Semi-Classical Quantization of the Many-Anyon System

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Abstract

We discuss the problem of $N$ anyons in harmonic well, and derive the semi-classical energy spectrum as an exactly solvable limit of the many-anyon Hamiltonian. The relevance of our result to the solution of the anyon-gas model is discussed.

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1. Introduction

The possibility of arbitrary statistics continuously interpolating between the Bose and the Fermi case in space dimensions less than three was suggested some years ago in a beautiful and pioneering paper [1] by Leinaas and Myrheim. Their analysis showed that the topology of a non-simply connected configuration space affects the behaviour of a quantum system of identical particles exactly like a multiply connected physical space determines effects of the Aharonov-Bohm type [2]. A physical model of particles with arbitrary, or fractional, statistics was put forward by Wilczek [3], who baptized such objects as “anyons”. Anyons are two-dimensional particles endowed with fictitious statistical charge and magnetic flux; their wavefunctions will in general be multivalued, due to the Aharonov-Bohm phases that the anyons acquire in winding around each other on the plane of motion. The relevance of the anyon model has been suggested for such important phenomena as the fractional quantum Hall effect and high temperature superconductivity [4], [5]. It is then crucial to acquire the deepest possible knowledge of the properties of the general $N$-body anyon system. Unfortunately, despite growing effort in the last few years, only the two-anyon problem has been exactly solved by Leinaas and Myrheim in their original paper [1]. For the three-anyon problem Wu has obtained a special set of exact solutions [6] which have been recently generalized to the arbitrary $N$-anyon system in a magnetic field by Dunne et al. [7]. Here we will not pursue this direction, but instead concentrate on a unified approximate treatment of the many-anyon problem in the framework of the semi-classical quantization.

2. Semi-classical approximation

In a previous paper [8], a semi-classical quantization method was put forward to study the two- and three-anyon systems. It was shown that the method yields the exact energy spectrum for two anyons. In the three-anyon case it gives a spectrum linear in the statistical parameter and in good agreement with the numerical one obtained by Sporre et al. [9],
except for a slight nonlinearity in some of the numerical levels. Here we briefly review
the main points of the argument. Consider a system of two non-interacting anyons in an
external harmonic oscillator potential of frequency $2\pi \omega$; the classical Hamiltonian for the
relative motion in polar coordinates $(r, \phi)$ reads

$$H = \frac{1}{2} p_r^2 + \frac{1}{2} \omega^2 r^2 + \frac{1}{2r^2} (p_\phi - \alpha \hbar)^2,$$

where $\alpha$ is the statistical parameter; Bose statistics is recovered for $\alpha = 0$ and Fermi
statistics for $\alpha = 1$. From the classical Hamiltonian one can build the classical partition
function

$$Z_{\text{class}} = \frac{1}{(2\pi \hbar)^2} \int dr dp_r d\phi dp_\phi e^{-\beta H}.$$  \tag{2}$$

The semi-classical strategy amounts now to a discretization of the relevant degrees of
freedom to obtain a partition function expressed as a sum of Boltzmann factors. By
identifying the energy eigenvalues in the exponents one can obtain the energy spectrum
of the system. The essential quantum behaviour of the anyon model is contained in the
angular degrees of freedom. Then, letting $p_\phi \to \hbar m$ and $\int dp_\phi \to \hbar \sum_m$, $m$ being the
angular momentum quantum number, the semi-classical partition function reads

$$Z_{s.-c.} = \frac{1}{2\beta \hbar \omega} \sum_m \exp \left( \beta \hbar \omega |m - \alpha| \right).$$  \tag{3}$$

The semi-classical quantization is most significant in the small quanta of action regime,
i.e. when $\beta \hbar \omega \ll 1$. In this limit it is correct to approximate $\beta \hbar \omega$ by $\sinh \beta \hbar \omega$ in the
denominator of (3). By use of a power series representation for $(2 \sinh \beta \hbar \omega)^{-1}$ one can
finally put $Z_{s.-c}$ in the desired form

$$Z_{s.-c.} = \sum_{m,n} \exp \left( -\beta (2n + 1 + |m - \alpha|) \hbar \omega \right),$$  \tag{4}$$

and read off the energy eigenvalues

$$E_{m,n} = (2n + 1 + |m - \alpha|) \hbar \omega.$$  \tag{5}$$
which turn out to be the exact ones [1]. Extension of the method to systems of more than two anyons meets the obstacle of the partition function being not factorizable in a product of one-dimensional integrals. To by-pass the problem, in [8] some further approximations were made. The result was a semi-classical partition function for the relative motion of three anyons expressed as the product of two partition functions for the relative motion of two anyons, with statistical parameters $\alpha$ and $2\alpha$ respectively. Going again through the semi-classical procedure sketched above, the energy spectrum for the relative motion of three anyons was found to be

$$E = (2n_1 + 1 + 2n_2 + 1 + |m_1 - \alpha| + |m_2 - 2\alpha|)\hbar\omega). \quad (6)$$

The spectrum (6) is drawn in Fig.1. It is again linear in $\alpha$, and the energy levels fall into two classes, one with slopes $\pm 1$, and the other with slopes $\pm 3$. Remarkably, this behaviour agrees with the numerical solution of the three-anyon problem [9], although the semi-classical levels with slopes $\pm 1$ appear to be slightly curved in the numerical solution. Furthermore, the semi-classical levels with slopes $\pm 3$ are equal to the exact eigenvalues obtained from the special solutions of Wu [6]. Thus the energy relations (5) and (6) seem to capture some of the essential features of the few-anyons physics. In this work we shall provide a deeper understanding of this fact, and a precise setting of the semi-classical quantization in terms of a limit of separability of the many-anyon Lagrangian. This connection explains the nature of the approximations made in deriving the semi-classical energy relation (6) and allows its immediate generalization to systems of arbitrarily many anyons. This semi-classical $N$-anyon energy spectrum and the numerical one are compared for the case $N = 4$ and are again shown to be in good agreement.

3. Separation of the many-anyon problem

We consider again eq. (6). After a moment’s thought one realizes that it is the sum of the energy spectra of two internal oscillators, one with statistical parameter $\alpha$, the other with statistical parameter $2\alpha$. This suggests that the approximation made in [8] to factorize
the partition function might simply amount to replace the exact relative hamiltonian with a separable one. This observation would also explain why the semi-classical spectrum is exact in the case of two anyons, since the exact two-anyon lagrangian is already separated.

We first recall that a non-relativistic system of \( N \) anyons in an external harmonic oscillator potential is described, in cartesian coordinates, by the Lagrangian

\[
L = \frac{1}{2} \sum_{i=1}^{N} (\dot{r}_i^2 - \omega^2 r_i^2) + \alpha \hbar \sum_{i<j} \dot{\phi}_{ij},
\]

(7)

where \( \mathbf{r}_i = (x_i, y_i) \), and the azimuthal angle \( \phi_{ij} \) is defined by

\[
\phi_{ij} = \arctan \frac{y_j - y_i}{x_j - x_i}.
\]

(8)

The separation of the center of mass motion can be achieved through the change of variables

\[
\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N \rightarrow \mathbf{R}, \mathbf{\rho}_1, \mathbf{\rho}_2, \ldots, \mathbf{\rho}_{N-1},
\]

(9)

with the center of mass coordinate \( \mathbf{R} \) and the \( N-1 \) Jacobi coordinates \( \{\mathbf{\rho}_k\} \) defined as

\[
\mathbf{R} = \frac{1}{\sqrt{N}} (\mathbf{r}_1 + \mathbf{r}_2 + \cdots + \mathbf{r}_N),
\]

\[
\mathbf{\rho}_1 = \frac{1}{\sqrt{2}} (\mathbf{r}_1 - \mathbf{r}_2),
\]

\[
\mathbf{\rho}_2 = \frac{1}{\sqrt{6}} (\mathbf{r}_1 + \mathbf{r}_2 - 2\mathbf{r}_3),
\]

\[
\vdots
\]

\[
\mathbf{\rho}_{N-1} = \frac{1}{\sqrt{N(N-1)}} (\mathbf{r}_1 + \mathbf{r}_2 + \cdots + \mathbf{r}_{N-1} - (N-1)\mathbf{r}_N).
\]

(10)

For the Jacobi coordinates the associated angles relative to the \( x \)-axis are

\[
\phi_k = \arctan \frac{y_1 + y_2 + \cdots + y_{k-1} + y_k - k y_{k+1}}{x_1 + x_2 + \cdots + x_{k-1} + x_k - k x_{k+1}}, \quad k = 1, 2, \ldots, N-1.
\]

(11)
Then, neglecting the center of mass motion and expressing the Jacobi coordinates in the polar form \( \{ \rho_k, \phi_k \} \) (with \( \rho_k = \sqrt{\rho_k \cdot \rho_k} \)), eq.(7) becomes for the relative motion

\[
L_{\text{rel}} = \frac{1}{2} \sum_{k=1}^{N-1} (\dot{\rho}_k^2 + \rho_k^2 \dot{\phi}_k^2) - \frac{1}{2} \sum_{k=1}^{N-1} \omega^2 \rho_k^2 + \alpha \hbar \sum_{i<k} \dot{\phi}_{ik}.
\] (12)

We now investigate the relation between the azimuthal angles \( \phi_{ik} \) and the polar Jacobi angles \( \phi_k \); consider first the situation in which the \( N \) anyons are arranged in “hierarchical clusters”, i.e.

\[
\rho_1 \ll \rho_2 \ll \rho_3 \ll \cdots \ll \rho_{N-1},
\] (13)

which represents a configuration where particle 3 is very far out off the area spanned by particles 1 and 2, particle 4 is very far out off the area spanned by particles 1, 2 and 3, and so on. In this limit, comparing eqns.(8) and (11) we have that

\[
\phi_{1j} = \phi_{2j} = \cdots = \phi_{j-1j} = \phi_{j-1},
\]

\[
j = 3, 4, \ldots, N.
\] (14)

The relation \( \phi_{12} = \phi_1 \) is always trivially true, independently of the cluster condition (13).

As an example, in the case \( N = 3 \) the clustering condition implies

\[
\phi_{13} = \phi_{23} = \phi_2,
\] (15)

while in the case of four anyons

\[
\phi_{13} = \phi_{23} = \phi_2,
\]

\[
\phi_{14} = \phi_{24} = \phi_{34} = \phi_3.
\] (16)

The Lagrangian (12) in the limit (13) is a function only of the polar Jacobi variables

\[
L_{\text{rel}} = \frac{1}{2} \sum_{k=1}^{N-1} (\dot{\rho}_k^2 + \rho_k^2 \dot{\phi}_k^2) - \frac{1}{2} \sum_{k=1}^{N-1} \omega^2 \rho_k^2 + \alpha \hbar \sum_{k=1}^{N-1} \dot{\phi}_k.
\] (17)
and the corresponding Hamiltonian reads

\[ H_{\text{rel}} = \frac{1}{2} \sum_{k=1}^{N-1} \left( P_{\rho_k}^2 + \frac{(P_{\phi_k} - k\alpha \hbar)^2}{\rho_k^2} + \omega \rho_k^2 \right). \]  

(18)

This is the first main result of our analysis: in the clustering limit, eq.(13), the \( N \)-anyon relative Hamiltonian is separated in the sum of \( N - 1 \) relative two-anyon oscillator Hamiltonians, with the respective statistical parameters \( \alpha, 2\alpha, \ldots, (N-1)\alpha \), and the associated Schroedinger problem can be solved exactly.

The quantum mechanics of the simplified hamiltonian system (18) and its connections with the complete solution of the many-anyon problem will not be discussed here. However, we want to remark that the situation described by eq.(13) is only a subcase of a much more general condition of separability of the Lagrangian (7). In fact, it is always correct to replace the azimuthal angles in (7) by the Jacobi polar angles according to the rule (14) as long as particle 3 winds up either around both particle 1 and particle 2 or around none of them, but not around particle 1 or particle 2 alone; particle 4 winds up around particles 1, 2, and 3 together or around none of them, but not around one of them or two of them alone, and so on (see Fig.2 for an illustration in the case \( N = 4 \)). This can be understood reminding that the topological term in the Lagrangian (7) can be anything that describes the correct windings of anyons, or, to put it differently, anything that belongs to the correct homotopy class of the problem: in the above described situation the winding orbits do not cross.

On the other hand, the azimuthal and the Jacobi angles are related by the property that the winding number of a Jacobi trajectory encircling \( m \) particles is \( m \) times the winding number of an azimuthal trajectory for two particles encircling each other; this means that since in the two-anyon case each anyon moving around the other picks up a phase factor \( 2\pi \alpha \hbar \), in the \( N \)-anyon case each anyon going around the other \( N - 1 \) anyons picks up a phase factor \( 2(N-1)\pi \alpha \hbar \). Therefore, the replacement of the Lagrangian (12) by the Lagrangian (17) is \textit{exact} for non-crossing windings (see Fig.2). The clustering condition (13) is then just an example of the many possible configurations leading to separability.

The energy spectrum for the quantum Hamiltonian (18) can be derived exactly. How-
ever, and this is our second main result, it is also straightforward to derive it applying the
semi-classical procedure described in the introduction. The result is the generalization of
\[ E = \sum_{k=1}^{N-1} (2n_k + 1 + |m_k - k\alpha|)\hbar\omega. \] (19)
The energy levels are linear in the statistical parameter \(\alpha\); the radial quantum numbers \(n_k\)
should be positive integers, while the angular quantum numbers \(m_k\) can be both positive
and negative. The correct degeneracy can be obtained only from the full quantum solution
for the Hamiltonian (18), and it will be discussed elsewhere (see ref. [11]).

4. Discussion and conclusions

Formula (19) predicts that all energy levels as function of \(\alpha\) should fall into different
classes, each with a positive and a negative slope. For \(N = 4\) the levels fall into four
different classes with slopes 0, \(\pm 2, \pm 4, \pm 6\). The corresponding spectrum is drawn in Fig.3.
Again, as in the case of three anyons, it reproduces the numerical results recently obtained
for the four-anyon system [10]. This is encouraging in view of a possible close connection
between the solution of the simplified model-Hamiltonian (18), and the solution of the
general anyon-gas model. Work is in progress in that direction [11].

To conclude, we stress that the procedure to derive the solvable Hamiltonian (18) and
the semi-classical spectrum (19) does not depend on the details of the external potential,
and therefore it can be easily applied to other related models, e.g. to the motion of anyons
in a magnetic field, which has a great physical relevance in connection with the fractional
quantum Hall effect. The semi-classical picture for the anyon-gas in external magnetic field
will be presented in a forthcoming paper [12].

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Figure Captions

**Fig.1.** Semi-classical energy levels for the three-anyon system in harmonic oscillator potential. Only the levels which equal the numerical intercepts are reproduced.

**Fig.2.** Different winding configurations for the 4-anyon problem. In the examples A and B the relative dynamics is exactly described either by the azimuthal or by the Jacobi variables, and the replacement of the Lagrangian (12) by the Lagrangian (17) is correct. This is not true in cases C and D.

**Fig.3.** The semi-classical energy spectrum of four anyons in harmonic oscillator potential.