"Groundstates of nonlinear Choquard equations: Hardy–Littlewood–Sobolev critical exponent"

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Abstract
We consider nonlinear Choquard equation \(-\Delta u + Vu = (I_{\alpha} * |u|^{(\alpha/N + 1)})|u|^{(\alpha/N - 1)} u\) where \(N \geq 3\), \(V \in L^\infty(\mathbb{R}^N)\) is an external potential and \(I_\alpha(x)\) is the Riesz potential of order \(\alpha \in (0, N)\). The power in the nonlocal part of the equation is critical with respect to the Hardy–Littlewood–Sobolev inequality. As a consequence, in the associated minimization problem a loss of compactness may occur. We prove that if \(\liminf_{x\to\infty} (1-V(x))|x|^2 > N^2(N - 2)/(4(N + 1))\) then the equation has a nontrivial solution. We also discuss some necessary conditions for the existence of a solution. Our considerations are based on a concentration compactness argument and a nonlocal version of Brezis–Lieb lemma.

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\[-\Delta u + Vu = (I_\alpha * |u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^N,\]
where \(N \geq 3\), \(V \in L^\infty(\mathbb{R}^N)\) is an external potential and \(I_\alpha(x)\) is the Riesz potential of order \(\alpha \in (0, N)\). The power \(\frac{N}{p} + 1\) in the nonlocal part of the equation is critical with respect to the Hardy-Littlewood-Sobolev inequality. As a consequence, in the associated minimization problem a loss of compactness may occur. We prove that if \(\liminf_{|x| \to \infty} (1 - V(x))|x|^2 > \frac{N^2}{4(N+2)}\) then the equation has a nontrivial solution. We also discuss some necessary conditions for the existence of a solution. Our considerations are based on a concentration compactness argument and a nonlocal version of Brezis-Lieb lemma.

1. Introduction and results

We consider a nonlinear Choquard type equation
\[(P)\quad -\Delta u + Vu = (I_\alpha * |u|^p)|u|^{p-2}u \quad \text{in } \mathbb{R}^N,\]
where \(N \in \mathbb{N}\), \(\alpha \in (0, N)\), \(p > 1\), \(I_\alpha : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}\) is the Riesz potential of order \(\alpha \in (0, N)\) defined for every \(x \in \mathbb{R}^N \setminus \{0\}\) by
\[I_\alpha(x) = \frac{\Gamma(N-\alpha)}{2^{\alpha} \pi^{N/2} \Gamma(\frac{\alpha}{2}) |x|^{N-\alpha}},\]
and \(V \in L^\infty(\mathbb{R}^N)\) is an external potential.

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For $N = 3$, $\alpha = 2$ and $p = 2$ equation (P) is the Choquard-Pekar equation which goes back to the 1954’s work by S. I. Pekar on quantum theory of a Polaron at rest [6, Section 2.1; 20] and to 1976’s model of P. Choquard of an electron trapped in its own hole, in an approximation to Hartree-Fock theory of one-component plasma [8]. In the 1990’s the same equation reemerged as a model of self-gravitating matter [7; 19] and is known in that context as the Schrödinger-Newton equation.

Mathematically, the existence and qualitative properties of solutions of Choquard equation (P) have been studied for a few decades by variational methods, see [8; 11; 12, Chapter III; 14] for earlier and [2; 5; 13; 16; 18] for recent work on the problem and further references therein.

The following sharp characterisation of the existence and nonexistence of nontrivial solutions of (P) in the case of constant potential $V$ can be found in [16].

**Theorem 1** (Ground states of (P) with constant potential [16, theorems 1 and 2]). Assume that $V \equiv 1$. Then (P) has a nontrivial solution $u \in H^1(\mathbb{R}^N) \cap L^{2N/\alpha}(\mathbb{R}^N)$ with $\nabla u \in H^{1,\alpha}_{\text{loc}}(\mathbb{R}^N) \cap L^{2N/\alpha}_{\text{loc}}(\mathbb{R}^N)$ if and only if $p \in (\frac{N}{\alpha} + 1, \frac{N+\alpha}{N-2})$.

If $p \in (\frac{N}{\alpha} + 1, \frac{N+\alpha}{N-2})$ then $H^1(\mathbb{R}^N) \subset L^{2N/\alpha}(\mathbb{R}^N)$ by the Sobolev inequality, and moreover, every $H^1$-solution of (P) belongs to $W^{2,p}_{\text{loc}}(\mathbb{R}^N)$ for any $p \geq 1$ by a regularity result in [17, proposition 3.1]. This implies that the Choquard equation (P) with a positive constant potential has no $H^1$-solutions at the end-points of the above existence interval.

In this note we are interested in the existence and nonexistence of solutions to (P) with nonconstant potential $V$ at the lower critical exponent $p = \frac{N}{\alpha} + 1$, that is, we consider the problem

$$(P) \quad -\Delta u + Vu = (I_\alpha * |u|^{\frac{\alpha+1}{\alpha}})|u|^{\frac{\alpha}{\alpha-1}} \quad \text{in } \mathbb{R}^N.$$ 

The exponent $\frac{\alpha}{\alpha} + 1$ is critical with respect to the Hardy-Littlewood-Sobolev inequality, which we recall here in a form of minimization problem

$$c_\infty = \inf \left\{ \int_{\mathbb{R}^N} |u|^2 \mid u \in L^2(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{\alpha+1}{\alpha}})|u|^{\frac{\alpha}{\alpha} + 1} = 1 \right\} > 0.$$ 

**Theorem 2** (Optimal Hardy-Littlewood-Sobolev inequality [9, theorem 3.1; 10, theorem 4.3]). The infimum $c_\infty$ is achieved if and only if

$$(1.1) \quad u(x) = C \left( \frac{\lambda}{\lambda^2 + |x-a|^2} \right)^{N/2},$$

where $C > 0$ is a fixed constant, $a \in \mathbb{R}^N$ and $\lambda \in (0, \infty)$ are parameters.

The form of minimizers in theorem 2 suggests that a loss of compactness in (P) may occur by translations and dilations.

In order to characterise the existence of nontrivial solutions for the lower critical Choquard equation (P) we define the critical level

$$c_* = \inf \left\{ \int_{\mathbb{R}^N} |
abla u|^2 + V|u|^2 \mid u \in H^1(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{\alpha+1}{\alpha}})|u|^{\frac{\alpha}{\alpha} + 1} = 1 \right\}.$$
It can be checked directly that if \( u \in H^1(\mathbb{R}^N) \) achieves the infimum \( c_* \), then a multiple of the minimizer \( u \) is a weak solution of Choquard equation \( (P_*) \).

Using a Brezis-Lieb type lemma for Riesz potentials [16, lemma 2.4] and a concentration compactness argument (lemma 10), we establish our main abstract result.

**Theorem 3** (Existence of a minimizer). Assume that \( V \in L^\infty(\mathbb{R}^N) \) and

\[
\liminf_{|x| \to \infty} |x|^2 > 0,
\]

If \( c_* < c_\infty \) then the infimum \( c_* \) is achieved and every minimizing sequence for \( c_* \) up to a subsequence converges strongly in \( H^1(\mathbb{R}^N) \).

The inequality for the existence of minimizers is sharp, as shown by the following lemma for constant potentials.

**Lemma 4.** If \( V \equiv 1 \), then \( c_* = c_\infty \).

Since problem \( (P_*) \) with \( V \equiv 1 \) has no \( H^1 \)-solutions, this shows that the strict inequality \( c_* < c_\infty \) is indeed essential for the existence of a minimizer for \( c_* \).

In fact, the strict inequality \( c_* < c_\infty \) is necessary at least for the strong convergence of all minimizing sequences.

**Proposition 5.** Let \( V \in L^\infty(\mathbb{R}^N) \). If

\[
\limsup_{|x| \to \infty} V(x) \leq 1,
\]

then

\[
c_* \leq c_\infty.
\]

In addition, if

\[
c_* = c_\infty,
\]

then there exists a minimizing sequence for \( c_* \) which converges weakly to 0 in \( H^1(\mathbb{R}^N) \).

Using Hardy-Littlewood-Sobolev minimizers (1.1) as a family of test functions for \( c_* \), we establish a sufficient condition for the strict inequality \( c_* < c_\infty \).

**Theorem 6.** Let \( V \in L^\infty(\mathbb{R}^N) \). If

\[
\liminf_{|x| \to \infty} (1 - V(x)) |x|^2 > \frac{N^2(N - 2)}{4(N + 1)},
\]

then \( c_* < c_\infty \) and hence the infimum \( c_* \) is achieved.

In particular, if \( N = 1, 2 \) then condition (1.3) reduces to

\[
\liminf_{|x| \to \infty} (1 - V(x)) |x|^2 > 0,
\]

that is, the potential \( 1 - V \) should not decay to zero at infinity faster then the inverse square of \( |x| \).

Employing a version of Pohožaev identity for Choquard equation \( (P_*) \) (see proposition 11 below), we show that a certain control on the potential \( V \) is indeed necessary for the strict inequality \( c_* < c_\infty \).
Proposition 7. Let $V \in C^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. If
\begin{equation}
\sup_{x \in \mathbb{R}^N} \left\{ \int_{\mathbb{R}^N} \frac{1}{2} (\nabla V(x)|x|) |\varphi(x)|^2 \, dx \mid \varphi \in C^1_c(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} |\nabla \varphi|^2 \leq 1 \right\} < 1, \tag{1.4}
\end{equation}
then Choquard equation $(P_\alpha)$ does not have a nonzero solution $u \in H^1(\mathbb{R}^N) \cap W^{2,2}_{\text{loc}}(\mathbb{R}^N)$.

In particular, combining (1.4) with Hardy’s inequality on $\mathbb{R}^N$, we obtain a simple nonexistence criterion.

Proposition 8. Let $N \geq 3$ and $V \in C^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. If for every $x \in \mathbb{R}^N$,
\begin{equation}
\sup_{x \in \mathbb{R}^N} |x|^2 (\nabla V(x)|x|) < \frac{(N-2)^2}{2}, \tag{1.5}
\end{equation}
then Choquard equation $(P_\alpha)$ does not have a nonzero solution $u \in H^1(\mathbb{R}^N) \cap W^{2,2}_{\text{loc}}(\mathbb{R}^N)$.

For example, for $N \geq 3$ and $\mu > 0$, we consider a model equation
\begin{equation}
- \Delta u + \left( 1 - \frac{\mu}{1 + |x|^2} \right) u = (I_\alpha * |u|^{\frac{N}{N-2}+1}) |u|^{\frac{N}{N-2}-1} u \quad \text{in } \mathbb{R}^N. \tag{1.6}
\end{equation}

Then proposition 8 implies that (1.6) has no nontrivial solutions for $\mu < \frac{(N-2)^2}{4}$, while for $\mu > \frac{N^2(N-2)}{4(N+1)}$, assumption (1.3) is satisfied and hence $(P_\alpha)$ admits a groundstate. We note that
\[
\frac{(N-2)^2}{4(N+1)} = 1 - \frac{N-2}{N^2},
\]
so that the two bounds are asymptotically sharp when $N \to \infty$. We leave as an open question whether (1.6) admits a ground state for $\mu \in \left[ \frac{(N-2)^2}{4}, \frac{N^2(N-2)}{4(N+1)} \right]$.

We emphasise that unlike the asymptotic sufficient existence condition (1.3), nonexistence condition (1.5) is a global condition on the whole of $\mathbb{R}^N$. For example, a direct computation shows that for $a = 0$ and every $\lambda > 0$, a multiple of the Hardy-Littlewood-Sobolev minimizer $(1.1)$ solves the equation
\begin{equation}
- \Delta u + \left( 1 + \frac{N(2|\lambda|^2 - N\lambda^2)}{(|x|^2 + \lambda^2)^2} \right) u = (I_\alpha * |u|^{\frac{N}{N-2}+1}) |u|^{\frac{N}{N-2}-1} u \quad \text{in } \mathbb{R}^N. \tag{1.7}
\end{equation}

Here (1.3) fails on an annulus centered at the origin, while $V(x) > 1$ and $(\nabla V(x)|x| < 0$ for all $|x|$ sufficiently large. Moreover,
\[
\lim_{|x| \to \infty} (1 - V(x)) |x|^2 = -2N < 0 \leq \frac{N^2(N-2)}{4(N+1)}.
\]

Note that the constructed solution $u_\lambda$ satisfies
\[
\int_{\mathbb{R}^N} |\nabla u_\lambda|^2 + V |u_\lambda|^2 = 0.
\]

In particular, we are unable to conclude that $c_* < c_\infty$. We do not know whether $u_\lambda$ is a groundstate of (1.7). However, if $u_\lambda$ was not a groundstate, then we would have $c_* < c_\infty$ and (1.7) would then have a groundstate by theorem 8.
2. Existence of minimizers under strict inequality: proof of theorem \( \underline{3} \)

In order to prove theorem \( \underline{3} \) we will use a special case of the classical Brezis-Lieb lemma \( \underline{1} \) for Riesz potentials.

**Lemma 9** (Brezis-Lieb lemma for the Riesz potential \( \underline{10} \) lemma 2.4)). Let \( N \in \mathbb{N} \), \( \alpha \in (0, N) \), and \( (u_n)_{n \in \mathbb{N}} \) be a bounded sequence in \( L^2(\mathbb{R}^N) \). If \( u_n \to u \) almost everywhere on \( \mathbb{R}^N \) as \( n \to \infty \), then

\[
\int_{\mathbb{R}^N} (I_\alpha * |u_n|^\frac{\alpha+1}{\alpha}) |u|^\frac{\alpha+1}{\alpha} = \lim_{n \to \infty} \int_{\mathbb{R}^N} (I_\alpha * |u_n|^\frac{\alpha+1}{\alpha}) |u_n|^\frac{\alpha+1}{\alpha} - \int_{\mathbb{R}^N} (I_\alpha * |u_n - u|^\frac{\alpha+1}{\alpha}) |u_n - u|^\frac{\alpha+1}{\alpha}.
\]

Our second result is a concentration type lemma.

**Lemma 10.** Assume that \( V \in L^\infty(\mathbb{R}^N) \) and \( \liminf_{|x| \to \infty} V(x) \geq 1 \). If the sequence \( (u_n)_{n \in \mathbb{N}} \) is bounded in \( L^2(\mathbb{R}^N) \) and converges in \( L^2_{\text{loc}}(\mathbb{R}^N) \) to \( u \) as \( n \to \infty \), then

\[
\int_{\mathbb{R}^N} |u|^2 \leq \liminf_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^2 - \int_{\mathbb{R}^N} |u_n - u|^2.
\]

**Proof.** Since the sequence \( (u_n)_{n \in \mathbb{N}} \) is bounded in \( L^2(\mathbb{R}^N) \) and converges in measure to \( u \), we deduce by the Brezis-Lieb lemma \( \underline{1} \) (see also \( \underline{10} \) theorem 1.9)) that

\[
\int_{\mathbb{R}^N} |u|^2 = \lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^2 - \int_{\mathbb{R}^N} |u_n - u|^2.
\]

Now, we observe that for every \( R > 0 \) and every \( n \in \mathbb{N} \),

\[
\int_{\mathbb{R}^N} (1 - V)|u_n - u|^2 \leq \int_{B_R} (1 - V)|u_n - u|^2 + (1 - \inf_{\mathbb{R}^N \setminus B_R} V) + \int_{\mathbb{R}^N} |u_n - u|^2.
\]

By the local \( L^2_{\text{loc}}(\mathbb{R}^N) \) convergence, we note that

\[
\lim_{n \to \infty} \int_{B_R} (1 - V)|u_n - u|^2 = 0.
\]

Since \( \lim_{R \to \infty} (1 - \inf_{\mathbb{R}^N \setminus B_R} V) = 0 \) and \( (u_n - u)_{n \in \mathbb{N}} \) is bounded in \( L^2(\mathbb{R}^N) \), we conclude that

\[
\limsup_{n \to \infty} \int_{\mathbb{R}^N} (1 - V)|u_n - u|^2 \leq 0;
\]

the conclusion follows. \( \square \)

**Proof of theorem \( \underline{3} \)** Let \( (u_n)_{n \in \mathbb{N}} \subset H^1(\mathbb{R}^N) \) be a minimizing sequence for \( c_* \), that is

\[
\int_{\mathbb{R}^N} (I_\alpha * |u_n|^\frac{\alpha+1}{\alpha}) |u_n|^\frac{\alpha+1}{\alpha} = 1
\]

and

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 + V|u_n|^2 \to c_*.
\]

In view of our assumption \( \underline{12} \) we observe that the sequence \( (u_n)_{n \in \mathbb{N}} \) is bounded in \( H^1(\mathbb{R}^N) \). So, there exists \( u \in H^1(\mathbb{R}^N) \) such that, up to a subsequence, the sequence \( (u_n)_{n \in \mathbb{N}} \) converges to \( u \) weakly in \( H^1(\mathbb{R}^N) \) and, by the classical Rellich-Kondrachov
compactly, strongly in $L^2_{\text{loc}}(\mathbb{R}^N)$. By the lower semi-continuity of the norm under weak convergence,

$$\int_{\mathbb{R}^N} |\nabla u|^2 + V|u|^2 \leq \lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 + V|u_n|^2 = c_*,$$

and by Fatou’s lemma

$$\int_{\mathbb{R}^N} (I_\alpha * |u_n|^\frac{N+1}{N})|u_n|^\frac{N}{N+1} \leq 1.$$

In order to conclude, it suffices to prove that equality is achieved in the latter inequality.

We observe that by lemma 10

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} (I_\alpha * |u_n - u|^\frac{N+1}{N})|u_n - u|^\frac{N}{N+1} = 1 - \int_{\mathbb{R}^N} (I_\alpha * |u|^\frac{N+1}{N})|u|^\frac{N}{N+1}$$

while by lemma 10 and by the lower-semicontinuity of the norm under weak convergence,

$$\int_{\mathbb{R}^N} |\nabla u|^2 + V|u|^2 \leq \lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 + \lim_{n \to \infty} \int_{\mathbb{R}^N} V|u_n|^2 - |u_n - u|^2$$

$$\leq \lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 + V|u_n|^2 - |u_n - u|^2$$

$$= c_* - \limsup_{n \to \infty} \int_{\mathbb{R}^N} |u_n - u|^2.$$

By definition of $c_\infty$, we have

$$\int_{\mathbb{R}^N} |u_n - u|^2 \geq c_\infty \left(\int_{\mathbb{R}^N} (I_\alpha * |u_n - u|^\frac{N+1}{N})|u_n - u|^\frac{N}{N+1}\right)^\frac{N}{N+1}.$$

Therefore, we conclude that

$$\int_{\mathbb{R}^N} |\nabla u|^2 + V|u|^2 \leq c_* - c_\infty \left(1 - \int_{\mathbb{R}^N} (I_\alpha * |u|^\frac{N+1}{N})|u|^\frac{N}{N+1}\right)^\frac{N}{N+1}.$$

In view of the definition of $c_\alpha$, this implies that

$$c_* \geq c_\infty \left(1 - \int_{\mathbb{R}^N} (I_\alpha * |u|^\frac{N+1}{N})|u|^\frac{N}{N+1}\right)^\frac{N}{N+1} + c_\alpha \left(\int_{\mathbb{R}^N} (I_\alpha * |u|^\frac{N+1}{N})|u|^\frac{N}{N+1}\right)^\frac{N}{N+1}.$$

Since by assumption $c_* < c_\infty$, we conclude that

$$\int_{\mathbb{R}^N} (I_\alpha * |u|^\frac{N+1}{N})|u|^\frac{N}{N+1} = 1,$$

and hence, by definition of $c_\alpha$,

$$\int_{\mathbb{R}^N} |\nabla u|^2 + V|u|^2 = c_\alpha,$$

that is the infimum $c_\alpha$ is achieved at $u$. Moreover, from (2.1) we conclude that $u_n \to u$ in $L^2(\mathbb{R}^N)$. Since $V \in L^\infty(\mathbb{R}^N)$, this implies that $Vu_n \to Vu$ in $L^2(\mathbb{R}^N)$. Using (2.1) again, we conclude that

$$\int_{\mathbb{R}^N} |\nabla u|^2 = \lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla u_n|^2.$$

Since $(u_n)_{n \in \mathbb{N}}$ converges to $u$ weakly in $H^1(\mathbb{R}^N)$, this implies that $(u_n)_{n \in \mathbb{N}}$ also converges to $u$ strongly in $H^1(\mathbb{R}^N)$. □
3. Optimality of the Strict Inequality

In this section we prove lemma 4 and proposition 5.

Proof of lemma 4. Let us denote by $\tilde{c}_\infty$ the infimum on the right-hand side. By density of the space $H^1(\mathbb{R}^N)$ in $L^2(\mathbb{R}^N)$ and by continuity in $L^2$ of the integral functionals involved in the definition of $c_\infty$, it is clear that $\tilde{c}_\infty \geq c_\infty$. We choose now $u \in H^1(\mathbb{R}^N)$ and define for every $x \in \mathbb{R}^N$ by

$$u_\lambda(x) = \lambda^{N/2} u(\lambda x).$$

We compute for every $\lambda > 0$ that

$$\int_{\mathbb{R}^N} (I_\alpha * |u_\lambda(\nabla) + 1)|u_\lambda(\nabla) + 1 = \int_{\mathbb{R}^N} (I_\alpha * |u(\nabla) + 1)|u(\nabla) + 1$$

and

$$\int_{\mathbb{R}^N} |\nabla u_\lambda|^2 + |u_\lambda|^2 = \frac{1}{\lambda^2} \int_{\mathbb{R}^N} |\nabla u|^2 + \int_{\mathbb{R}^N} |u|^2.$$

Hence,

$$\inf_{\lambda > 0} \int_{\mathbb{R}^N} |\nabla u_\lambda|^2 + |u_\lambda|^2 = \int_{\mathbb{R}^N} |u|^2,$$

and we conclude that $\tilde{c}_\infty \leq c_\infty$. □

Proof of proposition 5. For $\lambda > 0$, let

$$u_\lambda(x) = C \left( \frac{\lambda}{\lambda^2 + |x|^2} \right)^{N/2} = \lambda^{-N/2} u_1(\frac{x}{\lambda})$$

be a family of minimizers for $c_\infty$ as in (1.1). We observe that

$$\int_{\mathbb{R}^N} (I_\alpha * |u_\lambda(\nabla) + 1)|u_\lambda(\nabla) + 1 = c_\infty + \int_{\mathbb{R}^N} V \cdot |u_\lambda|^2$$

whereas by a change of variables,

$$\int_{\mathbb{R}^N} |\nabla u_\lambda|^2 + V |u_\lambda|^2 = \frac{1}{\lambda^2} \int_{\mathbb{R}^N} |\nabla u_1|^2 + \int_{\mathbb{R}^N} V \left( \frac{y}{\lambda} \right) \frac{C^2}{1 + |y|^2} dy.$$

By Lebesgue’s dominated convergence theorem

$$\limsup_{\lambda \to 0} \int_{\mathbb{R}^N} V \left( \frac{y}{\lambda} \right) \frac{C^2}{1 + |y|^2} dy \leq \int_{\mathbb{R}^N} \frac{C^2}{1 + |y|^2} dy = c_\infty,$$

so we conclude that $c_* \leq c_\infty$. If, in addition, $c_* = c_\infty$ then for any $\lambda_n \to 0$, $(u_{\lambda_n})_{n \in \mathbb{N}}$ is a minimizing sequence for $c_*$, and the conclusion follows. □

4. Sufficient Conditions for the Strict Inequality: Proof of Theorem 6

For $a \in \mathbb{R}^N$ and $\lambda > 0$, let

$$u_\lambda(x) = C \left( \frac{\lambda}{\lambda^2 + |x-a|^2} \right)^{N/2}$$

be a family of minimizers for $c_\infty$ as in (1.1). Then

$$\int_{\mathbb{R}^N} |\nabla u_\lambda|^2 + V |u_\lambda|^2 = c_\infty + \int_{\mathbb{R}^N} |\nabla u_\lambda|^2 + \int_{\mathbb{R}^N} (V - 1) |u_\lambda|^2.$$
Denote
\[ I_V(a, \lambda) := \lambda^2 \int_{\mathbb{R}^N} |\nabla u_\lambda|^2 + \lambda^2 \int_{\mathbb{R}^N} (V - 1)|u_\lambda|^2 < 0. \]

To obtain a sufficient conditions for \( c_* < c_\infty \) it is enough to show that for some \( a \in \mathbb{R}^N \),
\[(4.1) \quad \inf_{\lambda > 0} I_V(a, \lambda) < 0,\]

Proof of theorem 6. If \( N \leq 2 \), then by (1.3) there exists \( \mu > 0 \) such that
\[ \lim \inf_{|x| \to \infty} (1 - V(x))|x|^2 \geq \mu. \]

Therefore
\[ \lim_{\lambda \to \infty} \lambda^2 \int_{\mathbb{R}^N} (1 - V)|u_\lambda|^2 = \lim_{\lambda \to \infty} \lambda^2 \frac{(1 - V(\lambda x))}{(1 + |x|^2)^N} \] \[ \geq \int_{\mathbb{R}^N} \frac{\mu}{|x|^2(1 + |x|^2)^N} \] \[ = \infty. \]

Since for every \( \lambda > 0 \),
\[ \lambda^2 \int_{\mathbb{R}^N} |\nabla u_\lambda|^2 = \int_{\mathbb{R}^N} |\nabla u_1|^2 < \infty, \]

the condition (4.1) is satisfied.

If \( N \geq 3 \), we observe that for every \( \lambda > 0 \),
\[ \int_{\mathbb{R}^N} |\nabla u_\lambda|^2 = \frac{N^2(N - 2)}{4(N + 1)} \int_{\mathbb{R}^N} \frac{|u_\lambda(x)|^2}{|x|^2} \] \[ = \frac{N^2(N - 2)}{4(N + 1)} \int_{\mathbb{R}^N} \frac{1}{|x|^2(1 + |x|^2)^N} \] \[ = \infty. \]

This follows from the fact that
\[ \int_{\mathbb{R}^N} \frac{|x|^2}{(1 + |x|^2)^N + 2} dx = \frac{N - 2}{4(N + 1)} \int_{\mathbb{R}^N} \frac{1}{|x|^2(1 + |x|^2)^N} dx, \]

which can be proved by two successive integrations by parts. Then, after a transformation \( x = \lambda y + a \),
\[ I_V(a, \lambda) = \int_{\mathbb{R}^N} \frac{N^2(N - 2)}{4(N + 1)} |y|^2 - \lambda^2 (1 - V(a + \lambda y)) \] \[ \frac{C^2}{(1 + |y|^2)^N} dy, \]

and in view of (1.3), sufficient condition is (4.1) is satisfied for \( a = 0 \), so we conclude that \( c_* < c_\infty \). \( \Box \)

Note that if the function \( \lambda \to \lambda^2 (1 - V(a + \lambda y)) \) is nondecreasing for every \( y \in \mathbb{R}^N \),
then \( \lambda \mapsto I_V(a, \lambda) \) is nonincreasing. Therefore \( I_V(a, \lambda) \) admits negative values if and only if it has a negative limit as \( \lambda \to \infty \). The latter is ensured in theorem 6 via asymptotic condition (1.3). This explains that if the function \( \lambda \to \lambda^2 (1 - V(a + \lambda y)) \) is nondecreasing, like for instance, in the special case
\[ V(x) = 1 - \frac{\mu}{1 + |x|^2}, \]

then integral sufficient condition (1.1) is in fact equivalent to the asymptotic sufficient condition (1.3).
5. Pohožaev identity and necessary conditions for the existence

We establish a Pohožaev type identity, which extends the identities \([5.1]\) obtained previously for constant potentials \(V\) \([4]\) lemma 2.1; \([15]\) [16] proposition 3.1; \([17]\) theorem 3].

**Proposition 11.** Let \(N \geq 3\) and \(V \in C^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)\) and \(u \in W^{1,2}(\mathbb{R}^N)\). If
\[
\sup_{x \in \mathbb{R}^N} |(\nabla V(x))| x < \infty,
\]
and \(u \in W^{2,2}_{\text{loc}}(\mathbb{R}^N)\) satisfies Choquard equation \((P)\) then
\[
\int_{\mathbb{R}^N} |\nabla u|^2 = \frac{1}{2} \int_{\mathbb{R}^N} (\nabla V(x)| x) |u(x)|^2 \, dx.
\]

**Proof.** We fix a cut-off function \(\varphi \in C^1_c(\mathbb{R}^N)\) such that \(\varphi = 1\) on \(B_1\) and we test for \(\lambda \in (0, \infty)\) the equation against the function \(v_\lambda \in W^{1,2}(\mathbb{R}^N)\) defined for every \(x \in \mathbb{R}^N\) by
\[
v_\lambda(x) = \varphi(\lambda x)(\nabla u(x)| x)
\]
to obtain the identity
\[
\int_{\mathbb{R}^N} (\nabla u| \nabla v_\lambda) + \int_{\mathbb{R}^N} V uv_\lambda = \int_{\mathbb{R}^N} (I_\alpha * |u|^{\gamma + 1})|u|^{\gamma - 1} uv_\lambda.
\]
We compute for every \(\lambda > 0\), by definition of \(v_\lambda\), the chain rule and by the Gauss integral formula,
\[
\int_{\mathbb{R}^N} V uv_\lambda = \int_{\mathbb{R}^N} V(x)u(x)\varphi(\lambda x)| x \nabla u(x)| \, dx
\]
\[
= \int_{\mathbb{R}^N} V(x)\varphi(\lambda x)| x \nabla (\frac{|u|^2}{2})| \, dx
\]
\[
= - \int_{\mathbb{R}^N} (\nabla V(x) + (\nabla V(x)| x))\varphi(\lambda x) + V(x)\lambda x \nabla \varphi(\lambda x))| \frac{|u(x)|^2}{2} \, dx.
\]
Since \(\sup_{x \in \mathbb{R}^N} (\nabla V(x)| x) < \infty\), by Lebesgue’s dominated convergence theorem it holds
\[
\lim_{\lambda \to 0} \int_{\mathbb{R}^N} V uv_\lambda = -\frac{N}{2} \int_{\mathbb{R}^N} |u|^2 - \frac{1}{2} \int_{\mathbb{R}^N} (\nabla V(x)| x)| |u|^2.
\]
By Lebesgue’s dominated convergence again, since \(u \in W^{1,2}(\mathbb{R}^N)\), we have (see \([16]\) proof of proposition 3.1) for the details
\[
\lim_{\lambda \to 0} \int_{\mathbb{R}^N} (I_\alpha * |u|^{\gamma + 1})|u|^{\gamma - 1} uv_\lambda = -\frac{N}{2} \int_{\mathbb{R}^N} (I_\alpha * |u|^{\gamma + 1})|u|^{\gamma + 1}.
\]
We have thus proved the Pohožaev type identity
\[
(5.1) \quad \frac{N - 2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + \frac{N}{2} \int_{\mathbb{R}^N} V|u|^2 + \frac{1}{2} \int_{\mathbb{R}^N} (\nabla V(x)| x) |u(x)|^2 \, dx
\]
\[
= \frac{N}{2} \int_{\mathbb{R}^N} (I_\alpha * |u|^{\gamma + 1})|u|^{\gamma + 1}.
\]
If we test the equation against $u$, we obtain the identity
\[
\int_{\mathbb{R}^N} |\nabla u|^2 + \int_{\mathbb{R}^N} V |u|^2 = \int_{\mathbb{R}^N} (I_\alpha * |u|^{\frac{\alpha}{\alpha + 1}}) |u|^{\frac{\alpha}{\alpha + 1}};
\]
the combination of those two identities yields the conclusion. \(\Box\)

**Proof of propositions 7 and 8.** Proposition 7 is a direct consequence of proposition 11, while proposition 8 follows from proposition 11 and the classical optimal Hardy inequality on $\mathbb{R}^N$,
\[
\frac{(N-2)^2}{4} \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^2} \, dx \leq \int_{\mathbb{R}^N} |\nabla u|^2
\]
which is valid for all $u \in H^1(\mathbb{R}^N)$ (see for example [21, theorem 6.4.10 and exercise 6.8]). \(\Box\)

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