Counting States of Black Strings with Traveling Waves

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Abstract

We consider a family of solutions to string theory which depend on arbitrary functions and contain regular event horizons. They describe six dimensional extremal black strings with traveling waves and have an inhomogeneous distribution of momentum along the string. The structure of these solutions near the horizon is studied and the horizon area computed. We also count the number of BPS string states at weak coupling whose macroscopic momentum distribution agrees with that of the black string. It is shown that the number of such states is given by the Bekenstein-Hawking entropy of the black string with traveling waves.
I. INTRODUCTION

Recent progress in string theory has led to a statistical explanation of the Bekenstein-Hawking entropy for a certain class of black holes [1 - 8]. (For a recent review, see [9].) This class includes extremal and near extremal, four and five dimensional solutions. The microstates of these black holes turn out to be associated with fields moving on a circle in an internal spacetime direction. To see a closer connection between the quantum states and the classical spacetimes, it is convenient to rewrite the black hole solutions so as to explicitly include the internal direction. This results in black string solutions in one higher dimension. The simplest possibility (and the only one to be considered so far in connection with entropy calculations) is for the black string to be translationally invariant along the extra dimension.

In addition to the spacelike (and timelike) translational symmetry, the extremal limit of these black string solutions has a full 1 + 1 dimensional Poincare invariance, including in particular a null translational symmetry. It has been known for some time [10,11] that one can add traveling waves to any such solution, i.e., waves moving along the string. Since the initial wave profile can be specified arbitrarily, one obtains a class of solutions having event horizons and labeled by free functions. These solutions are quite remarkable in that the waves propagate indefinitely without radiating to infinity or falling into the horizon. Furthermore, the waves carry a finite energy and momentum per unit length along the string. We will show that that the Bekenstein-Hawking entropy of each such solution can be reproduced by counting string states, provided only that the free functions do not vary too rapidly. This restriction turns out to be necessary for a meaningful comparison of the two entropies. Our results greatly increase the class of solutions with horizons for which such a counting has been performed.

Black strings with traveling waves have recently been discussed in connection with a different approach to understanding black hole entropy. In [12] it was suggested that these waves should be viewed as “classical hair” and an attempt was made to count the number of states associated with this hair. This approach was further developed in [13,14]. The idea was that if the extra dimension was sufficiently small then this hair would not be directly visible but the degrees of freedom associated with it might account for the black hole entropy. Although we start with the same classical solutions, our interpretation is quite different. First of all, it was suggested in [12] that since the dependence of these traveling waves on the radial coordinate is fixed, one should view them as degrees of freedom on the horizon. But since the event horizon is a null surface, there is no sense in which a wave propagates along the horizon. We will study the global structure of these solutions and show that the horizon is always homogeneous. There is no local geometric quantity which distinguishes one point on the horizon from any other: There is no “classical hair” on the horizon. Secondly, in [13,14] it was proposed that this classical hair could be counted by studying a fundamental string propagating in the background of a black string without any waves. The (left-moving) states of this string correspond to linearized waves on the black string background. In contrast, we will study waves of finite amplitude. Finally, we will consider the regime where the size of one internal direction is large. Then for a fixed wave profile, the black string still has a horizon with finite area (which depends on the wave). The usual semiclassical arguments imply that it has a nonzero entropy which must still be explained. We will see that this entropy is naturally understood using the same methods that have been applied to the black
For simplicity we will consider a six dimensional black string carrying electric and magnetic charges associated with the Ramond-Ramond (RR) three form $H$. In the limit of weak string coupling, RR charges are carried by D-branes $[15][16]$, and the black string is described by a bound state of D-fivebranes and D-onebranes. We will study two different types of traveling waves. One corresponds to a wave $p(u)$ which just changes the momentum density along the string. The case of constant $p$ corresponds to the translationally invariant solution obtained by boosting the nonextremal black string in $[3]$. It has been shown that in the limit of weak string coupling (and for a single fivebrane), the states of the extremal black string are obtained by taking a certain set of fields on a circle (which can be viewed as representing the oscillations of the onebranes inside the fivebrane), and considering all right moving modes with the given total momentum. The number of such states agrees with the exponential of the Bekenstein-Hawking entropy in the limit of large charges and momentum $[1–3]$. To represent the states of the black string with a momentum wave, one would expect to keep the same fields on the circle, but now to consider only those states with momentum distribution given by $p(u)$. We will show that the number of such states is indeed the exponential of the Bekenstein-Hawking entropy.

The second type of wave we will consider consists of an oscillation $f^i(u)$ in a four dimensional compact internal space (so that, together with the six dimensions of the black string, one forms a ten dimensional spacetime). Unlike $p(u)$, we will show that this wave does not affect the area of the event horizon. This can also be understood from counting states at weak string coupling. When $f^i(u) \neq 0$, the black string has nonzero components of the momentum density in the four internal directions. In the weak coupling $1+1$ dimensional description, this corresponds to the field momentum $\pi$ canonically conjugate to the field $\chi$ that describes the fluctuations of the onebranes inside the fivebranes. We must thus count states with a given distribution of this field momentum, as well as a given distribution of the momentum along the string. While these internal waves increase the momentum distribution along the string (potentially increasing the number of available states), we will see that the constraint on the field momentum is a strong restriction on the allowed states. The net effect turns out to be that, to leading order for large charges, the number of available states is unaffected by this internal wave.

One can also turn this around. Suppose we start with the weakly coupled D-brane description in which BPS states are described by right moving modes in $1+1$ dimensions. One should be able to specify any macroscopic property of these states and increase the string coupling to form an event horizon. If one fixes the total momentum, one forms the usual translationally invariant black string. If one fixes the longitudinal momentum distribution, one forms the black string with traveling wave $p(u)$. If one fixes the field momentum, one forms the black string with the internal wave $f^i(u)$. In this way, one can see various properties of the black string microstates in the classical solutions.

Adding traveling waves to the six dimensional extremal black string is analogous to adding angular momentum (in two orthogonal planes with $J_1 = -J_2$ $[4]$). In both cases one does not break supersymmetry. The quantum states of the resulting solutions are a subset of the BPS states of the original nonrotating black string without traveling waves. One just restricts to states with given momentum distributions in the first case, or a given angular momentum in the second. However, while the angular momentum is specified by a single
number, the momentum distributions are given by arbitrary functions.

In the next section we describe the black string with traveling waves and study the structure of its event horizon; certain technical details are relegated to the appendix. In section three we count the corresponding states in string theory at weak coupling, and section four contains some discussion of the results.

II. THE BLACK STRING SOLUTIONS

We start with the Type IIB string theory in the ten dimensional Einstein frame, keeping only the metric, dilaton and RR three form $H$:

$$ S = \frac{1}{16\pi G_{10}} \int d^{10}x \sqrt{-g} \left( R - \frac{1}{2} (\nabla \phi)^2 - \frac{1}{12} e^{\phi} H^2 \right) \quad (2.1) $$

where $G_{10} = 8\pi^6 g^2$ is the ten-dimensional Newton’s constant in units with $\alpha' = 1$, $g$ denotes the string coupling, and the zero mode of $\phi$ is defined so that $\phi \to 0$ asymptotically. We wish to consider toroidal compactification to five dimensions with an $S^1$ of length $L$ and a $T^4$ of volume $V = (2\pi)^4 V$. We will assume that $L >> V^{1/4}$ so that the solutions resemble strings in six dimensions. This will be of use in section III. We also impose spherical symmetry in the five noncompact dimensions. The electric and magnetic charges associated with $H$ are defined by

$$ Q_1 = \frac{V}{4\pi^2 g} \int e^{\phi} * H, \quad Q_5 = \frac{1}{4\pi^2 g} \int H \quad (2.2) $$

and are normalized to be integers. Below, we will describe a family of black string solutions with two kinds of traveling waves and study the effect of each type of wave in turn on the horizon geometry.

A. Black Strings and Traveling Waves

A six parameter family of solutions to (2.1) with the desired properties and regular event horizons was found in [17,18]. Here we will be interested in the extremal limit of these solutions. We will also adjust one of the parameters so that the volume of the four torus is constant. The resulting solution is

$$ ds_0^2 = \left( 1 + \frac{r_0^2}{r^2} \right)^{-1} [-du dv + \frac{p}{r^2} du^2] + \left( 1 + \frac{r_0^2}{r^2} \right) (dr^2 + r^2 d\Omega_3^2) + dy_i dy^i \quad (2.3) $$

with

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1We follow the conventions of [17] so that T-duality sends $V$ to $1/V$ and S-duality sends $g$ to $1/g$.

2In the notation of [17], we set $\alpha = \gamma$. 

and $\phi = 0$. The coordinates $y^i$ label points on the $T^4$ and $u = t - z$, $v = t + z$ where $z$ is a coordinate on the $S^1$.

The black string (2.3) has an event horizon at $r = 0$ which is a smooth surface (for $p > 0$; see appendix) of area $2\pi^2 r_0^2 L V \sqrt{\rho}$. The string carries an ADM energy and momentum

$$E = \frac{L (2r_0^2 + p)}{\kappa^2}, \quad P = \frac{Lp}{\kappa^2}$$  \hspace{1cm} (2.5)

where

$$\kappa^2 = \frac{4 G_{10} V}{\pi^{10}} = \frac{2 \pi g^2}{V}$$  \hspace{1cm} (2.6)

Note that for $p < -2r_0^2$ the string carries negative energy and for $p < 0$ the compactified spacetime has closed timelike curves outside the horizon. In addition, the compactification introduces a conical singularity when $p = 0$ [19]. As a result, we will consider only the case $p > 0$.

Since the Killing vector field $\partial/\partial v$ is null, traveling waves can be added to this metric using the method of Garfinkle and Vachaspati [10,11]. The resulting solutions have been studied in [12,13]. The metrics take the form

$$ds^2 = \left(1 + \frac{r_0^2}{r^2}\right)^{-1} [-du dv + K(u,x,y)du^2] + \left(1 + \frac{r_0^2}{r^2}\right) dx_idx^i + dy_idy^i$$  \hspace{1cm} (2.7)

where we have introduced the cartesian coordinates $x^i$ in the four noncompact spatial dimensions, i.e. $r^2 = x_ix^i$ with the index $i$ being raised and lowered by the flat metric $\delta_{ij}$. Note that (2.7) again has a null Killing field $\partial/\partial v$ whose integral curves are null geodesics. The metric disturbance $K$ propagates along these curves.

The metric (2.7) satisfies the field equations with the same matter sources as (2.3) provided the waves $K(u,x,y)$ satisfy the flat space Laplace equation in the eight transverse coordinates $(x,y)$:

$$(\partial_x \partial_{x^i} + \partial_y \partial_{y^i}) K = 0.$$  \hspace{1cm} (2.8)

That is, the wave profiles are flat-space harmonic functions with arbitrary $u$ dependence. $K(u,x,y)$ may include terms with ‘sources’ at $r = 0$, since this is a coordinate singularity in (2.7). For monopole fields, we will see that the horizon remains well defined, with no actual sources present.

Many solutions to (2.8) simply correspond to gravitational waves superposed on the black string. This can be seen from the fact that they change little in the limit $r_0 \rightarrow 0$ in which the black string disappears. Such waves are not ‘anchored’ to the string in any way. Other waves will not be of interest as the resulting spacetimes do not have five asymptotically flat directions. However, there is a class of solutions (studied in [20] and [21] for the fundamental string) for which the metric remains asymptotically flat and the waves are anchored to the
string. Keeping spherical symmetry in the asymptotically flat directions, this is the class for which
\[ K(u, x, y) = \frac{p(u)}{r^2} - 2\ddot{f}_i(u)y^i. \] (2.9)

One could also consider the solution \( K = -2\ddot{h}_i(u)x^i \). These waves will be briefly discussed in section \([\text{??}]\) and in the appendix. The dots (‘’) above represent derivatives with respect to \( u \), and writing the waves in this way will simplify the notation below.

The waves described by \( p(u) \) and \( f_i(u) \) are rather different, both in their effect on the metric and in the way we will interpret them in terms of BPS states. As such, it is useful to give them different names. Those described by \( p(u) \) will be called ‘longitudinal waves,’ as they carry only momentum directed along the large \( S^1 \). Those described by \( f_i(u) \) will be called ‘internal waves’ as they carry momentum components in the internal directions.

With \( K \) given by (2.9) the metric appears neither asymptotically flat, nor translationally invariant in the internal four-torus. Both of these difficulties can be resolved by introducing new coordinates \((u, v', x, y')\) which are related to those above through
\[ v' = v + 2\ddot{f}_i y^i + \int^u \dot{f}^2 du \]
\[ y'^i = y^i + f^i \]
(2.10)

where \( \dot{f}^2 = \dot{f}_i\dot{f}^i \). The metric then takes the form
\[ ds^2 = \left(1 + \frac{r_0^2}{r^2}\right)^{-1} [-dudv' + \frac{p(u) + r_0^2\dot{f}^2}{r^2}du^2] + \left(1 + \frac{r_0^2}{r^2}\right)(dr^2 + r^2d\Omega_3^2) \]
\[ -\frac{2r_0^2}{r_0^2 + r^2}\dot{f}_idy'^i du + dy'^i dy'^i. \] (2.11)

It is in terms of these coordinates that we make the periodic identifications. The large \( S^1 \) is defined by the identification \( z \to z - L \), or \((u, v', x, y') \to (u + L, v' - L, x, y')\), and the small four-torus is defined by the identifications \((u, v', x, y') \to (u, v', x, y' + a_I)\) for an appropriate set of vectors \( a_I \), \( I = 1, 2, 3, 4 \). Clearly, \( p(u) \) and \( f_i(u) \) must be periodic with period \( L \). In addition, we will take \( f_i \) itself to be periodic. In the ten dimensional space before compactification, this amounts to choosing coordinates in which the string has no net momentum in the internal directions.

The asymptotic charges can be read directly from the metric (2.11). The black string carries a longitudinal momentum
\[ P = \frac{1}{\kappa^2} \int_0^L (p + r_0^2\dot{f}^2)du \] (2.12)
where \( \kappa^2 \) is given by (2.0). So \((p + r_0^2\dot{f}^2)/\kappa^2 \) is naturally interpreted as a momentum density along the string. Similarly, although the total momentum vanishes in the the internal \((y)\) directions, the oscillations give the string a nontrivial effective internal momentum distribution \( r_0^2\dot{f}_i/\kappa^2 \). Note that, in a weak coupling limit where localized energy-momentum is well defined, a string-shaped matter source with energy density \((2r_0^2 + p + r_0^2\dot{f}^2)/\kappa^2 \), momentum density \((p + r_0^2\dot{f}^2)/\kappa^2 \) in the string direction, and momentum density \((r_0^2\dot{f}_i)/\kappa^2 \) in the internal directions will produce a gravitational field with the same asymptotic behavior as (2.7).
As a result, (2.7) is the black string one would expect to obtain by taking a collection of D-strings and D-fivebranes with a fixed profile for the momentum components and turning up the coupling.

There is an important distinction between the black string with traveling waves described above and a fundamental string with traveling waves. It was shown in [21] that in order for the traveling wave (2.9) to match onto a fundamental string source, one must set \( p(u) = 0 \). The source string then carries a wave with profile \( f_i(u) \) (note that this fixes the normalization of \( f \)). Physically, one should expect \( p(u) \) to vanish since it represents the longitudinal momentum, and a fundamental string has only transverse degrees of freedom. More precisely, \( p(u) \) is the eigenvalue of the worldsheet stress energy operator \( T_{uu} \) which vanishes by reparametrization invariance. In [12], it was suggested that there is an analog of the corresponding level matching condition \( L_0 - \bar{L}_0 = 0 \) even for the black string. We find no such restriction here. We will see that the horizon is well defined for all waves (2.9).

**B. Longitudinal Waves Near the Horizon**

We begin our study of the horizon by setting \( f_i = 0 \) and considering the case of purely longitudinal waves. The effects of internal waves will be considered in section II C. The metric is thus (2.7) with \( K = p(u)/r^2 \). We wish in particular to compute the horizon area for comparison with a counting of weakly coupled string states. One might be tempted to compute the horizon area by setting \( t \) equal to a constant (where \( u = t - z \), \( v = t + z \)) and \( r = 0 \), to obtain the area \( 2\pi^2 r_0^2 \int_0^L \sqrt{p} \); however we will see that this is in general not the correct result. The above reasoning fails due to the fact that the constant \( t \) surfaces do not intersect the horizon. However, the exact expression will reduce to this result in an interesting limit.

For simplicity, we work here with only the leading order behavior of (2.7) near the horizon; a more complete analysis will be given in the appendix. While understanding the leading order behavior is not always sufficient, it does give the correct horizon area in this case and is much simpler than the full treatment. Near the horizon, the metric takes the form

\[
ds^2 = r_0^2 \left\{ R^{-2}( -dudv + dR^2 ) + \frac{p(u)}{r_0^4} du^2 + d\Omega_3^2 \right\} + dy_i dy^j
\]  

where \( R = r_0^2/r \) and we have neglected subleading terms in \( 1/R \). It is useful to realize that the (future) horizon lies not only at \( R = \infty \), but also at \( u = t - z \) while \( v = t + z \). More precisely, it may be shown that all geodesics (timelike, null, and spacelike) which reach \( R = \infty \) do so only as \( u \) and \( v \) diverge.

To study the horizon, let us suppose that there exist functions \( \sigma(u), G(u), \) and \( F(u) \) satisfying \( r_0^{-4} p = \sigma^2 + \dot{\sigma}, \sigma = \partial_u \log G, \) and \( G = (\dot{F})^{-1/2} \), where again the dot (\( \dot{\cdot} \)) denotes \( \partial_u \). Introducing new coordinates \( U = F(u) \), \( V = v - \sigma R^2 \), and \( W = G(u)/R \), the metric (2.13) takes the form

\[
ds^2 = r_0^2 \left\{ -W^2 dUdV + W^{-2} dW^2 + d\Omega_3^2 \right\} + dy_i dy^j
\]  

\[2.14\]
which is just three dimensional anti-deSitter space cross $S^3 \times T^4$. Note that the local geometry is completely independent of the wave profile $p(u)$.

In order for expression (2.14) to be valid, we must show that the functions $\sigma, G$, and $F$ in fact exist. Consider first the equation $r_0^{-4}p(u) = \sigma^2 + \dot{\sigma}$. Recall that we consider periodic $p$ with $p \geq 0$. Suppose that we try to solve this equation subject to the boundary condition $\sigma(0) = 0$. As long as $\sigma^2 < r_0^{-4}p$, we see that $\dot{\sigma} > 0$ and $\sigma$ is increasing. Similarly, when $\sigma^2 > r_0^{-4}p$, we have $\dot{\sigma} < 0$. Large departures of $\sigma$ from $r_0^{-4}p$ decay as $1/u$. As a result, the function $\sigma$ simply follows $r_0^{-2}\sqrt{p}$ and stays within a bounded region as $u$ ranges over the positive real line. A smooth solution $\sigma(u)$ exists for all $u$ and it is reasonable to assume that this solution asymptotically approaches a periodic function $\sigma_0(u) \geq 0$ which also solves the differential equation. From now on we will work with the periodic solution $\sigma(u) = \sigma_0(u)$. Independent of this assumption, one can always start with any periodic $\sigma(u)$, and consider the wave given by $p(u) = r_0^2(\sigma^2 + \dot{\sigma})$.

Since $\sigma \geq 0$ is continuous and $G = \exp \int \sigma \, du$, $G$ also exists and is positive for all $u$. In addition, $G$ diverges exponentially as $u \to \infty$. It follows that $F = \int G^{-2} \, du$ in fact converges so that the horizon is located at a finite value of $U$ which we will take to be $U = 0$. The set $(V, w = r_0 \log W, y)$ together with the angles $\theta$ on the 3-sphere define good coordinates on this surface and the metric on the horizon is just

$$
\mathrm{ds}_U^2 = dw^2 + r_0^2d\Omega_3^2 + dy_i dy^i.
$$

(2.15)

The area of the horizon depends on the range of $w$ which is determined as follows. The periodic identification $(u, v, x, y) \to (u + L, v - L, x, y)$ of the large $S^4$ induces the identifications $(u, v, R, \theta, y) \to (u + L, v - L, R, \theta, y)$ and $(U, V, w, \theta, y) \to (U + L, V - L, w + r_0 L \sigma, \theta, y)$ where $\sigma = L^{-1}\int_u^{u + L} \sigma(u') \, du'$ and $G^{-2}(u) = L^{-1}\int_u^{u + L} G^{-2}(u') \, du'$. Since $\sigma(u)$ is periodic, $\sigma$ is independent of $u$. Thus $w$ has period $r_0 L \sigma$, and the horizon of (2.13) is a smooth surface with area

$$
A = 2\pi^2 r_0^4 L V \sigma.
$$

(2.16)

Note that since $\sigma$ is periodic the total momentum is $P = \kappa^2 r_0^4 \int_0^L \sigma^2 \, du$. Also, by the Schwarz inequality we have

$$
L \sigma = \int_0^L \sigma \, du \leq \left( L \int_0^L \sigma^2 \, du \right)^{1/2}
$$

(2.17)

with equality when $\sigma$ is constant. This means that for a fixed total momentum $P$ the horizon area is maximized by the uniform distribution $\sigma = r_0^{-2} \sqrt{p} = \text{constant}$.

It is of interest to consider a limit in which $p$ is, in some sense, slowly varying. Note that when $\sigma^2 \gg \dot{\sigma}$, we have $\sigma = r_0^{-2} \sqrt{p}$ to leading order. Thus, the proper condition for $p$ to vary slowly is

$$
p^{3/2} \gg r_0^2 |\dot{p}|.
$$

(2.18)

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3Metrics with different longitudinal waves $p_1 \neq p_2$ are not diffeomorphic, but the difference appears only in the departure from the asymptotic form (2.13).
In this case, the area of the horizon is given by the naive result
\[ A = 2\pi^2 r_0^2 V \int_0^L \sqrt{p} du, \]
and the Bekenstein-Hawking entropy is
\[ S_{BH} = \frac{A}{4G_{10}} = \sqrt{2\pi Q_1 Q_5} \int_0^L \sqrt{p/\kappa^2} du. \]  

(2.19)

In particular, if \( p \) is constant, \( p/\kappa^2 = P/L \equiv 2\pi N/L^2 \), the Bekenstein-Hawking entropy reduces to the familiar form \( S_{BH} = 2\pi/\sqrt{Q_1 Q_5 N} \). We will mostly consider the case when (2.18) is satisfied.

Note that the horizon (at \( u = \infty \)) for the metric (2.13) is a smooth surface despite the fact that the periodic profile function \( p \) has no well-defined value for large \( u \). This is because, in the absence of identifications, the function \( p \) is pure gauge in the spacetime (2.13). It enters only in determining the periodic identification along the black string. For the full metric (2.17), the function \( p \) is again locally pure gauge near the horizon even though it carries physical information (e.g. the momentum distribution) far from the horizon. It is shown in the appendix that this spacetime is at least \( C^0 \) near the horizon and that the local horizon geometry is independent of \( p \).

As a final comment, note that since \( U = F(u) \) is an outgoing null coordinate and the horizon lies at \( U = 0 \), the disturbances near the horizon are not traveling along the horizon but are in fact following outward directed null geodesics. That these disturbances appear to be traveling along the horizon when described through the metric (2.13) is a result of the fact that angles are not Lorentz invariant and that natural coordinates near infinity differ from natural coordinates near the horizon by a divergent Lorentz boost. This results in an extreme version of the ‘headlight effect’ (familiar from radiation produced by a rapidly moving charge) in which radiation distributed across a large angle in one frame is in fact distributed across a small angle in another.

C. Internal Waves Near the Horizon

It is not difficult to extend this discussion to include internal waves \( f_i \) as well. Since the term involving \( p(u) \) clearly dominates the one involving \( f_i \) in (2.3) near the horizon \( r = 0 \), one might expect that the structure of the horizon is independent of the internal waves. We will see that this is indeed the case.

After shifting the origin of \( v \) in (2.4) by \( 2\dot{f}_i y^i \), we see that adding the internal waves corresponds to simply adding the term \( 2(1 + \frac{r_0}{L})^{-1} \dot{f}_i dy^i du \) to the leading order metric (2.13) near the horizon; again, we take \( f_i(u) \) to be periodic. In terms of the coordinates \((U, \bar{V}, W, \theta, \bar{y})\) this is just
\[ 2r_0^2 W^2 \dot{f}_i dy^i dU \]

(2.20)
to leading order in \( 1/R \). The term (2.20) is not smooth on the horizon since the periodic function \( \dot{f}_i \) is not well-defined at \( U = 0 \) \((u = \infty)\). However, the metric can be written in \( C^0 \) form (which is smooth enough for our purposes) by a further transformation to coordinates \( (U, \bar{V}, W, \theta, \bar{y}) \) where
\[ \bar{V} = V - W^2 r_0^2 \int_0^U \dot{f}_i dU \]
\[
\tilde{y}^i = y^i + r_0^2 W^2 \int^u \dot{f}^i du.
\] (2.21)

Note that \( f^U \dot{f}_i du = f^u \dot{f}_i G^{-2-du} \) converges as \( u \to \infty \) \((U \to 0)\) and so is \( C^0 \) at the horizon. The same is true of \( f^U \dot{f}^2 du \). The actual form of this metric is not particularly enlightening.

In achieving the \( C^0 \) form it was not necessary to replace the coordinate \( U \). As a result, the horizon still lies at \( U = 0 \). Furthermore, since \((2.20)\) is proportional to \( du \), it does not change the local geometry of any \( U = constant \) surface where it is well defined. Since the exact metric is \( C^0 \) in \( U \) at the horizon \( U = 0 \), it follows that the metric on the horizon is again given by

\[
ds^2 = dw^2 + r_0^2 d\Omega_3^2 + dy_i dy^i
\] (2.22)

and, in particular, it is still homogeneous. As before, this is due to the fact the that waves do not propagate along the horizon, but instead become purely outgoing near \( U = 0 \).

In addition, because the functions \( \tilde{f}_i \) are periodic, it may be verified that the internal waves effect the coordinate identifications on the horizon only through the coordinate \( V \); the identifications of the coordinates \((w, \theta, y)\) remain unchanged. However, \( \frac{\partial}{\partial V} \) is the null generator of the horizon and \( V \) plays no role in determining the area. As a result, the horizon area is completely independent of the internal wave profile \( f_i \).

### III. COUNTING STATES

Let us first recall the counting of string states for the black string without traveling waves i.e. constant \( p \) and \( f_i = 0 \). The idea is to go to weak string coupling where the RR charges are carried by D-branes. One starts with a ten dimensional flat spacetime with five directions compactified on a torus for which one circle of length \( L \) is much larger than the rest. The black string of the previous section corresponds to bound states of \( Q_5 \) fivebranes and \( Q_1 \) onebranes wrapped around the torus. Since one compact direction is much larger than the rest, the excitations of these D-branes are described by a 1 + 1 dimensional sigma model. Using the usual D-brane technology, one finds \( 4Q_1 Q_5 \) bosonic fields and an equal number of fermionic fields on the circle. (For the case \( Q_5 = 1 \), one can view this as arising from the oscillations of the \( Q_1 \) strings in the four internal directions.) The number of BPS states with total momentum \( P = 2\pi N/L \) is then \( e^S \) where \( S = 2\pi \sqrt{Q_1 Q_5 N} \) in agreement with the Bekenstein-Hawking entropy for the case of no traveling waves.

#### A. Longitudinal Waves

The weak coupling limit of the black string with longitudinal waves corresponds to states where the momentum distribution is fixed, not just the total momentum. Quantum mechanically, however, one cannot fix the exact density \( p(u) \) of momentum in a 1+1 field theory. The reason is simply that \( p(u) \) and \( p(u') \) do not commute; the Fourier modes of \( p \) satisfy the Virasoro algebra. We therefore take a ‘mesoscopic’ viewpoint of our fields. That is, we will imagine we use apparatus which can resolve the system only down to a ‘mesoscopic’ length scale \( l \) which is much larger than the ‘microscopic’ length scale (discussed below) on which...
quantum effects are relevant. We will therefore divide the spacetime into $M = L/l$ intervals $\Delta_a (a \in \{1, \ldots, M\})$ of length $l \ll L$. If our instruments find a momentum distribution $p(u)$, this means simply that the total momentum in the interval $\Delta_a$ is $P_a = \kappa^{-2} \int_{\Delta_a} p(u) du$; we cannot resolve $p(u)$ on smaller scales. Of course, it would be meaningless for us to assign a distribution $p(u)$ which has structure on scales of size $l$ or smaller. As a result, $l$ should be much smaller than $p/|\dot{p}|$, the ‘macroscopic’ length scale set by the variation of the wave profile $p$.

We can expect a state counting argument to yield the Bekenstein-Hawking entropy only when such a mesoscopic length scale actually exists. This will restrict us to a certain class of momentum distributions $p(u)$. We use the following heuristic argument to motivate the appropriate condition on $p$; it will be shown below that the resulting condition is in fact sufficient for the existence of our mesoscopic picture.

If quantum correlations between the intervals are to be irrelevant, the wavelength $\lambda$ of a typical excited mode should be much less than $l$. In the interval $\Delta_a$, each field carries a momentum of order $pl/Q_1Q_5\kappa^2$ so we expect $\lambda \sim Q_1Q_5\kappa^2/pl$. As a result, we require

$$p/|\dot{p}| \gg l \gg \sqrt{Q_1Q_5\kappa^2/p}.$$ (3.1)

Such an $l$ can exist only when

$$p^{3/2} \gg |\dot{p}|\sqrt{Q_1Q_5\kappa^2} = \sqrt{2\pi r_0^2|\dot{p}|},$$ (3.2)

so we consider only momentum distributions such that (3.2) holds. This is just the condition (2.18) that $p$ be ‘slowly varying’, under which the Bekenstein-Hawking entropy is given by (2.19). This argument is heuristic since it does not rule out the possibility that the momentum arises from a large number of quanta each with momentum of order $1/l$; we will show below that this does not occur. On the other hand, the above argument does show that whenever $p$ satisfies (2.18), one can choose $l$ so that $p$ is approximately constant on $l$ and the level number $N \sim pl^2/Q_1Q_5\kappa^2$ of a typical field is large in each interval.

If one could view states on the circle as consisting of a collection of $M$ independent systems of length $l$, then the number of states with momentum distribution $P_1, \ldots, P_M$ would be

$$S = 2\pi \sqrt{Q_1Q_5} \sum_{a=1}^M \sqrt{N_a}.$$ (3.3)

But $\sum \sqrt{2\pi N_a} = \int_0^L \sqrt{p/\kappa^2}$, so this agrees with the Bekenstein-Hawking entropy of the black string (2.19). We now justify our assumption that each interval acts like an independent system when $l$ satisfies (3.1).

To begin, consider a single scalar field $\chi$ on a circle of length $L$ parameterized by a (real) coordinate $z$. We want to count states with momentum distribution $p(z)$ satisfying (2.18). Let $P = \kappa^{-2} \int_0^L p$ be the total momentum and let $\Delta_a$ be the intervals introduced above. We now introduce a series of smooth cut-off functions $g_a(z)$ which will project operators into
each segment of our circle. These functions satisfy the following two properties: (1) \( g_a \) has support only in \( \Delta_a \) and (2) \( g_a = 1 \) except for a distance \( \epsilon \) from each endpoint where \( \epsilon < 1/P \).

The bound on \( \epsilon \) comes from the fact that any state with support in the region where \( g_a \) is not constant has momentum larger than the total momentum we are considering. Hence it will not contribute.

The tension of the D-strings is \( 1/g \) times the usual string tension \( (1/2\pi \text{ in units with } \alpha' = 1) \), so the action for \( \chi \) is

\[
S = -\frac{1}{4\pi g} \int (\partial \chi)^2. \tag{3.4}
\]

Since the field on the entire circle has independent left and right moving modes, it follows that in each segment there will also be independent left and right moving modes. Because we wish to count BPS states, we will consider only, say, the right moving modes, so \( \chi = \chi(u) \) with \( u = t - z \). The local momentum density is given by the stress energy tensor \( T_{uu} = (\partial_u \chi)^2/2\pi g \). The total momentum in segment \( \Delta_a \) is thus given by

\[
P_a = \int_{\Delta_a} dz \ g_a \ T_{uu}. \tag{3.5}
\]

We also define modes localized in each segment by

\[
\alpha_{a,n} = \frac{1}{\pi \sqrt{2g}} \int_{\Delta_a} dz \ g_a \ e^{i2\pi nu/l} \partial_u \chi, \tag{3.6}
\]

so that the field on \( \Delta_a \) can be represented in the form

\[
\partial_u \chi = \frac{\pi \sqrt{2g}}{l} \sum_{n=-\infty}^{\infty} \alpha_{a,n} e^{-i2\pi nu/l} + O(\epsilon). \tag{3.7}
\]

Using the equal time commutation relations \( [\partial_u \chi(z), \partial_u \chi(\tilde{z})] = (i\pi g) \delta' (\tilde{z} - z) \), one can show that \( [\alpha_{a,m}, \alpha_{b,n}] = m\delta_{a,b}\delta_{-m,n} + O(\epsilon) \). In other words, the modes in different intervals commute, and within each interval they satisfy the usual commutation relations up to small correction terms.

One can define a vacuum state in each interval \( \Delta_a \) by requiring that it be annihilated by all modes \( \alpha_{a,n} \) with positive mode number \( n \). Together with (3.7), the commutation relations imply that for negative \( n \) (and to leading order in \( \epsilon \)), the operators \( \alpha_{a,n} \) create states localized in region \( \Delta_a \) with momentum \( 2\pi n/l \). It also follows that

\[
P_a = \frac{2\pi}{l} \left( \frac{\alpha_{a,0}^2}{2} + \sum_{n=1}^{\infty} \alpha_{a,-n}\alpha_{a,n} \right) + O(\epsilon). \tag{3.8}
\]

This is just the same as the relation satisfied by the total momentum and the usual modes defined on the entire circle. (Recall that for the entire string \( P \) is the Virasoro operator \( L_0 \).) It then follows that the number of states in the interval \( \Delta_a \) with momentum \( P_a = 2\pi N_a/l \) is given by \( e^{S_a} \) where \( S_a = 2\pi \sqrt{N_a}/6 \) in the limit of large \( N_a \).

The above argument counts the number of states relative to the vacuum state in each interval. To complete the counting we need to relate these local vacua to the usual global
vacuum state on the circle. Note that the local modes \( \alpha_{a,n} \) are related to the global modes \( \alpha_n \) (defined on the entire circle of length \( L \)) through

\[
\alpha_{a,n} = \frac{1}{L} \sum_m \alpha_m \int_0^L g_a e^{i2\pi nu/L} e^{-i2\pi mu/L}.
\]

(3.9)

The vacuum defined on the interval \( \Delta_a \) is the state annihilated by \( \alpha_{a,n} \) for \( n > 0 \) while the actual vacuum of the system is annihilated by \( \alpha_n \) for \( n > 0 \). It can be shown from (3.9) that the contribution of the creation operators \( \alpha_m, m < 0 \) to the annihilation operators \( \alpha_{a,n}, n > 0 \) is \( O(1/n) \), so the vacua are essentially equivalent for the high modes. We can use the equipartition theorem to estimate the typical mode number \( n \): With overwhelming likelihood, the momentum in mode \( n \) should be roughly independent of \( n \) up to some cutoff \( n = \Lambda_a \). Suppose that this momentum per mode is \( \frac{2\pi}{\tau} \tilde{N}_a \). Then for \( n < \Lambda_a \) we have \( n\tilde{N}_{n,a} = \tilde{N}_a \), where \( \tilde{N}_{n,a} \) is the occupation number of the \( n^{th} \) mode of the \( a^{th} \) interval. Since the cutoff is set by the condition that the energy per mode \( \frac{2\pi \tilde{N}_a}{\tau^2} \) is insufficient to excite the next higher mode, we must have \( \Lambda_a \sim \tilde{N}_a \). It follows that the total level number satisfies \( \tilde{N}_a \sim \tilde{N}_a^2 \), and that a typical excited mode has \( n \sim \sqrt{\tilde{N}_a} \). Thus the vacuum for these local modes agrees with the actual vacuum to order \( \tilde{N}_a^{-1/2} \).

Combining these results, we see that for a single scalar field the log of the number of states with momentum distribution \( p(u) \) is given by

\[
S = 2\pi \sum \sqrt{\tilde{N}_a/6}.
\]

For the weak coupling limit of the black string with traveling waves, one has \( 4Q_1Q_5 \) bosonic fields and an equal number of fermionic fields. The entropy is thus given by

\[
S = 2\pi \sqrt{Q_1Q_5} \sum_{a=1}^M \sqrt{\tilde{N}_a} = \sqrt{2\pi Q_1Q_5} \int_0^L \sqrt{p/\kappa^2}
\]

(3.10)

in agreement with the Bekenstein-Hawking entropy (2.19).

Notice that since we have only considered right moving modes, these are all BPS states. The number of such states is independent of the string coupling, so we expect that we can extrapolate to strong coupling where the spacetime geometry is described by the black string with a longitudinal wave. The number of BPS states is also independent of all moduli such as the size of the circle \( L \). Although we assumed that \( L \) was large to facilitate the counting, the final answer holds for all values of \( L \) (provided (2.18) is satisfied).

**B. Internal Waves**

We now consider the weak coupling limit of a black string with internal waves. These waves carry a momentum \( r_0^2\dot{f}/\kappa^2 \) in the internal \( T^4 \), and a momentum \( (p + r_0^2\dot{f}^2)/\kappa^2 \) along

\[4\]This is also a consequence of the equipartition theorem, together with the fact that each fermion contributes as half a boson. If we had \( 6Q_1Q_5 \) independent bosonic fields with total level number \( N \), then it would be overwhelmingly likely that each field has a level number \( N/(6Q_1Q_5) \) and a corresponding entropy \( S_{\text{single field}} = 2\pi \sqrt{N/(6Q_1Q_5)} \). The total entropy is thus \( S = 2\pi \sqrt{Q_1Q_5N} \).
the string. Since there are no internal directions in our 1 + 1 dimensional description of the effective degrees of freedom for the black string, the question arises as to how to represent the internal momentum. There is a natural answer when \( Q_5 = 1 \). In this case the fields in the sigma model represent fluctuations of the \( Q_1 \) onebranes (D-strings) inside the fivebrane. Thus for each D-string and each internal direction, there is a 1+1 bosonic scalar field \( \chi \) representing the displacement of the string in that direction. As a result, the spacetime momentum of the string in an internal direction must be equal to the field momentum \( \dot{\chi}/2\pi g \), as both generate translations of the D-string in the internal direction. This is not to be confused with the momentum associated with translations in the \( z \)-direction; this is the longitudinal momentum \( T_{uu} \) that we have discussed previously. From (2.4) and (2.6) the internal momentum of the black string is

\[
\frac{r_0^2 \dot{f}^i}{\kappa^2} = Q_1 \frac{\dot{f}^i}{2\pi g}, \tag{3.11}
\]

which is indeed just the field momentum of \( Q_1 \) fields with \( f = \chi \). Since the normalizations of \( f \) and \( \chi \) are fixed independently, this agreement is further support for our interpretation of the entropy. The problem is now to count the number of BPS states in the 1 + 1 dimensional sigma model with field momentum constrained to be \( r_0^2 \dot{f}^i/\kappa^2 \) and longitudinal momentum \( (p + r_0^2 \dot{f}^2)/\kappa^2 \). (The field momentum associated with the fermions is unconstrained.) If we ignore the constraint on the field momentum, our previous argument shows that the number of states with longitudinal momentum \( (p + r_0^2 \dot{f}^2)/\kappa^2 \) is much greater than the number with momentum \( p/\kappa^2 \) when \( \dot{f}^i \) is large. We will see that when the constraint is included, the number of states becomes independent of \( \dot{f}^i \) in agreement with the Bekenstein-Hawking entropy.

Let us focus on one component of the internal momentum and drop the index \( i \). We proceed as before by dividing space into intervals \( \Delta_a \) on which \( \dot{f} \) and \( p \) are approximately constant. We argued above that each interval acts like an independent closed string. For each field \( \chi \), the total field momentum in this interval is given by the zero mode

\[
\frac{1}{2\pi g} \int_{\Delta_a} g_a \partial_a \chi = \frac{1}{\sqrt{2g}} \alpha_{a,0}. \tag{3.12}
\]

Since \( \dot{f} \) is assumed constant on the interval \( \Delta_a \), the contribution of the internal wave to the longitudinal momentum is

\[
\frac{r_0^2}{\kappa^2} \int_{\Delta_a} g_a f^2 = \frac{Q_1}{2\pi g} \int_{\Delta_a} g_a \dot{f}^2 = \frac{Q_1 \pi \alpha_{a,0}^2}{l}, \tag{3.13}
\]

which is just \( Q_1 \) times the usual contribution of the zero mode of \( f \) to the longitudinal momentum. Thus if there was only one field (\( Q_1 = 1 \)), the internal momentum \( r_0^2 \dot{f}/\kappa^2 \) would fix the zero mode, and the momentum left to distribute to the nonzero modes would be just \( p/\kappa^2 \), the same as it would be if \( \dot{f} = 0 \). (Previously, the zero mode was unconstrained and, as a result, most of the states had zero modes which were small compared to the total momentum.) Since the state of the zero mode is fixed, it contributes no entropy to the system. The counting of states thus proceeds just as in section \[\text{III A}\] and the entropy is \( 2\pi \sum_{a} \sqrt{N_a/6} \) with \( 2\pi N_a/l = p_a l/\kappa^2 \).
Since there are $Q_1$ different bosonic fields which contribute to the field momentum (in a given direction), only the total zero mode is fixed, not the zero modes of each field. To compute the entropy, we proceed as follows. Let us focus on one interval and drop the subscript ‘$a$’. Suppose the zero mode of the $A^{th}$ string is $\alpha^A_0$. Then from (3.11) and (3.12) our constraint is

$$\sum_A \alpha^A_0 = Q_1 \frac{\dot{f}}{\pi \sqrt{2g}}$$  \hspace{1cm} (3.14)

Let $k^2 = (\pi/l) \sum_A (\alpha^A_0)^2$ be the contribution of these zero modes to the longitudinal momentum $P$. Then for a given $k$, the total entropy consists of two contributions: one from the number of states with longitudinal momentum $P - k^2$, and the other from the number of ways of distributing the zero mode momentum among the strings so that their sum is given by (3.14) and the sum of their squares is $k^2$. The latter contribution can be estimated by considering the allowed volume of phase space. Our two conditions define a plane and a sphere in a $Q_1$ dimensional space. These surfaces do not intersect if $k^2 < k^2_0 \equiv Q_1 \dot{f}^2/2\pi g$. For $k^2 > k^2_0$ they intersect in a sphere of area less than $k^2Q_1$. Note that $k = k_0$ represents the point in phase space where the zero modes are all equal, so the field momentum is distributed evenly among all of the strings. Since the entropy coming from the nonzero modes with longitudinal momentum grows much faster than the entropy coming from the zero modes, one expects that the total entropy will be maximized when $k$ takes its minimum value. Setting $k^2 = k^2_0$ and using the fact that $k^2_0$ is precisely the contribution of the internal wave to the longitudinal momentum (3.13) we see that the entropy will be the same as that of a purely longitudinal momentum distribution $p$ in each interval. Thus the total entropy will be independent of the internal wave, in agreement with the Bekenstein-Hawking entropy.

More precisely, for a general $k$, the entropy is bounded by

$$S \leq \sqrt{2\pi Q_1 l(P - k^2)} + Q_1 \ln k/k_0$$  \hspace{1cm} (3.15)

where the additive constant $-Q_1 \ln k_0$ is chosen so that the zero modes contribute no entropy for $k = k_0$. Using the fact that $Pl >> Q_1$, the maximum of $S$ occurs for $k^2 \approx \sqrt{Q_1 P/2\pi l}$. Recall that $P = (p + r_0^2 \dot{f}^2)/l/\kappa^2$. If $p$ is not much larger than $r_0^2 \dot{f}^2$, the preferred value of $k^2$ is less than the minimum allowed value $k^2_0 \sim P$. So the entropy is maximized by $k^2 = k^2_0$ as expected. If $p$ is much larger than $r_0^2 \dot{f}^2$, the entropy is maximized for $k^2 > k^2_0$, but in this case the internal waves $\dot{f}$ make only a small correction to the entropy. The net result is that in all cases the leading order contribution to the entropy is independent of the internal wave in agreement with the calculations from the black string.

**IV. DISCUSSION**

We have shown that the Bekenstein-Hawking entropy of extremal black strings with traveling waves correctly counts the number of BPS states in weakly coupled string theory with momentum distributions which agree with the black string. This provides a new interpretation of the entropy of the black string without traveling waves: $p$ = constant, $\dot{f}_i = 0$. In previous discussions, the entropy of this black string was reproduced by counting all BPS
states with total momentum $P$. Here we interpret this solution as representing only a subset of these states; namely those for which the distribution of momentum is in fact uniform, at least on a macroscopic scale. There is no contradiction because the vast majority of states with total momentum $P$ do in fact have their momentum evenly distributed. This can be seen from the fact that among all momentum distributions $p(u)$ with total momentum $P$, the entropy (2.19) is maximized for constant $p(u)$.

There are additional solutions describing black strings with traveling waves that we have not yet discussed. For example, we have considered transverse waves in the internal directions, but not transverse waves in the asymptotically flat directions. These are described by $\tilde{K} = -2\tilde{h}_i(u)x^i$ cf (2.3). Black strings with these waves have nonzero momentum components in the four macroscopic directions. In the Appendix it is shown that these waves are similar to the internal waves in that they do not affect the horizon area. This can also be understood by counting string states. Let us again consider the case $Q_5 = 1$. In the weakly coupled D-brane description, there are four bosonic fields describing oscillations of the fivebrane in the noncompact directions. Such fields are not usually included in the counting of the entropy since they add only a small number to the $4Q_1$ fields arising from the oscillations of the D-strings. However, to reproduce the momentum in the macroscopic dimensions, one simply fixes the momentum distribution of this field to the specified value. As for the internal waves, the field momentum carried by the zero mode in each small region will compensate for the extra longitudinal momentum so that the number of states is independent of these transverse waves.

One can also consider solutions where $K$ is quadratic in the transverse coordinates. If $K$ satisfies Laplace’s equation, the matter fields remain unchanged and the solution describes a gravitational plane wave superposed on the black string. Alternatively, one can add plane waves to the matter fields. For example, if one adds an arbitrary function of $u$ to the dilaton $\phi$, the metric (2.7) remains a solution provided $K$ satisfies $\partial^2 K = 4\ddot{\phi}$. The solution for $K$ is again quadratic in the transverse coordinates. Since this is negligible compared to the $p$ term near the horizon, these plane waves should not affect the horizon area either. On the one hand, this is consistent with the fact that, in the absence of the black string, the plane wave solutions have no horizon and no gravitational entropy. On the other hand, it is clear that there is more than one microstate that can reproduce a given classical wave, so there should be some entropy associated with the wave. The resolution is simply that the entropy associated with the classical wave is much smaller than that of the black string. This can be seen as follows. In terms of the weakly coupled description, in addition to the D-brane states, one must now include closed string states to reproduce the macroscopic plane waves. Counting such states is much like our counting of states for a given internal wave on a single D-string (having $p(u) = 0$) in section III B. In each small region $\Delta_a$ (on which the macroscopic wave is roughly constant) all of the energy in the wave must go to the zero mode on $\Delta_a$; there is no extra (macroscopic) energy to be distributed randomly among the modes. As a result, the entropy of a matter wave is the entropy of ‘zero thermal energy’. This depends on the details of one’s measuring apparatus, but is independent of the wave; it is merely an additive constant. The same is true of a passing gravitational wave.

For the black string without traveling waves, it was shown in [3] that the entropy of the slightly nonextremal solution could also be reproduced by counting string states at weak coupling. The difference was simply that one must include left moving modes as well as right
moving modes with their respective momenta equal to the left and right moving momenta of the black string. It was argued that if \( L \) was sufficiently large, the energy of the modes carrying the entropy would be small enough so that interactions remain negligible even when the string coupling is increased to form the horizon. Since the black string with traveling waves corresponds to just a subset of the right moving modes considered previously, one can still add left moving modes (with an arbitrary momentum distribution) in the weak coupling description. What spacetime is produced when the coupling is now increased? This is puzzling since there does not appear to be a nonextremal analogue of a black string with traveling waves. The nonextremal black string does not have a null translational Killing field, so one expects that any wave present in the spacetime will either fall into the black hole or disperse to infinity. A possible explanation is that neither description is really time independent: The slow process of the nearly extremal black string absorbing the traveling wave might correspond to the slow process by which the weak but nonzero interactions between right- and left- movers homogenize the momentum distribution in the weakly coupled string.

In \cite{19} it was shown that the extremal black string (2.3) with no momentum (\( p = 0 \)) and no traveling waves has an unusual global structure. There is an event horizon at \( r = 0 \) but no singularity inside. In fact the region inside the horizon turns out to be another asymptotically flat spacetime which is isometric to the region outside.\(^5\) The addition of any momentum along the string changes this behavior. It is shown in the Appendix that the black string with nonzero momentum has a singularity inside the horizon. This should be expected since the dimensional reduction of one special case of this solution yields the five dimensional Reissner-Nordström metric, which is certainly not symmetric across the horizon.

The four dimensional Reissner-Nordström metric has been shown to arise from the dimensional reduction of a six dimensional black membrane which is similar to the black string discussed here \cite{13}. In particular, near the horizon the solution again reduces to the product of three dimensional anti-de Sitter space and \( S^3 \times T^4 \). The entropy of the translationally invariant solution (corresponding to a uniform momentum density) has been reproduced by counting states at weak string coupling. One can add traveling waves to this solution and we expect that the entropy will again correctly count the number of string states. This is currently under investigation.

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\(^5\)When the momentum is zero, the horizon area of the compactified spacetime vanishes and there is a conical singularity on the horizon resulting from the periodic identification of \( z = (v - u)/2 \).
APPENDIX A: DETAILS OF THE EXACT METRIC

As mentioned in section II, our study of the leading order behavior (2.13) of the black string metric is not really sufficient to show that the horizon is a smooth surface and, as a result, it is also insufficient to show that the area of the horizon is $2\pi^2 r_0^4 L\sqrt{\sigma}$. The point is that while a study of (2.13) shows that the leading order divergences in (2.7) are purely coordinate effects, any sub-leading divergences still remain to be addressed. In this appendix, we consider the exact metric, showing that the horizon is smooth when the momentum distribution is uniform along the string, that it is at least $C^0$ for a general metric of the form (2.7) with $K$ given by (2.3), and that in both cases the area of the horizon is given by $2\pi^2 r_0^4 L\sqrt{\sigma}$. In addition, we comment on the global structure of the homogeneous black string and we extend our treatment to include ‘transverse waves’ in the macroscopic directions (for which $K = -2h_i(u)x^i$), showing that the results stated above hold for these waves as well. From [21], the metric with such transverse waves can be placed in an asymptotically flat form by a coordinate transformation similar to (2.10).

Our discussion will parallel that of section II. We begin by writing the exact metric as

$$ds^2 = r_0^2 \left\{ -R^{-2}[dud\hat{v} - 2\hat{f}_i dy^i du - 2\hat{h}_i dx^i du] + \frac{\rho(u)}{r_0} du^2 + R^{-2}Z^{-3}dR^2 + Z^{-1}d^2\Omega_3 \right\} + dy_idy^i \tag{A1}$$

in terms of the coordinates $u$, $\hat{v} = v + 2\hat{f}_i y^i + 2\hat{h}_i x^i + r_0^{-2} \int^u p\, du$, and $R = r_0\sqrt{1 + \frac{r_0^2}{R^2}}$. Here, $Z(R) = \frac{R^2 - r_0^2}{R^2} = 1 - \frac{r_0^2}{R^2}$ takes the value 1 on the horizon; the departure of $Z$ from this value represents the terms neglected in the leading-order expression (2.13). Note the slight change from section II in the definition of $R$.

We now introduce new coordinates $U = F(u)$, $V = \hat{v} - R^2\sigma(u) - Y(u)$, and $W = G(u)/R$ where, as before, $\hat{F} = G^{-2}$, $\sigma(u) = G^{-1}\dot{G}$, and $\sigma^2 + \dot{\sigma} = r_0^{-4}p$; $Y(u)$ will be specified below. Recall that the horizon $u = \infty$ lies at a finite value of $U$; we again choose this value to be $U = 0$. The metric now becomes

$$ds^2 = r_0^2 \left\{ W^{-2}dUdV + 2W^2 \hat{f}_i dy^i dU + 2W^2\hat{h}_i dx^i dU - G^2[\sigma^2(1 - Z^{-3})G^2 + W^2\dot{Y}]dU^2 + dUdW[2W^{-1}\sigma G^2(1 - Z^{-3})] + W^{-2}Z^{-3}dW^2 + Z^{-1}d\Omega_3^2 \right\} + dy_idy^i. \tag{A2}$$

Since $G$ is roughly exponential in $u$, we see that the $dU^2$ and $dUdW$ terms have several potential divergences at the horizon $u = \infty$. Since $Z = 1 - W^2r_0^2G^{-2}$, $g_{UU}$ is of order $G^0$ while $g_{UU}$ is of order $G^2$. However, by choosing $Y(u)$ to satisfy $dY/du = 3r_0^2\sigma^2$ we can arrange that $g_{UU}$ is also of order $G^0$. For the case where $\hat{f}_i = 0$ and $p$ is independent of $u$, $\sigma = r_0^{-2}\sqrt{p}$ is a constant and the metric (A2) is smooth (and even analytic) at the horizon; the area of the horizon may be computed as in section II and agrees with the result stated there.

We can also see that (for $\hat{f}_i = 0$, $p = constant$) the global structure for $p \neq 0$ is much different from the $p = 0$ case, which has no singularity anywhere in the uncompactified spacetime [21]. In fact, it is symmetric across the horizon; passing through the horizon leads to an asymptotically flat region which is just like the original one. This is not the case for
the solution \( \text{[A2]} \). Note that the radius of the 3-spheres is \( r_0 Z^{-1/2} \). Now, when \( p \) is uniform, \( G = e^{\sqrt{\nu}/r_0^2} \) and \( U = -\frac{r_0^2}{2\sqrt{\nu}} \). As a result, \( Z = 1 + 2\sqrt{\nu}W^2 U \) is less than one outside the horizon \( (U < 0) \), but greater than one inside \( (U > 0) \). Thus, the 3-spheres continue to shrink as the horizon is crossed and a singularity forms when they reach zero size.

Returning to the general case, suppose that \( p \) is periodic in \( u \) but not constant. In this case \( p, \dot{f}, \text{and} \dot{h} \) (and therefore \( \sigma \)) oscillate infinitely many times before reaching the horizon at \( U = 0 \); as a result, the expression \( \text{[A2]} \) is not even continuous at the horizon. Another source of concern is the fact that

\[
\text{for which the} \ dU \ \text{term diverges for} \ U \to 0. \ \text{These are again coordinate effects. To see that this is so, it is simplest to work with the leading order (}\mathcal{O}(G^0)\text{) behavior of} \ \text{[A2]}. \ \text{Since} \ G \ \text{diverges at the horizon, any higher order terms vanish at} \ U = 0 \ \text{and so are already continuous there}^6. \ \text{We therefore write the metric} \ \text{[A2]} \ \text{in the form}
\]

\[
\begin{align*}
\text{(A3)}
\end{align*}
\]

\[
\text{where} \ ds_0^2 \ \text{contains only higher order} \ C^0 \ \text{terms. The metric can now be written in} \ C^0 \ \text{form by performing a final coordinate transformation. It is simplest to break this into a series of steps. For convenience, let} \ \beta_i = f^U \ \dot{f}_i dU. \ \text{Transforming to coordinates} \ (U, \tilde{V}, q, \theta, \tilde{y}) \ \text{where}
\]

\[
\begin{align*}
\tilde{V} &= V + 2W \left( \frac{x^i}{r} \right) \int_0^U \dot{h}_i \hat{G} dU' \\
q &= -\left( \frac{1}{2}W^{-2} + 3r_0^2 \int_0^U \sigma dU \right) \\
\tilde{y}^i &= y^i + r_0^2 W^2 \beta^i
\end{align*}
\]

\[
\text{(A5)}
\]

\[
\text{results in the metric}
\]

\[
\begin{align*}
ds^2 &= ds_0^2 - r_0^2 W^2 dU d\tilde{V} - r_0^2 W^4 (\dot{f}^2 + 3r_0^2 \sigma^2) dU^2 + r_0^2 d\Omega_3^2 + \\
&\quad + r_0^2 W^4 dq^2 + (d\tilde{y}^i - 2r_0^2 W^2 \beta^i dq - 6r_0^4 W^4 \beta^i \sigma dU)^2.
\end{align*}
\]

\[
\text{(A6)}
\]

\[
\text{Since} \ W \ \text{is a} \ C^0 \ \text{function of the new coordinates, a further transformation to coordinates} \ (U, \nu, q, \theta, z) \ \text{where}
\]

\[
\nu = \tilde{V} + \int_0^U r_0^4 W^4 (\dot{f}^2 + 3r_0^2 \sigma^2) dU'
\]

\[
\text{Moreover, they remain continuous under the coordinate transformations performed below.}
\]
\[ z^i = \tilde{y}^i - \int^U 6r_0^4 W^4 \sigma \beta^i dU' \]  

places the metric in \( C^0 \) form. In (A7), the integrals are performed at fixed \( q \) (not fixed \( W \)). Again the final form of the metric is not particularly enlightening. Whether or not the metric is in fact smooth at the horizon is a question which deserves further study.

Note that we have not replaced the coordinate \( U \) in obtaining the \( C^0 \) form. As a result, the horizon does indeed lie at \( U = 0 \) and the metric on the horizon can be read directly from (A2); it is just \( r_0^2 W^{-2} dW^2 + r_0^2 d\Omega_3^2 + dy_i dy^i \). It follows that the horizon is homogeneous and that it has area \( 2\pi^2 r_0^4 L V \sigma \) as claimed in section [I].
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