1 Introduction

Since Kashiwara introduced the theory of crystal base (3) in 1990, one of the most fundamental problems has been to describe the crystal base associated with the given integrable highest weight module as explicitly as possible. In order to answer this, many kinds of new combinatorial objects have been invented, e.g., in (9) some analogues of Young tableaux were introduced in order to describe the crystal base for classical Lie algebras and it is applied to generalize so-called the Littlewood-Richardson rule in (12). In (7)(8) we gave the new object 'perfect crystals' and applying it to describe the crystal bases of affine types, moreover, to solve the problems in mathematical physics. In (10)(11) Littelmann realized the crystal base for symmetrizable Kac-Moody Lie algebras by using 'path' and in (13) we also have done it for the nilpotent subalgebra $U_q^{-}(\mathfrak{g})$ by the 'polyhedral realization'.

The present paper is devoted to give the explicit feature of crystal bases for integrable highest weight modules in terms of 'polyhedral realization'. Here we introduce the formulation and background of (13) and this paper. Let $U_q(\mathfrak{g})$ be the quantum algebra associated with the Kac-Moody Lie algebra $\mathfrak{g}$ and $U_q^{-}(\mathfrak{g})$ be the nilpotent subalgebra given by the usual triangular decomposition as in 2.1 below. Furthermore, let $(L(\infty), B(\infty))$ be the crystal base of $U_q^{-}(\mathfrak{g})$ (see (3)). In (2), Kashiwara introduced the remarkable embedding of crystals $\Psi_\iota : B(\infty) \rightarrow \mathbb{Z}^\infty$ where $\iota$ is some infinite sequence from the index set $I$ and $\mathbb{Z}^\infty$ is the $\mathbb{Z}$-lattice of infinite rank (see 2.5). In (13), we tried to describe the exact image of the embedding $\Psi_\iota$ in $\mathbb{Z}^\infty$. This can be carried out by the unified method, called the polyhedral realization. Due to this method, (under some condition) we succeeded to present the explicit form of $B(\infty)$ as the set of lattice points of some polyhedral convex cone in the infinite vector space $\mathbb{Q}^\infty$, which is defined by the system of inequalities. This system of inequalities are determined only by the sequence $\iota$ and the Cartan data of $\mathfrak{g}$. In the present paper, we shall try to give the similar decription of the crystal $B(\lambda)$, where $B(\lambda)$ is the crystal (base) of the irreducible integrable highest weight module $V(\lambda)$ with the highest weight $\lambda$.

To mention our problem more precisely, let us introduce the object $R_\lambda$ which is a crystal consisting of the one element $r_\lambda$ ($\lambda$ is a weight) (see Example 2.4 (ii)
below and also [4]) and has the following remarkable property: The crystal $B(\infty)$ is connected as a crystal graph (see Definition 2.2), but in general, not so the crystal $B(\infty) \otimes R_\lambda$ is. Furthermore, the connected component including $u_\infty \otimes r_\lambda$ is isomorphic to the crystal $B(\lambda)$, where $u_\infty$ is the highest weight vector in $B(\infty)$ (Theorem 3.3). These properties guarantee the existence of the embedding of crystals $\Omega_\lambda : B(\lambda) \hookrightarrow B(\infty) \otimes R_\lambda$. (see also [4]). Combining $\Omega_\lambda$ and $\Psi_\iota$, we obtain the embedding of crystals $\Psi^{(\lambda)}_\iota := (\Psi_\iota \otimes \text{id}) \circ \Omega_\lambda : B(\lambda) \hookrightarrow \mathbb{Z}^\infty \otimes R_\lambda$. Here note that since $R_\lambda$ consists of one element, as a set $\mathbb{Z}^\infty \otimes R_\lambda$ can be identified with the infinite $\mathbb{Z}$-lattice $\mathbb{Z}^\infty$. Our goal is to give the explicit form of $\text{Im}(\Psi^{(\lambda)}_\iota)(\simeq B(\lambda))$ in the infinite $\mathbb{Z}$-lattice. To complete this, we shall introduce the set linear functions $\Xi_{\iota}[\lambda]$ (see (4.13)) which is uniquely determined by the Cartan data of $\mathfrak{g}$, the sequence $\iota$ and the highest weight $\lambda$. The set $\Sigma_{\iota}[\lambda]$ is the set of lattice points in the convex polyhedron defined by the system of inequalities: $\varphi(\bar{x}) \geq 0$ ($\bar{x} \in \mathbb{Z}^\infty$) for any $\varphi \in \Xi_{\iota}[\lambda]$. Finally, we can show $\text{Im}(\Psi^{(\lambda)}_\iota) = \Sigma_{\iota}[\lambda]$ under the assumption $\Sigma_{\iota}[\lambda] \ni 0 := (\cdots, 0, 0)$. We shall apply this to several explicit cases, namely, arbitrary rank 2 Kac-Moody algebras, $A_n$-case and $A^{(1)}_n$-case.

Now let us see the organization of this paper. The section 2 is devoted to preliminaries and reviews on the theory of crystals and crystal bases. In particular, in 2.1 the crystal $R_\lambda$ will be introduced and in 2.4 we shall give the results of [3]. In section 3, we shall introduce the surjective morphism $\Phi_\lambda : B(\infty) \otimes R_\lambda \twoheadrightarrow B(\lambda)$, the embeddings $\Omega_\lambda : B(\lambda) \hookrightarrow B(\infty) \otimes R_\lambda$ and $\Psi^{(\lambda)}_\iota : B(\lambda) \hookrightarrow \mathbb{Z}^\infty \otimes R_\lambda$. The section 4 is the main part of this paper. In 4.2 we shall construct the polyhedral realization of $B(\lambda)$ explicitly and in 4.3 it is applied to give the explicit description of the value $'\varepsilon_1^*''$. In section 5, we treat rank 2 Kac-Moody algebras. The explicit form of polyhedral realization of $B(\lambda)$ is given by using ‘Chebyshev polynomials’. In section 6, we consider the case $\mathfrak{g} = A_n$. Furthermore, in this section we shall give an example which does not satisfy the ‘positivity assumption’. Thus, our conjectural perspective in [3], 3.3, that the positivity assumption is satisfied automatically, turns out to be invalid. In section 7, we treat the higher rank affine case $\mathfrak{g} = A^{(1)}_{n-1}$.

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2 Crystals and Crystal Bases

2.1 Definition of crystals

Let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra over $\mathbb{Q}$ with a Cartan subalgebra $\mathfrak{t}$, a weight lattice $\mathcal{P} \subset \mathfrak{t}^*$, the set of simple roots $\{\alpha_i : i \in I\} \subset \mathfrak{t}^*$, and the set of coroots $\{h_i : i \in I\} \subset \mathfrak{t}$, where $I$ is a finite index set. Let $(h, \lambda)$ be the pairing
between $t$ and $t^*$, and $(\alpha, \beta)$ be an inner product on $t^*$ such that $(\alpha_i, \alpha_i) \in 2\mathbb{Z}_{\geq 0}$ and $\langle h_i, \lambda \rangle = \frac{2\langle \alpha_i, \lambda \rangle}{(\alpha_i, \alpha_i)}$ for $\lambda \in t^*$. Let $P^* = \{ h \in t : \langle h, P \rangle \subset \mathbb{Z} \}$ and $P_+ := \{ \lambda \in P : \langle h_i, \lambda \rangle \in \mathbb{Z}_{\geq 0} \}$. We call an element in $P_+$ a dominant integral weight.

The quantum algebra $U_q(\mathfrak{g})$ is an associative $Q(q)$-algebra generated by the $e_i, f_i \ (i \in I)$, and $q^h \ (h \in P^*)$ satisfying the usual relations (see e.g. [3] or [13]). The algebra $U_q(\mathfrak{g})$ is the subalgebra of $U_q(\mathfrak{g})$ generated by the $f_i \ (i \in I)$.

The following definition of a crystal is the one slightly modified those in [3, 13]. But there is no difference between their properties. In what follows we fix a finite index set $I$ and a weight lattice $P$ as above.

**Definition 2.1** A crystal $B$ is a set endowed with the following maps:

\[
\begin{align*}
\varepsilon_i : B &\rightarrow \mathbb{Z} \sqcup \{-\infty\}, \\
\varphi_i : B &\rightarrow \mathbb{Z} \sqcup \{-\infty\} \quad \text{for } i \in I, \\
\check{e}_i : B \sqcup \{0\} &\rightarrow B \sqcup \{0\}, \quad \check{f}_i : B \sqcup \{0\} \rightarrow B \sqcup \{0\} \quad \text{for } i \in I.
\end{align*}
\] (2.1)

(2.2)

(2.3)

Here 0 is an ideal element which is not included in $B$. These maps must satisfy the following axioms: for all $b, b_1, b_2 \in B$, we have

\[
\begin{align*}
\varphi_i(b) &= \varepsilon_i(b) + \langle h_i, \text{wt}(b) \rangle, \\
\text{wt}(\check{e}_i b) &= \text{wt}(b) + \alpha_i \text{ if } \check{e}_i b \in B, \\
\text{wt}(\check{f}_i b) &= \text{wt}(b) - \alpha_i \text{ if } \check{f}_i b \in B, \\
\check{e}_i b_2 &= b_1 \text{ if and only if } \check{f}_i b_1 = b_2, \\
\text{if } \varepsilon_i(b) = -\infty, \text{ then } \check{e}_i b &= \check{f}_i b = 0, \\
\check{e}_i(0) &= \check{f}_i(0) = 0.
\end{align*}
\] (2.4)

(2.5)

(2.6)

(2.7)

(2.8)

(2.9)

The above axioms allow us to make a crystal $B$ into a colored oriented graph with the set of colors $I$.

**Definition 2.2** The crystal graph of a crystal $B$ is a colored oriented graph given by the rule $b_1 \longrightarrow b_2$ if and only if $b_2 = \check{f}_i b_1 \ (b_1, b_2 \in B)$.

**Definition 2.3** (i) Let $B_1$ and $B_2$ be crystals. A strict morphism of crystals $\psi : B_1 \rightarrow B_2$ is a map $\psi : B_1 \sqcup \{0\} \rightarrow B_2 \sqcup \{0\}$ satisfying the following conditions: $\psi(0) = 0,$

\[
\begin{align*}
\text{wt}(\psi(b)) &= \text{wt}(b), \\
\varepsilon_i(\psi(b)) &= \varepsilon_i(b), \\
\varphi_i(\psi(b)) &= \varphi_i(b)
\end{align*}
\] (2.10)

if $b \in B_1$ and $\psi(b) \in B_2$,.

and the map $\psi : B_1 \sqcup \{0\} \rightarrow B_2 \sqcup \{0\}$ commutes with all $\check{e}_i$ and $\check{f}_i$.

(ii) An injective strict morphism is called an embedding of crystals. We call $B_1$ is a subcrystal of $B_2$, if $B_1$ is a subset of $B_2$ and becomes a crystal itself by restricting the data on it from $B_2$. 

3
It is well-known that $U_q(\mathfrak{g})$ has a Hopf algebra structure. Then the tensor product of $U_q(\mathfrak{g})$-modules has a $U_q(\mathfrak{g})$-module structure. Consequently, we can consider the tensor product of crystals: For crystals $B_1$ and $B_2$, we define their tensor product $B_1 \otimes B_2$ as follows:

\begin{align}
B_1 \otimes B_2 &= \{b_1 \otimes b_2 : b_1 \in B_1, b_2 \in B_2\}, \quad (2.11) \\
wt(b_1 \otimes b_2) &= wt(b_1) + wt(b_2), \quad (2.12) \\
\varepsilon_i(b_1 \otimes b_2) &= \max(\varepsilon_i(b_1), \varepsilon_i(b_2) - \langle h_i, wt(b_1) \rangle), \quad (2.13) \\
\varphi_i(b_1 \otimes b_2) &= \max(\varphi_i(b_1), \varphi_i(b_2) + \langle h_i, wt(b_2) \rangle), \quad (2.14) \\
\tilde{e}_i(b_1 \otimes b_2) &= \begin{cases} 
\varepsilon_i(b_1) + \varepsilon_i(b_2) & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\
\varepsilon_i(b_2) & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), 
\end{cases} \quad (2.15) \\
\tilde{f}_i(b_1 \otimes b_2) &= \begin{cases} 
\varepsilon_i(b_1) & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\
\varepsilon_i(b_2) - \langle h_i, wt(b_2) \rangle & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2). 
\end{cases} \quad (2.16)
\end{align}

Here $b_1 \otimes b_2$ is just another notation for an ordered pair $(b_1, b_2)$, and we set $b_1 \otimes 0 = 0 \otimes b_2 = 0$. Note that the tensor product of crystals is associative, namely, the crystals $(B_1 \otimes B_2) \otimes B_3$ and $B_1 \otimes (B_2 \otimes B_3)$ are isomorphic via $(b_1 \otimes b_2) \otimes b_3 \leftrightarrow b_1 \otimes (b_2 \otimes b_3)$.

The example of crystals below will be needed later.

**Example 2.4**

(i) For $i \in I$, the crystal $B_i := \{x_i : x \in \mathbb{Z}\}$ is defined by

\begin{align}
wt((x)_i) &= x\alpha_i, \quad \varepsilon_i((x)_i) = -x, \quad \varphi_i((x)_i) = x, \\
\varepsilon_j((x)_i) &= -\infty, \quad \varphi_j((x)_i) = -\infty \quad \text{for } j \neq i, \\
\tilde{e}_j(x)_i &= \delta_{i,j}(x+1)_i, \quad \tilde{f}_j(x)_i = \delta_{i,j}(x-1)_i,
\end{align}

(ii) (See also [4]) Let $R_\lambda := \{r_\lambda\} (\lambda \in P)$ be the crystal consisting of one-element given by

\begin{align}
wt(r_\lambda) &= \lambda, \quad \varepsilon_i(r_\lambda) = -\langle h_i, \lambda \rangle, \quad \varphi_i(r_\lambda) = 0, \quad \tilde{e}_i(r_\lambda) = \tilde{f}_i(r_\lambda) = 0.
\end{align}

**2.2 Crystal $B(\lambda)$**

In this subsection we review the crystal $B(\lambda)$ for a dominant integral weight $\lambda \in P_+$, which is our main object of study. All the results in this subsection are due to M.Kashiwara [3]. Let $V(\lambda)$ be the irreducible highest weight module of $U_q(\mathfrak{g})$ with the highest weight $\lambda \in P_+$. It can be defined by

\begin{equation}
V(\lambda) := U_q(\mathfrak{g}) \left/ \sum_i U_q(\mathfrak{g})c_i + \sum_i U_q(\mathfrak{g})f_i^{(h_i,\lambda)+1} + \sum_{h \in P^*} U_q(\mathfrak{g})(q^h - q^{(h,\lambda)}) \right. 
\end{equation}

(2.17)
It is well-known that as a $U_q(\mathfrak{g})$-module, there is the following natural isomorphism:

$$V(\lambda) \cong U_q(\mathfrak{g})/\sum_i U_q(\mathfrak{g})f_i^{(h_i,\lambda)+1}.$$  \hspace{1cm} (2.18)

Let $\pi_\lambda$ be a natural projection $U_q(\mathfrak{g}) \rightarrow V(\lambda)$ and set $u_\lambda := \pi_\lambda(1)$. This is the unique highest weight vector in $V(\lambda)$ up to constant.

For each $i \in I$, we have the decomposition: $V(\lambda) = \bigoplus_n f_i^{(n)}(\Ker e_i)$. Using this, we can define the endomorphisms $\tilde{e}_i$ and $\tilde{f}_i \in \text{End}(V(\lambda))$ by

$$\tilde{e}_i(f_i^{(n)}u) = f_i^{(n-1)}u, \text{ and } \tilde{f}_i(f_i^{(n)}u) = f_i^{(n+1)}u \text{ for } u \in \Ker e_i, \hspace{1cm} (2.19)$$

where we understand that $\tilde{e}_i u = 0$ for $u \in \Ker e_i$. Let $A \subset Q(\mathfrak{g})$ be the subring of rational functions regular at $q = 0$. We set

$$L(\lambda) := \sum_{i_j \in I, l \geq 0} A\tilde{f}_{i_0} \cdots \tilde{f}_{i_l} u_\lambda, \hspace{1cm} (2.20)$$

$$B(\lambda) := \{ \tilde{f}_{i_0} \cdots \tilde{f}_{i_l} u_\lambda \text{ mod } qL(\lambda) | i_j \in I, l \geq 0 \} \setminus \{0\}. \hspace{1cm} (2.21)$$

The pair $(L(\lambda), B(\lambda))$ is called \textit{crystal base} of $V(\lambda)$. It satisfies the following properties:

(i) $L(\lambda)$ is a free $A$-submodule of $V(\lambda)$ and $V(\lambda) \cong Q(\mathfrak{g}) \otimes_A L(\lambda)$.

(ii) $B(\lambda)$ is a basis of the $Q$-vector space $L(\lambda)/qL(\lambda)$.

(iii) $\tilde{e}_i L(\lambda) \subset L(\lambda)$ and $\tilde{f}_i L(\lambda) \subset L(\lambda)$.

By (iii) the $\tilde{e}_i$ and the $\tilde{f}_i$ act on $L(\lambda)/qL(\lambda)$ and

(iv) $\tilde{e}_i B(\lambda) \subset B(\lambda) \cup \{0\}$ and $\tilde{f}_i B(\lambda) \subset B(\lambda) \cup \{0\}$.

(v) For $u, v \in B(\lambda)$, $\tilde{f}_i u = v$ if and only if $\tilde{e}_i v = u$.

We define the weight function $wt : B(\lambda) \rightarrow P$ by $wt(b) := \lambda - \alpha_{i_1} - \alpha_{i_2} - \cdots - \alpha_{i_l}$ for $b = \tilde{f}_{i_0} \cdots \tilde{f}_{i_l} u_\lambda \text{ mod } qL(\lambda) \neq 0$. We define integer-valued functions $\varepsilon_i$ and $\varphi_i$ on $B(\lambda)$ by

$$\varepsilon_i(b) := \max\{ k : \tilde{e}_i^k b \neq 0 \}, \hspace{0.5cm} \varphi_i(b) := \max\{ k : \tilde{f}_i^k b \neq 0 \}.$$  \hspace{1cm} (2.22)

It is easy to verify that $B(\lambda)$ equipped with the operators $\tilde{e}_i$ and $\tilde{f}_i$, and with the functions $wt$, $\varepsilon_i$ and $\varphi_i$ becomes a crystal.

Let $(L(\infty), B(\infty))$ be the crystal base of the subalgebra $U_q(\mathfrak{g})$ (see [3],[13]). Here note that the functions $\varepsilon_i$ and $\varphi_i$ are given by

$$\varepsilon_i(b) := \max\{ k : \tilde{e}_i^k b \neq 0 \}, \hspace{0.5cm} \varphi_i(b) := \varepsilon_i(b) + \langle h_i, wt(b) \rangle.$$  \hspace{1cm} (2.23)
It is proved in [3] that the natural projection \( \pi : U_q^{-}(g) \to V(\lambda) \) sends \( L(\infty) \) to \( L(\lambda) \), and the induced map \( \tilde{\pi}_\lambda : L(\infty)/qL(\infty) \to L(\lambda)/qL(\lambda) \) sends \( B(\infty) \) to \( B(\lambda) \cup \{0\} \). The map \( \tilde{\pi}_\lambda \) has the following properties:

\[
\tilde{f}_i \circ \tilde{\pi}_\lambda = \tilde{\pi}_\lambda \circ \tilde{f}_i, \\
\tilde{e}_i \circ \tilde{\pi}_\lambda = \tilde{\pi}_\lambda \circ \tilde{e}_i, \text{ if } \tilde{\pi}_\lambda(b) \neq 0, \\
\tilde{\pi}_\lambda : B(\infty) \setminus \{\pi_\lambda^{-1}(0)\} \to B(\lambda) \text{ is bijective.} (2.26)
\]

Although the map \( \tilde{\pi}_\lambda \) has such nice properties, it is not a strict morphism of crystals. For instance, it does not preserve weights or does not necessarily commute with the action of \( \tilde{e}_i \) as in (2.25). We shall introduce a new morphism by modifying the map \( \tilde{\pi}_\lambda \) in 3.1.

### 2.3 Kashiwara Embedding

We define a \( \mathbb{Q}(q) \)-algebra anti-automorphism \( * \) of \( U_q(g) \) by:

\[
q^* = q, \quad e_i^* = e_i, \quad f_i^* = f_i, \quad (q^h)^* = q^{-h}.
\]

This anti-automorphism has the properties (see [4]):

\[
L(\infty)^* = L(\infty) \quad \text{and} \quad B(\infty)^* = B(\infty). (2.27)
\]

Then we can define \( \varepsilon_i^*(b) := \varepsilon_i(b^*) \) and \( \varphi_i^*(b) := \varphi_i(b^*) \).

Consider the additive group

\[
\mathbb{Z}^\infty := \{\cdots, x_k, \cdots, x_2, x_1) : x_k \in \mathbb{Z} \text{ and } x_k = 0 \text{ for } k \gg 0\}; (2.28)
\]

we will denote by \( \mathbb{Z}^\infty_{\geq 0} \subset \mathbb{Z}^\infty \) the subsemigroup of nonnegative sequences. To the rest of this section, we fix an infinite sequence of indices \( i = i_k, \cdots, i_2, i_1 \) from \( I \) such that

\[
i_k \neq i_{k+1} \text{ and } \sharp \{k : i_k = i\} = \infty \text{ for any } i \in I. \quad (2.29)
\]

We can associate to \( i \) a crystal structure on \( \mathbb{Z}^\infty \) and denote it by \( \mathbb{Z}^\infty_i \) (2.4).

**Proposition 2.5 ([4], See also [13])** There is a unique embedding of crystals

\[
\Psi_i : B(\infty) \hookrightarrow \mathbb{Z}^\infty_{\geq 0} \subset \mathbb{Z}^\infty_i, \quad (2.30)
\]

such that \( \Psi_i(u_\infty) = (\cdots, 0, \cdots, 0, 0) \).

We call this the **Kashiwara embedding** which is derived by iterating the following type of embeddings ([4]):

(i) For any \( i \in I \), there is a unique embedding of crystals

\[
\Psi_i : B(\infty) \hookrightarrow B(\infty) \otimes B_i, \quad (2.31)
\]

such that \( \Psi_i(u_\infty) = u_\infty \otimes (0)_i \).

(ii) For any \( b \in B(\infty) \), we can write uniquely \( \Psi_i(b) = b' \otimes \tilde{f}_i^m(0)_i \) where \( m = \varepsilon_i^*(b) \).
2.4 Polyhedral Realization of $B(\infty)$

In this subsection, we recall the main result of [13].

Consider the infinite dimensional vector space

$$Q^\infty := \{ \vec{x} = (\cdots, x_k, \cdots, x_2, x_1) : x_k \in \mathbb{Q} \text{ and } x_k = 0 \text{ for } k \gg 0 \},$$

and its dual $(Q^\infty)^* := \text{Hom}(Q^\infty, \mathbb{Q})$. We will write a linear form $\varphi \in (Q^\infty)^*$ as $\varphi(\vec{x}) = \sum_{k \geq 1} \varphi_k x_k$ ($\varphi_j \in \mathbb{Q}$).

Fix $\iota = (i_k)$ as in the previous section. For $\iota$ we set $k^+ := \min\{l : l > k \text{ and } i_k = i_l \}$ and $k^- := \max\{l : l < k \text{ and } i_k = i_l \}$ if it exists, or $k^- = 0$ otherwise. We set for $\vec{x} \in Q^\infty$, $\beta_0(\vec{x}) = 0$ and $\beta_k(\vec{x}) := x_k + \sum_{k < j < k^+} \langle h_{i_k}, \alpha_{i_j} \rangle x_j + x_k^+$. (2.32)

We define a piecewise-linear operator $S_k = S_{k, \iota}$ on $(Q^\infty)^*$ by

$$S_k(\varphi) := \begin{cases} \varphi - \varphi_k \beta_k & \text{if } \varphi_k > 0, \\ \varphi - \varphi_k \beta_k^- & \text{if } \varphi_k \leq 0. \end{cases} \quad (2.33)$$

Here we set

$$\Xi_\iota := \{ S_{j_1} \cdots S_{j_l} x_{j_l} \mid l \geq 0, j_0, j_1, \cdots, j_l \geq 1 \},$$

$$\Sigma_\iota := \{ \vec{x} \in \mathbb{Z}^\infty \subseteq Q^\infty \mid \varphi(\vec{x}) \geq 0 \text{ for any } \varphi \in \Xi_\iota \}.$$

We impose on $\iota$ the following positivity assumption:

if $k^- = 0$ then $\varphi_k \geq 0$ for any $\varphi(\vec{x}) = \sum_k \varphi_k x_k \in \Xi_\iota$.

**Theorem 2.6 ([13])** Let $\iota$ be a sequence of indices satisfying (2.29) and the positivity assumption, and $\Psi_\iota : B(\infty) \hookrightarrow \mathbb{Z}_+^\infty$ be the corresponding Kashiwara embedding. Then we have $\text{Im}(\Psi_\iota)(\simeq B(\infty)) = \Sigma_\iota$.

**Remark.** We shall show the example of the sequence $\iota$ which does not satisfy the positivity assumption. It will be given in the end of Sect.6.

2.5 Global crystal base

In this subsection, we recall several facts about global crystal bases (see [3], [6]).

We define a $\mathbb{Q}$-algebra automorphism $- : U_q(\mathfrak{g})$ by: $\mathfrak{t} = q^{-1}$, $\mathfrak{q}^{\pm h} = q^{\pm h}$, $e_i = e_i$, $f_i = f_i$.

Let $U_q^{-}(\mathfrak{g})$ be the sub-$\mathbb{Q}[q, q^{-1}]$-algebra of $U_q^{-}(\mathfrak{g})$ generated by $f_i(n) = f_i^n/[n]!$. and $V_{Q}(\lambda) := U_q^{-}(\mathfrak{g})u_{\lambda}$ for $\lambda \in P_+$. Let $p_\infty : L(\infty) \rightarrow L(\infty)/qL(\infty)$ (resp. $p_\lambda : L(\lambda) \rightarrow L(\lambda)/qL(\lambda)$) be the canonical projection.
Proposition 2.7 ([3]) The map \( p_\infty \) (resp. \( p_\lambda \)) gives rise to the \( \mathbb{Q} \)-linear isomorphism:

\[
U_{\mathbb{Q}}(\mathfrak{g}) \cap L(\infty) \cap \overline{L}(\infty) \xrightarrow{\sim} L(\infty)/qL(\infty) \quad \text{resp.} \quad V_{\mathbb{Q}}(\lambda) \cap L(\lambda) \cap \overline{L}(\lambda) \xrightarrow{\sim} L(\lambda)/qL(\lambda).
\] (2.34)

Let us denote the inverse of this isomorphism by \( G \). The set of inverse image of crystal base \( \{ G(b) \mid b \in B(\infty) \text{ (resp. } B(\lambda)) \} \) is called global (crystal) base of \( U_q^- (\mathfrak{g}) \) (resp. \( V(\lambda) \)). The global base holds the following remarkable property ([3, Theorem 7], [6, (6.3)]):

\[
f^n_{i} U_q^- (\mathfrak{g}) = \bigoplus_{b \in B(\infty), \varepsilon_i(b) \geq n} \mathbb{Q}(q) G(b).
\] (2.35)

As we have seen that the anti-automorphism \(*\) preserves \( U_q^- (\mathfrak{g}) \), and furthermore, has the property (2.27), which implies that the action of \(*\) commutes with \( p_\infty \) and then we have \( G(b^*) = G(b)^* \). Thus, applying \(*\) on (2.35) we obtain,

\[
U_q^- (\mathfrak{g}) f^n_{i} = \bigoplus_{b \in B(\infty), \varepsilon_i(b^*) \geq n} \mathbb{Q}(q) G(b).
\] (2.36)

For a dominant integral weight \( \lambda \), let \( \pi_\lambda \) be the projection \( U_q^- (\mathfrak{g}) \to V(\lambda) \) as in 2.2. By (2.17) we know that

\[
\text{Ker}(\pi_\lambda) = \sum_i U_q^- (\mathfrak{g}) f_{i}^{1+\langle h_i, \lambda \rangle}.
\] (2.37)

By virtue of (2.36) and (2.37) we have:

\[
\pi_\lambda(G(b)) = 0 \iff \varepsilon_i(b^*) > \langle h_i, \lambda \rangle \text{ for some } i \in I.
\] (2.38)

Proposition 2.8 For \( b \in B(\infty) \) and \( \lambda \in P_+ \), \( \hat{\pi}_\lambda(b) \neq 0 \) if and only if \( \varepsilon_i(b^*) \leq \langle h_i, \lambda \rangle \) for any \( i \in I \), where we set \( \varepsilon_i^*(b) := \varepsilon_i(b^*) \).

Proof. Since \( p_\lambda \circ \pi_\lambda = \hat{\pi}_\lambda \circ p_\infty \) (see [3]), it follows from Proposition 2.7 and (2.38) that \( \pi_\lambda(G(b)) = 0 \iff \hat{\pi}_\lambda(b) = 0 \). Thus, we get the desired result. \( \square \)

3 Embedding of \( B(\lambda) \)

In this section, \( \lambda \) is supposed to be a dominant integral weight.
3.1 Morphisms of Crystals

We shall introduce a new morphism of crystals by modifying the map \( \hat{\pi}_\lambda \). Let \( R_\lambda \) be the crystal defined in Example 2.4 (ii). Consider the crystal \( B(\infty) \otimes R_\lambda \) and define the map

\[
\Phi_\lambda : (B(\infty) \otimes R_\lambda) \cup \{0\} \rightarrow B(\lambda) \cup \{0\},
\]

by \( \Phi_\lambda(0) = 0 \) and \( \Phi_\lambda(b \otimes r_\lambda) = \hat{\pi}_\lambda(b) \) for \( b \in B(\infty) \). We set

\[
\tilde{B}(\lambda) := \{ b \otimes r_\lambda \in B(\infty) \otimes R_\lambda \mid \Phi_\lambda(b \otimes r_\lambda) \neq 0 \}.
\]

Theorem 3.1

(i) The map \( \Phi_\lambda \) becomes a surjective strict morphism of crystals \( B(\infty) \otimes R_\lambda \rightarrow B(\lambda) \).

(ii) \( \tilde{B}(\lambda) \) is a subcrystal of \( B(\infty) \otimes R_\lambda \), and \( \Phi_\lambda \) induces the isomorphism of crystals \( \tilde{B}(\lambda) \cong B(\lambda) \).

(iii) We have

\[
\tilde{B}(\lambda) = \{ b \otimes r_\lambda \in B(\infty) \otimes R_\lambda \mid \varepsilon_i^*(b) \leq \langle h_i, \lambda \rangle \text{ for any } i \in I \}.
\]

The proof of this theorem will be given in the next subsection.

Let us denote \( Z_i^\infty \otimes R_\lambda \) by \( Z_i^\infty[\lambda] \). Here note that since the crystal \( R_\lambda \) has only one element, as a set we can identify \( Z_i^\infty[\lambda] \) with \( Z_i^\infty \) but their crystal structures are different. By Theorem 3.1, we have the strict embedding of crystals (see also [1]):

\[
\Omega_\lambda : B(\lambda)(\cong \tilde{B}(\lambda)) \hookrightarrow B(\infty) \otimes R_\lambda.
\]

Combining \( \Omega_\lambda \) and the Kashiwara embedding \( \Psi_i \), we obtain the following:

Theorem 3.2 There exists the unique strict embedding of crystals

\[
\Psi_i^{(\lambda)} : B(\lambda) \xrightarrow{\Omega_\lambda} B(\infty) \otimes R_\lambda \xrightarrow{\Psi_i \otimes \text{id}} Z_i^\infty \otimes R_\lambda =: Z_i^\infty[\lambda],
\]

such that \( \Psi_i^{(\lambda)}(u_\lambda) = (\cdots, 0, 0, 0) \otimes r_\lambda \).

The main result of the present paper is an explicit description of the image of \( \Psi_i^{(\lambda)} (\cong \tilde{B}(\lambda)) \) as a part in \( B(\infty) \otimes R_\lambda \hookrightarrow Z_i^\infty[\lambda] \), which will be given in Sect.4.
3.2 Proof of Theorem 3.1
Before showing Theorem 3.1, we see the following lemmas:

Lemma 3.3 For $b \in B(\infty)$, suppose that $\tilde{e}_i b \neq 0$. Then we have

$$\varepsilon^*_j(\tilde{e}_i b) = \varepsilon^*_j(b) \ (i \neq j) \text{ and } \varepsilon^*_i(\tilde{e}_i b) \leq \varepsilon^*_i(b).$$

Proof. Let $\Psi_j : B(\infty) \to B(\infty) \otimes B_j \ (u_\infty \mapsto u_\infty \otimes (0)_j)$ be the strict embedding as in (2.31), which satisfies that for $b \in B(\infty)$ $\Psi_j(b) = b_1 \otimes f^m_j(0)_j$ where $m = \varepsilon^*_j(b)$ and $b_1 = (\tilde{e}_j^m b)^*$. If $i \neq j$, $\Psi_j(\tilde{e}_i b) = \varepsilon_i \Psi_j(b) = (\tilde{e}_i b_1) \otimes f^m_j(0)_j$ by (2.12) and then we have $\varepsilon^*_j(\tilde{e}_i b) = m = \varepsilon^*_j(b)$. In the case $i = j$, we have $\tilde{e}_i(b_1 \otimes f^m_j(0)_j) = \varepsilon_i b_1 \otimes f^m(0)_j$ or $b_1 \otimes f^{m-1}_j(0)_j$ ($m \geq 1$). This implies that $\varepsilon^*_i(\tilde{e}_i b) \leq \varepsilon^*_i(b)$.

Lemma 3.4 Suppose that $\tilde{\pi}_\lambda(b) \neq 0$ for $b \in B(\infty)$. Then $\tilde{e}_i \tilde{\pi}_\lambda(b) = 0$ if and only if $\tilde{e}_i b = 0$.

Proof. We assume $\tilde{e}_i \tilde{\pi}_\lambda(b) = 0$. Since $\tilde{e}_i \tilde{\pi}_\lambda(b) = \tilde{\pi}_\lambda(\tilde{e}_i b)$ if $\tilde{\pi}_\lambda(b) \neq 0$ by (2.22), we have $\tilde{\pi}_\lambda(\tilde{e}_i b) = 0$. If $\tilde{e}_i b \neq 0$, it follows from Lemma 3.3 that for any $j \in I$ $\varepsilon^*_j(\tilde{e}_i b) \leq \varepsilon^*_j(b) \leq (h_j, \lambda)$, which contradicts $\tilde{\pi}_\lambda(\tilde{e}_i b) = 0$ by Proposition 2.8. Hence, we have $\tilde{e}_i b = 0$. On the other hand, it is trivial that if $\tilde{e}_i b = 0$, then $\tilde{e}_i \tilde{\pi}_\lambda(b) = \tilde{\pi}_\lambda(\tilde{e}_i b) = 0$ by (2.22).

Proof of Theorem 3.1. The statement (iii) of the theorem is an immediate consequence of Proposition 2.8.

Let us show (i). The surjectivity follows from the one for the map $\tilde{\pi}_\lambda$. So we try to prove that $\Phi_\lambda$ is a strict morphism of crystals. To do this, according to Definition 2.3 (i) it suffices to show: for $u \in B(\infty) \otimes R_\lambda$,

\begin{enumerate}
  \item $wt(\Phi_\lambda(u)) = wt(u)$ if $\Phi_\lambda(u) \neq 0$,
  \item $\varepsilon_i(\Phi_\lambda(u)) = \varepsilon_i(u)$ for any $i$ if $\Phi_\lambda(u) \neq 0$,
  \item $\varphi_i(\Phi_\lambda(u)) = \varphi_i(u)$ for any $i$ if $\Phi_\lambda(u) \neq 0$,
  \item $\tilde{e}_i \Phi_\lambda(u) = \Phi_\lambda(\tilde{e}_i u)$ for any $i$,
  \item $\tilde{f}_i \Phi_\lambda(u) = \Phi_\lambda(\tilde{f}_i u)$ for any $i$.
\end{enumerate}

Let us show (1). For $u = b \otimes r_\lambda = (\tilde{f}_i \cdots \tilde{f}_{i_1} u_\infty) \otimes r_\lambda \in B(\infty) \otimes R_\lambda$, we have $\Phi_\lambda(u) = \tilde{\pi}_\lambda(b) = \tilde{f}_{i_1} \cdots \tilde{f}_i u_\infty$ since any $\tilde{f}_i$ commutes with $\tilde{\pi}_\lambda$. It follows immediately that $wt(u) = \lambda - \alpha_{i_1} - \cdots - \alpha_{i_1} = wt(\Phi_\lambda(u))$ if $\Phi_\lambda(u) \neq 0$.

In the case $\tilde{\pi}_\lambda(b) \neq 0$, it follows from Lemma 3.4 that $\tilde{e}_i b = 0$ if and only if $\tilde{e}_i \tilde{\pi}_\lambda(b) = 0$. This means if $\tilde{\pi}_\lambda(b) \neq 0$,

$$\varepsilon_i(b) = \varepsilon_i(\tilde{\pi}_\lambda(b)).$$

(3.4)
Furthermore, by (1), (2.4), (2.23) and (3.4) we have

\[ 0 \leq \varphi_i(\hat{\pi}_\lambda(b)) = \langle h_i, \lambda \rangle + \varphi_i(b). \]  

(3.5)

It follows from (2.13), (3.4) and (1.3) that

\[ \varepsilon_i(b \otimes r_\lambda) = \max(\varepsilon_i(b), \varepsilon_i(r_\lambda) - \langle h_i, wt(b) \rangle) = \max(\varepsilon_i(b), \varepsilon_i(b) - \varphi_i(b) - \langle h_i, \lambda \rangle) \]

\[ = \varepsilon_i(b) = \varepsilon_i(\hat{\pi}_\lambda(b)) = \varepsilon_i(\Phi(u)). \]

Now we obtained (2).

The statement (3) is derived immediately from (1), (2) and (2.4).

Let us show (4), namely, \( \hat{\varepsilon}_i(\hat{\pi}_\lambda(b)) = \max(\varepsilon_i(b), \varepsilon_i(r_\lambda) - \langle h_i, \lambda \rangle) \). Thus, it suffices to show

\[ \hat{\varepsilon}_i(b \otimes r_\lambda) = (\hat{\varepsilon}_i(b) \otimes r_\lambda). \]  

(3.6)

By (3.3) we have \( \varphi_i(b) = -\langle h_i, \lambda \rangle = \varepsilon_i(r_\lambda) \), which means (1.4) by (2.15).

Next, we consider the case \( \Phi(u) = \pi_\lambda(b) = 0 \). It suffices to show

\[ \Phi(u) = 0. \]  

(3.7)

If \( \hat{\varepsilon}_i(b \otimes r_\lambda) = 0 \), there is nothing to show. So we consider the case \( \hat{\varepsilon}_i(b \otimes r_\lambda) \neq 0 \).

Since \( \hat{\varepsilon}_i r_\lambda = 0 \), we have \( \hat{\varepsilon}_i(b \otimes r_\lambda) = (\hat{\varepsilon}_i b) \otimes r_\lambda \), which implies

\[ \varphi_i(b) \geq \varepsilon_i(r_\lambda) = -\langle h_i, \lambda \rangle, \]  

(3.8)

by (2.15). Now, assuming \( \Phi(u) = \pi_\lambda(b) = 0 \), we shall derive a contradiction. We have \( \hat{\varepsilon}_i(\pi_\lambda(b)) = \pi_\lambda(\hat{\varepsilon}_i(b)) = 0 \). Since \( \pi_\lambda(\hat{\varepsilon}_i(b)) = 0 \), we obtain \( \varphi_i(\pi_\lambda(b)) = 0 \) (see 2.22). Thus, taking into account (2.4), (2.5), (2.23) and (3.4), we have

\[ 0 = \varphi_i(\pi_\lambda(b)) = \langle h_i, wt(\pi_\lambda(b)) \rangle + \varepsilon_i(\hat{\pi}_\lambda(b)) = \langle h_i, \lambda + wt(b) + \alpha_i \rangle + \varepsilon_i(\hat{\pi}_\lambda(b)) \]

\[ = \langle h_i, \lambda \rangle + \langle h_i, wt(b) \rangle + 2 \varepsilon_i(b) - 1 = \langle h_i, \lambda \rangle + \varphi_i(b) - \varepsilon_i(b) + \varepsilon_i(b) + 1 \]

\[ = \langle h_i, \lambda \rangle + \varphi_i(b) + 1. \]

Thus, we have \( \varphi_i(b) = -\langle h_i, \lambda \rangle - 1 \leq \varepsilon_i(r_\lambda) \). Therefore, we have \( \hat{\varepsilon}_i(b \otimes r_\lambda) = 0 \) and completed to prove (4).

Finally, let us show (5). Since \( \tilde{f}_i \) commutes with \( \pi_\lambda \), for \( u = b \otimes r_\lambda \) we have \( \tilde{f}_i \Phi(u) = \tilde{f}_i \pi_\lambda(b) = \pi_\lambda(\tilde{f}_i b) \). Thus, if \( \tilde{f}_i u = (\tilde{f}_i b) \otimes r_\lambda \), we have \( \pi_\lambda(\tilde{f}_i b) = \Phi(\tilde{f}_i u) \), which means (5). So we consider the case \( \tilde{f}_i u = b \otimes \tilde{f}_i r_\lambda = 0 \). In this case, we shall try to show that \( \tilde{f}_i \Phi(u) = 0 \). It follows from (2.16) that we have

\[ \varphi_i(b) \leq \varepsilon_i(r_\lambda) = -\langle h_i, \lambda \rangle. \]  

(3.9)

If \( \Phi(u) = \pi_\lambda(b) = 0 \), there is nothing to show. So we may consider the case \( \Phi(u) = \pi_\lambda(b) \neq 0 \). Assuming

\[ \tilde{f}_i \Phi(u) = \tilde{f}_i \pi_\lambda(b) \neq 0, \]  

(3.10)
we shall derive a contradiction. The assumption \( (3.10) \) means \( \varphi_i(\pi_\lambda(b)) > 0 \). Thus, by \( (3.5) \) we have \( 0 < \langle h_i, \lambda \rangle + \varphi_i(b) \). It implies \( \varphi_i(b) > -\langle h_i, \lambda \rangle = \varepsilon_i(r_\lambda) \), which contradicts \( (3.9) \). Now we obtain \( f_i \Phi_\lambda(u) = 0 \). Thus, we have completed to show (5) and then (i).

Let us show (ii). The condition \( \Phi_\lambda(b \otimes r_\lambda) \neq 0 \) is equivalent to \( \pi_\lambda(b) \neq 0 \). Thus, the map \( \phi_\lambda := \Phi_\lambda|_{\tilde{B}(\lambda)} : \tilde{B}(\lambda) \longrightarrow B(\lambda) \) is bijective by \( (2.26) \). So if we show that \( \tilde{B}(\lambda) \) is stable by the actions of \( \tilde{e}_i \) and \( \tilde{f}_i \), it follows from (i) that the map \( \phi_\lambda \) is a strict morphism of crystals. Let us see the stability of \( \tilde{B}(\lambda) \), namely, that if \( \Phi_\lambda(\tilde{e}_i(b \otimes r_\lambda)) = 0 \) (resp. \( \Phi_\lambda(\tilde{f}_i(b \otimes r_\lambda)) = 0 \)) for \( b \otimes r_\lambda \in \tilde{B}(\lambda) \), then \( \tilde{e}_i(b \otimes r_\lambda) = 0 \) (resp. \( \tilde{f}_i(b \otimes r_\lambda) = 0 \)).

First, for \( b \otimes r_\lambda \in \tilde{B}(\lambda) \) suppose that \( \Phi_\lambda(\tilde{e}_i(b \otimes r_\lambda)) = 0 \), which implies \( \tilde{e}_i(\pi_\lambda(b)) = 0 \) by (i). Since \( \pi_\lambda(b) \neq 0 \), we have \( \tilde{e}_i b = 0 \) by Lemma 3.4. Thus, we obtain \( \tilde{e}_i(b \otimes r_\lambda) = 0 \) in view of \( (2.15) \) and \( \tilde{e}_i(r_\lambda) = 0 \). Next, for \( b \otimes r_\lambda \in \tilde{B}(\lambda) \) suppose that \( \Phi_\lambda(\tilde{f}_i(b \otimes r_\lambda)) = 0 \), which implies \( \tilde{f}_i(\pi_\lambda(b)) = 0 \). It follows from \( \pi_\lambda(b) \neq 0 \) that \( \varphi_i(\pi_\lambda(b)) = 0 \). Then, we have

\[
\varphi_i(\pi_\lambda(b)) = \varphi_i(b) + \langle h_i, \lambda \rangle = 0 \iff \varphi_i(b) = -\langle h_i, \lambda \rangle = \varepsilon_i(r_\lambda).
\]

This means \( \tilde{f}_i(b \otimes r_\lambda) = b \otimes \tilde{f}_i r_\lambda = 0 \). Now we have completed to prove (ii) and then Theorem 3.1.

**Example 3.5** Let us see the simplest example \( g = \mathfrak{sl}_2 \)-case. Let \( u_\infty \) be the highest weight vector in \( B(\infty) \). Then we have \( B(\infty) = \{ f^n u_\infty \} \). The crystal graph of \( B(\infty) \) is as follows:

![Crystal Graph](image)

where \( \tilde{2} = \tilde{f}_2 u_\infty \).

Next, let us see the crystal graph of \( B(\infty) \otimes R_m \) \((m \in \mathbb{Z}_{\geq 0}) \). We know that \( \varphi(f^n u_\infty) = -n \) and \( \varepsilon(r_m) = -m \). Then, by \( (2.16) \) we have

\[
\tilde{f}(f^n u_\infty \otimes r_m) = \begin{cases} f^{n+1} u_\infty \otimes r_m & \text{if } n < m, \\ f^n u_\infty \otimes \tilde{f}(r_m) = 0 & \text{if } n \geq m. \end{cases}
\]

Thus, the crystal graph of \( B(\infty) \otimes R_m \) is:

![Crystal Graph](image)

where \( \tilde{2} = \tilde{f}_2 u_\infty \otimes r_m \). The connected component including \( 0 = u_\infty \otimes r_m \) is isomorphic to the crystal \( B(m) \) associated with the \( m+1 \)-dimensional irreducible module \( V(m) \). In the subsequent section, we shall see how to remove the vectors cut off from \( B(\lambda) \) (in this case, the vectors \( \{ \tilde{2} x > m \} \)).

### 4 Polyhedral Realization of \( B(\lambda) \)

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4.1 Crystal structure of $\mathbb{Z}^\infty[\lambda]$

We shall give an explicit crystal structure of $\mathbb{Z}^\infty[\lambda]$ in a similar manner to \cite{13}. Fix a sequence of indices $i := (i_k)_{k \geq 1}$ satisfying the condition (2.28) and a weight $\lambda \in P$. (In this subsection, we do not necessarily assume that $\lambda$ is dominant.) As we stated in 3.1, we can identify $\mathbb{Z}^\infty$ with $\mathbb{Z}^\infty[\lambda]$ as a set. Thus $\mathbb{Z}^\infty[\lambda]$ can be regarded as a subset of $\mathbb{Q}^\infty$, and then we denote an element in $\mathbb{Z}^\infty[\lambda]$ by $\vec{x} = (\cdots, x_k, \cdots, x_2, x_1)$. For $\vec{x} = (\cdots, x_k, \cdots, x_2, x_1) \in \mathbb{Q}^\infty$ we define the linear functions

$$\sigma_k(\vec{x}) := x_k + \sum_{j > k} \langle h_{i_j}, \alpha_{i_j} \rangle x_j, \quad (k \geq 1) \quad (4.1)$$

$$\sigma_0^{(i)}(\vec{x}) := -\langle h_i, \lambda \rangle + \sum_{j \geq 1} \langle h_i, \alpha_{i_j} \rangle x_j, \quad (i \in I) \quad (4.2)$$

Here note that since $x_j = 0$ for $j \gg 0$ on $\mathbb{Q}^\infty$, the functions $\sigma_k$ and $\sigma_0^{(i)}$ are well-defined. Let $\sigma^{(i)}(\vec{x}) := \max_{i_k = i} \sigma_k(\vec{x})$, and

$$M^{(i)} = M^{(i)}(\vec{x}) := \{ k : i_k = i, \sigma_k(\vec{x}) = \sigma^{(i)}(\vec{x}) \}. \quad (4.3)$$

Note that $\sigma^{(i)}(\vec{x}) \geq 0$, and that $M^{(i)} = M^{(i)}(\vec{x})$ is a finite set if and only if $\sigma^{(i)}(\vec{x}) > 0$. Now we define the maps $\tilde{e}_i : \mathbb{Z}^\infty[\lambda] \cup \{ 0 \} \rightarrow \mathbb{Z}^\infty[\lambda] \cup \{ 0 \}$ and $\tilde{f}_i : \mathbb{Z}^\infty[\lambda] \cup \{ 0 \} \rightarrow \mathbb{Z}^\infty[\lambda] \cup \{ 0 \}$ by setting $\tilde{e}_i(0) = \tilde{f}_i(0) = 0,$ and

$$(\tilde{f}_i(\vec{x}))_k = x_k + \delta_{k, \min M^{(i)}} \text{ if } \sigma^{(i)}(\vec{x}) > \sigma_0^{(i)}(\vec{x}); \text{ otherwise } \tilde{f}_i(\vec{x}) = 0, \quad (4.4)$$

$$(\tilde{e}_i(\vec{x}))_k = x_k - \delta_{k, \max M^{(i)}} \text{ if } \sigma^{(i)}(\vec{x}) > 0 \text{ and } \sigma^{(i)}(\vec{x}) \geq \sigma_0^{(i)}(\vec{x}); \text{ otherwise } \tilde{e}_i(\vec{x}) = 0, \quad (4.5)$$

where $\delta_{i,j}$ is the Kronecker’s delta. We also define the weight function and the functions $\varepsilon_i$ and $\varphi_i$ on $\mathbb{Z}^\infty[\lambda]$ by

$$\text{wt}(\vec{x}) := \lambda - \sum_{j = 1}^\infty x_j \alpha_{i_j}, \quad \varepsilon_i(\vec{x}) := \max(\sigma^{(i)}(\vec{x}), \sigma_0^{(i)}(\vec{x}))$$

$$\varphi_i(\vec{x}) := \langle h_i, \text{wt}(\vec{x}) \rangle + \varepsilon_i(\vec{x}). \quad (4.6)$$

We will denote this crystal by $\mathbb{Z}^\infty[\lambda]$. Note that, in general, the subset $\mathbb{Z}^\infty[\lambda]_{\geq 0}$ is not a subcrystal of $\mathbb{Z}^\infty[\lambda]$ since it is not stable under the action of $\tilde{e}_i$’s.

4.2 The image of $\Psi_\lambda^{(\pm)}$

As in the previous sections, we fix a sequence of indices $i$ satisfying (2.28) and take a dominant integral weight $\lambda \in P$. For $k \geq 1$ let $k^{(\pm)}$ be the ones in 2.4. Let $\beta^{(\pm)}_k(\vec{x})$ be linear functions given by

$$\beta^{(+)}_k(\vec{x}) = \sigma_k(\vec{x}) - \sigma_k^{(+)}(\vec{x}), \quad (k \geq 1) \quad (4.7)$$

$$\beta^{(-)}_k(\vec{x}) = \begin{cases} \sigma_k^{(-)}(\vec{x}) - \sigma_k(\vec{x}) \quad \text{if } k^{(-)} > 0, \\ \sigma_0^{(i_k)}(\vec{x}) - \sigma_k(\vec{x}) \quad \text{if } k^{(-)} = 0, \end{cases} \quad (4.8)$$
where the functions $\sigma_k$ and $\sigma_k^{(i)}$ are defined by (4.1) and (4.2). Since $\langle h_i, \alpha_i \rangle = 2$ for any $i \in I$, we have

$$\beta^{(+)}_k(\vec{x}) = x_k + \sum_{k < j < k} \langle h_{k_j}, \alpha_{i_j} \rangle x_j + x_{k^{(+)}}, \quad (4.9)$$

$$\beta^{(-)}_k(\vec{x}) = \begin{cases} x_{k^{(-)}} + \sum_{k^{(-)} < j < k} \langle h_{k_j}, \alpha_{i_j} \rangle x_j + x_k & \text{if } k^{(-)} > 0, \\ -\langle h_{k_k}, \lambda \rangle + \sum_{1 \leq j < k} \langle h_{k_j}, \alpha_{i_j} \rangle x_j + x_k & \text{if } k^{(-)} = 0. \end{cases} \quad (4.10)$$

Here note that

$$\beta^{(+)}_k = \beta_k, \quad \beta^{(-)}_k = \beta^{(-)}_k \quad \text{if } k^{(-)} > 0.$$

Using this notation, for every $k \geq 1$, we define an operator $\hat{S}_k = \hat{S}_{k, i}$ for a linear function $\varphi(\vec{x}) = c + \sum_{k \geq 1} \varphi_k x_k (c, \varphi_k \in \mathbb{Q})$ on $\mathbb{Q}^\infty$ by:

$$\hat{S}_k (\varphi) := \begin{cases} \varphi - \varphi_k \beta^{(+)}_k & \text{if } \varphi_k > 0, \\ \varphi - \varphi_k \beta^{(-)}_k & \text{if } \varphi_k \leq 0. \end{cases} \quad (4.11)$$

An easy check shows that $(\hat{S}_k)^2 = \hat{S}_k$.

For the fixed sequence $\iota = (i_k)$, in case $k^{(-)} = 0$ for $k \geq 1$, there exists unique $i \in I$ such that $i_k = i$. We denote such $k$ by $i^{(i)}$, namely, $i^{(i)}$ is the first number $k$ such that $i_k = i$.

Here we set

$$\lambda^{(i)}(\vec{x}) := -\beta^{(-)}_{i^{(i)}}(\vec{x}) = \langle h_i, \lambda \rangle - \sum_{1 \leq j < i^{(i)}} \langle h_i, \alpha_{i_j} \rangle x_j - x_{i^{(i)}}. \quad (4.12)$$

For $\iota$ and a dominant integral weight $\lambda$, let $\Xi_\iota[\lambda]$ be the set of all linear functions generated by applying $\hat{S}_k = \hat{S}_{k, \iota}$ on the functions $x_j$ ($j \geq 1$) and $\lambda^{(i)}$ ($i \in I$), namely,

$$\Xi_\iota[\lambda] := \{ \hat{S}_{j_1} \cdots \hat{S}_{j_k} x_{j_0} : l \geq 0, j_0, \ldots, j_l \geq 1 \} \cup \{ \hat{S}_{j_k} \cdots \hat{S}_{j_1} \lambda^{(i)}(\vec{x}) : k \geq 0, i \in I, j_1, \ldots, j_k \geq 1 \}. \quad (4.13)$$

Now we set

$$\Sigma_\iota[\lambda] := \{ \vec{x} \in \mathbb{Z}_i^\infty[\lambda](\subset \mathbb{Q}^\infty) : \varphi(\vec{x}) \geq 0 \text{ for any } \varphi \in \Xi_\iota[\lambda] \}. \quad (4.14)$$

For a sequence $\iota$ and a dominant integral weight $\lambda$, a pair $(\iota, \lambda)$ is called ample if $\Sigma_\iota[\lambda] \ni \vec{0} = (\cdots, 0, 0)$.

**Theorem 4.1** Suppose that $(\iota, \lambda)$ is ample. Let $\Psi^{(\lambda)}_\iota : B(\lambda) \hookrightarrow \mathbb{Z}_i^\infty[\lambda]$ be the embedding as in (4.2). Then the image $\text{Im}(\Psi^{(\lambda)}_\iota) (\cong B(\lambda))$ is equal to $\Sigma_\iota[\lambda]$.
Proof. Taking into account of (2.21) and Theorem 3.2, the image \( \text{Im}(\Psi_{\lambda}) \) is a subcrystal of \( \mathbb{Z}^\infty_{\geq 0} \) obtained by applying \( \tilde{f}_i \)'s to \( \Psi_{\lambda} \)'s \( u_{\lambda} = (\cdots, 0, 0) \), that is,

\[
\text{Im}(\Psi_{\lambda}) = \{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_l} \Psi_{\lambda}(u_{\lambda}) | i_j \in I, l \geq 0 \} \setminus \{ 0 \}. \tag{4.15}
\]

By the explicit description of \( \tilde{f}_i \) in (4.14), we know that \( \text{Im}(\Psi_{\lambda}) \subset \mathbb{Z}^\infty_{\geq 0} \).

Since the pair \((i, \lambda)\) is ample, \( \Sigma_i[\lambda] \ni 0 \). Thus, the inclusion \( \text{Im}(\Psi_{\lambda}) \subset \Sigma_i[\lambda] \) follows from the fact that the set \( \Sigma_i[\lambda] \) is closed by the actions of \( \tilde{f}_i \)'s, namely, \( \tilde{f}_i \Sigma_i[\lambda] \subset \Sigma_i[\lambda] \cup \{ 0 \} \) for any \( i \in I \). Let us show this. For \( \vec{x} = (\cdots, x_2, x_1) \in \Sigma_i[\lambda] \) and \( i \in I \), suppose that \( \tilde{f}_i \vec{x} = (\cdots, x_k + 1, \cdots, x_2, x_1) \) (note that \( i_k = i \)). We shall show

\[
\varphi(\tilde{f}_i \vec{x}) \geq 0, \tag{4.16}
\]

for any \( \varphi(\vec{x}) = c + \sum \varphi_j x_j \in \Xi_i[\lambda] \). Since \( \varphi(\tilde{f}_i \vec{x}) = \varphi(\vec{x}) + \varphi_k \geq \varphi_k \), it suffices to consider the case \( \varphi_k \leq 0 \). By the definition of \( M_i(\vec{x}) \) in (4.13), we know that \( k \) is the minimum in \( M_i(\vec{x}) \). Thus, it follows from (4.14) that \( \sigma_k(\vec{x}) > \sigma_{k-1}(\vec{x}) \) if \( k > 0 \) or \( \sigma_k(\vec{x}) > \sigma_0(\vec{x}) \) if \( k = 0 \). Thus, by (4.15), we have \( \beta_k^{-}(\vec{x}) < 0 \). Therefore, since the function \( \beta_k^{-}(\vec{x}) \) takes an integer value for \( \vec{x} \in \mathbb{Z}^\infty \),

\[
\beta_k^{-}(\vec{x}) \leq -1. \tag{4.17}
\]

It follows from \( \hat{S}_k \varphi \in \Xi_i[\lambda] \) and \( \varphi_k < 0 \) that

\[
\varphi(\tilde{f}_i \vec{x}) = \varphi(\vec{x}) + \varphi_k \geq \varphi(\vec{x}) - \varphi_k \beta_k^{-}(\vec{x}) = (\hat{S}_k \varphi)(\vec{x}) \geq 0. \tag{4.18}
\]

Therefore, we get the inclusion \( \text{Im}(\Psi_{\lambda}) \subset \Sigma_i[\lambda] \).

Let us show the reverse inclusion \( \Sigma_i[\lambda] \subset \text{Im}(\Psi_{\lambda}) \). We first show that \( \Sigma_i[\lambda] \) is a subcrystal of \( \mathbb{Z}^\infty_{\geq 0} \). Since we have already shown that \( \tilde{f}_i \Sigma_i[\lambda] \subset \Sigma_i[\lambda] \cup \{ 0 \} \) for any \( i \in I \), it is enough to prove that \( \hat{e}_i \Sigma_i[\lambda] \subset \Sigma_i[\lambda] \cup \{ 0 \} \) for any \( i \in I \). For \( \vec{x} = (\cdots, x_2, x_1) \in \Sigma_i[\lambda] \) and \( i \in I \), suppose that \( \hat{e}_i \vec{x} = (\cdots, x_k - 1, \cdots, x_2, x_1) \), here note that \( i_k = i \). We have to show

\[
\varphi(\hat{e}_i \vec{x}) \geq 0. \tag{4.19}
\]

Since \( \varphi(\hat{e}_i \vec{x}) = \varphi(\vec{x}) - \varphi_k \geq -\varphi_k \), it suffices to consider the case \( \varphi_k > 0 \). Arguing similarly to the \( \tilde{f}_i \) case, by (4.13), (4.15) and (4.17), we have

\[
\beta_k^{(+)}(\vec{x}) \geq 1. \tag{4.18}
\]

It follows from \( \hat{S}_k \varphi \in \Xi_i[\lambda] \) and \( \varphi_k > 0 \) that

\[
\varphi(\hat{e}_i \vec{x}) = \varphi(\vec{x}) - \varphi_k \geq \varphi(\vec{x}) - \varphi_k \beta_k^{(+)}(\vec{x}) = (\hat{S}_k \varphi)(\vec{x}) \geq 0.
\]

Since \( \Sigma_i[\lambda] \) is included in \( \mathbb{Z}^\infty_{\geq 0} \) and is closed under the actions of \( \hat{e}_i \), for any \( \vec{x} \in \Sigma_i[\lambda] \) there exists \( l \gg 0 \) such that \( \hat{e}_i \vec{e}_{i_2} \cdots \hat{e}_{i_l} \vec{x} = 0 \) for any \( i_1, \cdots, i_l \in I \).
Remark. Let us define the set of linear forms $\Xi = \{ \xi \}$ from $\Xi$.

Theorem 4.2 Let $i$ be a sequence of indices satisfying (2.23) and the strict positivity assumption, and $\lambda$ be a dominant integral weight. Then for $i \in I$ and $\bar{x} \in \Sigma_i$ we have

$$\e_i^*(\bar{x}) = \max \{-\varphi(\bar{x}) | \varphi \in \Xi_i^{(i)} \} \quad (4.24)$$
Proof of Theorem 4.2. First, let us show that the ampleness is always satisfied under the strict positivity assumption. To do this, we see the following lemma:

**Lemma 4.3** Under the strict positivity assumption for $\iota$, we have

$$\hat{S}_{j_1} \cdots \hat{S}_{j_l} x_{j_0} = S_{j_1} \cdots S_{j_l} x_{j_0},$$

(4.25)

for any $l \geq 0$, $j_0, \cdots, j_l \geq 1$, and

$$\hat{S}_{j_1} \cdots \hat{S}_{j_l} \lambda^{(i)}(\bar{x}) = \langle h_i, \lambda \rangle + S_{j_1} \cdots S_{j_l} \xi^{(i)}(\bar{x}),$$

(4.26)

for any $l \geq 0$, $j_1, \cdots, j_l \geq 1$ and $i \in I$, if the L.H.S. of (4.26) is non-zero.

**Proof.** First we show (4.25). By the definition of $\beta_k^{(+)}$ and $\beta_k$ we know that if $k(\cdot) > 0$, $\hat{S}_k = S_k$. Furthermore, even if $k(\cdot) = 0$, under the positivity assumption, $\hat{S}_k = S_k$ because in this case their actions are given by using only $\beta_k^{(+)} = \beta_k$.

Next we shall see (4.26). Let us show it by the induction on $l$. If $l = 0$, (4.26) is just the equation $\lambda^{(i)} = \langle h_i, \lambda \rangle + \xi^{(i)}$ in (4.12). Now we assume (4.26) for $l > 0$ and write $\varphi(\bar{x}) = c + \sum_k \varphi_k x_k \neq 0$ for the both sides of (4.26). First, if $k(\cdot) \neq 0$, we have $\hat{S}_k \varphi = S_k \varphi$ since $\beta_k^{(+)} = \beta_k$ and $\beta_k^{(-)} = \beta_k(\cdot)$. If $k(\cdot) = 0$ and $\varphi \neq \lambda^{(i)}$, by the strict positivity assumption, we have $\varphi_k \geq 0$ and then

$$\hat{S}_k \varphi = \varphi - \varphi_k \beta_k^{(+)} = \varphi - \varphi_k \beta_k = S_k \varphi.$$

Finally, we consider the case $k(\cdot) = 0$ and $\varphi = \lambda^{(i)}$. In this case we have $k = \iota^{(j)}$ for some $j \in I$. By the explicit form of $\lambda^{(i)}$ we have $\varphi_i(\cdot) = -\langle h_i, \alpha_j \rangle \geq 0$ for $j \neq i$ in $\varphi = \lambda^{(i)}$. Thus, if $k = \iota^{(j)} (j \neq i)$, $\hat{S}_k \varphi = S_k \varphi$ by the fact $\beta_k^{(+)} = \beta_k$. If $k = \iota^{(j)}$, the coefficient of $x_{\iota^{(j)}}$ in $\lambda^{(i)}$ is $-1$. In this case, $S_k \varphi = \varphi$ and $\hat{S}_k \varphi = 0$. Then this is not the case of (4.26).

This lemma implies that under the strict positivity assumption, any linear function in $\Xi_i[\lambda]$ has a non-negative coefficient 0 or $\langle h_i, \lambda \rangle$, which means $(\iota, \lambda)$ is ample. Therefore, we have

$$\widetilde{B}(\lambda) = \{ \bar{x} \in B(\infty) \otimes R_{\lambda} \subset \mathbb{Z}_{\geq 0}[\lambda] \mid \langle h_i, \lambda \rangle + \varphi(\bar{x}) \geq 0 \text{ for any } i \in I \text{ and } \varphi \in \Xi_i^{(i)} \}.$$  

(4.27)

It follows from Proposition 2.8 and (4.27) that the condition $\varepsilon_i(\bar{x}) \leq \langle h_i, \lambda \rangle$ is equivalent to $-\varphi(\bar{x}) \leq \langle h_i, \lambda \rangle$ for any $\varphi \in \Xi_i^{(i)}$.

**Corollary 4.4** Let $\iota$ be the same one as in Theorem 4.2 and $\lambda$ be a dominant integral weight. Then we have:

$$B(\lambda) = \text{Im}(\Psi^{(\lambda)}_\iota) = \{ \bar{x} \in \Sigma_{\iota} \otimes R_{\lambda} \mid \langle h_i, \lambda \rangle + \varphi(\bar{x}) \geq 0 \text{ for any } i \in I \text{ and } \varphi \in \Xi_i^{(i)} \}.$$  

(4.28)
Furthermore, we also obtain the following combinatorial expression for the weight multiplicities and the tensor-product multiplicities as follows: The weight function of $Z^∞[\lambda]$ is described explicitly by (4.6): $\text{wt}(\vec{x}) = \lambda - \sum_k x_k \alpha_i$. Set $W(\lambda) := \{ \nu \in P | B(\lambda)_\nu \neq \emptyset \}$ and denote the weight multiplicity of $\nu$ in $B(\lambda)$ by $M_{\lambda,\nu}$. Any $\nu \in W(\lambda)$ is in the form $\lambda - \sum_i m_i \alpha_i$ ($m_i \in \mathbb{Z}_{\geq 0}$). Then we have

**Corollary 4.5** For $\nu = \lambda - \sum_i m_i \alpha_i \in W(\lambda)$, the weight multiplicity of $\nu$ is given by

$$M_{\lambda,\nu} = \# \{ \vec{x} \in \tilde{B}(\lambda) | m_i = \sum_{i_k = i} x_k \text{ for any } i \in I \}.$$  \hfill (4.29)

Now, we describe so-called the Littlewood-Richardson number $c_{\lambda,\mu}^\nu$. More precisely, for dominant integral weights $\lambda, \mu$ and $\nu$, let $c_{\lambda,\mu}^\nu$ be the number of irreducible components $V(\nu)$ in the tensor product $V(\lambda) \otimes V(\mu)$. Of course, it is same as the number of connected components $B(\nu)$ in tensor product $B(\lambda) \otimes B(\mu)$. To do this we need the following proposition similar to Proposition 3.2.1 [12]:

**Proposition 4.6** For dominant integral weights $\lambda$ and $\mu$, an element $u \otimes v \in B(\lambda) \otimes B(\mu)$ satisfies $\tilde{e}_i(u \otimes v) = 0$ for any $i \in I$ if and only if $\tilde{e}_i u = 0$ and $\tilde{e}_i^{(h_i,\lambda)+1} v = 0$ for any $i \in I$.

**Proof.** The argument in the proof of Proposition 3.2.1 [12] can be applied to any integrable highest weight modules for symmetrizable Kac-Moody Lie algebras. \hfill \Box

Here note that the condition $\tilde{e}_i^{(h_i,\lambda)+1} v = 0$ is equivalent to the one $\varepsilon_i(v) \leq \langle h_i, \lambda \rangle$ and the explicit form of $\varepsilon_i$ is given in (4.6). Here we set

$$E^{(i)} := \{ \sigma_k(\vec{x}) : i_k = i \} \cup \{ \sigma_0^{(i)}(\vec{x}) \}.$$  \hfill (4.30)

**Corollary 4.7** For dominant integral weight $\lambda, \mu$ and $\nu$, we have

$$c_{\lambda,\mu}^\nu = \# \{ \vec{x} \in \tilde{B}(\mu) | \text{wt}(\vec{x}) = \nu - \lambda \text{ and } \zeta(\vec{x}) \leq \langle h_i, \lambda \rangle \text{ for any } i \in I \text{ and } \zeta \in E^{(i)} \}.$$  \hfill (4.31)

## 5 Rank 2 case

In this section, we apply Theorem 4.2 and Corollary 4.4 to the case for the Kac-Moody algebras of rank 2. We adopt the same setting as in [13, Sect.4]. Without loss of generality, we can and will assume that $I = \{1,2\}$, and $\iota = \{ \cdots, 2, 1, 2, 1 \}$. The Cartan data is given by:

$$\langle h_1, \alpha_1 \rangle = \langle h_2, \alpha_2 \rangle = 2, \quad \langle h_1, \alpha_2 \rangle = - c_1, \quad \langle h_2, \alpha_1 \rangle = - c_2.$$
Here we either have $c_1 = c_2 = 0$, or both $c_1$ and $c_2$ are positive integers. We set $X = c_1 c_2 - 2$, and define the integer sequence $a_l = a_l(c_1, c_2)$ for $l \geq 0$ by setting $a_0 = 0$, $a_1 = 1$ and, for $k \geq 1$,
\[ a_{2k} = c_1 P_{k-1}(X), \quad a_{2k+1} = P_k(X) + P_{k-1}(X), \]  
(5.1)
where the $P_k(X)$ are Chebyshev polynomials given by the following generating function:
\[ \sum_{k \geq 0} P_k(X) z^k = (1 - X z + z^2)^{-1}. \]  
(5.2)

Here define $a'_l(c_1, c_2) := a_l(c_2, c_1)$. Let $l_{\max} = l_{\max}(c_1, c_2)$ be the minimal index $l$ such that $a_{l+1} < 0$ (if $a_l \geq 0$ for all $l \geq 0$, then we set $l_{\max} = +\infty$). By inspection, if $c_1 c_2 = 0$ (resp. 1, 2, 3) then $l_{\max} = 2$ (resp. 3, 4, 6). Furthermore, if $c_1 c_2 \leq 3$ then $a_{l_{\max}} = 0$ and $a_l > 0$ for $1 \leq l < l_{\max}$. On the other hand, if $c_1 c_2 \geq 4$, i.e., $X \geq 2$, it is easy to see from (5.2) that $P_k(X) > 0$ for $k \geq 0$, hence $a_l > 0$ for $l \geq 1$; in particular, in this case $l_{\max} = +\infty$.

**Theorem 5.1**  
(i) In the rank 2 case, for a dominant integral weight $\lambda = \lambda_1 \Lambda_1 + \lambda_2 \Lambda_2$ ($\lambda_1, \lambda_2 \in \mathbb{Z}_{\geq 0}$) the image of the embedding $\Psi_1^{(\lambda)}$ is given by
\[ \text{Im} \left( \Psi_1^{(\lambda)} \right) = \left\{ (\cdots, x_2, x_1) \in \mathbb{Z}_{\geq 0}^\infty : \begin{array}{l} x_k = 0 \text{ for } k > l_{\max}, \lambda_1 \geq x_1, \\ a_l x_l - a_{l-1} x_{l+1} \geq 0, \\ \lambda_2 + a'_l x_l - a'_{l+1} x_{l+1} \geq 0, \\ \text{for } 1 \leq l < l_{\max} \end{array} \right\}. \]  
(5.3)

(ii) For any $b \in B(\infty)$, writing $\Psi_1(b) = (\cdots, x_2, x_1)$, we have
\[ \varepsilon_1^*(b) = x_1, \quad \varepsilon_2^*(b) = \max_{1 \leq l \leq l_{\max}} \left\{ a'_l x_{l+1} - a'_{l+1} x_l \right\}. \]  
(5.4)

**Proof.** In order to apply Corollary 4.4 we shall describe the set of linear functions $\Xi^{(\infty)}_i$ and $\Xi^{(i)}_i$, and check that $i$ satisfies the strict positivity assumption.

The set $\Xi^{(\infty)}_i = \Xi$ has been given in 4.3. Lemma 4.2]. In particular, it is shown that the positivity assumption is satisfied. Hence, by Lemma 4.2 in 4.3, we have

**Lemma 5.2**  
(i) For $k \geq 1$ and $0 \leq l < l_{\max}$, we set
\[ \varphi_k^{(l)} = S_{k+l-1} \cdots S_{k+1} S_k x_k; \]  
(5.5)
in particular, $\varphi_k^{(0)} = x_k$. If $k$ is odd then $\varphi_k^{(l)} = a_{l+1} x_{k+l} - a_l x_{k+l+1}$; if $k$ is even then $\varphi_k^{(l)} = a'_{l+1} x_{k+l} - a'_{l} x_{k+l+1}$.  

(ii) If $c_1 c_2 \leq 3$, i.e., $l_{\max} < +\infty$, then $\varphi_k^{(l_{\max}-1)} = -x_k l_{\max}$. 

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(iii) The set \( \Xi^{(i)} \) consists of all linear forms \( \varphi_k^{(i)} \) with \( k \geq 1 \) and \( 0 \leq l < l_{\text{max}} \).

(iv) The positivity assumption for the sequence \( \iota \) is satisfied.

Now we return to the proof of Theorem 5.1. It is remained to describe
\[
\Xi^{(i)} = \{ S_{j_1} \cdots S_{j_k} \xi^{(i)}(\vec{x}) \mid k \geq 0, j_1, \ldots, j_k \geq 1 \}. \quad (i, 2)
\]
Here we see the explicit form of \( \xi^{(i)} \):
\[
\xi^{(1)} = -x_1, \quad \xi^{(2)} = c_2x_1 - x_2. \quad (5.6)
\]
It is evident that \( \Xi^{(1)} = \{-x_1\} \). The proof of the theorem is completed by the following lemma:

**Lemma 5.3**

(i) For \( 1 \leq l < l_{\text{max}} \), we set
\[
\eta_l = S_{l-1} \cdots S_2 S_1 (\xi^{(2)}); \quad (5.7)
\]
in particular, \( \eta_1 = \xi^{(2)} = c_2x_1 - x_2 \). Then we have \( \eta_l = a_{l+1}'x_l - a_l'x_{l+1} \).

(ii) If \( c_1 c_2 \leq 3 \), i.e., \( l_{\text{max}} = +\infty \), then \( \eta_{l_{\text{max}} - 1} = -x_{l_{\text{max}}} \).

(iii) The set \( \Xi^{(2)}_l \) consists of all linear forms \( \eta_l \) with \( 1 \leq l < l_{\text{max}} \).

(iv) Any element in \( \Xi^{(2)}_l \setminus \{\xi^{(2)}\} \) has non-negative coefficients for \( x_1 \) and \( x_2 \).

**Proof.** We can check (ii) by direct calculations for \( c_1 c_2 = 0, 1, 2, 3 \). The statement (iv) is immediate from (i) and (iii). Thus we shall show (i) and (iii).

Since \( a_l' \geq 0 \), we have
\[
S_{2k}(a_{2k+1}'x_{2k} - a_{2k}'x_{2k+1}) = a_{2k+1}'x_{2k} - a_{2k}'x_{2k+1} - a_{l+1}'x_l - a_l'x_{l+1} = (c_{2k}' - a_{2k}'x_{2k+1} + a_{2k}'x_{2k+2} = a_{2k+2}'x_{2k+1} - a_{2k+1}'x_{2k+2},
\]
where we use the relation \( a_{2k+2}' = c_2a_{2k+1}' - a_{2k}' \). Thus we get \( S_{2k} \eta_{2k} = \eta_{2k+1} \).

Similarly, we obtain \( S_{2k-1} \eta_{2k-1} = \eta_{2k} \), \( S_{2k+1} \eta_{2k} = \eta_{2k-1} \) and \( S_{2k} \eta_{2k-1} = \eta_{2k-2} \). We also have \( S_j \eta_k = \eta_k \) if \( j \neq k, k + 1 \). These imply (i) and also (iii).

Applying Lemma 5.2 and Lemma 5.3 to Corollary 4.4 we conclude that
\[
\text{Im} (\Psi^{(1)}_i) = \{(\cdots, x_2, x_1) \in \mathbb{Z}^\infty \mid \varphi_k^{(i-1)}(\vec{x}) \geq 0, \lambda_1 \geq x_1 \text{ and } \lambda_2 \geq -\eta_l(\vec{x}) \}
\]
for \( k \geq 1, 1 \leq l < l_{\text{max}} \). \( (5.8) \)

Comparing (5.8) with the desired answer (5.3), and using parts (i) and (ii) of Lemma 7.2 it only remains to show that the inequalities \( \varphi_k^{(i)} \geq 0 \) in (5.8) are redundant when \( k > 1 \) and \( l < l_{\text{max}} - 1 \), that is, they are consequences of the
remaining inequalities. This can be shown by the same way as in the proof of Theorem 4.1 in [13].

The proof of (ii) is evident from Theorem 4.2 and Lemma 5.3.

Note that the cases when \( \ell_{\text{max}} < +\infty \), or equivalently, the image \( \text{Im} (\Psi_i) \) is contained in a lattice of finite rank, just correspond to the Lie algebras \( g = A_1 \times A_1, A_2, B_2 \) or \( C_2, G_2 \).

In conclusion of this section, we illustrate Theorem 5.1 by the example when \( c_1 = c_2 = 2 \), i.e., \( g \) is the affine Lie algebra of type \( A^{(1)}_1 \), following to [13]. In this case, \( X = c_1c_2 - 2 = 2 \). It follows at once from (5.2) that \( P_k(2) = k + 1 \); hence, (5.1) gives \( a_l = l \) for \( l \geq 0 \). We see that for type \( A^{(1)}_1 \),

\[
B(\lambda) \cong \text{Im} (\Psi_i^{(\lambda)}) = \{ (\cdots, x_2, x_1) \in Z_\geq 0^n : lx_l - (l - 1)x_{l+1} \geq 0, \lambda_1 \geq x_1 \text{ and } \lambda_2 + (l + 1)x_l - lx_{l+1} \geq 0 \text{ for } l \geq 1 \},
\]

and for \( \bar{x} = (\cdots, x_2, x_1) \in \Sigma_i \) we have

\[
\varepsilon^*_i(\bar{x}) = x_1, \text{ and } \varepsilon^*_2(\bar{x}) = \max_{l \geq 1} \{ lx_l - (l + 1)x_l \}.
\]

6  \( A_n \)-case

We shall apply Theorem 4.2 and Corollary 4.4 to the case when \( g \) is of type \( A_n \). Let us identify the index set \( I \) with \( [1, n] := \{ 1, 2, \cdots, n \} \) in the standard way; thus, the Cartan matrix \( (a_{i,j} = \langle h_i, \alpha_j \rangle)_{1 \leq i, j \leq n} \) is given by \( a_{i,i} = 2, a_{i,j} = -1 \) for \( |i - j| = 1 \), and \( a_{i,j} = 0 \) otherwise. As the infinite sequence \( \ell \) let us take the following periodic sequence

\[
\ell = \cdots, n, \cdots, 2, 1, \cdots, n, \cdots, 2, 1, n, \cdots, 2, 1.
\]

Following to [13, Sect.5], we shall change the indexing set for \( Z^\infty \) from \( Z_\geq 1 \) to \( Z_\geq 1 \times [1, n] \), which is given by the bijection \( Z_\geq 1 \times [1, n] \to Z_\geq 1 \ ( (j; i) \to (j - 1)n + i) \). According to this, we will write an element \( \bar{x} \in Z^\infty \) as a doubly-indexed family \( (x_{j;i})_{j \geq 1, i \in [1, n]} \). We will adopt the convention that \( x_{j;i} = 0 \) unless \( j \geq 1 \) and \( i \in [1, n] \); in particular, \( x_{j;0} = x_{j;n+1} = 0 \) for all \( j \).

**Theorem 6.1** Let \( \lambda = \sum_{1 \leq i \leq n} \lambda_i \Lambda_i \ (\lambda_i \in Z_\geq 0) \) be a dominant integral weight.

(i) In the above notation, the image \( \text{Im} (\Psi_i^{(\lambda)}) \) is the set of all integer families \( (x_{j;i}) \) such that

\[
x_{1;i} \geq x_{2;i-1} \geq \cdots \geq x_{i;1} \geq 0 \text{ for } 1 \leq i \leq n
\]

\[
x_{j;i} = 0 \text{ for } i + j > n + 1,
\]

\[
\lambda_i \geq x_{j;i-j+1} - x_{j;i-j} \text{ for } 1 \leq j \leq i \leq n.
\]
(ii) For any \( b \in B(\infty) \), writing \( \Psi_i(b) = (\cdots, x_2, x_1) \) we have
\[
\varepsilon_i^*(b) = \max_{1 \leq j \leq i} \{ x_{j;i-j+1} - x_{j;i-j} \}. \tag{6.4}
\]

Proof. We will follow the proof of Theorem 5.1. So we first describe the set of linear functions \( \Xi_i(i) \) \((i = 1, \cdots, n, \infty)\), and check that \( \iota \) satisfies the strict positivity assumption. As in the previous section, we set
\[
\Xi_0(\infty) := \{ S_{j_0} \cdots S_{j_l} x_{j_0} \mid l \geq 0, j_0, j_1, \cdots, j_l \geq 1 \},
\]
\[
\Xi_i(i) := \{ S_{j_k} \cdots S_{j_l} \xi_i(x) \mid k \geq 0, j_0, j_1, \cdots, j_k \geq 1 \} (i \in I),
\]

The explicit description of the set \( \Xi_i(\infty) = \Xi_i \) is given in \[13\] Lemma 5.2 and it is shown that the sequence \( \iota \) satisfies the positivity assumption, that is, in this setting any linear form \( \varphi = \sum_k \varphi_i x_{j_i} \in \Xi_i \) has the property \( \varphi_i; i \geq 0 \) for any \( i = 1, \cdots, n \). Then \[13\] follows from Theorem 6.1 in \[13\].

Therefore, in order to complete the proof, it is sufficient for us to show that
\[
\Xi_i(i) = \{ -x_{j;i-j+1} + x_{j;i-j} \mid 1 \leq j \leq i \} \quad \text{(6.5)}
\]
Let us write \( F(i) \) for the R.H.S. of (6.5). By the definition, we have \( \xi_i(i) = -x_{1;i} + x_{1;i-1} \) and then \( \xi^i(i) \in F(i) \). Here for \( (j; i) \in \mathbb{Z}_{\geq 1} \times [1, n] \), we will write the piecewise-linear transformation \( S_{j-1} \) as \( S_j \); if \( (j; i) \notin \mathbb{Z}_{\geq 1} \times [1, n] \) then \( S_{j;i} \) is understood as the identity transformation. By the direct calculations, we obtain immediately,
\[
S_{p; q}(-x_{j;i-j+1} + x_{j;i-j}) = \begin{cases} 
-x_{j+1;i-j} + x_{j+1;i-j} & \text{if } (p; q) = (j; i-j) \text{ and } j < i, \\
x_{j-1;i-j+2} + x_{j-1;i-j+1} & \text{if } (p; q) = (j; i-j+1) \text{ and } j \neq 1, \\
x_{j;i-j+1} + x_{j;i-j} & \text{otherwise},
\end{cases} \tag{6.6}
\]
where note that if \( j = i \), \( -x_{j;i-j+1} + x_{j;i-j} = -x_{i;i} \). This implies that \( F(i) \) is closed by the action of \( S_{p; q} \) and all elements are obtained from \( \xi^i(i) \), which shows (6.3). The strict positivity assumption follows from (6.5) immediately. Thus, by virtue of Theorem 4.2 and Corollary 4.4 if \( \bar{x} = \Psi_i(b) \) we have
\[
\varepsilon_i^*(b) = \max \{ x_{j;i-j+1} - x_{j;i-j} \mid 1 \leq j \leq i \},
\]
which implies (ii) and then we have (i).

As we mentioned in 2.4, we give the example which does not satisfy the positivity assumption.

**Example 6.2** We consider the case \( g = \mathfrak{sl}_4 \) and take the sequence \( \iota = \cdots 2 1 3 2 1 \), where we do not need the explicit form of “…” in \( \iota \). For the simplicity, we
write \( \vec{x} = (\cdots, x_2, x_1) \) for an element \( \vec{x} \in \mathbb{Z}_n^{-\infty} \). In this setting, we have \( \beta_1 = x_1 - x_2 - x_4 + x_5, \beta_2 = x_2 - x_3 + x_4 \) and \( 5^{(-)} = 1 \). Then \( S_1(x_1) = x_1 - \beta_1 = x_2 + x_4 - x_5, S_2 S_1(x_1) = x_2 + x_4 - x_5 - \beta_2 = x_3 - x_5 \) and \( S_5 S_2 S_1(x_1) = x_3 - x_5 + \beta_1 = x_1 - x_2 + x_3 - x_4 \). Thus we see the form \( S_5 S_2 S_1(x_1) \) has the negative coefficient for \( x_2 \), which breaks the positivity assumption. Furthermore, this case is not ample. Fix \( \lambda \in P_+ \) with \( \langle h_2, \lambda \rangle > 0 \). Since \( \beta_2^{(-)} = -\langle h_2, \lambda \rangle + x_2 - x_1 \) and \( S_5 S_2 S_1(x_1) = S_5 S_2 S_1(x_1) \), we have
\[
\tilde{S}_2 \tilde{S}_5 \tilde{S}_2 \tilde{S}_1(x_1) = x_1 - x_2 + x_3 - x_4 + \beta_2^{(-)} = -\langle h_2, \lambda \rangle + x_3 - x_4,
\]
which implies \( \vec{0} = (\cdots, 0, 0) \not\in \Sigma_0[\lambda] \) because of \( \langle h_2, \lambda \rangle > 0 \).

7 \( A_{\frac{1}{n} - 1}^{(1)} \)-case

In this section we shall treat the affine Lie algebra \( g = A_{\frac{1}{n} - 1}^{(1)} \). We will assume that \( n \geq 3 \) since the case of \( A_{\frac{1}{1} - 1}^{(1)} \) was already treated in Sect.5. As in \([13]\) we will identify the index set \( I \) with \([1, n] \) in the way such that the Cartan matrix \( (a_{i,j} = \langle a_i, a_j \rangle)_{1 \leq i, j \leq n} \) is given by \( a_{i,i} = 2, \ a_{i,j} = -1 \) for \( |i-j| = 1 \) or \( |i-j| = n-1 \), and \( a_{i,j} = 0 \) otherwise. As the infinite sequence we take the following periodic sequence
\[
\iota = \cdots, n, \cdots, 2, 1, \cdots, n, \cdots, 2, 1, n, \cdots, 2, 1.
\]

In the rest of this section, we will use the notation \([13]\):
\[
j; i[k] := k - 1 + (j - 1)(n - 1) + i.
\]
Thus, the correspondence \( (j; i) \mapsto j; i[k] \) is a bijection from \( \mathbb{Z}_{\geq 1} \times [1, n - 1] \) to \( \mathbb{Z}_{\geq k} \). If there is no confusion, we shall use \( j; i \) for \( j; i[1] \). This bijection transforms the usual linear order on \( \mathbb{Z}_{\geq k} \) into the lexicographic order on \( \mathbb{Z}_{\geq 1} \times [1, n - 1] \) given by
\[
(j'; i') < (j; i) \text{ if } j' < j \text{ or } j' = j, i' < i.
\]
As in \([13]\) Sect 6] we consider integer “matrices” \( C = (c_{j; i}) \) indexed by \( \mathbb{Z}_{\geq 1} \times [1, n - 1] \), and such that \( c_{j; i} = 0 \) for \( j > 0 \). With every such \( C \) and any \( k \geq 1 \) we associate a linear form \( \varphi_{C[k]} \) on \( \mathbb{Z}^\infty \) given by \( \varphi_{C[k]} = \sum_{j; i} c_{j; i} x_{j; i[k]} \).

For any \( (j; i) \in \mathbb{Z}_{\geq 1} \times [1, n - 1] \), we set \( s_{j; i} = s_{j; i}(C) = c_{1; i} + c_{2; i} + \cdots + c_{j; i} \).

Definition 7.1 \([13]\) An integer matrix \( C \) indexed by \( \mathbb{Z}_{\geq 1} \times [1, n - 1] \) (and each of the corresponding forms \( \varphi_{C[k]} \)) is called admissible if it satisfies the following conditions (same as (6.2)–(6.5) in \([13]\)):
\[
s_{j; i} \geq 0 \text{ for } (j; i) \in \mathbb{Z}_{\geq 1} \times [1, n - 1].
\]
\[ s_{j;i} = \delta_{i,1} \text{ for } j \gg 0. \quad (7.2) \]
\[
\sum_{(j';i') \leq (j;i)} s_{j';i'} \leq j \text{ for any } (j;i), \text{ with the equality for } j \gg 0. \quad (7.3)
\]

If \( s_{j;i} > 0 \) then \( s_{j';i'} > 0 \) for some \( (j';i') \) with \( (j;i) < (j';i') \leq (j+1;i) \). \quad (7.4)

Let us denote the set of all admissible matrices by \( \mathcal{C} \) and \( C_0 \) for the matrix given by \( c_{j;i} = \delta_{j,i,1,1} \). Then we have \( \varphi_{C_0[k]} = x_k \). The following lemma is shown in \[13\], which is used repeatedly in the subsequent arguments.

**Lemma 7.2** *(Lemma 6.3 \[13\])* The matrix \( C_0 \) is the only admissible matrix with \( c_{1;1} = s_{1;1} > 0 \).

**Theorem 7.3** For \( \lambda = \sum_1 \lambda_i A_i \in P_+ \) and the sequence \( \iota \) as above, we have
\[
\text{Im} (\Psi_k^{(\lambda)}) = \ \left\{ \bar{x} \in Z^\infty[\lambda] \mid \begin{array}{l}
\varphi_{C[k]}(\bar{x}) \geq 0 \text{ for any } C \in \mathcal{C} \text{ and } k \geq 1, \\
\lambda_i \geq s_{j;i} - x_{j;i-1} \text{ for } j \geq 1 \text{ and } 1 \leq i \leq n - 1, \\
\lambda_n + \varphi_{C[0]}(\bar{x}) \geq 0 \text{ for any } C \in \mathcal{C} \setminus \{C_0\}, 
\end{array} \right\}. \quad (7.5)
\]

Here note that we treat the matrix \( C[0] \) in the third condition of \( (7.5) \). In this case there is no object corresponding to \( \varphi_{C_0[0]} = x_0 \), but it is removed from \( \mathcal{C} \). Furthermore, by Lemma 7.2, the matrix with non-trivial \( c_{1;1} \) is only \( C_0 \). Thus the R.H.S of \( (7.5) \) is well-defined.

**Proof.** Let \( \Xi_k \) be the set of linear forms obtained by applying \( S_j \)'s on the linear form \( x_k \) as in \[13\] Sect 6] and we denote \( \Xi_k \) by \( \Xi_k^{(\infty)} \). Then by Lemma 6.2 in \[13\], we have
\[
\Xi_k^{(\infty)} = \{ \varphi_{C[k]}(\bar{x}) \mid C \in \mathcal{C} \}. \quad (7.6)
\]
In order to complete the proof of the theorem, it suffices to show the following:

**Proposition 7.4** We have
\[
\Xi_k^{(1)} = \{-x_{1;1}\}, \quad (7.7)
\]
\[
\Xi_k^{(i)} = \{-x_{j;i} + x_{j;i-1} \mid j \geq 1, \ 1 < i \leq n - 1\}, \quad (7.8)
\]
\[
\Xi_k^{(n)} = \{ \varphi_{C[0]}(\bar{x}) \mid C \in \mathcal{C} \setminus \{C_0\} \}. \quad (7.9)
\]

**Proof of Proposition 7.4** The proof of \( (7.7) \) is trivial.
Let us show \( (7.8) \). Write \( F^{(i)} \) for the R.H.S. of \( (7.8) \) \((1 < i \leq n - 1)\). Using the double index \( j;i \), the linear form \( \beta_{j;i} \) can be written explicitly in the following form:
\[
\beta_{j;i}(\bar{x}) = x_{j;i} - x_{j;i+1} - x_{j+1,i} + x_{j+1,i+1},
\]
\[
\beta_{(j;i)(\cdots)}(\bar{x}) = \begin{cases}
  x_{j-1;i-1} - x_{j-1,i} - x_{j,i-1} + x_{j,i} & \text{if } (j;i) > (2;1), \\
  0 & \text{if } (1;1) \leq (j;i) \leq (2;1).
\end{cases} \quad (7.10)
\]
Here note that \( x_j; n \) means \( x_{j+1;1} \), \( x_j; 0 \) means \( x_{j-1; n-1} \) if \( j > 1 \) and \( x_j; i \) means \( 0 \) if \( j \leq 0 \). which is the different convention from the \( A_n \)-case. By the definition of \( \xi^{(i)} \) we have \( \xi^{(i)}(\vec{x}) = -x_{1;1} + x_{1; i-1} \) \( (1 < i \leq n - 1) \). Then \( \xi^{(i)}(\vec{x}) \in F^{(i)} \). By using the explicit form of \( \beta_j; i \) in (7.10), we obtain the similar formula to (6.6):

\[
S_{p; q}(x_j + x_{j;i-1}) = \begin{cases} 
-x_{j+1; i} + x_{j+1;i-1} & \text{if } (p; q) = (j; i - 1), \\
-x_{j-1; i} + x_{j-1;i-1} & \text{if } (p; q) = (j; i) > (2; 1), \\
-x_{j;i} + x_{j;i-1} & \text{otherwise}. 
\end{cases}
\tag{7.11}
\]

This implies that any form in \( F^{(i)} \) is generated from \( \xi^{(i)} \) and the set \( F^{(i)} \) is closed under the action of \( S_{p; q} \).

Before showing (7.9) we see the following lemma:

**Lemma 7.5** Suppose that \( C = (c_j; i) \in C \setminus \{C_0\} \) satisfies \( c_{2;2} < 0 \). Then we have \( c_{1;1} = c_{2;1} = 1, c_{2;2} = -1 \) and \( c_j; i = 0 \) for other \( j; i \).

**Proof of Lemma 7.5** By the definition of \( s_j; i \), we have \( c_{2; 2} = s_{1;1} - s_{1;2} \). Thus, our assumption \( c_{2; 2} < 0 \) implies

\[
0 \leq s_{2; 2} < s_{1;2}. \tag{7.12}
\]

It is obtained by (7.3) and (7.4) that

\[
s_{1;1} + s_{1;2} + \cdots + s_{1; n-1} \leq 1, \tag{7.13}
\]

\[
s_{1;1} + s_{1;2} + \cdots + s_{1; n-1} + s_{2;1} \leq 2, \tag{7.14}
\]

\[
s_j; i' > 0 \text{ for some } (1; 2) < (j'; i') \leq (2; 2). \tag{7.15}
\]

Since \( C \neq C_0 \), by Lemma 7.2 we have \( c_{1;1} = s_{1;1} = 0 \). Then by (7.1), (7.12) and (7.13), we get

\[
s_{1;2} = c_{1;2} = 1 \tag{7.16}
\]

\[
s_{1;3} = s_{1;4} = \cdots = s_{1; n-1} = s_{2;2} = 0. \tag{7.17}
\]

Then we have

\[
c_{2;2} = s_{2;2} - s_{1;2} = -1. \tag{7.18}
\]

Furthermore, by (7.14), (7.13) and (7.17) we have

\[
s_{2;1} = 1. \tag{7.19}
\]

Here we need the following lemma to complete the proof of Lemma 7.5:

**Lemma 7.6** If \( s_{j;1} = 1 \) and \( \sum_{(j'; i') \leq (j;1)} s_{j'; i'} = j \), we have \( s_{k; i} = \delta_{i,1} \) for \( (k; i) \geq (j; 1) \).
Proof. It follows from (7.3) and (7.4) that
\[ \sum_{(j';i') \leq (j;n-1)} s_{j';i'} \leq j, \]  
(7.20)
\[ \sum_{(j';i') \leq (j+1;1)} s_{j';i'} \leq j + 1, \]  
(7.21)
\[ s_{j';i'} > 0 \text{ for some } (j;1) < (j';i') \leq (j+1;1). \]  
(7.22)

Then by applying the assumption of the lemma to (7.20), (7.22) we have \( s_{j;2} = s_{j;3} = \cdots = s_{j;n-1} = 0 \) and \( s_{j+1;1} = 1 \), which implies \( \sum_{(j';i') \leq (j+1;1)} s_{j';i'} = j + 1 \). These are the assumption of the lemma replaced \( j \) by \( j + 1 \). Therefore, the induction proceeds and then we get \( s_{k;2} = s_{k;3} = \cdots = s_{k;n-1} = 0 \) and \( s_{k;1} = 1 \) for \( k \geq j \).

By (7.14), (7.16) and (7.19) we have \( s_{2;1} = 1 \) and \( \sum_{(j';i') \leq (2;1)} s_{j';i'} = 2 \). Then by applying the assumption of the lemma to (7.20)–(7.22) we have
\[ c_{j;i} = s_{j;1} - s_{j-1;1} = \begin{cases} 1 & \text{if } (j;i) = (1;2), (2;1), \\ -1 & \text{if } (j;i) = (2;2), \\ 0 & \text{otherwise}, \end{cases} \]
which is the desired result. Then we finished the proof of Lemma 7.3.

Let us show (7.9). We write \( F^{(n)} \) for the R.H.S. of (7.3). The explicit form of \( \xi^{(n)} \) is
\[ \xi^{(n)}(\vec{x}) = x_{1;1} + x_{1;n-1} - x_{2;1} = x_{1;2}[0] + x_{2;1}[0] - x_{2;2}[0]. \]  
(7.23)

Observing this form carefully, formally we can write
\[ \xi^{(n)}(\vec{x}) = S_{1;1}[0]x_{1;1}[0]. \]  
(7.24)

Of course there is nothing corresponding to \( x_{1;1}[0] = x_0 \). But, formally we have
\[ F^{(n)} = \{ S_{j;i}[0] \cdots S_{j;i}[0](S_{1;1}[0]x_{1;1}[0]) | l \geq 0, \ j k \geq 1, \ i k \in I \} \]  
(7.25)

By Lemma 7.2 we know that the form \( \varphi_{C[0]}(\vec{x}) = \sum c_{j;i} x_{j;i}[0] \ (C \in C) \) satisfying \( c_{1;1} \neq 0 \) is only \( x_{1;1}[0] = x_0 \) corresponding to the matrix \( C_0 \). Moreover, note that only the explicit form of \( S_{2;2}[0] \) is different from those of \( S_{2;1}[k] \) \( (k \geq 1) \), namely, \( (2;2)[0] = (2;1) = 0 \) and \( (2;2)[k] = 1;1[k] \) for \( k \geq 1 \) and then \( \beta_{2;2}[0] = 0 \) and \( \beta_{2;2}[k] = 1;1[k] \) for \( k \geq 1 \). But for \( k \geq 0 \) the form \( \varphi_{C[k]} = \sum c_{j;i} x_{j;i}[k] \) with \( c_{2;2} < 0 \) must be \( x_{1;2}[k] + x_{2;1}[k] - x_{2;2}[k] \) by Lemma 7.3. Then we have that if \( c_{2;2} < 0 \),
\[ S_{2;2}[k] \varphi_{C[k]}(\vec{x}) = \begin{cases} \varphi_{C[0]} = \xi^{(n)} = S_{1;1}[0]x_{1;1}[0], & \text{if } k = 0, \\ \varphi_{C_0[k]} = x_k, & \text{if } k \geq 1, \end{cases} \]  
(7.26)

which implies that \( \varphi_{C_0[0]} = x_0 \) never occurs in \( F^{(n)} \) and \( F^{(n)} \) is stable by the actions of \( S_{j;i}[0] \). Thus, by Lemma 6.2 in [3] we obtain
\[ F^{(n)} = \{ \varphi_{C[0]}(\vec{x}) | C \in C \setminus \{ C_0 \} \}. \]
Now, we completed the proof of Proposition 7.4.

**Proof of Theorem 7.3.** Let us see that the above proposition and lemma imply our theorem. In view of Corollary 4.4 and Proposition 7.4, it suffices to show that the sequence $i$ satisfies the strict positivity assumption. We have already shown this for $\Xi^{(\infty)}$ in (3). Next, we see $\Xi^{(i)} (i = 1, 2, \cdots, n)$. Since $\xi^{(1)} = -x_{1;1}$ and $\xi^{(i)} = -x_{1;i} + x_{1;i-1}$ for $1 < i \leq n - 1$, among $\bigcup_{1 \leq i \leq n-1}(\Xi^{(i)} \setminus \{\xi^{(i)}\})$ the only linear form which has non-trivial coefficients for $x_{1;i}$ ($1 \leq i \leq n - 1$) or $x_{2;1}$ is just $-x_{2;2} + x_{2;1}$, which has a positive coefficient for $x_{2;1}$. The remaining case is $i = n$. In this case, the explicit form of $\xi^{(n)}$ is given by $\xi^{(n)}(\vec{x}) = x_{1;1} + x_{1;n-1} - x_{2;1}$. So it suffices to show that any linear form $\varphi(\vec{x}) = \sum_{j; i} c_{j;i} x_{j;i}[0] \in \Xi^{(n)} \setminus \{\xi^{(n)}\}$ satisfies that $c_{1;2}, \cdots, c_{1;n-1}, c_{2;1}, c_{2;2} \geq 0$. Those for $c_{1;2}, \cdots, c_{1;n-1}$ are trivial from $s_{1;i} = c_{1;i}$ and (7.1). If we assume $c_{2;1} < 0$, we have $c_{1;1} = s_{2;1} - c_{2;1} > 0$. By Lemma 7.2, we get $c_{j;i} = \delta_{j;i,1,1}$. Now since $C \neq C_0$, we have $c_{2;1} \geq 0$. If we assume that $c_{2;2} < 0$, we have $\varphi C[0] = \xi^{(n)}$ by Lemma 7.3. Thus, we obtain the strict positivity assumption and then completed the proof of the theorem.

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