Research Article

Two-Weight, Weak-Type Norm Inequalities for Fractional Integral Operators and Commutators on Weighted Morrey and Amalgam Spaces

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1.1. Fractional Integral Operators. Let \( \mathbb{R}^n \) be the \( n \)-dimensional Euclidean space equipped with the Euclidean norm \( \| \cdot \| \) and the Lebesgue measure \( dx \). For given \( \gamma \), \( 0 < \gamma < n \), the fractional integral operator (or Riesz potential) \( I_\gamma \) with order \( \gamma \) is defined by (see [1] for the basic properties of \( I_\gamma \))

\[
I_\gamma f(x) := \frac{1}{\zeta(\gamma)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\gamma}} \, dy,
\]

\[
\zeta(\gamma) = \frac{\pi^{n/2} \Gamma (\gamma/2)}{\Gamma ((n-\gamma)/2)}.
\]

Weighted norm inequalities for fractional integral operators arise naturally in harmonic analysis and have been extensively studied by several authors. The study of two-weight problem for \( I_\gamma \) was initiated by Sawyer in his pioneer paper [2]. By a weight \( w \), we mean that \( w \) is a nonnegative and locally integrable function. In [2], Sawyer concerned the following question. Suppose that \( 1 < p \leq q < \infty \). For which pairs of weights \( (w, v) \) on \( \mathbb{R}^n \) is the fractional integral operator bounded from \( L^p(v) \) into \( \text{weak-}L^q(w) \)? A necessary and sufficient condition for the weak-type \((p, q)\) inequality was given by Sawyer. More specifically, he showed the following.

**Theorem 1** (see [2]). Let \( 0 < \gamma < n \) and \( 1 < p \leq q < \infty \). Given a pair of weights \((w, v)\) on \( \mathbb{R}^n \), the weak-type inequality

\[
\sigma \cdot w \left( \left\{ x \in \mathbb{R}^n : |I_\gamma f(x)| > \sigma \right\} \right)^{1/q} \leq C \left( \int_{\mathbb{R}^n} |f(x)|^p \, v(x) \, dx \right)^{1/p},
\]

holds for any \( \sigma > 0 \) if and only if

\[
\left( \int_Q \left[ I_\gamma (x_Q w)(x) \right]^p v(x)^{1/p} \, dx \right)^{1/p} \leq C \cdot w(Q)^{1/q} < \infty,
\]

for all cubes \( Q \) in \( \mathbb{R}^n \). Here, \( x_Q \) denotes the characteristic function of the cube \( Q \), \( p' = p/(p-1) \) denotes the conjugate index of \( p \), and \( C \) is a universal constant.

Sawyer’s result is interesting and important, and it promotes a series of research works on this subject (see, e.g.,...
[3, 4, 5–9]), but it has the defect that condition (3) involves the fractional integral operator $I_\gamma$ itself. In [3], Cruz-Uribe and Pérez considered the case when $q = p$ and found a sufficient $A_q$-type condition on a pair of weights $(w, v)$ which ensures the boundedness of the operator $I_\gamma$ from $L^p(v)$ into weak-$L^p(w)$, where $1 < p < \infty$. The condition (4) given below is simpler than (3) in the sense that it does not involve the operator $I_\gamma$ itself, and hence it can be more easily verified.

**Theorem 2** (see [3]). Let $0 < \gamma < n$ and $1 < p < \infty$. Given a pair of weights $(w, v)$ on $\mathbb{R}^n$, suppose that for some $r > 1$ and for all cubes $Q$ in $\mathbb{R}^n$,

$$|Q|^{\frac{n}{\gamma n} \cdot \left(\frac{1}{|Q|} \int_Q w(x)^{\gamma} dx\right)} \left(\frac{1}{|Q|} \int_Q v(x)^{-\frac{1}{p'}} dx \right)^{\frac{1}{p'}} \leq C < \infty. \quad (4)$$

This commutator was first introduced by Chanillo in [11]. In [6], Liu and Lu obtained a sufficient $A_p$-type condition for the commutator $[b, I_\gamma]$ to satisfy a two-weight weak-type $(p, p)$ inequality, where $1 < p < \infty$. That condition is an $A_p$-type condition in the scale of Orlicz spaces (see (7) given below).

**Theorem 3** (see [6]). Let $0 < \gamma < n$, $1 < p < \infty$ and $b \in BMO(\mathbb{R}^n)$. Given a pair of weights $(w, v)$ on $\mathbb{R}^n$, suppose that for some $r > 1$ and for all cubes $Q$ in $\mathbb{R}^n$,

$$|Q|^{\frac{n}{\gamma n} \cdot \left(\frac{1}{|Q|} \int_Q w(x)^{\gamma} dx\right)} \left(\frac{1}{|Q|} \int_Q v(x)^{-\frac{1}{p'}} dx \right)^{\frac{1}{p'}} \leq C < \infty, \quad (7)$$

where $\mathcal{A}(t) := t^{\frac{\gamma}{p}} (1 + \log^+ t)^{- \frac{1}{p'}}$ and $\log^+ t := \max\{\log t, 0\}$, that is,

$$\log^+ t = \begin{cases} \log t, & \text{as } t > 1; \\ 0, & \text{otherwise}. \end{cases} \quad (8)$$

Then, the linear commutator $[b, I_\gamma]$ satisfies the weak-type $(p, p)$ inequality

$$\sigma \cdot \mathcal{W} \left(\left\{ x \in \mathbb{R}^n : \left| [b, I_\gamma] f(x) \right| > \sigma \right\} \right)^{\frac{1}{p'}} \leq C \left(\int_{\mathbb{R}^n} |f(x)|^{\mathcal{A}(v(x))} dx \right)^{\frac{1}{p}}, \quad \text{for any } \sigma > 0, \quad (9)$$

where $C$ does not depend on $f$ nor on $\sigma > 0$.

In [7], Martell considered the case when $q > p$ and gave a verifiable condition which is sufficient for the two-weight, weak-type $(p, q)$ inequality for fractional integral operator $I_\gamma$. The condition (10) given below (in the Euclidean setting of [7]) is also simpler than the one in Theorem 1.

Then, the fractional integral operator $I_\gamma$ satisfies the weak-type $(p, p)$ inequality

$$\sigma \cdot \mathcal{W} \left(\left\{ x \in \mathbb{R}^n : \left| I_\gamma f(x) \right| > \sigma \right\} \right)^{\frac{1}{p'}} \leq C \left(\int_{\mathbb{R}^n} |f(x)|^{\mathcal{A}(v(x))} dx \right)^{\frac{1}{p}}, \quad \text{for any } \sigma > 0, \quad (5)$$

where $C$ does not depend on $f$ nor on $\sigma > 0$.

The proof of Theorem 2 is quite complicated. It depends on an inequality relating the Hardy–Littlewood maximal function and the sharp maximal function which is strongly reminiscent of the good-$\lambda$ inequality of Fefferman and Stein. For another, more elementary proof, see also [4]. This solves a problem posed by Sawyer and Wheeden in [8]. Moreover, in [5], Li improved this result by replacing the “power bump” in (4) by a smaller “Orlicz bump” (see also [10]). On the other hand, for given $0 < \gamma < n$, the linear commutator $[b, I_\gamma]$ generated by a suitable function $b$ and $I_\gamma$ is defined by

$$[b, I_\gamma]f(x) = b(x) \cdot I_\gamma f(x) - I_\gamma (bf)(x) = \frac{1}{\xi(\gamma)} \int_{\mathbb{R}^n} \frac{\{|b(x) - b(y)| \cdot f(y)|x - y|^{-\gamma}}{\int_{|x - y|^{\gamma - d}}} \, dy. \quad (6)$$

**Theorem 4** (see [7]). Let $0 < \gamma < n$ and $1 < p < q < \infty$. Given a pair of weights $(w, v)$ on $\mathbb{R}^n$, suppose that for some $r > 1$ and for all cubes $Q$ in $\mathbb{R}^n$,

$$|Q|^{\frac{n}{\gamma n+1/q-1/p}} \cdot \left(\frac{1}{|Q|} \int_Q w(x)^{\gamma} dx\right)^{\frac{1}{(\gamma n+1/q-1/p)}} \cdot \left(\frac{1}{|Q|} \int_Q v(x)^{-\frac{1}{p'}} dx \right)^{\frac{1}{p'}} \leq C < \infty, \quad (10)$$

Then, the fractional integral operator $I_\gamma$ satisfies the weak-type $(p, q)$ inequality

$$\sigma \cdot \mathcal{W} \left(\left\{ x \in \mathbb{R}^n : \left| I_\gamma f(x) \right| > \sigma \right\} \right)^{\frac{1}{q'}} \leq C \left(\int_{\mathbb{R}^n} |f(x)|^{\mathcal{A}(v(x))} dx \right)^{\frac{1}{p}}, \quad \text{for any } \sigma > 0, \quad (11)$$

where $C$ does not depend on $f$ nor on $\sigma > 0$.

Furthermore, in [9], Zhang sharpened Martell’s result by replacing the local $L^r$ norm by the outer edge of the smaller Orlicz space norm. On the other hand, by comparing Theorem 4 with Theorems 2 and 3, it is natural to conjecture that when $q > p$, there is a two-weight, weak-type $(p, q)$ inequality for the commutator $[b, I_\gamma]$ of fractional integral operator. By using the same method as in the proof of Theorem 1 in [12] and certain Orlicz norm, we are able to obtain the following sufficient condition on a pair of weights $(w, v)$ to ensure the $L^p(v) \rightarrow WL^q(w)$ boundedness of $[b, I_\gamma]$, whenever $b$ belongs to $BMO(\mathbb{R}^n)$. More specifically, the following statement is true.
Theorem 5. Let $0 < \gamma < n$, $1 < p < q < \infty$ and $b \in \text{BMO}(\mathbb{R}^n)$. Given a pair of weights $(w, v)$ on $\mathbb{R}^n$, suppose that for some $r > 1$ and for all cubes $Q$ in $\mathbb{R}^n$

\[
\|Q|^\gamma r^{1/r} 1/p \left( \frac{1}{|Q|} \int_Q w(x) y^{1/p} \right) \leq C < \infty,
\]

where $w$ is a weight on $\mathbb{R}^n$, $w(y) = 1 + |y|^p$. Then, the linear commutator $[b, I_r]$ satisfies the weak-type $(p, q)$ inequality

\[
\sigma \cdot w \left[ x \in \mathbb{R}^n : |[b, I_r]f(x) > \sigma \right]^{1/q} \leq C \left( \int_{|x|} \int_Q |f(x)|^p v(x) dx \right)^{1/p}, \quad \text{for any } \sigma > 0,
\]

where $C$ does not depend on $f$ nor on $\sigma > 0$.

The details are omitted here. Note that condition (12) reduces to condition (7) provided that $p = q$.

Question. In view of Theorems 2–5, it is a natural and interesting problem to find some sufficient conditions for which the two-weight, weak-type norm inequalities hold for the operators $I_r$ and $[b, I_r]$, in the endpoint case $p = 1$.

In this paper, we are mainly interested in the weighted Morrey spaces and weighted amalgam spaces. Let us recall their definitions.

1.2. Weighted Morrey Spaces. The classical Morrey space $L^{p, \lambda}(\mathbb{R}^n)$ was introduced by Morrey [12] in connection with elliptic partial differential equations. Let $1 \leq p < \infty$ and $0 \leq \lambda \leq n$. We recall that a real-valued function $f$ is said to belong to the space $L^{p, \lambda}(\mathbb{R}^n)$ on the $n$-dimensional Euclidean space $\mathbb{R}^n$, if the following norm is finite:

\[
\|f\|_{L^{p, \lambda}} := \sup_{(x, r) \in \mathbb{R}^n \times (0, \infty)} \left( r^{\lambda-n} \int_{B(x, r)} |f(y)|^p dy \right)^{1/p},
\]

where $B(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\}$ is the Euclidean ball with center $x \in \mathbb{R}^n$ and radius $r \in (0, \infty)$ as well as the Lebesgue measure $|B(x, r)| = v_n \cdot r^n$. Here, $v_n$ is the volume of the unit ball of $\mathbb{R}^n$. In particular, one has

\[
L^{p, 0}(\mathbb{R}^n) = L^p(\mathbb{R}^n),
\]

\[
L^{p, n}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n).
\]

In [13], Komori and Shirai considered the weighted case and introduced a version of weighted Morrey space, which is a natural generalization of weighted Lebesgue space.

Definition 1. Let $1 < p < \infty$ and $0 \leq \kappa < 1$. For two weights $w$ and $v$ on $\mathbb{R}^n$, the weighted Morrey space $L^{p, \kappa}(\mathbb{R}^n)$ is defined by

\[
L^{p, \kappa}(\mathbb{R}^n) := \left\{ f \in L^p_{\text{loc}}(\mathbb{R}^n) : \|f\|_{L^{p, \kappa}(\mathbb{R}^n)} < \infty \right\},
\]

where the norm is given by

\[
\|f\|_{L^{p, \kappa}(\mathbb{R}^n)} := \sup_{Q \in \mathbb{R}^n} \left( \frac{1}{w(Q)^{1/p}} \int_Q |f(x)|^p v(x) dx \right)^{1/p},
\]

and the supremum is taken over all cubes $Q$ in $\mathbb{R}^n$.

Definition 2. Let $1 < p < \infty$, $0 \leq \kappa < 1$, and $w$ be a weight on $\mathbb{R}^n$. We define the weighted weak Morrey space $W^{p, \kappa}(\mathbb{R}^n)$ as the set of all measurable functions $f$ satisfying

\[
\|f\|_{W^{p, \kappa}(\mathbb{R}^n)} := \sup_{Q \in \mathbb{R}^n} \sup_{\sigma > 0} \frac{1}{w(Q)^{1/p}} \int_Q |f(x)|^p w(x) dx \cdot \left( \frac{1}{w(|x - y| > \sigma)} \right)^{1/p} < \infty.
\]

By definition, it is clear that

\[
L^{p, 0}(\mathbb{R}^n) = L^p(\mathbb{R}^n),
\]

\[
W^{p, 0}(\mathbb{R}^n) = W^p(\mathbb{R}^n).
\]

1.3. Weighted Amalgam Spaces. Let $1 \leq p, s \leq \infty$, a function $f \in L^p_{\text{loc}}(\mathbb{R}^n)$ is said to be in the Wiener amalgam space $(L^p, L^s)(\mathbb{R}^n)$ of $L^p(\mathbb{R}^n)$ and $L^s(\mathbb{R}^n)$, if the function $\mu = |f(\cdot) \cdot \chi_{B(y, 1)}|_{L^p}$ belongs to $L^s(\mathbb{R}^n)$, where $B(y, 1)$ is an open ball in $\mathbb{R}^n$ centered at $y$ with radius $1$. $\chi_{B(y, 1)}$ is the characteristic function of the ball $B(y, 1)$, and $\|f\|_{L^p}$ is the usual Lebesgue norm in $L^p(\mathbb{R}^n)$. In [14], Fofana introduced a new class of function spaces $(L^p, L^s)^*(\mathbb{R}^n)$ which turned out to be the subspaces of $(L^p, L^s)(\mathbb{R}^n)$. More precisely, for $1 \leq p, s, \alpha \leq \infty$, we define the amalgam space $(L^p, L^s)^*(\mathbb{R}^n)$ of $L^p(\mathbb{R}^n)$ and $L^s(\mathbb{R}^n)$ as the set of all measurable functions $f$ satisfying $f \in L^p_{\text{loc}}(\mathbb{R}^n)$ and $\|f\|_{(L^p, L^s)^*(\mathbb{R}^n)} < \infty$, where

\[
\|f\|_{(L^p, L^s)^*(\mathbb{R}^n)} := \sup_{r > 0} \left\{ \int_{\mathbb{R}^n} \left[ |B(y, r)|^{\alpha-1/p} |f \cdot \chi_{B(y, r)}|_{L^p(\mathbb{R}^n)}^p \right] dy \right\}^{1/s},
\]

with the usual modification when $p = \infty$ or $s = \infty$, and $|B(y, r)|$ is the Lebesgue measure of the ball $B(y, r)$. As it was shown in [14] that the space $(L^p, L^s)^*(\mathbb{R}^n)$ is nontrivial if and only if $p \leq \alpha \leq s$, in the remaining of this paper, we will always assume that this condition $p \leq \alpha \leq s$ is satisfied. Let us consider the following two special cases:
(1) If we take \( p = s \), then \( p = \alpha = s \). By Fubini’s theorem, it is easy to check that

\[
\|B(y,r)\|^{-1/p}\|f \cdot X_{B(y,r)}\|_{L^p(R^n)} = \left[ \int_{\mathbb{R}^n} |B(y,r)|^{-1/2} \left( \int_{\mathbb{R}^n} |f(x)|^p \cdot X_{B(y,r)}^\alpha \right)^{1/p} \right]^{1/p} = \left( \int_{\mathbb{R}^n} |f(x)|^p \right)^{1/p},
\]

where the last equality holds since \( |B(y,r)|^{-1} \), \( |B(x,r)| \equiv 1 \). Hence, the amalgam space \( (L^p, L^s)^\alpha (\mathbb{R}^n) \) is equal to the Lebesgue space \( L^p(\mathbb{R}^n) \) with the same norms provided that \( p = \alpha = s \).

(2) If \( s = \infty \), then we can see that in such a situation, the amalgam space \( (L^p, L^s)^\alpha (\mathbb{R}^n) \) is equal to the classical Morrey space \( L^{p,\lambda}(\mathbb{R}^n) \) with equivalent norms, where \( \lambda = (pn)/\alpha \).

We are now in a position to state our main results. Let \( \nu, \mu \), and \( \lambda \) be three weights on \( \mathbb{R}^n \). We denote by \( (L^p, L^s)^\alpha (\nu, \mu) \) the weighted amalgam space, the space of all locally integrable functions \( f \) such that

\[
\|f\|_{(L^p, L^s)^\alpha (\nu, \mu)} := \sup_{\nu > 0} \left( \int_{\mathbb{R}^n} |\nu(x)|^{1/(\alpha - 1)} \|f \cdot X_{\nu(x)}\|^\alpha_{L\nu(x)} \right)^{1/\alpha}
= \sup_{\nu > 0} \left( \int_{\mathbb{R}^n} |\nu(x)|^{1/(\alpha - 1)} \|f \cdot X_{\nu(x)}\|^\alpha_{L\nu(x)} \right)^{1/\alpha}
\]

with \( \nu(x) = \int_{\mathbb{R}^n} \nu(x)dx \) and the usual modification when \( s = \infty \).

Note that in the particular case when \( \mu \equiv 1 \), this kind of weighted (weak) amalgam space was introduced by Feuto in [15] (see also [16]). We remark that Feuto [15] considered ball \( B \) instead of cube \( Q \) in his definition, but these two definitions are evidently equivalent. Also, note that when \( 1 \leq p \leq \alpha \) and \( s = \infty \), then \( (L^p, L^s)^\alpha (\nu, \mu) \) is just the weighted Morrey space \( L^{p,\lambda}(\nu, \mu) \) with \( \kappa = 1 - p/\alpha \), and \( (W(L^p, L^s)^\alpha (\nu, \mu) \) is just the weighted weak Morrey space \( W(L^{p,\lambda}(\nu, \mu) \) with \( \kappa = 1 - p/\alpha \).

Recently, in [17–19], the author studied the two-weight, weak-type \( (p, p) \) inequalities for fractional integral operator, as well as its commutators on weighted Morrey and amalgam spaces, under some \( A_p \)-type conditions (4) and (7) on the pair \((\nu, \mu)\). As a continuation of the works mentioned above, in this paper, we consider related problems about two-weight, weak-type \((p, q)\) inequalities for \( I_p \) and \([b, I_p] \), under some other \( A_p \)-type conditions (10) and (12) on \((\nu, \mu)\) and \( 1 < p < q \).

2. Statement of Our Main Results

We are now in a position to state our main results. Let \( p' \) be the conjugate index of \( p \) whenever \( p > 1 \); that is, \( 1/p + 1/p' = 1 \). First, we give the two-weight, weak-type norm inequalities for the fractional integral operator in the setting of weighted Morrey and amalgam spaces.
Theorem 6. Let $0 < \gamma < n$, $1 < p < q < \infty$ and $0 < \kappa < p/q$. Given a pair of weights $(w, \nu)$ on $\mathbb{R}^n$, suppose that for some $r > 1$ and for all cubes $Q$ in $\mathbb{R}^n$, 
\begin{align}
|Q|^{\gamma + 1/q - 1/p} \cdot \frac{1}{|Q|} \int_Q w(x)^r \, dx \right)^{1/(rq)} \cdot \left( \frac{1}{|Q|} \int_Q \nu(x)^{-\beta/r} \, dx \right)^{1/p} \leq C < \infty.
\end{align}

If $w \in A_\gamma$, then the fractional integral operator $I_\gamma$ is bounded from $L^{p, \infty}(\nu, w)$ into $WL^{q, (eq)/p}(w)$.

Theorem 7. Let $0 < \gamma < n$, $1 < p < q < \infty$ and $\mu \in A_\gamma$. Given a pair of weights $(w, \nu)$ on $\mathbb{R}^n$, assume that for some $r > 1$ and for all cubes $Q$ in $\mathbb{R}^n$, 
\begin{align}
|Q|^{\gamma + 1/q - 1/p} \cdot \frac{1}{|Q|} \int_Q w(x)^r \, dx \right)^{1/(rq)} \cdot \left( \frac{1}{|Q|} \int_Q \nu(x)^{-\beta/r} \, dx \right)^{1/p} \leq C < \infty.
\end{align}

If $p \leq \alpha < \beta < s \leq \infty$ and $w \in A_\alpha$, then the linear commutator $[b, I_\gamma]$ is bounded from $(L^p, L^s)^{\gamma}(\nu, w; \mu)$ into $(WL^q, L^p)^{\gamma}(\nu, w; \mu)$ with $1/\beta = 1/\alpha - (1/p - 1/q)$.

Moreover, for the extreme case $\kappa = p/q$ of Theorem 6, we will prove the following theorem, which could be viewed as a supplement of Theorem 6.

Theorem 8. Let $0 < \gamma < n$, $1 < p < q < \infty$. Given a pair of weights $(w, \nu)$ on $\mathbb{R}^n$, assume that for some $r > 1$ and for all cubes $Q$ in $\mathbb{R}^n$, 
\begin{align}
|Q|^{\gamma + 1/q - 1/p} \cdot \frac{1}{|Q|} \int_Q w(x)^r \, dx \right)^{1/(rq)} \cdot \left( \frac{1}{|Q|} \int_Q \nu(x)^{-\beta/r} \, dx \right)^{1/p} \leq C < \infty.
\end{align}

Next, we introduce the definition of the space of BMO $\mathbb{R}^n$. Suppose that for some $r > 1$ and for all cubes $Q$ in $\mathbb{R}^n$, 
\begin{align}
|Q|^{\gamma + 1/q - 1/p} \cdot \frac{1}{|Q|} \int_Q w(x)^r \, dx \right)^{1/(rq)} \cdot \left( \frac{1}{|Q|} \int_Q \nu(x)^{-\beta/r} \, dx \right)^{1/p} \leq C < \infty.
\end{align}

If $p \leq \alpha < \beta < s \leq \infty$ and $w \in A_\alpha$, then the linear commutator $[b, I_\gamma]$ is bounded from $(L^p, L^s)^{\gamma}(\nu, w; \mu)$ into BMO.

In addition, we will also discuss the extreme case $\beta = s$ of Theorem 7. In order to do so, we need to introduce the following new BMO-type space.

Definition 5. Let $1 \leq \gamma < \infty$ and $\mu \in A_\gamma$. The space $\text{BMO}(\mathbb{R}^n)$ is defined as the set of all locally integrable functions $f$ satisfying $\|f\|_\gamma < \infty$, where 
\begin{align}
\|f\|_\gamma := \sup_{\nu \in \mathcal{N}} \left\| \frac{1}{|Q|} \int_Q |f(x) - f_{\nu}(y)\, dx \right\|_{L_s(\nu)}.
\end{align}

Here, the $L_s(\nu)$-norm is taken with respect to the variable $y$. We also use the notation $f_{\nu}(y)$ to denote the mean value of $f$ on $Q(y, \ell)$. Observe that if $s = \infty$, then $(\text{BMO}, L_s^\infty)\gamma(\mu)$ is just the classical BMO space given above.

Now, we can show that $I_\gamma$ is bounded from $(L^p, L^s)^{\gamma}(\nu, w; \mu)$ into our new BMO-type space defined above. This new result may be viewed as a supplement of Theorem 7.

Theorem 11. Let $0 < \gamma < n$, $1 < p < q < \infty$, and $\mu \in A_\gamma$. Given a pair of weights $(w, \nu)$ on $\mathbb{R}^n$, assume that for some $r > 1$ and for all cubes $Q$ in $\mathbb{R}^n$, 
\begin{align}
|Q|^{\gamma + 1/q - 1/p} \cdot \frac{1}{|Q|} \int_Q w(x)^r \, dx \right)^{1/(rq)} \cdot \left( \frac{1}{|Q|} \int_Q \nu(x)^{-\beta/r} \, dx \right)^{1/p} \leq C < \infty.
\end{align}

If $p \leq \alpha < \beta < s \leq \infty$, $1/s = 1/\alpha - (1/p - 1/q)$, and $w \in A_\alpha$, then the fractional integral operator $I_\gamma$ is bounded from $(L^p, L^s)^{\gamma}(\nu, w; \mu)$ into $(\text{BMO}, L_s^\infty)\gamma(\nu, w; \mu)$. Theorem 10. Let $0 < \gamma < n$, $1 < p < q < \infty$. Given a pair of weights $(w, \nu)$ on $\mathbb{R}^n$, assume that for some $r > 1$ and for all cubes $Q$ in $\mathbb{R}^n$, 
\begin{align}
|Q|^{\gamma + 1/q - 1/p} \cdot \frac{1}{|Q|} \int_Q w(x)^r \, dx \right)^{1/(rq)} \cdot \left( \frac{1}{|Q|} \int_Q \nu(x)^{-\beta/r} \, dx \right)^{1/p} \leq C < \infty.
\end{align}

If $\kappa = p/q$ and $w \in A_\alpha$, then the fractional integral operator $I_\gamma$ is bounded from $L^{p, \infty}(\nu, w)$ into BMO.
3. Notation and Definitions

In this section, we recall some standard definitions and notation.

3.1. Weights. For given $y \in \mathbb{R}^n$ and $\ell > 0$, we denote by $Q(y, \ell)$ the cube centered at $y$ and has side length $\ell > 0$, and all cubes are assumed to have their sides parallel to the coordinate axes. Given a cube $Q(y, \ell)$ and $\lambda > 0$, $\lambda Q(y, \ell)$ stands for the cube concentric with $Q$ and having side length $\lambda \sqrt{n}$ times as long, i.e., $\lambda Q(y, \ell) = Q(y, \lambda \sqrt{n} \ell)$. A nonnegative function $w$ defined on $\mathbb{R}^n$ will be called a weight if it is locally integrable. For any given weight $w$ and any Lebesgue measurable set $E$ of $\mathbb{R}^n$, we denote the characteristic function of $E$ by $\chi_E$, the Lebesgue measure of $E$ by $|E|$, and the weighted measure of $E$ by $w(E)$, where $w(E) = \int_E w(x) \, dx$. We also denote $E^c = \mathbb{R}^n \setminus E$ the complement of $E$. Given a weight $w$, we say that $w$ satisfies the doubling condition, if there exists a finite constant $C > 0$ such that for any cube $Q$ in $\mathbb{R}^n$, we have

$$w(2Q) \leq C \cdot w(Q).$$

(34)

When $w$ satisfies this condition (34), we denote $w \in \Delta_2$ for brevity. A weight $w$ is said to belong to Muckenhoupt’s class $A_p$ for $1 < p < \infty$, if there exists a constant $C > 0$ such that

$$\left( \frac{1}{|Q|} \int_Q w(x) \, dx \right)^{1/p} \left( \frac{1}{|Q|} \int_Q w(x)^{-1/p} \, dx \right)^{1/p} \leq C,$$

(35)

holds for every cube $Q$ in $\mathbb{R}^n$. The class $A_{\infty}$ is defined as the union of the $A_p$ classes for $1 < p < \infty$, i.e., $A_{\infty} = \cup_{1 < p < \infty} A_p$. If $w$ is an $A_{\infty}$ weight, then we have $w \in \Delta_2$ (see [21]). Moreover, this class $A_\infty$ is characterized as the class of all weights satisfying the following property: there exists a number $\delta > 0$ and a finite constant $C > 0$ such that (see [21])

$$\frac{w(E)}{w(Q)} \leq C \left( \frac{|E|}{|Q|} \right)^{\delta},$$

(36)

holds for every cube $Q \subset \mathbb{R}^n$ and all measurable subsets $E$ of $Q$. Given a weight $w$ on $\mathbb{R}^n$ and for $1 \leq p < \infty$, the weighted Lebesgue space $L^p(w)$ is defined to be the collection of all measurable functions $f$ satisfying

$$\|f\|_{L^p(w)} := \left( \int_{\mathbb{R}^n} |f(x)|^p w(x) \, dx \right)^{1/p} < \infty.$$

(37)

For a weight $w$ and $1 \leq p < \infty$, define the distribution function of $f$ with $w$ by

$$d_f(\lambda) = w(\{ x \in \mathbb{R}^n : |f(x)| > \lambda \}),$$

(38)

where $\lambda$ is a positive number. We say that $f$ is in the weighted weak Lebesgue space $W(L^p)(w)$ if there exists a constant $C > 0$ such that

$$\|f\|_{W(L^p)(w)} := \sup_{\lambda > 0} \lambda \cdot d_f(\lambda)^{1/p} \leq C < \infty.$$

(39)

3.2. Orlicz Spaces. We next recall some basic facts from the theory of Orlicz spaces needed for the proofs of the main results. For more information about these spaces the reader may consult the book [22]. Let $\mathcal{A}$ : $[0, +\infty) \rightarrow [0, +\infty)$ be a Young function. That is, a continuous, convex, and strictly increasing function satisfying $\mathcal{A}(0) = 0$ and $\mathcal{A}(t) \rightarrow +\infty$ as $t \rightarrow +\infty$. For a Young function $\mathcal{A}$ and a cube $Q$ in $\mathbb{R}^n$, we will consider the $\mathcal{A}$-average of a function $f$ given by the following Luxemburg norm:

$$\|f\|_{\mathcal{A},Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q \mathcal{A}\left( \frac{|f(x)|}{\lambda} \right) \, dx \leq 1 \right\}.$$ 

(40)

In particular, when $\mathcal{A}(t) = t^p$ with $1 < p < \infty$, it is easy to see that

$$\|f\|_{\mathcal{A},Q} = \left( \frac{1}{|Q|} \int_Q |f(x)|^p \, dx \right)^{1/p},$$

(41)

that is, the Luxemburg norm in such a situation coincides with the normalized $L^p$ norm. The main examples that we are going to consider are $\mathcal{A}(t) = t^p (1 + \log^t)^p$ with $1 < p < \infty$.

Throughout the paper, $C$ always denotes a positive constant independent of the main parameters involved, but it may be different from line to line. We will use $A \preceq B$ to denote the equivalence of $A$ and $B$; that is, there exist two positive constants $C_1$ and $C_2$ independent of $A$ and $B$ such that $C_1 A \preceq B \preceq C_2 A$.

4. Proofs of Theorems 6 and 7

Proof of Theorem 6. Let $f \in L^{p_0}(w)$ with $1 < p < q < \infty$ and $0 < \kappa < p/q$. An arbitrary fixed cube $Q = Q(x_0, \ell)$ in $\mathbb{R}^n$, we decompose $f$ as

$$f = f_1 + f_2 \in L^{p_0}(w);$$

$$f_1 = f \cdot \chi_{2Q};$$

$$f_2 = f \cdot \chi_{(2Q)^c},$$

(42)

where $2Q := Q(x_0, 2\sqrt{n} \ell)$ and $\chi_{2Q}$ denotes the characteristic function of $2Q$. For any given $\sigma > 0$, we then write

$$\frac{1}{w(2Q)^{1/p} \cdot \sigma^{1/p}} \left( w\left( \left\{ x \in Q : |f_1(x)| > \sigma \right\} \right) \right)^{1/q} \leq \frac{1}{w(Q)^{1/p} \cdot \sigma^{1/p}} \left( w\left( \left\{ x \in Q : |f_1(x)| > \sigma \right\} \right) \right)^{1/q}$$

$$+ \frac{1}{w(Q)^{1/p} \cdot \sigma^{1/p}} \left( w\left( \left\{ x \in Q : |f_2(x)| > \sigma \right\} \right) \right)^{1/q}$$

$$= I_1 + I_2.$$

(43)

Let us consider the first term $I_1$. Using Theorem 4 and the condition $w \in \Delta_2$, we have
Moreover, we apply Hölder’s inequality again with exponent \( r > 1 \) to get
\[
w(2^{j+1}Q) = \left( \int_{2^{j+1}Q} |w(y)| dy \right)^{1/r} \left( \int_{2^{j+1}Q} |w(y)|^r dy \right)^{1/r}.
\]

This indicates that
\[
I_2 \leq C \|f\|_{L^p(\mathbb{R}^n)} \sum_{j=1}^{\infty} \frac{w(2^{j+1}Q)^{1/q'}}{w(Q)^{1/q'}} \int_{2^{j+1}Q} |w(y)|^{1/r} \left( \int_{2^{j+1}Q} |w(y)|^r dy \right)^{1/r}.
\]

The last inequality is obtained by the \( A_p \)-type condition (10) assumed on \((w, v)\). Furthermore, since \( w \in A_2 \), we can easily check that there exists a reverse doubling constant \( D = D(w) > 1 \) independent of \( Q \) such that (see [13], Lemma 4.1)
\[
w(2Q) \geq D \cdot w(Q),
\]
which implies that for any positive integer \( j \),
\[
w(2^{j+1}Q) \geq D^{j+1} \cdot w(Q),
\]
by induction principle. Hence,
\[
\sum_{j=1}^{\infty} \frac{w(2^{j+1}Q)^{1/q'}}{w(Q)^{1/q'}} \leq \sum_{j=1}^{\infty} \left( \frac{D^{j+1}}{D^{j+1}} \right)^{1/q'} \leq C,
\]
where the last series is convergent since the reverse doubling constant \( D > 1 \) and \( 1/q - \kappa/p > 0 \). Therefore, in view of (52), we get
\[
I_2 \leq C \|f\|_{L^p(\mathbb{R}^n)},
\]
which is our desired inequality. Combining the above estimates for \( I_1 \) and \( I_2 \) and then taking the supremum over all cubes \( Q \subset \mathbb{R}^n \) and all \( \sigma > 0 \), we complete the proof of Theorem 6. \( \square \)

**Proof of Theorem 7.** Let \( 1 < p \leq \alpha < s \leq \infty \) and \( f \in (L^p, L^s) \) \((v, w; \mu) \) with \( \mu \in \Delta_2 \) and \( \mu \in \Delta_2 \). For an arbitrary point \( y \in \mathbb{R}^n \), we set \( Q = Q(y, \ell) \) for the cube centered at \( y \) and of side length \( \ell \). Decompose \( f \) as
\[
\begin{cases}
  f = f_1 + f_2 \in (L^p, L^s) \quad (v, w; \mu);
  f_1 = f \cdot \chi_{2Q};
  f_2 = f \cdot \chi_{(2Q)^c},
\end{cases}
\]
where \( 2Q = Q(y, 2\sqrt{n} \ell) \). Then, for given \( y \in \mathbb{R}^n \) and \( \ell > 0 \), we write
\[
\begin{align*}
\omega(Q(y, \ell))^{1/\beta - 1/q - 1/n} & \left( I_y(f_y) \cdot \chi_{Q(y, \ell)} \right)_{\text{WL}^\infty(w)} \\
& \leq 2 \cdot \omega(Q(y, \ell))^{1/\beta - 1/q - 1/n} I_y(f_y) \cdot \chi_{Q(y, \ell)}_{\text{WL}^\infty(w)} \\
& \quad + 2 \cdot \omega(Q(y, \ell))^{1/\beta - 1/q - 1/n} I_y(f_y) \cdot \chi_{Q(y, \ell)}_{\text{WL}^\infty(w)} \\
& = I_1(y, \ell) + I_2(y, \ell).
\end{align*}
\]

Let us consider the first term \(I_1(y, \ell).\) According to Theorem 4, we get

\[
I_1(y, \ell) \leq 2 \cdot \omega(Q(y, \ell))^{1/\beta - 1/q - 1/n} \left( \int_{\mathbb{R}^n} |f_y(x)|^p \nu(x) \, dx \right)^{1/p} \\
\quad \leq C \cdot \omega(Q(y, \ell))^{1/\beta - 1/q - 1/n} \left( \int_{Q(y, 2\sqrt{n} \ell)} |f_y(x)|^p \nu(x) \, dx \right)^{1/p} \\
= C \cdot \omega(Q(y, \ell))^{1/\beta - 1/q - 1/n} \left( \int_{Q(y, 2\sqrt{n} \ell)} |f_y(x)|^p \nu(x) \, dx \right)^{1/p}.
\]

Observe that

\[
\frac{1}{\beta} - \frac{1}{q} - \frac{1}{s} = \frac{1}{\alpha} - \frac{1}{p} - \frac{1}{s}
\]

is valid by our assumption \(1/\beta = 1/\alpha - (1/p - 1/q).\) This allows us to obtain

\[
I_1(y, \ell) \leq C \cdot \omega(Q(y, \ell))^{1/\alpha - 1/p - 1/s} \left( \int_{Q(y, 2\sqrt{n} \ell)} |f_y(x)|^p \nu(x) \, dx \right)^{1/p} \\
= C \cdot \omega(Q(y, \ell))^{1/\alpha - 1/p - 1/s} \left( \int_{Q(y, 2\sqrt{n} \ell)} |f_y(x)|^p \nu(x) \, dx \right)^{1/p} \\
\times \frac{\omega(Q(y, \ell))^{1/\alpha - 1/p - 1/s}}{\omega(Q(y, 2\sqrt{n} \ell))^{1/\alpha - 1/p - 1/s}}.
\]

Moreover, since \(1/\alpha - 1/p - 1/s < 0\) and \(w \in \Delta_2,\) by doubling inequality (54), we get

\[
\frac{\omega(Q(y, \ell))^{1/\alpha - 1/p - 1/s}}{\omega(Q(y, 2\sqrt{n} \ell))^{1/\alpha - 1/p - 1/s}} \leq C.
\]

Substituting the above inequality (59) into (58), we thus obtain

\[
I_1(y, \ell) \leq C \cdot \omega(Q(y, 2\sqrt{n} \ell))^{1/\alpha - 1/p - 1/s} \left( \int_{Q(y, 2\sqrt{n} \ell)} |f_y(x)|^p \nu(x) \, dx \right)^{1/p}.
\]

We now estimate the second term \(I_2(y, \ell).\) Recall that by the definition of \(I_y,\) the following estimate holds for any \(x \in Q(y, \ell):\)

\[
\left| I_y(f_y)(x) \right| \leq C \sum_{j=1}^{\infty} \frac{1}{|Q(y, 2^{j+1} \sqrt{n} \ell)|^{1-\gamma/n}} \int_{Q(y, 2^{j+1} \sqrt{n} \ell)} |f(z)| \nu(z) \, dz.
\]

This pointwise estimate (61) along with Chebyshev’s inequality implies that

\[
I_2(y, \ell) \leq C \cdot \omega(Q(y, \ell))^{1/\beta - 1/q - 1/n} \left( \int_{Q(y, 2\sqrt{n} \ell)} |f_y(x)|^p \nu(x) \, dx \right)^{1/p} \\
\quad \leq C \cdot \omega(Q(y, \ell))^{1/\beta - 1/q - 1/n} \sum_{j=1}^{\infty} \frac{1}{|Q(y, 2^{j+1} \sqrt{n} \ell)|^{1-\gamma/n}} \int_{Q(y, 2^{j+1} \sqrt{n} \ell)} |f(z)| \nu(z) \, dz.
\]

A further application of Hölder’s inequality yields

\[
I_2(y, \ell) \leq C \cdot \omega(Q(y, \ell))^{1/\beta - 1/n} \sum_{j=1}^{\infty} \frac{1}{|Q(y, 2^{j+1} \sqrt{n} \ell)|^{1-\gamma/n}} \left( \int_{Q(y, 2^{j+1} \sqrt{n} \ell)} |f(z)| \nu(z) \, dz \right)^{1/p} \\
\quad \times \left( \int_{Q(y, 2^{j+1} \sqrt{n} \ell)} \nu(z)^{-\gamma/n} \, dz \right)^{1/p'} \\
= C \sum_{j=1}^{\infty} \frac{\omega(Q(y, 2^{j+1} \sqrt{n} \ell))^{1/\beta - 1/q - 1/n}}{\omega(Q(y, 2^{j+1} \sqrt{n} \ell))^{1/\beta - 1/q - 1/n}} \\
\quad \times \frac{\omega(Q(y, 2^{j+1} \sqrt{n} \ell))^{1/\beta - 1/q - 1/n}}{\omega(Q(y, 2^{j+1} \sqrt{n} \ell))^{1/\beta - 1/q - 1/n}} \\
\quad \times \left( \int_{Q(y, 2^{j+1} \sqrt{n} \ell)} \nu(z)^{-\gamma/n} \, dz \right)^{1/p'}.
\]
In addition, we apply Hölder’s inequality again with exponent $r > 1$ to get

\[
\begin{align*}
  w(Q(y, 2^{j+1} \sqrt{n} \ell)) &= \int_{Q(y, 2^{j+1} \sqrt{n} \ell)} w(z) \, dz \\
  \leq \left[ Q(y, 2^{j+1} \sqrt{n} \ell) \right]^{1/r} \left( \int_{Q(y, 2^{j+1} \sqrt{n} \ell)} w(z)^r \, dz \right)^{1/r}.
\end{align*}
\]  

(64)

Hence, in view of (64) and (57), we have

\[
\begin{align*}
  I_2(y, \ell) &\leq C \sum_{j=1}^{\infty} w(Q(y, 2^{j+1} \sqrt{n} \ell))^{1/\alpha - 1/p - 1/\alpha} \left\| f \cdot X_{Q(y, 2^{j+1} \sqrt{n} \ell)} \right\|_{L^p(\mathbb{R})} \cdot \frac{w(Q(y, \ell))^{1/\beta - 1/s}}{w(Q(y, 2^{j+1} \sqrt{n} \ell))^{1/\beta - 1/s}} \\
  &\quad \times \left( \int_{Q(y, 2^{j+1} \sqrt{n} \ell)} w(z)^r \, dz \right)^{1/\alpha} \left( \int_{Q(y, 2^{j+1} \sqrt{n} \ell)} w(z)^p \, dz \right)^{1/p} \\
  &\leq C \sum_{j=1}^{\infty} w(Q(y, 2^{j+1} \sqrt{n} \ell))^{1/\alpha - 1/p - 1/\alpha} \left\| f \cdot X_{Q(y, 2^{j+1} \sqrt{n} \ell)} \right\|_{L^p(\mathbb{R})} \cdot \frac{w(Q(y, \ell))^{1/\beta - 1/s}}{w(Q(y, 2^{j+1} \sqrt{n} \ell))^{1/\beta - 1/s}}.
\end{align*}
\]  

(65)

The last inequality is obtained by the $A_p$-type condition assumed on $(w, \nu)$. Furthermore, arguing as in the proof of Theorem 6, we know that for any positive integer $j$, there exists a reverse doubling constant $D = D(w) > 1$ independent of $Q(y, \ell)$ such that

\[
\begin{align*}
  w(Q(y, 2^{j+1} \sqrt{n} \ell)) &\geq D^{j+1} \cdot w(Q(y, \ell)).
\end{align*}
\]  

(66)

Hence, we compute

\[
\begin{align*}
  \sum_{j=1}^{\infty} \frac{w(Q(y, \ell))^{1/\beta - 1/s}}{w(Q(y, 2^{j+1} \sqrt{n} \ell))^{1/\beta - 1/s}} &\leq \sum_{j=1}^{\infty} \left( \frac{w(Q(y, \ell))}{D^{j+1} \cdot w(Q(y, \ell))} \right)^{1/\beta - 1/s} \\
  &\leq \sum_{j=1}^{\infty} \left( \frac{1}{D^{j+1}} \right)^{1/\beta - 1/s} \leq C,
\end{align*}
\]  

(67)

where the last series is convergent since the reverse doubling constant $D > 1$ and $1/\beta - 1/s > 0$. Therefore, by taking the $L^s(\mu)$-norm of both sides of (55) (with respect to the variable $y$) and then using Minkowski’s inequality, (60) and (65), we obtain

\[
\begin{align*}
  \left\| w(Q(y, \ell))^{1/\beta - 1/r} \right\|_{L^s(\mu)} &\leq \left\| I_1(y, \ell) \right\|_{L^s(\mu)} + \left\| I_2(y, \ell) \right\|_{L^s(\mu)} \\
  &\leq C \left\| w(Q(y, 2 \sqrt{n} \ell))^{1/\alpha - 1/p - 1/\alpha} \left\| f \cdot X_{Q(y, 2 \sqrt{n} \ell)} \right\|_{L^p(\mathbb{R})} \right\|_{L^s(\mu)} \\
  &\quad + C \sum_{j=1}^{\infty} \left\| w(Q(y, 2^{j+1} \sqrt{n} \ell))^{1/\alpha - 1/p - 1/\alpha} \left\| f \cdot X_{Q(y, 2^{j+1} \sqrt{n} \ell)} \right\|_{L^p(\mathbb{R})} \right\|_{L^s(\mu)} \\
  &\quad \times \frac{w(Q(y, \ell))^{1/\beta - 1/s}}{w(Q(y, 2^{j+1} \sqrt{n} \ell))^{1/\beta - 1/s}} \\
  &\leq C \left\| f \right\|_{(L^p(L^r)^\mu, \nu)} + C \left\| f \right\|_{(L^p(L^r)^\mu, \nu)} \sum_{j=1}^{\infty} \frac{w(Q(y, \ell))^{1/\beta - 1/s}}{w(Q(y, 2^{j+1} \sqrt{n} \ell))^{1/\beta - 1/s}} \leq C \left\| f \right\|_{(L^p(L^r)^\mu, \nu)}.
\end{align*}
\]  

(68)
where the last inequality follows from (67). Thus, by taking the supremum over all \( t > 0 \), we finish the proof of Theorem 7.

5. Proofs of Theorems 8 and 9

For the results involving commutators, we need the following properties of BMO \((\mathbb{R}^n)\), which can be found in [23, 24].

**Lemma 1.** Let be a function in BMO \((\mathbb{R}^n)\).

(i) For every cube \( Q \) in \( \mathbb{R}^n \) and for any positive integer \( j \),

\[
|b_{2^n+jQ} - b_Q| \leq C \cdot (j + 1)\|b\|.
\]

(ii) Let \( 1 < p < \infty \). For every cube \( Q \) in \( \mathbb{R}^n \) and for any \( w \in A_{\infty}, \)

\[
\left( \frac{1}{|Q|} \int_Q |b(x) - b_Q|^{\frac{p}{d}}w(x)\,dx \right)^{\frac{1}{p}} \leq C\|b\| \cdot w(Q)^{1/p}.
\]

Before proving our main theorems, we will also need a generalization of Hölder’s inequality due to O’Neil [25].

**Lemma 2.** Let \( \mathcal{A}, \mathcal{B}, \) and \( \mathcal{C} \) be Young functions such that for all \( t > 0, \)

\[
\mathcal{A}^{-1}(t) \cdot \mathcal{B}^{-1}(t) \leq \mathcal{C}^{-1}(t),
\]

where \( \mathcal{A}^{-1}(t) \) is the inverse function of \( \mathcal{A}(t) \). Then, for all functions \( f \) and \( g \) and all cubes \( Q \) in \( \mathbb{R}^n, \)

\[
\|f \cdot g\|_{\mathcal{B}Q} \leq 2\|f\|_{\mathcal{A}Q} \|g\|_{\mathcal{C}Q}.
\]

We are now ready to give the proofs of Theorems 8 and 9.

**Proof of Theorem 8.** Let \( f \in L^{p\times}(v, w) \) with \( 1 < p < q < \infty \) and \( 0 < \kappa < p/q \). For any given cube \( Q = Q(x_0, \ell) \subset \mathbb{R}^n \), as before, we decompose \( f \) as

\[
\begin{cases}
\hat{f} = f_1 + f_2 \in L^{p\times}(v, w); \\
\hat{f}_1 = \hat{f} : \chi_{2Q}; \\
\hat{f}_2 = \hat{f} : \chi_{(2Q)^c},
\end{cases}
\]

where \( 2Q = Q(x_0, 2\sqrt{n}\ell) \). Then, for any given \( \sigma > 0 \), one writes

\[
\frac{1}{w(Q)^{\frac{1}{p'\times}}} \sigma \cdot \left[w\left( x \in Q : \left|b, I_{\hat{f}}(f)(x) - \sigma\right| \right)\right]^{\frac{1}{q}}
\]

\[
\leq \frac{1}{w(Q)^{\frac{1}{p'\times}}} \sigma \cdot \left[w\left( x \in Q : \left|b, I_{\hat{f}_1}(f_1)(x) - \sigma\right| \right)\right]^{\frac{1}{q}}
\]

\[
+ \frac{1}{w(Q)^{\frac{1}{p'\times}}} \sigma \cdot \left[w\left( x \in Q : \left|b, I_{\hat{f}_2}(f_2)(x) - \sigma\right| \right)\right]^{\frac{1}{q}}
\]

\[
= J_1 + J_2.
\]

Applying Theorem 5 and doubling inequality (34), we have

\[
J_1 \leq C \cdot \frac{1}{w(Q)^{\frac{1}{p'\times}}} \left(\int_{\mathbb{R}^n} |f_1(x)|^{\frac{p}{d}}v(x)\,dx\right)^{\frac{1}{p}}
\]

\[
= C \cdot \frac{1}{w(Q)^{\frac{1}{p'\times}}} \left(\int_{2Q} |f(x)|^{\frac{p}{d}}v(x)\,dx\right)^{\frac{1}{p}}
\]

\[
\leq \|f\|_{L^{p\times}(v, w), w(Q)^{\frac{1}{p'\times}}} \cdot \frac{w(2Q)^{\frac{1}{p'\times}}}{w(Q)^{\frac{1}{p'\times}}}
\]

\[
\leq C\|f\|_{L^{p\times}(v, w), w}.\]

On the other hand, for any \( x \in Q \), from (6), it then follows that

\[
\left|\left[\left[ b, I_{\hat{f}_1}\right](f_1)(x)\right] \right| \leq C \int_{\mathbb{R}^n} \left|b(x) - b(y)\right| \cdot \left|f_2\left( y \right)\right| \frac{dy}{|x - y|^{\kappa + \gamma}}
\]

\[
\leq C \left|b(x) - b_Q\right| \cdot \int_{\mathbb{R}^n} \left|f_2\left( y \right)\right| \frac{dy}{|x - y|^{\kappa + \gamma}}
\]

\[
+ C \int_{\mathbb{R}^n} \left|b\left( y \right) - b_Q\right| \cdot \left|f_2\left( y \right)\right| \frac{dy}{|x - y|^{\kappa + \gamma}}
\]

\[
= \xi(x) + \eta(x).
\]

Thus, we can further split \( J_2 \) into two parts as follows:

\[
J_2 \leq \frac{1}{w(Q)^{\frac{1}{p'\times}}} \sigma \cdot \left[w\left( x \in Q : \xi(x) > \frac{\sigma}{4}\right)\right]^{\frac{1}{q}} + \frac{1}{w(Q)^{\frac{1}{p'\times}}} \sigma \cdot \left[w\left( x \in Q : \eta(x) > \frac{\sigma}{4}\right)\right]^{\frac{1}{q}}
\]

\[
= J_3 + J_4.
\]

Using the pointwise estimate (45) and Chebyshev’s inequality, we obtain that

\[
J_3 \leq \frac{C}{w(Q)^{\frac{1}{p'\times}}} \sigma \cdot \left[w\left( x \in Q : \xi(x) > \frac{\sigma}{4}\right)\right]^{\frac{1}{q}}
\]

\[
\leq \frac{C}{w(Q)^{\frac{1}{p'\times}}} \sigma \cdot \left[w\left( x \in Q : \xi(x) > \frac{\sigma}{4}\right)\right]^{\frac{1}{q}}
\]

\[
J_4 \leq \frac{C}{w(Q)^{\frac{1}{p'\times}}} \sigma \cdot \left[w\left( x \in Q : \eta(x) > \frac{\sigma}{4}\right)\right]^{\frac{1}{q}}
\]

\[
\leq \frac{C}{w(Q)^{\frac{1}{p'\times}}} \sigma \cdot \left[w\left( x \in Q : \eta(x) > \frac{\sigma}{4}\right)\right]^{\frac{1}{q}}
\]

\[
= J_3 + J_4.
\]
\[ J_s \leq \frac{4}{w(Q)^{\epsilon/p}} \left( \int_Q |\xi(x)|^p w(x) \, dx \right)^{1/q} \]
\[
\leq C \frac{1}{w(Q)^{\epsilon/p}} \left( \int_Q |b(x) - b_0|^p w(x) \, dx \right)^{1/q} \sum_{j=1}^{\infty} \frac{1}{|2^{j+1}Q|^{1-\gamma/n}} \int_{2^{j+1}Q} |f(y)| \, dy
\]
\[
\leq C\|b\| \cdot w(Q)^{1/q - \epsilon/p} \sum_{j=1}^{\infty} \frac{1}{|2^{j+1}Q|^{1-\gamma/n}} \int_{2^{j+1}Q} |f(y)| \, dy,
\]

where the last inequality is due to the assumption \( w \in A_\infty \) and Lemma 1 \((ii)\). By the same manner as in the proof of Theorem 6, we can also show that
\[
J_5 \leq C\|f\|_{L^{p^*}(y, w)}.
\]  

(79)

Similar to the proof of (45), for all \( x \in Q \), we can show the following pointwise inequality as well:
\[
|\eta(x)| \leq C \int_{(2Q)^c} \left\{ \frac{|b(y) - b_Q| \cdot |f(y)|}{|x_0 - y|^{\gamma}} \right\} \, dy,
\]
\[
\leq C \sum_{j=1}^{\infty} \frac{1}{|2^{j+1}Q|^{1-\gamma/n}} \int_{2^{j+1}Q} |b(y) - b_Q| \cdot |f(y)| \, dy.
\]  

(80)

This, together with Chebyshev’s inequality, yields
\[
J_5 \leq C \cdot w(Q)^{1/q - \epsilon/p} \sum_{j=1}^{\infty} \frac{1}{|2^{j+1}Q|^{1-\gamma/n}} \left( \int_{2^{j+1}Q} |f(y)|^p w(y) \, dy \right)^{1/p} \times \left( \int_{2^{j+1}Q} |b(y) - b_{2^{j+1}Q}|^p w(y)^{-p'/p} \, dy \right)^{1/p'}
\]
\[
\leq C\|f\|_{L^{p^*}(y, w)} \cdot w(Q)^{1/q - \epsilon/p} \sum_{j=1}^{\infty} \frac{w(Q)^{\epsilon/p}}{|2^{j+1}Q|^{1-\gamma/n}} \times |2^{j+1}Q|^{1/p'} \left\| [b - b_{2^{j+1}Q}] \cdot w^{-1/p} \right\|_{\ell, 2^{j+1}Q}
\]  

\[ \mathcal{B}(t) \approx t^{p^*} (1 + \log^* t)^{p'/q}, \]
\[ \mathcal{B}(t) \approx \exp \{ (1 - 1/q) t \}. \]  

(84)

Let \( \|b\|_{\exp L_Q} \) denote the mean Luxemburg norm of \( h \) on cube \( Q \) with Young function \( \mathcal{B}(t) \approx \exp \{ (1 - 1/q) t \} \). According to Lemma 2, we have
\[
\left\| [b - b_{2^{j+1}Q}] \cdot w^{-1/p} \right\|_{\ell, 2^{j+1}Q} \leq C\|b - b_{2^{j+1}Q}\|_{\exp L_Q} \times |2^{j+1}Q|^{1/p'} \left\| [b - b_{2^{j+1}Q}] \cdot w^{-1/p} \right\|_{\ell, 2^{j+1}Q},
\]

(85)
where in the last inequality, we have used the well-known fact that (see [23])
\[ \|b - b_Q\|_{L^p(Q)} \leq C\|b\|_L, \quad \text{for any cube } Q \subset \mathbb{R}^n. \] (86)

Indeed, the above inequality (86) is equivalent to the following inequality:

\[
J_5 \leq C\|f\|_{L^{p^*}(\mathbb{R}^n)} \sum_{j=1}^{\infty} \frac{w(Q)^{1/q - k/p}}{w(2^{j+1}Q)^{1/q - k/p}} \cdot \frac{w(2^{j+1}Q)^{1/q}}{|2^{j+1}Q|^{1/p - q/\kappa}} \cdot \|b\|_L \cdot \|y^{\cdot} - \|y\|_L^p\|_{L^{p^*}(\mathbb{R}^n)}
\] (88)

Since \( w \in A_{\kappa_0} \), we know that \( w \in A_2 \). Furthermore, by the \( A_p \)-type condition (12) on \( (w, \nu) \) and the estimate (52), we obtain
\[
J_5 \leq C\|f\|_{L^{p^*}(\mathbb{R}^n)} \sum_{j=1}^{\infty} \frac{w(Q)^{1/q - k/p}}{w(2^{j+1}Q)^{1/q - k/p}} \leq C\|f\|_{L^{p^*}(\mathbb{R}^n)}.
\] (89)

\[
J_6 \leq C \cdot w(Q)^{1/q - k/p} \sum_{j=1}^{\infty} \frac{(j + 1)}{2^{j+1}|Q|^{1/q - \kappa m}} \int_{2^{j+1}Q} |f(y)| dy
\]

\[
\leq C \cdot w(Q)^{1/q - k/p} \sum_{j=1}^{\infty} \frac{(j + 1)}{2^{j+1}|Q|^{1/q - \kappa m}} \left( \int_{2^{j+1}Q} |f(y)|^p \nu(y) dy \right)^{1/p} \left( \int_{2^{j+1}Q} \nu(y)^{-1/p'} dy \right)^{1/p'}
\] (90)

\[
\leq C\|f\|_{L^{p^*}(\mathbb{R}^n)} \cdot w(Q)^{1/q - k/p} \sum_{j=1}^{\infty} \frac{w(2^{j+1}Q)^{1/q}}{|2^{j+1}Q|^{1/q - \kappa m}} \left( \int_{2^{j+1}Q} \nu(y)^{-1/p'} dy \right)^{1/p'}
\]

\[
= C\|f\|_{L^{p^*}(\mathbb{R}^n)} \sum_{j=1}^{\infty} \frac{w(Q)^{1/q - k/p}}{w(2^{j+1}Q)^{1/q - k/p}} \cdot \frac{w(2^{j+1}Q)^{1/q}}{|2^{j+1}Q|^{1/q - \kappa m}} \left( \int_{2^{j+1}Q} \nu(y)^{-1/p'} dy \right)^{1/p'}
\]
Let $C(t)$ and $A(t)$ be the same as before. Clearly, $C(t) \leq A(t)$ for all $t > 0$; then, for any cube $Q$ in $\mathbb{R}^n$, one has $\|f\|_{L^Q} \leq \|f\|_{A(t)}$ by definition, which implies that condition (12) is stronger than condition (10). This fact together with (48) yields

$$J_5 \leq C \|f\|_{L^Q} \sum_{j=1}^{\infty} (j + 1) \cdot \frac{w(Q)^{1/q - \kappa/p}}{w(2^{j+1}Q)^{1/q - \kappa/p}} \cdot \left(\frac{2^{j+1}Q}{2^{j+1}Q}\right)^{1/p} \cdot 2^{j+1}Q^{1/q - \kappa/p}.$$

Moreover, by our hypothesis on $w$: $w \in A_{\infty}$, there exists a number $\delta > 0$ such that the inequality (36) holds, and hence we compute

$$\sum_{j=1}^{\infty} (j + 1) \cdot \frac{w(Q)^{1/q - \kappa/p}}{w(2^{j+1}Q)^{1/q - \kappa/p}} \leq C \sum_{j=1}^{\infty} (j + 1) \cdot \left(\frac{|Q|}{2^{j+1}Q}\right)^{1/(q - \kappa/p)}.$$

(92)

where the last series is convergent since the exponent $\delta(1/q - \kappa/p)$ is positive. This implies our desired estimate

$$J_5 \leq C \|f\|_{L^Q}.$$

(93)

Summarizing the estimates derived above and then taking the supremum over all cubes $Q \subset \mathbb{R}^n$ and all $\sigma > 0$, we conclude the proof of Theorem 8.

---

**Proof of Theorem 9.** Let $1 < p \leq s < s \leq \infty$ and $f \in (L^p, L^q)^{1/n}(v, w; \mu)$ with $w \in A_{\infty}$ and $\mu \in \Delta_2$. For any fixed cube $Q = Q(y, \ell)$ in $\mathbb{R}^n$, as usual, we decompose $f$ as

$$f = f_1 + f_2 \in (L^p, L^q)^{1/n}(v, w; \mu);$$

$$f_1 = f \cdot \chi_{Q};$$

$$f_2 = f \cdot \chi_{(2Q)^c},$$

where $2Q = Q(y, 2\sqrt{n} \ell)$. Then, for given $y \in \mathbb{R}^n$ and $\ell > 0$, we write

$$w(Q(y, \ell))^{1/p - 1/q - 1/s} ||| [b, I_1] (f) \cdot \chi_{Q(y, \ell)} |||_{W^{L^p} (w)}$$

$$\leq 2 \cdot w(Q(y, \ell))^{1/p - 1/q - 1/s} \left[ ||| [b, I_1] (f_1) \cdot \chi_{Q(y, \ell)} |||_{W^{L^p} (w)} + 2 \cdot w(Q(y, \ell))^{1/p - 1/q - 1/s} \left[ ||| [b, I_1] (f_2) \cdot \chi_{Q(y, \ell)} |||_{W^{L^p} (w)}$$

$$= J_1 (y, \ell) + J_2 (y, \ell).$$

(94)

Next, we shall calculate the two terms, respectively. According to Theorem 5, we get

$$J_1 (y, \ell) \leq 2 \cdot w(Q(y, \ell))^{1/p - 1/q - 1/s} \left[ ||| [b, I_1] (f_1) \cdot \chi_{Q(y, \ell)} |||_{W^{L^p} (w)}$$

$$\leq C \cdot w(Q(y, \ell))^{1/p - 1/q - 1/s} \int_{Q(y, 2\sqrt{n} \ell)} |f(x)|^p \chi_{Q(y, \ell)} dx \right]^{1/p}$$

$$= C \cdot w(Q(y, 2\sqrt{n} \ell))^{1/p - 1/q - 1/s} \left[ \int_{Q(y, 2\sqrt{n} \ell)} |f(x)|^p \chi_{Q(y, \ell)} dx \right]^{1/p}$$

$$\leq C \cdot w(Q(y, 2\sqrt{n} \ell))^{1/p - 1/q - 1/s} \left[ \int_{Q(y, 2\sqrt{n} \ell)} |f(x)|^p \chi_{Q(y, \ell)} dx \right]^{1/p}$$

(95)

where the last identity is due to (57). Moreover, since $w \in A_{\infty}$, we know that $w \in \Delta_2$, and hence by inequality (59),

$$J_1 (y, \ell) \leq C \cdot w(Q(y, 2\sqrt{n} \ell))^{1/p - 1/q - 1/s} \left[ \int_{Q(y, 2\sqrt{n} \ell)} |f(x)|^p \chi_{Q(y, \ell)} dx \right]^{1/p}$$

(96)

On the other hand, from (6), one can see that for any $x \in Q(y, \ell)$,

$$\left[ [b, I_1] (f_2) (x) \right] \leq |b(x) - b_{Q(y, \ell)}| \cdot |I_y (f_2) (x)| + |I_y (b_{Q(y, \ell)} - b) (f_2) (x)|$$

$$= \tilde{\eta} (x) + \eta (x).$$

(97)

Consequently, we can further divide $J_2 (y, \ell)$ into two parts:

$$J_2 (y, \ell) \leq 4 \cdot w(Q(y, \ell))^{1/p - 1/q - 1/s} \left[ \tilde{\eta} (x) \cdot \chi_{Q(y, \ell)} \right]_{W^{L^p} (w)} + 4 \cdot w(Q(y, \ell))^{1/p - 1/q - 1/s} \left[ \eta (x) \cdot \chi_{Q(y, \ell)} \right]_{W^{L^p} (w)}$$

(98)

$$= J_3 (y, \ell) + J_4 (y, \ell).$$

(99)
For the term $J_3(y, \ell)$, it follows directly from Chebyshev’s inequality and estimate (61) that

$$J_3(y, \ell) \leq 4 \cdot w(Q(y, \ell))^{1 - 1/q - 1/s} \left( \int_{Q(y, \ell)} |h(x)|^q w(x) dx \right)^{1/q}$$

$$\leq C \cdot w(Q(y, \ell))^{1 - 1/q - 1/s} \left( \int_{Q(y, \ell)} \left| b(x) - b_{Q(y, \ell)} \right|^q w(x) dx \right)^{1/q}$$

$$\times \sum_{j=1}^{\infty} \frac{1}{Q(y, 2^{j+1} \sqrt{n} \ell)} \int_{Q(y, 2^{j+1} \sqrt{n} \ell)} |f(z)| dz$$

$$\leq C \cdot w(Q(y, \ell))^{1 - 1/q - 1/s} \sum_{j=1}^{\infty} \frac{1}{Q(y, 2^{j+1} \sqrt{n} \ell)} \int_{Q(y, 2^{j+1} \sqrt{n} \ell)} \left| b(z) - b_{Q(y, \ell)} \right| \cdot |f(z)| dz,$$

where in the last inequality, we have used the fact that $w \in A_\infty$ and Lemma 1 (ii). Arguing as in the proof of Theorem 7, we can also obtain that

$$J_3(y, \ell) \leq C \sum_{j=1}^{\infty} w(Q(y, 2^{j+1} \sqrt{n} \ell))^{1 - 1/p - 1/s} \left\| f \cdot 1_{Q(y, 2^{j+1} \sqrt{n} \ell)} \right\|_{L^p} \cdot w(Q(y, \ell))^{1 - 1/s} \cdot w(Q(y, \ell))^{1 - 1/s}.$$

Let us now estimate the other term $J_4(y, \ell)$. As it was shown in Theorem 8 (see (81)), the pointwise estimate

$$\tilde{\eta}(x) = |I_{\gamma}([b_{Q(y, \ell)} - b] f_2)(x)| \leq C \sum_{j=1}^{\infty} \frac{1}{Q(y, 2^{j+1} \sqrt{n} \ell)} \int_{Q(y, 2^{j+1} \sqrt{n} \ell)} \left| b(z) - b_{Q(y, \ell)} \right| \cdot |f(z)| dz,$$

holds for any $x \in Q(y, \ell)$ by a routine argument. This, together with Chebyshev’s inequality, implies that

$$J_4(y, \ell) \leq 4 \cdot w(Q(y, \ell))^{1 - 1/q - 1/s} \left( \int_{Q(y, \ell)} |\tilde{\eta}(x)|^q w(x) dx \right)^{1/q}$$

$$\leq C \cdot w(Q(y, \ell))^{1 - 1/q - 1/s} \sum_{j=1}^{\infty} \frac{1}{Q(y, 2^{j+1} \sqrt{n} \ell)} \int_{Q(y, 2^{j+1} \sqrt{n} \ell)} \left| b(z) - b_{Q(y, \ell)} \right| \cdot |f(z)| dz$$

$$\leq C \cdot w(Q(y, \ell))^{1 - 1/q - 1/s} \sum_{j=1}^{\infty} \frac{1}{Q(y, 2^{j+1} \sqrt{n} \ell)} \int_{Q(y, 2^{j+1} \sqrt{n} \ell)} \left| b(z) - b_{Q(y, 2^{j+1} \sqrt{n} \ell)} \right| \cdot |f(z)| dz$$

$$+ C \cdot w(Q(y, \ell))^{1 - 1/q - 1/s} \sum_{j=1}^{\infty} \frac{1}{Q(y, 2^{j+1} \sqrt{n} \ell)} \int_{Q(y, 2^{j+1} \sqrt{n} \ell)} \left| b_{Q(y, 2^{j+1} \sqrt{n} \ell)} - b_{Q(y, \ell)} \right| \cdot |f(z)| dz$$

$$= J_5(y, \ell) + J_6(y, \ell).$$
An application of Hölder’s inequality leads to that

\[
J_5(y, \ell) \leq C \cdot w(Q(y, \ell))^{1/\beta - 1/\alpha} \sum_{j=1}^{\infty} \frac{1}{|Q(y, 2^{j+1} \sqrt{n} \ell)|^{1 - \gamma/n}} \left( \int_{Q(y, 2^{j+1} \sqrt{n} \ell)} |f(z)|^p v(z) \, dz \right)^{1/p} \\
\times \left( \int_{Q(y, 2^{j+1} \sqrt{n} \ell)} \left| b(z) - b_Q(y, 2^{j+1} \sqrt{n} \ell) \right|^p v(z)^{-p'/p} \, dz \right)^{1/p'} \\
= C \cdot w(Q(y, \ell))^{1/\beta - 1/\alpha} \sum_{j=1}^{\infty} \frac{1}{|Q(y, 2^{j+1} \sqrt{n} \ell)|^{1 - \gamma/n}} \left\| f \cdot \chi_{Q(y, 2^{j+1} \sqrt{n} \ell)} \right\|_{L^p(v)} \\
\times \left| Q(y, 2^{j+1} \sqrt{n} \ell) \right|^{1/p'} \left\| b - b_Q(y, 2^{j+1} \sqrt{n} \ell) \right\| \cdot v^{-1/\beta} \right\|_{\mathcal{F}(Q(y, 2^{j+1} \sqrt{n} \ell))},
\]

where \( \mathcal{C}(t) = t^{p'} \) is a Young function. Recall that the inequalities

\[
\left\| b - b_Q(y, 2^{j+1} \sqrt{n} \ell) \right\| \cdot v^{-1/\beta} \leq C \left\| b \right\| \cdot \left\| v^{-1/\beta} \right\|_{\mathcal{F}(Q(y, 2^{j+1} \sqrt{n} \ell))},
\]

hold by generalized Hölder’s inequality and the estimate (87), where

\[
\mathcal{A}(t) \approx t^{p'} (1 + \log^+ t)^{p'}, \\
\mathcal{B}(t) \approx \exp(t) - 1.
\]

Moreover, in view of (64) and (106), we can deduce that

\[
J_5(y, \ell) \leq C \cdot w(Q(y, \ell))^{1/\beta - 1/\alpha} \sum_{j=1}^{\infty} \frac{1}{|Q(y, 2^{j+1} \sqrt{n} \ell)|^{1 - \gamma/n}} \left\| f \cdot \chi_{Q(y, 2^{j+1} \sqrt{n} \ell)} \right\|_{L^p(v)} \cdot \left\| v^{-1/\beta} \right\|_{\mathcal{F}(Q(y, 2^{j+1} \sqrt{n} \ell))},
\]

The last inequality is obtained by the \( A_p \)-type condition (12) assumed on \((w, v)\). We now turn our attention to the last term \( J_6(y, \ell) \). Applying Lemma 1 \(i\) and Hölder’s inequality, we get
\[ J_6(y, \ell) \leq C \cdot w(Q(y, \ell))^{1/\beta - 1/s} \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \frac{(j+1)\|b\|}{|Q(y, 2^{j+1} \sqrt{n} \ell)|^{1-\frac{1}{p}} \int_{Q(y, 2^{j+1} \sqrt{n} \ell)} |f(z)| dz} \]

\[ \leq C \cdot w(Q(y, \ell))^{1/\beta - 1/s} \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \frac{(j+1)\|b\|}{|Q(y, 2^{j+1} \sqrt{n} \ell)|^{1-\frac{1}{p}} \left( \int_{Q(y, 2^{j+1} \sqrt{n} \ell)} |f(z)^p v(z)| dz \right)^{1/p}} \]

\[ \times \left( \int_{Q(y, 2^{j+1} \sqrt{n} \ell)} v(z)^{-p'/p} dz \right)^{1/p'} \]

\[ (108) \]

Also, observe that condition (12) is stronger than condition (10). Using this fact along with (64), we have

\[ J_6(y, \ell) \leq C\|b\| \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \frac{w(Q(y, 2^{j+1} \sqrt{n} \ell))^{1/\beta - 1/s}}{|Q(y, 2^{j+1} \sqrt{n} \ell)|^{1-\frac{1}{p}} \left( \int_{Q(y, 2^{j+1} \sqrt{n} \ell)} |f(z)| dz \right)^{1/p}} \]

\[ \times (j+1) \cdot \frac{w(Q(y, \ell))^{1/\beta - 1/s}}{w(Q(y, 2^{j+1} \sqrt{n} \ell))^{1-\frac{1}{p}}} \]

\[ \times \left( \frac{|Q(y, 2^{j+1} \sqrt{n} \ell)|^{1/(r-q)}}{|Q(y, 2^{j+1} \sqrt{n} \ell)|^{1-\frac{1}{p}}} \left( \int_{Q(y, 2^{j+1} \sqrt{n} \ell)} w(z)^q dz \right)^{1/(r-q)} \left( \int_{Q(y, 2^{j+1} \sqrt{n} \ell)} v(z)^{-p'/p} dz \right)^{1/p'} \]

\[ (109) \]

Summing up all the above estimates and taking into consideration (57), we conclude that

\[ J_2(y, \ell) \leq C \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \frac{w(Q(y, 2^{j+1} \sqrt{n} \ell))^{1/\beta - 1/s}}{|Q(y, 2^{j+1} \sqrt{n} \ell)|^{1-\frac{1}{p}} \left( \int_{Q(y, 2^{j+1} \sqrt{n} \ell)} |f(z)| dz \right)^{1/p}} \]

\[ \times (j+1) \cdot \frac{w(Q(y, \ell))^{1/\beta - 1/s}}{w(Q(y, 2^{j+1} \sqrt{n} \ell))^{1-\frac{1}{p}}} \]

\[ = C \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \frac{w(Q(y, 2^{j+1} \sqrt{n} \ell))^{1/\beta - 1/s}}{|Q(y, 2^{j+1} \sqrt{n} \ell)|^{1-\frac{1}{p}} \left( \int_{Q(y, 2^{j+1} \sqrt{n} \ell)} |f(z)| dz \right)^{1/p}} \]

\[ \times (j+1) \cdot \frac{w(Q(y, \ell))^{1/\beta - 1/s}}{w(Q(y, 2^{j+1} \sqrt{n} \ell))^{1-\frac{1}{p}}} \]

\[ (110) \]
Moreover, by our hypothesis on \( w: w \in A_\infty \) and inequality (36) with exponent \( \delta > 0 \), we compute

\[
\sum_{j=1}^{\infty} (j+1) \cdot \frac{w(Q(y,\ell))^{1/\beta-1/s}}{w(Q(y,2^{j+1}\sqrt{n}\ell))^{1/\beta-1/s}} \leq C \sum_{j=1}^{\infty} (j+1) \cdot \left( \frac{|Q(y,\ell)|}{|Q(y,2^{j+1}\sqrt{n}\ell)|} \right)^{\delta (1/\beta-1/s)}
\]

(111)

\[
= C \sum_{j=1}^{\infty} (j+1) \cdot \left( \frac{1}{2^{(j+1)n}} \right)^{\delta (1/\beta-1/s)} \leq C.
\]

Notice that the exponent \( \delta (1/\beta-1/s) \) is positive because \( \beta < s \), which guarantees that the last series is convergent. Thus, by taking the \( L^r(\mu) \)-norm of both sides of (96) (with respect to the variable \( y \)) and then using Minkowski’s inequality, (98) and (111), we finally obtain

\[
\left\| \left( \int_{Q} w(x)^r dx \right)^{1/r} \right\|_{L^1(\mu)} \leq C < \infty,
\]

where the last inequality is due to (112). We therefore conclude the proof of Theorem 9 by taking the supremum over all \( \ell > 0 \).

The higher order commutators formed by \( I_y \) and a symbol function \( b \) are usually defined by

\[
[b, I_y]_m f (x) = \frac{1}{\xi(y)} \int \frac{[b(x) - b(y)]^m f(y)}{|x - y|^{n+\gamma}} \, dx, \quad m = 1, 2, 3, \ldots
\]

(113)

Obviously, \([b, I_y]_1 = [b, I_y]\) which is the linear commutator (6), and

\[
[b, I_y]_m = [b, [b, I_y]_{m-1}], \quad m = 2, 3, \ldots
\]

(114)

By induction argument, we will then obtain the following conclusions.

**Theorem 12.** Let \( 0 < \gamma < n, 1 < p < q < \infty, \mu \in A_q, \) and \( b \in BMO(\mathbb{R}^n) \). Given a pair of weights \((w, \nu)\) on \( \mathbb{R}^n \), suppose that for some \( r > 1 \) and for all cubes \( Q \) in \( \mathbb{R}^n \),

\[
|Q|^{\gamma(rq+1)} \int_{Q} w(x)^r dx \left\| \left( \int_{Q} \nu^{1/p} \right)^{1/p} \right\|_{L^s(\nu)} \leq C < \infty,
\]

where \( A_m(t) = t^r (1 + \log^+ t)^m \), \( m = 2, 3, \ldots \). If \( w \in A_\infty \), then the higher order commutators \([b, I_y]_m \) are bounded from \( L^{r,\gamma}(\nu, \omega) \) into \( W^{r,\gamma}(\nu) \) (\( \nu \)).

**Theorem 13.** Let \( 0 < \gamma < n, 1 < p < q < \infty, \mu \in A_q, \) and \( b \in BMO(\mathbb{R}^n) \). Given a pair of weights \((w, \nu)\) on \( \mathbb{R}^n \), assume that for some \( r > 1 \) and for all cubes \( Q \) in \( \mathbb{R}^n \),

\[
|Q|^{\gamma(rq+1)} \int_{Q} w(x)^r dx \left\| \left( \int_{Q} \nu^{1/p} \right)^{1/p} \right\|_{L^s(\nu)} \leq C < \infty,
\]

where \( A_m(t) = t^r (1 + \log^+ t)^m \), \( m = 2, 3, \ldots \). If \( p \leq \alpha < \beta < \infty \) and \( w \in A_\infty \), then the higher order
commutators \([b, I_f]_m\) are bounded from \((L^p, L^q)^m(w, w; \mu)\) into \((W L^1, L^p)^m(w; \mu)\) with \(1/\beta = 1/\alpha - (1/p - 1/q)\).

6. Proofs of Theorems 10 and 11

In the last section, we will prove the conclusions of Theorems 10 and 11.

Proof of Theorem 10. Let \(f \in L^{p, c}(\gamma, w)\) with \(1 < p < q < \infty\) and \(\kappa = p/q\). For any given cube \(Q = Q(x_0, \ell)\) in \(\mathbb{R}^n\), it suffices to prove that the inequality

\[
\frac{1}{|Q|} \int_Q |I_y f(x) - (I_y f)_{Q}| \, dx \leq C\|f\|_{L^{p,c}(\gamma, w)},
\]

holds. Decompose \(f\) as \(f = f_1 + f_2\), where \(f_1 = f \cdot \chi_{(4Q)^c\ell}\), \(f_2 = f \cdot \chi_{(4Q)^c\ell}\), \(4Q = Q(x_0, 4\sqrt{n}\ell)\). By the linearity of the fractional integral operator \(I_y\), the left-hand side of (117) can be divided into two parts. That is,

\[
\frac{1}{|Q|} \int_Q |I_y f(x) - (I_y f)_{Q}| \, dx = \frac{1}{|Q|} \int_Q |I_y f_1(x) - (I_y f_1)_{Q}| \, dx + \frac{1}{|Q|} \int_Q |I_y f_2(x) - (I_y f_2)_{Q}| \, dx
\]

\(= I + II\).

For the first term \(I\), it follows directly from Fubini’s theorem that

\[
I \leq \frac{2}{|Q|} \int_Q |I_y f_1(x)| \, dx \leq \frac{C}{|Q|} \int_Q \left( \int_{4Q} \frac{1}{|y - \gamma|^{n+p-\gamma}} |f(y)| \, dy \right) \, dx
\]

\[
= \frac{C}{|Q|} \int_{4Q} \left( \int_Q \frac{1}{|y - \gamma|^{n+p-\gamma}} \, dx \right) |f(y)| \, dy.
\]

It is clear that

\[
|x - y| \leq |x - x_0| + |y - x_0| \leq \frac{5n\ell}{2},
\]

when \(x \in Q\) and \(y \in 4Q\). Using the transform \(x - y \rightarrow z\) and polar coordinates, one has

\[
\int_Q \frac{1}{|x - y|^{n+p-\gamma}} \, dx \leq \int_{|z| = (5n/2)\ell} \frac{1}{|z|^{n+p-\gamma}} \, dz
\]

\[
= w_{n-1} \int_0 \left( \frac{(5n/2)\ell}{|z|} \right)^{n-1} \frac{1}{z^{n+p-\gamma}} \, dz \]

\[
= w_{n-1} \left( \frac{5n}{2} \right)^{n-1}. \ell^{\gamma}.
\]

Here, we use \(w_{n-1}\) to denote the measure of the unit sphere in \(\mathbb{R}^n\). This indicates that

\[
I \leq \frac{C}{|Q|^{1-\gamma/n}} \int_{4Q} |f(y)| \, dy.
\]

Using Hölder’s inequality and noting the fact that \(\kappa = p/q\), we have

\[
I \leq \frac{C}{|Q|^{1-\gamma/n}} \left( \int_{4Q} |f(y)|^p \, dy \right)^{1/p} \left( \int_{4Q} (y)^{-(p-\gamma)/p} \, dy \right)^{1/(p-\gamma)}
\]

\[
\leq C\|f\|_{L^{p,c}(\gamma, w)} \cdot w(4Q)^{1/p} |Q|^{1-\gamma/n} \left( \int_{4Q} (y)^{-(p-\gamma)/p} \, dy \right)^{1/(p-\gamma)}.
\]

Moreover, it follows from condition (10) on \((\omega, \gamma)\) and (48) (consider 4Q instead of \(2^{j+1}Q\)) that

\[
I \leq C\|f\|_{L^{p,c}(\gamma, w)} \cdot \frac{|Q|^{1/(p-\gamma)}}{|Q|^{1-\gamma/n}} \left( \int_{4Q} (y)^{-(p-\gamma)/p} \, dy \right)^{1/(p-\gamma)}
\]

\[
\leq C\|f\|_{L^{p,c}(\gamma, w)} \cdot \frac{|Q|^{1/(p-\gamma)}}{|Q|^{1-\gamma/n}} \left( \int_{4Q} (y)^{-(p-\gamma)/p} \, dy \right)^{1/(p-\gamma)}.
\]

For the second term II, by (1), we have that for any \(x \in Q\),

\[
|I_y f_2(x) - (I_y f_2)_{Q}| = \frac{1}{|Q|} \int_Q |I_y f_2(x) - I_y f_2(y)| \, dy
\]

\[
= \frac{C}{|Q|} \int_Q \left( \int_{4Q} \frac{1}{|x - z|^{n+p-\gamma}} \frac{1}{|y - z|^{n+p-\gamma}} \, d(z) \right) \, dy
\]

\[
\leq \frac{C}{|Q|} \int_Q \left( \int_{4Q} \frac{1}{|x - z|^{n+p-\gamma}} - \frac{1}{|y - z|^{n+p-\gamma}} \, d(z) \right) \, dy.
\]

Since both \(x\) and \(y\) are in \(Q\), \(z \in (4Q)\), by a routine geometric observation, we must have \(|x - z| \geq |x - y|\) and \(|x - z| \geq |y - z|\). This fact along with the mean value theorem yields

\[
|I_y f_2(x) - (I_y f_2)_{Q}| \leq \frac{C}{|Q|} \int_Q \left( \int_{4Q} \frac{|x - y|}{|x - z|^{n+p-\gamma}} \cdot |f(z)| \, d(z) \right) \, dy
\]

\[
\leq \frac{C}{|Q|} \int_Q \left( \int_{4Q} \frac{\ell}{|z - x_0|^{n+p-\gamma+1}} \cdot |f(z)| \, d(z) \right) \, dy
\]

\[
\leq \frac{C}{|Q|} \int_{4Q} \frac{\ell}{|z - x_0|^{n+p-\gamma+1}} \cdot |f(z)| \, d(z)
\]

\[
\leq C \sum_{j=2}^{\infty} \frac{1}{|2^{j+1}Q|^{1-\gamma/n}} \int_{2^{j+1}Q} |f(z)| \, d(z).
\]

Another application of Hölder’s inequality gives that
\begin{align}
|I_y f_2(x) - (I_y f_2)_Q| &\leq C \sum_{j=1}^{\infty} \frac{1}{2^j} \cdot \frac{1}{2^{j+1}|Q|^{1-\gamma/n}} \\
& \quad \times \left( \int_{2^{j+1}Q} |f(z)|^p \nu(z) \, dz \right)^{1/p'} \left( \int_{2^{j+1}Q} \nu(z)^{-p'/p} \, dz \right)^{1/p'} \\
& \leq C \|f\|_{L^p(\nu,w)} \sum_{j=1}^{\infty} \frac{1}{2^j} \cdot \frac{w(2^{j+1}Q)^{\delta/p'}}{|2^{j+1}Q|^{1-\gamma/n}} \left( \int_{2^{j+1}Q} \nu(z)^{-p'/p} \, dz \right)^{1/p'} \\
& = C \|f\|_{L^p(\nu,w)} \sum_{j=1}^{\infty} \frac{1}{2^j} \cdot \frac{w(2^{j+1}Q)^{\delta/p'}}{|2^{j+1}Q|^{1-\gamma/n}} \left( \int_{2^{j+1}Q} \nu(z)^{-p'/p} \, dz \right)^{1/p'},
\end{align}

where the last equality is due to the fact that \( \kappa = p/q \). Moreover, we apply estimate (48) and condition (10) to get

\begin{align}
|I_y f_2(x) - (I_y f_2)_Q| &\leq C \|f\|_{L^p(\nu,w)} \sum_{j=1}^{\infty} \frac{1}{2^j} \cdot \frac{|2^{j+1}Q|^{1/\gamma}}{|2^{j+1}Q|^{1-\gamma/n}} \\
& \quad \times \left( \int_{2^{j+1}Q} w(z)^{\gamma} \, dz \right)^{1/(\gamma q)} \left( \int_{2^{j+1}Q} \nu(z)^{-p'/p} \, dz \right)^{1/p'} \\
& \leq C \|f\|_{L^p(\nu,w)} \sum_{j=1}^{\infty} \frac{1}{2^j} \leq C \|f\|_{L^p(\nu,w)}.
\end{align}

From the pointwise estimate (128), it readily follows that

\begin{equation}
II = \frac{1}{|Q|} \int_Q |I_y f_2(x) - (I_y f_2)_Q| \, dx \leq C \|f\|_{L^p(\nu,w)}. \tag{129}
\end{equation}

By combining the above estimates for \( I \) and \( II \), we are done. \( \square \)

**Proof of Theorem 11.** Let \( 1 < p \leq \alpha < s \leq \infty \) and \( f \in (L^p, L^s)^\Delta (\nu,w;\mu) \) with \( \nu \in \Delta_2 \) and \( \mu \in \Delta_2 \). For any fixed cube \( Q = Q(y,\ell) \) in \( \mathbb{R}^n \), we are going to estimate the following expression:

\begin{equation}
\frac{1}{|Q(y,\ell)|} \int_{Q(y,\ell)} |I_y f(x) - (I_y f)_Q(y,\ell)| \, dx. \tag{130}
\end{equation}

As usual, we decompose \( f \) as \( f = f_1 + f_2 \), where \( f_1 = f - \chi_{4Q}^c \), \( f_2 = f \cdot \chi_{(4Q)^c} \), \( 4Q = Q(y,\sqrt[n]{\ell}) \). By the linearity of the fractional integral operator \( I_y \), the above expression (130) can be divided into two parts. That is,

\begin{align}
\frac{1}{|Q(y,\ell)|} \int_{Q(y,\ell)} &|I_y f_1(x) - (I_y f_1)_Q(y,\ell)| \, dx \leq \frac{1}{|Q(y,\ell)|} \int_{Q(y,\ell)} |I_y f_1(x) - (I_y f_1)_Q(y,\ell)| \, dx \\
& + \frac{1}{|Q(y,\ell)|} \int_{Q(y,\ell)} |I_y f_2(x) - (I_y f_2)_Q(y,\ell)| \, dx \\
& = I(y,\ell) + II(y,\ell).
\end{align}

Let us first deal with the term \( I(y,\ell) \). Fubini's theorem allows us to obtain
\[ I(y, \ell) \leq \frac{2}{|Q(y, \ell)|} \int_{Q(y, \ell)} |I_y f_1(x)| \, dx \]

\[ \leq \frac{C}{|Q(y, \ell)|} \int_{Q(y, \ell)} \left( \int_{Q(y, 4\sqrt{n} \ell)} \frac{1}{|x-z|^{\alpha}} |f(z)| \, dz \right) \, dx \]

\[ = \frac{C}{|Q(y, \ell)|} \int_{Q(y, \ell)} \left( \int_{Q(y, 4\sqrt{n} \ell)} \frac{1}{|x-z|^{\alpha}} \, dx \right) |f(z)| \, dz \]

\[ \leq \frac{C}{|Q(y, \ell)|^{1-\gamma/n}} \int_{Q(y, 4\sqrt{n} \ell)} |f(z)| \, dz, \]

(132)

where we have invoked (121) in the last inequality. Moreover, by Hölder’s inequality, we can see that

\[ I(y, \ell) \leq \frac{C}{|Q(y, \ell)|^{1-\gamma/n}} \left( \int_{Q(y, 4\sqrt{n} \ell)} |f(z)|^p \, dz \right)^{1/p} \left( \int_{Q(y, 4\sqrt{n} \ell)} \nu(z)^{-p'/p} \, dz \right)^{1/p'} \]

\[ = C \cdot w(Q(y, 4\sqrt{n} \ell))^{1-\gamma/n} \left( \int_{Q(y, 4\sqrt{n} \ell)} \nu(z)^{-p'/p} \, dz \right)^{1/p'} \]

(133)

where in the last equality, we have used the hypothesis

\[ 1/s = 1/\alpha - (1/p - 1/q). \]

Taking into consideration (64) and condition (10) on \((w, \nu)\), we further obtain that

\[ I(y, \ell) \leq C \cdot w(Q(y, 4\sqrt{n} \ell))^{1-\gamma/n} \left( \int_{Q(y, 4\sqrt{n} \ell)} \nu(z)^{-p'/p} \, dz \right)^{1/p'} \]

\[ \leq C \cdot w(Q(y, 4\sqrt{n} \ell))^{1-\gamma/n} \left( \int_{Q(y, 4\sqrt{n} \ell)} \nu(z)^{-p'/p} \, dz \right)^{1/p'} \]

(134)

We now turn to estimate the second term \(II(y, \ell)\). From (1), it then follows that for any \(x \in Q(y, \ell)\),

\[ |I_y f_2(x) - (I_y f_2)_{Q(y, \ell)}| = \left| \frac{1}{|Q(y, \ell)|} \int_{Q(y, \ell)} \left[ I_y f_2(x) - I_y f_2(z) \right] \, dz \right| \]

\[ = \left| \frac{C}{|Q(y, \ell)|} \int_{Q(y, \ell)} \left\{ \int_{Q(y, 4\sqrt{n} \ell)} \frac{1}{|x-\zeta|^{\alpha-\gamma}} - \frac{1}{|z-\zeta|^{\alpha-\gamma}} \right\} f(\zeta) \, d\zeta \right| \]

\[ \leq \frac{C}{|Q(y, \ell)|} \int_{Q(y, \ell)} \left\{ \int_{Q(y, 4\sqrt{n} \ell)} \frac{1}{|x-\zeta|^{\alpha-\gamma}} - \frac{1}{|z-\zeta|^{\alpha-\gamma}} \right\} |f(\zeta)| \, d\zeta \, dz. \]

(135)
By the same reason as in the proof of Theorem 10 (see (126)), we can show that for any $x \in Q(y, \ell)$,

$$\left| I_{y, f_2}(x) - (I_{y, f_2})_{Q(y, \ell)} \right| \leq C \sum_{j=2}^{\infty} \frac{1}{2^j} \cdot \frac{1}{|Q(y, 2^{j+1} \sqrt{n} \ell)|^{1-\gamma/n}} \int_{Q(y, 2^{j+1} \sqrt{n} \ell)} |f(\zeta)| d\zeta. \quad (136)$$

Furthermore, by using Hölder’s inequality, the preceding expression in (136) can be estimated as follows:

$$\frac{1}{|Q(y, 2^{j+1} \sqrt{n} \ell)|^{1-\gamma/n}} \int_{Q(y, 2^{j+1} \sqrt{n} \ell)} |f(\zeta)| d\zeta \leq \frac{1}{|Q(y, 2^{j+1} \sqrt{n} \ell)|^{1-\gamma/n}} \left( \int_{Q(y, 2^{j+1} \sqrt{n} \ell)} |f(\zeta)|^p v(\zeta) d\zeta \right)^{1/p} \left( \int_{Q(y, 2^{j+1} \sqrt{n} \ell)} v(\zeta)^{-p'/p} d\zeta \right)^{1/p'} \quad (137)$$

where the last equality is also due to the fact that $1/\alpha - 1/p - 1/s = -1/q$. It then follows from (137) and (64) that

$$\left| I_{y, f_2}(x) - (I_{y, f_2})_{Q(y, \ell)} \right| \leq C \sum_{j=2}^{\infty} \frac{1}{2^j} \cdot \frac{w(Q(y, 2^{j+1} \sqrt{n} \ell))^{1-1/p - 1/s} \| f : \chi_{Q(y, 2^{j+1} \sqrt{n} \ell)} \|_{L^p(\gamma)} \left( \int_{Q(y, 2^{j+1} \sqrt{n} \ell)} w(\zeta)^{p'/p} d\zeta \right)^{1/p'}}{|Q(y, 2^{j+1} \sqrt{n} \ell)|^{1-\gamma/n}} \left( \int_{Q(y, 2^{j+1} \sqrt{n} \ell)} v(\zeta)^{-p'/p} d\zeta \right)^{1/p'} \quad (138)$$

Consequently,

$$II(y, \ell) = \frac{1}{|Q(y, \ell)|} \int_{Q(y, \ell)} \left| I_{y, f_2}(x) - (I_{y, f_2})_{Q(y, \ell)} \right| dx \leq C \sum_{j=2}^{\infty} \frac{1}{2^j} \cdot \frac{w(Q(y, 2^{j+1} \sqrt{n} \ell))^{1-1/p - 1/s} \| f : \chi_{Q(y, 2^{j+1} \sqrt{n} \ell)} \|_{L^p(\gamma)}}{|Q(y, \ell)|} \quad (139)$$
Therefore, by taking the $L^s'(μ)$-norm of (131) (with respect to the variable $y$) and then using Minkowski’s inequality, (134) and (139), we get

\[
\begin{align*}
\frac{1}{|Q(y,ℓ)|} \left\| \int_{Q(y,ℓ)} |f(x) - (I_1 f)(y,ℓ)| \, dx \right\|_{L^s'(μ)} & \\
& \leq \|I(y,ℓ)\|_{L^s'(μ)} + \|I(y,ℓ)\|_{L^s'(μ)} \\
& \leq C \|w(Q(y,4\sqrt{n} \ell))^{1/p-1/q}\|_{L^s'(μ)} \|f \cdot Q(y,4\sqrt{n} \ell)\|_{L^s'(μ)} \\
& \quad + C \sum_{j=2}^{\infty} \frac{1}{2^j} \left\| w(Q(y,2^{j+1} \sqrt{n} \ell))^{1/p-1/q}\|f \cdot Q(y,2^{j+1} \sqrt{n} \ell)\|_{L^s'(μ)} \right\|_{L^s'(μ)} \\
& \leq C \|f\|_{(L^s,L^s)^g(v,w,μ)} + C \|f\|_{(L^s,L^s)^g(v,w,μ)} \times \sum_{j=2}^{\infty} \frac{1}{2^j} \leq C \|f\|_{(L^s,L^s)^g(v,w,μ)}. \end{align*}
\]

We end the proof by taking the supremum over all $ℓ > 0$. 

**Data Availability**
No data were used to support this study.

**Conflicts of Interest**
The author declares that there are no conflicts of interest regarding the publication of this paper.

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