NECK PINCHES ALONG THE LAGRANGIAN MEAN CURVATURE FLOW OF SURFACES

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Abstract. Let \( L_t \) be a zero Maslov, rational Lagrangian mean curvature flow in a compact Calabi–Yau surface, and suppose that at the first singular time a tangent flow is given by the static union of two transverse planes. We show that in this case the tangent flow is unique, and that the flow can be continued past the singularity as an immersed, smooth, zero Maslov, rational Lagrangian mean curvature flow. Furthermore, if \( L_0 \) is a sphere that is stable in the sense of Thomas–Yau, then such a singularity cannot form.

1. Introduction

The question of the existence of special Lagrangian submanifolds is an important problem in complex and symplectic geometry. Special Lagrangians play a central role in the Strominger–Yau–Zaslow conjecture [25] on mirror symmetry, and are of interest in the variational problem of finding area-minimizing Lagrangians, studied extensively by Schoen–Wolfson [19]. Smoczyk [24] showed that the mean curvature flow preserves the class of Lagrangian submanifolds in Calabi–Yau manifolds, and so a natural expectation is that a suitable Lagrangian can be deformed into a special Lagrangian using the flow. The Thomas–Yau conjecture [28], motivated by mirror symmetry [27], predicts that this is indeed the case, assuming that the initial Lagrangian satisfies a certain stability condition. More recently Joyce [12] formulated a detailed conjectural picture relating singularity formation along the Lagrangian mean curvature flow to Bridgeland stability conditions on Fukaya categories.

It was shown by Neves [17] that singularities are, in a sense, unavoidable along the Lagrangian mean curvature flow, even if the initial Lagrangian is a small Hamiltonian perturbation of a special Lagrangian. At the same time, Neves [15] shows that for the flow of zero Maslov Lagrangians any tangent flow at a singular point is a union of special Lagrangian cones. This means that Type I singularities – which are typically easier to analyse – do not exist. In this paper we study the simplest kind of singularities, called neck pinches in [12, Conjecture 3.16], in the two-dimensional case. Our main result is the following, which we state in the setting of a compact ambient Calabi–Yau surface, though it also works in \( \mathbb{C}^2 \). Note that here, and throughout, we allow our Lagrangians to be immersed, which is important in the context of Lagrangian mean curvature flow.

\textbf{Theorem 1.1.} Let \( X \) be a compact Calabi–Yau surface, and \( L \subset X \) a zero Maslov, rational Lagrangian. Let \( L_t \) be the mean curvature flow starting from \( L \) for \( t \in [0,T) \), where \( T \) is the first finite singular time. Let \( (x_T, T) \) be a singular point, and suppose that a tangent flow at \( (x_T, T) \) is given by the transverse union of two multiplicity one planes. The tangent flow at \( (x_T, T) \) is then unique.

\textit{Date:} August 24, 2022.
In Theorem 1.1, the assumption is that one tangent flow is given by a union of multiplicity one transverse planes $P_1 \cup P_2$, with corresponding Lagrangian angles $\theta_1, \theta_2$. We note here that by Neves [16, Corollary 4.3] the flow cannot form a singularity unless $\theta_1 = \theta_2$. Therefore throughout the article we will only be concerned with the case when $P_1$ and $P_2$ have the same Lagrangian angle.

The uniqueness of tangent flows is a fundamental problem for analysing the singularities of mean curvature flow, and there have been several important results in this direction recently [20, 5, 3]. A major new difficulty in Theorem 1.1 is that it is the first example of uniqueness for a tangent flow that is singular. The proof crucially exploits several aspects of the Lagrangian setting, and does not apply in the general setting of mean curvature flow.

Theorem 1.1 allows us to analyse the behavior of the flow at the singularity. First, we have the following, showing that the flow can be continued past the singular time if all singularities at time $T$ are modelled on two transverse, multiplicity one planes. Recall that the grading of a zero Maslov Lagrangian corresponds to a global choice of function representing the Lagrangian angle, see Definition 2.1.

**Theorem 1.2.** Suppose that $X, L$ are as in Theorem 1.1 and assume that at each singular point $(x, T)$ a tangent flow is a static union of two multiplicity one, transverse planes. Then $L_t$ converges to an immersed Lagrangian $C^1$-submanifold $L_T$ in the sense of currents as $t \to T$, and the flow can be restarted as a smooth, zero Maslov, rational Lagrangian mean curvature flow with initial condition $L_T$. Furthermore, the extended flow is smooth (together with its grading) through the singular time, away from the singular points.

A slight extension of the ideas involved in proving uniqueness of the tangent flow also allows us to show that if the tangent flow is given by the union of two transverse planes, then close to the singularity the flow looks like the two transverse planes, desingularized by a Lawlor neck which “pinches off”: see Theorem 8.3. This has the following consequence.

**Theorem 1.3.** Suppose that $X, L$ are as in Theorem 1.1. For $t < T$ sufficiently close to the singular time we can write $L_t$ as a graded self-connected sum of an immersed Lagrangian $M$ at a self-intersection point.

If $M$ is not connected, then we can write it as a graded connected sum $M = M_1 \# M_2$ and the following holds:

$$\text{vol}(L) > \left| \int_{M_1} \Omega \right| + \left| \int_{M_2} \Omega \right|,$$

where $\Omega$ is the holomorphic volume form on $X$. If in addition $L$ is almost calibrated, then we also have

$$\phi(M_1), \phi(M_2) \subseteq (\inf_L \theta, \sup_L \theta),$$

where $\phi(M_i)$ is the “cohomological” Lagrangian angle of $M_i$ defined in (9.3). (See Section 9 for detailed definitions.)

This result provides some evidence for Thomas–Yau’s Conjecture 7.3 in [28]. Indeed, their conjecture states that if the flow has a finite time singularity, then $L$ can be decomposed into a graded connected sum $M_1 \# M_2$ satisfying the conditions in (1.1). Our result shows that this is one of the possible scenarios when the tangent flow at the first singular time is given by two transverse planes. In particular, the
decomposition as a graded connect sum is guaranteed if $L$ is a sphere. Note that Joyce’s conjectural picture [12] predicts that other singularities could still form, notably those with tangent flows given by two static planes meeting along a line. It is an important problem to understand what we can say about the flow in the presence of such singularities and some progress towards this was made in the authors’ previous work [14]. An optimistic expectation is that for a generic initial surface, the only tangent flows that appear at singularities are of these two types, i.e. two multiplicity one planes meeting either at a point or along a line.

1.1. Outline. To conclude this introduction we give a brief outline of the contents of the paper, along with some of the main ingredients of the proofs. In most of the paper we will consider the flow in $\mathbb{C}^2$ and we will only discuss the necessary changes in the case of a compact ambient space in Section 9.

The technical heart of our results is an analysis of rescaled Lagrangian mean curvature flows $M_\tau \subset \mathbb{C}^2$ which are close to the union $V = P_1 \cup P_2$ of two transverse planes on a time interval $t \in [0, 1]$, say. We introduce a distance function $D_V(M_\tau)$ from $V$ to $M_\tau$, which incorporates both an $L^2$-type distance, as well as the difference in Lagrangian angles. (See Definition 3.4 for the precise definition.) Although the non-compactness introduces some technical difficulties, the main difficulty when compared to earlier studies is the presence of the singularity at the origin. The first step for dealing with this is to observe that the function $|zw|^2$ satisfies a useful differential inequality along the flow: here $z, w$ are complex coordinates, for a suitable hyperkähler-rotated complex structure, such that $V = \{zw = 0\}$. The differential inequality is exploited in Lemma 3.5, allowing us to convert bounds on $D_V(M_\tau)$ to pointwise distance bounds at a later time.

In general the knowledge that $M_\tau$ is close to $V$ in the Hausdorff sense does not imply graphicality of $M_\tau$ over $V$, even on regions away from the origin, because of possible multiplicity. In Proposition 4.5 we show that good graphicality of $M_\tau$ over $V$ on a fixed annulus $B_2 \setminus B_1$ can be propagated out to larger annular regions $B_R \setminus B_r$ at later times, in the presence of our pointwise distance bounds. This graphicality estimate is then used in Proposition 4.6 to derive a crucial estimate

$$|\mathcal{A}(M_\tau)| \leq D_V(M_{\tau - 1})^{1+\alpha_1}$$

for the excess $\mathcal{A}$ (defined in (2.4)) in terms of the distance, where $\alpha_1 > 0$.

Our next task, in Sections 5 and 6, is to derive a three-annulus type lemma for the distance function $D_V$. The main result is Proposition 6.2 and the proof relies on an analysis of solutions of the drift heat equation on a plane with some mild singularities at the origin, together with the non-concentration estimates in Lemma 8.3.

The technical heart of the paper is the proof of the decay estimate, Proposition 7.5 for the distance function. Given a three-annulus lemma as in Proposition 6.2 the usual strategy for controlling the flow $M_\tau$ is to show that at each scale the flow must decay towards the “best fit” cone of the form $V' = P'_1 \cup P'_2$. The difficulty is that our estimates only apply when $V'$ is special Lagrangian, i.e. the planes $P'_1$ and $P'_2$ have the same Lagrangian angle. Since in general these angles might be different, the situation is somewhat reminiscent of the case of non-integrable tangent flows, which is typically dealt with using the Łojasiewicz–Simon inequality [22]. Our approach is quite different from this, and can be thought of as a
quantitative version of Neves’s result \cite[Corollary 4.3]{Neves} stating that at a singularity the tangent flow cannot be the union of two planes with different Lagrangian angles. These considerations lead to the alternative (ii) in Proposition 7.3.

We give the proofs of the main applications in Section 8. Given the decay estimate in Proposition 7.3, the uniqueness result, Theorem 1.1 follows standard arguments. The proof of the existence of a $C^1$ limiting surface $L_T$ in Theorem 1.2 is similar to \cite[Corollary 1.2]{Neves}. We can then restart the flow using the approach of Wang \cite{Wang}.

The main ingredient for proving Theorem 1.3 is to show that one can find small Lawlor necks, i.e. surfaces of the form \{zw = ±\epsilon\}, in the presence of a tangent flow $V$ given by two transverse planes. This relies on showing that if between different scales the flow stays close to possibly moving pairs of planes, then these planes have to stay very close to each other, similarly to how the uniqueness of the tangent flow is proved. A related result was shown by Edelen \cite[Theorem 13.1]{Edelen} in the context of minimal hypersurfaces.

Finally in Section 9 we will discuss the changes needed when working in a compact ambient space $X$ instead of in $\mathbb{C}^2$. We follow the approach found for instance in White \cite[Section 4]{White}, isometrically embedding $X \subset \mathbb{R}^N$, and writing the (rescaled) mean curvature flow in $X$ as a (rescaled) mean curvature flow in $\mathbb{R}^N$ with an additional forcing term. The quantities, such as $|zw|$, that we used in $\mathbb{C}^2$ can be defined by projecting to the tangent space $T_pX$ at the point $p$ where the singularity forms. Along the rescaled flow this projection as well as the forcing term introduces additional errors when compared to the calculations in $\mathbb{C}^2$, however these errors decay exponentially fast and so the geometric conclusions still hold.

1.2. **Acknowledgements.** We thank Dominic Joyce and Yang Li for their interest in this work and helpful comments. JDL and FS were partially supported by a Leverhulme Trust Research Project Grant RPG-2016-174. GSz was supported in part by NSF grant DMS-2203218.

2. **Preliminaries**

In this section we introduce various key definitions and notation that we shall require throughout the article.

2.1. **Lagrangians in $\mathbb{C}^2$.** We first recall some basic definitions concerning Lagrangian submanifolds in $\mathbb{C}^2$.

**Definition 2.1.** An oriented Lagrangian submanifold $L$ in $\mathbb{C}^2$ is zero *Maslov* if there exists a function $\theta$ on $L$ (called the *Lagrangian angle*) so that

$$H = J\nabla \theta,$$

where $H$ is the mean curvature vector of $L$ and $J$ is the complex structure on $\mathbb{C}^2$. The choice of function $\theta$ is called a *grading* of $L$. We further say that $L$ is *almost calibrated* if $\theta$ can be chosen so that, for some $\epsilon > 0$,

$$\sup \theta - \inf \theta \leq \pi - \epsilon.$$

**Definition 2.2.** An oriented Lagrangian $L$ in $\mathbb{C}^2$ is *exact* if there exists a function $\beta$ on $L$ so that

$$Jx^\perp = \nabla \beta,$$
where $x^\perp$ is the normal projection of the position vector $x \in \mathbb{C}^2$. Equivalently,
\begin{equation}
\frac{d\beta}{dt} = \lambda|_L,
\end{equation}
where $\lambda$ is the Liouville form on $\mathbb{C}^2$, which is a 1-form on $\mathbb{C}^2$ so that $\frac{1}{2}\lambda$ is a primitive for the Kähler form $\omega$ on $\mathbb{C}^2$. The Lagrangian $L$ is rational if the set $\lambda(H_1(L, \mathbb{Z}))$ is discrete in $\mathbb{R}$. Note that exact Lagrangians are rational.

2.2. **Lagrangian mean curvature flow.** In most of this article we will consider a smooth, zero Maslov solution to Lagrangian mean curvature flow (LMCF)
\[ [0, T) \ni t \mapsto L_t \subset \mathbb{C}^2 \]
which evolves with normal speed given by $H$. See Section [5] for the setting of a compact ambient space.

Throughout we will assume that the Lagrangian angle $\theta$ of $L_t$ along the flow is uniformly bounded: $|\theta| < C_0$. In addition we assume that $L_t$ has uniformly bounded area ratios, i.e. there exists $C_1 > 0$ such that
\[ \sup_{x, t} \mathcal{H}^2(L_t \cap B(x, r)) \leq C_1 r^2 \text{ for all } r > 0, \]
where $B(x, r)$ is the Euclidean ball of radius $r$ about $x \in \mathbb{C}^2$. We call
\[ M := \{ L_t \times \{ t \} | t \in [0, T) \} \subset \mathbb{C}^2 \times \mathbb{R} \]
the spacetime track of the flow, and write $M(t) = L_t$.

It will be useful to perform parabolic rescalings of our flows, so we shall introduce the following notation.

**Definition 2.3.** For $\lambda > 0$ we shall denote the parabolic rescaling
\[ D_\lambda : \mathbb{C}^2 \times \mathbb{R} \to \mathbb{C}^2 \times \mathbb{R}, (x, t) \mapsto (\lambda x, \lambda^2 t). \]
Note that for a (Lagrangian) mean curvature flow $\mathcal{M}$, it holds that $D_\lambda \mathcal{M}$ is again a (Lagrangian) mean curvature flow.

We recall Huisken’s monotonicity formula [10]:
\begin{equation}
\frac{d}{dt} \int_{L_t} f \rho_{x_0, t_0} \, d\mathcal{H}^2 = \int_{L_t} (\partial_t f - \Delta f) \rho_{x_0, t_0} \, d\mathcal{H}^2
- \int_{L_t} f \left| \frac{(x - x_0)^\perp}{2(t_0 - t)} \right|^2 \rho_{x_0, t_0} \, d\mathcal{H}^2,
\end{equation}
of $t < t_0$, where $f$ is a function on $L_t$ with polynomial growth (locally uniform in $t$), and
\[ \rho_{x_0, t_0}(x, t) = (4\pi(t_0 - t))^{-1} \exp \left( -\frac{|x - x_0|^2}{4(t_0 - t)} \right) \]
is the backwards heat kernel. The **entropy** $\lambda(L)$ defined by Colding-Minicozzi [4] is given by
\[ \lambda(L) = \sup_{x_0 \in \mathbb{C}^2, r > 0} \frac{1}{4\pi r} \int_{L} e^{-\frac{|x - x_0|^2}{4r^2}} \, d\mathcal{H}^2. \]
By virtue of Huisken’s monotonicity formula, $t \mapsto \lambda(L_t)$ is non-increasing along any 2-dimensional mean curvature flow in $\mathbb{C}^2$. 

We will be studying the behaviour of the flow close to a singularity at \((x_0, T)\). It is convenient to shift the flow in space-time such that \((x_0, T)\) is the origin \((0, 0)\), i.e. we consider instead the flow
\[
\tilde{M} := M - (x_0, T),
\]
defined for \(t \in [-T, 0)\). For ease of notation we will drop the tilde in the following.

A tangent flow at \((0, 0)\) is defined to be any weak limit of a sequence of rescalings \(D_{\lambda_k} M\), with \(\lambda_k \to \infty\). According to the structure theorem due to Neves [15, Theorem A], in our setting of a zero Maslov flow with bounded Lagrangian angle in two dimensions, the tangent flows are all given by unions of Lagrangian planes with multiplicities. In addition the Lagrangian angle \(\theta\) along the sequence of rescalings converges in a suitable sense to the angles of the planes. We recall that the assumptions of Theorem 1.1 mean that one tangent flow at \((0, 0)\) is given by a union of multiplicity one transverse planes \(P_1 \cup P_2\) with the same Lagrangian angle.

It turns out to be helpful to consider a further rescaling, which turns self-similarly shrinking solutions into static points of the flow.

**Definition 2.4.** The rescaled flow is
\[
\tau \mapsto M_\tau := e^{\tau}M(-e^{-\tau}) = e^{\tau}L_{-e^{-\tau}},
\]
which evolves with normal speed
\[
H + \frac{\mathbf{x}^2}{2}.
\]

In terms of the rescaled flow the tangent flows of \(M\) can be studied by taking limits of sequences \(M_{\tau_k}\) as \(\tau_k \to \infty\).

### 2.3. Set-up

For the majority of this article we will consider a rescaled Lagrangian mean curvature flow \(M_\tau\) in \(\mathbb{C}^2\), for \(\tau \in [T_0, T_1]\), close in a suitable sense to the static flow of the transverse union of two planes. The flow is assumed to have uniformly bounded Lagrangian angle and area ratios as above. Two additional conditions will play an important role, the first of which involves the following key quantity.

**Definition 2.5.** Let \(M\) be a graded Lagrangian in \(\mathbb{C}^2\). We define the excess of \(M\) to be
\[
A(M) = \int_M e^{-|\mathbf{x}|^2/4} - 2 \int_{\mathbb{R}^2} e^{-|\mathbf{x}|^2/4} + \inf_{\theta_0} \int_M |\theta - \theta_0|^2 e^{-|\mathbf{x}|^2/4}.
\]

Note that we allow \(A\) to be negative. We also observe that \(A\) is monotonically decreasing along the rescaled flow by Huisken’s monotonicity formula, since it is an infimum of a family of decreasing functions.

In the conditions below and throughout we let \(B_R(x)\) denote the ball of radius \(R\) about \(x \in \mathbb{C}^2\) and let \(B_R = B_R(0)\).

**Condition (‡).** We assume that there is a small \(c_0 > 0\) (to be chosen later) so that
\[
A(M_{T_0}) - A(M_{T_1}) < c_0.
\]
Condition (\(*\)). We assume that $M$ is exact in $B_1$, i.e. the Liouville form $\lambda$ satisfies
\[ \int_\gamma \lambda = 0 \]
for every closed loop $\gamma \subset M \cap B_1$. In addition we assume $M$ is connected in $B_1$, i.e. $M \cap B_1$ cannot be written as the union of two nonempty submanifolds.

We shall see later that Conditions (\(\dagger\)) and (\(*\)) hold along the rescaled flow when our original Lagrangian mean curvature flow develops a finite time singularity with tangent flow given by a special Lagrangian union of two transverse planes: see Lemma 8.1.

3. Distance to planes

We will now suppose that the flow $M_\tau$, defined for $\tau \in [T_0, T_1]$, is close to a special Lagrangian transverse union $V_0 = P_{0,1} \cup P_{0,2}$ of planes through 0 in $\mathbb{C}^2$, in a way that will be specified later.

3.1. Nearby pairs of planes and complex lines. It will be useful to restrict the set of pairs of planes we are considering to be those sufficiently close to $V_0$ in the following sense.

Definition 3.1. Fix a small number $c_1 > 0$ and let $V$ be the space of all special Lagrangian unions $V = P_1 \cup P_2$ of two planes through the origin, such that the angles between $P_i$ and $P_0$, $i=1,2$ are smaller than $c_1$, i.e. the cones in $V$ are small deformations of $V_0$. We further assume that $c_1$ is chosen sufficiently small so that elements $P_1 \cup P_2$ in $V$ are bounded away from the multiplicity two planes, i.e. there exists $c_2 > 0$ so the angle between $P_1$ and $P_2$ is at least $c_2$. Below we may further shrink $c_1$, however it will always be a constant that depends on the choice of $V_0$ only.

For any $V \in V$ we denote the Lagrangian angle of $V$ by $\theta_V$. We also let $V' \subset V$ be defined in the same way as $V$ but using the constant $c_1/2$ instead of $c_1$.

We can choose a hyperkähler rotation of the complex structure of $\mathbb{C}^2$ depending on $\theta_V$, and complex coordinates $z, w$, such that $V$ is defined by $zw = 0$. We have the following basic observation, where the notion of Lagrangian and Lagrangian angle refers to the original Calabi–Yau structure on $\mathbb{C}^2$.

Lemma 3.2. There is a constant $C > 0$, independent of $V \in V$, such that on any oriented Lagrangian 2-plane $P$ in $\mathbb{C}^2$ with Lagrangian angle $\theta$ we have
\[ |\nabla z \cdot \nabla w| \leq C|\theta - \theta_V|. \]

Proof. Let $\mathcal{L}$ denote the space of oriented Lagrangian 2-planes in $\mathbb{C}^2$, and let $\theta : \mathcal{L} \to S^1$ denote the Lagrangian angle function. Then $d\theta$ is nowhere vanishing.

Let $P \in \mathcal{L}$ with $\theta = \theta_V$. After the hyperkähler rotation which makes $V$ a pair of complex lines, $P$ also becomes a complex line on which $z, w$ and $zw$ are all holomorphic, and in particular harmonic. We therefore have
\[ 2\nabla z \cdot \nabla w = \Delta(zw) - z\Delta(w) - w\Delta(z) = 0. \]
Thus, $\nabla z \cdot \nabla w$ vanishes on the zero set of $\theta$.

Since $d\theta$ is nowhere vanishing, i.e. $\theta - \theta_V$ vanishes to order 1 along its zero set, the result follows. \(\square\)

We also record the following evolution equations.
Lemma 3.3. We have
\[(\partial_t - \Delta)|zw|^2 \leq 4|zw||\nabla z \cdot \nabla w|.
\]
Furthermore, for \(\delta > 0\) consider \(u_\delta := \sqrt{\delta^2 + |\theta - \theta_0|^2}\) and \(v_\delta := \sqrt{\delta^2 + |zw|^2}\).

Then
\[(\partial_t - \Delta)u_\delta \leq 0 \quad \text{and} \quad (\partial_t - \Delta)v_\delta \leq 2|\nabla z \cdot \nabla w|.
\]

Proof. We have that \(z, w\) satisfy the heat equation, so
\[(\partial_t - \Delta)zw = -2\nabla z \cdot \nabla w.
\]

Thus we can compute
\[\begin{align*}
(\partial_t - \Delta)\delta^2 = (\partial_t - \Delta)(zw) \\
= -2(\nabla z \cdot \nabla wzw + \nabla w \nabla z \cdot \nabla w) - 2\nabla(zw) \cdot \nabla(zw) \\
\leq 4|zw||\nabla z \cdot \nabla w|.
\end{align*}
\]

For \(v_\delta\) we compute using \(3.3\) and Kato’s inequality
\[\begin{align*}
(\partial_t - \Delta)v_\delta &= \frac{1}{2v_\delta}(\partial_t - \Delta)|zw|^2 + \frac{1}{4v_\delta^3}||\nabla|zw|^2||^2 \\
&= -\frac{1}{v_\delta} \left( (\nabla z \cdot \nabla wzw + \nabla w \nabla z \cdot \nabla w) + \nabla(zw) \cdot \nabla(zw) \right) \\
&\quad + \frac{|zw|^2}{v_\delta^3}||\nabla|zw||^2 \\
&\leq -\frac{1}{v_\delta} \left( (\nabla z \cdot \nabla wzw + \nabla w \nabla z \cdot \nabla w) \right) \leq 2|\nabla z \cdot \nabla w|.
\end{align*}\]

The computation for \(u_\delta\) is analogous. \(\square\)

3.2. Distance to planes and monotonicity. We now introduce two notions of distance to a pair of planes \(V \in \mathcal{V}\).

**Definition 3.4.** Given any \(V \in \mathcal{V}\), let \(d_V\) denote the distance function in \(C^2\) to \(V\) and define the \(L^2\)-distance \(I_V(M_\tau)\) of \(M_\tau\) to \(V\) by
\[I_V(M_\tau)^2 = \int_{M_\tau} \left(|x|^2d_V^2 + |\theta - \theta_0|^2\right)e^{-|x|^2/4}.
\]

It is convenient to introduce a variant \(D_V\) of this, which also encodes graphicality. Suppose that on the annulus \(B_2 \setminus B_1\) the surface \(M_\tau\) is the graph of a (vector-valued) function \(u\) over \(V\) with \(|u|, |\nabla u| \leq c_1\), for a small \(c_1\) as in Definition 3.1.

Then we set \(D_V(M_\tau) = I_V(M_\tau)\). If \(M_\tau\) is not such a small graph over \(V\) on the annulus, then we let \(D_V(M_\tau) = \infty\).

Lemma 3.3 leads to the following properties of \(I_V\).

**Lemma 3.5.** The \(L^2\)-distance to \(V \in \mathcal{V}\) has the following properties.

1. There is \(C > 0\), depending only on \(\mathcal{V}\), so that
\[I_V(M_{\tau+s}) \leq CI_V(M_\tau) \quad \text{for any} \ s \in [0, 1].
\]

2. For any \(0 < \delta \leq s \leq 1\) there are \(p > 1\) and \(C > 0\) (both depending on \(\delta\)), such that on \(M_{\tau+s}\) we have
\[|x|^2d_V^2 + |\theta - \theta_V|^2 \leq C\frac{\delta^2}{s^2}I_V(M_\tau)^2.
\]
(3) There is a $C > 0$ satisfying the following. For any $\gamma > 0$ there is a compact subset $K_\gamma \subset \mathbb{C}^2 \setminus \{0\}$ such that if

$$ \int_{K_\gamma \cap M_{\tau+1}} (|\nabla^2 z|^2 + |\theta - \theta_V|^2)e^{-|\mathbf{x}|^2/4} \leq C^2, $$

then

$$ I_V(M_{\tau+1}) \leq C(\epsilon + \gamma I_V(M_\tau)). $$

**Proof.** As in Lemma 3.2 we choose complex coordinates $z, w$ for a hyperkähler-rotated complex structure on $\mathbb{C}^2$ such that $V$ is given by $zw = 0$. It follows from Lemma 3.2 and (3.3) that

$$ \partial_t \Delta (|zw|^2 \leq 2|zw|\nabla z \cdot \nabla w |$$

$$ \leq C|zw| |\theta - \theta_V| $$

$$ \leq C(|zw|^2 + |\theta - \theta_V|^2), $$

and so

$$ (\partial_t - \Delta)(|zw|^2 + |\theta - \theta_V|^2) \leq C(|zw|^2 + |\theta - \theta_V|^2). $$

The monotonicity formula along the unrescaled flow then implies that for $-T \leq t_1 < t_2 < 0$

$$ \int_{L_{t_2}} (|zw|^2 + |\theta - \theta_V|^2) \rho_{t_1} \leq C(t_2 - t_1) \int_{L_{t_1}} (|zw|^2 + |\theta - \theta_V|^2) \rho_{t_0}. $$

Rescaling the flow parabolically such that $\tau$ corresponds to $t_1 = -1$ (i.e. translating $\tau$ to 0 for the rescaled flow) this implies that for the rescaled flow

$$ \int_{M_{\tau+s}} (e^{-2|zw|^2 + |\theta - \theta_V|^2} e^{-|\mathbf{x}|^2/4} \leq C \int_{M_\tau} (|zw|^2 + |\theta - \theta_V|^2) e^{-|\mathbf{x}|^2/4}. $$

To deduce property (1) we claim that $|\mathbf{x}|d_V$ is uniformly equivalent to $|zw|$, with a constant depending on $c_2$ in the definition of $V$. Indeed, the distance $d_V$ is uniformly equivalent to $\min\{\|z\|, |w|\}$, while $|\mathbf{x}|$ is uniformly equivalent to $\max\{\|z\|, |w|\}$, and $\min\{\|z\|, |w|\} \max\{\|z\|, |w|\} = |zw|$. Let $s > 0$. A sharper estimate as in property (1) is obtained by using Ecker’s log-Sobolev inequality [7, Theorem 3.4] along the rescaled flow $M_{\tau}$. We first note that, similar to above, we can use (3.2) and Lemma 3.2 to estimate

$$ (\partial_t - \Delta)(u_\delta + v_\delta) \leq C(u_\delta + v_\delta). $$

The results in [7] then imply that for some $p > 2$ depending on $s$, we have

$$ \left( \int_{M_{\tau+s/2}} (u_\delta + v_\delta)^{p_1} e^{-|\mathbf{x}|^2/4} \right)^{1/p_1} \leq C \left( \int_{M_\tau} (u_\delta + v_\delta)^2 e^{-|\mathbf{x}|^2/4} \right)^{1/2}, $$

for $C$ depending on $V$ and $\delta_0$. Letting $\delta \searrow 0$ then yields for $f = |zw|^2 + |\theta - \theta_V|^2$ and $p = \tilde{p}/2 > 1$ that

$$ \left( \int_{M_{\tau+s/2}} f^p e^{-|\mathbf{x}|^2/4} \right)^{1/p} \leq C \int_{M_\tau} f e^{-|\mathbf{x}|^2/4} \leq CI_V(M_\tau)^{p/2}. $$

Let us now consider the mean curvature flow $L_t$, with initial condition $L_{-1} = M_{\tau+s/2}$. Consider any $(x_0, t_0)$ with $t_0 \in (-1, 0)$ and let $\rho_{x_0,t_0}$ be the backwards
heat kernel centered at \((x_0, t_0)\). Using the monotonicity formula again for \(f\) with \(f = |zw| + |\theta - \theta_V|^2\), together with (3.4), we have the pointwise estimate

\[
(3.6) \quad f(x_0, t_0)^p \leq C \int_{L_{t-1}} f^p \rho_{x_0, t_0}.
\]

At the same time from (3.5) we have that

\[
(3.7) \quad \int_{L_{t-1}} f^p \rho_{0,0} \leq CI_V(M_\tau)^{2p}.
\]

To estimate \(f(x_0, t_0)^p\) we therefore need to bound \(\rho_{x_0, t_0} / \rho_{0,0}\) at \(t = -1\). We have

\[
(3.8) \quad \frac{\rho_{x_0, t_0}}{\rho_{0,0}}(x, -1) = C_{t_0} \exp \left( - \frac{|x - x_0|^2}{4(t_0 + 1)} + \frac{|x|^2}{4} \right),
\]

for a \(t_0\)-dependent constant \(C_{t_0}\) (which is uniformly bounded as long as \(t_0\) is bounded away from \(-1\)). Since \(\rho_{x_0, t_0} / \rho_{0,0}\) is assumed in the statement of property (3), we get

\[
(3.9) \quad (t_0 + 1)|x|^2 - |x - x_0|^2 = t_0|x + t_0^{-1}x_0|^2 - t_0^{-1}(t_0 + 1)|x_0|^2
\]

and \(t_0 \in (-1, 0)\), we have

\[
(3.10) \quad - \frac{|x - x_0|^2}{4(t_0 + 1)} + \frac{|x|^2}{4} \leq \frac{|x_0|^2}{4t_0}.
\]

It follows from (3.6), (3.7), (3.8) and (3.10) that

\[
f(x_0, t_0)^p \leq C_{t_0} \exp \left( - \frac{|x_0|^2}{4t_0} \right) \int_{L_{t-1}} f^p \rho_{0,0} \leq C_{t_0} \exp \left( - \frac{|x_0|^2}{4t_0} \right) I_V(M_\tau)^p.
\]

Scaling this estimate back to the rescaled flow \(M_\tau\) (i.e. scaling \(\bar{x} = (-t_0)^{-1/2}x\) at \(t = t_0\) and \(e^{-\tau} = -t_0\)) we then have the following estimate on \(M_{\tau + s}\):

\[
(e^{2s}|zw|^2 + |\theta - \theta_V|^2)^p \leq Ce^{\frac{|x|^2}{4p}}I_V(M_\tau)^{2p},
\]

and so

\[
|zw|^2 + |\theta - \theta_V|^2 \leq Ce^{\frac{|x|^2}{4p}}I_V(M_\tau)^2
\]

as claimed in property (2).

To see property (3) we use (3.5) again. With \(s = 2\) it implies that for any compact set \(K\), with suitable \(p, p' > 1\) we have

\[
(3.11) \quad \int_{M_{\tau + 1} \setminus K} f e^{-|x|^2/4} \leq \left( \int_{M_{\tau + 1} \setminus K} e^{-|x|^2/4} \right)^{1/p'} \left( \int_{M_{\tau + 1} \setminus K} f^p e^{-|x|^2/4} \right)^{1/p}
\]

\[
\leq \text{Vol}(M_{\tau + 1} \setminus K, e^{-|x|^2/4})^{1/p'} CI_V(M_\tau)^2.
\]

Together with the integral bound on \(K\) at \(\tau + 1\) assumed in the statement of property (3), if we choose \(K\) sufficiently large, depending on \(\gamma\), we get

\[
I_V(M_{\tau + 1})^2 \leq e^2 + C\gamma^2 I_V(M_\tau)^2,
\]

as required.

Next we control the growth of the distance \(D_V\).
Lemma 3.8. There is a constant $C > 0$ such that if the constant $c_0$ in Condition (1) and $D_V(M_\tau)$ are sufficiently small, then

\begin{equation}
D_V(M_{\tau+s}) \leq C D_V(M_\tau) \quad \text{for } s \in [0,1],
\end{equation}

as long as $\tau \in [T_0 + 1, T_1 - 2]$.

Proof. Given the growth bound for $I_V$ in Lemma 3.6 we only need to ensure that if $M_\tau$ is a $C^1$-small graph over $V$ on the annulus $B_2 \setminus B_1$, then so is $M_{\tau+s}$ for $s \in [0,1]$. We will show that this follows from the estimate for the excess, defined in Definition 2.4.

Recall that $\theta$ satisfies the heat equation along the Lagrangian mean curvature flow and $|\nabla \theta| = |H|$. It follows from Huisken’s monotonicity formula that we have

\begin{equation}
\mathcal{A}(M_0) - \mathcal{A}(M_t) \geq \int_{T_0}^{T_1} \int_{M_\tau} \left( 2 |H|^2 + \left| H + \frac{X^\perp}{2} \right|^2 \right) e^{-|x|^2/4}.
\end{equation}

We can then argue by contradiction, and suppose that we have a sequence of rescaled flows $M^i_\tau$, satisfying Condition (1) with corresponding constants $c_0 \to 0$, however the conclusion (3.12) does not hold. We can assume that we are working at $\tau = 0$ and that also $D_V(M^i_0) \to 0$.

As in Neves [16] Theorem A, up to choosing a subsequence, the flows $M^i_\tau$ converge to a static limit flow $M^\infty$ given by a union of planes $V'$ for $\tau \in (-1,2)$. The assumption that $D_V(M^i_0) \to 0$ implies that $M^i_0$ is the graph of a (vector-valued) function $u_i$ over $V$ on the annulus $B_2 \setminus B_1$, with $|u_i|, |\nabla u_i| < c_0$ and $|u_i| \to 0$. Note that this implies that $V' = V$. By White’s regularity theorem [30] the convergence is smooth on $(B_1 \setminus B_{1/2}) \times (-1/2,3/2)$. This implies that, for $i$ sufficiently large, $M^i_\tau$ is the graph of a function $|u_i|, |\nabla u_i| < c_0$ over $V$ on $B_2 \setminus B_1$ for all $t \in [0,1]$.

Thus Lemma 3.7 implies $D_V(M^i_\tau) \leq C D_V(M^i_0)$ for all $s \in [0,1]$ as required. \qed

Remark 3.7. From now on we will assume that $c_0 > 0$ in Condition (1) is small enough for Proposition 3.6 to hold.

For later use, we will need to compare the distances $D_V$ as we vary the cone $V \in \mathcal{V}$. For $V, V' \in \mathcal{V}$ let $d(V, V')$ denote the Hausdorff distance between $V \cap B_1$ and $V' \cap B_1$.

Lemma 3.8. There is a constant $C$ such that if $V, V' \in \mathcal{V}$ and $D_V(M_\tau)$ is sufficiently small, then

\begin{equation}
D_{V'}(M_\tau) \leq C(D_V(M_\tau) + d(V, V')).
\end{equation}

Proof. For any $x \in B_1$ we have $d_{V'}(x) \leq d_V(x) + d(V, V')$, so by scaling we have $d_{V'}(x) \leq d_V(x) + |x|d(V, V')$ for $x \in \mathbb{C}^2$. We also have $|\theta_V - \theta_{V'}| \leq C d(V, V')$ for a constant $C$. Combining these observations we get

\begin{equation}
I_{V'}(M_\tau) \leq C(I_V(M_\tau) + d(V, V')^2).
\end{equation}

To get the same estimate for $D_{V'}$ we just need to ensure that $M_\tau$ is graphical over $V'$ on $B_2 \setminus B_1$ for $u$ with $|u|, |\nabla u|$ sufficiently small as in Definition 3.4. This follows using Condition (1) as in the proof of Proposition 3.6. \qed
4. Graphicality

4.1. Graphicality and distance to planes. We want to see that an estimate for $D_V(M_{r-1})$ can be used to show graphicality of $M_r$ on a much larger region than just the fixed annulus $B_2 \setminus B_1$. For this we first need the following.

Lemma 4.1. Let $L_t$ be a mean curvature flow of surfaces in the ball $B_2 \subset \mathbb{R}^2$ for $t \in [-4, 0]$, with uniform area bound $\mathcal{H}^2(L_t) \leq C$. Suppose that $S$ is an embedded smooth surface passing through the origin, and with second fundamental form $g_{ij} \leq \delta$ for some $\delta > 0$. Assume that in addition we have the following.

1. $L_t$ is contained in the $\delta$-neighbourhood of $S$.
2. On the parabolic ball $[-1/4, 0] \times B_{1/2}$ the flow $L_t$ is the graph of a (vector-valued) function $u$ over $S$ with $|u| < \delta$ and $|\nabla u| < 1$.
3. We have the estimate

$$\int_{t_i}^0 \int_{L_t \cap B_2} |\mathbf{H}|^2 < \delta.$$  

Let $\varepsilon > 0$ be given. Then if $\delta$ is chosen sufficiently small, the flow $L_t$ is the graph of $u$ over $S$ on the parabolic ball $[-1, 0] \times B_1$, with $|\nabla u| \leq \varepsilon$.

Proof. We argue by contradiction. Suppose that we have a sequence of flows $L_t^i$ satisfying the assumptions with corresponding constants $\delta_i \to 0$, and surfaces $S_i$, but the claimed graphicality fails for all $i$. Up to choosing a subsequence we can assume that the $S_i$ converge to a plane $S_\infty$. We can furthermore assume that the flows $(L_t^i)_{-4 \leq t \leq 0}$ converge to a unit regular Brakke flow $(\mu_t)_{-4 \leq t \leq 0}$ on $B_2$, supported on $S_\infty$. The constancy theorem implies that $\mu_t$ agrees with $S_\infty$ up to an integer multiplicity, which is monotonically decreasing in time.

Note that the graphicality assumption (2) together with interior parabolic estimates implies that $L^i_t \to S_\infty$ smoothly on $[-1/4, 0] \times B_{1/2}$, and thus $\mu_t$ agrees with $S_\infty$ with multiplicity 1 on $[-1/4, 0] \times B_2$.

At the same time, for a cutoff function $\chi$ supported in $B_2$ we have

$$\left| \partial_t \int_{L_t^i} \chi \right| = \left| \int_{L_t^i} -\chi |\mathbf{H}|^2 + \langle D\chi, \mathbf{H} \rangle \right| \leq C \int_{L_t^i \cap B_2} |\mathbf{H}|^2 + |\mathbf{H}|,$$

so for any $t_0 < t_1$ we have

$$\left| \int_{L_t^i} \chi - \int_{L_t^1} \chi \right| \leq C \int_{-4}^0 \int_{L_t^i \cap B_2} |\mathbf{H}|^2 + |\mathbf{H}| \to 0.$$

It follows that $(\mu_t)$ is static on $[-4, 0]$ and thus agrees with $S_\infty$ with multiplicity one. From White’s regularity theorem [30] we deduce that for sufficiently large $i$ the flows $L^i_t$ converge smoothly on $[-1, 0] \times B_1$, which implies the required graphicality. \(\square\)

Using Lemma 4.1 repeatedly we can extend the graphicality of our flow to larger and larger regions, as long as it stays in a small neighbourhood of a smooth surface and we can control the integral of $|\mathbf{H}|^2$. For the latter we have the following result.

Lemma 4.2. There are constants $C, r_0, \alpha > 0$ and $p > 1$ satisfying the following. Suppose that $L_t$ is a Lagrangian mean curvature flow for $t \in [-1, 0]$, where $L_{-1} = M_\tau$ for some $\tau$. Then whenever $r \leq r_0$ and $t_0 \in (-3/4, -1/4)$ with $|t_0-r^2, t_0+r^2| \subset \{t \in [-1, 0] \mid L_t \subset B_2 \setminus B_1 \}$.
Let \( p \) be a function in \([ \alpha, r_0 \leq r_\alpha ]\) and

\[
D_V(M_{\tau - 1})^2 \exp \left( -\frac{|x_0|^2}{4 \rho t_0} \right) \leq 1,
\]

we have

\[
r^{-2} \int_{t_0 - r^2}^{t_0} \int_{L_{t_1} \cap B_r(x_0)} |H|^2 \leq C D_V(M_{\tau - 1})^\alpha.
\]

**Proof.** We consider the monotonicity formula applied with the backwards heat kernel \( \rho_{x_0, t_0 + r^2} \) centered at \((x_0, t_0 + r^2)\). Using \(|\nabla \theta|^2 = |H|^2\) we have that

\[
\int_{t_0 - r^2}^{t_0} \int_{L_{t_1}} 2|H|^2 \rho_{x_0, t_0 + r^2} \leq \int_{L_{t_1}} |\theta - \theta_V|^2 \rho_{x_0, t_0 + r^2}.
\]

By the pointwise estimate in Lemma 3.5 there is some \( p > 1 \) close to 1 and \( C > 0 \) such that on \( L_{t_1} \) we have

\[
|\theta - \theta_V|^2 \leq C e^{\frac{|x|^2}{4p}} D_V(M_{\tau - 1})^2.
\]

Therefore we need to estimate the integral

\[
\int_{L_{t_1}} \exp \left( \frac{|x|^2}{4p} - \frac{|x - x_0|^2}{4(t_0 + r^2 + 1)} \right).
\]

Let \( p_2 \in (1, p_1) \). Arguing as in (3.10) we find that

\[
\frac{|x|^2}{4p_2} - \frac{|x - x_0|^2}{4(t_0 + r^2 + 1)} \leq \frac{|x_0|^2}{4(p_2 - 1 - t_0 - r^2)}.
\]

The integral of \( \exp \left( \frac{|x|^2}{4p_2} - \frac{|x|^2}{4p_2} \right) \) on \( L_{t_1} \) is uniformly bounded, so combining (4.4), (4.4) and (4.0) gives

\[
\int_{t_0 - r^2}^{t_0} \int_{L_{t_1}} |H|^2 \rho_{x_0, t_0 + r^2} \leq C D_V(M_{\tau - 1})^2 \exp \left( \frac{|x_0|^2}{4(p_2 - 1 - t_0 - r^2)} \right).
\]

It remains to choose \( p \) close enough to 1 in the bound (4.2) and \( \alpha, r_0 > 0 \) small enough so that for \( r \leq r_0 \) we have

\[
D_V(M_{\tau - 1})^2 - \alpha \exp \left( \frac{|x_0|^2}{4(p_2 - 1 - t_0 - r^2)} \right) \leq 1.
\]

Using (4.2), this follows if

\[
\frac{2 - \alpha}{2} \frac{1}{4 \rho t_0} + \frac{1}{4(p_2 - 1 - t_0 - r^2)} < 0.
\]

Note that in the limiting case, when \( \alpha \) and \( r_0 \) are both 0 and \( p = 1 \), (4.2) reduces to the inequality \( p_2 > 1 \), which is satisfied. We can therefore arrange that (4.7) also holds for suitable \( \alpha, r_0, p \). Combining this with the fact that on \([t_0 - r^2, t_0] \times B_r(x_0)\) the function \( \rho_{x_0, t_0 + r^2} \) is bounded from below by a positive multiple of \( r^{-2} \) yields (4.3). \( \square \)
4.2. Graphicality scale. Note that under the correspondence between the mean curvature flow and its rescaled version, if $L_1 = M$, then $M_{t+1} = e^{1/2}L_{t-1}$. In particular, setting $t_0 = -e^{-1}$ and, for $x_0 \in L_{t-1}$, letting $\tilde{x}_0 = e^{1/2}x_0$, we see that $|x_0|^2/4p t_0 = |\tilde{x}_0|^2/4p$. This motivates the following definition.

**Definition 4.3.** We choose $p_0 > 1$ smaller than the $p > 1$ in both Lemma 3.5 for $s = 1$ and Lemma 4.2. For any (small) $d > 0$, we define $R_d > 0$ so that

$$(4.8) \quad d^2 e^{R_d^2/4p_0} = 1.$$  

The $R_d$ just defined will be the radius up to which we can obtain good graphicality of our flow. This motivates the following definition.

**Definition 4.4.** We say that $M_s$ has **good graphicality** on an annulus $A := B_{r_2} \setminus B_{r_1}$ for $0 < r_1 < r_2 < \infty$ over $V$ if $M_s \cap A$ is the graph of a (vector-valued) function $u$ over $V$ with $|u|, |\nabla u| \leq c_1$, where $c_1$ is as in Definition 3.1.

We have the following.

**Proposition 4.5.** Use the notation of Definition 4.3. There are constants $\epsilon, A > 0$ and $p > p_0$ such that if $D_V(M_{t-1}) = d < \epsilon$, then $M_{t+1}$ is the graph of a (vector-valued) function $u$ over $V$ on the annulus $B_{R_d} \setminus B_{Ad^{1/2}}$, satisfying the following estimates:

- for $1 < |x| < R_d$ we have $|u|, |\nabla u| \leq Ae^{d|x|^2/8pd}$;
- for $Ad^{1/2} < |x| < 2$ we have $|u|^{-1} |x|, |\nabla u| \leq Ad|x|^{-2}$.

**Proof.** Note that by Proposition 3.6 $M_s$ has good graphicality over $V$ (in the sense of Definition 4.3) on $B_{r_2} \setminus B_{r_1}$ for $s \in [\tau, \tau+1]$. In addition, property (2) in Lemma 3.5 shows that there is $C > 0$ and $p > p_0$ such that

$$(4.9) \quad d_V^2 \leq C|x|^{-2} e^{d|x|^2/4pd^2}$$

on $M_s$ for $s \in [\tau, \tau+1]$. Therefore, by the definition of $R_d$ in (4.8), if $1 < |x| < R_d$ we have

$$(4.10) \quad d_V^2 \leq Ce^{R_d^2(p_0-p)/4pp_0}.$$  

In particular, recalling that $p > p_0$, if we let $d$ be small, so that $R_d$ is large by (4.8), we can ensure that $d_V$ is as small as we like on the annulus $B_{R_d} \setminus B_1$ along the rescaled flow $M_s$ for $s \in [\tau, \tau+1]$.

Applying Lemma 1.2 with $t_0 = -e^{-1}$ we see that for all $x_0$ with $\exp(-\frac{|x_0|^2}{4p t_0}) \leq d^{-2}$ we have

$$(4.11) \quad r_0^{-2} \int_{t_0-r_0^2}^{t_0} \int_{L_{t}(\cap B_{r_0}(x_0))} |\mathbf{H}|^2 \leq Cd^\alpha.$$  

Rescaling the flow parabolically such that $L_{-1} = M_{\tau+1}$ (i.e. scaling parabolically by $e^{1/2}$) this implies that for any $x_0 \in B_{R_d}$ we have a backwards parabolic ball $[-1 - r_0^2, -1] \times B_{r_0}(x_0)$ on which the spacetime integral of $|\mathbf{H}|^2$ is bounded by $Cr_0^2d^\alpha$. Note that the constant changes by a controlled factor due to the rescaling.

Recall that the second fundamental form of $V$ vanishes and that $M_s$ has good graphicality over $V$ on $B_2 \setminus B_1$ for $s \in [\tau, \tau+1]$. Since $L_{-1} = M_{\tau+1}$ this implies that $L_{t}$ is the graph of a (vector-valued) function $\tilde{u}$ over $V$ on $\sqrt{-t}(B_2 \setminus B_1)$ with $|\tilde{u}| \leq \sqrt{-t}, c_1, |\nabla \tilde{u}| \leq c_1$ for $t \in [-e, -1]$. 


From (4.11), scaling the flow by $4r_0^{-1}$ and shifting the origin accordingly, by taking $d$ sufficiently small, we see for every $x_0 \in B_{4r_0^{-1} R_d} \setminus B_{4r_0^{-1}}$ that (4.11) implies that the hypotheses (1)–(3) of Lemma 4.1 (with $\varepsilon = c_1$) are satisfied. Thus, undoing the scaling and shifting of the origin, the good graphicality of $L_t$ for $t \in [-1-r_0^2, -1]$ over $V$ can be propagated out from the ball $\sqrt{-t}(B_2 \setminus B_1)$ if $d$ is sufficiently small, to the annulus $\sqrt{-t}(B_{R_d} \setminus B_1)$ on which $L_t$ is still in a sufficiently small neighbourhood of $V$. Note that once we have good graphicality of the flow on a parabolic ball, then on a smaller parabolic ball $|\nabla u|$ can be bounded in terms of $|u|$ by standard parabolic theory. This introduces the constant $A$ in the claimed estimates.

The argument for extending graphicality to the annulus $B_2 \setminus B_{Ad^{1/2}}$ is similar. Here the distance bound (4.29) implies that $d_V \leq C|x|^{-1}d$ on $L_t$ for $t \in [-1,-\varepsilon]$ (recalling that $L_{-1} = M_{\tau+1}$). Suppose that $x_0 \in V$ with $Ad^{1/2} \leq |x_0| < 2$, and let $r = A^{-1}|x_0|$. If $A$ is sufficiently large, then $V \cap B_r(x_0)$ is a plane, so it has zero second fundamental form. In addition for $t \in [-1,\varepsilon]$ the distance from $L_t$ to $V$ on $B_r(x_0)$ is bounded by $C|x_0|^{-1}d$, where $C$ can be chosen independent of sufficiently large $A$. After scaling up by $r^{-1}$ the distance from $r^{-1}L_t$ to $V$ on $r^{-1}B_r(x_0)$ is bounded above by $CA|x_0|^{-2} \leq CA^{-1}$ since $|x_0| \geq Ad^{1/2}$. By choosing $A$ large enough, we can again ensure that Lemma 4.1 can be applied repeatedly to extend the region of good graphicality. The required estimate for $|u|$ follows from the pointwise bound for $d_V$, while the estimate for $|\nabla u|$ in the annular region $B_2 \setminus B_{Ad^{1/2}}$ follows by standard parabolic theory and scaling parabolic balls of the form $[t-r^2, t] \times B_r(x)$ to unit size, where $r = |x|/2$.

4.3. Excess and distance. The graphicality bound from Proposition 4.4 implies the following estimate for the excess $A$ in (2.4) in terms of $D_V(M)$.

**Proposition 4.6.** There is a small $\alpha_1 > 0$ such that if $D_V(M_{\tau-1})$ is sufficiently small, then

$$|A(M_{\tau})| \leq D_V(M_{\tau-1})^{1+\alpha_1}.$$ 

**Proof.** By Definition 3.1 and Proposition 3.6 we have

$$\int_{M_{\tau}} |\theta - \theta_V|^2 e^{-|x|^2/4} \leq D_V(M_{\tau})^2 \leq CD_V(M_{\tau-1})^2.$$ 

Therefore, by formula (2.4) we only need to estimate the difference between the Gaussian areas of $M_{\tau}$ and $V$ (recalling that $V$ is a pair of planes).

Let $d = D_V(M_{\tau-1})$ and recall $p_0 > 1$ and $R_d > 0$ given in Definition 4.3. We also recall the constants $\varepsilon, A > 0$ from Proposition 4.5 and we assume that $d < \varepsilon$. We further assume that $d$ is sufficiently small so that $d^{1/10} > Ad^{1/2}$.

We study four regions separately.

(a) $|x| > R_d$. By the area growth bounds for $M_{\tau}$ we have constants $C, k > 0$ such that

$$\int_{M_{\tau} \setminus B_{R_d}} e^{-|x|^2/4} \leq CR_d^4 e^{-R_d^2/4}.$$ 

Once $R_d$ is sufficiently large,

$$CR_d^4 e^{-R_d^2/4} \leq e^{-R_d^2/4p_0} = d^2.$$ 

The same integral bound also holds on $V \setminus B_{R_d}$ so the required estimate (4.6) holds on this region with $\alpha_1 = 1$. 

(b) $1 < |x| < R_d$. Here Proposition 4.5 states that $M_r$ is the graph of $u$ over $V$, with $|u|, |\nabla u| \leq A e^{\frac{|x|^2}{8p} d}$, for some constant $p > 0$. By the definition of $R_d$ in (1.8), we have $A e^{\frac{R_d^2}{8p} d} \to 0$ as $d \to 0$. It follows that the area form $dA_{M_r}$ of $M_r$, pulled back to $V$, can be compared to the area form $dA_V$ of $V$ as follows:

$$\left| \frac{dA_{M_r}}{dA_V} - 1 \right| \leq C e^{\frac{|x|^2}{4p} d^2},$$

for some constant $C$. Note that the difference in the area forms is at least quadratic in $u$ since $V$ has zero mean curvature. Integrating, we have

$$\int_{M_r \cap (B_R \setminus B_1)} e^{-\frac{|x|^2}{4}} - \int_{V \cap (B_R \setminus B_1)} e^{-\frac{|x|^2}{4}} \leq C d^2 \int_{V \cap (B_R \setminus B_1)} e^{\frac{|x|^2}{4p} e^{-\frac{|x|^2}{4}}}. $$

Since $p > 1$ the last integral is bounded independently of $d$, and so the required estimate (4.6) holds on this region too, with $\alpha_1 = 1$.

(c) $d^{1/10} < |x| < 1$. Since we have assumed that $d^{1/10} > Ad^{1/2}$, Proposition 4.5 implies that on this region $M_r$ is the graph of $u$ over $V$, with $|x|^{-1}|u|, |\nabla u| \leq Ad|x|^{-2}$. Similarly to (b) we can again compare the area forms:

$$\left| \frac{dA_{M_r}}{dA_V} - 1 \right| \leq C d^2 |x|^{-4}. $$

Integrating, we have

$$\left| \int_{M_r \cap (B_1 \setminus B_{d^{1/10}})} e^{-\frac{|x|^2}{4}} - \int_{V \cap (B_1 \setminus B_{d^{1/10}})} e^{-\frac{|x|^2}{4}} \right| \leq C d^2 \int_{V \cap (B_1 \setminus B_{d^{1/10}})} |x|^{-4} e^{-\frac{|x|^2}{4}}$$

$$\leq C d^2 d^{-4/10} = C d^{8/5}. $$

The required estimate (4.6) therefore holds with $\alpha_1 = 3/5$.

(d) $|x| < d^{1/10}$. Let us write $r_0 = d^{1/10}$ for the radius of this ball for simplicity. As in (c), Proposition 4.5 implies that the cross section $M_r \cap \partial B_{r_0}$ is an exponential normal graph in the sphere $\partial B_{r_0}$ of a function $\tilde{u}$ over $V \cap \partial B_{r_0}$, where $\tilde{u}$ satisfies $r_0^{-1} |\tilde{u}|, |\nabla \tilde{u}| \leq A d r_0^{-2} = A d^{4/5}$. The cross section $V \cap \partial B_{r_0}$ of $V$ is minimal in the sphere (a union of geodesics), so the cross section of $M_r$ has length

$$(4.12) \quad \left| \mathcal{H}^1(M_r \cap \partial B_{r_0}) - \mathcal{H}^1(V \cap \partial B_{r_0}) \right| \leq C d^{8/5} r_0 = C d^{17/10}. $$

Let $V_r$ be the cone over $M_r \cap \partial B_{r_0}$. By (4.12), the area of $V_r$ then satisfies

$$(4.13) \quad \left| \mathcal{H}^2(V_r \cap B_{r_0}) - \mathcal{H}^2(V \cap B_{r_0}) \right| \leq C r_0 d^{17/10} = C d^{9/5}. $$

On $M_r$ we have $|\theta - \theta_V| \leq Cd$ by property (2) in Lemma 3.5, noting that the exponential factor in property (2) can be bounded independently of $d$ for $d$ small. Hence, up to rotating the holomorphic volume form $\Omega$ so that we can assume $\theta_V = 0$, we have that

$$\operatorname{Re} \Omega|_{M_r} \leq dA_{M_r} \leq (1 + C d^2) \operatorname{Re} \Omega|_{M_r}. $$
Therefore
\[ \left| \int_{M_r \cap B_{r_0}} e^{-|x|^2/4} dA_{M_r} - \int_{M_r \cap B_{r_0}} e^{-|x|^2/4} \text{Re} \Omega \right|_{M_r} \leq Cd^2 \left| \int_{M_r \cap B_{r_0}} e^{-|x|^2/4} \text{Re} \Omega \right|_{M_r}. \]

At the same time, there is a hypersurface \( U_r \) in \( B_{r_0} \) bounded by \( M_r \) and \( V_r \) so that
\[ \int_{M_r \cap B_{r_0}} e^{-|x|^2/4} \text{Re} \Omega = \int_{M_r \cap B_{r_0}} e^{-|x|^2/4} \text{Re} \Omega + \int_{U_r} d(e^{-|x|^2/4} \wedge \text{Re} \Omega), \]
\[ \text{since } d\Omega = 0. \]
We have \( d(e^{-|x|^2/4} \wedge \text{Re} \Omega) = -\frac{|x|}{2} e^{-|x|^2/4} d|x| \), so it follows that
\[ \int_{U_r} d(e^{-|x|^2/4} \wedge \text{Re} \Omega) \leq Cr_0 \mathcal{H}^3(U_r) = Cd^{10/11} \mathcal{H}^3(U_r). \]

Let us write \( U_r = U_{r,1} \cup U_{r,2} \), where
\[ U_{r,1} = U_r \cap B_{Ad^{1/2}} \quad \text{and} \quad U_{r,2} = U_r \setminus B_{Ad^{1/2}}. \]
Note that outside of \( B_{Ad^{1/2}} \) the surface \( M_r \) is still the graph of some \( u \) over \( V \) satisfying \( |x|^{-1} |u|, |\nabla u| \leq Ad|x|^{-2} \) by Proposition \[4.3\]. The cone \( V_r \) is also the graph of a function \( v \) over \( V \) with \( |x|^{-1} |v|, |\nabla v| \leq Cd^{4/5} \) by construction. It follows that, once \( d \) is small, \( M_r \) is the graph of some \( \tilde{u} \) over \( V_r \) with \( |x|^{-1} |\tilde{u}|, |\nabla \tilde{u}| \leq Ad|x|^{-2} \) (recalling that \( |x|^2 < d^{1/5} \) in the region under consideration). We can then choose \( U_r \) so that \( U_{r,2} \) is the hypersurface swept out by the graphs of \( s\tilde{u} \) over \( V_r \) for \( s \in [0,1] \). We estimate the volume of \( U_{r,2} \) by the integral
\[ \mathcal{H}^3(U_{r,2}) \leq C \int_{Ad^{1/2}}^d \frac{Ad}{r} r dr \leq Cd^{11/10}. \]

Finally, we choose \( U_{r,1} \) to minimize area, such that its boundary is given by the union of \( U_{r,2} \cap \partial B_{Ad^{1/2}}, M_r \cap B_{Ad^{1/2}} \) and \( V_r \cap B_{Ad^{1/2}} \). The isoperimetric inequality then implies that
\[ \mathcal{H}^3(U_{r,1}) \leq C \mathcal{H}^2(\partial U_{r,1})^{3/2}. \]
To estimate \( \mathcal{H}^2(\partial U_{r,1}) \), consider the three pieces of the boundary. In the sphere \( \partial B_{Ad^{1/2}} \) the cross sections of \( M_r \) and \( V_r \) are graphs of functions bounded by \( d^{3/2} \), so \( \mathcal{H}^2(U_{r,2} \cap \partial B_{Ad^{1/2}}) \leq Cd. \) From \[4.13\] we know that the boundary piece \( V_r \cap B_{Ad^{1/2}} \) also has area bounded by \( Cd \). To control \( \mathcal{H}^2(M_r \cap B_{Ad^{1/2}}) \) it is enough to use that \( M_r \cap B_{Ad^{1/2}} \) is almost calibrated, which follows from the fact that \(|\theta - \theta V| \leq Cd|d|\) and \( d \) is small. As in \[15\] Lemma 7.1 we have
\[ \mathcal{H}^2(M_r \cap B_{Ad^{1/2}}) \leq C \mathcal{H}^1(M_r \cap B_{Ad^{1/2}})^2 \leq Cd. \]
In sum, it follows that with this choice of \( U_r \) we have
\[ \mathcal{H}^3(U) \leq C(d^{3/2} + d^{11/10}) \leq Cd^{11/10}. \]
Therefore, using \[4.11\], \[4.13\] and \[4.16\], we have
\[ \int_{M_r \cap B_{r_0}} e^{-|x|^2/4} dA_{M_r} \leq (1 + Cd^2) \left( \int_{V \cap B_{r_0}} e^{-|x|^2/4} + Cd^{18/10} + Cd^{12/10} \right). \]
Here the term involving \( d^{18/10} \) is obtained from comparing the area forms of \( V_r \) and \( V \).
Combining our estimates on the different regions (a)-(d) we have
\[
\left| \int_{M_\tau} e^{-|x|^2/4} - \int_V e^{-|x|^2/4} \right| \leq C d^{12/10},
\]
and so the required estimate (4.9) for $A(M_\tau)$ holds with $\alpha_1 = 1/10$, once $d$ is sufficiently small. \hfill \square

5. Limiting solutions of drift heat equation

In this section we show that from a sequence of rescaled flows whose initial conditions are getting closer and closer to the pair of planes $V$, we can extract in the limit a solution to the drift heat equation which, after removing a leading order singular term, is defined on each plane $P_1, P_2$ in $V$. We also show that this solution will satisfy good estimates.

5.1. Sequences of rescaled flows. We will need to pass to limits along sequences of rescaled flows. It is here that the local exactness imposed in Condition (5.1) will begin to play a role. First we have the following, showing that Condition (3) is preserved along the flow as long as $D_V(M_\tau)$ stays sufficiently small.

**Lemma 5.1.** There is an $\epsilon > 0$ satisfying the following. Suppose the flow $M_\tau$ satisfies Condition (3) for $\tau \in [-1, T]$, with $T > 0$, and $D_V(M_\tau) < \epsilon$ for $\tau \in [-1, T]$. If $M_0$ satisfies Condition (3) then $M_\tau$ satisfies Condition (3) for $\tau \in [0, T]$.

**Proof.** If $\epsilon$ is sufficiently small, then $M_\tau$ is a smooth graph over $V$ on the annulus $B_2 \setminus B_{1/2}$ for $\tau \in [0, T]$. If follows that no additional component of the flow can appear in $B_1$ at any time $\tau > 0$ if $M_0 \cap B_1$ is connected. As for the exactness, if $\gamma$ denotes the evolution of a closed loop $\gamma$ along the (unrescaled) mean curvature flow, then $\partial_t \int_{\gamma(t)} \lambda = 0$. Moreover, by the graphicality statement above, any closed loop $\gamma$ in $M_\tau \cap B_2$ is homotopic to a closed loop in $M_\tau \cap B_1$, for $\tau \in [0, T]$. It follows from this that if $\int_{\gamma} \lambda = 0$ for any loop $\gamma \in B_1 \cap M_0$, then the same holds for any loop $\gamma \in B_1 \cap M_\tau$ for $\tau \in [0, T]$. In particular, $M_\tau$ satisfies Condition (3) for $\tau \in [0, T]$.

**Proposition 5.2.** There is a constant $C > 0$, depending only on $V$, satisfying the following. Let $T > 0$ and let $M_\tau^i$ be a sequence of flows defined for $\tau \in [-1, T + 2]$ which satisfy Condition (4), and such that $M_\tau^i$ satisfy Condition (3). Suppose that $D_V(M_\tau^0) =: d_i \to 0$.

1. There exist compact sets $K_i \subset \mathbb{C}^2 \setminus \{0\}$ exhausting $\mathbb{C}^2 \setminus \{0\}$, satisfying the following. For $s \in [1, T]$ the surface $M_i^s$ is the graph of $u_i(s)$ over $V$ on $K_i$ such that, up to choosing a subsequence, the $d_i^{1/2} u_i$ converge locally smoothly on $[1, T] \times V \setminus \{0\}$ to a solution $u(s)$ of the drift heat equation

\[
\partial_s u = \Delta u + \frac{1}{2} (u - x \cdot \nabla u).
\]

2. The limit $u$ can be identified with an exact 1-form on $V \setminus \{0\}$. We write $u = (u_1, u_2)$, where $u_j$ is the restriction of $u$ to the plane $P_j \setminus \{0\}$ in terms of $V = P_1 \cup P_2$.

We can further decompose

\[
u = e^s u_0 + \tilde{u} = e^s (a_1 d \ln |x|, a_2 d \ln |x|) + (\tilde{u}_1, \tilde{u}_2),
\]
where \( a_1, a_2 \) are constants such that \( |a_1|, |a_2| \leq C \), and the \( \tilde{u}_j \) extend smoothly across the origin.

(3) We have the following estimates at \( s = 1 \):

\[
\int_{V \setminus \{0\}} |x|^2 |u|^2 e^{-|x|^2/4} \leq C,
\]

\[
\sup_{B_r \cap V \setminus \{0\}} |x||u| + |d^*u| \leq C.
\]

Proof. It follows from Proposition 4.5 that \( M_i^s \) is the graph of \( u_i \) over \( V \) for \( s \in [1, T] \) on the annuli \( B_{R_i^s} \setminus B_{Ad_i^s/2} \), where \( R_i^s \to \infty \) as \( d_i \to 0 \) by Definition 4.3, and hence on larger and larger compact annuli \( K_i \). Moreover, on any fixed compact set \( K \subset V \setminus \{0\} \) we have uniform bounds for \( d_i^{-1}u_i \) and \( d_i^{-1}\nabla u_i \) as \( i \to \infty \). Standard parabolic estimates imply that, up to choosing a subsequence, the \( u_i \) converge locally smoothly to a solution of (5.1) on \( V \setminus \{0\} \).

Using that the \( M_i^s \) are Lagrangian it follows that \( u \) can be identified with a closed 1-form. Condition (b) implies that the integral of \( u \) along the two circles \( V \cap \partial B_1 \) vanishes, and so \( u \) is actually exact. Writing \( u = (u_1, u_2) \) as in (b), we then have \( u_j = df_j \) for functions \( f_j \) on \( P_2 \setminus \{0\} \).

The estimates (5.3) follow directly from the definition of \( I_V(M_i^t) \) together with the bounds given in Lemma 3.5. For the bound on \( d^*u \) note that when we locally view \( M_i^t \) as the graph of the 1-form \( u_i \) over a plane, then the difference \( \theta - \theta_V \) in the Lagrangian angle is given by \( d^*u_i \) up to lower order terms in \( u_i \).

It remains to show the claimed decomposition (5.2). For this we focus on one of the planes \( P = P_j \), and the corresponding solution \( u = df \) of the drift heat equation. By rescaling

\[
U(x,t) = \sqrt{tu}(x/\sqrt{-t}, -\ln(-t))
\]

we obtain a solution \( U \) of the heat equation on a time interval \( [T_0^s, T_1^s] \) on \( P \setminus \{0\} \). We have \( U = dF \) for \( F : P \setminus \{0\} \to \mathbb{R} \) and we can arrange that \( F \) also satisfies the heat equation. The bound \( |d^*u| \leq C \) on \( B_1 \setminus \{0\} \) implies that we also have a uniform bound \( |\Delta F| \leq C \) on \( [T_0^s, T_1^s] \times B_r(0) \setminus \{0\} \) for some \( r > 0 \). Since \( \Delta F \) also satisfies the heat equation, it follows that \( \Delta F \) extends smoothly across the origin in \( P \). Using that \( \Delta F = \partial_t F \) on \( P \setminus \{0\} \), for any \( t \in [T_0^s, T_1^s] \) we have

\[
F(t) - F(T_0^s) = v(t) \text{ on } P \setminus \{0\},
\]

where \( v(t) = \int_{T_0^s}^t \partial_s F \, ds \) is smooth across the origin. Since \( \Delta F(T_0^s) \) is smooth across the origin, there is a smooth function \( g \) such that \( F(T_0^s) - g \) is harmonic on \( P \setminus \{0\} \). Note that the bound \( |dF| \leq C|x|^{-1} \) near the origin implies that

\[
|F(T_0^s) - g| \leq C \ln |x|
\]

near the origin. This implies that \( F(T_0^s) - g = a \ln |x| \) for a constant \( a \) satisfying \( |a| \leq C \), up to modifying \( g \) by a smooth function. Using this in (5.4) we have that

\[
F(t) = a \ln |x| + \tilde{F}(t),
\]

where \( \tilde{F}(t) \) extends smoothly across the origin, and therefore solves the heat equation on all of \( P \). Scaling \( F \) and hence \( U \) back to give \( u = df \), this shows the required decomposition (5.2).  \( \square \)
5.2. Estimates for solutions of the drift heat equation. We will use the following estimates for the smooth part of the limiting solution of the drift heat equation obtained in Proposition 5.2.

**Proposition 5.3.** Suppose that \( u \) is an exact 1-form valued solution of the drift heat equation \((5.1)\) on \( \mathbb{R} \) on the time interval \([0, \infty)\).

1. Suppose that at \( s = 0 \) we have the bounds
   \[
   \int_{\mathbb{R}^2} |x|^2 |u|^2 e^{-|x|^2/4} \leq 1, \\
   |d^* u| \leq 1, \text{ on } B_1.
   \]
   Then there is a uniform constant \( C > 0 \) so that at \( s = 0 \) we also have
   \[
   \int_{\mathbb{R}^2} |u|^2 e^{-|x|^2/4} \leq C.
   \]

2. If at \( s = 0 \) we have \( \int_{\mathbb{R}^2} |u|^2 e^{-|x|^2/4} \leq 1 \), then at time \( s = 1 \) we have
   \[
   |u|^2, |\nabla u|^2, |\nabla^2 u|^2 \leq C e^{1/4p}
   \]
   for some constants \( C > 0 \) and \( p > 1 \).

**Proof.** To prove (1), it is enough to show that under the assumptions we have a uniform bound \( |u| \leq C \) on \( B_1 \) at \( s = 1 \). We can write \( u = df \), with \( |\Delta f| \leq 1 \) and \( f(p) = 0 \) for a basepoint \( p \in \partial B_1 \). Elliptic estimates for the system \( d u = 0 \), \( |d^* u| \leq 1 \) together with the integral bound for \( u \) imply that we have a uniform bound \( |u| \leq C \) on the annulus \( B_2 \setminus B_1/2 \), and thus \( f \) satisfies a gradient bound there. It follows that we have \( |f| \leq C \) on \( \partial B_1 \). Since \( |\Delta f| \leq 1 \) on \( B_1 \), the maximum principle then implies a uniform bound \( |f| \leq C \) on \( B_1 \) and so we also have a uniform gradient bound \( |df| \leq C \) on \( B_3/4 \). The required estimate for \( |u| \) follows.

To prove (2), we first argue as in the proof of Lemma 3.5 to obtain the pointwise estimate \( |u|^2 \leq C e^{1/4p} \) for \( s \in [1/4, 1] \) for some \( C > 0 \) and \( p > 1 \). In order to estimate \( \nabla u \), we can consider the evolution of \( f = |u|^2 + s|\nabla u|^2 \). In terms of the drift Laplacian \((5.5)\)
   \[
   L_0 = \Delta - \frac{1}{2} x \cdot \nabla
   \]
we have (recalling that \( s \geq 0 \))
   \[
   (\partial_s - L_0)(|u|^2 + s|\nabla u|^2) = |u|^2 - 2|\nabla u|^2 - 2s|\nabla^2 u|^2 + |\nabla u|^2 \leq |u|^2.
   \]
It follows, using an estimate analogous to \((6.3)\) (see also \cite{2} Theorem 1.6.2), that at time \( s = 1/2 \) we have a bound
   \[
   \int_{\mathbb{R}^2} |\nabla u|^{2p} e^{-|x|^2/4} \leq C,
   \]
for some \( p > 1 \). Arguing again as in the proof of Lemma 3.5 we obtain the required pointwise bounds for \( |\nabla u| \) at \( s = 1 \). The bound for \( |\nabla^2 u| \) is similar. \( \Box \)

6. Three-annulus lemmas

In this section we prove two versions of the 3-annulus lemma. The first is for solutions of the drift heat equation. The second is for our distance \( D_V \) to the planes \( V \).
6.1. Drift heat equation. We show the following 3-annulus lemma for solutions of the drift heat equation given by Proposition 5.2. Note that this is slightly stronger than log-convexity of the norm proved by Colding–Minicozzi [6]. The proof is similar to Simon [23, Lemma 3.3].

**Lemma 6.1.** There are small $0 < \lambda_1 < \lambda_2 < 1$ satisfying the following. Let $u = e^s a_0 \ln |x| + \tilde{u}$ be a solution of the drift heat equation (5.1) on $\mathbb{R}^2 \setminus \{0\}$, where $\tilde{u}$ extends smoothly across the origin. We define the norm

$$
\|u(\tau)\|^2 = |a_0|^2 e^{2\tau} + \int_{\mathbb{R}^2} |\tilde{u}(\tau)|^2 e^{-|x|^2/4}
$$

and observe that we have a decomposition

$$
u(s) = a_0 e^s \ln |x| + \sum_{i>0} a_i e^{\mu_i s} \phi_i,
$$

where the $\phi_i$ are orthonormal eigenfunctions of the drift Laplacian $\mathcal{L}_0$ in (5.3).

1. If $\|u(1)\| \geq e^{\lambda_1} \|u(0)\|$ then $\|u(2)\| \geq e^{\lambda_2} \|u(1)\|$.
2. If $u \neq 0$ has no homogeneous degree zero component, i.e. no term corresponding to $\mu_i = 0$ in (6.2), then we must have

either $\|u(2)\| \geq e^{\lambda_1} \|u(1)\|$ or $\|u(1)\| \leq e^{-\lambda_1} \|u(0)\|$.

**Proof.** By (6.2) and the definition of the norm, if we set $\mu_0 = 1$ then we have

$$
\|u(s)\|^2 = \sum_{i=0}^\infty a_i^2 e^{2\mu_i t}.
$$

Fix a small $\alpha > 0$ so that if $\mu_i \in [-10\alpha, 10\alpha]$ for some $i$, then $\mu_i = 0$. We have

$$
\|u(0)\|^2 = \sum a_i^2 e^{2\mu_i} - \sum a_i^2 e^{2\mu_i - 2\alpha},
$$

$$
\|u(1)\|^2 = \sum a_i^2 e^{2\mu_i} - \lambda \sum a_i^2 e^{2\mu_i - 4\alpha}.\n$$

It follows that

$$
\frac{1}{2} (\|u(0)\|^2 + e^{-4\alpha} \|u(2)\|^2) = \sum a_i^2 e^{2\mu_i - 2\alpha} \frac{1}{2} (e^{2\alpha - 2\mu_i} + e^{2\mu_i - 2\alpha})
$$

$$
= \sum a_i^2 e^{2\mu_i - 2\alpha} \cosh(2\mu_i - 2\alpha).
$$

By our choice of $\alpha$ we have $|2\mu_i - 2\alpha| \geq \alpha$ for all $i$, so we get

$$
\frac{1}{2} (\|u(0)\|^2 + e^{-4\alpha} \|u(2)\|^2) \geq (1 + 1/c) e^{-2\alpha} \|u(1)\|^2,
$$

for some $c > 0$. We choose $\lambda_1 < a < \lambda_2$ with $\lambda_j$ very close to $a$. If $\|u(1)\|^2 \geq \lambda_1 \|u(0)\|^2$ then from (6.3), we have

$$
(1 + c) e^{-2\alpha} \|u(1)\|^2 \leq \frac{1}{2} e^{-4\alpha} \|u(2)\|^2 + \frac{1}{2} e^{-2\lambda_1} \|u(1)\|^2.
$$

Rearranging this we have

$$
\|u(2)\|^2 \geq e^{2\alpha} (2(1 + c) - e^{2\alpha - 2\lambda_1}) \|u(1)\|^2 \geq e^{2\lambda_2} \|u(1)\|^2,
$$

if the $\lambda_j$ are sufficiently close to $a$. This shows (1). The proof of (2), choosing $\lambda_1, \lambda_2$ closer to zero if necessary, is similar. $\square$
6.2. Distance. Using Lemma 6.1 we can show a 3-annulus lemma for the distance $D_V$, by a contradiction argument.

**Proposition 6.2.** Let $\lambda_1, \lambda_2$ be as in Lemma 6.1. Let $0 < \lambda_1' < \lambda_2'$ be such that $\lambda_1' \in (\lambda_1, \lambda_2)$. There is a large $N_0 > 0$ satisfying the following. Given an integer $N > N_0$, suppose that the flow satisfies Condition (1) for $\tau \in [-1, 2N + 10]$, and $M_0$ satisfies Condition $\tau$. There is an $\epsilon > 0$ depending on $N$ such that if $D_V(M_0) < \epsilon$, then

$$D_V(M_N) \geq e^{\lambda_1 N} D_V(M_0) \implies D_V(M_{2N}) \geq e^{\lambda_2 N} D_V(M_N).$$

**Proof.** The proof is by contradiction, similar to the proof of [22, Lemma 2, p. 549], using property (3) in Lemma 3.5 to deal with the singularity of $V$. The proof is by contradiction, similar to the proof of [22, Lemma 2, p. 549], using property (3) in Lemma 3.5 to deal with the singularity of $V$ at the origin and its noncompactness. See [26, Proposition 5.12] for a related argument. Suppose that the result fails for a given large integer $N$, so that we have a sequence of flows $M^i_N$ with $D_V(M^i_N) \to 0$ such that the conclusion fails. By Proposition 3.6 we have $d_i = D_V(M^i_N) \to 0$ and our hypothesis can be written:

$$D_V(M^i_N) \leq e^{-\lambda_1 N} d_i,$$

(6.4)

$$D_V(M^i_{2N}) < e^{\lambda_2 N} d_i.$$  

In particular $d_i > 0$. Using Proposition 5.2 we can write $M^i_N$ as the graphs of $u_i(s)$ for $s \in [1, 2N + 8]$ over $V$ on larger and larger compact sets $K_i \subset V \setminus \{0\}$. The inequalities (6.4) and Proposition 5.6 imply that $D_V(M^i_N) \leq C_i d_i$ for $s \in [1, 2N + 8]$.

As in Proposition 5.2, up to choosing a subsequence, we can assume that the $d_i u_i$ converge locally smoothly to a limit solution of (5.1) on $V \setminus \{0\}$. We can write $u = e^a \ln |x| + \tilde{u}$ by Proposition 5.2 where $\tilde{u}$ is smooth across the origin and $a_0$ is constant on each plane in $V$. Using (5.3) and (6.4) the limit satisfies the estimates

$$\int_V |x|^2 |\tilde{u}(1)|^2 e^{-|x|^2/4} \leq C e^{-2\lambda_1 N}, \quad \int_V |x|^2 |\tilde{u}(2N + 1)|^2 e^{-|x|^2/4} \leq C e^{2\lambda_2 N},$$

for $C > 0$ (independent of $N$). In addition the $d \ln |x|$ component of $u$ satisfies $|a_0|^2 e^{2(N + 1)} \leq C e^{2\lambda_2 N}$, and so

$$|a_0| \leq C e^{(\lambda_2 - 2) N}.$$  

Recall the norm in (6.1). Proposition 5.2 Proposition 5.3 and (6.5) imply that

$$\|\tilde{u}(2)\| \leq C e^{-\lambda_1 N}, \quad \|\tilde{u}(2N + 2)\| \leq C e^{\lambda_2 N}.  \quad (6.6)$$

Let $\kappa > 0$. We now have the following using (6.6) together with Lemma 6.1

**Claim 6.3.** For $N$ sufficiently large (depending on $\kappa$) we have $\|\tilde{u}(N - 1)\| \leq \kappa$.

**Proof.** To see this note that we have two possibilities.

- If $\|\tilde{u}(k + 1)\| \leq e^{\lambda_1} \|\tilde{u}(k)\|$ for $k = 2, \ldots, N - 2$, then we have

$$\|\tilde{u}(N - 1)\| \leq e^{(N - 3)\lambda_1} \|\tilde{u}(2)\| \leq C e^{(N - 3)\lambda_1 - \lambda_1 N}.$$  

Since $\lambda_1 > \lambda_1 > 0$ this implies that $\|\tilde{u}(N - 1)\| \leq \kappa$ if $N$ is chosen sufficiently large.

- If $\|\tilde{u}(k + 1)\| \geq e^{\lambda_1} \|\tilde{u}(k)\|$ for some $k \leq N - 2$, then by Lemma 6.1 we have $\|\tilde{u}(k + 1)\| \geq e^{\lambda_2} \|\tilde{u}(k)\|$ for $k = N - 1, \ldots, 2N + 1$. This implies

$$\|\tilde{u}(N - 1)\| \leq e^{-(N + 3)\lambda_1} \|\tilde{u}(2N + 2)\| \leq C e^{-(N + 3)\lambda_1 - \lambda_1 N}.$$  

Since $\lambda_2 > \lambda_2 > 0$, we have $\|\tilde{u}(N - 1)\| \leq \kappa$ if $N$ is sufficiently large. □
Given Claim 6.3, let us assume therefore that $N$ is large enough that $\|\tilde{u}(N-1)\| \leq \kappa$. The estimates in Proposition 5.3 then imply that we have pointwise bounds
\begin{equation}
|\tilde{u}(N)|^2, |\nabla \tilde{u}(N)|^2 \leq C\kappa^2 e^{\|\alpha\|_2^2/4p},
\end{equation}
for some $C > 0$ and $p > 1$. The logarithmic component of $u$ also satisfies
\[ e^N a_0 \ln |x| \leq Ce^{(N-1)|x|^{-1}} \leq \kappa |x|^{-1}, \]
for large $N$ (we can assume that $\lambda_2 < 1$).

We now use the local smooth convergence of the $d_i^{-1} u_i$ to $u$. For any fixed compact set $K \subset \mathbb{C}^2 \setminus \{0\}$ this implies that, as $i \to \infty$, the functions $d_i^{-1} d_V$ and $d_i^{-1} (\theta - \Theta)$ on $M_i^h$ converge to $|u|$ and $d^* \tilde{u}$ on $V \cap K$. Using the estimates (6.7) and the fact that $p > 1$ it follows that for a given $K$, if we choose $i$ sufficiently large (depending on $K, \kappa$), we have (note that we can assume that $d_i^{-2} d_V$ differs from $|u|^2$ by at most $\kappa$ on $K$ for large $i$)
\begin{equation}
\int_{M_i^h \cap K} (|x|^2 d_V^2 + |\theta - \Theta|^2) e^{-|x|^2/4} \leq d_i^2 \kappa^2
\end{equation}
\begin{equation}
\quad + d_i^2 \int_{V \cap K} (|x|^2 |u|^2 + |d^* \tilde{u}|^2) e^{-|x|^2/4} \leq d_i^2 \kappa^2 + Cd_i^2 \kappa^2 \int_V (|x|^2 + 1) e^{-|x|^2/4} \leq Cd_i^2 \kappa^2.
\end{equation}

We will now apply part (3) of Lemma 5.5 to estimate $I_V(M_i^h)$ in terms of $I_V(M_{N-1}^h)$ together with the integral bound (6.8) for suitable $K$. Note first that by (6.3) and part (1) of Lemma 6.3, we have
\[ I_V(M_i^h) = D_V(M_{N-1}^h) \leq C_N d_i, \]
for an $N$-dependent constant, while $I_V(M_h^i) = D_V(M_h^i) = d_i$. Let $\gamma > 0$. Assuming $d_i > 0$ is sufficiently small (depending on $\gamma$), from Lemma 5.5 we have a compact set $K_{\gamma} \subset \mathbb{C}^2 \setminus \{0\}$ such that the integral estimate (6.8) on $K = K_{\gamma}$ implies
\begin{equation}
d_i = I_V(M_h^i) \leq C(d_i \kappa + \gamma C_N d_i).
\end{equation}

We first choose $\kappa$ such that $C\kappa < 1/4$. The choice of $\kappa$ determines an $N_0$, such that for $N > N_0$ we have the estimate $\|\tilde{u}(N-1)\| \leq \kappa$ from Claim 6.3. Choosing $N > N_0$ then determines the constant $C_N$, and we choose $\gamma$ such that $CC_N \gamma < 1/4$. This choice determines the set $K_{\gamma}$, and then for sufficiently large $i$ we have the estimate (6.8) on $K = K_{\gamma}$. For such large $i$ the inequality (6.9) holds, and it implies $d_i \leq d_i/2$, which is a contradiction as $d_i > 0$.

7. DECAY ESTIMATES

We first define a variant of the excess from Definition 2.5 and show that it satisfies a monotonicity formula.

**Definition 7.1.** Recall the excess $A(M)$ from Definition 2.5. For any $\alpha > 0$ we let
\[ A_{\alpha}(M) = |A(M)|^{\alpha - 1} A(M), \]
i.e. $|A_{\alpha}| = |A|^{\alpha}$, but $A_{\alpha}$ has the same sign as $A$. 
Lemma 7.2. For any \( \tau_1 < \tau_2 \) and for \( \alpha \in (0,1) \) we have
\[
A_\alpha(M_{\tau_1}) - A_\alpha(M_{\tau_2}) \geq \alpha \int_{\tau_1}^{\tau_2} |A(M_s)|^{\alpha-1} \int_M \left( 2|H|^2 + \left| H + \frac{x}{2} \right|^2 \right) e^{-|x|^2/4} ds.
\]

Proof. From Huisken’s monotonicity formula we know that \( A_\alpha(M_s) \) is monotonically decreasing with \( s \), being the infimum of a family of decreasing functions as we vary \( \theta_0 \) in the definition of \( A \) in (2.4). In particular \( A_\alpha(M_s) \) is differentiable almost everywhere, and at these points the derivative satisfies
\[
\frac{d}{ds} A_\alpha(M_s) \leq -\alpha |A(M_s)|^{\alpha-1} \int_M \left( 2|H|^2 + \left| H + \frac{x}{2} \right|^2 \right) e^{-|x|^2/4}.
\]

The required inequality follows by integrating with respect to \( s \in [\tau_1, \tau_2] \). \( \square \)

The main technical result of this section is the following. Recall that we defined \( V' \subset V \) in the same way as \( V \) in Definition 5.1, just with the constant \( c_1/2 \) instead of \( c_1 \) measuring closeness to the fixed pair of planes \( V_0 \).

**Proposition 7.3.** There are \( \epsilon_0, C, N_1 > 0 \) and \( \alpha \in (0,1) \) such that if \( N > N_1 \) is an integer, \( D_V(M_0) < \epsilon_0 \), \( M_1 \) satisfies Condition (3), and the flow exists for \( \tau \in [-1, 3N^2 + 2] \) satisfying Condition (4), then we have the following. If \( V \in V' \), then there is a \( V' \in V \) satisfying \( d(V, V') \leq CD_V(M_0) \) together with one of the following conditions:

(i) \( D_{V'}(M_N) \leq \frac{1}{2} D_V(M_0) \),

(ii) \( D_{V'}(M_N) \leq A_\alpha(M_{N-3}) - A_\alpha(M_N) \),

(iii) \( D_V(M_{N+2}) \geq \lambda_1 N^2 D_V(M_0) \),

where \( \lambda_1 \) is given by Proposition 6.2.

Proof. We prove the result by contradiction. Suppose that for some large integer \( N \) we have a sequence of flows \( M_i^\tau \) for \( \tau \in [-1, 3N^2 + 2] \), and \( D_{V_i}(M_0^\tau) = d_i \to 0 \) for some \( V_i \in V' \). We will show that if none of the conditions (i)–(iii) hold, then we reach a contradiction if \( N \) is sufficiently large. First note that up to choosing a subsequence we can replace the sequence \( V_i \) by a single \( V \).

Using Proposition 5.2 and Lemma 5.1 we know that for sufficiently large \( i \) the surfaces \( M_i^\tau \) satisfy Condition (4) for \( \tau \in [1, 3N^2 + 2] \). Using Proposition 6.2 we can find compact sets \( K_i \) exhausting \( C^2 \setminus \{0\} \) so that \( M_i^\tau \) is the graph of \( \tilde{u}_i \) over \( V \cap K_i \) for \( s \in [1, 3N^2] \). In addition, up to choosing a subsequence, the rescaled functions \( d_i^{-1} \tilde{u}_i \) converge locally smoothly to a solution \( u \) of the drift heat equation (5.1) on \([1, 3N^2] \times V \setminus \{0\} \).

Let \( u_0 \) be the static component of \( \tilde{u} \) in the decomposition (6.2), corresponding to the kernel of the drift Laplacian
\[
\mathcal{L}_0 + \frac{1}{2} = \Delta + \frac{1}{2}(1 - x \cdot \nabla).
\]

We can thus write \( u_0 = (df_1, df_2) \) for homogeneous degree 2 functions \( f_1, f_2 \) on \( \mathbb{R}^2 \) with respect to the splitting \( V = P_1 \cup P_2 \) into a pair of planes. We then let \( a_j = df_j \) for \( j = 1, 2 \) (which are constants), set \( a = \frac{1}{4}(a_1 - a_2) \) and define
\[
\begin{align*}
u_{00} &= (d(f_1 + a|x|^2), d(f_2 - a|x|^2)), \\
u_{01} &= (d(-a|x|^2), d(a|x|^2)).
\end{align*}
\]
again using the splitting $V = P_1 \cup P_2$. Note that
\[
\begin{align*}
d^*u_{00} &= (a_1 - 4a, a_2 + 4a) = \left(\frac{1}{2}(a_1 + a_2), \frac{1}{2}(a_1 + a_2)\right), \\
d^*u_{01} &= (4a, -4a) = \left(\frac{1}{2}(a_1 - a_2), \frac{1}{2}(a_2 - a_1)\right).
\end{align*}
\]
We have therefore decomposed $u$ as
\[(7.1) \quad u = u_0 + u_\perp = u_{00} + u_{01} + u_\perp.\]

The purpose of (7.1) is that we can deform the cone $V$ in the direction of $u_{00}$ while keeping it special Lagrangian, since $d^*u_{00} = \frac{1}{2}(a_1 + a_2)$ on both planes. The directions $u_{01}$ however correspond to deformations of $V$ into non-special Lagrangian directions, whenever $a_1 \neq a_2$. More precisely, let $V^i_t$ denote the graph of $d_iu_{00}$ over $V$. Using Lemma 5.3 we have $D_{V^i}(M_i^0) \leq C d_i$, and we can argue as above to write $M_i^0$ as the graph of $u'(s)$ over larger and larger subsets of $V^i_t$. The rescaled functions $d_i^{-1}u_i'$ then converge to $u' = u_{01} + u_\perp$ in the decomposition (7.1).

Fix a $\kappa > 0$. We first use (2) in Lemma 6.1 to show the following.

**Claim 7.4.** If $N$ is sufficiently large (depending on $\kappa$), then either $\|u_\perp(N - 4)\| \leq \kappa$ or (iii) holds.

**Proof.** Recall that we consider $u_\perp$ for $r \in [1, 3N^2]$ and that $u_\perp$ has no homogeneous degree zero component. Suppose that $\|u_\perp(N - 4)\| > \kappa$. If $N$ is sufficiently large, there must be some $k < N - 4$ for which $\|u_\perp(k + 1)\| > e^{-\lambda_1}\|u_\perp(k)\|$, since otherwise we would have $\|u_\perp(N - 4)\| \leq C e^{-(N-5)\lambda_1}$, which for large $N$ is less than $\kappa$. Lemma 6.1 (2) now implies that $\|u_\perp(k + 2)\| \geq e^{\lambda_1}\|u_\perp(k + 1)\|$. However, Lemma 6.1 (1) then implies that
\[
\|u_\perp(l + 1)\| \geq e^{\lambda_1}\|u_\perp(l)\| \quad \text{for } l \geq k + 2.
\]

It follows that $\|u_\perp(N - 3)\| \geq e^{\lambda_1 \kappa}$ and $\|u_\perp(l + 1)\| \geq e^{\lambda_2}\|u_\perp(l)\|$ for all $k \geq N - 3$. Iterating this, we find that
\[
\|u_\perp(N^2)\| \geq e^{\lambda_1 + (N^2 + 3 - N)\lambda_2} \kappa.
\]

We can split $u_\perp$ further, writing
\[(7.2) \quad u_\perp = e^s a_0 \ln|x| + \tilde{u}_\perp
\]
such that $\tilde{u}_\perp$ is a smooth solution to the drift heat equation on $V$. Ecker’s log-Sobolev inequality \cite{Ecker} then implies that there is $p > 2$ such that
\[
\left(\int_V (\tilde{u}_\perp(N^2))^p e^{-|x|^2/4} \right)^{1/p} \leq C \|\tilde{u}_\perp(N^2 - 1)\| \leq C \|\tilde{u}_\perp(N^2)\|.
\]

Combined with the estimates from Proposition 5.3 (2) for $\tilde{u}_\perp$ and (7.2), we deduce that there is $r_0 > 0$ such that
\[
\int_{V \cap (B_1 \setminus B_{r_0})} (u_\perp(N^2))^2 e^{-|x|^2/4} \geq \frac{1}{2} \|u_\perp(N^2)\|^2.
\]

The definition of $D_V$ then implies that for sufficiently large $i$ we have
\[
D_V(M_i^0) \geq C^{-1} d_i e^{\lambda_1 + (N^2 + 3 - N)\lambda_2} \kappa \geq e^{\lambda_1 N^2} d_i,
\]
for sufficiently large $N$, since $\lambda_2 > \lambda'_1$. Hence (iii) holds. \qed
Let us suppose from now on that (iii) does not hold. Suppose that \( \|u_{01}\| = \kappa_1 \) for some small \( \kappa_1 \geq 0 \) (note that \( u_{01} \) is \( s \)-independent). We have \( \kappa_1 < C \) for a uniform constant \( C \). We also assume that for a given small \( \kappa > 0 \), \( N \) is chosen large enough so that \( \|u_\perp(N-4)\| \leq \kappa \) by Claim 7.3. We now show the following, from which we will deduce that (i) holds if \( \kappa_1 \) is sufficiently small.

**Claim 7.5.** If \( i \) is sufficiently large,

\[
D_{V'}(M_{N-3}) \leq C(\kappa_1 + \kappa)d_i.
\]

**Proof.** By Proposition 3.6 together with Lemma 3.8 we have \( D_{V'}(M_{N-4}) \leq C_N d_i \) for \( C_N \) depending on \( N \). As in the proof of Proposition 6.2 we can write \( u = e^{s\alpha_0}d\ln|x| + \tilde{u} \) such that we have pointwise bounds of the form

\[
\|\tilde{u}(s)\|^2, |\nabla^k \tilde{u}(s)|^2 \leq C(\kappa_1 + \kappa)^2 e^{|x|^2/4p}, 
\]

for some \( C > 0 \) and \( p > 1 \). We can then use the local smooth convergence of \( d_i^{-1}u_i \to u \) together with property (3) of Lemma 3.5 to ensure that (3.6) holds. \( \square \)

It follows from Claim 7.5 using Proposition 3.6 that for a larger \( C \) we have \( D_{V'}(M_s^i) \leq C(\kappa_1 + \kappa)d_i \) for \( s \in [N-3, N] \). If now \( C\kappa_1 < 1/4 \), then by choosing \( \kappa = C^{-1}/4 \) we will have \( D_{V'}(M_N^i) \leq \frac{1}{2}d_i \), i.e. (i) holds for large enough \( i \).

We therefore assume further that (i) does not hold, \( \kappa_1 = \|u_{01}\| \geq C^{-1}/4 \), and we choose \( \kappa < \kappa_1 \). In particular, the value of \( d^*u_{01} \) on the two planes differs by at least \( C^{-1} \) for some \( C > 0 \).

In the rest of the proof our goal is to show that if \( \kappa \) is sufficiently small (i.e. \( N \) is large), and (ii) also does not hold, then we get a contradiction. The basic idea is that in this case the flow \( M_s^i \) for \( s \in [N-3, N] \) would have distance of order \( \kappa d_i \) from a pair of planes whose Lagrangian angles differ by \( \kappa_1 d_i \). If \( M_N^i \) is connected and \( \kappa \ll \kappa_1 \), then as in [15, Theorem B], one might expect that this leads to a contradiction. The difficulty is that we need a quantitative version of this idea, which works uniformly as \( d_i \to 0 \).

We are assuming that (ii) fails, therefore

\[
A_\alpha(M_{N-3}^i) - A_\alpha(M_N^i) < D_{V'}(M_N^i) \leq C_N d_i.
\]

Using Lemma 7.2 this implies

\[
\int_{N-3}^N |A(M_s^i)|^{\alpha-1} \int_{M_s^i} (|H|^2 + |x|^2)e^{-|x|^2/4} ds \leq \alpha^{-1} C d_i.
\]

From Proposition 4.6 there is a small \( \alpha_1 > 0 \) so that

\[
|A(M_s^i)| \leq D_{V'}(M_s^i)^{1+\alpha_1} \text{ for } s \in [N-3, N].
\]

Using this in (7.5) for \( \alpha \in (0, 1) \) we get

\[
\int_{N-3}^N \int_{M_s^i} (|H|^2 + |x|^2)e^{-|x|^2/4} ds \leq C_\alpha d_i^{1+(1-\alpha)(1+\alpha_1)},
\]

for an \( \alpha \)-dependent constant \( C_\alpha > 0 \). For \( \alpha \) sufficiently small and \( 0 < 2\alpha_2 < \alpha_1/2 \), we see that

\[
1 + (1-\alpha)(1+\alpha_1) > 2 + \frac{\alpha_1}{2} > 2 + 2\alpha_2.
\]
Therefore, for sufficiently large $i$ we have
\[
\int_{N-3}^N \int_{M^i \cap B_2} |\mathbf{H}|^2 + |x^\perp|^2 \, ds \leq d_i^{2+2\alpha_2},
\]
where we have removed the Gaussian weight by restricting to $M^i \cap B_2$.

Let $\sigma > 0$ be small, to be chosen later, independent of $i$. We can find $s_1^i, s_2^i \in [N - 2, N - 1]$ such that
\[
\frac{\sigma}{2} < s_2^i - s_1^i < \sigma,
\]
and, in addition, at $s_j^i$ for $j = 1, 2$ we have
\[
\int_{M^i \cap B_2} |\mathbf{H}|^2 + |x^\perp|^2 \leq d_i^{2+\alpha_2}
\]
for $i$ sufficiently large (depending on $\sigma$ as well).

Note that because of the bound $D_{V'}(M^i_{N-3}) \leq C_N d_i$ and Proposition 6.3 we have that $M^i_s$ has good graphicality over $V'_s$ on the annulus $B_{R_C d_i} \setminus B_{C_0 d_i^{1/2}}$ for $s \in [N - 2, 0]$. This graphicality and Condition (\ref{eq:graph}) implies that the integral of the Liouville form $\lambda$ vanishes on any loop in $M^i_s \cap B_{R_C d_i}$, so we can define primitives $\beta$ satisfying $d\beta = \lambda$ on $M^i_s \cap B_{R_C d_i}$. Restating Proposition 6.1 for the rescaled flow, we can choose the primitives $\beta$ along the flows $M^i_s$ such that $e^{-s}(\beta + 2\theta)$ satisfies the drift heat equation (7.7).

Our next goal is to estimate $\beta$ at the times $s_1^i, s_2^i$. First we consider what happens on the ball $B_{C_0 d_i^{1/2}}$. For simplicity we write $M$ for $M^i_{s_1}$ or $M^i_{s_2}$.

**Claim 7.6.** There is a constant $\beta_0$ (depending on $s, i$) such that $|\beta - \beta_0| < \kappa d_i$ on $M \cap B_{C_0 d_i^{1/2}}$.

**Proof.** Let $M = d_i^{-1/2}M$ and let $\tilde{\beta}$ on $\tilde{M}$ be given by $\tilde{\beta}(p) = d_i^{-1}\beta(d_i^{1/2}p)$. Then $\tilde{M}$ is a connected, almost calibrated Lagrangian, with uniform area bounds, satisfying
\[
\int_{\tilde{M} \cap B_{C_0}} |\nabla \tilde{\beta}|^2 = d_i^{-2} \int_{M \cap B_{C_0 d_i^{1/2}}} |\nabla \beta|^2 \leq d_i^{2\alpha_2},
\]
using (7.7) and $|\nabla \beta| = |x^\perp|$. (Recall that $\alpha_2 > 0$.) We also have a uniform bound
\[
|\nabla \tilde{\beta}(p)|_{\tilde{M}} = d_i^{-1/2} |\nabla \beta(d_i^{1/2}p)|_{M} \leq C
\]
for $p \in B_{C_0}$, so (5) Lemma 3.7), together with the connectivity assumption in Condition (\ref{eq:connectivity}), implies that osc $\tilde{\beta} \to 0$ as $i \to \infty$. It follows that on $M \cap B_{C_0 d_i^{1/2}}$ we have osc $(\beta) < \kappa d_i$ for sufficiently large $i$. Setting $\beta_0 = \beta(q)$ for some $q \in B_{C_0 d_i^{1/2}}$ yields the claim. \hfill \Box

Next we extend this pointwise bound on $B_{C_0 d_i^{1/2}}$ to an integral bound on the (larger) ball $B_1$, by using the integral estimate for $|\nabla \beta|^2$ from (7.7) again.

**Claim 7.7.** For sufficiently large $i$,
\[
\int_{M \cap B_1} |\beta - \beta_0|^2 \leq 9\kappa^2 d_i^2.
\]
Proof. Since on the annulus $B_1 \setminus B_{C_0d_i^{-1/2}}$ the surface $M$ has good graphicality over $V_i'$, we can view $\beta$ as a function $b$ on two copies of the annulus $B_1 \setminus B_{C_0d_i^{-1/2}} \subset \mathbb{R}^2$, where we use polar coordinates $r, \phi$. Using (7.8) we then have

$$\int_{B_1 \setminus B_{C_0d_i^{-1/2}}} |\nabla b|^2 \leq 2d_i^{2+\alpha_2}$$

for sufficiently large $i$. For each $r \in [C_0d_i^{-1/2}, 1]$ and $\phi \in [0, 2\pi]$ we have

$$|b(r, \phi) - \beta_0| \leq \kappa d_i + \int_{C_0d_i^{-1/2}} |\nabla b|(s, \phi) \, ds,$$

where $\beta_0$ is the constant given by Claim 7.6. Therefore,

$$\int_0^{2\pi} \int_{C_0d_i^{-1/2}} |b(r, \phi) - \beta_0|^2 \, r \, dr \, d\phi$$

$$\leq 2\pi \kappa^2 d_i^2 + 2 \int_0^{2\pi} \int_{C_0d_i^{-1/2}} \left( \int_{C_0d_i^{-1/2}} |\nabla b|(s, \phi) \, ds \right)^2 \, r \, dr \, d\phi.$$

Note that using Hölder’s inequality we have

$$\int_0^{2\pi} \int_{C_0d_i^{-1/2}} \left( \int_{C_0d_i^{-1/2}} |\nabla b|(s, \phi) \, ds \right)^2 \, r \, dr \, d\phi$$

$$\leq C \int_0^{2\pi} \int_{C_0d_i^{-1/2}} \int_{C_0d_i^{-1/2}} |\nabla b|^2(s, \phi) \, s \, ds \, r^2 \, dr \, d\phi$$

$$\leq C \int_0^{2\pi} \int_{C_0d_i^{-1/2}} |\nabla b|^2(s, \phi) \, ds \, d\phi$$

$$\leq C d_i^{2+\alpha_2} \leq \kappa^2 d_i^2,$$

once $i$ is large enough (so that $d_i$ is small). The result follows for $i$ sufficiently large, combined with Claim 7.6 and the uniform area ratio bounds. \hfill \Box

We now show that we can get a similar pointwise estimates to Claim 7.6 on $K_i \cap M$, where we recall the compact sets $K_i$ given at the start of the proof.

Claim 7.8. Up to replacing the compact sets $K_i$ by smaller sets (still exhausting $\mathbb{C}^2 \setminus \{0\}$ in the limit as $i \to \infty$), there is a constant $C > 0$ and $p > 1$ such that

$$|\beta - \beta_0| \leq C \kappa d_i e^{\epsilon |x|^2/8p}$$

on $K_i \cap M$.

Proof. Recall the decomposition of $u$ in (7.1). We now show that $\nabla \beta$ is of order $\kappa d_i$ on compact sets away from 0, because the term $u_{01}$ in (7.1) does not contribute to $x \perp$, being homogeneous of degree 1. More precisely, recall that $M_i'$ is the graph of $u_i(s)$ over $K_i \cap V$ and note that $d_i^{-1}x \perp$ on $M_i'$ converges locally smoothly as $i \to \infty$ to $u - x \cdot \nabla u$ on $V \setminus \{0\}$. Since $u_{001}, u_{01}$ have degree 1, it follows that they have no contribution to $u - x \cdot \nabla u$. Therefore, as $i \to \infty$ we have $d_i^{-1} |\nabla \beta| \to |u \perp - x \cdot \nabla u \perp|$ locally smoothly. At the same time, by Claim 7.8 we have $\|u \perp(N - 4)\| \leq C \kappa$ and so, by Proposition 5.3 for $s \in [N - 3, N]$ we also have pointwise bounds

$$|u \perp|, |\nabla u \perp| \leq C \kappa e^{\epsilon |x|^2/8p}$$

(7.11)
for some $C > 0$ and $p > 1$. Returning to the setting where $M = M'_i \setminus B_{\kappa_i}$ for $j = 1, 2$ and using (4.8), we can integrate the estimate we have for $|\nabla \beta|$ to find that up to replacing $K_i$ by smaller sets and decreasing $p$ we get (7.10).

We also need a more global estimate for $\beta$ and $\theta$, up to the good graphicality radius $R_{Cd_i}$ (from Definition 4.3) on $M'_i$ for $s \in [N - 3, N]$.

**Claim 7.9.** Recall Definition 4.3. There is $C > 0$ and $p > p_0 > 1$ such that for $s \in [N - 3, N]$ we have

\[
|\beta - \beta_0| \leq C d_i e^{8p} \quad \text{on} \quad M'_i \cap B_{R_{Cd_i} \setminus B_{1/2}},
\]

\[
|\theta - \theta_{V_i}| \leq C d_i e^{8p} \quad \text{on} \quad M'_i \cap B_{R_{Cd_i} \setminus B_{1/2}},
\]

once $i$ is sufficiently large.

*Proof.* By the smooth convergence of $d_{i-1}^{-1} u_i \to u$ on the annulus $B_2 \setminus B_{1/2}$, together with the bounds (7.11), we have estimates

\[
|\nabla \beta|, |\Delta \beta|, |\theta - \theta_{V_i}|, |\nabla \theta|, |\Delta \theta| \leq C d_i
\]
on $M'_i \cap B_2 \setminus B_{1/2}$ for $s \in [N - 3, N]$. Using the evolution equation for $\beta$ and (7.10) it then follows that

\[
|\beta - \beta_0| \leq C d_i \quad \text{on} \quad M'_i \cap B_2 \setminus B_{1/2}
\]

for $s \in [N - 3, N]$. To extend this estimate out to distance $R_{Cd_i}$, we first observe that, by Proposition 4.3 on $B_{R_{Cd_i} \setminus B_1}$ the surface $M'_i$ is the graph of $u_i$ over $V_i$ satisfying $|u_i|, |\nabla u_i| \leq C d_i e^{8p}$ for some $p > p_0 > 1$. Moreover, by the definition of $R_{Cd_i}$ in (4.8) and the fact that $p > p_0$ we know that $d_i e^{R_{Cd_i}/8p} \to 0$ as $i \to \infty$. Since the leading order term in $x^1$ is $u_i - x \cdot \nabla u_i$ for $i$ large, we see that on $M'_i \cap B_{R_{Cd_i} \setminus B_1}$ for $s \in [N - 3, N]$ we have (decreasing $p > p_0$ if necessary)

\[
|\nabla \beta| = |x^1| \leq C d_i e^{8p}
\]

for sufficiently large $i$. Integrating this and using our bound (7.13) on $B_2 \setminus B_{1/2}$ implies that for an even smaller $p > p_0 > 1$ we have the estimate (7.12) for $i$ sufficiently large and $s \in [N - 3, N]$. The bound for $\theta$ in (7.11) follows similarly, since to leading order $\theta - \theta_{V_i}$ is given by $d^* u_i$ for $i$ large.

Recall the times $s_2^j > s_1^j$ which satisfy (4.8) depending on some small $\sigma > 0$, which we are free to choose. We will now use that $e^{-s}(\beta + 2\theta)$ and $\theta$ satisfy the drift heat equation (4.3) to derive pointwise estimates in $B_2$ for $\theta$ at time $s_2^j$ in terms of an integral estimate for $\beta$ at time $s_1^j$.

**Claim 7.10.** Let

\[
h = e^{s_1^j - s}(\beta - \beta_0 + 2(\theta - \theta_{V_i})) - 2(\theta - \theta_{V_i})
\]
on $M'_i$ so that $h(s_1^j) = \beta - \beta_0$. There is some $C > 0$ independent of $i$ such that

\[
\sup_{M'_i \cap B_2} h^2 \leq C \kappa^2 d_i^2 \quad \text{at} \quad s = s_2^j
\]

and hence

\[
\text{osc} \theta \leq C d_i \quad \text{on} \quad M'_i \cap B_2 \setminus B_{1/2}.
\]
Proof. Since we have only defined \( \beta \) on the ball \( B_{R_{C_N}d_i} \), we need to incorporate a cutoff function. In fact even if \( \beta \) were defined globally we would need such a cutoff if we do not assume that \( \beta \) has uniform polynomial growth bounds.

To that end, let \( \chi_0 : [0, \infty) \to \mathbb{R} \) denote a cutoff function with \( \chi_0(t) = 1 \) for \( t < (R_{C_N}d_i - 1)^2 \) and \( \chi_0(t) = 0 \) for \( t > R_{C_N}^2d_i \). We can arrange that \( \chi, \chi', \chi'' \) are uniformly bounded independently of \( i \). Let \( \chi(x) = \chi_0(|x|^2) \). Note that \( \chi^2h^2 \) is then defined globally.

Recall \( \mathcal{L}_0 \) given in (6.1). Along the rescaled flow (for surfaces) we have

\[
(\partial_s - \mathcal{L}_0)|x|^2 = |x^T|^2 - 4
\]

and so

\[
(\partial_s - \mathcal{L}_0)e^{s\chi - \chi^2} |x|^2 \leq -2 \quad \text{for } s \in [s_1, s_2].
\]

We also have

\[
(\partial_s - \mathcal{L}_0)\chi = \chi'(|x|^2)(\partial_s - \mathcal{L}_0)|x|^2 - \chi''(|x|^2)|\nabla |x|^2|^2.
\]

Hence,

\[
(\partial_s - \mathcal{L}_0)\chi \leq C|x|^2
\]

and \((\partial_s - \mathcal{L}_0)\chi\) is supported on the set where \( R_{C_N}d_i - 1 < |x| < R_{C_N}d_i \).

Since \( h \) satisfies the drift heat equation (5.1), we may compute

\[
(\partial_s - \mathcal{L}_0)(\chi^2h^2) = 2\chi^2h^2(\partial_s - \mathcal{L}_0)\chi - 2|\nabla \chi|^2h^2 - \chi^2|h|^2 = 8\chi h \nabla \chi \cdot \nabla h.
\]

We use the inequality \( |8\chi h \nabla \chi \cdot \nabla h| \leq 2\chi^2|h|^2 + 8h^2|\nabla \chi|^2 \) and the estimate (7.18) to get

\[
(\partial_s - \mathcal{L}_0)(\chi^2h^2) \leq \begin{cases} 
C|x|^2h^2 & \text{for } R_{C_N}d_i - 1 < |x| < R_{C_N}d_i, \\
0 & \text{otherwise.}
\end{cases}
\]

By Claim (7.19) there is \( p > p_0 > 1 \) so that we also have the bound

\[
h^2 \leq Cd_i^2e^{R_{C_N}d_i/4p} \quad \text{for } 1 < |x| < R_{Cd_i}.
\]

We now define the function

\[
\Theta = \left(e^{s\chi - \chi^2} - e^{-\sigma(R_{Cd_i} - 1)^2}\right)_+,
\]

where \((\ldots)_+\) means the positive part of the function and \( \sigma \) is the constant in (7.10). Using (7.17) and \( s \in [s_1, s_2] \) we have that

\[
(\partial_s - \mathcal{L}_0)\Theta \leq \begin{cases} 
-2 & \text{when } e^{s\chi - \chi^2} > e^{-\sigma(R_{Cd_i} - 1)^2}, \\
0 & \text{otherwise},
\end{cases}
\]

in the distributional sense. Note that by (7.6) we have \( e^{s\chi - \chi^2} < 1 \) for \( s \in [s_1, s_2] \).

We deduce from (7.19), (7.20) and (7.21) that there is some \( C_1 > 0 \) so that

\[
(\partial_s - \mathcal{L}_0)\left(\chi^2h^2 + Cd_i^2e^{R_{C_N}d_i/4p}\right) \leq 0 \quad \text{for } s \in [s_1, s_2].
\]

At \( s = s_1 \) we have \( \chi^2h^2 = \chi^2(\beta - \beta_0)^2 \), and so using (7.8), (7.12), together with the uniform area ratio bounds for \( M_s^i \), we can ensure that

\[
\int_{M_{s_1}^i} \chi^2h^2 e^{-|x|^2/4} \leq C\kappa^2d_i^4,
\]
once $i$ is large enough. To estimate the integral of $\Theta$ at $s = s_1^i$, note that $\Theta(x) = 0$ if $s = s_1^i$ and $|x| < e^{-\sigma/2}(R_{Cd_d} - 1)$, which holds if $|x| < e^{-\sigma}R_{Cd_d}$ once $i$ is large. For all $x$ we have $\Theta(x) \leq |x|^2$ and so by our previous observation we have

$$
\int_{M_{s_1^i}} \Theta e^{-|x|^2/4} \leq \int_{M_{s_1^i} \setminus B_{e^{-\sigma}R_{Cd_d}}} |x|^2 e^{-|x|^2/4} \leq C e^{-2\sigma} R_{Cd_d}^2 e^{-e^{-2\sigma}R_{Cd_d}/4}.
$$

If $\sigma$ is chosen sufficiently small, so that $e^{2\sigma} < p$ for the $p > 1$ in Claim 7.9, then we will have

$$
(7.24) \quad \int_{M_{s_1^i}} d_i^2 e^{R_{Cd_d}(4\mu)} \Theta e^{-|x|^2/4} \leq C d_i^2 R_{Cd_d}^4 \exp \left( \frac{R_{Cd_d}^2}{4p} - \frac{R_{Cd_d}^2}{4e^{2\sigma}} \right) \leq \kappa^2 d_i^2
$$

for sufficiently large $i$. Combining this with (7.24) and using (7.22), we can apply the monotonicity formula to obtain pointwise estimates for $\chi^2 h^2$ at $s = s_2^i$. Since $s_2^i > \sigma/2$, and $\sigma$ only depends on $p$ in (7.24), we obtain the estimate (7.11) for $h^2$.

At $s = s_2^i$ we have

$$
h = e^{s_1^i - s_2^i} (\beta - \beta_0 + 2(\theta - \theta_{V'}) - 2(\theta - \theta_{V'})) = e^{s_1^i - s_2^i} \beta + 2(e^{s_1^i - s_2^i} - 1) \theta - e^{s_1^i - s_2^i} (\beta_0 + 2\theta_{V'}) + 2\theta_{V'}.
$$

and at this time the oscillation of $\beta$ on $B_2 \setminus B_1/2$ is bounded by $C\kappa d_i$ from (7.10) for $i$ large. Noting (7.9), it follows that the oscillation of $\theta$ is also bounded by $C\kappa d_i$ on this annulus for $i$ sufficiently large as claimed.

To complete the proof, recall the decomposition (7.1) and that on $B_2 \setminus B_1$ we know that $M_{s_2^i}$ is the graph of $u_i$ over $V_i'$, where $d_i^{-1} u_i \to u_{01} + u_\perp$, and so $d_i^{-1}(\theta - \theta_{V'}) \to d^*(u_{01} + u_\perp)$. From (7.11) we have $|d^* u_\perp| \leq C\kappa$ on $B_2 \setminus B_1$, while the value of $d^* u_{01}$ on the two planes differs by at least $C^{-1}$. Therefore the oscillation of $d^*(u_{01} + u_\perp)$ on $B_2 \setminus B_1$ is at least $C^{-1} - C\kappa > C^{-1}/2$ if we choose $\kappa$ sufficiently small. The oscillation of $\theta$ on $B_2 \setminus B_1/2$ is therefore at least $C^{-1} d_i/4$ for large $i$. This contradicts the bound (7.13) if $\kappa$ is sufficiently small.

7.1. **Closeness to planes.** We next show that if condition (iii) in Proposition 7.3 holds, then we can still arrange that the flow remains close to the original pair of planes $V_0$ as long as the change in the excess $A$ is controlled. From now on we fix $N > 0$ and $\alpha \in (0, 1)$ such that Proposition 7.3 applies, and we assume that $N > N_1$ is a fixed integer large enough so that Proposition 6.2 applies to $N^2$ and such that

$$
e^{-N^2} < 1/2.
$$

Again recall that $\lambda_1 > 0$ is given by Proposition 6.2.

**Proposition 7.11.** Let $\delta_1 > 0$. There is an $\epsilon_1 > 0$ depending on $\delta_1$ such that, if we have a flow $M_t$ satisfying Condition [1] for $t \in [-1, T + 10]$, $M_1$ satisfies Condition [4], and

1. $A_\alpha(M_0) - A_\alpha(M_T) < \epsilon_1$,
2. $D^V(M_0) < \epsilon_1$, for some $V \in V'$,
3. $D^V(M_N^2) \geq e^{\epsilon_1 N^2} D^V(M_0),$

then $D^V(M_T) < \delta_1$. 

Proof. Using assumptions (2), (3), if \( \epsilon_1 \) is sufficiently small then we can apply Proposition 6.2 to deduce \( D_\nu(M_{2N^2}) \geq e^{\lambda_i N^2} D_\nu(M_{N^2}) \). We can keep iterating this estimate, to get

\[
D_\nu(M_{iN^2}) \geq e^{\lambda_i N^2} D_\nu(M_{(i-1)N^2})
\]

for \( i = 3, 4, \ldots, k \), where \( k \) is the largest integer which still satisfies

\[
D_\nu(M_{(k-3)N^2}) < \epsilon_0, \quad \text{and} \quad kN^2 < T,
\]

for a constant \( \epsilon_0 \) that is smaller than the \( \epsilon \) in Proposition 6.2.

If \( (k+1)N^2 \geq T \), then Proposition 3.6 implies that \( D_\nu(M_T) \leq C \epsilon_0 \). If \( \epsilon_0 \) is chosen sufficiently small (and \( \epsilon_1 < \epsilon_0 \)), then this implies \( D_\nu(M_T) < \delta_1 \) as required.

We therefore assume that \( (k+1)N^2 < T \) and \( D_\nu(M_{(k-2)N^2}) \geq \epsilon_0 \). We have \( D_\nu(M_{(k-3)N^2}) \leq e^{-\lambda_i N^2} D_\nu(M_{(k-2)N^2}) \) so using (7.26) we have

\[
(7.26) \quad D_\nu(M_{(k-2)N^2}) - D_\nu(M_{(k-3)N^2}) \geq (1 - e^{-\lambda_i N^2}) D_\nu(M_{(k-2)N^2}) \geq \frac{\epsilon_0}{2}.
\]

We claim that (7.26) together with condition (1) and \( D_\nu(M_{(k-3)N^2}) < \epsilon_0 \) leads to a contradiction if \( \epsilon_0 \) is chosen small, and \( \epsilon_1 \) is sufficiently small depending on \( \epsilon_0 \).

Shifting \( \tau = (k-3)N^2 \) to \( \tau = 0 \), we can thus suppose that we have a sequence of flows \( M_i \) satisfying Condition (1) for \( \tau \in [-N^2, N^2] \), satisfying

\[
A_{\alpha}(M_{-N^2}^i) - A_{\alpha}(M_{N^2}^i) < \epsilon_{1,i},
\]

\[
D_\nu(M_{N^2}^i) - D_\nu(M_0^i) \geq \frac{\epsilon_0}{2},
\]

\[
D_\nu(M_{-N^2}^i) < \epsilon_0,
\]

for a sequence \( \epsilon_{1,i} \to 0 \). From this, together with the same argument as in Theorem A, we know that as \( i \to \infty \), along a subsequence, these flows converge to a static flow given by a union of planes. By the bound on \( D_\nu(M_{-N^2}^i) \) these planes must be given by some \( V' \in \mathcal{V} \) if \( \epsilon_0 \) is small enough. In particular for any \( \tau \in [0, N^2] \) the \( M_i^\tau \) converge locally smoothly to \( V' \) on compact sets in \( \mathbb{C}^2 \setminus \{0\} \).

Since \( M_{N^2}^i \) and \( M_0^i \) both converge to \( V' \), for any compact set \( K \subset \mathbb{C}^2 \setminus \{0\} \) we have as \( i \to \infty \):

\[
\int_{M_{N^2}^\tau \cap K} (|x|^2 d\nu + |\theta - \theta_V|^2) e^{-|x|^2/4} - \int_{M_0^\tau \cap K} (|x|^2 d\nu + |\theta - \theta_V|^2) e^{-|x|^2/4} \to 0.
\]

We can use the uniform bound \( D_\nu(M_{-N^2}^i) < \epsilon_0 \) and argue as in the proof of Proposition 6.2 in particular in 6.9, to show \( D_\nu(M_{N^2}^i) - D_\nu(M_0^i) \to 0 \) which gives our required contradiction.

We can now prove our main result controlling the distance of flows close to the union of two transverse planes by combining Propositions 7.3 and 7.11.

**Proposition 7.12.** Let \( \delta_2 > 0 \). There is an \( \epsilon_2 > 0 \) depending on \( \delta_2 \), such that if we have a flow \( M_\tau \) satisfying Condition (1) for \( \tau \in [-1, T + 10] \), \( M_1 \) satisfies Condition (4), and

1. \( A_{\alpha}(M_0) - A_{\alpha}(M_T) < \epsilon_2 \),
2. \( D_{\nu}(M_0) < \epsilon_2 \),

then \( D_{\nu}(M_T) < \delta_2 \).

Note that \( T \) is independent of the constants \( \delta_2, \epsilon_2 \) and, in particular, can be large.
Proof: We iterate the cases (i), (ii) in Proposition 7.3 as long as possible, starting with $V_0$. We obtain a sequence $V_1, \ldots, V_k \in \mathcal{V}$ together with numbers $e_i = D_{V_i}(M_{kN})$ such that $d(V_i, V_{i+1}) \leq C e_i$. We can continue this iteration and define $V_{k+1}$ unless one of the following occurs:

(a) $kN + 3N^2 + 2 > T$,
(b) $V_k \notin \mathcal{V}'$ or $D_{V_k}(M_{kN}) \geq \epsilon_0$,
(c) $D_{V_k}(M_{kN} + N^2) \geq e^{\lambda_1 N^2} D_{V_k}(M_{kN})$.

We show that if $\epsilon_2$ is sufficiently small, then (b) cannot occur before (a) or (c) does. To see this we can argue as in [26, Theorem 6.7] to control the sum of the $e_i$, to find

\begin{equation}
\sum_{i=0}^k e_i \leq 2 \epsilon_0 + 2C(A_0(M_0) - A_0(M_T)) \leq C\epsilon_2.
\end{equation}

In particular this implies that both $e_k$ and $d(V_0, V_k)$ are bounded above by $C\epsilon_2$, so we can ensure that (b) does not occur for $\epsilon_2$ sufficiently small. In addition, using Lemma 5.1 Condition (4) is preserved.

If (a) occurs first, then we have

\begin{equation}
D_{V_k}(M_T) \leq C D_{V_k}(M_{kN}) \leq C\epsilon_2
\end{equation}

by Proposition 3.6 and by Lemma 3.8 we get $D_{V_0}(M_T) \leq C\epsilon_2$. If (c) occurs first, then from Proposition 7.11 we conclude that $D_{V_k}(M_T) < \delta_1$ if we choose $\epsilon_2 < \epsilon_1$, and Lemma 3.8 implies $D_{V_0}(M_T) \leq C(\delta_1 + \epsilon_2)$.

If we choose $\delta_1 > 0$ sufficiently small (determining a value for $\epsilon_1 > 0$), and then choose $\epsilon_2 > 0$ small so that also $\epsilon_2 < \epsilon_1$, then in either case we will have $D_{V_0}(M_T) < \delta_2$ as required.

□

8. Neck pinches

In this section we give the main geometric applications of the estimates we have obtained. We suppose that $\mathcal{M}(t)$ is a rational, zero Maslov Lagrangian mean curvature flow in $\mathbb{C}^2$, with uniformly bounded area ratios and uniformly bounded Lagrangian angle.

8.1. Uniqueness of tangent flows. Our first result is the uniqueness of tangent flows given by a union of transverse planes. Note that by [16, Corollary 4.3] the two planes must have the same Lagrangian angle if a singularity forms. We first have the following, ensuring that Conditions (7) and (8) hold along the corresponding rescaled flow.

Lemma 8.1. Let $\mathcal{M}(t)$ be a mean curvature flow in $\mathbb{C}^2$, with initial condition given by a rational, zero Maslov Lagrangian with uniformly bounded area ratios and uniformly bounded Lagrangian angle. Suppose that the flow develops a singularity at $(0,0)$, with a tangent flow given by the static flow $V_0$, where $V_0$ is a special Lagrangian union of two transverse planes. Let $M_\tau = e^{\tau^2/2} \mathcal{M}(\tau^{-2})$ denote the corresponding rescaled flow. Then there is a sequence $\tau_i \to \infty$ satisfying

1. $D_{V_0}(M_{\tau_i}) \to 0$,
2. $M_{\tau}$ satisfies Condition (7) for $\tau \in [\tau_0, \infty)$,
3. $M_{\tau_{i+1}}$ satisfies Condition (8).
Proof. Note that the uniform bounds on area ratios and the bound for the Lagrangian angle is preserved along the flow. Condition (3) holds on \([\tau_0, \infty)\) for sufficiently large \(\tau_0\) by the monotonicity of \(A(M_\tau)\). The fact that \(D(V_0(M_\tau)) \to 0\) follows from the assumption that one tangent flow at \((0, 0)\) is given by \(V_0\).

It remains to show Condition (2) for \(M_{\tau_{i+1}}\) for large enough \(i\). Let us first consider the connectedness of \(B_1 \cap M_{\tau_{i+1}}\). Note that by the assumption \(D(V_0(M_\tau)) \to 0\) we have that \(M_{\tau_{i+1}}\) has good graphicality over \(V_0\) on \(B_2 \setminus B_1/3\) for large \(i\). For large \(i\) the pointwise bounds in Lemma 8.5 also imply that \(B_2 \cap M_{\tau_{i+1}}\) is almost calibrated. Since there are no compact almost calibrated Lagrangians in \(\mathbb{C}^2\), this implies that \(B_1 \cap M_{\tau_{i+1}}\) has either 1 or 2 connected components. If it has 2 components, then for sufficiently large \(i\) we can argue as in [16, Corollary 4.3] to show that in fact the original flow \(M(t)\) does not have a singularity at \((0, 0)\). Therefore for sufficiently large \(i\), \(B_1 \cap M_{\tau_{i+1}}\) is connected.

Finally consider the exactness part of Condition (2). The rationality assumption is preserved along the flow, see [15, Section 6]. It follows that there is a constant \(a > 0\) such that, after rescaling, for any loop \(\gamma \subset M_{\tau_{i+1}}\) we have

\[
\int_\gamma \lambda \in 2\pi ae^{\tau_{i+1}}\mathbb{Z}.
\]

Although we might only be able to define a multivalued function \(\beta\) satisfying \(d\beta = \lambda\) on \(M_\tau\), (8.1) implies that \(f = \sin(e^{-\tau_{i+1}}a^{-1}\beta)\) is single valued. Without loss of generality we can assume that \(f(x_0) = 0\) for a basepoint \(x_0 \in B_1 \cap M_{\tau_{i+1}}\). We have

\[
\nabla f = e^{-\tau_{i+1}}a^{-1} \cos(e^{-\tau_{i+1}}a^{-1}\beta)\nabla \beta,
\]

so using \(|\nabla \beta| = |x^{\perp}|\) we have \(|\nabla f| \leq e^{-\tau_{i+1}}a^{-1}\) on \(B_1\). For large \(i\), \(B_2 \cap M_{\tau_{i+1}}\) is almost calibrated, and so as in the proof of [15, Lemma 7.2], we have a uniform lower bound \(H^2(B_1(x_1, 1)) > K\) for the intrinsic unit balls in \(M_{\tau_{i+1}}\) centred at any \(x_1 \in B_1 \cap M_{\tau_{i+1}}\). We also have an upper bound for the area of \(B_1 \cap M_{\tau_{i+1}}\), using the bound for the area ratios. Together with the connectedness this implies that there is a uniform constant \(C_1\) such that for any \(x_1 \in B_1 \cap M_{\tau_{i+1}}\), the points \(x_0, x_1\) can be connected by a curve in \(B_1 \cap M_{\tau_{i+1}}\) of length at most \(C_1\), for large \(i\). The gradient bound for \(f\), together with \(f(x_0) = 0\), then implies that \(|f| \leq C_1 e^{-\tau_{i+1}}a^{-1}\) on \(B_1 \cap M_{\tau_{i+1}}\). For sufficiently large \(i\) we can then define a single-valued function \(\beta\) on \(B_1 \cap M_{\tau_{i+1}}\) satisfying \(d\beta = \lambda\), so \(\int_\gamma \lambda = 0\) for any loop \(\gamma \subset B_1 \cap M_{\tau_{i+1}}\).

Theorem 8.2. Suppose that \(M(t)\) satisfies the same assumptions as in Lemma 8.4. Then all tangent flows at \((0, 0)\) are given by \(V_0\).

Proof. Let \(M_\tau\) be the rescaled flow around \((0, 0)\). Using Lemma 8.1 we have that Condition (1) holds on a time interval of the form \([\tau_0, \infty)\), and in addition we have a sequence \(\tau_i \to \infty\) such that \(D(V_0(M_\tau)) \to 0\) and \(M_{\tau_{i+1}}\) satisfies Condition (2).

The uniqueness of the tangent flow then follows directly from Proposition 7.12.

Indeed, from the monotonicity of \(A_0\) from Lemma 7.2 we also have

\[
A_0(M_\tau) - \lim_{\tau \to \infty} A_0(M_\tau) \to 0 \quad \text{as} \quad i \to \infty.
\]

It follows from Proposition 7.12 that

\[
\lim_{i \to \infty} \sup_{\tau > \tau_i} D(V_0(M_\tau)) = 0,
\]

which implies that \(M_\tau \to V_0\) locally smoothly on compact sets in \(\mathbb{C}^2 \setminus \{0\}\) as \(\tau \to \infty\). \(\square\)
8.2. Lawlor necks. We now show that tangent flows given by a special Lagrangian union of two transverse planes can only form if for sufficiently small times before the singularity the flow looks locally like the two transverse planes, desingularised by a shrinking Lawlor neck. Recall that, given a special Lagrangian pair of transverse planes \( V_0 \) in \( \mathbb{C}^2 \), up to scale there are two (exact) Lawlor necks \( N_\pm \) asymptotic to these planes (corresponding to \( zw = \pm 1 \) in suitable complex coordinates \( (z, w) \) under hyperkähler rotation). Using ideas of Seidel (cf. [21]), one can see that \( N_\pm \) are not Hamiltonian isotopic (using compactly supported isotopies), but we shall not use this fact in our result below.

**Theorem 8.3.** Under the hypotheses of Theorem 8.2, for every \( \varepsilon > 0 \) there is \( r_0 > 0 \) and a smooth function \( r : [-r_0^2, 0) \to (0, r_0) \) with \( r(t) \to 0 \) as \( t \to 0 \) and points \( x_0(t) \to 0 \) such that \( M(t) \cap (B_{r_0}(0) \setminus B_{r(t)}(x_0(t))) \) is a \( C^1 \)-graph over \( V_0 \) with \( C^1 \)-norm bounded by \( \varepsilon \). Furthermore,

\[
(8.2) \quad r(t)^{-1}(M(t) - x_0(t))
\]

converge locally smoothly on \( \mathbb{C}^2 \) to a unique choice of Lawlor neck (either \( N_+ \) or \( N_- \)) asymptotic to \( V_0 \) at infinity of maximal neck size such that, outside of \( B_1(0) \), \( N \) can be written as a \( C^1 \)-graph over \( V_0 \) with \( C^1 \)-norm bounded by \( \varepsilon \).

To prove this result, we suppose \( M \) satisfies the hypotheses of Theorem 8.2 and consider basepoints \( X = (x, t) \in \mathbb{C}^2 \times [-1, 0] \) close to \((0, 0)\) at which to discuss closeness of \( M \) to some \( V \in \mathcal{V} \).

**Definition 8.4.** Let \( \varepsilon_0 \in (0, 1) \), \( \varepsilon \in (0, \varepsilon_0) \) and \( V \in \mathcal{V} \). We say that \( M \) is \( \varepsilon \)-close to \( V \) at \( X = (x, t) \) if the flow \( M' = M - X \) is a Lagrangian \( C^1 \)-graph with \( C^1 \)-norm bounded by \( \varepsilon \) over \( V \) on \( (B_{r_0}(0) \setminus B_{r}(0)) \times [-\varepsilon^2, -\varepsilon^2] \). We assume \( \varepsilon_0 \) is chosen sufficiently small (depending on \( \mathcal{V} \)) such that \( M' \cap ((B_{r_0}(0) \setminus B_{r}(0)) \times [-\varepsilon^2, -\varepsilon^2]) \) is the union of two disjoint embedded annuli.

**Remark 8.5.** We will assume further that \( \varepsilon_0 > 0 \) is sufficiently small such that pseudolocality [11] implies that for every \( \delta > 0 \) there is \( C_\delta \gg 1 \) and \( 0 < \varepsilon \leq \varepsilon_0 \) such that if \( M(t_0) \cap B_{C_\delta}(x) \) is a \( C^1 \)-graph over a 2-plane \( P \) with \( C^1 \)-norm bounded by \( \varepsilon \), then \( M(t) \cap B_1(x) \) is a \( C^1 \)-graph over \( P \) with \( C^1 \)-norm bounded by \( \delta \) for \( t \in [t_0, t_0 + 1] \cap [-1, 0] \).

Next we identify the range of scales at which the flow is close to some \( V \in \mathcal{V} \). We fix a small \( \varepsilon \in (0, \varepsilon_0) \).

**Definition 8.6.** Suppose that \( M \) is \( \varepsilon \)-close at \( X \) to some \( V \in \mathcal{V} \). We define \( \lambda_{\min}(X), \lambda_{\max}(X) \) to be the endpoints of the maximal interval

\[
1 \in (\lambda_{\min}(X), \lambda_{\max}(X)) \subseteq (0, \infty)
\]

such that for all \( \lambda \in (\lambda_{\min}(X), \lambda_{\max}(X)) \) we have that \( D_{\lambda}(M - X) \) is \( \varepsilon \)-close at \((0, 0)\) to \( V_\lambda \) for some \( V_\lambda \in \mathcal{V} \).

Note that \( \lambda_{\min}(X), \lambda_{\max}(X) \) are continuous in the base-point \( X \).

**Remark 8.7.** Since we can assume that all tangent flows of \( M \) at \((0, 0)\) are \( M_{V_0} \), for any sequence \( \lambda_i \searrow 0 \) we have

\[
(8.3) \quad M_i := D_{\lambda_i}(M) \to M_{V_0}.
\]

Note that along the sequence \( M_i \) this implies that for all points \( X = (x, t) \) sufficiently close to \((0) \times (-\infty, 0) \) we have that \( \lambda_{\min}(X) \to 0 \) and \( \lambda_{\max}(X) \to \infty \).
Lemma 8.8. Assume that for some \( X = (x, t) \) with \( t < 0 \) we have along \( M_i \) in \( \text{8.3} \) that \( \lambda_{\min}(X) = 0 \). Then, locally around \( X \), the flow \( M \) is the smooth flow of two immersed planes.

Proof. We first note that Remark \( \text{8.3} \) implies that we have smooth control on the flow forward in time in \( B_{\lambda_c}^t(x) \setminus B_{\lambda_c(t)\epsilon}(x) \) up to time \( t \) as one goes down with the scale \( \lambda \) from 1 down to \( \lambda_{\min} \), i.e., in \( B_{\lambda_c}^t(x) \setminus B_{\lambda_{\min}(C_t)}(x) \). So if \( \lambda_{\min}(X) = 0 \) we have that \( M_i(t) \) is locally around \( x \) the union of two smooth embedded flows, where each is a small \( C^1 \)-graph over the Lagrangian planes \( P_1 \) and \( P_2 \) respectively (for \( V_0 = P_1 \cup P_2 \)). Since the flow \( M_i \) is smooth, this has to be true forwards in time until \( t = 0 \), and thus there is no singularity at \((0,0)\) for \( M_i \) and thus for \( M \).

Given Lemma \( \text{8.8} \) we can thus always assume that \( \lambda_{\min}(X) > 0 \). We now argue that there is more or less a ‘unique’ point where \( \lambda_{\min} \) is minimised.

Lemma 8.9. For \( M_i \) as in \( \text{8.3} \) consider points \( X_i(t) = (x_i(t), t) \) which minimise \( \lambda_{\min} \) relative to other points \( X = (x, t) \). Then
\[
\lambda_{\min}(x, t) > \lambda_{\min}(X_i(t)) > 0
\]
for \( (x, t) \in \{ B_{\lambda_{\min}(X_i)(C_t)} (x_i(t)) \setminus B_{\lambda_{\min}(X_i)(C_t+\epsilon)} (x_i(t)) \} \times \{ t \} \), where \( C = C_\delta \) for a suitable \( \delta > 0 \) in Remark \( \text{8.2} \).

Proof. This follows by a similar argument to the proof of Lemma \( \text{8.8} \) since Remark \( \text{8.3} \) implies that we have smooth control on the flow forward in time in \( B_{\lambda_c}^t(x) \setminus B_{\lambda_{\min}(X_i)(C_t)}(x) \) up to time \( t \) as one goes down with the scale \( \lambda \) from \( \lambda_{\max} \) down to \( \lambda_{\min} \). □

Lemma 8.10. There is \( 0 < \epsilon_1 \leq \epsilon_0 \) such that for \( 0 < \epsilon \leq \epsilon_1 \) the following holds. If there is \( \lambda_0 \in (\lambda_{\min}(X), \lambda_{\max}(X)) \) such that we can choose \( V_{\lambda_0} = V_0 \), then \( V_{\lambda} \in V' \) for all \( \lambda \in (\lambda_{\min}(X), \lambda_0) \).

Proof. We consider the rescaled flow \( \hat{M}_{\lambda_0} \) for \( \hat{M} := D_{\lambda_0}^{-1} (M - X) \) and choose \( \delta_2 > 0 \) and \( \epsilon_0 > 0 \) small such that \( \hat{M}_{\lambda_0}^{-\log(\lambda) + 2\log(\lambda_0)}(X) \) for \( \lambda \in (\lambda_{\min}(X), \lambda_0) \) being \( \epsilon \)-close to \( V_{\lambda} \) for \( 0 < \epsilon \leq \epsilon_0 \) and \( D_{V_0}(\hat{M}_{\lambda_0}^{-\log(\lambda) + 2\log(\lambda_0)}(X)) \leq \delta_2 \) implies that \( V_{\lambda} \in V' \). This fixes \( \epsilon_2 > 0 \) in Proposition \( \text{8.12} \). We can then choose \( \epsilon_1 > 0 \) sufficiently small such that
\[
\hat{M}_{\lambda_0}, \hat{M}_{\lambda_0}^{-\log(\lambda_{\min}(X)) - 2\log(\lambda_0)}(X) \text{ is } \epsilon \text{-close to } V_0 \text{ for } \lambda 
\]
being \( \epsilon \)-close to \( V_0 \) and \( V_{\lambda_{\min}(X)} \) respectively implies that condition (1) in Proposition \( \text{8.12} \) is met. Applying Proposition \( \text{8.12} \) yields the statement. □

8.3. Finding Lawneck necks. We consider \( 0 < \epsilon \leq \epsilon_1 \), where \( \epsilon_1 \) as in Lemma \( \text{8.10} \). Consider a sequence \( \lambda_i \downarrow 0 \) and let \( M_i \) be as in \( \text{8.3} \). We fix \( t < 0 \) and let \( X_i(t) \) be as in Lemma \( \text{8.3} \). We consider the flows
\[
(8.4) \quad M_{i,t} := D_{\lambda_{\min}(X_i(t))}^{-1} (M_i - X_i(t)).
\]
We can assume that \( M_{i,t} \to \hat{M} \), where \( \hat{M} \) is an ancient unit-regular Brakke flow such that \( D_{\lambda_i}^{-1}\hat{M} \) is \( \epsilon \)-close to some \( V_{\lambda} \in V' \) for \( \lambda \in [1, \infty) \), but not for a sequence \( \lambda_i \not\to 1 \). Furthermore, \( \hat{M} \) is (locally) the limit of smooth, exact, almost calibrated
Lagrangian mean curvature flows with uniformly bounded Lagrangian angle and uniformly bounded area ratios.

**Lemma 8.11.** The flow $\hat{M}$ is a static Lawlor neck $N$ asymptotic to $V' \in V'$, where $V'$ is $\varepsilon$-close to $V_0$, where the centre and the scale of $N$ are such that there is no point $X = (x, t)$ with $\lambda_{\min}(X) < 1$.

**Proof.** We first note that $\hat{M}$ has entropy bounded by two. Furthermore, from the argument in the proof of Lemma 8.8 we see that outside of $(B_{C_3+1}(0) \times (-\infty, 0)) \cup \bigcup_{t<0} B_{\varepsilon/\sqrt{t}}(C_3+1)(0) \times \{t\}$ the flow is a smooth Lagrangian which is a controlled $C^1$-graph over $V_0$. Let $\hat{M}'$ be a tangent flow of $\hat{M}$ at $-\infty$. The discussion before implies that $D_{\lambda}\hat{M}'$ is $\varepsilon$-close to some $V_\lambda' \in V'$ for all $\lambda > 0$ as well as that $\hat{M}'$ is a smooth Lagrangian and controlled $C^1$-graph over $V_0$ outside of $\bigcup_{t<0} B_{\varepsilon/\sqrt{t}}(0) \times \{t\}$. Furthermore, the proof of [18, Theorem 3.1], see also [13, Theorem 3.1], directly extends to Brakke flows which are limits of smooth Lagrangian mean curvature flows with uniformly bounded Lagrangian angle and uniformly bounded area ratios. Since $\lambda_{\min}(X)$ cannot be zero, Lemma 8.11 follows. □

**Proof of Theorem 8.3.** Consider $0 < \varepsilon \leq \varepsilon_1$, where $\varepsilon_1$ as in Lemma 8.10. Replacing $M$ by $D_{\lambda_{-1}}M$ for some sufficiently small $\lambda > 0$ we can assume that for all $t \in [-1, 0)$ there are points $X(t)$ as in Lemma 8.2 such that $\lambda_{\min}(X(t)) < 1 \leq \lambda_{\max}(X(t))$, minimising $\lambda_{\min}$ in their time slice, and with $V_\lambda = V_0$ for $\lambda = 1$. Lemma 8.10 then implies that for each $t \in (0, 1]$ and $\lambda \in [\lambda_{\min}(X(t)), 1]$ we have that $D_{\lambda_{-1}}(M - X(t))$ is $\varepsilon$-close to $V_\lambda' \in V'$. Applying Remark 8.3 we see that $\lambda_{\min}(X(t)) < 1 \leq \lambda_{\max}(X(t))$, the rescaled flow $D_{\lambda_{\min}(X(t))^{-1}}(M - X(t))$ has to be a small $C^1$-graph over a Lawlor neck $N$ over a large compact set. Note that by continuity of the flow (and assuming sufficient $C^1$-closeness to $N$) this has to be either $N_+$ or $N_-$ for all $-\delta \leq t < 0$. The full convergence to $N$ as $t \nearrow 0$ then follows by considering a sequence $\varepsilon_i \to 0$. □

8.4. **Continuing the flow past the singularity.** We can now argue that the uniqueness of the tangent flow implies that at the singular time $t = 0$, in a neighbourhood of $0 \in \mathbb{C}^2$, the flow limits to the union of two Lagrangian graphs such that we can restart the flow as a Lagrangian mean curvature flow.

**Lemma 8.12.** There is $r_0 > 0$ such that on $B_{r_0}(0) \setminus \{0\}$ the flow $\mathcal{M}(t)$ converges as $t \to 0$ locally to two Lagrangian graphs $L_1, L_2$ over $P_1$ and $P_2$ respectively (where
$V_0 = P_1 \cup P_2$. Moreover, using Lagrangian neighbourhoods for $P_i$, we have that $L_i = \text{graph}_{P_i}(d f_i)$ for $i = 1, 2$, where $f_i \in C^2(P_i \cap B_{\epsilon_0}(0))$ is smooth away from 0 with $f_i(0) = d f_i(0) = \nabla_{P_i}(d f_i)(0) = 0$.

Proof. Note that Theorem 8.2 implies that there is $0 < \lambda_0 \leq 1$ and a continuous increasing function $\varepsilon : (0, \lambda_0] \to (0, 1)$ with $\varepsilon(\lambda) \to 0$ as $\lambda \to 0$ such that $D_{\lambda_0} M$ is $\varepsilon(\lambda)$-close to $V_0$. Combining this with Remark 8.3 as in the proof of Theorem 8.3 then yields that there is a smooth Lagrangian limiting surface $L_0$ on $B_{\epsilon_0}(0) \setminus \{0\}$ such that $\lambda^{-1} L$ is $\delta(\lambda)$-close to $V_0$ for some continuous decreasing function $\delta : (0, \lambda_0] \to (0, 1)$ with $\delta(\lambda) \to 0$ as $\lambda \to 0$. This implies the claimed convergence and that we can decompose $L = L_1 \cup L_2$ such that each $L_i$ is a small $C^1$-graph over $P_i \cap B_{\epsilon_0}(0)$ of a vector-valued function $u_i$, which is smooth away from 0 and $C^1$ across 0 with $u_i(0) = \nabla_{P_i} u_i(0) = 0$. Applying the Lagrangian neighbourhood theorem to each $P_i$, we see that $u_i$ may be identified with a closed 1-form on $P_i \cap B_{\epsilon_0}(0)$, which is then necessarily exact. The Poincaré lemma then gives a $C^2$-function $f_i$ with $d f_i = u_i$ as claimed.

**Proof of Theorem 8.1.** By Lemma 8.12 there can be only finitely many singularities at time $T$ where a tangent flow is a static union of two multiplicity one transverse planes. For simplicity of notation we can thus assume as before that there is one singularity and by shifting space and time that it occurs at $(0,0)$. Using Lemma 8.12 we see that $\mathcal{M}(t)$ converges as $t \nearrow 0$ to a $C^1$-immersed Lagrangian $L$, where the convergence (and $L$) is smooth away from 0, which is zero Maslov and rational. Furthermore, $L \cap B_{\epsilon_0}(0)$ is given as the union of two Lagrangian graphs as stated in Lemma 8.12.

We can use the decomposition given in Lemma 8.12 to approximate $L$ by smooth, zero Maslov, rational Lagrangians $L^i$ in $C^1$ (by approximating the $C^2$-functions giving $L$ as a graph by smooth functions). Furthermore, we can assume by the estimates of [29] that there is $T > 0$ and smooth, zero Maslov, rational solutions to Lagrangian mean curvature flow $(L^i_t)_{t \in [0,T]}$ with $L^i_0 = L^i$. By the $C^1$-convergence of $L^i_t \to L$ and interior estimates for higher codimension mean curvature flow (see [29] or [1, Appendix]), we see that the flows are uniformly controlled in $C^\infty$ for $t > 0$. Note that we can assume that the convergence $L^i_t \to L$ is smooth away from the singular points. Taking the limit we thus see that there is a smooth, zero Maslov, rational Lagrangian mean curvature flow $(L_t)_{t \in (0,T]}$ with $L_t \to L$ in $C^1$ (and smoothly away from 0) as $t \searrow 0$. This implies that the extended flow is smooth through the singular time, away from the singular points. Note that for $\varepsilon > 0$ there is $\delta > 0$ such that

$$\sup_{x \in L_t \cap B_{\delta}} |\theta(x, t) - \theta_0| \leq \varepsilon,$$

where $\theta_0$ is the Lagrangian angle of the special Lagrangian cone $V_0$. Thus the grading $\theta$ for the extended flow can be chosen that it is smooth as well through the singular time, away from the singular points. \qed

9. The flow in a compact ambient space

In this section we consider the Lagrangian mean curvature flow in a compact Calabi–Yau surface and we briefly explain the modifications needed to prove the results from Section 8 in this setting.
Let us suppose that $X$ is a compact complex surface with complex structure $J$, admitting a non-vanishing holomorphic volume form $\Omega$ and a Kähler metric $\omega$ with volume form $\frac{1}{2} \omega^2 = \Omega \wedge \overline{\Omega}$. Let $L \subset X$ be a Lagrangian submanifold. We say that $L$ is zero Maslov if there is a Lagrangian angle function $\theta : L \to \mathbb{R}$ satisfying

$$\Omega|_L = e^{i\theta} dA_L$$

in terms of the Riemannian area form $dA_L$ of $L$. Following the notion of rationality in Fukaya [9, Definition 2.2] we assume furthermore that $[\omega]$ defines an integral cohomology class in $H^2(X; \mathbb{R})$, and let $\xi$ denote a complex line bundle over $X$ together with a unitary connection $\nabla^\xi$ with curvature form $F_{\nabla^\xi} = 2\pi i \omega$. The connection $\nabla^\xi$ is then flat when restricted to $L$.

**Definition 9.1.** The Lagrangian $L$ in $X$ is rational if the holonomy group of $\nabla^\xi$ on $L$ is a finite subgroup of $U(1)$.

We will follow the approach of White [30, Section 4], viewing the mean curvature flow in $X$ as a mean curvature flow in a larger Euclidean space with an additional forcing term. More precisely, let $X \subset \mathbb{R}^N$ be an isometric embedding. The mean curvature flow $L_t$ in $X$ is equivalent to the flow

$$\frac{\partial}{\partial t} x = H + \nu,$$

where $H$ denotes the mean curvature vector inside $\mathbb{R}^N$ and $\nu(x, t) = -\text{tr} \Pi(x)|_{T_x L_t}$ is defined in terms of the second fundamental form $\Pi$ of $X$, restricted to the tangent space of $L_t$ at $x$. In particular $|\nu| \leq C$ for a constant independent of $x, t$. Note that $H + \nu$ is simply the mean curvature of $L_t$ in the ambient space $X$, and so we have $|\nabla \theta| = |H + \nu|$.

We need to recall the form of the monotonicity formula for the mean curvature flow with forcing term $\nu$: instead of (2.2) we have

$$\frac{d}{dt} \int_{L_t} f \rho_{x_0, t_0} dH^2 = \int_{L_t} (\partial_t f - \Delta f) \rho_{x_0, t_0} dH^2 + \int_{L_t} f \left| \nu \right|^2 \rho_{x_0, t_0} dH^2 - \int_{L_t} f \left| H - \frac{(x - x_0)^\perp}{2(t_0 - t)} \right|^2 \rho_{x_0, t_0} dH^2.$$

The estimate of Ecker [7, Theorem 3.4] also applies to subsolutions of the (drift) heat equation along the rescaled flow with a forcing term, with a slightly modified function $p(t)$ as in [7, Theorem 3.2].

For simplicity we assume the first singular time is at $t = 0$ and we are studying the tangent flow at $0 \in \mathbb{R}^N$. The corresponding rescaled flow $M_\tau$ is given by

$$\frac{\partial}{\partial \tau} x = H + \frac{x^\perp}{2} + \nu.$$

Note that $M_{\tau} \subset e^{\tau/2} X \subset \mathbb{R}^N$, and the forcing term $\nu$ is now given by $\nu(x, t) = -\text{tr} \Pi(x, t)|_{T_x M_\tau}$, where $\Pi(x, t)$ is the second fundamental form of $e^{\tau/2} X$. In particular we have the estimate $|\nu| \leq C e^{-\tau/2}$.

Let us write $J_0, \Omega_0, \omega_0$ for the complex structure, holomorphic volume form and symplectic form on $T_0 X$. In particular we identify $T_0 X = \mathbb{C}^2$, equipped with its standard structures. Note that $\lim_{\tau \to \infty} e^{\tau/2} X = T_0 X$ in the sense of $C^\infty$ convergence on compact sets, and so any tangent flow at $(0, 0)$, defined as a sequential limit of $M_\tau$ as $\tau \to \infty$, lives naturally in $T_0 X$. As in Section 2.3 we consider special
Lagrangian unions $V = P_1 \cup P_2$ contained in a neighbourhood $\mathcal{V}$ of a given tangent flow $V_0$. For any $V \in \mathcal{V}$ there is a hyperkähler rotation of the complex structure on $T_0 X$ such that $V$ is given by $\{ z w = 0 \}$ for complex coordinates $z, w$. Note that $z, w$ can be viewed as linear functions on $\mathbb{R}^N$, and in particular they define functions on $X \subset \mathbb{R}^N$. We have the following analogous result to Lemma 3.2

**Lemma 9.2.** There is a constant $C > 0$, depending on $X \subset \mathbb{R}^N$ and the choice of $\mathcal{V}$, such that on any Lagrangian subspace $P \subset T_x X$ with Lagrangian angle $\theta$ we have

$$|\nabla z \cdot \nabla w| \leq C (|\theta - \theta_P| + |x|).$$

**Proof.** In a small neighbourhood $U$ over $0 \in X$ we can use a Darboux chart to define a smooth projection map $\pi : U \to T_0 X$, such that $\pi^* \omega_0 = \omega$ and the derivative of $\pi$ is the identity map at $0$. It follows that the complex structures and holomorphic volume forms satisfy $|\pi^* J_0 - J|, |\pi^* \Omega_0 - \Omega| \leq C|x|$ for a constant $C > 0$. Similarly $|\pi^* z - z|, |\pi^* w - w| \leq C|x|$, and the derivatives of $z, w$ satisfy the same bounds. The required estimate on the neighbourhood $U$ then follows by applying Lemma 3.2 to the image $\pi(P) \subset T_0 X$. The estimate is clear outside of $U$ since $|x|$ is bounded away from zero on $X \setminus U$. \hfill \square

We define the excess $A(M_\tau)$ for a Lagrangian in $e^{\tau/2}X \subset \mathbb{R}^N$ as in (2.1), and Condition (1) is as before. Note that the condition of uniformly bounded area ratios is automatic in the compact case using the monotonicity formula, and the uniform bound for the Lagrangian angle is also preserved by the maximum principle. Let us record here the following consequence of the monotonicity formula, analogous to (5.13), recalling that $|\nabla \theta| = |H + \nu|:

$$A(M_{\tau_1}) - A(M_{\tau_2}) \geq \int_{\tau_1}^{\tau_2} \int_{M_\tau} \left( 2|H + \nu|^2 + |H + \frac{\nu}{2} + \frac{\nu'}{2}|^2 \right) e^{-|x|^4/2} \, dH^2 \, d\tau$$

$$- C \int_{\tau_1}^{\tau_2} \int_{M_\tau} |\nu|^2 e^{-|x|^2/4} \, dH^2 \, d\tau$$

$$\geq \frac{1}{16} \int_{\tau_1}^{\tau_2} \int_{M_\tau} (|H|^2 + |\nu|^2) \, dH^2 \, d\tau - C e^{-\tau_1},$$

for $\tau_1 < \tau_2$, where we also used the uniform bound on the Lagrangian angle.

We define $I_\nu(M_\tau)$ and $D_\nu(M_\tau)$ according to Definition 3.4. Note that the function $|x|d_\nu$ on $M_\tau$ is no longer uniformly equivalent to $|zw|$. In fact if we denote the orthogonal projection of $x$ onto $T_0 X$ by $\tilde{x}$, then $|zw(x)|$ is uniformly equivalent to $|\tilde{x}|d_\nu(x)$. At the same time, since on $X$ we have $|x - \tilde{x}| \leq C|x|^2$, it follows that on $M_\tau \subset e^{\tau/2}X$ we have

$$|x - \tilde{x}| \leq C e^{-\tau/2} |x|^2.$$

It follows from this, together with the bounds on the area ratios of $M_\tau$, that

$$C^{-1} \int_{M_\tau} |zw|^2 e^{-|x|^2/4} \leq \int_{M_\tau} |x|^2 d_\nu^2 e^{-|x|^2/4} \leq C \left( \int_{M_\tau} |zw|^2 e^{-x^2/4 + e^{-\tau/2}} \right).$$

In particular as long as $I_\nu(M_\tau) \geq e^{-\tau/2}$, we have that $I_\nu(M_\tau)$ is uniformly equivalent to the Gaussian $L^2$-norm of $|zw| + |\theta - \theta_P|$. We have the following.

**Lemma 9.3.** Suppose that $V \in \mathcal{V}$ and $I_\nu(M_\tau) \geq e^{-\tau/2}$. Then the conclusions (1), (2), (3) of Lemma 3.4 hold, with $K_\nu \subset \mathbb{R}^N \setminus \{0\}$ in (3).
Proof. In the proof of Lemma 3.5 we essentially applied the monotonicity formula to the function $f = |zw| + |\theta - \theta_V|$. Here we claim that the same argument works if instead we use the function

$$f = |zw| + |\theta - \theta_V| + e^{-r/2}|x| + e^{-r}.$$ 

Here we are thinking of $\tau$ as being fixed. Consider the solution of the mean curvature flow with forcing term as above with $L_{-1} = M_\tau$. Then using Lemma 5.2 and the bound $|\nu| \leq C e^{-\tau/2}$ for the forcing term, implies that

$$(\partial_t - \Delta)|zw| \leq C(|\theta - \theta_V| + e^{-\tau/2}|x|)$$

in the distributional sense. We also have

$$(\partial_t - \Delta)|\theta - \theta_V| \leq 0,$$

and combining these we find that $(\partial_t - \Delta)f \leq Cf$ for a constant $C > 0$. At the same time if $I_V(M_\tau) \geq e^{-\tau/2}$, then $I_V(M_\tau)$ is uniformly equivalent to the Gaussian $L^2$-norm of $f$. Using this the arguments in the proof of Lemma 3.5 can be followed verbatim.

Note that if $D_V(M_\tau) = d \geq e^{-\tau/2}$ and $\tau$ is large, then the region $|x| < R_d$ (with $R_d$ defined in (1.8)), where we expect good graphicality, is much smaller than the ball over which $e^{\tau/2}X$ is graphical over the tangent space $T_0X$. More precisely, we have the following under a stronger assumption for $D_V(M_\tau)$.

**Lemma 9.4.** Suppose that $D_V(M_\tau) = d \geq e^{-\tau/20}$ and $\tau$ is sufficiently large. Then on the region $|x| < R_d$, we can view $e^{\tau/2}X$ as the graph of $v$ over $T_0X$, where

$$(9.1) \quad |v|, |\nabla v| \leq C e^{-\tau/2}R_d = C e^{-\tau/2} |\ln d| \leq Cd^\theta.$$

Proof. Note that in a neighbourhood of $0$, $X$ is the graph of a function $V$ over $T_0X$ with $|V| \leq C|x|^2$ and $|\nabla V| \leq C|x|$. Rescaling this, and using that $R_d \ll e^{\tau/2}$ for large $\tau$, we find that on the ball $|x| < R_d$, $e^{\tau/2}X$ is the graph of $v$ over $T_0X$ such that $|v|, |\nabla v| \leq C e^{-\tau/2}R_d^2$. The required estimate follows from this.

Using this, the results following Lemma 3.5 hold in the present context, with minor adjustments of the proofs, as long as we always ensure that $D_V(M_\tau) \geq e^{-\tau/20}$, and that $\tau$ is sufficiently large. In Condition (4), to make sense of the condition $\int_\gamma \lambda = 0$ for loops $\gamma \in M_\tau \cap B_1$, we use a Darboux chart for $\omega$ on $B_1 \cap e^{\tau/2}X$ to define the Liouville form $\lambda$. Assuming $\tau$ is large, such a chart exists as in the proof of Lemma 9.2 above. Note that in a Darboux chart the holonomy of $\nabla^g$ around a loop $\gamma$ is given by

$$e^{-2\pi i \int_0^s \omega} = e^{-2\pi i \int_0^s \lambda},$$

where $\gamma = \partial D$. It follows that the rationality condition in Definition 9.1 coincides with the rationality condition in Definition 2.2 restricted to loops contained in the chart.

Proposition 9.5 takes the following form.

**Proposition 9.5.** Let $\delta_1 > 0$. There is an $\epsilon_2 > 0$, depending on $\delta_2$, such that if the flow $M_\tau$ satisfies Condition (4) for $\tau \in [\tau_0 - 1, \tau_0 + T + 10]$ with $T > 0$, $M_{\tau_0}$ satisfies Condition (4), $\tau_0$ is sufficiently large (depending on $\delta_2$), and
\[ A_\alpha(M_{\tau_0}) - A_\alpha(M_{\tau_0 + T}) < \epsilon_2, \]
\[ D_{V_0}(M_{\tau_0}) < \epsilon_2, \]
then \[ D_{V_0}(M_{\tau_0 + T}) < \delta_2. \]

Using this, the proofs of Theorems 1.1 and 1.2 from the Introduction follow the same arguments as the corresponding results in Section 8. Let us finally consider Theorem 1.3.

**Proof of Theorem 1.3.** Using Theorem 8.3, which also holds in the current setting, we know that we have a sequence \((x_k, t_k) \to (0, 0)\) and \(r_k \to 0\) such that the rescalings \(\tilde{\mathcal{L}}_k = r_k^{-1}(L_{t_k} - x_k)\) converge, smoothly on compact sets, to a Lawlor neck \(\tilde{\mathcal{L}}_\infty\) asymptotic to \(V_0\) at infinity. Here \(V_0\) is the (unique) tangent flow at the singularity \((0, 0)\). Note that by Condition \((\ast)\), this Lawlor neck is exact, and so up to scale there are only two possibilities. Changing the scales \(r_k\) if necessary, we can assume that in terms of suitable coordinates \(z, w\) for which \(V_0 = \{zw = 0\}\), we have \(\tilde{\mathcal{L}}_\infty = \{zw = \pm 1\}\). For sufficiently large \(k\), we can remove a large ball \(B_R\) from \(\tilde{\mathcal{L}}_k\), and replace it with a small Lagrangian perturbation of the two planes given by \(V_0\). Hence \(\tilde{\mathcal{L}}_k\) is an immersed Lagrangian where at the self-intersection point at the origin two sheets of the Lagrangian are intersecting transversely. This is exactly the reverse of the graded connected sum construction in [28, Section 3.1] (potentially taking the Lagrangians to be two sheets of the same Lagrangian in the connected sum). Hence \(\tilde{\mathcal{L}}_k\) is a graded self-connected sum and so is \(L_{t_k}\) as desired.

Suppose from now on that \(\tilde{\mathcal{L}}_k\) is not connected. Then we can write \(\tilde{\mathcal{L}}_k = \tilde{\mathcal{L}}_{1,k} \# \tilde{\mathcal{L}}_{2,k}\) as a graded connected sum. The choice of which component is \(\tilde{\mathcal{L}}_{1,k}\), respectively \(\tilde{\mathcal{L}}_{2,k}\), depends on which of the two possible Lawlor necks \(\tilde{\mathcal{L}}_\infty\) is. Note that the Lagrangian angles of \(\tilde{\mathcal{L}}_{1,k}, \tilde{\mathcal{L}}_{2,k}\) approach the constant \(\theta_{V_0}\) on the ball \(B_R\) as we let \(R, k \to \infty\) in the construction.

The upshot of this discussion is that after scaling back to the original flow, we can write \(L_{t_k} = M_{1,k} \# M_{2,k}\) as claimed. It now remains for us to show that \((1.1)\) holds and, if \(L\) is almost calibrated, that \((1.2)\) holds. Unless the initial Lagrangian \(L\) is special Lagrangian, in which case no singularity would form, we will have:

\[
\inf_{L} \theta < \inf_{M_{1,k}} \theta, \inf_{M_{2,k}} \theta; \]
\[
\sup_{L} \theta > \sup_{M_{1,k}} \theta, \sup_{M_{2,k}} \theta; \]
\[
\text{vol}(L) > \text{vol}(M_{1,k}) + \text{vol}(M_{2,k}),
\]
as long as \(k\) is sufficiently large. It follows from the last inequality that

\[
\text{vol}(L) > \left| \int \mathcal{M}_{1,k} \Omega \right| + \left| \int \mathcal{M}_{2,k} \Omega \right|,
\]
for sufficiently large \(k\). We deduce that \((1.1)\) holds.

Recall that in the almost calibrated case \(\phi(M_{i,k})\) is uniquely defined by

\[
\phi(M_{i,k}) = \arg \int_{M_{i,k}} \Omega = \arg \int_{M_{i,k}} e^{i\theta} d\mathcal{H}^2.
\]

Using \((1.2)\), this implies that \(\phi(M_{i,k}) \in (\inf_{L} \theta, \sup_{L} \theta)\),
for sufficiently large $k$. We thus conclude that \((1.2)\) holds in the almost calibrated case. \hfill \Box

References

[1] T. Begley and K. Moore, On short time existence of Lagrangian mean curvature flow, Math. Ann. 367 (2017), no. 3-4, 1473–1515. MR 3629231

[2] V. I. Bogachev, Gaussian measures, Mathematical Surveys and Monographs, vol. 62, American Mathematical Society, Providence, RI, 1998. MR 1642391

[3] O. Chodosh and F. Schulze, Uniqueness of asymptotically conical tangent flows, Duke Math. J. 170 (2021), no. 16, 3601–3657. MR 4332673

[4] T. H. Colding and W. P. Minicozzi, Generic mean curvature flow I: generic singularities, Ann. of Math. (2) 175 (2012), no. 2, 755–833. MR 2993752

[5] Uniqueness of blowups and Lojasiewicz inequalities, Ann. of Math. (2) 182 (2015), no. 1, 221–285. MR 3374960

[6] Parabolic frequency on manifolds, Int. Math. Res. Not. (2021), to appear.

[7] K. Ecker, Logarithmic Sobolev inequalities on submanifolds of Euclidean space, J. Reine Angew. Math. 522 (2000), 105–118. MR 1758578

[8] N. Edelen, Degeneration of 7-dimensional minimal hypersurfaces which are stable or have bounded index, arXiv:2103.13563.

[9] K. Fukaya, Galois symmetry on Floer cohomology, Turkish J. Math. 27 (2003), no. 1, 11–32. MR 1975330

[10] G. Huisken, Asymptotic behavior for singularities of the mean curvature flow, J. Differential Geom. 31 (1990), no. 1, 285–299. MR 1030675

[11] T. Ilmanen, A. Neves, and F. Schulze, On short time existence for the planar network flow, J. Differential Geom. 111 (2019), no. 1, 39–89. MR 3969904

[12] D. Joyce, Conjectures on Bridgeland stability for Fukaya categories of Calabi-Yau manifolds, special Lagrangians, and Lagrangian mean curvature flow, EMS Surv. Math. Sci. 2 (2015), no. 1, 1–62. MR 3354954

[13] B. Lambert, J. D. Lotay, and F. Schulze, Ancient solutions in Lagrangian mean curvature flow, Ann. Sc. Norm. Super. Pisa Cl. Sci. XXII (2021), 1169–1205. MR 4334316

[14] J. D. Lotay, F. Schulze, and G. Székelyhidi, Ancient solutions and translators of Lagrangian mean curvature flow, arXiv:2204.13836.

[15] A. Neves, Singularities of Lagrangian mean curvature flow: zero-Maslov class case, Invent. Math. 168 (2007), no. 3, 449–484. MR 2299559

[16] Recent progress on singularities of Lagrangian mean curvature flow, Surveys in geometric analysis and relativity, Adv. Lect. Math. (ALM), vol. 20, Int. Press, Somerville, MA, 2011, pp. 413–438. MR 2906935

[17] Finite time singularities for Lagrangian mean curvature flow, Ann. of Math. (2) 177 (2013), no. 3, 1029–1076. MR 3034293

[18] A. Neves and G. Tian, Translating solutions to Lagrangian mean curvature flow, Trans. Amer. Math. Soc. 365 (2013), no. 11, 5655–5680. MR 3091260

[19] R. Schoen and J. Wolfson, Minimizing area among Lagrangian surfaces: the mapping problem, J. Differential Geom. 58 (2001), no. 1, 1–86. MR 1895348

[20] F. Schulze, Uniqueness of compact tangent flows in mean curvature flow, J. Reine Angew. Math. 690 (2014), 163–172. MR 3200339

[21] A canonical way to deform a lagrangian submanifold, arXiv:dg-ga/9605005.

[22] A. Strominger, S.-T. Yau, and E. Zaslow, Mirror symmetry is $T$-duality, Nuclear Phys. B 479 (1996), no. 1-2, 243–259. MR 1429831

[23] G. Székelyhidi, Uniqueness of certain cylindrical tangent cones, arXiv:2012.02065.
[27] R. P. Thomas, Moment maps, monodromy and mirror manifolds, Symplectic geometry and mirror symmetry (Seoul, 2000), World Sci. Publ., River Edge, NJ, 2001, pp. 467–498. MR 1882337
[28] R. P. Thomas and S.-T. Yau, Special Lagrangians, stable bundles and mean curvature flow, Comm. Anal. Geom. 10 (2002), no. 5, 1075–1113. MR 1957663
[29] M.-T. Wang, The mean curvature flow smooths Lipschitz submanifolds, Comm. Anal. Geom. 12 (2004), no. 3, 581–599. MR 2128604
[30] B. White, A local regularity theorem for mean curvature flow, Ann. of Math. (2) 161 (2005), no. 3, 1487–1519. MR 2180405

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