Conditions for manipulation of a set of entangled pure states

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We derive a sufficient condition for a set of pure states, each entangled in two remote $N$-dimensional systems, to be transformable to $k$-dimensional-subspace equivalent entangled states ($k \leq N$) by same local operations and classical communication. If $k = N$, the condition is also necessary. This condition reveals the function of the relative marginal density operators of the entangled states in the entanglement manipulation without sufficient information of the initial states.

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The deep ways that quantum information differs from classical information involve the properties, implications, and uses of quantum entanglement [1]. As a useful physical resource of quantum information, entanglement plays a key role for quantum computation [2], quantum teleportation [3], quantum superdense coding [4] and certain types of quantum cryptography [5], etc. To accomplish these tasks, transformation between the input entanglement we possess and the target entanglement we require is necessary. Attempts [6–18] have been made to uncover the fundamental laws of the transformations under local quantum operations and classical communication (LQCC), that is, the different entangled parties may do whatever they wish to in their local system, and may communicate classically, but they cannot use quantum communication.

All previous entanglement manipulation protocols only consider a definite entangled state shared by distant observers. However, quantum information processing often has to work with insufficiently known initial states. It is therefore important to understand which processes work without full knowledge of initial states. In this letter, we address the question whether it is possible to manipulate a set of entangled pure states only by one LQCC protocol, just like quantum clone [19–22]. This problem is fundamentally and also practically important. An example may be in the disentangled process of quantum clone [23–24]. In deterministic state-dependent cloning process [21], although according to Nielsen Theorem [11] each of the two final states can be transformed to the disentangled state by LQCC respectively, it is impossible to separate the output by LQCC without knowing which one the initial state is [23]. The same result also exists in probabilistic telecloning process [24]. In Ref. [18], we showed that a local operation can enhance the entanglement of a set of two-level entangled states simultaneously. In this letter, we investigate the problem with some restrictions of the final states. The investigations here are for the finite (nonasymptotic) case, from which asymptotic results may be recovered by taking limits. The transformation process may be probabilistic, but not approximate. To present our questions and results, we first collect some useful Facts:

1. An arbitrary bipartite entangled pure state $|\Omega\rangle$ that Alice and Bob share can be written as [25] $|\Omega\rangle = (U_A \otimes U_B) \sum_{i=1}^{N} \sqrt{\lambda_i} |i_A\rangle |i_B\rangle$, where $U_A$ and $U_B$ are local unitary transformations by Alice and Bob respectively, $\sum_{i=1}^{N} \lambda_i = 1$, and $\{|i_A\rangle\}$ form an orthogonal basis for system $A$ ($B$). In this letter, we take $N$ as the maximum dimensions of the subsystem. If $|\Omega\rangle$ has $m$ nonzero eigenvalues, we call $|\Omega\rangle$ an $m$-dimensional entangled state. The $m$-dimensional maximally entangled state can be generally written as $|\Phi\rangle = (U_A \otimes U_B) \frac{1}{\sqrt{m}} \sum_{i=1}^{m} |i_A\rangle |i_B\rangle$.

The marginal density operator for Alice’s (Bob’s) subsystem is defined as $\rho_{A(B)}(|\Omega\rangle) = \text{Tr}_{B(A)} |\Omega\rangle \langle \Omega|$. Obviously $\lambda_i$ is the eigenvalue of $\rho_{A(B)} (|\Omega\rangle)$. Furthermore, we denote $|\alpha\rangle \sim |\beta\rangle$ if $|\alpha\rangle$ and $|\beta\rangle$ are the same up to local unitary operations by Alice and Bob. The Schmidt decomposition implies that $|\alpha\rangle \sim |\beta\rangle$ if and only if $\rho_{A} (|\alpha\rangle)$ and $\rho_{A} (|\beta\rangle)$ have the same spectrum of eigenvalues.

2. Denote $Q$ as an index set. We call that a set of entangled states $\{|\alpha_{\ell}\rangle, \ell \in Q\}$ are $k$-dimensional-subspace equivalent if and only if there exist no-zero

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constant $C_{\alpha}$ and non-zero Schmidt coefficients $\mu_i(|\alpha_\ell\rangle)$ \((1 \leq t \leq k)\) making $\mu_i(|\alpha_{\ell_0}\rangle) = C_{\alpha_\ell} \mu_i(|\alpha_\ell\rangle)$. Suppose $|\alpha\rangle$ and $|\beta\rangle$ are $N$ and $N'$ \((N' \leq N)\) dimensional entangled states respectively, we denote $\hat{F}(A_B)(|\alpha\rangle |\beta\rangle) = p\hat{A}(B)(|\alpha\rangle |\beta\rangle) - \frac{1}{2} |\alpha\rangle |\beta\rangle - \frac{1}{2} |\alpha\rangle |\beta\rangle$ as the relative marginal density operator of states $|\alpha\rangle$ and $|\beta\rangle$ for Alice’s \(Bob’s\) subsystem. $\hat{F}(A_B)$ describes the relation of the marginal density operators of the two states. For a set of relative marginal density operators $\{\hat{F}(A_B), \ell \in Q\}$, we denote that $\hat{F}(A_B)$ are similar about $I_k$ if and only if $\hat{F}(A_B)$ can be represented on the orthogonal basis $|i_A\rangle$ as

$$\hat{F}(A_B) = V \text{diag}(s \ell I_k, D_\ell) V,$$

where $s \ell > 0$, $I_k$ is the $k 	imes k$ unit matrix, $V$ is unitary and $D_\ell$ is a symmetric matrix.

3. Any operation $P$ Alice performs on the maximally entangled state $\frac{1}{\sqrt{N}} \sum_{k=1}^{N} |k_A\rangle |k_B\rangle$ is equal to the transposed operation $P^+$ performed by Bob \cite{26}, that is, \((P \otimes I) \sum_{k=1}^{N} |k_A\rangle |k_B\rangle = (I \otimes P^+) \sum_{k=1}^{N} |k_A\rangle |k_B\rangle\).

4. Given any pure bipartite state $|\Psi\rangle_{AB} = \sum_{i=1}^{N} \sqrt{\lambda_i} |i_A\rangle |i_B\rangle$ shared by Alice and Bob and any complete set of projection operators $\{P_{i}^{Bob}\}$’s by Bob, there exists a complete set of projection operators $\{P_{i}^{Alice}\}$’s by Alice and, for each outcome $l$, a direct product of local unitary transformations $U_l^A \otimes U_l^B$ such that, for each $l$ \cite{8}

$$(I \otimes P_{l}^{Bob}) |\Psi\rangle_{AB} = (U_l^A \otimes U_l^B) (P_{l}^{Alice} \otimes I) |\Psi\rangle_{AB}.$$ (2)

5. The most general scheme of entanglement manipulation of a bipartite entangled pure state involves local operations of respective system and two-way communication between Alice and Bob \cite{6}. The local operations can be represented as generalized measurements, described by operators $A_k$ and $B_k$ on each system, satisfying the condition $\sum_k A_k^+ A_k \leq I_N$ \(I_N - \sum_k A_k^+ A_k\) is positive semidefinite) and $\sum_k B_k^+ B_k \leq I_N$, where $I_N$ is the unit operator of Alice’s or Bob’s subsystem. The LQCC protocol we consider maps the initial state $|\phi\rangle \langle \phi|$ to the target state \cite{9},

$$|\varphi\rangle \langle \varphi| = \frac{\sum_{kl} A_k \otimes B_l |\phi\rangle \langle \phi| A_k^+ \otimes B_l^+}{Tr(\sum_{kl} A_k \otimes B_l |\phi\rangle \langle \phi| A_k^+ \otimes B_l^+)}.$$ (3)

The initial and final states are pure, it follows that

$$A_k \otimes B_l |\phi\rangle = \sqrt{p_{kl}} |\varphi\rangle,$$ (4)

with non-negative success probability $p_{kl}$ satisfying $p_{kl} = Tr(A_k \otimes B_l |\phi\rangle \langle \phi| A_k^+ \otimes B_l^+)$. Suppose Alice and Bob share a pure bipartite $N$-dimensional entangled state $|\phi_1\rangle$ they can convert to another entangled pure state $|\phi_1\rangle$ by a LQCC process with no-zero probability \cite{12}. Denote $S$ as an index set, our question is what property characterizes the set of entangled pure state $\{|\phi_1\rangle, |\phi_\nu\rangle, \nu \in S\}$ that can be transformed to the final states $\{|\varphi_1\rangle, |\varphi_\nu\rangle, \nu \in S\}$ by the same LQCC process if $|\varphi_\nu\rangle$ are $k$-dimensional-subspace equivalent to state $|\varphi_1\rangle$ \(k \leq N\). In this letter, we derive a sufficient condition for such manipulation. If $k = N$, we show that the condition is also necessary.

**Theorem 1:** A set of bipartite entangled pure states $\{|\phi_1\rangle, |\phi_\nu\rangle, \nu \in S\}$ can be probabilistic transformed to $k$-dimensional-subspace equivalent states by one LQCC protocol if the relative marginal density operators of states $|\varphi_1\rangle$ and $|\varphi_\nu\rangle$ are similar about $I_k$. As a simple application of the result, suppose Alice and Bob each possess a four-dimensional quantum system, with respectively orthonormal bases denoted by $|1\rangle, |2\rangle, |3\rangle$ and $|4\rangle$. The initial entangled state may be one of the following states

$$|\alpha\rangle = \frac{1}{\sqrt{4}} |11\rangle + \frac{1}{\sqrt{4}} |22\rangle + \frac{1}{\sqrt{16}} |33\rangle + \frac{1}{\sqrt{16}} |44\rangle,$$ (5)

$$|\beta\rangle = \frac{1}{\sqrt{4}} |11\rangle + \frac{1}{\sqrt{4}} |22\rangle + \frac{1}{\sqrt{2}} |33\rangle.$$ Obviosly the relative marginal density operator $F_A(|\alpha\rangle |\beta\rangle) = \text{diag}(1, 1, 1, 0)$ has two same eigenvalues. Alice can transform above two states to 2-dimensional maximally entangled state $|\Upsilon\rangle = \sqrt{\frac{1}{2}} (|11\rangle + |22\rangle)$ with local generalized measurement $P_1 = |1\rangle \langle 1| + |2\rangle \langle 2|$ satisfying $P_1^+ P_1 \leq I_4$.

**Proof of Theorem 1:**

Generally, the states to be transformed can be represented as $|\phi_1\rangle = \sum_{i=1}^{N} \sqrt{\lambda_i} |i_A\rangle |i_B\rangle$ and $|\phi_\nu\rangle = (U_A^\nu \otimes U_B^\nu) \sum_{i=1}^{N} \sqrt{\mu_i} |i_A\rangle |i_B\rangle$ with $\lambda_i > 0$. Suppose $|\phi_1\rangle$ is transformed to the state $|\varphi_1\rangle = \sum_{i=1}^{N} \sqrt{\gamma_i} |i_A\rangle |i_B\rangle$ by a LQCC process, the same LQCC should transform state $|\phi_\nu\rangle$ to state $|\varphi_\nu\rangle$ that has Schmidt coefficients $\eta'_i = c_{\nu} \gamma_i$, $i = 1, 2, ..., k$, where $c_{\nu}$ is a no-zero real number.

Denote $\lambda = \text{diag}(\lambda_1, \lambda_2, ..., \lambda_N)$, and $\mu'^{'}, \gamma, \eta''$ are of similar definitions. Obviously $\rho_A(|\phi_1\rangle) = \lambda, \rho_A(|\phi_\nu\rangle) = \lambda'$, and $\rho_B(|\phi_1\rangle) = \rho_B(|\phi_\nu\rangle) = \mu'^{'}$, $\gamma$, $\eta''$ are of similar definitions. Obviously $\rho_A(|\phi_1\rangle) = \lambda, \rho_A(|\phi_\nu\rangle) = \lambda'$, and $\rho_B(|\phi_1\rangle) = \rho_B(|\phi_\nu\rangle) = \mu'^{'}$, $\gamma$, $\eta''$ are of similar definitions.
$U_\nu^\dagger \mu^\nu (U_\nu^\dagger)^\dagger$. Since the relative marginal density operators $\{ P_\nu^\nu (|\varphi_1\rangle \langle \varphi_\nu|), \nu \in S \}$ are similar about $I_k$, applying Fact 2, we obtain

$$
\lambda^{-1/2} U_\nu^\nu \sqrt{\mu^\nu} = V \begin{pmatrix} \sqrt{s_\nu} I_k & 0 \\ 0 & \sqrt{D_\nu} \end{pmatrix} G_\nu,
$$

where $G_\nu$ is a unitary matrix. Suppose $P_1 = \sqrt{\nu} \sqrt{\nu}^\dagger + \sqrt{\lambda} I$. Since $P_1^\dagger P_1 = \varepsilon I + \sqrt{\lambda} I$, the choice of $\varepsilon$ can make $P_1^\dagger P_1 \leq I_N$, which means $P_1$ is a generalized measurement (Fact 5) independent of the initial states $|\phi_\nu\rangle$. According to Fact 3, $P_1$ acts on the states $|\phi_1\rangle$ and $|\phi_\nu\rangle$ as follows:

$$(P_1 \otimes I) |\phi_1\rangle = \sqrt{\varepsilon} (I \otimes V) \sum_{i=1}^N \sqrt{s_\nu} |i_\nu\rangle |i_B\rangle = \sqrt{\varepsilon} (I \otimes V) |\varphi_1\rangle,$$

$$(P_1 \otimes I) |\phi_\nu\rangle = \sqrt{\varepsilon s_\nu} (I \otimes U_B^\nu G_\nu^\nu V) (I \otimes H_\nu) \sum_{i=1}^N \sqrt{\nu} |i_\nu\rangle |i_B\rangle = \sqrt{\varepsilon s_\nu} (I \otimes U_B^\nu G_\nu^\nu V) |\varphi_\nu\rangle,$$

where the corresponding matrix of $H_\nu = \text{diag} \left( I_k, s_\nu^{-1} \sqrt{D_\nu} \right)$, $|\varphi_\nu\rangle = (I \otimes H_\nu) \sum_{i=1}^N \sqrt{\nu} |i_\nu\rangle |i_B\rangle$. Denote the normalized states of $|\varphi_\nu\rangle$ as $|\varphi_\nu\rangle$, they are $k$-dimensional-subspace equivalent to $|\varphi_1\rangle$. Thus we finish the proof of Theorem 1.

If the final states are $N$-dimensional-subspace equivalent, the above sufficient condition can be expressed in a more simple form with clear physical meaning. In fact, since $\text{Tr} \rho_A (|\phi_1\rangle \langle \phi_1|) = \text{Tr} \rho_A (|\phi_\nu\rangle \langle \phi_\nu|) = 1$, the above sufficient condition means that the relative minimal density operators $F_\nu^\nu (|\varphi_1\rangle \langle \varphi_\nu|) = I$, i.e., the marginal density operators $\rho_A (|\phi_1\rangle \langle \phi_1|) = \rho_A (|\phi_\nu\rangle \langle \phi_\nu|)$ and $|\varphi_\nu\rangle$ are also $N$-dimensional entangled pure states. In this case, such condition is also necessary.

**Theorem 2:** A set of $N$-dimensional entangled pure states $\{|\phi_1\rangle, |\phi_\nu\rangle, \nu \in S\}$ can be probabilistic transformed to $N$-dimensional-subspace equivalent states $\{|\varphi_1\rangle, |\varphi_\nu\rangle, \nu \in S\}$ by same LQCC protocol if and only if they share same marginal density operators for Alice’s or Bob’s subsystem.

**Proof of Theorem 2:**

We need only prove the necessity. Consider that a generalized measurement $A_k \otimes B_l$ transforms the $N$-dimensional states $|\phi_1\rangle$ and $|\phi_\nu\rangle$ to $N$-dimensional-subspace equivalent states $|\varphi_1\rangle$ and $|\varphi_\nu\rangle$. Obviously the Schmidt coefficients of states $|\varphi_1\rangle$ and $|\varphi_\nu\rangle$ are greater than zero and $|\varphi_1\rangle \sim |\varphi_\nu\rangle$. We first prove the necessity in the condition that only one side generalized measurement is performed. The one-side generalized measurement acts on the initial states as follows:

$$(P_1 \otimes I) \sum_{i=1}^N \sqrt{\lambda_i} |i_A\rangle |i_B\rangle = \sqrt{\nu} \sum_{i=1}^N \sqrt{\nu_i} |i_A\rangle |i_B\rangle,$$

$$
(P_1 U^\nu_A \otimes U^\nu_B) \sum_{i=1}^N \sqrt{\mu_i} |i_A\rangle |i_B\rangle = \sqrt{\tau_\nu} \sum_{i=1}^N \sqrt{\nu_i} |i_A\rangle |i_B\rangle,
$$

where $E_1 \otimes F_1$ and $E_\nu \otimes F_\nu$ are local unitary operations, $\varsigma$ and $\tau_\nu$ are the probabilities of success, $\kappa = \text{diag} \{ \kappa_1, \kappa_2, ..., \kappa_N \}$ is the eigenvalue matrix of the final states. Since the final states are $N$-dimensional-subspace equivalent, they must be $N$-dimensional entangled states (Fact 2) and $\nu_i > 0$ for $i = 1, 2, ..., N$. According to Fact 3, the above two equations can be represented with matrices as

$$P_1 \sqrt{\nu} = \sqrt{\nu} \nu E_1 \sqrt{\nu} F^+_1,$$

$$P_1 U^\nu_A \sqrt{\mu^\nu} (U^\nu_B)^+ = \sqrt{\nu} \nu E_\nu \sqrt{\nu} F^+_\nu.$$
where \( V_i^A, V_i^B, H_i^A \) and \( H_i^B \) are local unitary operations. The above two equations means that one-side generalized measurement \( A_k \otimes I \) can transform the initial states \( \left( V_i^A B_i \otimes I \right) \sum_{i=1}^{N} \sqrt{\lambda_i} |i_A \rangle |i_B \rangle \) and \( \left( U_A^{\nu} H_i^A B_i U_B^{\nu} \otimes I \right) \sum_{i=1}^{N} \sqrt{\mu_i} |i_A \rangle |i_B \rangle \) to \( N \)-dimensional-subspace equivalent states. Therefore \( \left( B_i \otimes I \right) \sum_{i=1}^{N} \sqrt{\lambda_i} |i_A \rangle |i_B \rangle \) and \( \left( B_i U_B^{\nu} \otimes I \right) \sum_{i=1}^{N} \sqrt{\mu_i} |i_A \rangle |i_B \rangle \) must also be \( N \)-dimensional-subspace equivalent states, which means the marginal density operators for Bob’s side of the input states must satisfy \( \rho_B (|\phi_1\rangle) = \rho_B (|\phi_{\nu}\rangle) \). So one of the two subsystems of the initial states must have same marginal density operators.

So far we have proven Theorem 1 and Theorem 2. In Theorem 1, we give a sufficient condition for that a set of entangled pure states can be probabilistic transformed to \( k \)-dimensional-subspace equivalent states by same LQCC protocol. We conjecture that this condition is also necessary. In fact, it is true if the generalized measurement is restricted in one side. In this case, the eigenvalue matrix of the final states in Eq. (10) is substituted by \( \kappa^{\nu} \). \( \kappa^{\nu} \) should have at least \( k_{\nu} \) \((k_{\nu} \geq k)\) no-zero eigenvalues \( \kappa^{\nu}_i = d_{\nu} \kappa_i, 1 \leq i \leq k_{\nu} \), where \( d_{\nu} \) is a constant dependent on \( \nu \). Eq. (11) should be rewritten as

\[
T^{\nu}_p \kappa T^{\nu}_p = \kappa^{\nu},
\]

where \( T^{\nu}_p \) is the same as that in Eq. (11). Eq. (14) means \( T^{\nu}_p \) can be represented as

\[
T^{\nu}_p = \begin{pmatrix}
\sqrt{d_{\nu}} M^{\nu}_{k_{\nu}} & 0 \\
0 & R_{\nu}
\end{pmatrix},
\]

where \( M^{\nu}_{k_{\nu}} \) is a \( k_{\nu} \times k_{\nu} \) unitary matrix and \( R_{\nu} \) may be any possible matrix. The unitarity of \( M^{\nu}_{k_{\nu}} \) yields

\[
\hat{F}_A (|\phi_1\rangle |\phi_{\nu}\rangle) = \lambda^{-\frac{1}{2}} U^{\nu}_A \mu^{\nu}_i (U^{\nu}_A)^+ \lambda^{-\frac{1}{2}}
\]

\[
= \frac{d_{\nu} T_{\nu}}{\kappa} F_1 \begin{pmatrix} k_{\nu} & 0 \\ 0 & \frac{1}{\kappa} R_{\nu} R^{\nu+}_A \end{pmatrix} F_1^+.\]

Since \( k_{\nu} \geq k \) and \( F_1 \) is independent of index \( \nu \), Eq. (16) means that the relative marginal density operators \( \hat{F}_A (|\phi_1\rangle |\phi_{\nu}\rangle) \) of the initial states \( |\phi_1\rangle \) and \( |\phi_{\nu}\rangle \) are similar about \( I_k \).

Theorem 2 shows that the sufficient condition in Theorem 1 is also necessary in a special case. The result means that Alice (Bob) cannot probabilistically transform \( N \)-dimensional entangled states that are different in her (his) local observation to \( N \)-dimensional-subspace equivalent states. Generally the ordered Schmidt coefficients of the states to be transformed must be the same, but these states need not be the same, there exist unitary transformations on both Alice’s and Bob’s sides. While arbitrary on Bob’s (Alice’s) side, the unitary operators on Alice’s (Bob’s) side must preserve the density matrix \( \rho_A (\rho_B) \), which means that only when there exist some coefficients satisfying \( \lambda_i = \lambda_{i+1} \), the unitary operators \( U^{\nu}_A (U^{\nu}_B) \) can be non-unit.

The above results can be directly applied to concentration of entanglement [6, 8], that is, transforming partial entanglement to maximally entanglement. Theorem 1 gives a sufficient condition for the concentration of a set of partial entanglement to the maximally entanglement (not necessary \( N \)-dimensional), while Theorem 2 shows that Alice (Bob) can probabilistic concentrate several different \( N \)-dimensional partial entangled states to \( N \)-dimensional maximally entangled states by same LQCC process if and only if the marginal density operators of these states are the same for her or his subsystem. In the proof of Theorem 2 we also showed the following important result:

Proposition 1: Different \( N \)-dimensional entangled states cannot be transformed to one \( N \)-dimensional entangled state by same LQCC protocol on individual pairs.

However, such result does not prohibit us from transforming different entangled states to one of lower dimension. An example is the states \( |\alpha\rangle \) and \( |\beta\rangle \) in Eqs. (5).

While one can always, with finite probability, bring an individual entangled pure state to a maximally entangled state using only LQCC [8], Linden et al. [9] have shown that it is impossible to purify a two-level mixed state to a maximally entangled state by any combination of LQCC acting on individual pairs. In this letter we generalize it to \( N \)-level mixed state as

Theorem 3: It is impossible to purify a \( N \)-dimensional mixed state to a \( N \)-dimensional maximally entangled state by LQCC on individual pairs.

Proof of Theorem 3:

Consider a given mixed state \( \rho \), generally we can use the spectral decomposition [25] of the state \( \rho = \sum_i p_i |\psi_i\rangle \langle \psi_i| \). Proposition 1 indicates that different decomposition terms \( |\psi_i\rangle \) of the mixed state \( \rho \) can never be transformed to one \( N \)-dimensional pure state by same LQCC, which means \( \rho \) cannot be concentrated into a \( N \)-dimensional maximally entangled pure state by LQCC on individual pairs. This result is surprising because we
expect entanglement to be a property of each pair individually rather than a global property of many pairs.

However, Theorem 3 does not mean that we cannot obtain lower dimensional entangled pure state from a mixed state by LQCC. For example, we can concentrate the mixed state $\rho = \frac{1}{2} |\alpha\rangle \langle \alpha | + \frac{1}{2} |\beta\rangle \langle \beta |$ to the maximally entangled pure state $|\Upsilon\rangle = \sqrt{\frac{1}{2}} (|11\rangle + |22\rangle)$, where $|\alpha\rangle$ and $|\beta\rangle$ are the states in Eqs. (5).

Another interesting application may be probabilistic quantum superdense coding. Suppose Bob has four choices to perform $U_B$ i.e. $\{I, \sigma_x, i\sigma_y, \sigma_z\}$ on the initial possessed partial entangled states, just like that in Ref. [4]. Alice still can transform the partial entangled state to the maximally entangled state with no-zero probability, although she does not know which $U_B$ Bob performs. Bob sends his particle to Alice after he has performed $U_B$. Alice’s task is then to identify the four Bell states and obtain the information.

The further application of these results need to be explored. Similar to quantum cloning process, although we lack sufficient information about the initial states, we still can make operations on them and extract information at the end. The indefinite initial entanglement may contain quantum information and our results may be useful in quantum cryptography and quantum communication.

In summary, we have shown that a set of entangled pure states $\{|\varphi_1\rangle, |\varphi_\nu\rangle, \nu \in S\}$ can be probabilistic transformed to $k$-dimensional-subspace equivalent states by same LQCC protocol if the relative marginal density operators $F_A(B) (|\varphi_1\rangle \langle \varphi_1|)$ are similar about $I_k$. In the case of that the final states are $N$-dimensional-subspace equivalent, the condition can be expressed as that the input states must share the same marginal density operators for Alice’s or Bob’s subsystem and it is both sufficient and necessary. As the application, we showed that it is impossible to purify a mixed state to a maximally entangled state of same dimension by LQCC on individual pairs and presented the probabilistic superdense coding.

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[26] Suppose $P = \sum_{i,j} p_{ij} |i\rangle \langle j|$, then $(P \otimes I) \sum_{k=1}^{N} |k\rangle_A |k\rangle_B = \sum_{i,k} p_{ik} |i\rangle_A |k\rangle_B = \sum_{i,k} p_{ki} |k\rangle_A |i\rangle_B = (I \otimes P^+) \sum_{k=1}^{N} |k\rangle_A |k\rangle_B$. 

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