ON CLASSIFICATION OF TYPICAL REPRESENTATIONS FOR GL_3(F).

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Abstract. Let $F$ be any non-Archimedean local field with residue field of cardinality $q_F$. In this article, we obtain a classification of typical representations for the Bernstein components associated to the inertial classes of the form $[GL_n(F) \times F^*, \sigma \otimes \chi]$ with $q_F > 2$, and for the principal series components with $q_F > 3$. With this we complete the classification of typical representations for $GL_3(F)$, for $q_F > 2$.

1. Introduction

Let $F$ be any non-Archimedean local field with residue field $k_F$ of cardinality $q_F$. Let $A_n$ be the set of isomorphism classes of irreducible smooth complex representations of $GL_n(F)$. The theory of Bernstein decomposition gives a natural partition of the set $A_n$:

$$A_n = \prod_{s \in B_n} A_n(s).$$

Here, the set $A_n(s)$ is defined in terms of parabolic induction. The parameter $s$ is the inertial class containing the cuspidal support of an irreducible smooth representation of $GL_n(F)$ (see Section 2). In the context of the local Langlands correspondence for $GL_n(F)$, the parameter $s$ determines the isomorphism class of the restriction to the inertia subgroup $I_F$ of the Weil–Deligne representation associated by the local Langlands correspondence.

The reciprocity map of the local class field theory gives an isomorphism between the abelianization of $I_F$ and $k_F^\times$, the group of units of the ring of integers of $F$. It is natural to ask for a relation between the representations of $I_F$, which can be extended to a Weil–Deligne representation, and the representations of the maximal compact subgroup $GL_n(k_F)$. One natural way would be to understand the cuspidal support of a smooth irreducible representation from its restriction to $GL_n(k_F)$. Indeed, in several arithmetic applications (see [BM02], [EG14]) it is desired to construct irreducible smooth representations $\tau_s$ of the maximal compact subgroup $GL_n(k_F)$ such that, for any irreducible smooth representation $\pi$ of $GL_n(F)$,

$$\text{Hom}_{GL_n(k_F)}(\tau_s, \pi) \neq 0 \Rightarrow \pi \in A_n(s).$$

Such a representation $\tau_s$ is called a typical representation for $s$.

The existence of typical representations, for any $s$, follows from the theory of types developed by Bushnell and Kutzko. For all $s \in B_n$, Bushnell and Kutzko explicitly constructed pairs of the form $(J_s, \lambda_s)$, where $J_s$ is a compact open subgroup of $GL_n(F)$ and $\lambda_s$ is an irreducible smooth representation of $J_s$ such that, for any irreducible smooth representation $\pi$ of $GL_n(F)$, we have

$$\text{Hom}_{J_s}(\lambda_s, \pi) \neq 0 \Leftrightarrow \pi \in A_n(s).$$

We may assume that $J_s \subseteq GL_n(k_F)$. It follows from Frobenius reciprocity that any irreducible subrepresentation of

$$\text{ind}_{J_s}^{GL_n(k_F)} \lambda_s$$

is a typical representation for $s$. In general, the representation (1) is not irreducible; therefore, we cannot expect to have a uniqueness result on typical representations for a general $s$. Now, it is natural to ask whether there exist any other typical representations which do not occur as subrepresentations of (1). Hence, the question we are interested in is the classification of all typical representations. In this article, we achieve this
classification for certain non-cuspidal inertial classes of $\text{GL}_n(F)$—including all non-cuspidal inertial classes of $\text{GL}_3(F)$—which give the classification of the typical representations for all inertial classes of $\text{GL}_3(F)$, when $q_F > 2$.

Henniart (see [BM02]) classified typical representations for all inertial classes of $\text{GL}_2(F)$. Later Paškūnas (see [Pas05]) classified typical representations occurring in the cuspidal representations of $\text{GL}_n(F)$, for $n \geq 3$. It turns out that there exists a unique typical representation occurring in each cuspidal representation. Typical representations for depth-zero inertial classes of $\text{GL}_n(F)$ are classified by the author in the article [Nad17]. We refer to the articles of Bushnell–Kutzko types. In fact, the monumental theory of Bushnell–Kutzko is the fundamental basis for this article.

It follows from the results of the article [Nad17] that this article.

**Theorem 1.0.1.** Let $n > 2$ be an integer and $q_F > 2$. Let $\sigma$ be an inertial class of the form $[\text{GL}_{n-1}(F) \times F^\times, \sigma \otimes \eta]$, where $\eta$ is a cuspidal representation of $\text{GL}_{n-1}(F)$ and $\eta$ is a character of $F^\times$. Any typical representation $\tau_s$ for $s$ is isomorphic to $\text{ind}_{J_s}^{\text{GL}_n(\varphi)}(\lambda_s)$, where $(J_s, \lambda_s)$ is a Bushnell–Kutzko type for $s$. Any non-cuspidal inertial class of $\text{GL}_3(F)$ is of the above form. Combined with the result of Paškūnas on the unicity of typical representation for cuspidal inertial classes, we prove the following theorem.

**Theorem 1.0.2.** Let $n > 2$ be an integer, and let $q_F > 3$ if $n \neq 3$ and let $q_F > 2$ if $n = 3$. Let $s$ be an inertial class of the form $[T, \chi]$, where $T$ is a maximal $F$-split torus contained in $\text{GL}_n(F)$ and $\chi$ is a smooth character of $T$. Any typical representation $\tau_s$ for $s$ is a subrepresentation of $\text{ind}_{J_s}^{\text{GL}_n(\varphi)}(\lambda_s)$, where $(J_s, \lambda_s)$ is a Bushnell–Kutzko type for $s$.

In our analysis we will also obtain a certain multiplicity result on the typical representations $\tau_s$.

We briefly explain the method of proof. Let $M$ be a Levi subgroup of an $F$-parabolic subgroup $P$ of $\text{GL}_n(F)$. Let $\sigma$ be a cuspidal representation of $M$. Let $\tau$ be the unique $M \cap \text{GL}_n(\varphi)$ typical representation contained in $\sigma$. The uniqueness of $\tau$ is a result of Paškūnas in the article [Pas05]. In order to classify typical representations, we begin by decomposing the representation

$$\text{res}_{\text{GL}_n(\varphi)}^{\text{GL}_n(F)}(\sigma).$$

It follows from the results of the article [Nad17] that

$$\text{res}_{\text{GL}_n(\varphi)}^{\text{GL}_n(F)}(\sigma) = \text{ind}_{\text{GL}_n(\varphi) \cap P}^{\text{GL}_n(\varphi)} \tau \oplus \Gamma,$$

where any irreducible $\text{GL}_n(\varphi)$-subrepresentation of $\Gamma$ is not a typical representation.

We then construct compact open subgroups of $\text{GL}_n(\varphi)$, denoted by $H_m$, for $m \geq 1$ such that

$$H_{m+1} \subset H_m,$$

for all $m \geq 1$.

and

$$\bigcap_{m \geq 1} H_m = P \cap \text{GL}_n(\varphi).$$

We will also show that $\tau$ extends as a representation of $H_m$ for $m \geq 1$. We will then show that any $\text{GL}_n(\varphi)$ subrepresentation of the representation $\text{ind}_{H_{m+1}}^{\text{GL}_n(\varphi)} \tau / \text{ind}_{H_m}^{\text{GL}_n(\varphi)} \tau$ is not typical. The group $H_1$ will be close enough to the compact subgroup $J_s$ in a Bushnell–Kutzko type $(J_s, \lambda_s)$. With some more additional work, similar to the above procedure, we complete the classification of typical representations. This requires the analysis of the induced representation $\text{ind}_{H_{m+1}}^{H_m} \text{id}$. We will also require some subtle aspects in the theory of Bushnell–Kutzko types. In fact, the monumental theory of Bushnell–Kutzko is the fundamental basis for this article.
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2. Preliminaries

For any ring $R$ with unity, we denote by $\text{Mat}_{n \times m}(R)$ the set of $n \times m$ matrices with entries in $R$. For any matrix $X$, we denote by $X^T$ the transpose of $X$. The identity matrix in $\text{Mat}_{n \times n}(R)$ is denoted by $\text{id}_n$ or by $I_n$.

Let $F$ be any non-Archimedean local field with its ring of integers $\mathfrak{o}_F$. Let $\mathfrak{p}_F$ be the maximal ideal of $\mathfrak{o}_F$, and $\varpi_f$ be an uniformiser of $F$. Let $k_F$ be the residue field of $F$. We denote by $q_F$ the cardinality of $k_F$. For any character $\chi$ of $F^\times$, we denote by $l(\chi)$ the level of $\chi$, i.e., the least positive integer $m$ such that $1 + p_F^m$ is contained in the kernel of $\chi$. Note that the level $l(\chi)$ of an unramified character $\chi$ is still 1. We denote by $\nu_F : F^\times \to \mathbb{Z}$ the normalised valuation of $F$.

All representations in this article are defined over complex vector spaces.

Let $G$ be the $F$-rational points of a connected reductive algebraic group $G$ defined over $F$. Let $\mathcal{R}(G)$ be the category of smooth representations of $G$. For any closed subgroup $H$ of $G$, we denote by $\text{ind}^G_H$ the compact induction functor from $\mathcal{R}(H)$ to $\mathcal{R}(G)$. Let $P$ be the group of $F$-rational points of any $F$-parabolic subgroup of $G$. Let $M$ be a Levi subgroup of $P$. We denote by $i_P^G$ the normalised parabolic induction functor from $\mathcal{R}(M)$ to $\mathcal{R}(G)$.

Let $H_1$ and $H_2$ be two groups and $\tau_1$ and $\tau_2$ be any representations of $H_1$ and $H_2$ respectively. We denote by $\tau_1 \boxtimes \tau_2$ the tensor product representation of $H_1 \times H_2$. If $H_1 = H_2$, then the representation $\tau_1 \otimes \tau_2$ is the tensor product representation of $H_1$.

For any positive integer $n$, the group $\text{GL}_n(F)$ is denoted by $G_n$ and the group $\text{GL}_n(\mathfrak{o}_F)$ is denoted by $K_n$. The principal congruence subgroup of $K_n$ of level $m$ is denoted by $K_n(m)$, for $m \geq 1$. Let $I$ be a sequence of positive integers $(n_1, n_2, \ldots, n_r)$ such that $n_1 + n_2 + \cdots + n_r = n$. For any ring $R$ with unity, we denote by $P_I(R)$ (resp. $P_I(R)$) the group of invertible block upper (resp. lower) triangular matrices of the type $I$. Let $U_I(R)$ (resp. $\bar{U}_I(R)$) be the group of block upper (lower) unipotent matrices of the type $I$. Let $M_I(R)$ be the block diagonal matrices of the type $I$. If $I = (1, 1, \ldots, 1)$, the groups $P_I(R)$, $M_I(R)$ and $U_I(R)$ are denoted by $B_n(R)$, $T_n(R)$ and $U_n(R)$ respectively. When $R = F$, we drop the symbol $R$, i.e., $P_I(R)$ will be denoted by $P_I$ etc. We have $P_I = M_I U_I$ and $P_I = M_I \bar{U}_I$.

As an example, when $I = (n-1, 1)$, the group $P_I$ in the block form is given by:

$$
\begin{pmatrix}
G_{n-1} & \text{Mat}_{n-1 \times 1}(F) \\
0 & F^\times
\end{pmatrix}.
$$

In the block form the groups $M_I$ and $U_I$ are given by:

$$
M_I = \begin{pmatrix} G_{n-1} & 0 \\ 0 & F^\times \end{pmatrix} \quad \text{and} \quad U_I = \begin{pmatrix} 1_{n-1} & \text{Mat}_{n-1 \times 1}(F) \\ 0 & 1 \end{pmatrix}.
$$

We identify the group $M_{(n-1,1)}$ with the group $G_{n-1} \times G_1$. Any irreducible smooth representation of $M_{(n-1,1)}$ is identified with $\sigma \boxtimes \chi$, where $\sigma$ is an irreducible smooth representation of $G_{n-1}$ and $\chi$ is a character of $G_1$.

We briefly recall the theory of Bernstein decomposition. Let $B(G)$ be the set of pairs $(M, \sigma)$, where $M$ is a Levi subgroup of an $F$-parabolic subgroup $P$ of $G$, and $\sigma$ is an irreducible cuspidal representation of $M$. The pairs $(M_1, \sigma_1)$ and $(M_2, \sigma_2)$ in $B(G)$ are said to be inertially equivalent if and only if there exist an element $g \in G$ and an unramified character $\chi$ of $M_2$ such that

$$
M_1 = gM_2g^{-1} \quad \text{and} \quad \sigma_1^g \simeq \sigma_2 \otimes \chi.
$$

We denote by $B_G$ the set of equivalence classes, called inertial classes.

Any irreducible smooth representation $\pi$ of $G$ occurs as a sub-representation of a parabolic induction $i_P^G(\sigma)$, where $\sigma$ is an irreducible cuspidal representation of a Levi subgroup $M$ of $P$. The pair $(M, \sigma)$ is
well determined up to $G$-conjugation. We call the class $s = [M, \sigma]$ the inertial support of $\pi$. We denote by $\mathcal{I}(\pi)$ the inertial support of $\pi$. For any inertial class $s = [M, \sigma]$, we denote by $\mathcal{R}_s(G)$ the full sub-category of $\mathcal{R}(G)$ consisting of smooth representations all of whose irreducible sub-quotients have inertial support $s$. It is shown by Bernstein in [Ber84] that the category $\mathcal{R}(G)$ decomposes as a direct product of $\mathcal{R}_s(G)$. The category $\mathcal{R}_s(G)$ is called a Bernstein component associated to the inertial class $s$. In particular, every smooth representation can be written as a direct sum of objects in the categories $\mathcal{R}_s(G)$. We denote by $\mathcal{A}_n(s)$ the set of isomorphism classes of irreducible representations in $\mathcal{R}_s(G_n)$.

**Definition 2.0.1.** Let $s$ be an inertial class for $G_n$. An irreducible smooth representation $\tau$ of $K_n$ is called a typical representation for $s$, if for any irreducible smooth representation $\pi$ of $G$, we have

$$\text{Hom}_{K_n}(\tau, \pi) \neq 0 \iff \mathcal{I}(\pi) = s.$$ 

A non-typical representation is called an atypical representation.

For any inertial class $s$ of $G_n$, the existence of a typical representation can be deduced from the theory of types developed by Bushnell and Kutzko in the book [BK93] and the article [BK99]. Bushnell and Kutzko constructed explicit pairs (called types) $(J_s, \lambda_s)$, where $J_s$ is a compact open subgroup of $GL_n(F)$, and $\lambda_s$ is an irreducible smooth representation of $J_s$. The pair $(J_s, \lambda_s)$ satisfies the condition that, for any irreducible smooth representation $\pi$ of $G$, we have

$$\text{Hom}_{J_s}(\pi, \lambda_s) \neq 0 \iff \mathcal{I}(\pi) = s.$$ 

The group $J_s$ can be arranged to be a subgroup of $GL_n(F)$ by conjugating with an element of $GL_n(F)$. Hence we assume that $J_s \subseteq GL_n(F)$. It follows from Frobenius reciprocity that any irreducible sub-representation of

$$\text{ind}_{J_s}^{GL_n(F)}(\lambda_s)$$

is a typical representation. The irreducible sub-representations of $[2]$ are classified by Schneider and Zink in [SZ99] Section 6, $T_{K, \lambda}$ functor.

For $s = [G_n, \sigma]$, Paškūnas in the article [Pas05] Theorem 8.1 showed that up to isomorphism there exists a unique typical representation for $s$. More precisely,

**Theorem 2.0.2** (Paškūnas). Let $n \geq 1$ be an integer and $\sigma$ be an irreducible cuspidal representation of $G_n$. Let $(J_s, \lambda_s)$ be a Bushnell-Kutzko type for the inertial class $s = [G_n, \sigma]$ with $J_s \subseteq K_n$. The representation

$$\text{ind}_{J_s}^{K_n}(\lambda_s)$$

is the unique typical representation for the inertial class $[G_n, \sigma]$. The representation $\text{ind}_{J_s}^{K_n}(\lambda_s)$ occurs with a multiplicity one in $\sigma$.

In this article, we classify typical representations for $GL_3(F)$ in terms of Bushnell–Kutzko types. We first obtain a classification of typical representations for the inertial classes $[M_{(n-1,1)}, \sigma \boxtimes \eta]$ and $[T_n, \chi]$, where $\eta$ and $\chi$ are characters of $F^\times$ and $T_n$ respectively. We will use some basic results from the article [Nad17] and we recall some of these results.

**Lemma 2.0.3.** Let $\chi$ be a character of $G_n$ and let $\tau$ be a typical representation for an inertial class $s = [M, \sigma]$ of $G_n$. The representation $\tau \otimes \chi$ is a typical representation for the inertial class $[M, \sigma \otimes \chi]$.

**Proof.** We refer to [Nad17] Lemma 2.7 for a proof. 

Let $P$ be any parabolic subgroup of $G_n$ with a Levi subgroup $M$ and $U$ be the unipotent radical of $P$. Let $\tilde{U}$ be the unipotent radical of the opposite parabolic subgroup of $P$ with respect to $M$. Let $J_1$ and $J_2$ be two compact open subgroups of $K_n$ such that $J_1$ contains $J_2$. Suppose $J_1$ and $J_2$ both satisfy the Iwahori decomposition with respect to $P$ and $M$. With $J_1 \cap U = J_2 \cap U$ and $J_1 \cap \tilde{U} = J_2 \cap \tilde{U}$. Let $\lambda$ be an irreducible smooth representation of $J_2$ which admits an Iwahori decomposition i.e. $J_2 \cap U$ and $J_2 \cap \tilde{U}$ are contained in the kernel of $\lambda$.

**Lemma 2.0.4.** The representation $\text{ind}_{J_1}^{J_1 \cap M}(\lambda)$ is the extension of the representation $\text{ind}_{J_2 \cap M}^{J_2}(\lambda)$ such that $J_1 \cap U$ and $J_1 \cap \tilde{U}$ are contained in the kernel of the extension.
Proof. The lemma is well known and frequently used when dealing with the formalism of $G$-covers. We refer to [Nad17] Lemma 2.6 for a proof.

Let $t_i = [M_i, \Theta_i]$ be an inertial class of $G_{n_i}$, for $1 \leq i \leq r$. Let $\sigma_i$ be a smooth representation from $\mathcal{R}_{t_i}(G_{n_i})$. We suppose

$$\res_{K_{n_i}} \sigma_i = \tau^0_i \oplus \tau^1_i,$$

for $1 \leq i \leq r$, such that irreducible $K_{n_i}$-subrepresentations of $\tau^1_i$ are atypical. We denote by $t$ the inertial class

$$[M_1 \times M_2 \times \cdots \times M_r, \Theta_1 \boxtimes \Theta_2 \boxtimes \cdots \boxtimes \Theta_r]$$

of $G_n$. The inertial class $t$ is independent of the choice of representatives $(M_i, \Theta_i)$. Let $\tau^0_T = \bigoplus_{i=1}^r \tau^0_i$ and $\sigma_T = \bigoplus_{i=1}^r (\sigma_i)$.

**Lemma 2.0.5.** The representation $\ind_{P_T \cap K_n}^G(\tau^0_T)$ admits a complement in $\res_{K_n} \ind_{P_T}^G(\sigma_T)$ with all its irreducible sub-representations atypical.

Proof. We refer to [Nad17] Proposition 2.3 for a proof.

In particular, if $\Theta_i = \sigma_i$ is a cuspidal representation, then from Theorem 2.0.2 we have $\res_{K_{n_i}} \sigma_i = \tau^0_i \oplus \tau^1_i$, where $\tau^0_i$ is the unique typical representation for the inertial class $[G_{n_i}, \sigma_i]$. Hence, any typical representation for the inertial class $t$ occurs as a sub-representation of $\ind_{P_T \cap K_n}^G \sigma_T$.

**Lemma 2.0.6.** Let $s = [M, \sigma]$ be any inertial class of $G_n$. Then there exists a partition $I$ of $n$ and a cuspidal representation $\sigma_I$ of $M_I$ such that $s = [M_I, \sigma_I]$.

Proof. We refer to [Nad17] Section 2.2, page no. 5 for a proof.

The following result is useful in understanding some stabilisers in the later part of this article. The space $\mathsf{Mat}_{n \times m}(k_F)$ is equipped with an action of $M_{(m,n)}(k_F) = \GL_m(k_F) \times \GL_n(k_F)$ given by $(g_1, g_2)U = g_2 U g_1^{-1}$, for $U \in \mathsf{Mat}_{n \times m}(k_F)$. We also have a $M_{(m,n)}(k_F)$ action on the set of matrices $\mathsf{Mat}_{m \times n}(k_F)$ by setting $(g_1, g_2)V = g_1 V g_2^{-1}$, for $V \in \mathsf{Mat}_{n \times m}(k_F)$. Let $\psi$ be a non-trivial character of the additive group $k_F$. We define a pairing $B$ between $\mathsf{Mat}_{m \times n}(k_F)$ and $\mathsf{Mat}_{n \times m}(k_F)$ by defining $B(V, U) = \psi \circ \tr(V U)$. Let $T$ be the map from $\mathsf{Mat}_{m \times n}(k_F)$ and $\mathsf{Mat}_{n \times m}(k_F)$ defined by

$$T(V)(U) = B(V, U).$$

**Lemma 2.0.7.** The map $T$ is an $M_{(m,n)}(k_F)$-equivariant isomorphism.

Let $s$ be any depth-zero inertial class $[M_I, \sigma_I]$ of $G_n$. The group $K_n \cap M_I$ acts on the space

$$\sigma_{I}^{K_n(1) \cap M_I},$$

and we denote this representation of $K_n \cap M_I$ by $\tau_I$. The pair $(K_n \cap M_I, \tau_I)$ is a Bushnell–Kutzko type for the inertial class $[M_I, \sigma_I]$ of $M_I$. Let $P_I(1)$ be the group $K_n(1)(P_I \cap K_n)$ and observe that $P_I(1) \cap M_I$ is equal to $K_n \cap M_I$. The representation $\tau_I$ extends as a representation of $P_I(1)$ such that $P_I(1) \cap U_I$ and $P_I(1) \cap U_I$ are contained in the kernel of this extension.

**Theorem 2.0.8.** Let $s = [M_I, \sigma_I]$ be any depth-zero inertial class of $G_n$. Any typical representation $\tau_s$ for $s$ occurs as a subrepresentation of $\ind_{P_I(1)}^{K_n(1)} \tau_I$. Moreover, we have

$$\dim_{\mathbb{C}} \Hom_{K_n}(\tau_s, \ind_{P_I(1)}^{K_n(1)} \tau_I) = \dim_{\mathbb{C}} \Hom_{K_n}(\tau_s, \ind_{P_I(1)}^{K_n(1)} \tau_I).$$

Proof. We refer to [Nad17] Theorem 3.2 for the proof.
3. The inertial class with Levi subgroup of type \((n - 1, 1)\)

Let \(n > 1\) be any positive integer. In this section we assume that \(I = (n - 1, 1)\). Let \(V\) and \(V_1\) be two \(F\)-vector spaces of dimensions \(n - 1\) and 1 respectively. Let \(P\) be the parabolic subgroup of \(GL(V \oplus V_1)\) fixing the flag \(V \subset V \oplus V_1\). We denote by \(M\) its Levi subgroup fixing the decomposition \(V \oplus V_1\). Hence, we have \(M = GL(V) \times GL(V_1)\). In this section, we are interested in the classification of typical representations for inertial classes \([M, \sigma \boxtimes \chi]\), where \(\sigma\) is a cuspidal representation of \(GL(V)\), and \(\chi\) is a character of \(GL(V_1)\). We will use the language of the book \([BK93]\) freely in this section. Let \((J(\mathfrak{A}, \beta), \lambda)\) be a maximal simple (Bushnell–Kutzko) type contained in the representation \(\sigma\). We recall certain important features of this type for our purpose.

3.1. Bushnell-Kutzko semi-simple type. We denote by \(A\) the algebra \(\text{End}_F(V)\). Let \([\mathfrak{A}, l, 0, \beta]\) be a simple stratum in \(A\) defining the maximal simple type \((J(\mathfrak{A}, \beta), \lambda)\). We denote by \(B\) the commutant of \(E = F[\beta]\) in \(A\). Let \(\mathfrak{B} = \mathfrak{A} \cap B\). Let \(\mathfrak{P}\) and \(\mathfrak{D}\) be the radicals of \(\mathfrak{A}\) and \(\mathfrak{B}\) respectively. Given any hereditary order \(\mathcal{O}\), we define the filtration \(U^i(\mathfrak{A})\) by setting

\[
U^i(\mathfrak{A}) = \text{id} + \mathfrak{P}^i
\]

for all \(i \geq 1\), and \(U^0(\mathfrak{A})\) is the set of units of \(\mathfrak{A}\). The type \((J(\mathfrak{A}, \beta), \lambda)\) is called maximal if \(\mathfrak{B}\) is a maximal hereditary order in \(B\).

The group \(J(\mathfrak{A}, \beta)\) contains \(U^0(\mathfrak{B})\). There is a normal subgroup \(J^1(\mathfrak{A}, \beta)\) of \(J(\mathfrak{A}, \beta)\) such that \(J^1(\mathfrak{A}, \beta) \cap U^0(\mathfrak{B}) = U^1(\mathfrak{B})\) and

\[
\frac{U^0(\mathfrak{B})}{U^1(\mathfrak{B})} \simeq J(\mathfrak{A}, \beta) / J^1(\mathfrak{A}, \beta)
\]

The group \(U^0(\mathfrak{B})/U^1(\mathfrak{B})\) is a general linear group of a vector space over a finite field. The representation \(\lambda\) is an irreducible representation which is given by a tensor product representation \(\kappa \otimes \rho\), where \(\kappa\) is a representation of \(J(\mathfrak{A}, \beta)\), called a \(\beta\)-extension (see \([BK93]\) Chapter 5, Definition 5.2.1), and \(\rho\) is a cuspidal representation of \(U^0(\mathfrak{B})/U^1(\mathfrak{B})\) (considered as a representation of \(J(\mathfrak{A}, \beta)\) through its quotient \(J(\mathfrak{A}, \beta)/J^1(\mathfrak{A}, \beta)\)). We refer to \([BK93]\) Chapter 5 for complete details of these constructions. For the precise definition and description see \([BK93]\) chapter 5, Definition 5.5.10. For simplicity, we will denote by \(J^0\) and \(J^1\) the groups \(J(\mathfrak{A}, \beta)\) and \(J^1(\mathfrak{A}, \beta)\) respectively.

We fix the following notations. Let \(e\) and \(f\) be the ramification index and inertial index of \(E\) respectively. We fix an \(\mathfrak{o}_E\)-lattice chain \(\mathcal{L}\) defining the hereditary \(\mathfrak{o}_E\)-order \(\mathfrak{B}\). Let \(\mathfrak{A}\) be the hereditary \(\mathfrak{o}_E\)-order defined by the lattice chain \(\mathcal{L}\), considering \(\mathcal{L}\) as an \(\mathfrak{o}_E\)-lattice chain. We fix a \(\mathfrak{o}_E\)-basis \((w_1, w_2, \ldots, w_{n-1}/ef)\) for the lattice chain \(\mathcal{L}\) (see \([BK93]\) Chapter 1, 1.1.7) and then a \(\mathfrak{o}_E\)-basis for \(\mathfrak{o}_E w_i\) for \(1 \leq i \leq ef\); hence, we obtain an \(F\)-basis \((v_1, v_2, \ldots, v_{n-1}, v_n)\) for the vector space \(V \oplus V_1\), where \(v_n \in V_1\). In this basis, we write all our endomorphisms as matrices of \(\text{Mat}_{n \times n}(F)\). With this basis we have \(J^0 \subseteq K_{n-1}\).

We are interested in the classification of typical representations for the inertial class \([M_I, \sigma \boxtimes \chi]\). By twisting with a character if necessary we may (and do) assume that \(\chi = \text{id}\) (see Lemma \([2.0.3]\)). Let \(\tau\) be any typical representation for the above inertial class. The representation \(\text{ind}_{K_{n-1}}^{K_n} \tau\) is a finitely generated representation of \(G_n\) and hence admits an irreducible quotient \(\pi\). Using the definition of a typical representation we see that \(\pi\) occurs as a sub quotient of \(i_{P_1}^{G_n}(\sigma \chi_1 \boxtimes \chi_2)\), where \(\chi_1\) and \(\chi_2\) are unramified characters of \(G_{n-1}\) and \(G_1\) respectively.

Hence, in order to classify typical representations for the inertial class \(s = [M_I, \sigma \boxtimes \text{id}]\), it is enough to examine which \(K_n\)-irreducible sub representations of \(\text{res}_{K_n} i_{P_1}^{G_n}(\sigma \boxtimes \text{id})\) are typical for the inertial class \(s\). Let \(\tau\) be the unique typical representation contained in the representation \(\sigma\). The representation

\[
\text{ind}_{P_1 \cap K_n}^{K_n} (\tau \boxtimes \text{id})
\]

has a complement in

\[
\text{res}_{K_n} i_{P_1}^{G_n}(\sigma \boxtimes \chi)
\]

whose irreducible sub representations are atypical (see Lemma \([2.0.5]\)).
Now we have to look for typical representations occurring in the representation
\[ \text{ind}_{P_I \cap K_n}^K (\tau \boxtimes \text{id}). \]
For this purpose, we will define compact subgroups \( H_m \subseteq K_n \), for \( m \geq N_0 \) (for some positive integer \( N_0 \)), such that

**Hypothesis 3.1.**

1. \( H_{m+1} \subseteq H_m \), for \( m \geq N_0 \) and \( \bigcap_{m \geq N_0} H_m = P_I (F_p) \).
2. The group \( H_m \) has the Iwahori decomposition with respect to \( P_I \) and its Levi subgroup \( M_I \).
3. The representation \( \tau \boxtimes \text{id} \) admits an extension to \( H_{N_0} \) such that \( H_{N_0} \cap \tilde{U}_I \) and \( H_{N_0} \cap U_I \) are contained in the kernel of this extension.

For any sequence \( \{H_m, m \geq N_0\} \) as above we have:
\[ \text{ind}_{P_I \cap K_n}^K (\tau \boxtimes \text{id}) \cong \bigcup_{m \geq N_0} \text{ind}_{H_m}^K (\tau \boxtimes \text{id}). \]

Before we start this construction it is instructive to first examine the Bushnell-Kutzko semi-simple type for the inertial class \([M_I, \sigma \boxtimes \text{id}]\).

Let us recall some standard material required from [BK99]. First, let us begin with lattice sequences. An \( \sigma_F \)-lattice sequence in a \( F \)-vector space, say \( V \), is a function \( \Lambda \) from \( \mathbb{Z} \) to the set of \( \sigma_F \)-lattices in \( V \) with the following conditions on \( \Lambda \):
\[ \Lambda(n + 1) \subseteq \Lambda(n), \quad \text{for all } n \in \mathbb{Z}; \]
and there exists an \( e(\Lambda) \in \mathbb{Z} \) such that
\[ \Lambda(n + e(\Lambda)) = p_F \Lambda(n), \quad \text{for all } n \in \mathbb{Z}. \]

An \( \sigma_F \)-lattice chain is an \( \sigma_F \)-lattice sequence with the strict inclusion between \( \Lambda(n + 1) \) and \( \Lambda(n) \), for all \( n \in \mathbb{Z} \). One extends the function \( \Lambda \) to the set of real numbers by setting
\[ \Lambda(r) = \Lambda([-r]), \]
for all \( r \in \mathbb{R} \). Here, \([x]\) is the greatest integer less than or equal to \( x \).

Given two \( \sigma_F \)-lattice sequences \( \Lambda_1 \) and \( \Lambda_2 \) in the vector spaces \( V_1 \) and \( V_2 \) over \( F \), Bushnell and Kutzko defined the notion of direct sum of \( \Lambda_1 \) and \( \Lambda_2 \). Let \( e = \text{lcm}(e(\Lambda_1), e(\Lambda_2)) \). The direct sum of \( \Lambda_1 \) and \( \Lambda_2 \), denoted by \( \Lambda \), is an \( \sigma_F \)-lattice sequence in the vector space \( V_1 \oplus V_2 \) given by
\[ \Lambda(e r) = \Lambda_1(e r) \oplus \Lambda_2(e r), \]
for any \( r \in \mathbb{R} \). Given an \( \sigma_F \)-lattice sequence \( \Lambda \) in a vector space \( V \) one can define a filtration \( \{a_r(\Lambda) \mid r \in \mathbb{R}\} \) on the algebra \( \text{End}_F(V) \) given by the equation
\[ a_r(\Lambda) = \{x \in \text{End}_F(V) \mid x \Lambda(i) \subseteq \Lambda(i + r) \forall i \in \mathbb{Z}\}. \]
Let \( u_0(\Lambda) \) be the group of units in the order \( a_0(\Lambda) \) and, for \( r > 0 \) and \( r \in \mathbb{Z} \), we set \( u_r(\Lambda) \) to be \( 1 + a_r(\Lambda) \).

Let \( (J_s, \lambda_s) \) be a Bushnell-Kutzko type for the inertial class \([M_I, \sigma \boxtimes \text{id}]\).

The group \( J_s \) satisfies the Iwahori decomposition with respect to the parabolic subgroup \( P_I \) and the Levi subgroup \( M_I \). Let \([\mathfrak{A}, l, 0, \beta]\) be a simple stratum defining the type \((J^0, \lambda)\) for the inertial class \([G_{n-1}, \sigma]\).

The order \( \mathfrak{A} \) is defined by a lattice chain \( \Lambda_1 \) with values in sub-lattices of \( \sigma_F^\mathfrak{A} \). We denote by \( \Lambda_2 \) the lattice chain defined by \( \Lambda_2(i) = p_F^{\mathfrak{A}}, \) for all \( i \in \mathbb{Z} \). We have

1. \( J_s \cap U_I = u_0(\Lambda_1 \oplus \Lambda_2) \cap U_I, \)
2. \( J_s \cap M_I = J^0 \times \sigma_F^\mathfrak{A}, \)
3. \( J_s \cap \tilde{U}_I = u_{l+1}(\Lambda_1 \oplus \Lambda_2) \cap \tilde{U}_I, \)
4. the restriction of \( \lambda_s \) to \( J_s \cap M_I \) is isomorphic to \( \lambda \boxtimes \text{id} \), and the groups \( J_s \cap \tilde{U}_I \) and \( J_s \cap U_I \) are contained in the kernel of \( \lambda_s \).
We refer to [BK99] Section 8, paragraph 8.3.1] for the construction of the above Bushnell–Kutzko type.

Now we make an explicit calculation of the groups $u_{l+1}(A_1 \oplus A_2) \cap \bar{U}_I$ and $u_0(A_1 \oplus A_2) \cap U_I$. Note that the period of the direct sum $A_1 \oplus A_2$ is the least common multiple of the period of the two lattice sequences $A_1$ and $A_2$. Hence, the period of the lattice sequence $\Lambda$ is $e$, where $e$ is the period of the lattice chain $A_1$. Let $t$ be an integer such that $0 \leq t \leq e - 1$. Let $L_0$ be the free $\mathfrak{o}_F$ module $\mathfrak{o}_F^{(n-1)/e}$. The lattice $\Lambda_l(t)$ is given by:

$$\Lambda_l(t) = (L_0 \oplus L_0 \oplus \cdots \oplus L_0) \oplus (\mathfrak{o}_F L_0 \oplus \mathfrak{o}_F L_0 \oplus \cdots \oplus \mathfrak{o}_F L_0),$$

where the $L_0$ is repeated $e - t$ times, and $\mathfrak{o}_F L_0$ is repeated $t$ times. Hence, the lattice chain $\Lambda$ is given by:

$$\Lambda(0) = \Lambda_l(0) \oplus \Lambda_2(0) = (L_0 \oplus L_0 \oplus \cdots \oplus L_0) \oplus \mathfrak{o}_F,$$

and

$$\Lambda(t) = \Lambda_l(t) \oplus \Lambda_2(t/e) = (L_0 \oplus L_0 \oplus \cdots \oplus L_0) \oplus (\mathfrak{o}_F L_0 \oplus \mathfrak{o}_F L_0 \oplus \cdots \oplus \mathfrak{o}_F L_0) \oplus \mathfrak{p}_F,$$

for $0 \leq t \leq e - 1$.

We note that $u_0(\Lambda) \cap U_I = U_I(\mathfrak{o}_F)$. We denote by $\bar{\mathfrak{n}}_I$ the lower nilpotent matrices of the type $(n - 1, 1)$, i.e. the Lie algebra of $U_I$. We then have:

$$u_{l+1}(\Lambda) \cap \bar{U}_I = \text{id} + (a_{l+1}(\Lambda) \cap \bar{\mathfrak{n}}_I).$$

Let $l + 1 = el' + r$, where $0 \leq r < e$. Since $\Lambda$ is a lattice chain of period $e$, we deduce that

$$a_{l+1}(\Lambda) \cap \bar{\mathfrak{n}}_I = \mathfrak{w}_F(a_r(\Lambda) \cap \bar{\mathfrak{n}}_I).$$

Finally, it remains to calculate the group $a_r(\Lambda) \cap \bar{\mathfrak{n}}_I$. We note that $a_r(\Lambda) \cap \bar{\mathfrak{n}}_I$ is the following set

$$\{x \in \text{Mat}_{n \times n}(F) \cap \bar{\mathfrak{n}}_I \ | \ x\Lambda(i) \subseteq \Lambda(i + r) \ \forall i \in \mathbb{Z}\}.$$

For $r \geq 1$, the $n^{th}$ row (in block form) of an element in $a_r(\Lambda) \cap \bar{\mathfrak{n}}_I$ is of the form $A = [M_1, M_2, \ldots, M_e, 0]$, where $M_i$ is a matrix of type $1 \times (n - 1)/e$, for $1 \leq i \leq e$ and:

1. $M_i \in \mathfrak{w}_F \text{Mat}_{1 \times (n-1)/e}(\mathfrak{o}_F)$, for $i \leq r - 1$,
2. $M_i \in \mathfrak{w}_F \text{Mat}_{1 \times (n-1)/e}(\mathfrak{o}_F)$, for $i > r - 1$.

If $r = 0$ and $e > 1$, then we know that $M_i \in \mathfrak{w}_F \text{Mat}_{1 \times (n-1)/e}(\mathfrak{o}_F)$, for $1 \leq i \leq e - 1$, and $M_e \in \text{Mat}_{1 \times (n-1)/e}(\mathfrak{o}_F)$. If $r = 0$ and $e = 1$, then we have $A \in \text{Mat}_{1 \times n}(\mathfrak{o}_F)$. This description is enough for the present purposes.

3.2. Some auxiliary groups. Let $m$ be a positive integer and $P_I(m)$ be the inverse image of the group $P_I(\mathfrak{o}_F/\mathfrak{p}_F^m)$ under the mod-$\mathfrak{p}_F^m$ reduction map

$$K_n \to \text{GL}_n(\mathfrak{o}_F/\mathfrak{p}_F^m).$$

There exists a positive integer $N_1$ such that the principal congruence subgroup of level $N_1$ is contained in the kernel of the representation $\tau$. The representation $\tau \boxtimes \text{id}$ of $M_I(\mathfrak{o}_F/\mathfrak{p}_F^{N_1})$ now extends to a representation of $P_I(N_1)$ by inflation. We note that $P_I(N_1) \cap \bar{U}_I$ and $P_I(N_1) \cap U_I$ are both contained in the kernel of this extension. Now the sequence of groups $H_m = P_I(m)$ and the representation $\tau \boxtimes \text{id}$, for $m \geq N_1$ satisfy the conditions in Hypothesis [5]. Hence, we get that

$$\text{ind}^K_{P_I(N_1)}(\tau \boxtimes \text{id}) \simeq \bigcup_{m \geq N_1} \text{ind}^K_{P_I(m)}(\tau \boxtimes \text{id}).$$

We conclude that typical representations occur as sub representations of

$$\text{ind}^K_{P_I(m)}(\tau \boxtimes \text{id}),$$

for some positive integer $m \geq N_1$.

For making Mackey decompositions easier and other reasons, it is convenient to work with a smaller subgroup $P_I^0(m)$ of $P_I(m)$. We begin by rewriting the representation

$$\text{ind}^K_{P_I^0(m)}(\tau \boxtimes \text{id}).$$

We also require to make $N_1$ explicit. We recall that $K_{n-1}(m)$ is the principal congruence subgroup of level $m$ of $G_{n-1}$. The group $J^{0}$ contains the group $U_{[1/2]+1}(\mathfrak{A})$ and the representation $\lambda$ restricted to the
group $U^{[l]/2}+1(\mathfrak{A})$ is a direct sum of copies of the same character $\psi_\beta$. The character $\psi_\beta$ is trivial on the group $U^{l+1}(\mathfrak{A})$. We also recall the notation that $l + 1 = e^l + r$, where $0 \leq r \leq e - 1$. We note that $U^{l+1}(\mathfrak{A}) = \id_{\mathfrak{A}} + \varphi_\mathfrak{A}^r$. If $r = 0$, then $K_{n-1}(1) \subseteq \mathfrak{P}^r_{\mathfrak{A}}$. If $r > 1$, then from the formulas \cite[2.5.2]{BK93} we get that $K_{n-1}(2) \subseteq \mathfrak{P}^r_{\mathfrak{A}}$, for $0 \leq r < e$. This shows that the representation $\lambda$ is trivial on $K_{n-1}(N_s)$, where $N_s$ is given by:

**Notation 3.2.1.** From now we fix $N_s = \lfloor (l+1)/e \rfloor + 1$ if $r = 0$ and $e > 1$. If $r = 0$ and $e = 1$, then $N_s = l + 1$. Finally, $N_s = \lfloor (l+1)/e \rfloor + 2$ if $r \geq 1$.

Let $\pi$ be the projection map

$$
\pi_t(\sigma_F) \to M_t(\sigma_F).
$$

For $m \geq N_s$, we denote by $P^0_t(m)$ the group $K_n(m)^{-1}(J^0 \times \sigma_F^m)$. Since $K_n(m) \cap P_t \subseteq \pi^{-1}(J^0 \times \sigma_F^m)$, the group $P^0_t(m)$ satisfies the Iwahori decomposition with respect to the subgroup $P_t$ and the Levi subgroup $M_t$. In particular, we have

$$
P^0_t(m) = (P^0_t(m) \cap U_t)(P^0_t(m) \cap M_t)(P^0_t(m) \cap \bar{U}_t).
$$

Here, $P^0_t(m) \cap U_t$ is equal to $U_t(\sigma_F)$, $P^0_t(m) \cap M_t$ is equal to $J^0 \times \sigma_F^m$, and $(P^0_t(m) \cap \bar{U}_t)$ is equal to $K_n(m) \cap \bar{U}_t$.

We observe that $\lambda \boxtimes \id$ extends as a representation of $P^0_t(m)$, for all $m \geq N_s$; the groups $P^0_t(m) \cap U_t$ and $P^0_t(m) \cap \bar{U}_t$ are contained in the kernel of this extension. Now the representation $\tau \boxtimes \id$ of $K_n(m) \cap \sigma_F^m$ is isomorphic to

$$
\{ \text{ind}^{K_n(m)}(\lambda) \} \boxtimes \id.
$$

Hence, we get that

$$
\text{ind}^{K_n(m)}(\tau \boxtimes \id) \simeq \text{ind}^{K_n(m)}(\lambda \boxtimes \id),
$$

for all $m \geq N_s$ (we apply Lemma \ref{Lem2.0.4} to the groups $J_1 = P_t(m)$ and $J_2 = P^0_t(m)$ and $\lambda = \lambda \boxtimes \id$). We get that

$$
\text{ind}^{K_n(m)}(\tau \boxtimes \id) \simeq \bigcup_{m \geq N_s} \text{ind}^{K_n(m)}(\lambda \boxtimes \id).
$$

Hence, any typical representation occurs as a sub-representation of

$$
\text{ind}^{K_n(m)}(\lambda \boxtimes \id),
$$

for some $m \geq N_s$.

We first have to understand the representation

$$
\text{ind}^{P^0_t(m)}(\id),
$$

for $m \geq N_s$. It is convenient to define a normal subgroup $R_t(m)$ of $P^0_t(m)$ such that $P^0_t(m)$ is equal to $R_t(m)P^0_t(m+1)$, and $R_t(m) \cap P^0_t(m+1) = R_t(m+1)$, for $m \geq N_s$. For any integer $m \geq N_s$, we define $R_t(m)$ to be the group $K_n(m)^{-1}(K_{n-1}(N_s) \times (1 + \mathfrak{P}^r_{\mathfrak{A}}))$. The group $R_t(m)$ has the Iwahori decomposition with respect to the parabolic subgroup $P_t$ and its Levi subgroup $M_t$.

**Lemma 3.2.1.** The group $R_t(m)$ is a normal subgroup of $P^0_t(m)$. The group $R_t(m+1)$ is a normal subgroup of $R_t(m)$, for all $m \geq N_s$.

**Proof.** By definition of the groups $R_t(m)$, we have $R_t(m) \cap U_t = P^0_t(m) \cap U_t$, and $R_t(m) \cap \bar{U}_t = P^0_t(m) \cap \bar{U}_t$.

To show the normality of $R_t(m)$ in $P^0_t(m)$, we have to verify that $P^0_t(m) \cap M_t$ normalize the group $R_t(m)$. But, $P^0_t(m) \cap M_t$ normalizes the group $R_t(m) \cap U_t = U_t(\sigma_F)$ and $R_t(m) \cap \bar{U}_t = \bar{U}_t(\varphi_\mathfrak{A}^r \sigma_F)$. The group $K_n(m) \cap M_t$ is a normal subgroup of $M_t(\sigma_F)$. Hence, $P^0_t(m) \cap M_t$ normalizes $R_t(m) \cap M_t$. This shows the first part of the lemma.

Since $R_t(m) \cap P_t = R_t(m+1) \cap P_t$, we have to check that $R_t(m) \cap \bar{U}_t$ normalizes the group $R_t(m+1)$. We note that $\bar{U}_t$ is abelian. Hence, we have to check that the conjugations $u^- j(u^-)^{-1}$ and $u^- u^+ (u^-)^{-1}$ belong to the group $R_t(m+1)$, for all $u^- \in R_t(m) \cap \bar{U}_t$, $j \in R_t(m+1) \cap M_t = R_t(m) \cap M_t$, and $u^+ \in R_t(m+1) \cap U_t = U_t(\sigma_F)$. Let us begin with the element $u^- j(u^-)^{-1}$. We have $u^- j(u^-)^{-1} = j^{-1} u^- j(u^-)^{-1}$. Let

$$
j = \begin{pmatrix} J_1 & 0 \\ 0 & J_1 \end{pmatrix},
\quad u^- = \begin{pmatrix} 1_{n-1} & 0 \\ U^- & 1 \end{pmatrix}.
$$
be the block diagonal form of $j$ and $u^{-}$; $J_{1} \in K_{n-1}(N_{s}), j_{1} \in 1 + pR N_{s}$ and $U^{-} \in \varpi_{F}^{m} \text{Mat}_{1 \times(n-1)}(\mathfrak{o}_{F})$. The element $j^{-1}u^{-}j(u^{-})^{-1}$ is of the form

$$
\begin{pmatrix}
1_{n-1} & 0 \\
j_{1}^{-1}U^{-}J_{1} - U^{-} & 1
\end{pmatrix},
$$

We note that the matrix $j_{1}^{-1}U^{-}J_{1} - U^{-}$ belongs to $\varpi_{F}^{m+1} \text{Mat}_{1 \times(n-1)}(\mathfrak{o}_{F})$. This shows that $j^{-1}u^{-}j(u^{-})^{-1} \in R_{I}(m + 1) \cap \bar{U}_{I}$. Hence we get that

$$
u^{-}j(u^{-})^{-1} = j\{j^{-1}u^{-}j(u^{-})^{-1}\} \in R_{I}(m + 1).
$$

We now consider the conjugation $u^{-}u^{+}(u^{-})^{-1}$. We write $u^{+}$ in its block matrix form as

$$
\begin{pmatrix}
1_{n-1} & U^{+} \\
0 & 1
\end{pmatrix}
$$

where $U^{+} \in \text{Mat}_{(n-1) \times 1}(\mathfrak{o}_{F})$. Now the conjugation $u^{-}u^{+}(u^{-})^{-1}$ in the block matrix from is as follows

$$
\begin{pmatrix}
1_{n-1} - U^{+}U^{-} & U^{+} \\
-U^{-}U^{+}U^{-} & U^{-}U^{+} + 1
\end{pmatrix}.
$$

Since $U^{-}U^{+}U^{-} \in \varpi^{m+1} \text{Mat}_{1 \times(n-1)}(\mathfrak{o}_{F})$, we conclude that $u^{-}u^{+}(u^{-})^{-1} \in R_{I}(m + 1)$. This ends the proof of this lemma.

Note that $P_{I}^{\delta}(m) = R_{I}(m)P_{I}^{\delta}(m + 1)$. Using Mackey decomposition, we get that

$$
\text{res}_{R_{I}(m)} \text{ind}_{P_{I}^{\delta}(m)}^{R_{I}(m)}(\text{id}) \simeq \text{ind}_{R_{I}(m+1)}^{R_{I}(m)}(\text{id}).
$$

It follows from the Iwahori decomposition with respect to $P_{I}$ and $M_{I}$ that the inclusion of $R_{I}(m) \cap \bar{U}_{I}$ in $R_{I}(m)$ induces an isomorphism between $R_{I}(m)/R_{I}(m + 1)$ and the abelian group

$$
\frac{R_{I}(m) \cap \bar{U}_{I}}{R_{I}(m + 1) \cap \bar{U}_{I}}.
$$

Hence, the representation $\text{ind}_{R_{I}(m+1)}^{R_{I}(m)}(\text{id})$ decomposes as a direct sum of characters $\eta_{k}$, for $1 \leq k \leq p$, where $\eta_{k}$ is trivial on $R_{I}(m + 1)$. The group $P_{I}^{\delta}(m)$ acts on these characters, and let $\{\eta_{n_{k}} \mid n_{k} \in \{1, 2, \ldots, p\}\}$ be a set of representatives for the orbits under this action. Let $Z(\eta_{k})$ be the $P_{I}^{\delta}(m)$-stabiliser of the character $\eta_{k}$, for $1 \leq k \leq p$. Now Clifford theory gives us the isomorphism

$$
\text{ind}_{P_{I}^{\delta}(m)}^{R_{I}(m+1)}(\text{id}) = \bigoplus_{\eta_{n_{k}}} \text{ind}_{Z(\eta_{n_{k}})}^{P_{I}^{\delta}(m)}(U_{\eta_{n_{k}}}),
$$

where $U_{\eta_{n_{k}}}$ is any irreducible representation of $Z(\eta_{n_{k}})$ such that $\text{res}_{R_{I}(m)}U_{\eta_{n_{k}}}$ contains $\eta_{n_{k}}$.

We have to bound the group $Z(\eta_{k})$. We note that $P_{I}^{\delta}(m)$ is equal to $(P_{I}^{\delta}(m) \cap M_{I})R_{I}(m)$. Hence, we have $Z(\eta_{k}) = (Z(\eta_{k}) \cap M_{I})R_{I}(m)$. To bound the group $Z(\eta_{k})$ we can only need to control $Z(\eta_{k}) \cap M_{I}$. Let $u^{-} \in R_{I}(m) \cap \bar{U}_{I}$ and

$$
\begin{pmatrix}
1_{n-1} & 0 \\
U^{-} & 1
\end{pmatrix}
$$

be the block form of $u^{-}$, where $U^{-}$ is a matrix in $\varpi_{F}^{m+1} \text{Mat}_{1 \times(n-1)}(\mathfrak{o}_{F})$. The map $u^{-} \mapsto \varpi_{F}^{-m}U^{-}$ induces an $M_{I}(\mathfrak{o}_{F})$-equivariant isomorphism between $\text{Mat}_{1 \times(n-1)}(k_{F})$ and the quotient

$$
\frac{R_{I}(m) \cap \bar{U}_{I}}{R_{I}(m + 1) \cap \bar{U}_{I}}.
$$

We also have an $M_{I}(\mathfrak{o}_{F})$-equivariant isomorphism between $\text{Mat}_{(n-1) \times 1}(k_{F})$ and $\text{Mat}_{1 \times(n-1)}(k_{F})$ (see Lemma 2.0.7). We note that $P_{I}^{\delta}(m) \cap M_{I} = \varphi^{0} \times \varphi_{F}^{\infty}$.

Let $\eta$ be a non-trivial character of $R_{I}(m)$ which is trivial on $R_{I}(m + 1)$. For the present purposes, it is enough to bound the subgroup $Z(\eta) \cap (\varphi^{0}(\mathfrak{B}) \times \varphi_{F}^{\infty})$, for $\eta \neq \text{id}$. Since we have a $M_{I}(\mathfrak{o}_{F})$-equivariant isomorphism between the group of characters on the quotient $R_{I}(m)/R_{I}(m + 1)$ with $\text{Mat}_{n-1 \times 1}(k_{F})$, we can as well study the group $Z(A) \cap (\varphi^{0}(\mathfrak{B}) \times \varphi_{F}^{\infty})$, where $Z(A)$ is the $M_{I}(\mathfrak{o}_{F})$-stabiliser of a non-zero matrix
A ∈ Mat\((n-1)×1\)\(k_F\). The action of the group \(M_f(\mathfrak{o}_F)\) factorizes through \(K_{n-1}(1) × (1 + p_F)\). Hence, the group \((\text{id}_{n-1} + \mathfrak{o}_F) × (1 + p_F)\) is contained in the kernel of the action of \(U^0(\mathfrak{B}) × \mathfrak{o}_F^\times\) obtained by restriction. Recall that \(\mathfrak{o}_F\) is the radical of \(\mathfrak{B}\).

This reduces our situation to the following setting. The group \(\text{GL}_{n-1}(k_F) × k_F^\times\) acts on \(\text{Mat}_{n-1×1}(k_F)\) by setting
\[
(g_1, g_2)A = g_1Ag_2^{-1},
\]
where \(g_1 ∈ \text{GL}_{n-1}(k_F)\), \(g_2 ∈ k_F^\times\), and \(A ∈ \text{Mat}_{n-1×1}(k_F)\). The basis \((v_1, v_2, \ldots, v_{n-1})\) we fixed for the vector space \(V\) at the beginning of this section, gives a basis of the \(k_F\)-vector space
\[
(\mathfrak{o}_E/\mathfrak{w}_E\mathfrak{o}_E)^{(n-1)/ef} = (\mathfrak{o}_E/p_E)^{(n-1)/ef}.
\]
Such a basis gives the inclusion
\[
\text{GL}_{(n-1)/ef}(\mathfrak{o}_E/p_E^e) = U^0(\mathfrak{B})/U^e(\mathfrak{B}) \hookrightarrow \text{GL}_{n-1}(k_F).
\]
Recall that \(\lambda = \kappa ⊗ \rho\), where \(\rho\) is a cuspidal representation of \(\text{GL}_{(n-1)/ef}(k_F) = U^0(\mathfrak{B})/U^1(\mathfrak{B})\). Hence, we are interested in the mod \(p_E\) reduction of the first projection of
\[
Z_{\text{GL}_{(n-1)/ef}(\mathfrak{o}_E/p_E^e) × k_F^\times}(A),
\]
for some non-zero matrix \(A\) in \(\text{Mat}_{n-1×1}(k_F)\). We set \(n_0 = (n-1)/ef\).

Let \(\mathfrak{w}_E\) be an uniformiser of \(\mathfrak{o}_E\). Let \(N\) be the operator on the \(k_E\)-vector space \(W := (\mathfrak{o}_E/p_E^e)^{n_0}\) given by
\[
N(w) = wEw \text{ for all } w ∈ W.
\]
Since \(\mathfrak{o}_E/\mathfrak{w}_E^e = k_E ⊕ k_E \mathfrak{w}_E ⊕ k_E \mathfrak{w}_E^2 ⊕ \cdots ⊕ k_E \mathfrak{w}_E^{e-1}\), we obtain a decomposition of \(W = W_1 ⊕ W_2 ⊕ \cdots ⊕ W_e\) such that \(N\) restricted to \(W_i\) is an isomorphism onto \(W_{i+1}\) for \(i < e\), and \(N\) acts trivially on \(W_e\). The mod \(p_E\)-reduction of \(W\) is the projection onto the first factor \(W_1\).

Any \(k_E[N]\)-linear map \(T\) is determined by its restriction to the space \(W_1\). Given a map \(T ∈ \text{Hom}_{k_E}(W_1, W)\) we obtain an extension \(\tilde{T} ∈ \text{End}_{k_E[N]}(W)\) by setting
\[
\tilde{T}(w) = N^{(i-1)}T(N^{-(i-1)}w),
\]
for all \(w ∈ W_1\) and \(1 ≤ i ≤ e\). The map \(T → \tilde{T}\) gives us an isomorphism of vector spaces
\[
\text{Hom}_{k_E}(W_1, W) ∼= \text{End}_{k_E[N]}(W, W).
\]
(4)
We may write \(W = W_1 ⊕ NW\). This shows that the mod \(p_E\) reduction map, denoted by \(π_E\), is given by sending \(\tilde{T}\) to \(p_1 \circ \tilde{T}|_{W_1}\), where \(p_1\) is the projection onto the first factor of the direct sum \(W_1 ⊕ W_2 ⊕ \cdots ⊕ W_e\). Now \(\text{End}_{k_E[N]}(W_1)\) is a subspace of \(\text{Hom}_{k_E}(W_1, W)\) and \(p_E\) reduction of \(\text{End}_{k_E[N]}(W_1)\) (the image of \(\text{End}_{k_E[N]}(W_1)\) under the map \(T → \tilde{T}\)) is identity on \(\text{End}_{k_E[N]}(W_1)\). Hence \(\text{Aut}_{k_E[N]}(W_1)\) is the semi-direct product \(\text{Aut}_{k_E[N]}(W_1) \text{ker}(\pi_E)\).

Let \(Q\) be a parabolic subgroup fixing the flag \(\mathcal{F}^i = \oplus_{j=1}^i W_i\), for \(1 ≤ i ≤ e\), and \(L\) be its Levi subgroup fixing the decomposition \(W_1 ⊕ W_2 ⊕ \cdots ⊕ W_e\). Now \(\text{Aut}_{k_E[N]}(W_1)\) diagonally embeds in \(L\) and \(\text{ker}(\pi_E)\) is a subgroup of the radical of \(Q\). The group \(\text{GL}_{n-1}(k_F) × k_F^\times\) acts on \(\text{Mat}_{n-1×1}(k_F)\) by the map \((g_1, g_2)A ↦ g_1Ag_2^{-1}\), where \(g_1 ∈ \text{GL}_{n-1}(k_F)\), \(g_2 ∈ k_F^\times\), and \(A ∈ \text{Mat}_{n-1×1}(k_F)\). We now have the action of \(\text{GL}_{n_0}(\mathfrak{o}_E/\mathfrak{w}_E^e) × k_F^\times\) on \(\text{Mat}_{n-1×1}(k_F)\) by restriction from the action of \(\text{GL}_{n-1}(k_F) × k_F^\times\). We are interested in
\[
(\mathfrak{p}_E × \text{id})(Z_{\text{GL}_{n_0}(\mathfrak{o}_E/\mathfrak{w}_E^e) × k_F^\times}(A)),
\]
for some \(A ∈ \text{Mat}_{n-1×1}(k_F)\). We first look at \(Z_{\mathfrak{Q} × k_F^\times}(A)\). Let \((A_{ij})\) be an element of \(Q\) in its block form. Let \((A_1, A_2, \ldots, A_e)^T\) be the block form of \(A\), where \(A_j\) is a block of size \(1 × n_0f\). If \(k\) is the largest positive integer such that \(A_k ≠ 0\) and \(A_k = 0\), then we get that
\[
A_{kk}A_k^{-1} = A_k.
\]
for all \(((A_{ij}), a) \in Z_{Q \times k_F^\times}(A)\). Hence \(|A_k| \setminus \{(A_{ij}), a \in Z_{Q \times k_F^\times}(A)\}\) is contained in a proper parabolic subgroup of \(\text{Aut}_{k_E}(W_k)\). If \(n_0f > 1\), we get that

\[
(\pi_E \times \text{id})\{Z_{\text{GL}_{n_0}(\mathbb{P}_E)} \times k_F^\times(A)\}
\]

is a subgroup of \(H \times k_F^\times\), where \(H\) is a subgroup of \(\text{Aut}_{k_E}(W_1)\) whose image under the inclusion map \(\text{Aut}_{k_E}(W_1) \hookrightarrow \text{Aut}_{k_E}(W_1)\) is contained in a proper \(k_{E,0}\)-parabolic subgroup of \(\text{Aut}_{k_E}(W_1)\), since \(A_k \neq 0\).

We recall the following proposition due to Paškūnas (see [Pas05] Definition 6.2, lemma 6.5, Proposition 6.8).

**Proposition 3.2.2** (Paškūnas). Let \(W\) be a \(k_E\)-vector space with (finite) dimension greater than one. Let \(\rho\) be a cuspidal representation of \(\text{Aut}_{k_E}(W)\). Let \(H\) be a subgroup of \(\text{Aut}_{k_E}(W)\) such that the image of \(H\) under the inclusion map \(\text{Aut}_{k_E}(W) \hookrightarrow \text{Aut}_{k_E}(W)\) is contained in a proper parabolic subgroup of \(\text{Aut}_{k_E}(W)\). For every \(H\)-irreducible sub-representation \(\xi\) of \(\text{res}_H(\rho)\) there exists an irreducible representation \(\rho'\) of \(\text{Aut}_{k_E}(W)\) such that \(\rho' \neq \rho\) and \(\text{Hom}_H(\xi, \rho') \neq 0\).

Going back to \(Z(\eta) \cap (U^0(\mathfrak{B}) \times \sigma_F^\times)\), for \(n_0 > 1\), we get that for every irreducible subrepresentation \(\xi\) of

\[
\text{res}_{Z(\eta) \cap (U^0(\mathfrak{B}) \times \sigma_F^\times)}((\kappa \otimes \rho) \boxtimes \text{id}),
\]

there exists an irreducible representation \(\rho'\) of \(U^0(\mathfrak{B})/U^1(\mathfrak{B})\) such that

\[
\text{Hom}_{Z(\eta) \cap (U^0(\mathfrak{B}) \times \sigma_F^\times)}(\xi, (\kappa \otimes \rho') \boxtimes \text{id}) \neq 0.
\]

For the case \(n_0 = 1\) and \(q_F > 2\), we have to look at

\[
(\pi_E \times \text{id})\{Z_{\mathbb{P}_E} \times k_F^\times(A)\}
\]

for some nonzero matrix \(A \in \text{Mat}_{n-1 \times 1}(k_F)\). We notice that the group \(\mathbb{G}\) is of the form \(\{(a, a) | a \in k_F^\times\}\) if \(k_E = k_F\). Let \(k_E\) be a proper extension of \(k_F\). If \((a, b)\) is an element of the centraliser \(\mathbb{G}\) then \(aA_kb^{-1} = A_k\) (\(A_k\) is defined in the previous paragraph). This shows that \(a\) lies in a proper parabolic subgroup of \(\text{GL}_E(k_F)\). This shows that the group \(\mathbb{G}\) is of the form \(\{(a, b) | a \in F^\times, b \in k_F^\times\}\) where \(F^\times\) is a proper sub-field of \(k_E^\times\).

In the case where \(n_0 = 1\) and \(k_E = k_F\), we consider a non-trivial character \(\phi\) of \(U^0(\mathfrak{B})/U^1(\mathfrak{B}) = k_F^\times\). We observe that

\[
\text{res}_{J^0 \times \sigma_F^\times}(A)(\lambda \phi \boxtimes (\phi)^{-1}) \simeq \text{res}_{J^0 \times \sigma_F^\times}(A)(\lambda \boxtimes \text{id}).
\]

Moreover, \([M_I, \sigma \boxtimes \text{id}]\) and \([M_I, \sigma' \boxtimes (\phi)^{-1}]\) are two distinct inertial classes for any cuspidal representation \(\sigma'\) containing \((J^0, \sigma \otimes \phi)\). Here, we will use the same notation \(\phi\) for the inflation of \(\phi\) to the group \(\sigma_F^\times\).

In the case where \(n_0 = 1\) and \(k_E\) is a proper extension of \(k_F\), we consider a non-trivial character \(\phi\) of \(k_E^\times\) which is trivial on \(F^\times\). We note that

\[
\text{res}_{J^0 \times \sigma_F^\times}(A)(\lambda \phi \boxtimes \text{id}) \simeq \text{res}_{J^0 \times \sigma_F^\times}(A)(\lambda \boxtimes \text{id})
\]

and moreover \([M_I, \sigma \boxtimes \text{id}]\) and \([M_I, \sigma' \boxtimes \text{id}]\) are two distinct inertial classes for any cuspidal representation \(\sigma'\) containing \((J^0, \lambda \otimes \phi)\). With this we finish our preliminaries.

### 3.3. Uniqueness of typical representations.

In this part, we will prove the uniqueness of typical representations for \([M_I, \sigma \boxtimes \text{id}]\). By Frobenius reciprocity, we get that \(\lambda \boxtimes \text{id}\) occurs with multiplicity one in \(\text{ind}_{P_I}^{\mathbb{P}_I}(N_a)(\lambda \boxtimes \text{id})\), for all \(m > N_a\). We denote by \(U^0_m(\lambda \boxtimes \text{id})\) the complement of \(\lambda \boxtimes \text{id}\) in \(\text{ind}_{P_I}^{\mathbb{P}_I}(N_a)(\lambda \boxtimes \text{id})\). We use the notation \(U_m(\lambda \boxtimes \text{id})\) for the representation

\[
\text{ind}_{P_I}^{\mathbb{P}_I}(N_a)(U^0_m(\lambda \boxtimes \text{id})_{\lambda \boxtimes \text{id}}).
\]

**Theorem 3.3.1.** Let \(\#k_F > 2\). The \(K_n\)-irreducible sub-representations of \(U_m(\lambda \boxtimes \text{id})\) are atypical, for all \(m > N_s\).
Proof. We prove the theorem by induction on the positive integer \( m > N_s \). We suppose the theorem is true for some positive integer \( m > N_s \). We will show the same for \( m + 1 \).

We first note that
\[
\text{ind}_{\rho_{P_l(m+1)}}(\lambda \boxtimes \text{id}) \simeq \text{ind}_{\rho_{P_l(m)}}(\text{ind}_{\rho_{P_l(m+1)}}(\text{id}) \otimes (\lambda \boxtimes \text{id})).
\]
From the decomposition \( K \) we get that
\[
\text{ind}_{\rho_{P_l(m+1)}}(\lambda \boxtimes \text{id}) \simeq \bigoplus_{\eta_{n_k}} \text{ind}_{\rho_{Z(\eta_{n_k})}}((\lambda \boxtimes \text{id}) \otimes U_{\eta_{n_k}}).
\]
Note that the above sum is taken over the orbits for the action of \( P_l(m) \) on the set of characters of \( R_l(m)/R_l(m+1) \). Since there is a unique orbit, among the characters \( \{\eta_k \mid 1 \leq k \leq p\} \), consisting the identity character, we get that
\[
\text{ind}_{\rho_{P_l(m+1)}}(\lambda \boxtimes \text{id}) \simeq \text{ind}_{\rho_{P_l(m)}}(\lambda \boxtimes \text{id}) \oplus \bigoplus_{\eta_{n_k} \neq \text{id}} \text{ind}_{\rho_{Z(\eta_{n_k})}}((\lambda \boxtimes \text{id}) \otimes U_{\eta_{n_k}}). \tag{6}
\]
Let \( \Gamma \) be an irreducible sub-representation of
\[
\text{ind}_{\rho_{Z(\eta_{n_k})}}((\lambda \boxtimes \text{id}) \otimes U_{\eta_{n_k}}). \tag{7}
\]
We have two cases \( n_0 = 1 \) and \( n_0 > 1 \). If \( n_0 = 1 \) we have seen that we can find a non-trivial character \( \phi \) of \( k_\mathcal{E} = U^0(\mathfrak{B})/U^1(\mathfrak{B}) \) such that
\[
\text{ind}_{\rho_{Z(\eta_{n_k})}}((\lambda \boxtimes \text{id}) \otimes U_{\eta_{n_k}}) \simeq \text{ind}_{\rho_{Z(\eta_{n_k})}}((\lambda \phi \boxtimes \phi^{-1}) \otimes U_{\eta_{n_k}})
\]
or
\[
\text{ind}_{\rho_{Z(\eta_{n_k})}}((\lambda \boxtimes \text{id}) \otimes U_{\eta_{n_k}}) \simeq \text{ind}_{\rho_{Z(\eta_{n_k})}}((\lambda \phi \boxtimes \text{id}) \otimes U_{\eta_{n_k}}).
\]
Hence in this case, the irreducible subrepresentations of
\[
\text{ind}_{\rho_{Z(\eta_{n_k})}}((\lambda \boxtimes \text{id}) \otimes U_{\eta_{n_k}}) \tag{8}
\]
ock as subrepresentations of \( \text{res}_{K_n} i^{G_n}_{P_l}(\sigma' \boxtimes \chi') \), where \( \sigma' \) is a cuspidal representation of \( G_{n-1} \) containing the type \((J^0, \lambda \otimes \phi)\). The inertial classes \([G_{n-1}, \sigma]\) and \([G_{n-1}, \sigma']\) are distinct. Hence, any irreducible subrepresentation of \( \bullet \) is atypical.

Now consider the case \( n_0 > 1 \). In this case, there exists an irreducible representation \( \xi \) of \((\pi_\mathcal{E} \times \text{id})\{Z(\eta) \cap (U^0(\mathfrak{B}) \times \sigma_\mathcal{F})\} \) such that \( \Gamma \) is a sub-representation of
\[
\text{ind}_{\rho_{Z(\eta_{n_k})}}((\xi \otimes \kappa) \boxtimes \text{id}) \otimes U_{\eta_{n_k}}. \tag{9}
\]
Now Proposition \( \ref{prop:12} \) gives us an irreducible representation \( \rho' \not\simeq \rho \) of \( U^0(\mathfrak{B}) \) obtained by inflation of an irreducible representation of \( U^0(\mathfrak{B})/U^1(\mathfrak{B}) \) such that \( \xi \) is contained in \( \rho' \). Now the representation \( \bullet \) is a sub-representation of
\[
\text{ind}_{\rho_{Z(\eta_{n_k})}}(((\rho' \otimes \kappa) \boxtimes \text{id}) \otimes U_{\eta_{n_k}}.
\]
The above representation is contained in
\[
\text{ind}_{\rho_{P_l(m+1)}}((\rho' \otimes \kappa) \boxtimes \text{id}) \simeq \text{ind}_{\rho_{P_l(m+1)}}(\tau' \boxtimes \text{id}), \tag{10}
\]
where \( \tau' \) is isomorphic to \(\text{ind}_{P_l}^{K_{n-1}}(\rho' \otimes \kappa) \). The representation \( \rho' \otimes \kappa \) is still irreducible (see \( \text{BK93} \) Chapter 5, Proposition 5.3.2(3))

We will show that irreducible subrepresentations of \( \Box \) are atypical for the inertial class \([M_l, \sigma \boxtimes \text{id}]\).

Any irreducible sub-representation of \( \Box \) occurs as a sub-representation of
\[
\text{ind}_{\rho_{P_l(m)}}(\gamma \boxtimes \text{id}),
\]
where \( \gamma \) is a \( K_{n-1} \)-irreducible subrepresentation of \( \tau' \). Now \( \gamma \) is contained in an irreducible smooth representation say \( \sigma_0 \) of \( G_{n-1} \). By Frobenius reciprocity this is possible only if the representation \( \rho' \otimes \kappa \) of \( J^0 \)}
is contained in \( \sigma_0 \). We have two possible situations either \( \rho' \) is cuspidal or otherwise. If \( \rho' \) is cuspidal, then the representation \( \sigma_0 \) is a cuspidal representation such that \( \sigma_0 \not\cong \sigma \). Hence the representation

\[
\text{ind}_{P_I(m+1)}^{K^n}(\gamma \boxtimes \text{id})
\]

occurs in

\[
\text{res}_{K^n}^{G^n}(\sigma_0 \boxtimes \text{id})
\]

with \( [G_{n-1} \times F^\times, \sigma_0 \boxtimes \text{id}] \not\cong [G_{n-1} \times F^\times, \sigma \boxtimes \text{id}] \). This shows that irreducible subrepresentations of \([10]\) are atypical representations.

Consider the case where \( \rho' \) is not cuspidal. If \((J^0, \rho' \otimes \kappa)\) is contained in an smooth irreducible representation \( \sigma_0 \), then \( \sigma_0 \) either contains a non-maximal simple-type \((J^0_1, \rho_1 \otimes \kappa_1)\) or contains a split type (see [BK93 Chapter 8, Theorem 8.3.5]). We also refer to the article [BH13 Lemma 2, Proposition 1] for quick reference. From this we conclude that \( \sigma_0 \) is not a cuspidal representation. Hence, the representation \([10]\) is contained in

\[
\text{res}_{K^n}^{G^n}(\sigma'),
\]

where \( P' \) is a parabolic subgroup \( G_n \) properly contained in \( P_I \), and \( \sigma' \) is a cuspidal representation of a Levi subgroup of \( P' \). Since \( \mathcal{I}(\sigma') \not\cong [M_f, \sigma \boxtimes \text{id}] \), we get that the irreducible subrepresentations of \([10]\) are atypical. \( \square \)

Recall the definition of the integer \( N_s \) from [3.2.1]. Now, any typical representation for \( s \) occurs as a subrepresentation of \( \text{ind}_{J_{\sigma_I}^0}^{K^n}(\lambda \boxtimes \text{id}) \), for some \( m \geq N_s \). For \( m > N_s \), we have

\[
\text{ind}_{J_{\sigma_I}^0}^{K^n}(\lambda \boxtimes \text{id}) = \text{ind}_{J_{\sigma_I}^0(N_s)}^{K^n}(\lambda \boxtimes \text{id}) \oplus U_m(\lambda \boxtimes \text{id}).
\]

From the above theorem we get that the typical representations for \( s \) occur as subrepresentations of

\[
\text{ind}_{J_{\sigma_I}^0(N_s)}^{K^n}(\lambda \boxtimes \text{id}).
\]

The above representation may still contain atypical representations. We will indeed show that this is the case and complete the classification.

The first observation is that the group \( J_s \) in a semisimple Bushnell-Kutzko type \((J_s, \lambda \boxtimes \text{id})\), for \( s = [M_f, \sigma \boxtimes \text{id}] \), contains the group \( P_{I1}^{(0)}(N_s) \). Hence we will try to decompose the representation

\[
\text{ind}_{J_{\sigma_I}^0(N_s)}^{J_{\sigma_I}^0}(\text{id}).
\]

We also note that \( P_{I1}^{(0)}(N_s) \cap P_I = J_s \cap P_I \). Let \( l + 1 = el' + r \), where \( 0 \leq r < e \). If \( r \leq 1 \), then \( J_s = P_{I1}^{(0)}(N_s) \) and hence, we have nothing further to analyse and Theorem [3.3.1] completes the classification of typical representations. From now we assume that \( e > 2 \) and \( r > 1 \). Note that the depth-zero case is already handled in [Nad17] (see Theorem [2.0.8]). We will first verify that the group \( U_I(\Phi \sigma) \) acts trivially on the representation \([11]\).

Let \( u^+ \) and \( u^- \) be two matrices from \( J_s \cap U_I = U_I(\Phi \sigma) \) and \( J_s \cap \bar{U}_I \) respectively. Let \( u^+ \) and \( u^- \) in block form be written as:

\[
\begin{pmatrix}
1_{n-1} & U^+ \\
0 & 1
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
1_{n-1} & 0 \\
0 & 1
\end{pmatrix}
\]

respectively. The block form of the conjugation \( u^- u^+ (u^-)^{-1} \) is given by

\[
\begin{pmatrix}
1_{n-1} - U^+ U^- & U^+ \\
-U^- U^+ U^- & U^- U^+ + 1
\end{pmatrix}.
\]

We have

\[
\begin{pmatrix}
0 & 0 \\
U & 0
\end{pmatrix} \in a_{l+1}(\Lambda) \cap \bar{\text{n}}_I = \varpi'_{\Phi}(a_r(\Lambda) \cap \bar{\text{n}}_I).
\]

If \( r \geq 1 \), then the valuation of each entry of a matrix in \( a_r(\Lambda) \cap \bar{\text{n}}_I \) is at least one. This shows that the valuation of each entry in \( U^- U^+ U^- \) is at least \( l' + 2 \). From which the conjugation \( u^- u^+ (u^-)^{-1} \) lies in the group \( P^{(0)}(N_s) \). If \( r = 0 \) and \( l' = 0 \), then we are in the case where \( \sigma \) is a level-zero cuspidal representation and in this case \( J_s = P^{(0)}(N_s) \). If \( r = 0 \) and \( l' > 0 \), then valuation of each entry in \( U^- U^+ U^- \) has valuation
\[ 2l' > l' + 1 \] and hence \( u^- u^+ (u^-)^{-1} \in P^0_k(N_s) \). Hence, the group \( U_{\mathfrak{f}_F} \) acts trivially on the representation (11).

From the Iwahori decomposition of the group \( J_s \), we get that \( J_s \) is equal to \( (J_s \cap \tilde{P}_I)P^0_k(N_s) \). Hence we have:

\[
\text{res}_{J_s \cap \tilde{P}_I} \text{ind}_{P^0_k(N_s) \cap \tilde{P}_I}^{J_s \cap \tilde{P}_I} (\text{id}) \simeq \text{ind}_{J_s \cap \tilde{P}_I}^{P^0_k(N_s) \cap \tilde{P}_I} (\text{id}).
\]

Note that \( J_s \cap \tilde{P}_I \) is a semi-direct product of the groups \( (J_s \cap M_I) \) and \( (J_s \cap \tilde{U}_I) \). Let \( \eta_k \), for \( 1 \leq k \leq t \), be all the characters of the group \( J_s \cap \tilde{U}_I \) which are trivial on the group \( P^0_k(N_s) \cap \tilde{U}_I \). The group \( J_s \cap \tilde{P}_I \) acts on these characters. Let \( \{\eta_k\} \) be a set of representatives for the orbits under this action. We denote by \( Z(\eta_k) \) the \( J_s \cap \tilde{P}_I \) stabiliser of the character \( \eta_k \). Let \( U_{\eta_k} \) be the isotypic component of the character \( \eta_k \) in the representation

\[
\text{ind}_{J_s \cap \tilde{P}_I}^{P^0_k(N_s) \cap \tilde{P}_I} (\text{id}).
\]

The space \( U_{\eta_k} \) has a natural action of \( Z(\eta_k) \). Now Clifford theory gives the decomposition

\[
\text{ind}_{J_s \cap \tilde{P}_I}^{P^0_k(N_s) \cap \tilde{P}_I} (\text{id}) \simeq \bigoplus_{\eta_k} \text{ind}_{Z(\eta_k)}^{K_s} (U_{\eta_k}).
\]

We note that the character id occurs with a multiplicity one in the list of characters \( \eta_k \).

If \( K_s \) is the kernel of the representation (11), then \( K_s \cap Z(\eta_k) \) acts trivially on \( U_{\eta_k} \). Hence we can extend the representation \( U_{\eta_k} \) to the group \( Z(\eta_k)K_s \) such that \( K_s \) acts trivially on the extension. Now consider the representation

\[
\pi = \text{ind}_{Z(\eta_k)K_s}^{K_s} U_{\eta_k}.
\]

Note that \( K_s \cap \tilde{P}_I \) is contained in the group \( Z(\eta_k) \cap \tilde{P}_I \) and moreover, \( U_{\mathfrak{f}_F} \) is contained in \( K_s \). Hence we have \( J_s = (J_s \cap \tilde{P}_I)Z(\eta_k)K_s \). From Mackey decomposition, we have

\[
\text{res}_{J_s \cap \tilde{P}_I} \text{ind}_{Z(\eta_k)K_s}^{J_s \cap \tilde{P}_I} U_{\eta_k} \simeq \text{ind}_{Z(\eta_k)K_s}^{J_s \cap \tilde{P}_I} (U_{\eta_k} \cap \tilde{P}_I) \simeq \text{ind}_{Z(\eta_k)K_s}^{J_s \cap \tilde{P}_I} (U_{\eta_k}).
\]

We hence have

\[
\text{ind}_{J_s \cap \tilde{P}_I}^{P^0_k(N_s)} (\text{id}) \simeq \bigoplus_{\eta_k} \text{ind}_{Z(\eta_k)K_s}^{J_s \cap \tilde{P}_I} U_{\eta_k}.
\]

(12)

Now using the decomposition (12) we get the decomposition

\[
\text{ind}_{P^0_k(N_s)}^{K_s} (\lambda \boxtimes \text{id}) \simeq \bigoplus_{\eta_k} \text{ind}_{Z(\eta_k)K_s}^{K_s} \{U_{\eta_k} \otimes (\lambda \boxtimes \text{id})\}.
\]

Note that the character id occurs with multiplicity one among the characters \( \eta_k \). Moreover, we have \( Z(\text{id})K_s = (J_s \cap \tilde{P}_I)K_s = J_s \) and we get that

\[
\text{ind}_{P^0_k(N_s)}^{K_s} (\lambda \boxtimes \text{id}) \simeq \text{ind}_{Z(\eta_k)K_s}^{K_s} (\lambda \boxtimes \text{id}) \oplus \bigoplus_{\eta_k \neq \text{id}} \text{ind}_{Z(\eta_k)K_s}^{K_s} \{U_{\eta_k} \otimes (\lambda \boxtimes \text{id})\}.
\]

(13)

Lemma 3.3.2. Let \( \# k_F > 2 \) and \( \eta_k \) be a non-trivial character. The irreducible sub representations of

\[
\text{ind}_{Z(\eta_k)K_s}^{K_s} \{U_{\eta_k} \otimes (\lambda \boxtimes \text{id})\}
\]

are atypical.

Proof. We observe that \( Z(\eta_k) = (Z(\eta_k) \cap M_I)(J_s \cap \tilde{U}_I) \). This shows that we have to bound the group \( Z(\eta_k) \cap M_I \), for \( \eta_k \neq \text{id} \). Recall that \( \eta_k \), for \( 1 \leq k \leq t \), are the characters of the quotient group

\[
\frac{(J_s \cap \tilde{U}_I)}{(P^0_k(N_s) \cap \tilde{U}_I)}.
\]

(14)

Now let \( u^- \) be a matrix from the group \( J_s \cap \tilde{U}_I \). In the block form the matrix \( u^- \) is of the form

\[
\begin{pmatrix}
1_{n-1} & 0 \\
U^- & 1
\end{pmatrix}
\]
where $U^− = [M_1, M_2, \ldots, M_e]$, $M_i$ is a matrix of size $(1 \times (n−1)/e)$. Let $δ = N_n − 1$. The map $Φ$

$[M_1, M_2, \ldots, M_e] \mapsto [\varpi^e_1 M_1, \varpi^e_2 M_2, \ldots, \varpi^e_e M_e]$

identifies the quotient $\{1\}$ with a subspace $t_1$ of $\text{Mat}_{n−1}(k_F)$. This identification commutes with the action of $M_I \cap J_s$, since $Φ$ is none other than conjugation by an element from the $Z(M_I)$ (The centre of $M_I$).

Let $t_2$ be the following space of column matrices:

$t_2 = \{0, 0, \ldots, 0, M_r, \ldots, M_e\}^{T} | M_j \in \text{Mat}_{n−1}(k_F) \forall r ≤ j ≤ e\}.$

The group $M_I(σ_F) ≃ K_{n−1} × K_1$ acts on the space $\text{Mat}_{n−1}(k_F)$ via the conjugation action of $M_I$ on $U_I$. The space $t_2$ is stable under the action of $U^0(\mathfrak{A}) × σ_F^\times \subseteq M_I$. The pairing $XY$, where $X ∈ t_2$ and $Y ∈ t_1$, gives a perfect pairing between $t_1$ and $t_2$. This pairing is equivariant for the action of $U^0(\mathfrak{A}) × σ_F^\times \subseteq M_I$. This gives an identification of the space of characters of $t_1$ with the space $t_2$ in a $U^0(\mathfrak{A}) × σ_F^\times$-equivariant way.

The group $M_I(σ_F)$ acts on $\text{Mat}_{n−1}(k_F)$ through its quotient $M_I(k_F)$. Now, the action of the group $(U^0(\mathfrak{B}) × σ_F^\times) \subseteq J_s \cap M_I$ on $t_2$ factors through its quotient by its subgroup $(1_n + \mathfrak{D}^\times) × (1 + p_F)$. Let $A$ be a non-zero matrix in $t_2$.

Recall that $n_0 = (n−1)/ef$. Now recall that we denote by $π_E$ by mod $p_E$ reduction map. We have seen that (the paragraph above the proposition $3.2.2$)

$(π_E \times \text{id})\{Z_{\text{GL}_{n_0}(p_E/\mathfrak{p}_E)^r} \times k_F^r}(A)\}

is a subgroup of $H \times k_F^r$, where $H$ is a subgroup of $\text{GL}_{n_0}(k_F)$ whose image under the inclusion map

$\text{GL}_{n_0}(k_F) \hookrightarrow \text{GL}_{n−1}(k_F)$

is contained in a proper $k_F$-parabolic subgroup of $\text{GL}_{n−1}(k_F)$. From the result of Paškūnas, stated as Proposition $3.2.2$, we get that for every irreducible representation $ξ$ of

$\text{res}_{Z(π_E)}(U_{k_p} \otimes ((κ \otimes \rho) \boxtimes \text{id}))$

we can find an irreducible representation $ρ' \not\sim \rho$ such that $ξ$ occurs in the representation

$\text{res}_{Z(π_E)}(U_{k_p} \otimes ((κ \otimes ρ') \boxtimes \text{id})).$

Hence irreducible subrepresentations of

$\text{ind}_{Z(π_E) K_s}^{K_n}(U_{k_p} \otimes (λ \boxtimes \text{id}))

occur as subrepresentations of

$\text{ind}_{Z(π_E) K_s}^{K_n}(U_{k_p} \otimes ((κ \otimes ρ') \boxtimes \text{id})).$

Now the above representation occurs as a sub-representation

$\text{ind}_{P_j(N_s)}^{K_n}(\{κ \otimes ρ' \boxtimes \text{id}) \simeq \text{ind}_{P_j(N_s)}^{K_n}(τ' \boxtimes \text{id})\},$

where $τ'$ is given by

$\text{ind}_{P_j(N_s)}^{K_n}(κ \otimes ν' \boxtimes \text{id})$

Any irreducible sub representation $γ$ of $τ'$ occurs in an irreducible smooth representation $σ_0$ of $\text{GL}_{n−1}(F)$. Assume that $ρ'$ is cuspidal. The representation $κ \otimes ρ'$ is contained in the representation $\text{res}_{P_j} γ$ and hence is contained in $σ_0$. This implies that $σ_0$ is cuspidal but not inertially equivalent to $σ$. If $ρ'$ is not cuspidal, then the representation $σ_0$ is not cuspidal. Hence, in every case, $σ_0$ is not inertially equivalent to $σ$. This shows that irreducible sub representations of

$\text{ind}_{P_j(N_s)}^{K_n}(γ \boxtimes \text{id})$

are atypical. We conclude the lemma. \hfill \square

Theorem 3.3.3. Let $n > 2$ and $q_F > 2$. Let $Γ$ be any typical representation for the inertial class $s = [M_I, σ \boxtimes χ]$. The representation $Γ$ is isomorphic to the representation

$\text{ind}_{J_n}^{K_n}(λ_s)$,
where \((J, \lambda)\) is any Bushnell-Kutzko semi-simple type for the inertial class \(s\). If \(P\) is a parabolic subgroup containing \(M_I\) as a Levi subgroup, then \(\Gamma\) occurs with a multiplicity one in the representation 

\[
\text{res}_{K_n} \ind^{G_n}_{I_n}(\sigma \boxtimes \chi).
\]

Proof. Let \(G\) be the group of \(F\)-rational points of a connected reductive group defined over \(F\). For any inertial class \(t = [L, \Theta]\) of \(G\), and for \(g \in N_G(L)\) we define \(t^g\) to be the inertial class \([L, \Theta^g]\). The map sending \(t\) to \(t^g\) is well defined. We denote by \(N_G(t)\) the group \(\{g \in N_G(L) \mid t^g = t\}\). The group \(N_G(t)\) clearly contains the group \(L\) and the quotient \(W_t = N_G(t)/L\) is finite. The cardinality of \(W_t\) does not depend on the choice of \(L\). We return to the case where \(G = G_n\) and \(t = s\). The intertwining of the representation \(\ind^{G_n}_{I_n}(\lambda)\) is bounded by the cardinality of \(W_s\). We have \(|W_s| = 1\) since \(n > 2\). Hence, the representation \(\ind^{G_n}_{I_n}(\lambda)\) is irreducible. We refer to [BK98] [Lemma 11.5] for these results. Hence the uniqueness of the typical representation. The multiplicity follows from the results 3.3.1 and 3.3.2. 

\[
\square
\]

4. Principal series components

Let \(I\) be the partition \((1, 1, \ldots, 1)\) of \(n\). Recall that we denote by \(B_n\) the group \(P_I\), \(U_n\) the group \(U_I\), and \(T_n\) the group \(M_I\) respectively. In this section, we will classify typical representations for the inertial classes \(s = [T_n, \chi]\), where \(\chi\) is a character of \(T_n\). Let \(\tau\) be a typical representation for the inertial class \(s\). The compact induction \(\ind^{G_n}_{I_n}(\lambda)\) is a finitely generated representation. Let \(\pi\) be an irreducible quotient of \(\ind^{G_n}_{I_n}(\lambda)\). By Frobenius reciprocity, the \(K_n\)-representation \(\tau\) occurs in the \(G_n\) representation \(\pi\).

Let \(B\) be any Borel subgroup of \(G_n\), \(T\) be a maximal split torus of \(G_n\) contained in \(B\), and \(\chi'\) be a character of \(T\). If \((T, \chi')\) and \((T_n, \chi)\) are inertially equivalent, then the representation \(\pi\) occurs as a sub-quotient of \(i_B^{G_n}(\chi'^\sigma)\), where \(\chi'^\sigma\) is obtained from \(\chi'\) by twisting with an unramified character of \(T\). For classifying typical representations it is enough to say which \(K_n\)-irreducible sub representations of \(i_B^{G_n}(\chi')\) are typical for the inertial class \([T_n, \chi]\).

Let \(\sigma\) be a permutation of the set \(\{1, 2, \ldots, n\}\). Let \(\chi = \bigotimes_{i=1}^{n} \chi_i \in \mathfrak{c}_{\sigma}^{n}\) be any character of \(T_n = \prod_{i=1}^{n} F^\times\). We denote by \(\chi^\sigma\) the character \(\bigotimes_{i=1}^{n} \chi_{\sigma(i)}\) of \(T_n\). We observe that the pairs \((T_n, \chi^\sigma)\) and \((T_n, \chi)\) are inertially equivalent. This implies that for a classifying typical representations we can classify typical representation occurring in \(i_B^{G_n}(\chi^\sigma)\), for any \(\sigma\). We will use a convenient permutation \(\sigma\) which satisfies the condition in the following lemma.

**Lemma 4.0.1.** Given any sequence of characters \(x_i = \chi_i\) of \(\mathfrak{c}_{\sigma}^{n}\), there exists a permutation \(\chi_{\sigma(i)}\) of \(\{x_i \mid 1 \leq i \leq n\}\) such that

\[
l(y_i y_k^{-1}) \geq \max\{l(y_i y_j^{-1}), l(y_j y_k^{-1})\},
\]

for all \(1 \leq i \leq j \leq k \leq n\).

**Proof.** For any ultrametric space \((X, d)\) and given any \(n\) points \(x_1, x_2, x_3, \ldots, x_n\) in \(X\) we may choose a permutation \(y_1, y_2, \ldots, y_n\) of the sequence \(\{x_i \mid 1 \leq i \leq n\}\) such that

\[
d(y_i, y_k) \geq \max\{d(y_i, y_j), d(y_j, y_k)\}
\]

for all \(1 \leq i \leq j \leq k \leq n\). Now apply this fact to the space \(X\) consisting of characters of \(\mathfrak{c}_{\sigma}^{n}\) and the distance function \(d(\chi_1, \chi_2)\) is defined as the level \(l(\chi_1 \chi_2^{-1})\) if \(\chi_1 \neq \chi_2\) and 0 otherwise. We point out that this ordering is not unique in general. We refer to [How97] Lemma 1 for a proof of these results. 

**Remark 4.0.2.** We note that the condition \(l(y_i y_k^{-1}) \geq \max\{l(y_i y_j^{-1}), l(y_j y_k^{-1})\}\) is equivalent to an equality since we always have

\[
l(y_i y_k^{-1}) \leq \max\{l(y_i y_j^{-1}), l(y_j y_k^{-1})\}.
\]

Given an inertial class \([T_n, \chi]\) we choose the representative \((T_n, \chi^\sigma)\) where \(\sigma\) is a permutation such that

\[
l(\chi_{\sigma(i)} \chi_{\sigma(j)}^{-1}) \geq \max\{l(\chi_{\sigma(i)} \chi_{\sigma(j)}^{-1}), l(\chi_{\sigma(j)} \chi_{\sigma(k)}^{-1})\}.
\]

From now on we assume that the pair \((T_n, \bigotimes_{i=1}^{n} \chi_i)\) satisfies the condition

\[
l(\chi_i \chi_k^{-1}) \geq \max\{l(\chi_i \chi_j^{-1}), l(\chi_j \chi_k^{-1})\},
\]

(15)
for all $i \leq j \leq k$.

In the following subsection we construct subgroups $H_m$, for $m \geq 1$ such that

1. $H_1 = J_s$, where $J_s$ is the compact open subgroup of a Bushnell–Kutzko type $(J_s, \chi)$ of $s$,
2. $H_{m+1} \subset H_m$, for all $m \geq 1$ and $\bigcap_{m \geq 1} H_m = B_n \cap K_n$,
3. The representation $\chi$ of $T_n \cap K_n$ extends to a representation of $H_m$ such that $H_m \cap U_n$ and $H_m \cap \bar{U}_n$ are contained in the kernel of this extension.

Such a construction gives the following equality:

$$\text{ind}_{K_n \cap B_n}^{K_n} \chi = \bigcup_{m \geq 1} \text{ind}_{H_m}^{K_n} \chi.$$  

Later, we show that any $K_n$-irreducible sub representation of $\text{ind}_{H_{m+1}}^{K_n} \chi / \text{ind}_{H_m}^{K_n} \chi$ is atypical.

4.1. Construction of compact open subgroups $H_m$. Let $A = (a_{ij})$ be a lower nilpotent matrix of size $n \times n$ such that $a_{ij}$ is non-negative, for $i > j$, and

$$a_{ki} = \max\{a_{ji}, a_{kj}\}, \quad (16)$$

for $1 \leq i < j < k \leq n$. We denote by $J(A)$ the set of $n \times n$ matrices $(m_{pq})$ such that $m_{pq} \in \mathfrak{o}_F$, for $p < q$, and $m_{pq} \in \mathfrak{P}_{F}^{\mathfrak{p}_q}$, for $p \geq q$. As a consequence of the condition $a_{ki} = \max\{a_{ji}, a_{kj}\}$ we get two important inequalities

$$a_{i1} \geq a_{i2} \geq \cdots \geq a_{ii-1} \geq \cdots \geq a_{ij}, \quad (17)$$

and

$$a_{j+1j} \leq a_{j+2j} \leq \cdots \leq a_{nj}. \quad (18)$$

The first is a consequence of $a_{ik} = \max\{a_{ik-1}, a_{ik}\}$, for $k < i$, and the second is a consequence of $a_{k+1j} = \max\{a_{k+1k}, a_{kj}\}$, for $j < k$.

**Lemma 4.1.1.** The set $J(A)$ is an order in $\text{Mat}_{n \times n}(\mathfrak{o}_F)$.

**Proof.** The set $J(A)$ is an additive group. We now check that the set $J(A)$ is closed under multiplication. Let $(m_{ij})$ and $(m'_{ij})$ be two matrices from $J(A)$. If $i > j$, then the $i \times j$ term in the product matrix $(m_{ij})(m'_{ij})$ can be split into three terms:

$$t_1 := m_{i1}m'_{1j} + m_{i2}m'_{2j} + \cdots + m_{ij}m'_{ij},$$

$$t_2 := m_{ij+1}m'_{j+1k} + \cdots + m_{ij}m'_{ij},$$

and

$$t_3 := m_{i+1j}m'_{i+1j} + \cdots + m_{nj}m'_{nj}.$$  

Observe that $\nu_F(m_{ik}m'_{kj}) \geq a_{ij}$, for $k \leq j$. This shows that $\nu_F(t_1) \geq \min\{a_{i1}, a_{i2}, \ldots, a_{ij}\}$ and

$$\min\{a_{i1}, \ldots, a_{ij}\} \geq a_{ij}.$$  

The valuation $\nu_F(m_{ik}m'_{kj}) \geq a_{ik} + a_{kj}$, for all $j \leq k \leq i$, and $a_{ik} + a_{kj}$ is greater or equal to $a_{ij}$. We get that $\nu_F(t_2) \geq a_{ij}$. Finally the valuation $\nu_F(m_{ik}m_{kj}) \geq a_{kj}$, for $k > i$. The valuation $\nu_F(t_3) \geq \min\{a_{i+1j}, \ldots, a_{nj}\}$ and $\min\{a_{i+1j}, \ldots, a_{nj}\} \geq a_{ij}$. Hence the additive group $J(A)$ is closed under multiplication. Since $J(A)$ is an $\mathfrak{o}_F$ lattice in $\text{Mat}_{n \times n}(F)$ we get that $J(A)$ is an order in $\text{Mat}_{n \times n}(\mathfrak{o}_F)$. \[\square\]

We denote by $J(A)$ the set of invertible elements of $J(A)$. The following are examples of $J(A)$.

1. If $A = 0$ then the group $J(A)$ is $K_n$,
2. If $A = (a_{ij})$ with $a_{ij} = 1$, for $i > j$, then $J(A)$ is the Iwahori subgroup with respect to the standard Borel subgroup $B_n$. 

The examples (1) and (2) satisfy Iwahori decomposition with respect to the standard Borel subgroup \( B_n \).

The next lemma concerns the Iwahori decomposition of \( J(A) \) in general.

Let \( A = (a_{ij}) \) be a lower nilpotent matrix such that \( a_{ki} = \max\{a_{kj}, a_{ji}\} \), for \( 1 \leq i < j < k \leq n \). We define an ordered partition \( I \) of \( n \) by induction on the set of positive integers \( m \leq n \). Let \( I_1 := (1) \) and if we know \( I_m = (n_1, n_2, \ldots, n_r) \), for some \( m \leq n - 1 \), then \( I_{m+1} \) is the following partition

\[
I_{m+1} = \begin{cases} 
(n_1, n_2, \ldots, n_r, 1) & \text{if } a_{m+1m} \neq 0 \\
(n_1, n_2, \ldots, n_r + 1) & \text{otherwise.}
\end{cases}
\]

(19)

We denote by \( I(A) \) the partition \( I_n \).

**Lemma 4.1.2.** The group \( J(A) \) satisfies Iwahori decomposition with respect to the parabolic subgroup \( P_I(A) \) and the Levi subgroup \( M_I(A) \). We have \( J(A) \cap M_I(A) = M_I(A)(\mathfrak{g}_F) \), \( J(A) \cap U_I(A) = U_I(A)(\mathfrak{g}_F) \).

**Proof.** We use induction on the positive integer \( n \). If \( n = 1 \) then \( J(A) \) is \( \mathfrak{g}_F^0 \) and the lemma is vacuously true. We assume that the lemma is true for all positive integers less than \( n \). Let \( I(A) \) be the ordered partition \( (n_1, n_2, \ldots, n_r) \). If \( r = 1 \) then the lemma is true by default. We suppose \( r > 1 \). We will show below that every element \( j \in J(A) \) can be written as a product \( u_1j_1 \) with \( u_1 \in J(A) \cap \mathbb{U}_{(n_1, n-n_1)} \) and \( j_1 \in J(A) \cap P_{(n_1, n-n_1)} \). Now \( j_1 \) can be written as \( j_2u_2^\pm \) where \( u_2^\pm \in U_{(n_1, n-n_1)}(\mathfrak{g}_F) \) and \( j_2 \in M_{(n_1, n-n_1)} \cap J(A) \). Now \( j_2 \) can be written as \( j_3u_3^\pm \) where \( j_3 \in J(A) \cap M_{(n_1, n-n_1)} \) and \( u_3^\pm \in U_I(A)(\mathfrak{g}_F) \). The group \( J(A) \cap M_{(n_1, n-n_1)} \) is equal to \( K_n \times J(A') \) where the nilpotent matrix \( A' = (a'_{ij}) \) is given by \( a'_{ij} = a_{i+n_1j+n_1} \). By induction hypothesis \( J(A') \) satisfies Iwahori decomposition with respect to the standard parabolic subgroup \( P_{I(A')} \) and its Levi subgroup \( M_{I(A')} \) and \( I(A') = (n_2, n_3, \ldots, n_r) \). Let \( j_3 = (j_3^0, j_3^1) \) where \( j_3^0 \in K_n \) and \( j_3^1 \in J(A') \). Now \( j_3^1 = u_3^j j_4^1 u_3^+ \) where \( u_3^- \in U_{I(A')} \cap J(A') \), \( u_3^+ \in U_{I(A')} \cap J(A') \) and \( j_4 \in M_{I(A')} \cap J(A') \). Hence \( j = u_1u_3^- (j_3^0, j_4)u_3^+ u_4^\pm \) (with a slight abuse of notation the elements \( u_3^- \) and \( u_3^+ \) are considered as elements of \( U_I(A) \)) respectively and \( (j_3^0, j_4) \) is an element of \( J(A) \cap M_{I(A)} = K_n \times (J(A') \cap M_{I(A')}) \).

We now prove that \( j \in J(A) \) can be written as a product \( u_1j_1 \) with \( u_1 \in J(A) \cap \mathbb{U}_{(n_1, n-n_1)} \) and \( j_1 \in J(A) \cap P_{(n_1, n-n_1)} \). Let \( j = (j_{pq}) \). Let \( C^i_j \) be the \( i^{th} \)-column of the first diagonal block (of size \( n_1 \times n_1 \)) on the diagonal. If every entry of \( C^i_j \) has positive valuation then, we claim that all the entries of the \( i^{th} \) column \( C^i_j \) have positive valuation. Suppose the \( k^{th} \) entry \( j_{ki} \) of \( C^i_j \) is an unit for some \( k > n_1 \). This shows that \( a_{ki} \) the \( k^{th} \)-entry of \( A \) is zero. Now the inequality \( \|\| \) gives \( a_{ki} \geq a_{kn_1} \) and this implies that \( a_{kn_1} = 0 \). Note now that \( a_{kn_1} \geq a_{n_1+1n_1} \) from the inequality \( \|\| \). This shows that \( a_{n_1+1n_1} \) is zero which gives a contradiction from the definition of \( I(A) \). We now deduce that \( j_{ki} \) is not invertible. This shows the claim. Since \( j \) is invertible we conclude that at least one entry of \( C^i_j \) is an unit. Let \( E_{ij}(c) = I_n + cE_{ij}(c) \) where \( E_{ij}(c) \) is the matrix with its \( ij \) entry \( c \) and all other entries 0. The left multiplication of \( E_{ij}(c) \) results in the row operation \( R_j + cR_i \). Since at least one entry of \( C^i_j \) is an unit we assume that its \( q^{th} \)-entry is an unit. We can perform row operations \( R_p + cR_q \) for all \( p \geq n_1 \) to make the \( p^{th} \)-entry trivial. We also note that the elementary matrix corresponding to this row operation also belongs to the group \( J(A) \) (note that \( q \leq n_1 \leq p \)). This completes the task of making \( j^* \) as the product \( u_1j_1 \). The uniqueness of the Iwahori decomposition is standard.

Let \( s = [T_n, \chi] \) be an inertial equivalence class. Let \( m \) be a positive integer and \( A_\chi(m) \) be the lower nilpotent matrix \( (a_{ij}^m) \in \text{Mat}_{n \times n}(\mathbb{Z}) \), where

\[
a_{ij}^m = l(\chi_i \chi_j^{-1}) + m - 1,
\]

for \( n \geq i > j \geq 1 \). As shown earlier, the representative \( (T_n, \chi = \sum_{i=1}^n \chi_i) \) for \( s \), can be chosen such that

\[
a_{ik} = \max\{a_{ij}, a_{jk}\},
\]

for all \( i < j < k \). We denote by \( J_\chi(m) \) the group \( J(A_\chi(m)) \). Note that \( J_\chi(m') \subseteq J_\chi(m) \), for all \( m' \geq m \). In our situation we have \( I(A_\chi(m)) \) is \( (1, 1, \ldots, 1) \), since none of \( a_{ij}^m \) are zero. Hence, using lemma 4.1.2 the group \( J_\chi(m) \) satisfies the Iwahori decomposition with respect to \( B_n \) and \( T_n \).

**Lemma 4.1.3.** The character \( \chi = \sum_{i=1}^n \chi_i \) of \( T_n \cap K_n \) extends to a character of \( J_\chi(1) \) such that \( J_\chi(1) \cap U_n \) and \( J_\chi(1) \cap U_n \) are contained in the kernel of the extension.
Proof. Let \( m = (m_{ij}) \) be an element of \( J_\chi(1) \). We define \( \tilde{\chi}(m) = \prod_{i=1}^n \chi_i(m_{ii}) \). The verification that \( \tilde{\chi} \) is a character of the group \( J_\chi(1) \) is very computational in nature and we sketch the proof here and for complete details see [Roc98, Section 3, Lemma 3.1, Lemma 3.2] or [How73, Pg 278-279]. The idea is to get an open normal subgroup \( U \) of \( J_\chi(1) \) such that \( J_\chi(1)/U \) is isomorphic to \( T(\mathfrak{o}_F)/T_\chi \) where \( T_\chi \) is an open subgroup of \( T(\mathfrak{o}_F) \) which is contained in the kernel of \( \tilde{\chi} \). The subgroup \( U \) is generated by \( J_\chi(1) \cap U_n \) and \( J_\chi(1) \cap U_n = U_n(\mathfrak{o}_F) \). One shows that \( U \) satisfies Iwahori decomposition with respect to the Borel subgroup \( B_n \) and \( U \cap T_n \) is given by \( \prod_{a \in \Phi} \alpha^\vee(1 + \Phi_F^{\alpha^\vee}) \) where \( \Phi \) is the set of roots of \( \text{GL}_n \) with respect to \( T_n \) and \( \alpha^\vee \) stands for the dual root. We observe that \( U \cap T_n \) is contained in the kernel of \( \tilde{\chi} \).

We are now ready to define the sequence of groups \( H_m \). We set \( H_m = J_\chi(m) \), for \( m \geq 1 \). We now get the equality

\[
\text{res}_{K_n} t_{B_n}^G(\chi) = \bigcup_{m \geq 1} \text{ind}_{J_\chi(m)}^K(\chi).
\]

For the purposes of proofs by induction, we need to construct some more compact open subgroups of \( K_n \).

We denote by \( A_\chi(1,m) \) the lower nilpotent matrix \( (a_{ij}) \) where \( a_{ij} = l(\chi_i \chi_j^{-1}) \) for \( j < i < n \), \( a_{nj} = l(\chi_n \chi_j^{-1}) + m - 1 \), for \( 1 \leq j < n - 1 \). Given a lower nilpotent matrix \( A = (a_{ij}) \) such that \( a_{ki} = \max \{ a_{kj}, a_{ji} \} \) we associated a compact subgroup \( J(A) \). The matrix \( A_\chi(1,m) \) need not satisfy this condition but, we can still associate the group \( J(A_\chi(1,m)) \) to the matrix \( A_\chi(1,m) \). We will prove this in the next Lemma.

**Lemma 4.1.4.** Let \( \mathcal{J}(A_\chi(1,m)) \) be the set consisting of matrices \((m_{ij}) \in \text{Mat}_{n \times n}(\mathfrak{o}_F) \) such that \( m_{ij} \in \Phi_F^{a_{ij}} \) for all \( i, j \). The set \( \mathcal{J}(A_\chi(1,m)) \) is an order in \( \text{Mat}_{n \times n}(\mathfrak{o}_F) \).

**Proof.** The set \( \mathcal{J}(A_\chi(1,m)) \) is a lattice in \( \text{Mat}_{n \times n}(F) \) and we have to verify that \( \mathcal{J}(A_\chi(1,m)) \) is closed under multiplication. Let \( (m_{ij}) \) and \( (m'_{ij}) \) be two elements of the set \( \mathcal{J}(A_\chi(1,m)) \). We suppose \( i > j \). The \( ij \)th-term of the product \((m_{ij})(m'_{ij})\) is the sum of the terms:

\[
\begin{align*}
t_1 &:= m_{i1} m'_{1j} + m_{i2} m'_{2j} + \cdots + m_{ij} m'_{ij}, \\
t_2 &:= m_{ij+1} m'_{j+1} + \cdots + m_{im} m'_{ij}, \\
t_3 &:= m_{ii+1} m'_{i+1j} + \cdots + m_{in} m'_{nj}.
\end{align*}
\]

Note that

\[
\nu_F(t_1) \geq \min \{ \nu_F(m_{ik} m'_{kj}) \mid \text{for all } 1 \leq k \leq j \},
\]

and

\[
\nu_F(m_{ik} m'_{kj}) = a_{ik} \quad \text{if } i < n, \quad a_{ik} = l(\chi_i \chi_k^{-1}) \quad \text{and} \quad a_{kj} = a_{ik}, \quad \text{for all } k \leq j < i.
\]

This shows that

\[
\nu_F(t_1) \geq a_{ij}.
\]

If \( i = n \), then we have

\[
a_{ik} = a_{nk} = l(\chi_n \chi_k^{-1}) + m - 1 \geq l(\chi_i \chi_j^{-1}) + m - 1 = a_{ij},
\]

for all \( k \leq j < n \). We conclude that in every possibility \( \nu_F(t_1) \geq a_{ij} \).

Consider the term \( t_2 \). We have

\[
\nu_F(t_2) \geq \{ \nu_F(m_{ik} m'_{kj}) \mid \text{for all } j < k \leq i \}
\]

and

\[
\nu_F(m_{ik} m'_{kj}) = a_{ik} + a_{kj}, \quad \text{for } j < k \leq i. \quad \text{If } i < n, \quad \text{then we have } a_{ik} = l(\chi_i \chi_k^{-1}) \quad \text{and} \quad a_{kj} = l(\chi_k \chi_j^{-1}).
\]

From our assumption on the arrangement of characters \( \chi_i, \chi_k \), for \( 1 \leq i \leq n \), we get that

\[
l(\chi_i \chi_j^{-1}) = \max \{ l(\chi_i \chi_k^{-1}), l(\chi_k \chi_j^{-1}) \}.
\]

At the same time \( i < n \) implies that \( a_{ij} = l(\chi_i \chi_j^{-1}) \). This shows that \( \nu_F(m_{ik} m'_{kj}) \) is equal to \( a_{ik} + a_{kj} \) and \( a_{ik} + a_{kj} \geq a_{ij} \). Consider the case \( i = n \) and in this case, \( a_{nk} = l(\chi_n \chi_k^{-1}) + m - 1 \). Now \( a_{kj} = l(\chi_k \chi_j^{-1}) \) and \( a_{nj} = l(\chi_n \chi_j^{-1}) + m - 1 \). From the equality \( l(\chi_i \chi_j^{-1}) = \max \{ l(\chi_i \chi_k^{-1}), l(\chi_k \chi_j^{-1}) \} \) we deduce that

\[
l(\chi_i \chi_k^{-1}) + l(\chi_k \chi_j^{-1}) > l(\chi_i \chi_j^{-1}).
\]

Now, adding \( m - 1 \) on both sides we get \( a_{ij} > a_{ik} + a_{kj} \). We conclude that \( \nu_F(t_2) \geq a_{ij} \).
We observe that $\nu_F(m_{ik}m'_{kj})$, for $i<k<n$ is equal to $a_{kj} = l(\chi_k\chi_j^{-1})$ and we have $l(\chi_k\chi_j^{-1}) \geq a_{ij}$. Note that $a_{nj} = l(\chi_n\chi_j^{-1}) + m - 1 \geq a_{ij}$ from which we conclude that $\nu_F(t_3) \geq a_{ij}$. This shows that
\[ \nu_F(t_1 + t_2 + t_3) \geq a_{ij} \] and we prove our lemma.
\[ \square \]

Let $J_\chi(1, m)$ be the group of units of $\mathcal{J}(A_\chi(1, m))$. We will need the structure of the representation
\[ \text{ind}^{J_\chi(1,m)}_{J_\chi(1,m+1)}(id). \]
We follow a similar strategy to that from previous section. Let $(a_{ij})$ be the matrix $A_\chi(1, m)$. Let $K_\chi(1, m)$ be the set of matrices $(m_{ij})$ such that $m_{ij} \in \mathfrak{p}_F$, for $i<j<n$, $m_{in} \in \mathfrak{o}_F$, for $i<n$, $m_{ii} \in 1 + \mathfrak{p}_F$, for $i\leq n$, $m_{ij} \in \mathfrak{p}_F^{\alpha_{ij}}$, for $i>j$. In the block form the group $K_\chi(1, m)$ is given by:
\[
\begin{pmatrix}
A_\chi(1) \cap K_{n-1}(1) & \text{Mat}_{(n-1)\times 1}(\mathfrak{p}_F) \\
\mathfrak{n} & 1 + \mathfrak{p}_F
\end{pmatrix},
\]
where $\mathfrak{n}$ is the lattice $(\mathfrak{p}_F^l(\chi_1x_n^{-1}), \ldots, \mathfrak{p}_F^l(\chi_{n-1}x_n^{-1}))$.

**Lemma 4.1.5.** The set $K_\chi(1, m)$ is a normal subgroup of $J_\chi(1, m)$.

**Proof.** We first check that $K_\chi(1, m)$ is closed under matrix multiplication. Let $(m_{ij})$ and $(m'_{ij})$ be two matrices from the set $K_\chi(1, m)$. Let $i<j<n$ the $i_j$th term is the sum of
\[
t_1 = m_{i1}m'_{1j} + m_{i2}m'_{2j} + \cdots + m_{ii}m_{ij},
\]
\[
t_2 = m_{i+1}m'_{i+1,j} + m_{i+2}m'_{i+2,j} + \cdots + m_{ij}m_{ij},
\]
and
\[
t_3 = m_{ij}m'_{i+1,j} + \cdots + m_{in}m_{nj}.
\]

Observe that $\nu_F(m_{ij}') \geq 0$, for $1 \leq i \leq n$ and hence, $\nu_F(t_1) > 0$. Now $\nu_F(m_{ik}) > 0$, for $i < k \leq n$ and we get that $\nu_F(t_2) > 0$. Note that $\nu_F(m_{ij}' \geq 0$, for $j < k \leq n$ and hence, the valuation $\nu_F(t_3) > 0$. This shows that $i_j$-term of the matrix product has positive valuation. The verifications on congruence conditions for the in-term are exactly the same as in Lemma 4.4. The existence of inverse for an element in $K_\chi(1, m)$ follows from Gaussian elimination.

Now we establish the normality of $K_\chi(1, m)$. The group $K_\chi(1, m)$ satisfies the Iwahori decomposition with respect to the subgroups $P_{(n-1,1)}$ and $M_{(n-1,1)}$. We also note that $K_\chi(1, m) \cap U_{(n-1,1)}$ is equal to $J_\chi(1, m) \cap U_{(n-1,1)}$ and $K_\chi(1, m) \cap \bar{U}_{(n-1,1)}$ is equal to $J_\chi(1, m) \cap \bar{U}_{(n-1,1)}$. To check the normality of $K_\chi(1, m)$ we have to check that $J_\chi(1, m) \cap M_{(n-1,1)}$ normalizes $K_\chi(1, m)$. This is equivalent to checking that $K_\chi(1, m) \cap M_{(n-1,1)}$ is a normal subgroup of $J_\chi(1, m) \cap M_{(n-1,1)}$.

We note that $J_\chi(1, m) \cap M_{(n-1,1)} = J_\chi'(1) \times \mathfrak{p}_F^\chi$ where $\chi' = \sum_{i=1}^{n-1} \chi_i$. Let $p_1$ be the projection of $J_\chi(1, m) \cap M_{(n-1,1)}$ onto $J_\chi'(1)$ and $\pi_1$ be the reduction mod $\mathfrak{p}_F$ map. Note that $K_\chi(1, m) \cap M_{(n-1,1)}$ is the kernel of $\pi_1 \circ p_1$.

From the above lemma, the group $K_\chi(1, m)$ is a normal subgroup of $J_\chi(1, m)$. We also note that $J_\chi(1, m) \cap \bar{U}_{(n-1,1)}$ is contained in $K_\chi(1, m)$. From this we conclude that $J_\chi(1, m) = K_\chi(1, m)J_\chi(1, m+1)$. From the Mackey decomposition we get that
\[
\text{res}_{K_\chi(1, m)} \text{ind}^{J_\chi(1, m)}_{J_\chi(1, m+1)}(id) \simeq \text{ind}_{K_\chi(1, m) \cap J_\chi(1, m+1)}^{K_\chi(1, m)}(id).
\]
From the definition of $K_\chi(1, m)$ we get that $K_\chi(1, m) \cap J_\chi(1, m+1) = K_\chi(1, m+1)$ and
\[
\text{res}_{K_\chi(1, m)} \text{ind}^{J_\chi(1, m)}_{J_\chi(1, m+1)}(id) \simeq \text{ind}_{K_\chi(1, m)}^{K_\chi(1, m+1)}(id).
\]

**Lemma 4.1.6.** The group $K_\chi(1, m+1)$ is a normal subgroup of $K_\chi(1, m)$.
Proof. The group $K_\chi(1, m)$ has the Iwahori decomposition with respect to $P_{(n-1,1)}$ and $M_{(n-1,1)}$. $K_\chi(1, m) \cap U_{(n-1,1)}$ is equal to $K_\chi(1, m+1) \cap U_{(n-1,1)}$ and $K_\chi(1, m) \cap M_{(n-1,1)}$ is equal to $K_\chi(1, m+1) \cap M_{(n-1,1)}$. We have to check that $u^{-1}j(u^{-1})$ and $u^{-u+}(u^{-1})$ belong to $K_\chi(1, m+1)$, for $u^- \in K_\chi(1, m) \cap \tilde{U}_{(n-1,1)}$, $j \in K_\chi(1, m) \cap M_{(n-1,1)}$ and $u^+ \in K_\chi(1, m) \cap U_{(n-1,1)}$.

We first consider the case $u^{-1}j(u^{-1})$. We can rewrite $u^{-1}j(u^{-1})$ as $j\{j^{-1}u^-j(u^{-1})\}$. Since $j \in K_\chi(1, m) \cap M_{(n-1,1)} = K_\chi(1, m+1) \cap M_{(n-1,1)}$, it is enough to show that $j^{-1}u^-j(u^{-1})$ belongs to the group $K_\chi(1, m+1)$. Let $j$ and $u^-$ be written in their block matrix form as follows.

$$j = \begin{pmatrix} J_1 & 0 \\ 0 & j_t \end{pmatrix} \quad u^- = \begin{pmatrix} 1_{n-1} & 0 \\ U^- & 1 \end{pmatrix}$$

The conjugation $j^{-1}u^-j(u^{-1})$ in its block form is given by

$$j^{-1}U^-j_1 - U^- = \begin{pmatrix} 1_{n-1} \\ j_1^{-1}U^-j_1 - U^- \end{pmatrix}$$

Let $U^- = [u_1, u_2, \ldots, u_{n-1}]$ and $J_1 = (j_{ij})$. The $k^{th}$ entry of the matrix $U^-j_1$ is the sum of

$$t_1 = u_1j_{1k} + u_2j_{2k} + \cdots + u_{k-1}j_{(k-1)k},$$

$$t_2 = u_kj_{kk},$$

and

$$t_3 = u_{k+1}j_{k+1k} + \cdots + u_{n-1}j_{n-1k}.$$ 

If $l(\chi_k\chi_n^{-1}) > 1$, then valuation $\nu_F(u_1j_{1k})$, for $t < k$, is at least $l(\chi_t\chi_n^{-1}) + m - 1 \geq l(\chi_k\chi_n^{-1}) + m$. Hence, we have

$$\nu_F(t_1) \geq l(\chi_k\chi_n^{-1}) + m - 1.$$ 

The valuation $\nu_F(u_tj_{tk})$, for $k < t$, is at least $l(\chi_t\chi_n^{-1}) + l(\chi_k\chi_n^{-1}) + m - 1 > l(\chi_k\chi_n^{-1}) + m - 1$. This shows that $t_1 + t_2 + t_3 = t_2 = u_kj_{kk} = u_k \mod p^l(\chi_k\chi_n^{-1} + m)$. We note that $j_1^{-1}u^-u \in p^lF$, for any $u \in p^lF$. Hence the matrix

$$\begin{pmatrix} 1_{n-1} \\ j_1^{-1}U^-j_1 - U^- \end{pmatrix}$$

is contained in $K_\chi(1, m+1) \cap \tilde{U}_{(n-1,1)}$.

Let us consider the conjugation $u^-u^+(u^{-1})$. We write $u^+$ in the block form as

$$\begin{pmatrix} 1_{n-1} & U^+ \\ 0 & 1 \end{pmatrix}$$

The conjugated matrix $u^-u^+(u^{-1})$ is given by

$$\begin{pmatrix} 1_{n-1} - U^-U^+ & U^+ \\ -U^-U^+U^- & U^-U^+ + 1 \end{pmatrix}.$$ 

Let $1_{n-1} - U^+U^- = (u_{ij})$. The valuation $\nu_F(u_{ij}) \geq l(\chi_n\chi_j^{-1})$, for $i > j$, and $l(\chi_n\chi_j^{-1})$ is greater or equal to $l(\chi_t\chi_j^{-1})$. From this we conclude that $u^-u^+(u^{-1}) \in K_\chi(1, m+1)$.

4.2. Calculation of some stabilisers. The inclusion map of $K_\chi(1, m) \cap \tilde{U}_n$ in $K_\chi(1, m)$ induces an isomorphism of the quotient $K_\chi(1, m)/K_\chi(1, m+1)$ with the abelian group

$$\frac{K_\chi(1, m) \cap \tilde{U}_{(n-1,1)}}{K_\chi(1, m+1) \cap \tilde{U}_{(n-1,1)}}.$$ 

Hence the representation $\text{ind}^{K_\chi(1, m)}_{K_\chi(1, m+1)}(\text{id})$ splits into direct sum of characters say $\{\eta_k \mid 1 \leq k \leq p\}$. The group $J_\chi(1, m)$ acts on these characters and let $\text{Z}(\eta_k)$ be the $J_\chi(1, m)$-stabiliser of the character $\eta_k$. From Clifford theory we get that

$$\text{ind}^{J_\chi(1, m)}_{J_\chi(1, m+1)}(\text{id}) \cong \bigoplus_{\eta_k} \text{ind}^{J_\chi(1, m)}_{\text{Z}(\eta_k)}(U_{\eta_k}),$$

(21)
where \( \eta_{n_k} \) is a representative for an orbit under the action of \( J_{\chi}(1, m) \) and \( U_{\chi n_k} \) is an irreducible representation of the group \( Z(\eta_{n_k}) \). Since

\[
J_{\chi}(1, m) = (J_{\chi}(1, m) \cap M_{(n-1,1)})K_{\chi}(1, m)
\]

we get that \( Z(\eta_k) = (Z(\eta_k) \cap M_{(n-1,1)})K_{\chi}(1, m) \).

The final step in our preliminaries is to understand the mod \( p_F \) reduction of the group

\[
Z(\eta_k) \cap M_{(n-1,1)}
\]

for some non-trivial character \( \eta_k \). The group \( J_{\chi}(1, m) \cap M_{(n-1,1)} \) (which is \( J_{\chi'}(1) \times \sigma_F^k \), for \( \chi' = \Xi_{i=1}^{n-1} \chi_i \)) acts on the quotient

\[
\frac{K_{\chi}(1, m) \cap \tilde{U}_{(n-1,1)}}{K_{\chi}(1, m + 1) \cap \tilde{U}_{(n-1,1)}}
\]

by conjugation. Let \( j \) and \( u^- \) be two elements from \( J_{\chi}(1, m) \cap M_{(n-1,1)} \) and \( K_{\chi}(1, m) \cap U \) respectively. We write the elements \( j \) and \( u^- \) written in their block diagonal form as

\[
\begin{pmatrix} J_1 & 0 \\ 0 & j_1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & -n_1 & 0 \\ 0 & U^- & 1 \end{pmatrix}
\]

respectively. The map \( u^- \mapsto \varphi_{-1}^{U^-}U^- \) induces an isomorphism between the group \([23]\) and \( \text{Mat}_{1 \times (n-1)}(k_F) \).

The map \( u^- \mapsto \varphi_{-1}^{U^-}U^- \) gives an \( J_{\chi}(1) \times \sigma_F^k \)-equivariant map between \( M_{1 \times (n-1)}(k_F) \) and the group \([23]\). We also have a \( M_{(n-1,1)}(k_F) \)-equivariant map between the group of characters of the quotient \([23]\) and the group \( M_{(n-1,1)}(k_F) \) (see Lemma 2.0.7). Hence we obtain a \( J_{\chi}(1) \times \sigma_F^k \)-equivariant map between the group of characters of the quotient \([23]\) and the group \( M_{(n-1,1)}(k_F) \), where \( J_{\chi}(1) \) acts through its (mod \( p_F \)) quotient \( B_{n-1}(k_F) \times k_F^\times \) and the action is \( (b, x)A = bAx^{-1} \). Hence to understand the group \( Z(\eta_k) \cap M_{(n-1,1)} \) for non-trivial \( \eta_k \), we first look at \( Z_{B_{n-1}(k_F) \times k_F^\times}(A) \) for some non-zero matrix \( A \) in \( M_{(n-1,1)}(k_F) \).

Let \( p \) be the projection of \( B_{n-1}(k_F) \times k_F^\times \) onto the diagonal torus

\[
T_{n-1}(k_F) \times k_F^\times = T_n(k_F),
\]

let \( p_i \) be the \( i \)th projection of \( T_n(k_F) \) onto \( k_F^\times \). The centraliser \( Z_{B_{n-1}(k_F) \times k_F^\times}(A) \) of a non-zero matrix \( A = [u_1, u_2, \ldots, u_{n-1}]^T \) satisfies the following property: there exists a \( j \) such that \( p_j(p(t)) = p_n(p(t)) \), for all \( t \in Z_{B_{n-1}(k_F) \times k_F^\times}(A) \) (see [Ad17] Lemma 3.8]). This shows that for any non-trivial character \( \eta_{n_k} \), \( Z(\eta_{n_k}) \cap T_n \) satisfies the property that

\[
p_j(t) = p_n(t) \mod p_F.
\]

The character \( \chi = \Xi_{i=1}^{n} \chi_i \) of \( J_{\chi}(1) \) occurs with multiplicity one in the representation

\[
\text{ind}_{J_{\chi}(1)}^{J_{\chi}(m)}(\chi).
\]

We denote by \( U_m^0(\chi) \) the complement of \( \chi \) in \( \text{ind}_{J_{\chi}(1)}^{J_{\chi}(m)}(\chi) \). We denote by \( U_m(\chi) \) the representation

\[
\text{ind}_{J_{\chi}(1)}^{J_{\chi}(m)}(U_m^0(\chi)).
\]

4.3. Elimination of atypical representations.

**Theorem 4.3.1.** Let \( q_F > 3 \). The irreducible sub representations of \( U_m(\chi) \) are atypical. If \( n = 3 \) and \( k_F > 2 \), then the irreducible sub representations of \( U_m(\chi) \) are atypical.

**Proof.** We prove the theorem by using induction on the positive integers \( n \) and \( m \). For \( n = 1 \) the representation \( U_m(\chi) \) is trivial and the theorem is vacuously true. Let \( n \) be a positive integer greater than one. We assume that the theorem is proved for all positive integers less than \( n \). We will use the induction hypothesis to show the theorem for \( n \).

We note that \( J_{\chi}(1, m) \) and \( J_{\chi}(m) \) satisfy the Iwahori decomposition with respect to the parabolic subgroup \( P_{(n-1,1)} \) and its Levi subgroup \( M_{(n-1,1)} \); we have \( J_{\chi}(1, m) \cap U_{(n-1,1)} = J_{\chi}(m) \cap U_{(n-1,1)} \) and \( J_{\chi}(1, m) \cap
We first note that the complement of the character \( \chi \) atypical and this proves the main theorem. We previously showed that the irreducible sub-representations of \( J_\chi(1, m) \) and \( J_\chi(1, m) \cap M_{(n-1,1)} \) are atypical. We now show that \( \chi \) occurs with multiplicity one in the representation \( \text{ind}^{J_\chi(1)}_{J_\chi(1, m)}(\chi) \). We denote by \( U_{1,m}(\chi) \) the complement of the character \( \chi \) in \( \text{ind}^{J_\chi(1)}_{J_\chi(1, m)}(\chi) \). We denote by \( U_{1,m}(\chi) \) the representation \( \text{ind}^{J_\chi(1)}_{J_\chi(1, m)}(\chi) \).

We first note that

\[
U_m(\chi) \simeq \text{ind}^{K_n}_{J_\chi(1, m)} \{ U_0^m(\chi') \otimes \chi_n \} \oplus U_{1,m}(\chi).
\]

We already showed that the \( K_n \)-irreducible sub-representations of the first summand on the right-hand side of the equation \((26)\) are atypical. We now show that \( K_n \)-irreducible sub-representations of \( U_{1,m}(\chi) \) are atypical and this proves the main theorem.

We first note that

\[
\text{ind}^{J_\chi(1)}_{J_\chi(1, m+1)}(\chi) \simeq \text{ind}^{J_\chi(1)}_{J_\chi(1, m)} \{ \text{ind}^{J_\chi(1, m+1)} \text{id} \} \otimes \chi.
\]

Using the decomposition \((22)\) we get that

\[
\text{ind}^{J_\chi(1)}_{J_\chi(1, m+1)}(\chi) \simeq \bigoplus_{\eta_{nk}} \text{ind}^{J_\chi(1)}_{J_{Z_{\eta_{nk}}}} \{ U_{\eta_{nk}} \otimes \chi \}.
\]
Recall that $\eta_{n_k}$ is a representative for the orbit under the action of the group $J_\chi(1, m)$ on the set of characters $\{\eta_k\mid 1 \leq k \leq p\}$ of the group $K_\chi(1, m)/K_\chi(1, m + 1)$, and $Z(\eta_{n_k})$ is the $J_\chi(1, m)$-stabiliser of the character $\eta_{n_k}$. There is exactly one orbit consisting of the identity character and hence

$$\text{ind}^{J_{\chi}(1)}_{J_{\chi}(1, m+1)}(\chi) \simeq \text{ind}^{J_{\chi}(1)}_{J_{\chi}(1, m)}(\chi) \bigoplus \text{ind}^{J_{\chi}(1)}_{Z(\eta_{n_k})}(U_{\eta_{n_k}} \otimes \chi).$$

(27)

Consider the representation

$$\text{ind}^{J_{\chi}(1)}_{Z(\eta_{n_k})}(U_{\eta_{n_k}} \otimes \chi)$$

for some representative $\eta_{n_k} \neq \text{id}$. Now, recall that $Z(\eta_{n_k}) \cap T_n$ is a subgroup of $T_n(o_F) = \prod_{i=1}^n o_F^\times$, and there exists a positive integer $j < n$ such that $p_j(t) \equiv p_m(t) \text{ mod } p_F$, for all $t \in Z(\eta_{n_k})$. Let $\kappa$ be a character of $F^\times$ such that $\kappa$ is ramified and $1 + Q_F$ is contained in the kernel of $\kappa$. Let $\chi^\kappa$ be the character

$$\chi_1 \boxtimes \chi_2 \boxtimes \chi_j \kappa \boxtimes \cdots \boxtimes \chi_n \kappa^{-1}.$$

We observe that $\text{res}_{Z(\eta_{n_k})}(\chi) = \text{res}_{Z(\eta_{n_k})}(\chi^\kappa)$ and hence

$$\text{ind}^{J_{\chi}(1)}_{Z(\eta_{n_k})}(U_{\eta_{n_k}} \otimes \chi) \simeq \text{ind}^{J_{\chi}(1)}_{Z(\eta_{n_k})}(U_{\eta_{n_k}} \otimes \chi^\kappa).$$

(28)

From the above paragraph we get that

$$U_{1,m+1}^0(\chi) \simeq U_{1,m}^0(\chi) \bigoplus \text{ind}^{J_{\chi}(1)}_{Z(\eta_{n_k})}(U_{\eta_{n_k}} \otimes \chi).$$

and from the above identity we conclude that

$$U_{1,m+1}(\chi) \simeq U_{1,m}(\chi) \bigoplus \text{ind}^{K_{\kappa}}_{Z(\eta_{n_k})}(U_{\eta_{n_k}} \otimes \chi).$$

(29)

From the equation (28) we get that

$$\text{ind}^{K_{\kappa}}_{Z(\eta_{n_k})}(U_{\eta_{n_k}} \otimes \chi) \simeq \text{ind}^{K_{\kappa}}_{Z(\eta_{n_k})}(U_{\eta_{n_k}} \otimes \chi^\kappa).$$

If we choose $\kappa$ such that $[T_n, \chi]$ and $[T_n, \chi^\kappa]$ are two distinct inertial classes, then we can conclude that irreducible subrepresentations of $U_{1,m}(\chi)$ are atypical. Hence, using the identity (29) recursively we get that irreducible subrepresentations of $U_{1,m}(\chi)$ are atypical representations, for all positive integers $m$.

To prove the theorem we have to justify that we can choose a character $\kappa$ as in the previous paragraph. Now for any character $\kappa$ non-trivial on $o_F^\times$ (such a character exists since $q_F > 2$) and trivial on $1 + p_F$, the equality of the inertial classes $[T_n, \chi]$ and $[T_n, \chi^\kappa]$ implies the equality of multiplicities of $\chi_j$ in the multisets $\{\chi_1, \chi_2, \ldots, \chi_n\}$ and $\{\chi_1, \chi_2, \ldots, \chi_j \kappa, \ldots, \chi_n \kappa^{-1}\}$. The equality of multiplicities implies $\chi_j \chi_n^{-1} = \kappa$. If $q_F > 3$, then we have at least two non-trivial tame characters and hence we can choose $\kappa$ distinct from a possibly tame character $\chi_j \chi_n^{-1}$.

If $l(\chi_i) = 1$, for $1 \leq i \leq n - 1$, and $l(\chi_n) > 1$, then we can always find $\kappa$ such that $[T, \chi]$ and $[T, \chi^\kappa]$ are distinct inertial classes. We note that the induction hypothesis here is supplied by depth-zero case stated as Theorem 2.0.8.

Consider the case where $n = 3, q_F = 3$ and $\eta$ is the non-trivial character of $k_F^\times$. We have the character $\chi = \chi_1 \boxtimes \chi_2 \boxtimes \chi_3$ of $T_3$. Assume that there exists $i \neq j$ and $i, j \in \{1, 2, 3\}$ such that $\chi_i \chi_j^{-1} = \eta$, with $l(\eta) = 1$. If such a pair $(i, j)$ does not exist, then our present proof goes through. Now, twisting with the character $\chi_j$ if necessary, and permuting the characters $\chi_1, \chi_2$ and $\chi_3$ if necessary, we may assume that $\chi_1 = \text{id}, \chi_2 = \eta$. This arrangement still satisfies the condition (15). If $l(\chi_3) = 1$, then we are depth-zero case and we refer to Theorem 2.0.8 for a proof of this result. If $l(\chi_3) > 1$, then $K_2$ irreducible subrepresentations of $U_m(\chi_1 \boxtimes \chi_2)$ are atypical, for $m \geq 1$ (we refer to 2.0.8 or [BM02, Appendix] for a proof). Now, the above proof shows that irreducible subrepresentations of $U_{1,m}(\chi)$ are atypical, for $m \geq 1$. This shows the theorem in the present case.  \qed
The pair $(J_s(1), \chi)$ is a Bushnell-Kutzko type for the inertial class $s$ (see [BK99 Section 8]). From the above theorem we deduce the following result:

**Theorem 4.3.2.** Let $q_F > 3$ if $n > 3$ and $q_F > 2$ if $n \in \{2, 3\}$. Let $\tau$ be a typical representation for the inertial class $s = [T_n, \chi]$ then $\tau$ is a subrepresentation of $\text{ind}_{J_s(1)}^{K_n}(\chi)$. Moreover we have

$$\dim \text{Hom}_{K_n}(\tau, \text{ind}_{J_s(1)}^{K_n}(\chi)) = \dim \text{Hom}_{K_n}(\tau, \text{ind}_{J_s(1)}^{K_n}(\chi))$$

**Remark 4.3.3.** When $\# k_F = 2$ and $n = 2$ Henniart showed in [BM02] [A.2.6, A.2.7] that the Bushnell-Kutzko type for the inertial class $s = [T_2, \chi_1 \boxtimes \chi_2]$, $\chi_1 \chi_2 \neq \text{id}$ has two typical representations one given by

$$\text{ind}_{J_s(1)}^{GL_2(F)}(\chi)$$

and the other representation turns out to be the complement (it follows from [Cas73 Proposition 1(b)] that there is a unique complement) of $\text{ind}_{J_s(1)}^{GL_2(F)}(\chi)$ in $\text{ind}_{J_s(2)}^{GL_2(F)}(\chi)$. But for $\# k_F > 2$ and $n > 3$ we expect that typical representations are precisely the irreducible subrepresentations of

$$\text{ind}_{J_s(1)}^{K_n}(\chi).$$

For $\# k_F = 2$ and $n > 2$ a typical representation may not be contained in the above representation as shown by Henniart for the case of $GL_2(F)$.

5. **Typical representations for $GL_3(F)$**

Any inertial class of the group $G_3$ belongs to one of the following classes (see Lemma 2.0.6):

1. $[G_3, \sigma]$, where $\sigma$ is a cuspidal representation of $G_3$
2. $[G_2 \times G_1, \sigma \boxtimes \chi]$, where $\sigma$ is a cuspidal representation of $G_2$ and $\chi$ is any character of $F^\times$.
3. $[T_3, \chi = \chi_1 \boxtimes \chi_2 \boxtimes \chi_3]$, where $\chi_1$, $\chi_2$ and $\chi_3$ are three characters of $F^\times$.

Typical representations for any inertial class of the form $s = [G_3, \sigma]$ are classified in the work of Paskunas [Pas05]. Up to isomorphism there exists a unique typical representation for $s$. Similarly, the theorem 3.3.3 shows that, if $q_F > 2$, then up to isomorphism there exists a unique typical representation for any inertial class of the form $[G_2 \times G_1, \sigma \boxtimes \chi]$. If $q_F > 2$, then for any inertial class $s = [T_3, \chi]$ we showed that any typical representation occurs as a subrepresentation of $\text{ind}_{J_s(1)}^{K_n}(\chi)$. The pair $(J_s(1), \chi)$ is a Bushnell–Kutzko type for $s$. Moreover, we also have the multiplicity result Theorem 4.3.2. We conclude the following result for $GL_3(F)$.

**Theorem 5.0.1.** Let $q_F > 2$ and $s$ be any inertial class of $G_3$. Typical representations for $s$ are precisely the irreducible subrepresentations of $\text{ind}_{J_s}^{K_3}(\lambda_s)$, where $(J_s, \lambda_s)$ is a Bushnell–Kutzko type for $s$.

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