Stability of piezoelectric beam with magnetic effect under (Coleman or Pipkin)–Gurtin thermal law

Mohammad Akil

Abstract. In this paper, we investigate the stabilization of a system of piezoelectric beams under (Coleman or Pipkin)–Gurtin thermal law with magnetic effect. First, we study the piezoelectric Coleman–Gurtin system and we obtain an exponential stability result. Next, we consider the piezoelectric Gurtin–Pipkin system and we establish a polynomial energy decay rate of type $t^{-1}$.

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1. Introduction

1.1. Piezoelectric Beam

It is known, since the nineteenth century that materials such as quartz, Rochelle salt and barium titanate under pressure produce electric charge/voltage, this phenomenon is called the direct piezoelectric effect and was discovered by brothers Pierre and Jacques Curie in 1880. These same materials, when subjected to an electric field, produce proportional geometric tension. Such a phenomenon is known as the converse piezoelectric effect and was discovered by Gabriel Lippmann in 1881.

Morris and Ozer proposed a piezoelectric beam model with a magnetic effect, based on the Euler–Bernoulli and Rayleigh beam theory for small displacement (the same equations for the model are obtained if Midlin–Timoshenko small displacement assumptions bare used ); they considered an elastic beam
covered by a piezoelectric material on its upper and lower surfaces, isolated by the edges and connected to a external electrical circuit to feed charge to the electrodes. As the voltage is prescribed at the electrodes, the following Lagrangian was considered:

$$\mathcal{L} = \int_0^T \left[ K - (P+E) + B + W \right] dt,$$

(1.1)

where $K$, $P+E$, $B$ and $W$ represent the (mechanical) kinetic energy, total stored energy, magnetic energy (electrical kinetic) of the beam and the work done by external forces, respectively. For a beam length $L$ to thickness $h$ and considering $v = v(x, t)$, $w = w(x, t)$ and $p = p(x, t)$ as functions that represent the longitudinal displacement of the center line, transverse displacement of the beam and the total load of the electric displacement along the transverse direction at each point $x$, respectively. So, one can assume that

$$P+E = \frac{h}{2} \int_0^L \left[ \alpha \left( v_x^2 + \frac{h^2}{12} w_{xx}^2 - 2\gamma\beta v_x p_x + \beta p_x^2 \right) \right] dx,$$

$$B = \frac{\mu h}{2} \int_0^L p_x^2 dx,$$

$$K = \frac{\rho h}{2} \int_0^L \left[ v_t^2 + \left( \frac{h^2}{12} + 1 \right) \omega_t^2 \right], \quad W = - \int_0^L p_x V(t) dx,$$

(1.2)

where $V(t)$ is the voltage applied at the electrode. From Hamilton’s principle for admissible displacement variations displacement variations $\{v, w, p\}$ of $L$ the zero and observing that the only external force acting on the beam is the voltage at the electrodes (the bending equation is decoupled) see [21,22], they got the system

$$\rho v_{tt} - \alpha v_{xx} + \gamma\beta p_{xx} = 0,$$

$$\mu p_{tt} - \beta p_{xx} + \gamma\beta v_{xx} = 0,$$

(1.3)

where $\rho, \alpha, \gamma, \mu$ and $\beta$ denote the mass density, elastic stiffness, piezoelectric coefficient, magnetic permeability, water resistance coefficient of the beam and the prescribed voltage on electrodes of beam, respectively, and in addition, the relationship

$$\alpha = \alpha_1 + \gamma^2 \beta.$$

(1.4)

They assumed that the beam is fixed at $x = 0$ and free at $x = L$, and thus, they got (from modelling) the following boundary conditions

$$v(0, t) = \alpha v_x(L, t) - \gamma\beta p_x(L, t) = 0,$$

$$p(0, t) = \beta p_x(L, t) - \gamma\beta v_x(L, t) = 0.$$

(1.5)

Then, the authors considered $V(t) = kp(t)$ (electrical feedback controller) in (1.5) and established strong stabilization for almost all system parameters and exponential stability for system parameters in a null measure set. In Ramos et al. [26] inserted a dissipative term $\delta v_t$ in the first equation of (1.3), where $\alpha > 0$ is a constant and considered the following boundary condition

$$v(0, t) = \alpha v_x(L, t) - \gamma\beta p_x(L, t) = 0,$$

$$p(0, t) = \beta p_x(L, t) - \gamma\beta v_x(L, t) = 0.$$

(1.6)

The authors showed, by using energy method, that the system’s energy decays exponentially. This means that the friction term and the magnetic effect work together in order to uniformly stabilize the system. In [1], the authors considered a one-dimensional dissipative system of piezoelectric beams with magnetic effect and localized damping. They proved that the system is exponential stable using a damping mechanism acting only on one component and on a small part of the beam. In [28], the authors considered a
one-dimensional piezoelectric beams with magnetic effect damped with a weakly nonlinear feedback in the presence of a nonlinear delay term. They established an energy decay rate under appropriate assumptions on the weight of the delay. In [4], the authors studied the stability of a piezoelectric beams with magnetic effects of fractional derivative type and with/without thermal effects of Fourier’s law; they obtained an exponential stability by taking two boundary fractional dampings and additional thermal effect.

1.2. (Coleman or Pipkin)—Gurtin thermal law

The theory of heat conduction under various non-Fourier heat flux laws has been developed since the 1940s. Let \( q \) be the heat flux vector. According to the Gurtin–Pipkin theory [16], the linearized constitutive equation of \( q \) is

\[
q(t) = -\int_{0}^{\infty} g(s)\theta_x(t - s)ds, \tag{1.7}
\]

where \( g \) is the heat conductivity relaxation kernel. The presence of convolution term in (1.7) entails finite propagation speed of heat conduction, and consequently, the equation is of hyperbolic type. Note that (1.7) reduces to the classical Fourier law when \( g \) is the Dirac mass at zero. Furthermore, if we take \( g \) as a prototype kernel

\[
g(t) = e^{-kt}, \quad k > 0, \tag{1.8}
\]

and differentiate (1.7) with respect to \( t \), we can (formally) arrive at the so-called Cattaneo–Fourier law.

\[
q_t(t) + kq = -\theta_x(t). \tag{1.9}
\]

On the other hand, when the heat conduction is due to the Coleman–Gurtin theory [9], the heat flux \( q \) depends on both the past history and the instantaneous of the gradient of temperature:

\[
q(t) = -\beta\theta_x(x, t) - \int_{0}^{\infty} g(s)\theta_x(x, t - s)ds, \tag{1.10}
\]

where \( \beta > 0 \) is the instantaneous diffusivity coefficient. The analysis on stabilization and controllability of the heat conduction equations under non-Fourier heat flux laws can be found in [8,14,15,23] and references therein.

Zhang [30] studied the stability of an interaction system comprised of a wave equation and a heat equation with memory. An exponential stability of the interaction system is obtained when the hereditary heat conduction is of Gurtin–Pipkin type and she showed the lack of uniform decay of the interaction system when the heat conduction law is of Coleman–Gurtin type. In [12], the authors studied the stability of a Timoshenko system with Gurtin–Pipkin thermal law; a necessary and sufficient condition for exponential stability is established in terms of the structural parameters of the equations. Later, in [13], the authors studied the asymptotic behavior of solutions of a one-dimensional coupled wave-heat system with Coleman–Gurtin thermal law. They proved an optimal polynomial decay rate of type \( t^{-2} \). In [11]; the author studied the stability of Bresse and Timoshenko systems with hyperbolic heat conduction. First, he studied the Bresse–Gurtin–Pipkin system, providing a necessary and sufficient condition for the exponential stability and the optimal polynomial decay rate when the condition is violated, also he studied the Timoshenko–Gurtin–Pipkin system and he finds the optimal polynomial decay rate.
1.3. Description of the model

Based on the description mentioned above piezoelectric beam and heat law, we design and propose to study the stability of the following system

\[
\begin{align*}
\rho u_{tt} - \alpha u_{txx} + \gamma \beta y_{xx} + \delta w_x &= 0, \quad (x, t) \in (0, L) \times (0, \infty), \\
\mu y_{tt} - \beta y_{xx} + \gamma \beta u_{xx} &= 0, \quad (x, t) \in (0, L) \times (0, \infty), \\
w_t - c(1 - m)w_{xx} - cm \int_{0}^{\infty} g(s)w_{xx}(x, t - s)ds + \delta u_{xt} &= 0, \\
u(0, t) &= y(0, t) = w(0, t) = w(L, t) = 0, \\
\alpha u_x(L, t) - \gamma y_x(L, t) &= \beta y_x(L, t) - \gamma \beta u_x(L, t). 
\end{align*}
\]

(P\textsubscript{CG})

The convolution kernel \( g : [0, \infty[ \to [0, \infty[ \) is a convex integrable function (thus non-increasing and vanishing at infinity) of unit total mass, taking the explicit form

\[
g(s) = \int_{s}^{\infty} \sigma(r)dr, \quad s \geq 0,
\]

where \( \sigma : (0, \infty) \to [0, \infty) \), called memory kernel, satisfying the following conditions

\[
\begin{align*}
\sigma &\in L^1((0, \infty)) \cap C^1((0, \infty)) \quad \text{with} \quad \int_{0}^{\infty} \sigma(r)dr = g(0) > 0, \quad \sigma(0) = \lim_{s \to 0} \sigma(s) < \infty, \\
\sigma &\text{satisfies the Dafermos condition } \sigma'(s) \leq -d_\sigma \sigma(s). 
\end{align*}
\]

(H)

Finally, we impose the initial conditions of the form

\[
\begin{align*}
(u(x, 0), u_t(x, 0), y(x, 0), y_t(x, 0)) &= (u_0(x), u_1(x), y_0(x), y_1(x)), \quad x \in (0, L), \\
(w(x, 0), w(x, -s)) &= (w_0(x), \phi_0(x, s)), \quad x \in (0, L), \quad s > 0, 
\end{align*}
\]

(1.11)

where \( u_0, u_1, y_0, y_1 \) are assigned data and \( c > 0 \). In particular, \( \phi_0 \) accounts for the so-called initial past history of \( w \). In the model (P\textsubscript{CG}), \( m \in [0, 1] \) is a fixed parameter and the temperatures obey the parabolic hyperbolic law introduced by Coleman and Gurtin in [9]. The limit cases:

- \( m = 0 \) corresponds to the piezoelectric Fourier law defined by:

\[
\begin{align*}
\rho u_{tt} - \alpha u_{xx} + \gamma \beta y_{xx} + \delta w_x &= 0, \quad (x, t) \in (0, L) \times (0, \infty), \\
\mu y_{tt} - \beta y_{xx} + \gamma \beta u_{xx} &= 0, \quad (x, t) \in (0, L) \times (0, \infty), \\
w_t - c w_{xx} + \delta u_{xt} &= 0, \\
u(0, t) &= y(0, t) = w(0, t) = w(L, t) = 0, \\
\alpha u_x(L, t) - \gamma y_x(L, t) &= \beta y_x(L, t) - \gamma \beta u_x(L, t). 
\end{align*}
\]

(P\textsubscript{F})

- \( m = 1 \) corresponds to the piezoelectric Gurtin–Pipkin law defined by:

\[
\begin{align*}
\rho u_{tt} - \alpha u_{xx} + \gamma \beta y_{xx} + \delta w_x &= 0, \quad (x, t) \in (0, L) \times (0, \infty), \\
\mu y_{tt} - \beta y_{xx} + \gamma \beta u_{xx} &= 0, \quad (x, t) \in (0, L) \times (0, \infty), \\
w_t - c \int_{0}^{\infty} g(s)w_{xx}(x, t - s)ds + \delta u_{xt} &= 0, \\
u(0, t) &= y(0, t) = w(0, t) = w(L, t) = 0, \\
\alpha u_x(L, t) - \gamma y_x(L, t) &= \beta y_x(L, t) - \gamma \beta u_x(L, t). 
\end{align*}
\]

(P\textsubscript{GP})

It is important to note that since \( y(x, t) = \frac{x}{0} D(\zeta, t)d\zeta \), where \( D(\zeta, t) \) represents the electric displacement in the direction \( z \), then \( y(0, t) = 0 \) and still \( y(L, t) \) may not be zero, because the boundary condition
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p(L, t) = 0 does not represent the fixation of the beam on both sides. In fact, the fixation is due to the boundary condition u(0, t) = u(L, t) = 0, where u is the transverse displacement of the beam.

This paper is organized as follows: In the first part, we study the well-posedness of system \((P_{CG})\). Next, we prove that the piezoelectric system with Coleman–Gurtin law is exponentially stable. In the last part, we consider the system of piezoelectric beam under Gurtin–Pipkin thermal law and we establish polynomial stability of type \(t^{-1}\).

2. Well-posedness

We start by introducing some notations and spaces used in this paper. First, we define

\[
H^1_L(0, L) = \{ u \in H^1(0, L); \ u(0) = 0 \}.
\]

It is easy to check that the space \(H^1_L(0, L)\) is a (complex) Hilbert space over \(\mathbb{C}\) equipped with the inner product

\[
\langle u, u^1 \rangle_{H^1_L(0, L)} = \langle u_x, u^1_x \rangle_{L^2(0, L)}.
\]

We also introduce the memory space \(W\), defined by

\[
W = L^2_\sigma(\mathbb{R}^+; H^1_0(0, L)),
\]

of \(H^1_0(0, L)\)–valued functions on \((0, \infty)\) which are square integrable with respect to the measure \(\sigma(s)ds\), endowed with the inner product

\[
(\eta_1, \eta_2)_W := cm \int_0^\infty \int_0^L \sigma(s)\eta_{1x}\eta_{2x}dsdx, \quad \forall \eta_1, \eta_2 \in W.
\]

We reformulate the \((P_{CG})\) using the history framework of Dafermos [10]. To this end, for \(s > 0\), we consider the auxiliary function

\[
\eta(x, s) = \int_0^s w(x, t - r)dr, \quad x \in (0, L), \ s > 0
\]

and we rewrite \((P_{CG})\) in the form

\[
\begin{aligned}
p\mu_{tt} - \alpha u_{xx} + \gamma \beta y_{xx} + \delta w_x &= 0, \quad (x, t) \in (0, L) \times (0, \infty), \\
\mu \gamma_{tt} - \beta y_{xx} + \gamma \beta u_{xx} &= 0, \quad (x, t) \in (0, L) \times (0, \infty), \\
w_t - c(1 - m)w_{xx} - cm \int_0^\infty \sigma(s)\eta_{xx}(s)ds + \delta u_{xt} &= 0, \\
\eta_t + \eta_s - w &= 0, \\
u(0, t) &= y(0, t) = w(0, t) = w(L, t) = 0, \\
\alpha u_x(L, t) - \gamma \beta y_x(L, t) &= \beta y_x(L, t) - \gamma \beta u_x(L, t) = 0, \quad (x, t) \in (0, L) \times (0, \infty) \times (0, \infty),
\end{aligned}
\]

\((P_{CG1})\)

with the initial conditions (1.11). The energy of system \((P_{CG1})\) is given by

\[
E_m(t) = E_{m,1}(t) + E_{m,2}(t) + E_{m,3}(t).
\]
where
\[ E_{m,1}(t) = \frac{1}{2} \int_0^L (\rho|u|^2 + \alpha_1|u_x|^2) \, dx, \quad E_{m,2}(t) = \frac{1}{2} \int_0^L (\mu|y|^2 + \beta|\gamma u_x - y_x|^2) \, dx \]
and
\[ E_{m,3}(t) = \frac{1}{2} \int_0^L |\omega|^2 \, dx + \frac{cm}{2} \int_0^\infty \int_0^L \sigma(s)|\eta_x|^2 \, ds \, dx. \]

**Lemma 2.1.** Let \( U = (u, u_t, y, y_t, w, \eta) \) be a regular solution of system \((\text{PCG1})\). Then, the energy \( E_{m}(t) \) satisfies the following estimation
\[ \frac{d}{dt} E_{m}(t) = c(m - 1) \int_0^L |\omega|^2 \, dx + cm \int_0^\infty \int_0^L \sigma'(s)|\eta_x|^2 \, ds \, dx. \] (2.2)

**Proof.** Multiplying the first and the second equation of \((\text{PCG1})\) by \( \overline{u_t} \) and \( \overline{y_t} \), respectively, integrating by parts over \((0, L)\), we get
\[ \frac{1}{2} \frac{d}{dt} \left( \rho \int_0^L |u|^2 \, dx + \alpha \int_0^L |u_x|^2 \, dx \right) - \gamma \beta \Re \left( \int_0^L y_x \overline{u_t} \, dx \right) + \delta \Re \left( \int_0^L \omega_x \overline{u_t} \, dx \right) = 0 \] (2.3)
and
\[ \frac{1}{2} \frac{d}{dt} \left( \mu \int_0^L |y|^2 \, dx + \beta \int_0^L |y_x|^2 \, dx \right) - \gamma \beta \Re \left( \int_0^L u_x \overline{y_t} \, dx \right) = 0. \] (2.4)

Adding (2.3) and (2.4), and using the fact that \( \alpha = \alpha_1 + \gamma^2 \beta \), we get
\[ \frac{1}{2} \frac{d}{dt} \left( \int_0^L (\rho|u|^2 + \alpha_1|u_x|^2 + \mu|y|^2 + \beta|\gamma u_x - y_x|^2) \, dx \right) + \delta \int_0^L \omega_x \overline{u_t} \, dx = 0. \] (2.5)

Now, multiplying the third equation of \((\text{PCG1})\) by \( \overline{w} \), integrating by parts over \((0, L)\), we get
\[ \frac{1}{2} \frac{d}{dt} \int_0^L |\omega|^2 \, dx + c(1 - m) \int_0^L |\omega_x|^2 \, dx + \Re \left( cm \int_0^\infty \int_0^L \sigma(s)|\eta_x|^2 \, ds \, dx \right) - \delta \int_0^L u_t \overline{w_x} \, dx = 0. \] (2.6)

Differentiating the fourth equation with respect to \( x \), we obtain
\[ \eta_x t + \eta_{xs} - w_x = 0. \] (2.7)

Multiplying (2.7) by \( cm \sigma(s)|\eta_x|^2 \), integrating over \((0, L) \times (0, \infty)\), we get
\[ \frac{1}{2} \frac{d}{dt} cm \int_0^L \int_0^\infty \sigma(s)|\eta_x|^2 \, ds \, dx - \frac{cm}{2} \int_0^L \int_0^\infty \sigma'(s)|\eta_x|^2 \, ds \, dx = \Re \left( cm \int_0^\infty \int_0^L \sigma(s)|\eta_x|^2 \, ds \, dx \right). \] (2.8)

Inserting (2.8) in (2.7), we get
\[ \frac{d}{dt} E_{m,3}(t) - \delta \int_0^L u_t \overline{w_x} \, dx = c(m - 1) \int_0^L |\omega|^2 \, dx + \frac{cm}{2} \int_0^\infty \int_0^L \sigma'(s)|\eta_x|^2 \, ds \, dx. \] (2.9)

Finally, adding (2.6) and (2.9), we get the desired result. The proof has been completed. \( \square \)
Now, we define the Hilbert space $\mathcal{H}$ (Energy space) by

$$
\mathcal{H} := (H^1_0(0, L) \times L^2(0, L))^2 \times L^2(0, L) \times W,
$$

equipped with the following inner product

$$
\langle U_1, U_2 \rangle_{\mathcal{H}} = \int_0^L \left[ \rho v_1 \ddot{v}_2 + \alpha_1 u_{1,x} \ddot{w}_{2,x} + \mu z_1 \dddot{w}_2 + \beta (\gamma u_{1,x} - y_{1,x}) (\gamma u_{1,x} - y_{1,x}) + w_1 \dddot{w}_2 \right] dx + (\eta_1, \eta_2) W,
$$

where $U_i = (u_i, v_i, y_i, z_i, w_i) \in \mathcal{H}$, $i = 1, 2$.

**Remark 2.2.** By using the fact that $\alpha = \alpha_1 + \gamma^2 \beta$, the boundary conditions at $L$ should be replaced by the Neumann conditions $u_x(L) = y_x(L) = 0$.

By introducing the state $U = (u, v, y, z, \omega, \eta(\cdot, s))^\top$, system $(P_{CG1})$ can be written as the following first-order evolution equation

$$
U_t = A_m U, \quad U(0) = U_0,
$$

where $A_m : D(A_m) \subset \mathcal{H} \rightarrow \mathcal{H}$ is an unbounded linear operator defined by

$$
A_m \begin{pmatrix} u \\ v \\ y \\ z \\ \omega \\ \eta \end{pmatrix} = \begin{pmatrix} v \\ 1/\rho (\alpha uu_{xx} - \gamma \beta y_{xx} - \delta \omega_x) \\ z \\ 1/\mu (\beta y_{xx} - \gamma \beta u_{xx}) \\ cA_{xx}^m - \delta v_x \\ -\eta_s + \omega \end{pmatrix}
$$

and

$$
D(A_m) = \left\{ U := (u, v, y, z, \omega, \eta) \in \mathcal{H}; \quad u, v, y, z, \omega, \eta \in H^1_0(0, L); \quad u, y \in H^2(0, L) \cap H^1_0(0, L), \quad w \in H^1_0(0, L), \quad A_{xx}^m \in H^1_0(0, L), \quad \eta_s \in W; \quad \eta(\cdot, 0) = 0 \quad \text{and} \quad u_x(L) = y_x(L) = 0. \right\}
$$

where $A_{xx}^m = (1 - m)\omega + m \int_0^\infty \sigma(s)\eta(s) ds$ and $U_0 = (u_0, u_1, y_0, y_1, \omega_0, \eta_0)^\top \in \mathcal{H}$ with $\eta_0 = \int_0^s \phi_0(x, r) dr$ for $x \in (0, L)$ and $s > 0$.

**Proposition 2.3.** Under the hypothesis $(H)$, the unbounded linear operator $A_m$ is $m$-dissipative in the energy space $\mathcal{H}$.

**Proof.** For all $U = (u, v, y, z, w, \eta(\cdot, s))^\top \in D(A_m)$, by using conditions $(H1)$, $(H2)$ and the fact that $m \in (0, 1)$, it’s easy to see that

$$
\Re(\langle A_m U, U \rangle_{\mathcal{H}}) = c(m - 1) \int_0^L |\omega_x|^2 dx + \frac{cm}{2} \int_0^L \int_0^\infty |\sigma'(s)| \eta_x^2 ds dx \leq 0,
$$

which implies that $A_m$ is dissipative. Now, let us prove that $A_m$ is maximal. For this aim, let $F = (f^1, f^2, f^3, f^4, f^5, f^6(\cdot, s))^\top \in \mathcal{H}$, we want to find $U = (u, v, y, z, w, \eta(\cdot, s))^\top \in D(A_m)$ unique solution of

$$
- A_m U = F.
$$
Equivalently, we have the following system

\[-v = f^1, \quad (2.13)\]
\[-\alpha u_{xx} + \gamma \beta y_{xx} + \delta \omega_x = \rho f^2, \quad (2.14)\]
\[-z = f^3, \quad (2.15)\]
\[-\beta y_{xx} + \gamma \beta u_{xx} = \mu f^4, \quad (2.16)\]
\[-c \Lambda^m_{xx} + \delta \nu_x = f^5, \quad (2.17)\]
\[-\eta_s - w = f^6. \quad (2.18)\]

Thanks to (2.13) and (2.15), it follows that, \(v, z \in H^1_0(0, L)\) and

\[v = -f^1 \quad \text{and} \quad z = -f^3. \quad (2.19)\]

From (2.18), we obtain

\[\eta(\cdot, s) = s w(x) + \int_0^s f^6(x, \tau) d\tau. \quad (2.20)\]

Then, from (2.13) and (2.17), we obtain

\[\Lambda^m_{xx} = -c^{-1} (f^5 + \delta f^1_x), \quad (2.21)\]

it yields,

\[\Lambda^m(x) = -c^{-1} \int_0^x \int_0^{x_1} (f^5 + \delta f^1_x) \, dx_2 \, dx_1 + c^{-1} \frac{x}{L} \int_0^L \int_0^{x_1} (f^5 + \delta f^1_x) \, dx_2 \, dx_1. \quad (2.22)\]

Now, using the definition of \(\Lambda^m\), (2.20) and (2.22), we get

\[w = \frac{1}{\tilde{m}} \left[ \Lambda^m(x) - m \int_0^\infty \sigma(s) \int_0^s f^6(\tau) d\tau ds \right], \quad (2.23)\]

where \(\tilde{m} = c(1 - m) + cm \int_0^\infty s \sigma(s) ds > 0\) and by using (H) \(\tilde{m} < \infty\). It is easy to see that \(w \in H^1_0(0, L)\).

It follows, from the previous result and Eqs. (2.18), (2.20) and (2.13), that

\[\eta, \eta_s \in W \quad \text{and} \quad \Lambda_{xx} \in L^2(0, L). \quad (2.24)\]

Now, let \(\phi^1, \phi^2 \in H^1_0(0, L)\) for all \(i = 1, 2\). Multiplying (2.14) and (2.16), respectively, by \(\phi^1\) and \(\phi^2\), integrating by parts over \((0, L)\), we get

\[\alpha \int_0^L u_x \phi^1_x \, dx - \gamma \beta \int_0^L y_x \phi^1_x \, dx = \int_0^L (\rho f^2 + f^1) \phi^1 \, dx - \delta \int_0^L w_x \phi^1 \, dx, \quad (2.25)\]
\[\alpha \int_0^L y_x \phi^2_x \, dx - \gamma \beta \int_0^L u_x \phi^2_x \, dx = \int_0^L (\mu f^4 + f^3) \phi^2 \, dx. \quad (2.26)\]

Adding (2.25) and (2.26), and using the fact that \(\alpha = \alpha_1 + \gamma^2 \beta\), we obtain

\[B((u, y), (\phi^1, \phi^2)) = L(\phi_1, \phi_2), \quad \forall (\phi^1, \phi^2) \in H^1_0(0, L) \times H^1_0(0, L), \quad (2.27)\]
where
\[ B((u, y), (\phi, \phi')) = \alpha_1 \int_0^L u_x \phi_x^2 \, dx + \beta \left[ \int_0^L \left( \gamma^2 u_x \phi^1 - \gamma y_x \phi_x^1 - \gamma u_x \phi^2 + y_x \phi_x^2 \right) \, dx \right] \]
and
\[ L(\phi_1, \phi_2) = \int_0^L (f^2 + f^1) \phi_x^2 \, dx - \delta \int_0^L w_x \phi^1 \, dx + \int_0^L (f^4 + f^3) \phi^2 \, dx. \]

It is easy to see that \( B \) is a sesquilinear, continuous and coercive form on \( (H^1_0(0, L) \times H^1_0(0, 1))^2 \) and \( L \) is a antilinear and continuous form on \( H^1_0(0, L) \times H^1_0(0, 1) \). Then, it follows by Lax–Milgram theorem that (2.27) admits a unique solution \((u, y) \in (H^1_0(0, L) \times H^1_0(0, 1))\). From (2.14), (2.16), (2.23) and the fact that, \( \alpha = \alpha_1 + \gamma^2 \beta \), we have
\[ -\alpha_1 u_{xx} = f^2 + \gamma f^4 - \frac{\delta}{m} \left[ \Lambda^m(x) - m \int_0^\infty \sigma(s) \int_0^s f^6(\tau) \, d\tau \, ds \right] x \in L^2(0, L) \]
and
\[ -\beta y_{xx} = \frac{\alpha}{\alpha_1} f^4 + \frac{\gamma \beta}{\alpha_1} \left[ f^2 - \frac{\delta}{m} \left[ \Lambda^m(x) - m \int_0^\infty \sigma(s) \int_0^s f^6(\tau) \, d\tau \, ds \right] \right] x \in L^2(0, L). \]

They follow that \( u, y \in H^2(0, L) \). Consequently, \( U = (u, -f^1, y, -f^3, w, \eta)^\top \in D(\mathcal{A}_m) \) is a unique solution of (2.12). Then, \( \mathcal{A}_m \) is an isomorphism and since \( \rho(\mathcal{A}_m) \) is open set of \( \mathbb{C} \) (see Theorem 6.7 (Chapter III) in [18]), we easily get \( R(\lambda I - \mathcal{A}_m) = \mathcal{H} \) for a sufficiently small \( \lambda > 0 \). This, together with the dissipativeness of \( \mathcal{A}_m \), implies that \( D(\mathcal{A}_m) \) is dense in \( \mathcal{H} \) and that \( \mathcal{A}_m \) is m-dissipative in \( \mathcal{H} \) (see Theorems 4.5, 4.6 in [24]). The proof is thus complete.

According to Lumer–Phillips theorem (see [24]), Proposition 2.3 implies that the operator \( \mathcal{A} \) generates a \( C_0 \)-semigroup of contractions \( e^{\mathcal{A}} \) in \( \mathcal{H} \) which gives the well-posedness of (2.10). Then, we have the following result:

**Theorem 2.4.** For all \( U_0 \in \mathcal{H} \), system \((\text{P}_{\text{CG}1})\) admits a unique weak solution
\[ U(t) = e^{\mathcal{A}} U_0 \in C^0([0, \infty), \mathcal{H}). \]
Moreover, if \( U_0 \in D(\mathcal{A}) \), then the system \((\text{P}_{\text{CG}1})\) admits a unique strong solution
\[ U(t) = e^{\mathcal{A}} U_0 \in C^0(\mathbb{R}^+, D(\mathcal{A})) \cap C^1(\mathbb{R}^+, \mathcal{H}). \]

### 3. Exponential stability of Piezoelectric with Coleman–Gurtin thermal law \((\text{P}_{\text{CG}})\)

In this section, we shall analyze the exponential stability of system \((\text{P}_{\text{CG}})\). The main result of this section is the following theorem:

**Theorem 3.1.** Assume that the conditions (H) hold and \( m \in (0, 1) \). Then the \( C_0 \)-semigroup of contractions \((e^{\mathcal{A}m})_{t \geq 0}\) is exponentially stable, i.e., there exist constants \( M \geq 1 \) and \( \epsilon > 0 \) independent of \( U_0 \) such that
\[ \|e^{\mathcal{A}m} U_0\|_\mathcal{H} \leq M e^{-\epsilon t}\|U_0\|_\mathcal{H}. \]
According to Huang [17] and Prüss [25], a $C_0$-semigroup of contractions $(e^{tA_m})_{t \geq 0}$ on $\mathcal{H}$ satisfy (3.1) if
\[ i \mathbb{R} \subset \rho(A_m) \] (E1)
and
\[ \sup_{\lambda \in \mathbb{R}} \| (i \lambda I - A_m)^{-1} \|_{\mathcal{L} (\mathcal{H})} = O(1) \] (E2)
hold. Let $(\lambda, U := (u, v, y, z, w, \eta)) \in \mathbb{R}^* \times D(A_m)$, such that
\[ (i \lambda I - A_m)U = F := (f^1, f^2, f^3, f^4, f^5, f^6) \in \mathcal{H}. \] (3.2)
That is
\[ i \lambda u - v = f^1 \quad \text{in} \quad H^1_0(0, L), \] (3.3)
\[ i \lambda \rho v - \alpha u_{xx} + \gamma \beta y_{xx} + \delta \omega_x = \rho f^2 \quad \text{in} \quad L^2(0, L), \] (3.4)
\[ i \lambda y - z = f^3 \quad \text{in} \quad H^1_0(0, L), \] (3.5)
\[ i \lambda u_{xx} - \beta y_{xx} + \gamma \beta u_{xx} = \mu f^4 \quad \text{in} \quad L^2(0, L), \] (3.6)
\[ i \lambda w - c \Lambda^m_{xx} + \delta v_x = f^5 \quad \text{in} \quad L^2(0, L), \] (3.7)
\[ i \lambda \eta + \eta_s - w = f^6(\cdot, s) \quad \text{in} \quad W. \] (3.8)
Here and below, we occasionally write $p \lesssim q$ to indicate that $p \leq Cq$ for some (implicit) constant $C > 0$.

The next Lemmas are a technical results to be used in the proof of Theorem 3.1.

**Lemma 3.2.** Assume that the conditions (H) hold and $m \in (0, 1)$. The solution $(u, v, y, z, w, \eta) \in D(A_m)$ of Eq. (3.2) satisfies the following estimates
\[ \int_0^L |\omega_x|^2 \, dx \leq K_1 \| F \|_{\mathcal{H}} \| U \|_{\mathcal{H}}, \] (3.9)
\[ \int_0^L \int_0^\infty \sigma(s)|\eta_x|^2 \, dsdx \leq K_2 \| F \|_{\mathcal{H}} \| U \|_{\mathcal{H}}, \] (3.10)
\[ \int_0^L |\omega|^2 \, dx \leq K_3 \| F \|_{\mathcal{H}} \| U \|_{\mathcal{H}}, \] (3.11)
\[ \int_0^L |\Lambda^m_{xx}|^2 \, dx \leq K_4 \| F \|_{\mathcal{H}} \| U \|_{\mathcal{H}}, \] (3.12)
where
\[ K_1 = \frac{1}{c(1 - m)}, \quad K_2 = \frac{2}{c \text{md}_\sigma}, \quad K_3 = c_p K_1 \quad \text{and} \quad K_4 = \frac{2(1 - m)}{c} + \frac{4g(0)}{c \text{md}_\sigma}. \]

**Proof.** First, taking the inner product of (3.2) with $U$ in $\mathcal{H}$, we get
\[ c(1 - m) \int_0^L |\omega_x|^2 \, dx - cm \int_0^L \int_0^\infty \sigma'(s)|\eta_x|^2 \, dsdx = -\Re(\langle A_m U^n, U^n \rangle_{\mathcal{H}}) \leq \| F_n \|_{\mathcal{H}} \| U \|_{\mathcal{H}}. \] (3.13)
From condition (H), we obtain
\[ \int_0^L \int_0^\infty \sigma(s)|\eta_x|^2 \, dsdx \leq \frac{1}{d_\sigma} \int_0^L \int_0^\infty \sigma'(s)|\eta_x|^2 \, dsdx. \]
Using the above estimation in (3.13), we get

\[
c(1 - m) \int_0^L |w_x|^2 dx + \frac{cmd_\sigma}{2} \int_0^\infty \int_0^L \sigma(s)|\eta_x|^2 ds dx = -\Re (\langle A, U, U \rangle_H) \leq \|F\|_H\|U\|_H, \tag{3.14}
\]

using the fact that \(m \in (0, 1)\) in (3.14), then we get (3.9) and (3.10). Using (3.9) and Poincaré inequality, we obtain (3.11). Finally, by using Cauchy–Schwarz inequality, we obtain

\[
\int_0^L |\Lambda_x^m|^2 dx \leq 2(1 - m)^2 \int_0^L |w_x|^2 dx + 2 \left( \int_0^\infty \int_0^L \sigma(s) ds \right) \int_0^\infty \int_0^L \sigma(s)|\eta_x(s)|^2 ds dx
\]

\[
\leq 2(1 - m)^2 \int_0^L |w_x|^2 dx + 2g(0) \int_0^\infty \int_0^L \sigma(s)|\eta_x(s)|^2 ds dx.
\]

Using (3.9) and (3.10) in the above inequality, we get (3.12); the proof is thus completed. \(\square\)

**Lemma 3.3.** Assume that the conditions (H) hold and \(m \in (0, 1)\). The solution \((u, v, z, w, \eta) \in D(A_m)\) of Eq. (3.2) satisfies the following estimation

\[
\int_0^L |u_x|^2 dx \leq (|\lambda|^{-1} + 1) \|F\|_H\|U\|_H + \|\lambda\|^{-1}\|A_m^2(U, U)\|_H \left(A_m^2(U, U) + A_m(U, U)\right), \tag{3.15}
\]

where \(A_m(U, F) = (|\lambda|^{\frac{1}{2}} + 1)\|U\|_H^{\frac{1}{2}} + \|F\|_H^{\frac{1}{2}}\).

**Proof.** From (3.3) and (3.7), we obtain

\[
i\lambda \delta u_x = -i\lambda w + c\Lambda_x^m + f^5 + \delta f_x^1. \tag{3.16}
\]

Multiplying (3.16) by \(-i\lambda^{-1}w\), integrating by parts over \((0, L)\), we get

\[
\delta \int_0^L |u_x|^2 dx = -\int_0^L w\Lambda_x^m dx + i\lambda^{-1}c \int_0^L \Lambda_x^m \Lambda_x^m dx + i\lambda^{-1}c\Lambda_x^m(0)\Lambda_x^m(0) - i\lambda^{-1} \int_0^L (f^5 + \delta f_x^1) \Lambda_x^m dx.
\]

It follows that,

\[
\delta \int_0^L |u_x|^2 dx \leq \int_0^L |w||u_x| dx + |\lambda|^{-1}c \int_0^L |\Lambda_x^m||u_x| dx + |\lambda|^{-1}c|\Lambda_x^m(0)||u_x(0)|
\]

\[
+ |\lambda|^{-1} \int_0^L |f^5 + \delta f_x^1||u_x| dx. \tag{3.17}
\]
Using Cauchy–Schwarz inequality, and the fact that \( \|u_x\|_{L^2(0,L)} \leq \frac{1}{\sqrt{\alpha_1}}\|U\|_{\mathcal{H}} \), \( \|f^1_x\| \leq \frac{1}{\sqrt{\alpha_1}}\|F\|_{\mathcal{H}} \), \( \|f_5\| \leq \|F\|_{\mathcal{H}} \) and (3.11), we get the following estimations

\[
\begin{align*}
\int_0^L |u|^2 dx + \frac{\delta}{2} \int_0^L |u_x|^2 dx & \leq \frac{K_3}{2\delta} \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} + \frac{\delta}{2} \int_0^L |u|^2 dx, \\
|\lambda|^{-1} \int_0^L |f^5| u_x dx & \leq K_5 |\lambda|^{-1} \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} \quad \text{where } K_5 = \frac{1}{\sqrt{\alpha_1}},
\end{align*}
\]

(3.18)

Now, using the fact that \( \alpha = \alpha_1 + \gamma^2 \beta \), (3.4) and (3.6), we get

\[
u_{xx} = \frac{1}{\alpha_1} \left[ i\lambda (\rho v + \gamma \mu z) + \delta w_x - \rho f^2 - \gamma \mu f^4 \right].
\]

(3.19)

Using the fact that \( \rho \int_0^L |v|^2 dx \leq \|U\|_{\mathcal{H}}^2 \), \( \mu \int_0^L |z|^2 dx \leq \|U\|_{\mathcal{H}}^2 \), \( \rho \int_0^L |f^2|^2 dx \leq \|F\|_{\mathcal{H}}^2 \), \( \mu \int_0^L |f|^4 dx \leq \|F\|_{\mathcal{H}}^4 \) and \( ab \leq a^2 + \frac{b^2}{4} \), in (3.19), we get

\[
\|u_{xx}\| \leq \frac{1}{\alpha_1} \left( |\lambda| (\sqrt{\rho} + \gamma \sqrt{\mu}) \|U\|_{\mathcal{H}} + \delta \sqrt{K_1} \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} + (\sqrt{\rho} + \gamma \sqrt{\mu}) \|F\|_{\mathcal{H}} \right)
\leq \frac{1}{\alpha_1} \max \left( \sqrt{\rho} + \gamma \sqrt{\mu}, \delta \sqrt{K_1} \right) \left( |\lambda| \|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}} \right)
\leq \frac{1}{\alpha_1} \max \left( \sqrt{\rho} + \gamma \sqrt{\mu}, \delta \sqrt{K_1} \right) \left( (|\lambda| + 1) \|U\|_{\mathcal{H}} + \frac{5}{4} \|F\|_{\mathcal{H}} \right).
\]

Hence, we get

\[
\|u_{xx}\| \leq K_7 ((|\lambda| + 1) \|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}),
\]

(3.20)

where \( K_7 = \frac{5}{4\alpha_1} \max \left( \sqrt{\rho} + \gamma \sqrt{\mu}, \delta \sqrt{K_1} \right) \). Using (3.12) and (3.20), we obtain

\[
|\lambda|^{-1} c \int_0^L |\Lambda_{mm}^n| u_x dx \leq K_8 |\lambda|^{-1} \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} \left( (|\lambda| + 1) \|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}} \right),
\]

(3.21)

where \( K_8 = cK_7 \sqrt{K_4} \). From (3.16) and the fact that \( \|f^5\| \leq \|F\|_{\mathcal{H}}, \|f^1_x\| \leq \alpha_1^{-\frac{1}{2}} \|F\|_{\mathcal{H}} \) and \( \|w\| \leq \|U\|_{\mathcal{H}} \), we get

\[
\|\Lambda_{mm}^n\| \leq K_9 \left( |\lambda| \|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}} \right),
\]

(3.22)

where \( K_9 = \left( \frac{\alpha}{\sqrt{\alpha_1}} + 1 \right) \). Using Gagliardo–Nirenberg inequality, we get

\[
|\Lambda_{mm}^n(0)| \leq C_1 \|\Lambda_{mm}^n\|^{\frac{1}{2}} \|\Lambda_{mm}^n\|^{\frac{1}{2}} + C_2 \|\Lambda_{mm}^n\|^{\frac{1}{2}} \leq \max(C_1, C_2) \|\Lambda_{mm}^n\|^{\frac{1}{2}} \left( \|\Lambda_{mm}^n\|^{\frac{1}{2}} + \|\Lambda_{mm}^n\|^{\frac{1}{2}} \right).
\]

(3.23)

Thanks to (3.22), (3.12) and (3.23), and the fact that \( \sqrt{a+b} \leq \sqrt{a} + \sqrt{b} \) and \( ab \leq a^2 + \frac{b^2}{4} \), we get

\[
|\Lambda_{mm}^n(0)| \leq K_{10} \|F\|_{\mathcal{H}} \left( \sqrt{|\lambda|} \|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}} \right) \leq K_{10} \|F\|_{\mathcal{H}} \left( \|U\|_{\mathcal{H}} \right) \leq K_{10} \|F\|_{\mathcal{H}} \left( A_\lambda(U, F) \right),
\]

(3.24)
where \( K_{10} = \frac{5}{4} \sqrt{\frac{K_4}{K_5}} \max(C_1, C_2) \max \left( \sqrt{\frac{K_3}{K_4}}, \sqrt{\frac{K_5}{K_4}} \right) \) and 
\[ A_\lambda(U, F) = \left( |\lambda|^{\frac{1}{2}} + 1 \right) \|U\|^\frac{3}{4}_\mathcal{H} + \|F\|^\frac{3}{4}_\mathcal{H}. \]

Again, using Gagliardo–Nirenberg inequality, we have

\[ |u_x(0)| \leq C_1 \|u_{xx}\|^\frac{3}{4}_\mathcal{H} + C_2 \|u_x\|. \]

Using (3.20) and the fact that \( \|u_x\| \leq \frac{1}{\sqrt{\alpha_1}} \|U\|_\mathcal{H} \), in the above inequality, we get

\[ |u_x(0)| \leq K_{11} \left( \sqrt{|\lambda| + 1} \|U\|_\mathcal{H} + \|F\|_\mathcal{H} \|U\|^\frac{3}{4}_\mathcal{H} + \|U\|_\mathcal{H} \right) \leq K_{11} \left( A_\lambda(U, F) \|U\|^\frac{3}{4}_\mathcal{H} + \|U\|_\mathcal{H} \right), \]

where \( K_{11} = \max \left( \frac{C_1 \sqrt{K_7}}{\alpha_1^2} \sqrt{K_2}, \frac{C_2}{\sqrt{\alpha_1}} \right) \). Using (3.23) and (2.5), we get

\[ |\lambda|^{-1} c |A_x^{m}(0)| u_x(0) \leq K_{17} \lambda^{-1} \|F\|_\mathcal{H} \left( A_\lambda^2(U, F) \|U\|^\frac{3}{4}_\mathcal{H} + A_\lambda(U, F) \|U\|^\frac{5}{4}_\mathcal{H} \right), \]

where \( K_{17} = cK_{10}K_{11} \). Finally, inserting (3.18), (3.21) and (2.5) in (3.17), we get

\[ \int_0^L |u_x|^2 \, dx \leq K_{13} \left( (|\lambda|^{-1} + 1) \|F\|_\mathcal{H} \|U\|_\mathcal{H} + |\lambda|^{-1} \|F\|_\mathcal{H} \|U\|^\frac{3}{4}_\mathcal{H} \|U\|_\mathcal{H} \right) \]

\[ + |\lambda|^{-1} \|F\|^\frac{4}{3}_\mathcal{H} \left( A_\lambda^2(U, F) \|U\|^\frac{3}{4}_\mathcal{H} + A_\lambda(U, F) \|U\|^\frac{5}{4}_\mathcal{H} \right), \]

where \( K_{13} = \frac{2}{\delta} \max \left( \left( K_1 + K_2 + \frac{K_3}{2\delta} \right), K_8, K_{17} \right) \). Hence, we obtain (3.15). The proof has been completed. \( \square \)

Inserting (3.3) in (3.4), we get

\[ \rho \lambda^2 u + \alpha u_{xx} - \gamma \beta y_{xx} - \delta w_x = -\rho f^2 - i\lambda f. \]

Lemma 3.4. The solution \((u, v, y, z, w, \eta) \in D(A_m)\) of Eq. (3.2) satisfies the following estimation

\[ \int_0^L |\lambda u|^2 \, dx \leq \int_0^L |u_x|^2 \, dx + \|u\|_\mathcal{H} |u_x| + (|\lambda|^{-1} + 1) \left( \|F\|_\mathcal{H} \|U\|_\mathcal{H} + \|F\|^\frac{2}{4}_\mathcal{H} \right). \]

Proof. Multiplying (3.27) by \( \pi \), integrating by parts over \((0, L)\), we get

\[ \rho \int_0^L |\lambda u|^2 \, dx = \alpha \int_0^L |u_x|^2 \, dx - \gamma \beta \int_0^L y_x \overline{u_x} \, dx - \delta \int_0^L w \overline{u_x} \, dx - \rho \int_0^L f^2 u \, dx - i\lambda \int_0^L f^1 \overline{u} \, dx. \]

It follows that,

\[ \rho \int_0^L |\lambda u|^2 \, dx \leq \alpha \int_0^L |u_x|^2 \, dx + \gamma \beta \int_0^L |y_x| |u_x| \, dx + \delta \int_0^L |w| |u_x| \, dx + \rho \int_0^L |f^2| |u| \, dx + |\lambda| \int_0^L |f^1| |u| \, dx. \]

Using the fact that \( \|y_x - \gamma u_x\| \leq \frac{1}{\sqrt{\beta}} \|U\|_\mathcal{H} \), we get

\[ \|y_x\| \leq \|y_x - \gamma u_x\| + \gamma \|u_x\| \leq \left( \frac{1}{\sqrt{\beta}} + \frac{\gamma}{\sqrt{\alpha_1}} \right) \|U\|_\mathcal{H}. \]

Using Cauchy–Schwarz inequality and (3.30), we get

\[ \gamma \beta \int_0^L |y_x| |u_x| \, dx \leq \gamma \beta \left( \frac{1}{\sqrt{\beta}} + \frac{\gamma}{\sqrt{\alpha_1}} \right) \|U\|_\mathcal{H} \|u_x\|. \]
Using (3.11) and Young’s inequality, we get
\[
\delta \int |w| |u_x| \, dx \leq \int |w|^2 \, dx + \frac{\delta^2}{4} \int |u_x|^2 \, dx \leq K_3 \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} + \frac{\delta^2}{4} \int |u_x|^2 \, dx.
\] (3.32)

From (3.3), the fact that \(\sqrt{\rho} \|v\| \leq \|U\|_{\mathcal{H}}\), and Poincaré inequality, we have
\[
\|\lambda u\| \leq \frac{1}{\sqrt{\rho}} \|U\|_{\mathcal{H}} + \frac{C_p}{\sqrt{\alpha_1}} \|f_x\|_{\mathcal{H}} \leq K_{14} \left(\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}\right),
\] (3.33)

where \(K_{14} = \max\left(\frac{1}{\sqrt{\rho}}, \frac{C_p}{\sqrt{\alpha_1}}\right)\). Using (3.33) and the fact that \(\rho \int \int |f_x|^2 \, dx \leq \|F\|^2_{\mathcal{H}},\) we get
\[
\rho \int \int |f_x|^2 \, dx \leq \sqrt{\rho} \|F\|_{\mathcal{H}} \|u\| \leq K_{15} \lambda^{-1} \left(\|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} + \|F\|^2_{\mathcal{H}}\right),
\] (3.34)

where \(K_{15} = \sqrt{\rho} K_{14}\). Using (3.33), Poincaré inequality and the fact that \(\alpha_1 \int \int |f_x|^2 \, dx \leq \|F\|^2_{\mathcal{H}},\) we get
\[
\rho |\lambda| \int \int |f|^2 \, dx + \rho |\lambda| \int \int |f|^2 \, dx \leq K_{16} \left(\|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} + \|F\|^2_{\mathcal{H}}\right),
\] (3.35)

where \(K_{16} = \frac{\rho C_p K_{14}}{\sqrt{\alpha_1}}\). Adding (3.34) and (3.35), we get
\[
\rho \int \int |f|^2 \, dx + \rho |\lambda| \int \int |f|^2 \, dx \leq K_{17} \left(\|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} + \|F\|^2_{\mathcal{H}}\right),
\] (3.36)

where \(K_{17} = \max(K_{15}, K_{16})\). Finally, inserting (3.31), (3.32) and (3.36), in (3.29), we get
\[
\int \int |\lambda u|^2 \, dx \leq K_{18} \int \int |u_x|^2 \, dx + K_{19} \|U\|_{\mathcal{H}} \|u_x\| + K_{20} \left(\|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} + \|F\|^2_{\mathcal{H}}\right),
\] (3.37)

where \(K_{18} = \rho^{-1} \left(\alpha + \frac{\delta^2}{4}\right), K_{19} = \rho^{-1} \gamma \beta \left(\frac{1}{\sqrt{\beta}} + \frac{2}{\sqrt{\alpha_1}}\right)\) and \(K_{20} = 2\rho^{-1} \max(K_{17}, K_{13})\). It follows that,
\[
\int \int |\lambda u|^2 \, dx \leq K_{21} \left(\int \int |u_x|^2 \, dx + \|U\|_{\mathcal{H}} \|u_x\| + \left(\|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} + \|F\|^2_{\mathcal{H}}\right)\right),
\] (3.37)

where \(K_{21} = \max(K_{18}, K_{19}, K_{20})\). Hence, we obtain (3.28). The proof is thus completed.

\[\square\]

**Lemma 3.5.** The solution \((u, v, y, z, w, \eta) \in D(A_m)\) of Eq. (3.2) satisfies the following estimation
\[
\int \int |y|^2 \, dx \leq \int \int |\lambda u|^2 \, dx + \int \int |u_x|^2 \, dx + \left(\|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} + \|F\|^2_{\mathcal{H}}\right)
\] (3.38)

and
\[
\int \int |\lambda y|^2 \, dx \leq \int \int |\lambda u|^2 \, dx + \int \int |u_x|^2 \, dx + \left(\|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} + \|F\|^2_{\mathcal{H}}\right).
\] (3.39)
The aim of this step is to prove the following estimation

\[ \frac{\gamma \beta}{2} \int_0^L |y_x|^2 \, dx \leq \frac{\rho^2}{\mu \gamma} \int_0^L |\lambda u|^2 \, dx + \frac{\mu \gamma}{4} \int_0^L |\lambda y|^2 \, dx + \frac{\alpha^2}{\gamma \beta} \int_0^L |u_x|^2 \, dx + \tilde{K} (|\lambda|^{-1} + 1) (\|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} + \|F\|^2_{\mathcal{H}}), \]  

(3.40)

where \( \tilde{K} \) is a positive constant which defined at the end of the proof of step 1. For this aim, multiplying (3.27) by \( \bar{y} \), integrating over \((0, L)\), we get

\[ \rho \lambda^2 \int_0^L u \bar{y} \, dx - \alpha \int_0^L u_x \bar{y_x} \, dx + \gamma \beta \int_0^L |y_x|^2 \, dx + \delta \int_0^L w \bar{y} \, dx = -\rho \int_0^L f \bar{y} \, dx - i \lambda \int_0^L f^1 \bar{y} \, dx. \]

It follows that,

\[ \gamma \beta \int_0^L |y_x|^2 \, dx \leq \rho \lambda^2 \int_0^L |u||y| \, dx + \alpha \int_0^L |u_x||y_x| \, dx + \delta \int_0^L |w||y| \, dx + \rho \int_0^L |f^2||y| \, dx + |\lambda| \int_0^L |f^1||y| \, dx. \]

(3.41)

Applying Young's inequality, we get

\[ \left\{ \begin{array}{l}
\rho \lambda^2 \int_0^L |u||y| \, dx \leq \frac{\rho^2}{\mu \gamma} \int_0^L |\lambda u|^2 \, dx + \frac{\mu \gamma}{4} \int_0^L |\lambda y|^2 \, dx, \\
\alpha \int_0^L |u_x||y_x| \, dx \leq \frac{\alpha^2}{\gamma \beta} \int_0^L |u_x|^2 \, dx + \frac{\gamma \beta}{4} \int_0^L |y_x|^2 \, dx.
\end{array} \right. \]

(3.42)

Using (3.11) and the fact that \( ab \leq a^2 + \frac{b^2}{4} \), we get

\[ \delta \int_0^L |w||y_x| \, dx \leq \frac{\delta^2}{\gamma \beta} \int_0^L |w|^2 \, dx + \frac{\gamma \beta}{4} \int_0^L |y_x|^2 \, dx \leq K_{22} \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} + \frac{\gamma \beta}{4} \int_0^L |y_x|^2 \, dx, \]

(3.43)

where \( K_{22} = \frac{\delta^2}{\gamma \beta} K_3 \). It is easy to see that

\[ \|f_3^3\| \leq \|f_3^3 - f_2^1\| + \gamma \|f_2^1\| \leq K_{23} \|F\|, \]

(3.44)

where \( K_{23} = \frac{1}{\sqrt{\beta}} + \frac{\gamma}{\sqrt{\alpha_1}} \). Using (3.44), Poincaré inequality and the fact that \( \mu \int_0^L |z|^2 \, dx \leq \|U\|^2_{H} \) in (3.5), we obtain

\[ \|\lambda y\| \leq \frac{1}{\sqrt{\mu}} \|U\| + C_p \|f_3^3\| \leq K_{24} (\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}), \]

(3.45)

where \( K_{24} = \max \left( \frac{1}{\sqrt{\mu}}, C_p K_{23} \right) \). Using (3.45), Poincaré inequality and the fact that \( \alpha_1 \int_0^L |f_2^1|^2 \, dx \leq \|F\|^2_{\mathcal{H}} \), we get

\[ |\lambda| \int_0^L |f^1||y| \, dx \leq C_p \|f_2^1\| \|\lambda y\| \leq K_{25} (\|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} + \|F\|^2_{\mathcal{H}}), \]

(3.46)
where $K_{25} = \frac{C_r K_{24}}{\sqrt{\alpha_1}}$. On the other hand, using (3.45) and the fact that $\rho \int_0^L |f^2|^2 dx \leq \|F\|_{\mathcal{H}}^2$, we get

$$\rho \int_0^L |f^2| |y| dx \leq |\lambda|^{-1} K_{26} \left( \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2 \right),$$

(3.47)

where $K_{26} = \sqrt{\rho} K_{24}$. Inserting (3.42), (3.43), (3.46) and (3.47) in (3.41), we get (3.40), such that \( \tilde{K} = 2 \max(K_{22}, K_{25}, K_{26}) \).

**Step 2.** The aim of this step is to prove the following estimation

$$\mu \int_0^L |\lambda y|^2 dx \leq \frac{5\beta}{4} \int_0^L |y_x|^2 dx + \gamma^2 \beta \int_0^L |u_x|^2 dx + \tilde{K}_1 \left( |\lambda|^{-1} + 1 \right) \left( \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2 \right),$$

(3.48)

where $\tilde{K}_1$ is a positive constant which is defined at the end of the proof of step 1. For this aim, inserting (3.5) in (3.6), we get

$$\mu \lambda^2 y + \beta y_{xx} - \gamma \beta u_{xx} = - \left( \mu f^4 + i \mu f^3 \right).$$

(3.49)

Multiplying (3.49) by $\bar{y}$ and integrating by parts over $(0, L)$, we get

$$\mu \int_0^L |\lambda y|^2 dx = \beta \int_0^L |y_x|^2 dx - \gamma \beta \int_0^L u_x \bar{y}_x dx - \mu \int_0^L |f^4| |y| dx - i \lambda \mu \int_0^L |f^3| |y| dx.$$

It follows that,

$$\mu \int_0^L |\lambda y|^2 dx \leq \beta \int_0^L |y_x|^2 dx + \gamma \beta \int_0^L |u_x||y_x| dx + \mu \int_0^L |f^4| |y| dx + |\lambda| \mu \int_0^L |f^3| |y| dx.$$

(3.50)

Using the fact that $ab \leq a^2 + \frac{b^2}{4}$, we get

$$\gamma \beta \int_0^L |u_x||y_x| dx \leq \gamma^2 \beta \int_0^L |u_x|^2 dx + \frac{\beta}{4} \int_0^L |y_x|^2 dx.$$

(3.51)

Using (3.45) and the fact that $\mu \int_0^L |f^4|^2 dx \leq \|F\|_{\mathcal{H}}^2$, we get

$$\mu \int_0^L |f^4| |y| dx \leq \sqrt{\mu} \|F\|_{\mathcal{H}} \|y\| \leq K_{27} |\lambda|^{-1} \left( \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2 \right),$$

(3.52)

where $K_{27} = \sqrt{\mu} K_{24}$. Using Poincaré inequality, (3.44) and (3.45), we get

$$|\lambda| \mu \int_0^L |f^3| |y| dx \leq \mu C_p |f_x|^3 \|\lambda y\| \leq K_{28} \left( \|F\|_{\mathcal{H}} \|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2 \right),$$

(3.53)

where $K_{28} = \mu C_p K_{23} K_{24}$. Inserting (3.51), (3.52) and (3.53) in (3.50), we obtain (3.48) with $\tilde{K}_1 = \max(K_{27}, K_{28})$. 


**Step 3.** The aim of this step is to prove (3.38) and (3.39). Inserting (3.48) in (3.40), we get

\[
\frac{3}{16} \gamma \beta \int_0^L |y_x|^2 dx \leq \frac{\rho^2}{\mu \gamma} \int_0^L |\lambda u|^2 dx + \frac{\gamma^3 \beta}{4} \int_0^L |u_x|^2 dx + \bar{K}_2 (|\lambda|^{-1} + 1) (\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2),
\]

where \( \bar{K}_2 = \frac{\gamma}{4} \bar{K}_1 + \bar{K} \). It follows that

\[
\int_0^L |y_x|^2 dx \leq K_{29} \left( \int_0^L |\lambda u|^2 dx + \int_0^L |u_x|^2 dx + (|\lambda|^{-1} + 1) (\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2) \right),
\]

(3.54)

where \( K_{29} = \frac{16}{3 \gamma \beta} \max \left( \frac{\rho^2}{\mu \gamma}, \frac{\gamma^3 \beta}{4} \bar{K}_2 \right) \). Hence, we obtain (3.38). Inserting (3.54) in (3.48), we get

\[
\int_0^L |\lambda u|^2 dx \leq K_{30} \left( \int_0^L |\lambda u|^2 dx + \int_0^L |u_x|^2 dx + (|\lambda|^{-1} + 1) (\|F\|_{\mathcal{H}}\|U\|_{\mathcal{H}} + \|F\|_{\mathcal{H}}^2) \right),
\]

(3.55)

where \( K_{30} = \frac{1}{\mu} \max \left( \frac{5 \beta}{4} K_{29} + \gamma^2 \beta, \frac{5 \beta}{4} K_{29} + \bar{K}_1 \right) \). Hence, we obtain (3.39). The proof is thus completed.

**Proof of Theorem 3.1.** First, we will prove (E1). Remark that it has been proved in Proposition (2.3) that \( 0 \in \rho(\mathcal{A}_m) \). Now, suppose (E1) is not true, then there exists \( \kappa \in \mathbb{R}^* \) such that \( i \kappa \notin \rho(\mathcal{A}_m) \). According to Remark A.3 in “Appendix A”, there exists

\[
\{ (\lambda_n, U^n := (u^n, v^n, y^n, z^n, w^n, \eta^n(s))) \}_{n \geq 1} \subset \mathbb{R}^* \times D(\mathcal{A}_m),
\]

with \( \lambda_n \to \kappa \) as \( n \to \infty \), \( |\lambda_n| < |\kappa| \) and \( \|U^n\|_{\mathcal{H}} = 1 \), such that

\[
(i \lambda_n I - \mathcal{A}_m) U^n = F_n := (f_1^n, f_2^n, f_3^n, f_4^n, f_5^n, f_6^n(\cdot, s)) \to 0 \text{ in } \mathcal{H}, \text{ as } n \to \infty.
\]

We will check (E1) by finding a contradiction with \( \|U^n\|_{\mathcal{H}} = 1 \) such as \( \|U^n\|_{\mathcal{H}} \to 0 \). Here and below take \( U = U^n, F = F_n \) and \( \lambda = \lambda_n \). According to Lemma 3.2, we get

\[
\int_0^L |w^n|^2 dx \to 0 \quad \text{and} \quad \int_0^\infty \int_0^L |\sigma(s)| \eta_{n_x}^2 |dsdx \to 0.
\]

(3.56)

According to Lemma 3.3 and using the facts that \( |\lambda_n| < |\kappa|, \|U^n\|_{\mathcal{H}} = 1 \) and \( \|F_n\|_{\mathcal{H}} \to 0 \), we have

\[
A_{\lambda_n}(U^n, F_n) \to |\kappa|^2 + 1, \quad \text{as} \quad n \to \infty.
\]

hence

\[
\int_0^L |u_x^n|^2 dx \to 0 \quad \text{as} \quad n \to \infty.
\]

(3.57)

Using (3.57) and the facts that \( |\lambda_n| < |\kappa|, \|U^n\|_{\mathcal{H}} = 1 \) and \( \|F_n\|_{\mathcal{H}} \to 0 \) in Lemma 3.4, we obtain

\[
\int_0^L |\lambda_n u^n|^2 dx \to 0 \quad \text{as} \quad n \to \infty.
\]

(3.58)

Using (3.57), (3.58) and the facts that \( |\lambda_n| < |\kappa|, \|U^n\|_{\mathcal{H}} = 1 \) and \( \|F_n\|_{\mathcal{H}} \to 0 \) in Lemma 3.5, we get

\[
\int_0^L |y_{x_z}^n|^2 dx \to 0 \quad \text{and} \quad \int_0^L |\lambda_n y^n|^2 dx \to 0 \quad \text{as} \quad n \to \infty.
\]

(3.59)
From (3.66)–(3.69), as $n \to \infty$, we get $\|U^n\|_{\mathcal{H}} \to 0$, which contradicts $\|U^n\|_{\mathcal{H}} = 1$. Thus, condition (E1) holds true. Next, we will prove (E2) by a contradiction argument. Suppose there exists 

$$\{(\lambda_n, U^n := (u^n, v^n, y^n, z^n, w^n(s)))\}_{n \geq 1} \subset \mathbb{R}^* \times D(A_m),$$

with $|\lambda_n| \geq 1$ without affecting the result, such that $|\lambda_n| \to \infty$, and $\|U^n\|_{\mathcal{H}} = 1$ and there exists a sequence $F_n = (f_{1n}, f_{2n}, f_{3n}, f_{4n}, f_{5n}, f_{6n}(s)) \in \mathcal{H}$, such that 

$$(i\lambda_n I - A_m)U^n = F_n \to 0 \text{ in } \mathcal{H}.$$ 

We use conventional asymptotic notation, including 'big O' and 'little o'. We will check (E2) by finding a contradiction with $\|U^n\|_{\mathcal{H}} = 1$ such as $\|U^n\|_{\mathcal{H}} = O(1)$. According to Lemma 3.2 and using the facts that $|\lambda_n| \to \infty$, $\|U^n\|_{\mathcal{H}} = 1$ and $\|F_n\|_{\mathcal{H}} \to 0$, we have

$$\int_0^L |w^n|^2 dx = o(1) \quad \text{and} \quad \int_0^\infty \int_0^L \sigma(s)|\eta^n_{\tau}|^2 ds dx = o(1). \quad (3.60)$$

According to Lemma 3.3 and using the facts that $|\lambda_n| \to \infty$, $\|U^n\|_{\mathcal{H}} = 1$ and $\|F_n\|_{\mathcal{H}} \to 0$, we have

$$A_{\lambda_n}(U^n, F_n) = \left( |\lambda_n|^{\frac{1}{2}} + 1 \right) \|U^n\|_{\mathcal{H}}^{\frac{3}{2}} + \|F_n\|_{\mathcal{H}}^2 \leq O(|\lambda_n|^{\frac{1}{2}}).$$

hence

$$\int_0^L |u^n_{x}|^2 dx = o(1). \quad (3.61)$$

Using (3.61) and the fact that $|\lambda_n| \to \infty$, $\|U^n\|_{\mathcal{H}} = 1$ and $\|F_n\|_{\mathcal{H}} \to 0$ in Lemma 3.4, we obtain

$$\int_0^L |\lambda_n u^n|^2 dx = o(1). \quad (3.62)$$

Using (3.61), (3.62) and the facts that $|\lambda_n| \to \infty$, $\|U^n\|_{\mathcal{H}} = 1$ and $\|F_n\|_{\mathcal{H}} \to 0$ in Lemma 3.5, we get

$$\int_0^L |y^n_{x}|^2 dx = o(1) \quad \text{and} \quad \int_0^L |\lambda_n y^n|^2 dx = o(1). \quad (3.63)$$

From (3.60)–(3.63), as $|\lambda_n| \to \infty$, we get $\|U^n\|_{\mathcal{H}} = o(1)$, which contradicts $\|U^n\|_{\mathcal{H}} = 1$. Thus, condition (E2) holds true. The result follows from Theorem A.4 (part (i)) in appendix section. The proof is thus complete.

Remark 3.6. In the limit case where $m \to 0$, the (P_{CG}) is transformed to piezoelectric with Fourier law (P_F). By proceeding with the same arguments as in the proof of Theorem 3.1, an exponential stability is obtained.

4. Piezoelectric with Gurtin–Pipkin thermal law (P_{GP})

In this section, we shall analyze the strong stability and the polynomial stability of system (P_{GP}).
4.1. Strong stability

In this subsection, we will prove the stability of system \((P_{GP})\). The main result of this section is the following theorem.

**Theorem 4.1.** Let \(m = 1\) and assume that (H) holds. Then, the \(C_0\)-semigroup of contractions \((e^{tA_1})_{t \geq 0}\) is strongly stable in \(\mathcal{H}\), i.e., for all \(U_0 \in \mathcal{H}\), the solution of (2.10) satisfies

\[
\lim_{t \to +\infty} \|e^{tA_1}U_0\|_\mathcal{H} = 0.
\]

According to Theorem A.2 in the appendix, to prove Theorem 4.1, we need to prove that the operator \(A_1\) has no pure imaginary eigenvalues and \(\sigma(A_1) \cap i\mathbb{R}\) is countable. The proof of Theorem 4.1 will be achieved from the following proposition.

**Proposition 4.2.** Let \(m = 1\) and assume that (H) holds, we have

\[
i\mathbb{R} \subseteq \rho(A_1).
\]

We will prove Proposition 4.2 by contradiction argument. Remark that, it has been proved in Proposition 2.3 that \(0 \in \rho(A_1)\). Now, suppose that (4.1) is false, then there exists \(l \in \mathbb{R}^*\) such that \(il \notin \rho(A_1)\). According to Remark A.3, let \(\{(\lambda^n, U^n := (u^n, v^n, y^n, z^n, w^n, \eta^n)^\top)\}_{n \geq 1} \subseteq \mathbb{R}^* \times D(A_1)\), with

\[
\lambda_n \to l \quad \text{as} \quad n \to \infty \quad \text{and} \quad |\lambda_n| < |l|,
\]

(NA1) and

\[
\|U^n\|_\mathcal{H} = \|(u^n, v^n, y^n, z^n, w^n, \eta^n)^\top\|_\mathcal{H} = 1,
\]

such that

\[
(\lambda_n l - A_1)U^n = F_n := (f_n^1, f_n^2, f_n^3, f_n^4, f_n^5, f_n^6(\cdot, s)) \to 0 \quad \text{in} \quad \mathcal{H}, \quad \text{as} \quad n \to \infty.
\]

Equivalently, from (4.3), we have

\[
i\lambda_n u^n - v^n = f_n^1 \quad \text{in} \quad H^1_0(0, L),
\]

\[
i\lambda_n \rho v^n - \alpha u^n_{xx} + \gamma \beta y^n_{xx} + \delta \omega_x^n = \rho f_n^2 \quad \text{in} \quad L^2(0, L),
\]

\[
i\lambda_n y^n - z^n = f_n^3 \quad \text{in} \quad H^1_0(0, L),
\]

\[
i\lambda_n \mu z^n - \beta y^n_{xx} + \gamma \beta u^n_{xx} = \mu f_n^4 \quad \text{in} \quad L^2(0, L),
\]

\[
i\lambda_n w^n - cA_{xx} z^n + \delta v_x^n = f_n^5 \quad \text{in} \quad L^2(0, L),
\]

\[
i\lambda_n \eta^n - \eta^n_{xx} = f_n^6(\cdot, s) \quad \text{in} \quad W.
\]

Then, we will proof condition (4.1) by finding a contradiction with (4.2) such as \(\|U^n\|_\mathcal{H} \to 0\). The proof of proposition 4.2 has been divided into several Lemmas.

**Lemma 4.3.** Let \(m = 1\) and assume that (H) holds. Then, the solution \((u^n, v^n, y^n, z^n, w^n, \eta^n) \in D(A_1)\) of (4.4)–(4.9) satisfies

\[
- \int_0^L \int_0^\infty |\eta^n_\cdot|^2dsdx \to 0 \quad \text{as} \quad n \to \infty.
\]

and

\[
\int_0^L \int_0^\infty |\eta^n_\cdot|^2dsdx \to 0.
\]
Proof. First, taking the inner product of (4.3) with $U^n$ in $\mathcal{H}$, we get
\[ -\frac{c}{2} \int_0^L \int_0^\infty \sigma'(s)|\eta^n_x|^2 dsdx = -\Re \langle \mathcal{A} U^n, U^n \rangle_{\mathcal{H}} \leq \| F_n \|_{\mathcal{H}} \| U^n \|_{\mathcal{H}} \xrightarrow{n \to \infty} 0. \]

Then, (4.10) holds. Using condition (H), we get
\[ \int_0^L \int_0^\infty \sigma(s)|\eta^n_x(\cdot, s)|^2 dsdx \leq -\frac{1}{d\sigma} \int_0^L \int_0^\infty \sigma'(s)|\eta^n_x(\cdot, s)|^2 dsdx. \]
Using (4.10) in the above inequality, we get (4.11). The proof has been completed. \hfill \Box

Lemma 4.4. Let $m = 1$ and assume that (H) holds. Then, the solution $(u^n, v^n, y^n, z^n, w^n, \eta^n) \in D(A_1)$ of (4.4)-(4.9) satisfies
\[ \int_0^L |\omega^n_x|^2 dx \xrightarrow{n \to \infty} 0 \quad \text{and} \quad \int_0^L |\omega^n|^2 dx \xrightarrow{n \to \infty} 0 \quad \text{(4.12)} \]

Proof. The proof of this Lemma is divided into two steps.

**Step 1.** First, we prove the following estimation
\[ \frac{g(0)}{2} \int_0^L |w^n_x|^2 dx \leq 2|\lambda|^2 \int_0^L \int_0^\infty \sigma(s)|\eta^n_x|^2 dsdx + \frac{\sigma(0)}{g(0)} \int_0^L \int_0^\infty (-\sigma'(s))|\eta^n_x|^2 dsdx \]
\[ + 2 \int_0^L \int_0^\infty \sigma(s)|(f^n_n)_x|^2 dsdx. \quad \text{(4.13)} \]

From (4.9), we have
\[ \omega^n_x = i\lambda \eta^n_x + \eta^n_{sx} -(f^n_n)_x. \quad \text{(4.14)} \]

Multiplying (4.14) by $\sigma(s)w^n_x$, integrating over $(0, \infty) \times (0, L)$, we get
\[ g(0) \int_0^L |w^n_x|^2 dx = i\lambda \int_0^L \int_0^\infty \sigma(s)\eta^n_x w^n_x dsdx + \int_0^L \int_0^\infty \sigma(s)\eta^n_{sx} w^n_x dsdx - \int_0^L \int_0^\infty \sigma(s)(f^n_n)_x w^n_x dsdx. \]

Using integration by parts with respect to $s$ in the above equation and the fact that $\eta^n(\cdot, 0) = 0$ in $(0, L)$, we get
\[ g(0) \int_0^L |w^n_x|^2 dx = i\lambda \int_0^L \int_0^\infty \sigma(s)\eta^n_x w^n_x dsdx - \int_0^L \int_0^\infty \sigma'(s)\eta^n_x w^n_x dsdx - \int_0^L \int_0^\infty \sigma(s)(f^n_n)_x w^n_x dsdx. \quad \text{(4.15)} \]

It follows that,
\[ g(0) \int_0^L |w^n_x|^2 dx \leq |\lambda| \int_0^L \int_0^\infty \sigma(s)|\eta^n_x||w^n_x| dsdx + \int_0^L \int_0^\infty -\sigma'(s)|\eta^n_x||w^n_x| dsdx \]
\[ + \int_0^L \int_0^\infty \sigma(s)|(f^n_n)_x||w^n_x| dsdx. \quad \text{(4.16)} \]
Applying Cauchy–Schwarz and Young’s inequality, we get
\[
|\lambda| \int_0^L \int_0^\infty \sigma(s)|\eta_x^n||w_x^n|dxds \leq \frac{\lambda^2}{2\varepsilon} \int_0^L \int_0^\infty |\sigma(s)|\eta_x^n|^2 d\sigma d\varepsilon + \frac{\varepsilon g(0)}{2} \int_0^L |w_x^n|^2 dx, \tag{4.17}
\]
\[
\int_0^L \int_0^\infty -\sigma'(s)|\eta_x^n||w_x^n|d\sigma ds \leq \sqrt{\sigma(0)} \left( \int_0^L |w_x^n|^2 dx \right)^{\frac{1}{2}} \left( \int_0^L \int_0^\infty -\sigma'(s)|\eta_x^n|^2 d\sigma ds \right)^{\frac{1}{2}} \tag{4.18}
\]
\[
\leq \frac{\varepsilon_1}{2} \int_0^L |w_x^n|^2 + \frac{\varepsilon(0)}{2\varepsilon_1} \int_0^L \int_0^\infty -\sigma'(s)|\eta_x^n|^2 d\sigma ds
\]
and
\[
\int_0^L \int_0^\infty \sigma(s)|(f^n_{x})_x||w_x^n|d\sigma ds \leq \frac{\varepsilon_2 g(0)}{2} \int_0^L |w_x^n|^2 dx ds + \frac{1}{2\varepsilon_2} \int_0^L \int_0^\infty \sigma(s)|(f^n_{x})_x|d\sigma ds. \tag{4.19}
\]

Inserting (4.17)–(4.19) in (4.16), we get
\[
\left( g(0) - \frac{\varepsilon g(0)}{2} \right) \int_0^L |w_x^n|^2 dx \leq \frac{\lambda^2}{2\varepsilon} \int_0^L \int_0^\infty |\sigma(s)|\eta_x^n|^2 d\sigma ds
\]
\[
+ \frac{\varepsilon(0)}{2\varepsilon_1} \int_0^L \int_0^\infty -\sigma'(s)|\eta_x^n|^2 d\sigma ds + \frac{1}{2\varepsilon_2} \int_0^L \int_0^\infty \sigma(s)|(f^n_{x})_x|d\sigma ds. \tag{4.20}
\]

Taking \(\varepsilon = \varepsilon_2 = \frac{1}{4}\) and \(\varepsilon_1 = \frac{g(0)}{2}\) in (4.20), we get (4.13).

**Step 2.** The aim of this step is to prove (4.12). For this aim, using Lemma (4.3) and the fact that \(\|f_0^n\|_W \to 0\) in \(H\) in (4.13), we get the first estimation in (4.12). Next, using Poincaré inequality, we get
\[
\int_0^L |w^n|^2 dx \leq c_p \int_0^L |w_x^n|^2 dx \quad \text{as } n \to \infty.
\]

The proof has been completed. \(\square\)

**Lemma 4.5.** Let \(m = 1\) and assume that (H) holds. Then, the solution \((u^n, v^n, y^n, z^n, w^n, \eta^n) \in D(A_1)\) of (4.4)–(4.9) satisfies
\[
\int_0^L |u_x^n|^2 dx \quad \text{as } n \to \infty \quad \text{and} \quad \int_0^L |v^n|^2 dx \to 0. \tag{4.21}
\]

**Proof.** The proof of this Lemma is divided into several steps.

**Step 1.** The aim of this step is to prove the following estimations:
\[
\|A_{1x}^{1,n}\|_{L^2(0,L)} \leq \lambda_n \|U^n\|_H + \|F^n\|_H, \tag{4.23}
\]
\[
\|u_{xx}^{1,n}\|_{L^2(0,L)} \leq \lambda_n \|U^n\|_H + \|w_x^n\|_{L^2(0,L)} + \|F^n\|_H. \tag{4.24}
\]

The proof of this Lemma is divided into several steps.
First, we prove (4.23). Using (4.8) and (4.4), we get
\[ c\Lambda_{xx}^{1,n} = i\lambda_n w^n + \delta i\lambda_n u^n - \delta (f_1^n)_x - f_5^n. \]
Using the fact that \( \|w^n\| \leq \|U^n\|_{\mathcal{H}}, \sqrt{\alpha_1} u^n_x \leq \|U^n\|_{\mathcal{H}}, \|f_5^n\| \leq \|F_n\|_{\mathcal{H}} \) and \( \sqrt{\alpha_1}(f_1^n)_x \leq \|F^n\|_{\mathcal{H}} \), we get
\[ c\|\Lambda_{xx}^{1,n}\|_{L^2(0,L)} \leq \left( 1 + \frac{\delta}{\sqrt{\alpha_1}} \right) (\|\lambda_n\|\|U^n\|_{\mathcal{H}} + \|F_n\|_{\mathcal{H}}). \]

Then, we get (4.23). In order to prove (4.24), using (4.5) and the fact that \( \alpha = \alpha_1 + \gamma^2 \beta \), we get
\[ \alpha_1 u^n_{xx} = i\lambda_n (\rho v^n + \gamma \mu z^n) + \delta w^n_x - \rho f_1^n - \gamma \mu f_4^n. \]
Using the fact that \( \sqrt{\rho} v^n \leq \|U^n\|_{\mathcal{H}}, \sqrt{\rho} z^n \leq \|U^n\|_{\mathcal{H}}, \sqrt{\rho} \|f_1^n\| \leq \|F^n\|_{\mathcal{H}} \) and \( \sqrt{\rho} \|f_4^n\| \leq \|F^n\|_{\mathcal{H}} \) in the above inequality, we obtain
\[ \alpha_1 \|u^n_{xx}\|_{L^2(0,L)} \leq (\sqrt{\rho} + \sqrt{\rho} \sqrt{\mu}) (\|\lambda_n\|\|U^n\|_{\mathcal{H}} + \|F_n\|_{\mathcal{H}} + \delta \|w^n_x\|). \]

Then, we get (4.24).

**Step 2.** The aim of this step is to prove (4.21). From (4.4) and (4.8), we have
\[ i\lambda_n \delta u^n_x = -i\lambda_n w^n + c\Lambda_{xx}^{1,n} + \delta (f_1^n)_x + f_5^n. \]
Multiplying the above equation by \(-i\lambda^{-1}_n \delta u^n_x\), integrating by parts over \((0,L)\), we get
\[ \delta \int_0^L |u^n_x|^2 dx = -\delta \int_0^L w^n |u^n_x| dx + i\lambda^{-1}_n c \int_0^L \Lambda_{xx}^{1,n} u^n_x dx + i\lambda^{-1}_n c\Lambda_{xx}^{1,n}(0)u^n_x(0) - i\lambda^{-1}_n \int_0^L (f_1^n)_x + f_5^n u^n_x dx. \]

It follows that,
\[ \delta \int_0^L |u^n_x|^2 dx \leq \delta \int_0^L |w^n| |u^n_x| dx + |\lambda_n|^{-1} c \int_0^L |\Lambda_{xx}^{1,n}| u^n_x dx + c |\lambda_n|^{-1} \|\Lambda_{xx}^{1,n}(0)\| |u^n_x(0)| \]
\[ + |\lambda_n|^{-1} \int_0^L |(f_1^n)_x| |u^n_x| dx + |\lambda_n|^{-1} \int_0^L |f_5^n| |u^n_x| dx. \]

Using the fact that \( \sqrt{\alpha_1} \|u^n_x\| \leq \|U^n\|_{\mathcal{H}}, (4.12) \), we get
\[ \int_0^L |w^n| |u^n_x| dx \overrightarrow{n \to \infty} 0. \]

Using the fact that \( \sqrt{\alpha_1}(f_1^n)_x \leq \|F_n\|_{\mathcal{H}} \to 0, \|f_5^n\| \leq \|F_n\|_{\mathcal{H}} \to 0, \sqrt{\alpha_1} \|u^n_x\| \leq \|U^n\|_{\mathcal{H}} \) and (CA1), we get
\[ |\lambda_n^{-1}| \delta \int_0^L |(f_1^n)_x| |u^n_x| dx \overrightarrow{n \to \infty} 0. \]

and
\[ |\lambda_n^{-1}| \int_0^L |f_5^n| |u^n_x| dx \overrightarrow{n \to \infty} 0. \]
Using (4.11), (4.12), (4.24), (4.2) and (CA1), we get
\[ |\lambda_n|^{-1} c \int_0^L |\Lambda_x^{1,n}| |u_{xx}^n| dx \rightarrow 0. \] (4.29)

Using Gagliardo–Nirenberg inequality, (4.23), (4.24), \( \|U^n\|_{\mathcal{H}} = 1 \), \( \|F_n\|_{\mathcal{H}} \rightarrow 0 \) and (CA1), we get
\[ |\Lambda_x^{1,n}(0)| \lesssim \left( \|\Lambda_x^{1,n}\|^{\frac{1}{2}} \|\Lambda_x^{1,n}\|^{\frac{1}{2}} + \|\Lambda_x^{1,n}\| \right) \rightarrow 0 \] (4.30)
and
\[ |u_{xx}^n(0)| \lesssim \left( \|u_{xx}^n\|^{\frac{1}{2}} \|u_{xx}^n\|^{\frac{1}{2}} + \|u_{xx}^n\| \right) \lesssim M. \] (4.31)
From (4.30) and (4.31), we obtain
\[ |\lambda_n^{-1} |\Lambda_x^n(0)| u_{xx}^n(0) | \rightarrow 0. \] (4.32)

Finally, inserting (4.26), (4.27), (4.28), (4.29) and (4.32) in (4.25), we get the desired result (4.21).

**Step 3.** The aim of this step is to prove (4.22). From (4.21) and Poincaré inequality, we get
\[ \int_0^L |u^n|^2 dx \rightarrow 0. \] (4.33)
From (4.4), we get
\[ \frac{L}{2} |v^n|^2 dx \leq 2|\lambda_n|^2 \int_0^L |u^n|^2 dx + 2c_p \int_0^L \left| \left( f_{1,n}^1 \right)_{xx} \right|^2 dx. \]
Passing to the limit in the above inequality and using (CA1), (4.33) and the fact that \( \sqrt{\alpha_1} \|f_{1,n}^1\| \leq \|F^n\|_{\mathcal{H}} \rightarrow 0 \), we get the desired result (4.22). The proof is thus completed.

Inserting (4.4) in (5), we get
\[ -\lambda^2 \rho u^n - \alpha u_{xx}^n + \gamma \beta y_{xx}^n + \delta w_{xx}^n = \rho f_{2,n}^2 + i\lambda_n \rho f_{1,n}^1. \] (4.34)

**Lemma 4.6.** Let \( m = 1 \) and assume that (H) holds. Then, the solution \( (u^n, v^n, y^n, z^n, w^n, \eta^n) \in D(A_1) \) of (4.4)–(4.9) satisfies the following estimation
\[ \int_0^L |y_{xx}^n|^2 dx \rightarrow 0 \] (4.35)
and
\[ \int_0^L |z^n|^2 dx \rightarrow 0. \] (4.36)

**Proof.** The proof of this lemma is divided into two steps.

**Step 1.** The aim of this step is to prove (4.35). For this aim, multiplying (4.34) by \(-\gamma\beta\) and integrating by parts over \((0, L)\), we get
\[ \gamma \beta \int_0^L |y_{xx}^n|^2 dx = \rho \lambda_n^2 \int_0^L u^n \bar{y}_x dx + \alpha \int_0^L u_{xx}^n \bar{y}_x dx - \delta \int_0^L w^n \bar{y}_x dx - \int_0^L (\rho f_{2,n}^2 + i\lambda_n \rho f_{1,n}^1) \bar{y} dx. \]
It follows that,
\[
\frac{\gamma}{\beta} \int_0^L |y^n_x|^2 \, dx \leq \rho \lambda_n^2 \int_0^L |u||y| \, dx + \alpha \int_0^L \left| u^n_x \right| \, dx + \delta \int_0^L |w^n| \, dx + \int_0^L \left( \rho |f^n_1| + \rho |\lambda_n||f^n_1| \right) |y^n| \, dx \quad (4.37)
\]

Using the fact that \(\lambda_n y^n\) is uniformly bounded in \(L^2(0, L)\), (4.33) and (CA1), we get
\[
\lambda_n^2 \int_0^L |u||y| \, dx \xrightarrow{n \to \infty} 0. \quad (4.38)
\]

Using the fact that \(y^n_x\) is uniformly bounded in \(L^2(0, L)\) and (4.21), we get
\[
\int_0^L \left| u^n_x \right| |y^n_x| \, dx \xrightarrow{n \to \infty} 0. \quad (4.39)
\]

Using the fact that \(y^n_x\) is uniformly bounded in \(L^2(0, L)\) and (4.12), we get
\[
\int_0^L \left| w^n \right| |y^n_x| \, dx \xrightarrow{n \to \infty} 0. \quad (4.40)
\]

Using the fact that \(\|F_n\|_H \to 0\) and the fact that \(\lambda_n y^n\) is uniformly bounded in \(L^2(0, L)\), we get
\[
\int_0^L \left( \rho |f^n_1| + \rho |\lambda_n||f^n_1| \right) |y^n| \, dx \xrightarrow{n \to \infty} 0. \quad (4.41)
\]

Inserting (4.38)–(4.41) in (4.37), we obtain (4.35).

**Step 2.** The aim of this step is to prove (4.36). From (4.35) and Poincaré inequality, we get
\[
\int_0^L |y^n|^2 \, dx \xrightarrow{n \to \infty} 0. \quad (4.42)
\]

From (4.6), we get
\[
\int_0^L |z^n|^2 \, dx \leq 2|\lambda_n|^2 \int_0^L |y^n|^2 \, dx + 2c_p \int_0^L |(f^n_3)_x|^2 \, dx.
\]

Passing to the limit in the above inequality and using (CA1), (4.42) and the fact that \(\|F_n\|_H \to 0\), we get the desired result (4.36). The proof is thus completed.

**Proof of Proposition 4.2.** From Lemmas 4.3–4.6, we obtain \(\|U^n\|_H \to 0\) as \(n \to +\infty\) which contradicts \(\|U^n\|_H = 1\). Thus, (4.1) holds true. The proof is thus complete.

**Proof of Theorem 4.1.** From Proposition 4.2, we have \(i\mathbb{R} \subset \rho(A)\) and consequently \(\sigma(A) \cap i\mathbb{R} = \emptyset\). Therefore, according to Theorem A.2 in appendix, we get the \(C_0\)-semigroup of contraction \(\left( e^{tA_1} \right)_{t \geq 0}\) is strongly stable. The proof is thus complete.
4.2. Polynomial stability

In this subsection, we will prove the polynomial stability of system \( (P_{GP}) \). The main result of this section is the following theorem.

**Theorem 4.7.** Let \( m = 1 \) and assume that \( (H) \) holds, then there exists \( C > 0 \) such that for every \( U_0 \in D(A_1) \), we have

\[
E_1(t) \leq \frac{C}{t} \| U_0 \|^2_{D(A_1)}, \quad t > 0.
\] (4.43)

According to Theorem A.4 in appendix, to prove Theorem 4.7, we still need to prove the following two conditions

\[
i \mathbb{R} \subset \rho(A_1), \quad \text{(POL1)}
\]

\[
\limsup_{|\lambda| \to \infty} \frac{1}{|\lambda|^2} \|(i\lambda - A_1)^{-1}\| < \infty, \quad \text{(POL2)}
\]

From Proposition 4.2, we obtain condition (POL1). Next, we will prove condition (POL2) by a contradiction argument. For this purpose, suppose that (POL2) is false, then there exists \( \{(\lambda^n, U^n) := (u^n, v^n, y^n, z^n, w^n, \eta^n(\cdot, s))\}_{n \geq 1} \subset \mathbb{R}^n \times D(A_1) \) with

\[
|\lambda^n| \to \infty \quad \text{and} \quad \| U^n \|_{\mathcal{H}} = \|(u^n, v^n, y^n, z^n, w^n, \eta^n(\cdot, s))\|_{\mathcal{H}} = 1,
\] (CA2)

such that

\[
\lambda_n^2 (i\lambda_n I - A_1) U^n = P_n := (f_n^1, f_n^2, f_n^3, f_n^4, f_n^5, f_n^6(\cdot, s))^\top \to 0 \quad \text{in} \quad \mathcal{H}.
\] (4.44)

For simplicity, we drop the index \( n \). Equivalently, from (4.44), we have

\[
i\lambda u - v = \lambda^{-2} f^1 \quad \text{in} \quad H^1_2(0, L),
\] (4.45)

\[
i\lambda v - \alpha u_{xx} + \gamma \beta y_{xx} + \delta \omega_x = \rho \lambda^{-2} f^2 \quad \text{in} \quad L^2(0, L),
\] (4.46)

\[
i\lambda y - z = \lambda^{-2} f^3 \quad \text{in} \quad H^1_2(0, L),
\] (4.47)

\[
i\lambda \mu z - \beta y_{xx} + \gamma \beta u_{xx} = \mu \lambda^{-2} f^4 \quad \text{in} \quad L^2(0, L),
\] (4.48)

\[
i\lambda w - c \Lambda_x^1 + \delta v_x = \lambda^{-2} f^5 \quad \text{in} \quad L^2(0, L),
\] (4.49)

\[
i\lambda \eta + \eta u = \lambda^{-2} f^6(\cdot, s) \quad \text{in} \quad W.
\] (4.50)

Here, we will check the condition (POL2) by finding a contradiction with (CA2) such as \( \| U \|_{\mathcal{H}} = o(1) \). For clarity, we divide the proof into several lemmas.

**Lemma 4.8.** Let \( m = 1 \) and assume that \( (H) \) holds. The solution \( (u, v, y, z, w, \eta) \in D(A_1) \) of (4.45)–(4.50) satisfies the following estimations

\[
- \int_0^L \int_0^s \sigma'(s) |\eta_x|^2 ds dx = o(\lambda^{-2}) \quad \text{and} \quad \int_0^L \int_0^s |\eta_x|^2 ds dx = o(\lambda^{-2}).
\] (4.51)

**Proof.** By proceeding the same argument used in Lemma 4.3, we get (4.51). \( \square \)

**Lemma 4.9.** Let \( m = 1 \) and assume that \( (H) \) holds. The solution \( (u, v, y, z, w, \eta) \in D(A_1) \) of (4.45)–(4.50) satisfies the following estimations

\[
\int_0^L |w_x|^2 dx = o(1) \quad \text{and} \quad \int_0^L |w|^2 dx = o(1).
\] (4.52)
Proof. By using the same techniques of step 1 in Lemma 4.4, we get
\[
\frac{g(0)}{2} \int_0^L |w_x|^2 \, dx \leq 2|\lambda|^2 \int_0^L \sigma(s)|\eta_x|^2 \, ds \, dx + \frac{\sigma(0)}{g(0)} \int_0^L (-\sigma'(s))|\eta_x|^2 \, ds \, dx + 2|\lambda|^{-4} \int_0^L \sigma(s)|(f^6)_x|^2 \, ds \, dx.
\]
(4.53)
Using the fact that \(\|F\|_H = o(1)\), Lemma 4.8 in (4.53), we get the first estimation in (4.53). Applying Poincaré inequality, we get the second estimation in (4.53). The proof has been completed. \(\square\)

**Lemma 4.10.** Let \(m = 1\) and assume that \((H)\) holds. The solution \((u,v,y,z,w,\eta) \in D(A_1)\) of (4.45)–(4.50) satisfies the following estimation
\[
\int_0^L |u_x|^2 \, dx = o(1).
\]
(4.54)

**Proof.** Similar to Lemma 4.5, we prove that
\[
\begin{cases}
\|\Lambda^1_x\|_{L^2(0,L)} \lesssim |\lambda|\|U\|_H + \|F\|_H \leq O(|\lambda|), \\
\|u_{xx}\|_{L^2(0,L)} \lesssim |\lambda|\|U\|_H + \|w_x\|_{L^2(0,L)} + \|F\|_H \leq O(|\lambda|).
\end{cases}
\]
(4.55)
Now, multiplying (4.48) by \(-i\lambda^{-1}\delta\eta_x\), integrating by parts over \((0,L)\) and using (4.45), we get
\[
\delta \int_0^L |u_x|^2 \, dx = -\delta \int_0^L \omega_0 dx + i\lambda^{-1}c \int_0^L \Lambda^1_x \omega dx + i\lambda^{-1}c \Lambda_x(0) u_x(0)
\]
\[
- i\lambda^{-3} \int_0^L (\delta(f^1_n)_x + f^5_n \omega_x) \, dx.
\]
(4.56)
Using the facts that \(u_x\) is uniformly bounded in \(L^2(0,L)\), Lemma 4.8, (4.52), (4.55) and \(\|F\|_H = o(1)\), we get
\[
\left| \int_0^L \omega_0 \, dx \right| = o(1), \quad \left| \lambda^{-1} \int_0^L \Lambda^1_x \omega \, dx \right| = o(|\lambda|^{-1}) \quad \text{and} \quad \left| \lambda^{-3} \int_0^L (\delta(f^1_n)_x + f^5_n \omega_x) \, dx \right| = o(|\lambda|^{-3}).
\]
(4.57)
Using Gagliardo–Nirenberg inequality, (4.55), (4.51) and the fact that \(u_x\) is uniformly bounded in \(L^2(0,L)\), we get
\[
|\lambda^{-1} \Lambda^1_x(0) u_x(0)| \lesssim |\lambda|^{-1} \left( \|\Lambda^1_x\| \|\Lambda^1_x\| \|u_x\| \right) \left( \|u_{xx}\| \|u_x\| \right) = o(1).
\]
(4.58)
Inserting (4.57) and (4.58) in (4.56), we get the desired result (4.54). The proof has been completed. \(\square\)

Inserting (4.45) in (4.46), we get
\[
- \lambda^2 \rho u - \alpha u_{xx} + \gamma \beta y_{xx} + \delta w_x = \rho \lambda^{-2} f^2 + i\lambda^{-1} \rho f^1.
\]
(4.59)
Lemma 4.11. Let \( m = 1 \) and assume that (H) holds. The solution \((u, v, y, z, w, \eta) \in D(A_1)\) of (4.45) – (4.50) satisfies the following estimation

\[
\int_0^L |\lambda u|^2 dx = o(1).
\]  

(4.60)

Proof. Multiplying (4.59) by \(-\pi\) integrating by parts over \((0, L)\), we get

\[
\rho \int_0^L |\lambda u|^2 dx = \alpha \int_0^L |u_x|^2 dx - \gamma \beta \int_0^L y x u_x dx + \delta \int_0^L w_x u dx - \int_0^L \left( \frac{\rho f^2}{\lambda^2} + \frac{i \rho f_1^1}{\lambda} \right) u dx.
\]

It follows that

\[
\rho \int_0^L |\lambda u|^2 dx \leq \alpha \int_0^L |u_x|^2 dx + \gamma \beta \int_0^L |y| u_x dx + \delta \int_0^L |w_x| |u| dx + \int_0^L \left( \frac{\rho |f^2|}{\lambda^2} + \frac{\rho |f_1^1|}{|\lambda|} \right) |u| dx.
\]  

(4.61)

Using the fact that \( y_x \) and \( \lambda u \) are uniformly bounded in \( L^2(0, L) \), (4.52), we get

\[
\int_0^L |y_x| |u_x| dx = o(1) \quad \text{and} \quad \int_0^L |w_x| |u| dx = \frac{o(1)}{\lambda}.
\]  

(4.62)

Using the fact that \( \|F\|_{\mathcal{H}} = o(1) \) and \( \lambda u \) is uniformly bounded in \( L^2(0, L) \), we get

\[
\int_0^L \left( \frac{\rho |f^2|}{\lambda^2} + \frac{\rho |f_1^1|}{|\lambda|} \right) |u| dx = \frac{o(1)}{\lambda^2}.
\]  

(4.63)

Inserting (4.62) and (4.63) in (4.61) and using (4.52), we obtain (4.60). The proof is thus completed. \(\square\)

Lemma 4.12. Let \( m = 1 \) and assume that (H) holds. The solution \((u, v, y, z, w, \eta) \in D(A_1)\) of (4.45) – (4.50) satisfies the following estimation

\[
\int_0^L |y_x|^2 dx = o(1).
\]  

(4.64)

Proof. The idea of proof is similar to Lemma 4.6. Multiplying (4.59) by \(-\bar{y}\), integrating by parts over \((0, L)\) and using the facts that \(\lambda y\) and \(y_x\) are uniformly bounded in \( L^2(0, L) \), (4.60), (4.54) and (4.52) and \( \|F\|_{\mathcal{H}} = o(1) \), we get

\[
\gamma \beta \int_0^L |y_x|^2 dx \leq \rho \lambda^2 \int_0^L |u| |y| dx + \alpha \int_0^L |u_x|^2 |y_x|^2 dx + \delta \int_0^L |w_x|^2 |y_x|^2 dx + \int_0^L \left( \rho |\lambda|^{-2} |f^2| + \rho |\lambda|^{-1} |f_1^1| \right) |y_x|^2 dx.
\]

The proof has been completed. \(\square\)

Lemma 4.13. Let \( m = 1 \) and assume that (H) holds. The solution \((u, v, y, z, w, \eta) \in D(A_1)\) of (4.45) – (4.50) satisfies the following estimation

\[
\int_0^L |\lambda y|^2 dx = o(1).
\]  

(4.65)
Proof. Inserting (4.6) in (4.7), we get

\[-\lambda^2 \mu y - \beta y_{xx} + \gamma \beta u_{xx} = \mu \frac{f^4}{\lambda^2} + i \mu \frac{f^3}{\lambda}.\]

Multiplying the above equation by \(-\bar{y}\) integrating by parts over \((0,L)\), we get

\[\mu \int_0^L |\lambda y|^2 dx = -\beta \int_0^L |y_x|^2 dx - \gamma \beta \int_0^L u_{x \bar{y}} dx - \int_0^L \left( \frac{\mu f^4}{\lambda^2} + i \mu \frac{f^3}{\lambda} \right) \bar{y} dx.\] (4.66)

Using (4.64), (4.54), \(\lambda y\) is uniformly bounded in \(L^2(0,L)\) and the fact that \(\|F^n\|_H = o(1)\), we get

\[\left| \int_0^L u_{x \bar{y}} dx \right| = o(1) \quad \text{and} \quad \left| \int_0^L \left( \frac{\mu f^4}{\lambda^2} + i \mu \frac{f^3}{\lambda} \right) \bar{y} dx \right| = o(1).\]

Inserting the above estimations in (4.66) and using (4.64), we get (4.65). The proof is thus completed.

\[\Box\]

**Proof of Theorem 4.7.** Using Lemmas 4.8–4.13, we obtain that \(\|U^n\|_H = o(1)\), which contradicts \(\|U^n\|_H = 1\). Thus, (POL2) holds true. The proof has been completed.

\[\Box\]

5. Conclusion

In this work, we studied the decay rate for one-dimensional piezoelectric beams with magnetic effect and heat equation with memory, where the hereditary heat conduction is due to Coleman–Gurtin law or Gurtin–Pipkin law. The exponential stability is obtained when the hereditary heat conduction is of Coleman–Pipkin type. Further, we show the polynomial stability of type \(t^{-1}\) when the heat conduction law is of Gurtin–Pipkin law. The table below summarizes the results of the paper: We conjecture

| Systems                               | Stability               |
|---------------------------------------|-------------------------|
| Piezoelectric with Coleman–Gurtin law (P_{CG}) | Exponential stability  |
| Piezoelectric with Fourier law (P_{F})    | Exponential stability  |
| Piezoelectric with Gurtin–Pipkin law (P_{GP}) | Polynomial stability of order \(t^{-1}\) |

that the polynomial energy decay rate obtained in Theorem 4.7 is optimal. The idea of the proof is to find a sequence \((\lambda_n)_n \subset \mathbb{R}_+^\ast\) with \(|\lambda_n| \to +\infty\) and a sequence of vectors \((U_n)_n \subset D(A_1)\) such that 

\[(i\lambda_n I - A_1)U_n = F_n\]

is bounded in \(H\) and

\[\lim_{n \to +\infty} \lambda_n^{-2+\varepsilon} \|U_n\|_H = \infty.\]

(see Theorem 3.1 in [29] and Theorem 5.1 in [2]). Depending on the boundary conditions, this approach and the construction of the vector \((U_n)_n\) is not feasible and the problem is still an open problem.

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Appendix A: Some notions and stability theorems

In order to make this paper more self-contained, we recall in this short appendix some notions and stability results used in this work.

**Definition A.1.** Assume that $A$ is the generator of a $C_0$–semigroup of contractions $(e^{tA})_{t \geq 0}$ on a Hilbert space $H$. The $C_0$–semigroup $(e^{tA})_{t \geq 0}$ is said to be

1. **Strongly stable** if
   $$\lim_{t \to +\infty} \|e^{tA}x_0\|_H = 0, \quad \forall x_0 \in H.$$
2. **Exponentially (or uniformly) stable** if there exists two positive constants $M$ and $\varepsilon$ such that
   $$\|e^{tA}x_0\|_H \leq Me^{-\varepsilon t}\|x_0\|_H, \quad \forall t > 0, \quad \forall x_0 \in H.$$
3. **Polynomially stable** if there exist two positive constants $C$ and $\alpha$ such that
   $$\|e^{tA}x_0\|_H \leq Ct^{-\alpha}\|x_0\|_H, \quad \forall t > 0, \quad \forall x_0 \in D(A).$$

For proving the strong stability of the $C_0$–semigroup $(e^{tA})_{t \geq 0}$, we will recall the result obtained by Arendt and Batty in [5].

**Theorem A.2.** (Arendt and Batty in [5]) Assume that $A$ is the generator of a $C_0$–semigroup of contractions $(e^{tA})_{t \geq 0}$ on a Hilbert space $H$. If $A$ has no pure imaginary eigenvalues and $\sigma(A) \cap i\mathbb{R}$ is countable, where $\sigma(A)$ denotes the spectrum of $A$, then the $C_0$–semigroup $(e^{tA})_{t \geq 0}$ is strongly stable.

There exists a second classical method based on Arendt and Batty theorem and the contradiction argument (see page 25 in [20]).

**Remark A.3.** Assume that the unbounded linear operator $A : D(A) \subset H \to H$ is the generator of a $C_0$–semigroup of contractions $(e^{tA})_{t \geq 0}$ on a Hilbert space $H$ and suppose that $0 \in \rho(A)$. According to (page 25 in [20], see also [3]), in order to prove that

$$i\mathbb{R} \equiv \{i\lambda \mid \lambda \in \mathbb{R}\} \subseteq \rho(A), \tag{A.1}$$

we need the following steps:

(i) It follows from the fact that $0 \in \rho(A)$ and the contraction mapping theorem that for any real number $\lambda$ with $|\lambda| < \|A^{-1}\|^{-1}$, the operator $i\lambda A - A = (i\lambda A - I)$ is invertible. Furthermore, $\|(i\lambda A - A)^{-1}\|$ is a continuous function of $\lambda$ in the interval $(-\|A^{-1}\|^{-1},\|A^{-1}\|^{-1})$.

(ii) If $\sup\{|(i\lambda I - A)^{-1}||\lambda| < \|A^{-1}\|^{-1}\} = M < \infty$, then by the contraction mapping theorem, the operator $i\lambda I - A = (i\lambda I - A)(I + i(\lambda - \lambda_0)(i\lambda I - A)^{-1})$ with $|\lambda_0| < \|A^{-1}\|^{-1}$ is invertible for $|\lambda - \lambda_0| < M^{-1}$. It turns out that by choosing $|\lambda_0|$ as close to $\|A^{-1}\|^{-1}$ as we can, we conclude that $\{\lambda \mid |\lambda| < \|A^{-1}\|^{-1} + M^{-1}\} \subseteq \rho(A)$ and $\|(i\lambda I - A)^{-1}\|$ is a continuous function of $\lambda$ in the interval $(-\|A^{-1}\|^{-1} - M^{-1},\|A^{-1}\|^{-1} + M^{-1})$.

(iii) Thus, it follows from the argument in (ii) that if (A.1) is false, then there is $\omega \in \mathbb{R}$ with $\|A^{-1}\|^{-1} \leq |\omega| < \infty$ such that $\{\lambda \mid |\lambda| < |\omega|\} \subseteq \rho(A)$ and $\sup\{|(i\lambda - A)^{-1}||\lambda| < |\omega|\} = \infty$. It turns out that there exists a sequence $\{(\lambda_n, U_n)\}_{n \geq 1} \subset \mathbb{R} \times D(A)$, with $\lambda_n \to \omega$ as $n \to \infty$, $|\lambda_n| < |\omega|$ and $\|U_n\|_H = 1$, such that

$$(i\lambda_n I - A)U_n = F_n \to 0 \quad \text{in} \quad H, \quad \text{as} \quad n \to \infty.$$ 

Then, we will prove (A.1) by finding a contradiction with $\|U_n\|_H = 1$ such as $\|U_n\|_H \to 0$. \qed
We now recall the following standard result which is stated in a comparable way [17, 25] for part (1) and [7] (see also [6, 19, 27]) for part (2).

**Theorem A.4.** Assume that $A$ is the generator of a strongly continuous semigroup of contractions $(e^{tA})_{t \geq 0}$ on $H$. If $i\mathbb{R} \subset \rho(A)$. Then;

1. The semigroup $e^{tA}$ is exponentially stable if and only if
   $$\limsup_{\lambda \in \mathbb{R}, \lambda \to \infty} \| (i\lambda I - A)^{-1} \| < \infty.$$  

2. The semigroup $e^{tA}$ is polynomially stable of order $\ell > 0$ if and only if
   $$\limsup_{\lambda \in \mathbb{R}, \lambda \to \infty} |\lambda|^{-\frac{1}{\ell}} \| (i\lambda I - A)^{-1} \| < \infty.$$  

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Mohammad Akil
Université Polytechnique Hauts-de-France
CÉRAMATHS/DEMAV, Le Mont Houy
59313 Valenciennes Cedex 9
France
e-mail: mohammad.akil@uphf.fr

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