HALL CONDUCTIVITY FOR TWO DIMENSIONAL MAGNETIC SYSTEMS

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A Kubo inspired formalism is proposed to compute the longitudinal and transverse dynamical conductivities of an electron in a plane (or a gas of electrons at zero temperature) coupled to the potential vector of an external local magnetic field, with the additional coupling of the spin degree of freedom of the electron to the local magnetic field (Pauli Hamiltonian). As an example, the homogeneous magnetic field Hall conductivity is rederived. The case of the vortex at the origin is worked out in detail. This system happens to display a transverse Hall conductivity ($P$ breaking effect) which is subleading in volume compared to the homogeneous field case, but diverging at small frequency like $1/\omega^2$. A perturbative analysis is proposed for the conductivity in the random magnetic impurity problem (Poissonian vortices in the plane). At first order in perturbation theory, the Hall conductivity displays oscillations close to the classical straight line conductivity of the mean magnetic field.

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1 Introduction : A classically free system

Consider a Poissonian distribution of infinitely thin vortices carrying a flux $\phi$, perpendicular to a plane (hereafter called magnetic impurities). One would like to ask about the way such a system could be equivalent to a homogeneous mean magnetic field, or, on the contrary, how it could localize electron wavefunctions, in such a way that their conductivity is altered. This question might seem academic at first sight. However, it is commonly believed that in Quantum Hall devices disorder does play a crucial role in the understanding of plateaus for the Hall conductivity as a function of $1/B$ or $N(E_F)$, the number of electrons, at integer (or fractional) values in unit of $e^2/h$. In the case of a homogeneous $B$ field, linear response of the system to a small electric field gives no hint of such a remarkable behavior, since all states are delocalised and have the same transverse conductivity which varies linearly with $1/B$ or $N(E_F)$ (classical straight line). Disorder is needed to explain why some states (in fact most of them) are localised in broadened Landau levels, thus the plateaus in

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the Hall conductivity, around the classical line. The random magnetic impurity model is a model where the disorder is contained in the definition of the magnetic field itself, and not added as an external random potential. After averaging over disorder, the model is translation invariant, as the usual Landau system. Translation invariance is at the origin of the infinite degeneracy of Landau levels, which in turn is at the root of the understanding of the macroscopic value of the Hall conductivity. We believe that the random magnetic impurity model encompasses at the same time generic features of a true Landau system (and in a certain limit both systems are indeed equivalent) and of a disordered system. Finally, it is in essence a topological model, without any classical counterpart. But topological considerations have been put forward in the past to explain why, in the Quantum Hall Effect, although most of the states are in principle localised, still the quantized Hall conductivity remains macroscopic.

It has been shown that two phases coexist in the random magnetic impurity model: a phase where the average density of states of a test particle (an electron) coupled to the random magnetic impurity distribution exhibits Landau level oscillations, and a phase with no oscillation. We refer to the former phase as an ordered phase, since the test particle "sees" a mean magnetic field irrespectively of disorder, and to the latter as a disordered phase, where the test particle "sees" impurities individually, and where disorder plays an important role. In the thermodynamic limit, the system is described by two parameters, \( \alpha = \phi/\phi_0 \), where \( \phi_0 \) is the flux quantum of the electron, and \( \rho \), the mean impurity density. The natural interval of definition of \( \alpha \) is \([-1/2, 1/2]\), since the system is unchanged when the fluxes are shifted by a multiple of the quantum of flux \( \alpha \to \alpha + 1 \). In the absence of a physical orientation to the plane, as coming from an exterior magnetic field for example, reversing the sign of the flux \( \alpha \to -\alpha \) does not change the system either. So, one can restrict the interval of definition of \( \alpha \in [0, 1/2] \). Note that the periodicity and the symmetry \( \alpha \to -\alpha \) in the spectrum imply that the partition function is unchanged when \( \alpha \to 1 - \alpha \), and therefore depends only on \( \alpha(1 - \alpha) \). For a Poissonian distribution, with magnetic impurities randomly dropped on the plane without correlations -the simplest distribution considering the impurities as a Bose gas at zero temperature and introducing no additional parameter in the model-, one finds that the partition function of the mean magnetic field \( \langle B \rangle = \hbar \rho \alpha \) is reproduced as a series in power of \( (\rho \alpha)^n \), i.e. in terms of the average problem at \( n \) impurities. This result is a little surprising since it amounts to say that a purely free classical system -an electron coupled to Aharonov-Bohm fluxes-leads, when quantized, to the quantum Landau problem. Strictly speaking, this limit is attained when \( \alpha \to 0, \rho \to \infty, \rho \alpha \) fixed, which is singular. It is
possible to organize in a systematic way perturbative corrections to the lead-
ing mean magnetic field expansion. It is sufficient to go at order $\rho^2 \alpha^4$, i.e. an
electron interacting 4 times with 2 impurities, to get a critical value $\alpha_c = 0.28$
below which one is certain that the average density of states has Landau level
oscillations.

What we are presently interested in is the transport properties of an elec-
tron, or a gas of electrons at zero temperature, coupled to random magnetic
impurities. We know for sure that in the mean magnetic field limit the con-
ductivity is the Landau conductivity (classical straight line for the transverse
conductivity). We would like to see if the corrections due to disorder are going
to produce oscillations -plateaus?- around the classical straight line. The pa-
per is organized as follows: in the first section we will present a Kubo inspired
formalism where the dynamical linear response of a general 2d Pauli magnetic
system to a small external electric field will be expressed in terms of the prop-
agators of the unperturbed Hamiltonian. We will in particular emphasize that
the time derivative of the dynamical conductivity is easier to compute than
the conductivity itself, due to the particular structure of the Pauli Hamil-
tonian. In the following section we will review, as a warm up exercise, the case
of a homogeneous magnetic field (Landau problem), examine the difficulties
inherent to the translation invariance of the system, and the zero width of the
Landau levels, and present a way to circumvent them. Next, we will con-
sider two examples of magnetic systems where the conductivity can be exactly com-
puted: i) one electron or a gas of electrons coupled to one vortex, ii) one
electron or a gas of electron coupled to a homogeneous magnetic field and one
vortex. These systems are of particular interest since they both are related to
the random magnetic impurity problem. In the next section, we will consider
the random magnetic impurity problem itself, and use perturbative methods
to get information on the Hall conductivity of this system. Finally, in the last
section a discussion will follow where general conclusions will be drawn from
the results obtained.

2 Conductivity for Pauli magnetic systems

By magnetic systems we mean in general the class of Hamiltonians for an elec-
tron minimally coupled to a vector potential $\vec{A}(\vec{r})$ with the additional coupling
of the electron spin up or down $\sigma_z = \pm 1$ to the local magnetic field $B(\vec{r})$ (we
set the electron mass $m_e = \hbar = 1$)

$$H = \frac{1}{2} \left( \vec{p} - e \vec{A}(\vec{r}) \right)^2 - \frac{eB(\vec{r})}{2} \sigma_z$$  \hspace{1cm} (1)
It rewrites
\[ \sigma_z = +1 \quad H_u = \frac{1}{2} \Pi_+ \Pi_+ \] \hspace{1cm} (2)
\[ \sigma_z = -1 \quad H_d = \frac{1}{2} \Pi_+ \Pi_- \] \hspace{1cm} (3)
where \( \Pi_\pm = (p_x - eA_x) \pm i(p_y - eA_y) \) are the covariant momentum operators.

In the homogeneous field case, the spin coupling is a trivial constant shift, but, in general, it has important effects. In the one vortex or magnetic impurity cases, it is a sum of \( \delta(\vec{r} - \vec{r}_i) \) functions, which is needed to define in a non ambiguous way the short distance behaviour of the wavefunctions at the location of the impurities \( \vec{r}_i \).

One would like to evaluate the linear response of such systems to a small homogeneous external electric field in the \( \vec{x} \) direction, i.e. to compute the longitudinal and transverse local conductivities \( \sigma_{xx}(\vec{r}, t) \) and \( \sigma_{yx}(\vec{r}, t) \). Conductivities characterize the non equilibrium dynamics of the system under the influence of an electric field. Linear response theory relates them to the equilibrium correlation functions of the Hamiltonian \( H \). In a time dependent formalism, one can start with an electric field \( \vec{E} = \delta(t) \vec{E}_o \), a single impulsion at initial time, to get a local current
\[ \vec{j}(\vec{r}) = \frac{e}{2} \{ \vec{v} | \vec{r} \rangle \langle \vec{r} | + | \vec{r} \rangle \langle \vec{r} | \vec{v} \} \] \hspace{1cm} (4)
(\( \vec{v} \) is the velocity operator \( \vec{v} = \vec{p} - e\vec{A} \)) proportional to the conductivity
\[ \sigma_{ij}(\vec{r}, t) = i\theta(t)e\text{Tr} \{ \rho [j_i(\vec{r}, t), r_j] \} \] \hspace{1cm} (5)
where \( \theta(t) \) is the Heaviside function. \( \text{Tr} \{ \rho \cdots \} \) is the thermal Boltzmann or Fermi-Dirac average. It can as well mean expectation value in a given quantum state if one wishes to compute the conductivity for an electron in this state, in which case \( \rho \) is the projector on this quantum state. \( j_i(\vec{r}, t) \) is the current density operator in the Heisenberg representation
\[ \vec{j}(\vec{r}, t) = e^{iHt} \vec{j}(\vec{r}) e^{-iHt} \] \hspace{1cm} (6)

We find convenient to compute conductivities in a propagator formalism. The propagator is defined as
\[ G_{\beta}(\vec{r}, \vec{r}') = \langle \vec{r} | e^{-\beta H} | \vec{r}' \rangle = \sum_n \varphi_n(\vec{r}) \varphi_n^*(\vec{r}') e^{-\beta E_n} \] \hspace{1cm} (7)
where \( \{ | \varphi_n \rangle \} \) is a complete set of eigenstates for the Hamiltonian \( H \). In the case of a thermal average for Boltzmann statistics, \( \rho \) is the usual density
operator $\rho = \exp(-\beta H)/Z_\beta$, with
\[
Z_\beta = \int d\vec{r} G_\beta(\vec{r}, \vec{r})
\] (8)
the partition function.

Consider the linear combinations
\[
\sigma^+(\vec{r}, t) = \sigma_{x\bar{x}}(\vec{r}, t) + i\sigma_{y\bar{x}}(\vec{r}, t), \quad \sigma^-(\vec{r}, t) = \sigma_{x\bar{x}}(\vec{r}, t) - i\sigma_{y\bar{x}}(\vec{r}, t)
\] (9)
The local thermal (Boltzmann) conductivity, \((\sigma \equiv \sigma^\pm)\)
\[
\sigma_\beta(\vec{r}, t) = \sum_n \sigma_n(\vec{r}, t)e^{-\beta E_n}/Z_\beta
\] (10)
where $\sigma_n(\vec{r}, t)$ is the local conductivity for one electron in the state $|\phi_n\rangle$, rewrites in terms of propagators
\[
\sigma^{\pm}_\beta(t) = \frac{1}{2i\theta(t)} \frac{e^2}{Z_\beta} \int d\vec{r}' \left( \Pi^{\pm}_\beta G_{it}(\vec{r}, \vec{r}')x' G_{-it}(\vec{r}', \vec{r}) + G_{it}(\vec{r}, \vec{r}')x'(\Pi^{\dagger\pm}_\beta)^* G_{-it}(\vec{r}', \vec{r}) - (it \to it + \beta) \right)
\] (11)
where $\Pi^{\dagger\pm}_\beta$ is the hermitian conjugate of $\Pi^{\pm}_\beta$. In (11) and in all formulae that follow, the differential operators $\Pi$ are always understood as acting on the $\vec{r}$ variable of the propagator that immediately follows them.

The local conductivity $\sigma_\beta(\vec{r}, t)$ leads upon space integration to the volume average global conductivity, for the global current density $\vec{j} = e\vec{v}/V$,
\[
\sigma^{\pm}(t) = i\theta(t) \frac{e^2}{V} \text{Tr} \{\rho[\Pi^{\pm}(t), x]\}
\] (12)
In terms of propagators it rewrites as
\[
\sigma^{\pm}_\beta(t) = \frac{1}{V} \int d\vec{r} \sigma^{\pm}_\beta(\vec{r}, t)
\] (13)
In the case of a system invariant by translation, the local conductivity $\sigma_\beta(\vec{r}, t)$ does not depend on $\vec{r}$, and both terms in (11) give the same contribution.

To deduce from (13) the conductivity of a gas of electrons at zero temperature and Fermi energy $E_F$, one uses the integral representation of the step function $\theta(E_F - H)$
\[
\sigma_{E_F}(t) = \lim_{\eta' \rightarrow 0^+} \int_{-\infty}^{\infty} \frac{dt'}{2i\pi t' - i\eta} Z_{\beta \rightarrow it' + \eta'} \sigma_{\beta \rightarrow it' + \eta'}(t)
\] (14)
where $\epsilon'$ and $\eta'$ are regulators which have to be set to zero at the end. The second-quantized conductivity rewrites as a sum of first quantized conductivities because, as defined in (3), it is a thermal average of a commutator of second quantized operators which are sums of first quantized operators. In the sequel we will refer to $\sigma_{EF}$ as given in (14) as the "Fermi transform" of $\sigma_\beta$.

One can rewrite (14) as

$$\frac{d\sigma_{EF}(t)}{dE} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt' e^{iE't} Z_{\beta \rightarrow \beta + \epsilon'} \sigma_\beta \rightarrow \beta + \epsilon'(t)$$

(15)

with

$$\sigma_{EF}(t) = \int_{0}^{E_F} \frac{d\sigma_{EF}(t)}{dE} dE$$

(16)

In (15), the integration starts at $E = 0$ since the spectrum is bounded from below at zero energy (Pauli Hamiltonian (1)).

In practice (Quantum Hall Effect devices), the number of electrons $N(E_F)$ may be fixed, implicitly determining $E_F$ by

$$N(E_F) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dt' e^{iE_F t'} Z_{it' + \epsilon'}$$

(17)

We also have for the density of states $\rho(E)$

$$\rho(E) \equiv \frac{dN(E)}{dE} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt' e^{iE't} Z_{it' + \epsilon'}$$

(18)

with accordingly

$$N(E_F) = \int_{0}^{E_F} \frac{dN(E)}{dE} dE$$

(19)

In general, it will be more convenient to calculate the derivative $\dot{\sigma}(t)$ of $\sigma(t)$ with respect to time, rather than $\sigma(t)$ itself. In the case of the thermal Boltzmann conductivity, one gets

$$\dot{\sigma}_\beta^\pm(t) = \frac{e^2}{V} \delta(t) \pm \theta(t) \frac{e^2}{2Z_\beta} \int d\vec{r} d\vec{r}' \left( e B(\vec{r}) \Pi_\pm G_{it}(\vec{r}, \vec{r}') x' G_{\beta - it} \Pi_\pm (\vec{r}', \vec{r}) -(it \rightarrow it + \beta) \right)$$

(20)

To derive (20), the identity

$$[H, \Pi_\pm] = \mp \left\{ eB(\vec{r}) \Pi_\pm + \frac{eB(\vec{r})}{2} \pm V(\vec{r}) \right\}$$

(21)
has been used, which is valid in general for an Hamiltonian $H = (\vec{p} - e\vec{A})^2/2 + V(\vec{r})$. But here, due to the particular structure of the Hamiltonian (1) where $V(\vec{r}) = -eB(\vec{r})\sigma_z/2$, we have

$$[H_u, \Pi_+] = -eB(\vec{r}) \Pi_+ \quad (22)$$

$$[H_d, \Pi_-] = eB(\vec{r}) \Pi_- \quad (23)$$

Therefore, (20) is valid if, when considering $H_u$, one computes $\dot{\sigma}^+$, and, when considering $H_d$, one computes $\dot{\sigma}^-$. In both cases, we wish to emphasize the appearance of the local magnetic field $B(\vec{r})$ in (20). In the homogeneous field case, it factors out from the space integrals. In the magnetic impurity case, it is a sum of $\delta(\vec{r} - \vec{r}_i)$ functions, greatly simplifying the space integrals.

We are looking at the Fourier transform of $\sigma(t)$ in frequency space, and more particularly at zero frequency $\omega = 0$ (i.e. the time independent or static response). The Fourier transform of $\sigma(t)$ is defined by

$$\sigma(\omega) = \lim_{\epsilon \to 0^+} \int_0^\infty dt \sigma(t) e^{(i\omega - \epsilon)t} \quad (24)$$

where $\epsilon$ is a regulator which has to be set to zero at the end. The Fourier transform of (20) rewrites

$$-(i\omega - \epsilon)\sigma_\beta^\pm(\omega) = e^2 V \left\{ 1 \pm \frac{1}{Z_\beta} \int_0^\infty dt e^{i(\omega - \epsilon)t} \int d\vec{r} d\vec{r}' \left( eB(\vec{r})\Pi \mp G_{it}(\vec{r}, \vec{r}')x'G_{it - \beta}(\vec{r}', \vec{r}) - (it \to it + \beta) \right) \right\} \quad (25)$$

and an analogous formula for the conductivity of a gas of electrons $\sigma_{E_F}^\pm(\omega)$

$$-(i\omega - \epsilon)\sigma_{E_F}^\pm(\omega) = e^2 V \left\{ N(E_F) \pm \frac{1}{2\pi i} \int_{-\infty}^\infty dt' \frac{e^{iE_Ft'}}{t' - i\eta} \int_0^\infty dt e^{i(\omega - \epsilon)t} \int d\vec{r} d\vec{r}' \left( eB(\vec{r})\Pi \mp G_{it}(\vec{r}, \vec{r}')x'G_{it + \epsilon - \beta}(\vec{r}', \vec{r}) - (it \to it + it' + \epsilon') \right) \right\} \quad (26)$$

Up to now, the Hamiltonian (1) has been used. However, one can simplify the expressions above by noting that a potential vector $\vec{A}(\vec{r})$, in 2d, can be always rewritten up to a gauge as

$$eA_i(\vec{r}) = \epsilon_{ij} \partial_j \phi(\vec{r}) \quad (27)$$

with the covariant momentum

$$\Pi_+ = -2i(\partial_x + \partial_y \phi) \quad (28)$$

$$\Pi_- = -2i(\partial_x - \partial_y \phi) \quad (29)$$
Let us redefine the wavefunctions as
\[ \psi(\vec{r}) = U(\vec{r}) \tilde{\psi}(\vec{r}) \]  
where the non unitary transformation \( U(\vec{r}) \) respectively read in the spin up and down cases
\[ U_u(\vec{r}) = e^{-\phi(\vec{r})} \]  
\[ U_d(\vec{r}) = e^{+\phi(\vec{r})} \]

One has
\[ U_u^{-1} \Pi_u U_u = \Pi_0^+ \]  
\[ U_u^{-1} \Pi_u U_u = \Pi_0^+ + 4i \partial_z \phi \]  
\[ U_d^{-1} \Pi_u U_d = \Pi_0^+ - 4i \partial_z \phi \]  
\[ U_d^{-1} \Pi_u U_d = \Pi_0^+ \]

where the potential vector disappears from the covariant \( U_u^{-1} \Pi_u U_u = \Pi_0^+ \) and \( U_d^{-1} \Pi_u U_d = \Pi_0^- \) operators, which narrow down to the free covariant momentum operators \( \Pi_0^+ = -2i \partial_z \) and \( \Pi_0^- = -2i \partial_z \). The Hamiltonian \( \tilde{H} = U^{-1} H U \) acting on \( \tilde{\psi} \) rewrites
\[ \tilde{H}_u = -2 \partial_z \partial_{\bar{z}} + 4 \partial \phi \partial_z \]  
\[ \tilde{H}_d = -2 \partial_z \partial_{\bar{z}} - 4 \partial \phi \partial_z \]

The propagator for the Hamiltonian \( H \) being defined in \( G_\beta(\vec{r}, \vec{r}') = \langle \vec{r}' | e^{-\beta H} | \vec{r} \rangle \), one has
\[ \tilde{G}_\beta(\vec{r}, \vec{r}') \equiv \langle \vec{r}' | e^{-\beta \tilde{H}} | \vec{r} \rangle = \frac{U(\vec{r}')}{U(\vec{r})} G_\beta(\vec{r}, \vec{r}') \]

It follows that the Hamiltonians \( H_u, \tilde{H}_u, \) on the one hand, \( H_d, \tilde{H}_d, \) on the other hand, are equivalent, and can be indifferently used for computing thermodynamical observables, i.e. traces of product of operators, as the partition function, the density of states, or the conductivity. The partition function rewrites as
\[ Z_\beta = \int d\vec{r} \tilde{G}(\vec{r}, \vec{r}) \]

and the conductivity becomes
\[ \sigma_\beta^\pm(t) = \frac{e^2}{V} \theta(t) \frac{1}{Z_\beta} \int d\vec{r} d\vec{r}' \left\{ U^{-1}(\vec{r}) \Pi_\pm U(\vec{r}) \tilde{G}_{it}(\vec{r}, \vec{r}') x' \tilde{G}_{-it}(\vec{r}', \vec{r}) - (it \to it + \beta) \right\} \]
\[ \sigma_\beta^\pm(t) = \frac{e^2}{V} \delta(t) \pm \frac{e^2}{V} \theta(t) \frac{1}{Z_\beta} \int d\vec{r} d\vec{r}' \left( eB(\vec{r}) U^{-1}(\vec{r}) \Pi_\pm U(\vec{r}) \tilde{G}_{\beta}(\vec{r}, \vec{r}') \tilde{G}_{\beta}(\vec{r}', \vec{r}) \right) \]

\[ - (it \to it + \beta) \]  

(42)

It is remarkable that the covariant momentum operator \( U^{-1}(\vec{r}) \Pi U(\vec{r}) \) appearing in \( \sigma \) or \( \dot{\sigma} \) in (41, 42) reduces to the free momentum provided that one considers \( \sigma^+ \) in the spin up case, and \( \sigma^- \) in the spin down case. It follows that it is appropriate, using \( \tilde{H} \), to compute \( \sigma^+ \) in the spin up case, and \( \sigma^- \) in the spin down case. In the sequel, we will essentially concentrate on the spin down coupling. However, one should keep in mind that computations for the spin up case should follow the same lines.

One finally extracts the real part of the transverse and longitudinal conductivity

\[ \text{Re} \sigma_{xx}(\omega) \pm i \text{Re} \sigma_{yx}(\omega) = \frac{\sigma^+ (\omega) + \sigma^+ (-\omega)}{2} \]  

(43)

(43) is a consequence of (4) and is valid both for the Boltzmann (23) or Fermi (24) cases.

### 3 Hall conductivity for a homogeneous magnetic field

As a warm up exercise, let us rederive the Hall conductivity \( \sigma_{EF}(\omega = 0) \) in the homogeneous magnetic field case. By convention, and without any loss of generality, we assume \( eB > 0 \), and denote by \( \omega_c = +eB/2 \) half the cyclotron frequency. The Landau Hamiltonian is

\[ H_L^u = \frac{1}{2} \Pi_L^u \Pi_L^+ = \frac{1}{2} \left( \vec{p} - \omega_c \vec{k} \times \vec{r} \right)^2 - \omega_c \]  

(44)

\[ H_L^d = \frac{1}{2} \Pi_L^d \Pi_L^+ = \frac{1}{2} \left( \vec{p} - \omega_c \vec{k} \times \vec{r} \right)^2 + \omega_c \]  

(45)

where \( \Pi_L^u = -2i(\partial_z - \frac{i}{2} \omega_c \tilde{z}) \) is the covariant Landau momentum. The constant spin coupling shift \( \mp \omega_c \) has clearly no influence on the conductivity. The shifted Landau propagator reads \((-(-))\) corresponds to spin \( u (d)\)

\[ G_L^\beta(\vec{r}, \vec{r}') = \frac{\omega_c e^{+ \beta \omega_c}}{2\pi \sinh(\beta \omega_c)} e^{-\frac{1}{2} (|z - z'|^2 \coth(\beta \omega_c) + \tilde{z} \tilde{z}' - \tilde{z}' \tilde{z})} \]  

(46)

Accordingly, the shifted partition function is

\[ Z_L^\beta = V \frac{\omega_c e^{+ \beta \omega_c}}{2\pi \sinh(\beta \omega_c)} \]  

(47)
Its Fermi transform is the Landau density of states, namely in the spin up case

\[ \rho^L(E) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt' e^{iEt'} Z_{it'+c}^L = V \frac{\omega_c}{\pi} \sum_{n=0}^{\infty} \delta(E - 2n\omega_c) \]  

(48)

and in the spin down case

\[ \rho^L(E) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt' e^{iEt'} Z_{it'+c}^L = V \frac{\omega_c}{\pi} \sum_{n=0}^{\infty} \delta(E - 2(n+1)\omega_c) \]  

(49)

As announced above, let us concentrate on the spin down case, i.e. on \( \sigma^- \) (with a homogeneous magnetic field, computing \( \sigma^+ \) would lead in the spin up case to the same result). To compute the thermal Boltzmann conductivity \( \Pi^L \), using

\[ \Pi^L G^L_{\beta} (\vec{r}, \vec{r}') = i\omega_c (\bar{z} - \bar{z}') (\coth(\beta\omega_c) + 1) G^L_{\beta} (\vec{r}, \vec{r}') \]  

(50)

one has to evaluate space integrals like

\[ \frac{1}{V} \int d\vec{r} d\vec{r}' (\bar{z} - \bar{z}') G^L_{\beta it} (\vec{r}, \vec{r'}) x' G^L_{\beta -it} (\vec{r'}, \vec{r}) \]  

(51)

Since, in the Landau case, \( G^L_{\beta it} (\vec{r}, \vec{r'}) G^L_{\beta -it} (\vec{r'}, \vec{r}) \) is a function of \( |z - z'| \) only, \( (51) \) might seem naively ill defined (infinite by power counting, vanishing by symmetry). Still, one can proceed if, looking at \( (11) \), one pays attention to first perform the \( \vec{r}' \) integral, leading to the local conductivity \( \sigma^L_{\beta} (\vec{r}, t) \), which upon integration on \( \vec{r} \), yields the global conductivity. In the homogeneous magnetic field case, the \( \vec{r}' \) integration in \( (51) \) gives, as it should, a result which is independent of \( \vec{r} \), the manifestation of the translation invariance of the system. The remaining \( \vec{r} \) integral simply factorizes a volume factor, which gives an unambiguous meaning to the divergence in \( (51) \). One finds

\[ \sigma^L_{\beta} (t) = \theta(t) \frac{e^2}{V} e^{2i\omega_c t} \]  

(52)

We would like to emphasize that \( (52) \) can be trivially recovered if one now looks at \( \dot{\sigma}^L_{\beta} (t) \), instead of \( \sigma^L_{\beta} (t) \). Indeed, in \( (20) \) \( B \) factors out from the space integrals. So, \( (20) \) rewrites as

\[ \dot{\sigma}^L_{\beta} (t) = \frac{e^2}{V} \delta(t) + 2i\omega_c \sigma^L_{\beta} (t) \]  

(53)

which can be directly integrated to \( (52) \), without any need of properly defining, in the thermodynamic limit, divergent space integrals.
In principle, one proceeds by taking the Fourier and Fermi transforms of (52). The former reads
\[
\sigma_L^- (\omega) = \frac{e^2}{V} \left\{ \mathcal{P} \frac{i}{\omega + 2\omega_c} + \pi \delta(\omega + 2\omega_c) \right\}
\]
and the latter rewrites
\[
\sigma_{E_F}^L (t) = \theta(t) \frac{e^2}{V} e^{2i\omega_c t} \int_{-\infty}^\infty dt' e^{iE_F t'} Z_{it'+\epsilon'}
\]
which yields the infinite series \( \sum_n \theta(E_F - 2\omega_c(n + 1)) \). But this series is meaningless, because \( E_F \) is not defined due to the zero width of the Landau levels.

However, coming back to (52) or (54), one sees that the thermal conductivity does not depend on \( \beta \), implying that the conductivity is the same for each Landau state. It follows that, for a gas of electrons at zero temperature, the conductivity is obtained by multiplying (52) or (54) by \( N(E_F) \). From (43), one infers the transverse conductivity
\[
\text{Re} \sigma_{E_F}^L (\omega) |_{yx} = -N(E_F) \frac{e^2}{V} \frac{2\omega_c}{4\omega_c^2 - \omega^2}
\]
In (56), the limit \( \omega_c \to 0 \) is properly defined only if one keeps \( \omega \neq 0 \), in which case it vanishes, as it should. The Hall conductivity finally reads
\[
\text{Re} \sigma_{E_F}^L (\omega = 0) |_{yx} = -N(E_F) \frac{e}{V B}
\]
This is the classical straight line, showing no plateaus in the Hall conductivity as a function of the number of electrons \( N(E_F) \), or of the inverse magnetic field \( 1/B \).

The longitudinal conductivity is
\[
\text{Re} \sigma_{E_F}^L (\omega) |_{xx} = N(E_F) \frac{e^2 \pi}{V} \left\{ \delta(\omega + 2\omega_c) + \delta(\omega - 2\omega_c) \right\}
\]

4 Hall conductivity for one vortex and for one vortex + a homogeneous \( B \) field

On the one hand, these two non trivial examples of magnetic systems are interesting because their conductivities can be entirely computed, thanks to the local magnetic field \( \delta(r) \) functions in the space integrals of \( \sigma_\beta(t) \). On the
other hand, they are respectively related, in the random magnetic impurity problem, to the conductivity of the mean magnetic field, and to perturbative corrections at first order in $\alpha$ to the conductivity of the mean magnetic field.

As in the Landau case, we set $eB > 0$, thus $\omega_c = +eB/2$. We consider a vortex at the origin with flux $\phi$. The coupling of the electron to the vortex is $e\phi/2\pi = \alpha$. This system is periodic in $\alpha$ of period 1, therefore $\alpha$ can always be chosen in the interval $[-1/2, 1/2]$. If $B = 0$, the interval of definition of $\alpha$ can be restricted to $[0, 1/2]$, since reversing the sign of the flux does not change the system. If $B \neq 0$, $\phi$ can be either parallel ($\alpha > 0$), or antiparallel ($\alpha < 0$) to the magnetic field, which imposes an orientation to the plane. Clearly, the physics depends on the sign of $\alpha$. We will focus on the physical situation where the flux and the magnetic field are parallel $\alpha \in [0, 1/2]$. Indeed, this is what happens in the random magnetic impurity problem, where the mean magnetic field $\langle B \rangle$ is built by the vortices $\phi$ carried by the impurities.

In the symmetric gauge, the Hamiltonian $H_d$ reads

$$H_d = \frac{1}{2} \left( \vec{p} - \omega_c \vec{k} \times \vec{r} - \alpha \frac{\vec{k} \times \vec{r}}{r^2} \right)^2 + (\pi \alpha \delta(\vec{r}) + \omega_c)$$

(59)

where $\vec{k}$ is the unit vector perpendicular to the plane.

The coupling of the electron to the infinite magnetic field inside the vortex defines in a non ambiguous way the short distance behaviour of the electron wavefunctions at the origin. This need for a proper characterisation of boundary conditions at the location of the vortex is related to the fact that perturbation theory in $\alpha$ is ill defined for the Hamiltonian (59) in the zero orbital momentum $m = 0$ sector. In fact, wavefunctions of zero orbital momentum $m = 0$ are affected by the short distance regularisation $\pi \alpha \delta(\vec{r})$. In the case of the spin down Hamiltonian $H_d$, the contact term is repulsive, indicating that the wavefunctions have to vanish when $r \to 0$ as quickly as $r^\alpha$. It follows that in the $m = 0$ sector, only regular wavefunctions at the origin should be retained, as in the standard Aharonov-Bohm prescription. It means that the electron cannot penetrate inside the vortex, a quite reasonable physical situation.

4.1 Partition function and density of states

The free and Landau partition functions diverge like the volume in the thermodynamic limit (continuous spectra), and corrections due to the vortex are subleading in volume. To give an unambiguous meaning to these corrections, one needs to regularize infrared divergences. This can be achieved by adding
to the Hamiltonian an harmonic regulator, $\frac{1}{2}\omega_o r^2$, which has to be set to zero $\omega_o \to 0$ at the end. The Hamiltonian (59) with the additional harmonic term has the spectrum

$$E_{n,m} = (2n + |m - \alpha| + 1)\omega_t - (m - \alpha)\omega_c + \omega_c$$  \hspace{1cm} (60)

$$E_{n,0} = (2n + \alpha + 1)\omega_t + \alpha\omega_c + \omega_c$$  \hspace{1cm} (61)

where $\omega_t^2 = \omega_c^2 + \omega_o^2$. The partition function rewrites

$$Z(\omega_o \beta) - Z(\beta) = e^{-\beta\omega_c} \left\{ \frac{1}{2} - e^{-\beta(\omega_t + \omega_c)} - e^{\beta(\omega_t - \omega_c)} - 1 \right\}$$  \hspace{1cm} (62)

The thermodynamic limit, $\omega_o \to 0$, i.e. $\omega_t - \omega_c \to 0$, yields

$$Z(\beta) - Z(0) = \frac{e^{-\beta\omega_c}}{2\sinh(\beta\omega_c)} \left\{ \alpha + \frac{\sinh(\beta\omega_c)}{\cosh(\beta\omega_c)} \right\}$$  \hspace{1cm} (63)

The Fermi transform of (63) gives the density of states

$$\rho(E, B, \alpha) - \rho(E, B, 0) = \sum_{n=0}^{\infty} \left( (n + 1 - \alpha) \delta(E - 2(n + 1)\omega_c) + \sum_{n=0}^{\infty} (n + 1) \delta(E - 2(n + 1 + \alpha)\omega_c) \right)$$  \hspace{1cm} (64)

In the limit $B \to 0$, (63) becomes

$$Z(0, \alpha) - Z(0, 0) = \frac{\alpha(\alpha - 1)}{2}$$  \hspace{1cm} (65)

and accordingly

$$\rho(E, 0, \alpha) - \rho(E, 0, 0) = \frac{\alpha(\alpha - 1)}{2} \delta(E)$$  \hspace{1cm} (66)

i.e. the usual Aharonov-Bohm depletion of states at the bottom of the spectrum with respect to the free density of states $\rho(E, 0, 0) \equiv \rho^0(E)$.

The physical meaning of (64) is that on each Landau level $E_n = 2(n+1)\omega_c$, $n + 1 - \alpha$ states states disappear and $n + 1$ appear at energy $2(n + \alpha + 1)\omega_c$. When $B = 0$, the sole effect of the vortex in (66) is that $\alpha(1 - \alpha)/2$ states have left the bottom of the spectrum.
4.2 Hall conductivity for one vortex

Let us first concentrate on the vortex alone. The propagator of (59) with $B = 0$ reduces to the standard Aharonov-Bohm propagator

$$G_{\beta}(\vec{r}, \vec{r}') = \frac{1}{2\pi \beta} e^{\frac{-\|\vec{r} - \vec{r}'\|^2}{\pi \beta}} \sum_{m=-\infty}^{+\infty} I_{|m|} \left( \frac{\|\vec{r} - \vec{r}'\|}{\beta} \right) e^{im(\theta - \theta')}$$

(67)

where the $I_m(z)$’s are the modified Bessel functions.

One expects that the Hall conductivity is antisymmetric with respect to $\alpha \to 1 - \alpha$, because of the antisymmetry $\alpha \to -\alpha$ (the transverse conductivity has to change its sign when the direction of the local magnetic field is reversed) and the periodicity. In particular it has to vanish when $\alpha = 1/2$. One has to compute (20), where $eB(\vec{r}) = 2\pi \alpha \delta(\vec{r})$ and $\Pi_- = -i(2\partial_z - \alpha/z)$. The $\delta(\vec{r})$ appearing in $\dot{\sigma}_{\beta}(-\alpha)$ greatly simplifies a computation otherwise untractable. Indeed, only the $m = 0$ and $m = 1$ orbital terms in the propagators eventually survive to yield

$$\dot{\sigma}_{\beta}(t) = \frac{e^2}{V} \delta(t) + \frac{e^2}{V} \theta(t) \frac{1}{\beta^2 Z_{\beta}} e^{i\pi \alpha} \sin(\pi \alpha) \pi (t^\alpha (t+i\beta)^{1-\alpha} - t^{1-\alpha} (t-i\beta)^\alpha)$$

(68)

Its Fourier transform reads

$$\sigma_{\beta}(\omega) = \frac{e^2}{V} \frac{i}{\omega + i\epsilon} \left\{ 1 - \frac{1}{Z_{\beta}} e^{i\pi \alpha} \sin(\pi \alpha) \pi \left( \Gamma(1 + \alpha) \Psi(1 + \alpha, 1; \beta(\omega + i\epsilon)) - \Gamma(2 - \alpha) \Psi(2 - \alpha, 1; -\beta(\omega + i\epsilon)) \right) \right\}$$

(69)

where the $\Psi(a, b; z)$’s are the unregular confluent hypergeometric functions. To get the Hall conductivity $\text{Re} \sigma(\omega)|_{yx}$ in the $\omega \to 0$ limit (static response), special attention must be paid to the $\epsilon \to 0^+$ limit, because of the logarithm that appears in the low $\beta \omega$ expansion of $\Psi(a, b; \beta(\omega + i\epsilon))$. One obtains (the electron mass, $m_e$, and $\hbar$ have been reintroduced)

$$\text{Re} \sigma_{\beta}(\omega)|_{yx} = \frac{\hbar e^2}{m_e^2 V^2} \frac{\sin(2\pi \alpha)}{\omega^2} \left( 1 + \frac{\alpha(1 - \alpha)}{2} (\beta \omega)^2 \ln(\beta \omega) + \cdots \right)$$

(70)

Thus, a single vortex in the plane is sufficient to display a Hall conductivity, an explicit $P$ violating effect (see and references therein for early tentatives on the subject). Of course, it is subleading in volume $\sigma \approx 1/V^2$ when compared to the homogeneous $B$-field case $\sigma \approx 1/V$. However, when, in the random magnetic impurity problem, comparaison will be made with the mean magnetic field,
the $1/V^2$ factor will be multiplied by the number of impurities $N$, yielding, in the thermodynamic limit, an adequate $\rho/V$ behavior. Also worth mentioning is the leading $1/\omega^2$ divergent behaviour of (70) in the $\omega \to 0$ limit, which will also be related to the dynamical conductivity of the mean magnetic field.

For a gas of electrons coupled to the vortex at zero temperature, one has to compute

$$-(i\omega - \epsilon)\sigma_{E_F}(\omega) = \frac{e^2}{V} \left\{ N(E_F) + \frac{1}{2\pi i} \int_\infty^{-\infty} dt' \frac{e^{iE_F t'}}{t' - i\eta} \int_0^\infty dt e^{(i\omega - \epsilon)t} \frac{1}{(it' + \epsilon)^2} \right\}$$

$$(71)$$

At small frequency, one should look at the leading high $t$ behaviour of (68) in which $\beta$ has been replaced by $it' + \epsilon'$. Since $t^\alpha (t - t' + i\epsilon')^{1-\alpha} - t^{1-\alpha} (t + t' - i\epsilon')^{\alpha} \approx -t'$, one finds a conductivity proportional to $N(E_F)$

$$\sigma_{E_F}(\omega \to 0) \approx N(E_F) \frac{e^2}{V} \left\{ \frac{1}{\epsilon - i\omega} + \frac{1}{V} 2ie^{i\pi\alpha} \sin(\pi\alpha) \frac{1}{(\epsilon - i\omega)^2} \right\}$$

$$(72)$$

Thus, in the limit $\omega \ll E_F$,

$$\text{Re} \sigma_{E_F}(\omega)|_{yx} = N(E_F) \frac{e^2}{V^2} \sin(2\pi\alpha) \frac{1}{\omega^2}$$

$$(73)$$

and

$$\text{Re} \sigma_{E_F}(\omega)|_{xx} = N(E_F) \frac{e^2}{V^2} 2 \sin^2(\pi\alpha) \frac{1}{\omega^2}$$

$$(74)$$

### 4.3 Hall conductivity for one vortex + a homogeneous magnetic field

The propagator of the Hamiltonian (59) (spin down, $\alpha \in [0, 1/2]$) is

$$G_\beta(\vec{r}, \vec{r}') = \frac{\omega_c e^{-\beta\omega_c}}{2\pi \sinh \beta\omega_c} e^{-\frac{\omega_c}{\beta} \cosh \beta \omega_c (r^2 + r'^2)} \sum_{m=-\infty}^{+\infty} I_{|m-\alpha|} \left( \omega_c \beta \omega_c \right) e^{i\omega_c (m-\alpha) e^{i\theta - \theta'}}$$

$$(75)$$

As in the one vortex case, and exactly for the same reason, only terms of angular momentum $m = 0$ and $m = 1$ survive in $\hat{\sigma}_\beta(t)$

$$\hat{\sigma}_\beta(t) = \frac{e^2}{V} \delta(t) + 2i\omega_c \sigma_\beta(t) + \frac{e^2}{V} \theta(t) \frac{\omega_c e^{-\beta\omega_c (\alpha + 1)}}{Z_\beta \sinh^2(\beta\omega_c)} \frac{\sin(\pi\alpha)}{\pi} G(t)$$

$$(76)$$

where

$$G(t) = \theta(t) \left\{ e^{it\omega_c (\sinh \beta - it)\omega_c} (\sinh(\beta - it)\omega_c)^{1-\alpha} - (it \rightarrow it + \beta) \right\}$$

$$(77)$$
One can check that the limit $\omega_c \to 0$ of $\dot{\theta}_\beta(t)$ in (76) gives (78), as it should.

Its Fourier transform is

$$
\sigma^-_\beta(\omega) = \frac{1}{\epsilon - i(\omega + 2\omega_c)} e^2 \left\{ 1 + i \frac{1}{Z_{\beta}} \frac{\omega_c e^{-\beta \omega_c (\alpha + 1)} \sin(\pi \alpha)}{\sinh^2(\beta \omega_c)} \right\} G(\omega)
$$

(78)

where $G(\omega)$ is the Fourier transform of $G(t)$. The thermal conductivity, splits in two terms, the usual Landau term for the magnetic field, and a term due to the vortex alone. Since $G(t)$ is periodic, one expects the conductivity to be regular at low frequency. One has

$$
G(\omega) = \frac{1}{1 - e^{2\pi \omega/\omega_c}} \int_0^{2\pi/\omega_c} dt G(t) e^{i\omega t}
$$

(79)

Since $G(t)^\ast = G(-t)$, $G(\omega)$ is purely imaginary. It follows that, in (78), no contribution to the real part of the longitudinal conductivity has to be expected from the vortex. Furthermore, $G(\omega)$ is finite when $\omega \to 0$, since

$$
\int_0^{2\pi/\omega_c} dt G(t) = 0
$$

These two points are to be contrasted with the pure vortex case where the longitudinal and transverse conductivities have been shown to diverge at low frequency (73,74). It means that, as in the pure Landau case, the limit $\omega_c \to 0$ is not smooth when $\omega \to 0$. In other words, if one wants to recover (73,74) from the present analysis, one should, while keeping $\omega \neq 0$, take the limit $\omega_c \to 0$, and then consider the small $\omega$ expansion. Here, we are considering the other situation where $\omega \to 0$, while $\omega_c$ is kept finite.

One is interested by the conductivity for a gas of electrons, i.e. the Fermi transform of (78), where $\beta$ has to be replaced by a complex time $\beta = it' + \epsilon'$. It happens that the small $\alpha$ expansion of $G(\omega = 0)$ when $\beta$ is imaginary has no term in $\alpha$

$$
G(\omega = 0) = i \frac{1}{2\omega_c} \sinh(i \omega_c t') + O(\alpha^2) + \ldots
$$

(80)

Note, as a remark, that, when $\alpha = 0, G(\omega) = i \frac{1}{2\omega_c} \sinh(i \omega_c t')$.

It follows that the small $\alpha$ expansion of the Fermi transform of the static thermal conductivity (8) reads

$$
\sigma^-_{Fr}(\omega = 0) = \int_{-\infty}^{+\infty} \frac{dt}{2i\pi} e^{iE_F t'} \left\{ Z_{i\nu} + \frac{Z_{iL}}{V} \left( -\frac{\pi \alpha}{\omega_c} + \pi \alpha^2 i t' + \ldots \right) \right\}
$$

(81)

where the partition function $Z_{i\nu}$ given in (8) should also be understood as expanded at first order in $\alpha$. It is not surprising that in (81), the first order correction to the conductivity appears as a mean magnetic field contribution.
of the vortex $\pi \alpha / V$ to the magnetic field $\omega_c$

$$\sigma_{EF}^{\omega = 0} = N(E_F) \frac{i e^2}{2 V} \left( \frac{1}{\omega_c} - \frac{\pi \alpha}{V \omega_c^2} \right) + \cdots = N(E_F) \frac{i e^2}{2 V} \omega_c + \frac{\pi \alpha / V}{\omega_c} + \cdots$$

(82)

to be compared to the Landau conductivity (54).

5 Hall conductivity for the magnetic impurity problem

5.1 The magnetic impurity problem

Let us now consider an electron coupled to the vector potential of a random distribution $\{ \vec{r}_i, i = 1, 2, \ldots, N \}$ of $N$ vortices at position $\vec{r}_i$, carrying a flux $\phi$, hereafter called magnetic impurity. The thermodynamic limit is $N \to \infty, V \to \infty$, with a finite impurity density $\rho = N / V$. The distribution is Poissonian, i.e. point magnetic impurities dropped on the plane randomly without correlation. It means that the infinitesimal probability $dP(\vec{r}_i)$ of finding an impurity at position $\vec{r}_i$ is

$$dP(\vec{r}_i) = \frac{d\vec{r}_i}{V}$$

(83)

One would like to see in which way this system could be compared with the naive mean field approximation where the local magnetic field $eB(\vec{r}) = 2\pi \alpha \sum_{i=1}^{N} \delta(\vec{r} - \vec{r}_i)$ is replaced by its mean value

$$\langle B \rangle V = N \phi \quad \text{i.e. in the thermodynamic limit} \quad e \langle B \rangle = 2\pi \rho \alpha$$

(85)

Two approaches have been used to study this problem: a path integral Brownian motion approach (or its discretised version, random walk in the plane, which allows for numerical simulations), which focuses on winding properties of Brownian paths in the plane, and a quantum mechanical Hamiltonian approach, which relies mainly on perturbative expansions of the average partition function (small $\alpha$, high temperature, etc). In the sequel, we will use the quantum mechanical Hamiltonian approach, that we summarize now.

---

If one takes a 2 dimensional impurity density of the order of the density of current carriers $\rho = 4 \times 10^{15} m^{-2}$, one obtains, for $\alpha = 1/2$, a magnetic field precisely in the experimental range of the Quantum Hall Effect

$$\langle B \rangle = \frac{\hbar \rho \alpha}{e} \simeq 10 T$$

(84)
In the symmetric gauge, the Hamiltonian for $N$ impurities is

$$H_d = \frac{1}{2} \left( \vec{p} - \sum_{i=1}^{N} \alpha \frac{\vec{k} \times (\vec{r} - \vec{r}_i)}{(\vec{r} - \vec{r}_i)^2} \right)^2 + \pi \alpha \sum_{i=1}^{N} \delta(\vec{r} - \vec{r}_i)$$

(86)

where $\alpha \in [0, 1/2]$, without any loss of generality. The need for a spin coupling term has already been discussed above. Recall that, physically, it corresponds to hard-core impurities where the electron is unable to penetrate.

Consider the nonunitary wavefunction redefinitions (32)

$$\psi(\vec{r}) = \prod_{i=1}^{N} |\vec{r} - \vec{r}_i|^\alpha \tilde{\psi}(\vec{r}) \equiv U_d(\vec{r})\psi(\vec{r})$$

(87)

(35, 36) become

$$U_d^{-1} \Pi_+ U_d = \Pi_0^0 - 2i\alpha \Omega$$

(88)

$$U_d^{-1} \Pi_- U_d = \Pi_0^0$$

(89)

where the impurity vector potential $\Omega = \sum_{i=1}^{N} 1/(\bar{z} - \bar{z}_i)$. The Hamiltonian $\tilde{H}$ acting on $\tilde{\psi}(\vec{r})$ rewrites

$$\tilde{H}_d = -2\partial_z \partial_{\bar{z}} - 2\alpha \sum_{i=1}^{N} \frac{1}{\bar{z} - \bar{z}_i} \partial_{\bar{z}}$$

$$= \frac{1}{2} \Pi_+^0 \Pi_-^0 - i\alpha \Omega \Pi_0^0$$

(90)

(91)

One can go a bit further by extracting $\tilde{\Pi}$ the mean value of $\prod_{i=1}^{N} |\vec{r} - \vec{r}_i|^\alpha$ which precisely yields, in the thermodynamic limit, the mean magnetic field Landau exponential factor $e^{\frac{1}{2} \langle \omega_c \rangle r^2}$ with $\langle \omega_c \rangle = e \langle B \rangle / 2 = \pi \rho \alpha$. Remember that $\alpha \in [0, 1/2]$, so that $e \langle B \rangle$, which has the sign of $\alpha$, is indeed positive. So let us redefine

$$\tilde{\psi}(\vec{r}) = e^{-\frac{\epsilon}{2} \langle \omega_c \rangle r^2} \prod_{i=1}^{N} |\vec{r} - \vec{r}_i|^\alpha \tilde{\psi}(\vec{r}) \equiv U_d'(\vec{r})\tilde{\psi}(\vec{r})$$

(92)

and obtain

$$(U_d')^{-1} \Pi_+ U_d' = \Pi_+^{<L>} - 2i\alpha(\Omega - \langle \Omega \rangle)$$

(93)

$$(U_d')^{-1} \Pi_- U_d' = \Pi_-^{<L>}$$

(94)
Thus,

\[ \hat{H}'_d = \frac{1}{2} \Pi_+^{\langle L \rangle} \Pi^{\langle L \rangle} - i\alpha (\Omega - \langle \Omega \rangle) \Pi^{\langle L \rangle} \]

(95)

where \( \Pi^{\langle L \rangle} = -2i (\partial_z - \langle \omega_c \rangle z/2) \) is the covariant momentum operator in the symmetric gauge for the Landau Hamiltonian of the mean magnetic field, and \( \langle \Omega \rangle = \pi \rho z \) is the mean value of \( \Omega \). Therefore, (95) is the Landau Hamiltonian for the mean field \( \langle B \rangle \) up to a \( \langle \omega_c \rangle \) shift, encoding the mean magnetic field contributions, plus a correction due to disorder, gathered in the interaction term \( \Omega' = -i\alpha (\Omega - \langle \Omega \rangle) \Pi^{\langle L \rangle} \).

The Hamiltonians \( H_d, \hat{H}_d \) and \( \hat{H}'_d \) are equivalent, and can be indifferently used for computing thermodynamical observables as the average partition function, the average density of states, or the average conductivity. However, quadratic interactions in the vector potential have disappeared from the Hamiltonians (90,95), making easier the average over disorder. In the case of the Poisson probability distribution (83), the average can be done using identities like

\[ \int d\bar{z} d\bar{z}' \frac{1}{\bar{z} - \bar{z}_i} = \pi z \]

(96)

\[ \int d\bar{z} d\bar{z}' \frac{1}{\bar{z} - \bar{z}_i} \frac{1}{\bar{z}' - \bar{z}_i} = \pi (\frac{z}{\bar{z}' - \bar{z}} + \frac{z'}{\bar{z} - \bar{z}'}) \]

(97)

What general information can be extracted from (86,95) after averaging over the Poissonian disorder?

i) because of the periodicity \( \alpha \rightarrow \alpha + 1 \), and, in the absence of any orientation to the plane, because of the symmetry \( \alpha \rightarrow -\alpha \), observables such as the partition function or the density of states, should be a function of \( \alpha (1 - \alpha) \) only. The transverse conductivity, on the other hand, because of the electric field, should change its sign when \( \alpha \rightarrow -\alpha \), implying that it vanishes when \( \alpha = 1/2 \).

ii) any observable can be expanded in power series in \( \alpha \) and \( \rho \) as \( \sum_{n,m} \rho^n \alpha^m, m \geq n \). The constraint \( m \geq n \) stems from the fact the number of vertex is necessarily larger than the number of impurities (an electron can interact at different times with the same impurity). This series can always be rewritten as \( \sum_{p=0}^{\infty} \alpha^p f_p(\rho) \).

iii) the \( \rho^n \alpha^n \) and \( (\rho \alpha)^n \) terms are exactly known: the former correspond to the entirely solvable one impurity problem (standard Aharonov-Bohm problem), whereas the latter is obtained, if, in the Hamiltonian \( \hat{H}'_d \), one sets \( \Omega = \langle \Omega \rangle \), thus reducing the problem to the average Landau problem. In
the absence of disorder, any observable is necessarily a power series in leading order \((\rho_\alpha)^n\) of the mean magnetic field.

iv) corrections coming from disorder are necessarily of the type \(\rho^m_\alpha, m > n\). To have information on these corrections, perturbation theory is needed. However, if perturbation theory is meaningful for the Hamiltonian \((86)\), still ultraviolet divergences occur in the perturbative computation of the spectrum, which are shown to cancel only if proper regulators are used. On the contrary, the Hamiltonians \(\hat{H}_d\) or \(\hat{H}^\prime_d\), \((90)\), yield a perturbation theory which is finite and easier to handle than whose of \((86)\). If one concentrates on the perturbative expansion for the Hamiltonian \(\hat{H}^\prime_d\), the unperturbed Hamiltonian is the Landau Hamiltonian for the mean magnetic field, that is to say \(\langle\omega_c\rangle = \pi \rho_\alpha\) is fixed. It follows that \(\rho^m_\alpha, m > n\) terms correspond to \(\alpha^{-n}\) corrections to the mean field. For instance, \(\rho^2_\alpha\) is actually of order \(\alpha^2\), meaning that a second order \(\alpha^2\) computation produces a first order \(\alpha\) correction. Conversely, the \(\alpha\) correction to the mean field will involve all the \(\rho^m_\alpha, m > n\) terms.

To develop a perturbative expansion for the partition function \((40)\) or for the conductivity \((41, 42)\), all what is needed is the perturbative expansion of the propagator \(\tilde{G}^{(2)}(\vec{r}, \vec{r}') = \langle\vec{r}|e^{-\beta \hat{H}_d'}|\vec{r}'\rangle\) of the Hamiltonian \((86)\). At order \(n\), one has

\[
\delta \tilde{G}^{(n)}_\beta(\vec{r}, \vec{r}') = (-1)^n \int_0^\beta d\beta_1 \int_0^{\beta_1} d\beta_2 \cdots \int_0^{\beta_{n-1}} d\beta_n \int d\vec{r}_1 \cdots \int d\vec{r}_n

G^{<L>}_{\beta, -\beta}(\vec{r}, \vec{r}_1) \tilde{V}'(\vec{r}_1) G^{<L>}_{\beta, -\beta}(\vec{r}_1, \vec{r}_2) \cdots \tilde{V}'(\vec{r}_n) G^{<L>}_{\beta, -\beta}(\vec{r}_n, \vec{r}')
\]

where \(\tilde{V}' = -i\alpha(\Omega - \langle\Omega\rangle)\Pi^{<L>}\) and \(G^{<L>}_{\beta, -\beta}(\vec{r}, \vec{r}')\) is the propagator of the mean magnetic field \((46)\) for the spin down case with \(\omega_c \rightarrow \langle\omega_c\rangle\). The only virtue of \(\langle\Omega\rangle\) in \(\tilde{V}'\) is to cancel the mean field \((\rho_\alpha)^n\) corrections coming from \(\Omega\) alone. This is why one can ignore \(\langle\Omega\rangle\) if one pays attention to keep only the non mean field contributions due to \(\Omega\). In term of Feynman diagrams, it means that after averaging over the positions of impurities, one should only retain the diagrams without isolated impurity legs. It follows that the correction in \(\alpha\) to the mean field expansion are obtained from the second order \(\rho_\alpha^2 \simeq \langle\omega_c\rangle\alpha\) corrections, with diagrams involving only one impurity.

Let us illustrate these considerations by computing the corrections to the average Landau partition function. One has to evaluate perturbatively the average second order correction to the average Landau propagator. \((88)\) yields the correction

\[
\delta \tilde{G}^{(2)}_\beta(\vec{r}, \vec{r}') = \langle i\alpha \rangle^2 \int_0^\beta d\beta_1 \int_0^{\beta_1} d\beta_2 \int d\vec{r}_1 d\vec{r}_2 G^{<L>}_{\beta, -\beta}(\vec{r}, \vec{r}_1)
\]
\[ \sum_{i=1}^{N} \frac{1}{z_1 - z_i} \Pi_{<L>} G_{\beta_1 - \beta_2} (\vec{r}_1, \vec{r}_2) \sum_{j=1}^{N} \frac{1}{\bar{z}_2 - \bar{z}_j} \Pi_{<L>} G_{\beta_2} (\vec{r}_2, \vec{r}') \]

\[ \sum_{i=1}^{N} \frac{1}{z_1 - z_i} \Pi_{<L>} G_{\beta_1 - \beta_2} (\vec{r}_1, \vec{r}_2) \sum_{j=1}^{N} \frac{1}{\bar{z}_2 - \bar{z}_j} \Pi_{<L>} G_{\beta_2} (\vec{r}_2, \vec{r}') \]

\[ \vec{r}' \xrightarrow{\alpha, \beta} \vec{r} \]

The average over impurity positions \( \vec{r}_i \) and \( \vec{r}_j \), using (96), produces either a term containing interaction with two different impurities \( i \neq j \), a \( \rho^2 \alpha^2 \) mean field contribution cancelled by \( \langle \Omega \rangle \), or a term containing two interactions with the same impurity \( i = j \), a \( \rho \alpha^2 \) term. It rewrites

\[ \langle \delta \tilde{G}_\beta^{(2)} (\vec{r}, \vec{r}') \rangle = \pi \rho (\alpha + 1) \langle \omega_c \rangle \]

\[ \langle \delta \tilde{G}_\beta^{(2)} (\vec{r}, \vec{r}') \rangle = \pi \rho (\alpha + 1) \langle \omega_c \rangle \]

Using the algebra of the average Landau operators \( \Pi_{<L>} \), \( \Pi_{+L} \), \( B_{<L>} \) and \( B_{+L} \), with

\[ B_{<L>} = -2i(\partial_z - \frac{1}{2} (\omega_c) z) \]

\[ B_{+L} = -2i(\partial_z + \frac{1}{2} (\omega_c) \bar{z}) \]

and the completeness relations for the propagators, the space integrations in (100) can be done to obtain

\[ \langle \delta \tilde{G}_\beta^{(2)} (\vec{r}, \vec{r}') \rangle = -\pi \rho (i\alpha + \frac{1}{2} \beta^2) \Pi_{<L>} \Pi_{+L} G_{\beta}^{<L>} (\vec{r}, \vec{r}') \]

\[ \langle \delta \tilde{G}_\beta^{(2)} (\vec{r}, \vec{r}') \rangle = -\pi \rho (i\alpha + \frac{1}{2} \beta^2) \Pi_{<L>} \Pi_{+L} G_{\beta}^{<L>} (\vec{r}, \vec{r}') \]
Therefore, the correction to the average partition function at order $\alpha$ is
\[ \delta Z_\beta = \alpha \frac{V \langle \omega_c \rangle}{\pi} \frac{\beta \langle \omega_c \rangle^2}{2 \sinh^2(\beta \langle \omega_c \rangle)} \] (105)

Accordingly, the correction to the average Landau density of states is
\[ \delta \rho(E) = \alpha \langle \omega_c \rangle \frac{d^2}{dE^2} \sum_{n=0}^{\infty} 2(n+1) \langle \omega_c \rangle \delta(E - 2(n+1) \langle \omega_c \rangle) \] (106)

One can check that (105) reproduces the correction to the mean field term in the one vortex problem, by simply taking the small $\langle \omega_c \rangle$ limit in (105)
\[ \alpha \frac{V \langle \omega_c \rangle}{\pi} \frac{1}{2} = \alpha \frac{2}{2} V \rho \] (107)
which is nothing but the $\alpha^2$ term in (105) if one drops the $\rho V = N$ factor.

Considering the conductivity, we have in general to evaluate
\[ \sigma_\beta(t) = \left\langle i \theta \left( \frac{e^2}{V} \int d\vec{r} d\vec{r}' \left( \Pi_{\langle L \rangle}^{-} G_{\alpha \mu}(\vec{r}, \vec{r}') x' G_{\beta \mu}^{-}(\vec{r}', \vec{r}) - (it \to it + \beta) \right) \right) \right\rangle \] (108)
or its Fermi transform
\[ \sigma_{E_F}(t) = \left\langle i \theta \left( \frac{e^2}{V} \int_{-\infty}^{\infty} \frac{dt'}{2\pi i} e^{iE_{F}t'} \int d\vec{r} d\vec{r}' \left( \Pi_{\langle L \rangle}^{-} G_{\alpha \mu}(\vec{r}, \vec{r}') x' G_{\beta \mu}^{-}(\vec{r}', \vec{r}) - (t \to t + t') \right) \right) \right\rangle \] (109)

At first order in perturbation theory, one has to compute
\[ \rho \times \alpha \quad \alpha \times \rho \quad \alpha \times \rho \]
\[ \Pi_{\langle L \rangle}^{-} \quad \Pi_{\langle L \rangle}^{-} \quad \Pi_{\langle L \rangle}^{-} \]
It is clearly a tedious task to evaluate perturbatively (108), already at first order in \( \alpha \). However, we will see below how to circumvent this computation.

5.2 \( \rho n \alpha n \) term: the one impurity case

Before doing so, let us check that, at leading order in \( \alpha \), the conductivity for the one vortex case coincides with the low \( \langle \omega_c \rangle \) expansion of the Landau conductivity for the mean magnetic field. This should be so, since we know that at leading order (\( \rho n \alpha n \)), the \( n \) random magnetic impurity system yields the Landau system (56) at order \( \langle \omega_c \rangle \).

\[
\text{Re} \sigma_{EF}^L(\omega)|_{yx} = N(E_F) \frac{e^2}{V} \frac{2 \langle \omega_c \rangle}{\omega} \left( 1 + \left( \frac{2 \langle \omega_c \rangle}{\omega} \right)^2 + \left( \frac{2 \langle \omega_c \rangle}{\omega} \right)^4 + \cdots \right) \quad (110)
\]

One has just to multiply the leading term in \( \alpha \) of the one vortex conductivity (73) by \( N \) to get the leading \( \rho n \alpha \) term

\[
\text{Re} \sigma_{EF}(\omega)|_{yx} \approx N(E_F) \frac{e^2}{V} 2\pi\rho\alpha \frac{1}{\omega^2} = N(E_F) \frac{e^2}{V} e\langle B \rangle \frac{1}{\omega^2} \quad (111)
\]

where the magnetic impurity density \( \rho = N/V, \) has factorized in \( \langle B \rangle \). One recovers the leading term in the expansion (110) of the conductivity of the Landau problem in the low mean magnetic field limit \( \langle \omega_c \rangle / \omega \to 0. \) If, in the single impurity case, the conductivity is divergent at small \( \beta \omega, \) in the random magnetic impurity case, this divergence is nothing but a manifestation of the small \( \langle \omega_c \rangle / \omega \) expansion of the mean Landau conductivity. On the other hand, considering the longitudinal conductivity for the Landau problem (58), its small \( \langle \omega_c \rangle \) expansion starts at order \( \langle \omega_c \rangle^2, \) a \( \rho^2 \) behavior which cannot be obtained from the single vortex problem (74).

5.3 \( \rho n \alpha n+1 \) terms: the one vortex + a homogeneous magnetic field

As advocated above, perturbation theory with (95) at first order in \( \alpha \) amounts to consider only the one impurity diagrams. It follows that, at this order, one can map the magnetic impurity perturbative problem on the exact one vortex + a homogeneous mean magnetic field problem. Consider again the Hamiltonian (54) for the one vortex + \( \langle B \rangle \) problem

\[
H_d = \frac{1}{2} \left( \vec{p} - \langle \omega_c \rangle \vec{k} \times \vec{r} - \alpha \frac{\vec{k} \times \vec{r}}{r^2} \right)^2 + (\pi \alpha \delta(r) + \langle \omega_c \rangle) \quad (112)
\]
and redefine the wavefunctions in order to extract only the short distance behaviour of the wavefunctions at the origin due to the vortex

\[ U_d(\vec{r}) = r^\alpha \] (113)

It follows that

\[ U_d^{-1} \Pi_+ U_d = \Pi_+^{<L>} - 2i\alpha \frac{1}{\tilde{z}} \] (114)
\[ U_d^{-1} \Pi_- U_d = \Pi_-^{<L>} \] (115)

The \( \tilde{H}_d \) Hamiltonian reads

\[ \tilde{H}_d = \frac{1}{2} \Pi_+^{<L>} \Pi_-^{<L>} - i\alpha \frac{1}{\tilde{z}} \Pi_-^{<L>} \] (116)

Comparing the Hamiltonian (113) with (116), one sees that they can be identified if, leaving aside the contribution of the average potential \( \langle \Omega \rangle \), one restricts (113) to one impurity located at the origin. This shows in particular that, apart for a factor \( N \), the second order \( \alpha^2 \) computation for the one vortex + \( \langle B \rangle \) problem is identical to the first order \( \rho \alpha^2 \simeq \langle \omega_c \rangle \alpha \) correction to the mean field in the magnetic impurity problem. One can check explicitly these considerations on the partition function. From (113), the \( \alpha^2 \) term reads

\[ \delta Z^{(2)}_{\beta}(\langle B \rangle, \alpha) = \alpha^2 \frac{(\beta \langle \omega_c \rangle)^2}{2 \sinh^2(\beta \langle \omega_c \rangle)} \] (117)

Multiplying it by \( N \) reproduces the perturbative result (108) of the magnetic impurity problem at order \( \alpha \).

5.4 Perturbative Hall conductivity

We now go back to the magnetic impurity problem, and instead of computing perturbatively (108), we rather consider in (113) the \( \alpha^2 \) correction multiplied by \( N \). One gets the conductivity for the magnetic impurity problem at first order in \( \alpha \)

\[ \sigma_{E_F}^{\omega = 0} = \int_{-\infty}^{+\infty} \frac{dt'}{2i\pi} \frac{e^{iE_F t'}}{t' - i\eta'} \left\{ \frac{i e^2}{2V \langle \omega_c \rangle} \right\} \left\{ Z_{it'} + \alpha \langle \omega_c \rangle i t' Z_{it'}^{<L>} + \cdots \right\} \] (118)
\[ = \frac{i e^2}{2V \langle \omega_c \rangle} \left\{ N(E_F) + \alpha \langle \omega_c \rangle dN^{<L>}(E_F) \right\} + \cdots \] (119)

where \( Z_{\beta}^{<L>} \) and \( N^{<L>}(E_F) \) are respectively the partition function and the number of electrons for the average Landau problem. \( N^{<L>}(E_F) \) is nothing
but the mean field contribution to the actual number of electrons in the system

\[ N(E_F) = N^{<L>}(E_F) + \alpha N^{(1)}(E_F) + \alpha^2 N^{(2)}(E_F) + \cdots \]

where the \( N^{(n)} \)'s are functions of \( \langle \omega_c \rangle \) and \( E_F \) only. Then, one can rewrite

\[
\sigma_{E_F}(\omega = 0) = \frac{i e^2}{2 V \langle \omega_c \rangle} N(E_F + \alpha \langle \omega_c \rangle)
\]

which is still valid at first order in \( \alpha \). The Hall conductivity is thus, at this order,

\[
\text{Re} \sigma_{E_F}(\omega = 0)|_{yx} = -N(E_F + \alpha \langle \omega_c \rangle) \frac{1}{V \langle B \rangle}
\]

a quite simple result (eventhough higher order perturbation theory might change this situation).

Also, from (78), one finds no correction for the longitudinal conductivity, which means that at this order, (58) is not affected by disorder.

6 Heuristic considerations and open questions

Let us consider the experimental situation where the magnetic field \( \langle B \rangle \) is kept fixed, and where the electron density varies. The Hall conductivity (121) is, basically, \( N(E_F + \alpha \langle \omega_c \rangle) \) as a function of \( N(E_F) \). In [4] the average density of states \( \rho(E) \) was estimated on analytical (approximation) and numerical grounds. In Figure 1, \( \rho(E) \) is given for \( \alpha = 0 \). This curve has essentially an \( \alpha \) behavior, with broadened Landau levels having a width of order \( \alpha \langle \omega_c \rangle \), a height of order \( 1/\alpha \), such that the total number of quantum states per unit surface in a broadened Landau level, \( \langle \omega_c \rangle / \pi \), is conserved whatever the disorder is.

In Figure 2, the Hall conductivity (121) is displayed (full line) as a function of the number of electrons \( N(E_F) \), while keeping the mean magnetic field fixed. The Hall conductivity has oscillations close to the classical straight line of the average Landau conductivity. We believe that the oscillations are smoothed because of the approximation made in (120) and in \( \rho(E) \), and consequently in \( N(E_F) \). In fact \( \rho(E) \) is too broad to get a stepwise Hall conductivity. On the other hand, we also considered that \( \rho(E) \) at small \( \alpha \) is the same object as \( \rho(E) \) at first order in \( \alpha \). But the expansion of the density of states around the mean field density of states is surely not analytical. To conclude, the curve given in Figure 2 should only be viewed as a crude first step towards a more complete computation. It is not impossible that higher order corrections, or a more general argument, will yield a behavior closer to the experimental curve (dashed lines in Figure 2, from [4]).
What kind of physical picture could support the model presented above? When the magnetic field is very strong, $\alpha \simeq 1/2$, the system is totally disordered, and cannot possibly transport any current. This is consistent with the symmetry argument which indicates that at $\alpha = 1/2$ the transverse conductivity has to vanish. On the other hand, when $\langle B \rangle$ decreases, Landau level oscillations appear, implying that some states conduct, thus the appearance of plateaus. When $\langle B \rangle$ becomes smaller and smaller, i.e. $\alpha \to 0$, a pure Landau system is reached with a classical Hall conductivity and no plateau at all.

Clearly, no edge or finite size effects have been considered whatsoever, and no interaction between electrons either. Also, the thermal average has forbidden any precise information on the possible localisation of quantum states in the random magnetic impurity distribution.

It would be certainly rewarding to know how to extract from (108) a more global (non perturbative) information, which would, like the Thouless topological argument, tell us for sure that, when considering the Fermi gas at zero temperature and Fermi energy $E_F$, the correction terms always oscillate around the classical straight line.

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Figure 1: Average density of states of the random magnetic impurity model for $\alpha = 0.01$
Figure 2: Hall conductivity in unit of $e^2/h$ of the random magnetic impurity model at first order in $\alpha$ for $\alpha = 0.01$ as a function of the filling factor $\nu = \frac{N(E_F)}{V} \frac{h}{\sigma_{xy}}$. Straight line = classical result, steps = experimental Integer Quantum Hall Effect (schematic), full line = perturbative result.