Combinatorial structure of a holonomic controlled phase gate

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Abstract

We investigate the combinatorial structures of a holonomic controlled quantum gate based on toric varieties. In particular, we in detail discuss the combinatorial structures of a two-qubit holonomic controlled quantum gate on a two-qubit, a three-qubit, and a four-qubit quantum states. Our results show interesting relations between toric varieties of qubit systems and action of a two-qubit holonomic quantum gate on multi-qubit states.

1 Introduction

Holonomic quantum computing is a model of computing which is based on geometrical properties of quantum systems and could e.g., result in a less error prone computational model. In the holonomic model a quantum gate is constructed from the geometric phase. The Hamiltonian is varied adiabatically through a closed curve in a parameter space. In this paper we are interested to visualized the action of a holonomic gate on product states based on toric varieties. A toric variety is an irreducible variety $X$ that satisfies the following conditions. First of all $T = (\mathbb{C}^*)^n$ is a Zariski open subset of $X$ and the action of $T$ on itself can extend to an action of $T$ on the variety $X$. We are interested in combinatorial structures of toric varieties which are discussed in [1, 2]. Next, we will consider the following Hamiltonian of an interacting two qubits spin system [3]

$$H = H_0 + H_{int}$$
$$= \hbar \omega_i S_{iz} \otimes I_j + \hbar \omega_j I_i \otimes S_{jz} + 2\pi \hbar J S_{iz} \otimes S_{jz}$$
$$= \frac{\hbar}{2} \text{diag}(\omega_i + \omega_j + \pi J, \omega_i - \omega_j - \pi J, -\omega_i + \omega_j - \pi J, -\omega_i - \omega_j + \pi J),$$

where diag denotes a diagonal matrix, $\omega_i > \omega_j$ are the transition angular frequency of the two spins, $S_{rz} = \sigma_{rz}/2$ is the Pauli matrix, and $J$ is the coupling constant. In the last part of the above equation the Hamiltonian is written in the basis $\{|\uparrow\rangle, |\downarrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle\}$. We assume that when the spin $S_j$ is in state $|\uparrow\rangle$, the transition frequency of spin $S_i$ is $\omega_i = \omega_i + \pi J$ and when the spin $S_j$
is in state $|\downarrow\rangle$, the transition frequency of spin $S_i$ is $\omega_+ = \omega_i - \pi J$. Next we define a Berry phase shift

$$
\gamma_+ + \gamma_- = \pi (\cos \theta_+ - \cos \theta_-) = \pi \left( \frac{\omega_+ - \omega}{\sqrt{(\omega_+ - \omega)^2 + \omega_+^2}} - \frac{\omega_- - \omega}{\sqrt{(\omega_- - \omega)^2 + \omega_-^2}} \right),
$$

where $\gamma_{\pm} = \mp \pi (1 - \cos \theta_{\pm})$ are the Berry phases acquired by spin $S_i$ when the spin $S_i$ is in the state $|\downarrow\rangle$ and $|\uparrow\rangle$. This phase shift gives the following transformation that is equivalent to the controlled phase gate

$$
U_{CPhase} = \begin{pmatrix}
e^{2i(\gamma_+ + \gamma_-)} & 0 & 0 & 0 \\
0 & e^{-2i(\gamma_+ + \gamma_-)} & 0 & 0 \\
0 & 0 & e^{-2i(\gamma_+ + \gamma_-)} & 0 \\
0 & 0 & 0 & e^{2i(\gamma_+ + \gamma_-)}
\end{pmatrix}
\tag{1}
$$

In the following sections we will use this holonomic quantum gate to investigate the geometrical structures quantum computing based on toric varieties.

## 2 Toric variety

Let $S \subset \mathbb{R}^n$ be finite subset, then a convex polyhedral cone is defined by $\sigma = \text{Cone}(S) = \left\{ \sum_{v \in S} \lambda_v v | \lambda_v \geq 0 \right\}$. In this case $\sigma$ is generated by $S$. In a similar way we define a polytope by $P = \text{Conv}(S) = \left\{ \sum_{v \in S} \lambda_v v | \lambda_v \geq 0, \sum_{v \in S} \lambda_v = 1 \right\}$.

We also could say that $P$ is convex hull of $S$. A convex polyhedral cone is called simplicial if it is generated by linearly independent set. Now, let $\sigma \subset \mathbb{R}^n$ be a convex polyhedral cone and $(u, v)$ be a natural pairing between $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$. Then, the dual cone of the $\sigma$ is define by

$$
\sigma^\vee = \left\{ u \in \mathbb{R}^n | \langle u, v \rangle \geq 0 \ \forall \ v \in \sigma \right\},
$$

where $\mathbb{R}^{*n}$ is dual of $\mathbb{R}^n$. We call a convex polyhedral cone strongly convex if $\sigma \cap (-\sigma) = \{0\}$.

The algebra of Laurent polynomials is defined by $\mathbb{C}[z, z^{-1}] = \mathbb{C}[z_1, z_1^{-1}, \ldots, z_n, z_n^{-1}]$, where $z_i = e^{\lambda_i}$. The terms of the form $\lambda \cdot z^\beta = \lambda_1^{\beta_1} \lambda_2^{\beta_2} \cdots \lambda_n^{\beta_n}$ for $\beta = (\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{Z}$ and $\lambda \in \mathbb{C}^*$ are called Laurent monomials. A ring $R$ of Laurent polynomials is called a monomial algebra if it is a $\mathbb{C}$-algebra generated by Laurent monomials. Moreover, for a lattice cone $\sigma$, the ring $R_\sigma = \{ f \in \mathbb{C}[z, z^{-1}] : \text{supp}(f) \subset \sigma \}$ is a finitely generated monomial algebra, where the support of a Laurent polynomial $f = \sum \lambda_i z^i$ is defined by $\text{supp}(f) = \{ i \in \mathbb{Z}^n : \lambda_i \neq 0 \}$.

Now, for a lattice cone $\sigma$ we can define an affine toric variety to be the maximal spectrum $X_\sigma = \text{Spec} R_\sigma$. A toric variety $X_\Sigma$ associated to a fan $\Sigma$ is the result of gluing affine varieties $X_\sigma = \text{Spec} R_\sigma$ for all $\sigma \in \Sigma$ by identifying $X_\sigma$ with the corresponding Zariski open subset in $X_\sigma'$ if $\sigma$ is a face of $\sigma'$. That is, first we take the disjoint union of all affine toric varieties $X_\sigma$ corresponding to the cones of $\Sigma$. Then by gluing all these affine toric varieties together we get $X_\Sigma$. 

2
3 Toric variety and holonomic quantum computation

3.1 Two qubit systems

First we consider a pair of qubits $|\Psi\rangle = \sum_{x_2=0}^{1} \sum_{x_1=0}^{1} \alpha_{x_1x_2} |x_1x_2\rangle$. For this two qubit state the separable state is given by the Segre embedding of $\mathbb{CP}^1 \times \mathbb{CP}^1 = \{(\alpha_0^1, \alpha_1^1), (\alpha_0^2, \alpha_1^2) \} : (\alpha_0^1, \alpha_1^1) \neq 0, (\alpha_0^2, \alpha_1^2) \neq 0$. Let $z_1 = \alpha_1^1(\alpha_0^1)^{-1}$ and $z_2 = \alpha_1^2(\alpha_0^2)^{-1}$. Then we can cover $\mathbb{CP}^1 \times \mathbb{CP}^1$ by four charts $X_{s_1} = \{(z_1, z_2)\}, X_{s_2} = \{(z_1^{-1}, z_2)\}, X_{s_3} = \{(z_1, z_2^{-1})\}, X_{s_4} = \{(z_1^{-1}, z_2^{-1})\}$. The fan $\Sigma$ for $\mathbb{CP}^1 \times \mathbb{CP}^1$ has edges spanned by $(0, 0), (0, 1), (1, 0), (1, 1)$. Now we consider the following holonomic controlled quantum gate

$$U_{C\text{Phase}} = \begin{pmatrix} e^{i\varphi_1} & 0 & 0 & 0 \\ 0 & e^{i\varphi_2} & 0 & 0 \\ 0 & 0 & e^{i\varphi_2} & 0 \\ 0 & 0 & 0 & e^{i\varphi_1} \end{pmatrix},$$

where $\varphi_2 = -\varphi_1 = -2(\gamma_+ + \gamma_-)$. If we apply this gate to a pure two-qubit product state $|\Psi\rangle = H \otimes H |00\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$, then we get

$$U_{C\text{Phase}}|\Psi\rangle = \frac{1}{2}(e^{i\varphi_1}(|00\rangle + |11\rangle) + e^{i\varphi_2}(|01\rangle + |10\rangle)).$$

In this case we can see directly that the phase factors $e^{i\varphi_1}$ and $e^{i\varphi_2}$ correspond to diagonal lines in the toric variety of a given two-qubit state. Thus the action of the holonomic controlled quantum gate can be seen from a toric variety, see Figure 1. We also can calculate the concurrence of the evolved state as follows

$$C(U_{C\text{Phase}}|\Psi\rangle) = 2|\alpha_{00}\alpha_{11} - \alpha_{01}\alpha_{10}| = 2|\frac{1}{4} e^{i2\varphi_1} - \frac{1}{4} e^{i2\varphi_2}| = |\sin 2\varphi_1|.$$  

3.2 Three qubit systems

Next, we will discuss a three-qubit state $|\Psi\rangle = \sum_{x_3,x_2,x_1=0}^{1} \alpha_{x_1x_2x_3} |x_1x_2x_3\rangle$. For this state the separable state is given by the Segre embedding of $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1 = \{(\alpha_0^1, \alpha_1^1), (\alpha_0^2, \alpha_1^2), (\alpha_0^3, \alpha_1^3)\} : (\alpha_0^1, \alpha_1^1) \neq 0, (\alpha_0^2, \alpha_1^2) \neq 0, (\alpha_0^3, \alpha_1^3) \neq 0$. The action of the holonomic controlled quantum gate on the toric variety of a given two-qubit state is given by the Segre embedding of $\mathbb{CP}^1 \times \mathbb{CP}^1$. Then we can cover $\mathbb{CP}^1 \times \mathbb{CP}^1$ by four charts $X_{s_1} = \{(z_1, z_2)\}, X_{s_2} = \{(z_1^{-1}, z_2)\}, X_{s_3} = \{(z_1, z_2^{-1})\}, X_{s_4} = \{(z_1^{-1}, z_2^{-1})\}$. The fan $\Sigma$ for $\mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1$ has edges spanned by $(0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1)$. Now we consider the following holonomic controlled quantum gate

$$U_{C\text{Phase}} = \begin{pmatrix} e^{i\varphi_1} & 0 & 0 & 0 \\ 0 & e^{i\varphi_2} & 0 & 0 \\ 0 & 0 & e^{i\varphi_2} & 0 \\ 0 & 0 & 0 & e^{i\varphi_1} \end{pmatrix},$$

In this case we can see directly that the phase factors $e^{i\varphi_1}$ and $e^{i\varphi_2}$ correspond to diagonal lines in the toric variety of a given two-qubit state. Thus the action of the holonomic controlled quantum gate can be seen from a toric variety, see Figure 1. We also can calculate the concurrence of the evolved state as follows

$$C(U_{C\text{Phase}}|\Psi\rangle) = 2|\alpha_{000}\alpha_{111} - \alpha_{001}\alpha_{110} - \alpha_{001}\alpha_{110} + \alpha_{011}\alpha_{100}| = 2|\frac{1}{4} e^{i2\varphi_1} - \frac{1}{4} e^{i2\varphi_2} - \frac{1}{4} e^{i2\varphi_1} + \frac{1}{4} e^{i2\varphi_2}| = |\sin 2\varphi_1|.$$  

Figure 1: Action of the holonomic controlled quantum gate on toric variety of a two-qubit state.
a three-qubit state. Figure 2: Action of the holonomic controlled quantum gate on toric variety of a three-qubit state.

Now, for example, let \( z_1 = \alpha_1^1/\alpha_0^1 \), \( z_2 = \alpha_1^2/\alpha_0^2 \), and \( z_3 = \alpha_1^3/\alpha_0^3 \). Then we can cover \( \mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1 \) by eight charts

\[
\begin{align*}
X_{\Delta_1} & = \{(z_1, z_2, z_3)\}, \quad X_{\Delta_2} = \{(z_1^{-1}, z_2, z_3)\}, \\
X_{\Delta_3} & = \{(z_1, z_2^{-1}, z_3)\}, \quad X_{\Delta_4} = \{(z_1^{-1}, z_2^{-1}, z_3)\}, \\
X_{\Delta_5} & = \{(z_1, z_2, z_3^{-1})\}, \quad X_{\Delta_6} = \{(z_1^{-1}, z_2, z_3^{-1})\}, \\
X_{\Delta_7} & = \{(z_1, z_2^{-1}, z_3^{-1})\}, \quad X_{\Delta_8} = \{(z_1^{-1}, z_2^{-1}, z_3^{-1})\}.
\end{align*}
\]

The toric polytope of \( X_\Sigma = \mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1 \) is a 3-cube. Now, if we consider the same holonomic controlled quantum gate \( U_{\text{CPhase}} \) acting on a pure three-qubit product state \( |\Psi\rangle = H \otimes H \otimes H |000\rangle = \frac{1}{\sqrt{2}} (|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle) \), then we get

\[
(U_{\text{CPhase}} \otimes I) |\Psi\rangle = \frac{1}{\sqrt{2}} (e^{i\varphi_1} |000\rangle + |001\rangle + |110\rangle + |111\rangle) + e^{i\varphi_2} (|010\rangle + |100\rangle + |011\rangle + |101\rangle).
\]

where \( I \) is 2-by-2 identity matrix. Now, we can see directly that phase factors \( e^{i\varphi_1} \) and \( e^{i\varphi_2} \) correspond to intersecting diagonal planes in the toric variety of a given three-qubit state. Moreover, we have

\[
(I \otimes U_{\text{CPhase}}) |\Psi\rangle = \frac{1}{2^3} (e^{i\varphi_1} (|000\rangle + |011\rangle + |100\rangle + |111\rangle) + e^{i\varphi_2} (|011\rangle + |010\rangle + |101\rangle + |110\rangle)).
\]

In this case the phase factors \( e^{i\varphi_1} \) and \( e^{i\varphi_2} \) correspond to intersecting diagonal planes in the toric variety of a three-qubit state, see Figure 2.

### 3.3 Four qubit systems

Finally, we will discuss a four-qubit state \( |\Psi\rangle = \sum_{x_4} \alpha_{x_1x_2x_3x_4} |x_1x_2x_3x_4\rangle \). In this case we can also shows that the toric variety \( X_\Sigma = \mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1 \times \mathbb{CP}^1 \) is a four hypercube following the same procedure. Now, we consider the same geometric quantum gate \( U_{\text{CPhase}} \) acting on a pure four-qubit product state \( |\Psi\rangle = (H \otimes H \otimes H \otimes H) |0000\rangle = \frac{1}{\sqrt{2^4}} (|0000\rangle + |0001\rangle + \cdots + |1111\rangle) \), then we
Figure 3: Action of the holonomic controlled quantum gate on toric variety of a four-qubit state.

get

\[
(U_{CPhase} \otimes I \otimes I)\vert \Psi \rangle = \frac{1}{\sqrt{2^4}} (e^{i\phi_1} (\vert 0000 \rangle + \vert 0010 \rangle + \vert 0111 \rangle + \vert 1100 \rangle + \vert 1101 \rangle + \vert 1110 \rangle + \vert 0011 \rangle + \vert 1111 \rangle)
+ e^{i\phi_2} (\vert 0100 \rangle + \vert 0101 \rangle + \vert 0110 \rangle + \vert 0111 \rangle + \vert 1000 \rangle + \vert 1001 \rangle + \vert 1010 \rangle + \vert 1011 \rangle))
\]

Here we can also see that phase factors \( e^{i\phi_1} \) and \( e^{i\phi_2} \) correspond to intersecting three cubes in the toric variety of a given four-qubit state. We are not going to discuss the other possible cases \( (I \otimes U_{CPhase} \otimes I)\vert \Psi \rangle \) and \( (I \otimes I \otimes U_{CPhase})\vert \Psi \rangle \). But in these cases also we will have intersecting three cubes in side a 4-hypercube, see Figure 3.

In this paper we have visualized the action of a holonomic controlled quantum gate based on toric varieties. The result can be generalized into multi-qubit states. In example, the holonomic quantum gate defined by equation (3) acts on a multi-qubit state as a two intersecting \((m - 1)\)-hypercubes \((m \geq 4)\) in side a \(m\)-hypercube which is the toric variety of the system. There is also another interesting relation between toric varieties and measures of entanglement for multi-qubits states such as \(m\)-tangle. And these combinatorial structures possibly have more applications in quantum information and quantum computing that need to be explored.

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