Non-unimodular reductions and $N = 4$ gauged supergravities

P.M. Petropoulos

Centre de Physique Théorique, CNRS, Ecole Polytechnique, 91128 Palaiseau Cedex, France

December 2007

ABSTRACT

We analyze the class of four-dimensional $N = 4$ supergravities obtained by gauging the axionic shift and axionic rescaling symmetries. These theories are formulated with the machinery of embedding tensors and shown to be deducible from higher dimensions using a Scherk–Schwarz reduction with a twist by a non-compact symmetry. This allows to evade the usual unimodularity requirement and completes the dictionary between heterotic gaugings and fluxes, at least for the “geometric sector”.

Based on works with J.-P. Derendinger and N. Prezas. To appear in the proceedings of the RTN workshop Constituents, fundamental forces and symmetries of the universe, Valencia, Spain, October 1 – 5 2007.

*marios@cpht.polytechnique.fr
†Unité mixte UMR7644.
Contents

1 Why gaugings and fluxes? 2
2 Gauged supergravities and the embedding tensor 3
3 The axionic transformations and their gaugings 5
4 Dynamics of the axionic gaugings 6
5 The higher-dimensional origin: non-unimodular Scherk–Schwarz reduction 8
6 Summary and outlook 10
1 Why gaugings and fluxes?

String compactifications share a set of usual caveats. First stands the issue of supersymmetry breaking. The original $N = 4$ or 8 supersymmetry of type II, heterotic or M theory vacua must be reduced to $N = 1$ at some reasonably low scale before being broken completely to match with TeV-scale phenomenology. A second major problem is the issue of moduli stabilization: many massless neutral scalars pollute the spectrum and disable any attempt of confrontation with low-energy physics. The third problem is that of the cosmological constant. It is obviously related to the previous ones, the common denominator of all these being the structure of the vacuum. It is not clear though to what extent string theory can shed light on this infrared problem.

The possibility to give vacuum expectation values to antisymmetric-tensor fields (NS–NS, R–R, spin connection) provides a tool for a better control of the situation. This was recognized long ago and has been reexamined extensively over the recent years (see [1] for a comprehensive review) with essentially two complementary approaches. The connection between these two methods (see [2] for a concise review) stems from the fact that the effective theories of flux compactifications are gauged supergravities with spontaneously broken supersymmetry, with scalars charged under (non-)Abelian gauge groups, and with moduli-dependent superpotential (and potential) – to be opposed to toroidal compactifications, which are ungauged supergravities with neutral scalars and flat potential. The first approach might be called “top-down” and consists in (i) understanding the generalized geometrical tools that describe the ten-dimensional theory in presence of fluxes, (ii) find admissible compactifications, and (iii) analyze the low-energy properties – in other words check the issues of stabilization and supersymmetry breaking, not a priori guaranteed. The second, somehow less popular (see [3, 4] and [5] for a review), has four dimensions as starting point. In this “bottom-up” scheme (i) one starts directly with phenomenologically relevant four-dimensional gauged supergravities, (ii) one translates the gauging parameters into fluxes, and (iii) one tries to reconstruct the fundamental theory. The latter point might be subtle because no systematic oxidation recipe exists and not all four-dimensional gauged $N = 4$ ($N = 8$) supergravities are heterotic, type-I or type-II-orientifold (M-theory) vacua. Despite this reservation, the method has been shown to capture a large variety of situations, including four-dimensional remnants of non-geometric string backgrounds or supersymmetric AdS$_4$ vacua of type II theories with stabilized main moduli [3, 6].

Here we will focus on four-dimensional $N = 4$ theories and remind the basics on the gauging procedure using the embedding tensor – outstanding tool described e.g. in [7, 8, 9, 10, 11, 12]. We will in particular analyze the gauging of axionic shifts and rescalings and trace its ten-dimensional origin. This is not straightforward: it requires a generalized, non-unimodular Scherk–Schwarz reduction with a twist by the scaling symmetries and relies on a duality between massive vectors and massive two-forms.
2 Gauged supergravities and the embedding tensor

The ungauged four-dimensional N = 4 supergravity has in general 1 gravitational and n vector multiplets. The bosonic content of the gravitational multiplet is 1 graviton, 6 graviphotons, and 2 real scalars combined into the axion-dilaton \( \tau = \chi + i \exp(-2\phi) \); the vector multiplet has 1 vector and 6 real scalars. The gauge group is Abelian, \( U(1)^{6+n} \), and all scalars are neutral and non-minimally coupled to the vectors (interaction terms of the type \( f(\text{scalars}) F^2 \)). There is no scalar potential.

The elimination of the auxiliary fields generates the scalar manifold:

\[
\mathcal{M} = \frac{SL(2,\mathbb{R})}{U(1)} \times \frac{SO(6,n)}{SO(6) \times SO(n)},
\]

which exhibits the global symmetries of the theory. The \( SL(2,\mathbb{R}) \times SO(6,n) \subset Sp(6,2n,\mathbb{R}) \) is realized as a U-duality symmetry of the full theory. In heterotic theory, only an \( A_{2,2} \times SO(6,n) \) is realized off-shell. The \( A_{2,2} \) generates the axionic rescaling and axionic shifts and does not mix electric and magnetic gauge fields. Genuine electric–magnetic duality transformations relate different Lagrangians written in different “symplectic frames”.

Although one set of vectors only describes propagating degrees of freedom – electric or magnetic or any combination depending on the choice of symplectic frame –, it is possible to include them all in a unified Lagrangian formulation. The latter comprises \( 12 + 2n \) fields \((\{A^{M+}\}, \{A^{M-}\}), M = 1, \ldots, 6 + n\), which form a \((2, \text{Vec})\) of \( SL(2,\mathbb{R}) \times SO(6,n) \) (i.e. a \( \text{Vec}\) of \( Sp(12+2n,\mathbb{R})\)), without kinetic term for \( \{A^{M-}\}\). It also includes extra two-form auxiliary fields dual to the scalars. The equations of motion for the magnetic vectors and the two-forms set the duality between the scalars and the two-forms, and between the electric and magnetic vectors, respectively. The presence of all auxiliary fields (magnetic vectors and two-forms) is also necessary for gauge invariance, which ensures the decoupling of all ghosts. The number of propagating degrees of freedom remains unaltered, as compared to the ungauged theory.

The gauging of a supergravity theory is a deformation which is compatible with supersymmetry. For \( N = 4 \) (and \( N = 8 \) which we will not discuss here) this is the only possible deformation. It is a promotion of a subgroup of the U-duality group to a local gauge symmetry supported by (part of) the existing \( U(1)^{n+6} \) vectors.

The generators of the duality group are \( T^{MN} = -T^{NM}, M, \ldots = 1, \ldots, 6 + n \) for the \( SO(6,n) \) subgroup and \( S_{\alpha\beta} = S^\alpha{}_{\beta}, \beta, \ldots = +, - \) for the \( SL(2,\mathbb{R}) \). They obey the following commutation relations:

\[
[T_{KL}, T_{IJ}] = \eta_{IJ}T_{KM} + \eta_{KM}T_{LI} - \eta_{KJ}T_{LM} - \eta_{LM}T_{KJ},
\]

\[
[S_{\alpha\beta}, S_{\gamma\delta}] = -\epsilon_{\alpha\gamma}S_{\beta\delta} - \epsilon_{\beta\delta}S_{\alpha\gamma} - \epsilon_{\alpha\delta}S_{\beta\gamma} - \epsilon_{\beta\gamma}S_{\alpha\delta},
\]

where \( \epsilon_{\alpha\beta} \) is the Levi-Civita symbol.
with η_{\ell J} being the SO(6, n)-invariant metric, and e^{+-} = 1 = e_{+-}.

The generators of the gauge algebra are

\[ \Xi_{aL} = \frac{1}{2} \left( \Theta_{aLMN} T^{MN} + \Theta_{aL\beta\gamma} S^{\beta\gamma} \right), \tag{3} \]

where \{Θ_{aLMN}, Θ_{aL\beta\gamma}\} \in (2, \text{Vec} \times \text{Adj}) + (2 \times 3, \text{Vec}) of SL(2, R) × SO(6, n) is the embedding tensor. Both electric and magnetic gauge transformations are included in this formalism. However, as we already emphasized, at most 6 + n \Xi's are independent. Put differently, the embedding tensor has maximal rank 6 + n and is therefore subject to constraints that we will now briefly discuss – for a comprehensive exposition, we recommend the already quoted literature.

Demanding supersymmetry leads to a set of linear constraints. This reduces the embedding tensor to (2, \text{Ant}_{[3]}^{[3]}) + (2, \text{Vec}):

\[ \Xi_{aL} = \frac{1}{2} \left( f_{aLMN} T^{MN} + η_{LQ} \xi_{aP} T^{QP} + e^{\gamma\alpha} ξ_{\beta\ell} S_{\gamma\alpha} \right), \tag{4} \]

where \(f_{aLMN} \equiv f_{a[LMN]}\) and \(\xi_{\beta\ell}\) are the irreducible blocks that define the embedding tensor. They allow for the complete determination of the gauge algebra and its commutators, the charges and covariant derivatives, the scalar potential and the mass matrices.

Since the vectors belong to the fundamental of Sp(12 + 2n, R), we must also impose that the latter contains the adjoint of the gauge algebra, and that this algebra closes. The minimal set of quadratic constraints that allow to fulfill these requirements is the following:

\[
\begin{align*}
\eta^{MN} \xi_{aM} \xi_{\beta N} & = 0 \\
\eta^{MN} \xi_{(aM f_{\beta} N)I} & = 0 \\
e^{\alpha\beta} \left( \xi_{\ell I} \xi_{\beta J} + \eta^{MN} \xi_{aM f_{\beta NIJ}} \right) & = 0 \\
\eta^{MN} f_{aMI} f_{f_{\beta KL} N} & = 1 \xi_{a[I} f_{\beta KL]I} \\
- \frac{1}{6} e^{\alpha\beta} e^{\gamma\delta} \xi_{\gamma I} f_{IJKL} + \frac{1}{2} \eta^{MN} \xi_{aM f_{\beta N[IK} \eta_{L]\gamma]} + \frac{1}{6} f_{a[KL} \xi_{\beta]} & = 0. \tag{5}
\end{align*}
\]

The ivth constraint is Jacobi-like. It should be stressed, however, that \(f_{aKL}\) are not necessarily structure constants of some algebra. The structure constants of the gauge algebra are given in general in terms of all gauging parameters i.e. \(f_{aKL}\) and \(\xi_{\beta\ell}\).

Most of the known solutions to the above set of constraints have vanishing \(\xi_{aL}\). In this

1Indices \(M, N, \ldots\) are lowered and raised with \(\eta_{\ell J}\) and \(\eta^{KM}\) (inverse matrix). With the present conventions for \(e_{a\beta}, e^{\gamma\alpha} e^{\gamma\delta} = \delta_\delta^{\alpha}\), we can raise and lower a-indices unambiguously as follows: \(A_a = A^\beta e_{\beta a}\) and \(B^\alpha = e^{a\beta} B_\beta\). This leads to \(A_+ = -A_-\) and \(A_- = A^+\). In particular, \(S_{++} = S^{--}, S_{+-} = -S^{-+}\) and \(S_{-+} = S^{-+}\).

2This means in particular that the components \(\Xi_{aLBM} \gamma^N\) of \(\Xi_{aL}\), as they can be read off from Eq. (4), are not entries of an antisymmetric \((12 + 2n) \times (12 + 2n)\) matrix. Consistency in gauge transformations is nevertheless ensured by the introduction of the auxiliary two-forms.
case only the iv\textsuperscript{th} constraint in Eqs. (5) survives. This “generalized Jacobi” identity can be solved by introducing \textit{de Roo–Wagemans phases} \cite{13} that relate the electric \((f_{+LMN})\) and magnetic \((f_{-LMN})\) gauging parameters \cite{12}. Hence, the solutions correspond to pure \(SO(6, n)\) gaugings, extensively studied in the literature, which lead to a variety of gauge algebras such as flat algebras. From a higher-dimensional perspective the latter turn out to be related to \textit{unimodular} Scherk–Schwarz reductions\cite{3} (see e.g. \cite{14}), in contrast to what we will be discussing in the following.

Our aim is here to elaborate on gaugings with non-vanishing parameters \(\xi_{\alpha L}\). Only a few isolated examples have been studied so far that fall in this class \cite{15,12} and their more systematic analysis will be the subject of the next chapters.

3 The axionic transformations and their gaugings

The axionic transformations are generated by a subgroup of the \(SL(2, \mathbb{R})\). The latter acts on the axion-dilaton as a Möbius transformation: \(\tau \rightarrow a\tau+b/\tau+c+d\). Since the axion-dilaton parameterizes the \(SL(2, \mathbb{R})/U(1)\) coset, we can define a \(2 \times 2\) matrix,

\[
(M^{\alpha \beta}) = \frac{1}{\text{Im} \tau} \begin{pmatrix}
1 & -\text{Re} \tau \\
-\text{Re} \tau & |\tau|^2
\end{pmatrix},
\]

where \(SL(2, \mathbb{R})\) acts linearly, with matrices \(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\). The generator of the genuine electric–magnetic duality is \(S^{--} = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}\), whereas axionic shifts \(\tau \rightarrow \tau + b\) and axionic rescalings \(\tau \rightarrow a^2 \tau\) are generated by \(S^{++} = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix}\) and \(S^{+-} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\), respectively.

It is clear from Eq. (4), that gauging the axionic symmetries requires an embedding tensor with \(\xi_{aM} \neq 0\). Such a gauging is necessarily accompanied by a partial gauging of \(SO(6, n)\).

As already advertised, our motivation is to proceed with electric gaugings, i.e. gaugings of the axionic shifts \(S^{++}\) and rescalings \(S^{+-}\) but not the electric–magnetic duality transformation \(S^{--}\). We must therefore set \(\xi_{-I} = 0\). Our further choice is \(f_{-LMN} = 0\). Although this is not compulsory for general electric gaugings, it simplifies considerably the quadratic constraints for \(\xi_{+I} f_{+LMN}\) (the “+” index will be dropped from now on). For simplicity, we focus on the case \(n = 6\) corresponding to 12 vectors in total, and use the light-cone-like convention: \(\{I\} \equiv \{i, i'\}\), \(\eta = \begin{pmatrix} 0 & 1 \\ \mathbb{I}_6 & 0 \end{pmatrix}\). \(\footnote{Flat groups were introduced in the reduction scheme proposed in \cite{16}, as the only solution to a double requirement: (i) scalar potential bounded from below and (ii) vanishing of the cosmological constant.} \)}}
A non-trivial solution to Eqs. (5) is captured by six real parameters, \( \{ \lambda_i, i = 1, \ldots, 6 \} \):

\[
\xi_i = \lambda_i, \quad \xi'_i = 0, \quad f_{ij} = f_{ji}, \quad f_{ijk} = -\lambda_i \delta_{jk}, \quad f_{ij} = f_{ij} = f_{ij} = 0. \tag{8}
\]

Several remarks are in order here. The gauging under consideration will be called “non-unimodular” for reasons that will become clear later, or “tracefull” since

\[
f_{ij} = -\frac{5}{2} \lambda_i. \tag{9}
\]

This is slightly misleading because the gauge algebra is traceless as a consequence of the full antisymmetry of its genuine structure constants. The latter are not \( f_{ij} \), which are not Lie-algebra structure constants, but specific combinations of \( f_{ijk} \) and \( \xi_i \), read off from the commutation relations of generators (4). We find in this way 8 independent generators out of \( 2 \times 12 \), as we see from the following:

\[
\begin{align*}
\Xi_{-i} &= \frac{\lambda_i}{2} S^{++} \equiv \lambda_i \Xi \\
\Xi_{-i'} &= 0 \\
\Xi_{+i} &= -\frac{\lambda_i}{2} \left( T_i^j + S^{+-} \right) \equiv \lambda_i Y \\
\Xi_{+i'} &= -\lambda_i T_i^j \equiv \Xi_i'
\end{align*} \tag{10}
\]

The commutation relations for \( \{ Y, \Xi, \Xi_i' \} \subset SL(2, \mathbb{R}) \times SO(6, 6) \) are

\[
\begin{align*}
[\Xi_{i'}, \Xi_{j'}] &= 0 \\
[\Xi, \Xi_{i'}] &= 0 \\
[\Xi_{i'}, Y] &= \Xi_i' \\
[\Xi, Y] &= -\Xi.
\end{align*} \tag{11}
\]

In contrast to the algebras obtained by standard Scherk–Schwarz reductions, the one under consideration is non-flat. The \( \{ Y, \Xi \} \) is the non-semi-simple subalgebra \( A_{2,2} \subset SL(2, \mathbb{R}) \) of axionic rescalings and axionic shifts. These axionic symmetries are gauged along with 6 Abelian generators \( \{ \Xi_i' \} \subset SO(6, 6) \), and \( \{ Y, \Xi_i' \} \) spans a seven-dimensional ideal. Notice finally that one could possibly introduce non-abelianity by switching on \( f_{ijk} \) provided \( f_{ijk} \lambda_{ij} = 0 \), as follows from (5).

## 4 Dynamics of the axionic gaugings

We would like now to discuss some dynamical aspects of the axionic gaugings. For this we need to describe the Lagrangian formulation of the system – including electric and mag-
netic components. Following [12], the bosonic sector of the Lagrangian associated with any consistent gauging, given in terms of the parameters \( f_{\alpha JK L} \) and \( \zeta_{\beta L} \), has three parts:

- \( \mathcal{L}_{\text{kin}} \): kinetic terms for graviton, electric vectors and scalars,
- \( \mathcal{L}_{\text{top}} \): auxiliary-field contributions (magnetic vectors and two-forms) necessary to maintain the correct number of propagating fields,
- the scalar potential:

\[
\mathcal{L}_{\text{pot}} = -\frac{e}{16} \left\{ f_{\alpha MNP} f_{\beta QRS} M^{\alpha \beta} \left( \frac{1}{3} M^{MQ} M^{NR} M^{PS} + \left( \frac{2}{3} \eta^{MQ} - M^{MQ} \right) \eta^{NR} \eta^{PS} \right) \right. \\
\left. - \frac{4}{5} f_{\alpha MNP} f_{\beta QRS} e^{\alpha \beta} M^{MNPQRS} + 3 \eta^M \zeta^N \eta^P \eta^R \right\},
\]  

(12)

where \( e \) is the vierbein, \( M^{\alpha \beta} \) is given in (6) and \( M^{MQ} \) parameterizes similarly the remaining \( 36 = 21 + 15 \) scalars of \( SO(6,6)/SO(6) \times SO(6) \):

\[
M^{MN} = \begin{pmatrix}
  h_{ij} & -h^{ik} b_{kj} \\
  b_{ik} h_{ij} & h_{ij} - b_{ik} h^{kl} b_{lj}
\end{pmatrix};
\]  

(13)

\( M^{MNPQRS} \) is another tensor build out of the 36 scalars (see [12] or [17] for precise expressions).

Working out the kinetic terms for the axionic gauging, it appears that among the original 12 vectors, 4 remain inert while \( 2 + 6 \) are embedded in \( SL(2, \mathbb{R}) \times SO(6,6) \) as generators of local symmetries – they enter in covariant derivatives acting on scalars. One can also obtain the specific scalar potential by inserting (9) into the general expression (12):

\[
\mathcal{L}_{\text{pot}} = \frac{1}{16} e^{2\phi} \lambda_i \left( 8h_{ij} - h^{ij} h^{kl} b_{km} h_{ln} b_{nk} + 2h_{ik} b_{km} h_{ln} b_{nr} h_{lj} \right) \lambda_j.
\]  

(14)

This is positive definite (as was the scalar potential in [15]).

The dynamics of the axion-dilaton requires a careful treatment. The kinetic term for this field is

\[
e^{-1} \mathcal{L}_{\text{kin;axion-dilaton}} = -D_\mu \phi D^\mu \phi - \frac{1}{4} e^{4\phi} D_\mu \chi D^\mu \chi,
\]  

(15)

where

\[
D_\mu \phi = \partial_\mu \phi - \frac{1}{2} Y_\mu \quad \text{and} \quad D_\mu \chi = \partial_\mu \chi + X_\mu + Y_\mu \chi
\]  

(16)

are the covariant derivatives involving the physical vectors

\[
Y_\mu = \lambda_i A^{i+}_\mu \quad \text{and} \quad X_\mu = \lambda_i A^{i-}_\mu.
\]  

(17)
It is worth stressing that these vectors are combinations of both electric and magnetic potentials. The gauging mixes the spurious and physical fields, keeping the number of propagating degrees of freedom fixed, though. The vectors $Y_\mu$ and $X_\mu$ are associated with the gauging of the symmetries generated by $Y$ (axion rescalings, $\chi \to a^2 \chi$, $\phi \to \phi - \log a$) and $\Xi$ (axion shifts, $\chi \to \chi + b$) respectively. Because of these local symmetries, the axion can be gauged away. In this process, $X_\mu$ acquires a mass via its Stückelberg coupling to $\chi$ and can be traded for a massive two-form $C_{\nu\rho}$.

The final bosonic content of the axionic gauging is as follows: the dilaton, $4 + 1 + 6$ vectors with Abelian algebra (4 inert, 1 associated with the axionic rescalings of $SL(2,\mathbb{R})$, 6 associated with maximal-Abelian-subalgebra transformations of $SO(6,6)$ -- translations), 1 massive two-form and 36 scalars minimally coupled to the $1 + 6$ vectors, with scalar potential (14). In general, depending on the specific values of the parameters $\lambda_i$, more scalars can be gauged away while vectors can simultaneously become massive thanks to their Stückelberg couplings. Some of the remaining scalars are massive, while other are massless.

5 The higher-dimensional origin: non-unimodular Scherk–Schwarz reduction

We will now perform a generalized dimensional reduction of heterotic ten-dimensional pure supergravity and show that the resulting effective theory belongs to the class of $N = 4$ four-dimensional gauged supergravities studied in Secs. 3 and 4.

The action of the heterotic ten-dimensional pure supergravity (bosonic sector) reads:

$$S = \int_{M_4} d^4x \int_{K_6} d^6y \sqrt{-G} e^{-\Phi} \left( R + G^{MN} \partial_M \Phi \partial_N \Phi - \frac{1}{12} H_{MNK} H^{MNK} \right),$$

(18)

where $\Phi$ is the dilaton, $G_{MN}$ the metric and $H = dB$ the NS-NS field strength, all in ten dimensions and in the sigma-model frame. Since $K_6$ is compact, from the four-dimensional viewpoint this action describes the dynamics of an infinitude of modes. A consistent reduction provides an effective theory on $M_4$ for a finite subset of modes. This requires a “good” choice for $K_6$ plus an ansatz for the $y$-dependence of all fields, which sets the mode-selection pattern. A necessary consistency condition is that $\mathcal{L}$ be $y$-independent.

For the standard reduction on a flat torus $T^6$, the ansatz is the absence of any $y$-dependence in the ten-dimensional fields. In this case, the bosonic spectrum consists of 1 graviton, $12 = 6 + 6$ Abelian vectors, $36 = 21 + 15$ scalars, 1 dilaton and 1 axion (dual to the NS–NS form), all massless and neutral.

The Scherk–Schwarz reduction [16] is an alternative that allows to generate (i) non-
Abelian gauge symmetries and corresponding charges, (ii) a scalar potential and (iii) a spontaneous breaking of (super)symmetries.

In ordinary Scherk–Schwarz reductions, the ansatz includes a precise $y$-dependance: around an internal cycle, the fields transform in a way that can be reabsorbed by the action of an internal symmetry. This is equivalent to the introduction of geometric (spin connection) fluxes $\gamma^i_{jk}$ appearing in the exterior differentials of the vielbeins,

$$d\theta^i = -\gamma^i_{jk} \theta^j \wedge \theta^k. \quad (19)$$

These fluxes satisfy the Bianchi identity,

$$\gamma^j_{[i[k} \gamma^j_{\ell m]} = 0, \quad (20)$$

and can alternatively be considered as structure constants

$$f^i_{jk} = 2\gamma^i_{jk} \quad (21)$$

of a group, the compact space being locally a group manifold. The identity (20) is therefore also a Jacobi identity. The unimodularity property,

$$\gamma^i_{ij} = 0 \quad (22)$$

is required for the consistency of the truncation (see e.g. [16, 20]). Many known examples fall in this class, which include non-semi-simple or semi-simple gauge groups – as the twisted tori leading to gaugings in $SO(6,6)$.

External Scherk–Schwarz reductions are more exotic. The symmetry which is used to guarantee the consistency of the $y$-dependence ansatz of the fields is external. A specific example is provided by the action (18), which is invariant under the following $SO(1,1)$ scaling symmetry:

$$\Phi \to \Phi + 4\lambda, \quad G_{MN} \to e^{\lambda}G_{MN}, \quad B_{MN} \to e^{\lambda}B_{MN}. \quad (23)$$

Using this “duality” symmetry, one can show that the following ansatz is consistent:

$$\Phi(x, y) = \Phi(x) + 4\lambda_i y^i, \quad G_{MN}(x, y) = e^{\lambda}G_{MN}(x), \quad B_{MN}(x, y) = e^{\lambda}B_{MN}(x). \quad (24)$$

The dynamics of this external Scherk–Schwarz reduction can be performed in detail using the following decomposition:

$$G_{MN} \to g_{\mu\nu}, A_{\mu k}, h_{ij}, \quad B_{MN} \to B_{\mu \nu}, B_{\mu k}, b_{ij}, \quad \phi = \Phi - \frac{1}{2} \log \det h. \quad (25)$$

---

5The reader is referred to [21] for a general discussion on reductions with duality twists.
Inserting (24) and (25) in the action (18), the \( y \)-dependence drops consistently and various new features emerge: (i) the vectors \( A_{\mu k} \) and \( B_{\mu k} \) carry Abelian gauge symmetry, (ii) the scalars \( h_{ij} \) are charged under \( A_{\mu k} \) with charges \( \lambda_k \), (iii) the scalars \( b_{ij} \) are charged under \( B_{\mu k} \) and \( \text{Stückelberg-coupled} \) to \( A_{\mu k} \), (iv) the four-dimensional dilaton \( \phi \) is \( \text{Stückelberg-coupled} \) to \( A_{\mu k} \) with charges \( \lambda_k \), (v) a scalar potential appears for \( h_{ij} \) and \( b_{ij} \).

After field redefinitions and integrations one vector drops and the two-form \( B_{\mu\nu} \) becomes massive, as a consequence of the \( \text{Stückelberg couplings} \). This is indicative of the gauging of a shift symmetry, and it is legitimate to compare the present “duality-twisted tori” reduction with the axionic gauging of four-dimensional \( N = 4 \) supergravity studied in last section. The matching of the Lagrangians is exact and the reduced theory at hand is precisely the gauged supergravity of Secs. 3 and 4.

6 Summary and outlook

In the framework of heterotic theory, the specific choice of generalized Scherk–Schwarz reduction, based on the ten-dimensional \( \text{SO}(1, 1) \) shift symmetry, Eq. (24), allows to turn on the four-dimensional gauging parameters \( \xi_i \) as ten-dimensional \( \text{SO}(1, 1) \) shift parameters \( \lambda_i \) along the torus one-cycles. This flux compactification is therefore equivalent to the gauging of the four-dimensional \( \text{SL}(2, \mathbb{R}) \) axionic shifts and rescalings, and makes it possible to evade the unimodularity property of more conventional geometric fluxes: as a consequence of (9) and (21),

\[
\gamma^j_{ij} = -\frac{5}{4}\lambda_i. \tag{26}
\]

As advertised, axionic gaugings are equivalent to non-unimodular geometric fluxes. It should be stressed that the distinction between unimodular (Eq. (22)) and non-unimodular (Eq. (26)) reductions is manifest in the sigma-model frame; in the Einstein frame all consistent reductions are unimodular. Nevertheless, this analysis elegantly demonstrates the power of the gauging procedure for describing diverse flux compactifications, and closes the chapter of characterizing a whole class of heterotic gaugings in terms of NS–NS and spin-connection fluxes. Further oxidation to M theory is also possible along the lines of [22, 23, 24].

Besides the precise relations that one can establish among ten-dimensional fluxes and four-dimensional gaugings, a fundamental and not yet unravelled question is the following: what are the geometrical features of the fundamental theory on the top that translate into the consistency constraints imposed to the embedding tensor from the bottom (Eqs. (5))? This question may not admit any answer, even in the framework of “generalized geometries”. Indeed, the analysis presented in [25, 26, 3, 4, 17] calls for further investigation of other classes of gaugings, related to the previous by duality transformations, and corresponding to possibly new fluxes. Following Sec. 2 the gauging parameters are \( f_{\alpha ljk}, \zeta_{aL} \). These are
464 real parameters, subject to the set of constraints (5). We can classify them according to their nature, using the light-cone-convention (7) and the subsequent splitting of the indices \( \{I\} \equiv \{i, i'\} \):

- \( f_{+ijk}, \xi_{+L} \): 232 electric parameters that include
  - the \( f_{+ijk} \)'s corresponding to NS–NS fluxes and the \( f_{+ijk'} \)'s which are spin-connection (unimodular and non-unimodular) fluxes,
  - their “non-geometric” counterparts: the \( f_{+ij'k} \)'s which are the T-dual NS–NS fluxes and the \( f_{+i'j'k} \)'s which are the T-dual spin-connection fluxes;

- \( f_{-ijk}, \xi_{-L} \): 232 magnetic-dual parameters which include similarly
  - the NS–NS- and spin-connection-like fluxes \( f_{-ijk} \) and \( f_{-ijk'} \),
  - their T-duals \( f_{-ij'k} \) and \( f_{-i'j'k} \).

We know that the parameters \( f_{+ijk} \), \( f_{+ijk'} \) and the corresponding \( \xi_{+i} \) have a clear higher-dimensional geometric interpretation as ten- or eleven-dimensional supergravity reductions. The other parameters can also be switched on though, leading to a well-defined gauged supergravity: the number of degrees of freedom is not altered but the algebra, its \( SL(2, \mathbb{R}) \times SO(6, n) \) embedding, the charges and the potential are. However, it is not clear that a higher-dimensional set up exists, which could reproduce all these gaugings upon dimensional reduction. This set up might simply not exist, or be a purely string-theory non-geometric vacuum, or some more exotic construction sitting between supergravity and string theory, like a double-torus compactification\(^{[27, 28, 29, 30, 31, 32]}\). Examples exist where this is indeed suspected to happen. These include four-dimensional \( N = 4 \) gauged supergravities that admit de Sitter vacua \(^{[33]}\) and are build using de Roo–Wagemans phases. As we pointed out in Sec. 2 de Roo–Wagemans phases are equivalent to switching on both electric (\( f_{+ijk} \)) and magnetic (\( f_{-ijk} \)) gauging parameters. Since it seems hard to oxidize de Sitter vacua to higher-dimensional supergravity \(^{[34]}\), we might conclude that the higher-dimensional origin of gauged supergravities with both electric and magnetic parameters is hard to achieve, without excluding its realization at the string level.

### Acknowledgements

The author would like to thank Jean-Pierre Derendinger and Nikolaos Prezas for nice collaboration, and Thomas Van Riet for stimulating scientific discussions during the Valencia meeting. This research was partially supported by the EU under contracts MEXT-CT-2003-509661, MRTN-CT-2004-005104 and MRTN-CT-2004-503369, by the French Agence Nationale pour la Recherche and by the Swiss National Science Foundation.
References

[1] M. Graña, “Flux compactifications in string theory: a comprehensive review”, Phys. Rept. 423 (2006) 91 [arXiv:hep-th/0509003].

[2] M. Trigiante, “Dual gauged supergravities”, arXiv:hep-th/0701218.

[3] J.-P. Derendinger, C. Kounnas, P.M. Petropoulos and F. Zwirner, “Superpotentials in IIA compactifications with general fluxes”, Nucl. Phys. B715 (2005) 211 [arXiv:hep-th/0411276].

[4] J.-P. Derendinger, C. Kounnas, P.M. Petropoulos and F. Zwirner, “Fluxes and gaugings: $N = 1$ effective superpotentials”, Fortsch. Phys. 53 (2005) 926 [arXiv:hep-th/0503229].

[5] J.-P. Derendinger, “Supergravity gaugings and moduli superpotentials”, Fortsch. Phys. 54 (2006) 366.

[6] C. Kounnas, D. Lüst, P.M. Petropoulos and D. Tsimpis, “AdS$_4$ flux vacua in type II superstrings and their domain-wall solutions”, JHEP 0709 (2007) 051 [arXiv:0707.4270 [hep-th]].

[7] B. de Wit, H. Samtleben and M. Trigiante, “On Lagrangians and gaugings of maximal supergravities”, Nucl. Phys. B655 (2003) 93 [arXiv:hep-th/0212239].

[8] B. de Wit, H. Samtleben and M. Trigiante, “Maximal supergravity from IIB flux compactifications”, Phys. Lett. B583 (2004) 338 [arXiv:hep-th/0311224].

[9] B. de Wit, H. Samtleben and M. Trigiante, “Gauging maximal supergravities”, Fortsch. Phys. 52 (2004) 489 [arXiv:hep-th/0311225].

[10] B. de Wit, H. Samtleben and M. Trigiante, “The maximal $D = 5$ supergravities”, Nucl. Phys. B716 (2005) 215 [arXiv:hep-th/0412173].

[11] B. de Wit, H. Samtleben and M. Trigiante, “Magnetic charges in local field theory”, JHEP 0509 (2005) 016 [arXiv:hep-th/0507289].

[12] J. Schön and M. Weidner, “Gauged $N = 4$ supergravities”, JHEP 0605 (2006) 034 [arXiv:hep-th/0602024].

[13] M. de Roo and P. Wagemans, “Gauge matter coupling in $N = 4$ supergravity”, Nucl. Phys. B262 (1985) 644.

[14] L. Andrianopoli, R. D’Auria, S. Ferrara and M. A. Lledo, “Gauging of flat groups in four dimensional supergravity”, JHEP 0207 (2002) 010 [arXiv:hep-th/0203206].
[15] G. Villadoro and F. Zwirner, “The minimal $N = 4$ no-scale model from generalized dimensional reduction”, JHEP 0407 (2004) 055 [arXiv:hep-th/0406185].

[16] J. Scherk and J.H. Schwarz, “How to get masses from extra dimensions”, Nucl. Phys. B153 (1979) 61.

[17] J.-P. Derendinger, P.M. Petropoulos and N. Prezas, “Axionic symmetry gaugings in $N = 4$ supergravities and their higher-dimensional origin”, Nucl. Phys. B785 (2007) 115 [arXiv:0705.0008 [hep-th]].

[18] F. Quevedo, “Duality beyond global symmetries: the fate of the $B_{\mu\nu}$ field”, arXiv:hep-th/9506081.

[19] P.K. Townsend, K. Pilch and P. van Nieuwenhuizen, Phys. Lett. 136B (1984) 38 [Addendum-ibid. 137B (1984) 443].

[20] C.M. Hull and R.A. Reid-Edwards, “Flux compactifications of string theory on twisted tori”, arXiv:hep-th/0503114.

[21] A. Dabholkar and C. Hull, “Duality twists, orbifolds, and fluxes”, JHEP 0309 (2003) 054 [arXiv:hep-th/0210209].

[22] G. Dall’Agata and N. Prezas, “Scherk–Schwarz reduction of M theory on $G_2$-manifolds with fluxes”, JHEP 0510 (2005) 103 [arXiv:hep-th/0509052].

[23] C.M. Hull and R.A. Reid-Edwards, “Flux compactifications of M theory on twisted tori”, JHEP 0610 (2006) 086 [arXiv:hep-th/0603094].

[24] C.M. Hull, “Generalised geometry for M theory”, JHEP 0707 (2007) 079 [arXiv:hep-th/0701203].

[25] J. Maharana and J.H. Schwarz, “Noncompact symmetries in string theory”, Nucl. Phys. B390 (1993) 3 [arXiv:hep-th/9207016].

[26] N. Kaloper and R.C. Myers, “The $O(d,d)$ story of massive supergravity”, JHEP 9905 (1999) 010 [arXiv:hep-th/9901045].

[27] C.M. Hull, “A geometry for non-geometric string backgrounds”, JHEP 0510 (2005) 065 [arXiv:hep-th/0406102].

[28] J. Shelton, W. Taylor and B. Wecht, “Nongeometric flux compactifications”, JHEP 0510 (2005) 085 [arXiv:hep-th/0508133].

[29] A. Dabholkar and C. Hull, “Generalised T-duality and non-geometric backgrounds”, JHEP 0605 (2006) 009 [arXiv:hep-th/0512005].
[30] C.M. Hull, “Doubled geometry and T-folds”, JHEP 0707 (2007) 080 [arXiv:hep-th/0605149].

[31] C.M. Hull and R.A. Reid-Edwards, “Gauge symmetry, T-duality and doubled geometry”, arXiv:0711.4818 [hep-th].

[32] G. Dall’Agata, N. Prezas, H. Samtleben and M. Trigiante, “Gauged supergravities from twisted doubled tori and non-geometric string backgrounds”, arXiv:0712.1026 [hep-th].

[33] M. de Roo, D.B. Westra and S. Panda, “De Sitter solutions in $N = 4$ matter coupled supergravity”, JHEP 0302 (2003) 003 [arXiv:hep-th/0212216].

[34] J.M. Maldacena and C. Nuñez, Int. J. Mod. Phys. A16 (2001) 822 [arXiv:hep-th/0007018].