \mathbb{R}, C \geq 0 \) and

\[ b \geq C \int_{0}^{+\infty} y \, \nu(dy), \]
where $\nu(dy)$ stands for the Lévy measure of $Z$. The variation of $Z$ is finite, so is the right side above. Recall, (1.5) with $Z$ being a Wiener process is the well known Vasiček equation, see [11]. Then the short rate is a Gaussian process, hence it takes negative values with positive probability. This drawback is eliminated by the jump version of the Vasiček equation (1.5).

This paper is devoted to the equation (1.1) with $d > 1$. The multidimensional setting makes the study of equation (1.1) more complicated. The reason is that, unlike as in the case $d = 1$, different generating equations may have identical solutions in the sense that the solutions’ generators are the same. Our first goal is to characterize the class of generators of solutions of generating equations and the second goal is to construct, for each element of this class, a related specific equation. In this way any short rate process given by (1.1) which generates an affine model becomes representable by a tractable equation. This approach seems to be useful for future applications.

Our solution of the problem is based on, rather abstract, result of Filipović [7], characterizing generators of a general Markovian non-negative short rate process. The contribution of this paper is making this characterization concrete for two classes of Lévy processes. In the first class the coordinates of the noise $Z_1(t), Z_2(t), ..., Z_d(t), t \geq 0,$ are independent Lévy processes being martingales of infinite variation. Their Laplace exponents are assumed to vary regularly at zero. We show that the solution of any generating equation with such a noise is the same as the solution of the equation

$$dR(t) = (aR(t) + b)dt + \sum_{k=1}^{g} d_k R(t-)^{1/\alpha_k} dZ^{\alpha_k}_k(t),$$

where $1 \leq g \leq d$, $a \in \mathbb{R}$, $b \geq 0$, $d_k > 0$, $2 \geq \alpha_1 > ... > \alpha_g > 1$, and $Z^{\alpha_k}_k$ is a stable process with index $\alpha_k$. The second class consists of spherical Lévy processes. We call a process $Z(t) := (Z_1(t), ..., Z_2(t))$ spherical if its Lévy measure $\nu(dy)$ admits the following representation

$$\nu(A) = \int_{S^{d-1}} \lambda(d\xi) \int_0^{+\infty} 1_A(r\xi) \gamma(dr), \quad A \in \mathcal{B}(\mathbb{R}^d).$$

(1.6)

Here $S^{d-1}$ is a unit sphere in $\mathbb{R}^d$, $\lambda(d\xi)$ is a finite measure on $S^{d-1}$ called a spherical part of $\nu$, $\gamma(dr)$ is a Lévy measure on $(0, +\infty)$ called a radial part of $\nu$. One can see that on each half-line in $\mathbb{R}^d$ starting from the origin, the Lévy measure is given in the same way by the radial part, up to multiplication by a nonnegative constant. An important example of a radial measure satisfying (4.1)-(4.2) is

$$\gamma(dr) = \frac{1}{r^{1+\alpha}} dr, \quad \alpha \in (1, 2).$$

(1.7)

Given this measure and any finite measure $\lambda(d\xi)$, the formula (1.6) corresponds to a stable process with index $\alpha \in (1, 2)$. This process has no Wiener part. The 2-stable process is the Wiener process. We prove, under mild conditions, that the solution of any generating equation with spherical noise is the same as the solution of equation (1.4).

Our results for each of the classes introduced above generalize the one dimensional results from [1], [2] and [3].

The structure of the paper is as follows. In Section 2 we introduce the probabilistic setting for the equation (1.1) and present a properly adapted version of the result from [7] characterizing the generator of a generating equation. In particular, we point out here the role of the projections of
Z along G, meant as processes $\sum_{i=1}^{d} G_i(x) Z_i(t)$, $x \geq 0$, for the generator of $R$. Using examples highlighting the differences between the one- and multidimensional case we justify the form of problem-stating described above. Section 3 is concerned with equation (1.1) driven by $Z$ with independent coordinates. The main result here is Theorem 3.1. Regularly varying Laplace exponents are described in terms of the Lévy measure in Subsection 3.2. We also describe all generating equations in the case $d = 2$ in Subsection 3.3 and provide an example showing the non-uniqueness of a generating equation when $d = 3$ in Subsection 3.4. The case when $Z$ is spherical is presented in Section 4. The main result here is Theorem 4.1. Its proof, presented in Subsection 4.3, requires a sequence of auxiliary results contained in Subsection 4.2.

2 Preliminaries

The problem of description of generating equations (1.1) in the multidimensional case will be handled in a different way than in the one-dimensional case. Basing on Proposition 2.1 in Subsection 2.2 and examples in Subsection 2.3 we explain here the formulation of the problem studied in the sequel.

2.1 Setup for the equation

Using the scalar product $\langle \cdot, \cdot \rangle$ in $\mathbb{R}^d$ we write (1.1) in the short form

$$dR(t) = F(R(t))dt + \langle G(R(t-)), dZ(t) \rangle, \quad R(0) = x \geq 0, \quad t > 0, \quad (2.1)$$

where $F : [0, +\infty) \rightarrow \mathbb{R}$, $G := (G_1, G_2, ..., G_d) : [0, +\infty) \rightarrow \mathbb{R}^d$ and $Z := (Z_1, Z_2, ..., Z_d)$ is a Lévy process in $\mathbb{R}^d$ with the characteristic triplet $(a, Q, \nu(dy))$. Recall, $a \in \mathbb{R}^d$ describes the drift part of $Z$, $Q$ is a non-negative, symmetric, $d \times d$ covariance matrix, characterizing the coordinates’ covariance of the Wiener part $W$ of $Z$, and $\nu(dy)$ is a measure on $\mathbb{R}^d \setminus \{0\}$ satisfying

$$\int_{\mathbb{R}^d} (|y|^2 \wedge 1) \nu(dy) < +\infty, \quad (2.2)$$

describing the jumps of $Z$ and called the Lévy measure of $Z$. Recall, $Z$ admits a representation as a sum of four independent processes of the form

$$Z(t) = at + W(t) + \int_{0}^{t} \int_{|y| \leq 1} y \tilde{\pi}(ds, dy) + \int_{0}^{t} \int_{|y| > 1} y \pi(ds, dy), \quad (2.3)$$

called the Lévy-Itô decomposition of $Z$. Above $\pi(ds, dy)$ and $\tilde{\pi}(ds, dy) := \pi(ds, dy) - ds \nu(dy)$ stand for the jump measure and the compensated jump measure of $Z$, respectively. We consider the case when $Z$ is a martingale and call it a Lévy martingale for short. Its drift and the Lévy measure are such that

$$\int_{|y| > 1} y \nu(dy) < +\infty, \quad a + \int_{|y| > 1} y \nu(dy) = 0. \quad (2.4)$$

Consequently, the characteristic triplet of $Z$ is

$$\left(-\int_{|y| > 1} y \nu(dy), Q, \nu(dy) \right), \quad (2.5)$$

and (2.3) takes the form

$$Z(t) = W(t) + X(t), \quad X(t) := \int_{0}^{t} \int_{\mathbb{R}^d} y \tilde{\pi}(ds, dy), \quad t \geq 0,$$
where $W$ and $X$ are independent. The martingale $X$ will be called the jump part of $Z$. Its Laplace exponent $J_\nu$, defined by the equation
\[ \mathbb{E} \left[ e^{-\langle \lambda, X(t) \rangle} \right] = e^{t J_\nu(\lambda)}, \] (2.6)
has the following representation
\[ J_\nu(\lambda) = \int_{\mathbb{R}^d} \left( e^{-\langle \lambda, y \rangle} - 1 + \langle \lambda, y \rangle \right) \nu(dy), \] (2.7)
and is finite for $\lambda \in \mathbb{R}^d$ satisfying
\[ \int_{|y| > 1} e^{-\langle \lambda, y \rangle} \nu(dy) < +\infty. \]
By the independence of $X$ and $W$ we see that
\[
\mathbb{E} \left[ e^{-\langle \lambda, Z(t) \rangle} \right] = \mathbb{E} \left[ e^{-\langle \lambda, W(t) \rangle} \right] \cdot \mathbb{E} \left[ e^{-\langle \lambda, X(t) \rangle} \right],
\]
so the Laplace exponent $J_Z$ of $Z$ equals
\[ J_Z(\lambda) = \frac{1}{2} \langle Q\lambda, \lambda \rangle + J_\nu(\lambda). \] (2.8)

### 2.2 Projections of the noise and problem formulation

For the function $G$ and the process $Z$ we consider the projections of $Z$ along $G$ given by
\[ Z^G(x)(t) := \langle G(x), Z(t) \rangle, \quad t \geq 0. \] (2.9)
For any $x \geq 0$, $Z^G(x)$ is a real-valued Lévy martingale. It follows from the identity
\[ \mathbb{E} \left[ e^{-\gamma Z^G(x)(t)} \right] = \mathbb{E} \left[ e^{-\gamma \langle G(x), Z(t) \rangle} \right], \quad \gamma \in \mathbb{R}, \]
and (2.8) that the Laplace exponent of $Z^G(x)$ equals
\[ J_{Z^G(x)}(\gamma) = J_Z(\gamma G(x)) = \frac{1}{2} \gamma^2 \langle QG(x), G(x) \rangle + \int_{|y| > 0} \left( e^{-\gamma \langle G(x), y \rangle} - 1 + \gamma \langle G(x), y \rangle \right) \nu(dy). \] (2.10)
Formula (2.10) can be written in a simpler form by using the Lévy measure $\nu_G(x)(dv)$ of $Z^G(x)$, which is the image of the Lévy measure $\nu(dy)$ under the linear transformation $y \mapsto \langle G(x), y \rangle$. This measure will be denoted by $\nu_G(x)(dv)$ and is given by
\[ \nu_G(x)(A) := \nu \{ y \in \mathbb{R}^d : \langle G(x), y \rangle \in A \}, \quad A \in B(\mathbb{R}). \]
Then we obtain from (2.10) that
\[ J_{Z^G(x)}(\gamma) = \frac{1}{2} \gamma^2 \langle QG(x), G(x) \rangle + \int_{|v| > 0} \left( e^{-\gamma v} - 1 + \gamma v \right) \nu_G(x)(dv). \] (2.11)
Thus the characteristic triplet of the projection $Z^G(x)$ has the form
\[ \left( - \int_{|v| > 1} v \nu_G(x)(dv), \langle QG(x), G(x) \rangle, \nu_G(x)(dv) \right) \] (2.12)
Above we used the restriction $\nu_{G(x)}(dv) \mid_{v \neq 0}$ by cutting off zero which may be an atom of $\nu_{G(x)}(dv)$.

In Proposition 2.1 below we provide a preliminary characterization of equations (2.1) generating affine models. The central role here is played by the law of $Z^G$. The result is deduced from Theorem 5.3 in [7], where the generator of a general non-negative Markovian short rate process for affine models was characterized. The result settles a starting point for proving the main results of the paper.

**Proposition 2.1** Let $Z$ be a Lévy martingale with characteristic triplet (2.12), $Z^{G(x)}$ be its projection (2.9) and $\nu_{G(x)}(dv)$ be the Lévy measure of $Z^{G(x)}$.

(A) Then equation (1.1) generates an affine model if and only if the following conditions are satisfied

a) For each $x \geq 0$ the support of $\nu_{G(x)}$ is contained in $[0, +\infty)$ which means that $Z^{G(x)}$ has positive jumps only, i.e. for each $t \geq 0$, with probability one,

$$\triangle Z^{G(x)}(t) := Z^{G(x)}(t) - Z^{G(x)}(t-) = (G(x), \triangle Z(t)) \geq 0.$$  \hspace{1cm} (2.13)

b) The jump part of $Z^{G(0)}$ has finite variation, i.e.

$$\int_{(0, +\infty)} v \nu_{G(0)}(dv) < +\infty.$$  \hspace{1cm} (2.14)

c) The characteristic triplet (2.12) of $Z^{G(x)}$ is linear in $x$, i.e.

$$\frac{1}{2} \langle QG(x), G(x) \rangle = cx, \quad x \geq 0,$$  \hspace{1cm} (2.15)

$$\nu_{G(x)}(dv) \mid_{(0, +\infty)} = \nu_{G(0)}(dv) \mid_{(0, +\infty)} + x\mu(dv), \quad x \geq 0,$$  \hspace{1cm} (2.16)

for some $c \geq 0$ and a measure $\mu(dv)$ on $(0, +\infty)$ satisfying

$$\int_{(0, +\infty)} (v \wedge v^2)\mu(dv) < +\infty.$$  \hspace{1cm} (2.17)

d) The function $F$ is affine, i.e.

$$F(x) = ax + b, \quad \text{where } a \in \mathbb{R}, \quad b \geq \int_{(1, +\infty)} (v - 1)\nu_{G(0)}(dv).$$  \hspace{1cm} (2.18)

(B) Equation (1.1) generates an affine model if and only if the generator of $R$ is given by

$$Af(x) = cf''(x) + \left[ax + b + \int_{(1, +\infty)} (1 - v)\{\nu_{G(0)}(dv) + x\mu(dv)\}\right]f'(x)$$

$$+ \int_{(0, +\infty)} [f(x + v) - f(x) - f'(x)(1 \wedge v)]\{\nu_{G(0)}(dv) + x\mu(dv)\}.$$  \hspace{1cm} (2.19)

for $f \in \mathcal{L}(\Lambda) \cup C^2_\mathcal{B}(\mathbb{R}_+)$, where $\mathcal{L}(\Lambda)$ is the linear hull of $\Lambda := \{f_\lambda := e^{-\lambda x}, \lambda \in (0, +\infty)\}$ and $C^2_\mathcal{B}(\mathbb{R}_+)$ stands for the set of twice continuously differentiable functions with compact support in $[0, +\infty)$. The constants $a, b, c$ and the measures $\nu_{G(0)}(dv), \mu(dv)$ are those from the part (A).
The proof of Proposition 2.1 is postponed to Appendix.

In the sequel we use an equivalent formulation of (2.14)-(2.17) with the use of Laplace exponents. Taking into account (2.11) we obtain the following.

**Remark 2.2** The conditions (2.14) and (2.16) are equivalent to

\[ J_{Z^{G(x)}}(b) = J_Z(bG(x)) = c b^2 x + J_{\nu_G(0)}(b) + x J_{\mu}(b), \quad b, x \geq 0, \]

where

\[ J_{\mu}(b) := \int_0^{+\infty} (e^{-bv} - 1 + bv) \mu(dv), \quad J_{\nu_G(0)}(b) := \int_0^{+\infty} (e^{-bv} - 1 + bv) \nu_G(0)(dv). \]

The part (A) states that (1.1) generates an affine model if \( F \) is affine and the projections \( Z^{G(x)}, x \geq 0 \), have characteristic triplets characterized by a constant \( c \geq 0 \) carrying information on the presence of the Wiener part and two measures \( \nu_G(0)(dv), \mu(dv) \) describing jumps. In view of part (B), the triplet \((c, \nu_G(0)(dv), \mu(dv))\) satisfying (2.14)-(2.17), together with \( F \), determine the generator of the short rate process. A pair \((G, Z)\) for which the projections \( Z^{G(x)} \) satisfy (2.14)-(2.17) will be called a *generating pair*. As \( F \) is of a simple affine form, the essential issue is to characterize the measures \( \nu_G(0)(dv) \) and \( \mu(dv) \). In the one-dimensional case the measures turn out to be such that either

- \( \nu_G(0)(dv) \) — is any measure on \((0, +\infty)\) of finite variation and \( \mu(dv) = 0 \),

or

- \( \nu_G(0)(dv) \equiv 0 \) and \( \mu(dv) = \frac{1}{v^{1+\alpha}} dv, v \geq 0, \alpha \in (1, 2) \), i.e. \( \mu(dv) \) is \( \alpha \)-stable.

Moreover, for each of the two situations above, there exists a unique, up to multiplicative constants, corresponding generating pair \((G, Z)\). For instance, if \( \mu(dv) \) is \( \alpha \)-stable then \( Z \) is also \( \alpha \)-stable and \( G(x) = C x^{1/\alpha} \), with some \( C > 0 \). This means that the triplet \((c, \nu_G(0)(dv), \mu(dv))\) corresponds to a unique equation (2.1), up to the choice of \( F \). The one-dimensional equations mentioned in Introduction can be characterized in terms of \((c, \nu_G(0)(dv), \mu(dv))\) in the following way.

a) \( c > 0, \nu_G(0)(dv) \equiv 0, \mu(dv) \equiv 0 \);

This case corresponds to the classical CIR equation (1.3).

b) \( c = 0, \nu_G(0)(dv) \equiv 0, \mu(dv) - \alpha \)-stable, \( \alpha \in (1, 2) \);

In this case (2.1) becomes the generalized CIR equation with \( \alpha \)-stable noise (1.4).

c) \( c = 0, \nu_G(0)(dv) \) — any measure on \((0, +\infty)\) of finite variation, \( \mu(dv) \equiv 0 \);

Here (2.1) becomes the generalized Vasićek equation (1.5).

In the case \( d > 1 \) one should not expect a one to one correspondence between the triplets \((c, \nu_G(0)(dv), \mu(dv))\) and the generating equations (2.1). The reason is that the distribution of the product \( \langle G(x), Z(t) \rangle \) does not determine the pair \((G, Z)\) in a unique way. If \((G, Z)\) and \((G', Z')\) are generating pairs satisfying (2.14)-(2.17) with an identical triplet \((c, \nu_G(0)(dv), \mu(dv))\), then it follows from part (B) that the corresponding equations (2.1) have solutions with the same generator. Our illustrating examples in Section 2.3 show a couple of different equations all providing the same short rate \( R \). Furthermore, it turns out that even for a fixed process \( Z \), the function \( G \) in the generating pair \((G, Z)\) does not need to be unique. For this reason we focus in this paper on the characterization of possible laws of projections \( Z^G \). As in the classes of Lévy processes under our consideration \( G(0) = 0 \), hence \( \nu_G(0)(dv) \) vanishes, the goal is to determine the measure \( \mu(dv) \) in the multidimensional case. Next, having such laws we provide corresponding generating equations which are of tractable form.
2.3 Examples

We present a couple of examples of generating pairs \((G, Z)\) such that the related projections \(Z^{G(x)}\) satisfy conditions \((2.14)-(2.17)\) with

\[ c = 0, \quad \nu_{G(0)} = 0, \quad \mu(dv) = 1_{\{v > 0\}} \frac{1}{v^{\alpha+1}} dv, \quad \alpha \in (1, 2). \]

Our goal is to illustrate the following features of generating pairs \((G, Z)\) which do not appear in the case \(d = 1\):

\(a\) for a given process \(Z\) the function \(G\) does not need to be unique, see Example 2.3,

\(b\) the coordinates of \(Z\) may be of infinite variation, nevertheless, \(G(0) \neq 0\), see Example 2.4.

In the case \(d = 1\) an important step in the proof of the form of (1.4) was to show that \(G(0) = 0\), see Step 4 in the Proof of Theorem 2.1 in [1], see also Proposition 3.2 in [3].

\(c\) The coordinates of \(Z\) do not need to be \(\alpha\)-stable, see Example 2.5.

We start with an \(\alpha\)-stable martingale in \(\mathbb{R}^d, d > 1\), with \(\alpha \in (1, 2)\) such that the spherical part \(\lambda\) of its Lévy measure is concentrated on \(S_{d-1}^\times := \{x \in \mathbb{R}^d : |x| = 1, x \geq 0\}\) (writing \(x \geq 0\) for \(x \in \mathbb{R}^d\) we mean that all coordinates of \(x\) are non-negative). The Laplace exponent of the jump part of \(Z\), identical with the Laplace exponent of \(Z\), admits the following representation:

\[
J_\nu(z) = \int_{S_{d-1}^\times} \lambda(d\xi) \int_0^{+\infty} \left(e^{-\langle z, r\xi \rangle} - 1 + \langle z, r\xi \rangle \right) \frac{1}{r^{1+\alpha}} dr \\
= \int_{S_{d-1}^\times} \lambda(d\xi) \int_0^{+\infty} \left(e^{-\langle z, \xi \rangle} - 1 + r\langle z, \xi \rangle \right) \frac{1}{r^{1+\alpha}} dr \\
= C_\alpha \int_{S_{d-1}^\times} \langle z, \xi \rangle^\alpha \lambda(d\xi), \tag{2.22}
\]

where \(C_\alpha := \Gamma(2 - \alpha)/(\alpha(\alpha - 1))\) and \(\Gamma\) stands for the Gamma function. Above we used the well known formula

\[
\int_0^{+\infty} \left(e^{-uy} - 1 + uy \right) \frac{1}{y^{1+\alpha}} dy = C_\alpha u^\alpha.
\]

The following example shows that the function \(G\) in a generating pair \((G, Z)\) is not unique.

Example 2.3 Let \(Z\) be an \(\alpha\)-stable martingale in \(\mathbb{R}^d\) with the Laplace exponent (2.22) and \(G : [0, +\infty) \to [0, +\infty]^d\), \(G(0) = 0\). Then a pair \((Z, G)\) generates an affine model if and only if the function \(G\) satisfies

\[
\int_{S_{d-1}^\times} (G(x), \xi)^\alpha \lambda(d\xi) = \frac{C}{C_\alpha} x, \quad x \geq 0, \tag{2.23}
\]

with \(C \geq 0\). We need to show that (2.23) is equivalent to (2.20) with some measure \(\mu(dv)\). Since \(Z\) has no Wiener part and \(\nu_{G(0)}(dv) \equiv 0\), we see that (2.20) takes the form

\[
J_Z(bG(x)) = J_\nu(bG(x)) = xJ_\mu(b), \quad x, b \geq 0.
\]
By (2.22)
\[ J_\nu(bG(x)) = C_\alpha b^\alpha \int_{S^{d-1}} (G(x), \xi) \lambda(d\xi) = C_\alpha b^\alpha \int_{S^{d-1}} (G(x), \xi) \lambda(d\xi). \]

Consequently,
\[ C_\alpha b^\alpha \int_{S^{d-1}} (G(x), \xi) \lambda(d\xi) = xJ_\mu(b), \]
holds if and only if
\[ J_\mu(b) = Cb^\alpha, \quad \int_{S^{d-1}} (G(x), \xi) \lambda(d\xi) = \frac{C}{C_\alpha} x, \]
for some \( C \geq 0 \). Hence, \( \mu \) is an \( \alpha \)-stable measure and \( G \) can be any function satisfying (2.23).

**Example 2.4** Let \( Z^\alpha(t) \) be a real valued \( \alpha \)-stable process with positive jumps only, \( \alpha \in (1, 2) \) and \( G^\alpha(x) := x^{1/\alpha} \). For
\[ Z(t) := (Z^\alpha(t), -Z^\alpha(t)), \quad t \geq 0, \]
\[ G(x) := (G^\alpha(x) + 1, -G^\alpha(x) + 1), \quad x \geq 0, \]
and \( F(x) \equiv 0 \) equation (2.1) becomes
\[
dR(t) = F(R(t))dt + \langle G(R(t)), dZ(t) \rangle \\
= \left( G^\alpha(R(t)) + 1, -G^\alpha(R(t)) + 1, 1, -1 \right) dZ^\alpha(t) \\
= 2R(t)^{1/\alpha} dZ^\alpha(t). \quad (2.24)
\]

By (1.4) we see that \((G, Z)\) is a generating pair. Although the coordinates of \( Z \) are of infinite variation, \( G(0) = (1, 1) \).

To see that \( \nu_{G(0)}(dv) \equiv 0 \) note that the Lévy measure of \( Z \) is supported by the half-line \( \{t(1, -1), t > 0\} \) and therefore
\[ \langle G(0), y \rangle = \langle (1, 1), (y_1, -y_1) \rangle = 0, \quad y \in \text{supp} \nu. \]

It follows that
\[ \nu_{G(0)}(A) = \nu \{ y \in \mathbb{R}^2 : \langle G(0), y \rangle \in A \} \\
= \nu \{ y \in \mathbb{R}^2 : y_1 + y_2 \in A \} = 0, \]
provided that \( 0 \notin A \).

Finally, we show that \( Z \) does not need to have stable components.

**Example 2.5** Let \( E \) be any Borel subset of \([0, +\infty)\) such that
\[ |E| = \int_E dr > 0, \quad \text{and} \quad |[0, +\infty) \setminus E| = \int_{[0, +\infty) \setminus E} dr > 0, \]
and \( Z_1 \) and \( Z_2 \) be two independent Lévy processes with the Lévy measures
\[ \nu_1(dr) = 1_E(r) \frac{dr}{r^{\alpha+1}}, \quad \nu_2(dr) = 1_{[0, +\infty) \setminus E}(r) \frac{dr}{r^{\alpha+1}}, \quad \alpha \in (1, 2). \]
Clearly, neither \( Z_1 \) nor \( Z_2 \) is stable, but \( Z_1 + Z_2 \) is, and has only positive jumps. Thus taking \( G(x) = (G_1(x), G_2(x)) = x^{1/\alpha}(1, 1) \), \( Z = (Z_1, Z_2) \) we get that the equation
\[
dR(t) = \langle G(R(t^-)), dZ(t) \rangle = R(t^-)^{1/\alpha}d(Z_1(t) + Z_2(t))
\]
generates an affine model.

Another example of a generating pair with \( Z \) of independent and non-stable coordinates is presented in Section 3.3.

2.4 The form of affine models

The following result has a supplementary character and shows how the functions \( A(\cdot), B(\cdot) \) of the affine model \( 1.2 \) are determined by a triplet \( (c, \nu_{G(0)}(dv), \mu(dv)) \) satisfying (2.14)-(2.17).

**Proposition 2.6** Let the equation (1.1) generate an affine model (1.2) with twice continuously differentiable functions \( A(\cdot), B(\cdot) \). Let the drift \( F(\cdot) \) and the projections \( Z^{G(\cdot)} \) satisfy (2.15), (2.16) and (2.18) with some constants \( a, b, c \) and measures \( \nu_{G(0)}(dv), \mu(dv) \). Then the functions \( A, B \) are solutions of the following differential equations
\[
B'(v) = aB(v) - \frac{1}{2}cB^2(v) - J_\mu(B(v)) + 1, \quad v \geq 0, \quad B(0) = 0,
\]
\[
A'(v) = bB(v) - J_{\nu_{G(0)}}(B(v)), \quad v \geq 0, \quad A(0) = 0.
\]

The proof of Proposition 2.6 is postponed to Appendix.

3 Noise with independent coordinates

This section deals with equation (2.1) in the case when the coordinates \( (Z_1, Z_2, \ldots, Z_d) \) of the martingale \( Z \) are independent processes. In view of Proposition 2.1 we are interested in characterizing possible distributions of projections \( Z^{G(z)} \) over all generating pairs \( (G, Z) \). By (2.13) the jumps of the projections are necessarily positive. As the coordinates of \( Z \) are independent, they do not jump together. Consequently, we see that, for each \( x \geq 0 \),
\[
\triangle Z^{G(x)}(t) = \langle G(x), \triangle Z(t) \rangle > 0
\]
holds if and only if, for some \( i = 1, 2, \ldots, d \),
\[
G_i(x) \triangle Z_i(t) > 0, \quad \triangle Z_j(t) = 0, \quad j \neq i.
\]

Condition 3.1 means that \( G_i(x) \) and \( \triangle Z_i(t) \) are of the same sign. We can consider only the case when both are positive, i.e.
\[
G_i(x) \geq 0, \quad i = 1, 2, \ldots, d, \quad x \geq 0, \quad \triangle Z_i(t) \geq 0, \quad t > 0,
\]
because the opposite case can be turned into this one by replacing \( (G_i, Z_i) \) with \( (-G_i, -Z_i) \), \( i = 1, \ldots, d \). The Lévy measure \( \nu_i(dv) \) of \( Z_i \) is thus concentrated on \((0, +\infty)\) and, in view of (2.8), the Laplace exponent of \( Z_i \) takes the form
\[
J_i(b) := \frac{1}{2}q_i b^2 + \int_0^{+\infty} (e^{-bv} - 1 + bv)\nu_i(dv), \quad b \geq 0, \quad i = 1, 2, \ldots, d.
\]
with \( q_{ii} \geq 0 \). Recall, \( q_{ii} \) stands on the diagonal of \( Q \) - the covariance matrix of the Wiener part of \( Z \). We will assume that \( J_i, i = 1, 2, ..., d \) are regularly varying at zero. Recall, that means that

\[
\lim_{x \to 0^+} \frac{J_i(bx)}{J_i(x)} = \psi_i(b), \quad b > 0, \quad i = 1, 2, ..., d,
\]

for some function \( \psi_i \). In fact \( \psi_i \) is a power function, i.e.

\[
\psi_i(b) = b^{\alpha_i}, \quad b > 0,
\]

with some \(-\infty < \alpha_i < +\infty\) and \( J_i \) is called to vary regularly with index \( \alpha_i \). A characterization of slowly varying Laplace exponent in terms of the corresponding Lévy measure is presented in Section 3.2.

### 3.1 Main results

The main result of this section is the following.

**Theorem 3.1** Let \( Z_1, ..., Z_d \) be independent components of the Lévy martingale \( Z \) in \( \mathbb{R}^d \). Assume that \( Z_1, ..., Z_d \) satisfy

\[
\triangle Z_i(t) \geq 0, \quad t > 0, \quad Z_i \text{ is of infinite variation}
\]

or

\[
\triangle Z_i(t) \geq 0, \quad t > 0, \quad \text{and } G(0) = 0.
\]

Further, let us assume that for all \( i = 1, ..., d \) the Laplace exponent \( \psi_i \) of \( Z_i \) is regularly varying at 0 and components of the function \( G \) satisfy

\[
G_i(x) \geq 0, \quad x \in [0, +\infty), \quad G_i \text{ is continuous on } [0, +\infty).
\]

Then \( (2.1) \) generates an affine model if and only if \( F(x) = ax + b, a \in \mathbb{R}, b \geq 0, \) and the Laplace exponent \( J_{ZG} \) of \( ZG(x) = (G(x), Z) \) is of the form

\[
J_{ZG}(b) = x \sum_{k=1}^{g} \eta_k b^{\alpha_k}, \quad \eta_k > 0, \quad \alpha_k \in (1, 2], \quad k = 1, 2, \ldots, g,
\]

with some \( 1 \leq g \leq d \) and \( \alpha_k \neq \alpha_j \) for \( k \neq j \).

Theorem 3.1 allows determining the form of the measure \( \mu(dv) \) in Proposition 2.1.

**Corollary 3.2** Let the assumptions of Theorem 3.1 be satisfied. If equation \( (2.1) \) generates an affine model then the function \( J_\mu \) defined in \( (2.21) \) takes the form

\[
J_\mu(b) = \sum_{k=1}^{g} \eta_k b^{\alpha_k}, \quad l \in \{1, 2\}, \quad \eta_k > 0, \quad \alpha_k \in (1, 2], \quad k = l, l + 1, \ldots, g,
\]

with \( 1 \leq g \leq d \), \( \alpha_k \neq \alpha_j, k \neq j \) (for the case \( l = 2, g = 1 \) we set \( J_\mu \equiv 0 \), which means that \( \mu(dv) \) disappears).

Theorem 3.1 specifies distributions of the projections \( ZG(x) \) of a generating pair \( (Z, G) \). As shown in Example 2.5, a given projection may correspond to many generating pairs \( (G, Z) \). This issue is also illustrated in Section 3.3 below, where all generating equations in the case \( d = 2 \) are described. Below we show a tractable generating equation with the law of \( ZG(x) \) required by Theorem 3.1.
Corollary 3.3 Let $R$ be the solution of (2.1) with $F, G, Z$ satisfying the assumptions of Theorem 3.1. Let $	ilde{Z} := (\tilde{Z}_1, \tilde{Z}_2, ..., \tilde{Z}_g)$ be a Lévy martingale with independent stable coordinates with indices $\alpha_k, k = 1, 2, ..., g$, respectively, and $\tilde{G}(x) = (d_1x^{1/\alpha_1}, ..., d_gx^{1/\alpha_g})$. Then

$$J_{ZG}(b) = J_{\tilde{Z}\tilde{G}}(b), \quad b, x \geq 0.$$  

Consequently, if $\tilde{R}$ is the solution of the equation

$$d\tilde{R}(t) = (a\tilde{R}(t) + b)dt + \sum_{k=1}^{g} d_k^{1/\alpha_k} \tilde{R}(t-)^{1/\alpha_k} d\tilde{Z}_k(t),  \quad (3.7)$$

where $d_k := (\eta_k/c_k)^{1/\alpha_k}, c_k = \frac{\Gamma(2-\alpha_k)}{\alpha_k(\alpha_k-1)}, k = 1, 2, ..., g$, then the generators of $R$ and $\tilde{R}$ are equal.

Proof: By (3.5) we need to show that

$$J_{\tilde{Z}\tilde{G}}(x)(b) = x \sum_{k=1}^{g} \eta_k b^{\alpha_k}, \quad b, x \geq 0.$$  

Recall, the Laplace exponent of $\tilde{Z}_k$ equals $J_k(b) = c_k b^{\alpha_k}, k = 1, 2, ..., g$. By independence and the form of $\tilde{G}$ we have

$$J_{\tilde{Z}\tilde{G}}(b) = \sum_{k=1}^{g} J_k(bG_k(x)) = \sum_{k=1}^{g} c_k b^{\alpha_k} \frac{\eta_k}{c_k} x = x \sum_{k=1}^{g} \eta_k b^{\alpha_k}, \quad b, x \geq 0,$$

as required. The second part of the thesis follows from Proposition 2.1(B). \qed

3.1.1 Proofs

The proofs of Theorem 3.1 and Corollary 3.2 are preceded by two auxiliary results, i.e. Proposition 3.4 and Proposition 3.5. The first one provides some useful estimation for the function

$$J_\rho(b) := \int_0^{+\infty} (e^{-bv} - 1 + bv) \rho(dv), \quad b \geq 0,  \quad (3.8)$$

where the measure $\rho(dv)$ on $(0, +\infty)$ satisfies

$$0 < \int_0^{+\infty} (v^2 \wedge v) \rho(dv) < +\infty.  \quad (3.9)$$

The second result shows that if all components of $Z$ are of infinite variation then $G(0) = 0$.

Proposition 3.4 Let $J_\rho$ be a function given by (3.8), where the measure $\rho$ satisfies (3.9). Then the function $(0, +\infty) \ni b \mapsto J_\rho(b)/b$ is strictly increasing and $\lim_{b \to 0+} J_\rho(b)/b = 0$, while the function $(0, +\infty) \ni b \mapsto J_\rho(b)/b^2$ is strictly decreasing and $\lim_{b \to +\infty} J_\rho(b)/b^2 = 0$. This yields, in particular, that, for any $b_0 > 0$,

$$\frac{J_\rho(b_0)}{b_0^2} < J_\rho(b) < \frac{J_\rho(b_0)}{b_0} b, \quad b \in (0, b_0).  \quad (3.10)$$

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Proof: Let us start from the observation that the function
\[
\frac{(1 - e^{-t})t}{e^{-t} - 1 + t}, \quad t \geq 0,
\]
is strictly decreasing, with limit 2 at zero and 1 at infinity. This implies
\[
(e^{-t} - 1 + t) < (1 - e^{-t})t < 2(e^{-t} - 1 + t), \quad t \in (0, +\infty),
\]
and, consequently,
\[
\int_0^{+\infty} (e^{-bv} - 1 + bv)\rho(dv) < \int_0^{+\infty} (1 - e^{-bv})bv \rho(dv) < 2 \int_0^{+\infty} (e^{-bv} - 1 + bv)\rho(dv), \quad b > 0.
\]
This means, however, that
\[
J_{\rho}(b) < bJ_{\rho}'(b) < 2J_{\rho}(b), \quad b > 0.
\]
So, we have
\[
\frac{1}{b} < \frac{J_{\rho}'(b)}{J_{\rho}(b)} = \frac{d}{db} \ln J_{\rho}(b) < \frac{2}{b}, \quad b > 0,
\]
and integration over some interval \([b_1, b_2]\), where \(b_2 > b_1 > 0\), yields
\[
\ln b_2 - \ln b_1 < \ln J_{\rho}(b_2) - \ln J_{\rho}(b_1) < 2 \ln b_2 - 2 \ln b_1
\]
which gives that
\[
\frac{J_{\rho}(b_2)}{b_2} > \frac{J_{\rho}(b_1)}{b_1}, \quad \frac{J_{\rho}(b_2)}{b_2^2} < \frac{J_{\rho}(b_1)}{b_1^2}.
\]
To see that \(\lim_{b \to 0^+} J_{\rho}(b)/b = 0\) it is sufficient to use de l’Hôpital’s rule, (3.9) and dominated convergence
\[
\lim_{b \to 0^+} \frac{J_{\rho}(b)}{b} = \lim_{b \to 0^+} \frac{J_{\rho}'(b)}{b} = \lim_{b \to 0^+} \int_0^{+\infty} (1 - e^{-bv})v \rho(dv) = 0.
\]
To see that \(\lim_{b \to +\infty} J_{\rho}(b)/b^2 = 0\) we also use de l’Hôpital’s rule, (3.9) and dominated convergence. If \(\int_0^{+\infty} v \rho(dv) < +\infty\), then we have
\[
\lim_{b \to +\infty} \frac{J_{\rho}(b)}{b^2} = \lim_{b \to +\infty} \frac{J_{\rho}'(b)}{2b} = \lim_{b \to +\infty} \frac{J_{\rho}''(b)}{2} = \frac{1}{2} \lim_{b \to +\infty} \int_0^{+\infty} e^{-bv}v^2 \rho(dv) = 0.
\]
If \(\int_0^{+\infty} v \rho(dv) = +\infty\) then we apply de l’Hôpital’s rule twice and obtain
\[
\lim_{b \to +\infty} \frac{J_{\rho}(b)}{b^2} = \lim_{b \to +\infty} \frac{J_{\rho}'(b)}{2b} = \lim_{b \to +\infty} \frac{J_{\rho}''(b)}{2} = \frac{1}{2} \lim_{b \to +\infty} \int_0^{+\infty} e^{-bv}v^2 \rho(dv) = 0.
\]
\[\square\]

Proposition 3.5 If \((G, Z)\) is a generating pair and all components of \(Z\) are of infinite variation then \(G(0) = 0\).

Proof: Let \((G, Z)\) be a generating pair. Since the components of \(Z\) are independent, its characteristic triplet (2.5) is such that \(Q = \{q_{i,j}\}\) is a diagonal matrix, i.e.
\[
q_{ii} \geq 0, \quad q_{i,j} = 0, \quad i \neq j, \quad i, j = 1, 2, ..., d,
\]

and the support of $\nu(dy)$ is contained in the positive half-axes of $\mathbb{R}^d$, see [10] p.67. On the $i^{th}$ positive half-axis

$$\nu(dy) = \nu_i(dy_i), \quad y = (y_1, y_2, \ldots, y_d),$$

for $i = 1, 2, \ldots, d$. The $i^{th}$ coordinate of $Z$ is of infinite variation if and only if its Laplace exponent (3.2) is such that $q_{ii} > 0$ or

$$\int_0^1 y_i \nu_i(dy_i) = +\infty,$$

see [8] Lemma 2.12. It follows from (2.15) that

$$G(0) = 0,$$

so if $q_{ii} > 0$ then $G_i(0) = 0$. If it is not the case, using (3.12) and (2.14) we see that the integral

$$\int_{(0, +\infty)} v \nu G(0)(dv) = \int_{\mathbb{R}^d_+} (G(0), y) \nu(dy)$$

is finite, so if (3.13) holds then $G_i(0) = 0$.

**Proof of Theorem 3.1** By assumption [8] and Proposition 3.5 or by assumption (3.4) we have $G(0) = 0$, so it follows from Remark 2.2 that

$$J_{Z^G(x)}(b) = J_1(bG_1(x)) + J_2(bG_2(x)) \ldots + J_d(bG_d(x)) = x\tilde{J}_\mu(b), \quad b, x \geq 0,$$

where $\tilde{J}_\mu(b) = cb^2 + J_\mu(b)$, $c > 0$ and $J_\mu(b)$ is given by (2.21). This yields

$$\frac{J_1(bG_1(x))}{J_1(G_1(x))} \cdot \frac{J_1(G_1(x))}{x} + \ldots + \frac{J_d(bG_d(x))}{J_d(G_d(x))} \cdot \frac{J_d(G_d(x))}{x} = \frac{J_\mu(b)}{x},$$

where in the case $G_i(x) = 0$ we set $\frac{J_i(bG_i(x))}{J_i(G_i(x))} = 0$. Without loss of generality we may assume that $J_1$, $J_2, \ldots, J_d$ are non-zero (thus positive for positive arguments). By assumption, $J_i, i = 1, 2, \ldots, d$ vary regularly at 0 with some indices $\alpha_i$, $i = 1, 2, \ldots, d, b > 0$

$$\lim_{y \to 0^+} \frac{J_i(b \cdot y)}{J_i(y)} = b^\alpha_i.$$ (3.16)

Assume that

$$\alpha_1 = \ldots = \alpha_{i(1)} > \alpha_{i(1)+1} = \ldots = \alpha_{i(2)} > \ldots > \alpha_{i(g-1)+1} = \ldots = \alpha_{i(g)} = \alpha_d,$$

where $i(g) = d$. Let us denote $i_0 = 0$ and

$$\eta_k(x) := \frac{J_{i(k-1)+1}(G_{i(k-1)+1}(x)) + \ldots + J_{i(k)}(G_{i(k)}(x))}{x}, \quad k = 1, 2, \ldots, g.$$ (3.17)
We can rewrite equation (3.15) in the form
\[
\sum_{k=1}^{g} \left( \sum_{i=(k-1)+1}^{i(k)} \frac{J_i(b \cdot G_i(x))}{J_i(G_i(x))} \cdot \frac{J_i(G_i(x))}{x} \right) = \tilde{J}_\mu(b),
\] (3.18)

By passing to the limit as \(x \to 0^+\), from (3.16) and (3.18) we get
\[
b^{\alpha_i(1)} \left( \lim_{x \to 0^+} \eta_1(x) \right) + \ldots + b^{\alpha_i(g)} \left( \lim_{x \to 0^+} \eta_g(x) \right) = \tilde{J}_\mu(b),
\] (3.19)

thus
\[
\tilde{J}_\mu(b) = \sum_{k=1}^{g} \eta_k b^{\alpha_i(k)},
\] (3.20)

providing that the limits \(\eta_k := \lim_{x \to 0^+} \eta_k(x), \ k = 1, 2, \ldots, g\), exist. Thus it remains to prove that for \(k = 1, 2, \ldots, g\) the limits \(\lim_{x \to 0^+} \eta_k(x)\) indeed exist and that \(\alpha_i(k) \in (1, 2]\).

First we will prove that \(\lim_{x \to 0^+} \eta_g(x)\) exists. Assume, by contrary, that this is not true, so
\[
\limsup_{x \to 0^+} \eta_g(x) - \liminf_{x \to 0^+} \eta_g(x) \geq \delta > 0.
\] (3.21)

It follows from (3.14) that
\[
\frac{J_1(G_1(x)) + J_2(G_2(x)) + \ldots + J_d(G_d(x))}{x} = \sum_{k=1}^{g} \eta_k(x) = \tilde{J}_\mu(1).
\]

Let now \(b_0 \in (0, 1)\) be small enough so that
\[
\tilde{J}_\mu(1)b_0^{\alpha_i(g-1)-\alpha_i(g)} < \frac{\delta}{6},
\] (3.22)

Let us set in (3.18) \(b = b_0\) and then divide both sides of (3.18) by \(b_0^{\alpha_i(g)}\). For \(x > 0\) sufficiently close to 0 we have
\[
\eta_g(x) - \frac{\delta}{6} \leq \frac{1}{b_0^{\alpha_i(g)}} \left( \sum_{i=(g-1)+1}^{i(g)} \frac{J_i(b_0 \cdot G_i(x))}{J_i(G_i(x))} \cdot \frac{J_i(G_i(x))}{x} \right) \leq \eta_g(x) + \frac{\delta}{6}
\]

and
\[
\frac{1}{b_0^{\alpha_i(g)}} \sum_{k=1}^{g-1} \left( \sum_{i=(k-1)+1}^{i(k)} \frac{J_i(b_0 \cdot G_i(x))}{J_i(G_i(x))} \cdot \frac{J_i(G_i(x))}{x} \right) \leq \sum_{k=1}^{g-1} 2b_0^{\alpha_i(k)-\alpha_i(g)} \eta_k(x)
\]
\[
\leq 2b_0^{\alpha_i(g-1)-\alpha_i(g)} \tilde{J}_\mu(1)
\]

thus from (3.18), two last estimates and (3.22)
\[
\eta_g(x) - \frac{\delta}{6} \leq \frac{\tilde{J}_\mu(b_0)}{b_0^{\alpha_i(g)}} \leq \eta_g(x) + \frac{\delta}{6} + 2\tilde{J}_\mu(1)b_0^{\alpha_i(g-1)-\alpha_i(g)} < \eta_g(x) + \frac{\delta}{2}.
\]

But this contradicts (3.21) since we must have
\[
\limsup_{x \to 0^+} \eta_g(x) \leq \frac{\tilde{J}_\mu(b_0)}{b_0^{\alpha_i(g)}} + \frac{\delta}{6}, \quad \liminf_{x \to 0^+} \eta_g(x) \geq \frac{\tilde{J}_\mu(b_0)}{b_0^{\alpha_i(g)}} - \frac{\delta}{2}.
\]
Having proved the existence of the limits \( \lim_{x \to 0^+} \eta_g(x), \ldots, \lim_{x \to 0^+} \eta_{g-m+1}(x) \) we can proceed similarly to prove the existence of the limit \( \lim_{x \to 0^+} \eta_{g-m}(x) \). Assume that \( \lim_{x \to 0^+} \eta_{g-m}(x) \) does not exist, so
\[
\limsup_{x \to 0^+} \eta_{g-m}(x) - \liminf_{x \to 0^+} \eta_{g-m}(x) \geq \delta > 0. \tag{3.23}
\]
Let \( b_0 \in (0, 1) \) be small enough so that
\[
\tilde{J}_\mu(1) b_0^{\alpha_i(g-m-1)-\alpha_i(g-m)} < \frac{\delta}{8}. \tag{3.24}
\]
Let us set in (3.18) \( b = b_0 \) and then divide both sides of (3.18) by \( b_0^{\alpha_i(g-m)} \). For \( x > 0 \) sufficiently close to 0 we have
\[
\frac{1}{b_0^{\alpha_i(g-m)}} \sum_{k=1}^{g-m-1} \left( \sum_{i=i(k-1)+1}^{i(k)} J_i(b_0 \cdot G_i(x)) \cdot \frac{J_i(G_i(x))}{x} \right) \leq \frac{g-m-1}{b_0^{\alpha_i(g-m)}} \sum_{k=1}^{g-m} 2 \mu_{i(k)} \eta_k(x)
\]
and
\[
\sum_{k=g-m+1}^{g} \frac{b_0^{\alpha_i(k)} \eta_k}{b_0^{\alpha_i(g-m)}} - \frac{\delta}{8} \leq \frac{1}{b_0^{\alpha_i(g-m)}} \sum_{k=g-m+1}^{g} \sum_{i=i(k-1)+1}^{i(k)} J_i(b_0 \cdot G_i(x)) \cdot \frac{J_i(G_i(x))}{x}
\]
thus from (3.18), last three estimates and (3.21)
\[
\eta_{g-m}(x) - \frac{\delta}{4} \leq \frac{J_\mu(b_0)}{b_0^{\alpha_i(g-m)}} - \sum_{k=g-m+1}^{g} \frac{b_0^{\alpha_i(k)} \eta_k}{b_0^{\alpha_i(g-m)}} \leq \eta_{g-m}(x) + \frac{\delta}{4} + 2 \tilde{J}_\mu(1) b_0^{\alpha_i(g-1)-\alpha_i(g)} < \eta_{g-m}(x) + \frac{\delta}{2}.
\]
But this contradicts (3.23).

Now we are left with the proof that for \( k = 1, 2, \ldots, g, \alpha_i(k) \in (1, 2] \). Since the Laplace exponent of \( Z_i \) is given by (3.2), by Proposition 3.4 we necessarily have that \( J_i \) varies regularly with index \( \alpha_i \in [1, 2], i = 1, 2, \ldots, d \). Thus it remains to prove that \( \alpha_i > 1, i = 1, 2, \ldots, d \). If it was not true we would have \( \alpha_i(g) = 1 \) in (3.20) and \( \eta_g > 0 \). Then
\[
\lim_{b \to 0^+} \tilde{J}_\mu(b)/b = \lim_{b \to 0^+} J_\mu(b)/b = \eta_g > 0,
\]
but, again, by Proposition 3.4 it is not possible. \( \square \)

**Proof of Corollary 3.2**: From Remark 2.2 and Theorem 3.1 we know that
\[
J_{Z_G(x)}(b) = xcb^2 + xJ_\mu(b) = x \sum_{k=1}^{g} \eta_k b^{\alpha_k},
\]

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where \(1 \leq g \leq d\), \(\eta_k > 0\), \(\alpha_k \in (1, 2]\), \(\alpha_k \neq \alpha_j\), \(k, j = 1, 2, \ldots, g\), \(c \geq 0\). Without loss of generality we may assume that \(2 \geq \alpha_1 > \alpha_2 > \ldots > \alpha_g > 1\). Thus, since the Laplace exponent is nonnegative, \(xJ_\mu(b)\) is of the form

\[
xJ_\mu(b) = x \sum_{k=1}^{g} \eta_k b^{\alpha_k}, \quad \text{if } c = 0, \tag{3.25}
\]
or

\[
xJ_\mu(b) = x \left[ (\eta_1 - c)b^2 + \sum_{k=2}^{g} \eta_k b^{\alpha_k} \right], \quad \text{if } 0 < c \leq \eta_1 \text{ and } \alpha_1 = 2. \tag{3.26}
\]

In the case (3.25) we need to show that \(\alpha_1 < 2\). If it was not true, we would have

\[
\lim_{b \to +\infty} \frac{J_\mu(b)}{b^{\alpha_1}} = \eta_1 > 0,
\]

but this contradicts Proposition 3.4. In the same way we prove that \(\eta_1 = c\) in (3.26). This proves the required representation (3.6). \(\square\)

### 3.2 Characterization of regularly varying Laplace exponents

In this section we reformulate the assumption that \(J_i, i = 1, \ldots, d\), vary regularly at zero in terms of the behaviour of the Lévy measures of \(Z_i, i = 1, \ldots, d\). As our considerations are componentwise, we write for simplicity \(\nu(dv) := \nu_i(dv)\) for the Lévy measure of \(Z_i\) and \(J := J_i\) for its Laplace exponent.

**Proposition 3.6** Let \(\nu(dv)\) be such that

\[
\int_{0}^{+\infty} (y^2 \land y) \nu(dy) < +\infty. \tag{3.27}
\]

Let \(\tilde{\nu}(dv)\) be the measure

\[
\tilde{\nu}(dv) := v^2 \nu(dv),
\]

and \(\tilde{F}\) its cumulative distribution function, i.e.

\[
\tilde{F}(v) := \tilde{\nu}((0, v)) = \int_{0}^{v} u^2 \nu(du), \quad v \geq 0.
\]

Then, for \(\alpha \in (1, 2]\), the following conditions are equivalent

\[
\lim_{x \to 0^+} \frac{J(bx)}{J(x)} = b^\alpha, \quad b \geq 0, \tag{3.28}
\]

\[
\lim_{y \to +\infty} \frac{\tilde{F}(by)}{\tilde{F}(y)} = b^{2-\alpha}, \quad b \geq 0.
\]

If, additionally, \(\nu(dv)\) has a density function \(g(v)\) such that

\[
\int_{0}^{+\infty} v^2 g(v) \nu(dv) = +\infty, \tag{3.29}
\]

then (3.28) is equivalent to the condition

\[
\lim_{y \to +\infty} \frac{g(by)}{g(y)} = b^{-\alpha-1}, \quad b > 0.
\]
Proof: Under (3.27) the function \( J \) given by (3.8) is well defined for \( b \geq 0 \), twice differentiable and

\[
J'(b) = \int_0^{+\infty} v(1 - e^{-bv})\nu(dv), \quad J''(b) = \int_0^{+\infty} v^2 e^{-bv}\nu(dv), \quad b \geq 0,
\]

see [9], Lemma 8.1 and Lemma 8.2. This implies that

\[
\lim_{x \to 0^+} \frac{J(bx)}{J(x)} = b \cdot \lim_{x \to 0^+} \frac{J'(bx)}{J'(x)} = b^2 \cdot \lim_{x \to 0^+} \frac{J''(bx)}{J''(x)}
\]

Consequently, by (3.28)

\[
\lim_{x \to 0^+} \frac{\int_0^{+\infty} e^{-bxv}v^2\nu(dv)}{\int_0^{+\infty} e^{-xv}v^2\nu(dv)} = b^{a-2}.
\]

Notice, that the left side is a quotient of two transforms of the measure \( \tilde{\nu}(dv) \). By the Tauberian theorem we have that (3.30) holds if and only if

\[
\frac{\tilde{F}(by)}{F(y)} \to b^{2-a}, \quad b \geq 0.
\]

If \( \nu(dv) \) has a density \( g(v) \) satisfying (3.29) then

\[
\lim_{y \to +\infty} \frac{\tilde{F}(by)}{F(y)} = \lim_{y \to +\infty} \frac{\int_0^y u^2 g(u)du}{\int_0^y u^2 g(u)du} = \lim_{y \to +\infty} \frac{b \cdot (by)^2 g(by)}{y^2 g(y)}
\]

It follows that

\[
\lim_{y \to +\infty} \frac{g(by)}{g(y)} = b^{-a-1}.
\]

which proves the result. \( \square \)

Remark 3.7 By general characterization of regularly varying functions we see that the functions \( \tilde{F} \) and \( g \) from Proposition 3.6 must be of the forms

\[
\tilde{F}(b) = b^{2-a} L(b), \quad b \geq 0,
\]

\[
g(b) = b^{-a-1} \tilde{L}(b), \quad b \geq 0,
\]

where \( L \) and \( \tilde{L} \) are slowly varying functions at +\( \infty \), i.e.

\[
\frac{L(by)}{L(y)} \to 1, \quad \frac{\tilde{L}(by)}{\tilde{L}(y)} \to 1.
\]
3.3 Generating equations on a plane

In this section we characterize all equations (2.1), with \( d = 2 \), which generate affine models. In view of Theorem 3.1 generating pairs \((G, Z)\) are such that

\[
J_1(bG_1(x)) + J_2(bG_2(x)) = x\tilde{J}_\mu(b), \quad b, x \geq 0, \tag{3.31}
\]

where \(\tilde{J}_\mu\) takes one of the two following forms

\[
\tilde{J}_\mu(b) = \eta_1 b^{\alpha_1}, \quad b \geq 0, \tag{3.32}
\]

or

\[
\tilde{J}_\mu(b) = \eta_1 b^{\alpha_1} + \eta_2 b^{\alpha_2}, \quad b \geq 0, \tag{3.33}
\]

where \(\eta_1, \eta_2 > 0\), \(2 \geq \alpha_1 > \alpha_2 > 1\). We deduce from (3.31) the form of \(G\) and characterize the noise \(Z\).

**Theorem 3.8** Let \(G(x) = (G_1(x), G_2(x))\) be continuous functions such that \(G_1(x) > 0, G_2(x) > 0, x > 0\) and \(\frac{G_2(x)}{G_1(x)} \in C^1(0, +\infty)\). Let \(Z(t) = (Z_1(t), Z_2(t))\) have independent coordinates of infinite variation with Laplace exponents varying regularly with indices \(\alpha_1, \alpha_2\), respectively, where \(2 \geq \alpha_1 \geq \alpha_2 > 1\).

I) If \(\tilde{J}_\mu\) is of the form (3.32) then \((G, Z)\) is a generating pair if and only if one of the following two cases holds:

a) \[G(x) = c_0 x^{1/\alpha_1} \cdot \left( \begin{array}{c} G_1 \\ G_2 \end{array} \right), \quad x \geq 0, \tag{3.34}\]

where \(c_0 > 0, G_1 > 0, G_2 > 0\) and the process

\[G_1Z_1(t) + G_2Z_2(t), \quad t \geq 0,\]

is \(\alpha_1\)-stable.

b) \(G(x)\) is such that

\[c_1 G_1^{\alpha_1}(x) + c_2 G_2^{\alpha_1}(x) = \eta_1 x, \quad x \geq 0, \tag{3.35}\]

with some constants \(c_1, c_2 > 0\), and \(Z_1, Z_2\) are \(\alpha_1\)-stable processes.

II) If \(\tilde{J}_\mu\) is of the form (3.33), then \((G, Z)\) is a generating pair if and only if

\[G_1(x) = \left( \frac{\eta_1}{c_1} x \right)^{1/\alpha_1}, \quad G_2(x) = \left( \frac{\eta_2}{c_2} x \right)^{1/\alpha_2}, \quad x \geq 0, \tag{3.36}\]

with \(c_1, c_2 > 0\) and \(Z_1\) is \(\alpha_1\)-stable, \(Z_2\) is \(\alpha_2\)-stable.

**Proof:** First let us consider the case when

\[
\left( \frac{G_2(x)}{G_1(x)} \right)' = 0, \quad x > 0. \tag{3.37}
\]
Then $G(x)$ can be written in the form

$$G(x) = g(x) \cdot \left( \frac{G_1}{G_2} \right), \quad x \geq 0,$$

with some function $g(x) \geq 0$, $x \geq 0$, and constants $G_1 > 0$, $G_2 > 0$. Equation (2.1) amounts then to

$$dR(t) = F(R(t)) + g(R(t-)) (G_1dZ_1(t) + G_2dZ_2(t)) = F(R(t)) + g(R(t-))d\tilde{Z}(t), \quad t \geq 0,$$

which is an equation driven by the one dimensional Lévy process $\tilde{Z}(t) := G_1Z_1(t) + G_2Z_2(t)$. It follows that $\tilde{Z}$ is $\alpha_1$-stable with $\alpha_1 \in (1, 2]$ and that $g(x) = c_0 x^{1/\alpha_1}, c_0 > 0$. Notice that $Z^G(x)(t) = c_0 x^{a_1} \tilde{Z}$, so $J_{2G(x)}(b) = C_{\alpha_1} (c_0 x^{a_1} b)^{\alpha_1} = xc_0^{\alpha_1} b^{\alpha_1}$. Hence (3.32) holds and this proves (Ia).

If (3.37) is not satisfied, then

$$(G_2(x)/G_1(x))' \neq 0, \quad x \in (\underline{x}, \bar{x}), \quad (3.38)$$

in some interval $(\underline{x}, \bar{x}) \subset (0, +\infty)$. In the rest of the proof we consider this case and prove (Ib) and (II).

(Ib) From the equation

$$J_1(bG_1(x)) + J_2(bG_2(x)) = x\eta_1 b^{\alpha_1}, \quad b \geq 0, \quad x \geq 0, \quad (3.39)$$

we explicitly determine unknown functions. Inserting $b/G_1(x)$ for $b$ yields

$$J_1(b) + J_2 \left( \frac{bG_2(x)}{G_1(x)} \right) = \eta_1 \frac{x}{G_1^{\alpha_1}(x)} b^{\alpha_1}, \quad b \geq 0, \quad x > 0. \quad (3.40)$$

Differentiation over $x$ yields

$$J_2' \left( \frac{bG_2(x)}{G_1(x)} \right) \cdot b \left( \frac{G_2(x)}{G_1(x)} \right)' = \eta_1 \left( \frac{x}{G_1^{\alpha_1}(x)} \right)' b^{\alpha_1}, \quad b \geq 0, \quad x > 0.$$

Using (3.38) and dividing by $\left( G_2(x)/G_1(x) \right)'$ leads to

$$J_2' \left( \frac{bG_2(x)}{G_1(x)} \right) \cdot b = \eta_1 \frac{x}{G_1^{\alpha_1}(x)} b^{\alpha_1}, \quad b \geq 0, \quad x \in (\underline{x}, \bar{x}).$$

By inserting $bG_1(x)/G_2(x)$ for $b$ one computes the derivative of $J_2$:

$$J_2'(b) = \eta_1 \frac{x}{G_1^{\alpha_1}(x)} \left( \frac{G_1(x)}{G_2(x)} \right)^{\alpha_1-1} \cdot b^{\alpha_1-1}, \quad b > 0, \quad x \in (\underline{x}, \bar{x}).$$

Fixing $x$ and integrating over $b$ provides

$$J_2(b) = c_2 b^{\alpha_1}, \quad b > 0, \quad (3.41)$$

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with some \( c_2 \geq 0 \). Actually \( c_2 > 0 \) as \( Z_2 \) is of infinite variation and \( J_2 \) can not disappear.

By the symmetry of (3.39) the same conclusion holds for \( J_1 \), i.e.

\[
J_1(b) = c_1b^{\alpha_1}, \quad b > 0,
\]

with \( c_1 > 0 \). Using (3.41) and (3.42) in (3.39) gives us (3.35). This proves (Ib).

(II) Solving the equation

\[
J_1(bG_1(x)) + J_2(bG_2(x)) = x(\eta_1b^{\alpha_1} + \eta_2b^{\alpha_2}), \quad b, x \geq 0,
\]

in the same way as we solved (3.39) yields that

\[
J_1(b) = c_1b^{\alpha_1} + c_2b^{\alpha_2}, \quad J_2(b) = d_1b^{\alpha_1} + d_2b^{\alpha_2}, \quad b \geq 0,
\]

with \( c_1, c_2, d_1, d_2 \geq 0, \ c_1 + c_2 > 0, d_1 + d_2 > 0 \). From (3.43) and (3.44) we can specify the following conditions for \( G \):

\[
c_1G_1^{\alpha_1}(x) + d_1G_2^{\alpha_1}(x) = \eta_1x,
\]

\[
c_2G_1^{\alpha_2}(x) + d_2G_2^{\alpha_2}(x) = \eta_2x.
\]

We show that \( c_1 > 0, c_2 = 0, d_1 = 0, d_2 > 0 \) by excluding the opposite cases.

If \( c_1 > 0, c_2 > 0 \), one computes from (3.45)–(3.46) that

\[
G_1(x) = \left( \frac{1}{c_1}(\eta_1x - d_1y^{\alpha_1}) \right)^{\frac{1}{\alpha_1}} = \left( \frac{1}{c_2}(\eta_2x - d_2y^{\alpha_2}) \right)^{\frac{1}{\alpha_2}}, \quad x \geq 0.
\]

This means that, for each \( x \geq 0 \), the value \( G_2(x) \) is a solution of the following equation in the \( y \)-variable

\[
\left( \frac{1}{c_1}(\eta_1x - d_1y^{\alpha_1}) \right)^{\frac{1}{\alpha_1}} = \left( \frac{1}{c_2}(\eta_2x - d_2y^{\alpha_2}) \right)^{\frac{1}{\alpha_2}},
\]

with \( y \in \left[ 0, \left( \frac{\eta_1x}{d_1} \right)^{\frac{1}{\alpha_1}} \right] \wedge \left( \frac{\eta_2x}{d_2} \right)^{\frac{1}{\alpha_2}} \). If \( d_1 = 0 \) or \( d_2 = 0 \) we compute \( y = y(x) \) from (3.48) and see that its positivity is broken close to zero or for large \( x \). We need to exclude the case \( d_1 > 0, d_2 > 0 \). However, in the case \( c_1, c_2, d_1, d_2 > 0 \) equation (3.48) has no solution because, for large \( x > 0 \), the left side of (3.48) is strictly less then the right side. This inequality follows from Proposition 3.9 proven below.

So, we proved that \( c_1 \cdot c_2 = 0 \) and similarly one proves that \( d_1 \cdot d_2 = 0 \). The case \( c_1 = 0, c_2 > 0, d_1 > 0, d_2 = 0 \) can be rejected because then \( J_1 \) would vary regularly with index \( \alpha_2 \) and \( J_2 \) with index \( \alpha_1 \), which is a contradiction. It follows that \( c_1 > 0, c_2 = 0, d_1 = 0, d_2 > 0 \) and in this case we obtain (3.36) from (3.45) and (3.46). \( \square \)

**Proposition 3.9** Let \( a, b, c, d > 0, \gamma \in (0, 1), 2 \geq \alpha_1 > \alpha_2 > 1 \). Then for large \( x > 0 \) the following inequalities are true

\[
\left( ax - (bx - cz)^{\gamma} \right)^{\frac{1}{\gamma}} - dz > 0, \quad z \in \left[ 0, \frac{b}{c} \right], \quad (3.49)
\]

\[
(bx - cy^{\alpha_1})^{\frac{1}{\alpha_1}} < (ax - dy^{\alpha_2})^{\frac{1}{\alpha_2}}, \quad y \in \left[ 0, \left( \frac{b}{c} \right)^{\frac{1}{\alpha_1}} \wedge \left( \frac{b}{d} \right)^{\frac{1}{\alpha_2}} \right]. \quad (3.50)
\]
Thus we obtain the following system of equations

\[ ax \geq (dz)^\gamma + (bx - cz)^\gamma =: h(z). \]  

(3.51)

Since

\[ h'(z) = \gamma \left( d'z^{\gamma-1} - c(bx - cz)^{\gamma-1} \right), \]

\[ h''(z) = \gamma(\gamma - 1)\left( d'z^{\gamma-2} + c^2(bx - cz)^{\gamma-2} \right) < 0, \quad z \in \left[ 0, \frac{b}{c} \right], \]

the function \( h \) is concave and attains its maximum at point

\[ z_0 := \theta x := \frac{bc \theta^{\gamma-1}}{d^{\gamma-1} + c^{\gamma-1}} x \in \left[ 0, \frac{b}{c} \right], \]

which is a root of \( h' \). It follows that

\[ h(z) \leq h(\theta x) = (\theta x)^\gamma + (bx - c\theta x)^\gamma = (\theta^\gamma + (b - c\theta)^\gamma)x^\gamma < ax. \]

The last strict inequality holds for large \( x \) and (3.49) follows. (3.50) follows from (3.49) by setting \( \gamma = \alpha_2/\alpha_1 \), \( z = y^{\alpha_1} \).

\[ \square \]

3.4 An example in higher dimensions

In Theorem 3.8 we showed that in the case \( d = 2 \) there exists a unique generating equation corresponding to the function

\[ \tilde{J}_\mu(b) = \eta_1 b^{\alpha_1} + \eta_2 b^{\alpha_2}, \quad b \geq 0, \]  

(3.52)

with \( \eta_1, \eta_2 > 0, 2 \geq \alpha_1 > \alpha_2 > 1 \). This function does not guarantee the uniqueness of generating equations in higher dimensions. Below we show a family of generating pairs \((G, Z)\) taking values in \( \mathbb{R}^3 \) such that \( J_{ZG(x)}(b) = x \tilde{J}_\mu(b) \).

**Example 3.10** Let us consider a process \( Z(t) = (Z_1(t), Z_2(t), Z_3(t)) \) with independent coordinates such that \( Z_1 \) is \( \alpha_1 \)-stable, \( Z_2 \) is \( \alpha_2 \)-stable, \( Z_3 \) is a sum of an \( \alpha_1 \)- and \( \alpha_2 \)-stable processes. Then

\[ J_1(b) = \gamma_1 b^{\alpha_1}, \quad J_2(b) = \gamma_2 b^{\alpha_2}, \quad J_3(b) = \gamma_3 b^{\alpha_1} + \tilde{\gamma}_3 b^{\alpha_2}, \quad b \geq 0, \]

where \( \gamma_1 > 0, \gamma_2 > 0, \gamma_3 > 0, \tilde{\gamma}_3 > 0 \). We are looking for non-negative functions \( G_1, G_2, G_3 \) solving the equation

\[ J_1(bG_1(x)) + J_2(bG_2(x)) + J_3(bG_3(x)) = x \tilde{J}_\mu(b), \quad x, b \geq 0, \]  

(3.53)

where \( \tilde{J}_\mu \) is given by (3.52). It follows from (3.53) that

\[ \gamma_1 b^{\alpha_1}(G_1(x))^{\alpha_1} + \gamma_2 b^{\alpha_2}(G_2(x))^{\alpha_2} + \gamma_3 b^{\alpha_1}(G_3(x))^{\alpha_1} + \tilde{\gamma}_3 b^{\alpha_2}(G_3(x))^{\alpha_2} = x [\eta_1 b^{\alpha_1} + \eta_2 b^{\alpha_2}], \quad x, b \geq 0, \]

and, consequently,

\[ b^{\alpha_1} [\gamma_1 G_1^{\alpha_1}(x) + \gamma_3 G_3^{\alpha_1}(x)] + b^{\alpha_2} [\gamma_2 G_2^{\alpha_2}(x) + \tilde{\gamma}_3 G_3^{\alpha_2}(x)] = x [\eta_1 b^{\alpha_1} + \eta_2 b^{\alpha_2}], \quad x, b \geq 0. \]

Thus we obtain the following system of equations

\[ \gamma_1 G_1^{\alpha_1}(x) + \gamma_3 G_3^{\alpha_1}(x) = x\eta_1, \]

\[ \gamma_2 G_2^{\alpha_2}(x) + \tilde{\gamma}_3 G_3^{\alpha_2}(x) = x\eta_2, \]

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which allows us to determine $G_1$ and $G_2$ in terms of $G_3$, that is

$$
G_1(x) = \left( \frac{1}{\gamma_1} (x\eta_1 - \gamma_3 G_3(x)) \right)^{\frac{1}{\alpha_1}},
$$

$$
G_2(x) = \left( \frac{1}{\gamma_2} (x\eta_2 - \tilde{\gamma}_3 G_3(x)) \right)^{\frac{1}{\alpha_2}}.
$$

(3.54)

(3.55)

The positivity of $G_1, G_2, G_3$ means that $G_3$ satisfies

$$
0 \leq G_3(x) \leq \left( \frac{\eta_1}{\gamma_3} x \right)^{\frac{1}{\alpha_1}} \wedge \left( \frac{\eta_2}{\tilde{\gamma}_3} x \right)^{\frac{1}{\alpha_2}}, \quad x \geq 0.
$$

(3.56)

It follows that $(G,Z)$ with any $G_3$ satisfying (3.56) and $G_1, G_2$ given by (3.54), (3.55) constitutes a generating pair.

4 Spherical Lévy noise

This section deals with equation (2.1) in the case when $Z$ is a spherical Lévy process with characteristic triplet $(a,Q,\nu(dy))$. Recall, the Lévy measure $\nu(dy)$ is described in terms of a finite spherical measure $\lambda(d\xi)$ on $S^{d-1}$ and a radial measure $\gamma(dr)$ on $(0, +\infty)$ by the formula (1.6). As $Z$ is a martingale, by (2.5),

$$
a = -\int_{|y|>1} |y| \nu(dy) = \int_{S^{d-1}} \lambda(d\xi) \int_1^{+\infty} r \xi \gamma(dr),
$$

and the integrability of $Z$ implies that

$$
\int_{|y|>1} |y| \nu(dy) = \int_{S^{d-1}} \lambda(d\xi) \int_0^{+\infty} r \xi \gamma(dr) = \lambda(S^{d-1}) \cdot \int_1^{+\infty} r \gamma(dr) < +\infty. \quad (4.1)
$$

The jump part of $Z$ is assumed to have infinite variation, which means that

$$
\int_{|y|\leq 1} |y| \nu(dy) = \lambda(S^{d-1}) \cdot \int_0^{+\infty} r \gamma(dr) = +\infty.
$$

Consequently, the radial measure is of infinite variation, i.e.

$$
\int_0^1 r \gamma(dr) = +\infty. \quad (4.2)
$$

Furthermore, the measure $\nu(dy)$ will be assumed to have its support not contained in any proper linear subspace of $\mathbb{R}^d$, i.e.

$$
\text{Linear span (supp } \lambda) = \mathbb{R}^d. \quad (4.3)
$$

Moreover, by Proposition (2.1)A(a)) and (1.6), $\lambda \{ \xi \in S^{d-1} : \langle G(0), \xi \rangle < 0 \} = 0$ which implies that

$$
\langle G(0), \xi \rangle \geq 0 \text{ for any } \xi \in \text{supp } \lambda. \quad (4.4)
$$

A consequence of (1.2), (1.3) and (1.4) is that if $(G,Z)$ is a generating pair, then

$$
G(0) = 0. \quad (4.5)
$$
Indeed, by Proposition \(2.1\text{A(b)}\), the jump part of \(Z^{(0)}\) is of finite variation. Therefore
\[
\int_0^{+\infty} v \nu_{G(0)}(dv) = \int_{\mathbb{R}^d} (G(0), y)v(dy) = \int_{S^{d-1}} \lambda(d\xi) \int_0^{+\infty} r \langle G(0), r\xi \rangle \gamma(dr)
\]
\[
= \int_{S^{d-1}} \langle G(0), \xi \rangle \lambda(d\xi) \int_0^{+\infty} r \gamma(dr) < +\infty,
\]
which, in view of (4.2), (4.3) and (4.4) implies (4.5).

### 4.1 Main results

In this section we prove the following theorem.

**Theorem 4.1** Let \(Z\) be a Lévy martingale with characteristic triplet \((a, Q, \nu(dy))\) such that \(\nu(dy)\) admits the decomposition (1.6) with spherical measure \(\lambda(d\xi)\) satisfying (4.3). Let us also assume that \(\gamma(dr)\) satisfies (4.2) or (4.5) holds. Moreover, let \(G : [0, +\infty) \to \mathbb{R}^d\) be a continuous function such that

\[
G_0 := \lim_{x \to 0^+} \frac{G(x)}{|G(x)|^\alpha},
\]

exists.

Then (2.1) generates an affine model if and only if \(F(x) = ax + b, a \in \mathbb{R}, b \geq 0\) and the measure \(\mu(dr)\) in Proposition (2.1A(c)) is \(\alpha\)-stable with \(\alpha \in (1, 2)\).

The proof of Theorem 4.1 is presented in Subsection 4.3 and is preceded by some auxiliary results presented in Subsection 4.2.

From Theorem 4.1 the following corollary follows.

**Corollary 4.2** Let the assumptions of Theorem 4.1 be satisfied. If \((Z, G)\) is a generating pair, then the continuous (Wiener) part of the process \(Z^{G(x)}\) vanishes for all \(x > 0\).

**Proof:** It follows from Proposition 2.1(c) that the Wiener part of \(Z^{G(x)}\) satisfies
\[
\frac{1}{2} \langle QG(x), G(x) \rangle = cx, \quad x \geq 0 \text{ for some } c \geq 0.
\]

Either directly by assumption (4.5) or by assumption (4.2) we get that \(G(0) = 0\). Therefore, by (2.20) and Theorem 4.1 the Laplace transform of the jump part of \(Z\) satisfies
\[
J_\nu(bG(x)) = xJ_\nu(b) = \gamma xb^\alpha, \quad x \geq 0 \text{ for some } \gamma > 0, \alpha \in (1, 2).
\]

Proposition 2.1A(a)) guarantees that \(\langle G_0, y \rangle \geq 0\) for any \(y \in \text{supp } \nu\) and condition (1.3) guarantees that \(y \mapsto \langle G_0, y \rangle, y \in \text{supp } \nu\), does not vanish, hence \(J_\nu(G_0) > 0\). Consequently, from (4.8) we obtain
\[
\lim_{x \to 0^+} \frac{\gamma x}{|G(x)|^\alpha} = \lim_{x \to 0^+} J_\nu \left( \frac{G(x)}{|G(x)|^\alpha} \right) = J_\nu(G_0) \in (0, +\infty).
\]

From this, \(\lim_{x \to 0^+} |G(x)| = 0\) and (4.7) we further have
\[
\langle QG_0, G_0 \rangle = \lim_{x \to 0^+} \frac{\langle QG(x), G(x) \rangle}{|G(x)|^2} = \lim_{x \to 0^+} \frac{\gamma x}{|G(x)|^\alpha} \frac{2\alpha}{|G(x)|^{2-\alpha}} = \begin{cases} 0 & \text{ if } c = 0; \\ +\infty & \text{ if } c > 0. \end{cases}
\]

Since \(\langle QG_0, G_0 \rangle \neq +\infty\), we necessarily have \(c = 0\) which, in view of (4.7), means that the continuous (Wiener) part of \(Z^{G(x)}\) vanishes.

\(\square\)
Remark 4.3 A generating pair \((G, Z)\) satisfying assumptions of Theorem 4.1 has projections \(Z^G(x), x \geq 0\) with the same law as the projections \(\tilde{Z}^G(x), x \geq 0\), where \(\tilde{G}(x) = (Cx)^\frac{1}{\alpha}, C > 0\), and \(\tilde{Z}\) is a one-dimensional \(\alpha\)-stable process with positive jumps. In view of Proposition 2.4 the short rates given by (2.1) and (1.4) have the same generator.

Remark 4.4 In the formulation of Theorem 4.1 the assumption (4.6) can be replaced by the existence of the limit \(\lim_{x \to +\infty} \frac{G(x)}{|G(x)|}\). Under the latter condition we were, however, unable to prove Corollary 4.2.

Remark 4.5 Our proof of Theorem 4.1 seems to work for more general measures, namely measures satisfying (1.6) with \(S^{d-1}\) replaced by the boundary \(\partial D\) of some convex set \(D\) in \(\mathbb{R}^d\), containing \(0\) in its interior, and the measure \(\lambda\) replaced by an appropriate finite measure on \(\partial D\).

4.2 Auxiliary results

Our first aim is to estimate, for a generating pair \((G, Z)\), the function \(J_\nu(bG(x)), b, x \geq 0\) with the use of the function \(J_\nu(bG_0), b \geq 0\) for \(x\) such that \(G(x)/|G(x)|\) is close to \(G_0\). The solution of this problem is presented in Lemma 4.6, Proposition 4.8 and Proposition 4.9.

Let \(\rho(dv)\) be a Lévy measure on \((0, +\infty)\) satisfying
\[
\int_0^{+\infty} (v^2 \wedge v) \rho(dv) < +\infty, \tag{4.9}
\]
and
\[
J_\rho(z) := \int_{(0, +\infty)} (e^{-zv} - 1 + zv) \rho(dv), \quad z \geq 0, \tag{4.10}
\]
Recall, Proposition 3.4 provides a growth estimation of the function \(J_\rho\). The second aim of this section is to provide sufficient conditions for \(J_\rho\) to be a power function. This problem is solved in Lemma 4.10 and Lemma 4.11.

Lemma 4.6 The function \(H : [0, +\infty) \to \mathbb{R}\) given by
\[
H(z) = e^{-z} - 1 + z,
\]
is convex, strictly increasing and
\[
\min\{1, t^2\} \cdot H(z) \leq H(tz) \leq \max\{1, t^2\} \cdot H(z), \quad z \geq 0, t > 0. \tag{4.11}
\]

Proof: Since \(H'(z) = 1 - e^{-z}\) the monotonicity and convexity of \(H\) follows. For \(t \geq 1\) it follows from the monotonicity of \(H\) that
\[
H(tz) \geq H(z) = \min\{1, t^2\} H(z).
\]
From (4.11) we obtain
\[
\frac{d}{ds} \ln H(s) = \frac{H'(s)}{H(s)} = \frac{1 - e^{-s}}{e^{-s} - 1 + s} \leq \frac{2}{s}, \quad s > 0,
\]
and, consequently, we obtain that for \(t \geq 1\):
\[
\ln H(tz) - \ln H(z) \leq \int_z^{tz} \frac{2}{s} ds = \ln t^2.
\]
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Proposition 4.8

Let $G$ generate an affine model and $\nu_n$ be such that for any $z = 1/n$ the function

$$\nu \left\{ y \in \mathbb{R}^d : \langle G, y \rangle < 0 \right\} = 0.$$  

Proof: Assume that

$$\nu \left\{ y \in \mathbb{R}^d : \langle G, y \rangle < 0 \right\} = \nu \left\{ y \in \mathbb{R}^d \setminus \{0\} : \langle G, y \rangle < 0 \right\} > 0.$$  

Then there exists a natural $n$ such that for

$$V_n := \left\{ y \in \mathbb{R}^d \setminus \{0\} : \langle G, y \rangle < -\frac{1}{n} \right\}$$

one has $\nu(V_n) > 0$.

Let $x$ be such that

$$\left| \frac{G(x)}{|G(x)|} - G_\infty \right| \leq \frac{1}{2n}.$$  

It follows from the Schwarz inequality that for any $y \in \mathbb{R}^d$,

$$\left| \langle G, y \rangle \right| \leq |y| \leq \frac{1}{2n} |y|.$$  

Let $y \in V_n$. From (4.14) and the definition of $V_n$ we estimate

$$\langle G, y \rangle \leq \langle G, y \rangle + \frac{1}{2n} |y| < -\frac{1}{n} |y| + \frac{1}{2n} |y| = -\frac{1}{2n} |y| < 0.$$  

Hence

$$\nu \left\{ y \in \mathbb{R}^d : \langle G, y \rangle < 0 \right\} \geq \nu(V_n) > 0$$

which is a contradiction with Proposition 2.1(A(a)).  

□

Corollary 4.7

Let $\rho(dv)$ be a Lévy measure on $(0, +\infty)$ satisfying $\int_0^{+\infty} (v^2 \wedge v) \rho(dv) < +\infty$. It follows from (4.11) and the formula

$$J_\rho(z) := \int_{(0, +\infty)} H(zv) \rho(dv) < +\infty$$

that the function $J_\rho$ satisfies

$$\min \left\{ 1, t^2 \right\} \cdot J_\rho(z) \leq \max \left\{ 1, t^2 \right\} \cdot J_\rho(z), \quad z \geq 0, t > 0. \quad (4.13)$$

Proposition 4.8

Let $Z$ be a Lévy process with characteristic triplet $(a, Q, \nu(dy))$. If $G$ generates an affine model and $G_\infty$ is an arbitrary limit point of the set

$$\left\{ \frac{G(x)}{|G(x)|} : x > 0 \right\}$$

then

$$\nu \left\{ y \in \mathbb{R}^d : \langle G, y \rangle < 0 \right\} = 0.$$  

Thus

$$\min\{1, t^2\} H(z) = H(z) \leq H(tz) \leq t^2 H(z) = \max\{1, t^2\} H(z). \quad (4.12)$$

Using the monotonicity of $H$ and (4.12) we see that for $t \in (0, 1)$,

$$H(tz) \leq H(z) = H \left( \frac{1}{t} tz \right) \leq \frac{1}{t^2} H(tz),$$

so also for $t \in (0, 1)$

$$\min\{1, t^2\} H(z) = t^2 H(z) \leq H(tz) \leq H(z) = \max\{1, t^2\} H(z).$$

□
Proposition 4.9 Let \((G, Z)\) be a generating pair where \(Z\) is a spherical Lévy process. Assume that \(\nu(dy)\) has the form \((4.16)\) and \((4.17)\) holds. Let \(G_\infty\) be any limit point of the set
\[
\left\{ \frac{G(x)}{|G(x)|} : x > 0 \right\}.
\]
Define
\[
M_{G_\infty}(b) := J_\nu (b \cdot G_\infty) := \int_{S^{d-1}} \int_{0}^{+\infty} H \left( b \langle G_\infty, r \cdot \xi \rangle \right) \gamma (dr) \lambda (d\xi),
\]
where \(H(z) := e^{-z} - 1 + z\). There exists a function \(\delta : (0, 1) \rightarrow (0, +\infty)\) such that for any \(\varepsilon_0 > 0\), any \(b \geq 0\) and \(x > 0\) such that \(|\frac{G(x)}{|G(x)|} - G_\infty| \leq \delta (\varepsilon_0)\) we have
\[
(1 - \varepsilon_0) M_{G_\infty}(b \langle G(x) \rangle) \leq J_\nu (b G(x)) \leq (1 + \varepsilon_0) M_{G_\infty}(b \langle G(x) \rangle).
\]
Proof: Let \(\varepsilon \in (0, 1)\) be such that
\[
(1 + \varepsilon)^2 \left( 1 + \frac{4\varepsilon}{(1 - \varepsilon)^2} \right) \leq 1 + \varepsilon_0 , \quad \frac{(1 - \varepsilon)^2}{(1 + \varepsilon^2)} \geq 1 - \varepsilon_0.
\]
Let us assume that
\[
\lambda \left\{ \xi \in S^{d-1} : \langle G_\infty, \xi \rangle > 0 \right\} = \lambda \left(S^{d-1}\right) - \lambda \left\{ \xi \in S^{d-1} : \langle G_\infty, \xi \rangle = 0 \right\} = 1,
\]
(we can assume this, multiplying \(\lambda (d\xi)\) by a positive constant, provided
\[
\lambda \left\{ \xi \in S^{d-1} : \langle G_\infty, \xi \rangle > 0 \right\} > 0,
\]
otherwise it follows from Proposition 4.8 that we get a degenerated case
\[
\lambda \left(S^{d-1}\right) = \lambda \left\{ \xi \in S^{d-1} : \langle G_\infty, \xi \rangle = 0 \right\}
\]
where \((4.18)\) is broken). Let \(\eta \in (0, 1)\) be such that
\[
\lambda \left\{ \xi \in S^{d-1} : 0 < \langle G_\infty, \xi \rangle < \eta \right\} \leq \varepsilon.
\]
Moreover, by Proposition 4.8
\[
\nu \left\{ y \in \mathbb{R}^d : \langle G_\infty, y \rangle < 0 \right\} = \lambda \left\{ \xi \in S^{d-1} : \langle G_\infty, \xi \rangle < 0 \right\} \cdot \gamma (\mathbb{R}_+) = 0.
\]
Let us define
\[
\forall_{\eta} = \left\{ \xi \in S^{d-1} : 0 < \langle G_\infty, \xi \rangle < \eta \right\}.
\]
Let \(x\) be such that
\[
\left| \frac{G(x)}{|G(x)|} - G_\infty \right| \leq \delta (\varepsilon_0) := \eta \cdot \varepsilon.
\]
From Lemma 4.6 for \(b, r \geq 0\) and \(\xi \in S^{d-1}\) such that \(\langle G_\infty, \xi \rangle \in [0, \eta)\) we estimate
\[
H \left( b \cdot r \left| G(x), \xi \right| \right) \leq H \left( b \cdot r \left| G(x) \right| \left( \langle G_\infty, \xi \rangle + \left| \frac{G(x)}{|G(x)|} - G_\infty, \xi \right| \right) \right)
\leq \max \left\{ H \left( b \cdot r \left| G(x) \right| 2 \langle G_\infty, \xi \rangle \right), H \left( b \cdot r \cdot \left| G(x) \right| 2 \left| \frac{G(x)}{|G(x)|} - G_\infty, \xi \right| \right) \right\}
\leq \max \left\{ H \left( b \cdot r \left| G(x) \right| 2\eta \right), H \left( b \cdot r \cdot \left| G(x) \right| 2\eta \cdot \varepsilon \right) \right\}
= H \left( b \cdot r \left| G(x) \right| 2\eta \right) \cdot H \left( b \cdot r \cdot \left| G(x) \right| 2\eta \cdot \varepsilon \right)
\leq 4H \left( b \cdot r \left| G(x) \right| \eta \right).
\]
Hence

\[
\int_{\mathcal{V}_n} \int_0^{+\infty} H(b \cdot r \langle G(x), \xi \rangle) \gamma(dy) \lambda(d\xi) \leq 4 \int_{\mathcal{V}_n} \int_0^{+\infty} H(b \cdot r |G(x)| \eta) \gamma(dy) \lambda(d\xi) \\
\leq 4\varepsilon \int_0^{+\infty} H(b \cdot r |G(x)| \eta) \gamma(dy) .
\]  

(4.19)

From Lemma 4.6, for \( b, r \geq 0 \) and \( \xi \in S^{d-1} \) such that \( \langle G_\infty, \xi \rangle \in [\eta, 1] \), we also estimate

\[
H(b \cdot r \langle G(x), \xi \rangle) \leq H \left( b \cdot r |G(x)| \left( \langle G_\infty, \xi \rangle + \left| \frac{G(x)}{|G(x)|} - G_\infty, \xi \right| \right) \right)
\leq H(b \cdot r |G(x)| (\langle G_\infty, \xi \rangle + (G_\infty, \xi) \varepsilon)) \\
\leq (1 + \varepsilon)^2 H(b \cdot r |G(x)| (G_\infty, \xi)) ,
\]

(4.20)

and

\[
H(b \cdot r \langle G(x), \xi \rangle) \geq H \left( b \cdot r |G(x)| \left( \langle G_\infty, \xi \rangle - \left| \frac{G(x)}{|G(x)|} - G_\infty, \xi \right| \right) \right)
\geq H(b \cdot r |G(x)| (\langle G_\infty, \xi \rangle - (G_\infty, \xi) \varepsilon)) \\
\geq (1 - \varepsilon)^2 H(b \cdot r |G(x)| (G_\infty, \xi)) .
\]

(4.21)

Notice that by (4.17) and (4.18), \( \lambda \left(S^{d-1} \setminus \mathcal{V}_\eta \right) \geq 1 - \varepsilon \). From (4.21) and then from \( \lambda \left(S^{d-1} \setminus \mathcal{V}_\eta \right) \geq 1 - \varepsilon \) and (4.19) we obtain

\[
\int_{S^{d-1} \setminus \mathcal{V}_\eta} \int_0^{+\infty} H(b \cdot r \langle G(x), \xi \rangle) \gamma(dy) \lambda(d\xi) \\
\geq \int_{S^{d-1} \setminus \mathcal{V}_\eta} \int_0^{+\infty} (1 - \varepsilon)^2 H(b \cdot r |G(x)| (G_\infty, \xi)) \gamma(dy) \lambda(d\xi) \\
\geq (1 - \varepsilon)^2 \int_{S^{d-1} \setminus \mathcal{V}_\eta} \int_0^{+\infty} H(b \cdot r |G(x)| \eta) \gamma(dy) \lambda(d\xi) \\
\geq (1 - \varepsilon)^2 (1 - \varepsilon) \int_{S^{d-1} \setminus \mathcal{V}_\eta} \int_0^{+\infty} H(b \cdot r |G(x)| \eta) \gamma(dy) \\
\geq \frac{(1 - \varepsilon)^3}{4\varepsilon} \int_{\mathcal{V}_\eta} \int_0^{+\infty} H(b \cdot r \langle G(x), \xi \rangle) \gamma(dy) \lambda(d\xi) .
\]

(4.22)

From (4.22) and (4.20) we estimate

\[
J_\nu (b G(x)) = \int_{S^{d-1} \setminus \mathcal{V}_\eta} \int_0^{+\infty} H(b \cdot r \langle G(x), \xi \rangle) \gamma(dy) \lambda(d\xi) + \int_{\mathcal{V}_\eta} \int_0^{+\infty} H(b \cdot r \langle G(x), \xi \rangle) \gamma(dy) \lambda(d\xi) \\
\leq \int_{S^{d-1} \setminus \mathcal{V}_\eta} \int_0^{+\infty} H(b \cdot r \langle G(x), \xi \rangle) \gamma(dy) \lambda(d\xi) \\
+ \frac{4\varepsilon}{(1 - \varepsilon)^3} \int_{S^{d-1} \setminus \mathcal{V}_\eta} \int_0^{+\infty} H(b \cdot r \langle G(x), \xi \rangle) \gamma(dy) \lambda(d\xi) \\
\leq (1 + \varepsilon)^2 \left( 1 + \frac{4\varepsilon}{(1 - \varepsilon)^3} \right) \int_{S^{d-1} \setminus \mathcal{V}_\eta} \int_0^{+\infty} H(b \cdot r |G(x)| (G_\infty, \xi)) \gamma(dy) \lambda(d\xi) \\
= (1 + \varepsilon)^2 \left( 1 + \frac{4\varepsilon}{(1 - \varepsilon)^3} \right) M_{G_\infty} (b \cdot r |G(x)|) .
\]
Hence

\[ J_\nu(bG(x)) \leq (1 + \varepsilon)^2 \left( 1 + \frac{4\varepsilon}{(1 - \varepsilon)^2} \right) M_{G_{\infty}}(b \cdot r | G(x)) \quad (4.23) \]

In order to get the lower bound let us notice that

\[
\int_{S^{d-1} \setminus V_\eta} \int_0^{+\infty} H(b \cdot r | G(x) | \langle G_{\infty}, \xi \rangle) \gamma(dr) \lambda(d\xi) \geq \int_{S^{d-1} \setminus V_\eta} \int_0^{+\infty} H(b \cdot r | G(x) | \eta) \gamma(dr) \lambda(d\xi) \\
\geq (1 - \varepsilon) \int_0^{+\infty} H(b \cdot r | G(x) | \eta) \gamma(dr).
\]

and

\[
\int_{V_\eta} \int_0^{+\infty} H(b \cdot r | G(x) | \langle G_{\infty}, \xi \rangle) \gamma(dr) \lambda(d\xi) \leq \int_{V_\eta} \int_0^{+\infty} H(b \cdot r | G(x) | \eta) \gamma(dr) \lambda(d\xi) \\
\leq \varepsilon \int_0^{+\infty} H(b \cdot r | G(x) | \eta) \gamma(dr).
\]

Hence

\[
\int_{S^{d-1} \setminus V_\eta} \int_0^{+\infty} H(b \cdot r | G(x) | \langle G_{\infty}, \xi \rangle) \gamma(dr) \lambda(d\xi) \\
\geq \frac{1 - \varepsilon}{\varepsilon} \int_{V_\eta} \int_0^{+\infty} H(b \cdot r | G(x) | \langle G_{\infty}, \xi \rangle) \gamma(dr) \lambda(d\xi), \quad (4.24)
\]

and from this we obtain

\[
\int_{S^{d-1}} \int_0^{+\infty} H(b \cdot r | G(x) | \langle G_{\infty}, \xi \rangle) \gamma(dr) \lambda(d\xi) = \int_{S^{d-1} \setminus V_\eta} \int_0^{+\infty} H(b \cdot r | G(x) | \langle G_{\infty}, \xi \rangle) \gamma(dr) \lambda(d\xi) \\
+ \int_{V_\eta} \int_0^{+\infty} H(b \cdot r | G(x) | \langle G_{\infty}, \xi \rangle) \gamma(dr) \lambda(d\xi) \\
\leq \left( 1 + \frac{\varepsilon}{1 - \varepsilon} \right) \int_{S^{d-1} \setminus V_\eta} \int_0^{+\infty} H(b \cdot r | G(x) | \langle G_{\infty}, \xi \rangle) \gamma(dr) \lambda(d\xi). \quad (4.25)
\]

From (4.21) and (4.25) we get

\[
J_\nu(bG(x)) \geq \int_{S^{d-1} \setminus V_\eta} \int_0^{+\infty} H(b \cdot r | G(x), \xi \rangle) \gamma(dr) \lambda(d\xi) \\
\geq (1 - \varepsilon)^2 \int_{S^{d-1} \setminus V_\eta} \int_0^{+\infty} H(b \cdot r | G(x) | \langle G_{\infty}, \xi \rangle) \gamma(dr) \lambda(d\xi) \\
\geq \frac{(1 - \varepsilon)^2}{\left( 1 + \frac{1}{1 - \varepsilon} \right)} \int_{S^{d-1}} \int_0^{+\infty} H(b \cdot r | G(x) | \langle G_{\infty}, \xi \rangle) \gamma(dr) \lambda(d\xi) \\
= \frac{(1 - \varepsilon)^2}{\left( 1 + \frac{1}{1 - \varepsilon} \right)} M_{G_{\infty}}(b \cdot r | G(x))).
\]

Hence

\[
J_\nu(bG(x)) \geq \frac{(1 - \varepsilon)^2}{\left( 1 + \frac{1}{1 - \varepsilon} \right)} M_{G_{\infty}}(b \cdot r | G(x))). \quad (4.26)
\]

Now (4.19) follows from (4.23), (4.24) and (4.19).
Lemma 4.10 Let $J_\rho$ be given by (4.10) with $\rho(dv)$ satisfying (1.9). Assume that

\[ J_\rho(\beta b) = \eta J_\rho(b), \quad b \geq 0, \]  
and

\[ J_\rho(\gamma b) = \theta J_\rho(b), \quad b \geq 0, \]  
for some $\beta > 1$, $\gamma > 1$ such that $\ln \beta/\ln \gamma \notin \mathbb{Q}$ and $\eta > 1$, $\theta > 1$. Then

\[ J_\rho(b) = Cb^\alpha, \quad b \geq 0, \]  
for some $C > 0$ and $\alpha \in (1, 2)$.

**Proof:** By iterative application of (4.27) and (4.28) we see that for any $m, n \in \mathbb{N}$

\[ J_\rho(\beta^m \gamma^n b) = \eta^m \theta^n J_\rho(b), \quad b \geq 0, \]  
which can be written as

\[ J_\rho(b e^{m \ln \beta + n \ln \gamma}) = e^{m \ln \eta + n \ln \theta} J_\rho(b), \quad b \geq 0. \]  
In Lemma 4.11 below we prove that the set

\[ D := \{m \ln \beta - n \ln \gamma; \quad m, n \in \mathbb{Z}\} \]

is dense in $\mathbb{R}$. So, for any $\delta > 0$ there exist $m, n \in \mathbb{Z}$, $m \neq 0$, such that

\[ |m \ln \beta - n \ln \gamma| < \delta, \]  
and then, by (4.13) and (4.30), we obtain that

\[ e^{-2\delta} \leq \frac{e^{m \ln \eta}}{e^{n \ln \theta}} = \frac{J_\rho(e^{m \ln \beta})}{J_\rho(e^{n \ln \gamma})} \leq e^{2\delta}. \]  
It follows from (4.31) that

\[ \frac{\ln \beta}{\ln \gamma} - \frac{n}{m} \leq \frac{\delta}{|m| \ln \gamma}, \]

and from (4.32) that

\[ \frac{\ln \eta}{\ln \theta} - \frac{n}{m} \leq \frac{2\delta}{|m| \ln \theta}. \]

Consequently,

\[ \frac{\ln \beta}{\ln \gamma} - \frac{\ln \eta}{\ln \theta} \leq \frac{\delta}{|m| \ln \gamma} + \frac{2\delta}{|m| \ln \theta} \leq \frac{\delta}{\ln \gamma} + \frac{2\delta}{\ln \theta}. \]

Letting $\delta \to 0$ yields

\[ \frac{\ln \beta}{\ln \gamma} = \frac{\ln \eta}{\ln \theta}. \]

Let us define

\[ \alpha := \frac{\ln \eta}{\ln \beta} = \frac{\ln \theta}{\ln \gamma} > 0, \]

and put $b = 1$ in (4.30). This gives

\[ J_\rho(e^{m \ln \beta + n \ln \gamma}) = J_\rho(1) \left(e^{m \ln \beta + n \ln \gamma}\right)^\alpha, \]

which means that $J_\rho(b) = J_\rho(1)b^\alpha$ for $b$ from the set $e^D$ which is dense in $[0, +\infty)$. As $J_\rho$ is continuous, (4.29) follows. Finally, by Proposition 3.4 it follows that $\alpha \in (1, 2)$. □

The following result is strictly related to Weyl’s equidistribution theorem, see [12].
Lemma 4.11 Let $p,q > 0$ be such that $p/q \notin \mathbb{Q}$. Let us define the set

$$G := \{ mp + nq; \quad m,n = 1,2,\ldots \}.$$ 

Then for each $\delta > 0$ there exists a number $M(\delta) > 0$ such that

$$\forall x \geq M(\delta) \quad \exists g \in G \quad \text{such that} \quad | x - g | \leq \delta.$$ 

Moreover, the set

$$D := \{ mp + nq; \quad m,n \in \mathbb{Z} \},$$

is dense in $\mathbb{R}$.

Proof: Since $p/q \notin \mathbb{Q}$, at least one of $p,q$, say $q$, is irrational. For simplicity assume that $p = 1$ and consider the sequence

$$r(jq), j = 1,2,\ldots \quad \text{where} \quad r(x) := x \mod 1,$$

of fractional parts of the numbers $jq, j = 1,2,\ldots$. Recall, Weyl’s equidistribution theorem states that

$$\lim_{N \to +\infty} \frac{\sharp \{ j \leq N : r(jq) \in [a,b] \}}{N} = b - a \quad (4.33)$$

for any $[a,b] \subseteq [0,1)$ if and only if $q$ is irrational.

For fixed $\delta > 0$ and $n$ such that $1/n < \delta$ let us consider a partition of $[0,1)$ of the form

$$[0,1) = \bigcup_{k=0}^{n-1} A_k, \quad A_k := [k/n,(k+1)/n).$$

For a natural number $N$ let us consider the set $R_N := \{ r(jq) : j = 1,2,\ldots,N \}$. By (4.33), for each $k = 0,1,\ldots,n-1$, there exists $N_k$ such that

$$R_{N_k} \cap A_k \neq \emptyset.$$ 

Then for $\bar{N} := \max\{ N_0, N_1,\ldots,N_{n-1} \}$ we have

$$R_{\bar{N}} \cap A_k \neq \emptyset, \quad k = 0,1,\ldots,n-1.$$ 

Let $M = M(\delta) := \bar{N}q$. Then, for $x \geq M$, there exists a number $N_x \leq \bar{N}$ such that

$$| r(N_xq) - r(x) | \leq \frac{1}{n} \quad (4.34)$$

Then for the number

$$g := [x] - [N_xq] + N_xg \in G$$

the following holds

$$| x - g | = | x - ([x] - [N_xq] + N_xq) |$$

$$= | [x] + r(x) - [x] + [N_xq] - N_xq |$$

$$= | r(x) - r(N_xq) | \leq 1/n < \delta,$$

where the last inequality follows from (4.34).

The density of $D$ is an immediate consequence of the first part of the Lemma. Indeed, for $x < M(\delta)$ and $g \in G$ such that $x + g > M(\delta)$ there exists $\tilde{g} \in G$ such that $| x + g - \tilde{g} | < \delta$.

The general case with $p \neq 1$ can be proven in the same way but requires a generalized version of Weyl’s theorem, which says that the numbers $r_p(nq), n = 1,2,\ldots$, where $r_p(x) := x \mod p$, are equidistributed on $[0,p)$ if and only if $p/q \notin \mathbb{Q}$. This can be proven by a straightforward modification of the original arguments of Weyl. \hfill \Box
4.3 Proof of Theorem 4.1

By (2.20), the Laplace transform $J_\nu$ of the the jump part of $Z$ satisfies

$$J_\nu(bG(x)) = J(bG(0)) + xJ_\mu(b), \quad b, x \geq 0,$$

(4.35)

with $J_\mu(b)$ given by (2.21), where $\mu(dv)$ is a measure satisfying conditions of Proposition 2.1(c).

By discussion at the beginning of this section we have $G(0) = 0$, hence (4.35) simplifies to

$$J_\nu(bG(x)) = xJ_\mu(b), \quad x \geq 0,$$

(4.36)

From the assumption that supp $\nu$ is not contained in any proper linear subspace of $\mathbb{R}^d$ and (4.36), we have that $J_\nu(y), J_\mu(b) > 0$, $G(x) \neq 0$, for $y \in \mathbb{R}^d \setminus \{0\}, b > 0, x > 0$.

Let $G_0 = \lim_{x \to 0^+} \frac{G(x)}{|G(x)|}$. From Proposition 4.9 it follows that there exists a function $\delta : (0, +\infty) \to (0, +\infty)$, such that for any $\varepsilon > 0$ from the inequality

$$\left| \frac{G(x)}{|G(x)|} - G_0 \right| \leq \delta(\varepsilon),$$

follows that for any $b \geq 0$

$$1 - \varepsilon \leq \frac{J_\nu \left( \frac{bG(x)}{|G(x)|} \right)}{J_\nu (bG_0)} \leq 1 + \varepsilon.$$

Thus for any $\varepsilon > 0$ there exists $m(\varepsilon) > 0$, such that for $x \in (0, m(\varepsilon))$

$$\left| \frac{G(x)}{|G(x)|} - G_0 \right| \leq \delta(\varepsilon),$$

and hence for any $b > 0$

$$1 - \varepsilon \leq \frac{J_\nu \left( \frac{bG(x)}{|G(x)|} \right)}{J_\nu (bG_0)} \leq 1 + \varepsilon.$$

Let us fix $\beta > 1$ and take $x_1, x_2$ satisfying $0 < x_1 \leq x_2 < m(\varepsilon), \beta |G(x_1)| = |G(x_2)| > 0$ (from the continuity of $G$ it follows that such $x_1$ and $x_2$ exist). Then for any $b > 0$ and $i = 1, 2$, by (4.36),

$$1 - \varepsilon \leq \frac{J_\nu \left( \frac{bG(x_i)}{|G(x_i)|} \right)}{J_\nu (bG_0)} = \frac{x_iJ_\nu \left( \frac{b}{|G(x_i)|} \right)}{J_\nu (bG_0)} \leq 1 + \varepsilon.$$

Hence for any $b > 0$, taking $\bar{b} = \beta |G(x_1)| b$ we get

$$\frac{1 - \varepsilon}{1 + \varepsilon} \cdot \frac{x_2}{x_1} \leq \frac{J_\mu \left( \frac{\bar{b}G(x_1)}{|G(x_1)|} \right)}{J_\mu \left( \frac{\bar{b}G(x_2)}{|G(x_2)|} \right)} = \frac{J_\mu (\beta b)}{J_\mu (b)} \leq \frac{1 + \varepsilon}{1 - \varepsilon} \cdot \frac{x_2}{x_1},$$

which yields

$$\frac{1 - \varepsilon}{1 + \varepsilon} \cdot \frac{J_\mu (\beta b)}{J_\mu (b)} \leq \frac{x_2}{x_1} \leq \frac{1 + \varepsilon}{1 - \varepsilon} \cdot \frac{J_\mu (\beta b)}{J_\mu (b)}.$$

Since $\varepsilon > 0$ is arbitrary, taking $\varepsilon \to 0$ and $x_1, x_2$ satisfying $0 < x_1 \leq x_2 < m(\varepsilon), \beta |G(x_1)| = |G(x_2)|$ we obtain that

$$\lim_{\varepsilon \to 0} \frac{x_2}{x_1} = \eta,$$

where
where $\eta = J_\mu(\beta b) / J_\mu(b) > 1$ is independent from $b > 0$. Hence, for all $b \geq 0$ we have

$$J_\mu(\beta b) = \eta J_\mu(b).$$

Similarly, take $\gamma > 1$ such that $\ln \beta / \ln \gamma \notin \mathbb{Q}$. Reasoning similarly as before we get that there exists $\theta > 1$, such that for all $b \geq 0$ we have

$$J_\mu(\gamma b) = \theta J_\mu(b).$$

Now the thesis follows from Lemma 4.10 and the one to one correspondence between Laplace transforms and measures on $[0, +\infty)$, see [6] p. 233.

5 Appendix

Proof of Proposition 2.1: (A) It was shown in [7, Theorem 5.3] that the generator of a general positive Markovian short rate generating an affine model is of the form

$$A f(x) = c x f''(x) + (\beta x + \gamma) f'(x)$$

$$+ \int_{(0, +\infty)} \left( f(x+y) - f(x) - f'(x)(1 \wedge y) \right) (m(dy) + \mu(dy)), \quad x \geq 0,$$

for $f \in \mathcal{L}(\Lambda) \cup C^2_\mathbb{C}([0, +\infty))$, where $\mathcal{L}(\Lambda)$ is the linear hull of $\Lambda := \{ f_\lambda := e^{-\lambda x}, \lambda \in (0, +\infty) \}$ and $C^2_\mathbb{C}([0, +\infty))$ stands for the set of twice continuously differentiable functions with compact support in $[0, +\infty)$. Above $c, \gamma \geq 0$, $\beta \in \mathbb{R}$ and $m(dy)$, $\mu(dy)$ are nonnegative Borel measures on $[0, +\infty)$ satisfying

$$\int_{(0, +\infty)} (1 \wedge y) m(dy) + \int_{(0, +\infty)} (1 \wedge y^2) \mu(dy) < +\infty. \quad (5.2)$$

The generator of the short rate process given by (2.1) equals

$$A_R f(x) = f'(x) F(x) + \frac{1}{2} f''(x) \langle Q G(x), G(x) \rangle$$

$$+ \int_{\mathbb{R}} \left( f(x + \langle G(x), y \rangle) - f(x) - f'(x) \langle G(x), y \rangle \right) \nu(dy)$$

$$= f'(x) F(x) + \frac{1}{2} f''(x) \langle Q G(x), G(x) \rangle$$

$$+ \int_{\mathbb{R}} \left( f(x + v) - f(x) - f'(x)v \right) \nu_{G(x)}(dv)$$

where $f$ is a bounded, twice continuously differentiable function.

By Proposition 5.1 below, the support of the measure $\nu_{G(x)}$ is contained in $[-x, +\infty)$, thus
it follows that

\[ \mathcal{A}_R f(x) = f'(x)F(x) + \frac{1}{2} f''(x)\langle QG(x), G(x) \rangle \\
+ \int_{(0, +\infty)} \left( f(x + v) - f(x) - f'(x)(1 \land v) \right) \nu_{G(x)}(dv) \\
+ f'(x) \int_{(0, +\infty)} \left( 1 \land v \right) \nu_{G(x)}(dv) \\
+ \int_{(-\infty, 0)} \left( f(x + v) - f(x) - f'(x)v \right) \nu_{G(x)}(dv) \]

Comparing (5.3) with (5.1) applied to a function \( cx\lambda \) with \( \lambda > 0 \) such that \( f_\lambda(x) = e^{-\lambda x} \) for \( x \geq 0 \), we get

\[ cx\lambda^2 - (\beta x + \gamma)\lambda \\
+ \int_{(0, +\infty)} \left( e^{-\lambda y} - 1 + \lambda(1 \land y) \right) (m(dy) + x\mu(dy)) \\
- \frac{1}{2} \lambda^2 \langle QG(x), G(x) \rangle + \left[ F(x) + \int_{(1, +\infty)} \left( 1 - v \right) \nu_{G(x)}(dv) \right] \lambda \\
- \int_{(0, +\infty)} \left( e^{-\lambda v} - 1 + \lambda(1 \land v) \right) \nu_{G(x)}(dv) \\
= \int_{[-x, 0]} \left( e^{-\lambda v} - 1 + \lambda v \right) \nu_{G(x)}(dv), \quad \lambda > 0, x \geq 0. \] (5.4)

Comparing the left and the right sides of (5.4) we see that the left side grows no faster than a quadratic polynomial of \( \lambda \) while the right side grows faster that \( de^{\lambda y} \) for some \( d, y > 0 \), unless the support of the measure \( \nu_{G(x)}(dv) \) is contained in \( [0, +\infty) \). It follows that \( \nu_{G(x)}(dv) \) is concentrated on \([0, +\infty)\), hence (a) follows, and

\[ cx\lambda^2 - (\beta x + \gamma)\lambda \\
- \frac{1}{2} \lambda^2 \langle QG(x), G(x) \rangle + \left[ F(x) + \int_{(1, +\infty)} \left( 1 - v \right) \nu_{G(x)}(dv) \right] \lambda \\
= \int_{(0, +\infty)} \left( e^{-\lambda y} - 1 + \lambda(1 \land y) \right) \left( \nu_{G(x)}(dy) - m(dy) - x\mu(dy) \right), \quad \lambda > 0, x \geq 0. \] (5.5)
Dividing both sides of the last equality by $\lambda^2$ and using the estimate
\[
\frac{e^{-\lambda y} - 1 + \lambda (1 \wedge y)}{\lambda^2} \leq \left( \frac{1}{2} y^2 \right) \wedge \left( \frac{e^{-\lambda y} - 1 + \lambda}{\lambda^2} \right)
\]
we get that the left side of (5.5) converges to $cx - \frac{1}{2} (QG(x), G(x))$ as $\lambda \to +\infty$, while the right side converges to 0. This yields (2.15), i.e.
\[
\frac{1}{2} (QG(x), G(x)) = cx \wedge (1 \wedge y), \quad x \geq 0.
\] (5.6)

Next, fixing $x \geq 0$ and comparing (5.3) with (5.1) applied to a function from the domains of both generators and such that $f(x) = f'(x) = f''(x) = 0$ we get
\[
\int_{(0, +\infty)} f(x + y)(m(dy) + x\mu(dy)) = \int_{(0, +\infty)} f(x + v)\nu_G(x)(dv)
\]
for any such a function, which yields
\[
\nu_G(x)(dv) \mid_{(0, +\infty)} = m(dv) + x\mu(dv), \quad x \geq 0.
\] (5.7)

This implies also
\[
\beta x + \gamma = F(x) + \int_{(1, +\infty)} \left( 1 - v \right)\nu_G(x)(dv), \quad x \geq 0.
\] (5.8)

(b) Setting $x = 0$ in (5.7) yields
\[
\nu_G(0)(dv) \mid_{(0, +\infty)} = m(dv).
\] (5.9)

To prove (2.14), by (5.2) and (5.9), we need to show that
\[
\int_{(1, +\infty)} v\nu_G(0)(dv) < +\infty.
\] (5.10)

It is true if $G(0) = 0$ and for $G(0) \neq 0$ the following estimate holds
\[
\int_{(1, +\infty)} v\nu_G(0)(dv) = \int_{\mathbb{R}^d} (G(0), y) \mathbf{1}_{[1, +\infty)}((G(0), y))\nu(dy)
\leq |G(0)| \int_{\mathbb{R}^d} |y| \mathbf{1}_{[1/|G(0)|, +\infty)}(|y|)\nu(dy),
\]
and (5.10) follows from (2.4).

(c) (2.16) follows from (5.7) and (5.9). To prove (2.17) we use (2.14), (2.14) and the following estimate for $x \geq 0$:
\[
\int_0^{+\infty} (v^2 \wedge v)\nu_G(x)(dv) = \int_{\mathbb{R}^d} (|G(x), y|^2 \wedge |G(x), y|)\nu(dy)
\leq \left( |G(x)|^2 \vee |G(x)| \right) \int_{\mathbb{R}^d} (|y|^2 \wedge |y|)\nu(dy) < +\infty.
\]

In the last line we used (2.4) and (2.4).
In particular, the support of the measure $\nu_G$.

Let us assume to the contrary, that for some $c > 0$ there exists $A \subseteq \mathbb{R}$ such that for some $\varepsilon > 0$:

$$x + \langle G(x), y \rangle < -\varepsilon$$

Consequently, (2.18) follows with

$$a := \left( \beta - \int_{(1, +\infty)} (1 - v)\,\mu(dv) \right), \quad b := \left( \gamma - \int_{(1, +\infty)} (1 - v)\,\nu_G(0)(dv) \right),$$

and $b \geq \int_{(1, +\infty)} (v - 1)\,\nu_G(0)(dv)$ because $\gamma \geq 0$.

Proposition 5.1 Let $G : [0, +\infty) \to \mathbb{R}^d$ be continuous. If the equation (2.1) has a non-negative strong solution for any initial condition $R(0) = x \geq 0$, then

$$\forall x \geq 0 \quad \nu\{y \in \mathbb{R}^d : x + \langle G(x), y \rangle < 0\} = 0.$$  \hfill (5.11)

In particular, the support of the measure $\nu_G(x)(dv)$ is contained in $[-x, +\infty)$.

Proof: Let us assume to the contrary, that for some $x \geq 0$

$$\nu\{y \in \mathbb{R}^d : x + \langle G(x), y \rangle < 0\} > 0.$$

Then there exists $c > 0$ such that

$$\nu\{y \in \mathbb{R}^d : x + \langle G(x), y \rangle < -c\} > 0.$$

Let $A \subseteq \{y \in \mathbb{R}^d : x + \langle G(x), y \rangle < -c\}$ be a Borel set separated from zero. By the continuity of $G$ we have that for some $\varepsilon > 0$:

$$x + \langle G(x), y \rangle < -\varepsilon, \quad x \in [(x - \varepsilon) \lor 0, x + \varepsilon], \quad y \in A.$$  \hfill (5.12)

Let $Z^2$ be a Lévy processes with characteristics $(0, 0, \nu^2(dy))$, where $\nu^2(dy) := 1_A(y)\nu(dy)$ and $Z^1$ be defined by $Z(t) = Z^1(t) + Z^2(t)$. Then $Z^1, Z^2$ are independent and $Z^2$ is a compound Poisson process. Let us consider the following equations

$$dR(t) = F(R(t))\,dt + \langle G(R(t)), dZ(t) \rangle, \quad R(0) = x,$$

$$dR^1(t) = F(R^1(t))\,dt + \langle G(R^1(t)), dZ^1(t) \rangle, \quad R^1(0) = x.$$
For the exit time \(\tau_1\) of \(R^1\) from the set \([(x-\varepsilon) \lor 0, x+\varepsilon]\) and the first jump time \(\tau_2\) of \(Z^2\) we can find \(T > 0\) such that \(\mathbb{P}(\tau_1 > T, \tau_2 < T) = \mathbb{P}(\tau_1 > T)\mathbb{P}(\tau_2 < T) > 0\). On the set \(\{\tau_1 > T, \tau_2 < T\}\) we have \(R(\tau_2-) = R^1(\tau_2-)\) and therefore
\[
R(\tau_2) = R^1(\tau_2-) + (G(R^1(\tau_2-)), \triangle Z^2(\tau_2)) < -\frac{c}{2}.
\]
In the last inequality we used (5.12). This contradicts the positivity of \(R\).

**Proof of Proposition 2.6**: The HJM condition for affine models takes the form
\[
J_Z(B(v)G(x)) = -A'(v) - [B'(v) - 1]x + B(v)F(x), \quad v, x \geq 0,
\]
(5.13)
for details see Proposition 3.2 in [1]. Using (2.18) we obtain
\[
J_Z(B(v)G(x)) = -A'(v) + bB(v) + [aB(v) - B'(v) + 1]x, \quad v, x \geq 0.
\]
(5.14)
Setting \(x = 0\) and using (2.20) yields
\[
-A'(v) + bB(v) = J_Z(B(v)G(0)) = J_{\nu_G(0)}(B(v)),
\]
which is (2.26). It follows from (5.14) that
\[
J_Z(B(v)G(x)) = J_Z(B(v)G(0)) + [aB(v) - B'(v) + 1]x, \quad v, x \geq 0.
\]
(5.15)
Using again (2.20) we obtain
\[
\frac{1}{2}B^2(v)c + J_{\nu_G(0)}(B(v)) + xJ_\mu(B(v)) = J_{\nu_G(0)}(B(v)) + [aB(v) - B'(v) + 1]x, \quad v, x \geq 0.
\]
Consequently,
\[
\frac{1}{2}B^2(v)c + J_\mu(B(v)) = aB(v) - B'(v) + 1, \quad v, x \geq 0,
\]
which finally yields (2.25).

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