Abstract

Given a Boolean function $f$, the (Hamming) weight $wt(f)$ and the nonlinearity $N(f)$ are well known to be important in designing functions that are useful in cryptography. The nonlinearity is expensive to compute, in general, so any shortcuts for doing that for particular functions $f$ are significant. It is clear from the definitions that $wt(f) \leq N(f)$. We give a useful and practical sufficient condition for the weight and nonlinearity to be equal, namely $wt(f) \leq 2^{n-2}$ if $f$ has $n$ variables. It is shown that this inequality cannot be improved, in general. This condition can also be used to actually compute the nonlinearity in some cases. As an application, we give a simple proof for the value of the nonlinearity of the well known majority function. Previous proofs relied on many detailed results for the Krawtchouk polynomials.

Key words: Hamming weight, nonlinearity, Boolean functions, majority function, Walsh transform.

1 Introduction

Boolean functions have many applications, particularly in coding theory and cryptography. A detailed account of the latter applications can be found in the book [3]. If we define $\mathbb{V}_n$ to be the vector space of dimension
n over the finite field $GF(2) = \{0, 1\}$, then an n variable Boolean function $f(x_1, x_2, \ldots, x_n) = f(x)$ is a map from $\mathbb{V}_n$ to $GF(2)$. Every Boolean function $f(x)$ has a unique polynomial representation (usually called the algebraic normal form [3, p. 8]), and the degree of $f$ is the degree of this polynomial. A function of degree 1 is called affine, and if the constant term is 0 such a function is called linear. We let $B_n$ denote the set of all Boolean functions in $n$ variables, with addition and multiplication done mod 2. If we list the $2^n$ elements of $\mathbb{V}_n$ as $v_0 = (0, \ldots, 0), v_1 = (0, \ldots, 0, 1), \ldots$ in lexicographic order (we abbreviate this as lexic order below), then the $2^n$-vector $(f(v_0), f(v_1), \ldots, f(v_{2^n-1}))$ is called the truth table of $f$. The weight (also called Hamming weight) $wt(f)$ of $f$ is defined to be the number of 1’s in the truth table for $f$. In many cryptographic uses of Boolean functions, it is important that the truth table of each function $f$ has an equal number of 0’s and 1’s; in that case, we say that the function $f$ is balanced.

The distance $d(f, g)$ between two Boolean functions $f$ and $g$ in the same number of variables is defined by

$$d(f, g) = wt(f + g),$$

where the polynomial addition is done mod 2. An important concept in cryptography is the nonlinearity $N(f)$ defined by

$$N(f) = \min_{a \text{ affine}} d(f, a).$$

In order for a Boolean function to be useful in a cryptographic application, it is usually necessary that the function have high nonlinearity (see, for example, [3, p. 122]). So-called Fourier analysis of Boolean functions (see [3, Chapter 2]) is very important in cryptography and other contexts. The efficient computation of values of the nonlinearity is important here, and for this a very important tool is the Walsh transform of a Boolean function $f_n$ in $n$ variables. The Walsh transform of $f_n$ is the map $W_f : \mathbb{V}_n \to \mathbb{R}$ defined by

$$W_f(w) = \sum_{x \in \mathbb{V}_n} (-1)^{f(x) + w \cdot x},$$

where the values of $f = f_n$ are taken to be the real numbers 0 and 1. This Walsh transform has also been important in physics and other sciences for at least 40 years. In this paper we want the Walsh transform because of the well known formula (see [3, Th. 2.21, p. 17], where $\hat{f}$ is used instead of $f$)

$$N(f_n) = 2^{n-1} - \frac{1}{2} \max_{u \in \mathbb{V}_n} |W_f(u)|$$

(1)
We shall use the obvious fact
\[ W_f(0) = 2^n - 2\text{wt}(f) \]
without comment in the rest of the paper.

2 Weight = nonlinearity theorem

It follows from the definitions that the nonlinearity is always \( \leq \) the weight. The following theorem gives a useful sufficient condition for the weight and nonlinearity to be equal. We use the notation \( C(f) \) for the complement of \( f \), that is, the function obtained by switching 0 to 1 and 1 to 0 for every entry in the truth table of \( f \). Similarly, if \( S \) is a bitstring, then \( C(S) \) is its complement.

**Theorem 1.** Let \( f \) be a Boolean function in \( n \) variables with \( \text{wt}(f) \leq 2^{n-2} \). Then \( \text{wt}(f) = N(f) \).

**Proof.** We shall prove \( \max_{u \in V_n} |W_f(u)| = W_f(0) \). Then (1) will give the theorem, as explained below.

Define
\[ T_{11}(u) = \{ v \in V_n | u \cdot v \equiv 1 \mod 2 \text{ and } f(v) = 1 \} \]
and
\[ T_{10}(u) = \{ v \in V_n | u \cdot v \equiv 1 \mod 2 \text{ and } f(v) = 0 \} \].

Similarly, define \( T_{00}(u) \) and \( T_{01}(u) \) by
\[ T_{0j}(u) = \{ v \in V_n | u \cdot v \equiv 0 \mod 2 \text{ and } f(v) = j \} \text{ for } j = 0, 1 \].

Let
\[ D_i(u) = \{ v | v \in V_n \text{ and } u \cdot v \equiv i \mod 2 \} \text{ for } i = 0, 1. \]

Then
\[ |D_1(u)| = T_{11}(u) + T_{10}(u) = 2^{n-1}, \text{ } u \neq 0 \quad (2) \]
and
\[ |D_0(u)| = T_{01}(u) + T_{00}(u) = 2^{n-1}, \text{ } u \neq 0. \quad (3) \]

We have
\[ W_f(u) = \sum_{v \in V_n} (-1)^{u \cdot v} (-1)^{f(v)} = -|T_{10}(u)| + |T_{11}(u)| + |T_{00}(u)| - |T_{01}(u)| \quad (4) \]
and (note $|T_{10}(0)| = |T_{11}(0)| = 0$, $|T_{01}(0)| = w(f)$, $|T_{00}(0)| = 2^n - w(f)$)

$$W_f(0) = |T_{00}(0)| - |T_{01}(0)| = 2^n - 2w(f) \geq 0. \quad (5)$$

Our assumption $w(f) \leq 2^{n-2}$ implies

$$|T_{11}(u)| \leq 2^{n-2} \text{ for all } u \neq 0. \quad (6)$$

Then (2) gives $|T_{10}(u)| \geq |T_{11}(u)|$ for $u \neq 0$. Hence (4) and (5) imply

$$W_f(u) - W_f(0) = |T_{11}(u)| - |T_{10}(u)| + |T_{00}(u)| - |T_{01}(u)| - 2^n + 2w(f). \quad (7)$$

Using (2) and (3) to eliminate $|T_{10}(u)|$ and $|T_{00}(u)|$, respectively, from (7) we obtain

$$W_f(u) - W_f(0) = 2(|T_{11}(u)| - |T_{01}(u)|) - 2^n + 2w(f). \quad (8)$$

Since $|T_{01}(u)| = w(f) - |T_{11}(u)|$, (5) implies

$$W_f(u) - W_f(0) = 4|T_{11}(u)| - 2^n \leq 0, \quad (9)$$

where the last inequality follows from (6). Now (5) and (9) give

$$W_f(u) + W_f(0) = 4|T_{11}(u)| + 2^n - 4w(f) \geq 0, \quad (10)$$

where the last inequality follows from our hypothesis $w(f) \leq 2^{n-2}$. Now (9) and (10) imply $\max_{u \in V_n} |W_f(u)| = W_f(0)$. Thus $N(f) = 2^{n-1} - \frac{1}{2}W_f(0) = 2^{n-1} - \frac{1}{2}(2^n - 2w(f)) = w(f). \quad \square$

If $w(f)$ is much smaller than the upper bound in Theorem 1, then it is possible to give a very simple proof that $w(f) = N(f)$ using the well known Parseval equation (see [3, Cor. 2.23, p. 17])

$$\sum_{x \in V_n} (W_f(w))^2 = 2^{2n}. \quad (11)$$

If

$$W_f(0) \geq 2^{n-(1/2)}, \quad (12)$$

then (11) trivially implies $\max_{u \in V_n} |W_f(u)| = W_f(0)$ and so the last sentence in the proof of Theorem 1 will give $w(f) = N(f)$. Since $W_f(0) = 2^n - 2w(f)$, (12) is equivalent to

$$w(f) \leq (1 - \frac{\sqrt{2}}{2})2^{n-1} = .146\ldots 2^n.$$
It does not seem possible to refine this kind of argument to get the much better constant .25 in Theorem 1. Theorem 1 is sharp in the sense that it is no longer true if we replace the upper bound on \( wt(f) \) by \( 2^{n-2} + 1 \). For example, the function with \( n = 3 \) having truth table \((0,0,0,1,0,0,1,1)\) (its algebraic normal form is \( x_1x_2x_3 + x_1x_2 + x_2x_3 \)) has weight 3 and nonlinearity 1. However, sometimes we can use Theorem 1 to evaluate the nonlinearity of functions \( g_n \) in \( n \) variables even if \( wt(g_n) \neq N(g_n) \), and even if \( wt(g_n) > 2^{n-2} \). An example of this (Theorem 6) is given in the next section.

3 Finding nonlinearity of the majority function

The Boolean majority function \( M_k(x) \) can be defined in several slightly different ways, but here we use the most common one, namely

\[ M_k(x) = 1 \text{ if and only if } wt(x) \geq k/2. \]

The majority function seems to have been first mentioned in a cryptographic context in the 1991 book [6, pp. 70-80], where only the case of \( k \) odd was considered. The function has been of special interest in cryptography for the last dozen years or so because this function and variants of it can be proven to have optimal algebraic immunity. We do not need to explain this technical concept here; interested readers can find a discussion of it in [3, pp. 174-176]. A useful recent survey of the applications of majority functions to algebraic immunity is given in [2]. The papers [1, 5, 7, 9, 10, 11] and others referenced in those papers all contain recent work on cryptographic applications of these functions. It is necessary to know the nonlinearity of the majority function to begin such studies.

A frequently quoted determination of the nonlinearity of the majority function was given in [5, Th. 3, p. 52], using many detailed results on the Krawtchouk polynomials (these date back to 1929; see [8, pp. 150-154] for a study of these polynomials and proofs of some of their properties). It is convenient to deal with the cases of odd and even \( k \) separately, so we define:

\[ M_{2n+1}(x) = 1 \text{ if and only if } wt(x) \geq n + 1 \]  \hspace{1cm} (13)

and

\[ M_{2n}(x) = 1 \text{ if and only if } wt(x) \geq n. \]  \hspace{1cm} (14)

For any \( k \), we let \( M(k) \) denote the truth table of \( M_k(x) \). We let \( A(k) \) and \( B(k) \) denote, respectively, the left and right halves of the truth table, so

\[ M(k) = A(k)B(k) \]
(juxtaposition of the truth tables, each one being thought of simply as a bitstring). We give a much simpler proof for the value of the nonlinearity of the majority function in Theorem 5 below, but first we need three preliminary lemmas. If $S$ is a bitstring, then $S^*$ denotes the string $S$ in reverse order.

**Lemma 2.** We have $M(2n + 1) = C(M(2n))^* M(2n)$ for $n \geq 2$.

*Proof.* By (13), we can describe the two halves of $M(2n + 1)$ by

$$A(2n + 1) = \{0y : wt(y) \geq n + 1, y \text{ in lexico order}\}$$  \(15\)

and

$$B(2n + 1) = \{1y : wt(y) \geq n, y \text{ in lexico order}\}. \quad (16)$$

Now (15) implies

$$A(2n + 1)^* = \{0y : wt(y) \geq n + 1, y \text{ in reverse lexico order}\},$$

so (16) gives

$$B(2n + 1) = C(A(2n + 1))^*. \quad (17)$$

It follows from (14) and (15) that

$$A(2n + 1) = \{C(1y) : wt(y) \leq n, y \text{ in lexico order}\}$$

$$= \{C(y1) : wt(y) \leq n, y \text{ in reverse lexico order}\}$$

$$= C(M(2n))^*$$

and then (17) gives $B(2n + 1) = M(2n)$. \[\square\]

Below is an example, in which we use the notation $a_i$ for a string of $i$ symbols $a = 0$ or 1.

**Example 1.** $M(5) = 071031013_31013017$

Note that it follows from (15) and (17) that $wt(M(2n + 1)) = 2^{2n}$, so $M_{2n+1}(x)$ is a balanced function.

**Lemma 3.** For any truth table $A$, the nonlinearity of the function with truth table $A \ C(A)^*$ is given by

$$N(A \ C(A)^*) = 2N(A).$$

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Proof. For any affine function \(a\) we have
\[
d(A, a) = d(C(A), C(a)) = d(C(A)^* , C(a)^*).
\]
Hence the lemma follows from the fact that \(aC(a)^*\) is an affine function whenever \(a\) is (see the well known Folklore Lemma [3, Lemma 2.2, p. 8]). \(\square\)

**Lemma 4.** We have \(wt(A(2n + 1)) = 2^{2n-1} - \frac{1}{2} \binom{2n}{n}\) for \(n \geq 2\).

Proof. If we let \(b_{2n}\) denote a bitstring of length \(2n\), then (15) gives
\[
wt(A(2n + 1)) = 2^{2n} - \left| \{ b_{2n} : wt(b_{2n}) \leq n \} \right| = 2^{2n} - \sum_{j=0}^{n} \binom{2n}{j}
\]
and now the Binomial Theorem implies the formula in the lemma. \(\square\)

Our next example illustrates Lemmas 2 and 4.

**Example 2.**
\[
M(6) = 0_710_3101_3 0_3101301_7 0_3101301_7 01_15
A(7) = 0_151 0_710_3101_3 0_710_3101_3 0_3101301_7
\]

Now the nonlinearity formulas for \(M_k(x)\) are given in the following theorem.

**Theorem 5.** The nonlinearity of the majority function for \(n \geq 2\) is given by
\[
N(M(2n + 1)) = 2wt(A(2n + 1)) = 2^{2n} - \binom{2n}{n} \quad (18)
\]
and
\[
N(M(2n)) = \frac{1}{2} N(M(2n + 1)) = 2^{2n-1} - \frac{1}{2} \binom{2n}{n}. \quad (19)
\]

Proof. The value \(2^{2n} - \binom{2n}{n}\) for \(M(2n + 1)\) (in a different notation) was obtained via (1) in [6, Th. 4.20, p. 74]. The proof uses only elementary properties of binomial coefficients but is long enough so that we will not repeat it here. Now Lemma 4 completes the proof of (18). Finally, Lemmas 2 and 3 imply (19). \(\square\)
Note that Lemma 2 gives $A(2n + 1) = C(M(2n))^*$ (see Example 2) and with (19) plus Lemma 4 we obtain
\[ N(A(2n + 1)) = N(M(2n)) = wt(A(2n + 1)), \]
which is a weight = nonlinearity result not implied by Theorem 1.

We can use Theorem 1 to obtain even more information about the majority functions, for example we can determine $N(B(2n))$, which is given in the next theorem. The next example, which illustrates how the truth tables of $M(i)$ change as $i$ increases, may be helpful in reading the proof.

**Example 3.**

\[
M(2n - 1) = A(2n - 1) \ B(2n - 1) \\
M(2n) = A(2n - 1) \ B(2n - 1) \ B(2n) \\
M(2n + 1) = Q1(2n + 1) \ M(2n - 1) \ M(2n - 1) \ B(2n)
\]

**Theorem 6.** The nonlinearity of $B(2n)$ for $n \geq 3$ is equal to
\[
wt(C(B(2n))^*) = 2^{2n-2} \sum_{j=n+1}^{2n-2} \binom{2n-2}{j} + \binom{2n-2}{n} 
\]
(20)
\[
= \sum_{j=n+1}^{2n-2} \binom{2n-2}{j} + 2^{2n-3} - \frac{1}{2} \binom{2n-2}{n-1}. 
\]
(21)

**Proof.** Let $Q1(2n+1)$ denote the first quarter of the truth table of $M(2n+1)$. It follows from Lemma 2 that
\[ Q1(2n + 1) = C(B(2n))^*. \]
(22)

If we let $Q1A(2n + 1)$ and $Q1B(2n + 1)$ denote, respectively, the left and right halves of the truth table for $Q1(2n + 1)$ and we let $b_{2n-2}$ denote a bitstring of length $2n - 2$, then
\[ Q1A(2n + 1) = \{000b_{2n-2} : wt(b_{2n-2}) \geq n + 1, b_{2n-2} \text{ in lexico order}\} \]
and
\[ Q1B(2n + 1) = \{001b_{2n-2} : wt(b_{2n-2}) \geq n, b_{2n-2} \text{ in lexico order}\}. \]

Therefore
\[ wt(Q1A(2n + 1)) = \sum_{j=n+1}^{2n-2} \binom{2n-2}{j} \]
(23)
and
\[ \text{wt}(Q1B(2n+1)) = \sum_{j=n}^{2n-2} \binom{2n-2}{j}. \] (24)

Now (20) follows from (22), (23) and (24), and (21) follows from (21) by elementary properties of binomial coefficients.

Simple estimates using (21) show that \( \text{wt}(C(B(2n))^*) < 2^{2n-2} \). Hence Theorem 1 gives \( \text{wt}(C(B(2n))^*) = N(C(B(2n))^*) = N(B(2n)) \).

The methods in Section 3 can be used to clarify various earlier results in the literature. For example, it is easy to show that the functions \( \phi_{2k} \) discussed in [4, p. 106] satisfy \( \phi_{2k} = M(2k) \).

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