GRIGORCHUK-GUPTA-SIDKI GROUPS AS A SOURCE FOR BEAUVILLE SURFACES

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Abstract. If $G$ is a Grigorchuk-Gupta-Sidki group defined over a $p$-adic tree, where $p$ is an odd prime, we study the existence of Beauville surfaces associated to the quotients of $G$ by its level stabilizers $st_G(n)$. We prove that if $G$ is periodic then the quotients $G/st_G(n)$ are Beauville groups for every $n \geq 2$ if $p \geq 5$ and $n \geq 3$ if $p = 3$. On the other hand, if $G$ is non-periodic, then none of the quotients $G/st_G(n)$ are Beauville groups.

1. Introduction

Groups acting on regular rooted trees have been widely studied since the 1980’s, when the first Grigorchuk group was defined by Rostislav Grigorchuk [12]. This group was designed to be a counterexample to the General Burnside Problem, so that it is a finitely generated, periodic and infinite group. More importantly, it was the first example of a group having intermediate word growth [13], answering the Milnor Problem [17]. Later on many different examples and generalizations came into the literature. The interest on this kind of groups resides in their odd properties, and because they have been useful to answer unsolved questions.

Some of the examples that came up were the Gupta-Sidki groups [14] and the second Grigorchuk group [12]. The Grigorchuk-Gupta-Sidki groups (GGS-groups for short) are a family of groups generalizing them. These groups act on the regular $p$-adic rooted tree where $p$ is an odd prime. More concretely, each of them is generated by two automorphisms: a rooted automorphism $a$ permuting the vertices hanging from the root according to the permutation $(1 \ 2 \ ... \ p)$, and a recursively defined automorphism $b$ which is defined according to a given vector $e = (e_1,\ldots,e_{p-1}) \in \mathbb{F}_p^{p-1}$. Vovkivsky [20] showed that $G$ is always infinite and it is a periodic group if and only if $\Sigma_{i=1}^{p-1} e_i = 0$, so some of them are also counterexamples for the General Burnside Problem.

A group $G$ acting faithfully on a regular rooted tree $T$ is always residually finite, and a way to analyze the structure of a residually finite group is by looking at its finite quotients. For these groups there is a very natural family of normal subgroups of finite index, which are the level stabilizers $st_G(n)$ for each $n \in \mathbb{N}$. The quotient $G/st_G(n)$ can be naturally seen as

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a subgroup of the group of automorphisms of the subtree $T_n$ consisting of the first $n$ levels, since the kernel of the action of $G$ on $T_n$ is $\text{st}_G(n)$. For the GGS-groups these quotients have been well studied. For instance, in [10] Fernández-Alcober and Zugadi-Reizabal have given the sizes of these quotients, and the profinite completion of each group has been compared to the completion with respect to the level stabilizers by Fernández-Alcober, Garrido and Uria-Albizuri in [8]. The aim of this paper is to determine whether these quotients are Beauville groups or not.

A finite group $G$ is called a Beauville group if it is a 2-generator group and there exists a pair of generating sets $\{x_1, y_1\}$ and $\{x_2, y_2\}$ of $G$ such that $\Sigma(x_1, y_1) \cap \Sigma(x_2, y_2) = 1$, where

$$\Sigma(x_i, y_i) = \bigcup_{g \in G} (\langle x_i \rangle^g \cup \langle y_i \rangle^g \cup \langle x_i y_i \rangle^g),$$

for $i = 1, 2$. Then we say that $\{x_1, y_1\}$ and $\{x_2, y_2\}$ form a Beauville structure for $G$.

Every Beauville group gives rise to a complex surface of general type which is known as a Beauville surface. Roughly speaking, a Beauville surface is a compact complex surface defined by taking a pair of complex curves $C_1$ and $C_2$ and letting a finite group $G$, which is called a Beauville group, act freely on their product to define the surface as the quotient $(C_1 \times C_2)/G$.

Beauville groups have been intensely studied in recent times; see surveys [5, 16]. For example, the abelian Beauville groups were classified by Catanese [4]: a finite abelian group $G$ is a Beauville group if and only if $G \cong C_n \times C_n$ for $n > 1$ with $\gcd(n, 6) = 1$. After abelian groups, the most natural class of finite groups to consider are nilpotent groups. The study of nilpotent Beauville groups is reduced to that of Beauville $p$-groups.

If $p$ is a prime, the state of the art on Beauville $p$-groups can be found in the survey papers [1], [3] and [6]. The smallest non-abelian Beauville $p$-groups were determined by Barker, Boston and Fairbairn in [1]. On the other hand, in [9], Fernández-Alcober and Gül extended Catanese’s criterion in the case of $p$-groups from abelian groups to finite $p$-groups having a ‘nice power structure’, including in particular $p$-groups of class $< p$.

Also in [1], it was shown that there are non-abelian Beauville $p$-groups of order $p^n$ for every $p \geq 5$ and every $n \geq 3$. The first explicit infinite family of Beauville 2-groups was constructed in [2]. Recently in [18], Stix and Vdovina constructed an infinite series of Beauville $p$-groups, for every prime $p$, by considering quotients of ordinary triangle groups. In particular this gives examples of non-abelian Beauville $p$-groups of arbitrarily large order. On the other hand, in [15], Gül showed that quotients by the terms of the lower $p$-central series in either the free group of rank 2 or in the free product of two cyclic groups of order $p$ are Beauville groups. In [4], quotients of the Nottingham group over $\mathbb{F}_p$ have been studied in order to construct more infinite families of Beauville $p$-groups, for an odd prime $p$.

Note that if $G$ is a GGS-group, then the quotients of $G$ by its level stabilizers $\text{st}_G(n)$ are finite $p$-groups generated by two elements of order $p$ but whose exponent can be arbitrarily high. As a consequence, they do not fit into the family of finite $p$-groups having a ‘nice power structure’. For this
reason, they are natural candidates to search for Beauville $p$-groups of a very different type from the ones in [9]. It turns out that the property of being Beauville for these quotients depends on whether $G$ is periodic or not.

The main results of this paper are as follows.

**Theorem A.** Let $G$ be a periodic GGS-group over the $p$-adic tree. Then the quotient $G/\mathrm{st}_G(n)$ is a Beauville group if $p \geq 5$ and $n \geq 2$, or $p = 3$ and $n \geq 3$.

**Theorem B.** Let $G$ be a non-periodic GGS-group over the $p$-adic tree. Then the quotient $G/\mathrm{st}_G(n)$ is not a Beauville group for any $n \geq 1$.

Theorem A shows that a periodic GGS-group is a source for the construction of an infinite series of Beauville $p$-groups. This gives yet another reason why GGS-groups constitute an important family in group theory.

**Notation.** If $G$ is a group, then we denote by $\mathrm{Cl}_G(x)$ the conjugacy class of the element $x \in G$. Also, if $p$ is a prime, then the exponent of a $p$-group $G$, denoted by $\exp G$, is the maximum of the orders of all elements of $G$.

### 2. Definitions and Preliminaries

In this section, we will establish a few properties of GGS-groups that will help us prove the main theorems of this paper. Before proceeding, we recall some facts about automorphisms of rooted trees.

If $d \geq 2$ is an integer and $X = \{1, \ldots, d\}$, the $d$-adic tree $T$ is the rooted tree whose set of vertices is the free monoid $X^*$, where the root corresponds to the empty word $\emptyset$, and a word $u$ is a descendant of $v$ if $u = vx$ for some $x \in X$. The set $L_n$ of all vertices of length $n$ is called the $n$th level of $T$, for every integer $n \geq 0$. If we consider only words of length $\leq n$, then we have a finite tree $T_n$, which we refer to as the tree $T$ truncated at level $n$.

An automorphism of $T$ is a bijection of the vertices that preserves incidence. The group of automorphisms of $T$ is denoted by $\Aut T$. The subgroup $\mathrm{st}(n)$ of $\Aut T$ consisting of the automorphisms that fix pointwise $L_n$ is called the $n$th level stabilizer. More generally, if $G \leq \Aut T$, we define $\mathrm{st}_G(n) = \mathrm{st}(n) \cap G$.

If an automorphism $g$ fixes a vertex $u$, then the restriction of $g$ to the subtree hanging from $u$ induces an automorphism $g_u$ of $T$. In particular, if $g \in \mathrm{st}(1)$ then $g_i$ is defined for every $i = 1, \ldots, d$, and we have an isomorphism

$$\psi: \mathrm{st}(1) \rightarrow \Aut T \times \cdots \times \Aut T$$

$$g \mapsto (g_1, \ldots, g_d).$$

An important automorphism of $T$ is the automorphism that permutes the $d$ subtrees hanging from the root rigidly according to the permutation $(12\ldots d)$. This is called a rooted automorphism and will be denoted by the letter $a$. Since $a$ has order $d$, it makes sense to write $a^k$ for $k \in \mathbb{Z}/d\mathbb{Z}$.

Now, given a non-zero vector $e = (e_1, \ldots, e_{d-1}) \in (\mathbb{Z}/d\mathbb{Z})^{d-1}$, we can define recursively $b \in \mathrm{st}(1)$ via

$$\psi(b) = (a^{e_1}, \ldots, a^{e_{d-1}}, b).$$

Then the subgroup $G = \langle a, b \rangle$ of $\Aut T$ is called the GGS-group corresponding to the defining vector $e$. 

Remark 2.2. For every positive integer $d$, there are only three essentially different defining vectors $e$, and they are $(1, 0), (1, 1)$ and $(1, 2)$. The corresponding groups are called the Fabrykowski-Gupta group, the Bartholdi-Grigorchuk group and the Gupta-Sidki group, respectively. If $d = 4$ then one example of GGS-group is the second Grigorchuk group, where the defining vector is $(1, 0, 1)$.

In the remainder of this paper, we will assume that $d = p$ for an odd prime $p$. Let $G = \langle a, b \rangle$ be a GGS-group with defining vector $e = (e_1, \ldots, e_{p-1})$. Observe that both $a$ and $b$ are of order $p$. For every integer $i$, we write $b_i = b^i$. Notice that $b_i = b_j$ if $i \equiv j \pmod{p}$. The images of the elements $b_i$ under the map $\psi$ can be described as:

$$\psi(b_0) = (a^{e_1}, \ldots, a^{e_{p-1}}, b),$$
$$\psi(b_1) = (b, a^{e_1}, \ldots, a^{e_{p-1}}),$$
$$\vdots$$
$$\psi(b_{p-1}) = (a^{e_2}, a^{e_3}, \ldots, b, a^{e_1}).$$

Here we collect some basic results regarding GGS-groups.

**Proposition 2.1.** [10, Theorem 2.1] If $G = \langle a, b \rangle$ is a GGS-group, then

(i) $\text{st}_G(1) = \langle b \rangle^G = \langle b_0, \ldots, b_{p-1} \rangle$, and $G = \langle a \rangle \rtimes \text{st}_G(1)$.

(ii) $\text{st}_G(2) \leq G' \leq \text{st}_G(1)$.

(iii) $|G : G'| = p^2$ and $|G : \gamma_3(G)| = p^3$.

**Remark 2.2.** For all $k \geq 1$ we have $\psi(\text{st}_G(k)) \subseteq \text{st}_G(k-1) \times \mathbb{F}_p \times \text{st}_G(k-1)$.

**Remark 2.3.** For every positive integer $n$, we can define an isomorphism $\psi_n$ from the stabilizer of the first level in $\text{Aut} T_n$ to the direct product $\text{Aut} T_{n-1} \times \cdots \times \text{Aut} T_{n-1}$, in the same way as $\psi$ is defined. Since $G_n = G / \text{st}_G(n)$ can be seen as a subgroup of $\text{Aut} T_n$, we can consider the restriction of $\psi_n$ to $\text{st}_G(n)$.

Then by the previous remark, we have

$$\psi_n(\text{st}_G(n)) \subseteq \text{st}_{G_{n-1}}(k-1) \times \cdots \times \text{st}_{G_{n-1}}(k-1).$$

For simplicity, throughout this paper we are not going to use bar notation for the elements in the quotient groups $G / \text{st}_G(n)$. However, the subscript $n$ at the map $\psi_n$ means that we are working in the quotient group $G / \text{st}_G(n)$.

If $e = (e_1, \ldots, e_{p-1})$ is the defining vector of a GGS-group, then we write $C(e, 0)$ for the circulant matrix $C(e_1, \ldots, e_{p-1}, 0)$ over $\mathbb{F}_p$.

**Proposition 2.4.** [10, Theorem 2.4] Let $G$ be a GGS-group with defining vector $e$, and put $C = C(e, 0)$. Then

(i) The dimension of $\text{st}_G(1)$ coincides with the rank $t$ of $C$.

(ii) $G_2$ is a $p$-group of maximal class of order $p^{t+1}$.

**Lemma 2.5.** [10, Lemma 2.7] Let $C = C(e_1, \ldots, e_{p-1}, 0)$ be a circulant matrix over $\mathbb{F}_p$. Then

(i) The rank of $C$ is $p - m$, where $m$ is the multiplicity of 1 as a root of the polynomial $f(X) = e_1 + e_2X + \cdots + e_{p-1}X^{p-2}$.

(ii) The rank of $C$ is strictly less than $p$ if and only if $\sum_{i=1}^{p-1} e_i = 0$. 

Proposition 2.6. [10] Theorems 2.13, 2.14] Let $G$ be a GGS-group. Then

(i) $\text{st}_G(1)^{\prime} \leq \text{st}_G(2)$.
(ii) $|G : \text{st}_G(1)^{\prime}| = p^{p+1}$.

We say that the defining vector $e = (e_1, \ldots, e_{p-1})$ of a GGS-group is symmetric if $e_i = e_{p-i}$ for all $i = 1, \ldots, p-1$.

Proposition 2.7. [10] Lemma 3.4, Theorem 3.7] Let $G$ be a GGS-group with non-symmetric defining vector $e$. Then

(i) $\psi(\text{st}_G(1)^{\prime}) = G' \times \ldots \times G'$.
(ii) If $G_n = G/\text{st}_G(n)$ then for every $n \geq 2$

\[
\log_p |G_n| = tp^{n-2} + 1,
\]

where $t$ is the rank of $C(e, 0)$.

3. Quotients of periodic GGS-groups

Let $G$ be a periodic GGS-group with defining vector $e = (e_1, \ldots, e_{p-1})$, where $p$ is an odd prime. In this section, we determine whether the quotients of $G$ by its level stabilizers $\text{st}_G(n)$ are Beauville groups.

We first deal with the quotient group $G_2 = G/\text{st}_G(2)$.

Theorem 3.1. Let $G$ be a periodic GGS-group. Then the quotient $G/\text{st}_G(2)$ is a Beauville group if and only if $p \geq 5$.

Proof. By Proposition 2.4(ii), we know that $G_2$ is a group of maximal class of order $p^{p+1}$, where $t$ is the rank of $C(e, 0)$. Then Lemma 2.5(ii) implies that $|G_2| \leq p^t$, and hence $G_2$ is regular by Lemma 3.13 in [19]. Since $G_2$ is generated by two elements of order $p$, this, together with being regular, implies that $\exp G_2 = p$, by Corollary 2.11 in [7]. Then according to Corollary 2.10 in [9], $G_2$ is a Beauville group if and only if $p \geq 5$.

We next deal with the quotient $G_3 = G/\text{st}_G(3)$. To this purpose, first we need to calculate the orders of $a^{-1}b$ and $ab$ for $1 \leq i \leq p-1$.

Lemma 3.2. Let $G$ be a periodic GGS-group. Then the orders of $a^{-1}b$ and $ab$ for $1 \leq i \leq p-1$ are $p^2$ in both $G$ and $G_3$.

Proof. Let $1 \leq i \leq p-1$. Then we have

\[(ab^i)^p = a^p(b^i)^{p-1} \cdots (b^i)^{a^{p-1}} = b_{p-1} \cdots b_i b_0,
\]

and hence

\[
\psi((ab^i)^p) = (a^{i(e_2+\cdots+e_{p-1})} b_i a^{i(e_3+\cdots+e_{p-1})} b_i a^{i(e_1+e_2)} \cdots, a^{i(e_1+\cdots+e_{p-1})} b_i).
\]

Then being $\Sigma_{i=1}^{p-1} e_i = 0$ implies that

\[
\psi((ab^i)^p) = ((b^i)^{a^{e_1}}, (b^i)^{a^{(e_1+e_2)}}, \ldots, b_i, b_i).
\]

Since each component in equation (3.1) is of order $p$, the order of $ab^i$ is $p^2$ for every $1 \leq i \leq p-1$. Observe that $(ab^i)^p \notin \text{st}_G(3)$, and hence also in $G_3$ the order of $ab^i$ is $p^2$.

On the other hand, since $(a^{-1}b)^{-1} = (ab^{-1})^b = (ab^{p-1})^b$, the order of $a^{-1}b$ is also $p^2$ in both $G$ and $G_3$. \qed
The next lemma shows the relation between the conjugates of $b$ by powers of a modulo $\text{st}_G(2)$.

**Lemma 3.3.** Let $G$ be a GGS-group. If

$$b^{a^i} \equiv b^{a^j} \pmod{\text{st}_G(2)}$$

for $0 \leq i, j \leq p - 1$, then $i = j$.

**Proof.** Suppose, on the contrary, that $i \neq j$. Then $G = \langle a^{j-i}, b \rangle$. The congruence $b^{a^i} \equiv b^{a^j} \pmod{\text{st}_G(2)}$ implies that $[b, a^{j-i}] \in \text{st}_G(2)$, and so $G/\text{st}_G(2)$ is abelian. However, by Lemma 2.5(i), the rank of the circulant matrix $C(e, 0)$ is at least 2. Thus, $G/\text{st}_G(2)$ is a $p$-group of maximal class of order at least $p^3$, and hence it is not abelian. \hfill $\square$

The following proposition is the key result for determining the existence of a Beauville structure for $G_3$.

**Proposition 3.4.** Let $G$ be a periodic GGS-group. If

$$\langle (ab^i)^p \rangle = \langle (ab^j)^p \rangle^g$$

for $1 \leq i, j \leq p - 1$ and $g \in G_3$, then $i = j$.

**Proof.** For simplicity we write $w_i = (ab^i)^p$ for every $1 \leq i \leq p - 1$. Then the equality $\langle w_i \rangle = \langle w_j \rangle^g$ implies that

$$(3.3) \quad w_i^k = w_j^g$$

for some $1 \leq k \leq p - 1$. By (3.1), we have

$$(3.3) \quad \psi_3(w_i) = (\langle b \rangle^{a^{r_1}}, \langle b \rangle^{a^{r_1+r_2}}, \ldots, b, b).$$

By Remark 2.3, each component in the equation (3.3) is an element of $G_2$.

Since $\text{st}_{G_2}(1)$ is abelian, the components of $\psi_3(w_i^g)$ are of the form $\langle b \rangle^{a^{m}}$, with $m \in \{0, \ldots, p - 1\}$. Also one of the components of $\psi_3(w_i^k)$ is $b^{ik}$. Then by equality (3.2), we have $\langle b \rangle^{a^{m}} = b^{ik}$ in $G_2$ for some $m$. Thus, $b^{ik-j} = [b, a^m] \in G_2^*$, and this is true only if $ik-j \equiv 0 \pmod{p}$. Therefore, $k \equiv i^{-1}j \pmod{p}$, and hence we have $w_i^{i^{-1}j} = w_j^g$. If we write $x_i = w_i^{-1}$ for every $1 \leq i \leq p - 1$, then we have

$$x_i = x_j^g.$$

Note that

$$\psi_3(x_i) = \langle b^{\alpha_{r_1}}, b^{\alpha_{r_1+r_2}}, \ldots, b, b \rangle$$

for every $1 \leq i \leq p - 1$.

Let $g = a^s h_s$ for some $0 \leq s \leq p - 1$ and $h_s \in \text{st}_{G_3}(1)$. Observe that

$$\psi_3(x_j^g) = \langle b^{\alpha_{r_1+r_2}}, \ldots, b, b, \ldots, b^{\alpha_{r_1+r_2}} \rangle,$$

where the first $b$ appears at the $(s - 1)$st component. Then the equality $x_i = x_j^g$ implies that

$$\psi_3(h_s) = \langle a^{i_{e_1}-j(e_1+\cdots+e_{p-(s-1)})} u_1, a^{l(e_1+e_2)-j(e_1+\cdots+e_{p-(s-2)})} u_2, \ldots, a^{-j(e_1+\cdots+e_{p-s})} u_p \rangle,$$

for some $u_1, u_2, \ldots, u_p \in \text{st}_G(2)$.
where \( u_\ell \in \text{st}_{G_2}(1) \) for all \( 1 \leq \ell \leq p \). We next define recursively elements \( h_{i-1} = h_0 b_{i-1}^{-j} \) for \( i = s, \ldots, 1 \). Now since \( G \) is periodic, we have \( e_1 + \cdots + e_{p-1} = 0 \) and consequently
\[
\psi_3(h_0) = (a^{(i-j)e_1}v_1, a^{(i-j)(e_1+e_2)}v_2, \ldots, v_{p-1}, v_p),
\]
with \( v_\ell \in \text{st}_{G_2}(1) \) for all \( 1 \leq \ell \leq p \).

Notice that we have
\[
\psi_3(h_0^j) \equiv \psi_3(h_0 b_{i-k}^{-j}) \pmod{\text{st}_{G_2}(1) \times \mathcal{P} \times \text{st}_{G_2}(1)}.
\]
Hence \( \psi_3(b_{i-k}^{-j}[h_0, a]) \in (\text{st}_{G_2}(1) \times \mathcal{P} \times \text{st}_{G_2}(1)) \cap \psi_3(\text{st}_{G_3}(1)) = \psi_3(\text{st}_{G_3}(2)). \)

Thus, \( b_{i-k}^{-j}[h_0, a] \in \text{st}_{G_3}(2) \), and hence \( b_{i-k}^{-j} \in \text{st}_{G_3}(2) \). This implies that \( i = j \).

We are now ready to prove that \( G_3 \) is a Beauville group. We deal separately with the cases \( p \geq 5 \) and \( p = 3 \).

**Theorem 3.5.** Let \( G \) be a periodic GGS-group over the \( p \)-adic tree. If \( p \geq 5 \) then the quotient \( G/\text{st}_G(3) \) is a Beauville group.

**Proof.** Note that \( \{a^{-2}, ab\} \) and \( \{ab^2, b\} \) are both systems of generators of \( G_3 = G/\text{st}_G(3) \). We claim that they yield a Beauville structure for \( G_3 \). If \( X = \{a^{-2}, ab, a^{-1}b\} \) and \( Y = \{ab^2, b, ab^3\} \), we have to see that
\[
\langle x \rangle \cap \langle y^g \rangle = 1
\]
for all \( x \in X \), \( y \in Y \), and \( g \in G_3 \). Observe that \( \langle x\Phi(G_3) \rangle \) and \( \langle y\Phi(G_3) \rangle \) have trivial intersection for every \( x \in X \) and \( y \in Y \), since \( a \) and \( b \) are linearly independent modulo \( \Phi(G_3) \). Thus, \( x \) and \( y^g \) lie in different maximal subgroups of \( G_3 \) in every case.

Assume first that \( x = a^{-2} \), which is an element of order \( p \). If \((3.4)\) does not hold, then \( \langle x \rangle \subseteq \langle y^g \rangle \), and consequently \( \langle x\Phi(G_3) \rangle = \langle y\Phi(G_3) \rangle \), which is a contradiction. The same argument holds if \( y = b \) since it is also of order \( p \).

The remaining elements in \( X \) and \( Y \) are of order \( p^2 \), by Lemma 3.2. Thus, in order to prove our claim, we need to show that
\[
\langle x^p \rangle \neq \langle y^p \rangle^g
\]
for all \( g \in G_3 \), and for those \( x \in X \) and \( y \in Y \). If \( x = ab \) and \( y = ab^2 \) or \( ab^3 \), then we apply Proposition 3.4 to conclude that \((3.5)\) holds.

It remains to deal with \( x = a^{-1}b \) and \( y = ab^2 \) or \( y = ab^3 \). Since \( (a^{-1}b)^{-1} = (ab^{-1})^b \), if \((3.5)\) does not hold, then \( \langle (ab^{-1})^b \rangle = \langle y^p \rangle^g \), that is,
\[
\langle (ab^{-1})^p \rangle = \langle y^p \rangle_{gb^{-1}}
\]
for some \( g \in G_3 \). This contradicts with Proposition 3.4 and hence \((3.5)\) holds. This completes the proof. \( \square \)

Now we assume that \( p = 3 \).

Since proportional nonzero vectors define the same GGS group, if \( G \) is periodic and \( p = 3 \), then the defining vector of \( G \) can only be \( e = (1, -1) \). So \( G \) is the Gupta-Sidki 3-group. In this case, observe that the rank of \( C(e, 0) \) is 2. Then Proposition (2.4(ii)), together with Proposition (2.4(iii)), implies that \( \text{st}_G(2) = \gamma_3(G) \), which is of index \( 3^3 \) in \( G \). By Proposition (2.7(ii)), the
Lemma 3.7. Let \( G = \langle a, b \rangle \) be the Gupta-Sidki 3-group. Then \( Z(G_3) \) is a subgroup of \( \text{st}_{G_3}(1)' \) of order 3. More precisely, if \( Z(G_3) = \langle z \rangle \) then
\[
\psi_3(z) = \langle [a, b], [a, b], [a, b] \rangle.
\]
Proof. First of all, we observe that \( Z(G_3) \leq \text{st}_{G_3}(1) \). Otherwise, if there were \( z \in Z(G_3) \) such that \( z \notin \text{st}_{G_3}(1) \), then \( G_3 = \langle z, b \rangle \) would be an abelian group, which is not true.

Since for every \( n \geq 1 \), \( \text{st}_G(n) \) is a subdirect product of \( 3^n \) copies of \( G \), this, together with
\[
\psi_3(\text{st}_G(n)) \subseteq G_2 \times G_2 \times G_2,
\]
implies that
\[
\psi_3(Z(G_3)) \subseteq Z(G_2) \times Z(G_2) \times Z(G_2) = G_2' \times G_2' \times G_2'.
\]
On the other hand, by Proposition 2.7 (i), we have
\[
\psi_3(\text{st}_{G_3}(1)') = G_2' \times G_2' \times G_2'.
\]
Thus, by (3.7) and (3.8), we have \( \psi_3(Z(G_3)) \leq \text{st}_{G_3}(1)' \). Let \( z \in Z(G_3) \). Since \( z = z^3 \), this yields that \( \psi_3(z) = (c, c, c) \) for some \( c \in G_2' = \langle [a, b] \rangle \). Hence \( |Z(G_3)| = 3 \) and \( Z(G_3) = \langle [a, b], [a, b], [a, b] \rangle \). \( \square \)

Lemma 3.7. Let \( G = \langle a, b \rangle \) be the Gupta-Sidki 3-group. Then
\[
Z(G_3) \cap \{ [b, g] \mid g \in G_3 \} = 1.
\]
Proof. Suppose that \( 1 \neq [b, g] \in Z(G_3) \). Then since \( Z(G_3) \leq \text{st}_{G_3}(1)' \), \( b \) and \( g \) commute modulo \( \text{st}_{G_3}(1)' \). We will first show that \( g \in \text{st}_{G_3}(1) \).

Since \( |G_3/\text{st}_{G_3}(1)' : \gamma_2(G_3/\text{st}_{G_3}(1)')| = 3^2 \) and \( \text{st}_{G_3}(1)/\text{st}_{G_3}(1)' \) is an abelian maximal subgroup of \( G_3/\text{st}_{G_3}(1)' \), the quotient group \( G_3/\text{st}_{G_3}(1)' \) is a 3-group of maximal class with an abelian maximal subgroup. Then being \( b \in \text{st}_{G_3}(1) \) yields that \( g \in \text{st}_{G_3}(1) \).

If \( g \in \text{st}_{G_3}(1)' \) then by (3.8), \( \psi_3(g) \in Z(G_2 \times G_2 \times G_2) \). This implies that \( \psi_3([b, g]) = (1, 1, 1) \), which is a contradiction. Hence \( g \in \text{st}_{G_3}(1) \setminus \text{st}_{G_3}(1)' \).

Write \( g = b_0^a b_1^a b_2^a c \) for some \( c \in \text{st}_{G_3}(1)' \). Then
\[
\psi_3([b, g]) = \left( [a, a^{10} b^i_1 a^{2i_2}], [a^2, a^{20+i_1} b^{i_2}], [b, b^{10} a^{2i_1+i_2}] \right) = \left( [a, a^{10} b^i_1 a^{2i_2}], b_2^{i_2} b_0^{i_1} b_1^{i_2}, b_0^{-1} b_2^{i_2} \right).
\]
Since \( [b, g] \in Z(G_3) \), it follows that \( \psi_3([b, g]) = ([a, b], [a, b], [a, b]) \). Note that in \( G_2 \), we have \( b_0 b_1 b_2 = 1 \), and thus \( [a, b] = b_1^{-1} b_0 = b_2^{-1} b_0 \). Then by the second and third components, we get \( i_1 = 0 \). Then the first component will be 1, which is a contradiction. \( \square \)

Lemma 3.8. Let \( G = \langle a, b \rangle \) be the Gupta-Sidki 3-group. Then the element \( v \in \text{st}_{G_3}(1)' \) such that \( \psi_3(v) = ([a, b], 1, 1) \) is not in the set \( \{ [a, g] \mid g \in G_3 \} \).

Proof. Since \( \psi_3(\text{st}_{G_3}(1)') = G_2' \times G_2' \times G_2' \), such an element \( v \) exists in \( \text{st}_{G_3}(1)' \). Suppose that \( v = [a, g] \) for some \( g \in G_3 \). If we write \( g = a^h \) for some \( h \in \text{st}_{G_3}(1) \) then \( v = [a, h] \). Write \( \psi_3(h) = (h_1, h_2, h_3) \). Then
\[
\psi_3((h^{-1})^a h) = (h_1^{-1} h_1, h_1^{-1} h_2, h_1^{-1} h_3) = ([a, b], 1, 1).
\]
This implies that $h_1 = h_2 = h_3$ in $G_2$. Then $[a, b] = h_3^{-1}h_1 = 1$ in $G_2$, which is a contradiction. Thus, $v \in \text{st}_{G_3}(1)^{t} \setminus \{[a, g] \mid g \in G_3\}$. \hfill \Box

In order to deal with the prime 3, we also need the following lemma.

**Lemma 3.9.** [Lemma 3.8] Let $G$ be a finite $p$-group and let $x \in G \setminus \Phi(G)$ be an element of order $p$. If $t \in \Phi(G) \setminus \{[x, g] \mid g \in G\}$ then

$$\left(\bigcup_{g \in G} \langle x \rangle^g\right) \cap \left(\bigcup_{g \in G} \langle xt \rangle^g\right) = 1.$$

**Theorem 3.10.** Let $G$ be the Gupta-Sidki 3-group. Then the quotient $G/\text{st}_G(3)$ is a Beauville group.

**Proof.** Let $1 \neq u \in Z(G_3)$ and let $v \in \text{st}_{G_3}(1)^t$ be such that $\psi_3(v) = ([a, b], 1, 1)$. We claim that $\{a, b\}$ and $\{av, b^2u\}$ form a Beauville structure for $G_3$. Let $X = \{a, b, ab\}$ and $Y = \{av, b^2u, avb^2u\}$.

If $x = a$, which is of order 3, and $y = b^2u$ or $avb^2u$ then $\langle x \rangle \cap \langle y \rangle^g = 1$ for all $g \in G_3$, as in the proof of Theorem 3.8. When $x = b$ and $y = av$ or $avb^2u$, the same argument applies. If we are in one of the following cases: $x = a$ and $y = av$, or $x = b$ and $y = b^2u = (bu)^2$, then the condition $\langle x \rangle \cap \langle y \rangle^g = 1$ follows from Lemma 3.9.

It remains to check the case $x = ab$ and $y \in Y$. If $y = b^2u$, which is of order 3, then we have $\langle x \rangle \cap \langle y \rangle^g = 1$, as in the previous paragraph. Now assume that $y = av$. Since $(av)^3 = v^{a^2}v^a v$, we have

$$\psi_3((av)^3) = ([a, b], [a, b], [a, b]).$$

By Lemma 3.6, $(av)^3 \in Z(G_3)$. On the other hand,

$$\psi_3((ab)^3) = (b^a, b, b),$$

and hence $(ab)^3 \notin Z(G_3)$. Thus, the condition $\langle x \rangle \cap \langle y \rangle^g = 1$ follows.

Finally, we have to take $x = ab$ and $y = avb^2u$. Since $v \in \text{st}_{G_3}(1)^t$ and $\text{st}_{G_3}(1)$ is of nilpotency class 2, $v \in Z(\text{st}_{G_3}(1))$. So, $y = ab^2vu$, and $y^3 = (ab^2v)^3$. Observe that

$$(ab^2v)^3 = b_2b_1b_0b_0v^{a^2}v^av.$$ 

By taking into account that $b_0b_1b_2 = 1$ in $G_2$, we get

$$\psi_3(y^3) = (b^{-1}, (b^{-1})^a, (b^{-1})^a).$$

If $\langle (ab)^3 \rangle = \langle y^3 \rangle^g$ for some $g \in G_3$, then $(ab)^3 = (y^3)^g$ for $i = 1$ or $-1$. Since $\text{st}_{G_2}(1)$ is abelian, the components of $\psi_3((y^3)^g)$ are of the form $(b^{-i})^m$, with $m \in \{0, 1, 2\}$. Also one of the components of $\psi_3((ab)^3)$ is $b$. Then by equality $(ab)^3 = (y^3)^g$, we have $(b^{-i})^m = b$ in $G_2$ for some $m$. Thus, $[a^m, b^i] = b^{1+i} \in G_2$, and this is true only if $1 + i \equiv 0 \pmod{3}$. Thus

$$(ab)^3 = (y^{-3})^g.$$ 

Note that

$$\psi_3((ab)^3) = (b_1, b_0, b_0) \quad \text{and} \quad \psi_3(y^{-3}) = (b_0, b_1, b_1).$$

We write $g = a^i h$ for some $h \in \text{st}_{G_3}(1)$ and $0 \leq i \leq 2$. Then the equality $(ab)^3 = (y^{-3})^g$ and (3.9) imply that $\psi_3(h)$ has to be congruent to one of the following modulo $\text{st}_{G_2}(1) \times \text{st}_{G_2}(1) \times \text{st}_{G_2}(1)$: $(a, a^2, a^3), (1, 1, a^2)$ or $(1, a, a^2)$.
(1, a², 1). However, since G is periodic, the product of the powers of a in the components of ψ₃(h) has to be 1. Thus, we conclude that there is no such h in G₃, and therefore \((ab)^3 \neq (y^3)^3\) for any \(g \in G₃\). This completes the proof. \(\square\)

The following result, which gives a sufficient condition to lift a Beauville structure from a quotient group, is Lemma 4.2 in [11].

**Lemma 3.11.** Let G be a finite group and let \(\{x₁, y₁\}\) and \(\{x₂, y₂\}\) be two sets of generators of G. Assume that, for a given \(N \leq G\), the following hold:

(i) \(\{x₁N, y₁N\}\) and \(\{x₂N, y₂N\}\) form a Beauville structure for \(G/N\).

(ii) \(o(g) = o(gN)\) for every \(g \in \{x₁, y₁, x₁y₁\}\).

Then \(\{x₁, y₁\}\) and \(\{x₂, y₂\}\) form a Beauville structure for G.

We are now ready to give the main result of this section.

**Theorem 3.12.** Let G be a periodic GGS-group over the p-adic tree. Then the quotient \(G/\text{st}_G(n)\) is a Beauville group if \(p \geq 5\) and \(n \geq 2\), or \(p = 3\) and \(n \geq 3\).

**Proof.** By Theorem 3.11, \(G/\text{st}_G(2)\) is a Beauville group if and only if \(p \geq 5\). Now assume that \(n \geq 3\). If \(p \geq 5\) then by Theorem 3.10, \(G/\text{st}_G(3)\) is a Beauville group with Beauville triples \(X = \{a⁻², ab, a⁻¹b\}\) and \(Y = \{ab², b, a³b\}\). Since \(o(ab\text{st}_G(n)) = o(a⁻¹b\text{st}_G(n)) = p²\) for any \(n \geq 3\), we can apply Lemma 3.11 and hence \(G/\text{st}_G(n)\) is a Beauville group for every \(n \geq 3\). Similarly, if \(p = 3\) then the Beauville structure of \(G/\text{st}_G(3)\) given in Theorem 3.10 is inherited by \(G/\text{st}_G(n)\) for any \(n \geq 3\). \(\square\)

4. Quotients of non-periodic GGS-groups

Let G be a non-periodic GGS-group with defining vector \(e = (e₁, \ldots, eₚ₋₁)\). In this section, we will prove that the quotients of G by its level stabilizers \(\text{st}_G(n)\) are not Beauville groups.

Since G is non-periodic, we have \(\Sigma_{i=1}^{p₋₁} e_i = α \neq 0\). Thus, by Lemma 2.4, the rank of the circulant matrix \(C(e, 0)\) is \(p\), and according to Proposition 2.2, \(G/\text{st}_G(2)\) is a \(p\)-group of maximal class of order \(p^{p+1}\). Since \(G/\text{st}_G(2) \cong \text{Aut} T/\text{st}(2) \cong C_p \wr C_p\), this implies that \(G/\text{st}_G(2) \cong C_p \wr C_p\), and thus it is of exponent \(p^2\).

Note that the \(p+1\) maximal subgroups of G are \(\langle a, G' \rangle\), \(\langle b, G' \rangle\) and \(M_i = \langle ab^i, G'_i \rangle\) for all \(1 \leq i \leq p - 1\). We write \(M_{n,i}\) instead of \(M_i/\text{st}_G(n)\), for every \(1 \leq i \leq p - 1\) and for \(n \geq 2\). Then \(M_{n,i} = \langle ab^i, G'_n \rangle\) is a maximal subgroup of \(G_n = G/\text{st}_G(n)\).

The following proposition gives the relation between the \(p^2\) powers of elements in \(M_{n,i} \setminus G'_n\) for all \(1 \leq i \leq p - 1\).

**Proposition 4.1.** Let \(G_n = G/\text{st}_G(n)\) for \(n \geq 2\). Then the following hold:

(i) All elements in \(M_{n,i} \setminus G'_n\) are of order \(p^n\) for every \(1 \leq i \leq p - 1\).

(ii) If \(g \in M_{n,i} \setminus G'_n\) is an element such that \(g = (ab^i)^k w\) for some \(w \in G'_n\) and for some \(1 \leq k \leq p - 1\), then \(g^{p^{n-1}} = (ab^i)^{kp^{n-1}}\).
(iii) Cyclic subgroups generated by \( p^{n-1} \)st powers of elements in \( \bigcup_{i=1}^{p-1} M_{n,i} \setminus G_n \) coincide.

Proof. We will first show the result for \( n = 2 \). Since \( G_2 \cong C_p \times C_p \), all elements in \( M_{2,i} \setminus G_2 \) are of order \( p^2 \) for \( 1 \leq i \leq p-1 \), and hence (i) holds. We know that for any element \( u \in M_{2,i} \setminus G_2 \), we have \( C_{\ell}\{u\} = uG'_2 \) and \( u^p \in Z(G_2) \). Thus, if \( g = (ab)^k \) for some \( w \in G_2 \) then \( g \) and \( (ab)^k \) are conjugate in \( G_2 \). Since \( (ab)^{np} \in Z(G_2) \), this implies that in \( G_2 \)
\[ g^p = (ab)^{kp} \]
and so (ii) holds. It remains to show that (iii) holds. Since all elements in \( \bigcup_{i=1}^{p-1} M_{2,i} \setminus G_2 \) are of order \( p^2 \), cyclic subgroups generated by their \( p \)th powers are equal to \( Z(G_2) \).

In order to prove the proposition we will use induction on \( n \). Before proceeding to the induction step, consider the element \( ab^i \) of \( G \) for \( 1 \leq i \leq p-1 \). We know that \( (ab)^p = b^{p-1} b_{p-2} \cdots b_0 \). Then the condition \( \Sigma_{i=1}^{p-1} e_i = \alpha \) implies that
\[ \psi((ab)^p) = (a^{(b^i) \alpha e_1}, a^{(b^i) \alpha (e_1 + e_2)}, \ldots, b^{i \alpha}, a^{i \alpha b}). \]
Then
\[ \psi((ab)^{kp}) = ((a^\alpha b^{ki}) u_1, (a^\alpha b^{ki}) u_2, \ldots, (a^\alpha b^{ki}) u_p), \]
for \( u_i \in G' \). Let \( g = (ab)^{kp} \) for some \( w \in G' \). By the previous paragraph we know that
\[ g^p = (ab)^{kp} \pmod{st_G(2)}. \]
Write \( g^p = (ab)^{kp} \) for some \( t \in st_G(2) \). By Proposition 2.3, we have \( st_G(2) = [st_G(1), st_G(1)] \), and hence \( \psi(t) \in G' \times \ldots \times G' \). Thus
\[ \psi(g^p) = ((a^\alpha b^{ki} w_1, (a^\alpha b^{ki} w_2, \ldots, (a^\alpha b^{ki} w_p), \]
where \( w_p \in G' \).

Now assume that the proposition holds for \( n \geq 2 \). Then
\[ \psi_{n+1}((ab)^{kp^n}) = (((a^\alpha b^{ki} u_1)^{p^{n-1}}, ((a^\alpha b^{ki} u_2)^{p^{n-1}}, \ldots, ((a^\alpha b^{ki} u_p)^{p^{n-1}}), \]
\[ \psi_{n+1}(g^{p^n}) = (((a^\alpha b^{ki} w_1)^{p^{n-1}}, ((a^\alpha b^{ki} w_2)^{p^{n-1}}, \ldots, ((a^\alpha b^{ki} w_p)^{p^{n-1}}), \]
where each component is an element of \( G_n \). By the induction hypothesis, all components are equal in \( G_n \), and of order \( p \). This completes the proof. \( \square \)

**Theorem 4.2.** Let \( G \) be a non-periodic GGS-group over the \( p \)-adic tree. Then the quotient \( G/\text{st}_G(n) \) is not a Beauville group for any \( n \geq 1 \).

**Proof.** Let \( G_n = G/\text{st}_G(n) \). Clearly, \( G_1 \) is not a Beauville group since it is cyclic of order \( p \). Thus, we assume that \( n \geq 2 \).

We argue by way of contradiction. Suppose \( \{x_1, y_1\} \) and \( \{x_2, y_2\} \) are two systems of generators of \( G_n \) such that \( \Sigma(x_1, y_1) \cap \Sigma(x_2, y_2) = 1 \). Since no two of the elements \( x_1 \), \( y_1 \) and \( x_1 y_1 \) can lie in the same maximal subgroup of \( G_n \), it follows from Proposition 4.1(i) that one of these elements, say \( x_1 \), is of order \( p^n \). Similarly, we may assume that the order of \( x_2 \) is also \( p^n \). Then by Proposition 4.1(iii), we conclude that \( \langle x_2^{p^n} \rangle = \langle x_2^{p^n} \rangle \), which is a contradiction. \( \square \)
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