Blow-up of non-radial solutions for the $L^2$ critical inhomogeneous NLS equation

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Abstract

We consider the $L^2$ critical inhomogeneous nonlinear Schrödinger (INLS) equation in $\mathbb{R}^N$

$$i\partial_t u + \Delta u + |x|^{-b}|u|^{\frac{4N}{N-2}} u = 0,$$

where $N \geq 1$ and $0 < b < 2$. We prove that if $u_0 \in H^1(\mathbb{R}^N)$ satisfies $E[u_0] < 0$, then the corresponding solution blows-up in finite time. This is in sharp contrast to the classical $L^2$ critical NLS equation where this type of result is only known in the radial case for $N \geq 2$.

1 Introduction

In this work we consider the initial value problem (IVP) for the $L^2$ critical inhomogeneous nonlinear Schrödinger (INLS) equation

$$\begin{cases}
i\partial_t u + \Delta u + |x|^{-b}|u|^{\frac{4N}{N-2}} u = 0, & x \in \mathbb{R}^N, \ t > 0, \\
u(0) = u_0 \in H^1(\mathbb{R}^N),
\end{cases} (1.1)$$

where $N \geq 1$ and $0 < b < 2$. For $b = 0$, (1.1) reduces to the IVP associated to the classical nonlinear Schrödinger (NLS) equation. This model is called $L^2$ critical since the scaling symmetry $u(x,t) \mapsto \lambda^\frac{N}{2} u(\lambda x, \lambda^2 t)$ leaves invariant the $L^2$ norm. The local well-posedness of the IVP (1.1) was obtained by Genoud and Stuart [8, Appendix K] (see also Guzmán [9]). Moreover, Genoud [7] also proved that this problem is globally well-posed below the ground state threshold.

The solutions of the IVP (1.1) satisfy mass and energy conservation laws given respectively by

$$M[u(t)] = \int |u(x,t)|^2 \, dx = M[u_0],$$

and

$$E[u(t)] = \frac{1}{2} \int |\nabla u(x,t)|^2 \, dx - \frac{N}{4 - 2b + 2N} \int |x|^{-b}|u(x,t)|^{\frac{4N}{N-2}} + 2 \, dx = E[u_0].$$

Our main result is the following.

**Theorem 1.1.** Let $N \geq 1$ and $0 < b < 2$. If $u_0 \in H^1(\mathbb{R}^N)$ and $E[u_0] < 0$, then the corresponding solution $u(t)$ to (1.1) blows-up in finite.

The previous theorem was first obtained by Ogawa and Tsutsumi [13] for the classical radial NLS equation when $N \geq 2$ and, combining a scaling argument, the same authors in [12] were able to improve this result for the non-radial NLS equation in dimension one. Applying the same ideas, Dinh [5] extended these results for the INLS model under identical conditions: radial for $N \geq 2$ and non-radial for $N = 1$. Here we refine the argument of Ogawa and Tsutsumi [13] to consider the non-radial INLS equation in all spatial dimensions, without relying on the scaling argument of [12]. We should point out that Theorem 1.1 is still unknown for the classical NLS equation in the non-radial when $N \geq 2$.

Recently, several papers reported results for the non-radial INLS equation that so far can only be obtained for the radial NLS equation [1], [2], [3], [10] and [11]. The present paper is another contribution in this direction. The new tool in the INLS setting is the decaying factor $|x|^{-b}$ which
implies a control, away from the origin, for the terms arising from the nonlinearity. For the NLS equation this type of control is usually made by an application of a radial Sobolev embedding due to Strauss [14, Lemma 1].

This paper is organized as follows. In Section 2, we introduce the basic notation and established a non-radial interpolation estimate. The last section is devoted to the proof of Theorem 1.1.

2 Notation and Preliminaries

In this section we introduce the basic notation used throughout the manuscript. The symbol $c$ will denote various positive constants and its exact value is not essential in our analysis. We write $a \lesssim b$ to denote $a \leq cb$ for some positive constant $c$. Similarly we define $a \gtrsim b$. The spaces $L^p(\mathbb{R}^N)$ and $H^1(\mathbb{R}^N)$ will be abbreviated as $L^p$ and $H^1$ with the norms denoted by $\| \cdot \|_p$ and $\| \cdot \|_{H^1}$ respectively. We also consider the functional space $W^{1,\infty} = \{ f \in L^\infty, \nabla f \in L^\infty \}$.

Next we obtain an interpolation estimate that will be very useful in the proof of our main result.

Lemma 2.1 (Non-radial interpolation estimate). Let $N \geq 1$, $0 < b < 2$ and $\phi$ be a positive real valued function.

- If $N \neq 2$ and $\phi^{\frac{1}{1-b}} \in W^{1,\infty}$, then for all $f \in H^1$ we have
  \[
  \int \phi|f|^{4-2b+2}dx \lesssim \left( \| \nabla \left( \phi^{\frac{1}{1-b}} \right) f \|_2 + \| \phi^{\frac{1}{1-b}} \nabla f \|_2 \right)^{2-b} \| f \|_2^{\frac{4+b(N-2)}{N}}.
  \] (2.1)

- If $N = 2$ and $\phi^{\frac{1}{1-b}} \in W^{1,\infty}$, then for all $f \in H^1$ we have
  \[
  \int \phi|f|^{4-b}dx \lesssim \left( \| \phi^{\frac{1}{1-b}} f \|_2 + \| \nabla (\phi^{\frac{1}{1-b}}) f \|_2 + \| \phi^{\frac{1}{1-b}} \nabla f \|_2 \right)^{2-b} \| f \|_2^{\frac{4+b}{N}}.
  \] (2.2)

Proof. For $N \geq 3$, we apply the Holder inequality and Sobolev embedding to obtain

\[
\int \phi|f|^{\frac{4-2b}{2}+2}dx = \int (\phi|f|^{2-b})(|f|^{\frac{4-2b}{2}+b})dx
\lesssim \| \phi |f|^{2-b} \|_2 N^{\frac{N}{N-2}} \| f \|_2^{\frac{4+b(N-2)}{N}} \| \phi |f|^{2-b} \|_2 \| f \|_2^{\frac{4+b(N-2)}{N}}
\lesssim \| \nabla (\phi^{\frac{1}{1-b}} f) \|_2^{2-b} \| f \|_2^{\frac{4+b(N-2)}{N}},
\]

which implies the desired inequality.

Next when $N = 1$, we first claim that

\[
\| \phi^{\frac{1}{1-b}} f \|_\infty \lesssim \| f \|_2^{1/2} \left( \| \left( \phi^{\frac{1}{1-b}} \right)' f \|_2 + \| \phi^{\frac{1}{1-b}} f' \|_2 \right)^{1/2}.
\] (2.3)

Indeed, by an approximation argument we may assume that $f$ has compact support and therefore

\[
\phi^{\frac{1}{1-b}} f^2(x) = \frac{1}{2} \left( \int_{-\infty}^x \phi^{\frac{1}{1-b}} f^2 \ ds + \int_x^{+\infty} \phi^{\frac{1}{1-b}} f^2 \ ds \right)
\lesssim \int \left( \phi^{\frac{1}{1-b}} \right)' f^2 ds + \int \phi^{\frac{1}{1-b}} f' f ds
\lesssim \| f \|_2 \left( \| \left( \phi^{\frac{1}{1-b}} \right)' f \|_2 + \| \phi^{\frac{1}{1-b}} f' \|_2 \right)
\]

and (2.3) is proved. Using this inequality we have

\[
\int \phi|f|^{4-2b+2}dx = \int \phi^{\frac{1}{1-b}} f^{\frac{4-2b}{2}} |f|^2 dx
\lesssim \| f \|_2^{2-b} \left( \| \left( \phi^{\frac{1}{1-b}} \right)' f \|_2 + \| \phi^{\frac{1}{1-b}} f' \|_2 \right)^{2-b} \| f \|_2^2,
\]
The main difference between inequalities (2.1) also holds in this case.

Finally, we consider the case $N = 2$ and use the following Sobolev embedding (see, for instance, Demengel and Demengel [4, Proposition 4.18])

$$\|f\|_{L^r} \leq c\|f\|_{H^1}, \text{ for all } r \in [2, +\infty).$$

together with the Holder inequality to obtain

$$\int \phi|f|^{4-b}dx = \int |\phi^{\frac{1}{b-2}}f|^{2-\frac{b}{2}}|f|^{2-\frac{b}{2}}dx$$

$$\lesssim \|\phi^{\frac{1}{b-2}}f\|^{2-\frac{b}{2}}\|f\|^{2-\frac{b}{2}}2^{-\frac{b}{2}}$$

$$= \|\phi^{\frac{1}{b-2}}f\|_{L^{2+b}}^{2-\frac{b}{2}}\|f\|_{L^{2+b}}^{2-\frac{b}{2}}$$

$$\lesssim \|\phi^{\frac{1}{b-2}}f\|_{H^1}^{2-\frac{b}{2}}\|f\|_{L^{2+b}}^{2-\frac{b}{2}}.$$  

Thus, from the definition of the $H^1$-norm we deduce inequality (2.2).  

\[\text{Remark 2.2.} \text{ The classical Gagliardo-Nirenberg (see for instance Weinstein [15], inequality (I.2))}
\]

$$\int |f|^{2\sigma+2}dx \leq C\|\nabla f\|_{L^2}^N\|f\|_{L^2}^{2+\sigma(2-N)}, \text{ if } 0 < \sigma < \frac{2}{N-2}$$

implies, for $\sigma = \frac{2-b}{N-2}$ and assuming $\phi^{\frac{1}{b-2}}f \in H^1$, that

$$\int \phi|f|^{\frac{4-2b}{N-2}+2}dx \leq C\|\nabla(\phi^{\frac{1}{b-2}}f)\|_{L^2}^{2-b}\|\phi^{\frac{1}{b-2}}f\|_{L^2}^{\frac{4+b(2-N)}{N}}. \quad (2.4)$$

The main difference between inequalities (2.1)-(2.2) and (2.4) is the power of the function $\phi$. As we will see later, to prove our main result we need this power to be greater then 1/2 and therefore inequality (2.4) is not enough to close the argument.

3 The proof of Theorem 1.1

Let $u_0 \in H^1$ such that $E[u_0] < 0$ and assume by contradiction that the corresponding solution $u(t)$ of (1.1) exists globally in time. For a bounded non-negative radial function $\phi \in C^\infty(\mathbb{R}^N)$, define $\phi_R(x) = R^2\phi\left(\frac{x}{R}\right)$ and

$$z_R(t) = \int \phi_R|u(t)|^2dx,$$

for $R > 0$ to be chosen later. It is clear that

$$z_R(t) \leq R^2\|\phi\|_{L^\infty}\|u_0\|^2,$$

by the mass conservation (1.2).

From direct computations (see, for instance, Proposition 7.2 in [6]), we have the following virial identities

$$z'_R(t) = 2i\text{Im} \int \nabla \phi_R \cdot \nabla u(t)\overline{\mu}(t) dx \quad (3.1)$$

and

$$z''_R(t) = 4\text{Re} \sum_{j,k=1}^N \int \partial_j u(t) \partial_k \overline{\mu}(t) \partial^2_{jk} \phi_R dx - \int |u(t)|^2 \Delta^2 \phi_R$$

$$- \frac{4-2b}{N+2-b} \int |x|^{-b}|u(t)|^{\frac{4-2b}{N+b}}+2 \Delta \phi_R dx$$

$$+ \frac{2N}{N+2-b} \int \nabla \left(|x|^{-b}\right) \cdot \nabla \phi_R|u(t)|^{\frac{4-2b}{N+b}}+2 dx. \quad (3.2)$$

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Recall that
\[ \partial_j = \frac{x_j}{r} \partial_r \text{ and } \partial_{kj} = \left( \frac{\delta_{kj} - x_k x_j}{r^3} \right) \partial_r + \frac{x_k x_j}{r^2} \partial_r^2, \]
where \( \partial_r \) denotes the radial derivative with respect to \( r = |x| \). From these relations, since \( \phi \) is radial, we deduce
\[
\sum_{j,k=1}^N \partial_j u \partial_k \nabla^2 \phi_R = \sum_{j,k=1}^N \partial_j u \partial_k \left[ \left( \frac{\delta_{kj} - x_k x_j}{r^3} \right) \partial_r \phi + \frac{x_k x_j}{r^2} \partial_r^2 \phi \right]
\]
\[= \frac{\partial_r \psi}{r} \nabla |u|^2 + \left( \frac{\partial^2 \phi}{r^2} - \frac{\partial_r \phi}{r^3} \right) \left( \sum_{j,k=1}^N (x_j \partial_j u)(x_k \partial_k u) \right)
\]
\[= \frac{\partial_r \psi}{r} \nabla |u|^2 + \left( \frac{\partial^2 \phi}{r^2} - \frac{\partial_r \phi}{r^3} \right) |x| \cdot \nabla u|^2.
\]
Moreover, it is easy to see that
\[
\Delta \phi = \frac{N-1}{r} \partial_r \phi + \partial_r^2 \phi
\]
and
\[
\nabla \left( |x|^{-b} \right) \cdot \nabla \phi = -b |x|^{-b-2} \frac{\partial_r \psi}{r},
\]
since \( \nabla \left( |x|^{-b} \right) = -b |x|^{-b-2} x \).

Therefore, we can rewrite the identities (3.1)-(3.2) as
\[
z'(t) = 2 \text{Im} \int \partial_r \phi_R \frac{x \cdot \nabla u(t)}{r} \bar{u}(t) \, dx
\]
and
\[
z''(t) = 4 \int \frac{\partial_r \phi_R}{r} |\nabla u(t)|^2 \, dx + 4 \int \left( \frac{\partial^2 \phi_R}{r^2} - \frac{\partial_r \phi_R}{r^3} \right) |x \cdot \nabla u(t)|^2 \, dx - \int |u(t)|^2 \Delta^2 \phi_R \, dx
\]
\[+ \frac{4 - 2b}{N + 2 - b} \int \left[ -\partial_r^2 \phi_R - \left( N - 1 + \frac{bN}{2-b} \right) \frac{\partial_r \phi_R}{r} \right] |x|^{-b} |u(t)| \frac{1}{r^{N-2+b}} \, dx.
\]
Continuing from above, we use the energy conservation (1.3) to obtain
\[
z''(t) = 2E[u_0] + K_1 + K_2 + K_3, \tag{3.3}
\]
where
\[
K_1 = -4 \int \left( 2 - \frac{\partial_r \phi_R}{r} \right) |\nabla u(t)|^2 \, dx - 4 \int \left( \frac{\partial^2 \phi_R}{r^2} - \frac{\partial_r \phi_R}{r^3} \right) |x \cdot \nabla u(t)|^2 \, dx,
\]
\[
K_2 = \frac{2}{N + 2 - b} \int \left[ (2-b)(2-\partial_r^2 \phi_R) + (2N - 2 + b) \left( 2 - \frac{\partial_r \phi_R}{r} \right) \right] |x|^{-b} |u(t)| \frac{1}{r^{N-2+b}} \, dx,
\]
\[
K_3 = -\int |u(t)|^2 \Delta^2 \phi_R \, dx.
\]
Now, we define a function \( \phi_R \) such that
\[
\partial_r \phi_R(r) - r \partial_r^2 \phi_R(r) \geq 0, \text{ for all } r = |x| \in \mathbb{R}. \tag{3.4}
\]
Indeed, inspired by the work of Ogawa and Tsutsumi [13], we first consider, for \( k \in \mathbb{N} \) to be chosen later, the following function
\[
v(r) = \begin{cases} 
2r, & \text{if } 0 \leq r \leq 1 \\
2r - 2(r-1)^k, & \text{if } 1 < r \leq 1 + \left( \frac{1}{k} \right)^{1-t} \\
\text{smooth and } v' < 0, & \text{if } 1 + \left( \frac{1}{k} \right)^{1-t} < r < 2 \\
0, & \text{if } r \geq 2.
\end{cases}
\tag{3.5}
\]
Remark 3.1. Note that the function \( f(r) = 2r - 2(r - 1)^k \) for \( r \geq 1 \) has an absolute maximum at \( r = 1 + \left( \frac{1}{k} \right)^{\frac{1}{k-1}} \) and therefore \( f' \left( 1 + \left( \frac{1}{k} \right)^{\frac{1}{k-1}} \right) = 0 \).

Define the radial function
\[
\phi(r) = \int_0^r v(s)ds
\]
Recall that \( \phi_R(r) = R^2 \phi \left( \frac{r}{R} \right) \), which implies
\[
\partial_r \phi_R(r) = Rv \left( \frac{r}{R} \right) \quad \text{and} \quad \partial_r^2 \phi_R(r) = v' \left( \frac{r}{R} \right). \tag{3.6}
\]
It is easy to see that inequality (3.4) holds for \( 0 < r \leq R \) and \( r \geq 2R \) by direct computation and for \( R \left( 1 + \left( \frac{1}{k} \right)^{\frac{1}{k-1}} \right) < r < 2R \) by (3.6) and the fact that \( v' < 0 \) and \( v \geq 0 \) in this region. It remains to consider the region \( R < r \leq R \left( 1 + \left( \frac{1}{k} \right)^{\frac{1}{k-1}} \right) \). By the definition of \( v \) (3.5) and relations (3.6), in this region we have
\[
\partial_r \phi_R(r) = 2R \left[ \frac{r}{R} - \left( \frac{r}{R} - 1 \right)^k \right] \quad \text{and} \quad \partial_r^2 \phi_R(r) = 2 - 2k \left( \frac{r}{R} - 1 \right)^{k-1}. \tag{3.7}
\]
Thus
\[
\partial_r \phi_R(r) - r \partial_r^2 \phi_R(r) = 2r \left( \frac{r}{R} - 1 \right)^{k-1} \left[ k - \frac{R}{r} \left( \frac{r}{R} - 1 \right)^k \right] \geq 2r \left( \frac{r}{R} - 1 \right)^{k-1} \left[ k - \left( \frac{1}{k} \right)^{\frac{1}{k-1}} \right] > 0.
\]
In view of inequality (3.4), the second integral in the definition of \( K_1 \) is positive and since \( \partial_r \phi_R(r) = 2r \) for \( 0 < r \leq R \) we obtain
\[
K_1 \leq - \int_{|x|>R} \Phi_{1,R} |\nabla u(t)|^2 dx, \tag{3.8}
\]
where \( \Phi_{1,R} = 4 \left( 2 - \frac{\partial_r \phi_R}{\Phi_R} \right) \).

Moreover, using the \( \partial_r^2 \phi_R(r) = 2 \) for \( 0 < r \leq R \), we have
\[
\text{supp} \left[ (2-b)(2-\partial_r^2 \phi_R) + (2N-2+b) \left( 2 - \frac{\partial_r \phi_R}{r} \right) \right] \subset (R, \infty),
\]
which implies
\[
K_2 = \int_{|x|>R} \Phi_{2,R} |x|^{-b} |u(t)|^{\frac{4-2b}{N}+2} dx,
\]
where \( \Phi_{2,R} = \frac{2}{N+2-b} \left[ (2-b)(2-\partial_r^2 \phi_R) + (2N-2+b) \left( 2 - \frac{\partial_r \phi_R}{r} \right) \right] \).

It is clear that \( \Phi_{1,R}(r), \Phi_{2,R}(r) \geq 0 \), since by definition \( \partial_r^2 \phi_R(r) \leq 2 \) and \( \partial_r \phi_R(r) \leq 2r \) for all \( r = |x| \in \mathbb{R} \).

Now we use the decay of \( |x|^{-b} \) away from the origin to estimate \( K_2 \). To fix the ideas we only consider the case \( N \neq 2 \). When \( N = 2 \) the proof is completely analogous just applying inequality (2.2) instead of (2.1) (see also Remark 3.2 below for more details). From the inequality (2.1) and Young's inequality, we deduce
\[
K_2 \lesssim \frac{1}{R^{\beta}} \int_{|x|>R} \Phi_{2,R} |u(t)|^{\frac{4-2b}{N}+2} dx
\]
\[
\lesssim \frac{1}{R^{\beta}} \left( ||\nabla (\Phi_{2,R})^{-1} u(t)||_2 + ||\Phi_{2,R}^{-1} \nabla u(t)||_2 \right)^{2-b} \left( \int \frac{u_0}{2} \right)^{\frac{4-2b+N}{N}}
\]
\[
\lesssim \varepsilon \left( ||\nabla (\Phi_{2,R})^{-1} u(t)||_2 + ||\Phi_{2,R}^{-1} \nabla u(t)||_2 \right)^{2} + \left( \int u_0 \right)^{\frac{2(2-2b+N)}{6N}} \varepsilon^{\frac{2b}{2b}} \frac{1}{R^2}.
\]
It is clear that $\Phi_{2,R} \in L^\infty$. We claim that
\[
\left| \nabla \left( \Phi_{2,R}^\frac{1}{b} (r) \right) \right| \lesssim \frac{1}{R}, \text{ for all } r = |x| \in \mathbb{R}. \tag{3.9}
\]
Indeed if $r \leq R$ and $r \geq 2R$, then $\nabla \left( \phi^{\frac{1}{b}} (r) \right) = 0$ and the desired inequality holds. In the intermediate region we first consider $R < r \leq R \left( 1 + \left( \frac{1}{r} \right)^{\frac{1}{b-b}} \right)$, where, in view of (3.7), we obtain
\[
\left| \nabla \left( \Phi_{2,R}^\frac{1}{b} (r) \right) \right| = \left| \partial_r \left( \left( \frac{r}{R} - 1 \right)^{\frac{b-1}{b}} \frac{4}{N+2-b} \left[ k(2-b) + (2N-2+b) \left( 1 - \frac{R}{r} \right) \right] \right) \right| \\
\lesssim \frac{1}{R} \left( \frac{r}{R} - 1 \right)^{\frac{b-1}{b} - 1} + \frac{R}{r^2} \left( \frac{r}{R} - 1 \right)^{\frac{b-1}{b}} \\
\lesssim \frac{1}{R},
\]
if we assume that $\frac{b-1}{b} - 1 > 0$ or $k > 3 - b$. Finally, when $R \left( 1 + \left( \frac{1}{r} \right)^{\frac{1}{b-b}} \right) < r < 2R$ we have that $\Phi_{2,R}(r) \gtrsim 1$ and from (3.6) we deduce
\[
\left| \nabla \left( \Phi_{2,R}^\frac{1}{b} (r) \right) \right| = \left| \partial_r \left( \left( \frac{2}{N+2-b} \left[ (2-b) \left( 2 - v' \left( \frac{r}{R} \right) \right) + (2N-2+b) \left( 2 - \frac{R}{r} v' \left( \frac{r}{R} \right) \right) \right] \right) \right|^{\frac{1}{b}} \\
\lesssim \Phi_{2,R}^\frac{1}{b} (r) \left( \frac{1}{R} \right)^{v'' \left( \frac{r}{R} \right)} + \frac{R}{r^2} \left( \frac{r}{R} \right) + \frac{1}{r} \left| v' \left( \frac{r}{R} \right) \right| \\
\lesssim \frac{1}{R},
\]
Returning to the bound of $K_2$ and using estimate (3.9) we get
\[
K_2 \lesssim \varepsilon \left( \left\| \nabla \left( \Phi_{2,R}^\frac{1}{b} \right) u(t) \right\|_2^2 + \left\| \Phi_{2,R}^\frac{1}{b} \nabla u(t) \right\|_2^2 \right) + \left\| u_0 \right\|_2^{\frac{2(4-2b+Nk)}{kn}} \frac{2}{\varepsilon^{\frac{2-b}{b}}} R^2 \\
\lesssim \varepsilon \int_{|x| > R} \Phi_{2,R}^\frac{2}{b} \left| \nabla u(t) \right|^2 \, dx + \frac{\varepsilon \left\| u_0 \right\|_2^{\frac{2(4-2b+Nk)}{kn}}}{\varepsilon^{\frac{2-b}{b}}} R^2. \tag{3.10}
\]
Finally, since $\|\Delta^2 \phi_R\|_\infty \lesssim 1/R^2$, we thus obtain from the mass conservation (1.2) the crude estimate
\[
K_3 \lesssim \frac{\left\| u_0 \right\|_2^2}{R^2}. \tag{3.11}
\]
Inserting estimates (3.8), (3.10) and (3.11) into the right hand side of (3.3), we infer that there exists $c > 0$ such that
\[
\varepsilon^2 R(t) \leq 2E[u_0] + \int_{|x| > R} \left( c \varepsilon \Phi_{2,R}^\frac{2}{b} - \Phi_{1,R} \right) \left| \nabla u(t) \right|^2 \, dx \\
+ c \frac{\varepsilon \left\| u_0 \right\|_2^{\frac{2(4-2b+Nk)}{kn}}}{\varepsilon^{\frac{2-b}{b}}} R^2 + c \frac{\left\| u_0 \right\|_2^{\frac{2(4-2b+Nk)}{kn}}}{\varepsilon^{\frac{2-b}{b}}} R^2. \tag{3.12}
\]
Next we claim that for sufficiently small $\varepsilon > 0$
\[
c \varepsilon \Phi_{2,R}^\frac{2}{b} (r) - \Phi_{1,R}(r) \leq 0, \text{ for all } r > R. \tag{3.13}
\]
Therefore there exists \( \epsilon > 0 \). The last two inequalities imply \( \Phi_2 \). Collecting estimates (3.12), (3.13) and standard arguments imply that the solution blows-up in finite time concluding the proof of Theorem 1.1.

We first consider the region \( R < r \leq R \left( 1 + \left( \frac{1}{k} \right)^{\frac{1}{k^2-1}} \right) \). By relations (3.7), in this region we have

\[
\epsilon \Phi_{2,R}(r) - \Phi_{1,R}(r) = \epsilon \left( \frac{r}{R} - 1 \right)^{\frac{2(b-1)}{2+b}} \left( \frac{4}{N+2-b} \left[ k(2-b) + (2N - 2 + b) \left( 1 - \frac{R}{r} \right) \right] \right)^{\frac{2}{b}} - 8 \frac{R}{r} \left( \frac{r}{R} - 1 \right)^k
\]

\[
= \left( \frac{r}{R} - 1 \right)^k \left[ \epsilon \left( \frac{r}{R} - 1 \right)^{\frac{b-2}{2+b}} \left( \frac{4}{N+2-b} \right) \left[ k(2-b) + (2N - 2 + b) \left( 1 - \frac{R}{r} \right) \right] ^{\frac{2}{b}} - 8 \frac{R}{r} \right]
\]

\[
\leq \left( \frac{r}{R} - 1 \right)^k \left[ \epsilon \left( \frac{1}{k} \right)^{\frac{2(b-1)}{2+b}} \left( \frac{4}{N+2-b} \right) \left[ k(2-b) + (2N - 2 + b) \left( 1 - \frac{R}{r} \right) \right] ^{\frac{2}{b}} - \frac{8}{1 + \left( \frac{1}{k} \right)^{\frac{1}{k^2-1}}} \right]
\]

if we assume that \( bk - 2 > 0 \) or \( k > 2/b \). So, we can chose \( \epsilon > 0 \) sufficiently small such that (3.13) holds in this case.

Now we turn our attention to the region \( R \left( 1 + \left( \frac{1}{k} \right)^{\frac{1}{k^2-1}} \right) < r \). Since \( v'(r/R) \leq 0 \) in this region from (3.6) we first have

\[
\frac{\partial_1 \phi_R}{r}(r) = \frac{R}{r} v \left( \frac{r}{1 + \left( \frac{1}{k} \right)^{\frac{1}{k^2-1}}} \right) = 2 - 2 \frac{\left( \frac{1}{k} \right)^{\frac{1}{k^2-1}}}{1 + \left( \frac{1}{k} \right)^{\frac{1}{k^2-1}}} \quad (3.14)
\]

and

\[
\left| \partial_1^2 \phi_R \right| = \left| v' \left( \frac{r}{R} \right) \right| \leq \| v' \|_{\infty}.
\]

The last two inequalities imply \( \Phi_{2,R}(r) \leq 1 \). Moreover, from (3.14), we have

\[
\Phi_{1,R}(r) = 4 \left( 2 - \frac{R}{r} v \left( \frac{r}{R} \right) \right) \geq 8 \frac{\left( \frac{1}{k} \right)^{\frac{1}{k^2-1}}}{1 + \left( \frac{1}{k} \right)^{\frac{1}{k^2-1}}}.
\]

Therefore there exists \( \epsilon > 0 \) such that (3.13) also holds in this case.

Note that in both cases considered above \( \epsilon > 0 \) was chosen independent of \( R > 0 \). Finally, collecting estimates (3.12), (3.13) and taking \( R > 0 \) sufficiently large we deduce

\[
\frac{d^2}{dt^2} \int \phi_R |u(t)|^2 \, dx = z''_R(t) \leq E[u_0] < 0,
\]

and standard arguments imply that the solution blows-up in finite time concluding the proof of Theorem 1.1.

**Remark 3.2.** In the case \( N = 2 \), the inequality (3.12) may be replaced by

\[
\frac{d^2}{dt^2} \int \phi_R |u(t)|^2 \, dx = z''_R(t) \leq 2E[u_0] + \int_{|x| > R} \left( \epsilon \Phi_{2,R}(r) - \Phi_{1,R}(r) \right) |\nabla u(t)|^2 \, dx
\]

\[
+ \frac{\epsilon \| u_0 \|_{L}^2}{R^2} + c \frac{\| u_0 \|_{L}^{2+b}{\epsilon}^{\frac{1}{k^2-1}} R^4} + c \frac{\| u_0 \|_{L}^{4-b}{\epsilon}^{\frac{1}{k^2-1}} R^4}.
\]

(3.15)
taking into account the inequality (2.2) to estimate $K_2$. The same arguments employed above implies
\[ ce^{-2 \frac{\varepsilon^2}{b^2}} (r) - \Phi_{1,R}(r) \leq 0, \text{ for all } r > R, \]
for sufficiently small $\varepsilon > 0$, independent of $R > 0$, as long as $k > 4/b$ for $v(r)$ given by (3.5). Moreover, the last three terms in the right hand side of (3.15) can be made small, for $R > 0$ sufficiently large, concluding the proof Theorem 1.1 also in this case.

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