CACTI AND CELLS

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Abstract. The goal of this paper is to construct an action of the cactus group of a Weyl group \( W \) on \( W \) that is nicely compatible with Kazhdan-Lusztig cells. The action is realized by the wall-crossing bijections that are combinatorial shadows of wall-crossing functors on the category \( O \).

1. Introduction

1.1. Cells. Let \( G \) be an adjoint semisimple algebraic group over \( \mathbb{C} \), \( \mathfrak{g} \) be its Lie algebra and \( W \) be the Weyl group. Introduce an independent variable \( v \) and consider the (equal-parameter) Hecke algebra \( \mathcal{H}_v \). Recall that it is defined as follows. It has a basis \( T_w \) with \( w \in W \) and the multiplication is recovered from

\[
T_s T_w = \begin{cases} 
T_{sw}, & \text{if } \ell(sw) > \ell(w), \\
(v - v^{-1})T_w + T_{sw}, & \text{else}.
\end{cases}
\]

The algebra \( \mathcal{H}_v \) admits another remarkable basis, the Kazhdan-Lusztig basis \( C_w, w \in W \). Using this basis one can introduce the left, right and two-sided pre-orders \( \preceq_L, \preceq_R, \preceq_{LR} \). For example, we write \( w \preceq_L w' \) if \( C_w \) belongs to the based left ideal generated by \( C_{w'} \). Equivalence classes for these pre-orders are called left, right and two-sided cells. The cells are of great importance for the representation theory of the universal enveloping algebra \( U(\mathfrak{g}) \) and also for that of the forms \( G(F_q) \) of \( G \) over the finite fields.

To cells one can assign different invariants. For example, to a two-sided cell \( c \) Lusztig assigned a finite group \( \overline{A}_c \), see [Lu1, Section 13.1]. To every left cell \( \sigma \subset c \) he assigned a subgroup \( H_\sigma \subset \overline{A}_c \), see [Lu2], defined up to conjugacy.

To finish our discussion of cells let us explain what happens when \( G = \text{PGL}_n(\mathbb{C}) \). Here the two-sided cells are in one-to-one correspondence with the partitions \( \lambda \) of \( n \). The left cells inside of the two-sided cell corresponding to \( \lambda \) are in one-to-one correspondence with the standard Young tableaux of shape \( \lambda \). The left and right cells containing \( w \in W = \mathfrak{S}_n \) are determined from the RSK correspondence that maps \( w \) to a pair of standard Young tableaux of the same shape. These Young tableaux parameterize the left and the right cells containing \( w \). The Lusztig group \( \overline{A}_c \) is trivial for any two-sided cell \( c \).

1.2. Cacti. Now let us recall the cactus group \( \text{Cact}_W \). Let \( D \) denote the Dynkin diagram of \( W \). The group \( \text{Cact}_W \) is generated by elements \( \tau_{D_1} \), where \( D_1 \subset D \) runs over all

MSC 2010: 05E10, 17B35, 18E30.
connected subdiagrams of $D$. The relations are as follows

$$\tau_{D_1}^2 = 1,$$

(1.1) $$\tau_{D_1} \tau_{D_2} = \tau_{D_2} \tau_{D_1}, \quad \text{if } D_1 \cup D_2 \text{ is disconnected},$$

$$\tau_{D_1} \tau_{D_2} = \tau_{D_2} \tau_{D_1}^*, \quad \text{if } D_1 \subset D_2,$$

where $D_1^*$ is obtained from $D_1$ by the involution of $D_2$ induced by the longest element $w_{D_2}$ of the parabolic subgroup $W_{D_2} \subset W$ corresponding to $D_2$.

Below we will often write $\text{Cact}_D$ instead of $\text{Cact}_W$.

According to [DJS, Theorem 4.7.2], the group $\text{Cact}_W$ is the orbifold fundamental group of the real locus in the stack $\mathcal{P}^{\text{reg}}_h / W$, where $\mathcal{P}^{\text{reg}}_h$ is the wonderful compactification of $\mathfrak{h}^{\text{reg}} := \mathfrak{h}^{\text{reg}} / \mathbb{C}^\times$, where $\mathfrak{h}^{\text{reg}}$ is the regular locus in the reflection representation $\mathfrak{h}$ of $W$.

The cactus group $\text{Cact}_W$ should be thought as a “crystal limit” of the braid group $\text{Br}_W$ of $W$ as justified, for example, by [HK]. Note that, similarly to the braid group, $\text{Cact}_W$ admits an epimorphism onto $W$ given by $\tau_{D_1} \mapsto w_{D_1}$.

1.3. Main result and motivations. Here is the main result of the present paper.

**Theorem 1.1.** There is an action of $\text{Cact}_W \times \text{Cact}_W$ on $W$ having the following properties.

(i) The first copy of $\text{Cact}_W$ preserves right cells and permutes left cells preserving the Lusztig subgroups.

(ii) The map $w \mapsto w^{-1}$ switches the actions of the first and the second copy.

(iii) Let $w \in W$ decompose as $w = w' w''$, where $w'' \in W_{D_1}$ and $w'$ is shortest in its right $W_{D_1}$-coset. Then $\tau_{D}(w' w'') = w' \tau_{D_1}^{\tau_{D_1}}(w'')$, where $\tau_{D_1}^{\tau_{D_1}}$ is the element corresponding to $D_1$ in $\text{Cact}_{D_1}$.

(iv) In type $A$, the orbits of the first copy of $\text{Cact}_W$ are precisely the right cells.

Note that outside type $A$ we always have two left cells inside of a single two-sided cell with different Lusztig subgroups. So (i) implies that an analog of (iv) fails outside of type $A$.

For us, there are two motivations for this theorem. One comes from a new (conjectural) approach to cells due to Bonnafe and Rouquier, [BR]. In that approach, the (left) cells are defined as orbits of a suitable Galois group action on $W$. It is expected that $\text{Cact}_W$ admits a homomorphism into that Galois group and so should act on $W$ preserving the left cells. Currently, it is a conjecture that the Bonnafe-Rouquier construction of cells is equivalent to the original one.

The other motivation comes from the work of the author and Bezrukavnikov, [BL]. A principle stated in Section 9 of that paper says that given a suitable braid group action on a category (or more generally, a “braid groupoid” action on a collection of categories) one should be able to take a crystal limit of this action and get an action of the corresponding cactus group(oid). A technical tool for this is to show that the functors corresponding to longest elements in parabolic subgroups are perverse equivalences. This is an approach that we use in the present paper.

1.4. Structure of the paper. In Section 2 we will recall some generalities on highest weight categories and on perverse equivalences. Section 3 deals with various facts from the representation theory of universal enveloping algebras that we need. There we review various versions of the category $\mathcal{O}$, the wall-crossing functors, and $W$-algebras. In Section 4 we prove Theorem 1.1. We first establish the perversity of wall-crossing functors corresponding to longest elements in the standard parabolic subgroups. This allows us to
define the bijections that constitute the action of the first copy of \( \text{Cact}_W \). Then we prove (iii) of Theorem 1.1. Next, we check the cactus relations. After that we prove (i) and (ii) of Theorem 1.1. We finish by proving (iv). Finally, in Section 5 we briefly describe several potential ramifications of our construction.

Acknowledgements. I would like to thank Roman Bezrukavnikov, Ben Elias, Pavel Etingof, Iain Gordon, Joel Kamnitzer, Victor Ostrik, Raphael Rouquier and Noah White for stimulating discussions. This work was partially supported by the NSF under grants DMS-1161584, DMS-1501558.

2. Preliminaries on categories and functors

2.1. Highest weight categories. Let \( \mathcal{C} \) be an abelian category equivalent to the category of finite dimensional modules over a finite dimensional algebra over \( \mathbb{C} \). Let us write \( \mathcal{T} \) for the set of irreducible objects in \( \mathcal{C} \). For \( \tau \in \mathcal{T} \), we write \( L(\tau) \) for the corresponding simple object and \( P(\tau) \) for its projective cover.

An additional structure of a highest weight category on \( \mathcal{C} \) is a partial order \( \leq \) on \( \mathcal{T} \) subject to a condition explained below. For \( \tau \in \mathcal{T} \), let \( \mathcal{C}_{\leq \tau} \) denote the Serre span of the simple objects \( L(\tau') \) with \( \tau' \leq \tau \). We write \( \Delta(\tau) \) for the projective cover of \( L(\tau) \) in \( \mathcal{C}_{\leq \tau} \). The objects \( \Delta(\tau), \tau \in \mathcal{T} \), are called standard. The condition on \( \leq \) is that the kernel of the natural epimorphism \( P(\tau) \to \Delta(\tau) \) is filtered by \( \Delta(\tau') \) with \( \tau' > \tau \).

The category \( \mathcal{C}^{opp} \) is highest weight with the same order on \( \mathcal{T} \). The standard objects for \( \mathcal{C}^{opp} \) are denoted by \( \nabla(\tau) \) are called costandard. An object of \( \mathcal{C}^{opp} \) is called tilting if it is both standardly filtered and costandardly filtered. Indecomposable tilting objects are labelled by \( \mathcal{T} \): there is a unique indecomposable tilting \( T(\tau) \) that admits a monomorphism from \( \Delta(\tau) \) such that the cokernel is filtered by \( \Delta(\tau') \) with \( \tau' < \tau \).

Now let \( \mathcal{T}' \) be a poset ideal of \( \mathcal{T} \). Let \( \mathcal{C}_{\mathcal{T}' \mathcal{T}} \) denote the Serre span of \( L(\tau), \tau \in \mathcal{T}' \). This is a highest weight category with respect to the restriction of \( \leq \) to \( \mathcal{T}' \) with standard objects \( \Delta(\tau) \) and costandard objects \( \nabla(\tau) \), where \( \tau \in \mathcal{T}' \). The quotient \( \mathcal{C}/\mathcal{C}_{\mathcal{T}'} \) is a highest weight category with respect to the restriction of \( \leq \) to \( \mathcal{T} \setminus \mathcal{T}' \). The natural functor \( D^b(\mathcal{C}_{\mathcal{T}'}) \to D^b(\mathcal{C}) \) is a full embedding. Further, the quotient \( D^b(\mathcal{C})/D^b(\mathcal{C}_{\mathcal{T}'}) \) coincides with \( D^b(\mathcal{C}/\mathcal{C}_{\mathcal{T}'}) \).

Let us finish by recalling the Ringel duality. Set \( T := \bigoplus_{\tau \in \mathcal{T}} T(\tau) \). Consider the category \( \mathcal{C}' := \text{End}_{\mathcal{C}}(T)^{opp \text{-mod}} \). Set \( \Delta'(\tau) := \text{Hom}(T, \nabla(\tau)) \). Then \( \mathcal{C}' \) is a highest weight category with respect to the poset \( \mathcal{T}^{opp} \). It is called the Ringel dual of \( \mathcal{C} \). The functor \( \mathcal{R} := R\text{Hom}(T, \bullet) \) is a derived equivalence \( \mathcal{R} : D^b(\mathcal{C}) \xrightarrow{\sim} D^b(\mathcal{C}') \) to be called the Ringel duality functor.

2.2. Perverse equivalences.

2.2.1. Definition and combinatorial data. Let \( \mathcal{T}^1, \mathcal{T}^2 \) be triangulated categories equipped with \( t \)-structures. Let \( \mathcal{C}^1, \mathcal{C}^2 \) denote the hearts of \( \mathcal{T}^1, \mathcal{T}^2 \), respectively. We are going to recall the notion of a perverse equivalence with respect to filtrations \( \mathcal{C}^i = \mathcal{C}^i_0 \supset \mathcal{C}^i_1 \supset \cdots \supset \mathcal{C}^i_k = \{0\} \) by Serre subcategories, see [R, Section 2.6] (there the definition is given for derived categories, but it generalizes to triangulated categories in a straightforward way). By definition, this is a triangulated equivalence \( \mathcal{T}^1 \to \mathcal{T}^2 \) subject to the following conditions:

- (P1) For any \( j \), the equivalence \( \mathcal{F} \) restricts to an equivalence \( \mathcal{T}^1_{\mathcal{C}^1_j} \to \mathcal{T}^2_{\mathcal{C}^2_j} \), where we write \( \mathcal{T}^i, i = 1, 2 \), for the category of all objects in \( \mathcal{T}^i \) with homology in \( \mathcal{C}^i_j \).
(P2) For $M \in C^1_j$, we have $H_\ell(FM) = 0$ for $\ell < j$.

(P3) The functor $M \mapsto H_j(FM)$ induces an equivalence $C^1_j/C^1_{j+1} \xrightarrow{\sim} C^2_j/C^2_{j+1}$ of abelian categories. Moreover, for $M \in C^1_j$ and $\ell > j$, we have $H_\ell(FM) \in C^2_{j+1}$.

To $F$ we assign its combinatorial data: the map $\varphi = (\varphi_b, \varphi_s) : \text{Irr}(C^1) \to \text{Irr}(C^2) \times \mathbb{Z}_{\geq 0}$. To $M \in \text{Irr}(C^1_j/C^1_{j+1}) \subset \text{Irr}(C^1)$ it assigns the pair $(H_j(FM), j)$, where the first component is viewed as an element of $\text{Irr}(C^2_j/C^2_{j+1}) \subset \text{Irr}(C^2)$. Note that $\varphi_b$ is a bijection. The second component of the map is called the homological shift.

Note, in particular, that any t-exact equivalence is perverse, where all homological shifts are 0.

2.2. Basic properties. The following lemma is trivial.

**Lemma 2.1.** Let $F : T^1 \to T^2$ be a perverse equivalence with perversity data $\varphi$. Further, let $G$ be a t-exact self-equivalence of $T^1$ inducing a bijection $\psi$ on $\text{Irr}(C^1)$ and $G'$ be a t-exact self-equivalence of $T^2$ inducing a bijection $\psi'$ on $\text{Irr}(C^2)$. Then the equivalences $F \circ G$ and $G' \circ F$ are perverse with perversity data $(\varphi_b \circ \psi, \varphi_s \circ \psi)$ and $(\psi' \circ \varphi_b, \varphi_s \circ \psi'^{-1})$.

The following important lemma is standard.

**Lemma 2.2.** Let $F, F' : T^1 \to T^2$ be two perverse equivalences with the same combinatorial data. Then there are t-exact self-equivalences $G'$ of $T^2$ inducing the identity map on $\text{Irr}(C^2)$ and $G$ of $T^1$ inducing the identity map on $\text{Irr}(C^1)$ with $F' = G' \circ F = F \circ G$.

**Corollary 2.3.** Let $F$ be a perverse equivalence $T^1 \to T^2$ and let $G$ be a t-exact self-equivalence of $T^1$ giving the identity on $\text{Irr}(C^1)$. Then there are t-exact self-equivalence $G'$ of $T^2$ giving the identity on $\text{Irr}(C^2)$ such that $G' \circ F = F \circ G$. A similar claim holds for $G, G'$ swapped.

**Proof.** Apply Lemma 2.2 to $F$ and $F \circ G$. \hfill $\square$

**Lemma 2.4.** Suppose that $C^1, C^2$ have finitely many irreducible objects. Let $F, F'$ be perverse equivalences $T^1 \to T^2$. If the induced maps $[F], [F'] : K_0(C^1) \to K_0(C^2)$ coincide, then do the bijections $\varphi_b, \varphi'_b : \text{Irr}(C^1) \to \text{Irr}(C^2)$.

**Proof.** Let $C^1 = C^1_0 \supset C^1_1 \supset \ldots \supset C^1_k = \{0\}, C^2 = C^2_0 \supset C^2_1 \supset \ldots \supset C^2_k = \{0\}$ be the filtrations for $F$. We will prove by the descending induction on $i$ that, for $L \in \text{Irr}(C^1_i)$, we have $\varphi_b(L) = \varphi'_b(L)$. The case $i = k$ is vacuous. Now suppose that we know that $\varphi_b(L_1) = \varphi'_b(L_1)$ for any $L_1 \in \text{Irr}(C^1_{i-1})$. Let $L \in \text{Irr}(C^1_i) \setminus \text{Irr}(C^1_{i+1})$. Then

$$[FL] = \pm[\varphi_b(L)] + \sum_{L_2 \in \text{Irr}(C^2_{i+1})} a_{L_2}[L_2]$$

with $a_{L_2} \in \mathbb{Z}$. On the other hand, $[\varphi'_b(L)]$ appears with nonzero coefficient in $[F'L]$. Since $[FL] = [F'L]$, we see that either $\varphi'_b(L) = \varphi_b(L)$ or $\varphi'_b(L) \in \text{Irr}(C^2_{i+1})$. But by the inductive assumption, any object $L_2 \in \text{Irr}(C^2_{i+1})$ is of the form $\varphi'_b(L_1)$ for $L_1 \in \text{Irr}(C^1_{i+1})$. We conclude that $\varphi_b(L) = \varphi'_b(L')$ and this finishes the induction step. \hfill $\square$

3. Preliminaries on the representation theory of $U(\mathfrak{g})$

3.1. Category $\mathcal{O}$. 
3.1.1. Definition of $\mathcal{O}$ and a highest weight structure. Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra and let $\mathfrak{b} \subset \mathfrak{g}$ be a Borel subalgebra containing $\mathfrak{h}$. We write $W$ for the Weyl group of $\mathfrak{g}$.

Set $\mathcal{U} = U(\mathfrak{g})$. We identify the center of $\mathcal{U}$ with $S(\mathfrak{h})^W$ by the Harish-Chandra isomorphism. So, for $\lambda \in \mathfrak{h}^*$, we can consider the central reduction $\mathcal{U}_\lambda$. Obviously, $\mathcal{U}_{w\lambda}$ is naturally isomorphic to $\mathcal{U}_\lambda$ for any $\lambda \in \mathfrak{h}^*$ and any $w \in W$.

We consider the category $\mathcal{O}_\lambda$ of all finitely generated $\mathcal{U}_\lambda$-modules with locally finite $\mathfrak{b}$-action. When $\lambda$ is regular (meaning that $\langle \lambda, \alpha^\vee \rangle \neq 0$ for all roots $\alpha$), the category $\mathcal{O}_\lambda$ is highest weight, its standard objects are Verma modules $\Delta(w\lambda)$, $w \in W$. The order is as follows: $w\lambda \preceq w'\lambda$ if there is a sequence of (non-necessarily simple) reflections $s_1, \ldots, s_k \in W$ such that $w' = s_k \cdots s_1 w$ and the difference $s_is_{i-1} \cdots s_1w\lambda - s_{i-1} \cdots s_1w\lambda$ is a positive multiple of a positive root for any $i$.

When $\lambda$ is regular, integral and anti-dominant, we identify the poset $W\lambda$ with $W$ by sending $w \in W$ to $w\lambda$. We write $\Delta_{\wedge}$ for $\Delta(w\lambda)$ and $L_w$ for $L(w\lambda)$. We will write $\mathcal{O}(W)$ for the corresponding category $\mathcal{O}_\lambda$ (that is independent of the choice of $\lambda$ up to a translation equivalence). Note that a highest weight order for $\mathcal{O}(W)$ is the Bruhat order on $W$.

To finish this section, we recall that the natural functor $D^b(\mathcal{O}_\lambda) \to D^b(\mathcal{U}_\lambda\text{-mod})$ is a fully faithful embedding provided $\lambda$ is regular.

3.1.2. Equivalences between categories $\mathcal{O}$ and HC bimodules. Let us start by introducing various versions of the category of Harish-Chandra (shortly, HC) bimodules.

By a HC $\mathcal{U}$-bimodule, we mean a finitely generated $\mathcal{U}$-bimodule with locally finite adjoint action of $\mathfrak{g}$. Pick $\lambda, \mu \in \mathfrak{h}^*$. We write $\text{HC}^{1,\infty}_{\lambda,\mu}(\mathcal{U})$ for the category of all HC $\mathcal{U}$-bimodules with central character $\lambda$ on the left and generalized central character $\mu$ on the right. The notations $\text{HC}^{1,\infty}_{\lambda,\mu}(\mathcal{U})$ or $\text{HC}^{\infty,1}_{\mu,\lambda}(\mathcal{U})$ have similar meanings. Note that the categories $\text{HC}^{1,\infty}_{\lambda,\mu}(\mathcal{U}), \text{HC}^{\infty,1}_{\mu,\lambda}(\mathcal{U})$ are naturally equivalent (by switching the left and right actions of $\mathcal{U}$ and twisting them by the antipode map for $\mathcal{U}$). We denote this equivalence by $X \mapsto X^{op}$.

Now let $\mathcal{O}^{\prime}_h$ denote the infinitesimal block of the BGG category $\mathcal{O}$ with generalized central character $\lambda$. The modules there are finitely generated over $\mathcal{U}$, have a locally finite $\mathfrak{b}$-action and diagonalizable action of Cartan. Again, for $\lambda$ regular and integral, we write $\mathcal{O}^{\prime}(W)$ for $\mathcal{O}^{\prime}_h$. For $\lambda$ regular, integral and dominant, a classical result of Bernstein and Gelfand, [BG], establishes an equivalence $\text{HC}^{\infty,1}_{\mu,\lambda}(\mathcal{U}) \cong \mathcal{O}^{\prime}_h$ that sends a HC bimodule $X$ to $X \otimes_{\mathcal{U}} \Delta(\lambda)$.

There is also an equivalence between $\mathcal{O}_\lambda$ and $\text{HC}^{1,\infty}_{\lambda,\mu}$ ($\mu$ is regular, integral and dominant) due to Soergel, [S]. It sends $X \in \text{HC}^{1,\infty}_{\lambda,\mu}$ to $\lim_{k \to \infty} X \otimes_{\mathcal{U}} \Delta^k(\mu)$, where $\Delta^k(\mu)$ stands for $\mathcal{U} \otimes_{U(\mathfrak{h})} U(\mathfrak{h})/\mathfrak{m}_\mu^k$ with $\mathfrak{m}_\mu$ denoting the maximal ideal of $\mu$ in $U(\mathfrak{h})$.

Composing the equivalences

$$\mathcal{O}_\lambda \cong \text{HC}^{1,\infty}_{\lambda,\lambda} \cong \text{HC}^{\infty,1}_{\lambda,\lambda} \cong \mathcal{O}^{\prime}_h$$

we get an equivalence $\mathcal{O}_\lambda \cong \mathcal{O}^{\prime}_h$ that sends $\Delta(w\lambda)$ to $\Delta(w^{-1}\lambda)$. Here $\lambda$ is regular, integral and antidominant.

We will need a corollary of this equivalence. Namely, pick a subdiagram $D_1 \subset D$ and let $W_1$ denote the corresponding parabolic subgroup. Pick a right $W_1$-coset, say $c$. This is an interval in the Bruhat order. Consider the highest weight subcategories $\mathcal{O}_{\leq c}(W) \subset \mathcal{O}_{\geq c}(W) \subset \mathcal{O}(W)$. Here we write $\mathcal{O}_{\leq c}(W)$ for the subcategory corresponding
to the poset ideal \( \{ w \in W | w \leq w' \text{ for some } w' \in c \} \), the subcategory \( \mathcal{O}_{\leq c}(W) \) is defined similarly. Form the quotient \( \mathcal{O}_c(W) := \mathcal{O}_{\leq c}(W)/\mathcal{O}_{< c}(W) \).

**Lemma 3.1.** There is an equivalence \( \mathcal{O}_c(W) \cong \mathcal{O}(W_1) \) which sends \( \Delta_{w^1} \) to \( \Delta_w \), where \( w^1 \) is the shortest element in \( c \).

**Proof.** Under the equivalence \( \mathcal{O}(W) \cong \mathcal{O}_\lambda' \), the interval \( c \) corresponds to an interval \( c' \) in \( W\lambda \) consisting of all \( \mu \) with fixed pairing with \( z(1) \), where \( I \) denotes the standard Levi subalgebra in \( g \) corresponding to \( D_1 \). Let \( \mu \) denote an element in \( c' \). Let \( p \) be the standard parabolic with Levi subalgebra \( t \). Consider the parabolic induction functor \( \Delta_p := U \otimes_{U(p)} : \mathcal{O}_\mu'(W_1) \rightarrow \mathcal{O}_\lambda' \). Its image lies in \( \mathcal{O}_{\lambda', \leq c'} \) and the composition \( \mathcal{O}_{\mu'}(W_1) \rightarrow \mathcal{O}_{\lambda', \leq c'}/\mathcal{O}_{\lambda', < c'} \) is easily seen to be an equivalence (a quasi-inverse functor is given by taking an appropriate eigenspace for \( z(1) \)).

This shows an equivalence \( \mathcal{O}_c(W) \cong \mathcal{O}(W_1) \). The claim about the images of Vermas follows from the corresponding statement for Soergel’s equivalences \( \mathcal{O}(W) \cong \mathcal{O}'(W_1) \), \( \mathcal{O}(W_1) \cong \mathcal{O}'(W) \). \( \square \)

### 3.1.3. Relation to Kazhdan-Lusztig bases and cells.

We identify \( K_0(\mathcal{O}(W)) \) with \( ZW \) by sending \( [\Delta_w] \) to \( w \). Recall that the Kazhdan-Lusztig conjecture (proved by Beilinson-Bernstein and Brylinski-Kashiwara) implies that \( [L_w] \) is the specialization of \( C_w \) to \( v = 1 \).

As a consequence, one has the following classical connection between the cells in \( W \) and simple modules in \( \mathcal{O}(W) \). Let us write \( J_w \) for the annihilator of \( L(-w\rho) \in \mathcal{O}_\rho \). Then the following is true:

- We have \( w \sim_L w' \) if \( J_w = J_{w'} \).
- We have \( w \sim_{LR} w' \) if the associated varieties of \( J_w, J_{w'} \) coincide.
- We have \( w \sim_R w' \) if \( w \sim_{LR} w' \) and there is \( X \in HC^{1,1}_L(U) \) such that \( L_w \) is a composition factor of \( X \otimes U L_w' \).

### 3.2. Wall-crossing functors.

#### 3.2.1. Localization theorems.

Recall the Beilinson-Bernstein (abelian and derived) localization theorems. Let \( \lambda \) be a regular element in \( \mathfrak{h}_* \). Recall that \( \lambda \) is called dominant if \( \langle \lambda, \alpha^\vee \rangle \notin \mathbb{Z}_{\leq 0} \). Let \( G \) be a semisimple algebraic group with Lie algebra \( g \) and \( B \subset G \) be the Borel subgroup corresponding to \( \mathfrak{b} \).

Form the sheaf \( D^\lambda_{G/B} \) of \( \lambda \)-twisted differential operators on \( G/B \). We have the global section functor \( \Gamma_\lambda : \text{Coh}(D^\lambda_{G/B}) \rightarrow U_\lambda \)-mod and its left adjoint, the localization functor \( \text{Loc}_\lambda := D^\lambda_{G/B} \otimes_{U_\lambda} \) \( \bullet \) (recall that \( \Gamma(D^\lambda_{G/B}) = U_\lambda \)). Further, we have the derived functors \( R\Gamma_\lambda : D^b(\text{Coh}(D^\lambda_{G/B})) \rightarrow D^b(U_\lambda \text{-mod}) \) and \( L \text{ Loc}_\lambda : D^- (U_\lambda \text{-mod}) \rightarrow D^- (\text{Coh}(D^\lambda_{G/B})) \).

The following result is due to Beilinson and Bernstein.

**Proposition 3.2.** The following is true.

1. The functors \( \Gamma_\lambda \) is an equivalence if and only if \( \lambda \) is regular and dominant. Its quasi-inverse is \( \text{Loc}_\lambda \).
2. The functors \( R\Gamma_\lambda \) is an equivalence if and only if \( \lambda \) is regular. Its quasi-inverse is \( L \text{ Loc}_\lambda \).
3.2.2. Wall-crossing functors and braid group action. Recall that $O_\lambda = O_{w_\lambda}$ for any $\lambda$. Assume that $\lambda$ is regular. For $w \in W$, let us write $\lambda w$ for $w^{-1} \lambda$. Let $W^\lambda$ be the subgroup of $W$ generated by reflections $s_\alpha$ such that $(\alpha^\vee, \lambda) \in \mathbb{Z}$ (a.k.a. the integral Weyl group of $\lambda$). Our goal here is to recall a self-equivalence $\mathcal{WC}_w, w \in W^\lambda,$ of $D^b(O_\lambda)$ (known as an intertwining functor, a twisting functor or a wall-crossing functor, we use the latter name) and list its properties. For details of the proofs, the reader may consult [Mi, Section L.3] or [BMR] Section 2) (that treats the positive characteristic case).

Assume that $w$ and $\lambda$ are such that $\lambda < \lambda w$ in the order recalled in 3.1.1. Consider the equivalences $L \text{Loc}_\lambda : D^b(U_{w_\lambda} \text{-mod}) \to D^b(\text{Coh}(D^\lambda_{G/B})), L \text{Loc}_{\lambda w} : D^b(U_{w_\lambda} \text{-mod}) \to D^b(\text{Coh}(D^\lambda_{G/B})).$ We also have an abelian equivalence $\mathcal{T}_{\lambda w \leftarrow w} : \text{Coh}(D^\lambda_{G/B}) \xrightarrow{\sim} \text{Coh}(D^{\lambda w}_{G/B})$ given by tensoring with the line bundle $O(\lambda w - \lambda)$.

**Proposition 3.3.** The following is true.

1. A derived self-equivalence $R \mathcal{T}_{\lambda w} \circ \mathcal{T}_{\lambda w \leftarrow w} \circ L \text{Loc}_\lambda$ of $D^b(U_{w_\lambda} \text{-mod})$ depends only on $w$ and $W^\lambda \lambda$, not on $\lambda$ itself. Denote it by $\mathcal{WC}_w$.

2. For simple roots $\alpha_1, \ldots, \alpha_k$ for $W^\lambda$, the map $T_{\alpha_i} \mapsto \mathcal{WC}_{s_i}$ (where $s_i$ here denotes the simple reflection in $W^\lambda$ corresponding to $\alpha_i$) gives a weak categorical action of $\text{Br}_{W^\lambda}$ on $D^b(O_\lambda)$. The element $T_w \in \text{Br}_{W^\lambda}$ maps to $\mathcal{WC}_w$. 

3.2.3. Behavior on $K_0$. Here we will compute the image of $\mathcal{WC}_w$ on $K_0(O_\lambda)$. Recall that $K_0(O_\lambda)$ is identified with $ZW$.

**Lemma 3.4.** The map $[\mathcal{WC}_w] : ZW \to ZW$ is given by $u \mapsto uw^{-1}$.

**Proof.** It is enough to check this on the generators $s_i$. In this case, the functor $\mathcal{WC}_{s_i}$ is known to coincide with the classical wall-crossing functor with respect to the wall $\alpha_i = 0$ and our claim follows. Recall that $\mathcal{WC}_{s_i}$ is defined as follows. Let $T_i$ be the translation functor to the wall $\alpha_i = 0$. Then $\mathcal{WC}_{s_i}(M)$ is quasi-isomorphic to the complex $M \to \mathcal{TT}^*(M)$. $\Box$

3.2.4. Wall-crossing bimodules. In fact, the functor $\mathcal{WC}_w$ can be realized as the derived tensor product with a Harish-Chandra bimodule. Basically, all constructions of this part can be found in [BPW] Sections 6.3, 6.4] in a more general setting.

Namely, suppose that $\lambda w$ is regular and dominant so that, automatically, $\lambda < \lambda w$. Lift $O_{\lambda w - \lambda}$ to a line bundle on $T^*(G/B)$. This line bundle admits a unique deformation to a $D^\lambda_{G/B} - D^\lambda_{G/B}$-bimodule to be denoted by $B^\lambda_{\lambda w \leftarrow \lambda}$. We have $\mathcal{T}_{\lambda w \leftarrow w} = B^\lambda_{\lambda w \leftarrow \lambda} \otimes D^\lambda_{G/B}$. We set $B_{\lambda w \leftarrow \lambda} := \Gamma(B^\lambda_{\lambda w \leftarrow \lambda})$. So we get

$$\mathcal{WC}_w = B_{\lambda w \leftarrow \lambda} \otimes_{U_{\lambda w}} \bullet.$$ 

Below we will need a deformation of $B_{\lambda w \leftarrow \lambda}$. Assume that $\lambda$ is integral and $\lambda w$ is dominant. Let $\mathfrak{q}_0 \subset \mathfrak{h}^*$ be a subspace fixed by $w$. Set $\mathfrak{q}_1 := \lambda + \mathfrak{q}_0, \chi := \lambda w - \lambda$. Consider the deformation $X_{\mathfrak{q}_0} := G \times_B (\mathfrak{h}_0 \oplus \mathfrak{n})$ of $T^*(G/B) = G \times_B \mathfrak{n}$, where $\mathfrak{n}$ is the nilpotent radical of $\mathfrak{b}$ and $\mathfrak{h}_0 \subset \mathfrak{h}$ corresponds to $\mathfrak{q}_0$ under an identification $\mathfrak{h} \cong \mathfrak{h}^*$ coming from the Killing form. We still have the line bundle $O_\chi$ on $X_{\mathfrak{q}_0}$, the lift of $O_\chi$ from $G/B$. Consider the quantizations $D^\mathfrak{q}_1_{G/B}, D^\mathfrak{q}_1_{G/B}$ of $X_{\mathfrak{q}_0}$. The bundle $O_\chi$ quantizes into the $D^\mathfrak{h}_{G/B} - D^\mathfrak{h}_{G/B}$-bimodule $B^\mathfrak{q}_1_{\chi^\lambda}$. We set $B_{\mathfrak{q}_1, \chi} := \Gamma(B^\mathfrak{q}_1_{\chi^\lambda})$.

A relation between the bimodules $B_{\lambda w \leftarrow \lambda}$ for $\lambda \neq \mathfrak{q}_1$ and $B_{\mathfrak{q}_1, \chi}$ is given by the following lemma that follows from [BPW] Proposition 6.25] combined with the Beilinson-Bernstein localization theorem.
Lemma 3.5. Suppose $\lambda w$ is regular and dominant. Then $B_{\lambda w} := B_{\lambda w} \otimes_{\mathbb{C}[\mathfrak{h}]} \mathbb{C}_{\lambda}$.

3.2.5. Long wall-crossing vs Ringel duality. Set $n := \dim \mathfrak{n} = \dim G/B$.

Lemma 3.6. Let $\lambda$ be regular and $w_0 \in W^\lambda$ denote the longest element. Then the functor $\mathcal{M}_{w_0} : D^b(\mathcal{O}_\lambda) \to D^b(\mathcal{O}_\lambda)$ is perverse, where both filtrations are by codimension of support: the subcategory $C_j^i \subset \mathcal{O}_\lambda$, $i = 1, 2$, consists of all modules $M$ such that $n - \text{GK-dim } M \geq j$. Moreover, there is an abelian equivalence $\mathcal{O}_\lambda \cong \mathcal{O}_\lambda$ that intertwines the inverse Ringel duality functor $\mathcal{R}^{-1} : \mathcal{O}_\lambda \to \mathcal{O}_\lambda$ with $\mathcal{M}_{w_0}$.

Proof. The functor $\mathcal{M}_{w_0}$ coincides with the homological duality functor $\text{RHom}_{\mathfrak{u}_\lambda}(\cdot, \mathcal{U}_\lambda)$ up to precomposing with an abelian equivalence. This is proved in [BFO] Section 4.3 in a slightly different setting (see also [BLa] Section 4), the proof used there can be adapted to our setting verbatim). That the homological duality functor is perverse is a bit implicit in [BFO] Section 4.3 and is more explicit in [L3] Lemma 2.5. That the homological duality functor coincides with the inverse Ringel duality (up to precomposing with an abelian equivalence) is checked in [L2] Section 4.1 (again, the proof carries to the present case verbatim).

3.3. $W$-algebras.

3.3.1. Construction and basic properties. Pick a nilpotent orbit $\mathcal{O} \subset \mathfrak{g}$ and an element $e \in \mathfrak{O}$. Consider the Slodowy slice $S \subset \mathfrak{g}$ that is a transverse slice to $\mathcal{O}$ in $\mathfrak{g}$. It is constructed as follows: we pick an $\mathfrak{sl}_2$-triple $(e, h, f)$ and set $S := e + \mathfrak{z}(f)$. This is an affine space equipped with a so called Kazhdan action of $\mathbb{C}^\times$ that contracts it to $e$, the action is given by $t.s := t^{-2}\gamma(t)s$, where $\gamma : \mathbb{C}^\times \to G$ corresponds to $h$. The algebra $\mathbb{C}[S]$ is graded Poisson with bracket of degree $-2$.

The algebra $\mathbb{C}[S]$ admits a distinguished quantization called the finite $W$-algebra that was first constructed by Premet in [P1]. Let us recall a construction in the version of Gan and Ginzburg, [GG].

The element $h$ induces a grading on $\mathfrak{g}$, $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$. Let $(\cdot, \cdot)$ denote the Killing form. We set $\chi := (e, \cdot)$ and $\mathfrak{g}(\leq i) := \bigoplus_{i \leq j} \mathfrak{g}(i)$. The space $\mathfrak{g}(-1)$ is symplectic with respect to the form $(x, y) \mapsto \langle \chi, [x, y] \rangle$. Pick a lagrangian subspace $\ell \subset \mathfrak{g}(-1)$ and set $\mathfrak{m} := \ell \oplus \mathfrak{g}(\leq -2)$. Let $M \subset G$ denote the corresponding connected subgroup, it is unipotent. The subgroup $M$ acts on $\mathfrak{g}^* \cong \mathfrak{g}$ in a Hamiltonian way, the moment map is just the restriction map $\mathfrak{g}^* \to \mathfrak{m}^*$. It is easy to see that $S \subset \mu^{-1}(\chi)$. It turns out that the action map $M \times S \to \mu^{-1}(\chi)$ is an isomorphism, see [GG] Lemma 2.1, so $\mathbb{C}[S] = [S(\mathfrak{g})/S(\mathfrak{g})\mathfrak{m}_\chi]^M$, where $\mathfrak{m}_\chi := \{x - \langle \chi, x \rangle | x \in \mathfrak{m} \}$ is viewed as a subspace in $\mathfrak{g} \oplus \mathbb{C}$.

This motivates the following definition: $\mathcal{W} := [U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{m}_\chi]^M$. This is a filtered associative algebra, the filtration is inherited from the so called Kazhdan filtration on $U(\mathfrak{g})$, where $\deg \mathfrak{g}(i) = i + 2$. We have $\text{gr } \mathcal{W} = \mathbb{C}[S]$ (where the grading on $\mathbb{C}[S]$ is introduced in a similar fashion), see [GG] Proposition 5.2. It was also shown in [GG] Section 5.5 that a natural homomorphism $[U/\mathcal{U}(\mathfrak{g}(\leq -2)\chi)]^{G(\leq -1)} \subset \mathcal{W}$ is an isomorphism. This gives rise to a Hamiltonian action of $Q := Z_G(e, h, f)$ on $\mathcal{W}$. The center of $U(\mathfrak{g})$ naturally maps into $\mathcal{W}$ and this map is an isomorphism onto the center of $\mathcal{W}$, see the footnote in [P2] Section 5.7. Because of this, for $\lambda \in \mathfrak{h}^*$, we have a central reduction $\mathcal{W}_\lambda$ of $\mathcal{W}$, quantizing $\mathbb{C}[S \cap \mathcal{N}]$, where $\mathcal{N}$ stands for the nilpotent cone in $\mathfrak{g}$.
Finally, let us mention Skryabin’s equivalence between the category of \(\mathcal{W}\)-modules and a suitable full subcategory of \(\text{Whittaker}\) modules (=modules, where \(m\) acts with generalized eigen-character \(\chi\)) in the category of \(U(\mathfrak{g})\)-modules. Namely, we have a functor \(M \mapsto M^{mx} : U(\mathfrak{g})\)-mod \(\rightarrow\) \(\mathcal{W}\)-mod that has the left adjoint \(Sk : N \mapsto [U(\mathfrak{g})/U(\mathfrak{g})m_{\chi}] \otimes_{\mathcal{W}} N\). Skryabin in the appendix to \([P1]\) has proved that these two functors are mutually inverse equivalences.

### 3.3.2. Restriction functor for HC bimodules.

There is a notion of a \(Q\)-equivariant HC \(\mathcal{W}\)-bimodule, \([L1]\) Section 2.5]. Let \(HC(\mathcal{W}, Q)\) denote the category of such bimodules. The notation \(HC^{\lambda,\mu}_{\lambda,\mu}(\mathcal{W}, Q)\) has the same meaning as for the universal enveloping algebras.

According to \([L1]\) Section 3.4, there is an exact tensor restriction functor \(HC(\mathcal{U}) \rightarrow HC(\mathcal{W}, Q)\) to be denoted by \(\bullet\). This functor is compatible with the associated varieties: \(V(X) = V(X) \cap S\). In particular, if \(V(X) = \emptyset\), then \(X\) is finite dimensional. Moreover, the restriction of \(\bullet\) to the full subcategory \(HC(\mathcal{U})_{\partial_{\emptyset}}\) of all HC bimodules \(X\) with \(V(X) = \emptyset\) is a quotient functor onto its image. The kernel is the category \(HC(\mathcal{U})_{\partial_{\emptyset}}\) of all HC bimodules \(X\) with \(V(X) \subset \partial\emptyset\). The image is closed under taking subquotients. In particular, if \(X \in HC(\mathcal{U})_{\partial_{\emptyset}}\) is simple, then \(X\) is a simple \(Q\)-equivariant finite dimensional \(\mathcal{W}\)-bimodule. Another useful property is that \(\bullet\) intertwines Tor’s and Ext’s, compare with \([L3]\) Lemma 3.11 and \([BLo]\) Section 5.5. Let us also point out that by the construction given in \([L1]\) Section 3.4, the functor \(\bullet\) maps \(HC^{\lambda,\mu}_{\lambda,\mu}(\mathcal{U})\) to \(HC^{\mu,\lambda}_{\mu,\lambda}(\mathcal{W}, Q)\), where \(?,! \in \{1, \infty\}\).

The functor \(\bullet_{\dagger} : HC(\mathcal{U})_{\emptyset} \rightarrow HC(\mathcal{W}, Q)_{\text{fin}}\) admits a right adjoint \(\bullet^\dagger\). The cokernel of \(\mathcal{B} \mapsto (\mathcal{B})_{\dagger}^\dagger\) is in \(HC(\mathcal{U})_{\partial_{\emptyset}}\) for any \(\mathcal{B} \in HC(\mathcal{U})_{\emptyset}\). The functor \(\bullet^\dagger\) maps \(HC^{\lambda,\mu}_{\lambda,\mu}(\mathcal{W}, Q)_{\text{fin}}\) to \(HC^{\lambda,\mu}_{\lambda,\mu}(\mathcal{U})_{\emptyset}\) for any \(\lambda, \mu\). For a \(Q\)-stable ideal \(\mathcal{I} \subset \mathcal{W}\) of finite codimension, we write \(\mathcal{I}_{\dagger}\) for the kernel of the natural map \(\mathcal{U} \rightarrow (\mathcal{W}/\mathcal{I})_{\dagger}\).

These constructions admit a ramification that will be used later. Namely, let \(\mathfrak{P}_1 \subset \mathfrak{h}^*\) be an affine subspace and let \(\chi \in \mathfrak{h}^*\). Consider the category \(HC^{\lambda,\mu}_{\lambda,\mu}(\mathcal{U})\) of all HC \(\mathcal{U}_{\mathfrak{P}_{1,\chi}}\)-bimodules, where the adjoint action of \(\mathfrak{h}\) (that maps naturally to both \(\mathbb{C}[\mathfrak{P}_1], \mathbb{C}[\mathfrak{P}_1 + \chi]\)) is given by \(\chi\). Here we set \(\mathcal{U}_{\mathfrak{P}_1} := \mathbb{C}[\mathfrak{P}_1] \otimes_{\mathbb{C}[\mathfrak{h}^*]} \mathcal{U}\) and \(\mathcal{U}_{\mathfrak{P}_1 + \chi} := \mathbb{C}[\mathfrak{P}_1 + \chi] \otimes_{\mathbb{C}[\mathfrak{h}^*]} \mathcal{U}\). Define the category \(HC^{\lambda,\mu}_{\lambda,\mu}(\mathcal{W}, Q)\) in a similar fashion. We consider the subcategory \(HC^{\lambda,\mu}_{\lambda,\mu}(\mathcal{U})_{\emptyset} \subset HC^{\lambda,\mu}_{\lambda,\mu}(\mathcal{U})\) consisting of all bimodules \(\mathcal{B}\) such that the intersection of \(V(\mathcal{B}) \subset \mathfrak{P}_0 \times_{\mathfrak{h}^*/W} \mathfrak{g}^0\) with \(\mathcal{N}\) lies in \(\emptyset\). Here we write \(\mathfrak{P}_0 \subset \mathfrak{h}^*\) for the associated vector subspace of \(\mathfrak{P}_1\). Similarly, define the subcategory \(HC^{\lambda,\mu}_{\lambda,\mu}(\mathcal{W}, Q)_{\text{fin}}\), it consists of all bimodules in \(HC^{\lambda,\mu}_{\lambda,\mu}(\mathcal{W}, Q)\) that are finitely generated over \(\mathbb{C}[\mathfrak{P}_1]\). We then have an exact functor \(HC^{\lambda,\mu}_{\lambda,\mu}(\mathcal{U}) \rightarrow HC^{\lambda,\mu}_{\lambda,\mu}(\mathcal{W}, Q)\) that restricts to \(HC^{\lambda,\mu}_{\lambda,\mu}(\mathcal{U})_{\emptyset} \rightarrow HC^{\lambda,\mu}_{\lambda,\mu}(\mathcal{W}, Q)_{\text{fin}}\). The restriction has a right adjoint \(\bullet^\dagger : HC^{\lambda,\mu}_{\lambda,\mu}(\mathcal{W}, Q)_{\text{fin}} \rightarrow HC^{\lambda,\mu}_{\lambda,\mu}(\mathcal{U})_{\emptyset}\). Note that, for \(\chi, \chi' \in \mathfrak{h}^*\), we have tensor product functors

\[
HC_{\lambda,\mu}^{\lambda,\mu}(\mathcal{U}) \times HC_{\lambda,\mu}^{\lambda,\mu}(\mathcal{U}) \rightarrow HC_{\lambda,\mu}^{\lambda,\mu}(\mathcal{U}),
\]

\[
HC_{\lambda,\mu}^{\lambda,\mu}(\mathcal{W}, Q) \times HC_{\lambda,\mu}^{\lambda,\mu}(\mathcal{W}, Q) \rightarrow HC_{\lambda,\mu}^{\lambda,\mu}(\mathcal{W}, Q).
\]

The functor \(\bullet_{\dagger}\) intertwines those functors.

The functor \(\bullet_{\dagger}\) also can be defined using the quantum Hamiltonian reduction, it maps a HC \(\mathcal{U}\)-bimodule \(X\) to \([X/Xm_{\chi}^M]\), \([P1]\) Section 3.5]. Note that \(Sk^{-1}(X \otimes_{U(\mathfrak{g})} Sk(N))\) is naturally identified with \(X_{\dagger} \otimes_{\mathcal{W}} N\). This is a special case of \([LO]\) Theorem 5.11.
3.3.3. Classification of finite dimensional irreducible representations. We are going to recall a classification of the finite dimensional irreducible modules over $\mathcal{W}_\lambda$, [L1, LO]. Let us start with results obtained in [L1].

Let $\Pr(\mathcal{U}_\lambda)$ denote the set of primitive ideals in $\mathcal{U}_\lambda$. Inside we have the subset $\Pr_0(\mathcal{U}_\lambda)$ of all primitive ideals $\mathcal{J}$ such that $V(\mathcal{U}_\lambda/\mathcal{J}) = \emptyset$. For $\mathcal{J} \in \Pr_0(\mathcal{U}_\lambda)$, the ideal $\mathcal{J}_1 \subset \mathcal{W}$ is $Q$-stable, has finite codimension, and is maximal with these properties. The maximal ideals containing $\mathcal{J}_1$ are $Q$-conjugate. This allows to assign an $A$-orbit in $\text{Irr}_{fin}(\mathcal{W}_\lambda)$ to $\mathcal{J}$, in fact, every finite dimensional irreducible $\mathcal{W}_\lambda$-module lies in one of these orbits. This gives rise to a bijection $\text{Irr}_{fin}(\mathcal{W}_\lambda)/A \xrightarrow{\sim} \Pr_0(\mathcal{U}_\lambda)$.

When $\lambda$ is integral, one can compute the $A$-orbit over a given primitive ideal, this was done in [LO]. Let us state the result in the case when $\lambda$, in addition, is regular. Namely, let $\text{Spr}_0$ denote the Springer $W \times A$-module corresponding to $0$. The set $\Pr_0(\mathcal{U}_\lambda)$ is non-empty if and only if $0$ is special and we will be assuming this from now on. So to $0$ we can assign the two-sided cell $c$. Then the Lusztig group $\tilde{A}$ can be defined as the quotient of $A$ by the kernel of the $A$-action on $\text{Hom}_\mathcal{W}([c], \text{Spr}_0)$, where we write $[c]$ for the two-sided cell bimodule viewed as a left $W$-module, see [LLO] Section 13.1. Recall that $\Pr_0(\mathcal{U}_\lambda)$ is in a bijection with the set of the left cells inside $c$. It was shown in [LO] Sections 6.4-6.8 that for a left cell $\sigma \subset c$, there is a unique (up to conjugacy) subgroup $H_\sigma \subset \tilde{A}$ such that $\mathbb{C}(\tilde{A}/H_\sigma) \cong \text{Hom}_\mathcal{W}([\sigma], \text{Spr}_0)$ (an isomorphism of $A$-modules). The main result of [LO], Theorem 1.1 there, is that the $A$-orbit corresponding to the primitive ideal indexed by $\sigma$ coincides with $\tilde{A}/H_\sigma$.

For brevity, let us write $Y_\lambda$ instead of $\text{Irr}_{fin}(\mathcal{W}_\lambda)$. Consider also the category $\mathcal{J}_0^\lambda$ of all semisimple objects in $\text{HC}^{1,1}_{\mathcal{W}_\lambda}(\mathcal{U})^{\mathbb{P}}/\text{HC}^{1,1}_{\mathcal{W}_\lambda}(\mathcal{U})^0$. This category is closed under taking the tensor products and internal Hom’s, see [LO] Section 5.3. The tensor category $\mathcal{J}_0^\lambda$ acts on the category $\mathcal{W}_\lambda\text{-mod}^{as}_{fin}$ that can be thought as the category of sheaves of finite dimensional vector spaces on $Y_\lambda$. The action is given by $X.V := X_1 \otimes_{\mathcal{W}_\lambda} V$. The category $\mathcal{J}_0^\lambda$ is identified with $\text{Sh}^A(Y_\lambda \times Y_\lambda)$ (where the tensor product is given by convolving the sheaves) in such a way that the action on $\mathcal{W}_\lambda\text{-mod}^{as}_{fin}$ becomes the convolution action of $\text{Sh}^A(Y_\lambda \times Y_\lambda)$ on $\text{Sh}(Y_\lambda)$, see [LO] Remark 7.7.

3.3.4. Localization theorems. Here we are going to recall the localization theorem for $W$-algebras, [G, DK]. Modules for $\mathcal{W}_\lambda$ localize to modules over a certain sheaf of non-commutative algebras on the Slodowy variety $\tilde{S}$. By definition, $\tilde{S}$ is the preimage of the Springer morphism $T^*(G/B) \rightarrow \mathfrak{g}^*$. This is a smooth symplectic variety that coincides with $\mu_{T^*(G/B)}^{-1}(\chi)/M$, where we write $\mu_{T^*(G/B)}$ for the moment map for the $M$-action on $T^*(G/B)$. Note that the natural morphism $\tilde{S} \rightarrow S \cap \mathcal{N}$ is a resolution of singularities.

The sheaf of non-commutative algebras of interest will be denoted by $\mathfrak{W}_\lambda$. It is a sheaf in the conical topology (the topology where “open” means Zariski open and $\mathbb{C}^x$-stable, where we consider the Kazhdan action on $\tilde{S}$). This sheaf is filtered in such a way that the filtration is complete and separated and the associated graded is $\mathcal{O}_{\tilde{S}}$ so that $\mathfrak{W}_\lambda$ is a filtered quantization of $\tilde{S}$. By definition, $\mathfrak{W}_\lambda$ is the reduction $[\mathcal{D}_{G/B}/\mathcal{D}_{G/B}^{1,1}m_\chi]^M$. Since the action of $M$ on $\mu_{T^*(G/B)}^{-1}(\chi)$ is free, we see that $\mathfrak{W}_\lambda$ is indeed a quantization of $\tilde{S}$. Note that there is a natural filtered algebra homomorphism $\mathcal{W}_\lambda \rightarrow \Gamma(\mathfrak{W}_\lambda)$. The associated graded of this homomorphism is an isomorphism $\mathbb{C}[S \cap \mathcal{N}] \xrightarrow{\sim} \mathbb{C}[\tilde{S}]$ and so we see that $\Gamma(\mathfrak{W}_\lambda) = \mathcal{W}_\lambda$. 

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We also remark that the isomorphism $\text{gr} \mathcal{M}_\lambda = \mathcal{O}_S$ together with $H^i(\tilde{S}, \mathcal{O}_S) = 0$ for $i > 0$ imply $H^i(\tilde{S}, \mathcal{M}_\lambda) = 0$ for $i > 0$.

Let us proceed to the localization theorems. First of all, the category of coherent $\mathcal{M}_\lambda$-modules is equivalent to the category $\text{Coh}^{m,\chi}(D^\lambda_{G/B})$ of twisted $(m, \chi)$-equivariant coherent $D^\lambda_{G/B}$-modules (the equivalence is again given by taking $m$-semiinvariants with character $\chi$). This is an easy consequence of the observation that the $M$-action on $\mu^{-1}_{T^*(G/B)}$ is free. Moreover, for $M \in \text{Coh}(\mathcal{M}_\lambda)$, we have

$$\Gamma(M)^{m,\chi} = \Gamma(M^{m,\chi}).$$

The following claim is a direct corollary of (3.2) and (1) of Proposition 3.2.

**Proposition 3.7.** Suppose $\lambda$ is regular and dominant. Then the global section functor $\Gamma_\lambda : \text{Coh}(\mathcal{M}_\lambda) \to \mathcal{W}_\lambda$-mod is a category equivalence. A quasi-inverse equivalence is given by the localization functor $\mathcal{M}_\lambda \otimes_{\mathcal{W}_\lambda} \bullet$.

Let us proceed to the derived localization. By [BLu. Theorem 1.6], the naive derived category $D^b(\text{Coh}^{m,\chi}(D^\lambda_{G/B}))$ is naturally isomorphic to the equivariant derived category $D^b_{m,\chi}(\text{Coh}(D^\lambda_{G/B}))$ because the $M$-action on $\mu^{-1}_{T^*(G/B)}(\chi)$ is free. Similarly, $D^b(\mathcal{W}_\lambda$-mod$)^{m,\chi} = D^b_{m,\chi}(\mathcal{W}_\lambda$-mod$)$. We deduce that if the functor $R\Gamma_\lambda : D^b(\text{Coh}(D^\lambda_{G/B})) \to D^b(\mathcal{W}_\lambda$-mod$)$ is an equivalence, then the same is true for $R\Gamma_\lambda : \text{Coh}^{m,\chi}(D^\lambda_{G/B}) \to D^b(\mathcal{W}_\lambda$-mod$)^{m,\chi}$.

Together with (3.2) and equivalences $\text{Coh}^{m,\chi}(D^\lambda_{G/B}) \cong \text{Coh}(\mathcal{M}_\lambda), \mathcal{U}_\lambda$-mod$^{m,\chi} \cong \mathcal{W}_\lambda$-mod$$ this yields the following result.

**Proposition 3.8.** Suppose that $\lambda$ is regular. Then the functor $R\Gamma_\lambda : D^b(\text{Coh}(\mathcal{M}_\lambda)) \to D^b(\mathcal{W}_\lambda$-mod$)$ is a category equivalence. A quasi-inverse functor is $L\text{Loc}_\lambda := \mathcal{M}_\lambda \otimes_{\mathcal{W}_\lambda} \bullet$.

3.3.5. **Wall-crossing bimodules for $W$-algebras.** Let $\lambda$ be regular and $w \in W^\lambda$ be such that $\lambda w$ is regular and dominant. So we have an equivalence $\mathcal{W}\mathcal{C}_{\lambda w \cdot \lambda} : D^b(\mathcal{W}_\lambda$-mod$) \to D^b(\mathcal{W}_\lambda$-mod$)$ defined similarly to the case of $\mathcal{U}_\lambda$, see [3.2.2]. For similar reasons, it is given by the tensor product with $\mathcal{B}_{\lambda w \cdot \lambda}^W \in \text{HC}^1_{\lambda w \cdot \lambda}(\mathcal{W}, Q)$ that is the global sections of a suitable quantized line bundle $\mathcal{B}_{\lambda w \cdot \lambda}^{W,\text{loc}}$, compare to 3.2.4.

**Lemma 3.9.** We have an isomorphism $\mathcal{B}_{\lambda w \cdot \lambda}^W \cong (\mathcal{B}_{\lambda w \cdot \lambda})_1$.

**Proof.** It is sufficient to establish an isomorphism $\mathcal{B}_{\lambda w \cdot \lambda}^W \otimes_{\mathcal{W}_\lambda} N \cong (\mathcal{B}_{\lambda w \cdot \lambda})_1 \otimes_{\mathcal{W}_\lambda} N$ for any $N \in \mathcal{W}_\lambda$-mod$. Recall [3.3.2] that the right hand side is $\text{Sk}^{-1}(\mathcal{B}_{\lambda w \cdot \lambda} \otimes_{\mathcal{U}_\lambda} \text{Sk}(N))$. Using that the localization and global section functors commute with the Skryabin equivalence, we reduce to checking the equality $\mathcal{B}_{\lambda w \cdot \lambda}^{W,\text{loc}} \cong [\mathcal{B}_{\lambda w \cdot \lambda}^{oc}/\mathcal{B}_{\lambda w \cdot \lambda}^{loc}]^M$. This follows from the observation that both quantize the line bundle $\mathcal{O}_\lambda$ on $\tilde{S}$ to a $\mathcal{W}_w \cdot \mathcal{W}_\lambda$-bimodule and that such a quantization is unique, see, e.g., [BPW. Section 5.1].

4. **Cactus group actions**

4.1. **Perversity of $\mathcal{W}\mathcal{C}_{D_1}$, statement.** We fix a subdiagram $D_1$ in the Dynkin diagram $D$ of $\mathfrak{g}$. Let $\mathfrak{h}_0$ denote the span of the fundamental weights corresponding to the vertices in $D \setminus D_1$. Pick a regular integral weight $\lambda$ such that $\lambda w_{D_1}$ is dominant and set $\mathfrak{h}_1 := \lambda + \mathfrak{h}_0$, this is an affine subspace in $\mathfrak{h}^*$ containing $\lambda$. Finally, set $\chi := \lambda w_{D_1} - \lambda$.

Define the algebras $\mathcal{U}_{\mathfrak{h}_1} := \mathbb{C}[\mathfrak{P}_1] \otimes_{\mathbb{C}[\mathfrak{h}^*]} U(\mathfrak{g}), \mathcal{W}_{\mathfrak{h}_1} := \mathbb{C}[\mathfrak{P}_1] \otimes_{\mathbb{C}[\mathfrak{h}^*]} \mathcal{W}$. Recall that we have a $\mathcal{U}_{\mathfrak{h}_1}$-bimodule $\mathcal{U}_{\mathfrak{h}_1, \lambda}$ defined as the global sections of the $D_{G/B}^{\mathfrak{h}_1} \cdot D_{G/B}^{\mathfrak{h}_1}$-bimodule...
quantizing the line bundle $\mathcal{O}_\lambda$. Recall also (Lemma 3.5) that when $\lambda_1 + \chi$ is regular dominant, the specialization $\mathcal{U}_{\lambda_1, \chi}$ coincides with the wall-crossing bimodule $\mathcal{B}_{\lambda_1, w_{D_1} - \lambda_1}$.

Further, set $\mathcal{W}_{\mathfrak{p}_1, \chi} := (\mathcal{U}_{\mathfrak{p}_1, \chi})^\dagger$.

We need to produce to produce chains of ideals in the algebras $\mathcal{U}_{\mathfrak{p}_0}, \mathcal{U}_{\mathfrak{p}_0'}$.

**Proposition 4.1.** There is a chain of ideals $\mathcal{U}_{\mathfrak{p}_1} = \mathcal{J}_{\mathfrak{p}_1} \supset \mathcal{J}_{\mathfrak{p}_2} \supset \ldots \supset \mathcal{J}_{\mathfrak{p}_n} = \{0\}$, where $n := \dim G/B$, with the following property: for a Weil generic point $\lambda_1 \in \mathfrak{p}_0$, the specialization $\mathcal{J}_{i, \lambda_1}$ is the minimal ideal with $\dim V(\mathcal{U}_{\lambda_1}/\mathcal{J}_{i, \lambda_1}) < 2i$.

**Proof.** Observe that, for any nilpotent orbit $\mathfrak{O}$ and any $\lambda_1 \in \mathfrak{h}^*$, the algebra $\mathcal{W}_{\lambda_1}$ has finite length as a bimodule over itself, see, e.g., [L4, Theorems 1.2, 1.3]. In particular, $\mathcal{W}_{\lambda_1}$ has a minimal ideal of finite codimension. Similarly to the proof of [L3, Lemma 5.1], one can show that there is an ideal $\mathcal{I}_{\mathfrak{p}_1} \subset \mathcal{W}_{\mathfrak{p}_1}$ such that $\mathcal{W}_{\mathfrak{p}_1}/\mathcal{I}_{\mathfrak{p}_1}$ is finitely generated over $\mathbb{C}[\mathfrak{p}_1]$ and, for a Weil generic $\lambda_1 \in \mathfrak{p}_1$, the specialization $\mathcal{I}_{\lambda_1}$ is the minimal ideal of finite codimension in $\mathcal{W}_{\lambda_1}$. Then we set $\mathcal{J}_{i, \mathfrak{p}_1} := \left( \bigcap_{i \geq 0} \mathcal{I}_{i, \mathfrak{p}_1}^{i, 0} \right)^i$, where the intersection is taken over all orbits $\mathfrak{O}$ with $\dim \mathcal{N} - \dim \mathfrak{O} < 2i$, $\mathcal{I}_{i, \mathfrak{p}_1}$ means the ideal $\mathcal{I}_{\mathfrak{p}_1}$ for the $\mathcal{W}$-algebra corresponding to $\mathfrak{O}$, and $i^{i, 0}$ has a similar meaning. Similarly to [L3, Lemma 5.2], the ideal $\mathcal{J}_{i, \mathfrak{p}_1}$ has required properties.

Note that, by the construction, $(\mathcal{J}_{i, \lambda_1})^2 = \mathcal{J}_{i, \lambda_1}$ when $\lambda_1$ is Weil generic. Therefore the equality is true when $\lambda_1$ is Zariski generic as well. It follows that the full subcategory $\mathcal{C}_i \subset \mathcal{O}_{\lambda_1}$ consisting of all modules annihilated by $\mathcal{J}_{n+1-i, \lambda_1}$ is a Serre subcategory.

**Theorem 4.2.** For a Zariski generic $\lambda_1 \in \mathfrak{p}_1$, the functor $\mathcal{F}(\lambda_1) := \mathcal{U}_{\lambda_1, \chi} \otimes_{\mathcal{U}_{\lambda_1}} \mathcal{B}_{\lambda_1}$ is a perverse self-equivalence of $D^b(\mathcal{O}_{\lambda_1})$ with respect to the filtrations defined by the ideals $\mathcal{J}_{i, \lambda_1}$.

In particular, we can always choose a regular integral $\lambda_1 \in \mathfrak{h}^*$ such that $\lambda_1 w_{D_1}$ is dominant and $\mathcal{F}(\lambda_1)$ is perverse with respect to the filtration above.

**4.2. Perversity of $\mathcal{W}\mathcal{C}_{D_1}$, proof.** The proof of Theorem 4.2 basically repeats the proof of [L3, Theorem 6.1]. We provide details here for readers convenience.

Let us write $\mathcal{B}_{\lambda_1}$ for $\mathcal{U}_{\lambda_1, \chi}$.

**Lemma 4.3.** For a Zariski generic $\lambda_1 \in \mathfrak{p}_1$, the following holds:

(a) For all $i, j$, we have $\mathcal{J}_{j, \lambda_1} \mathcal{U}_{i, \lambda_1} \mathcal{B}_{\lambda_1, \mathcal{U}_{i, \lambda_1}/\mathcal{J}_{j, \lambda_1}} = 0$.

(b) For all $i, j$, we have $\mathcal{U}_{i, \lambda_1} \mathcal{B}_{\lambda_1, \mathcal{U}_{i, \lambda_1}/\mathcal{J}_{j, \lambda_1}} \mathcal{J}_{j, \lambda_1} = 0$.

(c) We have $\mathcal{U}_{i, \lambda_1} \mathcal{B}_{\lambda_1, \mathcal{U}_{i, \lambda_1}/\mathcal{J}_{j, \lambda_1}} = 0$ for $i < n+1-j$.

(d) We have $\mathcal{J}_{j-1, \lambda_1} \mathcal{U}_{i, \lambda_1} \mathcal{B}_{\lambda_1, \mathcal{U}_{i, \lambda_1}/\mathcal{J}_{j, \lambda_1}} = \mathcal{U}_{i, \lambda_1} \mathcal{B}_{\lambda_1, \mathcal{U}_{i, \lambda_1}/\mathcal{J}_{j-1, \lambda_1}} = 0$ for $i > n+1-j$.

(e) Set $\mathcal{B}_{j, \lambda_1} := \mathcal{U}_{n+1-j} \mathcal{B}_{\lambda_1, \mathcal{U}_{i, \lambda_1}/\mathcal{J}_{j, \lambda_1}}$. The kernel and the cokernel of the natural homomorphism $\mathcal{B}_{j, \lambda_1} \otimes_{\mathcal{U}_{i, \lambda_1}} \mathcal{U}_{i, \lambda_1}/\mathcal{J}_{j, \lambda_1} \to \mathcal{U}_{i, \lambda_1}/\mathcal{J}_{j, \lambda_1}$ are annihilated by $\mathcal{J}_{j-1, \lambda_1}$ on the left and on the right.

(f) The kernel and the cokernel of the natural homomorphism $\mathcal{U}_{i, \lambda_1}/\mathcal{J}_{j, \lambda_1} \otimes_{\mathcal{U}_{i, \lambda_1}} \mathcal{B}_{j, \lambda_1} \to \mathcal{U}_{i, \lambda_1}/\mathcal{J}_{j, \lambda_1}$ are annihilated on the left and on the right by $\mathcal{J}_{j-1, \lambda_1}$.
Proof. This lemma is an analog of [L3, Proposition 6.2]. As in that proposition, the proof is in four steps. First, we prove (a),(b) for a Weil generic $\lambda_1$. Then we check the direct analogs of (c)-(f) for the ideal $\mathcal{I}_{\lambda_1} \subset \mathcal{W}$ (the minimal ideal of finite codimension) and the $\mathcal{W}_{\lambda_1}$-bimodule $B_{\lambda_1,i}$, where $\lambda_1$ is Weil generic. In Step 3 we establish (c)-(f) with an arbitrary $j$ and a Weil generic $\lambda_1$. Finally, we will prove the claims (a)-(f) for an arbitrary $j$ and a Zariski generic $\lambda_1$.

Step 1 is completely analogous to that of the proof of [L3, Proposition 6.2]. To prove Step 2, we first recall that $B_{\lambda_1,i} = B_{\lambda_1}^W$. Also note that, by the argument in [BL0, Section 4], we have $\text{Tor}^{\mathcal{W}_{\lambda_1}}(B_{\lambda_1}^W, N) = 0$ for any $\mathcal{W}_{\lambda_1}/\mathcal{I}_{\lambda_1}$-module $N$ and $i \neq \frac{1}{2} \dim \hat{S}$, and, moreover, the functor

$$N \mapsto \text{Tor}^{\mathcal{W}_{\lambda_1}}_{\frac{1}{2} \dim \hat{S}}(B_{\lambda_1}^W, \bullet)$$

is a self-equivalence of $\mathcal{W}_{\lambda_1}/\mathcal{I}_{\lambda_1}$-mod. Now the proof of Step 2 works in the same way as in [L3, Proposition 6.2].

The proofs of Steps 3 and 4 are the same as in loc. cit. \hfill \Box

The proof of Theorem 1.2 now repeats that of [L3, Theorem 6.1].

Let us remark that the self-equivalence of $\mathcal{C}_1/\mathcal{C}_1$ induced by $\mathcal{F}(\lambda_1)$ is given by taking the tensor product with the bimodule $B_{n+1-i,\lambda_1}$.

4.3. Proof of (iii) of Theorem 1.1. Note that, for a Zariski generic $\lambda_1 \in \mathbb{P}_1$, the functor $\mathcal{F}(\lambda_1)$ coincides with $\mathcal{M}_{\lambda_1,1}\sim \mathcal{C}_1$ provided $\lambda_1$ is regular and dominant (note that the dominance is not a Zariski generic condition). When $\lambda_1$ is integral, we have an identification $\text{Irr}(\mathcal{O}_{\lambda_1}) \cong W$. We write $\mathcal{M}_{D_1}$ for the self-bijection of $W$ induced by the perverse equivalence $\mathcal{M}_{\lambda_1}$.

Now we are going to prove (iii) of Theorem 1.1 the equality $\mathcal{M}_{D_1}(w'w'') = w'\mathcal{M}_{D_1}^{(D_1)}(w'')$, where $\mathcal{M}_{D_1}^{(D_1)}$ is the analogous bijection for the category $\mathcal{O}(W_{D_1})$.

Recall that $\mathcal{M}_{\lambda_1,1}\sim \mathcal{C}_1$ is the inverse Ringel duality provided $\lambda_1 \in \mathbb{P}_1$ is Weil generic, Lemma 3.6. It follows that $\mathcal{M}_{\lambda_1,1}\sim \mathcal{C}_1(\Delta(\mu)) = \nabla(\mu_{D_1})$ for any $\mu \in W\lambda_1$. From here we deduce that $\mathcal{F}(\lambda_1)(\Delta(w_{\lambda_1}))$ has no higher homology for a Zariski generic $\lambda_1 \in \mathbb{P}_1$. A consequence of this and Lemma 3.4 is that $\mathcal{M}_{w_{D_1}}$ preserves the subcategories $D^b(\mathcal{O}(W)_{s_c}) \subset D^b(\mathcal{O}(W)_{s_c})$ for any right coset $c$ of $W_{D_2}$ with $D_1 \subset D_2$.

Now let $c$ be a right coset for $W_{D_1}$. We see that, for $w \in W_{D_1}$, the functors $\mathcal{M}_{w}$ descend to the quotient $D^b(\mathcal{O}(W)_{s_c})$ (this is because they are compositions of $\mathcal{M}_{s_i}$, where $s_i$ is a simple reflection in $W_{D_1}$). Recall, Lemma 3.1, that we have established the equivalence $\mathcal{O}(W)_{s_c} \cong \mathcal{O}(W_{1})$, where we write $W_{1}$ for $W_{D_1}$. For $w \in W_{1}$, let us write $\mathcal{M}_{w}$ for the functor of $D^b(\mathcal{O}(W)_{s_c})$ induced by $\mathcal{M}_{w}$ on $D^b(\mathcal{O}(W))$ and $\mathcal{M}_{w}^1$ for the functor on $D^b(\mathcal{O}(W_{1}))$.

Lemma 4.4. For any $w \in W_{1}$, there are abelian self-equivalences $\mathcal{F}_{w}, \mathcal{F}_{w}'$ of $\mathcal{O}(W_{1})$ such that $\mathcal{M}_{w} = \mathcal{F}_{w} \circ \mathcal{M}_{w} = \mathcal{M}_{w}^1 \circ \mathcal{F}_{w}$ and $\mathcal{F}_{w}, \mathcal{F}_{w}'$ induce the identity map on $K_0$.

Proof. The proof is by induction on the length $\ell(w)$. The base is $\ell(w) = 1$, i.e., $w$ is a simple reflection $s_i, i \in D_1$. The endofunctor $\mathcal{M}_{s_i}$ of $D^b(\mathcal{O}(W))$ is a classical reflection functor, see the proof of Lemma 3.4. So there is only one filtration term, it is spanned by $L_{w}$ with $w s_i < w$. If $w s_i < w$, then $\mathcal{M}_{s_i} L_w = L_w[-1]$. Also by the paragraph preceding the present lemma it follows that $\mathcal{M}_{s_i}$ preserves the right $(1, s_i)$-cosets. It follows that the bijection induced by $\mathcal{M}_{s_i}$ is trivial. From here we deduce that the endofunctors...
\( \mathcal{W}_s, \mathcal{W}_s^1 \) are perverse equivalences with the same perversity data. Now we can use Lemma 2.2 to establish the existence of \( \mathcal{F}_i, \mathcal{F}_i' \).

To prove the induction step we use the equalities \( \mathcal{W}_{s_i} = \mathcal{W}_{s_i} \circ \mathcal{W}_{s_i}^1, \mathcal{W}_{s_i}^1 = \mathcal{W}_{s_i} \circ \mathcal{W}_{s_i}^1 \) and Corollary 2.3.

From Lemma 4.4 we conclude that \( \mathcal{W}_D, \mathcal{W}_D^1 \) are perverse self-equivalences of \( D^b(\mathcal{O}(W_1)) \) with the same combinatorial data. Recall that the equivalence \( \mathcal{O}(W_1) \cong \mathcal{O}(W) \) sends \( L_{w''} \) to \( L_{w''} \). This implies (iii) of the theorem.

4.4. Cactus group action. Here we will verify that the self-bijections \( \mathcal{W}_D \) of \( W \) satisfy the relations of \( \text{Cact}_W \), see (1.1).

4.4.1. Relation \( \mathcal{W}_D^2 = 1 \). Recall from Lemma 3.4 that the functor \( \mathcal{W}_D \) acts on \( K_0(\mathcal{O}(W)) \) by the right multiplication by \( w_D \) and so the action of \( \mathcal{W}_D^2 \) on the \( K_0 \) is trivial. Also note that, for \( L_w \in \mathcal{C}_\chi \bigcap \mathcal{C}_\chi^+ \), the image \( \mathcal{W}_D^2 L_w \) equals \( L_{w'\mid -2i} \) modulo \( D^b_{\mathcal{C}_\chi^+}(\mathcal{O}(W)) \), where \( w' = w_D \). These two observations imply the claim.

4.4.2. Relation \( \mathcal{W}_D \mathcal{W}_D = \mathcal{W}_D \mathcal{W}_D \). Thanks to (iii), we may assume that \( D = D_1 \sqcup D_2 \). Again, thanks to (iii), we have \( \mathcal{W}_D(w_1 w_2) = \mathcal{W}_D(w_1) w_2 \) and \( \mathcal{W}_D(w_1 w_2) = \mathcal{W}_D(w_1) w_2 \). Our claim follows.

4.4.3. Relation \( \mathcal{W}_D \mathcal{W}_D = \mathcal{W}_D \mathcal{W}_D^1 \). We can assume that \( D_2 = D \) thanks to (iii).

First of all, let us recall that

\[
(4.1) \quad w_D^{-1} w_D w_D = w_D^*.
\]

Note that the functor \( \mathcal{W}_D \) preserves the filtration \( D^b_{\leq i}(\mathcal{O}(W)) \subset D^b(\mathcal{O}(W)) \) by the dimension of support. This is because \( \mathcal{W}_D \) is given by taking the derived tensor product with a HC bimodule. So we have an auto-equivalence \( \mathcal{W}_D \) of \( D^b_{\leq i}(\mathcal{O}(W))/D^b_{\leq i-1}(\mathcal{O}(W)) \) that is perverse with respect to the t-structure induced from \( D^b(\mathcal{O}(W)) \). Note that the functor \( \mathcal{W}_D \) on \( D^b_{\leq i}(\mathcal{O}(W))/D^b_{\leq i-1}(\mathcal{O}(W)) \) is t-exact up to a shift. By Lemma 2.1 \( \mathcal{W}_D^1 \circ \mathcal{W}_D \circ \mathcal{W}_D^1 \) is a perverse self-equivalence of \( D^b_{\leq i}(\mathcal{O}(W))/D^b_{\leq i-1}(\mathcal{O}(W)) \) such that the corresponding bijection is \( \mathcal{W}_D^1 \circ \mathcal{W}_D \circ \mathcal{W}_D^1 \). By (4.1) the actions of \( \mathcal{W}_D^1 \circ \mathcal{W}_D \circ \mathcal{W}_D^1 \) on \( K_0(\mathcal{O}(W);\mathcal{O}(W)) \) coincide. Applying Lemma 2.1 we finish the proof of \( \mathcal{W}_D^1 \circ \mathcal{W}_D \circ \mathcal{W}_D = \mathcal{W}_D^1 \).

4.5. Cells and Lusztig subgroups. Here we will prove (i) and (ii) of Theorem 1.1.

Let \( \lambda \) be a generic enough integral element of \( \mathfrak{P} \) such that \( \lambda w_D \) is dominant. Recall that the subcategory \( \mathcal{C}_{i+1} \subset \mathcal{O}(W) \) consists of all objects annihilated by an ideal \( \mathcal{J}_{n+1-i} \subset \mathcal{U} \), while the self-equivalence \( \mathcal{C}_{i}/\mathcal{C}_{i+1} \) is induced from taking the tensor product with the bimodule \( B_{n+1-i,\lambda} \in \text{HC}^{1,1}_{\lambda,\lambda}(\mathcal{U}) \) mentioned in Lemma 4.3. This bimodule is annihilated by \( \mathcal{J}_{n+1-i} \) on the left and on the right. Inside \( \mathcal{C}_{i}/\mathcal{C}_{i+1} \) consider the Serre subcategory \( (\mathcal{C}_{i}/\mathcal{C}_{i+1})_{\pi} \) that is spanned by the images of all simples in \( \mathcal{C}_{i} \) whose associated variety is contained in \( \mathcal{U} \). We can define \( (\mathcal{C}_{i}/\mathcal{C}_{i+1})_{\pi} \) similarly. So we can form the quotient \( \mathcal{C}_{i}/\mathcal{C}_{i+1} \).

We note that, by the construction, the set \( \mathcal{P}_0(\mathcal{U}) \) splits into the union \( \mu \mathcal{P}_0(\mathcal{U}) \), such that the simples in \( (\mathcal{C}_{i}/\mathcal{C}_{i+1})_{\pi} \) are precisely those annihilated by the primitive ideals in \( \mathcal{P}_0(\mathcal{U}) \).

Since the equivalence \( \mathcal{C}_{i}/\mathcal{C}_{i+1} \to \mathcal{C}_{i}/\mathcal{C}_{i+1} \) is given by taking a tensor product (from the left) with \( B_{n+1-i,\lambda} \), it induces an equivalence \( (\mathcal{C}_{i}/\mathcal{C}_{i+1})_{\pi} \to (\mathcal{C}_{i}/\mathcal{C}_{i+1})_{\pi} \) by tensoring with an object in the subquotient \( \text{HC}^{1,1}_{\lambda,\lambda}(\mathcal{U}) \). This self-equivalence restricts to that of the
semisimple part \((C_i/C_{i+1})_{ss} \subset (C_i/C_{i+1})_0\). This semisimple part is equivalent (under the Beilinson-Bernstein equivalence) to the subcategory \(E_i\), where \(E_i := \sum_\mathcal{J} \mathcal{U}_\mathcal{J}/\mathcal{J}\), here the sum is taken over all \(\mathcal{J} \in \text{Pr}_\mathcal{D}(\mathcal{U}_\lambda)\). From here we conclude that \(\mathcal{B}_i := \mathcal{B}_{n+1-i,A_1} \otimes \mathcal{U}_\lambda E_i\) (the image of \(E_i\) under the equivalence) is in \(E_i \mathcal{D}_E E_i\). Tensoring from the left with this object coincides with the self-equivalence of \((C_i/C_{i+1})_{ss}\) of interest.

Recall the set \(Y\) of the finite dimensional irreducible representations of \(\mathcal{W}_\lambda\) that is equipped with an \(\tilde{A}\)-action such that \(Y/\tilde{A} = \text{Pr}_\mathcal{D}(\mathcal{U}_\lambda)\). We can split \(Y\) into the union \(Y = \bigsqcup_i Y_i\) according the decomposition of \(\text{Pr}_\mathcal{D}(\mathcal{U}_\lambda)\). Then the category \(E_i \mathcal{D}_E\) is identified with \(\text{Sh}^\tilde{A}(Y_i \times Y_i)\). The self-equivalence of this category is given by convolving with of \(\mathcal{B}_i\) viewed as an object in \(\text{Sh}^\tilde{A}(Y_i \times Y_i)\).

**Lemma 4.4.** Suppose the convolution with \(F \in \text{Sh}^\tilde{A}(Y_i \times Y_i)\) gives an autoequivalence of \(\text{Sh}^\tilde{A}(Y_i \times Y_i)\). Then the there is an \(\tilde{A}\)-equivariant bijection \(\sigma : Y_i \to Y_i\) such that the graph of \(\sigma\) coincides with the support of \(F\) and the fibers of \(F\) over the support all have dimension 1.

**Proof.** Let \(F'\) denote the inverse sheaf of \(F\). For any \(x, y \in Y_i\), we have

\[
\delta_{xy} = \sum_{z \in Y_i} \dim F'_{x,z} \dim F_{z,y}.
\]

It follows that, for any \(y \in Y_i\), there is exactly one \(z\) such that \(F_{z,y} \neq 0\) and in this case \(\dim F_{z,y} = 1\). The map \(y \mapsto z\) is a required bijection. It is \(\tilde{A}\)-equivariant because \(F\) is \(\tilde{A}\)-equivariant.

The claim that \(\text{wc}_D\) preserves the right cells follows from the fact that \(\text{wc}_D\) is given by a convolution on the left. The claim that \(\text{wc}_D\) permutes the left cells preserving the Lusztig subgroups follows from Lemma 4.5. This finishes the proof of (i) of Theorem 1.1.

Let us proceed to the proof of (ii). The construction above implies that the action of \(\text{wc}_{D_1}\) on a two-sided cell \(c\) is given by convolving with \(F := \bigoplus_i B_i \in \text{Sh}^\tilde{A}(Y \times Y)\) on the left. Convolving with \(F^*\) on the right defines a commuting \(\text{Cact}_W\)-action on \(c\). Clearly \((F \ast G)^* = G^* \ast F^*\). On the level of irreducible objects the involution \(G \mapsto G^*\) corresponds to \(w \mapsto w^{-1}\), see [Lu2] 3.1 (g) and (h)]. This completes the proof of (ii).

### 4.6. **Type A.**

Here we are going to prove (iv) of Theorem 1.1. For this we will explain how to compute \(\text{wc}_D\).

The following is a direct right handed analog of [Ma Theorem 3.1].

**Lemma 4.6.** There is a unique involution \(\sigma : W \to W\) such that \(C_w T_{w_0} = \alpha C_{\sigma(w)} + \sum_{w' \prec_{LR} w} \beta_{w'} C_{w'}\) in \(\mathcal{H}_w\), where \(\alpha\) is of the form \(\pm v^k\) for \(k \in \mathbb{Z}\).

**Lemma 4.7.** We have \(\text{wc}_D = \sigma\).

**Proof.** Let \(c_w \in \mathcal{C}W\) stand for the specialization of \(C_w\) to \(v = 1\) so that \(c_w = [L_w]\). We know that \([\mathcal{MC}_D L_w] \pm [L_{\text{wc}_D w}] \in I_{LR}\), where \(I_{LR}\) is the span of \(c_w\) with \(w' \prec_{LR} w\). But

\[
\mathcal{MC}_D L_w = [L_w] w_0 \text{ by Lemma 3.4}\]

By Lemma 1.6, \([L_w] w_0 \pm c_{\sigma(w)} \in I_{\prec_{LR}}\). This implies the claim of the present lemma.

Now let us assume that \(W = \mathfrak{S}_n\). Recall that \(\mathfrak{S}_n\) is in bijection (RSK) with the set of pairs \((P, Q)\) of standard Young tableaux of the same shape. Let us write \((P_w, Q_w)\) for the tableaux corresponding to \(w\). Recall that \(w \sim L w'\) if and only if \(Q_w = Q_{w'}\) and that
Now it follows from [Ma, Section 3.6] that \( Q_{\sigma(w)} = Q_{w}^* \), where \( \cdot^* \) denotes the Schützenberger involution uniquely specified by \( Q_{w}^* := Q_{w0w0w} \).

**Example 4.8.** Consider the case \( n = 3 \). Here we have four right cells: \( \{\text{id}\} \), \( \{(12), (231)\} \), \( \{(23), (312)\} \), \( \{31\} \). The Schützenberger involution swaps the two standard Young tableaux of shape 21. So it swaps the elements of the right cells with two elements.

The last example together with (iii) of Theorem 1.1 shows that the bijections \( wc_{i,i+1} \) are elementary Knuth transforms. It is known that one can get any element of a given right cell from another element of the same cell using a sequence of these transforms. This completes the proof of (iv).

5. **Ramifications**

5.1. **Unequal parameters and affine type.** An analog of Theorem 1.1 should be true for Hecke algebras with unequal parameters as well (though it is unclear whether Lusztig subgroups make sense in that generality). Lusztig has proved a direct analog of Lemma 4.6 in that setting, see [Lu3]. Then one can use (iii) of Theorem 1.1 to define the bijections \( wc_{D_1} \) for an arbitrary subdiagram \( D_1 \subset D \). The first two relations in (1.1) come for free and the last one should not be difficult to check.

The strategy of the first paragraph should work for \( D \) of affine type as well. The cactus group is defined in the same way (one considers subdiagrams \( D_1 \) of finite type only). In the equal parameter case, the action should have a categorical interpretation via affine categories \( \mathcal{O} \), as in Section 4.4.

5.2. **Categories \( \mathcal{O} \) for quantized symplectic resolutions and Rational Cherednik algebras.** Our techniques should generalize to the case of categories \( \mathcal{O} \) over quantizations of symplectic resolutions, see [BLPW], and, perhaps, to categories \( \mathcal{O} \) over Rational Cherednik algebras, [GGOR].

Let us elaborate on the symplectic resolution setting. For the definitions and details a reader is referred to [BPW], [BLPW]. An outcome is that we have two vector spaces, the \( \mathbb{C} \)-space \( p \) of quantization parameters with fixed rational form \( p \) and also a \( \mathbb{Q} \)-space \( s \) (of rational co-characters of a certain torus \( T \)). Both \( p \) and \( s \) come with a finite collection of codimension 1 subspaces (walls) that split them into chambers and with integral lattices \( p_{\mathbb{Z}}, s_{\mathbb{Z}} \).

For any \( \theta \in p_{\mathbb{Q}} \) and \( \nu \in s_{\mathbb{Q}} \) that lie inside their chambers and any \( \lambda \in p \), we have a highest weight category \( \mathcal{O}(\theta, \nu, \lambda) \). This category depends only on the chambers of \( \theta, \nu \) and \( \lambda + p_{\mathbb{Z}} \). The simple objects in these categories are labelled by the fixed points of \( T \) on a certain symplectic variety \( X \) (that is a symplectic resolution of interest). For fixed \( \lambda \), the categories \( \mathcal{O}(\theta, \nu, \lambda) \) are derived equivalent: there are wall-crossing functors that switch \( \theta \) that were introduced in [BPW, Section 6.4], and cross-walling functors introduced in [BLPW, Section 8.2]. The latter were proved to be equivalences in [L5, Theorem 6.3].

The long wall-crossing functor (that switches the chamber of \( \theta \) to the opposite) was shown to be a perverse equivalence in [BL0, Section 4]. Using this and techniques of [L3, Section 6] one can show that wall-crossing functors through faces are perverse equivalences. The corresponding bijections should satisfy the relations in the cactus groupoid for the hyperplane arrangement in \( p_{\mathbb{Q}} \).

The long cross-walling functor was shown in [L5] to be an inverse Ringel duality up to a homological shift. Since the inverse Ringel duality coincides with the long wall-crossing, this allows to define the long cross-walling bijection. To define these bijections
for arbitrary faces of the hyperplane arrangement in $\mathfrak{g}_C$ one can use [L5, Proposition 6.4].

Again, the resulting bijections should satisfy cactus relations.

One can define the notions of left and right cells in $X^T$ (this depends on $\lambda$), see [BLPW, Section 7.5]. By the definition there, the wall-crossing bijections preserve the right cells, while the cross-walling bijections should preserve the left cells.

The case considered in the present paper corresponds to $X = T^*(G/B)$, $T$ being a maximal torus of $B$ and integral $\lambda$. In this case, all categories $\mathcal{O}(\theta, \nu, \lambda)$ are naturally identified. In a subsequent paper, we will consider in detail the case when $X$ is a Nakajima quiver variety of affine type (the set $X^T$ has to do with multipartitions in this case).

Another setting, where one can define wall-crossing bijections, is for categories $\mathcal{O}$ for Rational Cherednik algebras $H_{1,c}(\Gamma)$, where $\Gamma$ is a complex reflection group. The labelling set for the simple objects in this case is $\text{Irr} \, \Gamma$. Wall-crossing functors (for a suitable hyperplane arrangement) were introduced in [L3]. It was proved in that case that wall-crossing through faces are perverse. The corresponding wall-crossing bijections should be viewed as generalizations and extensions of the Mullineux involution.

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