

**K-theory on tori and topological insulators**

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**ABSTRACT**

We discuss means to study topological properties of wavefunctions in a time reversal invariant crystalline system through $K$-groups. The well-known methods for calculating $K$-groups of $G$-bundles over spheres are extended using earlier results in order to deal with wavefunctions defined over toroidal Brillouin zones, following a method due to Nash. The recently discovered topological insulator is considered as an illustrative example.

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1 Introduction

Topological insulators are novel band insulators invariant under time reversal possessing strong spin-orbit coupling [1]. The topological nature of these insulators owes its origin to the presence of gap-less surface states. These states are robust against time reversal invariant weak disorders rendering topologically stable materials exhibiting striking properties [2–4] anticipated beforehand [5, 6]. Observation of such materials [7] has enkindled expectations of discovering newer materials with unusual properties or new stable coherent quantum structures through topological studies [8]. Condensed matter systems often exhibit interesting topological features simply because of having topologically non-trivial Brillouin zones arising from the underlying periodic structure.

The basic issue in all the approaches to spotting topological features in materials, say through Pfaffians [4, 9, 10] or by more intuitive reasoning [10], is to classify inequivalent configurations obtained by attaching a wavefunction to every point of the topologically non-trivial Brillouin zone. The inequivalent configurations are characterized by a certain discrete number, distinguishing the classes of wavefunctions so that states with different numbers cannot be deformed into each other continuously. The set of states over the Brillouin zone is thus partitioned into classes labelled by such numbers, which are robust under deformations. The assignment of class is dictated by the groups of symmetry of the system. For instance, states of a system invariant under $\mathbb{Z}_2$, the group with two elements, are arranged into two classes $+1$ and $-1$. A group $\mathbb{Z}$, on the other hand, calls for integral classes, which means that the system has a countably infinite number of inequivalent sectors. This scheme of assigning classes, associated to groups, is particularly simple when the groups of interest are Abelian.

Associating a wavefunction to every point of the Brillouin zone requires constructing a vector bundle over the Brillouin zone. The problem of determining all the inequivalent ways of making this association is the problem of classifying vector bundles, for which $\mathcal{K}$-theory is a standard tool. The relevance of $\mathcal{K}$-theory for classifying topological insulators has been noted earlier [4, 9, 12]. Our aim is to explain this in a self-contained manner. We show that a topological insulator is characterized by one or three $\mathbb{Z}_2$ groups in two or three dimensions, respectively. The approach adopted here was first developed by Nash [13] to classify Yang-Mills bundles over $T^4$ but can be easily used to classify bundles over a wide class of manifolds. This has been used to study persistent metastable currents in superfluid $^3$He in its $A$ phase in a torroidal container [14]. In view of the growing use of toplogical arguments in condensed matter physics it seems worthwhile to present a self-contained account of this powerful method illustrating its usefulness to classify the topological insulator.

In order to make our account reasonably self-contained we recall the time reversal invariant crystalline systems within the scope of Bloch theory in section 2. In sections 3 and 4 the basic ideas of principal bundles and $\mathcal{K}$-theory, respectively, indicating their relevance in studying such systems. In section 5 we review the general method of calculating $\mathcal{K}$-groups for principal bundles on tori, applying it for the present cases in section 6 before concluding in section 7 with a few remarks about the approach.
2 Time reversal symmetry in crystals

A $D$-dimensional crystalline system is a periodic structure specified by a lattice $\Lambda$ in the $D$-dimensional Euclidean space, $\Lambda \subset \mathbb{R}^D$. We simplify discussion by assuming $\Lambda = \mathbb{Z}^D$, the square lattice. Denoting the inner product with $\langle \cdot , \cdot \rangle : \Lambda \times \Lambda \to \mathbb{C}$, the dual lattice

$$\Lambda^\vee = \{ k \in \mathbb{R}^D | \langle k, \gamma \rangle \in 2 \pi \mathbb{Z}, \forall \gamma \in \Lambda \}$$

is called the reciprocal lattice. Its elements $k$ are crystal momenta, which label the irreducible representations of the Abelian group $\mathcal{T}$ of translations by vectors in $\Lambda$. Two crystal momenta differing by a reciprocal lattice vector correspond to the same representation. It thus suffices to consider crystal momenta $k$ valued in the smallest domain $\mathcal{B}$ obtained by bisecting the reciprocal lattice vectors with perpendicular planes. This is called the Brillouin zone, which is isomorphic to a torus, $\mathcal{B} \simeq \mathbb{T}^D = \mathbb{R}^D / \Lambda^\vee$.

Electronic states of a crystalline system are taken to be the eigenfunctions of an elliptic self-adjoint Hamiltonian

$$H = -\nabla^2 + V(x),$$

where $V$ is a sufficiently regular real-valued function on $\mathbb{R}^D / \Lambda$. As $H$ is invariant under $\mathcal{T}$, it possesses the same eigenfunctions as the translations in $\mathcal{T}$. Introducing an orthogonal set of basis vectors in the Hilbert space $\mathcal{H}$ underlying the irreducible representations of $\mathcal{T}$

$$\Psi_n(x, k) = e^{i(k, \gamma)} \psi_n(x, k),$$

$$\psi_n(x + \gamma, k) = \psi_n(x, k), \quad \gamma \in \Lambda,$$

called Bloch functions, the electronic states are given by a solution to the eigenvalue problem

$$H(k)\psi_n(x, k) = E_n(k)\psi_n(x, k).$$

The eigenvalue $E_n(k)$ of the Bloch Hamiltonian $H(k)$, called the $n$-th band function, is a continuous function of the crystal momentum $k \in \mathcal{B}$ for each $n = 0, 1, 2, \cdots$.

Topological insulators are crystalline systems invariant under time reversal. Electronic states of such systems, in addition to being irreducible representations of $\mathcal{T}$, form a representation of the finite group generated by the time reversal operator $\mathcal{T}$. Assuming that the time evolution of an eigenstate $\psi_n(x, k)$ of the Hamiltonian $H$ with eigenvalue $E$ is given by $e^{iEt} \psi_n(x, k)$, with $E$ positive, in conformity with the time-dependent Schrödinger equation, the time reversal operator is taken to be anti-unitary, that is an endomorphism of the Hilbert space $\mathcal{H}$ mapping the inner product to its complex conjugate. It changes the sign of the momentum and time, keeping the position of a state unaltered, that is

$$\mathcal{T} : \psi_n(x, k) \dashrightarrow \psi_n^\ast(x, -k),$$

where a $\ast$ denotes complex conjugation. On physical grounds a state is assumed to come back to itself up to a phase if acted on by $\mathcal{T}$ twice, so that

$$\mathcal{T}^2 = \alpha, \quad |\alpha| = 1.$$
The time reversal operator $\mathcal{T}$ is assumed to act on the Bloch Hamiltonian by conjugation as

$$\mathcal{T} H(k) \mathcal{T}^{-1} = H(-k),$$

(8)

The Hilbert space $\mathcal{H}$ then correspond to a chosen representation of $\mathcal{T}$.

If the spin of electrons is taken into consideration, then the Hamiltonian contains terms involving spin $1/2$ operators valued in a representation of the group $SU(2)$ furnished by the $2 \times 2$ Pauli matrices. The corresponding Bloch wavefunctions over $\mathcal{B}$, written as $\psi_n(x, k) = \begin{pmatrix} \psi_{n\uparrow}(x,k) \\ \psi_{n\downarrow}(x,k) \end{pmatrix}$, transform as vectors under the $SU(2)$, where the upper and lower components are referred to as the spin up and down states, respectively. By flipping the sign of momentum $\mathcal{T}$ changes the sign of angular momentum. The same is assumed for spin.

$$\mathcal{T} : \begin{pmatrix} \psi_{n\uparrow}(x,k) \\ \psi_{n\downarrow}(x,k) \end{pmatrix} \mapsto \begin{pmatrix} \psi_{n\downarrow}(x,-k) \\ -\psi_{n\uparrow}(x,-k) \end{pmatrix}.$$  

(9)

with $\mathcal{T}^2 = -1$. The identification of wavefunctions up to a sign under $\mathcal{T}^2$ breaks the group $SU(2)$ acting on them to $SU(2)/\mathbb{Z}_2 \simeq SO(3)$. Invariance under time reversal requires the existence of a state with energy $E(-k)$ if one with $E(k)$ exists, implying thereby, that states with $E(k_0) = E(-k_0)$ at a point $k_0$, called the Kramer point, are twofold degenerate, having both $\psi$ and $\mathcal{T}\psi$ as eigenfunctions, referred to as a Kramer pair. Generically, $k_0 = 0$ is a Kramer point as well as $k_0 = \pm 1$, as these two points are identified in $\mathcal{B}$. A Kramer pair at $k = 0$ may be associated to states on two sides of $k = 0$ in two inequivalent ways maintaining the continuity of Bloch wavefunctions as well as the band function. This is schematically shown in Figure 1, where band functions are plotted and the spinorial components of wavefunctions are indicated.

In one configuration, shown in the diagram on the left, the spin up and down components of the Kramer pair go over respectively to the up and down components of a Bloch wavefunction on both sides, resulting into two osculatory curves at the Kramer point. The second configuration is obtained as the up and down components go over respectively to the up and down components on one side but to a time-reversed wavefunction on the other, with up and down components exchanged according to (9), as indicated in the diagram on the right of Figure 1. This results in two intersecting curves with distinct
tangents at the Kramer point. Existence of distinct tangents mean that the band function forms a Dirac
cone at this point, since the dispersion in a neighborhood of the intersection point can be brought to
the form \( E = \pm k \). These are the only two types of band functions that may appear in a time reversal
invariant crystalline system.

3 Wavefunctions and bundles

In this section we describe Bloch wavefunctions on the Brillouin zone \( B \) in terms of principal \( G \)-bundles. A (principal) \( G \)-bundle \([15][16]\) over \( M \) is built by gluing together a fixed vector space at each point of a
smooth manifold \( M \), called a base space. Each vector space, called a fiber, is acted upon linearly by a
group \( G \). The action of \( G \) is assumed to be transitive, meaning every point of a fiber \( F \) can be reached
from some point in \( F \) through an action of \( G \) and there is no point held fixed by \( G \) under the action
of \( G \). The gluing procedure is best described by first covering the base space \( M \) with a collection of
possibly overlapping contractible open sets \( \{U_\alpha, U_\beta, \cdots \} \) with a choice of vector spaces \( \{F_\alpha, F_\beta, \cdots \} \)
on each open set, where \( \alpha, \beta, \cdots \) are valued in some countable index set. On an open set \( U_\alpha \) the bundle
\( V \) is simply the product of the spaces \( U_\alpha \) and \( F_\alpha \). The map from \( V \) to this product space is a coordinate
map \( \phi_\alpha \), which is said to describe the bundle \( V \) locally. If two open sets \( U_\alpha \) and \( U_\beta \) overlap then the
bundle \( V \) will have two coordinate descriptions in the overlapping region. A map given to link these
two descriptions consistently is called a transition function. Finally, a bundle \( V \) has a projection map
\( \pi : V \to M \). It describes the way in which overlapping descriptions of \( M \) must be joined together to
describe the global bundle space \( M \). A \( G \)-bundle is said to be of rank \( k \) if the dimension of the fiber
is \( k \) at every point of \( M \). Thus the bundle is constructed by gluing together spaces, that are locally
the product of one of the open sets that cover the base space \( M \) and the fiber space \( F \). The twists and
turns of the resultant bundle space are captured by the gluing procedure. The local product structure of

![Figure 2: Fiber Bundles: trivial (left) and non-trivial (right)](image)

a bundle \( V \) is called a local trivialization. Thus, a \( G \)-bundle of rank \( k \) over a smooth manifold \( M \) is a
smoothly varying locally trivial family of \( k \)-dimensional vector spaces endowed with an action of \( G \). If
the bundle \( V \) can be described as the direct product of \( M \) and a vector space \( F \) then the bundle is said
to be trivial. As an example, if \( M \) is a circle \( S^1 \) and the fiber at each point is an oriented line, then the
lines can be glued together maintaining their orientations to form a cylinder, which is a trivial bundle, or
with a twist, changing orientations along the circle such that the orientation is reversed upon traversing
the circle once, yielding a Möbius strip, which is not trivial. These are schematically shown in Figure 2.
Over a given base space vector bundles are topologically classified up to isomorphism. Two bundles on $M$ are isomorphic if on each open set $U_\alpha$ the vector spaces corresponding to these are isomorphic.

Let us discuss the connection between the time reversal invariant crystalline systems and $G$-bundles. A Bloch wavefunction $\psi(x, k)$ is an element of the Hilbert space $\mathcal{H}$ which varies over the Brillouin zone $\mathcal{B}$ parametrized by $k$. Hence the group of endomorphisms of $\mathcal{H}$, which form a moduli over $\mathcal{H}$ varies over $\mathcal{B}$ too. As the wavefunctions transform under an $SO(3) \simeq SU(2)/\mathbb{Z}_2$, as shown before, we are led to considering $SO(3)$-bundles over the torus $\mathcal{B}$. This corresponds to considering wavefunctions associated to two types of band functions discussed above. Whether or not the twisting can occur in a real system depends on the details of the Hamiltonian. The geometric picture we consider only classifies the different possibilities, of which there are but two, the first corresponding to a conventional insulator while the second to a topological one.

Since the isomorphism of vector spaces on an open set is required to be consistent with the action of the group $G$ on the fibers, a classification scheme for vector bundles depends on properties of $G$. The standard procedure is to determine the homotopy class of maps from the base space $M$ of $V$ to the classifying space $BG$ associated with $G$. Homotopy properties of $BG$ are known. In a homotopic classification two maps belong to the same class if they can be continuously deformed into each other and to different classes otherwise. Fortunately we do not need to get into the details of the space $BG$ since for vector bundles over a $D$-sphere $S^D$ the homotopy classes that need be computed depend only on the fiber group $G$. We also outline how $K$-theory fits in within this scheme. In order to classify vector bundles over a base space that is not a sphere we need to master a few simple specialized methods. This mastery is necessary as the problem of classifying topological insulators is, as we show, the same as the problem of classifying vector bundles over tori. We thus first explain how the $K$-theoretic method can be used to classify vector bundles over arbitrary base manifolds and then use this general method to classify the topological insulator.

4 $K$-theory

$K$-theory may be thought of as a generalized cohomology theory used to classify vector bundles. The method directly works with vector bundles and spots twists from the Abelian group structure introduced for adding bundles. The Abelian group operation between bundles starts by defining addition of bundles over the same base space simply as the usual addition of vector spaces. The fiber of the sum of two vector bundles on any open set of the base space is taken to be the sum of fibers of the summands on the same open set. A trick analogous to the definition of integers from natural numbers then leads to the notion of subtraction of bundles. This yields an Abelian group, known as the $K$-group, with vector bundles as its elements and addition of bundles as the group operation, its inverse operation being the subtraction. This is the $K$-group of the bundle. Topological information is fed into the structure by stating that a pair of vector bundles are isomorphic if they differ by the addition of a trivial bundle of appropriate rank, the latter taken to play the role identity like zero in the group of integers. A vector bundle $V^{(k)}$ of rank $k$ on a manifold $M$ of dimension $D$ is said to be in the stable range if $k > D$. Usefulness of the $K$-group comes from
**Theorem 4.1** (Nash [17]). If a vector bundle $V^{(k)}$ is in the stable range, then there is an isomorphism between the vector bundles $V^{(k)}$ and $V^m + I^{(k-m)}$ with $m < k$, where $V^{(m)}$ is a vector bundle of rank $m$ and $I^{(k-m)}$ is a trivial bundle of rank $(k - m)$ both defined over the base space $M$. We write $V^{(k)} = V^{(m)} + I^{(k-m)}$.

This means that the totality of topological information of a vector bundle of a sufficiently large rank is effectively encoded in a vector bundle of lower rank. This allows for the following equivalence relation between vector bundles.

**Definition 4.1.** Two vector bundles $V_1$ and $V_2$ are equivalent if $V_1 + I^\ell = V_2 + I^r$, where $\ell$ and $r$ are positive integers. We write $V_1 \sim V_2$.

We have the convenient

**Lemma 4.1.** For vector bundles $V_0$, $V_1$ and $V_2$, if $V_1 + V_0 = V_2 + V_0$, then $V_1 \sim V_2$.

The proof is based on the fact [17] that for a reasonable manifold $M$ it is always possible to find for a given vector bundle $V_0$ on $M$, a bundle $V'_0$, such that $V_0 + V'_0 = I$, $I$ being a trivial bundle. Thus $V_1 + V_0 = V_2 + V_0$ implies $V_1 + V_0 + V'_0 = V_2 + V_0 + V'_0$, or $V_1 + I = V_2 + I$, that is $V_1 \sim V_2$.

This equivalence relation partitions the set of vector bundles on a fixed base space into equivalence classes, which generalizes the isomorphism classes in $K$-theory. The usual notion of isomorphism follows by considering vector bundles of equal ranks. From now on a vector bundle will be taken to refer to a representative of the equivalence class to which it belongs. We have considered addition of vector bundles and their equivalence through the addition of a trivial bundle. Let us now discuss the subtraction.

The notion of subtraction of vector bundles parallels a particular feature of subtraction of natural numbers. Let us consider two pairs of natural numbers $\{v_1, v_2\}, \{v'_1, v'_2\}$ and write their pairwise differences as the ordered pairs, namely, $(v_1, v_2) = v_1 - v_2$ and $(v'_1, v'_2) = v'_1 - v'_2$. If the differences are equal, that is $(v_1, v_2) = (v'_1, v'_2)$, then we can also write $v_1 + v'_2 = v'_1 + v_2$, using the usual properties of subtraction as the inverse of addition for natural numbers, or, equivalently,

$$v_1 + v'_2 + v = v_2 + v'_1 + v \quad (10)$$

where $v$ is an arbitrary natural number. Reasoning backwards, subtraction of two natural numbers $(s, t)$ is now defined as the ordered pair so that $(v_1, v_2) = (v'_1, v'_2)$ for two pairs of natural numbers implies equation (10).

This notion is used to define the subtraction of vector bundles. Given a pair of vector bundles $V_1$ and $V_2$ on $M$ their difference, also considered to be a vector bundle on $M$, is defined to be an ordered pair $(V_1, V_2)$, such that if $(v_1, v_2) = (v'_1, v'_2)$ for two pairs of vector bundles, then

$$V_1 + V'_2 + V = V_2 + V'_1 + V, \quad (11)$$

for any vector bundle $V$ over $M$. The difference $(V_1, V_2)$ is also called a virtual bundle and denoted $V_1 - V_2$, with virtual dimension of $V_1 - V_2$ defined as the difference between the ranks of the individual bundles $V_1$ and $V_2$. Let us point out that the virtual dimension can be negative; it is zero if both the bundles have the same rank. As alluded to above, this notion of addition and subtraction of vector bundles endow the set of equivalence class of vector bundles with the structure of an Abelian group, known as the $K$-group.
5 Computing $K$-groups

In this section we discuss the computation of $K$-groups of $G$-bundles on tori in general. Let us first recall that two spaces are said to be homeomorphic if they are mapped to each other by a continuous one-to-one and onto mapping with a continuous inverse. Let us also recall that a sequence

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$$  \hspace{1cm} (12)

is said to be exact if the image of the set $A$ into the set $B$ under the injective mapping $f$ is mapped to the identity element of the set $C$ by the surjective mapping $g$, as illustrated in Figure 3. This is a manner of expressing the fact that $C$ is same as $B$ modulo $A$, which is shrunk to a point in $C$.

To compute $K$-groups of vector bundles on tori in terms of vector bundles on spheres the tori need be described in terms of spheres. We start by introducing certain combinations of topological spaces. Let $X$ and $Y$ be two (topological) spaces with base-points, that is each one has one point marked as distinguished, which we denote by $x_0$ and $y_0$, respectively. Then the Cartesian product $X \times Y$ of $X$ and $Y$ is defined to be the set of ordered pairs of points from $X$ and $Y$, namely

$$X \times Y = \{(x, y)| x \in X, y \in Y\}. \hspace{1cm} (13)$$

For example, a two-torus is (homeomorphic to) the Cartesian product to two circles, $T^2 \simeq S^1 \times S^2$, with $X = S^1$ and $Y = S^1$, as depicted in Figure 4. The wedge sum $X \vee Y$ is the disjoint union of $X$ and $Y$

$$X \vee Y = X \sqcup Y/(x_0 \sim y_0). \hspace{1cm} (14)$$

For example, the wedge sum of two circles is (homeomorphic to) a figure eight as indicated in Figure 5. The smash product $X \wedge Y$ of $X$ and $Y$ is defined as their Cartesian product with the subsets $(x_0, Y) = \cdots$
\{(x_0, y) | y \in Y\} and (X, y_0) = \{(x, y_0) | x \in X\} identified. Thus

\[ X \land Y = \frac{X \times Y}{(x_0, y) \sim (x, y_0)}, \forall x \in X, \forall y \in Y. \tag{15} \]

The subsets \(X\) and \(Y\) of \(X \times Y\) can be looked upon as the subspaces \(X \times \{y_0\}\) and \(\{x_0\} \times Y\), respectively, intersecting at \(x_0, y_0\), which is taken to be the base-point of \(X \times Y\). Thus their union can be identified with the wedge sum \(X \lor Y\). Hence the smash product can be expressed as a quotient

\[ X \land Y = \frac{(X \times Y)}{(X \lor Y)}, \tag{16} \]

which can also be described as the exact sequence

\[ 0 \longrightarrow X \lor Y \longrightarrow X \times Y \longrightarrow X \land Y \longrightarrow 0, \tag{17} \]

where the first map is injective taking a point, denoted 0, to the basepoint of \(X \lor Y\), while the last map is surjective collapsing the entire space \(X \land Y\) to a point, as depicted in Figure 3.

The smash product of two circles is (homeomorphic to) a sphere, \(S^2\). To see this we first “open up” one circle in each of \(S^1 \times S^1\) and \(S^1 \land S^1\) to an interval, ignoring the identification of endpoints. For the Cartesian product \(S^1 \times S^1\) this is depicted in Figure 6 where the radii of the circles along the interval is varied, decreasing from the center towards the end on both sides, without affecting the topology of the configuration. Upon identification of the end points it becomes the torus of Figure 4. Similarly, the figure eight wedge sum \(S^1 \lor S^1\) is opened up as indicated in the last diagram of Figure 5 with the two circles placed at the end points of an interval, which is contractible to a point. The figure eight is recovered upon identification of the end points of the interval. We can now describe the quotient \(S^1 \land S^1\) pictorially as the quotient of these two pictures, shrinking the interval in Figure 6 with the circles at its end points brought together to a point due to the identification by the configuration in the last diagram of Figure 5. This yields a sphere \(S^2\) as indicated in Figure 7. Coming back to the computation of \(K\)-groups, to an exact sequence of spaces is associated an exact sequence of the \(K\)-groups of the spaces in reverse order. Thus, an exact sequence of \(K\)-groups

\[ 0 \longrightarrow K(X \land Y) \longrightarrow K(X \times Y) \longrightarrow K(K \lor Y) \longrightarrow 0 \tag{18} \]

is associated to the exact sequence (17). Here and in the following we ignore the contribution of points, which are zero-dimensional, to the \(K\)-group thus restricting attention to the reduced \(K\)-group only, while continue denoting it with \(K\). Since the group operation of the \(K\)-group addition, as described in the previous section, it follows that

\[ K(X \times Y) = K(X \land Y) + K(X \lor Y). \tag{19} \]

\[ \begin{array}{c}
\bigcirc \lor \bigcirc \\
S^1 \quad S^1 \\
\end{array} \approx \begin{array}{c}
\bigcirc \land \bigcirc \\
S^1 \quad S^1 \\
\end{array} \approx \begin{array}{c}
\bigcirc \lor \bigcirc \\
S^1 \lor S^1 \\
\end{array} \approx \begin{array}{c}
\bigcirc \land \bigcirc \\
S^1 \land S^1 \\
\end{array} \]

Figure 5: Wedge sum of two circles
But for the wedge sum we have
\[ K(X \vee Y) = K(X) + K(Y). \] (20)
Combining these we have the
\[ K(X \times Y) = K(X \wedge Y) + K(X) + K(Y). \] (21)

The scope of \( K \)-theory is rather general in that it deals with bundles of arbitrary rank. The definition of \( K \)-groups can be restricted to \( G \)-bundles over spaces. The \( K \)-group of \( G \)-bundles over \( X \) will be denoted as \( K_G(X) \). The computation of \( K \)-groups on spheres is facilitated by their relation with homotopy groups of the structure group of the vector bundles. The reduced \( K \)-group of a \( G \)-bundle on a \( D \)-dimensional sphere is given by
\[ K_G(S^D) = \pi_{D-1}(G), \] (22)
where \( \pi_n(G) \) denotes the \( n \)-th homotopy group of \( G \). Let us sketch the proof of this for completeness. A sphere \( S^D \) can be represented topologically by its cover with two contractible spaces \( U_1 \) and \( U_2 \). These spaces overlap to form the topological space \( S^{D-1} \). As \( U_1 \) and \( U_2 \) are contractible spaces homeomorphic to \( \mathbb{R}^D \), bundles on them are trivial, that is direct products, \( U_1 \times F \) and \( U_2 \times F \), respectively, where \( F \) is the fiber space on which the group structure group \( G \) acts. The nature of bundles on \( S^D \) thus depends on the way of gluing the trivial bundles on \( U_1 \) and \( U_2 \) together. This in turn depends on the homotopy properties of the map from the intersection of \( U_1 \) and \( U_2 \) to \( G \), namely, \( S^{D-1} \simeq U_1 \cap U_2 \to G \). But this is just \( \pi_{D-1}(G) \), by definition.
6 \textit{K}-groups for time reversal invariant systems

We proceed to compute the reduced \( K \)-groups of \( G = SO(3) \)-bundles on the toroidal Brillouin zone \( B \) for a time reversal invariant crystalline system. For two- and three-dimensional systems \( B = T^2 \) and \( B = \mathbb{T}^3 \), respectively.

In the two-dimensional case by the identification of the Cartesian product of two circles with a two-torus and the smash product of two circles with a two sphere, we obtain,

\[
K_G(T^2) = K_G(S^1 \times S^1)
= K_G(S^1 \wedge S^1) + K_G(S^1) + K_G(S^1)
= K_G(S^2) + 2K_G(S^1),
\]

where the notation \( K(X) + K(X) = 2K(X) \) is used for \( K \)-groups in analogy with the natural numbers. Next, using equation (22) equation (23) yields

\[
K_G(T^2) = \pi_1(G) + 2\pi_0(G),
\]

where \( G = SO(3) \). Ignoring the last term in reduced \( K \)-theory we then obtain

\[
K_{SO(3)}(T^2) = \pi_1(SO(3)) = \mathbb{Z}_2.
\]

The two classes in the \( K \)-group represent the two types of bundles corresponding to the osculating and intersecting band functions discussed in section 2.

For the three-dimensional system we compute the reduced \( K \)-group of \( SO(3) \)-bundles on \( T^3 = S^1 \times S^1 \times S^1 \). Defining \( Y = S^1 \times S^1 \simeq T^2 \) we use (21) to obtain

\[
K_G(T^3) = K_G(S^1 \times Y)
= K_G(S^1 \wedge Y) + K_G(S^1) + K_G(Y)
= K_G(S^1 \wedge Y) + \pi_0(G) + K_G(T^2)
= K_G(S^1 \wedge Y) + \pi_1(G),
\]

where we have used the results for the two-torus and ignored \( \pi_0 \). In order to calculate \( K_G(S^1 \wedge Y) \) we need the

\textbf{Theorem 6.1} (James [18]). The \( r \)-th Betti number \( b_r \) of \( Y \) contributes a \( K_G(S^{r+1}) \) to the \( K \)-group of the wedge sum \( K_G(S^1 \wedge Y) \).

This arises from the fact that the \( r \)-th Betti number counts the number of \( r \)-th homology cycles of \( Y \) which are \( r \)-spheres. These contribute to the \( K \)-group in one higher dimension due to the extra \( S^1 \). Betti numbers of \( Y \simeq T^2 \) are \( b_0 = 1, b_1 = 2, b_2 = 1 \). Hence

\[
K_G(S^1 \wedge Y) = K_G(S^1) + 2K_G(S^2) + K_G(S^3).
\]

We thus have, up to \( \pi_0 \) of circles and spheres,

\[
K_G(T^3) = 3\mathbb{Z}_2 + K_G(S^3)
\]

in reduced \( K \)-theory. The last term requires special treatment not being in the stable range. We need the following
Theorem 6.2 (James & Thomas [19, Th. 1.6]). The map $[T^3, BSO(3)] \rightarrow [T^3, BSO] = K(T^3)$ is injective, and under this map the elements of $[T^3, BSO(3)]$ correspond to the subgroup of $K(T^3)$ with vanishing third Steifel Whitney class.

Here $T^3 = S^1 \times S^1 \times S^1$ and the Steifel Whitney class $H^3(S^3, \mathbb{Z}_2) = \mathbb{Z}_2$ is non-zero. Hence $K(T^3)$ is the trivial group. Thus

$$K_{SO(3)}(T^3) = 3\mathbb{Z}_2,$$

which agrees with previous results [4, 9, 10].

In section 2 we proved the existence of Dirac cones as a consequence of gap-less states appearing as band functions intersect. Let us briefly discuss how the appearance of gap-less states are understood from $K$-theory analysis. Taking into consideration the spinorial degrees of freedom of electrons necessitates incorporating spin-orbit interaction. This is done by adding a term proportional to the Dirac operator $D = \sum_{i=1}^{3} \sigma_i \partial_i$ in the Hamiltonian $H$. It is indeed the appropriate operator for the index theorem, relevant because the system has spin with an $SU(2)/\mathbb{Z}_2 \simeq SO(3)$ connection because of time reversal symmetry. We cannot use $SO(3)$ directly as $SO(3)$ does not act on spinors. The index of the Dirac operator $D$, which acts on smooth sections of the $SU(2)/\mathbb{Z}_2$-bundle on the torus $\mathcal{B}$, is defined as

$$\text{ind } D = \text{ch}(\ker D - \text{coker } D) = \text{ch}(\ker D) - \text{ch}(\text{coker } D),$$

where the Chern character map from the $K$-group of a space to its rational cohomology, namely,

$$\text{ch} : K(X) \rightarrow H^\bullet(X, \mathbb{Q})$$

is used to map the virtual bundle $\ker D - \text{coker } D$ in $K_G(\mathcal{B})$ to $H^\bullet(\mathcal{B}, \mathbb{Q})$. Since the $K$-group as computed above, in non-zero, either the kernel or its dual, the cokernel, of $D$ is non-void, implying the existence of zero modes, alias, gap-less states, at some point of $\mathcal{B}$.

7 Conclusion

We have outlined a general method of calculating $K$-groups for vector bundles over arbitrary base manifolds and have illustrated the method for the $SO(3)$-bundles on two- and three-dimensional tori. We have explained the relevance of $K$-groups for time reversal invariant crystalline systems, for example, topological insulators. The computation of $K$-groups are facilitated by reducing the calculation of $K$-groups for spaces $S \wedge Y$ in terms of the $K$-group of spheres [18,19]. Results thus obtained are consistent with previous results and clarify the occurrence of a trinity of $\mathbb{Z}_2$ groups rather than a possible quartet. The usefulness of $K$-theory lies in that it allows to bring out topological properties of bundles on tori, $S^1 \times S^1 \times S^1$ in a purely algebraic manner. Let us note that had the structure group of the bundle been $SU(k)$ for any $k$, i.e. had we have to deal with complex bundles, the $K$-groups would have been trivial [17]. Thus the time reversal symmetry which breaks $SU(2)$ to $SO(3)$ is essential in that there would have been no topological reason for the stability of the topological insulators in want of this symmetry.
The $K$-groups can be multiplied, forming a ring. While only the case of a single Dirac point has been dealt with above, multiple Dirac points can be treated similarly through a tensor product of vector bundles, whose $K$-group is the product of the $K$-groups of the factors, one for each Dirac point. This immediately implies triviality of $K$-groups for an even number of Dirac points and non-triviality of the same for an odd number as the product of an even number of $\mathbb{Z}_2$’s is trivial but is that of an odd number of $\mathbb{Z}_2$’s is not.

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