A Geometric Parametrization of the Cabibbo–Kobayashi–Maskawa Matrix and the Jarlskog Invariant

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Abstract

In this paper we give a geometric parametrization to the Cabibbo–Kobayashi–Maskawa (CKM) mixing matrix and the Jarlskog invariant, which is based on two flag manifolds $SU(3)/U(1)^2$.

To treat a fourth generation of quarks on CP violation we generalize the parametrization to one based on two flag manifolds $SU(4)/U(1)^3$.

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1 Introduction

CP violation plays a central role in the standard model. In this paper we revisit a problem of generation of quarks on CP violation ([1], [2], [3]) from the mathematical (geometric) point of view. As a quick introduction to the problem see for example [4].

We start with the paper [3] and assume for simplicity that the mass matrices $M$ and $M'$ are hermite and non-negative. Then $M$ and $M'$ can be diagonalized like

$$M = U \begin{pmatrix} m_u \\ m_c \\ m_t \end{pmatrix} U^\dagger, \quad M' = U' \begin{pmatrix} m_d \\ m_s \\ m_b \end{pmatrix} U'^\dagger \quad (1)$$

where $m_j$ is the mass of the quark $j$. From these we define

$$V = U^\dagger U' = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \equiv \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix}, \quad (2)$$

which is the famous Cabibbo–Kobayashi–Maskawa (CKM) mixing matrix.

The Jarlskog invariant which measures CP violation is given by

$$J = -i \det[M, M']/2TB \quad (3)$$

where

$$T = (m_t - m_u)(m_t - m_c)(m_c - m_u), \quad B = (m_b - m_d)(m_b - m_s)(m_s - m_d).$$

In terms of entries of $V$ in (2) $J$ is expressed as

$$J = \pm \text{Im} \left( V_{ij}V_{kl}\bar{V}_{il}\bar{V}_{kj} \right) \quad (4)$$

where Im denotes the imaginary part of complex number and $\bar{z}$ the complex conjugate of $z$. See the appendix. Usually Im ($V_{11}V_{22}\bar{V}_{12}\bar{V}_{21}$) is used, while Im ($V_{11}V_{33}\bar{V}_{13}\bar{V}_{31}$) is used in the following.

From (1) the forms are invariant under

$$U \rightarrow U \ \text{diag}(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}), \quad U' \rightarrow U' \ \text{diag}(e^{i\theta_1'}, e^{i\theta_2'}, e^{i\theta_3'}),$$

$$U \rightarrow U \ \text{diag}(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}), \quad U' \rightarrow U' \ \text{diag}(e^{i\theta_1'}, e^{i\theta_2'}, e^{i\theta_3'}),$$

$2$
so \( U \) and \( U' \) are considered as elements in a flag manifold \( U(3)/U(1)^2 \cong SU(3)/U(1)^2 \).

Since the dimension of \( SU(3)/U(1)^2 \) is six, \( U \) is usually parametrized as

\[
U = \begin{pmatrix}
  e^{i(\alpha+\beta)} & e^{i(\alpha-\beta)} & e^{-2i\alpha}
\end{pmatrix}
\begin{pmatrix}
  c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\
  -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\
  s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13}
\end{pmatrix}
\]

(5)

where \( c_{ij} = \cos \theta_{ij}, s_{ij} = \sin \theta_{ij} \) and \( \{\theta_{12}, \theta_{13}, \theta_{23}\} \) are three rotating angles and \( e^{i\delta} \) is a phase, see for example [5] and [6].

However, we don’t use this parametrization. Since \( SU(3)/U(1)^2 \) is a Kähler manifold, there is some deep geometric structure. As a general introduction to Geometry or Topology see [7] and [8] ([7] is particularly recommended).

Our aim is to give a geometric parametrization to \( V \) by use of a local coordinate of two flag manifolds \( SU(3)/U(1)^2 \) corresponding to \( U \) and \( U' \) (see the following diagram).

\[
\begin{pmatrix}
  SU(3) \\
  SU(3)/U(1)^2 \\
\end{pmatrix}
\]

We note that the argument above is based on the third generation of quarks, but it is easy to generalize to any generation except for calculation. In fact, we treat a fourth generation and consider two flag manifolds \( SU(4)/U(1)^3 \) in the following.

## 2 Geometric Parametrization

First of all we review how to parametrize the flag manifold \( SU(3)/U(1)^2 \cong U(3)/U(1)^3 \). See [9] as a good introduction and also [10] and [11].

The flag manifold of the second type (in our terminology) is the sequence of complex vector spaces defined by

\[
F_{1,1,1}(\mathbb{C}) = \{ \mathcal{V} \subset \mathcal{W} \subset \mathbb{C}^3 \mid \dim_{\mathbb{C}}\mathcal{V} = 1, \ \dim_{\mathbb{C}}\mathcal{W} = 2 \}.
\]

(6)
Then it is well–known that
\[ F_{1,1,1}(C) \cong U(3)/U(1)^3 \]  
(7)
and moreover
\[ U(3)/U(1)^3 \cong GL(3; C)/B_+ \]  
(8)
where \( B_+ \) is the (upper) Borel subgroup given by
\[
B_+ = \left\{ \begin{pmatrix} \alpha & * & * \\ 0 & \beta & * \\ 0 & 0 & \gamma \end{pmatrix} \in GL(3; C) \mid \alpha, \beta, \gamma \in GL(1; C) \equiv C^\times \right\}.
\]

In order to obtain the element of \( U(3)/U(1)^3 \) from an element in \( GL(3; C)/B_+ \) it is convenient to use the orthonormalization (method) by Gram–Schmidt. For the matrix
\[
F \equiv \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & z & 1 \end{pmatrix} \in GL(3; C)/B_+
\]  
(9)
we set
\[
V_1 = \begin{pmatrix} 1 \\ x \\ y \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 \\ 1 \\ z \end{pmatrix}, \quad V_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
\]

For \( \{V_1, V_2, V_3\} \) the Gramm–Schmidt orthogonalization is as follows:
\[
\hat{V}_1 = V_1, \quad \hat{V}_1 = \hat{V}_1(V_1^\dagger\hat{V}_1)^{-1/2} \implies P_1 = \hat{V}_1\hat{V}_1^\dagger : \text{projection}
\]
\[
\hat{V}_2 = (E - P_1)V_2, \quad \hat{V}_2 = \hat{V}_2(V_2^\dagger\hat{V}_2)^{-1/2} \implies P_2 = \hat{V}_2\hat{V}_2^\dagger : \text{projection}
\]
\[
\hat{V}_3 = (E - P_1 - P_2)V_3 = (E - P_1)(E - P_2)V_3, \quad \hat{V}_3 = \hat{V}_3(V_3^\dagger\hat{V}_3)^{-1/2}
\]
where \( E \) is the unit matrix in \( M(3; C) \). Explicitly,
\[
\hat{V}_1 = \begin{pmatrix} 1 \\ x \\ y \end{pmatrix} \Delta_1^{-1/2}, \quad \hat{V}_2 = \begin{pmatrix} -(\bar{x} + \bar{y}z) \\ 1 - (xz - y)\bar{y} \\ z + \bar{x}(xz - y) \end{pmatrix} (\Delta_1\Delta_2)^{-1/2}, \quad \hat{V}_3 = \begin{pmatrix} \bar{x}\bar{z} - \bar{y} \\ -\bar{z} \\ 1 \end{pmatrix} \Delta_2^{-1/2}
\]
where
\[ \Delta_1 = 1 + |x|^2 + |y|^2, \quad \Delta_2 = 1 + |z|^2 + |xz - y|^2. \]

Note that \( K = \log(\Delta_1 \Delta_2) \) is the Kähler potential and \( \omega = i\partial \bar{\partial} K \) is the Kähler two–form and the symplectic volume of the manifold is given by
\[
\Omega \equiv \omega \wedge \omega \wedge \omega = \frac{2}{3!} \prod_{j=1}^{3} \text{id}z_j \wedge d\bar{z}_j \tag{10}
\]
where we have set \( z_1 = x, \ z_2 = y, \ z_3 = z \) for simplicity, see for example [9].

Therefore we obtain the unitary matrix
\[
U = (\hat{V}_1, \hat{V}_2, \hat{V}_3) = \begin{pmatrix}
\frac{1}{\sqrt{\Delta_1}} & -\frac{(\bar{x} + \bar{y})}{\sqrt{\Delta_1 \Delta_2}} & \frac{\bar{z} - \bar{y}}{\sqrt{\Delta_2}} \\
\frac{x}{\sqrt{\Delta_1}} & \frac{1 - (xz - y)\bar{y}}{\sqrt{\Delta_1 \Delta_2}} & \frac{\bar{z}}{\sqrt{\Delta_2}} \\
\frac{y}{\sqrt{\Delta_1}} & \frac{z + \bar{x}(xz - y)}{\sqrt{\Delta_1 \Delta_2}} & \frac{1}{\sqrt{\Delta_2}}
\end{pmatrix} \begin{pmatrix}
1 & -\bar{x}z - \bar{y} \\
x & 1 - (xz - y)\bar{y} & -\bar{z} \\
y & z + \bar{x}(xz - y) & 1
\end{pmatrix} \begin{pmatrix}
\frac{1}{\sqrt{\Delta_1}} \\
\frac{1}{\sqrt{\Delta_1 \Delta_2}} \\
\frac{1}{\sqrt{\Delta_2}}
\end{pmatrix}. \tag{11}
\]

This is our geometric parametrization for \( U \).

A comment is in order. Our parametrization is not compatible with (5) because of the phases in the first. However, if we neglect them the correspondence is given by
\[
x \longleftrightarrow -\left( t_{12} \frac{c_{23}}{c_{13}} + s_{23}t_{13}e^{i\delta} \right), \quad y \longleftrightarrow \left( t_{12} \frac{s_{23}}{c_{13}} - c_{23}t_{13}e^{i\delta} \right), \quad z \longleftrightarrow -t_{23}
\]
where \( c_{ij} = \cos \theta_{ij}, \ s_{ij} = \sin \theta_{ij}, \ t_{ij} = \tan \theta_{ij} \) for simplicity.

Similarly, we parametrize \( U' \) in terms of \( (u, v, w) \) (in place of \( (x, y, z) \) in \( U \)) as
\[
U' = \begin{pmatrix}
1 & -\bar{u}w - \bar{v} \\
u & 1 - (uw - y)\bar{v} & -\bar{w} \\
v & w + \bar{u}(uw - v) & 1
\end{pmatrix} \begin{pmatrix}
\frac{1}{\sqrt{\Delta'_1}} \\
\frac{1}{\sqrt{\Delta'_1 \Delta'_2}} \\
\frac{1}{\sqrt{\Delta'_2}}
\end{pmatrix} \tag{12}
\]
where
\[
\Delta'_1 = 1 + |u|^2 + |v|^2, \quad \Delta'_2 = 1 + |w|^2 + |uw - v|^2.
\]
Therefore the CKM matrix $V = U^\dagger U'$ in (2) is parametrized as

\[
V = \left( \begin{array}{ccc}
\frac{1}{\sqrt{\Delta_1}} & & \\
\frac{1}{\sqrt{\Delta_1 \Delta_2}} & & \\
\frac{1}{\sqrt{\Delta_2}} & & \\
\end{array} \right) \left( \begin{array}{ccc}
1 & -(\bar{x} + \bar{y}z) & \bar{x}\bar{z} - \bar{y} \\
x & 1 - (xz - y)\bar{y} & -\bar{z} \\
y & z + \bar{x}(xz - y) & 1 \\
\end{array} \right)^\dagger \left( \begin{array}{ccc}
1 & -(\bar{u} + \bar{v}w) & \bar{u}\bar{w} - \bar{v} \\
u & 1 - (uw - y)\bar{v} & -\bar{w} \\
v & w + \bar{u}(uw - v) & 1 \\
\end{array} \right)
\times \left( \begin{array}{ccc}
\frac{1}{\sqrt{\Delta_1}} & & \\
\frac{1}{\sqrt{\Delta_1 \Delta_2}} & & \\
\frac{1}{\sqrt{\Delta_2}} & & \\
\end{array} \right) \left( \begin{array}{ccc}
f_{11} & f_{12} & f_{13} \\
f_{21} & f_{22} & f_{23} \\
f_{31} & f_{32} & f_{33} \\
\end{array} \right) \left( \begin{array}{ccc}
\frac{1}{\sqrt{\Delta_1}} & & \\
\frac{1}{\sqrt{\Delta_1 \Delta_2}} & & \\
\frac{1}{\sqrt{\Delta_2}} & & \\
\end{array} \right)
\]

(13)

where

\[
\begin{align*}
f_{11} &= 1 + \bar{x}u + \bar{y}v, \\
f_{12} &= -(\bar{u} + \bar{v}w) + \bar{x}\{1 - (uw - v)\bar{v}\} + \bar{y}\{w + \bar{u}(uw - v)\}, \\
f_{13} &= (\bar{u}\bar{w} - \bar{v}) - \bar{x}\bar{w} + \bar{y}, \\
f_{21} &= -(x + y\bar{z}) + \{1 - (\bar{x}\bar{z} - \bar{y})y\}u + \{\bar{z} + x(\bar{x}\bar{z} - \bar{y})\}v, \\
f_{22} &= (x + y\bar{z})(\bar{u} + \bar{v}w) + \{1 - (\bar{x}\bar{z} - \bar{y})y\}\{1 - (uw - v)\bar{v}\} + \{\bar{z} + x(\bar{x}\bar{z} - \bar{y})\}\{w + \bar{u}(uw - v)\}, \\
f_{23} &= -(x + y\bar{z})(\bar{u}\bar{w} - \bar{v}) - \{1 - (\bar{x}\bar{z} - \bar{y})y\}\bar{w} + \bar{z} + x(\bar{x}\bar{z} - \bar{y}), \\
f_{31} &= (xz - y) - zu + v, \\
f_{32} &= -(xz - y)(\bar{u} + \bar{v}w) - z\{1 - (uw - v)\bar{v}\} + w + \bar{u}(uw - v), \\
f_{33} &= (xz - y)(\bar{u}\bar{w} - \bar{v}) + zw + 1.
\end{align*}
\]

This is just our geometric parametrization to the CKM matrix. We believe that our parametrization is clear-cut.
From this it is easy to see that the Jarlskog invariant in (11) becomes

\[ J = \text{Im} \left( V_{11} V_{33} \bar{V}_{13} \bar{V}_{31} \right) \]
\[ = \text{Im} \left\{ (1 + \bar{x} u + \bar{y} v) (1 + z \bar{w} + (x z - y) (\bar{u} \bar{w} - \bar{v})) (\bar{x} \bar{z} - \bar{y} - z \bar{u} + \bar{v}) (u w - v - x w + y) \right\} \]
\[ \Delta_1 \Delta_2 \Delta'_1 \Delta'_2 \] (14)

We can of course expand the numerator of the equation. However, such a form is not so beautiful, so we omit it.

3 Generalization to a fourth generation

In the preceding section we studied some problems of the third generation of quarks. However, from the mathematical point of view there is no reason to stay at the point (situation). Therefore we try to generalize some results based on the third generation to ones based on a fourth generation of quarks.

The method is almost same. Namely, we have only to consider a flag manifold \( SU(4)/U(1)^3 \cong U(4)/U(1)^4 \) in place of \( SU(3)/U(1)^2 \) in the preceding section (see the following diagram).

\[
\begin{array}{ccc}
SU(4) & \rightarrow & U' \\
\uparrow & & \downarrow \\
SU(4)/U(1)^3 & \rightarrow & V = U^\dagger U'
\end{array}
\]

\[
\begin{array}{ccc}
SU(4)/U(1)^4 & \rightarrow & U' \\
\uparrow & & \downarrow \\
SU(4)/U(1)^3 & \rightarrow & V = U^\dagger U'
\end{array}
\]

In order to obtain the element of \( SU(4)/U(1)^3 \) from an element in \( GL(4; \mathbb{C})/B_+ \) (\( \cong U(4)/U(1)^4 \)) we consider the matrix

\[
F \equiv \begin{pmatrix}
1 & 0 & 0 & 0 \\
x_1 & 1 & 0 & 0 \\
x_2 & y_1 & 1 & 0 \\
x_3 & y_2 & z_1 & 1
\end{pmatrix} \in GL(4; \mathbb{C})/B_+ \] (15)
and set
\[
\begin{align*}
V_1 &= \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}, \\
V_2 &= \begin{pmatrix} 0 \\ 1 \\ y_1 \\ y_2 \end{pmatrix}, \\
V_3 &= \begin{pmatrix} 0 \\ 1 \\ z_1 \end{pmatrix}, \\
V_4 &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
\end{align*}
\]

For \{V_1, V_2, V_3, V_4\} the Gramm–Schmidt orthogonalization is as follows:
\[
\begin{align*}
\tilde{V}_1 &= V_1, \\
\hat{V}_1 &= \tilde{V}_1(\tilde{V}_1^\dagger\tilde{V}_1)^{-1/2} \Rightarrow P_1 = \hat{V}_1\hat{V}_1^\dagger : \text{projection} \\
\tilde{V}_2 &= (E - P_1)V_2, \\
\hat{V}_2 &= \tilde{V}_2(\tilde{V}_2^\dagger\tilde{V}_2)^{-1/2} \Rightarrow P_2 = \hat{V}_2\hat{V}_2^\dagger : \text{projection} \\
\tilde{V}_3 &= (E - P_1 - P_2)V_3 = (E - P_1)(E - P_2)V_3, \\
\hat{V}_3 &= \tilde{V}_3(\tilde{V}_3^\dagger\tilde{V}_3)^{-1/2} \Rightarrow P_3 = \hat{V}_3\hat{V}_3^\dagger : \text{projection} \\
\tilde{V}_4 &= (E - P_1 - P_2 - P_3)V_4 = (E - P_1)(E - P_2)(E - P_3)V_4, \\
\hat{V}_4 &= \tilde{V}_4(\tilde{V}_4^\dagger\tilde{V}_4)^{-1/2}
\end{align*}
\]

where \(E\) is the unit matrix in \(M(4; \mathbb{C})\).

We list the result (whose proof is not easy):
\[
\begin{align*}
\hat{V}_1 &= \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \frac{1}{\sqrt{\Delta_1}}, \\
\hat{V}_2 &= \begin{pmatrix} \Delta_1 - x_1 T \\ y_1 \Delta_1 - x_2 T \\ y_2 \Delta_1 - x_3 T \end{pmatrix} \frac{1}{\sqrt{\Delta_1 \Delta_2}} = \begin{pmatrix} -(\bar{x}_1 + \bar{x}_2 y_1 + \bar{x}_3 y_2) \\ 1 + \bar{x}_2 (x_2 - x_1 y_1) + \bar{x}_3 (x_3 - x_1 y_1) \\ y_1 - \bar{x}_1 (x_2 - x_1 y_1) - \bar{x}_3 (x_2 y_2 - x_3 y_1) \\ y_2 - \bar{x}_1 (x_3 - x_1 y_1) + \bar{x}_2 (x_2 y_2 - x_3 y_1) \end{pmatrix} \frac{1}{\sqrt{\Delta_1 \Delta_2}}
\end{align*}
\]

where

\[
T = \bar{x}_1 + \bar{x}_2 y_1 + \bar{x}_3 y_2,
\]

\(^1\) to calculate the norms \(|V_j|^2\) \((j = 2, 3, 4)\) is hard
and

\[
\hat{V}_3 = \begin{pmatrix}
  a_1 \\
  a_2 \\
  a_3 \\
  a_4 \\
\end{pmatrix} \frac{1}{\Delta_1 \Delta_2 \Delta_3}
\]

where

\[
\begin{align*}
  a_1 &= -(\bar{x}_2 + z_1 \bar{x}_3) \Delta_2 + \{(\bar{y}_1 \Delta_1 - \bar{x}_2 \bar{T}) + z_1 (\bar{y}_2 \Delta_1 - \bar{x}_3 \bar{T})\} T, \\
  a_2 &= -(\bar{x}_2 + z_1 \bar{x}_3) x_1 \Delta_2 - \{(\bar{y}_1 \Delta_1 - \bar{x}_2 \bar{T}) + z_1 (\bar{y}_2 \Delta_1 - \bar{x}_3 \bar{T})\} (\Delta_1 - x_1 T), \\
  a_3 &= \Delta_1 \Delta_2 - (\bar{x}_2 + z_1 \bar{x}_3) x_2 \Delta_2 - \{(\bar{y}_1 \Delta_1 - \bar{x}_2 \bar{T}) + z_1 (\bar{y}_2 \Delta_1 - \bar{x}_3 \bar{T})\} (y_1 \Delta_1 - x_2 T), \\
  a_4 &= z_1 \Delta_1 \Delta_2 - (\bar{x}_2 + z_1 \bar{x}_3) x_3 \Delta_2 - \{(\bar{y}_1 \Delta_1 - \bar{x}_2 \bar{T}) + z_1 (\bar{y}_2 \Delta_1 - \bar{x}_3 \bar{T})\} (y_2 \Delta_1 - x_3 T),
\end{align*}
\]

and

\[
\hat{V}_4 = \begin{pmatrix}
  -\bar{x}_3 + \bar{x}_1 \bar{y}_2 + \bar{x}_2 \bar{z}_1 - \bar{x}_1 \bar{y}_1 \bar{z}_1 \\
  -\bar{y}_2 + \bar{y}_1 \bar{z}_1 \\
  -\bar{z}_1 \\
  1
\end{pmatrix} \frac{1}{\sqrt{\Delta_3}} \equiv \begin{pmatrix}
  b_1 \\
  b_2 \\
  b_3 \\
  b_4
\end{pmatrix} \frac{1}{\sqrt{\Delta_3}}.
\]

Here we have used the notations

\[
\begin{align*}
  \Delta_1 &= 1 + |x_1|^2 + |x_2|^2 + |x_3|^2, \\
  \Delta_2 &= 1 + |y_1|^2 + |y_2|^2 + |x_2 - x_1 y_1|^2 + |x_3 - x_1 y_2|^2 + |x_2 y_2 - x_3 y_1|^2, \\
  \Delta_3 &= 1 + |z_1|^2 + |y_2 - y_1 z_1|^2 + |x_1 (y_2 - y_1 z_1) - (x_3 - x_2 z_1)|^2.
\end{align*}
\]

Therefore we have the unitary matrix parametrized by \((x_1, x_2, x_3, y_1, y_2, z_1)\)

\[
U = (\hat{V}_1, \hat{V}_2, \hat{V}_3, \hat{V}_4)
\]

\[
\begin{pmatrix}
  1 & -T & a_1 & b_1 \\
  x_1 & \Delta_1 - x_1 T & a_2 & b_2 \\
  x_2 & y_1 \Delta_1 - x_2 T & a_3 & b_3 \\
  x_3 & y_2 \Delta_1 - x_3 T & a_4 & b_4
\end{pmatrix} \frac{1}{\sqrt{\Delta_1 \Delta_2 \Delta_3}} \equiv \begin{pmatrix}
  \frac{1}{\sqrt{\Delta_1}} \\
  \frac{1}{\sqrt{\Delta_1 \Delta_2}} \\
  \frac{1}{\Delta_1 \sqrt{\Delta_2 \Delta_3}} \\
  \frac{1}{\sqrt{\Delta_3}}
\end{pmatrix}.
\]
Similarly, starting from

\[
F' \equiv \begin{pmatrix}
1 & 0 & 0 & 0 \\
u_1 & 1 & 0 & 0 \\
u_2 & v_1 & 1 & 0 \\
u_3 & v_2 & w_1 & 1
\end{pmatrix} \in GL(4, \mathbb{C})/B_+
\]  \hfill (17)

we have the unitary matrix parametrized by \((u_1, u_2, u_3, v_1, v_2, w_1)\)

\[
U' = \begin{pmatrix}
1 & -T' & a'_1 & b'_1 \\
u_1 & \Delta'_1 - u_1 T' & a'_2 & b'_2 \\
u_2 & v_1 \Delta'_1 - u_2 T' & a'_3 & b'_3 \\
u_3 & v_2 \Delta'_1 - u_3 T' & a'_4 & b'_4
\end{pmatrix} \begin{pmatrix}
\sqrt{\Delta'_1} \\
\Delta'_1 \Delta'_2 \\
\Delta'_1 \Delta'_2 \Delta'_3 \\
\sqrt{\Delta'_3}
\end{pmatrix}
\]  \hfill (18)

where \(T' = \bar{u}_1 + \bar{u}_2 v_1 + \bar{u}_3 v_2\) and etc.

As a result the CKM matrix \(V = U'^{\dagger} U'\) is given by

\[
V = \begin{pmatrix}
\frac{1}{\sqrt{\Delta'_1}} \\
\frac{1}{\sqrt{\Delta'_1 \Delta'_2}} \\
\frac{1}{\Delta'_1 \sqrt{\Delta'_2 \Delta'_3}} \\
\frac{1}{\Delta'_1 \sqrt{\Delta'_2 \Delta'_3}}
\end{pmatrix} \times
\begin{pmatrix}
\frac{1}{\sqrt{\Delta'_1}} \\
\frac{1}{\sqrt{\Delta'_1 \Delta'_2}} \\
\frac{1}{\Delta'_1 \sqrt{\Delta'_2 \Delta'_3}} \\
\frac{1}{\Delta'_1 \sqrt{\Delta'_2 \Delta'_3}}
\end{pmatrix}
\]  \hfill (19)

where

\[
\begin{pmatrix}
f_{11} & f_{12} & f_{13} & f_{14} \\
f_{21} & f_{22} & f_{23} & f_{24} \\
f_{31} & f_{32} & f_{33} & f_{34} \\
f_{41} & f_{42} & f_{43} & f_{44}
\end{pmatrix} = \begin{pmatrix}
1 & -T & a_1 & b_1 \\
x_1 & \Delta_1 - x_1 T & a_2 & b_2 \\
x_2 & y_1 \Delta_1 - x_2 T & a_3 & b_3 \\
x_3 & y_2 \Delta_1 - x_3 T & a_4 & b_4
\end{pmatrix} \begin{pmatrix}
1 & -T' & a'_1 & b'_1 \\
u_1 & \Delta'_1 - u_1 T' & a'_2 & b'_2 \\
u_2 & v_1 \Delta'_1 - u_2 T' & a'_3 & b'_3 \\
u_3 & v_2 \Delta'_1 - u_3 T' & a'_4 & b'_4
\end{pmatrix}.
\]
This is our geometric parametrization to the CKM matrix in the fourth generation of quarks. Though the form is a bit complicated, it is not avoidable.

A comment is in order. Jarlskog in [12], [13] has given another parametrization to $SU(n)$, which is based on the canonical coordinate of the second kind in the Lie group theory. See also [14] and [15]. However, the situation doesn’t become simpler.

4 Discussion

In the paper we revisited the Kabayashi–Maskawa theory in the standard model from the geometric point of view and generalized some basic facts based on the third generation of quarks on CP–violation to ones based on the fourth generation.

Though our method is of course not complete, to give a geometric insight to the standard model is very important.

To construct a unified model consisting of quarks and leptons we may treat a flag manifold $SU(6)/U(1)^5 \cong U(6)/U(1)^6$, see for example [5]. In our method, starting from

$$
F \equiv \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \in GL(6; \mathbb{C})/B_+
$$

we must perform the Gramm–Schmidt orthogonalization to obtain the unitary matrix $U$ likely in the text. However, the calculation becomes more and more hard. We will report it in the near future.

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Appendix  Jarlskog Determinant

In the appendix we calculate the determinant of the commutator $[M, M']$ in the general case. For mass matrices

$$M = UDU^\dagger, \quad M' = U'D'U'^\dagger$$

where $D = \text{diag}(m_1, m_2, \cdots, m_n)$ and $D' = \text{diag}(m'_1, m'_2, \cdots, m'_n)$, we want to calculate the Jarlskog determinant

$$\det[M, M'] = \det(DVD'(V^\dagger - VD'^\dagger D))$$

where $U^\dagger U' \equiv V = (V_{ij})$ is the general CKM matrix.

We set $X = DVD'(V^\dagger - VD'^\dagger D)$ for simplicity. $X$ is anti–hermite ($X^\dagger = -X$), so

$$\overline{\det(X)} = \det(X^\dagger) = \det(-X) = (-1)^n \det(X).$$

Therefore $\det(X)$ is real if $n$ is even, while $\det(X)$ is pure imaginary if $n$ is odd.

Explicitly,

$n = 2$ (real)

$$\det(X) = (m_2 - m_1)^2(m'_2 - m'_1)^2|V_{11}|^2|V_{21}|^2,$$

$n = 3$ (pure imaginary)

$$\det(X) = (m_3 - m_1)(m_3 - m_2)(m_2 - m_1)(m'_3 - m'_1)(m'_3 - m'_2)(m'_2 - m'_1) \times 2i \ \text{Im}(V_{11}V_{22}\bar{V}_{12}\bar{V}_{21}).$$

$n = 4$ (real) The calculation is not easy. In [13] Jarlskog tries to calculate the term

$$\text{Im}(V_{\alpha\beta}V_{\gamma\delta}V_{\gamma\delta}\bar{V}_{\alpha\beta}\bar{V}_{\gamma\delta}),$$

which is a “natural” generalization when taking the case of $n = 3$ into consideration. However, only such a term cannot be derived by the calculation above, [16]. We need further work.
References

[1] N. Cabibbo : Unitary Symmetry and Leptonic Decays, Phys. Rev. Lett. 10 (1963), 531.

[2] M. Kobayashi and T. Maskawa : CP–Violation in the Renormalizable Theory of Weak Interaction, Prog. Theor. Phys. 49 (1973), 652.

[3] C. Jarlskog : Commutator of the Quark Mass Matrices in the Standard Electroweak Model and a Measure of Maximal CP Nonconservation, Phys. Rev. Lett. 55 (1985), 1039.

[4] W. N. Cottingham and D. A. Greenwood : An Introduction to the Standard Model of Particle Physics, 1998, Cambridge University Press.

[5] G. W. Gibbons, S. Gielen, C. N. Pope and N. Turok : Naturalness of CP Violation in the Standard Model, arXiv:0810.4368 [hep-th].

[6] Particle Data Group : http://pdg.lbl.gov/

[7] M. Nakahara : GEOMETRY, TOPOLOGY AND PHYSICS (Second Edition), 2003, Taylor & Francis.

[8] K. Fujii : Introduction to Grassmann Manifolds and Quantum Computation, J. Applied Math, 2 (2002), 371, quant-ph/0103011.

[9] R. F. Picken : The Duistermaat–Heckman integration formula on flag manifolds, J. Math. Phys, 31 (1990), 616.

[10] M. Daoud and A. Jellal : Quantum Hall Effect on the Flag Manifold F_2, Int. J. Mod. Phys. A, 20 (2008), 3129, hep-th/0610157.

[11] K. Fujii and H. Oike : Reduced Dynamics from the Unitary Group to Some Flag Manifolds : Interacting Matrix Riccati Equations, to appear in Int. J. Geom. Methods Mod. Phys, 6 (2009), arXiv:0809.0165 [math-ph].
[12] C. Jarlskog : A recursive parametrization of unitary matrices, J. Math. Phys. 46 (2005), 103508, [math-ph/0504049](http://arxiv.org/abs/math-ph/0504049).

[13] C. Jarlskog : Recursive parameterisation and invariant phases of unitary matrices, J. Math. Phys. 47 (2006), 013507, [math-ph/0510034](http://arxiv.org/abs/math-ph/0510034).

[14] K. Fujii : Comment on “A Recursive Parametrisation of Unitary Matrices”, [quant-ph/0505047](http://arxiv.org/abs/quant-ph/0505047).

[15] K. Fujii, K. Funahashi and T. Kobayashi : Jarlskog’s Parametrization of Unitary Matrices and Qudit Theory, Int. J. Geom. Methods Mod. Phys, 3 (2006), 269, [quant-ph/0508006](http://arxiv.org/abs/quant-ph/0508006).

[16] T. Suzuki : in progress.