The emergence of universal relations in the AdS black holes thermodynamics

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Abstract

Our primary goal in this paper is to confirm new universal relations in black hole thermodynamics. We investigate the universal relations by selecting different black holes. First, we obtain the black holes thermodynamic relations assuming a new minor correction is added to the AdS part of the action. Then we confirm the universal relations by performing a series of direct calculations. It is noteworthy that according to each of the properties related to black holes, a new universal relation can be obtained according to this method. We confirm two different types of these universal relations for various black holes. Furthermore, we also consider black holes in AdS space surrounded by perfect fluid. We use a small correction to the action and obtain the modified thermodynamic quantities. We achieve two new universal relations which correspond to the parameters of perfect fluid and magnetic charge of the Bardeen AdS Black Hole. Finally, the new universal relation leads us to understand the charge-to-mass ratio, i.e., weak gravity conjecture like behavior. We also find that the weak gravity conjecture condition is satisfied for the black hole surrounded by perfect fluid.

1. Literature survey and motivation

In this paper we deduce new universal relations in black hole thermodynamics. Researchers have done much work to achieve universal relations in recent years. For example, Davida and Nian [1] investigated the universal entropy and Hawking radiation of near-extremal AdS$_3$ black holes. Chen, Hong, and Tao [2] also studied universal thermodynamic extremality relations for charged AdS black holes surrounded by quintessence.

In this paper, we want to study some universal relations between corrections to entropy and extremality bounds. Given the universal relations implications, universal thermodynamic extremality relations have not been fully explored in studies related to specific black holes. Therefore, in this work, we want to entirely evaluate several types of black holes with respect to these universal relations. Thus, by selecting several types of black holes in the AdS space, using a small constant correction added to the action, we obtain the modified thermodynamic relations and examine them to confirm the universal relations. The main point is using a small correction in action for making modified thermodynamic relations that this correction somehow connects to the AdS part. So it leads to modifying parameters such as mass, temperature, etc., in AdS black holes. With respect to the first law of thermodynamics, we can establish universal relations between some of them that we fully explain in the next section, which is due to this small correction that is added to the AdS part of the action. Also, we investigate some new universal relations connected to weak gravity conjecture (WGC). But in this paper, we want to challenge the new condition we pointed out earlier. One of the methods used in this theme, which is based on string theory, is the swampland program [3], which describes areas generally compatible with quantum gravity by stating some criteria. One of these criteria is the WGC, which has been widely studied in the last few years [4]. This conjecture
states that gravity is the weakest force, that is, there are states whose mass \( M \) is less than their charge \( Q \) which is \( \frac{Q}{M} \geq 1 \) [5–19]. Studies on this issue use black holes with different characteristics of mass, charge, etc. The important point is that this relation is not established for the black holes with naked singularity. Cosmic censorship is usually used to address this issue. One possible solution to this problem is to use a correction to the action that leads to modifying black hole solutions and, in a way, inverse the charge-to-mass ratio. For example, in [20], higher-derivative corrections are used for the mentioned issue. The result indicates that the charge-to-mass ratio is larger than 1 using these higher-derivative operators and satisfies the universal relation.

The idea of correcting higher derivatives has been widely used in the literature. One can find some works that have been done to limit effective theories using WGC in [21–33]. Concepts related to WGC and obtaining universal relations have been explored in many works in recent years. In a recent investigation, researchers prove the concept of WGC in flat space concerning Wald entropy [24]. The concepts related to entropy changes due to the charge and mass of black holes and higher-derivative corrections have been studied to prove the notion of WGC about black holes. Goon and Penco [34] also proposed a universal relation. This universal relation has also been studied for charged AdS black holes in four-dimensional space, which exhibits black hole WGC-like behavior [35]. Other work has also been done on the Goon-Penco universal relation, for example, in examining four-derivative corrections competencies to the geometry of charged AdS black holes. For further review of this relation, one can also look into [36–44]. Furthermore, In their previous work, the authors of these articles also examined black holes surrounded by quintessence dark energy and cloud of strings, which had beautiful results and led to the introduction of new universal relations [45]. We want to examine another form of universal relation for different black holes. We add small constant corrections to the corresponding action and obtain the modified thermodynamic quantities and relations. These modified quantities help us to perform a series of calculations and confirm the universal relations. We also see the effect of small corrections on the thermodynamic relations. The corresponding correction increases the charge-to-mass ratio, which indicates WGC-like behavior. To study thermodynamic relationships and the WGC, we select Reissner-Nordström-AdS, rotating Bardeen and Kerr-Newman-AdS black holes surrounded by perfect fluid matter. As we know, dark matter also includes cold dark matter (CDM), warm dark matter (WDM), scalar background dark matter (SFDM), and perfect fluid dark matter (PFDM). Here, we combine PFDM with a black hole solution and investigate a particular type of universal relation as in [45–65]. Therefore, in addition to PFDM, we consider black holes with string fluid and obtain the new universal relation as in [66–74].

The paper is organized as follows. In section 2, we discuss the universal relation for different models. For instance, in subsection 2.1, 2.2 and 2.3, we discuss the universal relations of AdS Schwarzschild black holes, charged rotating BTZ black holes, accelerating black holes, respectively. In subsection 2.4, we also confirm the universal thermodynamic relations for the Reissner-Nordström-AdS black hole with the perfect fluid dark matter due to a minor constant correction of the action with respect to confirming the WGC-like behavior. In subsection 2.5, we consider Kerr-Newman-AdS black hole surrounded by perfect fluid matter. Here also, we achieve the new universal relation and study the role of WGC on the corresponding system. In subsection 2.6, we consider rotating Bardeen black holes in AdS space surrounded by perfect fluid, and we confirm a new universal relation. Finally, we discuss and conclude in the last section 3.

2. The models

The corrections made on general relativity have always led to certain structures to be investigated in cosmology, among which the relationship between extremality bound and entropy can be mentioned. This issue has led to the creation of a series of universal relations for different structures of black holes. The most important universal relations that have recently been introduced in [34], are related to the thermodynamic relations of black holes. Considering the point that applying a series of perturbative corrections to GR leads to a change in entropy and extremality bound, we want to study it in this paper. Therefore, this issue can be a motivation to further investigate the results of these perturbative corrections. Of course, different types of these perturbations have been challenged in the literature, but the important point here is that adding even a very small modification to the AdS part of black holes also leads to the modification of many thermodynamic relations that lead to universal relations results. This very small correction is added to the AdS part. As a result, we should expect to calculate the corrected parameters and establish universal relations. It also leads to reducing the corrected parameters such as mass. This, in turn, leads to an increase in the charge-to-mass ratio or WGC-like behavior. Therefore, in this article, we intend to calculate these modified structures after applying a small correction that is added to the AdS part of the action of the corresponding black holes. Then, by performing a series of mathematical calculations specified in the following, we create universal relationships between different structures according to the first law of thermodynamics. The establishment of these universal relations in different formats, such as using higher derivatives, which we have already mentioned, has been investigated. Now we want to show that this
modification is minor and is added to the AdS part of the action. So, we selected several AdS black holes and will proceed with the calculations according to the mentioned process. Due to the modification of the action, the black hole solution is also modified. Therefore, some thermodynamic quantities of black holes will also be modified. Note that most of the time, one assumes the AdS radius to be constant, although exciting consequences have been obtained by mapping it to thermodynamic pressure. Now we want to consider the small shift for the AdS radius and, with this change in this parameter, challenge the proof of the thermodynamic universal relations. In this section, we selected several AdS black holes and will proceed with the calculations according to the mentioned process.

2.1. AdS Schwarzschild black holes
Concerning all the concepts discussed above, we examine the universal relations by considering different types of black holes. In this context, we first consider the AdS Schwarzschild black hole [39]. The action of an AdS de-Sitter Schwarzschild black hole is given by

\[ S = \frac{1}{16\pi} \int d^3x \sqrt{-g} \left( R - 2\Lambda \right), \]

where the gravitational constant \( G = 1 \) (for simplicity) and cosmological constant \( \Lambda = \frac{1}{\ell^2} \). The solution to the Einstein field equations for this black hole can be written as follows

\[ ds^2 = f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\Omega^2, \]

where \( f(r) \) has the following form:

\[ f(r) = 1 + \frac{r^2}{\ell^2} - \frac{2M_0}{r}. \]

Here \( M_0 \) characterizes the mass of the black hole. The outer and inner horizons associated with a black hole are calculated from \( f(r) = 0 \), and thus the values of temperature, entropy, etc., can be easily calculated using the thermodynamic relations of the black hole. Now, we implement a minimal constant correction by introducing \( \epsilon \) to the action as follows:

\[ S = \frac{1}{16\pi} \int d^3x \sqrt{-g} \left( R - (1 + \epsilon) \times 2\Lambda \right). \]

Due to the modification of the action, the black hole solution is also modified. Therefore, each of the thermodynamic values of the black holes will also be modified. Hence, the modified mass and temperature are obtained by considering a small constant correction \( \epsilon \) [39] as follows,

\[ M = \frac{\epsilon S(1 + \epsilon)}{128\ell^2\pi^3} - \frac{S}{8\pi}, \]

\[ T = \frac{3\epsilon S(1 + \epsilon)}{128\ell^2\pi^3} - \frac{1}{8\pi}. \]

For such black holes, the externality boundary is also modified. So with \( T = 0 \), the extremal entropy bounded by a small constant correction is given by,

\[ S = \frac{4}{\sqrt{3} \left( \frac{1}{\ell^2\pi^2} + \frac{\epsilon}{\ell^2\pi^2} \right)}. \]

Here, upon solving the temperature equation, we chose the real positive entropy. Also, by solving the equation (5), the constant correction takes the following value,

\[ \epsilon = \frac{128\ell^2M\pi^3 - 16\ell^2\pi^2S - S^3}{S^3}. \]

The derivative of \( \epsilon \) with respect to \( S \) is given by

\[ \frac{\partial \epsilon}{\partial S} = \frac{-16\ell^2\pi^2 - 3S^2(1 + \epsilon)}{S^3}. \]

By combining the equations (6), (7) and (9), we will have a relation at \( M \rightarrow M_{\text{ext}} \) as follows,

\[ -T \frac{\partial S}{\partial \epsilon} = \frac{1}{6\sqrt{3} \ell^2} \left( - \frac{f^2}{(1 + \epsilon)} \right)^{\frac{3}{2}}. \]
After calculating the above relation, according to the equations (5) and (7), $M_{\text{ext}}$ is given by

$$M_{\text{ext}} = \frac{1}{3\sqrt{3}} \left( \frac{l^2}{(1 + \epsilon)} \right)^{\frac{1}{2}}.$$  \hspace{1cm} (11)

The mass bound will increase with the constant correction parameter $\epsilon$, and in a way, this added correction can satisfy the conditions related to WGC. According to the concepts, we take the derivative of $M_{\text{ext}}$ with respect to this constant parameter, we have,

$$\frac{\partial M_{\text{ext}}}{\partial \epsilon} = \frac{1}{6\sqrt{3}l^2} \left( \frac{l}{(1 + \epsilon)} \right)^{\frac{1}{2}}.$$  \hspace{1cm} (12)

Incidentally, the right hand sides of the two (10) and (12) are precisely same. Hence we have first proved the Goon-Penco universal extremality relation [34] for the AdS Schwarzschild black hole. In the following, by selecting the other black holes and confirming the mentioned relation, we will examine the new universal relations that are somehow derived from the characteristics of black holes such as rotating ones.

### 2.2. Charged rotating BTZ black holes

In this section, we chose the charged rotating BTZ black holes [43, 44] in order to obtain the modified thermodynamic relations. We also consider the universal thermodynamic relations. First, we introduce the action and corresponding lapse function for the CR-BTZ black hole as follows,

$$S = \frac{1}{2\pi} \int d^3x \sqrt{-g} \left[ R + 2\Lambda - \frac{\pi}{2} F_{\mu\nu} F^{\mu\nu} \right].$$

$$f(R) = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2} - \frac{\pi}{2} Q^2 \ln r.$$

Now we consider a small correction in the action so that we will have,

$$S = \frac{1}{2\pi} \int d^3x \sqrt{-g} \left[ R + (1 + \epsilon) \times 2\Lambda - \frac{\pi}{2} F_{\mu\nu} F^{\mu\nu} \right].$$

Therefore, with reference to [40, 43, 44], the thermodynamic parameters such as mass, temperature, and angular velocity, as modified by the factor $1 + \epsilon$, are given by

$$M = \frac{4f^2 \pi^2}{S^2} + \frac{S^2(\epsilon + 1)}{16f^2 \pi^2} - \frac{1}{2} \frac{\pi Q^2 \log \left[ \frac{S}{4\pi} \right]}{S},$$  \hspace{1cm} (13)

$$T = -\frac{8f^2 \pi^2}{S^3} - \frac{\pi Q^2}{2S} + \frac{S(1 + \epsilon)}{8f^2 \pi^2},$$  \hspace{1cm} (14)

$$\Omega = \frac{8f^2 \pi^2}{S^2}.$$  \hspace{1cm} (15)

Now $T = 0$ from equation (14), the extremal entropy bound can be obtained. After solving this temperature equation, we choose the real positive entropy as

$$S = \sqrt{\frac{2\pi^2 Q^2}{1 + \epsilon} + \frac{2\pi^2 (16f^2 + \pi^2 Q^4 + 16f^2 \epsilon)^2}{1 + \epsilon}}.$$  \hspace{1cm} (16)

From equation (13), we obtain,

$$\epsilon = -\frac{64S^4 l^2 \pi^4}{S^4} + 16f^2 M^2 \pi^2 S^2 - S^4 + 8l^2 \pi^4 Q^2 S^4.$$  \hspace{1cm} (17)

The derivative with respect to $S$ leads to,

$$\frac{\partial \epsilon}{\partial S} = \frac{128l^2 f^2 \pi^4 + 8l^2 \pi^4 S^2 Q^2 - 2S^4(1 + \epsilon)}{S^3}.$$  \hspace{1cm} (18)

Exploiting relations (14), (16) and (18), we have a relation at $M \to M_{\text{ext}}$ as follows,

$$-T \frac{\partial S}{\partial \epsilon} = \frac{\pi Q^2 + \sqrt{l^4 \pi^4 Q^4 + 16f^2 l^2 (1 + \epsilon)}}{8l^2(1 + \epsilon)}.$$  \hspace{1cm} (19)

The expression for $M_{\text{ext}}$ in this case is given as,

$$M_{\text{ext}} = \frac{\sqrt{l^4 \pi^4 Q^4 + 16f^2 l^2 (1 + \epsilon)}}{4l^2} - \frac{\pi Q^2}{4} \log \left[ \frac{\pi Q^2 + \sqrt{l^4 \pi^4 Q^4 + 16f^2 l^2 (1 + \epsilon)}}{8l^2(1 + \epsilon)} \right].$$  \hspace{1cm} (20)
The derivative of $M_{\text{ext}}$ with respect to $\epsilon$ leads to,
\[
\frac{\partial M_{\text{ext}}}{\partial \epsilon} = \frac{\rho^2 Q^2 + \sqrt{\rho^2 Q^4 + 16J^2 F(1 + \epsilon)}}{8F(1 + \epsilon)}.
\]
(21)

The right hand sides of equations (19) and (21) are equal, so the universal relation for this black hole is also satisfied. The concepts presented and the rotating nature of this black hole lead us to examine another universal relation. We now study the new universal relation. Therefore, concerning equations (16) and (20), the shifting mass bound is given by,
\[
\frac{\partial M_{\text{ext}}}{\partial \epsilon} = \frac{S^2}{16\rho^2 \pi^2}.
\]
(22)

From equation (17), it is a matter of calculation only to show
\[
\frac{\partial J}{\partial \epsilon} = -\frac{S^4}{128\rho^4 \pi^4}.
\]
(23)

This expression further simplifies to
\[
-\Omega \frac{\partial J}{\partial \epsilon} = \frac{S^2}{16\rho^2 \pi^2},
\]
(24)

the right hand sides of the two equations (22) and (24) are the same. This relation is due to the rotating nature of black holes and, in this extremal limit, this equation is well proved. Hence, a new universal relation could be defined for a black hole based on its characteristics.

2.3. Accelerating black holes

In the previous two sections, we proved the universal relation for the Schwarzschild and charged BTZ black holes. In this section, we consider an accelerating black hole \([41]\) to check the universal relations. The metric for the accelerating black hole is given by,
\[
d^2 s = \frac{1}{\omega^2} \left[ f(r) dr^2 - f^{-1}(r) dr^2 - r^2 \frac{d\theta^2}{g(\theta)} + r^2 g(\theta) \sin^2 \theta d\phi^2 \right] K^2,
\]
(25)

where $K$ is the conical deficit and
\[
f(r) = (1 - A^2 r^2) \left(1 - \frac{2m}{r}\right) + \frac{r^2}{\rho^2}, \quad g(\theta) = 1 + 2mA \cos \theta, \quad \omega = 1 + Ar \cos \theta,
\]
(26)

here $A$ is the acceleration and $m$ is the mass scale of the black hole. We can modify the action with the small correction constant parameter. So with respect to a constant small correction $\epsilon$ and \([41]\), each of the thermodynamic values of the black holes also gets modified. Hence, the modified mass and temperature are obtained as follows:

\[
M = \frac{S^3 + f^2(16\pi^2 S(1 + \epsilon) - A^2 S^3)(1 + \epsilon)}{8F(16\pi^2 - A^2 \pi S^2)},
\]
(27)

\[
T = \frac{48\pi^2 S^2 - A^2 S^4 + f^2(-16\pi^2 + A^2 S^2)^2(1 + \epsilon)}{8F^2(-16\pi^2 + A^2 S^2)^2}.
\]
(28)

So with $T = 0$, the extremal entropy bounded by a small constant correction is given by,
\[
S = \sqrt{\frac{-24\pi^2 + 16A^2 F^2(1 + \epsilon) - 8\pi^2 \sqrt{9 - 8A^2 F^2(1 + \epsilon)}}{-A^2 + A^2 F^2(1 + \epsilon)}}.
\]
(29)

Here we chose the real positive entropy. After simplifying equation \((27)\), we get,
\[
\epsilon = \frac{S^3 - f^2(8M\pi - S)(16\pi^2 - A^2 S^2)}{f^2(-16\pi^2 S + A^2 S^2)}.
\]
(30)

Now, the derivative of $\epsilon$ with respect to $S$ yields,
\[
\frac{\partial \epsilon}{\partial S} = \frac{-48\pi^2 S + A^2 S^3}{f^2(-16\pi^2 + A^2 S^2)^2} - \frac{(1 + \epsilon)}{S}.
\]
(31)

Exploiting equations \((28)\), \((29)\) and \((31)\), we have a relation at $M \rightarrow M_{\text{ext}}$ as,
\[
-T \frac{\partial S}{\partial \epsilon} = \sqrt{\frac{f^2(1 + \epsilon)}{-6 + 4A^2 F^2(1 + \epsilon) + 2\sqrt{9 - 8A^2 F^2(1 + \epsilon)}}}.
\]
(32)
According to the equations \((27)\) and \((29)\), \(M_{\text{ext}}\) is given by,

\[
M_{\text{ext}} = \left( -1 + A^2 l^2(1 + \epsilon) \right) \left( -3 + \sqrt{9 - 8A^2 l^2(1 + \epsilon)} \right) \frac{1}{\sqrt{2} A^2 l^2(1 + \epsilon)}.
\]

\(33\)

This added correction can satisfy the conditions related to WGC. We calculate the derivative of the above relation with respect to the parameter \(\epsilon\) and have

\[
\frac{\partial M_{\text{ext}}}{\partial \epsilon} = \frac{A^2 l^2(1 + \epsilon)}{-6 + 4A^2 l^2(1 + \epsilon) + 2\sqrt{9 - 8A^2 l^2(1 + \epsilon)}}.
\]

\(34\)

The relations \((32)\) and \((34)\) coincide and the Goon-Penco universal extremality relation for this black hole is also justified. Notice that the relations are also valid for the other types of this black hole with some other parameters such as acceleration, cosmological constant, rotation, and charge.

### 2.4. Reissner-Nordström AdS black hole with PFDM

In this section, we wish to study the Goon-Penco universal extremality relation for Reissner-Nordström AdS black hole with PFDM and obtain the other new universal relation. So, we consider Reissner-Nordström AdS black hole. We use a small correction as \(\epsilon\) to the action, and obtain the modified thermodynamic quantities and relations. Here we first write the action which is given by [74],

\[
S = \int d^4x \sqrt{-g} \left[ \frac{R}{16 \pi G} - \frac{\Lambda}{8 \pi G} + \frac{1}{4} F_{\mu \nu} F^{\mu \nu} + L_{\text{DM}} \right],
\]

\(35\)

where \(G, \Lambda, F_{\mu \nu}\) and \(L_{\text{DM}}\) are the Newton gravity constant, cosmological constant, tensor of electromagnetic field and dark matter Lagrangian density, respectively. So, the solution of the action with PFDM is as follows [75],

\[
ds^2 = -f(r) dt^2 + f^{-1}(r) dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\]

\(36\)

where \(f(r)\) is

\[
f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{\Lambda r^2}{3} + \frac{\alpha}{r} \ln \left( \frac{r}{|\alpha|} \right).
\]

\(37\)

Here, the \(M, Q, \alpha\) are the mass, charge and the parameter describing PFDM density, respectively. \(f(r) = 0\) determine the outer and inner horizons. The thermodynamic relation such as temperature, entropy, etc can be easily calculated. The mass and temperature of Reissner-Nordström AdS black hole with PFDM are given by,

\[
M = \frac{\sqrt{\pi} Q^2}{\sqrt{S}} + \frac{\sqrt{S}}{2 \sqrt{2 \pi}} - \frac{S^2}{4 \sqrt{2} \pi^2} + \frac{1}{2} \alpha \log \left( \frac{\sqrt{S}}{\sqrt{2 \pi} \alpha} \right),
\]

\(38\)

and

\[
T = -\frac{\sqrt{\pi} Q^2}{\sqrt{S}} + \frac{1}{4 \sqrt{2 \pi}} - \frac{3 \sqrt{S}}{8 \sqrt{2} \pi^2} + \frac{\alpha}{4 S}.
\]

\(39\)

Here, we consider a small constant correction as \(\epsilon\) to the action. So, we have following modified action:

\[
S = \int d^4x \sqrt{-g} \left[ \frac{R}{16 \pi G} + (1 + \epsilon) \left( \frac{\Lambda}{8 \pi G} \right) + \frac{1}{4} F_{\mu \nu} F^{\mu \nu} + L_{\text{DM}} \right].
\]

\(40\)

With reference to [74] the expressions for mass and temperature as modified by the factor \(1 + \epsilon\) take the form

\[
M = \frac{\sqrt{\pi} Q^2}{\sqrt{S}} + \frac{\sqrt{S}}{2 \sqrt{2 \pi}} - \frac{S^2 (1 + \epsilon)}{4 \sqrt{2} \pi^2} + \frac{1}{2} \alpha \log \left( \frac{\sqrt{S}}{\sqrt{2 \pi} \alpha} \right),
\]

\(41\)

and

\[
T = -\frac{\sqrt{\pi} Q^2}{\sqrt{S}} + \frac{1}{4 \sqrt{2 \pi}} - \frac{3 \sqrt{S} (1 + \epsilon)}{8 \sqrt{2} \pi^2} + \frac{\alpha}{4 S}.
\]

\(42\)

We see here that the modified mass and temperature of the black hole are written in terms of entropy \(S\), charge \(Q\), \(\alpha\) and correction parameter \(\epsilon\). By solving equation \((41)\), we obtain constant correction parameter \(\epsilon\) as,

\[
\epsilon = \frac{4 \sqrt{\pi} \alpha Q^2 - 8 \pi^2 M \pi^2 \sqrt{S} + 2 \sqrt{S} P S \pi - \sqrt{S} S^2 + 4 \pi^2 \sqrt{S} \alpha \pi \log \left( \frac{\sqrt{S}}{\sqrt{2 \pi} \alpha} \right)}{\sqrt{S}^2}.
\]

\(43\)
Now, we take derivative with respect to $S$, which yields,

$$
\frac{\partial \varepsilon}{\partial S} = \frac{l^2 \pi (-8\pi Q^2 - 2S + \sqrt{2\pi S} (6M + \alpha - 3\alpha \log \left( \frac{S}{\sqrt{2\pi \alpha}} \right))}{S^3}.
$$

(44)

We use the equations (42) and (44) and then the corresponding limit is taken, which simplifies equation as follows,

$$
-T \frac{\partial S}{\partial \varepsilon} = -\frac{S^2}{4\sqrt{2} l^2 \pi^2}.
$$

(45)

To obtain the second part of universal relation, we solve the temperature equation and obtain the corresponding entropy. We use equations (41), (42) and corresponding entropy, to obtain

$$
\frac{\partial M_{\text{ext}}}{\partial \varepsilon} = -\frac{S^2}{4\sqrt{2} l^2 \pi^2}.
$$

(46)

We see that the equations (45) and (46) are exactly same. So, we have confirmed the Goon-Penco universal extremality relation [34] for Reissner-Nordström AdS black hole with PFDM. In the second step, we investigate another universal relation. So, we use the relation (43) and calculate,

$$
\frac{\partial \varepsilon}{\partial Q} = \frac{8l^2 \pi^2 Q}{S^2}.
$$

(47)

By considering equation (47), electric potential $\Phi = \frac{\partial \varepsilon}{\partial S}$ and extremality bound, we get,

$$
-\Phi \frac{\partial Q}{\partial \varepsilon} = -\frac{S^2}{4\sqrt{2} l^2 \pi^2}.
$$

(48)

Here, two also equations (48) and (46) are the same.

In the following, we try to find another universal relationship between mass and pressure $(P = \frac{3}{8\pi l^2} = -\frac{\alpha}{8\pi})$. So with respect to equation (41) we will have,

$$
\frac{\partial P}{\partial \varepsilon} = 4\sqrt{2} P \frac{S^2}{3S^2(1 + \epsilon)}.
$$

(49)

Therefore, according to $V = -\frac{1}{3} \sqrt{\frac{\pi}{\alpha}} S^3 (1 + \epsilon)$ and extremal bound, we have the following expression,

$$
-V \frac{\partial P}{\partial \varepsilon} = -\frac{S^2}{4\sqrt{2} l^2 \pi^2},
$$

(50)

the right hand sides of the two (48) and (50) are exactly same and universal relation is proved. Now our goal is to get a new universal relation, the meaning of being new is to be related to a newly introduced parameter describing PFDM density ($\alpha$). To achieve such new universal relation, we use (43) and obtain the following equation:

$$
\frac{\partial \varepsilon}{\partial \alpha} = -\frac{4l^2 \sqrt{5} \pi^4 + 4l^2 \sqrt{3} \pi^4 \log \left( \frac{\sqrt{\pi}}{\sqrt{2\alpha}} \right)}{\sqrt{2} S^2}.
$$

(51)

Hence, by using equation (51) and expression $\xi = -\frac{1}{2} + \frac{1}{2} \log \left( \frac{\sqrt{\pi}}{\sqrt{2\alpha}} \right)$, one can obtain,

$$
-\frac{\partial \alpha}{\partial \varepsilon} = -\frac{S^2}{4\sqrt{2} l^2 \pi^2},
$$

(52)

we see here the right hand sides of the two (52) and (50) are exactly same. In fact, we obtained a new universal relation for the Reissner-Nordström AdS black hole with PFDM, so this relationship is confirmed correctly. Now we draw some figures and compare the mass charge ratio and also thermodynamic quantities in the modified and unmodified modes. We fix some parameters and draw figures of the mass-to-charge, to have three cases as charge-mass ratio be bigger, equal and smaller than one. So, when we have extremality the charge-mass ratio is one which is shown in the figure 1 by dashed line. For different radius $l$ of AdS in the uncorrected mass mode, the mass-to-charge ratio is greater than one as shown in the figure 1 (a). For the modified case we have figure 1(b), when the constant correction is always a negative value, the mass of the black hole decreases, and when the small correction is positive, the mass also increases. Therefore, when we consider a small negative correction, the charge-to-mass ratio always increases. As we mentioned in the text, it can behave like WGC. So, we see that the small correction and its parameter in the metric background play an important role for the existence of WGC. In the next section, we calculate the same universal relation for the Kerr-Newman AdS black hole surrounded by perfect fluid matter and also obtain a new universal relation. Then we compare the corresponding results of the two sections with respect to each other.
2.5. Kerr-Newman AdS black hole surrounded by perfect fluid matter

In this section, we want to study the universal relations for Kerr-Newman AdS black hole surrounded by perfect fluid matter in Rastall gravity [76]. The concepts of general relativity and its modification have always been discussed in various branches of physics. Some researchers use its modified form in concepts of conservation condition of the energy-momentum tensor. One of these modified theories was also introduced by Rastall. Rastall gravity is based on the hypothesis introduced by $T_{\mu \nu} = \lambda R_{\mu \nu}$, where $T_{\mu \nu}$ and $\lambda$ are energy-momentum tensor and the Rastall parameter, respectively [77, 78]. The Kerr-Newman AdS black hole solution is given by

$$ds^2 = \frac{\Sigma^2}{f(r)} dr^2 + \frac{\Sigma^2}{f(\theta)} d\theta^2 + \frac{f(\theta)}{\Sigma^2} \left( \frac{dt}{\Xi} - \left( r^2 + a^2 \right) \frac{d\phi}{\Xi} \right)^2,$$

where $f(r) = r - 2M + a^2 + Q^2 - \frac{\Lambda}{3} r^2 (r^2 + a^2) - \alpha r^{\frac{-3}{\lambda+1}}$, $f(\theta) = 1 + \frac{\Lambda}{3} a^2 \cos^2 \theta$, $\Xi = 1 + \frac{\Lambda}{3} a^2$,

$$M = \frac{a^2 \sqrt{\pi}}{\sqrt{S}} \frac{\sqrt{\pi} Q^2}{\sqrt{S}} + \frac{\sqrt{S}}{4 \sqrt{\pi}} + \frac{a^2 \sqrt{S}}{4L \sqrt{\pi}} + \frac{S^2}{16L^2 \pi^2} - 2 \frac{a^2 \sqrt{\pi}}{\sqrt{S}},$$

$$T = \frac{a^2 \sqrt{\pi}}{\sqrt{S}} - \frac{\sqrt{\pi} Q^2}{2S^2} + \frac{1}{8 \sqrt{\pi} S} + \frac{a^2}{8L \sqrt{\pi} S} + \frac{3 \sqrt{S}}{32L^2 \pi^2} + \frac{3 \times 2^{-\frac{2}{\lambda+1}}}{\sqrt{S}} \frac{a^2 \sqrt{\pi}}{\sqrt{S}},$$

and

$$\Omega = \frac{2a \sqrt{\pi}}{\sqrt{S}} + \frac{a \sqrt{\pi}}{2L \sqrt{\pi}}.$$

Now, by considering a small constant correction $\epsilon$ for Kerr-Newman AdS black hole surrounded perfect fluid and [76], we calculate the modified thermodynamic quantities as mass, temperature and angular velocity are given by,
From here, the right hand sides of the two equations (59) and (60), we have,

$$-T \frac{\partial S}{\partial \epsilon} = \frac{\sqrt{S} (4a^2 \pi + S)}{16\pi^2}.$$

To get the second part of the universal relation, we set $T = 0$. By solving for the temperature, obtaining the entropy and using equations (59) and (60), we obtain following equation:

$$\frac{\partial M_{ext}}{\partial \epsilon} = \frac{\sqrt{S} (4a^2 \pi + S)}{16\pi^2}.$$

Here, the right hand sides of the two equations (64) and (65) are exactly same. So, we confirmed the Goon-Penco universal extremality relation. To investigate another universal relation, we use the equation (62) to obtain following relation:

$$\frac{\partial \epsilon}{\partial Q} = -\frac{2\sqrt{\pi} Q}{\sqrt{S} \left( \frac{a^2}{4\pi} + \frac{S^2}{16\pi^2} \right)}.$$

From (66) and electric potential $\Phi$ as well as extremality bound, we have,

$$-\Phi \frac{\partial Q}{\partial \epsilon} = \frac{\sqrt{S} (4a^2 \pi + S)}{16\pi^2}.$$

Also here, the right hand sides of the two equations (67) and (65) are same, so the universal relation is also proved. Therefore, with respect to pressure $P = \frac{3}{8\pi^2} = -\frac{\Lambda}{8\pi}$ and equation (62), we have,
The above equation and extremal bound lead us to obtain following equation:

$$\frac{\partial P}{\partial \varepsilon} = -\frac{P^2 \left( \frac{2}{3} a^2 \sqrt{\pi S} + \frac{3^2}{\sqrt{\pi}} \right)}{\sqrt{4\alpha^2 + S(1 + \epsilon)}}. \quad (68)$$

The right hand sides of the two equations (69) and (65) are exactly same. We also study the other universal relation, hence according to relation (62) and using the equation (61) one can obtain,

$$-\frac{\partial \alpha}{\partial \varepsilon} = \sqrt{S} \frac{(4\alpha^2 + S)}{16\pi^2}, \quad (70)$$

we see, the right hand sides of the two equations (70) and (65) are exactly same. Now we will study the new universal relation with respect to $\eta$ which is conjugate to the perfect fluid parameter $\alpha$. So, according to equation (62),

$$\frac{\partial \varepsilon}{\partial \alpha} = 2 \frac{1}{4\alpha^2 + S(1 + \epsilon)} \times \frac{\frac{\beta^{3/2} S}{2\alpha^2 + S(1 + \epsilon)} + \frac{3}{16\pi^2}}{\beta^{3/2} S + \frac{3}{16\pi^2}}. \quad (71)$$

Now by using the $\eta$ and the equation (71), we confirmed another universal relation which is calculated by,

$$-\eta \frac{\partial \alpha}{\partial \varepsilon} = \sqrt{S} \frac{(4\alpha^2 + S)}{16\pi^2} = \frac{\partial M_{pot}}{\partial \varepsilon}. \quad (72)$$

According to the mass charge plot of the modified thermodynamic relations for the Kerr Newman AdS black hole as shown in figure 2, we want to describe the concepts of WGC. We compare the mass charge ratio and thermodynamic quantities of modified and unmodified cases to each other. The charge and mass are equal in the extremal state or the mass charge ratio is unit as shown in the figure by dashed line. For the uncorrected mass mode, the mass-to-charge ratio is greater than one, we have plotted for different radii $l$ of AdS, as shown in the figure 2 (a). Now we look at the modified state and use the corrected thermodynamic parameters. We evaluate the amount of change in this correction as shown in figure 2 (b). When the constant correction is negative, the mass of the black hole decreases, and when the small correction is positive, the mass increases. Therefore, when we consider small negative correction, the charge-to-mass ratio always increases. So, the negative correction to the action show us this black hole can behave like WGC. Thus, we confirm that the corrections play a very important role in the concept of WGC.

### 2.6. Rotating Bardeen black holes in AdS space surrounded by perfect fluid

In this section, we check universal relations for the rotating Bardeen black holes in AdS space surrounded by perfect fluid [79]. We note here that black holes have singularities [80] and in such cases, densities and curvatures tend to infinity [81, 82]. The physical predictions in these points face serious problems and these singularities always cause serious problems for general relativity. But the Bardeen black hole solution includes a new description of black holes without singularity [79, 83–86]. Now, in order to investigate a new universal relation
Here we introduce the magnetic charge $\vartheta$ to the theory and see the effect of such parameter to the universal relation. The rotating Bardeen black holes in AdS space surrounded by perfect fluid is described by,

$$ds^2 = f(r)dt^2 - f^{-1}(r)dr^2 - r^2d\Omega^2,$$

with

$$f(r) = r^2 + a^2 - \frac{2Mr^4}{(r^2 + \vartheta^2)^2} + \frac{r^2}{f^2} + \alpha r \ln \frac{r}{|\alpha|},$$

where $M$, $\alpha$ and $\vartheta$ denote mass, rotational parameter, perfect fluid parameter and magnetic charge parameter, respectively. With respect to the entropy of a black hole, we investigate the thermodynamic relation of rotating Bardeen black holes in AdS space surrounded by perfect fluid. Here, we consider a small constant correction and with respect to $[79]$, we calculate the modified thermodynamic quantities as given by,

$$M = \frac{(S + \pi \vartheta^2)^2}{2S^2\sqrt{\pi}}((\alpha^2 \pi + S) + \frac{2(1 + \epsilon)}{f^2} + \frac{1}{\sqrt{\alpha}} \log \frac{\sqrt{S}}{\alpha \sqrt{\pi}}),$$

$$T = \frac{(S + \pi \vartheta^2)^2}{2S^2\sqrt{\pi}}\left(-1 - \frac{2\alpha^2 \pi}{S} + \frac{\sqrt{\pi} \alpha}{2\sqrt{S}} + \frac{2(\alpha^2 \pi + S)}{2(S + \pi \vartheta^2)} + \frac{(S - 2\pi \vartheta^2)(1 + \epsilon)}{2f^2(S + \pi \vartheta^2)} - \frac{3\vartheta^2 \alpha \pi \log \frac{\sqrt{S}}{\alpha \sqrt{\pi}}}{2\sqrt{S}(S + \pi \vartheta^2)} \right),$$

and

$$\Omega = \frac{a\sqrt{\pi}(S + \pi \vartheta^2)}{S^2}.$$  

Due to thermodynamic relations, we consider $\eta = \frac{(S + \pi \vartheta^2)^2}{2S^2\sqrt{\pi}}(\alpha^2 \pi + S) - \frac{\log \frac{\sqrt{S}}{\alpha \sqrt{\pi}}}{\sqrt{\pi}}$ conjugate to PFDM density parameter $\alpha$, volume $V = \frac{4\sqrt{\pi}(S + \pi \vartheta^2)(1 + \log \frac{\sqrt{S}}{\alpha \sqrt{\pi}})}{2S^2\sqrt{\pi}}$ conjugate to $P = -\frac{\lambda}{4\pi}$ and $\zeta = \frac{3Ms_0}{S + \pi \vartheta^2}$ conjugate to $\vartheta$. To study the universal relation we solve the equation (75). So the constant correction parameter is given by,

$$\epsilon = -1 + \frac{2M S \sqrt{\pi}}{(S + \pi \vartheta^2)^2} - \frac{F\left(\alpha^2 \pi + S + \sqrt{\pi} S \alpha \log \frac{\sqrt{S}}{\alpha \sqrt{\pi}}\right)}{S},$$

The derivative of $\epsilon$ with respect to $S$ gives,

$$\frac{\partial \epsilon}{\partial S} = -\frac{2MS \sqrt{\pi}}{(S + \pi \vartheta^2)^2} + \frac{2S^2 \sqrt{\pi}}{(S + \pi \vartheta^2)^2} - \frac{\alpha \sqrt{S} + \alpha \sqrt{S} \log \frac{\sqrt{S}}{\alpha \sqrt{\pi}}}{2S^2\sqrt{\pi}}.$$  

By combining equations (76) and (79), one can obtain,

$$-T \frac{\partial S}{\partial \epsilon} = \frac{(S + \pi \vartheta^2)^2}{2f^2S\sqrt{\pi}}.$$  

By setting $T = 0$ and using equations (75) and (76), we get,

$$\frac{\partial M_{\text{ext}}}{\partial \epsilon} = \frac{(S + \pi \vartheta^2)^2}{2f^2S\sqrt{\pi}},$$

the right hand sides of the two equations (80) and (81) are exactly same. Now, we use the pressure $P = \frac{S}{8\pi f^2} = -\frac{\lambda}{8\pi}$ and equation (78) to obtain,

$$\frac{\partial \epsilon}{\partial P} = \frac{3}{8f^2} + \frac{\alpha \log \frac{\sqrt{S}}{\alpha \sqrt{\pi}}}{8f^2},$$

According to volume and extremal bound, we have,

$$-V \frac{\partial P}{\partial \epsilon} = \frac{(S + \pi \vartheta^2)^2}{2f^2S\sqrt{\pi}},$$

here, the right hand sides of the two equations (83) and (81) are exactly same.
Also with respect to equation (78), we get,

$$\frac{\partial \epsilon}{\partial a} = -\frac{2a l^2 \pi}{S}. \tag{84}$$

So from (77) and (84), we obtain following relation:

$$-\Omega \frac{\partial a}{\partial \epsilon} = \frac{(S + \pi \Omega^2)^2}{2l^2 S \sqrt{\pi}}, \tag{85}$$

we see here, the right hand sides of the two equations (83) and (85) are exactly same. We use equation (78) for obtaining the another universal relation. Here, we have

$$\frac{\partial \epsilon}{\partial \alpha} = \frac{l^2 \sqrt{\pi}}{\sqrt{S}} - \frac{l^2 \sqrt{\pi} \log \left( \frac{\sqrt{\pi}}{\alpha \sqrt{S}} \right)}{\sqrt{S}}. \tag{86}$$

So by using the $\eta$ and (86), we obtain the other universal relation as given by,

$$-\eta \frac{\partial \alpha}{\partial \epsilon} = \frac{(S + \pi \Omega^2)^2}{2l^2 S \sqrt{\pi}}, \tag{87}$$

also here, the right hand sides of the two equations (87) and (85) are equal. Again we use relation (78) and obtain,

$$\frac{\partial \epsilon}{\partial \theta} = -\frac{6l^2 M \pi \Omega^2 S \theta}{(S + \pi \Omega^2)^2}. \tag{88}$$

So, with respect the above equation and $\zeta$, we obtain the new universal relation of rotating Bardeen black hole as given by,

$$-\zeta \frac{\partial \theta}{\partial \epsilon} = \frac{(S + \pi \Omega^2)^2}{2l^2 S \sqrt{\pi}} = \frac{\partial M_{ext}}{\partial \epsilon}. \tag{89}$$

The most important thing here is that the universal relation related to magnetic charge is also confirmed. We have thoroughly investigated the universal relations for the three different types of black holes in the AdS space surrounded by perfect fluid, such as Kerr-Newman, rotating Bardeen and Reissner-Nordström. We calculated the universal relationships of each black hole separately. Furthermore, we introduced new universal relations with respect to the concepts of perfect fluid and string fluid. We also observed that when a small correction constant is added to the action, the modified thermodynamic quantities and relations can be calculated. This constant correction can lead to a decrease of mass and increase the charge-to-mass ratio, which is a clue of WGC behaviour. The concepts in this paper can be evaluated for other black holes with different properties, as well as considering the higher dimensions and black holes of different structures.

### 3. Conclusions

Researchers have acquired new universal relations from multiple methods for black holes in the last few years. These universal relations can be an excellent impetus for integrating different sciences and possibly a great solution to the path of quantum gravity for physicists. In this paper, we confirmed new universal relations for black hole thermodynamics. We investigated each of these universal relations by selecting different black holes such as AdS de Sitter-Schwarzschild, charged BTZ, charged rotating BTZ, accelerating and charged accelerating black holes, and AdS black hole surrounded by perfect fluid. First, we obtained the modified thermodynamic relations of the black holes assuming a small correction to the action. We confirmed the universal relations by performing a series of direct calculations. It is noteworthy that according to each of the properties related to black holes, such as rotating, charged, accelerating, etc., a new universal relation can be obtained according to this method. So, we have confirmed two different types of these universal relations for various black holes. One of the most valuable results is that using the unique feature of black holes, a new universal relation between different black hole thermodynamics can be investigated. For Kerr-Newman, rotating Bardeen and Reissner-Nordström black holes in the AdS space surrounded by perfect fluid we also considered a small constant correction to the action and computed modified thermodynamic quantities and relations. Then, by using a series of calculations, we obtained the universal relations of these black holes. We have investigated new universal relations related to the parameter of perfect fluid and magnetic charge. Also, according to the constant correction and universal relations, we have studied the effect of perfect fluid and magnetic parameters on the charge-to-mass ratio. Here, we have seen that the WGC-like behavior is satisfied by the black hole system. This work will also be interesting in investigating the universal relation of the black hole in higher dimensions. Also, it may be interesting to make a similar correction to the Einstein-Gauss-Bonnet action and obtain the new modified thermodynamic relations. Given the relationship between the universal relation and the WGC, it may
be interesting to obtain the relation between the correction parameter to the action and the Gauss–Bonnet parameter.

Data availability statement

No new data were created or analysed in this study.

Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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