Energy spectrum for the $cr^2 + br - \frac{a}{r}$ interaction

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Abstract

Energy spectrum of the Schrödinger equation for the interaction $cr^2 + br - \frac{a}{r}$ is obtained in the form of three different cases such as (i) a quantized $c$, (ii) a quantized $b$ and (iii) a quantized $a$ in the non-linear regime. Energy spectrum of the first is concave-up, and energy spectrum of the second is concave-down with given $L$ in the area where $0 \leq N \leq L + 5$ roughly. And energy spectrum of the third is a linearly increasing straight line. Here, $L$ is an angular momentum quantum number, and $N = 0, 1, 2, \cdots$ is the quantum number counting the radial nodes.

Keywords: Heun’s equation, Three term recurrence relation, Polynomial, quark mass, Confinement

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1. Introduction

The radial wave functions satisfy the equation

$$\left[ -\frac{\hbar^2}{2m} \left( \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} \right) + \frac{\hbar^2}{2m} L(L+1) + V(r) \right] R(r) = ER(r) \quad (1)$$

with

$$V(r) = cr^2 + br - \frac{a}{r} \quad (2)$$

where $0 \leq r < \infty$, $E$ is the eigenvalue, $L$ is the rotational quantum number, $c > 0$ and $a, b \in \mathbb{R}$.

Eq. (2), consisting of a Coulombic term and a quark confining potential, appears in the quark dynamics in the concepts of asymptotic freedom and quark confinement in non-Abelian gauge theories such as non-relativistic quark-antiquark bound states described by a Schrödinger equation. [1,13,15] Eq. (2) has also been studied by Gupta and Khare. [16]

Seeking solution in the form $R(r) = r^f(r)$ with putting $r = a\tilde{r}$, eq. (1) becomes

$$\frac{d^2f(\tilde{r})}{d\tilde{r}^2} + \frac{2(L+1)}{\tilde{r}} \frac{df(\tilde{r})}{d\tilde{r}} + \left( \tilde{E} - \tilde{c}^2\tilde{r}^2 - \tilde{b}\tilde{r} + \tilde{a} \right) f(\tilde{r}) = 0 \quad (3)$$

with $\tilde{E} = aE$, $\tilde{c}^2 = a^3c$, $\tilde{b} = a^2b$, $\tilde{a} = a$ and $a = \frac{\hbar^2}{2m}$. If one requires solutions in the form $f(\tilde{r}) = \exp \left(-\frac{\tilde{c}^2}{2}\tilde{r}^2 - \frac{\tilde{b}}{2}\tilde{r} + \frac{\tilde{a}}{\tilde{r}} \right) y(\tilde{r})$ in eq. (3), and then making the substitution $\rho = \sqrt{\tilde{c}}\tilde{r}$ into the new
equation \([1]\), it becomes
\[ \rho^2 \frac{d^2 y(\rho)}{d \rho^2} + \left( -2 \rho^2 - \frac{\hat{b}}{2 \rho} + 2(L+1) \right) \frac{dy(\rho)}{d \rho} + \left( \frac{1}{\gamma} \left( \hat{E} + \frac{\hat{b}^2}{4 \gamma^2} - (2L+3) \right) \right) \rho + \frac{\hat{a}}{\gamma} = 0 \] (4)

Comparing eq. (4) with eq. (A.1), the former is the special case of the latter with \( z = \rho, \mu = -2, \ v = -\frac{\hat{b}}{2 \gamma}, \ \omega = L+1 - \frac{\hat{d}}{\rho} \) and \( \Omega = \frac{1}{\gamma} \left( \hat{E} + \frac{\hat{b}^2}{4 \gamma^2} - (2L+3) \right) \).

Let’s investigate an asymptotic behaviour of the radial wave function \( R(r) \) in eq. (1) as a variable \( r = a \tilde{r} \) goes to positive infinity. We assume that \( y(\rho) \) is an infinite series in eq. (4), and substitute eq. (B.15) into \( R(r) = r^\lambda \exp \left( -\frac{\tilde{c}}{2} \tilde{r}^2 - \frac{\hat{b}}{2 \tilde{r}} \right) y(\rho) \)

\[ R(r) \sim A (a \tilde{r})^\lambda \exp \left( -\frac{\tilde{c}}{2} \tilde{r}^2 - \frac{\hat{b}}{2 \tilde{r}} \right) \tilde{r}^{b-\gamma} \exp(\tilde{z}) \]

\[ = A \tilde{r}^\lambda \left( \sqrt{\frac{\tilde{c}}{\lambda}} \right)^{\frac{b}{2}} \exp \left( \frac{1}{2} \sqrt{\frac{\tilde{c}}{\lambda}} \tilde{r}^2 - \frac{\hat{b}}{2 \sqrt{\frac{\tilde{c}}{\lambda}}} \right) \exp(\tilde{z}) \] (5)

In eq. (5), if \( r \rightarrow \infty \), then \( R(r) \rightarrow \infty \). It is unacceptable that wave function \( R(r) \) is divergent as \( r \) goes to infinity in the quantum mechanical point of view. Therefore the function \( y(\rho) \) must to be polynomial in eq. (4) in order to make the wave function \( R(r) \) being convergent even if \( r \) goes to infinity.

We observed the energy spectrum in the linear regime where both \( |\frac{\hat{b}}{\sqrt{\tilde{c}}}|, |\frac{\hat{d}}{\sqrt{\tilde{c}}}| \sim O(1) \) [10]. Particularly, we derived the non-linear energy spectrum where \( |\frac{\hat{b}}{\sqrt{\tilde{c}}}|, |\frac{\hat{d}}{\sqrt{\tilde{c}}}| \leq O(1) \) with \( \tilde{a}, \tilde{b}, \tilde{c} \neq 0 \).

2. Quantization of \( \tilde{c} \)

It has been believed that we can build the wave function normalizable whatever form is the Schrödinger equation by tuning the energy eigenvalue. However, for the BCH equation, it is not possible. Actually, we need to fine tune one more parameter apart from the energy in order to build polynomial solution for the Heun equations and its confluent forms. Because their series expansions consist of a three term recurrence relation. On the other hand, hypergeometric-type functions are composed of a two term recursive relation in which case we can construct normalizable polynomial solution by tuning the single parameter, energy; but, for the three term case, we need two parameters in order to make polynomials, for example, \( \tilde{c} (\tilde{b} \text{ or } \tilde{a}) \) and \( \tilde{E} \) in eq. (4).

Actually, the necessary and sufficient condition for constructing polynomials with a single parameter (the energy eigenvalue) is that its power series should be reduced to the two term recurrence relation. But, for the Heun case including its confluent forms, we cannot reduce its recursive relation to the two term. We show why polynomials of the BCH equation cannot be described with a single parameter in the next subsection. With believing our argument at this moment, we build polynomials with two parameters \( \tilde{c} \) and \( \tilde{E} \).

The spectral polynomial of the BCH equation around the origin, heretofore has been obtained by applying a power series with unknown coefficients [2] [12] [18] [28]. The BCH polynomial comes from the BCH equation that has a fixed integer value of \( \gamma - \alpha - 2 \), just as it has a fixed value of \( \delta \). They left a polynomial equation of the \((N+1)\)th order for the determination of a parameter \( \delta \) as the determinant of \((N+1) \times (N+1)\) matrices [3] [11].

For polynomials of eq. (4) around \( \rho = 0 \), we treat \( \tilde{a}, \tilde{b} \) as free variables; consider \(-\Omega/\mu = \frac{1}{\gamma} (\tilde{E} + \frac{\tilde{b}^2}{4 \gamma^2} - (2L+3) \tilde{c}) \) to be a positive integer; and treat \( \tilde{c} \) as a fixed value. Through eq. (A.3),
we are able to observe that a series expansion eq. (A.2) becomes a polynomial of degree $N$ by imposing two conditions in the form

$$ B_{N+1} = d_{N+1} = 0 \quad \text{where } N \in \mathbb{N}_0 \tag{6} $$

Eq. (6) is sufficient to give $d_{N+2} = d_{N+3} = d_{N+4} = \cdots = 0$ successively and the solution to eq. (4) becomes a polynomial of order $N$.

The general expression of a power series of eq. (4) about $\rho = 0$ for the polynomial and its algebraic equation for the determination of an accessory parameter $\tilde{c}$ are given by: (i) For $N = 0$, Eq. (6) gives $B_1 = \frac{\tilde{c}}{2L+3} = 0$ and $d_1 = A_0 d_0 = \frac{\tilde{c}}{2L+3} d_0 = 0$. If we choose $d_0 = 0$ the whole series solution vanishes. Therefore $\tilde{c} = \tilde{b}/\tilde{a}$ with $\tilde{E} = \frac{\tilde{b}^2}{\tilde{a}} - \frac{\tilde{b}^4}{4\tilde{a}^2}$ at $L = 0$. Its eigenfunction is $y(\rho) = \sum_{n=0}^0 d_n \rho^n = 1$ where $d_0 = 1$ chosen for simplicity.

(ii) For $N = 1$, $B_2 = \frac{\tilde{c}^2}{(2L+3)^2}$ and $d_2 = A_1 d_1 + B_1 d_0 = (A_0 A_1 + B_1) d_0 = \left( \frac{\tilde{c}^2}{(2L+3)^2} - \frac{\tilde{c}}{2L+3} \right) d_0$.

Requesting both $B_2$ and $d_2$ to be zero, we get $\tilde{E} = (2L+5)\tilde{c} - \frac{\tilde{c}^2}{2L}$ with $L = 0, 1$, and $\tilde{c}$ is the roots of a quadratic equation such as $2(2L + 2)\tilde{c}^3 - \tilde{b}^2 \tilde{a}^2 \tilde{c}^2 + \tilde{a} \tilde{b} (2L + 3) \tilde{c} - \tilde{b}^2 (L + 1) (L + 2) = 0$. In this case, $y(\rho) = \sum_{n=0}^1 d_n \rho^n = 1 - \frac{b(L+1)}{2L+3} \rho$.

(iii) For $N \geq 2$, the eigenvalue is obtained by letting $B_{N+1} = 0$ and it gives $\tilde{E} = 2\tilde{c} \left( N + L + \frac{1}{2} \right) - \frac{\tilde{c}^2}{2L}$ where $N \in \mathbb{N}_0$ and $L = 0, 1, 2, \cdots, N$. Roots of $\tilde{c}$’s is obtained by letting $d_{N+1} = 0$, and its eigenfunction is $\sum_{n=0}^N d_n \rho^n$: We are only interested in real roots because of their physical conditions.

We obtain

(I) The ground state energy and its eigenfunction with a quantized $\tilde{c}$ where $L = 0$ are

$$ \begin{align*}
\tilde{E} &= 3\tilde{c} - \frac{\tilde{c}^2}{2L} \\
\tilde{c} &= \frac{1}{\sqrt{2}} \\
R(r) &= \exp\left( -\frac{1}{2} \tilde{c}^2 - \frac{\tilde{c}}{2L} r \right)
\end{align*} \tag{7} $$

(II) The first excited state energy and its eigenfunction with a quantized $\tilde{c}$ where $L = 0$ are given by

$$ \begin{align*}
\tilde{E} &= 5\tilde{c} - \frac{\tilde{c}^2}{2L} \\
\tilde{c} &= \frac{1}{\sqrt{3}} \left( 2\tilde{b}^2 \tilde{b}^2 + 36\tilde{a} \tilde{b} + \tilde{a}^4 - 36\tilde{a} \tilde{b} \right) \left( \tilde{a}^6 - 54\tilde{a}^4 \tilde{b} + 432\tilde{a}^2 \tilde{b}^2 + 6\sqrt{5184\tilde{b}^6 - 3\tilde{b}^2} \right)^{1/3} \\
R(r) &= \exp\left( -\frac{1}{2} \tilde{c}^2 - \frac{\tilde{c}}{2L} r \right)
\end{align*} \tag{8} $$

(III) The first excited state energy and its eigenfunction with a quantized $\tilde{c}$ where $L = 1$ are obtained by

$$ \begin{align*}
\tilde{E} &= 7\tilde{c} - \frac{\tilde{c}^2}{2L} \\
\tilde{c} &= \frac{1}{\sqrt{5}} \left( 2\tilde{b}^2 \tilde{b}^2 + 180\tilde{a} \tilde{b} + 5184\tilde{b}^2 + 12\sqrt{62208\tilde{b}^6 - 320\tilde{b}^2} \right)^{1/3} \\
R(r) &= r \left( 1 + \frac{1}{2} \left( \tilde{c} - \frac{\tilde{c}^2}{2L} r \right) \right) \exp\left( -\frac{1}{2} \tilde{c}^2 - \frac{\tilde{c}}{2L} r \right) \tag{9}
\end{align*} $$

If $\tilde{a} = 0$, $\tilde{b}$ should be zero for the ground state, because a polynomial equation of degree 1 for the determination of a parameter $\tilde{c}$ is $\tilde{a} \tilde{c} = \tilde{b}$. Therefore, $\tilde{c}$ is not a fixed value but a free variable only for the ground state. But we are only interested in the case of a quantized $\tilde{c}$ rather than a free variable. we conclude that there is no solution with radial nodal number $N = 1$. The first
Eq. (10) is a confluent hypergeometric equation and its solution is well-known. Its eigenvalue
state. The first excited state energy and its eigenfunction with a quantized \( \tilde{c} \) are given by putting \( \tilde{a} = 0 \) in eq. (8) and eq. (9).

If \( \tilde{a} = \tilde{b} = 0 \) in eq. (8) with allowing \( \xi = \rho^2 \),

\[
\xi^2 \frac{d^2y(\xi)}{d\xi^2} + \left( L + \frac{3}{2} - \xi \right) \frac{dy(\xi)}{d\xi} + \left( \frac{\tilde{E}}{4\xi^2} - \frac{2L + 3}{4} \right)y(\xi) = 0
\]  

(10)

Eq. (10) is a confluent hypergeometric equation and its solution is well-known. Its eigenvalue
is \( \tilde{E} = 2\xi (N + L + 3/2) \) where \( N = 0, 1, 2, \ldots \). We can observe that \( \tilde{c} \) is not a fixed value any longer but a free variable.

There are no such analytic solution for \( \tilde{c} = 0 \) and \( \tilde{a}, \tilde{b} \neq 0 \), because its infinite series is divergent as \( r \to \infty \). And there is no way to make a polynomial of its series expansion. In order to make a polynomial of its solution, \( \tilde{b} \) should be zero, and its series expansion is a associated Laguerre polynomial; the mathematical structure of its differential equation is equivalent to the equation of hydrogen-like atoms.

2.1. The shooting method for modified BCH polynomials

We show why polynomials of the BCH equation cannot be described with a single parameter \( \tilde{E} \) by applying the shooting method in general. For \( \tilde{a} = 2/5 \) and \( \tilde{b} = 1 \) in eq. (7), \( \tilde{E} = E_0 = 7.46 \) and \( \tilde{c} = c_0 = 2.5 \) for the ground state and its polynomial is the unity.

Assume that we are able to construct a polynomial of eq. (4) for the ground state with only a single variable \( \tilde{E} \) without quantizing \( \tilde{c} \). Then, no matter what value \( \tilde{c} \) has, it will have any fixed value \( \tilde{c} \) to take the polynomial of the equation. For \( \tilde{c} = c_0 + 1.0 \), we look for a proper value of \( \tilde{E} \) with initial conditions \( y(0) = d_0 = 1 \) and \( y'(0) = 0 \). Then we try to construct a normalizable solution such as \( y(\rho) = 1 \) by shooting method.

| Table 1: \( \tilde{E} \) of \( y(\rho) \) where \( \tilde{c} = c_0 + 1.0 \) |
|-----------------|-----------------|
| (1) \( E = E_0 - 0.19 \) | (11) \( E = E_0 - 0.17552989106006135211 \) |
| (2) \( E = E_0 - 0.15 \) | (12) \( E = E_0 - 0.1755298910600613521149071 \) |
| (3) \( E = E_0 - 0.18 \) | |
| (4) \( E = E_0 - 0.16 \) | |
| (5) \( E = E_0 - 0.17553 \) | |
| (6) \( E = E_0 - 0.17552 \) | |
| (7) \( E = E_0 - 0.1755298911 \) | |
| (8) \( E = E_0 - 0.1755298910 \) | |
| (9) \( E = E_0 - 0.175529891060662 \) | |
| (10) \( E = E_0 - 0.17552989106066135211 \) | |
| (11) \( E = E_0 - 0.17552989106006135211 \) | |
| (12) \( E = E_0 - 0.1755298910600613521149071 \) | |

Fig. 1 shows how the trial wave functions approach to the unity as we increase the precision of the eigenvalue \( \tilde{E} \). The functions (1), (3), (5), (7) and (9) are under-shot solutions and the
functions (2), (4), (6), (8), (10), (11) and (12) are over-shoted ones. Starting from a under-shoted solution (1) at \( E = E_0 - 0.19 \), one can increase the precision of \( E \) by increasing minimal amount in the next digit to obtain the over-shoted solution. Again, starting from a over-shoted solution (2) at \( E = E_0 - 0.15 \), one can increase the precision of \( E \) by decreasing minimal amount in the next digit to get the under-shoted solution. After a number of iterations, the solutions stop to approach to the unity although we increase the precision by alternating the over- and under-shoting. We can observe that there is a limit to pushing the function value to the right any longer.

On the contrary, Fig. 2 tells us that \( y(\rho) \) is pushed to the right more and more as \( \tilde{E} \) approaches 7.46 with \( \tilde{c} = 2.5 \). And we can see from Fig. 2 that if \( E \) is exactly 7.46, \( y(\rho) \approx 1 \). And we can easily check that if \( \tilde{E} = 7.46 \) exactly, \( y(\rho) \approx 1 \) numerically also. This phenomenon occurs because a series expansion of eq.(4) consists of a three term recurrence relation, which require two quantized parameters \((E, \tilde{c})\) to create a polynomial.

As \( \tilde{a} = \tilde{b} = 0 \) in eq.(4), its differential equation is a confluent hypergeometric equation which its series expansion is a two term recurrence relation. And \( \tilde{c} \) is not a quantized value but a free parameter. Again, we are interested in a proper numerical value of \( E \) by applying the shooting method with initial conditions \( y'(0) = 0 \) and \( y(0) = 1 \) in eq.(10).

| \( E \)   | \( y \)       |
|---------|--------------|
| 7.4     | \( E = 7.4 \) |
| 7.50    | \( E = 7.50 \) |
| 7.49    | \( E = 7.49 \) |
| 7.501   | \( E = 7.501 \) |
| 7.49999 | \( E = 7.49999 \) |
| 7.50001 | \( E = 7.50001 \) |

Table 3: \( E \) of \( y(\xi) \) where \( \tilde{c} = c_0 = 2.5 \).

| \( E \)   | \( y \)       |
|---------|--------------|
| 10.4    | \( E = 10.4 \) |
| 10.6    | \( E = 10.6 \) |
| 10.49   | \( E = 10.49 \) |
| 10.51   | \( E = 10.51 \) |
| 10.4999 | \( E = 10.4999 \) |
| 10.5001 | \( E = 10.5001 \) |

Table 4: \( E \) of \( y(\xi) \) where \( \tilde{c} = c_0 + 1.0 = 2.6 \).

Fig. 3 shows us that \( y(\xi) \) approaches the unity for the ground state with \( \tilde{c} = c_0 = 2.5 \) as \( \tilde{E} \) reaches to 7.5. In Fig. 4, we can observe that \( y(\xi) \) approaches the unity for the ground state for
\( \tilde{c} = c_0 + 1.0 \) as \( \tilde{E} \) reaches to 10.5. These various examples tells us that we need two parameters (\( \tilde{E} \) and \( \tilde{c} \)) in order to make a polynomial when a power series solution have of a three term recurrence relation. On the other hand, we only need a single parameter (\( \tilde{E} \)) to have a polynomial when a series solution consists of a two term recursion relation.

\[ \begin{align*}
\text{Figure 3: } y(\xi) \text{ with a fixed } \tilde{c} = c_0 = 2.5 \text{ and unfixed } \tilde{E}'s \text{ as } \tilde{a} = \tilde{b} = 0 \\
\text{Figure 4: } y(\xi) \text{ with a fixed } \tilde{c} = c_0 + 1.0 \text{ and unfixed } \tilde{E}'s \text{ as } \tilde{a} = \tilde{b} = 0
\end{align*} \]

\subsection{The discrete energy spectrum}

The Abel-Ruffini theorem tells that it is really hard to derive the roots of the characteristic polynomial for more than degree five by hands without a computer system generally. \( \tilde{b} = \alpha \frac{a^2 b}{2m} \) factor. We set \( \tilde{a} = 1 \) and \( \tilde{b} = 1/10 \) for determination of real values of \( \tilde{c} \). We are only interested in the smallest numeric real values of \( \tilde{c} \) for given \( N \) and \( L \) because the smallest ones make eigenvalue \( \tilde{E} \) to be minimized. For an example, there are 6 possible real values of \( \tilde{c} \) as \( N = L = 10 \) such as 0.0744463, 0.086344, 0.104144, 0.13491, 0.20725 and 1.57137. We choose 0.0744463 as our numeric real value of \( \tilde{c} \).

\[ \begin{align*}
\text{Figure 5: } \text{Real values of } \tilde{c}'s \text{ for } N = 1, 2, 3, \ldots \text{ & } L = 0, 1, 2, \ldots, \text{ as } \tilde{a} = 2/5 \text{ and } \tilde{b} = 1 \\
\text{Figure 6: The magnification of Fig.5}
\end{align*} \]

Fig. [5] tells us that the smallest real values \( \tilde{c} \) with given \( N \)'s. The lowest point indicates a numeric value of \( \tilde{c} \) in each \( N \) where \( L = 0 \). The next point tells us that numeric one of \( \tilde{c} \) where \( L = 1 \). The last point which is the top of points in each \( N \) gives us numeric one of \( \tilde{c} \) where \( L = N \). We observe that real values of \( \tilde{c} \) decreases as \( N \) increases with given \( L \)'s. And as we see Table [5] the size of a gap between the lowest point of \( \tilde{c} \)'s and the top point in each \( N \) decreases.
very slowly as $N$ increases. $\tilde{c} \to 0$ with given $L$ as $N \to \infty$. Fig. 7 shows us that the gap between two successive points decreases with given $N$ as $L$ increases.

| $N$ | $L=1$ | $L=10$ | $L=100$ | $L=1000$ |
|-----|-------|--------|---------|----------|
| 10  | 0.0125212 | 0.0220397 | 0.0236173 | 0.0249429 |
| 11  | 0.0215948 | 0.0257187 | 0.025  | 0.0280503 |
| 12  | 0.0220397 | 0.0236173 | 0.0249429 | 0.0257187 |
| 13  | 0.0220397 | 0.0236173 | 0.0249429 | 0.0257187 |
| 14  | 0.0220397 | 0.0236173 | 0.0249429 | 0.0257187 |
| 15  | 0.0220397 | 0.0236173 | 0.0249429 | 0.0257187 |
| 16  | 0.0220397 | 0.0236173 | 0.0249429 | 0.0257187 |
| 17  | 0.0220397 | 0.0236173 | 0.0249429 | 0.0257187 |
| 18  | 0.0220397 | 0.0236173 | 0.0249429 | 0.0257187 |
| 19  | 0.0220397 | 0.0236173 | 0.0249429 | 0.0257187 |
| 20  | 0.0220397 | 0.0236173 | 0.0249429 | 0.0257187 |

Table 5: Size of a gap between the lowest point of $\tilde{c}$’s where $L = 0$ and the top point where $L = N$ with given $N$.

Figure 7: Fitting of $\tilde{c}$ by $0.0652/\sqrt{N^{10} + (\frac{11}{10})^{L^{1/5} + (\frac{12}{5})}}$ as functions of $L$ with a few fixed values of $N$ as $\tilde{n} = 1$ and $\tilde{b} = 1/10$

Figure 8: Fitting of $\tilde{c}$ by $0.0652/\sqrt{N^{10} + (\frac{11}{10})^{L^{1/5} + (\frac{12}{5})}}$ as functions of $N$ with a few fixed values of $L$ as $\tilde{n} = 1$ and $\tilde{b} = 1/10$

\[
\tilde{c} = \frac{0.0652/\sqrt{N^{10} + (\frac{11}{10})^{L^{1/5} + (\frac{12}{5})}}}{N^{1/10} + (\frac{3}{4})} \]

is a least-squares fit line to a list of data as a linear combination of $\tilde{c}$ of variables $N$ and $L$; We choose 231 different values of $\tilde{c}$’s at $(N, L)$ where $1 \leq N \leq 20$ & $0 \leq L \leq N$.

Fig. 7 shows us that each $\tilde{c}$’s points are positioned on fit lines $0.0652/\sqrt{N^{10} + (\frac{11}{10})^{L^{1/5} + (\frac{12}{5})}}$ with given $N$’s; the lowest fit line is for $N = 20$; the next fit line is for $N = 19$; the top one is for $N = 1$.

Fig. 8 tells us that each $\tilde{c}$’s points lies on fit lines $0.0652/\sqrt{N^{10} + (\frac{11}{10})^{L^{1/5} + (\frac{12}{5})}}$ with given $L$’s; the lowest fit line is for $L = 0$; the next fit line is for $L = 1$; the top one is for $L = 19$.

Figure 9: Fitting of $\tilde{c}$ by $0.02582/\sqrt{N^{10} + (\frac{11}{10})^{L^{1/30} + (\frac{13}{10})}}$ as functions of $L$ with a few fixed values of $N$ as $\tilde{n} = 1$ and $\tilde{b} = 1/30$

Figure 10: Fitting of $\tilde{c}$ by $0.02582/\sqrt{N^{10} + (\frac{11}{10})^{L^{1/30} + (\frac{13}{10})}}$ as functions of $N$ with a few fixed values of $L$ as $\tilde{n} = 1$ and $\tilde{b} = 1/30$
Figs. 9 through 14 shows us that each $\tilde{c}$’s points are positioned on fit lines of $\tilde{c}$’s approximately with various values of $\tilde{a}$ and $\tilde{b}$; (i) for fit lines with given $N$’s, the lowest fit line is for $N = 20$; the next fit line is for $N = 19$; the top one is for $N = 1$; (ii) for fit lines with given $L$’s, the lowest fit line is for $L = 0$; the next fit line is for $L = 1$; the top one is for $L = 19$.

As $\tilde{a} = 1$ and $\tilde{b} = 1/30$, $\tilde{c} = \frac{0.02582N^{33/29} + \frac{1}{4}L^{41/29} + \frac{11}{4}}{N^{13/29} + \frac{1}{4}L^{22/29} + \frac{11}{4}}$ is a least-squares fit to a list of data as a linear combination of $\tilde{c}$ of variables $N$ and $L$: We choose 231 different values of $\tilde{c}$’s at $(N, L)$ where $1 \leq N \leq 20$ & $0 \leq L \leq N$. As $\tilde{a} = 1$ and $\tilde{b} = 1/50$, we get a fit line $\tilde{c} = \frac{0.01621N^{33/29} + \frac{1}{4}L^{41/29} + \frac{11}{4}}{N^{13/29} + \frac{1}{4}L^{22/29} + \frac{11}{4}}$.

As $\tilde{a} = 1$ and $\tilde{b} = 1/100$, we obtain a fit line $\tilde{c} = \frac{0.008996N^{33/29} + \frac{1}{4}L^{41/29} + \frac{11}{4}}{N^{13/29} + \frac{1}{4}L^{22/29} + \frac{11}{4}}$.

The eigenvalue is given by

$$\hat{E} = 2\tilde{c}\left(N + L + \frac{3}{2}\right) - \frac{\tilde{b}^2}{4\tilde{c}^2}$$  \hspace{1cm} (11)

and putting a fit line of $\tilde{c}$’s with given $\tilde{a}$ and $\tilde{b}$ into eq. (11), we obtain discrete energy spectrum with various values of $\tilde{a}$ and $\tilde{b}$ in Figs. 15 through 18; the lowest fit line is for $L = 0$; the
second fit line is for $L = 1$; the third fit one is for $L = 2$. It tells us that energy spectrum has non-linear behaviors. As $\tilde{b}$ becomes smaller, the energy reduction gradually disappears in the area where $N$ is smaller. Roughly, as $N \geq 40$, $\tilde{E}$ have linear behaviors.

3. Quantization of $\tilde{b}$

3.1. Polynomial solutions

In the previous section, We observed that both $\tilde{c}$ and $\tilde{E}$ are quantized in order to have a polynomial solution eq.(4) when we have three term recurrence relation. Now, we consider the case of quantization of $\tilde{b}$ and $\tilde{E}$. For better simple notations, eq.(4) becomes

$$\rho \frac{d^2 y(\rho)}{d\rho^2} + \left(-2\rho^2 - B\rho + 2(L + 1)\right) \frac{dy(\rho)}{d\rho} + \left(\left(E + \frac{B^2}{4} - (2L + 3)\right)\rho + A - B(L + 1)\right) y(\rho) = 0$$ (12)

where $A = \frac{\tilde{a}}{\tilde{c}}$, $B = \frac{\tilde{b}}{\tilde{c}^2}$ and $E = \tilde{E}$.

Comparing eq.(12) with eq.(A.1), the former is the special case of the latter with $z = \rho$, $\mu = -2$, $e = -B$, $\nu = 2(L + 1)$, $\omega = L + 1 - \frac{A}{B}$ and $\Omega = \left(E + \frac{B^2}{4} - (2L + 3)\right)$.
For polynomials of eq. (12) around $\rho = 0$, we treat $A$ as a free variable; consider $-\Omega (\mu = \frac{1}{L} (E + \frac{B^2}{4} - (2L + 3)))$ to be a positive integer; and treat $B$ as a fixed value. Through eq. (A.3), we are able to observe that a series expansion eq. (A.2) becomes a polynomial of degree $N$ by imposing two conditions in the form

$$B_{N+1} = d_{N+1} = 0 \quad \text{where } N \in \mathbb{N}_0 \quad (13)$$

Eq. (13) gives successively $d_{N+2} = d_{N+3} = d_{N+4} = \cdots = 0$.

The general expression of a power series of eq. (12) about $\rho = 0$ for the polynomial and its algebraic equation for the determination of an accessory parameter $B$ are given by:

(i) For $N = 0$, Eq. (13) gives $B_1 = \frac{-\Omega}{2(L+3)} = 0$ and $A_1d_0 = \frac{B(L+1)}{2(L+3)}d_0 = 0$. If we choose $d_0 = 0$ the whole series solution vanishes. Therefore $B = A$ with $E = 3 - \frac{A^2}{4}$ at $L = 0$. Its eigenfunction is $y(\rho) = \sum_{n=0}^{0} d_n \rho^n = 1$ where $d_0 = 1$ chosen for simplicity.

(ii) For $N = 1$, $B_2 = \frac{-\Omega^2}{2(L+4)}$ and $d_2 = A_1d_1 + B_1d_0 = (A_1B_1 + d_1) = \left(\frac{B(L+1)A_1(L+2)}{4(L+1)(2L+3)} - \frac{\Omega}{2(L+3)}\right)d_0$.

Requesting both $B_2$ and $d_2$ to be zero, we get $B = \frac{A(2L+3) + \sqrt{A^2 + 16L+1^2(L+2)}}{2(L+1)(L+2)}$ and $E = 2L + 5 - \frac{1}{4} \left(\frac{A(2L+3) + \sqrt{A^2 + 16L+1^2(L+2)}}{2(L+1)(L+2)}\right)^2$ with $L = 0, 1$. In this case, $y(\rho) = \sum_{n=0}^{N} d_n \rho^n = 1 + \frac{\Omega^2}{(L+1)(L+2)}$.

(iii) For $N \geq 2$, the eigenvalue is obtained by letting $B_{N+1} = 0$ and it gives $E = 2 \left(\frac{N + L + \frac{3}{2}}{2}\right)$\r

$B = \frac{5.87593A^{1/3}(\frac{\mu}{4L+6})^{1/6}}{N^{1/3}(\frac{\mu}{4L+6})^{1/6}}$ is a least-squares fit line to a list of data as a linear combination of $B$ of variables $N$ and $L$: We choose 288 different values of $B$’s at $(N, L)$ where $1 \leq N \leq 22$ & $0 \leq L \leq N$.

Figure 19: Fitting of $B$ by $\frac{5.87593A^{1/3}(\frac{\mu}{4L+6})^{1/6}}{N^{1/3}(\frac{\mu}{4L+6})^{1/6}}$ as functions of $L$ with a few fixed values of $N$ as $A = 1$.

Figure 20: Fitting of $B$ by $\frac{5.87593A^{1/3}(\frac{\mu}{4L+6})^{1/6}}{N^{1/3}(\frac{\mu}{4L+6})^{1/6}}$ as functions of $N$ with a few fixed values of $L$ as $A = 1$.
Fig. 19 shows us that each B's points are positioned on fit lines \( \frac{5.87593(N^{1/10} + \frac{1}{2})}{N + \frac{L}{L^{1/10} + \frac{1}{4}}} \) with given \( N \)'s; the lowest fit line is for \( N = 0 \); the next fit line is for \( N = 1 \); the top one is for \( N = 22 \).

Fig. 20 tells us that each B's points lies on fit lines \( \frac{5.87593(N^{1/10} + \frac{1}{2})}{N + \frac{L}{L^{1/10} + \frac{1}{6}}} \) with given \( L \)'s; the lowest fit line is for \( L = 21 \); the next fit line is for \( L = 20 \); the top one is for \( L = 0 \).

![Figure 21: Fitting of B by \( \frac{5.74785(N^{1/10} + \frac{1}{2})}{N^{1/10} + \frac{L}{L^{1/10} + \frac{1}{2}}} \) as functions of \( N \) with a few fixed values of \( A = 1/50 \)](image)

![Figure 22: Fitting of B by \( \frac{5.74785(N^{1/10} + \frac{1}{2})}{N^{1/10} + \frac{L}{L^{1/10} + \frac{1}{6}}} \) as functions of \( N \) with a few fixed values of \( L = 20 \)](image)

![Figure 23: Fitting of B by \( \frac{5.1482(N^{1/10} + \frac{1}{2})}{N^{1/10} + \frac{L}{L^{1/10} + \frac{1}{4}}} \) as functions of \( L \) with a few fixed values of \( N = 22 \)](image)

![Figure 24: Fitting of B by \( \frac{5.1482(N^{1/10} + \frac{1}{2})}{N^{1/10} + \frac{L}{L^{1/10} + \frac{1}{6}}} \) as functions of \( L \) with a few fixed values of \( N = 20 \)](image)

Figs. 21, 22, 23 and 24 shows us that each B's points are positioned on fit lines approximately with various values of \( A \):(i) for fit lines with given \( N \)'s, the lowest fit line is for \( N = 0 \); the next fit line is for \( N = 1 \); the top one is for \( N = 22 \); (ii) for fit lines with given \( L \)'s, the lowest fit line is for \( L = 21 \); the next fit line is for \( L = 20 \); the top one is for \( L = 0 \).

As \( A = 1/50 \), \( B = \frac{5.74785(N^{1/10} + \frac{1}{2})}{N^{1/10} + \frac{L}{L^{1/10} + \frac{1}{6}}} \) is a least-squares fit to a list of data as a linear combination of \( B \) of variables \( N \) and \( L \): We choose 288 different values of \( B \)'s at \((N, L)\) where \( 1 \leq N \leq 22 \) & \( 0 \leq L \leq N \). As \( A = 2.5 \), we obtain a fit line \( B = \frac{5.1482(N^{1/10} + \frac{1}{2})}{N^{1/10} + \frac{L}{L^{1/10} + \frac{1}{4}}} \).
The eigenvalue is given by

$$\mathcal{E} = 2\left(N + L + \frac{3}{2}\right) - \frac{B^2}{4} \tag{14}$$

and putting a fit line of $B$'s with given $A$ into eq. (14), we obtain discrete energy spectrum with various values of $A$ in Figs. 25, 26, and 27; the lowest fit line is for $L = 0$; the second fit is for $L = 1$; the third fit one is for $L = 2$. It tells us that energy spectrum has non-linear behaviors in the area where $N \leq 20$. Roughly, as $N \geq 40$, $\tilde{E}$ have linear behaviors.

![Figure 25: Fitting of $\mathcal{E}$ as functions of $N$ with a few fixed values of $L$ as $A = 1$](image)

![Figure 26: Fitting of $\mathcal{E}$ as functions of $N$ with a few fixed values of $L$ as $A = 1/50$](image)

![Figure 27: Fitting of $\mathcal{E}$ as functions of $N$ with a few fixed values of $L$ as $A = 2.5$](image)

### 3.3. Conclusion

Figs. 15, 16, 17, and 18 tells us that discrete energy spectrum is concave-down for quantization of $\tilde{c}$ with given $L$ in the area where $0 \leq N \leq L + 5$ roughly. Figs. 25, 26, and 27 shows us that energy spectrum is concave-up for quantization of $\tilde{b}$ (or $B$) with given $L$ in the area where $0 \leq N \leq L + 5$ approximately. Energy spectrum is straight increasing line with various $N$ at given $L$ for quantization of $\tilde{a}$ (or $A$) in eq. (14), because its spectrum does not depend on $A$. By observing energy spectrum, we can decide which coefficient is quantized among $\tilde{a}$, $\tilde{b}$, and $\tilde{c}$.

### 4. Application: Quark-antiquark system with scalar interaction

In 1974, Wilson showed how the string theory meet with an approximation to strongly interacting regime of QCD by the lattice gauge theory having a computable strong-coupling limit to
QCD. In 1975, Eguchi depicted that two quarks squeeze together and formed a bilocal linear structure with one quark at the end and a diquark at the other end through the string approximation with high rotational excitation. A year later, Johnson and Thorn following the bag model for a baryon structure showed to QCD that the elongated bag model whose structure is controlled by tubes of color flux lines, stretched in a rotationally excited baryon, has an nearly linear Regge trajectory. And they calculated the universal Regge slope \( \alpha = \frac{1}{2\pi} = 0.88\,\text{(GeV)}^{-2}. \)

Lichtenberg and collaborators observed the semi-relativistic Hamiltonian (the “Krolikowski” type second order differential equation) in order to calculate meson and baryon masses in 1982.

In the center of mass of a two particle system, the relativistic energy \( E \) of two free particles of masses \( m_1 \) and \( m_2 \), and three-momentum \( \vec{p} \) is

\[
H = \sqrt{\vec{p}^2 + m_1^2} + \sqrt{\vec{p}^2 + m_2^2}
\]

(15)

We suggest that \( S \) is an interaction which is a Lorentz scalar and \( V \) is an interaction which is a time component of a Lorentz vector. Then, let us allow to incorporate the \( V \) and \( S \) into eq. (15) by making the replacements

\[
H \rightarrow H - V, \quad m_i \rightarrow m_i + \frac{1}{2}S, \quad i = 1, 2.
\]

(16)

Letting \( m_1 = m_2 = m \), \( V = 0 \) followed by eq. (16), and suggesting the scalar potential \( S = br \), Gürsey et al. obtained the spin free Hamiltonian involving only scalar potential in the meson \((q - \bar{q})\) system

\[
H^2 = 4 \left( E^2 + P_r^2 + \frac{L(L+1)}{r^2} \right)
\]

(17)

where \( P_r = -\frac{\partial}{\partial r} - \frac{i}{m} \), \( m = \) mass, \( b = \) real positive, and \( L = \) angular momentum quantum number. Since they neglected the mass of quark in their supersymmetric wave equation, they noticed that its differential equation is equivalent to confluent hypergeometric equation having a 2-term recurrence relation between successive coefficients in its classical formal series. And they obtain the following eigenvalue such as

\[
E^2 = 4b(\frac{1}{4} + L + \frac{3}{2})
\]

(18)

where \( N_r = 0, 1, 2, \ldots \) is the radial quantum number, therefore they obtain Regge trajectories of slope \( \frac{1}{2b} \) when they make plots of \( L \) versus \( M^2 \); their theory are consistent with experiment.

We suggest that a radial Schrödinger wave function \( R(r) = \exp \left( -\frac{b}{2} \left( r + \frac{2m}{b} \right)^2 \right) r^l y(r) \) acts on both sides of eq. (17) without neglecting \( m \), it becomes

\[
r \frac{d^2 y}{dr^2} + \left( -br^2 - 2mr + 2(L+1) \right) \frac{dy}{dr} + \left( \frac{E^2}{4} - b \left( L + \frac{3}{2} \right) \right) r = 2m(L+1) y = 0
\]

(19)

Comparing eq. (19) with eq. (17), the former is the special case of the latter with \( z = r, \mu = -b, \varepsilon = -2m, \nu = 2(L+1), \omega = L + 1 \) and \( \Omega = \frac{E^2}{4} - b \left( L + \frac{3}{2} \right) \).

Let’s investigate an asymptotic behaviour of the radial wave function \( R(r) \) in eq. (19) as a variable \( r \) goes to positive infinity. We assume that \( y(r) \) is an infinite series in eq. (19), and substitute eq. (B.15) into \( R(r) = \exp \left( -\frac{b}{2} \left( r + \frac{2m}{b} \right)^2 \right) r^l y(r) \)
\[
\lim_{n \to \infty} R(r) \approx A r^L \left( \frac{b}{2} r^2 \right)^{L} \exp \left( \frac{b}{4} r^2 - m \left( r + \frac{m}{b} \right) \right)
\]

In eq. (20) if \( r \to \infty \), then \( \lim R(r) \to \infty \). It is unacceptable that wave function \( R(r) \) is divergent as \( r \) goes to infinity in the quantum mechanical point of view. Therefore the function \( y(r) \) must to be polynomial in eq. (19) in order to make the wave function \( R(r) \) being convergent even if \( r \) goes to infinity.

For polynomials of eq. (19) around \( r = 0 \), we treat \( b \) and \( E \) as quantized values. The general expression of a power series of eq. (19) about \( r = 0 \) for the polynomial and its algebraic equation for the determination of an accessory parameter \( b \) are taken by

1. For \( N = 0 \), Eq. (6) gives \( B_1 = \frac{1}{2} \left( \frac{1}{3} - 3 \right) = 0 \) and \( d_1 = 0 \). If we choose \( d_0 = 0 \) the whole series solution vanishes. Therefore there is no solution unless \( m = 0 \), in which case the solution is reduced to that of the confluent Hypergeometric case \( E^2 = 4b(L + 3/2) \). Since we are considering the case \( m \neq 0 \), we conclude that there is no solution as \( N = 0 \).

2. For \( N \geq 1 \), the energy eigenvalue is determined from \( B_{N+1} = 0 \), giving \( E^2 = 4b \left( N + L + \frac{1}{2} \right) \) with \( L = 0, 1, 2, \ldots, N \). Quantized allowed values of \( b \)'s are obtained from \( d_{N+1} = 0 \). Its eigenfunction is \( N \)-th order polynomial \( y(r) = \sum_{n=0}^{N} d_{n} r^{n} \).

An algebraic equation of degree \( N/2 \) for the determination of \( b/m^2 \) for \( N = \) even number has \( N/2 \) real roots with a given \( N \). And a characteristic equation of degree \( (N + 1)/2 \) for \( N = \) odd one has \( (N + 1)/2 \) real roots with a given \( N \). We obtain the numeric every real values of \( b/m^2 \) with given \( N \) and \( L \) to solve algebraic equations of \( b/m^2 \).

\[
\begin{array}{l|l}
K & 0 & 2.17476L+1.23455N+3.13165 \\
& 1 & 2.0736L+1.0252N+2.642736 \\
& 2 & 2.0198L+1.3965N+4.197853 \\
& 3 & 1.9968L+1.3552N+6.129672 \\
& 4 & 1.9712L+1.4135N+8.158113 \\
& 5 & 1.9591L+1.4668N+10.194577 \\
& 6 & 1.9509L+1.5467N+12.047167 \\
& 7 & 1.9469L+1.4748N+14.252067 \\
& 8 & 1.9400L+1.4907N+16.283441 \\
& 9 & 1.93656L+1.5160N+18.314824 \\
& 10 & 1.9335L+1.5212N+20.346212 \\
\end{array}
\]

Table 6: Fit lines of \( b/m^2 \) of variables \( N \) and \( L \) where \( N \geq 2K + 1 \)

In Table 6, as \( K = 0 \), \( b/m^2 = 2.17476L+1.23455N+3.13165 \) for \( N \) and \( L \) is a least-squares fit line to a list of data as a linear combination of \( b/m^2 \) of variables \( N \) and \( L \). We choose 350 different smallest values of \( b/m^2 \)’s at \((N,L)\) with given \( N \) and \( L \) where \( 1 \leq N \leq 25 \) & \( 0 \leq L \leq N \): For
instance, there are 5 possible real values of \( b/m^2 \) as \( N = L = 10 \) such as 0.366018, 0.579236, 1.03967, 2.35494 and 9.45702. We choose 0.366018 as our numeric real value of \( b/m^2 \).

As \( K = 1 \), \( b/m^2 = \frac{(22/17) - (11/44)}{(N-2) + (17/9 + 12/7/4) L + 22K + 13/4} \), for \( N \) and \( L \) is a least-squares fit line to a list of data as a linear combination of \( b/m^2 \) of variables \( N \) and \( L \): We choose 345 different second smallest values of \( b/m^2 \)'s at \( (N, L) \) with given \( N \) and \( L \) where \( 3 \leq N \leq 25 \) & \( 0 \leq L \leq N \).

As \( K = 2 \), \( b/m^2 = \frac{(1.93544L+1.5212(N-20)+34.6212)}{(N-20)^2+0.0054395(N-20)-0.00247279} \), for \( N \) and \( L \) is a least-squares fit line to a list of data as a linear combination of \( b/m^2 \) of variables \( N \) and \( L \): We choose 336 different third smallest values of \( b/m^2 \)'s at \( (N, L) \) with given \( N \) and \( L \) where \( 5 \leq N \leq 25 \) & \( 0 \leq L \leq N \).

As \( K = 10 \), \( b/m^2 \) is a least-squares fit line to a list of data as a linear combination of \( b/m^2 \) of variables \( N \) and \( L \): We choose 120 different smallest values of \( b/m^2 \)'s at \( (N, L) \) with given \( N \) and \( L \) where \( 21 \leq N \leq 25 \) & \( 0 \leq L \leq N \).

From Table 3, we obtain the following general least-squares fit line of \( b/m^2 \) of variables \( N \) and \( L \) with given \( K \):

\[
b/m^2 = \frac{(22/17) - (11/44)}{(N-2) + (17/9 + 12/7/4) L + 22K + 13/4} \tag{21}
\]

where \( N \geq 2K + 1 \) and \( L = 0, 1, 2, \cdots, N \).

![Figure 28: Fitting of \( b/m^2 \) by eq \( 21 \) as functions of \( N \) with a few fixed values of \( N \) as \( K = 4 \)](image)

Fig. 28 shows us that each \( b/m^2 \)'s points are positioned on fit lines eq \( 21 \) with given \( N \)'s as \( K = 4 \); the lowest fit line is for \( N = 25 \); the next fit is for \( N = 24 \); the top one which has the most steep slop is for \( N = 13 \). Fig. 29 shows us that each \( b/m^2 \)'s points are positioned on fit lines eq \( 21 \) with given \( L \)'s as \( K = 4 \); the lowest fit line is for \( L = 0 \); the next fit is for \( L = 1 \); the top one is for \( L = 24 \).

Fig. 30 shows us that each \( b/m^2 \)'s points are positioned on fit lines eq \( 21 \) with given \( N \)'s as \( K = 5 \); the lowest fit line is for \( N = 25 \); the next fit is for \( N = 24 \); the top one which has the most steep slop is for \( N = 14 \). Fig. 31 shows us that each \( b/m^2 \)'s points are positioned on fit lines eq \( 21 \) with given \( L \)'s as \( K = 5 \); the lowest fit line is for \( L = 0 \); the next fit is for \( L = 1 \); the top one is for \( L = 24 \). Similarly, Fig. 32 and Fig. 33 tell us that each \( b/m^2 \)'s points are positioned on fit lines eq \( 21 \) as \( K = 6 \). Fig. 34 and Fig. 35 are the case of \( K = 7 \).
By substituting eq (21) into $E^2 = 4b(N + L + \frac{1}{2})$, we get the experimental fit to the eigenvalue $E^2$

$$E^2 \approx 4 \left( \frac{22}{13} - \frac{11/4}{K^2} \right) (N - 2K) + \left( \frac{17}{9} + \frac{1/2}{K^2/4} \right) L + \frac{22}{7} K + \frac{13}{4} \right) \left( N + L + \frac{3}{2} \right) m^2$$  (22)
where \( N \geq 2K + 1 \) and \( L = 0, 1, 2, \cdots, N \).

The mass spectrum which is roughly given by eq.(22) can not be linear in \( N \) unlike \( m = 0 \) case given in Eq.(18). Because \( b \) is determined by other parameters, which in turn introduces extra dependence of \( E^2 \) on \( N, L \) and \( K \) through that of \( b \).

Many great scholars noticed that the excited meson including baryon conduct like a elongated bilocal linear structure hold by the gluon flux tube behaving as a scalar linear potential for high rotational excitation at large separation. Gürsey and collaborators wrote the semi-relativistic Hamiltonian for the \( q - \bar{q} \) system neglecting small mass of quarks\[6\], suggested by Lichitenberg et al.\[24\]. In this system, QCD forces are flavor independent, the strong-coupling potential like a Coulomb potential is negligible and the confining part of QCD potential is spin independent. They detected that their wave equation neglecting the mass of quark is equivalent to a confluent hypergeometric differential equation in which the recursive relation involves two terms in its power series expansion. They only obtained one mass formula for meson with the universal Regge slope, and their formula is consistent with experiment.

However, since we include the mass of quark into their spin free Hamiltonian involving only scalar potential, a modified form of BCH equation arises. Its recurrence relation consists of three terms in the formal series solution. For the two term recursive relation in a formal series, there is an only single quantized parameter \( (E^2) \) required to make a polynomial solution. But, for the three term case, we need two parameters to construct a polynomial; a tension \( b \) and an energy \( E^2 \) should be quantized.

As we compare Gürsey’s mass formula with eq.(22), we notice that if \( m \) is non-zero, a tension \( b \) is determined by \( m, N \) and \( L \) with given \( K \)-dependence, and linearity of a spectrum is disappeared, on other hands, it has non-linearity behaviour. The model is inconsistent with hadron spectrum phenomenology. So, to fit our theory to the data, current quark’s mass should be zero. If \( m = 0 \), \( b \) is not determined by \( m \) and has an intrinsic scale by itself. Its spectrum has linearity behaviour by its scale. Therefore, Gürsey’s assumption in which he neglects the mass of current quark in his mass formula is correct. The current quark does not have its own mass. And we can say that chiral symmetry is induced by the color confinement.

Appendix A. Biconfluent Heun equation

\[
z \frac{d^2 y}{dz^2} + \left( \mu z^2 + \varepsilon z + \nu \right) \frac{dy}{dz} + \left( \Omega z + \varepsilon \omega \right) y = 0 \tag{A.1}
\]

Eq.(A.1) is a modified form of Biconfluent Heun (BCH) equation where \( \mu, \varepsilon, \nu, \Omega \) and \( \omega \) are real or imaginary parameters. It has a regular singularity at the origin and an irregular singularity at the infinity of rank 2. BCH equation is derived, the special case of the modified BCH equation, by putting coefficients \( \mu = 1 \) and \( \omega = -q/e \). [25] Recently, its equation starts to appear the Schrödinger equation for the second exton potential and the power potentials. [19][21]

Since a formal series with unknown coefficients is substituted into the hypergeometric equation having three regular singular points, a 2-term recursion relation between successive coefficients in its power series solution starts to appear. Confluent types of the hypergeometric equation is derived when two or more singularities coalesce into an irregular singularity. This

---

1For the canonical form of BCH equation [26], replace \( \mu, \varepsilon, \nu, \Omega \) and \( \omega \) by \(-2, -\beta, 1 + \alpha, \gamma - \alpha - 2 \) and \( 1/2(\delta/\beta + 1 + \alpha) \) in eq.(A.1).
type includes equations of Legendre, Laguerre, Kummer, Bessel, Jacobi and etc, whose analytic solutions in compact forms are already constructed by great many scholars extensively including their definite or contour integrals.

For the Heun equation having four regular singular points, the recurrence relation in its Frobenius solution involves 3 terms. The Heun equation generalizes all well-known equations of Spheroidal Wave, Lame, Mathieu, hypergeometric type and etc. Until now, its series solutions in which coefficients are given fully and clearly have been unknown because of its complex mathematical computations; their numerical calculations are still ambiguous. Of course, for definite or contour integrals of Heun equation, no analytic solutions have been constructed regrettably.

Like deriving of confluent hypergeometric equation from the hypergeometric equation, 4 confluent types of Heun equation can be derived from merging two or more regular singularities to take an irregular singularity in Heun equation. These types include such as (1) Confluent Heun (two regular and one irregular singularities), (2) Doubly Confluent Heun (two irregular singularities), (3) Biconfluent Heun (one regular and one irregular singularities), (4) Triconfluent Heun equations (one irregular singularity).

\(y(z)\) has a series expansion of the form around the origin

\[
y(z) = \sum_{n=0}^{\infty} a_n z^n \tag{A.2}
\]

Plug eq. (A.2) into eq. (A.1).

\[
c_{n+1} = A_n d_n + B_n d_{n-1} \quad ; \ n \geq 1 \tag{A.3}
\]

where \(A_n = \frac{\epsilon(n+\nu)}{(n+1)(n+\nu)} \) \(B_n = -\frac{\Omega(n-1)}{(n+1)(n+\nu)} \) and \(d_1 = A_0 d_0\).

Appendix B. Asymptotic behavior of a modified BCH function

Let’s investigate a function \(y(z)\) as \(z\) go to infinity. Assume that a \(y(z)\) is an infinite series and its series expansion of eq. (A.3) is given by

\[
y(z) = \sum_{m=0}^{\infty} y_m(z) = \sum_{n=0}^{\infty} \left( \frac{\Omega}{\Omega} \right)_n z^n + \epsilon \sum_{n=0}^{\infty} \frac{(i_0 + \frac{\nu}{2})}{(i_0 + \frac{1}{2}) i(i_0 - \frac{1}{2} + \gamma)} (1)_n (\gamma)_n \\
\times \sum_{n=2}^{\infty} \left( \frac{\Omega}{\Omega} \right)_n z^{2i} + \epsilon \sum_{n=2}^{\infty} \frac{(i_0 + \frac{\nu}{2})}{(i_0 + \frac{1}{2}) i(i_0 - \frac{1}{2} + \gamma)} (1)_n (\gamma)_n \\
\times \prod_{k=1}^{\infty} \left\{ \frac{(i_0 + \frac{\nu}{2})_n (1 + \frac{\nu}{2})_n (\gamma + \gamma)_n}{(i_0 + \frac{1}{2})_n (1 + \frac{1}{2})_n (\gamma + \gamma)_n} \right\} \\
\times \sum_{n=2}^{\infty} \left( \frac{\Omega}{\Omega} \right)_n z^{2i} + \epsilon \sum_{n=2}^{\infty} \frac{(i_0 + \frac{\nu}{2})}{(i_0 + \frac{1}{2}) i(i_0 - \frac{1}{2} + \gamma)} (1)_n (\gamma)_n \\
\times \prod_{k=1}^{\infty} \left\{ \frac{(i_0 + \frac{\nu}{2})_n (1 + \frac{\nu}{2})_n (\gamma + \gamma)_n}{(i_0 + \frac{1}{2})_n (1 + \frac{1}{2})_n (\gamma + \gamma)_n} \right\} \tag{B.1}
\]
where
\[
\begin{align*}
\hat{z} &= -\frac{1}{2} \mu z^2 \\
\hat{\nu} &= -\frac{1}{2} \epsilon z \\
\gamma &= \frac{1}{2} (1 + \nu)
\end{align*}
\]

On the above, \(y_n(z)\) is a \(m\)-tuple series; \(y_0(z)\) is a single series, \(y_1(z)\) is a double series, \(y_2(z)\) is a triple series, etc. In this article Pochhammer symbol \((x)_n\) is used to represent the rising factorial: \((x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}\).

There is a generalized hypergeometric function which is given by
\[
\left(\begin{array}{c}
\frac{\Omega}{2}\mu + \frac{j}{2} + i_{j-1} \\
\frac{\Omega}{2}\mu + \frac{j}{2} + i_{j-1} + \sigma
\end{array}\right) = \sum_{l=0}^{\infty} \frac{\left(\frac{\Omega}{2}\mu + \frac{j}{2}\right)_l \left(\frac{\Omega}{2}\mu + \frac{j}{2} - 1\right)_{l+1}}{\Gamma\left(l + \frac{\sigma}{2}\right)} (1 - q)^l
\]

By using integral form of beta function,
\[
B\left(\tilde{z}n, \frac{a}{2}, \frac{b}{2}\right) = \int_0^1 dt f(t) \left(1 - t\right)^{\frac{a}{2}} \left(1 - t\right)^{\frac{b}{2}}
\]

where \(j = 1, 2, 3, \ldots\)

Substitute eq. (B.3a) and eq. (B.3b) into eq. (2.2), and divide \((i_{j-1} + \frac{1}{2})(i_{j-1} - 1 + \gamma + \frac{1}{2})\) into the new eq. (B.2).

\[
I_j = \frac{1}{1 + \gamma + \frac{1}{2}} \sum_{l=0}^{\infty} \frac{\left(\frac{\Omega}{2}\mu + \frac{j}{2}\right)_l \left(\frac{\Omega}{2}\mu + \frac{j}{2} - 1\right)_{l+1}}{\Gamma\left(l + \frac{1}{2}\right)} (1 - q)^l
\]

Here, \(M(a, b, z) = \sum_{n=0}^{\infty} \frac{a_n(b, z)^n}{n!}\) is a Kummer function of the first kind. The asymptotic behavior of Kummer’s solution as the real part of \(z\) goes to positive infinity is \(M(a, b, z) \sim \frac{\text{Gamma}(b)}{\text{Gamma}(a)} e^{-z} \exp(z)\).

The asymptotic function of Kummer’s solution in eq. (B.4) is written as
\[
M\left(\frac{\Omega}{2\mu} + \frac{j}{2} + i_{j-1}, 1, \tilde{z}(1 - t)(1 - u)\right) \sim \frac{\tilde{z}(1 - t)(1 - u)^{\frac{\Omega}{2\mu} + \frac{j}{2} + i_{j-1}}}{\Gamma\left(\frac{\Omega}{2\mu} + \frac{j}{2} + i_{j-1}\right)} \exp\left(\tilde{z}(1 - t)(1 - u)\right)
\]

(B.5)
Plug eq. (B.5) into eq. (B.3).

\[
I_j \sim \frac{z^{\frac{3}{2} - 1 + \frac{1}{2j-1}}}{\Gamma\left(\frac{3}{2j} + \frac{1}{2} + i_j\right)} \sum_{i=0}^{\infty} \frac{\zeta^i}{i!} \left(1 + \frac{\Omega}{2\mu} + \frac{j}{2} + i_j + k\right) B\left(\gamma - 1 + \frac{j}{2} + i_j, \frac{\Omega}{2\mu} + \frac{j}{2} + i_j + k\right)
\]

We know

\[
\sum_{k=0}^{\infty} \frac{B(a,b+k)B(c,b+k)}{k!} \zeta^k = B(a,b)B(c,b) \quad 2F_2(b,b,a+b,c+b;c;z) \quad (B.7)
\]

Applying eq. (B.7) into eq. (B.6), we obtain the following \( I_j \):

\[
I_j \sim \frac{z^{\frac{3}{2} - 1 + \frac{z^2}{2j-1}}}{\Gamma\left(\frac{3}{2j} + \frac{1}{2} + i_j\right)} \sum_{i=0}^{\infty} \frac{\zeta^i}{i!} \left(1 + \frac{\Omega}{2\mu} + \frac{j}{2} + i_j + k\right) B\left(\gamma - 1 + \frac{j}{2} + i_j, \frac{\Omega}{2\mu} + \frac{j}{2} + i_j + k\right)
\]

The asymptotic behavior of a \( 2F_2 \) function for large \( |z| \) is

\[
2F_2(a_1,a_2;b_1,b_2;z) \sim \frac{\Gamma(b_1)\Gamma(b_2)}{\Gamma(a_1)\Gamma(a_2)} z^{a_1-a_2-b_1-b_2} e^z \quad (B.9)
\]

Substituting eq. (B.9) into eq. (B.8), we have the following asymptotic function of \( I_j \):

\[
I_j \sim \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\gamma - 1 + \frac{j}{2} + i_j\right)}{\Gamma\left(\frac{3}{2j} + \frac{1}{2} + i_j\right)} z^{\frac{2}{2j} - 1} \zeta^\frac{1}{2j} e^\frac{1}{2} \quad (B.10)
\]

The asymptotic behavior of \( y_0(z) \) (Kummer function of the first kind) for large \( |z| \) is

\[
y_0(z) \sim \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2j} + \frac{1}{2} + i_j\right)} \zeta^\frac{1}{2j} e^\frac{1}{2} \quad (B.11)
\]

Put \( j = 1 \) into eq. (B.10), and take the new eq. (B.10) into \( y_1(z) \) in eq. (B.1):

\[
y_1(z) \sim \left(\frac{-e}{\sqrt{-2\mu}}\right) \sum_{n=0}^{\infty} \frac{\left(i_0 + \frac{j}{2}\right)\Gamma\left(i_0 + \frac{1}{2}\right)\Gamma\left(i_0 + \gamma - \frac{1}{2}\right)}{\Gamma\left(i_0 + \frac{3}{2} + \frac{j}{2}\right)\Gamma\left(i_0 + \gamma - \frac{1}{2}\right)} \frac{\left(\frac{\Omega}{2\mu}\right)^n}{\Gamma\left(i_0 + \frac{3}{2} + \frac{j}{2}\right)} (1)_n (y)_n \quad (B.12)
\]
Put \( j = 2 \) into eq. (B.10), and take the new eq. (B.10) into \( y_2(z) \) in eq. (B.1)

\[
y_2(z) \sim \left( -\frac{e}{\sqrt{-2\mu}} \right)^2 \sum_{i=0}^\infty \frac{(i_0 + \frac{\omega}{2})}{(i_0 + \frac{1}{2}) \Gamma(i_0 + \gamma - \frac{1}{2})} \left( \frac{\Omega}{\gamma} \right)_{i_0} \\
\times \sum_{i_1=0}^\infty \frac{(i_1 + \frac{1}{2} + \frac{\omega}{2}) \Gamma(i_1 + 1) \Gamma(i_1 + \gamma) \left( \frac{\Omega}{\gamma} \right)_{i_1}}{\Gamma(i_1 + \frac{3}{2} + 1) \left( \frac{\Omega}{\gamma} \right)_{i_1}} \frac{(\gamma + \frac{1}{2})}{\gamma} \frac{\epsilon^\omega}{\gamma} (B.13)
\]

Similarly, putting \( j = 3 \) into eq. (B.10), and take the new eq. (B.10) into \( y_3(z) \) in eq. (B.1)

\[
y_3(z) \sim \left( -\frac{e}{\sqrt{-2\mu}} \right)^3 \sum_{i=0}^\infty \frac{(i_0 + \frac{\omega}{2})}{(i_0 + \frac{1}{2}) \Gamma(i_0 + \gamma - \frac{1}{2}) (\gamma)_0} \\
\times \sum_{i_1=0}^\infty \frac{(i_1 + \frac{1}{2} + \frac{\omega}{2}) \Gamma(i_1 + 1) \Gamma(i_1 + \gamma) \left( \frac{\Omega}{\gamma} \right)_{i_1}}{\Gamma(i_1 + \frac{3}{2} + 1) \left( \frac{\Omega}{\gamma} \right)_{i_1}} \frac{(\gamma + \frac{1}{2})}{\gamma} \frac{\epsilon^\omega}{\gamma} (B.14)
\]

By repeating this process for all higher terms of asymptotic functions of sub-summation \( y_m(z) \) terms where \( m \geq 4 \), we obtain every asymptotic forms of \( y_m(x) \) terms. Since we substitute eq. (B.11), eq. (B.12), eq. (B.13), eq. (B.14) and including all asymptotic forms of \( y_m(z) \) terms where \( m \geq 4 \) into eq. (B.14), we obtain the following asymptotic behavior of \( y(z) \):

\[
y(z) \sim \mathcal{A} \xi^\Omega \exp(\xi) (B.15)
\]

where

\[
\mathcal{A} = \frac{\Gamma(\Omega)}{\Gamma(\frac{3}{2})} \left( -\frac{e}{\sqrt{-2\mu}} \right)^3 \sum_{i=0}^\infty \frac{(i_0 + \frac{\omega}{2}) \Gamma(i_0 + \frac{1}{2}) \Gamma(i_0 + \gamma - \frac{1}{2}) \left( \frac{\Omega}{\gamma} \right)_{i_0}}{\Gamma(i_0 + \frac{3}{2} + 1) \left( \frac{\Omega}{\gamma} \right)_{i_0}} \\
\times \sum_{i_1=0}^\infty \frac{(i_1 + \frac{1}{2} + \frac{\omega}{2}) \Gamma(i_1 + 1) \Gamma(i_1 + \gamma) \left( \frac{\Omega}{\gamma} \right)_{i_1}}{\Gamma(i_1 + \frac{3}{2} + 1) \left( \frac{\Omega}{\gamma} \right)_{i_1}} \frac{(\gamma + \frac{1}{2})}{\gamma} \frac{\epsilon^\omega}{\gamma} (B.14)
\]

\( \mathcal{A} \) is just a fixed constant, depended on \( \mu, e, \nu, \Omega \) and \( \omega \) (real or imaginary) parameters.
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