AUTOMORPHISMS OF $\mathbb{C}^2$ WITH NON-RECURRENT SIEGEL CYLINDERS

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Abstract. A non-recurrent Siegel cylinder is an invariant, non-recurrent Fatou component $\Omega$ of an automorphism $F$ of $\mathbb{C}^2$ satisfying: (1) The closure of the $\omega$-limit set of $F$ on $\Omega$ contains an isolated fixed point, (2) there exists a univalent map $\Phi$ from $\Omega$ into $\mathbb{C}^2$ conjugating $F$ to the translation $(z, w) \mapsto (z + 1, w)$, and (3) every limit map of $\{F^n\}$ on $\Omega$ has one-dimensional image. In this paper we prove that the existence of non-recurrent Siegel cylinders. In fact, we provide an explicit class of maps having such Fatou components, and show that examples in this class can be constructed as compositions of shears and overshears.

1. Introduction

Let $F$ be a holomorphic self-map of a complex manifold $X$. The Fatou set of $F$ is the set of points $p \in X$ for which there exists an open neighborhood $U \ni p$ such that the sequence of iterates $\{F^n\}$ form a normal family on $U$. The connected components of the Fatou set of $F$ are called Fatou components of $F$. A Fatou component $\Omega \subset X$ of $F$ is invariant if $F(\Omega) = \Omega$. Following Bedford-Smillie [5], an invariant Fatou component is called recurrent if it contains an orbit which is relatively compact in $\Omega$, and non-recurrent otherwise.

For rational functions in one complex variable there is a complete description of all possible Fatou components and the dynamics of the map on such components is quite well understood. In particular, the invariant non-recurrent Fatou components are “Leau-Fatou petals” at a parabolic fixed point. All orbits in such petals converge to the fixed point and on such petals the map is conjugated to a translation via the so-called “Fatou coordinate”.

2000 Mathematics Subject Classification. 32H02, 32H50, 37F50, 37F99.

Key words and phrases. holomorphic dynamics; local dynamics; automorphisms; Fatou components.

† Supported by the SIR grant “NEWHOLITE - New methods in holomorphic iteration” no. RBSI14CFME and by the research program P1-0291 from ARRS, Republic of Slovenia.

†† Partially supported by the MIUR Excellence Department Project awarded to the Department of Mathematics, University of Rome Tor Vergata, CUP E83C18000100006 and PRIN Real and Complex Manifolds: Topology, Geometry and holomorphic dynamics n.2017JZ2SW5.
Despite significant recent progress, including the construction of wandering domains \[2\] and the classification of invariant Fatou components \[3, 9\], the situation is not nearly as well understood in \(\mathbb{C}^2\).

Let \(F\) be an automorphism of \(\mathbb{C}^2\). If \(\Omega \subset \mathbb{C}^2\) is a Fatou component of \(F\), we say that a holomorphic map \(h : \Omega \to \mathbb{C}^2 \cup \{\infty\}\) is a limit map of \(F\) on \(\Omega\) if there exists a sequence \(\{F^{\circ n_k}\}\) which converges uniformly on compacta of \(\Omega\) to \(h\) — here, for the sake of uniformizing notation, we let \(h \equiv \infty\) in case \(\{F^{\circ n_k}\}\) compactly diverges to \(\infty\).

If \(F\) is a polynomial automorphism, the Jacobian determinant \(\delta\) is necessarily constant and different from 0. When \(|\delta| = 1\) all Fatou components \(\Omega\) of \(F\) are recurrent, the so-called Siegel domains, and \(h(\Omega) = \Omega\) for any limit map \(h\). This does not complete the description, as it remains an open question whether \(\Omega\) must be topologically trivial, see for example \[4\].

In the case \(|\delta| < 1\) the orbits in a recurrent Fatou component converge exponentially fast to either an attracting fixed point or to a 1-dimensional properly embedded Riemann surface \(\Sigma \subset \Omega\), see \[5\]. In the latter case, which could be called a recurrent Siegel cylinder, the action of \(f\) on the invariant set \(\Sigma\) is that of an irrational rotation, and \(\Sigma\) is equivalent to either the disk or an annulus. Whether an annulus can actually occur is a pressing open question.

The non-recurrent case has been described in \[9\], under the additional assumption \(|\delta| < \frac{1}{\deg(f)}\). In this case all orbits converge to a parabolic-attracting fixed point, and the component is biholomorphic to \(\mathbb{C}^2\), by a result of Ueda \[15\], and \(F\) is conjugate on \(\Omega\) to a map \((z, w) \mapsto (z + 1, w)\).

Very little is known about which other phenomena can occur when considering non-polynomial automorphisms of \(\mathbb{C}^2\). An invariant Fatou component is attracting if all the orbits in the component converge to the same (necessarily fixed) point \(p \in \mathbb{C}^2\). By \[10, 13\], a recurrent attracting Fatou component \(\Omega\) is necessarily biholomorphic to \(\mathbb{C}^2\), the spectrum of \(dF_p\) is contained in the (open) unit disk and \(F\) is conjugate to a polynomial triangular map on \(\Omega\).

In \[7\] (see also \[11\] for the construction of several “petals”) the authors constructed an attracting non-recurrent Fatou component biholomorphic to \(\mathbb{C} \times \mathbb{C}^*\), where the map is semi-conjugated to a translation over \(\mathbb{C}\). It is an open question if these two possibilities exhaust all possible cases of attracting invariant Fatou components for automorphisms of \(\mathbb{C}^2\).

In \[8\], Jupiter and Lilov considered invariant non-recurrent Fatou components of automorphisms of \(\mathbb{C}^2\). They showed that when \(\Omega\) is an invariant non-recurrent non-attracting Fatou component of \(F\) such that all limit maps of \(F\) on \(\Omega\) are constants, the union of the images of the limit maps is uncountable, has no isolated points and it is contained in a one dimensional subvariety of fixed points of \(F\). Moreover, on each such a point, the
spectrum of $dF$ is $\{1, e^{2\pi i \theta}\}$ with $\theta \in \mathbb{R}$ not diophantine. Whether such invariant non-recurrent non-attracting Fatou component with constant limit maps can actually exist is presently unknown.

Jupiter and Lilov also proved that if an invariant non-recurrent Fatou component has more than one limit map with one dimensional image, then the intersection of the images is either open in both images or empty. To the best of our knowledge it is not known whether invariant non-recurrent Fatou component with two limit maps with disjoint (or even merely distinct) one dimensional images exist. It is also an open question whether there exist non-recurrent invariant Fatou components having at the same time both 0- and 1-dimensional limit maps.

In the same paper (see [8, Section 5.2.2]), Jupiter and Lilov produce examples of invariant non-recurrent Fatou component with one-dimensional limit maps.

As a matter of notation, in this paper, we write $g(z, w) = O(f(z, w))$ if there exists a constant $C > 0$ such that $|g(z, w)| \leq C|f(z, w)|$ in a neighborhood of the origin.

Jupiter and Lilov prove that if $F$ is an automorphism of $\mathbb{C}^2$ of the form

$\begin{align*}
F(z, w) &= (z + z^2 + O(z^3, z^4w, z^6w^2), w - \frac{z^2w}{2} + O(z^3, z^3w, z^3w^2)),
\end{align*}$

(and such automorphisms exist as finite composition of shears and overshears), then there exists an invariant, non-recurrent, Fatou-component $\Omega$, having the $w$-axis on the boundary such that $\{F^n\}$ converges uniformly on compacta of $\Omega$ to a surjective holomorphic map $h : \Omega \to \{0\} \times \mathbb{C}$. Post-composing the previous $F$ with the automorphism $(z, e^{2\pi i \theta}w)$, with $\theta \in \mathbb{R} \setminus \mathbb{Q}$, one can produce an example of an automorphism of $\mathbb{C}^2$ with an invariant, non-recurrent Fatou component with an isolated fixed point of $F$ on the boundary and having different one-dimensional limit maps (but all with the same image $\{0\} \times \mathbb{C}$).

The aim of this paper is to better understand the dynamics of invariant, non-recurrent Fatou components having one isolated fixed point on the boundary and one-dimensional limit maps. In particular, as a consequence of our result, we show that Jupiter and Lilov’s example gives rise to a “Siegel cylinder” where the dynamics of the map is very well understood.

In order to properly set our result, we need a definition. We recall that a point $p \in \mathbb{C}^2$ is said to lie in the $\omega$-limit set of $F$ on $\Omega$ if there exist $z \in \Omega$ and a subsequence $\{n_k\}$ converging to $\infty$ such that $\lim_{k \to \infty} F^{n_k}(z) = p$.

**Definition 1.1.** Let $F$ be an automorphism of $\mathbb{C}^2$. An invariant non-recurrent Fatou component $\Omega$ is called a non-recurrent Siegel cylinder if

1. the closure of the $\omega$-limit set of $F$ on $\Omega$ contains an isolated fixed point,
2. there exists a univalent map $\Phi : \Omega \to \mathbb{C}^2$, conjugating $F$ to the translation $(z, w) \mapsto (z + 1, w),$
3. all limit maps of $F$ on $\Omega$ have dimension one.
The term Siegel cylinder suggests an irrational rotation, hence the choice of the normal form \((z, w) \mapsto (z + 1, w)\) may seem surprising. However, note that if \(\tilde{\Phi} : \Omega \to \Phi(\Omega)\) is a biholomorphism satisfying
\[
(\tilde{\Phi} \circ F \circ \tilde{\Phi})(z, w) = T(z, w) := (z + 1, \lambda w)
\]
for some \(\lambda \in \mathbb{C} \setminus \{0\}\), then the overshear \(\Theta(z, w) := (z, \lambda^{-z} w)\), determined by any fixed choice of \(\log(\lambda)\), is a biholomorphism from \(\Phi(\Omega)\) onto its image, and
\[
(\Theta \circ T \circ \Theta^{-1})(z, w) = (z + 1, w).
\]
Hence from an “intrinsic” point of view, rotations in the second coordinates are not seen, and the normal form in our definition of a non-recurrent Siegel cylinder contains no rotational term.

The main result of this paper is the following:

**Theorem 1.2.** Let \(F\) be an automorphism of \(\mathbb{C}^2\) of the form
\[
F(z, w) = (z + f(w)z^2 + O(z^3, z^2w), e^{2\pi i \theta} w + g(w)z + O(z^2, z^2w)),
\]
where \(\theta \in \mathbb{R} \setminus \mathbb{Q}\) is Diophantine, \(f(0) \neq 0\), and \(g(w) = O(w^2)\). Then there exists a non-recurrent Siegel cylinder \(\Omega\) for \(F\) biholomorphic to \(\mathbb{C}^2\). Moreover, every limit map of \(F\) on \(\Omega\) has image \(\{0\} \times \mathbb{C}\).

In the last section we show that maps \(F\) satisfying the constraints in the theorem can actually be constructed as finite compositions of shears and overshears. Recall that such compositions form a dense subset of all automorphisms of \(\mathbb{C}^2\) in the compact-open topology \([1]\). An immediate consequence of the above theorem, which can also be observed directly, is that \(F\) cannot have constant Jacobian determinant, and therefore cannot be approximated by compositions of shears only.

We note that the terms \(g(w)z\) in the second coordinate of (1.1), which do not occur in the examples by Jupiter and Lilov, significantly complicate the analysis. A consequence of the parabolic behavior in the \(z\)-coordinate and rotational behavior in the \(w\)-coordinate is that the terms \(g(w)z\) are not absolutely summable. Controlling the behavior caused by these terms is an important step in our proof.

In a forthcoming paper J. Reppekus \([12]\), exploiting the example in \([7]\) and blowing-up, shows that there exist non-recurrent Siegel cylinders biholomorphic to \(\mathbb{C} \times \mathbb{C}^*\).

The previous discussion and results give rise to the following open questions:

(a) Let \(\Omega\) be an invariant non-recurrent Fatou component of an automorphism \(F\) of \(\mathbb{C}^2\) having an isolated fixed point on the boundary which is in the closure of the \(\omega\)-limit set of \(F\) in \(\Omega\). Is \(\Omega\) biholomorphic to \(\mathbb{C}^2\) or \(\mathbb{C} \times \mathbb{C}^*\)?

(b) In case \(\Omega\) is non-attracting, is \(\Omega\) necessarily a non-recurrent Siegel cylinder?

Note that, by Jupiter and Lilov, in the hypothesis of the previous question, if \(\Omega\) is non-attracting, there necessarily exists a one-dimensional limit map.
The outline of the paper is the following. In section 2 we introduce coordinate changes to be used in section 3, where it is shown in Proposition 3.4 that the map $F$ can be locally conjugated to a map $H(u, w) = (u + 1 + \frac{4}{u}, \lambda w) + O(\frac{1}{u^2})$. In these coordinates it follows that $F$ has a Fatou component on which all orbits converge to \{ $z = 0$ \}. Section 3 concludes by showing that the limit map $h : \Omega \to \{ z = 0 \}$ is surjective, and that the image \{ $z = 0$ \} lies in $\partial \Omega$, showing that $\Omega$ is non-recurrent.

In section 4 it is shown that $\Omega$ is biholomorphically equivalent to $\mathbb{C}^2$, and that the map $F : \Omega \to \Omega$ is holomorphically conjugate to the linear normal form. The paper ends with a short construction in section 5 of an explicit automorphism $F$ of the required form ($\mathbb{I}$).

2. Preliminaries

Throughout this paper we will use $\lambda = e^{2\pi i \theta}$ where $\theta \in \mathbb{R} \setminus \mathbb{Q}$ is diophantine, i.e. there exist $c, r > 0$ such that $|\lambda^n - 1| \geq cn^{-r}$ for every $n \geq 1$. Such numbers form a dense subset of the unit circle with full measure. Note that if $\lambda$ is diophantine then $\lambda^{-1}$ is also diophantine and satisfies the same estimates.

We will be using the following notation. Let $u(x)$ and $v(x)$ be two functions. By writing $u(x) = O(v(x))$ we mean that there exist a constant $C > 0$ such that $|u(x)| \leq C|v(x)|$ for all $x$ in a neighborhood of the origin where $u$ and $v$ are defined. The notation $u(x) = o(v(x))$ as $x \to a$ means that $u(x)/v(x) \to 0$ as $x \to a$. In case of a sequence $u_n$ of complex numbers, the notation $u_n = O(v(n))$ has to be understood as $|u_n| \leq C|v(n)|$ for all $n \in \mathbb{N}$.

The following was used in [3], we repeat the proof for convenience of the reader.

**Lemma 2.1.** There exist constants $C, r > 0$ such that for every integer $n \geq 1$ and for every $m \geq 0$,

$$\left| \sum_{j=m}^{\infty} \lambda^n j \right| < C n^r.$$ 

**Proof.** Let $N \geq m$. Since $\lambda = e^{2\pi i \theta}$ where $\theta \in \mathbb{R} \setminus \mathbb{Q}$ is diophantine, there exist $c, r > 0$ such that $|\lambda^n - 1| \geq cn^{-r}$ for all $n$. This gives the bound

$$\left| \sum_{j=m}^{N} \lambda^n j \right| = \left| \sum_{j=m}^{N} \frac{\lambda^{n(j+1)} - \lambda^n j}{\lambda^n - 1} \right| = \left| \frac{1}{\lambda^n - 1} \sum_{j=m}^{N} (\lambda^{n(j+1)} - \lambda^n j) \right| < \left| \frac{2}{\lambda^n - 1} \right| < C n^r,$$

and we are done. \qed

For $R \in \mathbb{R}$ and $\delta > 0$ we let

$$K_R := \{ u \in \mathbb{C} \mid \arg(u - R) \in [-3\pi/4, 3\pi/4] \},$$

and

$$U_{R, \delta} := \{ (u, w) \in \mathbb{C}^2 \mid u \in K_R \text{ and } |w| < \delta \}.$$
Note that $U_{R_1,\delta_1} \subset U_{R_2,\delta_2}$ when $R_2 \leq R_1$ and $\delta_1 \leq \delta_2$.

**Lemma 2.2.** Let $R > 0$. There exist $r > 0$ and $\tilde{C} > 0$ such that for every integer $n \geq 1, m \geq 0$ and $u \in K_R$,

\[
\left| \sum_{j=m}^{\infty} \frac{\lambda^{nj}}{u+j} \right| < \frac{\tilde{C}n^r}{|u+m|}.
\]

In particular, the series $\sum_{j=0}^{\infty} \frac{\lambda^{nj}}{u+j}$ is converging uniformly on compacta of $K_R$ for every $n \geq 1$.

**Proof.** Let $N \geq m$. Lemma 2.1 gives

\[
\left| \sum_{j=m}^{N} \frac{\lambda^{nj}}{u+j} \right| = \left| \frac{1}{u+N} \sum_{j=m}^{N} \lambda^{nj} - \sum_{j=m}^{N-1} \left( \frac{1}{u+j+1} - \frac{1}{u+j} \right) \sum_{k=m}^{j} \lambda^{nk} \right|
\]

\[
< \frac{Cn^r}{|u+N|} + Cn^r \cdot \sum_{j=m}^{N-1} \frac{1}{(u+j+1)(u+j)} < \frac{\tilde{C}n^r}{|u+m|},
\]

with the constant $\tilde{C}$ chosen to be independent from $N$ and $u$, and we are done. \(\square\)

**Lemma 2.3.** Let $g : \mathbb{C} \to \mathbb{C}$ be an entire function such that $g(0) = g'(0) = 0$. Let $g(w) = \sum_{\ell=2}^{\infty} d_{\ell} w^{\ell}$ be its expansion at 0. Then for every $\delta > 0$, there exists $R = R(\delta) > 0$, which depends continuously on $\delta$, such that the map

\[
\Phi(u, w) := \left( u, w + \lambda^{-1} \sum_{\ell=2}^{\infty} \left( d_{\ell} w^{\ell} \sum_{k=0}^{\infty} \frac{\lambda^{(\ell-1)k}}{u+k} \right) \right)
\]

is univalent on $U_{R,\delta}$. Moreover $\Phi(u, w) = (u, w + O \left( \frac{1}{u} \right))$ and for every $\delta > 0$ and $R > 0$ there exists $R' \geq R$ such that $\Phi$ is univalent on $U_{R',\delta}$ and $\Phi(U_{R',\delta}) \subset U_{R,2\delta}$. Also, for every $\delta > 0$ there exists $R'' \geq R(\delta)$ such that $U_{R'',\delta/2} \subset \Phi(U_{R(\delta),\delta})$.

**Proof.** Let $\varepsilon > 0$. By Lemma 2.2, the map $\Phi(u, w)$ is a well defined holomorphic map on $\{ (u, w) \in \mathbb{C}^2 \mid u \in K_{\varepsilon} \}$ and $\Phi(u, w) = \left( u, w + O \left( \frac{1}{u} \right) \right)$. In order to check injectivity, first observe that $\Phi(u, w) = \Phi(w', w')$ implies $u = u'$. Therefore $\Phi(u, w) = \Phi(u', w')$ if and only if

\[
w - w' + \lambda^{-1} \sum_{\ell=2}^{\infty} d_{\ell}(w^{\ell} - w'^{\ell}) \sum_{k=0}^{\infty} \frac{\lambda^{(\ell-1)k}}{u+k} = 0.
\]

Assuming that $w \neq w'$ we can divide this equation by $w - w'$ to obtain

\[
1 + \lambda^{-1} \sum_{\ell=2}^{\infty} d_{\ell} \left( \frac{w^{\ell} - w'^{\ell}}{w - w'} \right) \sum_{k=0}^{\infty} \frac{\lambda^{(\ell-1)k}}{u+k} = 0.
\]
Since $\left| \frac{\ell^t}{w^{\ell}} - \frac{w^{\ell}}{w} \right| \leq \ell \delta^{\ell-1}$, taking into account Lemma 2.2 and that $g$ is entire, we can choose $R$ large enough so that

$$\sum_{\ell=2}^{\infty} d_\ell \left( \frac{w^{\ell}}{w} - \frac{w^{\ell}}{w'} \right) \sum_{k=0}^{\infty} \frac{\lambda^{\ell-1}k}{u+k} < \frac{C}{|u|} \sum_{\ell=2}^{\infty} |d_\ell| \delta^{\ell-1} \ell (\ell - 1)^r < 1$$

everywhere on $U_{R,\delta}$, and hence $\Phi(u, w) = \Phi(u', w')$ implies $(u, w) = (u', w')$. This last inequality also implies that $R$ depends continuously on $\delta$.

In order to prove the final statement, let

$$h(u, w) := \lambda^{-1} \sum_{\ell=2}^{\infty} \left( d_\ell w^{\ell} \sum_{k=0}^{\infty} \frac{\lambda^{\ell-1}k}{u+k} \right) = O \left( \frac{1}{u} \right),$$

and let $R'' \geq R(\delta)$ be such that $|h(u, w)| < \frac{\delta}{2}$ for $u \in K_{R''}$ and $|w| \leq \delta$. Let $u_0 \in K_{R''}$ and let $|w_0| \leq \delta/2$. Let $C := \{ w \in \mathbb{C} : |w - w_0| = \delta/2 \}$. By the triangular inequality $|w| \leq \delta$ for all $w \in C$. Thus, for all $w \in C$,

$$|w - w_0| = \frac{\delta}{2} > |h(u_0, w)|.$$

Hence, by the Rouché theorem, the functions $w \mapsto w - w_0$ and $w \mapsto w + h(u_0, w) - w_0$ have the same number of zeros in $\{ w \in \mathbb{C} : |w - w_0| < \delta/2 \}$. In particular, there exists $w_1 \in \mathbb{C}$ such that $|w_1| < \delta$ and $w_1 + h(u_0, w_1) = w_0$. Therefore, $(u_0, w_0) \in \Phi(U_{R(n), \delta})$. By the arbitrariness of $(u_0, w_0)$, this proves that $U_{R''(\delta)/2} \subset \Phi(U_{R(\delta), \delta})$.

**Lemma 2.4.** Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that $f(0) = 1$. Let $f(w) = 1 + \sum_{\ell=1}^{\infty} d_\ell w^{\ell}$ be its expansion at 0. The map $\Psi : \mathbb{C}^2 \to \mathbb{C}^2$ defined as

$$\Psi(u, w) := \left( u + \sum_{\ell=1}^{\infty} \frac{d_\ell}{\lambda^{\ell-1}} w^{\ell}, w \right)$$

is a holomorphic automorphism of $\mathbb{C}^2$. Moreover, for every $\delta > 0$ there exists $C_1 > 0$ such that $\|\Psi(u, w) - (u, w)\| < C_1$ for all $(u, w) \in \mathbb{C} \times \{ w \in \mathbb{C} : |w| \leq \delta \}$. In particular, for every $\delta > 0$ there exists $M_\delta > 0$ such that for all $R \geq M_\delta$

$$\Psi(U_{2R, \delta}) \subset U_{R, \delta}.$$

**Proof.** Clearly the series $\sum_{\ell=1}^{\infty} d_\ell w^{\ell}$ is absolutely convergent on compacta of $\mathbb{C}$. Recall that by our assumption $\lambda$ satisfies the condition $|\lambda^n - 1| > cn^{-r}$, for some $c, r > 0$. It follows that

$$\sum_{\ell=1}^{\infty} \frac{d_\ell}{\lambda^{\ell-1}} w^{\ell} < C \sum_{\ell=1}^{\infty} |d_\ell| |w|^{\ell}.$$
From this last inequality we can deduce that the series which appears in the first coordinate of the map Ψ is absolutely convergent on compacta in $\mathbb{C}$. Therefore Ψ is an automorphism of $\mathbb{C}^2$.

The last statement follows at once setting $C_1 := C \sum_{\ell=1}^{\infty} |d_\ell| \ell r^\ell$.

**Lemma 2.5.** Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function with $f(0) = 0$. Let $f(w) = \sum_{\ell=1}^{\infty} d_\ell w^\ell$ be its expansion at 0. Then for every $\delta > 0$, there exists $R > 0$ such that the map

$$
\tau(u, w) := \left( u - \sum_{k=0}^{\infty} \frac{f(\lambda^k w)}{u + k}, w \right)
$$

is univalent on $U_{R, \delta}$. Moreover $R$ depends continuously on $\delta$ and $\tau(U_{R, \delta}) \subset U_{R, \delta}$. Also, for every $\delta > 0$ there exists $R' \geq R(\delta)$ such that $U_{R', \delta} \subset \tau(U_{R(\delta), \delta})$.

**Proof.** We first prove that the map is well defined, i.e., we prove that for every $\varepsilon > 0$ the series

$$
\sum_{k=0}^{\infty} \frac{f(\lambda^k w)}{u + k}
$$

converges uniformly on compacta of $\{(u, w) \in \mathbb{C}^2 \mid u \in K_\varepsilon\}$. By Lemma 2.3,

$$
\left| \sum_{k=0}^{N} \frac{f(\lambda^k w)}{u + k} \right| \leq \sum_{k=0}^{N} \left| \sum_{j=1}^{\infty} \frac{\lambda^k d_j w^j}{u + k} \right| = \sum_{j=1}^{\infty} d_j w^j \sum_{k=0}^{N} \frac{\lambda^k}{u + k} < \frac{C}{|u|} \sum_{j=1}^{\infty} |d_j||w|^j j^r.
$$

Since $f$ is entire, the last series converges uniformly on compacta, and so does the series (2.2).

In order prove injectivity, we first observe that $\tau(u, w) = \tau(u', w')$ implies that $w = w'$. If $u \neq u'$ then $\tau(u, w) = \tau(u', w')$ if and only if

$$
u - u' - \sum_{k=0}^{\infty} \frac{f(\lambda^k w)}{u + k} - \frac{f(\lambda^k w)}{u' + k} = 0.
$$

Dividing this equation by $u - u'$ we obtain

$$
1 + \sum_{k=0}^{\infty} \frac{f(\lambda^k w)}{(u + k)(u' + k)} = 0.
$$

Given $\delta > 0$, we can find $R$ large enough such that for all $u, u' \in K_R$ we have

$$
\sum_{k=0}^{\infty} \frac{1}{|(u + k)(u' + k)|} < \frac{1}{\sup_{|w| < \delta} |f|}.
$$
Therefore, (2.3) cannot be satisfied in $U_{R,\delta}$, and hence $\tau$ is injective in $U_{R,\delta}$.
By the previous considerations it follows also that $\tau(u, w) = \left(u + O\left(\frac{1}{u}\right), w\right)$.
The last statement follows by applying Rouché’s theorem as in Lemma 2.3.

3. Non-recurrent Fatou component

Let $F$ be a holomorphic automorphism of $\mathbb{C}^2$ of the form

$$F(z, w) = \left(z + f(w)z^2 + O(z^3), \lambda w + g(w)z + O(z^2)\right),$$

where $f$ and $g$ are entire functions in $\mathbb{C}$, $f(0) \neq 0$ and $g(w) = O(w^2)$. Notice that the inverse $F^{-1}$ has the same form as $F$. From now on we assume without loss of generality that $f(0) = 1$, since otherwise we can simply conjugate $F$ with a dilatation in the first factor.

Let $\delta' > 0$ and let $R' > 0$. We define the holomorphic map $\Theta : U_{R',\delta'} \to \mathbb{C}^2$ as

$$\Theta(u, w) := \left(-\frac{1}{u}, w\right).$$

Note that $\Theta$ is univalent and hence a biholomorphism onto its image.

Looking at (3.1), it is clear that there exists $\tilde{R}' = R'_0(\delta') > 0$ such that

$$\tilde{F}(u, w) := (\Theta^{-1} \circ F \circ \Theta)(u, w) = \left(u + f(w) + O\left(\frac{1}{u}\right), \lambda w - \frac{g(w)}{u} + O\left(\frac{1}{u^2}\right)\right)$$

is well defined and univalent on $U_{R',\delta'}$ for all $R' \geq R_0$. Moreover, given $R'' > 0$, we can find $R' \geq R'_0$ such that $\tilde{F}(U_{R',\delta'}) \subset U_{R'',2\delta'}$.

Let $g(w) = \sum_{\ell=2}^{\infty} d_{\ell}w^{\ell}$ be the expansion of $g$ at 0 and let $\Phi$ be as in Lemma 2.3.

Fix $\delta > 0$. By Lemma 2.3 there exists $R'' > 0$ such that $\Phi^{-1}$ is well defined and univalent on $U_{R'',4\delta}$. By the previous considerations, there exists $R' \geq R'_0(\delta)$ such that $\tilde{F}(U_{R',2\delta}) \subset U_{R',4\delta}$ and finally, by Lemma 2.3 there exists $R > 0$ such that $\Phi$ is univalent on $U_{R,\delta}$ and $\Phi(U_{R,\delta}) \subset U_{R,2\delta}$. Thus,

$$G := \Phi^{-1} \circ \tilde{F} \circ \Phi$$

is well defined and univalent on $U_{R,\delta}$.

Lemma 3.1. For $(u, w) \in U_{R,\delta}$, we have

$$G(u, w) = \left(u + f(w) + O\left(\frac{1}{u}\right), \lambda w + O\left(\frac{1}{u^2}\right)\right).$$

Proof. Let us write $(u_1, w_1) := G(u, w)$. First observe that

$$(\tilde{F} \circ \Phi)(u, w) = \left(u + f(w) + O\left(\frac{1}{u}\right), \lambda w + \sum_{\ell=2}^{\infty} d_{\ell}w^{\ell} \sum_{k=0}^{\infty} \frac{\lambda^{(\ell-1)k}}{u + k} - \frac{1}{u} \sum_{\ell=2}^{\infty} d_{\ell}w^{\ell} + O\left(\frac{1}{u^2}\right)\right).$$
Since
\[ \Phi^{-1}(u, w) = \left( u, w - \lambda^{-1} \sum_{\ell=2}^{\infty} d_{\ell} w^{\ell} \sum_{k=0}^{\infty} \frac{\lambda^{(\ell-1)k}}{u + k} + O\left( \frac{1}{u^2} \right) \right), \]
it follows that
\[ u_1 = u + f(w) + O\left( \frac{1}{u} \right) \]
\[ w_1 = \lambda w + \sum_{\ell=2}^{\infty} d_{\ell} w^{\ell} \left( -\frac{1}{u} + \sum_{k=0}^{\infty} \frac{\lambda^{(\ell-1)k}}{u + k} - \sum_{k=1}^{\infty} \frac{\lambda^{(\ell-1)k}}{u + k + (f(w) - 1) + O\left( \frac{1}{u} \right)} \right) + O\left( \frac{1}{u^2} \right) \]

\[ = \lambda w + \sum_{\ell=2}^{\infty} d_{\ell} w^{\ell} \sum_{k=1}^{\infty} \frac{\lambda^{(\ell-1)k} (f(w) - 1) + O\left( \frac{1}{u} \right)}{(u + k) (u + k + (f(w) - 1) + O\left( \frac{1}{u} \right))} + O\left( \frac{1}{u^2} \right) \]

\[ = \lambda w + (f(w) - 1) \sum_{\ell=2}^{\infty} d_{\ell} w^{\ell} \sum_{k=1}^{\infty} \frac{\lambda^{(\ell-1)k}}{(u + k) (u + k + (f(w) - 1) + O\left( \frac{1}{u} \right))} + O\left( \frac{1}{u^2} \right) \]

\[ = \lambda w + O\left( \frac{1}{u^2} \right). \]

The last equality follows from the fact that
\[ \left| \sum_{k=1}^{\infty} \frac{\lambda^{(\ell-1)k}}{(u + k) (u + k + (f(w) - 1) + O\left( \frac{1}{u} \right))} \right| < C(\ell - 1)^{\rho} \]
on \( U_{R,\delta} \) for some \( C > 0 \), which follows similarly as Lemma 2.2.

Let us write \( f(z) = 1 + \sum_{k=1}^{\infty} d_k w^k \) and let
\[ \Psi(u, w) = \left( u + \sum_{k=1}^{\infty} \frac{d_k}{\lambda^{k-1}} w^k, w \right) \]
be the map defined in Lemma 2.4. By Lemma 2.4 there exists \( C_1 > 0 \) such that
\[ \left| \sum_{k=1}^{\infty} \frac{d_k}{\lambda^{k-1}} w^k \right| < C_1 \]
for all \( |w| \leq \delta \).

In particular, there exists \( R' > R \) with \( \Psi(U_{R',\delta}) \subset U_{R,\delta} \) for which
\[ \tilde{G} := \Psi^{-1} \circ G \circ \Psi \]
is a well defined univalent map on \( U_{R',\delta} \).

In order to avoid burdening notations, we still denote \( R' \) by \( R \), so that \( \tilde{G} \) is univalent on \( U_{R,\delta} \).
Lemma 3.2. Given $\delta > 0$, we can choose $R$ sufficiently large so that there exists $h : \{w \in \mathbb{C} : |w| < \delta\} \to \mathbb{C}$ holomorphic such that for $(u, w) \in U_{R, \delta}$,

\begin{equation}
\tilde{G}(u, w) = \left( u + 1 + \frac{h(w)}{u}, \lambda w \right) + O\left( \frac{1}{u^2} \right).
\end{equation}

Namely, there exist holomorphic maps $A, B : U_{R, \delta} \to \mathbb{C}$ and a constant $K > 0$ such that for all $(u, w) \in U_{R, \delta}$,

\[ u' = \lambda w + B(u, w) = \lambda w + O\left( \frac{1}{u^2} \right). \]

Proof. First of all, note that, by Lemma 2.4, if $g(u)$ is a holomorphic function on $U_{R, \delta}$ with $g(u) = O(1/u^k)$ for some natural number $k \geq 1$, then $g(\Psi(u, w)) = O(1/u^k)$.

Taking this into account and writing $(u', w') = \tilde{G}(u, w)$, by Lemma 3.1 we have

\[ w' = \lambda w + B(u, w) = \lambda w + O\left( \frac{1}{u^2} \right). \]

Next, again by Lemma 3.1 we can write

\[ G(u, w) = \left( u + f(w) + \frac{h(w)}{u} + O\left( \frac{1}{u^2} \right), \lambda w + O\left( \frac{1}{u^2} \right) \right), \]

for some holomorphic function $h : \{w \in \mathbb{C} : |w| < \delta\} \to \mathbb{C}$. Again by the same token as before, we have

\[ u' = u + \sum_{k=1}^{\infty} \frac{d_k}{\lambda^k - 1} w^k + f(w) - \sum_{k=1}^{\infty} \frac{d_k}{\lambda^k - 1} (\lambda w + B_n(u, w))^k \]

\[ + \frac{h(w)}{u + \sum_{k=1}^{\infty} \frac{d_k}{\lambda^k - 1} w^k} + O\left( \frac{1}{u^2} \right). \]

By Lemma 2.4 the function $q(w) := \sum_{k=1}^{\infty} \frac{d_k}{\lambda^k - 1} w^k$ is an entire function, therefore we can choose $R$ so large that $|q(w)| < R$ for all $|w| \leq \delta$. Thus,

\[ \frac{h(w)}{u + q(w)} = \frac{h(w)}{u} \frac{1}{1 + \frac{1}{u} q(w)} = \frac{h(w)}{u} + O\left( \frac{1}{u^2} \right), \]
Hence,
\[
\begin{aligned}
u' &= u + \frac{h(w)}{u} + \sum_{k=1}^{\infty} \frac{d_k}{\lambda^k - 1} w^k + f(w) - \sum_{k=1}^{\infty} \frac{d_k}{\lambda^k - 1} (\lambda w + B(u, w))^k + O\left(\frac{1}{u^2}\right) \\
&= u + 1 + \frac{h(w)}{u} + \sum_{k=1}^{\infty} \frac{d_k}{\lambda^k - 1} w^k + \sum_{k=1}^{\infty} d_k w^k - \sum_{k=1}^{\infty} \frac{d_k}{\lambda^k - 1} (\lambda w + B(u, w))^k + O\left(\frac{1}{u^2}\right) \\
&= u + 1 + \frac{h(w)}{u} + \sum_{k=1}^{\infty} \frac{d_k}{\lambda^k - 1} (\lambda^k w^k - (\lambda w + B(u, w))^k) + O\left(\frac{1}{u^2}\right) \\
&= u + 1 + \frac{h(w)}{u} + O\left(\frac{1}{u^2}\right),
\end{aligned}
\]
where we used the fact that \(B(u, w) = O\left(\frac{1}{u^2}\right)\).
\[\square\]

Remark 3.3. It follows from (3.3) that for any \(R'' > 0\) there exists \(R' \geq R\) such that \(\tilde{G}(U_{R', \delta}) \subset U_{R'', 2\delta}\).

Taking into account (3.3), let us denote \(A := h(0)\). Let
\[
(3.4) \quad \tau(u, w) = \left(u - \sum_{k=0}^{\infty} \frac{h(\lambda^k w) - A}{u + k}, w\right).
\]

By Lemma 2.5, there exists \(R'' > 0\) such that \(\tau^{-1}\) is well defined on \(U_{R'', 2\delta}\). By Remark 3.3 we can choose \(R' \geq R\) such that \(\tilde{G}(U_{R', \delta}) \subset U_{R'', 2\delta}\). Finally, we can choose \(R_1 > 0\) such that \(\tau\) is well defined and univalent on \(E_{R_1, \delta}\) and \(\tau(U_{R_1, \delta}) \subset U_{R', \delta}\).

Once again, in order to simplify notation, we will write \(R\) instead of \(R_1\), so that
\[
(3.5) \quad H(u, w) := \tau^{-1} \circ \tilde{G} \circ \tau
\]
is well defined and univalent on \(U_{R, \delta}\).

Proposition 3.4. For \((u, w) \in U_{R, \delta}\), we have \(H(u, w) = (u + 1 + \frac{A}{u} + O\left(\frac{1}{u^2}\right), \lambda w + O\left(\frac{1}{u^2}\right))\).

Proof. By Lemma 2.5, \(\tau(u, w) = (u + O\left(\frac{1}{u}\right), w)\). This, together with (3.3), implies
\[
\tilde{G} \circ \tau(u, w) = \left(u + 1 + \frac{h(w)}{u} - \sum_{k=0}^{\infty} \frac{h(\lambda^k w) - A}{u + k} + O\left(\frac{1}{u^2}\right), \lambda w + O\left(\frac{1}{u^2}\right)\right).
\]

Next observe that
\[
\tau^{-1}(u, w) = \left(u + \sum_{k=0}^{\infty} \frac{h(\lambda^k w) - A}{u + k} + O\left(\frac{1}{u^2}\right), w\right).
\]
Let us write \((u', w') = H(u, w)\) and observe that 
\[w' = \lambda w + O \left( \frac{1}{u^2} \right)\].
Let us write \(w' = \lambda w + \alpha(u)\) where \(\alpha(u) = O \left( \frac{1}{u^2} \right)\) is a holomorphic function (with coefficients depending on \(w\)). In the first coordinate we get

\[
u' = u + 1 + \frac{h(w)}{u} \left( u + \sum_{k=0}^{\infty} \frac{h(\lambda^k w)}{u + k} - A \right) \left( u + \sum_{k=1}^{\infty} \frac{h(\lambda^{k+1} w + \lambda^k \alpha(u)) - A}{u + k + O \left( \frac{1}{u} \right)} \right) + O \left( \frac{1}{u^2} \right)
\]

\[
u' = u + 1 + \frac{A}{u} \left( u + \sum_{k=0}^{\infty} \frac{h(\lambda^k w)}{u + k} - A \right) \left( u + \sum_{k=1}^{\infty} \frac{h(\lambda^k w + \lambda^{-1} \alpha(u)) - h(\lambda^k w)}{u + k + O \left( \frac{1}{u} \right)} \right)
\]

\[- \sum_{k=1}^{\infty} \frac{h(\lambda^k w) O \left( \frac{1}{u} \right)}{(u + k + O \left( \frac{1}{u} \right))(u + k)} + O \left( \frac{1}{u^2} \right).\]

We are going to show that both of the infinite sums in the above expression are of order \(O \left( \frac{1}{u^2} \right)\).

Clearly

\[
\left| \sum_{k=1}^{\infty} \frac{h(\lambda^k w) O \left( \frac{1}{u} \right)}{(u + k + O \left( \frac{1}{u} \right))(u + k)} \right| \leq \sup_{|w| \leq \delta} |h(w)| \cdot O \left( \frac{1}{|u|} \right) \sum_{k=1}^{\infty} \frac{1}{|(u + k + O \left( \frac{1}{u} \right))(u + k)|} = O \left( \frac{1}{|u^2|} \right).
\]

As before we can write convergent power series:

\[
h(w + \lambda^{-1} \alpha(u)) - h(w) = \sum_{\ell=1}^{\infty} \sum_{j=0}^{\infty} \frac{(\ell + j)! \lambda^{-\ell}}{j!} b_{j+\ell} w^j (\alpha(u))^\ell,
\]

where \(h(w) = \sum_{j=0}^{\infty} b_j w^j\). Now observe that the same computation as in Lemma 2.2 tells us that the sum

\[
\sum_{k=0}^{\infty} \frac{\lambda^{k+\ell}}{u + k + O \left( \frac{1}{u} \right)}
\]

converges and that

\[
\left| \sum_{k=0}^{\infty} \frac{\lambda^{k+\ell}}{u + k + O \left( \frac{1}{u} \right)} \right| \leq \frac{C(j + \ell)^r}{|u|}.
\]
Therefore
\[
\left| \sum_{k=0}^{\infty} h(\lambda^k w + \lambda^{k-1} \alpha(u)) - h(\lambda^k w) \right| = \left| \sum_{\ell=1}^{\infty} \left( \sum_{j=0}^{\infty} \frac{(\ell + j)!\lambda^{-\ell}}{j!} b_{\ell+j\ell} w^j (\alpha(u))^\ell \right) \sum_{k=0}^{\infty} \frac{\lambda^{k+\ell}}{u + k + O\left( \frac{1}{u} \right)} \right|
\]
\[
= \frac{C}{|u|} \sum_{\ell=1}^{\infty} \left( \sum_{j=0}^{\infty} \frac{(\ell + j)!}{j!} \left( \frac{j + \ell}{\lambda^{\ell + j}} \right) \right) |b_{\ell+j\ell} w^j (\alpha(u))^\ell|
\]
\[
= O\left( \frac{1}{u^2} \right).
\]

\[\square\]

Fix \((u_0, w_0) \in U_{R,\delta}\). For every \(n \geq 1\) we write \((u_n, w_n) := H^n(u_0, w_0)\).

**Lemma 3.5.** Given \(\delta > 0\) there exists \(T_\delta \geq R\) such that for every \(T \geq T_\delta\) and \((u_0, w_0) \in U_{T,\frac{\delta}{2}}\), we have
\[
(u_n, w_n) \in U_{T + \frac{n}{2}, \delta}
\]
for every \(n \geq 1, j \geq 0\). Moreover given \(0 < \varepsilon \leq \frac{\delta}{2}\), there exists \(T_\varepsilon \geq T_\delta\) such that
\[
|w_n - \lambda^n w_0| < \varepsilon
\]
for every \((u_0, w_0) \in U_{T_\varepsilon, \frac{\delta}{2}}\) and for every \(n \geq 1, j \geq 0\).

**Proof.** Fix \(0 < \varepsilon \leq \delta/2\). By Proposition 3.4 we can choose \(T \geq R\) and \(C > 0\) such that \(|u_1 - u_0 - 1| < \frac{1}{2}\) and \(|w_1 - \lambda w_0| < \frac{C}{n^2}|w_0|\) on \(U_{T,\delta}\) and such that \(\sum_{n=0}^{\infty} \frac{C}{(T + \frac{n}{2})^2} < \varepsilon\). Using induction it is easy to see that \(\text{Re}(u_n) > T + \frac{n}{2}\) and \(|w_n - \lambda^n w_0| < \varepsilon\). \(\square\)

**Lemma 3.6.** Let \(\delta > 0\) and \(T_\delta\) be as in Lemma 3.5. For every compact subset \(K \subset U_{T,\frac{\delta}{2}}\), there exists a constant \(C > 0\) so that for every \((u_0, w_0) \in K\) and every \(n \geq 1\) we have
\[
\frac{1}{|u_n|} \leq \frac{C_1}{n} \quad \text{and} \quad \left| \frac{1}{u_n} - \frac{1}{n} \right| \leq C \log \frac{n}{n^2}.
\]

**Proof.** The first inequality follows directly from \(\text{Re}(u_n) > T + \frac{n}{2}\). As for the second inequality first observe that
\[
\left| \frac{1}{u_n} - \frac{1}{n} \right| \leq \frac{C_1 |u_n - n|}{n^2}.
\]

(3.6)
Recall that \((u', w') := H(u, w) = (u + 1 + \frac{A}{u}, \lambda w) + O(1/u^2)\) where \(|u' - u + 1 + \frac{A}{u}| \leq \frac{C'}{|u'|}\) for some \(C' > 0\). Using the inequality (3.6) we can now deduce that

\[
|u_n - n| \leq |u_0 + n + A \left( \frac{1}{u_0} + \ldots + \frac{1}{u_{n-1}} \right) - n| + C' C_1 \sum_{k=1}^{n} \frac{1}{k^2}
\]

\[
\leq |u_0| + |A| C_1 \sum_{k=1}^{n} \frac{1}{k^2} + C'' C_1 \sum_{k=1}^{n} \frac{1}{k^2}
\]

\[
\leq |u_0| + C_2 \log n + C_3
\]

where \(C_2, C_3 > 0\). Finally we obtain

\[
\left| \frac{1}{u_n} - \frac{1}{n} \right| \leq \frac{C_1(|u_0| + C_2 \log n + C_3)}{n^2} = O\left( \frac{\log n}{n^2} \right).
\]

\[\square\]

**Proposition 3.7.** Let \(F\) be an automorphism of the form (3.1). Then \(F\) has an invariant non-recurrent Fatou component \(\Omega\) with \(\omega\)-limit set \(\{0\} \times \mathbb{C} \subset \partial \Omega\).

**Proof.** Let \(\delta > 0\) and \(R > 0\). Let \(\Theta\) be the map defined in (3.2). Let \(V_{R,\delta} := \Theta(U_{R,\delta})\) and observe that \(V_{R,\delta} = \Lambda_R \times \mathbb{D}_\delta\), where \(\Lambda_R := \{z \in \mathbb{C} | -\frac{1}{R} < z < \frac{1}{R}\}\) and \(\mathbb{D}_\delta := \{w \in \mathbb{C} : |w| < \delta\}\).

**Step 1:** For every \(\delta > 0\) there exists \(R(\delta) > 0\) such that \(\{F^n\}\) is a normal family on \(V_{R(\delta),\delta/4}\).

Let \(\delta > 0\). Let \(\tau\) be the map defined in (3.4) and let \(\Phi\) be the one defined in Lemma 2.3.

Let \(M_\delta > 0\) be given by Lemma 2.4. By Lemma 2.3, Lemma 2.4 and Lemma 2.5, for any \(T \geq M_\delta\) there exists \(R(\delta, T) > 0\) such that

\[
\tau^{-1} \circ \Psi^{-1} \circ \Phi^{-1} \circ \Theta^{-1}(V_{R(\delta,T),\delta/4}) \subset \Theta^{-1}(V_{2T,\delta/2}) = U_{2T,\delta/2}
\]

and

\[
\Phi \circ \Psi \circ \tau(U_{2T,\delta}) \subset U_{T,2\delta},
\]

for every \(n \geq 1\).

Therefore, if \(T_\delta > 0\) is given by Lemma 3.5 take \(T := \max\{M_\delta, T_\delta\}\) and \(R(\delta) := R(\delta, T)\).

Since

\[
F^n = \Theta \circ \Phi \circ \Psi \circ \tau \circ H^n \circ \tau^{-1} \circ \Psi^{-1} \circ \Phi^{-1} \circ \Theta^{-1},
\]

Lemma 3.5 implies at once that the family \(\{F^n\}\) is normal on \(V_{R(\delta),\delta/4}\).

**Step 2:** Let \(g\) be a limit map of \(\{F^n\}\) on \(V_{R(\delta),\delta/4}\). Then

\[
\{0\} \times \mathbb{D}_\delta \subset g(V_{R(\delta),\delta/4}) \subset \{0\} \times \mathbb{C}.
\]
Let $\varepsilon = \delta/16$. Let $T_\varepsilon$ be as in Lemma 3.3 and let $R \geq \max\{M_{\delta}, T_\varepsilon\}$. Denote by $\pi_j : \mathbb{C}^2 \rightarrow \mathbb{C}$, $j = 1, 2$ the projection on the $j$-th component, that is, $\pi_1(z, w) = z$, $\pi_2(z, w) = w$. It follows from Lemma 3.3 that for all $(z, w) \in V_{R, \delta/4}$ and $n \geq 1$:

$$|\pi_2 \circ \Psi \circ \tau \circ H^n \circ \tau^{-1} \circ \Psi^{-1} \circ \Phi^{-1} \circ \Theta^{-1}(z, w) - \lambda \pi_2 \circ \Phi^{-1} \circ \Theta^{-1}(z, w)| < \frac{\delta}{16}.$$  

Taking into account that, by Lemma 2.8, $\pi_2 \circ \Phi(u, w)$ and $\pi_2 \circ \Phi^{-1}(u, w)$ are of the form $w + O(1/u)$, the previous equation and (3.7) imply that there exist $R' \geq R$ such that

$$(3.8) \quad |\pi_2 \circ F^n(z, w) - \lambda^n w| < \frac{\delta}{10}$$

for all $(z, w) \in V_{R', \delta/4}$ and $n \geq 1$.

Moreover, again by Lemma 3.3, given $\eta > 0$, there exists $n_0$ such that for all $n \geq n_0$,

$$(3.9) \quad |\pi_1 \circ F^n(z, w)| < \eta$$

for all $(z, w) \in V_{R', \delta/4}$.

Let $\{n_j\}$ be any increasing sequence for which $F^{n_j}$ converges uniformly on compacta of $V_{R(\delta), \delta/4}$ to a holomorphic function $g$ and $\lambda^{n_j}$ converges to some $\mu \in \partial \mathbb{D}$. It follows from (3.9) that $g(z, w) = (0, g_2(z, w))$ for all $(z, w) \in V_{R(\delta), \delta/4}$, where $g_2 : V_{R(\delta), \delta/4} \rightarrow \mathbb{C}$ is holomorphic. Moreover, by (3.8),

$$(3.10) \quad |g_2(z, w) - \mu w| < \frac{\delta}{10}$$

for all $(z, w) \in V_{R', \delta/4}$.

Fix $w_0 \in \mathbb{C}$, $|w_0| < \delta/16$. Let $z_0 \in \mathbb{C}$ be such that $-1/z_0 \in K_{R'}$. Hence, $(z_0, w_0) \in V_{R', \delta/4}$. Moreover, $\{z_0\} \times \{w \in \mathbb{C} : |w - w_0| < \delta/8\} \subset V_{R', \delta/4}$ and (3.10) holds on $|w - w_0| = \delta/8$. Therefore, by Rouché’s theorem, $\overline{\partial}g_2(z_0, w) - \overline{\partial}w_0$ and $w - \overline{\partial}w_0$ have the same number of zeros in $\{w \in \mathbb{C} : |w - w_0| < \delta/8\}$. Since $|\overline{\partial}w_0 - w_0| \leq 2|w_0| < \delta/8$, it follows that there exists $w_1$, with $|w_1 - w_0| < \delta/16$ such that $\overline{\partial}g_2(z_0, w_1) = w_0$. By the arbitrariness of $w_0$, it follows that $D_{\delta/16} \subset g_2(V_{R(\delta), \delta/4})$, which completes step 2.

**Step 3:** There exists an invariant Fatou component $\Omega$ such that the image of any limit map of $\{F^n\}_{\Omega}$ is $\{0\} \times \mathbb{C}$.

Let $\{\delta_m\}$ be an increasing sequence of positive real numbers which converges to $+\infty$. We can choose $R(\delta_{m+1}) \geq R(\delta_m)$ for all $n \geq 0$. Let $V_m := V_{R(\delta_m), \delta_m/4}$, $n \geq 0$. Hence,

$$\{(z, w) : |w| < \frac{\delta_m}{4}\} \subset V_m.$$ 

Therefore $V := \cup_{m \geq 0} V_m$ is open and connected. Since $\{F^n\}$ is a normal family on $V_m$ for all $m \geq 0$ by Step 1, the previous equation and a diagonal argument imply that $\{F^n\}$ is
a normal family on $V$ and, hence, $\{F^n\}$ is normal on

$$V := \bigcup_{n=0}^{\infty} F^n(V).$$

By (3.9) and (3.8), $F^n(V) \cap V \neq \emptyset$ for every $n \geq 1$. Hence $V$ is a $F$-forward invariant, open, connected set on which $\{F^n\}$ is normal. Therefore, there exists an invariant Fatou component $\Omega$ which contains $V$.

Now, let $g$ be a limit map of $\{F^n\}$ on $\Omega$. Hence, $g|_{V_m}$ is a limit map of $\{F^n|_{V_m}\}$ for all $m \geq 0$. Then, it follows from Step 2 that $g(V) = \{0\} \times \mathbb{C}$. Since $V$ is open in $V$, it follows as well that $g(\Omega) = \{0\} \times \mathbb{C}$.

Step 4: $\Omega$ is non-recurrent.

Observe that $F^n(0, w) = (0, \lambda^nw)$ for all $w \in \mathbb{C}$. Equation (3.11) therefore implies that

$$\frac{\partial^2 \pi_1(F^n)}{\partial z^2}(0, w) = 2 \sum_{k=0}^{n-1} f(\lambda^kw) = 2 \sum_{k=0}^{n-1} (f(\lambda^kw) - 1) + 2n.$$

Let us first observe that $\sum_{k=0}^{n-1} (f(\lambda^kw) - 1)$ is uniformly bounded in $w$ with respect to $n$. We have

$$\left| \sum_{k=0}^{n-1} (f(\lambda^kw) - 1) \right| = \sum_{k=0}^{n-1} \sum_{\ell=1}^{\infty} d_\ell \lambda^{k\ell} w^\ell$$

$$= \sum_{\ell=1}^{\infty} d_\ell w^\ell \sum_{k=0}^{n-1} \lambda^{k\ell}$$

$$\leq \sum_{\ell=1}^{\infty} |d_\ell| |w^\ell| \left| \sum_{k=0}^{n-1} \lambda^{k\ell} \right|$$

$$\leq C \sum_{\ell=1}^{\infty} |d_\ell| |w^\ell|.$$

Since $f(w)$ is an entire function, the last sum converges on uniformly on compacta of $\mathbb{C}$. This implies that

$$\lim_{n \to \infty} \left| \frac{\partial^2 \pi_1(F^n)}{\partial z^2}(0, w) \right| = \infty.$$

Therefore, $(0, w)$ cannot be contain in any Fatou component of $F$ for all $w \in \mathbb{C}$. Thus $\{0\} \times \mathbb{C} \subset \partial \Omega$, which completes the proof. \qed

Remark 3.8. The set $V$ defined in (3.11) depends on the sequence $\{\delta_n\}$ and on the choice of $\{R(\delta_n)\}$. In particular, given any $\eta > 0$ one can construct the open set $V$ of the form
Proof. If by construction.

\[ \circ \]

\[ F \]

\[ V \]

\[ \text{Lemma 4.1.} \]

\[ \text{Let } (z_0, w_0) \in \mathbb{C}^2 \setminus \{(0) \times \mathbb{C}\} \text{ be such that } \lim_{n \to \infty} F^{\circ n}(z_0, w_0) = (0, \zeta) \text{ for some } \zeta \in \mathbb{C}. \text{ Then for every set } V \text{ as before, there exists } n_0 = n_0(V, z_0, w_0) \text{ such that } F^{\circ n}(z_0, w_0) \in V \text{ for all } n \geq n_0. \text{ In particular, } \Omega = V^c. \]

Proof. If \( F^{\circ n}(z, w) \in V \) for some \( n_0 \), then \( F^{\circ n}(z, w) \in V \) for all \( n \geq n_0 \) since \( F(V) \subset V \) by construction.

Therefore, we assume by contradiction that \( F^{\circ n}(z, w) \notin V \) for all \( n \geq 0 \).

As we already notice, \( F^{-1} \) has the same form of \( F \), that is,

\[ F^{-1}(z, w) = (z + \tilde{f}(w)z^2 + O(z^3), \bar{w} + \tilde{g}(w)z + O(z^2)), \]

where \( \tilde{f}, \tilde{g} : \mathbb{C} \to \mathbb{C} \) are holomorphic, \( \tilde{f}(0) = -1 \) (since we assumed \( f(0) = 1 \)) and \( \tilde{g}(w) = O(w^2) \). Let \( \chi(z, w) = (-z, w) \). The automorphism \( \chi \circ F^{-1} \circ \chi \) has the same form as \( F^{-1} \), but the coefficient of \( z^2 \) in the first coordinate is 1. Hence, by Proposition [3.7] there exists an invariant non-recurrent Fatou component \( \Omega^- \) of \( \chi \circ F^{-1} \circ \chi \) with \( \{0\} \times \mathbb{C} \subset \partial \Omega^- \).
and \( \Omega^- \) contains a connected open set \( \tilde{V}^- \) of the same form as (3.11). Moreover, by Remark 3.8 we can assume that \( |z| < |z_0| \) for all \( (z, w) \in \tilde{V}^- \).

In particular, \( \chi(\Omega^-) \) is a non-recurrent Fatou component of \( F^{-1} \) with \( \{0\} \times \mathbb{C} \) on the boundary, contains the open set \( V^- := \chi(\tilde{V}^-) \), and \( |z| < |z_0| \) for all \( (z, w) \in V^- \), that is, \((z_0, w_0) \notin \mathcal{V}^-\). By the very definition of \( V \) and \( V^- \), it follows that \( V \cup V^- \cup \{0\} \times \mathbb{C} \) is a neighborhood of \( \{0\} \times \mathbb{C} \). Therefore, since \( \lim_{n \to \infty} F^n(z_0, w_0) = (0, 0) \) and \( F^n(z_0, w_0) \notin V \), the sequence \( \{F^n(z, w)\}_{n \in \mathbb{Z}} \) has to be eventually contained in \( V^- \).

However, since \( F^{-1}(V^-) \subset V^- \), it follows that the entire orbit \( \{F^n(z, w)\}_{n \in \mathbb{Z}} \) is contained in \( V^- \), hence \((z_0, w_0) \in V^- \), a contradiction. \( \square \)

For natural numbers \( n + 1 > j \geq 1 \), we let 
\[
Q_n(u, w) = (u - n - A \log n, \lambda^{-n} w)
\]
and let 
\[
\varphi_n = Q_n \circ H^n,
\]
where the \( H \) is defined in (3.5).

For \( n > 0 \) one can easily verify the following equality
\[
(4.1) \quad \varphi_n \circ H = \chi_n \circ \varphi_{n+1},
\]
where \( \chi_n(u, w) = (u + 1 + A \log(1 + \frac{1}{u}), \lambda w) \).

Lemma 4.2. For every \( \delta > 0 \) there exists \( S_{\delta} > 0 \) such that the sequence \( \{\varphi_n\}_{n \in \mathbb{N}} \) converges uniformly on compacta of \( U_{S_{\delta},\delta/4} \) to a univalent map \( \varphi : U_{S_{\delta},\delta/4} \to \mathbb{C}^2 \) such that
\[
(4.2) \quad \varphi \circ H = \chi \circ \varphi,
\]
where \( \chi(u, w) = (u + 1, \lambda w) \) and
\[
(4.3) \quad \varphi(u, w) = (u - A \log(u) + o(1), w + o(1))
\]
as \( \text{Re}(u) \to \infty \). Moreover, given any increasing sequence \( \{\delta_m\} \) of positive real numbers converging to \( \infty \), \( \varphi : \bigcup_{m \geq 0} U_{S_{\delta_m},\delta_m/4} \to \mathbb{C}^2 \) is univalent.

Proof: Fix \( \delta > 0 \) and let \( T_{\delta} \) be given by Lemma 3.3. Let \((u_0, w_0) \in U_{T_{\delta},\delta/2} \) and set \((u_n, w_n) := H^n(u_0, w_0) \). By Lemma 3.3 we have \((u_n, w_n) \in U_{T_{\delta},\delta} \) for all \( n \). Hence, by Proposition 3.4,
\[
(u_{n+1}, w_{n+1}) = \left(u_n + 1 + \frac{A}{u_n} + O\left(\frac{1}{u_n^2}\right), \lambda w_n + O\left(\frac{1}{u_n^2}\right)\right),
\]
where the bounds in the \( O \)'s are uniform in \( n \). Let us write
\[
(u', w') = \varphi_{n+1}(u_0, w_0) - \varphi_n(u_0, w_0).
\]
Observe that
\[ w' = \lambda^{-(n+1)} \left( \lambda w_n + O \left( \frac{1}{u_n^2} \right) \right) - \lambda^{-n} w_n = O \left( \frac{1}{u_n^2} \right) \]
and
\[ u' = u_n + 1 + \frac{A}{u_n} + O \left( \frac{1}{u_n^2} \right) - (n + 1) - A \log(n + 1) - u_n + n + A \log n \]
\[ = A \left( \frac{1}{u_n} - \frac{1}{n} \right) + A \left( \frac{1}{n} - \log(1 + \frac{1}{n}) \right) + O \left( \frac{1}{u_n^2} \right) . \]
By Lemma 3.6 we have
\[ \left| \frac{1}{u_n} - \frac{1}{n} \right| = O \left( \frac{\log n}{n^2} \right) \]
and
\[ O \left( \frac{1}{|u_n|^2} \right) = O \left( \frac{1}{n^2} \right) . \]
Next observe that \( (\frac{1}{n} - \log(1 + \frac{1}{n})) = O(\frac{1}{n^2}) \). Since all bounds are uniform on compact subsets of \( U_{T, \delta/2} \) and independent from \( n \) it follows that \( \sum_{n=1}^{\infty} (\varphi_{n+1} - \varphi_n) \) converges absolutely, hence the sequence \( \{\varphi_n\} \) converges uniformly on compacta of \( U_{T, \delta/2} \) to a map \( \varphi \).

Fix \( \varepsilon > 0 \). By Lemma 3.3 \((u_n, w_n) \in U_{T, \delta/2}^{\frac{n}{n}}\) for every \((u_0, w_0) \in U_{T, \delta/2}^{\frac{n}{n}}\). Therefore, by the same lemma, there exists \( n_0 \) such that for all \( n \geq n_0 \) and all \((u, w) \in U_{T, \delta/2}^{\frac{n}{n}}\),
\[ |\varphi(u, w) - \varphi_n(u, w)| < \varepsilon . \]
Since all maps \( \varphi^n \) are univalent it follows that \( \varphi \) is also univalent on \( U_{T, \delta/2}^{\frac{n}{n}} \). Setting \( S_\delta := T_{\delta/2} \) we have the first result.

Since \( \varphi_n(u, w) = \sum_{k=0}^{n-1} (\varphi_{k+1}(u, w) - \varphi_k(u, w)) \), (4.3) follows immediately from the previous computations. The functional equation (4.2) follows from (4.1) passing to the limit. Finally observe that given any increasing sequence \( \{\delta_m\} \) of positive real numbers converging to \( \infty \) there exist a sequence of positive real numbers \( \{S_{\delta_m}\} \) so that \( \varphi \) is univalent on \( \bigcup_{m \geq 0} U_{S_{\delta_m}, \delta_m/4} \).

Let \( \{\delta_m\} \) be an increasing sequence of positive real numbers converging to \( \infty \) and let \( \{S_{\delta_m}\} \) be the sequence given by Lemma 1.2 Let \( \{R_m\} \) be an increasing sequence of positive real numbers such that \( R_m \geq \max\{R(\delta_m), S_{\delta_m}\} \) (where, as before, the \( R(\delta_m) \)'s are defined in Step 1 of the proof of Proposition 3.7). Let \( V_m := V_{R_m, \delta_m/4} \).

We define
\[ P = (\Theta \circ \Phi \circ \Psi \circ \tau \circ \varphi^{-1})^{-1} . \]
By Lemma 4.2 \( P \) is a univalent map defined on \( \bigcup_{m=1}^{\infty} V_m \).
By (3.7) and (4.2),

\[ F = P^{-1} \circ \chi \circ P \]
on \bigcup_{m=1}^{\infty} V_m \text{ for all } n \geq 0.

**Proposition 4.3.** The Fatou component \( \Omega \) is biholomorphic to \( \mathbb{C}^2 \) and there exists a univalent map \( Q \) defined on \( \Omega \) such that

\[ Q \circ F = \chi \circ Q \]

where \( \chi(u, w) = (u + 1, \lambda w) \).

**Proof.** Let \( V_m \) as before. Let \( V := \bigcup_{n \geq 0} \bigcup_{m \geq 0} F^{on}(V_m) \). By Lemma 4.1, \( \Omega = \bigcup_{k=0}^{\infty} F^{-k}(V) \).

We extend \( P \) to a univalent map \( Q \) defined on \( \Omega \) as follows. If \( (z, w) \in \Omega \), there exists a natural number \( n \) such that \( F^n(z, w) \in V_m \) for some \( m \). Hence, we set

\[ Q(z, w) = (\chi^{-1})^n \circ P \circ F^{on}(z, w). \]

By (4.4), this definition is well posed and \( Q : \Omega \to \mathbb{C}^2 \) is univalent. The functional equation (4.5) therefore follows from (4.4).

Now we prove that \( Q(\Omega) = \mathbb{C}^2 \). Let \( \Omega_n := \bigcup_{k=0}^{n} (F^{-1})^{on}(V) \). Observe that

\[ Q(\Omega) = \bigcup_{n=0}^{\infty} (\chi^{-1})^n \circ P \circ F^{on}(\Omega_n) = \bigcup_{n=0}^{\infty} (\chi^{-1})^n \circ P(V). \]

From the definition of the maps \( \Theta, \Phi, \Psi \) and \( \tau \) and the set \( V \) we can find a sequence of \( \rho_n \to \infty \) satisfying

\[ \bigcup_{k=0}^{\infty} \{ u \in \mathbb{C} : \text{Re}(u) > \rho_k \} \times \mathbb{D}_k \subset \tau^{-1} \circ \Psi^{-1} \circ \Phi^{-1} \circ \Theta^{-1}(V). \]

It follows

\[ \varphi \left( \bigcup_{k=0}^{\infty} \{ u \in \mathbb{C} : \text{Re}(u) > \rho_k \} \times \mathbb{D}_k \right) \subset P(V). \]

Therefore

\[ \bigcup_{n=0}^{\infty} \bigcup_{k=0}^{\infty} (\chi^{-1})^n \circ \varphi \left( \{ u \in \mathbb{C} : \text{Re}(u) > \rho_k \} \times \mathbb{D}_k \right) \subset Q(\Omega). \]

Equation (4.3) shows that for every \( k \) we can find \( r_k \geq \rho_k \) such that

\[ \{ u \in \mathbb{C} : \text{Re}(u - r_k) > |\text{Im}(u)| \} \times \mathbb{D}_{k/2} \subset \varphi \left( \{ u \in \mathbb{C} : \text{Re}(u) > \rho_k \} \times \mathbb{D}_k \right), \]

hence

\[ \mathbb{C}^2 = \bigcup_{n=0}^{\infty} \bigcup_{k=0}^{\infty} (\chi^{-1})^n \left( \{ u \in \mathbb{C} : \text{Re}(u - r_k) > |\text{Im}(u)| \} \times \mathbb{D}_{k/2} \right) \subset Q(\Omega), \]

and we are done.

\[ \square \]

**Theorem 1.2** now follows from Proposition 3.7 and Proposition 4.3.
5. AN EXAMPLE

Using shears and overshears we can construct an explicit automorphism of the form (3.1). We first define automorphisms

\[\begin{align*}
F_1(z, w) &= (z, \lambda w + z), \\
F_2(z, w) &= (ze^w, w), \\
F_3(z, w) &= (z, w - z), \\
F_4(z, w) &= (ze^{-w}, w), \\
F_5(z, w) &= (z, we^z),
\end{align*}\]

and finally

\[F(z, w) := (F_5 \circ F_4 \circ F_3 \circ F_2 \circ F_1)(z, w).\]

Quick computation shows that

\[F(z, w) = \left(z + e^{\lambda w}z^2 + O(z^3), \lambda w - \sum_{k=2}^{\infty} \frac{\lambda^k}{k!}w^k + O(z^2)\right)\]

and

\[F^{-1}(z, w) = \left(z - e^{w}z^2 + O(z^3), \lambda^{-1}w + \lambda^{-1}z \sum_{k=2}^{\infty} \frac{1}{k!}w^k + O(z^2)\right).\]

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