STABILITY AND BIFURCATION WITH SINGULARITY FOR A GLYCOLYSIS MODEL UNDER NO-FLUX BOUNDARY CONDITION

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ABSTRACT. In this paper, a glycolysis model subject to no-flux boundary condition is considered. First, by discussing the corresponding characteristic equation, the stability of constant steady state solution is discussed, and the Turing’s instability is shown. Next, based on Lyapunov-Schmidt reduction method and singularity theory, the multiple stationary bifurcations with singularity are analyzed. In particular, under no-flux boundary condition we show the existence of nonconstant steady state solution bifurcating from a double zero eigenvalue, which is always excluded in most existing works. Also, the stability, bifurcation direction and multiplicity of the bifurcation steady state solutions are investigated by the singularity theory. Finally, the theoretical results are confirmed by numerical simulations. It is also shown that there is no Hopf bifurcation on basis of the condition (C).

1. Introduction. Glycolysis, which occurs in the cytosol, is thought to be the archetype of a universal metabolic pathway for cellular energy requirement. The wide occurrence of glycolysis indicates that it is one of the most ancient known metabolic pathways and a common way of providing limited energy for the organism in living nature. However, its significance lies in that it can supply the energy not only with a rapid speed, but more importantly under oxygen free conditions such as strenuous exercise and high altitude hypoxia. On the other hand, it is the main way of providing energy for the tumor cells [1, 20, 35].

Glycolysis model turns out to be a classic and representative system in biochemical reaction. All glycolysis models are based on the same reaction scheme. The different models stem from the difference in the reaction mechanism for key enzyme. The first model of glycolysis was proposed by Higgins [15], whose reaction

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mechanism is
\[ A \rightarrow X, \]
\[ X + Y \rightarrow 2Y, \]
\[ Y \rightarrow P \text{ (with saturation).} \]
The existence, stability and multiplicity of steady state solutions and the existence, stability of spatially inhomogeneous periodic solutions are studied in [5, 6, 9, 24, 27] and [38], respectively. The second glycolysis model was presented by Sel’kov [29] with the mechanism
\[ A \rightarrow X, \]
\[ X + pY \rightarrow (1 + p)Y, \]
\[ Y \rightarrow P, \]
which was further analyzed for the existence of nonconstant positive steady state solutions and periodic solutions with homogeneous Neumann boundary condition in [8, 11, 13, 19, 23, 33] and the non-existence and existence of positive steady state solutions with homogeneous Dirichlet boundary condition in [25].
In the present paper, we deal with the following reaction mechanism
\[ A \rightarrow X, \]
\[ B + X \rightarrow Y, \]
\[ X + 2Y \rightarrow 3Y, \]
\[ Y \rightarrow P \]
in the later stage of glycolysis reaction, whose corresponding dimensionless system subject to no-flux boundary condition in the one-dimension spatial domain is
\[
\begin{aligned}
\frac{u_t}{d_1} &= u_{xx} + \delta - ku - uv^2, & x \in (0, l), \ t > 0, \\
\frac{v_t}{d_2} &= v_{xx} + ku - v + uv^2, & x \in (0, l), \ t > 0, \\
u_x = v_x &= 0, & x = 0, l, \ t > 0, \\
u(x, 0) &= u_0(x) \geq 0, \ v(x, 0) &= v_0(x) \geq 0, & x \in (0, l).
\end{aligned}
\]
(1)
Here, \( u \) and \( v \) represent the chemical concentrations, \( d_1 \) and \( d_2 \) are the diffusion coefficients, \( \delta \) is the dimensionless input flux, and \( k \) is the dimensionless rate constant for the low activity state. For \( k = 0 \), the model is Sel’kov model studied in [8, 11, 13, 19, 23, 25, 33]. Throughout this paper we assume that all constants \( d_1, d_2, \delta \) and \( k \) are positive and \( 0 < k < 1/8 \) for the range of rational experiment data. The reaction mechanism and the generic class of model (1) are given in [22, 28, 32]. Concerning this model for a two cell system without spatial diffusion, the spatially homogeneous periodic solution is obtained in [10], which shows the sequential structure of time in the later stage of biochemical reaction. In view of spatial diffusion, there are some existence and stability results of constant steady states (see [3, 21]). Under the fixed Dirichlet boundary condition, the authors [36] discuss the stability of constant steady state solution and the existence of non-constant steady state solutions not only from a simple eigenvalue, but more difficultly from a double one, which are confirmed by numerical simulations.
Based on the following condition \((C)\), no Hopf bifurcation can occur for the model (1) by taking the bifurcation parameter \( d_1 \). Then, we aim to continue the stationary bifurcation of (1) for no-flux boundary condition. However, the steady state bifurcation mostly focuses on the case of the simple eigenvalue, such as [4, 14, 16, 17, 18, 34, 37, 38], but rather less seems to be known about the non-simple case.
The corresponding method for the simple bifurcation is the traditional theory, see [7, 26]. Just as [4, 14, 16, 17, 18, 34, 37, 38], the bifurcation analysis may be from either a simple or non-simple zero eigenvalues, but the latter case is excluded in the most existing works. Motivated by this, we are interested in the formation of steady state spatially inhomogeneous solutions bifurcating from the double zero eigenvalue. This results in the steady-state solution with mode interaction of \( \cos \frac{\pi j}{l} x \) and \( \cos \frac{\pi m}{l} x \), which is investigated in Section 3 and confirmed by numerical simulations in Section 5. We also show that no steady state solutions bifurcating from the higher eigenvalue can exist for model (1).

The main contribution of this paper is to focus on the singularity bifurcation under the no-flux boundary condition, i.e. the bifurcation where one or more of the hypotheses of the traditional theory fail. In this case, the involved methods are Lyapunov-Schmidt reduction and singularity theory. The singularity theory as developed by Golubitsky and Schaeffer [12] allows one to classify bifurcation into equivalence classes. It offers an extremely useful approach to the bifurcation problem, whose principal advantage is that it adapts well to the singularity bifurcation, such as the double bifurcation. Here, it is worth noting that Lyapunov-Schmidt reduction and singularity theory are also applied to analyze the bifurcation direction, multiplicity and stability of the solutions bifurcating from the simple eigenvalue.

The rest of the present paper is organized as follows. In Section 2, we first briefly analyze the Turing’s instability of constant steady state solution, and then study bifurcation direction and multiplicity of the steady state solutions of (1) from simple bifurcation. In Section 3, we mainly discuss the bifurcation from a double eigenvalue for the no-flux boundary condition. In this case, the classical Crandall-Rabinowitz theorem can no longer be applied, and we investigate the conditions in detail for the bifurcation structure by use of Lyapunov-Schmidt procedure and singularity theory. In Section 4, we discuss the stability of the solutions from simple and double bifurcations by the singularity theory. Finally, in Section 5, numerical simulations are performed to confirm the analytic results.

2. Steady state solutions when \( d_1^{(j)} \neq d_m^{(j)} \) for any \( m \neq j \). Obviously, \((u^*, v^*) = (\frac{\delta}{l}, \frac{\delta}{l})\) is the unique constant steady state solution of the PDE system (1) and the corresponding ODE system of (1). For the bifurcation from the constant solution, it is essential to analyze the Turing instability of the constant solution.

It is well known that the eigenvalue problem

\[
\begin{cases}
-\phi_{xx} = \lambda \phi, & x \in (0, l), \\
\phi_x = 0, & x = 0, l
\end{cases}
\]

has eigenvalues \( \lambda_i = (\pi i/l)^2, i = 0, 1, 2, \ldots \) with corresponding normalized eigenfunctions

\[
\phi_i(x) = \begin{cases} 
1/\sqrt{l}, & i = 0, \\
\sqrt{2/l}\cos(\pi ix/l), & i > 0.
\end{cases}
\]

Throughout this paper, we always assume

\[
(C) \quad \delta^2 \in (k, \frac{1 - 2k - \sqrt{1 - 8k}}{2}) \cup (\frac{1 - 2k + \sqrt{1 - 8k}}{2}, \infty).
\]

Let

\[
d_1^{(i)} = \frac{g_0(1 + d_2 \lambda_i)}{\lambda_i(g_1 - d_2 \lambda_i)}, \quad 1 \leq i \leq \Lambda,
\]
where
\[ g_0 = k + \delta^2 > 0, \quad g_1 = \frac{\delta^2 - k}{k + \delta^2} > 0, \]
\[ \Lambda = \Lambda(l, d_2, \delta, k) \]
is defined to be the largest positive integer such that \( \lambda_i < \frac{g_1}{d_2} \), and
then denote \( d_{1\min} = \min\{d_1^{(i)} : 1 \leq i \leq \Lambda\} \).

Setting
\[ \tilde{u} = u - u^*, \quad \tilde{v} = v - v^*, \]
we still denote \( \tilde{u}, \tilde{v} \) by \( u, v \).
Letting
\[ X = \{ (u, v) : u, v \in C^2[0, l], \quad u = v = 0 \text{ at } x = 0, l \}, \]
\[ Y = C^0[0, l] \times C^0[0, l], \]
we define the smooth mapping \( F : X \times (0, \infty) \to Y \) as
\[ F(u, v, \lambda) = L \begin{pmatrix} u \\ v \end{pmatrix} + N(u, v), \]
where \( \lambda = d_1 - d_1^{(j)} \).

Here, the linear part is
\[ L \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} (\lambda + d_1^{(j)}) \frac{\partial^2}{\partial x^2} + g_0 f_0 & f_1 \\ g_0 & d_2 \frac{\partial^2}{\partial x^2} + g_1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \]
and the nonlinear part is
\[ N(u, v) = \left( \frac{\delta}{k + \delta^2} u^2 + 2\delta uv + uv^2 \right) \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \] (3)
where
\[ f_0 = -k - \delta^2 < 0, \quad f_1 = -\frac{2\delta^2}{k + \delta^2} < 0. \]

Then the stationary solutions of (1) are corresponding to the solutions of the elliptic problem
\[ F(w, \lambda) = 0, \quad w = (u, v), \quad x \in (0, l) \]
with the boundary condition
\[ u_x = v_x = 0, \quad x = 0, l. \]
Thus we have \( F(0, \lambda) = 0 \) and take \( \lambda \) instead of \( d_1 \) as bifurcation parameter for the later discussions.

Based on the condition \((C)\), it follows that
(i) no Hopf bifurcation occurs for taking \( \lambda \) (or \( d_1 \)) as the bifurcation parameter.
(ii) the model (1) is an activator-substrate system where the activator \( u \) consume the substrate \( v \).
(iii) the unique constant solution \((u^*, v^*)\) of (1) is diffusion-free stable.

The following lemma shows that the Turing instability is found for the glycolysis model (1).

**Lemma 2.1.** Assume \((C)\) holds. If \( \lambda_1 \geq \frac{g_1}{d_2} \) or \( \lambda_1 < \frac{g_1}{d_2} \) \( \text{and} \ 0 < d_1 < d_{1\min} \), then the constant steady state solution \((0, 0)\) of (1) is asymptotically stable. If \( \lambda_1 < \frac{g_1}{d_2} \) \( \text{and} \ d_1 > d_{1\min} \), then \((0, 0)\) is unstable.

According to Eq. (2), it is readily found that at most two elements in \( \{d_1^{(i)} : 1 \leq i \leq \Lambda\} \) are equal. So, taking arbitrary integer \( m \) for the fixed \( j \) such that \( 1 \leq m, j \leq \Lambda \), there are only two possibilities such as \( d_1^{(j)} \neq d_1^{(m)} \) and \( d_1^{(j)} = d_1^{(m)} \), which correspond to different bifurcation analysis. Without loss of generality, we
fix $a_1^{(j)}$ for $j \in [1, A]$ to be the main discussed mode and divide our discussions into two cases: (i) $a_1^{(j)} \neq a_1^{(m)}$ for any $m \neq j$ in the present section; (ii) $a_1^{(j)} = a_1^{(m)}$ for some $m \neq j$ in the next section.

Denote the inner product of $Y$ by

$$\langle U_1, U_2 \rangle = \langle u_1, u_2 \rangle_{L^2(0, l)} + \langle v_1, v_2 \rangle_{L^2(0, l)}, \quad U_1 = (u_1, v_1), \quad U_2 = (u_2, v_2) \in Y$$

and the linearized operator by

$$L_0 = \begin{pmatrix} d_1^{(j)} \frac{\partial^2}{\partial x^2} + f_0 & f_1 \\ g_0 & d_2 \frac{\partial^2}{\partial x^2} + g_1 \end{pmatrix}.$$ 

Set the space decompositions $Y = N(L_0) \oplus R(L_0)$ and $X = N(L_0) \oplus X_1$, where

$$N(L_0) = \text{span}\{\Phi_j\}, \quad \Phi_j = \begin{pmatrix} a_j \\ 1 \end{pmatrix} \phi_j, \quad a_j = \frac{d_2 \lambda_j - g_1}{g_0} < 0,$$

and $X_1 = X \cap R(L_0)$. We further define the operator $P$ on $Y$ by

$$PU = \langle U, \Phi_j^* \rangle \Phi_j,$$

where $\Phi_j^*$ satisfies $N(L_0^* \Phi_j) = \text{span}\{\Phi_j^*\}$ and is defined as

$$\Phi_j^* = \frac{1}{1 + a_j a_j^*} \begin{pmatrix} a_j^* \\ 1 \end{pmatrix} \phi_j, \quad a_j^* = \frac{d_2 \lambda_j - g_1}{f_1} > 0.$$

Then $R(P) = N(L_0)$, and it is easy to verify that $P^2 = P$ which implies that $P$ is the projection onto $N(L_0)$. Thus, $Q = I - P$ is a projection onto $R(L_0)$ in $Y$. Hence, the system (4) can be transformed into

$$PF(w, \lambda) = 0, \quad QF(w, \lambda) = 0. \tag{5}$$

According to the decomposition of $X$, we rewrite $w \in X$ as $w = s \Phi_j + W$, where $s \in R$ and $W \in X_1$. It follows from the implicit theorem that the second equation of (5) is uniquely solvable for $W$ in the neighborhood of zero. Let $W(s, \lambda) := W(s \Phi_j, \lambda)$ denotes this solution for all small $s$ and $\lambda$. It is obvious that $W_s(0, 0) = 0$, $W(0, \lambda) \equiv 0$ for small $\lambda$, and then $W_{\lambda}(0, 0) = W_{\lambda}(0, 0) = \cdots = 0$. Substituting $W(s, \lambda)$ into the first equation of (5), we can get

$$PF(s \Phi_j + W(s, \lambda), \lambda) = 0. \tag{6}$$

Rewrite $L = L_0 + \lambda M$ with $M = \begin{pmatrix} \frac{\partial^2}{\partial x^2} & 0 \\ 0 & 0 \end{pmatrix}$. Furthermore, owing to the definition of $P$, (6) can be rewritten in the reduced problem

$$h(s, \lambda) := \langle \Phi_j^* H(s \Phi_j + W(s, \lambda), \lambda) \rangle = 0, \tag{7}$$

where

$$H(w, \lambda) := \lambda M w + N(w). \tag{8}$$

Therefore, the solutions of (4) are further determined by the zeros of the reduced equation (7).

In order that the given reduction problem (that is, the algebraic equation which results from the original partial differential equation by the Lyapunov-Schmidt reduction method) is equivalent to such a normal form, certain conditions have to be met by the singularity theory.

It is obvious that $h_s(0, 0) = 0$ from $W_s(0, 0) = 0$ and $h_\lambda(0, 0) = h_{\lambda\lambda}(0, 0) = \cdots = 0$ from $H(0, \lambda) = 0$. To begin the following discussions, by straightforward
calculations, we further show that the second and third derivatives of $H$ at the origin are given as
\[ \frac{\partial^2 H}{\partial s \partial \lambda} = M \Phi_j, \quad \frac{\partial^2 H}{\partial s^2} = d^2 N(\Phi_j, \Phi_j), \]
\[ \frac{\partial^3 H}{\partial s^3} = 3d^2 N(\Phi_j, W_{ss}(0,0)) + d^3 N(\Phi_j, \Phi_j). \] (9)

From (3) we further get the second and third Fréchet derivatives of $N$ are separately given by
\[ d^2 N(\Phi_i, \Phi_m) = 2[\delta(\Phi_{i1}\Phi_{m2} + \Phi_{i2}\Phi_{m1}) + \frac{\delta}{k + \delta^2} \Phi_{i2}\Phi_{m2}] \left( \begin{array}{c} -1 \\ 1 \end{array} \right) \] (10)
and
\[ d^3 N(\Phi_i, \Phi_m, \Phi_n) = 2[\Phi_{i1}\Phi_{m2}\Phi_{n2} + \Phi_{i2}\Phi_{m1}\Phi_{n2} + \Phi_{i2}\Phi_{m2}\Phi_{n1}] \left( \begin{array}{c} -1 \\ 1 \end{array} \right). \] (11)

According to the second equation of (5), the second order derivative of $W$ with respect to $s$ at the origin denoted by $W_{ss}$ meets
\[ L_0 W_{ss} = -Q d^2 N(\Phi_j, \Phi_j) = -e_j \left( \frac{1}{\sqrt{t}} \phi_0 + \frac{1}{\sqrt{2l}} \phi_{2j} \right), \]
where
\[ e_j = 2(2\delta a_j + \frac{\delta}{k + \delta^2}) = \frac{2\delta}{k + \delta^2} (2d_2\lambda_j - 2g_1 + 1). \]

Setting $W_{ss} = \sum_{i=0}^{\infty} \left( \frac{c_i}{d_i} \right) \phi_i$, we have
\[ L_0 W_{ss} = \sum_{i=0}^{\infty} B_i \left( \frac{c_i}{d_i} \right) \phi_i = -e_j \left( \frac{1}{\sqrt{t}} \phi_0 + \frac{1}{\sqrt{2l}} \phi_{2j} \right), \]
where
\[ B_i = \begin{pmatrix} f_0 - d_4^{(j)} \lambda_i & f_1 \phi_0 \\
0 & g_1 - d_2 \lambda_i \end{pmatrix}. \] (12)

Thus we obtain
\[ W_{ss} = -\frac{e_j}{l} \left[ B_0^{-1} \left( \begin{array}{c} -1 \\ 1 \end{array} \right) + B_2^{-1} \left( \begin{array}{c} -1 \\ 1 \end{array} \right) \cos \frac{2\pi j}{l} x \right]. \]

Due to $|B_i| = (\lambda_j - \lambda_i)[g_1 - d_2(\lambda_j + \lambda_i + d_2 \lambda_i \lambda_j)]g_0$ and $B_i^{-1} = \frac{B_i^*}{|B_i|}$, we have
\[ W_{ss} = -\frac{e_j}{l} \left( \frac{1}{0} \right) + \frac{g_1 - d_2 \lambda_j}{3[g_1 - d_2 \lambda_j(5 + 4d_2 \lambda_j)]} \left( -4d_2 \lambda_j - 1 \right) \left( \begin{array}{c} -1 \\ 1 \end{array} \right) \cos \frac{2\pi j}{l} x. \] (13)

Hence, from (9)-(11) and (13), it is easy to get
\[ h_{\lambda s}(0,0) = \langle \Phi^*_s, M \Phi_j \rangle = -\frac{\lambda_j a_j}{1 + a_j a_j^*} > 0, \quad h_{ss}(0,0) = h^{(4)}_{\lambda s}(0,0) = 0, \]
\[ h_{sss}(0,0) = \left\{ \Phi^*_s, \frac{1}{2} d^2 N(\Phi_j, W_{ss}) + \frac{1}{2} d^3 N(\Phi_j^3) \right\} \]
\[ = \frac{e_j \delta[(1 - a_j^*)g_1 - d_2 \lambda_j(21 + 8g_1 + 16d_2 \lambda_j) + 4]}{6 \delta g_0 (1 + a_j a_j^*)(g_1 - d_2 \lambda_j(5 + 4d_2 \lambda_j))^2} + \frac{3a_j(1 - a_j^*)}{2l(1 + a_j a_j^*)}. \] (14)
\[
\begin{align*}
&= \left\{ -\frac{(g_1 + 1)(2d_2\lambda_j - 2g_1 + 1)[g_1 - d_2\lambda_j(21 + 8g_1 + 16d_2\lambda_j) + 4]}{3[g_1 - d_2\lambda_j(5 + 4d_2\lambda_j)]} \\
&\quad + 3(d_2\lambda_j - g_1)\right\} \frac{1 - a_j^*}{2l(g_0(1 + a_ja_j^*)}. \\
\end{align*}
\]

Thus, for \( h_{ss}(0,0) \neq 0 \), \( h(s,\lambda) \) is equivalent to \( g(s,\lambda) = \text{sgn}(h_{ss}(0,0))s^3 + \lambda s \) from [12], and so the system (4) has two non-trivial solutions and the bifurcation direction is subcritical (see Fig. 1(a)) for \( h_{ss}(0,0) > 0 \) and supercritical (see Fig. 1(b)) for \( h_{ss}(0,0) < 0 \). However, for \( h_{ss}(0,0) = 0 \), the fifth order derivative is the particular number, the calculation of which is more tedious and omitted here. Therefore, the system (1) has at least two non-constant steady state solutions in a neighborhood of the simple bifurcation point \((u^*, v^*, d_1^{(j)})\).

![Figure 1](image)

**Figure 1.** Local bifurcation of \( g(s,\lambda) = 0 \) at \((s,\lambda) = (0,0)\). Here, \( (a) \) \( g(s,\lambda) = s^3 + \lambda s \); \( (b) \) \( g(s,\lambda) = -s^3 + \lambda s \).

To sum up, we find that each local solution \( s, \lambda \) of the reduced problem (7) corresponds to a bifurcation solution of (1) with the form

\[ (u,v)^T = (u^*,v^*)^T + s\Phi_j + W(s,\lambda), \]

where \( W(s,\lambda) \) satisfies \( W_s(0,0) = 0, W(0,\lambda) \equiv 0 \) and \( W_\lambda(0,0) = W_{\lambda\lambda}(0,0) = \ldots = 0 \) for small \( \lambda \). The multiplicity of the solutions is determined by the number of zeros of \( h(s,\lambda) = 0 \).

**Remark 1.** From [30], we know that \( \lambda_j'(0) = 0 \) on the basis of \( h_{ss}(0,0) = 0 \), where \( \lambda(s) \) is decided by (7).

3. **Steady state solutions when \( d_1^{(j)} = d_1^{(m)} \) for some \( m \neq j \).** From (2), one may verify that \( d_1^{(j)} = d_1^{(m)} \) for some \( m \neq j \) if and only if

\[
d_2 = \frac{\sqrt{(\lambda_j + \lambda_m)^2 + 4g_1\lambda_j\lambda_m} - (\lambda_j + \lambda_m)}{2\lambda_j\lambda_m}, \]

which is the basic assumption in this section. It leads to \( d_1^{(j)} = d_1^{(m)} = \frac{g_0}{d_2\lambda_j\lambda_m} \), \( g_1 - d_2\lambda_j = d_2\lambda_m(1 + d_2\lambda_j) \) and \( g_1 - d_2\lambda_m = d_2\lambda_j(1 + d_2\lambda_m) \). Without loss of generality, we assume \( j < m \).
In \( \{d^{(i)}_1 : 1 \leq i \leq \Lambda \} \), the case \( d^{(j)}_1 \neq d^{(m)}_1 (m \neq j) \) has been discussed in the previous section, and the case \( d^{(i)}_1 = d^{(m)}_1 (m \neq j) \) will be analyzed in this section. We remark that if \( d^{(j)}_1 = d^{(m)}_1 \) for some \( m \neq j \), then 0 is not a simple eigenvalue of \( L_0 \) and the Crandall-Rabinowitz theorem is no longer applied. In this case, 0 is a double eigenvalue of \( L_0 \), and the Lyapunov-Schmidt reduction technique and singularity theory will be used to investigate whether or not the bifurcation occurs.

For \( d^{(j)}_1 = d^{(m)}_1 \), \( m \neq j \), we have
\[
N(L_0) = \text{span} \{ \Phi_j, \Phi_m \}, \quad N(L^*_0) = \text{span} \{ \Phi^*_j, \Phi^*_m \},
\] (17)
where
\[
\Phi_i = \left( \frac{a_i}{1} \right) \phi_i, \quad a_i = \frac{d_2 \lambda_i - g_1}{g_0} < 0,
\]
\[
\Phi^*_i = \frac{1}{1 + a_i a^*_i} \left( \frac{a_i}{1} \right) \phi_i, \quad a^*_i = \frac{d_2 \lambda_i - g_1}{f_1} > 0, \quad i = j, m,
\]
which are normalized as \( \langle \Phi_j, \Phi^*_n \rangle = \delta_{jn}, i, n = j, m \). For the later discussions, we point out that \( 1 + a_i a^*_i > 0, 1 - a^*_i > 0, i = j, m \).

Now we assume the decompositions \( Y = N(L_0) \oplus R(L_0) \) and \( X = N(L_0) \oplus X_1 \), where \( N(L_0) \) is given by (17) and \( X_1 = X \cap R(L_0) \). By letting \( P \) be the projection of \( Y \) onto \( N(L_0) \), we must take the form of \( P \) as
\[
PU = \langle U, \Phi^*_j \rangle \Phi_j + \langle U, \Phi^*_m \rangle \Phi_m, \quad U \in Y.
\]
and then \( Q = I - P \) is a projection onto \( R(L_0) \) in \( Y \). Hence, (4) is determined by a pair of equations
\[
PF(w, \lambda) = 0, \quad QF(w, \lambda) = 0.
\] (18)

Here, we decompose \( w \in X \) in the form \( w = s\Phi_j + \tau\Phi_m + W \in X \), where \( (s, \tau) \in R^2 \) and \( W \in X_1 \). For the second equation of (18), it follows from the implicit theorem that there exists a unique smooth function \( W(s, \tau, \lambda) := W(s\Phi_j + \tau\Phi_m, \lambda) \) near the origin, which satisfies \( W(0, 0, 0) = 0 \) and \( QF(s\Phi_j + \tau\Phi_m + W(s, \tau, \lambda), \lambda) = 0 \). It is easy to get \( W_0(0, 0, 0) = 0, W_\tau(0, 0, 0) = 0, W(0, 0, \lambda) \equiv 0, W_\lambda(0, 0, 0) = W_{\lambda\lambda}(0, 0) = \cdots = 0 \). Substituting \( W(s, \tau, \lambda) \) into the first equation of (18), we obtain
\[
PF(s\Phi_j + \tau\Phi_m + W(s, \tau, \lambda), \lambda) = 0.
\]

According to the definition of \( P \), the solutions of (4) are in one-to-one correspondence with the zeros of the reduced problem
\[
\begin{align}
\zeta(s, \tau, \lambda) := & \left( \langle \Phi^*_j, H(s\Phi_j + \tau\Phi_m + W(s, \tau, \lambda), \lambda) \rangle \right) = 0, \\
\vartheta(s, \tau, \lambda) := & \left( \langle \Phi^*_m, H(s\Phi_j + \tau\Phi_m + W(s, \tau, \lambda), \lambda) \rangle \right) = 0,
\end{align}
\] (19)
where \( H(w, \lambda) \) is defined as (8). Thus when (19) is solvable for \( s, \tau, \lambda \), the solution of (4) has the form \( w = s\Phi_j + \tau\Phi_m + W(s\Phi_j + \tau\Phi_m, \lambda) \). It follows from [2] that (4) inherits a symmetric structure, and the above reduced form can be rewritten as
\[
\begin{align}
\zeta(s, \tau, \lambda) := & \left( sa(s, \bar{\tau}, \lambda) + sm^{-1} \tau \bar{b}(\bar{s}, \bar{\tau}, \lambda) \right), \\
\vartheta(s, \tau, \lambda) := & \left( rc(s, \bar{\tau}, \lambda) + sm^{-1} \tau \bar{d}(\bar{s}, \bar{\tau}, \lambda) \right),
\end{align}
\] (20)

In view of \( H(0, 0, 0) = 0 \), we see that \( \zeta_{00n} = \vartheta_{00n} = 0, n = 1, 2, \ldots \). Clearly, we have \( \zeta_{100} = \vartheta_{100} = \zeta_{010} = \vartheta_{010} = 0 \) from \( W_0(0, 0, 0) = 0 \) and \( W_\tau(0, 0, 0) = 0 \).

The second derivatives of \( H \) at the origin are given as
\[
\frac{\partial^2 H}{\partial s_i \partial \lambda} = M \Phi_i, \quad \frac{\partial^2 H}{\partial s_i \partial s_n} = d^2 N(\Phi_i, \Phi_n), \quad i, n = j, m
\] (21)
with \( s_j = s, s_m = \tau \). From (10) and (21), we have

\[
a_{001} = \langle \Phi_j^*, M\Phi_j \rangle = -\check{\lambda}_j a_j^* a_j > 0,
\]

\[
c_{001} = \langle \Phi_m^*, M\Phi_m \rangle = -\check{\lambda}_m a_m^* a_m > 0.
\]

The following vectors defined at the origin play a central role in the computations of Taylor coefficients and are exhibited here.

\[
H_{300} = \frac{1}{2} \frac{d^2 N(\Phi_j, W_{ss})}{\partial s^2} + \frac{1}{3!} \frac{d^3 N(\Phi_3^j)}{\partial s^3},
\]

\[
H_{030} = \frac{1}{2} \frac{d^2 N(\Phi_m, W_{\tau\tau})}{\partial \tau^2} + \frac{1}{3!} \frac{d^3 N(\Phi_3^m)}{\partial \tau^3},
\]

\[
H_{120} = \frac{1}{2} \frac{d^2 N(\Phi_j, W_{\tau\tau})}{\partial s^2} + \frac{1}{2} \frac{d^2 N(\Phi_m, W_{s\tau})}{\partial s \partial \tau} + \frac{1}{2} \frac{d^3 N(\Phi_j, \Phi_m^2)}{\partial s^2 \partial \tau},
\]

\[
H_{210} = \frac{1}{2} \frac{d^2 N(\Phi_m, W_{ss})}{\partial \tau^2} + \frac{1}{2} \frac{d^2 N(\Phi_j, W_{s\tau})}{\partial s \partial \tau} + \frac{1}{2} \frac{d^3 N(\Phi_j^2, \Phi_m)}{\partial s^2 \partial \tau},
\]

where

\[
H_{ijk} := \frac{1}{i!j!k!} \frac{\partial^{i+j+k}}{\partial s^i \partial \tau^j \partial \lambda^k} N(0, 0),
\]

and the second and third Fréchet derivatives of \( N \) are given by (10) and (11), respectively.

Combining (19) with (20), we obtain

\[
c_{010} = \langle \Phi_m^*, H_{030} \rangle = \left\langle \Phi_m^*, \frac{1}{2} \frac{d^2 N(\Phi_m, W_{\tau\tau})}{\partial \tau^2} \right\rangle + \left\langle \Phi_m^*, \frac{1}{3!} \frac{d^3 N(\Phi_3^m)}{\partial \tau^3} \right\rangle.
\]

(22)

From the second equation of (5), the second order derivative of \( W \) with respect to \( \tau \) at the origin denoted by \( W_{\tau\tau} \) meets

\[
L_0 W_{\tau\tau} = -Q \frac{d^2 N(\Phi_m, \Phi_m)}{\partial \tau^2} = -\left[ e_m \left( \begin{array}{c} -1 \\ 1 \end{array} \right) \phi_m^2 - P e_m \left( \begin{array}{c} -1 \\ 1 \end{array} \right) \phi_m^2 \right]
\]

\[
= -e_m \left( \begin{array}{c} -1 \\ 1 \end{array} \right) \phi_m^2
\]

\[
= -e_m \left( \begin{array}{c} -1 \\ 1 \end{array} \right) \left( \frac{1}{\sqrt{l}} \phi_0 + \frac{1}{\sqrt{2l}} \phi_{2m} \right),
\]

where

\[
e_i = \frac{2\delta}{k + \delta^2} (2d_2 \lambda_i - 2g_1 + 1).
\]

Letting \( W_{\tau\tau} = \sum_{i=0}^{\infty} \left( \begin{array}{c} c_i \\ d_i \end{array} \right) \phi_i \), we have

\[
L_0 W_{\tau\tau} = \sum_{i=0}^{\infty} B_i \left( \begin{array}{c} c_i \\ d_i \end{array} \right) \phi_i = -e_m \left( \begin{array}{c} -1 \\ 1 \end{array} \right) \left( \frac{1}{\sqrt{l}} \phi_0 + \frac{1}{\sqrt{2l}} \phi_{2m} \right),
\]

where \( B_i \) is given by (12). Thus, we get

\[
W_{\tau\tau} = -e_m \left[ \frac{1}{\sqrt{l}} B_0^{-1} \left( \begin{array}{c} -1 \\ 1 \end{array} \right) \phi_0 + \frac{1}{\sqrt{2l}} B_{2m}^{-1} \left( \begin{array}{c} -1 \\ 1 \end{array} \right) \phi_{2m} \right]
\]

\[
= -e_m \left[ \frac{1}{l} B_0^{-1} \left( \begin{array}{c} -1 \\ 1 \end{array} \right) + B_{2m}^{-1} \left( \begin{array}{c} -1 \\ 1 \end{array} \right) \frac{2\pi m}{l} \right] x,
\]
where $B_i^{-1}$ can be obtained by $B_i^{-1} = \frac{B_i^*}{|B_i|}$ and $B_i^*$ is the adjoint matrix of $B_i$. In view of $f_0 + g_0 = 0$, $f_1 + g_1 = -1$ and

$$|B_i| = \frac{(\lambda_j - \lambda_i)(\lambda_m - \lambda_i)}{\lambda_j \lambda_m} g_0,$$

we have

$$W_{\tau \tau} = -\frac{\varepsilon_m}{l g_0} \left[ \begin{array}{c} 1 \\ 0 \end{array} \right] + \frac{\lambda_j}{3(\lambda_2m - \lambda_j)} \left( 1 + d_2\lambda_2m \right) \cos \frac{2\pi m}{l} x. \quad (23)$$

From (22) and (23), the first and second terms of $c_{010}$ are respectively given by

$$c_{010}^1 = \langle \Phi_m^*, \frac{1}{2} d^2 N(\Phi_m, W_{\tau \tau}) \rangle = -\varepsilon_m (1 - a_m^*) \Gamma_{j,2m},$$

$$c_{010}^2 = \frac{3a_m(1 - a_m^*)}{2l(1 + a_m a_m^*)},$$

where $\varepsilon_i = c_i \delta = (g_1 + 1)(2d_2\lambda_i - 2g_1 + 1)$, $\Gamma_{i,n} = \frac{\lambda_i - 6\lambda_n - 2d_2\lambda_i\lambda_n + \frac{4}{a^2}}{3(\lambda_i - \lambda_n)}$. Thus, we conclude

$$c_{010} = c_{010}^1 + c_{010}^2 = \frac{1 - a_m^*}{2l g_0 (1 + a_m a_m^*)} C_0,$$

$$C_0 = 3(d_2\lambda_m - g_1) - \varepsilon_m \Gamma_{j,2m}.$$

### 3.1. The case $(j, m) = (1, 2)$

For this case, we have

$$d_2 = \frac{\sqrt{25 + 16g_1} - 5}{8\lambda_1}.$$

From [2], if $a_{001} c_{010} b_0 d_0 c_{010} \neq 0$, then the reduced problem (20) is equivalent to the normal form

$$\begin{pmatrix}
-s(\tau + \varepsilon_1 \lambda) \\
\tau (\varepsilon_3 \tau^2 + \varepsilon_4 \lambda) + \varepsilon_2 s^2
\end{pmatrix} \quad (24)$$

where

$$\varepsilon_1 = -\text{sgn} a_{001}, \quad \varepsilon_2 = -\text{sgn}(b_0 d_0), \quad \varepsilon_3 = \text{sgn} c_{010}, \quad \varepsilon_4 = \text{sgn} c_{001}.$$

In this case, we have

$$a_{001} = \langle \Phi_1^*, M \Phi_1 \rangle = -\frac{\lambda_1 a_1 a_1^*}{1 + a_1 a_1^*} > 0,$$

$$c_{001} = \langle \Phi_2^*, M \Phi_2 \rangle = -\frac{\lambda_2 a_2 a_2^*}{1 + a_2 a_2^*} > 0,$$

$$b_0 = \langle \Phi_1^*, d^2 N(\Phi_1, \Phi_2) \rangle = \frac{(1 - a_1^*)(e_1 + e_2)}{2\sqrt{2l}(1 + a_1 a_1^*)},$$

$$d_0 = \frac{1}{2} \langle \Phi_2^*, d^2 N(\Phi_1, \Phi_1) \rangle = \frac{(1 - a_2^*) e_1}{2\sqrt{2l}(1 + a_2 a_2^*)},$$

$$c_{010} = \frac{1 - a_2^*}{2l g_0 (1 + a_2 a_2^*)} C.$$
where
\[ C = \left( \sqrt{25 + 16g_1} - 2g_1 - 4 \right) \left\{ \frac{3}{2} + \frac{g_1 + 1}{45} \left[ \frac{2}{g_1} - 4 \right] \sqrt{25 + 16g_1} - 75 + \frac{10}{g_1} \right\} - 3. \]

Based on \( a_{001} > 0 \) and \( c_{001} > 0 \), we know \( \varepsilon_1 = -1, \varepsilon_4 = 1 \) in (24). Due to \( g_1 < 1 \), from (25) we need \( e_1 + e_2 = 0, e_1 = 0 \) and \( C = 0 \) for \( b_0 = 0, d_0 = 0 \) and \( c_{010} = 0 \), respectively. This implies that \( g_1 = \frac{5\sqrt{57} - 9}{32} < 1 \), \( g_1 = \sqrt{\frac{3}{8}} < 1 \) and \( g_1 \neq \hat{c} \) (see Fig. 2).

![Figure 2. The zero of \( C = 0 \).](image)

**Theorem 3.1.** If \( g_1 \neq \frac{5\sqrt{57} - 9}{32}, g_1 \neq \sqrt{\frac{3}{8}} \) and \( g_1 \neq \hat{c} \), then the reduced problem (20) is equivalent to the normal form
\[
\begin{align*}
-s(\tau - \lambda) &= 0, \\
\tau(\varepsilon_3 \tau^2 + \lambda) + \varepsilon_2 s^2 &= 0,
\end{align*}
\]
where
\[
\varepsilon_2 = \begin{cases} 
-1, & g_1 \in \left( 0, \sqrt{\frac{3}{8}} \right) \cup \left( \frac{5\sqrt{57} - 9}{32}, 1 \right), \\
+1, & g_1 \in \left( \sqrt{\frac{3}{8}}, \frac{5\sqrt{57} - 9}{32} \right)
\end{cases}
\]
and \( \varepsilon_3 = \begin{cases} 
+1, & g_1 \in (0, \hat{c}), \\
-1, & g_1 \in (\hat{c}, 1).
\end{cases} \)

Hence, the reduced problem (20) can be solved by equivalence from Theorem 3.1. Then, for \( d_1^{(1)} = d_1^{(2)} \), (1) has the solution as the following form
\[
\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u^* \\ v^* \end{pmatrix} + s \begin{pmatrix} a_1 \\ 1 \end{pmatrix} \phi_1 + \tau \begin{pmatrix} a_2 \\ 1 \end{pmatrix} \phi_2 + W(s, \tau, \lambda),
\]
where \( W(s, \tau, \lambda) \) satisfies \( W(0, 0, \lambda) = W_\lambda(0, 0, 0) = W_{\lambda\lambda}(0, 0) = \cdots = 0 \), \( W_s(0, 0, 0) = 0 \) and \( W_\tau(0, 0, 0) = 0 \). The solution is characterized by two modes \( \phi_1 \) and \( \phi_2 \), and the corresponding double bifurcation is the first bifurcation.
Remark 2. From Theorem 3.1 above, it turns out that there exist either two or four new solutions bifurcating from the constant solution at the double eigenvalue.

Remark 3. From (16), we know that when \( j \) and \( m \) are the adjacent integer, then the corresponding double bifurcation is the first bifurcation.

3.2. The case \( j > 1 \). For this case, we define the determinants \( D_j (j = 1, 2, 3) \) by

\[
D_1 = a_{100}c_{010} - a_{010}c_{100}, \quad D_2 = a_{100}c_{001} - a_{001}c_{100}, \quad D_3 = a_{010}c_{001} - a_{001}c_{010}.
\]

From [2], if \( a_{100}a_{001}c_{010}c_{001}D_1D_2D_3 \neq 0 \), then the reduced problem (20) is equivalent to the normal form

\[
\begin{align*}
& (s(\varepsilon_1 s^2 + \rho r^2 - \varepsilon_3 \lambda + \mu_1 s^{m-2}r)) \\
& (\tau(\varepsilon_2 r^2 + \kappa s^2 - \varepsilon_4 \lambda + \mu_2 s^m r^{j-2}))
\end{align*}
\]

with

\[
\varepsilon_1 = \text{sgn} a_{100}, \quad \varepsilon_2 = \text{sgn} a_{010}, \quad \varepsilon_3 = -\text{sgn} a_{001}, \quad \varepsilon_4 = -\text{sgn} c_{001},
\]

\[
\rho = \frac{a_{010}}{a_{100}} \frac{c_{100}}{c_{010}}, \quad \kappa = \frac{c_{100}}{c_{010}} \frac{a_{010}}{a_{100}}.
\]

The parameters \( \mu_1, \mu_2 \) are determined as follows:

\[
\begin{align*}
\mu_1 &= \text{sgn} d_0, \quad \mu_2 = \frac{d_0}{|b_0|} \left| \frac{c_{010}}{a_{100}} \frac{a_{001}}{c_{001}} \right|^2 & \text{if } b_0 \neq 0, \\
\mu_1 &= 0, \quad \mu_2 = \text{sgn} d_0 & \text{if } b_0 = 0,
\end{align*}
\]

\[
\mu_1 = \mu_2 = 0 & \text{ if } b_0 = d_0,
\]

i.e. we have to require that \( b_0 \neq 0 \) for \((j, m) = (2, 3)\) and \( d_0 \neq 0 \) for \( m = 3 \).

In this case, we have

\[
\begin{align*}
a^{100} &= \langle \Phi^*_j, H_{300} \rangle = \left( \Phi^*_j, \frac{1}{2} d^2 N(\Phi_j, W_{ss}) \right) + \left( \Phi^*_j, \frac{1}{3!} d^3 N(\Phi_j^3) \right), \\
a^{010} &= \langle \Phi^*_j, H_{120} \rangle = \left( \Phi^*_j, \frac{1}{2} d^2 N(\Phi_j, W_{\tau \tau}) + \frac{1}{2} d^3 N(\Phi_j, \Phi_j^3) + d^2 N(\Phi_j, W_{\tau \tau}) \right), \\
c^{100} &= \langle \Phi^*_j, H_{210} \rangle = \left( \Phi^*_m, \frac{1}{2} d^2 N(\Phi_m, W_{ss}) + \frac{1}{2} d^3 N(\Phi_j^3, \Phi_m) + d^2 N(\Phi_j, W_{\tau \tau}) \right).
\end{align*}
\]

In the same way, the derivative \( W_{ss} \) meets

\[
L_0 W_{ss} = -Q d^2 N(\Phi_j, \Phi_j)
\]

\[
= -\left[ e_j \left( -1 \right) \phi^2_j - Pe_j \left( -1 \right) \phi^2_j \right] = \left\{ \begin{array}{ll}
- e_j \left( -1 \right) \phi^2_j, & m \neq 2j, \\
- e_j \left( -1 \right) \phi^2_j + \frac{e_j(1 - a^*_m)}{\sqrt{2l}(1 + a_m a^*_m)} \left( a_m \right) \phi_m, & m = 2j \\
- e_j \left( -1 \right) \left( \frac{1}{\sqrt{l}} \phi_0 + \frac{1}{\sqrt{2l}} \phi_{2j} \right), & m \neq 2j, \\
- e_j \left( -1 \right) \phi_0 + \frac{e_j(1 + a_{2j})}{2l(1 + a_{2j} a^*_{2j})} \left( -a^*_{2j} \right) \phi_{2j}, & m = 2j.
\end{array} \right.
\]
Letting $W_{ss} = \sum_{i=0}^{\infty} \left( \frac{c_i}{d_i} \right) \phi_i$, we conclude

$$L_0 W_{ss} = \sum_{i=0}^{\infty} B_i \left( \frac{c_i}{d_i} \right) \phi_i = \left\{ \begin{array}{ll}
- e_j \left( \frac{1}{1} \right) \left( \frac{1}{\sqrt{1}} \phi_0 + \frac{1}{\sqrt{2l}} \phi_{2j} \right), & m \neq 2j, \\
- e_j \left( \frac{1}{1} \right) \phi_0 + \frac{e_j(1 + a_{2j})}{\sqrt{2l}(1 + a_{2j}a_{2j}^*)} \left( \frac{1}{-a_{2j}^*} \right) \phi_{2j}, & m = 2j.
\end{array} \right.$$  

When $m \neq 2j$, we have

$$W_{ss} = - e_j \left[ \frac{1}{\sqrt{1}} B_0^{-1} \left( \frac{1}{1} \right) \phi_0 + \frac{1}{\sqrt{2l}} B_{2j}^{-1} \left( \frac{1}{1} \right) \phi_{2j} \right] = - e_j \left[ \frac{B_0^{-1} \left( \frac{1}{1} \right) \phi_0 + \frac{1}{\sqrt{2l}} B_{2j}^{-1} \left( \frac{1}{1} \right) \phi_{2j} \right] = - e_j \left[ \frac{1}{l \phi_0} + \frac{\lambda_m}{3(l \lambda_m - \lambda_m)} \left( \frac{1 + d_2 \lambda_{2j}}{-a_{1j}^*} \right) \cos \frac{2\pi j}{l} x \right].$$

When $m = 2j$, we know

$$d_2 = \frac{\sqrt{25 + 16q_1}}{8\lambda_j}.$$  

According to

$$B_0 \left( \frac{c_0}{d_0} \right) \phi_0 + B_{2j} \left( \frac{c_{2j}}{d_{2j}} \right) \phi_{2j} = - e_j \left( \frac{1}{1} \right) \phi_0 + \frac{e_j(1 + a_{2j})}{\sqrt{2l}(1 + a_{2j}a_{2j}^*)} \left( \frac{1}{-a_{2j}^*} \right) \phi_{2j},$$

from $B_i^{-1} = \frac{B_i^*}{|B_i|}$ we have

$$\left( \frac{c_0}{d_0} \right) = - e_j \left( \frac{1}{1} \right) \phi_0 = \frac{1}{\lambda_j \phi_0} \left( \frac{1}{0} \right).$$

But it is worth noting that $|B_{2j}| = 0$, so we cannot solve $\left( \frac{c_{2j}}{d_{2j}} \right)$ as above. Based on the equation

$$B_{2j} \left( \frac{c_{2j}}{d_{2j}} \right) = \frac{e_j(1 + a_{2j})}{\sqrt{2l}(1 + a_{2j}a_{2j}^*)} \left( \frac{1}{-a_{2j}^*} \right),$$

we solve algebraically for $\left( \frac{c_{2j}}{d_{2j}} \right)$ to get infinitely many solutions

$$\left( \frac{c_{2j}}{d_{2j}} \right) = \left( \frac{e_j(1 + a_{2j})}{\sqrt{2l}(1 + a_{2j}a_{2j}^*)(f_0 - d_{1j}^* \lambda_{2j})} + a_{2j} \right) + k \left( \frac{a_{2j}}{1} \right)$$

for any number $k$. Then it follows that

$$W_{ss} = - \frac{e_j}{\sqrt{l \phi_0}} \left( \frac{1}{0} \right) \phi_0 + \left[ \frac{e_j(1 + a_{2j})}{\sqrt{2l}(1 + a_{2j}a_{2j}^*)(f_0 - d_{1j}^* \lambda_{2j})} + a_{2j} \right] \phi_{2j}.$$
However, we note that $W_{ss} \in X_1$ is an important point to be considered here, and $W_{ss}$ subtracts off an appropriate multiple of $\Phi$ and/or $\Phi_m$ to be the solution to $L_0W_{ss} = -Qd^2N(\Phi_j, \Phi_j)$ in $X_1$. Then we can find the unique solution $W_{ss}$ in $X_1$.

It is obvious that the first term of $W_{ss}$ is in $X_1$, and then we only need the second term, denoted by $W_2$, belongs to $X_1$. We know that $W_2 - \langle W_2, \Phi_j \rangle \Phi_j - \langle W_2, \Phi_m \rangle \Phi_m \in X_1$. Then for $m = 2j$, it follows that

$$W_2 - \langle W_2, \Phi_j \rangle \Phi_j - \langle W_2, \Phi_m \rangle \Phi_m = W_2 - \langle W_2, \Phi_j \rangle \Phi_{2j} = \frac{e_j(1 + a_{2j})}{\sqrt{2l}(1 + a_{2j}a_{2j}^*)^2(f_0 - d_1^{(j)}\lambda_{2j})} \left( \frac{1}{-a_{2j}^*} \right) \phi_{2j}.$$ 

Thus, when $m = 2j$, we get

$$W_{ss} = -\frac{e_j}{lg_0} \left[ \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \phi_0 + \frac{e_j(1 + a_{2j})}{\sqrt{2l}(1 + a_{2j}a_{2j}^*)^2(f_0 - d_1^{(j)}\lambda_{2j})} \left( \frac{1}{-a_{2j}^*} \right) \phi_{2j} \right] = -\frac{e_j}{lg_0} \left[ (f_0 + g_1 - d_2\lambda_{2j})(f_0 - d_1^{(j)}\lambda_{2j}) \left( \frac{1}{-a_{2j}^*} \right) \cos \frac{2\pi j}{l}x \right].$$

Therefore, by the calculations, we obtain

$$a_{100}^1 = \left\langle \Phi_j^*, \frac{1}{2} d^2 N(\Phi_j, W_{ss}) \right\rangle = \begin{cases} -\hat{e}_j(1 - a_j^*)\Gamma_{m,2j}, & m \neq 2j, \\ \frac{2lg_0(1 + a_ja_j^*)}{2l}(l_{2j,j} + 2), & m = 2j, \end{cases}$$

$$a_{100}^2 = \left\langle \Phi_j^*, \frac{1}{3} d^3 N(\Phi_j^3) \right\rangle = \frac{3a_j(1 - a_j^*)}{2l(1 + a_ja_j^*)},$$

where $l_{i,n} = \frac{(f_0 + g_1 - d_2\lambda_i)(f_0 - \frac{g_0\lambda_i}{d_2\lambda_i, \lambda_{2j}} + d_2\lambda_n - g_1 + 1)}{(f_0 - \frac{g_0\lambda_i}{d_2\lambda_i, \lambda_{2j}} + g_1 - d_2\lambda_i)^2}$. Then we have

$$a_{100} = a_{100}^1 + a_{100}^2 = \begin{cases} 1 - a_j^* \frac{2lg_0(1 + a_ja_j^*)}{A_1}, & m \neq 2j, \\ \frac{1 - a_j^*}{2l}(1 + a_ja_j^*) \hat{A}_1, & m = 2j, \end{cases}$$

where

$$A_1 = 3(d_2\lambda_j - g_1) - \tilde{e}_j\Gamma_{m,2j},$$

$$\hat{A}_1 = 3(d_2\lambda_j - g_1) - \tilde{e}_j(l_{2j,j} + 2).$$

Similarly, the second order derivative of $W$ with respect to $s$ and $\tau$ at the origin denoted by $W_{ss\tau}$ meets

$$L_0W_{ss\tau} = -Qd^2N(\Phi_j, \Phi_m) = -\left[ e \left( \begin{array}{c} -1 \\ 1 \end{array} \right) \phi_j\phi_m - Pc \left( \begin{array}{c} -1 \\ 1 \end{array} \right) \phi_j\phi_m \right].$$
Due to

we can obtain

with $e = \frac{e_j + e_m}{2}$.

In the same way as above, for $m \neq 2j$, we have

$$W_{sr} = -\frac{e}{\sqrt{2l}} \left[ B_{m-j}^{-1} \left( \begin{array}{c} -1 \\ 1 \end{array} \right) + \phi_{m-j} B_{m+j}^{-1} \left( \begin{array}{c} -1 \\ 1 \end{array} \right) \phi_{m+j} \right]$$

$$= -\frac{e}{l} \left[ B_{m-j}^{-1} \left( \begin{array}{c} -1 \\ 1 \end{array} \right) \cos \frac{\pi(m-j)}{l} x + B_{m+j}^{-1} \left( \begin{array}{c} -1 \\ 1 \end{array} \right) \cos \frac{\pi(m+j)}{l} x \right]$$

$$= -\frac{e}{l \gamma_0} \left[ \lambda_j \lambda_m \left( 1 + d_2 \lambda_{m-j} \right) \cos \frac{\pi(m-j)}{l} x + \lambda_j \lambda_m \left( 1 + d_2 \lambda_{m+j} \right) \cos \frac{\pi(m+j)}{l} x \right]$$

with $\nu = (\lambda_j - \lambda_i)(\lambda_m - \lambda_i)$ by the Fourier transformation techniques.

For $m = 2j$, in view of

$$B_{3j} \left( \begin{array}{c} c_{3j} \\ d_{3j} \end{array} \right) \phi_{3j} + B_j \left( \begin{array}{c} c_j \\ d_j \end{array} \right) \phi_j = -\frac{e}{\sqrt{2l}} \left( \begin{array}{c} -1 \\ 1 \end{array} \right) \phi_{3j} + \frac{e(1 + a_j)}{\sqrt{2l}(1 + a_j a_j^*)} \left( \begin{array}{c} 1 \\ -a_j^* \end{array} \right) \phi_j,$$

we can obtain

$$\left( \begin{array}{c} c_{3j} \\ d_{3j} \end{array} \right) = -\frac{e}{\sqrt{2l}} B_{3j}^{-1} \left( \begin{array}{c} -1 \\ 1 \end{array} \right) = -\frac{e}{10 \sqrt{2l} \gamma_0} \left( \begin{array}{c} 1 + d_2 \lambda_{3j} \\ -d_1^{(j)} \lambda_{3j} \end{array} \right).$$

Due to $|B_j| = 0$, we solve for $\left( \begin{array}{c} c_j \\ d_j \end{array} \right)$ from

$$B_j \left( \begin{array}{c} c_j \\ d_j \end{array} \right) = \frac{e_j(1 + a_j)}{\sqrt{2l}(1 + a_j a_j^*)} \left( \begin{array}{c} 1 \\ -a_j^* \end{array} \right)$$

to get infinitely many solutions

$$\left( \begin{array}{c} c_j \\ d_j \end{array} \right) = \left( \frac{e(1 + a_j)}{\sqrt{2l}(1 + a_j a_j^*) (f_0 - d_1^{(j)} \lambda_j)} + a_j \right) + k \left( \begin{array}{c} a_j \\ 1 \end{array} \right)$$

for any number $k$. Since $W_{sr} \in X_1$, for $m = 2j$, by the same method as above we get the only solution in $X_1$ as

$$W_{sr} = -\frac{e}{10 \sqrt{2l} \gamma_0} \left( \begin{array}{c} 1 + d_2 \lambda_{3j} \\ -d_1^{(j)} \lambda_{3j} \end{array} \right) \phi_{3j} + \frac{e(1 + a_j)}{\sqrt{2l}(1 + a_j a_j^*)^2 (f_0 - d_1^{(j)} \lambda_j)} \left( \begin{array}{c} 1 \\ -a_j^* \end{array} \right) \phi_j.$$
where

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Hence, we have

\[ A = \text{Similarly, we conclude} \]

\[ \tilde{\gamma} = \left( \begin{array}{c} \gamma_1 \\dot{\gamma} \end{array} \right) \]

\[ = 2(\gamma_1 + \gamma_2) \]

\[ \left( \begin{array}{ccc} 2 & \ldots & 2 \\ \text{etc.} \\ 2 & \ldots & 2 \\ \end{array} \right) \]

\[ a_{010} = \left\{ \begin{array}{ll} -\left( 1 - \alpha^2 \right) & \text{if } m \neq 2j, \\ \frac{1 - \alpha^2}{l_0} \right\}_0, \quad \text{if } m = 2j, \]

\[ \Lambda_{i,n} = \frac{\lambda_i \lambda_{m-j} (1 + d_2 \lambda_m) + \lambda_j \lambda_m (1 + d_2 \lambda_{m-j}) - \lambda_{m,j} \lambda_{m+j}}{\nu_{m-j} + \nu_{m+j}} \]

\[ \right\} \frac{\lambda_i \lambda_{m-j} (1 + d_2 \lambda_m) + \lambda_j \lambda_m (1 + d_2 \lambda_{m-j}) - \lambda_{m,j} \lambda_{m+j}}{\nu_{m-j} + \nu_{m+j}} \]

\[ \text{where } \hat{e} = e \delta = \hat{e}_j + \hat{e}_m = \frac{2}{2} \]

\[ a_{010} = a_{010} + a_{010} + a_{010} \]

\[ = \left\{ \begin{array}{ll} \frac{1 - \alpha^2}{l_0} \right\}_0, \quad \text{if } m \neq 2j, \\ \frac{1 - \alpha^2}{l_0} \right\}_0, \quad \text{if } m = 2j, \]

\[ \text{where} \]

\[ A_0 = 2(d_2 \lambda_m - d_1 - d_2 \lambda_j - d_1 - \hat{e}_m - \hat{e} \Lambda_{j,m}, \]

\[ \hat{A}_0 = 2(d_2 \lambda_m - d_1 - d_2 \lambda_j - d_1 - \hat{e}_m - \hat{e} \left[ \frac{1 - \alpha^2}{l_0} \right] \text{if } m \neq 2j, \\ \end{array} \right. \]

\[ \frac{1 - \alpha^2}{l_0} \right\}_0 \text{ if } m = 2j, \]

\[ \text{Similarly, we conclude} \]

\[ c_{100} = \left\{ \begin{array}{ll} -\left( 1 - \alpha^2 \right) & \text{if } m \neq 2j, \\ \frac{1 - \alpha^2}{l_0} \right\}_0, \quad \text{if } m = 2j, \]

\[ = \left\{ \begin{array}{ll} -\left( 1 - \alpha^2 \right) & \text{if } m \neq 2j, \\ \frac{1 - \alpha^2}{l_0} \right\}_0, \quad \text{if } m = 2j, \]
Thus, we have

$$
C_1 = 2(d_2 \lambda_j - g_1) + d_2 \lambda_m - g_1 - \hat{c}_j - \hat{c} \Lambda_{m,j},
$$

$$
\hat{C}_1 = 2(d_2 \lambda_j - g_1) + d_2 \lambda_m - g_1 - \hat{c}_j - \hat{c} \left[ l_{j,j} + \frac{1}{40} \left( 40 + 72d_2 \lambda_j - \frac{9}{d_2 \lambda_j} \right) \right].
$$

Thus, we have

$$
D_1 = a_{100} c_{010} - a_{010} c_{100} = \frac{(1 - a_j^*)(1 - a_m^*)}{4l^2 g_0^2(1 + a_j a_j^*)(1 + a_m a_m^*)} \left\{ \begin{array}{ll}
p_1, & m \neq 2j, \\
\tilde{p}_1, & m = 2j,
\end{array} \right.
$$

$$
D_2 = a_{100} c_{001} - a_{001} c_{100} = \frac{(g_1 - d_2 \lambda_j)(g_1 - d_2 \lambda_m)}{2d_2^2 f_1^2 g_0^2(1 + a_j a_j^*)(1 + a_m a_m^*)} \left\{ \begin{array}{ll}
p_2, & m \neq 2j, \\
\tilde{p}_2, & m = 2j,
\end{array} \right.
$$

$$
D_3 = a_{010} c_{001} - a_{001} c_{100} = \frac{(g_1 - d_2 \lambda_j)(g_1 - d_2 \lambda_m)}{2d_2^2 f_1^2 g_0^2(1 + a_j a_j^*)(1 + a_m a_m^*)} \left\{ \begin{array}{ll}
p_3, & m \neq 2j, \\
\tilde{p}_3, & m = 2j,
\end{array} \right.
$$

where

$$
p_1 = A_1 C_0 - 4A_0 C_1, \quad \tilde{p}_1 = \tilde{A}_1 C_0 - 4\tilde{A}_0 \tilde{C}_1,
$$

$$
p_2 = A_1 (g_1 - d_2 \lambda_m) - 2C_1 (g_1 - d_2 \lambda_j), \quad \tilde{p}_2 = \tilde{A}_1 (g_1 - d_2 \lambda_m) - 2\tilde{C}_1 (g_1 - d_2 \lambda_j),
$$

$$
p_3 = 2A_0 (g_1 - d_2 \lambda_m) - C_0 (g_1 - d_2 \lambda_j), \quad \tilde{p}_3 = 2\tilde{A}_0 (g_1 - d_2 \lambda_m) - \tilde{C}_0 (g_1 - d_2 \lambda_j).
$$

**Theorem 3.2.** If $A_1 C_0 p_1 p_2 p_3 \neq 0$ for $m \neq 2j$ and $\tilde{A}_1 C_0 \tilde{p}_1 \tilde{p}_2 \tilde{p}_3 \neq 0$ for $m = 2j$, then the reduced problem (20) is equivalent to the normal form

$$
\left( s(\varepsilon_1 s^2 + \rho \tau^2 + \lambda + \mu_1 s^{m-2} \tau^j) \right)
\left( \tau(\varepsilon_2 \tau^2 + \kappa s^2 + \lambda + \mu_2 s^{m+2} \tau^{j-2}) \right)
$$

with the $\varepsilon_i$ and $\rho, \kappa$ given by

$$
\varepsilon_1 = \begin{cases} 
\text{sgn} A_1, & m \neq 2j, \\
\text{sgn} \tilde{A}_1, & m = 2j,
\end{cases}
$$

$$
\rho = \begin{cases} 
\frac{4|A_0|C_1}{|A_1 C_0|}, & m \neq 2j, \\
\frac{4|\tilde{A}_0|\tilde{C}_1}{|\tilde{A}_1 C_0|}, & m = 2j,
\end{cases}
$$

$$
\kappa = \begin{cases} 
\frac{4|A_0|C_1}{|A_1 C_0|}, & m \neq 2j, \\
\frac{4|\tilde{A}_0|\tilde{C}_1}{|\tilde{A}_1 C_0|}, & m = 2j.
\end{cases}
$$

The parameters $\mu_1, \mu_2$ are determined as follows:

$$
\mu_1 = \text{sgn} b_0, \quad \mu_2 = \begin{cases} 
\frac{d_0 |C_0|}{|b_0 A_1|} q, & m \neq 2j, \\
\frac{d_0 |C_0|}{|b_0 \tilde{A}_1|} q, & m = 2j
\end{cases}
$$

if $b_0 \neq 0$,

$$
\mu_1 = 0, \quad \mu_2 = \text{sgn} d_0 \text{ if } b_0 = 0, \quad d_0 \neq 0,
$$

$$
\mu_1 = \mu_2 = 0 \text{ if } b_0 = d_0,
$$

where

$$
q = \frac{\lambda_j (g_1 - d_2 \lambda_j)^2 (f_0 - d_1^{(j)} \lambda_m + g_1 - d_2 \lambda_m)}{\lambda_m (g_1 - d_2 \lambda_m)^2 (f_0 - d_1^{(j)} \lambda_j + g_1 - d_2 \lambda_j)}.
$$
Remark 4. From Theorem 3.2, for $j > 1$, the solution of double bifurcation has the form
\[
\begin{pmatrix}
u \\
v
\end{pmatrix} = \begin{pmatrix} u^* \\ v^*
\end{pmatrix} + s\begin{pmatrix} a_j \\ 1
\end{pmatrix} \phi_j + \tau\begin{pmatrix} a_m \\ 1
\end{pmatrix} \phi_m + W(s, \tau, \lambda),
\] (27)
where $W(0, 0, \lambda) = 0$, $W_\lambda(0, 0, 0) = W_{\lambda\lambda}(0, 0) = \cdots = 0$, $W_s(0, 0, 0) = 0$ and $W_\tau(0, 0, 0) = 0$.

4. Stability of bifurcation solutions. In this section, we discuss the stability of the solution (15) of simple bifurcation and the solution (27) of double bifurcation, in which there is an open problem for the stability of the solution from first double bifurcation such as (26). To facilitate the discussion, we denote $d_1^{(m_1)} := \min\{d_1^{(j)} : \lambda_j < \frac{g_1}{d_2}\}$.

**Theorem 4.1.** Assume that $j \neq m_1$. Then $L_0$ has a positive eigenvalue, and the solutions of simple and double bifurcations are unstable.

**Proof.** Suppose that $\mu$ is an eigenvalue of $L_0$ with a corresponding eigenfunction $\phi(x), \psi(x)$. Then
\[
d_1^{(j)} \frac{\partial^2 \phi}{\partial x^2} + (f_0 - \mu)\phi + f_1\psi = 0, \quad d_2 \frac{\partial^2 \psi}{\partial x^2} + g_0\phi + (g_1 - \mu)\psi = 0.
\]
Letting $\phi(x) = \sum_{i=0}^{\infty} c_i \phi_i$, $\psi(x) = \sum_{i=0}^{\infty} d_i \phi_i$, we have
\[
\sum_{i=0}^{\infty} \left( f_0 - d_1^{(j)} \lambda_i - \mu \right. \left. \begin{array}{c} f_1 \\ g_0 \\ g_1 - d_2 \lambda_i - \mu \end{array} \right) \begin{pmatrix} c_i \\ d_i \\ c_i \\ d_i 
\end{pmatrix} \phi_i = 0.
\]
It follows that the eigenvalues of $L_0$ are given by
\[
\mu^2 + P_i(d_1^{(j)}) \mu + Q_i(d_1^{(j)}) = 0, \quad i \geq 0,
\]
where
\[
P_i(d_1^{(j)}) = (d_1^{(j)} + d_2)\lambda_i - (f_0 + g_1),
\]
\[
Q_i(d_1^{(j)}) = -d_1^{(j)} \lambda_i (g_1 - d_2 \lambda_i) + g_0 (1 + d_2 \lambda_i).
\]
By the condition (C), we have $P_i(d_1^{(j)}) > 0, \quad i \geq 0$. When $j \neq m_1$, $Q_m(d_1^{(j)}) < 0$ from (29), and then $L_0$ has a positive eigenvalue. Thus, according to the perturbation theory of linearized operator, the bifurcation solutions of simple and double bifurcation described by (15) and (27) are unstable for $j \neq m_1$. The proof is completed. \hfill \Box

**Lemma 4.2.** Suppose that $j = m_1$ and $d_1^{(j)} \neq d_1^{(m)}$ for any integer $m \neq j$. Then 0 is a simple eigenvalue of $L_0$ with the largest real part, and all the other eigenvalues of $L_0$ lie in the left half complex plane.

**Proof.** For $j = m_1$, it follows that
\[
N(L_0) = \text{span}\{\Phi_{m_1}\}, \quad N(L_0^*) = \text{span}\{\Phi_{m_1}^*\}
\]
with $(\Phi_{m_1}, \Phi_{m_1}^*) = 1 > 0$, which implies $\Phi_{m_1} \notin R(L_0)$ by the Fredholm alternative, and then 0 a simple eigenvalue of $L_0$. From (28) and (29), we have that for all $i,
\[
P_i(d_1^{(m_1)}) > 0, \quad Q_{m_1}(d_1^{(m_1)}) = 0, \quad Q_i(d_1^{(m_1)}) > 0, \quad i = 0, 1, 2, \cdots, m_1 - 1, m_1 + 1, \cdots.
\]
Hence, 0 is a simple eigenvalue of $L_0$ with the largest real part, and all the other eigenvalues of $L_0$ lie in the left half complex plane. Thus we complete the proof. \hfill \Box
Therefore, from (14), we have the following stability result for the solution (15) of simple bifurcation for $j = m_1$ on the basis of Lemma 4.2 and [12], where the stability theorem in [31] is not valid.

**Theorem 4.3.** Suppose that $j = m_1$ and $d^{(j)}_1 \neq d^{(m)}_1$ for any integer $m \neq j$. If $h_{sss}(0, 0) < 0$ ($> 0$), then the bifurcation solution (15) is stable (unstable) for both $s < 0$ and $s > 0$.

5. **Numerical results.** The goal of this section is to present the numerical simulations for complementing the analytic results in the previous sections. We perform the initial-boundary-value problem (1) numerically by use of a standard implicit method, that is, the Crank-Nicholson scheme. Here, we transform the spatial domain from $0 < x < l$ to $0 < \hat{x} < 1$ by putting $\hat{x} = x/l$, and still denote $\hat{x}$ by $x$. The numerical results presented below are plotted at sufficiently large times so that the system reaches a steady state.

![Figure 3. The graph of (2) with $k = 0.1, \delta = 3.0$ and $l = 6.0$. Here, (a) $d_2 = 0.105$; (b) $d_2 = \frac{\sqrt{(\lambda_3 + \lambda_4)^2 + 4\lambda_1 \lambda_4 (\lambda_3 + \lambda_4)}}{2\lambda_3 \lambda_4} = 0.1200$.](image)

![Figure 4. Numerical simulations of the steady state solution characterized by $\phi_4$ for system (1) with $k = 0.1, \delta = 3.0, l = 6.0, d_2 = 0.105$ and $d_1 = 5.8560$.](image)
(1) For the system (1), we choose fixed values for $k, \delta$ and $l$ in all simulations, namely $k = 0.1, \delta = 3.0$ and $l = 6$. By taking $d_2 = 0.105$, Fig. 3 (a) is obtained by (2), where $i < \sqrt{\frac{q_1 l^2}{d_2}}$ is decided by $\lambda_i < \frac{q_1}{d_2}$. There are five simple bifurcation

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{Concentration profiles for $u$ and $v$ of (1) for $k = 0.1, \delta = 3.0, l = 6.0, d_2 = 0.105$ and $d_1 = 5.8560$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure6.png}
\caption{Numerical simulations of the steady state solution involved two models $\phi_3$ and $\phi_4$ for system (1) with $k = 0.1, \delta = 3.0, l = 6.0, d_2 = 0.1200$ and $d_1 = 7.0093$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure7.png}
\caption{Concentration profiles for $u$ and $v$ of (1) for $k = 0.1, \delta = 3.0, l = 6.0, d_2 = 0.1200$ and $d_1 = 7.0093$.}
\end{figure}
points $d_1^{(i)}$ ($i = 1, 2, 3, 4, 5$) with $d_{1\text{min}} = d_1^{(4)} = 5.8558$. When we choose $d_2 = \sqrt{(\lambda_3 + \lambda_4)^2 + 4g_1\lambda_3\lambda_4 - (\lambda_3 + \lambda_4)} = 0.1200$, there are still five bifurcation points $d_1^{(i)}$ ($i = 1, 2, 3, 4, 5$), but there exists a double bifurcation point $d_1^{(3)} = d_1^{(4)} = 7.0084$ (see Fig. 3 (b)). It is noting that the minimum value $d_{1\text{min}} = d_1^{(3)} = d_1^{(4)} = 7.0084$ is the first double bifurcation points.

(2) Based on Fig. 3 (a) and Section 2, by choosing $d_1 = 5.8560 > d_{1\text{min}}$, it follows that the $\lambda_4$-mode is the most unstable mode, and the system (1) has a spatially inhomogeneous steady-state structure characterized by $\phi_4$, see Figs. 4 and 5.

(3) Based on Fig. 3 (b) and Section 3, by taking $d_1 = 7.0093 > d_1^{(3)}$ (or $d_1^{(4)}$), there exists steady state solution from double bifurcation involving the couple of mode $\phi_4$ and $\phi_5$, which is shown in Figs. 6 and 7.

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