Assouad-Nagata dimension of $C'(1/6)$ groups

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Abstract

We prove that the Assouad-Nagata dimension of any finitely generated (but not necessarily finitely presented) $C'(1/6)$ group is at most 2. Then, we apply this result to show that for any natural numbers $n, k$ with $n \geq 3$, there exists a finitely generated group with asymptotic dimension $n$ and Assouad-Nagata dimension $n+k$.

Asymptotic dimension (asdim) is a coarse invariant of metric spaces, first defined by Gromov to serve as a large-scale analogue of the Lebesgue covering dimension of a topological space [1]. Since then, asymptotic dimension has been studied extensively, and has proven to be a useful invariant in group theory. Famously, finitely generated groups of finite asymptotic dimension satisfy Yu’s Property A (equivalently, are $C^*$-exact), and thus admit a coarse embedding into a Hilbert space and satisfy the Novikov Conjecture [2].

Another notion of dimension is the asymptotic Assouad-Nagata dimension (asdim$_{AN}$) of a metric space. That asdim$_{AN}(X) \leq n$ is by definition a stronger condition than that asdim$(X) \leq n$, and the former implies some interesting properties of the metric space that the latter does not. For instance, if a metric space $(X,d)$ satisfies asdim$_{AN}(X) \leq n$, then for a sufficiently small $\varepsilon > 0$ the scaled metric space $(X,d^\varepsilon)$ admits a quasi-isometric embedding into a product of $n+1$ locally finite trees [3]. The topological dimension of any asymptotic cone of a metric space does not exceed its asymptotic Assouad-Nagata dimension [4]. An analogue of the Morita theorem holds for asymptotic Assouad-Nagata dimension, that is, whenever $X$ is a proper metric space with finite asymptotic Assouad-Nagata dimension, we have asdim$_{AN}(X \times \mathbb{R}) = asdim_{AN}(X) + 1$ [5], though at the time of writing it is unknown whether the same is true for asymptotic dimension. For discrete metric spaces, such as groups with the word metric, asymptotic Assouad-Nagata dimension is equivalent to Assouad-Nagata dimension, so we will use the shorter moniker when talking about groups.

Though not a coarse invariant, asymptotic Assouad-Nagata dimension is a quasi-isometry invariant, and thus a legitimate tool for studying finitely generated groups. As group invariants, asymptotic dimension and Assouad-Nagata dimension have a lot in common, and indeed much work has been done to distinguish the two. Brodskiy, Dydak, and Lang proved that asdim$(\mathbb{Z}_2 * \mathbb{Z}) = 2$ but asdim$_{AN}(\mathbb{Z}_2 * \mathbb{Z}^2) = \infty$ [6]; and Higes proved that for each $n,k \in \mathbb{N}$, there exists a countable, infinitely generated abelian group $G$ and a proper left-invariant metric on $G$ such that, with respect to this metric, asdim$(G) = n$ but asdim$_{AN}(G) = n + k$ [7].
It is natural to ask whether this can hold for finitely generated groups as well, that is, whether there exists a finitely generated group of finite asymptotic dimension, and finite but greater Assouad-Nagata dimension. The answer is yes, and in this paper we prove the following.

**Theorem 1.** For any \( n, k \in \mathbb{N} \) with \( n \geq 3 \), there exists a finitely generated group \( G \) such that \( \text{asdim}(G) = n \) and \( \text{asdim}_{AN}(G) = n + k \).

Small cases when \( n \leq 2 \) are discussed separately. When \( n = 2 \) it is possible to prove something similar, but the control on \( \text{asdim}_{AN}(G) \) is not as precise.

Two essential ingredients are needed in the construction of such a group: Higes’ group and metric, and the following theorem, which we believe is of independent interest.

**Theorem 2.** Every finitely generated \( C'(1/6) \) group has Assouad-Nagata dimension at most 2.

The group \( G \) in Theorem 1 is then constructed from a short exact sequence

\[ 1 \to K \to G \to H \to 1 \]

where \( H \) is a finitely generated \( C'(1/6) \) group and \( K \), with the restriction of the word metric on \( G \), is quasi-isometrically isomorphic to Higes’ group. Here the distinction between finitely presented and infinitely presented \( C'(1/6) \) groups must be emphasized. A \( C'(1/6) \) group is hyperbolic if and only if it is finitely presented. The importance of Theorem 2 is that it holds for infinitely presented groups, and indeed for the proof of Theorem 2 it is necessary that \( H \) be infinitely presented.

Theorem 2 classifies the Assouad-Nagata dimension of all finitely presented \( C'(1/6) \) groups. Since a finitely presented group has asymptotic dimension 1 if and only if it is virtually free \([10, 11]\), Theorem 2 implies that the Assouad-Nagata dimension of a finitely presented \( C'(1/6) \) group is 1 if the group is virtually free and 2 otherwise. In the finitely presented case, this result was likely already known to experts. Although apparently not in the literature, a MathOverflow post by Agol \([12]\) shows how to obtain that \( \text{asdim}(G) \leq 2 \) for \( G \) a finitely presented \( C'(1/6) \) group, using a theorem of Buyalo and Lebedeva that \( \text{asdim}(G) = \text{dim}(\partial G) + 1 \) when \( G \) is hyperbolic \([13]\). One might then wish to derive Theorem 2 for infinitely presented groups using the same result for finitely presented groups, but this approach cannot work in general. This is because in \([14]\), Osajda constructs a sequence of groups and surjective homomorphisms \( G_0 \to G_1 \to G_2 \to \cdots \) such that \( \text{asdim}(G_n) = 2 \) for all \( n \in \mathbb{N} \), but the inductive limit of the sequence has infinite asymptotic dimension.

A technique in many proofs relating hyperbolicity and finiteness of asymptotic dimension (see for example \([15, 18]\)) uses the fact that geodesics in some hyperbolic spaces satisfy a certain finiteness property. In \([15]\), Bowditch associates a set of *tight* geodesics to every two points in the curve graph of a surface of positive complexity. The cardinality of each cross-section of the set of tight geodesics between two points has a finite upper bound which is uniform, depending only on the space itself. Bowditch uses this property to prove acylindricity of the action of the mapping class group on the curve graph \([15]\), and in \([16]\) Bell and Fujiwara use it to show that the curve graph has finite asymptotic dimension. In this paper we use a similar technique. Although infinitely presented \( C'(1/6) \) groups are not hyperbolic, they are “hyperbolic enough” to be susceptible to a kind of tight geodesics argument. This stems from the fact that geodesic triangles in \( C'(1/6) \) groups have a limited number of specific forms, a result due to Strebel \([19]\). Our proof appears to be the first application of a tight geodesics argument in a non-hyperbolic setting.
The paper is organized as follows. In Section 1 we review the definitions of asymptotic dimension and asymptotic Assouad-Nagata dimension, and give a version of the Hurewicz mapping theorem for asymptotic Assouad-Nagata dimension used in the next section. In Section 2 we introduce the notion of an \((\varepsilon,k)\)-tight geodesic combing for \(\varepsilon > 0\) and \(k \in \mathbb{N}\), and show that a geodesic metric space admitting a \((\varepsilon,k)\)-tight geodesic combing for some \(\varepsilon > 0\) has asymptotic Assouad-Nagata dimension at most \(k\). In Section 3 we give some preliminaries on van Kampen diagrams and the classical small cancellation condition \(C'(1/6)\). We also review the classification of van Kampen diagrams over simple geodesic triangles in \(C'(1/6)\) groups due to Strebel, the essential tool needed in the proof of Theorem 2. In Section 4 we use Strebel’s classification to prove that \(C\) and the classical small cancellation condition and asymptotic Assouad-Nagata dimension at most \(k\). In Section 5 is dedicated to proving a necessary technical lemma about central extensions of \(C'(1/6)\) groups for \(\lambda < 1/12\). In Section 6, we construct a short exact sequence \(1 \to K \to G \to H \to 1\), where \(G\) is a central extension of an infinitely presented \(C'(1/6)\) group \(H\). We prove that \(K\) is quasi-isometrically isomorphic to Higes’ example of a group with asymptotic dimension 0 and Assouad-Nagata dimension \(m\), and thus \(G\) is a finitely generated group satisfying \(\text{asdim}(G) \leq 2\) and \(m + 1 \leq \text{asdim}_{AN}(G) \leq m + 2\). After this, taking the free product of \(G\) or \(G \times \mathbb{Z}\) with an appropriately-chosen free abelian group yields Theorem 1.

1 Preliminaries on asymptotic dimension

1.1 Asymptotic dimension and asymptotic Assouad-Nagata dimension

In this paper, \(0 \in \mathbb{N}\). The set of positive integers is \(\mathbb{Z}^+\). The set of positive real numbers is denoted \(\mathbb{R}^+\), and the set of non-negative real numbers is \(\mathbb{R}_0^+\). The letter \(d\) always stands for a metric, on whatever set makes sense in context.

Let \(X\) be a metric space. The open ball of radius \(r > 0\) about a point \(x \in X\) is denoted \(B(x, r)\). If \(A, B \subseteq X\), then \(d(A, B)\) is defined to be \(\inf\{d(a, b) \mid a \in A, b \in B\}\), and we write \(d(A, B)\) for \(d(\{A\}, B)\). Define \(\text{diam}(A) = \sup\{d(a, a') \mid a, a' \in A\}\).

For \(N > 0\) and \(V \subseteq X\), we say that \(V\) is \(N\)-bounded if \(\text{diam}(V) \leq N\). A family \(\mathcal{V}\) of subsets of \(X\) is uniformly bounded by \(N\) or uniformly \(N\)-bounded if \(\text{diam}(V) \leq N\) for all \(V \in \mathcal{V}\). For \(r > 0\), the \(r\)-multiplicity of \(\mathcal{V}\) is the maximum, over all \(x \in X\), of the number of elements of \(\mathcal{V}\) intersected by \(B(x, r)\), if there is a finite maximum: otherwise, we write that the \(r\)-multiplicity of \(\mathcal{V}\) is \(\infty\).

If \(A \subseteq X\) and \(s > 0\), an \(s\)-path in \(A\) is a finite sequence of points \((a_0, \ldots, a_n)\) such that \(a_i \in A\) and \(d(a_i, a_{i+1}) < s\) for each \(i\). A set \(A' \subseteq A\) is called \(s\)-connected if every two points in \(A'\) can be connected by an \(s\)-path. An \(s\)-component of \(A\) is a maximal \(s\)-connected subset of \(A\).

There are many equivalent definitions of the asymptotic dimension of a metric space: we present two of them here. The reason for giving both definitions will become clear at the end of this section.

**Definition 1.1.** Let \(X\) be a metric space, \(n \in \mathbb{N}\). The asymptotic dimension of \(X\) is at most \(n\), written \(\text{asdim}(X) \leq n\), if one (equivalently, both) of the following conditions hold:

(a) For every \(r > 0\), there exists an \(N(r) > 0\) and a cover \(\mathcal{V}\) of \(X\) such that \(\mathcal{V}\) has \(r\)-multiplicity at most \(n + 1\) and is uniformly bounded by \(N(r)\).
(b) For every $s > 0$, there exists an $M(s) > 0$ and a cover $\{X_1, \ldots, X_{n+1}\}$ of $X$ such that the $s$-components of each $X_i$ are $M(s)$-bounded.

The asymptotic dimension of $X$, denoted $\text{asdim}(X)$, is the least $n \in \mathbb{N}$ such that $\text{asdim}(X) \leq n$, if such an $n$ exists. Otherwise, we say that the asymptotic dimension of $X$ is infinite and write $\text{asdim}(X) = \infty$.

In Definition 1.1, $r$ and $s$ are thought of as large numbers. The function $M : \mathbb{R}^+ \to \mathbb{R}^+$ or $N : \mathbb{R}^+ \to \mathbb{R}^+$ is called an $n$-dimensional control function for $X$. If $\mathcal{V}$ has $r$-multiplicity at most $n + 1$ then $\mathcal{V}$ also has $r'$-multiplicity at most $n + 1$ for all $r' < r$; likewise, an $s'$-component of $X_i$ is a subset of an $s$-component of $X_i$ for all $s' < s$. Therefore we assume without loss of generality that any $n$-dimensional control function is nondecreasing.

Asymptotic Assouad-Nagata dimension is a version of asymptotic dimension in which the control function is required to be linear.

**Definition 1.2.** [8] Let $X$ be a metric space, $n \in \mathbb{N}$. Then the asymptotic Assouad-Nagata dimension of $X$ is at most $n$, written $\text{asdim}_{AN}(X) \leq n$, if there exist $a, b > 0$ such that either

(a) $N(r) = ar + b$ is an $n$-dimensional control function by Definition 1.1 (a), or

(b) $M(s) = as + b$ is an $n$-dimensional control function by Definition 1.1 (b).

We define $\text{asdim}_{AN}(X)$ to be the least $n \in \mathbb{N}$ such that $\text{asdim}(X) \leq n$, or $\infty$ if no such $n$ exists.

In this paper we will often consider norms on groups. We denote the identity element of an arbitrary group by 1, and the identity element of an abelian group by 0.

**Definition 1.3.** Let $G$ be a group. A norm on $G$ is a function $\| \cdot \| : G \to \mathbb{R}_0^+$ satisfying all of the following conditions:

- $\|g\| = 0$ if and only if $g = 1$.
- $\|g\| = \|g^{-1}\|$ for all $g \in G$.
- $\|gh\| \leq \|g\| + \|h\|$ for all $g, h \in G$.

A norm on $G$ is proper if $\{g \in G \mid \|g\| < r\}$ is finite for all $r > 0$. Norms and left-invariant metrics on $G$ are in bijection via the identification $d(g, h) = \|g^{-1}h\|$, and a norm is proper if and only if its associated metric is proper. The word norm on a finitely generated group is an example of a proper norm. Every countable group admits a proper norm, and any two proper norms on the same countable group are coarsely equivalent. Therefore for any countable group $G$ we define $\text{asdim}(G)$ to be the asymptotic dimension of $G$ with respect to any proper norm.

Though not a coarse invariant, it is easy to verify that asymptotic Assouad-Nagata dimension is a quasi-isometry invariant. Therefore for a finitely generated group $G$ we define $\text{asdim}_{AN}(G)$ to be the asymptotic Assouad-Nagata dimension of $G$ equipped with the word norm with respect to any
finite generating set. In general, given any norm \( \| \cdot \| \) on a group \( G \) it is also reasonable to define \( \text{asdim}_{\text{AN}}(G, \| \cdot \|) \), but what number this is will depend on the norm: see \( \text{[7]} \).

The proof of the main result on \( C'(1/6) \) groups uses the Hurewicz mapping theorem for asymptotic Assouad-Nagata dimension. In order to state it, we need to present some definitions from \( \text{[8]} \) that extend the notions of control functions and asymptotic Assouad-Nagata dimension to maps between metric spaces.

**Definition 1.4.** \( \text{[8]} \) Let \( X, Y \) be metric spaces, \( f : X \to Y \), and \( n \in \mathbb{N} \). Then \( D_f : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \) is an \( n \)-dimensional control function for \( f \) if one (equivalently, both) of the following conditions hold:

(a) For all \( r, K > 0 \), if \( f(A) \) is \( K \)-bounded then there exists a cover \( V \) of \( A \) such that \( V \) has \( r \)-multiplicity at most \( n + 1 \) and is uniformly bounded by \( D_f(r, K) \).

(b) For all \( s, K > 0 \), if \( f(A) \) is \( K \)-bounded then there exists a cover \( \{ A_1, \ldots, A_{n+1} \} \) of \( A \) such that the \( s \)-components of each \( A_i \) are \( D_f(s, K) \)-bounded.

In \( \text{[8]} \), Brodskiy, Dydak, Levin and Mitra define the Assouad-Nagata dimension of a map between metric spaces, in a sense that agrees with part (b) of all previous definitions. They then state and prove the Hurewicz mapping theorem for asymptotic Assouad-Nagata dimension in terms of this definition. In Section \( \text{2} \) we will want to apply the Hurewicz mapping theorem to a function which has finite asymptotic Assouad-Nagata dimension in a sense that agrees with part (a) of all previous definitions. This is why we have given two definitions at each step. Here is the definition from \( \text{[8]} \).

**Definition 1.5.** \( \text{[8]} \) Let \( X, Y \) be metric spaces, \( f : X \to Y \), and \( n \in \mathbb{N} \). Then \( \text{asdim}_{\text{AN}}(f) \leq n \) if there exist constants \( a, b, c > 0 \) such that \( D_f(s, K) = as + bK + c \) is an \( n \)-dimensional control function for \( f \) by Definition 1.4 (b).

**Definition 1.6.** Let \( X, Y \) be metric spaces. A function \( f : X \to Y \) is asymptotically Lipschitz if there exist constants \( a, b > 0 \) such that \( d(f(x), f(x')) \leq a(d(x, x')) + b \) for all \( x, x' \in X \).

The following is known as the Hurewicz mapping theorem for asymptotic Assouad-Nagata dimension.

**Theorem 1.7.** Let \( f : X \to Y \) be an asymptotically Lipschitz map between metric spaces. Then \( \text{asdim}_{\text{AN}}(X) \leq \text{asdim}_{\text{AN}}(f) + \text{asdim}_{\text{AN}}(Y) \).

In order to use the Hurewicz mapping theorem in our case, we will need the following lemma.

**Lemma 1.8.** Let \( f : X \to Y \) be a map between metric spaces. Then \( \text{asdim}_{\text{AN}}(f) \leq n \) if and only if there exist constants \( a, b, c > 0 \) such that \( D_f(r, K) = ar + bK + c \) is an \( n \)-dimensional control function for \( f \) by Definition 1.4 (a).

This may seem obvious to experts in asymptotic dimension theory, but to the author’s knowledge it does not appear in the literature. For the sake of giving a complete proof, we establish this fact in the next subsection. As a consequence we also prove the equivalences of parts (a) and (b) of all previous definitions. The reader who is already convinced may skip to Section \( \text{2} \).
1.2 Proof of the equivalence of two definitions

Lemma 1.9. Let \( X \) be a metric space with \( \text{asdim}(X) \leq n \). Let \( M(s) \) be an \( n \)-dimensional control function for \( X \) in the sense of Definition 1.1 (b), and let \( N(r) \) be an \( n \)-dimensional control function in the sense of Definition 1.2 (a). Then

(a) \( N'(r) := M(2r) \) is an \( n \)-dimensional control function for \( X \) by Definition 1.1 (a).

(b) \( M'(s) := N(2^n s) + 2^{n+1} s \) is an \( n \)-dimensional control function for \( X \) by Definition 1.1 (b).

Proof. Let \( r > 0 \) be given. Applying Definition 1.1 (b) to \( M(s) \) with \( s = 2r \), we have that there exist sets \( X_1, \ldots, X_{n+1} \) such that the \( 2r \)-components of each \( X_i \) are \( M(2r) \)-bounded. For each \( i \in \{1, \ldots, n+1\} \), let \( V_i \) be the set of \( 2r \)-components of \( X_i \), and let \( V = \bigcup_{i=1}^{n+1} V_i \). Then \( V \) covers \( X \).

Let \( x \in X \). If \( B(x,r) \) meets more than \( n + 1 \) elements of \( V \), then by pigeonhole \( B(x,r) \) must meet two distinct \( V_i, V_i' \in V_i \) for some \( i \). But this is impossible since \( V_i, V_i' \) are distinct \( 2r \)-components of \( X_i \). Thus the \( r \)-multiplicity of \( V \) is at most \( n + 1 \). Setting \( N'(r) = M(2r) \), we have that \( N' \) is an \( n \)-dimensional control function for \( X \) by Definition 1.1 (a). This proves part (a).

For part (b), let \( s > 0 \) be given. Applying Definition 1.1 (a) with \( r = 2^n s \), there exists a cover \( V \) of \( X \) such that \( V \) is uniformly bounded by \( N(2^n s) \) and has \( 2^n s \)-multiplicity at most \( n + 1 \). Let \( X_0 = \emptyset \). For \( i \in \{1, \ldots, n+1\} \), define \( X_i \) inductively as follows:

\[
X_i = \{ x \in X \setminus \bigcup_{j=0}^{i-1} X_j \mid B(x, 2^n i + 1) \text{ meets exactly } n - i + 2 \text{ elements of } V \}\.
\]

Let \( x \in X \). First we prove by induction on \( i \) that if \( x \notin \bigcup_{j=1}^{i} X_j \), then \( B(x, 2^{n-i+1} s) \) meets fewer than \( n - i + 2 \) elements of \( V \). For the base case, if \( i = 1 \) then the claim is that if \( x \notin X_1 \), then \( B(x, 2^n s) \) meets fewer than \( n + 1 \) elements of \( V \). But this follows directly from the fact that \( V \) has \( 2^n s \)-multiplicity at most \( n + 1 \). Suppose now that the claim is true for \( i - 1 \), and that \( x \notin \bigcup_{j=1}^{i-1} X_j \).

Then of course \( x \notin \bigcup_{j=1}^{i} X_j \) so by the induction hypothesis \( B(x, 2^{n-i+2}) \) meets fewer than \( n - i + 3 \) elements of \( V \). Therefore \( B(x, 2^{n-i+1} s) \) also meets fewer than \( n - i + 3 \) elements of \( V \), and does not meet \( n - i + 2 \) elements of \( V \) since \( x \notin X_i \) by assumption. Therefore \( B(x, 2^{n-i+1} s) \) meets fewer than \( n - i + 2 \) elements of \( V \). This completes the induction step.

In particular, if \( x \notin \bigcup_{j=1}^{n+1} X_j \), then \( B(x, s) \) meet fewer than \( n - (n+1) + 2 = 1 \) elements of \( V \). But \( B(x, s) \) meets at least one element of \( V \) since \( V \) covers \( X \). Thus \( \bigcup_{j=1}^{n+1} X_j \) covers \( X \).

By our definition of \( X_i \), for each \( x \in X \), there is a unique subset of \( V \) of cardinality \( n - i + 2 \) that witnesses that \( x \in X_i \). Call this set \( V_i(x) \). Define an equivalence relation \( \sim \) on \( X_i \) by declaring that \( x \sim y \) if \( V_i(x) = V_i(y) \), and let \([x]\) denote the \( \sim \) equivalence class of \( x \). We claim that the \( s \)-components of each \( X_i \) are contained in the \( \sim \) equivalence classes of each \( X_j \).

Suppose otherwise. Then for some \( i \) there exist \( x, y \in X_i \) such that \([x] \neq [y]\) and \( d(x,y) < s \). Then \( x, y \notin \bigcup_{j=1}^{i-1} X_j, V_i(x) \neq V_i(y) \), and \( |V_i(x)| = |V_i(y)| = n - i + 2 \). Let \( V \in V_i(y) \setminus V_i(x) \). Then \( d(x,V) \leq d(x,y) + d(y,V) < s + 2^{n-i+1} s \leq 2^{n-i+2} s \). But then \( B(x, 2^{n-i+2} s) \) meets \( V \) and every element of \( V_i(x) \), so \( B(x, 2^{n-i+2} s) \) meets at least \( n - i + 3 \) elements of \( V \). This means that \( x \in \bigcup_{j=1}^{i-1} X_j \), a contradiction.
Next we show that each \( \sim \) equivalence class is uniformly bounded by \( N(2^n s) + 2^{n+1} s \). Let \( y \in [x] \), and let \( V \in V_i(x) \). Then \( d(x, V), d(y, V) < 2^{n-i+1} s \leq 2^n s \). Therefore

\[
d(x, y) \leq d(x, V) + \text{diam}(V) + d(V, y) < 2^n s + \text{diam}(V) + 2^n s = \text{diam}(V) + 2^{n+1} s \leq N(2^n s) + 2^{n+1} s
\]

since each \( V \in V \) is \( N(2^n s) \)-bounded. Since each \( s \)-component of each \( X_i \) is contained in some \( \sim \) equivalence class, the \( s \)-components of each \( X_i \) are all \( N(2^n s) + 2^{n+1} s \)-bounded.

Putting all this together, we have that \( \{X_1, \ldots, X_n\} \) covers \( X \), and the \( s \)-components of each \( X_i \) are uniformly bounded by \( N(2^n s) + 2^{n+1} s \). Therefore setting \( M'(s) = N(2^n s) + 2^{n+1} s \), we have that \( M' \) is an \( n \)-dimensional control function for \( X \) in the sense of Definition 1.1(b). Thus part (b) is proved.

\( \square \)

**Corollary 1.10.** Let \( X \) be a metric space with \( \text{asdim}(X) \leq n \). Let \( L(t) \) be an \( n \)-dimensional control function for \( X \) by either Definition 1.1 (a) or (b). Then \( L(2^n t) + 2^{n+1} t \) is an \( n \)-dimensional control function for \( X \) by both definitions.

**Corollary 1.11.** Let \( X, Y \) be metric spaces \( f : X \rightarrow Y, n \in \mathbb{N} \), and suppose that \( D_f(t, K) \) is an \( n \)-dimensional control function for \( f \) by Definition 1.4 (a) or (b). Then \( D_f(2^n t, K) + 2^{n+1} t \) is an \( n \)-dimensional control function for \( f \) according to both definitions.

**Proof.** For a fixed \( K > 0 \), define \( D_{f,K} : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) by \( D_{f,K}(t) = D_f(t, K) \). Note that an equivalent formulation of the statement that \( D_f \) is an \( n \)-dimensional control function for \( f \) by Definition 1.4 (a) or (b) is that, for all \( A \subseteq X \) such that \( f(A) \) is \( K \)-bounded, \( D_{f,K}(t) \) is an \( n \)-dimensional control function by Definition 1.1(a) or (b), respectively, for \( A \) as a subspace of \( X \). By Corollary 1.10, for all \( A \subseteq X \) with \( f(A) \) \( K \)-bounded, \( D_{f,K}(2^n t) + 2^{n+1} t \) is an \( n \)-dimensional control function for \( A \) in both senses. Therefore \( D_f(2^n t, K) + 2^{n+1} t \) is an \( n \)-dimensional control function for \( f \) according to both definitions.

\( \square \)

In particular, if \( D_f(t, K) \) is linear in both coordinates, then so is \( D_f(2^n t, K) + 2^{n+1} t \). Therefore Lemma 1.8 is proved.

## 2 Tight geodesic combings

Let \( X \) be a metric space. A subspace \( Y \subseteq X \) is called **cobounded** if there exists a constant \( c > 0 \) such that \( d(x, Y) \leq c \) for all \( x \in X \).

**Definition 2.1.** Let \( X \) be a geodesic metric space with base point \( x \in X \). Then a **geodesic combing** of the pointed space \( (X, x) \) is a set \( T = \{T_y \mid y \in Y\} \), where \( Y \) is a cobounded subset of \( X \) and \( T_y \) is a geodesic from \( x \) to \( y \) for each \( y \in Y \).

**Example 2.2.** Suppose that \( \Gamma \) is a connected graph with the combinatorial metric, and let \( x \in V(\Gamma) \) be a base point. A geodesic tree based at \( x \) is a subgraph \( T \) of \( \Gamma \) such that \( T \) is a tree, and for all \( y \in V(\Gamma) \), the unique path from \( x \) to \( y \) in \( T \) is geodesic in \( \Gamma \). If \( T \) is a geodesic tree based at \( x \) and \( V(T) = V(\Gamma) \), then we call \( T \) a geodesic spanning tree based at \( x \). If \( T \) is a geodesic spanning tree based at \( x \) and \( y \in V(\Gamma) \), let \( [x, y] \) be the path from \( x \) to \( y \) in \( T \). Then \( \{[x, y] \mid y \in V(\Gamma)\} \) is a geodesic combing of \( (\Gamma, x) \).
Suppose that \( \{T_y \mid y \in Y\} \) is a geodesic combing of a pointed geodesic metric space \((X, x)\). For each \( y \in Y \) and \( s > 0 \), let
\[
T(y, s) = \bigcup \{T_{y'} \mid y' \in Y \cap B(y, s)\}
\]
and for each \( t \geq 0 \), let
\[
S(t) = \{x' \in X \mid d(x, x') = t\}
\]
be the sphere of radius \( t \) in \( X \) centered at \( x \).

**Definition 2.3.** Let \((X, x)\) be a pointed geodesic metric space, \( Y \) a cobounded subset of \( X \), and \( T = \{T_y \mid y \in Y\} \) a geodesic combing of \((X, x)\). Let \( \varepsilon > 0 \) and \( k \in \mathbb{N} \). Then we say that \( T \) is \((\varepsilon, k)\)-tight if for all \( r > 0, y \in Y \), and \( t \leq d(x, y) - r \), we have \( |T(y, \varepsilon r) \cap S(t)| \leq k \).

Figure 1 illustrates this definition.

![Figure 1: An \((\varepsilon, k)\)-tight geodesic combing.](image)

**Proposition 2.4.** Let \((X, x)\) be a pointed geodesic metric space. If \( X \) admits an \((\varepsilon, k)\)-tight geodesic combing for some \( \varepsilon > 0 \), then \( \text{asdim}_{AN}(X) \leq k \).

**Proof.** Suppose that \( Y \) is a cobounded subset of \( X \) and \( T = \{T_y \mid y \in Y\} \) is a \((\varepsilon, k)\)-tight geodesic combing. Let \( d_x : Y \to \mathbb{R}^+_0 \) be defined by \( d_x(y) = d(x, y) \). For any \( n \in \mathbb{N} \) and \( r > 0 \), let
\[
A(n, r) = \{y \in Y \mid nr \leq d(x, y) \leq (n + 2)r\} = d_x^{-1}([nr, (n + 2)r])
\]
be the \( n \)th annulus of width \( 2r \) in \( Y \).

We claim that for each \( n \in \mathbb{N} \) and \( r \geq 0 \), there exists a cover \( \mathcal{V}(n, r) \) of \( A(n, r) \) which has \( \varepsilon r \)-multiplicity at most \( k \) and is uniformly bounded by \( 6r \). To see this, define an equivalence relation \( \sim \) on \( A(n, r) \) by declaring that \( y \sim y' \) if \( T_y \) and \( T_{y'} \) pass through the same element of \( S((n - 1)r) \). Let \( \mathcal{V}(n, r) \) be the set of \( \sim \) equivalence classes. Clearly \( y \sim y' \) implies that there is a path in \( T_y \cup T_{y'} \) from \( y \) to \( y' \) of length at most \( 6r \), hence \( \mathcal{V}(n, r) \) is uniformly \( 6r \)-bounded. Furthermore, since \( T \) is \((\varepsilon, k)\)-tight, for each \( y \in A(n, r) \) we have that \( |T(y, \varepsilon r) \cap S((n - 1)r)| \leq k \), hence any open ball of radius \( \varepsilon r \) in \( A(n, r) \) can meet at most \( k \) equivalence classes.
Now we claim that $\text{asdim}_{AN}(d_x) \leq k - 1$. Let $s, K > 0$ be given. Now fix $r = \max(\frac{1}{\varepsilon}s, K)$. Let $A \subseteq Y$ be such that $d_x(A)$ is $K$-bounded. Then $A \subseteq A(n, K) \subseteq A(n, r)$. By the previous argument, there exists a cover $V(n, r)$ of $A(n, r)$ (and thus of $A$) with $\varepsilon r$-multiplicity at most $k$ which is uniformly bounded by $6r$. Therefore $V(n, r)$ has $s$-multiplicity at most $k$ and is uniformly bounded by $6r = 6\max(\frac{1}{\varepsilon}s, K) \leq \frac{6}{\varepsilon}s + 6K$. Thus $D_{d_x}(s, K) := \frac{6}{\varepsilon}s + 6K$ is a $(k - 1)$-dimensional control function for $d_x$ that is linear in both $s$ and $K$ and we have $\text{asdim}_{AN}(d_x) \leq k - 1$.

It is easy to check that $\text{asdim}_{AN}(\mathbb{R}^d_+) \leq 1$, and that $d_x$ is $1$-Lipschitz and therefore asymptotically Lipschitz. Therefore by the Hurewicz mapping theorem for asymptotic Assouad-Nagata dimension,

$$\text{asdim}_{AN}(Y) \leq \text{asdim}_{AN}(d_x) + \text{asdim}_{AN}(\mathbb{R}^d_+) = (k - 1) + 1 = k.$$ 

Since $Y$ is quasi-isometric to $X$, $\text{asdim}_{AN}(X) \leq k$. □

Though Proposition 2.4 as stated is sufficient for our purposes, it is possible to relax slightly its hypotheses. For example, if $X$ is not geodesic but quasi-geodesic, and $T$ is a set of quasi-geodesics which satisfies a quasified version condition of $(\varepsilon, k)$-tightness, then the proof that $X$ has finite asymptotic Assouad-Nagata dimension will still hold with minor changes. However, every quasi-geodesic metric space is quasi-isometric to a graph, so there is no real loss of generality.

We conclude this section by showing that every connected graph has a geodesic spanning tree. Hence Example 2.2 shows that every connected graph has a geodesic combing, which may or may not be $(\varepsilon, k)$-tight for some $\varepsilon > 0$ and $k \in \mathbb{N}$. Although the argument is elementary, it is set down here for the sake of completeness.

**Lemma 2.5.** Let $\Gamma$ be a connected graph, $x \in V(\Gamma)$. Then there exists a geodesic spanning tree of $\Gamma$ based at $x$.

**Proof.** Let $\mathcal{T}$ be the set of all subgraphs of $\Gamma$ that are geodesic trees based at $x$. Note that $\mathcal{T}$ is nonempty and partially ordered by inclusion. By Zorn’s Lemma, $\mathcal{T}$ contains a maximal element, call it $T$.

Suppose that $T$ is not a spanning tree, i.e. there exists some vertex $y \in V(\Gamma) \setminus V(T)$. Let $[x, y]$ be a geodesic. Let $z$ be the vertex of $[x, y] \cap T$ which is farthest away from $x$. Let $[y, z]$ be the subpath of $[x, z]$ from $y$ to $z$. Then $T \cap [y, z] = \{y\}$ and so $T \cup [y, z]$ is a tree which contains both $T$ and $z$, a contradiction to the supposed maximality of $T$. Therefore $T$ must be a spanning tree. □

Clearly if $\Gamma$ is a connected graph, $x \in V(\Gamma)$, and $(\Gamma, x)$ admits a $(\varepsilon, k)$-tight geodesic combing for some $\varepsilon > 0$ and $k \in \mathbb{N}$, then we can assume without loss of generality that it is given by a geodesic spanning tree. Therefore if $T$ is a geodesic spanning tree based at $x$, we say that $T$ is $(\varepsilon, k)$-tight if the combing it induces is $(\varepsilon, k)$-tight. In Section 4 we show that if $\Gamma$ is the Cayley graph of a finitely generated $C'(\frac{1}{6})$ group with respect to any finite generating set, then any geodesic spanning tree of $(\Gamma, 1)$ is $(\frac{1}{9}, 2)$-tight.
3 Preliminaries on van Kampen diagrams and small cancellation

Here we fix notation, review the definition of the $C'(1/6)$ condition and that of a van Kampen diagram, and present a classification of van Kampen diagrams over simple geodesic triangles in $C'(1/6)$ groups. This result, due to Strebel, is the essential tool used in the proof of the main theorem. We also prove some lemmas regarding the geometry of simple geodesic triangles in $C'(1/6)$ groups, which are used repeatedly in the next section.

3.1 The classical small cancellation condition

Let $S$ be a set. Let $S^{-1}$ be the set of formal inverses of $S$, let 1 be a new symbol not in $S$, and declare $1^{-1} = 1$. Let

\[ S_1 = S \cup \{1\} \]
\[ S_0 = S \cup S^{-1} \cup \{1\} \]  \hspace{2cm} (1)

The length of a word $w$ in the free monoid $S^*$ is denoted $|w|$. A word $w \in S_0^*$ is reduced if $w$ does not contains a subword of the form $1, ss^{-1}$, or $s^{-1}s$ for any $s \in S$, and cyclically reduced if every cyclic shift of $w$ (including $w$ itself) is reduced.

Let $R$ be a language over $S_0$, that is, $R \subseteq S_0^*$. Then $R_*$ denotes the closure of $R$ under taking cyclic shifts and formal inverses of elements. We say that $R$ is reduced if every element of $R$ is reduced, and cyclically reduced if $R_*$ is reduced. For us, a group presentation is a pair $\langle S \mid R \rangle$ where $R \subseteq S_0^*$ is cyclically reduced. The phrase $G = \langle S \mid R \rangle$ abbreviates the statement that $\langle S \mid R \rangle$ is a presentation and $G \cong F(S)/\langle\langle R \rangle\rangle$, where $\langle\langle R \rangle\rangle$ is the normal closure of $R$ as a subset of the free group with basis $S$. Whenever $S$ is a generating set of a group $G$ there is a natural monoid homomorphism $S_0^* \rightarrow G$, and for $w \in S_0^*$ and $g \in G$ we write $w =_G g$ to mean that $g$ is the image of $w$ under this homomorphism.

Given a language $R \subseteq S_0^*$ and $u \in S_0^*$, we say that $u$ is a piece (of $r$ and of $r'$) if $u$ is a common prefix of two distinct words $r, r' \in R_*$.  

Definition 3.1. Let $0 < \lambda < 1$. Then $R$ satisfies $C'(\lambda)$ if $|u| < \lambda|r|$ whenever $u$ is a piece of $r$.

If $G$ is a group and $G = \langle S \mid R \rangle$ for some $R$ satisfying $C'(\lambda)$, then $\langle S \mid R \rangle$ is called a $C'(\lambda)$ presentation and $G$ is called a $C'(\lambda)$ group.

3.2 van Kampen diagrams

By convention, if $\Gamma$ is a graph and we write $x \in \Gamma$, we mean that $x \in V(\Gamma)$; similarly if $\alpha$ is a combinatorial path in a graph, then $x \in \alpha$ means $x \in V(\alpha)$. From now on, ‘path’ will mean combinatorial path.

Let $\Gamma$ be any directed graph, and suppose that $\text{Lab} : E(\Gamma) \rightarrow S_1$ (see (1) above) is a function which assigns labels from $S_1$ to the edges of $\Gamma$. Then we extend $\text{Lab}$ to a map from the set of all paths in $\Gamma$ to $S_0^*$ in the following natural way:
• If $e = (x, y)$ is a directed edge labeled $s$, then $\text{Lab}(x, e, y) = s$ and $\text{Lab}(y, e, x) = s^{-1}$.

• If $\sigma = (x_0, e_1, x_1, \ldots, x_{n-1}, e_n, x_n)$ is a path, then

$$\text{Lab}(\sigma) = \text{Lab}(x_0, e_1, x_1) \text{Lab}(x_1, e_2, x_2) \cdots \text{Lab}(x_{n-1}, e_n, x_n).$$

Paths are allowed to have repeated edges and vertices. For a path $\sigma$ we define $\ell(\sigma)$, the length of $\sigma$, to be the number of edges traversed by $\sigma$, counting multiplicity. Equivalently, $\ell(\sigma) := |\text{Lab}(\sigma)|$.

A graph is planar if it admits a topological embedding into $\mathbb{R}^2$, and a plane graph if it comes already equipped with a specific embedding. If $M$ is a plane graph, a face of $M$ is the closure of a connected component of $\mathbb{R}^2 \setminus M$. Every finite plane graph has exactly one unbounded face any number of bounded faces. Let $F$ be a face of a finite directed plane graph $M$ with edges labeled by elements of $S_1$. Choosing a base point $x \in \partial F$ and an orientation counterclockwise ($+$) or clockwise ($-$), there is a unique path which traverses $\partial F$ exactly once. This is called the boundary path and denoted $(\partial F, x, \pm)$. Choosing a different base point yields a label which is a cyclic shift of the original, and choosing the opposite orientation yields a label which is the formal inverse of the original. If all properties of $(\partial F, x, \pm)$ that we care about are preserved under cyclic shifts and inverses, we leave these choices out of the notation and write $\partial F$. We write $\partial M$ instead of $\partial F$ if $F$ is the unbounded face. The boundary label of $F$ is $\text{Lab}(\partial F, x, \pm)$, or sometimes just $\text{Lab}(\partial F)$.

**Definition 3.2.** A van Kampen diagram over a presentation $\langle S \mid R \rangle$ is a finite, connected, directed plane graph $M$ with edges labeled by elements of $S_1$, such that if $F$ is a bounded face of $M$, then either $\text{Lab}(\partial F) \in R_*$ or $\text{Lab}(\partial F) =_{F(S)} 1$.

By convention, all faces are assumed to be bounded unless otherwise stated. A face $F$ is called essential if $\text{Lab}(\partial F) \in R_*$ and inessential if $\text{Lab}(\partial F) =_{F(S)} 1$. A face with boundary label $r \in R$ is called an $r$-face. An edge is essential if it is labeled by an element of $S$, and inessential if it is labeled by 1. We call a van Kampen diagram bare if it contains no inessential faces, and padded otherwise. Because $R$ is cyclically reduced, any inessential face must border an inessential face, so a bare van Kampen diagram also has no inessential edges. In this section and the next, we will only need to consider bare van Kampen diagrams, but we will need padded ones in Section 5.

Let $M$ be a van Kampen diagram, and suppose $F$ and $F'$ are distinct faces of $M$. Then we say that $F$ cancels with $F'$ if there exists an edge $e = (x, y)$ in $\partial F \cap \partial F'$ such that $\text{Lab}(\partial F, x, +) = \text{Lab}(\partial F', x, -)$. Then we have the following geometric interpretation of the $C'(\lambda)$ condition, which follows immediately from the definition.

**Lemma 3.3.** Let $\langle S \mid R \rangle$ be a presentation where $R$ satisfies $C'(\lambda)$, and let $M$ be a van Kampen diagram over $\langle S \mid R \rangle$. Suppose that $F, F'$ are essential faces of $M$ and $\sigma$ is a common subpath of $\partial F$ and $\partial F'$. Then either $F$ and $F'$ cancel, or $\ell(\sigma) < \lambda \min(\ell(\partial F), \ell(\partial F'))$.

A van Kampen diagram is called reduced if no two of its faces cancel. We say a van Kampen diagram $M$ is minimal if $M$ has the minimum number of essential faces among all van Kampen diagrams having the same boundary label, and among all those that have the same number of essential faces, $M$ has the minimum number of inessential faces. If a van Kampen diagram is minimal, then it is bare and reduced [21].

Whenever $G$ is a group generated by $S$, the Cayley graph of $G$ with respect to $S$ is denoted $\Gamma(G, S)$. 

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Lemma 3.4 (van Kampen Lemma).\[21\] Let $G = \langle S \mid R \rangle$ and $w \in S^*_G$. Then $w =_{G1}$ if and only if there exists a van Kampen diagram $M$ over $\langle S \mid R \rangle$ and $x \in \partial M$ such that $\text{Lab}(\partial M, x, +) = w$. Furthermore, given $g \in G$, there exists a combinatorial map $f : M \to \Gamma(G, S)$ preserving labels and orientations of edges, such that $f(x) = g$. In particular, $f$ does not increase distances.

If $\sigma = (g_0, e_1, g_1, \ldots, g_{n-1}, e_n, g_0)$ is a loop in $\Gamma(G, S)$ and we write that $M$ is a van Kampen diagram for $\sigma$, we mean that $\text{Lab}(\partial M, x, +) = \text{Lab}(\sigma)$ and we choose the combinatorial map that sends $x$ to $g_0$. Thus for our purposes the combinatorial map is unique. If in addition $\sigma$ is a simple closed curve, then $f|_{\partial M} : \partial M \to \sigma$ is a bijection.

3.3 Simple geodesic triangles

Let $a, b, c$ be distinct elements of $G = \langle S \mid R \rangle$, and let $[a, b], [b, c], [c, a]$ be fixed geodesics between them in $\Gamma(G, S)$. Then $[a, b] \cup [b, c] \cup [c, a]$ is called a geodesic triangle and denoted $\Delta(a, b, c)$. We say that $\Delta(a, b, c)$ is a simple geodesic triangle if the boundary path $\partial \Delta(a, b, c) := [a, b] \ast [b, c] \ast [c, a]$ is a simple closed curve in $\Gamma(G, S)$.

If $\Gamma$ is a directed graph, the underlying graph of $\Gamma$ is the undirected graph obtained by removing the orientation of every edge of $\Gamma$. If $\Gamma$ is a graph and $e = (x, y)$ is an edge of $\Gamma$, then subdividing $e$ means adding a vertex $z$ and edges $(x, z)$ and $(z, y)$ to $\Gamma$, and removing $e$. A subdivision of $\Gamma$ is a graph obtained from $\Gamma$ by a finite sequence of subdivisions of edges.

Theorem 3.5.\[19\] Suppose that $G = \langle S \mid R \rangle$, $S$ is finite, $R$ satisfies $C'(1/6)$, $\Delta$ is a simple geodesic triangle in $\Gamma(G, S)$, and $M$ is a minimal van Kampen diagram over $\langle S \mid R \rangle$ for $\partial \Delta$. Then the underlying graph of $M$ is a subdivision of a member of one of the four infinite families of plane graphs depicted in Figure 2.

In Figure 2, the blue edges and dots signify a sequence of parallel edges which may or may not be present. Vertices are located at the corners and at every juncture of edges. Our notation is slightly different from Strebel’s notation in \[19\]: our I-II encompasses Strebel’s I$^2$, I$^3$ and II, as well as the van Kampen diagram consisting of a single face, and our III is Strebel’s III$^1$.

For the remainder of this section, suppose that $G = \langle S \mid R \rangle$, $S$ is finite, $R$ satisfies $C'(1/6)$, $\Delta = \Delta(a, b, c)$ is a simple geodesic triangle in $\Gamma(G, S)$, $M$ is a minimal van Kampen diagram for $\partial \Delta$, $f : M \to \Gamma(G, S)$ is the combinatorial map, $\alpha = [b, c], \beta = [c, a]$, and $\gamma = [a, b]$. All lemmas in this section are proved under these assumptions. Note that $f|_{\partial M} : \partial M \to \Delta$ is bijective, and isometric when restricted to each of the subpaths of $\partial M$ corresponding to $\alpha, \beta$, or $\gamma$. Thus without
harm we blur the distinction between $\partial M$ and $\partial \Delta$, and refer to vertices, edges, paths etc. in $\partial M$ by their images in $\partial \Delta$.

A face $F$ of $M$ is called extremal if $F$ contains $a$, $b$, or $c$. A side of $F$ is a maximal subpath of $\partial F$ whose internal vertices all have degree 2 and do not include $a$, $b$, or $c$. A side is called exterior if it is contained in $\partial M$ and interior otherwise. An exterior side is necessarily a subpath of $\alpha$, $\beta$, or $\gamma$, so all exterior sides are geodesic. Let $i(F)$ denote the number of interior sides of $F$. We call $F$ triangular if $F$ has exactly three sides, quadrilateral if $F$ has exactly four sides, etc. A triangular face is always extremal, but an extremal face need not be triangular. The figure below shows an example of a van Kampen diagram of type V with two triangular faces, four quadrilateral faces, and two pentagonal faces. One of the extremal faces is quadrilateral.

![Diagram showing extremal faces, interior sides, and exterior sides.]

**Lemma 3.6.** Let $F$ be a face of $M$ and $\sigma$ an exterior side of $F$. Then

$$\sum \{\ell(\tau) \mid \tau \text{ is a side of } F \text{ other than } \sigma\} \geq \frac{1}{2} \ell(\partial F).$$

In particular,

$$\sum \{\ell(\tau) \mid \tau \text{ is an exterior side of } F \text{ other than } \sigma\} > \left(\frac{1}{2} - \frac{i(F)}{6}\right) \ell(\partial F).$$

*Proof.* If $\sigma$ is an exterior side, then $\sigma$ is geodesic, from which the first inequality follows. The second inequality follows from the first inequality and Lemma 3.3. \qed

**Definition 3.7.** Let $A$ be the union of all faces $F$ of $M$ such that $\partial F$ does not share an edge with $\alpha$, if at least one such $F$ exists; otherwise, set $A = \{a\}$. We call $A$ the $a$-corner of $M$. Similarly define $B$ and $C$, the $b$-corner and $c$-corner of $M$. A face which is not included in a corner, i.e. one that shares at least one edge each with $\alpha$, $\beta$, and $\gamma$, is called a middle face. The middle face is unique if it exists, and is denoted $D$. Thus $A$, $B$, $C$, $D$ divide $M$ into three or four regions depending on the existence of $D$, and these may overlap: see Figure 3.

A corner may contain no faces if $M$ is of type I-II. A corner which contains at least one face contains an extremal face, which is either triangular, or possibly quadrilateral if $M$ is of type IV or V. The extremal face is possibly followed by a sequence of quadrilateral faces; which is possibly followed by a pentagonal face if $M$ is of type III, IV or V; which is possibly followed by two pentagonal faces each with one exterior side if $M$ is of type IV. Figure 3 illustrates where the corners and middle faces are in van Kampen diagrams of various types.
Divide the boundary of each corner into three parts
\[ \alpha_B = \partial B \cap \alpha \quad \gamma_B = \partial B \cap \gamma \quad \iota_B = \partial B \setminus (\alpha \cup \gamma) \]
and assign similar notation for the other corners. The next proposition shows that \( \alpha_B, \gamma_B, \) and \( \iota_B \) are of comparable length, and if one is small, then the entire corner is small. The first three inequalities are used extensively in the next section; the last two are easy consequences of the proof, included because they may be of independent interest.

**Proposition 3.8.** The following inequalities hold, and analogous inequalities hold after switching the roles of \( a, b, \) and \( c. \)

(a) \( \ell(\iota_B) < 2 \min(\ell(\alpha_B), \ell(\gamma_B)) \). If \( M \) has a middle face, \( \ell(\iota_B) < \min(\ell(\alpha_B), \ell(\gamma_B)) \).

(b) \( \max(\ell(\alpha_B), \ell(\gamma_B)) < 3 \min(\ell(\alpha_B), \ell(\gamma_B)) \). If \( M \) has a middle face, \( \max(\ell(\alpha_B), \ell(\gamma_B)) < 2 \min(\ell(\alpha_B), \ell(\gamma_B)) \).

(c) If \( F \) is the middle face of \( M \) or \( F \) is the pentagonal face of \( A \) that borders \( B \) and \( C \), then \( \ell(\partial F \cap (\iota_B \cup \alpha_D \cup \iota_C)) < \ell(\alpha) \).

(d) \( \text{diam}(B) < 4 \min(\ell(\alpha_B), \ell(\gamma_B)) \). If \( M \) has a middle face, then \( \text{diam}(B) < 2 \min(\ell(\alpha_B), \ell(\gamma_B)) \).

(e) \( \text{diam}(M) < 2 \max(\ell(\alpha), \ell(\beta), \ell(\gamma)) \).

**Proof.** Assume that \( M \) is of type IV, the most complicated case. If \( M \) is of a different type the arguments are analogous but shorter. Assume without loss of generality that \( \ell(\alpha_B) \leq \ell(\gamma_B) \).

If \( B = \{b\} \), then the statement is trivial. Therefore let \( B = \bigcup_{i=0}^{k+3} B_i \), where

- \( B_0 \) is the extremal face containing \( b \).
- \( B_1, \ldots, B_k \) is a (possibly empty) sequence of quadrilateral faces such that \( B_{j-1} \) borders \( B_j \) for all \( j \in \{1, \ldots, k\} \).
- \( B_{k+1} \) is the pentagonal face with two exterior sides, if it exists: otherwise, \( B_{k+1} = B_0 \).
- \( B_{k+2} \) and \( B_{k+3} \) are the two pentagonal faces with one exterior side.
We assign the following labels in order to streamline notation: see Figure 4.

\[ \alpha_i = \partial B_i \cap \alpha \quad \text{for } i \in \{0, \ldots, k+1\}, \text{ and } \alpha_{k+2} = \partial B_{k+3} \cap \alpha \]
\[ \gamma_i = \partial B_i \cap \gamma \quad \text{for } i \in \{0, \ldots, k+2\} \]
\[ \iota_i = \partial B_i \cap \partial B_{i+1} \quad \text{for } i \in \{0, \ldots, k+2\}, \text{ and } \iota_{k+3} = \partial B_{k+2} \cap \partial B_{k+3} \]

Let \( i \in \{0, \ldots, k\} \). Then applying Lemma 3.6 to \( \gamma_i \), we obtain \( \ell(\alpha_i) > \frac{1}{6}\ell(\partial B_i) \). Since \( \iota_i \) is an interior side of \( B_i \), \( \ell(\iota_i) < \frac{1}{6}\ell(\partial B_i) \) by the \( C^0(1/6) \) condition. Therefore

\[ \ell(\iota_i) < \ell(\alpha_i) \quad \text{for } i \in \{0, \ldots, k\}. \tag{2} \]

Now consider \( B_{k+1} \). Applying Lemma 3.6 to \( \gamma_{k+1} \) yields \( \ell(\alpha_{k+1}) + \ell(\iota_k) > \frac{1}{6}\ell(\partial B_{k+1}) \). We know by (2) that \( \ell(\iota_k) < \ell(\alpha_k) \), so \( \frac{1}{6}\ell(\partial B_{k+1}) < \ell(\alpha_k) + \ell(\alpha_{k+1}) \). But both \( \iota_{k+1} \) and \( \iota_{k+2} \) are interior sides of \( B_{k+1} \). Therefore

\[ \ell(\iota_{k+1}) < \ell(\alpha_k) + \ell(\alpha_{k+1}) \]
\[ \ell(\iota_{k+2}) < \ell(\alpha_k) + \ell(\alpha_{k+1}) \tag{3} \]

Notice that \( \partial B_{k+3} \) consists of four interior sides and \( \alpha_{k+2} \). Therefore \( \ell(\alpha_{k+2}) > \frac{1}{6}\ell(\partial B_{k+3}) \). Since \( \iota_{k+3} \) is an interior side of \( \ell(\partial B_{k+3}) \), we have \( \ell(\iota_{k+3}) < \frac{1}{6}\ell(\partial B_{k+3}) \). Similar observations about \( \iota_{k+1}, \iota_{k+2}, \) and \( \gamma_{k+2} \) yield
\[
\ell(t_{k+3}) < \frac{1}{2}\ell(\alpha_{k+2}) \quad \ell(t_{k+3}) < \frac{1}{2}\ell(\gamma_{k+2})
\]
\[
\ell(t_{k+2}) < \frac{1}{2}\ell(\alpha_{k+2}) \quad \ell(t_{k+1}) < \frac{1}{2}\ell(\gamma_{k+2}).
\]
\tag{4}

Now \(\iota_B\) consists of two interior sides of \(B_{k+2}\) and two interior sides of \(B_{k+3}\). Thus
\[
\iota_B < \frac{1}{3}\ell(\partial B_{k+2}) + \frac{1}{3}\ell(\partial B_{k+3}).
\]
\tag{5}

On the other hand, by Lemma 3.6 applied to \(\gamma_{k+2}\) we have that \(\ell(t_{k+1}) + \ell(t_{k+3}) > \frac{1}{6}\ell(\partial B_{k+2})\). Applying Lemma 3.6 to \(\alpha_{k+2}\), we find that \(\ell(t_{k+2}) + \ell(t_{k+3}) > \frac{1}{6}\ell(\partial B_{k+3})\). Therefore
\[
\frac{1}{3}\ell(\partial B_{k+2}) + \frac{1}{3}\ell(\partial B_{k+3}) < 2\ell(t_{k+1}) + 2\ell(t_{k+2}) + 4\ell(t_{k+3}).
\]
\tag{6}

Combining inequalities (3)-(6), we have
\[
\ell(\iota_B) < 2\ell(\alpha_k) + 2\ell(\alpha_{k+1}) + 2\ell(\alpha_{k+2}) \leq 2\ell(\alpha_B).
\]

If \(M\) has a middle face, then \(\iota_B = \iota_k\) and inequality (2) gives that \(\iota_B < \alpha_B\). This proves part (a). Since \(\gamma_B\) is geodesic,
\[
\ell(\gamma_B) \leq \ell(\iota_B) + \ell(\alpha_B) < 2\ell(\alpha_B) + \ell(\alpha_B) = 3\ell(\alpha_B),
\]
and if \(M\) has a middle face this is lowered to \(2\ell(\alpha_B)\). This establishes part (b).

For part (c), let \(\alpha_D = \alpha \cap D\). If \(F = D\), then \(\partial F \cap \iota_B = t_k\) in Figure 4. If \(F\) is instead the pentagonal face of \(A\), then \(\partial F \cap \iota_B = t_{k+1}\) (similarly for \(t_C\)). In either case we have \(\ell(\partial F \cap (\iota_B \cup \alpha \cup t_C)) < \ell(\alpha_B) + \ell(\alpha_D) + \ell(\alpha_C) = \ell(\alpha)\).

For (d) and (e), let \(z\) be the center point of \(M\), at the juncture of \(t_{k+3}\) and \(\iota_B\) in Figure 4. Let \(\hat{c}\) be the midpoint of \(\gamma_{k+2}\). Let \(\hat{\gamma}_{k+2}\) be the half of \(\gamma_{k+2}\) between \(c'\) and \(b\). Let \(B_\gamma = \hat{\gamma}_{k+2} \cup \cdots \cup \gamma_0 \cup t_0 \cup \cdots \cup t_{k+1} \cup t_{k+3}\). Define \(\hat{a}, \hat{\alpha}_{k+2}\), and \(B_\alpha\) symmetrically. Note that every two points in \(B_\gamma\) are connected by a subpath of a path of one of the following forms:
\[
\iota_i * \gamma_{i+1} * \cdots * \gamma_j * \iota_j,
\]
\[
\iota_i * \gamma_{i+1} * \cdots * \gamma_{k+1} * \hat{\gamma}_{k+2},
\]
\[
\iota_i * \gamma_{i+1} * \cdots * \gamma_{k+1} * t_{k+1} * t_{k+3}, \text{ or}
\]
\[
\hat{\gamma}_{k+2} * t_{k+1} * t_{k+3}.
\]

The first has length less than \(\ell(\gamma_B) + \ell(t_j) < \ell(\gamma_B) + \ell(\alpha_B)\) by (2). Also by (2), the second path has length less than \(\gamma_B\). The third and fourth paths have length less than \(\gamma_B\) by (2) and (4). Thus \(\text{diam}(B_\gamma) < \gamma_B + \alpha_B < 4\alpha_B\). The last two paths witness that every point in \(B_\gamma\) has a path to \(z\) of length less than \(\gamma_B\). Arguing similarly for \(B_\alpha\), we have that \(\text{diam}(B_\alpha) < \alpha_B\) and for all \(x \in B_\gamma\) and \(y \in B_\alpha\), \(d(x,y) \leq d(x,z) + d(z,y) < \gamma_B + \alpha_B < 4\alpha_B\). Now suppose that \(x \in B\) and \(y \in B \setminus (B_\gamma \cup B_\alpha)\); then \(y \in \partial B\). Since \(\ell(\partial B) < \ell(\alpha_B) + \ell(t_B) + \ell(\gamma_B) < 6\alpha_B\), we have \(\text{diam}(\partial B) < 3\alpha_B\). And clearly every point of \(B\) is at distance at most \(\ell(\iota_i) < \ell(\alpha_B)\) for some \(i \in \{0, \ldots, k+3\}\), therefore \(d(x,y) < \alpha_B + \text{diam}(\partial B) < 4\alpha_B\). This proves part (d).
Now divide \( M \) up into six regions, two for each corner, as in the preceding paragraph. Let \( x, y \in M \). There are two cases depending on whether or not the subscripts of the regions of \( x \) and \( y \) agree. In case they do not, say without loss of generality that \( x \in C_\alpha \) and \( y \in A_\beta \). Then we have already shown that \( d(x, z) < \ell(\alpha \gamma) \) and \( d(z, y) < \ell(\beta \gamma) \), therefore \( d(x, y) < \ell(\alpha \gamma) + \ell(\beta \gamma) \leq \ell(\alpha) + \ell(\beta) \). Otherwise, say \( x \in B_\alpha \) and \( y \in C_\alpha \). Then \( d(x, y) < \ell(\alpha \gamma) + \ell(\beta \gamma) \leq \ell(\alpha) + \ell(\alpha_{k-2}) \leq 2\ell(\alpha) \). Thus \( d(x, y) \) is always less than the sum of the lengths of two sides of \( M \), so (e) holds. \( \square \)

If \( M \) consists of a single face and \( \Delta(a, b, c) \) is equilateral, then \( \text{diam}(M) = \frac{3}{2} \max(\ell(\alpha), \ell(\beta), \ell(\gamma)) \), so inequality (e) is probably the best possible.

## 4 Assouad-Nagata dimension of finitely generated \( C'(1/6) \) groups

In this section we prove the following proposition.

**Proposition 4.1.** Let \( G = \langle S \mid R \rangle \), where \( S \) is finite and \( R \) satisfies \( C'(1/6) \). Then any geodesic spanning tree of \( \Gamma(G, S) \) is \((1/6, 2)\)-tight.

We divide this section into two parts. In the first part we fix all notation and assumptions, and give a description of a van Kampen diagram which is obtained by fixing a geodesic spanning tree of \( \Gamma(G, S) \) based at 1 and assuming it is not \((\varepsilon, 2)\)-tight. All lemmas in the second part are proved under the assumptions stated in the first. We determine \( \varepsilon \) along the way, choosing at each stage an \( \varepsilon \) small enough to make the lemmas work. In the end we reach a contradiction with any \( \varepsilon \leq \frac{1}{9} \), meaning that the spanning tree must have been \((1/6, 2)\)-tight all along.

### 4.1 Construction of a van Kampen diagram

Let \( G = \langle S \mid R \rangle \) be a finitely generated \( C'(1/6) \) group. Fix a geodesic spanning tree \( T \) of \( \Gamma(G, S) \) based at 1 as per Lemma 2.5. Let \( \| \cdot \| \) be the word norm on \( G \) with respect to \( S \), and let \( d \) be the word metric, so \( \| g \| = d(1, g) \). For each \( g \in G \), let \([1, g]\) be the unique path from 1 to \( g \) in \( T \).

Suppose to the contrary that \( T \) is not \((\varepsilon, 2)\)-tight. Let \( r \in \mathbb{N} \) witness that \( T \) is not \((\varepsilon, 2)\)-tight. Then there exists an \( x \in G \) such that \( \| x \| \geq r \) and \( B(x, \varepsilon r) \) contains two elements \( y, y' \) such that the geodesics \([1, x], [1, y] \), and \([1, y'] \) each pass through different elements of the sphere of radius \( \| x \| - r \) in \( \Gamma(G, S) \). Because \( T \) is a tree, for every distinct \( g, h \in G \), there is a unique vertex of \( \Gamma(G, S) \) where the geodesics \([1, g]\) and \([1, h]\) diverge. Let \( a \) be the point at which \([1, x]\) diverges from \([1, y]\), and let \( a' = \) the point at which \([1, x]\) diverges from \([1, y']\). Then we have that \( d(a, x), d(a', x) \geq r \) and \( d(a, y), d(a', y') > r - \varepsilon r \). Without loss of generality suppose that \( \| a' \| \geq \| a \| \).

Let \([x, y]\) and \([x, y']\) be arbitrarily chosen geodesics. Let \( \Delta(1, x, y) \) be the geodesic triangle in \( \Gamma(G, S) \) with sides \([1, x], [1, y] \) and \([x, y]\); similarly define \( \Delta(1, x, y') \). Note that \( \Delta(1, x, y) \) is not a tripod, since \( \ell([x, y]) < \varepsilon r < 2(r - \varepsilon r) \leq \ell([x, a]) + \ell([a, y]) \). Therefore \( \Delta(1, x, y) \) contains exactly one maximal simple geodesic triangle, and \( a \) is the vertex of this triangle which is closest to 1. Let \( \Delta = \Delta(a, b, c) \) be the maximal simple geodesic triangle in \( \Delta(1, x, y) \), where \( a, b, \) and \( c \) are the points closest to 1, \( x, \) and \( y \), respectively. Similarly let \( \Delta' = \Delta(a', b', c') \) be the maximal simple
geodesic triangle of \( \Delta(1, x, y') \) where \( a', b', \) and \( c' \) are the vertices of \( \Delta(a', b', c') \) which are closest to \( 1, x, \) and \( y' \), respectively. Note that \( d(b, x), d(b', x) < \varepsilon r \), and \( d(a', x) > r \), thus we have that \( \|a\| \leq \|a'\| < \|b\| \leq \|b'\| \) or \( \|a\| \leq \|a'\| < \|b'\| \leq \|b\| \).

![Figure 5: \( N = M \cup_{[1,x]} M' \)](image)

Let \( M \) and \( M' \) be minimal van Kampen diagrams for \( \Delta \) and \( \Delta' \), respectively. Attaching the appropriate geodesic segments and gluing \( M \) and \( M' \) along \( [1, x] \), we obtain a van Kampen diagram, call it \( N \), over the loop \( [1, y] * [y, x] * [x, y'] * [y', 1] \). Thus \( N \) is the diagram depicted in Figure 5, allowing that \( b, b' \) may appear in either order along \( [1, x] \), and that \( M \) and \( M' \) may take any of the forms depicted in Figure 2. We retain all notation used in the previous section to describe the geometry of the simple geodesic triangles \( \Delta \) and \( \Delta' \) and their van Kampen diagrams \( M \) and \( M' \), using 's where appropriate. Thus \( \alpha \) is the geodesic opposite \( a \), \( C' \) is the \( c' \)-corner, which is opposite \( \gamma' \), etc. The vertices labeled \( h \) and \( h' \) in Figure 5 are the vertices of \( \gamma, \gamma' \) at the extremities of the \( b \) and \( b' \) corner, respectively.

Since \( M \) is minimal, no two faces of \( M \) cancel: similarly for \( M' \). However, in principle a face of \( M \) may cancel with a face of \( M' \), so \( N \) may or may not be reduced. If \( F \) is a face of \( M \), then we refer to the number of sides of \( F \) with respect to \( M \), not \( N \). Thus for example if \( F \) is a quadrilateral face in \( M \) we will still refer to it as a quadrilateral face even though a side of \( \partial F \) might be split into multiple sides in \( N \). Whether a side of \( F \) is interior or exterior will also be decided with respect to \( M \) rather than \( N \); likewise for faces of \( M' \).

Note that the combinatorial map \( f \) may not be injective when restricted to \( \partial M \cup \partial M' \); for example it may happen that \( f(\alpha') \) intersects \( f(\alpha) \) or \( f(\beta) \) in \( \Gamma(G, S) \). However, the important thing to note is that \( [1, x] \), \( [1, y] \) and \( [1, y'] \) do not intersect at any vertex of \( \Gamma(G, S) \) farther from the identity than \( a' \), so \( f \) is injective when restricted to \( \beta \cup \beta' \cup \gamma \cup \gamma' \).
4.2 Proof that a certain geodesic spanning tree is tight

All lemmas in this subsection are proved under the standing assumptions described in the previous subsection, which are not restated. The argument is as follows. First, we examine how faces of $M$ and $M'$ may line up along their common boundary, and determine that there is a face of $M'$ that shares more than a third of its boundary with $\gamma$ and does not cancel with any face of $M$. Playing around with inequalities provided by the $C'(1/6)$ condition, we find that this situation implies that $\varepsilon > \frac{1}{6}$, the desired contradiction.

Lemma 4.2. Let $h, h'$ be the vertices of $\gamma_B, \gamma'_B$, respectively, which are closest to $a'$. Then $\min(d(a', h), d(a', h')) > (1 - 3\varepsilon)r$.

Proof. By Proposition 3.8, $\ell(\gamma_B) < 3\ell(\alpha_B)$ and so $d(h, x) = \ell(\gamma_B) + d(a, x) < 3\ell(\alpha_B) + d(a, x) \leq 3(\ell(\alpha) + d(a, x)) = 3d(b, x) < 3d(y, x) < 3\varepsilon r$. Since $d(a, x) > r$, we have $d(a, h) > (1 - 3\varepsilon)r$. Similarly for $h'$.

Now we examine how faces of $M$ and $M'$ may meet up along $\gamma \cup \gamma'$. We say that a face $F$ of $N$ cancels if there is some face $F'$ of $N$ such that $F$ and $F'$ cancel. If $F, F'$ are faces of $N$, we say that $F'$ subsumes $F$ if $F$ does not cancel with $F'$ but $(F \cap \gamma) \subseteq (F' \cap \gamma')$. We say that $F$ is subsumed if there is some face $F'$ that subsumes $F$.

Lemma 4.3. Let $F, F'$ be faces of $M, M'$. If $F$ cancel with $F'$, then $\partial F \cap \gamma = \partial F' \cap \gamma'$.

Proof. Suppose that $F$ cancel with $F'$, but $\partial F \cap \gamma \neq \partial F' \cap \gamma'$. Let $\partial F \cap \gamma = [p, q]$ and $\partial F' \cap \gamma' = [p', q']$. Then either $p \neq p'$ or $q \neq q'$. Suppose that $||p|| < ||p'||$: the other cases are similar. Let $\sigma$ be the side of $F$ (in $N$) which is incident to $p'$ and is not contained in $\partial F'$. Let $\tau'$ be the side of $F'$ incident to $p'$ which is not contained in $\gamma'$: see Figure 6. Then $\sigma \ast \tau'$ is a subpath of either a face bordering $F'$ or the geodesic $[1, y']$. Since $F$ and $F'$ cancel, if $\text{Lab}(\sigma)$ ends with a letter $s$, then $\text{Lab}(\tau')$ begins with $s^{-1}$. Therefore either the boundary label of some face is not freely reduced, or $\text{Lab}([1, y'])$ is not freely reduced. The former contradicts the fact that $R$ is cyclically reduced, and the latter contradicts that $[1, y']$ is geodesic.

Lemma 4.4. Let $F$ be a face of $M$ such that either $F$ is triangular, $F$ is quadrilateral, or $F$ is pentagonal with only one exterior side. Then $F$ is not subsumed.
Proof. Let \( \sigma = \partial F \cap \gamma \), and let \( \tau \) be the other exterior side of \( \partial F \) if there is one, either \( \tau = \partial F \cap \alpha \) or \( \tau = \partial F \cap \beta \). If \( F \) is triangular or quadrilateral, then applying Lemma 3.6 to \( \tau \) yields that \( \ell(\sigma) > \frac{1}{6} \ell(\partial F) \). If \( F \) is pentagonal and has only one exterior side, then \( \sigma \) is the only exterior side of \( F \), so \( \ell(\sigma) > \ell(\partial F) - \frac{4}{9} \ell(\partial F) = \frac{1}{3} \ell(\partial F) \). In all cases, if \( \sigma \) is also a subpath of the boundary of a face \( F' \) which does not cancel with \( F \), then this contradicts the \( C'(1/6) \) condition.

**Corollary 4.5.** If a face \( F \) of \( N \) is subsumed, then either \( F \) is the middle face, or \( F \) is a pentagonal face with two exterior sides. In either case \( F \) borders a face in \( B \), so \( h \in \partial F \).

In Lemmas 4.6-4.11, \( E' \) is the extremal face at \( a' \), and

\[
\rho' = \partial E' \cap \gamma', \\
\sigma' = \partial E' \cap \beta', \\
\tau' = \partial E' \setminus (\rho' \cup \sigma').
\]

**Lemma 4.6.** If \( \varepsilon < \frac{1}{6} \), then \( \ell(\tau') < \frac{1}{6} \ell(\partial E') \).

Proof. There are three cases to consider: either \( E' \) is triangular, \( E' \) is quadrilateral, or \( A' \) contains no faces and \( E' \) is the middle face (see Figure 7). In Case 1, \( \tau' \) is an interior side and the result is immediate. In Cases 2 and 3, we may apply Proposition 3.8 to get that \( \ell(\tau') < \ell(\alpha') < \varepsilon r \). Also, in these cases \( E' \) borders \( B' \), so \( h \in E' \). Since \( a' \in E' \) by definition, \( [a', h'] \subseteq \ell(\rho') \) and so \( \ell(\partial E') > 2\ell(\rho') \geq 2d(a', h') > 2(1 - 3\varepsilon)r \). Therefore \( \ell(\tau') < \frac{\varepsilon r}{2(1 - 3\varepsilon)} \ell(\partial E') = \frac{\varepsilon r}{2 - 6\varepsilon} \ell(\partial E') \). Solving \( \frac{\varepsilon}{2 - 6\varepsilon} \leq \frac{1}{5} \) gives \( \varepsilon \leq \frac{1}{6} \).

Applying Lemma 3.6 to \( \rho' \) and \( \sigma' \) in turn, we obtain the following.

**Corollary 4.7.** \( \ell(\rho') > \frac{1}{3} \ell(\partial E') \) and \( \ell(\sigma') > \frac{1}{3} \ell(\partial E') \).

**Lemma 4.8.** \( E' \) does not cancel.

Proof. Suppose that \( E' \) cancels with some face \( F \) of \( M \). Since \( F \cap \gamma = E' \cap \gamma' \) by Lemma 4.3 and \( a' \notin B \), we have that \( F \) is not a face of \( B \). Therefore \( F \) borders \( \beta \). Now there are two cases. Either
there is an interior side of $F$ which is incident to both $\gamma$ and $\beta$ (Case 1 in Figure 8), or $F$ is the extremal face at $a$ and $a = a'$ (Case 2). In Case 1, let $\theta$ be the side of $F$ incident to $a'$ and $\beta$. In Case 2, let $\theta$ be the edge of $\partial F \cap \beta$ which is incident to $a'$.

In either case, let $\theta'$ be the path starting from $a'$ which is a subpath of $\partial E'$ and has label $\text{Lab}(\theta)$. By Corollary 4.7, $\theta'$ is a subpath of $\sigma'$. Let $p, p'$ be the endpoints of $\theta, \theta'$, respectively. Since $\text{Lab}(\theta) = \text{Lab}(\theta')$ and the combinatorial map $f$ is label-preserving, $f(p) = f(p')$. Note that $p \in \beta$ by definition, and $p' \in \sigma' \subseteq \beta'$. Since $T$ is a tree, $f$ is injective when restricted to $\beta \cup \beta'$, so this is a contradiction.

**Lemma 4.9.** $\ell(\rho') > (1 - 3\varepsilon)r$.

**Proof.** Since $\|a'\| \geq \|a\|$, we have that either $\rho'$ is a subpath of $\gamma$ or $\rho'$ extends beyond $\gamma$. If the latter is the case, then $a', b \in \rho'$ and so $\ell(\rho') \geq d(a', b) > (1 - \varepsilon)r > (1 - 3\varepsilon)r$.

Suppose then that $\rho'$ is a subpath of $\gamma$. Since $E'$ does not cancel, the $C'(1/6)$ condition implies that each face of $M$ bordering $E'$ must cover less than one sixth of $\partial E'$. Recall that $\ell(\rho') > \frac{1}{3}\ell(\partial E')$ by Corollary 4.7. Therefore $E'$ must border at least three faces of $M$, so $E'$ subsumes some face $F$. Since $F$ is subsumed, $(\partial F \cap \gamma) \subseteq \rho'$ and $h \in (\partial F \cap \gamma)$ by Corollary 4.5. Since $\rho'$ contains $a'$ as well, we have that $\ell(\rho') \geq d(a', h) > (1 - 3\varepsilon)r$.

**Corollary 4.10.** $\ell(\rho' \cap [a', h]) > \frac{1 - 6\varepsilon}{9 - 9\varepsilon}\ell(\partial E')$.

**Proof.** First, observe that $\ell([h, x]) < 3\varepsilon r$ and $\ell(\rho') > (1 - 3\varepsilon)r$. Therefore $\ell([h, x]) < \frac{3\varepsilon}{1 - 3\varepsilon}\ell(\rho')$, so

$$\ell(\rho \cap [a', h]) = \ell(\rho' \setminus [h, x]) \geq \ell(\rho') - \ell([h, x]) > \ell(\rho') - \left(\frac{3\varepsilon}{1 - 3\varepsilon}\right)\ell(\rho') = \left(\frac{1 - 6\varepsilon}{1 - 3\varepsilon}\right)\ell(\rho').$$

By Corollary 4.7, $\ell(\rho') > \frac{1}{3}\ell(\partial E')$. Therefore

$$\ell(\rho' \cap [a', h]) > \left(\frac{1 - 6\varepsilon}{1 - 3\varepsilon}\right)\ell(\rho') > \left(\frac{1 - 6\varepsilon}{3 - 9\varepsilon}\right)\ell(\partial E').$$
Lemma 4.11. If \( \varepsilon \leq \frac{1}{9} \), there is a face \( F \) of \( M \) satisfying all of the following conditions:

(a) \( F \) is subsumed by \( E' \).
(b) \( F \) is either the middle face of \( M \) or the pentagonal face in \( A \).
(c) \( \ell(\partial F \cap \partial E') > (\frac{1-6\varepsilon}{3-9\varepsilon} - \frac{1}{6})\ell(\partial E') \).
(d) \( \ell(\partial F) < 6\varepsilon r \).

Proof. From the previous corollary we know that more than \( \frac{1-6\varepsilon}{3-9\varepsilon} \) of \( \partial E' \) must be covered by faces which are not in \( B \). Since \( E' \) does not cancel, if \( \frac{1-6\varepsilon}{3-9\varepsilon} \geq \frac{1}{6} \), then \( E' \) subsumes some face \( F \) of \( M \) which is not in \( B \). Solving \( \frac{1-6\varepsilon}{3-9\varepsilon} \geq \frac{1}{6} \) yields \( \varepsilon \leq \frac{1}{9} \) so choose \( \varepsilon \leq \frac{1}{9} \) and part (a) follows. Lemma [4.4] shows that \( F \) can only be the middle face or the pentagonal face of \( A \), giving part (b). Furthermore, no other faces of \( M \setminus B \) can be subsumed by \( E' \). Now \( E' \) subsumes \( F \) implies that \( \ell(\partial F \cap \partial E') < \frac{1}{6}\ell(\partial E') \), so there is still a subpath of \( \rho' \) of length more than \( (\frac{1-6\varepsilon}{3-9\varepsilon} - \frac{1}{6})\ell(\partial E') \geq 0 \) to be covered. Therefore \( E' \) must border one additional face of \( M \) which is not contained in \( B \). Call this face \( E \). Since \( E \) cannot be subsumed by \( E' \), we have that part of \( \partial E \) extends beyond \( \gamma' \), so \( a' \in \partial E \). Therefore we have the situation depicted in Figure 9.

![Figure 9](image)

Notice that \( \ell(\partial E \cap \partial E') < \frac{1}{6}\ell(\partial E') \) and \( \ell(\partial F \cup \partial E) \cap \partial E' > \ell(\rho') > \frac{1-6\varepsilon}{3-9\varepsilon}\ell(\partial E') \). Therefore \( \ell(\partial F \cap \partial E') > (\frac{1-6\varepsilon}{3-9\varepsilon} - \frac{1}{6})\ell(\partial E') \). This proves part (c).

Let \( \rho = \partial F \cap \gamma \) and \( \sigma = \partial F \cap \beta \). Then \( \ell(\rho) < \frac{1}{6}\ell(\partial F) \) since \( E' \) subsumes \( F \). Since \( F \) is either the middle face or the pentagonal face of \( A \), \( F \) has exactly one interior side, call it \( \tau \), which does not border either \( \alpha, B, \) or \( C \). The sum of the lengths of the other sides of \( F \) is less than \( \ell(\alpha) < \varepsilon r \) by Proposition [3.8]. By Lemma [3.6] applied to \( \sigma \), we have that \( \ell(\rho) + \ell(\tau) + \varepsilon r \geq \frac{1}{2}\ell(\partial F) \). But \( \max(\ell(\rho), \ell(\tau)) < \frac{1}{6}\ell(\partial F) \), so we have \( \varepsilon r > \frac{1}{6}\ell(\partial F) \), or \( \ell(\partial F) < 6\varepsilon r \). This proves part (d).

We return to Proposition [4.1] which we are now ready to prove.
Proof of Proposition 4.1. Suppose that $T$ is a geodesic spanning tree of $\Gamma(G,S)$ based at 1, and $T$ is not $(\epsilon,2)$-tight. If $\epsilon \leq \frac{1}{6}$, then all of the previous lemmas hold. But then, in the notation of Lemma 4.11 we have

$$6\epsilon r > \ell(\partial F) > 6\ell(\partial F \cap \partial E') > 6\left(\frac{1}{6} - \frac{1}{36\epsilon}\right)\ell(\partial E') > \frac{9\epsilon - 3}{3}2\ell(\rho') = \frac{18\epsilon - 2}{3}(1 - 3\epsilon)r = (18\epsilon - 2)r.$$  

Thus $6\epsilon > 18\epsilon - 2$ or $\epsilon > \frac{1}{6}$, a contradiction. Therefore $T$ is $(\frac{1}{6},2)$-tight. 

Combining this with Proposition 2.4, we have the following theorem.

**Theorem 4.12.** If $G$ is a finitely generated $C'(\frac{1}{6})$ group, then $\operatorname{asdim}_{AN}(G) \leq 2$.

For infinitely generated groups $G$, it is possible to define $\operatorname{asdim}(G)$ to be the supremum of the asymptotic dimensions of its finitely generated subgroups, although this definition doesn’t make sense for Assouad-Nagata dimension: see [22]. Of course languages over infinite alphabets can also satisfy $C'(\frac{1}{6})$, so the notion of a $C'(\frac{1}{6})$ group extends to infinitely generated groups. Therefore we can also say the following.

**Corollary 4.13.** If $G$ is a $C'(\frac{1}{6})$ group, then $\operatorname{asdim}(G) \leq 2$.

5 A technical lemma on certain central extensions of small cancellation groups

In order to prove the main result on asymptotic and Assouad-Nagata dimension, we will need the following technical lemma. Given a set $S$ and a language $R \subseteq S^*$, we say that $R$ is cyclically minimal if there do not exist distinct $r, r' \in R$ such that $r'$ is a cyclic shift of $r$ or $r^{-1}$. Given a group presentation $\langle S \mid R \rangle$, we may assume that $R$ is cyclically minimal and cyclically reduced without changing the resulting group. Recall that for constants $K, C > 0$, a path $\alpha : [0, \ell(\alpha)] \to \Gamma(G,S)$ is $(K,C)$-quasi-geodesic if $d(\alpha(s),\alpha(t)) \leq K|s-t| + C$ for all $s,t \in [0,\ell(\alpha)]$. We say that a word $u \in S^*_c$ is $(K,C)$-quasi-geodesic in $G$ if any path in $\Gamma(G,S)$ with label $u$ is $(K,C)$-quasi-geodesic.

**Lemma 5.1.** Let $H$ be a group given by a presentation $\langle S \mid R_H \rangle$, such that $R_H$ is cyclically minimal, cyclically reduced, does not contain any $s \in S$, and satisfies $C'(\lambda)$ for some $0 < \lambda < \frac{1}{12}$. Suppose also that for some $a \in S$, we have that $a^{|r|}$ is not a prefix of any cyclic shift of $r^{\pm 1}$ for all $r \in R$. Let $G$ be the central extension of $H$ defined by

$$G = \langle S \mid R_G \rangle = \langle S \mid [s,r], r^{k(r)} : s \in S, r \in R_H \rangle$$

where $k(r) \geq 2$ for all $r \in R_H$. Let $r_1, \ldots, r_m \in R_H$ be distinct relations of equal length, and suppose that $k(r_1) = \cdots = k(r_m) =: k$. Let $u \in S^*_c$ be a word of the form

$$u = a_1^{r_1} \cdots a_m^{r_m}$$

where $|a_j| \in \{0,\ldots,\lfloor k/2 \rfloor\}$ for all $j \in \{1,\ldots,m\}$. Then $u$ is $(\frac{4}{1-12\lambda},0)$-quasi-geodesic in $G$. 

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Lemma 5.1 is a generalized version of Lemma 5.3 of [23], which was used to exhibit examples of finitely generated groups with circle-tree asymptotic cones. The proof of Lemma 5.1 actually shows something more general, but for the sake of clarity we avoid making the most general statement.

The strategy to prove Lemma 5.1 is the following. Suppose that $u = G v$, where $v \in S_\epsilon^*$ is geodesic in $G$. Then $wv^{-1} = G 1$, so there exists a van Kampen diagram $M_G$ over $\langle S \mid R_G \rangle$ with boundary path $\alpha * \beta$, where $\text{Lab}(\alpha) = u$ and $\text{Lab}(\beta) = v^{-1}$. We apply a series of operations on $M_G$ to get a new van Kampen diagram $M_H$ over $\langle S \mid R_H \rangle$ with the same boundary path (Lemma 5.14). We then apply another series of operations to $M_H$ to get another van Kampen diagram $M_H'$ over $\langle S \mid R_H' \rangle$ with boundary path $\alpha' * \beta$, where $\alpha'$ is a shortened version of $\alpha$ (Lemma 5.15). After this we use some properties of van Kampen diagrams over $C'(\frac{1}{6})$ presentations to bound $\ell(\alpha)$ in terms of $\ell(\alpha')$ and $\ell(\alpha')$ in terms of $\ell(\beta)$. In the end this gives us that $\ell(\alpha) \leq \frac{1}{12g} \ell(\beta)$ and thus $|u| \leq \frac{4}{12g} |v|$, the desired result.

In Section 5.1, we describe various operations on van Kampen diagrams and how they affect the boundary label and a quantity we call the signed $r$-cell count. We also collect various lemmas about van Kampen diagrams over $C'(\frac{1}{6})$ groups that we will need to construct $M_H$ and $M_H'$. In Section 5.2 we provide, in detail, the proof of Lemma 5.1 outlined in the preceding paragraph. The reader who is only interested in how to use Lemma 5.1 to prove the main results on asymptotic and Assouad-Nagata dimension may skip to Section 6.

5.1 Operations on van Kampen diagrams and the signed $r$-cell count

When performing surgery on one van Kampen diagram to get another, one only needs to check that the operation does not disconnect the graph, produces a planar embedding of the new graph, and leaves the combinatorial map well-defined. In our case, we will also need to keep track of the boundary label and a quantity we call the signed $r$-cell count. When we say that the boundary path of a van Kampen diagram is unaffected by an operation, we mean that it consists of the same sequence of edges, even though the operation may change the image of the boundary path topologically.

Recall that if $F$ is a face of a van Kampen diagram over a presentation $\langle S \mid R \rangle$, $x$ is a vertex of $\partial F$, and $w$ is a word in $S_\epsilon^*$, then $\text{Lab}(\partial F, x, +) = w$ means that the boundary path $\partial F$, read counterclockwise from $x$, reads $w$ (replacing $+$ with $-$ changes ‘counterclockwise’ to ‘clockwise’).

**Definition 5.2.** Let $M$ be a van Kampen diagram over a presentation $\langle S \mid R \rangle$, where $R$ is cyclically minimal. Then we define the signed $r$-cell count $\sigma_r(M)$ for each $r \in R$ as follows:

- If $F$ is a face of $M$, then
  $$\sigma_r(F) = \begin{cases} 
  1 & \text{if } \text{Lab}(\partial F, x, +) = r \text{ for some } x \in \partial F \\
  -1 & \text{if } \text{Lab}(\partial F, x, -) = r \text{ for some } x \in \partial F \\
  0 & \text{otherwise.}
  \end{cases}$$

- $\sigma_r(M) = \sum \{ \sigma_r(F) \mid F \text{ is a face of } M \}$.

The assumption that $R$ is cyclically minimal ensures that each face contributes to the signed $r$-cell count for at most one $r \in R$. Note that if $F$ and $F'$ cancel, then $\sigma_r(F) = -\sigma_r(F')$ for all $r \in R$.  

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**Operation 5.3** (Removing an inessential edge). Suppose that \( e = (x, y) \) is an inessential edge of a van Kampen diagram \( M \). If \( x \neq y \), then contract \( e \) to remove it, since edge contractions preserve connectedness and planarity. Otherwise if \( x = y \), then simply deleting \( e \) will leave the graph connected. Since an inessential edge can only belong to an inessential face, deleting such a loop can only merge two inessential faces. Thus it does not affect \( \sigma_r(M) \) for any \( r \in R \), and does not affect \( \partial M \) if \( \text{Lab}(\partial M) \) is reduced.

A subdiagram of a van Kampen diagram \( M \) is a simply connected union of faces of \( M \). We call a face \( F \) or a subdiagram \( D \) a disk if it is homeomorphic to a closed disk, that is, if \( \partial F \) or \( \partial D \) has no self-intersections.

**Operation 5.4** (Removing a disk subdiagram with trivial boundary). Suppose that a van Kampen diagram \( M \) over \( \langle S \mid R \rangle \) contains a disk subdiagram \( D \) such that \( \text{Lab}(\partial D) = F(S) \). Applying Operation 5.3, we can assume that \( \partial D \) contains no inessential edges and thus \( \partial D = \alpha_+\alpha_- \), where \( \text{Lab}(\alpha_-) = \text{Lab}(\alpha_+)^{-1} \). We may then remove \( D \) by replacing \( D \) with an inessential disk face \( F \) and deforming \( \alpha_+ \) onto \( \alpha_- \) through the interior of \( F \). This does not affect the boundary path of \( M \).

Perhaps surprisingly, this operation does not always preserve the signed \( r \)-cell count, as the following example shows.

**Example 5.5.** Figure 10 depicts a van Kampen diagram over the presentation \( \langle a, b \mid a^2, aba^{-1}b \rangle \) with boundary label \( bb^{-1} \), such that \( \sigma_{aba^{-1}b} = 2 \).

![Diagram](image)

Figure 10

However, Operation 5.4 does preserve the signed \( r \)-cell count of van Kampen diagrams over \( C'(1/6) \) presentations. This is because \( C'(1/6) \) presentations are aspherical. The definition of a spherical van Kampen diagram is the same as that of a van Kampen diagram with \( \mathbb{R}^2 \) replaced by \( S^2 \), in particular every face is bounded. A presentation \( \langle S \mid R \rangle \) is aspherical if every bare spherical van Kampen diagram over \( \langle S \mid R \rangle \) contains a pair of faces that cancel. The following lemma is a special case of Theorem 31.1 of [24]; a brief proof is given here for the reader’s convenience.

**Lemma 5.6** ([24], Theorem 31.1). Let \( \langle S \mid R \rangle \) be an aspherical presentation, and suppose that \( M \) is a van Kampen diagram over \( \langle S \mid R \rangle \) with boundary label \( w \), where \( w = F(S) \). Then \( \sigma_r(M) = 0 \) for all \( r \in R \).

**Proof.** Assume that \( M \) is a counterexample with at least one essential face, that minimizes first number of faces and then number of edges. Suppose that \( M \) contains an inessential face, call it \( F \).
If $\partial F$ and $\partial M$ have a common subpath, then we may delete this common subpath to obtain a new van Kampen diagram with trivial boundary label, the same signed $r$-cell count for each $r \in R$, and one face fewer, contradicting minimality. If $\partial F$ and $\partial M$ do not share a common subpath, let $\alpha$ be a shortest path from $\partial F$ to $\partial M$. Now cut along $\alpha$ to obtain a new van Kampen diagram with boundary label $w_1 \text{Lab}(\alpha)^{-1}\text{Lab}(F)\text{Lab}(\alpha)w_2$, where $w_1w_2 = w$. This van Kampen diagram has trivial boundary label, the same signed $r$-cell count for all $r \in R$, and one face fewer than $M$, again contradicting minimality. Therefore $M$ contains no inessential faces, i.e. $M$ is bare.

Since $\text{Lab}(\partial M) =_{F(S)} 1$, there exists a subpath $(y_-, e_-, x, e_+, y_+)$ of $\partial M$ such that $\text{Lab}(e_+) = \text{Lab}(e_-)^{-1}$. If $y_- \neq y_+$, then we may deform $e_-$ through the unbounded face so that $y_-$ is identified with $y_+$ and $e_-$ is identified with $e_+$. This reduces the number of edges in $M$ without increasing the number of faces, contradicting minimality of $M$. Therefore $y_- = y_+ =: y$, so the path $(y, e_-, x, e_+, y)$ encloses a subdiagram $D$ of $M$ with boundary label $ss^{-1}$ for some $s \in S$. If $M \supseteq D$, then $M = M' \cup D$ for some subdiagram $M'$. Thus if $\sigma_r(M) \neq 0$ for some $r \in R$, we have either $\sigma_r(D) \neq 0$ or $\sigma_r(M') \neq 0$, and both $D$ and $M'$ have trivial boundary label and are strictly smaller than $M$. This contradicts minimality of $M$, so $M = D$.

Now embed $D$ into the sphere $S^2$, where we consider $S^2$ to be the one-point compactification of $\mathbb{R}^2$ with the point at infinity lying in the unbounded face of $D$. Either $D$ is a disk, if $x \neq y$, or $D$ is the wedge of two disks, if $x = y$. Deform $e_-$ and $e_+$ onto each other so that they meet along the equator, if $D$ is the wedge of two disks, or half the equator if $D$ is a disk. Since $(S \setminus R)$ is aspherical and $D$ is bare, there are two faces $F$ and $F'$ of $D$ that cancel. If $\alpha$ is a common subpath between them, then delete $\alpha$, replacing $F \cup F'$ with an inessential face $F''$. Since $\sigma_r(F) = -\sigma_r(F')$ for all $r \in R$, this operation preserves $\sigma_r(D)$ for all $r \in R$. Declaring that the point at infinity lies in the interior of $F''$, we obtain a new van Kampen diagram $D'$ with trivial boundary label and two faces fewer than $D$. This contradicts minimality of $D = M$, completing the proof.

**Operation 5.7** (Padding a vertex). Suppose that $x$ is a vertex of $M$ which appears twice in the boundary path $\partial F$ for some essential face $F$ of $M$. Choose $\varepsilon$ small enough so that $B(x, \varepsilon) \subset \mathbb{R}^2$ contains only the ends of edges incident to $x$. Now $B(x, \varepsilon) \setminus M$ consists of finitely many connected components: let $C_1, \ldots, C_k$ be the components of $B(x, \varepsilon) \setminus M$ which do not belong to the unbounded face or any inessential face. For each $i \in \{1, \ldots, k\}$, insert a clone $x_i$ of $x$ into $C_i$ and join it to $x$ with an inessential edge. Then duplicate the edges on either side of $x_i$, attaching the endpoint meant for $x$ to $x_i$ instead: see Figure 11. The resulting graph has the same essential faces and boundary path as $M$, and one fewer vertex that is a point of self-intersection of the boundary of an essential face. Each new inessential face is a triangle with boundary label $1ss^{-1}$ for some $s \in S$.

Figure 11

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Operation 5.8 (Quotienting disk faces). Suppose that \( G = \langle S \mid R_G \rangle \) and \( H = \langle S \mid R_H \rangle \) is a quotient of \( G \), so every word in \( R_G \) represents the identity element of \( H \). Suppose that \( M_G \) is a van Kampen diagram over \( \langle S \mid R_G \rangle \). Let \( F \) be a disk face of \( M_G \), and let \( M_F \) be a chosen van Kampen diagram over \( \langle S \mid R_H \rangle \) with boundary label \( \text{Lab}(\partial F) \). Then we may quotient \( F \) to a copy of \( M_F \) without affecting the boundary path of \( M_G \): see Figure 12. Applying this operation once produces a van Kampen diagram over \( \langle S \mid R_G \cup R_H \rangle \). If \( F \) is the last face of \( M_G \) with label in \( R_G \setminus R_H \), then this results in a van Kampen diagram over \( \langle S \mid R_H \rangle \). Thus, if this operation can be applied to every essential face of \( M_G \) in sequence, then we obtain a “quotient van Kampen diagram” \( M_H \) over \( \langle S \mid R_H \rangle \) with the same boundary path as \( M_G \).

![Figure 12](image)

Operation 5.9 (Excising a subpath of \( \partial M \)). Let \( M \) be a van Kampen diagram over a presentation \( \langle S \mid R \rangle \). Suppose that \( \partial M = \alpha \ast \beta \), and that \( \alpha = \alpha_1 \ast \rho \ast \alpha_2 \), where \( \rho \) is a path from \( x \) to \( y \) with label \( r \pm 1 \) for some \( r \in R \). Then we may contract \( x \) to \( y \) through the unbounded face to obtain a new van Kampen diagram \( \hat{M} \), in which \( \rho \) encloses a bounded face with label \( r \ominus 1 \): see Figure 13. Then \( \partial \hat{M} = \hat{\alpha} \ast \beta \), where \( \hat{\alpha} = \alpha_1 \ast \alpha_2 \). In particular, \( \ell(\hat{\alpha}) = \ell(\alpha) - |r| \), and \( \beta \) is unaffected. This operation adds 1 to \( \sigma_r(M) \) if \( \text{Lab}(\rho) = r^{-1} \), and subtracts 1 from \( \sigma_r(M) \) if \( \text{Lab}(\rho) = r \).

![Figure 13](image)

Under mild assumptions about the presentation, Operation 5.9 affects the boundary label of certain van Kampen diagrams in a predictable way.

Lemma 5.10. Let \( S \) be a set, \( a \in S \), and let \( R \subset S^* \) be a cyclically minimal language satisfying \( \text{C}'(1/2) \). Suppose also that for each \( r \in R \), \( a^{\lvert r \rvert}/2 \) is not a prefix of any cyclic shift of \( r \pm 1 \). Let \( u = a^n u_1 \ldots u_k \), where \( u_i = r_i \pm 1 \) for some \( r_i \in R \) for all \( i \in \{1, \ldots, k\} \). Suppose that \( t \) is a subword of \( u \), \( t = r \pm 1 \) for some \( r \in R \), and \( \hat{u} \) is \( u \) with \( t \) deleted. Then
\[
\hat{u} = a^n u_1 \ldots u_{i-1} u_{i+1} \ldots u_k.
\]
Thus, there is a subpath of $\partial F$.

Remark. Then every essential face of $M$.

If instead $t = r$, then $t$ is a subword of some $u_i$, some $u_i$ is a subword of $t$, $t$ is a subword of $u_i u_{i+1}$ for some $i$, or $t$ is a subword of $a^n u_1$.

If $t$ is a subword of $u_i$ for some $i$, then $u_i = ptq$ for some words $p$ and $q$, thus $r_i^{\pm 1} = prq$. Let $\tilde{r}_i r \in R_*$. Then $\tilde{r}_i, r \in R_*$ have a common prefix $r$. By the $C'(1/2)$ condition, $\tilde{r}_i = r$. Thus $|t| = |r| = |\tilde{r}_i| = |u_i|$, since $t$ is a subword of $u_i$, and the result is immediate. The argument when $u_i$ is a subword of $t$ is similar.

Suppose that $t$ is a subword of $u_i u_{i+1}$. This means we can write $u_i u_{i+1}$ as $p(p'q)q'$, where $u_i = r_i^{\pm 1} = pp'$, $u_{i+1} = r_i^{\pm 1} = qq'$, and $t = r = p'q$. One of $p'$ or $q$, without loss of generality say $p'$, has length at least $\frac{1}{2} |r|$. Let $\tilde{r}_i = p'p$. Then $\tilde{r}_i, r \in R_*$ have a common prefix $p'$ of length at least $\frac{1}{2} |r|$, so $\tilde{r}_i = r$ by the $C'(1/2)$ condition. Since $r$ is a cyclic shift of $r_i^{\pm 1}$ and $R$ is cyclically minimal, $r = r_i^{\pm 1}$. We want to show that deleting $t$ from $u_i u_{i+1}$ has the same effect on $u$ as deleting $u_i$, i.e. that $p'q' = qq'$. Notice however that $p'p = \tilde{r}_i = r = p'q$, hence $p = q$. Therefore $p'q' = qq'$.

If instead $t$ is a subword of $a^n u_1$, then we can write $a^n u_1$ as $pp'qq'$, where $a^n = pp'$, $u_1 = r_1 = qq'$, and $t = r = p'q$. Since $p'$ is a power of $a$, we must have that $|q| \geq \frac{1}{2} |r|$. At this point the argument is similar to the one in the preceding paragraph.

Difficulties that can occur when working with van Kampen diagrams are that a face is not a disk, or that two faces (which may be disks) intersect in more than one common subpath. In van Kampen diagrams over $C'(1/6)$ presentations, the former problem does not occur, and the latter is easily resolved. These facts are consequences of the Greendlinger Lemma.

Lemma 5.11 (Greendlinger Lemma). [21] Let $M$ be a bare and reduced van Kampen diagram over a $C'(\lambda)$ presentation, where $\lambda \leq 1/6$. Then there exists a face $F$ of $M$ such that $\partial F$ and $\partial M$ share a common subpath of length more than $\frac{1}{2} |\partial F|$.

Lemma 5.12. [21] Let $M$ be a van Kampen diagram over a cyclically reduced $C'(1/6)$ presentation. Then every essential face of $M$ is a disk.

Remark. Lemma 5.12 implies in particular that in a $C'(1/6)$ presentation, no proper subword of a relation represents the identity. In particular, if $H = \langle S \mid R \rangle$ is a $C'(1/6)$ presentation, $s \in S$, and $s \not\in R$, then $s \neq H 1$.

Lemma 5.13. Let $M$ be a bare van Kampen diagram over a $C'(1/6)$ presentation. Suppose that $M$ is not reduced. Then there exist two faces of $M$ that cancel, such that their boundaries intersect in a single common subpath.

Proof. Suppose that $M$ is a counterexample with the minimum number of faces. Then $M$ contains two faces, say $F$ and $F'$, that cancel, and $\partial F$ does not intersect $\partial F'$ in a single common subpath. Thus, there is a subpath of $\partial F$ and a subpath of $\partial F'$ that together enclose a disk subdiagram $D$: see Figure [14].
If $D$ is not reduced, then by minimality of $M$ there are two faces of $D$ that cancel and whose boundaries intersect in a single common subpath, and we are done. Therefore assume $D$ is reduced. By the Greendlinger Lemma, there is a face $E$ of $D$ such that $E$ shares a common subpath of length at least $\frac{1}{2} \ell(\partial E)$ with $\partial D$. But then $E$ shares a common subpath of length at least $\frac{1}{4} \ell(\partial E)$ with one of $F$ or $F'$, without loss of generality say with $F$. Thus $E$ cancels with $F$. If $\partial E$ does not intersect $\partial F$ in a single common subpath, then $F \cup D$ is a subdiagram of $M$ which provides a counterexample to the statement, but does not contain $F'$ and therefore contains strictly fewer faces than $M$, contradicting minimality of $M$. Therefore $\partial E$ intersects $\partial F$ in a single common subpath, also a contradiction, and we are done.

\section*{5.2 Proof that a certain word is quasi-geodesic}

At this point the reader may want to review the statement of Lemma 5.1: all lemmas in this section are proved with the same notation and standing assumptions.

Let $v$ be a geodesic representative of $u$. Let $M_G$ be a van Kampen diagram over $\langle S \mid R_G \rangle$ such that $\partial M_G = \alpha \ast \beta$, where $\text{Lab}(\alpha) = u$ and $\text{Lab}(\beta) = v^{-1}$.

\textbf{Lemma 5.14.} There exists a van Kampen diagram $M_H$ over $\langle S \mid R_H \rangle$ such that all of the following conditions hold. Let $\sigma_j$ denote the signed $r_j$-cell count of $M_H$ for each $j \in \{1, \ldots, m\}$.

(a) $\partial M_H = \alpha \ast \beta$.

(b) $\sigma_j$ is a multiple of $k$ for each $j$.

(c) $M_H$ is bare and reduced.

\textit{Proof.} Start with $M_G$. For each face $F$ of $M_G$, let $M_F$ be a van Kampen diagram over $\langle S \mid R_H \rangle$ with boundary label $\text{Lab}(\partial F)$, of one of the forms shown in Figure 15. Then for all faces $F$ of $M_G$...
and all \( r \in R_H \),

\[
\sigma_r(M_F) = \begin{cases} 
\pm k(r) & \text{if } \text{Lab}(\partial F, x, \pm) = r^{k(r)} \text{ for some } x \in M \\
0 & \text{otherwise.}
\end{cases}
\]

By repeatedly padding vertices, we may assume that all essential faces of \( M_G \) are disks. Now take an essential disk face \( F \) of \( M_G \), and quotient it to \( M_F \). This may introduce self-intersections among essential faces in \( M_G \). Pad vertices again until all essential faces of \( M_G \) are disks, and repeat as many times as necessary to quotient all essential faces that were originally in \( M_G \). Since padding vertices and quotienting disk faces preserve the boundary path, we obtain a van Kampen diagram \( M_H \) over \( \langle S \mid R_H \rangle \), possibly with many inessential faces, such that \( \partial M_H = \alpha \ast \beta \). In addition, for all \( r \in R_H \),

\[
\sigma_r(M_H) = \sum \{ \sigma_r(M_F) \mid F \text{ is a face of } M_G \},
\]

which is a multiple of \( k(r) \). In particular, \( \sigma_j \) is a multiple of \( k \). Thus (a) and (b) hold.

Each inessential face created in the process of padding vertices is, at the time it is added, a triangle with boundary label \( 1ss^{-1} \) for some \( s \in S \). During the quotienting process, these triangles may acquire self-intersections. By the remark following Lemma 5.12, \( s \neq H \) for all \( s \in S \). Therefore the only possibilities for inessential faces in \( M_H \) are the ones shown in Figure 16.

![Figure 15](image1)

![Figure 16](image2)

We contract all inessential edges that are not loops. Thus, at this stage, every inessential face is contained in a disk subdiagram with boundary label \( ss^{-1} \) for some \( s \in S \). It may happen that, during the process of quotienting faces of \( M_G \), an inessential edge comes to enclose a subdiagram with several essential faces: this is the situation depicted in the diagram on the right-hand side of Figure 14. In theory, removing such a disk subdiagram could change \( \sigma_j \). However, since \( \langle S \mid R_H \rangle \)
is aspherical, by Lemma 5.6 we have that any subdiagram with boundary label $ss^{-1}$ has $\sigma_j = 0$. Therefore we can apply Operation 5.4 to remove all disk subdiagrams with boundary label $ss^{-1}$ from $M_H$ without affecting $\sigma_j$. Thus we may assume that $M_H$ is bare.

To ensure that $M_H$ is reduced, note that if $F,F'$ are two faces of $M_H$ that cancel, then they are disks by Lemma 5.12 and we may assume that they intersect in a single common subpath by Lemma 5.13. Therefore $F \cup F'$ is a disk subdiagram of $M_H$ with trivial boundary label, and we may apply Operation 5.4 to remove $F \cup F'$ from $M$ without affecting $\partial M_H$ or $\sigma_j$. Thus (c) holds, completing the construction. \hfill \Box

**Lemma 5.15.** There exist integers $p_1,\ldots,p_m$ and a van Kampen diagram $M'_H$ over $\langle S \mid R_H \rangle$, such that all of the following conditions hold. Here $\sigma_j, \sigma'_j$ are the signed $r_j$-cell counts of $M_H, M'_H$, respectively, and $\kappa_j, \kappa'_j$ are the total (unsigned) number of $r_j$-faces of $M_H, M'_H$, respectively, for each $j \in \{1,\ldots,m\}$.

(a) For all $j$, the sign of $p_j$ agrees with the sign of $a_j$, and $|p_j| \leq |a_j| \leq \lfloor k/2 \rfloor$.

(b) $\sigma'_j = \sigma_j - p_j$ for all $j$.

(c) $\partial M'_H = \alpha' * \beta$, where $\text{Lab}(\alpha') = u' := a^{a_1 r_1^{a_1} \cdots r_m^{a_m}}$.

(d) $M'_H$ is bare and reduced.

(e) $\kappa'_j = \kappa_j - |p_j|$ for all $j$, and $\ell(\alpha) - \ell(\alpha') = |r_j| (\sum_{j=1}^{m} |p_j|)$.

(f) If $F$ is a face of $M'_H$, then $\partial F$ does not intersect $\alpha'$ in a common subpath of length at least $2\lambda(\partial F)$.

(g) If $F$ is a face of $M'_H$ and $\partial F$ shares at least one edge with $\alpha'$, then $\partial F$ intersects $\alpha'$ in a single common subpath.

**Proof.** Start with $M_H$. Suppose that $F$ is a face of $M_H$ which intersects $\alpha$ in a common subpath of length at least $2\lambda(\partial F)$. Then since $\text{Lab}(\alpha) = u = a^{a_1 r_1^{a_1} \cdots r_m^{a_m}}$, and $\partial F$ cannot share a common subpath of length at least $\lambda(\partial F)$ with the subpath of $\alpha$ labeled $a^n$, we have that for some $j$ there is a subpath $\tau$ of $\partial F$ with label equal to a subword of $r_j^{a_j}$ of length at least $\lambda(\partial F)$, where $\pm$ is the sign of $a_j$. For simplicity, suppose that $a_j$ is positive. By the $C'(\lambda)$ condition, $\text{Lab}(\partial F) = r_j$. Moreover, there exists a subpath $\rho$ of $\partial M_H$ such that $\tau$ is a subpath of $\rho$ and $\text{Lab}(\rho) = r_j$.

Now apply Operation 5.9 to $\rho$ to obtain a new van Kampen diagram $\hat{M}_H$ with boundary path $\hat{\alpha} * \beta$, where $\hat{\alpha} = \alpha$ with $\rho$ excised. Then the signed $r_j$-cell count of $\hat{M}_H$ is $\sigma_j - 1$, so $\hat{M}_H$ satisfies (b) with $p_j = 1$. Since $\text{Lab}(\hat{\alpha}) = \text{Lab}(\alpha)$ with a subword of $r_j$ deleted, $\hat{M}_H$ also satisfies (c) with $p_j = 1$ by Lemma 5.10.

Let $F'$ be the face which is now enclosed by $\rho$ in $M_H$. Since $\tau$ is still a common subpath of $F$ and $F'$ of length at least $\lambda(\partial F)$, we have that $F$ and $F'$ cancel. Since $M_H$ was reduced, it must be the case that $F$ and $F'$ are the only faces of $\hat{M}_H$ that cancel, and therefore $F$ intersects $\partial \hat{M}_H$ or $\sigma_j$. Thus $\hat{M}_H$ satisfies (d). We added one $r_j$-face and removed two, so $\hat{M}_H$ satisfies (e) with $\sum_{j=1}^{m} |p_j| = 1$. Clearly $\hat{M}_H$ has one fewer face than $M_H$ which fails satisfy condition (f). Therefore, repeating $p_j$ times for each $j$, where $p_j$
is the number of faces of $M_H$ with label $r_j$ failing to satisfy (f), we obtain a van Kampen diagram $M_H'$ over $\langle S \mid R_H \rangle$ satisfying (a)-(f).

It remains to prove (g). Suppose that $F$ is a face of $M_H'$ such that $F$ shares at least one edge with $\alpha'$, but $\partial F$ does not intersect $\alpha'$ in a single common subpath. Then there exist subpaths of $\alpha$ and $\partial F$ that together enclose a disk subdiagram $D$ of $M_H'$. Since $M_H'$ is reduced, by the Greendlinger Lemma there exists a face $F'$ of $D$ such that $\partial F'$ shares a common subpath of length at least $\frac{1}{2}\ell(\partial F')$ with $\partial D$. Thus $\partial F$ intersects either $\partial F$ or $\alpha$ in a common subpath of length at least $\frac{1}{4}\ell(\partial F')$. Since we chose $\lambda < \frac{1}{12}$, this means either that $F$ cancels with $F'$, contradicting (d), or $\partial F'$ shares a common subpath with $\alpha'$ of length at least $2\lambda\ell(\partial F')$, contradicting (f). Therefore $M_H'$ satisfies (g).

**Definition 5.16.** For a van Kampen diagram $M$, the perimeter sum of $M$, denoted $\text{PS}(M)$, is defined by

$$\text{PS}(M) = \sum \{\ell(\partial F) \mid F \text{ is a face of } M\}.$$

The following lemma allows us to estimate $\ell(\alpha')$ in terms of $\ell(\beta)$ and $\ell(\alpha)$ in terms of $\ell(\alpha')$: it is the final puzzle piece in the proof.

**Lemma 5.17.** [21] Let $M$ be a bare and reduced van Kampen diagram over a $C'(\lambda)$ presentation, where $\lambda \leq \frac{1}{6}$. Then $(1 - 6\lambda) \text{PS}(M) \leq \ell(\partial M)$.

**Lemma 5.18.** $\ell(\alpha') < 3\ell(\beta)$.

**Proof.** Note that any common subpath of $\alpha'$ and the boundary path of a face $F$ of $M_H'$ is really a common subpath of $\alpha' \prec \beta$ and $\partial F$. This is because $\partial M_H' = \alpha' * \beta$, so any common edge of $\alpha'$ and $\beta$ appears twice on the unbounded face, and thus does not border any face of $M_H'$. Condition (g) then implies that

$$\text{PS}(M_H') > \frac{1}{2\lambda}\ell(\alpha' \prec \beta).$$

On the other hand, $(1 - 6\lambda) \text{PS}(M_H') \leq \ell(\partial M_H') = \ell(\alpha') + \ell(\beta)$ by Lemma 5.17. Thus we have

$$\frac{1}{1-6\lambda}(\ell(\alpha') + \ell(\beta)) < \frac{1}{2\lambda}\ell(\alpha' \prec \beta) \leq \frac{1}{2\lambda}(\ell(\alpha') - \ell(\beta))$$

$$2\lambda(\ell(\alpha') + \ell(\beta)) < (1 - 6\lambda)\ell(\alpha') - \ell(\beta)$$

$$\ell(\alpha') < \frac{1 - 4\lambda}{1 - 8\lambda}\ell(\beta) < \frac{1}{1 - 8\lambda}\ell(\beta) < 3\ell(\beta)$$

since $0 < \lambda < \frac{1}{12}$.

We are now ready to prove Lemma 5.1.

**Proof of Lemma 5.1.** Since $\alpha$ is a path with label $u$ and $\beta$ is a path with label $v^{-1}$, it suffices to show that $\ell(\alpha) < \frac{4}{1 - 12\lambda}\ell(\beta)$.

Since $|p_j|$ faces with boundary label $r_j$ are removed from $M_H$ to form $M_H'$, it follows that $\kappa_j \geq |p_j|$ for each $j$. Now, $\sigma_j$ is a multiple of $k$ for each $j$, and $|p_j| \leq \lceil k/2 \rceil$. Therefore $\kappa_j \geq 2|p_j|$ for each $j$. Let $\kappa = \sum_{j=1}^m \kappa_j$. Then $\kappa \geq \sum_{j=1}^m 2|p_j| = \frac{2}{|p_j|}(\ell(\alpha) - \ell(\alpha'))$. On the other hand, $\text{PS}(M_H) \geq \kappa|r_j|$. Therefore by Lemma 5.17 we have

$$2(\ell(\alpha) - \ell(\alpha')) \leq \kappa|r_j| \leq \text{PS}(M_H) \leq \frac{1}{1 - 6\lambda}\ell(\alpha) + \ell(\beta).$$

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Since $\ell(\alpha') < 3\ell(\beta)$ and $\lambda < \frac{1}{12}$,
\[
2(1 - 6\lambda)(\ell(\alpha) - \ell(\alpha')) \leq \ell(\alpha) + \ell(\beta) \\
(1 - 12\lambda)\ell(\alpha) \leq (2 - 12\lambda)\ell(\alpha') + \ell(\beta) \\
< \ell(\alpha') + \ell(\beta) < 4\ell(\beta).
\]
Hence $\ell(\alpha) < \frac{4}{1 - 12\lambda}\ell(\beta)$, as desired. \qed

6 Finitely generated groups of differing asymptotic and Assouad-Nagata dimension

This section is devoted to proving the following theorem.

**Theorem 6.1.** For any $m \in \mathbb{N}$, there exists a finitely generated group $G$ such that
\[
1 \leq \text{asdim}(G) \leq 2 \\
m + 1 \leq \text{asdim}_{AN}(G) \leq m + 2.
\]

In [7], Higes constructs, for any $\ell, m \in \mathbb{N}$, a countable abelian group $A$ and a proper norm $\| \cdot \|_A$ such that, with respect to the metric induced by this norm, $\text{asdim}(\mathbb{Z}^\ell \times A) = \ell$ and $\text{asdim}_{AN}(\mathbb{Z}^\ell \times A) = \ell + m$. We prove Theorem 6.1 by constructing a short exact sequence
\[
1 \to K \to G \to H \to 1
\]
such that $G$ is a finitely generated group and $H$ is a $C'(1/6)$ group, and such that $K$ and $\langle K, a \rangle$, with respect to the restriction of the word norm on $G$, are quasi-isometrically isomorphic to $A$ and $A \times \mathbb{Z}$, respectively. The result then follows from the extension theorems for asymptotic and Assouad-Nagata dimension. By taking free products with appropriately-chosen free abelian groups, we obtain the following as a corollary.

**Theorem 6.2.** For all $n, k \in \mathbb{N}$ with $n \geq 3$, there exists a finitely generated group $G$ such that
\[
\text{asdim}(G) = n \\
\text{asdim}_{AN}(G) = n + k.
\]

6.1 Review of a construction of Higes

Let $m$ be a fixed positive integer. For the remainder of this paper, $i$ will be an index in $\mathbb{Z}^+$ and $j$ will be an index in $\{1, \ldots, m\}$. Unless otherwise stated, subscripts of $i$ and $j$ are quantified over all $i \in \mathbb{Z}^+$ and $j \in \{1, \ldots, m\}$. Thus $(a_{ij})$ is a matrix or sequence doubly indexed by $\mathbb{Z}^+ \times \{1, \ldots, m\}$, $(r_{ij})$ is a set indexed by $\mathbb{Z}^+ \times \{1, \ldots, m\}$ and not a singleton, $(s_i)$ is a sequence indexed by $\mathbb{Z}^+$, and so on.

Let $(k_i)$ be an increasing sequence of positive integers. As a group,
\[
A = \bigoplus_{i=1}^{\infty} \mathbb{Z}_{k_i}^m = \bigoplus_{i=1}^{\infty} \bigoplus_{j=1}^{m} \mathbb{Z}_{k_i}
\]
and so a typical element of $A$ is $(a_{ij})$, where each entry $a_{ij} \in \mathbb{Z}_{k_i}$. Let $A_i = \mathbb{Z}^{m}_{k_i}$. Assuming $A_i$ is given the obvious generating set, the word norm on $A_i$ is given by

$$
\|(a_{i1}, \ldots, a_{im})\|_{A_i} = \sum_{j=1}^{m} \min(a_{ij}, k_i - a_{ij}).
$$

There is a natural family of projections $\pi_i : A \to A_i$ defined by $\pi_i((a_{ij})) = (a_{i1}, \ldots, a_{im})$. Define $h : A \setminus \{0\} \to \mathbb{Z}^+$ by

$$
h((a_{ij})) = \max\{i \in \mathbb{Z}^+ | \pi_i((a_{ij})) \neq 0\}.
$$

Now inductively define a sequence of positive real scaling constants $(s_i)$ satisfying

$$
s_1 \geq 1,
$$

$$
s_{i+1} \geq 1 + s_i \text{diam}(A_i) = 1 + s_i m[k_i/2].
$$

Under these assumptions, Higes proved the following. Recall that for two normed spaces $(X, \| \cdot \|_X)$ and $(Y, \| \cdot \|_Y)$, the $\ell^1$ product norm is the norm $\| \cdot \|$ on $X \times Y$ defined by $\|(x, y)\| = \|x\|_X + \|y\|_Y$.

**Theorem 6.3 ([7], Theorem 4.9).** The function $\| \cdot \|_{A}$ defines a proper norm on $A$. Let $\ell \in \mathbb{N}$, and suppose that $\mathbb{Z}^\ell$ is given the word norm with respect to the usual generating set, $A$ is given the norm $\| \cdot \|_{A}$, and $\mathbb{Z}^\ell \times A$ is given the $\ell^1$ product norm. Then

$$
\text{asdim}(\mathbb{Z}^\ell \times A) = \ell,
$$

$$
\text{asdim}_{AN}(\mathbb{Z}^\ell \times A) = \ell + m.
$$

### 6.2 Construction of the group

Let $\lambda$ be a positive real number with $\lambda < 1/12$, and let $S = \{a, b\}$. Start with a sequence of natural numbers $(k_i)$ and a language $\{r_{ij}\} \subset S^*$ satisfying all of the following conditions:

(a) $(k_i)$ is increasing and $k_1 \geq 2$.

(b) $a^{\lambda|r_{ij}|}$ is not a prefix of $r_{ij}^{\pm 1}$ for any $i, j$.

(c) $|r_{ij}| = |r_{ij'}|$ for all $j, j'$.

(d) $|r_{(i+1)j}| > \frac{8}{1 - 12\lambda} mk_i |r_{ij}|$ for all $i, j$.

(e) $\{r_{ij}\}$ is cyclically minimal and cyclically reduced, and satisfies $C'(\lambda)$.

Such a pair $(k_i), \{r_{ij}\}$ exists, as the following example shows.
Example 6.4. Suppose that \( \lambda = 1/24 \), so chosen to make the arithmetic easy. Let \((k_i)\) be the sequence \(((25m)^{i+m})\), and suppose that \( |r_{i(i+1)}| = 25mk_i |r_{ij}| = (25m)^{(i+1)+m} |r_{ij}| \) to satisfy (b).
Thus if \( |r_{ij}| = 2(25m)^{1+m} \), we require that \( |r_{ij}| = 2(25m)^{i} \sum_{k=1}^{m} k^{i+m} = 2(25m)^{(i+1)/2+i+m} \).
For convenience let \( m_i = (i(i+1)/2) + im \). Now let
\[
\begin{align*}
    r_{i1} &= \langle a \rangle^{(25m)^{m_i-1}} \langle b \rangle^{(25m)^{m_i-1}} 25m \\
    r_{i2} &= \langle a \rangle^{(25m)^{m_i-2}} \langle b \rangle^{(25m)^{m_i-2}} 25m^2 \\
    &\vdots \\
    r_{im} &= \langle a \rangle^{(25m)^{m_i-m}} \langle b \rangle^{(25m)^{m_i-m}} 25m^m.
\end{align*}
\]
Now \( \{r_{ij}\} \) satisfies (b) since the maximum subword of any \( r_{ij}^{+1} \) which is a power of \( a \) is of length at most \( \frac{1}{25} |r_{ij}| \). Also \( \{r_{ij}\} \) satisfies (c) since \( |r_{ij}| = 2(25m)^m \) depends only on \( i \), and satisfies (d) by our choice of \( m_i \). That \( \{r_{ij}\} \) is cyclically minimal and cyclically reduced is obvious. To see that \( \{r_{ij}\} \) satisfies \( C'(1/24) \), note that a maximal piece of a relation \( r_{ij} \) has the form \( \langle a \rangle^{(25m)^{m_i-1}} \langle b \rangle^{(25m)^{m_i-j}} \). But \( |a^{(25m)^{m_i-1}} b^{(25m)^{m_i-j}}| \leq |a^{(25m)^{m_i-1}} b^{(25m)^{m_i-j}}| = 2(25m)^{m_i-1} \leq \frac{1}{25} (2(25m)^m) \leq \frac{1}{25} |r_{ij}| < \frac{1}{25} |r_{ij}|. \)
Let \( H \) be the group given by a presentation \( \langle a, b \mid \{r_{ij}\} \rangle \) where \( \{r_{ij}\} \) satisfies (a)-(e) above, abbreviated hereafter by
\[
H = \langle a, b \mid r_{ij} \rangle.
\]
Now define a sequence of positive natural numbers \( (s_i) \) by declaring that
\[
s_i = |r_{ij}|
\]
for all \( i \in \mathbb{Z}^+ \), and construct \( A \) and \( \| \cdot \|_A \) with respect to the sequences \((k_i)\) and \((s_i)\) as in the previous section.
Condition (c) implies that the sequence \((s_i)\) is well defined. Condition (b) implies that \( \{r_{ij}\} \) is infinite and rules out any possible trivial \( C'(\lambda) \) presentations such as \( \langle a, b \mid a^2, b \rangle \). Conditions (c) and (e) imply that \( s_1 > 12 \), since any two distinct words have a piece consisting of at least one letter. And since \( m \geq 1, s_i \geq 1 \) and \( k_i \geq 2 \) for all \( i \) by (a), condition (d) guarantees that inequalities (7) are satisfied. Thus we are under the assumptions of Higgins' theorem, so \( \text{asdim}_A(A, \| \cdot \|_A) = 0 \) and \( \text{asdim}_{AN}(A, \| \cdot \|_A) = m \).
Now let \( G \) be the group with presentation \( \langle a, b \mid \bigcup \{ [a, r_{ij}], [b, r_{ij}], r_{ij}^{k_i} \} \mid i \in \mathbb{Z}^+, j \in \{1, \ldots, m\} \} \), which we abbreviate hereafter by
\[
G = \langle a, b \mid [a, r_{ij}], [b, r_{ij}], r_{ij}^{k_i} \rangle.
\]
Let \( \pi : G \to H \) be the natural epimorphism, and let \( K = \text{Ker}(\pi) \). In order to avoid confusing words in \( S_0^* \) with group elements, let \( g_{ij} \) be the element of \( G \) represented by the word \( r_{ij} \). Then \( K \) is the subgroup of \( G \) generated by \( \{g_{ij} \mid i \in \mathbb{Z}^+, j \in \{1, \ldots, n\} \} \), and each cyclic subgroup \( \langle g_{ij} \rangle \) has order dividing \( k_i \) by definition of \( G \). Since \( K \) is central in \( G \) by definition, we have an epimorphism \( \phi : \mathbb{Z} \times A \to \langle a, K \rangle \) defined by
\[
\phi(n_i, a_{ij}) = a^n \prod_{i=1}^{h} \prod_{j=1}^{n_i} g_{ij}^{a_{ij}}
\]
where \( h = h((a_{ij})) \).
Lemma 6.5. Let \( \| \cdot \| \) be the \( \ell^1 \) product norm on \( \mathbb{Z} \times A \), where \( \mathbb{Z} \) is given the usual norm \( | \cdot | \), and \( A \) is given the norm \( \| \cdot \|_A \). Then \( \phi : (\mathbb{Z} \times A, \| \cdot \|) \to ((K, a), \| \cdot \|_G) \) is bi-Lipschitz, hence \( \phi \) is both a quasi-isometry and an isomorphism.

Proof. Note that \( \|g_{ij}\|_G \leq |r_{ij}| = s_i \). A straightforward induction argument using condition (b) shows that \( s_h \geq \frac{1}{12} \sum_{i=1}^{h-1} s_im[k_i/2] \) whenever \( h \geq 2 \). Therefore if \( h = h((a_{ij})) \geq 2 \) we have

\[
\|\phi(n, (a_{ij}))\|_G \leq |n| + \sum_{i=1}^{h} \sum_{j=1}^{m} |g_{ij}^{a_{ij}}|_G \leq |n| + \sum_{i=1}^{h} \sum_{j=1}^{m} |r_{ij}| \min(a_{ij}, k_i - a_{ij})
\]

\[
= |n| + \sum_{i=1}^{h} s_i \|\pi_i((a_{ij}))\|_{A_i} = |n| + \|\phi(a_{ij})\|_A + \sum_{i=1}^{h-1} s_i \|\pi_i((a_{ij}))\|_{A_i}
\]

\[
\leq |n| + \|\phi(a_{ij})\|_A + \sum_{i=1}^{h-1} s_i m[k_i/2] \leq |n| + \|\phi(a_{ij})\|_A + \frac{1-12\lambda}{8} s_h \leq 2\|\phi(n, (a_{ij}))\|
\]

and of course the same result holds if \( h((a_{ij})) = 1 \) or \( (a_{ij}) = 0 \), so \( \phi = 2 \)-Lipschitz.

For the lower bound, we claim that \( \|\phi(n, (a_{ij}))\|_G \geq \frac{1-12\lambda}{8} \|\phi(n, (a_{ij}))\|. \) Suppose without loss of generality that \( (a_{ij}) \) is such that \( |a_{ij}| \leq \lfloor k_i/2 \rfloor \) for all \( i, j \). Suppose furthermore that \( \pi_i((a_{ij})) \neq 0 \) for exactly one \( i \). For this \( i \), we have \( u := a^n r_{i,1} \ldots r_{im} = G \phi(n, (a_{ij})) \). By Lemma 5.1, \( u \) is \( \bigg( \frac{1-12\lambda}{8}, 0 \bigg) \)-quasi-geodesic. Thus \( \|\phi((a_{ij}))\|_G \geq \frac{1-12\lambda}{4} |u| = \frac{1-12\lambda}{4} \bigg( n + \sum_{j=1}^{m} |a_{ij}| \|r_{ij}\| \bigg) \). In other words, for each fixed \( i \) we have

\[
\left\| a^n \prod_{j=1}^{m} g_{ij}^{a_{ij}} \right\|_G \geq \left( \frac{1-12\lambda}{4} \right) \left( |n| + s_i \|\pi_i((a_{ij}))\|_{A_i} \right).
\]

Thus for an arbitrary element \( (n, (a_{ij})) \) of \( \mathbb{Z} \times A \) we have

\[
\|\phi((a_{ij}))\|_G = \left\| a^n \prod_{i=1}^{h} \prod_{j=1}^{m} g_{ij}^{a_{ij}} \right\|_G \geq \left\| a^n \prod_{j=1}^{m} g_{ij}^{a_{ij}} \right\|_G \geq \left( \frac{1-12\lambda}{4} \right) \left( |n| + \|\phi(a_{ij})\|_A \right) \geq \left( \frac{1-12\lambda}{8} \right) s_h
\]

\[
\geq \left( \frac{1-12\lambda}{8} \right) \|\phi(n, (a_{ij}))\| G \leq 2\|\phi(n, (a_{ij}))\|
\]

Therefore for all \( i, j \),

\[
\frac{1-12\lambda}{8} \|\phi(n, (a_{ij}))\| \leq \|\phi(n, (a_{ij}))\|_G \leq 2\|\phi(n, (a_{ij}))\|
\]

and \( \phi \) is bi-Lipschitz, as desired. \( \square \)
The extension theorems for asymptotic and Assouad-Nagata dimension are the final ingredients in the proof of Theorem 6.1.

**Theorem 6.6.** [8][9] Let \( 1 \to K \to G \to H \to 1 \) be a short exact sequence, where \( G \) and \( H \) are finitely generated, and \( K \) is equipped with the restriction of the word norm on \( G \). Then

\[
\operatorname{asdim}(G) \leq \operatorname{asdim}(H) + \operatorname{asdim}(K)
\]

\[
\operatorname{asdim}_{AN}(G) \leq \operatorname{asdim}_{AN}(H) + \operatorname{asdim}_{AN}(K).
\]

**Proof of Theorem 6.1.** By Lemma 6.5, we have a bi-Lipschitz map \( \phi : (\mathbb{Z} \times A, \| \cdot \|) \to (\langle a, K \rangle, \| \cdot \|_G) \) which is a group epimorphism. Restricting to \( A \), we also have \( \phi|_A : (A, \| \cdot \|_A) \to (K, \| \cdot \|_G) \) is bi-Lipschitz. Hence \( \phi \) and \( \phi|_A \) are quasi-isometric isomorphisms. Thus by Higes’ Theorem, with respect to the restriction of the word norm on \( G \), we have

\[
\operatorname{asdim}(K) = 0 \quad \quad \operatorname{asdim}(\langle a, K \rangle) = 1
\]

\[
\operatorname{asdim}_{AN}(K) = m \quad \quad \operatorname{asdim}_{AN}(\langle a, K \rangle) = m + 1.
\]

In particular \( \operatorname{asdim}(G) \geq 1 \) and \( \operatorname{asdim}_{AN}(G) \geq m + 1 \). By definition of \( G, H, \) and \( K \), we have a short exact sequence

\[
1 \to K \to G \to H \to 1
\]

and \( H \) is a \( C'(1/\phi) \) group, hence \( \operatorname{asdim}(H) \leq \operatorname{asdim}_{AN}(H) \leq 2 \) by Theorem 4.12. Therefore by the extension theorems for asymptotic and Assouad-Nagata dimension,

\[
1 \leq \operatorname{asdim}(G) \leq 2
\]

\[
m + 1 \leq \operatorname{asdim}_{AN}(G) \leq m + 2.
\]

With the free product theorems for asymptotic and Assouad-Nagata dimension, it is possible increase the lower bound on \( \operatorname{asdim}(G) \).

**Theorem 6.7.** [25][26] Let \( A, B \) be finitely generated, infinite groups. Then

\[
\operatorname{asdim}(A \ast B) = \max\{\operatorname{asdim}(A), \operatorname{asdim}(B)\}
\]

\[
\operatorname{asdim}_{AN}(A \ast B) = \max\{\operatorname{asdim}_{AN}(A), \operatorname{asdim}_{AN}(B)\}.
\]

Thus if \( G_0 \) is a finitely generated group with \( \operatorname{asdim}(G_0) \leq 2 \) and \( m + 1 \leq \operatorname{asdim}_{AN}(G_0) \leq m + 2 \), and \( 2 \leq n \leq m + 1 \), then \( \mathbb{Z}^n \ast G_0 \) satisfies \( \operatorname{asdim}(G) = n \) and \( m + 1 \leq \operatorname{asdim}_{AN}(G) \leq m + 2 \). Therefore there are finitely generated groups \( G \) with arbitrarily large asymptotic dimension and arbitrarily larger Assouad-Nagata dimension. It is worth mentioning that, if one does not care about controlling the asymptotic and Assouad-Nagata dimension of \( G \) precisely, it is possible to give an explicit presentation of \( G \).

**Example 6.8.** Let \( m, n \) be natural numbers with \( 2 \leq n \leq m + 1 \). Let \( G_0 \) be the group constructed as in the proof of Theorem 6.1 with \( \{r_{ij}\} \) as in Example 6.4. Then if \( G = \mathbb{Z}^n \ast G_0 \), we have

\[
G = \langle a, b, c_1, \ldots, c_n \mid \{c_k, c_\ell\} \text{ for all } k, \ell \in \{1, \ldots, n\}; \]

\[
[a, (a^{(25m)^{m_i-j}} b^{(25m)^{m_i-j}})^{(25m)^j}], [b, (a^{(25m)^{m_i-j}} b^{(25m)^{m_i-j}})^{(25m)^j}],
\]

\[
((a^{(25m)^{m_i-j}} b^{(25m)^{m_i-j}})^{(25m)^{j+m}} \text{ for all } i \in \mathbb{Z}^+, j \in \{1, \ldots, m\})
\]

satisfies \( \operatorname{asdim}(G) = n \) and \( m + 1 \leq \operatorname{asdim}_{AN}(G) \leq m + 2 \).
To control the asymptotic dimension and Assouad-Nagata dimension more precisely, we use the Morita-type theorem for Assouad-Nagata dimension.

**Theorem 6.9.**\(^5\) Let \(G\) be a finitely generated group. Then \(\text{asdim}_{AN}(G \times \mathbb{Z}) = \text{asdim}_{AN}(G) + 1\).

Suppose that for natural numbers \(n, k\), we want a finitely generated group of asymptotic dimension \(n\) and Assouad-Nagata dimension \(n + k\). Assume that \(n + k \geq 2\). Applying Theorem 6.1 with \(m = n + k - 2\), there exists a group \(G_0\) with \(\text{asdim}(G_0) \leq 2\) and \(\text{asdim}_{AN}(G_0) = n + k\) or \(n + k - 1\).

If \(\text{asdim}_{AN}(G_0) = n + k - 1\), then by Theorem 6.9 we have that \(\text{asdim}_{AN}(G_0 \times \mathbb{Z}) = n + k\). Hence there exists a group \(G_1\), either \(G_0\) or \(G_0 \times \mathbb{Z}\), such that \(\text{asdim}(G_1) \leq 3\) and \(\text{asdim}_{AN}(G_1) = n + k\).

Let \(G = \mathbb{Z}^n \ast G_1\). If we suppose that \(n \geq 3\), then

\[
\begin{align*}
\text{asdim}(G) &= \max\{\text{asdim}(\mathbb{Z}^n), \text{asdim}(G_1)\} = n \\
\text{asdim}_{AN}(G) &= \max\{\text{asdim}_{AN}(\mathbb{Z}^n), \text{asdim}_{AN}(G_1)\} = n + k,
\end{align*}
\]

establishing Theorem 6.2.

### 7 Open Questions

Theorem 6.2 requires that \(n \geq 3\). When \(n \leq 2\) the situation is somewhat more mysterious. If \(n = 2\) then, following the construction in the previous paragraph, we may take \(G = G_0 \ast \mathbb{Z}^2\) and obtain \(\text{asdim}(G) = 2\) and \(\text{asdim}_{AN}(G) = k + 1\) or \(k + 2\). We can put \(\text{asdim}_{AN}(G)\) within 1 of where we want, but can we hit the bullseye?

**Question 7.1.** Given any \(k \in \mathbb{N}\), does there exist a finitely generated group \(G\) with \(\text{asdim}(G) = 2\) and \(\text{asdim}_{AN}(G) = k + 2\)?

By results of Gentimis \(^{10}\) and Fujiwara and Whyte \(^{11}\), we know that a finitely presented group has asymptotic dimension 1 if and only if it is virtually free. Thus given \(k \in \mathbb{Z}^+\) it is not possible to construct a finitely presented group \(G\) such that \(\text{asdim}(G) = 1\) and \(\text{asdim}_{AN}(G) = 1 + k\). However, the question remains for infinitely presented groups.

**Question 7.2.** For any \(k \in \mathbb{N}\), is there a finitely generated group \(G\) such that \(\text{asdim}(G) = 1\) and \(\text{asdim}_{AN}(G) = 1 + k\)?

Likewise, we know that if \(G\) is a finitely presented \(C'(1/6)\) group, then \(\text{asdim}(G) = \text{asdim}_{AN}(G) = 1\) if \(G\) is virtually free and \(\text{asdim}(G) = \text{asdim}_{AN}(G) = 2\) otherwise. However, for infinitely presented \(C'(1/6)\) groups, the question remains.

**Question 7.3.** Suppose \(G\) is a finitely generated, infinitely presented \(C'(1/6)\) group. When is \(\text{asdim}(G)\) or \(\text{asdim}_{AN}(G)\) equal to 1, and when is it equal to 2?

Another natural question is the following.

**Question 7.4.** In Theorem 6.1 can ‘finitely generated’ be replaced with ‘finitely presented’?
References

[1] M. Gromov, *Asymptotic invariants of infinite groups*, Geometric group theory, Vol. 2 (Sussex, 1991), 1993, pp. 1–295.

[2] G. Yu, *The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space*, Invent. Math. 139 (2000), no. 1, 201–240, DOI 10.1007/s002220000032.

[3] U. Lang and T. Schlichenmaier, *Nagata dimension, quasisymmetric embeddings, and Lipschitz extensions*, Int. Math. Res. Not. 58 (2005), 3625–3655, DOI 10.1155/IMRN.2005.3625.

[4] J. Dydak and J. Higes, *Asymptotic cones and Assouad-Nagata dimension*, Proc. Amer. Math. Soc. 136 (2008), no. 6, 2225–2233, DOI 10.1090/S0002-9939-08-09149-1.

[5] A. N. Dranishnikov and J. Smith, *On asymptotic Assouad-Nagata dimension*, Topology Appl. 154 (2007), no. 4, 934–952, DOI 10.1016/j.topol.2006.10.010.

[6] N. Brodskiy, J. Dydak, and U. Lang, *Assouad-Nagata dimension of wreath products of groups*, Canad. Math. Bull. 57 (2014), no. 2, 245–253, DOI 10.4153/CMB-2014-024-8.

[7] J. Higes, *Assouad-Nagata dimension of locally finite groups and asymptotic cones*, Topology Appl. 157 (2010), no. 17, 2635–2645, DOI 10.1016/j.topol.2010.07.015.

[8] N. Brodskiy, J. Dydak, M. Levin, and A. Mitra, *A Hurewicz theorem for the Assouad-Nagata dimension*, J. Lond. Math. Soc. (2) 77 (2008), no. 3, 741–756, DOI 10.1112jlms/djn005.

[9] G. C. Bell and A. N. Dranishnikov, *A Hurewicz-type theorem for asymptotic dimension and applications to geometric group theory*, Trans. Amer. Math. Soc. 358 (2006), no. 11, 4749–4764, DOI 10.1090/S0002-9947-06-04088-8.

[10] T. Gentimis, *Asymptotic dimension of finitely presented groups*, Proc. Amer. Math. Soc. 136 (2008), no. 12, 4103–4110, DOI 10.1090/S0002-9939-08-08973-9.

[11] K. Fujiwara and K. Whyte, *A note on spaces of asymptotic dimension one*, Algebr. Geom. Topol. 7 (2007), 1063–1070, DOI 10.2140/agt.2007.7.1063.

[12] I. Agol, *Answer to “Asymptotic dimension of $C'(1/6)$ small cancellation groups”* (2015), https://mathoverflow.net/questions/195489/asymptotic-dimension-of-c1-6-small-cancellation-groups.

[13] S. V. Buyalo and N. D. Lebedeva, *Dimensions of locally and asymptotically self-similar spaces*, Algebra i Analiz 19 (2007), no. 1, 60–92, DOI 10.1090/S1061-0022-07-00985-5.

[14] D. Osajda, *Small cancellation labellings of some infinite graphs and applications*, arXiv preprint 1406.5015 (2014).

[15] B. H. Bowditch, *Tight geodesics in the curve complex*, Invent. Math. 171 (2008), no. 2, 281–300, DOI 10.1007/s00222-007-0081-y.

[16] G. C. Bell and K. Fujiwara, *The asymptotic dimension of a curve graph is finite*, J. Lond. Math. Soc. (2) 77 (2008), no. 1, 33–50, DOI 10.1112jlms/djm090.

[17] J. Roe, *Hyperbolic groups have finite asymptotic dimension*, Proc. Amer. Math. Soc. 133 (2005), no. 9, 2489–2490, DOI 10.1090/S0002-9939-05-08138-4.

[18] D. Osin, *Asymptotic dimension of relatively hyperbolic groups*, Int. Math. Res. Not. 35 (2005), 2143–2161, DOI 10.1155/IMRN.2005.2143.

[19] É. Ghys and P. de la Harpe, *Sur les groupes hyperboliques d’après Mikhael Gromov*, Progress in Mathematics, vol. 83, Birkhäuser Boston, Inc., Boston, MA, 1990.

[20] G. C. Bell and A. N. Dranishnikov, *Asymptotic dimension*, Topology Appl. 155 (2008), no. 12, 1265–1296, DOI 10.1016/j.topol.2008.02.011.

[21] R. C. Lyndon and P. E. Schupp, *Combinatorial group theory*, Classics in Mathematics, Springer-Verlag, Berlin, 2001. Reprint of the 1977 edition.

[22] G. C. Bell and A. N. Dranishnikov, *Asymptotic dimension in Bedlewo*, Topology Proc. 38 (2011), 209–236.

[23] A. Yu. Ol’shanskii, D. V. Osin, and M. V. Sapir, *Lacunary hyperbolic groups*, Geometry & Topology 13 (2009), no. 4, 2051–2140.

[24] A. Yu. Olshanskii and P. E. Schupp, *Geometry of Defining Relations in Groups*, Classics in Mathematics, Springer-Verlag, Berlin, 2001. Reprint of the 1977 edition.

[25] A. N. Dranishnikov, *On asymptotic dimension of amalgamated products and right-angled Coxeter groups*, Algebr. Geom. Topol. 8 (2008), no. 3, 1281–1293, DOI 10.2140/agt.2008.8.1281.

[26] N. Brodskiy and J. Higes, *Assouad-Nagata dimension of tree-graded spaces*, arXiv preprint 0910.2378 (2009).