Universal ratios of critical physical quantities of charged AdS black holes

Jie-Xiong Mo, Gu-Qiang Li and Xiao-Bao Xu

Institute of Theoretical Physics, Lingnan Normal University, Zhanjiang, 524048, Guangdong, China

E-mail: mojiexiong@gmail.com, zsgqli@hotmail.com, xbxu789@163.com

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Abstract. We investigate the ratios of critical physical quantities related to the $T-S$ criticality of charged AdS black holes. It is shown that the ratio $\frac{T_c S_c}{Q_c}$ is universal while $\frac{T_c r_c}{Q_c}$ is not. This finding is quite interesting considering the former observation that both the $T-S$ graph and $T-r_+$ graph exhibit reverse van der Waals behavior. It is also worth noting that the value of $\frac{T_c S_c}{Q_c}$ differs from that of $\frac{P_c V_c}{T_c}$ for $P-V$ criticality. Moreover, we discuss ratios for the $P-V$ criticality and $Q-\Phi$ criticality. By introducing the dimensional analysis technique, we successfully interpret the phenomenon that the ratio $\frac{\Phi_c Q_c}{T_c}$ is not universal and construct two universal ratios for the $Q-\Phi$ criticality instead. It is expected that the dimensional analysis technique can be generalized to probe the universal ratios for $Y-X$ criticality in future research.

Keywords: GR black holes, gravity

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1 Introduction

Phase transitions and critical phenomena of charged AdS black holes have long been fascinating topics ever since the pioneer work of Chamblin et al. \cite{1,2} disclosed the amazing close relation between charged AdS black holes and van der Waals liquid-gas system. Regarding the cosmological constant $\Lambda$ as a variable \cite{3-8} and identifying it as the thermodynamic pressure, the $P-V$ criticality revolution initiated by Kubizňák and Mann \cite{9} further enhances this relation. One can read the most recent review \cite{10} and references therein to gain an elegant and unified picture.

Recently, $T-S$ criticality of AdS black holes also attracts much attention. Ref. \cite{11} first presented the exact construction of the Maxwell equal area law in the $T-S$ plane of charged AdS black hole. It was argued that the behavior of the $T-S$ plane is dual to that of the $P-V$ plane and this duality is analogous to the $T$-duality of string theory. This intriguing work was soon generalized to entanglement entropy-temperature plane creatively \cite{12-14}. From then on, the phase structure of various black holes have been investigated in either the $T-S$ plane or entanglement entropy-temperature plane, including five-dimensional RN-AdS black holes \cite{13,15}, Born-Infeld AdS black holes \cite{16}, quintessence black holes \cite{17}, $f(\mathcal{R})$ AdS black holes \cite{18}, Gauss-Bonnet black holes \cite{19,20}, dilaton black holes \cite{21}, dimensionally continued AdS black holes \cite{22} and black holes in massive gravity \cite{23}.

However, the discussion of universal ratios of critical quantities related to the $T-S$ criticality was unfortunately missed in all these literatures (they only calculated critical quantities). The term universal here refers to the property that the quantity being considered is independent of the parameters $(m, q, l)$ of black holes. Probing the universal ratios of critical quantities is of great physical significance. In classical thermodynamics, the ratio $\frac{P_{c}V_{c}}{kT_{c}}$ is a universal number for all van der Waals fluids. It was shown in the pioneering work of $P-V$ criticality \cite{9} that four-dimensional RN-AdS black holes with arbitrary charge share exact the same universal ratio as the van der Waals fluids. Namely, $\frac{P_{c}V_{c}}{kT_{c}} = \frac{3}{8}$. Ref. \cite{24} further showed that this ratio was worked out to be $\frac{2d-5}{4d-8}$ for the $d$-dimensional charged black holes and it is also independent of the parameter $q$ which is related to the charge of black holes.

Considering the duality between the behavior of the $T - S$ plane and that of the $P - V$ plane, we believe there also exist universal ratios of critical quantities related to the $T - S$ criticality. In this paper, we will attack this problem under the background of $d$-dimensional charged AdS black hole spacetime. The $T - S$ criticality of $d$-dimensional charged black holes has not been covered in literature yet to the best of our knowledge (As stated in the second paragraph, only four and five-dimensional charged black holes were covered. Ref. [25] studied the coexistence line, the Maxwell area law and the heat engine of $d$-dimensional black holes but it did not study their critical quantities for the $T - S$ criticality). On the other hand, ref. [26] recently investigated the $Q - \Phi$ criticality of $d$-dimensional charged black holes and argued that the ratio $\frac{\Phi_c}{Q_c}$ is not universal. To explain this phenomenon, we will also carry out some discussion on the universal ratios for $P - V$ criticality and $Q - \Phi$ criticality in this paper. And we will pave the way to construct universal ratios for the $Q - \Phi$ criticality. We believe that the technique we used in this paper can be generalized to probe the universal ratios for $Y - X$ criticality in future research.

The organization of this paper is as follows. In section 2 we will investigate the $T - S$ criticality of $d$-dimensional charged AdS black holes and search for universal ratios of critical quantities related to the $T - S$ criticality. Section 3 will be devoted to the discussions on the universal ratios for $P - V$ criticality and $Q - \Phi$ criticality. In the end, a brief conclusion will be presented in section 4.

2 Universal ratios for $T - S$ criticality of charged AdS black holes

2.1 Setup

The bulk action of the $d$-dimensional ($d > 3$) charged AdS black hole reads

$$I_{EM} = -\frac{1}{16\pi} \int_M d^d x \sqrt{-g} \left( R - F^2 - 2\Lambda \right), \quad (2.1)$$

where $\Lambda$ is the cosmological constant which is related to the characteristic length scale $l$ through $\Lambda = -\frac{(d-1)(d-2)}{2l^2}$.

The corresponding solution of this action have been reviewed in ref. [24] as

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega_{d-2}^2,$$

$$F = dA, \quad A = -\sqrt{\frac{d-2}{2(d-3)}} \frac{q}{r^{d-3}} dt,$$  \quad (2.2)

where

$$f = 1 - \frac{m}{r^{d-3}} + \frac{q^2}{r^{2(d-3)}} + \frac{r^2}{l^2}. \quad (2.3)$$

The ADM mass and the electric charge of the black hole has been identified as [1]

$$M = \frac{\omega_{d-2}(d-2)}{16\pi} m, \quad (2.4)$$

$$Q = \frac{\omega_{d-2}\sqrt{2(d-2)(d-3)}}{8\pi} q,$$  \quad (2.5)

where $\omega_d = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2} + 1)}$ denotes the volume of the unit $d$-sphere.
The corresponding Hawking temperature, entropy and electric potential have been reviewed in ref. [24] as

\[
T = \frac{f'(r_+)}{4\pi} = \frac{d - 3}{4\pi r_+} \left(1 - \frac{q^2}{r_+^{2(d-3)}} + \frac{d - 1}{d - 3} \frac{r_+^2}{l^2}\right),
\]  
(2.6)

\[
S = \frac{\omega_{d-2} r_+^{d-2}}{4},
\]  
(2.7)

\[
\Phi = \sqrt{\frac{d - 2}{2(d - 3)}} \frac{q}{r_+^{d-3}}.
\]  
(2.8)

2.2 Ratios for critical quantities of \(T - S\) criticality

The possible critical point of \(T - S\) graph is determined by the following equations

\[
\left(\frac{\partial T}{\partial S}\right)_{q=q_c, S=S_c} = 0,
\]  
(2.9)

\[
\left(\frac{\partial^2 T}{\partial S^2}\right)_{q=q_c, S=S_c} = 0,
\]  
(2.10)

where the subscript “c” denotes the values of physical quantities at the critical point.

The analytic expressions for the critical quantities can be obtained via tedious calculation as

\[
S_c = \frac{\omega_{d-2}}{4} \left[ \frac{(d - 3) l}{\sqrt{(d - 2)(d - 1)}} \right]^{d-2},
\]  
(2.11)

\[
q_c = \left[ (d - 3) l \right]^{(d-3)} \frac{1}{\left(2d - 5\right)\left(d - 2\right)^{d-2}(d - 1)^{d-3}},
\]  
(2.12)

\[
T_c = \frac{(d - 3) \sqrt{(d - 2)(d - 1)}}{(2d - 5)l\pi}.
\]  
(2.13)

The above solutions exhibit such a concise manner that the physics behind this certainly deserves to be probed. It is worth noting that it is not always possible to obtain analytical results. The charged AdS black hole here is one of the few cases that analytical results can be obtained. In this sense, the charged AdS black holes may serve as a toy model to probe the deep connection between black holes and ordinary thermodynamic systems. It is one of the motivations of our research to further enhance this connection. As an example, we plot the \(T - S\) graph in figure 1a and the phase diagram in figure 1b for \(d = 5, l = 1\). When \(q < q_c\), the corresponding \(T - S\) curve can be divided into three branches by the minimum temperature and maximum temperature. And the temperature increases monotonically when \(q > q_c\).

It can be witnessed from the phase diagram that the coexistence curve ends at the critical point. Note that we have utilized the equal area law to depict the phase diagram.

One can further obtain the critical charge \(Q_c\) and the critical horizon radius \(r_c\) as

\[
Q_c = \frac{(d - 3)^{2d-5}}{4\sqrt{4d-10}[(d - 2)(d - 1)]^{\frac{d-3}{2}}} l^{d-2} \omega_{d-2},
\]  
(2.14)

\[
\frac{(d - 3)}{\sqrt{(d - 1)(d - 2)}}.
\]  
(2.15)
From eqs. (2.11)–(2.15), one can see clearly that these critical quantities depend on both the dimensionality \( d \) and the characteristic length scale \( l \).

With these critical quantities on hand, we can calculate various ratios of these critical physical quantities. The results are listed as

\[
\frac{T_{cS}}{q_{c}} = \frac{(d-3)^2 \omega_{d-2}}{4\pi} \sqrt{\frac{d-2}{2d-5}},
\]

(2.16)

\[
\frac{T_{cS}}{Q_{c}} = \frac{\sqrt{2}(d-3)^{3/2}}{\sqrt{2d-5}},
\]

(2.17)

\[
\frac{T_{cr}}{q_{c}} = \frac{(d-3)^{5-d}[(d-1)(d-2)]^{1/2}}{\pi l^{d-3} \sqrt{(d-1)(2d-5)}},
\]

(2.18)

\[
\frac{T_{cr}}{Q_{c}} = \frac{4 \sqrt{2}(d-3)^{3/2}[(d-1)(d-2)]^{1/2}}{\omega_{d-2} l^{d-3} \sqrt{2d-5}}.
\]

(2.19)

It can be seen clearly that the ratios \( \frac{T_{cS}}{q_{c}} \) and \( \frac{T_{cS}}{Q_{c}} \) only depend on the dimensionality \( d \) while the ratios \( \frac{T_{cr}}{q_{c}} \) and \( \frac{T_{cr}}{Q_{c}} \) depend on both the dimensionality \( d \) and the characteristic length scale \( l \). In this sense, the ratios \( \frac{T_{cS}}{q_{c}} \) and \( \frac{T_{cS}}{Q_{c}} \) are universal (independent of \( l \)) while the other two ratios are not. This finding is quite interesting considering the former observation \([18]\) that both the \( T-S \) graph and \( T-r_+ \) graph exhibit reverse van der Waals behavior. It is also worth noting that the universal ratio \( \frac{T_{cS}}{q_{c}} \) for \( T-S \) criticality differs from that for \( P-V \) criticality \([24]\). The latter one reads \( \frac{P_{Vc}}{T_{Vc}} = \frac{2d-5}{4d-8} \) \([24]\).

### 2.3 Ratios for generic choice of parameters

Concerning the critical quantities at the critical point, we have shown above the ratios \( \frac{T_{cS}}{q_{c}} \) and \( \frac{T_{cS}}{Q_{c}} \) only depend on the dimensionality \( d \) and independent of parameters \( m, q, l \). Here, we would like to probe the ratios \( \frac{T_{S}}{q_{c}} \) and \( \frac{T_{S}}{Q_{c}} \) for generic \( (m, q, l) \).

With eqs. (2.6) and (2.7), it is quite easy to derive these two ratios. However, it is difficult to obtain the explicit expression of \( r_+ \) as the function of \( m \) by solving the equation

\[
\]
Table 1. The ratio $\frac{T S}{q}$ of four-dimensional black holes for generic $(m, q, l)$.

| $l$ | $q$ | $m$ | $T$   | $S$   | $\frac{T S}{q}$ |
|-----|-----|-----|-------|-------|-----------------|
| 0.1 | 0.1 | 1.0 | 4.97523 | 0.120690 | 6.009785       |
| 1.0 | 0.1 | 1.0 | 0.276535 | 1.436159 | 3.971485       |
| 10  | 0.1 | 1.0 | 0.082666 | 3.019514 | 2.496111       |
| 1.0 | 0.2 | 1.0 | 0.266675 | 1.353301 | 1.804458       |
| 1.0 | 0.3 | 1.0 | 0.245991 | 1.200541 | 0.984408       |

$f(r_+) = 0$. Then it is hard to express these two ratios into the function of $(m, q, l)$. As an alternative approach, we would like to probe the ratios via numerics. The case of four-dimensional black holes is taken as an example here and the results are listed in table 1. Note that here we only consider a set of possible values of the entropy $S$ for convenience (There exists more than one root of $r_+$ for given $m$). Also note that for $d = 4$, we have $Q = q$ and the ratios $\frac{T S}{q}$ and $\frac{T S}{Q}$ would be identical. So we only need to calculate the ratio $\frac{T S}{q}$.

As can be witnessed from table 1, the ratio varies with $q$ and $l$, in distinction to the case at criticality. This observation will definitely deepen our understanding of universal ratios.

3 Discussions on universal ratios for $P - V$ criticality and $Q - \Phi$ criticality of charged AdS black holes

3.1 Ratios for critical quantities of $P - V$ criticality

$P - V$ criticality of $d$-dimensional ($d > 3$) charged AdS black hole has been investigated elegantly in ref. [24]. The critical quantities have been obtained as [24]

\[ v_c = \frac{1}{\kappa} \left[ q^2 (d-2)(2d-5) \right]^{1/[2(d-3)]}, \]

\[ T_c = \frac{(d-3)^2}{\pi \kappa v_c (2d-5)}, \]

\[ P_c = \frac{(d-3)^2}{16 \pi \kappa^2 v_c^2}. \]

The specific volume $v$ is related to the horizon radius $r_+$ through $r_+ = \kappa v$, where $\kappa = \frac{d-2}{4}$. It was shown that the ratio $\frac{P_c v_c}{T_c}$ is universal (independent of the parameter $q$) [24].

Here, we would like to discuss on another two ratios which are also related to the $P - V$ criticality of charged AdS black holes. The results are listed as

\[ \frac{P_c r_c}{T_c} = \frac{2d-5}{16}, \]

\[ \frac{P_c V_c}{T_c} = \frac{(2d-5)(d-2)(2d-5)q^2}{16(d-1)} \frac{d-2}{d-1} \omega_{d-2}. \]

Note that $r_c$ and $V_c$ is obtained via eq. (3.1) utilizing the relations among these three quantities, which read $r_+ = \kappa v$ and $V = \frac{\omega_{d-2} r_c^{d-1}}{d-1}$.

It can be seen that the ratio $\frac{P_c v_c}{T_c}$ is also universal since it does not depend on the parameter $q$. However, the ratio $\frac{P_c V_c}{T_c}$ is not universal and depends on both the dimensionality
3.2 Ratios for critical quantities of $Q - \Phi$ criticality

Ref. [26] studied $Q - \Phi$ criticality of $(n + 1)$-dimensional (Note that $n = d - 1$) charged AdS black hole. The critical quantities have been presented as [26]

$$\Phi_c = \frac{\pi^{n/2-1}}{4\Gamma(n/2)} \sqrt{\frac{n-1}{2n-3}},$$

$$T_c = \frac{n-2}{\pi(2n-3)} (2\Lambda)^{1/2},$$

$$Q_c = \left[ -\frac{(n-2)^2}{2\Lambda} \right]^{(n-2)/2} \frac{((n-1)(2n-3))^{-1/2}}{2n^2 - 5n + 3\Gamma\left(\frac{n}{2}\right)}.$$  \hspace{1cm} (3.8)

And the ratio $\frac{\Phi_c Q_c}{T_c}$ was obtained as

$$\rho_c = \sqrt{-\Lambda^2 - \frac{n^2}{4}(n-1)\pi^{n/2} - \frac{(n-2)^2}{4\Lambda}} \sqrt{\frac{n-1}{2n-3} \sqrt{2n^2 - 5n + 3\Gamma\left(\frac{n}{2}\right)}}.$$ \hspace{1cm} (3.9)

Based on the above equation, the authors of ref. [26] argued that this critical ratio depends on both $\Lambda$ and $n$, which is different from the $P - V$ criticality.

However, $(V, P), (\Phi, Q)$ and $(T, S)$ are conjugate pairs determined by the first law of black hole thermodynamics $dM = TdS + \Phi dQ + VdP$. In this sense, they should behave similarly. In fact, it has been shown that the duality of descriptions in $P - V$ plane and $T - S$ plane is analogous to the $T$-duality of string theory [11]. So we believe that there also exist universal ratios for $Q - \Phi$ criticality, just as for $T - S$ criticality and $P - V$ criticality. The phenomenon that the ratio $\frac{\Phi_c Q_c}{T_c}$ fails to show the universal property may be attributed to the fact that this ratio is not dimensionless. One can easily show that this ratio has dimension $[\text{length}^{d-2}]$ via dimensional analysis. The same technique can also be used to explain why the ratio $\frac{P_c V_c}{T_c}$ is not universal.

Here, we construct two ratios for the $Q - \Phi$ criticality of charged AdS black holes as follows

$$\frac{\Phi_c Q_c^{\frac{1}{n-1}}}{T_c} = \frac{((2n-3)(n-1))^{\frac{n-1}{2}}\pi^{n/2}}{4(n-2)^2\Gamma\left(\frac{n}{2}\right)},$$ \hspace{1cm} (3.10)

$$\frac{\Phi_c Q_c}{T_c^{2n-1}} = \frac{(n-2)^2(2n-3)1^{-n}\pi^{1-n/2}}{4\Gamma\left(\frac{n}{2}\right)}.$$ \hspace{1cm} (3.11)

These two ratios only depend on $n$, which is related to the dimensionality $d$ via $d = n + 1$. In this sense, they are universal.

4 Conclusions

In this paper, we investigate the ratios of critical physical quantities related to the $T - S$ criticality of charged AdS black holes. It is shown that the ratios $\frac{T_c S_c}{Q_c}$ and $\frac{T_c S_c}{Q_c}$ only depend on
the dimensionality $d$ while the ratios $\frac{T_{c}S_{c}}{Q_{c}}$ and $\frac{T_{c}r_{c}}{Q_{c}}$ depend on both the dimensionality $d$ and the characteristic length scale $l$. In this sense, the ratios $\frac{T_{c}S_{c}}{Q_{c}}$ and $\frac{T_{c}Q_{c}}{Q_{c}}$ are universal (independent of $l$) while the other two ratios are not. This finding is quite interesting considering the former observation that both the $T−S$ graph and $T−r_{+}$ graph exhibit reverse van der Waals behavior. It is also worth noting that the universal ratio $\frac{T_{c}S_{c}}{Q_{c}}$ for $T−S$ criticality differs from $\frac{P_{c}V_{c}}{T_{c}}$ for $P−V$ criticality [24]. The value of the latter one was shown to be $\frac{2d−5}{4d−8}$ [24].

Moreover, we discuss on ratios for $P−V$ criticality and $Q−\Phi$ criticality of charged AdS black holes. It is shown that the ratio $\frac{P_{c}r_{c}}{T_{c}}$ is also universal while the ratio $\frac{P_{c}V_{c}}{T_{c}}$ is not. This finding further supports the former observation that the specific volume $v$ rather than the thermodynamic volume $V$ plays the crucial role in the $P−V$ criticality of charged AdS black holes [9]. Recently, ref. [26] showed that the ratio $\frac{\Phi_{c}Q_{c}}{T_{c}}$ fails to show the universal property. We attribute this phenomenon to the fact that the ratio $\frac{\Phi_{c}Q_{c}}{T_{c}}$ is not dimensionless. And we successfully construct two universal ratios for the $Q−\Phi$ criticality of charged AdS black holes. The dimensional analysis technique we used in this paper can be utilized to probe the universal ratios for $Y−X$ criticality in future research. Furthermore, it is expected that there also exists universal ratios for the Hawking temperature-Entanglement entropy criticality. And we will return to this issue in the near future.

To conclude, the dimensional analysis technique is a powerful tool to probe the universal ratios for various criticality of AdS black holes. Only dimensionless quantities could be candidates for universal ones. Quantities with dimensions can not be considered as possibly universal.

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