Central extensions and Riemann-Roch theorem on algebraic surfaces *

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Abstract

We study canonical central extensions of the general linear group over the ring of adeles on a smooth projective algebraic surface $X$ by means of the group of integers. By these central extensions and adelic transition matrices of a rank $n$ locally free sheaf of $\mathcal{O}_X$-modules we obtain the local (adelic) decomposition for the difference of Euler characteristics of this sheaf and the sheaf $\mathcal{O}_X^n$. Two various calculations of this difference lead to the Riemann-Roch theorem on $X$ (without the Noether formula).

1 Introduction

This paper is about locally free sheaves, adeles and the Riemann-Roch theorem on algebraic surfaces.

But first, we briefly recall the well-known case of algebraic curves.

Recall that two vector subspaces $A$ and $B$ in a vector space $V$ over a field $k$ are commensurable (see [20]), i.e. $A \sim B$, iff

$$\dim_k (A + B)/(A \cap B) < \infty.$$ 

For such $A$ and $B$ we denote their relative dimension

$$[A \mid B] = \dim_k B/(A \cap B) - \dim_k A/(A \cap B).$$

We fix a $k$-vector subspace $K$ in $V$ and any $k$-vector subspaces $D_1$ and $D_2$ in $V$ such that $D_1 \sim D_2$ and

$$H^0(D_i) = D_i \cap K \quad H^1(D_i) = V/(D_i + K)$$

are finite-dimensional $k$-vector spaces for $i \in \{1, 2\}$. We denote for $i \in \{1, 2\}$

$$\chi(D_i) = \dim_k H^0(D_i) - \dim_k H^1(D_i).$$

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Then we have (see, e.g., [6, § 14.14, Exer. 14.59–14.61]) “the abstract Riemann-Roch theorem”:

$$\chi(D_1) - \chi(D_2) = [D_2 \mid D_1].$$  \hspace{1cm} (1)

Let $n \geq 1$ be an integer.

For a smooth projective curve $S$ over $k$ we consider

$$V = A^n_S,$$

where $A_S = \prod_{p \in S} K_p$ is the space of adeles of the curve $S$, and $K_p$ is the field of fraction of the completion $\mathcal{O}_p$ of the local ring $\mathcal{O}_p$ at a (closed) point $p \in S$. We consider $K = k(S)^n$, where $k(S)$ is the field of rational functions on $S$.

Now we consider a locally free sheaf $\mathcal{F}$ of $\mathcal{O}_S$-modules of rank $n$ on $S$. The stalk of $\mathcal{F}$ at the generic point $\text{Spec} k(S)$ of $S$ is a $k(S)$-vector space. We fix a basis $e_0$ of this vector space. For any (closed) point $p \in S$, the completion of the stalk $\mathcal{F}_p$ of $\mathcal{F}$ at $p$ is a free $\mathcal{O}_p$-module. We fix a basis $e_p$ of this module. For all (closed) points $p \in S$, we consider the transition matrices $\gamma_{01,p} \in GL_n(K_p)$ defined by the equality $e_0 = \gamma_{01,p} e_p$, which is calculated in the $K_p$-vector space $\mathcal{F}_p \otimes_{\mathcal{O}_p} K_p$. The element given by the collection of matrices

$$\gamma_{01,\mathcal{F}} = \prod_{p \in S} \gamma_{01,p} \in GL_n\left(\prod_{p \in S} K_p\right),$$

belongs to the subgroup $GL_n(A_S)$.

We note that the chosen and fixed basis $e_0$ gives an embedding of $\mathcal{F}$ to a constant sheaf $K$ on $S$. Therefore we can associate by $\mathcal{F}$ a $k$-vector subspace $D_F$ in $V$ explicitly given as

$$D_F = \gamma_{01,\mathcal{F}} D,$$

where $D = (\prod_{p \in S} \mathcal{O}_p)^n$ is a $k$-vector subspace of $V$.

We note that $D_{\mathcal{F}} \sim D$.

Now, applying formula (1) for the $k$-vector subspaces $D_{\mathcal{F}}$ and $D$ and using the adelic complex for $\mathcal{F}$ on $S$ that gives

$$H^i(D_{\mathcal{F}}) = H^i(S, \mathcal{F}) \quad \text{and} \quad H^i(D) = H^i(S, \mathcal{O}_S^n)$$

for $i \in \{0, 1\}$, we obtain the Riemann-Roch theorem for $\mathcal{F}$ on $S$:

$$\chi(\mathcal{F}) - n \chi(\mathcal{O}_S) = c_1(\mathcal{F}),$$  \hspace{1cm} (2)

where $\chi(\mathcal{F})$ is the Euler characteristic of the sheaf $\mathcal{F}$ on $S$. And the first Chern number $c_1(\mathcal{F})$ is obtained from the homomorphism of groups:

$$\text{deg} : GL_n(A_S) \rightarrow \mathbb{Z}, \quad a \mapsto [aD \mid D]$$  \hspace{1cm} (3)

such that $c_1(\mathcal{F}) = \text{deg}(\gamma_{01,\mathcal{F}})$.
This can be called the *local (or adelic) decomposition* for the difference of Euler characteristics of the sheaf $\mathcal{F}$ and the sheaf $\mathcal{O}_S^2$.

Now let $X$ be a smooth projective algebraic surface over $k$. Then there is the Parshin-Beilinson adelic ring $\mathbb{A}_X$ of $X$ (see [14] and Section 2 below). We have

$$\mathbb{A}_X = \prod_{x \in C} K_{x,C} \subset \prod_{x \in C} K_{x,C},$$

where the (“two-dimensional”) adelic product is taken over all pairs $x \in C$, where $C$ is an irreducible curve on $X$, and $x$ is a point on $C$, and an Artinian ring $K_{x,C}$ is a finite direct product of two-dimensional local fields such that this product consists of one field provided that $x$ is smooth on $C$. (Here a point $x$ is a usual closed point.) Every two-dimensional local field which appear here is isomorphic to a field of iterated Laurent series $k((u))((t))$, where a field $k'$ is a finite extension of $k$.

Besides, instead of the homomorphism (3) we have now a canonical central extension (see more in Section 2.1 below):

$$0 \rightarrow \mathbb{Z} \rightarrow \hat{GL}_n(\mathbb{A}_X) \rightarrow GL_n(\mathbb{A}_X) \rightarrow 1.$$

The goal of this paper is to connect this central extension with the Riemann-Roch theorem for a locally free sheaf of $\mathcal{O}_X$-modules of rank $n$ on $X$ using the transition matrices for this sheaf, where these transition matrices are obtained from the bases of the completions of the stalks of the sheaf at scheme points of $X$.

From this central extension one obtains canonically another central extension $\hat{GL}_n(\mathbb{A}_X)$ (see Section 2.2) of $GL_n(\mathbb{A}_X)$ by $\mathbb{Z}$ such that by transition matrices $\alpha_{ij} \in GL_n(\mathbb{A}_X)$ ($i \neq j$ are from $\{0, 1, 2\}$) for a rank $n$ locally free sheaf of $\mathcal{O}_X$-modules on $X$ and by the central extension $\hat{GL}_n(\mathbb{A}_X)$ it is possible to obtain the second Chern number $c_2$ of this sheaf. This was done in [14], see also Remark 9. Besides, transition matrices $\alpha_{ij}$ are analogs for a locally free sheaf of $\mathcal{O}_X$-modules on the surface $X$ of a transition matrix $\gamma_{01}$ for a locally free sheaf of $\mathcal{O}_S$-modules on the curve $S$, see the reasoning above. (We have omitted here the indication on the sheaf in the notation for transition matrices.)

To solve the above tasks we use canonical lifts of transition matrices $\alpha_{ij}$ of a sheaf from the group $GL_n(\mathbb{A}_X)$ to the groups $\hat{GL}_n(\mathbb{A}_X)$. Thus we obtain the *local (or adelic) decomposition* for the differences of Euler characteristics of a rank $n$ locally free sheaf of $\mathcal{O}_X$-modules and the sheaf $\mathcal{O}_n^X$, see Remark 10.

We can say also that the main ingredient of this paper is the calculation of the integer $\tilde{\alpha}_{02} \cdot \tilde{\alpha}_{21} \cdot \tilde{\alpha}_{10}$, where $\tilde{\alpha}_{ij}$ is the canonical lift of $\alpha_{ij}$ from $GL_n(\mathbb{A}_X)$ to $\hat{GL}_n(\mathbb{A}_X)$. This integer does not depend on the choice of transition matrices $\alpha_{ij}$.

We do this calculation in two ways. The first way leads to Theorem 1 and uses adelic complexes for rank $n$ locally free sheaves of $\mathcal{O}_X$-modules on $X$, see Proposition 1. For the second way we suppose that a basic field $k$ is perfect, and we use the “self-duality” of the adelic space $\mathbb{A}_X$ based on reciprocity laws on $X$ for residues of differential two-forms on two-dimensional local fields introduced and studied in [14]. This another way leads in Theorem 2 to an answer, which uses also another invariants of a sheaf and $X$. 3
The comparison of these two answers (after Theorems 1 and 2) gives the Riemann-Roch theorem for a rank \( n \) locally free sheaf of \( \mathcal{O}_X \)-modules on \( X \) (without the Noether formula).

We note that the relation of the Riemann-Roch theorem on an algebraic surface with the local constructions was also discussed in [2, 4, 17], but without using an adelic ring on a surface.

The paper is organized as follows.

In Section 2.1 we very briefly recall on adeles on algebraic surfaces and also recall the construction of the central extension \( \widetilde{GL}_n(\mathbb{A}_X) \) of the group \( GL_n(\mathbb{A}_X) \) by the group \( \mathbb{Z} \).

In Section 2.2 we recall the construction of the central extension \( \hat{GL}_n(\mathbb{A}_X) \) of the group \( GL_n(\mathbb{A}_X) \) by the group \( \mathbb{Z} \). This central extension is obtained from the central extension \( GL_n(\mathbb{A}_X) \). In Remark 2 we compare these central extensions from the point of view of group cohomology.

In Section 3 we construct special elements in \( \mathbb{Z} \)-torsors. These elements are related to the special \( k \)-vector subspaces and subrings in \( \mathbb{A}_X \). The subspaces and subrings are connected with points and irreducible curves on \( X \).

In Section 4 we discuss various formulas (formula (16) and Proposition 2) for the intersection index of divisors on \( X \) related with special elements constructed in Section 3 and with the central extension \( GL_1(\mathbb{A}_X) \).

In Section 5 we construct canonical splittings of the central extension \( \widetilde{GL}_n(\mathbb{A}_X) \) over special subgroups of the group \( GL_n(\mathbb{A}_X) \). We discuss the properties of these splittings. In Remark 7 we recall the corresponding splittings and their properties for the central extension \( GL_1(\mathbb{A}_X) \) from [11].

In Section 6 we introduce the transition matrices \( \alpha_{ij} \in GL_n(\mathbb{A}_X) \) for a rank \( n \) locally free sheaf \( \mathcal{E} \) of \( \mathcal{O}_X \)-modules on \( X \) and canonical lifts \( \tilde{\alpha}_{ij} \) of \( \alpha_{ij} \) to \( GL_n(\mathbb{A}_X) \) by means of canonical splittings from Section 5.

In Section 7 we calculate the integer

\[
f_{\mathcal{E}} = \tilde{\alpha}_{02} \cdot \tilde{\alpha}_{21} \cdot \tilde{\alpha}_{10} = (\chi(\mathcal{E}) - n\chi(\mathcal{O}_X)) - 2\text{ch}_2(\mathcal{E})
\]

in the first way, where the rational number \( \text{ch}_2(\mathcal{E}) = \frac{1}{2}c_1(\mathcal{E})^2 - c_2(\mathcal{E}) \).

In Section 8 when the field \( k \) is perfect we calculate the integer

\[
f_{\mathcal{E}} = -\frac{1}{2}K \cdot c_1(\mathcal{E}) - \text{ch}_2(\mathcal{E})
\]

in the second way, where \( K \simeq \mathcal{O}_X(\omega), \ \omega \in \Omega^2_{k(X)/k}, \ \omega \neq 0 \). We derive the Riemann-Roch theorem for \( \mathcal{E} \).

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2 Constructions of central extensions of $GL_n(\mathbb{A}_X)$

2.1 Adelic ring on a surface and a first central extension

As we have already mentioned in § 1 (Introduction), throughout the article, $X$ is a smooth projective algebraic surface over a field $k$. But for the constructions in § 2, it is not important that $X$ is projective.

We recall (see, e.g., survey [8] and also [11, § 2.1]) that the adelic ring of $X$ is

$$A_X = A_X = \prod_{x \in C} K_{x,C} \subset \prod_{x \in C} K_{x,C}$$

and every ring $K_{x,C} = \prod_i K_i$ is the finite direct product of two-dimensional local fields $K_i$ such that every two-dimensional local field $K_i$ corresponds to the formal branch of $C$ at $x$.

We define a subring $A_{12} \subset A_X$:

$$A_{12} = A_X \cap \prod_{x \in C} \mathcal{O}_{K_{x,C}} \subset \prod_{x \in C} K_{x,C},$$

where $\mathcal{O}_{K_{x,C}} = \prod_i \mathcal{O}_{K_i}$ is the finite direct product of discrete valuation rings $\mathcal{O}_{K_i}$ of $K_i$. (If $K_i$ is isomorphic to $k_i((u))(t))$, then $\mathcal{O}_{K_i}$ is isomorphic to $k_i((u))[[t]]$.)

For any locally linearly compact $k$-vector space (or, in other words, a Tate vector space) $U$ we recall canonical construction of $\mathbb{Z}$-torsor $\mathrm{Dim}(U)$ from [7]. As a set, $\mathrm{Dim}(U)$ consists of all maps $d$ (which are called “dimension theories” in [2]) from the set of all open linearly compact $k$-vector subspaces of $U$ to $\mathbb{Z}$ with the property

$$d(Z_2) = d(Z_1) + [Z_1 \mid Z_2],$$

where $Z_1, Z_2$ are any open linearly compact $k$-vector subspaces of $U$. The group $\mathbb{Z}$ acts on $\mathrm{Dim}(U)$ in the following way:

$$(m + d)(Z_1) = d(Z_1) + m, \quad \text{where} \quad m \in \mathbb{Z}.$$

For any exact sequence of locally linearly compact $k$-vector spaces

$$0 \longrightarrow U_1 \overset{\phi_1}{\longrightarrow} U_2 \overset{\phi_2}{\longrightarrow} U_3 \longrightarrow 0,$$

where $\phi_1, \phi_2$ are continuous maps and $\phi_1$ is a closed embedding, we have a canonical isomorphism:

$$\mathrm{Dim}(U_1) \otimes_\mathbb{Z} \mathrm{Dim}(U_3) \longrightarrow \mathrm{Dim}(U_2),$$

$$d_1 \otimes d_3 \longmapsto d_2, \quad d_2(Z) = d_1(Z \cap U_1) + d_3(\phi_2(U_1)),$$

where $Z$ is an open linearly compact $k$-vector subspace of $U_2$. 
For any locally linearly compact \( k \)-vector space \( U \) we define the locally linearly compact \( k \)-vector space \( \hat{U} \) as a \( k \)-vector subspace of the dual \( k \)-vector space \( U^* \) in the following way:

\[
\hat{U} = \bigcup_W W^\perp,
\]

where \( W \) runs over all open linearly compact \( k \)-vector subspaces of \( U \), the \( k \)-vector subspace \( W^\perp \subset U^* \) is the annihilator of \( W \) in \( U^* \), and \( W^\perp \), which is the dual vector space to the discrete vector space \( U/W \), is an open linearly compact \( k \)-vector subspace of \( \hat{U} \). In other words, \( \hat{U} \) is the continuous dual space, i.e., it consists of all continuous linear functionals. We have a canonical isomorphism:

\[
\dim(U) \otimes \mathbb{Z} \dim(\hat{U}) \simeq \mathbb{Z}, \quad d_1 \otimes d_2 \mapsto d_1(Z) + d_2(Z^\perp), \tag{6}
\]

where \( Z \subset U \) is a linearly compact \( k \)-vector subspace, and the result does not depend on the choice of \( Z \).

Let \( D = \sum_i a_i C_i \) be any divisor on \( X \), where \( C_i \) are irreducible curves on \( X \). We denote

\[
\mathbb{A}_{12}(D) = \mathbb{A}_X \cap \prod_{x \in C} t_C^{-\nu_C(D)} \mathcal{O}_{K_x,c},
\]

where the intersection is taken inside \( \prod_{x \in C} K_{x,c} \), and \( t_C = 0 \) is an equation of an (irreducible) curve \( C \) in some open subset of \( X \), \( \nu_C(D) \) equals \( a_i \) when \( C = C_i \) and zero otherwise. (The definition of \( \mathbb{A}_{12}(D) \) does not depend on the choice of \( t_C \).)

We note that

\[
\mathbb{A}_X = \lim_{\rightarrow} \lim_{D_1 \geq D_2} \mathbb{A}_{12}(D_2)/\mathbb{A}_{12}(D_1),
\]

and \( \mathbb{A}_{12}(D_2)/\mathbb{A}_{12}(D_1) \) is a locally linearly compact \( k \)-vector space, and for any divisors \( D_1 \geq D_2 \geq D_3 \) on \( X \) the exact sequence

\[
0 \rightarrow \mathbb{A}_{12}(D_2)/\mathbb{A}_{12}(D_1) \rightarrow \mathbb{A}_{12}(D_3)/\mathbb{A}_{12}(D_1) \rightarrow \mathbb{A}_{12}(D_3)/\mathbb{A}_{12}(D_2) \rightarrow 0
\]

is of type \([4]\), see, e.g., \([3]\) § 2.2.3, \([11]\) § 2.3.

Let \( n \geq 1 \) be an integer.

**Definition 1.** A \( k \)-vector subspace \( E \) of \( \mathbb{A}^n_X \) is called a lattice if and only if there are divisors \( D_1 \) and \( D_2 \) on \( X \) such that

\[
\mathbb{A}_{12}(D_1)^n \subset E \subset \mathbb{A}_{12}(D_2)^n
\]

and the image of \( E \) in \( \mathbb{A}_{12}(D_2)^n/\mathbb{A}_{12}(D_1)^n \) is a closed \( k \)-vector subspace.

If \( E_1 \subset E_2 \) are lattices, then \( E_2/E_1 \) is a locally linearly compact \( k \)-vector space with the quotient and induced topology from the locally linearly compact \( k \)-vector space \( \mathbb{A}_{12}(D_2)^n/\mathbb{A}_{12}(D_1)^n \). In this case we define a \( \mathbb{Z} \)-torsor

\[
\dim(E_1 \mid E_2) = \dim(E_2/E_1).
\]

6
If \( E_1 \subset E_2 \subset E_3 \) are lattices, then an exact sequence
\[
0 \rightarrow E_2/E_1 \rightarrow E_3/E_1 \rightarrow E_3/E_2 \rightarrow 0
\]
is of type (4).

Now for arbitrary lattices \( E_1 \) and \( E_2 \) we define a \( \mathbb{Z} \)-torsor
\[
\operatorname{Dim}(E_1 | E_2) = \lim_{\rightarrow E} \operatorname{Hom}_\mathbb{Z}(\operatorname{Dim}(E_1/E), \operatorname{Dim}(E_2/E)),
\]
where the direct limit is taken over all lattices \( E \subset \mathbb{A}_X^n \) such that \( E \subset E_i \) for \( i = 1 \) and \( i = 2 \). Here we use the following isomorphisms of \( \mathbb{Z} \)-torsors for lattices \( E \supset E' \) and \( i = 1 \), \( i = 2 \):
\[
\operatorname{Dim}(E_i/E) \otimes \operatorname{Dim}(E/E') \rightarrow \operatorname{Dim}(E_i/E'),
\]
so that the transition maps in this direct limit are given as
\[
f \mapsto f', \quad f'(a \otimes c) = f'(a) \otimes c.
\]

Obviously, for any lattices \( E_1, E_2, E_3 \) we have canonical isomorphism of \( \mathbb{Z} \)-torsors
\[
\operatorname{Dim}(E_1 | E_2) \otimes \operatorname{Dim}(E_2 | E_3) \rightarrow \operatorname{Dim}(E_1 | E_3)
\]
which satisfies the associativity diagram for four lattices.

It is not difficult to see that for any \( g \in \text{GL}_n(\mathbb{A}_X) \) and any lattice \( E \), the \( k \)-vector subspace \( gE \) is again a lattice. For any lattices \( E_1, E_2 \) we have an obvious isomorphism
\[
\operatorname{Dim}(E_1 | E_2) \rightarrow \operatorname{Dim}(gE_1 | gE_2), \quad d \mapsto g(d).
\]

This all leads to the construction of the central extension
\[
0 \rightarrow \mathbb{Z} \rightarrow \widehat{\text{GL}_n(\mathbb{A}_X)} \xrightarrow{\Theta} \text{GL}_n(\mathbb{A}_X) \rightarrow 1,
\]
where \( \widehat{\text{GL}_n(\mathbb{A}_X)} \) consists of all pairs \( (g, d) \), where \( g \in \text{GL}_n(\mathbb{A}_X) \) and \( d \in \operatorname{Dim}(\mathbb{A}^n_{12} | g\mathbb{A}^n_{12}) \).

The group operation and the map \( \Theta \) are the following
\[
(g_1, d_1)(g_2, d_2) = (g_1g_2, d_1 \otimes g_1(d_2)), \quad \Theta((g, d)) = g.
\]

**Remark 1.** The above construction of the central extension \( \widehat{\text{GL}_n(\mathbb{A}_X)} \) is a special case of a more general construction. Let \( B \) be a \( C_2 \)-space over \( k \) (or, with a little bit more restrictions, a \( 2 \)-Tate vector space over \( k \)), see [9]. Then there is a canonical central extension (see [12] § 5.5, Remark 15])
\[
0 \rightarrow \mathbb{Z} \rightarrow \widehat{\text{Aut}_{C_2}(B)}_E \rightarrow \text{Aut}_{C_2}(B) \rightarrow 1,
\]
which depends on the choice of a lattice \( E \subset B \), and where \( \text{Aut}_{C_2}(B) \) is the automorphism group of \( B \) as an object of the category of \( C_2 \)-spaces. Now, \( \mathbb{A}_X^n \) is a \( C_2 \)-space over \( k \) and there is an embedding of groups \( \text{GL}_n(\mathbb{A}_X) \subset \text{Aut}_{C_2}(\mathbb{A}_X^n) \), see [9]. Then the central extension \( \widehat{\text{GL}_n(\mathbb{A}_X)} \) is the restriction of the central extension \( \widehat{\text{Aut}_{C_2}(B)}_E \) under the last embedding of groups when \( B \) is \( \mathbb{A}_X^n \) and \( E \) is \( \mathbb{A}_{12}^n \).
2.2 A second central extension

From central extension (8) we will obtain another central extension (see also more in [11, § 3.1]). We note that

\[ GL_n(\mathbb{A}_X) = SL_n(\mathbb{A}_X) \rtimes \mathbb{A}_X^*, \]

where the group of invertible elements \( \mathbb{A}_X^* \) of the ring \( \mathbb{A}_X \) acts on \( SL_n(\mathbb{A}_X) \) by conjugations, i.e. by inner automorphisms \( h \mapsto aha^{-1} \), where \( h \in SL_n(\mathbb{A}_X) \) and we embed \( \mathbb{A}_X^* \) into \( GL_n(\mathbb{A}_X) \) as \( a \mapsto \text{diag}(a,1,\ldots,1) \).

Now we define \( \hat{GL_n(\mathbb{A}_X)} = \Theta^{-1}(SL_n(\mathbb{A}_X)) \rtimes \mathbb{A}_X^* \),

where \( \mathbb{A}_X^* \) acts on \( \Theta^{-1}(SL_n(\mathbb{A}_X)) \) by inner automorphisms in the group \( \tilde{GL_n(\mathbb{A}_X)} \) via lifting of elements from \( \mathbb{A}_X^* \) to \( \tilde{GL_n(\mathbb{A}_X)} \), i.e. \( a(g) = a'ga'^{-1} \), where \( g \in \Theta^{-1}(SL_n(\mathbb{A}_X)) \) and \( a' \in \tilde{GL_n(\mathbb{A}_X)} \) is any element such that \( \Theta(a') = \text{diag}(a,1,\ldots,1) \).

Clearly, we obtain the central extension:

\[ 0 \longrightarrow \mathbb{Z} \longrightarrow \hat{GL_n(\mathbb{A}_X)} \longrightarrow GL_n(\mathbb{A}_X) \longrightarrow 1. \quad (10) \]

This central extension restricted to the subgroup \( SL_n(\mathbb{A}_X) \subset GL_n(\mathbb{A}_X) \) coincides with the central extension \( \tilde{GL_n(\mathbb{A}_X)} \) restricted to this subgroup. Besides, by construction, central extension (10) canonically splits over the subgroup \( \mathbb{A}_X^* \).

Remark 2. We explain what the transition from central extension (8) to central extension (10) means in terms of group cohomology.

Recall (see [3, 1.7. Construction]) that a central extension \( \hat{C} \) of a group \( C = G \rtimes H \) by a group \( A \) is equivalent to the following data:

1) a central extension of \( H \) by \( A \);
2) a central extension \( 1 \longrightarrow A \longrightarrow \hat{G} \longrightarrow G \longrightarrow 1; \)
3) an action of the group \( H \) on the group \( \hat{G} \) such that this action lifts the action of \( H \) on \( G \) and is trivial on \( A \).

And an isomorphism of central extensions corresponds to an isomorphism of data.

Using this description and that \( H \) is a subgroup in \( C \) and \( H \) is a quotient group of \( C \), we obtain that

\[ H^2(C, A) = H^2(H, A) \oplus T, \quad (11) \]

and that there is an exact sequence

\[ 0 \longrightarrow H^1(H, \text{Hom}(G, A)) \longrightarrow T \xrightarrow{\psi} H^2(G, A)^H \xrightarrow{\phi} H^2(H, \text{Hom}(G, A)), \quad (12) \]

which we will now explain. (Here the actions of \( H \) on \( \text{Hom}(G, A) \) and on \( H^2(G, A) \) are obtained in the usual way from the action on \( G \).)
We will use that for any central extension
\[ 1 \to A \to \hat{G} \to G \to 1 \] (13)
the group \( \text{Hom}(G, A) \) is canonically isomorphic to the automorphism group of central extension (13), i.e. it is isomorphic to the group of automorphisms of the group \( \hat{G} \) such that these automorphisms induce identically action on the subgroup \( A \) and on the quotient group \( G \).

We consider any central extension \( I \) from \( H^2(G, A)^H \). We have a group \( L(I) \) which consists of lifts of the actions of elements of \( H \) to actions on the group corresponding to the central extension \( I \), with the trivial action on the subgroup \( A \). The group \( L(I) \) is an extension (in general, non-central) of the group \( H \) by the automorphism group \( \text{Hom}(G, A) \) of the central extension \( I \). The isomorphism class of this non-central extension is \( \phi(I) \). If \( \phi(I) = 0 \), then the action of the whole group \( H \) can be lifted to an action on the group corresponding to the central extension \( I \), with the trivial action on the subgroup \( A \).

An element \( t \) from \( T \) is an isomorphism class of the following data: a central extension \( \psi(t) \) together with a lift of the action of the group \( H \) to an action on the group corresponding to this central extension, with the trivial action on the subgroup \( A \).

Correspondingly, for any central extension \( I \) from \( \text{Ker} \phi \) we have that \( \psi^{-1}(I) \) is the set of equivalence classes of group sections \( s \) of the natural homomorphism \( L(I) \to H \). Group sections \( s \) and \( s' \) are equivalent if \( s = \sigma_a s' \), where \( \sigma_a \) is an inner automorphism of the group \( L(I) \), given by an element \( a \) from the automorphism group \( \text{Hom}(G, A) \) of the central extension \( I \). Hence we obtain that \( \psi^{-1}(I) \) is an \( H^1(H, \text{Hom}(G, A)) \)-torsor.

We note that another, computational construction and proof of exact sequence (12) and its continuation to the right was given in [19].

Now the transition from central extension (8) to central extension (10) is the projection from \( H^2(C, A) \) to \( T \) in (11).

We note that if a group \( G \) is perfect, i.e. \([G, G] = G\), then \( \text{Hom}(G, A) = 0 \). Therefore, in this case we obtain from (11)-(12) that
\[ H^2(C, A) = H^2(H, A) \oplus H^2(G, A)^H. \]

This condition is satisfied, for example, when \( C = GL_n(F), H = F^*, G = SL_n(F) \), where \( F \) is an infinite field (for example, \( F \) is an \( n \)-dimensional local field). Then we have (cf. [10 § 2.2])
\[ H^2(SL_n(F), A)^{F^*} = \text{Hom}(H_2(SL_n(F), \mathbb{Z})_{F^*}, A) = \text{Hom}(K_2(F), A). \]

### 3 Specially constructed elements of \( \mathbb{Z} \)-torsors

For an irreducible curve \( C \) on \( X \) let a field \( K_C \) be the completion of the field \( k(X) \) of rational functions on \( X \) with respect to the discrete valuation given by \( C \).
For a point \( x \) of \( X \) let \( K_x = k(X) \cdot \hat{O}_{x,X} \) be a subring of the fraction field \( \text{Frac} \hat{O}_{x,X} \), where \( \hat{O}_{x,X} \) is the completion of the local ring of \( x \) on \( X \).

We have natural diagonal embeddings:

\[
\prod_{C \subset X} K_C \hookrightarrow \prod_{x \in C} K_{x,C} \quad \text{and} \quad \prod_{x \in X} K_x \hookrightarrow \prod_{x \in C} K_{x,C}.
\] (14)

There are the following subrings of the adelic ring \( \mathbb{A}_X \):

\[
\mathbb{A}_{01} = \left( \prod_{C \subset X} K_C \right) \cap \mathbb{A}_X \quad \text{and} \quad \mathbb{A}_{02} = \left( \prod_{x \in X} K_x \right) \cap \mathbb{A}_X,
\]

where the intersection is taken inside the ring \( \prod_{x \in C} K_{x,C} \).

Let \( n \geq 1 \) be an integer.

For lattices \( E_1 \subset E_2 \) of \( \mathbb{A}_X^n \) (see Definition 1) we define a \( k \)-vector subspace

\[
\tilde{\mu}_{E_1,E_2} = (E_2 \cap \mathbb{A}_0^n_{02})/(E_1 \cap \mathbb{A}_0^n_{02}) \subset E_2/E_1.
\]

It is easy to see that for any divisors \( D_1 \leq D_2 \) on \( X \) the \( k \)-vector subspace

\[
(A_{12}(D_2) \cap \mathbb{A}_0^n_{02})/(A_{12}(D_1) \cap \mathbb{A}_0^n_{02})
\]

is an open linearly compact \( k \)-vector subspace of \( \mathbb{A}_{12}(D_2)/\mathbb{A}_{12}(D_1) \). Hence \( \tilde{\mu}_{E_1,E_2} \) is an open linearly compact \( k \)-vector subspace of \( E_2/E_1 \).

Now for arbitrary lattices \( E_1 \) and \( E_2 \) of \( \mathbb{A}_X^n \) we introduce an element

\[
\mu_{E_1,E_2} \in \text{Dim}(E_1 \mid E_2)
\]

uniquely defined by the following two rules:

1. If \( E_1 \subset E_2 \), then \( \mu_{E_1,E_2} \) is a “dimension theory” that equals 0 on a \( k \)-vector subspace \( \tilde{\mu}_{E_1,E_2} \subset E_2/E_1 \).
2. For arbitrary lattices \( F_1,F_2,F_3 \) of \( \mathbb{A}_X^n \) we have

\[
\mu_{F_1,F_2} \otimes \mu_{F_2,F_3} = \mu_{F_1,F_3}
\]

with respect to isomorphism (7).

**Remark 3.** To construct an element \( \mu_{E_1,E_2} \subset \text{Dim}(E_1 \mid E_2) \) it is not important that \( X \) is a projective surface.

Analogously, for lattices \( E_1 \subset E_2 \) of \( \mathbb{A}_X^n \) we define a \( k \)-vector subspace

\[
\tilde{\nu}_{E_1,E_2} = (E_2 \cap \mathbb{A}_0^n_{01})/(E_1 \cap \mathbb{A}_0^n_{01}) \subset E_2/E_1.
\]

The \( k \)-vector subspace \( \tilde{\nu}_{E_1,E_2} \subset E_2/E_1 \) is a discrete subspace such that the quotient space of \( E_2/E_1 \) by this subspace is a linearly compact space. (This fact is first easy to see for \( E_1 = \mathbb{A}_{12}(D_1), \ E_2 = \mathbb{A}_{12}(D_2) \), where \( D_1 \) and \( D_2 \) are divisors on \( X \). The latter follows,
since for any projective curve $C$ on $X$ the field of rational functions $k(C)$ is a discrete subspace in the adelic space of $C$ and the quotient space of this adelic space by $k(C)$ is a linearly compact $k$-vector space, and this follows, for example, from adelic complex of $C$ and that the cohomology spaces of coherent sheaves on $C$ are finite-dimensional $k$-vector spaces.)

Now for arbitrary lattices $E_1$ and $E_2$ of $\mathbb{A}_X^n$ we introduce an element
\[
\nu_{E_1, E_2} \in \dim(E_1 \mid E_2)
\]
uniquely defined by the following two rules.

1. If $E_1 \subset E_2$, then $\nu_{E_1, E_2} \in \dim(E_2/E_1)$ is defined from exact sequence
\[
0 \longrightarrow \tilde{\nu}_{E_1, E_2} \longrightarrow E_2/E_1 \longrightarrow (E_2/E_1)/\tilde{\nu}_{E_1, E_2} \longrightarrow 0
\]
where the first non-zero term is a discrete space and the last non-zero term is a linearly compact space, and by (7) there is a canonical isomorphism
\[
\dim(\tilde{\nu}_{E_1, E_2}) \otimes \mathbb{Z} \cong \dim((E_2/E_1)/\tilde{\nu}_{E_1, E_2})
\]
now $\nu_{E_1, E_2} = \nu_1 \otimes \nu_2$, where $\nu_1 \in \dim(\tilde{\nu}_{E_1, E_2})$ is the “dimension theory” that equals 0 on the zero-subspace of the discrete space $\tilde{\nu}_{E_1, E_2}$ and $\nu_2 \in \dim((E_2/E_1)/\tilde{\nu}_{E_1, E_2})$ is the “dimension theory” that equals 0 on the whole linearly compact $k$-vector space $(E_2/E_1)/\tilde{\nu}_{E_1, E_2}$.

2. For arbitrary lattices $F_1, F_2, F_3$ of $\mathbb{A}_X^n$ we have
\[
\nu_{F_1, F_2} \otimes \nu_{F_2, F_3} = \nu_{F_1, F_3}
\]
with respect to isomorphism (7).

**Remark 4.** To construct an element $\nu_{E_1, E_2} \subset \dim(E_1 \mid E_2)$ it is important that $X$ is a projective surface.

We note that any rank $n$ locally free subsheaf of $\mathcal{O}_X$-modules $\mathcal{E} \subset k(X)^n$ on $X$ gives a lattice $\mathbb{A}_{12}(\mathcal{E}) \subset \mathbb{A}_X^n$ (see [9, § 2.2.3]) which generalizes the case $\mathbb{A}_{12}(\mathcal{O}_X^n) = \mathbb{A}_{12}^{n_2}$ and the case $\mathbb{A}_{12}(\mathcal{O}_X(D)) = \mathbb{A}_{12}(D)$ for $n = 1$, where $D$ is a divisor on $X$.

**Proposition 1.** For any rank $n$ locally free subsheaves of $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$ of the constant sheaf $k(X)^n$ on $X$ we have
\[
\nu_{\mathbb{A}_{12}(\mathcal{F}), \mathbb{A}_{12}(\mathcal{G})} - \mu_{\mathbb{A}_{12}(\mathcal{F}), \mathbb{A}_{12}(\mathcal{G})} = \chi(\mathcal{G}) - \chi(\mathcal{F}),
\]
where the subtraction in the left hand side of the formula makes sense, because it is applied to the elements of the $\mathbb{Z}$-torsor $\dim(\mathbb{A}_{12}(\mathcal{F}) \mid \mathbb{A}_{12}(\mathcal{G}))$, and for any Zariski sheaf $\mathcal{E}$ on $X$
\[
\chi(\mathcal{E}) = H^0(X, \mathcal{E}) - H^1(X, \mathcal{E})
\]
is its Euler characteristic on $X$.  

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Proof. From the properties of the left hand side and the right hand side of formula (15) we see that it is enough to suppose that \( F \subset G \). For any quasicoherent sheaf \( \mathcal{E} \) of \( \mathcal{O}_X \)-modules on \( X \) there is the adelic complex \( \mathcal{A}_X(\mathcal{E}) \) which has non-zero terms only in degrees 0, 1, 2 and \( H^i(X, \mathcal{E}) = H^i(\mathcal{A}_X(\mathcal{E})) \) (see, e.g., [8]).

We have canonical embedding of \( \mathcal{A}_X(F) \) to \( \mathcal{A}_X(G) \) such that the quotient complex looks as following:

\[
\tilde{\nu}_{\mathcal{A}_12(\mathcal{F})} \oplus \tilde{\mu}_{\mathcal{A}_12(\mathcal{F})} \longrightarrow \mathcal{A}_{12}(\mathcal{G})/\mathcal{A}_{12}(\mathcal{F})
\]

\[
x \oplus y \longmapsto x + y.
\]

Hence we obtain formula (15), since the Euler characteristic of the latest complex equals to \( \chi(\mathcal{A}(\mathcal{G})) - \chi(\mathcal{A}(\mathcal{F})) \).

\( \square \)

4 Intersection index of divisors

Now we recall how it is possible to obtain the intersection index of divisors on \( X \) by means of central extension (8) when \( n = 1 \):

\[
0 \longrightarrow \mathbb{Z} \longrightarrow GL_1(A_X) \longrightarrow A_X^* \longrightarrow 1.
\]

We consider a bimultiplicative and anti-symmetric map

\[
\langle \cdot, \cdot \rangle : A_X^* \times A_X^* \longrightarrow \mathbb{Z}
\]

given for any \( x, y \in A_X^* \) as following

\[
\langle x, y \rangle = [x', y'] = x'y'x'^{-1}y'^{-1},
\]

where \( x', y' \in GL_1(A_X) \) are any elements such that \( \Theta(x') = x \) and \( \Theta(y') = y \). This definition does not depend on the choice of \( x' \) and \( y' \).

We consider a divisor \( D \) on \( X \).

For any irreducible curve \( C \) on \( X \) let \( j_C^D \in K_C^* \) be a local equation of the restriction of \( D \) to \( \text{Spec} \mathcal{O}_{K_C} \) under the natural morphism \( \text{Spec} \mathcal{O}_{K_C} \to X \), where \( \mathcal{O}_{K_C} \) is the discrete valuation ring of \( K_C \). Note that \( j_C^D \cdot v \), where \( v \in K_C^* \), is again such a local equation. Then under the first diagonal embedding from (14) we obtain that the collection \( j_{1,D} = \prod_C j_C^D \), where \( C \) runs over all irreducible curves \( C \) of \( X \), belongs to \( A_{01}^* \).

For any point \( x \) on \( X \) let \( j_x^D \in K_x^* \) be a local equation of the restriction of \( D \) to \( \text{Spec} \mathcal{O}_{x,X} \) under the natural morphism \( \text{Spec} \mathcal{O}_{x,X} \to X \). Note that \( j_x^D \cdot w \), where \( w \in K_{x,X}^* \), is again such a local equation. Then under the second diagonal embedding from (14) we obtain that the collection \( j_{2,D} = \prod_x j_x^D \), where \( x \) runs over all points \( x \) of \( X \), belongs to \( A_{02}^* \).

For any divisors \( S \) and \( T \) on \( X \) we have (see [11], Proposition 2] which is based on [15, § 2.2]) that

\[
\langle j_{2,S}, j_{1,T} \rangle = -(S,T), \tag{16}
\]
where \((S, T) \in \mathbb{Z}\) is the intersection index of \(S\) and \(T\) on \(X\).

Now we give another presentation for the intersection index of divisors based on formula \([16]\) and special lifts of elements as in Section 3.

**Proposition 2.** For any divisors \(S\) and \(T\) on \(X\) we have
\[
(S, T) = (\mu_{A_{12},A_{12}(-T)} \otimes \nu_{A_{12}(-T),A_{12}(-T-S)}) - (\nu_{A_{12},A_{12}(-S)} \otimes \mu_{A_{12}(-S),A_{12}(-S-T)}) =
\]
\[
= (\mu_{A_{12},A_{12}(T)} \otimes \nu_{A_{12}(T),A_{12}(T+S)}) - (\nu_{A_{12},A_{12}(S)} \otimes \mu_{A_{12}(S),A_{12}(S+T)}),
\]
where the subtraction in these formulas makes sense, because it is applied to elements of the \(\mathbb{Z}\)-torsor
\[
\dim(A_{12} \mid A_{12}(-S - T)) \quad \text{or} \quad \dim(A_{12} \mid A_{12}(S + T))
\]
using isomorphism \([7]\).

**Proof.** The second equality follows from the first equality, because \((S, T) = (-S, -T)\).

We prove the first equality. We note that for any divisor \(D\) on \(X\) we have \(j_{1,D} \cdot j_{2,D} \in A_{12}\) and equality of \(k\)-vector subspaces in \(A_{X}\):
\[
j_{1,D}A_{12} = j_{2,D}A_{12} = A_{12}(-D).
\]

Therefore we can take the following special lifts of elements \(j_{1,T}\) and \(j_{2,S}\) to \(GL_{1}(\mathbb{A}_{X})\) to calculate \(\langle j_{1,T}; j_{2,S} \rangle:\)
\[
j_{1,T} \mapsto (j_{1,T}; \mu_{A_{12},A_{12}(-T)}), \quad j_{2,S} \mapsto (j_{2,S}; \nu_{A_{12},A_{12}(-S)}).
\]

Immediately from constructions we have that
\[
\begin{align*}
\dot{j}_{1,T}(\nu_{A_{12},A_{12}(-S)}) &= \nu_{A_{12}(-T),A_{12}(-T-S)} \quad \text{and} \quad \dot{j}_{2,S}(\mu_{A_{12},A_{12}(-T)}) = \mu_{A_{12}(-S),A_{12}(-T-S)}. \\
\end{align*}
\]

Now from formula \([16]\) we have the following equality in the group \(GL_{1}(\mathbb{A}_{X})\):
\[
(j_{1,T}; \mu_{A_{12},A_{12}(-T)})(j_{2,S}; \nu_{A_{12},A_{12}(-S)}) = (j_{2,S}; \nu_{A_{12},A_{12}(-S)})(j_{1,T}; \mu_{A_{12},A_{12}(-T)})(S, T),
\]
where we consider the intersection index \((S, T)\) as an element of the central subgroup \(\mathbb{Z} \subset GL_{1}(\mathbb{A}_{X})\). Now, using definition of the group operation in \(GL_{1}(\mathbb{A}_{X})\) and formulas \([18]-[19]\), we obtain the statement of the proposition.

\[\square\]

**Remark 5.** Analogous to \([17]\) lift of elements was considered in \([13]\ § 5) for divisors \(S\) and \(T = (\omega) - S\), where \(\omega \in \Omega_{k(X)/k}^{2}\).
5 Canonical splittings

We construct canonical splittings of central extensions (8) and (10) over some subgroups of $GL_n(A_X)$ and investigate the properties of these splittings.

We consider central extension (8).

Proposition 3. We consider central extension (8).

1. The map
   
   \[ GL_n(A_{12}) \ni g \mapsto (g, 0) \in \widetilde{GL_n(A_X)}, \tag{20} \]
   
   where 0 is the zero “dimension theory”, gives a splitting of this central extension over the subgroup $GL_n(A_{12})$.

2. The map
   
   \[ GL_n(A_{02}) \ni g \mapsto (g, \mu_{A_{12}, gA_{12}^n}) \in \widetilde{GL_n(A_X)} \tag{21} \]
   
   gives a splitting of this central extension over the subgroup $GL_n(A_{02})$.

3. The map
   
   \[ GL_n(A_{01}) \ni g \mapsto (g, \nu_{A_{12}, gA_{12}^n}) \in \widetilde{GL_n(A_X)} \tag{22} \]
   
   gives a splitting of this central extension over the subgroup $GL_n(A_{01})$.

4. Splittings (20) and (22) coincide over the subgroup $GL_n(\prod C \mathcal{O}_{K_C})$. Splittings (20) and (21) coincide over the subgroup $GL_n(\prod \hat{O}_{X,X})$. Splittings (22) and (21) coincide over the subgroup $GL_n(k(X))$. (Here we use the diagonal embeddings of subgroups, see (14).)

Remark 6. For the construction of splittings (20)-(21) it is not important that $X$ is projective, but for the construction of splitting (22) it is important.

Proof. Item 1 is evident, since $gA_{12}^n = A_{12}^n$ for any $g \in GL_n(A_{12})$.

Item 2 and 3 follow from the equalities:

\[ g\tilde{\mu}_{E_1,E_2} = \tilde{\mu}_{gE_1,gE_2}, \quad h\tilde{\nu}_{E_1,E_2} = \tilde{\nu}_{hE_1,hE_2}, \]

where $E_1 \subset E_2$ are any lattices in $\mathbb{A}_X^n$, an element $g \in GL_n(A_{02})$, an element $h \in GL_n(A_{01})$.

In item 4 the only non-evident statement is that splittings (22) and (21) coincide over the subgroup $GL_n(k(X))$. To prove this statement we note that for any $g \in GL_n(k(X))$ we have $gA_{12}^n = A_{12}(g\mathcal{O}_X^n)$ and by Proposition 1 we have

\[ \nu_{A_{12}(g\mathcal{O}_X^n), A_{12}(g\mathcal{O}_X^n)} - \mu_{A_{12}(g\mathcal{O}_X^n), A_{12}(g\mathcal{O}_X^n)} = \chi(g\mathcal{O}_X^n) - \chi(\mathcal{O}_X^n). \]

We note that the sheaf $g\mathcal{O}_X^n$ is isomorphic to the sheaf $\mathcal{O}_X^n$. Therefore we have $\chi(\mathcal{O}_X^n) = \chi(g\mathcal{O}_X^n)$. Hence

\[ \nu_{A_{12}(g\mathcal{O}_X^n)} = \mu_{A_{12}(g\mathcal{O}_X^n)}. \]
and two splittings coincide. (We note that we could apply Proposition 1 only for the case $n = 1$, since $GL_n(k(X)) = SL_n(k(X)) \rtimes k(X)^*$ and the group $SL_n(k(X))$ is perfect that implies that any two sections of any central extension of $SL_n(k(X))$ coincide.)

Let $n = n_1 + n_2$, where $n_1$ and $n_2$ are positive integers. We consider a parabolic subgroup

$$P_{n_1,n_2} = \left\{ \begin{pmatrix} GL_{n_1}(\mathbb{A}_X) & \ast \\ 0 & GL_{n_2}(\mathbb{A}_X) \end{pmatrix} \right\} \subset GL_n(\mathbb{A}_X).$$

Let $p_i : GL_n(\mathbb{A}_X) \to P_{n_i}$, where $i = 1$ or $i = 2$, be natural homomorphisms.

**Proposition 4.** The pullback of the central extension $\widetilde{GL}_n(\mathbb{A}_X)$ via the embedding $P_{n_1,n_2} \hookrightarrow GL_n(\mathbb{A}_X)$ is isomorphic to the Baer sum of pullbacks $p_1^*(\widetilde{GL}_{n_1}(\mathbb{A}_X))$ and $p_2^*(\widetilde{GL}_{n_2}(\mathbb{A}_X))$. Besides the splittings (over special subgroups) from Proposition 3 are compatible with respect to this isomorphism.

**Proof.** Construction of isomorphism is directly obtained from exact triple

$$0 \to \mathbb{A}^{n_1}_{X} \to \mathbb{A}^{n}_{X} \to \mathbb{A}^{n_2}_{X} \to 0,$$

the fact that the action of $P_{n_1,n_2}$ preserves this triple, from construction of the group $\widetilde{GL}_n(\mathbb{A}_X)$ and from isomorphism (5).

The compatibility of sections follows from exact triples

$$0 \to \mathbb{A}^{n_1}_{01} \to \mathbb{A}^{n}_{01} \to \mathbb{A}^{n_2}_{01} \to 0$$

and that the corresponding groups $P_{n_1,n_2} \cap GL_n(\mathbb{A}_{01})$ or $P_{n_1,n_2} \cap GL_n(\mathbb{A}_{02})$ act on these triples.

**Remark 7.** A direct analog of Proposition 3 is valid for central extension (10) (see [11, Prop. 4]), where we use that $\widetilde{GL}_n(\mathbb{A}_X) = \Theta^{-1}(SL_n(\mathbb{A}_X)) \rtimes \mathbb{A}_X^*$, and we take the splittings over the intersections of $SL_n(\mathbb{A}_X)$ with the corresponding subgroups that come from Proposition 3 and the identity splitting over the intersection of $\mathbb{A}_X^*$ with the corresponding subgroups.

The behaviour of the pullback of central extension (10) via the map $P_{n_1,n_2} \hookrightarrow GL_n(\mathbb{A}_X)$ differs from what is described in Proposition 4 for central extension (8), see [11, § 3.3] and Remark 9 below.

6 Trivializations of locally free sheaves

We describe trivializations of locally free sheaves of $\mathcal{O}_X$-modules over completions of local rings of scheme points of $X$. We consider a point $x \in X$ as a closed scheme point of $X$, the generic point of an irreducible curve $C$ on $X$ as a non-closed point of $X$, and also consider the generic point of $X$.
Let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}_X$-modules of rank $n$ on $X$.
For a (closed) point $x$ of $X$, the completion of the stalk of $\mathcal{E}$ at $x$ is a free $\hat{\mathcal{O}}_{x,X}$-module. Let $e_x$ be a basis of this module. We will also call such a basis a basis of $\mathcal{E}$ restricted to $\text{Spec} \hat{\mathcal{O}}_{x,X}$.

For an irreducible curve $C$ on $X$, the completion of the stalk of $\mathcal{E}$ at the generic point of $C$ is a free $\mathcal{O}_{K_C}$-module. Let $e_C$ be a basis of this module. We will also call such a basis a basis of $\mathcal{E}$ restricted to $\text{Spec} \mathcal{O}_{K_C}$.

The stalk of $\mathcal{E}$ at the generic point of $X$ is a $k(X)$-vector space. Let $e_0$ be a basis of this vector space. We will call such a basis a basis of $\mathcal{E}$ restricted to $\text{Spec} k(X)$.

By embedding of the completions of stalks of $\mathcal{E}$ at scheme points of $X$ to the tensor products (over the local rings of the points) of the stalks of $\mathcal{E}$ and the corresponding rings: $K_x$, $K_C$ or $K_{x,C}$, we obtain transition matrices

$$\alpha_{02,x} \in GL_n(K_x), \quad \alpha_{01,C} \in GL_n(K_C), \quad \alpha_{21,x,C} \in GL_n(\mathcal{O}_{K_{x,C}})$$

defined by the following equalities

$$e_0 = \alpha_{02,x} e_x, \quad e_0 = \alpha_{01,C} e_C, \quad e_x = \alpha_{21,x,C} e_C.$$ 

Denote collections of matrices $\alpha_{02} = \prod_x \alpha_{02,x}$, where $x$ runs over all closed points of $X$, $\alpha_{01} = \prod_C \alpha_{01,C}$, where $C$ runs over all irreducible curves $C$ on $X$, $\alpha_{21} = \prod_{x \in C} \alpha_{21,x,C}$, where $x \in C$ runs over all pairs $x \in C$ with $x$ a closed point on an irreducible curve $C$ on $X$.

Define now $\alpha_{ij} = \alpha_{ji}^{-1}$, where $i \neq j$ from the set $\{0, 1, 2\}$. Then for any $i \neq j \neq k$ from the set $\{0, 1, 2\}$ we have an equality (cocycle identity) in $GL_n\left(\prod_{x \in C} K_{x,C}\right)$, where we use diagonal embeddings (14),

$$\alpha_{ij} \cdot \alpha_{jk} \cdot \alpha_{ki} = 1. \quad (23)$$

Note that if we change the chosen bases, then the matrices will change in the following way

$$\alpha_{02} \mapsto \alpha_0 \cdot \alpha_{02} \cdot \alpha_2^{-1}, \quad \alpha_{01} \mapsto \alpha_0 \cdot \alpha_{01} \cdot \alpha_1^{-1}, \quad \alpha_{21} \mapsto \alpha_2 \cdot \alpha_{21} \cdot \alpha_1^{-1}, \quad (24)$$

where $\alpha_0 \in GL_n(k(X)) \subset GL_n(\mathcal{A}_X)$, $\alpha_1 \in GL_n\left(\prod_{C} \mathcal{O}_{K_C}\right) \subset GL_n(\mathcal{A}_X)$,

$$\alpha_2 \in GL_n\left(\prod_x \hat{\mathcal{O}}_{x,X}\right) \subset GL_n(\mathcal{A}_X).$$

Using diagonal embeddings (13) and by means of non-difficult reasonings, one can show (see [11 § 3.3]) that

$$\alpha_{02} \in GL_n(\mathcal{A}_{q_2}), \quad \alpha_{01} \in GL_n(\mathcal{A}_{q_1}), \quad \alpha_{21} \in GL_n(\mathcal{A}_{12}). \quad (25)$$

We will use notation $\alpha_{ij,\mathcal{E}}$ instead of $\alpha_{ij}$ when it is not clear from the context which bundle this notation is associated with.
Remark 8. If \( n = 1 \), then \( \mathcal{E} = \mathcal{O}_X(D) \) for some divisor \( D \) on \( X \). We have that (see Section 4)

\[
j_{2,D} = \alpha_{02, \mathcal{E}} \quad \text{and} \quad j_{1,D} = \alpha_{01, \mathcal{E}}.
\]
Therefore from formula (16) we obtain that for invertible sheaves \( \mathcal{F} \) and \( \mathcal{G} \) on \( X \), their intersection index \((\mathcal{F}, \mathcal{G})\) is

\[
(\mathcal{F}, \mathcal{G}) = \langle \alpha_{01, \mathcal{G}}, \alpha_{02, \mathcal{F}} \rangle = \langle \alpha_{02, \mathcal{F}}, \alpha_{10, \mathcal{G}} \rangle = \langle \alpha_{12, \mathcal{F}}, \alpha_{01, \mathcal{G}} \rangle = \langle \alpha_{12, \mathcal{F}}, \alpha_{20, \mathcal{G}} \rangle,
\]

where we used also splittings of central extension (8) over certain subgroups from Proposition 3, whence it follows that \( \langle \cdot, \cdot \rangle \) restricted to these subgroups equals 0.

Definition 2. Let \( \mathcal{E} \) be a locally free sheaf of \( \mathcal{O}_X \)-modules of rank \( n \) on \( X \). For any \( i \neq j \) from \( \{0, 1, 2\} \) let \( \tilde{\alpha}_{ij} \) be the canonical lift of \( \alpha_{ij} \) for \( \mathcal{E} \) to \( \hat{GL}_n(\mathbb{A}_X) \) by splitting over the corresponding subgroup, see Proposition 3 and formula (25). We define

\[
f_\mathcal{E} = \tilde{\alpha}_{02} \cdot \tilde{\alpha}_{21} \cdot \tilde{\alpha}_{10} \in \mathbb{Z}.
\]

This definition is correct, because from formula (23) it follows that \( f_\mathcal{E} \in \mathbb{Z} \), and by Proposition 3 and formula (24), the integer \( f_\mathcal{E} \) depends only on \( \mathcal{E} \), i.e., this integer does not depend on the choice of bases \( e_0, \{e_x\}, \{e_C\} \) for \( \mathcal{E} \).

As we have already mentioned before, the goal of this paper is to calculate \( f_\mathcal{E} \) and to relate the calculations of this integer, given by two different ways, to the Riemann-Roch theorem for \( \mathcal{E} \) on \( X \) (without the Noether formula for \( \mathcal{O}_X \)).

Remark 9. For central extension (10)

\[
0 \longrightarrow \mathbb{Z} \longrightarrow \hat{GL}_n(\mathbb{A}_X) \longrightarrow GL_n(\mathbb{A}_X) \longrightarrow 1
\]

let \( \hat{\alpha}_{02}, \hat{\alpha}_{21}, \hat{\alpha}_{10} \) be the corresponding canonical lifts of elements \( \alpha_{02}, \alpha_{21}, \alpha_{10} \) for \( \mathcal{E} \) to \( GL_n(\mathbb{A}_X) \) by corresponding splittings over special subgroups of \( GL_n(\mathbb{A}_X) \), see Remark 7. Then the following integer does not depend on the choice of bases \( e_0, \{e_x\}, \{e_C\} \) and by [11, Theorem 1] we have

\[
\hat{\alpha}_{02} \cdot \hat{\alpha}_{21} \cdot \hat{\alpha}_{10} = c_2(\mathcal{E}).
\]

7 The first way to calculate \( f_\mathcal{E} \)

Let \( \mathcal{E} \) be a locally free sheaf of \( \mathcal{O}_X \)-modules of rank \( n \) on \( X \). We calculate the integer \( f_\mathcal{E} \) (see Definition 2) in the first way.

Proposition 5. The following properties are satisfied.

1. Consider an exact triple of finite rank locally free sheaves of \( \mathcal{O}_X \)-modules:

\[
0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_2 \longrightarrow 0.
\]

We have \( f_\mathcal{E} = f_{\mathcal{E}_1} + f_{\mathcal{E}_2} \).
2. Let $\pi: Y \rightarrow X$ be the blow-up of a point on $X$. We have $f_E = f_{\pi^*(E)}$.

Proof. 1. The integers $f_{E_i}$, where $i \in \{1, 2, 3\}$, do not depend on the choice of bases of $E_i$ at completions of local rings of scheme points of $X$. Therefore we can choose, first, the bases for $E_1$ and then complete them to the bases of $E$. Hence all the transition matrices $\alpha_{ij}$ belong to the subgroup $P_{n_1, n_2} \subset GL_n(\mathbb{A}_X)$, where $n_1$ or $n_2$ are the corresponding ranks of $E_1$ or $E_2$. Now we apply Proposition 4.

2. (Cf. the proof of [11, Theorem 1].) Let $\pi$ be the blow-up of a point $x \in X$, and $\pi^{-1}(x) = R$. We note that

\[ \mathbb{A}_Y = \mathbb{A}_X \times \prod_{p \in R} K_{p,R}. \]

(27)

Since $f_E$ and $f_{\pi^*(E)}$ do not depend on the choice of bases for $E$ and $\pi^*(E)$, we choose the special bases. Fix a trivialization of $E$ on an open neighbourhood of $x$ on $X$. This trivialization induces the bases $e_0$, $e_x$, $e_R$ and $e_p$, where $p \in R$. We have a canonical isomorphism

\[ \delta: \widehat{GL_n(\mathbb{A}_X)} \rightarrow \gamma^*(\widehat{GL_n(\mathbb{A}_Y)}), \]

where the embedding $\gamma: GL_n(\mathbb{A}_X) \hookrightarrow GL_n(\mathbb{A}_Y)$ is induced by decomposition (27). Now for any $i \neq j$ from $\{0, 1, 2\}$ we have

\[ \gamma(\alpha_{ij,E}) = \alpha_{ij,\pi^*(E)} \quad \text{and} \quad \delta(\tilde{\alpha}_{ij,E}) = \alpha_{ij,\pi^*(E)}, \]

where we consider $\alpha_{ij,\pi^*(E)}$ as elements of the group $\gamma^*(GL_n(\mathbb{A}_Y))$. This implies the statement. \qed

In the sequel, we denote also the intersection index $E \cdot F = (E, F) \in \mathbb{Z}$ or $c_1(E)^2 = E \cdot E = (E, E) \in \mathbb{Z}$, where $E$ and $F$ are invertible sheaves on $X$.

Theorem 1. Let $E$ be a locally free sheaf of $\mathcal{O}_X$-modules of rank $n$ on a smooth projective surface $X$ over a field $k$. We have

\[ f_E = (\chi(E) - n\chi(O_X)) - c_1(E)^2 + 2c_2(E) = (\chi(E) - n\chi(O_X)) - 2\text{ch}_2(E). \]

Proof. The second equality is just reformulation of the first equality. Therefore we prove the first equality.

The left hand side and the right hand side of the equality are additive with respect to the short exact sequences of locally free sheaves of $\mathcal{O}_X$-modules and are preserved by blow-ups of points (see Proposition 5 for the left hand side of the equality, in the right hand side of the equality it is clear for $\text{ch}_2$, and for the difference of Euler characteristics this follows from the simple case $\chi(F) - \chi(E) = \chi(F/E)$, where $F$ is a locally free sheaf of $\mathcal{O}_X$-modules of rank $n$, $E \subset F$ and the sheaf $F$ coincides with the sheaf $E$ in a neighbourhood of a point which is blew up). Therefore, by the splitting principle for locally free sheaves on smooth surfaces (cf. the proof of [11, Theorem 1]), it is enough to prove the equality for the case $n = 1$.

So, we suppose $n = 1$. The chosen basis $e_0$ of $E$ at the generic point of $X$ gives the embedding of $E$ to the constant sheaf $k(X)$ on $X$. (Therefore $E = \mathcal{O}_X(D)$ for some
divisor $D$ on $X$.) We fix also the other bases for $E$ and hence the transitions matrices $\alpha_{ij}$ for $E$, where $i \neq j$ from $\{1, 2\}$, as in Section 6.

We have to prove that

$$f_E = (\chi(E) - n\chi(O_X)) - (E, E). \quad (28)$$

From formula (26), which describes the intersection index of invertible sheaves as the commutator of lifting of corresponding elements to $\widetilde{GL_1(A_X)}$, we have

$$\tilde{\alpha}_{10} \cdot \tilde{\alpha}_{02} = \tilde{(E, E)} \cdot \tilde{\alpha}_{02} \cdot \tilde{\alpha}_{10}, \quad (29)$$

where we consider $-(E, E)$ as an element of the central subgroup $Z \subset \widetilde{GL_1(A_X)}$.

Besides, using conjugation, we obtain

$$f_E = \tilde{\alpha}_{02} \cdot \tilde{\alpha}_{21} \cdot \tilde{\alpha}_{10} = \tilde{\alpha}_{10} \cdot \tilde{\alpha}_{02} \cdot \tilde{\alpha}_{21} \cdot \tilde{\alpha}_{10}^{-1} = \tilde{\alpha}_{10} \cdot \tilde{\alpha}_{02} \cdot \tilde{\alpha}_{21}.$$

Thus, we have

$$\tilde{\alpha}_{10} \cdot \tilde{\alpha}_{02} = f_E \cdot \tilde{\alpha}_{12}, \quad (30)$$

where we consider $f_E$ as an element of the central subgroup $Z \subset \widetilde{GL_1(A_X)}$.

From formulas (29)–(30) we have that formula (28) will follow from the following lemma.

**Lemma 1.** Let $E$ be an invertible sheaf on $X$. Then we have

$$\tilde{\alpha}_{02} \cdot \tilde{\alpha}_{10} = (\nu_{A_{12}, A_{12}(E)} - \mu_{A_{12}, A_{12}(E)}) \cdot \tilde{\alpha}_{12} = (\chi(E) - \chi(O_X)) \cdot \tilde{\alpha}_{12},$$

where we consider $\nu_{A_{12}, A_{12}(E)} - \mu_{A_{12}, A_{12}(E)}$ and $\chi(E) - n\chi(O_X)$ as elements of the central subgroup $Z \subset \widetilde{GL_1(A_X)}$, and the chosen and fixed basis $e_0$ of $E$ at the generic point of $X$ gives the embedding of $E$ to the constant sheaf $k(X)$ on $X$.

We note that since $\alpha_{02}$ and $\alpha_{10}$ are from $A_X^*$, we have that $\alpha_{02} \cdot \alpha_{10} = \alpha_{10} \cdot \alpha_{02} = \alpha_{12}$.

We prove now this lemma. The second equality in the statement of the lemma follows immediately from the first equality and Proposition 1. Therefore we prove the first equality.

Let $c \in Z \subset \widetilde{GL_1(A_X)}$ such that $\tilde{\alpha}_{02} \cdot \tilde{\alpha}_{10} = c \cdot \tilde{\alpha}_{12}$. Then we have

$$\tilde{\alpha}_{10} = c \cdot \tilde{\alpha}_{20} \cdot \tilde{\alpha}_{12}. \quad (31)$$

We note that $\alpha_{12}A_{12} = A_{12}$. Therefore $\tilde{\alpha}_{12} = (\alpha_{12}, 0)$, where $0 \in Z = \text{Dim}(A_{12} \mid A_{12})$. Hence, product with $\tilde{\alpha}_{12}$ in the right hand side of formula (31) does not affect the “dimension theory”.

Hence (see also Remark 8, where transition matrices for an invertible sheaf are explicitly given) and by direct calculation we obtain that

$$c = \nu_{A_{12}, A_{12}(E)} - \mu_{A_{12}, A_{12}(E)}.$$

This finishes the proof of the lemma and, consequently, the proof of the theorem.
Remark 10. From Theorem 1 Remark 9 and formula (26) we obtain the following “local (adelic) decomposition” for the difference of Euler characteristics for a rank $n$ locally free sheaf $\mathcal{E}$ of $\mathcal{O}_X$-modules and the sheaf $\mathcal{O}_X^n$, using central extensions (8) and (10) and transition matrices for $\mathcal{E}$:

$$\chi(\mathcal{E}) - n\chi(\mathcal{O}_X) = \tilde{\alpha}_{02} \cdot \tilde{\alpha}_{21} \cdot \tilde{\alpha}_{10} - 2\tilde{\alpha}_{02} \cdot \tilde{\alpha}_{21} \cdot \tilde{\alpha}_{10} + \langle \det(\alpha_{02}), \det(\alpha_{10}) \rangle.$$

This formula generalizes formulas (2)-(3) from §1 (Introduction) from the case of smooth projective algebraic curves to the case of smooth projective algebraic surfaces.

8 The second way to calculate $f_E$ and the Riemann-Roch theorem

There is another way to calculate $f_E$ for a locally free sheaf $\mathcal{E}$ of $\mathcal{O}_X$-modules of rank $n$ on $X$. We will do this way in this section. This another way leads to an answer, which uses also another invariants of $\mathcal{E}$ and $X$, see Theorem 2 below. And the comparison of this answer with the answer obtained in Theorem 1 immediately gives us the Riemann-Roch theorem for $\mathcal{E}$ on $X$ (without the Noether formula), see Corollary 1.

In the sequel, we suppose that the basic field $k$ is perfect (it is important for the theory of two-dimensional residues used below). The idea for the new calculation of $f_E$ is to use the fact that $\mathbb{A}_X$ is self-dual as a $C_2$-space over $k$ (or a 2-Tate vector space) and to make some calculations as on the “dual side”.

More exactly, the self-duality of $\mathbb{A}_X$ is given by the following pairing (cf. [13, §2]). We fix $\omega \in \Omega^2_{k(X)/k}$ such that $\omega \neq 0$. Then by $\omega$ we construct a bilinear symmetric non-degenerate pairing by means of the residues on two-dimensional local fields (see more about residues on $n$-dimensional local fields in [13, 21])

$$\mathbb{A}_X \times \mathbb{A}_X \longrightarrow k : \{f_{x,C}\} \times \{g_{x,C}\} \longmapsto \sum_{x \in C} \text{Tr}_{k(x)/k} \circ \text{res}_{x,C}(f_{x,C} g_{x,C} \omega),$$

where $\{f_{x,C}\}$ and $\{g_{x,C}\}$ are from $\prod_{x \in C} K_{x,C} = \mathbb{A}_X$, and if $K_{x,C} = \prod_i K_i$, then $\text{res}_{x,C} : \Omega^2_{K_{x,C}/k} \longrightarrow k(x)$ equals $\sum_i \text{Tr}_{k_i/k(x)} \circ \text{res}_{K_i}$, where

$$\text{res}_{K_i} : \Omega^2_{K_i/k(x)} \longrightarrow \Omega^2_{K_i/k_i} \longrightarrow \tilde{\Omega}^2_{K_i/k_i} \longrightarrow k_i,$$

where $K_i \simeq k_i((u))((t))$ and this isomorphism is a homeomorphism under natural topologies. The first map in formula (33) is the natural map, the second map is the map to “continuous differential forms”, i.e. to the quotient module by a $K_i$-submodule generated by elements $f_1 df_2 \wedge df_3 - f_1 df_2 \wedge (\frac{\partial f_3}{\partial u} du + \frac{\partial f_3}{\partial t} dt)$, and the last map is

$$\sum_{j,l} a_{j,l} u^j t^l du \wedge dt \longmapsto a_{-1,-1},$$

where $a_{j,l} \in k_i$. We note also that the sum (32) is finite.

We now give a theorem.

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Theorem 2. Let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}_X$-modules of rank $n$ on a smooth projective surface $X$ over a perfect field $k$. We have

$$f_\mathcal{E} = -\frac{1}{2} K \cdot c_1(\mathcal{E}) - \frac{1}{2} c_1(\mathcal{E})^2 + c_2(\mathcal{E}) = -\frac{1}{2} K \cdot c_1(\mathcal{E}) - \operatorname{ch}_2(\mathcal{E}),$$

where $K \simeq \mathcal{O}_X(\omega)$, $\omega \in \Omega^2_{k(X)/k}$, $\omega \neq 0$.

From Theorem 1 and Theorem 2 we immediately obtain a corollary.

**Corollary 1** (Riemann-Roch theorem). We have

$$\chi(\mathcal{E}) - n\chi(\mathcal{O}_X) = \frac{1}{2} c_1(\mathcal{E}) \cdot (c_1(\mathcal{E}) - K) - c_2(\mathcal{E}).$$

**Remark 11.** If $\mathcal{E}$ is an invertible sheaf on $X$, then a number

$$p_a(\mathcal{E}) = 1 + \frac{1}{2} \mathcal{E} \cdot (\mathcal{E} + K)$$

is called a *virtual arithmetic genus* of $\mathcal{E}$ (see, e.g., [18, Ch. IV, § 2.8]). Therefore, in this case we have

$$f_\mathcal{E} = 1 - p_a(\mathcal{E}).$$

**Proof.** It is enough to prove the first equality in the statement of Theorem 2. By the same arguments as in the beginning of the proof of Theorem 1 we obtain that it is enough to suppose that $n = 1$, which we will do.

For any $k$-vector subspace $V \subset \mathbb{A}_X$ we denote by $V^\perp$ the annihilator of $V$ in $\mathbb{A}_X$ with respect to the pairing (32). From the reciprocity laws on $X$ for residues of differential two-forms on two-dimensional local fields (these reciprocity laws are “along an irreducible curve” and “around a point”), it is possible to obtain (see [14]) that for any divisor $D$ on $X$

$$A_{12}(D)^\perp = A_{12}((\omega) - D), \quad A_{02}^\perp = A_{02}, \quad A_{01}^\perp = A_{01}. \quad (34)$$

There is a canonical isomorphism of groups (see [12, § 5.5.5])

$$\varphi : GL_1(\overline{\mathbb{A}_X}) \longrightarrow GL_1(\overline{\mathbb{A}_X})_{A_{12}^\perp},$$

where the central extension $GL_1(\overline{\mathbb{A}_X})_{A_{12}^\perp}$ is constructed similar to the central extension $GL_1(\overline{\mathbb{A}_X})$, but starting from the lattice $A_{12}^\perp$ instead of the lattice $A_{12}$ (cf. Remark 11). More precisely, $GL_1(\overline{\mathbb{A}_X})_{A_{12}^\perp}$ consists of all pairs $(g, d)$, where $g \in GL_1(\overline{\mathbb{A}_X})$ and $d \in \operatorname{Dim}(A_{12}^\perp | gA_{12})$ with the multiplication law as in formula (39). Explicitly, isomorphism $\varphi$ is given as

$$\varphi((g, d)) = (g^{-1}, d),$$

where we use canonical isomorphism $\operatorname{Dim}(A_{12} | gA_{12}) \simeq \operatorname{Dim}(A_{12}^\perp | g^{-1}A_{12}^\perp)$, which is based on the equality $g^{-1}A_{12}^\perp = (gA_{12})^\perp$ and on a canonical isomorphism.
Dim($E_1 \mid E_2$) \simeq Dim($E_1^+ \mid E_2^+$), where $E_1, E_2$ are lattices in $\mathbb{A}_X$. The last isomorphism comes from the following chain of isomorphisms, where $E_3$ is a lattice such that $E_3 \subset E_1, E_3 \subset E_2$:

\[
\text{Dim}(E_1 \mid E_2) \simeq \text{Dim}(E_1 \mid E_3) \otimes_{\mathbb{Z}} \text{Dim}(E_3 \mid E_2) \simeq \text{Dim}(E_1/E_3)^* \otimes_{\mathbb{Z}} \text{Dim}(E_2/E_3) \simeq \\
\simeq \text{Dim}(E_3^+/E_1^+) \otimes_{\mathbb{Z}} \text{Dim}(E_3^+/E_2^+)^* \simeq \text{Dim}(E_1^+ \mid E_2^+) ,
\]

where * means the dual $\mathbb{Z}$-torsor and we used canonical isomorphisms $[5]$ and $[6]$.

We note that isomorphism $\varphi$ restricted to the central subgroup $\mathbb{Z} \subset GL_1(\mathbb{A}_X)$ is the identity morphism to the central subgroup $\mathbb{Z} \subset GL_1(\mathbb{A}_X)$.

The chosen basis $e_0$ of $\mathcal{E}$ at the generic point of $X$ gives the embedding of $\mathcal{E}$ to the constant sheaf $k(X)$ on $X$. Therefore $\mathcal{E} = \mathcal{O}_X(D)$ for some divisor $D$ on $X$. We fix also the other bases for $\mathcal{E}$ and hence the transitions elements $\alpha_{ij}$ for $\mathcal{E}$, where $i \neq j$ from $\{1, 2\}$, as in Section $[6]$.

Therefore, we have

\[ f_\mathcal{E} = \varphi(f_\mathcal{E}) = \varphi(\tilde{a}_{02} \cdot \tilde{a}_{21} \cdot \tilde{a}_{10}) = \varphi(\tilde{a}_{02}) \cdot \varphi(\tilde{a}_{21}) \cdot \varphi(\tilde{a}_{10}). \]

Hence, the proof of the theorem will follow from the proof of the following formula

\[ \varphi(\tilde{a}_{01}) \cdot \varphi(\tilde{a}_{12}) \cdot \varphi(\tilde{a}_{20}) = \tilde{a}_{02} \cdot \tilde{a}_{21} \cdot \tilde{a}_{10} + (\mathcal{E}, \mathcal{E}) + (\mathcal{E}, \mathcal{O}_X(\omega)), \]

where we used that

\[-(\tilde{a}_{02} \cdot \tilde{a}_{21} \cdot \tilde{a}_{10}) = \tilde{a}_{01} \cdot \tilde{a}_{12} \cdot \tilde{a}_{20}.\]

From formula $[26]$ for the intersection index of invertible sheaves we have

\[ (\mathcal{E}, \mathcal{E}) = (\tilde{a}_{01}, \tilde{a}_{02}) = \tilde{a}_{01} \cdot \tilde{a}_{02} \cdot \tilde{a}_{10} \cdot \tilde{a}_{20}. \]

Therefore we obtain

\[ \tilde{a}_{02} \cdot \tilde{a}_{21} \cdot \tilde{a}_{10} + (\mathcal{E}, \mathcal{E}) = \tilde{a}_{02} \cdot \tilde{a}_{21} \cdot \tilde{a}_{10} \cdot \tilde{a}_{01} \cdot \tilde{a}_{02} \cdot \tilde{a}_{10} \cdot \tilde{a}_{20} = \\
= \tilde{a}_{02} \cdot (\tilde{a}_{21} \cdot \tilde{a}_{02} \cdot \tilde{a}_{10}) \cdot \tilde{a}_{02}^{-1} = \tilde{a}_{21} \cdot \tilde{a}_{02} \cdot \tilde{a}_{10} = \\
= \tilde{a}_{12} \cdot (\tilde{a}_{21} \cdot \tilde{a}_{02} \cdot \tilde{a}_{10}) \cdot \tilde{a}_{12}^{-1} = \tilde{a}_{02} \cdot \tilde{a}_{10} \cdot \tilde{a}_{21}, \]

where we used that the conjugation does not change the result. Hence, to prove the theorem it is enough to prove the formula

\[ \varphi(\tilde{a}_{01}) \cdot \varphi(\tilde{a}_{12}) \cdot \varphi(\tilde{a}_{20}) = \tilde{a}_{02} \cdot \tilde{a}_{10} \cdot \tilde{a}_{21} = (\mathcal{E}, \mathcal{O}_X(\omega)). \]

Let $c = \tilde{a}_{02} \cdot \tilde{a}_{10} \cdot \tilde{a}_{21}$. Hence $c \cdot \tilde{a}_{12} = \tilde{a}_{02} \cdot \tilde{a}_{10}$. Therefore, by Lemma $[11]$ we have

\[ c = \nu_{A_{12}, A_{12}}(\mathcal{E}) - \mu_{A_{12}, A_{12}}(\mathcal{E}). \]

Now we calculate $d = \varphi(\tilde{a}_{01}) \cdot \varphi(\tilde{a}_{12}) \cdot \varphi(\tilde{a}_{20}) = \varphi(\tilde{a}_{12}) \cdot \varphi(\tilde{a}_{20}) \cdot \varphi(\tilde{a}_{01})$. We have

\[ d \cdot \varphi(\tilde{a}_{21}) = \varphi(\tilde{a}_{20}) \cdot \varphi(\tilde{a}_{01}) \quad (35) \]
From the construction of $\varphi$ and formulas (34) we have

$$\varphi(\alpha_{01}) = \varphi((\alpha_{01}, \nu_{A_{12}}, \alpha_{A_{12}})) = (\alpha_{10}, \nu_{A_{12}}, \alpha_{A_{12}}) = (\alpha_{10}, \nu_{A_{12}}, \alpha_{A_{12}} + D)$$

$$\varphi(\alpha_{02}) = \varphi((\alpha_{02}, \mu_{A_{12}}, \alpha_{A_{12}})) = (\alpha_{20}, \mu_{A_{12}}, \alpha_{A_{12}}) = (\alpha_{20}, \mu_{A_{12}}, \alpha_{A_{12}} + D)$$

$$\varphi(\alpha_{21}) = \varphi((\alpha_{21}, 0)) = (\alpha_{12}, 0) .$$

From these formulas and formula (35), by the same reason as in calculation of $c$ above (see the proof of Lemma [1]) we obtain that

$$d = \nu_{A_{12}}((\omega), A_{12}) - \mu_{A_{12}}((\omega), A_{12}) + D .$$

Now we have

$$d - c = (\nu_{A_{12}}((\omega), A_{12}) - \mu_{A_{12}}((\omega), A_{12})) - (\nu_{A_{12}} - \mu_{A_{12}})(D) =$$

$$= (\nu_{A_{12}}((\omega), A_{12}) - \mu_{A_{12}}((\omega), A_{12})) + (\nu_{A_{12}} - \mu_{A_{12}})(D) +$$

$$+ (\nu_{A_{12}}(D), A_{12}((\omega)) - (\nu_{A_{12}}(D), A_{12}((\omega)) + (\nu_{A_{12}}((\omega), A_{12}) - \mu_{A_{12}}((\omega), A_{12}))) =$$

$$= (\nu_{A_{12}}(D), A_{12}((\omega))) + (-\nu_{A_{12}}(D), A_{12}((\omega)) + \mu_{A_{12}}((\omega), A_{12}((\omega)) + D) =$$

$$= (\nu_{A_{12}}(D), A_{12}((\omega))) + (-\nu_{A_{12}}(D), A_{12}((\omega)) + \mu_{A_{12}}((\omega), A_{12}((\omega)) + D) .$$

Hence and by Proposition [2] we obtain $d - c = (D, (\omega)) . \square$

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