An interaction-driven topological insulator in fermionic cold atoms on an optical lattice: A design with a density functional formalism

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We design an interaction-driven topological insulator for fermionic cold atoms in an optical lattice, i.e., we pose a question whether we can realize in a continuous space a spontaneous symmetry breaking induced by the inter-atom interaction into a topological Chern insulator. Such a state, sometimes called topological Mott insulator (TMI), has yet to be realized in solid-state systems, since this requires, in the tight-binding model, large off-site interactions on top of a small on-site interaction. Here we overcome the difficulty by introducing a spin-dependent potential, where a spin-selective occupation of fermions in $A$ and $B$ sublattices makes the on-site interaction absent, while a sizeable inter-site interaction is achieved by a shallow optical potential with a large overlap between neighboring Wannier orbitals. This puts the system away from the tight-binding model, so that we adopt the density functional theory for cold-atoms, here extended to accommodate non-collinear spin structures emerging in the topological regime, to quantitatively demonstrate the phase transition to TMI.

Introduction.— There are growing fascinations with topological phases in condensed-matter physics\cite{1,2}. The topological systems are characterized by various topological invariants\cite{3,4}, e.g., Chern numbers\cite{5,6}, as exemplified by the quantum Hall system, and $Z_2$ numbers\cite{7,8} for the topological insulator. Such topologically non-trivial phases emerge from one-body physics: A non-zero Chern number arises when the time-reversal symmetry is broken, e.g., by a strong external magnetic field. A non-zero $Z_2$ number can be realized without breaking the time-reversal symmetry, while typically a large spin-orbit coupling is required.

Recently, a class of topological phases that do not have such one-body terms but that can still be made topological has been proposed, in terms of spontaneous symmetry breaking due to many-body interactions\cite{9,10}. Such systems, so-called the “topological Mott insulator” (TMI), accompany interaction-driven loop currents, which act as an effective magnetic field or spin-orbit coupling. These have been suggested to arise from repulsive inter-site interactions, mainly from Fock-term contributions, and existence of such anomalous topological phases has been proposed for the tight-binding extended Hubbard model on various lattices\cite{9-13}. Although some experimental schemes or candidate materials are suggested\cite{11,13-17}, condensed-matter realization of such topological phases has yet to be achieved. The difficulty here is that the TMI often requires large inter-site interactions to trigger the desired symmetry breaking, while other interactions with closer sites should be suppressed to avoid competing instabilities.

On the other hand, ultracold atom systems, now attracting much interests as quantum simulators for various physical phenomena\cite{18,19}, provide clean and tunable systems as a platform for exploring exotic topological phenomena\cite{20-24}. Hence there is a chance of realizing the TMI in cold-atom systems, although some trick to encode large inter-site interaction is required: Typically we manipulate interactions by changing an $s$-wave scattering length via the Feshbach resonance\cite{24}, while it is short-ranged and thus usually provides too small inter-site interactions. Several studies\cite{9,11,14} propose schemes with other interactions, e.g., molecular dipole-dipole interactions, which are experimentally challenging and still lack quantitative estimates.

Thus we propose in the present paper an optical lattice system that is tuned to exhibit, in spite of employing an $s$-wave scattering, the interaction-driven topological phase transition, and demonstrate quantitatively the existence of a significant topological gap. In the proposal, a spin-dependent optical lattice potential, which constitutes a square optical lattice, resolves the primary difficulty: spin-dependent potential minima yield a spin-selective occupation of fermions in $A$ and $B$ sublattices, which changes the leading interaction derived from $s$-wave scattering from on-site interactions to nearest-neighbor (NN) interactions\cite{25}. First we discuss this trick in detail in terms of a tight-binding limit, and shall show that the limit corresponds to a checkerboard tight-binding model studied by Sun et al.\cite{10}. On the other hand, sufficient breadth of Wannier orbitals are required for the strong inter-site interaction, where the tight-binding picture is inadequate: Thus we employ the density-functional theory (DFT) for cold-atom systems\cite{26}, here extended to accommodate non-collinear spin-density functionals to describe topological spin structures. We confirm quantitatively that the proposed cold-atom system in a continuous space does indeed exhibit a topological phase transition from a semimetallic phase to a Chern insulator, as the repulsive interaction is increased.

Basic idea.— In the present work, we consider ultracold fermions of spin-$1/2$ in a continuous space in the presence of an optical lattice potential, with a Hamilto-
nian,
\[ \hat{H}_{\text{OL}} = \sum_{\sigma, \sigma'} \int d\mathbf{r} \hat{\psi}^\dagger_\sigma(\mathbf{r}) \left[ -\frac{\hbar^2}{2M} \delta_{\sigma\sigma'} \nabla^2 + V_{\sigma\sigma'}(\mathbf{r}) \right] \hat{\psi}_{\sigma}(\mathbf{r}) + \int d\mathbf{r} d\mathbf{r}' \hat{\psi}^\dagger_\uparrow(\mathbf{r}) \hat{\psi}^\dagger_\downarrow(\mathbf{r}') U(\mathbf{r} - \mathbf{r}') \hat{\psi}_\downarrow(\mathbf{r}') \hat{\psi}_\uparrow(\mathbf{r}). \]

Here \( \hat{\psi}_\sigma(\mathbf{r}) \) is the fermion field operator, \( V_{\sigma\sigma'}(\mathbf{r}) = W(\mathbf{r}) \delta_{\sigma\sigma'} + B(\mathbf{r}) \cdot s_{\sigma\sigma'} \) the spin-dependent optical lattice potential, consisting of a periodic potential \( W(\mathbf{r}) \) and a periodic Zeeman field \( B(\mathbf{r}) \), with the Pauli matrix \( s_{\sigma\sigma'} \), and \( U(\mathbf{r} - \mathbf{r}') \) the hard-core fermion-fermion interaction with an \( s \)-wave scattering length \( a_s \).

In tight-binding models\(^9\)\,-\(^{13}\), realization of the TMI is shown to require repulsive interactions extending to neighboring sites and a semimetallic band structure in the non-interacting case. However, even in cold atoms with a short-range interaction, we can still generate off-site interactions by employing a spin-dependent optical potential as discussed below. It turns out that a form of the potential,
\[ V_{\uparrow\uparrow}(\mathbf{r}) = -V_A \cos \frac{\pi(y - z)}{d} - V_B \cos \frac{\pi(y + z)}{d} + V_T(x), \]
\[ V_{\downarrow\downarrow}(\mathbf{r}) = V_B \cos \frac{\pi(y - z)}{d} + V_A \cos \frac{\pi(y + z)}{d} + V_T(x), \]
\[ V_{\uparrow\downarrow}(\mathbf{r}) = V_{\downarrow\uparrow}(\mathbf{r}) = V_C \left( \sin \frac{\pi y}{d} + \sin \frac{\pi z}{d} \right), \]
where \( d \) is a lattice constant, accomplishes the desired natures, semimetallic band structure and sizeable NN interactions. Here \( V_T(x) \) is a trapping potential along \( x \), taken to have a form \( V_T(x) = V_x \sin^2(\pi x/2d) \) with a cutoff of \( x \) to \([-d, d]\) for simplicity. Spatial patterns at \( x = 0 \) are shown in Fig.1(a-c).

Although we work in a continuous space for cold atoms, in order to explain why we adopt the form of the potential Eqs.(2-4), we can first have an intuitive look at what the system would look like in the tight-binding limit. Let us first focus on the spin-diagonal components, where the optical lattice sites (the minima of \( V_{\uparrow\uparrow}(\mathbf{r}) \) and \( V_{\downarrow\downarrow}(\mathbf{r}) \)) constitute square lattices (see Fig.1(a,b)). Because the positions of the minima are here spin-dependent unlike ordinary lattice models, fermions with spin-up occupy \( A \) sublattice sites while spin-down \( B \) sites. We can thus regard the system as that of spinless fermions, if we translate the spin into the sublattice index.

In this situation on-site interactions are absent, so that the leading interaction is the NN (density-density) one, which is caused by overlapping tails of neighboring Wannier orbitals. While this idea of encoding NN interactions is adopted from Ref.25, where a kagomé lattice with NN interactions is described in a tripod scheme of resonant transitions between atomic levels (atoms in three levels correspond to those in the three sublattices in kagomé), the scheme we adopt is fairly different as it employs off-resonant lasers\(^{27}\), although we finally get a spin-dependent potential\(^{28,29}\).

In the situation so far, NN hopping is absent in the model. We can induce it by adding the spin-off-diagonal part, Eq.(4), as a perturbation \( V_C \ll V_A, V_B \), which indeed induces transitions between spin-up (\( A \) site) and spin-down (\( B \) orbitals. Since Eq.(4) is a Zeeman field along \( x \)-axis in a staggered form, the hopping amplitude takes a real value with alternating signs (see Fig.1(c)).

Then a corresponding tight-binding model with NN interactions can be depicted as shown in Fig.1(d), where \( t_A \) (\( t_B \)) denotes hopping amplitude from tunneling through potential barriers in Eqs.(2,3), while \( t \) denotes those from the off-diagonal part, Eq.(4) (all parameters are taken as positive). We can make the tight-binding model simpler by performing a unitary transformation\(^{27}\) which puts the model into the one depicted in Fig.1(e). The transformed model is equivalent to an extended Hubbard model for spinless fermions on a checkerboard lattice, with alternating signs for the second-neighbor hoppings. The tight-binding model is semimetallic at half-filling in the non-interacting case, and theoretically reported to have a non-zero Chern number for infinitesimal NN repulsions due to a spontaneous breaking of the time-reversal symmetry\(^{10}\).

Hence we can expect that the Hamiltonian in the original continuous space \( \hat{H}_{\text{OL}} \) may realize a Chern insulating state driven by repulsive interactions.

In passing, we remark on the symmetry of the present system: The time reversal symmetry, defined as a physical symmetry (which inverts spin directions), is explicitly
broken by the Zeeman fields \((B_z^2)\). However, \(B_x\)-component is absent so that the Hamiltonian is real, and the system has a symmetry against the complex conjugation. The complex conjugation corresponds to the time-reversal symmetry in the spinless system (which does not invert sublattice indices), where the breaking of that symmetry signifies the topological phase transition\(^{35}\). Hence the ordered phase of the present system should accompany a (staggered) magnetization along \(y\)-axis inducing a magnetic field along \(y\)-axis, i.e., the imaginary spin-offdiagonal part of the mean-field potential\(^{27}\). In terms of the spinless tight-binding model, that translates into a complex NN hopping amplitude and is consistent with spontaneous loop currents in the topological phase\(^{10}\).

In short, the present idea is summarized as follows: (i) By employing the spin-dependent potential minima, the \(s\)-wave scattering translates into the NN interaction of a spinless square lattice system, which is a key ingredient of the TMI. (ii) By the staggered Zeeman field along \(x\), a checkerboard pattern of the second-neighbor hopping with alternating signs is realized (after a unitary transformation), which accomplish another important feature, a semimetallic behavior for the non-interacting case.

**Formulation.** — Now we go back to the original problem in the continuous space. The whole point is that, although the reduced tight-binding model has a desired form, there is no guarantee that this reduction is adequate. In fact this becomes especially serious, since we have to employ, for a realization of a TMI, a shallow lattice potential to enhance the NN interaction from a large overlap between neighboring Wannier orbitals. The shallow potential will then also enhance the longer-distance hoppings and effects of excited bands, which may well degrade the desired situation. This is precisely why we have to employ the DFT to directly solve \(\tilde{H}_{\text{OL}}\). The exchange-correlation functional for ultracold fermions within the local spin-density approximation (LSDA) has been formulated in a pioneering work by Ma et al.\(^{26}\), where it is reported that LSDA quantitatively reproduces the total energy of the shallow optical lattice system as estimated from the diffusion Monte Carlo method.

If we apply the DFT to the present system, however, the potential \(V_{\sigma\sigma'}(r)\) is not spin-diagonal, so that we have to deal with *non-collinear* spin DFT\(^{30,31}\), where the spatial pattern of spin directions are allowed to have general structures. We adopt here the local approximation, i.e., we employ the Hartree exchange-correlation functional of the form \(E_{\text{HXC}}[n(r), \langle m(r) \rangle]\), where \(n(r)\) is the atomic density and \(\langle m(r) \rangle\) the magnetization. It can be obtained from the collinear functional given in Ref.\(^{26}\), \(E_{\text{HXC}}[n_1(r), n_1(r)]\), by replacing the collinear spin density \(n_1(r)\) with \(n(r) \pm \langle m(r) \rangle\), as is done in electron systems\(^{31}\). The resulting Kohn-Sham potential then reads \(V_{\sigma\sigma'}(r) + \delta E_{\text{HXC}}/\delta n(r) \delta_{\sigma\sigma'} + \delta E_{\text{HXC}}/\delta \langle m(r) \rangle \langle m(r) \rangle^{-1} \langle m(r) \rangle \cdot s_{\sigma\sigma'}\).

We can mention that the non-collinear LSDA formalism should be particularly appropriate to the TMI in cold-atom systems: While for the long-ranged Coulombic interaction, the Fock term, which is essential for the topological transition, is non-local and may be difficult to capture with LDA, the term in the present system can be expressed explicitly as the non-collinear LSDA functional\(^{30}\). Further, the topological phase suggested in the checkerboard lattice is estimated to occur in a weakly-correlated regime, where the DFT should be adequate.

**Results.** — Now we present the density-functional results. We consider the periodic boundary condition to employ Bloch wavefunctions. The number of \(k\)-points is taken as \(1 \times 32 \times 32\), and each Bloch function is represented by \(9 \times 21 \times 21\) plane waves. We set the parameters of the lattice potential as \(V_A = 0.8\), \(V_B = 1.2\), \(V_C = 0.25\) and \(V_z = 10\), all in units of \(E_R = h^2 \pi^2/(4M d^2)\).

In the non-interacting case, \(a_s = 0\), we have a band structure as depicted in Fig.\(2(a)\). Due to the staggered fields, a unit cell contains four lattice sites (two A and two B sites), so that the bottom four bands correspond to those in the tight-binding model. Two dispersive bands (second and third from bottom) touch with each other at the corner of the Brillouin zone, which is called a quadratic band-crossing point (QBCP), associated with the symmetry against complex conjugation. At half-filling, the lowest two bands are fully-occupied, and the system is a semimetal. When the interaction is switched on in Fig.\(2(b)\), however, we can see that a gap opens at QBCP, where a spontaneously broken symmetry makes the system an insulator. The size of the gap as a function of \(a_s\) in Fig.\(2(d)\) shows the gap grows with the interaction.
FIG. 3: (Color online) (a,b) Spatial patterns of the magnetization (arrows) for the non-interacting system (a) and for an interacting one with \( a_s = 0.25d \) (b). The color represents the \( z \)-component magnetization up (red) or down (blue). The direction of the arrows are rotated by \( \pi/2 \) around \( z \)-axis (a) and \( y \)-axis (b) for convenience of viewing. Direction of spins as we sweep the unit cell is depicted in the top insets. (c) The \( y \)-component of the magnetization (order parameter for the topological phase) for the state depicted in (b). (d) Spatial pattern of the atomic area density for the case Fig.2(e) with \( a_s = 0.2d \). (e) The same as (d) for a larger \( a_s = 0.27d \), for which a site-nematic order coexists with a topological gap.

with a threshold behavior.

The gap is indeed a topological gap, which is verified from the Chern number. Figure 2(e) shows the Chern density for the lowest two bands, where we can see conspicuous magnitudes around the QBCPs. The Chern number as an integrated value turns out to be \(-1\), and we can conclude that the system is a Chern insulator driven by spontaneous symmetry breaking from a semimetallic phase.

The spin structure of the system in continuous space is depicted in Fig.3(a-c). As we have noted above in the discussion on the symmetry, the order parameter for the present system is the staggered magnetization along \( y \)-axis. Fig.3(a) shows the non-interacting case, where the spatial spin structure comes from the Zeeman field in the \( x-z \) plane, so that the \( y \)-component is trivially absent. In the interacting case, by sharp constrast, \( y \)-component magnetization spontaneously emerges, as most clearly seen in the Bloch sphere inset, and we can identify the insulator to be topological. Accordingly, the spatial behavior of the spins in the periodic system change from two-dimensional vortices to three-dimensional ones (Fig.3(b,c)). Hence we conclude that the designed system does indeed realize, in the continuous space, the mechanism for the emergence of the topological phase conceived for tight-binding models.

Figure 3(c) indicates that the order parameter has large amplitudes around \( (y, z) = (\pm d/2, \pm d/2) \), where the atomic density gives the upper limit for \( m_y \). Hence we can enlarge the topological gap by enhancing the density around these positions. This can be achieved by controlling the anisotropy of the potential barrier separating adjacent A sites (or B sites) (i.e., reducing \( V_A - V_B \)), as shown in Fig.2(e-c), where \( V_A - V_B \) has an effect of increasing the gap. Figure 3(d) shows that the density around \( (y, z) = (\pm d/2, \pm d/2) \) is sizeable.

We can notice a round-off of the gap at stronger interactions in Fig.2(e), which we identify to come from another phase transition into a site-nematic order (i.e., a spontaneous imbalance of the filling between the two sublattices), which is reported for a tight-binding model in Ref.10. Coexistence of the topological and nematic orders occurs above \( a_s \sim 0.23d \) in the present setup (see Fig.3(e)).

**Discussions.**—We should mention we have neglected some factors that may work against the topological transition: (i) thermal fluctuations, (ii) a Zeeman splitting accompanying the Feshbach resonance, (iii) three-body scattering processes, which induce an instability toward a dimerized phase and (iv) non-local/dynamical correlations. Let us discuss (i)/(ii) in detail. The critical temperature should have an order of magnitude of the topological gap, which is scaled by the bandwidth. The bandwidth is larger in shallower optical lattices, so that we can expect that the critical temperature can be made accessible. If we introduce a uniform Zeeman field, \( V_M \frac{s_x}{n_r}/2 \), we can estimate the upper bound of the magnetic field to be \( V_M - 2C \), at which the Zeeman splitting makes \( V_M(r) \) non-staggered. As shown in Fig.2(f), the topological order, although reduced, still remains in a magnetic field of \( V_M = 0.8C \).

As for experimental detections, the phase transition can be detected from the behavior of the magnetization. There is also a theoretical suggestion to detect the Chern number. Also, the chiral edge states, the inherent topological feature, should emerge along the boundaries of the phase domains and edges of the trapped system, where the latter may be difficult to distinguish from the metallic states arising from the depleted region in the trapped system.

To summarize, the present design is the first example of a realistic model in a continuous space that exhibits an interaction-induced spontaneous symmetry breaking toward the Chern insulator. Compared to the other proposals on the TMI in cold-atom systems, our proposal has some advantages: it employs only a simple and established scheme for cold atoms, i.e., the s-wave Feshbach resonance and the electric dipole transition between hyperfine states induced by off-resonant...
lasers, along with shallow lattice potentials, which tend to enhance the transition temperature.

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35 Namely, the present many-body Hamiltonian has an anti-unitarity symmetry $K$ with $K^2 = 1$ and belongs to class AI, where topological phases are absent in two-dimensional non-interacting systems. Nevertheless, many-body effects allow the system to be a Chern insulator, because the Kohn-Sham Hamiltonian of the system after the spontaneous symmetry breaking can be class A.
36 The Hartree-Fock energy, $(\pi^2 a_s / M) \int \, d\mathbf{r} (n^2 - m^2)$, for contact interactions indeed coincides with that for hard-core interactions in the dilute limit.
37 We have calculated the Chern number as an integral of the Chern density for the lowest two bands, $-2Im \sum_{n=1,2} \int d\mathbf{k} \int d\mathbf{r} [\partial u^{*}_{nk}(\mathbf{r}) / \partial k_x] [\partial u_{nk}(\mathbf{r}) / \partial k_y]$, since they cross with each other.

Appendix A: Implementation of the lattice potential

Here we indicate how we can realize the lattice potential given in Eqs.(2-4). We consider a situation where the laser electric fields $\sum_\omega \mathbf{E}(\omega, \mathbf{r}) e^{i\omega t} + c.c.$ are imposed to atoms with a hyperfine structure. If all the laser frequencies ($\omega$'s) are off-resonant with the hyperfine splitting, the effect of the laser field is represented as additional terms in the Hamiltonian$^{1,2}$, a potential $\propto |\mathbf{E}(\omega, \mathbf{r})|^2$ and a Zeeman term $\propto -i\mathbf{E}^*(\omega, \mathbf{r}) \times \mathbf{E}(\omega, \mathbf{r}) \cdot \mathbf{\hat{F}}$ with $\omega$-dependent coefficients, where $\mathbf{\hat{F}}$ is the total angular momentum operator. For the present system we focus on $F = 1/2$ multiplets (then $\mathbf{\hat{F}} = s_{\sigma'}$).

First, we consider a pair of confronting circularly-polarized lasers along $z$-axis, and one linearly-polarized laser along $y$-axis, all with a frequency $\omega$,

$$
\mathbf{E}(\omega, \mathbf{r}) \propto e^{\pi i y/d} e_x + \sum_{\xi = \pm} \xi e^{\pi i z/d} e_x + i \xi e_y \sqrt{2},
$$

(A1)
which can be shown to realize the $W$- and $B^z$-components of $V_{\sigma\sigma}(r)$:

$$|E(\omega, r)|^2 \propto 3 + \sqrt{2} \cos \frac{\pi(y - z)}{d} - \sqrt{2} \cos \frac{\pi(y + z)}{d},$$  \hspace{2cm} (A2)$$

$$-iE^*(\omega, r) \times E(\omega, r) \propto \left( \cos \frac{\pi(y - z)}{d} + \cos \frac{\pi(y + z)}{d} \right) e_z.$$  \hspace{2cm} (A3)

We further superpose four linearly-polarized lasers of a frequency $\omega' = \omega/\sqrt{2}$ with a spatial part\(^4\),

$$E(\omega', r) \propto \sum_{\xi = \pm} \left[ e^{\xi \pi i(y - z)/2d} (e_y + e_z) - \xi e^{\xi \pi i(y + z)/2d} (e_y - e_z) \right],$$  \hspace{2cm} (A4)

which realizes the $W$- and $B^z$-components:

$$|E(\omega', r)|^2 \propto 2 + \cos \frac{\pi(y - z)}{d} - \cos \frac{\pi(y + z)}{d},$$  \hspace{2cm} (A5)$$

$$-iE^*(\omega', r) \times E(\omega', r) \propto \left( \sin \frac{\pi y}{d} + \sin \frac{\pi z}{d} \right) e_x.$$  \hspace{2cm} (A6)

If we combine laser fields in Eqs.(A1,A4), we end up with the desired Hamiltonian, $\hat{H}_{\text{OL}}$. Schematic pictures of these configurations are given in Fig.4(a,b).

While we can change the strengths of the fields Eq.(A1) and (A4), we have three parameters, $V_A$, $V_B$, and $V_C$, to adjust. If the tuning of the two strengths does not attain the desired parameters, we can introduce additional, linearly-polarized lasers, e.g.,

$$E(\omega'', r) \propto \sum_{\xi = \pm} \left[ e^{\xi \pi i(x/2+y+z)/2d} + \xi e^{\xi \pi i(x/2-y+z)/2d} \right] (e_z - 2e_x),$$  \hspace{2cm} (A7)$$

$$|E(\omega'', r)|^2 \propto 2 - \cos \frac{\pi(y - z - x/2)}{d} + \cos \frac{\pi(y + z + x/2)}{d},$$  \hspace{2cm} (A8)$$

$$-iE^*(\omega'', r) \times E(\omega'', r) = 0.$$  \hspace{2cm} (A9)

**Appendix B: Equivalence of the tight-binding limit to the checkerboard lattice model**

Here we describe how the present Hamiltonian $\hat{H}_{\text{OL}}$ is related to the checkerboard lattice model\(^3\) in the tight-binding limit. As we have discussed in the main text, the tight-binding limit is depicted in Fig.1(d), whose Hamiltonian would read

$$\hat{H} = -\sum_{i,j \in A} (t_A a_{i+1,j-1}^\dagger a_{i,j} + \text{h.c.}) - \sum_{i,j \in B} (t_B b_{i+1,j-1}^\dagger b_{i,j} + \text{h.c.}) - t \sum_{i,j \in A} \sum_{\xi = \pm 1} \xi(-1)^j (b_{i+\xi,j}^\dagger b_{i,j} + \text{h.c.}) + \text{h.c.} + V \sum_{i,j \in A} \sum_{i',j' \in B} a_{i,j}^\dagger b_{i',j'}^\dagger b_{i',j} a_{i,j},$$  \hspace{2cm} (B1)
where \( \hat{a}_{i,j} \) annihilates a spin-up (A site) fermion at \((y,z) = (i,j)\) with \(d = 1\), while \( \hat{b}_{i,j} \) is for a spin-down (B) fermion.

In order to relate Eq.(B1) to the checkerboard lattice, we can perform a unitary transformation,
\[
\hat{a}'_{i,j} = (-1)^{(i-j)/2}\hat{a}_{i,j},
\]
\[
\hat{b}'_{i,j} = (-1)^{(i+j-1)/2}\hat{b}_{i,j}.
\]

Then the transformed model reads
\[
\hat{H} = -\sum_{i,j\in A} (-t_A\hat{a}'_{i+1,j}^\dagger + t_B\hat{a}'_{i+1,j+1}^\dagger)\hat{a}_{i,j} + \text{h.c.} - \sum_{i,j\in B} (t_B\hat{b}'_{i,j}^\dagger - t_A\hat{b}'_{i,j+1}^\dagger)\hat{b}_{i,j} + \text{h.c.} - t\sum_{i,j\in A} \sum_{\xi = \pm 1} (\hat{b}_{i+1,j+\xi}^\dagger + \hat{b}_{i,j+\xi}^\dagger)\hat{a}_{i,j} + \text{h.c.} + V\sum_{i,j\in A} \sum_{i',j'\in B} \hat{a}_{i,j}^\dagger \hat{b}_{i',j'}^\dagger \hat{b}_{i',j'} \hat{a}_{i,j},
\]
which precisely coincides with the spinless checkerboard lattice model, Fig.1(e). In the derivation we have used that \((-1)^{i-j} = 1\) for \((i,j) \in A\).

**Appendix C: Order parameter in the mean-field description**

Here we discuss the order parameter of the optical lattice system from the corresponding mean-field description of the tight-binding checkerboard lattice model. The order parameter of the Chern insulating phase in the checkerboard (which precisely coincides with the spinless checkerboard lattice model, Fig.1(e). In the derivation we have used that \((-1)^{i-j} = 1\) for \((i,j) \in A\).

\[
\langle \hat{a}_{i,j}^\dagger \hat{b}_{i',j'}^\dagger \rangle = \begin{cases} i\phi & (i',j') = (i \pm 1,j), \\ -i\phi & (i',j') = (i,j \pm 1). \end{cases}
\]

We can readily go back to the tight-binding description of the optical lattice with Eqs.(B2,B3), and the order parameter emerges as
\[
\langle \hat{a}_{i,j}^\dagger \hat{b}_{i',j'}^\dagger \rangle = \begin{cases} i\phi(-1)^j & (i',j') = (i + 1,j), (i,j - 1), \\ -i\phi(-1)^j & (i',j') = (i,j + 1), (i - 1,j). \end{cases}
\]

While this is the order parameter for the optical lattice system expressed in terms of the tight-binding picture, we can introduce an alternate, basis-independent observable appropriate to continuous problems. We can start with an observation that the Fock term corresponding to Eq.(C2) in the mean-field decoupling is
\[
-iV\phi \sum_{i,j\in A} \sum_{\xi = \pm 1} \xi(-1)^j(\hat{b}_{i+1,j+\xi}^\dagger - \hat{b}_{i,j+\xi}^\dagger)\hat{a}_{i,j} + \text{h.c.},
\]
as depicted in Fig.4(c). As explained in the main text, NN hopping with real amplitudes is obtained by the \(B^x\)-component, while that with imaginary amplitudes is given by the \(B^y\)-component, since it realizes spin-offdiagonal potential \(V_{\pm 1} = -V_{\mp 1} = iB^y\). The staggered pattern in Eq.(C3) can be realized by a sinusoidal
\[
B^y(r) \propto \sin \frac{\pi y}{d} - \sin \frac{\pi z}{d},
\]
which enables us to define the order parameter as the staggered magnetization along \(y\)-axis, proportional to Eq.(C4):
\[
\sum_{\sigma,\sigma'} \int dr \psi_{\sigma}^\dagger(r) \left( \sin \frac{\pi z}{d} - \sin \frac{\pi y}{d} \right) s_{\sigma\sigma'}^y \psi_{\sigma'}(r).
\]

**Appendix D: Enhancement of the order parameter by a lattice anisotropy**

As discussed in the main text, the atomic density around \((y,z) = (\mp d/2, \pm d/2)\) gives the upper limit for the order parameter, and can be controlled by lattice anisotropy. It is clearly described in Fig.5, where \(A + B\) and \(C\) are fixed to \(2E_R\) and \(0.25E_R\), respectively. The phase boundary is here displayed as the place where the topological gap opens with vanishing density of states, while the spontaneous magnetization emerges prior to that to open a gap at the QBCP with an initially overlapping bands for smaller repulsions.
FIG. 5: (a) A phase diagram against $A - B$ and $a_s$. SM indicates a semimetallic phase, and TMI a insulating phase with a spontaneous topological gap. (b-d) The atomic area density of the non-interacting system for various value of $A - B$.

\( \text{SM} \)

\( \text{TMI} \)

\( (A - B)/E_R \)

\( a_s/d \)

\( \text{bandgap } [E_R] \)

\( n_{2D}(r)d^2 \)

\( y/d \)

\( z/d \)

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4. In place of $\omega/\sqrt{2}$ we can employ $\omega\sqrt{2 + p^2}/2$, for which the laser field is obtained by substituting $y \rightarrow y + px$ in Eq.(A4). It has the desired form on the system plane, $x = 0$. 