$\varepsilon$-regularity criteria in anisotropic Lebesgue spaces and regularity in one direction to the 3D Navier-Stokes equations

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Abstract

In this paper, we derive some $\varepsilon$-regularity criteria in anisotropic Lebesgue spaces for suitable weak solutions to the 3D Navier-Stokes equations as follows

$$
\limsup_{\varepsilon \to 0} \varepsilon \frac{1}{p} \frac{1}{q} \frac{1}{r} \frac{1}{s} \|u\|_{L^p_t L^q_x L^r_y L^s_z (Q(\varepsilon))} \leq \varepsilon, \quad \frac{2}{p} + \frac{1}{q} + \frac{1}{r} + \frac{1}{s} \leq 2, \quad \text{with } q, r, s > 2;
$$

$$
\limsup_{\varepsilon \to 0} \varepsilon \frac{1}{p} \frac{1}{q} \frac{1}{r} \frac{1}{s} \|\nabla_1 u\|_{L^p_t L^q_x L^r_y L^s_z (Q(\varepsilon))} \leq \varepsilon, \quad \frac{2}{p} + \frac{1}{r} + \frac{1}{s} \leq 2, \quad \text{with } r, s > 2;
$$

$$
\|u\|_{L^p_t L^q_x L^r_y L^s_z (Q(1))} + \|\Pi\|_{L^1_t (Q(1))} \leq \varepsilon, \quad \frac{2}{p} + \frac{1}{q} + \frac{1}{r} + \frac{1}{s} < 2, \quad \text{with } q, r, s > 2,
$$

which extends previous corresponding results in [16, 17, 19, 25, 26, 35]. As a by-product, this allows us to obtain local regularity criteria in terms of $\nabla_1 u$, namely,

$$
\nabla_1 u \in L^p_1 L^q_{x_1} L^r_{x_2} L^s_{x_3} (Q(\varepsilon)), \quad \frac{2}{p} + \frac{1}{q} + \frac{1}{r} + \frac{1}{s} = 2, \quad 2 < q, r, s \leq 3, 2 \leq p < 4. \quad (0.2)
$$

This generalizes the recent result in [23], where the range of $(p, q = r = s)$ is that $\frac{3}{4} < q < 3$ and $2 < p < 3$.

More importantly, the proof utilized in (0.1) together with the result of [16] implies

$$
\limsup_{\varepsilon \to 0} \varepsilon \frac{2}{p} \frac{3}{q} \frac{1}{s} \|\nabla_1 u\|_{L^p_t L^q_x L^s_z (Q(\varepsilon))} \leq \varepsilon, \quad 2 \leq \frac{2}{p} + \frac{3}{q} \leq 3, \quad 1 \leq p, q \leq \infty. \quad (0.3)
$$

This gives an improvement of known results obtained in [1, 16]. It is worth remarking that (0.3) yields full range of $q$ in (0.2) with $q = r = s$.

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1 Introduction

We study the following incompressible Navier-Stokes equations in three-dimensional space

\[
\begin{align*}
  u_t - \Delta u + u \cdot \nabla u + \nabla \Pi &= 0, \\
  \text{div } u &= 0, \\
  u|_{t=0} &= u_0,
\end{align*}
\]  

(1.1)

where \(u\) stands for the flow velocity field, the scalar function \(\Pi\) represents the pressure. The initial velocity \(u_0\) satisfies \(\text{div } u_0 = 0\).

We are concerned with the regularity of suitable weak solutions satisfying local energy inequality to the Navier-Stokes system (1.1). A point \((x, t)\) is said to be a regular point if \(|u|\) is bounded at some neighbourhood of this point. Otherwise, \((x, t)\) is singular point. The local energy inequality of (1.1) is due to Scheffer in [31–33]. In this direction, a milestones result that one dimensional Hausdorff measure of the possible space-time singular points of suitable weak solutions to the 3D Navier-Stokes equations is zero was obtained by Caffarelli, Kohn and Nirenberg in [1]. This result relies on the following two \(\epsilon\)-regularity criteria in [1] to the suitable weak solutions of (1.1). One holds at one scale: \((0, 0)\) is regular point provided

\[
\|u\|_{L^3(Q(1))} + \|u\Pi\|_{L^1(Q(1))} + \|\Pi\|_{L_{t}^{5/4}L_{x}^{1}(Q(1))} \leq \epsilon.
\]  

(1.2)

The other needs infinitely many scales and an alternative assumption of (1.2) is that

\[
\limsup_{\varrho \to 0} \varrho^{-\frac{3}{2}} \|\nabla u\|_{L^2(Q(\varrho))} \leq \epsilon.
\]  

(1.3)

We list some known \(\epsilon\)-regularity criteria at infinitely many scales

- Tian and Xin [35],

  \[
  \limsup_{\varrho \to 0} \varrho^{-\frac{3}{2}} \|\text{curl } u\|_{L^2(Q(\varrho))} \leq \epsilon \quad \text{and} \quad \limsup_{\varrho \to 0} \varrho^{-\frac{3}{2}} \|u\|_{L^3(Q(\varrho))} \leq \epsilon.
  \]  

  (1.4)

- Gustafson, Kang and Tsai [16]

  \[
  \limsup_{\varrho \to 0} \varrho^{1-\frac{3}{p}+\frac{3}{q}} \|u - \text{tr}_u\|_{L_{t}^{p}L_{x}^{q}(Q(\varrho))} \leq \epsilon, \quad 1 \leq 2/p + 3/q \leq 2, \quad 1 \leq p, q \leq \infty;
  \]  

  (1.5)

  \[
  \limsup_{\varrho \to 0} \varrho^{2-\frac{3}{p}+\frac{3}{q}} \|\nabla u\|_{L_{t}^{p}L_{x}^{q}(Q(\varrho))} \leq \epsilon, \quad 2 \leq 2/p + 3/q \leq 3, \quad 1 \leq p, q \leq \infty;
  \]  

  (1.6)

  \[
  \limsup_{\varrho \to 0} \varrho^{2-\frac{3}{p}+\frac{3}{q}} \|\text{curl } u\|_{L_{t}^{p}L_{x}^{q}(Q(\varrho))} \leq \epsilon, \quad 2 \leq 2/p + 3/q \leq 3, \quad 1 \leq p, q \leq \infty, \quad (p, q) \neq (1, \infty).
  \]  

  (1.7)

- Mahalov, Nicolaenko and Seregin [27], (Deformation tensor \(D(u) = \frac{1}{2}(\nabla u + \nabla u^T)\))

  \[
  \limsup_{\varrho \to 0} \varrho^{-\frac{3}{4}} \|D(u)\|_{L^2(Q(\varrho))} \leq \epsilon.
  \]  

  (1.8)

The extension of (1.8) can be found in [37].
• Seregin [34]
\[
\limsup_{\theta \to 0} \theta^{-\frac{1}{2}} \| \nabla u \|_{L^2(Q(\theta))} \leq M \quad \text{and} \quad \liminf_{\theta \to 0} \theta^{-\frac{1}{2}} \| \nabla_3 u \|_{L^2(Q(\theta))} \leq \varepsilon(M); \\
\limsup_{\theta \to 0} \theta^{-\frac{2}{3}} \| u \|_{L^3(Q(\theta))} \leq M \quad \text{and} \quad \liminf_{\theta \to 0} \theta^{-\frac{2}{3}} \| u \|_{L^3(Q(\theta))} \leq \varepsilon(M). 
\]

(1.9)

For progress concerning (1.9), the reader may refer to [36] by Wang and Zhang and [23, 24] by Kukavica and Rusin and Ziane.

• Wolf [42]
\[
\limsup_{\theta \to 0} \theta^{-\frac{1}{2}} \| \omega \times u \|_{L^2(Q(\theta))} \leq \varepsilon.
\]

More $\varepsilon$-regularity criteria via $\omega \times \frac{u}{|u|}$ and $u \times \frac{\omega}{|\omega|}$ may be found in [28].

• Wang and Wu [38]
\[
\limsup_{\theta \to 0} \theta^{-\frac{1}{2}} \| \nabla h u \|_{L^2(Q(\theta))} \leq \varepsilon. 
\]

(1.10)

• Choe, Wolf and Yang [10]
\[
\limsup_{\theta \to 0} \theta^{-\frac{2}{3}} \| \nabla u \|_{L^2(Q(\theta))} \leq \varepsilon.
\]

Besides suitable weak solutions, there exists other kind of weak solutions equipping energy inequality to the Navier-Stokes equations (1.1). This kind of weak solutions are called Leray-Hopf weak solutions. A number of papers have been devoted to the study of regularity of Leray-Hopf weak solutions and many sufficient regularity conditions are established (see for example, [2, 3, 5–9, 13, 14, 18, 21, 29, 30, 40, 43–45]). In particular, utilizing the anisotropic Lebesgue spaces, Zheng first studied anisotropic regularity criterion in terms of one velocity component in [43]. Later, Qian [30]; Guo, Caggio and Skalak [13]; Guo, Kucera and Skalak [14], further considered regularity condition in anisotropic Lebesgue spaces to the Leray-Hopf weak solutions in system (1.1). It is worth pointing out that Sobolev-embedding theorem in anisotropic Lebesgue space was established in these works. For the details, see Lemma 2.1 in Section 2. Inspired by recent works [13, 14, 30, 43], we investigate $\varepsilon$-regularity criteria to the 3D Navier-Stokes equations in anisotropic Lebesgue space. Now we formulate our result as follows

**Theorem 1.1.** Let $(u, \Pi)$ be a suitable weak solutions to (1.1) in $Q(\theta)$. Then $(0, 0)$ is regular point provided, one of the following conditions holds

(1) There exists a positive constant $\varepsilon_1$ such that $u \in L^p_t L^q_x L^2_r L^s_\theta(Q(\theta))$ with
\[
\limsup_{\theta \to 0} \theta^{1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r} - \frac{1}{s}} \| u - \bar{u}_\theta \|_{L^p_t L^q_x L^2_r L^s_\theta(Q(\theta))} \leq \varepsilon_1, 
\]
where satisfying
\[
\frac{2}{p} + \frac{1}{q} + \frac{1}{r} + \frac{1}{s} \leq 2, \quad \text{with} \ q, r, s > 2, 1 \leq p < 4.
\]

(1.11)
There exists a positive constant $\varepsilon_1$ such that $\nabla u \in L^p_tL^q_xL^r_xL^s_x(Q(\rho))$ with

$$
\limsup_{\rho \to 0} \rho^{1-\frac{2}{p}-\frac{1}{q}-\frac{1}{r}-\frac{1}{s}}\|\nabla u\|_{L^p_tL^q_xL^r_xL^s_x(Q(\rho))} \leq \varepsilon_1, \quad (1.13)
$$

where satisfying

$$
\frac{2}{p} + \frac{1}{q} + \frac{1}{r} + \frac{1}{s} \leq 2, \quad \text{with } q, r, s > 2, 1 \leq p < 2. \quad (1.14)
$$

Remark 1.1. Theorem 1.1 is a extension of (1.4)-(1.6).

Remark 1.2. We would like to point out that the range $1 \leq p < 4$ corresponds to the limiting case $\frac{2}{p} + \frac{1}{q} + \frac{1}{r} + \frac{1}{s} = 2$ in (1.12). By means of Hölder’s inequality, the range can be generalized to

$$
\frac{2}{p} + \frac{1}{q} + \frac{1}{r} + \frac{1}{s} = \begin{cases} 2-\delta & \text{with } \frac{1}{2} - \delta \leq \frac{2}{p} \leq 2-\delta \quad (0 \leq \delta \leq \frac{1}{2}), \\
2-\delta & \text{with } \frac{2}{2-\delta} \leq p \leq \infty \quad (\frac{1}{2} \leq \delta \leq 1).
\end{cases}
$$

Remark 1.3. As said above, in the light of the Hölder inequality, one can extend the range of $p$ in (1.14) to

$$
\frac{2}{p} + \frac{1}{q} + \frac{1}{r} + \frac{1}{s} = 2-\delta \quad \text{with } \quad 1-\delta < \frac{2}{p} \leq 2-\delta \quad (0 \leq \delta \leq 1).
$$

The absolute continuity of Lebesgue integral immediately yields the following result.

Corollary 1.2. Suppose that $(u, \Pi)$ is a suitable weak solution to (1.1). If there exists a constant $\rho$ such that

$$
\nabla u \in L^p_tL^q_xL^r_xL^s_x(Q(\rho)), \quad \text{with } \frac{2}{p} + \frac{1}{q} + \frac{1}{r} + \frac{1}{s} = 2, \quad 2 < q, r, s \leq 3, 2 \leq p < 4.
$$

then $(0, 0)$ is regular point.

Remark 1.4. Compared with the regularity criteria proved by Kukavica, Rusin and Ziane in [23], we not only establish the local regularity criteria via $\nabla u$ in anisotropic Lebesgue spaces but also extend the range of $(p, q)$.

Motivated by the proof of Theorem 1.1, we derive from (1.5) that the following result

Theorem 1.3. Suppose that $(u, \Pi)$ is a suitable weak solution to (1.1). If there exists a constant $\rho$ such that

$$
\nabla u \in L^p_tL^q_x(Q(\rho)), \quad \text{with } \frac{2}{p} + \frac{3}{q} = 2, \quad \frac{3}{2} < q \leq \infty,
$$

then $(0, 0)$ is regular point.

Remark 1.5. The borderline case $L^\infty_tL^{3/2}_x$ in (1.15) should read $\nabla u \in L^\infty_tL^{3/2}_x(Q(\rho))$ and $\|\nabla u\|_{L^\infty_tL^{3/2}_x(Q(\rho))}$ sufficiently small.
Remark 1.6. To the knowledge of authors, even for Leray-Hopf solutions in $\mathbb{R}^3$, the known range of $q$ is $[\frac{3\sqrt{7}}{4} - 3, 3]$, see e.g. [3, 21, 44]. It should be pointed out that the range of $q$ in (1.15) is full. Theorem 1.3 is a significant generalization of the results in [23].

As said before, proving Theorem 1.3 reduces to the following theorem.

**Theorem 1.4.** Let $(u, \Pi)$ be a suitable weak solutions to (1.1) in $Q(\rho)$. Then $(0, 0)$ is a regular point provided

$$\limsup_{\rho \to 0} \rho^{2 - \frac{2}{p} - \frac{2}{q}} \| \nabla_1 u \|_{L^p_t L^q_x (Q(\rho))} \leq \varepsilon_1, \quad 2 \leq \frac{2}{p} + \frac{3}{q} \leq 3, \quad 1 \leq p, q \leq \infty.$$  \hfill (1.16)

**Remark 1.7.** A special case of (1.16) is that

$$\limsup_{\rho \to 0} \rho^{-\frac{1}{2}} \| \nabla_1 u \|_{L^2 (Q(\rho))} \leq \varepsilon,$$

which improves the classical result (1.3).

Next we turn attentions to the $\varepsilon$-regularity criteria at one scale in the type of (1.2). In particular, Choi and Vasseur [11], Guevara and Phuc [17] improved (1.2) to

$$\| u \|_{L^\infty_t L^2_x (Q(1))} + \| \nabla u \|_{L^2_t (Q(1))} + \| \Pi \|_{L^1_t (Q(1))} \leq \varepsilon.$$  \hfill (1.17)

Recently, Guevara and Phuc [17] found that (1.2) can be replaced by the follows

$$\| u \|_{L^p_t L^{3q/(2p)} (Q(1))} + \| \Pi \|_{L^p_t L^q (Q(1))} \leq \varepsilon, \quad 3/q + 2/p = 7/2 \quad \text{with} \quad 1 \leq p \leq 2.$$  \hfill (1.18)

Authors in [19] further extended (1.18) to

$$\| u \|_{L^p_t L^q (Q(1))} + \| \Pi \|_{L^1_t (Q(1))} \leq \varepsilon, \quad 1 \leq \frac{2}{p} + \frac{3}{q} \leq 2, \quad 1 \leq p, q \leq \infty.$$  \hfill (1.19)

Very recently, an alternative proof of (1.19) was presented by Dong and Wang [12]. Moreover, for a short summary on $\varepsilon$-regularity criteria at one scale we refer the reader to [19] and references therein. In addition, for the $\varepsilon$-regularity criterion without pressure to local suitable weak solutions of the Navier-Stokes equations at one scale (see the works [4, 20, 39, 41]).

The last result concerns $\varepsilon$-regularity criteria in anisotropic Lebesgue spaces at one scale, which generalizes the corresponding results in (1.19).

**Theorem 1.5.** Let the pair $(u, \Pi)$ be a suitable weak solution to the 3D Navier-Stokes system (1.1) in $Q(1)$. There exists an absolute positive constant $\varepsilon$ such that if the pair $(u, \Pi)$ satisfies

$$\| u \|_{L^p_t L^q_x L^r_t L^s_x (Q(1))} + \| \Pi \|_{L^1_t (Q(1))} \leq \varepsilon,$$  \hfill (1.20)

where satisfying

$$\frac{2}{p} + \frac{1}{q} + \frac{1}{r} + \frac{1}{s} < 2, \quad \text{with} \quad q, r, s > \frac{4}{\frac{2}{p} + \frac{1}{q} + \frac{1}{r} + \frac{1}{s}},$$  \hfill (1.21)

then, $u \in L^\infty (Q(1/2))$.

This paper is organized as follows. In the second section, we localize the Sobolev-embedding theorem in anisotropic Lebesgue spaces. Then various inequalities in anisotropic Lebesgue spaces and decay estimates for the scaling invariant quantities are established for the proof of Theorem 1.1 and Theorem 1.5. To prove Theorem 1.5 we also require the pressure decomposition developed in [19]. In the third section, we follow the path of [16] to complete the proof of Theorem 1.1 and Theorem 1.4. In Section 4 in the spirit of iterative approach utilized in [17, 19] and the pressure decomposition, we prove Theorems 1.5.
2 Notations and some auxiliary lemmas

For \( p \in [1, \infty] \), the notation \( L^p((0, T); X) \) stands for the set of measurable functions on the interval \((0, T)\) with values in \( X \) and \( \|f(t, \cdot)\|_X \) belongs to \( L^p(0, T) \).

For simplicity, we write
\[
\|f\|_{L^p_t L^q_x(Q(\varrho))} := \|f\|_{L^p(-\varrho^2, 0; L^q(B(\varrho)))} \quad \text{and} \quad \|f\|_{L^p(Q(\varrho))} := \|f\|_{L^p(0, \varrho)},
\]
where \( Q(\varrho) = B(\varrho) \times (-\varrho^2, 0) \) and \( B(\varrho) \) denotes the ball of center 0 and radius \( \varrho \). A function \( f \) belongs to the anisotropic Lebesgue spaces \( L^q_1 L^2_2 L^\infty_3(\Omega) \) if
\[
\|f\|_{L^q_1 L^2_2 L^\infty_3(\Omega)} = \left\| \left\| u \right\|_{L^q_1(\{x_1: x_1 \in \Omega\})} \left\| \right\|_{L^2_2(\{x_2: x_2 \in \Omega\})} \left\| \right\|_{L^\infty_3(\{x_3: x_3 \in \Omega\})} < \infty.
\]
Denote the average of \( f \) on the ball \( B(\varrho) \) by \( \overline{f}_\varrho \). To consider the function in anisotropic Lebesgue spaces and apply Poincaré-Wirtinger inequality in one-dimensional space, we set
\[
\overline{u}_\varrho = \frac{1}{2\varrho} \int_{|x_1|<\varrho} u dx_1.
\]
Moreover, for the convenience of the reader, we state a fact which will be frequently used below
\[
\Omega_1 \subseteq \Omega_2 \subseteq \Omega_3. \tag{2.1}
\]
where
\[
\Omega_1 = \{ x : |x| < 1 \}, \Omega_2 = \{ x : |x_1|, |x_2|, |x_3| < 1 \}, \Omega_3 = \{ x : |x| < \sqrt{3} \}.
\]
Similar fact also was used in the proof of (1.10) in [8]. The classical Sobolev space \( W^{k,2}(\Omega) \) is equipped with the norm \( \|f\|_{W^{k,2}(\Omega)} = \sum_{\alpha=0}^k \|D^\alpha f\|_{L^2(\Omega)} \). Let \( W_0^{k,2}(\Omega) \) be the completion of \( C_0^\infty(\Omega) \) in the norm of \( W^{k,2}(\Omega) \). We denote by \( \tilde{H}^s \) homogeneous Sobolev spaces with the norm \( \|f\|_{\tilde{H}^s}^2 = \int_{\mathbb{R}^3} |\xi|^{2s} |\hat{f}(\xi)|^2 d\xi \). We will also use the summation convention on repeated indices. \( C \) is an absolute constant which may be different from line to line unless otherwise stated.

Now, for the convenience of readers, we recall the definition of suitable weak solution to the Navier-Stokes system (1.1).

**Definition 2.1.** A pair \((u, \Pi)\) is called a suitable weak solution to the Navier-Stokes equations (1.1) provided the following conditions are satisfied,
\[
\begin{align*}
(1) & \quad u \in L^\infty(-T, 0; L^2(\mathbb{R}^3)) \cap L^2(-T, 0; \tilde{H}^1(\mathbb{R}^3)), \Pi \in L^{3/2}(-T, 0; L^{3/2}(\mathbb{R}^3)); \\
(2) & \quad (u, \Pi) \text{ solves (1.7) in } \mathbb{R}^3 \times (-T, 0) \text{ in the sense of distributions;}
\end{align*}
\]
\[
(3) \quad (u, \Pi) \text{ satisfies the following inequality, for a.e. } t \in [-T, 0],
\]
\[
\int_{\mathbb{R}^3} |u(x, t)|^2 \phi(x, t) dx + 2 \int_{-T}^t \int_{\mathbb{R}^3} |\nabla u|^2 \phi dx ds \\
\leq \int_{-T}^t \int_{\mathbb{R}^3} |u|^2 (\partial_s \phi + \Delta \phi) dx ds + \int_{-T}^t \int_{\mathbb{R}^3} u \cdot \nabla \phi (|u|^2 + 2\Pi) dx ds, \tag{2.2}
\]
where non-negative function \( \phi(x, s) \in C_0^\infty(\mathbb{R}^3 \times (-T, 0)) \).
We recall the Sobolev embedding theorem in anisotropic Lebesgue space in the full three-dimensional space. We refer to [13] for the proof of the following result.

**Lemma 2.1.** Let $q, r, s \in (2, \infty]$ and $1/q + 1/r + 1/s \leq 1$. Then there exists a constant $C$ such that

$$
\|f\|_{L^q((\mathbb{R}^3)^3)} \leq C\|\partial_1 f\|_{L^2(\mathbb{R}^3)}\|\partial_2 f\|_{L^2(\mathbb{R}^3)}\|\partial_3 f\|_{L^2(\mathbb{R}^3)}\|f\|_{L^2(\mathbb{R}^3)}^{1-(1/q+1/r+1/s)}
$$

$$
\leq C\|\nabla f\|_{L^q(\mathbb{R}^3)}\|f\|_{L^2(\mathbb{R}^3)}^{1-(1/q+1/r+1/s)}. \quad (2.3)
$$

In what follows, for the sake of simplicity of presentation, we define $\tau = 1/q + 1/r + 1/s$, where $q, r, s \in (2, \infty]$. We can state the local version of the above lemma.

**Lemma 2.2.** Let $q, r, s \in (2, \infty]$ and $\tau \geq 1$. Then, for $\varrho > 0$ and $0 < \xi < \eta$, there exists a constant $C$ such that

$$
\|f\|_{L^q(B(\varrho))} \leq C\|\nabla f\|_{L^q(B(\sqrt{\varrho}))}\|f\|_{L^q(B(\sqrt{\varrho}))}^{1-\tau} + C\varrho^{-\tau}\|f\|_{L^q(B(\sqrt{\varrho}))}, \quad (2.4)
$$

$$
\|f\|_{L^q(B(\eta))} \leq C\|\nabla f\|_{L^q(B(\eta))}\|f\|_{L^q(B(\eta))}^{1-\tau} + C(\eta - \xi)^{-\tau}\|f\|_{L^q(B(\eta))}. \quad (2.5)
$$

**Proof.** Let $\phi(x)$ be non-negative smooth function supported in $B(\sqrt{\varrho})$ such that $\phi(x) \equiv 1$ on $B(\varrho)$, $0 \leq \phi(x) \leq 1$ and $|\nabla \phi| \leq C/\varrho$.

Making use of (2.1), the Hölder inequality and (2.3), we see that

$$
\|f\|_{L^q(B(\varrho))} \leq \|f\phi\|_{L^q((\mathbb{R}^3)^3)} \leq C\|\nabla (f\phi)\|_{L^2(\mathbb{R}^3)}\|\nabla f\phi\|_{L^2(\mathbb{R}^3)}^{1-\tau}
$$

$$
\leq C\left(\|\nabla f\|_{L^2(\mathbb{R}^3)} + \|f\|_{L^q(\mathbb{R}^3)}\right)^\tau\|\nabla f\|_{L^2(\mathbb{R}^3)}
$$

$$
\leq C\|\nabla f\|_{L^q(B(\sqrt{\varrho}))}\|f\|_{L^q(B(\sqrt{\varrho}))}^{1-\tau} + C\varrho^{-\tau}\|f\|_{L^q(B(\sqrt{\varrho}))},
$$

which means (2.4).

Along the exact same lines as the above proof, we have (2.5). This achieves the proof of the desired estimate. □

Before we present the decay type lemmas, by the natural scaling property of Navier-Stoke equations (1.1), we introduce the following dimensionless quantities,

$$
E_*(\varrho) = \frac{1}{\varrho} \int_{Q(\varrho)} |\nabla u|^2 dx dt,
$$

$$
E(\varrho) = \sup_{-\varrho^2 \leq t < 0} \frac{1}{\varrho} \int_{B(\varrho)} |u|^2 dx,
$$

$$
E_p(\varrho) = \frac{1}{r^{ex-p}} \int_{Q(r)} |u|^p dx dt,
$$

$$
P_{3/2}(\varrho) = \frac{1}{r^2} \int_{Q(r)} |\Pi - \Pi_{B(\varrho)}|^3 \frac{3}{2} dx dt.
$$

According to the Hölder inequality, it suffices to prove Theorem 1.1 for the case $2/p + \tau = 2$. Therefore, we introduce the dimensionless quantities below

$$
E_{p, q, r, s, t}(u, \varrho) = r^{-1} \|u - \bar{u}_t\|_{L^p(-\varrho^2, 0; L^q_L^s L^s_3(B(\varrho)))},
$$

$$
G_{p, q, r, s, t}(u, \varrho) = r^{-1} \|\nabla u\|_{L^p(-\varrho^2, 0; L^q_L^s L^s_3(B(\varrho)))}.
$$
Lemma 2.3. For $0 < \sqrt{6}\mu \leq \rho$, there is an absolute constant $C$ independent of $\mu$ and $\rho$, such that

$$E_3(\mu) \leq C \left( \frac{\rho}{\mu} \right)^2 E_{p,q,r}(\rho) \left( E^{1-\frac{\rho}{\mu}}(\rho)E^{\frac{1}{\mu}}(\rho) + E(\rho) \right) + C \left( \frac{\mu}{\rho} \right) E_3(\rho), \quad (2.6)$$

$$E_3(\mu) \leq C \left( \frac{\rho}{\mu} \right)^2 G_{p,1,r}(\sqrt{3}\rho) \left( E^{1-\frac{1}{\sqrt{3}}}\left( -\frac{\rho}{\sqrt{3}} \right)E^{\frac{1}{3\sqrt{3}}}+\frac{1}{\sqrt{3}}(\rho) + E(\rho) \right) + C \left( \frac{\mu}{\rho} \right) E_3(\rho). \quad (2.7)$$

Proof. In view of the Hölder inequality and (2.4) in Lemma 2.1, we see that, for $q, r, s > 2$,

$$\int_{B(\rho)} |u|^3 \, dx \leq \|u\|_{L^1_1L^2_2L^3_3(B(\rho))} \|u\|_{L^2(B(\rho))} \leq C \|u\|_{L^1_1L^2_2L^3_3(B(\rho))} \left( \|\nabla u\|_{L^2(B(\sqrt{\rho}\theta))} \|u\|_{L^2(B(\sqrt{\rho}\theta))} + \|\nabla u\|_{L^2(B(\sqrt{\rho}\theta))} \|u\|_{L^2(B(\sqrt{\rho}\theta))} \right) \leq C \|u\|_{L^1_1L^2_2L^3_3(B(\rho))} \left( \|\nabla u\|_{L^2(B(\sqrt{\rho}\theta))} \|u\|_{L^2(B(\sqrt{\rho}\theta))} + \|\nabla u\|_{L^2(B(\sqrt{\rho}\theta))} \|u\|_{L^2(B(\sqrt{\rho}\theta))} \right).$$

Integrating with respect to the time, Hölder’s inequality, and $2/p + \tau = 2$, we infer that

$$\int_{Q(\rho)} |u|^3 \, dx \leq C \|u\|_{L^2(B(\rho))} \left( \|\nabla u\|_{L^2(B(\sqrt{\rho}\theta))} \|u\|_{L^2(B(\sqrt{\rho}\theta))} + \|\nabla u\|_{L^2(B(\sqrt{\rho}\theta))} \|u\|_{L^2(B(\sqrt{\rho}\theta))} \right). \quad (2.8)$$

In the light of (2.1), we deduce that

$$\|u\|_{L^2(B(\rho))} \leq \|u\|_{L^2(|x_1|,|x_2|,|x_3|<\theta)} = \|u\|_{L^2_1L^2_2L^2_3(|x_1|,|x_2|,|x_3|<\theta)}.$$

Replacing $u$ by $u - \bar{u}_\theta^1$ and by the last inequality, we know that

$$\|u - \bar{u}_\theta^1\|_{L^2(B(\rho))} \leq \|u - \bar{u}_\theta^1\|_{L^2(|x_1|<\theta)} \|\nabla u\|_{L^2(|x_2|<\theta)} \|u\|_{L^2(|x_3|<\theta)} \leq \|u\|_{L^2(|x_1|,|x_2|,|x_3|<\theta)} \leq \|u\|_{L^2(B(\sqrt{\rho}\theta))}, \quad (2.9)$$

where we have used (2.1).

Inequalities (2.8) and (2.9) entail that

$$\int_{Q(\rho)} |u - \bar{u}_\theta^1| \sqrt{\rho} \, dx \leq C \int_{Q(\rho)} \|u - \bar{u}_\theta^1\|^2 \sqrt{\rho} \, dx \leq C \int_{Q(\rho)} \|\nabla u\|^2 \sqrt{\rho} \, dx + \|u - \bar{u}_\theta^1\|^2 \sqrt{\rho}$$

By virtue of the triangle inequality, we have

$$\int_{B(\mu)} |u|^3 \, dx \leq C \int_{B(\mu)} |u - \bar{u}_\rho/\sqrt{\rho}|^3 \, dx + C \int_{B(\mu)} |\bar{u}_\rho/\sqrt{\rho}|^3 \, dx$$. 

8
\begin{align*}
\leq & C \int_{B(\rho/\sqrt{3})} |u - \bar{u}_\rho| |\bar{u}_\rho|^{3/4} dx + C \frac{\mu^3}{\rho^3} \left( \int_{B(\rho)} |u|^3 dx \right) \\
\leq & C \int_{B(\rho/\sqrt{3})} |u - \bar{u}_\rho/\sqrt{3}|^3 dx + C \frac{\mu^3}{\rho^3} \left( \int_{B(\rho)} |u|^3 dx \right).
\end{align*}

Inserting (2.10) into the latter inequality, we arrive at
\begin{equation}
\int_{Q(\rho)} |u|^3 dx dt \leq C \|u - \bar{u}_\rho\|_{L^1_t L^4_x L^3(Q(\rho))} \left( \|\nabla u\|_{L^2(Q(\rho))}^{2/3} + \|u\|_{L^4(Q(\rho))} \right) + C \frac{\mu^3}{\rho^3} \left( \int_{Q(\rho)} |u|^3 dx dt \right),
\end{equation}
which means (2.11).

Taking advantage of the Hölder inequality and the Poincaré-Wirtingers inequality on $I = (-\rho, \rho)$, we see that
\begin{equation}
\|u - \bar{u}_\rho\|_{L^p_t L^q_x L^3(B(\rho))} \leq C \rho^{1/q} \|u - \bar{u}_\rho\|_{L^p_t L^\infty_x L^\infty_I((x_1, x_2, x_3) < \rho)} \\
\leq C \rho^{1/q} \|
abla u\|_{L^p_t L^2_x L^2_I((x_1, x_2, x_3) < \rho)} \\
\leq C \rho^{1/q} \|
abla u\|_{L^p_t L^2_x L^2_I(B(\sqrt{3} \rho))}.
\end{equation}

This together with (2.11) yields (2.12).

**Lemma 2.4.** For $0 < 4\sqrt{6} \mu \leq \rho$, there exists an absolute constant $C$ independent of $\mu$ and $\rho$ such that
\begin{align}
P_{3/2}(\mu) & \leq C \left( \frac{\mu}{\rho} \right)^2 E_{p,q,r,s,t}(\rho) \left( E^{1 - \frac{3}{4}}(\rho) E^{\frac{3}{4}}(\rho) + E(\rho) \right) + C \left( \frac{\mu}{\rho} \right)^{\frac{3}{2}} P_{3/2}(\rho), \tag{2.13} \\
P_{3/2}(\mu) & \leq C \left( \frac{\mu}{\rho} \right)^2 G_{p,1,r,s,t}(\sqrt{3} \rho) \left( E^{1 - \frac{3}{4}}(\rho) E^{\frac{3}{4}}(\rho) + E(\rho) \right) + C \left( \frac{\mu}{\rho} \right)^{\frac{3}{2}} P_{3/2}(\rho). \tag{2.14}
\end{align}

**Proof.** We consider the usual cut-off function $\phi \in C_0^\infty(B(\frac{\rho}{\sqrt{6}}))$ such that $\phi \equiv 1$ on $B(\frac{3\rho}{4\sqrt{6}})$ with $0 \leq \phi \leq 1$, $|\nabla \phi| \leq C \rho^{-1}$ and $|\nabla^2 \phi| \leq C \rho^{-2}$.

Because the incompressible condition, the pressure equation can be written
\[ \partial_t \partial_i (\Pi \phi) = - \phi \partial_t \partial_j U_{i,j} + 2 \partial_i \phi \partial_i \Pi + \Pi \partial_i \partial_i \phi, \]
where $U_{i,j} = (u_j - \bar{u}_{\rho,\sqrt{3}})(u_i - \bar{u}_{\rho,\sqrt{3}})$.

from which it follows that, for $x \in B(\frac{3\rho}{4\sqrt{6}})$
\begin{equation}
\Pi(x) = \Phi \ast \left\{ - \phi \partial_t \partial_j U_{i,j} + 2 \partial_i \phi \partial_i \Pi + \partial_i \partial_i \phi \right\} \\
= - \partial_t \partial_j \Phi \ast (\phi U_{i,j}) \\
+ 2 \partial_i \Phi \ast (\partial_j \phi U_{i,j}) - \Phi \ast (\partial_j \partial_j \phi U_{i,j}) \\
- 2 \partial_i \Phi \ast (\partial_i \phi \Pi) - \Phi \ast (\partial_i \partial_i \phi \Pi) \\
:= P_1(x) + P_2(x) + P_3(x),
\end{equation}
where Φ stands for the standard normalized fundamental solution of Laplace equation in \( \mathbb{R}^3 \).

Since \( \phi(x) = 1 \), where \( x \in B(\mu) \) \( (0 < \mu \leq \frac{\rho}{2\sqrt{6}}) \), we have

\[
\Delta (P_2(x) + P_3(x)) = 0.
\]

According to the interior estimate of harmonic function and the H"older inequality, we thus have, for every \( x_0 \in B(\frac{\rho}{4\sqrt{6}}) \),

\[
|\nabla (P_2 + P_3)(x_0)| \leq \frac{C}{\rho^4} \|(P_2 + P_3)\|_{L^1(B_{x_0}(\frac{\rho}{4\sqrt{6}}))}
\]

\[
\leq \frac{C}{\rho^4} \|(P_2 + P_3)\|_{L^1(B(\frac{\rho}{2\sqrt{6}}))}
\]

\[
\leq \frac{C}{\rho^3} \rho^{3(1 - \frac{q}{2})} \|(P_2 + P_3)\|_{L^q(B(\frac{\rho}{2\sqrt{6}}))}.
\]

We infer from (2.16) that

\[
\|\nabla p_2\|_{L^q(B(\frac{\rho}{4\sqrt{6}}))} \leq C \rho^{-(n+q)} \|p_2\|_{L^q(B(\frac{\rho}{2\sqrt{6}}))}.
\]

(2.17)

Using the mean value theorem and (2.17), for any \( \mu \leq \frac{\rho}{4\sqrt{6}} \), we arrive at

\[
\|(P_2 + P_3) - (P_2 + P_3)\|_{L^{3/2}(B(\mu))} \leq C \mu^3 \|\nabla (P_2 + P_3)\|_{L^{\infty}(B(\mu))}
\]

\[
\leq C \mu^3 (2\mu)^q \|\nabla (P_2 + P_3)\|_{L^{\infty}(B(\frac{\rho}{2\sqrt{6}}))}
\]

\[
\leq C \left( \frac{\mu}{\rho} \right)^q \|(P_2 + P_3)\|_{L^{3/2}(B(\frac{\rho}{2\sqrt{6}}))}.
\]

(2.18)

By time integration, we get

\[
\|(P_2 + P_3) - (P_2 + P_3)\|_{L^{3/2}(Q(\mu))} \leq C \left( \frac{\mu}{\rho} \right)^q \|(P_2 + P_3)\|_{L^{3/2}(Q(\frac{\rho}{2\sqrt{6}}))}.
\]

As \( (P_2 + P_3) - (P_2 + P_3)_{B(\frac{\rho}{2\sqrt{6}})} \) is also a Harmonic function on \( B(\frac{\rho}{2\sqrt{6}}) \), we deduce that

\[
\|(P_2 + P_3) - (P_2 + P_3)_{B(\frac{\rho}{2\sqrt{6}})}\|_{L^{3/2}(Q(\mu))}
\]

\[
\leq C \left( \frac{\mu}{\rho} \right)^q \|(P_2 + P_3) - (P_2 + P_3)_{B(\frac{\rho}{2\sqrt{6}})}\|_{L^{3/2}(Q(\frac{\rho}{2\sqrt{6}}))}.
\]

(2.18)

The triangle inequality guarantees that

\[
\|(P_2 + P_3) - (P_2 + P_3)_{B(\frac{\rho}{2\sqrt{6}})}\|_{L^{3/2}(Q(\frac{\rho}{2\sqrt{6}}))}
\]

\[
\leq \|P - \overline{P}_{(\frac{\rho}{2\sqrt{6}})}\|_{L^{3/2}(Q(\frac{\rho}{2\sqrt{6}}))} + \|P_1 - \overline{P}_1(\frac{\rho}{2\sqrt{6}})\|_{L^{3/2}(Q(\frac{\rho}{2\sqrt{6}}))}
\]

\[
\leq \|P - \overline{P}_0\|_{L^{3/2}(Q(\frac{\rho}{2\sqrt{6}}))} + \|P_1\|_{L^{3/2}(Q(\frac{\rho}{2\sqrt{6}}))}.
\]
which leads to that
\[
\| (P_2 + P_3) - (P_2 + P_3)_\mu \|= \left( \frac{\mu}{\rho} \right)^{3/2} \left( \| \Pi - \Pi_\rho \|_{L^{3/2}(Q(\rho))} \right)
\]
\[
\leq C \left( \frac{\mu}{\rho} \right)^{3/2} \left( \| \Pi - \Pi_\rho \|_{L^{3/2}(Q(\rho))} + \| P_1 \|_{L^{3/2}(Q(\rho))} \right)
\]
\[
\leq C \left( \frac{\mu}{\rho} \right)^{3/2} \left( \| \Pi - \Pi_\rho \|_{L^{3/2}(Q(\rho))} + \| P_1 \|_{L^{3/2}(Q(\rho))} \right).
\]
(2.19)

By virtue of the Hölder inequality and the argument in (2.10), we get
\[
\int_{B(\rho/\sqrt{\rho})} |u - \bar{u}_\rho| dx \leq C \int_{B(\rho/\sqrt{\rho})} |u - \bar{u}_\rho| dx
\]
\[
\leq C \| u - \bar{u}_\rho \|_{L^{3/2}(B(\rho))} \left( \| \nabla u \|_{L^2(B(\rho))} \right)^{2\tau} + \rho^{-\tau} \| u \|_{L^2(B(\rho))}^{2\tau}.
\]
The classical Calderón-Zygmund Theorem and the latter inequality implies that
\[
\int_{B(\rho/\sqrt{\rho})} |P_1(x)| dx \leq C \int_{B(\rho/\sqrt{\rho})} |u - \bar{u}_\rho| dx
\]
\[
\leq C \| u - \bar{u}_\rho \|_{L^{3/2}(B(\rho))} \left( \| \nabla u \|_{L^2(B(\rho))} \right)^{2\tau} + \rho^{-\tau} \| u \|_{L^2(B(\rho))}^{2\tau},
\]
(2.20)
and
\[
\int_{B(\rho)} |P_1(x)| dx \leq C \| u - \bar{u}_\rho \|_{L^{3/2}(B(\rho))} \left( \| \nabla u \|_{L^2(B(\rho))} \right)^{2\tau} + \rho^{-\tau} \| u \|_{L^2(B(\rho))}^{2\tau}.
\]
(2.21)
The inequality (2.19)-(2.21) allows us to deduce that
\[
\int_{B(\rho)} |\Pi - \Pi_\rho| dx \leq C \int_{B(\rho)} |P_1 - (P_1)_\rho| dx + \left( \| P_2 + P_3 - (P_2 + P_3)_\rho \| \right)^2 dx
\]
\[
\leq C \| u - \bar{u}_\rho \|_{L^{3/2}(B(\rho))} \left( \| \nabla u \|_{L^2(B(\rho))} \right)^{2\tau} + \rho^{-\tau} \| u \|_{L^2(B(\rho))}^{2\tau} + C \left( \frac{\mu}{\rho} \right)^{3/2} \| \Pi - \Pi_\rho \|^{3/2} dx.
\]
(2.22)
We readily get
\[
\int_{Q(\rho)} |\Pi - \Pi_\rho| dx \leq C \left( \frac{\mu}{\rho} \right)^{3/2} \| u - \bar{u}_\rho \|_{L^{3/2}(B(\rho))} \left( \| \nabla u \|_{L^2(B(\rho))} \right)^{2\tau} + \rho^{-\tau} \| u \|_{L^2(B(\rho))}^{2\tau} + C \left( \frac{\mu}{\rho} \right)^{3/2} \| \Pi - \Pi_\rho \|^{3/2} dx,
\]
(2.23)
which leads to
\[
P_{3/2}(\mu) \leq C \left( \frac{\rho}{\mu} \right)^2 \left( E_{p,q,r,s,t}(\rho) \left( E_{1-\left( \frac{1}{2\tau} + \frac{1}{2\tau} + \frac{1}{2\tau} \right)} + E_{\left( \frac{1}{2\tau} + \frac{1}{2\tau} + \frac{1}{2\tau} \right)} + E(\rho) \right) + C \left( \frac{\mu}{\rho} \right)^{3/2} P_{3/2}(\rho) \right),
\]
(2.24)
We derive from (2.24) and (2.12) that (2.14). The proof of this lemma is completed. \( \square \)
Moreover, there holds

where

Under the hypotheses of Theorem 1.5, we write

\[ \alpha = \frac{2}{p + \frac{1}{q} + \frac{1}{r} + \frac{1}{s}} > 1. \]  

With (2.5) at our disposal, we have

**Lemma 2.6.** Let \( \alpha \) given in (2.29). For \( 0 < \xi < \eta \), there is an absolute constant \( C \) such that

\[ \|u\|_{L^3(\mathbb{R}^3)}^3 \leq C \rho^3(\alpha - 1)/2 \|u\|_{L^q_1 L^{r_1}_2 L^{s_1}_3(Q(\eta))} \left[ \left( 1 + \frac{\eta^{\alpha \tau}}{(\eta - \xi)^{\alpha \tau}} \right) \|u\|_{L^{2, \infty}(Q(\eta))}^3 + \|\nabla u\|_{L^2(Q(\eta))}^3 \right]. \]  

**Proof.** Combining the Hölder inequality and , we conclude

\[
\begin{align*}
\int_{B(\frac{\xi}{4})} |u|^3 dx dt &= \int_{B(\frac{\xi}{4})} \rho^3 \alpha |u|^3 dx dt \\
&\leq \|u\|_{L^3(B(\frac{\xi}{4}))}^3 \left( \int_{B(\frac{\xi}{4})} \alpha \frac{|u|^3 dx dt}{\rho^3} \right) \\
&\leq \rho^3(\alpha - 1)/2 \|u\|_{L^q_1 L^{r_1}_2 L^{s_1}_3(B(\xi))} \left( \|\nabla u\|_{L^2(B(\eta))}^3 \right) + \|u\|_{L^2(B(\eta))}^3 \left( \int_{B(\frac{\xi}{4})} \alpha \frac{|u|^3 dx dt}{\rho^3} \right).
\end{align*}
\]
Integrating between \(-\left(\frac{5 + 3n}{4}\right)^2\) and 0 yields
\[
\int_Q \left|u^3\right| dx dt 
\leq C \rho^{3(\alpha-1)/2} \left\|u^{\alpha} \right\|_{L^1_t L^2_x L^2_y(Q(\rho))} \left(\left\|\nabla u\right\|_{L^\infty_x L^2_y(Q(\rho))} \left\|u\right\|_{L^\infty_x L^2_y(B(\rho))}^{1-\alpha} \right) + C \left(\frac{\rho^{\alpha}}{(\eta - \xi)^{\alpha}} \left\|u\right\|_{L^\infty_x L^2_y(B(\rho))} \left\|u\right\|_{L^2_x(B(\eta))}^{2-\alpha}\right)
\leq \rho^{3(\alpha-1)/2} \left\|u^{\alpha} \right\|_{L^1_t L^2_x L^2_y(Q(\rho))} \left(1 + \left\|\nabla u\right\|_{L^\infty_x L^2_y(Q(\rho))}^{3-\alpha} \right) + \left\|\nabla u\right\|_{L^2_x(Q(\rho))}^{3-\alpha},
\]
where the Young inequality was used.

The proof of this lemma is completed.

\[\square\]

3 Regularity criteria in anisotropic Lebesgue space at infinitely many scales

Inspired by [16], we present the proof of Theorem 1.1 by Lemma 2.3 and Lemma 2.4. Then combining the method in the proof of (2.7) and the known \(\varepsilon\)-regularity criteria (1.5), we finish the proof of Theorem 1.1.

Proof of Theorem 1.1. (1) From (1.11), we know that there is a constant \(\varrho_0\) such that, for any \(\varepsilon \leq \varrho_0\),
\[
\varrho^{1-\frac{3}{p} - \frac{1}{q} - \frac{1}{r} - \frac{1}{s} - \frac{1}{t}} \left\|u - u_0\right\|_{L^1_t L^2_x L^2_y(Q(\varrho))} \leq \varepsilon_1.
\]

By the Young inequality and local energy inequality (2.2), we have
\[
E(\rho) + E_\varepsilon(\rho) \leq C \left[E_{3/3}^2(2\rho) + E_{3}(2\rho) + P_{3/2}(2\rho)\right] 
\leq C \left[1 + E_{3}(2\rho) + P_{3/2}(2\rho)\right]. \tag{3.1}
\]

From (2.6) in Lemma 2.3, we see that, for \(2\sqrt{6}\mu \leq \rho\),
\[
E_{3}(\mu) \leq C \left(\frac{\rho}{\mu}\right)^2 E_{p,q,r,s,t}(\rho/2) \left(E_{3}^{1-\frac{1}{2}}(\rho/2) + E_{3}(\rho/2)\right) + C \left(\frac{\mu}{\rho}\right) E_{3}(\rho/2) 
\leq C \left(\frac{\rho}{\mu}\right)^2 E_{p,q,r,s,t}(\rho/2) \left(1 + E_{3}(\rho) + P_{3/2}(\rho)\right) + C \left(\frac{\mu}{\rho}\right) E_{3}(\rho/2) \tag{3.2}
\leq C \left(\frac{\rho}{\mu}\right)^2 E_{p,q,r,s,t}(\rho) \left(1 + E_{3}(\rho) + P_{3/2}(\rho)\right) + C \left(\frac{\mu}{\rho}\right) E_{3}(\rho).
\]

It follows from (2.13) in Lemma 2.4 that, for \(8\sqrt{6}\mu \leq \rho\),
\[
P_{3/2}(\mu) \leq C \left(\frac{\rho}{\mu}\right)^2 E_{p,q,r,s,t}(\rho) \left(1 + E_{3}(\rho) + P_{3/2}(\rho)\right) + C \left(\frac{\mu}{\rho}\right)^{5/2} P_{3/2}(\rho). \tag{3.3}
\]

Before going further, we set
\[
F(\mu) = E_{3}(\mu) + P_{3/2}(\mu).
\]

13
With the help of (3.2) and (3.3), we conclude that
\[
F(\mu) \leq C \left( \frac{\rho}{\mu} \right)^2 E_{p,q,r,s,t}(\rho) F(\rho) + \left( \frac{\rho}{\mu} \right)^2 E_{p,q,r,s,t}(\rho) + C \left( \frac{\mu}{\rho} \right) F(\rho)
\]
\[
\leq C_1 \lambda^{-2} \varepsilon_1 F(\rho) + C_2 \lambda^{-2} \varepsilon_1 + C_3 \lambda F(\rho),
\]
where \( \alpha > 0 \), \( \lambda = \frac{\mu}{\rho} \leq \frac{1}{4\sqrt{6}} \) and \( \rho \leq \rho_0 \).

Choosing \( \lambda, \varepsilon_1 \) such that \( q = 2C_3 \lambda < 1 \) and \( \varepsilon_1 = \min \left\{ \frac{q \lambda^2}{2C_1}, \frac{(1-q) \lambda^2 \varepsilon}{2C_2 \lambda^2} \right\} \), we see that
\[
F(\lambda \rho) \leq q F(\rho) + C_2 \lambda^{-2} \varepsilon_1.
\] (3.4)

We iterate (3.4) to get
\[
F(\lambda^k \rho) \leq q^k F(\rho) + 1 \frac{\lambda^2 \varepsilon}{2}.
\]

According to the definition of \( F(r) \), for a fixed \( \rho_0 > 0 \), we know that there exists a positive number \( K_0 \) such that
\[
q^{K_0} F(\rho_0) \leq 2 \frac{M(\|u\|_{L^\infty L^2}, \|u\|_{L^2 W^{1,2}} \|\Pi\|_{L^{3/2} L^{3/2}})}{\rho_0^2} q^{K_0} \leq 1 \frac{\varepsilon \lambda^2}{2}.
\]

We denote \( \rho_1 := \lambda^{K_0} \rho_0 \). Then, for all \( 0 < \rho \leq \rho_1 \), \( \exists k \geq K_0 \), such that \( \lambda^{k+1} \rho_0 \leq \rho \leq \lambda^{k} \rho_0 \), there holds
\[
E_3(\rho) + P_{3/2}(\rho) = \frac{1}{\rho^2} \int_{Q(\rho)} |u|^3 dx dt + \frac{1}{\rho^2} \int_{Q(\rho)} |\Pi - \Pi_{\rho}|^3 dx dt
\]
\[
\leq \frac{1}{(\lambda^{k+1} \rho_0)^2} \int_{Q(\lambda^{k+1} \rho_0)} |u|^3 dx dt + \frac{1}{(\lambda^{k+1} \rho_0)^2} \int_{Q(\lambda^{k+1} \rho_0)} |\Pi - \Pi_{\lambda^{k} \rho_0}|^3 dx dt
\]
\[
\leq \frac{1}{\lambda^2} F(\lambda^{k} \rho_0)
\]
\[
\leq \frac{1}{\lambda^2} (q^{k-K_0} q^{K_0} F(\rho_0) + 1 \frac{\lambda^2 \varepsilon}{2})
\]
\[
\leq \varepsilon.
\]

This together with (1.2) completes the proof of first part of Theorem 1.1.

(2). With (2.7) and (2.14) in hand, by an argument completely analogous to that adopted in the proof of Theorem 1.1, we can complete the second part of the proof of Theorem 1.1.

We now turn to the proof of Theorem 1.4.

Proof of Theorem 1.4. By the Hölder inequality and the Poincaré-Wirtingers inequality on
By (1.5), we complete the proof of Theorem 1.4.

Therefore,
\[ \frac{1}{q} - \frac{2}{p} - \frac{3}{q} \| u - \bar{u}_0 \|_{L^q_tL^3_x(Q(\varrho))} \leq \frac{2}{q} - \frac{2}{p} - \frac{3}{q} \| \nabla_1 u \|_{L^q_tL^3_x(Q(\sqrt{\varrho})))}. \]

By (1.5), we complete the proof of Theorem 1.4.

4 Regularity criteria in anisotropic Lebesgue space at one scale

In this final section, we show Theorem 1.5. This may be easily proved by taking advantage of (1.19) and of the following proposition. The proof of this proposition is based on ideas used in [19].

**Proposition 4.1.** Let \( \alpha \) be given in (2.29). Suppose that \((u, \Pi)\) is a suitable weak solution to the Navier-Stokes equations in \( Q(R) \). Then there holds, for any \( R > 0 \)

\[
\| u \|_{L^\infty L^2(Q(R/2))}^2 + \| \nabla u \|_{L^2(Q(R/2))}^2 \leq \frac{C}{R^{(3-3\alpha)/\alpha}} \| u \|_{L^q_tL^3_xL^3_z(Q(R))}^2 + \frac{C}{R^{(5-3\alpha)/(\alpha-1)}} \| u \|_{L^q_tL^3_xL^3_z(Q(R))}^{2\alpha/(\alpha-1)} + \frac{C}{R^6} \| \Pi \|_{L^1(Q(R))}^2. \tag{4.1}
\]

**Proof.** Consider \( 0 < R/2 \leq \xi < \frac{3\xi + \eta}{4} < \frac{\xi + \eta}{2} < \frac{\xi + 3\eta}{4} < \rho \leq R \). Let \( \phi(x, t) \) be non-negative smooth function supported in \( Q(\frac{\xi + \eta}{2}) \) such that \( \phi(x, t) \equiv 1 \) on \( Q(\frac{3\xi + \eta}{4}) \), \( |\nabla \phi| \leq C/(\eta - \xi) \) and \( |\nabla^2 \phi| + |\partial_t \phi| \leq C/(\eta - \xi)^2 \).

The local energy inequality (2.2) and the decomposition of pressure in Lemma 2.5 ensure that

\[
\int_{B(\frac{3\xi + \eta}{4})} |u(x, t)|^2 \phi(x, t) dx + 2 \int_{Q(\frac{5\xi + \eta}{4})} |\nabla u|^2 \phi dx ds \leq I + II + III + IV + V, \tag{4.2}
\]
where

\[ I = \frac{C}{(\eta - \xi)^2} \int_{Q(2\pi \delta)} |u|^2 dx ds; \]
\[ II = \frac{C}{(\eta - \xi)} \int_{Q(2\pi \delta)} |u|^3 dx ds; \]
\[ III = \frac{C}{(\eta - \xi)} \int_{Q(2\pi \delta)} u \Pi_1 dx ds; \]
\[ IV = \frac{C}{(\eta - \xi)} \int_{Q(2\pi \delta)} u \Pi_2 dx ds; \]
\[ V = \frac{C}{(\eta - \xi)} \int_{Q(2\pi \delta)} u \Pi_3 dx ds. \]

The Hölder inequality and (2.26)–(2.28) entail

\[ I \leq \frac{C J^{5/3}}{(\eta - \xi)\eta^{2/3}} \left( \int_{Q(2\pi \delta)} |u|^3 dx ds \right)^{2/3} \] (4.3)
\[ III \leq \frac{C}{(\eta - \xi)} \|\Pi_1\|_{L^3(Q(2\pi \delta))} \|u\|_{L^3(Q(2\pi \delta))} \leq \frac{C}{(\eta - \xi)} \|u\|_{L^3(Q(2\pi \delta))}^3, \] (4.4)
\[ IV \leq \frac{C}{(\eta - \xi)} \|\Pi_2\|_{L^3(Q(2\pi \delta))} \|u\|_{L^3(Q(2\pi \delta))} \leq \frac{C J^{3/4}}{(\eta - \xi)\eta^{3/4}} \|u\|_{L^3(Q(2\pi \delta))}^3, \] (4.5)
\[ V \leq \frac{C}{(\eta - \xi)} \|\Pi_3\|_{L^2,Q(2\pi \delta)} \|u\|_{L^2,Q(2\pi \delta)} \leq \frac{C J^{3/2}}{(\eta - \xi)\eta^{3/2}} \|\Pi\|_{L^1(Q(2\pi \delta))} \|u\|_{L^2,Q(2\pi \delta)}^3. \] (4.6)

After inserting (2.30) into (4.3)–(4.5), we conclude that, by the Young inequality,

\[ I \leq \frac{C J^{3+2/\alpha}}{(\eta - \xi)\eta^{3+2/\alpha}} \|u\|_{L^p,Q(2\pi \delta)}^2 + \frac{1}{5} \left( \|u\|_{L^2,Q(2\pi \delta)}^2 + \|\nabla u\|_{L^2,Q(2\pi \delta)}^2 \right), \]
\[ II + III \leq \frac{C J^{3}}{(\eta - \xi)^{3/2+\alpha}} \|u\|_{L^p,Q(2\pi \delta)}^{2\alpha/(\alpha-1)} + \frac{1}{5} \left( \|u\|_{L^2,Q(2\pi \delta)}^2 + \|\nabla u\|_{L^2,Q(2\pi \delta)}^2 \right), \]
\[ IV \leq \frac{C J^{3+2/\alpha}}{(\eta - \xi)^{8/\alpha}} \|u\|_{L^p,Q(2\pi \delta)}^2 + \frac{1}{5} \left( \|u\|_{L^2,Q(2\pi \delta)}^2 + \|\nabla u\|_{L^2,Q(2\pi \delta)}^2 \right). \]

By means of the Young inequality again, we get

\[ L_5 \leq \frac{C J^{3}}{(\eta - \xi)^8} \|\Pi\|_{L^1,Q(2\pi \delta)}^2 + \frac{1}{5} \|u\|_{L^2,Q(2\pi \delta)}^2. \]

All the above estimates allow us to obtain

\[ \|u\|_{L^2,Q(2\pi \delta)}^2 + \|\nabla u\|_{L^2,Q(2\pi \delta)}^2 \]
\[ \leq \frac{C J^{3+2/\alpha}}{(\eta - \xi)^{3/2+\alpha}} \|u\|_{L^p,L^2,L^2,L^2,Q(2\pi \delta)}^2 + \frac{C J^{3}}{(\eta - \xi)^{3/2+\alpha}} \|u\|_{L^p,L^2,L^2,L^2,Q(2\pi \delta)}^{2\alpha/(\alpha-1)} \]
\[ + \frac{C J^{3+2/\alpha}}{(\eta - \xi)^{8/\alpha}} \|u\|_{L^p,L^2,L^2,L^2,Q(2\pi \delta)}^{2\alpha/(\alpha-1)} + \frac{C J^{3}}{(\eta - \xi)^{8/\alpha}} \|\Pi\|_{L^1,Q(2\pi \delta)}^2 + \frac{4}{5} \left( \|u\|_{L^2,Q(2\pi \delta)}^2 + \|\nabla u\|_{L^2,Q(2\pi \delta)}^2 \right). \]

We derive from the last inequality and iteration Lemma 15, Lemma V.3.1, p.161 get (4.1). 

16
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