COLLISION-FREE FLOCKING FOR A TIME-DELAY SYSTEM

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Abstract. The co-existence of collision avoidance and time-asymptotic flocking of multi-particle systems with measurement delay is considered. Based on Lyapunov stability theory and some auxiliary differential inequalities, a delay-related sufficient condition is established for this system to admit a time-asymptotic flocking and collision avoidance. The estimated range of the delay is given, which may affect the flocking performance of the system. An analytical expression was proposed to quantitatively analyze the upper bound of this delay. Under the flocking conditions, the exponential decay of the relative velocity of any two particles in the system is characterized. Particularly, the collision-free flocking conditions are also given for the case without delay. This work verifies that both collision avoidance and flocking behaviors can be achieved simultaneously in a delay system.

1. Introduction. “Flocking” refers to a type of self-organizing behavior that is prevalent in biological populations, such as the ballet in the air of European starlings, the collective predation of prairie wolves, the gathering and migration of marine fish schools and so on. To reveal the cooperative mechanism of swarm intelligence, biologists take a flock of birds as an example, by observing their activity and analyzing the real-time data, a variety of different movement models have been constructed. Among them, Cucker and Smale first proposed a second-order dynamic mathematical model with Newton-type interaction function, the Cucker-Smale model, to describe this aggregation behavior in their pioneering creations [8, 9]. The authors’ extremely creative and crucial idea is that each particle updates its state based on the interaction of the weights of its relative distance. The Cucker-Smale model is described as follows

\[
\begin{align*}
\dot{x}_i(t) &= v_i(t) \\
\dot{v}_i(t) &= \sum_{j=1}^{N} \psi(r_{ij})(v_j(t) - v_i(t)), \quad i = 1, 2, \ldots, N,
\end{align*}
\]

where \(x_i(t), v_i(t) \in \mathbb{R}^d\) represent the position and velocity of the particle \(i\) at time \(t\); \(d\) is a positive integer representing the dimension of the space; \(r_{ij} = \|x_i - x_j\|\), \(i, j \in \mathcal{N} := \{1, 2, \ldots, N\}\); \(\|\cdot\|\) represents the 2-norm of \(\mathbb{R}^d\); \(\psi(r_{ij})\) depicts the interaction strength between particle \(j\) and particle \(i\); In [8, 9], \(\psi(r) = K(\sigma^2 + r^2)^{-\beta}\) with \(K > 0, \sigma > 0, \beta \geq 0\) are constant. The Cucker-Smale model depicts a ubiquitous

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mechanism of biological population behavior evolution and has aroused the interest of many scholars in various fields. With the extensive application of cluster technology, higher requirements have been placed on the flexibility, coordination, and robustness of cluster intelligent systems. A large number of practical problems need to be solved urgently. The most typical of these are the impact of conflicts within the group and time delay on the evolution of system behavior.

Given the above mentioned application requirements and the theoretical framework proposed by Cucker and Smale in [8, 9], some scholars have separately studied the internal collision avoidance and delay effect of multi-particle aggregation, and accumulated abundant research results. For the collision avoidance problem, the theoretical methods and conclusions of the research are relatively mature, see [11, 16, 7, 1, 2] and the references therein. Considering the time delay, much of the work that has been done so far has focused on transmission delay and processing delay, see [15, 4, 10, 17, 5, 19, 3, 6]. Most of these works are performed with a sufficiently small delay, and the asymptotic flocking condition is independent of the delay. Beyond that, little research has been done on achieving conflict-free in time-delay systems. Therefore, the development of this work is of great theoretical value.

Regarding the internal conflict avoidance of a delay system and inspired by [3], both measurement delay and conflict-free are considered in (1). Based on a wealth of available research results, a delay model with non-singular influence function and external forces is constructed, as shown below

\[
\begin{align*}
\dot{x}_i(t) &= v_i(t), \quad i \in \mathcal{N}, \\
\dot{v}_i(t) &= \alpha \sum_{j=1}^{N} I(r_{ij})(v_j(t) - v_i(t)) + \sum_{j=1, j \neq i}^{N} f(r_{ij}) (x_i(t) - x_j(t)),
\end{align*}
\]  

subject to initial data

\[
\begin{align*}
x_i(\theta) &= \varphi_i(\theta), \quad v_i(\theta) = \phi_i(\theta), \quad \theta \in [-\tau, 0],
\end{align*}
\]  

where \((\varphi_i, \phi_i) \in \mathcal{C}^2 := \mathcal{C} \times \mathcal{C}(i \in \mathcal{N})\) and \(\mathcal{C} := \mathcal{C}([-\tau, 0], \mathbb{R}^d)\) is the Banach space of all continuous functions mapping the interval \([-\tau, 0]\). In (2), \(\alpha > 0\) indicates the adjustable coupling strength; the measurement delay \(\tau > 0\); \(I(\cdot) : [0, +\infty) \to (0, +\infty)\) is a suitable communication rate with the symmetric case (e.g., \(I(r) = (1 + r^2)^{-\beta}\) with \(\beta \geq 0\)). In addition, \(I(\cdot)\) is further assumed to satisfy the following conditions.

**Assumption 1.** \(I(\cdot)\) is non-negative, non-increasing, Lipschitz continuous and bounded on \([0, +\infty)\). Without loss of generality (if necessary, re-parameterize the time),

\[
0 < I(r_{ij}) \leq 1, \quad \text{for all } i, j \in \mathcal{N}.
\]  

Inspired by the general collision-free framework in [7], the repulsion term \(f : (\delta, +\infty) \to [0, +\infty)\) is non-increasing function, local Lipschitz, and satisfies

\[
\int_{d}^{+\infty} f(r)dr < +\infty, \quad d \geq \delta \geq 0.
\]

We still follow the definition of time-asymptotic flocking proposed in [12].

**Definition 1.1.** Let \((x_i(t), v_i(t))\) for \(i \in \mathcal{N}\) be the solution to (2)-(3), a time-asymptotic flocking can be achieved if and only if the system satisfies the following:

\[
\sup_{t>0} \|x_i(t) - x_j(t)\| < +\infty \quad \text{and} \quad \lim_{t \to +\infty} \|v_i(t) - v_j(t)\| = 0, \quad \text{for all } i, j \in \mathcal{N}.
\]
The rest of this paper is organized as follows. We provide the global solution existence proof of System (2)-(3) in Section 2. The relative velocity exponential decay conditions are stated in Section 3. Section 4 is devoted to exploring appropriate collision avoidance and time-asymptotic flocking conditions for a delay-free system. Some numerical simulations are given in Section 5 to clarify the availability of theoretical results.

2. Existence of global solutions. A rigorous proof of the existence of the global solution of System (2)-(3) is given below. The method is mainly based on the extension theorem of solution and some auxiliary differential inequalities.

Lemma 2.1. [18] Let $\Omega$ be a open subset of $\mathbb{R} \times C([-\tau,0], \mathbb{R}^d)$, $F(t,x_i) \in C(\Omega, \mathbb{R}^d)$. If $x(t)$ is a solution to the functional differential equation

$$
\begin{align*}
\dot{x}(t) &= F(t,x_i), \\
\dot{x}_t &= \varphi, \ t \in [-\tau,0],
\end{align*}
$$

with a maximal interval of existence $[-\tau,T)$, $T < +\infty$. Then, for any compact set $V \subseteq \Omega$ with $(t,\varphi) \in V$, $t \in [-\tau,0]$, there is a $t_V$ such that $(t,x_i) \notin V$ for all $t \in [t_V,T)$.

Lemma 2.2. [14] Suppose $a \in \mathbb{R}$ and $b \in \mathbb{R}$ are non-negative, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$ab \leq \varepsilon a^p + C_r b^q,$$

where $\varepsilon > 0$ is sufficiently small and $C_r > 0$ is sufficiently large.

To prove the existence of global solutions, as in [10], a assumption is required on the matrix of communication rates. Let $A = (a_{ij})_{N \times N}$ be a nonnegative, symmetric matrix, its Laplacian matrix is $L$, its smallest positive eigenvalue, also called Fiedler number and denoted by $\lambda > 0$. The assumption is the following: there exists $\lambda_0 > 0$ such that $\lambda \geq \lambda_0$. This assumption is guaranteed for instance if the communication rates are uniformly bounded away from zero, i.e., if there exists $I_\ast > 0$ such that $I(\cdot) \geq I_\ast > 0$, see Corollary 1 in [9]. For another, we define the following macro variables,

$$
\Gamma(x(t))^2 := \sum_{i=1}^{N} \sum_{j=1}^{N} \|x_i(t) - x_j(t)\|^2, \ \Lambda(v(t))^2 := \sum_{i=1}^{N} \sum_{j=1}^{N} \|v_i(t) - v_j(t)\|^2.
$$

Therefore, to prove that the solution of System (2)-(3) satisfies (6), it just need to testify that

$$
\sup_{t \geq 0} \Gamma(x(t)) < +\infty, \text{ and } \lim_{t \to +\infty} \Lambda(v(t)) = 0.
$$

The following lemma shows a differential inequality satisfied by $\Gamma(x(t))$ and $\Lambda(v(t))$, which assists in proving the existence of global solutions.

Lemma 2.3. Let $(x_i(t),v_i(t))$ for $i \in N$ be a solution to (2)-(3) on $(-\tau,T)$ for some $T \in (0, +\infty)$, then $\Gamma(x(t))$ and $\Lambda(v(t))$ on $(-\tau,T)$ satisfy

$$
\left| \frac{d}{dt} \Gamma(x(t)) \right| \leq N \Lambda(v(t)), \ a.e. \ t \in (-\tau,T),
$$

where $\Gamma(x(t))$ and $\Lambda(v(t))$ are shown as (8).
Proof. For one thing,

\begin{equation}
\left| \frac{d\Gamma(x(t))}{dt} \right|^2 = 2 \left| \sum_{i=1}^{N} \sum_{j=1}^{N} (x_i(t) - x_j(t))^T (v_i(t) - v_j(t)) \right| \\
\leq 2 \sum_{i=1}^{N} \sum_{j=1}^{N} \|x_i(t) - x_j(t)\| \cdot \|v_i(t) - v_j(t)\| \\
\leq 2N\Gamma(x(t)) \cdot \Lambda(v(t)), \text{ a.e. } t \in (-\tau, T),
\end{equation}

where we used the following fact,

\begin{equation}
\left( \sum_{i=1}^{N} \sum_{j=1}^{N} \|x_i(t) - x_j(t)\| \cdot \|v_i(t) - v_j(t)\| \right)^2 \\
\leq N^2 \sum_{i=1}^{N} \sum_{j=1}^{N} \|x_i(t) - x_j(t)\|^2 \cdot \|v_i(t) - v_j(t)\|^2 \\
\leq N^2 \sum_{i=1}^{N} \sum_{j=1}^{N} \|x_i(t) - x_j(t)\|^2 \cdot \sum_{i=1}^{N} \sum_{j=1}^{N} \|v_i(t) - v_j(t)\|^2.
\end{equation}

For another,

\begin{equation}
\left| \frac{d\Gamma(x(t))}{dt} \right|^2 = 2\Gamma(x(t)) \left| \frac{d\Gamma(x(t))}{dt} \right|, \text{ a.e. } t \in (-\tau, T).
\end{equation}

Combining (11) and (12) yields (10). This completes the proof of Lemma 2.3. \qed

Below we will present the main results of this section. Prior to this, a Lyapunov function is first proposed to complete the existence of the global solution of the system (2)-(3),

\begin{equation}
Q(t) = \sum_{i=1}^{N} v_i(t)x_i(t) + \xi \sum_{i=1}^{N} \int_{-\tau}^{0} v_i(t+s)x_i(t+s)ds \\
+ \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \int_{||x_i - x_j||^2}^{+\infty} f(r)dr,
\end{equation}

where \(\xi > 0\) is a constant.

**Theorem 2.4.** Assume that \(I(\cdot)\) satisfies Assumption 1 and let \((x_i(t), v_i(t)), i \in \mathcal{N}\) be a local solution to the system (2)-(3). Suppose that the initial configurations (3) satisfy \(||\varphi_i(\theta) - \varphi_j(\theta)||^2 > \delta \geq 0\) for \(i \neq j \in \mathcal{N}, \theta \in [-\tau, 0]\), moreover,

\begin{equation}
Q(0) < \min \left\{ \left( \int_{\Gamma(x(0))}^{+\infty} I(s)ds \right)^2, \frac{1}{2} \int_{\delta}^{+\infty} f(r)dr \right\}.
\end{equation}

Then, the solution is global in time. In addition, there exists \(d_0 > \delta \geq 0\) and \(0 < B_0 < +\infty\) such that \(0 < d_0 \leq \|x_i(t) - x_j(t)\| \leq \Gamma(x(t)) \leq B_0 < +\infty\) for all \(t \geq 0\) and \(i \neq j \in \mathcal{N}\), that is, System (2)-(3) will implement collision avoidance.

Proof. Let \((x_i(t), v_i(t))\) for \(i \in \mathcal{N}\) be the solution to (2)-(3) and denote \([-\tau, T)\) \((T \in (0, +\infty))\) be the maximal existence interval. This proof will be divided into the following three steps. First, we will show that System (2)-(3) will not collide...
in any finite interval \([-\tau, T]\). And then, we prove that \(T = +\infty\). Finally, the boundedness of the relative positions of any two particles is proved.

**Step One.** Prove that System (2)-(3) will not collide in a finite time, that is, the system can achieve collision avoidance within \([-\tau, T]\).

According to our assumptions on the initial configuration, i.e., \(\|\varphi_i(t) - \varphi_j(t)\|^2 > \delta \geq 0\) for \(i \neq j \in \mathcal{N}\), we only need to investigate that there is no collision on \([0, T]\). It follows the symmetry of indicators \(i\) and \(j\) that

\[
\sum_{i=1}^{N} \sum_{j=1,j\neq i}^{N} f(r_{ij}^2)(x_i - x_j)^T(v_i - v_j) = 2 \sum_{i=1}^{N} \sum_{j=1,j\neq i}^{N} f(r_{ij}^2)v_i(t)^T(x_i - x_j).
\]

The derivative of \(Q(t)\) along the solutions of (2) is figured out as

\[
\frac{dQ}{dt}|_2 = \frac{2\alpha}{N} \sum_{i,j=1}^{N} I(r_{ij})v_i(t)^Tv_j(t - \tau) - \frac{2\alpha}{N} \sum_{i,j=1}^{N} I(r_{ij})v_i(t)^Tv_i(t) + \xi \sum_{i=1}^{N} v_i(t)^Tv_i(t) - \xi \sum_{i=1}^{N} v_i(t - \tau)^Tv_i(t - \tau).
\]

From Lemma 2.2, there exist \(\varepsilon > 0\) and \(C_\varepsilon > 0\), satisfying \(C_\varepsilon > \varepsilon\), such that (15) is further rewritten to get

\[
\frac{dQ}{dt}|_2 \leq \frac{\alpha}{N} \varepsilon \sum_{i,j=1}^{N} v_i(t)^Tv_i(t) + \frac{\alpha}{N} C_\varepsilon \sum_{i,j=1}^{N} v_j(t - \tau)^Tv_j(t - \tau) + \xi \sum_{i=1}^{N} v_i(t - \tau)^Tv_i(t - \tau) - \frac{2\alpha}{N} \sum_{i,j=1}^{N} I(r_{ij})v_i(t)^Tv_i(t).
\]

 Appropriately select \(\xi\), which satisfies \(\frac{\alpha}{N} C_\varepsilon < \xi < \frac{\alpha}{N} (2 - \varepsilon)\), and then it follows from (16) that

\[
\frac{dQ}{dt}|_2 \leq -\frac{2\alpha}{N} \sum_{i,j=1}^{N} (I(r_{ij}) - \rho) v_i(t)^Tv_i(t),
\]

where \(\rho := \frac{\xi}{2\alpha} + \frac{\varepsilon}{2}\). Select appropriate adjustable parameters \(\varepsilon\) that satisfies \(\rho < I_\ast\), it shows from (17) that \(Q(t)\) is non-increasing and \(Q(t) \leq Q(0)\) for \(t \in [0, T]\).

Based on the above preparations, we now show that \(\|x_i(t) - x_j(t)\|^2 > \delta\) for \(t \in [0, T]\) and \(i \neq j \in \mathcal{N}\). Otherwise, there exists \(i_\ast \neq j_\ast \in \mathcal{N}\) and \(t_\ast \in (0, T]\) such that \(\|x_{i_\ast}(t_\ast) - x_{j_\ast}(t_\ast)\|^2 = \delta\) and \(\|x_i(t) - x_j(t)\|^2 > \delta\) for all \(i \neq j \in \mathcal{N}\) and \(t \in [0, t_\ast]\). It follows from \(Q(t_\ast) \leq Q(0)\) that

\[
\int_{\|x_{i_\ast}(t_\ast) - x_{j_\ast}(t_\ast)\|^2}^{+\infty} f(r)dr = \int_{\delta}^{+\infty} f(r)dr \leq 2Q(t_\ast) \leq 2Q(0),
\]

which contradicts with (14). Therefore, \(\|x_i(t) - x_j(t)\|^2 > \delta\) for \(t \in [0, T]\) and \(i \neq j \in \mathcal{N}\). Further, combining (14), (15) and the monotonicity of \(Q(t)\) produces

\[
\int_{\|x_i(t) - x_j(t)\|^2}^{+\infty} f(r)dr \leq 2Q(t) \leq 2Q(0) < \int_{\delta}^{+\infty} f(r)dr.
\]
Hence, there exists $d_0 > 0$ such that $0 < \delta \leq d_0^2 \leq \|x_i(t) - x_j(t)\|^2$ for all $t \in [0, T)$ and $i \neq j \in \mathcal{N}$, that is, collision avoidance within the System (2) on $[0, T)$ can be achieved.

**Step Two.** Prove that $T = +\infty$, that is, the solution $(x_i(t), v_i(t)) (i = 1, 2, \ldots, N)$ is global in time.

If not, then $T < +\infty$. According to (13) and (17), one can be obtained that $\|v_i(t)\|^2 \leq Q(t) \leq Q(0)$, thus $\|v_i(t)\| \leq \sqrt{Q(0)}$ for $t \in [0, T)$. Since $\dot{x}(t) = v(t)$ for $t \in [0, T)$, we have

$$\|x_i(t)\| \leq \|\varphi_i(0)\| + \int_0^t \|v_i(s)\| ds \leq \|\varphi_i(0)\| + 2T\sqrt{Q(0)} =: X_0.$$  

Therefore, $(x_i(t), v_i(t))$ for $t \in [0, T)$ and $i \in \mathcal{N}$ lies in the compact set $V$, which is defined as follows

$$V = \left\{(x_i, v_i) : \|x_i - x_j\|^2 \geq d_0, \text{ for } i \neq j, \text{ and } \|x_i\| \leq X_0, \|v_i\| \leq \sqrt{Q(0)} \right\}$$

$$\subseteq \left\{(x_i, v_i) \in \mathbb{R}^d \times \mathbb{R}^d : \|x_i - x_j\|^2 > \delta, i \neq j \right\} =: \Omega,$$

where $\Omega$ is open in $\mathbb{R}^d \times \mathbb{R}^d$. It is straightforward to show that the solution $(x_i(t), v_i(t)) \in V$ for $t \in [-\tau, T)$. This leads to a contradiction with Lemma 2.1. Hence, we get $T = +\infty$, i.e., the solutions of System (2)-(3) exist globally.

**Step Three.** Prove that the relative positions of any two particles in System (2) are bounded. Namely, there exists $0 < B_0 < +\infty$ such that $\|x_i(t) - x_j(t)\| \leq B_0$ for all $i, j \in \mathcal{N}$ and $t \in [0, +\infty)$.

If not, then there exists $T_* < +\infty$ such that $\Gamma(x(T_*)) = +\infty$. From **Step One**, it can easily be checked that for all $i, j \in \mathcal{N}$ and $t \in [0, T) \subseteq [0, T_*)$,

$$\|x_i(t) - x_j(t)\|^2 \leq \Gamma(x(t))^2, \quad \Lambda(v(t))^2 \leq 4N\sum_{i=1}^N \|v_i(t)\|^2 \leq 4NQ(0),$$

(22)

where we used the fact from (17) that $\sum_{i=1}^N v_i(t)^T v_i(t) \leq Q(0)$ for all $t \in [0, T)$ and $i \in \mathcal{N}$. $\Gamma(x(t))$ and $\Lambda(v(t))$ are shown in (8). Using (17), (22) and integrating on both sides of (17) from 0 to $t \in [0, T)$ produces

$$Q(t) - Q(0) \leq -\frac{2\alpha}{N} \sum_{i=1}^N \sum_{j=1}^N \int_0^t (I(r_{ij}) - \rho) v_i(s)^T v_i(s) ds$$

$$\leq -\frac{2\alpha}{N} \int_0^t (I(\Gamma(x)) - \rho) \sum_{i=1}^N v_i(s)^T v_i(s) ds$$

$$\leq -\frac{\alpha}{2N} \int_0^t (I(\Gamma(x)) - \rho) \Lambda(v(s))^2 ds$$

$$\leq -\frac{\alpha}{2N} \int_0^t I(\Gamma(x))\Lambda(v(s))^2 ds + 2\alpha\rho Q(0)T_*,$$

(23)

which implies that

$$0 \leq \frac{\alpha}{2N} \int_0^t I(\Gamma(x))\Lambda(v(s))^2 ds \leq (1 + 2\alpha\rho T_*)Q(0) \text{ for all } t \in (0, T).$$
Hence, it can be calculated very directly from the above inequality and $\Lambda(v(t)) \leq 2\sqrt{NQ(0)}$ for $0 < t < T$ that

$$\frac{\alpha}{N} \int_0^t I(\Gamma(x)) \|\Lambda(s)\|ds \leq (1 + 2\alpha\rho T_*) \sqrt{\frac{Q(0)}{N}} \quad \text{for } t \in (0, T). \tag{24}$$

According to Lemma 2.3, we have $\frac{d}{dt}\Gamma(x(t)) \leq \alpha\Lambda(v)$ and (24) is further reduced to

$$\alpha \int_{\Gamma(x(0))}^{\Gamma(x(t))} I(u)du \leq (1 + 2\alpha\rho T_*) \sqrt{\frac{Q(0)}{N}}. \tag{25}$$

(1) If $T_* \leq \frac{\alpha\sqrt{N} - 1}{2\alpha\rho}$, then let $t \to T_*$ in (25) and combining the boundedness of $T_*$, which produces a contradiction with (14).

(2) If $\frac{\alpha\sqrt{N} - 1}{2\alpha\rho} < T_* < +\infty$, then there exists a positive integer $m > 1$ such that $T_* = m\frac{\alpha\sqrt{N} - 1}{2\alpha\rho} + m_0$, where $0 < m_0 < \frac{\alpha\sqrt{N} - 1}{2\alpha\rho}$. Let $t \to T_*$ in (25), we get

$$\alpha \int_{\Gamma(x(0))}^{T_+} I(u)du \leq (1 + 2\alpha\rho T_*) \sqrt{\frac{Q(0)}{N}}$$

$$\leq (2\alpha\rho m_0 + 1 + m(\alpha\sqrt{N} - 1)) \sqrt{\frac{Q(0)}{N}}$$

$$< \alpha(m + 1) \sqrt{Q(0)}. \quad \tag{26}$$

Since $m$ is a definite finite positive integer and corresponds to $T_*$. Therefore, by adjusting the system parameters $\alpha$ properly, $\alpha(m + 1) < 1$ can always be established. Similarly, (26) also leads to contradictions a contradiction with (14).

Hence, there exists $0 < B_0 < +\infty$ such that $\|x_i(t) - x_j(t)\| \leq B_0$ for all $i, j \in \mathcal{N}$ and $t \in [0, +\infty)$. That is, the relative positions of any two particles in System (2) are bounded. The proof of Theorem 2.4 has been completed. \qed

**Corollary 1.** Suppose that the conditions of Theorem 2.4 are satisfied, then there exists $M_1$ and $M_2$ such that $M_1 \leq \|\dot{v}_i(t)\| \leq M_2$ for all $t \geq 0$ and $i \in \mathcal{N}$.

**Proof.** As a direct application of the above theorem, we have $I(B_0) \leq I(\cdot) \leq I(d_0)$ and $f(B_0^2) \leq f(\cdot) \leq f(d_0^2)$). Note the fact that $\|v_i(t)\| \leq \sqrt{Q(0)}$ and $d_0 \leq \|x_i(t) - x_j(t)\| \leq B_0$ for all $i, j \in \mathcal{N}$ and $t \geq 0$. Using the second equation of (2) produces

$$\|\dot{v}(t)\| = \left\| \frac{\alpha}{N} \sum_{j=1}^N I(r_{ij})(v_j(t - \tau) - v_i(t)) + \sum_{j=1}^N f(r_{ij}^2)(x_i(t) - x_j(t)) \right\|$$

$$\geq \left\| \sum_{j=1}^N f(r_{ij}^2)(x_i(t) - x_j(t)) \right\| - \left\| \frac{\alpha}{N} \sum_{j=1}^N I(r_{ij})(v_j(t - \tau) - v_i(t)) \right\| \tag{27}$$

$$\geq Nf(B_0)d_0 - 2\alpha I(d_0)\sqrt{Q(0)} =: M_1.$$
For another,
\[
\| \dot{v}(t) \| = \left\| \frac{\alpha}{N} \sum_{j=1}^{N} I(r_{ij})(v_j(t - \tau) - v_i(t)) + \sum_{j=1}^{N} f(r_{ij}^2)(x_i(t) - x_j(t)) \right\| \\
\leq \left\| \frac{\alpha}{N} \sum_{j=1}^{N} I(r_{ij})(v_j(t - \tau) - v_i(t)) \right\| + \left\| \sum_{j=1}^{N} f(r_{ij}^2)(x_i(t) - x_j(t)) \right\| \\
\leq 2\alpha I(d_0) \sqrt{Q(0)} + NB_0 f(d_0^2) =: M_2.
\]

The proof is completed. \(\square\)

3. The exponential decay of relative velocity. This section will characterize quantitatively the relative velocity decay of any two agents in System (2)-(3). The main results of this part are given below.

**Theorem 3.1.** Suppose that the conditions of Theorem 2.4 are satisfied and let \((x_i(t), v_i(t)), i \in \mathcal{N}\) be the solution to (2)-(3). If the measurement delay \(\tau\) satisfies
\[
0 \leq \tau < \tau_0 \quad \text{with} \quad C\tau_0 - e^{-\tau_0} = 0,
\]
where
\[
C = \frac{\alpha}{2I(B_0)} + 9\alpha^2.
\]
Then System (2) converges to a flock over time. In addition,
\[
\| v_i(t) - v_j(t) \| \leq C_0 e^{-\alpha I(B_0)t} \quad \text{for} \ i, j \in \mathcal{N},
\]
where \(B_0\) is defined in Theorem 2.4 and \(C_0 > 0\).

**Proof.** Combining with the results in Theorem 2.4, we just need to prove that the relative velocity of any two agents in System (2) converges asymptotically to 0. Motivated by the work of [12], for System (2)-(3), taking the candidate Lyapunov functional
\[
E(t) = \frac{1}{2N} \left\| w(t) \right\|^2 + \int_{t-\tau}^{t} e^{-(t-s)} \int_{s}^{t} \sum_{i=1}^{N} \left\| \dot{v}_i(\delta) \right\|^2 d\delta ds \\
+ \frac{\gamma}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \int_{0}^{+\infty} f(r) dr \| x_i - x_j \|^2,
\]
where \(\gamma > 0\) is a constant to be determined; \(w(t) = (w_1(t), \cdots, w_N(t))^T\) with \(w_i(t) = v_i(t) - \frac{1}{N} \sum_{i=1}^{N} v_i(t)\) for \(i \in \mathcal{N}\). It follows directly that \(\sum_{i=1}^{N} w_i(t) = 0\) and \(\sum_{i=1}^{N} \sum_{j=1}^{N} \| w_i(t) - w_j(t) \|^2 = 2N \| w(t) \|^2\) for all \(t \geq \tau\). Further, the derivative of \(E(t)\) along the solution trajectory of (2)-(3) is described as follows
\[
\frac{dE}{dt}(2) = \frac{1}{2N} \frac{d}{dt} \left\| w(t) \right\|^2 - \int_{t-\tau}^{t} e^{-(t-s)} \int_{s}^{t} \sum_{i=1}^{N} \left\| \dot{v}_i(\sigma) \right\|^2 d\sigma ds
\\
- e^{-\tau} \int_{t-\tau}^{t} \sum_{i=1}^{N} \left\| \dot{v}_i(\sigma) \right\|^2 d\sigma + \int_{t-\tau}^{t} e^{-(t-s)} \sum_{i=1}^{N} \left\| \dot{v}_i(t) \right\|^2 ds
\\
- \frac{\gamma}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} f(r_{ij}^2) \| x_i - x_j \| \frac{dr_{ij}}{dt}.
\]
Moreover,
\[
\frac{dE}{dt} \bigg|_{(2)} = \frac{1}{2N} \frac{d}{dt} \|w(t)\|^2 - \int_{t-\tau}^{t} e^{-(t-s)} \int_{s}^{t} \sum_{i=1}^{N} \|\dot{v}_{i}(\sigma)\|^2 d\sigma + \sum_{i=1}^{N} \|\dot{v}_{i}(t)\|^2 (1 - e^{-\tau})
\]
\[
- e^{-\tau} \int_{t-\tau}^{t} \sum_{i=1}^{N} \|\dot{v}_{i}(\sigma)\|^2 d\sigma + \sum_{i=1}^{N} \|\dot{v}_{i}(t)\|^2 (1 - e^{-\tau})
\]
\[
- \gamma \sum_{i=1}^{N} \sum_{j=1}^{N} f(r_{ij}^2) \|x_i - x_j\| \frac{dr_{ij}}{dt}
\]
\[
\leq \frac{1}{2N} \frac{d}{dt} \|w(t)\|^2 - e^{-\tau} \int_{t-\tau}^{t} \sum_{i=1}^{N} \|\dot{v}_{i}(\sigma)\|^2 d\sigma + \sum_{i=1}^{N} \|\dot{v}_{i}(t)\|^2
\]
\[
- \gamma \sum_{i=1}^{N} \sum_{j=1}^{N} f(r_{ij}^2) \|x_i - x_j\| \frac{dr_{ij}}{dt}.
\]

To simplify \(\frac{dE}{dt} \bigg|_{(2)}\), we divide the calculation of (32) into the following three steps.

**Step one.** The estimate for \(\frac{d}{dt} \|w(t)\|^2\).

Using the second equation of (2), it can be obtained by a simple calculation that
\[
\dot{w}_{i}(t) = \dot{v}_{i}(t) - \frac{1}{N} \sum_{k=1}^{N} \dot{v}_{k}(t)
\]
\[
= \alpha \sum_{j=1}^{N} I(r_{ij})(w_j(t) - w_i(t)) + \alpha \sum_{j=1}^{N} I(r_{ij})(v_j(t - \tau) - v_i(t)) + \sum_{j=1}^{N} f(r_{ij}^2)(x_i(t) - x_j(t)) - \frac{1}{N} \sum_{k=1}^{N} \sum_{j=1}^{N} f(r_{kj}^2)(x_k(t) - x_j(t))
\]
\[
- \frac{\alpha}{N^2} \sum_{k=1}^{N} \sum_{j=1}^{N} I(r_{kj})(w_j(t) - w_k(t))
\]
\[
- \frac{\alpha}{N^2} \sum_{k=1}^{N} \sum_{j=1}^{N} I(r_{kj})(v_j(t - \tau) - v_j(t)).
\]

Following the fact that \(\sum_{i=1}^{N} w_i(t) = 0\) for \(t \geq 0\), multiplying the two ends of (33) by \(w_i(t)\) and then summing \(i\) from 1 to \(N\) produces
\[
\frac{1}{2N} \frac{d}{dt} \|w(t)\|^2 = -\frac{\alpha}{2N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} I(r_{ij}) \|w_j(t) - w_i(t)\|^2
\]
\[
+ \frac{\alpha}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} I(r_{ij})(v_j(t - \tau) - v_j(t))w_i(t)
\]
\[
- \frac{1}{2N} \sum_{i=1}^{N} \sum_{j=1}^{N} f(r_{ij}^2) (x_i(t) - x_j(t))(w_i(t) - w_j(t)).
\]
For the sake of simplicity, denote $\xi_i(t) := \sum_{j=1, j \neq i}^N I(r_{ij})(v_j(t - \tau) - v_j(t))$ for $i \in N$ and $\xi(t) = (\xi_1(t), \cdots, \xi_N(t))^T \in \mathbb{R}^{dN}$. Due to $I(\cdot) < 1$ and the Cauchy inequality, we have

$$\left\| \frac{\alpha}{N^2} \sum_{i=1}^N \sum_{j=1}^N I(r_{ij})(v_j(t - \tau) - v_j(t))w_i(t) \right\|$$

$$\leq \frac{\alpha}{N^2} \left\| \sum_{i=1}^N \xi_i(t)w_i(t) \right\|$$

$$\leq \frac{\alpha}{N^2} \|\xi(t)\| \|w(t)\|,$$

where

$$\|\xi(t)\| \leq \sum_{i=1}^N \|\xi_i(t)\| \leq \sum_{i=1}^N \sum_{j=1}^N \int_{t-\tau}^t \|\dot{v}_j(s)\|ds = N \int_{t-\tau}^t \sum_{j=1}^N \|\dot{v}_j(s)\|ds.$$ 

For another, according to $f(B_0^0) \leq f(\cdot) \leq f(d_0^2)$ and $\Gamma(x(t)) \leq B_0$ from Theorem 2.4, one can be obtained,

$$\left\| \frac{1}{2N} \sum_{i=1}^N \sum_{j=1}^N f(r_{ij}^2)(x_i(t) - x_j(t))(w_i(t) - w_j(t)) \right\|$$

$$\leq \frac{1}{2N} f(d_0^2) \sqrt{N(N - 1)} \Gamma(x(t)) \cdot \sqrt{2N} \|w(t)\|$$

$$= \frac{\sqrt{2(N - 1)}}{2} f(d_0^2) B_0 \|w(t)\|.$$ 

Notice that $I(B_0) \leq I(\cdot) \leq I(d_0)$ again and it follows from (34) that

$$\frac{1}{2N} \frac{d\|w(t)\|^2}{dt} \leq -\alpha \frac{1}{2N^2} \sum_{i=1}^N \sum_{j=1}^N I(r_{ij}) \|w_j(t) - w_i(t)\|^2$$

$$+ \frac{\alpha}{N} \sum_{j=1}^N \int_{t-\tau}^t \|\dot{v}_j(s)\|ds \cdot \|w(t)\|$$

$$+ \frac{\sqrt{2(N - 1)}}{2} f(d_0^2) B_0 \|w(t)\|,$$

Moreover, using the Young inequality produces that

$$\frac{1}{2N} \frac{d\|w(t)\|^2}{dt} \leq -\alpha I(B_0) \|w(t)\|^2 + \frac{\sqrt{2(N - 1)}}{2} f(d_0^2) B_0 \|w(t)\|$$

$$+ \frac{\alpha \delta}{2N} \|w(t)\|^2 + \frac{\alpha \tau}{2d} \sum_{j=1}^N \int_{t-\tau}^t \|\dot{v}_j(s)\|^2 ds,$$

with $\delta > 0$. Let $\delta = I(B_0) > 0$ and $P_\tau(t) := \sum_{i=1}^N \int_{t-\tau}^t \|\dot{v}_j(s)\|^2 ds$. Further (37) is reduced to

$$\frac{1}{2N} \frac{d\|w(t)\|^2}{dt} \leq -\alpha I(B_0) \|w(t)\|^2 + \frac{\sqrt{2(N - 1)}}{2} f(d_0^2) B_0 \|w(t)\|$$

$$+ \frac{\alpha \tau}{2I(B_0)} P_\tau(t).$$
Step Two. The estimate for $\sum_{i=1}^{N} \| \dot{v}_i(t) \|^2$.

From the second equation of (2), we have

$$\dot{v}_i(t) = \frac{\alpha}{N} \sum_{j=1}^{N} I(r_{ij})(v_j(t) - v_i(t)) + \sum_{j=1, j \neq i}^{N} f(r_{ij}^2)(x_i - x_j)$$

$$= \frac{\alpha}{N} \sum_{j=1}^{N} I(r_{ij})(w_j(t) - w_i(t)) + \frac{\alpha}{N} \sum_{j=1}^{N} I(r_{ij})(v_j(t - \tau) - v_j(t))$$

$$+ \sum_{j=1, j \neq i}^{N} f(r_{ij}^2)(x_i - x_j)$$

$$= \frac{\alpha}{N} \sum_{j=1}^{N} I(r_{ij})(w_j(t) - w_i(t)) - \frac{\alpha}{N} \sum_{j=1}^{N} I(r_{ij}) \int_{t-\tau}^{t} \dot{v}_j(s) ds$$

$$+ \sum_{j=1, j \neq i}^{N} f(r_{ij}^2)(x_i - x_j).$$

(39)

Reviewing $I(r) < 1$, $f(r_{ij}^2) \leq f(d_0^2)$ and (39), it further follows from the inequality of arithmetic and geometric means that

$$\| \dot{v}_i(t) \|^2 \leq \frac{3\alpha^2}{N^2} \left( \sum_{j=1}^{N} \| w_j(t) - w_i(t) \| \right)^2 + \frac{3\alpha^2}{N^2} \left( \sum_{j=1}^{N} \| \dot{v}_j(s) \| ds \right)^2$$

$$+ 3f(d_0^2)^2 \left( \sum_{j=1}^{N} \| x_i - x_j \| \right)^2.$$

(40)

Thus, we have

$$\sum_{i=1}^{N} \| \dot{v}_i(t) \|^2 \leq \frac{3\alpha^2}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \| w_j(t) - w_i(t) \|^2 + 3\alpha^2 \tau \sum_{j=1}^{N} \int_{t-\tau}^{t} \| \dot{v}_j(s) \|^2 ds$$

$$+ 3Nf(d_0^2)^2 \sum_{i=1}^{N} \sum_{j=1}^{N} \| x_i - x_j \|^2$$

$$\leq 6\alpha^2 \| w(t) \|^2 + 3\alpha^2 \tau \cdot P_x(t) + 3NB^2 f(d_0^2)^2.$$

(41)

Step Three. The estimate for $\gamma \sum_{i=1}^{N} \sum_{j=1}^{N} f(r_{ij}^2)(x_i(t) - x_j(t))^T(v_i(t) - v_j(t))$.

It can be checked easily that

$$\gamma \sum_{i=1}^{N} \sum_{j=1}^{N} f(r_{ij}^2)(x_i(t) - x_j(t))^T(v_i(t) - v_j(t))$$

$$\leq \gamma f(d_0^2) \sum_{i=1}^{N} \sum_{j=1}^{N} \| x_i(t) - x_j(t) \| \cdot \| v_i(t) - v_j(t) \|$$

$$\leq \sqrt{2} \gamma NB_0 f(d_0^2) \| w(t) \|,$$

(42)
where we used the following fact

\[ \gamma \sum_{i=1}^{N} \sum_{j=1}^{N} f(r_{ij}^2) ||x_i(t) - x_j(t)|| \frac{dr_{ij}}{dt} = \gamma \sum_{i=1}^{N} \sum_{j=1}^{N} f(r_{ij}^2)(x_i(t) - x_j(t))^T (v_i(t) - v_j(t)). \]

Combining (38), (41) and (42), (32) is written as

\[ \frac{dE}{dt} \bigg|_{(2)} \leq - \frac{\alpha I(B_0)}{2N} ||w(t)||^2 + \frac{\sqrt{2(N - 1)}}{2} f(d_0^2)B_0 ||w(t)|| + \frac{\alpha \tau}{2I(B_0)} P_\tau(t) \]

\[ + 6\alpha^2 ||w(t)||^2 + 3\alpha^2 \tau \cdot P_\tau(t) + 3NB_0^2 f(d_0^2)^2 \]

\[ + \sqrt{2\gamma NB_0f(d_0^2)} ||w(t)|| - e^{-\tau} P_\tau(t). \]

Moreover,

\[ \frac{dE}{dt} \bigg|_{(2)} \leq - \frac{\alpha I(B_0)}{2N} ||w(t)||^2 \]

\[ + \left( \frac{\sqrt{2(N - 1)}}{2} f(d_0^2)B_0 + \sqrt{2\gamma NB_0f(d_0^2)} \right) ||w(t)|| \]

\[ + \left( \frac{\alpha \tau}{2I(B_0)} + 3\alpha^2 \tau - e^{-\tau} \right) P_\tau(t) + 3NB_0^2 f(d_0^2)^2 \]

\[ \leq - \frac{\alpha I(B_0)}{2N} ||w(t)||^2 + \left( \frac{\alpha \tau}{2I(B_0)} + 3\alpha^2 \tau - e^{-\tau} \right) P_\tau(t) \]

\[ + 24\alpha^2 NQ(0) + 3NB_0^2 f(d_0^2)^2 \]

\[ + f(d_0^2)B_0 \sqrt{Q(0)N} \left( \sqrt{2(N - 1)} + 2\sqrt{2\gamma N} \right), \]

where we used \( ||v_i(t)|| \leq \sqrt{Q(0)} \) and

\[ ||w(t)||^2 = \sum_{i=1}^{N} ||w_i(t)||^2 = \sum_{i=1}^{N} ||v_i(t) - v_c(t)||^2 \leq 4NQ(0). \]

For the sake of simplicity, denote \( c_1 := \alpha I(B_0)/(2N) \), \( c_2 := \alpha \tau/(2I(B_0)) + 3\alpha^2 \tau - e^{-\tau} \), \( c_3 := 24\alpha^2 NQ(0) + 3NB_0^2 f(d_0^2)^2 + f(d_0^2)B_0 \sqrt{Q(0)N} \left( \sqrt{2(N - 1)} + 2\sqrt{2\gamma N} \right) \).

Further, (43) is rewritten as

\[ \frac{dE}{dt} \bigg|_{(2)} \leq -c_1 ||w(t)||^2 + c_2 P_\tau(t) + c_3. \]

(44)

According to (29), we have \( c_2 < 0 \). It can easily be checked from Corollary 1 that \( NM_1^2 t^2 \leq P_\tau(t) \leq N M_2^2 t^2 \) for all \( t \geq 0 \). Further, \( M_1^2 \tau N c_2 < -c_3 \) can be guaranteed by an appropriate \( \gamma \), which depends on \( \tau \). Hence, (44) can be reduced to

\[ \frac{dE}{dt} \bigg|_{(2)} < -c_1 ||w(t)||^2. \]

(45)

Further,

\[ \frac{1}{2N} ||w(t)||^2 \leq E(t) = E(0) + \int_0^t \dot{E}(s)ds \leq E(0) - \int_0^t c_1 ||w(s)||^2 ds, \]

(46)

that is,

\[ ||w(t)||^2 \leq 2NE(0) - 2Nc_1 \int_0^t ||w(s)||^2 ds. \]

(47)
According to Gronwall inequality, we have \( \|v(t)\|^2 \leq 2NE(0)e^{-2Nc_1t} \), that is, \( \|v(t) - v_j(t)\|^2 = \|v_0(t) - w_j(t)\|^2 \leq E(0)e^{-2Nc_1t} =: C_0e^{-\alpha t}(R_0)^t \) for all \( t \geq 0 \) and \( i, j \in \mathcal{N} \). Therefore, the velocity of System (2)-(3) is asymptotically synchronized.

**Remark 1.** With our observations, a conclusion can be drawn from (29), that is, the upper bound of the allowable measurement delay decreases as the diameter of the particle swarm position increases. This is consistent with the real scene. It is well known that when the diameter or number of particle groups keeps increasing, a common and effective method is to control the size of the time lag to emerge a flock. Obviously, the smaller the time lag, the more conducive to the occurrence of flocking.

**Remark 2.** It obviously gets from Theorem 2.4 and Theorem 3.1 that, if the conditions in both theorems are satisfied, then System (2)-(3) will converge to a flock asymptotically. Specifically, the relative velocities among particles in this population converge exponentially to 0.

4. **Flocking analysis without measurement delay.** We next consider System (2) without delay, that is, \( \tau = 0 \). Further, (2) degenerates into the following form,

\[
\begin{align*}
\dot{x}_i(t) &= v_i(t), \quad i \in \mathcal{N}, \\
v_i(t) &= \frac{1}{N} \sum_{j=1}^{N} I(r_{ij})(v_j(t) - v_i(t)) + \sum_{j=1, j \neq i}^{N} f(r_{ij})(x_i(t) - x_j(t)),
\end{align*}
\]

(48)

where \( \alpha > 0 \), \( r_{ij} := \|x_i - x_j\| \), \( i, j \in \mathcal{N} \); The communication function \( I(\cdot) \) satisfies Assumption 1; \( f(\cdot) \) is the same as in (2).

Denote the mean velocity and the motion path of the center of the population by

\[
v_c(t) = \frac{1}{N} \sum_{i=1}^{N} v_i(t), \quad x_c(t) = \frac{1}{N} \sum_{i=1}^{N} x_i(t).
\]

It is derived from the second equation of (48) with the symmetry of indicators \( i \) and \( j \), \( \sum_{i=1}^{N} v_i(t) = 0 \), which implied that \( v_c(t) \) for \( t > 0 \) is time-invariant. Further, the motion path of the center is a straight line. That is, \( x_c(t) = x_c(0) + v_c(0)t \).

Define the error variables \( \overline{x}_i(t) := x_i(t) - x_c(t) \) and \( \overline{v}_i(t) := v_i(t) - v_c(t) \), moreover, it can be checked easily that \( \sum_{i=1}^{N} \overline{x}_i(t) = 0 \) and \( \sum_{i=1}^{N} \overline{v}_i(t) = 0 \). Due to the equalities, \( \overline{x}_i(t) - \overline{x}_j(t) = x_i(t) - x_j(t) \) and \( \overline{v}_i(t) - \overline{v}_j(t) = v_i(t) - v_j(t) \), \( \overline{x}_i \) and \( \overline{v}_i \) satisfy the error system

\[
\begin{align*}
\dot{\overline{x}}_i(t) &= \overline{v}_i(t), \quad i \in \mathcal{N}, \\
\dot{\overline{v}}_i(t) &= \frac{1}{N} \sum_{j=1}^{N} I(r_{ij})(\overline{v}_j(t) - \overline{v}_i(t)) + \sum_{j=1, j \neq i}^{N} f(r_{ij})(\overline{x}_i(t) - \overline{x}_j(t)).
\end{align*}
\]

(49)

Throughout this section, “the error System (49) subject to some certain initial conditions” is abbreviated as “System (48)’. For System (48), denote \( \|x(t)\|^2 := \sum_{i=1}^{N} \|x_i(t)\|^2 \) and \( \|v(t)\|^2 := \sum_{i=1}^{N} \|v_i(t)\|^2 \).

Taking the following candidate Lyapunov function, moreover, the collision-free flocking conditions are stated in Theorem 4.1.

\[
\mathcal{L}(x(t), v(t)) = \|v(t)\|^2 + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \int_{t}^{+\infty} f(\|x_i(t) - x_j(t)\|^2) f(r) dr, \quad t > 0.
\]

(50)
Theorem 4.1. Suppose that \((x_i(t), v_i(t))\) for \(i \in \mathcal{N}\) be a solution to System (48) with the initial condition \((x_i(0), v_i(0))\), \(i \in \mathcal{N}\) satisfying \(\|x_i(0) - x_j(0)\| > d_0\) for \(i \neq j\) and \(v(0) \neq 0\),

\[
\mathcal{E}(0) < \min \left\{ \frac{1}{2} \int_{1}^{+\infty} f(r) dr, \alpha \int_{\|x(0)\|}^{+\infty} I(\sqrt{2}r) dr \right\}. \tag{51}
\]

Then, this population converges to a flock and achieves collision avoidance. Furthermore, the asymptotic velocity \(v_\infty = v_i(0)\).

To complete the proof of Theorem 4.1, we start introducing a few auxiliary results and stating a well-known result.

Lemma 4.2. \cite{12} Let \((x_i(t), v_i(t))\), \(i \in \mathcal{N}\) be a solution to System (48). Then we have

\[
\frac{d}{dt} \|x(t)\| \leq \|v(t)\|, \ a.e. \ t \in (0,T).
\]

Lemma 4.3. Let \((x_i(t), v_i(t))\) for \(i \in \mathcal{N}\) be a solution to System (48). Then we have

\[
\frac{d}{dt} \|v(t)\|^2 \leq -2\alpha I(\sqrt{2}\|x\|)\|v(t)\|^2 + \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} f(r_{ij}^2)(v_i(t) - v_j(t), x_i(t) - x_j(t)), \ a.e. \ t \in (0,T). \tag{52}
\]

Proof. For \(t \in (0,T)\) and it follows from the second equation of (49) that

\[
\frac{d}{dt} \|v(t)\|^2 = 2 \sum_{i=1}^{N} \left\langle v_i(t), \alpha \sum_{j=1}^{N} I(r_{ij})(v_j(t) - v_i(t)) \right\rangle
\]

\[
+ 2 \sum_{i=1}^{N} \left\langle v_i(t), \sum_{j=1, j \neq i}^{N} f(r_{ij}^2)(x_i(t) - x_j(t)) \right\rangle
\]

\[
= -\alpha \sum_{i=1}^{N} \sum_{j=1}^{N} \|v_i(t) - v_j(t)\|^2
\]

\[
+ \sum_{i=1}^{N} \left\langle v_i(t) - v_j(t), \sum_{j=1, j \neq i}^{N} f(r_{ij}^2)(x_i(t) - x_j(t)) \right\rangle
\]

\[
\leq -\alpha \sum_{i=1}^{N} \sum_{j=1}^{N} \|v_i(t) - v_j(t)\|^2
\]

\[
+ \sum_{i=1}^{N} \left\langle v_i(t) - v_j(t), \sum_{j=1, j \neq i}^{N} f(r_{ij}^2)(x_i(t) - x_j(t)) \right\rangle
\]

\[
\leq -2\alpha I(\sqrt{2}\|x\|)\|v(t)\|^2
\]

\[
+ \sum_{i=1}^{N} \left\langle v_i(t) - v_j(t), \sum_{j=1, j \neq i}^{N} f(r_{ij}^2)(x_i(t) - x_j(t)) \right\rangle.
\]

The proof is completed. \qed
Lemma 4.4. \[13\] Let \( f(t, x) \) be piecewise continuous in \( t \) and locally Lipschitz in \( x \) for all \( t > t_0 \) and all \( x \) in a domain, which is open and connected set \( D \subseteq \mathbb{R}^n \). Let \( x(t) \) be the solution of the differential equation
\[
\dot{x}(t) = f(t, x), \quad x(t_0) = x_0
\]
with maximal interval of existence \([t_0, T)\) with \( T < \infty\). Then, for any compact set \( W \subseteq D \) with \( x_0 \in W \), there is some \( t_1 \in [t_0, T) \) such that \( x(t_1) \notin W \).

Proof of Theorem 4.1. The method of this proof is similar to that of Theorem 2.4. Therefore, the proof is omitted here, for more details see the Appendix.

5. Numerical simulation. Several examples will be given to verify that the measurement delay does affect the formation of a flock for System (2)-(3) and demonstrate the validity of the theoretical results.

Consider a system (2) with 7 particles, fix \( \alpha = 1, \) \( I(r) = (1 + r^2)^{-\beta} \) with \( \beta \geq 0 \) and the external force function is set to \( f(r) = r^{-2} \), which satisfies the condition (5) with \( \delta = 10^{-5} \). For the sake of discussion, the initial conditions are both random numbers over the interval \([0, 1]\). For the above initial configurations, it can be checked easily that they meet (14) in Theorem 2.4. In all numerical experiments, two variables are used to characterize the asymptotic behavior of System (2), i.e., the velocity and position of each particle. Therefore, for System (2)-(3), the system asymptotically converges to form a flock if and only if the velocity of each particle will gradually converge and the relative position between any two particles is bounded.

Example 1. \( \beta = 0.5, \tau = 0.01 \) in (2).

**Figure 1.** \( \beta=0.5, \tau=0.01 \), the relative position of any two particles in System (2)-(3) is always bounded, and the velocity will asymptotically converge, i.e., the emergence of flocking.

**Analysis from theoretical results:** It can calculates directly from (13) and to get that \( Q(0) = 14.035, B_0^2 = 22025 \) and \( I(B_0) = (1 + B_0^2)^{-\beta} = 1/148 \), where the selection method of \( B_0 \) originates from the Step Three in the proof of Theorem 2.4. Further, the parameter \( C = 83 \) in Theorem 3.1 can be calculated directly. It follows that \( \tau - e^{-\tau} < 0 \) based on \( \tau = 0.01 \) and \( C = 83 \). Since \( \tau_0 \) satisfies \( C\tau_0 - e^{-\tau_0} = 0 \), we can get that \( 0 < \tau = 0.01 < \tau_0 \), which is satisfied (29) in Theorem 3.1. This shows that for the above parameters, the theoretical results of this paper show that the system (2)-(3) can converge asymptotically to form a flock.
Analysis of numerical simulation results: It can be seen intuitively from Figure 1 that the relative positions of any two particles are bounded and the velocity of each particle is asymptotically convergent. This shows that the simulation results are consistent with the theoretical results. Therefore, this confirms the validity of the theoretical results.

Example 2. $\beta = 0.6, \tau = 0.01$ in (2).

![Figure 2](image1.png)

**Figure 2.** $\beta=0.6, \tau=0.01$, the relative position of any two particle in System (2)-(3) is always bounded, and the velocity will asymptotically converge, i.e., the emergence of flocking.

Similarly to Example 1, the corresponding parameters can be calculated, that is, $Q(0) \approx 14.035, B^2_0 \approx 11, I(B_0) \approx 0.78, C \approx 9.64$. Hence, $C\tau - e^{-\tau} < 0$. Combining with $\tau_0$ satisfying $C\tau_0 - e^{-\tau_0} = 0$ yields that $0 < \tau = 0.01 < \tau_0$, which is satisfied (29) in Theorem 3.1. This shows that for the above parameters, the theoretical results indicate that System (2)-(3) can converge to a flock asymptotically. For another, the simulation results are shown in Figure 2 and it can be seen intuitively that the relative positions of any two particles are bounded and the velocity of each particle is asymptotically convergent. This also shows that the theoretical results are consistent with the simulation and verifies the validity of the theoretical results.

Example 3. $\beta = 0.6, \tau = 0.1$ in the system (2).

![Figure 3](image2.png)

**Figure 3.** $\beta=0.6, \tau=0.1$, System (2)-(3) fail to form a flock.
In this case, the corresponding parameter $Q(0) = 14.35, B_0^2 = 2, I(B_0) = 0.2, C = 11.5$ can be calculated. Further, $C \tau - e^{-\tau} > 0$. While $C \tau_i - e^{-\tau_i} = 0$, it follows that $\tau = 0.1 > \tau_i$, which means that (29) in Theorem 3.1 is destroyed. It is pointed out that for the above parameters, according to the theoretical results, System (2)-(3) will fail to form a flock. In fact, the simulation results shown in Figure 3 also confirm that flocking behavior will not occur. This also shows that the conclusions are theoretically and numerically consistent.

6. Conclusions. Aiming at the problem of collision-free flocking of time-delay systems, the two realistic factors of measurement delay and avoiding collisions within the group are considered in the classic Cucker-Smale model. For this time-delay flock model, the asymptotic flocking conditions which are closely related to the time-delay is established. The simulation results also confirm the influence of time delay on the evolution of group behavior. This work explains in detail and strictly proves that collision avoidance and time asymptotic flocking can coexist in a class of time-delay systems.

Appendix. Proof of Theorem 4.1

Proof. This proof will be divided into the following four steps. First, System (48) will not collide in any finite interval $[-\tau, T]$ with $T \in (0, +\infty]$. Second, $T$ cannot be finite, that is, $T = +\infty$. Then, the relative position for any two particles is bounded. Finally, the velocity of the group will converge asymptotically.

For any initial condition $(x_i(0), v_i(0))$, $i \in N$ satisfying $\|x_i(0) - x_j(0)\| > d_0$ for $i \neq j$ and $v(0) \neq 0$, the solutions for System (48) exist locally and uniquely. Let $[0, T)$ be the maximal existence interval of the solution of System (48) starting at $(x_i(0), v_i(0))$, $i \in N$.

Step One. Prove that System (48) will not collide in any finite interval $[-\tau, T)$ with $T \in (0, +\infty]$.

Using Lemma 4.2 yields the derivative of $L(t)$ along the solution trajectory of System (48), which is described specifically as

$$\frac{dL}{dt} \bigg|_{(48)} = \frac{d}{dt}\|v(t)\|^2 - \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} f(r_{ij}^2) (x_i - x_j, v_i - v_j)$$

$$\leq -2\alpha I(\sqrt{2}\|x_i\|\|v(t)\|)^2.$$  (53)

It implies that $\Sigma(t)$ is a decreasing function of $t$ along the solution to System (48), that is, for $t \in [0, T)$, we have $\Sigma(t) \leq \Sigma(0)$. Further, it follows from (50) that

$$\|v(t)\|^2 + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \int_{v_t^2}^{+\infty} f(r) dr \leq \Sigma(0),$$  (54)

which shows that

$$\frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \int_{v_t^2}^{+\infty} f(r) dr < \Sigma(0), \text{ for } t \in [0, T),$$  (55)

Similar to the proof of Theorem 2.4 and combining with (51), there exists $d_0 > \delta \geq 0$ such that $0 < d_0 \leq \|x_i(t) - x_j(t)\|$ for all $t \in [0, T)$ and $i \neq j \in N$, that is, collision avoidance within System (48) on $[0, T)$ can be achieved.
Step Two. Prove that \( T = +\infty \).

If not, then for \( t \in [0, T) \) with \( T < +\infty \), it follows from (53) that
\[
\|v(t)\|^2 \leq \mathcal{L}(0).
\] (56)
Hence \( \|v(t)\| \leq \sqrt{\mathcal{L}(0)} =: v_0 \) for \( t \in [0, T) \). Moreover,
\[
\|x(t)\| \leq \|x(0)\| + \int_0^t \|v(s)\|ds \leq \|x(0)\| + \int_0^T \|v(s)\|ds \leq \|x(0)\| + v_0 T =: x_0,
\]
which means that \( (x(t), v(t)) \) for all \( t \in [0, T) \) lies in the following compact set
\[
\mathcal{W} := \left\{ (x,v) \mid \|x(t)\| \leq x_0, \; \|v(t)\| \leq v_0, \; \|x_i(t) - x_j(t)\| \geq d_0, \; i \neq j \in \mathcal{N} \right\}
\subset \left\{ (x,v) \mid \|x_i(t) - x_j(t)\| \geq d_0, \; i \neq j \in \mathcal{N} \right\}.
\]
This is in contradiction with Lemma 4.4. Therefore, we have \( T = +\infty \), which implies that the solution exists for \( t \geq 0 \) and its extension is unique.

Step Three. Prove that the relative position for any two particles is bounded.

Using (53) again and integrating on both sides of (53) from 0 to \( t \in [0, T) \) produces
\[
\mathcal{L}(t) - \mathcal{L}(0) \leq -2\alpha \int_0^t I(\sqrt{2}\|x\|)\|v(s)\|^2ds.
\] (57)
It is straightforward to show from (56) that there exists \( M > 0 \) such that for all \( i \in \mathcal{N} \) and \( t > 0 \), we have \( \|v_i(t)\| \leq M \). Combining with (57) yields
\[
2\alpha M \int_0^t I(\sqrt{2}\|x\|)\|v(s)\|ds \leq \mathcal{L}(0).
\] (58)
It is easy to verify the following fact that
\[
\left| \frac{d}{dt}\|x(t)\| \right| \leq \|v(t)\|.
\] (59)
We get
\[
2\alpha M \int_\|x(0)\|^{\|x(t)\|} I(\sqrt{2}y)dy \leq \mathcal{L}(0), \; \text{for all } t > 0.
\] (60)
Assume that \( \|x(t)\| \) is not bounded with \( t \), which means
\[
2\alpha M \int_{\|x(0)\|}^{+\infty} I(\sqrt{2}y)dy \leq \mathcal{L}(0).
\] (61)
It is in contradiction with (51). Therefore, there exists \( B_0 \) such that \( \|x(t)\| \leq B_0 \) for \( t > 0 \). Further, for \( i \neq j \in \mathcal{N} \), we have \( \|x_i(t) - x_j(t)\| \leq \sqrt{2}\|x(t)\| \leq \sqrt{2}B_0 \).

Step Four. Prove that the velocity of the group will converge asymptotically.

Notice that for all \( i \neq j \in \mathcal{N}, d_0 \leq \|x_i(t) - x_j(t)\| \leq \sqrt{2}B_0 \) and the monotonicity of \( I(\cdot) \) and \( f(\cdot) \), we have \( I(B_0) \leq I(r_{ij}) \leq I(d_0) \) and \( f(B_0) \leq f(r_{ij}) \leq f(d_0) \). Further, it follows from Lemma 4.3 that
\[
\left| \frac{d}{dt}\|v(t)\|^2 \right| \leq 2\alpha I(\sqrt{2}\|x\|)\|v(t)\|^2 + f(d_0^2)\sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} \|v_i(t) - v_j(t), (x_i(t) - x_j(t))\|
\leq 2\alpha I(d_0)\|v(t)\|^2 + f(d_0^2)NB_0\sqrt{2(N-1)}\|v(t)\|.
\]
Moreover,
\[
\frac{d}{dt} \|v(t)\| \leq 2\alpha I(d_0) \|v(t)\| + f(d_0^2) NB_0 \sqrt{2(N - 1)}
\]
\[
\leq 2\alpha I(d_0) \sqrt{E(0)} + f(d_0^2) NB_0 \sqrt{2(N - 1)}.
\]

Apply Barbalat Lemma to \(\int_0^t \|v(s)\| ds\), it can be checked easily that \(\left(\int_0^t \|v(s)\| ds\right)''\) exists and is bounded. Therefore, \(\lim_{t \to +\infty} \|v(t)\| = 0\), which implied that for all \(i \in \mathcal{N}\), \(\lim_{t \to +\infty} \|v_i(t)\| = v_c(t) \equiv v_c(0)\).

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