Exact solution of the Klein Gordon equation in the presence of a minimal length

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Abstract

We obtain exact solutions of the (1 + 1) dimensional Klein Gordon equation with linear vector and scalar potentials in the presence of a minimal length. Algebraic approach to the problem has also been studied.

Keywords: Klein Gordon; Minimal length

PACS: 03.65-w; 03.65.Ge; 03.65Pm

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1 Introduction

During the past few years, there have been growing interest in obtaining exact solutions of relativistic
wave equations. In particular exact solutions of the Klein Gordon equation with various vector and scalar
potentials have been obtained by a number of authors [1, 2]. In all these cases the models have been
studied within the context of point particles.

On the other hand the concept of a minimal length has emerged from studies on quantum gravity [3],
perturbative string theory [4], black holes [5] etc. In such a scenario the standard Heisenberg uncertainty
relation gets modified and this causes UV/IR mixing. As a consequence it is meaningful to study quantum
mechanics in the presence of a minimal length [6]. In particular exact as well as approximate solutions
of various quantum mechanical problems have been obtained in the presence of a minimal length [6, 7].

It may be noted that Klein Gordon equation which is solvable for different vector and scalar potentials
without minimal length may not be exactly solvable when considered in the presence of a minimal length.
Here we shall consider (1 + 1) dimensional Klein Gordon equation with (unequal) linear scalar and vector
potentials in the presence of a minimal length and it will be shown that the problem admits exact
analytical solutions. More precisely we shall treat the problem in two ways: first we obtain the solutions
by solving the Schrödinger like equation in momentum space and secondly we shall obtain the solutions
in a purely algebraic fashion by utilizing the shape invariance symmetry of the problem [8].

2 Klein Gordon equation in the presence of minimal length

In one dimensional quantum mechanics in the presence of a minimal length the canonical commutation
relation between position and momentum becomes deformed. Out of the various deformed commutation
relations we shall consider here the simplest one and it is given by [6]

$$[\hat{x}, \hat{p}] = i\hbar(1 + \beta p^2)$$

where $\beta \geq 0$ is a small parameter. A representation of $\hat{x}$ and $\hat{p}$ satisfying (1) are given by [6]

$$\hat{x} = i\hbar[(1 + \beta p^2)\frac{\partial}{\partial p} + \gamma p], \quad \hat{p} = p$$

Also as a consequence of (1) the Heisenberg uncertainty relation gets modified and the generalized
uncertainty relation reads

$$\Delta \hat{x}\Delta \hat{p} \geq \frac{\hbar}{2}[1 + \beta(\Delta \hat{p})^2]$$

From (3) it follows that there also exist a minimal length given by

$$(\Delta \hat{x})_{\text{min}} = \hbar \sqrt{\beta}$$
In the space where position ($\hat{x}$) and momentum ($\hat{p}$) are given by (2) the associated scalar product (which ensures hermiticity of $\hat{x}$ and $\hat{p}$) is defined by

$$\langle \phi(p)|\psi(p) \rangle = \int \frac{\phi^*(p)\psi(p)}{(1 + \beta p^2)^{1/2}} dp$$  \hspace{1cm} (5)

We note that the form of the (1+1) dimensional Klein Gordon equation in the presence of a minimum length is similar to the standard one \[1\] except that the position and momentum are now given by (2):

$$\left[ c^2 p^2 + (mc^2 + S(\hat{x}))^2 \right] \psi = [E - V(\hat{x})]^2 \psi$$  \hspace{1cm} (6)

We now choose the vector and the scalar potential to be of the form

$$V(\hat{x}) = \mu \hat{x}, \quad S(\hat{x}) = \lambda \hat{x}$$  \hspace{1cm} (7)

where we have taken $\lambda^2 > \mu^2$ so as to avoid complex eigenvalues. Now using the representation (2) the Klein Gordon equation (6) can be written in momentum space as

$$\left[ -f(p) \frac{d^2}{dp^2} + g(p) \frac{d}{dp} + h(p) \right] \psi = \epsilon \psi$$  \hspace{1cm} (8)

where the functions $f(p), g(p)$ and $h(p)$ are given by

$$f(p) = (1 + \beta p^2)^2, \quad g(p) = -2(1 + \beta p^2) \left[ p(\beta + \gamma) - \frac{ig(\mu E + mc^2\lambda)}{\hbar(\lambda^2 - \mu^2)} \right],$$
$$h(p) = -p^2 \left[ \gamma(\beta + \gamma) - \frac{c^2}{\hbar^2(\lambda^2 - \mu^2)} + \frac{2igp(\mu E + mc^2\lambda)}{\hbar(\lambda^2 - \mu^2)} \right]$$  \hspace{1cm} (9)

$$\epsilon = \gamma + \frac{(E^2 - m^2c^4)}{\hbar^2(\lambda^2 - \mu^2)}$$

Eq. (8) can be solved by performing a transformation involving a change of variable as well as wavefunction:

$$\psi(p) = \rho(p)\phi(p), \quad q = \int \frac{1}{\sqrt{f(p)}} dp$$  \hspace{1cm} (10)

where

$$\rho(p) = e^\int \chi(p) dp, \quad \chi(p) = \frac{f' + 2g}{4f}$$  \hspace{1cm} (11)

Under the above transformation Eq. (8) can be written in the form of a Schrödinger equation

$$\left[ -\frac{d^2}{dq^2} + V(q) \right] \phi = \epsilon \phi$$  \hspace{1cm} (12)

where the potential $V(q)$ and the energy $\epsilon$ are given by

$$V(q) = \frac{c^2}{\beta \hbar^2(\lambda^2 - \mu^2)} \sec^2(\sqrt{\beta}q), \quad -\frac{\pi}{2\sqrt{\beta}} < q < \frac{\pi}{2\sqrt{\beta}}$$
$$\epsilon = \frac{\gamma}{\beta} + \frac{c^2}{\hbar^2(\lambda^2 - \mu^2)} + \frac{\hbar^2(\lambda^2 - \mu^2)\beta}{\hbar^2(\lambda^2 - \mu^2)}$$  \hspace{1cm} (13)

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The potential appearing above is a standard solvable potential whose energy and eigenfunctions (apart from a normalization factor) are given by [9]

\[ e_n = (A + \sqrt{\beta} n)^2, \quad A = \frac{\sqrt{\beta} + \sqrt{\beta + \frac{4c^2}{\hbar^2(\lambda^2 - \mu^2)}}}{2}, \quad n = 0, 1, 2, \ldots \]  

(14)

\[ \phi_n(q) = (\cos \sqrt{\beta} q)^\frac{1}{A} P_n^{\left(\frac{\lambda^2}{\lambda^2 - \mu^2}, \frac{\lambda^2}{\lambda^2 - \mu^2} - 1\right)} (\sin \sqrt{\beta} q) \]  

(15)

Now using the relations [9] and [10] we finally obtain

\[ E_n = -\mu mc^2 \lambda + \frac{\hbar (\lambda^2 - \mu^2)}{\lambda} \sqrt{\beta(n^2 + n + \frac{1}{2}) + \beta(n + \frac{1}{2}) \sqrt{1 + \frac{4c^2}{\hbar^2 \beta^2(\lambda^2 - \mu^2)}}} \]  

(16)

\[ \psi_n(p) = e^{i\frac{(mc^2 E_n + \hbar p)}{\lambda \sqrt{\lambda^2 - \mu^2}} \tan^{-1}(\sqrt{\beta} p)} (1 + \beta p^2)^{-\frac{\lambda^2}{\lambda^2 - \mu^2}} P_n^{\left(\frac{\lambda^2}{\lambda^2 - \mu^2}, \frac{\lambda^2}{\lambda^2 - \mu^2} - 1\right)} (\sin \sqrt{\beta} p) \]  

(17)

where \( P_n^{(a,b)} \) denotes Jacobi polynomials. It can be shown that the eigenfunctions are orthogonal with respect to the scalar product [5]. We note that although (16) is an exact result, nevertheless it is sometimes useful to separate the \( \beta \) dependent contribution to the spectrum. So expanding (16) in powers of \( \beta \) we find

\[ E_n \approx -\mu mc^2 \lambda + \frac{\hbar (\lambda^2 - \mu^2)}{\lambda} \sqrt{2n + 1} + \frac{\hbar^3}{4c^2} (\lambda^2 - \mu^2) \frac{\sqrt{2n + 1}}{2n + 1} \beta + O(\beta^2) \]  

(18)

where the first two terms are the standard \( \beta \) independent contribution [1] while the rest depends on \( \beta \).

3 Algebraic approach

In the last section we obtained exact solutions of the Klein Gordon oscillator. Since the problem is exactly solvable it is natural to examine it’s underlying symmetry. Here we shall show that the problem has shape invariance symmetry and use the approach suggested in [8] to obtain exact solutions in a purely algebraic fashion. We recall that two Hamiltonians \( H_1(\lambda) \) and \( H_2(\lambda) \) are said to be shape invariant if [9]

\[ H_2(\lambda_1) = H_1(\lambda_2) + R(\lambda_1) \]  

(19)

where \( \lambda_1 \) is a set of parameters, \( \lambda_2 \) is a function of \( \lambda_1 \) and \( R \) is a function independent of \( p \). Then it can be shown that the spectrum is given by

\[ E_n = \sum_{i=1}^{n} R(\lambda_i) \]  

(20)

To utilize the shape invariance property we shall now try to factorize the following Hamiltonian in the form

\[ H = -f(p) \frac{d^2}{dp^2} + g(p) \frac{d}{dp} + h(p) + c_1 = CB \]  

(21)
where $c_1$ is a constant (to be determined later) and the operators $B$ and $C$ are not necessarily self-adjoint.

We consider these operators to be of the form

$$B = F(p) \frac{d}{dp} + W(p) + \Omega(p)$$
$$C = -F(p) \frac{d}{dp} + W(p) - \Omega(p)$$

From (22) it follows that

$$CB = -F^2(p) \frac{d^2}{dp^2} - F(p)[F'(p) + 2\Omega(p)] \frac{d}{dp} - F(p)[W'(p) + \Omega'(p)] + W^2(p) - \Omega^2(p)$$

Comparing (21) and (23) we get $F(p), g(p)$ and an equation for $W(p)$:

$$F(p) = (1 + \beta p^2)$$
$$\Omega(p) = \gamma p - \frac{i(\mu E + mc^2\lambda)}{\hbar(\lambda^2 - \mu^2)}$$

$$(1 + \beta p^2) \frac{dW}{dp} + W^2(p) = \frac{c^2 p^2}{\hbar^2(\lambda^2 - \mu^2)} + \gamma - \frac{(\mu E + mc^2\lambda)^2}{\hbar^2(\lambda^2 - \mu^2)^2} + c_1$$

To solve this equation we now consider an ansatz for $W(p)$:

$$W(p) = c_2 p$$

It can be shown that the Ricatti equation (25) is satisfied if

$$c_1 = \frac{(\mu E + mc^2)^2}{\hbar^2(\lambda^2 - \mu^2)^2} - c_2 - \gamma$$
$$c_2 = \frac{\beta + \sqrt{\beta^2 + \frac{4\lambda^2}{\hbar^2(\lambda^2 - \mu^2)^2}}}{2} p$$

Then after some calculations it can be shown that

$$B(\lambda_1)C(\lambda_1) = C(\lambda_2)B(\lambda_2) + R(\lambda_1)$$

where we have taken

$$\lambda_1 = c_2, \quad \lambda_i = c_2 + (i - 1)\beta, \quad R(\lambda_i) = 2\lambda_i + \beta$$

Now iterating (28) it follows that

$$\epsilon + c_1 = \sum_{i=0}^{n} R(\lambda_i)$$

Finally using (9) and (27) one obtains $E_n$ and they are the same as in (16).

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1 Note that one can prove shape invariance of Eq. (8) using shape invariance property of the potential (13) and the inverse transformation of (10).
4 Discussion

Here we have obtained exact solutions of the Klein Gordon equation with linear vector and scalar potentials. We feel it would be interesting to search for other vector and scalar potentials for which exact or approximate solutions of the Klein Gordon equation can be obtained. We would like to mention that apart from solving the Klein Gordon equation we have also exploited the shape invariance symmetry of the problem to obtain the spectrum. We feel it would be interesting to investigate other symmetries e.g., Lie algebraic symmetry of this class of problems whenever the transformation (10) is invertible. It may also be noted here that we have obtained the trigonometric potential (13) as a consequence of the relation (1) and the choice of $S(\hat{x}), V(\hat{x})$. However this is not the only deformation of the canonical commutation relation involving $\hat{x}$ and $\hat{p}$. It may be interesting to search for some other choice of $[\hat{x}, \hat{p}], S(\hat{x})$ and $V(\hat{x})$ which may eventually lead to other exactly solvable potentials e.g., hyperbolic Pöschl Teller potential. Finally we would like to mention that higher dimensional analogue of the system considered here may have applications in phenomenology e.g., in the study of meson spectrum [10].

References

[1] G. Chen, Z. Chen and Z. Lou, Phys.Lett A331, (2004) 374
G. Chen, Phys.Lett A339, (2005) 300
G. Chen, Z. Chen and P. Xuan, Phys.Lett A352, (2006) 317
A. de Souza Dutra and G. Chen, Phys.Lett A349, (2006) 297
T. Jana and P. Roy, Phys.Lett A361, (2007) 55.

[2] A. S. de Castro, Phys.Lett A338, (2005) 81
L. Z. Yi, Y. F. Diao, J. Y. Liu and C. S. Jia, Phys.Lett A333, (2004) 212
Y. F. Diao, L. Z. Yi and C. S. Jia, Phys.Lett A332, (2004) 157
A. D. Alhaidari, H. Bahliouli and A. Al-Hasan, Phys.Lett A349, (2006) 47.

[3] L.J. Garay, Int.J.Mod.Phys A10, (1995) 145.

[4] D. J. Gross and P. F. Mende, Nucl. Phys. B 303, (1988) 407.

[5] M. Maggiore, Phys.Lett B304, (1993) 65.

[6] A. Kempf, J.Math.Phys 35, (1994) 4483
A. Kempf, G. Mangano and R.B. Mann, Phys.Rev D52, (1995) 1108
A. Kempf, J.Phys A30, (1997) 2093
H. Hinrichsen and A. Kempf, J.Math.Phys 37, (1996) 2121.

[7] L.N Chang et al., Phys.Rev D65, (2002) 125027
   I. Dadič et al, Phys.Rev D67, (2003) 087701
   K. Gemba et al, Preprint hep-th/0712.2078
   F. Brau, J.Phys A32, (1999) 7691
   T.V. Fityo et.al, J.Phys A39, (2006) 2143
   R. Akhoury and Y.-P. Yao, Phys.Lett B572, (2003) 37
   S. Benczik et.al, Phys.Rev A72, (2005) 012014
   K. Nouicer, J.Math.Phys 47, (2006) 122102
   C. Quesne and V.M. Tkachuk, J.Phys A38, (2005) 1747
   K. Nouicer, J.Phys A39, (2006) 5125
   C. Quesne and V.M. Tkachuk, SIGMA 3, (2007) 016
   M.S. Hossain and S.B. Faruque, Phys.Scr 78, 035006 (2008) 035006.

[8] D. Spector, J. Math. Phys. 49, (2008) 082101.

[9] F. Cooper et al, Supersymmetry in Quantum Mechanics , World Scientific Publishing Co, 2002.

[10] J.S. Kang and H.J. Schnitzer, Phys.Rev D12, (1979) 841.