q-Bessel Functions and Rogers-Ramanujan Type Identities

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Abstract

We evaluate q-Bessel functions at an infinite sequence of points and introduce a generalization of the Ramanujan function and give an extension of the m-version of the Rogers-Ramanujan identities. We also prove several generating functions for Stieltjes-Wigert polynomials with argument depending on the degree. In addition we give several Rogers-Ramanujan type identities.

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1 Introduction

The Rogers–Ramanujan identities are

\[
\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} = \frac{1}{(q, q^4; q^5)_{\infty}}
\]

(1.1)

\[
\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q; q)_n} = \frac{1}{(q^2, q^3; q^5)_{\infty}},
\]

where the notation for the \(q\)-shifted factorials is the standard notation in [9], [11]. References for the Rogers–Ramanujan identities, their origins and many of their applications are in [1], [2], and [4]. In particular we recall the partition theoretic interpretation of the first Rogers–Ramanujan identity as the partitions of an integer \(n \mod 5 \equiv 1 \text{ or } 4\) are equinumerous with the partitions of \(n\) into parts where any two parts differ by at least 2.

Garrett, Ismail, and Stanton [8] proved the \(m\)-version of the Rogers-Ramanujan identities

\[
\sum_{n=0}^{\infty} \frac{q^{n^2+mn}}{(q; q)_n} = \frac{(-1)^m q^{-(\frac{m}{2})} a_m(q)}{(q, q^4; q^5)_{\infty}} - \frac{(-1)^m q^{-(\frac{m}{2})} b_m(q)}{(q^2, q^3; q^5)_{\infty}},
\]

(1.2)

where

\[
a_m(q) = \sum_j q^{j^2+j} \left[ \begin{array}{c} m-j-2 \\ j \end{array} \right]_q,
\]

(1.3)

\[
b_m(q) = \sum_j q^{j^2+2} \left[ \begin{array}{c} m-j-1 \\ j \end{array} \right]_q.
\]

The polynomials \(a_m(q)\) and \(b_m(q)\) were considered by Schur in conjunction with his proof of the Rogers–Ramanujan identities, see [1] and [8] for details. We shall refer to \(a_m(q)\) and \(b_m(q)\) as the Schur polynomials. The closed form expressions for \(a_m\) and \(b_m\) in (1.3) were given by Andrews in [3], where he also gave a polynomial generalization of the Rogers–Ramanujan identities. In this paper we give a family of Rogers–Ramanujan type identities involving the evaluation of \(q\)-Bessel and allied functions at special points. We also give the partition theoretic interpretation of these identities. In Section 2 we define the functions and polynomials used in our analysis. In Section 3 we present our Rogers–Ramanujan type identities. They resemble the \(m\) form in (1.2).

In a series of papers from 1903 till 1905 F. H. Jackson introduced \(q\)-analogues of Bessel functions. The modern notation for the modified \(q\)-Bessel functions, that is
$q$-Bessel functions with imaginary argument, is, \[10\],

\begin{equation}
I^{(1)}_\nu(z; q) = \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} \sum_{n=0}^{\infty} \frac{(z/2)^{\nu+2n}}{(q, q^{\nu+1}; q)_n}, \quad |z| < 2,
\end{equation}

\begin{equation}
I^{(2)}_\nu(z; q) = \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} \sum_{n=0}^{\infty} \frac{q^{n(\nu+1)}}{(q, q^{\nu+1}; q)_n} (z/2)^{\nu+2n},
\end{equation}

\begin{equation}
I^{(3)}_\nu(z; q) = \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} \sum_{n=0}^{\infty} \frac{q^n}{(q, q^{\nu+1}; q)_n} (z/2)^{\nu+2n}.
\end{equation}

The functions $I^{(1)}_\nu$ and $I^{(2)}_\nu$ are related via

\begin{equation}
I^{(1)}_\nu(z; q) = \frac{I^{(2)}_\nu(z; q)}{(z^2/4; q)_\infty},
\end{equation}

[11, Theorem 14.1.3]. Formula (1.7) analytically continues $I^{(1)}_\nu$ to a meromorphic function in the complex plane. The Stieltjes–Wigert polynomials [11], [18], are defined by

\begin{equation}
S_n(x; q) = \frac{1}{(q; q)_n} \sum_{k=0}^{n} \binom{n}{k}_q q^{k^2} (-x)^k = \frac{1}{(q; q)_n} \sum_{k=0}^{n} \frac{(q^{-n}; q)_k}{(q; q)_k} q^{\frac{k(k+1)}{2}} (xq^n)^k,
\end{equation}

respectively. Ismail and C. Zhang [13] proved the following symmetry relation for the Stieltjes–Wigert polynomials

\begin{equation}
q^{n^2} (-t)^n S_n(q^{-2n}/t; q) = S_n(t; q).
\end{equation}

Section 2 contains the evaluation of $I^{(2)}_\nu$ at an infinite number of special points. These new sums seem to be new. In Section 3 we introduce a generalization of the Ramanujan function

\begin{equation}
A_q(z) := \sum_{n=0}^{\infty} \frac{(-z)^n}{(q; q)_n} q^{n^2},
\end{equation}

which S. Ramanujan introduced and studied many of its properties in the lost notebook [17]. It was later realized that this is an analogue of the Airy function. In Section 4 we introduce a function $B^\alpha_q$ prove some identities it satisfies then use them to derive several Rogers-Ramanujan type identities. The function $B^\alpha_q$ is also a generalization of the Ramanujan function and is expected to lead to numerous new Rogers-Ramanujan type identities. The Stieltjes–Wigert polynomials satisfy a second order $q$-difference equation of polynomial coefficients of the for

\[ f(x)y(qx) + g(x)y(x) + h(x)y(x/q) = 0. \]

In Section 5 we evaluate $y(q^n x)$ in terms of $y(x)$ and $y(x/q)$ with explicit coefficients. Section 6 contains miscellaneous properties of the Stieltjes–Wigert polynomials.
2 \textit{q-Bessel Sums}

Our first result is the following theorem.

\textbf{Theorem 2.1.} The function \(I^{(2)}\) has the representation

\begin{equation}
I^{(2)}_{\nu}(2) = \frac{z^\nu}{(q;q)_\infty} \phi_1(z^2; 0; q, q^{\nu+1}).
\end{equation}

In particular \(I^{(2)}\) takes the special values

\begin{equation}
I^{(2)}_{\nu}(2q^{-n}/2; q) = \frac{q^{\nu n/2} S_n(-q^{-\nu-n}; q)}{(q^{n+1}; q)_\infty},
\end{equation}

and

\begin{equation}
I^{(2)}_{\nu}(2q^{-n}/2; q) = \frac{q^{-\nu n/2} S_n(-q^{-\nu-n}; q)}{(q^{n+1}; q)_\infty}.
\end{equation}

\textbf{Proof.} Recall the Heine transformation \cite{9} (III.2)]

\begin{equation}
2\phi_1 \left( \frac{A, B}{C} \bigg| q, Z \right) = \frac{(C/B, BZ; q)_\infty}{(C, Z; q)_\infty} 2\phi_1 \left( \frac{ABZ/C, BZ}{q, C} \bigg| q, \frac{C}{B} \right).
\end{equation}

The left-hand side of (2.1) is

\begin{equation}
\frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} z^\nu \sum_{k=0}^\infty \frac{q^{k^2+2k\nu} z^{2k}}{(q^{\nu+1}; q)_k} = \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} z^\nu \lim_{a,b \to \infty} 2\phi_1 \left( \frac{a, b}{q, q^{\nu+1}z^2} \bigg| q, \frac{q^{\nu+1}z^2}{ab} \right)
\end{equation}

which implies (2.1). When \(z = q^{-n/2}\) and in view of (1.8), the left-hand side of (2.2) equals its right-hand side. Formula (2.3) follows from the symmetry relation (1.9) \(\square\)

The results (2.2)–(2.3) of Theorem 2.1 when written as a series becomes

\begin{equation}
\sum_{k=0}^\infty \frac{q^{k(k+\nu-n)}}{(q, q^{\nu+1}; q)_k} = \frac{q^{\nu n}}{(q^{\nu+1}; q)_\infty} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k^2} q^{-k(\nu+n)}
\end{equation}

\begin{equation}
= \frac{1}{(q^{\nu+1}; q)_\infty} \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k^2} q^{k(\nu-n)}.
\end{equation}

Another way to prove (2.2) for integer \(\nu\) is to use the generating function

\begin{equation}
\sum_{-\infty}^{\infty} q^{(n^2)} I^{(2)}_m(z; q) t^n = (-t^2/2, -qz/2t; q)_\infty.
\end{equation}

Carlitz \cite{6} did this for \(n = 0, 1\) and used this to give another proof of the Rogers–Ramanujan identities.
Theorem 2.2. \[1\] The \( q \)-binomial coefficient \( \binom{n}{k}_q \) is the generating function for integer partitions whose Ferrers diagrams fit inside a \( k \times (n-k) \) rectangle.

Recall that
\[
I^{(j)}_{\nu}(z; q) = e^{-\nu \pi i/2} J^{(j)}_{\nu}(e^{\pi i/2} z; q), j = 1, 2.
\]

Chen, Ismail, and Muttalib \[7\] established an asymptotic series for \( J^{(2)}_{\nu}(z; q) \). Their main term for \( r > 0 \) is
\[
I^{(2)}_{\nu}(r; q) = \left( r/2 \right)^{\nu} \left( q^{1/2}; q \right)_\infty \frac{2(q; q)_\infty}{(r q^{(\nu+1)/2}/2; q^{1/2})_\infty + (-r q^{(\nu+1)/2}/2; q^{1/2})_\infty},
\]
as \( r \to +\infty \). This determines the large \( r \) behavior of the maximum modulus of \( I^{(2)}_{\nu} \).

We next derive a Mittag–Leffler expansion for \( I^{(1)}_{\nu}(z; q) \).

Theorem 2.3. We have the expansion
\[
I^{(1)}_{\nu}(z; q) = \left( \frac{z^2}{2} \right)^{\nu} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n+1/2} S_n(-q^{\nu-n}; q)}{(1 - z^2 q^n/4)}.
\]

Using residues it is easy to see that the difference between \( I^{(1)}_{\nu}(z; q) / (z^2; q)_{\infty} \) and the right-hand side of (2.9) is entire. We give a direct proof that this difference is zero.

Proof of Theorem 2.3. Use (1.8) to see that the sum on the right-hand side of (2.9) is
\[
\sum_{n=0}^{\infty} \frac{(-1)^n q^{n+1/2}}{(1 - z^2 q^n/4)} \sum_{k=0}^{n} q^{k^2 + k(\nu-n)} (q; q)_k (q; q)_{n-k} = \sum_{k=0}^{\infty} \frac{(-1)^k q^{k^2 + (k+1)/2}}{(q; q)_k (1 - z^2 q^k/4)} \phi_1(z^{2}q^k/4; z^2q^{k+1}/4; q, q).
\]

Now apply (III.4) of \[9\] with \( a = z^2 q^k/4, b = 1, c = 0, z = q \) to see that the above sum is \((q; q)_\infty / (z^2 q^{k+1}/4; q)_{\infty}\). This shows that the right-hand side of (2.9) is given by
\[
\left( \frac{z/2}{q, z^2/4; q}_{\infty} \right) \phi_1(z^2/4; 0; q, q^{\nu+1}),
\]
and the result follows from (2.11) and (1.7). \( \square \)
3 A Generalization of the Ramanujan Function

The Rogers-Ramanujan identities evaluate $A_q$ at $z = -1, -q$. The result (1.2) evaluates $A_q(-q^m)$. This motivated us to consider the function

$$u_m(a, q) := \sum_{n=-\infty}^{\infty} \frac{q^{n^2+mn}}{(aq; q)_n},$$

as a function of $q^m$. When $a = 1$ we get the Rogers-Ramanujan function. It is clear that

$$q^{m+1}u_{m+2}(a, q) = \sum_{n=-\infty}^{\infty} \frac{(1-aq^n)}{(aq; q)_n} q^{n^2+mn}$$

Therefore

$$q^{m+1}u_{m+2}(a, q) = u_m(a, q) - au_{m+1}(a, q).$$

Let $u_m(a, q) = q^{-\binom{m}{2}}(-1)^m \tilde{u}_m(a, q)$. Then $\{\tilde{u}_m(a, q)\}$ satisfy the difference equation

$$y_{m+1} = q^{m-1} y_{m-1} + ay_m, \quad m > 0.$$  

We now solve (3.3) using generating functions. The generating function $Y(t) := \sum_{n=0}^{\infty} y_n t^n$ satisfies

$$Y(t) = \frac{y_0 + t(y_1 - ay_0)}{1-at} + \frac{t^2}{1-at} Y(qt),$$

whose solution is

$$Y(t) = \sum_{n=0}^{\infty} \frac{q^n(n-1)t^{2n}}{(at; q)_{n+1}} [y_0 + tq^n(y_1 - ay_0)].$$

We now need two initial conditions, so choose two solutions $\{c_m(a, q)\}$ and $\{d_m(a, q)\}$

$$c_0(a, q) = 1, c_1(a, q) = 0, \quad c_0(a, q) = 0, d_1(a, q) = 1.$$  

**Theorem 3.1.** The polynomials $\{c_m(a, q)\}$ and $\{d_m(a, q)\}$ have the generating functions

$$\sum_{n=0}^{\infty} c_n(a, q)t^n = \sum_{n=0}^{\infty} \frac{q^{n(n-1)}}{(at; q)_n} t^{2n},$$

$$\sum_{n=0}^{\infty} d_n(a, q)t^n = \sum_{n=0}^{\infty} \frac{q^{n^2}t^{2n+1}}{(at; q)_{n+1}}.$$  

It is clear from the initial conditions (3.4) and the recurrence relation (3.3) that both $\{c_n(a, q)\}$ and $\{d_n(a, q)\}$ are polynomials in $a$ and in $q$. 

6
Theorem 3.2. The polynomials \( \{c_n(a,q)\} \) and \( \{d_n(a,q)\} \) have the explicit form

\[
\begin{align*}
\text{(3.7)} \quad c_n(a,q) & = \sum_{j=0}^{\lfloor(n-2)/2\rfloor} q^{j(j+1)} \binom{n-j-2}{j} a^{n-2j-2}, \\
\text{(3.8)} \quad d_n(a,q) & = \sum_{j=0}^{\lfloor(n-1)/2\rfloor} q^{j^2} \binom{n-j-1}{j} a^{n-2j-1}.
\end{align*}
\]

The proof follows form equations (3.5) and (3.6); and the \( q \)-binomial theorem.

Theorem 3.3. We have

\[
\begin{align*}
\text{(4.1)} \quad \sum_{n=-\infty}^{\infty} \frac{q^{n^2+mn}}{(aq;q)_n} &= (-1)^m q^{-\binom{m}{2}} \times \left[ c_m(a,q) \sum_{n=-\infty}^{\infty} \frac{q^{n^2}}{(aq;q)_n} + d_m(a,q) \sum_{n=-\infty}^{\infty} \frac{q^{n^2+n}}{(aq;q)_n} \right] \quad \text{for} \quad a \neq 1.
\end{align*}
\]

The case \( a = 1 \) is the \( m \)-version of the Rogers-Ramanujan identities in (1.2) first proved by Garret, Ismail, and Stanton [8].

4 Rogers-Ramanujan Type Identities

In this section we prove several identities of Rogers-Ramanujan type. One of the proofs uses the Ramanujan \( \psi_1 \) sum [9 (II.29)]

\[
\begin{align*}
\text{(4.1)} \quad \sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} z^n &= \frac{(q,b/a,az,q/az;q)_\infty}{(b,q/a,z,b/az;q)_\infty}, \quad \left| \frac{b}{a} \right| < |z| < 1.
\end{align*}
\]

Throughout this section we define \( \rho \) by

\[
\text{(4.2)} \quad \rho = e^{2\pi i/3}.
\]

Lemma 4.1. For nonnegative integer \( j, k, \ell, m, n \) and \( \rho = e^{2\pi i/3} \) we have

\[
\begin{align*}
\text{(4.3)} \quad \sum_{k=0}^{n} \frac{(a;q)_k}{(q;q)_k} \frac{(a;q)_{n-k}}{\left( q^2; q^2 \right)_m} (-1)^k &= \begin{cases} 
0 & n = 2m + 1 \\
\frac{(a^2;q^2)_m}{\left( q^2; q^2 \right)_m} & n = 2m
\end{cases}, \\
\text{and} \\
\text{(4.4)} \quad \sum_{j+k+\ell = n} \frac{(a;q)_j}{(q;q)_j} \frac{(a;q)_k}{(q;q)_k} \rho^{j+k+2\ell} &= \begin{cases} 
0 & 3 \nmid n \\
\frac{(q^3;q^3)_m}{\left( q^3; q^3 \right)_m} & n = 3m
\end{cases}.
\end{align*}
\]
For $j, k, m, \ell, n \in \mathbb{Z}$, we have

\[(4.5) \quad \sum_{j+k=n} \frac{(a;q)_j (a;q)_k (-1)^k}{(b;q)_j (b;q)_k} = \begin{cases} 0 & n = 2m+1 \\ \frac{(a,b/a,-b,-q/a;q)_\infty (a^2;q^2)_m}{(-q,-b/a,b/a;q)_{\infty} (b^2;q^2)_m} & n = 2m \end{cases} \]

and

\[(4.6) \quad \sum_{j+k+\ell=n} \frac{(a;q)_j (a;q)_k (a;q)_\ell \rho^{k+2\ell}}{(b;q)_j (b;q)_k (b;q)_\ell} = 0 \]

for $3 \nmid n$,

\[(4.7) \quad \sum_{j+k+\ell=3m} \frac{(a;q)_j (a;q)_k (a;q)_\ell \rho^{k+2\ell}}{(b;q)_j (b;q)_k (b;q)_\ell} = \frac{(q,b/a;q)_\infty^3 (b^3,q^3a^{-3};q^3)_\infty (a^3;q^3)_m}{(b,q/a;q)_\infty^3 (q^3,b^3a^{-3};q^3)_\infty (b^3;q^3)_m} \]

**Proof.** Formula (4.3) follows from

\[\frac{(at;q)_\infty (-at;q)_\infty}{(t;q)_\infty (-t;q)_\infty} = \frac{(a^2t^2;q^2)_\infty}{(t^2;q^2)_\infty}, \quad |t| < 1,\]

while (4.4) follows from

\[\frac{(at;q)_\infty (apt;q)_\infty (ap^2t;q)_\infty}{(t;q)_\infty (pt;q)_\infty (p^2t;q)_\infty} = \frac{(a^3t^3;q^3)_\infty}{(t^3;q^3)_\infty}, \quad |t| < 1.\]

For $|ba^{-1}| < |x| < 1$, apply the Ramanujan $\psi_1$ sum (4.11) to the identity

\[\frac{(ax,q/(ax);q)_\infty (-ax,-q/(ax);q)_\infty}{(x,b/(ax);q)_\infty (-x,-b/(ax);q)_\infty} = \frac{(a^2x^2,q^2/(a^2x^2);q^2)_\infty}{(x^2,b^2/(a^2x^2);q^2)_\infty},\]

to derive (4.5). Similarly we apply (4.1) to

\[\frac{(a^3x^3,q^3/(a^3x^3);q^3)_\infty}{(x^3,b^3/(a^3x^3);q^3)_\infty} = \frac{(axp^2,q/(axp^2);q)_\infty (axp, -q/(axp);q)_\infty (ax,q/(ax);q)_\infty}{(xp^2,b/(axp^2);q)_\infty (xp, -b/(axp);q)_\infty (x,b/(ax);q)_\infty},\]

and establish (4.3)-(4.7). \[\square\]

It must be noted that (4.3) is essentially the evaluation of a continuous $q$-ultraspherical polynomial at $x = 0$, \cite[(12.2.19)]{11}. 

8
For $\alpha > 0$, let

$$A^{(\alpha)}_q(a; t) = \sum_{n=0}^{\infty} \frac{(a; q)_n q^{\alpha n^2} t^n}{(q; q)_n},$$

in particular,

$$A^{(1)}_q(q; t) = \omega(t; q), \quad A^{(2)}_q(q^2; t^2) = \omega(t^2; q^4), \quad A^{(1)}_q(0; t) = A_q(-t),$$

where

$$\omega(v; q) = \sum_{n=0}^{\infty} q^{n^2} v^n.$$

**Theorem 4.2.** Let $\alpha \geq 0$, then

$$A^{(2\alpha)}_q(q^2; t^2) = \sum_{j=0}^{\infty} \frac{(a; q)_j q^{2\alpha j^2}}{(q; q)_j} A^{(\alpha)}_q(a; t^2 q^{2\alpha j}).$$

For $\rho = e^{2\pi i/3}$ we have

$$A^{(3\alpha)}_q(q^3; t^3) = \sum_{j,k=0}^{\infty} \frac{(a; q)_j (a; q)_k \rho^k q^{\alpha(j+k)^2} t^{j+k}}{(q; q)_j (q; q)_k} A^{(\alpha)}_q(a; \rho^2 q^{2\alpha(j+k)} t).$$

**Proof.** These two identities can be proved by applying (4.3) and (4.4) and straightforward series manipulation.

We now consider the following generalization of the $\psi_1$ function. For $\alpha \geq 0$, define $B^{(\alpha)}_q$ by

$$B^{(\alpha)}_q(a, b; x) = \sum_{n=-\infty}^{\infty} \frac{(a; q)_n q^{\alpha n^2} x^n}{(b; q)_n},$$

**Theorem 4.3.** We have

$$\frac{(-b, -q/a, q, b/a; q)_\infty}{(-q, -b/a, b, q/a; q)_\infty} B^{(2\alpha)}_q(a^2, b^2; x^2) = \sum_{j=-\infty}^{\infty} \frac{(a; q)_j q^{\alpha j^2} (-x)^j}{(b; q)_j} B^{(\alpha)}_q(a, b; xq^{2\alpha j}).$$

and

$$B^{(3\alpha)}_q(a^3, b^3; x^3) = \frac{(b, q/a; q)_3^\infty}{(q, b/a; q)_3^\infty} \frac{(q^3, b^3 a^{-3}; q^3)}{(b^3, q^3 a^{-3}; q^3)} \sum_{j,k=-\infty}^{\infty} \frac{(a; q)_j (a; q)_k \rho^k q^{\alpha(j+k)^2} x^{j+k}}{(b; q)_j (b; q)_k} B^{(\alpha)}_q(a, b; xq^{2\alpha(j+k)}).$$
The proof follows from (4.5), (4.6) and (4.7) and straightforward series manipulation.

**Corollary 4.4.** The following Rogers-Ramanujan type identities hold

\[
\begin{align*}
(4.14) \quad & \frac{(-a, -q/a, q, q; q)_\infty}{(a, q/a, -q, -q; q)_\infty} \sum_{n=-\infty}^{\infty} \frac{q^{4n^2} x^{2n}}{1 - a^2 q^{2n}} = \sum_{j,k=-\infty}^{\infty} \frac{q^{(j+k)^2} (-1)^j x^{j+k}}{(1 - aq^j) (1 - aq^k)}, \\
(4.15) \quad & \frac{(q, q; q)_\infty}{(-q, -q; q)_\infty} \sum_{n=-\infty}^{\infty} \frac{q^{4n^2} x^{2n}}{1 + q^{2n+1}} = \sum_{j,k=-\infty}^{\infty} \frac{q^{(j+k)^2} (-1)^j x^{j+k}}{(1 + iq^{j+1/2}) (1 + iq^{k+1/2})}.
\end{align*}
\]

**Proof.** Formula (4.14) is the special case \(\alpha = 1\) and \(b = aq\) of (4.12) while (4.15) is the special case \(a = -q^{1/2}i\) of (4.14). \(\square\)

The special choice \(\alpha = 1\) and \(b = aq\) in (4.13) establishes

\[
(4.16) \quad \sum_{n=-\infty}^{\infty} \frac{q^{9n^2} x^{3n}}{1 - a^3 q^{3n}} = \frac{(q^3; q^3)_\infty^2}{(q^3; q^3)_\infty} \frac{(a, q/a; q)_3^3}{(a^3, q^3a^{-3}; q^3)_\infty} \times \sum_{j,k,\ell=-\infty}^{\infty} \frac{q^{k+2\ell} q^{(j+k+\ell)^2} x^{j+k+\ell}}{(1 - aq^j) (1 - aq^k) (1 - aq^\ell)}.
\]

Two special case of (4.16) are worth noting. First when \(a = q^{1/3}\) we find that

\[
(4.17) \quad \frac{(q; q)_7^7}{(q^3; q^3)_\infty^2} \frac{(q^{1/3}, q^{2/3}, q)_3^3}{(q^3; q^3)_\infty} \sum_{n=-\infty}^{\infty} \frac{q^{9n^2} x^{3n}}{1 - q^{3n+1}} = \sum_{j,k,\ell=-\infty}^{\infty} \frac{q^{k+2\ell} q^{(j+k+\ell)^2} x^{j+k+\ell}}{(1 - q^{j+1/3}) (1 - q^{k+1/3}) (1 - q^{\ell+1/3})}.
\]

With \(a = -q^{1/3}\) in (4.16) we conclude that

\[
(4.18) \quad \sum_{n=-\infty}^{\infty} \frac{q^{9n^2} x^{3n}}{1 + q^{3n+1}} = \frac{(q^3; q^3)_\infty^2}{(q; q)_\infty^6} \frac{(-q^{1/3}, -q^{2/3}; q)_3^3}{(-q^2, -q; q^3)_\infty} \times \sum_{j,k,\ell=-\infty}^{\infty} \frac{q^{k+2\ell} q^{(j+k+\ell)^2} x^{j+k+\ell}}{(1 + q^{j+1/3}) (1 + q^{k+1/3}) (1 + q^{\ell+1/3})}.
\]

It is clear that one can generate other identities by specializing the parameters in the master formulas.
5  \( q \)-Lommel Polynomials

Iterating the three term recurrence relation of the \( q \)-Bessel function leads to

\[
q^{n\nu+n(n-1)/2} J^{(2)}_{\nu+n}(x; q) = h_{n,\nu} \left( \frac{1}{x}; q \right) J^{(2)}_{\nu}(x; q) - h_{n-1,\nu+1} \left( \frac{1}{x}; q \right) J^{(2)}_{\nu-1}(x; q),
\]

where \( h_{n,\nu} (x; q) \) are the \( q \)-Lommel polynomials introduced in [10], [11], §14.4. It is more convenient to use the polynomials

\[
p_{n,\nu}(x; q) := e^{-i\pi n/2} h_{n,\nu}(ix) = \sum_{j=0}^{[n/2]} \frac{(q^n; q; q)_{n-j}}{(q, q^n; q)_{j}(q; q)_{n-2j}}(2x)^{n-2j} q^{j(\nu+1)}.
\]

The identity (5.1) expressed in terms of \( I_{\nu} \)'s is

\[
(-1)^n q^{n\nu+n(n-1)/2} I^{(2)}_{\nu+n}(x; q) = p_{n,\nu}(1/x; q) I^{(2)}_{\nu}(x; q) - p_{n-1,\nu+1}(1/x; q) I^{(2)}_{\nu-1}(x; q),
\]

When \( x = 2q^{-k/2} \) we obtain, after replacing \( \nu \) by \( \nu + k \),

\[
(-1)^n q^{n(2\nu+k-1)/2} S_k ( -q^{\nu+n}; q ) = p_{n,\nu+k}(q^{k/2}/2; q) S_k ( -q^{\nu}; q ) - q^{k/2} p_{n-1,\nu+k+1}(q^{k/2}/2; q) S_k ( -q^{\nu-1}; q ).
\]

We now rewrite this as a functional equation in the form

\[
y^n q^{n(n+k-1)/2} S_k ( yq^n; q ) = u_n(q^{k/2}; -yq^k; q) S_k ( y/q; q ) - q^{k/2} u_{n-1}(q^{k/2}; -yq^{k+1}; q) S_k ( y/q; q ).
\]

with

\[
u_n(x, y) = \sum_{j=0}^{[n/2]} \frac{(y; q; q)_{n-j}}{(q, y; q)_{j}(q; q)_{n-2j}} x^{n-2j}.
\]

Therefore

\[
S_k ( y; q ) = \frac{y^n q^{n(n+k-1)/2} u_n(q^{k/2}; -yq^{k+1}; q)}{\Delta_n} S_k ( yq^n; q ) - \frac{y^{n+1} q^{(n+1)(n+k)/2} u_{n+1}(q^{k/2}; -yq^{k+1}; q)}{\Delta_n} S_k ( -q^{\nu+n+1}; q ),
\]

where

\[
\Delta_n = u_n(q^{k/2}; -yq^{k+1}; q) u_n(q^{k/2}; -yq^k; q) - u_{n+1}(q^{k/2}; -yq^k; q) u_{n-1}(q^{k/2}; -yq^{k+1}; q).
\]
6 Identities Involving Stieltjes–Wigert Polynomials

In this section we state several identities involving Stieltjes–Wigert polynomials and the Ramanujan function.

\begin{equation}
(xt, -t; q)_\infty = \sum_{n=0}^{\infty} q^{\binom{n}{2}} t^n S_n(xq^{-n}; q).
\end{equation}

\begin{equation}
\frac{q^{\binom{n}{2}} x^n}{(q; q)_n} = \sum_{k=0}^{n} \frac{(-1)^k q^{\binom{k}{2}}}{(q; q)_{n-k}} S_k(xq^{-k}; q),
\end{equation}

\begin{equation}
S_n(x) = \sum_{k=0}^{\infty} q^{\binom{k+1}{2}} (xq^n)^k A_q(xq^k),
\end{equation}

\begin{equation}
S_n(ab; q) = b^n \sum_{k=0}^{n} \frac{(b^{-1}; q)_k (-q^{1-n})^k q^{\binom{k}{2}}}{(q; q)_k} S_{n-k}(aq^k; q),
\end{equation}

\begin{equation}
S_n(a; q) = \frac{(-aq; q)_{\infty}}{(q, -aq; q)_n} \sum_{k=0}^{\infty} \frac{q^{k^2} (-a)^k}{(q, -aq^{n+1}; q)_k},
\end{equation}

\begin{equation}
S_{2n+1}(q^{-2n-1}; q) = 0, \quad S_{2n}(q^{-2n}; q) = \frac{(-1)^n q^{-n^2}}{(q^2; q^2)_n}.
\end{equation}

\begin{equation}
S_n(-q^{-n+1/2}; q) = \frac{q^{-(n^2-n)/4}}{(q^{1/2}; q^{1/2})_n},
\end{equation}

\begin{equation}
S_n(-q^{-n-1/2}; q) = \frac{q^{-(n^2+n)/4}}{(q^{1/2}; q^{1/2})_n},
\end{equation}

\begin{equation}
A_q(wz) = (wq; q)_{\infty} \sum_{n=0}^{\infty} \frac{q^{n^2} w^n}{(wq; q)_n} S_n(zq^{-n}; q).
\end{equation}

\begin{equation}
A_q(z) = (q; q)_m \sum_{n=0}^{\infty} \frac{q^{n^2+mn} (-z)^n}{(q; q)_n} S_m(zq^n; q).
\end{equation}

Proofs. Formula (6.1) follows from the definition (1.8) and Euler’s identities. Dividing both sides of (6.1) by \((-t; q)_{\infty}\) then expand \(1/(-t; q)_{\infty}\) on the right-hand side implies (6.2). The expansion (6.3) follows from (1.10), and the \(q\)-binomial theorem in the form

\begin{equation}
(x; q)_n = \sum_{j=0}^{n} \binom{n}{j}_q (-x)^j q^{\binom{j}{2}}.
\end{equation}

To prove (6.4) start with (6.1) as

\[ \sum_{n=0}^{\infty} q^{\binom{n}{2}} t^n S_n(abq^{-n}; q) = (abt, -t; q)_{\infty} = (abt, -bt; q)_{\infty} \frac{(-t; q)_{\infty}}{(-bt; q)_{\infty}}, \]
then expand the first product in \( S_k(aq^{-k}; q) \) and the second term using the \( q \)-binomial theorem. The proof of (6.5) consists of writing \((-aq; q)\infty/(-aq; q)_n(-aq^{n+1}; q)_k\) as \(-aq^{n+k+1}; q)\infty then expand this infinite product and use (6.11). The special values in (6.6) follow from letting \( x = 1 \) in (6.1) then equate like powers of \( t \). Similarly the special values in (6.7) and (6.8) follow from putting \( x = -q^{-1/2} \) in (6.1). Replace \( x \) by \( z \) in then multiply by \((-w)^nq^{(n+1)}/q)\infty then expand this infinite product and use (6.11). To prove (6.9) we expand the right-hand side in powers of \( z \) and realize that the coefficient of \((-z)^n\) is

\[
\frac{q^{n^2+mn}}{(q; q)_n} {}_2\phi_1(q^{-m}, q^{-n}; 0; q, q).
\]

By the \( q \)-Chu-Vandermonde sum \([9, (II.6)]\) the \( {}_2\phi_1 \) equals \( q^{-mn} \).

We note that the polynomials \( \{S_n(xq^{-n}; q)\} \) are related to the \( q^{-1} \)-Hermite polynomials, \([5], [12]\), which are defined by

\[
(6.12) \quad h_n(\sinh \xi \mid q) = \sum_{k=0}^{n} \frac{(q; q)_n}{(q; q)_k(q; q)_{n-k}} (-1)^k q^{k(k-n)} e^{(n-2k)\xi}.
\]

Indeed

\[
(6.13) \quad S_n(e^{-2\xi}q^{-n}; q) = \frac{1}{(q; q)_n} h_n(\sinh \xi \mid q).
\]

In fact (6.1) is equivalent to the generating function for the \( q^{-1} \)-Hermite polynomials, \([11]\). Moreover (6.13) and the generating function \([11, Theorem 21.3.1] \) lead to

\[
(6.14) \quad \sum_{n=0}^{\infty} \frac{(q; q)_n q^{n^2/4}}{(\sqrt{q}; \sqrt{q})_n} t^n S_n(zq^{-n}; q) = \frac{(-tq^{1/4}, -tq^{1/4}z; \sqrt{q})_\infty}{(-t^2z; q)\infty}.
\]

The Poisson kernel of \( q^{-1} \)-Hermite polynomials, \([11, Theorem 21.2.3] \) implies

\[
(6.15) \quad \sum_{n=0}^{\infty} (q; q)_n q^{n^2/2} t^n S_n(zq^{-n}; q) S_n(\zeta q^{-n}; q) = \frac{(-t, -t\zeta, tz, t\zeta; q)\infty}{(t^2z/\zeta; q)\infty}.
\]

Similarly one can derive other generating relations.

It must be noted that (6.7) and (6.8) when written in terms of the \( q^{-1} \)-Hermite polynomials are the evaluation of \( h_n(0 \mid q) \), see \([11, Corollary 21.2.2] \). It is easy to see that the evaluations (6.7) and (6.8) are equivalent to the identity in the following theorem.

**Theorem 6.1.** We have

\[
(6.16) \quad A_{q^2} (-b^2) = (b\sqrt{q}; q)\infty \sum_{n=0}^{\infty} \frac{q^{n^2/2}b^n}{(q, b\sqrt{q}; q)_n},
\]

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