Spin-orbit coupled fermions in ladder-like optical lattices at half-filling

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We study the ground-state phase diagram of two-component fermions loaded in a ladder-like lattice at half filling in the presence of spin-orbit coupling. For repulsive fermions with unidirectional spin-orbit coupling along the legs we identify a Néel state which is separated from rung-singlet and ferromagnetic states by Ising phase transition lines. These lines cross for maximal spin-orbit coupling and a direct Gaussian phase transition between rung-singlet and ferro phases is realized. For the case of Rashba-like spin-orbit coupling, besides the rung singlet phases two distinct striped ferromagnetic phases are formed. In case of attractive fermions with spin-orbit coupling at half-filling for decoupled chains we identify a dimerized state that separates a singlet superconductor and a ferromagnetic states.

I. INTRODUCTION

The possibility of inducing synthetic electromagnetism in ultra-cold gases has attracted recently a large deal of attention. In spite of the electric charge neutrality of an atom, synthetic magnetic field may be induced by a proper laser arrangement [1]. Interestingly, unidirectional spin-orbit coupling (USOC) resulting from an equal superposition of Rashba [2] and linear Dresselhaus [3] terms, has been realized for both spinor Bose [4] and Fermi gases [5, 6] with the help of counter-propagating Raman lasers. Recently this technique has allowed for the observation of superfluid Hall effect [7], Zitterbewegung [8], and the spin-Hall effect in a quantum gas [9]. Several theory works have discussed the creation of pure Rashba or Dresselhaus SOC with optical [10] and magnetic means [11], and even proposed methods to generate a three-dimensional SOC [12].

The presence of a synthetic SOC is expected to lead to a rich physics for atoms loaded in optical lattices. For two-dimensional Hubbard models at half filling the effects of a Rashba-like SOC were studied both for two-component bosons and fermions, for which exotic spin textures in the ground state such as coplanar spiral waves and stripes as well as non-coplanar vortex/antivortex configurations have been predicted [13–16]. Note, however, that the SOC introduces frustration, invalidating quantum Monte Carlo (MC) approaches, and hence most studies have relied on classical MC calculations.

In this paper we analyze the effects of SOC in a two-component Fermi gas loaded in an optical lattice in the Mott-insulator regime. Since we are interested in the quantum spin-1/2 phases in the presence of SOC, we can not rely on classical MC, and must hence employ exact diagonalization or density-matrix renormalization group (DMRG) techniques. We employ the latter in our paper, restricting our analysis to the minimal system where the non-Abelian character of the vector potential may be manifested allowing non-trivial effects of SOC without the need of breaking the time-reversal invariance, namely a two-leg ladder-like optical lattice, which may be created by incoherently combining a 1D lattice and a two-well potential. By a combination of numerical

II. EFFECTIVE SPIN MODEL FOR TWO-COMPONENT FERMIONS WITH USOC

Recent experiments have realized an USOC characterized by a Hamiltonian of the form [2]:

\[ H_{\text{USOC}} = \frac{1}{2m}(\mathbf{p}\sigma^0 - \mathbf{A})^2 + \frac{\delta}{2}\sigma^z - \frac{\hbar}{2}\sigma^x, \]  

where \( \sigma^x, \sigma^y, \sigma^z \) are Pauli matrices, \( \sigma^0 \) is the identity matrix, and the effective vector potential for counter-propagating Raman lasers on the \( xy \) plane is given by \( \mathbf{A} = -\hbar k_0 \sigma^z \), with \( k_0 = (k_0^x, k_0^y, 0) \). Here the eigenvectors of \( \sigma^x \) correspond to atomic hyperfine components, the term \( \frac{\delta}{2}\sigma^z \) is due to detuning from resonance, and \( \hbar \) is the Rabi

FIG. 1: Two-leg ladder lattice of \( s = 1/2 \) spins \( S_{a,j} \), where \( \alpha = 1, 2 \) enumerates the ladder legs, and \( j = 1, 2, 3, \ldots, L \) labels the ladder rungs.

DMRG results, bosonization techniques and strong rung-coupling expansions, we obtain the spin quantum phases for both a USOC with different orientations with respect to the ladder, and the isotropic SOC.

The paper is organized as follows. In Sec. III we introduce the effective spin model for a Mott state of two-component fermions with USOC in a ladder-like lattice. In Sec. IV we review the phases for the case of decoupled one-dimensional lattices. Section V deals with the quantum phases of an USOC discussing the different orientations between the USOC and the ladder legs. In Sec. VI we analyze the case of an isotropic SOC. We finally summarize in Sec. VII.
coupling. Crucially, $A$ cannot be completely gauged out, since it does not commute with the scalar potential $\Phi = \frac{i}{2} \sigma^z - \frac{h}{2} \sigma^x$.

We consider a two-component Fermi gas loaded in a ladder-like optical lattice of inter-site spacing $a$, with the ladder legs oriented along $x$ and the rungs along $y$. Projecting on the lowest lattice band [17], one obtains, in absence of SOC, the two-component Fermi-Hubbard model:

$$H_{FH} = - \sum_{(i,i'),\sigma,\sigma'} t_{i,i'} \sigma_0^{\sigma,\sigma'} a_{i,\sigma}^\dagger a_{i',\sigma'} + \frac{U}{2} \sum_i n_i (n_i - 1),$$

(2)

where $a_{i,\sigma}$ is the annihilation operator of fermions with spin $\sigma = \uparrow, \downarrow$ on site $i$, $n_i = \sum_{\sigma} a_{i,\sigma}^\dagger a_{i,\sigma}$. $U$ characterizes the on-site interaction, and $t_{i,i'}$ are the hopping amplitudes along the bonds connecting nearest-neighbor sites $(i, i')$, with the hopping along the legs (the rungs) given by $t_{i,i'} = t_x$ ($t_y$). The presence of SOC results in the Peierls substitution $t_{i,i'} \sigma_0 \rightarrow t_{i,i'} e^{i \alpha_{i,i'} \sigma_0}$. In the strong coupling limit, $U \rightarrow \infty$, and considering half-filling (i.e. we consider a Mott phase with one fermion per site [18, 19]), the Fermi-Hubbard model may be rewritten as an effective spin-1/2 model of the form:

$$H = J_{\parallel} \sum_{\alpha,\beta} \left\{ \cos(2k_0^{\alpha} a) S_{\alpha,j} S_{\alpha,j+1} + \sin^2(k_0^{\alpha} a) (S_{\alpha,j} \times S_{\alpha,j+1})^z \right\}$$

$$+ 2 \sin^2(k_0^{\alpha} a) S_{\parallel,j}^z S_{\perp,j+1} + \sin(2k_0^{\alpha} a) [S_{\alpha,j} \times S_{\alpha,j+1}]^z \right\}$$

$$+ J_{\perp} \sum_j \left\{ \cos(2k_0^{\parallel} a) S_{\parallel,j} S_{\perp,j} + \sin^2(k_0^{\parallel} a) (S_{\parallel,j} \times S_{\perp,j})^z \right\}$$

$$+ \delta \sum_{\alpha,\beta} S_{\alpha,j}^z - h \sum_{\alpha,\beta} S_{\alpha,j}^z,$$

(3)

where $J_{\parallel} = 4t_x^2/U$, $J_{\perp} = 4t_y^2/U$, and

$$S_{\alpha,j} = (a_{\alpha,j,\uparrow}^\dagger a_{\alpha,j,\downarrow})^\sigma \left( a_{\alpha,j,\uparrow} a_{\alpha,j,\downarrow}^\dagger \right).$$

(4)

are the spin operators associated to the leg $\alpha = 1, 2$, and the rung $j$ (see Fig. 1), with the site index $i$ in Eq. 2 split into leg and rung indices: $i \rightarrow (\alpha, j)$. The value of $k_0^{\alpha}$ and $k_0^{\parallel}$ is provided by the orientation between the Raman lasers creating the USOC and the ladder axis. Note that the scalar potential $\Phi$ produces the last two terms in Eq. 3, whereas the vector potential $A$ produces Dzyaloshinskii-Moriya (DM) terms [21, 22], $\sim [S_{\alpha,j} \times S_{\alpha',j}]^z$, as well as easy-axis anisotropy (EAA) along $e_z$.

![Fig. 2: (Color online) Ground states of a 1D spin-1/2 chain with USOC and transverse magnetic field obtained using DMRG for 96 sites. The magnetic field is in units of $J_\parallel$. LL denotes a luttinger liquid phase and F stands for ferromagnetic state.](image-url)

**III. DECOUPLED CHAINS**

**A. Repulsive interactions**

We first discuss the case of decoupled chains, $J_{\perp} = 0$ (i.e. $t_y = 0$), which results in the 1D Hamiltonian

$$H_{1D} = J_{\parallel} \sum_j \left( S_{\parallel,j}^x S_{\parallel,j+1}^x + \cos 2k_0^{\parallel} a (S_{\parallel,j}^z S_{\parallel,j+1}^z + S_{\parallel,j}^y S_{\parallel,j+1}^y) + \sin 2k_0^{\parallel} a (S_{\parallel,j}^y S_{\parallel,j+1}^x - S_{\parallel,j}^x S_{\parallel,j+1}^y) \right) - h \sum_j S_{\parallel,j}^z. \quad (5)$$

For $k_0^{\parallel} = 0$ Eq. 3 describes an SU(2)-symmetric spin-1/2 antiferromagnetic chain in external magnetic field, which is exactly solvable by means of Bethe ansatz [23].

The ground state is a gapless Luttinger liquid (LL) for $h < 2J_{\parallel}$, and a fully polarized state for $h > 2J_{\parallel}$. These two phases are separated by a commensurate-incommensurate (C-IC) phase transition.

In order to discuss the effects of the USOC it is convenient to introduce a gauge transformation that renders exchange interactions explicitly SU(2) invariant, $H_{1D} \rightarrow U H_{1D} U^\dagger = H_{1D}$, where $U = \prod_j e^{-2ik_0^{\parallel} a_j} S_{\parallel,j}^y$. The spin operators transform as

$$\tilde{S}_{\parallel,j}^x = \cos (2k_0^{\parallel} a_j) S_{\parallel,j}^x - \sin (2k_0^{\parallel} a_j) S_{\parallel,j}^y,$n$$

$$\tilde{S}_{\parallel,j}^y = \cos (2k_0^{\parallel} a_j) S_{\parallel,j}^y + \sin (2k_0^{\parallel} a_j) S_{\parallel,j}^x.$$

(6)

and $\tilde{S}_{\parallel,j}^z = S_{\parallel,j}^z$, and the Hamiltonian becomes

$$\tilde{H}_{1D} = J_{\parallel} \sum_j \tilde{S}_{\parallel,j} \tilde{S}_{\parallel,j+1} - \sum_j h_j (k_0) \tilde{S}_{\parallel,j}. \quad (7)$$
where the effect of the USOC is entirely absorbed into an external magnetic field, \( \mathbf{h}_j(\mathbf{k}_0) = h (\cos(2k_0^x a_j), \sin(2k_0^x a_j), 0) \), that spirals on the \( xy \) plane.

For \( k_0^x a = \pi/2 \), \( \mathbf{h}_j = (-1)^j h \mathbf{e}_x \), i.e. a staggered effective magnetic field. A staggered field constitutes a relevant perturbation (in the renormalization group sense) as it couples to the Néel order, which is one of the leading instabilities in a 1D antiferromagnetic chain. As a result of that, a gap in the excitation spectrum, \( \Delta E \sim h^{3/4} \), opens for any arbitrary coupling \( h \). The low-energy behavior is described by a massive sine-Gordon model where one of the breather modes is degenerate with soliton and antisoliton excitations \[24\]. In the gauge transformed variables the ground state develops Néel order, which after un-doing the gauge transformation results for the original spin operators into an uniformly magnetized state, i.e. a ferromagnetic (F) state, although magnetization is never fully saturated for \( k_0^x \neq 0 \).

For \( 0 < k_0^x a < \pi/2 \), \( \mathbf{h}_j(\mathbf{k}_0) \) is incommensurate and hence the gapless LL phase survives up to a finite \( h \) value at which the F phase is reached. We have employed the matrix product formulation \[26\] of DMRG method \[27, 28\] to obtain numerically the phase diagram for arbitrary values of the USOC (see Fig. 2). This phase diagram confirms the existence of a gapless LL and a gapped F phase separated by a C-IC transition. Note that correlation functions, which decay algebraically in the LL phase and exponentially in the F phase, are generically incommensurate due to the DM anisotropy and the vector product of two neighbouring spins has finite expectation value \( \langle [S_j \times S_{j+1}]^z \rangle \sim -\sin(2k_0^x a) \) as depicted in Fig. 3(a). Its magnetic field dependence is presented in Fig. 3(b).

\section{B. Attractive interactions}

For the decoupled chains we have also studied the case of two-component fermions with attractive interactions. The most interesting ground-state physics occurs at half-filling in the vicinity of the maximal USOC, \( k_0^a \simeq \pi/2 \). In this case, after particle-hole transformation the 1D Fermi-Hubbard model becomes dual to the repulsive ionic-Hubbard model \[22\], being characterized by the existence of a dimerized (D) phase between a superconducting (SC) phase and the F state. With increasing magnetic field the SC phase undergoes a Kosterlitz-Thouless (KT) transition into the D state, where translational symmetry is spontaneously broken. Further increasing the magnetic field results in a D-F Ising transition. We characterized the D phase in our numerical simulations by means of the dimerization order parameter, which in a chain with \( L \) sites is defined as:

\[ D = \sum_j (\frac{-1}{L})^j \left( a_j^\dagger a_{j+1\downarrow} - a_j^\dagger a_{j+1\uparrow} + h.c. \right). \]

\section{IV. TWO-LEG LADDER WITH USOC}

We consider now the case of coupled chains with nonzero hoppings \( t_{x,y} \). As mentioned above, the value of \( k_0^x \) and \( k_0^y \) depends on the orientation of the USOC lasers and the ladder axis. In the following we consider separately the case in which the USOC is along the rungs and that in which the USOC is along the legs.

\subsection{A. USOC along the ladder rungs}

We analyze first the case of an USOC along the ladder rungs, i.e. \( k_0^y = 0 \) in Eq. (4). For \( k_0^y = 0 \) the magnetic field introduces two C-IC phase transitions: from a rung-singlet (RS) into a LL and then from the LL into the fully polarized F state. As in our discussion of Sec. III it is
phase boundaries are obtained after finite-size extrapolation of a 1D superconductor. The magnetic field is in units of $H_{\parallel}$. FIG. 4: Phase diagram, for an attractive two-component Fermi Hubbard model on a chain at half filling with maximal USOC, where D denotes a dimerized phase, and SC stands for a 1D superconductor. The magnetic field is in units of $t_x$. The phase boundaries are obtained after finite-size extrapolation from data obtained for 128, 256, 512 and 1024 sites.

![Graph](image)

convenient to introduce the gauge transformation

$$
\tilde{S}_{\alpha,j}^z = \cos (2k_0^y a) S_{\alpha,j}^z - \sin (2k_0^y a) S_{\alpha,j}^y,
$$

$$
\tilde{S}_{\alpha,j}^y = \cos (2k_0^y a) S_{\alpha,j}^y + \sin (2k_0^y a) S_{\alpha,j}^x
$$

and $\tilde{S}_{\alpha,j}^z = S_{\alpha,j}^z$. For the case of the maximal USOC, $k_0^y a = \pi/2$, the gauge transformed Hamiltonian becomes:

$$
\tilde{H} = J_{||} \sum_{\alpha=(1,2), j} \tilde{S}_{\alpha,j} \tilde{S}_{\alpha,j+1} + J_\perp \sum_{j} \tilde{S}_{1,j} \tilde{S}_{2,j}
$$

$$
- h \sum_{\alpha=(1,2), j} (-1)^\alpha \tilde{S}_{\alpha,j}^x.
$$

In the strong rung-coupling limit, $J_\perp \gg J_{||}$, the ground state becomes a rung-product state of the form:

$$
|\tilde{R}S\rangle = \prod_{j} \left( |\tilde{1}_{1,j}\rangle \otimes |\tilde{1}_{2,j}\rangle - \beta |\tilde{1}_{1,j}\rangle \otimes |\tilde{2}_{2,j}\rangle \right) / \sqrt{1 + \beta^2},
$$

where $\{\tilde{1}, \tilde{2}\}$ refer to the eigenstates of $\tilde{S}^z$. For $h = 0$, $\beta = 1$ and the ground-state is a product state of singlets along the rungs. With increasing magnetic field $\beta$ decreases gradually tending to zero. For $\beta = 0$ the ground-state after undoing the gauge transformation translates into the F state. Hence, for $k_0^y a = \pi/2$ the magnetic field just results in an adiabatic evolution of $|RS\rangle$ into the F state.

To address the general case $0 < k_0^y a < \pi/2$ we consider the case of weak USOC, $k_0^y a \ll 1$, closely following the strong rung-coupling derivation of Ref. [29]. For $h = 0$ the ground state is well approximated by a direct product of singlets along the rungs, and the energy gap to the lowest rung triplet excitation is $J_\perp$. The external magnetic field splits linearly the rung triplet excitations, and the energy of the state where both spins of the rung point in the direction of the field approaches that of the RS state for $h \sim J_\perp$. Identifying the RS state on a rung with an effective spin-1/2 pointing down, and the $S^x = 1$ component of the rung triplet state with the spin-1/2 pointing up, the effective pseudo-spin-1/2 model in the strong rung-coupling limit for $h \sim J_\perp$ takes the form of an XXZ model in a tilted uniform magnetic field:

$$
H_x = J_{||} \sum_j \left( \tau^x_{j,j+1} + \tau^y_{j,j+1} + \tau^z_{j,j+1} \right)
$$

$$
- h_x \sum_j \tau^x_j - h_y \sum_j \tau^y_j
$$

where $\tau^{x,y,z}$ are the pseudo-spin-1/2 operators, $h_x = h - J_\perp \cos 2k_0^y a + J_\perp (1 - \cos 2k_0^y a)/4 - J_{||}/2$, and $h_y = J_\perp \sin 2k_0^y a/\sqrt{2}$. With varying $h_x$ the model (12) undergoes changes in three ground-state phases [30]: two F phases separated by Ising transitions from an intermediate Néel phase in $\tau^z$ state. One of the F phases of the effective model (12) translates to the RS phase of the ladder, whereas the Néel phase and the second F phase of (12) translates into identical ladder phases. Note that it is the DM interaction that in the leading order breaks in Eq. (12) the $U(1)$ rotation symmetry in the $yz$ plane allowing for the Néel ordering.

We consider at this point weakly coupled chains, $J_\perp \ll J_{||}$, again for weak USOC, $k_0^y a \ll 1$. For this case we can use bosonization mapping [31] with the convention:

$$
S_{\alpha,j}^x \rightarrow \partial_x \phi_\alpha + (-1)^j \sin \sqrt{2\pi} \phi_\alpha,
$$

$$
S_{\alpha,j}^y \rightarrow (-1)^j \sin \sqrt{2\pi} \phi_\alpha + \cdots,
$$

$$
S_{\alpha,j}^z \rightarrow (-1)^j \cos \sqrt{2\pi} \phi_\alpha + \cdots
$$

where $x = ja$, the dots denote sub-leading fluctuations of uniform components, and we have introduced two pairs of dual bosonic fields, $[\delta_\alpha(x), \partial_y \phi_\alpha] = i \delta_\alpha \omega_\alpha(x - y)$. It is convenient to introduce the symmetric and antisymmetric combinations of the original bosonic fields, $\theta_\pm = (\phi_1 \pm \phi_2)/\sqrt{2}, \phi_\pm = (\phi_1 \pm \phi_2)/\sqrt{2}$. We treat $J_\perp$ as a perturbation of the two decoupled systems retaining only the relevant contributions that it generates. We obtain
we monitor the following Hamiltonian density:

\[ H = \sum_{\nu = \pm} v_\nu \left[ (\partial_x \phi_\nu)^2 + (\partial_y \phi_\nu)^2 \right] - h \partial_x \phi_+ / \sqrt{\pi} \]

\[ + J_1 (2 \cos \sqrt{4\pi} \phi_- - \cos \sqrt{4\pi} \phi_+ + \cos \sqrt{4\pi} \phi_-) \]

\[ + d_\perp (\cos \sqrt{4\pi} \phi_+ \sin \sqrt{4\pi} \theta_+ \sin \sqrt{4\pi} \phi_- \cos \sqrt{4\pi} \theta_- \]

\[ - \sin \sqrt{4\pi} \phi_+ \cos \sqrt{4\pi} \phi_+ \cos \sqrt{4\pi} \phi_- \sin \sqrt{4\pi} \theta_-) \]

\[ + d_\perp (\cos \sqrt{4\pi} \theta_- + \cos \sqrt{4\pi} \theta_+) \]

where

\[ v_\pm \approx \frac{J_{||} a \pi}{2} \left( 1 \pm \frac{J_{||} \cos(2k_0^\parallel a)}{J_0 \pi^2} \right) \]

and \( J_\perp \sim J_1 \cos(2k_0^\parallel a) \). The DM anisotropy generates the terms with the coupling constant \( d_\perp \sim J_1 \sin(2k_0^\parallel a) \), whereas the EEA induces the terms with the pre factor \( d_\perp \sim J_1 (1 - \cos(2k_0^\parallel a)) \). All three factors \( J_\perp, d_\perp \) and \( d_\parallel \) depend as well on a short-distance cut-off. Note that the magnetic field, \( h \), just couples to the symmetric sector.

For \( k_0^\parallel \ll 1 \), \( d_\perp \ll J_1 \), and the antisymmetric sector remains gapped with \( \langle \theta_- \rangle = \sqrt{\pi}/2 \). Note that when the magnetic field suppresses the dominant coupling in the symmetric sector, \( J_\perp \cos \sqrt{4\pi} \phi_+ \), the EEA term induces the leading instability.

The symmetric sector can be solved by means of the Jordan-Wigner mapping and a subsequent Bogoliubov transformation. It supports two Ising phase transitions with increasing magnetic field that separate three different ground-state phases. In the original spin variables, these phases translate into the RS phase, a Néel state degenerate with the ground state in the thermodynamic limit, whereas the RS and F states have unique gapped ground-states. Hence a simple way to obtain the boundary of the Néel state is to follow the closing of the gap between the ground-state and the first excited state (that becomes degenerate with the ground state in the thermodynamic limit in the Néel phase). We plot the behavior of the gap in Fig. 8(b). The gaps close linearly with the magnetic field when approaching the quantum phase transition points, as expected from the Ising character.

We note finally, that the vector product of two neighboring spins has a finite expectation value along rungs, \( \langle S_{1,j} \times S_{2,j} \rangle \sim -\sin(2k_0^\parallel a) \), in all phases, becoming zero only deep in the F phase for large values of \( h \).

### B. 2-leg ladder with USOC along legs

We consider now that the USOC is oriented along the ladder legs, as depicted in Fig. 9 and hence \( k_0^\parallel = 0 \) in Eq. (3). For \( k_0^\parallel = 0 \) the external magnetic field induces two consecutive C-IC phase transitions: a first one from RS into the gapless LL phase, and a second one from LL into the fully polarized F state. For \( k_0^\parallel \neq 0 \) we may perform a gauge transformation similar to the ones discussed above, which for the maximal USOC, \( k_0^\parallel = \pi/2 \), results in a model similar to Eq. (19) but in this case with a field that couples uniformly to spins belonging to the same rung and it alternates from rung to rung, \(-h \sum_{\alpha,j} (-1)^j S_{\alpha,j}^z \). Using bosonization in the weak rung-coupling limit, \( J_\perp \ll J_{||} \), and in opposite limit \( J_\perp \gg J_{||} \) employing strong rung-coupling expansion it has been
determined that such magnetic field introduces Gaussian criticality between two gapped phases of the antiferromagnetic spin ladder \[33\], that for our original spin variables corresponds to RS and F states.

The difference at maximal USOC between the case with USOC along the ladder legs and that with USOC along the rungs can be easily understood in the limit \( J_\perp \gg J_\parallel \). For the case of USOC along the ladder legs the magnetic field couples uniformly to the spins on the same rung, and hence it favors a triplet state on each rung with both spins pointing in the same direction, which alternates from one rung to the next. This state is orthogonal to the RS configuration. In contrast, in Eq. (10) the magnetic field couples in a staggered way to the spins in the same rung, and the ground-state favored by a strong magnetic field is not orthogonal to the RS state. As a result, for the USOC along rungs the RS state can be adiabatically connected to the F state, whereas for the USOC along legs this is not possible.

Based on the previous discussion we hence expect that the two C-IC phase transition points for \( k_0^a = 0 \) have to evolve into a single Gaussian point for \( k_0^a = \pi/2 \). As for the case of USOC along rungs, we can employ bosonization to understand this evolution of the critical points in the limit \( J_\perp \ll J_\parallel \). The leading instability once the magnetic field suppresses the RS phase is again the EAA, however now in exchange interactions along the chains and DM anisotropy induces incommensurability \[34\]. In contrast to the relevant couplings produced by the USOC along the rungs, in Eq. (10) the USOC along ladder legs produces a marginal perturbation, \( \sim \cos \sqrt{4\pi\theta_-} \cos \sqrt{4\pi\theta_+} \). After mean-field decoupling between the symmetric and antisymmetric sectors the weak rung-coupling bosonic Hamiltonian for the USOC along legs is equivalent to Eq. (10), and hence the ground-states and phase transitions will be similar to the previous case of USOC along rungs. Thus C-IC phase transition points evolve into Ising lines for \( k_0^a > 0 \) and at \( k_0^a = \pi/2 \) these two Ising lines merge in a Gaussian
USOC along the ladder legs, $k_0^x = 0$. For $k_0^x = 0(\pi)$ there are two C-IC transitions with increasing magnetic field: a first one from RS to LL, and a second one from LL into the fully polarized state (both LL and fully polarized states are indicated by bold lines). For $0 < k_0^x < \pi/2$ instead of the LL state a Néel state is realized, being separated from the RS and F states by Ising phase transition lines. These lines cross at $k_0^x = \pi/2$ resulting in a direct Gaussian transition from RS to F. The magnetic field is in units of $J$.

Criticality due to the enhanced symmetry.

Our numerical results for $n^2$ and the excitation gap are depicted in Figs. 10. Note that finite size effects are more pronounced at the RS to Néel transition, whereas for the Néel to F transition finite-size effects are negligible. For the RS to Néel transition we have carefully performed finite-size scaling of the order parameter and determined the critical field $h_c$ corresponding to the phase transition from the intersection of the order parameter curves for different system sizes. The collapse of order parameter for different system sizes in the vicinity of $h_c$ on the single curve according to the Ising law is depicted in the inset of Fig. 11(a).

Note finally that the vector product of two neighboring spins has finite expectation value along the chains $\langle S_{\alpha,j} \times S_{\alpha,j+1}\rangle = -\sin(2k_0^x a)$. Its magnetic field dependence is similar to the curve of Fig. 11(b), and it vanishes quickly in the F phase.

The ground-state phase diagram for the USOC along the ladder legs is depicted in Fig. 11. As mentioned above, the C-IC phase transition points (corresponding to $U(1)$ symmetry at $k_0^x = 0$) transform into Ising transitions (for $0 < k_0^x < \pi/2$ the system does not have continuous symmetry), and then they combine into a Gaussian point at $k_0^x a = \pi/2$ (where $U(1)$ symmetry is revived). So in both cases, USOC either along ladder rungs or along ladder legs, the system presents three possible phases, RS, Néel, and F. For a general orientation of the USOC and the ladder legs, $k_0^x \neq 0$ and $k_0^y \neq 0$, we hence expect these three phases as well.

V. TWO-LEG LADDER WITH NON-ABELIAN VECTOR POTENTIAL

We consider at this point a non-Abelian vector potential of the form, $\mathbf{A} = (-\hbar k_0^x \sigma^x, -\hbar k_0^y \sigma^y)$. Contrary to the case of USOC the magnetic field, $h$, is not necessary to ensure the non-trivial character of SOC. We hence consider the time-reversal symmetric case, $h = 0$, and a balanced mixture of up and down spin fermions. The effective spin model in this case acquires the form:

$$H = J_\parallel \sum_{\alpha,j} \left\{ \cos(2k_0^x a) S_{\alpha,j} S_{\alpha,j+1} + \sin(2k_0^y a) [S_{\alpha,j} \times S_{\alpha,j+1}]^z \right\} + J_\perp \sum_j \left\{ \cos(2k_0^x a) S_{1,j} S_{2,j} + \sin(2k_0^y a) [S_{1,j} \times S_{2,j}]^y \right\}$$

(16)

FIG. 11: (Color online) Numerical phase diagram for the USOC along the ladder legs, $k_0^x = 0$. For $k_0^x = 0(\pi)$ there are two C-IC transitions with increasing magnetic field: a first one from RS to LL, and a second one from LL into the fully polarized state (both LL and fully polarized states are indicated by bold lines). For $0 < k_0^x < \pi/2$ instead of the LL state a Néel state is realized, being separated from the RS and F states by Ising phase transition lines. These lines cross at $k_0^x = \pi/2$ resulting in a direct Gaussian transition from RS to F. The magnetic field is in units of $J$.

FIG. 12: Ground states for spin-ladder with Rashba SOC, see text. The numerical results correspond to DMRG calculations for $L = 48$ rungs.
In the vicinity of \((k_0^a, k_0^b) = (\pi/2, \pi/4)\), which is characterized by a ferromagnetic \(S_{\alpha,j}^x S_{\alpha,j+1}^x\) coupling along legs, and an antiferromagnetic \(S_{1,j}^y S_{2,j}^y\) exchange along rungs. \(S_{x,y}\) states are dual to each other with respect to the interchange of leg and rung directions and \(S^x\) and \(S^y\) components. Our numerical simulations suggest that similarly to the USOC case all phase transitions for the case of the non-Abelian vector potential are of second-order Ising nature. This is natural, since the system does not enjoy in general any continuous symmetry, and striped phases break spontaneously discrete \(Z^2\) symmetries: \(S_{x,y}\) breaks translation symmetry along the chains direction whereas \(S_{x,y}\) breaks the parity symmetry associated with the exchange of ladder legs. Both striped phases break as well time reversal symmetry.

The RS and \(\tilde{R}S\) phases present different parity symmetry for an odd number of rungs, whereas the RS phase is antisymmetric the RS is symmetric; As a result both phases cannot connect adiabatically. We could not determine numerically whether RS and \(\tilde{R}S\) states can be connected adiabatically for an even number of rungs in the parameter space \((k_0^a, k_0^b)\) in Fig. 12. In particular, the string order, defined for the pair of spins across the ladder diagonal, is finite for both RS and \(\tilde{R}S\) states and vanishes in the striped phases. However in the thermodynamic limit we expect the behavior of odd and even number of rungs to converge, and hence it is most likely that in the model given in Eq. (10) RS and \(\tilde{R}S\) states are always separated by a phase transition (indicated by dashed lines in Fig. 12).

**VI. CONCLUSIONS**

In our work we have discussed the quantum spin phases and the associated quantum phase transitions for a two-component Fermi lattice gas, focusing on the case of a two-leg ladder-like lattice at half-filling, a minimal system to study the non-Abelian character of the vector potential. We have shown that for an USOC along the ladder rungs an Néel state phase is located within a RS-F phase, in which a rung-singlet may be adiabatically connected to a ferromagnetic phase in the parameter space of \(h\) and the SOC. In contrast, for the USOC along the ladder legs the RS and F states cannot be adiabatically connected, and are separated by an intermediate Néel state, which disappears at a maximal SOC to lead to a direct Gaussian RS-F quantum phase transition. The case of a Rashba-like SOC is characterized by the appearance of rung-singlet and striped phases. Compared to the classical spin phases predicted for fermions on a square lattice with SOC the only striped configurations of the 2D lattice have identical quantum counterparts on the ladder. On the contrary, the Néel and spiral waves are substituted by gapped rung-singlet states, whereas non-coplanar configurations such as vortex/antivortex tex-
tures are not stabilized.

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