New developments for modern celestial mechanics – I. General coplanar three-body systems. Application to exoplanets

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ABSTRACT

Modern applications of celestial mechanics include the study of closely packed systems of exoplanets, circumbinary planetary systems, binary–binary interactions in star clusters and the dynamics of stars near the Galactic centre. While developments have historically been guided by the architecture of the Solar System, the need for more general formulations with as few restrictions on the parameters as possible is obvious. Here, we present clear and concise generalizations of two classic expansions of the three-body disturbing function, simplifying considerably their original form and making them accessible to the non-specialist.

Governing the interaction between the inner and outer orbits of a hierarchical triple, the disturbing function in its general form is the conduit for energy and angular momentum exchange and as such, governs the secular and resonant evolution of the system and its stability characteristics. Focusing here on coplanar systems, the first expansion is one in the ratio of inner to outer semimajor axes and is valid for all eccentricities, while the second is an expansion in eccentricity and is valid for all semimajor axis ratios, except for systems in which the orbits cross (this restriction also applies to the first expansion). Our generalizations make both formulations valid for arbitrary mass ratios. The classic versions of these appropriate to the restricted three-body problem are known as Kaula’s expansion and the literal expansion, respectively. We demonstrate the equivalence of the new expansions, identifying the role of the spherical harmonic order $m$ in both and its physical significance in the three-body problem, and introducing the concept of principal resonances.

Several examples of the accessibility of both expansions are given including resonance widths, and the secular rates of change of the elements. Results in their final form are gathered together at the end of the paper for the reader mainly interested in their application, including a guide for the choice of expansion.

Key words: chaos – methods: analytical – celestial mechanics – planets and satellites: dynamical evolution and stability.

1 INTRODUCTION

The recent explosion in the study of exoplanets is bringing together specialists from diverse backgrounds. On the observational side, spectroscopic and photometric techniques for binary star observations have been refined to the point where radial velocity and eclipse contrast measurements at the 50 cm s$^{-1}$ and 10$^{-4}$ level, respectively, are possible (with the HARPS spectrograph and the Kepler satellite; Borucki et al. 2011; Pepe et al. 2011; Dumusque et al. 2012; Batalha et al. 2013), while the field is breathing new life into observational techniques ranging from direct imaging (Kalas et al. 2008; Marois et al. 2008, 2010; Lagrange et al. 2009), to infrared and even X-ray photometry and spectroscopy (Richardson et al. 2006; Fortney & Marley 2007; Pillitteri et al. 2010). As a direct consequence of the ability to make high-precision estimates of the masses and radii of transiting planets, as well as to observe their atmospheres directly (Seager & Deming 2010, and references therein), geophysical and atmospheric scientists are finding a place in this booming field, with new centres for planet characterization and habitability springing up around the world.

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On the theoretical side, the discovery of 51 Peg b in its 4.23 d orbit (Mayor & Queloz 1995) immediately reinvigorated existing theories of planet formation, especially the question of the role of planet migration which until then (and still) had specialists wondering why the giant planets in the Solar system apparently migrated so little (Goldreich & Tremaine 1979; Lin & Papaloizou 1979; Lin, Bodenheimer & Richardson 1996). The discovery of significantly eccentric exoplanets (Holman, Touma & Tremaine 1997; Naef et al. 2001) and the detection of misaligned and even retrograde planetary orbits (Traud et al. 2010; Winn et al. 2010) revived ideas about the secular evolution of inclined systems (Kozai 1962; Kiseleva, Eggleton & Mikolkova 1998; Fabrycky & Tremaine 2007), as well as dynamical interactions resulting in scattering during the formation process (Rasio & Ford 1996). The discovery of dynamically packed multiplanet systems (Lissauer et al. 2011; Lovis et al. 2011) reminds us of the age-old quest to understand the stability of the Solar system (Laskar 1996), and indeed as Poincaré well appreciated (Poincaré 1892), the quest to understand stability in the ‘simpler’ three-body problem (Wisdom 1980; Robutel 1995; Mardling 2008, 2013).

Large surveys have yielded a rich harvest of planets by now (Udry & Santos 2007), enabling a comparison with the results of Monte Carlo-type simulations of the planet-formation process (Mordasini et al. 2009). It is now becoming clear that the planet mass distribution continues to rise towards lower masses with a hint of a deficit at 30 M⊕ (Mayor et al. 2011), and with almost no objects in the range 25–45 M⊕ at the high-mass end of the distribution (Sahlmann et al. 2011). The latter is the so-called brown dwarf desert and it supports the idea that at least two distinct mechanisms operate for the formation of planets and stars (Udry & Santos 2007).

Until very recently, the notion that planets might form in a circumbinary disc was purely theoretical,1 with no clear consensus on whether or not the strong fluctuating gravitational field of the binary pair would prevent their formation (Meschiari 2012a,b; Paardekooper et al. 2012; Pelupessy & Portegies Zwart 2013). The Kepler survey has revealed that Nature does, indeed, accomplish this feat (Doyle et al. 2011; Orosz et al. 2012a,b; Welsh et al. 2012), with planet–binary period ratios almost as low as they can be stability-wise [the current range is 5.6 for Kepler 16 (Doyle et al. 2011) to 10.4 for Kepler 34 (Welsh et al. 2012)]. With our understanding of the growth of planetesimals in a relatively laminar environment still in its infancy (Meisner, Wurm & Teiser 2012; Okuzumi et al. 2012; Takeuchi & Ida 2012), the very fact that planets exist in such ‘hostile’ environments puts strong constraints on how and where planetesimals form.

At every step of the way, some knowledge of celestial mechanics and the dynamics of small-N systems is essential. A short list might include using stability arguments to place limits on the masses in a multiplanetary system observed spectroscopically (Mayor et al. 2009); including dynamical constraints in orbit-fitting algorithms (Lovis et al. 2011; Laskar, Boué & Correia 2012); modelling the planet–planet interaction in a resonant system to infer the presence of a low-mass planet (Rivera et al. 2005; Correia et al. 2010); using transit timing variations (TTVs) to infer the presence of unseen companions (Ballard et al. 2011; Torres et al. 2011; Nesvorný et al. 2012) or to estimate the masses of planets in a multitransiting system (where no spectroscopic data are available: Holman et al. 2010; Lissauer et al. 2011; Fabrycky et al. 2012; Steffen et al. 2012); understanding the origin of resonant and near resonant systems (Papaloizou 2011; Lithwick & Wu 2012; Batygin & Morbidelli 2013; Mardling & Udry, in preparation); deciphering the complex light curves of multitransiting systems (Lissauer et al. 2011) and eclipsing binaries with circumbinary planets (Orosz et al. 2012a); inferring the true orbit of a planet observed astrometrically (McArthur et al. 2010); understanding the influence of stellar and planetary tides on a system which includes a short-period planet (Wu & Goldreich 2002; Mardling 2007, 2010); inferring information about the internal structure of such a planet (Wu & Goldreich 2002; Batygin, Bodenheimer & Laughlin 2009; Mardling 2010). Most of these examples involve orbit–orbit interactions in systems with arbitrary mass ratios, eccentricities and inclinations, and as such many researchers resort to expensive (time-wise) direct integrations. Moreover, numerical studies offer shrouded insight into parameter dependence and physical processes. The availability of a generalized and easy-to-use disturbing function therefore seems timely; this is the focus of the present paper.

1.1 The disturbing function

As the interaction potential for a hierarchically arranged triple system of bodies, the disturbing function (la fonction perturbatrice of Le Verrier 1855) is developed as a Fourier series expansion whose harmonic angles are linear combinations of the various orbital phase and orientation angles in the problem (except for the inclinations), and whose coefficients are functions of the masses, semimajor axes, eccentricities and inclinations. With the Solar system as the only planetary system known until 1992 (Wolszczan & Frail 1992), many expansions of the disturbing function from Laplace and Le Verrier till now (see Murray & Dermott 2000 for other references) rely on the smallness of eccentricities, inclinations and mass ratios, with ranges for Solar system planets being 0.007–0.093, 0–3.4° and 3.2 × 10−7–10−3, respectively (not including Mercury and Pluto). Moreover, since the period ratios between adjacent pairs of planets are not very large, varying between 1.5 and 2.5 (including the mean period of the asteroids), such expansions tend to place no restriction on the ratio of semimajor axes except that the orbits should not cross. These include the literal expansions which are written in terms of Laplace coefficients (Murray & Dermott 2000, and Appendix C of this paper). They are often presented in somewhat long and unwieldy form [for example, the paper by Le Verrier (1855) is 84 pages long], resulting in the reluctance of non-specialists to use them. Moreover, such presentations often obscure the dependence of the disturbing function on the various system parameters, making their use somewhat doubtful in an era where Morse’s law still holds.

Many elegant Hamiltonian formulations exist, for example, Laskar & Robutel (1995) which uses Poincaré canonical heliocentric variables to study the planetary three-body problem. The Hamiltonian form is used, for example, when one wishes to exploit the integrability

1 Unless one considers the HD 202060 system to be in this category; it is composed of an inner pair with masses 1.13 and 0.017 M☉, and an outer planet with mass 2.44 MJ (Correia et al. 2005).
of the underlying subsystems in the case that they are weakly interacting. The aim of Laskar & Robutel (1995) was to use this form to study
the stability of the planetary three-body problem (Robutel 1995) in the framework of the famous KAM theorem (Kolmogorov 1954; Arnol’d
1963; Moser 1962). While the Laskar & Robutel (1995) formulation does not make any assumptions about the mass ratios, their dependence
is not explicit in the resulting expressions and it is hard to get a feel for the dependence on the eccentricities and inclinations. Moreover, it
has the drawback that except for the innermost orbit, ellipses described by the orbital elements are not tangential to the actual motion (they
are not ‘osculating’) because the dominant body is used as the origin. In contrast, Jacobi coordinates which refer the motion of a body to the
centre of mass of the subsystem to its interior ensure that this is true. Jacobi coordinates are used here.

The study of systems with more substantial eccentricities and inclinations, for example comet orbits perturbed by the giant planets, or
stellar triples and quadruples, etc., has largely relied on numerical integrations and continues to do so since the discovery of significantly
eccentric exoplanets. On the analytical side, Legendre expansions in the ratio of semimajor axes generally use orbit averaging to produce
a secular disturbing function which is suitable for the study of systems in which resonance does not play a role. Examples include the
formulations of Innanen et al. (1997), Ford, Kozinsky & Rasio (2000), Laskar & Boué (2010) and Naoz et al. (2011). Each of these uses a
Hamiltonian approach, each is valid for arbitrary eccentricities and inclinations, and all but Innanen et al. (1997) are valid for arbitrary mass
ratios. Innanen et al. (1997) includes only quadrupole terms \( l = 2 \), Ford et al. (2000) and Naoz et al. (2011) include quadrupole and octopole
terms \( l = 2, 3 \), while Laskar & Boué (2010) includes terms up to \( l = 5 \) for inclined systems and \( l = 10 \) for coplanar systems. Naoz et al.
(2011) demonstrate the importance of including the full mass dependence, especially for systems for which significant angular momentum is
transferred between the inner and outer orbits.

While secular expansions give significant insight into the long-term evolution of arbitrary configurations, care should be taken to
determine that such an expansion is appropriate for any given system. The orbit averaging technique effectively involves discarding all terms
which depend on the fast-varying mean longitudes. However, even if resonance does not play a role, that is, if no major harmonic angles
are librating, the influence of some non-secular harmonics on the secular evolution can be significant. In effect, all harmonics in a Fourier
expansion of the disturbing function force all other harmonics at some level (Mardling 2013, hereafter Paper II in this series). On the other
hand, the idea behind the averaging technique is that this influence is only short term in nature, and has no effect on the long-term evolution
(see Wisdom 1982 for an excellent historical and insightful discussion of the use of the averaging principle).\(^2\) In fact, as with all forced
non-linear oscillatory systems (see, for example, Neyfah & Mook 1979), the characteristic frequencies are modulated by forcing at some
level, and for secularly varying triple configurations this becomes more pronounced the closer to the stability boundary the system is. A
demonstration of this is given in Giuppone et al. (2011) who study the secular variation of a test particle orbiting the primary in a binary star
system. In particular, they demonstrate that predictions from the usual ‘first-order’ secular theory consistently underestimate the frequency
and overestimate the equilibrium value of the forced eccentricity, with relative errors as large as 80 and 40 per cent, respectively, when the
ratio of binary to test particle semimajor axes is 10. Note that they take a binary mass ratio of 0.25 and a binary eccentricity of 0.36. Using
a technique called Hori’s averaging process (Hori 1966), they go on to calculate the ‘second-order’ corrections to the secular frequency and
amplitude, reducing the errors to a few per cent. While the authors do not identify the nature of the expansion parameter (and claim that
their expressions are too complicated to write down), in effect they are using the neglected harmonics to force the secular system, and in so
doing obtain corrected frequencies. In fact, Hori’s averaging process is simply a version of the better-known Lindstedt–Poincaré method for
correcting the frequencies of a forced non-linear oscillator (see Neyfah 1973 for some simple applications including the Duffing equation).

Here, we present two new formulations of the hierarchical three-body problem which are accessible to anyone interested in the short-
and long-term evolution of small-\( N \) systems, be they stellar or planetary systems or a mixture of both (for example, circumbinary planets).
The two formulations are valid for arbitrary mass ratios, and are distinguished by their choice of expansion parameter and hence their
range of validity in those parameters. For closely packed systems, our generalization of the literal expansion\(^3\) (Le Verrier 1855; Murray &
Dermott 2000, and references therein) with its lack of constraint on the period ratio (except that the orbits should not cross) and its use of
the eccentricities as expansion parameters is appropriate, while more widely spaced systems are best studied with the spherical harmonic
expansion, a generalization of the work of Kaula (1962) (also see Murray & Dermott 2000), which exploits the properties of spherical
harmonics and which is valid for all eccentricities. Inclined systems will be studied in a future paper in this series.

Throughout the paper, we refer to ’moderate mass ratio systems’. Our formal definition of such a system is the one whose stability
characteristics are governed by the interaction of \( N : 1 \) harmonics (Paper II). In practice, a conservative lower bound for both mass
ratios \( m_2/(m_1 + m_2) \) and \( m_3/(m_1 + m_2 + m_3) \) is around 0.05, where \( m_1 \geq m_2 \), however, there are important exceptions to these estimates
(Paper II).

The paper is arranged as follows.

2. Spherical harmonic expansion

Section 2.1. Derivation. Section 2.2. Practical application: dominant terms. Section 2.3. Resonance widths and stability. Section 2.3.1.
Libration frequency. Section 2.4. The secular disturbing function in the spherical harmonic expansion.

\(^2\) The averaging principle relies on adiabatic invariance in the system; see Landau & Lifshitz (1969) for a thorough discussion of this concept.

\(^3\) The origin of the use of the word ‘literal’ in this context is unclear, but one should take it to indicate that the dependence on the ratio of semimajor axes is via
Laplace coefficients.
3. Literal expansion

Section 3.1. Derivation. Section 3.2. Eccentricity dependence. Section 3.2.1. Power-series representations of Hansen coefficients and the choice of expansion order. Section 3.3. Dependence on the mass and semimajor axis ratios. Section 3.3.1. Summary of leading terms in $\alpha$. Section 3.3.2. Coefficients when $m_2/m_1 \to 0$. 3.4. The spherical harmonic order $m$ and principal resonances. Section 3.4.1. ‘Zeeman splitting’ of resonances. Section 3.5. A second-order resonance. Section 3.6. First-order resonances. Section 3.7. Resonance widths using the literal expansion. Section 3.7.1 Libration frequency. 3.7.2. Widths of first-order resonances. Section 3.8. The secular disturbing function in the literal expansion.

4. Comparison of formulations to leading order in eccentricities

5. Equivalence of formulations

6. Comparisons with classic expansions

7. Conclusion and highlights of new results

8. Quick reference

Section 8.1. Harmonic coefficients for the semimajor axis expansion. Section 8.1.1. Secular disturbing function to octopole order. Section 8.2. Harmonic coefficients for the eccentricity expansion. Section 8.2.1. The secular disturbing function to second order in the eccentricities. Section 8.2.2. Widths and libration frequencies of $[n' : n](m)$ resonances.

Appendices

Appendix A. Spherical harmonics. Appendix B. Hansen coefficients. Appendix C. The Laplace coefficients. Appendix D. Lagrange’s planetary equations for the variation of the elements. Appendix E. The mean longitude at epoch. Appendix F. Notation.

2 SPHERICAL HARMONIC EXPANSION

2.1 Derivation

Both expansions presented in this paper make use of three-body Jacobi or hierarchical coordinates and their associated osculating orbital elements to describe the dynamics of the system (e.g. Murray & Dermott 2000). Illustrated in Fig. 1(a), these are used for systems for which the motion of two of the bodies is predominantly Keplerian about their common centre of mass (the ‘inner orbit’), with the third body executing predominantly Keplerian motion about the centre of mass of the inner pair (the ‘outer orbit’). Note that the word ‘hierarchical’ need not imply that the orbits are necessarily well spaced, but rather, that the orbits retain their identities for at least several outer periastron passages, although the osculating orbital elements may vary dramatically from orbit to orbit if the system is unstable. With this broad definition, even systems involving the exchange of the outer body with one of the inner pair can be considered as hierarchical. Note also that one of the many advantages of using these coordinates is that the osculating semimajor axes are constant on average when resonance does not play a role. When resonance does play a role, it is then easy to define the energy exchanged between the orbits.

Using Jacobi coordinates, the equations of motion for the inner and outer orbits are

\[ \mu_3 \ddot{r} + \frac{G m_3 m_2}{r^2} \dot{r} = \frac{\partial R}{\partial r} \tag{1} \]

and

\[ \mu_o \ddot{\mathbf{R}} + \frac{G m_1 m_3}{R^2} \dot{\mathbf{R}} = \frac{\partial \mathbf{R}}{\partial \mathbf{R}} \tag{2} \]

where $r$ is the position of body 2 relative to body 1. $R$ is the position of body 3 relative to the centre of mass of the inner pair, $m_1$ and $m_2$ are the masses of the bodies forming the inner orbit, $m_3$ is the mass of the outer body, $m_{12} = m_1 + m_2$, $\mu_1 = m_1 m_2 / m_{12}$ and $\mu_o = m_{12} m_3 / m_{123}$ are the inner and outer reduced masses with $m_{123} = m_1 + m_2 + m_3$. $\dot{r}$ and $\dot{\mathbf{R}}$ are unit vectors in the $r$ and $\mathbf{R}$ directions, respectively, $r = |r|$, $R = |\mathbf{R}|$ and

\[ R = -\frac{G m_{12} m_3}{R} + \frac{G m_3 m_2}{|R - \beta_1 r|} + \frac{G m_1 m_3}{|R - \beta_2 r|} \tag{3} \]

is the disturbing function or interaction energy, with $\beta_1 = m_1 / m_{12}$ and $\beta_2 = -m_3 / m_{12}$.\(^4\) In general, we will use the subscripts i and o to represent quantities associated with the inner and outer orbits respectively. The notation $\partial / \partial r$ refers to the gradient with respect to the outer orbit.

\(^4\) Note that the use of Jacobi coordinates and the fact that the disturbing function has the units of energy and not energy per unit mass means that it is only necessary to define a single disturbing function. In contrast, in the classic expansions one distinguishes between an ‘internal’ and ‘external’ disturbing function, each composed of a ‘direct’ component as well as individual ‘indirect’ components (Murray & Dermott 2000). Similarly, there is no distinction between ‘internal’ and ‘external’ resonances when the general disturbing function is used.
spherical polar coordinates \((\beta, r, \theta, \varphi)\) associated with the position of body 2 relative to the centre of mass of bodies 1 and 2, and similarly for \(\frac{\partial}{\partial R}\) for the position of body 3 with spherical polar coordinates \((R, \theta_3, \varphi_3)\) relative to the same origin; see Fig. 1(b). (1) and (2) written in this form clearly demonstrate the perturbed Keplerian nature of a hierarchical triple system when \(\mathcal{R}\) and its gradients are small,\(^5\) which is the case when \(r \ll R\). Note that the total energy is given by

\[
E = E_i + E_o - \mathcal{R},
\]

where

\[
E_i = \frac{1}{2} \mu_i \mathbf{r} \cdot \mathbf{r} - \frac{GM_1 m_2}{r}
\]

and

\[
E_o = \frac{1}{2} \mu_o \mathbf{R} \cdot \mathbf{R} - \frac{GM_{12} m_3}{R}
\]

are the instantaneous binding energies of the inner and outer orbits, respectively.

A common way to expand terms of the form of the last two in equation (3) is in terms of Legendre polynomials. In this case, we have for \(s = 1, 2\)

\[
\frac{1}{|\mathbf{R} - \beta|} = \frac{1}{R} \sum_{l=0}^{\infty} \left(\frac{\beta \cdot \mathbf{R}}{R}\right)^l P_l(\cos \psi),
\]

where \(\cos \psi = \mathbf{r} \cdot \mathbf{R}\) and \(P_l\) is a Legendre polynomial. A disadvantage of this is that the angles associated with each individual orbit do not appear explicitly. The use of spherical harmonics overcomes this problem as follows.\(^6\) Using the addition theorem for spherical harmonics (e.g. Jackson 1975), one can write

\[
P_l(\cos \psi) = \sum_{m=-l}^{l} \frac{4\pi}{2l+1} Y_{lm}(\theta_i, \varphi_i) Y_{lm}^*(\theta_o, \varphi_o),
\]

where \(Y_{lm}\) is a spherical harmonic of degree \(l\) and order \(m\) with \(Y_{lm}^*\) its complex conjugate. Using equations (6) and (7), the disturbing function (3) becomes

\[
\mathcal{R} = GM_1 m_3 \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \left(\frac{4\pi}{2l+1}\right) \mathcal{M}_l \left(\frac{r^l}{R^{l+1}}\right) Y_{lm}(\theta_i, \varphi_i) Y_{lm}^*(\theta_o, \varphi_o),
\]

where the mass factor \(\mathcal{M}_l\) is given by

\[
\mathcal{M}_l = \frac{m_1^{l-1} + (-1)^l m_2^{l-1}}{m_3^{l+1}} = \beta_1^{l-1} - \beta_2^{l-1}.
\]

Note that \(\mathcal{M}_2 = 1\) for any masses, while for equal masses \(\mathcal{M}_l = 0\) when \(l\) is odd. In this paper, we focus on coplanar systems for which we take \(\theta_i = \theta_o = \pi/2\), \(\varphi_i = f_i + \sigma_i\) and \(\varphi_o = f_o + \sigma_o\), where \(f_i\) and \(f_o\) are the true anomalies of the inner and outer orbits, respectively, with \(\sigma_i\) and \(\sigma_o\) the corresponding longitudes of periastron. With the definition of the spherical harmonic used in Jackson (1975),

\[
Y_{lm}(\theta, \varphi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos \theta) e^{im\varphi}.
\]

\(^5\) Relative to the binding energy of the individual orbits and the other terms in equations (1) and (2), respectively.

\(^6\) The advantage of using spherical harmonics becomes particularly apparent when inclined systems are considered; this will be done in a future paper in this series.
where $P_i^m$ is an associated Legendre function, the disturbing function becomes

$$\mathcal{R} = G\mu \tau_3 \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \frac{1}{2} c_{lm}^2 M_l e^{i\nu_m} \left( \frac{e^{i\phi}}{R^{l+1}} \right) \left( \frac{\zeta e^{i\phi}}{R^{l+1}} \right),$$

(11)

where

$$c_{lm}^2 = \frac{8\pi}{2l+1} \left[ Y_{lm}(\pi/2, 0)^2 + c_{lm}^2 \right].$$

(12)

A closed-form expression for these constants is given in Appendix A with specific values given in Table B2. Note that the sum over $m$ is in steps of 2 because $Y_{lm}(\pi/2, 0) \propto \cos \theta$ when $l - m$ is odd so that $Y_{lm}(\pi/2, \varphi) = 0$ in this case.

For stable systems including those near the stability boundary, the expressions in the last two pairs of brackets in equation (11) associated with the inner and outer orbits are nearly periodic in their respective orbital periods, and therefore can be expressed in terms of Fourier series of the inner and outer mean anomalies, $M_i = v_i t + M_i(0)$ and $M_o = v_o t + M_o(0)$, respectively, with $M_i(0)$ and $M_o(0)$ their values at $t = 0$ and $v_i$ and $v_o$ the associated orbital frequencies (mean motions). Thus,

$$r' e^{i\phi} = a_i \left( \frac{1}{1 + e_i \cos f_i} \right)^l e^{i\phi_{\text{int}}} = a_i \sum_{n=-\infty}^{\infty} X_n^{l,m}(e_i) e^{i\phi_{\text{int}}},$$

(13)

and

$$e^{-i\phi_{\text{int}}} = a_o^{-l+1} \left( \frac{1}{1 + e_o \cos f_o} \right)^{l+1} e^{-i\phi_{\text{int}}} = a_o^{-(l+1)} \sum_{n'=0}^{\infty} X_{n'}^{l+1,m}(e_o) e^{-i\omega_{n'}},$$

(14)

where $a_i$ and $a_o$ are the inner and outer semimajor axes, $e_i$ and $e_o$ are the corresponding eccentricities, the Fourier coefficients $X_n^{l,m}(e) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f}{\tau_i} e^{i\nu_m} e^{-i\nu_{n}} dM_i = \mathcal{O}(e_i^{m-n})$ and $X_{n'}^{l+1,m}(e_o) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f}{\tau_o} e^{-i\nu_m} e^{-i\nu_{n'}} dM_o = \mathcal{O}(e_o^{m-n'})$ are called Hansen coefficients (Hughes 1981) and we have indicated the order of the leading terms (see Appendix B for graphical representations, closed-form expressions and approximations, MATHEMATICA programs and in particular, Appendix B2 for general expansions which demonstrate the form of the leading terms). Note that since the real part of the integrands of equations (15) and (16) are even and the imaginary parts are odd, the integrals are real so that

$$X_n^{l,m} = X_{n'}^{l,m} = X_n^{l,m},$$

(17)

and

$$X_{n'}^{l+1,m} = X_{n'}^{l+1,m} = X_{n'}^{l+1,m},$$

(18)

justifying the notation used in equation (16). The disturbing function (11) can then be expressed as

$$\mathcal{R} = G\mu \tau_3 \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \frac{1}{2} c_{lm}^2 M_l a_i^l X_n^{l,m}(e_i) X_{n'}^{l+1,m}(e_o) e^{i\phi_{\text{int}}},$$

(19)

and

$$\mathcal{R} = G\mu \tau_3 \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \frac{1}{2} c_{lm}^2 M_l a_i^l X_n^{l,m}(e_i) X_{n'}^{l+1,m}(e_o) e^{i\phi_{\text{int}}},$$

(20)

where

$$\phi_{\text{int}} = n M_i - n' M_o + m(\sigma_i - \sigma_o)$$

(21)

is a harmonic angle,

$$\zeta_m = \begin{cases} 1/2, & m = 0 \\ 1, & \text{otherwise} \end{cases} \quad \text{and} \quad m_{\text{min}} = \begin{cases} 0, & l \text{ even} \\ 1, & l \text{ odd} \end{cases}$$

(22)

In going from equation (19) to (20), we have used the properties (17) and (18) and have grouped together terms with the same value of $|m|$ (thus the factor 1/2 in the definition of $\zeta_m$). Writing the harmonic angle in terms of longitudes only (in anticipation of employing Lagrange’s planetary equations for the rates of change of the elements), (21) becomes

$$\phi_{\text{int}} = n \lambda_i - n' \lambda_o + (m - n)\sigma_i - (m' - n')\sigma_o,$$

(23)

Recall that even functions have the property that $\int_{-\infty}^{\infty} f(x)dx = 2 \int_0^{\infty} f(x)dx$ while odd functions have the property that $\int_{-\infty}^{\infty} f(x)dx = 0$. 

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where \( \lambda_i = M_i + \sigma_i \) and \( \lambda_o = M_o + \sigma_o \) are the inner and outer mean longitudes, respectively. Note that the harmonic angle should be invariant to a rotation of the coordinate axes. Since such a rotation changes all longitudes by the same amount, their coefficients should add up to zero thereby satisfying the d'Alembert relation (Murray & Dermott 2000), which indeed equation (23) does.

Expression (20) for the disturbing function may be compared with that derived by Kaula (1962) for the case where \( m_2 \) is a test particle (see also Murray & Dermott 2000, p 232). Note that the Kaula expression is valid for arbitrary inclinations; this case will be considered in a future paper in this series.

Defining the coefficient of \( \cos \phi_{mn\nu} \) as \( R_{mn\nu} \), it is desirable to change the order of summation of \( l \) and \( m \) so that \( m \) no longer depends on \( l \) (i.e. it becomes a free index independent of any other index). The simplest way to see how this works is to write out the first few terms, grouping them appropriately. Thus,

\[
\sum_{l=2}^{\infty} \sum_{m=m_{\text{min}}2}^{\infty} T_{lm} = [T_{20} + T_{22}] + [T_{31} + T_{33}] + [T_{40} + T_{42} + T_{44}] + [T_{51} + T_{53} + T_{55}] + \ldots
\]

or

\[
= [T_{20} + T_{40} + \ldots] + [T_{31} + T_{51} + \ldots] + [T_{22} + T_{42} + \ldots] + [T_{33} + T_{53} + \ldots] + \ldots
\]

\[
= \sum_{m=0}^{\infty} \sum_{l=l_{\text{min}}2}^{\infty} T_{lm},
\]

(24)

where

\[
l_{\text{min}} = \begin{cases} 
2, & m = 0 \\
3, & m = 1 \\
m, & m \geq 2
\end{cases}
\]

(25)

so that the disturbing function (20) becomes

\[
R = \sum_{m=0}^{\infty} \sum_{m=-\infty}^{m} \sum_{n=-\infty}^{\infty} R_{mn\nu} \cos \phi_{mn\nu}
\]

(26)

with

\[
R_{mn\nu} = \frac{G\mu_{m1}}{a_o} \sum_{l=m_{\text{min}}2}^{\infty} \zeta_{m} c_{lm}^2 M_{l} e_{l} X_{n}^{l,m} (e_{o}) X_{n}^{l+1,m} (e_{o})
\]

(27)

\[
= \frac{G\mu_{m1}}{R_p} \sum_{l=m_{\text{min}}2}^{\infty} \zeta_{m} c_{lm}^2 M_{l} e_{l} X_{n}^{l,m} (e_{o}) Z_{n}^{l+1,m} (e_{o}).
\]

(28)

Here, \( \rho = a_o/R_p \) with \( R_p = a_o(1 - e_o) \) the outer periastron distance, and we will refer to

\[
Z_{n}^{l+1,m} (e_{o}) = (1 - e_{o})^{l+1} X_{n}^{l+1,m} (e_{o})
\]

(29)

as a modified Hansen coefficient. The form (28) is especially useful for systems with high outer eccentricity since \( X_{n}^{l+1,m} (e_{o}) \) is singular at \( e_{o} = 1 \) while \( Z_{n}^{l+1,m} (e_{o}) \) is not. Note that the summation over \( l \) in (27) is in steps of 2. Moreover, note that there are only three independent indices associated with each harmonic for a coplanar system, although usually an additional one is included erroneously (see discussion in Section 6). This makes sense because there are three independent frequencies in the problem, that is, the two orbital frequencies and the rate of change of the relative orientation of the orbits. We will refer to the quantity \( R_{mn\nu} \) as the harmonic coefficient associated with the harmonic angle \( \phi_{mn\nu} \). Moreover, the classical nomenclature for terms associated with \( l = 2 \) and \( l = 3 \) is ‘quadrupole’ and ‘octopole’ (or ‘octupole’) respectively.

2.2 Practical application: dominant terms

The spherical harmonic expansion is significantly simpler to use than the literal expansion when the accuracy required can be achieved with only one or two values of \( l \), and hence is recommended for use in preference to the latter except for the very closest systems (period-ratio-wise). In Section 4, we compare the two expansions to leading order in the eccentricities with the aim of determining the minimum period ratio for which the spherical harmonic expansion is acceptably accurate when only the two lowest values of \( l \) are included. Figs 2 and 3 suggest that this minimum is around 2 (panel a of Fig. 2), although the expansion is still reasonably accurate for a period ratio as low as 1.5 (panel a of Fig. 3).

The questions then arise: Which harmonics in the triple-infinite series (26) should one include for a given application? How does one know whether or not resonant harmonics play a role? If they do not, is it only necessary to include the secular terms in equation (26), that is, terms which do not depend on the mean longitudes (those with \( n = n' = 0 \); see Section 2.4)? What about non-resonant, non-secular harmonics? A few general comments can be offered here, however, in general the answers depend on the questions being asked, on the time-scales of interest and of course on the configuration itself.
Time-scales on which point-mass three-body systems which are coplanar and non-orbit crossing vary can generally be arranged according to the following hierarchy:

\[ P_1 < T_p \leq P_o < P_{lib} < P_{sec} \lesssim \tau_{stab}, \]  

(30)

where \( P_1 \) and \( P_o \) are the inner and outer orbital periods, \( T_p \equiv (1 - e_o)^{3/2}P_o \) is the ‘time of periastron passage’ of the outer body, a time-scale of interest when the outer orbit is significantly eccentric and the orbit–orbit interaction is effective only around periastron, \( P_{lib} = 2\pi/\omega_{lib} \) is the libration period in case that the system is in (or near) resonance, with \( \phi_{lib} \) given by equation (44), \( P_{sec} \) is the period on which the eccentricities vary secularly and \( \tau_{stab} \) is the dynamical stability time-scale in case that the system is unstable to the escape of one of the bodies (Lagrange instability). If one is interested in studying short-period variations on time-scales up to a few times \( P_o \), non-secular harmonics whose coefficients are zeroth and/or first order in \( e_i \) are generally included, independent of the value of the inner eccentricity. However, the selection from amongst such harmonics depends on the value of the outer eccentricity, and these are not necessarily those which are low order in \( e_o \), except when \( e_o \) is small. Inspection of Fig. B2 shows that for significant values of \( e_o \), harmonics spanning a wide range of values of \( n' \) have similar amplitudes so that in principle, many harmonics should be included in such cases. One can avoid this by using overlap integrals, in that case, for each value of \( n \) (and \( l \) and \( m \)); this technique will be discussed in a future paper in this series.

While the eccentricities and semimajor axes of resonant or near resonant systems vary on the time-scale of the orbital periods, these variations tend to accumulate on the libration time-scale and it is the resonant harmonics which govern the behaviour. For stable systems with significant outer eccentricities, it is usually adequate to include only one term in the analysis, that term being the \([N : 1](2)\) harmonic with \( N \simeq \sigma \equiv \nu_l/\nu_o \) as discussed in Section 2.3 and Paper II. Here and later the notation \([n' : n](m)\) refers to the harmonic term associated with the angle \( \phi_{mnn} \) and coefficient \( R_{mnn} \) (see Section 3.4).

Unstable coplanar systems are also governed by resonant harmonics, but it is their interaction with ‘neighbouring’ non-resonant terms which results in the chaotic behaviour of the system. In this case, it is usually sufficient to include only the resonant harmonic \([N : 1](2)\) and its neighbour \([N + 1 : 1](2)\) (Section 2.3), although one often needs to take into account the forced and secular variation of the eccentricities (Paper II).

For stable systems, one is often interested in the long-term secular variation of the elements, in which case it is usually sufficient only to include the secular \([0 : 0](m)\) harmonics, that is, those which do not depend on the mean longitudes. In addition, it is normally only necessary to include the first two of these, that is, \( m = 0 \) and \( m = 1 \). However, for stable systems which are relatively close period-ratio-wise, it may be necessary to include the forcing effect of some non-secular harmonics; this is discussed in Section 1.1.

On the subject of the secular variation of the elements, it is worth mentioning here that while the purely secular coplanar three-body system governed by the single angle \( \phi_{COP} = \sigma_l - \sigma_o \) is integrable and therefore not admitting of chaotic solutions, non-coplanar secular three-body systems as well as coplanar (and non-coplanar) higher order (four-body, etc.) systems are governed by two or more independent angles and hence do admit chaotic solutions (see, for example, Laskar 1988 for a numerical study of the long-term secular evolution of the Solar system).

2.3 Resonance widths and stability

Amongst systems with moderate mass ratios, stable systems tend to have significant period ratios because strong mutual interactions between the bodies tend to destabilize closer systems. The stability of a system can be studied using the heuristic resonance overlap stability criterion which involves calculating the widths of resonances (Chirikov 1979; Wisdom 1980; Mardling 2008; Paper II). For systems with moderate mass ratios, it is the \([n' : 1](2)\) resonances which govern the exchange of energy between the orbits and hence it is these which are responsible for the stability of the system. In this section, we summarize the derivation of the simple expression for the widths of these resonances, the full derivation of which can be found in Paper II where a thorough study of resonance and stability in hierarchical systems with moderate mass ratios is presented.

The general harmonic angle has the form \( \phi_{mnn} = n\lambda_i - n'\lambda_o + (m - n)\sigma_i - (m - n')\sigma_o \), and unless \( n'/n \) is sufficiently close to the period ratio \( \nu_l/\nu_o \), this angle will circulate, that is, it will pass through all values \([0, 2\pi]n \) because there is no commensurability between the rates of change of the individual angles making up \( \phi_{mnn} \). In fact, were there no (non-linear) coupling between the inner and outer orbits, \( \phi_{mnn} \) would circulate no matter how close \( n'/n \) was to \( \nu_l/\nu_o \) (except if \( \phi_{mnn} = 0 \) precisely). But because the orbits are able to exchange energy, the period ratio changes slightly each outer orbit allowing for the possibility of libration of \( \phi_{mnn} \) when the energy is coherently transferred (i.e. when conjunction occurs at almost the same place in the orbit; see, for example, Peale (1976) for a general discussion). In that case, \( \phi_{mnn} \) will oscillate between two values such that \( \int \cos \phi_{mnn} d\phi_{mnn} \neq 0 \), where the integral is taken over one libration cycle. Then, we refer to the harmonic angle in question as a resonance angle and say that the system is in resonance.\(^8\) The period of libration may be tens to hundreds or even thousands of outer orbital periods, depending on the system parameters; an expression for this is given below in terms of ‘distance’ from exact commensurability in dimensionless units of period ratio.

\(^8\)Note that it is possible for the harmonic angle to librate and for the system to be not in resonance; formally, the latter requires the existence of a hyperbolic point in the phase space \( (\phi_{mnn}, \dot{\phi}_{mnn}) \) and this may not be the case when the eccentricities are very small (Delisle et al. 2012), which is never the case when at least one of the mass ratios is significant (except when the period ratio is very large).
In order to study resonant behaviour, we use the pendulum model for resonance (see, for example, Chirikov 1979; Wisdom 1980; Murray & Dermott 2000), which involves deriving a pendulum-like differential equation for $\phi_{\text{res}}$. To do this, we need to take into account the dependence of the orbital frequency on time. Following Brouwer & Clements (1961, p. 285), the mean longitude is defined in terms of the orbital frequency such that

$$\lambda_i = M_i + \sigma_i = \int_{T_0}^{t} v_i(t') \, dt' + M_i(T_0) + \sigma_i = \int_{T_0}^{t} v_i(t') \, dt' + \epsilon_i,$$

where $\epsilon_i$ is the mean longitude at epoch $t = T_0$.\(^\dagger\) With a similar expression for $\lambda_o$, the rate of change of a harmonic angle is then

$$\dot{\phi}_{\text{res}} = n v_i - n' v_o + n \dot{\epsilon}_i - n' \dot{\epsilon}_o + (m - n) \dot{\sigma}_i - (m - n') \dot{\sigma}_o.$$\(^\dagger\)

Except for systems with very small eccentricities, we have in general that $\dot{\sigma} \ll \dot{\nu}$ and $\dot{\sigma} \ll \dot{\nu}$ (see Section 3.4.1 for an example which illustrates this). Moreover, for all systems, $\dot{\epsilon} \ll \dot{\nu}$ and $\dot{\epsilon} \ll \dot{\nu}$. Since some eccentricity is always induced\(^\dagger\) and this is only small when the mass ratios are very small, for systems with moderate mass ratios it is a reasonable approximation to take

$$\dot{\phi}_{\text{res}} \simeq n v_i - n' v_o.$$\(^\dagger\)

Now consider the $[n': n](m) = [N : 1](2)$ harmonic, where $N$ is an integer close to the period ratio $v_i/v_o$. Libration of the angle $\phi_{2N} \equiv \phi_N$ will occur when

$$\dot{\phi}_N = v_i - N v_o \simeq 0.$$\(^\dagger\)

Thus, one can ask: How close to exact commensurability should the system be for this angle to librate? This is equivalent to asking for the width of the resonance. We can get a good answer to this question by showing that $\phi_N$ satisfies approximately a pendulum equation of the form

$$\ddot{\phi}_N = -\omega_N^2 \sin \phi_N,$$

where $\omega_N^2$ depends on the parameters of the system. Note that the $[N : 1](2)$ resonance librates about $\phi_N = 0$ as we show below. Once $\omega_N^2$ is known, the range of values of $\phi_N$ for which $\phi_N$ librates is determined from the equation for the pendulum separatrix, that is,

$$\dot{\phi}_N = \pm 2 \omega_N \cos \left( \frac{\phi_N}{2} \right),$$

so that libration occurs if $\dot{\phi}_N < 2 \omega_N$ when $\phi_N = 0$. In order to determine $\omega_N$, we start by writing

$$\ddot{\phi}_N = v_i - N v_o = v_o \left( \frac{v_i}{v_o} - N \frac{v_i}{v_o} \right) = -\frac{3}{2} v_o \left( \frac{\sigma}{a_i} - N \frac{\dot{\omega}}{a_o} \right),$$

where $\sigma = v_i/v_o$ is the period ratio, and we have used Kepler’s third law to replace $v_i/v_o$ by $-\frac{3}{2} \dot{a}_i/a_i$ and similarly for the outer orbit. The rates of change of the semimajor axes are given by Lagrange’s planetary equation (D4). For the latter, consider a reduced disturbing function which contains only the $[n': 1](2)$ harmonics truncated at $l = 2$; one might call this the quadrupole contribution to the disturbing function, although it does not contain terms with $n = 0$ or $n \neq 1$. Referring to this as $R_q$, we have from equations (26) and (27) that

$$R_q = \frac{3}{4} \frac{G \mu m_1}{a_o} \sigma^2 X^{2,2}(e_i) \sum_{n=-\infty}^{\infty} X_{n'}^{-3,2}(e_o) \cos \phi_{n'},$$

where

$$\phi_{n'} = \lambda_i - n' \lambda_o + \sigma_i - (n' - 2) \sigma_o$$

and we have put C12 = 3/4 and $M_1 = 1$. Now suppose that the harmonic angle with $n' = N$ librates and for now, assume that it is unaffected by all the harmonics contributing to $R_q$ except itself. Retaining only the $[N : 1](2)$ harmonic in equation (38), equation (37) together with Lagrange’s planetary equation (D4) gives

$$\ddot{\phi}_N = -\omega_N^2 \sin \phi_N = \frac{9}{4} v_o \left[ \frac{m_1}{m_{123}} + N^{2/3} \left( \frac{m_{12}}{m_{123}} \right)^{2/3} \left( \frac{m_{123}}{m_{12}} \right) \right] X_{n'}^{2,2}(e_i) X_{n'}^{-3,2}(e_o) \sin \phi_N.$$

\(^9\) Another model used to study resonance is the second fundamental model of resonance of Henrard & Lemaitre (1983). The associated Hamiltonian was designed specifically for the study of resonance capture, although it is possible to study this phenomenon without the Hamiltonian formalism using the pendulum model (Mardling & Udry, in preparation).

\(^\dagger\) See Appendix D for a discussion of the this orbital element.

\(^1\) An expression for the induced eccentricity is given in Paper II.
where we have replaced \( \sigma \) by \( N \) and used Kepler’s third law to replace \( \alpha \) by \((m_{12}/m_{123})^{1/3}N^{-2/3}\). If we further replace the Hansen coefficients by the approximations given in Table B1 and equation (B5), we obtain

\[
\nu = -v_0^2 \left\{ \frac{9H_{22}}{2\pi} \left[ \left( \frac{m_1}{m_{123}} \right) + N^{2/3} \left( \frac{m_{12}}{m_{123}} \right)^{2/3} \left( \frac{m_{123}m_1}{m_{123}^2} \right) \right] \left( e_o \right) \left( 1 - \frac{13}{24} e_o^2 \right) \left( 1 - e_o^2 \right)^{3/4} N^{3/4} e^{-N(e_o)} \right\} \sin \phi_N, \tag{41}
\]

where \( \xi(e_o) = \cos^{-1}(1/e_o) - \sqrt{1 - e_o^2} \) and from Table B1 \( H_{22} = 0.71 \), giving us an expression for \( \omega_N \) and hence the range of values of \( \phi_N \) for which \( \phi_N \) librates. Moreover, we see that \( \phi_N \) does indeed librate around \( \phi_N = 0 \) due to the fact that \( X_1^2(e_o) < 0 \) for all \( 0 < e_o \leq 1 \).

A more practical definition of the resonance width is in terms of the ‘distance’ from exact commensurability in dimensionless units of period ratio. Rewriting equation (34) and incorporating the libration condition, we have that libration occurs when

\[
\phi_N = v_o(\sigma - N) < 2\omega_N \tag{42}
\]

so that the width of the \([N : 1]/2\) resonance is approximately

\[
\Delta\sigma_N = 2\omega_N/v_o = \frac{6H_{22}}{(2\pi)^{1/2}} \left[ \left( \frac{m_1}{m_{123}} \right) + N^{2/3} \left( \frac{m_{12}}{m_{123}} \right)^{2/3} \left( \frac{m_{123}m_1}{m_{123}^2} \right) \right]^{1/2} \left( e_o \right) \left( 1 - \frac{13}{24} e_o^2 \right)^{1/2} \left( 1 - e_o^2 \right)^{3/8} N^{3/4} e^{-N(e_o)/2}, \tag{43}
\]

where \( \lim_{\varepsilon_o \to 0} \Delta\sigma_N \) is infinite for \( N = 1 \), finite for \( N = 2 \) and zero for \( N \geq 3 \). Note the steep dependence on the quantity \( N\xi(e_o) \); since \( \xi(e_o) \) is a monotonically decreasing function of \( e_o \), the widths of high-\( N \) resonances are only significant when \( e_o \) is also high. Note also that \( \Delta\sigma_N = 0 \) when \( e_o = 0 \); this implies that systems with circular inner orbits are always stable which is most certainly not the case. In fact, one needs to know how much eccentricity is induced dynamically to calculate the true resonance width, and moreover one needs to know the maximum inner eccentricity the system acquires during a secular cycle to study its stability. This is thoroughly addressed in Paper II in which stability maps are plotted which clearly demonstrate the success of equation (43) as a predictor of instability using the concept of resonance overlap. Simple algorithms are also provided for determining the stability of any moderate-mass ratio hierarchical triple.

Note that our definition of the resonance width does not involve the usual concept of ‘internal’ and ‘external’ resonance (Murray & Dermott 2000), in the same way that the present formulation does not involve separate internal and external disturbing functions.

### 2.3.1 Libration frequency

While the libration frequency of a pendulum depends on the amplitude, for small amplitudes it is independent of amplitude and is given approximately by \( \omega_N \). Thus, the libration frequency of the angle \( \phi_N \) is

\[ \omega_N = v_o \Delta\sigma_N/2. \tag{44} \]

For example, for an equal mass system with \( e_1 = 0.1, e_o = 0.5 \) and \( \sigma = 20 \), the libration period is 1000 outer orbital periods, while increasing \( e_o \) to 0.6 decreases this to only 66 outer orbital periods (with the same factor increase in the resonance width).

### 2.4 The secular disturbing function in the spherical harmonic expansion

Keeping in mind the caveats discussed in the Introduction, one can use the averaging principle to eliminate fast-varying terms from the disturbing function (26), a process involving integrating over the two mean longitudes individually (as if they were independent) for an orbital period of each. In practice, this is achieved simply by retaining only the \( n = n' = 0 \) terms in equation (26). Using the notation \( \tilde{R} \) for the averaged disturbing function, we obtain

\[
\tilde{R} = \sum_{m=0}^{\infty} \tilde{R}_m \cos \{m(\sigma_1 - \sigma_o)\}, \tag{45}
\]

where

\[
\tilde{R}_m = \frac{G_1\mu_1 m_3}{a_o} \sum_{l=0}^{\infty} \sum_{l'=0}^{\infty} \xi_m \zeta_{m'} \alpha \beta X_{l,m}(e_o) X^{l'_{-1},m'}(e_o). \tag{46}
\]

Closed-form expressions exist for \( X_{l,m}(e_o) \) and \( X^{l'_{-1},m'}(e_o) \) when \( n = n' = 0 \); these are given in Appendix B, with some explicit forms given in Table B2. Expanding to octopole order, the disturbing function (45) becomes

\[
\tilde{R} = \frac{G_1\mu_1 m_3}{a_o} \left[ \frac{1}{4} \left( \frac{a_1}{a_o} \right)^2 \left( 1 + \frac{3}{2} e_o^2 \right)^{1/2} - \frac{15}{16} \left( \frac{a_1}{a_o} \right)^3 \left( \frac{m_1}{m_{12}} \right)^{1/2} \left( 1 - e^2_o \right)^{3/2} \cos(\sigma_1 - \sigma_o) \right]. \tag{47}
\]
For coplanar secular systems, only the rates of change of the eccentricities and longitudes of the periastra are of interest. From Lagrange’s planetary equations (Appendix D), these are

\[
\frac{de_i}{dt} = -v_i \frac{15}{16} \left( \frac{m_3}{m_{12}} \right) \left( \frac{m_1 - m_2}{m_{12}} \right) \left( \frac{a_i}{a_o} \right)^3 \frac{e_o(1 + \frac{3}{2}e_i^2)}{e_i (1 - e_i^2)^{3/2}} \sin(\sigma_i - \sigma_o),
\]

\[
\frac{d\sigma_i}{dt} = v_i \left( \frac{m_3}{m_{12}} \right) \left[ 3 \left( \frac{a_i}{a_o} \right)^2 \left( 1 - e_i^2 \right)^{1/2} - 15 \left( \frac{m_1 - m_2}{m_{12}} \right) \left( \frac{a_i}{a_o} \right)^3 \frac{e_o(1 + \frac{3}{2}e_i^2)}{e_i (1 - e_i^2)^{3/2}} \sin(\sigma_i - \sigma_o) \right],
\]

\[
\frac{de_o}{dt} = v_o \left( \frac{m_1 m_2}{m_{12}^2} \right) \left( \frac{m_1 - m_2}{m_{12}} \right) \left( \frac{a_i}{a_o} \right)^2 \frac{e_i(1 + \frac{3}{2}e_i^2)}{e_i (1 - e_i^2)^2} \sin(\sigma_i - \sigma_o),
\]

\[
\frac{d\sigma_o}{dt} = v_o \left( \frac{m_1 m_2}{m_{12}^2} \right) \left( \frac{m_1 - m_2}{m_{12}} \right) \left( \frac{a_i}{a_o} \right)^3 \left[ \frac{3}{4} \left( \frac{a_i}{a_o} \right) \left( 1 + \frac{3}{2}e_i^2 \right) - \frac{15}{16} \right] \left( \frac{m_1 - m_2}{m_{12}} \right) \left( \frac{a_i}{a_o} \right)^3 \frac{e_i(1 + \frac{3}{2}e_i^2)}{e_i (1 - e_i^2)^{3/2}} \cos(\sigma_i - \sigma_o).
\]

Thus, for example, it is clear that for systems with \( m_1 = m_2 \), there is no secular variation in the eccentricities at this level of approximation and consequently, the inner and outer rates of apsidal motion are constant.

### 3 Literal Expansion

#### 3.1 Derivation

The original literal expansions (for example, Le Verrier 1855) were especially devised to study planetary orbits in the Solar system, and in particular, to take advantage of the small planet-to-star mass ratios, small eccentricities and inclinations, while putting essentially no restrictions on the ratio of semimajor axes except that they should not cross. Our aims here are to generalize the formulation so that no assumptions about the mass ratios are made, and to present the formulation in a clear and concise way which makes it easy to use to any order in the eccentricities and for any appropriate application. Again, for this paper we consider coplanar configurations only, and for clarity of presentation, we will repeat the definition of some quantities already defined in Section 2.

We start by writing down the disturbing function in a form which is useful for the coming analysis:

\[
R = -\frac{Gm_1 m_3}{R} + \frac{Gm_2 m_3}{|R - \beta_1 r|} + \frac{Gm_1 m_3}{|R - \beta_2 r|}
\]

\[
= \frac{Gm_1 m_3}{a_o} \left[ \beta_1^{-1} \beta_2^{-1} \left( \frac{a_o}{R} \right) + \beta_1^{-1} \frac{a_o}{|R - \beta_1 r|} - \beta_2^{-1} \frac{a_o}{|R - \beta_2 r|} \right],
\]

where again, \( \beta_1 = m_1/m_{12} \) and \( \beta_2 = -m_2/m_{12} \). Now consider the second and third terms in equation (52) and write

\[
\frac{a_o}{|R - \beta_1 r|} = \frac{a_o}{\sqrt{R^2 - 2\beta_1 r \cdot R + (\beta_1 r)^2}} = \frac{a_o}{R} \frac{1}{\sqrt{1 - 2x_o \cos \psi + x_o^2}},
\]

with \( s \) being 1 or 2, \( x = \beta_i r/R \) and \( \cos \psi = \hat{r} \cdot \hat{R} \) so that for coplanar systems,

\[
\psi = f_i + \sigma_i - f_o - \sigma_o,
\]

where again, \( f_i \) and \( f_o \), and \( \sigma_i \) and \( \sigma_o \) are the inner and outer true anomalies and longitudes of periastron respectively.

The literal expansion involves (1) a Taylor series expansion about the circular state \( x_i = \alpha_i = \beta_i \epsilon \) with small parameter \( \epsilon \) related to the eccentricities, followed by (2) a Fourier expansion in the angle \( \psi \); this step introduces the Laplace coefficients, then (3) binomial expansions of powers of \( \epsilon \); (4) Fourier series in the mean anomalies, and finally (5) combining the three contributions to the disturbing function in a simple expression. This procedure is set out in Murray & Dermott (2000) for the case of the restricted problem; here, we present a significantly more compact formulation for the general problem. Now let

\[
g(x_i, \psi) = \left[ 1 - 2x_i \cos \psi + x_i^2 \right]^{-1/2}.
\]

Writing

\[
x_i = \frac{\beta_i r}{R} = \beta_i \left( \frac{a_i}{a_o} \right) \frac{(r/a_i)}{(R/a_o)}
\]

\[
= \alpha_i \left( \frac{1 - e_i^2}{1 + e_i \cos f_i} \right) \frac{1 + e_o \cos f_o}{1 - e_o^2}
\]

\[
= \alpha_i (1 + e),
\]

\[
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\]
where
\[ \epsilon = (r/a_i)/(R/a_o) - 1 \] (57)
is first order in the eccentricities, the first two steps described above are
\[ [1 - 2x_r \cos \psi + x_r^2]^{-1/2} = g(\alpha_i + \alpha_i \epsilon, \psi) \]
\[ \frac{\partial^j g}{\partial x_{ij}} \bigg|_{x_i = a_i} \quad \ldots \text{Step 1} \]
\[ = \sum_{j=0}^{\infty} \frac{\partial^j}{\partial \alpha_i} g(\alpha_i, \psi) \]
\[ = \sum_{j=0}^{\infty} \frac{\partial^j}{\partial \alpha_i} \left[ \sum_{m=-\infty}^{\infty} \frac{1}{2} b^{(m)}(\alpha_i) e^{im\psi} \right] \quad \ldots \text{Step 2} \]
\[ = \sum_{j=0}^{\infty} \sum_{m=-\infty}^{\infty} e^{im(\epsilon - \epsilon_0)} \frac{1}{2} b^{(j,m)}(\alpha_i) e^{im(\psi - \psi_0)}, \] (58)

where we have used equation (54) for \( \psi \) in the last step. The Fourier coefficient in \( \text{Step 2} \) is a Laplace coefficient defined by
\[ \frac{1}{2} b^{(j,m)}(\alpha_i) = \frac{1}{2 \pi} \int_0^{2\pi} \frac{e^{-im\psi}}{\sqrt{1 - 2\alpha_i \cos \psi + \alpha_i^2}} \psi \] (59)

(Murray & Dermott 2000), with the factor \( 1/2 \) (as opposed to the subscript 1/2) introduced to obtain the standard definition of \( b^{(m)}_{1/2} \). Note that since the real part of the integrand is even and the imaginary part is odd, the integral is real so that
\[ b^{(m)}_{1/2}(\alpha_i) = \left[ b^{(m)}_{1/2}(\alpha_i) \right]^* = b^{(m)}_{1/2}(\alpha_i). \] (60)
The function in equation (58) involving the \( j \)th derivative of the Laplace coefficient is
\[ \mathcal{B}^{(j,m)}_{1/2}(\alpha_i) = \frac{\alpha_i^j}{j!} \frac{\partial^j}{\partial \alpha_i} b^{(m)}_{1/2}(\alpha_i). \] (61)

General properties of Laplace coefficients and their derivatives are given in Appendix C. Note also that we have used \( m \) for the Fourier summation index because in fact it corresponds to the spherical harmonic order \( m \) (see Section 5 where the equivalence of the two formulations is demonstrated).

Referring to equation (53) and introducing the factor \( a_o/R \), the next step in the procedure is a binomial expansion of \( \epsilon \), so that
\[ (a_o/R) \left[ 1 - 2x_r \cos \psi + x_r^2 \right]^{-1/2} = \sum_{j=0}^{\infty} \sum_{m=-\infty}^{\infty} e^{im(\epsilon - \epsilon_0)} \frac{1}{2} \mathcal{B}^{(j,m)}_{1/2}(\alpha_i) \left[ \frac{r/a_i}{(R/a_o)} - 1 \right]^j \left( \frac{a_o}{R} \right)^n e^{im(\psi - \psi_0)} \] (62)
\[ = \sum_{j=0}^{\infty} \sum_{m=-\infty}^{\infty} e^{im(\epsilon - \epsilon_0)} \frac{1}{2} \mathcal{B}^{(j,m)}_{1/2}(\alpha_i) \sum_{k=0}^{j} \binom{j}{k} (-1)^{j-k} \left( r/a_i \right)^k e^{im\psi_0} \left[ \frac{e^{-imf_o}}{(R/a_o)^{k+1}} \right], \] (63)
\[ \ldots \text{Step 3} \]

where we have gathered together in the square brackets quantities associated with the inner and outer orbits in preparation for the next step. As functions of the eccentricities and the sine and cosine of the true anomalies, these terms can be expanded in Fourier series with period \( 2\pi \) such that
\[ (r/a_i)^k e^{imf_i} = \left[ \frac{1 - e_i^2}{1 + e_i \cos f_i} \right]^k e^{imf_i} = \sum_{n=-\infty}^{\infty} X_{n}^{k,m} (e_i) e^{imM_i}, \] (64)

and
\[ e^{-imf_o}/(R/a_o)^{k+1} = \left[ \frac{1 + e_o \cos f_o}{1 - e_o^2} \right]^{k+1} e^{-imf_o} = \sum_{n=-\infty}^{\infty} X_{n}^{-(k+1),m} (e_o) e^{-imM_o}, \] (65)

\[ \ldots \text{Step 3} \]

See footnote 7.
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where \( X^{k,m}_n(\varepsilon) \) and \( X^{-(k+1),m}_{n'}(\varepsilon) \) are again Hansen coefficients given by equations (15) and (16) (recall that \( X^{-(k+1),m}_{n'} = X^{-(k+1),m}_n \)).

Substituting equations (64) and (65) into (63) and gathering together the angles, we obtain for \( s = 1, 2 \)

\[
\frac{a_o}{[R - \beta_r]} = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \sum_{n'=-\infty}^{\infty} \sum_{j=0}^{\infty} \frac{1}{2} B^{(j,m)}_1(a_r) \left( \sum_{k=0}^{j} (-1)^{j-k} \binom{j}{k} X^{k,m}_n(\varepsilon) X^{-(k+1),m}_{n'}(\varepsilon) \right) e^{i \phi_{mn'}}
\]

(66)

... Step 4

where

\[
F^{(j)}_{mn'}(\varepsilon, \varepsilon_0) = \sum_{k=0}^{j} (-1)^{j-k} \binom{j}{k} X^{k,m}_n(\varepsilon) X^{-(k+1),m}_{n'}(\varepsilon),
\]

(68)

\[
\phi_{mn'} = nM_1 - n'M_0 + m(\sigma_1 - \sigma_0)
\]

(69)

is again a harmonic angle, and \( \eta_m = 1/2 \) when \( m = 0 \) and 1 otherwise. As with the spherical harmonic formulation, in going from equation (66) to (67) we have paired together terms with positive and negative values of \( m \) to make the expression manifestly real.

The final step involves writing down the full literal expansion for the disturbing function. Substituting equation (67) into (52), one obtains

\[
\mathcal{R} = \frac{G\mu m_1}{a_o} \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} \sum_{n'=0}^{\infty} \sum_{j=0}^{\infty} \mathcal{A}_{jm}(\alpha; \beta_2) F^{(j)}_{mn'}(\varepsilon, \varepsilon_0) \cos \phi_{mn'}
\]

... Step 5

(70)

where the harmonic coefficient associated with the angle \( \phi_{mn'} \) is

\[
\mathcal{R}_{mn'} = \frac{G\mu m_1}{a_o} \sum_{j=0}^{\infty} \mathcal{A}_{jm}(\alpha; \beta_2) F^{(j)}_{mn'}(\varepsilon, \varepsilon_0),
\]

(71)

with

\[
\mathcal{A}_{jm}(\alpha; \beta_2) = \eta_m \left[ \beta_1^{-1} B^{(j,m)}_1(\alpha_1) - \beta_2^{-1} B^{(j,m)}_2(\alpha_2) \right]
\]

(72)

for all \( j, m \) except when \( j = m = 0 \) in which case

\[
\mathcal{A}_{00}(\alpha; \beta_2) = \frac{1}{2} \left[ \beta_1^{-1} B^{(0,0)}_1(\alpha_1) - \beta_2^{-1} B^{(0,0)}_2(\alpha_2) \right] + \beta_1^{-1} \beta_2^{-1}.
\]

(73)

Recall here that \( -\beta_2 = m_2/m_1 = 1 = -\beta_1, \alpha = a_1/a_2 \) and \( \alpha_s = \beta_s \alpha, s = 1, 2 \). Note that the order of the expansion is given by the number of terms included in the summation in equation (71), that is, it is given by the maximum value of \( j \). The equivalence of the literal and spherical harmonic formulations is demonstrated in Section 5.

In contrast to the classical literal expansion which is valid for \( m_2/m_1 \ll 1 \) (see Section 3.3.2) and involves Laplace coefficients as functions of the ratio of semimajor axes \( \alpha \), equation (70) is valid for any mass ratios and expresses the disturbing function in terms of Laplace coefficients whose arguments are \( \alpha_s = \beta_s \alpha \), that is, the ratio of semimajor axes scaled by the mass ratios \( m_i/m_{12}, s = 1, 2 \).

By calculating the harmonic coefficients for a second-order resonance as well as those for general first-order resonances and for the secular harmonics, and also by calculating resonance widths, we demonstrate in Sections 3.5–3.8, the ease with which this form of the literal expansion can be used to any required order in eccentricity.

3.2 Eccentricity dependence

The dependence of the disturbing function on the eccentricity is via \( F^{(j)}_{mn'}(\varepsilon, \varepsilon_0) \), defined in equation (68). Although each term in the finite summation over \( k \) is \( O(\varepsilon^{2|m-n|} \varepsilon_0^{n-n'}) \), the leading order of \( F^{(j)}_{mn'}(\varepsilon, \varepsilon_0) \) will be either \( j \) or \( j + 1 \) when \( j > |m - n| \) or \( |m - n'| \equiv \eta \). This results from the fact that from equation (62), \( F^{(j)}_{mn'}(\varepsilon, \varepsilon_0) \propto \varepsilon^j \) with \( \varepsilon = O(\max(\varepsilon, \varepsilon_0)) \), and also that the Hansen coefficients which make
up that \(F_{\text{max}}(e, e_0)\) are either odd or even functions of the eccentricity (so their power-series expansions have only odd or even powers). In particular,

\[
F_{\text{max}}^{(j)}(e, e_0) = \begin{cases} \ell(e_i^{m-n} e_0^{m-n'}), & j \leq \eta \\ \mathcal{O}(e_i^{m-n} e_0^{m-n'}), & j > \eta, \quad j - \eta \text{ even} \\ \mathcal{O}(e_i^{m-n} e_0^{m-n'}), & j > \eta, \quad j - \eta \text{ odd} \end{cases}
\]

(74)

Thus, for example, the leading-order term in an expansion of \(F_{\text{max}}^{(j)}(e, e_0)\) is \(O(e_i^n e_0^n)\) as it is for \(F_{\text{max}}^{(j)}(e, e_0)\), while the leading-order terms of both of \(F_{\text{max}}^{(j)}(e, e_0)\) and \(F_{\text{max}}^{(j)}(e, e_0)\) are \(O(e_i^n e_0^n)\) and \(O(e_i^n e_0^n)\), respectively. As a consequence, if one requires an expansion of the disturbing function which is correct to order \(j_{\text{max}}\) in the eccentricities, then one should include terms up to and including \(j = j_{\text{max}}\) in equation (71), and moreover, expand each of \(F_{\text{max}}^{(j)}(e, e_0)\), \(j \leq j_{\text{max}}\), to order \(j_{\text{max}}\) in the eccentricities. Similarly, one should only include harmonics which are such that \(|m - n| + |m - n'| \leq j_{\text{max}}\). Conversely, if one is particularly interested in a term whose harmonic angle is \(\phi_{\text{max}}\), one should include terms at least up to \(j = |m - n| + |m - n'|\) in order to obtain a non-zero coefficient for such a term.

For practical applications, it is usually most efficient to evaluate the functions \(F_{\text{max}}^{(j)}(e, e_0)\) using MATHEMATICA (Section B4) or similar for the particular range of values of \(m, n, n', j\) of interest, summing over the various Hansen coefficients which contribute. However, it is of considerable interest to examine the functional dependence of the power-series representations of the Hansen coefficients, not only on the eccentricity, but also on the associated indices \(n, j\) and \(m\).

### 3.2.1 Power-series representations of Hansen coefficients and the choice of expansion order

Power-series expansions for \(X_{j, m}^e(e)\) are given in Appendix B2 for arbitrary \(j\) and \(m\), and for \(m = n \pm p, p = 0, 1, 2, 3, 4\), with the choice of values for \(m\) being guided by the use of Hansen coefficients in the study of resonance, since the order of a resonance is \(|m - n| + |m - n'|\) (see Section 3.5). Two features in particular emerge from these general series expansions. First one sees that their leading terms are indeed proportional to \(e_i^{m-n}\), and secondly that the coefficients of the leading terms contain contributions which are \(O(j^p)\) and \(O(n^p)\). This means that \(X_{j, m}^e(e)\) is in fact \(O(\max[\{n(e_i^{m-n}), (j e_i^{m-n})\}], \) which turn has implications for the radius of convergence of the series (the range of values of each of the eccentricities for which the series converges), and the order at which one should truncate the series for given eccentricities. This should be kept in mind when using these expansions, especially for systems with period ratio close to 1. The effect is evident when one compares, for example, the first-order Hansen coefficients in Figs (B4) and (B5). In both of these figures, numerically evaluated integrals (solid curves) are compared with their fourth-order correct series approximations (dashed curves). For example, the series representation of \(X_{1, 0}^{e, 2}(e) = O(e_i)\) in panel (a) of Fig. (B4) approximates the actual function well for \(0 \leq e_i \lesssim 0.8\), while that of \(X_{0, 7}^{e, 0}(e) = O(6e_i)\) in panel (a) of Fig. (B5) is only accurate for \(0 \leq e_i \lesssim 0.1 \zeta 0.8/6 = 0.13\).

While it may be tempting to avoid the series expansions of Hansen coefficients with high values of \(n\), one should remember that any particular harmonic coefficient is only accurate to order \(j_{\text{max}} \geq |m - n| + |m - n'|\), even if individual Hansen coefficients [and hence \(F_{\text{max}}^{(j)}(e, e_0)\)] are evaluated accurately. On the other hand, if computational efficiency is required, it is best to calculate individual series expansions of \(F_{\text{max}}^{(j)}(e, e_0)\) to adequate order in the eccentricities. A MATHEMATICA program is provided in Appendix B4 for this purpose.

While we do not attempt a formal convergence analysis here, the most straightforward way to gain confidence in any particular expansion is to compare its predictions with direct numerical integration of the equations of motion.

### 3.3 Dependence on the mass and semimajor axis ratios

The coefficient of any particular harmonic term in the Fourier expanded disturbing function (70) is given by equation (71), and this itself may be expressed as an infinite series of terms in increasing orders of eccentricity. Each term contributing to a coefficient depends on the mass ratio \(m_2/m\) and the semimajor axis ratio \(\alpha\) though the factor \(A_{j/m}(\alpha; \beta)\). To obtain an idea of this dependence, we can use the expansion (C12) for \(B_{1/2}^{(j/m)}(\alpha)\) in equation (72). Noting that the leading term in this expansion depends on whether \(j \leq m\) or \(j > m\), we have in the first case that

\[
A_{j/m}(\alpha; \beta) = \zeta_m \sum_{p=0}^{\infty} E_{j,m}^{(j,m)} \left[p_1^{m+2p-1} - p_2^{m+2p-1}\right] \alpha^{m+2p}
\]

\[
= \zeta_m E_{0,1}^{(j,m)} \left[\frac{m_1^{m-1} + (-1)^m m_2^{m-1}}{m_1^{m-1}}\right] \alpha^m + \ldots
\]

\[
= \zeta_m E_{0,1}^{(j,m)} \left[1 + (m - 1)(m_2/m_1) + \ldots + (-1)^m(m_2/m_1)^{m-1} + \ldots\right] \alpha^m + \ldots, \quad j \leq m, \quad m \geq 2,
\]

(75)

where \(E_{0,1}^{(j,m)}\) is given by equation (C15). However, the leading term is zero when \(m = 1\) in which case

\[
A_{j/1}(\alpha; \beta) = E_{1,1}^{(j,1)} \left[m_1 - m_2 \right] \alpha^3 + \ldots
\]

\[
= E_{1,1}^{(j,1)} \left[1 - 2(m_2/m_1) + \ldots\right] \alpha^3 + \ldots, \quad j = 0, 1,
\]

(76)
while for \( m = j = 0 \) we have from equation (73) that
\[
\mathcal{A}_{00}(\alpha; \beta_2) = \frac{1}{2} E_1^{(0,0)} \alpha^2 + \frac{1}{2} E_1^{(0,0)} \left( \frac{m_1^3 + m_2^3}{m_{12}^3} \right) \alpha^4 + \ldots
\]
\[
= \frac{1}{4} \alpha^2 + \frac{9}{64} (1 - 3(m_2/m_1) + \cdots) \alpha^4 + \ldots
\]  
(77)

The fact that there are no monopole or dipole terms (i.e. no power of \( \alpha \) less than 2) is consistent with the spherical harmonic expansion (27).

When \( j > m \),
\[
\mathcal{A}_{jm}(\alpha; \beta_2) = \xi_m \sum_{p=p_1}^{\infty} E_{p}^{(j,m)} \left[ \beta_1^{m+2p-1} - \beta_2^{m+2p+1} \right] \alpha^{m+2p}
\]
\[
= \left\{ \begin{array}{ll}
\xi_0 E_{p_1}^{(j,m)} \left[ \frac{m_{1}^{j-1} + (-1)^j m_{2}^{j-1}}{m_{12}^{j}} \right] \alpha^{j} + \ldots, & j > m, \ j - m \ even, \\
\xi_0 E_{p_1}^{(j+1,m)} \left[ \frac{m_{1}^{j+1} + (-1)^j m_{2}^{j+1}}{m_{12}^{j}} \right] \alpha^{j+1} + \ldots, & j > m, \ j - m \ odd.
\end{array} \right.
\]  
(78)

where \( p_1 = \lfloor (j - m + 1)/2 \rfloor \) with \( \lfloor \ \rfloor \) denoting the nearest lowest integer and \( E_{p}^{(j,m)} \) is given by equation (C16). Note that since the leading terms of the harmonic coefficient \( \mathcal{R}_{\text{mn}} \) are of order \( |m - n| + |m - n'| \) in eccentricity, they will be such that \( 0 \leq j \leq m \) (see example in the next section).

### 3.3.1 Summary of leading terms in \( \alpha \)

We can summarize the above as follows. For \( \mathcal{A}_{jm} \) we have
\[
\mathcal{A}_{jm}(\alpha; \beta_2) = \begin{cases} 
O(\alpha^m), & m \geq 2, \ j \leq m \\
O(\alpha^j), & j > m, \ j - m \ even \\
O(\alpha^{j+1}), & j > m, \ j - m \ odd \\
O(\alpha^2), & m = 0, \ j = 0 \\
O(\alpha^3), & m = 1, \ j = 0, 1,
\end{cases}
\]  
(79)

so that the harmonic coefficients are such that
\[
\mathcal{R}_{\text{mn}} = \begin{cases} 
O(\alpha^2), & m = 0 \\
O(\alpha^3), & m = 1 \\
O(\alpha^m), & m \geq 2.
\end{cases}
\]  
(80)

### 3.3.2 Coefficients when \( m_2/m_1 \to 0 \)

The standard literal expansion is derived assuming that one or other of the mass ratios \( m_2/m_1 \) and \( m_3/m_1 \) is zero (Murray & Dermott 2000). Putting \( \beta_1 = 1 \) and \( \beta_2 = 0 \) in equations (72) and (73), we have
\[
\lim_{\beta_2 \to 0} \mathcal{A}_{jm} = \xi_0 E_{1/2}^{(j,m)}(\alpha)
\]  
(81)

for all \( j, m \) except when \( m = 1, j = 0, 1 \) and when \( j = m = 0 \). In these cases, from equations (C2) and (C12), we have
\[
\lim_{\beta_2 \to 0} A_{01} = b_{1/2}^{(1)}(\alpha) - \alpha,
\]  
(82)
\[
\lim_{\beta_2 \to 0} A_{11} = \alpha \frac{d b_{1/2}^{(1)}(\alpha)}{d \alpha} - \alpha
\]  
(83)

and
\[
\lim_{\beta_2 \to 0} A_{00} = \frac{1}{2} b_{1/2}^{(0)}(\alpha) - 1
\]  
(84)

(recall that \( b_{1/2}^{(0)}(0) = 2 \); see equations C2 and C3). Using these approximations makes the disturbing function zeroth-order correct in the mass ratio \( m_2/m_1 \) (times the factor \( m_2/m_3 \) when it has dimensions of energy as in the formulations presented here), so that the rates of change of the elements are first order in \( m_3/m_1 \) for the inner elements, and first order in \( m_2/m_1 \) for the outer elements (see Sections 2.4 and 3.8).
3.4 The spherical harmonic order $m$ and principal resonances

For coplanar systems, there are three labels, $m$, $n$ and $n'$, associated with each harmonic. In turn, each label is associated with an independent frequency of the system: $n$ and $n'$ are associated with the inner and outer orbital frequencies, respectively, while $m$ is associated with the difference in the rates of apsidal advance $\sigma_i - \sigma_o$; see the definition of the harmonic angle (69). In Section 5, we demonstrate the equivalence of the spherical harmonic and literal expansions, where the index $m$ in the literal expansion is shown to correspond to the spherical harmonic order $m$. This plays an important role in many physical systems, and the three-body problem is no exception. For example, we will show in a future paper in this series that one can define the concept of ‘modes of oscillation of a binary’ that are excited in the presence of a triple companion, in analogy with the modes of oscillation of a star which are excited in the presence of a binary companion. The spherical harmonic order $m$ acts as an azimuthal mode number in the formalism, with the analogy between the two physical problems revealing a rich vein of exploration.

First note that $m$ distinguishes resonant states with the same values of $n$ and $n'$. In general, a harmonic coefficient is $O(e_i^{m-n}e_o^{m-n'})$, and since

$$|m-n| + |m-n'| = \begin{cases} n' - n, & n \leq m \leq n', \\ |2m - n - n'|, & \text{otherwise,} \end{cases}$$

(85)

the order in eccentricity is minimized at $n' - n$ when $n \leq m \leq n'$. Using the notation $[n':n](m)$ introduced in Section 2.2 to emphasize the association of the harmonic angle $\phi_{n'n}$ with the $n':n$ resonance of spherical harmonic order $m$, we refer to the $[n':n](m)$ resonances, $n \leq m \leq n'$, as the principal resonances or principal harmonics of the $n':n$ resonance. For example, the two principal 2 : 1 resonances are $[2 : 1](1)$ and $[2 : 1](2)$, with harmonic angles $\phi_{112} = \lambda_i - 2\lambda_o + \sigma_o$ and $\phi_{212} = \lambda_i - 2\lambda_o + \sigma_o$, respectively, and with harmonic coefficients $\mathcal{R}_{112} = O(e_o)$ and $\mathcal{R}_{212} = O(e_i)$. Similarly, there are four principal 5 : 2 resonances, each third-order in eccentricity, namely $[5 : 2](3)$, $[5 : 2](4)$ and $[5 : 2](5)$, with resonance angles $\phi_{225} = 2\lambda_i - 5\lambda_o + 3\sigma_o$, $\phi_{325} = 2\lambda_i - 5\lambda_o + 3\sigma_o$, $\phi_{425} = 2\lambda_i - 5\lambda_o + 2\sigma_o$ and $\phi_{525} = 2\lambda_i - 5\lambda_o + 3\sigma_o$, and harmonic coefficients $\mathcal{R}_{225} = O(e_i^2)$, $\mathcal{R}_{325} = O(e_i^2e_o)$, $\mathcal{R}_{425} = O(e_o^3)$ and $\mathcal{R}_{525} = O(e_o^3)$. In general, there are $n' - n + 1$ principal harmonics associated with the $n':n$ resonance.

The 2 : 1 resonance is referred to as a first-order resonance because the minimum order in eccentricity of either of the principal harmonic coefficients is first order. Using the nomenclature introduced here, we can be more definite and say that in general, a resonance is $p$th order if the principal resonances are $p$th order in eccentricity.

It is sometimes desirable to express the ‘largeness’ or otherwise of the values of $n$ and $n'$, especially for first-order resonances. The author is aware that the term resonance degree is occasionally used for this purpose; however, in the context of spherical harmonics the words ‘degree’ and ‘order’ are associated with the indices $l$ and $m$, respectively. In hindsight, this is unfortunate because $m$ could have been used for this purpose had the word ‘order’ not already referred to the value of $n' - n$. Moreover, one correctly refers to a polynomial’s degree rather than order when describing its highest power (although the latter is often used), and had degree been adopted for describing the order in eccentricity of a resonance, all would be consistent. But history takes precedence for words in common use, and the 7 : 6 resonance continues to be a seventh-degree first-order resonance.

Finally, recall from Section 3.3.1 that $R_{n'str} = O(a^n)$, $m \geq 2$, while $R_{Out} = O(a^2)$ and $R_{Int} = O(a^3)$. This implies that in general, unless $e_o \ll e_i$, it is the principal resonance with $m = n$ which tends to make the largest contribution to the disturbing function, except when $n = 1$ in which case the $m = 2$ harmonic tends to make the largest contribution.

3.4.1 ‘Zeeman splitting’ of resonances

Just as a magnetic field introduces fine structure to atomic energy levels (Zeeman splitting), apsidal advance of the inner and outer orbits introduces fine structure in the positions of the centres of resonances relative to exact commensurability. In both cases, it is the spherical harmonic order $m$ which labels the associated frequencies and moreover physically, it is the introduction of one or more distinguished directions (magnetic field or third body) which breaks the otherwise symmetric state of the system. The slow rotation of the system about directions (magnetic field or third body) which breaks the otherwise symmetric state of the system. The slow rotation of the system about
where we have used the expansions for the Hansen coefficients in Section B.2. If all of \( \sigma_2 - \sigma_o \), \( \phi_{212} \) and \( \phi_{112} \) librate, the rates of apsidal advance are then

\[
\dot{\sigma}_1 = \frac{3}{4} v_o \left( \frac{m_o}{m_s} \right) \alpha^3 \left[ 1 - \frac{5}{4} \alpha \left( \frac{e_o}{e_i} \right) \cos(\sigma_1 - \sigma_o) - \left( \frac{3}{e_i} \right) \cos(\lambda_i - 2\lambda_o + \sigma_o) \right]
\]

(88)

and

\[
\dot{\sigma}_o = \frac{3}{4} v_o \left( \frac{m_i}{m_s} \right) \alpha^2 \left[ 1 - \frac{5}{4} \alpha \left( \frac{e_i}{e_o} \right) \cos(\sigma_2 - \sigma_o) + \left( \frac{3}{2e_o} \right) \cos(\lambda_i - 2\lambda_o + \sigma_o) \right].
\]

(89)

Whether or not a particular angle contributes on average to \( \dot{\sigma}_1 \) and \( \dot{\sigma}_o \) depends on whether it librates or not, that is, whether or not the average value of its cosine is non-zero. If it does librate, the sign of its contribution will depend on whether it does so around zero or \( \pi \) (or some other angle in some cases). Using equation (102), one can show that for small eccentricities, when \( \phi_{212} \) and \( \phi_{112} \) librate, they do so around zero and \( \pi \), respectively. One may then ask whether it is possible for both angles to librate at the same time. It is possible to show that if the harmonic angle \( \phi_{212} \) librates, then all other angles of the form \( \phi_{212} + n' \pi, n' \neq N \), must circulate. This is not necessarily true for a set of principal resonances because for any two angles from the set, labelled, say, by \( m_1 \) and \( m_2 \) (not to be confused with the masses),

\[
\phi_{m_1n1} = \phi_{m_2n'1} + (m_2 - m_1)(\sigma_1 - \sigma_o).
\]

(90)

Thus, if one resonance angle librates and in addition, \( \sigma_1 - \sigma_o \) librates, then all other associated principal resonance angles will librate.\(^\text{13}\)

Now, the angle \( \sigma_1 - \sigma_o \) will librate if the eccentricities are small enough [see, for example, Mardling (2007) for a study of the libration and circulation of this angle in the case of secular evolution]. If this occurs, then since \( \phi_{212} - \phi_{112} = \sigma_1 - \sigma_o \), then \( \sigma_1 - \sigma_o \) must librate around \( \pi \) (because \( \phi_{212} \) librates around zero and \( \phi_{112} \) librates around \( \pi \)) so that the average value of \( \cos(\sigma_1 - \sigma_o) \) is \(-1\), while the average values of \( \cos(\phi_{212}) \) and \( \cos(\phi_{112}) \) are both \(-1\), respectively (at exact resonance). From equations (88) and (89), the average rates of apsidal advance in this case are therefore

\[
\dot{\sigma}_1 = \frac{3}{4} v_o \left( \frac{m_o}{m_s} \right) \alpha^3 \left[ 1 - \frac{3}{2} \alpha \frac{e_i}{e_o} \right] \simeq -\frac{9}{4} v_o (m_o/m_s) \sigma^{-1} e_i^{-1}
\]

(91)

and

\[
\dot{\sigma}_o = \frac{3}{4} v_o \left( \frac{m_i}{m_s} \right) \alpha^2 \left[ 1 - \frac{5}{4} \alpha \frac{e_i}{e_o} \right] \simeq -\frac{9}{8} \frac{v_o (m_i/m_s)^2 \sigma^{-4/3} e_o^{-1}},
\]

(92)

where \( \sigma \) is the period ratio and the approximations hold for small to moderate eccentricities. In such cases, the rates of change of the two \( 2:1 \) principal resonances are, from equations (69),

\[
\dot{\phi}_{212} = v_o \left[ (\sigma - 2) - \frac{9}{4} (m_o/m_s) \sigma^{-1} e_i^{-1} \right]
\]

(93)

and

\[
\dot{\phi}_{112} = v_o \left[ (\sigma - 2) - \frac{9}{8} \frac{m_i}{m_s} \sigma^{-4/3} e_o^{-1} \right].
\]

(94)

For two Jupiter-mass planets orbiting a solar-mass star, the positions of exact resonance (that is, the value of \( \sigma \) for which \( \phi_{212} = 0 \) or \( \phi_{112} = 0 \)) are therefore approximately a distance

\[
\delta \sigma_{212} = 0.0011 e_i^{-1} \quad \text{and} \quad \delta \sigma_{112} = 0.0004 e_o^{-1}
\]

(95)

away from exact commensurability. These can be significant for small eccentricities, and this should be remembered when deciding whether or not an observed system is likely to be in resonance (subject to the caveat discussed in footnote 8).

It is interesting to note here that for non-coplanar systems, there are five independent labels including \( n, n' \) and \( m \), and an additional two spherical harmonic \( m's \) which we denote by \( m_i \) and \( m_o \) (non-coplanar systems will be studied in a future paper in this series). The harmonic angle becomes

\[
\phi_{m_i,m_o,n,n'} = n\lambda_i - n'M_o + m_i \omega_i - m_o \omega_o + m(\Omega_i - \Omega_o)
\]

\[= n\lambda_i - n'M_o + (m_i - n')\sigma_1 - (m_o - n')\sigma_o + (m - m_i)\Omega_i - (m - m_o)\Omega_o,
\]

(96)

with, respectively, \( \omega_i \) and \( \omega_o \), and \( \Omega_i \) and \( \Omega_o \), the arguments of periastron and the longitudes of the ascending nodes of the inner and outer orbits. Note that for coplanar systems, \( m_i = m_o = m \). The additional labels reflect the extra frequencies introduced when the problem becomes three dimensional. The three frequencies associated with \( m_i, m_o \), and \( m_o \) are, respectively, the difference in the rates of precession of the orbital planes about the total angular momentum vector, and the rates of change of the inner and outer arguments of periastron. We note also that the additional fine structure introduced when the orbits are not coplanar has its own analogy with Zeeman splitting. Before the latter phenomenon was understood in the context of quantum mechanics, physicists referred to energy level splittings which were accurately predicted by the

\(^{13}\) In fact, all \( n' : n \) resonance angles will librate, not just the principal angles.
classical theory of Lorentz as ‘normal’ and those which were not as ‘anomalous’. Once the quantum mechanical concept of electron spin was introduced, it became clear that the additional source of angular momentum was responsible for the ‘anomalous’ fine structure, with states not involving electron spin remaining ‘normal’. Suffice to say here that orbital precession introduces ‘anomalous’ fine structure, with ‘normal’ fine structure associated with principal resonances for which \( m_i = m_o = m \).

As well as illustrating the phenomenon of resonance splitting, the fine structure calculation (86) to (95) serves to further demonstrate the ease with which the spherical harmonic expansion can be applied. We now consider some specific applications which make more explicit the power of the literal expansion.

### 3.5 A second-order resonance

Written in the form (70) with (71), it is easy to include terms to any order in the eccentricities and mass ratios. For example, say one wanted to study the 5 : 3 resonance for which \( n = 3 \) and \( n' = 5 \). The principal resonance angles are (from 69) \( \phi_{355} = 3 \lambda_i - 5 \lambda_o + 2 \sigma_o \), \( \phi_{535} = 3 \lambda_i - 5 \lambda_o + \sigma_i + \sigma_o \) and \( \phi_{535} = 3 \lambda_i - 5 \lambda_o + 2 \sigma_i \), with \( \mathcal{R}_{355} = \mathcal{O}(e_i^2), \mathcal{R}_{353} = \mathcal{O}(e_i e_o) \) and \( \mathcal{R}_{535} = \mathcal{O}(e_o^2) \). If we choose to include terms, say, up to fourth order in the eccentricities, we would include in the summation in (71) terms up to \( j = 4 \). Note that for this resonance, all terms will be even-ordered in eccentricity so that, for example, \( j = 3 \) terms will actually be fourth order. For instance, \( F_{355}^{(3)} = \frac{9}{2} e_i^2 e_o^2 \). Thus, to third order in eccentricity, the terms in the disturbing function associated with the 5 : 3 resonance are

\[
\mathcal{R} = \frac{G \mu_{353}}{a_o} \left\{ \ldots + \left( \frac{67}{8} A_{03} + \frac{9}{4} A_{13} + \frac{1}{4} A_{23} \right) e_o^2 \cos(3 \lambda_i - 5 \lambda_o + 2 \sigma_o) \right. \\
- \left( 18 A_{04} + \frac{9}{2} A_{14} + \frac{1}{2} A_{24} \right) e_i e_o \cos(3 \lambda_i - 5 \lambda_o + \sigma_i + \sigma_o) \\
+ \left( \frac{75}{8} A_{05} + \frac{9}{4} A_{15} + \frac{1}{4} A_{25} \right) e_i^2 \cos(3 \lambda_i - 5 \lambda_o + 2 \sigma_i) + \ldots \right\}.
\]

Note for this example, however, that systems which exist stably in the 5 : 3 resonance tend to have very small values of the mass ratios \( m_2/m_1 \) and \( m_3/m_1 \) (generally of the order of \( 10^{-4} \)), in which case the approximations in Section 3.3.2 are reasonable. One then obtains

\[
\mathcal{R} = \frac{G \mu_2 m_3}{a_o} \left\{ \ldots + \left( \frac{67}{8} b_{1/2}^{(3)}(\alpha) + \frac{9}{4} \alpha \frac{db_{1/2}^{(3)}}{d\alpha} + \alpha^2 \frac{d^2 b_{1/2}^{(3)}}{d\alpha^2} \right) e_o^2 \cos(3 \lambda_i - 5 \lambda_o + 2 \sigma_0) \\
- \left( 18 b_{1/2}^{(4)}(\alpha) + \frac{9}{2} \alpha \frac{db_{1/2}^{(4)}}{d\alpha} + \alpha^2 \frac{d^2 b_{1/2}^{(4)}}{d\alpha^2} \right) e_i e_o \cos(3 \lambda_i - 5 \lambda_o + \sigma_i + \sigma_o) \\
+ \left( \frac{75}{8} b_{1/2}^{(5)}(\alpha) + \frac{9}{4} \alpha \frac{db_{1/2}^{(5)}}{d\alpha} + \alpha^2 \frac{d^2 b_{1/2}^{(5)}}{d\alpha^2} \right) e_i^2 \cos(3 \lambda_i - 5 \lambda_o + 2 \sigma_i) + \ldots \right\}.
\]

On the other hand, one may wish, for example, to estimate the width of any of the 5 : 3 resonances for some arbitrary configuration (which may or may not be stable) in which case the expression (97), valid for arbitrary mass ratios, should be used.

As discussed in the previous section, in general there are \( n' - n + 1 \) distinct principal resonances associated with the \( n' : n \) resonance, and each of these has \( n' - n + 1 \) terms contributing to the lowest order in eccentricity.

### 3.6 First-order resonances

Now consider those harmonic terms in the expansion which are first order in eccentricity, that is, those terms for which \( j = 0 \) or 1 and for which

\[
|m - n| + |m' - n'| = n' - n = 1
\]

so that \( m = n \) or \( m = n + 1 \). The relevant terms in the disturbing function are then

\[
\mathcal{R} = \frac{G \mu_{353}}{a_o} \left\{ \ldots + \frac{1}{2} \left[ (2n + 1) A_{0n} + A_{1n} \right] e_o \cos(n \lambda_i - (n + 1) \lambda_o + \sigma_o) \\
- \left[ (n + 1) A_{0n+1} + \frac{1}{2} A_{1n+1} \right] e_i \cos(n \lambda_i - (n + 1) \lambda_o + \sigma_i) + \ldots \right\},
\]

and when \( m_2 \ll m_1 \), this reduces to

\[
\mathcal{R} = \frac{G m_2 m_3}{a_o} \left\{ \ldots + \frac{1}{2} \left[ (2n + 1) b_{1/2}^{(n)}(\alpha) + \alpha \frac{db_{1/2}^{(n)}}{d\alpha} - 4 \alpha \delta_{n1} \right] e_o \cos(n \lambda_i - (n + 1) \lambda_o + \sigma_o) \\
- \left[ (n + 1) b_{1/2}^{(n+1)}(\alpha) + \frac{\alpha}{2} \frac{db_{1/2}^{(n+1)}}{d\alpha} \right] e_i \cos(n \lambda_i - (n + 1) \lambda_o + \sigma_i) + \ldots \right\},
\]

which is consistent with Papaloizou (2011). Here, the term involving \( \delta_{n1} \) comes from equations (82) and (83).
3.7 Resonance widths using the literal expansion

The literal expansion is especially suited to the study of systems with period ratios close to unity. Stable systems in this category tend to have small mass ratios and at most modest eccentricities, with the induced (forced) contributions to the latter being of order $m_3/m_1$ and $m_2/m_1$ for the inner and outer eccentricities, respectively (Paper II). In this section, we derive an expression for the width of a general \( [n':n](m) \) resonance; however, we will assume that the eccentricities are not so small that the apsidal advance of one or both orbits contributes significantly to the resonance width; this case will be considered elsewhere. Having said this, one should keep in mind the convergence issues discussed in the previous section.

Following the analysis in Section 2.3 for the resonance width in the case of the spherical harmonic expansion, including the assumptions that the dynamics is dominated by a single harmonic (not always true when the period ratio is close to 1) and that $\dot{\nu}_n$ and $\dot{\sigma}_n$ are negligible compared to $v_o$, the librating angle $\phi_{n'n'}$ is governed by

\[
\dot{\phi}_{n'n'} = n\dot{\nu}_1 - n'\dot{\nu}_0 = -\frac{3}{2} v_0 \left( \frac{n\dot{\alpha}_1 - n'\dot{\alpha}_0}{a_0} \right),
\]

\[
= v_0 \left\{ 3 n^2 \left[ \alpha \sigma^2 \left( \frac{m_3}{m_{12}} \right) + \left( \frac{n'}{n} \right)^2 \left( \frac{m_1m_2}{m_{12}} \right) \right] \sum_{j=0}^{j_{\text{max}}} A_{jm} F_{n'n'}^{(j)} \right\} \sin \phi_{n'n'},
\]

\[
= -2n \sigma_{n'n'} \sin \phi_{n'n'},
\]

\[
\text{(102)}
\]

where $j_{\text{max}} \geq |m - n| + |m - n'|$ and is chosen to equal the highest order in eccentricity required. Libration is around $\phi_{n'n'} = 0$ if $\sigma_{n'n'}^2 > 0$ and about $\phi_{n'n'} = \pi$ if $\sigma_{n'n'} < 0$. The angle $\phi_{n'n'}$ will librate when

\[
\dot{\phi}_{n'n'} = n\nu_0 (\sigma - n'/n) < 2\sigma_{n'n'},
\]

\[
\text{(103)}
\]

so that the width of the $[n':n](m)$ resonance, that is, the maximum excursion of $\sigma$ away from $n'/n$ is given by

\[
\Delta \sigma_{n'n'} = 2\sqrt{3} \left( \frac{n'}{n} \right) \left[ \alpha \left( \frac{m_3}{m_{12}} \right) + \left( \frac{n'}{n} \right)^2 \left( \frac{m_1m_2}{m_{12}} \right) \right] \left[ \sum_{j=0}^{j_{\text{max}}} A_{jm} F_{n'n'}^{(j)} \right]^{1/2},
\]

\[
\text{(104)}
\]

where we have put $\sigma = n'/n$ and $\alpha$ should be replaced by its value at exact commensurability.

3.7.1 Libration frequency

As discussed in Section 2.3.1, the libration frequency of a resonant harmonic is given by

\[
\omega_{n'n'} = v_o \Delta \sigma_{n'n'}/2.
\]

\[
\text{(105)}
\]

3.7.2 Widths of first-order resonances

The widths of the two principal first-order resonances are then

\[
\Delta \sigma_{n+n+1} = 2\sqrt{3} \left( \frac{n+1}{n} \right) \left[ \frac{1}{2} \left[ \alpha \left( \frac{m_3}{m_{12}} \right) + \left( \frac{m_1m_2}{m_{12}} \right) \right] \right] \left[ (2n+1) A_{0n} + A_{1n} \right] \epsilon_0^{1/2},
\]

\[
\text{(106)}
\]

and

\[
\Delta \sigma_{n+1n+1} = 2\sqrt{3} \left( \frac{n+1}{n} \right) \left[ \alpha \left( \frac{m_3}{m_{12}} \right) + \left( \frac{m_1m_2}{m_{12}} \right) \right] \left[ (n+1) A_{0n+1} + \frac{1}{2} A_{1n+1} \right] \epsilon_0^{1/2}.
\]

\[
\text{(107)}
\]

3.8 The secular disturbing function in the literal expansion

As for the spherical harmonic expansion (see Section 2.4), the secular disturbing function is obtained by retaining the $n = n' = 0$ terms only in equation (70). Again using the notation $\bar{R}$ for the averaged disturbing function, we obtain

\[
\bar{R} = \sum_{m=0}^{\infty} \bar{R}_m \cos [m(\sigma_n - \sigma_o)],
\]

\[
\text{(108)}
\]

where now

\[
\bar{R}_m \equiv \bar{R}_{m00} = \frac{G\mu m_3}{a_0} \sum_{j=0}^{\infty} A_{jm}(\alpha; \beta_2) F_{m0j}^{(j)}(\epsilon_0, \epsilon_0).
\]

\[
\text{(109)}
\]
As before, \(A_{00}\) and \(F_{00}^{(i)}\) are given by equations (72) and (68), respectively, but note that unlike for general \(n, n'\), closed-form expressions exist for the Hansen coefficients \(X_{0}^{2,0} (e)\) and \(X_{0}^{(k+1),m} (e)\); these are given in Appendix B. However, since the literal formulation for the disturbing function involves an expansion in the eccentricities, it is still only correct to order \(f_{\text{max}}\) in the eccentricities, where \(f_{\text{max}}\) is the highest value of \(j\) included in the expansion (see discussion in Section 3.2). Note that since \(|m-n| + |m-n'| = 2m\) is even, all terms in the secular expansion are even-ordered in eccentricities (including products of odd powers).

Recall from Section 2.4 on the spherical harmonic secular disturbing function that to octopole order (i.e. including \(l = 2\) with \(m = 0, 2\) and \(l = 3\) with \(m = 1, 3\), only terms with \(m = 0\) and \(m = 1\) are non-zero because \(X_{0}^{2,2} (e) = 0\) and \(X_{0}^{3,3} (e) = 0\). Thus, the only secular harmonic angle appearing in the spherical harmonic development up to octopole level is \(\phi_{100} = \sigma_{1} - \sigma_{0}\). From the point of view of the literal expansion, the coefficients of the ‘quadrupole’ and ‘octopole’ harmonic angles \(\phi_{200} = 2 (\sigma_{0} - \sigma_{0})\) and \(\phi_{300} = 3 (\sigma_{0} - \sigma_{0})\) are \(O(e_{1}^{2} e_{0}^{2})\) and \(O(e_{1}^{2} e_{0}^{3})\), respectively, and are non-zero because values of \(j\) in addition to \(j = 2\) and \(j = 3\) contribute to them. The literal planar secular disturbing function to second order in the eccentricities and correct for any mass ratios is then

\[
\mathcal{R} = \frac{G \mu m_{3}}{a_{o}} \left[ A_{00} + \frac{1}{2} (e_{1}^{2} e_{0}^{2}) (A_{11} + A_{21}) + \frac{1}{2} e_{0} e_{0} (A_{01} - A_{11} - A_{21}) \cos (\sigma_{1} - \sigma_{0}) \right]
\]

so that to first order in the eccentricities, the rates of change of the eccentricities and longitudes of the periastra are, from Lagrange’s planetary equations (D1) and (D2),

\[
\frac{de}{dt} = \frac{1}{2} v_{1} \left( \frac{m_{1}}{m_{12}} \right) e_{1} \alpha (A_{01} - A_{11} - A_{21}) \sin (\sigma_{1} - \sigma_{0}) \]

\[
\frac{d\sigma_{1}}{dt} = \frac{1}{2} v_{1} \left( \frac{m_{1}}{m_{12}} \right) \left[ 2 \alpha (A_{10} + A_{20}) + \left( \frac{e_{0}}{e_{1}} \right) \alpha (A_{01} - A_{11} - A_{21}) \cos (\sigma_{1} - \sigma_{0}) \right]
\]

\[
\frac{de}{dt} = -\frac{1}{2} v_{0} \left( \frac{m_{1}}{m_{12}} \right) e_{1} (A_{01} - A_{11} - A_{21}) \sin (\sigma_{1} - \sigma_{0})
\]

\[
\frac{d\sigma_{0}}{dt} = \frac{1}{2} v_{0} \left( \frac{m_{1}}{m_{12}} \right) \left[ 2 (A_{10} + A_{20}) + \left( \frac{e_{0}}{e_{1}} \right) (A_{01} - A_{11} - A_{21}) \cos (\sigma_{1} - \sigma_{0}) \right]
\]

In the limiting case that \(m_{2}/m_{1} \ll 1\), the disturbing function to second order in the eccentricities becomes

\[
\mathcal{R} = \frac{G m_{2} m_{3}}{a_{o}} \left[ \frac{1}{2} b_{10}^{(i)} (\alpha) - \frac{1}{8} \left( e_{1}^{2} + e_{0}^{2} \right) (2 \alpha D + a^{2} D^{2}) b_{10}^{(i)} (\alpha) + \frac{1}{2} e_{0} e_{0} (2 - 2 \alpha D - a^{2} D^{2}) b_{10}^{(i)} (\alpha) \cos (\sigma_{1} - \sigma_{0}) \right]
\]

where \(D = d/dt\). This is consistent with equations 6.164–6.168 in Murray & Dermott (2000) except for the additional term \(-G m_{2} m_{3}/a_{o}\) which corresponds to their indirect term. Since in the secular case this term is constant, it contributes nothing to the secular dynamics.

### 4 COMPARISON OF FORMULATIONS TO LEADING ORDER IN ECCENTRICITIES

In this section, we compare the two formulations in terms of the mass parameter \(|\beta_{3}| = m_{2}/m_{12}\) and the ratio of semimajor axes \(\alpha\). The parameter \(\beta_{3}\) is chosen because it is taken to be zero in the classic literal expansion and is introduced here without restriction. For each harmonic coefficient considered, the eccentricity dependence is factored out to leading order and the resulting functional dependence on \(|\beta_{3}|\) is compared. The dependence on \(\alpha\) of the resulting expression is therefore exact in the literal case (to leading order in eccentricity), while in the spherical harmonic case it will depend on the number of terms included in the summation over \(l\).

We start by defining the function \(S_{\text{max}} (\alpha, \beta_{2})\) such that the harmonic coefficients are given to leading order in the eccentricities by

\[
R_{\text{max}} \simeq \frac{G \mu m_{3}}{a_{o}} S_{\text{max}} (\alpha, \beta_{2}) e_{0}^{(m-n)} e_{0}^{(m-n')}
\]

where for the spherical harmonic expansion,

\[
S_{\text{max}} (\alpha, \beta_{2}) = \sum_{l_{\text{min}}}^{l_{\text{max}}} \zeta_{n,l} e_{l_{m}}^{(m-n)} \mathcal{M}_{l_{m}} \alpha^{l_{m}} x_{n,l_{m}}^{m-n} = \sum_{l_{\text{min}}}^{l_{\text{max}}} \zeta_{n,l} e_{l_{m}}^{(m-n)} [(1 - \beta_{2})^{l_{m} - 1} - \beta_{2}^{l_{m} - 1}] \alpha^{l_{m}} x_{n,l_{m}}^{m-n}
\]

with \(l_{\text{min}}\) given by (25) and \(l_{\text{max}} = l_{\text{min}}\) or \(l_{\text{min}} + 2\) (cases I and II), while for the literal expansion,

\[
S_{\text{max}} (\alpha, \beta_{2}) = \sum_{j=0}^{[m-n]+[m-n']} A_{j} (\alpha, \beta_{2}) f_{j}^{(i)}
\]
\[ x_n^{l,m} = \lim_{e_i \to 0} X_n^{l,m}(e_i) e_{i}^{-|m-n|}, \quad x_n^{-(l+1),m} = \lim_{e_o \to 0} X_n^{-(l+1),m}(e_o) e_{o}^{-|m-n|} \]  

(119)

and

\[ f^{(j)}_{mn} = \lim_{e_i \to 0} \lim_{e_o \to 0} F^{(j)}_{mn}(e_i, e_o) e_{i}^{-|m-n|} e_{o}^{-|m-n|} \]  

(120)

are the coefficients of \( e_{i}^{-|m-n|} e_{o}^{-|m-n|} \) and \( e_{i}^{-|m-n|} e_{o}^{-|m-n|} \) in expansions of \( X_n^{l,m}(e_i) \), \( X_n^{-(l+1),m}(e_o) \) and \( F^{(j)}_{mn}(e_i, e_o) \), respectively. Fig. 2 compares \( |S_{\text{max}}(\alpha, \beta_2)| \) for these three cases for various \( 2 : 1, 3 : 1, 5 : 1 \) and \( 10 : 1 \) principal resonances, while Fig. 3 does the same for the two \( 3 : 2 \) and \( 4 : 3 \) principal resonances (see Section 3.4 for their definition). The biggest errors incurred are associated with \( |\beta_2| = 0 \), and these decrease to zero for \( |\beta_2| = 0.5 \). The main conclusion one draws from these comparisons is that even for systems with period ratios as low as 1.5, including only the first two values of \( l \) in the spherical harmonic expansion produces quite accurate estimates of the harmonic coefficients. Since the spherical harmonic expansion is the simplest to use of the two expansions, it is recommended for use in preference to the literal expansion except when the period ratio is less than, say, 2.

Finally, while one might hope that the error incurred truncating the spherical harmonic expansion at \( a_{\text{max}} \) is \( O(a_{\text{max}}^{-2}) \), this is by no means guaranteed. In fact, for the harmonics plotted in Figs 2 and 3, the error appears to be more like \( O(a_{\text{max}}^{-1}) \) for \( \beta_2 = 0 \), decreasing with increasing \( \beta_2 \). For example, for the \( 5 : 1 \) harmonic shown in Fig. 2(c), \( \alpha = 5^{2/3} = 0.34 \) so that including only the \( l = 2 \) term in the spherical harmonic expansion (blue dashed line associated with \( m = 2 \)) incurs an error at \( \beta_2 = 0 \) of \( |\Delta S_{155}/S_{155}| = O(\alpha^2) \approx 0.012 \). Thus \( |\Delta S_{155}| \approx 0.6 \), consistent with the figure.

5 Equivalent of Formulations

We now demonstrate the equivalence of the spherical harmonic and literal expansions, which amounts to demonstrating the equivalence of the individual coefficients \( R_{\text{max}} \) given by equations (27) and (71). Doing this will involve changing summation orders as well as a change of variable. Throughout, one should keep in mind that the indices \( m, n, n' \) are fixed.

Our aim is to show that

\[
R_{\text{max}} = \frac{G \mu M}{a_0} \sum_{l=0}^{\infty} c_{lm}^2 M_l \alpha' X_n^{l,m}(e_i) X_{n'}^{-(l+1),m}(e_o) = \frac{G \mu M}{a_0} \sum_{j=0}^{\infty} A_{jm}(\alpha; \beta_2) F^{(j)}_{mn}(e_i, e_o),
\]

(121)

where again,

\[
l_{\text{min}} = \begin{cases} 
2, & m = 0 \\
3, & m = 1 \\
2 + m, & m \geq 2.
\end{cases}
\]

(122)

Both \( A_{jm} \) and \( F^{(j)}_{mn} \) can be expressed as series given, respectively, by equations (72) and (73) with (C12) and (C13), and (68). Noting the form of equation (122), consider first \( m \geq 2 \). Distinguishing the left- and right-hand sides of equation (121) by \( R^{(L)}_{\text{max}} \) and \( R^{(R)}_{\text{max}} \), we have

\[
R^{(L)}_{\text{max}} = \frac{G \mu M}{a_0} \sum_{l=2}^{\infty} \sum_{m=0}^{\infty} \left( \frac{-1}{2} \right)^{l+m} (l+m)! \left( \frac{2(l+m)}{|l-m|} \right) \left( \beta_1^{l+1} - \beta_2^{l+1} \right) \alpha' X_n^{l,m}(e_i) X_{n'}^{-(l+1),m}(e_o),
\]

(123)

where we have used, respectively, equations (A5) and (9) to replace \( c_{lm}^2 \) and \( M_l \), and

\[
R^{(R)}_{\text{max}} = \frac{G \mu M}{a_0} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} F_p^{(j)} \left[ \beta_1^{m+2p-1} - \beta_2^{m+2p-1} \right] \alpha^{m+2p} F^{(j)}_{mn}(e_i, e_o),
\]

(124)

with \( \rho_{mn} = \max \left( 0, \left\lfloor \frac{1}{2} (j - m + 1) \right\rfloor \right) \) and \( \lfloor \rfloor \) denoting the nearest lowest integer. Referring to Fig. 4(a), in the next step we change the order of summation of \( j \) and \( p \) in equation (124), then make a change of variable for \( p \) putting \( j = m + 2p \) so that

\[
R^{(R)}_{\text{max}} = \frac{G \mu M}{a_0} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \sum_{j=0}^{\infty} 2[j+2m+2p]! \left[ \beta_1^{m+2p-1} - \beta_2^{m+2p-1} \right] \alpha^{m+2p} F^{(j)}_{mn}(e_i, e_o),
\]

(125)
The two 2:1 principal resonances

The three 3:1 principal resonances

The first two 5:1 PRs

The ‘quadrupole’ 10:1 PR

The two 3:2 principal resonances

The two 4:3 principal resonances

Figure 2. Comparison of the dependence of harmonic coefficients on the inner mass ratio for the two expansions for various harmonics. The function $S_{mn}(\alpha, \beta_2)$ is given by equation (117) for the spherical harmonic expansion and by equation (118) for the literal expansion. One (blue dashed curves) and two (red dashed curves) values of $l$ are included in the spherical harmonic expansion, while the literal expansion (black curves) is exact to the leading order in eccentricity. For an $N:1$ harmonic, $\alpha = N^{-2/3}$ so that (a) $\alpha = 0.63$, (b) $\alpha = 0.48$, (c) $\alpha = 0.40$ and (d) $\alpha = 0.22$.

Figure 3. Similar to Fig. 3 for two first-order harmonics. Two values of $l$ are inadequate for such close systems when $|\beta_2| = m_2/m_{12}$ is small. Here, (a) $\alpha = 0.76$ and (b) $\alpha = 0.83$.

$$
\begin{align*}
\frac{G\mu m_3}{a_o} & \sum^{\infty}_{l=m+2} \sum^{l}_{j=0} \frac{(l+m)! l! (l-m)!}{2^{l-j} j![(l-m)/2]!(l+m)/2)!} \left[ \beta_1^{l-1} - \beta_2^{l-1} \right] \alpha^l F^{(j)}_{\Delta mn}(e_i, e_o) \\
& = \frac{G\mu m_3}{a_o} \sum^{\infty}_{l=m+2} \sum^{l}_{j=0} \frac{c_{lm}^2 \left[ \beta_1^{l-1} - \beta_2^{l-1} \right] \alpha^l \sum^{l}_{j=0} \frac{l!}{j!(l-j)!} F^{(j)}_{\Delta mn}(e_i, e_o)}{l^{l} e^{l}_{\Delta})}
\end{align*}
$$

(126)

$$
\begin{align*}
& = \frac{G\mu m_3}{a_o} \sum^{\infty}_{l=m+2} \sum^{l}_{j=0} c_{lm}^2 \left[ \beta_1^{l-1} - \beta_2^{l-1} \right] \alpha^l F^{(j)}_{\Delta mn}(e_i, e_o) \\
& \equiv \frac{G\mu m_3}{a_o} \sum^{\infty}_{l=m+2} c_{lm}^2 \left[ \beta_1^{l-1} - \beta_2^{l-1} \right] \alpha^l F^{(j)}_{\Delta mn}
\end{align*}
$$

(127)
Note that \((-1)^{l+m} = 1\) because \(l + m\) is always even in the coplanar case. Comparing equation (126) with (121), it remains to show that the expression in the large square brackets in equation (126) which we have defined as \(\chi^l_{j0}\) in equation (127) is equal to \(X^l_{\ell m}(e_i)X^{-(l+1)m}_{\ell 0}(e_o)\). Using the definition of \(F^{(i)}_{mon}(e_i, e_o)\) from equation (68), we have

\[
\chi^l_{j0} = \sum_{k=0}^{l} \sum_{j=0}^{j} (-1)^{j-k} \frac{l!}{j!(l-j)!(j-k)!} X^k_{j}(e_i)X^{-(k+1)m}_{j 0}(e_o).
\]

We therefore need to reduce the double summation over \(j\) and \(k\) to a single term. With that aim in mind, the next step involves gathering together the coefficients of each individual product \(X^k_{j}(e_i)X^{-(k+1)m}_{j 0}(e_o)\) by changing the order of summation of \(j\) and \(k\). Referring to Fig. 4(b), we then have

\[
\chi^l_{j0} = \sum_{k=0}^{l} \sum_{j=0}^{j} \delta_{jk} \frac{l!}{j!(l-j)!(j-k)!} X^k_{j}(e_i)X^{-(k+1)m}_{j 0}(e_o)
\]

\[
= \sum_{k=0}^{l} \frac{\delta_{jk}}{k!} X^k_{j}(e_i)X^{-(k+1)m}_{j 0}(e_o)
\]

\[
= X^l_{j}(e_i)X^{-(l+1)m}_{j 0}(e_o),
\]

decreasingly verifying the equivalence of the formulations for \(m \geq 2\). When \(m = 1\), \(\beta_1^{l-1} - \beta_2^{l-1} = 0\) so that the summation over \(l\) starts at \(l = 3\), consistent with equation (123). When \(m = 0\), there is no contribution from \(l = 0\) because of the additional term in the definition of \(A_{00}\) (see equations 73 and 77). Thus, the formulations are equivalent for all values of \(m\).

6 COMPARISONS WITH CLASSIC EXPANSIONS

Kaula expansion

Using our notation and setting the inclinations equal to zero, the Kaula (1962) expression for the disturbing function with \(m_2 = 0\) in units of energy per unit mass, is

\[
\mathcal{R}_K = \sum_{l=2}^{\infty} \mathcal{R}_l,
\]

where \(l\) is the spherical harmonic degree, \(\mathcal{R}_l\) is of the form (see his equation 10)

\[
\mathcal{R}_l = \frac{Cm}{a_0} \left( \sum_{m=0}^{l} \sum_{m=0}^{\infty} \sum_{m=-\infty}^{\infty} C_{lm} X_{j-m+\pi}(e_i) X_{j-m+\pi}(e_o) \cos \phi_{lm\pi} \right).
\]

\(C_{lm}\) (which is different to our \(c_{lm}\)) is a constant involving lengthy expressions from Kaula (1961), and the harmonic angle is

\[
\phi_{lm\pi} = (l - m + \pi)M_i - (l - m + \pi)M_o + (l - m)(\theta_i - \theta_o).
\]

Note that in this form, \(l\) appears in the harmonic angle which is not the case in our spherical harmonic expansion (see equation 21). In particular, the harmonic angle has four indices while ours has three, so that all four cannot be independent.
Of special importance in our formulation is the simple form of the general harmonic angle and the clear relationship between the indices and the natural frequencies in the problem (see Section 3.4 for a discussion of this point). Moreover, by swapping the order of summation over the spherical harmonic indices $l$ and $m$ (equation 24), they become effectively decoupled with $m$ taking on the role of independent harmonic label (together with $s$ and $n$), while $l$ retains the role of expansion index.

The new spherical harmonic expansion benefits especially from its general dependence on the mass ratios, as well as its simple and evincing dependence on the eccentricities via power-series and asymptotic approximations. For the inner eccentricity dependence, power-series expansions of the Hansen coefficients associated with the dominant terms are accurate for most eccentricities less than unity (see Table B1 and Fig. B1 a), while for the outer eccentricity dependence, asymptotic expressions demonstrate explicitly the exponential falloff of the harmonic coefficients with period ratio and eccentricity (equations B5 and B7).

**Literal expansion**

Basing their analysis on the work of Le Verrier (1855), a lengthy derivation of the literal expansion of the direct and indirect parts of the disturbing function for the restricted problem is given in Murray & Dermott (2000) to second order in the eccentricities and inclinations, and for either $m_2 = 0$ or $m_1 = 0$. No general expression for the harmonic coefficients is given, but rather several tables of the harmonic angles and their coefficients up to fourth order in those elements are provided.

One of the features of our formulation which simplifies the analysis is the fact that the expression for the general harmonic coefficient (71) involves derivatives of $b_{l1}^{(s)}$ only, rather than $b_{l1}^{(m)}$ for many values of the half-integer index $s$ (although the derivatives themselves involve values of $s$ other than $s = 1/2$ via equations C8 and C9). This together with simple expressions for the eccentricity functions makes it straightforward to write down any harmonic coefficient (see Appendix B which gives series approximations for Hansen coefficients as well as a short Mathematica program which generates a power series for $F_{mn}^{(j)}(e_1, e_2)$ to the order of the expansion). We have demonstrated in Sections 3.5 and 3.6 the ease with which this can be done.

It has previously been assumed that it is not possible to use Jacobi coordinates for a literal expansion of the disturbing function without performing an expansion in the mass ratios as well (see, for example, the discussion in Laskar & Robutel 1995, p. 195). Here, we avoid such an expansion by taking advantage of the symmetry in the mass ratios $m_1/m_{12}$ and $m_2/m_{12}$ in our form of the disturbing function (52). The novelty is in the use of two mass-weighted ratios of the semimajor axes as the arguments of Laplace coefficients. Note that the usual form for the disturbing function (in units of energy per unit mass) is

$$ R_i = Gm_i \left( \frac{1}{R - r} - \frac{r \cdot R}{R^3} \right), $$

where $m_i$ is $m_2 \ll m_1$ or $m_3 \ll m_1$, that is, there are two distinct disturbing functions, each with a ‘direct’ and ‘indirect’ term (the first and second terms, respectively).

### 7 Conclusion and Highlights of New Results

The aim of this paper has been to provide new general expansions of the disturbing function which are clear and accessible to anyone contributing to the rapidly expanding field of exoplanets, as well as to many other fields of astrophysics. The expansions are applicable to systems with arbitrary mass ratios, eccentricities and period ratios, making them suitable for the study of any of the diverse stellar and planetary configurations now being discovered in great numbers by surveys such as HARPS and Kepler.

The many applications include determining the rates of change of the orbital elements of both secular and resonant systems; calculating the widths of resonances for the purpose of studying libration cycles and their effect on TTVs, or for the purpose of studying the stability characteristics of arbitrary configurations (not just those with small eccentricities and masses); deriving analytical constraints for use in orbit fitting procedures; and calculating the dynamical characteristics of circumbinary planetary systems.

Several new results and concepts have been introduced here including the following.

(i) Arbitrary dependence of the disturbing function on the mass ratios for both the spherical harmonic and literal expansions.

(ii) Simple general expressions for all harmonics to arbitrary order in the ratio of semimajor axes (spherical harmonic expansion) and the eccentricities (literal expansion).

(iii) Accurate and simple approximations for Hansen coefficients for $0 \leq e_1 \leq 1$ and $0 \leq e_2 \leq 1$ for the dependence of both expansions on eccentricities, including asymptotic expressions (Paper II) for the outer eccentricity which reveal the exponential dependence of the disturbing function on $n'$ and $e_2$.

(iv) The fact that for a given level of accuracy, the order in eccentricity at which one truncates the series depends on the configuration being studied. For example, given the eccentricities, one requires fewer terms when studying the $2 : 1$ resonance than one does for the $7 : 6$ resonance, even though they are both first-order resonances.

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14 A general expression for the coefficients of a hybrid Kaula-literal expansion (Ellis & Murray 2000) is given, with many conditions on the maximum and minimum values of the summation indices. The derivation of this expression is not given, and a promise of such a derivation does not appear to have been met. In addition, there is no discussion of the errors associated with the expansion. Note that there appears to be a typographical error in equation 52 of Ellis & Murray (2000) which has carried over to equation 6.113 of Murray & Dermott (2000): the index $j$ on the Laplace coefficient should be $k$. 
(v) Widths of resonances of arbitrary order.
(vi) The equivalence of the spherical harmonic and literal expansions revealing the role of the spherical harmonic order \( m \) in the literal expansion.
(vii) The concept of ‘principal resonances’ and the physical importance of the spherical harmonic order \( m \) including ‘Zeeman splitting’ of resonances.
(viii) Comparison of the two expansions showing that the simpler spherical harmonic expansion can be used for problems with period ratios as low as 2.

This work has revealed that the link between the three-body problem and spherical harmonics is more than just a convenient way to label Fourier terms. Via analogy with other physical problems involving spherical harmonics, the analysis presented here has the potential to expose deep symmetries in this rich problem.

8 QUICK REFERENCE

This section provides a quick reference to the main results for readers mainly interested in their application. Equation numbers corresponding to the main text are provided.

The paper derives two expansions, one in the ratio of semimajor axes (the spherical harmonic expansion), and the other in the eccentricities (the literal expansion). Both are valid for any masses. The choice of which to use depends on the configuration being studied as well as the application, and there is no clear boundary between them. In general, one uses the expansion in eccentricity for systems with period ratios less than, say, 2 or 3 and for which the eccentricities are small, while the expansion in semimajor axis ratio is best for wider eccentric systems.

For both expansions the disturbing function is expressed as a triple Fourier series over the indices \( m, n \) and \( n' \), where the frequencies associated with \( n \) and \( n' \) are those of the inner and outer orbits, and the frequency associated with \( m \) is the difference in the apsidal motion rates. We write this as

\[
\mathcal{R} = \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} \sum_{n'=-\infty}^{\infty} \mathcal{R}_{mnn'}(e, e_0, a_1, a_2; m_1, m_2, m_3) \cos \phi_{mnn'},
\]

where the harmonic angle \( \phi_{mnn'} \) can be written in terms of the mean anomalies of the inner and outer orbits, \( M_i \) and \( M_o \), or the corresponding mean longitudes \( \lambda_i \) and \( \lambda_o \), as well as the longitudes of periastron \( \sigma_i \) and \( \sigma_o \), so that

\[
\phi_{mnn'} = n M_i - n' M_o + m(\sigma_i - \sigma_o)
\]

\[
= n\lambda_i - n'\lambda_o + (m - n)\sigma_i - (m - n')\sigma_o.
\]

The harmonic coefficients \( \mathcal{R}_{mnn'} \) depend on the other parameters in the problem, namely the inner and outer eccentricities \( e \) and \( e_0 \), the semimajor axes \( a_1 \) and \( a_2 \), and the masses \( m_1, m_2 \) and \( m_3 \), and are given below for the semimajor axis and eccentricity expansions, respectively.

8.1 \( \mathcal{R}_{mnn'} \) for the semimajor axis expansion

In this case, the harmonic coefficients are given by

\[
\mathcal{R}_{mnn'} = \frac{G\mu m_3}{a_0} \sum_{l=0}^{l_{\text{max}}} \sum_{m=0}^{m_{\text{max}}} \zeta_m c_l^2 \mathcal{M}_l \alpha' X_{n'}^l m(e_1) X_{n}^{l+1} m(e_0)
\]

\[
= \frac{G\mu m_3}{a_0} \sum_{l=0}^{l_{\text{max}}} \sum_{m=0}^{m_{\text{max}}} \zeta_m c_l^2 \mathcal{M}_l \rho' Y_{n'}^l m(e_1) Z_{n}^{l+1} m(e_0),
\]

where \( \alpha = a_1/a_2 \), \( \rho = a_1/R_o \) with \( R_o \) the outer periastron distance, \( l_{\text{max}} = m \) for \( m \geq 2 \) and 2 or 3 if \( m = 0 \) or 1, respectively, \( l_{\text{max}} \) is chosen according to the accuracy required, the notation \( \sum_{l=0}^{l_{\text{max}}} \) means the summation is in steps of 2, \( \zeta_m \) takes on the values 1/2 or 1 according to whether \( m \) is zero or not zero, respectively, \( c_{20}^2 = 1/2, c_{22}^2 = 3/4, c_{31}^2 = 3/8, c_{33}^2 = 5/8 \) with a general expression given by (12), and

\[
\mathcal{M}_l = m_{l-1}^2 + (-1)^l m_{l-1}^2 / (m_1 + m_2)^2.
\]

The eccentricity functions \( X_{n'}^l m(e_1) \) and \( X_{n}^{l+1} m(e_0) \) are Hansen coefficients and \( Z_{n}^{l+1} m(e_0) = (1 - e_0)^{l+1} X_{n}^{l+1} m(e_0) \) is a modified Hansen coefficient. These are defined and discussed in Appendix B. The leading terms in their Taylor expansions are \( O(e_i^{m-n}) \) and \( O(e_o^{m-n'}) \), respectively. They may be evaluated numerically with as much precision as one requires, by closed-form expression when \( n = n' = 0 \), or approximately by series expansion (Appendices B2 and B4) or closed-form asymptotic approximation (equation B5).
8.1 Secular disturbing function to octopole order

The secular disturbing function \( \ddot{\mathcal{R}} \) is given by (27) with \( n = n' = 0 \). To octopole order this is

\[
\ddot{\mathcal{R}} = \frac{G\mu m_3}{a_o} \left[ \frac{1}{4} \left( \frac{a_1}{a_o} \right)^2 \left( 1 + \frac{3}{2} \frac{e^2}{e_o^2} \right) - \frac{15}{16} \left( \frac{m_1 - m_2}{m_{12}} \right) e_i e_o \left( 1 + \frac{3}{4} \frac{e^2}{e_o^2} \right) \cos(\sigma_i - \sigma_o) \right].
\]

(47)

The secular rates of change of the eccentricities and apsidal longitudes are given by equations (48)–(51). Note that care should be taken when using secular expansions; see the discussion in Section 1.1.

8.1.2 Dominant non-secular terms

The dominant non-secular harmonics for systems well represented by the semimajor axis expansion tend to be those with \( m = 2 \) and \( n = 1 \). Including only \( l = 2 \) in equation (27) and using Table B1 for \( X_i^{2,l}(e) \) and the asymptotic approximation (B5) for \( X_i^{2,0}(e) \) (with accuracies indicated in Table B1 and Fig. B3, respectively), their coefficients are approximately

\[
\ddot{\mathcal{R}}_{21n} = -\frac{G\mu m_3}{a_o} \left( \mathcal{H}_{22} \frac{e}{\sqrt{2\pi}} \right) \alpha^2 \left[ 3e_i - \frac{13}{8} e_o^3 \right] \left( 1 - e_o^2 \right)^{3/4} n^{3/2} e^{-n'\xi(e_o)} e_o^{-2},
\]

(135)

where \( \xi(e_o) = \cosh^{-1}(1/e_o) - \sqrt{1 - e_o^2} \) and \( \mathcal{H}_{22} = 0.71 \). Note the steep dependence on \( n' \xi(e_o) \). Note also that at this order of the expansion (quadrupole) there is no dependence on the inner mass ratio (apart from the factor \( \mu_m \)) because \( M_o = 1 \).

Most systems down to a period ratio of around 2 are well approximated by the spherical harmonic expansion with only one or two values of \( i \) included (see Section 4).

8.1.3 Widths and libration frequencies of \( [N : 1](2) \) resonances

The spherical harmonic expansion is especially useful for studying the stability properties of eccentric systems with moderate mass ratios (Paper II). Such systems tend to have significant period ratios and hence can be quite accurately truncated at the quadrupole level. For a system with period ratio close to the integer value \( N \), the harmonic angle of interest is the one corresponding to \( n = 1, n' = N \) and \( m = 2 \), that is,

\[
\phi_{21N} = \lambda_i - N\lambda_o + \sigma_i + (N - 2)\sigma_o.
\]

(136)

Using the general notation \( [n':n](m) \) for an \( n' : n \) resonance of spherical harmonic order \( m \), the width of the \( [N : 1](2) \) resonance is approximately

\[
\Delta\sigma_N = \frac{6\mathcal{H}_{22}^{1/2}}{(2\pi)^{3/2}} \left[ \left( \frac{m_1}{m_{12}} \right) + N^{2/3} \left( \frac{m_{12}}{m_{123}} \right)^{2/3} \left( \frac{m_{12}}{m_{123}} \right) \right]^{1/2} \left( \frac{1}{e_o^2} \right)^{1/2} \left( 1 - e_o^2 \right)^{3/8} N^{3/4} e^{-N\xi(e_o)/2}.
\]

(43)

Note that \( \lim_{e_o \to 0} \Delta\sigma_N \) is infinite for \( N = 1 \), finite for \( N = 2 \) and zero for \( N \geq 3 \). The corresponding libration frequency \( \omega_N \) is

\[
\omega_N = v_o \Delta\sigma_N / 2,
\]

(44)

where \( v_o \) is the outer orbital frequency.

8.2 \( \mathcal{R}_{mne} \) for the eccentricity expansion

In this case, the harmonic coefficients are given by

\[
\mathcal{R}_{mne} = \frac{G\mu m_3}{a_o} \sum_{j=0}^{j_{\text{max}}} \mathcal{A}_{jm}(\alpha; \beta_2) F_{mne}^{(j)}(e_i, e_o).
\]

(71)

where \( j_{\text{max}} \) is the order in eccentricity of the expansion, the eccentricity functions \( F_{mne}^{(j)}(e_i, e_o) \) are finite sums of products of Hansen coefficients given by

\[
F_{mne}^{(j)}(e_i, e_o) = \sum_{k=0}^{l} (-1)^{l-k} \binom{l}{k} X_k^{j,m}(e_i) X^{(k+1),m}(e_o),
\]

(68)

with the latter defined and discussed in Appendix B, and the \( \mathcal{A}_{jm} \) depend on the semimajor axis ratio \( \alpha \) and the mass ratios \( \beta_1 = m_1/m_{12}, \beta_2 = -m_2/m_{12} \) through

\[
\mathcal{A}_{jm}(\alpha; \beta_2) = \xi_m \left[ \beta_1^{-1} B_{j/2}^{(l,m)}(\alpha_1) - \beta_2^{-1} B_{j/2}^{(l,m)}(\alpha_2) \right]
\]

(72)

\(^{15}\text{Note the additional scaling factors indicated in Fig. B3 for } n' \leq 10.\)
for all \( j, m \) except when \( j = m = 0 \) in which case
\[
\mathcal{A}_{00}(\alpha; \beta_2) = \frac{1}{2} \left[ \beta_1^{-1} b_{1/2}^{(0)}(\alpha_1) - \beta_2^{-1} b_{1/2}^{(0)}(\alpha_2) \right] + \beta_1^{-1} \beta_2^{-1}.
\]

Here, \( \alpha_1, \alpha_2, s = 1, 2, b_{1/2}^{(0)}(\alpha_s) \) is a Laplace coefficient defined in equation (59) and \( b_{1/2}^{(j,m)}(\alpha_s) = (\alpha_s / j !) (d^j / d\alpha_s^j) b_{1/2}^{(m)}(\alpha_s) \). The evaluation of these is discussed in Appendix C, while a Mathematica program for series expansions of \( F_{\text{max}}^{(j)}(e_1, e_2) \) is given in Appendix B4.

Examples are presented in Sections 3.5 and 3.6 in which terms contributing to the \( 5 : 3 \) resonance and general first-order resonances are calculated.

### 8.2.1 The secular disturbing function to second order in the eccentricities

To second order in the eccentricities the secular disturbing function is
\[
\mathcal{R} = \frac{G \mu_{m3}}{a_0} \left[ \mathcal{A}_{00} + \frac{1}{2} (e_1^2 + e_2^2)(\mathcal{A}_{10} + \mathcal{A}_{20}) + \frac{1}{2} e_1 e_2 (\mathcal{A}_{01} - \mathcal{A}_{11} - \mathcal{A}_{21}) \cos(\varpi_1 - \varpi_2) \right] + \mathcal{O}(e_1^3 e_2^2)
\]

which reduces to equation (115) in the limit that \( m_2/m_{12} \to 0 \). Equations governing the rates of change of the elements are given in equations (111)–(114). Note that care should be taken when using secular expansions; see the discussion in Section 1.1.

### 8.2.2 Widths and libration frequencies of \([n': n](m)\) resonances

The widths of the principal harmonics of the \( n' : n \) resonance (those which are lowest order in the eccentricities) are given by
\[
\Delta \sigma_{\text{max}} = 2\sqrt{3} \left( \frac{n}{n'} \right) \left[ \alpha \left( \frac{m_1}{m_{12}} \right) + \left( \frac{m_1 m_2}{m_{12}} \right) \right] \sum_{j=0}^{\text{max}} A_{jm} F_{\text{max}}^{(j)} \frac{1}{2},
\]

where \( n \leq m \leq n' \). The corresponding libration frequencies are
\[
\omega_{\text{max}} = \varpi_0 \Delta \sigma_{\text{max}} / 2.
\]

To first order in the eccentricities, the widths of the two principal first-order resonances are
\[
\Delta \sigma_{n,n+1} = 2\sqrt{3} \left( \frac{n + 1}{n} \right) \left[ \alpha \left( \frac{m_1}{m_{12}} \right) + \left( \frac{m_1 m_2}{m_{12}} \right) \right] \left[ (n + 1)A_{0n} + \frac{1}{2} A_{1n} \right] e_0 \frac{1}{2},
\]

and
\[
\Delta \sigma_{n+1,n,n+1} = 2\sqrt{3} \left( \frac{n + 1}{n} \right) \left[ \frac{1}{2} \alpha \left( \frac{m_1}{m_{12}} \right) + \left( \frac{m_1 m_2}{m_{12}} \right) \right] \left[ (2n + 1)A_{0n+1} + A_{1n+1} \right] e_0 \frac{1}{2}.
\]

Note that expressions (104)–(107) do not include contributions from \( \tilde{\omega}_1 \) and/or \( \tilde{\omega}_n \) which can be significant for first-order resonances when the eccentricities are very small.

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### DEDICATION

For SDU.

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APPENDIX A: SPHERICAL HARMONICS

Using the definition of $Y_{lm}(\theta, \phi)$ (Jackson 1975)

$$Y_{lm}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}} \frac{(l-m)!}{(l+m)!} P_l^m(\cos \theta) e^{im\phi} \quad (A1)$$

with

$$P_l^m(x) = \frac{(-1)^m}{2^l!} (1-x^2)^{l/2} \frac{d^{l+m}}{dx^{l+m}} (x^2 - 1)^l = \frac{(-1)^m}{2^l!} (1-x^2)^{l/2} \frac{d^{l+m}}{dx^{l+m}} \sum_{j=0}^l \binom{l}{j} (-1)^{l-j} x^{2j}$$

$$= \frac{(-1)^m}{2^l!} (1-x^2)^{l/2} \sum_{j=[(l+m+1)/2]}^l (-1)^{l-j} \binom{l}{j} \frac{(2j)!}{(2j-l-m)!} x^{2j-l-m}, \quad (A2)$$

where $[\ ]$ denotes the nearest lowest integer, we have

$$Y_{lm}(\pi/2, f) = \sqrt{\frac{2l+1}{4\pi}} \frac{(l-m)!}{(l+m)!} P_l^m(0) e^{imf} = \sqrt{\frac{2l+1}{4\pi}} c_{lm} e^{imf} \quad (A3)$$

where from equation (A2),

$$P_l^m(0) \equiv (-1)^{l+m/2} \binom{l}{(l+m)/2} \frac{(l+m)!}{2^l!}, \quad l+m \text{ even} \quad (A4)$$

and zero otherwise, so that

$$c_{lm}^2 = \frac{(l-m)!(l+m)!}{2^{2l-1} \{(l+m)/2\!/[!(l-m)/2!]} \quad (A5)$$

Note that the association of non-zero values of $P_l^m(0)$ with even $l+m$ is consistent with the sum in equation (11) being in steps of 2. Some values of $c_{lm}^2$ are listed in Table B2.

APPENDIX B: HANSEN COEFFICIENTS

The two formulations presented in this paper are distinguished by the expansion parameter; for the spherical harmonic expansion the parameter is the ratio of semimajor axes and it places no restrictions on the two eccentricities, while for the literal expansion the parameters are the eccentricities, with no restriction on the ratio of semimajor axes (except that the orbits should not cross). For the spherical harmonic expansion, we therefore require expressions for the Hansen coefficients which are accurate for all eccentricities, while for the literal expansion, power-series representations are appropriate because the expansion is valid only to order $j_{\text{max}}$ in the combined powers of the eccentricities.
Figure B1. Hansen coefficients $X_{n}^{l,m}(e_i)$ for quadrupole ($l = 2, m = 0, 2$) and octopole ($l = 3, m = 1, 3$) values of $l$ and $m$, and for various values of $n$, with dominant values labelled. Notice the $O(e_i^{[m-n]})$ behaviour for small $e_i$. The black curves are for $n = 1, 2, \ldots, 10$, the blue curves are for $n = -10, -9, \ldots, -1$ and the pink curves are for $n = 0$ for which closed-form expressions are given in Table B2. The red dashed curves are for the polynomial approximations $X_{n}^{l,m}(e_i) \approx -3e_i + \frac{11}{3}e_i^3$, and $X_{n}^{l,1}(e_i) \approx 1 + 2e_i^2 - \frac{11}{3}e_i^4$ (see Table B1).

### B1 Hansen coefficients relevant for the spherical harmonic expansion

Hansen coefficients are defined such that

$$X_{n}^{l,m}(e_i) = \frac{1}{2\pi} \int_0^{2\pi} (r/a_l)^{l-1} e^{-im\xi} dM_l$$

and

$$X_{n}^{-(l+1),m}(e_i) = \frac{1}{2\pi} \int_0^{2\pi} e^{-im\xi} (R/\alpha_o)^{l+1} e^{il\xi} dM_o.$$  \hspace{1cm} (B1) \hspace{1cm} (B2)

Figs B1 and B2 show the numerically integrated functions $X_{n}^{l,m}(e_i)$ and $Z_{n}^{-(l+1),m}(e_i) = (1 - e_o)^{l+1} X_{n}^{-(l+1),m}(e_o)$ for quadrupole and octopole values of $l$ and $m$, and for the first 10 positive values of $n'$ and $n$ (see equations 15 and 16). The scaling factor $(1 - e_o)^{l+1}$ replaces $a_o$ with the outer periastron separation $R_p = a_o(1 - e_o)$ in the definition (16) of $X_{n}^{-(l+1),m}(e_o)$, factoring out the singularity at $e_o = 1$. We refer to the $Z_{n}^{-(l+1),m}(e_o)$ as modified Hansen coefficients. While no closed-form expressions for these integrals exist (except for $n' = n = 0$; see Section B3), for many applications it is reasonable (CPU-wise) to integrate them numerically. However, simple approximations exist as outlined below, the analytic form of which provides insight into the behaviour of the physical variables which depend on them.

In Section B2, we give general power-series expansions which are correct to fourth order in the eccentricity. Amongst other things, these expressions demonstrate that the leading terms are such that

$$X_{n}^{l,m}(e_i) = O(e_i^{[m-n]}) \quad \text{and} \quad X_{n}^{-(l+1),m}(e_o) = O(e_o^{[m-n]})$$

consistent with Figs B1 and B2.

For most applications for which a spherical harmonic expansion of the disturbing function is appropriate, it suffices to know expressions for the Hansen coefficients associated with the $[n':1](2)$ harmonics for $l = 2, 4$ (see, for example, Sections 2.2 and 2.3), that is, $X_{1}^{2,2}(e_i)$, $X_{1}^{3,2}(e_i)$, $X_{4}^{3,2}(e_i)$ and $X_{4}^{5,2}(e_i)$, and perhaps those associated with the $[2n'+1:2](2)$ harmonics (those half-way between the $[n':1](2)$ harmonics), that is, $X_{2}^{2,2}(e_i)$, $X_{2}^{3,2}(e_i)$, $X_{2n'+1}^{3,2}(e_o)$ and $X_{2n'+1}^{5,2}(e_o)$. Table B1 gives the first few terms of the series expansions of these Hansen
Figure B2. The modified Hansen coefficients $Z^{-(l+1),m}_n(e_o) = (1 - e_o)^{l+1} X^{-(l+1),m}_n(e_o)$ for quadrupole ($l = 2, m = 0, 2$) and octopole ($l = 3, m = 1, 3$) values of $l$ and $m$, and for various values of $n$, with dominant values labelled. The black curves are for $n = 1, 2, \ldots, 10$, the blue curves are for $n = -10, -9, \ldots, -1$ (only visible in panel d), and the pink curves are for $n = 0$ for which closed-form expressions are given in Table B2.

Table B1. Hansen coefficients for $n = 1$ and $n = 2$, errors $E_{e_i} \equiv |\delta X^{l,m}_n(e_i)|/\max_{e_i}[X^{l,m}_n]$ and scale factors.

| $l$ | $m$ | $X^{l,m}_n(e_i)$ | $E_{0.7}$, $E_{0.9}$ | $X^{l,m}_2(e_i)$ | $E_{0.7}$, $E_{0.9}$ | $H_{lm}$ |
|-----|-----|----------------|----------------|----------------|----------------|----------|
| 2   | 2   | $-3e_i + 13\pi e_i^3 + \frac{\pi}{15} e_i^5$ | 0.006, 0.05 | $1 - 2e_i^2 + 23\pi e_i^4 - \frac{64}{15} e_i^6$ | 0.002, 0.025 | 0.71 |
| 4   | 2   | $-4e_i - 3e_i^3 + \frac{79}{39} e_i^5$ | 0.006, 0.025 | $1 + e_i^2 - \frac{42}{16} e_i^4 + \frac{35}{8} e_i^6$ | 0.007, 0.06 | 1.44 |
| 3   | 1   | $1 + 2e_i^2 - \frac{41}{13} e_i^4 - \frac{57}{8\pi} e_i^6$ | 0.0007, 0.014 | $\frac{1}{8} e_i^2 + \frac{35}{8\pi} e_i^4 + \frac{23}{8}\pi e_i^6$ | 0.002, 0.006 | 1.91 |

coefficients for which $e_i$ is the argument, as well as the error at $e_i = 0.7$ and 0.9, defined such that

$$E_{e_i} \equiv \left| \frac{\delta X^{l,m}_n(e_i)}{\max_{e_i}[X^{l,m}_n]} \right|$$

where $\delta X^{l,m}_n(e_i)$ is the difference between the numerically integrated expression and the series approximation at the given value of $e_i$, and $\max_{e_i}[X^{l,m}_n]$ is the maximum value of $X^{l,m}_n$ over the interval $0 \leq e_i \leq 1$. The expansions are correct to $O(e_i^6)$, except for $X^{3,1}_n(e_i)$ which is correct to $O(e_i^8)$ because the errors are an order of magnitude smaller with the extra term. Note that the error decreases monotonically with decreasing $e_i$ for each approximation. The functions $X^{1,2}_n(e_i)$ and $X^{3,1}_n(e_i)$ are compared to the numerically calculated integrals in Fig. B1, panels (a) and (d), respectively.

A good approximation for modified Hansen coefficients governing the dependence of the disturbing function on the outer eccentricity is given by the expression (Paper II)

$$Z^{-(l+1),m}_n(e_o) \approx (1 - e_o)^{l+1} \frac{4H_{l+2}}{3\sqrt{2\pi}} \left(1 - e_o^2\right)^{l/4} n^{3/2} e^{-\xi(e_o)} Z^{-(l+1),m}_n$$

where

$$\xi(e_o) = \cosh^{-1}(1/e_o) - \sqrt{1 - e_o^2}$$

$$\left| \frac{\delta X^{l,m}_n(e_i)}{\max_{e_i}[X^{l,m}_n]} \right|$$

$$\left| \frac{\delta X^{l,m}_n(e_i)}{\max_{e_i}[X^{l,m}_n]} \right|$$

where $\delta X^{l,m}_n(e_i)$ is the difference between the numerically integrated expression and the series approximation at the given value of $e_i$, and $\max_{e_i}[X^{l,m}_n]$ is the maximum value of $X^{l,m}_n$ over the interval $0 \leq e_i \leq 1$. The expansions are correct to $O(e_i^6)$, except for $X^{3,1}_n(e_i)$ which is correct to $O(e_i^8)$ because the errors are an order of magnitude smaller with the extra term. Note that the error decreases monotonically with decreasing $e_i$ for each approximation. The functions $X^{1,2}_n(e_i)$ and $X^{3,1}_n(e_i)$ are compared to the numerically calculated integrals in Fig. B1, panels (a) and (d), respectively.

A good approximation for modified Hansen coefficients governing the dependence of the disturbing function on the outer eccentricity is given by the expression (Paper II)

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where

$$\xi(e_o) = \cosh^{-1}(1/e_o) - \sqrt{1 - e_o^2}$$
and the constant $H_{22}$ is an empirical scaling factor given in Table B1, determined by comparing the maxima of $Z_{20}^{-3,2}(e_o)$ and $\tilde{Z}_{20}^{-3,2}(e_o)$ and scaling the latter so that the values of their maxima are the same. All other $Z_{n}^{-3,2}(e_o)$ are then scaled by the same factor, except for $n \leq 10$ which involve additional scale factors listed in panel (a) of Fig. B3. Note that \( \lim_{n\to\infty} e^{-\nu \xi \tilde{e}_o} \tilde{e}_o^{-2} = 0 \), \( n' \geq 3 \), while for \( n'=2 \) the limit is \( e^2/2 \). However, for small values of \( e_o \) it may be preferable to use a power-series approximation for $X_{n}^{-3,2}(e_o)$.

The derivation of equation (B5) uses the method of steepest descents to evaluate the integrals and is therefore referred to as an asymptotic approximation. They are closely related to overlap integrals which quantify the strength of the interaction between the orbits (Paper II). Scale factors are necessary because approximations made in the analysis rely on the values of $e_o$ and $\sigma$ (and hence $n'$) being high, and accuracy is lost when they are not (although the shape of the curves is preserved). Fig. B3 compares numerically evaluated integrals with their asymptotic approximations for general $l, m \geq 0$ and $n' \geq 2$.

The asymptotic approximation for for general $l, m \geq 0$ and $n' \geq 2$ is

$$Z_{n'}^{-3,2}(e_o) \approx (1-e_o)^{j+1} \cdot \frac{\mathcal{H}_{nm} \cdot 2^{m}}{\sqrt{2\pi} (l+m-1)! \cdot (1-e_o)^{3m-l-1/4} e_{o}^{m} n^{l+m-1/2} e^{-n' \xi \tilde{e}_o}} \equiv \tilde{Z}_{n'}^{-3,2}(e_o).$$

(B7)

Note that $\tilde{Z}_{n'}^{-3,2}(e_o) = O(e_o^{n'-m})$, $n' \geq 2$, consistent with equation (B3).

### B2 Expansions up to fourth order in eccentricity for use in the literal expansion

Using MATHEMATICA or similar, it is easy to derive general power-series expansions for $X_{n}^{j,m}(e)$ valid for any integers $j$ and $n$ and for specific values of $m$. These series contain either even or odd powers of $e$ only. Writing $v = n \text{sgn}(m-n)$, we have for $m = n, m = n \pm 1, m = n \pm 2, m = n \pm 3$ and $m = n \pm 4$ correct to $O(e^4)$.

$$X_{n}^{j,m}(e) = 1 + \frac{1}{4} \left[ (j+1) - 4v^2 \right] e^2 + \frac{1}{64} \left[ (j+1)(j+1)(j-1) + v(16v^2 - (8j^2 + 9)) \right] e^4 + \cdots,$$

(B8)

$$X_{n}^{j,n \pm 1}(e) = -\frac{1}{2} \left[ (j+2) + 2v \right] e + \frac{1}{16} \left[ -j(j+2)(j-1) + v(8v^2 + 2v(2j + 7) - j(2j - 3) + 6) \right] e^3 + \cdots,$$

(B9)

$$X_{n}^{j,n \pm 2}(e) = \frac{1}{8} \left[ (j+2)(j+3) + v(4v + 4v + 11) \right] e^2$$

$$+ \frac{1}{96} \left[ (j+3)(j+2)(j-1)(j-2) - 2v(8v^3 + 34v^2 + 44v + 13) + jv(4j^2 - 3 - 47 - 16v(v + 3)) \right] e^4 + \cdots.$$

(B10)
2:1 resonance

Figure B4. Some Hansen coefficients associated with the first-order 2:1 resonance. Plotted are $X_{j,m}^{1}(e_i)$ and $Z_{-1}^{1} - (j+1,m) (e_o)$ for $m = 1$ (red) and $m = 2$ (blue), and for $j = 0$ (panels a and b) and $j = 2$ (panels c and d). Solid curves: 'exact' integration, dashed curves: fourth-order correct series expansions. Accuracy is good up to at least $e_{i,0} = 0.4$ for all Hansen coefficients shown here, this value increasing up to 1 in some cases (for example, $Z_{-1}^{1,1} (e_i)$). These should be compared with series expansions for the first-order 7:6 resonance which are less accurate given the higher values of $n$ and $n'$ (see Fig. B5).

\[ X_{j,m}^{1}(e_i) = \frac{1}{48} \left[ (j+4)(j+3)(j+2) + 2v(4v^2 + 21v + 31) + 3jv(4v + 2j + 13) \right] e^{\frac{3}{4}} + \cdots, \]  

\[ X_{j,m}^{2}(e_i) = \frac{1}{384} \left[ (j+5)(j+4)(j+3)(j+2) + v(16v^3 + 136v^2 + 379v + 394) + 2jv(4j^2 + 45j + 165 + v(16v + 12j + 96)) \right] e^{\frac{5}{4}} + \cdots. \]  

Note that these are consistent with $X_{j,m}^{l}(e_i) = O(e^{m-n})$. Comparison of these approximations with numerically evaluations of equations (15) and (16) are shown in Figs B4–B6. Note that for the outer eccentricity functions, modified Hansen coefficients are calculated.

Comparison of Fig. B4 with Fig. B5 for the 2:1 and 7:6 resonances, respectively, suggests that convergence of the series (B8)–(B12) is slower for the 7:6 resonance. Inspection of the dependence of these series on $n$ and $j$ shows that the magnitude of the term proportional to $e_{q}^{n}$ is $O(e^{m-n})$ or $O(j e)^{n}$, whichever is the greatest, although in some cases the errors may cancel to come extent (see, for example, the $m = 1$ curve in panel d of Fig. B4 which plots $Z_{-1}^{1,1} (e_i) = (1 - e_{o})X_{2}^{1,1} (e_i)$, with the series expansion of $X_{2}^{1,1} (e_i)$ given by equation B9).

B3 Closed-form expressions

Hughes (1981) has provided closed-form expressions for Hansen coefficients with $n = 0$ and $n' = 0$. Those associated with the inner orbit are

\[ X_{j,m}^{l}(e_i) = \frac{1}{2\pi} \int_{0}^{2\pi} (r/a_i)^{l} e^{m+l} dM_i \]  

\[ = \left( \frac{e_i}{2} \right)^{m} \left( \begin{array}{c} l+m+1 \\ m \end{array} \right) F \left( \frac{m-l-1}{2}, \frac{m-l}{2}; m+1; e_i^2 \right). \]
Some Hansen coefficients associated with the first-order 7 : 6 resonance. Plotted are $X_j^m(e_i)$ and $Z_n^m(e_o)$ for $m = 6$ (red) and $m = 7$ (blue), and for $j = 0$ (panels a and b) and $j = 4$ (panels c and d). Solid curves: ‘exact’ integration, dashed curves: fourth-order correct series expansions. While errors in the series expansions (B8)–(B12) are formally of the order of the fifth power of the eccentricity, inspection shows that errors are in fact $O(e^5)$. Putting $6^5e^5 = 0.1$, the expansions for this case are accurate only for $e_i, e_o \ll 0.1$ for a 10 per cent error.

where $F(\cdot)$ is a hypergeometric function given here by (see Gradstein & Ryzhik 1980 for a general definition)

$$F\left(\frac{m-l-1}{2}, \frac{m-l}{2}; m+1; e_i^2\right) = 1 + \sum_{j=1}^{\lfloor(m-l-1)/2\rfloor} \prod_{k=0}^{l-1} \frac{[(l+m+1)/2-k] [(l-m)/2-k]}{(m+k+1)(k+1)} e_i^{2j}.$$  

with $F(-1/2, 0; m+1; e_i^2) = 1$ when $m = l$. Hansen coefficients associated with the outer orbit are

$$X_0^{-(l+1),m}(e_o) = \frac{1}{2\pi} \int_0^{2\pi} e^{-imf_i} \left(\frac{R/a_o}{y^+}\right)^m dM_i$$

$$= \left(\frac{e_o}{2}\right)^m (1-e_o^2)^{-(2l-1)/2} \sum_{j=0}^{\lfloor(l-m-1)/2\rfloor} \binom{l-1}{2j+m} \binom{2j+m}{j} \left(\frac{e_o^2}{2}\right)^j,$$

where $\lfloor \cdot \rfloor$ denotes the nearest lowest integer.

Quadrupole and octopole eccentricity functions are listed in Table B2.

B4 MATHEMATICA programs for Hansen coefficients and $F_m^{i,j}(e_i, e_o)$

It is the experience of the author that most colleagues in Astronomy have access to the software package MATHEMATICA (Wolfram Research, Inc. 2010), but many are not familiar with the syntax. For this reason, the following short programs are included here. The first two calculate series approximations for the Hansen coefficients

$$X_n^m(e_i) = \frac{1}{2\pi} \int_0^{2\pi} r e^{imf_i} e^{-inM_i} dM_i = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{E}_n^{i,j} e^{-inM_i} dE_i$$
Some Hansen coefficients associated with the third-order $4 : 1$ resonance. Plotted are $X_j^{\pm m}(e_i)$ and $Z_j^{-(j+1),m}(e_o)$ for $m = 1$ (red), $m = 2$ (blue), $m = 3$ (purple) and $m = 4$ (magenta), and for $j = 0$ (panels a and b) and $j = 3$ (panels c and d). Solid curves: ‘exact’ integration, dashed curves: fourth-order correct series expansions. While approximations are accurate for $e_i \lesssim 0.5$ for $n = 1$ (panels a and c), they are accurate only for $e_o \lesssim 0.15$ for $n = 4$. As in the case of the $7 : 6$ resonance in Fig. B5, this is because the error in the expansions is $O[n^5 e^5]$. 

Table B2. Secular Hansen coefficients and $c_{lm}^2$.

| $l$ | $m$ | $X_0^{l,m}(e_i)$ | $X_0^{-(l+1),m}(e_o)$ | $c_{lm}^2$ |
|-----|-----|-----------------|----------------------|---------|
| 2   | 2   | $\frac{1}{2}e_i^2$ | 0                     | $\frac{1}{3}$ |
|     | 0   | $1 + \frac{1}{2}e_i^2$ | $(1 - e_o^2)^{-3/2}$ | $\frac{1}{2}$ |
| 3   | 3   | $-\frac{3}{2}e_i^3$ | 0                     | $\frac{1}{6}$ |
|     | 1   | $-\frac{1}{6}e_i(4 + 3e_i^2)$ | $e_o(1 - e_o^2)^{-5/2}$ | $\frac{1}{7}$ |

and

$$X_{n}^{-(l+1),m}(e_o) = \frac{1}{2\pi} \int_{0}^{2\pi} e^{-imf_o} e^{i\omega'M_i} dM_i = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{[e^{-if_o}]^m}{R^{l+1}} e^{i\omega'M_i} dE_o,$$

(B19)

where $E_i$ is the eccentric anomaly with $r = 1 - e_i \cos E_i$, $M_i = E_i - e_i \sin E_i$, $c_{f_i} = (\cos E_i - e_i)/(1 - e_i \cos E_i)$, $E_o = \sqrt{1 - e_o^2} \sin E_i/(1 - e_i \cos E_i)$, and similarly for $R$, $M_o$, $c_{f_o}$ and $\sin f_o$. The third program calculates $F_{\text{max}}^{(j)}(e_i, e_o)$ according to equation (68). Recall that if $j_{\text{max}}$ is the order of the literal expansion (i.e. the highest combined powers of the eccentricities), then according to Section 3.2 $F_{\text{max}}^{(j)}(e_i, e_o)$ should be expanded to $j_{\text{max}}$. 

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MATHEMATICA program to calculate $X_1^{2,2}(e_i) = -3e_i + \frac{13}{8}e_i^3 + \frac{5}{192}e_i^5 + \mathcal{O}(e_i^7)$

\begin{verbatim}
l = 2;
m = 2;
n = 1;
ne = 5;
r = 1 - e Cos[EA];
M = EA - e Sin[EA];
cosf = (Cos[EA] - e)/(1 - e Cos[EA]);
sinf = Sqrt[1 - e**2] Sin[EA]/(1 - e Cos[EA]);
 expernM = Cos[n M] - I Sin[n M];
Xlmn = Normal[Series[r**(l + 1) (cosf + I sinf)**m expernM, {e, 0, ne}]]; Xlmnav = Simplify[Integrate[Xlmn/(2 Pi), {EA, 0, 2 Pi}]]
\end{verbatim}

MATHEMATICA program to calculate $X_2^{3,2}(e_o) = 1 - \frac{5}{2}e_o^2 + \frac{11}{16}e_o^4 + \mathcal{O}(e_o^6)$

\begin{verbatim}
l = 2;
m = 2;
n = 2;
ne = 4;
R = 1 - e Cos[EA];
M = EA - e Sin[EA];
cosf = (Cos[EA] - e)/(1 - e Cos[EA]);
sinf = Sqrt[1 - e**2] Sin[EA]/(1 - e Cos[EA]);
 expernM = Cos[n M] + I Sin[n M];
Xlmn = Normal[Series[(cosf - I sinf)**m expernM/R**l, {e, 0, ne}]]; Xlmnav = Simplify[Integrate[Xlmn/(2 Pi), {EA, 0, 2 Pi}]]
\end{verbatim}

MATHEMATICA program to calculate $F_{235}^{(2)}(e_i, e_o) = -\frac{1}{2}e_i e_o - \frac{21}{16}e_i^3 e_o - \frac{97}{16}e_i^5 e_o + \mathcal{O}(e_o^6)$ correct to fourth order. Note that the leading term is of order $j = 2$, and that all possible combinations of the powers of $e_i$ and $e_o$ are present given the constraints that only odd powers of $e_i \geq |m - n| = 1$ and $e_o \geq |m - n'| = 1$ can appear. Terms of $O(e_o^6)$ should be discarded.

\begin{verbatim}
jmax=4;
j = 2;
m = 4;
n = 3;
no = 5;
nei = Max[Abs[m - n], jmax - Abs[m - no]];
neo = Max[Abs[m - no], jmax - Abs[m - n]];
r = (1 - ei Cos[EA]);
M = EA - ei Sin[EA];
cosf = (Cos[EA] - ei)/(1 - ei Cos[EA]);
sinf = Sqrt[1 - ei**2] Sin[EA]/(1 - ei Cos[EA]);
 expernM = Cos[n M] - I Sin[n M];
R = (1 - eo Cos[E Ao]);
Mo = E Ao - eo Sin[E Ao];
cosfo = (Cos[E Ao] - eo)/(1 - eo Cos[E Ao]);
sinfo = Sqrt[1 - eo**2] Sin[E Ao]/(1 - eo Cos[E Ao]);
 experMo = Cos[no Mo] + I Sin[no Mo];
Fjmnnp = Expand[Simplify[Sum[(-1)**(j - k) Binomial[j, k] Integrate[
    Normal[Series[r**(k + 1) (cosf + I sinf)**m expernM, {ei, 0, nei}]]/(2 Pi),
    {EA, 0, 2 Pi}],
    Integrate[
    Normal[Series[(cosfo - I sinfo)**m experMo/R**k, {eo, 0, neo}]]/(2 Pi),
    {E Ao, 0, 2 Pi}],
    {k, 0, j}]]]
\end{verbatim}
APPENDIX C: THE LAPLACE COEFFICIENTS

C1 Series expansion

The general Laplace coefficient is defined as

\[ b^{(m)}_s(x) = \frac{1}{\pi} \int_0^{2\pi} \frac{e^{-im\psi}}{(1 - 2x \cos \psi + x^2)^{s/2}} d\psi, \quad \text{(C1)} \]

where \( s \) is a positive half-integer. A series expansion for this is

\[ b^{(m)}_s(x) = \sum_{p=0}^{\infty} C^{(m,s)}_p x^{2p} = O(x^m), \quad \text{(C2)} \]

where \( s \geq 1/2 \) is a half-integer and

\[ C^{(m,s)}_p = \frac{2\Gamma(s + p)\Gamma(s + m + p)}{\Gamma(s)^2 p!(m + p)!}. \quad \text{(C3)} \]

Recall that

\[ \Gamma(n) = (n - 1)!, \quad \text{(C4)} \]

and

\[ \Gamma\left(n + \frac{1}{2}\right) = \frac{(2n)!}{2^{2n}n!} \sqrt{\pi}. \quad \text{(C5)} \]

For \( s = 1/2 \), we have

\[ C^{(m,1/2)}_p = \frac{2(2p)! (m + 2p)!}{2^{2p+2m} [p!(m + p)!]^2}. \quad \text{(C6)} \]

By the ratio test, we have that the series (C2) converges as long as \( x > 1 \).

C2 Derivatives

The following formulae are taken from Brouwer & Clements (1961, p. 502) and Murray & Dermott (2000) who take their formulae from Brouwer & Clements (1961). Defining the operator

\[ D = \frac{d}{dx}, \quad \text{(C7)} \]

we have

\[ \mathcal{D} b^{(m)}_s(x) = s \left[ b^{(m-1)}_{s+1} - 2x b^{(m)}_{s+1} + b^{(m+1)}_{s+1} \right] \quad \text{(C8)} \]

and in general,

\[ \mathcal{D}^n b^{(m)}_s(x) = s \left[ \mathcal{D}^{n-1} b^{(m-1)}_{s+1} - 2x \mathcal{D}^{n-1} b^{(m)}_{s+1} + \mathcal{D}^{n-1} b^{(m+1)}_{s+1} - 2(n-1) \mathcal{D}^{n-2} b^{(m)}_{s+1} \right], \quad n \geq 2. \quad \text{(C9)} \]

C2.1 Leading terms

Recalling the definition from equation (61)

\[ B^{(j,m)}_{1/2}(x) = \frac{x^j}{j!} \frac{d^j}{dx^j} \left[ b^{(m)}_{1/2}(x) \right], \quad \text{(C10)} \]

we have from equation (C2) that

\[ B^{(j,m)}_{1/2}(x) = \sum_{p=p_{\text{min}}}^{\infty} \frac{(m + 2p)!}{(m + 2p - j)!} C^{(m,1/2)}_p x^{m+2p} \quad \text{(C11)} \]

\[ = \sum_{p=p_{\text{min}}}^{\infty} E^{(j,m)}_p x^{m+2p}, \quad \text{(C12)} \]

where \( p_{\text{min}} = \max \{ 0, \lfloor j/2 \rfloor (j - m + 1) \} \) with \( \lfloor \cdot \rfloor \) denoting the nearest lowest integer, and

\[ E^{(j,m)}_p = \binom{m + 2p}{j} C^{(m,1/2)}_p = \frac{2(2m + 2p)!(m + 2p)!(2p)!}{4^{p+m} j!(m + 2p - j)!p!(m + p)!}. \quad \text{(C13)} \]
The leading term of $B_{1/2}^{(j,m)}(x)$ is therefore

$$B_{1/2}^{(j,m)}(x) = \begin{cases} E_{0}^{(j,m)} x^m + \ldots, & j \leq m \\ E_{p_*}^{(j,m)} x^j + \ldots, & j \geq m, \ j - m \text{ even} \\ E_{p_*+1}^{(j,m)} x^{j+1} + \ldots, & j > m, \ j - m \text{ odd} \end{cases}$$  

(C14)

where

$$E_{0}^{(j,m)} = \frac{2(2m)!}{4^m j!(m-j)!m!},$$  

(C15)

and

$$E_{p_*}^{(j,m)} = \frac{2(j+m)!(j-m)!}{4^m [(j-m)/2]!(j+m)/2)!}$$  

(C16)

with $p_* = [(j - m + 1)/2]$. Note that by the ratio test, the series (C12) converges as long as $x < 1$.

C3 Graphical representations

Fig. C1 shows $b_{1/2}^{(m)}(x)$ and $B_{1/2}^{(j,m)}(x)$ for $j = 1$, both for $m = 0, 1, \ldots, 20$. Notice how, for most values of $m$, $B_{1/2}^{(1,m)}(x) > b_{1/2}^{(m)}(x)$. In general, $b_{1/2}^{(j,m)}(x) > B_{1/2}^{(j,m)}(x)$ when $j_1 > j_2$. Note that $b_{1/2}^{(0)}(0) = 2, b_{1/2}^{(m)}(0) = 0, m \geq 1$ and $B_{1/2}^{(j,m)}(0) = 0$ for all $m$.

APPENDIX D: LAGRANGE’S PLANETARY EQUATIONS FOR THE VARIATION OF THE ELEMENTS

Because Lagrange’s equations for the variation of the elements were developed for the restricted three-body problem (the mass of one of the three bodies of interest is negligible compared to the other two so that one orbit is fixed), they are normally given in terms of a disturbing function which has the dimensions of energy per unit mass (Brouwer & Clements 1961; Murray & Dermott 2000). Here, the disturbing function has the dimensions of energy, and as a result the same function can be used for the rates of change of the inner and outer orbital elements, that is, there is no need to define separate inner and outer disturbing functions. Note also that Lagrange’s ‘planetary’ equations hold for any mass ratios, and in particular, there is no assumption about the smallness of $R$. In spite of the fact that in most applications the disturbing function acts to perturb the Keplerian orbits from invariant elliptical motion, the derivation of Lagrange’s equations does not involve a perturbation technique. Rather it uses the method of variation of parameters, those parameters being the orbital elements which are constant when there is no interaction between the orbits (i.e. when $R = 0$), and vary once the orbits are allowed to interact via a non-zero $R$.

For reference, the relevant Lagrange equations for the rates of change of the elements for coplanar systems are

$$\frac{de}{dr} = -\frac{\varepsilon (1 - \varepsilon)}{\mu v a^e e} \frac{\partial R}{\partial \lambda} - \frac{\varepsilon}{\mu v a^e} \frac{\partial R}{\partial \sigma},$$  

(D1)

$$\frac{d\sigma}{dr} = \frac{\varepsilon}{\mu v a^e} \frac{\partial R}{\partial e}$$  

(D2)

$$\frac{de}{dr} = -\frac{2}{\mu v a} \frac{\partial R}{\partial a} + \frac{\varepsilon (1 - \varepsilon)}{\mu v a^e} \frac{\partial R}{\partial e}$$  

(D3)
and
\[ \frac{da}{dt} = \frac{2}{\mu \nu a} \frac{\partial R}{\partial \lambda}, \tag{D4} \]

where \( \varepsilon = \sqrt{1 - e^2} \), and with the set \( \{a, e, \varpi, \epsilon; \mu, \nu\} \) representing either of the inner orbit \( \{a_i, e_i, \varpi_i, \epsilon_i; \mu_i, \nu_i\} \) or the outer orbit \( \{a_o, e_o, \varpi_o, \epsilon_o; \mu_o, \nu_o\} \). Note that in this form, \( \lambda \) and \( \epsilon \) are related through \( \lambda = \int_0^t \nu \, dt + \epsilon \) rather than the usual \( \lambda = \nu t + \epsilon \) [see Brouwer & Clements (1961, p. 285) for a discussion of this point, in particular that in using this definition one need not consider \( \nu \) to be a function of \( a \) when evaluating \( \partial R / \partial a \)]. The former definition should always be used when resonance plays a role; however, note that the two definitions are equivalent when it does not (because in that case the semimajor axes are constant).

**APPENDIX E: THE MEAN LONGITUDE AT EPOCH**

Finally, a note on the mean longitude at epoch, \( \epsilon \equiv M(T_0) + \varpi \), where \( T_0 \) is the time (‘epoch’) at which the osculating elements are determined and \( M(T_0) \) is the mean anomaly at that time. For some applications, \( T_0 \) is taken to be the time at periastron passage, while for others it corresponds to some given time, for example, the mid-time of a particular set of observations. While it is clear what it means for \( \varpi \) to vary, one might reasonably ask what it means for \( M(T_0) \) to vary.

To answer this question, consider the usual definition of the mean anomaly. This is given in terms of the orbital frequency (mean motion) \( \nu \) and the time at periastron passage \( T_p \) as
\[ M = \nu (t - T_p), \tag{E1} \]
or in terms of a general epoch \( T_0 \),
\[ M = \nu (t - T_0) + \nu (T_0 - T_p) = \nu (t - T_0) + M(T_0). \tag{E2} \]
Either way, \( M(T_0) = 0 \). Referring to Fig. E1, imagine at \( t = 0 \) a force acts on the system in such a way that only the eccentricity is changed. If the eccentricity increases, the time to periastron passage, \( T_p \), decreases, and since \( \nu \) remains unchanged and \( T_0 \) is fixed, the mean anomaly at epoch, \( M(T_0) = \nu (T_0 - T_p) \), must increase. The change in the mean anomaly at epoch must therefore be proportional to the change in the time at periastron, that is,
\[ \delta M(T_0) = \nu (T_{p,1} - T_{p,2}), \tag{E3} \]
where \( T_{p,1} \) and \( T_{p,2} \) are the times to periastron before and after the perturbation is applied. In general, an arbitrary force acting on the system will cause the mean anomaly at epoch to change by an amount (Pollard 1966, p. 36)
\[ \delta M(T_0) = \left[ \frac{3}{2} M - r \sqrt{1 - e^2} / (e \sin f) \right] (\delta a/a) + \sqrt{1 - e^2} \cot f (\delta e/e), \tag{E4} \]
that is, only changes to the osculating eccentricity and semimajor axis affect this quantity (as reflected in Lagrange’s planetary equation D3 for the rate of change of \( \epsilon \)).

\[ \delta M(T_0) = \nu (T_{p,1} - T_{p,2}) \]

**Figure E1.** Illustration of the effect on the mean anomaly at epoch, \( M(T_0) = \nu (T_0 - T_p) \), of increasing the eccentricity while holding the semimajor axis constant. If the perturbation is applied at \( t = 0 \) when \( M = -\nu T_p \), then although at the instant the force is applied there is no change in the true anomaly \( f \) (the true position in the orbit), there is a change in the mean anomaly and hence the mean anomaly at epoch (since \( T_0 \) is a fixed time). This is given by \( \delta M(T_0) = \nu (T_{p,1} - T_{p,2}) \), where \( T_{p,1} \) and \( T_{p,2} \) are the times to periastron before and after the perturbation is applied.
## APPENDIX F: NOTATION

| Symbol | Description | Page |
|--------|-------------|------|
| $f_i, f_o$ | Inner and outer true anomalies | 13 |
| $M_i, M_o$ | Inner and outer mean anomalies | 13 |
| $\lambda_i, \lambda_o$ | Inner and outer mean longitudes | 15 |
| $\sigma_i, \sigma_o$ | Inner and outer longitudes of periastron | 13 |
| $e_i, e_o$ | Inner and outer mean longitudes at epoch | 20 |
| $a_i, a_o$ | Inner and outer eccentricities | 14 |

| Symbol | Description | Page |
|--------|-------------|------|
| $\alpha = a_i/a_o$ | Inner and outer semimajor axes | 15 |
| $\alpha_s = \beta_s \alpha$, $s = 1, 2$ | | 26 |
| $P_i, P_o$ | Inner and outer orbital periods | 17 |
| $v_i, v_o$ | Inner and outer orbital frequencies (mean motions) | 13 |
| $\sigma = P_o/P_i = v_i/v_o$ | Period ratio | 18 |
| $r$ | Position of body 2 relative to body 1 | 11 |
| $R$ | Position of body 3 relative to the centre of mass of bodies 1 and 2 | 11 |

| Symbol | Description | Page |
|--------|-------------|------|
| $m_{12} = m_1 + m_2$ | | 11 |
| $m_{123} = m_1 + m_2 + m_3$ | | 11 |
| $\beta_1 = m_1/m_{12}$ | | 11 |
| $\beta_2 = -m_2/m_{12}$ | | 11 |
| $\mathcal{A}_l = \beta_{1}^{l-1} - \beta_{2}^{l-1}$ | Harmonic coefficient | 13 |
| $\mu_i = m_1m_2/m_{12}$ | Reduced mass of inner binary | 11 |
| $\mu_o = m_{12}m_3/m_{123}$ | Reduced mass of outer binary | 11 |
| $\mathcal{R}$ | The disturbing function (interaction energy) | 11 |
| $\mathcal{R}$ | The orbit-averaged disturbing function | 24 |
| $\mathcal{R}_{mn}$ | Harmonic coefficient | 16, 29 |
| $\phi_{mn}$ | Harmonic associated with the angle $\phi_{mn}$ | 15, 28 |
| $[n': n](m)$ | Harmonic associated with the angle $\phi_{mn}$ | 18 |
| $\Delta \sigma_N$ | Width of $[N : 1](2)$ resonance | 23 |
| $\Delta \sigma_{mn}$ | Width of $[n' : n](m)$ resonance | 42 |
| $\omega_N, \omega_{mn}$ | Libration frequencies | 21, 42 |
| $Y_l^m(\theta, \phi)$ | A spherical harmonic of degree $l$ and order $m$ | 12 |
| $c_l^m$ | Coefficients in the semimajor axis expansion | 13 |
| $\xi_l$ | | 15 |
| $m_{min}$ | | 15 |
| $l_{min}$ | | 16 |
| $X_l^m(r_i)$ | A Hansen coefficient | 14 |
| $Z_l^{(l+1)m}(r_i)$ | A modified Hansen coefficient | 16 |
| $F_l^{(l)}(e_i, e_o)$ | Linear combination of Hansen coefficients | 28 |
| $b_{l}^{(12)}(e_i, e_o)$ | A Laplace coefficient | 27 |
| $A_{lm}(\alpha; \beta_2)$ | Coefficient in eccentricity expansion | 29 |

Numbers in the right-hand column refer to the page where the variable is defined.