Nonholonomic constraints in time-dependent mechanics

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The constraint reaction force of ideal nonholonomic constraints in time-dependent mechanics on a configuration bundle $Q \to \mathbb{R}$ is obtained. Using the vertical extension of Hamiltonian formalism to the vertical tangent bundle $VQ$ of $Q \to \mathbb{R}$, the Hamiltonian of a nonholonomic constrained system is constructed.

I. INTRODUCTION

This work addresses the geometric theory of nonholonomic constraints in time-dependent mechanics. We refer the reader to Refs. 1-7 for the autonomous case. We follow the approach based on the D’Alembert principle because the variational methods with Lagrange multipliers are not always appropriate to nonholonomic constraints (see Refs. 2,5,6,8).

Let the jet manifold $J^1Q$ be a velocity phase space of time-dependent mechanics on a configuration bundle $Q \to \mathbb{R}$. The most general nonholonomic constraints considered in the literature are given by codistributions $S$ or, accordingly, by distributions $\text{Ann}(S)$ on the jet manifold $J^1Q$. Distributions on a configuration space $Q$ and submanifolds of the jet manifold $J^1Q$ can be also seen as nonholonomic constraints.

Dealing with nonholonomic constraints in time-dependent mechanics, one usually studies the following problem. Let $\xi$ be a second order dynamic equation on $Q$ and $S$ a codistribution on $J^1Q$ whose annihilator $\text{Ann}(S)$ is treated as a nonholonomic constraint. The goal is to find a decomposition,

$$\xi = \tilde{\xi} + r,$$

where $\tilde{\xi}$ is a second order dynamic equation obeying the condition

$$\tilde{\xi} \subset \text{Ann}(S).$$

One can think of $\tilde{\xi}$ as describing a mechanical system subject to the nonholonomic constraint $S$, while $(-r)$ is the constraint reaction acceleration. The decomposition (1) however is not unique. In the case of Newtonian systems, including nondegenerate Lagrangian systems, we

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obtain the decomposition (1) which satisfies the D’Alembert principle for ideal nonholonomic constraints. We construct the Hamiltonian counterpart of the constrained equation of motion (2). We show that this can be seen as Hamilton equations in the framework of the vertical extension of Hamiltonian formalism to the configuration space $VQ$ which is the vertical tangent bundle of $Q \to \mathbb{R}$. This may be a step towards the functional integral formulation of nonholonomic time-dependent mechanics and its further quantization.

II. GEOMETRIC INTERLUDE

All manifolds throughout the paper are real, finite-dimensional, second-countable (hence, paracompact) and connected.

We refer the reader to Refs. 8-11, 13-15 for the geometric formulation of Lagrangian and Hamiltonian time-dependent mechanics. In accordance with this formulation, a configuration space of time-dependent mechanics is an $(m+1)$-dimensional fiber bundle $Q \to \mathbb{R}$, coordinatized by $(t, q^i)$. Its base $\mathbb{R}$ is treated as a time axis provided with the Cartesian coordinate $t$. With this coordinate, $\mathbb{R}$ is equipped with the standard vector field $\frac{\partial}{\partial t}$ and the standard 1-form $dt$. For the sake of convenience, we will also utilize the compact notation $q^\lambda$, where $q^0 = t$. Obviously, any fiber bundle $Q \to \mathbb{R}$ is trivial, but it cannot be canonically identified to a product $\mathbb{R} \times M$ in general. Different trivializations $Q \cong \mathbb{R} \times M$ correspond to different reference frames.

The velocity phase space of time-dependent mechanics is the first order jet manifold $J^1Q$ of $Q \to \mathbb{R}$, coordinatized by $(t, q^i, \dot{q}^i)$. There is the canonical imbedding,

$$\lambda : J^1Q \hookrightarrow TQ, \quad (t, q^i, \dot{q}^i) \mapsto (t, q^i, \dot{q}^i = q^i_\tau),$$

(3)
of $J^1Q$ onto the affine sub-bundle of the tangent bundle $TQ$ of $Q$ which is modelled over the vertical tangent bundle $VQ$ of $Q \to \mathbb{R}$. From now on we will identify the jet manifold $J^1Q$ with its image in $TQ$.

Similarly, we have the imbeddings,

$$J^2Q \hookrightarrow J^1J^1Q \hookrightarrow TJ^1Q,$$

$$(t, q^i, q^i_\tau, q^i_{\tau\tau}) \mapsto (t, q^i, q^i_\tau, \dot{q}^i = q^i_\tau, \ddot{q}^i = q^i_{\tau\tau}),$$

where $J^2Q$, coordinatized by $(q^\lambda, q^\lambda_\tau, q^\lambda_{\tau\tau})$, is the second order jet manifold of the fiber bundle $Q \to \mathbb{R}$. The affine bundle $J^2Q \to J^1Q$ is modelled over the vertical tangent bundle,

$$V_QJ^1Q \cong J^1Q \times VQ,$$

(4)
of the affine jet bundle $J^1Q \to Q$.

The jet manifold $J^1Q$ is provided with the canonical tangent-valued form,

$$\hat{v} = \theta^i \otimes \partial^i_t,$$
where $\theta^i = dq^i - q_t^i dt$ are the contact forms. We have the corresponding endomorphism,

\[ \hat{v}(\partial_t) = -q_t^i \partial_i, \quad \hat{v}(\partial_i) = \partial_t^i, \quad \hat{v}(\partial_t^i) = 0, \]

of the tangent bundle $TJ^1Q$ and that,

\[ \hat{v}(dt) = 0, \quad \hat{v}(dq^i) = 0, \quad \hat{v}(dq_t^i) = \theta^i, \]

of the cotangent bundle $T^*J^1Q$ of $J^1Q$. The nilpotent rule $\hat{v}^2 = 0$ holds.

Due to the imbeddings (3), any connection,

\[ \Gamma = dt \otimes (\partial_t + \Gamma_i \partial_i), \]

on a fiber bundle $Q \to \mathbb{R}$ can be identified with a nowhere vanishing horizontal vector field,

\[ \Gamma = \partial_t + \Gamma_i \partial_i, \tag{5} \]

on $Q$ which is the horizontal lift of the standard vector field $\partial_t$ on $\mathbb{R}$ by means of $\Gamma$. Conversely, any vector field $\Gamma$ on $Q$ such that $dt|\Gamma = 1$ defines a connection on $Q \to \mathbb{R}$. Accordingly, the covariant differential,

\[ D_\Gamma : J^2Q \to VQ, \quad \dot{q}^i \circ D_\Gamma = \dot{q}_t^i - \Gamma_i, \tag{7} \]

associated with a connection $\Gamma$ on $Q \to \mathbb{R}$, takes its values into the vertical tangent bundle $VQ$ of $Q \to \mathbb{R}$.

**Remark:** From the physical viewpoint, a connection (3) sets a reference frame. There is one-to-one correspondence between these connections and the equivalence classes of atlases of local constant trivializations of the fiber bundle $Q \to \mathbb{R}$, i.e., such that transition functions $q^i \to q'^i$ of the corresponding bundle coordinates are independent of $t$, and $\Gamma = \partial_t$ with respect to these coordinates. In particular, every trivialization of $Q$ defines a complete connection $\Gamma$ on $Q \to \mathbb{R}$, and vice versa.

A connection $\xi$ on the jet bundle $J^1Q \to \mathbb{R}$ is said to be holonomic if it is a section,

\[ \xi = \partial_t + q_t^i \partial_i + \xi^i \partial^i \]

of the holonomic sub-bundle $J^2Q \to J^1Q$ of the affine jet bundle $J^1J^1Q \to J^1Q$. Holonomic connections make up an affine space modelled over the linear space of vertical vector fields on the affine jet bundle $J^1Q \to Q$, i.e., which live in $VQJ^1Q$. Every holonomic connection $\xi$ defines the corresponding covariant differential on the jet manifold $J^1Q$:

\[ D_\xi : J^2Q \to VQJ^1Q \subset VJ^1Q, \]

\[ \dot{q}^i \circ D_\xi = 0, \quad \dot{q}_t^i \circ D_\xi = \dot{q}_t^i - \xi^i, \tag{7} \]
which takes its values into the vertical tangent bundle \( V_Q J^1 Q \) of the affine jet bundle \( J^1 Q \to Q \). Any integral section \( \tau : \mathbb{R} \ni (t) \to J^1 Q \) for a holonomic connection \( \xi \) is holonomic, i.e., \( \tau = \dot{c} \) where \( c \) is a curve in \( Q \).

A second order dynamic equation (or simply a dynamic equation) on a configuration bundle \( Q \to \mathbb{R} \) is defined as the kernel,

\[
q_{tt} = \xi^i(t, q^j, \dot{q}^j),
\]

of the covariant differential (4) for some holonomic connection \( \xi \) on the jet bundle \( J^1 Q \to \mathbb{R} \). Therefore, holonomic connections are also called dynamic equations. By a solution of the dynamic equation (8) is meant a curve \( c \) in \( Q \) whose second order jet prolongation \( \ddot{c} \) lives in (8). Any integral section \( \tau \) for the holonomic connection \( \xi \) is the jet prolongation \( \dot{\tau} \) of a solution \( c \) of the dynamic equation (8), and vice versa.

III. NONHOLONOMIC CONSTRAINTS

Let \( S \) be an \( n \)-dimensional codistribution on the velocity phase space \( J^1 Q \). Its annihilator \( \text{Ann} (S) \) is treated as a nonholonomic constraint. Let the codistribution \( S \) be locally spanned by the 1-forms,

\[
s^a = s^a_0 dt + s^a_i dq^i + \dot{s}^a_i dq^i_t,
\]
on the jet manifold \( J^1 Q \). Then a dynamic equation \( \tilde{\xi} \) on the configuration bundle \( Q \to \mathbb{R} \) is said to be compatible with the nonholonomic constraint \( S \) if

\[
s^a(\tilde{\xi}) = \tilde{\xi} | s^a = s^a_0 + s^a_i \dot{q}^i + \dot{s}^a_i \tilde{\xi}^i = 0.
\]

This equation is algebraically solvable for \( n \) components of \( \tilde{\xi} \) iff the \( n \times m \) matrix \( \dot{s}^a_i (q^\lambda, \dot{q}^\lambda_i) \) has everywhere maximal rank \( n \leq m \). Therefore, we restrict our consideration to the nonholonomic constraints, called admissible, such that \( \dim S = \dim \hat{v}(S) \).

If a nonholonomic constraint is admissible, there exists a local \( m \times n \) matrix \( \dot{s}^a_i (q^\lambda, \dot{q}^\lambda_i) \) such that

\[
\dot{s}^a_i \dot{s}^b_j = \delta^b_a.
\]

Then the local decomposition (11) of a dynamic equation \( \xi \) can be written in the form

\[
\xi^i = \tilde{\xi}^i + \dot{s}^a_i s^a(\xi).
\]

The global decomposition (11) exists by virtue of the following lemma.

Lemma 1: The intersection

\[
W = J^2 Q \cap \text{Ann} (S)
\]
is an affine bundle over $J^1Q$, modelled over the vector bundle

$$\mathbf{W} = V_QJ^1Q \cap \text{Ann}(S).$$

**Proof:** $\mathbf{W}$ consists of the vertical vectors $v^i\partial_t^i \in V_QJ^1Q$ which fulfill the conditions

$$\dot{s}^a_i(q^\lambda, q^j_t)v^i = 0.$$ 

Since the nonholonomic constraint $S$ is admissible, every fiber of $\mathbf{W}$ is of dimension $m - n$, i.e., $\mathbf{W}$ is a vector bundle, while $W$ is an affine bundle.

The affine structure of $W \to J^1Q$ implies that it has a global section $\tilde{\xi}$.

To construct the global decomposition (1), one usually perform a splitting of the vertical tangent bundle,

$$V_QJ^1Q = W \oplus J^1Q \mathcal{V}, \tag{10}$$

and obtain the corresponding splitting of the second order jet manifold,

$$J^2Q = W \oplus J^1Q \mathcal{V}. \tag{11}$$

Here $\mathcal{V} \to J^1Q$ should be interpreted as the bundle of possible constraint reaction accelerations.

If an admissible nonholonomic constraint $S$ is of dimension $n = m$, a dynamic equation $\xi$ is decomposed in a unique fashion. If $n < m$, the decomposition (1) is not unique. Different variants of this decomposition lead to different constraint reaction forces which, from the physical viewpoint, characterize different types of nonholonomic constraints. In next Section, we will construct the decomposition of dynamic equations of Newtonian systems which corresponds to ideal nonholonomic constraints.

Now, let us consider some important examples of nonholonomic constraints.

Let $N$ be a closed imbedded submanifold of the velocity phase space $J^1Q$, defined locally by the equations

$$f^a(q^\lambda, q^i_t) = 0, \quad a = 1, \ldots, n < m.$$ 

One can treat $N$ as a nonholonomic constraint given by the codistribution $S = \text{Ann}(TN)$ on $J^1Q |_N$. This codistribution is locally spanned by the 1-forms

$$s^a = df^a = \partial_t f^a dt + \partial_j f^a dq^j + \partial_t^j f^a dq^j_t.$$ 

The nonholonomic constraint $N$ is admissible iff the matrix $(\partial_j^i f^a)$ is of maximal rank $n$. It follows that $N$ is a fibred submanifold of the affine jet bundle $J^1Q \to Q$. 

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A nonholonomic constraint \( N \) is said to be linear if it is an affine sub-bundle of the affine jet bundle \( J^1Q \to Q \). Locally, a linear constraint \( N \) is given by the equations

\[
f^a = f^a_0(q^\lambda) + f^a_i(q^\lambda)q^i = 0,
\]

where the matrix \( f^a_i \) is of maximal rank. A linear constraint is always admissible. Since \( N \) is an affine sub-bundle of \( J^1Q \to Q \), it has a global section \( \Gamma \) which is a connection on the configuration bundle \( Q \to \mathbb{R} \), called the constraint reference frame. Then, the constraint equations (12) take the form

\[
f^a_i(q^\lambda)(q^i - \Gamma^i) = 0.
\]

We can say that the linear constraint is immovable with respect to the constraint reference frame \( \Gamma \). Then, one can think of \( \dot{q}^i = q^i - \Gamma^i \), satisfying the equation (13), as virtual velocities relative to the linear constraint \( N \).

Let now a configuration space \( Q \) admit a composite fibration \( Q \to \Sigma \to \mathbb{R} \), where \( \pi_{Q\Sigma} : Q \to \Sigma \) is a fiber bundle, and let \((t, \sigma^r, q^a)\) be coordinates on \( Q \), compatible with this fibration. Given a connection,

\[
B = dt \otimes (\partial_t + B^a_a \partial_a) + d\sigma^r \otimes (\partial_r + B^a_r \partial_a),
\]

on the fiber bundle \( Q \to \Sigma \), we have the corresponding horizontal splitting of the tangent bundle \( TQ \). Restricted to the jet manifold \( J^1Q \subset TQ \), this splitting reads

\[
J^1Q = B(\pi_{Q\Sigma}^* J^1\Sigma) \oplus V_\Sigma Q,
\]

\[
\partial_t + \sigma^r \partial_r + q^a \partial_a = [(\partial_t + B^a \partial_a) + \sigma^r(\partial_r + B^a \partial_a)] + [q^a - B^a - \sigma^r B^a] \partial_a,
\]

where \( \pi_{Q\Sigma}^* J^1\Sigma \) is the pull-back of the affine jet bundle \( J^1\Sigma \to \Sigma \) onto \( Q \). It is readily observed that

\[
N = B(\pi_{Q\Sigma}^* J^1\Sigma)
\]

is an affine sub-bundle of the affine jet bundle \( J^1Q \to Q \), defined locally by the equations

\[
q^a - \sigma^r B^a(q^\lambda) - B^a(q^\lambda) = 0.
\]

This sub-bundle yields a linear nonholonomic constraint.\(^{16,17}\) The corresponding codistribution \( S = \text{Ann}(TN) \) is locally spanned by the 1-forms,

\[
s^a = -(\partial_t B^a + \sigma^r \partial_r B^a)dt - (\partial_s B^a + \sigma^r \partial_r B^a) d\sigma^s - (\partial_b B^a + \sigma^r \partial_r B^a) dq^b + dq^a - B^a \sigma^r.
\]
With the connection (14), we also have the splitting of the vertical tangent bundle $VQ$ of $Q \to \mathbb{R}$ and the corresponding splitting of the vertical tangent bundle $VQJ^1Q$ which reads

$$VQJ^1Q = \overline{W} \oplus \mathcal{V},$$

$$\dot{\sigma}_t^r \partial_r^j + \dot{q}_t^a \partial_a^j = \dot{\sigma}_t^r (\partial_r^j + B_r^a \partial_a^j) + (\dot{q}_t^a - B_r^a \sigma_t^r) \partial_a^j.$$  \hspace{1cm} (16)

It is readily observed that $\overline{W}|_N$ consists of vertical vectors which are the annihilators of the codistribution (15). The splitting (16) yields the corresponding splitting (11) of the second order jet manifold $J^2Q$. Then we obtain the decomposition (1) of every dynamic equation $\xi$ on $J^1Q$ as

$$\tilde{\xi}^r = \xi^r, \quad \tilde{\xi}^a = \xi^a - s^a(\xi).$$

IV. NEWTONIAN SYSTEMS WITH NONHOLONOMIC CONSTRAINTS

Let $Q \to \mathbb{R}$ be a fiber bundle together with (i) a non-degenerate fiber metric,

$$\m: J^1Q \to V^*Q \otimes V^*Q, \quad \m = \frac{1}{2} m_{ij} \overline{dq}^i \wedge \overline{dq}^j,$$

in the fiber bundle $VQJ^1Q \to J^1Q$ which satisfies the symmetry condition,

$$\partial_k^j m_{ij} = \partial_j^k m_{ik},$$  \hspace{1cm} (17)

and (ii) a dynamic equation $\xi$ (6) on the jet bundle $J^1Q \to \mathbb{R}$, related to the fiber metric $\m$ by the compatibility condition,

$$2 \xi^j dm_{ij} + m_{ik} \partial_j^k \xi^k + m_{jk} \partial_i^k \xi^k = 0.$$ \hspace{1cm} (18)

The triple $(Q, \m, \xi)$ is called a Newtonian system. \hspace{1cm} (15) A Newtonian system is said to be standard if $\m$ is the pull-back of a fiber metric in the vertical tangent bundle $VQ$ in accordance with the isomorphism (4). In this case, $\m$ is independent of the velocity coordinates $q_t^i$.

The notion of a Newtonian system generalizes the second Newton law of particle mechanics. Indeed, the dynamic equation for a Newtonian system is equivalent to the equation

$$m_{ik} (\dot{q}_t^k - \xi^k) = 0.$$ \hspace{1cm} (19)

Therefore, $\m$ is called a mass metric.

There are two main reasons in order to consider Newtonian systems. From the physical viewpoint, with a mass metric, we can introduce the notion of an external force, defined as
a section of the vertical cotangent bundle $V^\ast Q J^1 Q \to J^1 Q$. Let $(Q, \widehat{m}, \xi)$ be a Newtonian system and $F$ an external force. Then

$$\xi^i_F = \xi^i + (m^{-1})^{ik} F_k,$$

is a dynamic equation, but the triple $(Q, \widehat{m}, \xi_F)$ is a Newtonian system only if $F$ possesses the property

$$\partial_i F_j + \partial_j F_i = 0. \quad (20)$$

From the mathematical viewpoint, the equation (19) is the kernel of an Euler–Lagrange-type operator. By an appropriate choice of a mass metric, one may hope to bring it into Lagrange equations. This is the well-known inverse problem in time-dependent mechanics.

Here, we consider Newtonian systems because they provide the vertical tangent bundle $V^\ast Q J^1 Q$ with a nondegenerate fiber metric $\widehat{m}$. Let us assume that $\widehat{m}$ is a Riemannian metric. With this metric, we immediately obtain the splitting (13), where $\mathcal{V}$ is the orthocomplement of $\mathcal{W}$. Then the corresponding decomposition (11) takes the form,

$$\xi^i = \tilde{\xi}^i + \tilde{m}_{ab} m^{ij} \dot{s}^a_j s^b_\xi, \quad (21)$$

where $\tilde{m}_{ab}$ is the inverse matrix of

$$\tilde{m}^{ab} = \dot{s}^a_i s^b_j m^{ij}.$$

It is readily observed that the decomposition (21) satisfies the generalized D’Alembert principle. The constraint reaction acceleration,

$$- r^i = -\tilde{m}_{ab} m^{ij} \dot{s}^a_j s^b_\xi, \quad (22)$$

is orthogonal to every element of $V^\ast Q J^1 Q \cap \text{Ann}(S)$ with respect to the mass metric $\widehat{m}$. Since elements of $V^\ast Q J^1 Q \cap \text{Ann}(S)$ can be treated as the virtual accelerations relative to the nonholonomic constraint $S$, the constraint reaction acceleration (22) characterizes $S$ as an ideal constraint.

The Gauss principle is also fulfilled as follows. Given a dynamic equation $\xi$ and the above-mentioned fiber metric $\widehat{m}$, let us define a positive function $G(w)$ on $J^2 Q$ as

$$G(w) = \widehat{m} \left( \xi(\pi^2_1(w)) - w, \xi(\pi^2_2(w)) - w \right),$$

$$G(q^i, \dot{q}^i, q^{\lambda}_t, \dot{q}^{\lambda}_t) = m_{ij}(q^\lambda, \dot{q}^\lambda_t) (\xi^i(q^\lambda, \dot{q}^\lambda_t) - q^{\lambda}_t) (\xi^j(q^\lambda, \dot{q}^\lambda_t) - q^{\lambda}_t).$$

We say that $\|w\| = G(w)^{1/2}$ is a norm of $w \in J^2 Q$.

**Proposition 2:** Among all dynamic equations compatible with a nonholonomic constraint, the dynamic equation $\xi$ defined by the decomposition (21) is that of least norm.
Proof: Let \( \zeta \) be another dynamic equation which takes its values into \( W \). Then \( \tilde{\xi} - \zeta \in W \) and

\[
\tilde{m}(\tilde{\xi} - \zeta, \xi - \tilde{\xi}) = 0.
\]

Hence, we obtain

\[
\|\zeta\| = \tilde{m}(\xi - \tilde{\xi} + \tilde{\xi} - \zeta, \xi - \tilde{\xi} + \tilde{\xi} - \zeta) = \|\tilde{\xi}\| + \tilde{m}(\tilde{\xi} - \zeta, \tilde{\xi} - \zeta).
\]

In next Section, we will show that, in the case of nondegenerate Lagrangian systems and linear nonholonomic constraints, the decomposition (21) satisfies the traditional D’Alembert principle.

IV. LAGRANGIAN SYSTEMS WITH NONHOLONOMIC CONSTRAINTS

Nondegenerate Lagrangian systems are particular Newtonian systems.
A Lagrangian is defined as a horizontal density,

\[
L = \mathcal{L}dt, \quad \mathcal{L} : J^1Q \to \mathbb{R},
\]

on the velocity phase space \( J^1Q \). Here, we apply in a straightforward manner the first variational formula.\(^{13,15}\)

Let us consider a projectable vector field

\[
u = u^t\partial_t + u^i\partial_i, \quad u^t = 0, 1,
\]
on the configuration bundle \( Q \to \mathbb{R} \) and calculate the Lie derivative of the Lagrangian (23) along the jet prolongation,

\[
\bar{\nu} = u^t\partial_t + u^i\partial_i + \partial_t(u^i\partial_i^t),
\]
of \( u \), where \( \partial_t = \partial_t + q_i\partial_i + \cdots \) is the operator of formal derivative. We obtain

\[
\mathbf{L}_{\bar{\nu}}L = (\pi][d\mathcal{L})dt = (u^t\partial_t + u^i\partial_i + \partial_t(u^i\partial_i^t))\mathcal{L}dt.
\]

The first variational formula provides the following canonical decomposition of the Lie derivative (24) in accordance with the variational problem:

\[
\bar{\nu}[d\mathcal{L} = (u^t - u^tq_i^t)\mathcal{E}_i + \partial_t(u][H_L)
\]

where

\[
H_L = \bar{v}(dL) + L = \pi_idq^i - (\pi_iq_i^t - \mathcal{L})dt
\]
is the Poincaré–Cartan form and
\[
\mathcal{E}_L : J^2Q \to V^*Q, \\
\mathcal{E}_L = \mathcal{E}_i \theta^i = (\partial_i - d_i \partial_t^i)\mathcal{L}\theta^i,
\]
is the Euler–Lagrange operator for \(L\). We will use the notation
\[
\pi_i = \partial_t^i \mathcal{L}, \quad \pi_{ji} = \partial_t^j \partial_t^i \mathcal{L}.
\]
A Lagrangian \(L\) is called nondegenerate if \(\det \pi_{ji} \neq 0\) everywhere on the velocity phase space \(J^1Q\).

The kernel \(\text{Ker} \mathcal{E}_L \subset J^2Q\) of the Euler–Lagrange operator \((27)\) defines the system of second order differential equations,
\[
(\partial_i - d_i \partial_t^i)\mathcal{L} = 0,
\]
on \(Q\), called the Lagrange equations. Their solutions are (local) section \(c\) of the fiber bundle \(Q \to \mathbb{R}\) whose second order jet prolongations \(\tilde{c}\) live in \((28)\).

A holonomic connection on the jet bundle \(J^1Q \to \mathbb{R}\) is said to be a Lagrangian connection \(\xi_L\) for the Lagrangian \(L\) if it takes its values in the kernel \((28)\) of the Euler–Lagrange operator \(\mathcal{E}_L\). Every Lagrangian connection \(\xi_L\) defines a dynamic equation on the configuration space \(Q\) whose solutions are also solutions of the Lagrange equations \((28)\). If \(L\) is non-degenerate, the Lagrange equation \((28)\) can be algebraically solved for the second order derivatives, and they are equivalent to the dynamic equation,
\[
q_{tt}^i = \xi_{tt}^i, \quad \xi_L^i = (\pi^{-1})^{ij} \mathcal{E}_j + q_{tt}^i,
\]
called the Lagrange dynamic equation.

Every Lagrangian \(L\) on the jet manifold \(J^1Q\) yields the Legendre map,
\[
\hat{\mathcal{L}} : J^1Q \to V^*Q, \quad p_i \circ \hat{\mathcal{L}} = \pi_i,
\]
where \((t, q^i, p_i)\) are holonomic coordinates on the vertical cotangent bundle \(V^*Q\). As is well known, the Legendre map \((30)\) is a local diffeomorphism iff \(L\) is nondegenerate. A Lagrangian \(L\) is called hyperregular if the Legendre map \(\hat{\mathcal{L}}\) is a diffeomorphism.

The vertical tangent map \(V\hat{\mathcal{L}}\) to the Legendre map \(\hat{\mathcal{L}}\) reads
\[
V\hat{\mathcal{L}} : V_QJ^1Q \to VV^*Q \cong V^*Q \times V^*Q.
\]
It yields the linear fibred morphism \(V_QJ^1Q \to V_QJ^1Q\) and the corresponding mapping,
\[
J^1Q \to V^*_QJ^1Q \otimes V^*_QJ^1Q, \quad m_{ij} = \pi_{ij}.
\]
If a Lagrangian $L$ is nondegenerate, then (31) is a mass metric which satisfies the symmetry condition (17) and the compatibility condition (18) for the Lagrange dynamic equation (29).

Thus, every nondegenerate Lagrangian $L$ defines a Newtonian system. Moreover, a non-degenerate Lagrangian system plus an external force which fulfills the condition (20) is also a Newtonian system. Conversely, every standard Newtonian system can be seen as a Lagrangian system with the Lagrangian,

$$L = \frac{1}{2} m_{ij} (q^i_t - \Gamma^i)(q^j_t - \Gamma^j) dt,$$

where $\Gamma$ is a reference frame, plus an external force.

Given a nondegenerate Lagrangian $L$ with a Riemannian mass metric $m_{ij} = \pi_{ij}$, let now $S$ be an admissible nonholonomic constraint on the velocity phase space $J^1Q$. Since this is a particular Newtonian system, we obtain the dynamic equation

$$q^i_{tt} = \xi^i L - \bar{m}_{ab} m_{ij} \dot{s}^a_j (s^b_k \dot{q}^k_j + s^b_0),$$

$$\xi^i L = m^{ij} (-\partial_t \pi_j - \partial_k \pi_j q^k_t + \partial_j \mathcal{L}),$$

which is compatible with the constraint $S$, treated as an ideal nonholonomic constraint. This is the Lagrange dynamic equation in the presence of the additional constraint reaction force

$$F_i = -\bar{m}_{ab} \dot{s}^a_i s^b (\xi_L).$$

Let us consider the energy conservation law in the presence of this force.

The energy conservation law in Lagrangian time-dependent mechanics is deduced from the first variational formula (25) when the vector field $u = \Gamma$ is a reference frame. On the shell $\mathcal{E}_i = 0$, this formula leads to the weak identity,

$$\mathbf{L}_T L \approx -d_i (\pi_i \dot{q}^i - \mathcal{L}),$$

where $\dot{q}^i = q^i_t - \Gamma^i$ is a relative velocity and

$$T \Gamma = \pi_i \dot{q}^i - \mathcal{L}$$

is the energy function with respect to the reference frame $\Gamma$. In the presence of an external force $F$, i.e., on the shell $\mathcal{E}_i = -F_i$, the weak identity (35) is modified as

$$\mathbf{L}_T L - \dot{q}^i_i F_i = -d_i T \Gamma.$$

It is readily observed that, if a nonholonomic constraint is linear and $\Gamma$ is a constraint reference frame, the constraint reaction force (34) does not contribute to the energy conservation law. It follows that, in this case, the standard D’Alembert principle holds, while the equation
describes a motion in the presence of an ideal nonholonomic constraint in the spirit of this principle.

The constrained equation of motion (33) is neither Lagrange equations nor a dynamic equation of a Newtonian system. In Section VI, we aim to show that it can be seen as a part of Hamilton equations in the framework of the Hamiltonian formalism extended to the configuration space $VQ$.

V. VERTICAL EXTENSION OF HAMILTONIAN FORMALISM

This Section provides a brief exposition of Hamiltonian formalism of time-dependent mechanics on a configuration bundle $Q \to \mathbb{R}$ and its extension to the vertical configuration space $VQ$. We consider this extension because any first order dynamic equation on the momentum phase space $V^*Q$ can be seen as a Hamilton equation in the framework of the extended Hamiltonian formalism. This extension is also of interest in the path-integral formulation of mechanics.\(^{19,20}\)

Given a mechanical system on a configuration bundle $Q \to \mathbb{R}$, its momentum phase space is the vertical cotangent bundle $V^*Q$ of $Q \to \mathbb{R}$, equipped with the holonomic coordinates $(t, q^i, p, p_i)$.\(^{13-15}\) The momentum phase space $V^*Q$ is endowed with the canonical exterior 3-form,

$$\Omega = dp_i \wedge dq^i \wedge dt.$$  \(\text{(33)}\)

Let us consider the cotangent bundle $T^*Q$ of $Q$ with the holonomic coordinates $(t, q^i, p, p_i)$. It admits the canonical Liouville form

$$\Xi = pdt + p_i dq^i.$$  \(\text{(37)}\)

An exterior 1-form $H$ on the momentum phase space $V^*Q$ is called a Hamiltonian form if it is the pull-back

$$H = h^*\Xi = p_i dq^i - \mathcal{H}dt$$  \(\text{(38)}\)

of the Liouville form $\Xi$ \((\text{37})\) by a section $h$ of the fiber bundle

$$\zeta : T^*Q \to V^*Q.$$  \(\text{(39)}\)

Remark: With respect to a trivialization $Q \cong \mathbb{R} \times M$, the Hamiltonian form \((\text{38})\) is the well-known integral invariant of Poincaré–Cartan, where $\mathcal{H}$ is a Hamiltonian. The peculiarity of Hamiltonian time-dependent mechanics issues from the fact that Hamiltonians are not scalar functions under time-dependent transformations, but make up an affine space modelled over the linear space of functions on $V^*Q$.  

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For instance, every connection $\Gamma$ on a configuration bundle $Q \rightarrow \mathbf{R}$ is an affine section, 

$$p \circ \Gamma = -p_i \Gamma^i,$$

of the fiber bundle \([39]\), and defines the Hamiltonian form 

$$H_\Gamma = p_i dq^i - p_i \Gamma^i dt.$$

It follows that any Hamiltonian form on the momentum phase space $V^*Q$ admits the splitting,

$$H = H_\Gamma - \tilde{H}_\Gamma dt = p_i dq^i - (p_i \Gamma^i + \tilde{H}_\Gamma) dt,$$

where $\Gamma$ is a connection on $Q \rightarrow \mathbf{R}$ and $\tilde{H}_\Gamma$ is a real function on $V^*Q$, called the Hamiltonian function. The following assertions are basic facts in the Hamiltonian formulation of time-dependent mechanics.\(^{14,15}\)

** Proposition 3:** Every Hamiltonian form $H$ on the momentum phase space $V^*Q$ defines the associated Hamiltonian map,

$$\tilde{H} : V^*Q \rightarrow J^1 Q, \quad q^i_t \circ \tilde{H} = \partial^i \mathcal{H}.$$

** Proposition 4:** Given a Hamiltonian form $H$ on the momentum phase space $V^*Q$, there exists a unique connection

$$\gamma_H = \partial_t + \partial^i \mathcal{H} \partial_i - \partial_i \mathcal{H} \partial^i \quad (40)$$

on $V^*Q \rightarrow \mathbf{R}$, called a Hamiltonian connection, such that

$$\gamma_H \Omega = dH.$$

The kernel of the covariant differential of the Hamiltonian connection \(\gamma_H\) defines the Hamilton equations,

$$q^i_t = \partial^i \mathcal{H}, \quad (41a)$$

$$p_{ti} = -\partial_i \mathcal{H}, \quad (41b)$$

for the Hamiltonian form $H$. Their solutions are integral curves for the Hamiltonian connection $\gamma_H$ \(\text{[40]}\).
Now let us consider the vertical tangent bundle \( VQ \) of the fiber bundle \( Q \to \mathbb{R} \), coordinated by \((t, q^i, \dot{q}^i)\). It can be seen as a new configuration space, called the vertical configuration space. The corresponding vertical momentum phase space is the vertical cotangent bundle \( V^*VQ \) of \( VQ \to \mathbb{R} \). The vertical momentum phase space \( V^*VQ \) is canonically isomorphic to the vertical tangent bundle \( V^*Q \to \mathbb{R} \) coordinated by \((t, q^i, p_i, \dot{q}^i, \dot{p}_i)\). It is easily seen from the transformation laws that \((q^i, \dot{p}_i)\) and \((\dot{q}^i, p_i)\) are canonically conjugate pairs.

The vertical momentum phase space \( V^*VQ \) is endowed with the canonical 3-form,

\[
\Omega_V = [dp_i \wedge dq^i + dp \wedge d\dot{q}^i] \wedge dt.
\]

For the sake of brevity, one can write \( \Omega_V = \partial_V \Omega \), where \( \partial_V = \dot{q}^i \partial_i + \dot{p}_i \partial^i \) is the vertical derivative.

The notions of a Hamiltonian connection, a Hamiltonian form, etc., on the vertical momentum phase space \( V^*Q \cong V^*VQ \) are introduced similarly to those on the ordinary momentum phase space \( V^*Q \). In particular, a Hamiltonian form on \( V^*VQ \) reads

\[
H_V = \dot{p}_i dq^i + p_i d\dot{q}^i - \mathcal{H}_V dt.
\]

Since Hamiltonian forms are determined modulo exact forms and the function \( p_i \dot{q}^i \) is globally defined on \( V^*VQ \), we will write

\[
H_V = \dot{p}_i dq^i - \dot{q}^i dp_i - \mathcal{H}_V dt. \tag{42}
\]

The corresponding Hamilton equations read

\[
\begin{align*}
\gamma^i &= q^i_t = \dot{q}^i \mathcal{H}_V, \tag{43a} \\
\gamma_i &= p_{ti} = -\partial_i \mathcal{H}_V, \tag{43b} \\
\gamma &= \dot{q}^i = \dot{q}^i \mathcal{H}_V, \tag{43c} \\
\mathcal{H} &= \dot{p}_{ti} = -\partial_i \mathcal{H}_V, \tag{43d}
\end{align*}
\]

where

\[
\gamma = \partial_t + \gamma^i \partial_i + \gamma_i \partial^i + \gamma \dot{q}^i + \mathcal{H} \dot{p}_i
\]

is a Hamiltonian connection on the vertical momentum phase space \( V^*Q \to \mathbb{R} \).

There is the following relation between Hamiltonian formalisms on \( V^*Q \) and \( V^*VQ \).\(^{13,15}\) Let \( VT^*Q \) be the vertical tangent bundle of the cotangent bundle \( T^*Q \to \mathbb{R} \) is equipped with holonomic coordinates \((t, q^i, p_i, \dot{q}^i, \dot{p}_i, \dot{p})\) and endowed with the canonical form,

\[
\Xi_V = \dot{p} dt + \dot{p}_i dq^i - \dot{q}^i dp_i.
\]
Proposition 5: Let $\gamma_H$ be a Hamiltonian connection on the ordinary momentum phase space $V^*Q \to \mathbb{R}$ for a Hamiltonian form,

$$H = h^*\Xi = p_i dq^i - \mathcal{H}dt. \quad (44)$$

Then the connection

$$V\gamma_H : VV^*Q \to VJ^1V^*Q \cong J^1VV^*Q,$$

$$V\gamma_H = \partial_t + \gamma^i \partial_i + \gamma^i \partial_t \gamma_i + \partial_t \gamma_i \dot{\gamma}_i + \partial_t \gamma_i \dot{\gamma}_i, \quad (45)$$
on the vertical momentum phase space $VV^*Q \to \mathbb{R}$ is a Hamiltonian connection for the Hamiltonian form,

$$H_V = (Vh)^*\Xi_V = \dot{p}_i dq^i - \dot{q}_i dp_i - \partial_t \mathcal{H}dt,$$

$$\partial_t \mathcal{H} = (\dot{q}_i \partial_i + \dot{p}_i \partial_t) \mathcal{H}, \quad (46)$$

where $Vh : VV^*Q \to VT^*Q$ is the vertical tangent map to $h$.

The corresponding Hamilton equations read

$$\gamma^i = \dot{\partial}_i \mathcal{H} = \dot{\mathcal{H}}, \quad (48a)$$

$$\gamma_i = -\dot{\partial}_i \mathcal{H} = -\partial_i \mathcal{H}, \quad (48b)$$

$$\dot{\gamma}^i = \dot{\partial}_i \dot{\mathcal{H}} = \partial_v \dot{\mathcal{H}}, \quad (48c)$$

$$\dot{\gamma}_i = -\partial_i \mathcal{H} \dot{\mathcal{H}} = -\partial_v \partial_i \mathcal{H}. \quad (48d)$$

It is easily seen that the equations $(48a) - (48b)$ are exactly the Hamilton equations $(41a) - (41b)$ for the Hamiltonian form $H$.

Remark: In order to clarify the physical meaning of the Hamilton equations $(48c) - (48d)$, let $r(t)$ be a solution of the Hamilton equations $(48a) - (48b)$. Let $\dot{r}(t)$ be a Jacobi field, i.e., $r(t) + \varepsilon \dot{r}(t)$ is also a solution of the same Hamilton equations modulo terms of order two in $\varepsilon$. Then it is readily observed that the Jacobi field $\dot{r}(t)$ fulfills the Hamilton equations $(48c) - (48d)$.

The following assertion plays a prominent role in the sequel.$^{13,15}$

Proposition 6: Any connection $\gamma$ on the momentum phase space $V^*Q \to \mathbb{R}$ gives rise to the Hamiltonian connection,

$$\gamma^i = \gamma^i, \quad \gamma_i = \gamma_i, \quad \dot{\gamma}^i = \dot{p}_j \partial^j \gamma^i - \dot{q}_j \partial^j \gamma_i, \quad \dot{\gamma}_i = -\dot{p}_j \partial_i \gamma^j + \dot{q}_j \partial_i \gamma_j, \quad (49)$$

for the Hamiltonian form,

$$H_V = \dot{p}_i (dq^i - \gamma_i dt) - \dot{q}_i (dp_i - \gamma_i dt) = \dot{p}_i dq^i - \dot{q}_i dp_i - (\dot{p}_i \gamma^i - \dot{q}_i \gamma_i) dt,$$
on the vertical momentum phase space $V^*Q$.

In particular, if $\gamma$ is a Hamiltonian connection on the fiber bundle $V^*Q \to R$, then (49) is exactly the connection $V\gamma$ (15).

It follows that every first order dynamic equation on the momentum phase space $V^*Q$ can be seen as the Hamilton equations (43a) \& (43b) for a suitable Hamiltonian form on the vertical momentum phase space.

VI. HAMILTONIAN SYSTEMS WITH NONHOLONOMIC CONSTRAINTS

Let $L$ be a hyperregular Lagrangian with a Riemannian mass metric $\hat{m}$. In this case, Hamiltonian and Lagrangian formalisms of time-dependent mechanics are equivalent. There exists a unique associated Hamiltonian form $H$ (38) on $V^*Q$ such that

$$\hat{H} = \hat{L}^{-1}, \quad p_i \equiv \pi_i(q^\lambda, \partial^i \mathcal{H}(q^\lambda, p_k)), \quad q_i^j \equiv \partial^j \mathcal{H}(q^\lambda, \pi_j(q^\lambda, q_k^i)), \quad (50a)$$

$$\mathcal{L} \circ \hat{H} \equiv \gamma_H | H = p_i \partial^i \mathcal{H} - \mathcal{H}. \quad (50b)$$

As an immediate consequence of (50a), we have $J^1\hat{H} = (J^1\hat{L})^{-1}$, where the jet prolongations of the Hamiltonian and Legendre maps read

$$J^1\hat{H} : J^1V^*Q \to J^1J^1Q, \quad (q^\lambda, q_i^j, q_i^{j(t)}) \circ J^1\hat{H} = (q^\lambda, \partial^i \mathcal{H}, q_i^j, d_t \partial^i \mathcal{H}),$$

$$J^1\hat{L} : J^1J^1Q \to J^1V^*Q, \quad (q^\lambda, p_i, q_i^j, p_i^{j(t)}) \circ J^1\hat{L} = (q^\lambda, \pi_i, q_i^j, d_t \pi_i).$$

Then, using (50a) \& (50b), we obtain

$$\gamma_H = J^1\hat{L} \circ \xi_L \circ \hat{H}.$$ 

Let introduce the notation $M^{ij} = \partial^i \partial^j \mathcal{H}$. There are the relations

$$M^{ik}(m_{kj} \circ \hat{H}) = \delta^i_j, \quad m_{kj}(M^{ik} \circ \hat{L}) = \delta^j_i, \quad m_{ij} = \pi_{ij}.$$ 

It follows that $M$ is a fiber metric in the vertical tangent bundle $V_QV^*Q$ of the fiber bundle $V^*Q \to Q$.

Given a codistribution $\mathbf{S}$ on $J^1Q$, let us consider the pull-back codistribution $\hat{H}^*\mathbf{S}$ on $V^*Q$, spanned locally by 1-forms

$$\beta^a = \hat{H}^*s^a = (s_0^a + s_j^a \partial_t \partial^j \mathcal{H})dt + (s_i^a + s_j^a \partial_i \partial^j \mathcal{H})d^j + \hat{s}_i^a M^{ij} dp_j =$$

$$\beta_0^a dt + \beta_i^a dq^i + \hat{\beta}_{ai} dp_i.$$ 

This codistribution defines a nonholonomic constraint on the momentum phase space $V^*Q$.

Given a Hamiltonian connection $\gamma_H$ (40), let us find its splitting

$$\gamma_H = \tilde{\gamma} + \vartheta \quad (51)$$
where $\tilde{\gamma}$ is a connection on $V^*Q \to \mathbb{R}$ which satisfies the condition

$$\tilde{\gamma} \subset \text{Ann}(\tilde{H}^*S).$$

(52)

The connection $\tilde{\gamma}$ (52) obviously defines a first order dynamic equation on the momentum phase space $V^*Q$ which is compatible with the nonholonomic constraint $\tilde{H}^*S$. The decomposition (51) is not unique. Let us construct it as follows.

Given a Hamiltonian connection $\gamma_H$, we consider the codistribution $S_H$ on $V^*Q$, spanned locally by the 1-forms $dq^i - \gamma^i_H dt$. Its annihilator $\text{Ann}(S_H)$ is an affine sub-bundle of the affine jet bundle $J^1V^*Q \to V^*Q$, modelled over the vertical tangent bundle $V_QV^*Q$. The Hamiltonian connection $\gamma_H$ is a section of this sub-bundle. Let us take the intersection

$$W = \text{Ann}(S_H) \cap \text{Ann}(\tilde{H}^*S).$$

**Lemma 7:** $W$ is an affine bundle over $V^*Q$, modelled over the vector bundle,

$$\mathcal{W} = V_QV^*Q \cap \text{Ann}(\tilde{H}^*S).$$

**Proof:** The intersection $\mathcal{W}$ consists of elements $v = v_i\partial^i$ of $V_QV^*Q$ which fulfill the conditions

$$v_i\dot{\beta}^{ai} = 0.$$

Since the nonholonomic constraint $S$ is admissible and the matrix $M^{ij}$ is nondegenerate, every fiber of $\mathcal{W}$ is of dimension $m - n$, i.e., $\mathcal{W}$ is a vector bundle, while $W$ is an affine bundle.

Then, using the fiber metric $M$ in $V_QV^*Q$, we obtain the splitting

$$V_QV^*Q = \mathcal{W} \oplus \mathcal{V},$$

where $\mathcal{V}$ is the orthocomplement of $\mathcal{W}$, and the associated splitting

$$\text{Ann}(S_H) = W \oplus \mathcal{V}.$$

The corresponding decomposition (51) reads

$$\tilde{\gamma} = \gamma_H - \tilde{M}_{ab}M_{ij}\dot{\beta}^{ai}\dot{\beta}^{bj}(\gamma_H)\partial^j,$$

(53)

where $\tilde{M}_{ab}$ is the inverse matrix of

$$\tilde{M}^{ab} = \dot{\beta}^{ai}\dot{\beta}^{bj}M_{ij}.$$
The splitting (53) is the Hamiltonian counterpart of the splitting (21). We have the relations
\[ \tilde{m}^{ab} = \tilde{M}^{ab} \circ \tilde{H}, \quad \beta^a(\gamma_H) = s^a(\xi_L) \circ \tilde{H}, \]
and as a consequence
\[ \tilde{\gamma} = J^1 L \circ \tilde{\xi} \circ \tilde{H}. \]

Remark: The above procedure can be extended in a straightforward manner to any standard Newtonian system, seen as a Lagrangian system with the Lagrangian (32) and an external force. Following this procedure, one may also study a nonholonomic Hamiltonian system, without appealing to its Lagrangian counterpart.

The connection (53) defines the system of first order dynamic equations,
\[ \dot{q}_i^t = \partial^i \mathcal{H}, \quad p_{ti} = -\partial_i \mathcal{H} - \tilde{M}_{ab} M_{ij} \beta^{ai} \beta^{b}(\gamma_H), \tag{54} \]
on the momentum phase space \(V^*Q\), which are not Hamilton equations. Nevertheless, in accordance with Proposition 3, one can restate the constrained equations of motion (54) as the Hamilton equations (48a) – (48b) for the Hamiltonian form,
\[ H_V = \dot{p}_i dq^i - \dot{q}^i dp_i - \partial_{V} \mathcal{H} dt - \dot{q}^i \tilde{M}_{ab} M_{ij} \beta^{ai} \beta^{b}(\gamma_H) dt, \]
on the vertical momentum phase space \(VV^*Q\), where the last term can be written in brief as \(\dot{\omega}_V \Omega\).

The Hamiltonian form of the constrained equations of motion may be important in connection with the following speculations.

Given a Hamiltonian form \(H_V\) (42) on the vertical momentum phase space \(VV^*Q\), let us consider the Lagrangian
\[ L_H = \dot{p}_i q^i - \dot{q}^i p_{ti} - H_V \tag{55} \]
on the first order jet manifold \(J^1 V V^*Q\) of the fiber bundle \(V V^*Q \to \mathbb{R}\). It is readily observed that the corresponding Lagrange equations are exactly the Hamilton equations (13a) – (13d) for the Hamiltonian form \(H_V\). In particular, let \(H\) be a Hamiltonian form on an ordinary momentum phase space \(V^*Q\) and \(H_V = \partial_{V} H\). In this case, the Lagrangian (55) reads
\[ L_H = \dot{p}_i (q^i - \partial^i H) - \dot{q}^i (p_{ti} + \partial_i H). \]

It is easily seen that this Lagrangian vanishes on solutions of the Hamilton equations for the Hamiltonian form \(H\). By this reason, it is applied to the functional integral formulation of mechanics.\(^{19,20}\)
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