Specht polynomials and modules over the Weyl algebra

Ibrahim Nonkané

Received: 14 May 2016 / Accepted: 27 November 2018 / Published online: 8 December 2018
© African Mathematical Union and Springer-Verlag GmbH Deutschland, ein Teil von Springer Nature 2018

Abstract
In this paper we study the decomposition structure of a direct image of a polynomial ring under certain map.

Keywords Algebraic geometry · D-modules · Direct image · Representation theory · Specht Polynomials · Symmetric groups

Mathematics Subject Classification 13N10 · 20C30

1 Introduction

The D-module direct image of an irreducible holonomic module with regular singularities under a proper map is semi-simple according to the decomposition theorem (see [7], and the references there). The simplest case is when the map \( \pi : X = \text{spec} B \rightarrow Y = \text{spec} A \) is finite, and the module is the structure sheaf \( B = O_X \). Then an elementary and wholly algebraic proof exists, using essentially the ordinary Galois group \( G \) of the function field extension corresponding to \( \pi \). This proof uses that the irreducible D-submodules of \( \pi_+ O_X \) are in one-to-one correspondence with irreducible representations of \( G \).

We study the decomposition in the case of the invariants of the symmetric group, \( B = \mathbb{C}[x_1, \ldots, x_n] \supset A = \mathbb{C}[x_1, \ldots, x_n]^{S_n} = \mathbb{C}[y_1, \ldots, y_n] \). There are several well-known descriptions of \( B \) as an \( A \)-module, and our main result is that one of these—the Specht polynomials (introduced combinatorially by several authors [2,3,17]), are adapted to the D-module structure. Ordinary Specht polynomials give generators of all irreducible D-module factors of \( \pi_+ O_X \), and the higher give a basis.
2 Decomposition

2.1 Notation

Let $k$ be an algebraically closed field of characteristic $0$. As a general setup we will use the following notation. We have an inclusion of smooth $k$-algebras $A \subseteq B$, where $B$ is a finite flat module over $A$.

Correspondingly, on spectra, we have the finite map $\pi: X \to Y$ of affine smooth varieties of finite type over an algebraically closed field of characteristic $0$. Denote for a $k$-algebra $B$, by either $\mathcal{T}_B/k$ or $\mathcal{T}_X/k$ the $k$-linear derivations (or vector fields) of $B$. For simplicity, we will assume that both $\mathcal{T}_B/k$ and $\mathcal{T}_A/k$ are free modules over respective ring. Since $B$ is smooth this will be a locally free $B$-module, that generates the algebra $\mathcal{D}_X$ of differential operators on $B$. There is a canonical map $d\pi: \mathcal{T}_B/k \to B \otimes_A \mathcal{T}_A/k$, which is an isomorphism if the extension $A \to B$ is étale. In that case there is a unique map $\phi: \mathcal{D}_A/k \to \mathcal{T}_B/k$ that is a splitting such that $\phi(D)(a) = D(a)$, for $D \in \mathcal{T}_A/k$, $a \in A$. Extending $d\pi$ there is a the map $d\pi: \mathcal{D}_X \to \pi^*\mathcal{D}_Y$, defined by the property that

$$d\pi(P)(f) = P(f),$$

(2.1)

if $f \in A$ and $P \in \mathcal{D}_X$. If the extension $A \to B$ is étale this is an isomorphism (since the ring of differential operators of a smooth variety is generated by derivations), and there is an unique lifting $\phi: \mathcal{D}_Y \to \mathcal{D}_X$, such that $d\pi \circ \theta = 1d$.

The map $A \to B$ is étale outside the closed set $F$ in $X$ defined by the Jacobian, i.e. the ideal that is locally determined by the determinant of the map $d\pi$, (choosing basis for $\mathcal{T}_B/k$ and $\mathcal{T}_A/k$). In particular, on the function fields $K \subseteq L$ corresponding to $A \to B$, we always have a lifting $\phi: \mathcal{T}_K/k \to \mathcal{T}_L/k$.

Denote the image of $F$ by $D$. We are mostly interested in the extension $B = \mathbb{C}[x_1, \ldots, x_n] \supseteq A = \mathbb{C}[x_1, \ldots, x_n]^{\mathfrak{m}} = \mathbb{C}[y_1, \ldots, y_n]$. In that case the Jacobian is given by the van der Monde determinant $\Delta = \prod_{i<j}(x_i - x_j)$ in the variables $x_j$, and the discriminant $D$ by $\Delta^2 \in A$. We will in general denote the complements of the branch locus and the discriminant by $U$ and $V$ respectively, and then by the above we have an étale map $U \to V$, and an isomorphism $d\pi: \mathcal{D}_U \to \pi^*\mathcal{D}_V$. As a consequence $d\pi: \mathcal{D}_X \to \pi^*\mathcal{D}_Y$ is an injection since $\mathcal{D}_X \subseteq \mathcal{D}_U$. Furthermore we also get a unique lifting $\phi: \mathcal{D}_V \to \mathcal{D}_U$ of $d\pi$ satisfying $d\pi \circ \theta = 1d$.

2.2 Constants

The condition on $k$ to be algebraically closed, is motivated by the needed fact that then $k = B^\mathcal{T}_B/k := \{b \in B \mid \partial(b) = 0, \partial \in \mathcal{T}_B/k\}$. This is a consequence of the following lemma.

Lemma 2.1 $C \to B$ be an inclusion of integral domains such that the extension of fraction fields $K(B)/K(C)$ is separable and finitely generated. Let $\mathcal{T}_B/C$ be the $C$-linear derivations and $\bar{C}$ be the integral closure of $C$ in $B$. Then

$$\bar{C} = B^\mathcal{T}_B/C := \{b \in B \mid \partial(b) = 0, \partial \in \mathcal{T}_B/C\}.$$ 

Proof It is easy to see that $\bar{C} \subseteq B^\mathcal{T}_B/C$. If $b \notin \bar{C}$ then there exists a chain of separable field extensions $K(C) \subseteq L \subseteq K(B)$ where $\text{tr. deg } L/K(C) \geq 1$, $b \in L = K(C)(b)$, and $\mathcal{T}_L/K(C)(b) \neq \{0\}$. Since $K(B)/L$ is separable, it follows that the map $T_{K(B)/K(C)} \to$
$K(B) \otimes_L T_{L/K(C)}$ is surjective. This implies that there exists an element $\partial \in T_{K(B)/K(C)}$ such that $\partial(b) \neq 0$ in $K(C)$. Multiplying by a suitable element $c \in B$ we get $c\partial \in T_{B/C}$ and $c\partial \cdot b \neq 0$ in $B$, hence $b \notin B^{T_{B/C}}$.

2.3 The direct image of the structure sheaf

We temporarily lift our condition that $\pi$ has to be finite in order to prove the following result.

**Lemma 2.2** Let $\pi : X \to Y$ and $\text{Mod}^\pi(D_X)$ be the category of finitely generated $D_X$-modules such that the map $\text{supp } M \to Y$ is finite. Then $\pi_+ : \text{Mod}^\pi(D_X) \to \text{Mod}(D_Y)$ is an exact functor.

**Proof** For a closed embedded the direct image functor is exact by Kashiwaras theorem. Factorising into a closed embedding and a projection we can reduce to the case when $\pi$ is a projection of relative dimension 1, $X = C \times Y \to Y$. Let $t$ be the parameter corresponding to the fibre. Since the map $\text{supp } M \to Y$ is affine, it is clear that the sheaf functor $\pi_+$ is exact. We will use that the direct image is the cohomology of the relative de Rham complex $O_X \otimes_{O_X} M \to O_X dt \otimes_{O_X} M$ (with map $f \otimes m \mapsto f dt \otimes \partial_m$), in degree 1(see[13]). Hence

$$\pi_+(M) = \pi_+(M/\partial_t M)$$

and by diagram chasing it suffices to show that for any module $N$ with finite relative support $N^{\partial_t} = \{m \in N|\partial_t m = 0\}$ is zero. Since for any $m \in N$, the support of $m$ over $Y$ is finite, there is a monic polynomial of positive degree in $t$, say $p(t) \in O_Y[t]$ such that $p(t)m = 0$. Assuming that $p(t)$ is of minimal positive degree in $t$, and that $\partial_t m = 0$, we get that $0 = \partial_t(pm) = \partial_t(p)m$, implying that $\partial_t(p) = 0$, and hence (using that $p(t)$ is monic) that $m = 0$. \(\square\)

In particular, reverting to our assumption that $\pi : X \to Y$ is a finite morphism, $\pi_+$ is exact on the category of coherent $D$-modules. Hence the inclusion of $D_X$-modules $O_X \subseteq O_U$, gives an injection

$$\pi_+ O_X \to \pi_+ O_U \quad (2.2)$$

Now note that $O_U$ has a $D_Y$-structure, since we may lift all differential operators on $V$ to differential operators on $U$. With this structure it is actually isomorphic to $\pi_+ O_U$, as we can see by checking the definition:

$$\pi_+ O_X := \omega_Y^{-1} \otimes_{O_Y} (\omega_X \otimes_{D_X} \pi^* D_Y),$$

in the following way. The $O_Y$-algebra $D_Y^\pi$ of liftable differential operators in $Y$ is defined as $d\pi(D_X) \cap D_Y$. Through the injection $d\pi$ we may identify $\pi^*(D_Y^\pi)$ with a subalgebra of $D_X$. If we now in the definition of direct image, instead take the tensor product with $\pi^*(D_Y^\pi) \subseteq \pi^*(D_Y)$, we get

$$\omega_Y^{-1} \otimes_{O_Y} \left(\omega_X \otimes_{\pi^*(D_Y^\pi)} \pi^* (D_Y^\pi)\right) = \omega_{X/Y},$$

(when $\omega_{X/Y} = \pi^*(\omega_Y^{-1}) \otimes_{O_X} \omega_X$). The relative canonical module thus maps to a submodule of $\pi_+ O_X$. If $\pi : X \to Y$ is étale, so that $D_Y^\pi = D_X$, we have equality $\omega_{X/Y} = \pi_+ O_X$. In particular this holds for $\pi : U \to V$.

Assume now that $U, V$ are such that the respective canonical modules are generated by volume forms $dx$, and $dy$, related by $dx = \Delta^{-1} dy$, where $\Delta$ is the Jacobian of $\pi$.
Lemma 2.3  (i) The relative canonical bundle $\omega_{X/Y}$ is a $D_Y^\pi-$submodule

$$\omega_{X/Y} \subset \pi_+ O_X \subset \pi_+ O_U = \omega_{U/Y}$$

(ii) There is an isomorphism of $D_V$-modules

$$T : \pi_+ O_U \cong O_U, \quad r(dy^{-1} \otimes dx) \mapsto r \Delta^{-1}.$$  

(Here $O_U$ is a $D_V$-module structure by the unique lifting $\phi : D_V \to D_U$ of $d\pi$.)

(iii) $T$ induces an isomorphism $\omega_{X/Y} \to O_X$, of $D_Y^\pi$-submodules. Hence

$$O_X \subset T(\pi_+ O_X) \subset O_U,$$

and $T(\pi_+ O_X)$ is isomorphic as a $D_V$-module to $\pi_+ O_X$.

Proof  That $T$ is a $D_V$-isomorphism is a consequence of the relation between the volume forms $dx = \Delta^{-1}dy$. The rest was proved above.

It is more convenient to study $T(\pi_+ O_X) \cong \pi_+ O_X$, as a submodule of $O_U$, than using the definition of $\pi_+ O_X$. We will soon see that $T(\pi_+ O_X)$ may be characterized as the minimal $D_Y$-module containing $O_X$.

2.4 The correspondence between $G$-representations and $D$-modules

There is a general correspondence between representations of the differential Galois group of a $D_X$-module $M$, defined using a Picard-Vessiot extension and the category of modules generated by $M$ (for the case of one variable see [18]). In our situation this correspondence takes a very simple form. Because of lack of a good reference, we include the (short) proofs.

Say that a $T_{K/k}$-module $M$ is $L$-trivial if $L \otimes_K M \cong L^n$ as $T_{L/k}$-modules. Denote by $\text{Mod}^L(T_{K/k})$ the full subcategory of finitely generated $T_{K/k}$-modules that are $L$-trivial. It is immediate that it is closed under taking submodules and quotient modules. Using the lifting $\phi$, $L$ may be thought of as a $T_{K/k}$-module.

If $G$ is a finite group let $\text{Mod}(k[G])$ be the category of finite-dimensional representations of $k[G]$. Let now $k \to K \to L$ be a tower of fields such that $K = L^G$. Note that the action of $T_{K/k}$ commutes with the action of $G$. If $V$ is a $k[G]$-module, $L \otimes_k V$ is a $T_{K/k}$-module by $D(l \otimes v) = D(l) \otimes v, \quad D \in T_{K/k}$, and $(L \otimes_k V)^G$ is a $T_{K/k}$-submodule.

Proposition 2.4 The functor

$$\Delta : \text{Mod}(k[G]) \to \text{Mod}(T_{K/k}), \quad V \mapsto (L \otimes_k V)^G$$

is fully faithful, and defines an equivalence of categories,

$$\text{Mod}(k[G]) \to \text{Mod}^L(T_{K/k}).$$

The quasi-inverse of $\Delta$ is the functor,

$$\text{Loc} : \text{Mod}^L(T_{K/k}) \to \text{Mod}(k[G]), \quad \text{Loc}(M) = (L \otimes_K M)^{\phi(T_{K/k})}.$$  

Proof  We will use the normal basis theorem, which says that there is an isomorphism $L = K[G]$ as a $K$-vector space and as a $G$-module. It follows from this that an arbitrary finite-dimensional $L$-vector space $V$ with a $G$-action that is compatible with the $L$-vector space structure, satisfies that $V \cong L \otimes_K V^G$, by the canonical map $L \otimes_K V^G \to V, \quad l \otimes v \mapsto lv$.  

(This is Speiser’s lemma. (See e.g. [11, Lemma 2.3.8]) If we apply this to $\Delta(V) = (L \otimes_k V)^G$,  

\[\text{Mod}^L(T_{K/k})\]
we get that $L \otimes_k V = L \otimes_K (L \otimes_k V)^G$. This isomorphism respects the $T_K/k$-module structure of the two sides, and shows that $\Delta$ takes values in $\text{Mod}^L(T_K/k)$. Furthermore, since $L^{\phi(T_K/k)} = k$, we get that,

$$\text{Loc}(\Delta(V)) = (L \otimes_K (L \otimes_k V)^G)^{\phi(T_K/k)} = (L \otimes_k V)^{\phi(T_K/k)} = V.$$ 

By definition, if $M \in \text{Mod}^L(T_K/k)$ then,

$$\text{Loc}(M) = (L \otimes_K M)^{\phi(T_K/k)} = (L^n)^{\phi(T_K/k)} = k^n,$$

so we get a finite-dimensional representation of $G$ in $\text{Mod}(k[G])$. The map (given by multiplication),

$$L \otimes_k (L^n)^{\phi(T_K/k)} \to L^n \cong L \otimes_K M,$$

is clearly an isomorphism. Then,

$$\Delta(\text{Loc}(M)) = (L \otimes_k k^n)^G = (L \otimes_K M)^G = M,$$

which concludes the proof of the proposition. \hfill \Box

Since $\text{Mod}(k[G])$ is a semi-simple category, this is also true for $\text{Mod}^L(T_K/k)$.

**Example 2.5** Every right $G$-module $V$ induces a left $G$-module $V^r$ with action $gv := g^{-1}v$. We may give $k[G]$ the left action induced by right multiplication, and call it $k[G]^r$. This will then be a module with two commuting left actions of $G$.

By definition $\Delta(k[G]^r) = (L \otimes_k k[G]^r)^G$. If $x = \sum_{g \in G} l_g \otimes g$ is invariant, then $l_{g^{-1}} \otimes g^{-1} = g(l_e \otimes e)$ so $l_{g^{-1}} = g(l_e)$, and $x = \sum_{g \in G} g(l_e) \otimes g^{-1}$. Hence there is a $T_K/k$-isomorphism,

$$\Delta(k[G]^r) \to L, \quad \sum_{g \in G} g(l_e) \otimes g^{-1} \mapsto l_e.$$

Hence $L$ itself corresponds to the (right) regular representation of $G$, and as a $T_K/k$-module belongs to $\text{Mod}^L(T_K/k)$.

**Corollary 2.6**

$$\text{Hom}_{T_K/k}(L, L) \cong k[G]$$

**Proof** By the preceding example, and the category equivalence of the proposition the endomorphism ring is isomorphic to $\text{Hom}_{k[G]}(k[G]^r, k[G]^r)$. But there is an isomorphism,

$$\text{Hom}_{k[G]}(k[G]^r, k[G]^r) \cong k[G], \quad \theta \mapsto \theta(1).$$

Note that since we are really dealing with right modules, $\eta \circ \theta \mapsto \eta(\theta(1)) = \eta(1)\theta(1)$, and hence if the composition of $\eta, \theta \in \text{Hom}_{k[G]}(k[G], k[G])$ is defined by $\eta \circ \theta$ we have an isomorphism with $k[G]$. It is easy to check that this is compatible with the canonical inclusion $k[G] \to \text{Hom}_{T_K/k}(L, L)$, which is given by the action of $G$ on $L$. This follows by,

$$h \left( \sum_{g \in G} g(h(l_e)) \otimes gh^{-1} \right) \otimes g^{-1} \mapsto h(l_e).$$

\hfill \Box
2.5 Direct and inverse images under finite maps

The above equivalence,

\[ \text{Mod}(T_K) \xrightarrow{\text{Loc}} \text{Mod}(G) \xrightarrow{\Delta} \text{Mod}(T_K), \]

can be extended to a Galois correspondence. Fix \( K \) and \( L \) and consider intermediate fields \( L \subset E \subset K \). Given two such fields \( E_1 \subset E_2 \), we have the categories \( \text{Mod}^L(T_{E_1/k}) \) and \( \text{Mod}^L(T_{E_2/k}) \). The map \( \pi : \text{spec} E_1 \rightarrow \text{spec} E_2 \) induces an isomorphism \( E_1 \otimes_{E_2} T_{E_2/k} \cong T_{E_1/k} \), in particular a canonical lifting \( D_{E_2} \rightarrow D_{E_1} \). Corresponding to this ring homomorphism we have the usual pair of adjoint functors. First the inverse image:

\[ \pi^+ : \text{Mod}^L(T_{E_2/k}) \rightarrow \text{Mod}^L(T_{E_1/k}), \]

given by,

\[ M \rightarrow E_1 \otimes_{E_2} M. \]

It is immediate by \( L \otimes_{E_1} (E_1 \otimes_{E_2} M) \cong L \otimes_{E_2} M \) that the image of the inverse image lies in \( \text{Mod}^L(T_{E_1/k}) \). Secondly we have the direct image functor \( \pi_+ \), between the same categories, given by restricting the action on \( M \) to \( T_{E_2/k} \) using the canonical lifting \( T_{E_2/k} \rightarrow T_{E_1/k} \). The direct image is right adjoint to the inverse image.

That the direct image lands in \( \text{Mod}^L(T_{E_2/k}) \) is clear e.g. in the following way. By the proposition in the preceding section, it suffices to prove this for \( E_1 \), since it then follows for any direct factor. Now the category \( \text{Mod}^L(T_{L/k}) \) is closed under submodules and quotients, and \( L \otimes_{E_2} E_1 \) is a submodule of \( L \otimes_{E_2} L \), which is a quotient of \( L \otimes_K L \in \text{Mod}^L(T_{L/k}) \). So \( L \otimes_{E_2} E_1 \in \text{Mod}^L(T_{E_2/k}) \).

We have the usual inclusion reversing correspondence of lattices of subgroups of a Galois group with the lattice of intermediate fields: \( H_1 \subset H_2 \subset G \),

\[ H_i \mapsto E_i := L^{H_i}, \quad i = 1, 2. \]

Between the two categories of left modules of the respective groups, we have a restriction functor,

\[ \text{Res}^{H_2}_{H_1} : \text{Mod}(k[H_2]) \rightarrow \text{Mod}(k[H_1]), \]

and two induction functors: the usual induction,

\[ \text{Ind}^{H_2}_{H_1} : \text{Mod}(k[H_1]) \rightarrow \text{Mod}(k[H_2]), \quad V \rightarrow k[H_2] \otimes_{k[H_1]} V, \]

and coinduction,

\[ \text{Coind}^{H_2}_{H_1} : \text{Mod}(k[H_1]) \rightarrow \text{Mod}(k[H_2]), \quad V \rightarrow \text{Hom}_{k[H_1]}(k[H_2], V). \]

(The left \( k[H_2] \)-action on the last module is given by \((gf)(h) = f(hg) \) iff \( g \in \text{Hom}_{k[H_1]}(k[H_2], V), \quad g \in G_2 \). The two induction functors are left respectively right adjoint to the restriction functor. They are actually isomorphic, so we will use the induction functor later on.

This means that,

\[ \text{Hom}_{k[H_1]}(M, \text{Res}^{H_2}_{H_1} N) = \text{Hom}_{k[H_2]}(\text{Ind}^{H_2}_{H_1} M, N) \]

\[ \text{Hom}_{k[H_1]}(\text{Res}^{H_2}_{H_1} N, M) = \text{Hom}_{k[H_2]}(N, \text{Coind}^{H_2}_{H_1} M). \]

(These are consequences of ordinary adjointness between hom and tensor products)
By the proposition there are equivalences of categories $\text{Mod}(k[H_1]) \rightarrow \text{Mod}(T_{E_1/k})$, and we now want to express the direct and inverse images of $D$-modules in terms of the corresponding group representation categories and functors.

**Proposition 2.7**  
(i) Define $\pi^+ : \text{Mod}(k[H_2]) \rightarrow \text{Mod}(k[H_1])$ as the restriction associated to the injection $k[H_1] \subset k[H_2]$. Then,  
\[
\pi^+(\Delta(V)) = \Delta(\pi^+(V)).
\]

(ii) Define $\pi_+ : \text{Mod}(k[H_1]) \rightarrow \text{Mod}(k[H_2])$ as the coinduction associated to the injection $k[H_1] \subset k[H_2]$, defined in the following way:  
\[
\pi_+(V) = \text{Hom}_{k[H_1]}(k[H_2], V).
\]

Then,  
\[
\pi_+(\Delta(V)) = \Delta(\pi_+(V)).
\]

**Proof**  
(i) Now restriction on the group side corresponds to inverse image of $D$-modules:  
\[
\pi^+(\Delta N) = E_1 \otimes_{E_2} (L \otimes_k N)^{H_2} \cong (L \otimes_k \text{Res}_{H_1}^{H_2} N)^{H_1} = \Delta\left(\text{Res}_{H_1}^{H_2} N\right).
\]

The last map is given by the injection $E_1 \otimes_{E_2} (L \otimes_k N)^{H_2} \rightarrow L \otimes_{E_1} (L \otimes_k N)^{H_2}$ composed with the map that takes $l_1 \otimes l_2 \otimes n \mapsto l_1 l_2 \otimes n$ into $L \otimes N$, and is hence injective by Speiser’s lemma. The image is clearly invariant under $H_1$. Furthermore for $N = k[H_2]$ the modules $(L \otimes_k \text{Res}_{H_1}^{H_2} N)^{H_1}$ and $E_1 \otimes_{E_2} (L \otimes_k N)^{H_2}$ are easily seen to have the same dimension over $K$, and so the map is surjective, and hence has to be surjective for all direct summands, i.e. for all modules. That it is a natural equivalence follows directly from its construction.

(ii) We have a natural map,  
\[
L \otimes_k \text{Hom}_{k[H_2]}(k[H_1], V)) \rightarrow L \otimes_k \text{Hom}_{k[H_1]}(k[H_1], V)) = L \otimes_k V,
\]

that induces a map  
\[
\Delta\left(\text{Ind}_{H_1}^{H_2} V\right) = (L \otimes_k \text{Hom}_{k[H_2]}(k[H_1], V))^{H_2} \rightarrow \pi_+(L \otimes_k V)^{H_1}.
\]

Also it commutes with multiplication by elements in $E_2$, as well as elements in $T_{K/k}$. Since the map is easily seen to be an isomorphism for $V = k[H_1]$ it follows that it will be an isomorphism for all summands, i.e. all modules.

\[\square\]

### 2.6 Reduction to the generic case

Use of the preceding result enables us to reduce the study of the decomposition factors of $\pi_+ O_X$ to the behavior of the direct image over the complement to the branch locus, or even over the generic point. Let $j : U \subset X$ and $i : V \subset Y$ be the inclusions.

**Proposition 2.8** Let $\pi : X \rightarrow Y$ be a finite map. Then  

(i) $\pi_+ O_X$ is semi-simple as a $D_Y$-module.

(ii) If $\pi_+ O_X = \oplus M_i$, $i \in I$ is a decomposition into simple (non-zero) $D_Y$-modules, then $\pi_+ O_U = \oplus i^* M_i$, $i \in I$, is a decomposition of $\pi_+ O_U$ into simple (non-zero) $D_Y$-modules.
The second part of the proposition follows from the fact that if $M$ is a torsion free irreducible $DY$-module then $i^*M$ is an irreducible non-zero $DV$-module (see e.g. [1, Prop.2.3]).

If the finite extension of function fields $K \subset L$ corresponding to $A \subset B$ is Galois, we saw in the preceding sections that $\pi_+ L$ is a semi-simple $DK$-module. This remains true for an arbitrary finite extension, as can be seen by taking a Galois extension $K \subset L \subset W$, and noting that $L \in Mod^W(T_L)$, so that the results on direct images in the preceding section apply.

By Lemma 2.3, $\pi_+ O_X$ is a torsion free $O_Y$-submodule of $i_* i^* \pi_+ O_X \cong i_* \pi_+ O_U$. We know that $\pi_+ O_X$ has a finite decomposition series, since it is holonomic. Let $S \subset \pi_+ O_X$ be the socle. By Proposition 2.4, $i^* \pi_+ O_X \cong \pi_+ O_U = \sum N_i$, is semi-simple (after possibly shrinking $V$). By localizing and using torsion freeness $S_i := \text{soc} D_i i^* N_i$ is simple, and $N_i / S_i$ torsion. Furthermore $i_* N_i$ contains an element from $\pi_+ O_X$, and hence its socle has to lie in $\pi_+ O_X$. All in all, this means that we have a short exact sequence,

$$0 \to S \to \pi_+ O_X \to \pi_+ O_X / S \to 0,$$

in which the cokernel is torsion. Now dualize:

$$0 \leftarrow D(S) \leftarrow D(\pi_+ O_X) \leftarrow D(\pi_+ O_X / S) \leftarrow 0.$$

First $D(\pi_+ O_X) \cong \pi_+ O_X$ by [13, Theorem 2.7.2], and so this module is torsion free. Second, the functor $D$ preserves support (by [13, Prop 2.6.12].), and hence $D(\pi_+ O_X / S)$ is torsion, and so zero. This implies that $\pi_+ O_X / S = 0$ and proves the first part of the proposition.

As a consequence we have that $T(\pi_+ O_X) \subset O_U$ is the $DY$-module generated by $O_X \subset O_U$.

### 2.7 Construction of simple D-modules

In this section we consider only the case $B = \mathbb{C}[x_1, \ldots, x_n]$ and $A = \mathbb{C}[x_1, \ldots, x_n]^S_n$. It follows from the equivalence between $S_n$-modules and $L$-trivial $T_K$ modules, that there is an irreducible module corresponding to each Specht module $S^\lambda$. We now identify these as submodules of $\pi_+ B$.

If $I \subset \{1, \ldots, n\}$ define,

$$\Delta(J) = \prod_{i,j \in I, \ i < j} (x_i - x_j).$$

Let $J = P_1 \cup \cdots \cup P_k$ be a partition of $\{1, 2, \ldots, n\}$ as a set. Associated to it is the subgroup $S_J = S_{P_1} \times \cdots \times S_{P_k} \subset S_n$ and $Y_J = \text{spec} \mathbb{C}[x_1, \ldots, x_n]^{S_J}$. The maps $\pi_{0,J} : X \to Y_J$ and $\pi_{1,J} : Y_J \to Y$ are the obvious ones, and we clearly have maps $\pi_{I,J} : Y_I \to Y_J$, whenever $I$ is a refinement of $J$. The jacobian of $\pi_{0,J}$ is,

$$p_J = \prod_{i=1}^k \Delta(P_i),$$

a product of van der Monde determinants on each of the set of variables in the $P_i$. Clearly $p_J$ belongs to $O_{Y_J}$.

Consider $O_X \subset \pi_{0,J} O_X \subset O_U$. 

\[ Springer \]
Lemma 2.9 The discriminant of $\pi_{0,J}$ is $p_J^2 \subset O_{Y_J}$. Let $V_J \subset Y_J$ be the complement to the discriminant. Then,

$$O_{V_J} p_J \cong D_{Y_J} p_J \subset O_U,$$

is an irreducible $D_{Y_J}$-module of rank one. Furthermore, considering the module over the generic point $\text{spec} L J, \text{Loc}(D_{L_J} p_J) \cong -1$, the sign representation of $S_J$.

**Proof** It is clear that,

$$D (p_J^2) \in O_{Y_J} \implies D (p_J) = D (p_J^2) p_J \in O_{Y_J} p_J,$$

and hence that $O_{Y_J} p_J$ is a $D_{Y_J}$-module, irreducible since it has rank 1. 

Define $F_{J} := D_{Y_J} p_J \subset O_{U}$, and $M_{J} := \pi_{1,J} F_{J}$. (Note that, using an earlier lemma, this gives a submodule isomorphic to $O_{V_J}$ of $\pi_{0,J} O_{X_J}$.) Let the numerical partition of $n$, associated to $J$ be $\lambda(J)$. $J$ corresponds to a tableau of shape $\lambda(J)$, where each row is increasing (from the left). If the columns also are increasing (top to bottom) this tableau is called standard.

Associated to $J$ there is an irreducible $S_n$-module, called the Specht module $S_{J}$. It may be realized, as a submodule of $\pi_{+} B$, as the $C[Sn]$-module generated by the Specht polynomial $p_J$ (see [14,15]). Its isomorphism class only depends on $\lambda(J)$. We have a corresponding irreducible $DA$-module of $\pi_{+} B$.

**Theorem 2.10**

(i) $MT := D_{A} p_{T}$ is an irreducible $D_{A}$-module of $\pi_{+} B$.

(ii) There is a direct sum decomposition

$$\pi_{+} B = \bigoplus_\lambda \bigoplus_{T \in ST_{\lambda}} M_{T}$$

(2.3)

where the first sum runs over all partitions $\lambda$ of $n$, and the second over all standard tableau of shape $\lambda$.

(iii) Generically

$$\Delta(S_{\lambda}) \cong M_{T}|_{K}.$$

**Proof** This now follows from standard representation theory of the symmetric group. Associated to each standard tableau $T$ is a Young symmetrizer $e_{T}$. Up to a scalar factor these are idempotent and mutually orthogonal. Furthermore $S_{T} := C[Sn]e_{T}$ is an irreducible module, isomorphic to $S_{\lambda}$ if and only if $S$ and $T$ have the same shape, and they form a spanning set of irreducible submodules of $k[Sn] = \bigoplus k[Sn]_{S}T$ (see e.g.[9]). In particular the number $f_{\lambda}$ of standard tableau of shape $\lambda$ is the same as the dimension of one of these irreducible modules, and as a consequence $\sum f_{\lambda}^{2} = n!$. Now $e_{T} p_{T} = c p_{T}$, where $c$ is a non-zero scalar, and (hence) $e_{T} p_{S} = 0$ if $S \neq T$ (see [14]). Thus $e_{T} D_{A} p_{T} = D_{A} p_{T}$ is non-zero, and $e_{T} D_{A} p_{S} = 0$ if $S \neq T$. This implies that the sum decomposition in (2.3) is direct. Hence $\dim_k\text{Hom}_{(D_{A})}(\pi_{+} B, \pi_{+} B) \geq n!$ with equality by Schur’s lemma if and only if all $D_{A} p_{T}$ are irreducible. But we calculated this homomorphism group in Corollary 2.6 (generically, but this does not influence the result), and from that see that we have equality. For the third part of the theorem, consider the right $k[Sn]$-module $V = e_{T} k[Sn]$ where $T$ has shape $\lambda$. It is the image of the map $k[Sn] \to k[Sn]$ given by multiplication with $e_{T}$. Then changing this map into a map $k[Sn]^r \to k[Sn]^r$ we get as image a module isomorphic to $S_{\lambda}$. It induces a map,

$$\Delta(k[Sn]^r) \to \Delta(S_{T}) \subset \Delta(k[Sn]^r).$$
that according to a previous example is given by multiplication of $e_T$. Hence $\Delta(S_T)$ contains $e_T L$ and thus must equal this module. \hfill \Box

The usual description of $S_\lambda$ is as the image of the unique non-zero map,

$$(-1)^{n_i} \frac{S^n}{S_i} \to 1 \frac{S^n}{S_i}.$$ 

In our context this has the interpretation that,

$$M_T = \pi_{1,T} O_{Y_T} \cap \pi_{1,T'} + \left( O_{V_T}, p_{T'} \right) \subset \pi_{+,O_X},$$

where $T'$ is the transpose tableau to $T$.

\section{2.8 A $K$-basis of $M_\lambda$ using higher Specht polynomials}

Let $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$ be a partition of $n$ and $S$ a standard $\lambda$-tableau. One associates an index tableau $i(S)$ of the same form as the following way. Define the word $w(S)$ reading $S$ from bottom up in consecutive columns, from the left to the right. Then give the number 1 in $w(S)$ the index 0. If the number $k$ in the word has index $p$, then $k+1$ has index $p$ if it is to the right of $k+1$ and $p+1$ if it is to the left. For example, if, $S = 12435$, then $w(S) = 3110522041$. Changing a number to its index in the tableau $S$, one obtains the index tableau $i(S)$ of $S$.

If $T$ is a $\lambda$-tableau, $R(T)$ and $C(T)$ the row-stabilizer and column-stabilizer of $T$ respectively, consider the Young symmetrizer,

$$e_T = \sum_{\sigma \in R(T)} \sum_{\tau \in C(T)} (\text{sgn} \tau) \tau \sigma,$$

in $\mathbb{C}[S_n]$. Terasoma et Yamada [17] have defined $F_T^S$ by,

$$F_T^S = (x_1, \ldots, x_n) = e_T \left( \lambda^{i(S)}_T \right),$$

for all $\lambda$-tableau $S$. For the standard tableau $S_0$ of form $\lambda$ in which the cells are numbered from left to right on successive rows beginning top left, $F_T^{S_0}$ is proportional to the Specht polynomial associated to $T$. The polynomials $F_T^S$ are called higher Specht polynomials. For a standard tableau $T \in \text{Tab}(\lambda)$ the polynomial $F_T^S$ is called standard. One has then $n!$ standard higher Specht polynomials $F = \{ F_T^S; S, T \in \text{Tab}(\lambda), \lambda \vdash n \}$. The following gives their fundamental property.

\textbf{Theorem 2.11} [17]. The higher Specht polynomials in $F$ form a basis of the $A := \mathbb{C}[x_1, \ldots, x_n]^{S_n}$-module $B := \mathbb{C}[x_1, \ldots, x_n]$.

As a corollary we get

\textbf{Theorem 2.12} $(M_T)_\mathcal{W} = \bigoplus A_\Delta F_T^S$, where the sum runs over all standard tableaux of form $\lambda$. \hfill \square
Proof A consequence of the definition of the higher Specht polynomials and the cited theorem is that \( e_I B = \bigoplus_{S \in ST(\lambda)} AF_S^I \). Since we also know from the proof of Theorem 2.10 that \( e_I B \subset D_A p_T \) and that \( e_I D_A p_T = D_A p_T \) it follows that \( e_I B \) has full rank in the simple module \( D_A p_T \). If we localize \( e_I B \) at the open set \( V \), it becomes an \( D_A V \)-module and hence coincides with the (still irreducible irreducible see e.g. [1, Prop.2.3]) module \( D_A V p_T \). 

2.9 Example

We consider the case \( n = 2, 3 \).

For \( n = 2 \) the two possible Specht polynomials are \( p_1 = 1 \) and \( p_2 = (x_2 - x_1) \). We have that,

\[
\pi_+ B = A \oplus A_{\Delta^2}(x_1 - x_2),
\]

are rank 1 \( D_Y \)-modules generated by these Specht polynomials, corresponding to the two possible standard tableau.

For \( n = 3 \) the Specht polynomials corresponding to standard tableau are 1 and,

\[
p_1 = (x_2 - x_1), \quad p_2 = (x_3 - x_1),
\]

and,

\[
p_3 = (x_2 - x_1)(x_3 - x_1)(x_3 - x_2).
\]

In addition there are two higher Specht polynomials,

\[
p_1^2 = (x_2 - x_1)x_3, \quad p_2^2 = (x_3 - x_1)x_2,
\]

associated to \( p_1 \), \( p_2 \) respectively.

Correspondingly we have that,

\[
\pi_+ B = A \oplus D_Y(x_2 - x_1) \oplus D_Y(x_3 - x_1) \oplus D_Y(x_2 - x_1)(x_3 - x_1)(x_3 - x_2),
\]

and \( D_Y(x_2 - x_1) \) is isomorphic to \( D_Y(x_3 - x_1) \), has rank 2 and generically \( p_1, p_1^2 \) are a basis of it over the quotient field of \( A \).

2.10 Liftable derivations

We have described the decomposition of \( \pi_+ B \) as a \( D_Y \)-module. It is natural to ask for results on the algebra of differential operators that preserve \( B \). Liftable differential operators will act on the submodule \( B \subset T(\pi_+ B) \). The liftable derivations form a free module over \( A \) with basis given by,

\[
T_j = \sum_{i=1}^{n} x_i^{j+1} \partial_{x_i}, \quad j = -1, \ldots, n - 2.
\]

It is easily seen that these are actually derivations in \( D_Y \), and it follows from Saito’s criterion that they are a free basis of all derivations in \( D_X \) that preserve \( \Delta \). Let now \( F_Y \subset D_Y \) be the subalgebra generated by \( A \) and \( T_j, j = -1, \ldots, n - 2 \). Then after localization \( F_Y|_V = D_V \), and \( F_Y \) preserves \( B \). So \( B \) splits as a direct sum of factors \( M_T = B \cap D_Y p_T \). We have verified, up to \( n = 5 \), that intriguingly enough

\[
B = \sum F_Y p_T
\]
(the sum over all standard tableau) but ignore whether this is true in general. E.g. for \( n = 3 \),
\( x_1 - x_2 \), and \( T_1(x_1 - x_2) = x_1^2 - x_2^2 \) give an \( A \)-basis of the rank 2 module \( B \cap D_Y(x_1 - x_2) \).
It would be interesting to know whether it is possible to generate a basis for the coinvariants
in general, in this way, by using the simple derivations \( T_i \).

Acknowledgements
This article is based on part of the author’s Ph.D. Dissertation written under the supervision of Prof. Rikard Bøgvad. I deeply thankful to R. Källstrom and Prof R. Bøgvad for instructive comments during the writing of this paper. This paper is financially supported by the International Science Program (ISP).

References
1. Abebaw, T., Bøgvad, R.: Decomposition of D-modules over a hyperplane arrangement in the plane. Ark. Mat. 48(2), 211–229 (2010)
2. Allen, E.: A conjecture of Procesi and a new basis for the decomposition of the graded left regular representation of \( S_n \). Adv Math. 1100(2), 262–292 (1993)
3. Ariki, S., Terasoma, T., Yamada, H.: Higher Specht polynomials. Hiroshima Math. 27(1), 177–188 (1997)
4. Bergeron, F.: Algebraic combinatorics and coinvariant spaces. CMS Treatises in Mathematics. Canadian Mathematical Society, Ottawa (2009) (A K Peters Ltd., Wellesley )
5. Björk, J.-E.: Rings of Differential Operators. North-Holland Publishing Co., Amsterdam (1979)
6. Björk, J.-E.: Analytic D-modules and Applications, vol. 247. Kluwer Academic Publishers Group, Dordrecht (1993)
7. de Cataldo, M.A.A., Migliorini, L.: The decomposition theorem, perverse sheaves and the topology of algebraic maps. Bull. Am. Math. Soc. (N.S.) 46(4), 535–633 (2009)
8. Coutinho, S.C.: A Primer of Algebraic D-Modules, vol. 33. Cambridge University Press, Cambridge (1995)
9. Fulton, W., Harris, J.: Representation Theory: A First Course, Graduate Texts in Mathematics. Readings in Mathematics, vol. 129. Springer, New York (1991)
10. Garsia, A., Procesi, C.: On certain graded \( S_n \)-modules and the \( q \)-Kostka polynomials. Adv. Math. 94(1), 82–138 (1992)
11. Gille, P., Szamuely, T.: Central Simple Algebras and Galois Cohomology, vol. 101. Cambridge Studies in Advanced Mathematics, Cambridge (2006)
12. Hartshorne, R.: Residues and Duality. Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20. Springer, Berlin, New York (1966)
13. Hotta, R., Takeuchi, K., Tanisaki, T.: D-Modules, Perverse Sheaves, and Representation Theory. Progress in Mathematics, 236. Birkhäuser Boston, Inc., Boston (2008)
14. Peel, M.H.: Specht modules and symmetric groups. J. Algebra 36(1), 88–97 (1975)
15. Sagan, B.E.: The Symmetric Group. Representations, Combinatorial Algorithms, and Symmetric Functions, 2nd edn. Graduate Texts in Mathematics, 203. Springer, New York (2001)
16. Serre, J.-P.: Linear Representations of finite Groups, Graduate Texts in Mathematics, vol. 42. Springer, New York, Heidelberg (1977)
17. Terasoma, T., Yamada, H.: Higher Specht polynomials for the symmetric group. Proc. Jpn. Acad. Ser. A Math. Sci. 69, 41–44 (1993)
18. van der Put, M., Singer, M.S.: Galois Theory of Linear Differential Equations. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 328. Springer, Berlin (2003)