EVOLUTION OF COMPLETE NONCOMPACT GRAPHS BY POWERS OF CURVATURE FUNCTION

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Abstract. This paper concerns the evolution of complete noncompact locally uniformly convex hypersurface in Euclidean space by curvature flow, for which the normal speed $\Phi$ is given by a power $\beta \geq 1$ of a monotone symmetric and homogeneous of degree one function $F$ of the principal curvatures. Under the assumption that $F$ is inverse concave and its dual function approaches zero on the boundary of positive cone, we prove that the complete smooth strictly convex solution exists and remains a graph until the maximal time of existence. In particular, if $F = K^{s/n}G^{1-s}$ for any $s \in (0, 1]$, where $G$ is a homogeneous of degree one, increasing in each argument and inverse concave curvature function, we prove that the complete noncompact smooth strictly convex solution exists and remains a graph for all times.

1. Introduction

Let $\Sigma_0$ be a complete noncompact hypersurface embedded in $\mathbb{R}^{n+1}$ and $X_0: M^n \to \mathbb{R}^{n+1}$ be a smooth immersion with $X_0(M) = \Sigma_0$. We consider a one-parameter family of smooth immersions $X: M \times [0, T) \to \mathbb{R}^{n+1}$ satisfying the following evolution equation

$$\begin{cases}
\frac{\partial}{\partial t}X(x,t) = -\Phi(F(W(x,t)))\nu(x,t), \\
X(\cdot,0) = X_0(\cdot),
\end{cases}$$

where $\nu(x,t)$ is the unit outward normal of the evolving hypersurface $\Sigma_t = X(M,t)$ at the point $X(x,t)$, $W$ is the matrix of Weingarten map of $\Sigma_t$, $\Phi(F) = F^{\beta}$ ($\beta \geq 1$) and function $F(W)$ satisfies the following conditions:

**Condition 1.1.**

(i) $F(W) = f(\lambda(W))$, where $\lambda(W)$ gives the eigenvalues of $W$ and $f$ is a smooth, symmetric function defined on the positive cone $\Gamma_+ = \{\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n : \lambda_i > 0, i = 1, \cdots, n\}$;

(ii) $f$ is strictly increasing in each argument: $\frac{\partial f}{\partial \lambda_i} > 0$ on $\Gamma_+$, $\forall i = 1, \cdots, n$;

(iii) $f$ is homogeneous of degree one: $f(k\lambda) = kf(\lambda)$ for any $k > 0$;

(iv) $f$ is strictly positive on $\Gamma_+$ and is normalized such that $f(1, \cdots, 1) = 1$;

(v) $f$ is inverse concave, that is, the function

$$f(\lambda_1, \cdots, \lambda_n) = f(\lambda_1^{-1}, \cdots, \lambda_n^{-1})^{-1}$$

is concave;

(vi) $f$ approaches zero on the boundary of $\Gamma_+$.

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For compact convex hypersurface, problem (1.1) has been widely studied in the last decades. In [27], Huisken showed that any closed convex hypersurface evolving by the mean curvature flow contracts to a point in finite time, and become spherical in shape as the limit is approached. Later, this behavior were established for a wide range of flows where the speed is homogeneous of degree one in the principal curvatures, see [2, 3, 17, 18, 24, 28]. For higher homogeneity, contracting flows and constrained curvature flows were considered and studied in [5, 7–10, 12, 25, 30, 31, 34, 38].

However, much less results are known when initial hypersurface is complete noncompact. In two fundamental papers [20] and [21], Ecker and Huisken studied the evolution of entire graph by the mean curvature. In [20], they proved that if the initial hypersurface is a graph of locally Lipschitz continuous function and has linear growth rate for its height function, the solution exists for all times. They obtained some interior estimates in [21] and applied them to prove that the hypothesis of linear growth in [20] is not necessary. Later, Stavrou [36] proved the convergence to a selfsimilar profile of Lipschitz graphs having a unique cone at infinity, while Rasul [32] obtained a convergence result under a weaker oscillation condition than in [20].

The result in [21] can be extended to different ambient spaces. Unterberger [39, 40] proved the flow by the mean curvature of locally Lipschitz continuous entire radial graph over $\mathbb{S}^n_+$ in hyperbolic space $\mathbb{H}^{n+1}$ has a smooth solution for all times, and each evolving hypersurface is an entire radial graph. Recently, in warped product space, Borisenko and Miquel [11] considered the flow by the mean curvature of a locally Lipschitz continuous graph on complete Riemannian manifold, and proved that the flow exists for all times and evolving hypersurface remains a graph for all times.

The evolution of complete noncompact graphs by other special homogeneous function of degree equal to one has been considered, including $E^{1/k}_k$ [26] and $\frac{E_k}{E_{k-1}}$ [15], where $E_k$ is the elementary symmetric polynomial of degree $k$. In [26], Holland derived gradient and curvature estimates for strictly $k$-convex solutions, and proved long time existence of the flow for $k$-convex initial data under assumption that initial graph function $w_0(x) \to \infty$ as $|x| \to \infty$. Under the weak convexity assumption, Choi and Daskalopoulos [15] proved the long time existence of complete convex solution for $\frac{E_k}{E_{k-1}}$-flow. Recently, Alessandroni and Sinestrari [1] considered the evolution of entire convex graph by a general symmetric function $F$ of principal curvatures. If velocity $F$ is concave and inverse concave, they proved the solution exists for all times provided $F \geq \varepsilon H$ holds for some positive constant $\varepsilon$.

While for special homogeneous curvature function with higher degree, there are several results on curvature problems (1.1) for complete noncompact initial hypersurfaces. Under the assumption that initial graph is convex and satisfies a mild condition on the oscillation of the normal, Schnürrer and Urbas [33] proved long time existence of convex graphs evolving by powers of the Gauss curvature. A similar result was obtained by Franzen [22] for the flow by powers of the mean curvature. Very recently, Choi, Daskalopoulos, Kim and Lee [16] considered the evolution of complete noncompact locally uniformly convex hypersurface by powers of Gauss curvature. Based on some a prior estimates for principal curvatures, they proved that the solution of flow (1.1) exists and remains a graph for all times, without any assumption on the oscillation of the normal speed. We remark that the evolution of strictly
mean convex entire graphs over $\mathbb{R}^n$ by inverse mean curvature flow was also considered by Daskalopoulos and Huisken in [19], and they established the global existence of star-shaped entire graphs with superlinear growth at infinity. More recently, Choi and Daskalopoulos [14] studied the evolution of complete non-compact convex hypersurfaces in $\mathbb{R}^{n+1}$ by the inverse mean curvature flow. They established the long time existence of solutions and provided the characterization of the maximal time of existence in terms of the tangent cone at infinity of the initial hypersurface.

In this paper, we consider the evolution (1.1) of complete noncompact locally uniformly convex hypersurfaces by a power of curvature function satisfying Condition 1.1. In order to formulate the main result of this work, it is necessary to recall some definitions as in [15, 16].

**Definition 1.1.** We use $C^2_H(\mathbb{R}^{n+1})$ to denote the class of second-order differentiable complete (either closed or non-compact) hypersurfaces embedded in $\mathbb{R}^{n+1}$. Given any complete convex hypersurface $\Sigma$ and a point $p \in \Sigma$, we define the smallest principal curvature of $\Sigma$ at point $p$ by

$$\lambda_{\min}(\Sigma)(p) = \sup \{\lambda_{\min}(\Xi)(p) : p \in \Xi \in C^2_H(\mathbb{R}^{n+1}), \Sigma \subset \text{the convex hall of } \Xi\},$$

and we say that

(i) $\Sigma$ is strictly convex, if $\lambda_{\min}(\Sigma)(p) > 0$ holds for all $p \in \Sigma$;
(ii) $\Sigma$ is uniformly convex, if there is a constant $\varepsilon > 0$ such that $\lambda_{\min}(\Sigma)(p) \geq \varepsilon$ for all $p \in \Sigma$;
(iii) $\Sigma$ is locally uniformly convex, if for any compact subset $\Omega \subset \mathbb{R}^{n+1}$, there is a constant $\varepsilon_\Omega > 0$ such that $\lambda_{\min}(\Sigma)(p) \geq \varepsilon_\Omega$ for all $p \in \Sigma \cap \Omega$.

The first main result of this work is stated as follows:

**Theorem 1.1.** Suppose curvature function $F$ satisfies Condition 1.1. Let $\Sigma_0$ be a complete non-compact and locally uniformly convex hypersurface embedded in $\mathbb{R}^{n+1}$. Suppose $X_0 : M^n \to \mathbb{R}^{n+1}$ is an immersion such that $\Sigma_0 = X_0(M)$. Then for any $\beta \in [1, \infty)$, there exists a complete non-compact smooth and strictly convex solution $\Sigma_t = X(M^n, t)$ of (1.1), which is the graph of some smooth and strictly convex function for all $t \in (0, T)$, where $T$ is the maximal time of existence of (1.1).

In particular, if $\Sigma_0$ is an entire graph over $\mathbb{R}^n$, then the smooth strictly convex solution $\Sigma_t$ exists and remains a graph for all times $t \in (0, \infty)$.

In addition, for particular inverse concave curvature function $F = K^{s/n}G^{1-s}$ (See Remark 2.1), by constructing an appropriate barrier to guarantee each solution remains as a graph over the same domain, we have the following long time existence of solution to (1.1) for all times.

**Theorem 1.2.** Suppose curvature function $G$ satisfies Condition 1.1 and $F = K^{s/n}G^{1-s}$ for any $s \in (0, 1]$. Let $\Sigma_0$ be a complete non-compact and locally uniformly convex hypersurface embedded in $\mathbb{R}^{n+1}$. Suppose $X_0 : M^n \to \mathbb{R}^{n+1}$ is an immersion such that $\Sigma_0 = X_0(M)$. Then for any $\beta \in [1, \infty)$, there exists a complete non-compact smooth and strictly convex solution $\Sigma_t = X(M^n, t)$ of (1.1), which is the graph of some smooth and strictly convex function for all times $t \in (0, \infty)$. 

Remark 1.1. The case $s = 1$ of Theorem 1.2 reduces to Theorem 1.1 in [16]. Compared with Theorem 1.1 in [16], the power restriction $\beta \geq 1$ in Theorem 1.2 comes from the estimation of the local lower bound on the principal curvatures for general curvature function $F$ in Proposition 3.2.

As a byproduct of Theorem 1.2, we have the long time existence of a smooth solution $w : \Omega \times (0, \infty) \to \mathbb{R}$ to the following fully nonlinear parabolic equation (see formula (2.4) with $\Phi = K^{s\beta/n}G^{(1-s)\beta}$)

$$\begin{cases}
\frac{\partial w}{\partial t} = \left(\frac{\det D^2 w)^{s\beta/n}}{(1+|Dw|^2)^{(n+2)\beta/2}}\right) G^{(1-s)\beta}(D^2 w, Dw, w, x, t), \\
\lim_{t \to 0} w(x, t) = w_0(x),
\end{cases}$$

where each $w(x, t)$ satisfies the conditions in Theorem 2.1 (see Section 2) and curvature function $G$ satisfies Condition 1.1.

The rest of the paper is organized as follows. First we recall some notations, known results and some basic evolution equations in Section 2. In Section 3, local a priori estimates for gradient function and the principal curvatures are established. We also prove the interior estimates for all derivatives of the second fundamental form by the inverse concavity of curvature function. Based on the interior estimates in previous section, Section 4 is devoted to the proof of the existence of complete noncompact smooth solution, and the long time existence of solution for special inverse concave curvature function.

2. Notations and preliminary results

Let $X : M \to \mathbb{R}^{n+1}$ be a hypersurface of $\mathbb{R}^{n+1}$. The second fundamental form and the Weingarten map are denoted by $A = \{h_{ij}\}$ and $W = \{g^{ik}h_{kj}\} = \{h^i_j\}$ respectively. The eigenvalues $\lambda_i, i \in \{1, \cdots, n\}$ of $W$ are called the principal curvatures of $X(M)$ with respect to the induced metric $g = \{g_{ij}\}$. The trace of $W$ with respect to $g$ is the mean curvature $H$, and the Gauss curvature is

$$K = \det(W) = \det(h^i_j) = \frac{\det(h_{ij})}{\det(g_{ij})} = \prod_{i=1}^{n} \lambda_i.$$

For a curvature function $F$ in Section 1, we shall use $\dot{F}^{kt}$ to indicate the matrix of the first order partial derivatives with respect to the components of its argument

$$\frac{d}{ds} F(A + sB)\bigg|_{s=0} = \dot{F}^{kl} A B_{kl}.$$

Similarly the second order partial derivatives of $F$ are given by

$$\frac{d^2}{ds^2} F(A + sB)\bigg|_{s=0} = \ddot{F}^{kl,rs} A B_{kl} B_{rs}.$$

We also use the notations

$$\dot{f}^i(\lambda) = \frac{\partial f}{\partial \lambda_i}(\lambda) \quad \text{and} \quad \ddot{f}^{ij}(\lambda) = \frac{\partial^2 f}{\partial \lambda_i \partial \lambda_j}(\lambda).$$
to denote the first and second derivatives of \( f \) respect to \( \lambda \). In what follows, we will drop the arguments when derivatives of \( F \) and \( f \) are evaluated at \( W \) and \( \lambda(W) \) respectively. At any diagonal matrix \( A \) with distinct eigenvalues, the second derivative \( \ddot{F} \) in direction \( B \in \text{Sym}(n) \) can be expressed as follows (see \([2, 4]\)):

\[
\ddot{F}^{ij,kl}B_{ij}B_{kl} = \sum_{i,k} \dddot{f}^{ik}B_{ii}B_{kk} + 2\sum_{i>k} \dot{f}^i - \dot{f}^k \lambda^i - \lambda^k B^2_{ik}.
\] (2.1)

This formula makes sense as a limit in the case of any repeated values of \( \lambda_i \).

The following properties of inverse concave functions shall be needed.

**Lemma 2.1** ([4, 8]). If \( f \) is inverse concave, then \( \sum_{i=1}^n \dot{f}^i \lambda_i^2 \geq f^2 \), and

\[
\frac{\dot{f}^k - \dot{f}^l}{\lambda_k - \lambda_l} + \frac{\dot{f}^k}{\lambda_k} + \frac{\dot{f}^l}{\lambda_l} \geq 0, \quad \forall k \neq l.
\] (2.2)

**Remark 2.1.** There are many examples of inverse concave function with the dual function approaching zero on the boundary of positive cone, for example, \( F = E_1^k/k \) (\( k = 1, \ldots, n \)), the power means \( F = (\frac{1}{n} \sum_i \lambda_i^r)^{\frac{1}{r}} \) (\( r > 0 \)), and convex function \( F \). More examples can be constructed as follows: If curvature functions \( G_1 \) and \( G_2 \) satisfy Condition 1.1, then \( F = G_1^s G_2^{1-s} \) satisfies Condition 1.1 for any \( s \in [0, 1] \) (see \([4, 6]\) for more examples).

In order to prove the main results, we need some extra notations as in \([15, 16]\).

**Notation 2.1.**

(i) For set \( \Sigma \subset \mathbb{R}^{n+1} \), we denote the convex hull of \( \Sigma \) by

\[
\text{Conv}(\Sigma) = \{ \varepsilon x + (1 - \varepsilon)y : x, y \in \Sigma, \varepsilon \in [0, 1] \}.
\]

(ii) Given a convex complete (either non-compact or closed) hypersurface \( \Sigma \), if set \( V \) is a subset of \( \text{Conv}(\Sigma) \), we say \( V \) is enclosed by \( \Sigma \) and use the notation \( V \preceq \Sigma \). In particular, if \( V \cap \Sigma = \emptyset \) and \( V \subset \Sigma \), we use \( V \prec \Sigma \).

(iii) For a convex hypersurface \( \Sigma \) and constant \( \varepsilon > 0 \), we use \( \Sigma^\varepsilon \) to denote its \( \varepsilon \)-envelope

\[
\Sigma^\varepsilon = \{ Y \in \mathbb{R}^{n+1} : d(Y, \Sigma) = \varepsilon, Y \notin \text{Conv}(\Sigma) \},
\]

where \( d \) is the distance function.

For a locally uniformly convex hypersurface, we have the following theorem of Wu in \([41]\).

**Theorem 2.1** ([41]). Let \( \Sigma \) be a complete and locally uniformly convex hypersurface embedded in \( \mathbb{R}^{n+1} \), then there exists a function \( w : \Omega \to \mathbb{R} \) defined on a convex open domain \( \Omega \subset \mathbb{R}^n \) such that \( \Sigma = \text{graph } w \) and

(i) \( w \) attains its minimum in \( \Omega \) and \( \inf_{\Omega} w \geq 0 \);

(ii) if \( \Omega \neq \mathbb{R}^n \), then \( \lim_{x \to x_0} w(x) = +\infty \) for all \( x_0 \in \partial \Omega \);

(iii) if \( \Omega \) is unbounded, then \( \lim_{r \to +\infty} \inf_{\Omega \setminus B_r(0)} w = \infty \).
Let hypersurface $\Sigma$ be a graph given by function $w : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}$, that is, 
$$\Sigma = \{(x, w(x)) : x \in \Omega\}.$$ 
Then the induced metric $g_{ij}$ and its inverse are given by 
$$g_{ij} = \delta_{ij} + w_i w_j \quad \text{and} \quad g^{ij} = \delta^{ij} - \frac{w^i w^j}{1 + |Dw|^2},$$ 
where $w_i$ is the partial derivatives of $w$. In addition, the unit outward normal is 
$$\nu = \frac{1}{\sqrt{1 + |Dw|^2}}(Dw, -1). \quad (2.3)$$ 
The sign of the unit outward normal is chosen such that $\Sigma$ is convex if and only if the hessian 
of its graph representation $w(\cdot, t)$ is nonnegative. After a standard computation, the second 
fundamental form can be expressed as 
$$h_{ij} = \frac{w_{ij}}{\sqrt{1 + |Dw|^2}},$$ 
which implies 
$$h^i_j = \frac{w_{jk}}{\sqrt{1 + |Dw|^2}} \left( \delta^{ik} - \frac{w^i w^k}{1 + |Dw|^2} \right).$$ 
It follows from (1.1) and (2.3) that the parabolic system (1.1) is, up to tangential diffeomorphisms, equivalent to the following equation 
$$\begin{cases} 
\frac{\partial w}{\partial t} = \sqrt{1 + |Dw|^2} \Phi(D^2 w, Dw, w, x, t), \\
\lim_{t \to 0} w(x, t) = w_0(x).
\end{cases} \quad (2.4)$$ 
To ensure that evolving hypersurface stays a graph, we have to estimate $\langle \nu, \omega \rangle$ from below 
for some fixed vector $\omega \in \mathbb{R}^{n+1}, |\omega| = 1$. Let us choose $\omega = -e_{n+1}$, and define the gradient function 
$$v = \langle \nu, -e_{n+1} \rangle^{-1} = \sqrt{1 + |Dw|^2},$$ 
and the height function 
$$u(x, t) = \langle X(x, t), e_{n+1} \rangle.$$ 
We conclude this section by showing some evolution equations for important geometric quantities.
Lemma 2.2. Let $\Sigma_t$ be a complete strictly convex graph solution of (1.1). Then the following evolution equations hold.

\[
\begin{align*}
\partial_t g_{ij} &= -2\Phi h_{ij}, & \partial_t \nu &= X_\ast(\nabla \Phi), \\
\partial_t \Phi &= \mathcal{L} \Phi + \Phi \dot{h}_{ij} h_{ij}^k, \\
\partial_t h_{ij} &= \mathcal{L}_t h_{ij} + \dot{h}_{ij} h_{kl} \nabla_i h_{kl} \nabla_j h_{mn} + \dot{h}_{ij} h_{ks} h_{rs} + (\beta + 1) \Phi h_{ij} h_{ij}^k, \\
\partial_t b_{ij} &= \mathcal{L} \dot{b}_{ij} - 2b_{ij} h_{kq} h_{pq} \dot{h}_{ij} - b_{ij} b_{pq} \dot{h}_{pq} \nabla_i h_{kl} \nabla_j h_{rs} - b_{ij} b_{pq} \dot{h}_{pq} \nabla_i h_{kl} \nabla_j h_{rs} - b_{pq} b_{ij} b_{pq} + (\beta + 1) \Phi g_{ij}, \\
\partial_t u &= \mathcal{L} u + (1 - \beta) \Phi v^{-1}, & \partial_t v &= \mathcal{L} v - 2v^{-1} |\nabla v|^2 - v \dot{h}_{ij} h_{ij}^k,
\end{align*}
\]

where $b_{ij} = h_{ij}^{-1}$, $\mathcal{L} = \dot{h}_{ij} h_{kl} \nabla_i h_{kl} \nabla_j h_{mn}$ and $|\nabla v|^2 = \dot{h}_{ij} h_{ij}^k \nabla_i \nabla_j v$.

Proof. The first four evolution equations under flow (1.1) follow from straightforward computations as in §3 of [27] (see also [23, 30]). Now we prove the evolution equation for $b_{ij}$. The identity $b_{ik} h_{kj} = \delta_j^i$ implies

\[
\partial_t b_{pq} = -b_{pq} b_{ij} \partial_t h_{ij} \quad \text{and} \quad \nabla b_{pq} = -b_{pq} b_{ij} \nabla h_{ij}.
\]

Therefore

\[
\nabla_i \nabla_j b_{pq} = -b_{pq} b_{ij} \nabla_i \nabla_j \nu + 2b_{ij} b_{pq} \nabla_i \nabla_j \nu h_{kl},
\]

which implies

\[
\mathcal{L} b_{pq} = -b_{pq} b_{ij} \mathcal{L} h_{ij} + 2b_{ij} b_{pq} \nabla_i \nabla_j h_{kl}.
\]

Combination of the above formulae with (2.6) gives

\[
\begin{align*}
\partial_t b_{pq} &= -b_{pq} b_{ij} \left( \mathcal{L} h_{ij} + \dot{h}_{ij} h_{kl} \nabla_i h_{kl} \nabla_j h_{mn} + \dot{h}_{ij} h_{ks} h_{rs} + (\beta + 1) \Phi h_{ij} h_{ij}^k \right) \\
&= \mathcal{L} b_{pq} - 2b_{ij} b_{pq} \dot{h}_{ij} h_{kl} \nabla_i \nabla_j h_{kl} - b_{ij} b_{pq} \dot{h}_{pq} \nabla_i h_{kl} \nabla_j h_{rs} - b_{pq} b_{ij} \dot{h}_{ij} h_{ij}^k + (\beta + 1) \Phi g_{ij},
\end{align*}
\]

which is equation (2.7).

Next, we give the proof of (2.8). By direct computations we have

\[
\partial_t u = \langle \partial_t X_\ast, e_{n+1} \rangle = -\Phi \langle \nu, e_{n+1} \rangle,
\]

and

\[
\nabla_i \nabla_j u = \langle \nabla_i \nabla_j X_\ast, e_{n+1} \rangle = -h_{ij} \langle \nu, e_{n+1} \rangle.
\]

Then equation (2.8) follows from above two equations.

Last, we prove the evolution equation for gradient function $v$. From the evolution equation for $\nu$, we have

\[
\partial_t v = -\partial_t \nu (\nu, e_{n+1})^{-1} = v^2 \langle \nabla \Phi, e_{n+1} \rangle,
\]
and
\[ \nabla_i \nabla_j v = \nabla_i \left( v^2 \langle \nabla_j \nu, e_{n+1} \rangle \right) \\
= 2v \nabla_i v \langle \nabla_j \nu, e_{n+1} \rangle + v^2 \langle \nabla_i \nabla_j \nu, e_{n+1} \rangle \\
= 2v^{-1} \nabla_i v \nabla_j v + v^2 \langle \nabla h_{ij}, e_{n+1} \rangle + vh_{ik}h^k_j, \]
which implies formula (2.9). \[ \square \]

3. Local estimates

In this section, we will deduce interior a priori estimates for the gradient function, the principal curvatures and all the derivatives of the second fundamental form for solutions to flow (1.1), under assumption that the initial hypersurface is smooth.

We begin by defining cut-off functions
\[ \varphi_\gamma = \left( R - u(x,t) - \gamma t \right)_+ \quad \text{and} \quad \varphi = \left( R - u(x,t) \right)_+ \]
for some positive constants \( R \) and \( \gamma \). First, we have the following local gradient estimate.

**Proposition 3.1.** Assume \( \Sigma_0 \) is a complete locally uniformly convex smooth hypersurface in \( \mathbb{R}^{n+1} \), and let \( \Sigma_t \) be a complete strictly convex smooth graph solution of (1.1) defined on \( M^n \times [0,T] \), for some \( T > 0 \). Then, for some constants \( \gamma > 0 \) and \( R \geq \gamma \), we have
\[ v(x,t) \varphi_\gamma(x,t) \leq R \max \left\{ \sup_{\Sigma_0} v(x,0), \frac{\beta - 1}{\gamma} \right\}, \]
where \( \bar{\Sigma}_0 = \{ x \in \Sigma_0 : u(x,0) \leq R \} \).

**Proof.** We derive from (2.8) and the definition of \( \varphi_\gamma \) that
\[ \partial_t \varphi_\gamma = \mathcal{L} \varphi_\gamma + (\beta - 1) \Phi v^{-1} - \gamma. \]
Combining this with (2.9) we obtain
\[ \partial_t (\varphi_\gamma v) = \varphi_\gamma (\mathcal{L} v - 2v^{-1} |\nabla v|^2 - v \dot{\Phi}^{ij} h_{ik} h^k_j) + v (\mathcal{L} \varphi_\gamma + (\beta - 1) \Phi v^{-1} - \gamma) \\
= \mathcal{L} (\varphi_\gamma v) - 2 \langle \nabla \varphi_\gamma, \nabla v \rangle \mathcal{L} - 2 \varphi_\gamma v^{-1} |\nabla v|^2 - v \varphi_\gamma \dot{\Phi}^{ij} h_{ik} h^k_j + (\beta - 1) \Phi - \gamma v \\
= \mathcal{L} (\varphi_\gamma v) - 2v^{-1} \langle \nabla v, \nabla (\varphi_\gamma v) \rangle \mathcal{L} + (\beta - 1) \Phi - \varphi_\gamma v \dot{\Phi}^{ij} h_{ik} h^k_j - \gamma v. \]
It follows from Theorem 2.1 that the cut-off function \( \varphi_\gamma \) is compactly supported. Assume the function \( \varphi_\gamma v \) attains its maximum at point \( (x_0, t_0) \). If \( t_0 = 0 \), then the result follows. Now let us assume \( t_0 > 0 \). Using \( \beta \geq 1 \), we have the following inequality by the weak parabolic maximum principle
\[ (\beta - 1) \Phi \geq \gamma v + \varphi_\gamma v \dot{\Phi}^{ij} h_{ik} h^k_j. \]
Multiplying above inequality by \( R \Phi^{-1} \) and using \( R \geq \gamma \) we have
\[ (\beta - 1) R \geq \gamma v (RF^{-\beta} + \beta \varphi_\gamma \dot{\Phi}^{ij} h_{ik} h^k_j F^{-1}) \]
\[ \geq \gamma v (RF^{-\beta} + \beta \varphi_\gamma F) \]
\[ \geq \gamma v \varphi_\gamma \left( \frac{F^{-\beta}}{\beta + 1} + \frac{\beta F}{\beta + 1} \right) \geq \gamma v \varphi_\gamma, \]
where Lemma 2.1 and Young’s inequality are used in the second and last inequality, respectively. Then the assertion follows. □

Now we will show local lower bounds on the principal curvatures in terms of the initial data. Here, a Pogorelov type computation, which was introduced by Sheng, Urbas and Wang in [35] for the elliptic setting, appeared in [15, 16] is used. We begin by recall the following known Euler’s formula.

**Lemma 3.1 ([16]).** Let $\Sigma$ be a smooth strictly convex hypersurface. Assume smooth immersion $X : M \to \mathbb{R}^{n+1}$ satisfies $\Sigma = X(M)$. Then, for all $x \in M$ and $i \in \{1, \cdots, n\}$, the following inequality holds

$$\frac{b_{ii}(x)}{g^{ii}(x)} \leq \frac{1}{\lambda_{\min}(x)},$$

where $\{b^i\}$ is the inverse matrix of the second fundamental form $\{h_{ij}\}$.

**Proposition 3.2.** Assume $\Sigma_0$ is a complete locally uniformly convex smooth hypersurface in $\mathbb{R}^{n+1}$, and let $\Sigma_t$ be a complete strictly convex smooth graph solution of (1.1) defined on $M^n \times [0, T]$, for some $T > 0$. Given a positive constant $R$, then for any $\sigma \in (0, 1)$, the following estimate holds

$$\inf_{\{x \in \Sigma_t : u(x,t) \leq \sigma R, t \in [0,T]\}} \varphi \lambda_{\min}(x,t) \geq \inf_{\{x \in \Sigma_0 : u(x,0) \leq \sigma R\}} \varphi \lambda_{\min}(x,0).$$

**Proof.** Since cut-off function $\varphi$ is compactly supported by Theorem 2.1, then for fixed $T > 0$, the function $\varphi^{-1} \lambda_{\min}^{-1}$ attains its maximum on

$$\{\Sigma_t : u(x,t) \leq \sigma R, t \in [0,T]\}$$

at point $(x_0, t_0)$. If $t_0 = 0$, the result follows. In what follows, we assume $t_0 > 0$.

Choose a chart $(U, \Psi)$ with $x_0 \in \Psi(U)$ such that the covariant derivatives $\{\nabla_i X(x_0, t_0)\}_{i=1,\cdots,n}$ form an orthonormal basis of $(T\Sigma_0)_{X(x_0,t_0)}$ satisfying

$$g_{ij}(x_0,t_0) = \delta_{ij}, \quad h_{ij}(x_0,t_0) = \delta_{ij} \lambda_i(x_0,t_0), \quad \lambda_1(x_0,t_0) = \lambda_{\min}(x_0,t_0).$$

Then at point $(x_0, t_0)$, we have

$$b^{11}(x_0,t_0) = \lambda_{\min}^{-1}(x_0,t_0), \quad g^{11}(x_0,t_0) = 1.$$

Let us define the function

$$\rho = \varphi^{-1} b^{11}.$$ 

For any point $(x, t) \in \Psi(U) \times [0, T]$, the following inequality holds by Lemma 3.1

$$\rho(x,t) \leq \varphi^{-1} \lambda_{\min}^{-1}(x,t) \leq \varphi^{-1} \lambda_{\min}^{-1}(x_0,t_0) = \rho(x_0,t_0),$$

which implies $\rho$ attains its maximum at point $(x_0, t_0)$. From the evolution equation (2.8), we have

$$\partial_t \varphi = \mathcal{L} \varphi + (\beta - 1) \Phi v^{-1}. \quad (3.1)$$
By the definition of $\rho$ and $\nabla g^{11} = 0$, we derive that

$$\mathcal{L}\rho = \dot{\Phi}^i \nabla_i \nabla_j \left( \varphi^{-1} \frac{b_1^{11}}{g^{11}} \right)$$

$$= \dot{\Phi}^i \nabla_i \left( -\varphi^{-2} \frac{\nabla_j \varphi \rho^{11}}{g^{11}} + \varphi^{-1} \frac{\nabla_j b^{11}}{g^{11}} \right)$$

$$= \dot{\Phi}^i \left( -\varphi^{-2} \frac{b_1^{11}}{g^{11}} \nabla_i \nabla_j \varphi + 2\varphi^{-2} \frac{b_1^{11}}{g^{11}} \nabla_j \varphi \nabla_i \varphi - 2\varphi^{-2} \frac{\nabla_j \varphi \nabla_j b^{11}}{g^{11}} + \varphi^{-1} \frac{\nabla_i \nabla_j b^{11}}{g^{11}} \right)$$

$$= -\varphi^{-2} \frac{b_1^{11}}{g^{11}} \mathcal{L}\varphi + \varphi^{-1} \frac{g^{11}}{b_1^{11}} \mathcal{L}b^{11} - 2\varphi^{-1} \langle \nabla \varphi, \nabla \rho \rangle \mathcal{L}. \quad (3.2)$$

Then at point $(x_0, t_0)$, from equations (2.7), (3.1) and (3.2), it follows that

$$\partial_t \rho = -\varphi^{-2} \frac{b_1^{11}}{g^{11}} \partial_t \varphi + \varphi^{-1} \frac{g^{11}}{b_1^{11}} \varphi^{-1} \partial_t b^{11} - \varphi^{-1} \frac{b_1^{11}}{(g^{11})^2} \partial_t g^{11}$$

$$= -\varphi^{-2} \frac{b_1^{11}}{g^{11}} \left( \mathcal{L}\varphi + (\beta - 1) \Phi u^{-1} \right) + \varphi^{-1} \frac{g^{11}}{b_1^{11}} \left( -b_1^{11} \dot{\Phi}^{kl} h_{kp} h_l^p + (1 + \beta) \Phi g^{11} \right)$$

$$+ \varphi^{-1} \left( \mathcal{L}b^{11} - 2b_1^{kp} b_1^{ql} \dot{\Phi}^{rs} \nabla_r p q \nabla_s h_{kl} - b_1^{kp} b_1^{ql} \dot{\Phi}^{pq,rs} \nabla_k h_{pq} \nabla_l h_{rs} \right)$$

$$- 2\varphi^{-1} \frac{b_1^{11}}{(g^{11})^2} \Phi h^{11}$$

$$= \mathcal{L}\rho + 2\varphi^{-1} \langle \nabla \varphi, \nabla \rho \rangle \mathcal{L} - \varphi^{-2} (\beta - 1) \Phi u^{-1} b^{11} + (\beta - 1) \Phi \varphi^{-1} - b_1^{11} \varphi^{-1} \dot{\Phi}^{kl} h_{kp} h_l^p$$

$$- \varphi^{-1} (b_1^{11})^2 \left( 2b_1^{kl} \dot{\Phi}^{kl} \nabla_1 h_{ik} \nabla_1 h_{jl} + \dot{\Phi}^{pq,rs} \nabla_1 h_{pq} \nabla_1 h_{rs} \right). \quad (3.3)$$

Now we estimate the terms on the last line of (3.3) by a trick that appeared in the proof of Theorem 3.2 in [9]. To make use the inverse concavity of $f$, let $\tau_i = \frac{1}{\lambda_i}$ and $f_*(\tau) = f(\lambda)^{-1}$. We can compute that

$$\dot{j}^k = \frac{1}{f_*^2} \partial f_* \frac{1}{\lambda_k^2} \frac{\dot{j}^k}{f_*^2} = \frac{\dot{j}^k}{f_*^2} \frac{1}{\lambda_k^2},$$

and

$$\dot{j}^{kl} = -\frac{\dot{j}^{kl}}{f_*^2} \frac{1}{\lambda_l \lambda_k} + 2 \frac{1}{f_*^2} \dot{j}^k \dot{j}^l - 2f_* \frac{1}{f_*^2} \lambda_k \delta_{kl}$$

$$= -\frac{\dot{j}^{kl}}{f_*^2} \frac{1}{\lambda_l \lambda_k} + 2 \frac{\dot{j}^{kl}}{f_*^2} \delta_{kl}. \quad (3.4)$$
Therefore by equations (2.1) and (3.4), we have
\[
2\beta^{ij} \hat{F}^{kli} \nabla_k h_{i1} \nabla_l h_{j1} + \hat{\Phi}^{pq,rs} \nabla_1 h_{pq} \nabla_1 h_{rs}
\]
\[
= 2\beta F^\beta - 1 b^{ij} \hat{F}^{kli} \nabla_k h_{i1} \nabla_l h_{j1} + \beta F^{\beta - 1} \hat{F}^{pq,rs} \nabla_1 h_{pq} \nabla_1 h_{rs} + \beta (\beta - 1) F^{\beta - 2} (\nabla_1 F)^2
\]
\[
= 2\beta F^\beta - 1 b^{ij} \hat{F}^{kli} \nabla_k h_{i1} \nabla_l h_{j1} + \beta (\beta - 1) F^{\beta - 2} (\nabla_1 F)^2 + \beta F^{\beta - 1} \sum_{k \neq l} \frac{\dot{j}_k - \dot{j}_l}{\lambda_k - \lambda_l} (\nabla_1 h_{kl})^2
\]
\[
= 2\beta F^\beta - 1 b^{ij} \hat{F}^{kli} \nabla_k h_{i1} \nabla_l h_{j1} + \beta (\beta - 1) F^{\beta - 2} (\nabla_1 F)^2 + \beta F^{\beta - 1} \sum_{k \neq l} \frac{\dot{j}_k - \dot{j}_l}{\lambda_k - \lambda_l} (\nabla_1 h_{kl})^2
\]
\[
\geq 2\beta F^\beta - 1 b^{ij} \hat{F}^{kli} \nabla_k h_{i1} \nabla_l h_{j1} - \beta F^{\beta + 1} j^{*}_s \frac{1}{\lambda_k^2 \lambda_l^2} (\nabla_1 h_{kk})^2 - 2\beta F^{\beta - 1} \sum_{k \neq l} \frac{\dot{j}_k}{\lambda_l} (\nabla_1 h_{kl})^2 + \beta (\beta + 1) F^{\beta - 2} (\nabla_1 F)^2
\]
\[
= 2\beta F^\beta - 1 b^{ij} \hat{F}^{kli} \nabla_k h_{i1} \nabla_l h_{j1} - \beta F^{\beta + 1} j^{*}_s \frac{1}{\lambda_k^2 \lambda_l^2} (\nabla_1 h_{kk})^2 - 2\beta F^{\beta - 1} \sum_{k \neq l} \frac{\dot{j}_k}{\lambda_l} (\nabla_1 h_{kl})^2 + \beta (\beta + 1) F^{\beta - 2} (\nabla_1 F)^2
\]
\[
= -\beta F^{\beta + 1} j^{*}_s \frac{1}{\lambda_k^2 \lambda_l^2} (\nabla_1 h_{kk})^2 - \beta (\beta + 1) F^{\beta - 2} (\nabla_1 F)^2 \geq 0,
\]
where inequality (2.2) is used in the first inequality. By the fact
\[
\hat{F}^{kli} h_{k} h_{l}^p \geq h_{11} F,
\]
equation (3.3) can be rewritten as follows
\[
\partial_t \rho \leq \mathcal{L} \rho + 2 \phi^{-1} (\nabla \phi, \nabla \rho)_{\mathcal{E}} - \phi^{-2} (\beta - 1) \Phi \phi^{-1} b^{11} + (\beta - 1) \Phi \phi^{-1} b^{11} \phi^{-1} \hat{F}^{kli} h_{k} h_{l}^p
\]
\[
\leq \mathcal{L} \rho + 2 \phi^{-1} (\nabla \phi, \nabla \rho)_{\mathcal{E}} + \phi^{-2} (1 - \beta) \Phi \phi^{-1} b^{11} - \Phi \phi^{-1}.
\]
Thus at maximum point \((x_0, t_0)\), we have
\[
0 \leq \partial_t \rho \leq \phi^{-2} (1 - \beta) \Phi \phi^{-1} b^{11} - \Phi \phi^{-1} < 0,
\]
which is a contradiction. Hence \(t_0 = 0\) and the result holds.

\[\square\]

**Remark 3.1.** In [16], Choi, Daskalopoulos, Kim and Lee obtained local lower bounds on the principal curvature of strictly convex complete smooth graph solution of \(K^\beta\)-flow for all \(\beta > 0\) by considering the compactly supported function \(\phi^{\beta} \cdot \nabla \phi^{\beta} \cdot \frac{b^{11}}{\Phi \phi^{-1}}\). However, for general inverse-concave curvature function \(F\), the term \(\hat{\Phi}^{pq,rs} \nabla_1 h_{pq} \nabla_1 h_{rs}\) appeared in (3.3) cannot
be estimated accurately as in [16]. Here we choose another compactly supported function \( \varphi^{-1/2} \), and the power \( \beta \geq 1 \) cannot be weaken to \( \beta > 0 \).

Next we will derive local upper bounds on the principal curvatures by using the assumption that dual function \( F_* \) approaches zero on the boundary of positive cone. For this purpose, the local upper bound on velocity \( F \) is needed.

**Proposition 3.3.** Assume \( \Sigma_0 \) is a complete locally uniformly convex smooth hypersurface in \( \mathbb{R}^{n+1} \), and let \( \Sigma_t \) be a complete strictly convex smooth graph solution of (1.1) defined on \( M^n \times [0, T] \), for some \( T > 0 \). Then, given a constant \( R > 0 \), the following holds

\[
\frac{t}{1+t} F \varphi^2 \leq C_0 \theta^{1+\frac{1}{4}},
\]

where \( C_0 = 2^{1+\frac{1}{4}} (2\beta \Lambda (1 + 4\beta(\theta + 1)) + R^2 + 2(\beta - 1)R) \) and \( \theta, \Lambda \) are given by

\[
\theta = \sup_{\{\Sigma_t; u(x, t) \leq R, t \in [0, T]\}} v^2, \quad \Lambda = \sup_{\{\Sigma_t; u(x, t) \leq R, t \in [0, T]\}} \lambda_{\min}.
\]

**Proof.** From equations (2.5) and (2.9), we infer that

\[
\partial_t \Phi^2 = \mathcal{L} \Phi^2 - 2|\nabla \Phi|^2_L + 2\Phi^2 \dot{\Phi}^i_j h^i_k h^k_j,
\]

\[
\partial_t v^2 = \mathcal{L} v^2 - 6|\nabla v|^2_L - 2v^2 \dot{\Phi}^i_j h^i_k h^k_j.
\]

Using an idea of Caffarelli, Nirenberg and Spruck in [13] (see also [15, 16, 21]), we define function \( \eta = \eta(v^2) \) by

\[
\eta(v^2) = \frac{v^2}{2\theta - v^2}.
\]

Then the evolution equation for \( \eta \) is

\[
\partial_t \eta = \eta' \partial_t v^2 = \eta'(\mathcal{L} v^2 - 6|\nabla v|^2_L - 2v^2 \dot{\Phi}^i_j h^i_k h^k_j)
= \mathcal{L} \eta - \eta''|\nabla v|^2_L - 6\eta'|\nabla v|^2_L - 2v^2 \eta' \dot{\Phi}^i_j h^i_k h^k_j.
\]

Hence we have

\[
\partial_t (\Phi^2 \eta) = \Phi^2 \left( \mathcal{L} \eta - \eta''|\nabla v|^2_L - 6\eta'|\nabla v|^2_L - 2v^2 \eta' \dot{\Phi}^i_j h^i_k h^k_j \right)
+ \eta \left( \mathcal{L} \Phi^2 - 2|\nabla \Phi|^2_L + 2\Phi^2 \dot{\Phi}^i_j h^i_k h^k_j \right)
= \mathcal{L} (\Phi^2 \eta) - 2\eta|\nabla \Phi|^2_L - \Phi^2 (4\eta''v^2 + 6\eta')|\nabla v|^2_L
+ 2\Phi^2 (\eta - \eta'v^2) \dot{\Phi}^i_j h^i_k h^k_j - 2(\nabla \eta, \nabla \Phi^2)_L.
\]

The last term can be estimated by

\[
-2(\nabla \eta, \nabla \Phi^2)_L = -\langle \nabla \eta, \nabla \Phi^2 \rangle_L - \langle \nabla \eta, \eta^{-1} \nabla (\Phi^2 \eta) \rangle_L + \eta^{-1} \Phi^2 |\nabla \eta|^2_L
\leq -\eta^{-1} (\nabla \eta, \nabla (\Phi^2 \eta))_L + \frac{3}{2} \eta^{-1} \Phi^2 |\nabla \eta|^2_L + 2\eta |\nabla \Phi^2 |_L.
\]

Thus the evolution equation for \( \Phi^2 \eta \) can be rewritten as

\[
\partial_t (\Phi^2 \eta) \leq \mathcal{L} (\Phi^2 \eta) - \eta^{-1} (\nabla \eta, \nabla (\Phi^2 \eta))_L + 2\Phi^2 (\eta - \eta'v^2) \dot{\Phi}^i_j h^i_k h^k_j
- \Phi^2 (4\eta''v^2 + 6\eta' - 6\eta^{-1}(\eta')^2v^2) |\nabla v|^2_L.
\]
From the expression of $\eta$, we have

$$\eta - \eta' v^2 = -\eta^2, \quad \eta^{-1} \nabla \eta = 4\theta \eta v^{-3} \nabla v,$$

and

$$4\eta'' v^2 + 6\eta' - 6\eta^{-1} (\eta')^2 v^2 = \frac{4\theta \eta}{(2\theta - v^2)^2}.$$

Substituting these identities into the evolution equation for $\psi = \Phi^2 \eta$ implies

$$\partial_t \psi \leq \mathcal{L} \psi - 4\theta \eta v^{-3} \langle \nabla v, \nabla \psi \rangle_{\mathcal{L}} - \frac{4\theta \psi}{(2\theta - v^2)^2} |\nabla v|^2_{\mathcal{L}} - 2\eta \psi \hat{\Phi}^k h_k h_j.$$

It follows from (3.1) that

$$\partial_t \varphi^{4\beta} = \mathcal{L} \varphi^{4\beta} - 4\beta (4\beta - 1) \varphi^{4\beta - 2} \nabla \varphi |\nabla \varphi|^2_{\mathcal{L}} + 4\beta (\beta - 1) \varphi^{4\beta - 1} \Phi v^{-1}.$$ 

Hence the following inequality holds

$$\partial_t (\psi \varphi^{4\beta}) \leq \mathcal{L} (\psi \varphi^{4\beta}) - 4\theta \eta v^{-3} \varphi^{4\beta} \langle \nabla v, \nabla \psi \rangle_{\mathcal{L}} - \frac{4\theta \psi \varphi^{4\beta}}{(2\theta - v^2)^2} |\nabla v|^2_{\mathcal{L}}$$

$$- 2\eta \varphi^{4\beta} \hat{\Phi}^k h_k h_j - 4\beta (4\beta - 1) \psi \varphi^{4\beta - 2} |\nabla \varphi|^2_{\mathcal{L}}$$

$$+ 4\beta (\beta - 1) \varphi^{4\beta - 1} \Phi v^{-1} \psi - 2 \langle \nabla \psi, \nabla \varphi^{4\beta} \rangle_{\mathcal{L}}.$$ 

The last term can be rewritten as

$$-2 \langle \nabla \psi, \nabla \varphi^{4\beta} \rangle_{\mathcal{L}} = -2 \varphi^{-4\beta} \langle \nabla \varphi^{4\beta}, \nabla (\psi \varphi^{4\beta}) \rangle_{\mathcal{L}} + 32\beta^2 \varphi^{4\beta - 2} \psi |\nabla \varphi|^2_{\mathcal{L}}.$$ 

We also estimate

$$-4\theta \eta v^{-3} \varphi^{4\beta} \langle \nabla v, \nabla \psi \rangle_{\mathcal{L}}$$

$$= -4\theta \eta v^{-3} \langle \nabla v, \nabla (\psi \varphi^{4\beta}) \rangle_{\mathcal{L}} + 16\beta^2 \eta v^{-3} \psi \varphi^{4\beta - 1} \langle \nabla v, \nabla \varphi \rangle_{\mathcal{L}}$$

$$\leq -4\theta \eta v^{-3} \langle \nabla v, \nabla (\psi \varphi^{4\beta}) \rangle_{\mathcal{L}} + \frac{4\theta \psi \varphi^{4\beta}}{(2\theta - v^2)^2} |\nabla v|^2_{\mathcal{L}}$$

$$+ 16\beta^2 (2\theta - v^2)^2 \theta \psi v^{-6} \eta^2 \varphi^{4\beta - 2} |\nabla \varphi|^2_{\mathcal{L}}$$

$$= -4\theta \eta v^{-3} \langle \nabla v, \nabla (\psi \varphi^{4\beta}) \rangle_{\mathcal{L}} + \frac{4\theta \psi \varphi^{4\beta}}{(2\theta - v^2)^2} |\nabla v|^2_{\mathcal{L}}$$

$$+ 16\beta^2 \theta \psi v^{-2} \varphi^{4\beta - 2} |\nabla \varphi|^2_{\mathcal{L}}.$$ 

Combining above inequalities gives

$$\partial_t (\psi \varphi^{4\beta}) \leq \mathcal{L} (\psi \varphi^{4\beta}) - \langle 4\theta \eta v^{-3} \nabla v + 2 \varphi^{-4\beta} \nabla \varphi^{4\beta}, \nabla (\psi \varphi^{4\beta}) \rangle_{\mathcal{L}}$$

$$+(32\beta^2 + 16\beta^2 \theta v^{-2} - 4\beta (4\beta - 1)) \psi \varphi^{4\beta - 2} |\nabla \varphi|^2_{\mathcal{L}}$$

$$+ 4\beta (\beta - 1) \varphi^{4\beta - 1} \Phi v^{-1} \psi - 2 \eta \varphi \varphi^{4\beta} \hat{\Phi}^k h_k h_j.$$ 

On the other hand, by

$$\nabla_i \varphi = -\nabla_i u = -\langle \nabla_i X, e_{n+1} \rangle,$$
we have

\[ |\nabla \varphi|_L^2 = \beta F^{\beta - 1} \dot{F}^{ij} \nabla_i \varphi \nabla_j \varphi \]

\[ = \beta F^{\beta - 1} \dot{F}^{ij} \langle \nabla_i X, e_{n+1} \rangle \langle \nabla_j X, e_{n+1} \rangle \]

\[ \leq \sum_{p=1}^{n+1} \beta F^{\beta - 1} \dot{F}^{ij} \langle \nabla_i X, e_p \rangle \langle \nabla_j X, e_p \rangle \]

\[ = \beta F^{\beta - 1} \dot{F}^{ij} g_{ij} \leq \frac{\beta}{\lambda_{\min}} F^\beta \leq \beta \Lambda F^\beta, \]

which implies that, by \( v \geq 1 \)

\[ (32\beta^2 + 16\beta^2 \theta v^{-2} - 4\beta(4\beta - 1)) \psi \varphi^{4\beta - 2} |\nabla \varphi|_L^2 \leq 4\Lambda \beta^2 (1 + 4\beta(\theta + 1)) \psi \varphi^{4\beta - 2} F^\beta. \]

Thus the evolution equation for \( \psi \varphi^{4\beta} \) can be rewritten as

\[ \partial_t (\psi \varphi^{4\beta}) \leq \mathcal{L}(\psi \varphi^{4\beta}) - \langle 4\theta \eta v^{-3} \nabla v + 2\varphi^{4\beta} \nabla \varphi^{4\beta}, \nabla (\psi \varphi^{4\beta}) \rangle_L \]

\[ + 4\Lambda \beta^2 (1 + 4\beta(\theta + 1)) \psi \varphi^{4\beta - 2} F^\beta - 2\eta \psi \varphi^{4\beta} \dot{F}^{ij} h_{ik} h_{kj} \]

\[ + 4\beta(\beta - 1) \varphi^{4\beta - 1} \Phi v^{-1} \psi. \]

Let \( \xi = \frac{1}{1+t} \), then we have, by \( \partial_t \xi = \frac{1}{(1+t)^2} \leq 1 \),

\[ \partial_t (\psi \varphi^{4\beta} \xi^{2\beta}) \leq \mathcal{L}(\psi \varphi^{4\beta} \xi^{2\beta}) - \langle 4\theta \eta v^{-3} \nabla v + 2\varphi^{4\beta} \nabla \varphi^{4\beta}, \nabla (\psi \varphi^{4\beta} \xi^{2\beta}) \rangle_L \]

\[ - 2\eta \psi \varphi^{4\beta} \xi^{2\beta} \dot{F}^{ij} h_{ik} h_{kj} + 4\Lambda \beta^2 (1 + 4\beta(\theta + 1)) \psi \varphi^{4\beta - 2} F^\beta \xi^{2\beta} \]

\[ + 4\beta(\beta - 1) \varphi^{4\beta - 1} \Phi v^{-1} \psi \xi^{2\beta} + 2\beta \xi^{2\beta - 1} \psi \varphi^{4\beta}. \]

Since cut-off function \( \varphi \) is compactly supported and \( \xi = 0 \) when \( t = 0 \), function \( \psi \varphi^{4\beta} \xi^{2\beta} \) attains its maximum at some point \( (x_0, t_0) \) for \( t_0 > 0 \). Thus the weak parabolic maximum principle implies, by the fact \( \eta \geq \frac{1}{2\beta} \) and Lemma 2.1 (i),

\[ \frac{\beta}{\theta} \psi \varphi^{4\beta} \xi^{2\beta} F^{\beta + 1} \leq 2\eta \psi \varphi^{4\beta} \xi^{2\beta} \dot{F}^{ij} h_{ik} h_{kj} \]

\[ \leq 4\Lambda \beta^2 (1 + 4\beta(\theta + 1)) \psi \varphi^{4\beta - 2} F^\beta \xi^{2\beta} + 2\beta \xi^{2\beta - 1} \psi \varphi^{4\beta} \]

\[ + 4\beta(\beta - 1) \varphi^{4\beta - 1} \Phi v^{-1} \psi \xi^{2\beta}. \] (3.5)

Multiplying by \( \frac{\beta}{\theta} \varphi^{-4\beta + 2} \psi^{-1} \xi^{-2\beta + 1} F^{-\beta} \) and noticing that \( v \geq 1, \xi \leq 1 \) and \( \varphi \leq R \), we have

\[ \varphi^2 \xi F \leq 4\beta \Lambda (1 + 4\beta(\theta + 1)) \theta \xi + 2\theta \varphi^2 F^{-\beta} + 4(\beta - 1) \theta \varphi \xi v^{-1} \]

\[ \leq 4\beta \Lambda (1 + 4\beta(\theta + 1)) \theta + 2\theta \varphi^2 F^{-\beta} + 4(\beta - 1) \theta R \]

\[ = 4\beta \Lambda (1 + 4\beta(\theta + 1)) \theta + 2\theta (\varphi^2 \xi F)^{-\beta} \varphi^{2\beta + 2} + 4(\beta - 1) \theta R \]

\[ \leq 4\beta \Lambda (1 + 4\beta(\theta + 1)) \theta + 2\theta R^2 (\varphi^2 \xi F)^{-\beta} R^{2\beta} + 4(\beta - 1) \theta R. \]

If \( \varphi^2 \xi F \geq R^2 \), above inequality shows

\[ \varphi^2 \xi F \leq 4\beta \Lambda (1 + 4\beta(\theta + 1)) \theta + 2\theta R^2 + 4(\beta - 1) \theta R. \]
Otherwise we can obtain, by \( \theta \geq 1 \),
\[
\varphi^2 \xi F \leq R^2 \leq 4 \beta \Lambda (1 + 4 \beta (\theta + 1)) \theta + 2 \theta R^2 + 4 (\beta - 1) \theta R.
\]
Thus at point \((x_0, t_0)\), the following inequality holds
\[
\varphi^2 \xi F \leq 4 \beta \Lambda (1 + 4 \beta (\theta + 1)) \theta + 2 \theta R^2 + 4 (\beta - 1) \theta R.
\]
Using that \( \frac{1}{2 \eta} \leq \eta \leq 1 \) and \( \psi \varphi^{4 \beta \xi 2 \beta} \) attains its maximum at point \((x_0, t_0)\), we conclude that, for all \((x, t) \in \mathcal{M} \times [0, T]\), the following holds
\[
\frac{1}{2 \theta} (F \varphi^2 \xi)^{2 \beta} \leq \psi \varphi^{4 \beta \xi 2 \beta} = F^{2 \beta} \eta \varphi^{4 \beta \xi 2 \beta} \leq (F \varphi^2 \xi)^{2 \beta} \leq (2 \beta \Lambda (1 + 4 \beta (\theta + 1))) + R^2 + 2 (\beta - 1) R)^{2 \beta} (2 \theta)^{2 \beta}.
\]
From this, the assertion follows. \( \square \)

To establish the existence of smooth complete noncompact solution of flow (1.1), the local estimates for all the derivatives of the second fundamental form are needed. Here we apply the Gauss map parametrization of convex hypersurface, which has been used widely in [8, 9], and write the flow (1.1) as a parabolic equation of the support function which is concave with respect to its arguments.

**Proposition 3.4.** Assume \( \Sigma_0 \) is a complete locally uniformly convex smooth hypersurface in \( \mathbb{R}^{n+1} \), and let \( \Sigma_t \) be a complete strictly convex smooth graph solution of (1.1) defined on \( M^n \times [0, T] \), for some \( T > 0 \). Given a constant \( R > 0 \), for any \( k \geq 0 \) and \( \sigma \in (0, 1) \), we have
\[
\sup_{\{\Sigma_t : u(x, t) \leq \sigma R, t \in [0, T]\}} ||\nabla^k A|| \leq C(n, k, \sigma, \beta, \sup_{\Sigma_0} v, \sup_{\Sigma_0} F, \inf_{\Sigma_0} \lambda_{\min}),
\]
where \( \Sigma_0 = \{\Sigma_t : u(x, 0) \leq R\} \).

**Proof.** By Proposition 3.2, we have principal curvatures are local bounded from below. And the local upper bounds for velocity \( F \) can be obtained by Proposition 3.3, which implies the dual function \( F_* \) is bounded from below by a positive constant. Since \( f_* \) approaches zero on the boundary of positive cone \( \Gamma_+ \), there exits a positive constant \( c \) such that \( \frac{1}{\lambda_i} \geq c \) for all \( i \), that is, all the principal curvatures are local bounded from above. Notice that \( \Sigma_t = \{\Sigma_t : u(x, t) \leq \sigma R\} \) is a closed convex hypersurface. Its support function is given by
\[
S(z, t) = \sup\{(x, z) : x \in \hat{\Sigma}_t, z \in \mathbb{S}^n\},
\]
where \( \hat{\Sigma}_t \) is a convex body enclosed by \( \Sigma_t \). Then hypersurface \( \Sigma_t \) can be given by the embedding (ref. [8])
\[
X(z, t) = S(z, t) z + DS(z, t),
\]
where \( D \) is the gradient with respect to the standard metric \( \sigma_{ij} \) and connection on \( \mathbb{S}^n \). The derivative of this map is given by
\[
\partial_i X = \tau_{ik} \sigma^{kl} \partial_l z,
\]
where \( \tau_{ij} \) has the form
\[
\tau_{ij} = D_i D_j S + \sigma_{ij} S.
\]
In particular the eigenvalues of $\tau_{ij}$ with respect to the metric $\sigma_{ij}$ are the inverses of the principal curvatures, or the principal radii of curvature.

Therefore the solution of (1.1) is given, up to a time dependent tangential diffeomorphism, by solving the following scalar parabolic equation on $\mathbb{S}^n$

$$\partial_t S = -F_*^{-\beta}(\tau_{ij}) \triangleleft G(D^2 S, DS, S, z, t),$$

for the support function $S$. By the local bounds for principal curvatures, we already have local $C^2$ estimates on the support function $S$ and above formula is uniformly parabolic. Straightforward computations give

$$\dot{G}^{ij} = \frac{\partial G}{\partial (D^2_{ij} S)} = \beta F_*^{-\beta-1} \dot{F}^{pq} \frac{\partial \tau_{pq}}{\partial (D^2_{ij} S)} = \beta F_*^{-\beta-1} \dot{F}^{ij},$$

and

$$\ddot{G}^{ij,kl} = -\beta(\beta + 1) F_*^{-\beta-2} \dot{F}^{ij} \dot{F}^{kl} + \beta F_*^{-\beta-1} \ddot{F}^{ij,kl}. \tag{3.6}$$

By the concavity of $F_*$ and (3.6), we have operator $G$ is concave with respect to $D^2 S$. From the local $C^2$ estimates on $S$ in space-time, we can apply the H"older estimates of $[29,37]$ to obtain the $C^{2,\alpha}$ estimate on $S$ and $C^\alpha$ estimate on $\partial_t S$ in space-time. Therefore, by standard parabolic theory, we have all derivatives of the second fundamental form are bounded. \qed

### 4. Existence of Complete Noncompact Solution

Based on the local estimates in Section 3, in this section, we will establish the existence of the complete noncompact solution of (1.1) and the long time existence of solution for special inverse concave curvature function $F = K^{s/n} G^{1-s}$ for any $s \in (0,1]$.

Given a locally uniformly convex hypersurface $\Sigma_0$, which is a graph of function $w$ defined on a convex open domain $\Omega \subset \mathbb{R}^n$, the following proposition shows the existence of complete noncompact solution $\Sigma_t$ of (1.1) on time interval $[0,T)$, where $T$ depends on the given domain $\Omega$.

**Proposition 4.1.** Let $\Sigma_0$ be a complete noncompact and locally uniformly convex hypersurface embedded in $\mathbb{R}^{n+1}$. Suppose $X_0 : M^n \to \mathbb{R}^{n+1}$ is an immersion such that $X_0(M) = \Sigma_0$. If $B_r(x_0) \subset \Omega_0$ for some $r > 0$ and $x_0$, then there exists a solution $X(x,t) : M \times (0,T) \to \mathbb{R}^{n+1}$ of (1.1) for some $T \geq (\beta + 1)^{-1} r^{\beta+1}$ such that, for each $t \in (0,T)$, the image $\Sigma_t = X(M,t)$ is a strictly convex smooth complete graph of function $w(\cdot,t) : \Omega_t \to \mathbb{R}$ defined on a convex open $\Omega_t \subset \Omega_0$, and also $w(\cdot,t)$ and $\Omega_t$ satisfy the conditions of $w_0$ and $\Omega_0$ determined by $\Sigma_0$ in Theorem 2.1.

**Proof.** The proof follows similar arguments as that of the proof of Theorem 5.1 in [16]. For the convenience of readers, we sketch the proof.

We first construct approximating sequence $\Gamma_t^i$. Let $w_0$ and $\Omega_0$ be determined by $\Sigma_0$ in Theorem 2.1, and assume $\inf_{\Omega_0} w_0 = 0$. For each $i \in \mathbb{N}$, let us define the approximate function

$$\tilde{w}_0^i(x) = w_0(x) + 2/i$$
with corresponding graph \( \tilde{\Sigma}_0^i = \{(x, \tilde{w}_0^i(x)) : x \in \Omega_0\} \). Let \( \tilde{\Gamma}_0\) denote the reflection of \( \tilde{\Sigma}_0^i \cap (\mathbb{R}^n \times [0, i]) \) over the \( i \)-level hyperplane, that is,
\[
\tilde{\Gamma}_0 = \{(x, h) \in \mathbb{R}^{n+1} : h \in (\tilde{w}_0^i(x), 2i - \tilde{w}_0^i(x)) , \ x \in \Omega_0, \tilde{w}_0^i(x) \leq i\}.
\]

It follows from the locally uniform convexity of \( \Sigma_0 \) that \( \tilde{\Gamma}_0 \) is a uniformly convex closed hypersurface. Since \( \tilde{\Gamma}_0 \) fails to be smooth at its intersection with the hyperplane \( \mathbb{R}^n \times \{i\} \), we approximate \( \tilde{\Gamma}_0 \) by its \( \frac{1}{i} \)-envelope \( \Gamma_0^i \), which is a uniformly convex closed hypersurface of class \( C^{1,1} \). By Theorem 5 in \([8]\) and approximation arguments, we can obtain that there exists a unique closed convex solution \( \Gamma_t^i \) of (1.1) with initial data \( \Gamma_0^i \) defined for \( t \in (0, T_i) \), where \( T_i \) is the maximal time of existence. In addition, the symmetry of \( \Gamma_t^i \) with respect to the hyperplane \( \mathbb{R}^n \times \{i\} \) can be obtained by the uniqueness of solution. Thus its lower half \( \Sigma_t^i = \Gamma_t^i \cap (\mathbb{R}^n \times [0, i]) \) is a graph of some function \( w^i(\cdot, t) \) defined on a convex set \( \Omega_t^i \), that is,
\[
\Sigma_t^i = \Gamma_t^i \cap (\mathbb{R}^n \times [0, i]) = \{(x, w^i(x, t)) : x \in \Omega_t^i\}.
\]

Let
\[
\Sigma_t = \partial \{ \bigcup_{i \in \mathbb{N}} \text{Conv}(\Gamma_t^i) \}, \quad \Omega_t = \bigcup_{i \in \mathbb{N}} \Omega_t^i, \quad t \in [0, T)
\]
where \( T = \sup_{i \in \mathbb{N}} T_i \).

By Proposition 6.3 in \([15]\), we can regularize \( \Gamma_0^i \) by convolving its support function with some compactly supported mollifiers on \( S^n \). Exactly as in \([16]\), we can prove that the interior estimates in Section 3 hold in \( \Gamma_t^i \) for cut-off function \( \varphi_{\gamma} = (R - u(x, t) - \gamma t)_+ \) with \( R < i \). Therefore, \( \Sigma_t^i \) is a strictly convex smooth graph of function \( w^i(\cdot, t) \) for \( t \in (0, T_i) \).

On the other hand, by the definition of \( \Gamma_0^i \), we have \( \Gamma_0^i \leq \Gamma_{i+1}^i \). The comparison principle gives that
\[
\Gamma_t^i \leq \Gamma_{i+1}^i \leq \Sigma_0,
\]
which implies
\[
w_0(x) \leq w^{i+1}(x, t) \leq w^i(x, t). \tag{4.1}
\]

Thus \( \bigcup_{i \in \mathbb{N}} \text{Conv}(\Gamma_t^i) \) is a convex body and \( \Sigma_t = \partial (\bigcup_{i \in \mathbb{N}} \text{Conv}(\Gamma_t^i)) \) is a complete convex hypersurface embedded in \( \mathbb{R}^{n+1} \). From inequality (4.1), we have
\[
w(x, t) = \lim_{i \to \infty} w^i(x, t), \quad t \in (0, T).
\]

By the same manner as in the proof of Theorem 5.1 in \([16]\), we derive that \( \Sigma_t \) is a complete noncompact strictly convex smooth graph solution. At last, the lower bound on the existence time \( T \) can be achieved by the comparison principle. \( \square \)

**Proof of Theorem 1.1.** It follows from the locally uniformly convexity of \( \Sigma_0 \) that there exists function \( w_0 \) defined on a convex open domain \( \Omega_0 \subset \mathbb{R}^n \) such that \( \Sigma_0 = \text{graph } w_0 \) by Theorem 2.1. As a result, there exist point \( x_0 \) and \( r > 0 \) such that \( B_r(x_0) \subset \Omega_0 \), and the first part of Theorem 1.1 follows by Proposition 4.1.

If initial hypersurface \( \Sigma_0 \) is an entire graph over \( \mathbb{R}^n \), then for any \( r > 0 \), we have \( B_r(x_0) \subset \mathbb{R}^n \). And the long time existence of solution follows by the lower bound on the existence time \( T \geq (\beta + 1)^{-1}r^{-\beta+1} \) in Proposition 4.1. \( \square \)
To prove the long time existence of complete noncompact solution of (1.1) for special inverse concave curvature function $F = K^{s/n} |G|^{1-s}$ in Theorem 1.2, we will construct an appropriate barrier (see Theorem 5.4 in [16]) to guarantee that each $\Sigma_t$ remains as a graph over the same domain $\Omega$ for all $t \in (0, T)$, which implies $T = \infty$ independently from the domain $\Omega$.

**Theorem 4.1.** Suppose curvature function $G$ satisfies Condition 1.1 and $F = K^{s/n} |G|^{1-s}$ for any $s \in (0, 1)$. Let $\Sigma_0$ be a complete noncompact and locally uniformly convex hypersurface embedded in $\mathbb{R}^{n+1}$. Assume that $\Sigma_t = \{(x, w(x, t)) : x \in \Omega_t, t \in (0, T)\}$ is a strictly convex smooth complete graph solution of (1.1) such that $\Omega_t, w(\cdot, t)$ and $\Sigma_t$ satisfy the conditions of $\Omega_0$ and $w_0$ determined by $\Sigma_0$ in Theorem 2.1. Then for any closed ball $B_{R_0}(x_0) \subset \Omega_0$ and any $t_0 \in (0, T)$, we have $B_{R_0}(x_0) \subset \Omega_{t_0}$.

**Proof.** Without loss of generality, we may assume that $x_0 = 0$ and $R_0 < 1$. From $\overline{B_{R_0}(0)} \subset \Omega_0$, it follows that there exists a constant $l_0 \geq 0$ such that $\overline{B_{R_0}(0)} \leq \ell_{l_0}(\Sigma_0) = B_{l_0}(0)$. For given constants $l \geq l_0 + 1$, $\sigma \in (0, 1)$ and $\delta > 0$ sufficiently small such that

$$\delta + 2^{(1-s+s/n)\beta + 2}(1-\sigma)^{\beta} n^{(1-s)\beta} R_0^{-\beta} \delta \leq 2^{n} \left(1 + \frac{\pi^2}{\delta}\right) R_0^{-\beta} \delta \leq \sigma R_0,$$

let us define the function $\phi^{\delta,l} : [l-1, l] \times [0, t_0] \to \mathbb{R}$ by

$$\phi^{\delta,l}(h, t) = R_0 - \delta(h-l)^2 - 2^{(1-s+s/n)\beta + 2}(1-\sigma)^{\beta} n^{(1-s)\beta} R_0^{-\beta} \delta \leq \sigma R_0,$$

We denote the inverse function of $\phi(\cdot, t)$ by $\phi^{-1}(\cdot, t)$, and denote the graph of the rotationally symmetric function $\phi^{-1}(|x|, t)$ by

$$\Psi_t^{\delta,l} = \{(x, h) : |x| = \phi^{\delta,l}(h, t)\}.$$ 

It is easy to see that $\Psi_{t}^{\delta,l} \subset \Sigma_t$. We claim that $\Psi_t^{\delta,l}$ defines a supersolution of (1.1).

In fact, let $\phi^{-1}_r$ and $\phi^{-1}_{rr}$ denote the first and second derivatives of $\phi^{-1}(r, t)$ with respect to $r$. Then the Gauss curvature $K$ and the mean curvature $H$ of $\Psi_t^{\delta,l}$ satisfy the following inequalities respectively

$$K = \frac{|\phi^{-1}_{rr}|^{n-1}}{r^{n-1}(1 + |\phi^{-1}_r|^2)^{\frac{n+1}{2}}},$$

$$\leq \frac{\phi^{-1}_{rr}}{((1-\sigma)R_0)^{n-1}(1 + |\phi^{-1}_r|^2)^{\frac{3}{2}}},$$

$$= \frac{\phi_{hh}}{(1-\sigma)^{n-1}R_0^{-1}(1 + \phi_h^2)^{\frac{3}{2}}},$$

$$\leq 2R_0^{-n}(1-\sigma)^{-n}\delta,$$
and
\begin{align*}
H = & \frac{1}{\sqrt{1 + (\phi_r^{-1})^2}} \left( (n - 1) \frac{\phi_r^{-1}}{r} + \frac{\phi_{rr}^{-1}}{1 + (\phi_r^{-1})^2} \right) \\
= & \frac{1}{\sqrt{1 + \phi_h^2}} \left( (n - 1) \frac{1}{r} - \frac{\phi_{hh}}{1 + \phi_h^2} \right) \\
\leq & \frac{n - 1}{(1 - \sigma)R_0} + 2\delta \leq \left( \frac{n - 1}{1 - \sigma} + 2 \right) R_0^{-1} \\
\leq & \frac{2n}{(1 - \sigma)R_0},
\end{align*}

where \( \delta \leq \sigma R_0 \leq 1/R_0 \) is used in above inequality. Therefore, the following inequality holds
\[ F = K^{s/n}G^{1-s} \leq K^{s/n}H^{1-s} \leq \left( \frac{2^{1/n}\delta^{1/n}}{(1 - \sigma)R_0} \right)^s \left( \frac{2n}{(1 - \sigma)R_0} \right)^{1-s} = \frac{2^{1-s+s/n}n^{1-s}}{(1 - \sigma)R_0^{1+s/n}}. \]

In addition, the gradient function \( v \) of \( \phi^{-1} \) on \( L_{[l-1,l]}(\Psi_{\delta,l}^t) = B_{\phi^{-1}([l-1,l])}(0) \) can be estimated as follows
\[ v = \sqrt{1 + (\phi_r^{-1})^2} = \sqrt{1 + \frac{1}{4\delta^2(h - l)^2}} = \frac{1 + 4\delta^2(h - l)^2}{2\delta(h - l)} \leq \frac{2}{\delta(l - h)}, \]

where \( \delta \leq \sigma R_0 \leq 1 \) is used in the last inequality. On the other hand, by
\[ \phi^{-1}(\phi(h, t), t) = h, \]
we have
\[ \partial_t(\phi^{-1}) = -\phi_r^{-1}\partial_r \phi = \frac{2^{1-s+s/n}n^{1-s}}{(1 - \sigma)R_0^{1-s}} \frac{2\delta}{\delta(l - h)} \geq F^{\beta}v. \]

Thus \( \Psi_{\delta,l}^t \) is a supersolution of (1.1).

Once the supersolution \( \Psi_{\delta,l}^t \) is obtained, we can conclude with the arguments as in Theorem 5.4 in [16] that \( B_{R_0}(x_0) \subset \Omega_{t_0} \) for any \( t_0 \in (0, T) \).

**Proof of Theorem 1.2.** It follows from Remark 2.1 that \( F = K^{s/n}G^{1-s} \) satisfies the Condition 1.1 for any \( s \in (0, 1] \). Then by Theorem 1.1, there exists complete noncompact smooth strictly convex solution \( \Sigma_t \) to (1.1), which remains the graph for \( t \in (0, T) \). By Theorem 4.1, we have \( \Sigma_t \) remains as a graph over the same domain \( \Omega_0 \) for all \( t \in (0, T) \). As a conclusion, \( T = \infty \) and the assertion follows. \( \square \)

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