A characterization of $Q$-polynomial distance-regular graphs

Aleksandar Jurišić, Paul Terwilliger, Arjana Žitnik

August 3, 2009

Abstract

We obtain the following characterization of $Q$-polynomial distance-regular graphs. Let $\Gamma$ denote a distance-regular graph with diameter $d \geq 3$. Let $E$ denote a minimal idempotent of $\Gamma$ which is not the trivial idempotent $E_0$. Let $\{\theta_i^*\}_{i=0}^d$ denote the dual eigenvalue sequence for $E$. We show that $E$ is $Q$-polynomial if and only if (i) the entry-wise product $E \circ E$ is a linear combination of $E_0$, $E$, and at most one other minimal idempotent of $\Gamma$; (ii) there exists a complex scalar $\beta$ such that $\theta_i^* - 1 - \beta \theta_i^* + \theta_i^* + 1$ is independent of $i$ for $1 \leq i \leq d - 1$; (iii) $\theta_i^* \neq \theta_0^*$ for $1 \leq i \leq d$.

1 Introduction

In this paper we give a new characterization of the $Q$-polynomial property for distance-regular graphs. In order to motivate and describe our result, we first recall some notions. Let $\Gamma$ denote a distance-regular graph with diameter $d \geq 3$ and vertex set $X$ (see Section 2 for formal definitions). Recall that there exist a minimal idempotent $E_0$ of $\Gamma$ such that $E_0 = |X|^{-1}J$, where $J$ is the all 1’s matrix. We call $E_0$ trivial. Let $\{E_i\}_{i=1}^d$ denote an ordering of the nontrivial minimal idempotents of $\Gamma$. It is known that for $0 \leq i, j \leq d$ the entry-wise product $E_i \circ E_j$ is a linear combination of the minimal idempotents of $\Gamma$, so that

$$E_i \circ E_j = \frac{1}{|X|} \sum_{h=0}^d q_{ij}^h E_h.$$ 

The coefficients $q_{ij}^h$ are called the Krein parameters of $\Gamma$. They are real and nonnegative; see for example [2, p. 48–49]. Now consider when is a Krein parameter zero. Note that $J \circ E_j = E_j$ for $0 \leq j \leq d$, so $q_{ij}^h = \delta_{hj}$ for $0 \leq h, j \leq d$. The ordering $\{E_i\}_{i=1}^d$ is called $Q$-polynomial whenever for all $0 \leq i, j \leq d$ the Krein parameter $q_{ij}^1$ is zero if $|i - j| > 1$ and nonzero if $|i - j| = 1$. Let $E$ denote a nontrivial minimal idempotent of $\Gamma$. We say that $E$ is $Q$-polynomial whenever there exists a $Q$-polynomial ordering $\{E_i\}_{i=1}^d$ of the nontrivial minimal idempotents of $\Gamma$ such that $E = E_1$. We now explain the $Q$-polynomial property in terms of representation diagrams. Let $E$ denote a nontrivial minimal idempotent of $\Gamma$, and for notational convenience write $E = E_1$. The representation diagram $\Delta_E$ is the undirected graph with vertex set $\{0, \ldots, d\}$ such that
vertices $i,j$ are adjacent whenever $i \neq j$ and $q_{ij}^1 \neq 0$. By our earlier comments, $q_{0j}^1 = \delta_{1j}$ for $0 \leq j \leq d$. Therefore, in $\Delta_E$ the vertex 0 is adjacent to the vertex 1 and no other vertex, see Figure 1(a). Observe that $E$ is $Q$-polynomial if and only if $\Delta_E$ is a path, and in this case the natural ordering 0, 1, \ldots of the vertices in $\Delta_E$ agrees with the $Q$-polynomial ordering associated with $E$. See Figure 1(c).

Let $E$ denote a nontrivial minimal idempotent of $\Gamma$. In this paper we give a condition that is necessary and sufficient for $E$ to be $Q$-polynomial. We now describe the condition, which has three parts.

The first part has to do with the representation diagram $\Delta_E$. According to Lang [5], $E$ is a tail whenever $E \circ E$ is a linear combination of $E_0$, $E$, and at most one other minimal idempotent of $\Gamma$. In terms of the diagram $\Delta_E$, and writing $E = E_1$ for notational convenience, $E$ is a tail if and only if vertex 1 is adjacent to at most one vertex besides vertex 0, see Figure 1(b). Note that if $E$ is $Q$-polynomial, then $E$ is a tail.

The next part of our condition involves the dual eigenvalue sequence $\{\theta^*_i\}_{i=0}^d$ for $E$. This sequence satisfies $E = |X|^{-1} \sum_{i=0}^d \theta^*_i A_i$, where $\{A_i\}_{i=0}^d$ are the distance matrices of $\Gamma$. Following [5], we say that $E$ is three-term recurrent (in short TTR) whenever there exists a complex scalar $\beta$ such that $\theta^*_{i-1} - \beta \theta^*_i + \theta^*_{i+1}$ is independent of $i$ for $1 \leq i \leq d - 1$. If $E$ is $Q$-polynomial, then $E$ is TTR by [2, Theorem 8.1.2], cf. [7].

The third part of our condition involves both the diagram $\Delta_E$ and the dual eigenvalues $\{\theta^*_i\}_{i=0}^d$. By [2, Proposition 2.11.1], $\Delta_E$ is connected if and only if $\theta^*_i \neq \theta^*_0$ for $1 \leq i \leq d$. These equivalent statements hold if $E$ is $Q$-polynomial, since in this case $\Delta_E$ is a path.

We now state our main result.

**Theorem 1.1.** Let $\Gamma$ denote a distance-regular graph with diameter $d \geq 3$. Let $E$ denote a nontrivial minimal idempotent for $\Gamma$ and let $\{\theta^*_i\}_{i=0}^d$ denote the corresponding dual eigenvalue sequence. Then $E$ is $Q$-polynomial if and only if

(i) $E$ is a tail,

(ii) $E$ is TTR,

(iii) $\theta^*_i \neq \theta^*_0$ for $1 \leq i \leq d$. 

2
In Section 1, we discuss the minimality of assumptions (i)–(iii) of Theorem 1.1. We show that in general, no proper subset of (i)–(iii) is sufficient to imply that $E$ is $Q$-polynomial. Theorem 1.1 gives a characterization of the $Q$-polynomial distance-regular graphs. A similar characterization, where assumption (i) is replaced by some equations involving the dual eigenvalues and intersection numbers, is given by Pascasio [8].

2 Preliminaries

In this section we review some definitions and basic concepts. See Brouwer, Cohen and Neumaier [2] and Terwilliger [9] for more background information. Let $\mathbb{C}$ denote the complex number field and $X$ a nonempty finite set. Let $\text{Mat}_X \mathbb{C}$ denote the $\mathbb{C}$-algebra consisting of all matrices whose rows and columns are indexed by $X$ and whose entries are in $\mathbb{C}$. Let $V = \mathbb{C}^X$ denote the vector space over $\mathbb{C}$ consisting of column vectors whose coordinates are indexed by $X$ and whose entries are in $\mathbb{C}$. Observe that $\text{Mat}_X \mathbb{C}$ acts on $V$ by left multiplication. For all $y \in X$, let $\hat{y}$ denote the element of $V$ with 1 in the $y$-th coordinate and 0 in all other coordinates.

From now on $\Gamma$ denotes a finite, undirected, connected graph, without loops or multiple edges, with vertex set $X$, the shortest path-length distance function $\partial$ and diameter $d := \max\{\partial(x,y) \mid x,y \in X\}$. For a vertex $x \in X$ and integer $i \geq 0$ define

$$\Gamma_i(x) = \{y \in X \mid \partial(x,y) = i\}.$$  

For notational convenience abbreviate $\Gamma(x) = \Gamma_1(x)$. For an integer $k \geq 0$, the graph $\Gamma$ is said to be regular with valency $k$ whenever $|\Gamma(x)| = k$ for all $x \in X$. The graph $\Gamma$ is said to be distance-regular whenever for all integers $h, i, j$ (0 ≤ $h, i, j \leq d$) and vertices $x, y \in X$ with $\partial(x,y) = h$, the number $p_{ij}^h = |\Gamma_i(x) \cap \Gamma_j(y)|$ is independent of $x, y$. The constants $p_{ij}^h$ are called the intersection numbers of $\Gamma$. From now on assume $\Gamma$ is distance-regular with diameter $d \geq 3$. Note that $\Gamma$ is regular with valency $k = p_{11}^0$.

We recall the Bose-Mesner algebra of $\Gamma$. For 0 ≤ $i \leq d$ let $A_i$ denote the matrix in $\text{Mat}_X \mathbb{C}$ with $(x,y)$-entry equal to 1 if $\partial(x,y) = i$ and 0 otherwise. We call $A_i$ the $i$-th distance matrix of $\Gamma$. Note that $A_i$ is real and symmetric. We observe that $A_0 = I$, where $I$ is the identity matrix, and abbreviate $A = A_1$. We observe that $\sum_{i=0}^{d} A_i = J$ and $A_iA_j = \sum_{h=0}^{d} p_{ij}^h A_h$ for 0 ≤ $i, j \leq d$. Let $M$ denote the subalgebra of $\text{Mat}_X \mathbb{C}$ generated by $A$. By [2, p. 127] the matrices $\{A_i\}_{i=0}^{d}$ form a basis for $M$. We call $M$ the Bose-Mesner algebra of $\Gamma$. By [2, p. 45], $M$ has a basis $\{E_i\}_{i=0}^{d}$ such that (i) $E_0 = |X|^{-1} J$; (ii) $I = \sum_{i=0}^{d} E_i$; (iii) $E_iE_j = \delta_{ij} E_i$ for 0 ≤ $i, j \leq d$. By [1, p. 59, 64] the matrices $\{E_i\}_{i=0}^{d}$ are real and symmetric. We call $\{E_i\}_{i=0}^{d}$ the minimal idempotents of $\Gamma$. We call $E_0$ trivial. Since $\{E_i\}_{i=0}^{d}$ form a basis for $M$, there exist complex scalars $\{\theta_i\}_{i=0}^{d}$ such that

$$A = \sum_{i=0}^{d} \theta_i E_i.$$  \hspace{1cm} (1)
By (1) and since \( E_i E_j = \delta_{ij} E_i \) we have

\[
AE_i = E_i A = \theta_i E_i \quad (0 \leq i \leq d).
\]

(2)

We call the scalar \( \theta_i \) the eigenvalue of \( \Gamma \) corresponding to \( E_i \). Note that the eigenvalues \( \{ \theta_i \}_{i=0}^d \) are mutually distinct since \( A \) generates \( M \). Moreover \( \{ \theta_i \}_{i=0}^d \) are real, since \( A \) and \( \{ E_i \}_{i=0}^d \) are real. Let \( E \) denote a minimal idempotent of \( \Gamma \). Since \( \{ A_i \}_{i=0}^d \) form a basis for \( M \), there exist complex scalars \( \{ \theta_i^* \}_{i=0}^d \) such that

\[
E = \frac{1}{|X|} \sum_{i=0}^d \theta_i^* A_i.
\]

(3)

We call \( \theta_i^* \) the \( i \)-th dual eigenvalue of \( \Gamma \) corresponding to \( E \). Note that \( \{ \theta_i^* \}_{i=0}^d \) are real, since \( E \) and \( \{ A_i \}_{i=0}^d \) are real. Let \( \circ \) denote the entry-wise product in Mat\(_X\)\(\mathbb{C}\). Observe \( A_i \circ A_j = \delta_{ij} A_i \) for \( 0 \leq i, j \leq d \), so \( M \) is closed under \( \circ \). Therefore there exist complex scalars \( q_{ij}^h \) such that

\[
E_i \circ E_j = \frac{1}{|X|} \sum_{h=0}^d q_{ij}^h E_h \quad (0 \leq i, j \leq d).
\]

(4)

We call the \( q_{ij}^h \) the Krein parameters of \( \Gamma \). These parameters are real and nonnegative [2, p. 48–49]. For the moment fix integers \( h, i, j \) \( (0 \leq h, i, j \leq d) \). By construction \( q_{ij}^h = q_{ji}^h \). By [2, Lemma 2.3.1] we have \( m_h q_{ij}^h = m_i q_{jh}^h = m_j q_{hi}^h \). Therefore

\[
q_{ij}^h = 0 \text{ iff } q_{ij}^i = 0 \text{ iff } q_{hi}^i = 0.
\]

(5)

We recall the dual Bose-Mesner algebra of \( \Gamma \) [9, p. 378]. For the rest of this section, fix a vertex \( x \in X \). For \( 0 \leq i \leq d \) let \( E_i^* = E_i^*(x) \) denote the diagonal matrix in Mat\(_X\)\(\mathbb{C}\) with \((y, y)\)-entry

\[
(E_i^*)_{yy} = \begin{cases} 1 & \text{if } \partial(x, y) = i \\ 0 & \text{if } \partial(x, y) \neq i \end{cases} \quad (y \in X).
\]

We call \( E_i^* \) the \( i \)-th dual idempotent of \( \Gamma \) with respect to \( x \). We observe that \( I = \sum_{i=0}^d E_i^* \) and \( E_i^* E_j^* = \delta_{ij} E_i^* \) for \( 0 \leq i, j \leq d \). Therefore the matrices \( \{ E_i^* \}_{i=0}^d \) form a basis for a commutative subalgebra \( M^* = M^*(x) \) of Mat\(_X\)\(\mathbb{C}\). We call \( M^* \) the dual Bose-Mesner algebra of \( \Gamma \) with respect to \( x \).

For \( 0 \leq i \leq d \) let \( A_i^* = A_i^*(x) \) denote the diagonal matrix in Mat\(_X\)\(\mathbb{C}\) with \((y, y)\)-entry

\[
(A_i^*)_{yy} = |X| (E_i)_{xy} \quad (y \in X).
\]

We call \( A_i^* \) the dual distance matrix corresponding to \( E_i \). By [9, p. 379] the matrices \( \{ A_i^* \}_{i=0}^d \) form a basis for \( M^* \). Select an integer \( \ell \) \( (1 \leq \ell \leq d) \) and set \( E := E_\ell \), \( A^* := A_\ell^* \). Let \( \{ \theta_i^* \}_{i=0}^d \) denote the dual eigenvalues corresponding to \( E \). Then using (3) we find

\[
A^* = \sum_{i=0}^d \theta_i^* E_i^*.
\]

(6)
Moreover
\[ A^* E_i^* = E_i^* A^* = \theta_i^* E_i^* \quad (0 \leq i \leq d). \] (7)

We now recall how \( M \) and \( M^* \) are related. By the definition of the distance matrices and dual idempotents, we have
\[ E_h A_i E_j^* = 0 \quad \text{if and only if} \quad p_{ij}^h = 0 \quad (0 \leq h, i, j \leq d). \] (8)

By [9, Lemma 3.2],
\[ E_h A_i^* E_j = 0 \quad \text{if and only if} \quad q_{ij}^h = 0 \quad (0 \leq h, i, j \leq d). \] (9)

Let \( T = T(x) \) denote the subalgebra of \( \text{Mat}_X \mathbb{C} \) generated by \( M \) and \( M^* \). We call \( T \) the subconstituent algebra or Terwilliger algebra of \( \Gamma \) with respect to \( x \) [9, p. 380]. By a \( T \)-module we mean a subspace \( W \subseteq V \) such that \( BW \subseteq W \) for all \( B \in T \). Let \( W \) denote a \( T \)-module. Then \( W \) is said to be irreducible whenever \( W \) is nonzero and contains no \( T \)-modules other than \( 0 \) and \( W \).

We mention a special irreducible \( T \)-module. Let \( \hat{\theta}_i \) denote the expression in (10).

3 The main result

In this section we prove our characterization of \( Q \)-polynomial distance-regular graphs.

**Lemma 3.1.** Let \( \Gamma \) denote a distance-regular graph with diameter \( d \geq 3 \). Let \( E \) denote a nontrivial minimal idempotent of \( \Gamma \) that is TTR and let \( \{\theta_i^*\}_{i=0}^d \) be the corresponding dual eigenvalues. Let \( \beta \) and \( \gamma^* \) denote complex scalars such that \( \theta_i^* - \beta \theta_i^* + \theta_i^* = \gamma^* \) for \( 1 \leq i \leq d - 1 \). Then the expression
\[ \theta_{i-1}^* - \theta_i^* \theta_i^* + \theta_i^* - \gamma^*(\theta_{i-1}^* + \theta_i^*) \] (10)
is independent of \( i \) for \( 1 \leq i \leq d \).

**Proof.** Let \( \delta_i^* \) denote the expression in (10). For \( 1 \leq i \leq d - 1 \) the difference \( \delta_i^* - \delta_{i+1}^* \) is equal to
\[ (\theta_{i-1}^* - \theta_{i+1}^*) (\theta_{i-1}^* - \beta \theta_i^* + \theta_{i+1}^* - \gamma^*), \]
and is therefore 0. It follows that \( \delta_i^* \) is independent of \( i \) for \( 1 \leq i \leq d \). \( \square \)
Lemma 3.2. Let $\Gamma$ denote a distance-regular graph with diameter $d \geq 3$. Let $E$ denote a nontrivial minimal idempotent of $\Gamma$ that is TTR. Fix a vertex $x$ of $\Gamma$ and let $A^* = A^*(x)$ denote the dual distance matrix corresponding to $E$. Then

$$0 = [A^*, A^{*2}A - \beta A^* AA^* + AA^{*2} - \gamma^*(AA^* + A^*A) - \delta^* A],$$

(11)

where $\beta$ and $\gamma^*$ are from Lemma 3.1 and $\delta^*$ it the common value of (10). Here $[r, s]$ means $rs - sr$.

Proof. Let $C$ denote the expression on the right in (11). We show that $C = 0$. Observe

$$C = ICI = \left( \sum_{i=0}^{d} E_i^* \right) C \left( \sum_{j=0}^{d} E_j^* \right) = \sum_{i=0}^{d} \sum_{j=0}^{d} E_i^* C E_j^*.$$

To show $C = 0$ it suffices to show $E_i^* C E_j^* = 0$ for $0 \leq i, j \leq d$. For notational convenience define a polynomial $P$ in two variables

$$P(u, v) = u^2 - \beta uv + v^2 - \gamma^*(u + v) - \delta^*.$$

For $0 \leq i, j \leq d$ we have

$$E_i^* C E_j^* = E_i^* A E_j^* P(\theta_i^*, \theta_j^*) (\theta_i^* - \theta_j^*)$$

by (7), where $\{\theta_i^*\}_{i=0}^{d}$ are the dual eigenvalues corresponding to $E$. By (8) we find $E_i^* A E_j^* = 0$ if $|i - j| > 1$. By Lemma 3.1 we find $P(\theta_i^*, \theta_j^*) = 0$ if $|i - j| = 1$. Of course $\theta_i^* - \theta_j^* = 0$ if $i = j$. Therefore $E_i^* C E_j^* = 0$ as desired. We have now shown $C = 0$. \qed

In (9) we gave a characterization of a vanishing Krein parameter. We will need a variation on that result. We will obtain this variation after a few lemmas. The first lemma follows directly from the definitions of $\hat{x}$ and the dual adjacency matrices.

Lemma 3.3. Let $\Gamma$ denote a distance-regular graph with diameter $d \geq 3$. Let $E$ denote a nontrivial minimal idempotent of $\Gamma$. Fix a vertex $x$ of $\Gamma$ and let $A^* = A^*(x)$ denote the dual distance matrix corresponding to $E$. Then $A^* v = |X|(E \hat{x}) \circ v$ for all $v \in V$. \qed

Lemma 3.4. Let $\Gamma$ denote a distance-regular graph with diameter $d \geq 3$ and let $\{E_i\}_{i=0}^{d}$ denote the minimal idempotents of $\Gamma$. Fix $x \in X$. Then for $0 \leq h, i, j \leq d$,

$$E_h A_i^* E_j^* \hat{x} = q_{ij}^h E_h \hat{x},$$

where $A_i^* = A_i^*(x)$.

Proof. From Lemma 3.3 we find $A_i^* E_j^* \hat{x} = |X|(E_i \hat{x}) \circ (E_j \hat{x})$, and observe that this equals $|X|(E_i \circ E_j) \hat{x}$. Now

$$E_h A_i^* E_j^* \hat{x} = |X| E_h (E_i \circ E_j) \hat{x} = E_h \sum_{\ell=0}^{d} q_{ij}^\ell E_\ell \hat{x} = q_{ij}^h E_h \hat{x}. \quad \Box$$
Corollary 3.5. Let $\Gamma$ denote a distance-regular graph with diameter $d \geq 3$ and let $\{E_i\}_{i=0}^d$ denote the minimal idempotents of $\Gamma$. Fix $x \in X$ and let $V_0$ denote the primary module for $T(x)$. Then the following (i), (ii) are equivalent for $0 \leq h, i, j \leq d$.

(i) $q_{ij}^h = 0$.

(ii) $E_h A_i^* E_j$ vanishes on $V_0$, where $A_i^* = A_i^*(x)$.

Suppose (i), (ii) fail. Then $E_h A_i^* E_j V_0 = E_h V_0$.

Proof. (i) $\implies$ (ii) $E_h A_i^* E_j$ is zero by [9] and hence vanishes on $V_0$.

(ii) $\implies$ (i) Observe $\hat{x} \in V_0$ so $E_h A_i^* E_j \hat{x} = 0$. Therefore $q_{ij}^h E_h \hat{x} = 0$ in view of Lemma 3.4. The vector $E_h \hat{x}$ is a basis for $E_h V_0$ so $E_h \hat{x} \neq 0$. Thus $q_{ij}^h = 0$.

Suppose (i), (ii) fail. Then $E_h A_i^* E_j V_0$ is a nonzero subspace of the one-dimensional space $E_h V_0$ and is therefore equal to $E_h V_0$.

We are now ready to prove our main result.

Proof of Theorem 1.1 First suppose that $E$ is $Q$-polynomial. Then condition (i) holds by definition of a tail and the definition of the $Q$-polynomial property, condition (ii) holds by [2] Theorem 8.1.2], cf. Leonard [7], and condition (iii) holds by [2] Proposition 4.1.8].

To obtain the converse, assume that $E$ satisfies (i)–(iii). We show $E$ is $Q$-polynomial. To do this we consider the representation diagram $\Delta_E$ from the introduction. We will show that $\Delta_E$ is a path.

Let $\{E_i\}_{i=1}^d$ denote an ordering of the nontrivial minimal idempotents of $\Gamma$ such that $E = E_1$. Let $X$ denote the vertex set of $\Gamma$. Fix $x \in X$ and let $A_i^* = A_i^*(x)$. We first show that $\Delta_E$ is connected. To do this we follow an argument given in [11] Theorem 3.3]. Suppose that $\Delta_E$ is not connected. Then there exists a nonempty proper subset $S$ of $\{0, 1, \ldots, d\}$ such that $i, j$ are not adjacent in $\Delta_E$ for all $i \in S$ and $j \in \{0, 1, \ldots, d\} \setminus S$. Invoking [9] we find $E_i A^* E_j = 0$ and $E_j A^* E_i = 0$ for $i \in S$ and $j \in \{0, 1, \ldots, d\} \setminus S$. Define $F := \sum_{i \in S} E_i$ and observe

$$A^* F = I A^* F = \left( \sum_{i=0}^d E_i \right) A^* F = F A^* F.$$ 

By a similar argument $F A^* = F A^* F$, so $A^*$ commutes with $F$. Since $F \in M$ there exist complex scalars $\{\alpha_i\}_{i=0}^d$ such that $F = \sum_{i=0}^d \alpha_i A_i$. We have

$$0 = A^* F - F A^* = \sum_{i=1}^d \alpha_i (A^* A_i - A_i A^*).$$

(12)

We claim that the matrices $\{A^* A_i - A_i A^* | 1 \leq i \leq d\}$ are linearly independent. To prove the claim, for $1 \leq i \leq d$ define $B_i = A^* A_i - A_i A^*$, and observe $B_i \hat{x} = (\theta_i^* - \theta_0^*) A_i \hat{x}$. The vectors $\{A_i \hat{x}\}_{i=1}^d$ are linearly independent and $\theta_i^* \neq \theta_0^*$ for $1 \leq i \leq d$ so the vectors $\{B_i \hat{x}\}_{i=1}^d$ are...
linearly independent. Therefore the matrices \( \{B_i\}_{i=1}^d \) are linearly independent and the claim is proved. By the claim and (12) we find \( \alpha_i = 0 \) for \( 1 \leq i \leq d \). Now \( F = \alpha_0 I \). But \( F^2 = F \) so \( \alpha_0^2 = \alpha_0 \). Thus \( \alpha_0 = 0 \), in which case \( S = \emptyset \), or \( \alpha_0 = 1 \), in which case \( S = \{0, 1, \ldots, d\} \). In either case we have a contradiction so \( \Delta_E \) is connected.

As we mentioned in the introduction, in \( \Delta_E \) the vertex 0 is adjacent to vertex 1 and no other vertex of \( \Delta_E \). By the definition of a tail and since \( \Delta_E \) is connected, vertex 1 is adjacent to vertex 0 and exactly one other vertex in \( \Delta_E \). To show that \( \Delta_E \) is a path, it suffices to show that each vertex in \( \Delta_E \) is adjacent to at most two other vertices in \( \Delta_E \). We assume this is not the case and obtain a contradiction. Let \( v \) denote a vertex in \( \Delta_E \) that is adjacent to more than two vertices of \( \Delta_E \). Of all such vertices, we pick \( v \) such that the distance to 0 in \( \Delta_E \) is minimal. Call this distance \( i \). Note that \( 2 \leq i \leq d - 1 \) by our above comments and the construction. For notational convenience and without loss of generality we may assume that the vertices of \( \Delta_E \) are labelled such that for \( 1 \leq j \leq i - 1 \) the vertex \( j \) is adjacent to \( j - 1 \) and \( j + 1 \) and no other vertex in \( \Delta_E \). By construction the chosen vertex \( v \) is labelled \( i \). This vertex is adjacent to \( i - 1 \) and at least two other vertices in \( \Delta_E \). Let \( t \) denote a vertex in \( \Delta_E \) other than \( i - 1 \) that is adjacent to vertex \( i \). Since \( E \) is TTR there exists \( \beta \in \mathbb{C} \) such that \( \theta_{j-1}^* - \beta \theta_j^* + \theta_{j+1}^* \) is independent of \( j \) for \( 1 \leq j \leq d - 1 \). We claim that

\[
\theta_i - (\beta + 1) \theta_i + (\beta + 1) \theta_{i-1} - \theta_{i-2} = 0. \tag{13}
\]

To prove the claim we consider the equation (11). In that equation we expand the right-hand side to get

\[
0 = A^3 - (\beta + 1)(A^2 A^* - A^* A^2) - AA^3 - \gamma^*(A^2 A - AA^2) - \delta^*(A^* A - AA^*).
\]

In this equation we multiply each term on the left by \( E_{i-2} \) and on the right by \( E_t \). To help simplify the results we make some comments. Using (2) we find \( E_{i-2} A^3 E_t = E_{i-2} A^3 E_t \). Using (9) we find

\[
E_{i-2} A^3 E_t = E_{i-2} A^* \left( \sum_{r=0}^d E_r \right) A^* \left( \sum_{s=0}^d E_s \right) A^* E_t = E_{i-2} A^* E_{i-1} A^* E_t A^* E_t.
\]

Similarly we calculate

\[
E_{i-2} A^2 A^* E_t = E_{i-2} A^* E_{i-1} A^* E_t A^* E_t \theta_i,
\]

\[
E_{i-2} A^* A A^2 E_t = E_{i-2} A^* E_{i-1} A^* E_t A^* E_t \theta_{i-1},
\]

\[
E_{i-2} A A^3 E_t = E_{i-2} A^* E_{i-1} A^* E_t A^* E_t \theta_{i-2}
\]

and

\[
E_{i-2} A^2 A E_t = 0, \quad E_{i-2} A A^2 E_t = 0,
\]

\[
E_{i-2} A^* A E_t = 0, \quad E_{i-2} A A^* E_t = 0.
\]

From these comments we find

\[
0 = E_{i-2} A^* E_{i-1} A^* E_t \left( \theta_t - (\beta + 1) \theta_i + (\beta + 1) \theta_{i-1} - \theta_{i-2} \right). \tag{14}
\]
We show that $E_{i-2}A^*E_{i-1}A^*E_iA^*E_t \neq 0$. By the last assertion of Corollary 3.5 and since the sequence $(i-2, i-1, i, t)$ is a path in $\Delta_E$, we find $E_iA^*E_tV_0 = E_iV_0$ and $E_{i-1}A^*E_iV_0 = E_{i-1}V_0$ and $E_{i-2}A^*E_{i-1}V_0 = E_{i-2}V_0$. Therefore

$$E_{i-2}A^*E_{i-1}A^*E_iA^*E_tV_0 = E_{i-2}V_0.$$ 

Observe $E_{i-2}V_0 \neq 0$ so $E_{i-2}A^*E_{i-1}A^*E_iA^*E_t \neq 0$, as desired. By this and (14) we obtain (13).

By (13) the scalar $\theta_t$ is uniquely determined. The scalars $\theta_0, \ldots, \theta_d$ are mutually distinct so $t$ is uniquely determined, for a contradiction. We have shown that $\Delta_E$ is a path and therefore $E$ is $Q$-polynomial.

4 Remarks

In this section we make some remarks concerning the three conditions in Theorem 1.1. Throughout the section assume $\Gamma$ is a distance-regular graph with diameter $d \geq 3$ and eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$. Pick a nontrivial minimal idempotent $E = E_j$ of $\Gamma$ and let $\{\theta_i^*\}_{i=0}^d$ denote the corresponding dual eigenvalue sequence. Abbreviate $\theta = \theta_j$.

Note 4.1. [2, pp. 142–143, 161] Pick an integer $i$ ($1 \leq i \leq d$). Then $\theta_i^* = \theta_0^*$ if and only if at least one of the following holds.

(i) $\Gamma$ is bipartite, $i$ is even, and $j = d$,

(ii) $\Gamma$ is antipodal, $i = d$, and $j$ is even.

Note 4.2. The graph $\Gamma$ is imprimitive if and only if $\Gamma$ is bipartite or antipodal [2, Theorem 4.2.1]. Thus if $\Gamma$ is primitive then $\theta_i^* \neq \theta_0^*$ for $1 \leq i \leq d$.

Lemma 4.3. Assume that $\theta \neq -1$ and one of the following occurs:

(i) $d = 3$,

(ii) $d = 4$, $\Gamma$ is antipodal, and $j$ is even,

(iii) $d = 5$, $\Gamma$ is antipodal, and $j$ is even.

Then $E$ is TTR.

Proof. We have $(\theta_i^* - \theta_j^*)kb_1 = (k - \theta_0)(1 + \theta)$ by [4, Lemma 2.2], so $\theta_i^* \neq \theta_j^*$. Define $\beta \in \mathbb{C}$ such that $\beta + 1 = (\theta_0^* - \theta_j^*)/(\theta_1^* - \theta_2^*)$. By the construction $\theta_{i-1}^* - \beta \theta_i^* + \theta_{i+1}^*$ is independent of $i$ for $i = 1, 2$. We are done in case (i), so assume we are in cases (ii) or (iii). By [2, p. 142] we have $\theta_i^* = \theta_{d-i}^*$ for $0 \leq i \leq d$. Therefore $\theta_{i-1}^* - \beta \theta_i^* + \theta_{i+1}^*$ is independent of $i$ for $1 \leq i \leq d - 1$. In other words $E$ is TTR. \qed
Note 4.4. Referring to the conditions (i)–(iii) of Theorem 1.1 we show that no proper subset of (i)–(iii) implies that $E$ is $Q$-polynomial.

- Conditions (ii), (iii) are not sufficient. Assume $\Gamma$ is the generalized hexagon of order $(2, 1)$ [2, p. 200, 425]. It is primitive with diameter $d = 3$ and eigenvalues $4, 1 + \sqrt{2}, 1 - \sqrt{2}, -2$. Pick $j = 1$. Then $E$ satisfies condition (ii) by Lemma 4.3 and $E$ satisfies condition (iii) by Note 4.2. But $E$ does not satisfy (i) by [2, p. 413, 425]. In particular $E$ is not $Q$-polynomial.

- Conditions (i), (iii) are not sufficient. Assume $\Gamma$ is the dodecahedron, which is antipodal with diameter $d = 5$ and eigenvalues $3, \sqrt{5}, 1, 0, -2, -\sqrt{5}$, see [2, p. 417]. Pick $j = 1$. Then $E$ satisfies condition (i) by [2, p. 413, 417] and $E$ satisfies condition (iii) by Lemma 4.3. By [4, Lemma 2.2] we find $\theta^*_0 = 3, \theta^*_1 = \sqrt{5}, \theta^*_2 = 1, \theta^*_3 = -1, \theta^*_4 = -\sqrt{5}, \theta^*_5 = -3$ and using this one verifies that $E$ does not satisfy condition (ii). In particular $E$ is not $Q$-polynomial.

- Conditions (i), (ii) are not sufficient. Assume $\Gamma$ is the Wells graph [2, p. 421], which is antipodal with diameter $d = 4$ and eigenvalues $5, \sqrt{5}, 1, -\sqrt{5}, -3$. Pick $j = 2$. Then $E$ satisfies condition (i) by [2, p. 413] and $E$ satisfies condition (ii) by Lemma 4.3, but $E$ does not satisfy condition (iii) by Note 4.1. In particular $E$ is not $Q$-polynomial.

Note 4.5. Assume $\Gamma$ is bipartite, and consider the conditions (i)–(iii) of Theorem 1.1. If $E$ satisfies conditions (i), (iii) then $E$ is $Q$-polynomial by [5, proof of Theorem 5.4]. If $E$ satisfies conditions (ii), (iii) then $E$ is $Q$-polynomial by [6, Theorem 10.5].

References

[1] Bannai, E. and T. Ito, *Algebraic Combinatorics I: Association Schemes*. Benjamin-Cummings Lecture Note Ser. 58, The Benjamin/Cumming Publishing Company, Inc., London (1984).

[2] A. E. Brouwer, A. M. Cohen and A. Neumaier, *Distance-Regular Graphs*, Springer-Verlag, Berlin, Heidelberg, 1989.

[3] P. Delsarte, An algebraic approach to the association schemes of coding theory, *Philips Research Reports Suppl.* 10 (1973).

[4] A. Jurčič, J. Koolen and P. Terwilliger, Tight distance-regular graphs, *J. Algebraic Combin.* 12 (2000), 163–197.

[5] M. S. Lang, Tails of bipartite distance-regular graphs, *European J. Combin.* 23 (2002), 1015–1023.

[6] M. S. Lang, A new inequality for bipartite distance-regular graphs, *J. Combin. Theory Ser. B* 90 (2004), 55–91.

[7] D. A. Leonard, Orthogonal polynomials, duality and association schemes, *SIAM J. Math. Anal.* 13 (1982), 656-663.

[8] A. A. Pascasio, A characterization of $Q$-polynomial distance-regular graphs, *Discrete Math.* 308 (2008), 3090–3096.

[9] P. Terwilliger, The subconstituent algebra of an association scheme, I., *J. Algebraic Combin.* 1 (1992), 363–388.

[10] P. Terwilliger, The subconstituent algebra of an association scheme, III., *J. Algebraic Combin.* 2 (1993), 177–210.

[11] P. Terwilliger, A new inequality for distance-regular graphs, *Discrete Math.* 137 (1995), 319–332.