Weighted models for level statistics across the many–body localization transition

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(Dated: August 9, 2018)

Level statistics across the many-body localization transition are studied in detail for a random disorder. The gap ratio statistics reveals characteristic inter-sample randomness reflecting fluctuations in localization properties of the system, in particular Griffiths-like rare events. Defining a mean gap ratio for a single realization of disorder we show that it has a broad, system specific distribution across the whole transition. That explains the necessity of introducing weighted random matrix ensembles that correctly grasp the sample-to-sample variation of system properties including the rare events. We consider two such approaches. One is a weighted short-range plasma model, the other a weighted power-law random banded matrix model. Treating the single sample gap ratio distribution as input, the considered weighted models yield a very good agreement both for spacing distribution including its exponential tail and the number variance up to tens of level spacings. We show explicitly that our weighted models describe the level statistics across the whole ergodic to many-body localized transition much more faithfully than earlier predictions. The remaining deviations for long-range spectral correlations are discussed and attributed mainly to the intricacies of level unfolding.

I. INTRODUCTION

It is 90 years already since Wishart in a seminal paper introduced the concept of random matrices into science. His original aim was to generalize the chi-squared distribution to multiple dimensions, random symmetric non negative matrices played then the role of random variables. The corresponding Wishart distribution found many applications from modern random matrix theory to various applications in physics, wireless communications, financial data for large portfolios etc.

The next big step came with the introduction of Gaussian ensembles and the realization of Wigner and others that spectra for usually unknown complex nuclear Hamiltonians may be understood statistically using properties of these ensembles obeying appropriate symmetries. It became a textbook knowledge that there exist in fact exactly three universality classes the Gaussian Orthogonal Ensemble (GOE) corresponds to systems invariant with respect to (generalized) time-reversal, the Gaussian Unitary Ensemble corresponds to systems with broken time-reversal invariance and the symplectic ensemble corresponds to half-integer spin systems with preserved time-reversal invariance and no other symmetries present. Thus since the sixties the common knowledge developed that spectra of many-body interacting systems are statistically well described by random matrix theory (RMT). Further justifications of successes of RMT come from the theory of Dyson yielding the gaussian ensembles from an appropriate statistical mechanics description.

An interesting development appeared in the eighties – the conjecture that statistical properties of spectra of systems chaotic in the classical limit are faithful to random matrix predictions. This came as a surprise - even simple single particle Hamiltonians containing no randomness and represented by large, very sparse (due to strong selection rules in appropriately chosen bases) matrices were statistically faithful to RMT predictions as revealed e.g. in the study of hydrogen atom spectrum in the presence of strong magnetic field inducing the so called quadratic Zeeman coupling. More precisely, after unfolding the levels (obtaining mean density equal to unity) the remaining fluctuations were faithfully represented by predictions of RMT as shown on nearest neighbor spacing distribution, $P(s)$, the so called number variance (i.e. the variance of the number of levels in the interval of length $L$), correlation functions etc. The same measures indicated, however, that the transition from the chaotic to integrable situation (described by Poisson ensemble of uncorrelated levels for systems of large dimensions) seems system specific and determined by the structure of the underlying classical mechanics in the mixed phase space.

Recent years provided another important example of such a transition between ergodic (describable by standard gaussian RMT) and integrable behavior - the appearance of many-body localization (MBL). While for weak disorder many-body interacting systems behave as expected for a long time (see above) being ergodic and following gaussian RMT predictions for a sufficiently strong disorder a gradual (for finite systems sizes) transition to localized situation occurs. This behavior at-
tracted enormous interest in last 10 years as it provides a robust example of non-ergodic behavior in a complex many-body system. Instead of effectively thermalizing (as suggested by the eigenvector thermalization hypothesis (ETH)\textsuperscript{21}) such a strongly disordered systems often remember their initial state as manifested in a series of spectacular experiments\textsuperscript{22–24}. Already early theoretical studies\textsuperscript{25} showed that a transition to MBL situation is accompanied in a change of level statistics from that corresponding to GOE to Poissonian-like for MBL. Importantly, a new measure was also introduced, the gap ratio, i.e. the ratio of consecutive energy spacings. As a dimensionless quantity the ratio does not need level unfolding that is a difficult and often not unique task\textsuperscript{26,27}. The mean ratio may be found analytically both for Poisson and GOE regime (for small matrices)\textsuperscript{28} and become a common tool to characterize a gradual transition from GOE-like ergodic, metallic situation to Poisson-like, MBL case\textsuperscript{25,29–35}.

Importantly, it has been suggested that MBL phase is indeed integrable\textsuperscript{36,37}, namely in MBL phase a complete set of local integrals of motions (LIOMS) may be defined. On one side finding LIOMs provides information about the system for a given disorder realization (LIOMs are disorder realization dependent) – on the other side the very existence of LIOMs explains the Poissonian statistics observed deep in the localized phase.

While the two extremal situations - the metallic, GOE-like ergodic behavior for a weak disorder and the full MBL phase seem to be presently quite well understood it is desirable to understand and describe the nature of MBL-ergodic transition. The problem is not simple - it has been found, in particular, that the nature of the disorder may play a decisive role in the character of the transition\textsuperscript{38}. Two types of disorder are commonly considered, a genuine random disorder as well as a quasi-periodic disorder easily realized in cold atom experiments with lasers of incommensurate frequencies\textsuperscript{22,24}. In the following we shall consider a random disorder only as it seems more elegant from the theoretical perspective.

Our aim is to demonstrate that the weighted short–range plasma model (wSRPM) model introduced\textsuperscript{39} describes faithfully the level statistics during the whole ergodic to MBL transition for a random disorder. We claim that the model proposed grasps correctly not only the bulk properties of $P(s)$ but also its exponential tails and correctly reproduces the number variance at $L$ of the order of tens level spacings. Remaining discrepancies are discussed in details providing insight into the long range spectral correlations during the transition. Furthermore, we show that the wSRPM is universal working across the MBL transition not only in spin models but also in different bosonic and fermionic systems. The gap ratio analysis\textsuperscript{39} is used to show the impact of the intra-sample fluctuations on the level statistics in MBL transition. In addition also an alternative description of level statistics in MBL transition based on power-law random banded matrix model\textsuperscript{40,41} is discussed. We compare our results with earlier propositions\textsuperscript{27,42–45} showing that the model proposed by us represents the data much more faithfully.

## II. LEVEL STATISTICS IN MBL TRANSITION

Level statistics of a system which undergoes MBL transition change from GOE statistics (for time-reversal invariant systems) to Poisson statistics (PS) of fully many-body localized system. A number of models for intermediate statistics in the MBL transition have been proposed\textsuperscript{27,43–45}. In this section we compare level spacing distributions $P(s)$ and number variances $\Sigma^2(L)$ predicted by those models with data for the standard model of MBL – XXZ spin chain described by the Hamiltonian

$$H = J \sum_{i=1}^{K} \vec{\sigma}_i \cdot \vec{\sigma}_{i+1} + \sum_{i=1}^{K} h_i \sigma_i^z, \quad (1)$$

where $J = 1$ and $h_i \in [-W, W]$ is random magnetic field. Periodic boundary conditions are assumed so that $\sigma_{K+1} = \sigma_1$. This model becomes MBL at $W_C \approx 3.7$. Fig. 1 shows level spacing distribution $P(s)$ and number variance $\Sigma^2(L)$ for the system (1) at disorder strength $W = 2.1$ compared with predictions of different proposed models\textsuperscript{43–45} of MBL transition supplemented by data for the short-range plasma model (SRPM)\textsuperscript{46}. The numerical data for spacing distribution and the number variance for XXZ spin chain are fitted with those models. Mean field plasma model. The work\textsuperscript{43} describes the flow of level statistics across the MBL transition. Close to the ergodic regime a mean field plasma model\textsuperscript{47} with an effective power–law interaction between energy levels is proposed. It predicts the level spacing distribution and the number variance to be

$$P(s) = C_1 s^\beta e^{-C_2 s^{2-\gamma}} \quad \text{and} \quad \Sigma_2(L) \propto L^\gamma \quad (2)$$

with $C_{1,2}$ determined by normalization conditions $\langle s \rangle = 1$ and $\langle \sigma \rangle = 1$. The exponents $\beta$, $\gamma$ reflect a local repulsion of energy levels and an effective range of interactions between energy levels. They are treated as fitting parameters which vary across the transition. Note that for $\gamma = 1$ the eigenvalues are interacting only locally leading to semi–Poisson statistics

$$P(s) \propto s^\beta e^{-(\beta+1)s} \quad \text{and} \quad \Sigma_2(L) \propto \frac{1}{\beta+1} L. \quad (3)$$

In the MBL transition when $\gamma < 1$ (2) predicts that tail of the level spacing distribution decays faster than exponentially with $s$ and also that the number variance $\Sigma^2(L)$ increases as a power law of $L$. The level spacing distribution and the number variance predicted by this model are denoted by the solid violet line in Fig. 1 – the values of $\beta$ and $\gamma$ are obtained by the least square fit to the bulk of $P(s)$ and the multiplicative factor in front of the $\Sigma^2(L)$ is treated as the third fitting parameter. While the bulk of the level spacing distribution is nicely recovered, the
Another work\textsuperscript{45,46} demonstrates that the level spacing distributions decay logarithmically to enable the comparison of tails of the distributions, inset shows the data in doubly linear scale; RP – the Rosenzweig-Porter (RP) ensemble can be written as the probability distribution function (JPDF) of eigenvalues. A JPDF for a Random Matrix Ensemble can be written as the probability distribution of a one-dimensional gas of classical particles with total energy $W(E_1, \ldots, E_n)$

$$P(E_1, \ldots, E_N) = Z_N^{-1} \exp(-\beta W(E_1, \ldots, E_n)),$$ \hfill (5)

where $Z_N$ is a normalization constant and the total energy

$$W(E_1, \ldots, E_n) = \sum_i U(E_i) + \sum_{i<j} V(|E_i - E_j|)$$ \hfill (6)

is determined by the trapping potential $U(E)$ and inter-particle interactions $V(|E - E'|)$. For instance, for harmonic trapping potential $U(E) \propto E^2$, and logarithmic interactions $V(|E - E'|) = -\log(|E - E'|)$ and $\beta = 1$ one recovers form (5) the JPDFs for GOE, for which the interactions in (5) are between all pairs of eigenvalues which reflects the long range spectral correlations of the GOE ensemble.

One way of constructing an ensemble with statistical properties intermediate between GOE and PS is to put a rational $\beta \in [0, 1]$ into JPDF (5) – in such a way a $\beta$-Gaussian ensemble ($\beta$GE) arises. A recent work\textsuperscript{45} uses $\beta$GE to describe the $P(s)$ and the gap ratio distribution in the MBL transition. Setting up appropriate tridiagonal matrices\textsuperscript{49} of size $n = 10^5$ and diagonalizing them, we obtain $P(s)$ and $\Sigma^2(L)$ for this ensemble – denoted by the green line with squares in Fig. 1. The agreement of this model with XXZ numerical data in the bulk of the $P(s)$ is not perfect. The disagreement in the tail of the $P(s)$ and the number variance is even more pronounced. Long–range correlations of eigenvalues in $\beta$GE are visible in the spectral rigidity of the spectrum – for the acquired data the number variance grows only logarithmically, just like in the GOE case, in a violent disagreement with the XXZ data. Thus, contrary to statements in\textsuperscript{45} based on short range correlations only, the $\beta$GE does not seem to describe the MBL transition.

**Rosenzweig-Porter ensemble.** Another work\textsuperscript{44} suggest that Rosenzweig-Porter (RP) ensemble can be appropriate to describe the MBL transition. Multifractal properties of eigenvectors of this model, which is defined as an ensemble of real symmetric (for $\beta = 1$ orthogonal class relevant for us) random matrices $M = (M_{ij})$ of size $n \times n$ with matrix elements being independent Gaussian variables with zero average values $M_{ij} = 0$ and

$$\langle M_{ii}^2 \rangle = 1, \quad \text{and} \quad \langle M_{ij}^2 \rangle = \sigma/2$$ \hfill (4)

were studied in\textsuperscript{48}. The dotted line in Fig. 1 shows the obtained level spacing distribution and the number variance which fits best the data for the XXZ spin chain at $W = 2.1$. The presented data are for $n = 3000$ and $\sigma = 0.0016$ and are in rather poor agreement even regarding the bulk of the $P(s)$ distributions. Moreover, at $L \gtrsim 3$ the number variance bends abruptly upwards – a feature which we do not observe for the $W = 2.1$ data.

**Short-range plasma models.** Another way of constructing intermediate level statistics is to restrict the range of the logarithmic interactions in (5) to a finite number $h$ which leads to a family of short-range plasma models (SRPMs)\textsuperscript{46}. Consider $N \rightarrow \infty$ particles in a ring geometry $E_0 < E_1 < \ldots < E_N < E_{N+1}$, $E_{N+1+k} = E_k$ mod $N$ with logarithmic interaction among $h$ neighbor-
ing eigenvalues so that the JPDF is given by

$$P_h^\beta(E_1, ..., E_N) = Z_N^{-1} \prod_{i=0}^{N} |E_i - E_{i+1}|^\beta \cdot |E_i - E_{i+h}|^\beta. \quad (7)$$

For integer values of $h$ and $\beta$ this model can be analytically solved yielding the level spacing distribution

$$P_h^\beta(s) = s^\beta W(s)e^{-(h\beta+1)s} \quad (8)$$

where $W(s)$ is a polynomial. The corresponding number variance has asymptotically linear behavior:

$$\Sigma_{h,\beta}^2(L) \xrightarrow{L \to \infty} \frac{L}{h\beta + 1}. \quad (9)$$

For rational values of $\beta$ (for us the particularly interesting interval is $\beta \in [0, 1]$) one has to resort to numerical methods to find $P_h^\beta(s)$ and $\Sigma_{h,\beta}^2(L)$. Those two asymptotic behaviors are exactly the two features of the XXZ data presented in Fig. 1. While grasping the bulk of $P(s)$ accurately, the SRPM model does not outperform the mean field model (2), (3). One still does not obtain the correct tails of the level spacing $P(s)$ or the correct slope of the number variance $\Sigma^2(L)$ – see the line with triangles in Fig. 1.

**III. THE WEIGHTED SRPM MODEL**

The models of level statistics presented in the preceding section are clearly insufficient to grasp the level statistics in MBL transition beyond purely local correlations reflected by the bulk of the level spacing distribution. Our aim is to propose such a model.

Results of Refs. 38, 39 indicate that large inter-sample randomness is an inherent feature of the MBL transition in systems with purely random disorder. It manifests itself in shape of a distribution $P(r_S)$ of the gap ratio for a single disorder realization $r_S = \{r_{i}\}$ (the average is over a certain part of spectrum) which significantly broadens in the regime of MBL transition. The broadening of $P(r_S)$ shows that system which has predominantly ergodic features becomes more localized for certain disorder realizations – the converse statement for mostly localized system is also true. The small fraction of events for which the system is more localized than usually reveals itself in the tail of the level spacing distribution and in the number variance. For instance consider an ensemble of matrices created in such a way that with probability $1 - p$ the matrix is taken from GOE and with probability $p \ll 1$ it has the Semi-Poisson level statistics $P_{\beta=1}^\beta$. The bulk of the level spacing distribution of such an ensemble will be very close to the Wigner distribution $P_{GOE}(s)$ of the GOE matrix ensemble (as $p \ll 1$). However, for large level spacings the distribution will be dominated by exponentially decaying tail of the level spacing distribution $P(s)$ from the small fraction of matrices with Semi-Poisson statistics. Analogously, the number variance $\Sigma^2(L)$ will be a sum of logarithmically growing number variance for GOE and linearly increasing number variance for Semi-Poisson statistics. Hence, it will be dominated by the latter and increase linearly with $L$ with a very good approximation.

This leads us to a question whether the inter-sample randomness can be responsible for the exponential tails of level spacing distribution and a linear number variance in the MBL transition via the mechanism described above. To verify this, we examine level statistics of XXZ system at certain disorder strength but accept only disorder realizations for which the $r_S$ belongs to a certain narrow interval – results for $W = 2.5$ are presented in Fig. 2. The procedure of selecting $r_S$ affects significantly the resulting level statistics. For disorder realizations with $r_S \in [0.515, 0.535]$ one could expect to extract the contribution to level statistics from ergodic systems. However, the level spacing still has exponentially decaying tail and the number variance (not shown) is linear. Therefore, even though the effects of inter-sample randomness are significant, another mechanism also affects the level statistics in MBL transition.

We assume that the other mechanism is associated with intra-sample randomness. Namely, that correlation properties of eigenvalues change significantly also within a single spectrum for a given disorder realization. The intra-sample variance, $V_I = \langle r_S^2 - \langle r_S \rangle^2 \rangle_{dis}$ (the average is taken over disorder realizations) increases in the MBL transition and reaches maximum in the MBL regime indirectly supports our assumption.

This leads us to the formulation of the weighed short-range plasma model (wSRPM) which, by definition, has JPDF given by

$$P_{gSRMP}(E_1, ..., E_N) = \sum_i c_i P_{h_i}^\beta(E_1, ..., E_N) \quad (10)$$

![Figure 2. Level spacing distribution $P(s)$ for XXZ spin chain (1) of size $K = 16$ at disorder strength $W = 2.5$ (solid blue line) – left: lin-lin scale, right: lin-log scale to facilitate comparison of tails of $P(s)$. Selecting disorder realizations for which $r_S$ is from a given interval results in statistics with properties which vary between those of an ergodic and nearly localized system.](image)
where $h_i$ and $\beta_i$ range over an appropriate set of values and $c_i$ are weight coefficients ($\sum c_i = 1$). By integrating the JPDF for wSRPM with $\delta(s - |E_k - E_{k-1}|)$ one gets the level spacing distribution

$$P_{wSRPM}(s) = \sum_i c_i P_{h_i}^{\beta_i}(s)$$

which is a linear combination of the level spacing distributions $P_{h_i}^{\beta_i}(s)$. An analogous expression holds for the number variance

$$\Sigma^2_{wSRPM}(L) = \sum_i c_i \Sigma^2_{\beta_i,h_i}(L),$$

which stems from the formula $\Sigma^2(L) = L - \int_0^L dE(L - E)(1 - \mathcal{R}_2(E))$ and the fact that the two-level correlation function $\mathcal{R}_2(E)$ for wSRPM is a linear combination of two-level functions of SRPMs $P_{h_i}^{\beta_i}$. The wSRPM defined by (10) allows one to incorporate the effects of inter- and intra-sample on level statistics in a straightforward way – simply by choosing appropriate weights $c_i$ which determine the contribution of $P_{h_i}^{\beta_i}(E_1, ..., E_N)$ to the overall level statistics.

Let us note that the family of SRPM is well suited for construction of the “mixed” level statistics as a single $P_{h_i}^{\beta_i}(E_1, ..., E_N)$ already nicely describes the bulk of the level spacing distribution in MBL transition. Moreover, SRPM allow to do “mixing” (10) of level statistics in a consistent way operating only with JPDF of the form (7).

### IV. LEVEL SPACING DISTRIBUTION AND NUMBER VARIANCE IN MBL transition

The wSRPM model, defined by (10) depends on a large number of parameters – one needs to specify JPDFs of the SRPMs $P_{h_i}^{\beta_i}$ which contribute to the full JPDF of the generalized model $P_{wSRPM}$ and find appropriate weight coefficients $c_i$. To complete this task we utilize the $P(r_S)$ distributions which encode the inter-sample randomness across the MBL transition. Distributions of $r_S$ for individual SRPMs $P_{h_i}^{\beta_i}(r_S)$ are to a good approximation Gaussians centered around $r_S^{h_i}$ which depends on $h_i$ and $\beta_i$ parameters. The corresponding distribution for wSRPM reads $P_{wSRPM}(r_S) = \sum_i c_i P_{h_i}^{\beta_i}(r_S)$ and the $h_i$, $\beta_i$ and $c_i$ parameters are chosen so that the $P_{wSRPM}(r_S)$ reproduces the $P(r_S)$ distribution for XXZ spin chain (1) most faithfully. Results are presented in Fig. 3. Distributions $P(r_S)$ are indeed recovered in the whole transition region. The oscillations of the $P_{wSRPM}(r_S)$, particularly visible for the $W = 2.5, 2.9$ stem from the fact that at most four contributions $P_{h_i}^{\beta_i}$ to JPDF of wSRPM were considered. In this way a wSRPM level statistics which reproduces the bulk of level spacing distribution as well as its exponential tail across the MBL transition arises.

Figure 3. The fit of $P(r_S)$ distributions. Distributions $P(r_S)$ of the sample-averaged gap ratio $r_S$ for the XXZ spin chain Eq. (1) are denoted by markers. The corresponding wSRPM fits are denoted with solid lines.

Figure 4. Top panel: level spacing distributions $P(s)$ during MBL transition in XXZ spin chain (1) of size $K = 16$ for various disorder strengths $W$ are denoted by markers, wSRPM results denoted by solid lines and gray dashed lines denote the level spacing distributions in the limiting GOE and PS cases; Middle panel: same as above, but vertical axis in logarithmic scale to facilitate comparison between tails of level spacing distributions; Bottom panel: dashed lines with markers – number variance $\Sigma^2(L)$ for XXZ spin chain, solid lines – results for wSRPM model. All of the quantities for SRPMs were obtained in Monte Carlo integration of (7).
Table I. Coefficients $h_i$, $\beta_i$ and $c_i$ used in wSRPM model to describe level statistics during MBL transition in XXZ spin chain (1) across the MBL transition are presented in Fig. 4. Values of the mean gap ratio $\tau$ and spectral compressibility $\chi$ for XXZ spin chain are compared with predictions of wSRPM model: $\tau_{wSRPM}$ and $\chi_{wSRPM}$.

| $W$ | $h_0$ | $h_1$ | $c_0$ | $h_1$ | $h_2$ | $h_2$ | $h_3$ | $h_3$ | $c_1$ | $c_2$ | $c_3$ | $\tau$ | $\chi$ | $\tau_{wSRPM}$ | $\chi_{wSRPM}$ |
|-----|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|----------------|----------------|
| 1.5 | 25    | 1     | 0.60  | 7     | 1     | 0.40  | 1     | 0.4   | 0.02  |
| 1.9 | 5     | 1     | 0.83  | 1     | 1     | 0.11  | 1     | 0.65  | 0.04  | 1     | 0.4   | 0.40  | 0.5306 | 0.0975          | 0.5281(13)       | 0.0765(11)       |
| 2.1 | 4     | 1     | 0.63  | 1     | 0.90  | 0.22  | 1     | 0.60  | 0.11  | 1     | 0.25  | 0.04  | 0.5219 | 0.259           | 0.5181(28)       | 0.231(16)        |
| 2.5 | 3     | 1     | 0.31  | 1     | 0.75  | 0.31  | 1     | 0.35  | 0.30  | 1     | 0.15  | 0.07  | 0.5092 | 0.358           | 0.5057(21)       | 0.342(18)        |
| 2.9 | 3     | 1     | 0.12  | 1     | 0.65  | 0.33  | 1     | 0.20  | 0.44  | 1     | 0.05  | 0.11  | 0.4720 | 0.523           | 0.4733(19)       | 0.546(21)        |
| 3.5 | 1     | 0.95  | 0.04  | 1     | 0.5   | 0.08  | 1     | 0.2   | 0.41  | 1     | 0.06  | 0.47  | 0.4107 | 0.774           | 0.4131(21)       | 0.859(17)        |
| 4.5 | 1     | 0.55  | 0.02  | 1     | 0.25  | 0.11  | 1     | 0.05  | 0.40  | 1     | 0.07  | 0.47  | 0.3938 | 0.857           | 0.3951(26)       | 0.951(15)        |

V. UNIVERSALITY

The wSRPM model has so far been used to describe level statistics in the standard model of MBL – the XXZ spin chain (1). It has already been noted in\textsuperscript{50} that there are differences in level statistics across the MBL transition in systems of hard-core bosons and fermions. In this section we demonstrate that wSRPM can faithfully reproduce level statistics in MBL transition in a system of disordered Bose-Hubbard model\textsuperscript{34} as well as in disordered Fermi-Hubbard model\textsuperscript{50}.

The system of disordered bosons is described by the Bose-Hubbard Hamiltonian

$$H_B = -J \sum_{\langle i,j \rangle} \hat{a}_i^\dagger \hat{a}_j + \frac{U}{2} \sum_i \hat{n}_i (\hat{n}_i - 1) + \sum_i \mu_i \hat{n}_i,$$

where $\hat{a}_i^\dagger, \hat{a}_i$ are bosonic creation and annihilation operators respectively, the tunneling amplitude $J = 1$ sets the energy scale, $U$ is interaction strength and the chemical potential $\mu_i$ is distributed uniformly in an interval...
This model have been shown to be MBL above a critical disorder strength $W_B$ which depends on the interaction strength $U$. The Hamiltonian for disordered fermions reads

$$H_{F0} = -J \sum_{i, \sigma = \uparrow, \downarrow} (\hat{c}_{i, \sigma} \hat{c}_{i+1, \sigma} + h.c) + U \sum_i n_{i \uparrow} n_{i \downarrow} + \sum_i \mu_i \hat{n}_i$$

where $c_i$ and $\hat{c}_i$ are fermionic creation and annihilation operators respectively; $J = 1$ and $U$ are tunneling and interaction amplitudes and $\mu_i \in [-W; W]$ is uncorrelated disorder. To avoid integrability in the absence of disorder it is sufficient to add the next-to-nearest neighbor tunneling terms

$$H_1 = -J' \sum_{i, \sigma} (\hat{c}_{i, \sigma} \hat{c}_{i+2, \sigma} + h.c)$$

and an additional symmetry breaking term

$$H_{SB} = h_B (n_{i \uparrow} - n_{i \downarrow}) + \mu_B (n_{i L \uparrow} + n_{i L \downarrow}).$$

Transition between GOE and PS statistics for the system with the full Hamiltonian

$$H_F = H_{F0} + H_1 + H_{BS}$$

has been observed in.

Level statistics across the MBL transition for the bosonic and fermionic models together with wSRPM fits are presented in Fig. 5. Similarly as in the case of XXZ spin chain, the level spacing distributions $P(s)$ and their tails are reproduced by wSRPM across the whole MBL transition with a very good accuracy. The same holds for the number variance $\Sigma^2(L)$ up to $L \lesssim 20$. Small deviations from wSRPM predictions appear only at larger scale $L \approx 40$. It is notable that the deviations are different in the two studied systems – there is no super-linear growth of number variance at large $L$ for bosons as opposed to fermions. This suggests that the long-range correlations which are beyond wSRPM are model specific. On the other hand, the exponential tails of level
spacing distributions and the finite spectral compressibility that appear already deeply in the metallic phase were observed for the XXZ spin chain as well as in the bosonic and fermionic systems. It seems that these are universal features of level statistics in MBL transition. The wSRPM model is able to grasp all of those features which provides an argument in favor of its generality. Moreover, distinct numbers of rare events occur in various systems during MBL transition which reveals itself in dissimilar correspondences between the bulk of level spacing distribution and its tail as well as the number variance – as an example compare data for fermionic system (17) for $W = 7$ and data for bosons (13) at $W = 10$. In general, different systems are characterized by different inter-sample randomness during the MBL transition – compare for instance the shape of $P(r_S)$ distributions displayed in Fig. 6 with data for the XXZ spin chain in Fig. 3. This demonstrates that the wSRPM model is in the sense of number parameters minimal to grasp MBL transition.

VI. WEIGHTED POWER-LAW RANDOM BANDED MATRICES

The wSRPM describes faithfully level statistics in MBL transition. However, it provides no information on properties of eigenstates. One particularly interesting property is multifractality of matrix elements of local operators in such states. Therefore, an identification of a random matrix model which could provide some information about eigenvectors in MBL transition can be productive.

In this section we examine an ensemble of power-law random banded matrices (PLRBM) which is the ensemble of $N \times N$ symmetric real matrices with matrix elements $H_{ij}$ being independent random Gaussian variables with

$\langle H_{ij} \rangle = 0$ and $\langle H_{ij}^2 \rangle = (1+\delta_{ij}) \left(1 + (|i-j|/B)^2 \mu \right)^{-1}$. \hspace{1cm} (18)

This ensemble interpolates between GOE statistics for $B \gg 1$, $\mu < 1$ and PS statistics which arises for $\mu > 1$ in $N \rightarrow \infty$ limit. In the special case of $\mu = 1$ and large $B$ the model can be solved by a mapping onto an effective $\sigma$ model. Numerical calculations of level statistics of the PLRBM model at the critical line $\mu = 1$ were carried out in.

We consider PLRBM of size $N = 1000$, accumulating 10000 matrices for each set of parameters $(\mu, B)$. Let us note that the exact values of the $(\mu, B)$ coefficients are strongly dependent on size $N$ of matrix from PLRBM. With growing $N$ a flow of level statistics in this model occurs – points $(\mu, B)$ with $\mu < 1$ correspond to statistics closer and closer to GOE and analogously – for $\mu > 1$ statistics flow towards PS. Calculating the $P(r_S)$ distribution for PLRBM model we have verified that $P(r_S)$ remains Gaussian in large region of parameter space $(\mu, B)$. Moreover, there exist a region of parameters for which the level spacing distributions $P_{ij}(s)$ decay exponentially and the number variance $\Sigma_{ij}(s)$ is asymptotically linear. Therefore, a similar extension as in the case of SRPM can be proposed in which the inter-sample randomness encoded in the $P(r_S)$ distribution is mimicked by considering a mixture of PLRBM with various $\mu$, and $B$, to describe the level statistics in given point of MBL transition. More precisely, matrices for
Table II. Parameters used in fitting of weighted PLRBM to level statistics of XXZ spin chain Eq. (1) across MBL transition. Values of the mean gap ratio $\tau$ and spectral compressibility $\chi$ for XXZ spin chain are compared with predictions of wSRPM model: $\tau_{wSRPM}$ and $\chi_{wSRPM}$.

| $W$ | $\mu_0$ | $c_0$ | $\mu_1$ | $c_1$ | $\mu_2$ | $c_2$ | $\mu_3$ | $c_3$ | $\tau$ | $\chi$ | $\tau_{wSRPM}$ | $\chi_{wSRPM}$ |
|-----|---------|-------|---------|-------|---------|-------|---------|-------|-------|-------|-------------|--------------|
| 1.5 | 0.75    | 1.0   | 0.95    | 0.08  | 1.025   | 0.06  | 1.1      | 0.08  | 0.5306| 0.0975| 0.5306(4)  | 0.0992(5)   |
| 1.9 | 0.85    | 0.86  | 0.975   | 0.30  | 1.025   | 0.07  | 1.1      | 0.15  | 0.5219| 0.259 | 0.5223(5)  | 0.237(21)   |
| 2.1 | 0.85    | 0.55  | 0.975   | 0.30  | 1.025   | 0.07  | 1.1      | 0.15  | 0.5092| 0.358 | 0.5089(32) | 0.314(25)   |
| 2.5 | 0.875   | 0.30  | 1.025   | 0.30  | 1.15    | 0.25  | 1.3      | 0.15  | 0.4720| 0.523 | 0.4741(21) | 0.473(36)   |
| 2.9 | 0.925   | 0.12  | 1.075   | 0.28  | 1.2     | 0.20  | 1.4      | 0.40  | 0.4390| 0.638 | 0.4391(14)| 0.635(30)   |
| 3.5 | 1.05    | 0.05  | 1.2     | 0.18  | 1.4     | 0.37  | 1.625    | 0.40  | 0.4107| 0.774 | 0.4093(26)| 0.766(28)   |
| 4.5 | 1.2     | 0.04  | 1.3     | 0.06  | 1.5     | 0.14  | 1.925    | 0.76  | 0.3938| 0.857 | 0.3942(31)| 0.890(27)   |

Table II. Parameters used in fitting of weighted PLRBM to level statistics of XXZ spin chain Eq. (1) across MBL transition. Values of the mean gap ratio $\tau$ and spectral compressibility $\chi$ for XXZ spin chain are compared with predictions of wSRPM model: $\tau_{wSRPM}$ and $\chi_{wSRPM}$.

A weighted PLRBM (wPLRBM) model are drawn from PLRBM ensemble characterized by parameters $(\mu_i, B_i)$ with probability $c_i$ with $i = 0, 1, 2, 3$. Analogously to the wSRPM model, the parameters $\mu_i, B_i$ and $c_i$ are chosen in such a way that the $\langle r_S \rangle$ distribution is faithfully recovered as presented in Fig. 7. As we have verified, for $N = 1000$ it suffices to restrict oneself to $B_i = 0.35$ and vary $\mu_i \in [0.85, 2]$ to obtain the results. The specific values of parameters $\mu_i$ and $c_i$ together with comparison of values of the average gap ratio $\tau$ and spectral compressibility $\chi$ are presented in Tab. II. Let us note that this model gives very good agreement at the level of ten level spacings, particularly, the predicted values of $\tau$ reproduce within the error bars the values for XXZ spin chain. Certain deviations from the XXZ spin chain data are visible in the spectral compressibility $\chi$. Typically, the weighted PLRBM model gives underestimated value of the spectral compressibility – which is different from the wSRPM case.

The PLRBM model was introduced as a model for studies of critical properties of Anderson localization. In its direct interpretation the model (18) describes a single particle on one dimensional sample with disorder and with long-range hopping – tunneling amplitude decays according to a power-law with distance. Our results show that the PLRBM can be used also in MBL transition provided the weighted mixture of matrices is considered. One way of interpreting this result is that MBL can be thought of as a single particle localization in a ‘Fock-space lattice’ with complex geometry$^{56,57}$ (reflecting the quantum many-body character of the phenomenon). Another approach is to view wPLRBM as the Hamiltonian of the system at late stages of diagonalization flow$^{58–60}$ so that the diagonal entries represent random eigenvalues associated with soon-to-be LIOMs and the quickly decaying off-diagonal elements account for still present interactions which become weaker and weaker close to the MBL phase.

If the latter is true, then to get the multifractal properties of matrix elements of local operators$^{51,52}$ one has to know transformation between the $\sigma_i^z$ eigenbasis (in which the Hamiltonian matrix is straightforwardly computed) and the basis in which the Hamiltonian becomes the banded matrix. This would also be the basis in which an interesting relation between the multifractal dimension $D_1$ and the spectral compressibility $\chi$ holds. This is beyond the scope of the present paper.

Let us mention for completeness that although we have compared MBL transition statistics for bosons and fermions with wSRPM in Section V a similar accuracy one obtains also for the wPLRBM approach which thus provides a promising alternative description of MBL transition. We discuss advantages and drawbacks of both these approaches more in the conclusions section.

VII. LONG-RANGE SPECTRAL CORRELATIONS

The number variance $\Sigma^2(L)$ at large $L$ reflects correlations between energy levels which lie far apart in the spectrum of a system. Such long-range correlations between eigenvalues are strong in the GOE ensemble, resulting in the so called spectral rigidity which is apparent in the asymptotic behavior of the number variance $\Sigma^2_{GOE}(L) \rightarrow \log(L)$ at $L \gg 1$. The spectral rigidity of GOE is associated with the fact that the logarithmic interactions act between all pairs of eigenvalues in the JPDF for GOE (6). And it is the spectral rigidity of $\beta$-Gaussian model which causes the large discrepancy between its prediction and the number variance for XXZ spin chain in Fig. 1. On the other hand, the SRPMs describe interactions only among a finite number $h$ of neighboring eigenvalues which results in the spectral compressibility of those models $\Sigma^2_{GOE}(L) \rightarrow \chi(L) \rightarrow L \gg 1$, with $0 < \chi < 1$. The resulting spectral compressibility of the wSRPM model allows to grasp the linear behavior of number variance in the MBL transition. The similar behavior can be also obtained with wPLRBM as presented in the preceding section.

To be able to compare statistical properties of eigenvalues from different parts of spectra of various systems, one has to perform the unfolding of energy levels$^{11}$ – the procedure of setting mean level spacing to unity. Unfortunately, the number variance $\Sigma^2(L)$ is very sensitive to details of the unfolding$^{26}$ which has already been a source of discrepancies in descriptions of level statistics in the MBL transition$^{27,43}$. Consider a set of eigenvalues $\{E_i\}$...
ordered in an ascending manner. During the unfolding, a
level staircase function \( \sigma(E) = \sum_i \Theta(E-E_i) \) is separated
into smooth and fluctuating parts \( \sigma(E) = \bar{\sigma}(E) + \delta \sigma(E) \)
and the eigenvalues are mapped via

\[ E_i \rightarrow \epsilon_i = \bar{\sigma}(E_i). \]  

(19)

The difficulty of unfolding lies in an ambiguity of the def-
dinition of the smooth part \( \bar{\sigma}(E) \) of the staircase function.
The most common way is to fit the staircase function \( \sigma(E) \) for each disorder realization with a polynomial of a
small degree which determines the smooth part \( \bar{\sigma}(E) \).

In our case, a set of \( n = 400 \) consecutive eigenvalues is
gathered and the resulting level staircase is fitted with a
straight line which defines the smooth part \( \bar{\sigma}(E) \) used in
the unfolding of energy levels. For each disorder realization
7 non-overlapping sets of \( n = 400 \) eigenvalues from the
middle of spectrum are taken – effectively employing \( \approx 20\% \) of the spectrum to the analysis as the matrix
size for \( K = 16 \) is equal to 12870. The finite size \( n \) of the set of eigenvalues introduces a correction \(-a_2 L^2/n\) to
the number variance\(^{35}\). Carrying out the unfolding with
\( n = 50, 100, 200, 400, 800 \) we verify that it is indeed the
case. We perform a quadratic fit to \( \Sigma^2(L) \) in the interval
\( L \in [10, 70] \) and obtain the coefficient \( a_2 \) which is nearly
constant. Therefore, in order to eliminate the quadratic
correction and thus to get rid of the finite \( n \) effects we
subtract the \(-a_2 L^2/n\) term from the number variance
data. Let us note that unfolding with finite number \( n \) of
energy levels can have two consequences. For eigenval-
ues which are strongly correlated at large distances (e.g.
GOE), it destroys level correlations at approximately \( n \)
level spacings meaning that at this ranges the eigenvalues
become uncorrelated. Hence, the number variance be-
comes overestimated at \( L \approx n \). The converse is true for
uncorrelated energy levels – unfolding based on \( n \) energy
levels introduces correlations between them at a certain
scale – and the number variance is underestimated. We
have checked that our unfolding procedure (together with
the \(-a_2 L^2/n\) term subtraction) allows us to get correct
number variances in the two limiting cases of GOE and
PS statistics up to \( L \approx 100 \).

The number variances for the XXZ spin chain (1) at
various disorder strengths \( W \) together with the wSRPM
results from the preceding section are presented in Fig. 9.
Nearly perfect agreement between the XXZ spin chain
data and the predictions of wSRPM visible in Fig. 4
for \( L \in [0, 10] \) is lost. Small deviations from the linear
behavior of the number variance predicted by wSRPM
appear at larger scales which was already indicated by the
slight discrepancies between spectral compressibility \( \chi \)
of the data and the prediction of wSRPM. There are
two distinct regimes. For metallic systems with disor-
der strengths \( W < 2.1 \) the number variance obtained
from the wSRPM is smaller than the result for the XXZ
spin chain. This indicates that there exists a regime
(for \( L \gtrsim 20 \)) where the number variance grows faster
than linearly which was interpreted in\(^{37}\) as a signature of
anomalous Thouless energy in the system\(^{41}\). We indicate
below that this behavior of the number variance for large
\( L \) has to be examined with an uttermost caution. The
second regime arises as the disorder strength increases
above \( W \approx 2.5 \). Then, the number variance predicted
by wSRPM slightly overestimates the number variance
for the XXZ spin chain. As we have checked this effect
diminishes as one changes the system size from \( K = 14 \)
through \( K = 16 \) to \( K = 18 \) and thus it is likely a fi-
nite size effect. However, we cannot completely exclude
the possibility that there are some remaining long-range
correlations between eigenvalues in the system which are
not grasped within the wSRPM.

Large \( L \) behavior of the number variance \( \Sigma^2(L) \) for
XXZ spin chain is compared with results for wPLRB model in Fig. 10. The distinctive feature of the
wPLRB model is that it underestimates the number
variance \( \Sigma^2(L) \) for the considered system. While the
quality of wSRPM and wPLRB predictions close to the
ergodic regime (up to \( W \lesssim 2.5 \)) is similar, the wPLRB model performs better at larger disorder strengths. One

\[ \Sigma^2(L) \approx 20 40 60 80 100 \]

\[ L \approx 10 \]

\[ 20 40 60 80 100 \]

\[ n \approx 400 \]

\[ 20 40 60 80 100 \]

\[ L \approx 100 \]

\[ 20 40 60 80 100 \]

\[ n \approx 400 \]

\[ 20 40 60 80 100 \]

\[ L \approx 100 \]

\[ 20 40 60 80 100 \]

\[ n \approx 400 \]

\[ 20 40 60 80 100 \]

\[ L \approx 100 \]

\[ 20 40 60 80 100 \]

\[ n \approx 400 \]

\[ 20 40 60 80 100 \]

\[ L \approx 100 \]
where the exponent $\gamma$ acquires values up to $\gamma \approx 1.4$. The number variance obtained by us for $W = 1.9$ has clearly some region in which it increases faster than linearly, but such a power-law growth is not observed by us. This discrepancy has its root in the unfolding. Unfolding employed in$^{27}$ relies on assumption that the shape of mean density of states obtained for the system at given disorder strength can be used (after appropriate linear transformations) to unfold large portions of spectrum of the system taking $n \approx 6000$ consecutive energy levels for $K = 18$. The fluctuations of density of eigenvalues on the scales of tens or hundreds of eigenvalues which are different for different disorder realizations are not incorporated in the $\sigma(E)$ as it is determined by the mean density of states in which such fluctuations are averaged out.

Fig. 11 compares the number variances obtained after the local linear unfolding with $n = 400$ consecutive eigenvalues and after the unfolding of$^{27}$. The results agree up to $L \approx 15$. In order to show that the difference between the results stems from the density fluctuations we introduce a particular density modulation to the data from the local linear unfolding. Namely, the unfolding is modified so that the eigenvalues are mapped via

$$E_i \rightarrow \epsilon_i = \sigma(E_i) + a(E_i - E_C)^2,$$  \hspace{1cm} (20)

where $E_C$ lies in the middle of the energy interval which is unfolded. The $a(E_i - E_C)^2$ term mimics the density fluctuations which were not incorporated into $\sigma(E)$, for $a = 0$ (20) reduces to the local linear unfolding (19). Such a density modulation does not alter $P(s)$ at all, however, it modifies the number variance exactly in the manner which allows us to reproduce the result of$^{27}$ and showing that the density fluctuations are the mechanism which causes the power-law growth of the number variance.

In conclusion, the behavior of the number variance $\Sigma^2(L)$ suggests that long-range spectral correlations might be present in the level statistics of XXZ spin chain during MBL transition. This feature of MBL transition lies beyond the scope of wSRPM, however, as we demonstrate in the next section it is model dependent. It is not clear whether the unfolding employed in$^{27}$ is justified. As we have indicated, it does not take into account variations of density of eigenvalues at scales of tens and hundreds of level spacings for a given disorder realization. Let us point out that the situation differs starkly from the usual RMT where a random matrix depends on the number of random entries which scales as square of its size whereas the number of random entries in the Hamiltonian of the XXZ spin chain scales only as logarithm of the size of the Hilbert space of the system. Therefore one may expect that while the density fluctuations average out for RMT and using Wigner’s semi-circle to unfold GOE is a good idea it may not be the case for the many body quantum systems which undergo MBL transition.

![Figure 11](image.png)

Figure 11. The number variance $\Sigma^2(L)$ for XXZ spin chain (1) of size $K = 18$ for $W = 1.5$. The dashed line show result for local linear unfolding with $n = 400$, the solid line for unfolding based on mean density of states$^{27}$ with $n = 6000$. The number variance $\Sigma^2(L)$ obtained after introducing of fluctuations of density of eigenvalues with parameter $a$ are denoted with lines with markers.
VIII. CONCLUSION

Examining the bulk and the tail of level spacing distribution together with the number variance we have demonstrated that the proposed models of spectral statistics across MBL transition\textsuperscript{27,43–45} are insufficient to grasp level statistics on the level of tens of level spacings. We have proposed two possible candidates for statistical ensembles that correctly grasp the inter-sample and intra-sample randomness of the statistical data and revealed by broad distributions of the so called gap ratio. Let us underline that those broad distributions are not due to finite size effects but rather are manifestations of rare Griffiths-like events. Their existence necessitates the introduction of weighted ensembles that are the statistical mixtures of well known random matrix models. The first proposition is based on short range plasma models (and has been referred to as wSRPM). It generalizes the short-range plasma model\textsuperscript{46} and describes faithfully the flow of level statistics during MBL transition. According to wSRPM the intermediate spectral statistics across the MBL transition stem from correlations between eigenvalues that are present only at a finite range $h$. In the ergodic phase the range $h$ diverges resulting in GOE statistics and as the system flows towards MBL phase the range of correlations diminishes. At certain point the interactions become local ($h = 1$), finally in the vicinity of MBL phase the level repulsion vanishes ($\beta \rightarrow 0$) resulting in the Poisson statistics. The wSRPM can be used to model level statistics across MBL transition in a variety of spin, bosonic and fermionic systems with random disorder. It assumes that there are no correlations between eigenvalues at ranges larger than $h$ predicting finite spectral compressibility $\chi \in [0,1]$ across the transition. The latter seems to be approximately true for the studied systems albeit small deviations from the linear behavior of the number variance have been noticed. This may be either an artifact of the unfolding procedure or could also stem from weak long-range interactions which are size and model dependent and are also present in those systems during MBL transition.

The second proposition is a weighted ensemble of power law banded random matrices. An appropriate mixture of PLBRM (again necessitated by a broad distribution of gap ratio in physical samples) seems to be at least competitive to wSRPM leading even to smaller deviations of the fitted model from the data for large range correlations. Even large statistical samples used in these work are insufficient to rule out weighted PLBRM or wSRPM scenarios. Both approaches have their advantages. While for SRPMs the eigenvalues may be generated by brute force Monte Carlo integration of the JPDF, a softer analytic approach, working at any range of of eigenvalues interaction, $h$, is possible following the path shown by Bogomolny and coworkers\textsuperscript{46}. It provides semi-analytic expressions for the level spacing distribution $P(s)$ and, more importantly, gives analytical formulas for asymptotic behavior of the number variance $\Sigma^2(L)$ as well as for the tails of $P(s)$. Moreover, the wSRPM proposes a concrete microscopic description of correlations between eigenvalues across the whole MBL transition. On the other hand that approach provides us with no clue on the eigenvectors behavior. On the contrary PLBRM model provides access to both eigenvalues and eigenvectors by a direct (although costly) diagonalization of a large number of matrices from the ensemble. The drawback of this approach is that the parameters of the model ($B$ and $\mu$, compare Eq. 18) are random matrix size dependent. In addition, the wPLRB predictions are dependent on details of the unfolding. There are no analytical results for this model at finite $N$ or $\mu \neq 1$ so a clear picture of correlations between eigenvalues is not available.

IX. ACKNOWLEDGMENTS

We acknowledge fruitful and enlightening discussions with D. Delande as well as exchanges on unfolding procedures with A. M. Garcia-Garcia and M. Sieber. This work was performed with the support of EU via Horizon2020 FET project QUIC (nr. 641122). Numerical results were obtained with the help of PL-Grid Infrastructure. We acknowledge support of the National Science Centre (PL) via project No.2015/19/B/ST2/01028 (P.S.) and the QuantERA programme No. 2017/25/Z/ST2/03029 (J.Z.)

\footnotesize{jakub.zakrzewski@uj.edu.pl

1. J. Wishart, Biometrika 20A, 32 (1928).
2. R. Janik and M. Nowak, J. Phys. A Math. Gen. 36, 3629 (2003).
3. C. W. J. Beenakker, Phys. Rev. Lett. 70, 1155 (1993).
4. J. M. Verbaarschot and I. Zahed, Phys. Rev. Lett. 70, 3852 (1993).
5. Y. V. Fyodorov and H.-J. Sommers, J. Math. Phys. 38, 1918 (1997).
6. Y. V. Fyodorov and B. A. Khoruzhenko, Phys. Rev. Lett. 83, 65 (1999).
7. A. Zanella, M. Chiani, and M. Z. Win, IEEE Transactions on Communications 57, 1050 (2009).
8. J.-P. Bouchaud and M. Potters, Theory of Financial Risks (Cambridge University Press, Cambridge, 2001).
9. ed. by C.E. Porter, Statistical Theory of Spectra: Fluctuations (Academic, New York, 1965).
10. M. L. Mehta, Random Matrices (Revised and Enlarged Second Edition) (Elsevier, 1990).
11. F. Haake, Quantum Signatures of Chaos (Springer, Berlin, 2010).
12. F. J. Dyson, J. Math. Phys. 3, 140 (1962).}
