Uniform $W^{1,p}$ Estimates and Large-Scale Regularity for Dirichlet Problems in Perforated Domains

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Abstract

In this paper we study the Dirichlet problem for Laplace’s equation in a domain $\omega_{\varepsilon, \eta}$ perforated periodically with small holes in $\mathbb{R}^d$, where $\varepsilon$ represents the scale of the minimal distances between holes and $\eta$ the ratio between the scale of sizes of holes and $\varepsilon$. We establish $W^{1,p}$ estimates for solutions with bounding constants depending explicitly on $\varepsilon$ and $\eta$. The proof relies on a large-scale Lipschitz estimate for harmonic functions in perforated domains. The results are optimal for $d \geq 2$.

Keywords: Uniform Estimates; Large-scale Regularity; Perforated Domain; Homogenization.

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1 Introduction

This paper continues the study of the Dirichlet problem for Laplace’s equation,

\[
\begin{cases}
-\Delta u = F + \text{div}(f) & \text{in } \omega_{\varepsilon, \eta}, \\
\quad u = 0 & \text{on } \partial \omega_{\varepsilon, \eta},
\end{cases}
\]

in a domain $\omega_{\varepsilon, \eta}$ perforated with small holes, where $\varepsilon$ represents the scale of the minimal distances between holes and $\eta$ the ratio between the sizes of holes and $\varepsilon$. We are interested in the $W^{1,p}$ estimates,

\[
\|\nabla u\|_{L^p(\omega_{\varepsilon, \eta})} \leq A_p(\varepsilon, \eta) \|f\|_{L^p(\omega_{\varepsilon, \eta})} + B_p(\varepsilon, \eta) \|F\|_{L^p(\omega_{\varepsilon, \eta})},
\]

and

\[
\|u\|_{L^p(\omega_{\varepsilon, \eta})} \leq C_p(\varepsilon, \eta) \|f\|_{L^p(\omega_{\varepsilon, \eta})} + D_p(\varepsilon, \eta) \|F\|_{L^p(\omega_{\varepsilon, \eta})},
\]

for $1 < p < \infty$, with bounding constants $A_p(\varepsilon, \eta)$, $B_p(\varepsilon, \eta)$, $C_p(\varepsilon, \eta)$ and $D_p(\varepsilon, \eta)$ depending explicitly on the small parameters $\varepsilon, \eta \in (0, 1]$.

To state our main results, we let $Y = [-1/2, 1/2]^d$ be a closed unit cube in $\mathbb{R}^d$ and $T$ the closure of an open subset of $Y$. Throughout the paper we shall assume that $Y \setminus T$ is connected and that

\[
B(0, c_0) \subset T \quad \text{and} \quad \text{dist}(\partial T, \partial Y) \geq c_0 > 0
\]

for some $c_0 > 0$. Define

\[
\omega_{\varepsilon, \eta} = \mathbb{R}^d \setminus \bigcup_{k \in \mathbb{Z}^d} \varepsilon(k + \eta T),
\]

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where $0 < \varepsilon, \eta \leq 1$. Roughly speaking, the periodically perforated domain $\omega_{\varepsilon, \eta}$ is obtained from $\mathbb{R}^d$ by removing a hole $\varepsilon(k + \eta T)$ of size $\varepsilon \eta$ from each cube $\varepsilon(k + Y)$ of size $\varepsilon$. The distances between holes are bounded below by $c_0 \varepsilon$.

The following are the main results in this paper. The first theorem deals with the case $d \geq 3$, while the second treats the case $d = 2$.

**Theorem 1.1.** Suppose $d \geq 3$ and $1 < p < \infty$. Let $\omega_{\varepsilon, \eta}$ be given by (1.5), where $T$ is the closure of an open subset of $Y$ with $C^1$ boundary. For any $f \in L^p(\omega_{\varepsilon, \eta}; \mathbb{R}^d)$ and $F \in L^p(\omega_{\varepsilon, \eta})$, the Dirichlet problem (1.1) has a unique solution $u$ in $W^{1,p}_0(\omega_{\varepsilon, \eta})$. Moreover, the solution satisfies the estimate,

$$
\|\nabla u\|_{L^p(\omega_{\varepsilon, \eta})} \leq \begin{cases} 
C\eta^{-d}\frac{1}{2} - \frac{1}{p} \|f\|_{L^p(\omega_{\varepsilon, \eta})} + C\varepsilon \eta^{1-d}\|F\|_{L^p(\omega_{\varepsilon, \eta})} & \text{for } 1 < p < 2, \\
C\eta^{-d}\frac{1}{2} - \frac{1}{p} \|f\|_{L^p(\omega_{\varepsilon, \eta})} + C\varepsilon \eta^{1-d+\frac{2}{p}}\|F\|_{L^p(\omega_{\varepsilon, \eta})} & \text{for } 2 \leq p < \infty,
\end{cases}
$$

and

$$
\|u\|_{L^p(\omega_{\varepsilon, \eta})} \leq \begin{cases} 
C\varepsilon \eta^{1-d}\|f\|_{L^p(\omega_{\varepsilon, \eta})} + C\varepsilon \eta^{2-d}\|F\|_{L^p(\omega_{\varepsilon, \eta})} & \text{for } 1 < p < 2, \\
C\varepsilon \eta^{1-d}\|f\|_{L^p(\omega_{\varepsilon, \eta})} + C\varepsilon \eta^{2-d}\|F\|_{L^p(\omega_{\varepsilon, \eta})} & \text{for } 2 \leq p < \infty,
\end{cases}
$$

where $C$ depends on $d$, $p$ and $T$. Furthermore, the estimates (1.6)–(1.7) are sharp.

**Theorem 1.2.** Suppose $d = 2$ and $1 < p < \infty$. Let $\omega_{\varepsilon, \eta}$ be given by (1.5), where $T$ is the closure of an open subset of $Y$ with $C^1$ boundary. For any $f \in L^p(\omega_{\varepsilon, \eta}; \mathbb{R}^d)$ and $F \in L^p(\omega_{\varepsilon, \eta})$, the Dirichlet problem (1.1) has a unique solution $u$ in $W^{1,p}_0(\omega_{\varepsilon, \eta})$. Moreover, the solution satisfies the estimate,

$$
\|\nabla u\|_{L^p(\omega_{\varepsilon, \eta})} \leq \begin{cases} 
C\eta^{-2}\frac{1}{2} - \frac{1}{p} |\ln(\eta/2)|^{-\frac{1}{2}} \|f\|_{L^p(\omega_{\varepsilon, \eta})} + C\varepsilon |\ln(\eta/2)|^{1}\|F\|_{L^p(\omega_{\varepsilon, \eta})} & \text{for } 1 < p < 2, \\
\|f\|_{L^2(\omega_{\varepsilon, \eta})} + C\varepsilon |\ln(\eta/2)|^{1} \|F\|_{L^2(\omega_{\varepsilon, \eta})} & \text{for } p = 2, \\
C\eta^{-2}\frac{1}{2} - \frac{1}{p} |\ln(\eta/2)|^{-\frac{1}{2}} \|f\|_{L^p(\omega_{\varepsilon, \eta})} + C\varepsilon \eta^{-1+\frac{2}{p}}\|F\|_{L^p(\omega_{\varepsilon, \eta})} & \text{for } 2 < p < \infty,
\end{cases}
$$

and

$$
\|u\|_{L^p(\omega_{\varepsilon, \eta})} \leq \begin{cases} 
C\varepsilon \eta^{1-\frac{2}{p}}\|f\|_{L^p(\omega_{\varepsilon, \eta})} + C\varepsilon^2 |\ln(\eta/2)|\|F\|_{L^p(\omega_{\varepsilon, \eta})} & \text{for } 1 < p < 2, \\
C\varepsilon |\ln(\eta/2)|^{1} \|f\|_{L^p(\omega_{\varepsilon, \eta})} + C\varepsilon^2 |\ln(\eta/2)|\|F\|_{L^p(\omega_{\varepsilon, \eta})} & \text{for } 2 \leq p < \infty,
\end{cases}
$$

where $C$ depends on $p$ and $T$. Furthermore, the estimates (1.8)–(1.9) are sharp.

We point out that the estimates (1.6)–(1.9) are sharp in $\varepsilon$ and $\eta$. Indeed, if $d \geq 3$ and the estimates (1.2) and (1.3) hold for some constants $A_p(\varepsilon, \eta)$, $B_p(\varepsilon, \eta)$, $C_p(\varepsilon, \eta)$ and $D_p(\varepsilon, \eta)$, then

$$
A_p(\varepsilon, \eta) \geq c \eta^{-d}\frac{1}{2} - \frac{1}{p},
$$

$$
D_p(\varepsilon, \eta) \geq c \varepsilon^2 \eta^{2-d},
$$

for $1 < p < \infty$,

$$
B_p(\varepsilon, \eta) \geq \begin{cases} 
\epsilon \varepsilon \eta^{1-\frac{2}{p}} & \text{for } 1 < p \leq 2, \\
\epsilon \varepsilon \eta^{1-d+\frac{2}{p}} & \text{for } 2 < p < \infty,
\end{cases}
$$

and

$$
C_p(\varepsilon, \eta) \geq \begin{cases} 
\epsilon \varepsilon \eta^{1-\frac{2}{p}} & \text{for } 1 < p \leq 2, \\
\epsilon \varepsilon \eta^{1-d+\frac{2}{p}} & \text{for } 2 < p < \infty,
\end{cases}
$$

for $1 < p < \infty$.}
where $c > 0$ depends only on $d$, $p$ and $T$. Similar statements hold for the case $d = 2$; the lower bounds for $A_p(\varepsilon, \eta)$, $B_p(\varepsilon, \eta)$, $C_p(\varepsilon, \eta)$ and $D_p(\varepsilon, \eta)$ are given by the corresponding constants in (1.8)-(1.9) (with a different $c > 0$). The powers of $\varepsilon$ in the estimates (1.6)- (1.9) are due to scaling. In fact, by rescaling, it suffices to prove Theorems 1.1 and 1.2 for the case (1.8)-(1.9) (with a different $A$ bounds for constant $D$). In [10] we also established estimates (1.2) and (1.3) in a general non-periodic setting with a sharp estimate for $\eta$ was proved by the first author in [10], using a classical method of test functions and a Poincaré inequality for $L^p$ functions that vanish on $Q_1 \setminus \eta T$, where $Q_R = (-R/2, R/2)^d$. The main results in this paper provide a complete solution in the periodic setting for $d \geq 2$.

We now describe our approach to Theorems 1.1 and 1.2. By rescaling we assume $\varepsilon = 1$. Our starting point is the estimates (1.7) and (1.9) for $u$ in the case $2 \leq p < \infty$. The estimates were proved in a general non-periodic setting in [10], using a classical method of test functions and a Sobolev space for Laplace’s equation $-\Delta u = F + \text{div}(f)$ in an exterior domain $\mathbb{R}^d \setminus T$. By localization and rescaling, this allows us to control the $L^p$ norm of $\nabla u$ in each cell $k + Q_1$ by the $L^p$ norm of $u$ in a slightly larger cell $k + (1 + c_0)Q_1$.

Next, to bound the localization error, we construct a corrector $\psi_\eta \in W^{1,2}(Q_1)$ such that

\begin{align*}
-\Delta \psi_\eta &= F_\eta + \text{div}(f_\eta) \quad \text{in} \ Q_1 \setminus \eta T, \\
\psi_\eta &= 0 \quad \text{in} \ \eta T, \\
\psi_\eta &= 1 \quad \text{in} \ Q_1 \setminus B(0, 1/3),
\end{align*}

(1.13)

where $F_\eta$ and $f_\eta$ satisfy the condition $\|f_\eta\|_\infty + \|F_\eta\|_\infty \leq C\eta^{d-2}$ for $d \geq 3$. The construction of $\psi_\eta$, which is motivated by the correctors used in [1], uses a solution to the exterior problem,

\begin{align*}
\Delta \phi_\ast &= 0 \quad \text{in} \ \mathbb{R}^d \setminus T, \\
\phi_\ast &= 0 \quad \text{in} \ T, \\
\phi_\ast &\to 1 \quad \text{as} \ |x| \to \infty.
\end{align*}

(1.14)

See (6.5) for $d \geq 3$ and (6.6) for $d = 2$. We apply a localization argument to the solution $u - \alpha \psi_\eta$ in $(1 + c_0)Q_1 \setminus \eta T$, with

\[ \alpha = \int_{(1 + c_0)Q_1 \setminus B(0, 1/3)} u. \]

With sharp estimates for $\|u\|_{L^p(\omega_1, \eta)}$ and $\psi_\eta$, this reduces the $L^p$ estimate of $\nabla u$ to the $L^p$ estimate for the operator $S_{\varepsilon, \eta}$, defined by

\[ S_{\varepsilon, \eta}(F, f)(x) = \left( \int_{x + \varepsilon Q_2} |\nabla u|^2 \right)^{1/2}, \]

(1.15)

for $p > 2$ ($u$ is extended by zero into the holes). Note that $\|S_{\varepsilon, \eta}(F, f)\|_{L^2(\mathbb{R}^d)} = \|\nabla u\|_{L^2(\mathbb{R}^d)}$. By a real-variable argument in [8], to establish the $L^p$ boundedness of the operator for $p > 2$, it suffices to prove a (weak) reverse Hölder inequality in a cube $Q$ for solutions of the Dirichlet problem (1.1) with $F = 0$ and $f = 0$ in $4Q$. 

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Finally, we note that the desired reverse H"{o}lder inequality in $L^p$ follows from a large-scale Lipschitz estimate,
\[
\sup_{1 \leq r \leq R} \left( \int_{Q_r} |\nabla u|^2 \right)^{1/2} \leq C \left( \int_{Q_R} |\nabla u|^2 \right)^{1/2},
\]
for harmonic functions in perforated domains. By exploiting the periodicity of the domain $\omega_{1,\eta}$, we are able to establish the large-scale Lipschitz estimate, using an approach taken from [4]. The proof relies on a Caccioppoli inequality as well as a discrete Sobolev inequality in $\mathbb{Z}^d$. The constant $C$ in (1.16) depends only on $d$ and $c_0$ in (1.4).

The paper is organized as follows. In Section 2 we establish a large-scale $L^\infty$ estimate for harmonic functions in $Q_R \cap \omega_{1,\eta}$ that vanish on $Q_R \cap \partial \omega_{1,\eta}$. The large-scale Lipschitz estimate (1.16) is proved in Section 3. The $L^p$ bound with $2 < p < \infty$ for the operator $S_{\varepsilon,\eta}$ in (1.15) is obtained in Section 4. In Sections 5 and 6 we present the localization argument for solutions in $(1 + c_0)Q_1 \setminus \eta T$. The argument relies on some weighted estimates in [3] for an exterior problem and utilizes the corrector $\psi_\eta$ mentioned before. Finally, the proofs of Theorems 1.1 and 1.2 are given in Section 7.

2 Large-scale $L^\infty$ estimates

Throughout this section we assume that $\omega_{\varepsilon,\eta}$ is given by (1.5), where $T$ is the closure of an open subset of $Y$ with Lipschitz boundary. Let $Q_R = (-R/2, R/2)^d$. Our goal is to prove the following theorem.

**Theorem 2.1.** Let $u \in W^{1,2}(Q_R)$ for some $R \geq \varepsilon$. Suppose that
\[
\Delta u = 0 \quad \text{in} \quad Q_R \cap \omega_{\varepsilon,\eta} \quad \text{and} \quad u = 0 \quad \text{in} \quad Q_R \setminus \omega_{\varepsilon,\eta}.
\]
Then, for $\varepsilon \leq r \leq R$,
\[
\left( \int_{Q_r} |u|^2 \right)^{1/2} \leq C \left( \int_{Q_R} |u|^2 \right)^{1/2},
\]
where $C$ depends on $d$.

The proof of Theorem 2.1, as well as the proof of Theorem 3.1 in the next section, is based on an approach found in [4] and relies on a Caccioppoli inequality for solutions of (2.1).

For $u \in L^1(\mathbb{R}^d)$ and $z \in \mathbb{Z}^d$, define
\[
\hat{u}(z) = \int_{z + Q_1} u(x) \, dx.
\]

**Lemma 2.2.** Let $u \in W^{1,2}(Q_{r+2})$, where $r \geq 1$. Then
\[
\left( \int_{Q_r} |u|^2 \right)^{1/2} \leq C \max_{z \in \mathbb{Z}^d \cap Q_{r+2}} |\hat{u}(z)| + C \left( \int_{Q_{r+2}} |\nabla u|^2 \right)^{1/2},
\]
where $C$ depends only on $d$.

**Proof.** Let $z \in \mathbb{Z}^d \cap Q_r$. By Poincaré’s inequality,
\[
\int_{z + Q_1} |u|^2 \, dx \leq C|\hat{u}(z)|^2 + C \int_{z + Q_1} |\nabla u|^2 \, dx,
\]

where $C$ depends only on $d$. Let $\ell \geq 1$ be an odd integer. By summing (2.5) over $z \in \mathbb{Z}^d \cap Q_\ell$, we obtain
\[
\left( \int_{Q_\ell} |u|^2 \right)^{1/2} \leq C \max_{z \in \mathbb{Z}^d \cap Q_\ell} |\hat{u}(z)| + C \left( \int_{Q_\ell} |\nabla u|^2 \right)^{1/2}. \tag{2.6}
\]
Finally, for any $r \geq 1$, choose an odd integer $\ell$ such that $r \leq \ell \leq r + 2$. It is not hard to see that (2.4) follows from (2.6). \hfill $\square$

For a function $g$ defined in $\mathbb{R}^d$ or $\mathbb{Z}^d$, let
\[
\Delta_j g(x) = g(x + e_j) - g(x)
\]
for $1 \leq j \leq d$, where $e_j = (0, \ldots, 1, \ldots, 0)$ with 1 in the $j$th position. For a multi-index $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_d)$, we use the notation $\Delta^\gamma g = g$ if $\gamma = 0$, and
\[
\Delta^\gamma g = \Delta^\gamma_1 \Delta^\gamma_2 \cdots \Delta^\gamma_d g
\]
if $|\gamma| \geq 1$. Let $\partial^\ell g = (\Delta^\gamma g)|_{|\gamma| = \ell}$ and
\[
|\partial^\ell g| = \left( \sum_{|\gamma| = \ell} |\Delta^\gamma g|^2 \right)^{1/2}
\]
for an integer $\ell \geq 0$. It is not hard to see that
\[
|\partial^{\ell+1} \hat{u}(z)| \leq \left( \int_{z + 3Q_1} |\nabla \partial^\ell u|^2 \, dx \right)^{1/2}
\]
for any $z \in \mathbb{Z}^d$.

The next lemma provides a discrete Sobolev inequality in $\mathbb{Z}^d$.

**Lemma 2.3.** Let $g$ be a function on $\mathbb{Z}^d$. Then, for $R \geq 3d$,
\[
\max_{z \in \mathbb{Z}^d \cap Q_R} |g(z)| \leq C \sum_{\ell=0}^{N} R^\ell \left( \frac{1}{R^d} \sum_{z \in \mathbb{Z}^d \cap Q_{2R}} |\partial^\ell g(z)|^2 \right)^{1/2}, \tag{2.10}
\]
where $N = [d/2] + 1$ and $C$ depends only on $d$.

**Proof.** This follows from [11, Lemma 2.6]. \hfill $\square$

The following lemma gives the Caccioppoli inequality for solutions of $-\Delta u = F$ in $Q_r \cap \omega_{1,\eta}$ with $u = 0$ on $Q_r \setminus \omega_{1,\eta}$.

**Lemma 2.4.** Let $u \in W^{1,2}(Q_r)$ for some $r \geq 1$. Suppose that $-\Delta u = F$ in $Q_r \cap \omega_{1,\eta}$ and $u = 0$ in $Q_r \setminus \omega_{1,\eta}$. Then
\[
\left( \int_{Q_{sr}} |\nabla u|^2 \right)^{1/2} \leq \frac{C}{(t - s)r} \left( \int_{Q_{ts}} |u|^2 \right)^{1/2} + C(t - s)r \left( \int_{Q_{tr}} |F|^2 \right)^{1/2}
\]
for $(1/2) \leq s < t \leq 1$, where $C$ depends only on $d$.\hfill $\square$
Proof. The proof is exactly the same as that for the usual Caccioppoli inequality. Choose a cut-off function \( \varphi \in C^\infty_0(Q_{tr}) \) such that \( \varphi = 1 \) in \( Q_{sr} \) and \( |\nabla \varphi| \leq C(t-s)^{-1}r^{-1} \). Note that

\[
\int_{Q_{tr}} \nabla u \cdot \nabla (u\varphi^2) \, dx = \int_{Q_{tr} \cap \omega_{1,\eta}} \nabla u \cdot \nabla (u\varphi^2) \, dx = \int_{Q_{tr} \cap \omega_{1,\eta}} F(u\varphi^2) \, dx = \int_{Q_{tr}} F(u\varphi^2) \, dx.
\]

Hence,

\[
\int_{Q_{tr}} |\nabla u|^2 \varphi^2 \, dx = -2 \int_{Q_{tr}} \varphi (\nabla u \cdot \nabla \varphi) u \, dx + \int_{Q_{tr}} F(u\varphi^2) \, dx,
\]

which, by the Cauchy inequality, yields (2.11).

Proof of Theorem 2.1. By rescaling we may assume \( \varepsilon = 1 \). Let \( u \in W^{1,2}(Q_R) \) for some \( R \geq 1 \). Suppose that \( \Delta u = 0 \) in \( Q_R \cap \omega_{1,\eta} \) and \( u = 0 \) in \( Q_R \setminus \omega_{1,\eta} \). To prove (2.2), without loss of generality, we may assume \( R \geq \delta^{-2} \), where \( \delta = \delta(d) > 0 \) is sufficiently small (the case \( 1 \leq R \leq \delta^{-2} \) is trivial). Let \( 1 \leq r \leq \delta R \). By applying the discrete Sobolev inequality (2.10) to \( g(z) = \hat{u}(z) \), we obtain

\[
\max_{z \in \mathbb{Z}^d \cap Q_{r+2}} |\hat{u}(z)| \leq \max_{z \in \mathbb{Z}^d \cap Q_{2R}} |\hat{u}(z)| \leq C \sum_{\ell=0}^N R^\ell \left( \frac{1}{R^d} \sum_{z \in \mathbb{Z}^d \cap Q_{4R}} |\partial^\ell \hat{u}(z)|^2 \right)^{1/2}, \quad (2.12)
\]

where \( N = [d/2] + 1 \) and we have used (2.9) for the last inequality.

Next, let \( v = \Delta^\gamma u \), where \( \gamma \) is a multi-index with \( 1 \leq |\gamma| = \ell \leq d \). Observe that \( \Delta v = 0 \) in \( Q_{R-3\ell} \cap \omega_{1,\eta} \) and \( v = 0 \) in \( Q_{R-3\ell} \setminus \omega_{1,\eta} \). By applying Lemma 2.4 to \( v \) we see that

\[
\left( \int_{Q_{\rho}} |\nabla \partial^\ell u|^2 \right)^{1/2} \leq C \frac{\rho}{\rho} \left( \int_{Q_{2\rho}} |\partial^\ell u|^2 \right)^{1/2} \leq C \frac{\rho}{\rho^\ell} \left( \int_{Q_{2\rho+3}} |\nabla \partial^{\ell-1} u|^2 \right)^{1/2}, \quad (2.13)
\]

for \( 1 \leq \rho \leq (R - 3d)/2 \), where, for the last step, we have used the inequality

\[
\left( \int_{y+Q_1} |\Delta_j u|^2 \right)^{1/2} \leq C \left( \int_{y+3Q_1} |\nabla u|^2 \right)^{1/2}. \quad (2.14)
\]

By induction it follows that

\[
\left( \int_{Q_{\rho}} |\nabla \partial^\ell u|^2 \right)^{1/2} \leq C \frac{\rho^\ell}{\rho} \left( \int_{Q_{C\rho}} |\nabla u|^2 \right)^{1/2}, \quad (2.14)
\]

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where \(0 \leq \ell \leq d\) and \(C\) depends only on \(d\). This, together with Lemma 2.2 and (2.12), shows that for any \(1 \leq r \leq \delta R\),

\[
\left(\int_{Q_r} |u|^2\right)^{1/2} \leq C \left(\int_{Q_R} |u|^2\right)^{1/2} + CR \left(\int_{Q_{R/2}} |\nabla u|^2\right)^{1/2} + C \left(\int_{Q_{3r}} |\nabla u|^2\right)^{1/2} \\
\leq C \left(\int_{Q_R} |u|^2\right)^{1/2} + \frac{C}{r} \left(\int_{Q_{4r}} |u|^2\right)^{1/2}.
\]

It follows that

\[
\sup_{s \leq r \leq R} \left(\int_{Q_r} |u|^2\right)^{1/2} \leq C \left(\int_{Q_R} |u|^2\right)^{1/2} + \frac{C}{s} \sup_{s \leq r \leq R} \left(\int_{Q_r} |u|^2\right)^{1/2},
\]

for any \(1 < s \leq R\), where \(C\) depends only on \(d\). By choosing \(s = s(d) > 1\) sufficiently large we obtain

\[
\sup_{s \leq r \leq R} \left(\int_{Q_r} |u|^2\right)^{1/2} \leq C \left(\int_{Q_R} |u|^2\right)^{1/2}.
\]

Finally, we note that

\[
\sup_{1 \leq r \leq s} \left(\int_{Q_r} |u|^2\right)^{1/2} \leq C_s \left(\int_{Q_s} |u|^2\right)^{1/2} \leq C \left(\int_{Q_R} |u|^2\right)^{1/2},
\]

where we have used (2.15) for the last step. This, together with (2.15), gives (2.2). \(\square\)

### 3 Large-scale Lipschitz estimates

In this section we establish a large-scale Lipschitz estimate. Recall that \(Q_R = (-R/2, R/2)^d\). As in the last section, we assume \(\omega_{\varepsilon, \eta}\) is given by (1.5), where \(T\) is the closure of an open subset of \(Y\) with Lipschitz boundary.

**Theorem 3.1.** Let \(u \in W^{1,2}(Q_R)\) for some \(R \geq \varepsilon\). Suppose that

\[
\Delta u = 0 \quad \text{in} \quad Q_R \cap \omega_{\varepsilon, \eta} \quad \text{and} \quad u = 0 \quad \text{in} \quad Q_R \setminus \omega_{\varepsilon, \eta}.
\]

Then, for \(\varepsilon \leq r \leq R\),

\[
\left(\int_{Q_r} |\nabla u|^2\right)^{1/2} \leq C \left(\int_{Q_R} |\nabla u|^2\right)^{1/2},
\]

where \(C\) depends on \(d\) and \(c_0\) in (1.4).\(\square\)

We begin with a Poincaré inequality.

**Lemma 3.2.** Suppose that \(u \in W^{1,2}(Q_R)\) and \(u = 0\) on \(Q_R \setminus \omega_{1, \eta}\), where \(R \geq 1\) is an odd integer. Then

\[
\int_{Q_R} |u|^2 \, dx \leq C_\eta^{2-d} \int_{Q_R} |\nabla u|^2 \, dx
\]

for \(d \geq 3\), and

\[
\int_{Q_R} |u|^2 \, dx \leq C |\ln(\eta/2)| \int_{Q_R} |\nabla u|^2 \, dx
\]

for \(d = 2\), where \(C\) depends only on \(d\) and \(c_0\).
Proof. The case $R = 1$ is well known. See e.g. [2], p.270 or [10], Lemma 2.1] for a proof. The
general case follows by covering $Q_R$ with unit cubes $\{k + Q_1 : k \in \mathbb{Z}^d \cap Q_R\}$. \hfill \square

Following [6], we introduce a $Y$-periodic function $\chi_\eta$ in $\mathbb{R}^d$ that satisfies

$$- \Delta \chi_\eta = \eta^{d-2} \text{ in } \omega_{\epsilon, \eta} \text{ and } \chi_\eta = 0 \text{ in } \mathbb{R}^d \setminus \omega_{\epsilon, \eta}.$$  

(3.5)

Let $H^1_{\text{per}}(Q_1)$ denote the closure in $H^1(Q_1)$ of the set of smooth $Y$-periodic functions in $\mathbb{R}^d$. The existence and uniqueness of $\chi_\eta$ may be proved by using the Lax-Milgram Theorem on a Hilbert

space $H$, given by the closure of $\{u \in H^1_{\text{per}}(Q_1) : u = 0 \text{ on } \eta T\}$ in $H^1_{\text{per}}(Q_1)$.

Lemma 3.3. Let $\chi_\eta$ be given by (3.5). Then

$$C_1 \eta^{\frac{d-2}{2}} \leq \left( \int_{Q_1} |\nabla \chi_\eta|^2 \right)^{1/2} \leq C_2 \eta^{\frac{d-2}{2}},$$  

(3.6)

for $d \geq 3$, and

$$C_1 |\ln(\eta/2)|^{\frac{1}{2}} \leq \left( \int_{Q_1} |\nabla \chi_\eta|^2 \right)^{1/2} \leq C_2 |\ln(\eta/2)|^{\frac{1}{2}},$$  

(3.7)

for $d = 2$, where $C_1, C_2 > 0$ depend only on $d$ and $c_0$.

Proof. See [6] or [10], Lemma 4.4. \hfill \square

Lemma 3.4. Let $u \in W^{1,2}(Q_R)$ for some $R \geq 100d$. Then, for any $r \in [1, R/100]$,

$$\left( \int_{Q_r} |u - \hat{u}(0)|^2 \right)^{1/2} \leq Cr \sum_{\ell=0}^N R^\ell \left( \int_{Q_{R/2}} |\nabla \partial^\ell u|^2 \right)^{1/2} + C \left( \int_{Q_{3r}} |\nabla u|^2 \right)^{1/2},$$  

(3.8)

where $N = [d/2] + 1$ and $C$ depends only on $d$.

Proof. In view of Lemma 2.2, it suffices to show that if $z \in \mathbb{Z}^d \cap Q_{3r}$, then $|\hat{u}(z) - \hat{u}(0)|$ is bounded by the first term in the right-hand side of (3.8). To this end we use the observation,

$$\max_{z \in \mathbb{Z}^d \cap Q_{3r}} |\hat{u}(z) - \hat{u}(0)| \leq C r \max_{z \in \mathbb{Z}^d \cap Q_{3r}} |\partial \hat{u}(z)|.$$

By applying the discrete Sobolev inequality (2.10) with $g(z) = \partial \hat{u}(z)$, we obtain

$$\max_{z \in \mathbb{Z}^d \cap Q_{3r}} |\hat{u}(z) - \hat{u}(0)| \leq C r \sum_{\ell=0}^N R^\ell \left( \frac{1}{R^d} \sum_{z \in \mathbb{Z}^d \cap Q_{R/4}} |\partial^{\ell+1} \hat{u}(z)|^2 \right)^{1/2}$$

$$\leq C r \sum_{\ell=0}^N R^\ell \left( \int_{Q_{R/2}} |\nabla \partial^\ell u|^2 \right)^{1/2},$$

where we have used (2.9) for the last inequality. \hfill \square

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We are now in a position to give the proof of Theorem 3.1.

**Proof of Theorem 3.1.** By rescaling we may assume \( \varepsilon = 1 \). Let \( u \in W^{1, 2}(Q_R) \) be a solution of (3.1) for some \( R \geq 1 \). To prove (3.2), we may assume that \( R \geq \delta^{-2} d \), where \( \delta = \delta(d) > 0 \) is sufficiently small. We may also assume that \( R \) is an odd integer (the general case follows by choosing an odd integer \( \ell \) so that \( R/2 \leq \ell \leq R \)).

Let \( w = u - \alpha \chi_{\eta} \), where \( \chi_{\eta} \) is given by (3.5) and \( \alpha \in \mathbb{R} \) is chosen so that \( \dot{w}(0) = 0 \). Since \( \dot{u}(0) = \alpha \dot{\chi}_{\eta}(0) \), by Lemma 3.3 we have

\[
|\alpha| \leq \begin{cases} 
C |\dot{u}(0)| & \text{for } d \geq 3, \\
C |\ln(\eta/2)|^{-1} |\dot{u}(0)| & \text{for } d = 2,
\end{cases}
\]

(3.9)

where \( C \) depends only on \( d \) and \( c_0 \). Let \( r \in [1, \delta R] \). By applying Lemma 3.4 to \( w \) we obtain

\[
\left( \int_{Q_r} |w|^2 \right)^{1/2} \leq C r \sum_{\ell=0}^N R^{\ell} \left( \int_{Q_{100dR}} |\nabla \partial^\ell w|^2 \right)^{1/2} + C \left( \int_{Q_{2r}} |w|^2 \right)^{1/2}.
\]

(3.10)

Next, we note that \( -\Delta w = -\alpha \eta \partial^d w \) in \( Q_R \cap \omega_{1, \eta} \) and \( w = 0 \) on \( Q_R \setminus \omega_{1, \eta} \). It follows by the Caccioppoli inequality in Lemma 2.4 that

\[
\left( \int_{Q_\rho} |\nabla w|^2 \right)^{1/2} \leq C \rho \left( \int_{Q_{2\rho}} |w|^2 \right)^{1/2} + C |\alpha| \eta^{d-2} \rho
\]

(3.11)

for \( 1 \leq \rho \leq R/2 \). Also, observe that if \( \ell \geq 1 \), then \( \Delta(\partial^\ell w) = 0 \) in \( Q_R \setminus 3\ell \cap \omega_{1, \eta} \) and \( \partial^\ell w = 0 \) on \( Q_{R-3\ell} \setminus \omega_{1, \eta} \). Hence, by the proof of (2.14),

\[
\left( \int_{Q_\rho} |\nabla \partial^\ell w|^2 \right)^{1/2} \leq C \rho \left( \int_{Q_{2\rho}} |\nabla w|^2 \right)^{1/2}
\]

(3.12)

for \( 1 \leq \ell \leq d \) and \( 1 \leq \rho \leq \delta R \), where \( C \) depends only on \( d \). It follows from (3.11) and (3.10) that for \( 1 \leq r \leq \delta R/2 \),

\[
\left( \int_{Q_r} |\nabla w|^2 \right)^{1/2} \leq C \sum_{\ell=0}^N R^{\ell} \left( \int_{Q_{100dR}} |\nabla \partial^\ell w|^2 \right)^{1/2} + C \left( \int_{Q_{6r}} |\nabla w|^2 \right)^{1/2} + C |\alpha| \eta^{d-2} r
\]

\[
\leq C \left( \int_{Q_R} |\nabla w|^2 \right)^{1/2} + C \frac{\rho}{r} \left( \int_{Q_{6r}} |\nabla w|^2 \right)^{1/2} + C |\alpha| \eta^{d-2} R,
\]

where \( C \) depends only on \( d \) and we have used (3.12) for the last inequality. Since \( w = u - \alpha \chi_{\eta} \), this yields,

\[
\left( \int_{Q_r} |\nabla u|^2 \right)^{1/2} \leq C \left( \int_{Q_R} |\nabla u|^2 \right)^{1/2} + C \frac{\rho}{r} \left( \int_{Q_{6r}} |\nabla u|^2 \right)^{1/2}
\]

(3.13)

\[
+ C |\alpha| \left( \int_{Q_1} |\nabla \chi_{\eta}|^2 \right)^{1/2} + C |\alpha| \eta^{d-2} R,
\]

for any \( 1 \leq r \leq R/6 \) (the case \( \delta R/2 \leq r \leq R/6 \) is trivial). We point out that the periodicity of \( \nabla \chi_{\eta} \) is also used for (3.13).
Suppose $d \geq 3$. We consider two cases. If $R \leq \eta^{\frac{2-d}{2}}$, in view of (3.3) and (3.9), we have
\[
|\alpha| \left( \int_{Q_1} |\nabla \chi_n|^2 \right)^{1/2} + |\alpha|\eta^{d-2}R \leq C\eta^{\frac{d-2}{2}} |\tilde{u}(0)| \leq C\eta^{\frac{d-2}{2}} \left( \int_{Q_1} |u|^2 \right)^{1/2}
\leq C\eta^{\frac{d-2}{2}} \left( \int_{QR} |u|^2 \right)^{1/2}
\leq C \left( \int_{QR} |\nabla u|^2 \right)^{1/2},
\]
where we have used the large-scale $L^\infty$ estimate (2.2) for the third inequality and the Poincaré inequality (3.3) for the last step. The assumption that $R$ is an odd integer is also used here. This, together with (3.13), gives
\[
\left( \int_{QR} |\nabla u|^2 \right)^{1/2} \leq C \left( \int_{QR} |\nabla u|^2 \right)^{1/2} + C \left( \int_{Q_{\eta R}} |\nabla u|^2 \right)^{1/2}
\]
for any $r \in [1, R/6]$, where $C$ depends only on $d$ and $c_0$. As in the proof of Theorem 2.1, the large-scale Lipschitz estimate (3.2) with $R \leq \eta^{\frac{2-d}{2}}$ follows readily from (3.14).

Suppose $d \geq 3$ and $R > \eta^{\frac{2-d}{2}}$. Let $\eta^{\frac{2-d}{2}}/2 \leq r \leq R/2$. We use the Caccioppoli inequality (2.11) to obtain
\[
\left( \int_{Q_r} |\nabla u|^2 \right)^{1/2} \leq C \left( \int_{Q_{2r}} |u|^2 \right)^{1/2} \leq C\eta^{\frac{d-2}{2}} \left( \int_{Q_{2r}} |u|^2 \right)^{1/2}
\leq C\eta^{\frac{d-2}{2}} \left( \int_{QR} |u|^2 \right)^{1/2} \leq C \left( \int_{QR} |\nabla u|^2 \right)^{1/2},
\]
where we have used the large-scale $L^\infty$ estimate (2.2) for the third inequality and the Poincaré inequality for the last step. As a result, we deduce that if $R > R_\eta = \eta^{\frac{2-d}{2}}$,
\[
\sup_{1 \leq r \leq R} \left( \int_{Q_r} |\nabla u|^2 \right)^{1/2} \leq \sup_{1 \leq r \leq R_\eta} \left( \int_{Q_r} |\nabla u|^2 \right)^{1/2} + \sup_{R_\eta < r \leq R} \left( \int_{Q_r} |\nabla u|^2 \right)^{1/2}
\leq C \sup_{R_\eta \leq r \leq R} \left( \int_{Q_r} |\nabla u|^2 \right)^{1/2}
\leq C \left( \int_{QR} |\nabla u|^2 \right)^{1/2},
\]
where we have used the large-scale Lipschitz estimate for the case $R = R_\eta$ for the second inequality.

The proof for the case $d = 2$ is similar. Again, we consider two cases. If $R \leq |\ln(\eta/2)|^{1/2}$, in view of (3.7) and (3.9), we have
\[
|\alpha| \left( \int_{Q_1} |\nabla \chi_n|^2 \right)^{1/2} + |\alpha|\eta^{d-2}R \leq C|\ln(\eta/2)|^{\frac{1}{2}} |\tilde{u}(0)| \leq C|\ln(\eta/2)|^{\frac{1}{2}} \left( \int_{Q_1} |u|^2 \right)^{1/2}
\leq C|\ln(\eta/2)|^{\frac{1}{2}} \left( \int_{QR} |u|^2 \right)^{1/2}
\leq C \left( \int_{QR} |\nabla u|^2 \right)^{1/2},
\]
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where we have used the large-scale $L^\infty$ estimate (2.11) for the third inequality and the Poincaré inequality (3.4) for the last step. As in the case $d \geq 3$, this gives (3.14), which leads to (3.2). If $R > R_\eta = |\ln(\eta/2)|^{1/2}$ and $R_\eta/2 \leq r \leq R/2$, we use the Caccioppoli inequality (2.11) and large-scale $L^\infty$ estimate (2.12) to obtain

\[
\left( \int_{Q_r} |\nabla u|^2 \right)^{1/2} \leq C \sup_{Q_{2r}} \left( \int_{Q_{2r}} |u|^2 \right)^{1/2} \leq C |\ln(\eta/2)|^{1/2} \left( \int_{Q_{2r}} |u|^2 \right)^{1/2} \leq C |\ln(\eta/2)|^{1/2} \left( \int_{Q_R} |u|^2 \right)^{1/2} \leq C \left( \int_{Q_R} |\nabla u|^2 \right)^{1/2},
\]

where we have used the Poincaré inequality (3.4) for the last step. This, combined with the estimate for the case $R \leq R_\eta$, yields (3.2) and completes the proof of Theorem 3.1.

\[d \geq 3, \quad \text{and} \quad (4.2)\]

\[\int_{\omega_{\varepsilon, \eta}} |u|^2 \, dx \leq C \varepsilon^2 |\ln(\eta/2)| \int_{\omega_{\varepsilon, \eta}} |\nabla u|^2 \, dx\]

for $d = 2$, where $\varepsilon, \eta \in (0, 1]$ and $C$ depends only on $d$ and $c_0$. Let $W^{1,p}(\omega_{\varepsilon, \eta})$ denote the closure of $C_0^\infty(\omega_{\varepsilon, \eta})$ in $W^{1,p}(\omega_{\varepsilon, \eta})$. It follows from (4.1)-(4.2) by the Lax-Milgram Theorem that for any $F \in L^2(\omega_{\varepsilon, \eta})$ and $f \in L^2(\omega_{\varepsilon, \eta}; \mathbb{R}^d)$, the Dirichlet problem (1.1) has a unique solution in $W^{1,2}(\omega_{\varepsilon, \eta})$. Moreover, the solution satisfies

\[\|\nabla u\|_{L^2(\omega_{\varepsilon, \eta})} \leq \|f\|_{L^2(\omega_{\varepsilon, \eta})} + C \varepsilon \eta^{1-\frac{d}{2}} \|F\|_{L^2(\omega_{\varepsilon, \eta})}\]

for $d \geq 3$, and

\[\|\nabla u\|_{L^2(\omega_{\varepsilon, \eta})} \leq \|f\|_{L^2(\omega_{\varepsilon, \eta})} + C \varepsilon |\ln(\eta/2)|^{1/2} \|F\|_{L^2(\omega_{\varepsilon, \eta})}\]

for $d = 2$. The constants $C$ in (4.3)-(4.4) depend only on $d$ and $c_0$.

Choose $u \in W^{1,2}(\omega_{\varepsilon, \eta})$ be a weak solution of (4.1). Define

\[S_{\varepsilon, \eta}(F, f)(x) = \left( \int_{x+\varepsilon Q_2} |\nabla u|^2 \right)^{1/2},\]

where we have extended $u$ to $\mathbb{R}^d$ by zero. It is easy to see that

\[\|S_{\varepsilon, \eta}(F, f)\|_{L^p(\mathbb{R}^d)} = \|\nabla u\|_{L^p(\mathbb{R}^d)},\]

The following theorem gives the $L^p$ boundedness of $S_{\varepsilon, \eta}$ for $p \geq 2$.

**Theorem 4.1.** Let $2 \leq p < \infty$ and $\omega_{\varepsilon, \eta}$ be given by (1.5), where $T$ is the closure of an open subset of $Y$ with Lipschitz boundary. Then, for any $f \in C_0^\infty(\mathbb{R}^d; \mathbb{R}^d)$ and $F \in C_0^\infty(\mathbb{R}^d)$,

\[\|S_{\varepsilon, \eta}(F, f)\|_{L^p(\mathbb{R}^d)} \leq \begin{cases} C \|f\|_{L^p(\mathbb{R}^d)} + C \varepsilon \eta^{1-\frac{d}{2}} \|F\|_{L^p(\mathbb{R}^d)} & \text{for } d \geq 3, \\ C \|f\|_{L^p(\mathbb{R}^d)} + C \varepsilon |\ln(\eta/2)|^{1/2} \|F\|_{L^p(\mathbb{R}^d)} & \text{for } d = 2, \end{cases}\]

where $C$ depends on $d$, $p$ and $c_0$.
The case $p = 2$ follows readily from (4.6) and (1.3)-(4.1). To prove Theorem 4.1 for $p > 2$, we use a real-variable argument and the large-scale Lipschitz estimate obtained in the last section.

An operator $S$ is called sublinear if there exists a constant $K$ such that

$$|S(f + g)| \leq K \{ |S(f)| + |S(g)| \}. \tag{4.8}$$

**Theorem 4.2.** Let $S$ be a bounded sublinear operator from $L^2(\mathbb{R}^d; \mathbb{R}^m)$ to $L^2(\mathbb{R}^d)$ with $\|S\|_{L^2 \to L^2} \leq C_0$. Let $q > 2$. Suppose that

$$\left( \int_B |S(g)|^q \right)^{1/q} \leq N \left\{ \left( \int_{2B} |S(g)|^2 \right)^{1/2} + \sup_{B' \supset B} \left( \int_B |g|^2 \right)^{1/2} \right\} \tag{4.9}$$

for any ball $B$ in $\mathbb{R}^d$ and for any $g \in C^\infty_0(\mathbb{R}^d; \mathbb{R}^m)$ with $\text{supp}(g) \subset \mathbb{R}^d \setminus 4B$. Then for any $f \in C^\infty_0(\mathbb{R}^d; \mathbb{R}^m)$,

$$\|S(f)\|_{L^p(\mathbb{R}^d)} \leq C_p \|f\|_{L^p(\mathbb{R}^d)}, \tag{4.10}$$

where $2 < p < q$ and $C_p$ depends at most on $p$, $q$, $C_0$, $N$ and $K$ in (4.8).

**Proof.** See [8] or [9, pp.79-80].

Observe that by linearity,

$$S_{\epsilon,\eta}(F,f) \leq S_{\epsilon,\eta}(F,0) + S_{\epsilon,\eta}(0,f). \tag{4.11}$$

We first treat the case $S_{\epsilon,\eta}(0,f)$.

**Lemma 4.3.** Let $2 < p < \infty$ and $S_{\epsilon,\eta}$ be defined by (4.5). Then

$$\|S_{\epsilon,\eta}(0,f)\|_{L^p(\mathbb{R}^d)} \leq C\|f\|_{L^p(\mathbb{R}^d)} \tag{4.12}$$

for any $f \in C^\infty_0(\mathbb{R}^d; \mathbb{R}^d)$, where $C$ depends only on $d$, $p$ and $c_0$ in (1.4).

**Proof.** By rescaling we may assume $\epsilon = 1$. Let $S(f) = S_{1,\eta}(0,f)$. Note that $S$ satisfies (4.8) with $K = 1$ and that $\|S\|_{L^2 \to L^2} \leq 1$. Let $Q$ be a cube in $\mathbb{R}^d$. We will show that if $g \in C^\infty_0(\mathbb{R}^d; \mathbb{R}^d)$ with $\text{supp}(g) \subset \mathbb{R}^d \setminus 4Q$, then

$$\|S(g)\|_{L^\infty(Q)} \leq C \left( \int_{2Q} |S(g)|^2 \right)^{1/2}, \tag{4.13}$$

where $C$ depends only on $d$ and $c_0$. By covering a ball $B = B(x_0,r)$ with non-overlapping cubes of side length $c_0r$, it is not hard to deduce (4.13) from (4.11). As a result of the right-hand side of (4.9) is not needed). As a result, we obtain (4.12) for any $f \in C^\infty_0(\mathbb{R}^d; \mathbb{R}^d)$.

Let $Q = Q(x_0,\ell)$ be a cube centered at $x_0$ and with side length $\ell$. Suppose that $-\Delta u = \text{div}(g)$ in $\omega_{1,\eta}$ and $u = 0$ in $\mathbb{R}^d \setminus \omega_{1,\eta}$, where $g \in C^\infty(\mathbb{R}^d; \mathbb{R}^d)$ and $\text{supp}(g) \subset \mathbb{R}^d \setminus 4Q$. To show (4.13), we use Theorem 2.1 as well as the observation,

$$\left( \int_{2Q} |S(g)|^2 \right)^{1/2} = \left( \frac{1}{(2\ell)^d} \int_{Q(x_0,2+2\ell)} |\nabla u(y)|^2 |Q(y,2) \cap Q(x_0,2\ell)| dy \right)^{1/2}. \tag{4.14}$$

We consider two cases. In the first case we assume $0 < \ell \leq 2$. Note that

$$S(g)(x) \leq \left( \int_{Q(x_0,2+2\ell)} |\nabla u(y)|^2 dy \right)^{1/2}. \tag{4.15}$$
for any \( x \in Q(x_0, \ell) \). Since \( |Q(y, 2) \cap Q(x_0, 2\ell)| \geq c \ell^d \) for \( y \in Q(x_0, 2 + \ell) \), we obtain (4.13) from (4.14) and (4.15), with \( C \) depending only on \( d \).

In the second case we assume \( \ell > 2 \). Note that \( \Delta u = 0 \) in \( \omega_{1, \eta} \cap Q(x_0, 4\ell) \) and \( u = 0 \) in \( \mathbb{R}^d \setminus \omega_{1, \eta} \). It follows by Theorem 3.1 that

\[
\left( \int_{Q(x, \ell)} |\nabla u|^2 \right)^{1/2} \leq C \left( \int_{Q(x, \ell)} |\nabla u|^2 \right)^{1/2}
\]

for any \( x \in Q(x_0, \ell) \), where \( C \) depends only on \( d \) and \( c_0 \). Hence, for any \( x \in Q(x_0, \ell) \),

\[
S(g)(x) \leq C \left( \int_{Q(x, \ell)} |\nabla u|^2 \right)^{1/2}
\leq C \left( \int_{Q(x_0, 2\ell)} |\nabla u|^2 \right)^{1/2},
\]

where we have used the fact \( Q(x, \ell) \subset Q(x_0, 2\ell) \) for \( x \in Q(x_0, \ell) \). This shows that

\[
\|S(g)\|_{L^\infty(Q)} \leq C \left( \int_{Q(x_0, 2\ell)} |\nabla u|^2 \right)^{1/2}
\leq C \left( \int_{2Q} |S(g)|^2 \right)^{1/2},
\]

where, for the last inequality, we have used (4.14) and the observation that \( |Q(y, 2) \cap Q(x_0, 2\ell)| \geq c \) for any \( y \in Q(x_0, 2\ell) \). Consequently, we have proved (4.13) for any cube \( Q \).

Next, we deal with the operator \( S_{\varepsilon, \eta}(F, 0) \).

**Lemma 4.4.** Let \( 2 < p < \infty \) and \( S_{\varepsilon, \eta} \) be defined by (4.5). Then

\[
\|S_{\varepsilon, \eta}(F, 0)\|_{L^p(\omega_{1, \eta})} \leq \begin{cases} C\varepsilon \eta^{-\frac{d}{2}} \|F\|_{L^p(\omega_{1, \eta})} & \text{for } d \geq 3, \\ C\varepsilon \ln(\eta/2)^{\frac{1}{2}} \|F\|_{L^p(\omega_{1, \eta})} & \text{for } d = 2, \end{cases} (4.16)
\]

for any \( F \in C_0^\infty(\mathbb{R}^d) \), where \( C \) depends only on \( d, p \) and \( c_0 \).

**Proof.** As before, we may assume \( \varepsilon = 1 \) by rescaling. Define

\[
S(F) = \begin{cases} \eta^{\frac{d}{2}-1} S_{1, \eta}(F, 0) & \text{for } d \geq 3, \\ |\ln(\eta/2)|^{-\frac{1}{2}} S_{1, \eta}(F, 0) & \text{for } d = 2. \end{cases}
\]

Then \( S \) satisfies (4.8) with \( K = 1 \), and \( \|S\|_{L^2 \to L^2} \leq C_0 \) by (4.3) - (4.4). Let \( u \) be a weak solution of \( -\Delta u = G \) in \( \omega_{1, \eta} \) with \( u = 0 \) on \( \partial\omega_{1, \eta} \), where \( G \in C_0^\infty(\mathbb{R}^d) \) and \( \text{supp}(G) \subset \mathbb{R}^d \setminus 4Q \). Since \( \Delta u = 0 \) in \( \omega_{1, \eta} \cap 4Q \) and \( u = 0 \) in \( \mathbb{R}^d \setminus \omega_{1, \eta} \), the same argument as in the proof of Lemma 4.3 yields the estimate (4.13). As a result, by Theorem 4.2 we obtain

\[
\|S(F)\|_{L^p(\omega_{1, \eta})} \leq C \|F\|_{L^p(\omega_{1, \eta})}
\]

for any \( 2 < p < \infty \), where \( C \) depends only on \( d, p \) and \( c_0 \). This gives (4.16) with \( \varepsilon = 1 \).

**Proof of Theorem 4.1.** In view of (4.11), the estimates in (4.7) follow readily from (4.12) and (4.16). 

\[ 13 \]
5 Estimates in an exterior domain

In this section we establish \( W^{1,p} \) estimates for solutions with compact support of Laplace’s equation in the exterior domain \( \mathbb{R}^d \setminus T \), where \( T \) is the closure of a bounded \( C^1 \) domain in \( \mathbb{R}^d \) with connected boundary. We assume that \( B(0,c_0) \subset T \).

We begin with \( W^{1,p} \) estimates for Laplace’s equation in a bounded Lipschitz or \( C^1 \) domain.

**Theorem 5.1.** Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^d \). There exists \( \delta > 0 \), depending on \( d \) and the Lipschitz character of \( \Omega \), such that if

\[
\left| \frac{1}{p} - \frac{1}{2} \right| < \begin{cases} 
\frac{1}{6} + \delta & \text{for } d \geq 3, \\
\frac{1}{4} + \delta & \text{for } d = 2,
\end{cases}
\]  

(5.1)

the Dirichlet problem, \( -\Delta u = F \) in \( \Omega \) and \( u = 0 \) on \( \partial \Omega \), has a unique solution in \( W^{1,p}_0(\Omega) \) for any \( F \in W^{-1,p}(\Omega) \). Moreover, the solution satisfies the estimate,

\[
\|\nabla u\|_{L^p(\Omega)} \leq C_p\|F\|_{W^{-1,p}(\Omega)},
\]  

(5.2)

where \( C_p \) depends on \( d \), \( p \) and the Lipschitz character of \( \Omega \). Furthermore, if \( \Omega \) is a bounded \( C^1 \) domain, the results above hold for \( 1 < p < \infty \).

**Proof.** The estimate (5.2) for \( 1 < p < \infty \) is well known if \( \Omega \) is a \( C^{1,\alpha} \) domain. For Lipchitz and \( C^1 \) domains, the theorem was proved in [3]. \( \square \)

The next theorem is on the solvability of the Dirichlet problem in a weighted Sobolev space in the exterior domain \( \mathbb{R}^d \setminus T \),

\[-\Delta u = F \quad \text{in } \mathbb{R}^d \setminus T \quad \text{and} \quad u = 0 \quad \text{on } \partial T.\]  

(5.3)

We first introduce some notations. For \( 1 < p < \infty \) and \( p \neq d \), let

\[ X^{1,p}(\mathbb{R}^d \setminus T) = \left\{ u \in W^{1,p}_{loc}(\mathbb{R}^d \setminus T) : (1 + |x|)^{-1}u \in L^p(\mathbb{R}^d \setminus T) \right\}, \]  

(5.4)

with its natural norm,

\[
\|u\|_{X^{1,p}(\mathbb{R}^d \setminus T)} = \|(1 + |x|)^{-1}u\|_{L^p(\mathbb{R}^d \setminus T)} + \|\nabla u\|_{L^p(\mathbb{R}^d \setminus T)}. \]  

(5.5)

If \( p = d \), let

\[ X^{1,d}(\mathbb{R}^d \setminus T) = \left\{ u \in W^{1,d}_{loc}(\mathbb{R}^d \setminus T) : ((1 + |x|)\ln(2 + |x|))^{-1}u \in L^d(\mathbb{R}^d \setminus T) \right\}, \]  

(5.6)

and

\[
\|u\|_{X^{1,d}(\mathbb{R}^d \setminus T)} = \|((1 + |x|)\ln(2 + |x|))^{-1}u\|_{L^d(\mathbb{R}^d \setminus T)} + \|\nabla u\|_{L^d(\mathbb{R}^d \setminus T)}. \]  

(5.7)

It follows from [3] Theorem 1.1] that for \( u \in X^{1,p}(\mathbb{R}^d \setminus T) \),

\[
\inf_{r \in \mathbb{R}} \|u - r\|_{X^{1,p}(\mathbb{R}^d \setminus T)} \leq C\|\nabla u\|_{L^p(\mathbb{R}^d \setminus T)} \quad \text{if } 1 < p < d, \]  

(5.8)

\[
\inf_{\alpha \in \mathbb{R}} \|u - \alpha\|_{X^{1,p}(\mathbb{R}^d \setminus T)} \leq C\|\nabla u\|_{L^p(\mathbb{R}^d \setminus T)} \quad \text{if } d \leq p < \infty. \]  

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Moreover, the solution is given by
\[ X_0^{1,p}(\mathbb{R}^d \setminus T) = \{ u \in X^{1,p}(\mathbb{R}^d \setminus T) : u = 0 \text{ on } \partial T \}, \tag{5.9} \]
and \( X^{-1,p}(\mathbb{R}^d \setminus T) \) be the dual of \( X_0^{1,p'}(\mathbb{R}^d \setminus T) \), where \( p' = \frac{p}{p-1} \). It is known that \( C_0^\infty(\mathbb{R}^d) \) is dense in \( X^{1,p}(\mathbb{R}^d \setminus T) \) and \( C_0^\infty(\mathbb{R}^d \setminus T) \) is dense in \( X_0^{1,p}(\mathbb{R}^d \setminus T) \) \[3\].

Let
\[ V_0^p(\mathbb{R}^d \setminus T) = \{ w \in X_0^{1,p}(\mathbb{R}^d \setminus T) : \Delta w = 0 \text{ in } \mathbb{R}^d \setminus T \}. \tag{5.10} \]

**Theorem 5.2.** Let \( d \geq 2 \) and \( 2 \leq p < \infty \). Let \( T \) be the closure of a bounded \( C^1 \) domain in \( \mathbb{R}^d \) with connected boundary. Then, for any \( F \in X^{-1,p}(\mathbb{R}^d \setminus T) \), the Dirichlet problem \[5.3\] has a unique solution in \( X_0^{1,p}(\mathbb{R}^d \setminus T)/V_0^p(\mathbb{R}^d \setminus T) \). Moreover, the solution satisfies
\[ \inf_{w \in V_0^p(\mathbb{R}^d \setminus T)} \| u - w \|_{X^{1,p}(\mathbb{R}^d \setminus T)} \leq C \| F \|_{X^{-1,p}(\mathbb{R}^d \setminus T)}, \tag{5.11} \]
where \( C \) depends on \( d, p \) and \( T \).

**Proof.** This was proved in \[3\] Theorem 2.10] under the assumption that \( \partial T \) is \( C^{1,1} \). With the \( W^{1,p} \) estimates in Theorem 5.1 for bounded \( C^1 \) domains, an inspection of the proof shows that Theorem 2.10 in \[3\] continues to hold under the condition that \( \partial T \) is \( C^1 \). \qed

A few remarks are in order.

**Remark 5.3.** If \( d \geq 3 \) and \( 2 \leq p < d \), or \( d = p = 2 \), then
\[ V_0^p(\mathbb{R}^d \setminus T) = \{0\}. \tag{5.12} \]
As a result, the solution of \[5.3\] is unique in \( X_0^{1,p}(\mathbb{R}^d \setminus T) \) and satisfies
\[ \| u \|_{X^{1,p}(\mathbb{R}^d \setminus T)} \leq C \| F \|_{X^{-1,p}(\mathbb{R}^d \setminus T)}. \tag{5.13} \]

**Remark 5.4.** Suppose \( d \geq 3 \) and \( p \geq d \). Then
\[ V_0^p(\mathbb{R}^d \setminus T) = \{ \alpha \phi_* : \alpha \in \mathbb{R} \}, \tag{5.14} \]
where \( \phi_* \) is the unique solution of the exterior problem,
\[ \left\{ \begin{array}{ll} \Delta \phi_* = 0 & \text{in } \mathbb{R}^d \setminus T, \\ \phi_* = 0 & \text{on } \partial T, \\ \phi_*(x) \to 1 & \text{as } |x| \to \infty. \end{array} \right. \tag{5.15} \]
Moreover, the solution is given by
\[ \phi_*(x) = 1 - \int_{\partial T} \frac{g_*(y)}{|x-y|^{d-2}} d\sigma(y) \]
for some \( g_* \in L^2(\partial T) \) \[12, 3\]. It follows that if \( 0 \in T \),
\[ \left\{ \begin{array}{ll} \phi_*(x) = 1 - c_* |x|^{2-d} + O(|x|^{1-d}), \\ \nabla \phi_*(x) = -c_* \nabla (|x|^{2-d}) + O(|x|^{-d}), \\ \nabla^2 \phi_*(x) = O(|x|^{-d}) \end{array} \right. \tag{5.16} \]
as \( |x| \to \infty \), where
\[ c_* = \int_{\partial T} g_*(y) d\sigma(y) \neq 0. \]

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Remark 5.5. If \( d = 2 \) and \( p > 2 \), then
\[
V_0^p(\mathbb{R}^d \setminus T) = \{ \alpha \phi_s : \alpha \in \mathbb{R} \},
\] (5.17)
where \( \phi_s \) is a harmonic function in \( \mathbb{R}^2 \setminus T \) with the properties that \( \phi_s = 0 \) on \( \partial T \) and
\[
\begin{align*}
\phi_s(x) &= -c_s \ln |x| + O(|x|^{-1}), \\
\nabla \phi_s(x) &= -c_s \nabla \ln |x| + O(|x|^{-2}), \\
\nabla^2 \phi_s(x) &= O(|x|^{-2}),
\end{align*}
\] (5.18)
as \( |x| \to \infty \) [12] [3].

Theorem 5.6. Let \( d \geq 2 \) and \( 2 < p < \infty \). Let \( u \in W^{1,p}(\mathbb{R}^d \setminus T) \) be a solution of
\[
- \Delta u = F + \text{div}(f) \quad \text{in} \quad \mathbb{R}^d \setminus T \quad \text{and} \quad u = 0 \quad \text{on} \; \partial T.
\] (5.19)
Suppose that \( T \subset B(0, R) \) and \( \text{supp}(u), \text{supp}(F), \text{supp}(f) \subset B(0, R) \) for some \( R \geq 2 \). Then
\[
\| \nabla u \|_{L^p(\mathbb{R}^d \setminus T)} \leq C \Phi_p(R) \left\{ \| f \|_{L^p(\mathbb{R}^d \setminus T)} + R \| F \|_{L^p(\mathbb{R}^d \setminus T)} \right\},
\] (5.20)
where
\[
\Phi_p(R) = \begin{cases}
1 & \text{if } d \geq 3 \text{ and } 2 < p < d, \\
\left( \ln R \right)^{1 - \frac{1}{p}} & \text{if } d \geq 3 \text{ and } p = d, \\
R^{1 - \frac{3}{p}} & \text{if } d \geq 3 \text{ and } d < p < \infty, \\
R^{1 - \frac{3}{p}} \left( \ln R \right)^{-1} & \text{if } d = 2 \text{ and } 2 < p < \infty,
\end{cases}
\] (5.21)
and \( C \) depends only on \( d, p \) and \( T \).

Proof. Note that \( W^{1,p}(\mathbb{R}^d \setminus T) \subset X_0^{1,p}(\mathbb{R}^d \setminus T) \), and that for any \( \psi \in X_0^{1,p'}(\mathbb{R}^d \setminus T) \),
\[
\left| \int_{\mathbb{R}^d \setminus T} F \psi \, dx \right| \leq \| F \|_{L^p(\mathbb{R}^d)} \| \psi \|_{L^{p'}(\mathbb{R}^d)} \leq C R \| F \|_{L^p(\mathbb{R}^d)} \| \psi \|_{X^{1,p'}(\mathbb{R}^d \setminus T)},
\]
where we have used the facts that \( \text{supp}(F) \subset B(0, R) \) and \( p' \neq d \). It follows that
\[
\| F + \text{div}(f) \|_{X^{-1, p}(\mathbb{R}^d \setminus T)} \leq C \left\{ \| f \|_{L^p(\mathbb{R}^d \setminus T)} + R \| F \|_{L^p(\mathbb{R}^d \setminus T)} \right\}.
\] (5.22)
This allows us to apply Theorem 5.2 to obtain
\[
\inf_{w \in V_0^p(\mathbb{R}^d \setminus T)} \| u - w \|_{X_0^{1,p}(\mathbb{R}^d \setminus T)} \leq C \left\{ \| f \|_{L^p(\mathbb{R}^d \setminus T)} + R \| F \|_{L^p(\mathbb{R}^d \setminus T)} \right\}.
\] (5.23)

Suppose \( d \geq 3 \) and \( 2 < p < d \). Then \( V_0^p(\mathbb{R}^d \setminus T) = \{ 0 \} \). It follows from (5.23) that
\[
\| \nabla u \|_{L^p(\mathbb{R}^d \setminus T)} \leq C \left\{ \| f \|_{L^p(\mathbb{R}^d \setminus T)} + R \| F \|_{L^p(\mathbb{R}^d \setminus T)} \right\}.
\] (5.24)
Let \( d \geq 3 \) and \( d < p < \infty \). Then, by Remark 5.4 \( V_0^p(\mathbb{R}^d \setminus T) = \{ \alpha \phi_s : \alpha \in \mathbb{R} \} \), where the harmonic function \( \phi_s \) satisfies (5.16). Let
\[
\inf_{w \in V_0^p(\mathbb{R}^d \setminus T)} \| u - w \|_{X_0^{1,p}(\mathbb{R}^d \setminus T)} = \| u - \alpha_0 \phi_s \|_{X_0^{1,p}(\mathbb{R}^d \setminus T)}.
\] (5.25)
for some $\alpha_0 \in \mathbb{R}$. Since $u = 0$ in $\mathbb{R}^d \setminus B(0, R)$, it follows by (5.23) that

$$|\alpha_0| \left\| x^{-1} \phi_\epsilon \right\|_{L^p(\mathbb{R}^d \setminus B(0, R))} \leq C \left\{ \|f\|_{L^p(\mathbb{R}^d \setminus T)} + R\|F\|_{L^p(\mathbb{R}^d \setminus T)} \right\}.$$  

Since $\phi_\epsilon \sim 1$ for $|x|$ large, this yields

$$|\alpha_0| \leq CR^{1 - \frac{d}{p}} \left\{ \|f\|_{L^p(\mathbb{R}^d \setminus T)} + R\|F\|_{L^p(\mathbb{R}^d \setminus T)} \right\}.$$

Hence,

$$\|\nabla u\|_{L^p(B(0, R) \setminus T)} \leq \|\nabla (u - \alpha_0 \phi_\epsilon)\|_{L^p(\mathbb{R}^d \setminus T)} + |\alpha_0| \left\| \nabla \phi_\epsilon \right\|_{L^p(\mathbb{R}^d \setminus T)} \leq C \left\{ \|f\|_{L^p(\mathbb{R}^d \setminus T)} + R\|F\|_{L^p(\mathbb{R}^d \setminus T)} \right\} + C|\alpha_0| \quad \text{(5.26)}$$

If $d \geq 3$ and $p = d$, a similar argument shows that

$$|\alpha_0| \leq CR^{1 - \frac{d}{p}} \left( \ln R \right)^{-1} \left\{ \|f\|_{L^p(\mathbb{R}^d \setminus T)} + R\|F\|_{L^p(\mathbb{R}^d \setminus T)} \right\},$$

and

$$\|\nabla u\|_{L^p(B(0, R) \setminus T)} \leq C \left( \ln R \right)^{-1} \left\{ \|f\|_{L^p(\mathbb{R}^d \setminus T)} + R\|F\|_{L^p(\mathbb{R}^d \setminus T)} \right\}. \quad \text{(5.27)}$$

The argument above works equally well for $d = 2$ and $2 < p < \infty$. In this case, using (5.18), we obtain

$$|\alpha_0| \leq CR^{1 - \frac{d}{p}} \left( \ln R \right)^{-1} \left\{ \|f\|_{L^p(\mathbb{R}^d \setminus T)} + R\|F\|_{L^p(\mathbb{R}^d \setminus T)} \right\},$$

and

$$\|\nabla u\|_{L^p(B(0, R) \setminus T)} \leq CR^{1 - \frac{d}{p}} \left( \ln R \right)^{-1} \left\{ \|f\|_{L^p(\mathbb{R}^d \setminus T)} + R\|F\|_{L^p(\mathbb{R}^d \setminus T)} \right\}. \quad \text{(5.28)}$$

This completes the proof. \hfill \Box

**Corollary 5.7.** Let $d \geq 2$ and $2 < p < \infty$. Let $u$ be a solution of $-\Delta u = F + \text{div}(f)$ in $R\tilde{Y} \setminus T$ with $u = 0$ on $\partial T$, where $\tilde{Y} = (1 + c_0)Q_1$. Then, for $R \geq 3$,

$$\|\nabla u\|_{L^p(Q_R \setminus T)} \leq C\Phi_p(R) \left\{ \|f\|_{L^p(R\tilde{Y} \setminus T)} + R\|F\|_{L^p(R\tilde{Y} \setminus T)} + R^{\frac{d}{p} - \frac{d}{2} - 1}\|u\|_{L^2(R\tilde{Y} \setminus B(0, R/3))} \right\}, \quad \text{(5.29)}$$

where $\Phi_p(R)$ is given by (5.21) and $C$ depends only on $d$, $p$ and $T$.

**Proof.** Choose a cut-off function $\varphi \in C_0^\infty((1 + c_0/3)Q_R)$ such that $\varphi = 1$ in $Q_R$ and $|\nabla \varphi| \leq CR^{-1}$, $|\nabla^2 \varphi| \leq CR^{-2}$. Note that $u\varphi = 0$ on $\partial T$ and

$$-\Delta(u\varphi) = F\varphi + \text{div}(f \varphi) - f \cdot \nabla \varphi - 2\text{div}(u \nabla \varphi) + u\Delta \varphi$$

in $\mathbb{R}^d \setminus T$. It follows by Theorem 5.6 that

$$\|\nabla u\|_{L^p(Q_R \setminus T)} \leq \|\nabla (u\varphi)\|_{L^p(\mathbb{R}^d \setminus T)} \leq C\Phi_p(R) \left\{ \|f\|_{L^p(R\tilde{Y} \setminus T)} + R\|F\|_{L^p(R\tilde{Y} \setminus T)} + R^{-1}\|u\|_{L^p((1 + c_0/3)Q_R \setminus Q_R)} \right\}. \quad \text{(5.30)}$$

where $\Phi_p(R)$ is given by (5.21). Using interior estimates for Laplace’s equation, one may show that

$$\|u\|_{L^p((1 + c_0/3)Q_R \setminus Q_R)} \leq CR^{\frac{d}{p} - \frac{d}{2}}\|u\|_{L^2(R\tilde{Y} \setminus B(0, R/3))} + CR\|f\|_{L^p(Y \setminus T)} + CR^2\|F\|_{L^p(Y \setminus T)},$$

which, together with (5.30), yields (5.29). \hfill \Box
6 Local estimates in a cell

In this section we establish $W^{1,p}$ estimates for solutions of

$$
\begin{cases}
-\Delta u = F + \text{div}(f) & \text{in } \tilde{Y} \setminus \eta T, \\
u = 0 & \text{on } \partial(\eta T),
\end{cases}
$$

(6.1)

where $\tilde{Y} = (1 + \alpha_0)Q_1$ and $\eta \in (0,(4d)^{-1})$. Throughout the section, unless indicated otherwise, we assume that $T$ is the closure of a bounded $C^1$ subdomain of $Y$ and satisfies (1.4). Let $\Phi_p(R)$ be given by (5.21). Our goal is to prove the following.

**Theorem 6.1.** Let $2 < p < \infty$. Suppose that $u$ is a solution of (6.1) with $F \in L^p(\tilde{Y} \setminus \eta T)$ and $f \in L^p(\tilde{Y} \setminus \eta T; \mathbb{R}^d)$. Let $\alpha \in \mathbb{R}$. Then, for $d \geq 3$,

$$
\|\nabla u\|_{L^p(\tilde{Y} \setminus \eta T)} \leq C|\alpha|\eta^p + C\Phi_p(\eta^{-1}) \left( \int_{\tilde{Y} \setminus \eta T} \left( |f|^p + |F|^p \right) dx \right)^{1/p} + C\Phi_p(\eta^{-1}) \left( \int_{\tilde{Y} \setminus (0,1/3)} |u - \alpha|^2 dx \right)^{1/2},
$$

(6.2)

and for $d = 2$,

$$
\|\nabla u\|_{L^p(\tilde{Y} \setminus \eta T)} \leq C|\alpha|\eta^{p-1} \ln \eta^{-1} + C\eta^{p-1} \ln \eta^{-1} \left( \int_{\tilde{Y} \setminus \eta T} \left( |f|^p + |F|^p \right) dx \right)^{1/p} + C\eta^{p-1} \ln \eta^{-1} \left( \int_{\tilde{Y} \setminus (0,1/3)} |u - \alpha|^2 dx \right)^{1/2},
$$

(6.3)

where $C$ depends only on $d$, $p$ and $T$.

**Lemma 6.2.** Let $2 < p < \infty$. Let $u$ be the same as in Theorem 6.1. Then

$$
\|\nabla u\|_{L^p(\tilde{Y} \setminus \eta T)} \leq C\Phi_p(\eta^{-1}) \left\{ \|u\|_{L^2(\tilde{Y} \setminus (0,1/3))} + \|f\|_{L^p(\tilde{Y} \setminus \eta T)} + \|F\|_{L^p(\tilde{Y} \setminus \eta T)} \right\},
$$

(6.4)

where $\Phi_p$ is given by (5.21) and $C$ depends only on $d$, $p$ and $T$.

**Proof.** This follows readily from Corollary 5.7 by a simple rescaling argument. Indeed, suppose $-\Delta u = F + \text{div}(f)$ in $\tilde{Y} \setminus \eta T$. Let $v(x) = u(\eta x)$. Then $-\Delta v = G + \text{div}(g)$ in $R\tilde{Y} \setminus T$, where $R = \eta^{-1}$, $G(x) = \eta^2 F(\eta x)$ and $g(x) = \eta f(\eta x)$.

Note that if $u$ is a solution of (6.1) and $\alpha \neq 0$, then $u - \alpha$ is not a solution of (6.1) since it does not satisfy the boundary condition on $\partial(\eta T)$. To prove Theorem 6.1, we need to construct a corrector $\psi_\eta$ such that $\psi_\eta = 0$ on $\partial T$ and $\psi_\eta = 1$ on $(1 + \alpha_0)Y \setminus B(0,1/3)$.

Let $d \geq 3$. Let $\phi_*$ be defined by (5.15). For each $\eta \in (0,1/(4d))$, we introduce a function $\psi_\eta$ in $Y$, defined by

$$
\psi_\eta(x) = \begin{cases}
1 & \text{if } x \in Y \setminus B(0,1/3), \\
\phi_*(x/\eta) & \text{if } x \in B(0,1/4) \setminus \eta T, \\
0 & \text{if } x \in \eta T,
\end{cases}
$$

(6.5)
and \( \psi_\eta \) is the harmonic function in \( B(0,1/3) \setminus B(0,1/4) \) such that \( \psi_\eta = 1 \) on \( \partial B(0,1/3) \) and \( \psi_\eta(x) = \phi_\eta(x/\eta) \) on \( \partial B(0,1/4) \). In the case \( d = 2 \), we define \( \psi_\eta \) by

\[
\psi_\eta(x) = \begin{cases} 
1 & \text{if } x \in Y \setminus B(0,1/3), \\
\ln |x| - \ln(\eta x) & \text{if } x \in B(0,1/3) \setminus B(0,\eta), \\
0 & \text{if } x \in B(0,\eta).
\end{cases}
\]

(6.6)

Since \( \psi_\eta = 1 \) on \( \partial Y \), we may extend \( \psi_\eta \) to \( \mathbb{R}^d \) periodically. Thus, \( \psi_\eta \) is \( Y \)-periodic, i.e., \( \psi_\eta(x+k) = \psi_\eta(x) \) for any \( x \in \mathbb{R}^d \) and \( k \in \mathbb{Z}^d \). Note that \( 0 \leq \psi_\eta \leq 1 \) for \( d = 2 \). By the maximum principle, the same is true for \( d \geq 3 \).

**Lemma 6.3.** Let \( \psi_\eta \) be defined by (6.5)–(6.6). If \( d \geq 3 \),

\[
\left( \int_Y |\nabla \psi_\eta|^p \, dx \right)^{1/p} \approx \begin{cases} 
\eta^{d-1} & \text{if } d' < p < \infty, \\
\eta^{d-2} |\ln \eta|^\frac{1}{p} & \text{if } p = d', \\
\eta^{d-2} & \text{if } 1 < p < d',
\end{cases}
\]

(6.7)

where \( d' = \frac{d}{d-1} \). If \( d = 2 \), we have \( \|\nabla \psi_\eta\|_{L^p(Y)} \approx \eta^{\frac{1}{2}d-1} |\ln \eta|^{-1} \) for \( 2 < p < \infty \), \( \|\nabla \psi_\eta\|_{L^p(Y)} \approx |\ln \eta|^{-1/2} \) for \( p = 2 \), and \( \|\nabla \psi_\eta\|_{L^p(Y)} \approx |\ln \eta|^{-1} \) for \( 1 < p < 2 \).

**Proof.** The case \( d = 2 \) follows by a direct calculation. Consider the case \( d \geq 3 \). Since \( \psi_\eta(x) = \phi_\eta(x/\eta) \) in \( B(0,1/4) \setminus \eta T \), we have

\[
\int_{B(0,1/4) \setminus \eta T} |\nabla \psi_\eta|^p \, dx = \eta^{d-p} \int_{B(0,(4\eta)^{-1}) \setminus T} |\nabla \phi_\eta|^p \, dx \\
\approx \begin{cases} 
\eta^{d-p} |\ln \eta| & \text{if } p = d', \\
\eta^{d-2} & \text{if } 1 < p < d',
\end{cases}
\]

(6.8)

where we have used (5.16). We also used the fact that \( |\nabla \phi_\eta| \in L^p(2T \setminus T) \) for any \( 1 < p < \infty \), under the assumption that \( \partial T \) is \( C^1 \).

To bound \( \nabla \psi_\eta \) on \( B(0,1/3) \setminus B(0,1/4) \), we observe that \( w = \psi_\eta - 1 \) is harmonic in \( B(0,1/3) \setminus B(0,1/4) \) and \( w = 0 \) on \( \partial B(0,1/3) \), \( w = \phi_\eta(x/\eta) - 1 \) on \( \partial B(0,1/4) \). By (5.16) and regularity estimates for harmonic functions, we obtain \( |\nabla \psi_\eta| = |\nabla w| \leq C\eta^{-d+2} \) in \( B(0,1/3) \setminus B(0,1/4) \). This, together with (6.8), gives (6.7).

**Lemma 6.4.** Let \( \psi_\eta \) be defined by (6.5)–(6.6) and extended periodically to \( \mathbb{R}^d \). Then

\[
\begin{aligned}
-\Delta \psi_\eta &= F_\eta + \text{div}(f_\eta) & \text{in } \omega_{1,\eta}, \\
\psi_\eta &= 0 & \text{in } \mathbb{R}^d \setminus \omega_{1,\eta},
\end{aligned}
\]

(6.9)

where \( F_\eta \) and \( f_\eta \) are \( Y \)-periodic functions satisfying

\[
|F_\eta| + |f_\eta| \leq C\eta^{-d+2} \quad \text{in } Y \setminus \eta T
\]

(6.10)

for \( d \geq 3 \), and

\[
|F_\eta| + |f_\eta| \leq C|\ln \eta|^{-1} \quad \text{in } Y \setminus \eta T
\]

(6.11)

for \( d = 2 \). The constant \( C \) depends only on \( d \) and \( T \).
Proof. We first consider the case $d \geq 3$. Let $\varphi$ be a $Y$-periodic $C^\infty$ function in $\mathbb{R}^d$ such that $\varphi = 0$ in $\mathbb{R}^d \setminus \omega_{1,\eta}$. We need to show that

$$
\int_Y \nabla \psi_\eta \cdot \nabla \varphi \, dx = \int_Y F_\eta \varphi \, dx - \int_Y f_\eta \cdot \nabla \varphi \, dx
$$

for some $F_\eta$ and $\eta_\eta$ satisfying (6.10). To this end, observe that

$$
\int_Y \nabla \psi_\eta \cdot \nabla \varphi \, dx = \int_{B(0,1/3) \setminus B(0,1/4)} \nabla \psi_\eta \cdot \nabla \varphi \, dx + \int_{B(0,1/4) \setminus \eta T} \nabla \psi_\eta \cdot \nabla \varphi \, dx
$$

$$
= I_1 + I_2.
$$

For $I_1$, recall that

$$
|\nabla \psi_\eta(x)| \leq C \eta^{d-2} \quad \text{for } x \in B(0,1/3) \setminus B(0,1/4).
$$

(6.12)

To handle $I_2$, using $\varphi = 0$ on $\partial(\eta T)$, we may write

$$
I_2 = \int_{\partial B(0,1/4)} \frac{\partial \psi_\eta}{\partial n} \varphi \, d\sigma,
$$

(6.13)

where we also used the fact that $\psi_\eta$ is harmonic in $B(0,1/4) \setminus \eta T$. Let

$$
g = \frac{\partial \psi_\eta}{\partial n} = \eta^{-1} n \cdot \nabla \phi_*(x/\eta)
$$
on $\partial B(0,1/4)$. By (5.16), $|g| + |\nabla g| \leq C \eta^{d-2}$. Hence, there exists $G \in C^1(B(0,1/4))$ such that $G = g$ on $\partial B(0,1/4)$ and $|G| + |\nabla G| \leq C \eta^{d-2}$ in $B(0,1/4)$. It follows that

$$
\int_{\partial B(0,r)} g \varphi \, d\sigma = \frac{1}{r} \int_{B(0,r)} \{dG + x \cdot \nabla G\} \varphi \, dx + \frac{1}{r} \int_{B(0,r)} G(x \cdot \nabla \varphi) \, dx,
$$

where $r = (1/4)$. This, together with (6.12), yields (6.9) and (6.10).

The proof for the case $d = 2$ is similar. Indeed, note that

$$
\int_Y \nabla \psi_\eta \cdot \nabla \varphi \, dx = \int_{B(0,1/3) \setminus B(0,d\eta)} \nabla \psi_\eta \cdot \nabla \varphi \, dx
$$

$$
= \int_{\partial B(0,1/3)} \frac{\partial \psi_\eta}{\partial n} \varphi \, d\sigma
$$

$$
= \frac{3}{\ln(1/3) - \ln(d\eta)} \int_{\partial B(0,1/3)} \varphi \, d\sigma
$$

$$
= \frac{9}{\ln(1/3) - \ln(d\eta)} \int_{B(0,1/3)} (2 \varphi + x \cdot \nabla \varphi) \, dx,
$$

which yields the estimate (6.11).

We are now in a position to give the proof of Theorem 6.1.

Proof of Theorem 6.1. Let $u$ be a solution of (6.1). Let $\psi_\eta$ be defined by (6.5)–(6.6). Note that for any $\alpha \in \mathbb{R}$, we have $u - \alpha \psi_\eta = 0$ on $\partial(\eta T)$ and

$$
-\Delta (u - \alpha \psi_\eta) = (F - \alpha F_\eta) + \text{div}(f - \alpha f_\eta)
$$
in $\tilde{Y} \setminus \eta T$. It follows by Lemma 6.2 that

$$
\left( \int_{\tilde{Y} \setminus \eta T} |\nabla u|^p \, dx \right)^{1/p} \leq |\alpha| \left( \int_{\tilde{Y} \setminus \eta T} |
abla \psi_\eta|^p \, dx \right)^{1/p} + C\Phi_p(\eta^{-1}) |\alpha| (\|F\|_{L^p} + \|f\|_{L^p})
$$

(6.14)

$$
+ C\Phi_p(\eta^{-1}) \left( \int_{\tilde{Y} \setminus \eta T} (|F|^p + |f|^p) \, dx \right)^{1/p}
$$

$$
+ C\Phi_p(\eta^{-1}) \left( \int_{\tilde{Y} \setminus B(0,1/3)} |u - \alpha|^2 \, dx \right)^{1/2}
$$

where we have used the fact $\psi_\eta = 1$ in $\tilde{Y} \setminus B(0,1/3)$. By Lemmas 6.3 and 6.4 if $d \geq 3$, the first two terms in the right-hand side of (6.14) are bounded by

$$
C|\alpha| \eta^{-d/p} + C|\alpha| \Phi_p(\eta^{-1}) \eta^{-d-2} \leq C|\alpha| \eta^{-d/p-1}.
$$

This, together with (6.14), gives (6.2). Similarly, if $d = 2$, the first two terms in the right-hand side of (6.14) are bounded by

$$
C|\alpha| \eta^{2/p-1} |\ln \eta|^{-1},
$$

which yields (6.3). \hfill \Box

7 Proofs of Theorems 1.1 and 1.2

We begin with an estimate for $\|u\|_{L^p(\omega, \eta)}$.

Lemma 7.1. Let $1 < p < \infty$. For any $F \in L^p(\omega, \eta)$ and $f \in L^p(\omega, \eta; \mathbb{R}^d)$, the Dirichlet problem (7.1) has a unique solution in $W^{1,p}_0(\Omega, \eta)$. Moreover, if $2 \leq p < \infty$, the solution satisfies

$$
\|u\|_{L^p(\omega, \eta)} \leq C \left\{ \varepsilon^2 \eta^{2-d} \|F\|_{L^p(\omega, \eta)} + \varepsilon \eta^{1-\frac{d}{2}} \|f\|_{L^p(\omega, \eta)} \right\}
$$

(7.1)

for $d \geq 3$, and

$$
\|u\|_{L^p(\omega, \eta)} \leq C \left\{ \varepsilon^2 |\ln(\eta/2)| \|F\|_{L^p(\omega, \eta)} + \varepsilon |\ln(\eta/2)|^{1/2} \|f\|_{L^p(\omega, \eta)} \right\}
$$

(7.2)

for $d = 2$. The constant $C$ depends only on $d$, $p$ and $c_0$.

Proof. The existence and uniqueness of the solution are known [7] [10]. The estimates (7.1) - (7.2) for $2 \leq p < \infty$ were proved in [10, Theorem 3.3] in a general non-periodic setting. In particular, the $C^1$ assumption on $T$ is not needed. \hfill \Box

Next, we consider the case $F = 0$.

Theorem 7.2. Let $1 < p < \infty$. For any $f \in L^p(\omega, \eta; \mathbb{R}^d)$, the solution of the Dirichlet problem

$$
- \Delta u = \text{div}(f) \quad \text{in } \omega, \eta \quad \text{and} \quad u = 0 \quad \text{on } \partial \omega, \eta,
$$

(7.3)

in $W^{1,p}_0(\omega, \eta)$ satisfies the estimate,

$$
\|\nabla u\|_{L^p(\omega, \eta)} \leq C \eta^{-d\frac{1}{2} - \frac{1}{p}} \|f\|_{L^p(\omega, \eta)},
$$

(7.4)

for $d \geq 3$, and

$$
\|\nabla u\|_{L^p(\omega, \eta)} \leq C \eta^{-2\frac{1}{2} - \frac{1}{p}} |\ln(\eta/2)|^{-\frac{1}{2}} \|f\|_{L^p(\omega, \eta)},
$$

(7.5)

for $d = 2$ and $p \neq 2$, where $C$ depends only on $d$, $p$ and $T$.
Proof. By rescaling and duality we may assume that $\varepsilon = 1$ and $p > 2$. Moreover, we only need to prove the estimates (7.4)–(7.5) for $\eta > 0$ sufficiently small.

We first consider the case $d \geq 3$. Let $u$ be a solution of (7.3) with $\varepsilon = 1$. It follows by Theorem 6.1 that

$$\int_{k+\tilde{Y}\setminus\eta T} |\nabla u|^p \, dx \leq C|\alpha|^p \eta^{d-p} + C[\Phi_p(\eta^{-1})]^p \int_{k+\tilde{Y}\setminus\eta T} |f|^p \, dx$$

$$+ C[\Phi_p(\eta^{-1})]^p \left( \int_{k+\tilde{Y}\setminus B(0,1/3)} |u - \alpha|^2 \, dx \right)^{p/2}$$

for any $k \in \mathbb{Z}^d$ and $\alpha \in \mathbb{R}$. Choose

$$\alpha = \int_{k+\tilde{Y}\setminus B(0,1/3)} u \, dx.$$ 

By using the Poincaré inequality we obtain

$$\int_{k+\tilde{Y}\setminus\eta T} |\nabla u|^p \, dx \leq C \eta^{d-p} \int_{k+\tilde{Y}\setminus\eta T} |u|^p \, dx + C[\Phi_p(\eta^{-1})]^p \int_{k+\tilde{Y}\setminus\eta T} |f|^p \, dx$$

$$+ C[\Phi_p(\eta^{-1})]^p \left( \int_{k+\tilde{Y}\setminus\eta T} |\nabla u|^2 \right)^{p/2}$$

$$\leq C \eta^{d-p} \int_{k+\tilde{Y}\setminus\eta T} |u|^p \, dx + C[\Phi_p(\eta^{-1})]^p \int_{k+\tilde{Y}\setminus\eta T} |f|^p \, dx$$

$$+ C[\Phi_p(\eta^{-1})]^p \int_{k+Y} |S_{1,\eta}(0,f)|^p,$$

where the operator $S_{1,\eta}$ is defined by (4.5). By summing over $k \in \mathbb{Z}^d$ we deduce that

$$\|\nabla u\|_{L^p(\omega_{1,\eta})} \leq C \eta^{d-1} \|u\|_{L^p(\omega_{1,\eta})} + C[\Phi_p(\eta^{-1})] \left\{ \|f\|_{L^p(\omega_{1,\eta})} + \|S_{1,\eta}(0,f)\|_{L^p(\mathbb{R}^d)} \right\}$$

$$\leq C \eta^{-d(\frac{1}{2} - \frac{1}{p})} \|f\|_{L^p(\omega_{1,\eta})},$$

where we have used (7.1) and (4.7) as well as the observation $\Phi_p(\eta^{-1}) \leq C \eta^d$. In the last inequality this gives (7.4) with $\varepsilon = 1$ and $p > 2$ for the case $d \geq 3$.

The proof for the case $d = 2$ is similar. Using (6.3), we obtain

$$\|\nabla u\|_{L^p(\omega_{1,\eta})} \leq C \eta^{d-1} \ln \eta^{-1} \left\{ \|u\|_{L^p(\omega_{1,\eta})} + \|f\|_{L^p(\omega_{1,\eta})} + \|S_{1,\eta}(0,f)\|_{L^p(\mathbb{R}^d)} \right\}.$$ 

The desired estimate then follows from (7.2) and (4.7). \hfill \Box

We now consider the case $f = 0$.

Theorem 7.3. Let $1 < p < \infty$. For any $F \in L^p(\omega_{\varepsilon,\eta})$, the solution of the Dirichlet problem,

$$- \Delta u = F \text{ in } \omega_{\varepsilon,\eta} \text{ and } u = 0 \text{ on } \partial \omega_{\varepsilon,\eta},$$

(7.6)

in $W^{1,p}_0(\omega_{\varepsilon,\eta})$ satisfies the estimate,

$$\|\nabla u\|_{L^p(\omega_{\varepsilon,\eta})} \leq \begin{cases} C \varepsilon \eta^{1 - \frac{d}{p}} \|F\|_{L^p(\omega_{\varepsilon,\eta})} & \text{for } 1 < p \leq 2, \\ C \varepsilon \eta^{1 - d + \frac{d}{p}} \|F\|_{L^p(\omega_{\varepsilon,\eta})} & \text{for } 2 < p < \infty \end{cases}$$

(7.7)
for $d \geq 3$, and

$$\|\nabla u\|_{L^p(\omega_{\varepsilon,\eta})} \leq \begin{cases} C \varepsilon |\ln(\eta/2)|^{\frac{1}{2}} \|F\|_{L^p(\omega_{\varepsilon,\eta})} & \text{for } 1 < p \leq 2, \\
C \varepsilon \eta^{1+\frac{d}{p}} \|F\|_{L^p(\omega_{\varepsilon,\eta})} & \text{for } 2 < p < \infty \end{cases} \tag{7.8}$$

for $d = 2$.

Proof. The case $1 < p \leq 2$ for $d \geq 2$ was proved in [10, Theorem 5.2] in a general non-periodic setting. To treat the case $2 < p < \infty$, we may assume $\varepsilon = 1$ by rescaling. Suppose $d \geq 3$ and $u$ is a solution of (7.6). As in the proof of Theorem 7.2, using Theorem 6.1, we may deduce by summation that

$$\|\nabla u\|_{L^p(\omega_{1,\eta})} \leq C \eta^{\frac{d}{p}} \|u\|_{L^p(\omega_{1,\eta})} + C \Phi_p(\eta^{-1}) \left\{ \|F\|_{L^p(\omega_{1,\eta})} + \|S_{1,\eta}(F,0)\|_{L^p(\omega_{1,\eta})} \right\} \leq C \eta^{1-d+\frac{d}{p}} \|F\|_{L^p(\omega_{1,\eta})} + C \Phi_p(\eta^{-1}) \left\{ \|F\|_{L^p(\omega_{1,\eta})} + \eta^{-1} \right\} \|F\|_{L^p(\omega_{1,\eta})} \leq C \eta^{1-d+\frac{d}{p}} \|F\|_{L^p(\omega_{1,\eta})},$$

where we have used (7.1) and (4.16) for the second inequality. The proof for the case $d = 2$ is similar. By Theorem 6.1, we obtain

$$\|\nabla u\|_{L^p(\omega_{1,\eta})} \leq C \eta^{\frac{2-d}{p}} \ln\eta^{-1} \|u\|_{L^p(\omega_{1,\eta})} + C \eta^{\frac{2-d}{p}} \ln\eta^{-1} \left\{ \|F\|_{L^p(\omega_{1,\eta})} + \|S_{1,\eta}(F,0)\|_{L^p(\omega_{1,\eta})} \right\} \leq C \eta^{2-d} \|F\|_{L^p(\omega_{1,\eta})},$$

which, together with (7.2) and (1.10), yields (7.8) for $2 < p < \infty$. \hfill \Box

For $1 < p < \infty$ and $\varepsilon, \eta \in (0,1]$, let $A_p(\varepsilon,\eta)$, $B_p(\varepsilon,\eta)$, $C_p(\varepsilon,\eta)$ and $D_p(\varepsilon,\eta)$ be the smallest constants for which the inequalities (1.2) and (1.3) hold. Clearly, $A_2(\varepsilon,\eta) \leq 1$. By duality, $C_p(\varepsilon,\eta) = B_p'(\varepsilon,\eta)$, where $p' = \frac{p}{p-1}$ (see [10]). It follows from Theorems 7.2 and 7.3 that

$$A_p(\varepsilon,\eta) \leq \begin{cases} C \eta^{-d+\frac{d}{p}} & \text{if } d \geq 3, \\
C \eta^{-2+\frac{d}{p}} |\ln(\eta/2)|^{-\frac{1}{2}} & \text{if } d = 2 \text{ and } p \neq 2, \end{cases} \tag{7.9}$$

and

$$B_p(\varepsilon,\eta) = C_p'(\varepsilon,\eta) \leq \begin{cases} C \varepsilon \eta^{-\frac{d}{p}} & \text{if } d \geq 3 \text{ and } 1 < p \leq 2, \\
C \varepsilon |\ln(\eta/2)|^{\frac{1}{2}} & \text{if } d = 2 \text{ and } 1 < p \leq 2, \\
C \eta^{-1+\frac{d}{p}} & \text{if } d \geq 2 \text{ and } 2 < p < \infty, \end{cases} \tag{7.10}$$

where $C$ depends only on $d$, $p$ and $T$. Furthermore, it was proved in [10] that

$$D_p(\varepsilon,\eta) \leq \begin{cases} C \varepsilon^2 \eta^{2-d} & \text{if } d \geq 3, \\
C \varepsilon^2 |\ln(\eta/2)| & \text{if } d = 2. \end{cases} \tag{7.11}$$

Proofs of Theorems 1.1 and 1.2. The estimates (1.6) and (1.8) follow from (7.9) and (7.10) by linearity, while (1.7) and (1.9) follow from (7.10) and (7.11). As we mentioned in the introduction, the sharpness of the estimates (7.9)-(7.11) was proved in [10]. \hfill \Box

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