Remarks on Azarov’s work on soluble groups of finite rank

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Received: 11 September 2015 / Accepted: 10 December 2015 / Published online: 9 January 2016
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Abstract We present proofs of D. N. Azarov’s recent three theorems determining precisely when a soluble group of finite rank is residually a finite $\pi$-group for a specified finite set $\pi$ of primes. Our proofs seem to be substantially shorter; they also apply to groups with a somewhat weaker notion of finite rank.

Keywords Soluble group · Finite rank · Residually finite

Mathematics Subject Classification 20F16 · 20E26

According to its English summary Azarov’s paper [1] is devoted to proving the following. If $\pi$ is a finite set of primes, then a soluble group of finite rank is a finite extension of a residually finite $\pi$-group for some finite set $\pi$ of primes if and only if it is a reduced FA TR group with no $\pi$-divisible elements of infinite order (see below or see [3] for definitions). Further he proves that a soluble group of finite rank is residually a finite $\pi$-group for some finite set $\pi$ of primes if and only if it is a reduced FATR group. (This is effectively an old result of D. J. S. Robinson, see [3] 5.3.8 or [4] Page 138.) We give below what presumably are much shorter proofs of these interesting results. In fact we prove the following. (Below $\tau(G)$ denotes the maximal periodic normal subgroup of a group $G$, $\text{Fitt}G$ its Fitting subgroup and $\{\zeta_\alpha(G)\}$ its upper central series.)

Theorem Let $G$ be a finite extension of a soluble FAR group and let $\pi$ be some finite set of primes. The following are equivalent.

(a) $G$ is a finite extension of a residually finite $\pi$-group.
(b) $\tau(G)$ is finite and $\zeta_1(\text{Fitt}H)$ is $\pi$-reduced for some normal subgroup $H$ of $G$ of finite index.
(c) $G$ is a finite extension of a residually finite nilpotent $\pi$-group.
(d) $G$ is a finite extension of a reduced soluble FATR group with no $\pi$-divisible elements of infinite order.

**Corollary** Suppose $G$ is a finite extension of a soluble FAR group. The following are equivalent.

(a) There exists a finite set $\pi$ of primes such that $G$ is residually a finite $\pi$-group.
(b) $G$ is reduced and a finite extension of an FATR group.
(c) $\tau(G)$ is finite and $\zeta_1(\text{Fitt}G)$ is reduced.

Soluble FAR and FATR groups are defined in [3]. An equivalent definition, often more convenient, is the following. A soluble group $G$ is FAR if it has finite Hirsch number and satisfies $\text{min-p}$ for every prime $p$. (A group $G$ has Hirsch number $h$ if $G$ has a series of finite length with exactly $h$ of the factors infinite cyclic, the remaining factors of the series being locally finite; $G$ satisfies $\text{min-p}$ if it satisfies the minimal condition on $p$-subgroups.)

It is elementary that to within some normal subgroup of finite index, locally finite factors can essentially be moved down a series past torsion-free abelian factors of finite rank and finite factors can be moved up past torsion-free abelian factors of finite rank and past periodic abelian factors satisfying $\text{min-p}$ for all primes $p$. Further divisible abelian factors in a periodic FAR group sink to the bottom (e.g. [2] 3.18). Thus it is elementary to see that a group $G$ is a finite extension of a soluble FAR group if and only if it has a characteristic series

$$\langle 1 \rangle = G_0 \leq G_1 \leq \cdots \leq G_r \leq \cdots \leq G_s \leq G,$$

where $G_1$ is periodic, divisible, abelian and satisfies $\text{min-p}$ for all primes $p$, $G_{i+1}/G_i$ for $1 \leq i < r$ is infinite periodic abelian with all its primary components (i.e. its Sylow subgroups) finite, $G_{i+1}/G_i$ for $r \leq i < s$ is torsion-free abelian of finite rank and $G/G_s$ is finite. The soluble FATR groups are exactly those $G$ above with $G_1$ involving only finitely many primes, with $r = 1$ and, of course, with $G/G_s$ soluble.

Suppose $G$ is a finite extension of a soluble FAR group with its maximal periodic normal subgroup $\tau(G)$ finite. Then from the above, the following hold.

(a) $G$ is (torsion-free)-by-finite and
(b) $G$ is a finite extension of a soluble FATR group. Further
(c) $\text{Fitt}G$ is nilpotent and $G/\text{Fitt}G$ is abelian-by-finite (see [3] 5.2.2) and
(d) $G/\text{Fitt}G$ is a finite extension of a free abelian group of finite rank (see [3] 5.2.3).

Note that in general $\text{Fitt}G_s = G_s \cap \text{Fitt}G$, $\text{Fitt}G/\text{Fitt}G_s$ is finite and $\text{Fitt}G$ is nilpotent if and only $\text{Fitt}G_s$ is nilpotent. Also the soluble groups $G$ of finite rank discussed by Azarov in [1] are exactly FAR groups above with $G/G_s$ soluble and $G_r$ of finite rank.

**The proofs**

We use the following elementary results.

**Lemma 1** Let $A$ be a torsion-free abelian group.

(a) If $\pi$ is any set of primes then $A$ is residually a finite $\pi$-group if and only if $A$ is $\pi$-reduced.
(b) If $A$ is reduced and of finite rank, then for some finite set $\pi$ of primes, $A$ is residually a finite $\pi$-group.

For example see [3] 5.3.4 and 5.3.5. We also use the following, see [4] 9.38.
Lemma 2  Let $G$ be a nilpotent FAR group and $\pi$ any set of primes. Then $G$ is residually a finite $\pi$-group if and only if $\zeta_1(G)$ is $\pi$-reduced.

Thus in Lemma 2 if $\zeta_1(G)$ is $\pi$-reduced, then $G$ is residually a finite $\pi$-group, so each $G/\zeta_i(G)$ is also residually a finite $\pi$-group by Learner’s Lemma. Hence each $\zeta_{i+1}(G)/\zeta_i(G)$ is $\pi$-reduced, something that is easy to see directly.

The proof of the theorem  (a) implies (b). Let $H$ be a normal subgroup of $G$ that is residually a finite $\pi$-group. Then $\zeta_1(FittH)$ is $\pi$-reduced by Lemma 1. Also $\tau(H)$ is a $\pi$-group. If $P$ is a $p$-subgroup of $\tau(H)$ then $P$ is Chernikov, residually finite and hence finite, and $\pi$ is finite. Consequently $\tau(H)$ is finite. Clearly $\tau(G)/\tau(H)$ is finite, so $\tau(G)$ is finite.

(b) implies (c). (This is actually the core of the proof of the Theorem.) There exist normal subgroups $N \leq L \leq H$, $G/L$ finite, $L/N$ free abelian of finite rank and $N = FittL$ torsion-free nilpotent of finite rank. Clearly $N \leq FittH$. By (b) and Lemma 2 $FittH$ is residually a finite $\pi$-group, so $N$ is residually a finite $\pi$-group.

Set $q = \Pi_{p \in \pi} p$ and $M = \cap_{i \geq 1} \langle \zeta_{i+1}(N)/\zeta_i(N) \rangle$. Clearly $N \leq M \leq L$ and $G/M$ is finite. We claim that $M$ is residually a finite nilpotent $\pi$-group. If so then (c) holds. Now $M/N$ is free abelian, so $M/N$ at least is residually a finite nilpotent $\pi$-group.

Let $x \in N \setminus \langle 1 \rangle$. Since $N$ is residually a finite $\pi$-group, there exists a power $m = q^\mu$ of $q$ with $x \notin N^m$. Now $N/N^m$ is finite and $M/N$ is polycyclic, so there exists a torsion-free normal subgroup $T/N^m$ in $M/N^m$ with $M/T$ finite. Also $N/N^m$ lies in the hypercentre of $M/N^m$ and $M/N$ is abelian. Hence $M/N^m$ is nilpotent, as therefore is its finite image $M/T$. Let $S/T$ denote the Hall $\pi'$-subgroup of $M/T$. Then $M/S$ is a finite nilpotent $\pi'$-group. Further $x \notin S$, since $x$ is a non-trivial $\pi$-element modulo $N/m$. Now $M/N$ is free abelian, so $M/N$ at least is residually a finite nilpotent $\pi$-group.

(c) implies (a). This is trivial. Thus (a), (b) and (c) are equivalent.

(a) and (b) imply (d). $G$ is residually finite, so $G$ is reduced. Also $\tau(G)$ is finite, so $G$ is a finite extension of a residually finite-$\pi$, reduced FATR group $H$. Since $H$ is residually finite-$\pi$, $H$ contains no $\pi$-divisible elements of infinite order. Consequently neither does $G$.

(d) implies (b). By (d) $G$ has a reduced soluble normal FATR subgroup $H$ of finite index. Then $H$ is (torsion-free)-by-finite, $\tau(H)$ is finite and consequently $\tau(G)$ is finite. Further we may choose $H$ torsion-free. Then $H$ has no non-trivial $\pi'$-divisible elements by (d) and hence $\zeta_1(Fitt\tau(G))$ is $\pi$-reduced. Thus (b) holds.

The proof of the corollary  If (a) holds, then so does (b) by the Theorem. Clearly (b) implies (c). Suppose (c) holds. By Lemma 1 there exists a finite set $\kappa$ of primes such that $\zeta_1(FittG)$ is $\kappa$-reduced. Hence by the Theorem, (b) implies (a), there exists a normal subgroup $H$ of $G$ of finite index that is residually a finite $\kappa$-group. But then $G$ is residually a finite $\pi$-group for $\pi = \kappa \cup \{\text{all prime divisors of } (G:H)\}$. Thus (a) holds.

Remark  In a special but still quite general case there is a slicker but less elementary proof of (b) implies (c), the main implication of the theorem.

With $N$ as in the original proof let $N_p$ denote the finite-$p$ residual of $N$. Then $\cap_{p \in \pi} N_p = \langle 1 \rangle$. The upper central factors of $N/N_p$ are $p$-reduced (Lemma 2). If they are actually (torsion-free)-by-(a $p'$-group)-by-finite, then $G/N_p$ embeds into $GL(n, J)$ for some integer $n$ and $J$ the integers localized at $p$. Thus $G/N_p$ is a finite extension of a residually finite $p$-group and consequently $G$ is a finite extension of a residually finite nilpotent $\pi$-group.
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