ADE functional dilogarithm identities
and integrable models

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Abstract

We describe a new infinite family of multi-parameter functional equations for the Rogers dilogarithm, generalizing Abel’s and Euler’s formulas. They are suggested by the Thermodynamic Bethe Ansatz approach to the renormalization group flow of 2D integrable, ADE-related quantum field theories. The known sum rules for the central charge of critical fixed points can be obtained as special cases of these. We conjecture that similar functional identities can be constructed for any rational integrable quantum field theory with factorized S-matrix and support it with extensive numerical checks.

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1 Introduction

The Rogers’ dilogarithm function [1, 2] has many intriguing properties with important implications in many branches of mathematics and physics. Recently, in the context of integrable two-dimensional quantum field theories, it has been observed that the effective central charge $\tilde{c}$ of the conformal field theory describing ultraviolet (UV) or infrared (IR) fixed points of the renormalization group flow can be expressed through the Rogers dilogarithm $L(x)$ evaluated at certain algebraic numbers $x_i$ leading to sum-rules of the type [3]-[18]

$$\sum_i L(x_i) = \frac{\pi^2}{6} \tilde{c} \quad (1.1)$$

In this way it has been found, besides known identities, a large class of new sum-rules. Many of them are, strictly speaking, only conjectures, even if they are strongly supported by extensive numerical checks. Some of them have also been proven using character identities of Rogers-Ramanujan type [18] or other analytic means [4].

The aim of this paper is to show that these identities are special cases of new functional equations, written in Eqs (2.13), (2.14) and (2.15), where the $x_i$ are rational functions of a set of variables. In the ADE-related models the number of independent variables can be expressed in terms of the rank of the corresponding Lie algebras.

Before attempting to build up these new identities it is helpful to briefly describe the framework where these identities arise and their physical motivations. A formal proof is postponed in the next section.

The RG evolution of a two-dimensional integrable theory is described, in the Thermodynamic Bethe Ansatz approach (TBA), by the ground state energy $E(R)$ of the system on an infinitely long cylinder of radius $R$. The equations known (or conjectured) to give $E(R)$ are of the form

$$E(R) = -\frac{1}{2\pi} \sum_{a=0}^{a=N} \int_{-\infty}^{\infty} d\theta \, \nu_a(\theta) \log (1 + Y_a(\theta)) \quad , \quad (1.2)$$

where the $Y_a(\theta)$ are $R$ dependent functions determined by a set of coupled integral equations known as TBA equations; the $\nu_a(\theta)$ are suitable elementary functions (usually $\nu_a(\theta) = c_a \exp(\pm \theta)$ or $\nu_a = c_a \cosh(\theta)$) describing the asymptotic behaviour of the solutions: $Y_a(\theta) \to \nu_a(\theta)$ for $\theta \to \pm \infty$. 

1
A crucial observation of ref. [11] was that any solution \( \{Y_a(\theta)\} \) of the TBA equations satisfies a set of simple functional algebraic equations, called the Y-system. Conversely it is easy to show that a set of entire functions satisfying the Y-system with a suitable asymptotic behaviour is a solution of the TBA equations, thus the Y-system encodes all the dynamical properties of the model.

In [6, 10, 11, 16, 17] a large class of TBA systems classified according to the ADET Dynkin’s diagrams was proposed to describe integrable perturbed coset theories [19, 20, 21] like the \( G_k \times G_l \) model with \( G \) a group associated to one of the ADE simply-laced algebra, the \( \frac{G_{k+1}}{U(1)} \) generalized parafermionic models, the \( SU(n)_{2k}, \frac{SO(2n)_{k}}{SO(2n)_{2k}}, \frac{(E_6)_k}{SU(8)_k}, \frac{(E_7)_k}{SO(10)_k} \) perturbed theories, particular points of the fractional super-sine-Gordon models etc. [29, 30]. The Y-systems associated to all these models can be written in terms of an ordered pair \( G \times H \) of ADET Dynkin diagrams in the following form

\[
Y^b_a\left(\theta + \frac{i\pi}{\tilde{g}}\right) Y^b_a\left(\theta - \frac{i\pi}{\tilde{g}}\right) = \prod_{c=1}^{r_G} \left(1 + Y^b_c(\theta)\right)^{G_{ac}} \prod_{d=1}^{r_H} \left(1 + \frac{1}{Y^d_a(\theta)}\right)^{-H_{bd}} \quad (1.3)
\]

where \( G_{ac} \) and \( H_{bd} \) are the adjacency matrices of the corresponding ADET Dynkin diagram, \( \tilde{g} \) is the dual Coxeter number of \( G \), \( r_G \) and \( r_H \) are the ranks of the corresponding algebras. The functional equations (1.3) are universal in the sense that using different IR boundary conditions, they describe different theories or different regimes of the same theory.

These algebraic equations may be used as recursion relations: starting from a set of arbitrary values assigned to the functions \( Y^b_a \) at \( \theta \) and \( \theta - \frac{i\pi}{\tilde{g}} \) it is possible to evaluate the same functions at \( \theta + \frac{i\pi}{\tilde{g}} \).

The main property of these recursion relations is that they generate periodic functions. More precisely, denoting with \( \tilde{h} \) the dual Coxeter number of \( H \), for whatever choice of the initial values of \( Y^b_a \) one can verify by direct successive substitutions or, in the high rank cases, by numerical computations that [11, 22]

\[
Y^b_a\left(\theta + i\pi - \frac{\tilde{h} + \tilde{g}}{\tilde{g}}\right) = Y^{\tilde{b}}_{\tilde{a}}(\theta) \quad , \quad (1.4)
\]

where \( \tilde{a} \) and \( \tilde{b} \) denote the nodes of the Dynkin diagram conjugate to \( a \) and \( b \); for instance, in the \( A_n \) diagrams one has \( \tilde{a} = n + 1 - a \). This periodicity has
many important consequences and is in relation with the conformal dimension of the perturbing operator in the UV region \[1\].

Near the fixed points \((R \to 0 \text{ or } R \to \infty)\) \(E(R)\) tends to the asymptotic form \[23\]

\[ E(R) \sim -\frac{\pi \tilde{c}}{6R} , \quad (1.5) \]

where \(\tilde{c}\) is the aforesaid central charge, and the \(Y\)'s approach constants \(Y^b_a(\theta) \to y^b_a\) which can be determined, for consistency, by the algebraic equations obtained from the \(Y\)-system dropping out the \(\theta\) dependence. Combining Eq.s \((1.2)\) and \((1.5)\) with the \(\theta\) independent form of the \(Y\)-system, one gets \[2, 6, 16, 17\]

\[
\begin{align*}
\sum_{a=1}^{r_G} \sum_{b=1}^{r_H} L \left( \frac{y^b_a}{1 + y^b_a} \right) &= \frac{\pi^2 r_G r_H \tilde{g}}{6 \tilde{h} + \tilde{g}} \\
\end{align*}
\]

\[ (1.6) \]

This equation represents a wide class of identities of the type \((1.4)\). In the next section we shall describe a functional extension of them.

We conclude this section outlining the physical motivations suggesting the existence of such a generalization. It can be understood in a heuristic way in two steps.

First, note that the effective central charge \(\tilde{c}\) of the Casimir energy of Eq.\((1.3)\) measures in some sense the number of massless degrees of freedom of the theory \[3, 24, 25\]. As a consequence, if such a theory flows to a non-trivial IR limit, \(\tilde{c}_{IR}\) is a RG invariant quantity, because these degrees of freedom are not washed away by the process of renormalization.

In order to take a step forward, note that this RG invariant should be expressed in terms of the \(Y_a(\theta)'\)s functions, as they encode the dynamical properties of the system everywhere in \(R\). Note also that the RG trajectory connecting the UV to the IR fixed points is by no means unique. Instead of connecting these varying the radius of a regular cylinder, we may interpolate them with any one-parameter family of surfaces describing a homotopic deformation of the UV cylinder into the IR one. It is conceivable that some of these trajectories correspond to integrable flows. We assume, as it seems reasonable, that these are described by the same \(Y\)-system with a modified asymptotic behaviour, which becomes a function of the trajectory. On the other hand, the RG invariant \(\tilde{c}_{IR}\) should not depend on the choice of the trajectory, of course. We are then faced to an apparent dilemma: deforming
a RG trajectory produces a modification of the $Y_a(\theta)$'s keeping invariant $\tilde{c}_{IR}$, which is a function of them. A possible way out is to guess that $\tilde{c}_{IR}$ is given by a sort of topological invariant of the Y-system: its value should be determined by the fact that the $Y_a(\theta)$'s satisfy the recursion relations of Eq.(1.3) and not by the analytic properties of them.

Then one expects a generalization of Eq.(1.6) where the constants $y_a$ are suitably replaced by an arbitrary set of solutions $Y_a(\theta)$ of the Y-system.

## 2 New functional dilogarithm identities

The Rogers dilogarithm $L(x)$ with $0 \leq x \leq 1$ is the unique function that is three times differentiable and satisfies the following five term relationship known also as the Abel functional equation (see for instance ref. [2])

$$L(x) + L(1 - xy) + L\left(\frac{1 - y}{1 - xy}\right) + L\left(\frac{1 - x}{1 - xy}\right) = 3 L(1),$$

with $0 \leq x, y \leq 1$ and the normalization $L(1) = \pi^2/6$.

To begin with, we observe that this relation has an hidden pentagonal symmetry. Indeed, denoting the five arguments of $L$ in the same order as they appear in Eq.(2.7) by $a_n$ with $n = 0, 1 \ldots 4$, one can verify at once that they satisfy for arbitrary values of $x$ and $y$ the following recursion relation

$$a_{n-1}a_{n+1} = 1 - a_n,$$

which has an intriguing $\mathbb{Z}_5$ symmetry:

$$a_{n+5} = a_n.$$

One can easily recognize a similarity between such a periodic recursion relation and the Y-systems of Eq.(1.3). Such a similarity can be made even more strict by reshuffling the variables in the following way: taking $b_1 = a_3$, $b_2 = a_1$, $b_3 = a_4$, $b_4 = a_2$, $b_5 = a_5$ and putting $b_n = \frac{Y_n}{1 + Y_n}$ we get the following set of periodic recursion relations

$$Y_{n-1}Y_{n+1} = 1 + Y_n.$$
where one gets again $\mathcal{Y}_{n+5} = \mathcal{Y}_n$. Actually Eq. (2.10) is a slightly disguised form of the $A_1 \times A_2$ $Y$-system describing the RG flow of the tricritical Ising fixed point to the critical Ising one. Indeed, putting $i(n) = 2 + (-1)^n$, comparison with Eq. (1.3) yields
\[
\mathcal{Y}_n = 1/Y_1^{i(n)} \left( \theta + \frac{n \pi}{2} \right) , \quad (2.11)
\]
Showing that the five term functional relation may be viewed as a dilogarithm identity satisfied by an arbitrary solution of the $A_1 \times A_2$ $Y$-system. Thus one is naturally led to conjecture that a similar property should hold also for the other $Y$-systems. In order to write explicitly this new family of identities it is useful to simplify the notation by taking
\[
\mathcal{Y}_b^a(n) = Y_b^a \left( \theta + \frac{n \pi}{\tilde{g}} \right) , \quad n = 0, 1, 2, \ldots \quad (2.12)
\]
Then the recursive relations of the $Y$-system of Eq. (1.3) can be written as
\[
\mathcal{Y}_b^a(n+1)\mathcal{Y}_a^b(n-1) = \prod_{c=1}^{r_G} (1 + \mathcal{Y}_c^b(n))^{G_a} \prod_{d=1}^{r_H} \left( 1 + \frac{1}{\mathcal{Y}_d^a(n)} \right)^{-H_b^d} . \quad (2.13)
\]
According to our conjecture, any solution of such recursive equations must satisfy the following identity
\[
\sum_{a=1}^{r_G} \sum_{b=1}^{r_H} \sum_{n=0}^{\tilde{h}+\tilde{g}-1} H(\mathcal{Y}_a^b(n)) = r_G r_H \tilde{g} \quad (2.14)
\]
where we introduced for our notational convenience the function
\[
H(x) = \frac{6}{\pi^2} L \left( \frac{x}{1 + x} \right) . \quad (2.15)
\]
Before attempting to prove these new identities it is useful to note that they are multi-parameter generalizations of known identities, indeed putting the $n$-independent solution of Eq. (2.13) into Eq. (2.14) one gets at once Eq. (1.6). Notice also that the $Y$-systems belonging to the subset of those lacking of the tadpole diagrams of the type $T_n$ can be split in a pair of independent algebraic systems. As a consequence the identity (2.14) is split accordingly in a pair of equivalent identities, of course. For instance in a $A_l \times A_m$ $Y$-system
we can constrain the triple sum of Eq. (2.14) to the subset in which \( a + b + n \) is an even number; consistently we have to divide by two the right-hand side of the equation. In particular the simplest identity, associated to the \( A_1 \times A_1 \) system, gives

\[
H(x) + H\left(\frac{1}{x}\right) = 1 ,
\]

which is nothing but the Euler identity \( L(y) + L(1 - y) = L(1) \). The Abel equation can be written within these notations as

\[
\sum_{n=0}^{4} H(\gamma_n) = 3 .
\]

The simplest new identity of the type (2.14) is associated to the \( Y \)-system \( A_1 \times A_3 \). In order to prove it, let us indicate for convenience the \( \gamma(n) \) associated to the central node of \( A_3 \) by \( Z_n \) and those associated to the other two by \( X_n \) and \( \bar{X}_n \). Then we can write the following periodic recursive equations with underlying \( \mathcal{Z}_6 \) symmetry

\[
\begin{align*}
X_n X_{n+2} &= 1 + Z_{n+1} \\
\bar{X}_n \bar{X}_{n+2} &= 1 + Z_{n+1} \\
Z_{n-1} Z_{n+1} &= (1 + X_n)(1 + \bar{X}_n),
\end{align*}
\]

(2.18)

where the mentioned splitting between odd and even variables is made evident. One can also verify at once that \( X_n = \bar{X}_{n+6} \) and \( Z_n = Z_{n+6} \).

The corresponding identity can be written as

\[
\sum_{m=0}^{2} \left( H(X_{2m}) + H(Z_{2m+1}) + H(\bar{X}_{2m}) \right) = 6 ,
\]

(2.19)

or in more explicit way, solving the recursion relations in terms of the three free parameters \( x = X_0, y = Z_1 \) and \( \bar{x} = \bar{X}_0 \), as

\[
\begin{align*}
H(x) &+ H\left(\frac{1 + y}{x}\right) + H\left(\frac{1 + x + y + \bar{x} + \bar{xx}}{y\bar{x}}\right) + \\
H(y) &+ H\left(\frac{(1 + x + y)(1 + y + \bar{x})}{xy\bar{x}}\right) + H\left(\frac{(1 + x)(1 + \bar{x})}{y}\right) + \\
H(\bar{x}) &+ H\left(\frac{1 + y}{\bar{x}}\right) + H\left(\frac{1 + x + y + \bar{x} + \bar{xx}}{yx}\right) = 6 .
\end{align*}
\]

(2.20)
Eq.\( (2.18) \) tells us that the three triplets \( \{\bar{X}_0, Z_1, X_2\} \), \( \{X_2, Z_3, X_4\} \) and \( \{\bar{X}_4, Z_5, X_0\} \), can be considered as the first three terms \( Y_0, Y_1, Y_2 \) of Eq.\( (2.10) \), so we can use Eq.\( (2.17) \) to replace the sum of the nine terms of the left-hand side of Eq.\( (2.19) \) to the following sum of six \( H \) terms

\[
9 - H \left( \frac{y\bar{x}}{1 + y + \bar{x}} \right) - H \left( \frac{x(1 + \bar{x})}{1 + y + \bar{x}} \right) - H \left( \frac{1 + y + \bar{x}}{x(1 + \bar{x})} \right)
- H \left( \frac{y}{1 + \bar{x}} \right) - H \left( \frac{1 + \bar{x}}{y} \right) - H \left( \frac{1 + y + \bar{x}}{y\bar{x}} \right)
\]

which can be split in three pairs of the form \( H(z) + H(\frac{1}{z}) \), so using Eq.\( (2.10) \) we get that the LHS is equal to 6 as it was stated.

Similarly one can prove these identities for other low rank systems. As the rank increases the direct algebraic manipulations become rather involved and we not succeeded in finding a general recursive proof. For higher rank algebras, we did extensive numerical checks.

We conclude with two important observations.

There is a class of integrable models where the TBA equations or the corresponding Y-system is not apparently related to the Dynkin diagrams. This class should include the sine-Gordon model at rational points and its quantum reduced models as well all the reduced models of the affine Toda field theories at imaginary coupling constant \( [26, 27] \). The physical motivations described in §1 suggest that also in these cases identities of the kind \( (2.14) \) should hold. In order to support this conjecture we used the TBA equations proposed in \( [16, 28, 30] \) and others associated to models with purely elastic S-matrix (see for instance \( [31, 32] \)) and we verified the corresponding dilogarithm identity for the associated Y-system by extensive numerical checks.

Finally, note that the periodicity of the Y-system, although it plays a fundamental role in many properties of the integrable models, it has not been fully understood. Actually it has not been proven in a general way, but it has been verified by direct algebraic manipulations only for algebras of low rank. Yet the validity of the Rogers dilogarithm identities discussed above are strictly related to it: if the \( \Upsilon \) variables were not periodic Eq.\( (2.14) \) would be meaningless. It would be interesting to understand this property in a more general way.
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