Multiplicative loops of 2-dimensional topological quasifields

Abstract

We determine the algebraic structure of the multiplicative loops for locally compact 2-dimensional topological connected quasifields. In particular, our attention turns to multiplicative loops which have either a normal subloop of positive dimension or which contain a 1-dimensional compact subgroup. In the last section we determine explicitly the quasifields which coordinatize locally compact translation planes of dimension 4 admitting an at least 7-dimensional Lie group as collineation group.

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1. Introduction

Locally compact connected topological non-desarguesian translation planes have been a popular subject of geometrical research since the seventies of the last century ([18], [2]-[9], [13], [15]). These planes are coordinatized by locally compact quasifields $Q$ such that the kernel of $Q$ is either the field $\mathbb{R}$
of real numbers or the field $\mathbb{C}$ of complex numbers (cf. [11], IX.5.5 Theorem, p. 323). If the quasifield $Q$ is 2-dimensional, then its kernel is $\mathbb{R}$.

The classification of topological translation planes $\mathcal{A}$ was accomplished by reconstructing the spreads corresponding to $\mathcal{A}$ from the translation complement which is the stabilizer of a point in the collineation group of $\mathcal{A}$. In this way all planes $\mathcal{A}$ having an at least 7-dimensional collineation group have been determined ([3]-[8], [15]).

Although any spread gives the lines through the origin and hence the multiplication in a 2-dimensional quasifield $Q$ coordinatizing the plane $\mathcal{A}$, to the algebraic structure of the multiplicative loop $Q^*$ of a proper quasifield $Q$ is not given special attention apart from the facts that the group topologically generated by the left translations of $Q^*$ is the connected component of $\text{GL}_2(\mathbb{R})$, the group topologically generated by the right translations of $Q^*$ is an infinite-dimensional Lie group (cf. [14], Section 29, p. 345) and any locally compact 2-dimensional semifield is the field of complex numbers ([17]).

Since in the meantime some progress in the classification of compact differentiable loops on the 1-sphere has been achieved (cf. [10]), we believe that loops could have more space in the research concerning 4-dimensional translation planes. Using the images of differentiable sections $\sigma : G/H \to G$, where $H = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, a > 0, b \in \mathbb{R} \right\}$, we classify the $C^1$-differentiable
multiplicative loops $Q^*$ of 2-dimensional locally compact quasifields $Q$ by functions, the Fourier series of which are described in [10].

The multiplicative loops $Q^*$ of 2-dimensional locally compact left quasifields $Q$ for which the set of the left translations of $Q^*$ is the product $\mathcal{T}\mathcal{K}$ with $|\mathcal{T}\cap\mathcal{K}| \leq 2$, where $\mathcal{T}$ is the set of the left translations of a 1-dimensional compact loop and $\mathcal{K}$ is the set of the left translations of $Q^*$ corresponding to the kernel $K_r$ of $Q$, form an important subclass of loops, that we call decomposable loops. Namely, if $Q^*$ has a normal subloop of positive dimension or if it contains the group $\text{SO}_2(\mathbb{R})$, then $Q^*$ is decomposable. Moreover, we show that any 1-dimensional $C^1$-differentiable compact loop is a factor of a decomposable multiplicative loop of a locally compact connected quasifield coordinatizing a 4-dimensional translation plane. A 2-dimensional locally compact quasifield $Q$ is the field of complex numbers if and only if the multiplicative loop $Q^*$ contains a 1-dimensional normal compact subloop.

Till now mainly those simple loops have been studied for which the group generated by their left translations is a simple group. If the group generated by the left translations of a loop $L$ is simple, then $L$ is also simple (cf. Lemma 1.7 in [14]). The multiplicative loops $Q^*$ of 2-dimensional locally compact quasifields show that there are many interesting 2-dimensional locally compact quasi-simple loops for which the group generated by their left translations has a one-dimensional centre.
In the last section we use Betten’s classification to determine in our framework the multiplicative loops $Q^*$ of the quasifields which coordinatize the 4-dimensional non-desarguesian translation planes $\mathcal{A}$ admitting an at least seven-dimensional collineation group and to study their properties. The results obtained there yield the following

**Theorem** Let $\mathcal{A}$ be a 4-dimensional locally compact non-desarguesian translation plane which admits an at least 7-dimensional collineation group $\Gamma$. If the quasifield $Q$ coordinatizing $\mathcal{A}$ is constructed with respect to two lines such that their intersection points with the line at infinity are contained in the 1-dimensional orbit of $\Gamma$ or contain the set of the fixed points of $\Gamma$, then the multiplicative loop $Q^*$ of $Q$ is decomposable if and only if one of the following cases occurs:

(a) $\Gamma$ is 8-dimensional, the translation complement $C$ is the group $\text{GL}_2(\mathbb{R})$ and acts reducibly on the translation group $\mathbb{R}^4$;

(b) $\Gamma$ is 7-dimensional, the translation complement $C$ fixes two distinct lines of $\mathcal{A}$ and leaves on one of them, one or two 1-dimensional subspaces invariant;

(c) $\Gamma$ is 7-dimensional, the translation complement $C$ fixes two distinct lines $\{S, W\}$ through the origin and acts transitively on the spaces $P_S$ and $P_W$ but does not act transitively on the product space $P_S \times P_W$, where $P_S$ and $P_W$ are the sets of all 1-dimensional subspaces of $S$, respectively of $W$. 

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2. Preliminaries

A binary system \((L, \cdot)\) is called a quasigroup if for any given \(a, b \in L\) the equations \(a \cdot y = b\) and \(x \cdot a = b\) have unique solutions which we denote by \(y = a \backslash b\) and \(x = b/a\). If a quasigroup \(L\) has an element 1 such that \(x = 1 \cdot x = x \cdot 1\) holds for all \(x \in L\), then it is called a loop and 1 is the identity element of \(L\). The left translations \(\lambda_a : L \to L, x \mapsto a \cdot x\) and the right translations \(\rho_a : L \to L, x \mapsto x \cdot a, a \in L\), are bijections of \(L\).

Two loops \((L_1, \circ)\) and \((L_2, \ast)\) are called isotopic if there exist three bijections \(\alpha, \beta, \gamma : L_1 \to L_2\) such that \(\alpha(x) \circ \beta(y) = \gamma(x \circ y)\) holds for all \(x, y \in L_1\).

A binary system \((K, \cdot)\) is called a subloop of \((L, \cdot)\) if \(K \subset L\), for any given \(a, b \in K\) the equations \(a \cdot y = b\) and \(x \cdot a = b\) have unique solutions in \(K\) and \(1 \in K\). The kernel of a homomorphism \(\alpha : (L, \cdot) \to (L', \ast)\) of a loop \(L\) into a loop \(L'\) is a normal subloop \(N\) of \(L\), i.e. a subloop of \(L\) such that

\[
x \cdot N = N \cdot x, \ (x \cdot N) \cdot y = x \cdot (N \cdot y), \ (N \cdot x) \cdot y = N \cdot (x \cdot y)
\]

hold for all \(x, y \in L\). A loop \(L\) is called simple if \(\{1\}\) and \(L\) are its only normal subloops.

A loop \(L\) is called topological, if it is a topological space and the binary operations \((a, b) \mapsto a \cdot b, (a, b) \mapsto a \backslash b, (a, b) \mapsto b/a : L \times L \to L\) are continuous. Then the left and right translations of \(L\) are homeomorphisms of \(L\). If \(L\) is a connected differentiable manifold such that the loop multiplication and
the left division are continuously differentiable mappings, then we call $L$ an almost $C^1$-differentiable loop. If also the right division of $L$ is continuously differentiable, then $L$ is a $C^1$-differentiable loop. A connected topological loop is quasi-simple if it contains no normal subloop of positive dimension.

Every topological, respectively almost $C^1$-differentiable, connected loop $L$ having a Lie group $G$ as the group topologically generated by the left translations of $L$ corresponds to a sharply transitive continuous, respectively $C^1$-differentiable section $\sigma : G/H \to G$, where $G/H = \{xH | x \in G\}$ consists of the left cosets of the stabilizer $H$ of $1 \in L$ such that $\sigma(H) = 1_G$ and $\sigma(G/H)$ generates $G$. The section $\sigma$ is sharply transitive if the image $\sigma(G/H)$ acts sharply transitively on the factor space $G/H$, i.e. for given left cosets $xH, yH$ there exists precisely one $z \in \sigma(G/H)$ which satisfies the equation $zxH = yH$.

A (left) quasifield is an algebraic structure $(Q, +, \cdot)$ such that $(Q, +)$ is an abelian group with neutral element 0, $(Q \setminus \{0\}, \cdot)$ is a loop with identity element 1 and between these operations the (left) distributive law $x \cdot (y + z) = x \cdot y + x \cdot z$ holds. A locally compact connected topological quasifield is a locally compact connected topological space $Q$ such that $(Q, +)$ is a topological group, $(Q \setminus \{0\}, \cdot)$ is a topological loop, the multiplication $\cdot : Q \times Q \to Q$ is continuous and the mappings $\lambda_a : x \mapsto a \cdot x$ and $\rho_a : x \mapsto x \cdot a$ with $0 \neq a \in Q$ are homeomorphisms of $Q$. If for any given $a, b, c \in Q$ the
equation $x \cdot a + x \cdot b = c$ with $a + b \neq 0$ has precisely one solution, then $Q$ is called planar. A translation plane is an affine plane with transitive group of translations; this is coordinatized by a planar quasifield (cf. [16], Kap. 8).

The kernel $K_r$ of a (left) quasifield $Q$ is a skewfield defined by

$$K_r = \{ k \in Q; \ (x+y) \cdot k = x \cdot k + y \cdot k \text{ and } (x \cdot y) \cdot k = x \cdot (y \cdot k) \text{ for all } x,y \in Q \}.$$  

In this paper we consider left quasifields $Q$. Then $Q$ is a right vector space over $K_r$. Moreover, for all $a \in Q$ the map $\lambda_a : Q \to Q, x \mapsto a \cdot x$ is $K_r$-linear. According to [12], Theorem 7.3, p. 160, every quasifield that has finite dimension over its kernel is planar.

Let $F$ be a skewfield and let $V$ be a vector space over $F$. A collection $B$ of subspaces of $V$ with $|B| \geq 3$ is called a spread of $V$ if for any two different elements $U_1, U_2 \in B$ we have $V = U_1 \oplus U_2$ and every vector of $V$ is contained in an element of $B$.

If $S$ and $W$ are different subspaces of the spread $B$, then $V$ can be coordinatized in such a way that $S = \{0\} \times X$ and $W = X \times \{0\}$. Any spread of $V = X \times X$ can be described by a collection $M$ of linear mappings $X \to X$ satisfying the following conditions:

$(M_1)$ For any $\omega_1 \neq \omega_2 \in M$ the mapping $\omega_1 - \omega_2$ is bijective.

$(M_2)$ For all $x \in X \setminus \{0\}$ the mapping $\phi_x : M \to X : \omega \mapsto \omega(x)$ is surjective.

Namely, if $M$ is a collection of linear mappings satisfying $(M_1)$ and $(M_2)$,
then the sets \( U_\omega = \{(x, \omega(x)), x \in X\} \) and \( \{0\} \times X \) yield a spread of \( V = X \times X \). Conversely, every component \( U \in \mathcal{B} \setminus \{S\} \) of \( V \) is the graph of a linear mapping \( \omega_U : W \to S \) and the set of \( \omega_U \) gives a collection \( \mathcal{M} \) of linear mappings of \( X \) satisfying \((M_1)\) and \((M_2)\) (cf. [13], Proposition 1.11.). The mapping \( \omega_W \) is the zero mapping. For this reason any collection \( \mathcal{M} \) of linear mappings of \( X \) a spread set of \( X \).

Every translation plane can be obtained from a spread set of a suitable vector space \( V = X \times X \) (cf. [13], Theorem 1.5, p. 7, and [1]). As every translation plane can be coordinatized by a quasifield and a quasifield contains 0 and 1, the associated spread set contains the zero endomorphism and the identity map. This is not true for arbitrary spread sets \( \mathcal{M} \), but if \( \omega_0, \omega_1 \in \mathcal{M} \) are distinct, then \( \mathcal{M}' = \{(\omega - \omega_0)(\omega_1 - \omega_0)^{-1}, \omega \in \mathcal{M}\} \) is a normalized spread of \( X \) which contains the zero and the identity map and the translation planes obtained from \( \mathcal{M} \) and \( \mathcal{M}' \) are isomorphic (cf. [13], Lemma 1.15, p. 13).

Let \( \mathcal{M} \) be a normalized spread of \( X \), \( e \in X \setminus \{0\} \) and let \( \phi_e : \mathcal{M} \to X \) be defined by \( \phi_e(\omega) = \omega(e) \). Then the multiplication \( \circ : X \times X \to X \) defined by \( m \circ x = (\phi_e^{-1}(m))(x) \) yields a multiplicative loop of a left quasifield \( Q \) coordinatizing the translation plane \( \mathcal{A} \) belonging to the spread \( \mathcal{M} \) of \( X \).

If we fix a basis of \( Q \) over its kernel \( K_r \) and identify \( X \) with the vector space of pairs \( \{(x, y), x, y \in K_r\} \), then the set \( \mathcal{M} \) consists of matrices \( C(\alpha, \beta, \gamma, \delta) = \)
\[
\begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix}, \alpha, \beta, \gamma, \delta \in K_r.
\]
If \( e = (1, 0)^t \), then we get \( \phi_e(C(\alpha, \beta, \gamma, \delta)) = C(\alpha, \beta, \gamma, \delta)(e) = (\alpha, \gamma)^t \). Since \( \mathcal{M} \) is a spread of \( X \) the set of vectors \((\alpha, \gamma)^t\) consists of all vectors of \( X \). Hence if \((\alpha, \gamma)^t\) is an element of \( X \), then there exists a unique matrix of \( \mathcal{M} \) having \((\alpha, \gamma)^t\) as the first column.

We consider multiplicative loops of locally compact connected topological quasifields \( Q \) of dimension 2 coordinatizing 4-dimensional non-desarguesian topological translation planes. Then the kernel \( K_r \) of \( Q \) is isomorphic to the field of the real numbers, \((Q, +)\) is the vector group \( \mathbb{R}^2 \) and the multiplicative loop \((Q\setminus\{0\}, \cdot)\) is homeomorphic to \( \mathbb{R} \times S^1 \), where \( S^1 \) is the circle.

4. Multiplicative loops of 2-dimensional quasifields

Let \((Q, +, \cdot)\) be a real topological (left) quasifield of dimension 2. Let \( e_1 \) be the identity element of the multiplicative loop \( Q^\ast = (Q \setminus \{0\}, \cdot) \) of \( Q \), which generates the kernel \( K_r = \mathbb{R} \) of \( Q \) as a vector space and let \( B = \{e_1, e_2\} \) be a basis of the right vector space \( Q \) over \( K_r \). Once we fix \( B \), we identify \( Q \) with the vector space of pairs \((x, y)^t \in \mathbb{R}^2 \) and \( K_r \) with the subspace of pairs \((x, 0)^t \). The element \((1, 0)^t \) is the identity element of \( Q^\ast \). According to [14], Theorem 29.1, p. 345, the group \( G \) topologically generated by the left translations of \( Q^\ast \) is the connected component of the group \( \mathrm{GL}_2(\mathbb{R}) \). As \( \dim Q^\ast = 2 \) and the stabilizer \( H \) of the identity element of \( Q^\ast \) in \( G \) does
not contain any non-trivial normal subgroup of $G$ we assume that $H$ is the subgroup
\[ \begin{cases} \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, a > 0, b \in \mathbb{R} \end{cases} \]. The elements $g$ of $G$ have a unique
decomposition as the product
\[
g = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} k & l \\ 0 & k^{-1} \end{pmatrix}
\]
with suitable elements $u \in \mathbb{R}\setminus\{0\}$, $k > 0$, $l \in \mathbb{R}$, $t \in [0, 2\pi)$. Hence the loop
$Q^*$ corresponds to a continuous section $\sigma : G/H \to G$;
\[
\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} H \mapsto \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} a(u, t) & b(u, t) \\ 0 & a^{-1}(u, t) \end{pmatrix}
\]
where the pair of continuous functions $a(u, t), b(u, t) : \mathbb{R}\setminus\{0\} \times \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:
\[
a(u, t) > 0, \quad a(1, 2\pi k) = 1, \quad b(1, 2\pi k) = 0 \quad \text{for all} \quad k \in \mathbb{Z}.
\]
As $Q$ is a left quasifield, any $(x, y)^t \in Q^*$ induces a linear transformation
$M(x, y) \in \sigma(G/H)$. More precisely one has
\[
\begin{pmatrix} x \\ y \end{pmatrix} = M(x, y) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix} \begin{pmatrix} a(r, \varphi) & b(r, \varphi) \\ 0 & a^{-1}(r, \varphi) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix},
\]
where $x = r \cos(\varphi)a(r, \varphi)$, $y = -r \sin(\varphi)a(r, \varphi)$. The kernel $K_r$ of $Q$ consists
of $(0, 0)^t$ and $(ra(r, 0), 0)^t$, $r \in \mathbb{R}\setminus\{0\}$, such that the matrices corresponding
to the elements $(ra(r, 0), 0)^t$ have the form
\[
M(ra(r, 0), 0) = \begin{pmatrix} ra(r, 0) & rb(r, 0) \\ 0 & ra^{-1}(r, 0) \end{pmatrix}.
\]
The identity matrix $I$ corresponds to the identity $(1, 0)^t$ of $Q^*$. Since to each real number $ra(r, 0)$ corresponds precisely one matrix $M(ra(r, 0), 0)$, the function $f(r) = ra(r, 0)$ is strictly monotone. If the function $a(r, 0)$ is differentiable, then for every $r \in \mathbb{R} \setminus \{0\}$ the derivative $a(r, 0) + ra'(r, 0)$ is either always positive or negative. This is equivalent to the fact that the derivative $[\ln(a(r, 0))]'$ is always greater or smaller than $-r^{-1}$.

**Remark 1.** The set $K = \{M(ra(r, 0), 0); r \in \mathbb{R} \setminus \{0\}\}$ of the left translations of $Q^*$ corresponding to the kernel $K_r$ of $Q$ is

$$\begin{cases}
\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}, r \in \mathbb{R} \setminus \{0\}
\end{cases}
$$

if and only if one has $a(r, 0) = 1$, $b(r, 0) = 0$ for all $r \in \mathbb{R} \setminus \{0\}$.

The section $\sigma$ given by (2) is sharply transitive precisely if for all pairs $(u_1, t_1), (u_2, t_2)$ in $\mathbb{R} \setminus \{0\} \times [0, 2\pi)$ there exists precisely one $(u, t) \in \mathbb{R} \setminus \{0\} \times [0, 2\pi)$ and $k > 0$, $l \in \mathbb{R}$ such that

$$\begin{pmatrix}
 u \cos t & u \sin t \\
 -u \sin t & u \cos t
\end{pmatrix}
\begin{pmatrix}
 a(u, t) & b(u, t) \\
 0 & a^{-1}(u, t)
\end{pmatrix}
\begin{pmatrix}
 u_1 \cos t_1 & u_1 \sin t_1 \\
 -u_1 \sin t_1 & u_1 \cos t_1
\end{pmatrix}
= 
\begin{pmatrix}
 u_2 \cos t_2 & u_2 \sin t_2 \\
 -u_2 \sin t_2 & u_2 \cos t_2
\end{pmatrix}
\begin{pmatrix}
 k & l \\
 0 & k^{-1}
\end{pmatrix}. 
\tag{4}
$$

As the determinant of the matrices on both sides of (4) are equal we get that $u = u_1^{-1}u_2$. Therefore the system (4) of equations is uniquely solvable if and
only if for any fixed \( u \in \mathbb{R} \setminus \{0\} \) the mapping

\[
\sigma_u : \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} H \mapsto \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} a(u, t) & b(u, t) \\ 0 & a^{-1}(u, t) \end{pmatrix}
\]

determines a quasigroup \( F_u \) homeomorphic to \( S^1 \). One may take as the points of \( F_u \) the vectors \( (ua(u, t)a^{-1}(u, 0) \cos t, -ua(u, t)a^{-1}(u, 0) \sin t)^t \) and as the section the mapping

\[
\sigma_u : \begin{pmatrix} ua(u, t)a^{-1}(u, 0) \cos t \\ -ua(u, t)a^{-1}(u, 0) \sin t \end{pmatrix} \mapsto \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} a(u, t)a^{-1}(u, 0) & b(u, t) \\ 0 & a^{-1}(u, t)a(u, 0) \end{pmatrix} = \begin{pmatrix} a(u, t)a^{-1}(u, 0) \cos t & b(u, t) \cos t + a^{-1}(u, t)a(u, 0) \sin t \\ -a(u, t)a^{-1}(u, 0) \sin t & -b(u, t) \sin t + a^{-1}(u, t)a(u, 0) \cos t \end{pmatrix}.
\]

(5)

In this way we see that the quasigroup \( F_u \) has the right identity \((u, 0)^t\) since

\[
\sigma_u \begin{pmatrix} ua(u, t)a^{-1}(u, 0) \cos t \\ -ua(u, t)a^{-1}(u, 0) \sin t \end{pmatrix} \cdot \begin{pmatrix} u \\ 0 \end{pmatrix} = \begin{pmatrix} ua(u, t)a^{-1}(u, 0) \cos t \\ -ua(u, t)a^{-1}(u, 0) \sin t \end{pmatrix}.
\]

The quasigroup \( F_u \) is a loop, i.e. \((u, 0)^t\) is the left identity of \( F_u \), if and only if

\[
\sigma_u \begin{pmatrix} u \\ 0 \end{pmatrix} = \begin{pmatrix} a(u, 0)a^{-1}(u, 0) \cos 0 & b(u, 0) \cos 0 \\ 0 & a^{-1}(u, 0)a(u, 0) \cos 0 \end{pmatrix} = \begin{pmatrix} 1 & b(u, 0) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

which means \( b(u, 0) = 0 \) for all \( u \in \mathbb{R} \setminus \{0\} \). The almost \( C^1 \)-differentiable loop \( Q^* \) belonging to the sharply transitive \( C^1 \)-differentiable section \( \sigma \) given by (2) is \( C^1 \)-differentiable precisely if the mapping \((xH, yH) \mapsto z : G/H \times G/H \to \sigma(G/H)\) determined by \( zxH = yH \) is \( C^1 \)-differentiable (cf. [14], p. 32), i.e. the solutions \( u \in \mathbb{R} \setminus \{0\}, \ t \in [0,2\pi) \) of the matrix equation (4) are continuously differentiable functions of \( u_1, u_2 \in \mathbb{R} \setminus \{0\}, \ t_1, t_2 \in \)
[0, 2\pi). The function \( u = u_1^{-1}u_2 \) is continuously differentiable. If for each fixed \( u \in \mathbb{R} \setminus \{0\} \) the section \( \sigma_u \) given by (5) yields a 1-dimensional \( C^1 \) -differentiable compact loop, then the function \( t(u_1,u_2,t_1,t_2) = t_{(u_1,u_2)}(t_1,t_2) \) is continuously differentiable (cf. [14], Examples 20.3, p. 258). Indeed, the function \( t_{(u_1,u_2)}(t_1,t_2) \) is determined implicitly by equations which depend continuously differentiably also on the parameters \( u_1 \) and \( u_2 \). Applying the above discussion we can prove the following:

**Theorem 2.** Let \( Q^* \) be the \( C^1 \) -differentiable multiplicative loop of a locally compact 2-dimensional connected topological quasifield \( Q \). Then \( Q^* \) is diffeomorphic to \( S^1 \times \mathbb{R} \) and belongs to a \( C^1 \) -differentiable sharply transitive section \( \sigma \) of the form

\[
\begin{pmatrix}
 u \cos t & u \sin t \\
 -u \sin t & u \cos t
\end{pmatrix}
\begin{pmatrix}
 a(u,t) & b(u,t) \\
 0 & a^{-1}(u,t)
\end{pmatrix}.
\]

with \( b(u,0) = 0 \) for all \( u \in \mathbb{R} \setminus \{0\} \) if and only if for each fixed \( u \in \mathbb{R}\setminus\{0\} \) the function \( a_u^{-1}(t) := a(u,0)a^{-1}(u,t) \) has the shape

\[
a_u^{-1}(t) = e^t(1 - \int_0^t R(s)e^{-s} \, ds)
\]

where \( R(s) \) is a continuous function, the Fourier series of which is contained in the set \( \mathcal{F} \) of Definition 1 in [10] and converges uniformly to \( R \). Moreover, \( b_u(t) := b(u,t) \) is a periodic \( C^1 \) -differentiable function with \( b_u(0) = b_u(2\pi) = 0 \) such that
\[ b_u(t) > -a_u(t) \int_0^t \frac{(a_u^2(s) - a_u^2(s))}{a_u^4(s)} \, ds \quad \text{for all} \quad t \in (0, 2\pi). \]

**Proof.** The section \( \sigma_u \) given by (5) yields a 1-dimensional \( C^1 \)-differentiable compact loop having the group \( SL_2(\mathbb{R}) \) as the group topologically generated by its left translations if and only if for each fixed \( u \in \mathbb{R}\setminus\{0\} \) the continuously differentiable functions \( a(u,0)a^{-1}(u,t) := \bar{a}_u(t), \quad -b(u,t) := \bar{b}_u(t) \) satisfy the conditions

\[
\bar{a}_u'(t) + \bar{b}_u(t)\bar{a}_u'(t) + \bar{b}_u'(t)\bar{a}_u(t) - \bar{a}_u^2(t) < 0, \quad \bar{b}_u'(0) < 1 - \bar{a}_u^2(0) \tag{6}
\]

(cf. [14], Section 18, (C), p. 238, [10], pp. 132-139). The solution of the differential inequalities (6) is given by Theorem 6 in [10], pp. 138-139. This proves the assertion. \( \square \)

**Proposition 3.** Let \( Q^* \) be the \( C^1 \)-differentiable multiplicative loop of a locally compact 2-dimensional connected topological quasifield \( Q \). Assume that for each fixed \( u \in \mathbb{R}\setminus\{0\} \) the function \( a_u(t) := a^{-1}(u,0)a(u,t) \) is the constant function 1 and that \( b(u,0) = 0 \) is satisfied for all \( u \in \mathbb{R}\setminus\{0\} \). Then \( Q^* \) belongs to a \( C^1 \)-differentiable sharply transitive section \( \sigma \) of the form (2) if and only if for each fixed \( u \in \mathbb{R}\setminus\{0\} \) one has \( b_u(t) := b(u,t) > -t \) for all \( 0 < t < 2\pi \).

**Proof.** If for each fixed \( u \in \mathbb{R}\setminus\{0\} \) the function \( a(u,0)a^{-1}(u,t) = a_u^{-1}(t) = \bar{a}_u(t) \) is constant with value 1, then the section \( \sigma_u \) given by (5) yields a \( C^1 \)-
differentiable compact loop $L$ if and only if for each fixed $u \in \mathbb{R}\{0\}$ the continuously differentiable function $\bar{b}_u(t) := -b_u(t)$ satisfies the differential inequality $\bar{b}_u'(t) < 1$ with the initial condition $\bar{b}_u'(0) < 1$ (cf. (6)). This is the case precisely if one has $b_u(t) > -t$ for all $0 < t < 2\pi$. \hfill \square

**Proposition 4.** Let $Q^*$ be the $C^1$-differentiable multiplicative loop of a locally compact 2-dimensional connected topological quasifield $Q$. Assume that for each fixed $u \in \mathbb{R}\{0\}$ the function $b(u,t)$ is the constant function 0. Then $Q^*$ belongs to a $C^1$-differentiable sharply transitive section $\sigma$ of the form (2) precisely if for each fixed $u \in \mathbb{R}\{0\}$ one has $e^{-t} < a(u,t)a^{-1}(u,0) < e^t$ for all $0 < t < 2\pi$.

**Proof.** If for each fixed $u \in \mathbb{R}\{0\}$ the function $b(u,t) = -\bar{b}_u(t)$ is constant with value 0, then the section $\sigma_u$ given by (5) determines a $C^1$-differentiable compact loop $L$ if and only if for each fixed $u \in \mathbb{R}\{0\}$ the following inequalities are satisfied:

$$(\bar{a}_u'(t) - \bar{a}_u(t))(\bar{a}_u'(t) + \bar{a}_u(t)) < 0, \quad 0 < 1 - \bar{a}_u^2(0),$$

where $\bar{a}_u(t) = a(u,0)a^{-1}(u,t)$. This is the case precisely if either one has $\bar{a}_u'(t) - \bar{a}_u(t) < 0$ and $\bar{a}_u'(t) + \bar{a}_u(t) > 0$ or one has $\bar{a}_u'(t) - \bar{a}_u(t) > 0$ and $\bar{a}_u'(t) + \bar{a}_u(t) < 0$. Now we consider the first case. Then the function $\bar{a}_u(t)$ determines a loop if and only if for each fixed $u \in \mathbb{R}\{0\}$ it is a subfunction of a differentiable function $h_u(t) := h(u, t)$ with $h_u(0) = 1$, $h_u^2(0) = 1$. 15
\[ h'_u(t) = h_u(t) \] and an upper function of a differentiable function \( l_u(t) := l(u, t) \)

with \( l_u(0) = 1, l'_u(0) = 1, l'_u(t) = -l_u(t) \) (cf. [19], p. 66). Hence for each fixed \( u \in \mathbb{R} \setminus \{0\} \) the function \( \bar{a}_u(t) \) is a subfunction of the function \( e^t \) and an upper function of the function \( e^{-t} \) for all \( t \in (0, 2\pi) \). Therefore, any continuously differentiable function \( \bar{a}_u(t) \) such that for each fixed \( u \in \mathbb{R} \setminus \{0\} \) and for all \( t \in (0, 2\pi) \) one has \( e^{-t} < \bar{a}_u(t)^{-1} < e^t \) determines a \( C^1 \)-differentiable compact loop \( L \).

In the second case an analogous consideration as in the first case gives that for all fixed \( u \in \mathbb{R} \setminus \{0\} \) the function \( a(u, t)a^{-1}(u, 0) \) must be a subfunction of the function \( e^{-t} \) and an upper function of the function \( e^t \) for all \( t \in (0, 2\pi) \). Hence in this case the function \( a(u, t)a^{-1}(u, 0) \) does not exist.

**Proposition 5.** Let

\[
\begin{pmatrix}
  u \cos t & u \sin t \\
  -u \sin t & u \cos t
\end{pmatrix}
\quad H \mapsto
\begin{pmatrix}
  u & 0 \\
  0 & u
\end{pmatrix}
\begin{pmatrix}
  \cos t & \sin t \\
  -\sin t & \cos t
\end{pmatrix}
\begin{pmatrix}
  a(1, t) & b(u, t) \\
  0 & a^{-1}(1, t)
\end{pmatrix}, \quad u \in \mathbb{R} \setminus \{0\}, \quad t \in \mathbb{R} \quad (7)
\]

with \( b(u, 0) = 0 \) for all \( u \in \mathbb{R} \setminus \{0\} \) be a section belonging to a multiplicative loop \( Q^* \) of a locally compact 2-dimensional connected topological quasifield \( Q \). Then \( Q^* \) contains for any \( u \in \mathbb{R} \setminus \{0\} \) a 1-dimensional compact subloop.

**Proof.** The image of the section (7) acts sharply transitively on the point set \( \mathbb{R}^2 \setminus \{(0, 0)^t\} \). Since the subgroup \( \left\{ \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}, \quad u \in \mathbb{R} \setminus \{0\} \right\} \) leaves any line
through $(0,0)^t$ fixed, the subset

$$
\mathcal{T} = \left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} a(1,t) & b(u,t) \\ 0 & a^{-1}(1,t) \end{pmatrix}, t \in \mathbb{R} \right\}
$$

acts sharply transitively on the oriented lines through $(0,0)^t$ for any $u \in \mathbb{R} \setminus \{0\}$. Therefore $\mathcal{T}$ corresponds to a 1-dimensional compact loop since $b(u,0) = 0$ for all $u \in \mathbb{R} \setminus \{0\}$.

As $\mathcal{T}$ given by (8) is the image of a section corresponding to a 1-dimensional compact subloop of $Q^*$, every element of $\mathcal{T}$ is elliptic.

**Proposition 6.** Every element of the set $\mathcal{T}$ given by (8) is elliptic if and only if the following holds:

1) if for all $t \in \mathbb{R}$ and $u \in \mathbb{R} \setminus \{0\}$ one has $b(u,t) = 0$, then the function $a(1,t)$ satisfies the inequalities:

$$
\frac{1 - |\sin(t)|}{|\cos(t)|} \leq a(1,t) \leq \frac{1 + |\sin(t)|}{|\cos(t)|},
$$

(9)

2) if the function $b(u,t)$ is different from the constant function 0, then for $\sin(t) > 0$ one has

$$
\frac{(a(1,t) + a(1,t)^{-1}) \cos(t) - 2}{\sin(t)} < b(u,t) < \frac{(a(1,t) + a(1,t)^{-1}) \cos(t) + 2}{\sin(t)},
$$

(10)

for $\sin(t) < 0$ we have

$$
\frac{(a(1,t) + a(1,t)^{-1}) \cos(t) + 2}{\sin(t)} < b(u,t) < \frac{(a(1,t) + a(1,t)^{-1}) \cos(t) - 2}{\sin(t)}.
$$

(11)
Proof. Any element of (8) is elliptic if and only if the inequality

\[ |\cos(t)(a(1, t) + a(1, t)^{-1}) - \sin(t)b(u, t)| \leq 2 \tag{12} \]

holds, where the equality sign occurs only for \( t = k\pi, k \in \mathbb{Z} \). If \( b(u, t) = 0 \), then inequality (12) reduces to \( a^2(1, t)\cos(t) - 2a(1, t) + |\cos(t)| \leq 0 \) which is equivalent to inequalities (9). If \( b(u, t) \neq 0 \), then inequality (12) is equivalent for all \( t \neq k\pi, k \in \mathbb{Z} \), to

\[ (a(1, t) + a(1, t)^{-1})^2 \cos^2(t) - 2(a(1, t) + a(1, t)^{-1}) \sin(t) \cos(t)b(u, t) + \sin^2(t)b^2(u, t) < 4. \tag{13} \]

Solving the quadratic equation

\[ (a(1, t) + a(1, t)^{-1})^2 \cos^2(t) - 2(a(1, t) + a(1, t)^{-1}) \sin(t) \cos(t)x + \sin^2(t)x^2 = 4 \tag{14} \]

we get

\[ x = \frac{2(a(1, t) + a(1, t)^{-1}) \cos(t) \sin(t) \pm 4 \sin(t)}{2 \sin^2(t)} = \frac{(a(1, t) + a(1, t)^{-1}) \cos(t) \pm 2}{\sin(t)}. \]

Comparing (13) and (14) one obtains

\[ \left( b(u, t) - \frac{(a(1, t) + a(1, t)^{-1}) \cos(t) - 2}{\sin(t)} \right) \left( b(u, t) - \frac{(a(1, t) + a(1, t)^{-1}) \cos(t) + 2}{\sin(t)} \right) < 0, \]

which yields inequalities (10) and (11). \( \square \)

**Proposition 7.** The multiplicative loop \( Q^* \) of a locally compact connected topological quasifield \( Q \) of dimension 2 is the field \( \mathbb{C} \) of complex numbers if and only if it contains a 1-dimensional compact normal subloop.

Proof. If \( Q \) is the field of complex numbers, then \( Q^* \) is the group \( \mathrm{SO}_2(\mathbb{R}) \times \mathbb{R} \) and the assertion is true. Assume that the loop \( Q^* \) contains a 1-dimensional compact normal subloop. If \( Q^* \) is a proper loop, then the group topologically generated by its left translations is the connected component \( \mathrm{GL}_2^+(\mathbb{R}) \) of \( \mathrm{GL}_2(\mathbb{R}) \) (cf. [14], Theorem 29.1, p. 345). By Lemma 1.7, p. 19, in [14], the left translations of a normal subloop of \( Q^* \) generate a normal subgroup \( N \) of
GL_2^+(\mathbb{R}) which can be only the group SL_2(\mathbb{R}). This contradiction proves the assertion.

Lemma 8. If the multiplicative loop Q* of a 2-dimensional locally compact connected topological quasifield Q is not quasi-simple, then the set \( K = \{ M(ra(r,0),0); r \in \mathbb{R} \setminus \{0\} \} \) of the left translations of Q* corresponding to the kernel \( K_r \) of Q has the form \( K = \left\{ \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}, 0 \neq r \in \mathbb{R} \right\} \), which is a normal subgroup of the set \( \Lambda Q^* \) of all left translations of Q*.

Proof. If Q is the field of complex numbers, then the assertion is true. If the loop Q* is proper and not quasi-simple, then the set \( \Lambda Q^* \) of the left translations of Q* must contain the group \( K < GL_2^+(\mathbb{R}) \) as a normal subgroup. 

Assume that the set \( K \) of the left translations of the loop Q* having \((1,0)^t\) as identity corresponding to the elements of the kernel \( K_r \) of Q has the form given in Lemma 8. According to (3) the element

\[
\begin{pmatrix}
ra(r,\varphi)\cos\varphi & rb(r,\varphi)\cos\varphi + ra^{-1}(r,\varphi)\sin\varphi \\
-ra(r,\varphi)\sin\varphi & -rb(r,\varphi)\sin\varphi + ra^{-1}(r,\varphi)\cos\varphi
\end{pmatrix}
\]

corresponds to the left translation of \((ra(r,\varphi)\cos\varphi, -ra(r,\varphi)\sin\varphi)^t\). Let \( N^* \) be the subgroup of Q* corresponding to the normal subgroup \( K \) of \( \Lambda Q^* \). We show that \( N^* := \{(s,0)^t, s \in \mathbb{R} \setminus \{0\}\} \) is normal in Q*. For all elements \( x := (\cos\varphi, -\sin\varphi)^t, y := (u, v)^t \) of Q* the condition \((N^*x)*y = N^*(x*y)\)
of (1) is satisfied if and only if we have
\[
\begin{bmatrix}
    s \\
    0
\end{bmatrix} * \begin{bmatrix}
    \cos \varphi \\
    -\sin \varphi
\end{bmatrix} * \begin{bmatrix}
    u \\
    v
\end{bmatrix} = \begin{bmatrix}
    s' \\
    0
\end{bmatrix} * \begin{bmatrix}
    \cos \varphi \\
    -\sin \varphi
\end{bmatrix} * \begin{bmatrix}
    u \\
    v
\end{bmatrix}
\]
for all \( \varphi \in \mathbb{R}, \ (u, v) \in \mathbb{R}^2 \setminus \{(0, 0)\} \) with suitable \( s, s' \in \mathbb{R} \setminus \{0\} \). This is the case precisely if one has
\[
\begin{bmatrix}
    \cos \varphi & -\sin \varphi \\
\end{bmatrix} * \begin{bmatrix}
    s \\
    0
\end{bmatrix} * \begin{bmatrix}
    u \\
    v
\end{bmatrix} = \begin{bmatrix}
    \cos \varphi & -\sin \varphi \\
\end{bmatrix} * \begin{bmatrix}
    s' \\
    0
\end{bmatrix} * \begin{bmatrix}
    u \\
    v
\end{bmatrix}
\]
for all \( \varphi \in \mathbb{R}, \ (u, v) \in \mathbb{R}^2 \setminus \{(0, 0)\} \) with suitable \( s, s' \in \mathbb{R} \setminus \{0\} \). This is the case precisely if one has
\[
\begin{bmatrix}
    \cos \varphi & -\sin \varphi \\
\end{bmatrix} * \begin{bmatrix}
    s \\
    0
\end{bmatrix} * \begin{bmatrix}
    u \\
    v
\end{bmatrix} = \begin{bmatrix}
    \cos \varphi & -\sin \varphi \\
\end{bmatrix} * \begin{bmatrix}
    s' \\
    0
\end{bmatrix} * \begin{bmatrix}
    u \\
    v
\end{bmatrix}
\]

The last equation holds if and only if
\[
a(s, \varphi)a^{-1}(1, \varphi) - a^{-1}(s, \varphi)a(1, \varphi) = 0, b(s, \varphi)b^{-1}(1, \varphi) - b^{-1}(s, \varphi)b(1, \varphi) = 0.
\]

As \( a(s, \varphi) \) is positive we have \( a(s, \varphi) = a(1, \varphi) \) and \( b(s, \varphi) = b(1, \varphi) \) for all \( s \in \mathbb{R} \setminus \{0\}, \ \varphi \in \mathbb{R} \). By (1) the group \( N^* \) is a normal subgroup of \( Q^* \) if and only if for all \( \varphi \) and all \((u, v) \in \mathbb{R}^2 \setminus \{(0, 0)\} \) one has
\[
\begin{bmatrix}
    \cos \varphi \\
    -\sin \varphi
\end{bmatrix} * \begin{bmatrix}
    s \\
    0
\end{bmatrix} * \begin{bmatrix}
    u \\
    v
\end{bmatrix} = \begin{bmatrix}
    \cos \varphi \\
    -\sin \varphi
\end{bmatrix} * \begin{bmatrix}
    s' \\
    0
\end{bmatrix} * \begin{bmatrix}
    u \\
    v
\end{bmatrix}
\]
\[
\begin{pmatrix}
  sa(1, \varphi)a(1, \varphi) \cos \varphi & sa(1, \varphi)b(1, \varphi) \cos \varphi + ss \sin \varphi \\
  -sa(1, \varphi)a(1, \varphi) \sin \varphi & -sa(1, \varphi)b(1, \varphi) \sin \varphi + ss \cos \varphi \\
  a(1, \varphi) \cos \varphi & b(1, \varphi) \cos \varphi + a^{-1}(1, \varphi) \sin \varphi \\
  -a(1, \varphi) \sin \varphi & -b(1, \varphi) \sin \varphi + a^{-1}(1, \varphi) \cos \varphi
\end{pmatrix}
\begin{pmatrix}
u \\
v
\end{pmatrix}
= \\
\begin{pmatrix}
  t \cos \varphi \\
  u \\
  s' \cos \varphi \\
  s' \sin \varphi
\end{pmatrix}
\begin{pmatrix}
  u \\
  v \\
  s' u \\
  s' v
\end{pmatrix}
\]
for suitable \( s, s' \in \mathbb{R} \setminus \{0\} \). This is equivalent to
\[
\begin{pmatrix}
  sua(1, \varphi) \cos \varphi + sv[a(1, \varphi)b(1, \varphi) \cos \varphi + \sin \varphi] \\
  -sua(1, \varphi) \sin \varphi + sv[-a(1, \varphi)b(1, \varphi) \sin \varphi + \cos \varphi] \\
  us'a(1, \varphi) \cos \varphi + s'v[b(1, \varphi) \cos \varphi + a^{-1}(1, \varphi) \sin \varphi] \\
  -us'a(1, \varphi) \sin \varphi + s'v[-b(1, \varphi) \sin \varphi + a^{-1}(1, \varphi) \cos \varphi]
\end{pmatrix}
\]
A direct computation yields that
\[
[ua(1, \varphi)^2 \cos \varphi + va(1, \varphi)b(1, \varphi) \cos \varphi + v \sin \varphi] - [-ua(1, \varphi) \sin \varphi - vb(1, \varphi) \sin \varphi + va^{-1}(1, \varphi) \cos \varphi] = \\
[-ua(1, \varphi)^2 \sin \varphi - va(1, \varphi)b(1, \varphi) \sin \varphi + v \cos \varphi] \cdot [ua(1, \varphi) \cos \varphi + vb(1, \varphi) \cos \varphi + va^{-1}(1, \varphi) \sin \varphi].
\]
Using Proposition 7, Lemma 8 and the discussion above we have the following

**Theorem 9.** The multiplicative loop \( Q^* \) of a locally compact 2-dimensional quasifield \( Q \) with \((1, 0)^t \) as identity of \( Q^* \) is not quasi-simple if and only if for all \( r \in \mathbb{R} \setminus \{0\} \), \( \varphi \in \mathbb{R} \) one has \( a(r, 0) = 1 \), \( b(r, 0) = 0 \), \( a(r, \varphi) = a(1, \varphi) \) and \( b(r, \varphi) = b(1, \varphi) \). Then \( Q^* \) is a split extension of a 1-dimensional normal subgroup \( N^* \) by a subloop homeomorphic to the 1-sphere. Moreover, one has
a) \( N^* \) is isomorphic to \( \mathbb{R} \) or to \( \mathbb{R} \times Z_2 \), where \( Z_2 \) is the group of order 2.
b) This extension is the direct product precisely if \( Q \) is the field \( \mathbb{C} \).
Proof. We have only to prove a) and b). According to Lemma 8 and the above discussion the only possibility for a normal subloop of positive dimension is the group $N^*$. The intersection of a compact subloop of $Q^*$ with $N^*$ has cardinality at most 2 (cf. Proposition 5 and Lemma 8). Hence the claim a) is proved. The claim of b) follows from Proposition 7.

The set $\Lambda_{Q^*}$ of the left translations of $Q^*$ with a normal subloop of positive dimension has the form

$$\left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} ua(1,t) & ub(1,t) \\ 0 & ua^{-1}(1,t) \end{pmatrix}, u \in \mathbb{R}\{0\}, t \in [0, 2\pi) \right\}.$$  \hfill (15)

5. Decomposable multiplicative loops of 2-dimensional quasifields

**Definition 1.** We call the multiplicative loop $Q^*$ of a locally compact connected topological 2-dimensional quasifield $Q$ decomposable, if the set of all left translations of $Q^*$ is a product $\mathcal{T}\mathcal{K}$ with $|\mathcal{T} \cap \mathcal{K}| \leq 2$, where $\mathcal{T}$ is the set of all left translations of a 1-dimensional compact loop of the form (8) and $\mathcal{K}$ is the set of all left translations of $Q^*$ corresponding to the kernel $K_r$ of $Q$.

If the loop $Q^*$ is decomposable, then it contains compact subloops for any $u \in \mathbb{R}\{0\}$ corresponding to the section (7). From now on we choose $u = 1$. 22
Then one has
\[
\begin{pmatrix}
\cos t a(1, t) & \cos t b(1, t) + \sin t a^{-1}(1, t) \\
-\sin t a(1, t) & -\sin t b(1, t) + \cos t a^{-1}(1, t)
\end{pmatrix}
\begin{pmatrix}
ra(r, 0) & rb(r, 0) \\
0 & ra^{-1}(r, 0)
\end{pmatrix}
\begin{pmatrix}
1 \\
0
\end{pmatrix}
= 
\begin{pmatrix}
 r \cos t a(r, t) & r \cos t b(r, t) + r \sin t a^{-1}(r, t) \\
-r \sin t a(r, t) & -r \sin t b(r, t) + r \cos t a^{-1}(r, t)
\end{pmatrix}
\begin{pmatrix}
 1 \\
 0
\end{pmatrix}.
\]
(16)

Equation (16) yields that \( a(r, t) = a(1, t) a(r, 0) \).

Now we give sufficient and necessary conditions for the loop \( Q^* \) to be decomposable.

**Proposition 10.** The multiplicative loop \( Q^* \) of a locally compact connected topological 2-dimensional quasifield \( Q \) with \((1, 0)^t\) as identity of \( Q^* \) is decomposable if and only if for all \( r \in \mathbb{R} \setminus \{0\}, t \in \mathbb{R} \) one has
\[
a(r, t) = a(1, t) a(r, 0) \text{ and } b(r, t) = a(1, t) b(r, 0) + a^{-1}(r, 0) b(1, t).
\]

**Proof.** The point \((x, y)^t\) is the image of the point \((1, 0)^t\) under the linear mapping \( M(x, y) \) and the set \( \{M(x, y); (x, y)^t \in Q^*\} \) acts sharply transitively on \( Q^* \). The matrix equation
\[
\begin{pmatrix}
\cos t a(1, t) & \cos t b(1, t) + \sin t a^{-1}(1, t) \\
-\sin t a(1, t) & -\sin t b(1, t) + \cos t a^{-1}(1, t)
\end{pmatrix}
\begin{pmatrix}
ra(r, 0) & rb(r, 0) \\
0 & ra^{-1}(r, 0)
\end{pmatrix}
\begin{pmatrix}
u \cos \varphi a(u, \varphi) \\
-u \sin \varphi a(u, \varphi)
\end{pmatrix}
= 
\begin{pmatrix}
r \cos t a(r, t) & r \cos t b(r, t) + r \sin t a^{-1}(r, t) \\
-r \sin t a(r, t) & -r \sin t b(r, t) + r \cos t a^{-1}(r, t)
\end{pmatrix}
\begin{pmatrix}
u \cos \varphi a(u, \varphi) \\
-u \sin \varphi a(u, \varphi)
\end{pmatrix}
\]
(17)
holds precisely if the identities of the assertion are satisfied. \(\square\)

**Theorem 11.** If the multiplicative loop \( Q^* \) of a locally compact connected topological 2-dimensional quasifield \( Q \) is not quasi-simple, then \( Q^* \) is decomposable.
Proof. By Theorem 9 the loop $Q^*$ is not quasi-simple if and only if for all $r \in \mathbb{R} \setminus \{0\}$, $t \in \mathbb{R}$ one has $a(r, 0) = 1$, $b(r, 0) = 0$, $a(r, t) = a(1, t)$ and $b(r, t) = b(1, t)$. Therefore the identities given in the assertion of Proposition 10 are satisfied.

If $Q^*$ is decomposable, then $|T \cap K| = 1$ if and only if one has $a(1, 0) = a(-1, 0) = a(1, \pi) = 1$ and $b(1, 0) = b(-1, 0) = b(1, \pi) = 0$, since $a(-1, \pi) = a(-1, 0)a(1, \pi) = 1$ as well as $b(-1, \pi) = b(-1, 0)a(1, \pi) = 0$. In this case the set of all left translations of $Q^*$ is a product $TW$ with $T \cap W = I$, where $W$ is the set of all left translations corresponding to the connected component of the kernel $K_r$ of $Q$. We say in this case that $Q^*$ is positively decomposable.

**Proposition 12.** The $C^1$-differentiable multiplicative loop $Q^*$ of a locally compact connected topological 2-dimensional quasifield $Q$ is decomposable precisely if for the inverse function $\bar{a}(1, t) = a^{-1}(1, t)$ and for $\bar{b}(1, t) = -b(1, t)$ the differential inequalities

$$\bar{a}''(1, t) + \bar{b}(1, t)\bar{a}'(1, t) + \bar{b}'(1, t)\bar{a}(1, t) - \bar{a}^2(1, t) < 0, \quad \text{and}$$

$$\bar{b}'(1, 0) < 1 - \bar{a}^2(1, 0) \quad (18)$$

are satisfied.

Proof. If $Q^*$ is a $C^1$-differentiable multiplicative loop of a quasifield $Q$, then the continuously differentiable functions $a(u, t) = \bar{a}^{-1}(u, t)$, $b(u, t) = -\bar{b}(u, t)$
satisfy the conditions in (6). The set of all left translations of \( Q^* \) is a product \( T\mathcal{K} \) if and only if \( a(u,t) = a(u,0)a(1,t) \) and \( b(u,t) = a(1,t)b(u,0) + a^{-1}(u,0)b(1,t) \) (cf. Proposition 10). Putting this into (6) we get

\[
a'^2(1,t) + b(1,t)a'(1,t)a^2(1,t) - b'(1,t)a^3(1,t) - a^2(1,t) < 0 \quad \text{and} \quad b'(1,0) > a'^2(1,0) - 1. \tag{19}
\]

Inequalities (19) are equivalent to the inequalities (18) with \( \bar{a}(1,t) = a^{-1}(1,t) \) and \( \bar{b}(1,t) = -b(1,t) \).

**Corollary 13.** Let \( T \) be any 1-dimensional \( C^1 \)-differentiable connected compact loop such that the set \( T \) of its left translations has the form (8) and let \( \mathcal{K} \) be any set of matrices of the form

\[
\mathcal{K} = \left\{ \begin{pmatrix} ua(u,0) & ub(u,0) \\ 0 & ua^{-1}(u,0) \end{pmatrix} , 0 \neq u \in \mathbb{R} \right\},
\]

where \( a(u,0) > 0 \) and \( b(u,0) \) are continuously differentiable functions such that \( ua(u,0) \) is strictly monotone. Then the product \( T\mathcal{K} \) is the set of all left translations of a \( C^1 \)-differentiable decomposable multiplicative loop \( Q^* \) of a 2-dimensional locally compact connected quasifield \( Q \).

**Proof.** As

\[
\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} a(1,t) & b(1,t) \\ 0 & a(1,t)^{-1} \end{pmatrix} \begin{pmatrix} ua(u,0) & ub(u,0) \\ 0 & ua^{-1}(u,0) \end{pmatrix}
\]
\[
\begin{pmatrix}
\cos t & \sin t \\
-\sin t & \cos t
\end{pmatrix}
\begin{pmatrix}
a(u,0)a(1,t) & ub(u,0)a(1,t) + ub(1,t)a^{-1}(u,0) \\
0 & ua^{-1}(u,0)a(1,t)^{-1}
\end{pmatrix}
= 
\begin{pmatrix}
\cos t & \sin t \\
-\sin t & \cos t
\end{pmatrix}
\begin{pmatrix}
a(u,t) & ub(u,t) \\
0 & ua^{-1}(u,t)
\end{pmatrix}
\]

and for the continuously differentiable functions \(a(1,t), b(1,t)\) the inequalities (19) hold, for each fixed \(u \in \mathbb{R} \setminus \{0\}\) the continuously differentiable functions \(\bar{a}^{-1}(u,t) = a(u,t) = a(u,0)a(1,t), \quad \bar{b}(u,t) = b(u,t) = b(u,0)a(1,t) + b(1,t)a^{-1}(u,0)\) satisfy inequalities (6). Hence the product \(\mathcal{TK}\) given in the assertion is the image of a \(C^1\)-differentiable section of a multiplicative loop \(Q^*\) of a quasifield \(Q\).

**Proposition 14.** The set \(\Lambda_{Q^*}\) of all left translations of the multiplicative loop \(Q^*\) for a locally compact connected topological 2-dimensional quasifield \(Q\) contains the group \(SO_2(\mathbb{R})\) if and only if \(\Lambda_{Q^*}\) has the form

\[\Lambda_{Q^*} = \left\{ \begin{pmatrix}
\cos t & \sin t \\
-\sin t & \cos t
\end{pmatrix}
\begin{pmatrix}
a(u,0) & ub(u,0) \\
0 & ua^{-1}(u,0)
\end{pmatrix}, u > 0, t \in [0, 2\pi) \right\}\]

(20)

where \(a(u,0), b(u,0)\) are arbitrary continuous functions with \(a(u,0) > 0\) such that \(ua(u,0)\) is strictly monotone. In this case \(Q^*\) is positively decomposable.

**Proof.** If the set \(\Lambda_{Q^*}\) contains the group \(SO_2(\mathbb{R})\), then for each fixed \(u \in \mathbb{R} \setminus \{0\}\) the function \(a_u(t)\) is constant with value 1 and the function \(b_u(t)\) is constant with value 0. So the functions \(a(u,t) = a(u,0), b(u,t) = b(u,0)\)
do not depend on the variable $t$. Hence the identities in Proposition 10 are satisfied and the set $\Lambda_{Q^*}$ has the form as in the assertion.

Conversely, if $ua(u, 0)$ is a strictly monotone continuous function, then for arbitrary continuous functions $a(u, 0), b(u, 0)$ with $a(u, 0) > 0$ the set given by (20) is the set $\Lambda_{Q^*}$ of all left translations of the multiplicative loop $Q^*$ of a locally compact quasifield such that $\Lambda_{Q^*}$ contains the group $\text{SO}_2(\mathbb{R})$.

Furthermore, $Q^*$ is positively decomposable because $a(1, \pi) = 1$, $b(1, \pi) = 0$, $a(-1, \pi) = a(-1, 0)a(1, \pi) = 1$ and $b(-1, \pi) = b(-1, 0)a(1, \pi) = 0$. □

6. Betten’s classification of 4-dimensional translation planes

Using 2-dimensional spreads, Betten in [3], [4], [5], [6], [7], [8], see also [13] and [15], has classified all locally compact 4-dimensional translation planes which admit an at least 7-dimensional collineation group. His normalized 2-dimensional spreads are images of sharply transitive sections $\sigma' : G/H' \to G$, where $G$ is the connected component of the group $\text{GL}_2(\mathbb{R})$, $H'$ is the subgroup

$$\left\{ \begin{pmatrix} 1 & c \\ 0 & d \end{pmatrix} \mid d > 0, c \in \mathbb{R} \right\}$$

(cf. [2], [3]) and $\sigma'(G/H')$ consists of the elements

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} ra(r, t) & 0 \\ 0 & r^{-1}a^{-1}(r, t) \end{pmatrix} \begin{pmatrix} 1 & b(r, t)a^{-1}(r, t) \\ 0 & r^2 \end{pmatrix}.$$
With respect to the stabilizer \( H = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}, a > 0, b \in \mathbb{R} \right\} \) the sharply transitive section \( \sigma' \) transforms into a sharply transitive section \( \sigma : G/\mathbb{H} \rightarrow G \) given by (2), because the elements of \( \sigma'(G/\mathbb{H}') \) coincide with

\[
\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} a(r,t) & b(r,t) \\ 0 & a^{-1}(r,t) \end{pmatrix}.
\]

**Proposition 15.** Let \( A \) be a 4-dimensional non-desarguesian translation plane admitting an 8-dimensional collineation group such that \( A \) is coordinatized by the locally compact topological quasifield \( Q \). Then the multiplicative loop \( Q^* \) can be given by one of the following sets \( \Lambda_{Q^*} \) of the left translations of \( Q^* \):

a) \( \Lambda_{Q^*} \) has the form (15) with \( a(1,t) = 1 \) and \( b(1,t) = 0 \) for \( 0 \leq t \leq \pi \), \( a(1,t) = 1/\sqrt{\cos^2 t + \sin^2 t} \) and \( b(1,t) = a(1,t) \frac{1-w}{w} \sin t \cos t \) for \( \pi < t < 2\pi \).

The quasifields \( Q_w, w > 1 \), correspond to a one-parameter family of planes \( A_w \).

b) \( \Lambda_{Q^*} \) is the range of the section given by (2) such that for \( \alpha \geq -\frac{3\beta^2}{4} \) one has

\( a(r,t) = \sqrt{\frac{\alpha^2 + \beta^2}{\alpha^2}} \) and \( b(r,t) = \varepsilon \beta \frac{(-\alpha+1)}{\alpha^2 + \beta^2} \), where \( \varepsilon = 1 \) for \( \alpha + \beta^2 > 0 \) and \( \varepsilon = -1 \) for \( \alpha + \beta^2 < 0 \) with \( r \cos(t) = \alpha \sqrt{\frac{(\alpha + \beta^2)^2}{\alpha^2 + \beta^2}} \), \( r \sin(t) = -\beta \sqrt{\frac{(\alpha + \beta^2)^2}{\alpha^2 + \beta^2}} \).

For \( \alpha < -\frac{3\beta^2}{4} \) we have \( a(r,t) = 3 \sqrt{\frac{\alpha^2 + \beta^2}{\alpha^2}} \) and \( b(r,t) = \beta \sqrt{\frac{\alpha^2 + \beta^2}{\alpha^2}} + \frac{\beta \alpha}{3 \sqrt{\alpha^2 (\alpha^2 + \beta^2)}} \) with \( r \cos(t) = \frac{\alpha}{3} \sqrt{\frac{\alpha^2}{\alpha^2 + \beta^2}} \), \( r \sin(t) = -\frac{\beta}{3} \sqrt{\frac{\alpha^2}{\alpha^2 + \beta^2}} \). The quasifield \( Q \) coordinates a single plane.
c) \( \Lambda_Q \) is the range of the section given by (2) such that
\[
a(r,t) = \sqrt{\frac{v^2+s^2}{s^4+s^2v+v^2}},
\]
\[
b(r,t) = \frac{-s\sqrt{\frac{v^4+s^2v+v^2}{s^4+s^2v+v^2}}}{s^3} \quad \text{with}
\]
\[
r \cos(t) = \sqrt{\frac{1}{s^4+s^2v+v^2}} \quad r \sin(t) = -s \sqrt{\frac{1}{s^4+s^2v+v^2}}.
\]

The quasifield \( Q \) coordinatizes a single plane.

In case a) the multiplicative loop \( Q_w^* \) is positively decomposable and a split extension of the normal subgroup \( N^* \cong \mathbb{R} \) corresponding to the connected component of \( K = \begin{cases} \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}, 0 \neq r \in \mathbb{R} \end{cases} \) with a subloop homeomorphic to the 1-sphere. In cases b) and c) the set of the left translations of \( Q^* \) corresponding to the kernel \( K_r \) of the quasifield \( Q \) has the form \( K \). The multiplicative loops \( Q^* \) are not decomposable and quasi-simple.

Proof. If the translation complement of \( \mathcal{A} \) is the group \( \text{GL}_2(\mathbb{R}) \) and acts reducibly on \( \mathbb{R}^4 \), then one obtains the one-parameter family \( \mathcal{A}_w, w > 1 \), of the non-desarguesian translation planes corresponding to the following spreads:

\[
\{S\} \cup \left\{ \begin{pmatrix} s & -v \\ v & s \end{pmatrix}, s, v \in \mathbb{R}, v \geq 0 \right\} \cup \left\{ \begin{pmatrix} s & \frac{-v}{w} \\ v & s \end{pmatrix}, s, v \in \mathbb{R}, v < 0 \right\},
\]

\( w > 1 \) (cf. [3], Satz 5, p. 144). Any such spread coincides with the set \( \Lambda \) in (15) with \( a(1,t) \) and \( b(1,t) \) as in assertion a). By Theorem 9 the multiplicative loop \( Q_w^* \) is a split extension of a normal subgroup \( N^* \) with a
1-dimensional compact loop. By Theorem 11 the loop \( Q^*_w \) is decomposable. As \( a(\pm 1, 0) = a(1, \pi) = 1 \), \( b(\pm 1, 0) = b(1, \pi) = 0 \) the loop \( Q^*_w \) is positively decomposable. Hence \( N^* \) has the form as in the assertion.

If the translation complement \( \text{GL}_2(\mathbb{R}) \) acts irreducibly on \( \mathbb{R}^4 \), then one obtains a single plane \( \mathcal{A} \) generated by the spread

\[
\{ S \} \cup \left\{ \begin{pmatrix} \alpha & -\alpha \beta - \beta^3 \\ \beta & \alpha + \beta^2 \end{pmatrix}, \alpha, \beta \in \mathbb{R}, \alpha \geq \frac{-3\beta^2}{4} \right\} \cup \left\{ \begin{pmatrix} \alpha & \frac{1}{2} \alpha \beta - \frac{3}{4} \\ \beta & \frac{3}{2} \beta^2 + \frac{1}{4} \end{pmatrix}, \alpha, \beta \in \mathbb{R}, \alpha < \frac{-3\beta^2}{4} \right\},
\]

(21) (cf. \cite{5}, Satz, p. 553).

If the translation complement is solvable, then one gets a single plane \( \mathcal{A} \) generated by the spread

\[
\{ S \} \cup \left\{ \begin{pmatrix} v & -\frac{s^3}{3} \\ s & s^2 + v \end{pmatrix}, s, v \in \mathbb{R} \right\},
\]

(22) (cf. \cite{4}, Satz 2 (b), p. 331).

The spread (21), respectively (22) coincides with the image of the section \( \sigma \) in (2) with the well defined functions \( a(r, t) \) and \( b(r, t) \) given in assertion b), respectively c). Since in both cases one has \( a(r, 0) = 1, b(r, 0) = 0 \), Remark 1 gives the form \( \mathcal{K} \) of the assertion.

For decomposable \( Q^* \), the identity \( a(r, t) = a(1, t) \) holds for all \( r \in \mathbb{R} \setminus \{0\}, t \in \mathbb{R} \) (cf. Proposition 10). In case b) for \(-3 \leq \alpha \leq 1\) one has \( a(1, t) = \sqrt{\alpha^3 - \alpha + 1} \) which yields a contradiction. In case c) we have \( a(r, \frac{\pi}{4}) = \sqrt{\frac{2}{1-s+\frac{s^2}{4}}}, s \in \mathbb{R} \setminus \{0\} \) and the condition \( a(r, \frac{\pi}{4}) = a(1, \frac{\pi}{4}) \) gives a contradiction. Hence in both cases \( Q^* \) is not decomposable and therefore quasi-simple (cf. Theorem 11).

\( \square \)
If the translation complement of a 4-dimensional topological plane $A$ has dimension 3, then the point $\infty$ of the line $S = \{(0, 0, u, v), u, v \in \mathbb{R}\}$ is fixed under the seven-dimensional collineation group $\Gamma$ of $A$.

**Proposition 16.** Let $Q$ be a 2-dimensional quasifield coordinatizing a 4-dimensional locally compact translation plane $A$ such that the 7-dimensional collineation group $\Gamma$ of $A$ acts transitively on the points of $W \setminus \{\infty\}$, where $W$ is the translation axis of $A$ and the kernel of the action of the translation complement on the line $S$ has dimension 1. Then the multiplicative loop $Q^*$ can be given by one of the following sets $\Lambda_{Q^*}$ of the left translations of $Q^*$:

a) $\Lambda_{Q^*}$ is the range of the section (2) such that

$$a(r, t) = \sqrt{\frac{s^2 + v^2}{s^2u + v^2 + \frac{s^4}{3} + s^2}}$$

and

$$b(r, t) = \frac{s^3 - s^3v}{(s^2u + v^2 + \frac{s^4}{3} + s^2)(s^2 + v^2)}$$

with $r \cos(t) = v \sqrt{\frac{s^2u + v^2 + \frac{s^4}{3} + s^2}{s^2 + v^2}}$, $r \sin(t) = -s \sqrt{\frac{s^2u + v^2 + \frac{s^4}{3} + s^2}{s^2 + v^2}}$. The quasifield $Q$ corresponds to a single plane.

b) $\Lambda_{Q^*}$ is the range of the section given by (2) such that

$$a(r, t) = \sqrt{\frac{v^2 + u^2 + 2\gamma^2(1 - \cos(u)) - 2\gamma v \sin(u) - 2\gamma u \cos(u) + 2\gamma^2}{v^2 + u^2 - 2\gamma^2 + 2\gamma^2 \cos(u)}}$$

and

$$b(r, t) = \frac{-2\gamma \gamma \sin u + 2\gamma \gamma \cos u - 2\gamma}{\sqrt{v^2 + u^2 + 2\gamma^2(1 - \cos(u)) - 2\gamma v \sin(u) - 2\gamma u \cos(u) + 2\gamma^2 \cos(u) + u^2 - 2\gamma^2(1 - \cos(u))}}$$

with

$$r \cos(t) = (v - \gamma \sin(u)) \sqrt{\frac{v^2 + u^2 - 2\gamma^2 + 2\gamma^2 \cos u}{v^2 + u^2 + 2\gamma^2(1 - \cos(u)) - 2\gamma v \sin(u) - 2\gamma u \cos(u) + 2\gamma^2 \cos(u) + u^2 - 2\gamma^2(1 - \cos(u))}}$$

and

$$r \sin(t) = (u - \gamma (\cos(u) - 1)) \sqrt{\frac{v^2 + u^2 - 2\gamma^2 + 2\gamma^2 \cos u}{v^2 + u^2 + 2\gamma^2(1 - \cos(u)) - 2\gamma v \sin(u) - 2\gamma u \cos(u) + 2\gamma^2 \cos(u) + u^2 - 2\gamma^2(1 - \cos(u))}}$$

The quasifields $Q_\gamma$ coordinatize a one-parameter family of planes $A_\gamma, 0 < |\gamma| \leq 1.$
In all cases the multiplicative loop \( Q^* \) is not decomposable and quasi-simple.

The set \( K \) of the left translations of \( Q^* \) corresponding to the kernel of the quasifield \( Q \) has the form

\[
\left\{ \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}, 0 \not= r \in \mathbb{R} \right\}.
\]

**Proof.** If the translation complement \( C \) leaves a 1-dimensional subspace of \( S \) invariant, then one obtains a single plane \( \mathcal{A} \) which corresponds to the following spread:

\[
\{S\} \cup \left\{ \begin{pmatrix} v & -s^3 - s \\ s & s^2 + v \end{pmatrix}, s, v \in \mathbb{R} \right\}
\]

(cf. [18], 73.10., [4], pp. 330-331).

If the translation complement acts transitively on the 1-dimensional subspaces of \( S \), then one gets a one-parameter family \( E_\gamma, 0 < |\gamma| \leq 1 \), of planes which are generated by the normalized spread

\[
\{S\} \cup \left\{ \begin{pmatrix} v - \gamma \sin u & u + \gamma(\cos u - 1) \\ \gamma(\cos u - 1) - u & v + \gamma \sin u \end{pmatrix}, u, v \in \mathbb{R} \right\},
\]

([8], Satz, p. 128, [13], Proposition 5.8). The spread (23), respectively (24) coincides with the image of the section \( \sigma \) in (2) such that the well defined functions \( a(r, t) \) and \( b(r, t) \) are given in assertion a), respectively b). Since in both cases one has \( a(r, 0) = 1, b(r, 0) = 0 \), Remark 1 gives the form of \( K \). Moreover, in case a) one has \( a(r, \frac{\pi}{4}) = \sqrt{\frac{2}{2+v+\frac{\pi}{4}}}, \; v \in \mathbb{R} \setminus \{0\} \). In case b) for \( v = 1 \) we get

\[
a(r_j, t_j) = \sqrt{\frac{1 + u^2 + 2\gamma^2(1 - \cos u) - 2\gamma \sin u - 2\gamma u \cos u + 2\gamma u}{1 + u^2 - 2\gamma^2 + 2\gamma^2 \cos u}}, \quad a(1, t_j) = 1.
\]

For decomposable \( Q^* \) one has \( a(r, t) = a(1, t) \) for all \( r \in \mathbb{R} \setminus \{0\}, t \in \mathbb{R} \) (cf.
Proposition 10) which yields a contradiction. Thus in both cases $Q^*$ is not decomposable and hence quasi-simple (cf. Theorem 11).

Proposition 17. Let $Q$ be a 2-dimensional quasifield coordinatizing a 4-dimensional locally compact translation plane $A$ such that the translation complement $C$ of the 7-dimensional collineation group $\Gamma$ of $A$ has an orbit of dimension 1 on $W \setminus \{0\}$, $C$ leaves in the set of lines through the origin only $S$ fixed and the kernel of its action on $S$ has positive dimension. Then the multiplicative loop $Q^*$ can be given by one of the following sets $\Lambda_{Q^*}$ of the left translations of $Q^*$:

a) $\Lambda_{Q^*}$ is the range of the section (2) such that for $\beta \geq 0$ one has

$$a(r, t) = \frac{\alpha^2 + \beta^2}{\alpha^2 + 2s\alpha + w\beta}, \quad b(r, t) = \frac{w\alpha^2 + \alpha\beta + z\beta^2}{\sqrt{\alpha^2 + \beta^2}}$$

with $r \cos(t) = \alpha \frac{\alpha^2 + s\alpha + w\beta}{\alpha^2 + \beta^2}$, $r \sin(t) = -\beta \frac{w\alpha + \alpha\beta + z\beta^2}{\alpha^2 + \beta^2}$.

For $\beta < 0$ one gets

$$a(r', t) = \frac{\alpha^2 + \beta^2}{\alpha^2 + q\alpha(-\beta)} + p(-\beta)$$

$$\sqrt{\alpha^2 + \beta^2}$$

and $b(r', t) = \frac{p\alpha(-\beta) + q\alpha(-\beta)}{\sqrt{\alpha^2 + \beta^2}}$.

with

$$r' \cos(t) = \alpha \frac{\alpha^2 + q\alpha(-\beta) + p(-\beta)}{\alpha^2 + \beta^2}$$

and $r' \sin(t) = -\beta \frac{\alpha^2 + q\alpha(-\beta) + p(-\beta)}{\alpha^2 + \beta^2}$.

The quasifields $Q_{s,w,z,p,q}$ coordinatize a family of planes $A_{s,w,z,p,q}$ such that the parameters $s, w, z, p, q$ satisfy the conditions $0 < s < 1$, $z^2 + 4w(1 - s^2) \leq 0$, $q^2 - 4p(1 - s^2) \leq 0$. 

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b) $\Lambda_Q$ is the range of the section (2) such that for $\beta \geq 0$ we have

$$a(r, t) = \sqrt{\alpha^2 + \beta^2 - w^2 + x^2 - w^2 + 2w^2 \ln(\beta) + x^2 \ln(\beta) + \beta^2(\ln(\beta))^2}$$

and

$$b(r, t) = \sqrt{\frac{(w + 1)\alpha^2 + x^2 - xw \ln(\beta) - \alpha \beta(\ln(\beta))^2 + 2\beta^2 \ln(\beta)}{\alpha^2 + \beta^2}}$$

with

$$r \cos(t) = \alpha \sqrt{\frac{\alpha^2 + x^2 - w^2 + 2w^2 \ln(\beta) + x^2 \ln(\beta) + \beta^2(\ln(\beta))^2}{\alpha^2 + \beta^2}}$$

$$r \sin(t) = -\beta \sqrt{\frac{\alpha^2 + x^2 - w^2 + 2w^2 \ln(\beta) + x^2 \ln(\beta) + \beta^2(\ln(\beta))^2}{\alpha^2 + \beta^2}}$$

For $\beta < 0$ we obtain

$$a(r', t) = \sqrt{\frac{\alpha^2 - q^2 + \beta^2 + (2\beta - q\beta^2) \ln(-\beta) + \beta^2(\ln(-\beta))^2}{\alpha^2 + \beta^2}}$$

and

$$b(r', t) = \sqrt{\frac{(1 - \beta^2)\alpha^2 - q^2 + (2\beta - q\beta^2) \ln(-\beta) - \beta \ln(-\beta))^2}{\alpha^2 + \beta^2}}$$

with

$$r' \cos(t) = \alpha \sqrt{\frac{\alpha^2 - q^2 + \beta^2 + (2\beta - q\beta^2) \ln(-\beta) + \beta^2(\ln(-\beta))^2}{\alpha^2 + \beta^2}}$$

$$r' \sin(t) = -\beta \sqrt{\frac{\alpha^2 - q^2 + \beta^2 + (2\beta - q\beta^2) \ln(-\beta) + \beta^2(\ln(-\beta))^2}{\alpha^2 + \beta^2}}$$

The quasifields $Q_{w,z,p,q}$ coordinatize a family of planes $A_{w,z,p,q}$ such that for the parameters $w, z, p, q$ the relations $\left(\frac{z}{\sqrt{2}}\right)^2 \leq -w - 1, \left(\frac{q}{\sqrt{2}}\right)^2 \leq p - 1$ hold.

c) $\Lambda_Q$ is the range of the section given by (2) such that $a(r, 0) = 1 = a(r, \pi)$ and $b(r, 0) = 0 = b(r, \pi)$ with $r = \beta$ for $t = 0$ and $r = -\beta$ for $t = \pi$.

For $\beta > 0$, we get

$$a(r, t) = \frac{\cos^2(t)(2uw + 2u + 2z) + \sin(t)(1 - u - w^2 - z^2 - wz) - (u + z + wz)}{\sqrt{\sin^2(t)(2uw + 2u + 2z) + \sin(t)(1 - u - w^2 - z^2 - wz) - (u + z + wz)}}$$

and

$$b(r, t) = \frac{u^2 + uz - w}{\sqrt{u^2 + \sin^2(t)(2uw + 2u + 2z) + \cos^2(t)(1 - (u + z + wz))}$$

with

$$r \cos(t) = \beta \left( u - (u + 1) \sin(t) \right) + z \sin(t)$$

and

$$r \sin(t) = \beta \left( u \sin^2(t) + z \sin(t) \right) - \cos^2(t)$$

where $l = \frac{1}{\beta} \ln(\beta)$. For $\beta < 0$ one gets

$$a(r', t') = \frac{u^2 + \sin^2(t')(2uw + 2u + 2q) + \cos^2(t') + (2u + 2q - 2up) \sin(t') \cos(t')}{\sqrt{u^2 + \sin^2(t')(2uw + 2u + 2q) + \cos^2(t') + (2u + 2q - 2up) \sin(t') \cos(t')}}$$

and

$$b(r', t') = \frac{\sin(t') \cos(t')(1 - 2uw - p + q) + \sin^2(t')(2q + 2u - 2up) + (up - q - u)}{\sqrt{u^2 + \sin^2(t')(2uw + 2u + 2q) + \cos^2(t') + (2q + 2u - 2up) \sin(t') \cos(t')}}$$
the parameters \( k, w, z, p, q \) spaces of \( S \).

If the translation complement \( C \) fixes normalized spreads have the form:

\[
\begin{align*}
\alpha \beta w q \gamma & \in S \cup \left\{ \begin{array}{c}
\alpha \\ \beta \\
\gamma
\end{array} \right\}, \quad \alpha, \beta, \gamma \in \mathbb{R}, \\
\alpha \beta w q \gamma & \in S \cup \left\{ \begin{array}{c}
\alpha \\ \beta \\
\gamma
\end{array} \right\}, \quad \alpha, \beta, \gamma \in \mathbb{R}, \\
\alpha \beta w q \gamma & \in S \cup \left\{ \begin{array}{c}
\alpha \\ \beta \\
\gamma
\end{array} \right\}, \quad \alpha, \beta, \gamma \in \mathbb{R}, \\
\alpha \beta w q \gamma & \in S \cup \left\{ \begin{array}{c}
\alpha \\ \beta \\
\gamma
\end{array} \right\}, \quad \alpha, \beta, \gamma \in \mathbb{R}, \\
\alpha \beta w q \gamma & \in S \cup \left\{ \begin{array}{c}
\alpha \\ \beta \\
\gamma
\end{array} \right\}, \quad \alpha, \beta, \gamma \in \mathbb{R}, \\
\alpha \beta w q \gamma & \in S \cup \left\{ \begin{array}{c}
\alpha \\ \beta \\
\gamma
\end{array} \right\}, \quad \alpha, \beta, \gamma \in \mathbb{R}, \\
\alpha \beta w q \gamma & \in S \cup \left\{ \begin{array}{c}
\alpha \\ \beta \\
\gamma
\end{array} \right\}, \quad \alpha, \beta, \gamma \in \mathbb{R},
\end{align*}
\]

(26) (cf. Satz 2, [6], pp. 418-419).

If the translation complement \( C \) acts transitively on the 1-dimensional subspaces of \( S \), then we have a family of translation planes \( A_{k,w,z,p,q} \) such that the normalized spreads belonging to these planes have the form

\[
\{ S \} \cup \left\{ \begin{array}{c}
1 \\ 0 \\
0 \\
1
\end{array} \right\}, \quad \beta \in \mathbb{R}
\]

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where \( l = \frac{1}{k} \ln \beta, \ l_1 = \frac{1}{k} \ln(-\beta) \) (cf. [15], Proposition 4.1, p. 6, and [6], Satz 3, pp. 422-423). The spreads (25), respectively (26), respectively (27) coincide with the image of the section \( \sigma \) in (2) such that the well defined functions \( a(r,t) \) and \( b(r,t) \) are given in assertion a), respectively b), respectively c). Since in all three cases we have \( a(r,0) = 1, b(r,0) = 0 \), Remark 1 shows that \( \mathcal{K} \) has the form as in the assertion. In case a), respectively b) for \( \beta > 0 \) one gets \( a(r, \frac{\pi}{4}) = \sqrt{\frac{2\beta}{2-\beta^2}}, \) respectively \( a(r, \frac{\pi}{4}) = \sqrt{\frac{2}{1-z-w-2\ln \beta+2\ln \beta+\ln \beta^*}}. \) In case c) for \( u = 0, \beta > 0 \) we get that \( a(1,t_j) \) is constant. These relations give a contradiction to the condition \( a(r,t) = a(1,t) \) of Proposition 10. Hence in all cases \( Q^* \) is not decomposable and quasi-simple (cf. Theorem 11).

**Proposition 18.** Let \( Q \) be a 2-dimensional quasifield coordinatizing a 4-dimensional locally compact translation plane \( \mathcal{A} \) such that the translation complement \( C \) of the 7-dimensional collineation group \( \Gamma \) of \( \mathcal{A} \) has an orbit of dimension 1 on \( W \setminus \{0\} \), \( C \) leaves only \( S \) in the set of lines through the origin fixed and the kernel of its action on \( S \) is zero-dimensional. Then the set \( \Lambda_{Q^*} \) of all left translations of the multiplicative loop \( Q^* \) is given by the range of the section (2) defined as follows: For \( \alpha \geq -\frac{\beta^2}{2} \) one has

\[
a(r,t) = \sqrt{\frac{\alpha^2 + \beta^2}{\sqrt{\frac{\alpha^2 + \beta^2}{\alpha^2 + \frac{\beta^2}{4}} + \left(\alpha + \frac{\beta^2}{4}\right) \left(\alpha + \frac{\beta^2}{4} - \frac{\alpha^2}{\left(\alpha + \frac{\beta^2}{4}\right)^2}\right)}}
\]

\[
b(r,t) = \frac{\sqrt{\alpha^2 + \frac{\beta^2}{4}} \left(\alpha + \frac{\beta^2}{4}\right) \left(\alpha + \frac{\beta^2}{4} - \frac{\alpha^2}{\left(\alpha + \frac{\beta^2}{4}\right)^2}\right) - \frac{\alpha^2 + \beta^2}{\sqrt{\alpha^2 + \frac{\beta^2}{4}} \left(\alpha + \frac{\beta^2}{4}\right) \left(\alpha + \frac{\beta^2}{4} - \frac{\alpha^2}{\left(\alpha + \frac{\beta^2}{4}\right)^2}\right)}}{\sqrt{\alpha^2 + \frac{\beta^2}{4}} \left(\alpha + \frac{\beta^2}{4}\right) \left(\alpha + \frac{\beta^2}{4} - \frac{\alpha^2}{\left(\alpha + \frac{\beta^2}{4}\right)^2}\right) - \frac{\alpha^2 + \beta^2}{\sqrt{\alpha^2 + \frac{\beta^2}{4}} \left(\alpha + \frac{\beta^2}{4}\right) \left(\alpha + \frac{\beta^2}{4} - \frac{\alpha^2}{\left(\alpha + \frac{\beta^2}{4}\right)^2}\right)}}.
\]

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with

\[
\begin{align*}
\cos(t) & = \alpha \sqrt{\frac{\alpha^2 + \beta^2}{\alpha^2 + \beta^2} + \left(\alpha + \frac{\alpha^2 + 3\beta^2}{\alpha^2 + \beta^2}\right) \left(\alpha + \frac{\alpha^2 + 3\beta^2}{\alpha^2 + \beta^2}\right) - \frac{\beta^2}{\alpha^2 + \beta^2}}, \\
\sin(t) & = -\beta \sqrt{\frac{\alpha^2 + \beta^2}{\alpha^2 + \beta^2} + \left(\alpha + \frac{\alpha^2 + 3\beta^2}{\alpha^2 + \beta^2}\right) \left(\alpha + \frac{\alpha^2 + 3\beta^2}{\alpha^2 + \beta^2}\right) - \frac{\beta^2}{\alpha^2 + \beta^2}}.
\end{align*}
\]

For \( \alpha < -\frac{\beta^2}{2} \) we get

\[
\begin{align*}
a(r, t) & = \frac{\sqrt{\alpha^2 + \beta^2}}{\sqrt{\alpha^2 + \beta^2} - \left(\alpha + \frac{\alpha^2 + 3\beta^2}{\alpha^2 + \beta^2}\right) \left(\alpha + \frac{\alpha^2 + 3\beta^2}{\alpha^2 + \beta^2}\right) - \frac{\beta^2}{\alpha^2 + \beta^2} \left(-\alpha - \frac{\beta^2}{2}\right)^2} \\
b(r, t) & = \frac{\sqrt{\alpha^2 + \beta^2} \left(-\alpha - \frac{\beta^2}{2}\right)^2 + \frac{\beta^2}{2} \left(-\alpha^2 - \beta^2\right) + \left(\frac{\beta^2}{2} \alpha \beta - \frac{\beta^2}{4}\right) \left(\alpha + \frac{\alpha^2 + 3\beta^2}{\alpha^2 + \beta^2}\right) - \frac{\alpha^2 \beta^2 + \beta^4}{\alpha^2 + \beta^2}}{\sqrt{\alpha^2 + \beta^2} \left(\alpha + \frac{\alpha^2 + 3\beta^2}{\alpha^2 + \beta^2}\right) \left(\alpha + \frac{\alpha^2 + 3\beta^2}{\alpha^2 + \beta^2}\right) - \frac{\beta^2}{\alpha^2 + \beta^2} \left(-\alpha - \frac{\beta^2}{2}\right)^2}.
\end{align*}
\]

with

\[
\begin{align*}
\cos(t) & = \alpha \sqrt{\frac{\alpha^2 + \beta^2}{\alpha^2 + \beta^2} + \left(\alpha + \frac{\alpha^2 + 3\beta^2}{\alpha^2 + \beta^2}\right) \left(\alpha + \frac{\alpha^2 + 3\beta^2}{\alpha^2 + \beta^2}\right) - \frac{\beta^2}{\alpha^2 + \beta^2}}, \\
\sin(t) & = -\beta \sqrt{\frac{\alpha^2 + \beta^2}{\alpha^2 + \beta^2} + \left(\alpha + \frac{\alpha^2 + 3\beta^2}{\alpha^2 + \beta^2}\right) \left(\alpha + \frac{\alpha^2 + 3\beta^2}{\alpha^2 + \beta^2}\right) - \frac{\beta^2}{\alpha^2 + \beta^2}}.
\end{align*}
\]

The quasifields \( Q_{w,z,p,q} \) coordinatize a family of planes \( A_{w,z,p,q} \) such that the parameters \( w, z, p, q \) satisfy \((3w)^2 \leq -16z(z+1), (3p)^2 \leq 16(q-1), q > 0, z < 0 \) and \((w, z, p, q) \neq (0, -\frac{1}{3}, 0, 3)\).

The multiplicative loops \( Q^*_{w,z,p,q} \) of the quasifields \( Q_{w,z,p,q} \) are not decomposable and quasi-simple. The left translations of \( Q^*_{w,z,p,q} \) corresponding to the kernel of \( Q_{w,z,p,q} \) have the form \( \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \), \( 0 \neq r \in \mathbb{R} \), if and only if \( w = p = 0 \).

Proof. By Satz 5 in [6], the planes \( A_{w,z,p,q} \) are determined by the normalized spreads which have the form

\[
\begin{align*}
\{ S \} \cup \left\{ \begin{array}{c}
\alpha \beta \left(\frac{\alpha}{\beta}^2 + \frac{3\beta}{\alpha^2 + \beta^2}\right)^2 + \frac{\beta^2}{\alpha^2} \left(\alpha + \frac{\alpha^2}{\alpha^2 + \beta^2}\right) - \frac{\beta^2}{\alpha^2 + \beta^2} - \frac{\beta^2}{\alpha^2 + \beta^2} \end{array} \right\}.
\end{align*}
\]

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\[
\left\{ \begin{array}{l}
\alpha - \gamma \alpha + \frac{\gamma}{\beta} \left( -\alpha - \frac{\gamma^2}{\beta} \right)^2 + \frac{(\alpha + \frac{\gamma^2}{\beta})}{\beta} \left( \alpha + \frac{\gamma^2}{\beta} \right) - \frac{\gamma^2}{\beta} \\
\frac{\gamma}{\beta} \alpha + \frac{\gamma^2}{\beta} - \frac{\gamma}{\beta} \left( \alpha + \frac{\gamma^2}{\beta} \right)
\end{array} \right\}, \beta \in \mathbb{R}, \alpha < -\frac{\beta^2}{2}
\]

These spreads coincide with the image of the section \( \sigma \) in (2) such that the well defined functions \( a(r, t) \) and \( b(r, t) \) are given in the assertion. For \( \beta > 2 \) we obtain

\[
a(r, \frac{\pi}{4}) = \frac{\sqrt{2w^2}}{\sqrt{\frac{\alpha^4}{\beta} - \frac{\alpha^3}{\beta} + \left( \frac{\alpha^2}{\beta} - \beta \right) \left( \frac{\alpha^2}{\beta} - 2 \beta - \frac{\alpha^2}{\beta} - \beta \right) - \frac{\alpha^2}{\beta} \left( \frac{\alpha^2}{\beta} - \beta \right)^2}}
\]

The loop \( Q_{w,z,p,q}^* \) is not decomposable since we have a contradiction to the condition \( a(r, \frac{\pi}{4}) = a(1, \frac{\pi}{4}) \) for \( r < 0 \) (cf. Proposition 10). Hence \( Q_{w,z,p,q}^* \) is quasi-simple (cf. Theorem 11). As \( a(r, 0) = 1 \) and \( b(r, 0) = 0 \) holds precisely if \( w = p = 0 \) the last assertion follows. \( \square \)

**Proposition 19.** Let \( Q \) be a 2-dimensional quasifield coordinatizing a 4-dimensional locally compact translation plane \( A \) such that the translation complement \( C \) of the 7-dimensional collineation group \( A \) fixes two distinct lines \( \{S, W\} \) through the origin and leaves on \( S \) one or two 1-dimensional subspaces invariant. Then the multiplicative loop \( Q^* \) can be given by one of the following sets \( \Lambda_{Q^*} \) of the left translations of \( Q^* \) having the form (20):

a) \( a(r, 0) = r \frac{1-w}{1+w}, \ b(r, 0) = c \left( \frac{w-1}{r w + 1} - r \frac{1-w}{1+w} \right), \)

with \( r = s \frac{w+1}{2}, \ s > 0, \ t = -\varphi \), where \( s \) and \( \varphi \) are variables of the spreads (28). The quasifields \( Q_{w,c} \) coordinatize a family of planes \( A_{w,c} \) such that for the parameters \( w \neq 1, c \) one has \( 0 < w \) and \( (w-1)^2 c^2 \leq 4w \).

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b) 

\[ a(r, 0) = 1, \quad b(r, 0) = \frac{\ln r}{d}, \]

with \( r = e^s, t = -\varphi, \) where \( s \) and \( \varphi \) are variables of the spreads (29). The quasifields \( Q_d \) coordinatize a one-parameter family of planes \( A_d \) such that \( 4d^2 \geq 1 \).

In both cases \( Q^* \) is positively decomposable and contains the group \( \text{SO}_2(\mathbb{R}) \).

Proof. If the group \( C \) fixes two 1-dimensional subspaces of \( S \), respectively only one 1-dimensional subspace of \( S \), then one obtains a family of translation planes corresponding to the normalized spreads

\[ \{S, W\} \cup \left\{ \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix} \begin{pmatrix} s & c(s^w - s) \\ 0 & s^w \end{pmatrix}, \ s, \varphi \in \mathbb{R}, s > 0 \right\} \tag{28} \]

(cf. [7], Satz 1 and [9], p. 15), respectively

\[ \{S, W\} \cup \left\{ \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix} \begin{pmatrix} e^s & e^s \frac{s}{d} \\ 0 & e^s \end{pmatrix}, \ s, \varphi \in \mathbb{R} \right\}, \tag{29} \]

(cf. [7], Satz 2 and [9], p. 15). In both cases these spreads coincide with the set \( \Lambda = \text{SO}_2(\mathbb{R})K \) given in (20) such that the set \( K \) corresponding to the kernel \( K_r \) of \( Q \) is determined by the functions \( a(r, 0), b(r, 0) \) as in assertion a), respectively b).

Remark 20. In [2] D. Betten constructed 4-dimensional locally compact non-desarguesian planes \( A_f \) corresponding to continuous, non-linear, strictly
monotone functions \( f \) defined for \( 0 \leq u \in \mathbb{R} \) with \( f(0) = 0 \) and \( \lim_{u \to \infty} f(u) = \infty \). The planes \( \mathcal{A}_f \) are determined by the normalized spreads

\[
\begin{align*}
\left\{ \begin{array}{l}
\left( u \cos \varphi - \frac{f(u)\sin \varphi}{f(1)} \right), \quad u > 0, \quad \varphi \in [0, 2\pi) \\
u \sin \varphi - \frac{f(u)\cos \varphi}{f(1)}
\end{array} \right\}.
\end{align*}
\]

These spreads coincide with the set \( \Lambda = \text{SO}_2(\mathbb{R}) \mathcal{K} \) given in (20) such that the set \( \mathcal{K} \) corresponding to the kernel \( K_r \) of the quasifield \( Q_f \) coordinatizing \( \mathcal{A}_f \) is determined by the functions \( a(r, 0) = \sqrt{u f(1)} / f(u) \), \( b(r, 0) = 0 \) with \( r = \sqrt{u f(u) / f(1)} \), \( t = -\varphi, \ u \neq 0 \). For \( f(u) = f(1)u^w \) these planes are planes in Proposition 19 a) with \( c = 0 \) and \( a(r, 0) = r^{1-w/2} \). Otherwise the full collineation group of the planes \( \mathcal{A}_f \) has dimension 6.

**Proposition 21.** Let \( Q \) be a 2-dimensional quasifield coordinatizing a 4-dimensional locally compact translation plane \( \mathcal{A} \) such that the translation complement \( C \) of the 7-dimensional collineation group of \( \mathcal{A} \) fixes two distinct lines \( \{S, W\} \) through the origin and acts transitively on the spaces \( P_S \) and \( P_W \) of all 1-dimensional subspaces of \( S \), respectively \( W \). Then the multiplicative loop \( Q^* \) of \( Q \) can be given by one of the following sets \( \Lambda_{Q^*} \) of the left translations of \( Q^* \):

a) \( \Lambda_{Q^*} \) is the range of the sections \((2) \) with

\[
a(r, u) = \frac{D}{a} \left[ e^{2(qt-p\theta)} + e^{2q\pi} + e^{q\theta-p\theta+q\pi} \left( 2d \cos s \cos t + (c^2 + 1 + d^2) \sin s \sin t \right) \right],
\]

\[
b(r, u) = \frac{e^{2(qt-p\theta)} \left( -c^2 - 1 + d^2 \right) \cos s \cos t - c \left( c^2 + 1 + d^2 \right) \sin s \sin t \left( 2d \cos s \cos t + (d^2 + c^2 + 1) \sin s \sin t \right) \left( e^{2q\pi} + e^{q\theta-p\theta+q\pi} \left( 2d \cos s \cos t + (d^2 + c^2 + 1) \sin s \sin t \right) \right)}{a}.
\]

such that

\[
r \cos u = \frac{e^{qt-p\theta} \left( \cos s \cos t + \cos t \cos s \cos s \sin t \sin s + e^{q\pi} \right) - 1}{1 + e^{q\pi}} a^{-1}(r, u),
\]

\[
r \sin u = \frac{-e^{qt-p\theta} \left( \cos s \cos t - \sin s \cos t - c \cos s \sin s \sin t \right) \left( 1 + e^{q\pi} \right) a^{-1}(r, u)}{1 + e^{q\pi}},
\]

\[
D = e^{2(qt-p\theta)} \left( \cos s \cos t \left( 1 + d^2 \right) \sin s \sin t \right) + e^{2q\pi} + 2e^{qt-p\theta+q\pi} \left( \cos s \cos t + c \cos s \sin s \sin t + d \cos s \cos s \sin t \sin s \sin t \right).
\]
The quasifields $Q_{p,q,c,d}$ coordinatize a family of planes $A_{p,q,c,d}$ such that the parameters $p, q, c, d$ satisfy the conditions

\begin{align*}
    p &= q > 0 \\
    q > 0, \quad p &= \frac{k-1}{k}, \quad k = 1, 2, 3, \ldots \\
    -1 &\leq d < 0, \quad \text{and} \\
    p &= q > 0, \quad p = k - \frac{1}{k+1} q, \quad k = 1, 2, 3, \ldots \quad \text{and} \\
    d &> 0.
\end{align*}

\[-(q + p)^2 A + (q - p)^2 B - 4AB \geq 0, \quad \text{where} \quad A = \frac{(d - 1)^2 + c^2}{4d} \quad \text{and} \quad B = \frac{(d + 1)^2 + c^2}{4d}.
\]

The multiplicative loops $Q^*$ of the quasifields $Q_{p,q,c,d}$ are not decomposable and quasi-simple.

b) $\Lambda_{Q^*}$ has the form (15) with

\[
    a(1,u) = \sqrt{(\cos nt + c \sin nt)^2 + d^2 \sin^2 nt}, \quad b(1,u) = \frac{\sin nt \cos nt (d^2 - 1 - c^2) - c \sin^2 nt (d^2 + 1 + c^2)}{d \sqrt{(\cos nt + c \sin nt)^2 + d^2 \sin^2 nt}}
\]

such that

\[
    r \cos u = \frac{s \cos nt \cos mt + c \sin nt \sin mt}{\sqrt{(\cos nt + c \sin nt)^2 + d^2 \sin^2 nt}}, \quad r \sin u = \frac{s \sin nt \cos mt - c \sin nt \sin mt}{\sqrt{(\cos nt + c \sin nt)^2 + d^2 \sin^2 nt}}
\]

and $s \geq 0$.

The quasifields $Q_{m,n,c,d}$ coordinatize a family of planes $A_{m,n,c,d}$ such that the parameters $m, n \in \mathbb{Z}$, $(m,n) = 1$, $c, d \in \mathbb{R}$ satisfy the conditions

\begin{align*}
    m &= n = 1 \quad \text{and} \quad -1 \leq d < 0 \\
    m &= 1, 2, 3, \ldots \quad n = m + 1 \quad \text{and} \quad d > 0 \\
    m &= 1, 3, 5, \ldots \quad n = m + 2 \quad \text{and} \quad d > 0
\end{align*}

\[-(n - m)^2 B \geq (n + m)^2 A, \quad \text{where} \quad A = \frac{(d - 1)^2 + c^2}{4d} \quad \text{and} \quad B = \frac{(d + 1)^2 + c^2}{4d}.
\]

The loops $Q^*_{m,n,c,d}$ are split extensions of the normal subgroup $N^* \cong \mathbb{R}$ corresponding to the connected component of \( \left\{ \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \right\} \) with a subloop homeomorphic to the 1-sphere.

**Proof.** If the translation complement $C$ acts transitively on the product space $P_S \times P_w$, then there is a family of translation planes corresponding to the
normalized spreads

\[ \{S, W\} \cup \left\{ \begin{pmatrix} \frac{\alpha(s,t) + e^{q\pi}}{1 + e^{q\pi}} & \frac{\gamma(s,t) - \alpha(s,t)}{d(1 + e^{q\pi})} \\ \frac{\beta(s,t)}{1 + e^{q\pi}} & \frac{\delta(s,t) - c\beta(s,t) + d e^{q\pi}}{d(1 + e^{q\pi})} \end{pmatrix}, s, t \in \mathbb{R} \right\} \]

such that

\[ \alpha(s,t) = e^{q t - p s} (\cos s \cos t + c \sin t \cos s + d \sin t \sin s), \]

\[ \beta(s,t) = e^{q t - p s} (d \cos s \sin t - \sin s \cos t - c \sin s \sin t), \]

\[ \gamma(s,t) = e^{q t - p s} (d \cos t \sin s - \sin t \cos s + c \cos t \cos s), \]

\[ \delta(s,t) = e^{q t - p s} (d \cos t \cos s + \sin t \sin s - c \cos t \sin s) \] (cf. [7], Satz 3, pp. 135-136). These spreads coincide with the image of the section \( \sigma \) in (2) with the well defined functions \( a(r, u) \) and \( b(r, u) \) as in assertion a). For \( s = 0 \) we get a contradiction to the condition \( a(r_j, u_j) = a(r_j, 0) a(1, u_j) \) which must hold for decomposable \( Q^* \). It follows that \( Q^* \) is not decomposable and hence quasi-simple (cf. Theorem 11).

If the translation complement \( C \) does not act transitively on the product space \( P_S \times P_W \), then there is a family of translation planes which correspond to the normalized spreads

\[ \{S, W\} \cup \left\{ \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} \begin{pmatrix} a_{11}(t) & -\frac{e}{d} a_{11}(t) + \frac{1}{d} a_{21}(t) \\ a_{12}(t) & -\frac{e}{d} a_{12}(t) + \frac{1}{d} a_{22}(t) \end{pmatrix}, s \geq 0, t \in \mathbb{R} \right\} \]

with

\[ a_{11}(t) = \cos nt \cos mt + c \sin nt \cos mt + d \sin nt \sin mt, \]

\[ a_{12}(t) = d \sin nt \cos mt - \cos nt \sin mt - c \sin nt \sin mt, \]

\[ a_{21}(t) = d \cos nt \sin mt - \sin nt \cos mt + c \cos nt \sin mt, \]

\[ a_{22}(t) = d \cos nt \cos mt + \sin nt \sin mt - c \cos nt \sin mt \] (cf. [7], Satz 4, pp. 42
142-144). These spreads coincide with the set Λ in (15) such that the periodic functions $a(1,t)$ and $b(1,t)$ are given in assertion b). As in the proof of Proposition 15 a) it follows that the loop $Q_{m,n,c,d}^*$ is a split extension as in the assertion.

**Corollary 22.** Let $\mathcal{A}$ be a 4-dimensional locally compact non-desarguesian topological plane which admits an at least 7-dimensional collineation group $\Gamma$. If the quasifield $Q$ coordinatizing $\mathcal{A}$ is constructed with respect to two lines such that their intersection points with the line at infinity are contained in the 1-dimensional orbit of $\Gamma$ or contain the set of the fixed points of $\Gamma$, then for the multiplicative loop $Q^*$ of $Q$ one of the following holds:

a) $Q^*$ is quasi-simple and not decomposable. Such quasifields $Q$ are described by Propositions 15 b), 15 c), 16), 17), 18) and in Proposition 21 a).

b) $Q^*$ is quasi-simple but decomposable and it is a product $SO_2(\mathbb{R})B$, where $B$ is a 1-dimensional loop homeomorphic to $\mathbb{R}$. The quasifields $Q$ of this type are described in Proposition 19.

c) $Q^*$ is a split extension of the group $N^* \cong \mathbb{R}$ with a loop homeomorphic to the 1-sphere. The quasifields of this type are described in Propositions 15 a) and 21 b).

**Proof.** A locally compact topological quasifield coordinatizing the translation plane $\mathcal{A}$ and constructed with respect to two lines satifying the assumptions
is isotopic to a quasifield given in Betten’s classification (cf. [11], p. 321, [3] Satz 5). For isotopic loops $Q^*_1$ and $Q^*_2$ the following holds: The group generated by their left translations, every subgroup and all nuclei of them are isomorphic (cf. [14], Lemmata 1.9, 1.10, p. 20). From these facts we get: If $Q_1$ is quasisimple and not decomposable, then also $Q_2$ is quasisimple and not decomposable. If $Q_1$ contains the subgroup $SO_2(\mathbb{R})$, then also $Q_2$ contains the group $SO_2(\mathbb{R})$. If $Q_1$ is a split extension of $N^*$ with a 1-dimensional compact loop, then the same holds for $Q_2$. \hfill \square

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