A REMARK ON ASYMPTOTIC OF HIGHEST WEIGHTS IN TENSOR POWERS OF A REPRESENTATION

KIUMARS KAVEH

Abstract. We consider the semigroup $S$ of highest weights appearing in tensor powers $V^{\otimes k}$ of a finite dimensional representation $V$ of a connected reductive group. We describe the cone generated by $S$ as the cone over the weight polytope of $V$ intersected with the positive Weyl chamber. From this we get a description for the asymptotic of the number of highest weights appearing in $V^{\otimes k}$ in terms of the volume of this polytope.

Contents

1. Introduction 1
2. Semigroups of integral points and convex bodies 3
3. Main result 4
4. Relation with moment polytope 6
References 7

Key words: Reductive group representation, tensor power, highest weight, semigroup of integral points, Newton-Okounkov body, weight polytope, moment polytope, PRV theorem.

AMS subject classification: 05E10, 20G05.

1. Introduction

Let $G$ be a connected reductive algebraic group over an algebraically closed filed $k$ of characteristic 0, and let $V$ be a finite dimensional $G$-module. We consider the semigroup of dominant weights

$$S = S(V) = \{(k, \lambda) \mid V_\lambda \text{ appears in } V^{\otimes k}\},$$

where $V_\lambda$ is the irreducible representation with highest weight $\lambda$. In this note we describe the cone $C(S)$ of this semigroup, i.e. the smallest closed convex cone (with apex at the origin) containing $S$ (in other words, the closure of the convex hull of $S \cup \{0\}$). We use this to describe the asymptotic of the number of highest weights $\lambda$ appearing in $V^{\otimes k}$. 
This work is in the spirit of the theory of semigroups of integral points and Newton-Okounkov bodies developed in [Kaveh-Khatami09, Part I].

Let $A$ denote the finite set of highest weights in $V$, i.e. the dominant weight $\lambda$ for which $V\lambda$ appears in $V$. Consider the union of all the Weyl group orbits of $\lambda \in A$ and let $P^+(V)$ be its convex hull intersected with the positive Weyl chamber. We show that the slice of the cone $C(S)$ at $k = 1$ coincides with the polytope $P^+(V)$ (Theorem 3.1). Our main tool in the proof will be the PRV theorem on the tensor product of irreducible representations.

Let $H_V(k)$ denote the number of dominant weights $\lambda$ where $V\lambda$ appears in $V^\otimes k$. From general statements about semigroups of integral points we then conclude that $H_V(k)$ grows of degree $q = \dim(P^+(V))$, i.e. the limit

$$a_q = \lim_{k \to \infty} H_V(k)/k^q$$

exists and is non-zero. In addition $a_q$ is equal to the (properly normalized) volume of the polytope $P^+(V)$.

In the last section we discuss the connection between the semigroup $S(V)$, its associated polytope $P^+(V)$ with the moment polytope of $G$-varieties.

At the end, we would like to mention the related paper of Tate and Zelditch [Tate-Zelditch04] in which the authors address the (more difficult) question of describing the asymptotic of multiplicities of irreducible representations appearing in tensor powers of an irreducible representation.

**Acknowledgement:** the author would like to thank Askold Khovanskii and Kevin Purbhoo for helpful discussions, as well as Shrawan Kumar for helpful email correspondence.

**Notation:** Throughout the paper we will use the following notation. $G$ denotes a connected reductive algebraic group over an algebraically closed field $k$ of characteristic 0.

- We fix a Borel subgroup $B$ and a maximal torus $T$ in $G$. $R = R(G, T)$ is the root system with $R^+ = R^+(G, T)$ the subset of positive roots for the choice of $B$. The Weyl group of $(G, T)$ is denoted by $W$. It contains a unique longest element denoted by $w_0$.
- $\Lambda$ denotes the weight lattice of $G$ (that is, the character group of $T$), and $\Lambda^+$ is the subset of dominant weights (for the choice of $B$). Put $\Lambda_R = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$. Then the convex cone generated by $\Lambda^+$ in $\Lambda_R$ is the positive Weyl chamber $\Lambda^+_R$.
- For a weight $\lambda \in \Lambda$, the irreducible $G$-module corresponding to $\lambda$ will be denoted by $V_\lambda$ and a highest weight vector in $V_\lambda$ will be denoted by $v_\lambda$. Finally for a dominant weight $\lambda$, we put $\lambda^* = -w_0(\lambda)$ which is again a dominant weight. One has $V_{\lambda^*} \cong V_\lambda^*$ as $G$-modules.
2. Semigroups of integral points and convex bodies

Let $S \subset \mathbb{Z}_{\geq 0} \times \mathbb{Z}^n$ be a semigroup of integral points (i.e. $S$ is closed under addition).

Let $C(S)$ be the smallest closed convex cone (with apex at the origin) containing $S$. Also let $G(S)$ be the subgroup of $\mathbb{Z}^{n+1}$ generated by $S$ and $L(S)$ the linear subspace of $\mathbb{R}^{n+1}$ spanned by $S$. The sets $C(S)$ and $G(S)$ lie in $L(S)$. To $S$ we associate its \textit{regularization} which is the semigroup $\text{Reg}(S) = C(S) \cap G(S)$. The regularization $\text{Reg}(S)$ is a simpler semigroup with more points which contains the semigroup $S$. In [Kaveh-Khovanskii09, Section 1.1] it is proved that the regularization $\text{Reg}(S)$ asymptotically approximates the semigroup $S$. More precisely:

\textbf{Theorem 2.1} (Approximation theorem). Let $C' \subset C(S)$ be a strongly convex cone which intersects the boundary (in the topology of the linear space $L(S)$) of the cone $C(S)$ only at the origin. Then there exists a constant $N > 0$ (depending on $C'$) such that each point in $G(S) \cap C'$ whose distance from the origin is bigger than $N$ belongs to $S$.

Let $\pi : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ denote the projection on the first factor. We call a semigroup $S \subset \mathbb{Z}_{\geq 0} \times \mathbb{Z}^n$ a \textit{non-negative semigroup} if it is not contained in the hyperplane $\pi^{-1}(0)$. If in addition the cone $C(S)$ intersects the hyperplane $\pi^{-1}(0)$ only at the origin, $S$ is called a \textit{strongly non-negative semigroup}.

Let $S_k = S \cap \pi^{-1}(k)$ be the set of points in $S$ at level $k$. For simplicity throughout this section we assume that $S_1 \neq \emptyset$.

We denote the group $G(S) \cap \pi^{-1}(0)$ by $\Lambda(S)$ and call it the \textit{lattice associated to the non-negative semigroup} $S$. Finally, the number of points in $S_k$ is denoted by $H_S(k)$. $H_S$ is called the \textit{Hilbert function of the semigroup} $S$.

\textbf{Definition 2.2} (Newton-Okounkov convex set). We call the projection of the convex set $C(S) \cap \pi^{-1}(1)$ to $\mathbb{R}^n$ (under the projection on the second factor $(1,x) \mapsto x$), the \textit{Newton-Okounkov convex set of the semigroup} $S$ and denote it by $\Delta(S)$. In other words,

$$\Delta(S) = \text{conv}(\bigcup_{k>0} \{x/k \mid (k,x) \in S_k\}).$$

If $S$ is strongly non-negative then $\Delta(S)$ is compact and hence a convex body.

Let $\Lambda \subset \mathbb{R}^n$ be a lattice of full rank $n$. Let $E \subset \mathbb{R}^n$ be a subspace of dimension $q$ which is rational with respect to $\Lambda$. The \textit{Lebesgue measure normalized with respect to the lattice} $\Lambda$ in $E$ is the Lebesgue measure $d\gamma$ in $E$ normalized such that the smallest measure of a $q$-dimensional parallelepiped with vertices in $E \cap \Lambda$ is equal to 1. The measure of a subset $A \subset E$ will be called its \textit{normalized volume} and denoted by $\text{Vol}_q(A)$ (whenever the lattice $\Lambda$ is clear from the context).

Let $H_S$ and $H_{\text{Reg}(S)}$ be the Hilbert functions of $S$ and its regularization respectively. From Theorem 2.1 it follows that $H_S(k)$ and $H_{\text{Reg}(S)}(k)$ have the same asymptotic as $k$ goes to infinity. Thus the Newton-Okounkov
convex set $\Delta(S)$ is responsible for the asymptotic of the Hilbert function of $S$ ([Kaveh-Khovanskii09 Section 1.4]):

**Theorem 2.3.** The function $H_S(k)$ grows like $a_qk^q$ where $q$ is the dimension of the convex body $\Delta(S)$. This means that the limit

$$a_q = \lim_{k \to \infty} H_S(k)/k^q$$

exists and is non-zero. Moreover, the $q$-th growth coefficient $a_q$ is equal to $\text{Vol}_q(\Delta(S))$, where the volume is normalized with respect to the lattice $\Lambda(S)$.

Finally we prove a proposition which will be used later in proof of the main result (Theorem 3.1).

**Proposition 2.4.** Let $S \subset \mathbb{Z}_{\geq 0} \times \mathbb{Z}^n$ be a non-negative semigroup and $C = C(S)$ the cone associated to $S$. Let $C' \subset C$ be a convex cone of full dimension (centered at the origin) and $S' = S \cap C'$ the subsemigroup consisting of all the points of $S$ contained in $C'$. Then the cone $C(S')$ associated to $S'$ coincides with $C'$.

**Proof.** Clearly $C(S') \subset C'$. By contradiction suppose $C(S')$ is not equal to $C'$. Then there is a convex cone $\tilde{C} \subset C'$ of full dimension which does not intersect $C(S')$ and moreover intersects the boundary of $C'$ (in the topology of the subspace $L(S)$) only at the origin. Since $\tilde{C}$ has full dimension it contains a rational point (with respect to the lattice $\Lambda(S)$) which then implis that it contains a point in $\Lambda(S)$. Now applying Theorem 2.1 we see that the convex cone $\tilde{C}$ should contain a point in $S'$ which contradicts that $C(S') \cap \tilde{C} = \emptyset$. \qed

In the rest of the paper we will deal with semigroups and convex polytopes naturally associated to a reductive group $G$ and its representations.

3. **Main result**

Let $V$ be a finite dimensional $G$-module. Define the set $S(V) \subset \mathbb{Z}_{\geq 0} \times \Lambda^+$ by

$$S(V) = \{(k, \lambda) \mid V_\lambda \text{ appears in } V^\otimes k\}.$$ 

If $v_\lambda$ and $v_\mu$ are highest weight vectors in $V^\otimes k$ and $V^\otimes \ell$ with weights $\lambda$ and $\mu$ respectively, then $v_\lambda \otimes v_\mu$ is a highest weight vector in $V^\otimes k+\ell$ of weight $\lambda + \mu$. It follows that $S(V)$ is a semigroup with respect to addition. Let $\Delta(V)$ denote the Newton-Okounkov body of the semigroup $S(V)$. In other words,

$$\Delta(V) = \text{conv}\left(\bigcup_{k > 0}\{\lambda/k \mid V_\lambda \text{ appears in } V^\otimes k\}\right).$$

Also let $A$ be the collection of $\gamma$ where $V_\gamma$ appears in $V$. Then the weight polytope of $V$ is defined by $P(V) = \text{conv}\{w(\gamma) \mid w \in W, \gamma \in A\}$, i.e. the convex hull of the union of Weyl orbits of $\gamma \in A$. We will denote the intersection of $P(V)$ with the positive Weyl chamber $\Lambda^+_R$ by $P^+(V)$ and call it the positive weight polytope.
Lemma 3.2. Let $\lambda_1, \lambda_2$ be dominant weights and let $V_\gamma$ appear in $V_{\lambda_1} \otimes V_{\lambda_2}$. Then $\gamma = \lambda_1 + \lambda_2 - \sum_{\alpha \in R^+} c_\alpha \alpha$ where $c_\alpha \geq 0$. From this it follows that $\gamma$ belongs to $P^+(\lambda_1 + \lambda_2)$.

Our main tool to prove Theorem 3.1 will be the well-known PRV theorem regarding the tensor product of two irreducible representations. It was conjectured by K. Parthasarathy, R. Ranga Rao and V. Varadarajan in [PRV67]. Later it was proved by S. Kumar in [Kumar88]. We briefly recall its statement.

Theorem 3.3 (PRV theorem). Let $\lambda_1, \lambda_2 \in \Lambda^+$ be two dominant weights. Suppose for two Weyl group elements $w_1, w_2 \in W$ we have $\gamma = w_1(\lambda_1) + w_2(\lambda_2)$ is a dominant weight. Then $V_\gamma$ appears in the decomposition of the tensor product $V_{\lambda_1} \otimes V_{\lambda_2}$ into irreducible representations.

Define the set $\tilde{S}(V) \subset \mathbb{Z}_{\geq 0} \times \Lambda$ by:

$$\tilde{S}(V) = \{(k, w(\lambda)) \mid w \in W, V_\lambda \text{ appear in } V^{\otimes k}\}.$$ 

Roughly, speaking $\tilde{S}(V)$ is the union of $W$-orbits of $\lambda$ for which $V_\lambda$ appears in some tensor power $V^{\otimes k}$. Notice that $S(V) = \tilde{S}(V) \cap (\mathbb{Z}_{\geq 0} \times \Lambda^+)$. The following is a straight forward corollary of Theorem 3.3.

Corollary 3.4. 1) $\tilde{S}(V)$ is a semigroup. 2) The convex body $\Delta(\tilde{S}(V))$ associated to this semigroup coincides with $P(V)$.

Proof. 1) Let $(k, w_1(\lambda)), (\ell, w_2(\lambda_2))$ be two elements in $\tilde{S}(V)$. We can write $w_1(\lambda_1) + w_2(\lambda_2)$ as $w(\lambda)$ for some $\lambda \in \Lambda$, $w \in W$. By Theorem 3.3, $V_\lambda$ appears in $V_{\lambda_1} \otimes V_{\lambda_2}$ and hence it appears in $V^{\otimes \ell + k}$. This shows that $(\ell + k, w_1(\lambda_1) + w_2(\lambda_2)) = (\ell + k, w(\lambda))$ belongs to $\tilde{S}(V)$ which proves 1). 2) By Lemma 3.2 for any integer $k > 0$, the convex hull of $\tilde{S}(V)_k$ is $kP(V)$ and hence $\Delta(\tilde{S}(V))$ coincides with $P(V)$. □

Proof of Theorem 3.1. From Proposition 2.4 the convex body $\Delta(V)$ associated to the semigroup $S(V) \subset \tilde{S}(V)$ is just the intersection of $\Delta(\tilde{S}(V))$ with $\Lambda^+_R$. By Corollary 3.4(2) we know that $\Delta(\tilde{S}(V))$ is the weight polytope $P(V)$ which finishes the proof. □

Corollary 3.5. Let $H_V(k)$ be the number of $\lambda$ such that $V_\lambda$ appears in $V^{\otimes k}$. Then $H_V(k)$ grows of degree $q = \dim P^+(V)$. That is, the limit

$$a_q = \lim_{k \to \infty} \frac{H_V(k)}{k^q}$$

exists and is non-zero. Moreover, $a_q$ is equal to $\text{Vol}_q(P^+(V))$, where volume is the Lebesgue measure in $\Lambda_R^+$ normalized with respect to the lattice $\Lambda(S(V)) \subset \Lambda$.

Proof. Follows directly from Theorem 3.1 and Theorem 2.8. □
4. Relation with moment polytope

In this section we see how the positive weight polytope $P^+(V)$ appears as a moment polytope for the action of $G \times G$ on $G$.

Let $V$ be a finite dimensional $G$-module, and $X \subset \mathbb{P}(V)$ an irreducible closed $G$-invariant subvariety. Let $R = \bigoplus_{k \geq 0} R_k$ denote the homogeneous coordinate ring of $X$. It is a graded $G$-algebra. Following Brion, one defines the moment polytope $\Delta(X)$ to be:

$$\Delta(X) = \text{conv}\left(\bigcup_{k>0} \{\lambda/k \mid V_\lambda \text{ appears in } R_k\}\right).$$

One shows that $\Delta(X) \subset \Lambda^+_R \times \Lambda^+_R$ is a polytope (see [Brion87]). Moreover, when $k = \mathbb{C}$ and $X$ is smooth, the polytope $\Delta(X)$ can be identified with the moment polytope of $X$ regarded as a Hamiltonian space for the action of a maximal compact subgroup $K$ of $G$ and the symplectic structure induced from the projective space (see for example [Guillemin-Sternberg84]).

Let $G \times G$ act on $G$ via multiplication from left and right. Let $k[G]$ denote the algebra of regular functions on the variety $G$. It is a rational $G \times G$-module. It is well-known that for each dominant weight $\lambda$, the $(\lambda^*, \lambda)$-isotypic component $k[G]_{(\lambda^*, \lambda)}$ is isomorphic to $V_{\lambda^*} \otimes V_\lambda$. Moreover, any isotypic component of $k[G]$ is of this form for some $\lambda$. In fact any $G \times G$-isotypic component $k[G]_{(\lambda^*, \lambda)}$ in $k[G]$ is the linear span of the matrix entries corresponding to the representation of $G$ in $V_\lambda$ (see [Kraft85]).

Now let $\pi : G \to \text{GL}(V)$ be a finite dimensional representation. Then $\text{End}(V)$ is naturally a $G \times G$-module where $G \times G$ acts via $\pi$ by multiplication from left and right. Let $\tilde{\pi} : G \to \mathbb{P}(\text{End}(V))$ be the induced map to projective space and let $X$ be the closure of the image of $G$ in $\mathbb{P}(\text{End}(V))$. It is a $G \times G$-invariant closed irreducible subvariety.

From (1) one can see the following:

**Proposition 4.1.** Let $R = \bigoplus_k R_k$ denote the homogeneous coordinate ring of $X$ in $\mathbb{P}(\text{End}(V))$. Then for $k > 0$ we have: $V_{(\lambda^*, \lambda)}$ appears in $R_k$ if and only if $V_\lambda$ appears in $V^{\otimes k}$. It follows that, under the projection on the second factor, $\Delta(X) \subset \Lambda^+_R \times \Lambda^+_R$ identifies with $P^+(V)$.
Remark 4.2. The relation between the moment polytope of $X$ (i.e. a group compactification) and the weight polytope $P^+(V)$ has also been shown in [Kazarnovskii87] using methods from symplectic geometry.

REFERENCES

[Brion87] Brion, M. Sur l'image de l'application moment. Sminaire d'algbre Paul Dubreil et Marie-Paule Malliavin (Paris, 1986), 177–192, Lecture Notes in Math., 1296, Springer, Berlin, 1987.

[Guillemin-Sternberg84] Guillemin, V.; Sternberg, S. Geometric quantization and multiplicities of group representations. Invent. Math. 77 (1984), 533–546.

[Kaveh-Khovanskii09] Kaveh, K.; Khovanskii, A. G. Newton-Okounkov convex bodies, semigroups of integral points, graded algebras and intersection theory. Preprint: arXiv:0904.3350v1

[Kazarnovskii87] Kazarnovskii, B. Newton polyhedra and the Bezout formula for matrix-valued functions of finite dimensional representations. Functional Analysis and its applications, v. 21, no. 4, 73–74 (1987).

[Kraft85] Kraft, H. Geometrische Methoden in der Invariantentheorie. (Geometrical methods in invariant theory). Aspects of Mathematics, D1. Friedr. Vieweg & Sohn, Braunschweig, 1984.

[Kumar88] Kumar, S. Proof of the Parthasarathy-Ranga Rao-Varadarajan conjecture. Invent. Math. 93 (1988), no. 1, 117–130.

[PRV67] Parthasarathy, K. R.; Ranga Rao, R.; Varadarajan, V. S. Representations of complex semi-simple Lie groups and Lie algebras. Ann. of Math. (2) 85 (1967) 383–429.

[Tate-Zelditch04] Tate, T.; Zelditch, S. Lattice path combinatorics and asymptotics of multiplicities of weights in tensor powers. J. Funct. Anal. 217 (2004), no. 2, 402–447.