MEAN CURVATURE FLOW AND BERNSTEIN-CALABI RESULTS FOR SPACELIKE GRAPHS

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Abstract

This is a survey of our work on spacelike graphic submanifolds in pseudo-Riemannian products, namely on Heinz-Chern and Bernstein-Calabi results and on the mean curvature flow, with applications to the homotopy of maps between Riemannian manifolds.

1 Introduction

It has been an important issue in geometry and in topology to determine when a map $f : \Sigma_1 \to \Sigma_2$ between manifolds can be homotopically deformed to a constant one. If each $\Sigma_i$ has a Riemannian structure $g_i$, the curvature of these spaces may give an answer. This is particularly more complex if $\Sigma_i$ are both compact. For $\Sigma_i$ noncompact, by a famous result due to Gromov (8), $\Sigma_i$ admits a Riemannian metric of negative sectional curvature and also one of positive sectional curvature. In each of these cases, if $\Sigma_i$ is complete and simply connected, then $\Sigma_i$ is diffeomorphic to a contractible space, by the Cartan-Hadmard theorem and by a result of Cheeger and Gromoll, respectively (see in 41). If this is the case for one of the $\Sigma_i$, then $f$ is obviously homotopically trivial.

A deformation problem of an initial map can be handled using some geometric evolution equation, obtaining homotopic deformations of a certain type and with geometrical and analytical meaning, namely, giving at infinite time a solution of a certain partial differential equation. We recall the great discovery of Eells and
Sampson ([7]), a first example of this kind, on using the heat flow to deform a map to an harmonic one:

**Theorem 1 (Eells and Sampson (1964))**  If $\Sigma_1$ and $\Sigma_2$ are closed and $\Sigma_2$ has non-positive sectional curvature then $f$ is homotopic to a harmonic map $f_\infty$. Furthermore, if the Ricci tensor of $\Sigma_1$ is nonnegative then $f_\infty$ is totally geodesic and if it is positive somewhere, then $f_\infty$ is constant.

The last part of this theorem can be seen as a Bernstein-type theorem, and it was obtained from a Weitzenböck formula for the Laplacian of $\|df_\infty\|^2$. We recall that Bernstein-type theorems are theorems that give conditions that ensures that a solution of certain P.D.Es. with geometrical meaning, must be a a ”trivial” solution, as for example a totally geodesic or a constant map.

In this note, a survey of our main results in [10, 11, 12], we will show how to use the mean curvature flow and a Bernstein-Calabi type result for spacelike graphs to obtain a deformation of a map between Riemannian manifolds to a totally geodesic or a constant one.

The Bernstein-Calabi result is obtained by computing the Laplacian of a positive geometric quantity, the hyperbolic cosine of the hyperbolic angle of a spacelike graph, and analyzing the sign of this Laplacian, based on an idea of Chern [5] of computing a similar quantity in the Riemannian case.

Furthermore, we also will show that under somehow more general curvature conditions as in the above theorem, we can obtain a direct proof of the homotopy to a constant map, with no need to use a Bernstein-type result. This approach was started by Wang [14] for the graph $\Gamma_f$ of the map $f$, considered as a submanifold of the Riemannian product $\Sigma_1 \times \Sigma_2$ of closed spaces with constant sectional curvature, and take its mean curvature flow and show that under certain conditions the flow preserves the graphic structure of the submanifold and converges to the graph of a constant map. The main difference with our approach is that we consider the pseudo-Riemannian structure on $\Sigma_1 \times \Sigma_2$ instead the Riemannian one. Our assumption on $\Gamma_f$ to be a spacelike submanifold is essentially identical to the assumptions on the eigenvalues of $f^* g_2$ imposed in [14] in the corresponding Riemannian setting. Our advantage is that the pseudo-Riemannian setting turns out to be a more natural one, since it allows less restrictive assumptions on the curvature tensors (and that include the case of any negative sectional curvature for $\Sigma_2$) and on the map $f$ itself after a suitable rescaling of the metric of $\Sigma_2$, and long time existence and convergence of the flow are easier obtained. In [14] it is necessary to use a White’s regularity theorem, based on a monocity formula due to Huisken, to detect possible singularities of the mean curvature flow, while in the
pseudo-Riemannian case, because of good signature in the evolution equations, we have better regularity. This will become clear in equations (1), (3) and (4) below.

Let \((\Sigma_1, g_1)\) and \((\Sigma_2, g_2)\) be Riemannian manifolds of dimension \(m \geq 2\) and \(n \geq 1\) respectively, and of sectional curvatures \(K_i\) and Ricci tensors \(Ricci_i\). On \(\mathcal{M} = \Sigma_1 \times \Sigma_2\) we consider the pseudo-Riemannian metric \(\bar{g} = g_1 - g_2\). We assume \(\Sigma_1\) oriented. Given a map \(f\), we assume the graph, \(\Gamma_f : \Sigma_1 \to \mathcal{M}, \Gamma_f(p) = (p, f(p))\), is a spacelike submanifold that is, \(g := \Gamma_f^* \bar{g} = g_1 - f^* g_2\) is a Riemannian metric on \(\Sigma_1\). Thus, the eigenvalues of \(f^* g_2\), at \(p \in M\), \(\lambda_1^2 \geq \ldots \geq \lambda_m^2 \geq 0\), are bounded from above by \(1 - \delta(p)\), where \(0 < \delta(p) \leq 1\) is a constant depending on \(p\). The hyperbolic angle \(\theta\) of \(\Gamma_f\) is given by one of the equivalent definitions:

\[
\cosh \theta = \left( \prod_i (1 - \lambda_i^2) \right)^{-1/2} = \frac{Vol_{\Sigma_1}(\pi_1(e_1), \ldots, \pi_1(e_m))}{Vol_{(\Sigma_1, g_1)}} = \frac{Vol_{(\Sigma_1, g_1)}}{Vol_{(\Sigma_1, g)}}
\]

where \(\pi_1 : \mathcal{M} \to \Sigma_1\) is the projection and \(e_i\) is a direct o.n. basis of \(\Gamma_f\), and \(Vol_{(\Sigma_1, g)}\) is the volume element of \((\Sigma_1, g)\). Then \(\cosh \theta = 1\) iff \(f\) is constant, that is \(\Gamma_f\) is a slice.

### 2 Bernstein-Calabi and Heinz-Chern type results

The classic Bernstein theorem says that an entire minimal graph in \(\mathbb{R}^3\) is a plane. This result was generalized to codimension one graphs in \(\mathbb{R}^{m+1}\) for \(m \leq 7\), and for higher dimensions and codimensions under additional conditions by many other authors. Calabi \((\text{[2]})\) considered the same problem for the maximal (the mean curvature \(H = 0\)) spacelike hypersurfaces \(M\) in the Lorentz-Minkowski space \(\mathbb{R}^{m+1}\) with the metric \(ds^2 = \sum_{i=1}^m (dx_i)^2 - (dx_{m+1})^2\). If \(M\) is given by the graph of a function \(f\) on \(\mathbb{R}^m\) with \(|Df| < 1\), the equation \(H = 0\) has the form

\[
\sum_{i=1}^m \frac{\partial}{\partial x_i} \left( \frac{\partial f/\partial x_i}{\sqrt{1 - |Df|^2}} \right) = 0.
\]

Calabi showed that for \(m \leq 4\), the graph of any entire solution to the above equation is a hyperplane. The same conclusion was established by Cheng and Yau \((\text{[3]})\) for any \(m\). A further generalization of this problem to \(\mathbb{R}^{m+n}\) has been obtained by some authors (see for instance in \(\text{[9]}\)).
Another natural generalization is to consider maximal spacelike graphic submanifolds in a non flat ambient space and in higher codimension. We consider a spacelike graph \( \Gamma_f \), for a map \( f : \Sigma_1 \to \Sigma_2 \).

We can take \( a_i \) an orthonormal basis of \( T_p \Sigma_1 \) and \( e_\alpha \) of \( T_{f(p)} \Sigma_2 \), \( 1 \leq i \leq m \), \( m + 1 \leq \alpha \leq m + n \), such that \( df(a_i) = -\lambda_i a_{m+i} \) (\( \lambda_i = 0 \) if \( i > n \)). Then \( e_i = (1 - \lambda_i^2)^{-1/2}(a_i + \lambda_i a_{m+i}) \) and \( e_{m+i} = (1 - \lambda_i^2)^{-1/2}(a_{m+i} + \lambda_i a_i) \), \( e_\alpha = a_\alpha \) if \( \alpha > 2m \), define o.n.b.s. of \( T_{(p,f(p))}\Gamma_f \), and of the normal bundle at \( (p,f(p)) \) respectively. Assuming \( \Gamma_f \) has parallel mean curvature, in this basis we have

\[
\Delta \cosh \theta = \cosh \theta \left\{ ||B||^2 + 2 \sum_{k} \sum_{i<j} \lambda_i \lambda_j h_{ik}^{m+i} h_{jk}^{m+j} - 2 \sum_{k} \sum_{i<j} \lambda_i \lambda_j h_{ik}^{m+i} h_{jk}^{m+j}
\right.
\]

\[
+ \sum_i \left( \frac{\lambda_i^2}{1-\lambda_i^2} Ricci_1(a_i, a_i) + \sum_{j \neq i} \frac{\lambda_i^2 \lambda_j^2}{(1-\lambda_i^2)(1-\lambda_j^2)} [K_1(P_{ij}) - K_2(P'_{ij})] \right) \right\}
\]  

(1)

where \( P_{ij} = \text{span}\{a_i, a_j\} \) and \( P'_{ij} = \text{span}\{a_{m+i}, a_{m+j}\} \). Here \( h_{ij}^\alpha \) are the components of the second fundamental form \( B \) of \( \Gamma_f \) in the basis \( e_i, e_\alpha \).

**Theorem 2 ([12, 10])** Let \( M = \Gamma_f \) be a spacelike graphic submanifold of \( \overline{M} \) with parallel mean curvature vector. We assume for each \( p \in \Sigma_1 \), \( Ricci_1(p) \geq 0 \) and for any two-dimensional planes \( P \subset T_p \Sigma_1 \), \( P' \subset T_{f(p)} \Sigma_2 \), \( K_1(P) \geq K_2(P') \). We have:

(i) If \( n = 1 \) and \( \cosh \theta \leq o(r) \) when \( r \to +\infty \), where \( r \) is the distance function to a point \( p \in (\Sigma_1, g_1) \), and \( \Sigma_1 \) is complete, then \( M \) is maximal.

(ii) If \( M \) is compact, then it is totally geodesic. Moreover, if \( Ricci_1(p) > 0 \) at some point, then \( M \) is a slice, that is \( f \) is constant;

(iii) If \( M \) is complete, noncompact, and \( K_1, K_2 \) and \( \cosh \theta \) are bounded, then \( M \) is maximal.

(iv) If \( M \) is a complete maximal spacelike surface, then \( M \) is totally geodesic. Moreover, (a) if \( K_1(p) > 0 \) at some point \( p \in M \), then \( M \) is a slice; (b) If \( \Sigma_1 = \mathbb{R}^2 \) and \( \Sigma_2 = \mathbb{R}^n \), then \( M \) is a plane; (c) if \( \Sigma_1 \) is flat and \( K_2 < 0 \) at some point \( f(p) \), then either \( M \) is a slice or the image of \( f \) is a geodesic of \( \Sigma_2 \).

We obtain (i) by applying a Heinz-Chern inequality derived in [12], for the absolute norm of \( H \)

\[
m ||H|| \leq \sup \cosh \theta \ h(B_r(p)),
\]

where \( h(B_r(p)) = \inf D V_{m-1}(\partial D)/V_m(D) \), is the Cheeger constant of the open geodesic ball of center \( p \) and radius \( r \), where \( D \) runs all over the bounded domains.
of the ball with smooth boundary $\partial D$. Since $Ricci_1 \geq 0$, $h(B_r(p)) \leq C/r$, when $r \to +\infty$, where $C > 0$ is a constant. For $\Sigma_1$ the $m$-hyperbolic space (with non-zero Cheeger constant), we give examples in [12] of foliations of $\mathbb{H}^m \times \mathbb{R}$ by complete spacelike graphic hypersurfaces with bounded hyperbolic angle and with constant mean curvature any real $c$, the same for all leaves, or parameterized by the leaf.

The proof of (ii) and (iii) consists on showing that, under the curvature conditions, one has $\Delta \ln \cosh \theta \geq \delta \|B\|^2$, where $\delta > 0$ is a constant that does not depend on $p$ and in (iii) showing that the Ricci tensor of $M$ is bounded from below, and applying the Omori-Cheng-Yau maximum principle for noncompact manifolds. (i) and (iii) are obtained by different approaches. If $M$ is a maximal Riemannian surface, (iv) gives a generalization of the Bernstein type theorem of Albujer-Alías [1] for maximal graphic spacelike surfaces in a Lorentzian three manifold $\Sigma_1 \times \mathbb{R}$ to higher codimension. As in [5, 1] the proof is based on a parabolicity argument for surfaces with nonnegative Gauss curvature. In fact, in this case, we have that $\Delta (\frac{1}{\cosh \theta}) \leq 0$ and the Gauss curvature of $M$ satisfies

$$K_M = \frac{1}{(1-\lambda_1^2)(1-\lambda_2^2)} [K_1 - \lambda_1^2 \lambda_2^2 K_2(a_3, a_4)] + \sum_\alpha [(h_{11}^\alpha)^2 + (h_{12}^\alpha)^2] \geq 0.$$ 

The conclusion that $B = 0$ comes from analyzing the vanishing of the term involving the components of $B$ in the expression of $\Delta (1/\cosh \theta)$. Our proof for (iv)(b), gives a simpler proof of the same result of Jost and Xin [9] for the case of surfaces, but using their result that any entire maximal graph in $\mathbb{R}^{m+n}$ is complete.

We also derive in [11] a Simons’ type identity for the absolute norm of the second fundamental form $\|B\|^2$ of a spacelike submanifold $M$ of any pseudo-Riemannian manifold $\overline{M}$,

$$\Delta \|B\|^2 = 2\|\nabla B\|^2 + \sum_{ij\alpha} 2h_{ij}^\alpha H_{ij}^\alpha - \sum_{ij\alpha} 2h_{ij}^\alpha \sum_k (\bar{\nabla}_j \bar{R})_{kik} + \sum_k (\bar{\nabla}_j \bar{R})_{ikj}$$

$$+ \sum_{ijkl\alpha} 2 \sum_k (4\bar{R}_{ijkl}^\alpha h_{ij}^\alpha - \bar{R}_{kij} h_{ij}^\alpha h_{ij}^\alpha) + \bar{R}_{i\beta}^\alpha h_{ij}^\beta h_{ij}^\alpha$$

$$+ \sum_{ijkl\alpha} 4(\bar{R}_{ijkl}^\alpha h_{ij}^\alpha h_{kl}^\alpha + \bar{R}_{ikj}^\alpha h_{ij}^\alpha h_{kl}^\alpha) - \sum_{ijkl\alpha\beta} 2h_{ij}^\alpha h_{ik}^\alpha h_{jk}^\beta h_{ij}^\alpha$$

$$+ 2\sum_{ijkl\alpha\beta} \left( \sum_k (h_{ik}^\alpha h_{jk}^\beta - h_{ik}^\beta h_{jk}^\alpha)^2 \right) + 2\sum_{ijkl\alpha\beta} \left( \sum_k (h_{ik}^\alpha h_{jk}^\beta - h_{ik}^\beta h_{jk}^\alpha)^2 \right).$$

5
3 The mean curvature flow

The mean curvature flow of an immersion $F_0 : M \rightarrow \overline{M}$ is a family of immersions $F_t : M \rightarrow \overline{M}$ defined in a maximal interval $t \in [0, T)$ evolving according to

$$\begin{align*}
\frac{d}{dt}F(x,t) &= H(x,t) = \Delta_{g_t}F_t(x) \\
F(\cdot,0) &= F_0
\end{align*}$$

(2)

where $H_t$ is the mean curvature of $M_t = F_t(M) = (M, g_t = F_t^*\bar{g})$. The mean curvature flow of hypersurfaces in a Riemannian manifold has been extensively studied. Recently, mean curvature flow of submanifolds with higher co-dimensions has been paid more attention. In [14], the graph mean curvature flow is studied in Riemannian product manifolds, and it is proved long-time existence and convergence of the flow under suitable conditions. When $M$ is a pseudo-Riemannian manifold, it is considered the mean curvature flow of spacelike submanifolds. This flow for spacelike hypersurfaces has also been largely studied, but very little is known on mean curvature flow in higher codimensions except in a flat space $\mathbb{R}^{n+m}$ [15]. In [11] we consider (2) with $M$ any pseudo-Riemannian manifold and $F_0$ any spacelike submanifold, and we derive the evolution of the following quantities at a given point $(x,t)$ with respect to an o.n. frame $e^\alpha$ of the normal bundle of $M_t$ and a coordinate chart $x^i$ of $M$, normal at $x$ relatively to the metric $g_t$

$$\frac{d}{dt}g_{ij} = 2H^\alpha h^\alpha_{ij},$$

$$\frac{d}{dt}Vol_{M_t} = ||H||^2Vol_{M_t},$$

$$\frac{d}{dt}||B||^2 = \Delta ||B||^2 - 2||\nabla B||^2 + \sum_{ij}2h^\alpha_{ij} \left( \sum_k (\bar{\nabla}_j \bar{R})^\alpha_{kk} + (\bar{\nabla}_k \bar{R})^\alpha_{jk} \right) -\sum_{ijk\alpha\beta}2(4\bar{R}^\alpha_{\beta\kappa} h^\beta_{ij} h^\alpha_{kj} - \bar{R}^\alpha_{k\beta} h^\alpha_{ij} h^\beta_{kj}) + \sum_{ijkl\alpha\beta}4(\bar{R}^\alpha_{ij} h^\alpha_{kl} + \bar{R}^\beta_{ij} h^\beta_{kl} + \bar{R}^i_{kij} h^\alpha_{ij}) -2\sum_{ijk\alpha\beta} \left( \sum_k (h^\alpha_{ik} h^\beta_{jk} - h^\beta_{ik} h^\alpha_{jk}) \right)^2 -2\sum_{ij\alpha\beta} \left( \sum_j (h^\alpha_{ij} h^\beta_{ij} - h^\beta_{ij} h^\alpha_{ij}) \right)^2.$$
This means that either $K_1(p) \geq K_2^+(q) = \max\{K_2(q), 0\}$, or $\text{Ricci}_1(p) \geq 0$ and $K_1(p)(P) < 0$ for some two-plane $\bar{P}$ and $K_1(p) \geq K_2(q)$, with $K_2(q) < 0$, $\forall p, q$.

We recall the main steps of [11].

For $t > 0$ sufficiently small, $F_t$ is near $F_0$ and so it is a spacelike graph with $\lambda_i^2(t) \leq 1 - \delta(t)$. We derive the evolution of the hyperbolic angle

$$
\frac{d}{dt} \ln(\cosh(\theta)) = \Delta \ln(\cosh(\theta)) - \left\{ \|B\|^2 - \sum_{k,i} \lambda_i^2(h_{ik}^m)^2 - 2 \sum_{k,j} \lambda_i \lambda_i h_{ik}^m h_{jk}^m \right\} \geq \delta(t)\|B\|^2
$$

$$
- \sum_{i} \lambda_i^2 \left( \frac{1}{1 - \lambda_i^2} \text{Ricci}_i(e_i, e_i) \right) \geq 0
$$

$$
\sum_{i,j} \lambda_i^2 \left( \frac{\lambda_j^2}{1 - \lambda_i^2} \right) \left( K_1(P_{ij}) - K_2(P'_{ij}) \right) \geq 0
$$

Therefore, $\frac{d}{dt} \ln(\cosh(\theta)) \leq \Delta \ln(\cosh(\theta)) - \delta(t)\|B\|^2 \leq \Delta \ln(\cosh(\theta))$, and by the maximum principle for parabolic equations, $\max_{\Sigma_t} \cosh(\theta)$ is a nondecreasing function on $t$, and in particular $F_t$ remains a spacelike graph $F_t = \Gamma_{f_t}$ for a smooth map $f_t : \Sigma_t \to \bar{M}$, $\forall t$. On what follows, $c_i$ denotes positive constants. We may take a uniform bound $\delta = \delta(0)$, such that $\lambda_i^2(t) \leq 1 - \delta$ for all $t$ as long as the flow exists. Consequently $g_t = g_1 - f_t g_2$ are uniformly equivalent metrics on $\Sigma_t$ and $\text{Vol}_{M_t}$ are uniformly bounded, and from the above evolution equations $\text{Vol}_{M_t} = e^{\int_0^t |H_t|^2 ds} \text{Vol}_{M_0}$, what implies $\int_0^T \sup_{\Sigma_t} |H_t|^2 dt < c_0$. From the evolution equations one gets

$$
\frac{d}{dt}\|B\|^2 \leq \Delta\|B\|^2 + c_1\|B\| + c_2\|B\|^2 - \frac{2}{n}\|B\|^4 \leq \Delta\|B\|^2 - \frac{1}{n}\|B\|^4 + c_3.
$$

This is the point where regularity theory is better in the pseudo-Riemannian setting than the Riemannian one (note the negative coefficient of the highest power of $\|B\|$, that holds in the pseudo-Riemannian case and not in the Riemannian case). From the above inequality we may use a result of Ecker and Huisken [6] to conclude that $\|B\|^2$ is uniformly bounded. From this inequality we may apply an interpolation formula due to Hamilton and applying parabolic maximum principles we conclude $\|\nabla^k B\|^2$ is uniformly bounded for all $k$.

For each $t$ it is defined on $\bar{M}$ a Riemannian metric $\bar{g}_t = \bar{g}_{\|P_t\|} - \bar{g}_{\|P_t\|}$ that makes $e_i, e_\alpha$ an orthonormal basis. These metrics defined along the flow are uniformly equivalent to the natural Riemannian metric $\bar{g}^+ = g_1 + g_2$ of $\bar{M} = \Sigma_1 \times \Sigma_2$. 


for we have some positive constants $c(\delta)$ and $c'(\delta)$, depending only on $\delta$, such that $c(\delta)\bar{g}_+ \leq \bar{g} \leq c'(\delta)\bar{g}_+$ holds. We observe that the Levi-Civita connections of $(\bar{M}, \bar{g}_+)$ and of $(\bar{M}, \bar{g})$ are the same and $\|\bar{\nabla}B\|^2 \leq c_2 \|B\|^4 + \|\nabla B\|^2$. By induction on $k$ we see that $\bar{\nabla}^k B$ are $\bar{g}$ and so $\bar{g}_+$-uniformly bounded for all $k \geq 0$, that is all derivatives of $B$ in $\bar{M}$ are also bounded for the Riemannian structure. Then we can apply Schauder theory, by embedding isometrically $(\Sigma_t, g_t)$ into an Euclidean space $\mathbb{R}^N$. The spaces $C^{k+\sigma}(\Sigma_1, \bar{M})$, $k \in \mathbb{N}$, $0 \leq \sigma < 1$ are Banach manifolds and can be seen as closed subsets of the Banach space $C^{k+\sigma}(\Sigma_1, \mathbb{R}^{N_1+N_2})$ with the Hölder norms. Equation (2) in local coordinates is of the form

$$\sum_{ij} a_{ij} \frac{\partial^2 F^a}{\partial x_i \partial x_j} - \sum_k b_k \frac{\partial F^a}{\partial x_k} = \bar{G}(x,t)^a + \frac{dF^a}{dt}$$

where $a_{ij} = g^{ij}$, $b_k = g^{ij} \Gamma_{ij}^k$, $\bar{G}(x,t)^a = (\Gamma_a^b \circ F_i) \frac{\partial F^b}{\partial x_i} \frac{\partial F^c}{\partial x_j}$. From the uniform bounds of $\bar{\nabla}^k B$ and of $\bar{\nabla}^k H$ we have that the coefficients $a_{ij}, b_j$ are $C^{k-1+\sigma}(\Sigma_1)$-uniformly bounded, and if $F_t$ lies on a compact set of $\bar{M}$ then

$$\|F(\cdot,t)\|_{C^{3+\sigma}(\Sigma_1, \bar{M})} \leq c_{-1}, \quad \|F(\cdot,t)\|_{C^{2+k+\sigma}(\Sigma_1, \bar{M})} \leq c_k, \quad k \geq 0$$

for some positive constants $c_t$ that do not depend on $t$. Standard use of Ascoli-Arzela’s theorem to $F_t$ leads to the conclusion that $T = +\infty$ (by assuming $T < +\infty$ one has $F_t = F_0 + \int_0^t H$ lies in a compact set and gets an extension of the maximal solution $F_t$ to $t = T$, what is a contradiction). We also note that the assumption of $R_2$ and its derivatives to be bounded is necessary to guarantee the existence of a maximal solution of the flow, as well the trick of DeTurck can also be applied in the pseudo-Riemannian case like in the Riemannian case, to reparametrize $F_t$ in a suitable way to convert the above system in one of strictly parabolic equations (see [16] p. 17). This is necessary since the coefficients $b_k$ also depend on the second derivatives of $F_t$, and so it can give a degenerated system.

**Theorem 3 ([11])** *The mean curvature flow of the spacelike graph of $f$ remains a spacelike graph of a map $f_t : \Sigma_1 \to \Sigma_2$ and exists for all time $t \geq 0$.*

Since $\int_0^{\infty} \sup_{\Sigma_1} \|H_t\|^2 dt \leq c_{12}$, then $\exists N \to +\infty$ such that $H_{Nt} \to 0$. Assuming $f_t$ lies in a compact set of $\Sigma_2$ we obtain a subsequence $F_{tn}$ that $C^\infty$-converges at infinity to a map $F_\infty \in C^\infty(\Sigma_1, \bar{M})$, necessarily a spacelike graph of a map $f_\infty \in C^\infty(\Sigma_1, \Sigma_2)$, and maximal, for $H_\infty = 0$. From Bernstein theorem 2, we conclude
Theorem 4 ([11]) If $\Sigma_2$ is also compact there is a sequence $t_n \to +\infty$ such that the sequence $\Gamma_{f_{tn}}$ of the flow converges at infinity to a spacelike graph $\Gamma_{f_\infty}$ of a totally geodesic map $f_\infty$, and if $\text{Ricci}_1(p) > 0$ at some point $p \in \Sigma_1$, the sequence converges to a slice.

Finally we consider the case $\text{Ricci}_1 > 0$ everywhere. In this case we will see that we can drop the compactness assumption of $\Sigma_2$. From (3)

$$\frac{d}{dt} \ln(\cosh \theta) \leq \Delta \ln(\cosh \theta) - c_{15} \sum_i \lambda_i^2,$$

what implies $\frac{d}{dt} \ln(\cosh \theta) \leq \Delta \ln(\cosh \theta) - c_{15} \left(1 - \frac{1}{\cosh^2 \theta}\right)$, and consequently,

$$\begin{cases} 1 \leq \max_{\Sigma_1} \cosh \theta \leq 1 + c_{16}e^{-2c_{15}t} \\
\lambda_i^2(p,t) \leq \frac{c_{16}e^{-2c_{15}t}}{(1 + c_{16}e^{-2c_{15}t})} \leq c_{16}e^{-2c_{15}t} =: (1 - \delta(t)) \end{cases} \tag{5}$$

that is, we have for each $t$ a constant $\delta(t)$ explicitly defined, and that approaches one in an exponentially decreasing way, and

$$\frac{d}{dt} \cosh \theta \leq \Delta \cosh \theta - \delta(t) \cosh \theta \|B\|^2.$$

Setting $p(t) = \frac{1}{\sqrt{c_{16}}} e^{c_{15}t}$ and $\psi = e^{\frac{1}{2}c_{15}t} \cosh^{p(t)} \theta \|B\|^2$, we have

$$\frac{d}{dt} \psi \leq \Delta \psi - 2 \cosh^{-p} \theta \nabla \cosh^p \theta \nabla \psi - c_{17} \left\{ e^{\frac{1}{2}c_{15}t} \psi^2 - e^{\frac{1}{4}c_{15}t} \psi^\frac{3}{2} - \psi \right\}.$$

In [11] we show this implies $\|B\| \leq c_{18} e^{-\tau t}$, where $\tau$ is a positive constant. Since $F_t = F_0 + \int_0^t H$ and the mean curvature is exponentially decreasing we can conclude that $F_t(p)$ lies on a compact region of $M$, and for any sequence $t_N \to +\infty$ we obtain a subsequence $t_n$ such that $F_{t_n}$ converges uniformly to a spacelike graph of a map $f_\infty$. By (5) this map must be constant. Furthermore, in this case the limit is the same, for any sequence $t_N \to +\infty$ we take. This gives the next theorem, obtained with no need of using Bernstein results:

Theorem 5 ([11]) If $\text{Ricci}_1 > 0$ everywhere and $K_1 \geq K_2$, $\Sigma_2$ not necessarily compact, all the flow converges to a unique slice.
4 Homotopy to a constant map

We will give some applications of theorem 5. We assume in this section \( \Sigma_1 \) is closed and \( \Sigma_2 \) is complete with \( R_2 \) bounded and its derivatives. We also assume either \( K_1 > 0 \) everywhere, or \( \text{Ricci}_1 > 0 \) and \( K_2 \leq -c < 0 \) everywhere.

Given a constant \( \rho > 0 \) we consider a new metric \( g_1 - g_2' \) on \( \Sigma_1 \times \Sigma_2 \) where \( g_2' = \rho^{-1} g_2 \). Now if \( f : \Sigma_1 \rightarrow \Sigma_2 \) satisfies \( f^* g_2 < \rho g_1 \), means \( \Gamma_f \) is a timelike submanifold w.r.t. \( g_1 - g_2' \). Then the curvature conditions in theorem 5 demands \( K_1 \geq \rho K_2 \), that can be translated in the following

**Theorem 6 ([11])** There exist a constant \( 0 \leq \rho \leq +\infty \), such that any map \( f : \Sigma_1 \rightarrow \Sigma_2 \) satisfying \( f^* g_2 < \rho g_1 \) is homotopically trivial. If \( K_1 > 0 \) everywhere we may take \( \rho \leq \min_{\Sigma_1} K_1 / \sup_{\Sigma_2} K_2^+ \). For \( K_2 \leq -c \) everywhere, we may take \( \rho = +\infty \).

Note that, for \( \text{Ricci} > 0, K_2 \leq -c < 0 \) everywhere, then \( \rho \geq \max_{\Sigma_1} K_1^- / \inf_{\Sigma_2} K_2 \), where \( K^- = \max\{-K, 0\} \). This means we may take \( \rho = +\infty \) if \( K_2 \leq -c \) as in case \( \sup_{\Sigma_2} K_2^+ = 0 \) and \( K_1 > 0 \). This is the case \( n = 1 \). The homotopy is given by the flow, namely, since \( F_t(p) = (\phi_t(p), f_t(\phi_t(p))) \), where \( \phi : \Sigma_1 \rightarrow \Sigma_1 \) is a diffeomorphism with \( \phi_0 = \text{id}_{\Sigma_1} \), then \( K(t, p) = f_t(\phi_t(p)) \) is the homotopy. This gives a new proof of the classic Cartan-Hadmard theorem:

**Corollary 1** If \( K_2 \leq 0, m \geq 2, \) any map \( f : S^m \rightarrow \Sigma_2 \) is homotopically trivial.

The condition given in [14], \( det(g_1 + f^* g_2) < 2 \) implies \( \sum_i \lambda_i^2 + 1 \leq \prod_i (1 + \lambda_i^2) < 2 \) and so \( \Gamma_f \) is a spacelike submanifold. The next theorem, obtained in the Riemannian setting, can be seen as a reformulated corollary of theorem 5:

**Theorem 7 ([14, 13])** Assume both \( \Sigma_i \) are closed and with constant sectional curvature \( K_i \) and satisfying \( K_1 \geq |K_2|, K_1 + K_2 > 0 \).

1. If \( det(g_1 + f^* g_2) < 2 \), then \( \Gamma_f \) can be deformed by a family of graphs to the one of a constant map.

2. If \( f \) is an area decreasing map, that is \( \lambda_1 \lambda_j < 1 \) for \( i \neq j \), then it is homotopically trivial.

The area decreasing condition is a slightly more general condition than spacelike graph for \( n \geq 2 \). In case \( n = 1 \) any map is area decreasing, but it is included in the case \( K_2 \leq 0 \). We note that in the previous theorem it is used the Riemannian structure, and in this setting \( K_2 \) cannot be given arbitrarily negative, a somehow artificial condition, that can be dropped if one uses the pseudo-Riemannian structure of the product.
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