The Isgur-Wise Function: A Lattice Determination from Pseudoscalar → Pseudoscalar Form Factors

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Form factors for pseudoscalar → pseudoscalar decays of heavy-light mesons are found in quenched lattice QCD with heavy-quark masses in the range of approximately 1-2 GeV. The Isgur-Wise function, \( \xi(\omega) \), is extracted from these form factors. Results are in good agreement with \( \xi(\omega) \) derived from CLEO measurements for \( B \to D^* \mu \nu \).

1. THE ISGUR-WISE FUNCTION

Matrix elements are parameterized in terms of two form factors \( h_\pm \)

\[
\langle B(\bar{p}_b)|V^\mu|A(\bar{p}_a)\rangle = \frac{h_+(\omega; m_a, m_b)(v_a + v_b)\mu + h_-(\omega; m_a, m_b)(v_a - v_b)\mu}{\sqrt{m_am_b}}
\]

where \( v_a \) and \( v_b \) are the meson four-velocities and \( \omega = v_a \cdot v_b \).

In the heavy quark limit, \( m_{Q,a,b} \to \infty \), form factor \( h_+ \) tends to zero while \( h_- \) approaches \( \xi(\omega) \), the universal Isgur-Wise form factor\(^1\).

At finite heavy-quark mass, \( h_\pm \) are still related to \( \xi(\omega) \) although there are now both short-distance perturbative corrections and nonperturbative corrections in powers of \( 1/m_Q \). Neglecting the power law corrections,

\[
h_+(\omega) = \left[ \hat{C}_1 + \frac{\omega + 1}{2} (\hat{C}_2 + \hat{C}_1) \right] \xi_{\text{ren}}(\omega) \quad (2)
\]

\[
h_-(\omega) = \frac{\omega + 1}{2} \left[ \hat{C}_2 - \hat{C}_3 \right] \xi_{\text{ren}}(\omega) \quad (3)
\]

where the Wilson coefficients \( \hat{C}_i \) have been computed at next-to-leading order by Neubert\(^2\).

2. METHODOLOGY

An \( O(a) \)-improved fermion action\(^3\) was used to generate fermion propagators for 60 quenched gauge configurations on a \( 24^3 \times 48 \beta = 6.2 \) lattice\(^4\). The three light-quark masses, \( m_q \), and the four heavy-quark masses, \( m_{Q,a,b} \), used here are also used in our study of \( f_D \) and \( f_g \) on these same configurations\(^5\). Estimating the heavy-quark mass by the spin-average of the heavy-light pseudoscalar (P) and vector (V) meson masses, in the \( m_q \to 0 \) limit, we find, \( m_Q \approx 1.5, 1.9, 2.1, \) and 2.4 GeV. The light-quark masses in ratio to strange quark mass are \( m_q/m_s \approx 0.41, 0.68, \) and 1.3.

We study euclidean three-point correlation functions

\[
G^\mu(0, t; m_Q, m_a, m_b, \bar{p}_a, \bar{p}_b) = \sum_{\vec{x}, \vec{y}} e^{i \vec{p}_a \cdot \vec{x}} e^{i \vec{q} \cdot \vec{y}} \langle P_a(\vec{x}, t) V^\mu(\vec{y}, 0) P_a^\dagger(\vec{0}, 0) \rangle \quad (4)
\]

where \( \vec{q} = \vec{p}_b - \vec{p}_a \). Operator \( P_a^\dagger \) creates a \( Q_a \bar{q} \) pseudoscalar and \( P_a \) annihilates a \( Q_b \bar{q} \) pseudoscalar. The current \( V^\mu \) is a local \( O(a) \)-improved vector current\(^6\).

We set \( t_b = 24 \) and symmetrize correlators about this time. Correlators have lattice momenta \( \vec{k}_b = (12a/\pi)\vec{p}_b = (0, 0, 0), (1, 0, 0), \) and \( 0 \leq |\vec{k}_a|^2 \leq 2 \). Quark mass \( m_{Q,b} \) can be either 2.4 or 1.9 GeV.

The ratio of a matrix element to the temporal component of the forward matrix element of the flavor-conserving current is extracted by taking the ratio of three-point functions

\[
\frac{G^\mu(0, t; m_Q, m_a, m_b, \bar{p}_a, \bar{p}_b)}{G^4(0, t; m_Q, m_a, m_b, \bar{p}_a, \bar{p}_b)} \rightarrow \frac{Z_a(\bar{p}_a) E_b}{Z_b(\bar{p}_b) E_a} \times \langle B(\bar{p}_a)|V^\mu|A(\bar{p}_a)\rangle e^{-\delta t}. \quad (5)
\]

For all Lorentz components in the ratio that
are non-zero, a single minimal $\chi^2$ fit, using the 
full correlation matrix, is found for the $t$ dependence 
in Eqn. 3. Field normalizations $Z_{a,b}$, energies $E_{a,b}$, and $\delta E = E_a - E_b$ are constrained 
to values obtained in fits to the meson propagators. $\xi$ 
Equation 3 is used with Eqn. 3 to find 
h_+ (\omega; m_a, m_b) / h_+ (1; m_b, m_b). After extrapolating 
h_+ to the $m_q \to 0$ limit, relation Eqn. 3 is 
the Isgur-Wise function $\xi (\omega)$ from 
h_+ (\omega).

For flavor-conserving matrix elements, $h_-$ 
should be exactly zero. To test this, we allow 
both $h_{\pm}$ to be free parameters in the $\chi^2$ fit. For 
m_Q = 1.5 GeV and $m_q \to 0$ we find $|h_-| \lesssim 0.1$ 
which is within $1 \sigma$ of zero. We then constrain 
h_- to zero in fits for flavor-conserving matrix 
elements.

3. RESULTS

- **Slope Parameter** The slope parameter, $\rho^2 = -\xi'(1)$, is extracted by finding a minimum $\chi^2$ fit 
of the lattice $\xi (\omega)$ to some possible forms for the 
Isgur-Wise function

$$\xi_{BSW} (\omega) = \frac{2}{\omega + 1} \exp \left( 1 - 2 \rho^2_{BSW} \left( \frac{\omega - 1}{\omega + 1} \right) \right)$$

$$\xi_{pole} (\omega) = \left( \frac{2}{\omega + 1} \right)^{2 \rho^2_{pole}}$$

$$\xi_{ISGW} (\omega) = \exp \left( - \rho^2_{ISGW} (\omega - 1) \right)$$

as discussed in References [4], [5], and [6] respectively. Values obtained for $\rho^2$ should be relatively 
insensitive to the choice of parameterization since 
Equations 4, 5, and 6 differ only at $O ((\omega - 1)^2)$.

In the continuum limit, the forward matrix element 
of the flavor-conserving vector current has a 
known normalization. On the lattice, matrix elements 
are normalized by $\langle B(p_b) | V^4 | B(p_b) \rangle$ to reduce lattice artifacts and to cancel the 
local vector current renormalization $Z_V$. It is important to test the consistency of this normalization 
method. We fit lattice form factors to the function 
$N \xi_{BSW} (\omega)$ with both $\rho^2$ and the normalization, $N$, determined by the $\chi^2$ fit. Typically, $N$ 
differs from one by $\lesssim 3\%$ which is within $1 \sigma$. We then constrained $N$ to one when finding $\rho^2$.

Label as mass set $\mathcal{A}$ the combination of quark 
masses: $m_{Qb} = 2.4$ GeV, $m_{Qs} = m_{Qb}$ and $m_{Qd} = 0$. Values for $\rho^2$ obtained for this set of masses and Equations 4-6 are shown in Tab. 1. The table also shows $\rho^2_{linear}$ from $\xi_{linear} = 1 - \rho^2_{linear} (\omega - 1)$. Uncertainty estimates are obtained 
by a bootstrap procedure with only statistical uncertainties shown. The values obtained agree 
with other determinations [7] and our earlier results [10].

- **Measured Form Factors** In Fig. 1 we compare the lattice form factor for mass set $\mathcal{A}$ with 
$|V_{cb}| \xi (\omega)$ derived from CLEO [11] data for $B \to D^* \mu \nu$. A fit of the CLEO data to 
$|V_{cb}| \xi_{BSW} (\omega)$ with $\rho^2$ constrained to the lattice value $\rho^2_{BSW}$ of Tab. 1 yields

$$|V_{cb}| = 0.034 \pm 0.034 \sqrt{1.49 \text{ ps}}.$$

Table 1

| $\rho^2$ vs $\xi (\omega)$ model for parameter set $\mathcal{A}$ |
|-----------------|-------|---------|-------|
| $\xi_{BSW}$ pole | ISGW  | linear |
| 0.92 (±0.20)    | 0.89 (±0.17) | 0.83 (±0.13) | 0.73 (±0.10) |
The first error is the $\Delta \chi^2 = 1$ error in the fit to the experimental data and the second error reflects the uncertainty in $\rho_{\text{BSW}}$. Statistical uncertainties in the lattice form factor are of the same size as the errors in the experimental form factor.

The figure shows the lattice $\xi(\omega)$ from $P \to P$ transitions and $\xi(\omega)$ from CLEO $P \to V$ decay data to be remarkably similar in shape. Further studies of heavy quark spin symmetry using $P \to V$ three-point functions are underway[12,13].

\begin{table}[h]
\centering
\begin{tabular}{lcc}
\hline
$\rho^2_{\text{BSW}}$ vs $m_Q$ (GeV). & \\
\hline
$m_Q$ & 1.9 & 2.4 \\
$\rho^2$ & $0.91 (^{+41}_{-20})$ & $1.06 (^{+66}_{-34})$ \\
\hline
\end{tabular}
\caption{Table 2}
\end{table}

The table shows values for $\rho^2_{\text{BSW}}$ from separate analyses of flavor-conserving matrix elements with $m_Q = 1.9$ and 2.4 GeV. The errors are large and the change in $\rho^2$ with $m_Q$ is only about 0.5$\sigma$ over the range of $m_Q$ studied.

The $O(1/m_Q)$ corrections to Eqn. 2 that relates $h_+(\omega)$ to $\xi(\omega)$ may be small since, by Luke’s theorem[14], there can be at most $O(1/m_Q^2)$ corrections to this relation at $\omega = 1$.

For mass set $A$ with $|\vec{k}_b| = 0$, the variations in the values of $\xi(1)$ extracted from $h_+(1; m_a, m_b)$ are $< 0.5\%$ as $m_{Q_a}$ is varied over the four possible values of $m_Q$. The differences are smaller than the statistical uncertainties. For $|\vec{k}_b| = 1$, the variations in $\xi(1)$ values are now as much as ten times larger than for the zero momentum case. However, the differences are still within $1\sigma$ of zero.

Tests using the relation in Eqn. 3 which is not protected from $O(1/m_Q)$ corrections by Luke’s theorem, are more sensitive indicators of $m_Q$ effects[13]. A study of $h_-$ may then help characterize the nonperturbative power law corrections to $\xi(\omega)$ at finite $m_Q$.

\begin{table}[h]
\centering
\begin{tabular}{lcc}
\hline
$\rho^2_{\text{BSW}}$ vs $m_{q/m_s}$. & \\
\hline
$m_{q/m_s}$ & 0.41 & 0.68 & 1.3 \\
$\rho^2$ & $1.09 (^{+24}_{-11})$ & $1.19 (^{+17}_{-10})$ & $1.31 (^{+15}_{-6})$ \\
\hline
\end{tabular}
\caption{Table 3}
\end{table}

A study of $h_-$ also helps compare the value of $\rho^2_{\text{BSW}}$ in Tab. 2 to values in the chiral limit. The trend is for $\rho^2$ to decrease with decreasing light-quark mass. Further work is necessary to understand the chiral behavior of $\xi(\omega)$.

\section{4. CONCLUSION}

Using heavy quark symmetry and the lattice is an effective way to study $B \to D$ decays.

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