On electromagnetic momentum of an electric dipole in a magnetic field

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Abstract

The total linear electromagnetic field momentum $P_{em}$ of a stationary electric dipole $p$ in a static magnetic field $B$ is considered. The expression $P_{em} = \frac{1}{2} B \times p$, which has previously reported to hold in all static magnetic field situations, is not valid in general. The contribution of the electromagnetic momentum of the fringing fields of the dipole is discussed. When either the static magnetic field or the electric dipole moment is changed, the mechanical impulse on the system equals $-\Delta P_{em}$. The role of “hidden momentum” is discussed.
I. INTRODUCTION

In classical electromagnetism, electric ($E$) and magnetic ($B$) fields store linear momentum in the form of electromagnetic momentum density $\epsilon_0 (E \times B)$ [SI units are used throughout this paper]. It is therefore possible for static electric and magnetic fields to store electromagnetic momentum (although the total momentum of the system must zero, as is discussed later). To illustrate the consequences of the electromagnetic momentum on a specific system, several authors have studied the electromagnetic momentum due to electric and magnetic dipoles in the presence of static magnetic and electric fields.

The total electromagnetic momentum of a system is the integral of the electromagnetic momentum density,

$$P_{em} = \epsilon_0 \int E(r) \times B(r) \, dr \quad (1)$$

For a static electric dipole in the presence of a static magnetic field and a static magnetic dipole in the presence of a static electric field, it has been reported (see Ref. and references therein) that

$$P_{em} = \frac{1}{2} B \times p, \quad (2a)$$

$$P_{em} = \frac{1}{c^2} E \times m, \quad (2b)$$

where $c$ is the speed of light, $p$ ($m$) is the electric (magnetic) dipole moment and $B$ ($E$) is the magnetic (electric) field at the position of dipole. However, these two results are incompatible with each other, as illustrated in the following example.

Assume there is a magnetic moment $m = m\hat{z}$ at the origin and an electric dipole $p = p\hat{x}$ at $R\hat{x}$ (where carets indicate unit vectors), as shown in Fig. 1. The magnetic field at a displacement $r \neq 0$ from the magnetic moment is (see e.g., Ref. , p. 255)

$$B = \frac{\mu_0}{4\pi R^3} [3(m \cdot \hat{r})\hat{r} - m]. \quad (3)$$

The electric field at a displacement $r$ from the electric dipole is obtained by replacing $m$ by $p$ and $\mu_0$ by $\epsilon_0^{-1}$ in the above equation. The magnetic field at $R\hat{x}$ due to the magnetic dipole is $B(r = \hat{x}R) = -\mu_0 m/(4\pi R^3) \hat{z}$, and the electric field at the origin due to the electric dipole is $E(0) = p/(2\pi\epsilon_0 R^3) \hat{x}$. These together with Eqs. (2a), (2b) and $c^{-2} = \mu_0\epsilon_0$ give

From Eq. (2a): $P_{em} = -\frac{1}{8} \frac{\mu_0 mp}{\pi R^3} \hat{y}; \quad (4a)$

From Eq. (2b): $P_{em} = -\frac{1}{2} \frac{\mu_0 mp}{\pi R^3} \hat{y}. \quad (4b)$
Since $P_{em}$ evaluated using Eqs. (2a) and (2b) do not agree, at least one of these expressions is not valid in general. It turns out that Eq. (2a) was derived for locally uniform magnetic fields produced by a long solenoid and a spinning sphere with a uniform surface charge density, and in Ref. ? , Griffiths inadvertently omitted to mention that the result is only valid for locally uniform magnetic fields. However, we shall see in this paper that Eq. (2a) is not in general valid even in cases where the magnetic field is locally uniform, and that $P_{em}$ cannot in general be written in terms of $p$ and $B$ at the position of the dipole.

We shall also investigate the impulse imparted on an electric dipole in a magnetic field when either the electric dipole moment or magnetic field is reduced to zero. We shall see that the total electromagnetic momentum stored in the system is in general equal to the impulse that is imparted to the system in either case, and it is not necessary to take into account hidden momentum to obtain conservation of momentum in these systems.

II. $P_{em}$ FOR STATIONARY ELECTRIC (MAGNETIC) DIPOLES IN STATIC MAGNETIC (ELECTRIC) FIELDS

The total electromagnetic momentum $P_{em}$ for the case in which the electric and magnetic fields $E$ and $B$ are due to stationary charges distribution $\rho(r)$ and steady current distributions $J(r)$ which are local (do not extend to infinity), can also be expressed as

$$P_{em} = \frac{1}{c^2} \int V(r) J(r) \, dr,$$  \hspace{1cm} (5a)

where $V(r)$ is the scalar potential,

$$V(r) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(r')}{|r-r'|} \, dr'.$$  \hspace{1cm} (5b)

In Appendix [A] the derivation of Eq. (5a) is reproduced, and circumstances in which the locality of $J$ can be relaxed are discussed.

The electromagnetic momentum can alternatively be expressed as

$$P_{em} = \int \rho(r) A(r) \, dr.$$  \hspace{1cm} (6a)

where $A(r)$ is the vector potential in the Coulomb gauge ($\nabla \cdot A = 0$) for a static current source,

$$A(r) = \frac{\mu_0}{4\pi} \int \frac{J(r')}{|r-r'|} \, dr'.$$  \hspace{1cm} (6b)
It is easy to see that both expressions Eqs. (5a) and (6a) for $P_{em}$ are equivalent, by substituting Eq. (5b) into (5a) and Eq. (6b) into (6a), and using $c^{-2} = \epsilon_0 \mu_0$.

### A. Magnetic dipole in a static electric field

First, we confirm that Eq. (2b) is valid for a magnetic dipole in the presence of a static electric field. The vector potential for a magnetic dipole at $r_m$ in the Coulomb gauge is

$$A = \mu_0 m \times (r - r_m)/(4\pi|r - r_m|^3),$$

so Eq. (6a) gives

$$P_{em} = \mu_0 \int d\mathbf{r} \rho(\mathbf{r}) \frac{m \times (r - r_m)}{4\pi|r - r_m|^3} = \mu_0 \epsilon_0 \left[ -\int d\mathbf{r} \frac{\rho(\mathbf{r}) (r_m - r)}{4\pi\epsilon_0|r - r_m|^3} \right] \times m. \quad (7)$$

Since the term in the square parentheses is the electric field at the position of the magnetic dipole and $\mu_0 \epsilon_0 = c^{-2}$, this gives Eq. (2b).

### B. Electric dipole in a static magnetic field

We now present four expressions for the total electromagnetic field momentum for an electric dipole $p$ in the presence of a static current density $J(\mathbf{r})$ which produces a static magnetic field $B(\mathbf{r})$ and a corresponding vector potential in the Coulomb gauge $A(\mathbf{r})$. In these expressions, the gradients and the magnetic fields are evaluated at the position $\mathbf{r}$ of the electric dipole. The expressions are

$$P_{em} = (p \cdot \nabla)A(\mathbf{r}) \quad (8a)$$

$$= -\frac{\mu_0}{4\pi} \int \frac{[p \cdot (\mathbf{r} - \mathbf{r}')] J(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \, d\mathbf{r}' \quad (8b)$$

$$= \mathbf{B} \times \mathbf{p} + \nabla(p \cdot A) \quad (8c)$$

$$= \mathbf{B} \times \mathbf{p} - \frac{\mu_0}{4\pi} \int \frac{[p \cdot J(\mathbf{r}')]}{|\mathbf{r} - \mathbf{r}'|^3} \, d\mathbf{r}' \quad (8d)$$

Expression (8a), which was given in Ref. ?, can be derived by taking a point dipole $\mathbf{p}$ at a position $\mathbf{r}$ to be the limit of point charges $-q$ at $\mathbf{r}$ and $q$ at $\mathbf{r} + \mathbf{1}$ in which $|\mathbf{l}| \equiv l \to 0$ and $q \to \infty$, with the product $\mathbf{l}q = \mathbf{p}$ being finite. Using this in Eq. (6a) and the expansion $A(\mathbf{r} + \mathbf{1}) \approx A(\mathbf{r}) + (\mathbf{l} \cdot \nabla)A(\mathbf{r})$, results in $P_{em} = \lim_{q \to p} q[A(\mathbf{r} + \mathbf{1}) - A(\mathbf{r})] = \lim_{q \to p} (q\mathbf{l} \cdot \nabla)A$, which gives expression (8a).
Combining Eq. (8a) with Eq. (6b) and using the relationship (where $\nabla$ is the gradient with respect the variable $r$)

$$\nabla \frac{1}{|r - r'|} = -\frac{r - r'}{|r - r'|^3},$$

(9)

[or alternatively using the scalar potential for a point dipole, $V(r) = p \cdot r/(4\pi \varepsilon_0 r^3)$, in Eq. (5a)] yields expression (8b).

Expression (8c) is obtained by using the vector identity [see, e.g., Refs. ? or ? ]

$$\nabla (p \cdot A) = p \times (\nabla \times A) + A \times (\nabla \times p) + (p \cdot \nabla)A + (A \cdot \nabla)p,$$

together with $\nabla \times A = B$ and $p$ not having any spatial dependence, gives expression (8c). Finally, using Eq. (6b) in Eq. (8c) and utilizing Eq. (9) yields expression (8d).

The differences between the forms of the expressions for $P_{em}$ for a magnetic dipole in an electric field and for an electric dipole in a magnetic field are due to the differences between the sources of static electric and magnetic field. Static magnetic fields are caused by electric currents and static electric fields are caused by electric charges. Therefore, despite the fact that the magnetic and electric fields for magnetic and electric dipoles have the same form (away from the dipoles themselves) the expressions for $P_{em}$ are different. For more details, see Ref. ?.

III. $P_{em}$ FOR VARIOUS CASES OF ELECTRIC DIPOLES IN A MAGNETIC FIELD

We use the expressions Eqs. (8a) – (8d) to evaluate $P_{em}$ for a static electric dipole in a static magnetic field. We obtain the correct result for the magnetic and dipole configuration shown in Fig. 1. We then consider cases of an electric dipole in a uniform magnetic field produced by a spinning uniformly charged spherical shell and a uniform circular solenoid, where we reproduce the result $\frac{1}{2}B \times p.$ ?? We see then that in the case of a uniform field produced by electric currents in two parallel plates, $P_{em}$ is not $\frac{1}{2}B \times p$. A case where $P_{em} \neq \frac{1}{2}B \times p$ for purely local currents is also discussed. This is followed by a discussion of the reasons for the differences in $P_{em}$ in these cases.
A. Configuration shown in Fig. 1

We now re-evaluate $P_{em}$ for the example shown in Fig. 1. The vector potential for a magnetic dipole $m = m\hat{z}$ at the origin in the Coulomb gauge, in spherical and cartesian coordinates, is\(^7\)

$$\mathbf{A}_m(r) = \frac{\mu_0 m \times r}{4\pi r^3} = \frac{\mu_0 m \sin \theta}{4\pi r^2} \mathbf{\hat{\phi}} = \frac{\mu_0 m}{4\pi} \frac{-y\mathbf{\hat{x}} + x\mathbf{\hat{y}}}{(x^2 + y^2 + z^2)^{3/2}}. \quad (10)$$

Therefore, for an electric dipole $p = p\mathbf{\hat{x}}$ at position $R\mathbf{\hat{x}}$ [or in spherical coordinates, $r = R$, $\theta = \pi/2$ and $\phi = 0$] in the presence of the magnetic dipole at the origin, Eq. (8a) gives

$$P_{em} = p \left[ \frac{\partial}{\partial r} \left( \frac{\mu_0 m \sin \theta}{4\pi r^2} \right) \right] \mathbf{\hat{\phi}} = -\frac{\mu_0 mp}{2\pi R^3} \mathbf{\hat{\phi}}, \quad (11)$$

which agrees with the result for $P_{em}$ using Eq. (2b), since $\mathbf{\hat{\phi}} = \mathbf{\hat{y}}$ when $\phi = 0$. In this particular case, $P_{em} = 2B \times p$.

Alternatively, we can also use Eq. (8c). In this case, it is more convenient to us the Cartesian-coordinate expression for $\mathbf{A}_m(r)$, which gives

$$\left[ \nabla (p \cdot \mathbf{A}_m) \right]_{(R, 0, 0)} = \frac{\mu_0 mp}{4\pi} \left[ \frac{-y}{(x^2 + y^2 + z^2)^{3/2}} \right]_{(R, 0, 0)} = -\frac{\mu_0 mp}{4\pi R^3} \mathbf{\hat{y}} = B \times p. \quad (12)$$

Using this in Eq. (8c) also gives $P_{em} = 2B \times p$.

B. Constant magnetic field inside uniformly charged spinning spherical shell

McDonald\(^7\) and Babson et al.\(^7\) evaluated Eq. (1) for a spherical capacitor and an electric dipole $p$, respectively, at the center of a uniformly charged spinning spherical shell (or equivalently a uniformly magnetized sphere) producing a uniform field $B$ inside the sphere. They obtained the result $P_{em} = \frac{1}{2}B \times p$. We will see that this result is reproduced by the expressions given in the previous section, and is in fact independent of the position of the dipole (or equivalently capacitor) within the sphere.

The vector potential in the Coulomb gauge is\(^7\) $\mathbf{A}_{in}(r) = \frac{1}{3} \mu_0 \sigma (\mathbf{\omega} \times r)$ and this gives a uniform magnetic field inside the sphere of $B = \frac{2}{3} \mu_0 \sigma R \mathbf{\omega}$. Thus, for this case the vector potential inside the sphere can be written as

$$\mathbf{A}_{in}(r) = \frac{1}{2} B \times r. \quad (13)$$
Since \( p \cdot A_{\text{in}} = \frac{1}{2} p \cdot (B \times r) = -\frac{1}{2} r \cdot (B \times p) \), Eq. (8c) gives

\[
P_{\text{em}} = B \times p + \frac{1}{2} \nabla (p \cdot (B \times r)) = \frac{1}{2} B \times p,
\]

independent of the position of the electric dipole inside the sphere. By linear superposition, this result holds for any distribution of dipoles within the sphere. This explains why in Ref. ? obtained result Eq. (14) for the electromagnetic momentum of a spherical-cap capacitor inside a uniformly magnetized sphere (which is equivalent to a rotating uniformly charged spherical surface), where \( p \) is the total dipole moment of the charged capacitor. In fact, so long as the capacitor is wholly contained within the sphere, the capacitor can any shape and the result would still hold.

C. Constant magnetic field created by a cylindrical solenoid

We now consider the case of an electric dipole in a region of constant magnetic field created by an infinite cylindrical solenoid of radius \( R \) that is centered along the \( z \)-axis. The vector potential in the Coulomb gauge in cylindrical coordinates is

\[
A(r) = \begin{cases} 
\frac{1}{2} B s \hat{\phi} = \frac{1}{2} B \times r & \text{if } s \leq R, \\
\frac{B R^2}{2s} \hat{\phi}, & \text{if } s \geq R
\end{cases}
\]  

(15)

where \( s \) is the distance from the \( z \)-axis. These give \( B = B \hat{z} \) inside the solenoid and \( B = 0 \) outside. We now evaluate \( P_{\text{em}} \) for a dipole inside and outside the solenoid.

1. Dipole within the solenoid

Since the vector potential inside the cylinder is the same form as \( A_{\text{in}}(r) \) for the rotating sphere, the calculation yields the same result.\(^7\)\(^7\) As in the case of the dipole within the sphere, this result is independent of the position of the dipole inside the solenoid, and therefore even a macroscopic object, such as a capacitor, with a total dipole moment \( p \) within the solenoid gives \( P_{\text{em}} = \frac{1}{2} B \times p \).

At this stage, one might surmise that even though \( P_{\text{em}} = \frac{1}{2} B \times p \) does not hold for non-uniform magnetic fields, perhaps it is always true for uniform magnetic fields. The next and subsequent examples show that this is not the case.
2. Dipole outside the solenoid

For a dipole outside the solenoid, the electromagnetic momentum is

\[ P_{em} = (p \cdot \nabla) A = \frac{BR^2}{2} \left( p_s \frac{\partial}{\partial s} + p_\phi \frac{\partial}{\partial \phi} + p_z \frac{\partial}{\partial z} \right) \frac{\hat{\phi}(\phi)}{s} \]

\[ = -\frac{BR^2}{2s^2} \left( p_s \hat{\phi} + p_\phi \hat{s} \right), \tag{16} \]

where \( \frac{\partial \hat{\phi}}{\partial \phi} = -\hat{s} \) has been used. Since the magnetic field outside the solenoid is zero, this result is clearly not equal to \( \frac{1}{2} B \times p \).

Eq. (16) can be reproduced relatively easily in the limit of an infinitely thin solenoid carrying a magnetic flux \( \pi BR^2 \) along the \( z \)-axis. The electromagnetic momentum of a point charge \( q \) that is a distance \( s \) from the \( z \)-axis is, in cylindrical coordinates,

\[ P_{em} = \frac{qBR^2}{2s} \hat{\phi}. \tag{17} \]

Consider now a dipole which consists of two charges \( q \) and \( -q \) separated by a distance \( l \), in the limit \( l \to 0 \) and \( ql \to p \). For \( p \) in the \( z \)-direction, the contributions to \( P_{em} \) from the two charges in the dipole cancel each other, so \( P_{em} = 0 \). For \( p = p_s \hat{s} \), given by charges \( -q \) and \( q \) at \( s \hat{s} \) and \( (s+l) \hat{s} \) respectively, Eq. (17) yields

\[ P_{em} = \frac{BR^2}{2} \hat{\phi} \left[ \lim_{l \to 0} \left( -\frac{q}{s} + \frac{q}{s+l} \right) \right] = \frac{BR^2}{2} \hat{\phi} \lim_{ql \to p} \left[ -\frac{ql}{s^2} \right] = -\frac{BR^2}{2s^2} p_s \hat{\phi}. \tag{18} \]

For \( p = p_\phi \hat{\phi} \), given by charges \( q \) and \( -q \) are at the same distance \( s \) from the \( z \)-axis but with azimuthal angles that are different by an angle \( \Delta \phi = l/s \), Eq. (17) yields

\[ P_{em} = \frac{BR^2}{2s} \lim_{ql \to p_\phi} \left[ q\hat{\phi}(\phi + \Delta \phi) - q\hat{\phi}(\phi) \right] = \frac{BR^2}{2s} \lim_{ql \to p_\phi} \left[ l \frac{\partial \hat{\phi}}{s \partial \phi} \right] \]

\[ = -\frac{BR^2}{2s^2} p_\phi \hat{s}. \tag{19} \]

D. Two parallel plates with counter-propagating currents

Assume that there are a pair thin conducting plates are in the \( x-y \) plane and they are at \( z = \pm a \). The plate on the top carries a uniform current density \( j \) in the \(+x\) direction and the plate at the bottom carries the an equal magnitude of current in the \(-x\) direction, as shown in Fig. 2. The calculation of the vector potential using Eq. (6b), is analogous to that
of the scalar potential from parallel plate capacitors with a uniform charge distribution, and it gives a vector potential of

$$\mathbf{A}(\mathbf{r}) = \begin{cases} Bz\hat{\mathbf{x}}, & \text{if } -a < z < a, \\ \pm Ba\hat{\mathbf{x}}, & \text{if } z \gtrless a, \end{cases}$$

(20)

where $B = \mu_0 j$, and a magnetic field of $B\hat{\mathbf{y}}$ for $|z| < a$ and $B = 0$ for $|z| > a$.

The $\mathbf{P}_{em}$ for an electric dipole outside the plates is zero, since $\mathbf{A}$ is constant. For an electric dipole $\mathbf{p}$ in between the plates, using Eq. (8a), the total electromagnetic momentum is

$$\mathbf{P}_{em} = \left( p_x \frac{\partial}{\partial x} + p_y \frac{\partial}{\partial y} + p_z \frac{\partial}{\partial z} \right) Bz\hat{\mathbf{x}} = p_z B\hat{\mathbf{x}}.$$  

(21)

Thus, only the component of $\mathbf{p}$ perpendicular to the plates contributes to $\mathbf{P}_{em}$. This provides an example where there is a non-zero uniform magnetic field, but $\mathbf{P}_{em} \neq \frac{1}{2} \mathbf{B} \times \mathbf{p}$.

Note that in this case, even though the currents and hence the magnetic fields extend to infinity, the expression Eq. (6a) is convergent and the surface term in Eq. (A1) in Appendix A goes to zero in the limit when the volume of integration goes to infinity. Therefore, Eq. (8a) [which is based on Eq. (6a)] can be used to obtain $\mathbf{P}_{em}$. In Appendix B the above result for $\mathbf{P}_{em}$ is also obtained by evaluating $\varepsilon_0 \int d\mathbf{r} \mathbf{E} \times \mathbf{B}$ directly.

To circumvent any problems which might be associated with infinite currents and fields, one can assume that the current-carrying plates are large but finite, and the top and bottom plates are connected at the ends where the currents flow to and from. In this case, there will be contributions to $\mathbf{P}_{em}$ from the fringing fields, but these can be made arbitrarily small by increasing the size of the plates.

E. Example of $\mathbf{P}_{em} \neq \frac{1}{2} \mathbf{B} \times \mathbf{p}$ for local currents

To confirm that the previous example is not a consequence of the unbounded currents and magnetic fields (which in some cases gives ill-defined physical quantities), let us consider a case where the magnetic field around the electric dipole is locally uniform, and the currents which produce the magnetic field do not extend to infinity, but $\mathbf{P}_{em} \neq \frac{1}{2} \mathbf{B} \times \mathbf{p}$.

Consider a spinning sphere with a uniform surface charge, as in Refs. ? and ? and in Section III B together with a torus which contains magnetic field. The magnetic field in the torus is caused by a surface current on the torus, and hence both current and field are local.
In the limit where the torus is infinitesimally small, the vector potential of the magnetic of the torus in the Coulomb gauge has the same form as the magnetic field of a point dipole; namely, in spherical coordinates (with the z-axis perpendicular to the plane of the torus)

\[
A_{\text{tor},r} = \frac{2C}{r^3} \cos \theta \\
A_{\text{tor},\theta} = \frac{C}{r^3} \sin \theta \\
C = \frac{\mu_0 V_{\text{tor}} I}{16\pi^2}
\]

where \( V_{\text{tor}} \) is the volume of the torus and \( I \) is the current times number of windings of the wire around the torus.

The spinning sphere creates a uniform magnetic field \( B_s \) inside the sphere, and the electromagnetic momentum due to the electric dipole and the sphere is \( P_{\text{em},s} = \frac{1}{2} B_s \times p \). For simplicity, let the dipole be at a distance \( R \) from the torus and on the axis that passes through the torus, as shown in Fig. 3. The contribution to the electromagnetic momentum of the magnetic field inside the torus and the electric field of the dipole \( p = p_r \hat{r} + p_\theta \hat{\theta} \) (where the origin is taken to be the position of the torus), is \( P_{\text{em},\text{tor}} = CR^{-4}(-6p_r \hat{r} + p_\theta \hat{\theta}) \), by Eq. (8a). Therefore, the total electromagnetic momentum is \( P_{\text{em},s} + P_{\text{em},\text{tor}} \neq \frac{1}{2} B_s \times p \), in general.

**F. Which gauge of \( A(\mathbf{r}) \) should be used?**

Eqs. (8a) and (8c) indicate that \( P_{\text{em}} \) for an electric dipole in a constant magnetic field is dependent on form of the vector potential \( A(\mathbf{r}) \) at the dipole. In the case of a constant magnetic field \( \mathbf{B} = B\hat{y} \), both \( A(\mathbf{r}) = \frac{B}{2} \hat{y} \times \mathbf{r} = \frac{B}{2}(z\hat{x} - x\hat{z}) \) and \( A(\mathbf{r}) = Bz\hat{x} \) (or for that matter, \( A = B[(1 - \alpha)z\hat{x} - \alpha x\hat{z}] \) for any constant \( \alpha \)) satisfy \( \nabla \times A = \mathbf{B} \hat{y} \) and \( \nabla \cdot A = 0 \).

As shown above, the value of \( P_{\text{em}} \) depends on which \( A(\mathbf{r}) \) is used. So, which of these is the correct one to use? The Helmholtz theorem states that the appropriate \( A \) in the Coulomb gauge \( (\nabla \cdot A = 0) \) is determined by \( \mathbf{B} = \nabla \times A \) over all space. Provided that \( \mathbf{B} \) goes to zero sufficiently fast as \( r \to \infty \), \( A(\mathbf{r}) \) is given by Eq. (6b), where the current density is given by the differential form of Ampere’s law, \( \mathbf{J} = \mu_0^{-1} \nabla \times \mathbf{B} \). So, the currents that are source of the magnetic field determine which form of \( A(\mathbf{r}) \) should be used. The \( A = \frac{B}{2} \hat{y} \times \mathbf{r} \) is appropriate for the interior regions of uniform cylindrical solenoids and spinning spheres, whereas \( A = \hat{x} By \) is appropriate for the interior region of parallel plates.
From the partial differential equation point of view, the equations $\nabla \cdot A = 0$, $\nabla \times A = B$ and $\nabla \times B = \mu_0 J$ are equivalent to $\nabla^2 A = -\mu_0 J$; i.e., each component of $A$ satisfies Poisson’s equation. In cases where the magnetic field is produced by surface currents, the appropriate $A$ is obtained by matching boundary conditions across each surface current in the same way that the scalar potential must be matched across surface charges (see e.g., Example 3.9 on page 142 in Ref. ?). That is to say, each component of the magnetic vector potential must be continuous across the boundary, and for a surface current density $K$, $\partial A_{\parallel, \text{in}} / \partial r_\perp - \partial A_{\parallel, \text{out}} / \partial r_\perp = \mu_0 K$. This, together with Laplace’s equation for each component of $A$ in the current-free regions, $\nabla^2 A = 0$, and the fact that $A \rightarrow 0$ for $r \rightarrow \infty$, give a unique $A$.

G. Contribution of the fringing electric fields of a capacitor to $P_{\text{em}}$

A capacitor can be modeled as a superposition of point dipoles. Since the $P_{\text{em}}$ is linear in $E$, which in turn is linear in $p$, the results for $P_{\text{em}}$ for a point dipole are equally valid for a capacitor. For cylindrical solenoids and spinning spheres of uniform surface charge, the factor of $\frac{1}{2}$ in the result $P_{\text{em}} = \frac{1}{2} B \times p$ implies $P_{\text{em}} = \frac{1}{2} P_{\text{capacitor}}$, where $P_{\text{capacitor}}$ is the electromagnetic momentum that is stored in between the capacitor plates. Obviously, the fringing electric fields of the capacitor contain a contribution of $-\frac{1}{2} P_{\text{capacitor}}$. On the other hand, for a constant field generated by a “solenoid” consisting of parallel conducting plates with counter-propagating currents, if the capacitor plates are oriented in the same way as the current-carrying plates, the $P_{\text{em}} = P_{\text{capacitor}}$, but if the capacitor plates are perpendicular to the plates carrying the current, $P_{\text{em}} = 0$. What causes these differences in the total electromagnetic momentum?

Consider the cases shown in Fig. 4(a) – (c), in which the capacitors can be approximated as an electric dipole moment is the positive $z$-direction, and the magnetic field is in the positive $y$-direction. In these cases, the $y$- and $z$-components of the total electromagnetic momentum $P_{\text{em}}$ are zero, so to determine the contribution to $P_{\text{em}}$, we need only consider the $z$-component of the electric field due to the dipole (since the electromagnetic momentum density is $\epsilon_0 E \times B$, and here $B$ is in the $y$-direction.) Fig. 4(d) indicates the regions where $E_z$ is positive and where it is negative around the electric dipole. Since the $z$-component of the electric field for a dipole at the origin is $E_z(r) = \frac{2}{4\pi r^3} (3 \cos^2 \theta - 1) - \frac{2}{3\epsilon_0} \delta(r)$, $E_z > 0$
in the cones that make an angle of \( \cos^{-1}(1/\sqrt{3}) \approx 55^\circ \) with respect to the positive and negative \( z \)-axes, as indicated by the shaded regions in Fig. 4(d). When \( \mathbf{B} \) in the positive \( y \)-direction, these regions will give a negative contribution to \( P_{em,x} \), while conversely, the regions within approximately 35\(^\circ\) of the “equator” of the dipole, the electric field gives a positive contribution to \( P_{em,x} \).

In the fringing field regions around the capacitors where the magnetic field exists, the volume of the regions where \( E_z > 0 \) (\( E_z < 0 \)) increases (decreases) in the configurations shown in Figs. 4(b), (a) and (c), respectively. This results in a corresponding increase in the contribution of the fringing fields, in that order. In Fig. 4(b), apparently the fringing field contributions to \( P_{em} \) of the \( E_z > 0 \) and \( E_z < 0 \) regions cancel, and the result of the total electromagnetic momentum is the same as that contained within the capacitor plates. Figs. 4(c) is at the other extreme of the cases considered here – the regions where \( E_z > 0 \) in the fringing fields dominate to the extent that it completely cancels the electromagnetic momentum contained within the capacitor plates (where \( E_z < 0 \)). Figs. 4(a) is in between (b) and (c).

**IV. MOMENTUM IMPARTED TO THE SYSTEM WHEN CURRENT OR DIPOLE IS CHANGED**

In this section, we will see that when the dipole moment or the magnitude of the current producing the electric field is changed, the momentum imparted to the system is equal to the change in the total electromagnetic momentum.

**A. Current changed**

When the current is changed the vector potential changes by \( \Delta \mathbf{A}(\mathbf{r}) \), which induces a change in the electric field \( \mathbf{E} = -\partial \mathbf{A}/\partial t \) (the Faraday effect). The momentum imparted to a charge distribution \( \rho(\mathbf{r}) \) as the current changes is

\[
\tilde{I}_p = \int dt \int d\mathbf{r} \, \rho(\mathbf{r}) \, \mathbf{E}(\mathbf{r}, t) = \int d\mathbf{r} \, \rho(\mathbf{r}) \left[ -\int dt \, \frac{\partial \mathbf{A}}{\partial t} \right] \\
= -\int d\mathbf{r} \, \rho(\mathbf{r}) \, \Delta \mathbf{A}(\mathbf{r}) = -\Delta P_{em},
\]

where the last equality comes from Eq. (6a).
Thus, when the magnitude of the current is changed, the change in the electromagnetic momentum is equal and opposite to the impulse given to the charges. Since this is true for any arbitrary distribution of charges, it is also true for the momentum imparted on the dipole.

B. Dipole moment changed

When the dipole moment is changed, there is an electric current density \( J_{\text{dip}} \) due to the transfer of charge within the dipole which experiences a Lorentz force \( F_{\text{dip}} = J_{\text{dip}} \times B \). In addition, \( J_{\text{dip}} \) itself produces a magnetic field, \( B_{\text{dip}} \), which imparts a Lorentz force on the current in the solenoid. Let us examine the impulses exerted by these forces and compare these to the change in the electromagnetic momentum.

Assuming the dipole is at the origin, the current distribution due to the change in the dipole moment is
\[
J_{\text{dip}}(r, t) = \dot{p}(t) \delta(r).
\] (24)
The impulse on the dipole, which is equal to the momentum imparted on the dipole, is given by
\[
\mathbf{I}_{\text{dip}} = \int dt \int J_{\text{dip}}(r, t) \times B(r) \, dr
= \int dt \dot{p}(t) \times B(0) = \Delta p \times B(0).
\] (25)

The current \( J_{\text{dip}}(t) \) produces a magnetic field at position \( r' \) and time \( t \) of
\[
B_{\text{dip}}(r', t) = -\frac{\mu_0}{4\pi} \left[ \frac{r' \times [\dot{p} + (r'/c)\ddot{p}]}{r'^3} \right]_{\text{ret}},
\] (26)
where “ret” indicates that \( \dot{p} \) and \( \ddot{p} \) are evaluated at retarded time \( t_{\text{ret}} = t - |r'|/c \). The impulse imparted on the current in the solenoid is
\[
\mathbf{I}_{\text{sol}} = \int dt \int J(r') \times B_{\text{dip}}(r', t) \, dr'
= -\frac{\mu_0}{4\pi} \int J(r') \times \left( \frac{r' \times \Delta p}{r'^3} \right) \, dr'
= \frac{\mu_0}{4\pi} \int \left[ -\frac{r'(J(r') \cdot \Delta p)}{r'^3} + \Delta p \left( \frac{J(r') \cdot r'}{r'^3} \right) \right] \, dr',
\] (27)
where the identity $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$ has been used in the last equality.

The $\dot{\mathbf{p}}$ term in $\mathbf{B}_{\text{dip}}(\mathbf{r}, t)$ does not contribute because $\Delta \mathbf{p} = 0$, since $\mathbf{p}$ is constant before and after it is changed.) The second term in the last expression in Eq. (27) is zero because

$$
\int \frac{\mathbf{J}(\mathbf{r}')}{{r'}^3} \, d\mathbf{r}' = - \int \mathbf{J}(\mathbf{r}') \cdot \nabla' \left( \frac{1}{{r'}^2} \right) \, d\mathbf{r}' = \int \frac{1}{{r'}^3} \nabla' \cdot \mathbf{J}(\mathbf{r}') - \nabla' \cdot \left( \frac{\mathbf{J}(\mathbf{r}')}{{r'}^3} \right) = 0 \quad (28)
$$

since $\nabla' \cdot \mathbf{J}(\mathbf{r}') = 0$ for static currents, and from the divergence theorem, $\int \nabla' \cdot (\mathbf{J}(\mathbf{r}')/{r'}) \, d\mathbf{r}' = \int_S \mathbf{J}(\mathbf{r}')/r' \cdot d\mathbf{a}' = 0$, because $\mathcal{S}$ is a surface at infinity and $\mathbf{J}(\mathbf{r}')$ is be localized. Therefore,

$$
\vec{I}_{\text{sol}} = -\frac{\mu_0}{4\pi} \int \frac{\mathbf{r}'(\mathbf{J}(\mathbf{r}') \cdot \Delta \mathbf{p})}{{r'}^3} \, d\mathbf{r}' \quad (29)
$$

This implies that the total impulse given to the system due to the change $\Delta \mathbf{p}$ in the dipole moment at the origin is

$$
\vec{I}_{\text{dip}} + \vec{I}_{\text{sol}} = \Delta \mathbf{p} \times \mathbf{B}(0) - \frac{\mu_0}{4\pi} \int \frac{\mathbf{r}'(\mathbf{J}(\mathbf{r}') \cdot \Delta \mathbf{p})}{{r'}^3} \, d\mathbf{r}' \quad (30)
$$

But from Eq. (8d), this is exactly the negative of the change in the electromagnetic momentum for an electric dipole at $\mathbf{r} = 0$, so

$$
\vec{I}_{\text{dip}} + \vec{I}_{\text{sol}} + \Delta \mathbf{P}_{\text{em}} = 0, \quad (31)
$$

which shows that the momentum imparted to the system when the electric dipole moment changes is equal to the change in field electromagnetic momentum.

V. IMPULSE IMPARTED IN VARIOUS CASES

We now examine several cases of electric dipoles in magnetic fields created by different sources, specifically by a magnetic dipole, and by cylindrical, spherical and parallel-plate solenoids. In each case, we show explicitly that when either the electric dipole or the magnetic-field-producing current is changed, the impulse on the system is equal to the change in electromagnetic momentum.

A. Magnetic and electric dipole at arbitrary orientation and displacement

For an arbitrary placement of an magnetic moment $\mathbf{m}$ and an electric dipole $\mathbf{p}$ displaced by $\mathbf{r} \neq 0$ from the magnetic moment, the electromagnetic momentum is:

$$
\mathbf{P}_{\text{em}} = \frac{\mu_0}{4\pi r^3} \left( (\mathbf{m} \times \mathbf{p}) - \frac{3}{r^2} (\mathbf{p} \cdot \mathbf{r}) (\mathbf{m} \times \mathbf{r}) \right) . \quad (32)
$$
Let us assume that \( \mathbf{m} \) is at the origin and \( \mathbf{p} \) is at \( \mathbf{r} \).

**Magnetic dipole changed** — When the magnetic dipole is changed by \( \Delta \mathbf{m} \), the change in the electromagnetic momentum from Eq. (32) is

\[
\Delta \mathbf{P}_{\text{em}} = \frac{\mu_0}{4\pi r^3} \left( (\Delta \mathbf{m} \times \mathbf{p}) - \frac{3}{r^2}(\mathbf{p} \cdot \mathbf{r})(\Delta \mathbf{m} \times \mathbf{r}) \right). \tag{33}
\]

The change in the dipole also produces an electric field

\[
\mathbf{E}_{\text{dip}}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \left[ \mathbf{r} \times \left[ \dot{\mathbf{m}} + (r/c)\ddot{\mathbf{m}} \right] \right]_\text{ret}. \tag{34}
\]

The force on the electric dipole is \( \nabla (\mathbf{p} \cdot \mathbf{E}_{\text{dip}}) \), so the momentum imparted on the electric dipole due to a change in the magnetic moment is

\[
\mathbf{I} = \int \nabla (\mathbf{p} \cdot \mathbf{E}_{\text{dip}}) \, dt = \frac{\mu_0}{4\pi} \int \nabla \left( \frac{\mathbf{r} \times (\dot{\mathbf{m}} + (r/c)\ddot{\mathbf{m}})}{r^3} \right) \times \mathbf{r} \, d\mathbf{r} = \frac{\mu_0}{4\pi} \mathbf{r} \times \Delta \mathbf{m}, \tag{35}
\]

where we have used \( \int \ddot{\mathbf{m}} \, dt = \Delta \dot{\mathbf{m}} = 0 \). Eqs. (33) and (35) give \( \Delta \mathbf{P}_{\text{em}} + \mathbf{I} = 0 \).

It is easy to generalize the above from a point electric dipole to an arbitrary distribution of static charges \( \rho(\mathbf{r}) \). The electric field \( \mathbf{E}_{\text{dip}}(\mathbf{r}, t) \) induced by the change in \( \mathbf{m} \) imparts an impulse on \( \rho(\mathbf{r}) \) of (using Eq. (34), Coulomb’s law for the electric field and \( \mu_0\epsilon_0 = c^{-2} \))

\[
\mathbf{I} = \int d\mathbf{r} \rho(\mathbf{r}) \left[ \int dt \mathbf{E}_{\text{dip}}(\mathbf{r}, t) \right] = \int d\mathbf{r} \frac{\mu_0}{4\pi} \left[ \frac{\rho(\mathbf{r}) \mathbf{r} \times \Delta \mathbf{m}}{r^3} \right] = -\frac{1}{c^2} \mathbf{E} \times \Delta \mathbf{m}, \tag{36}
\]

where \( \mathbf{E} \) is electric field at the origin; i.e., the position of the magnetic moment. Since \( \Delta \mathbf{P}_{\text{em}} = c^{-2} \mathbf{E} \times \Delta \mathbf{m} \), this implies that \( \mathbf{I} + \Delta \mathbf{P}_{\text{em}} = 0 \) when \( \mathbf{m} \) is changed for an arbitrary distribution of static charges.

**Electric dipole changed** — If the electric dipole is changed by \( \Delta \mathbf{p} \), the change in the electromagnetic momentum from Eq. (32) is

\[
\Delta \mathbf{P}_{\text{em}} = \frac{\mu_0}{4\pi r^3} \left( \mathbf{m} \times \Delta \mathbf{p} - \frac{3}{r^2}(\Delta \mathbf{p} \cdot \mathbf{r})(\mathbf{m} \times \mathbf{r}) \right). \tag{37}
\]

The impulse of the magnetic field on the charge current in the electric dipole is, using Eq. (25) and Eq. (3) for the field of the magnetic dipole,

\[
\mathbf{I}_{\text{e.dip.}} = \Delta \mathbf{p} \times \mathbf{B} = \frac{\mu_0}{4\pi r^3} \left[ \frac{3(\mathbf{m} \cdot \mathbf{r})(\Delta \mathbf{p} \times \mathbf{r})}{r^2} - \Delta \mathbf{p} \times \mathbf{m} \right]. \tag{38}
\]

In addition, the current of the electric dipole at \( \mathbf{r} \) induces a magnetic field at \( \mathbf{r}' \) of

\[
\mathbf{B}_{\text{e.dip.}}(\mathbf{r}', t) = -\frac{\mu_0}{4\pi} \left[ \frac{(\mathbf{r}' - \mathbf{r}) \times (\dot{\mathbf{p}} + (r/c)\ddot{\mathbf{p}})}{|\mathbf{r}' - \mathbf{r}|^3} \right]_\text{ret}. \tag{39}
\]
The impulse on the magnetic dipole is

\[
\vec{\mathcal{I}}_{m,\text{dip}} = \int \nabla'[\mathbf{m} \cdot \mathbf{B}_{e,\text{dip}}(\mathbf{r}',t)]_{r'=0} \, dt = -\frac{\mu_0}{4\pi} \nabla \left( \frac{\mathbf{m} \cdot [(\mathbf{r}' - \mathbf{r}) \times \Delta \mathbf{p}]}{|\mathbf{r}' - \mathbf{r}|^3} \right)_{r'=0}
\]

\[
= -\frac{\mu_0}{4\pi} \nabla' \left( \frac{(\mathbf{r}' - \mathbf{r}) \cdot (\Delta \mathbf{p} \times \mathbf{m})}{|\mathbf{r}' - \mathbf{r}|^3} \right)_{r'=0}
\]

\[
= -\frac{\mu_0}{4\pi} \left[ \frac{\Delta \mathbf{p} \times \mathbf{m}}{r^3} - \frac{3\mathbf{r} \cdot (\Delta \mathbf{p} \times \mathbf{m})\mathbf{r}}{r^5} \right].
\]  

(40)

Therefore, the total impulse on the system is

\[
\vec{\mathcal{I}} = \vec{\mathcal{I}}_{e,\text{dip}} + \vec{\mathcal{I}}_{m,\text{dip}}
\]

\[
= \frac{\mu_0}{4\pi r^5} \left[ 3(\mathbf{m} \cdot \mathbf{r})(\Delta \mathbf{p} \times \mathbf{r}) + 2(\mathbf{m} \times \Delta \mathbf{p})\mathbf{r}^2 + 3\mathbf{r} \cdot (\Delta \mathbf{p} \times \mathbf{m})\mathbf{r} \right]
\]

\[
= \frac{\mu_0}{4\pi r^5} \left[ -(\mathbf{m} \times \Delta \mathbf{p})\mathbf{r}^2 + 3(\Delta \mathbf{p} \cdot \mathbf{r})(\mathbf{r} \times \mathbf{m}) \right]
\]  

(41)

where in the last equality we use the identity

\[
(\mathbf{m} \cdot \mathbf{r})(\Delta \mathbf{p} \times \mathbf{r}) + (\mathbf{r} \cdot \mathbf{r})(\mathbf{m} \times \Delta \mathbf{p}) + (\Delta \mathbf{p} \cdot \mathbf{r})(\mathbf{r} \times \mathbf{m}) + [\mathbf{r} \cdot (\Delta \mathbf{p} \times \mathbf{m})]\mathbf{r} = 0.
\]  

(42)

Eqs. (37) and (41) give \(\Delta \mathbf{P}_{em} + \vec{\mathcal{I}} = 0\).

**B. Electric dipole on the axis of a cylindrical solenoid and at the center of a rotating charged spherical shell**

The cases in which an electric dipole is on the axis of a cylindrical solenoid and at the center of a rotating charged uniform spherical shell were considered by Babson et al.\(^7\). In both cases, \(\mathbf{P}_{em} = \frac{1}{2} \mathbf{B} \times \mathbf{p}\). Furthermore the impulse imparted on the system when the dipole strength or the magnetic field is changed is also the same in both cases.

**Magnetic field changed** – When the current that is the source of the magnetic field is changed by \(\Delta \mathbf{B}\), a straightforward generalization of the calculation given in Babson et al.\(^7\) shows that the impulse given to the dipole is \(-\frac{1}{2} \Delta \mathbf{B} \times \mathbf{p}\). Since \(\Delta \mathbf{P}_{em} = \frac{1}{2} \Delta \mathbf{B} \times \mathbf{p}\), this implies \(\Delta \mathbf{P}_{em} + \vec{\mathcal{I}} = 0\).

**Electric dipole changed** – When the electric dipole is changed by \(\Delta \mathbf{p}\), a straightforward generalization of the calculation of Ref. \(^?\) shows that the impulse to the dipole is \(\vec{\mathcal{I}}_{\text{dip}} = -\mathbf{B} \times \Delta \mathbf{p}\) and the impulse to the solenoid is \(\vec{\mathcal{I}}_{\text{sol}} = \frac{1}{2} \mathbf{B} \times \Delta \mathbf{p}\), giving a total impulse of \(\vec{\mathcal{I}} = \vec{\mathcal{I}}_{\text{dip}} + \vec{\mathcal{I}}_{\text{sol}} = -\frac{1}{2} \mathbf{B} \times \Delta \mathbf{p}\). Again, \(\Delta \mathbf{P}_{em} + \vec{\mathcal{I}} = 0\).
C. Electric dipole in a “solenoid” of parallel counter-propagating currents

In the case where the magnetic field is created by currents in parallel conducting plates in the configuration shown in Fig. 2 and considered in Sec. III D, the electromagnetic momentum is given by Eq. (21).

Magnitude of current in the solenoid is changed

When the current density is changed by \( \Delta j = \pm \Delta j \hat{x} \) in the plate at \( z = \pm a \), the magnetic field is changed by \( \Delta B = \Delta B \hat{y} \) in between the plates, and therefore Eq. (21) yields

\[
\Delta P_{em} = p_z \Delta B \hat{x}.
\]  \( (43) \)

The change in the magnetic field induces an electric field, which can be deduced by Amperian loops. From the symmetry of the problem, the electric field must be in the \( \hat{x} \) direction, and independent of the \( x \)-coordinate. Taking rectangular amperian loops which have a length \( L \) in the \( x \)-direction and \( d \) in the \( z \)-direction, the area through the loop is \( Ld \) and the change in flux \( \Delta B Ld \) is equal to \( \int \mathbf{E} \cdot d\mathbf{l} \) around the loop. This shows that \( \int [E_x(z) - E_x(z + d)] \, dt = \Delta B d \). Thus, the momentum of a dipole inside the solenoid \( |z| < a \), has momentum imparted of the form

\[
\vec{I} = \int q \left[ E_x(z + d) - E_x(z) \right] \, dt \, \hat{x} = -qd \Delta B \hat{x} = -\Delta B p_z \hat{x}.
\]  \( (44) \)

This gives \( \Delta P_{em} + \vec{I} = 0 \).

Electric dipole changed

When the electric dipole changes, Eq. (21) gives

\[
\Delta P_{em} = \Delta p_z B \hat{x}.
\]  \( (45) \)

The change in the \( p \) results in a current density \( \mathbf{j} = \delta(\mathbf{r} - \mathbf{r}_{dip}) \dot{\mathbf{p}} \) (where \( \mathbf{r}_{dip} \) is the position of the dipole) and \( \int \mathbf{p} \, dt = \Delta \mathbf{p} \). As in previous instances considered, when \( \mathbf{p} \) changes, there are two impulses on the system. First, there is the Lorentz force of the magnetic field \( \mathbf{B} \) on the current in the dipole, which gives an impulse of

\[
\vec{I}_{dip} = \int dt \int d\mathbf{r} \mathbf{j} \times \mathbf{B} = \int dt \mathbf{p} \times \mathbf{B} = \Delta \mathbf{p} \times \mathbf{B}.
\]  \( (46) \)

Then, there is the Lorentz force of the magnetic field induced by the change in the dipole (see Eq. (38)) on the currents in the parallel-plate solenoid. We treat each of the three different components of the dipole, \( \mathbf{p} = p_x \hat{x}, p_y \hat{y} \) and \( p_z \hat{z} \), individually.
\(\Delta p_z\): When the \(z\)-component of \(p\) is changes by \(\Delta p_z\), the impulse on the the dipole due to the magnetic field is \(\vec{I}_{\text{dip}} = \Delta p_z B (\hat{z} \times \hat{y}) = -\Delta p_z B \hat{x}\). The magnetic fields induced by the current in the dipole are in the azimuthal direction and symmetric with respect to the azimuthal angle, so by symmetry the total impulse \(\vec{I}_{\text{sol}}\) due to these magnetic fields on the currents at \(z = \pm a\) integrates to zero. Hence, \(\vec{I} = \vec{I}_{\text{sol}} + \vec{I}_{\text{dip}} = -\Delta p_z B \hat{x}\).

\(\Delta p_y\): When the \(y\)-component of \(p\) is changed by \(\Delta p_y\), the impulse due to the magnetic field on the current in the dipole \(\vec{I}_{\text{dip}} = \Delta p \times B = 0\). By symmetry, the magnetic fields produced by the change in the electric dipole has \(x\) and \(z\) components. The \(x\)-component of the magnetic field does not impart an impulse on the currents at \(z = \pm a\), and the impulses from the \(z\)-components cancel due to symmetry. Therefore, \(\vec{I} = \vec{I}_{\text{sol}} + \vec{I}_{\text{dip}} = 0\).

\(\Delta p_x\): For \(\Delta p = \Delta p_x \hat{x}\), the impulse on the current in the dipole is \(\vec{I}_{\text{dip}} = p \times B = \Delta p_x \hat{B} \hat{x}\). If we let \(r'\) be the position with respect to the dipole, then the current in the dipole produces a magnetic field \(B_{e,\text{dip}}(r', t)\) given by Eq. (39) [with \(r = 0\) in that equation]. The current at \(z = +a\) has a surface current density of \(j = j \hat{x}\), so the force per unit area is \(j \hat{x} \times B_{e,\text{dip}}(r', t)\). The impulse on the entire sheet is

\[
\vec{I}_{\text{sol}+a} = \int dx' \int dy' \int dt \ j \hat{x} \times B_{e,\text{dip}}(r', t)
\]

\[
= \frac{\mu_0 j}{4\pi} \int dx' \int dy' \ j \hat{x} \times \left( \frac{\Delta p \times r'}{r'^3} \right) 
\]

\[
= \frac{\mu_0 j}{4\pi} \Delta p \int dx' \int dy' \ \frac{\hat{x} \cdot r'}{r'^3} - r' 
\]

\[
= - \frac{\mu_0 j}{4\pi} \Delta p \int dx' \int dy' \ y' \hat{y} + z' \hat{z} \frac{1}{r'^3}, \quad (47)
\]

where \(z' > 0\) is the \(z\)-coordinate of the current sheet at \(z = +a\) relative to the electric dipole. The \(\hat{y}\) component vanishes because the integrand is anti-symmetric with respect to \(y'\). Changing variables of the integral \(\int dx' \int dy'\) to \(2\pi \int ds' s'\) (taking advantage of azimuthal symmetry of the integrand) gives, using a change of variables \(\zeta = s'/z'\),

\[
\vec{I}_{\text{sol}+a} = - \frac{\mu_0 j}{2} \Delta p \int_0^\infty ds' s' \frac{z'}{(z'^2 + s'^2)^{3/2}} \hat{x} 
\]

\[
= - B \Delta p \int_0^\infty d\zeta \frac{\zeta}{(1 + \zeta^2)^{3/2}} \hat{x} = - \frac{B \Delta p_x}{2} \hat{x}, \quad (48)
\]
where we have used $B = \mu_0 j$. For the current in the plane at the $z = -a$, the calculation is identical except that the the current is $\mathbf{j}$ is in the opposite direction, and $z' < 0$ so the change of variables $\zeta = s'/|z'| = -s'/(z')$ introduces a negative sign. These two sign changes negate each other, so $\vec{I}_{\text{sol},-a} = -\frac{1}{2} B \Delta p \hat{x}$, and the total impulse on the solenoid is

$$\vec{I}_{\text{sol}} = \vec{I}_{\text{sol},+a} + \vec{I}_{\text{sol},-a} = -B \Delta p_x \hat{x}. \quad (49)$$

Therefore $\vec{I}_{\text{sol}} + \vec{I}_{\text{dip}} = 0$.

Comparing these results for $\vec{I}$ caused by $\Delta p_x$, $\Delta p_y$ and $\Delta p_z$ with the change in the field momentum Eq. (45), we see that $\Delta \mathbf{P}_{\text{em}} + \vec{I} = 0$ for $\Delta \mathbf{p}$ in any direction.

**VI. CONCLUSION AND DISCUSSION**

An electric dipole in a static magnetic field and a magnetic dipole in a static electric field both generally give rise to a non-zero electromagnetic field momentum $\mathbf{P}_{\text{em}} = \varepsilon_0 \int dr \mathbf{E} \times \mathbf{B}$. While the $\mathbf{P}_{\text{em}}$ for a point magnetic dipole in a static electric field can be expressed in terms of magnetic moment and the local electric field at the position of the moment, namely Eq. (2b), the same cannot be said for a point electric dipole in a static magnetic field; Eq. (2a) is not true in general, even for locally uniform magnetic fields. To determine $\mathbf{P}_{\text{em}}$ for a point electric dipole in a static magnetic field, one needs to know the full current distribution which produces the magnetic field, or the appropriate derivatives of the magnetic vector potential in the Coulomb gauge at the position of the electric dipole; see Eqs. (8a) – (8d). The difference between electric dipoles in static magnetic fields and magnetic dipoles in static electric fields is due the difference in the sources of static electric and magnetic fields; static electric fields are produced by charges whereas static magnetic fields are produced by charge currents.

It has also been shown that the impulse on the system is equal to the negative of the change of the electromagnetic momentum for an electric dipole in the magnetic field, when either electric dipole moment or the current producing the magnetic field is changed (and also for a magnetic dipole in an electric field, when the magnetic dipole is changed). The motivation for showing this comes from Babson et al.\textsuperscript{7} , who stated in the conclusion section that “there is no reason why this impulse should equal the momentum originally stored
in the fields” in these situations. Babson et al. seemed to imply (although it was not their intention?) that it is necessary to take into account the hidden momentum in the system in order for the total momentum to be conserved. Hidden momentum is a form of mechanical momentum that associated with internally moving parts in the presence of a potential gradient, and is a relativistic effect. As shown in Ref. ?, the hidden momentum must be included for the total momentum to be equal to zero, as is required since the “center-of-energy” in the systems considered are stationary. The amount of hidden momentum does change when the electric dipole or the magnetic field is changed, but it simply is transformed from one form of mechanical momentum to another; for example, from the hidden momentum of the charge carriers of the solenoid into the structure of the solenoid and vice versa. The hidden momentum however does not play any role in the transfer of momentum from electromagnetic to mechanical or vice versa.

Finally, note also that the results presented here more apply to situations that are more general than point dipole charges. In fact, any charge distribution \( \rho \) which is neutral overall can be considered to consist of a polarization distribution \( \vec{\mathcal{P}}(\mathbf{r}) \), where \( -\nabla \cdot \vec{\mathcal{P}} = \rho \). Since the results are linear in the dipole moment, by linear superposition the total \( \mathbf{P}_{em} \) is given by the integral of Eqs. (8a) – (8d) over \( \vec{\mathcal{P}}(\mathbf{r}) \).

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Appendix A: Derivation of Eq. (8a)

To see under what conditions Eq. (8a) holds, let us first derive this for a finite volume \( \mathcal{V} \), with surface \( \mathcal{S} \). For static electric and magnetic fields,

\[
\mathbf{P}_{em} = \epsilon_0 \int_\mathcal{V} d\mathbf{r} \mathbf{E} \times \mathbf{B} = -\epsilon_0 \int_\mathcal{V} d\mathbf{r} \nabla V \times \mathbf{B} \\
= \epsilon_0 \int_\mathcal{V} d\mathbf{r} \left[ V(\nabla \times \mathbf{B}) - \nabla \times (V\mathbf{B}) \right] = \frac{1}{c^2} \int_\mathcal{V} d\mathbf{r} \ VJ - \epsilon_0 \int_\mathcal{S} d\mathbf{a} \ V \ (\hat{n} \times \mathbf{B}).
\] (A1)
where we have used $E = -\nabla V$, $\nabla \times (VB) = \nabla V \times B + V \nabla \times B$, $\nabla \times B = \mu_0 J$, $\mu_0 \epsilon_0 = c^{-2}$, and $\int_V d\mathbf{r} \nabla \times \mathbf{A} = \int_S \mathbf{n} \times \mathbf{A}$, where $\mathbf{n}$ is the unit vector perpendicular to the surface.

Assuming $\int_V d\mathbf{r} \mathbf{E} \times \mathbf{B}$ converges when $V \to \infty$, then $P_{em} = c^{-2} \int_V d\mathbf{r} V J$ if $\epsilon_0 \int_S d\mathbf{a} V (\mathbf{n} \times \mathbf{B}) \to 0$ when $S \to \infty$. Note that this is true even if the current $\mathbf{J}$ is not bounded, so long as the integral of the product $VJ$ over all space is convergent.

Appendix B: Alternative derivation for $P_{em}$ due to electric dipole in between antiparallel current sheets

The result Eq. (21) for the momentum of an electric dipole in a constant magnetic field created by conducting parallel plates can also be obtained by integrating $\epsilon_0 \mathbf{E} \times \mathbf{B}$ over all space. To do this, we divide the volume integral in between the parallel current-carrying plates into an integral across the entire $x$-$y$ plane, followed by an integral from $z = -a$ to $z = +a$; i.e.,

$$P_{em} = \epsilon_0 \int_{-a}^{a} dz' \mathbf{E}(z') \times \mathbf{y};$$

(B1a)

where

$$\mathbf{E}(z') = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \mathbf{E}(x, y, z').$$

(B1b)

is the integral of the electric field over the $x$-$y$ plane at $z = z'$. Below, we evaluate $\mathbf{E}(z')$ for physical dipoles $p = ql$, with a finite $q$ and $l$, and then take the point dipole limit $q \to \infty$, $l \to 0$ and $ql \to p$ in the end. We show here that $\mathbf{E}(z')$ is zero for any physical dipole except when plane at $z = z'$ lies in between the positive and negative charges of the dipole.

By symmetry, the component $\mathbf{E}(z')$ along the surface of the plane due to a static point charge is be zero. (For example, for a point charge on the $z$-axis, $E_{x,y}(x, y, z') = -E_{x,y}(-x, -y, z')$, so when $E_x$ or $E_y$ are integrated over the $x$-$y$, the contributions from $x, y$ and $-x, -y$ cancel each other.) By linear superposition, this is also true for electric fields due to static dipoles. Therefore, the only possible non-zero component is the component of $\mathbf{E}$ that is perpendicular to the surface, which is electric flux through the surface. By Gauss’ law, the magnitude of the electric field flux through the plane due to a point charge $q$ is $|q|/(2\epsilon_0)$, independent of the distance of the charge from the plane. (This is because half of the flux lines that emanate from the charge will pass through the plane – see Fig. 5(a).
When both the charges of a dipole are on the same side of the plane, the contributions from the positive and negative charges cancel. The only case when $E(z')$ is non-zero is when the plane straddles the charges; see Fig. 5(b)). So, if the dipole consists of a charge $-q$ at $(x_0, y_0, z_0)$ and $q$ at $(x_0 + l_x, y_0 + l_y, z_0 + l_z)$, then $E(z')$ is equal to $-q/\varepsilon_0 \hat{z}$ for $z_0 < z' < z_0 + l_z$, and zero otherwise. Therefore,

$$P_{em} = \varepsilon_0 \int_{z_0}^{z_0 + l_z} dz' \left( -\frac{qB}{\varepsilon_0} \right) \hat{z} \times \hat{y} = ql_z B \hat{x} = p_z B \hat{x},$$

reproducing Eq. (21)
FIG. 1: Configuration of electric (p) at the origin and magnetic (m) dipole at R\hat{x} which demonstrates that Eqs. 2a and 2b give inconsistent results for P_{em}. The dotted line labeled E is the electric field due to p at m, and the dotted line labeled B is the magnetic field due to m at p.

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FIG. 2: Electric dipole in a uniform magnetic field produced by currents in parallel plates. The plates are in the $x$-$y$ plane at $z = \pm a$. The current in the top (bottom) layer is in the positive (negative) $x$-direction, which produces a magnetic field in the positive $y$-direction. The lateral extent of the plates is assumed to be much, much larger than the separation $2a$ of the plates.

FIG. 3: A situation where the electric current does not extend to infinity, and $\mathbf{P}_{em} \neq \frac{1}{2} \mathbf{B} \times \mathbf{p}$, where $\mathbf{B}$ is the locally uniform magnetic field around electric dipole $\mathbf{p}$. The electric dipole (indicated by the arrow) is in a uniform magnetic field created by a spinning sphere with a uniform surface charge, and is a distance $R$ away from and on the axis of a torus that contains magnetic flux.
FIG. 4: Capacitors in the uniform magnetic fields produced by (a) infinite cylindrical solenoid with uniform current density around the circumference (b) infinite parallel plates with currents in opposite directions, and perpendicular the dipole of the capacitor and (c) same as (b) but with currents densities parallel and anti-parallel to the electric dipole moment of the capacitor. In these three cases, the electromagnetic momenta $P_{em}$ is (a) $\frac{1}{2}B \times p$, (b) $B \times p$ and (c) 0, where $p$ is the electric dipole moment of the capacitor and $B$ is the magnetic field. Figure (d) shows the regions around an electric dipole that is oriented in the positive $z$-direction where the $z$-component of the electric field of the dipole is positive (shaded regions) and negative (white regions). Thus, when the magnetic field is in the positive $y$-direction (i.e., into the paper), the $x$-component of the electromagnetic momentum is negative in the shaded regions and positive in the white regions.
FIG. 5: (a) Electric flux lines from a positive point charge piercing an infinite plane. Since the plane is infinite, all the electric field lines with a component that is directed towards the plane will pierce the plane. This implies that half the electric field lines that emanate from the point charge will pierce the plane. Therefore, from Gauss’ law, for an infinite plane (and a point charge $q$ which is not on the plane itself), the flux through the plane $\int \mathbf{E} \cdot d\mathbf{r} = \frac{1}{2}q/\varepsilon_0$. When there are two charges of equal magnitude and opposite sign on the same side of the plane, the the contributions of the $+$ and $-$ charges cancel each other and the electric flux through the plane is zero,. (b) Electric flux lines for an infinite plane in between a positive (black dot) and negative (white dot) charges of equal magnitude. The contribution for each charge to the flux is equal, and therefore the total flux is $q/\varepsilon_0$. 