A notion of seminormalization for real algebraic varieties

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Abstract
The seminormalization of an algebraic variety \( X \) is the biggest variety linked to \( X \) by a finite, birational, and bijective morphism. In this paper, we introduce a variant of the seminormalization, suited for real algebraic varieties, called the R-seminormalization. This object has a universal property of the same kind as the one of the seminormalization but related to the real closed points of the variety. In a previous paper, the author studied the seminormalization of complex algebraic varieties using rational functions that extend continuously to the closed points for the Euclidean topology. We adapt some of those results here to the R-seminormalization, and we provide several examples. We also show that the R-seminormalization modifies the singularities of a real variety by normalizing the purely complex points and seminormalizing the real ones.

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INTRODUCTION

The present paper is devoted to the introduction of the R-seminormalization of real algebraic varieties. It can be seen as the real version of the previous paper [4] of the author about seminormalization and complex regulus functions.

The operation of seminormalization was formally introduced around 50 years ago in the case of analytic spaces by Andreotti and Norguet [2]. For algebraic varieties, the seminormalization $X^+$ of $X$ is the biggest intermediate variety between $X$ and its normalization, which is bijective with $X$. Recently, the concept of seminormalization appears in the study of singularities of algebraic varieties, in particular in the minimal model program of Kollár and Kovács (see [9] and [10]). The seminormalization has the property of having “multicross” singularities in codimension 1 (see [12]), which means that they are locally analytically isomorphic to the union of linear subspaces of affine space meeting transversally along a common linear subspace.

Around 1970, Traverso [14] introduced the notion of the seminormalization $A^+_B$ of a commutative ring $A$ in an integral extension $B$. The idea is to glue together the prime ideals of $B$ lying over the same prime ideal of $A$. The seminormalization $A^+_B$ has the property that it is the biggest extension $C$ of $A$ in $B$, which is subintegral that is, such that the map Spec($C$) $\rightarrow$ Spec($A$) is bijective and equiresidual (it gives isomorphisms between the residue fields). We refer to Vitulli [15] for a survey on seminormality for commutative rings and algebraic varieties.

In real algebraic geometry, the seminormalization was first studied in 1975 [1] in the case of real analytic sets. However, in 1981, Marinari and Raimondo [13] showed that the construction of Traverso with real spectrum has no natural universal property. Recently, Fichou, Monnier, and Quarez defined in [7] a real seminormalization called the “central” seminormalization, but whose universal property does not rely on all the real closed points of the variety. The R-seminormalization is equipped with such a universal property, and this is what motivated the introduction of this object.

The idea of the construction of the R-seminormalization is the following: since the seminormalization of a variety is obtained by gluing together the complex points that have been separated in the normalization, one may want to do the same thing but with real points of real varieties. Unfortunately, gluing the real points of the normalization in the fibers of the real points of a variety can lead to some problems because the restriction to the real closed points of a finite morphism of real varieties is not necessarily surjective. For example, the normalization of the real variety $X := \text{Spec} (\mathbb{R}[x, y]/ \langle y^2 - x^2(x - 1) \rangle)$ is $X' := \text{Spec} (\mathbb{R}[x, y]/ \langle y^2 - x + 1 \rangle)$, and we have $(\pi')^{-1} (\{0\}) = \emptyset$ where $\pi'$ is the normalization morphism. In fact, for a general variety $X$, one cannot find a variety $Y$ that would be maximal for the property of having a morphism $\pi : Y \rightarrow X$ that is finite, birational, and such that $\pi_{|R}$ is bijective. A proof that such a variety does not exist in general can be found in [7, Example 5.6]. This lack of surjectivity led the authors of [7] to consider the “central” seminormalization where they glue the real central points (i.e., the points in the Euclidean closure of the regular locus) of the normalization above the real central points of the variety. Another way to counter this lack of surjectivity is to consider all the complex points in the normalization lying over the real points of the variety. This is the idea behind the R-seminormalization.
Recently, the present author highlighted in [4] the correlation between seminormalization and continuous rational functions for the Euclidean topology on complex affine varieties and on any algebraic variety over a field of characteristic 0, together with Fichou, Monnier, and Quarez in [5]. The spirit of the work presented here is to get an analog of the results of [4] for the R-seminormalization while getting a better understanding of this new object.

This paper is organized as follows: In Section 1, we give a generalization of Traverso’s construction of the seminormalization. This allows us to define and provide a universal property of the R-seminormalization. This universal property states that the R-seminormalization of a ring $A$ in an integral extension of $A$ is the biggest subextension such that there exists a unique prime ideal above each real prime ideal of $A$ and their residue fields are isomorphic. In Section 2, we look at the R-seminormalization of real algebraic varieties and prove a geometrical universal property for this object. We show that the R-seminormalization is the biggest variety between a variety $X$ and its normalization such that there exists a unique complex closed point above each element of $X(\mathbb{R})$. In Section 3, we identify, in several ways, as made in the paper [4], the ring of functions on $X(\mathbb{R})$ that becomes polynomials on the closed points of the R-seminormalization. On the normalization, those functions correspond to the polynomials that are constant on the complex fibers over the real closed points of $X$. They also correspond to the integral rational functions, which are continuous on $X(\mathbb{R})$ for the topology of the space where the rational function is well defined in $X(\mathbb{C})$. More precisely, we get the following version of Theorem 4.13 of [4] for the R-seminormalization.

**Proposition (3.7).** Let $X$ be a real affine variety, and let $f : X(\mathbb{R}) \to \mathbb{R}$. Then $f$ becomes polynomial on the R-seminormalization if and only if it satisfies the following properties.

1. The function $f$ is an extension of a rational function of $X$.
2. The function $f$ is integral over $\mathbb{R}[X]$.
3. For all $x \in X(\mathbb{R})$, for all $(z_n)_n \subset \text{Dom}_C(f) \cup X(\mathbb{R})$ such that $z_n \to x$, then $f(z_n) \to f(x)$.

Where $\text{Dom}_C(f)$ is the domain, in $X(\mathbb{C})$, of the rational function that $f$ is extending. We give a third characterization of those functions using their graphs in the same spirit as Theorem 4.20 of [4]. In Section 4, we look at the functions on $X(\mathbb{R})$ that are the restriction of continuous rational functions on $X(\mathbb{C})$. This ring of functions corresponds to the coordinate ring of the seminormalization $X^+$. Then, we present several examples in order to compare the seminormalization, the R-seminormalization, the central seminormalization, and the central weak-normalization. Finally, in Section 5, we prove that the R-seminormalization $X^{R+}$ of a real variety $X$ is related to its seminormalization $X^+$ and to its biregular normalization $X^b$, which has been introduced by Fichou, Monnier, and Quarez in [6]. Briefly, the biregular normalization is the biggest variety that is linked to $X$ by a birational, finite, and biregular morphism.

**Theorem (5.8).** Let $X$ be a real affine variety. Then

$$X^{R+} \simeq (X^+)^b \simeq (X^b)^+$$

This result allows us to see how the R-seminormalization modifies the singularities of a real algebraic variety: it normalizes the purely complex points and seminormalizes the real points.
1  |  GENERALIZATION OF THE SEMINORMALIZATION FOR GENERAL RINGS

In this section, we present a generalization of Traverso’s construction of the seminormalization. This generalization will allow us to provide a universal property of the R-seminormalization but also of the R-Max-seminormalization, which will be convenient in Section 2.

Let $A$ be a ring. We say that an ideal $I$ of $A$ is real if it has the following property: If a sum of square elements $\sum a_i^2 \in \sum A^2$ belongs to $I$, then every element $a_i$ belongs to $I$. We denote by $\text{Spec}(A)$ (resp. $\text{R-Max}(A)$) the set of real prime (resp. maximal) ideals of $A$. Let $F$ be one of the functors $\text{Spec}$, $\text{Max}$, $\text{R-Max}$, or $\text{R-Spec}$. In particular, $F : \text{Ann} \to \text{Top}$ is a contravariant subfunctor of Spec. It means that for all $A \in \text{Ann}$ we have $F(A) \subseteq \text{Spec}(A)$, and for all extensions of rings $A \hookrightarrow B$, we get a continuous application $F(B) \to F(A)$ given by $\mathfrak{p} \mapsto \mathfrak{p} \cap A$. One may want to take a general subfunctor of Spec, but we will need, at some point in this section, the specificity of the four considered subfunctors. See the remark after Definition 1.2 for more details.

**Definition 1.1.** Let $A \hookrightarrow B$ be an integral extension of rings. We define

$$A_B^F := \{b \in B \mid \forall \mathfrak{p} \in F(A), \ b_\mathfrak{p} \in A_\mathfrak{p} + \text{Rad}(B_\mathfrak{p})\},$$

where $B_\mathfrak{p}$ is the localization of $B$ by $A \setminus \mathfrak{p}$ and $b_\mathfrak{p}$ is the image of $b$ in $B_\mathfrak{p}$. Moreover, $\text{Rad}(B_\mathfrak{p})$ is the Jacobson radical of $B_\mathfrak{p}$.

**Remark.** If $F = \text{Spec}$, the ring $A_B^F$ is the seminormalization $A_B^+$ of $A$ in $B$.

**Definition 1.2.** Let $A \hookrightarrow B$ be an integral extension of rings. The extension $A \hookrightarrow B$ is called $F$-subintegral if it satisfies the following conditions.

1. For all $\mathfrak{p} \in F(A)$, there exists a unique $\mathfrak{q} \in \text{Spec}(B)$ such that $\mathfrak{q} \cap A = \mathfrak{p}$.
2. For such $\mathfrak{p}$ and $\mathfrak{q}$, we have $\mathfrak{q} \in F(B)$, and the induced map $\kappa(\mathfrak{p}) \hookrightarrow \kappa(\mathfrak{q})$ on the residue fields is an isomorphism.

**Remark.** It is important to see that if $\mathfrak{p} \in F(A)$ and $\mathfrak{q} \in \text{Spec}(B)$ are such that $\mathfrak{q} \cap A = \mathfrak{p}$, then the condition $\kappa(\mathfrak{p}) \simeq \kappa(\mathfrak{q})$ and the fact that the extension is integral implies that $\mathfrak{q} \in F(B)$. It is really specific to the fact that an ideal is real if and only if its residual field is a real field.

We give here a first geometric property of $F$-subintegral extensions.

**Notation.** Throughout this paper, we will add the prefix “Z-” before a topological property when it refers to the Zariski topology. Moreover, for a ring $A$ and an ideal $I$ of $A$, we denote by $\mathcal{V}(I)$ the $Z$-closed set defined by

$$\mathcal{V}(I) = \{\mathfrak{p} \in \text{Spec}(A) \mid I \subseteq \mathfrak{p}\}.$$

**Proposition 1.3.** Let $A \hookrightarrow B$ be an $F$-subintegral extension of rings, and $\pi : F(B) \to F(A)$ be the induced map. Then $\pi$ is a $Z$-homeomorphism for the induced topology.
Proof. The morphism Spec(B) → Spec(A) is Z-continuous, so is its restriction to F(B). Thus, we just have to show that π is Z-closed. Let p ∈ F(A) and q ∈ F(B) be such that q ∩ A = p. We have

\[ \pi(V(q)) = \{q' \cap A \mid q' \subseteq q' \in F(B)\}, \]

so \( \pi(V(q)) \subseteq V(q \cap A) = V(p) \). If \( p' \in V(p) \cap F(A) \), then the going-up property says that there exists \( q' \in \text{Spec}(B) \) such that \( q' \cap A = p' \) and \( q \subseteq q' \). Since the extension is F-subintegral, the ideal \( q' \) belongs to F(B). Hence \( p' \in \pi(V(q)) \) and finally \( V(p) \subseteq \pi(V(q)) \), so \( \pi(V(q)) = V(p) \) which is Z-closed.

The goal of this section is to prove the following universal property: The F-seminormalization of \( A \) in \( B \) is the biggest F-subintegral extension of \( A \hookrightarrow B \).

**Proposition 1.4.** Let \( A \hookrightarrow C \hookrightarrow B \) be integral extensions of rings. Then the following statements are equivalent.

1. The extension \( A \hookrightarrow C \) is F-subintegral.
2. The image of \( C \hookrightarrow B \) is a subring of \( A_{F}^{B} \).

We prove the universal property through a series of propositions. But first, let us recall the important property of “going-up” verified for integral extensions.

**Proposition 1.5** ([3]). Let \( A \hookrightarrow B \) be an integral extension of rings. Then

1. The associated map Spec(B) → Spec(A) is surjective.
2. The inverse image of Max(A) under the map Spec(B) → Spec(A) is Max(B).

**Proposition 1.6.** Let \( A \hookrightarrow B \) be an integral extension of rings. Then

\( A \hookrightarrow A_{F}^{B} \) is F-subintegral.

**Proof.** Proceeding similarly to the proof of Proposition 2.10 of [4], one shows that for all \( p \in F(A) \), there exists a unique \( q \in \text{Spec}(A_{F}^{B}) \) such that \( q \cap A = p \) and \( \kappa(p) \simeq \kappa(q) \). Because of the special nature of F (see remark following Definition 1.2), the equiresiduality implies that \( q \in F(A_{F}^{B}) \) and so \( A \hookrightarrow A_{F}^{B} \) is F-subintegral. \( \square \)

**Proposition 1.7.** Let \( A \hookrightarrow C \hookrightarrow B \) be integral extensions of rings. Then the following properties are equivalent.

1. The extension \( A \hookrightarrow B \) is F-subintegral.
2. The extensions \( A \hookrightarrow C \) and \( C \hookrightarrow B \) are F-subintegral.

**Proof.** We prove (1) implies (2). Let \( p \in F(C) \) and \( q_1, q_2 \in \text{Spec}(B) \) be such that \( q_1 \cap C = q_2 \cap C = p \). We have \( q_1 \cap A = q_2 \cap C \cap A = p \cap A \) and the same is true for \( q_2 \). Since \( A \hookrightarrow B \) is F-subintegral and \( p \cap A \in F(A) \), then

\[ q_1 \cap A = q_2 \cap A \Rightarrow q_1 = q_2. \]
Moreover, $\kappa(p \cap A) \hookrightarrow \kappa(p) \hookrightarrow \kappa(q_i)$ and $\kappa(p \cap A) \simeq \kappa(q_i)$. So, $\kappa(p) \simeq \kappa(q_i)$. Now, let us consider $p \in F(A)$ and $p_1, p_2 \in \text{Spec}(C)$ such that $p_1 \cap A = p_2 \cap A = p$. We know that $\text{Spec}(B) \to \text{Spec}(C)$ is surjective, so we can find $q_1, q_2 \in \text{Spec}(B)$ such that $q_1 \cap C = p_1$ and $q_2 \cap C = p_2$. Then $q_1 \cap A = p$ and $q_2 \cap A = p$. Since $A \hookrightarrow B$ is $F$-subintegral and $p \in F(A)$, we get $q_1 = q_2$ and so $p_1 = p_2$. Moreover, $\kappa(p) \hookrightarrow \kappa(p_i) \hookrightarrow \kappa(q_i)$ and $\kappa(p) \simeq \kappa(q_i)$. So, $\kappa(p) \simeq \kappa(p_i)$.

We now show that (2) implies (1). Let us suppose that $A \hookrightarrow C$ and $C \hookrightarrow B$ are $F$-subintegrals. Let $p \in F(A)$, then there exists a unique element $p' \in \text{Spec}(C)$ such that $p' \cap A = p$ and $p' \in F(C)$. Moreover, $\kappa(p') \simeq \kappa(p)$. Then, since $C \hookrightarrow B$ is $F$-subintegral, there exists a unique element $p'' \in \text{Spec}(B)$ such that $p'' \cap C = p'$ and $\kappa(p'') \simeq \kappa(p')$. So $p''$ is the unique element of $\text{Spec}(B)$ such that $p'' \cap A = p$ and $p'' \in F(B)$. Moreover, $\kappa(p'') \simeq \kappa(p)$. Hence, $A \hookrightarrow B$ is $F$-subintegral. □

**Proposition** (Proposition 1.4). Let $A \hookrightarrow C \hookrightarrow B$ be integral extensions of rings. Then the following statements are equivalent.

(1) The extension $A \hookrightarrow C$ is $F$-subintegral.
(2) The image of $C \hookrightarrow B$ is a subring of $A_F^B$.

**Proof.** One shows that (1) implies (2) proceeding similarly to the proof of Proposition 2.4 of [4]. The converse is also very similar: suppose that we have $A \hookrightarrow C \hookrightarrow A_F^B \hookrightarrow B$. Those extensions are integral, and by Proposition 1.6, the extension $A \hookrightarrow A_F^B$ is $F$-subintegral. Then Lemma 1.7 tells us that $A \hookrightarrow C$ is $F$-subintegral. □

Let us conclude this section by rewriting Proposition 1.4 in the form of a universal property theorem.

**Theorem 1.8** (Universal property of the $F$-seminormalization). Let $A \hookrightarrow B$ be an integral extension of rings. For every intermediate extension $C$ of $A \hookrightarrow B$ such that $A \hookrightarrow C$ is $F$-subintegral, the image of $C$ by the injection $C \hookrightarrow B$ is contained in $A_F^B$.

**Remark.** Let $A \hookrightarrow B$ be an integral extension. We have that $A \hookrightarrow A_F^B$ is $F$-subintegral by Proposition 1.6. So, we can apply the universal property in the following way:
Thus, there is an injection from $A_B^F$ to $A_B^P$. Since, by definition, $A_B^F$ is included in $A_B^P$, we get the following idempotency property:

$$A_B^F = A_B^F.$$ 

### 2. THE R-SEMINORMALIZATION FOR GEOMETRIC RINGS

Let $X = \text{Spec}(A)$ be an affine algebraic variety with $A$ an $\mathbb{R}$-algebra of finite type. Let $\mathbb{R}[X] := A$ denote the coordinate ring of $X$. We have $\mathbb{R}[X] \simeq \mathbb{R}[x_1,\ldots,x_n]/I$ for an ideal $I$ of $\mathbb{R}[x_1,\ldots,x_n]$. We will say that $X$ is a real affine variety if $I$ is a real ideal. A morphism $\pi : Y \to X$ between two real varieties induces the morphism $\pi^* : \mathbb{R}[X] \hookrightarrow \mathbb{R}[Y]$ that is injective if and only if $\pi$ is dominant. We say that $\pi$ is of finite type (resp. is finite) if $\pi^*$ makes $\mathbb{R}[Y]$ an $\mathbb{R}[X]$-algebra of finite type (resp. a finite $\mathbb{R}[X]$-module). The space $X$ is equipped with the Zariski topology, for which the closed sets are of the form $\mathfrak{V}(I) := \{p \in \text{Spec}(\mathbb{R}[X]) \mid I \subset p\}$ where $I$ is an ideal of $\mathbb{R}[X]$. We define $X(\mathbb{R}) := \{p \in \text{Max}(\mathbb{R}[X]) \mid \mathfrak{V}(p) \cap \mathbb{R}[X] = \mathfrak{V}(p)\}$. The Real Nullstellensatz gives a bijection between $X(\mathbb{R})$ and the algebraic set $\mathfrak{Z}(\mathbb{R})(I) := \{x \in \mathbb{R}^n \mid \forall f \in I, f(x) = 0\} \subset \mathbb{R}^n$. For any real variety, we can look at its complexification whose coordinate ring is given by the change of coordinate $\mathbb{C}[X] := \mathbb{R}[X] \otimes_{\mathbb{R}} \mathbb{C}$. Hence, we can consider the set of its closed points $X(\mathbb{C})$, and if $X$ is a real variety, we have that $X(\mathbb{R})$ is dense for the Zariski topology in $X(\mathbb{C})$. We write $\mathcal{K}(X) := \text{Frac}(\mathbb{R}[X])$.

The goals of this section are to provide a geometric universal property of the R-seminormalization and to see that the R-seminormalization coincides with the R-Max-seminormalization for real algebraic varieties. The following theorem gives a reinterpretation of the R-subintegrality from a geometric point of view.

**Theorem 2.1.** Let $\pi : Y \to X$ be a finite morphism between real affine varieties. The following properties are equivalent.

1. The extension $\pi^* : \mathbb{R}[X] \hookrightarrow \mathbb{R}[Y]$ is R-subintegral.
2. The extension $\pi^* : \mathbb{R}[X] \hookrightarrow \mathbb{R}[Y]$ is R-Max-subintegral.
3. The restriction $\tilde{\pi} : \pi^{-1}(X(\mathbb{R})) \to X(\mathbb{R})$ of the morphism $\pi_c : Y(\mathbb{C}) \to X(\mathbb{C})$ is bijective.
4. The morphism $\pi_R : Y(\mathbb{R}) \to X(\mathbb{R})$ is bijective and $\pi^{-1}(X(\mathbb{R})) = Y(\mathbb{R})$.

**Proof.**

1. $\Rightarrow$ (2). Suppose that $\mathbb{R}[X] \hookrightarrow \mathbb{R}[Y]$ is R-subintegral. Then for all $p \in \text{R-Spec}(\mathbb{R}[X])$, there exists a unique $q \in \text{Spec}(\mathbb{R}[Y])$ such that $q \cap \mathbb{R}[X] = p$. Moreover, $q \in \text{R-Spec}(\mathbb{R}[Y])$. In particular, it is true if $p$ is maximal, and so, we get Property 2.

2. $\Rightarrow$ (1). Suppose that for all $m \in \text{R-Max}(\mathbb{R}[X])$, we have a unique element $m' \in \text{Max}(\mathbb{R}[Y])$ such that $m' \cap \mathbb{R}[X] = m$ and $m' \in \text{R-Max}(\mathbb{R}[Y])$. Let $p \in \text{R-Spec}(\mathbb{R}[X])$ and $q \in \text{Spec}(\mathbb{R}[Y])$ be such that $q \cap \mathbb{R}[X] = p$. We define $V := \mathfrak{V}(p)$ and $W := \mathfrak{V}(q)$. Then $\dim V = \dim W$ since $\pi$ is finite. Moreover, $\dim V(\mathbb{R}) = \dim V(\mathbb{C})$ because $p$ is a real ideal, and we also have $\dim V(\mathbb{R}) = \dim W(\mathbb{R})$ since $\pi_R$ is bijective. We get

$$\dim W(\mathbb{R}) = \dim V(\mathbb{R}) = \dim V(\mathbb{C}) = \dim V = \dim W = \dim W(\mathbb{C}).$$

So, $q \in \text{R-Spec}(\mathbb{R}[Y])$. Let us show that $q$ is unique. Suppose that there exists $q' \in \text{Spec}(\mathbb{R}[Y])$ such that $q' \cap \mathbb{R}[X] = p$. By the previous arguments, we have $q' \in \text{R-Spec}(\mathbb{R}[Y])$. Let $m \in$
R-Max(\(\mathbb{R}[Y]\)) be such that \(q \subset m\), then \(q \cap \mathbb{R}[X] \subset m \cap \mathbb{R}[X]\). Moreover, by the going-up property, we can consider \(m' \in \text{Max}(\mathbb{R}[Y])\) such that \(q' \subset m'\) and \(m' \cap \mathbb{R}[X] = m \cap \mathbb{R}[X]\). Then, by assumption, we get \(m = m'\). So, \(\mathbb{Z}_m(q) = \mathbb{Z}_m(q')\) and the real Nullstellensatz gives us \(q = q'\).

It remains to see that \(\kappa(p') \simeq \kappa(p)\). If we write \(V = \text{Spec}(\mathbb{R}[X]/p)\) and \(W = \text{Spec}(\mathbb{R}[Y]/p')\), we get the following commutative diagram:

![Diagram](Diagram)

The double tip in the arrows means that the map is surjective. As \(\mathbb{R}[Y]\) is a finite \(\mathbb{R}[X]\)-module, we have that \(\mathbb{R}[W]\) is a finite \(\mathbb{R}[V]\)-module. Thus, \(\pi_W\) is a finite morphism between two irreducible varieties. Therefore, we get \(n := [K(W) : K(V)] < +\infty\). Since the characteristic is zero, the extension \(K(V) \subseteq K(W)\) is separable and finite. Hence, we can consider a primitive element \(a \in K(W)\) such that \(K(W) = K(V)(a)\). Let \(F\) be the minimal polynomial of \(a\). Then \(\deg F = n\) and \(\text{disc}(F) \neq 0\), where \(\text{disc}(F)\) is the discriminant of \(F\). Thus, the set \(\{\text{disc}(F) \neq 0\}\) is a nonempty \(Z\)-open set of \(V\). Since \(p \in R-\text{Spec}(\mathbb{R}[X])\), we have that \(V(\mathbb{R})\) is \(Z\)-dense in \(V\), and so, there exists \(y \in V(\mathbb{R})\) such that \(\text{disc}(F)(y, .) \neq 0\). This means

\[
\#\pi^{-1}_c(y) = \#\{\text{complex roots of } F(y, .)\} = \deg F = n.
\]

By assumption, there is a unique element of \(Y(\mathbb{C})\) above every element of \(X(\mathbb{R})\). So, \(\#\pi^{-1}_c(y) = n = 1\) and so \(\kappa(p) \simeq \kappa(p')\).

Now, let us see that the properties 2–4 are equivalent.

(2) \(\Rightarrow\) (3). Property 2 implies that for all \(m \in \text{R-Max}(\mathbb{R}[X])\), there exists a unique \(m' \in \text{Spec}(\mathbb{R}[Y])\) such that \(m' \cap \mathbb{R}[X] = m\). By the going-up property, we have \(m' \in \text{Max}(\mathbb{R}[Y])\). So, by the Nullstellensatz, we get the third property.

(3) \(\Rightarrow\) (4). Let \(x \in X(\mathbb{R})\) and \(z \in Y(\mathbb{C})\) be such that \(\pi_c(z) = x\). Then \(\pi_c(\bar{z}) = x\). Since \(z\) is supposed to be unique, we get \(z = \bar{z}\) and so \(z \in Y(\mathbb{R})\).

(4) \(\Rightarrow\) (2). Suppose 4 and take \(m \in \text{R-Max}(\mathbb{R}[X])\). By the going-up property, we know that there is a finite number of prime ideals in \(\mathbb{R}[Y]\) lying over \(m\) and that those ideals are maximal. Since we have \(\pi_c^{-1}(X(\mathbb{R})) = Y(\mathbb{R})\), then all the \(m'\) are real. So, the morphism \(\pi_c\) being bijective, we get that there is a unique prime ideal \(m'\) of \(\mathbb{R}[Y]\) lying over \(m\). Moreover, \(m' \in \text{R-Max}(\mathbb{R}[Y])\) so \(\kappa(m) \simeq \kappa(m') \simeq \mathbb{R}\), and we get the second property.

Let \(A \hookrightarrow B\) be an integral extension of rings. We write

\[
A_{B}^{\text{R-Spec}} = A_{B}^{R^+} \quad \text{and} \quad A_{B}^{\text{R-Max}} = A_{B}^{R^+_{\text{max}}}.
\]

Let us see that if \(A\) is a coordinate ring, then \(A^{R^+}\) and \(A^{R^+_{\text{max}}}\) are also the coordinate rings of some real varieties.
Proposition 2.2. Let $\pi : Y \to X$ be a finite morphism between real affine varieties. Let $A$ be an intermediate ring between $\mathbb{R}[X]$ and $\mathbb{R}[Y]$. Then there exists a unique affine variety $Z$ such that $A = \mathbb{R}[Z]$. Moreover, if $X$ and $Y$ are real varieties and $\pi$ is birational, then $Z$ is a real affine variety.

Proof. We have that $A$ is an $\mathbb{R}[X]$-module because it is an $\mathbb{R}[X]$-submodule of $\mathbb{R}[Y]$. Thus, it is an $\mathbb{R}$-algebra of finite type because so is $\mathbb{R}[X]$. If $\pi$ is also birational, then we get $\mathbb{R}[X] \hookrightarrow \mathbb{R}[Z] \hookrightarrow \mathbb{R}[X]'$, and by $[6]$ Lemma 2.8, the ring $\mathbb{R}[Z]$ is real. □

This leads us to define, for every real variety, a new variety called its $R$-seminormalization.

Definition 2.3. Let $\pi : Y \to X$ be a finite morphism between two affine varieties over $\mathbb{R}$. The affine variety defined by

$$X^R_Y := \text{Spec} \left( \mathbb{R}[X]^{R+}_{\mathbb{R}[Y]} \right)$$

is called the $R$-seminormalization of $X$ in $Y$.

The $R$-subintegrality being equivalent to the $R$-Max-subintegrality for affine rings, we naturally get that the $R$-Max-seminormalization corresponds to the $R$-seminormalization. Note that, for the central seminormalization defined in $[7]$, this property is not true, and we get two different varieties: the central seminormalization $X^{\text{sc}}$ and the central weak-normalization $X^{\text{wc}}$.

Corollary 2.4. Let $\pi : Y \to X$ be a finite morphism between two real affine varieties. Then

$$\mathbb{R}[X]^{R+\max}_{\mathbb{R}[Y]} = \mathbb{R}[X]^{R+}_{\mathbb{R}[Y]}$$

Proof. First, the inclusion $\mathbb{R}[X]^{R+}_{\mathbb{R}[Y]} \subset \mathbb{R}[X]^{R+\max}_{\mathbb{R}[Y]}$ is clear. Now, by Proposition 1.6, we know that $\mathbb{R}[X]^{R+\max}_{\mathbb{R}[Y]}$ is $R$-Max-subintegral, so by Theorem 2.1, it is also $R$-subintegral. Then the universal property of $\mathbb{R}[X]^{R+\max}_{\mathbb{R}[Y]}$ gives us $\mathbb{R}[X]^{R+\max}_{\mathbb{R}[Y]} \subset \mathbb{R}[X]^{R+}_{\mathbb{R}[Y]}$. □

We can now rewrite the universal property of the $R$-seminormalization for the geometric case. It is the biggest variety such that there is a unique complex closed point in the fiber of every real closed point.

Theorem 2.5 (Universal property of the $R$-seminormalization). Let $Y \to Z \to X$ be finite morphisms between real affine varieties. Then the restriction $\widetilde{\pi}^{-1}_Z : \pi^{-1}_Z(X(\mathbb{R})) \to X(\mathbb{R})$ of the morphism $\pi_Z : Z(\mathbb{C}) \to X(\mathbb{C})$ is bijective if and only if there exits a morphism $\pi^+_Z : X^R_Y \to Z$ such that $\pi_Z \circ \pi^+_Z = \pi^+$. Moreover, $\pi^+_Z$ is unique, and the restriction $\widetilde{\pi}^+_Z : (\pi^+_Z)^{-1}(Z(\mathbb{R})) \to Z(\mathbb{R})$ of the morphism $\pi^+_Z : X^R_Y(\mathbb{C}) \to Z(\mathbb{C})$ is bijective.
Proof. We have the following equivalences:

\[ \text{The morphism } \tilde{\pi}_Z \text{ is bijective } \iff \mathbb{R}[X] \hookrightarrow \mathbb{R}[Z] \text{ is } R\text{-subintegral (by Theorem 2.1)} \]
\[ \iff \mathbb{R}[Z] \hookrightarrow \mathbb{R}[X^+_{\mathbb{R}[Y]}] \text{ (by Proposition 1.4)} \]
\[ \iff \exists \pi^+_Z : X^+_Y \to Z \text{ dominant, such that } \pi^+_Z \circ \pi_Z = \pi^+. \]

We get the uniqueness of \( \pi^+_Z \) by injectivity of \( \pi_Z \). Since \( \mathbb{R}[X] \hookrightarrow \mathbb{R}[X^+_{\mathbb{R}[Y]}] \) is R-subintegral, then Lemma 1.7 says that \( \mathbb{R}[Z] \hookrightarrow \mathbb{R}[X^+_{\mathbb{R}[Y]}] \) is also R-subintegral. So, by Theorem 2.1, the morphism \( \tilde{\pi}_Z \) is bijective.

\[ \square \]

3 | FUNCTIONS WHICH BECOME POLYNOMIAL ON THE R-SEMINORMALIZATION

Let \( X \) be a real variety. The normalization \( X' \) of a real variety is defined by the coordinate ring \( \mathbb{R}[X]'_{\mathcal{K}(X)} \), and its seminormalization \( X^+ \) is defined by the coordinate ring \( \mathbb{R}[X]^+_{\mathbb{R}[X]}' \). Moreover, we have \( \mathbb{C}[X]' = \mathbb{R}[X'] \otimes_{\mathbb{R}} \mathbb{C} \) and \( \mathbb{C}[X]^+ = \mathbb{R}[X^+] \otimes_{\mathbb{R}} \mathbb{C} \) by [8] Corollary 5.7.

For a reduced affine variety \( X \) with a finite number of irreducible components, the total ring of fractions \( \mathcal{K}(X) \) is a product of fields. Moreover, in order to look at the normalization of \( \mathbb{R}[X] \) in \( \mathcal{K}(X) \), one can look at the normalization of each irreducible component in its field of fractions.

The R-seminormalization \( X_{\mathbb{R}^+} \) of \( X \) is the variety \( X_{\mathbb{R}^+} \) introduced in Definition 2.3. The R-seminormalization \( X_{\mathbb{R}^+} \) comes with a finite, birational morphism \( \pi_{\mathbb{R}^+} : X_{\mathbb{R}^+} \to X \) such that \( \pi_{\mathbb{R}^+} \) is bijective and \( (\pi_{\mathbb{R}^+})^{-1}(X(\mathbb{R})) = X_{\mathbb{R}^+}(\mathbb{R}) \). Its universal property is given by Theorem 2.5.

We have the following extensions of rings:

\[ \mathbb{R}[X] \hookrightarrow \mathbb{R}[X^+] \hookrightarrow \mathbb{R}[X_{\mathbb{R}^+}] \hookrightarrow \mathbb{R}[X'] \hookrightarrow \mathcal{K}(X). \]

Throughout this section, we will consider real varieties with real irreducible components. The set of real closed points \( X(\mathbb{R}) \) can be seen as a subset of \( \mathbb{R}^n \) for some \( n \). Hence, we will be able to use the Euclidean topology on \( X(\mathbb{R}) \) induced by \( \mathbb{R}^n \).

In the same spirit as in [4], we want to study the set \( \mathcal{K}^{\mathbb{R}^+}(X(\mathbb{R})) \) of functions defined on \( X(\mathbb{R}) \) that become polynomials on the R-seminormalization. This can be useful, for example, to construct the R-seminormalization of a given real variety. We start by identifying the elements of \( \mathbb{R}[X]' \), which come from an element of \( \mathbb{R}[X^+_{\mathbb{R}[Y]}] \). Then, we show that the elements of \( \mathcal{K}^{\mathbb{R}^+}(X(\mathbb{R})) \) are integral rational functions that are continuous on \( X(\mathbb{R}) \) for the topology of the space where they are well defined in \( X(\mathbb{C}) \). Finally, we provide a last characterization of the elements of \( \mathcal{K}^{\mathbb{R}^+}(X(\mathbb{R})) \) with properties concerning their graph.

Definition 3.1. Let \( X \) be a real variety and \( \pi_{\mathbb{R}^+} \) be its R-seminormalization morphism. We will denote by \( \mathcal{K}^{\mathbb{R}^+}(X(\mathbb{R})) \) the ring of functions \( f : X(\mathbb{R}) \to \mathbb{R} \) such that \( f \circ \pi_{\mathbb{R}^+} \in \mathbb{R}[X^+] \).

Remark. All along the paper, we will make an abuse of notation between the functions \( f : X(\mathbb{R}) \to \mathbb{R} \) of \( \mathcal{K}^{\mathbb{R}^+}(X(\mathbb{R})) \) and the rational function \((f|_{\text{Dom}_\mathbb{R}(f)}, \text{Dom}_\mathbb{R}(f))\) they extend. The couple
(f |\text{Dom}_R(f)). \text{Dom}_R(f)) being the maximal rational representation among all the Z-open Z-dense sets \( U \) of \( X(\mathbb{R}) \) such that there exists \( p, q \in \mathbb{R}[X] \) with \( f|_{U} = p/q|_{U} \).

We start by identifying the elements of \( \mathbb{R}[X'] \), which come from an element of \( \mathbb{R}[X^R+] \). Note that this identification is interesting but does not identify the ring \( \mathcal{K}^R+(X(\mathbb{R})) \) independently from the variety \( X^R+ \).

**Proposition 3.2.** Let \( X \) be a real affine variety, and let \( \pi' : X' \to X \) be the normalization morphism of \( X \). Then

\[
\mathbb{R}[X^R+] \simeq \{p \in \mathbb{R}[X] | \forall z_1, z_2 \in \pi'^{-1}_c(X(\mathbb{R})), \pi'_c(z_1) = \pi'_c(z_2) \Rightarrow p_c(z_1) = p_c(z_2)\}.
\]

**Proof.** We will denote by \( \pi : X' \to X^R+ \) the morphism induced by the extension \( \mathbb{R}[X^R+] \hookrightarrow \mathbb{R}[X'] \). Let \( q \in \mathbb{R}[X^R+] \) and \( x \in X(\mathbb{R}) \). We want to show that for all \( z_1, z_2 \in \pi'^{-1}_c(x) \), we have \( q \circ \pi'_c(z_1) = q \circ \pi'_c(z_2) \). Let \( \mathbb{R}[X']_{m_x} \) be the localization of \( \mathbb{R}[X'] \) by \( \mathbb{R}[X] \setminus m_x \). This ring has a finite number of maximal ideals and they are of the form \( m_i \mathbb{R}[X']_{m_x} \), where the \( m_i \) are the ideals of \( \mathbb{R}[X'] \) lying over \( m_x \). In other words, we have

\[
\mathbb{R}[X]_{m_x} \twoheadrightarrow \mathbb{R}[X']_{m_x},
\]

\[
m_x \mathbb{R}[X]_{m_x} \leftrightarrow m_1 \mathbb{R}[X']_{m_x} \twoheadrightarrow \cdots \twoheadrightarrow m_n \mathbb{R}[X']_{m_x},
\]

where the \( m_i \in \text{Max}\mathbb{R}[X'] \) are such that \( m_i \cap \mathbb{R}[X] = m_x \). By definition of \( \mathbb{R}[X^R+] \), we can write \( (q \circ \pi'_c)_x = \alpha \circ \pi'_c + \beta \) with \( \alpha \in \mathbb{R}[X]_x \) and \( \beta \in \text{Rad}(\mathbb{R}[X']_x) \). Then,

\[
\forall z \in \pi'^{-1}_c(\{x\}) \quad (q \circ \pi'_c)_x(z) = \alpha \circ \pi'_c(z) + \beta_c(z) = \alpha_c \circ \pi'_c(z) + \beta_c(z) = \alpha_c(x) + \beta_c(z).
\]

Moreover, if we write \( \pi'^{-1}_c(\{x\}) = \{x_1, \ldots, x_k, z_{k+1}, \ldots, z_n, z_{n+k} \} \), then we have

\[
\beta_c \in \bigcap_{i=1}^{k} m_i \mathbb{C}[X']_{m_x} = \bigcap_{i=1}^{k} m_{x_i} \mathbb{C}[X']_{m_x} \cap \bigcap_{i=k+1}^{n} (m_{z_i} \cap m_{z_{n+k}}) \mathbb{C}[X']_{m_x}.
\]

So, for all \( z \in \pi'^{-1}_c(\{x\}) \), we get \( \beta_c(z) = 0 \) and finally

\[
\forall z \in \pi'^{-1}_c(\{x\}) \quad q \circ \pi'_c(z) = \alpha(x).
\]

Conversely, let \( p \in \mathbb{R}[X'], x \in X(\mathbb{R}) \) and \( \pi'^{-1}_c(\{x\}) = \{x_1, \ldots, x_k, z_{k+1}, \ldots, z_n, z_{n+k}\} \) with \( x_i \in X'(\mathbb{R}) \) and \( z_i \in X'(\mathbb{C}) \setminus X'(\mathbb{R}) \). Suppose that there exists \( \alpha \in \mathbb{C} \) such that, for all \( z \in \pi'^{-1}_c(\{x\}) \), we have \( p_c(z) = \alpha \). Since \( p(z) = p(\overline{z}) = \overline{p(\overline{z})} = \alpha \), we get that \( \alpha \in \mathbb{R} \) and so \( \alpha_x \in \mathbb{R}[X]_{m_x} \). Then, we can write

\[
p_x = \alpha_x + (p_x - \alpha_x) \in \mathbb{R}[X']_{m_x}.
\]
Moreover, for all $z \in \pi^{-1}(x)$, we have $(p - \alpha)_C \in m_x \mathbb{C}[X']$. So,

$$(p_x - \alpha_x)_C \in \mathbb{R}[X']_{m_x} \cap \bigcap_{z \in \pi^{-1}(x)} m_x \mathbb{C}[X']_{m_x}$$

$$= \mathbb{R}[X']_{m_x} \cap \bigcap_{i=1}^{k} m_{x_i} \mathbb{C}[X']_{m_x} \cap \bigcap_{i=k+1}^{n} (m_{z_i} \cap m_{\bar{z}_i}) \mathbb{C}[X']_{m_x}$$

$$= \bigcap_{i=1}^{k} m_{x_i} \mathbb{R}[X'] \cap \bigcap_{i=k+1}^{n} (m_{z_i} \cap m_{\bar{z}_i}) \mathbb{R}[X']$$

$$= \text{Rad}(\mathbb{R}[X']_{m_x}).$$

So, $p_x - \alpha_x \in \text{Rad}(\mathbb{R}[X']_{m_x})$. □

**Corollary 3.3.** Let $X$ be a real affine variety, and $\pi' : X' \to X$ be the normalization morphism. Then

$$\mathcal{K}^{R+}(X(\mathbb{R})) \simeq \left\{ p \in \mathbb{R}[X'] \mid \forall z_1, z_2 \in \pi'^{-1}(X(\mathbb{R})), \pi'(z_1) = \pi'(z_2) \Rightarrow p_c(z_1) = p_c(z_2) \right\}.$$

**Remark.** Note that if $f \in \mathcal{K}^{R+}(X(\mathbb{R}))$ and $p$ is the element of $\mathbb{R}[X']$ above $f$, then for all $x \in X(\mathbb{R})$ and for all $z \in \pi'^{-1}(x)$, we have $p_c(z) = f(x) \in \mathbb{R}$.

In the paper [7], the authors introduced the central seminormalization (resp. weak-normalization) of real algebraic varieties. The idea behind those constructions is to glue together the central points of the normalization over the central points of the variety (resp. over the maximal central points). The central locus of a real variety is the Euclidean closure of its real regular locus. They have shown that, for a real algebraic variety $X$, the coordinate ring of its central weak-normalization $X_{wc}$ corresponds to the integral closure of $\mathbb{R}[X]$ in the ring $\mathcal{K}^0(\text{Cent}(X))$ of continuous rational functions on the central points of $X(\mathbb{R})$. Also, the coordinate ring of its central seminormalization $X_c$ corresponds to the integral closure of $\mathbb{R}[X]$ in the ring $\mathcal{K}^0(\text{Cent}(X))$ of regulous functions on the central points of $X(\mathbb{R})$. Those are the continuous rational functions that stay rational by restriction to the real closed points of a real subvariety of $X$. The elements of $\mathcal{K}^0(X(\mathbb{R}))$ and $\mathcal{R}^0(X(\mathbb{R}))$ have been extensively studied in real algebraic geometry. One can find more information about regulous and rational continuous functions in the survey of Kucharz and Kurdyka [11].

The following Lemmas 3.4, 3.5, and 3.6 show that the elements of $\mathcal{K}^{R+}(X(\mathbb{R}))$ are regulous integral functions, so that they become polynomial on $\mathbb{R}[X_{wc}]$ and $\mathbb{R}[X_{wc}]$.

**Lemma 3.4.** Let $X$ be a real affine variety and $f \in \mathcal{K}^{R+}(X(\mathbb{R}))$. Then $f$ is rational, and there exists a monic polynomial $P(t) \in C[X][t]$ such that $P(f) = 0$ on $X(\mathbb{R})$.

**Proof.** Let $f : X(\mathbb{R}) \to \mathbb{R}$ be such that $f \circ \pi^{R+}_R \in \mathbb{R}[X^{R+}]$. First, we have $f \in \mathcal{K}(X)$ because $\pi^{R+}$ is birational. Now, since $f \circ \pi^{R+}_R \in \mathbb{R}[X^{R+}] \Rightarrow f \circ \pi' \in \mathbb{R}[X']$, then there exists a monic polynomial
\( P(t) \in \mathbb{R}[X][t] \) such that
\[
P(f \circ \pi_R') = (f \circ \pi_R')^n + (a_{n-1} \circ \pi_R') (f \circ \pi_R')^{n-1} + \cdots + (a_0 \circ \pi_R') = 0.
\]

Since we have an injection \( \mathbb{R}[X^R+] \hookrightarrow \mathbb{R}[X'] \), we get
\[
(f \circ \pi_R^+)^n + (a_{n-1} \circ \pi_R^+) (f \circ \pi_R^+)^{n-1} + \cdots + (a_0 \circ \pi_R^+) = 0.
\]

Now, by Proposition 1.5, we have \( \pi_R^+ \) surjective. So,
\[
f^n + a_{n-1}f^{n-1} + \cdots + a_0 = 0.
\]

Thus, \( f \) is integral over \( \mathbb{R}[X] \).

\[\square\]

**Lemma 3.5.** Let \( X \) be a real affine variety and \( f \in \mathcal{K}^R+(X(\mathbb{R})) \). Then \( f \) is continuous for the Euclidean topology on \( X(\mathbb{R}) \).

**Proof.** Let \( f \in \mathcal{K}^R+(X(\mathbb{R})) \) and \( F \) be an Euclidean closed set of \( \mathbb{R} \). Since \( f \circ \pi_R^+ \in \mathbb{R}[X^R+] \), we have that \( (\pi_R^+)^{-1}(f^{-1}(F)) = (f \circ \pi_R^+)^{-1}(F) \) is closed. By [7] Lemma 3.1, the function \( \pi_R^+ \) is closed for the Euclidean topology. Then,
\[
\pi_R^+((\pi_R^+)^{-1}(f^{-1}(F))) = f^{-1}(F) \text{ is closed},
\]
and so, \( f \) is continuous.

\[\square\]

**Lemma 3.6.** Let \( X \) be a real affine variety. Let \( V \subset X \) be a real subvariety of \( X \) and \( f \in \mathcal{K}^R+(X(\mathbb{R})) \). Then
\[
f|_V(\mathbb{R}) \text{ is rational}.
\]

**Proof.** Let \( V \) be a subvariety of \( X \). Then there exists \( p \in \text{R-Spec}(\mathbb{R}[X]) \) such that \( \mathbb{R}[V] \simeq \mathbb{R}[X]/p \). So, there is a unique \( q \in \text{Spec}(\mathbb{R}[X^R+]) \) such that \( q \cap \mathbb{R}[X] = p \). We note \( W \) the subvariety of \( X^R+ \) such that \( \mathbb{R}[W] \simeq \mathbb{R}[X^R+]/q \). Then, we get the following commutative diagram:

\[
\begin{array}{ccc}
\mathbb{R}[X] & \xrightarrow{(\pi_R^+)^*} & \mathbb{R}[X^R+] \\
\vert & \downarrow & \vert \\
\mathbb{R}[V] & \xrightarrow{(\pi_{IV}^+)^*} & \mathbb{R}[W] \\
\vert & \downarrow & \vert \\
\mathcal{K}(V) & \xrightarrow{=} & \mathcal{K}(W)
\end{array}
\]

We have \( f \circ \pi_R^+ \in \mathbb{R}[X^R+] \), so \( f|_V(\mathbb{R}) \circ \pi_{IW}(\mathbb{R}) = (f \circ \pi_R^+)|_W(\mathbb{R}) \in \mathbb{R}[W] \). Then, \( \frac{f|_V(\mathbb{R}) \circ \pi_{IW}(\mathbb{R})}{1} \in \mathcal{K}(W) \) and so \( f|_V(\mathbb{R}) \in \mathcal{K}(V) \).

\[\square\]
Remark. We have shown that the elements of $K^R_+(X(\mathbb{R}))$ are regulous functions and that they are integral over $\mathbb{R}[X]$. So,

$$K^R_+(X(\mathbb{R})) \subset \mathbb{R}[X]' \subset \mathbb{R}[X]^0_0(X(\mathbb{R})).$$

In the same spirit as for $X^\times$ and $X^u$, one may wonder if the ring $K^R_+(X(\mathbb{R}))$ can also be seen as the integral closure of $\mathbb{R}[X]$ in a ring of functions over $X(\mathbb{R})$. Moreover, we might expect this ring to be the set of functions on $X(\mathbb{R})$ that are continuous for the Euclidean topology of $X(\mathbb{C})$. Indeed, the idea behind the R-seminormalization is to glue together the points of the normalization lying over the same real point of $X$, and its coordinate ring corresponds to the ring of rational functions that are continuous on $X(\mathbb{C})$ for the Euclidean topology of $X(\mathbb{C})$. Since the idea behind the R-seminormalization is to glue together the points of the normalization lying over the same real point of $X$, the continuity on $X(\mathbb{R})$ for the topology of $X(\mathbb{C})$ seems to be the right notion to consider. There is, in fact, a little subtlety. The elements of $K^R_+(X(\mathbb{R}))$ are only defined on $X(\mathbb{R})$, and they cannot necessarily be extended to all $X(\mathbb{C})$. This is why the correct type of functions to use are rational functions that satisfy condition 3 of the next proposition.

**Remark.** Let us recall that there is an abuse of notation between the elements $f: X(\mathbb{R}) \to \mathbb{R}$ of $K^R_+(X(\mathbb{R}))$ and the rational function $(f|_{\text{Dom}_{\mathbb{R}}(f)}, \text{Dom}_{\mathbb{R}}(f))$ they extend. An important point is that a rational representation $(p/q, U(\mathbb{R}))$ of an element $f$ of $K^R_+(X(\mathbb{R}))$ can be extended to $U(\mathbb{C})$, so $f$ can be extended to $\text{Dom}_c(f) \cup X(\mathbb{R})$, but it does not necessarily extend to $X(\mathbb{C})$.

**Proposition 3.7.** Let $X$ be a real affine variety, and let $f: X(\mathbb{R}) \to \mathbb{R}$. Then $f \in K^R_+(X(\mathbb{R}))$ if and only if it satisfies the following properties.

1. The function $f$ is rational.
2. The function $f$ is integral over $\mathbb{R}[X]$.
3. For all $x \in X(\mathbb{R})$, for all $(z_n)_n \subset \text{Dom}_c(f) \cup X(\mathbb{R})$ such that $z_n \to x$, then $f(z_n) \to f(x)$.

**Notation.** Let $q \in \mathbb{R}[X]$ or $\mathbb{C}[X]$. We denote by $D(q)$ the $\mathbb{Z}$-open set $\mathcal{V}(q)^c$ of $X$ defined by $q$. For $q_c: X(\mathbb{C}) \to \mathbb{C}$, we have $D(q_c) = \{x \in X(\mathbb{C}) | q_c(x) \neq 0\}$.

**Proof.** Let $f: X(\mathbb{R}) \to \mathbb{R}$ be integral over $\mathbb{R}[X]$, and suppose that there exist $p, q \in \mathbb{R}[X]$ such that $f = p/q$ on $D(q)$. Then there exists $g \in \mathbb{R}[X']$ such that $f \circ \pi' = g$ on $(\pi')^{-1}(D(q)) = D(q \circ \pi')$. Note that $f$ can be extended to $D(q_c) \cup X(\mathbb{R})$. Let $x \in X(\mathbb{R})$. By Proposition 3.2 and Lemma 3.4, we just have to show that the following propositions are equivalent.

1. For all $z'_1, z'_2 \in (\pi')^{-1}(x)$, we have $g_c(z'_1) = g_c(z'_2) = f(x)$.
2. For all $(z_n)_n \subset D(q_c) \cup X(\mathbb{R})$ such that $z_n \to x$, then $f(z_n) \to f(x)$.

We prove 1 $\Rightarrow$ 2. Let $(z_n)_n \subset D(q_c) \cup X(\mathbb{R})$ be such that $z_n \to x$ as in Figure 1, and suppose that $(f(z_n))_n$ is convergent. We can consider an open ball $B(x, \varepsilon) \subset X(\mathbb{C})$ and an integer $N \in \mathbb{N}$ such that for all $n > N$, we have $z_n \in B(x, \varepsilon)$. In particular, it is also true for $B := \overline{B(x, \varepsilon)}$, which is compact. By [4, Lemma 4.8], the map $\pi'$ is a proper map, and so, $(\pi')^{-1}(B)$ is compact. Now, if for all $n > N$, we consider $z'_n \in (\pi'_c)^{-1}(z_n)$, then we obtain a sequence $(z'_n)_{n>N}$ whose elements are contained in the compact set $(\pi'_c)^{-1}(B)$. So, there exists a convergent subsequence $(z'_{n_k})_{n_k>N}$ and its limit, that we note $l$, belongs to $X'(\mathbb{C})$ because this set is closed for the Euclidean
Figure 1. Illustration of condition 3 in Proposition 3.7.

topology. Moreover, since π′_C is continuous, we have z_nk = π′_C(z_nk) → π′_C(l). So, π′_C(l) = x. Then, by continuity of g, we have f(z_nk) = g(z'_nk) → g(l) = f(x). This means that the limit of (f(z_nk))_n is f(x).

Now suppose that (f(z_nk))_n is not necessarily convergent. Since, for all n > N, f o π′_C(z'_nk) = g(z'_nk), then \{f(z_nk)\}_n>N ⊂ g (π′_C)^{-1}(B), which is a compact set because g is continuous. Then, by applying the arguments of the preceding paragraph, we can show that every convergent subsequence of (f(z_nk))_n>N admits the same limit, which is f(x). So, the sequence (f(z_nk))_n>N is convergent, and we get f(z_nk) → f(x).

Now we prove 2 ⇒ 1. Let z ∈ (π′_C)^{-1}(x). Since D(q o π′_C) is dense in X'(C), we can consider a sequence (z_n) ∈ D(q o π′_C)^N such that z_n → z. By continuity of π′_C, we have π′_C(z_n) → π′_C(z) = x, so f(π′_C(z_n)) → f(π′_C(z)), by assumption on f. The function g being continuous, we have g(z_n) → g(z), and since for all n ∈ N, g(z_n) = f o π′_C(z_n), we get that g(z) = f(π′_C(z)) = f(x).

□

In the case of real curves, the poles of f are small enough for f to be well defined in a neighborhood of X(ℝ) in X(C). Thus, we get the following corollary.

**Corollary 3.8.** Let X be a real curve. Then a function f : X(ℝ) → ℝ belongs to K^R+(X(ℝ)) if and only if it is rational, integral, and if it admits a rational extension F to an open neighborhood Ω ⊂ X(C) of X(ℝ) that is continuous at the points of X(ℝ).

*Proof.* Let f ∈ K^R+(X(ℝ)). Since X is a curve, then Dom_C(f) = X(C) \ {finite number of points}. So, for all x ∈ X(ℝ), there exists ε > 0 such that B_{X(C)}(x, ε) ∩ Dom_C(f)^C = x. By the previous proposition, for all (z_n)_n ∈ B_{X(C)}(x, ε)^N such that z_n → x, then f(z_n) → f(x). So, f is continuous at x in X(C).

□

We use this characterization in order to identify K^R+(X(ℝ)) as the integral closure of ℝ[X] in a ring of functions.
Definition 3.9. Let \( f : X(\mathbb{R}) \to \mathbb{R} \) be a function. We say that \( f \) is \( R \)-continuous if there exists a Zariski dense open set \( U \) such that \( f \) extends to \( U(\mathbb{C}) \cup X(\mathbb{R}) \) to a continuous function for the topology of \( U(\mathbb{C}) \cup X(\mathbb{R}) \). We denote by \( C^R_+(X(\mathbb{R}), \mathbb{R}) \) the set of \( R \)-continuous functions on \( X(\mathbb{R}) \).

Remark. If \( f \) is a rational function, then condition 3 in Proposition 3.7 is equivalent to \( f \in C^R_+(X(\mathbb{R}), \mathbb{R}) \). The interest of this definition is that, unlike condition 3, it can apply to nonnecessarily rational functions.

Theorem 3.10. Let \( X \) be a complex algebraic variety. Then
\[
K^R_+(X(\mathbb{R})) = \mathbb{R}[X]'_{C^R_+(X(\mathbb{R}), \mathbb{R})}.
\]

Proof. The inclusion
\[
K^R_+(X(\mathbb{R})) \subseteq \mathbb{R}[X]'_{C^R_+(X(\mathbb{R}), \mathbb{R})}
\]
comes from Proposition 3.7. Let \( f \in \mathbb{R}[X]'_{C^R_+(X(\mathbb{R}), \mathbb{R})} \). Then \( f \) extends continuously, for the complex topology, to a Zariski dense open set \( U(\mathbb{C}) \). We can suppose that \( U \) is affine, and then, we apply [4, Corollary 3.18]. Hence, \( f \in K^0(U(\mathbb{C})) \). In particular, \( f \) is rational, and Proposition 3.7 implies \( f \in K^R_+(X(\mathbb{R})) \).

We now want to give a characterization of the elements of \( K^R_+(X(\mathbb{R})) \) of the same kind as [4, Theorem 4.20], concerning the elements of \( K^0(X(\mathbb{C})) \). More precisely, we want to find conditions on the graph of an integral rational function that imply that the function belongs to \( K^R_+(X(\mathbb{R})) \). This will allow us, for instance, to provide nontrivial examples of such functions. We start by proving two necessary conditions on the graph for a function to be in \( K^R_+(X(\mathbb{R})) \).

Lemma 3.11. Let \( X \) be a real affine variety and \( f \in K^R_+(X(\mathbb{R})) \). Then the graph \( \Gamma_f \) is Zariski closed in \( X(\mathbb{R}) \times A^1(\mathbb{R}) \).

Proof. Let \( f \in K^R_+(X(\mathbb{R})) \) and \( \pi^R_+ : X^R_+ \to X \) be the \( R \)-seminormalization morphism. By Theorem 2.1, we have that \( \pi^R_+ \) is bijective. So,
\[
\Gamma_f = \{(x; f(x)) \mid x \in X(\mathbb{R})\} = \left\{ \left( \pi^R_+(x); f \circ \pi^R_+(x) \right) \mid x \in X^R_+(\mathbb{R}) \right\}.
\]
But since \( f \circ \pi^R_+ \) is a polynomial and, by Proposition 1.3, the morphism \( \pi^R_+ \times \text{Id} \) is a \( Z \)-homeomorphism, we get that
\[
\left\{ \left( \pi^R_+(x); f \circ \pi^R_+(x) \right) \mid x \in X^R_+(\mathbb{R}) \right\} = \pi^R_+ \times \text{Id}\left( \Gamma_{f \circ \pi^R_+} \right)
\]
is \( Z \)-closed, and so that \( \Gamma_f \) is a \( Z \)-closed subset of \( X(\mathbb{R}) \times A^1(\mathbb{R}) \).

Notation. We will denote by \( \overline{\Gamma_f}^C \) the Zariski closure of \( \Gamma_f \) in the set \( X(\mathbb{C}) \times A^1(\mathbb{C}) \).

Lemma 3.12. Let \( X \) be a real affine variety and \( f \in K^R_+(X(\mathbb{R})) \). Then, for all \( x \in X(\mathbb{R}) \), we have
\[
\overline{\Gamma_f}^C \cap \left\{ \{x\} \times A^1(\mathbb{C}) \right\} = \left\{ (x; f(x)) \right\}.
\]
Proof. Let \( f \in \mathcal{K}^{R+}(X(\mathbb{R})) \). We can consider \( p, q \in \mathbb{R}[X] \) such that \( f|_{D(q)} = p/q \). Let \((x, t) \in \Gamma_f^C \cap (\{x\} \times \mathbb{A}^1(\mathbb{C}))\) with \( x \in X(\mathbb{R}) \) and consider the Z-dense Z-open set \( D(q) \subset \Gamma_f^C \). Since it is also dense for the Euclidean topology, we can consider a sequence \((z_n, t_n) \in D(q) \cap \mathbb{N}\) such that \((z_n, t_n) \to (x, t)\). Moreover, we have

\[
\Gamma_f^C \subset \{q C t - p_C\} \subset X(\mathbb{C}) \times \mathbb{A}^1(\mathbb{C}),
\]

and so, we get, for all \( n \in \mathbb{N} \), that \((z_n, t_n) = (z_n, p(z_n)/q(z_n))\). Then, by Proposition 3.7, we have

\[
(z_n, t_n) = (z_n, f(z_n)) \to (x, f(x)).
\]

This means that \( \Gamma_f^C \cap (\{x\} \times \mathbb{A}^1(\mathbb{C})) = \{ (x; f(x)) \} \), which concludes the proof. \( \square \)

We can now show that if an integral rational function on \( X(\mathbb{R}) \) satisfies Lemmas 3.11 and 3.12, then it is an element of \( \mathcal{K}^{R+}(X(\mathbb{R})) \).

**Theorem 3.13.** Let \( X \) be a real affine variety, and let \( f : X(\mathbb{R}) \to \mathbb{R} \). Then \( f \in \mathcal{K}^{R+}(X(\mathbb{R})) \) if and only if it satisfies the following properties.

1. \( f \in \mathcal{K}(X) \).
2. There exists a monic polynomial \( P(t) \in \mathbb{R}[X][t] \) such that \( P(f) = 0 \) on \( X(\mathbb{R}) \).
3. The graph \( \Gamma_f \) is Zariski closed in \( X(\mathbb{R}) \times \mathbb{A}^1(\mathbb{R}) \).
4. For all \( x \in X(\mathbb{R}) \), we have \( \Gamma_f^C \cap (\{x\} \times \mathbb{A}^1(\mathbb{C})) = \{ (x; f(x)) \} \).

**Proof.** Let \( f \in \mathcal{K}^{R+}(X(\mathbb{R})) \). Then, by Lemmas 3.4, 3.11, and 3.12, it satisfies the four properties of the proposition. Conversely, suppose that \( f : X(\mathbb{R}) \to \mathbb{R} \) satisfies the four properties above. We consider the map

\[
\psi : \mathbb{R}[X][t] \to \mathcal{K}(X)
\]

\[
Q(t) \mapsto Q(f),
\]

and write \( \mathbb{R}[Y] \simeq \mathbb{R}[X][t]/\ker \psi \simeq \mathbb{R}[X][f] \) with \( \pi : Y \to X \) the morphism induced by \( \mathbb{R}[X] \hookrightarrow \mathbb{R}[Y] \). We then have

\[
\mathbb{R}[X] \hookrightarrow \mathbb{R}[Y] \simeq \mathbb{R}[X][f] \subset \mathcal{K}(X).
\]

So \( \mathcal{K}(X) \simeq \mathcal{K}(Y) \), and \( \pi \) is birational. Moreover, \( \mathbb{R}[Y] \) is a finite \( \mathbb{R}[X] \)-module because so is \( \mathbb{R}[X][t]/<P(t)> \) and

\[
\mathbb{R}[Y] \simeq \mathbb{R}[X][t]/\ker \psi \simeq (\mathbb{R}[X][t]/<P(t)>)/(\ker \psi/ <P(t)>).
\]

Hence, \( \pi : Y \to X \) is a finite birational morphism. We want to show that the restriction \( \pi_c : \pi^{-1}(X(\mathbb{R})) \to X(\mathbb{R}) \) of the morphism \( \pi_c \) is bijective. By assumption, we can consider the real ideal \( I_f := I(\Gamma_f) \), and we have \( \Gamma_f = \mathcal{Z}(I_f) \). One can see that \( I_f \subset \ker \psi \) because

\[
\forall Q \in I_f \quad \forall x \in X(\mathbb{R}) \quad Q(x, f(x)) = 0.
\]
So,

\[ Y(\mathbb{C}) = \mathcal{Z}_c(\ker \psi) \subseteq \mathcal{Z}_c(I_f) = \overline{\Gamma_f^C}. \]

Let \( x \in X(\mathbb{R}) \). By the fourth condition, we have

\[ \pi_c^{-1}(x) = Y(\mathbb{C}) \cap (\{x\} \times \mathbb{A}^1(\mathbb{C})) \subseteq \overline{\Gamma_f^C} \cap (\{x\} \times \mathbb{A}^1(\mathbb{C})) = \{(x; f(x))\} \]

and since \( \pi_c : Y(\mathbb{C}) \to X(\mathbb{C}) \) is finite, then it is surjective, and we get \( \pi_c^{-1}(x) = \{(x; f(x))\} \). We have shown that \( \pi \) is a finite birational morphism and \( \bar{\pi} \) is bijective. From the universal property of the R-seminormalization, we get

\[ \mathbb{R}[X] \leftrightarrow \mathbb{R}[Y] \leftrightarrow \mathbb{R}[X^R]. \]

So, \( f \circ \pi^{R+}_R \in \mathbb{R}[X^R]. \)

**Example 3.13.1.** Let \( X = \text{Spec}(\mathbb{R}[x, y]/\langle y^3 - x^2y^2 + yx^2(x + 1) - x^4(x + 1) \rangle) \) and consider

\[ f = \begin{cases} y/x & \text{if } x \neq 0 \\ 0 & \text{otherwise} \end{cases}. \]

We have that \( f \) is a root of the polynomial \( P_f(t) = t^3 - xt^2 + t(x + 1) - x(x + 1) \). Since 0 is the only real root of \( P_f \) when we substitute \( x = 0 \), we get that \( \Gamma_f \) is a Z-closed subset of \( X(\mathbb{R}) \times \mathbb{A}^1(\mathbb{R}) \) given by

\[ \Gamma_f = \begin{cases} y^3 - x^2y^2 + yx^2(x + 1) - x^4(x + 1) = 0, \\ xt - y = 0, \\ t^3 - xt^2 + t(x + 1) - x(x + 1) = 0. \end{cases} \]

But it does not satisfy the fourth condition because \( \overline{\Gamma_f^C} \cap (\{0\} \times \mathbb{A}^1(\mathbb{C})) = \{(0; 0), (0; \pm i)\} \).

4 | **RATIONAL FUNCTIONS EXTENDING CONTINUOUSLY TO THE COMPLEX POINTS**

In the case of complex varieties, it has been shown in [4] that the ring of polynomial functions to the seminormalization corresponds to the ring \( \mathcal{K}^0(X(\mathbb{C})) \) of rational functions that extend continuously to \( X(\mathbb{C}) \). The purpose of this section is to show that, in the case of real varieties, the real-valued functions that are the restriction to \( X(\mathbb{R}) \) of an element of \( \mathcal{K}^0(X(\mathbb{C})) \) correspond to the polynomial functions on the seminormalization. Those functions can also be seen as the elements \( f \in \mathcal{K}^0(X(\mathbb{C})) \) such that \( \overline{f(z)} = f(z) \), for all \( z \in X(\mathbb{C}) \). We end this section by giving a characterization of those functions with their graphs, as in Theorem 4.20 of [4] or like Theorem 3.13. This will allow us to construct examples of different functions that become polynomials on \( X^+, X^{R+}, X^\times, \) or \( X^w \).
Definition 4.1. Let $X$ be a real affine variety. We define the set

$$\mathcal{K}^+(X(\mathbb{R})) := \{ f : X(\mathbb{R}) \to \mathbb{R} \mid \exists g \in \mathcal{K}^0(\mathbb{C}) \text{ such that } g|_{X(\mathbb{R})} = f \}.$$

**Remark.** By Proposition 3.7, it is clear that $\mathcal{K}^+(X(\mathbb{R})) \subset \mathcal{K}^R(\mathbb{R})$. Note also that those rings inject in $\mathcal{K}(X)$, and so, even if their elements are defined on all $X(\mathbb{R})$, they are uniquely represented by the equivalent class of a rational representation.

Every real rational function can be extended to a complex rational function. The next lemma says that, for an element of $\mathcal{K}^+(X(\mathbb{R}))$, this extension is a rational representation of an element of $\mathcal{K}^0(\mathbb{C})$.

**Lemma 4.2.** Let $f : X(\mathbb{R}) \to \mathbb{R}$ and $p, q \in \mathbb{R}[X]$ be such that $f = p/q$ at the points at which $q \neq 0$. Then the following statements are equivalent:

1. The function $f$ belongs to $\mathcal{K}^+(X(\mathbb{R}))$.
2. The rational function $p_{\mathbb{C}}/q_{\mathbb{C}}$ extends continuously to an element $f_{\mathbb{C}} \in \mathcal{K}^0(\mathbb{C})$.

**Proof.** Suppose that $f \in \mathcal{K}^+(X(\mathbb{R}))$, then there exists $g = p'/q' \in \mathcal{K}^0(\mathbb{C})$ such that $g|_{X(\mathbb{R})} = f$. Then we have $p'q_{\mathbb{C}} = q'p_{\mathbb{C}}$ on $X(\mathbb{R})$. Thus, $p'q_{\mathbb{C}} = q'p_{\mathbb{C}}$ on $X(\mathbb{C})$ because $X(\mathbb{R})$ is Z-dense in $X(\mathbb{C})$. So, we get $g = f_{\mathbb{C}}$ on $D(q_{\mathbb{C}}) = D(q')$ and finally $g = f_{\mathbb{C}}$ on $X(\mathbb{C})$ by continuity. Conversely, it is clear that if $f_{\mathbb{C}} \in \mathcal{K}^0(\mathbb{C})$, then $f = (f_{\mathbb{C}})|_{X(\mathbb{R})} \in \mathcal{K}^+(X(\mathbb{R}))$. $\square$

By definition, we know that for each element $f \in \mathcal{K}^+(X(\mathbb{R}))$, there is an extension $f_{\mathbb{C}}$ that belongs to $\mathcal{K}^0(\mathbb{C})$. One can ask the reverse question: which elements of $\mathcal{K}^0(\mathbb{C})$ are the extension of an elements of $\mathcal{K}^+(X(\mathbb{R}))$? The answer is given by the next proposition.

Notation. Let $E \subset \mathcal{P}(X(\mathbb{C}); \mathbb{C})$ and $f \in E$. We note $\sigma f$ the function defined by $\sigma f(z) = \overline{f(\overline{z})}$, for all $z \in X(\mathbb{C})$. We will say that $f$ is $\sigma$-invariant if $\sigma f = f$ and we note $\sigma E$ the elements of $E$ that are $\sigma$-invariant.

**Proposition 4.3.** Let $X$ be a real algebraic variety. Then we have the following isomorphism:

$$\psi : \mathcal{K}^0(\mathbb{C}) \sim \mathcal{K}^+(X(\mathbb{R}))$$

$$g \mapsto g|_{X(\mathbb{R})}.$$

**Proof.** First of all, the morphism $\psi$ is well defined because if $g \in \mathcal{K}^0(\mathbb{C})$, then for all $x \in X(\mathbb{R})$, we have $g(x) = \overline{\sigma g(x)} = \overline{g(x)} = g(\overline{x})$ and so $g(x) \in \mathbb{R}$. The morphism $\psi$ is surjective because, by Lemma 4.2, if $f \in \mathcal{K}^+(X(\mathbb{R}))$, then $f_{\mathbb{C}} \in \mathcal{K}^0(\mathbb{C})$ and it is clear that $f_{\mathbb{C}}$ is $\sigma$-invariant. Let us see now that it is injective. Let $g_1, g_2 \in \mathcal{K}^0(\mathbb{C})$ be such that $g_1|_{X(\mathbb{R})} = g_2|_{X(\mathbb{R})}$. We can write $g_1 = p_1/q_1$ on $D(q_1)$ and $g_2 = p_2/q_2$ on $D(q_1)$ for $p_1, q_1 \in \mathbb{C}[X]$. Then $q_1q_2g_1 = q_1q_2g_2$ on $X(\mathbb{R})$, so $p_1q_2 = p_2q_1$ on $X(\mathbb{R})$. Since $X(\mathbb{R})$ is Zariski dense in $X(\mathbb{C})$, the relation extends to $X(\mathbb{C})$, and so, $p_1/q_1 = p_2/q_2$ on $D(q_1q_2)$. Hence, $g_1 = g_2$ by continuity. $\square$

We obtain a real version of Theorem 4.13 of [4] saying that the polynomial functions on the seminormalization correspond to the rational functions on $X$ that extend continuously to $X(\mathbb{C})$. 


Corollary 4.4. Let $X$ be a real algebraic variety. Then, we have the following isomorphism:

$$\varphi : \mathcal{K}^+(X(\mathbb{R})) \sim \mathbb{R}[X^+]$$

$$f \mapsto f \circ \pi^+_R.$$

Proof. By [4, Theorem 4.13], we have the following isomorphism:

$$\mathcal{K}^0(X(\mathbb{C})) \sim \mathbb{C}[X^+]$$

$$f \mapsto f \circ \pi^+_C.$$

Let us show that it induces an isomorphism on the $\sigma$-invariant elements

$$\sigma \mathcal{K}^0(X(\mathbb{C})) \sim \sigma \mathbb{C}[X^+]$$

$$f \mapsto f \circ \pi^+_C.$$

Let $f \in \sigma \mathcal{K}^0(X(\mathbb{C}))$ and $z \in X(\mathbb{C})$. Since $\sigma \pi^+_C = \pi^+_C$, we get $\pi^+_C(z) = \sigma \pi^+_C(z) = \pi^+_C(z)$, and so,

$$f \circ \pi^+_C(z) = \sigma(f \circ \pi^+_C(z)) = \sigma(f(\pi^+_C(z))) = f(\pi^+_C(z)).$$

Since $\sigma \mathbb{C}[X^+] \simeq \mathbb{R}[X^+]$, we deduce from the previous proposition that

$$\mathcal{K}^+(X(\mathbb{R})) \sim \sigma \mathcal{K}^0(X(\mathbb{C})) \sim \sigma \mathbb{C}[X^+] \sim \mathbb{R}[X^+]$$

$$f \mapsto f_C \mapsto f_C \circ \pi^+_C \mapsto (f_C \circ \pi^+_C)_{|X(\mathbb{R})}.$$

One can see that $(f_C \circ \pi^+_C)_{|X(\mathbb{R})} = f \circ \pi^+_R$, so it concludes the proof. \hfill \Box

We now give a characterization of the elements of $\mathcal{K}^+(X(\mathbb{R}))$ with their graphs. It can be seen as the real version of Theorem 4.20 of [4].

Lemma 4.5. Let $X$ be a real affine variety and $f \in \mathcal{K}^+(X(\mathbb{R}))$. Let $P \in \mathbb{R}[X][t]$ be such that $P(f) = 0$ on $X(\mathbb{R})$. Then $P(f_C) = 0$ on $X(\mathbb{C})$.

Proof. Let $p$ and $q$ be such that $f = p/q$ on $D(q)$, and let $P \in \mathbb{C}[X][t]$ be such that $P(f) = 0$ on $X(\mathbb{R})$. In particular, $P(p/q) = 0$ on $D(q)$. Then, $q_{C}^{deg(P)}P(p_C/q_C) = 0$ on $X(\mathbb{R})$. Since $X(\mathbb{R})$ is $Z$-dense in $X(\mathbb{C})$, we have $q_{C}^{deg(P)}P(p_C/q_C) = 0$ on $X(\mathbb{C})$. Thus, $P(f) = P(p_C/q_C) = 0$ on $D(q_C)$, and finally, by continuity, we get $P(f) = 0$ on $X(\mathbb{C})$. \hfill \Box

Theorem 4.6. Let $X$ be a real affine variety and $f : X(\mathbb{R}) \to \mathbb{R}$. Then, $f \in \mathcal{K}^+(X(\mathbb{R}))$ if and only if it satisfies the following properties.

1. $f \in \mathcal{K}(X)$.
2. There exists a monic polynomial $P \in \mathbb{R}[X][t]$ such that $P(f) = 0$ on $X(\mathbb{R})$.
3. The graph $\Gamma_f$ is Zariski closed in $X(\mathbb{R}) \times \mathbb{A}^1(\mathbb{R})$.
4. For all $z \in X(\mathbb{C})$, we have $\# \left( \Gamma_f^C \cap \{z\} \times \mathbb{A}^1(\mathbb{C}) \right) = 1$. 

Remark. If \( f \in \mathcal{K}^+(X(\mathbb{R})) \), then the last property says that \( \overline{\Gamma_f \cap C} = \Gamma_{f_c} \).

Proof. Let \( f \in \mathcal{K}^+(X(\mathbb{R})) \). Then \( f \in \mathcal{K}^{R+}(X(\mathbb{R})) \), and by Theorem 3.13, the function \( f \) satisfies conditions (1)–(3). By Lemma 4.2, we have \( f_c \in \mathcal{K}^0(X(\mathbb{C})) \), and by [4, Theorem 4.20], we get that \( \Gamma_{f_c} \) is Z-closed. Since \( \Gamma_f \subset \Gamma_{f_c} \), we have \( \overline{\Gamma_f \cap C} \subset \overline{\Gamma_{f_c} \cap C} \). Moreover, the graph of \( f \) being a Zariski closed set, its elements are the common roots of a finite number of polynomials of \( \mathbb{R}[X][t] \). Thus, by Lemma 4.5, the elements \( (x, f_c(x)) \in \Gamma_{f_c} \subset X(\mathbb{C}) \times \mathbb{A}(\mathbb{C}) \) are also roots of those polynomials. So, if \( I_f \) is a real ideal such that \( \mathcal{Z}(I_f) = \Gamma_f \), then \( \Gamma_{f_c} \subset \mathcal{Z}(I_f) = \overline{\Gamma_f \cap C} \). We obtain \( \Gamma_{f_c} = \overline{\Gamma_f \cap C} \), and so, \( f \) satisfies condition (4).

Conversely, let \( f : X(\mathbb{R}) \to \mathbb{R} \) be a function that satisfies all four conditions. Thanks to condition (4), we can define a function \( g : X(\mathbb{C}) \to \mathbb{C} \) such that

\[
\forall z \in X(\mathbb{C}) \quad \{ (z; g(z)) \} = \overline{\Gamma_f \cap C} \cap \{ z \} \times \mathbb{A}^1(\mathbb{C}).
\]

Then, \( \varGamma_g = \overline{\Gamma_f \cap C} \), so \( g \) is rational, integral over \( \mathbb{C}[X] \), and \( \varGamma_g \) is Z-closed. By [4, Theorem 4.20], this means that \( g \in \mathcal{K}^0(X(\mathbb{C})) \). Moreover, we have \( \Gamma_f \subset \Gamma_g \), so \( g|_{X(\mathbb{R})} = f \), and we get \( f \in \mathcal{K}^+(X(\mathbb{R})) \).

Thanks to Theorems 3.13 and 4.6, we give several examples of functions belonging to the rings \( \mathcal{K}^+(X(\mathbb{R})) \) and \( \mathcal{K}^{R+}(X(\mathbb{R})) \). In particular, we show that the following inclusions are strict in general:

\[
\mathbb{R}[X] \subset \mathcal{K}^+(X(\mathbb{R})) \subset \mathcal{K}^{R+}(X(\mathbb{R})) \subset \mathbb{R}[X]_{\mathcal{R}^0(Cent(\mathbb{X}))} \subset \mathbb{R}[X]_{\mathcal{K}^0(Cent(\mathbb{X}))}.
\]

**Example 4.6.2.** We give an example of a function in \( \mathcal{K}^+(X(\mathbb{R})) \setminus \mathbb{R}[X] \). Let \( X = \text{Spec}(\mathbb{R}[x, y]/ (y^2 - x^3)) \) and consider the function

\[
f = \begin{cases} \frac{y}{x} & \text{if } x \neq 0 \\ 0 & \text{otherwise.} \end{cases}
\]

We have that \( f \) is a root of the polynomial \( P(t) = t^2 - x \). Since 0 is the only complex root of \( P \) when we substitute \( x = 0 \), we get that \( \varGamma_f \) is Z-closed in \( X(\mathbb{R}) \times \mathbb{A}^1(\mathbb{R}) \) and that \( \varGamma_{f_c} \) is Z-closed in \( X(\mathbb{C}) \times \mathbb{A}^1(\mathbb{C}) \). So, by Theorem 4.6, we have \( f \in \mathcal{K}^+(X(\mathbb{R})) \).

**Example 4.6.3.** We give an example of a function in \( \mathcal{K}^{R+}(X(\mathbb{R})) \setminus \mathcal{K}^+(X(\mathbb{R})) \). Consider \( X = \text{Spec}(\mathbb{R}[x, y]/ (y^2 - x^3(x^2 + 1)^2)) \) and the function

\[
f = \begin{cases} \frac{y}{x(x^2 + 1)} & \text{if } x \neq 0 \\ 0 & \text{otherwise.} \end{cases}
\]

We have that \( f \) is a root of the polynomial \( P(t) = t^2 - x \). Since 0 is the only complex root of \( P \) when we substitute \( x = 0 \), we get that \( \varGamma_f \) is Z-closed in \( X(\mathbb{R}) \times \mathbb{A}^1(\mathbb{R}) \) and that, for all \( x \in X(\mathbb{R}) \), we have \( \varGamma_f \cap \{ \{x\} \times \mathbb{A}^1(\mathbb{C}) \} = \{ (x; f(x)) \} \). So, by Theorem 3.13, we get \( f \in \mathcal{K}^{R+}(X(\mathbb{R})) \). However, if we substitute \( x = \pm i \), then \( P(t) \) has two distinct complex roots. So, \( \varGamma_f \) is not the graph
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**Figure 2** The R-seminormalization of \( \{y^2 - x^3(x^2 + 1)^2 = 0\} \).

of a function defined on \( X(\mathbb{C}) \) and this implies \( f \notin \mathcal{K}^+(X(\mathbb{R})) \). Hence, if we add \( f \) to the coordinate ring of \( X \), we obtain the R-seminormalization of \( X \) (see Figure 2) that is different from its seminormalization.

**Example 4.6.4.** We give an example of a function in \( \mathbb{R}[X]^\prime \mathcal{K}^0(\text{Cent}(X)) \setminus \mathcal{K}^R(X(\mathbb{R})) \)). Consider the real variety \( X = \text{Spec}(\mathbb{R}[x, y]/ <y^4 - x(x^2 + y^2)>) \) and the function

\[
\begin{align*}
  f &= \begin{cases} 
    y^2/x & \text{if } x \neq 0 \\
    0 & \text{otherwise,}
  \end{cases}
\end{align*}
\]

which is a root of the polynomial \( P(x, y)(t) = t^2 - t - x \). Moreover, \( f \) is continuous, so \( f \in \mathbb{R}[X]^\prime \mathcal{K}^0(\text{Cent}(\mathbb{R})) \). Since \( X \) is a curve, we have \( \mathcal{K}^0(X(\mathbb{R})) = \mathcal{K}^0(X(\mathbb{R})) \), and so, \( f \in \mathbb{R}[X]^\prime \mathcal{K}^0(\text{Cent}(\mathbb{R})) \). However, since \( P \) has two roots if we substitute \( x = 0 \), then the graph \( \Gamma_f \subset \mathbb{Z}(<y^4 - x(x^2 + y^2); xt - y^2; t^2 - t - x>) \) is not \( \mathbb{Z} \)-closed. By Lemma 3.12, we get that \( f \notin \mathcal{K}^R(X(\mathbb{R})) \).

**Example 4.6.5.** We give an example of a function in \( \mathbb{R}[X]^\prime \mathcal{K}^0(\text{Cent}(X)) \setminus \mathbb{R}[X]^\prime \mathcal{K}^0(\text{Cent}(X)) \). Consider the variety \( X = \text{Spec}(\mathbb{R}[x, y]/ <x^3 - y^3(1 + z^2)>) \) and the function

\[
\begin{align*}
  f &= \begin{cases} 
    x/y & \text{if } y \neq 0 \\
    3 \sqrt[3]{1 + z^2} & \text{otherwise.}
  \end{cases}
\end{align*}
\]

We have that \( f \) is continuous, rational and is a root of the polynomial \( P(t) = t^3 - 1 - z^2 \). But it is not in \( \mathcal{K}^0(\text{Cent}(X)) \).

To conclude, we summarize this section with the following diagram:

```
\[
\begin{array}{cccccc}
\mathbb{R}[X^+] & \xrightarrow{\cong} & \mathbb{R}[X^+R] & \xrightarrow{\cong} & \mathbb{R}[X^+] & \xrightarrow{\cong} & \mathbb{R}[X^+] \\
\mathcal{K}^+(X(\mathbb{R})) & \xrightarrow{\cong} & \mathcal{K}^+(X(\mathbb{R})) & \xrightarrow{\cong} & \mathcal{K}^+(X(\mathbb{R})) & \xrightarrow{\cong} & \mathcal{K}^+(X(\mathbb{R})) \\
\mathbb{R}[X'^+] & \xrightarrow{\cong} & \mathbb{R}[X'^+R] & \xrightarrow{\cong} & \mathbb{R}[X'^+] & \xrightarrow{\cong} & \mathbb{R}[X'^+]
\end{array}
\]
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5 | BIREGULAR NORMALIZATION AND R-SEMINORMALIZATION

Let \( A \) be a ring. We say that \( A \) satisfies the condition (mp) if \( A \) is reduced and has a finite number of minimal primes that are all real ideals. Note that the ring of polynomial functions on an algebraic
set always satisfies condition \((\text{mp})\). Moreover, by \([6, \text{Lemma 2.8}]\), if \(A\) satisfies \((\text{mp})\) and \(B\) is such that \(A \hookrightarrow B \hookrightarrow A'\), then \(B\) satisfies \((\text{mp})\). For such a ring, we have the inclusion

\[
A \hookrightarrow A/p_1 \times \ldots \times A/p_n \hookrightarrow \kappa(p_1) \times \ldots \times \kappa(p_n) \simeq \mathcal{K}(A),
\]

where the \(p_i\) are the minimal primes of \(A\). Remark that the rings \(A/p_i\) are real rings, and so, \(A\) is also a real ring.

In the paper \([6]\), the authors introduced the biregular normalization \(A^b\) of a ring \(A\) that is defined in the following way: Let \(A\) be a ring satisfying condition \((\text{mp})\) and let \(T(A)\) be the multiplicative subset \(1 + \sum A^2\) of \(A\), which does not contain any zero divisors since \(A\) is real. Then we can consider \(O(A) := T(A)^{-1}A\), and the biregular normalization \(A^b\) is defined as the integral closure of \(A\) in \(O(A)\). This ring can also be defined as the biggest biregular subextension of \(A \hookrightarrow A'\). We recall here Proposition 4.13 from \([6]\).

**Proposition.** Let \(A\) be a ring that satisfies \((\text{mp})\) and let \(B\) such that \(A \hookrightarrow B \hookrightarrow A'\). Then the following statements are equivalent.

1. The extension \(A \hookrightarrow B\) is biregular.
2. For all \(\mathfrak{p} \in R-\text{Max}(A)\), there exists a unique \(m' \in \text{max}(B)\) such that \(m' \cap A = m\). Moreover, the morphism \(A_m \to B_{m'}\) is an isomorphism.
3. For all \(\mathfrak{p} \in R-\text{Spec}(A)\), there exists a unique \(q \in \text{Spec}(B)\) such that \(q \cap A = p\). Moreover, the morphism \(A_p \to B_q\) is an isomorphism.

We will say that a ring \(A\) is a real affine ring if it is the coordinate ring of a real affine variety. In this case, it satisfies the condition \((\text{mp})\) and we have shown in Corollary 2.4 that \(A^{R+\text{max}} = A^{R+}\). Moreover, we also have \(A^{+\text{max}} = A^+\) by \([4, \text{Corollary 3.7}]\).

The goal of this section is to compare the notions of biregular normalization, seminormalization, and R-seminormalization. We start by giving two lemmas that will lead us to the first comparison.

**Lemma 5.1.** Let \(A \hookrightarrow B\) be an integral extension of rings such that \(A\) satisfies \((\text{mp})\). If \(A \hookrightarrow B\) is biregular, then \(A \hookrightarrow B\) is R-subintegral.

**Proof.** Let \(p \in R-\text{Spec}(A)\). Then, by \([6]\) Proposition 4.13, there exists \(q \in \text{Spec}(B)\) such that \(q \cap A = p\) and \(A_p \simeq B_q\). So we get

\[
\kappa(p) \simeq A_p/pA_p \simeq B_q/qB_q \simeq \kappa(q),
\]

and finally \(A \hookrightarrow B\) is R-subintegral. \(\square\)

**Remark.** As a consequence, we get \(A^b \subset A^{R+}\). More precisely, we have the following commutative diagram:

\[
\begin{array}{ccc}
A^{+\text{max}} & \overset{\ll}{\longrightarrow} & A^{R+\text{max}} \overset{\ll}{\longrightarrow} A' \\
\ll & & \ll \\
A^+ & \overset{\ll}{\longrightarrow} & A^{R+} \\
\searrow & & \searrow \\
& A^b \subset &
\end{array}
\]
Lemma 5.2. Let $A \hookrightarrow B$ be an integral extension of rings, and let $A \hookrightarrow C \hookrightarrow A_B^F$ be a subextension of the $F$-seminormalization of $A$ in $B$. Then

$$A_B^F = C_B^F.$$ 

Proof. By Proposition 1.6, the extension $C \hookrightarrow A_B^F$ is $F$-subintegral. So, by the universal property of $C_B^F$, we get

$$A \hookrightarrow C \hookrightarrow A_B^F \hookrightarrow C_B^F \hookrightarrow B,$$

and the first three extensions are $F$-subintegral. This implies that $A \hookrightarrow C_B^F$ is $F$-subintegral, and so, by the universal property of $C_B^F$, we obtain $C_B^F \hookrightarrow A_B^F$. \hfill $\Box$

By applying this lemma to the subextensions of the previous diagram, we obtain the first comparisons between the considered notions.

Proposition 5.3. Let $A$ be a real affine ring. Then

1. $(A_b)^{R+} = (A^{R+})^b = A^{R+}$.
2. $(A^+)^{R+} = (A^{R+})^+ = A^{R+}$.
3. $(A_b)^+ \subset A^{R+}$.
4. $(A^+)^b \subset A^{R+}$.

Proof. By applying the previous proposition with $F = R$-Spec to the subextension $A \hookrightarrow A_b \hookrightarrow A^{R+}$, we get $(A_b)^{R+} = A^{R+}$. Moreover, $A^{R+} \hookrightarrow (A^{R+})^b \hookrightarrow (A^{R+})^{R+}$, so $(A^{R+})^b = A^{R+}$, and (1) is proved. One can do the exact same thing with the subextension $A \hookrightarrow A^+ \hookrightarrow A^{R+}$ to get (2). The two inclusions (3) and (4) follow because we have

$$A_b \hookrightarrow (A^b)^+ \hookrightarrow (A^b)^{R+} = A^{R+}$$

and

$$A^+ \hookrightarrow (A^+)^b \hookrightarrow (A^+)^{R+} = A^{R+}.$$ \hfill $\Box$

We now prove that the inclusion (3) of the previous proposition is an equality. It will be the first part of the main theorem of this section. The reverse inclusion of (4) will be proved in Proposition 5.7.

Proposition 5.4. Let $A$ be a real affine ring. Then

$$(A^b)^+ = A^{R+}.$$ 

Proof. By Proposition 5.3, we have $(A^b)^+ \subset (A^b)^{R+}$. We want to show that the extension $A^b \hookrightarrow (A^b)^{R+}$ is subintegral. Since we deal with affine rings, by Theorem 2.1, it is enough to show that it is $R$-subintegral.

Let $m^b \in \text{Max}(A^b)$ and let us consider $m := m^b \cap A \in \text{Max}(A)$. If $m \in \text{R-Max}(A)$, then $m^b$ is the unique prime ideal of $A^b$ above $m$ and $m^b$ is real. This is because $A \hookrightarrow A^b$ is biregular. But since $A^b \hookrightarrow (A^b)^{R+}$ is $R$-subintegral, there exists a unique prime ideal of $(A^b)^{R+}$ lying over $m^b$.
Now, if \( m \not\in \text{R-Max}(A) \), then we have

\[
(A^b)_m \hookrightarrow ((A^b)^{R^+})_m \hookrightarrow (A^b)'_m = A'_m.
\]

By [6, Proposition 4.6], we have \( (A^b)_m = A'_m \). It follows that \( (A^b)_m = ((A^b)^{R^+})_m \). So, if we consider \( q_1, q_2 \in \text{Max}(A^{R^+}) \) such that \( q_1 \cap A^b = q_2 \cap A^b = m^b \), then \( q_1(A^{R^+})_m = q_2(A^{R^+})_m \). Suppose that there exists \( a \in q_1 \cap q_2 \). Then \( \frac{a}{1} \in q_1(A^{R^+})_m = q_2(A^{R^+})_m \). Thus, there exists \( b \in q_2 \) and \( s \in A \setminus m \) such that \( \frac{a}{1} = \frac{b}{s} \), and so, there exists \( u \in A \setminus m \) such that \( au = bu \). We have \( a \not\in q_2 \) by assumption and also \( su \notin q_2 \) because otherwise \( su \) would belong to \( A \cap q_2 = m \). Since \( bu \in q_2 \) and \( q_2 \) is prime, we obtain a contradiction. This means that \( a \) does not exist, and so \( q_1 = q_2 \). □

It is shown in [6] that the biregular normalization of a real variety can be seen as a normalization of its nonreal points. We want to understand what the R-seminormalization does locally to the real points and to the nonreal points of a variety. This description will also be useful to prove the second part of the main theorem.

**Lemma 5.5.** Let \( A \) be a real affine ring and \( m \in \text{Max}(A) \). We have that

1. If \( m \in \text{R-Max}(A) \), then \( (A^{R^+})_m = A^+_m \).
2. If \( m \not\in \text{R-Max}(A) \), then \( (A^{R^+})_m = A'_m \).

**Proof.** By Proposition 5.4, we have \( (A^b)^+ = A^{R^+} \). So, for \( m \in \text{Max}(A) \), we get \( ((A^b)^+_m = (A^{R^+})_m \) and since the seminormalization commutes with the localization (see [14], e.g.), we get \( ((A^b)^+_m = ((A^b)_m)^+ \). By [6, Proposition 4.6], we have \( (A^b)_m = A_m \) if \( m \) is real and \( (A^b)_m = A'_m \) else. It follows that \( (A^{R^+})_m = A^+_m \) if \( m \) is real and \( (A^{R^+})_m = (A'_m)^+ = A'_m \) else. □

**Proposition 5.6.** Let \( A \) be a real affine ring. Then

\[
A^{R^+} = \bigcap_{m \in \text{R-Max}(A)} A^+_m \cap \bigcap_{m \in \text{Max}(A) \setminus \text{R-Max}(A)} A'_m.
\]

**Proof.** We have

\[
A^{R^+} \subset \bigcap_{m \in \text{Max}(A)} (A^{R^+})_m \subset \bigcap_{m \in \text{Max}(A^{R^+})} (A^{R^+})_m.
\]

Since \( A^{R^+} \) is an affine ring, the last term is equal to \( A^{R^+} \). So, we get

\[
A^{R^+} = \bigcap_{m \in \text{Max}(A^{R^+})} (A^{R^+})_m,
\]

and we conclude the statement using Lemma 5.5. □

**Remark.** Let \( X \) be a real reduced algebraic variety with real irreducible components. From the previous proposition, we see that \( X^{R^+} \) is the normalization of the nonreal locus of \( X \) and the seminormalization of its real locus.
Using the previous property, we show the second part of the main theorem of this section.

**Proposition 5.7.** Let $A$ be a real affine ring. Then

$$(A^+)^b = A^{R+}.$$ 

**Proof.** By [6, Proposition 4.8], we have

$$(A^+)^b = \bigcap_{m \in \text{Max}(A^+)} (A^+)_m \cap \bigcap_{m \not\in \text{Max}(A^+)} (A^+)_m'.$$

By Proposition 5.3, we have $(A^+)^{R+} = A^{R+}$, so

$$A^{R+} = (A^+)^{R+} = \bigcap_{m \in \text{Max}(A^+)} (A^+)_m \cap \bigcap_{m \not\in \text{Max}(A^+)} (A^+)_m'.$$

Hence, $A^{R+} = (A^+)^b$. \qed

From Propositions 5.7 and 5.4, we get the following theorem.

**Theorem 5.8.** Let $X$ be a real affine variety. Then

$$\mathbb{R}[X^{R+}] \cong \mathbb{R}[X^+]^b \cong \mathbb{R}[X^b]^+. $$

We illustrate Theorem 5.8 with the following example.
Example 5.8.6. Let \( X = \text{Spec}\left( \mathbb{R}[x, y]/<y^2 - x^6(x-1)^3(x-2)^2(x^2 +1)^2(x^2 +4)^3 > \right) \). This variety has at least seven singularities: The real isolated point \((0,0)\) is a nonseminormal point, the real point \((1,0)\) is a nonseminormal point, the point \((2,0)\) is the crossing of two real lines, the two complex conjugate points \((\pm i, 0)\) are the crossing of two complex lines, and the two complex conjugate points \((\pm 2i, 0)\) are nonseminormal points. We depict the closed points of \( X, X^b, X^+ \), and \( X^\mathbb{R}^+ \) in Figure 3.

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