The spinor string in a Clifford substructure of space-time

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Abstract

We factorize the space-time coordinates of Minkowski space into Weyl spinors with components in a split Clifford algebra. Poisson brackets are defined for Clifford-valued canonical variables and applied to the quantization of the point particle and string. In particular, we obtain the Lorentz algebra for the quantum string and show that the string can have half-integral spin. The Clifford algebra is augmented with the octonions through an R-algebra tensor product, and we apply the results of Manogue, Schray and Dray on octonionic Lorentz transformations to obtain a Lorentz invariant string action in ten dimensions.

1 Spinor substructure of Minkowski space

At first glance there seems to be no connection between the quantum interference between alternative space-time paths and the double homomorphism \( SL(2, \mathbb{C}) \Rightarrow SO(1,3) \) of the Lorentz group. However, if space-time is equipped with a spinorial substructure, the interference terms in the transition probabilities can be interpreted as single amplitudes in the underlying \( SL(2, \mathbb{C}) \) space which reproduces space-time twice [1]. This suggests that the non-locality of Quantum Mechanics is an artifact caused by describing amplitudes relative to an \( SO(1,3) \) base space. It is therefore of interest to examine the quantization of particles and strings in a spinorial Clifford space.

As shown in [2], the space-time coordinates \( x^\mu \) of four-dimensional Minkowski space can be resolved into Weyl spinors according to

\[
x^{AB} = c^A \bullet c^B, \quad c^A \bullet c^B = 0, \quad x^{AB} \equiv \sigma^{\mu AB} x^\mu, \quad u \bullet v \equiv \frac{1}{2} \{u, v\},
\]

where the \( \sigma \)'s are the Hermitian Pauli matrices [3, 4] and the components of \( c^A \) belong to the complexified generating space of the split Clifford Algebra \( Cl(4, 4, \mathbb{R}) \). The product \( \bullet \) is the inner product of the Clifford Algebra. A similar factorization is well known from the factorization of the Lorentz metric \( \eta_{\mu\nu} = \frac{1}{2} \{ \gamma_\mu, \gamma_\nu \} = \gamma_\mu \bullet \gamma_\nu \), where the \( \gamma \)'s generate the Clifford algebra \( Cl(1, 3, \mathbb{R}) \). In general, an even-dimensional real Clifford algebra can be written in complex form by a decomposition of the complexified generating space (a polarization) [5]. This turns the algebra into a dual system, in our case exemplified by \( c \) and
\(c^*,\) and is well known from the algebra of creation and annihilation operators for a system of fermions. The dotted and undotted capital letters refer to the transformation properties of the Weyl spinors under \(SL(2,\mathbb{C}).\)

Complex Weyl spinors can only generate a four-dimensional Minkowski space, but there are reasons to believe that four dimensions do not suffice to accommodate the symmetries of the Standard Model. Within the framework of Weyl spinors, our only option of increasing the dimension is to replace the complex numbers with a higher dimensional normed division algebra. It has been conjectured that there is a connection between the octonions and ten-dimensional Minkowski space \([6, 7, 8, 9]\). One of the objects of this paper is to create a model which exemplifies this connection.

The octonions \([6]\) \(\mathcal{O}\) is a non-commutative normed division algebra which is alternative, but not associative. Alternative means that the product of three numbers is associative if at least two of them differ by only a real factor. An octonion \(z\) can be written as

\[
z = x_0 e_0 + \sum_{i=1}^{7} x_i e_i, \quad x_i \in \mathbb{R}, \quad e_0 = 1, \quad (e_i)^2 = -1,
\]

where \(e_i, i = 1, \ldots, 7\) are the seven anti-commuting imaginary units.

Consider the tensor product

\[
T = \mathcal{O} \otimes_{\mathbb{R}} Cl(2,2,\mathbb{R})
\]

of the two \(\mathbb{R}\)-algebras \(\mathcal{O}\) and \(Cl(2,2,\mathbb{R})\). Since \(\mathcal{O}\) is non-commutative and non-associative and \(Cl(2,2,\mathbb{R})\) is non-commutative and associative, their tensor product will be a non-commutative, non-associative \(\mathbb{R}\)-algebra. The octonionic conjugation \(*\) is trivially extended to \(T\) by

\[
(z \otimes u)^* \equiv z^* \otimes u,
\]

and the inner product of \(Cl(2,2,\mathbb{R})\) defines the non-commutative product in \(T\)

\[
(z_1 \otimes u_1) \times (z_2 \otimes u_2) \overset{def}{=} (z_1 z_2) \otimes (u_1 \cdot u_2), \quad (2)
\]

with the conjugation

\[
(v_i \times v_j)^* = v_j^* \times v_i^*, \quad v_i, v_j \in T, \quad (3)
\]

When \(u_1\) and \(u_2\) belong to the generating space of \(Cl(2,2,\mathbb{R})\), then \(u_1 \cdot u_2 = r \mathbb{1}, r \in \mathbb{R}\) and \((3)\) becomes

\[
(z_1 \otimes u_1) \times (z_2 \otimes u_2) = (z_1 z_2 r) \otimes \mathbb{1}, \quad (4)
\]

In this case we may leave out \(\otimes \mathbb{1}\) and regard \((4)\) as the octonion \(z_1 z_2 r\).
To extend the factorization (1) to the octonion case, we consider the $2 \times 2$ matrix

$$H_{ij} = v_i \times v_j^*, \ v_i \in T, \ i, j = 1, 2,$$

which according to (3) is identically Hermitian. If the Clifford components of $v_i$ and $v_j$ belong to the generating space of $\text{Cl}(2, 2, \mathbb{R})$ so that their inner products are real, $H$ becomes an octonionic Hermitian matrix. The following proposition shows that any $2 \times 2$ octonionic Hermitian matrix can be obtained in this manner:

**Proof.** Consider the two vectors

$$v_i = \sum_{k=1}^{2} a_{ik} \otimes e_k + b_{ik} \otimes f_k, \ i = 1, 2,$$

$$\{e_i, e_j\} = \delta_{ij}, \ \{f_i, f_j\} = -\delta_{ij}, \ \{e_i, f_j\} = 0, \ i, j = 1, 2.$$  

(6)

where $a_{ik}$ and $b_{ik}$ are octonions. Then (5) becomes

$$H_{11} = |a_{11}|^2 - |b_{11}|^2 - |a_{12}|^2 - |b_{12}|^2,$$

(7)

$$H_{22} = |a_{22}|^2 - |b_{22}|^2 + |a_{21}|^2 - |b_{21}|^2,$$

(8)

$$H_{1,2} = a_{11}^* b_{21}^* - b_{11} a_{21}^* + a_{12} b_{22}^* - b_{12} a_{22}^*.$$  

(9)

Setting $b_{11} = a_{11}$ and $b_{22} = a_{22}$, equations (7) and (8) can be solved with respect to $a_{12}$ and $a_{21}$ so that $a_{21}^* - b_{21}^* \neq 0$ and $a_{12} - b_{12} \neq 0$. These solutions will not contain $a_{11}$ and $a_{22}$, and therefore equation (9) can be solved with respect to either of them. Any $n \times n$ complex Hermitian matrix can be factorized in the stronger form (1) [2].

Applying this proposition to a ten-dimensional Minkowski space, we get

$$x^{AB} = c^A \times c^B, \ x^{AB} \overset{\text{def}}{=} \sigma_\mu^{AB} x^\mu, \ c^A = \sum_{k=1}^{2} a^A_k \otimes e_k + b^A_k \otimes f_k,$$

(10)

where $\sigma^\mu$ are the ten octonionic Hermitian Pauli matrices.

The determinant of an octonionic Hermitian matrix is well-defined

$$\det \begin{pmatrix} a & c \\ c^* & b \end{pmatrix} \overset{\text{def}}{=} ab - cc^*,$$

and as in four-dimensional Minkowski space, we have $\det(X) = x_\mu x^\mu$, where $X$ is the octonionic Hermitian matrix with entries $x^{AB}$.

The octonionic spinors fall into two classes corresponding to the Lorentz transformations

$$X^A \rightarrow S^A_{\ B} X^B, \ \psi_A \rightarrow -S^*_{\ A} B \psi_B, \ S^A_{\ B} \overset{\text{def}}{=} \epsilon^{BE} S^E_{\ F} \epsilon_{FA},$$

(11)
where $S$ contains only octonions from a single complex subspace of $O$. In case both $S$, $\chi^A$ and $\psi_A$ belong to the same complex subspace, the Lorentz transformations of $\chi^A$ and $\psi^*_E F^A$ are the same. As we shall see, spinors transforming like $\chi^A$ and $\psi_A$ generate coordinates and momenta respectively. Under a Lorentz transformation of $c^A$, the coefficients in (10) transform according to

$$a_A^k \to S^A_B a_B^k, \quad b_A^k \to S^A_B b_B^k, \quad a_k \to S a_k, \quad b_k \to S b_k,$$

which makes

$$x^{A\dot{B}} = \sum_{k=1}^{2} a_A^k a^*_B b^*_k b_k - b_A^k b^*_B a_k a_k.$$

transform as

$$X \to X' = \sum_{k=1}^{2} (S a_k)(S a_k)^\dagger - (S b_k)(S b_k)^\dagger. \quad (12)$$

Manogue and Dray [10] found that the compatibility condition

$$(S v)(S v)^\dagger = S(v v^\dagger)S^\dagger$$

between the spinor and vector representations is satisfied iff $S$ contains only octonions from a single complex subspace and $\det(S) \in \mathbb{R}$. If therefore we assume that $\det(S) = \pm 1$, (12) becomes

$$X \to X' = S X S^\dagger. \quad (13)$$

There is no associativity ambiguity because the octonions are alternative and only one complex subspace is used in each transformation. The determinant of $X'$ is

$$\det(X') = \det(S S^\dagger) \det(X),$$

and since $\det(S S^\dagger) = \det(S) \det(S)^* = 1$, it follows that $\det(X)$ is preserved and that (13) therefore generates a Lorentz transformation of $x^\mu$. Arbitrary finite Lorentz transformations are obtained by consecutive (nested) application of transformations corresponding to different complex subspaces [7].

In general, the contraction $\chi^A \psi^*_A$ of the two octonionic spinors in (11) is not Lorentz invariant. However, the real part of their contraction is Lorentz invariant when $\det(S) = 1$. In the appendix we prove that

$$(S^A_E X^E)(-\psi^*_E S^F_A) + o.c. = \det(S) \chi^A \psi^*_A + o.c., \quad \det(S) \in \mathbb{R}, \quad (14)$$

and since the basic types of $SO(1,9)$ transformations can be obtained from a single transformation with $\det(S) = 1$ or from two consecutive ones with $\det(S) = -1$ [10], it follows that $\text{Re}(\chi_A^A \psi^*_A)$ is Lorentz invariant. For two elements in $T$ of the form

$$c^A = \sum_i \chi_i^A \otimes u_i, \quad d_A^j = \sum_j \psi_A^j \otimes v_j, \quad (15)$$

where $u_i, v_j$ belong to the generating space, it follows that $\text{Re}(c^A \times d_A^j)$ is Lorentz invariant.
2 The classical point particle

The Point particle in Clifford space is described by Clifford coordinates and momenta $c^A(\tau)$ and $d_A(\tau)$ of the form (15), where $\tau$ is a parameter time. They determine the space-time coordinates and momenta through

$$x^{AB} \overset{\text{def}}{=} c^A \times c^B, \quad p_{AB} \overset{\text{def}}{=} d_B \times d_A^*.$$  \hfill (16)

The Hamiltonian action takes the form

$$I = \sqrt{|l|m} \int d\tau \left( \frac{dc^A}{d\tau} \times d_A^* + \text{o.c.} - \sqrt{\frac{\tau}{m}} \mathcal{H}(x^\mu, p^\nu) \right),$$  \hfill (17)

where $\mathcal{H}(x^\mu, p^\nu)$ is the Hamiltonian density and $l$ and $m$ are constants with the dimension of length and mass respectively. For a free particle with mass $M$, the Hamiltonian density is

$$\mathcal{H}(x^\mu, p^\nu) = \frac{e(\tau)}{2M} (p^\mu p_\mu - M^2),$$

where $e(\tau)$ is an einbein.

In the following we shall restrict ourselves to the complex case with a four-dimensional Minkowski space. The product $\times$ of $T$ is hereby replaced by the inner product $\cdot$ of $\text{Cl}(4, 4, \mathbb{R})$ written in complex form. This leads to conventional quantization with complex quantum amplitudes. Differentiation with respect to Clifford-valued variables is defined through

$$\delta f = \frac{1}{2} \{ \partial f / \partial c^A, \delta c^A \},$$

and leads to the differentiation rules

$$\partial c^A / \partial c^B = \delta^A_B, \quad \partial (d_A^* \cdot c^A) / \partial c^B = d_A^*, \quad \partial f(x) / \partial c^A = c^B \partial f(x) / \partial x^{AB}.$$  \hfill (18)

The Poisson bracket in Clifford space can then be defined as the ‘Clifford bracket’

$$\{N, M\}_{C.B.} \overset{\text{def}}{=} \frac{1}{2\sqrt{|l|m}} \left( \{ \partial N / \partial c^A, \partial M / \partial d_A^* \} + \{ \partial N / \partial c^A, \partial M / \partial d_A^* \} ight)$$

$$- \{ \partial M / \partial c^A, \partial N / \partial d_A^* \} - \{ \partial M / \partial c^A, \partial N / \partial d_A^* \},$$

which is skew-symmetric in $N$ and $M$ and real when $N$ and $M$ are real. By means of this bracket, the equations of motion obtained from (17) by independent variation of $c$ and $d$ can be written as

$$\frac{d}{d\tau} c^A = l \{ c^A, \mathcal{H} \}_{C.B.}, \quad \frac{d}{d\tau} d_A^* = l \{ d_A^*, \mathcal{H} \}_{C.B.},$$  \hfill (18)

$$\frac{d}{d\tau} x^\mu = l \{ x^\mu, \mathcal{H} \}_{C.B.}, \quad \frac{d}{d\tau} p_\mu = l \{ p_\mu, \mathcal{H} \}_{C.B.}. \quad (19)$$
The action (17) has a global $SL(2,\mathbb{C})$ and $U(1)$ gauge symmetry with the conserved Noether charges

$$\mathcal{J}_{AB} = d_A^* \cdot c_B + d_B^* \cdot c_A, \quad j = i(d_A^* \cdot c^A - d_A \cdot c^A).$$

To obtain a pure space-time system where the equations of motion (19) only contain $x$ and $p$ themselves, we must require that they vanish

$$d_A^* \cdot c_B + d_B^* \cdot c_A = 0, \quad d_A^* \cdot c^A - d_A \cdot c^A = 0,$$

which is equivalent to

$$d_A^* \cdot c^B = \mu(\tau) \delta^B_A, \quad \mu(\tau) \in \mathbb{R}. \quad (20)$$

In this case the Clifford bracket becomes proportional to the ordinary Poisson bracket:

$$\{N(x,p), M(x,p)\}_{C.B.} = \frac{1}{\sqrt{lm}} \left( \frac{\partial N}{\partial x^\mu} \frac{\partial M}{\partial p_\nu} - \frac{\partial M}{\partial x^\mu} \frac{\partial N}{\partial p_\nu} \right) \left( \frac{1}{8} \sigma^\mu_{AB} \sigma^\nu_{EF} \{c^B, d^F\} \right) + c.c. \quad \Rightarrow \quad \mu(\tau) \sqrt{lm} \{N(x,p), M(x,p)\}_{P.B.}. \quad (21)$$

When $\mu(\tau) \neq 0$ it can be absorbed into proper time $\tau$ by

$$\frac{d\tau}{d\tau} = \sqrt{\frac{l}{m}} \mu(\tau), \quad (22)$$

which turns (19) into the usual space-time canonical equations of motion with proper time $\tau$

$$\frac{d}{d\tau} x^\mu = \{x^\mu, \mathcal{H}(x,p)\}_{P.B.}, \quad \frac{d}{d\tau} p_\mu = \{p_\mu, \mathcal{H}(x,p)\}_{P.B.}.$$ 

and ‘hides’ the Clifford substructure. In general $\mu(\tau)$ has a zero. For a free particle with the parametrization $e(\tau) = 1$, it follows from the equations of motion (18) that

$$\frac{d}{d\tau} \mu(\tau) = \frac{d}{d\tau} \left( \frac{1}{2} d^*_{E} \cdot c^{E} \right) = \frac{1}{2} \sqrt{\frac{l}{m}} M. \quad (23)$$

which yields the proper time

$$\tau = \frac{lM}{4m} (\tau - \tau_0)^2 + \tau_0.$$

Hence there is a minimum proper time where the path in Clifford space starts to reproduce the space-time path for the second time.
3 Matrix mechanics

As a first step towards quantization, we shall describe the motion of the classical point particle by \(N\) integral curves in Clifford coordinate-momentum space and show that they form a unitarily invariant system. Let \(c^A_i(\tau), d^A_i(\tau), i = 1, \ldots, N\) be \(N\) solutions to the equations of motion (18) and let us assemble them into two \(N\)-dimensional ket- and bra-vectors \(\tilde{C}\) and \(\tilde{D}\). We shall assume that their components belong to \(\text{Cl}(4N, 4N, \mathbb{R})\) and that all inner products between coordinates and momenta on different integral curves vanish. This means that the two Hermitian \(N \times N\) matrices

\[
X^{AB} = \tilde{C}^A \cdot \tilde{C}^B, \quad P_{AB} = \tilde{D}_B \cdot \tilde{D}_A, \tag{24}
\]

are diagonal. The diagonal entries are the space-time coordinates and momenta of the integral curves. Since \(X\) and \(P\) are diagonal, they trivially commute with each other

\[
[X^\mu, X^\nu] = [P_\mu, P_\nu] = [X^\mu, P_\nu] = 0. \tag{25}
\]

The Noether charge condition (20) can be written as

\[
\tilde{C}^A \cdot \tilde{D}_B = \mu(\tau) \delta^A_B \cdot 1, \quad \mu(\tau) \in \mathbb{R}. \tag{26}
\]

The \(N\) integral curves may be regarded as a single solution \(\tilde{C}, \tilde{D}\) to the equations of motion for an action describing \(N\) independent identical point particles. If the Hamiltonian is a polynomial expression in \(x\) and \(p\) this action can be written as

\[
\sqrt{\text{Im}} \int d\tau \text{Tr} \left( \frac{d}{d\tau} \tilde{C}^A \cdot \tilde{D}_A + \text{h.c.} - \sqrt{\frac{i}{m}} \mathcal{H}(X, P) \right). \tag{27}
\]

The system (25), (26) and (27) is preserved by the global \(U(N)\) transformations

\[
\tilde{C}^A \to U \tilde{C}^A, \quad \tilde{D}_A \to D_A U^\dagger, \quad X^\mu \to UX^\mu U^\dagger, \quad P_\mu \to UP_\mu U^\dagger,
\]

which produce artificial couplings between different integral curves. Conversely, given this unitarily invariant system, \(X\) and \(P\) can be diagonalized and we regain the \(N\) integral curves. The appearance of a \(U(N)\) symmetry should not come as a surprise since it is already present in the complex form of the generating algebra of \(\text{Cl}(4N, 4N, \mathbb{R})\).

The observables of the system are the coordinates and momenta of the original \(N\) integral curves. They can be characterized in a unitarily invariant manner as the eigenvalues of \(X\) and \(P\). If the point particle is restricted to move along one of the integral curves, we can define the ‘state’ of the particle as the integral
curve along which it is moving. This information can be encoded in a ‘state vector’ \( |s\rangle \) which selects the integral curve in question. The state vector is defined as determining the expectation value \( E \)

\[
E(C^A) \overset{\text{def}}{=} |s\rangle \langle C^A|
\]

(28)

which is unitarily invariant when \( |s\rangle \) transforms like a bra-vector. When \( |s\rangle \) is set equal to the \( i \)’th eigenvector of \( X \) (and \( P \)), the expectation value is the Clifford coordinate corresponding to the space-time eigenvalue on the \( i \)’th integral curve. The equation of motion of \( |s\rangle \) is \( d/d\tau |s\rangle = 0 \), and the expectation value (28) will move through the Clifford coordinates corresponding to the eigenvalues on the integral curve selected by \( |s\rangle \).

To quantize the classical point particle system, we must translate the Clifford brackets \( \{ c^A, M(x,p) \}_{C.B.} \) and \( \{ d^*_A, M(x,p) \}_{C.B.} \) in such a way that the Poisson bracket becomes a commutator. This is accomplished by

\[
\begin{align*}
  c^A &\rightarrow C^A, \quad d^*_A \rightarrow D_A, \quad x^{AB} \rightarrow C^A \bullet C^B, \quad p_{AB} \rightarrow D_B \bullet D_A, \\
  \{ c^A, M(x,p) \}_{C.B.} &\rightarrow \frac{1}{i\hbar \sqrt{lm}} [X^{AB}, M(X,P)] D_B, \\
  \{ d^*_A, M(x,p) \}_{C.B.} &\rightarrow \frac{1}{i\hbar \sqrt{lm}} C^B [P_{AB}, M(X,P)],
\end{align*}
\]

which by use of the Noether charge condition (26) gives

\[
\{ N(x,p), M(x,p) \}_{C.B.} \rightarrow \frac{1}{i\hbar \sqrt{lm}} [N(X,P), M(X,P)],
\]

and from the proportionality (21) becomes

\[
\{ N(x,p), M(x,p) \}_{P.B.} \rightarrow \frac{1}{i\hbar} [N(X,P), M(X,P)].
\]

In the quantum matrix system, \( X \) and \( P \) are conjugate variables. Consequently, it is no longer possible to turn them into a set of integral curves through diagonalization. The state vector which used to describe the state of the system as a choice of integral curve now takes on a more abstract character. After a measurement of an eigenvalue has been performed, the expectation value will not remain a Clifford coordinate corresponding to an eigenvalue, but develop in time into a complex linear combination of Clifford coordinates corresponding to different eigenvalues. The measurements will no longer be predictable, but become stochastic in accordance with the Born rule. The equation of motion of the state vector is the same: \( d/d\tau |s\rangle = 0 \), as in the classical system, and in the non-relativistic limit it can be turned into the matrix form of the Schrödinger equation by a local unitary transformation (the Schrödinger picture) [2]. Regardless of its abstract character, the formal resemblance of the quantum mechanical wave function to the classical state vector lends support to the view that it is an information-carrying object rather than a primary physical variable.
4 The classical string

We consider a string carrying the spinor fields $c^A(\tau, \sigma)$ and $\dot{d}_A(\tau, \sigma)$ which are two-component spinors, both in relation to space-time and in relation to the worldsheet. The worldsheet spinor indices will be suppressed. The spinor components belong to an infinite-dimensional split Clifford algebra written in complex form. We follow the convention that $\mu, \nu, \ldots$ denote the space-time indices and $\alpha, \beta, \ldots$, the worldsheet indices. Differentiation with respect to the worldsheet parameters $\sigma^\alpha = \tau, \sigma$ is written as $\partial^\alpha$. The spinors $c^A$ and $\dot{d}_A$ determine the space-time coordinates and the space-time momentum current according to

$$x^{AB} = \bar{c}^B \cdot c^A,$$

$$p^\alpha_{AB} = \bar{d}^A \cdot \rho^\alpha d_B,$$

$$\rho^0 \overset{def}{=} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho^1 \overset{def}{=} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \{\rho^a, \rho^b\} = 2\eta^{ab}, \quad \eta^{ab} \overset{def}{=} \text{diag}(1, -1),$$

$$\bar{\psi} \overset{def}{=} \psi\rho^0, \quad \rho^a \overset{def}{=} \rho^0 (\rho^a)^\dagger \rho^0 = \rho^a.$$

$\rho^0$ and $\rho^1$ are the Dirac matrices in $1+1$ dimensions \cite{11}, and $x^\mu$ and $p_\mu^a$ are worldsheet scalars and vectors respectively.

For a string residing in space-time, the Lorentz metric $\eta_{\mu\nu}$ induces a metric on the worldsheet through the tangent derivatives $\partial^\alpha x^\mu$. In Clifford space, these are replaced by the complex vectors

$$V^\mu_{\alpha} \overset{def}{=} \sigma^\mu_{AB} \partial^\alpha c^B \cdot c^A$$

with the real part $\partial^\alpha x^\mu$. The Hermitian tensor

$$g_{\alpha\beta} \overset{def}{=} V^\mu_{\alpha} V^\nu_{\beta} \eta_{\mu\nu}, \quad g^*_{\alpha\beta} = g_{\beta\alpha},$$

can be split into a metric $h_{\alpha\beta}$ and a scalar field $\phi$. The metric allows the space-time momentum current \cite{22} to be written in covariant form as a current density

$$p^\alpha_{AB} = e \bar{d}_A \cdot \rho^\alpha d_B,$$

$$h_{\alpha\beta} = e^a_{\alpha} e^b_{\beta} \eta_{ab}, \quad e \overset{def}{=} \text{det}(e^a_{\alpha}), \quad \rho^a \overset{def}{=} e^a_{\alpha} \rho^\alpha.$$

The simplest string action is

$$\sqrt{lm} \int d\tau d\sigma e(\bar{d}_A \cdot \rho^a \nabla_\alpha c^A + \text{c.c.} - \frac{\sqrt{lm}}{m^2}(\bar{d}_A \cdot d_B)(\bar{d}^B \cdot d^A) + \phi R(h_{\alpha\beta})), \quad (31),$$

where $l$ and $m$ are constants with dimension of length and mass respectively. The equations of motion obtained by independent variation of $c^A, d_B, \phi$ and
choose a parametrization in which $\Gamma$ have a clear physical interpretation. By means of the identity
\[\nabla_a \nabla^2 \phi - 2e^\beta \nabla_a \nabla_\beta \phi + \partial L_M / \partial e^\alpha = 0, \quad R(h_{\alpha \beta}) = 0,\]
where the covariant derivative $\nabla_\alpha$ satisfies the tetrad postulate \[12\]
\[\nabla_\alpha e^\beta_a \equiv \partial_\alpha e^\beta_a + \Gamma^\beta_{\gamma \alpha}(h)e^\gamma_a - \omega^\beta_{ab}e^\beta_b = 0,\]
with $\Gamma$ as a metrical connection. Since the scalar curvature vanishes, we can choose a parametrization in which $h_{\alpha \beta} = \eta_{\alpha \beta}$, $e^\alpha_a = \delta^\alpha_a$ and $\nabla_\alpha = \partial_\alpha$. From \[36\] it follows that the space-time momentum current density \[40\] is conserved.

The equations of motion for $c^A$ and $\bar{d}_A$ can be written in the covariant canonical form
\[
\dot{\varphi} c^A = \partial \mathcal{H} / \partial \dot{d}_A, \quad \nabla_\alpha \varphi \equiv -\partial \mathcal{H} / \partial c^A (= 0),
\]
\[
\varphi \equiv c^\alpha_a \rho^a \partial_\alpha, \quad \mathcal{H} \equiv \sqrt{\frac{m}{\hbar}} (\bar{d}_A \cdot \bar{d}_B)(\bar{d}^A \cdot \bar{d}^B),
\]
but as in De Donder-Weyl theory \[13\] \[14\] \[15\], the scalar Hamiltonian does not have a clear physical interpretation.

We shall find the class of solutions where $\bar{d}_A \cdot \bar{d}_B$ is constant and invertible (not a null vector). Applying $\rho^a \partial_\alpha$ to both sides of \[32\] and using \[33\], we find that $c^A$ satisfies the wave equation and can be expanded according to
\[
c^A = c^A(0) + t^A + \sum_{n \neq 0} a^A(n)e^{i \sqrt{4n(\tau + \sigma)}}, \quad 0 \leq \sigma \leq \pi. \quad (34)
\]
By means of the identity
\[H_{A\bar{E}}H^{B\bar{E}} \equiv \frac{1}{2} H_{E\bar{E}}H^{E\bar{E}} \delta_A^B\]
for Hermitian second rank spinors, \[32\] can be solved with respect to $\bar{d}_B$, giving
\[
d_{\bar{B}} = k_{AB} \rho^a \partial_\alpha c^A, \quad k_{AB} \equiv \frac{m^2}{\sqrt{\hbar m}} (\bar{d}_E \cdot \bar{d}_F \bar{d}^E \cdot \bar{d}^F)^{-1} \bar{d}_A \cdot \bar{d}_B, \quad (35)
\]
with the expansion
\[
d_{\bar{B}} = k_{AB} \rho^a (t^A \delta^0_\alpha + \frac{i}{2} \sum_{n \neq 0} na^A(n)k^L_a e^{i \sqrt{4n(\tau + \sigma)}}, \quad (36)
\]
k\[L\] def \[= (1, 1), \quad k\[R\] def \[= (1, -1),
\]
10
Since the Clifford algebra is infinite-dimensional, we may assume that the components of Fourier coefficients corresponding to different modes are orthogonal to each other

\[ a \cdot b^\dagger = 0, \quad a(n) \cdot a^\dagger(m) = b(n) \cdot b^\dagger(m) = 0 \quad \text{for} \quad n \neq \pm m. \]

By using that the wave vectors \( k^L \) and \( k^R \) are null vectors, we get

\[ \overline{d}_A \cdot d_B = -k_{AB}(\overline{t}^E \cdot t^E)k_{EB}, \]  

(37)

which is constant in accordance with our ansatz.

The conserved \( SL(2,\mathbb{C}) \) Noether current density corresponding to the string action \( S \) is

\[ j_{AB}^\gamma \overset{\text{def}}{=} \sqrt{\text{Im}} e (\overline{d}_A \cdot \rho^\gamma c_B + \overline{d}_B \cdot \rho^\gamma c_A), \]

which contains the angular momentum current density. From (34) and (36), we obtain the expansions of the space-time momentum and angular moment current densities

\[ p^\gamma_{AB} = k_{AB}(\overline{t}^E \cdot \rho^0 \rho^0 t^E + \sum_{n \neq 0} \frac{n^2}{4}(\overline{\alpha}^F(n) \cdot k^L_{\alpha} \rho^\gamma \rho^\gamma k^R_{\beta} \rho^\beta a^E(n) \]

\[ + \overline{\beta}^F(n) \cdot k^R_{\alpha} \rho^\gamma \rho^\gamma k^L_{\beta} \rho^\beta b^E(n)) - \overline{\alpha}^F(n) \cdot k^L_{\alpha} \rho^\gamma \rho^\gamma k^R_{\beta} \rho^\beta a^E(n)e^{in(\tau + \sigma)} \]

\[ - \overline{\beta}^F(n) \cdot k^R_{\alpha} \rho^\gamma \rho^\gamma k^L_{\beta} \rho^\beta b^E(n)e^{in(\tau - \sigma)} )k_{EB}, \]  

(38)

\[ j_{AB}^\gamma = -\frac{i}{2} \sqrt{\text{Im}} k_{AB} \sum_{n \neq 0} n\overline{\alpha}^F(n) \cdot k^L_{\alpha} \rho^\beta \rho^\beta a_B(n) + n\overline{\beta}^F(n) \cdot k^R_{\alpha} \rho^\gamma \rho^\gamma b_B(n) \]

\[ + n\overline{\beta}^F(n) \cdot k^R_{\alpha} \rho^\gamma \rho^\gamma a_B(-n)e^{-in(\tau + \sigma)} + n\overline{\beta}^F(n) \cdot k^L_{\alpha} \rho^\beta \rho^\beta b_B(-n)e^{-in(\tau - \sigma)} + A \leftrightarrow B \]  

(39)

The boundary condition says that there is no flow of momentum and angular momentum at the endpoints of the string

\[ p_{AB}^1(\tau, 0) = p_{AB}^1(\tau, \pi) = 0, \quad j_{AB}^1(\tau, 0) = j_{AB}^1(\tau, \pi) = 0, \]

which gives the relations between the \( a- \) and \( b \) coefficients:

\[ \overline{\alpha}^B(n) \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} a^A(\pm n) = \overline{\beta}^B(n) \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} b^A(\pm n), \]

\[ \overline{\alpha}^B(n) \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} a^A(\pm n) = \overline{\beta}^B(n) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} b^A(\pm n), \]

and the restriction on the constant momentum flow

\[ \overline{t}^E \cdot \rho^0 t^E = 0. \]

(40)
This simplifies the expansions (38) and (39) into

\[ p^0_{AB} = k_{AF} \left( -\overline{F} \cdot \overline{\rho} \overline{E} + \sum_{n \neq 0} 2n^2 \overline{a}^E(n) \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} a^E(n) \right) \]

\[ + 2n^2 \overline{a}^E(n) \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} a^E(-n)e^{-in\tau \cos(n\sigma)} ) k_{EB}, \]

\[ p^1_{AB} = -k_{AF} \left( \sum_{n \neq 0} 2in^2 \overline{a}^E(n) \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} a^E(-n)e^{-in\tau \sin(n\sigma)} \right) k_{EB}, \]

\[ j^0_{AB} = -2i \sqrt{\text{Im}} k_{AF} \sum_{n \neq 0} n\overline{a}^E(n) \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} a_B(n) \]

\[ + n\overline{a}^E(n) \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} a_B(-n) e^{-in\tau \sin(n\sigma)} + A \leftrightarrow B, \quad (41) \]

\[ j^1_{AB} = -2i \sqrt{\text{Im}} k_{AF} \sum_{n \neq 0} n\overline{a}^E(n) \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} a_B(-n) e^{-in\tau \sin(n\sigma)} + A \leftrightarrow B. \]

The boundary condition \( \partial_x \sigma = 0 \) gives

\[ \overline{a}^B(n) \cdot a^A(\pm n) = \overline{a}^B(n) \cdot b^A(\pm n), \]

and turns the expansion of the space-time coordinates into a standing wave

\[ x^{AB} \overset{\text{def}}{=} \overline{c}^B \cdot c^A = \overline{c}^B(0) \cdot c^A(0) + \overline{c}^B \cdot l^A \tau^2 \]

\[ + 2 \sum_{n \neq 0} \overline{c}^B(n) \cdot a^A(n) + \overline{c}^B(n) \cdot a^A(-n)e^{-in\tau \cos(n\sigma)}. \quad (42) \]

When all Fourier amplitudes vanish, the string should move like a free point particle. However, the constants of integration \( l^E \) determine three independent four-vectors \( \overline{F} \cdot \overline{\rho} l^E \) and \( \overline{F} \cdot l^E \), whereas the space-time description only needs one (the space-time momentum). When we add the condition

\[ \overline{F} \cdot \overline{\rho} l^E = -\overline{F} \cdot l^E \]

to (10), the four three-vectors are reduced to the single one \( \overline{F} \cdot l^E \). From (35) and (37) it follows that \( \overline{F} \cdot l_A \) and \( k_{AB} \) are proportional so that the total space-time momentum can be written as

\[ \rho^{tot}_{AB} \overset{\text{def}}{=} \int_0^{\pi} d\sigma_{p_{AB}} = -\pi k_{AF} \overline{F} \cdot \overline{\rho} l^E k_{EB} \]

\[ = \pi k_{AF} (\overline{F} \cdot l^E ) k_{EB} = \frac{\pi}{2} k_{EF} k_{EF} \overline{F} \cdot l_A. \]
yielding the correct space-time inertial motion
\[
\frac{d}{d\tau} x_{AB} = \frac{p^\text{tot}_{AB}}{M},
\]
\[
d\tau \overset{\text{def}}{=} \left( \eta_{\mu\nu} dx^\mu dx^\nu \right)^{1/2} = \left( \frac{1}{2} (\mathbf{T} \cdot \mathbf{F}) \right)^{1/2} dt - \frac{1}{2} \lambda_{A\beta} k_{A\beta} \left( \frac{1}{2} (\mathbf{F} \cdot \mathbf{E}) \right)^{1/2} M_{\text{def}} = \left( \eta_{\mu\nu} p^\text{tot}_\mu p^\text{tot}_\nu \right)^{1/2} = \pi / k_A^1.
\]

The octonionic generalization of the string action (31) is
\[
\sqrt{\text{Im}} \int d\tau d\sigma e (\rho^a_\alpha \nabla_\alpha e^a \times d_{\text{A}a} + \text{o.c.} - \sqrt{\text{Im}} / m^2 \det (d_{\text{A}a} \times \overline{d}_{\text{A}a}) + \phi R(h_{\alpha\beta})),
\]
where we have written the worldsheet spinor indices \( a \) and \( b \) explicitly because the matrix spinor notation conflicts with the correct order of the octonion components. We assume that \( c \) and \( d \) are of the form (15) so that the \( \times \) products in (13) are octonionic. Since \( \rho^0 \) and \( \rho^1 \) are real, \( \overline{d}_{\text{A}a} \) transforms like \( d^a \) and \( \rho^\alpha \nabla_\alpha c^A \) like \( c^A \). The first two terms in the Lagrangian therefore transform like the real part of an octonionic spinor contraction and, according to (14), are Lorentz invariant. The Hermitian octonionic matrix \( d_{\text{A}a} \times \overline{d}_{\text{A}a} \) transforms like \( X \) in (13) with \( S \) and \( S^\dagger \) switched around, and its determinant is therefore Lorentz invariant.

In the octonionically generated ten-dimensional space-time, each of the seven imaginary units generates a four-dimensional space-time. Our universe appears to be confined to one of these, and therefore we need a mechanism for dimensional reduction. It has been proposed that the octonions could provide alternatives to compactification of the extra dimensions [16]. One possibility is to restrict the endpoints of open strings to contain only one imaginary unit. This would confine them to a four-dimensional space-time and would limit the interaction between open strings with endpoints in different four-dimensional space-times. The question is whether such a boundary condition can be justified in other respects. Closed strings are not subject to boundary conditions, and therefore all open strings are able to interact gravitationally (assuming that closed strings mediate gravity like in Superstring theory). If the ten-dimensional space-time is homogenous, then six out of seven strings that a string can interact with gravitationally, reside in the six other four-dimensional space-times. This is interesting since dark matter is thought to outweigh visible matter roughly six to one.

## 5 The quantum string

Let \( \Gamma \) be a space-like curve connecting two fixed points on the boundaries of the worldsheet and let \( \sigma^\alpha(u), \sigma^\alpha(u') \) and \( \sigma^\alpha(u'') \) be three points on this curve. We can then define the worldsheet spinors
\[
d_B(u) \overset{\text{def}}{=} e^{\nu_\alpha} \epsilon_{\beta\alpha} \rho^\beta d_B(u), \quad d_A(u) \overset{\text{def}}{=} e^{\nu_\alpha} \epsilon_{\beta\alpha} d_A(u) \rho^\beta, \quad v^\alpha \overset{\text{def}}{=} \frac{d\sigma^\alpha}{du}.
\]
and the Clifford bracket

\[ \{N', M''\}_{C.B.} \]

\[
def \frac{1}{2\sqrt{\ell m}} \int_{\Gamma} du \left( \{\partial N'/\partial c^G, \partial M''/\partial \bar{d}_G\} + \{\partial N'/\partial \bar{c}^G, \partial M''/\partial d_G\} \\ - \{\partial M''/\partial c^G, \partial N'/\partial \bar{d}_G\} - \{\partial M''/\partial \bar{c}^G, \partial N'/\partial d_G\} \right) \]

for two functions \(N\) and \(M\) of \(c^A\) and \(d_B\) and their Dirac conjugates. \(N\) and \(M\) are worldsheet scalars; so therefore their derivatives in the Clifford bracket are worldsheet spinors and their Dirac conjugates. It is understood that these spinors are to be contracted with each other so that the Clifford bracket becomes a worldsheet scalar. Unprimed variables depend on \(u\), and variables with a single prime or a double prime depend on \(u'\) and \(u''\) respectively.

The space-time angular momentum defines the worldsheet scalars

\[
j_{AB}(u) \defeq \sqrt{\ell m} (\bar{D}_A(u) \cdot c_B(u) + \bar{D}_B(u) \cdot c_A(u)),
\]

with the Clifford brackets

\[
\{j^{\prime}_{AB}, j^{\prime\prime}_{EF}\}_{C.B.} = \left( (j^{\prime}_{AE} \epsilon_{FB} + A \leftrightarrow B) + E \leftrightarrow F \right) \delta(u' - u'').
\]

The quantization

\[
c^A \to \hat{c}^A, \quad d^*_A \to \hat{c}_A, \quad J_{AB} \to \hat{J}_{AB},
\]

\[
\{N, M\}_{C.B.} \to \frac{1}{i\hbar} [N, M],
\]

turns single spinors into infinite sets of spinors transforming like ket and bra vectors under a unitary symmetry. \(J_{AB}\) is hereby turned into an infinite-dimensional matrix \(J_{AB}\) with the commutation relations

\[
[J_{AB}^{\prime}, J_{EF}^{\prime\prime}] = i\hbar \left( (J_{AE}^{\prime} \epsilon_{FB} + A \leftrightarrow B) + E \leftrightarrow F \right) \delta(u' - u''). \tag{44}
\]

Integrating both sides of (44) with respect to both \(u'\) and \(u''\), we get the commutation relations for the total Noether charges

\[
[J_{AB}^{\text{tot}}, J_{EF}^{\text{tot}}] = i\hbar \left( (j_{AE}^{\text{tot}} \epsilon_{FB} + A \leftrightarrow B) + E \leftrightarrow F \right), \quad [J_{AB}^{\text{tot}}, J_{EF}^{\text{tot}}] = 0, \tag{45}
\]

\[
J_{AB}^{\text{tot}} \defeq \int_{\Gamma} d\sigma^\alpha \epsilon_{\beta\alpha} j_{AB}^{\beta}, \quad j_{AB}^{\beta} = \sqrt{\ell m} (\bar{D}_A \cdot \rho^\beta C_B + \bar{D}_B \cdot \rho^\beta C_A),
\]

where for notational reasons, we have omitted the ket- and bra tags on top of \(C\) and \(D\). Since \(J_{AB}^{\text{tot}}\) is a conserved current density, its total charge is independent of the path of integration \(\Gamma\).

In terms of

\[
N_1^{\dagger} \defeq \frac{i}{4} (j_{22}^{\text{tot}} - j_{11}^{\text{tot}}), \quad N_2^{\dagger} \defeq \frac{1}{4} (j_{11}^{\text{tot}} + j_{22}^{\text{tot}}), \quad N_3^{\dagger} \defeq \frac{i}{2} j_{12}^{\text{tot}},
\]

14
becomes the Lorentz algebra
\[ [N_i, N_j] = i\hbar\epsilon_{ijk}N_k, \quad [N_i^\dagger, N_j^\dagger] = i\hbar\epsilon_{ijk}N_k^\dagger, \quad [N_i, N_j^\dagger] = 0. \]  
(46)

In tensor notation, the Lorentz generators can be written as
\[ J_{\mu\nu} = 4\sigma_\mu^A\sigma_\nu^B\varepsilon_{E\dot{F}}(J_{AB}^\dot{E}\dot{F} + J_{\dot{E}F}^\dagger\epsilon_{AB}). \]

The states of the string are therefore representations of the Lorentz group, which is a minimum requirement for identifying them with elementary particles.

It is well known that the space-time orbital angular momentum operator has only integral eigenvalues [17]. We shall show that the total angular momentum of the Clifford-space string also supports half-integral eigenvalues. We consider only the first term \((n = 1)\) in (41) corresponding to the spinning string:
\[ J_{0,\text{tot}}^{AB} \overset{\text{def}}{=} \int_0^{2\pi} d\sigma J_{AB}^0 = -2i\pi\sqrt{\text{lm}}k_{A\dot{E}}\pi^\dagger \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} a_B + A \leftrightarrow B \]
\[ = -2i\pi\sqrt{\text{lm}}k_{A\dot{E}}a_1^\dagger \cdot a_{2,B} + A \leftrightarrow B, \quad a_B = \begin{pmatrix} a_{1,B} \\ a_{2,B} \end{pmatrix}. \]  
(47)

The quantized form of (47) is
\[ J_{AB}^{0,\text{tot}} = -2i\pi\sqrt{\text{lm}}K_{A\dot{E}}A_{2,B} \cdot A_1^\dagger + A \leftrightarrow B, \]  
(48)

where \(J^0\) and \(K\) are infinite-dimensional matrices, and \(A_1\) and \(A_2\) are infinite-dimensional vectors. Any infinite-dimensional matrix \(M_{ij}\) can be decomposed according to
\[ M_{ij} = v_i \cdot v_j^* + i u_i \cdot u_j^*, \]  
(49)

by factorizing its Hermitian and anti-Hermitian parts. When the components of \(v\) and \(u^*\) are chosen to be orthogonal to each other, (49) can be written as
\[ M_{ij} = (v_i + i u_i) \cdot (v_j^* + u_j^*), \]

and therefore any matrix can be written on the form \(A_2 \cdot A_1\). When \(K_{A\dot{E}}\) is invertible, it follows that (48) imposes no algebraic constraint on the total angular momentum of the string. The only constraint is the Lorentz algebra which permits both integral and half-integral eigenvalues. Note that this would not have been the case if the Clifford coordinates and momenta had been worldsheet scalars instead of worldsheet spinors.

6 Redshift

The average motion of the free string given by (42) is \(x^\mu = x^\mu(0) + l^\mu\tau^2\), which shows that there are two different measures of time involved in the description of the string: parameter time \(\tau\) and coordinate time \(t = t^0 + l^0\tau^2\). In the point
particle model, parameter time could be absorbed into proper time by using the proportionality between Clifford brackets and conventional Poisson brackets. This is not possible in the string model, where the internal dynamics of the string is described in terms of parameter time, while the external dynamics (like inertial motion) is described in terms of coordinate time. Coordinate time has a minimum where parameter time starts to reproduce it for the second time.

If a photon could be modeled as a string state in Clifford space, we would expect its frequency to be defined in terms of parameter time $\tau$ and to remain fixed as long as the string is moving freely in a flat background. Externally, however, we measure its frequency in terms of coordinate time $t$ corresponding to its inertial motion. If the photon is emitted at $t_{\text{emit}}$ and observed at $t_{\text{obs}}$ and the minimum of $t$ is set at $t = 0$, we would observe a redshift of

$$z \overset{\text{def}}{=} \frac{f_{\text{emit}}}{f_{\text{obs}}} - 1 = \frac{d\tau}{dt}(t_{\text{emit}})/\frac{d\tau}{dt}(t_{\text{obs}}) - 1 = \sqrt{t_{\text{obs}}/t_{\text{emit}}} - 1 = \sqrt{T/(T - \Delta t)} - 1, \quad (50)$$

where $T$ is the time which has elapsed since the minimum coordinate time, and $\Delta t$ is the lookback time to the light-emitting object [18]. For small values of $z$ where the redshift-distance relation is linear, this new redshift would increase the optically determined Hubble constant by $1/2T$, and $T$ must therefore be significantly greater than a Hubble-time. To resolve the Hubble tension [19], $T$ would have to be of the order of magnitude of $70 - 100$ billion years. The application of the redshift (50) to cosmology is however limited by the fact that the string model is based on a Minkowski space.

7 Conclusion

We have examined the quantization of the point particle and string in a Clifford space which forms a substructure of Minkowski space. The classical point particle has been described by a unitarily invariant system of integral curves in Clifford space. This paves the way for quantization in a more direct manner than the usual space-time based procedure. We have obtained the Lorentz algebra for the quantum string from Poisson brackets in Clifford space and shown that that a Clifford string can have half-integral spin. By enlarging the Clifford algebra to an R-algebra tensor product with the octonions, we have obtained a Lorentz invariant string action in ten-dimensional Minkowski space. This lends support to the mathematical hypothesis that there is a connection between the dimension of space-time and the normed division algebras.
8 Appendix

To prove (14), we expand $S$, $\chi$ and $\psi$ in terms of the octonion units:

$$S^E_A = S^E_A(0) + S^E_A(k)e_k,$$

$$\chi^E = \chi^E(0) + \sum_{i=1}^7 \chi^E(i)e_i, \quad \psi^*_F = \psi^*_F(0) + \sum_{j=1}^7 \psi^*_F(j)e_j,$$

where $S$ lies in the complex subspace corresponding to $e_k$. When $\text{det}(S)$ is real, it follows from the general formula for determinants

$$\epsilon_{AB}S^A_ES^B_F = \text{det}(S)\epsilon_{EF}, \quad \text{or} \quad S^A_ES^A_F = -\text{det}(S)\delta^F_E,$$

that

$$S^A_E(0)S^F_A(0) - S^A_E(k)S^F_A(k) = -\text{det}(S)\delta^F_E,$$

$$S^A_E(0)S^F_A(k) + S^A_E(k)S^F_A(0) = 0. \quad (51)$$

Terms with imaginary units contribute to the expansions in (14) only if the product of these units is $\pm 1$. Terms with two or four imaginary units can be computed as if $e_ne_m = -\delta_{nm}$ and $(e_ke_i)(e_je_k) = \delta_{ij}$.

The terms of the l.h.s of (14) with no imaginary units are

$$-S^A_E(0)S^F_A(0)\chi^E(0)\psi^*_F(0) + o.c. \quad (52)$$

The terms with two imaginary units become

$$S^A_E(k)\chi^E(0)\psi^*_F(0)S^F_A(0) + S^A_E(k)\chi^E(k)\psi^*_F(0)S^F_A(k)$$

$$+ S^A_E(k)\chi^E(0)\psi^*_F(k)S^F_A(0) + S^A_E(0)\chi^E(0)\psi^*_F(k)S^F_A(k)$$

$$+ \sum_{j=1}^7 S^A_E(0)\chi^E(j)\psi^*_F(j)S^F_A(0) + S^A_E(k)\chi^E(0)\psi^*_F(0)S^F_A(k) + o.c.,$$

which, by use of (51) are reduced to

$$S^A_E(0)S^F_A(0)\sum_{j=1}^7 \chi^E(j)\psi^*_F(j) + S^A_E(k)S^F_A(k)\chi^E(0)\psi^*_F(0) + o.c. \quad (53)$$

The terms with three imaginary units become

$$-S^A_E(0)\sum_{i,j=1}^7 \chi^E(i)\psi^*_F(j)S^F_A(k)e_i(e_je_k)$$

$$- S^A_E(k)\sum_{i,j=1}^7 \chi^E(i)\psi^*_F(j)S^F_A(0)(e_ke_i)e_j + o.c., \quad (54)$$
(terms with two $e_k$'s vanish). $e_i(e_j e_k)$ only contributes if $e_j e_k = \pm e_i$, in which case we have $e_i(e_j e_k) = (e_k e_j)e_j$. Together with (54), this makes (54) vanish. Terms with four imaginary units become

$$-S^A_E(k)S^F_A(k)\sum_{j=1}^7 \chi^E(j)\psi^*_F(j) + o.c. \quad (55)$$

The total contribution to the expansion of the l.h.s. of (14) is obtained by adding (52), (53) and (55), which gives

$$\text{det}(S)\left(\chi^E(0)\psi^*_E(0) - \sum_{j=1}^7 \chi^E(j)\psi^*_E(j)\right) + o.c.,$$

and is readily seen to be the same as the expansion of the r.h.s. of (14).

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