Algorithms for Deforming and Contracting Simply Connected Discrete Closed Manifolds (III)

Li Chen
Department of Computer Science and Information Technology
University of the District of Columbia
Email: lchen@udc.edu

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Abstract

In a recent paper, L. Chen, Algorithms for Deforming and Contracting Simply Connected Discrete Closed Manifolds (II), we discussed two algorithms for deforming and contracting a simply connected discrete closed manifold to a discrete sphere. The first algorithm was the continuation of the work initiated in L. Chen, Algorithms for Deforming and Contracting Simply Connected Discrete Closed Manifolds (I); the second algorithm was more direct treatment of contraction for discrete manifolds. In this short paper, we clarify that we can use the same method to the standard piecewise linear (PL) complexes from triangulation of general smooth manifolds. Our discussion is based on the triangulation techniques invented by Cairns, Whitehead and Whitney more than half-century ago.

Note that, in this paper, we try to use PL or simplicial complexes to replace some concepts of discrete manifolds. However, some details in original paper, might also need to be slightly modified. This paper also adds more details and corrects several typos in the theorem proof in the first paper above that was done in a timely manner.

We will post the detailed algorithm / procedure of practical triangulations next.
1 Introduction

According to the Whitney Embedding Theorem and the Nash Embedding Theorem, any smooth real $m$-dimensional manifold can be smoothly embedded in $\mathbb{R}^{2m}$ and any $m$-manifold with a Riemannian metric (Riemannian manifold) can be isometrically imbedded to an $n$-Euclidean space $\mathbb{E}^n$ where $n \leq c \cdot m^3$.

Therefore, we only need to deal with a smooth manifold $\mathcal{M}$ in $\mathbb{E}^n$. In 1940, based on Cairns’s work, Whitehead first proved the following theorem [14]: Every smooth manifold admits an (essentially unique) compatible piecewise linear (PL) structure.

A PL structure is a triple $(K, \mathcal{M}, \cdot\cdot\cdot)$ where $K$ is a PL-complex, and $f : K \to \mathcal{M}$ is piecewise differentiable. In addition, Whitehead proved (roughly speaking) that there is a sequence of homeomorphic mapping $f_i$ on the finite complex $K_i$ that are arbitrarily good approximations of the $f$, where $K_i$ is a collection of cells each of them is a subdivision of the some cells in $K$.

In 1957, Whitney designed an simpler procedure to get a triangulation [15]. In 1960, Cairns found an even simpler way for compact smooth manifolds[7, 8]: A compact smooth manifold admits a finite triangulation, i.e. there is a finite complex that is diffeomorphic to the original manifold. A finite complex means that a complex that has finite number of simplices or (PL) cells. For simplicity, a compact smooth manifold can be triangulated by a finite complex $\mathcal{K}$ (with perserving diffeomorphism between them).

For the most important case, 3-manifolds of this paper, a more general domain based on the research of the Combinatorial Triangulation Conjecture can also be used. The Combinatorial Triangulation Conjecture is stated as follows:

Every compact topological manifold can be triangulated by a PL manifold.

Rado proved it in 1925: For dimension two, the conjecture holds. Moise proved it for dimension three. However, it is false for in dimensions $\geq 4$ [9].

In [11], we discussed two algorithms for deforming and contracting a simply connected discrete closed manifold to a discrete sphere. The first algorithm was the continuation of the work initiated in [2]. And, the second algorithm was more direct treatment of contraction for discrete manifolds.

This paper will focus on the second algorithm and its relationship to smooth manifolds. We clarify that we can use the same method to the standard PL complexes from triangulation of general smooth manifolds. Our discussion is based on the triangulation techniques invented by Cairns, Whitehead and Whitney more than half-century ago.

In this paper, we like to use $\mathcal{M}$ to denote a smooth or general topological manifold in Euclidean space, use $K$ to denote a complex that is the collection of cells based on a triangulation or decomposition. Use $M$ to represent a restored manifold of $K$ in Euclidean space.
$M$ is essentially $K$ but $M$ is a discrete (-like) manifold that will be better for our discussion.

We also assume in this paper, a manifold is orientable.

## 2 Some Basic Concepts of Manifolds

A topological manifold is defined as follows [9]: A topological space $M$ is a manifold of dimension $m$ if (1) $M$ is Hausdorff, and (2) $M$ is second countable, and (3) $M$ is locally Euclidean of dimension $m$.

We also know that every manifold is locally compact. It means that every point has a neighborhood contained in a compact set.

Let $x$ be a point in $M$ and an open set containing $x$ is a neighborhood $\theta(x)$. Mapping $\theta(x)$ locally homeomorphic to a $E^m$ called a chart, a collection of charts whose domain ($\theta(.)$) covers $M$ is called an atlas of $M$.

Roughly, a manifold is smooth if for any point $x \in M$ and each nearby point $y$ of $x$, the chart transition (called transition-map) from $\theta(x)$ to $\theta(y)$ is smooth. This means that $chart^{-1}_y(chart_x(\theta(x) \cap \theta(y)))$ is smooth.

We omit the formal definition here for simplicity of the presentation. We just want to have the property of local compactness in this paper.

On the other hand, in [2], we observed that the Chen-Krantz’s paper, (L. Chen and S. Krantz, A Discrete Proof of The General Jordan-Schoenflies Theorem, http://arxiv.org/abs/1504.05263) actually proved the following result: A simply connected (orientable) manifold $M$ in discrete space $U$. If $M$ is a supper submanifold, the dimension of $M$ is smaller than the dimension of $U$ by one, in such a case, we can use Jordan’s theorem to first separate the $U$ into two components. The deformation becomes the pure contraction. This result can be obtained directly from the Chen-Krantz’s paper.

However, if the dimension of $U$ is much bigger than the dimension of $M$, we might need other method, for instance, we could fill an $(m+1)$-manifold bounded by $M$, where $m$ is the dimension of $M$. Some algorithms have been discussed in [2]. But these algorithms may not work for some cases. In [1], we continued the discussions by (1) we refined the algorithm by introducing a tree structure in filling, and (2) designed another direct method of contraction.

This short paper is to continue the second part of [1]. In this paper, we will attempt to use the concepts such as the simplicial complex in combinatorial topology instead of discrete manifold as we used in [2, 1]. Which makes more readable to readers in mathematics. We also attempt to make a stronger connection our results to the Poincare Conjuncture.

According to the official description of the Poincare Conjuncture made by John Milnor [11]:

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Question. If a compact three-dimensional manifold \( M^3 \) has the property that every simple closed curve within the manifold can be deformed continuously to a point, does it follow that \( M^3 \) is homeomorphic to the sphere \( S^3 \)?

According to Moise, any 3D topological manifold is homeomorphic equivalent to a compact smooth manifold \( M^3 \). There is only one diff-structure for any \( M^3 \). So this question is to determine a compact smooth manifold as well, say the same as \( M^3 \).

Morgan wrote "Now to the notion of equivalence. For topological manifolds it is homeomorphism. Namely, two topological manifolds are equivalent (and hence considered the same object for the purposes of classification) if there is a homeomorphism, i.e., a continuous bijection with a continuous inverse, between them. Two smooth manifolds are equivalent if there is a diffeomorphism, i.e., a continuous bijection with a continuous inverse with the property that both the map and its inverse are smooth maps, between the manifolds. Milnors examples were of smooth 7-dimensional manifolds that were topologically equivalent but not smoothly equivalent to the 7-sphere. That is to say, the manifolds were homeomorphic but not diffeomorphic to the 7-sphere. Fortunately, these delicate issues need not concern us here, since in dimensions two and three every topological manifold comes from a smooth manifold and if two smooth manifolds are homeomorphic then they are diffeomorphic. Thus, in studying 3-manifolds, we can pass easily between two notions. Since we shall be doing analysis, we work exclusively with smooth manifolds." \[12\]

Lott explained the Poincare conjecture as \[10\]: "The version stated by Poincare is equivalent to the following.

Poincare conjecture: A simply-connected closed (= compact boundaryless) smooth 3-dimensional manifold is diffeomorphic to the 3-sphere."

So both Milnor and Lott used the compact manifold. It can be sure that there is an equivalence between the compact manifold and general manifold for simply-connected smooth 3D manifold.

In other to make more general coverage, we will not restrict our discussion to 3D compact manifolds. We like just directly to use the property of locally compactness of a topological manifold \[9\]. We like to have a sequence of PL complexes that is approaching the original manifold in any degree of approximation. Using this method, we can extend the results to any type of high dimensions of smooth manifolds.

3 Some Previous Results and Concepts in Discrete Cases

In this short paper, we want to make some modifications to the previous paper \[11\] that was mainly based on discrete manifolds. Now, we like to just use cell complexes especially PL complexes in order to show the same results to readers in mathematics.

Let \( K \) be a cell complexes that is a decomposition (triangulation) of a compact smooth \( m \)-manifold \( \mathcal{M} \) in the \( n \)-dimensional Euclidean space \( E^n \). \( K \) contains only finite number of cells \[7\]. \( M \) is the union of all cells of \( K \). \( M \)
can be viewed as a polyhedron in Euclidean space. \( M \) is a discrete manifold in \( \mathbb{R}^2 \). \( M \) is a discrete manifold where each cell in \( M \) can also be found in \( K \).

So \( K \) is a set of 0-cells, 1-cells, \( \ldots \), \( m \)-cells, where the set of \( i \)-cells is called the \( i \)-skeleton of \( M \). We must assume that there is no \((m+1)\)-cell. We also assume that each \( i \)-cell in \( M \) is an \( i \)-simplex, \( i \)-cube, or an \( i \)-convex polyhedron.

Without losing generality, we can assume that the intersection of any two cells is empty or a \( j \)-cell. An \( i \)-cell is an \( i \)-complex that is a simply connected closed \((i−1)\)-manifold plus the inner part of this manifold (\( i \)-cell).

In other words, an \( i \)-cell is bounded by a set of \((i−1)\)-cells. It contains the inner part of the cell in \( i \)-dimension. The inner part of an \( i \)-cell is usually refer to an open \( i \)-cell in \( E^n \). \( K \) contains all inner part of cells contained in \( K \).

It is in fact that \( K \) only contains inner part of \( i \)-cell when \( i \geq 1 \). When we attach all 0-cells (points), it becomes a \( K \). In order to make our discussion easier, let \( K \) contain all open and closed cells from the triangulation of \( M \).

Let \( a, b, c \) be 0-cells. To represent a complex in computers or any other explicit form, we can use \([a, b, c, \ldots]\) to express a cell that contains boundary of the cell. \((a, b, c, \ldots)\) is the open set, and that is the inner part of \([a, b, c, \ldots]\). \(< a, b, c, \ldots >\) express the boundary of \([a, b, c, \ldots]\). So we have

\[
\partial ([a, b, c, \ldots]) = < a, b, c, \ldots > = [a, b, c, \ldots] \setminus (a, b, c, \ldots),
\]
and

\[
[a, b, c, \ldots] = < a, b, c, \ldots > \cup (a, b, c, \ldots).
\]

Because each element of \( K \) in \( E^n \), such as \([a, b, c, \ldots]\), \((a, b, c, \ldots)\), or \(< a, b, c, \ldots >\) has the real meaning, we can assume \( K \) contains all not only the open cells for convenience.

A curve in \( M \) is a path of 1-cells in \( K \). So we can define simple connectivity of \( M \). \( K \) contains the finite number of cells. Usually no cell can be split into two or more if it is not specifically specified. That means when \( K \) is done, we will not make any changes (usually).

Since every pair of two vertices in the finite \( K \) (0-cells) have a certain distance in \( E^n \), so

**Proposition 3.1**: \( M \) is locally flat in \( E^n \).

**Proposition 3.2**: Every \( i \)-submanifold \( M(i) \) determined by \( i \)-skeleton of \( K \) is locally flat in \( E^n \).

This local flatness here is not the same as *discrete local flatness* in \([1]\). When we really need discrete local flatness, we could use the barycentric subdivision of triangles to make one. However, since we use the embedding in \( E^n \), there is no need to do that. We know that the inner part of a \( j \)-cell \((j > 0)\) can be determined in \( E^n \) as the centroid of the cell (or the counterclockwise direction of the \((j−1)\)-cell in the \( j \)-cell.). (Unlike it is in pure discrete space, there is no centroid point in discrete space, it only contains a rotation direction.)

Therefore, the case of \( K \) with embedding to \( E^n \) is much simpler than the discrete manifolds in pure discrete
One of the concepts we have developed in the Chen-Krantz paper is the cell-distance of two points. It plays an important role in Chen-Krantz paper (L. Chen and S. Krantz, A Discrete Proof of The General Jordan-Schoenflies Theorem, http://arxiv.org/abs/1504.05263). Two points \( p \) and \( q \neq q \) in \( K \) or \( M \) have a \( j \)-cell-distance \( k \) meaning that the length of the shortest \( j \)-cell-path that contains both \( p \) and \( q \) is \( k \).

Again, we assume all manifolds discussed in this paper is orientable.

### 4 Contraction of the Finite PL Complex \( K \)

Let \( M \) be a simply connected closed manifold in \( E^n \). All partitioned PL cells of \( M \) are included in \( K \) where \( K \) is a finite simplicial complex or PL complex of \( M \). We will only deal with \( K \) and its elements. We do not deal with every point in \( M \).

Based on a theorem (Theorem 5.1) proved by Chen-Krantz, we concluded that a simply connected closed \((m - 1)\)-manifold \( S \) split a simply connected closed \( m \)-manifold into two components, each of which is simply connected. We require both \( S \) and \( M \) are locally flat. That are both true for continuous case. For instance, \( M \) in \( E^n \) is locally flat.

The key here is that allowing simplexes or cells with open and closed-cells in \( E^n \). We do not need to consider locally flat for discrete cases.

We restate this theorem as follows:

**Theorem 4.1** (The Jordan Theorem for the closed surface on a 3D manifold) Let \( M \) be a simply connected 3D manifold (discrete or PL); a closed discrete surface \( S \) (with local flatness) will separate \( M \) into two components. Here \( M \) can be closed.

Note that \( M \) in Theorem 4.1 is slightly different from \( M \) in this paper. \( M \) in Theorem 4.1 is a discrete manifold it is the combined form of \( M \) and \( K \) in this paper. Local flatness is required in Theorem 4.1 (Theorem 5.1 in Chen-Krantz paper) but in this paper, \((M,K)\) would be naturally locally flat. This is because (finite) \( K \) can be subdivided finite times, and every cell (simplex) except 0-cell must have a centroid point that is distinguished from the boundary of the cell. When we consider \((M,K)\) in this paper, any simple curve (no 0-cell appears twice in the curve in \( K \)) or simple manifold is locally flat. Therefore, based on Theorem 4.1, we can easily get:

**Theorem 4.2** Let \( M \) be a simply connected closed 3D manifold with associated the finite complex \( K \). A simple closed surface \( S \) of \( M \), where each 2-cell of the surface is a 2-cell in \( K \), will separate \( M \) into two components.
Based on this theorem we will design a procedure that will generate a homeomorphic mapping for a component in \( M - S \) to a 3-disk. So if \( M \) is closed then \( M \) is homeomorphic to a 3-sphere.

**Theorem 4.3** If \( M \) is closed in Theorem 4.2, we can algorithmically make \( M \) to be homeomorphic to a 3-sphere.

**The Algorithmic Proof:**

According to Theorem 4.2, a simply connected closed 2-manifold (orientable) \( B \) will split the 3-manifold \( M \) into two components \( D \) and \( D' \).

Part 1: We now first show that each of the components will be simply connected.

In fact, each of the two components will be simply connected. This is because that if \( M \) is simply connected. A simply closed curve \( C \) is contractible to a point \( p \in C \) on \( M \). Let \( \Omega \) be the contraction sequence in discrete case, we call gradually variation in \([3]\). We might as well assume \( p \) is not on \( B \).

The contraction sequence \( \Omega \) may contain some point on \( B \), we can modify the contraction by using \( \Omega \cap B \) to replace \( \Omega \) to get a new contraction sequence. So this theorem is valid. A curve in \( \Omega \) may intersect with \( C \) but will not cross-over \( C \).

A curve started in a component \( D \) will pass (have both enter to \( B \) and out of \( B \) to another component called a pass) even times on \( B \) and also finite number of times in discrete case. When \( C' \in \Omega \) pass \( B \) and it will enter \( B \) from \( D' \), we can find a curve \( C_B'(0) \) on \( B \) two link two points (the last point of leaving of \( B \) and the fist point entering \( B \), since \( B \) is simply connected.) Use \( C_B'(0) \) to replace that corresponding arc in \( D' \). If there are multiple passes, we can use \( C_B'(1),...,C_B'(i) \) to replace all. So we get a new \( C'_B \) that only contains points in \( D \cup B \). Use the same process for all curves in \( \Omega \), we will get a \( \Omega_B \). That is the contraction to \( p \). So \( D \) is simply connected and, so is \( D' \).

Note that If the curve only stay on \( B \) and back to the original component \( D \) will not be counted as a pass.

Part 2: We design an algorithm to show that \( D \) is homeomorphic to an \( m \)-disk.

Again, according to the definition of simple connectedness, a closed 1-cycle will be contractible on \( M \). since we assume that there is no edge in \( M \), \( M \) is closed and orientable, then there is no holes in \( M \). (if there is a hole, then the edge of a hole will be a 1-cycle and it is not contractible. ). For instance, if \( M \) is a torus, some cycles are not contractible.

Let \( m = 3 \) for now, i.e. \( M \) is a 3-manifold. We will see that \( m \) can be any number.

Step (1): Remove an \( m \)-cell \( e \) from \( M_m = M \) will result a \((m - 1)\)-cycle on the boundary, called \( B \). This cycle \( B \) is always simply connected as well. This is because the boundary is the boundary of \( e \). We use this property (plus the theorem we discussed above). (Note: We can assume that any closed simply connected \((m - 1)\)-manifold is an \((m - 1)\)-cycle that is homeomorphic to \((m - 1)\)-sphere when we prove for \( m \).) Let \( D = M_m - e \). Note that this subtraction is to remove the inner part of the \( m \)-cell \( e \) not its \((m - 1)\) edges (faces).
Step (2): We will remove more \( m \)-cells of \( M_m \) if they have an edge(face) on \( B \). Algorithmically, \( B \) is changing when time is going on) we remove another \( e \in D \) and \( e \) has an \((m - 1)\)-edge(face) in \( B \). We also denote the new edge of \( D \) to be \( B \) as well. \( B \) will be a new edge set of \( D = M - \{ \text{removed } m\text{-cells} \} \). \( B \) is still the \((m - 1)\)-cycle (pseudo-manifold in \([5]\)) since use the boundary of new \( e \) in \( D - B \) to replace \( e \cap B \).

Step (2'): For actual design of the algorithm, we will calculate the cell-distances to all points in \( M_m \) from a fix point \( o \), \( e \neq o \), determine beforehand. We always select a new \( e \) that is adjacent to \( B \) (has an edge on \( B \)) has the greatest cell distance to \( o \). It means that one 0-cell in \( e \) has the greatest \( m \)-cell-distance to \( o \). When more of such cells are found, we will select the cell \( e \) (\( e \cap B \) is an \((m - 1)\)-cell) has most of the greatest distance points (0-cells) to \( o \). This is a strategy for the balanced selection of the new \( e \). We will always attempt to make the equal distance cycle \( B \) respect to \( o \) (in terms of each point on \( B \)).

Step (3): Since \( B \) is an \((m - 1)\)-cycle, so \( D \) is always simply connected based on Theorem 4.2. \( M_m \) only contains finite number of \( m \)-cells, It mean that this process will end at the \( \text{Star}(o) \) in \( M_m \). Therefore, we algorithmically showed that \( M_m, D = M_m - e, D \leftarrow D - e_{\text{new}}, \ldots, \) is continuously shrinking (homeomorphic) to \( \text{Star}(o) \cap M_m \). Since \( \text{Star}(o) \cap M_m \) is an \( m \)-disk. So, the reversed steps determine a homeomorphic mapping from an \( m \)-disk to \( M_m - e \) where \( e \) was the first removed cell.

Please note that \( M_m - e \) is to remove the inner part of \( e \), i.e. \( (e) \).

We also know that \( e \) is also homeomorphic to \( m \)-disk. The connected-sum of \( e \) and \( M_m - e \) is homeomorphic to \( m \)-sphere where \( m = 3 \). This is the end of the algorithmic proof. (End of Proof)

Despite of Theorem 4.1 and 4.2, in Theorem 4.3, we actually proved that for 3D compact smooth manifolds the Jordan Separation Theorem implies the Poincare Conjuncture. We can do this because we have used the property of finiteness of cells in complex \( K \). And the contraction will be based on a procedure described in the above proof. This type of algorithms can only be performed in finite cases and in discrete-like manifolds.

**Corollary 4.1** For the 3D closed smooth manifold that admits the finite simplicial complex decomposition, the Jordan Separation Theorem (Theorem 4.2) implies the Poincare Conjuncture in 3D.

Using this result, we can design the algorithms or procedure to show that for cell (simplical) complex, a closed simply connected smooth \( m \)-manifold that has a finite cell (simplical) complex decomposition (meaning admits a finite triangulation) is homeomorphic to \( m \)-sphere.

**Theorem 4.4** *(The Jordan Theorem of Discrete m-manifolds)* Let \( M_m \) be a simply connected discrete or PL \( m \)-manifold; a closed simply connected discrete \((m - 1)\)-manifold \( B \) (with local flatness) will separate \( M_m \) into two components. Here \( M_m \) can be closed.

With the recursive assumption, \( B \) can be assumed to be homeomorphic to an \((m - 1)\)-sphere. We can change the terminology to cell complexes,
Theorem 4.5 (The Jordan Theorem of m-manifolds that admit the finite triangulation) Let $M_m$ be a simply connected (finite-cell) PL $m$-manifold; a closed simply connected $(m-1)$-manifold $B$ in $M_m$ separates $M_m$ into two components. Here $M_m$ can be closed.

Theorem 4.6 If $M_m$ is closed in Theorem 4.5, we can algorithmically make $M_m$ to be homeomorphic to a $m$-sphere.

Some discussions: In Step (2) in the proof of Theorem 4.3, if $B$ is not a simple cycle, then $M$ is not simply connected according to the theorem above. We will use this property to determine if $M$ is simply connected, see [1].

In Step (3), since $M$ contains finite number of $m$-cells, there will be definite always to reach the end. It is also possible to use gradually varied ”curve” or $(m-1)$-cycle on $M_m$ of $B$ to replace $B$ to reach the maximum number of removal in practice. However, we have to keep the removal balanced to a certain point meaning that we try always to remove one that has the furthermore distance on the edge to the fixed point $o$ (which we contract to).

This entire process of the algorithm determined a homeomorphism to the $m$-sphere. Therefore, we would like to say that with finite cell complex triangulation, a closed-orientable simply connected smooth $m$-manifold is homomorphic to an $m$-sphere.

Corollary 4.2 With finite PL complex decomposition or triangulation, a closed-orientable compact simply connected smooth $m$-manifold is homomorphic to an $m$-sphere.

In paper (I) [2] and paper (II) [1], we have used filling techniques to fill a closed manifold to be a deformed disk. As an equivalent statement, we observed that a closed-orientable $m$-simply connected manifold $M_m$ is a homeomorphic to $m$-sphere if and only if there is $(m+1)$-deformed disk that is simply connected and has the boundary that is $M_m$.

5 Existence of the Approximating Sequence of Triangulations of a Smooth Manifold

In the above sections, we dealt with the compact smooth manifold that admits the finite triangulation or PL complex decomposition (the decomposition is diffeomorphic to the original manifold). If the manifold is not compact, we might not have a finite cell complex decomposition. This section we will discuss this problem.

According to the definition of topological manifolds, we know that it must be locally compact. That means for any smooth manifold, there is a finite triangulation or simplicial decomposition for any given point and its neighbor. Our strategy is to find a sequence of PL complexes that will approach the original smooth manifold in any approximating degree.
To do this, we need to use the axiom of choice. Since $M$ is smooth, there is no point where the total curvature of the point is infinitive. We can select a point that has the smallest curvature, then we use Whitney’s triangulation method to obtain the triangulation. It must be finite. Now the problem is that this process is not constructive since we used the axiom of choice.

For any very small number $\varepsilon > 0$, we like to make a triangulation based on $\varepsilon > 0$ where each simplex will be involved in an $\varepsilon-$cube. We will have a sequence of triangulations. Since based on the local compactness, there must be a $\varepsilon$ that will be the one who can have a finite cover to each point that has the smallest curvature.

We will discuss more for this in the next paper.

6 Conclusion

In this short paper, we want to make some connections from discrete mathematics to combinatorial topology. Then we attempted to deal with smooth manifolds.

We also added some details and corrected several typos for the proof of a theorem in [1]. There might be some issues that we still need to continue for further discussions. We believed that the current research direction is valuable for understanding some of the essential problems in geometric topology.

7 Appendix: Some Concepts of Cell Distances

The concepts of this paper are in [3]. We also use some concepts from the following two papers: L. Chen A Concise Proof of Discrete Jordan Curve Theorem, [http://arxiv.org/abs/1411.4621](http://arxiv.org/abs/1411.4621) and L. Chen and S. Krantz, A Discrete Proof of The General Jordan-Schoenflies Theorem, [http://arxiv.org/abs/1504.05263](http://arxiv.org/abs/1504.05263).

In a graph, we refer to the distance as the length of the shortest path between two vertices. The concept of graph-distance in this paper is the edge distance, meaning how many edges are needed from one vertex to another. We usually use the length of the shortest path in between two vertices to represent the distance in graphs. In order to distinguish from the distance in Euclidean space, we use graph-distance to represent lengths in graphs in this paper.

Therefore graph-distance is edge-distance or 1-cell-distance. It means how many 1-cells are needed to travel from $x$ to $y$. We can generalize this idea to define 2-cell-distance by counting how many 2-cells are needed from a point (vertex) $x$ to point $y$. In other words, 2-cell-distance is the length of the shortest path of 2-cells that contains $x$ and $y$. In this path, each adjacent pair of 2-cells shares a 1-cell. (This path is 1-connected.)

We can define $d^{(k)}(x,y)$, the $k$-cell-distance from $x$ to $y$, as the length of the shortest path of (or the minimum
of number of k-cells in such a sequence) where each adjacent pair of two k-cells shares a \((k-1)\)-cell. (This path is \((k-1)\)-connected.)

We can see that \(d^{(1)}(x,y)\) is the edge-distance or graph-distance. We write \(d(x,y) = d^{(1)}(x,y)\)

(We can also define \(d_i^{(k)}(x,y)\) to be a k-cell path that is i-connected. However, we do not need to use such a concept in this paper.)

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