TOPOLOGICAL ENTROPY FOR SET-VALUED MAPS

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Abstract. In this paper we define and study the topological entropy of a set-valued dynamical system. Actually, we obtain two entropies based on separated and spanning sets. Some properties of these entropies resembling the single-valued case will be obtained.

1. Introduction. The concept of entropy was introduced in thermodynamics by Clausius ([13], [14]), in statistical and quantum mechanics by Boltzmann and von Neumann ([7], [34]) and in information theory by Shannon, Khinchin and McMillan ([31], [20], [23]). In 1958, Kolmogorov [21] defined metric entropy for K-systems and Sinai [29] extended his definition to all measure-preserving transformations. Adler, Konheim and McAndrews [1] defined topological entropy on compact topological spaces whereas Bowen [15] and Dinaburg [15] obtained an equivalent definition in the metrizable case. Katok [18] deduced the metric entropy of ergodic transformations from spanning sets. Basic references including extended historical notes are [16], [18], [30].

On the other hand, the study of the set-valued dynamical systems has been increasing along these years following the success of the single-value case. For instance, [22] investigated the existence of endpoints for set-valued dynamical systems through the notion of stable sets. The results are formulated by means of the inverse of a Pareto-minimal point of a vectorial Lyapunov function. In [5] it is introduced the notion of invariant measure for set-valued dynamical systems, generalizing the same concept in the single-valued case. Further notions of invariant or coincidence

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measures are given in [24], [25] and [32]. In [36] it is studied the asymptotic behavior of the trajectories of a set-valued dynamical system. In particular, necessary and sufficient conditions for the existence of global attractors for dispersive systems as well as motivations coming from economic models are given. In [26] and [27] there were analyzed the notion of pseudo-orbit and inverse tracing properties for set-valued dynamical systems. Continuous time set-valued dynamical systems and its relationship with economic models were defined and studied in [11]. See also the monographs [2], [4], [3], [10] for further information about set-valued analysis.

In this paper we will introduce and study the notion of topological entropy for set-valued maps on metric spaces. Indeed, we obtain two entropies given by separated and spanning sets respectively. We prove that they keep a number of well-known properties of the classical topological entropy of single-valued maps [1], [8], [15].

Our motivation comes not only from the aforementioned works but also from the hope that the notion of entropy would play an important role in the analysis of chaotic set-valued systems (as in the single-valued case).

This paper is organized as follows. In Section 2 we define the separated and spanning entropies for a discrete set-valued dynamical system. In Section 3 we prove the main results dealing with properties of these entropies. In Section 4 we present some related examples. In Section 5 we present some conclusions.

2. Definition. Let \( X \) be a metric space. Given \( A, B \subset X \) we define
\[
d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}.
\]

Denote by \( 2^X \) the set formed by the subsets of \( X \). By a \textit{set-valued map} on \( X \) we mean a map \( f : X \to 2^X \). We say that \( f \) is \textit{single-valued} if \( \text{card}(f(x)) = 1 \) for all \( x \in X \), where \( \text{card} \) denotes cardinality. There is an obvious correspondence between single-valued maps \( f : X \to 2^X \) and maps \( f : X \to X \). Hereafter we shall assume that every set-valued map \( f \) is \textit{strict}, i.e., \( f(x) \neq \emptyset \) for every \( x \in X \).

Let \( f \) be a set-valued map of \( X \). For all \( n \in \mathbb{N}^+ \) we define the map \( d_n : X \times X \to \mathbb{R}^+ \) by
\[
d_n(x, y) = \inf \left\{ \max_{0 \leq i \leq n-1} d(x_i, y_i) : (x_i)_{i=0}^n, (y_i)_{i=0}^n \text{ are sequence satisfying (1)} \right\}
\]

\[
x_0 = x, y_0 = y, x_{i+1} \in f(x_i) \text{ and } y_{i+1} \in f(y_i) \text{ for all } i \text{ with } 0 \leq i \leq n-1 \right\}.
\]

The notation \( d_n^f \) indicates dependence on \( f \). These maps are metrics in the single-valued case. In general they are only semimetrics [6], [17] (see Example 4.5).

Define the \( \epsilon \)-balls centered at \( x \in X \) with respect to \( d_n \),
\[
B_n[x, \epsilon] = \{ y \in X : d_n(x, y) \leq \epsilon \}.
\]

Again the notation \( B_n^f[x, \epsilon] \) indicates dependence on \( f \).

Given \( \epsilon > 0 \) and \( F \subset X \) we say that \( F \) is \((n, \epsilon)\)-separated (for \( f \)) if
\[
B_n[x, \epsilon] \cap F = \{x\}, \quad \forall x \in F.
\]

Define \( s(n, \epsilon) = \sup\{\text{card}(F) : F \text{ is } (n, \epsilon)\text{-separated}\} \). We write \( s(n, \epsilon, f) \) to emphasize \( f \). It is possible that \( s(n, \epsilon) = \infty \). Define
\[
h_{se}(f, \epsilon) = \limsup_{n \to \infty} \frac{\log s(n, \epsilon)}{n}.
\]
Since any \((n, \epsilon)-\)separated set is \((n, \epsilon')-\)separated for all \(0 < \epsilon' \leq \epsilon\), we obtain \(s(n, \epsilon) \leq s(n, \epsilon')\) and so \(h_{se}(f, \epsilon) \leq h_{se}(f, \epsilon')\) for \(0 < \epsilon' \leq \epsilon\). Then, the limit
\[
\lim_{\epsilon \to 0} h_{se}(f, \epsilon) = \sup_{\epsilon > 0} h_{se}(f, \epsilon)
\]
extists and the following definition is given.

**Definition 2.1.** The separated topological entropy of \(f\) is defined by
\[
h_{se}(f) = \lim_{\epsilon \to 0} h_{se}(f, \epsilon).
\]

As in the single-valued case ([8], [15]) one can also consider a topological entropy for set-valued maps via spanning sets.

Given \(n \in \mathbb{N}^+\), \(\epsilon > 0\) and \(E \subset X\) we say that \(E\) is \((n, \epsilon)-\)spanning for \(f\) if
\[
X = \bigcup_{x \in E} B_n[x, \epsilon].
\]
Define \(r(n, \epsilon) = \min\{\text{card}(E) : E\text{ is } (n, \epsilon)-\text{spanning}\}\) and write \(r(n, \epsilon, f)\) to emphasize \(f\). It is possible that \(r(n, \epsilon) = \infty\).

Define
\[
h_{sp}(f, \epsilon) = \limsup_{n \to \infty} \frac{\log r(n, \epsilon)}{n}.
\]

Since any \((n, \epsilon')-\)spanning set is \((n, \epsilon)-\)spanning for \(0 < \epsilon' \leq \epsilon\), we obtain \(r(n, \epsilon) \leq r(n, \epsilon')\) and so \(h_{sp}(f, \epsilon) \leq h_{sp}(f, \epsilon')\) for \(0 < \epsilon' \leq \epsilon\). Then, the limit
\[
\lim_{\epsilon \to 0} h_{sp}(f, \epsilon) = \sup_{\epsilon > 0} h_{sp}(f, \epsilon)
\]
extists and the following definition is given.

**Definition 2.2.** The spanning topological entropy of \(f\) is defined by
\[
h_{sp}(f) = \lim_{\epsilon \to 0} h_{sp}(f, \epsilon).
\]

As the reader can see, these definitions emulate the similar ones for single-valued maps by Bowen [8] and Dinaburg [15]. In particular, they both reduce to the classical topological entropy in the single-valued case. We then write \(h(f) = h_{se}(f) = h_{sp}(f)\) for single-valued maps \(f\).

Notice that, unlike [8], [15], we do not assume any hypothesis of continuity for the involved maps. In the single-valued case, Ciklová [12] obtained a number of properties for the topological entropy without such a hypothesis (including the equality between the separated and spanning entropies, see Proposition 3.4 of p. 624 in [12]).

### 3. Properties

In this section we prove the main results of this paper dealing with properties of the separated and spanning entropies.

Let \(f\) be a set-valued map on a metric space \(X\). Given \(A \subset X\) we define
\[
f(A) = \bigcup_{x \in A} f(x).
\]

We say that \(A\) is invariant if \(f(A) \subset A\). For such sets there is an induced set-valued map \(f|_A\) defined by \(f|_A(x) = f(x)\) for all \(x \in X\). Our first property is about the separated entropy of invariant sets.

**Theorem 3.1.** Let \(f\) be a set-valued map on a metric space \(X\). If \(X = \bigcup_{i=1}^m A_i\) where each \(A_i\) is an invariant set of \(f\), then \(h_{se}(f) = \max_{1 \leq i \leq m} h_{se}(f|_{A_i})\).
Proof. We assert that $h_{sc}(f|_A) \leq h_{sc}(f)$ for any invariant set $A \subset X$. Indeed, for any $F \subset A$, $n \in \mathbb{N}^*$, $\epsilon > 0$ and $x \in F$ one has $B_n^{f,x}[x, \epsilon] \cap F = B_n^{f,x}[x, \epsilon] \cap F$. It follows that every $(n, \epsilon)$-separated set $F$ of $f|_A$ is $(n, \epsilon)$-separated for $f$. Therefore, $s(n, \epsilon, f|_A) \leq s(n, \epsilon, f)$ from which the assertion follows.

By the assertion we get $h_{sc}(f) \geq \max_{0 \leq i \leq m} h_{sc}(f|_{A_i})$. The reverse inequality follows as in Theorem 7.5 p. 172 of [35]. \hfill \Box

Our second property is about the natural inclusion order in the set of set-valued maps defined by $f \leq g$ if and only if $f(x) \subset g(x)$ for all $x \in X$.

**Theorem 3.2.** Both the separated and spanning entropies reverse the inclusion order on the set of set-valued maps.

**Proof.** Let $f, g$ be two set-valued maps of a metric space $X$ with $f \leq g$. Let $F$ be an $(n, \epsilon)$-separated set of $g$. We claim that $F$ is also $(n, \epsilon)$-separated for $f$. Indeed, take distinct $x, y \in F$ and sequences $(x_i)_{i=1}^n$, $(y_i)_{i=0}^n$ satisfying $x_0 = x$, $y_0 = y$, $x_{i+1} \in f(x_i)$ and $y_{i+1} \in f(y_i)$ for every $i$ with $0 \leq i \leq n-1$. Since $f \leq g$, we obtain $x_{i+1} \in g(x_i)$ and $y_{i+1} \in g(y_i)$ for all $i$ with $0 \leq i \leq n-1$. As $F$ is $(n, \epsilon)$-separated for $g$, we conclude that there is an $i$ with $0 \leq i \leq n-1$ such that $d(x_i, y_i) > \epsilon$. Then, $F$ is $(n, \epsilon)$-separated for $f$ as claimed. From this we obtain $s(n, \epsilon, g) \leq s(n, \epsilon, f)$ for every $(n, \epsilon) \in \mathbb{N}^+ \times [0, \infty[$, thus

$$\limsup_{n \to \infty} \frac{\log s(n, \epsilon, g)}{n} \leq \limsup_{n \to \infty} \frac{\log s(n, \epsilon, f)}{n}, \quad \forall \epsilon > 0.$$ 

Taking limits as $\epsilon \to 0$ above we obtain $h_{sc}(g) \leq h_{sc}(f)$. The proof that the spanning entropy reverses the set-valued map order is similar. \hfill \Box

Recall that a selection of a set-valued map $f : X \to 2^X$ is any map $s : X \to X$ satisfying $s(x) \in f(x)$ for all $x \in X$. Selections always exist under the axiom of choice. A direct consequence of Theorem 3.2 is the following.

**Corollary 3.3.** If $f$ is a set-valued map on a metric space, then $h_{sc}(f) \leq h(s)$ for every selection $s$ of $f$.

Every single-valued map on a metric space satisfies $r(n, \epsilon) \leq s(n, \epsilon) \leq r(n, \frac{\epsilon}{2})$. It is natural to expect the same inequalities in the set-valued case. However, we only obtain the first of these inequalities as reported below. The proof of the second in the single-valued case depends on the fact that the maps $d_n$ in (1) are metrics, a fact which is false in general by Example 4.5.

**Lemma 3.4.** If $f$ is a set-valued map on a metric space, then $r(n, \epsilon) \leq s(n, \epsilon)$ for every $n \in \mathbb{N}^+$ and $\epsilon > 0$.

**Proof.** If $s(n, \epsilon) = \infty$ there is nothing to prove. Then, we can assume $s(n, \epsilon) < \infty$. Hence, there is an $(n, \epsilon)$-spanning set $E$ such that $\text{card}(E) = s(n, \epsilon)$. In particular $\text{card}(E) < \infty$. If we prove that $E$ is $(n, \epsilon)$-spanning, then $r(n, \epsilon) \leq \text{card}(E) = s(n, \epsilon)$ and we are done. Otherwise, we can arrange

$$y \in K \setminus \left( \bigcup_{x \in E} B_n[x, \epsilon] \right).$$ 

In particular, $y \not\in E$, and so $\text{card}(E \cup \{y\}) > \text{card}(E)$ because $\text{card}(E) < \infty$. If $x \in E$, (2) implies $B_n[x, \epsilon] \cap (E \cup \{y\}) = B_n[x, \epsilon] \cap E = \{x\}$. Moreover, if there were some $x \in B_n[y, \epsilon] \cap E$ then $y \in B_n[x, \delta]$ for some $x \in E$ against (2). Therefore,
Proof. It suffices to show that $B_n[y, \epsilon] \cap E = \emptyset$ which proves $B_n[y, \epsilon] \cap (E \cup \{y\}) = \{y\}$. Thus, $E \cup \{y\}$ is $(n, \epsilon)$-separated. By (2) we have $y \in K$ thus $E \cup \{y\} \subset K$ contradicting $\text{card}(E) = s(n, \epsilon)$. This proves the result.

Our third property compares the separated and spanning entropies.

Theorem 3.5. The spanning entropy is less than or equal to the separated entropy.

Proof. The proof is a direct consequence of Lemma 3.4. \hfill \Box

The next property gives a sufficient condition for the separated and spanning entropies of a set-valued dynamical system to be equal.

Theorem 3.6. Both the separated and spanning entropies coincide when the maps $d_n$ in (1) are metrics for all $n$ large.

Proof. The proof is similar to the single-valued case [35]. Indeed, by Theorem 3.5 it suffices to show that $h_{se}(f) \leq h_{sp}(f)$ and, for this, we only need to prove that $s(n, \epsilon) \leq r(n, \frac{\epsilon}{2})$ for any $n$ large and any $\epsilon > 0$. This is done as follows.

Take $N \in \mathbb{N}^+$ such that $d_n$ in (1) is a metric for $n \geq N$. Let $F$ and $E$ be an $(n, \epsilon)$-separated set and an $(n, \frac{\epsilon}{2})$-spanning set respectively for $n \geq N$ and $\epsilon > 0$. Then, there is a map $\phi : F \to E$ satisfying $x \in B_n[\phi(x), \frac{\epsilon}{2}]$ for $x \in F$. This map is injective. Indeed, if $\phi(x) = \phi(x')$ for some $x, x' \in F$, then $d_n(x, x) \leq d_n(x, \phi(x)) + d_n(\phi(x), x') \leq \epsilon$ because $n \geq N$ (and so $d_n$ is metric). This implies $x' \in B_n[x, \epsilon] \cap F$ and so $x = x'$. It follows that $\text{card}(F) \leq \text{card}(E)$ which proves $s(n, \epsilon) \leq r(n, \frac{\epsilon}{2})$. \hfill \Box

We observe that the hypothesis of the above result are not valid in general (e.g. Example 4.5).

Next we discuss the invariance of the entropies under topological conjugacy. Observe that, for any pair of sets $X$ and $Y$, every map $H : X \to Y$ induces a map $H : 2^X \to 2^Y$ given by $H(A) = \{H(a) : a \in A\}$ for $A \subset X$. About this map we have the following proposition.

Proposition 3.7. Let $f$ and $g$ be set-valued maps of metric spaces $X$ and $Y$ respectively. If there is a uniformly continuous surjective map $H : X \to Y$ satisfying $H \circ f \leq g \circ H$, then $h_{se}(f) \geq h_{se}(g)$ for $* = se, sp$.

Proof. First we prove $h_{se}(f) \geq h_{se}(g)$. We claim that for every $\epsilon > 0$ there is $\delta > 0$ such that for every $n \in \mathbb{N}^+$ and every $(n, \epsilon)$-separated set $F'$ of $f$ there is an $(n, \delta)$-separated set $F''$ of $f$ such that $\text{card}(F) = \text{card}(F'')$.

Since $H$ is uniformly continuous, for every $\epsilon > 0$ there is $\delta > 0$ such that whenever $a, b \in X$ one has

$$d(H(a), H(b)) > \epsilon \implies d(a, b) > \delta.$$  

Let $F$ be an $(n, \epsilon)$-separated set of $g$. Since $H$ is onto, we can choose an injective map $\phi : F \to X$ such that $H \circ \phi = \text{Id}$ where $\text{Id}$ is the identity. Let us prove that $F'' = \phi(F)$ is $(n, \epsilon)$-separated for $f$.

Take distinct $x, y \in F'$ and sequences $(x_i)_{i=0}^n$, $(y_i)_{i=0}^n$ such that $x_0 = x$, $y_0 = y$, $x_{i+1} \in f(x_i)$ and $y_{i+1} \in f(y_i)$ for all $i$ with $0 \leq i \leq n-1$.

Since $x \neq y$, we have $H(x) \neq H(y)$. In fact, $x = \phi(a)$ and $y = \phi(b)$ for some $a, b \in F$. As $x \neq y$ one has $a \neq b$ thus $H(x) = H(\phi(a)) = a \neq b = H(\phi(b)) = H(y)$. Now, the sequences $(H(x_i))_{i=0}^n$ and $(H(y_i))_{i=0}^n$ satisfy $H(x_0) = H(x)$, $H(y_0) = H(y)$, $H(x_{i+1}) \in H(f(x_i)) \subset g(H(x_i))$ and $H(y_{i+1}) \in H(f(y_i)) \subset g(H(y_i))$ for all $i$ with $0 \leq i \leq n-1$. As $H(x), H(y) \in F$ and $F$ is $(n, \epsilon)$-separated for $g$, there is an
i_0 with 0 \leq i_0 \leq n - 1 such that d(H(x_{i_0}), H(y_{i_0})) > \epsilon. Then, d(x_{i_0}, y_{i_0}) > \delta and the claim follows because \text{card}(F') = \text{card}(\phi(F)) = \text{card}(F).

It follows from the claim that for every \epsilon > 0 there is \delta > 0 such that \text{card}(F') \leq \text{card}(E').

Next we prove \text{h}_{sp}(f) \geq \text{h}_{sp}(g). We claim that for every \epsilon > 0 there is \delta > 0 such that for every n \in \mathbb{N}^+ and every (n, \delta) \text{-spanning set} E' of f there is an (n, \epsilon) \text{-spanning set} E of g such that \text{card}(E) \leq \text{card}(E').

Fix \epsilon > 0 and take \delta > 0 such that
\[
d(a, b) \leq \delta \implies d(H(a), H(b)) \leq \epsilon.
\]

Now fix n \in \mathbb{N}^+ and an (n, \delta) \text{-spanning set} E' of f, i.e.,
\[
X = \bigcup_{x \in E'} B_n[x, \delta],
\]

Because H is surjective one has
\[
Y = \bigcup_{x \in E'} H(B_n[x, \delta]).
\]

We assert that \text{H}(B_n[x, \delta]) \subset B_n[H(x), \epsilon]. Take y \in B_n[x, \delta]. It follows that there are sequences (x_i)_{i=0}^n and (y_i)_{i=0}^n satisfying x_0 = x, y_0 = y, x_{i+1} = f(x_i), y_{i+1} = f(y_i) and d(x_i, y_i) \leq \delta for all i with 0 \leq i \leq n - 1. Then, the sequences (H(x_i))_{i=0}^n and (H(y_i))_{i=0}^n satisfy \text{H}(x_0) = H(x), \text{H}(y_0) = H(y), \text{H}(x_{i+1}) \in H(f(x_i)) \subset g(H(x_i)) and \text{H}(y_{i+1}) \in H(f(y_i)) \subset g(H(y_i)) for all i with 0 \leq i \leq n - 1. Moreover, we also have d(H(x_i), H(y_i)) \leq \epsilon for all i with 0 \leq i \leq n - 1 by the choice of \delta. Therefore \text{H}(y) \in B_n[H(x), \epsilon] proving the assertion.

It follows from the assertion that
\[
Y = \bigcup_{x \in E'} B_n[H(x), \epsilon].
\]

Then, E = H(E') is (n, \epsilon) \text{-spanning for} g. Clearly \text{card}(E) \leq \text{card}(E') and the claim follows.

The claim implies that for every \epsilon > 0 there is \delta > 0 such that r(n, \delta, f) \geq r(n, \epsilon, g) for every n \in \mathbb{N}^+. Indeed, fix \epsilon > 0 and take \delta as in the claim. We can assume that r(n, \delta, f) < \infty. Otherwise we are done. Then, there is an (n, \delta) \text{-spanning set of} f such that \text{card}(E') = r(n, \delta, f). By the claim we can choose an (n, \epsilon) \text{-spanning set} E of g such that \text{card}(E') \geq \text{card}(E). Then, r(n, \delta, f) = \text{card}(E') \geq \text{card}(E) \geq r(n, \epsilon, g). From this we obtain that for every \epsilon > 0 there is \delta > 0 such that
\[
\text{h}_{sp}(f) \geq \limsup_{n \to \infty} \frac{\log r(n, \delta, f)}{n} \geq \limsup_{n \to \infty} \frac{\log r(n, \epsilon, g)}{n}.
\]

As \epsilon is arbitrary, we get \text{h}_{sp}(f) \geq \text{h}_{sp}(g). \qed

We say that the set-valued maps f and g of the respective metric spaces X and Y are \textit{topologically conjugated} if there is a uniform homeomorphism \text{H} : X \to Y such that g \circ \text{H} = \text{H} \circ f. Correspondingly, two metrics d and d' of X are said to be \textit{uniformly equivalent} if both \text{Id} : (X, d) \to (X, d') and \text{Id} : (X, d') \to (X, d) are uniformly continuous, where \text{Id}(x) = x is the identity.
Our next property is the invariance of the separated and spanning entropy under conjugacies or equivalent metrics.

**Theorem 3.8.** The separated and spanning entropies are invariant under topological conjugacy. Moreover, both entropies are independent from uniformly equivalent metrics.

**Proof.** The first part is a direct consequence of Proposition 3.7 while the second follows from the first.

**Remark 3.9.** Theorem 3.8 implies that the topological entropy is also an invariant for any single-valued map, whether continuous or not. This extends the single-valued result in Proposition 3.7 of p. 625 in Ciklová [12].

We define the composition $g \circ f$ of set-valued maps $f, g$ of $X$ by

$$(g \circ f)(x) = g(f(x)) \quad \forall x \in X.$$ 

We define $f^0$ by $f^0(x) = \{x\}$ for all $x \in X$. Inductively we define $f^k = f \circ f^{k-1}$ for $k \geq 1$.

**Lemma 3.10.** For every set-valued map $f$ of a metric space one has

$$d^f_n \leq d^f_{kn} \quad \forall n, k \in \mathbb{N}^+.$$ 

**Proof.** We can assume that $k \in \mathbb{N}^+$. Fix $n \in \mathbb{N}^+$ and $x, y \in X$. Given $\gamma > 0$ there are sequences $(x_i)_{i=0}^{kn}$ and $(y_i)_{i=0}^{kn}$ such that $x_0 = x$, $y_0 = y$, $x_{i+1} \in f(x_i)$, $y_{i+1} \in f(y_i)$ and $d(x_i, y_i) \leq d^f_{kn}(x, y) + \gamma$ for all $i$ with $0 \leq i \leq kn - 1$. Define $\hat{x}_i = x_{ki}$ and $\hat{y}_i = y_{ki}$ for all $i$ with $0 \leq i \leq n$. Then, the resulting sequences $(\hat{x}_i)_{i=0}^n$ and $(\hat{y}_i)_{i=0}^n$ satisfy $\hat{x}_0 = x$, $\hat{y}_0 = y$, $\hat{x}_{i+1} \in f^k(\hat{x}_i)$, $\hat{y}_{i+1} \in f^k(\hat{y}_i)$ and $d(\hat{x}_i, \hat{y}_i) \leq d^f_{kn}(x, y) + \gamma$ for all $i$ with $0 \leq i \leq n - 1$. Then, $d^f_n(x, y) \leq d^f_{kn}(x, y) + \gamma$. As $\gamma$ is arbitrary we obtain the result.

Lemma 3.10 implies the following corollary.

**Corollary 3.11.** For any set-valued map $f$ of a metric space $X$ one has

$$B^f_{kn}[x, \epsilon] \subset B^f_n[x, \epsilon], \quad \forall x \in X, n \in \mathbb{N}^+, \epsilon > 0.$$ 

We say that a set-valued map $f$ of a metric space $X$ is **uniformly continuous** if for every $\epsilon > 0$ there is $\delta > 0$ such that $d(f(x), f(y)) < \epsilon$ whenever $x, y \in X$ satisfy $d(x, y) < \delta$. This definition reduces to the usual uniform continuity in the single-valued case.

The next lemma is a kind of converse of the above corollary in the uniformly continuous case.

**Lemma 3.12.** Let $f$ be a uniformly continuous set-valued map on a metric space. Then, for every $\epsilon > 0$ there is $\delta > 0$ such that

$$B^f_n[x, \delta] \subset B^f_{kn}[x, \epsilon] \quad \forall x \in X, n \in \mathbb{N}^+, k \in \mathbb{N}.$$ 

**Proof.** Fix $\epsilon > 0$. Since $f$ is uniformly continuous, there are $0 < \delta_k < \delta_{k-1} < \cdots < \delta_1 < \delta_0 = \epsilon$ such that for every $r$ satisfying $0 \leq r \leq k$ and every $a, b \in X$ one has

$$d(a, b) \leq \delta_r \quad \Rightarrow \quad d(f(b), f(a)) < \delta_{r-1}. \quad (3)$$

Let us prove that $\delta = \delta_k$ satisfies the conclusion of the lemma.

Fix $x \in X, n \in \mathbb{N}^+$ and $k \in \mathbb{N}$. We can assume that $k \in \mathbb{N}^+$. 

Take $y \in B^f_k[x, \delta]$, i.e., there are sequences $(x_i)_{i=0}^n$ and $(y_i)_{i=0}^n$ such that $x_0 = 0$, $y_0 = y$, $x_{i+1} \in f^k(x_i)$, $y_{i+1} \in f^k(y_i)$ and $d(x_i, y_i) \leq \delta$ for all $i$ with $0 \leq i \leq n - 1$.

Given $i$ with $0 \leq i \leq n - 1$ we construct the sequences $(x^i_j)_{j=0}^{k-1}$ and $(y^i_j)_{j=0}^{k-1}$ as follows:

Define $x^i_0 = x_i$ and $y^i_0 = y_i$. Then, $d(x^i_0, y^i_0) = d(x_i, y_i) \leq \delta = \delta_k$. Applying (3) there are $x^i_1 \in f(x^i_0)$ and $y^i_1 \in f(y^i_0)$ such that $d(x^i_1, y^i_1) < \delta_{k-1}$. Again by (3) there are $x^i_2 \in f(x^i_1)$ and $y^i_2 \in f(y^i_1)$ such that $d(x^i_2, y^i_2) < \delta_{k-2}$. Repeating the process we obtain $x^i_j$ and $y^i_j$ for $1 \leq j \leq k - 1$ such that $x^i_{j+1} \in f(x^i_j)$, $y^i_{j+1} \in f(y^i_j)$ for all $j$ with $0 \leq j \leq k - 2$ and $d(x^i_j, y^i_j) \leq \delta_{k-j}$ for all $i$ with $0 \leq j \leq k - 1$. This completes the construction.

Next we define the sequences $(\hat{x}_i)_{i=0}^{kn}$ and $(\hat{y}_i)_{i=0}^{kn}$ by $\hat{x}_{ik+j} = x^i_j$ and $\hat{y}_{ik+j} = y^i_j$ for all $i, j$ with $0 \leq i \leq n - 1$ and $0 \leq j \leq k - 1$ (resp.).

It follows from these choices that $\hat{x}_0 = x$, $\hat{y}_0 = y$, $\hat{x}_l+1 \in f(\hat{x}_l)$, $\hat{y}_l+1 \in f(\hat{y}_l)$ and $d(\hat{x}_l, \hat{y}_l) \leq \epsilon$ for all $l$ with $0 \leq l \leq kn - 1$. Then, $y \in B^f_{kn}[x, \epsilon]$ and the lemma follows.

Our next property is a power inequality for set-valued maps corresponding to the power formula $h(f^k) = k \cdot h(f)$ for single-valued maps $f$.

**Theorem 3.13.** Every uniformly continuous set-valued map $f$ of a metric space satisfies

$$h_*(f) \leq h_*(f^k) \leq k \cdot h_*(f) \quad \forall k \in \mathbb{N}^+ \text{ where } * = sp, se.$$

**Proof.** We first prove the result for $* = se$. Fix $k \in \mathbb{N}^+$.

Let $n \in \mathbb{N}^+$, $\epsilon > 0$ and $F$ be an $(n, \epsilon)$-separated set for $f^k$. By Corollary 3.11 we obtained

$$F \cap B^f_{kn}[x, \epsilon] \subset F \cap B^f_n[x, \epsilon] = \{x\} \quad \forall x \in F,$$

hence $F$ is $(kn, \epsilon)$-separated for $f$. This implies $s(n, \epsilon, f^k) \leq s(kn, \epsilon, f)$ and so $h_{se}(f^k) \leq k \cdot h_{se}(f)$.

To prove $h_{se}(f) \leq h_{se}(f^k)$, let $\epsilon > 0$ and take $\delta$ from Lemma 3.12. Take $n \in \mathbb{N}$ and let $F$ be $(kn, \epsilon)$-separated set of $f$. Then, Lemma 3.12 implies

$$F \cap B^f_{kn}[x, \epsilon] \subset F \cap B^f_n[x, \epsilon] = \{x\} \quad \forall x \in F,$$

hence $F$ is $(n, \epsilon)$-separated for $f^k$. This implies $s(kn, \epsilon, f) \leq s(n, \delta, f^k)$. As $s(n, \epsilon, f)$ is increasing in $n$, we get $s(n, \epsilon, f) \leq s(n, \delta, f^k)$. Then, $h_{se}(f) \leq h_{se}(f^k)$.

Next we prove the result for $* = sp$. Fix $k \in \mathbb{N}^+$.

Let $n \in \mathbb{N}^+$, $\epsilon > 0$ and $E$ be an $(kn, \epsilon)$-spanning set for $f$. By Corollary 3.11 we obtained

$$X = \bigcup_{x \in E} B^f_{kn}[x, \epsilon] \subset \bigcup_{x \in E} B^f_n[x, \epsilon]$$

and so $E$ is $(n, \epsilon)$-spanning for $f^k$. This implies $r(n, \epsilon, f^k) \leq r(kn, \epsilon, f)$ hence $h_{sp}(f^k) \leq k \cdot h_{sp}(f)$.

To prove $h_{sp}(f) \leq h_{sp}(f^k)$, let $\epsilon > 0$ and take $\delta$ from Lemma 3.12. Take $n \in \mathbb{N}$ and let $E$ be a $(n, \delta)$-spanning for $f^k$. Then, Lemma 3.12 implies

$$X = \bigcup_{x \in E} B^f_{kn}[x, \delta] \subset \bigcup_{x \in E} B^f_n[x, \delta]$$

and so $E$ is $(kn, \epsilon)$-spanning for $f$. This implies $r(kn, \epsilon, f) \leq r(n, \delta, f^k)$. But again $r(n, \epsilon, f)$ is increasing in $n$, so $h_{sp}(f) \leq h_{sp}(f^k)$. $\square$
Next we prove the following lemma.

**Lemma 3.14.** Let $f$ be a uniformly continuous set-valued map of a metric space $X$. Then, for every $n \in \mathbb{N}^+$ and $\epsilon > 0$ there is $\delta > 0$ such that $d_n(x,y) < \epsilon$ whenever $x,y \in X$ satisfy $d(x,y) < \delta$.

**Proof.** Fix $\epsilon > 0$. By uniform continuity there are $0 < \delta_n < \cdots < \delta_1 < \delta_0 = \epsilon$ such that $d(f(w),f(z)) < \delta_{i-1}$ whenever $d(z,w) < \delta_i$ and $1 \leq i \leq n$. Now take $\delta = \delta_n$. \hfill $\square$

From this lemma we obtain the following property.

**Lemma 3.15.** If $f$ is a uniformly continuous set-valued map on a compact metric space $X$, then $s(n,\epsilon) < \infty$ for every $n \in \mathbb{N}$ and $\epsilon > 0$.

**Proof.** Otherwise there are $n \in \mathbb{N}^+$, $\epsilon > 0$ and a sequence $F_k$ of $(n,\epsilon)$-separated sets satisfying $\text{card}(F_k) \to \infty$ as $k \to \infty$. For these $n$ and $\epsilon$ we fix $\delta > 0$ as in Lemma 3.14. By compactness there are $k$ large and distinct points $x,y \in F_k$ with $d(x,y) < \delta$. Then, Lemma 3.14 implies $y \in B_n[x,\epsilon] \cap F_k$ which contradicts that $F_k$ is $(n,\epsilon)$-separated. \hfill $\square$

We say that a set-valued map $f$ of a metric space $X$ is **equicontinuous** if for every $\epsilon > 0$ there is $\delta > 0$ such that for all $x,y \in X$ with $d(x,y) < \delta$ there are sequences $(x_i)_{i \in \mathbb{N}}$ and $(y_i)_{i \in \mathbb{N}}$ such that $x_0 = x$, $y_0 = y$, $x_{i+1} \in f(x_i)$, $y_{i+1} \in f(y_i)$ and $d(x_i,y_i) < \epsilon$ for every $i \in \mathbb{N}$. This definition is the natural extension of the corresponding definition in the single-valued case [9]. Another related concept but in the set-valued setting is Definition 3.1 in [22].

The last property is a generalization of a well known fact in the single-valued case.

**Theorem 3.16.** Both separated and spanning entropies vanish for equicontinuous set-valued maps on compact metric spaces.

**Proof.** Equicontinuity is equivalent to the property that for every $\epsilon > 0$ there is $\delta > 0$ such that $B[x,\delta] \subset B_n[x,\epsilon]$ for every $x \in X$ and $n \in \mathbb{N}$. This implies $s(n,\epsilon) \leq s(1,\delta)$ for all $n \in \mathbb{N}^+$. On the other hand, it is obvious that $s(1,\delta) > 0$ and we also have $s(1,\delta) < \infty$ by Lemma 3.15. Then, $h_{se}(f) = 0$ and so $h_{sp}(f) = 0$ by Theorem 3.5. \hfill $\square$

4. **Examples.** In this section we present some related examples. For this we need the following facts.

**Lemma 4.1.** If $f$ is a set-valued map on a metric space $X$, then

$$d_2(a,b) = \max\{d(a,b),d(f(a),f(b))\}, \quad \forall a, b \in X.$$

**Proof.** We derive the result from the two assertions below:

- If $d(f(a),f(b)) \leq d(a,b)$, then $d_2(a,b) = d(a,b)$.
- If $d(a,b) < d(f(a),f(b))$, then $d_2(a,b) = d(f(a),f(b))$.

It follows from the definition of $d_2$ that

$$d_2(a,b) = \inf\{\max\{d(a,b),d(a_1,b_1)\} : a_1 \in f(a), b_1 \in f(b)\}.$$ 

In particular, $d_2 \geq d$. First we prove the first assertion. If $\gamma > 0$, there are $a_1 \in f(a)$, $b_1 \in f(b)$ such that $d(a_1,b_1) < d(a,b) + \gamma$. For this particular choice one
has \( \max(d(a,b), d(a_1,b_1)) < d(a,b) + \gamma \) so \( d_2(a,b) < d(a,b) + \gamma \). As \( \gamma \) is arbitrary, \( d_2(a,b) \leq d(a,b) \) hence \( d_2(a,b) = d(a,b) \).

For the second assertion, if \( d(a,b) < d(f(a),f(b)) \), then \( \max(d(a,b), d(a_1,b_1)) = d(a_1,b_1) \) for all \( a_1 \in f(a), b_1 \in f(b) \). Then, \( d_2(a,b) = d(f(a),f(b)) \). \hfill \Box

Using this lemma we obtain the following proposition.

**Proposition 4.2.** Let \( f \) be set-valued map on a metric space \( X \). If there are \( a,b,c \in X \) satisfying

\[
d(f(a), f(b)) = d(f(b), f(c)) = 0 \quad \text{and} \quad \max(d(a,b), d(b,c)) < \frac{1}{2} d(f(a), f(c)),
\]

then \( d_2 \) is not a metric.

**Proof.** It follows at once from Lemma 4.1 that \( d_2(a,b) = d(a,b) \) and \( d_2(b,c) = d(b,c) \). On the other hand,

\[
d(a,c) \leq d(a,b) + d(b,c) < \frac{1}{2} d(f(a), f(c)) + \frac{1}{2} d(f(a), f(c)) = d(f(a), f(c)).
\]

From this and Lemma 4.1 we obtain \( d_2(a,c) = d(f(a), f(c)) \). Then,

\[
d_2(a,c) = d(f(a), f(c)) = \frac{1}{2} d(f(a), f(c)) + \frac{1}{2} d(f(a), f(c)) > d(a,b) + d(b,c)
\]

and so \( d_2 \) does not satisfies the triangle inequality. Then, \( d_2 \) is not a metric and the result follows. \hfill \Box

Now we present some examples. The first is the so-called subdifferential of the Euclidean norm \( f(x) = |x| \) of \( \mathbb{R} \) (see p. 215 in [28]).

**Example 4.3.** Consider the set-valued map \( \partial f \) of \( \mathbb{R} \) defined by

\[
\partial f(x) = \begin{cases} 
1 & \text{if } x > 0 \\
[-1,1] & \text{if } x = 0 \\
{-1} & \text{if } x < 0
\end{cases}
\]

By putting \( a = \frac{1}{2}, b = 0 \) and \( c = -\frac{1}{2} \) in Proposition 4.2 we obtain that \( d_2 \) is not a metric.

The second example is the following.

**Example 4.4.** It is easy to see that the set-valued map \( f \) of \([0,1]\) defined by

\[
f(x) = \begin{cases} 
\{2x\} & \text{if } 0 \leq x < \frac{1}{2} \\
[0,1] & \text{if } x = \frac{1}{2} \\
\{2x-1\} & \text{if } \frac{1}{2} < x \leq 1
\end{cases}
\]

is uniformly continuous. Nevertheless, by putting \( a < b = \frac{1}{2} < c \) with \( d(a,c) \) small in Proposition 4.2 we obtain that \( d_2 \) is not a metric in this case as well.

The next example is the following.

**Example 4.5.** For every \( k \in \mathbb{N}^+ \) there is a set-valued map \( f \) on the sphere \( S^k = \{x \in \mathbb{R}^{k+1} : \|x\| = 1\} \) of \( \mathbb{R}^{k+1} \) for which \( d_n \) is not a metric for every \( n \geq 2 \).
Example 4.6. Let $a, b, c \in S^k$ and three subsets $A, B, C \subset S^k$ satisfying the following three hypotheses:

- $d(A, B) = d(B, C) = 0$;
- $\max(d(a, b), d(b, c)) < \frac{1}{2}d(A, C)$;
- $\{a, b, c\} \cap (A \cup B \cup C) = 0$.

Since $a \neq b \neq c \neq a$, the set-valued map $f$ of $S^k$ defined by

$$f(x) = \begin{cases} A & \text{if } x = a \\ B & \text{if } x = b \\ C & \text{if } x = c \\ \{x\} & \text{if } x \notin \{a, b, c\} \end{cases}$$

is well-defined. It follows from the two first hypothesis and Proposition 4.2 that $d_2$ is not a metric for this $f$. Using the third hypothesis we obtain $d_n = d_2$ for every $n \geq 2$, and so, $d_n$ is not a metric for $n \geq 2$. This finishes the construction. \hfill \Box

Another example is the following one.

Example 4.7. Let $f$ be a set-valued map on a metric space $X$ satisfying $x \in f(x)$ for all $x \in X$. It follows that $d(f(x), f(y)) \leq d(x, y)$ for all $x, y \in X$. Then, $d_2 = d$ is a metric by Lemma 4.1. Now fix $\delta > 0$ and define the set-valued map $f$ on $\mathbb{R}^2$ by $f(x) = B(x, \delta)$ for all $x \in \mathbb{R}^2$. Then, $f$ is not single-valued but $d_2$ is a metric. These maps have both separating and spanning entropies equal to zero (see Example 4.7).

To finish we present some example where the separated and spanning entropies can be computed.

Example 4.8. Let $f$ be a set-valued map on a metric space $X$ satisfying $x \in f(x)$ for all $x \in X$. Then, $h_{se}(f) = h_{sp}(f) = 0$. Indeed, in this case the identity $Id$ is a continuous selection of $f$ so $h_{se}(f) \leq h(Id) = 0$ by Corollary 3.3. This can be also proved by noticing that all such set-valued maps are equicontinuous and then both entropies vanish by Theorem 3.16. In particular, given $\delta \geq 0$ the set-valued map $f(x) = B[x, \delta]$ for $x \in X$ has zero separated and spanning topological entropies.

For the next three examples we will consider the unit circle

$$S^1 = \{z \in \mathbb{C} : |z| = 1\}.$$ 

Define $g : S^1 \to S^1$ by $g(z) = z^2$. It follows that $h(g) = \log 2$. Furthermore $g$ is an expanding map, i.e., there is $\lambda > 1$ such that $\|Dg(z)\| \geq \lambda$ for all $z \in S^1$.

Example 4.8. Endow the unit interval $[0, 1]$ with the Euclidean metric. Define the set-valued map $f$ of $[0, 1]$ by

$$f(x) = \begin{cases} \{2x\}, & \text{if } 0 \leq x < \frac{1}{2} \\ \{0, 1\}, & \text{if } x = \frac{1}{2} \\ \{2x - 1\}, & \text{if } \frac{1}{2} < x \leq 1 \end{cases}$$

It follows that $h_{se}(f) = \log 2$.

Proof. Let $H : [0, 1] \to S^1$ be defined by $H(x) = e^{2\pi ix}$. It follows that $H$ is a continuous surjection. One can check easily that $g \circ H = H \circ f$ thus we have $h_{se}(f) \geq h_{se}(g) = h(g) = \log 2$ by Proposition 3.7. On the other hand, the map

$$s(x) = \begin{cases} 2x, & \text{if } 0 \leq x \leq \frac{1}{2} \\ 2x - 1, & \text{if } \frac{1}{2} < x \leq 1 \end{cases}$$

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is a selection of \( f \). It follows that \( h_{se}(f) \leq h(s) \) by Corollary 3.3. As \( h(s) = \log 2 \), we obtain \( h_{se}(f) \leq \log 2 \) whence \( h_{se}(f) = \log 2 \).

**Example 4.9.** Given \( \delta > 0 \) we define the set-valued map \( f \) of \( S^1 \) by \( f(z) = B[g(z), \delta] \) for all \( z \in S^1 \). Since \( g \) is expanding, there is \( n \in \mathbb{N}^+ \) depending on \( \delta \) only such that \( f^n(z) = S^1 \) for all \( z \in S^1 \). In particular, \( z \in f^n(z) \) for all \( z \in S^1 \) and so \( h_{se}(f^n) = 0 \). On the other hand, \( f \) is clearly uniformly continuous so \( h_{se}(f) \leq h_{se}(f^n) = 0 \) by Theorem 3.13. Therefore, \( h_{se}(f) = 0 \). From this and Theorem 3.5 we obtain \( h_{sp}(f) = 0 \).

The following is an example of a genuine (i.e. not single-valued) map for which the separated and spanning entropies coincide and the common value is positive.

**Example 4.10.** Let \( f \) be the set-valued map on \([0, 1]\) defined in Example 4.8. By Theorem 3.5 one has \( h_{sp}(f) \leq h_{se}(f) = \log 2 \). On the other hand, we have already seen that \( h_{sp}(f) \geq h_{sp}(g) \). As \( h_{sp}(g) = h_{se}(g) = \log 2 \) we conclude that \( h_{sp}(f) = h_{se}(f) = \log 2 \).

The last example is the following.

**Example 4.11.** The set-valued map \( f \) on \([0, 1]\) defined in Example 4.4 has both spanning and separating entropies equal to \( \log 2 \).

**Proof.** Clearly \( f \) is bigger than the set-valued map in Example 4.8 which, in turns, has spanning entropy less than or equal to \( \log 2 \). Then, \( h_{sp}(f) \leq \log 2 \) by Theorem 3.5. On the other hand, by taking suitable closed intervals \( I_1, I_2 \) at each side of \( \frac{1}{2} \) we obtain a compact invariant set \( A = \bigcap_{n \in \mathbb{N}} f^{-n}(I_1 \cup I_2) \) for which \( f|_A \) has topological entropy equals to \( \log 2 \). Then, \( \log 2 \leq h_{sp}(f) \) by Theorem 3.1 and we are done. Similarly we obtain \( h_{se}(f) = \log 2 \).

5. **Conclusions.** In this paper we used the single-valued approach of separated and spanning sets \([8], [15]\) to define the spanning and separated topological entropies for set-valued maps. We proved that these entropies satisfy some properties resembling to the single-valued case. These include the kind of sub-additivity property in Theorem 3.1 (similar to the single-valued case), that they reverse natural inclusion orders for set-valued maps (not available in the single-valued case), that the spanning entropy is less than the separated one (and that they both coincide when the induced semimetrics are metrics), that they are topological invariants (similar to the single-valued case), that they satisfy a power inequality closely related to the power formula in the single-valued case, and that they both vanish for equicontinuous set-valued maps (again as in the single-valued case). We also computed them in some genuine (i.e. not single-valued) examples.

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