Distributional cosmological quantities solve the paradox of soft singularity crossing

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Abstract. Both dark energy models and modified gravity theories could lead to cosmological evolutions different from either the recollapse into a Big Crunch or exponential de Sitter expansion. The newly arising singularities may represent true endpoints of the evolution or alternatively they can allow for the extension of geodesics through them. In the latter case only the components of the Riemann tensor representing tidal forces diverge. A subclass of these soft singularities, the Sudden Future Singularity (SFS) occurs at finite time, finite scale factor and finite Hubble parameter, only the deceleration parameter being divergent. In a Friedmann universe evolving in the framework of general relativity they are realized by perfect fluids with regular energy density and diverging pressure at the SFS. A particular SFS, the Big Brake occurs when the energy density vanishes and the expansion arrives at a full stop at the singularity. Such scenarios are generated by either a particular scalar field (the tachyon field) or the anti-Chaplygin gas. By adding any matter (in particular the simplest, the dust) to these models, an unwanted feature appears: at the finite scale factor of the SFS the matter energy density remains finite, implying (for a spatially flat universe) a finite Hubble parameter, hence finite expansion rate, rather then full stop. The universe would then further expand through the singularity, this nevertheless seems forbidden as the energy density of the tachyonic field / anti-Chaplygin gas would become ill-defined. This paradox is relieved in the case of the anti-Chaplygin gas by redefining its energy density and pressure in terms of distributions peaked on the singularity. The regular cosmological quantities which are continuous across the SFS are then the energy density and the square of the Hubble parameter; those allowing for a jump at the SFS are the Hubble parameter and expansion rate (both being mirror-symmetric). The pressure and the deceleration parameter will contain Dirac delta-function contributions peaked on the SFS, however this is no disadvantage as they anyhow diverge at the singularity.

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INTRODUCTION

General relativity and the Copernican principle, combined with observations on the Hubble redshift of the galaxies and modelling the present baryonic content of the Universe by a pressureless perfect fluid (dust) together with a minor contribution from radiation leads to a Universe born from a Big Bang singularity at finite time in the past. The Big Bang is characterized by infinite values of the energy density \( \rho \), pressure \( p \) and temperature \( T \). The scale factor \( a \) characterizing the size of the Universe vanishes, leading to a diverging scalar curvature, a true singularity.

The study of the rotation curves of galaxies, the stability of galaxy clusters and the formation of structure in the Universe all imply the existence of a dark matter component, manifesting itself only through the gravitational interaction and dominating over baryonic matter by approximately a factor of ten. Dark matter can be either cold or warm (but not hot), and its inclusion into the past cosmological evolution does not eliminate the Big Bang singularity. It is also confirmed by the very existence of the cosmic microwave background and light element abundancies in the Universe.

The future can be either continued expansion (still persisting after infinite time or just asymptoting to a stop) or an expansion arriving to a halt after finite time, followed by a contraction phase, leading eventually to a Big Crunch singularity, which is very similar to the Big Bang. The actual scenario is selected by the amount of combined dark and baryonic matter densities, as compared to the critical density. In all these scenarios the future evolution is decelerated due to gravitational attraction.

Modern cosmological observations (distant supernovae of type Ia, the cosmic microwave background, gravitational lensing) confirm on one hand that the Universe is quite close to the critical energy density \( \kappa = 0 \), however they imply the necessity of an accelerated expansion in the recent past \([1]\), eventually disruling the three above mentioned scenarios for the future of the Universe. Introducing dark energy, accounted for 73% of the energy content of the Universe (dark and bary-
monic matter contributing with 23% and 4%, respectively) leads to the observed accelerated expansion as it violates the strong energy condition. The energy density $\rho$ and the pressure $p$ of the dark energy satisfies $\rho + 3p < 0$, the condition for accelerated expansion imposed by the Raychaudhuri equation

$$\frac{\dot{a}}{a} = -\frac{1}{2}(\rho + 3p)$$

(1)

(we chose units $c = 1$ and $8\pi G/3 = 1$).

In its simplest variant dark energy is the cosmological constant $\Lambda$, with negligible contribution to the dynamics of the Universe in the past, however modifying its future. In the $\Lambda$CDM (cosmological constant and cold dark matter) model the Universe asymptotes to an exponentially expanding de Sitter universe. Although of appealing simplicity, a cosmological constant would conflict by many orders of magnitude the outcome of all variants of calculation of the vacuum expectation energy. A dynamic dark energy model would be clearly more satisfactory and perfectly compatible with observations (which settle but the present value of this field). For a review on dark energy models see [2]. There are many dark energy candidates, their common feature being that they change the future of the Universe in a drastic manner.

In Section 2 we enlist and succinctly characterize the possible outcomes of such dynamic dark energy dominated evolutions together with certain unconventional evolutions in modified gravity theories, leading to various exotic singularities. In Section 3 we concentrate on a particular type of evolution, leading to a Big Brake singularity. By adding ordinary matter to the model, the Big Brake singularity is generalized to a Sudden Future Singularity (SFS). This is still a soft and traversable singularity, however the future evolution is obstructed by the dark energy becoming ill-defined. A possible way of overcoming this difficulty is by generalizing the cosmological quantities in a distributional sense.

**A COMPENDIUM OF EXOTIC COSMOLOGICAL SINGULARITIES**

The common characteristic of all dark energy induced, novel type of singularities is that they occur in finite time. Despite certain components of the Riemann curvature tensor diverging some of these singularities remain traversable. The classification below based on traversability is consistent with Królik’s definition of the strongness of a singularity [3].

### Strong singularities

These are the singularities of type I. and III. in the classification of Ref. [3].

Singlatures of type I. occur for phantom dark energy models (with barotropic index $w = p/\rho$ slightly smaller than $-1$). These models have the counter-intuitive feature that the energy density increases with the expansion of the Universe. The singularity, dubbed Big Rip or Doomsday [5] occurs at finite time and infinite scale factor $a$ and is characterized by diverging Hubble parameter $H = \dot{a}/a$ and a diverging $\dot{H}$. Due to the Raychaudhuri equation (1) and the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \rho$$

(2)

then both $\rho$ and $p$ also diverge. The energy density and pressure thus behave similarly as in the Big Bang or Big Crunch, however this happens at infinite, rather than vanishing scale factor.

Singlatures of type III. are very similar, $H$, $\dot{H}$, $\rho$ and $p$ diverge, however this occurs at finite scale factor. Therefore the singularities of type III. are also known as Finite Scale Factor singularities[1]. Note that although this singularity is strong according to Królik’s definition, it shows up as weak according to Tipler’s definition [6], which seems then less adequate to characterize the strongness of a singularity. The singularities of type III. are compatible with available cosmological observations [7].

A particular singularity of type III. is the Big Freeze, occurring in the evolution of the generalized phantom Chaplygin gas [8].

### Weak singularities

Pure kinematical investigations of evolutions in a Friedmann universe lead to the possibility of Sudden Future Singularity (SFS) occurrence [9]. Such singularities are of type II in the classification of Ref. [4] and are characterized by finite scale factor $a$ and finite Hubble parameter $H$, while $\dot{H}$ diverges. Hence at these singularities the energy density is finite, while the pressure diverges. As the metric contains only the scale factor, the geodesic equations will contain but $H$, hence point particles may pass through this singularity, generating afterwards the new geometry. The diverging $\dot{H}$ appears only in the deviation equation, generating infinite tidal forces at the SFS

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1 Nevertheless, a finite scale factor is also characteristic for other singularities. In fact all weak (soft) singularities to be mentioned in this paper occur not only at finite time, but also finite (but non-vanishing) scale factor.
crossing, but only for an infinitesimally short time [10]. SFS are weak in both the Królak and Tippler’s definitions.

A particular SFS occurs when a full stop is realized at the singularity. Such Big Brake singularities could be produced by the dynamics of an anti-Chaplygin gas or by a particular tachyonic scalar field showing superluminal evolution over certain periods of its existence [11]. The tachyonic model does not violate causality due to the continued homogeneity and isotropy of the Universe and was shown to be in agreement with observations on the supernovae of type Ia [12]. The Big Brake occurs after a time comparable with the present age of the Universe and was also shown explicitly to be traversable and to eventually evolve into a Big Crunch [13]. In Refs. [13] the solutions of the Wheeler-DeWitt equation for the quantum state of the universe in the presence of the Big Brake singularity was studied.

A time-reversed version of the Big Brake singularity is the Big Démarrage [8], when the Universe starts expanding from a state of infinite pressure but finite energy density.

There are also weak singularities characterized by vanishing pressure and vanishing energy density, however their ratio, the barotropic index $w$ being divergent. Both the singularities of type IV. from the classification of Ref. [4], where the time derivatives of rank three or higher of the scale factor diverge; and the $w$-singularities with completely regular scale factor introduced in Ref. [14] belong here. These singularities are quite soft, they do not harm in any way the evolution of the Universe or standard matter, rather manifest themselves only in the dark energy model, possibly signaling its breakdown.

**Exotic brane-world singularities**

In brane-worlds the Einstein equation is replaced by the effective Einstein equation. Beside the cosmological constant term and the energy-momentum tensor this equation has additional source terms: i) a quadratic source in the energy-momentum tensor (which becomes important only at high energy densities or pressures), ii) a pull-back to the brane of non-standard model fields acting in 5 dimensions, iii) the asymmetric embedding of the brane. All these are reviewed in detail in [16].

A singularity very similar to the SFS, dubbed quiescent singularity arises in brane cosmology, in which $\rho$ and $H$ remain finite, but all higher derivatives of the scale factor diverge as the cosmological singularity is approached [17].

Brane-world dynamics however, in particular the presence of the energy-momentum squared term among the source terms allows for the appearance of even stranger singularities, which are characterized by diverging $\rho$ and $\rho$, nevertheless regular evolutions of the scale factor. Such an example is provided by the collapse of a perfect fluid metamorphosing into dark energy [18].

Another such singularity arises in the context of brane-world flat Swiss-cheese cosmologies, in the presence of a huge cosmological constant [19]. At this singularity the scale factor, its first, second and all higher derivatives stay regular. This universe forever expands and decelerates, as its general relativistic analogue, the Einstein-Straus model [20]. However after a finite time the pressure diverges to plus infinity. This smooth pressure singularity is different from the case when both the pressure and the second derivative of the scale factor diverge, the latter stays regular. The accompanying energy density turns negative shortly before reaching the singularity and becomes ill-defined there. The asymmetric embedding enhances the apparition of such a singularity. There is a critical value of the asymmetry in the embedding, above which these singularities necessarily appear [21].

If one combines the cosmological constant, the energy-momentum and the energy-momentum squared source terms into an effective fluid, it turns out that this is dust, following the standard evolution of an Einstein-Straus model. The effective energy density evolves through positive values through the singularity, towards reaching asymptotically zero, as the universe expands. In terms of the effective dust source it is quite natural that the singularity can be passed through. Nevertheless the pressure of the physical fluid diverges and its energy density becomes ill-defined. The singularity is induced by the brane dynamics non-linear in the energy-momentum, modified as compared to GR.

**SFS CROSSING**

The Big Brake singularity is the simplest SFS and phenomenological models, like a tachyonic scalar field or anti-Chaplygin gas were found, which evolve into a Big Brake [11]. Although the tachyonic scalar field has a subluminal evolution at present and mimics well dark energy [12], also displays a dust-like (dark matter like) behaviour in the more distant past [13], a more comprehensive cosmological model would certainly include baryonic matter as well, customarily modelled by dust. The addition of dust to the tachyonic scalar field however induces a paradox. Its energy density at any finite scale factor being positive, by virtue of the Friedmann equation the Hubble parameter will not vanish at the singularity. The Big Brake is replaced by a SFS exhibiting a finite expansion rate. The paradox arises from allowing for further expansion: for larger scale factor than the one characterizing the SFS the tachyonic field becomes ill-defined. The same paradox also arises when the dust is
added to the anti-Chaplygin gas.

In Ref. [22], based on certain distributional identities we have worked out the details of including a distributional contribution to the pressure of the anti-Chaplygin gas (and equivalently to $\dot{H}$), centered on the SFS:

$$P_{AcH} = \sqrt{\frac{A}{6H_0 SFS - t}} + \frac{4}{5} H_0 \delta(t_{SFS} - t), \quad (3)$$

$$\dot{H} = -2H_{SFS} \delta(t_{SFS} - t) - \sqrt{\frac{3A}{8H_{SFS} \rho_{SFS}^4}} \frac{\text{sgn}(t_{SFS} - t)}{\sqrt{|t_{SFS} - t|}}. \quad (4)$$

Then $\dot{H}$ may have a jump (the derivative of the Heaviside function being a delta function). In order to keep the energy density continuous, $\dot{H}^2$ should not have a jump, thus when crossing the SFS, the Hubble parameter should obey a $Z_2$-symmetry. If the Universe arrives to the SFS with the expansion rate $\dot{a}_{SFS}$, after crossing it it will have the expansion rate $-\dot{a}_{SFS}$. The respective equations are:

$$H(t) = H_{SFS} \text{sgn}(t_{SFS} - t) + \sqrt{\frac{3A}{2H_{SFS} \rho_{SFS}^4}} \frac{\text{sgn}(t_{SFS} - t)}{\sqrt{|t_{SFS} - t|}}, \quad (5)$$

In order to preserve the anti-Chaplygin gas equation of state $p = A/\rho$ a delta function also enters the denominator of the energy density. Alternatively, $\rho$ may be kept regular, but then the equation of state should be generalized into a distributional relation.

There is a full analogy with a ball bouncing back from a wall or a tennis / squash racquet. A simple description of the process includes a sudden reversal of the normal velocity. A detailed description instead requires to allow for modelling the ball deformation. After being compressed, the ball will reach a full stop, before bouncing back. A description of the SFS crossing without distributions would require to deform the equation of state in the 2-component fluid, such that $H = 0$ occurs at the SFS.

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