Aggregate Power Flexibility in Unbalanced Distribution Systems

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Abstract—With a large-scale integration of distributed energy resources (DERs), distribution systems are expected to be capable of providing support for the transmission grid. To effectively harness the collective flexibility from massive DER devices, this paper studies distribution-level power aggregation strategies for transmission-distribution interaction. In particular, this paper proposes a method to model and quantify the aggregate power flexibility, i.e., the net power injection achievable at the substation, in unbalanced distribution systems over time. Incorporating the network constraints and multi-phase unbalanced modeling, the proposed method obtains an effective approximate feasible region of the net power injection. For any aggregate power trajectory within this region, it is proved that there exists a feasible disaggregation solution. In addition, a distributed model predictive control (MPC) framework is developed for practical implementation of the transmission-distribution interaction. At last, we demonstrate the performances of the proposed method via numerical tests on a real-world distribution feeder with 126 multi-phase nodes.

Index Terms—Power aggregation, distributed energy resources, unbalanced power flow, distributed optimization.

I. INTRODUCTION

Conventionally, a distribution grid is treated as an equivalent passive load in the operation of transmission systems due to its non-dispatchability [1]. In the past decade, a rapid proliferation of distributed energy resources (DERs) has been witnessed in distribution systems, especially photovoltaic (PV) generation, energy storage and demand response. As the penetration of DERs deepens, significant flexibility has been introduced to electricity distribution, making the distribution systems evolve from passive to active state [2]. In particular, the coordinated dispatch of ubiquitous DER devices enables the active participation of distribution systems in the grid operation and even the provision of reverse power flow from distribution to transmission. Through transmission-distribution coordinated operation, the power grid can fully exploit its flexibility and achieve greater efficiency and resilience.

In practice, managing a large population of DER devices for system-wide control and operation is challenging due to the computational complexity. References [3]–[5] apply decomposition methods to incorporate transmission and distribution in economic dispatch and reactive power optimization. However, these methods require a number of iterations and boundary information exchanges between these two systems, and are thus arduous for practical applications. To address this challenge, as a promising alternative, power aggregation has received considerable attention for leveraging the available flexibility from the distribution side. As illustrated in Figure [1] the generation or power consumption of each DER device can be captured by a certain feasible set, which is specified by its own operational constraints and dynamics. Power aggregation is to model and qualify the aggregate flexibility at the substation, which is the achievable net power injection to the distribution feeder. By reporting this concise and compact feasible region, a distribution grid can actively participate in transmission system operation and control as a virtual power plant. Hence, this paper focuses on developing novel power aggregation methods for the distribution system, which takes into account the models and constraints of both the network and DER devices.

![Fig. 1. Illustration of power aggregation in distribution networks.](image)

Basically, power aggregation can be regarded as a projection of the high-dimensional operational constraints onto the feasible region of the net substation loads. However, considering tens of thousands of electric devices and multiple time steps, procuring this projection for an exact aggregate feasible region is computationally intensive and impractical. Therefore, most research efforts are devoted to building inner or outer approximations of the exact feasible set. Reference [6] utilizes a series of time-moving ellipsoids to model the aggregate P-Q feasible domain over time, and follows a data-driven system identification procedure to obtain the corresponding parameters. In [7]–[9], individual flexibility of each DER is represented by a polytopic feasible set, then the aggregate flexibility is calculated as the Minkowski sum of the individual polytopes. Reference [10] applies the polytopic projection to procure the aggregate flexibility, and formulates an optimization problem to approximate this projection for a tractable solution. In [11] [12], robust optimization models are developed for estimating and optimally scheduling the aggregate reserve capacities.

Many researches above focus on a single type of DERs, e.g., thermostatically controlled loads (TCL) or heating, ventilation
and air-conditioning (HVAC) systems. Hence, their flexibility modeling and aggregation approaches may not be able to handle a variety of DERs in distribution systems. In addition, most existing studies disregard the network constraints, such as voltage limits and line thermal constraints, which are crucial to the system-level power aggregation. Furthermore, since distribution networks are intrinsically unbalanced due to non-symmetrical conductors, untransposed lines, and unequal interphase power injection \[\text{[13]}\], the multi-phase modeling of networks and DER devices is required.

To address these challenges, this paper proposes an power aggregation method to quantify the aggregate flexibility from different types of DERs in unbalanced distribution systems. In particular, we approximate the exact feasible set of the net power injection at the substation with an inner-box region, and the upper and lower operational trajectories are defined to specify the aggregate flexibility over time. Then two multi-period optimization models are established for system-level power aggregation: one aims to evaluate the maximal flexibility level of the distribution systems, and the other is to co-optimize the base operational trajectory and flexibility reserve capacities. The main contributions of this paper are summarized as follows.

1. A distribution-level power aggregation method is proposed, which incorporates both the network constraints and multi-phase unbalanced modeling. And the existence of power disaggregation solution is guaranteed with our method (see Proposition \[\text{[1]}\].

2. To protect the privacy of DER facilities and enable scalable application, we develop a distributed model predictive control (MPC) framework for practical implementation of the distribution-transmission interaction.

The remainder of this paper is organized as follows: Section II introduces the multi-phase network and DER models. Section III presents the inner-box approximation method and these two optimization models. Section IV develops the distributed MPC framework for transmission-distribution interaction. Numerical tests are carried out on a real feeder system in Section V, and conclusions are drawn in Section VI.

II. NETWORK AND DER MODELS

\textbf{Notation:} Upper-case (lower-case) boldface letters are used for matrices (column vectors). For a given vector \(\mathbf{x} \in \mathbb{R}^N\), \(\text{diag}(\mathbf{x})\) returns the \(N \times N\) matrix with the elements of \(\mathbf{x}\) in its diagonal. \(|\cdot|\) takes entry-wise absolute value of a vector or denotes the cardinality of a set. Symbol \(\mathbf{1}_N\) denotes the \(N \times 1\) vector with all ones. We use \((\cdot)^\top\) for transposition, \((\cdot)^{-1}\) for matrix inverse and \((\cdot)^*\) for complex conjugate. \(\mathbb{R}\{\cdot\}\) and \(\mathbb{I}\{\cdot\}\) represent the real and imaginary parts of a complex value respectively, and \(j := \sqrt{-1}\).

A. Network Model

Consider a multi-phase distribution network described by the graph \(G(\mathcal{N}_0, \mathcal{E})\), where \(\mathcal{N}_0\) denotes the set of buses and \(\mathcal{E} \subset \mathcal{N}_0 \times \mathcal{N}_0\) represents the set of distribution lines. Let \(\mathcal{N}_0 := \mathcal{N} \cup \{0\}\) with \(\mathcal{N} := \{1, 2, \ldots, N\}\), and bus-0 is the slack bus of the distribution network, i.e., the substation interface to the transmission grid.

As shown in Figure 2 each electric device can be multi-phase wye-connected or delta-connected \[\text{[13]}\] to the distribution network. Let \(\phi_Y := \{a, b, c\}\) and \(\phi_\Delta := \{ab, bc, ca\}\). Then we denote the concrete connection manner of an electric device by notation \(\psi\) with either \(\psi \subseteq \phi_Y\) or \(\psi \subseteq \phi_\Delta\). For instance, \(\psi = \{a\}\) if the device is wye-connected in phase A and only has the power injection \(s^a\), while \(\psi = \{ab, bc\}\) if it is delta-connected in phase AB and BC with power injection \(s^{ab}\) and \(s^{bc}\).

Denote \(\mathcal{N}_Y\) and \(\mathcal{N}_\Delta\) as the number of buses via wye- and delta-connection respectively with \(\mathcal{N}_Y + \mathcal{N}_\Delta = \mathcal{N}\). Let vector \(s_Y := p_Y + jq_Y \in \mathbb{C}^{3\mathcal{N}_Y}\) and \(s_\Delta := p_\Delta + jq_\Delta \in \mathbb{C}^{3\mathcal{N}_\Delta}\) collect the three-phase power injections via wye- and delta-connection respectively. For compact expression, we stack all power injections into a long vector \(\mathbf{x} := [p_Y^a, q_Y^a, p_Y^b, q_Y^b, p_Y^c, q_Y^c, p_\Delta^ab, q_\Delta^ab, p_\Delta^bc, q_\Delta^bc, p_\Delta^{ca}, q_\Delta^{ca}]^\top\).

Let vector \(\mathbf{v} \in \mathbb{C}^{3\mathcal{N}_Y}\) gather the three-phase nodal voltage for all buses \(i \in \mathcal{N}\), and vector \(\mathbf{i}_L \in \mathbb{C}^{3|\mathcal{E}|}\) gather the three-phase line current for all lines \(ij \in \mathcal{E}\). Vector \(\mathbf{p}_0 \in \mathbb{R}^3\) denotes the three-phase active power injection at the substation bus.

![Fig. 2. Illustration of wye-connection and delta-connection.](image)

Based on a given operation point \(\{\hat{\mathbf{v}}, \hat{s}_Y, \hat{s}_\Delta\}\), we can derive the linear multi-phase power flow model \([\text{14}]\) as follows with the fixed-point linearization method \([\text{16}]\).

\[
\begin{align*}
|\mathbf{v}| &= \mathbf{A}\mathbf{x} + \mathbf{a} & (1a) \\
|\mathbf{i}_L| &= \mathbf{B}\mathbf{x} + \mathbf{b} & (1b) \\
\mathbf{p}_0 &= \mathbf{D}\mathbf{x} + \mathbf{d} & (1c)
\end{align*}
\]

Here, \(|\mathbf{v}|\) and \(|\mathbf{i}_L|\) denote the corresponding voltage magnitudes and line current magnitudes respectively. Matrices \(\mathbf{A}, \mathbf{B}, \mathbf{D}\) and vectors \(\mathbf{a}, \mathbf{b}, \mathbf{d}\) are system parameters, whose definitions are provided in Appendix A. See \([\text{17}]\) for more details on the deviation of this linear power flow model.

\textbf{Remark:} For exposition and notation simplicity, we outline the power flow model for a three-phase system. However, the proposed framework is clearly applicable to the systems with a mix of three-, double-, single-phase connections. For example, if a electric device is just double-phase or single-phase integrated, we fix the entries of the missing phases as zero in \(s_Y\) (or \(s_\Delta\)) and the corresponding line impedance matrix.

The linear multi-phase power flow model \([\text{14}]\) captures all possible connection manners of electric devices, which is applicable to both meshed and radial distribution networks. Accordingly, the network constraints \([\text{2}]\), involving the voltage
Here, \( \psi \) denotes the state of charge (SOC) at time \( t \), energy storage (ES) devices, directly controllable loads and HVAC systems. Their operational models are formulated as follows.

1) PV Units: \( \forall i \in \mathcal{N}_{pv}, \ t \in T \)
\[
P_{i,t}^{\psi} \leq P_{i,t}^{\psi} \leq \bar{P}_{i,t}^{\psi} \quad (3a)
\]
\[
\left( P_{i,t}^{\psi} \right)^{2} + \left( Q_{i,t}^{\psi} \right)^{2} \leq \left( \bar{S}_{i,t}^{\psi} \right)^{2} \quad (3b)
\]

Here, \( P_{i,t}^{\psi} \) and \( Q_{i,t}^{\psi} \) are the active and reactive PV generation power in phase \( \psi \) of bus \( i \) at time \( t \), respectively. \( \bar{P}_{i,t}^{\psi} \) and \( \bar{S}_{i,t}^{\psi} \) are the upper and lower limits of active generation power, and \( \bar{S}_{i,t}^{\psi} \) denotes the apparent power capacity of PV units.

2) Energy Storage Devices: \( \forall i \in \mathcal{N}_{es}, \ t \in T \)
\[
P_{i,t}^{\psi} \leq P_{i,t}^{\psi} \leq \bar{P}_{i,t}^{\psi} \quad (4a)
\]
\[
\left( P_{i,t}^{\psi} \right)^{2} + \left( Q_{i,t}^{\psi} \right)^{2} \leq \left( \bar{S}_{i,t}^{\psi} \right)^{2} \quad (4b)
\]
\[
E_{i,t+1} = E_{i,t} - \delta_t \sum_{\psi} P_{i,t}^{\psi} \quad (4c)
\]
\[
E_i \leq E_{i,t} \leq \bar{E}_i \quad (4d)
\]

where \( P_{i,t}^{\psi} \) and \( Q_{i,t}^{\psi} \) are the active and reactive ES power outputs in phase \( \psi \) of bus \( i \) at time \( t \), respectively; \( \bar{P}_{i,t}^{\psi} \) and \( \bar{S}_{i,t}^{\psi} \) are the corresponding upper and lower limits of active power outputs, and \( \bar{S}_{i,t}^{\psi} \) is the apparent power capacity of the ES devices; \( E_{i,t} \) denotes the state of charge (SOC) at time \( t \), while \( \delta_t \) and \( \bar{E}_i \) are the corresponding upper and lower limits; \( \delta_t > 0 \) is the power-energy conversion coefficient related to the length of each time interval \([t, t + 1]\). Here, \( P_{i,t}^{\psi} \) can be either positive (discharging) or negative (charging), and we assume unit charging/discharging efficiency for simplification.

3) Directly Controllable Loads: \( \forall i \in \mathcal{N}_{dcl}, \ t \in T \)
\[
P_{i,t}^{d,\psi} \leq P_{i,t}^{d,\psi} \leq \bar{P}_{i,t}^{d,\psi} \quad (5a)
\]
\[
Q_{i,t}^{d,\psi} = \eta_i^d \cdot P_{i,t}^{d,\psi} \quad (5b)
\]

where \( P_{i,t}^{d,\psi} \) and \( Q_{i,t}^{d,\psi} \) are the active and reactive controlled loads in phase \( \psi \) of bus \( i \) at time \( t \), respectively. \( \bar{P}_{i,t}^{d,\psi} \) and \( \bar{S}_{i,t}^{\psi} \) are the upper and lower limits for active load adjustment. Here, we assume constant \( \eta_i^d \) that indicates fixed power factors.

4) HVAC Systems: \( \forall i \in \mathcal{N}_{hvac}, \ t \in T \)
\[
0 \leq P_{i,t}^{h,\psi} \leq \bar{P}_{i,t}^{h,\psi} \quad (6a)
\]
\[
Q_{i,t}^{h,\psi} = \eta_i^h \cdot P_{i,t}^{h,\psi} \quad (6b)
\]
\[
T_{i,t}^{in} = T_{i,t}^{in} = T_{i+1}^{in} + \alpha_i \cdot (T_{i,t+1}^{out} - T_{i,t}^{in}) + \beta_i \sum_{\psi} P_{i,t}^{h,\psi} \quad (6c)
\]
\[
T_{i} \leq T_{i,t}^{in} \leq \bar{T}_{i} \quad (6d)
\]

Here, \( P_{i,t}^{h,\psi} \) and \( Q_{i,t}^{h,\psi} \) are the active and reactive HVAC power demands in phase \( \psi \) of bus \( i \) at time \( t \), respectively. \( T_{i,t}^{in} \) and \( T_{i,t}^{out} \) represent the temperatures inside and outside the buildings at time \( t \), and the comfortable temperature zone is described as \([\bar{T}_{i}, T_{i}]\). Equation (6c) formulates the indoor temperature evolution, where \( \alpha_i \in (0, 1) \) and \( \beta_i \) are the parameters specifying the thermal characteristics of the buildings and the environment. \( \beta_i > 0 \) if the HVAC appliance works as a heater, and \( \beta_i < 0 \) if it works as a cooler. See [18] for a detailed description. To facilitate the subsequent proof of disaggregation feasibility, we assume that the sign of \( \beta_i \) keeps fixed during the given time period, which implies that the HVAC appliances do not change their operation modes between cooling and heating.

Consequently, the network model (1)-(5) and the DER models (6)-(8) can be equivalently rewritten as the following compact form.
\[
P_{0,t} = 1_3^T \cdot p_{0,t} = 1_3^T \cdot D \cdot x_t + 1_3^T \cdot d \quad t \in T \quad (7a)
\]
\[
g_r (x_t) \leq 0 \quad t \in T \quad (7b)
\]
\[
\sum_{\tau=1}^{T} H_{r} x_{\tau} \leq h \quad t \in T \quad (7c)
\]

where \( x_t \) is a realization of \( x \) at time \( t \), scalar \( P_{0,t} \) is the total net power injection to the substation bus, \( g_r (\cdot) \) is a vector-valued convex function, and matrix \( H_r \) and vector \( h \) collect appropriate system parameters.

Equation (7a) is the formula of the net load or net power injection at the substation, which is obtained by adding (13) over the three phases. Equation (7b) captures the time-decoupled operational constraints. It is noted that the power flow equalities (14) and (15) are reformulated in an equivalent unified form as inequalities, which are also contained in equation (7b). Equation (7c) gathers the time-coupled constraints, involving the ES SOC limits (4c)-(4d) and the HVAC comfortable temperature limits (6c)-(6d).

III. POWER AGGREGATION METHODOLOGY

In this section, we measure the aggregate flexibility using an inner-box approximation method. Then two optimization models are developed to implement power aggregation: one aims to evaluate the maximal flexibility level of the distribution systems, while the other is to optimally schedule the flexibility reserve and the base-case power dispatch for minimization of the net operation cost.

A. Inner-box Approximation Method

Given both the network and DER constraints, the goal is to determine the net power injection achievable at the substation
over time, which constructs the aggregated power feasible region. Since it is computationally impractical to procure the exact feasible region with massive DER devices, we quantify the aggregate flexibility at the substation with an inner-box approximation method. As illustrated in Figure 3(a), we define a power interval \( [P_{0,t}^\wedge, P_{0,t}^\vee] \) to restrict the net power injection for each time \( t \in \mathcal{T} \), which forms a box-shape feasible region \( \mathcal{S} \) in the power coordinates:

\[
\mathcal{S} = [P_{0,t}^\wedge, P_{0,t}^\vee] \times [P_{0,t}^\wedge, P_{0,t}^\vee] \times \cdots \times [P_{0,t}^\wedge, P_{0,t}^\vee]
\]  

(8)

When mapped to the time coordinate, the feasible region \( \mathcal{S} \) is specified by the upper power trajectory \( \Delta \) where \( T \) is the time horizon.

The inner-box approximation for the aggregated power feasible region, which illustrates the case with two time steps; (b) depicts how the actually deployed real trajectories of net power injection at the substation should be in the shade area given by the upper and lower ones. We further define the aggregate flexibility \( E_{af} \) for the time horizon \( \mathcal{T} \) as

\[
E_{af}(\mathcal{T}) := \sum_{t \in \mathcal{T}} (P_{0,t}^\wedge - P_{0,t}^\vee) \cdot \Delta t
\]  

(9)

where \( \Delta t \) is the time granularity. \( E_{af} \) has the unit of energy, which can be interpreted as the potential energy flexibility level of the distribution system.

The proposed inner-box approximation method integrates three main merits: 1) The feasibility region \( \mathcal{S} \) is a conservative approximation of the real one, so that any arbitrary trajectory within \( \mathcal{S} \) is achievable by properly coordinating the DER devices while respecting the operation constraints. 2) The feasibility region is defined in a time-decoupled manner, which facilitates practical applications. 3) With the interpretation of a multi-dimensional box, it is straightforward to quantify the "volume" of this feasibility region.

**Remark:** Although the proposed method is illustrated for the case where the summation of net power injection across the phases is considered, it is capable of quantifying the power flexibility of each phase.

In the following aggregation models, we use the superscripts "\( \wedge \)", "\( \vee \)", "\( \psi \)" to denote the upper, base, lower trajectories, respectively. \( P_{0,t}^\wedge, P_{0,t}^\vee, P_{0,t}^\psi \) are the net power injection to the substation for these three trajectories, which are achieved by the power injection \( x_t^\wedge, x_t^\vee, x_t^\psi \) with \( P_{0,t}^{\wedge,\psi,\vee} = 1/T \cdot D \cdot x_t^{\wedge,\psi,\vee} + 1/T \cdot d \) respectively.

**B. Maximal-flexibility Power Aggregation Model**

With these definitions above in place, we aim to find the optimal approximate feasible region that achieves the largest aggregate flexibility. Accordingly, the following maximal-flexibility power aggregation (MPA) model \( (10)-(12) \) is built to solve the optimal upper and lower operational trajectories.

1) **Objective Function:**

\[
\max_{x_t^\wedge, x_t^\vee} \sum_{t \in \mathcal{T}} (P_{0,t}^\wedge - P_{0,t}^\psi) \cdot \Delta t = \sum_{t=1}^{\mathcal{T}} 13 \cdot \mathbf{D} \cdot (x_t^\wedge - x_t^\vee) \cdot \Delta t
\]  

(10)

Objective function \( (10) \) is formulated to maximize the aggregate flexibility \( E_{af} \) of the distribution system.

2) **Individual Constraints:**

\[
\sum_{t=1}^{\mathcal{T}} \mathbf{H}_t \cdot x_t^\wedge, x_t^\vee \leq \mathbf{h}_t \cdot g_t(x_t^\wedge) \leq 0 \quad \forall t \in \mathcal{T}
\]  

(11a)

\[
\sum_{t=1}^{\mathcal{T}} \mathbf{H}_t \cdot x_t^\wedge, x_t^\vee \leq \mathbf{h}_t \cdot g_t(x_t^\vee) \leq 0 \quad \forall t \in \mathcal{T}
\]  

(11b)

Identical to equation \( (7b) \) and \( (7c) \), equation \( (11a) \) and \( (11b) \) depict the network and DER constraints for the upper and lower operational trajectories respectively, where the inequalities are taken entry-wise. Since there is no overlapping term between the upper and lower trajectories in \( (11a) \) and \( (11b) \), we call them "individual constraints".

3) **Joint Constraints:**

\[
P_{0,t}^\wedge \leq P_{0,t}^\vee \quad \forall t \in \mathcal{T}
\]  

(12a)

\[
P_{i,t}^{\psi,\wedge} \leq P_{i,t}^{\psi,\vee} \quad \forall i \in \mathcal{N}_{es}, t \in \mathcal{T}
\]  

(12b)

\[
P_i^{h,\wedge} \leq P_i^{h,\vee} \quad \forall i \in \mathcal{N}_{heac}, t \in \mathcal{T}
\]  

(12c)

where \( P_{i,t}^\psi, P_{i,t}^{\psi,\wedge}, P_{i,t}^{\psi,\vee} \) and \( P_i^{h,\psi,\wedge}, P_i^{h,\psi,\vee} \) are the expected power injection associated with the upper and lower trajectories respectively. Equation \( (12) \) collects all the “joint constraints” corresponding to the upper and lower trajectories, which is utilized to guarantee that any aggregated power trajectory between the upper and lower ones is achievable. In particular, equation \( (12a) \) indicates the upper aggregated power should always be larger than the lower one, so that these two trajectories do not intersect with each other and the aggregate feasible region is well defined. Equation \( (12b) \) implies that the ES power injection associated with the upper trajectory should always be smaller than that at the lower trajectory. Intuitively, this equation is imposed to make any trajectories in between satisfy the SOC limits \( (4) \). Similarly, equation \( (12c) \) is utilized to guarantee that the comfortable temperature limits \( (6d) \) will not be violated by any interior trajectories. See Appendix B for detailed explanations.

As a consequence, the proposed MPA model \( (10)-(12) \) is formulated as a convex programming problem. Through solving this model, we can procure the largest inner-box approximation of the aggregated power feasible region, together with the optimal upper and lower operational trajectories. In addition, the disaggregation feasibility of any trajectories within this approximate region can be guaranteed, which is restated formally as the following proposition.
Proposition 1. Suppose that $\{P_{0,t}^\vee\}_t \in T$ and $\{P_{0,t}^\land\}_t \in T$ are the lower and upper aggregated power trajectories that respect the constraints (7) with there exists a disaggregation solution $(x_i^\land)_{t \in T}$ satisfying the operational constraints (7) with $P_{0,t}^0 = 1/3 \cdot x_i^\land + 1/3 \cdot d^0$. The proof of Proposition 1 is provided in Appendix B.

Remark: Proposition 1 indicates a generic property for all lower and upper aggregated power trajectories respecting the constraints (11)-(12), which is independent of the objective function. Therefore, this property also works on the optimal lower and upper aggregated power trajectories $(P_{0,t}^\land, P_{0,t}^\vee)_{t \in T}$ that solve the MPA model (10)-(12).

C. Economic Power Aggregation Model

Providing reserve services is one of the major schemes to offer flexibility support to the transmission grid. In the electricity market, the reward of flexibility reserve is based on the available power capacity instead of the actual regulated power. Accordingly, we define three operational trajectories (upper, lower and base) for power aggregation in the distribution system. The base trajectory is associated with the economic dispatch of the DER facilities, while the upper and lower trajectories are utilized to specify the reserves of upward and downward flexibility respectively.

The following economic power aggregation (EPA) model (13) is established to optimally schedule the power dispatch and flexibility reserve:

$$\text{Obj.} \quad \min_{x_i^\land, x_i^\vee, x_i^\ast} \quad \sum_{t=1}^{T} \left[ C_t \left( x_i^\land \right) - R_t \left( P_{0,t}^\land, P_{0,t}^\vee, P_{0,t}^\ast \right) \right] \quad (13a)$$

s.t. Equation (11) (12b) (12c)

$$\sum_{\tau = 1}^{t} H_i(t) x_i^\land(t) \leq h_i, \quad g_i(t) x_i^\ast(t) \leq 0 \quad \forall t \in T \quad (13c)$$

$$P_{0,t}^\land \leq P_{0,t}^\ast \leq P_{0,t}^\vee \quad \forall t \in T \quad (13d)$$

In objective function (13a), $C_t(\cdot)$ is the cost function associated with the base trajectory, and $R_t(\cdot)$ is the reward function for the flexibility reserve. The detailed formulation of function $C_t$ is

$$C_t := \sum_{i \in N_{pv}} c_i^p \cdot (P_{0,t}^\land)^2 + \sum_{i \in N_{hvac}} c_i^h \cdot (T_{i,t}^{min} - T_{i,t}^{ef})^2 + \sum_{i \in N_{pv}} \left[ c_{i,1}^p \cdot P_{0,t}^\land + c_{i,2}^p \cdot (P_{0,t}^\land)^2 \right] + p_t \cdot P_{0,t}$$

where the first term captures the damaging effect of charging/discharging to the ES systems. The second term describes the HVAC disutility of deviating from the most comfortable temperature $T_{i,t}^{ef}$. The third term denotes the operational cost of PV units. And $c_i^p, c_i^h, c_{i,1}^p, c_{i,2}^p$ are the corresponding cost coefficients. The last term is the cost of purchasing electricity from the transmission grid with the real-time price $p_t$. On the other hand, the reward function $R_t$ can be expressed as

$$R_t := r_t^\land \cdot (P_{0,t}^\land - P_{0,t}) + r_t^\vee \cdot (P_{0,t}^\vee - P_{0,t}) \quad (15)$$

where $r_t^\land$ and $r_t^\vee$ are the reward coefficients for upward and downward flexibility reserve at time $t$ respectively.

Comparing with the MPA model, the EPA model aims to minimize the net operation cost of the distribution system, and we supplement constraints (13c) and (13d) to restrict the base operational trajectory in a similar way. Equation (12b) (12c) are duplicated from the MPA model, which implies that the base ES and HVAC power trajectories are not required to lie between the upper and lower ones. By solving the EPA model (13), distribution system operators can obtain the economically optimal power dispatch schemes $(x_i^\ast)_{t \in T}$ and flexibility reserve intervals $[P_{0,t}^\land, P_{0,t}^\ast]_{t \in T}$ simultaneously. Then the distribution system provides reserve services for the transmission grid by reporting its flexibility intervals. According to Proposition 1, the distribution system is capable of tracking any transmission power regulation signals within the flexibility intervals.

IV. DISTRIBUTED MPC FRAMEWORK FOR TRANSMISSION-DISTRIBUTION INTERACTION

This section presents the transmission-distribution interaction framework with the proposed power aggregation method. To protect private information of participating DER facilities and enable scalable application, we develop a distributed MPC solver for the practical implementation of this framework.

A. Transmission-Distribution MPC Interaction Framework

In essence, power aggregation is the prediction of future flexibility trajectories of distribution systems. Since the operations of energy storage devices and HVAC facilities are highly time-coupled, the predicted flexibility trajectories may be significantly changed by executing the upcoming regulation commands from the transmission, so that a new round of power aggregation is required to update the prediction after each transmission-distribution interaction. Besides, it is arduous to accurately forecast the renewable generation and uncontrollable power demand for a long term. Hence, we implement the transmission-distribution interaction in the MPC manner [19]. In particular, the distribution systems perform power aggregation for a fairly long time horizon, while only receive and execute the regulation commands for the current few time slots. Then the time horizon is shifted one step (the executed time length) forward and the distribution repeats this process with updated system conditions. In this way, future time slots are taken into consideration when making the current decisions, and latest updated information can be utilized in each interaction.

Based on the work in [12], the framework for transmission-distribution interaction is proposed as the following procedure:

1) The transmission broadcasts the time-varying electricity price $p_t$ and reward coefficients $r_t^\land, r_t^\vee$ for the next $T_p$ time steps to each connected distribution system.

2) Based on the broadcast information, each distribution system performs the EPA model (13) w.r.t. the next $T_p$ time steps, and report its solved flexibility interval $[P_{0,t}^\land, P_{0,t}^\ast]$ to the transmission.
3) After gathering the flexibility intervals and the generators’ information, the transmission determines the optimal power dispatch schemes and the regulation commands \( P_{0,t}^{reg} \in [P_{0,t}^{\min}, P_{0,t}^{\max}] \) in the next \( T_d \) time steps (\( T_d \) is typically much smaller than \( T_p \)) for each distribution system.

4) Once receiving the regulation command, each distribution system solves the following power disaggregation (PD) problem (16), and executes the decisions to optimally track the regulation command and dispatch its DER facilities for the next \( T_d \) time steps.

\[
\begin{align*}
\text{Obj.:} & \quad \min_{x_t} \sum_{t=1}^{T_d} \left[ C_t(x_t) + \rho_t \cdot (P_{0,t} - P_{0,t}^{reg})^2 \right] \\
\text{s.t.:} & \quad \text{Equation (7)}
\end{align*}
\]

where \( \rho_t \) is the penalty coefficient for deviating from the regulation command.

5) Move \( T_d \) time steps forward and repeat from step 1) with the updated DER conditions.

The above framework is illustrated in Figure 4. In this way, the flexibility from massive DER devices in the distribution side can be exploited by the transmission system.

### B. Distributed Solution Algorithm

To perform the above framework, centralized solvers necessitate the efforts to gather the real-time operational information of all DER facilities and solve a global optimization problem. It not only carries huge computational and communication burdens, but also may violate the privacy of each participated DER stakeholder. To address these issues, we develop a distributed solver for the MPA, EPA and PD models based on the PCPM algorithm [14].

Let \( y_0 \) and \( \{y_k\}_{k \in K} \) denote the local variables of the system aggregator (SA) and \( k \)-th DER facility for the considered period \( T_e \), which are defined as follows.

\[
\begin{align*}
y_0 &= \left\{ y_0^{\text{voltage}}, y_0^{\text{power flow response}} \right\} \in \mathcal{Y}_0 \\
y_k &= \left\{ y_k^{\text{voltage}} \right\} \in \mathcal{Y}_k \quad \forall k \in K
\end{align*}
\]

where \( K := \{1, 2, \ldots, K\} \) is the index set of DER facilities. \( \mathcal{Y}_0 \) and \( \mathcal{Y}_k \) are the feasible sets for the corresponding local variables, which are specified by their own operational constraints.

Leveraging their separable structure, the MPA, EPA and PD model can be rewritten as the general compact form (18).

\[
\begin{align*}
\text{Obj.:} & \quad \min_{y_0 \in \mathcal{Y}_0, y_k \in \mathcal{Y}_k} f_0(y_0) + \sum_{k=1}^{K} f_k(y_k) \\
\text{s.t.:} & \quad y_0 = \sum_{k=1}^{K} W_k y_k + w 
\end{align*}
\]

where \( f_0 \) and \( f_k \) captures the corresponding objective functions for the SA and \( k \)-th DER facility. Constraint (18b) stacks the power flow equations (1), where matrix \( W_k \) and vector \( w \) are network parameters.

![Fig. 5. Schematic of the distributed solution algorithm.](image)

Introducing the dual variable \( \mu \) and virtual dual variable \( \nu \), we develop the distributed solution algorithm as Algorithm 1. The implementation procedure is illustrated as Figure 5. In each iteration, it requires two-way communications between the SA and each DER facility, and they solve their individual optimization problems in parallel.

**Algorithm 1**: Distributed Solution Algorithm

1. Initialization: \( l \leftarrow 0 \). Each DER facility \( k \in K \) sets its initial \( y_0^0 \) and sends to the system aggregator (SA). The SA sets the initial \( y_0^0, \mu^0 \), step length \( \rho \) and tolerance \( \epsilon \).

2. Update Virtual Dual Variables: The SA updates the virtual dual variables by

\[
\nu^{l+1} = \mu^l + \rho \left( \sum_{k=1}^{K} W_k y_k^l + w - y_0^l \right)
\]

then broadcast \( \nu^{l+1} \) to every DER.

3. Parallel Optimization:

For SA: solve the power aggregation problem

\[
y_0^{l+1} = \arg\min_{y_0 \in \mathcal{Y}_0} f_0(y_0) - \nu^{l+1} \top y_0 + \frac{1}{2\rho} ||y_0 - y_0^l||^2
\]

For DER \( k \): solve its DER operation problem

\[
y_k^{l+1} = \arg\min_{y_k \in \mathcal{Y}_k} f_k(y_k) + \nu^{l+1} \top W_k y_k + \frac{1}{2\rho} ||y_k - y_k^l||^2
\]

and send \( y_k^{l+1} \) to SA.

4. Update Dual Variables: The SA updates the dual variables by

\[
\mu^{l+1} = \mu^l + \rho \left( \sum_{k=1}^{K} W_k y_k^{l+1} + w - y_0^{l+1} \right)
\]

5. Check Convergence: if \( ||\mu^{l+1} - \mu^l|| \leq \epsilon \), terminate. Otherwise \( l \leftarrow l + 1 \) and go back to step 2.

**V. NUMERICAL TESTS**

**A. Simulation Setup**

Numerical tests are carried out on a real distribution feeder located within the territory of Southern California Edison.
This feeder contains 126 multi-phase buses with a total of 366 single-phase connections. The nominal voltage at the substation is 12kV (1 p.u.), and we set the upper and lower limits of voltage magnitude as 1.02 p.u. and 0.98 p.u. Dispatchable DERs include 33 PV units, 28 energy storage devices and 5 HVAC systems. The real data of power consumption from industrial, commercial, and residential loads are applied, as well as real solar irradiance profiles. The total amounts of uncontrollable loads and PV available power from 9:00 to 18:00 are presented as Figure 6. The simulation time is discretized with the granularity of 20 minutes. We set the initial SOC of the energy storage devices to 50%.

All the programming and numerical tests are implemented in Matlab 2017b. While we use the CVX package [21], [22] to model and solve the convex programs.

**B. Implementation of Maximal-flexibility Power Aggregation**

We implemented the MPA model (10)-(12) to evaluate the maximal flexibility of the test system. The optimal upper and lower trajectories of the net power injection at the substation are presented as Figure 7. The aggregate flexibility, i.e., the area between the upper and lower trajectories, is calculated as $E_{af} = 37.1 \text{ MW-h}$. The corresponding DER power dispatch schemes are shown as Figure 8. It is observed that the upper trajectory corresponds to less power injection and larger amount of loads comparing with the lower one, which is consistent with the intuition.

**C. Implementation of Transmission-Distribution Interaction**

We further carry out the transmission-distribution interaction framework for the test system, which is implemented with the proposed MPC setting. At each time instant, the EPA model is performed for the next 12 time steps (4 hours), while only the flexibility intervals of the first time step are reported, i.e., $T_p = 12$ and $T_d = 1$, before the procedure moves one time step forward. The regulation commands $P_{reg0, t}$ sent by the transmission are generated randomly and independently across time $t$ around the optimal operating points $P_{0, t}^\star$, with Gaussian distribution, and are projected onto $[P_{0, t}^{\land, \star}, P_{0, t}^{\lor, \star}]$. We set the penalty coefficient $\rho_t$ as a very large value for simplification, so that the test feeder system strictly tracks the regulation commands.

Figure 9 illustrates the upper, implemented, lower trajectories of the net power injection at the substation. The dashed red and blue curves represent the reported flexibility intervals $[P_{0, t}^{\lor, \star}, P_{0, t}^{\land, \star}]$, and the solid green curve denotes the eventually implemented power trajectory, i.e., the regulation commands $P_{0, t}^{\land, \star}$. These trajectories are accumulated step by step in the receding horizon. Figure 10 presents the DER power dispatch schemes associated with the three trajectories. It is observed that the implemented DER trajectories do not always lie between their upper and lower trajectories, while it is still ensured that the implemented aggregate power is within the calculated flexibility intervals.

**D. Convergence of Distributed Solver**

In the above simulations, the distributed solver developed in Section IV-B is used to solve the MPA, EPA and PD models. Taking the MPA model as an example, the convergence of the distributed solver is shown in Figure 11. It is observed that...
This paper develops a method to model and quantify the aggregate flexibility for different types of DERs in unbalanced distribution systems. With the inner-box approximation approach, we use the upper and lower aggregated power trajectories to specify the feasible region of the net power injection at the substation. Two convex optimization models are established to implement distribution-level power aggregation, which incorporate the network constraints and multi-phase unbalanced modeling. In addition, a distributed MPC framework is proposed to implement the transmission-distribution interaction in a scalable and privacy-preserving manner. The effectiveness of our proposed method are validated via the numerical tests on a real distribution feeder.

**APPENDIX A**

**LINEAR MULTI-PHASE POWER FLOW MODEL**

Define $x_Y := [p_Y^T, q_Y^T]^T$ and $x_\Delta := [p_\Delta^T, q_\Delta^T]^T$. Based on a given operation point $\{v, x_Y, x_\Delta\}$, the three-phase nodal complex voltage $v$ can be derive as the following linear formulation with the fixed-point linearization method \[16\].

$$v = M_Y x_Y + M_\Delta x_\Delta + m$$ (23)

where $M_Y$, $M_\Delta$ and $m$ are defined as

$$M_Y := [Y_{LL}^{-1}\text{diag}(\hat{v}^*)^{-1}, -j \cdot Y_{LL}^{-1}\text{diag}(\hat{v}^*)^{-1}]$$

$$M_\Delta := [Y_{LL}^{-1}L^T\text{diag}(L\hat{v}^*)^{-1}, -j \cdot Y_{LL}^{-1}L^T\text{diag}(L\hat{v}^*)^{-1}]$$

$$m := -Y_{LL}^{-1}Y_{L0}v_0$$

Here, $v_0 \in \mathbb{C}^3$ denotes the three-phase complex voltage at the substation bus. $Y_{LL} \in \mathbb{C}^{3N \times 3N}$ is the sub-matrix of the three-phase admittance matrix

$$Y := \begin{bmatrix} Y_{00} & Y_{0L} \\ Y_{L0} & Y_{LL} \end{bmatrix} \in \mathbb{C}^{3(N+1) \times 3(N+1)}$$

$L$ is a block-diagonal matrix defined by

$$L := \begin{bmatrix} T & \cdot & \cdot \\ \cdot & T & \cdot \\ \cdot & \cdot & T \end{bmatrix}, \quad T := \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

The three-phase nodal voltage magnitude $|v|$ can be further derived as

$$|v| = Ax + a$$ (24)

where $A := [A_Y, A_\Delta]$ and $x := [x_Y^T, x_\Delta^T]^T$ with

$$A_Y := \text{diag}(\hat{v})^{-1}R \{\text{diag}(\hat{v}^*)M_Y\}$$

$$A_\Delta := \text{diag}(\hat{v})^{-1}R \{\text{diag}(\hat{v}^*)M_\Delta\}$$

$$a := |\hat{v}| - A_Y \hat{x}_Y - A_\Delta \hat{x}_\Delta$$

According to Kirchhoff’s laws, we can also derive matrices $B, D$ and vectors $b, d$. See \[17\] for a detailed description.

**APPENDIX B**

**PROOF OF PROPOSITION 1**

Suppose that $\{P_{0,t}^\geq\}_{t \in \mathcal{T}}$ and $\{P_{0,t}^\leq\}_{t \in \mathcal{T}}$ are the lower and upper aggregated power trajectories that respect the constraints \[11\] \[12\], which are associated with the operational trajectories $\{x^\geq_t\}_{t \in \mathcal{T}}$ and $\{x^\leq_t\}_{t \in \mathcal{T}}$ respectively. For any given aggregated
power trajectory \( \{P_{0,t}^\alpha\}_{t \in \mathcal{T}} \) with \( P_{0,t}^\alpha \leq P_{0,t}^\nu \leq P_{0,t}^\wedge \) for all \( t \in \mathcal{T} \), we define auxiliary coefficients \( \lambda_t \in [0, 1] \) as
\[
\lambda_t = \frac{P_{0,t}^\nu - P_{0,t}^\alpha}{P_{0,t}^\nu - P_{0,t}^\wedge}
\]
and hence
\[
P_{0,t}^\alpha = \lambda_t P_{0,t}^\nu + (1 - \lambda_t) P_{0,t}^\wedge.
\]
Then we claim that \( \{x_t^\nu\}_{t \in \mathcal{T}} \) defined as
\[
x_t^\nu = \lambda_t x_t^\wedge + (1 - \lambda_t) x_t^\alpha
\]
is a feasible disaggregation solution for \( \{P_{0,t}^\nu\}_{t \in \mathcal{T}} \), which satisfies the operational constraints (14) with \( x_{0,t}^\nu = 1 \frac{3}{2} D \cdot x_t^\nu + 1 \frac{3}{2} d \).

Firstly, we show that
\[
P_{0,t}^\alpha = (1 - \lambda_t) \left( (1 \frac{3}{2} D \cdot x_t^\wedge + 1 \frac{3}{2} d) \right) + \lambda_t \left( (1 \frac{3}{2} D \cdot x_t^\alpha + 1 \frac{3}{2} d) \right) = 1 \frac{3}{2} D \cdot x_t^\nu + 1 \frac{3}{2} d.
\]

Then, due to the convexity of \( g_t(\cdot) \), we prove that
\[
g_t(x_t^\nu) = g_t(\lambda_t x_t^\wedge + (1 - \lambda_t) x_t^\alpha) \leq \lambda_t g_t(x_t^\wedge) + (1 - \lambda_t) g_t(x_t^\alpha) \leq 0
\]
Next, we verify that \( \{x_t^\nu\}_{t \in \mathcal{T}} \) also satisfies the time-coupled constraints (26), which involve the SOC limits (24)-(25) and the HAVC comfort limits (28). For the SOC constraints (28a)-(28b), we can reformulate them equivalently as
\[
E_{i,0}^\nu - \overline{E}_i \leq \sum_{t=1}^{t} P_{i,t}^\nu E_{i,0}^\nu - \overline{E}_i \forall t \in \mathcal{T}
\]
where \( E_{i,0}^\nu \) is the initial SOC at time 0. By constraint (12b), we obtain
\[
\sum_{t=1}^{t} P_{i,t}^\nu = \sum_{t=1}^{t} \left[ \lambda_t P_{i,t}^\nu + (1 - \lambda_t) P_{i,t}^\wedge \right]
\]
\[
= \sum_{t=1}^{t} \left[ P_{i,t}^\wedge + \lambda_t (P_{i,t}^\nu - P_{i,t}^\wedge) \right] \geq \sum_{t=1}^{t} P_{i,t}^\wedge \geq \frac{E_{i,0}^\nu - \overline{E}_i}{\delta}
\]
\[
\sum_{t=1}^{t} P_{i,t}^\wedge = \sum_{t=1}^{t} \left[ P_{i,t}^\wedge - (1 - \lambda_t) (P_{i,t}^\nu - P_{i,t}^\wedge) \right] \leq \sum_{t=1}^{t} P_{i,t}^\nu \leq \frac{E_{i,0}^\nu - \overline{E}_i}{\delta}
\]
Hence, \( \{x_t^\nu\}_{t \in \mathcal{T}} \) satisfies the SOC limit constraints (24)-(25). Using a similar argument, we can prove that \( \{x_t^\nu\}_{t \in \mathcal{T}} \) also satisfies the HAVC comfort constraints (28a)-(28b).

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