Reflected Advanced Backward Stochastic Differential Equations with Default

N. Agram\(^1,2\), S. Labed\(^2\), B. Mansouri\(^2\) & M. A. Saouli\(^2\)

20 March 2018

Abstract

We are interested on reflected advanced backward stochastic differential equations (RABSDE) with default. By the predictable representation property and for a Lipschitz driver, we show that the RABSDE with default has a unique solution in the enlarged filtration. A comparison theorem for such type of equations is proved. Finally, we give a connection between RABSDE and optimal stopping.

Keywords: Reflected Advanced Backward Stochastic Differential Equations, Single Jump, Progressive Enlargement of Filtration.

1 Introduction

Reflected advanced backward stochastic differential equations (RABSDE) appear in their linear form as the adjoint equation when dealing with the stochastic maximum principle to study optimal singular control for delayed systems, we refer for example to Øksendal and Sulem [10] and also to Agram \textit{et al} [11] for more general case. This is a natural model in population growth, but also in finance, where people’s memory plays a role in the price dynamics.

After the economic crises in 2008, researchers started to include default in banks as a part of their financial modelling. This is why we are interested on RABSDE also in the context of enlargement of filtration. In order to be more precise, let us consider a random

\(^1\)Department of Mathematics, University of Oslo, P.O. Box 1053 Blindern, N–0316 Oslo, Norway. Email: naciraa@math.uio.no.

This research was carried out with support of the Norwegian Research Council, within the research project Challenges in Stochastic Control, Information and Applications (STOCONINF), project number 250768/F20.

\(^2\)Department of Mathematics, University of Biskra, Algeria.

Emails: labed.saloua@yahoo.fr, mansouri.badreddine@gmail.com, saoulimoustapha@yahoo.fr.
time \( \tau \) which is neither an \( \mathbb{F} \)-stopping time nor \( \mathcal{F}_T \)-measurable. Examples of such random times are default times, where the reason for the default comes from outside the Brownian model. We denote \( H_t = 1_{\tau \leq t}, \ t \in [0, T] \), and consider the filtration \( \mathbb{G} \) obtained by enlarging progressively the filtration \( \mathbb{F} \) by the process \( H \), i.e., \( \mathbb{G} \) is the smallest filtration satisfying the usual assumptions of completeness and right-continuity, which contains the filtration \( \mathbb{F} \) and has \( H \) as an adapted process. The RABSDE related with, we want to study is the following:

\[
\begin{align*}
Y_t & = \xi + \int_t^T f(s, Y_s, Z_s, \mathbb{E}[Y_{s+\delta}|\mathbb{G}_s], \mathbb{E}[Z_{s+\delta}|\mathbb{G}_s], U_s)ds - \int_t^T Z_s dW_s \\
& \quad - \int_t^T U_s dH_s + K_T - K_t, \quad t \in [0, T], \\
Y_T & = \xi, \quad t \geq T, \\
Z_t & = U_t = 0, \quad t > T.
\end{align*}
\]

By saying that the RBSDE is advanced we mean that driver at the present time \( s \) may depend not only on present values of the solution processes \( (Y, Z, K) \), but also on the future values \( s+\delta \) for some \( \delta > 0 \). To make the system adapted, we take the conditional expectation of the advanced terms.

We will see that by using the predictable representation property (PRP) the above system is equivalent to a RABSDE driven by a martingale, consisting of the Brownian motion \( W \) and the martingale \( M \) associated to the jump process \( H \), as follows:

\[
\begin{align*}
Y_t & = \xi + \int_t^T F(s, Y_s, Z_s, \mathbb{E}[Y_{s+\delta}|\mathbb{G}_s], \mathbb{E}[Z_{s+\delta}|\mathbb{G}_s], U_s)ds - \int_t^T Z_s dW_s \\
& \quad - \int_t^T U_s dM_s + K_T - K_t, \quad t \in [0, T], \\
Y_T & = \xi, \quad t \geq T, \\
Z_t & = U_t = 0, \quad t > T.
\end{align*}
\]

Our aim in this paper is not to find solutions in the Brownian filtration by using the decomposition approach, as it has been done in Kharroubi and Lim [6] for BSDE and in Jeanblanc et al [9] for ABSDE. However, we want to find solutions under the enlarged filtration rather than the Brownian one as in the previous works.

In Dumitrescu et al [3], [4], [5], the authors consider directly BSDE and RBSDE driven by general filtration generated by the pair \( (W, M) \).

We will extend the recent works by Dumitrescu et al [3], [4], [5], to the anticipated case and we will explain how such an equations appear by using the PRP.

We will extend also the comparison theorem for ABSDE in Peng and Yang [14] to RABSDE with default and finally, we give a link between RABSDE with default and optimal stopping as it has been done in El Karoui et al [6] and Øksendal and Zhang [13].

For more details about ABSDE with jumps coming from the compensated Poisson random measure which is independent of the Brownian motion, we refer to Øksendal et al [12], [11]. For RBSDE with jumps, we refer to Quenez and Sulem [15] and for more details about enlargement progressive of filtration, we refer to Song [16].
2 Framework

Let \((\Omega, \mathcal{G}, P)\) be a complete probability space. We assume that this space is equipped with a one-dimensional standard Brownian motion \(W\) and we denote by \(\mathcal{F} := (\mathcal{F}_t)_{t \geq 0}\) the right continuous complete filtration generated by \(W\). We also consider on this space a random time \(\tau\), which represents for example a default time in credit risk or in counterparty risk, or a death time in actuarial issues. The random time \(\tau\) is not assumed to be an \(\mathcal{F}\)-stopping time. We therefore use in the sequel the standard approach of filtration enlargement by considering \(\mathcal{G}\) the smallest right continuous extension of \(\mathcal{F}\) that turns \(\tau\) into a \(\mathcal{G}\)-stopping time (see e.g. Chapter 4 in [2]). More precisely \(\mathcal{G} := (\mathcal{G}_t)_{t \geq 0}\) is defined by

\[
\mathcal{G}_t := \bigcap_{\varepsilon > 0} \tilde{\mathcal{G}}_{t+\varepsilon},
\]

for all \(t \geq 0\), where \(\tilde{\mathcal{G}}_s := \mathcal{F}_s \vee \sigma(1_{\tau \leq u}, u \in [0, s])\), for all \(s \geq 0\).

We denote by \(\mathcal{P}(\mathcal{G})\) the \(\sigma\)-algebra of \(\mathcal{G}\)-predictable subsets of \(\Omega \times [0, T]\), i.e. the \(\sigma\)-algebra generated by the left-continuous \(\mathcal{G}\)-adapted processes.

We then impose the following assumptions, which are classical in the filtration enlargement theory.

\((H)\) The process \(W\) is a \(\mathcal{G}\)-Brownian motion. We observe that, since the filtration \(\mathcal{F}\) is generated by the Brownian motion \(W\), this is equivalent with the fact that all \(\mathcal{F}\)-martingales are also \(\mathcal{G}\)-martingales. Moreover, it also follows that the stochastic integral \(\int_0^t X_s dW_s\) is well defined for all \(\mathcal{P}(\mathcal{G})\)-measurable processes \(X\) such that \(\int_0^t |X_s|^2 ds < \infty\), for all \(t \geq 0\).

- The process \(M\) defined by

\[
M_t = H_t - \int_0^{t \wedge \tau} \lambda_s ds, \quad t \geq 0,
\]

is a \(\mathcal{G}\)-martingale with single jump time \(\tau\) and the process \(\lambda\) is \(\mathcal{F}\)-adapted, called the \(\mathcal{F}\)-intensity of \(\tau\).

- We assume that the process \(\lambda\) is upper bounded by a constant.

- Under \((H)\) any square integrable \(\mathcal{G}\) martingale \(Y\) admits a representation as

\[
Y_t = y + \int_0^t \varphi_s dW_s + \int_0^t \gamma_s dM_s,
\]

where \(M\) is the compensated martingale of \(H\), and \(\varphi, \gamma\) are square-integrable \(\mathcal{G}\)-predictable processes. (See Theorem 3.15 in [2]).

Throughout this section, we introduce some basic notations and spaces.
\[ S^2_G \] is the subset of \( \mathbb{R} \)-valued \( \mathcal{G} \)-adapted càdlàg processes \((Y_t)_{t \in [0,T]}\), such that
\[ \|Y\|_{S^2}^2 := \mathbb{E}[ \sup_{t \in [0,T]} |Y_t|^2 ] < \infty. \]

\( K^2 \) is a set of real-valued nondecreasing processes \( K \) with \( K_{0^-} = 0 \) and \( \mathbb{E}[K_t] < \infty \).

\( H^2_G \) is the subset of \( \mathbb{R} \)-valued \( \mathcal{P}(\mathcal{G}) \)-measurable processes \((Z_t)_{t \in [0,T]}\), such that
\[ \|Z\|_{H^2}^2 := \mathbb{E}[\int_0^T |Z_t|^2 dt] < \infty. \]

\( L^2(\lambda) \) is the subset of \( \mathbb{R} \)-valued \( \mathcal{P}(\mathcal{G}) \)-measurable processes \((U_t)_{t \in [0,T]}\), such that
\[ \|U\|_{L^2(\lambda)}^2 := \mathbb{E}[\int_0^{T \wedge \tau} \lambda_t |U_t|^2 dt] < \infty. \]

3 Existence and Uniqueness

We study the RABSDE with default
\[
\left\{
\begin{array}{l}
Y_t = \xi + \int_t^T f(s, Y_s, Z_s, \mathbb{E}[Y_{s+\delta} | \mathcal{G}_s], \mathbb{E}[Z_{s+\delta} | \mathcal{G}_s], U_s) ds - \int_t^T Z_s dW_s \\
- \int_t^T U_s dH_s + K_T - K_t, \quad t \in [0, T], \\
Y_{t^-} = \xi, \quad t \geq T, \\
Z_{t^-} = U_t = 0, \quad t > T,
\end{array}
\right.
\tag{3.1}
\]

where \( f \) is \( \mathcal{G}_t \otimes \mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbb{R}^5) \)-measurable, and the terminal condition \( \xi \) is \( \mathcal{G}_T \)-measurable. Moreover

- \( Y_t \geq S_t \), for each \( t \geq 0 \) a.s.
- \( K_t \) is càdlàg, increasing and \( \mathcal{G} \)-adapted process with \( K_{0^-} = 0 \).
- \( \int_0^T (Y_t - S_t) dK_t = 0 \) and \( \triangle K_t^d = - \Delta Y_t 1_{\{Y_t = S_t\}} \), where denote the continuous and discontinuous parts of \( K \) respectively.

\((S_t)_{t \geq 0}\) is the obstacle which is a càdlàg, increasing and \( \mathcal{G} \)-adapted process.

We call the quadruplet \((Y, Z, U, K)\) solution of the RABSDE \eqref{3.1}.

Let us impose the following set of assumptions.
(i) Assumption on the terminal condition:

- \( \xi \in L^2(\Omega, \mathcal{G}_T) \).

(ii) Assumptions on the generator function \( f : \Omega \times [0, T] \times \mathbb{R}^5 \to \mathbb{R} \) is such that
• $\mathcal{G}$-predictable and satisfies the integrability condition, such that

$$\mathbb{E}\left[\int_0^T |f(t, 0, 0, 0, 0)|^2 dt\right] < 0,$$

(3.2)

for all $t \in [0, T]$.

• Lipschitz in the sense that, there exists $C > 0$, such that

$$|f(t, y, z, \mu, \pi, u) - f(t, y', z', \mu', \pi', u')|$$

$$\leq C(|y - y'| + |z - z'| + |\pi - \pi'| + |\mu - \mu'| + |\lambda_t| |u - u'|),$$

(3.3)

for all $t \in [0, T]$ and all $y, y', z, z', \mu, \mu', \pi, \pi', u, u' \in \mathbb{R}$.

We give the existence of the solution to a RABSDE in the enlarged filtration $\mathcal{G}$. The existence follows from the PRP as we can also say, the property of martingale representation (PMR), and a standard approach like any classical RBSDE.

Under our assumptions we know that equation (3.1) is equivalent to

$$\begin{cases}
Y_t &= \xi + \int_t^T F(s, Y_s, Z_s, \mathbb{E}[Y_{s+\delta}|\mathcal{G}_s], \mathbb{E}[Z_{s+\delta}|\mathcal{G}_s], U_s)ds - \int_t^T Z_s dW_s \\
& \quad - \int_t^T \mathbb{E}[K_s|\mathcal{G}_s]ds + K_T - K_t, \quad t \in [0, T], \\
Y_T &= \xi, \quad t \geq T, \\
Z_t &= U_t = 0, \quad t > T,
\end{cases}$$

(3.4)

with $dH_s = dM_s + \lambda_s \mathbb{1}_{s < T} ds$, and

$$F(s, y, z, \mu, \pi, u) := f(s, y, z, \mu, \pi, u) - \lambda_s(1 - H_s)u.$$

By assumption, the process $\lambda$ is bounded.

In order to get existence and uniqueness for the RABSDE (3.4), let us check that the generator $F$ satisfies the same assumption as $f$ : The function $F : \Omega \times [0, T] \times \mathbb{R}^5 \to \mathbb{R}$ is such that

(i) $\mathcal{G}$-predictable and integrable in the sense that, for all $t \in [0, T]$, by inequality (3.2), we have

$$\mathbb{E}\left[\int_0^T |F(t, 0, 0, 0, 0)|^2 dt\right] = \mathbb{E}\left[\int_0^T |f(t, 0, 0, 0, 0)|^2 dt\right] < 0.$$

(ii) Lipschitz in the sense that there exists a constant $C' > 0$, such that

$$|F(t, y, z, \mu, \pi, u) - F(t, y', z', \mu', \pi', u')|$$

$$\leq |f(t, y, z, \mu, \pi, u) - f(t, y', z', \mu', \pi', u') - \lambda_t(1 - H_t)(u - u')|$$

$$\leq |f(t, y, z, \mu, \pi, u) - f(t, y', z', \mu', \pi', u')| + \lambda_t(1 - H_t)|u - u'|$$

$$\leq C(|y - y'| + |z - z'| + |\mu - \mu'| + |\pi - \pi'| + \lambda_t(1 - H_t)|u - u'|) + \lambda_t(1 - H_t)|u - u'|$$

$$\leq C'(|y - y'| + |z - z'| + |\mu - \mu'| + |\pi - \pi'| + \lambda_t|u - u'|),$$
for all $t \in [0, T]$ and all $y, z, u, \pi, \mu, y', z', u', \pi', \mu' \in \mathbb{R}$, where we have used the Lipschitzianity of $f$ \((3.3)\).

(iii) The terminal value: $\xi \in L^2(\Omega, \mathcal{G}_T)$.

**Theorem 3.1** Under the above assumptions (i)-(iii), the RABSDE \((3.4)\) admits a unique solution $(Y, Z, U, K) \in S^2_\mathcal{G} \times H^2_\mathcal{G} \times L^2(\lambda) \times \mathcal{K}^2$.

**Proof.** We define the mapping

$$
\Phi : H^2_\mathcal{G} \times H^2_\mathcal{G} \times L^2(\lambda) \rightarrow H^2_\mathcal{G} \times H^2_\mathcal{G} \times L^2(\lambda),
$$

for which we will show that it is contracting under a suitable norm. For this we note that for any $(Y, Z, U, K) \in H^2_\mathcal{G} \times H^2_\mathcal{G} \times L^2(\lambda) \times \mathcal{K}^2$ there exists a unique quadruple $(\tilde{Y}, \tilde{Z}, \tilde{U}, \tilde{K}) \in S^2_\mathcal{G} \times H^2_\mathcal{G} \times L^2(\lambda) \times \mathcal{K}^2$, such that

$$
\dot{\tilde{Y}}_t = \xi + \int^T_t F(s, Y_s, Z_s, \mathbb{E}[Y_{s+\delta}|\mathcal{G}_s], \mathbb{E}[Z_{s+\delta}|\mathcal{G}_s], U_s) ds - \int^T_t \tilde{Z}_s dW_s - \int^T_t \tilde{U}_s dM_s - \int^T_t \tilde{K}_s, \quad t \in [0, T],
$$

\((3.5)\)

Let $\Phi(Y, Z, U) := (\tilde{Y}, \tilde{Z}, \tilde{U})$. For given $(Y^i, Z^i, U^i) \in H^2_\mathcal{G} \times H^2_\mathcal{G} \times L^2(\lambda)$, for $i = 1, 2$, we use the simplified notations:

$$(\tilde{Y}^i, \tilde{Z}^i, \tilde{U}^i) := \Phi(Y^i, Z^i, U^i),$$

$$(\tilde{Y}, \tilde{Z}, \tilde{U}) := (\tilde{Y}^1, \tilde{Z}^1, \tilde{U}^1) - (\tilde{Y}^2, \tilde{Z}^2, \tilde{U}^2),$$

$$(\tilde{Y}, \tilde{Z}, \tilde{U}) := (Y^1, Z^1, U^1) - (Y^2, Z^2, U^2).$$

The triplet of processes $(\tilde{Y}, \tilde{Z}, \tilde{U})$ satisfies the equation

$$
\tilde{Y}_t = \int^T_t \{ F(s, Y^1_s, Z^1_s, \mathbb{E}[Y^1_{s+\delta}|\mathcal{G}_s], \mathbb{E}[Z^1_{s+\delta}|\mathcal{G}_s], U_s^1) - F(s, Y^2_s, Z^2_s, \mathbb{E}[Y^2_{s+\delta}|\mathcal{G}_s], \mathbb{E}[Z^2_{s+\delta}|\mathcal{G}_s], U_s^2) \} ds
$$

$$- \int^T_t \tilde{Z}_s dW_s - \int^T_t \tilde{U}_s dM_s - \int^T_t \tilde{K}_s, \quad t \in [0, T].$$

We have that $M_t = H_t - \int^t_0 \lambda_s ds$ which is a pure jump martingale. Then,

$$
[M]_t = \sum_{0 \leq s \leq t} (\triangle M_s)^2 = \sum_{0 \leq s \leq t} (\triangle H_s)^2 = H_t,
$$

and

$$
\langle M \rangle_t = \int^t_0 \lambda_s ds,
$$

$$\int^t_0 |\tilde{U}_s|^2 d\langle M \rangle_s = \int^t_0 \lambda_s |\tilde{U}_s|^2 ds.$$

Applying Itô’s formula to $e^\beta t |\tilde{Y}_t|^2$, taking conditional expectation and using the Lipschitz condition, we get

$$
6$$
\[ \mathbb{E}\left[ \int_0^T e^{\beta s} (\beta |\tilde{Y}_s|^2 + |\tilde{Z}_s|^2 + \lambda_s |\tilde{U}_s|^2) ds \right] \leq 10\rho C^2 \mathbb{E}\left[ \int_0^T e^{\beta s} |\tilde{Y}_s|^2 ds \right] + \frac{1}{2\rho} \mathbb{E}\left[ \int_0^T e^{\beta s} \{ |\tilde{Z}_s|^2 + \lambda_s^2 |\tilde{U}_s|^2 \} ds \right], \]

where we have used that
\[ \tilde{Y}_s dK^{1,c}_s = (Y^1_s - S_s) dK^{1,c}_s - (Y^2_s - S_s) dK^{1,c}_s = -(Y^2_s - S_s) dK^{1,c}_s \leq 0 \text{ a.s.}, \]

and by symmetry, we have also \( \tilde{Y}_s dK^{2,c}_s \geq 0 \) a.s. For the discontinuous case, we have as well
\[ \tilde{Y}_s dK^{1,d}_s = (Y^1_s - S_s) dK^{1,d}_s - (Y^2_s - S_s) dK^{1,d}_s = -(Y^2_s - S_s) dK^{1,d}_s \leq 0 \text{ a.s.}, \]

and by symmetry, we have also \( \tilde{Y}_s dK^{2,d}_s \geq 0 \) a.s.

Since \( \lambda \) is bounded, we get that \( \lambda^2 \leq k\lambda \) and by choosing \( \beta = 1 + 10\rho C^2 \) we obtain
\[ ||(\tilde{Y}, \tilde{Z}, \tilde{U})||^2 \leq \frac{1}{2\rho} ||(\bar{Y}, \bar{Z}, \bar{U})||^2 \]

which means for \( \rho \geq 1 \), there exists a unique fixed point that is a solution for our RABSDE (3.4). \( \square \)

### 4 Comparison Theorem for RABSDE with Default

In this section we are interested in a subclass of RABSDE where the driver only depend on future values of \( Y \) and is not allowed to depend on future values of \( Z \), as follows:

\[
\begin{align*}
Y_t &= \xi + \int_t^T g(s, Y_s, Z_s, \mathbb{E}[Y_{s+\delta} | \mathcal{G}_s], U_s) ds - \int_t^T Z_s dW_s \\
&- \int_t^T U_s dM_s + K_T - K_t, \quad t \in [0, T], \\
Y_t &= \xi, \quad t \geq T, \\
Z_t &= U_t = 0, \quad t > T,
\end{align*}
\]

such that

- \( Y_t \geq S_t \), for each \( t \geq 0 \) a.s.
- \( K_t \) is càdlàg, increasing and \( \mathcal{G} \)-adapted process with \( K_{0^-} = 0 \).
- \( \int_0^T (Y_t - S_t) dK_t^c = 0 \) and \( \Delta K_t^d = -\Delta Y_t 1_{(Y_t - S_t) < 0} \), where denote the continuous and discontinuous parts of \( K \) respectively.
- \( (S_t)_{t \geq 0} \) is the obstacle which is a càdlàg, increasing and \( \mathcal{G} \)-adapted process.
We impose the following set of assumptions.

(a) The driver \( g : \Omega \times [0, T] \times \mathbb{R}^4 \to \mathbb{R} \) is \( \mathcal{G} \)-predictable, and satisfies
\[
\mathbb{E} \left[ \int_0^T |g(t, 0, 0, 0)|^2 dt \right] < 0,
\]
\[
|g(t, y, z, \mu, u) - g(t, y', z', \mu', u')| \\
\leq C(|y - y'| + |z - z'| + |\mu - \mu'| + \lambda_t |u - u'|),
\]
for all \( t \in [0, T] \) and all \( y, y', z, z', \mu, \mu', u, u' \in \mathbb{R} \).

(b) The terminal condition: \( \xi \in L^2 (\Omega, \mathcal{G}_T) \).

Let us first state the comparison theorem for RBSDE with default which relies on the comparison theorem for BSDE with default done by Dumitrescu et al. [4], Theorem 2.17.

**Theorem 4.1** Let \( g^1, g^2 : \Omega \times [0, T] \times \mathbb{R}^3 \to \mathbb{R} \), \( \xi^1, \xi^2 \in L^2 (\Omega, \mathcal{G}_T) \) and let the quadruplet
\[
(Y^j_i, Z^j_i, U^j_i, K^j_i)_{j=1,2}
\]
be the solution of the RBSDE with default
\[
\begin{cases}
Y^j_i = \xi^j_i + \int_t^T g^j_i(s, Y^j_i, Z^j_i, U^j_i)ds - \int_t^T Z^j_i dW_s \\
- \int_t^T U^j_i dM_s + \int_t^T dK^j_i,
\end{cases}
\]
\( t \in [0, T] \),
\[
Y^j_i = \xi^j_i, \quad t \geq T,
\]
\[
Z^j_i = U^j_i = 0, \quad t > T.
\]
The drivers \( (g^j_i)_{j=1,2} \) satisfies assumptions (a)-(b). Suppose that there exists a predictable process \( (\theta_t)_{t \geq 0} \) with \( \theta_t \lambda_t \) bounded and \( \theta_t \geq -1 \ dt \otimes dP \ a.s. \) such that
\[
g^1(t, y, z, u) - g^1(t, y, z, u') \geq \theta_t (u - u') \lambda_t.
\]
Moreover, suppose that
\begin{itemize}
\item \( \xi^1 \geq \xi^2, \ a.s. \)
\item For any \( t \in [0, T], S^1_t \geq S^2_t, \ a.s. \)
\item \( g^1(t, y, z, u) \geq g^2(t, y, z, u), \ \forall t \in [0, T], y, z, u \in \mathbb{R} \).
\end{itemize}

Then
\[
Y^1_t \geq Y^2_t, \quad \forall t \in [0, T].
\]
Theorem 4.2 Let $g^1, g^2 : \Omega \times [0, T] \times \mathbb{R}^4 \to \mathbb{R}$, $\xi^1, \xi^2 \in L^2(\Omega, \mathcal{G}_T)$ and let the quadruplet $(Y^j, Z^j, U^j, K^j)_{j=1,2}$ be the solution of the RABSDE

\[
\begin{aligned}
Y^j_t &= \xi^j + \int_t^T g^1(s, Y^j_s, Z^j_s, \mathbb{E}[Y^j_{s+\delta}|\mathcal{G}_s], U^j_s)ds - \int_t^T Z^j_s dW_s \\
&\quad - \int_t^T U^j_s dM_s + \int_t^T dK^j_s, \quad t \in [0, T], \\
Y^j_t &= \xi^j, \quad t \geq T,
\end{aligned}
\]

\[
\begin{aligned}
Z^j_t &= U^j_t = 0, \quad t > T.
\end{aligned}
\]

The drivers $(g^j)_{j=1,2}$ satisfies assumptions (a)-(b). Moreover, suppose that:

(i) For all $t \in [0, T], y, z, u \in \mathbb{R}$, $g^2(t, y, z, \cdot, u)$ is increasing with respect to $Y_{t+\delta}$ in the sense that

\[
g^2(t, y, z, Y_{t+\delta}, u) \geq g^2(t, y, z, Y'_{t+\delta}, u),
\]

for all $Y_{t+\delta} \geq Y'_{t+\delta}$.

(ii) $\xi^1 \geq \xi^2$, a.s.

(iii) For each $t \in [0, T]$, $S^1_t \geq S^2_t$, a.s.

(iv) Suppose that there exists a predictable process $(\theta_t)_{t \geq 0}$ with $\theta_t \lambda_t$ bounded and $\theta_t \geq -1$ dt $\otimes$ dP a.s., such that

\[
g^1(t, y, z, Y_{t+\delta}, u) - g^1(t, y, z, Y_{t+\delta}, u') \geq \theta_t(u - u') \lambda_t.
\]

(v) $g^1(t, y, z, Y_{t+\delta}, u) \geq g^2(t, y, z, Y_{t+\delta}, u), \forall t \in [0, T], y, z, Y_{t+\delta}, u \in \mathbb{R}$.

Then, we have

\[
Y^1_t \geq Y^2_t, \quad \text{a.e., a.s.}
\]

Proof. Consider the following RABSDE

\[
\begin{aligned}
Y^3_t &= \xi^2 + \int_t^T g^2(s, Y^3_s, Z^3_s, \mathbb{E}[Y^1_{s+\delta}|\mathcal{G}_s], U^3_s)ds - \int_t^T Z^3_s dW_s \\
&\quad - \int_t^T U^3_s dM_s + \int_t^T dK^3_s, \quad t \in [0, T], \\
Y^3_t &= \xi^2, \quad t \geq T,
\end{aligned}
\]

\[
\begin{aligned}
Z^3_t &= U^3_t = 0, \quad t > T.
\end{aligned}
\]

From Proposition 3.2 in Dumitrescu et al. [5], we know there exists a unique quadruplet of $\mathcal{G}$-adapted processes $(Y^3, Z^3, U^3, K^3) \in S^2_\mathcal{G} \times \mathcal{H}^2_\mathcal{G} \times L^2(\lambda) \times \mathcal{K}^2$ satisfies the above RBSDE since the advanced term is considered as a parameter.

Now we have by assumptions (iii)-(v) and Theorem 4.1 that

\[
Y^1_t \geq Y^3_t, \quad \text{for all } t, \text{ a.s.}
\]

Set

\[
\begin{aligned}
Y^4_t &= \xi^2 + \int_t^T g^2(s, Y^4_s, Z^4_s, \mathbb{E}[Y^3_{s+\delta}|\mathcal{G}_s], U^4_s)ds - \int_t^T Z^4_s dW_s \\
&\quad - \int_t^T U^4_s dM_s + \int_t^T dK^4_s, \quad t \in [0, T], \\
Y^4_t &= \xi^2, \quad t \geq T,
\end{aligned}
\]

\[
\begin{aligned}
Z^4_t &= U^4_t = 0, \quad t > T.
\end{aligned}
\]
By the same arguments, we get \( Y^3_t \geq Y^4_t \), a.e., a.s.

For \( n = 5, 6, \ldots \), we consider the following RABSDE

\[
\begin{cases}
Y^n_t &= \xi^2 + \int_t^T g^2(s, Y^n_s, Z^n_s, E[Y^n_s G_s], U^n_s) ds - \int_t^T Z^n_s dW_s \\
&\quad - \int_t^T U^n_s dM_s + \int_t^T dK^n_s, \\
Y^n_t &= \xi^2, \quad t \geq T, \\
Z^n_t &= U^n_t = 0, \quad t > T.
\end{cases}
\]

We may remark that it is clear that \( Y^{n-1}_{s+\delta} \) is considered to be knowing on the above RABSDE.

By induction on \( n > 4 \), we get

\[
Y^4_t \geq Y^5_t \geq Y^6_t \geq \cdots \geq Y^n_t \geq \cdots, \quad \text{a.s.}
\]

If we denote by

\[
\begin{align*}
\bar{Y} &= Y^n - Y^{n-1}, \\
\bar{Z} &= Z^n - Z^{n-1}, \\
\bar{U} &= U^n - U^{n-1}, \\
\bar{K} &= K^n - K^{n-1}.
\end{align*}
\]

By similar estimations as in the proof of Theorem 3.1, we can find that \((Y^n, Z^n, U^n, K^n)\) converges to \((Y_{s+\delta}, Z_{s+\delta}, U_{s+\delta}, K_{s+\delta})\) as \( n \to \infty \).

Iterating with respect to \( n \), we obtain when \( n \to \infty \), that \((Y^n, Z^n, U^n, K^n)\) converges to \((Y, Z, U, K) \in S^2_G \times H^2_G \times L^2(\lambda) \times K^2\), such that

\[
\begin{cases}
Y_t &= \xi^2 + \int_t^T g^2(s, Y_s, Z_s, E[Y_s G_s], U_s) ds - \int_t^T Z_s dW_s \\
&\quad - \int_t^T U_s dM_s + \int_t^T dK_s, \quad t \in [0, T], \\
Y_t &= \xi^2, \quad t \geq T, \\
Z_t &= U_t = 0, \quad t > T.
\end{cases}
\]

By the uniqueness of the solution (Theorem 3.1), we have that \( Y_t = Y_t^2 \), a.s.

Since for all \( t, Y^1_t \geq Y^3_t \geq Y^4_t \geq \ldots \geq Y_t \), a.s. it hold immediately for a.a. \( t \)

\[
Y^1_t \geq Y^2_t, \quad \text{a.s.}
\]

\[\square\]

## 5 RABSDE with Default and Optimal Stopping

We recall here a connection between RABSDE and optimal stopping problems. The following result is essentially due to El Karoui et al [6] under the Brownian filtration and to Øksendal and Zhang [13]:

**Definition 5.1** • Let \( F : \Omega \times [0, T] \times \mathbb{R}^5 \to \mathbb{R} \) be a given function such that:

- \( F \) is \( \mathcal{G} \)-adapted and \( \mathbb{E}[\int_0^T |F(t, 0, 0, 0, 0)|^2 dt] < 0 \).
Let $S_t$ be a given $\mathbb{G}$-adapted continuous process such that $\mathbb{E}[\sup_{t \in [0,T]} S_t^2] < \infty$.

- The terminal value $\xi \in L^2(\Omega, \mathbb{G}_T)$ is such that $\xi \geq S_T$ a.s.

We say that a $\mathbb{G}$-adapted triplet $(Y, Z, K)$ is a solution of the reflected ABSDE with driver $F$, terminal value $\xi$ and the reflecting barrier $S_t$ under the filtration $\mathbb{G}$, if the following hold:

1. $\mathbb{E}[\int_0^T |F(s, Y_s, Z_s, \mathbb{E}[Y_{s+\delta}\mid \mathcal{G}_s], \mathbb{E}[Z_{s+\delta}\mid \mathcal{G}_s], U_s)|^2 dt] < \infty$,

2. $Y_t = \xi + \int_t^T F(s, Y_s, Z_s, \mathbb{E}[Y_{s+\delta}\mid \mathcal{G}_s], \mathbb{E}[Z_{s+\delta}\mid \mathcal{G}_s], U_s)ds - \int_t^T dK_s - \int_t^T Z_s dB_s - \int_t^T U_s dM_s, t \in [0, T]$, 
   or, equivalently, 
   $Y_t = \mathbb{E}[\xi + \int_t^T F(s, Y_s, Z_s, \mathbb{E}[Y_{s+\delta}\mid \mathcal{G}_s], \mathbb{E}[Z_{s+\delta}\mid \mathcal{G}_s], U_s)ds - \int_t^T dK_s\mid \mathcal{G}_t], t \in [0, T]$,

3. $K_t$ is nondecreasing, $\mathbb{G}$-adapted, càdlàg process with $\int_0^T (Y_t - S_t)dK_t = 0$ and $\Delta K_t^\delta = -\Delta Y_t 1_{\{Y_t = S_t\}}$, where denote the continuous and discontinuous parts of $K$ respectively,

4. $Y_t \geq S_t$ a.s., $t \in [0, T]$.

**Theorem 5.2** For $t \in [0, T]$ let $\mathcal{T}_{[t,T]}$ denote the set of all $\mathbb{G}$-stopping times $\tau : \Omega \mapsto [t, T]$. Suppose $(Y, Z, U, K)$ is a solution of the RABSDE above.

(i) Then $Y_t$ is the solution of the optimal stopping problem

$$Y_t = \text{ess sup}_{\tau \in \mathcal{T}_{[t,T]}} \{\mathbb{E}[\int_0^\tau F(s, Y_s, Z_s, \mathbb{E}[Y_{s+\delta}\mid \mathcal{G}_s], \mathbb{E}[Z_{s+\delta}\mid \mathcal{G}_s], U_s)ds + S_{\tau} 1_{\tau < T} + \xi 1_{\tau = T}\mid \mathcal{G}_t], \ t \in [0, T]\}.$$

(ii) Moreover the solution process $K(t)$ is given by

$$K_T - K_{T^-} = \max_{s \leq t} \{\xi + \int_{T^-}^T F(r, Y_r, Z_r, \mathbb{E}[Y_{r+\delta}\mid \mathcal{G}_r], \mathbb{E}[Z_{r+\delta}\mid \mathcal{G}_r], U_r)dr - \int_{T^-}^T Z_r dB_r - S_{T^-}\}, \ t \in [0, T],$$

where $x^- = \max(-x, 0)$ and an optimal stopping time $\hat{\tau}_t$ is given by

$$\hat{\tau}_t : = \inf \{s \in [t, T] \mid Y_s \leq S_s\} \wedge T$$

$$= \inf \{s \in [t, T] \mid K_s > K_t\} \wedge T.$$  

(iii) In particular, if we choose $t = 0$ we get that

$$\hat{\tau}_0 : = \inf \{s \in [0, T] \mid Y_s \leq S_s\} \wedge T$$

$$= \inf \{s \in [0, T] \mid K_s > 0\} \wedge T$$

solves the optimal stopping problem

$$Y_0 = \sup_{\tau \in \mathcal{T}_{[0,T]}} \mathbb{E}[\int_0^\tau F(s, Y_s, Z_s, \mathbb{E}[Y_{s+\delta}\mid \mathcal{G}_s], \mathbb{E}[Z_{s+\delta}\mid \mathcal{G}_s], U_s)ds + S_{\tau} 1_{\tau < T} + \xi 1_{\tau = T}], \ t \in [0, T].$$

**Acknowledgement.** We would like to thank Professors Bernt Øksendal and Shiqi Song for helpful discussions.
References

[1] Agram, N., Bachouch, A., Øksendal, B., & Proske, F. (2018). Singular control and optimal stopping of memory mean-field processes. arXiv preprint arXiv:1802.05527.

[2] Aksamit, A., & Jeanblanc, M. (2017). Enlargement of filtration with finance in view. Springer.

[3] Dumitrescu, R., Quenez, M. C., & Sulem, A. (2017). Game options in an imperfect market with default. SIAM Journal on Financial Mathematics, 8(1), 532-559.

[4] Dumitrescu, R., Quenez, M. C., & Sulem, A. (2016). BSDEs with default jump. arXiv preprint arXiv:1612.05681.

[5] Dumitrescu, R., Quenez, M. C., & Sulem, A. (2017). American Options in an Imperfect Complete Market with Default. ESAIM: Proceedings and Surveys, 1-10.

[6] El Karoui, N., Kapoudjian, C., Pardoux, É., Peng, S., & Quenez, M. C. (1997). Reflected solutions of backward SDE’s, and related obstacle problems for PDE’s. the Annals of Probability, 702-737.

[7] Jeulin, T. (2006). Semi-martingales et grossissement d’une filtration (Vol. 833). Springer.

[8] Kharroubi, I., & Lim, T. (2014). Progressive enlargement of filtrations and backward stochastic differential equations with jumps. Journal of Theoretical Probability, 27(3), 683-724.

[9] Jeanblanc, M., Lim, T., & Agram, N. (2017). Some existence results for advanced backward stochastic differential equations with a jump time. ESAIM: Proceedings and Surveys, 56, 88-110.

[10] Øksendal, B., & Sulem, A. (2012). Singular stochastic control and optimal stopping with partial information of Itô–Lévy processes. SIAM Journal of Control and Optimization, 50(4), 2254-2287.

[11] Øksendal, B., & Sulem, A. (2016). Optimal control of predictive mean-field equations and applications to finance. In Stochastics of Environmental and Financial Economics (pp. 301-320). Springer, Cham.

[12] Øksendal, B., Sulem, A., & Zhang, T. (2011). Optimal control of stochastic delay equations and time-advanced backward stochastic differential equations. Advances in Applied Probability, 43(2), 572-596.

[13] Øksendal, B. & Zhang, T.: Backward stochastic differential equations with respect to general filtrations and applications to insider finance. Communications on Stochastic Analysis (COSA) Vol 6, No 4 (2012).
[14] Peng, S., & Yang, Z. (2009). Anticipated backward stochastic differential equations. The Annals of Probability, 37(3), 877-902.

[15] Quenez, M. C., & Sulem, A. (2014). Reflected BSDEs and robust optimal stopping for dynamic risk measures with jumps. Stochastic Processes and their Applications,

[16] Song, S. (2015). An introduction of the enlargement of filtration. arXiv preprint arXiv:1510.05212