Technical Report: Infinite Horizon Discrete-Time Linear Quadratic Gaussian Tracking Control Derivation

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1 Introduction

This technical report is meant to accompany [4], which relies on the optimal controller for an infinite horizon LQG tracking problem in discrete time. Although this controller can be derived with standard methods, we were unable to find it in the existing literature, and we therefore derive it here. The notation and the general approach taken in this report are adapted from the work in [2, Chapters 1,4,5]. We adapt that work to derive the optimal controller, which is not explicitly derived in that reference. Below, we consider stochastic discrete-time systems of the form

\[ x_{k+1} = f_k(x_k, u_k, w_k), \quad k = 0, 1, \ldots, N - 1, \]

where \( N \) is the number of timesteps in the control problem. The state \( x_k \) is an element of a space \( S_k \), the control \( u_k \) is an element of a space \( C_k \), and the random disturbance \( w_k \) is an element of a space \( D_k \). The control \( u_k \) is constrained to take values in a given nonempty subset \( U_k(x_k) \subset C_k \) which depends on the current state \( x_k \); that is, \( u_k \in U_k(x_k) \) for all \( x_k \in S_k \) and \( k \). We consider the class of policies that consist of a sequence of functions

\[ \pi = \{ \mu_0, \ldots, \mu_{N-1} \}, \]

where \( \mu_k \) maps states \( x_k \) into controls via \( u_k = \mu_k(x_k) \), and is defined such that \( \mu_k(x_k) \in U_k(x_k) \) for all \( x_k \in S_k \).

The total cost to be minimized is

\[ E \left\{ g_N(x_N) + \sum_{k=1}^{N-1} g_k(x_k, u_k, w_k) \right\}, \]

in which \( g_N(x_N) \) is the terminal cost and \( g_k(x_k, u_k, w_k) \) is the cost incurred at time step \( k \). The expected value is with respect to the joint distribution of the random variables involved, which will be detailed below.

The main technique used in this report is dynamic programming which is based on the principle of optimality introduced by Bellman [1], which we will briefly introduce now; see [2, Page 18] for a more complete exposition.

**Principle of Optimality:** Let \( \pi^* = \{ \mu_0^*, \mu_1^*, \ldots, \mu_{N-1}^* \} \) be an optimal policy for the basic problem in Equation (1), and assume that when using \( \pi^* \), the state \( x_i \) occurs at time \( i \) with positive probability. Consider the subproblem in which the state is \( x_i \) at time \( i \) and we wish to minimize the “cost-to-go” from time \( i \) to time \( N \), namely,

\[ E \left\{ g_N(x_N) + \sum_{k=i}^{N-1} g_k(x_k, \mu_k(x_k), w_k) \right\}. \]
The principle of optimality states that the truncated policy \( \{ \mu^*_0, \mu^*_1, \ldots, \mu^*_{N-1} \} \) is optimal for this sub-problem.

We now formally define the dynamic programming algorithm.

**DP Algorithm:** For every initial state \( x_0 \), the optimal cost \( J^* (x_0) \) of the basic problem is equal to \( J_0 (x_0) \), given by the last step of the following algorithm, which proceeds backward in time from period \( N - 1 \) to period 0:

\[
J_N (x_N) = g_N (x_N),
\]

\[
J_k (x_k) = \min_{u_k \in U_k(x_k)} E \left\{ g_k (x_k, u_k, w_k) + J_{k+1} (f_k (x_k, u_k, w_k)) \right\}, \quad k = 0, 1, \ldots, N - 1,
\]

where the expectation is taken with respect to the probability distribution of \( w_k \), which can depend on \( x_k \) and \( u_k \). Furthermore, if \( u^* = \mu^* (x_k) \) minimizes the right side of Equation (4) for each \( x_k \) and \( k \), the policy \( \pi^* = \{ \mu^*_0, \mu^*_1, \ldots, \mu^*_{N-1} \} \) is optimal.

## 2 Basic Problem with Imperfect State Information

Consider the basic problem introduced in Section [I] where the controller has access to the noisy observations \( z_k \) of the form

\[
\begin{align*}
z_k &= h_k (x_k, u_{k-1}, v_k), \quad k = 1, 2, \ldots, N - 1 \\
z_0 &= h_0 (x_0, v_0).
\end{align*}
\]

The observation \( z_k \) belongs to a given observation space \( Z_k \). The observation noise \( v_k \) belongs to a given space \( V_k \) which is characterized by a given probability distribution

\[
P_{v_k} (\cdot \mid x_k, \ldots, x_0, u_{k-1}, \ldots, u_0, w_k, v_{k-1}, \ldots, v_0).
\]

The initial state \( x_0 \) is random and is characterized by \( P_{x_0} \). The probability distribution of the stochastic disturbances in the system, which is denoted by \( P_{w_k} (\cdot \mid x_k, u_k) \), is given, and \( w_k \) does not depend on prior disturbances \( w_0, \ldots, w_{k-1}, v_0, \ldots, v_{k-1} \), but it may depend explicitly on \( x_k \) and \( u_k \). The control \( u_k \) is constrained to take values from a given nonempty subset \( U_k \) of the control space \( C_k \). We are assuming that this subset does not depend on \( x_k \).

The information available to the controller at time \( k \) is denoted \( I_k \) and is called the information vector, formally defined as

\[
I_k = (z_0, z_1, \ldots, z_k, u_0, u_1, \ldots, u_{k-1}), \quad k = 1, 2, \ldots, N - 1 \\
I_0 = z_0.
\]

We consider the class of policies consisting of a sequence of functions \( \pi = \{ \mu_0, \mu_1, \ldots, \mu_{N-1} \} \), where each function \( \mu_k \) maps the information vector \( I_k \) into the control space \( C_k \), and

\[
\mu_k (I_k) \in U_k \text{ for all } I_k, \quad k = 0, 1, \ldots, N - 1,
\]

i.e., such policies are admissible. We want to find the policy \( \pi = \{ \mu_0, \mu_1, \ldots, \mu_{N-1} \} \) that minimizes the cost function

\[
J_\pi = \min_{x_0, u_0, \ldots, u_k} \left\{ g_N (x_N, \bar{x}_N) + \sum_{k=0}^{N-1} g_k (x_k, \mu_k (I_k), \omega_k, \bar{x}_k) \right\}
\]

subject to the system dynamics

\[2\]
\[ x_{k+1} = f_k (x_k, \mu_k (I_k), u_k), \quad k = 1, 2, \ldots, N - 1 \]

and the measurement equation
\[ z_k = h_k (x_k, \mu_{k-1} (I_{k-1}), v_k), \quad k = 1, 2, \ldots, N - 1 \]
\[ z_0 = h_0 (x_0, v_0). \]

2.1 Reformulation as Perfect State Information Problem

In this section, we will show the reduction from imperfect (i.e., noisy state information) to perfect state information. We define a system whose state at time \( k \) is the set of all variables whose values can benefit the controller when making the \( k^{th} \) decision. By definition,

\[ I_{k+1} = (I_k, z_{k+1}, u_k), \quad k = 1, 2, \ldots, N - 2, \]
\[ I_0 = z_0. \]  \hspace{1cm} (7)

These equations can be viewed as a new system where the state of the system is \( I_k \), control is \( u_k \), and the measurement \( z_{k+1} \) can be treated as a random disturbance. We then have

\[ P (z_{k+1} \mid I_k, u_k) = P (z_{k+1} \mid I_k, u_k, z_0, z_1, \ldots, z_k). \]

Therefore, since we already have \( z_0, z_1, \ldots, z_k \) as part of the information vector, the probability distribution of \( z_{k+1} \) depends explicitly on the state \( I_k \) and control of the new system, \( u_k \), and not on the prior disturbances \( z_0, z_1, \ldots, z_k \). In order to reformulate the cost function in terms of the variables of the new system we write

\[ E \{ g_k (x_k, u_k) \} = E \left\{ E_{x_k, u_k} \{ g_k (x_k, u_k, w_k) \mid I_k, u_k \} \right\}. \]

Hence, the cost per stage as a function of the new state \( I_k \) and \( u_k \)

\[ \tilde{g}_k (I_k, u_k) = E_{x_k, u_k} \{ g_k (x_k, u_k, w_k) \mid I_k, u_k \}. \]

The dynamic programming algorithm for \( k = 0, 1, \ldots, N - 2 \) is then written as

\[ J_k (I_k) = \min_{u_{k-1} \in U_{k-1}} \left\{ E_{x_k, u_k, z_{k+1}} \{ g_k (x_k, u_k, w_k) + J_{k+1} (I_k, z_{k+1}, u_k) \mid I_k, u_k \} \right\}, \]  \hspace{1cm} (8)

where \( J_{k+1} (I_k, z_{k+1}, u_k) = J_{k+1} (I_{k+1}) \). In Equation [8], it is important to note that the expected value is taken with respect to the observation \( z_{k+1} \) as well, since the information vector \( I_{k+1} \) is explicitly dependent on \( z_{k+1} \). The cost function for stage \( k = N - 1 \) is

\[ J_{N-1} (I_{N-1}) = \min_{u_{N-1} \in U_{N-1}} \left\{ E_{x_{N-1}, u_{N-1}} \{ g_N (f_{N-1} (x_{N-1}, u_{N-1}, w_{N-1})) + g_{N-1} (x_{N-1}, u_{N-1}, w_{N-1}) \mid I_{N-1}, u_{N-1} \} \right\}, \]  \hspace{1cm} (9)

where \( g_N (f_{N-1} (x_{N-1}, u_{N-1}, w_{N-1})) = g_N (x_N) \).
2.2 Linear Quadratic Gaussian Tracking Controller Derivation

We now apply the DP principle to solve for the linear quadratic cost function. Consider the linear system equation

\[ x_{k+1} = A_k x_k + B_k u_k + w_k, \quad k = 1, 2, ..., N - 1, \]

and the quadratic cost function

\[ J \left( I_{N-1} \right) = \min_{u_{N-1}} \left[ E \left\{ \sum_{k=0}^{N-1} \left( (x_k - \bar{x}_k)^T Q_N (x_k - \bar{x}_k) + u_k^T R_k u_k \right) \right\} \right], \]

where the reference trajectory \( \bar{x}_k \) is an element of space \( T_k \). At the beginning of period \( k \) we get an observation \( z_k \) of the form

\[ z_k = C_k x_k + v_k, \]

where the vectors \( v_k \) are mutually independent, and independent from \( w_k \) and \( x_0 \) as well. The noise terms \( w_k \) are assumed to be independent, zero mean and have a finite variance. \( C_k \) is a given \( s \times n \) matrix. We also assume \( Q_N \succ 0 \) and \( R_N \succ 0 \). Then, the cost function for step \( N - 1 \) can be expressed as

\[ J_{N-1} (I_{N-1}) = \min_{u_{N-1}} \left[ E \left\{ \sum_{k=0}^{N-1} \left( (x_{N-1} - \bar{x}_{N-1})^T Q_{N-1} (x_{N-1} - \bar{x}_{N-1}) + u_{N-1}^T R_{N-1} u_{N-1} \right) \right\} \right]. \]

Since we have \( E \{ w_{N-1} \mid I_{N-1} \} = E \{ w_{N-1} \} = 0 \), we can write

\[ J_{N-1} (I_{N-1}) = \min_{u_{N-1}} \left[ E \left\{ (x_{N-1} - \bar{x}_{N-1})^T Q_{N-1} (x_{N-1} - \bar{x}_{N-1}) + (Ax_{N-1} + Bu_{N-1} + w_{N-1} - \bar{x}_N)^T Q_N (Ax_{N-1} + Bu_{N-1} + w_{N-1} - \bar{x}_N) \mid I_{N-1}, u_{N-1} \right\} \right]. \]

In order to minimize the cost for the last period, we set \( \frac{dJ_{N-1}(I_{N-1})}{du_{N-1}} = 0 \) and find

\[ u^*_N = \mu^*_N (I_{N-1}) = - (B^T Q_N B + R_{N-1})^{-1} B^T Q_N (Ax_{N-1} \mid I_{N-1} \} - \bar{x}_N). \]
\[ J_{N-1}(I_{N-1}) = \mathcal{E}_{x_{N-1}} \left\{ (x_{N-1} - \bar{x}_{N-1})^T Q_{N-1} (x_{N-1} - \bar{x}_{N-1}) \right. \\
+ (Ax_{N-1} - \bar{x}_{N})^T Q_N (Ax_{N-1} - \bar{x}_{N}) \\
+ \left( (B^T Q_N B + R_{N-1})^{-1} B^T Q_N (AE \{x_{N-1} \mid I_{N-1}\} - \bar{x}_{N}) \right)^T \\
. (B^T Q_N B + R_{N-1}) (B^T Q_N B + R_{N-1})^{-1} B^T Q_N (AE \{x_{N-1} \mid I_{N-1}\} - \bar{x}_{N}) \\
- 2 (AE \{x_{N-1} \mid I_{N-1}\} - \bar{x}_{N})^T Q_N B (B^T Q_N B + R_{N-1})^{-1} B^T Q_N (AE \{x_{N-1} \mid I_{N-1}\} - \bar{x}_{N}) \mid I_{N-1} \\
+ E_{w_{N-1}} \left\{ w_{N-1}^T Q_N w_{N-1} \right\}. \]

Toward completing the square, we add and subtract \( \mathcal{E}_{x_{N-1}} \left\{ (Ax_{N-1} - \bar{x}_{N})^T P_{N-1} (Ax_{N-1} - \bar{x}_{N}) \right\} \) to get

\[ J_{N-1}(I_{N-1}) = \mathcal{E}_{x_{N-1}} \left\{ (x_{N-1} - \bar{x}_{N-1})^T Q_{N-1} (x_{N-1} - \bar{x}_{N-1}) + (Ax_{N-1} - \bar{x}_{N})^T Q_N (Ax_{N-1} - \bar{x}_{N}) \mid I_{N-1} \right\} \\
+ \mathcal{E}_{x_{N-1}} \left\{ (AE \{x_{N-1} \mid I_{N-1}\} - \bar{x}_{N})^T P_{N-1} (AE \{x_{N-1} \mid I_{N-1}\} - \bar{x}_{N}) \right\} \\
- 2 (Ax_{N-1} - \bar{x}_{N})^T P_{N-1} (AE \{x_{N-1} \mid I_{N-1}\} - \bar{x}_{N}) \mid I_{N-1} \right\} \\
+ \mathcal{E}_{x_{N-1}} \left\{ (Ax_{N-1} - \bar{x}_{N})^T P_{N-1} (Ax_{N-1} - \bar{x}_{N}) \right\} \\
+ \mathcal{E}_{w_{N-1}} \left\{ w_{N-1}^T Q_N w_{N-1} \right\}, \]

where we define \( P_{N-1} \overset{\Delta}{=} Q_N B (B^T Q_N B + R_{N-1})^{-1} B^T Q_N \). Completing the square, we find

\[ J_{N-1}(I_{N-1}) = \mathcal{E}_{x_{N-1}} \left\{ (x_{N-1} - \bar{x}_{N-1})^T Q_{N-1} (x_{N-1} - \bar{x}_{N-1}) \\
+ (Ax_{N-1} - \bar{x}_{N})^T (Q_N - P_{N-1}) (Ax_{N-1} - \bar{x}_{N}) \mid I_{N-1} \right\} \\
+ \mathcal{E}_{x_{N-1}} \left\{ [(Ax_{N-1} - \bar{x}_{N}) - (AE \{x_{N-1} \mid I_{N-1}\} - \bar{x}_{N})]^T P_{N-1} \right. \\
. [(Ax_{N-1} - \bar{x}_{N}) - (AE \{x_{N-1} \mid I_{N-1}\} - \bar{x}_{N})] \mid I_{N-1} \\
+ \left. \mathcal{E}_{w_{N-1}} \left\{ w_{N-1}^T Q_N w_{N-1} \right\}. \right. \] (16)

Here, the terms containing \( \bar{x}_{N} \) cancel each other out in the 2\textsuperscript{nd} and 3\textsuperscript{rd} line. Inductively, we write the cost function for step \( N - 2 \) as
\[ J_{N-2}(I_{N-2}) = \min_{u_{N-2}} \left[ E \left\{ (x_{N-2} - \bar{x}_{N-2})^T Q_{N-2} (x_{N-2} - \bar{x}_{N-2}) \right. \right. \\
\left. \left. + u_{N-2}^T R_{N-2} u_{N-2} + J_{N-1}(I_{N-1}) \mid I_{N-2}, u_{N-2} \right\} \right] , \]  
(17)

and substituting for \( J_{N-1} \) from Equation (16) gives

\[ J_{N-2}(I_{N-2}) = E \left\{ (x_{N-2} - \bar{x}_{N-2})^T Q_{N-2} (x_{N-2} - \bar{x}_{N-2}) \mid I_{N-2} \right\} \\
+ \min_{u_{N-2}} \left[ u_{N-2}^T R_{N-2} u_{N-2} + E \left\{ x_{N-1}^T (Q_{N-1} + A^T Q_N A - A^T P_{N-1} A) x_{N-1} \right. \right. \\
\left. \left. + \bar{x}_{N-1} Q_{N-1} \bar{x}_{N-1} + \bar{x}_{N} (Q_N - P_{N-1}) \bar{x}_{N} \right) \right. \\
\left. \left. - 2 \bar{x}_{N-1}^T (Q_{N-1} x_{N-1} - 2 \bar{x}_{N} (Q_N - P_{N-1}) A x_{N-1} \mid I_{N-1} \right) \right. \\
\left. \left. + E \left\{ [A (x_{N-1} - E \{ x_{N-1} \mid I_{N-1} \})]^T P_{N-1} [A (x_{N-1} - E \{ x_{N-1} \mid I_{N-1} \})] \mid I_{N-1} \right\} \right. \\
\left. \left. + E \left\{ w_{N-1}^T Q_N w_{N-1} \right\} . \right\} \]  
(18)

It is important to note that based on \textit{Lemma 5.2.1 in [3]}, the term in line 4 of Equation (18) is not in the above minimization with respect to \( u_{N-2} \). Next, define

\[ K_{N-1} \triangleq Q_{N-1} + A^T Q_N A - P_{N-1} \]
(19)

\[ P_{N-1} \triangleq A^T Q_N B \left( B^T Q_N B + R_{N-1} \right)^{-1} B^T Q_N A, \]
(20)

and then set \( \frac{d J_{N-2}(I_{N-2})}{da_{N-2}} = 0 \) to find

\[ 2 u_{N-2}^T R_{N-2} \]
\[ + E \left\{ 2 (A x_{N-2} + B u_{N-2} + w_{N-2})^T K_{N-1} B - 2 B^T Q_{N-1} \bar{x}_{N-1} - 2 B^T A^T (Q_N - P_{N-1}) \bar{x}_{N} \mid I_{N-1} \right\} = 0, \]

where we used \( x_{N-1} = A x_{N-2} + B u_{N-2} + w_{N-2} \). Simplifying this equation results in

\[ u_{N-2}^T R_{N-2} \]
\[ + u_{N-2}^T B^T K_{N-1} B + B^T K_{N-1} A E \{ x_{N-1} \mid I_{N-1} \} - B^T Q_{N-1} \bar{x}_{N-1} - B^T A^T (Q_N - P_{N-1}) \bar{x}_{N} \mid I_{N-1} = 0, \]

and solving for \( u_{N-2}^* \) provides

\[ u_{N-2}^* = - \left( R_{N-2} + B^T K_{N-1} B \right)^{-1} B^T \left( K_{N-1} A E \{ x_{N-1} \mid I_{N-1} \} - Q \bar{x}_{N-1} - A (Q_N - P_{N-1}) \bar{x}_{N} \right). \]

Continuing this strategy and putting it into recursive form, we have the optimal control

\[ u_k^* = - \left( R_k + B^T K_{k+1} B \right)^{-1} B^T \left( K_{k+1} A E \{ x_k \mid I_k \} + g_{k+1} \right), \]  
(21)

where
\[ g_k = A^T \left[ I - K_{k+1} B (R + B^T K_{k+1} B)^{-1} B^T \right] g_{k+1} - Q \bar{x}_k \]  \hspace{1cm} (22)

\[ g_N = -Q_N \bar{x}_N \]  \hspace{1cm} (23)

\[ K_N = Q_N \]  \hspace{1cm} (24)

\[ P_k = A^T K_{k+1}^T B (B^T K_{k+1} B + R_k)^{-1} B^T K_{k+1} A \]  \hspace{1cm} (25)

\[ K_k = A^T K_{k+1} A - P_k + Q_k. \]  \hspace{1cm} (26)

The authors in [3] investigate a closely related problem where they assume access to non-stochastic state knowledge and absence of process noise. They derive a closed form solution to the linear quadratic tracking control problem using the minimum principle of Pontryagin, and the results in this report are consistent with the results in [3].

3 Stochastic Reference Tracking LQG Over Infinite Horizon

A common extension to the problem solved in Section 2 is the case where the problem is solved in steady state across an infinite horizon. In particular, we are interested in infinite horizon LQG tracking control problems in which the reference trajectory is a stochastic process. First we solve the problem for the finite horizon case and then we extend the results to infinite horizon.

3.1 Stochastic Reference Tracking LQG for Finite Time

The cost function for finite time horizon LQG tracking a stochastic reference trajectory is defined as

\[ E \left\{ (x_N - \tilde{x}_N)^T Q_N (x_N - \tilde{x}_N) + \sum_{k=0}^{N-1} \left( (x_k - \tilde{x}_k)^T Q_N (x_k - \tilde{x}_k) + u_k^T R_k u_k \right) \right\}, \]

where the stochastic reference trajectory \( \tilde{x}_k \) is defined as

\[ \tilde{x}_k = \bar{x}_k + \bar{w}_k. \]

The random disturbances \( \bar{w}_k \) are assumed to be independent, zero mean, have a finite variance, and independent from \( w_k, v_k \) and \( x_0 \).

Following the derivation in Section 2 the cost function for timestep \( N - 1 \) can be written as

\[ J_{N-1} (I_{N-1}) = \min_{u_{N-1}} \left[ \sum_{x_{N-1}, w_{N-1}, \bar{w}_{N-1}, \bar{w}_{N-1}} E \left\{ (x_{N-1} - \tilde{x}_{N-1})^T Q_{N-1} (x_{N-1} - \tilde{x}_{N-1}) + u_{N-1}^T R_{N-1} u_{N-1} \right. \right. \]

\[ + (A x_{N-1} + B u_{N-1} + w_{N-1} - \tilde{x}_N)^T Q_N (A x_{N-1} + B u_{N-1} + w_{N-1} - \tilde{x}_N) \mid I_{N-1}, u_{N-1} \left. \right\}. \]
Since \( E \{ \bar{w}_k \} = 0 \), we can take the similar steps as in Section 2 to find

\[
J_{N-1}(I_{N-1}) = E_{x_{N-1}} \left\{ (x_{N-1} - \bar{x}_{N-1})^T Q_{N-1} (x_{N-1} - \bar{x}_{N-1}) + (Ax_{N-1} - \bar{x}_N)^T Q_N (Ax_{N-1} - \bar{x}_N) | I_{N-1} \right\}
\]

\[
+ \min_{\bar{u}_{N-1}} \left\{ u_{N-1}^T (B^T Q_N B + R_{N-1}) u_{N-1} + 2 (AE \{ x_{N-1} | I_{N-1} \} - \bar{x}_N)^T Q_N B u_{N-1} \right\}
\]

\[
+ E_{w_{N-1}} \left\{ \bar{w}_{N-1}^T Q_N w_{N-1} \right\}
\]

\[
+ E_{\bar{w}_N} \left\{ \bar{w}_N^T Q_N \bar{w}_N \right\}.
\] (27)

Hence, whether tracking a deterministic reference signal (Equation 14), or whether tracking a stochastic reference signal (Equation 27) the terms over which the minimization is carried out remain the same. Therefore, tracking a stochastic reference trajectory with \( \bar{w}_k \) is immaterial to the controller for \( k = N - 1 \). Inductively, this can be shown for each timestep \( k = 0, \ldots, N \) and therefore the controller which was developed in Section 2, which is the optimal controller for an LQG tracking control (LQGTC) problem, is equal to the optimal controller in LQG tracking control problem in which the reference signal is a stochastic process (LQGSTC). Similarly, this result can be extended to the infinite time horizon case which we study below.

Comparing the increase in cost in Equation (27) with the cost in Equation (14), and proceeding inductively, it can be seen that the total cost increases by \( E \{ \bar{w}_k^T Q_k \tilde{w}_k \} \) for each timestep \( k \) when we are tracking a stochastic reference trajectory. Therefore, we can write

\[
J_{LQGSTC}(x_k, u_k, \bar{x}_k) = J_{LQGTC}(x_k, u_k, \bar{x}_k) + \sum_{i=1}^{N} E \{ \bar{w}_k^T Q_k \tilde{w}_k \}. \] (28)

Having quantified the effects of adding noise to the reference trajectories for finite time, we complement these results by applying them to the infinite horizon case.

### 3.2 Stochastic Reference Tracking LQG Over Infinite Horizon

In this section, we further develop the LQG tracking control problem by analyzing the increase in the cost due to the added noise to the reference trajectories. We assume that \( \bar{x} \) has a limiting value i.e.,

\[
\lim_{k \to \infty} \bar{x}_k = \bar{x}
\]

is finite. The cost function in this case is defined as

\[
\lim_{N \to \infty} \frac{1}{N} E_{x_k, \bar{x}} \left\{ \sum_{k=0}^{N-1} (x_k - \bar{x})^T Q (x_k - \bar{x}) + u_k^T R u_k \right\}, \] (29)

where \( \bar{x} = \bar{x} + \bar{w} \) and \( \bar{w} \) is a noise term. From Equation (28) the cost function for the infinite horizon LQG stochastic tracking control (IHLQGSTC) can be expressed as

\[
J_{IHLQGSTC}(x_k, u_k, \bar{x}) = \lim_{N \to \infty} \frac{1}{N} \left( J_{LQGTC}(x_k, u_k, \bar{x}) + \sum_{i=1}^{N} E \{ \bar{w}_i^T Q \bar{w}_i \} \right). \] (30)

Finally, simplifying Equation (30) gives

\[
J_{IHLQGSTC}(x_k, u_k, \bar{x}) = \lim_{N \to \infty} \left( \frac{1}{N} J_{LQGTC}(x_k, u_k, \bar{x}) \right) + E \{ \bar{w}^T Q \bar{w} \}. \] (31)
The optimal controller for this cost in then given by
\[
    u_k^* = -(R + B^T KB)^{-1} B^T (K AE \{ x_k \mid I_k \} + g),
\]  
(32)

where
\[
    g = A^T \left[ I - KB (R + B^T KB)^{-1} B^T \right] g - Q \hat{x},
\]  
(33)

and where $K$ is the unique positive semi-definite solution to the algebraic Riccati equation
\[
    K = A^T KA - A^T K^T B (B^T KB + R)^{-1} B^T KA + Q.
\]  
(34)

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