ABSTRACT

It is shown how some results on harmonic maps within the chiral model can be carried over to self-dual gravity. The WZW-like action for self-dual gravity is found.
1. Introduction

Some years ago Ward [1] and Husain [2] showed that self-dual gravity could be considered as a principal chiral model within the $\text{su}(\infty)$ or the Poisson algebras.

In our previous paper [3] the analogous point of view has been presented but we have considered self-dual gravity to be the $\hbar \to 0$ limit of the principal chiral model in the Moyal bracket algebra. This last approach seems to be natural and convenient as the associative $\ast$-algebra defining the Moyal bracket algebra has a transparent interpretation.

I.A.B. Strachan and T. Matos suggested (private communication) that the results of [3] should be very closely related to the considerations of Fairlie, Fletcher and Zachos [4] on the $\text{su}(N)$ algebra as embedded in the Moyal bracket algebra and also to some results on the harmonic maps, especially to ones given by Uhlenbeck [5] (see also [6,7,8]).

The aim of the present paper is to give a partial answer to those suggestions.

In Section 2 we recall the main points of the previous work [8]. In Section 3, using the embedding of $\text{su}(N)$ in the Moyal bracket algebra [4], the construction of solutions to Husain’s heavenly equation [2] is given which, in fact, appears to be the Fourier expansion with the coefficients satisfying $\text{SU}(N)$ principal chiral equations for $N \to \infty$.

Section 4 is devoted to indicating how Uhlenbeck’s considerations on harmonic maps into finite Lie groups [5] can be carried over to the case of the groups defined by the Moyal $\ast$-product. Finally, in Section 5 we consider the Wess-Zumino-Witten-like (WZW-like) action [9,10] within the Moyal formalism and it is shown that the action for Husain’s heavenly equation is the $\hbar \to 0$ limit of the WZW-like action in the Moyal algebra.
2. The Moyal deformation of Husain’s heavenly equation

In this section we recall some results of our previous paper [3].

Let $\mathbf{G}$ be a matrix Lie group and $\mathcal{G}$ its Lie algebra. Consider the $\mathbf{G}$ principal chiral model in a connected and simply connected submanifold $\Omega$ of the two-dimensional Euclidean space $\mathbb{R}^2$.

The principal chiral equations read

$$\partial_x A_y - \partial_y A_x + [A_x, A_y] = 0,$$  \hspace{1cm} (2.1a)

$$\partial_x A_x + \partial_y A_y = 0,$$  \hspace{1cm} (2.1b)

where $A_\mu \in \mathcal{G} \otimes C^\infty(\Omega)$, $\mu \in \{x, y\}$, stand for the chiral potentials and $(x, y)$ are the Cartesian coordinates in $\Omega$. Then, from (2.1b) it follows that there exists a function $\theta = \theta(x, y) \in \mathcal{G} \otimes C^\infty(\Omega)$ such that

$$A_x = -\partial_y \theta, \quad \text{and} \quad A_y = \partial_x \theta.$$  \hspace{1cm} (2.2)

Inserting (2.2) into (2.1a) one gets the principal chiral equations to read

$$\partial_x^2 \theta + \partial_y^2 \theta + [\partial_x \theta, \partial_y \theta] = 0.$$  \hspace{1cm} (2.3)

Equivalently, we can proceed as follows. From (2.1a) it follows that $A_\mu, \mu \in \{x, y\}$, is of the pure gauge form, i.e., there exists a $\mathbf{G}$-valued function $g = g(x, y)$ such that
\[ A_\mu = g^{-1} \partial_\mu g. \]  

(2.4) Substituting (2.4) into (2.1b) we get another form of the principal chiral equations

\[ \partial_\mu (g^{-1} \partial_\mu g) = 0 \]  

(2.5) (Summation over \( \mu \) is assumed.).

Eq. (2.5) means that \( g : \Omega \to G \) is a harmonic map from \( \Omega \subset \mathbb{R}^2 \) into the Lie group \( G \) (or, simply, \( g : \Omega \to G \) is a \( G \) harmonic map).

Comparing (2.2) and (2.4) one quickly finds the relations

\[ g^{-1} \partial_x g = -\partial_y \theta \quad \text{and} \quad g^{-1} \partial_y g = \partial_x \theta. \]  

(2.6)

Then, the Lagrangians leading to (2.3) or (2.5) read (under the assumption that the algebra \( \mathcal{G} \) is semisimple)

\[ \mathcal{L}_{Ch} := \alpha \text{Tr} \left\{ \frac{1}{3} \theta [\partial_x \theta, \partial_y \theta] - \frac{1}{2} (\partial_x \theta)^2 + (\partial_y \theta)^2 \right\} \]  

(2.7) or

\[ \mathcal{L}'_{Ch} = -\beta \text{Tr} \left\{ (g^{-1} \partial_\mu g)(g^{-1} \partial_\mu g) \right\} \]

\[ = \beta \text{Tr} \left\{ (\partial_\mu g)(\partial_\mu g^{-1}) \right\}, \]  

(2.8)

respectively; where \( \alpha > 0 \) and \( \beta > 0 \) are some constants.

Now, let \( \hat{G} \) be some Lie group of linear operators acting on the Hilbert space \( L^2(\mathbb{R}^1) \) and let \( \hat{\mathcal{G}} \) be the Lie algebra of \( \hat{G} \). Consider the \( \hat{G} \) principal chiral model. The principal chiral equations read now
\[ \partial_x \hat{A}_y - \partial_y \hat{A}_x + [\hat{A}_x, \hat{A}_y] = 0, \]  
\[ \partial_x \hat{A}_x + \partial_y \hat{A}_y = 0, \]  
where \( \hat{A}_\mu = \hat{A}_\mu(x,y) \in \hat{G} \otimes C^\infty(\Omega), \mu \in \{x,y\} \).

Analogously as before, from (2.9b) it follows that there exists \( \hat{\theta} = \hat{\theta}(x,y) \in \hat{G} \otimes C^\infty(\Omega) \) such that

\[ \hat{A}_x = -\partial_y \hat{\theta}, \quad \text{and} \quad \hat{A}_y = \partial_x \hat{\theta}. \]  
(2.10)

Then, (2.9a) under (2.10) gives

\[ \partial_x^2 \hat{\theta} + \partial_y^2 \hat{\theta} + [\partial_x \hat{\theta}, \partial_y \hat{\theta}] = 0. \]  
(2.11)

Equivalently, from (2.9a) one infers that

\[ \hat{A}_\mu = \hat{g}^{-1} \partial_\mu \hat{g}, \quad \mu \in \{x,y\}, \]  
(2.12)

where \( \hat{g} = \hat{g}(x,y) \) is some \( \hat{G} \)-valued function on \( \Omega \). Substituting (2.12) into (2.9b) we get

\[ \partial_\mu (\hat{g}^{-1} \partial_\mu \hat{g}) = 0. \]  
(2.13)

It is convenient to define a new operator-valued function \( \hat{\Theta} = \hat{\Theta}(x,y) \) by

\[ \hat{\Theta} := i\hbar \hat{\theta}. \]  
(2.14)

Thus, by (2.11), \( \hat{\Theta} \) satisfies the following equation
\[ \partial_x^2 \hat{\Theta} + \partial_y^2 \hat{\Theta} + \frac{1}{i\hbar} [\partial_x \hat{\Theta}, \partial_y \hat{\Theta}] = 0. \] (2.15)

The Weyl correspondence \( \mathcal{W}^{-1} \) leads from \( \hat{\Theta} = \hat{\Theta}(x, y) \) to the function \( \Theta = \Theta(x, y, p, q) \), \( \Theta = \mathcal{W}^{-1}(\hat{\Theta}) \), defined on \( \Omega \times \mathbb{R}^2 \) according to the formula

\[ \Theta = \Theta(x, y, p, q) := \int_{-\infty}^{+\infty} \exp \left( \frac{i p \xi}{\hbar} \right) \left< q - \frac{\xi}{2} | \hat{\Theta}(x, y) | q + \frac{\xi}{2} \right> d\xi. \] (2.16)

Then \( \Theta \) satisfies the Moyal deformation of Husain’s heavenly equation

\[ \partial_x^2 \Theta + \partial_y^2 \Theta + \{\partial_x \Theta, \partial_y \Theta\}_M = 0, \] (2.17)

where \( \{\cdot, \cdot\}_M \) denotes the Moyal bracket i.e.,

\[ \{f_1, f_2\}_M := \frac{1}{i\hbar} (f_1 * f_2 - f_2 * f_1) = f_1 \frac{2}{\hbar} \sin \left( \frac{\hbar}{2} \mathcal{P} \right) f_2, \]

\[ \mathcal{P} := \frac{\partial}{\partial q} \frac{\partial}{\partial p} - \frac{\partial}{\partial p} \frac{\partial}{\partial q}; \quad f_1 = f_1(x, y, p, q), \quad f_2 = f_2(x, y, p, q). \] (2.18)

The Moyal \(*\)-product is defined by

\[ f_1 * f_2 := f_1 \exp \left( \frac{i \hbar}{2} \mathcal{P} \right) f_2. \] (2.19)

If the functions \( f_1 \) and \( f_2 \) are independent of \( \hbar \), then
\[
\lim_{\bar{h} \to 0} f_1 \ast f_2 = f_1 f_2, \quad \lim_{\bar{h} \to 0} \{f_1, f_2\}_M = \{f_1, f_2\}_P := f_1 \overleftarrow{\mathcal{P}} f_2,
\]  \tag{2.20}

where \(\{\cdot, \cdot\}_P\) denotes the Poisson bracket.

Equivalently, defining the function

\[
g = g(x, y, p, q) := W^{-1}(\hat{g}(x, y)) \tag{2.21}
\]

one gets from (2.13)

\[
\partial_\mu \left( g^{-1} \ast \partial_\mu g \right) = 0, \tag{2.22}
\]

where \(g^{-1}\) denotes the inverse of \(g\) in the sense of the Moyal \(\ast\)-product i.e.,

\[
g^{-1} \ast g = g \ast g^{-1} = 1. \tag{2.23}
\]

Comparing (2.22) with (2.5) we can say that the function \(g = W^{-1}(\hat{g}(x, y))\) defines a harmonic map from \(\Omega\) into the Lie group \(G_* := W^{-1}(\hat{G})\).

As it has been shown in [3] the Lagrangians leading to Eq. (2.17) or Eq. (2.22) read

\[
\mathcal{L}^{(M)}_{SG} = -\frac{1}{3} \Theta \ast \{\partial_x \Theta, \partial_y \Theta\}_M + \frac{1}{2} \left( (\partial_x \Theta) \ast (\partial_x \Theta) + (\partial_y \Theta) \ast (\partial_y \Theta) \right) \tag{2.24}
\]

or

\[
\text{or}
\]
\[ L'_{SG} := -\frac{\hbar^2}{2} (g^{-1} \ast \partial_\mu g) \ast (g^{-1} \ast \partial_\mu g), \] (2.25)

respectively. To derive Eq. (2.22) one can use \( L'_{SG} \) or equivalently, the Lagrangian \( L''_{SG} \) defined by

\[ L''_{SG} = \frac{\hbar^2}{2} (\partial_\mu g) \ast (\partial_\mu g^{-1}). \] (2.26)

(Compare with (2.8)).

Note also the relations which follow from (2.10), (2.12), (2.14) and from the definitions of \( \Theta \) and \( g \). Namely

\[ \partial_y \Theta = -i\hbar g^{-1} \ast \partial_x g \quad \text{and} \quad \partial_x \Theta = i\hbar g^{-1} \ast \partial_y g. \] (2.27)

Assume now that the function \( \Theta \) is analytic in \( \hbar \), i.e., [11]

\[ \Theta = \sum_{n=0}^{\infty} \hbar^n \Theta_n, \] (2.28)

where \( \Theta_n = \Theta_n(x, y, p, q), n = 0, 1, \ldots, \) are independent of \( \hbar \). If \( \Theta \) is a solution of (2.17), then by (2.20) one concludes that the function \( \Theta_0 \) satisfies Husain's heavenly equation [2]

\[ \partial_x^2 \Theta_0 + \partial_y^2 \Theta_0 + \{ \partial_x \Theta_0, \partial_y \Theta_0 \}_P = 0. \] (2.29)

Moreover, the Lagrangian \( L_{SG} \) leading to Eq. (2.29) can be quickly found to read
\[ \mathcal{L}_{SG} = \lim_{\hbar \to 0} \mathcal{L}_{SG}^{(M)} = -\frac{1}{3} \Theta_0 \{ \partial_x \Theta_0, \partial_y \Theta_0 \} P + \frac{1}{2} \left( (\partial_x \Theta_0)^2 + (\partial_y \Theta_0)^2 \right). \] (2.30)

(The Lagrangians of this type were first considered by Boyer, Finley and one of us (J.F.P.) [12] in the context of the first and second heavenly equations.)

Therefore, self-dual gravity appears to be the \( \hbar \to 0 \) limit of the principal chiral model for the Moyal bracket algebra.

One can interpret self-dual gravity to be the principal chiral model for the Poisson bracket algebra. [1,2].

These two approaches are evidently equivalent. However, the first one seems to be much more convenient as the Moyal bracket is defined in a natural manner as the commutator in the associative algebra defined by the Moyal \( \ast \)-product (see (2.18) and (2.19)). Thus, in a sense the Moyal bracket algebra can be considered to be an infinite dimensional matrix Lie algebra. Especially interesting is the case when the group \( G_\ast \) is a subgroup of \( U_\ast \), where

\[ U_\ast := \{ f = f(p, q) \in C^\infty(\mathbb{R}^2); f \ast \bar{f} = \bar{f} \ast f = 1 \}; \] (2.31)

(the bar stands for the complex conjugation.).

It means that \( \hat{G} = \mathcal{W}(G_\ast) \) is a subgroup of the group \( \hat{U} \) of unitary operators acting on \( L^2(\mathbb{R}^1) \).

Now one quickly finds that if \( g = g(x, y, p, q) \) is an \( U_\ast \)-valued function, then \( g^{-1} \ast \partial_\mu g = \bar{g} \ast \partial_\mu g \) is pure imaginary, and consequently, by (2.27), the function \( \Theta \) can be chosen so as to be real.

Finally, we recall the method proposed in [3] of searching for solutions of Husain’s heavenly equation (2.29).

Let \( \Psi : \mathcal{G} \to \hat{G} \) be a Lie algebra homomorphism, and let (summation over \( a! \))
\[ \theta = \theta(x, y) = \theta_a(x, y) \tau_a \in \mathcal{G} \otimes C^\infty(\Omega) \]  

be some solution of the principal chiral equation (2.3); \( \tau_a \in \mathcal{G}, \ a = 1, 2, \ldots, \text{dim} \ \mathcal{G} \) constitute a basis of \( \mathcal{G} \). Then

\[ \hat{\Theta} = \hat{\Theta}(x, y) = i \hbar \theta_a(x, y) \hat{X}_a \]

\[ \hat{X}_a := \Psi(\tau_a) \]  

satisfies Eq. (2.15). Therefore, the function \( \Theta \) defined by (see (2.16))

\[ \Theta = \Theta(x, y, p, q) = i \hbar \theta_a(x, y) X_a(p, q) \]

\[ X_a(p, q) := \mathcal{W}^{-1}(\hat{X}_a) \]  

fulfills the *Moyal deformation of Husain’s heavenly equation*, (2.17).

Consequently, if \( \Theta \) is of the form (2.28) then \( \Theta_0 \) satisfies *Husain’s heavenly equation*, (2.29). Moreover, if the Lie group \( \hat{\mathcal{G}} \) defined by the Lie algebra \( \hat{\mathcal{G}} \) appears to be a subgroup of the group \( \hat{\mathcal{U}} \) of unitary operators in \( L^2(\mathbb{R}^1) \) then the functions \( \Theta \) and \( \Theta_0 \) are real.

[Remark:]

In contrary to the case of finite matrix groups the equivalence between (2.11) and (2.13), or (2.17) and (2.22) are not proved to hold in all generality.

We have the implications
but the inverse implications, in general, may not be true.

This problem should be considered elsewhere.

3. The construction of solutions of Husain’s heavenly equation from $SU(N)$ chiral fields

Consider the $SU(N)$ principal chiral model. In this case $A_\mu = A_\mu(x, y) \in su(N) \otimes C^\infty(\Omega)$, $\mu \in \{x, y\}$, and consequently $\theta = \theta(x, y)$ defined by (2.2) belongs to $su(N) \otimes C^\infty(\Omega)$.

In a distinguished paper by Fairlie, Fletcher and Zachos [4] the basis of the Lie algebra $su(N)$ is defined which appears to be very useful in our further considerations.

The elements of this basis are denoted by $L_m$, $L_n$, etc., $m = (m_1, m_2)$, $n = (n_1, n_2)$, etc., and $m, n, \ldots \in I_N \subset \mathbb{Z} \times \mathbb{Z} - \{(0, 0) \text{ mod } N_q\}$ where $q$ is any element of $\mathbb{Z} \times \mathbb{Z}$. The basic vectors $L_m$, $m \in I_N$, are the $N \times N$ matrices satisfying the following commutation relations

$$[L_m, L_n] = \frac{N}{\pi} \sin\left(\frac{\pi}{N} m \times n\right) L_{m+n \mod N_q},$$

where $m \times n := m_1 n_2 - m_2 n_1$.

Then any solution of the $SU(N)$ principal chiral equations (2.3) can be written in the form

$$\theta = \theta(x, y) = \sum_{m \in I_N} u_m(N; x, y) L_m.$$

Now we let $N$ tend to infinity. In this case $I_\infty \equiv I = \mathbb{Z} \times \mathbb{Z} - \{(0, 0)\}$ and the
commutation relations (3.1) read

\[ [L_m, L_n] = m \times n \ L_{m+n}. \]  

(3.3)

Consider the set \( \{e_m\}_{m \in I}, \ e_m := \exp[i(m_1p + m_2q)] \). One quickly finds that

\[ \{e_m, e_n\}_P = m \times n \ e_{m+n}. \]  

(3.4)

Thus the mapping \( F : L_m \mapsto e_m, \ m \in I \), defines the isomorphism

\[ \text{su}(\infty) \cong \text{the Poisson algebra on } T^2 \cong \text{sdiff}(T^2), \]  

(3.5)

where \( T^2 \) is the 2-torus.

Let now there exists the limit

\[ v_m = v_m(x, y) := \lim_{N \to \infty} v_m(N; x, y) \]  

(3.6)

for every \( m \in I \). Then it is evident that the function

\[ \Theta_0 = \Theta_0(x, y, p, q) = \sum_{m \in I} v_m(x, y) \exp[i(m_1p + m_2q)] \]  

(3.7)

is a solution of Husain’s heavenly equation (2.29). The self-dual vacuum metric \( ds^2 \) is determined by \( \Theta_0 \) as follows [2]
\[ ds^2 = dx(\Theta_{0, xp} dp + \Theta_{0, xq} dq) + dy(\Theta_{0, yp} dp + \Theta_{0, yq} dq) + \frac{1}{(\Theta_{0, x}, \Theta_{0, y})^p} [((\Theta_{0, xp} dp + \Theta_{0, xq} dq)^2 + (\Theta_{0, yp} dp + \Theta_{0, yq} dq)^2] , \] (3.8)

where \( \Theta_{0, xp} \equiv \partial_p \partial_x \Theta_0, \Theta_{0, yp} \equiv \partial_p \partial_y \Theta_0, \ldots \), etc.

The method presented here leads to the construction of the self-dual vacuum metrics from the solutions of the SU(N) principal chiral equations. (Compare this with Ward’s question: “Can one construct a sequence of SU(N) chiral fields, for \( N = 2, 3, \ldots \), tending to a curved space in the limit?” [1].)

4. The gravitational unitor

The aim of the present section is to show how the considerations concerning the harmonics maps into the Lie group U(N) given in an excellent paper by Uhlenbeck [5] (See also Pohlmeyer [13], Ward [6] and Anand [7]) can be carry over to sel f-dual gravity. Our results are very far from that to be complete; nevertheless, we intend to indicate the direction of further investigations. First, one can find the Lax pair for the Moyal deformation of Husain’s heavenly equation, (2.17). To this end, by the analogy to [5-7], we consider the following system of linear partial differential equations on the function \( E = E(\lambda, x, y, p, q) \in C^\infty(\mathbb{C}^* \times \Omega \times \Sigma), \mathbb{C}^* := \mathbb{C} - \{0\}, \Sigma \subset \mathbb{R} e^2, \)

\[ \text{i} \hbar \partial_z E_\lambda = (1 - \lambda) E_\lambda \ast a_\bar{z}, \]

\[ \text{i} \hbar \partial_{\bar{z}} E_\lambda = (1 - \lambda^{-1}) E_\lambda \ast a_z, \] (4.1)

where \( \lambda \in \mathbb{C}^*, E_\lambda = E_\lambda(x, y, p, q) := E(\lambda, x, y, p, q) \) and \( z := x + iy, \bar{z} := x - \ldots \)
\[ iy, \partial_z = \frac{1}{2}(\partial_x - i\partial_y), \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y); \text{ moreover } a_z = a_z(x, y, p, q) \text{ and } a_{\bar{z}} = a_{\bar{z}}(x, y, p, q) \text{ are independent of } \lambda. \]

The integrability conditions of the system (4.1) read

\[ \lambda^0 \rightarrow \partial_z a_z - \partial_{\bar{z}} a_{\bar{z}} + 2\{a_z, a_{\bar{z}}\}_M = 0 \] (4.2a)

\[ \lambda^1 \rightarrow \partial_z a_z + \{a_z, a_{\bar{z}}\}_M = 0 \] (4.2b)

\[ \lambda^{-1} \rightarrow \partial_{\bar{z}} a_z + \{a_{\bar{z}}, a_z\}_M = 0. \] (4.2c)

One quickly finds that any two of the above conditions imply the third.

Adding (4.2b) and (4.2c) we get

\[ \partial_z a_z + \partial_{\bar{z}} a_{\bar{z}} = 0. \] (4.3)

Hence, there exists a function \( \Theta = \Theta(x, y, p, q) \) such that

\[ a_z = -\frac{i}{2} \partial_z \Theta \quad \text{and} \quad a_{\bar{z}} = \frac{i}{2} \partial_{\bar{z}} \Theta. \] (4.4)

Subtraction of (4.2b) and (4.2c) gives (4.2a), and with the use of (4.4) we get finally

\[ i\partial_z \partial_{\bar{z}} \Theta + \frac{1}{2}\{\partial_z \Theta, \partial_{\bar{z}} \Theta\}_M = 0. \] (4.5)

One can easily find that Eq. (4.5) written in terms of the coordinates \((x, y)\) is exactly the Moyal deformation of Husain’s heavenly equation, (2.17).
Concluding: the system (4.1) constitutes the Lax pair for Eq. (2.17).

Comparing (4.4) with (2.27) we obtain

\[ a_z = \frac{i\hbar}{2} g^{-1} \ast \partial_z g \quad \text{and} \quad a_{\bar{z}} = \frac{i\hbar}{2} g^{-1} \ast \partial_{\bar{z}} g, \]  

(4.6)

where \( g = g(x, y, p, q) \) is a harmonic map from \( \Omega \) into \( C^\infty(\Sigma) \) i.e., the equation

\[ \partial_z (g^{-1} \ast \partial_z g) + \partial_{\bar{z}} (g^{-1} \ast \partial_{\bar{z}} g) = 0 \]  

(4.7)

or equivalently Eq. (2.22), hold.

In what follows, if \( g = g(x, y, p, q) \in C^\infty(\Omega \times \Sigma) \) satisfies Eq. (4.7) we will say that \( g \) is a *-harmonic map from \( \Omega \) into \( C^\infty(\Sigma) \). In particular, consider the case when

\[ g^{-1} = \bar{g}. \]  

(4.8)

In this case we say that \( g \) is *-unitary and the group of *-unitary functions will be denoted by \( U_*(\Omega \times \Sigma) \) (compare with (2.31)). Define \( a_x = a_x(x, y, p, q) \) and \( a_y = a_y(x, y, p, q) \) in a natural manner

\[ a_z =: \frac{1}{2}(a_x - ia_y) \quad \text{and} \quad a_{\bar{z}} =: \frac{1}{2}(a_x + ia_y). \]  

(4.9)

Then by (4.4) we get

\[ a_x = -\frac{1}{2}\partial_y \Theta \quad \text{and} \quad a_y = \frac{1}{2}\partial_x \Theta, \]  

(4.10)

and by (4.6)
\[ a_x = \frac{i\hbar}{2} g^{-1} \ast \partial_x g \quad \text{and} \quad a_y = \frac{i\hbar}{2} g^{-1} \ast \partial_y g. \] (4.11)

Therefore, if \( g \) is \( \ast \)-unitary \( \text{i.e.} \) (4.8) holds, then from (4.11) one infers that \( a_x \) and \( a_y \) are real functions. Consequently, by (4.10) we can choose \( \Theta \) to be real. Then, in the present case

\[ \overline{a_z} = a_{\overline{z}}. \] (4.12)

Now we can prove the theorem which is an analogue of the Theorem 2.2 given in Uhlenbeck’s paper [5] (see also [6,7]).

**Theorem 4.1**

Let \( g = g(x, y, p, q) \) be a \( \ast \)-harmonic map from \( \Omega \) into \( C^\infty(\Sigma) \) and let \( g(x_0, y_0, p, q) = h(p, q) \), where \( h = h(p, q) \) is some given function on \( \Sigma \) and \( (x_0, y_0) \in \Omega \) is a given point. Moreover, let for every \( \lambda \in \mathbb{C}^\ast \) there exists a unique solution \( E_\lambda = E_\lambda(x, y, p, q) \) of the Lax pair (4.1) with (4.6), such that \( E_\lambda(x_0, y_0, p, q) = h(p, q) \) and \( E_\lambda \) has the \( \ast \)-inverse \( E_\lambda^{-1}. \)

Then

(i) \( E_1 = h(p, q), \)

(ii) \( E^{-1} = g, \)

(iii) If \( g \in U_\ast(\Omega \times \Sigma) \) then

\[ E_\lambda^{-1} = \overline{E_{\overline{\lambda}}^{-1}} \quad \text{for every} \quad \lambda \in \mathbb{C}^\ast. \] (4.13)

**Proof:**

The proof of (i) and (ii) is trivial.

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To prove \((iii)\) observe that if \(g \in \mathbf{U}_s(\Omega \times \Sigma)\) then the relation (4.12) holds, and consequently the complex conjugation of Eqs. (4.1) gives

\[-i\hbar \partial_{\bar{z}} E_{\bar{\lambda}} = (1 - \bar{\lambda}) a_{\bar{z}} \ast \overline{E_{\lambda}},\]

\[-i\hbar \partial_{\bar{z}} \overline{E_{\lambda}} = (1 - \bar{\lambda}^{-1}) a_{\bar{z}} \ast \overline{E_{\lambda}}.\]  \hspace{1cm} (4.14)

By simple manipulations one quickly finds that the system (4.14) can be equivalently rewritten as follows

\[i\hbar \partial_{z} \overline{E_{\lambda}}^{-1} = (1 - \bar{\lambda}) \overline{E_{\lambda}}^{-1} \ast a_{z},\]

\[i\hbar \partial_{\bar{z}} \overline{E_{\lambda}}^{-1} = (1 - \bar{\lambda}^{-1}) \overline{E_{\lambda}}^{-1} \ast a_{\bar{z}}.\]  \hspace{1cm} (4.15)

Then, changing \(\lambda \leftrightarrow \bar{\lambda}^{-1}\) we get

\[i\hbar \partial_{z} \overline{E_{\lambda^{-1}}}^{-1} = (1 - \lambda^{-1}) \overline{E_{\lambda^{-1}}}^{-1} \ast a_{z},\]

\[i\hbar \partial_{\bar{z}} \overline{E_{\lambda^{-1}}}^{-1} = (1 - \lambda) \overline{E_{\lambda^{-1}}}^{-1} \ast a_{\bar{z}}.\]  \hspace{1cm} (4.16)

Comparing (4.16) with (4.1), and employing also the assumption of the existence and the uniqueness of the solution to Eqs. (4.1) one obtains (4.13). ♦

It is evident that Theorem 4.1 is a “weak” version of the Uhlenbeck’s Theorem 2.2 as we assume and don’t prove the existence and uniqueness of solution to the Lax pair. In our case this problem is rather subtle and we have not succeeded in solving it in all generality.
We now prove a theorem which corresponds exactly to Uhlenbeck’s Theorem 2.3 [5].

**Theorem 4.2**

Let the function \( E = E(\lambda, x, y, p, q) \in C^\infty(\mathbb{C}^* \times \Omega \times \Sigma) \) be such that \( E_1 = E(1, x, y, p, q) = f(p, q) \) and the expressions \( \frac{E_1^{-1} \cdot \partial_z E_\lambda}{1-\lambda} \) and \( \frac{E_1^{-1} \cdot \partial_z E_\lambda}{1-\lambda^{-1}}, \lambda \in \mathbb{C}^* \), are independent of \( \lambda \). Then \( g = E_1 \) is a \( \ast \)-harmonic map from \( \Omega \) into \( C^\infty(\Sigma) \).

**Proof:**

Define

\[
  a_{\bar{z}} := \frac{i\hbar E_1^{-1} \ast \partial_{\bar{z}} E_\lambda}{1-\lambda} \quad \text{and} \quad a_z := \frac{i\hbar E_1^{-1} \ast \partial_z E_\lambda}{1-\lambda^{-1}} \quad \lambda \in \mathbb{C}^*. \tag{4.17}
\]

By the assumption, \( a_z \) and \( a_{\bar{z}} \) are independent of \( \lambda \). Thus, in fact, we arrive at the Lax pair (4.1). Consequently, Eq. (4.3) is satisfied. From (4.17) one quickly finds that

\[
  a_{\bar{z}} := \frac{i\hbar}{2} E_{-1}^{-1} \ast \partial_z E_{-1} \quad \text{and} \quad a_z := \frac{i\hbar}{2} E_{-1}^{-1} \ast \partial_z E_{-1}. \tag{4.18}
\]

Inserting (4.18) into (4.3) we get the thesis. ◊

Now we are in a position to prove the fundamental theorem

**Theorem 4.3**

Let the function \( E = E(\lambda, x, y, p, q) \in C^\infty(\mathbb{C}^* \times \Omega \times \Sigma) \) be as in the Theorem 4.2, and let \( \Theta = \Theta(x, y, p, q) \) be defined by

\[
  \partial_{\bar{z}} \Theta = \hbar E_{-1}^{-1} \ast \partial_z E_{-1} \quad \text{and} \quad \partial_z \Theta = -\hbar E_{-1}^{-1} \ast \partial_{\bar{z}} E_{-1}. \tag{4.19}
\]

Then \( \Theta \) satisfies the Moyal deformation of Husain’s heavenly equation, (2.17) or (4.5). Moreover, if \( E_\lambda \) fulfills the \( \ast \)-unitary condition (4.13) then \( \Theta \) can be chosen so as to be real.
Proof:

Straightforward by the Theorem 4.2 and by the relations (4.4). ♦

From the previous considerations (see (2.28)) it follows that if the function
\( \Theta = \Theta(x, y, p, q) \) defined by (4.19) is analytic in \( \hbar \) then \( \Theta_0 \) satisfies Husain’s heavenly equation (2.29). Given a \(*\)-harmonic map \( g \) from \( \Omega \) into \( C^\infty(\Sigma) \) we define the \textit{extended solution corresponding to} \( g \) to be the solution \( E = E(\lambda, x, y, p, q) \in C^\infty(\mathcal{BF}C^* \times \Omega \times \Sigma) \) of the Lax pair (4.1) with \( a_z \) and \( \bar{a}_z \) defined by (4.6).

Then by the analogy to Uhlenbeck’s definition [5] (see also [6,7]) one has

\textbf{Definition 4.1}

Let \( g = g(x, y, p, q) \) be a \(*\)-harmonic form \( \Omega \) into \( C^\infty(\Sigma) \) such that it has an extended solution \( E_\lambda \) of the form

\[ E_\lambda = \sum_{k=0}^{n} T_k \lambda^k, \quad T_k \in C^\infty(\Omega \times \Sigma), \quad (4.20) \]

satisfying the \(*\)-unitary condition (4.13). The self-dual metric (if it exists) defined by \( g \) is called the \textit{n-gravitational uniton}.

The question of existence and of the interpretation of the gravitational unitons will be considered in a separate paper. Here we have only shown that the Moyal bracket algebra, enables one to consider self-dual gravity as the principal chiral model.
5. The self-dual gravity action as the WZW-like action

In this section we are going to show that the action leading to the Moyal deformation of Husain’s heavenly equation (2.17) can be written in the WZW-like form within the Moyal bracket algebra formalism. Consequently, the action for Husain’s heavenly equation (2.29) appears to be the $\hbar \to 0$ limit of the WZW-like action in the Moyal bracket algebra. To this end consider the $\text{SU}(N)$ chiral model. Let $\Omega = S^2$ with the radius $r \to \infty$ and let $B$ be a three dimensional manifold such that $\Omega$ is the boundary of $B$ i.e., $\partial B = \Omega$. The Cartesian coordinates on $\Omega$ will be denoted by $\sigma^\mu$, $\mu = 1, 2$. The coordinates on $B$ will be denoted by $t^i$, $i = 1, 2, 3$. Then employing the formula (2.7) which defines the Lagrangian leading to the principal chiral equations (2.3) one quickly finds that Eqs. (2.3) can be derived from the following action

$$I(\theta) = -\frac{\alpha}{2} \int_\Omega d^2\sigma \text{Tr}[(\partial_\mu \theta)(\partial_\mu \theta)] + \frac{\alpha}{3} \int_B \text{Tr}(d\theta \wedge d\theta \wedge d\theta)$$

$$= -\frac{\alpha}{2} \int_\Omega d^2\sigma \text{Tr}[(\partial_\mu \theta)(\partial_\mu \theta)] + \frac{\alpha}{3} \int_B d^3t \epsilon^{ijk} \text{Tr}[(\partial_i \theta)(\partial_j \theta)(\partial_k \theta)], \quad (5.1)$$

where in the integral over $B$ the function $\theta$ is considered to be extended over $B$. The action (5.1) evidently resembles the WZW action [9,10].

Analogously, keeping in mind the Lagrangian (2.24) one obtains the action for the Moyal deformation of Husain’s heavenly equation (2.17) to be

$$I_{SG}^{(M)}(\Theta) = \frac{1}{2} \int_{\Omega \times \Sigma} d^2\sigma dpdq (\partial_\mu \Theta)^* (\partial_\mu \Theta) - \frac{1}{3i\hbar} \int_{B \times \Sigma} d^3tdpdq\epsilon^{ijk} \left[(\partial_i \Theta)^* b_{ijg}(\partial_j \Theta)^* (\partial_k \Theta)\right]. \quad (5.2)$$

Then we rewrite (5.2) in the following form
\[ I^{(M)}_{SG}(\Theta) = \frac{1}{2} \int_{\Omega \times \Sigma} d^2 \sigma dp dq (\partial_\mu \Theta) \ast (\partial_\mu \Theta) - \frac{1}{6} \int_{B \times \Sigma} d^3 t dp dq e^{ijk} \left[ \partial_i \Theta \ast \{ \partial_j \Theta, \partial_k \Theta \}_M \right]. \]

Consequently, taking the \( \hbar \to 0 \) limit of (5.3) one gets the WZW-like action for Husain’s heavenly equation to be

\[ I_{SG}(\Theta_0) = \frac{1}{2} \int_{\Omega \times \Sigma} d^2 \sigma dp dq (\partial_\mu \Theta_0) (\partial_\mu \Theta_0) - \frac{1}{6} \int_{B \times \Sigma} d^3 t dp dq e^{ijk} \left[ \partial_i \Theta_0 \{ \partial_j \Theta_0, \partial_k \Theta_0 \}_P \right]. \]

The fundamental question is how one can interpret geometrically the “Wess-Zumino” term in (5.2). We intend to consider this question soon.

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