Floer theoretic invariants for 3- and 4-manifolds

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Abstract

Seiberg-Witten (Floer) theory, Ozsvath-Szabo’s Heegaard Floer theory, Hutchings’s embedded contact homology, in different stages of development, define (or are expected to define) packages of invariants for 3- and 4-manifolds (including manifolds with boundary and manifolds with certain types of corners). We describe what are known about their relationship, what are expected, and raise some questions along the way.

1 Floer homologies for 3-manifolds: $HF$, $HM$, and $ECH$

Let $(M, s)$ be a closed spin-c manifold. Three Floer-theoretic invariants (of differential structure) are associated to $(M, s)$: the Heegaard Floer homology of Ozsvath-Szabo, $HF^+ (M, s)$, the Seiberg-Witten-Floer homology (aka the monopole Floer homology) of Kronheimer-Mrowka, $HM (M, s, c_b)$, and, given in addition a contact structure $\xi$ on $M$, the embedded contact homology of Hutchings-Taubes, $ECH (-M, [\xi, s])$. Here, $[\xi, s]$ denotes an element of $H_1 (M; \mathbb{Z})$ determined by the pair $\langle \xi, s \rangle$ described in the following manner: By definition, a contact structure on $M$ is an oriented 2-plane field. Meanwhile, on an oriented 3-manifold, a spin-c structure can be identified with an equivalence class of oriented 2-plane...
fields. (See e.g. the last section of [HL1] or [HL2]). In view of this, a contact structure on $M$ determines a spin-c structure $s_\xi$. Recall that the set of spin-c structures on $M$ is a torsor over $H^2(M;\mathbb{Z}) \simeq H_1(M;\mathbb{Z})$. By assigning $0 \in H_1(M;\mathbb{Z})$ to the spin-c structure $s_\xi$, one defines an isomorphism from $H_1(M;\mathbb{Z})$ to the set of spin-c structures that intertwines with the $H^2(M;\mathbb{Z})$ action on both sides.

It is now known that these three Floer homologies are all isomorphic: the first equivalence, $\overline{HM}(M, s) \simeq ECH(-M, [\xi, s])$, is established by Taubes [Te]; the second equivalence, $\overline{HM}(M, s) \simeq HF^+(M, s)$, by Kutluhan-Lee-Taubes [KLT]; the third equivalence, $HF^+(M, s) \simeq ECH(-M, [\xi, s])$, by Colin-Ghiggini-Honda [CGH]. These invariants come equipped with rich algebraic structures: they take value in a cyclically-graded module over the graded ring $\mathbb{A}_1(M) := \mathbb{Z}[U] \otimes \wedge^* H_1(M;\mathbb{Z})/\text{Tor}$, where $U$ is of degree $-2$ and elements in $H_1(M;\mathbb{Z})/\text{Tor}$ are of degree $-1$.

Among the three Floer theories, the first two carry the following additional features: The Heegaard Floer $HF^+$ and the Seiberg-Witten $\overline{HM}$ are respectively one of a system of four flavors of Floer homologies in either theory. These four flavors are also modules over $\mathbb{A}_1(M)$, and they are related by two fundamental long exact sequences. In the Heegaard Floer case, the four flavors are denoted $HF^-, HF^\infty, HF^+$ and $\overline{HF}$, and the fundamental sequences are:

$$
\cdots \to HF^-(M, s) \to HF^\infty(M, s) \to HF^+(M, s) \to \cdots
$$

The map $HF^-(M, s) \to HF^-(M, s)$ in the second sequence above is the $U$-action; thus the second exact sequence is essentially the defining sequence of the flavor $\overline{HF}(M, s)$ from the $\mathbb{Z}[U]$-module structure of $HF^-$. The pair $HF^-$ and $HF^+$ satisfies certain duality property.

In parallel, the four flavors of Seiberg-Witten-Floer homologies corresponding to $HF^-, HF^\infty, HF^+$, $\overline{HF}$ are respectively denoted $\overline{HM}, \overline{HM}, \overline{HM}$, and $HM$, and they satisfy the same fundamental long exact sequences and duality property as the Heegaard Floer homology described above.

The aforementioned equivalence theorems are useful mostly due of the very different geometric origins of the three Floer homologies. These theorems will be stated more precisely in next section. In preparation, here we give a minimal sketch of the three Floer homologies’ respective setup and background. For some
representative applications of these equivalence theorems, see Section 4 below.

The Seiberg-Witten invariants for closed 4-manifolds were discovered and underwent rapid development during the second half of the 1990’s. Their relative simplicity in comparison to its predecessor, the Yang-Mills theory, led to significantly shortened proofs of many major theorems originally obtained via the latter, as well as important new results. See e.g. [D2] and [Ti] for surveys. Most notably, Taubes was able to establish a surprising equivalence of the Seiberg-Witten invariant and a version of Gromov invariant for closed 4-manifolds [T], which have very different constructions. (This is referred to as “Taubes’s SW = Gr theorem” below). Very roughly speaking, the Seiberg-Witten invariant of a closed 4-manifold $X$ is defined by counting (equivalence classes) of solutions to the Seiberg-Witten equation on $X$, while the Gromov invariant is defined by counting pseudo-holomorphic curves in $X$ when $X$ is equipped with a symplectic form $\omega$.

Recall that given a spin-c structure $s$ on $X$, the Seiberg-Witten equation takes the following form:

$$F^+_A - (\Psi^\dagger \tau \Psi - i \varpi) = 0 \text{ and } D^+_A \Psi = 0,$$

(2)

where $A$ is a Hermitian connection on the line bundle $\text{det}(S^+)$, $S = S^+ \oplus S^-$ being the spinor bundle associated to $s$, and $\Psi$ is a section of $S^+$. The term $\Psi^\dagger \tau \Psi$ stands for the bilinear map from $S^+$ to $i \Lambda^+$ that is defined using the Clifford multiplication, and $D^+_A : \Gamma(S^+) \to \Gamma(S^-)$ and $D^-_A : \Gamma(S^-) \to \Gamma(S^+)$ are the 4-dimensional Dirac operators on $X$ defined by the metric and the chosen connection $A$. Lastly, $\varpi$ is a self-dual 2-form on $X$. It is often referred to as the “perturbation form” in the equation. When $(X, \omega)$ is symplectic, one may choose the metric $g$ on $X$ so that $\omega$ is self-dual (and hence harmonic). This defines an almost complex structure $J$ so that $g(\cdot, \cdot) = \omega(\cdot, J \cdot)$. Taubes considered perturbation forms of the type

$$\varpi = 2r \omega - i F^+_A,$$

(3)

where $r \in \mathbb{R}^+$, and $A_K$ is the induced connection on the canonical line bundle $K$ associated to $J$. Use Clifford multiplication by $\omega$ to split $S^+$ as a sum of eigen-bundles $E \oplus E \otimes K^{-1}$, and let $\alpha$ denote the $E$-component of $\Psi$ under this decomposition. It was shown that as one takes $r \to \infty$, the zero locus of $\alpha$ approaches, in certain technical sense, a (possibly disconnected) $J$-holomorphic curve in a homology class determined by $s$. 

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If instead of closed 4-manifolds, one considers gauge-theoretic equations on 4-dimensional cylinders \( \mathbb{R} \times M \), one may in principle construct an associated Floer-homology for 3-manifolds. The Seiberg-Witten version of this, \( HM \), while long expected, has rather technical actual construction and was not fully written down until almost a decade later \([KM]\). Nevertheless, in comparison to the Yang-Mills version, usually called the instanton Floer homology, Kronheimer-Mrowka’s monopole Floer homology is defined for all closed, oriented 3-manifolds. (The instanton Floer homology has only been successfully defined for rational homology spheres, except some special cases. However, unlike the closed 4-manifold Seiberg-Witten invariant, which is in most cases independent of the perturbation form \( \varpi \), the Seiberg-Witten Floer homology depends on the cohomology class of the perturbation form in the Seiberg-Witten equation: Let \((M, s)\) be a closed spin-c 3-manifold and \( X = \mathbb{R} \times M \), and take the form \( \varpi \) in (2) to be \( \mu + ds \wedge *_3 \mu \) for a closed 2-form on \( M \), where \( s \) denotes the affine coordinate on the \( \mathbb{R} \) factor of \( X = \mathbb{R} \times M \), and \(*_3\) denotes the 3-dimensional Hodge dual. The spin-c structure \( s \) on \( M \) determines a spin-c structure on \( X \), which we denote by the same notation. Denote the associated spinor bundles on \( M \) and \( X \) respectively by \( S_X = S_X^+ \oplus S_X^- \) and \( S \). Clifford multiplication by \( ds \) determines an isomorphism \( S_X^+ \simeq S_X^- \simeq S \). The equation (2) can be interpreted as a formal gradient flow equation on \( \text{Conn}(M) \times \Gamma(S) \), where \( \text{Conn}(M) \) denotes the space of connections on \( \det S \):

\[
\frac{d}{ds}(B, \Phi) = -\left( *_3 (F_B - \Phi^\dagger \Phi + i\mu), D_B \Phi \right).
\]

Here, \((B(s), \Phi(s))\) denotes a path in \( \text{Conn}(M) \times \Gamma(S) \) parametrized by \( s \in \mathbb{R} \). The gauge group \( G := C^\infty(M, U(1)) \) acts on \( \text{Conn}(M) \times \Gamma(S) \) by \( u \cdot (B, \Phi) = (B - 2u^{-1}du, u\Phi) \ \forall u \in G \) and \((B, \Phi) \in \text{Conn}(M) \times \Gamma(S) \). The Seiberg-Witten equation (4) is invariant under the gauge action, and thus has an interpretation as the (formal) flow equation of the dual vector field to a closed 1-form on \( B := (\text{Conn}(M) \times \Gamma(S)) / G \). Note that \( \pi_1(B) = H_1(B) = H^1(M; \mathbb{Z}) \), and the cohomology class of the aforementioned closed 1-form is given by

\[
2\pi^2 c_1(s) - \pi[\mu] \in H^2(M; \mathbb{R}) \overset{P.D.}{\simeq} \text{Hom}(H^1(M; \mathbb{Z}); \mathbb{R}) \simeq H^1(B; \mathbb{R}).
\]

The Seiberg-Witten-Floer homology associated to (4) is modelled on the Morse-Novikov theory associated this closed 1-form on \( B \), and thus for this heuristic
reason is expected to depend on the classes $[\mu]$ and $c_1(s)$ but not other parameters. This is indeed verified by a lengthy argument in [KM]. With $s$ fixed, the class $-\pi[\mu]$ is called the “period class” of the Seiberg-Witten-Floer homology built from (4). Meanwhile, it follows from the Atiyah-Patodi-Singer theorem on spectral flows that this Floer homology group has a relative grading by the cyclic abelian group $\mathbb{Z}/c_s$, where $c_s \in 2\mathbb{Z}$ is the gcd of the values of $c_s$, viewed as a linear map from $H^1(M; \mathbb{Z}) \overset{P.D.}{\simeq} H_2(M; \mathbb{Z})$ to $\mathbb{Z}$. The grading group is $\mathbb{Z}$ when $c_1(s)$ is torsion, namely when $c_s = 0$. For technical reasons that we shall not explain here, given an arbitrary pair of $c_1(s)$ and periodic class $c$, the corresponding Seiberg-Witten Floer homology is only defined for coefficient rings $\Lambda$ that satisfy certain completeness conditions, called “$c$-completeness” in [KM]. This Floer homology is denoted by $\overline{HM}(M, s, c; \Lambda)$ in [KM]. The coefficient ring $\Lambda$ is assumed to be $\mathbb{Z}$ when $\Lambda$ is omitted from the notation, and the period class $c$ is assumed to be 0 when it is omitted.

Remark. To be more precise, for $HM$ to be well-defined, due to transversality issues one should allow certain additional abstract perturbations to (4). Cf. [KM] Chapter 10. We ignore this technical issue in this article.

$ECH$ is constructed as a Floer-homology extension of Taubes’s Gromov invariant $Gr$ in his $SW = Gr$ theorem [1] for closed 4-manifolds. An equivalence of $HM$ and $ECH$, in parallel to Taube’s theorem for closed 4-manifolds is thus expected from the very beginning of the development of $ECH$. Let $(M, a)$ be a contact 3-manifold with a contact 1-form $a$, and choose the metric on $M$ so that $a$ is co-closed. In analogy to the type of perturbations (3) used in Taubes’s proof of his $SW = Gr$ theorem, choose the 2-form $\mu$ in (4) to be of the form $\mu = 2r \ast_3 a - iF_{B_K}$ and consider the associated $HM$ as $r \to \infty$. Here, $B_K$ is the connection on $K^{-1}$ induced from the metric, and $K := \ker a \subset TM$ is equipped with the complex structure given by Clifford multiplication by $a$. As the $Gr$-side analog of $HM$, the chain modules of $ECH$ are generated by certain union of (weighted) orbits of the Reeb flow of $a$ (called “orbit sets”), and the entries of the differential as a matrix with respect to the basis consisting of orbit sets are defined by counting (disjoint, weighted) pseudo-holomorphic curves in $\mathbb{R} \times M$ asymptotic to the relevant orbit sets. It turns out that the actual proof of the equivalence of $HM$ and $ECH$ requires essential new ideas in addition to those in [1]. See
For a more precise statement of this equivalence theorem, and for full details.

Ozsváth-Szabó’s $HF$ may be viewed another $Gr$ counterpart of $HM$, albeit in a less straightforward manner. Its motivation comes from a variant of the Atiyah conjecture [A], in addition to Taubes’s philosophy of $SW$-$Gr$ correspondence. Fix a Heegaard decomposition of a closed 3-manifold $M$. Let $f : M \to \mathbb{R}$ be a self-indexing Morse function adapted to this Heegaard decomposition. By this we mean that $f$ has unique maximum and minimum, and $G$-pairs of index 2 and index 1 critical points, where $G$ is the genus of the Heegaard surface $\Sigma = f^{-1}(3/2)$. Let $\alpha := \{\alpha_1, \alpha_2, \ldots, \alpha_G\}$ denote the set of descending cycles from index 2 critical points on $\Sigma$, and let $\beta := \{\beta_1, \ldots, \beta_G\}$ denote the set of ascending cycles from index 1 critical points on $\Sigma$. We call the triple $(\Sigma, \alpha, \beta)$ a Heegaard diagram of $M$. The idea of the Atiyah conjecture is to relate the Floer homology of $M$ to a Lagrangian Floer homology of $(M, T_\alpha, T_\beta)$, where $M$ is a symplectic manifold typically coming from the moduli space of a suitable dimensional reduction of the relevant gauge equation on 3-manifolds, and $T_\alpha$ and $T_\beta$ are Lagrangian submanifolds (typically with singularities) defined from moduli spaces of solutions to the gauge equation on $f^{-1}[3/2, \infty)$ and $f^{-1}(-\infty, 3/2]$ respectively. In the setting of Seiberg-Witten theory, heuristic reasoning from Taubes’s philosophy predicts that for certain large $r$ perturbation involving $*df$, the corresponding triple $(\mathcal{M}, T_\alpha, T_\beta)$ should be $(\text{Sym}^G(\Sigma), \alpha_1 \times \cdots \alpha_G, \beta_1 \times \cdots \times \beta_G)$. The Heegaard Floer homology is a variant of Lagrangian Floer homology associated to $(\text{Sym}^G(\Sigma), \alpha_1 \times \cdots \alpha_G, \beta_1 \times \cdots \times \beta_G)$, with one extra key ingredient: a choice of a base point $z_0 \in \Sigma - \bigcup_i \alpha_i \cup \bigcup_j \beta$. This is used to define a filtration on the relevant Heegaard Floer complex. It is somewhat long and complicated to explain the relation of $HF$ with $HM$ with Taubes’s type of perturbations; the interested reader is referred to [L].

### 2 Equivalences of Floer homologies: theorems and questions

We may now state the isomorphism theorem of [KLT] more precisely:

**2.1 Theorem ([KLT] Theorem V.1.4)** Let $M$ be a closed, oriented 3-manifold, and $s$ be a spin-c structure on $M$. Then there exists a system of isomorphisms from $HF^\circ_s(M, s)$, $\circ = -, \infty, +, \wedge$, respectively to $HM^\circ_s(M, s, c_b)$, $\circ = \wedge, -, \vee, \sim$, as
\( \mathbb{Z}/c_b\mathbb{Z} \)-graded \( \mathbb{A}_1(M) \)-modules, which is natural with respect to the fundamental exact sequences of the Heegaard and monopole Floer homologies.

Here, \( c_b \) stands for a “balanced perturbation” in the terminology of [KM], and refers to the case when the cohomology class \( \delta \) is 0. Among all periodic classes, the balanced case is strongest in the following sense: it is one for which the associated \( HM \) can be defined over \( \mathbb{Z} \), and the \( c_b \)-completeness condition required for the coefficient ring of \( HM \) is vacuous. Thus, \( HM(M, s, c; \Lambda) \) of other local coefficients \( \Lambda \) may be computed via the universal coefficient theorem, together with results in [KM]’s Chapter 31 relating monopole Floer homologies associated to proportional \( \delta \). It is also the only class for which \( \hat{HM} \) is nonvanishing, and consequently by the fundamental exact sequences for \( HM \), the only class that the two flavors of \( HM \), \( \hat{HM} \) and \( \hat{HM} \), differ.

A subtle point worth noting is that in [KM] as well as in other literature, the monopole Floer homology frequently refers to the “bullet version” (or completed version) \( HM_* \) instead of the “star version” (or pre-completed version) \( HM_* \) appearing in the statement of the preceding Theorem. The former version, \( HM_* \), uses coefficients that are completed with respect to the \( U \)-action, and therefore is slightly weaker than the latter version, \( HM_* \). For example, \( \hat{HM}_* \) vanishes while \( \hat{HM}_* \) is nontrivial in the example computed in [KM] Equation (35.4). Working with \( HM_* \) is in particular more convenient in discussions involving maps between monopole Floer homologies induced from cobordisms between two 3-manifolds. However, to be able to define the Floer chain complex with polynomial (in \( U \)) coefficients (instead of power series coefficients), certain strong compactness results are necessary. For \( HM \) and \( HF \), these are guaranteed respectively via the balanced perturbation condition and the “strong admissibility” assumption on the Heegaard diagram.

In comparison, the aforementioned finiteness/compactness results are missing in \( ECH \). This nevertheless is consistent with Taubes’s \( HM = ECH \) theorem, because according to [KM], \( \tilde{HM}(M, s) \simeq \tilde{HM}_*(M, s, c_b) = \tilde{HM}_*(M, s, c_b) \).

Here, \( \tilde{HM}(M, s) \) stands for the version of Seiberg-Witten-Floer homology when the perturbation form is exact. In this case, except for the case when \( c_1(s) = 0 \), \( \tilde{HM}(M, s) = \tilde{HM}(M, s) \), and both of them are finite rank \( \mathbb{Z} \)-modules.

Meanwhile, just as (the original) \( ECH \) is an analog of \( \tilde{HM} \simeq HF^+ \), one
may define another flavor of $ECH$, called $\widehat{ECH}$, as an analog of $\widehat{HM} \simeq \widehat{HF}$.

(See e.g. [CGHH]). The pair $ECH$ and $\widehat{ECH}$ fits in an analog of the second long exact sequence in Equation (1) above, also called the fundamental sequence of $ECH$. It follows directly from Taubes’s proof of $HM = ECH$, together with the definitions of $ECH(-M, [\xi, s])$ in [CGHH] and $HM(M, s)$ in [KLT] part V, that the latter is equivalent to $\widehat{HM}$.

To summarize in a more precise fashion:

2.2 Theorem Let $M, s$ be as in the previous theorem, and let $\xi$ be a contact structure on $M$. Then there is a pair of isomorphisms $ECH(-M, [\xi, s]) \simeq \widehat{HM}(M, s); \ ECH(-M, [\xi, s]) \simeq \widehat{HM}(M, s)$, which are natural with respect to the fundamental sequences on both sides. I

As a consequence of Theorems 2.1 and 2.2 one has

2.3 Theorem Let $M, s$ and $\xi$ be as in the previous theorem. Then there is a system of isomorphisms $ECH(-M, [\xi, s]) \simeq HF^+(M, s); \ ECH(-M, [\xi, s]) \simeq HF^-(M, s)$, which are natural with respect to the fundamental exact sequences on both sides.

Alternatively, [CGH] has a “purely symplectic” proof of the preceding theorem without going through Seiberg-Witten theory. Comparing Theorems 2.1, 2.2 and 2.3 above, one naturally asks:

2.4 Question Is there an $ECH$ analog of (pre-completed) $\widehat{HM}$, or $\widehat{HF}$, which is defined from Floer chain complexes of $\mathbb{Z}[U]$-modules?

If such an analog can be shown to be equivalent to $\widehat{HM}$ or $\widehat{HF}$, the aforementioned finiteness properties of the latter Floer homologies might help answer questions related to finiteness of certain types of Reeb orbits on contact 3-manifolds.

In a different direction, Taubes’s $HM = ECH$ theorem has a sister version for 3-dimensional mapping tori, where the contact form is replaced by a harmonic, nowhere-vanishing 1-form. A variant of $ECH$, dubbed “PFH” (periodic Floer homology) by Hutchings, is shown to be equivalent to (a different version) of $HM$ by the present author and Taubes:
Let \((F, w_F)\) denote a closed oriented surface \(F\) equipped with a volume form \(w_F\), and \(\varphi\) is a volume preserving automorphism of \(F\). Let \(M_\varphi\) denote the mapping torus of \(\varphi\) and \(w_\varphi\) the closed 2-form on \(M_\varphi\) induced from \(w_F\). Let \(K^{-1} \subset TM_\varphi\) denote the subbundle consisting of tangent vectors to the fibers of the bundle \(M_\varphi \to S^1\), and \(c_1(K^{-1})\) its Euler class. An element in \(\Gamma \subset H_1(M_\varphi; \mathbb{Z})\) is said to be monotone when \([w_\varphi] = -\lambda (c_1(K^{-1}) + 2P.D.(\Gamma))\) for a real number \(\lambda\). It is said to be positive monotone when \(\lambda > 0\), and negative monotone with \(\lambda < 0\). The periodic Floer homology of \(M_\varphi\) in the class \(\Gamma\) is denoted by \(HP^* (\varphi : (F, w_F) \otimes, \Gamma)\) in \([LT]\). See §1.1 therein for details of the definition. In §1.2 of the same paper, a spin-c structure \(s_\Gamma\) and a closed 2-form \(\varpi_r := 2rw_\varphi + \varphi\) is assigned to each pair \((\varphi : (F, w_F) \otimes, \Gamma)\). Here, \(r\) is a sufficiently large real number, and \(\varphi\) is a closed 2-form in the cohomology class \(2\pi c_1(s_\Gamma)\). The precise choice of \(r\) and \(\varphi\) turns out to be immaterial.

2.5 Theorem ([LT], Theorem 1.1) Let \((F, w_F), \varphi\) be as above, and let \(\Gamma\) denote either a positive or negative monotone class. Then \(HP(\varphi : (F, w_F) \otimes, \Gamma) \simeq HM(M_\varphi, s_\Gamma, [\varpi_r])\).

The \(HM\) on the right hand side of the isomorphism above stands for either \(\hat{HM}\) or \(\overline{HM}\), which are the same under this particular setting. According to \([KM]\), for a monotone \(\Gamma\), the periodic class \([\varpi_r]\) is what is called “positive/negative monotone” with respect to the spin-c structure \(s_\Gamma\) for all sufficiently large \(r\), precisely when \(\Gamma\) is positive/monotone. In other words, when \(\Gamma\) is monotone,

\[
HP(\varphi : (F, w_F) \otimes, \Gamma) \simeq HM(M_\varphi, s_\Gamma) \quad \text{if} \quad d_\Gamma := \langle P.D.[F], \Gamma \rangle < g - 1, \quad \text{and} \\
HP(\varphi : (F, w_F) \otimes, \Gamma) \simeq HM(M_\varphi, s_\Gamma, c_-) \quad \text{if} \quad d_\Gamma > g - 1.
\]

In the above, \(c_\pm\) respectively denotes a positive/negative periodic class for \(HM\). According to \([KM]\), \(HM(M_\varphi, s_\Gamma, c_+ \simeq HM(M_\varphi, s_\Gamma)\) and \(HM(M_\varphi, s_\Gamma, c_-)\) is related to \(HM(M_\varphi, s_\Gamma)\) and \(\overline{HM}(M_\varphi, s_\Gamma, c_b)\) via a long exact sequence, and therefore is always different from the version of monopole Floer homology appearing in the \(HM = ECH\) theorem, Theorem 2.2 (See discussions following Corollaries 1.4 and 1.5 in \([LT]\), as well as references given therein).

The contact structure appearing in Theorem 2.2 and the mapping tori structure appearing in Theorem 2.5 are both special cases of the so-called “stable Hamiltonian structure” (see e.g. \([HT3]\)). It is therefore natural to ask:
2.6 Question Is there a version of ECH for 3-manifolds with stable Hamiltonian structure, that encompasses both the PFH for mapping tori and the ECH for contact 3-manifolds as special cases?

As hinted in [CFP], this might require more than the analytic techniques in [HT]. Assuming that the preceding question has been positively answered,

2.7 Question Prove a version of $HM = ECH$ for this generalized ECH that encompasses $HM = ECH$ of [Te] and $HM = PFH$ of [LT] as special cases.

This is likely a very difficult question that requires essential new ideas. In spite of many similarities in the proofs of the sister theorems $HM = ECH$ and $HM = PFH$ in [Te] and [LT], one major difference is the “energy bound” (roughly, a certain $L^1$-bound of curvature) that forms the starting point of the proof for geometric convergence of Seiberg-Witten solutions giving rise to holomorphic curves. In the case of $HM = PFH$, it follows mainly from topological reasons; while in the case of $HM = ECH$, it follows from a more delicate spectral flow estimate. Unfortunately, there is no easy way of combining these two very different arguments in general.

Some partial results towards this direction appear in [KLT]. (See in particular papers IV and V of [KLT]). As will be explained in Section 3 below in more detail, this series of articles defines a variant of $ECH$ for certain manifolds with stable Hamiltonian structure, and aspects of its relation to $HM$ are established. The proofs of these results are long and hard, and in fact constitute the technical core of the proof of Theorem 2.1. However, the method therein work only for a very special type of stable Hamiltonian structure. (More will be said about this stable Hamiltonian structure in Section 4 below). Brute force is used to amalgamate the key energy bounds in the proofs of $HM = ECH$ and $HM = PFH$, that are obtained via very different methods. In these articles, the relevant stable Hamiltonian structure “splits” along certain simple surfaces, such that one side has a mapping torus structure, and the other a specific contact structure studied extensively by Taubes. (See e.g. [TZ]). The 3-manifold is then stretched very long along the splitting surface so that the estimates on both sides may be performed essentially separately.
3 Local coefficients and general perturbation classes

No additional work is required to define local coefficients versions for the Floer homologies $HM$, $HF$, $ECH$. In fact, the compactness results needed to define a Floer homology are typically weaker for local coefficients with suitable completeness properties. In Heegaard Floer theory, this manifests itself in that only “weakly admissible” Heegaard diagrams are required to define Heegaard Floer complexes with twisted coefficients.

The isomorphism theorems, Theorem 2.1, 2.2 and hence 2.3 extend to isomorphisms in any corresponding local coefficients, once it is verified that the chain maps inducing the isomorphisms in Floer homologies intertwine with the $A_\dag$-action. E.g., this is done in paper V of [LT] for the isomorphism $HM=ECH$.

(Very roughly, this is because the $H_1(M)/\text{Tor}$ part of the $A_\dag$ acts reflects the action of the relevant “fundamental group” on Floer complexes. For an explanation in more precise terms, see e.g. Section 6 of [LT]).

A little more needs to be said about defining and extending the aforementioned isomorphism theorems to general “perturbation classes”. (What is what was called the “periodic class” in Seiberg-Witten theory).

On the $ECH$ side, a contact form is exact, and the “perturbation class” is always trivial if one restricts to contact 3-manifolds. “Perturbed” versions of $ECH$ enter the scene in the more general realm of stable Hamiltonian structures. In particular, $PFH$ can be defined for arbitrary $\Gamma$ and a range of necessarily nontrivial perturbation classes and local coefficients that satisfy certain completeness conditions. The isomorphism of these with the corresponding Seiberg-Witten-Floer homology, extending Theorem 2.5 above, is stated precisely as Theorem 6.5 in [LT]. The generalized $ECH$ relevant to the proof of Theorem 2.1 in [KLT] is another example of $ECH$ with nontrivial perturbation class.

On the Heegaard Floer side, a Heegaard Floer homology $HF^\circ(M, s, \eta)$ corresponding to other perturbation class $\eta \in H^2(M, \mathbb{R})$ is defined in [OS] §11.0.1, see also e.g. [Wu] for more details. This “perturbed Heegaard Floer homology” has as coefficients the Novikov ring $\Lambda_A$ described presently. This ring consists of formal power series $\sum_{r \in \mathbb{R}} a_r T^r$ with $a_r \in \mathbb{R}$ and $T$ a formal variable, satisfying the condition that for any fixed $N \in \mathbb{R}$, the number of $a_r \neq 0, r < N$ is finite. Its multiplication law is given by $(\sum_{r \in \mathbb{R}} a_r T^r) \cdot (\sum_{s \in \mathbb{R}} b_s T^s) = \sum_{r \in \mathbb{R}} \sum_{k \in \mathbb{R}} a^k b^{r-k} T^r$. 

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As a pay off for working with this complicated completed coefficient ring, no ad-
missibility condition on the Heegaard diagram is required for the associated Hee-
gaard Floer complex to be well defined. The cohomology $\eta \in H^2(M, \mathbb{R})$ enters
the definition of the boundary map $\partial$ of this perturbed Heegaard Floer complex
via the area assigned to each domain in the Heegaard diagram counted in $\partial$.

It takes only superficial changes (modification of Lemma 1.1 and Lemma 1.2
in paper II) to the proof of Theorem 2.1 in [KLT] to show that:

3.1 Theorem  Let $(M, s)$ be as in Theorem 2.1. There is a system of isomorphisms
from $HF^\circ(M, s, \eta)$, $\circ = -, \infty, +, \wedge$, respectively to $HM^\circ(M, s, c_\eta; \Lambda_A)$, $\circ =$
$\wedge, -, \vee, \sim$, as $\mathbb{Z}/c_\mathbb{Z}$-graded $\mathbb{A}_3(M)$-modules, which is natural with respect to the
fundamental exact sequences of the Heegaard and monopole Floer homologies.

4 Applications of the equivalence theorems

The equivalence theorems in Section 1.1 are useful because of the very different
geometric origins of the three Floer homologies.

Among the aforementioned equivalence theorems, Theorem 2.2 has more suc-
cess in finding applications so far. Most notably, the 3-dimensional Weinstein con-
jecture and its analogs. See e.g. [Tw], [Tw2], [HT2], [HT3]. In another direction,
Hutchings defined an ECH version of symplectic capacities, which provides com-
plete obstructions to symplectically embedding 4-dimensional ellipsoids to each
other. See e.g. [Hs]. These results make use of the fact that expected properties
or definitions in ECH are often difficult to carry out directly, and thus its relation
with $HM$ enables one to appeal to the more fully developed Seiberg-Witten the-
ory. For example, The proof of Weinstein conjecture type results indirectly make
use of the easy computatin of $HM$; the proof of the Arnold’s chord conjecture
makes use of the surgery exact sequences in Seiberg-Witten-Floer homology. The
definition and key properties of ECH capacities go through the definitions and
properties of maps induced by cobordisms in Seiberg-Witten-Floer theory.

As immediate consequences of Theorem 2.1 the equivalence $HM = HF$ can
be used to compute one Floer homology of specific 3-manifolds in terms of the
other, depending on which is simpler. For example, the intricate computations of
$HF$ for mapping tori done in [JM, JM2] and the computation of $HF^\infty$ in [M]
follow easily from the corresponding computation on the Seiberg-Witten side.
The Seiberg-Witten-Floer homology of Seibert-fibered spaces has been carried out for general Seifert fibered spaces in [MOY], while the computation of $HF$ are only done for various special cases of Seifert fibered spaces in the literature, using different methods. On the other hand, it has been shown that $\hat{HM}$ is purely combinatorial (see e.g. [SW]), and thus the equivalence $HM = HF$ implies that $\hat{HM}$ can likewise be computed purely combinatorially.

To find more interesting application of Theorem 2.1 will most likely require extending the equivalence of Floer homologies of 3-manifolds both sides to other aspects of Seiberg-Witten and Ozsvath-Szabo theories. For example, like the aforementioned applications of $HM = ECH$, it is important to know that whether the system of isomorphisms constructed in [KLT] is natural with respect to the surgery exact sequences and TQFT structures on both sides. In the rest of this article, we discuss some possible amplifications of the arguments in [KLT] towards this goal. These are emphatically not straightforward, in particular because the approach adopted in [KLT] is rather indirect. In preparation, the next section gives a brief outline of the proof in [KLT]. For a fuller summary, see paper I of [KLT].

5 Outlining the proof of Theorem 2.1: motivation and strategy

Surprisingly, the formally similar algebraic structures on $HF$ and $HM$ have completely different origins. For example, The mechanism that gives rise to the four flavors $\circ = -, \infty, +, \wedge$ in $HF^\circ$, stem from a filtration on the (simplest flavor of) the Heegaard Floer complex. Roughly speaking, the Floer complex $CF^\infty$ can be viewed as a chain complex with local coefficients $\mathbb{Z}[\mathbb{Z}] \simeq \mathbb{Z}[U, U^{-1}]$. Alternatively, viewing it as an analog of the chain complex of a $\mathbb{Z}$-covering of a finite-dimensional space, the $U$-action corresponds to deck transformations. A key consequence of the geometric setup in constructing $HF$ is that a basis of the chain module $CF^\infty$ may be found so that with respect to which, the boundary map $\partial^\infty$ has the form of a matrix with coefficients in the polynomial ring $\mathbb{Z}[U]$. This equips $CF^\infty$ with a filtration by $\mathbb{Z}[U]$-subcomplexes

$$\cdots CF^- \subset U^{-1} \cdot CF^- \subset U^{-2} \cdot CF^- \subset \cdots \subset CF^\infty$$

The four flavors of $HF^\circ$ are defined as various homology groups associated to this filtered chain complex. For example, the first fundamental sequence is the
relative long exact sequence induced from the short exact sequence:

\[ 0 \to CF^- \to CF^\infty \to CF^+ := CF^\infty / CF^- \to 0. \]

In contrast, the Seiberg-Witten Floer homology is modelled on an \( S^1 \)-equivariant homology theory. From this point of view, the module structure over \( \mathbb{Z}[U] \) in \( HM \) reflects the module-structure of \( S^1 \)-equivariant homologies. (\( \mathbb{Z}[U] \) is the cohomological ring of the classifying space of \( S^1 \) actions, \( \mathbb{C}P^\infty \)). The four flavors of \( HM \) arise as various versions of \( S^1 \)-equivariant homologies. For example, the first fundamental sequence is modelled on a well-known long exact sequence in \( S^1 \)-equivariant homology theory relating the Borel equivariant homology, the Tate version of equivariant homology, and the so-called co-Borel version.

To resolve the seemingly irreconcilable differences in the foundations of Heegaard and Seiberg-Witten Floer homologies, the strategy of [KLT] is to go through a third, intermediate version of Floer homology, denoted \( ech^o \) in paper I of [KLT]. The isomorphisms in [2.1] will follow from composing a number of isomorphisms among several Floer homologies.

Consider \( \tilde{HM}(M, s, c_b) \) and \( HF^\infty(M, s) \) from the statement of Theorem 2.1. The relevant intermediate Floer homology \( ech^o \) is defined for an auxiliary 3-manifold \( Y := Y_M \). Given a pointed Heegaard diagram used to define \( HF^\infty(M, s) \), let \( f \) be a self-indexing Morse function associated with this Heegaard decomposition. Suppose this \( f \) has one pair of index 0 and index 3 critical points, and \( G \) pairs of index 1 and index 2 critical points. \( Y_M \) is built from \( M \) and by doing a 0-dimensional surgery along the aforementioned \( G + 1 \) copies of \( S^0 \)'s (i.e. pairs of critical points). Denote the copy of \( I \times S^2 \subset Y_M \) resulted from surgery along the pair of index 0 and index 3 critical points by \( \mathcal{H}_0 \), and denote by \( \mathcal{H}_i \), \( i \in \Lambda := \{1, \ldots, G\} \), each of the \( G \) copies of \( I \times S^2 \subset Y_M \) resulted from surgery along a pair of index 2 and index 1 critical points. This \( Y_M \) is then assigned with a special stable Hamiltonian structure, \( a \in \Omega^1(Y_M) \) and \( w \in \Omega^2(Y_M) \), such that \( a \land w \) is nowhere vanishing, \( dw = 0 \), and \( da = \lambda w \) for a scalar function \( \lambda \) on \( Y_M \), \( \lambda \geq 0 \). Choose a metric on \( Y_M \) such that \( a \) agrees with the Hodge star of \( w \). The salient features of this special stable Hamiltonian structure includes:

- The metric is such that the interval factor \( I \) in \( \mathcal{H}_i \simeq I \times S^2 \) is very long cylinder.
• $\lambda$ “approximates” the zero function in the interior of $Y_M - \bigcup_{i \in \Lambda} \mathcal{H}_i$. In other words, both $a$ and $w$ are “almost harmonic” in this region.

• $\lambda$ “approximates” a nonzero constant function in the interior of each $\mathcal{H}_i$, $i \in \Lambda$. In other words, the 1-form $a$ is “almost contact” on each $\mathcal{H}_i$.

• Over the long cylinder $\mathcal{H}_0$, $a$ approximates the harmonic form $dt$ on $\mathbb{R} \times S^2$, where $t$ is an affine parameter of the first factor $\mathbb{R}$.

• Over the interior of $Y_M - \bigcup_{i \in \{0\} \cup \Lambda} \mathcal{H}_i$, $a$ approximates a constant multiple of $df$.

• Over each long cylinder $\mathcal{H}_i \simeq I \times S^2$, $i \in \Lambda$, $a$ approximates the following well-known contact form $a_0$ on $\mathbb{R} \times S^2$:

$$a_0 = (1 - 3 \cos^2 \theta) dt - \sqrt{6} \cos \theta \sin \theta^2 d\phi,$$

where $(t, \theta, \phi)$ is the standard cylindrical coordinates of $\mathbb{R} \times S^2$.

The contact form $a_0$ above has an analog on the closed 3-manifold $S^1 \times S^2$, in which case $t \in S^1$ instead of $\mathbb{R}$. The p-holomorphic curves in the symplectization of this contact 3-manifold have previously been studied extensively by Taubes, see e.g. [Tz]. The contact manifold $S^1 \times S^2$ is of particular interest because its symplectization serves as an asymptotic model for the (real blow-up of) tubular neighborhoods of connected components of zero loci of generic self-dual harmonic 2-forms on closed 4-manifolds. A harmonic 2-form of this type is regarded as a generalization of symplectic forms; it is sometimes called a “singular symplectic form”. Unlike symplectic forms, it exists on any closed 4-manifolds with $b_2^+ > 0$. See e.g. [Ti] for an explanation of the relevance of such harmonic 2-forms in 4-manifold topology.

Roughly speaking, singular symplectic forms arise in the setup of [KLT] in the following way: Let $\overline{M}$ denote the 3-manifold obtained from $M$ by performing a 0-dimensional surgery along the pair of index 0 and index 3 critical points of $f$. The real-valued Morse function $f$ extends naturally to an $S^1$-valued Morse function $\overline{f}$ on $\overline{M}$, which has no extrema. A result of Calabi asserts that a metric on $\overline{M}$ may be found with respect to which $\overline{f}$ become harmonic. This naturally gives rise to a singular symplectic form on the cylinder $\mathbb{R} \times \overline{M}$ with product metric, namely

$$\omega = ds \wedge df + *_3 df,$$
where $s$ is an affine coordinate for the $\mathbb{R}$-factor of the cylinder $\mathbb{R} \times M$, and $*_{3}$ denotes the 3-dimensional Hodge star. In view of the above discussion, the construction of $Y_M$ and the special Hamiltonian structure $(a, w)$ should be understood as doing real blow-ups of $M$ along critical points of $f$, then “connect pairs of ends” of this blown-up manifold to get a closed manifold. There is a good reason for choosing to work with closed 3-manifolds instead of manifold with cylindrical ends: The analysis required for constructing Floer homologies of noncompact manifolds is almost always intractible.

Fix a spin-c structure $s$ on $M$. There is a natural way to choose a corresponding spin-c structure on $Y_M$. As explained e.g. in [HL2], one may represent $s$ by a set of mutually disjoint embedded arcs $\{\gamma_i\}_{i \in \{0\} \cup \Lambda}$, where $\partial \gamma_i \simeq S^0$ is the attaching sphere for $H_i$. Meanwhile, by “pinching” along the boundary 2-sphere of a tubular neighborhood of each $\gamma_i$, $Y_M$ may be expressed as a connected sum of $M \# G + 1$ copies of $S^1 \times S^2$, one for each $H_i$. The spin-c structure on $Y_M$ corresponding to $s$ is, with respect to this connected sum decomposition, the connected sum of $s$ for the $M$-summand, together with $s_0$ for each copy of $S^1 \times S^2$ corresponding to $i \in \Lambda$, and with $s_K$ on the copy of $S^1 \times S^2$ corresponding to $i = 0$. Here, $s_0$ denotes the trivial spin-c structure, namely, the spin-c structure with trivial first Chern class. The spin-c structure $s_K$ is the one represented by the oriented 2-plane field $\ker(dt)$, $t \in S^1$ being an affine parameter of the first factor of $S^1 \times S^2$. Fix a choice of the arcs $\{\gamma_i\}_i$, and write the corresponding connected sum decomposition of $Y_M$ as

$$Y_M \simeq M \#_{s_0}(S^1 \times S^2) \#_{\gamma_0}(S^1 \times S^2) \cdots \#_{\gamma_G}(S^1 \times S^2).$$

With respect to this connected sum decomposition, the graded algebra $A^*(Y_M)$ is identified with a tensor product of the graded algebras $A^*(M) \otimes \bigotimes_{i \in \{0\} \cup \Lambda} \Lambda_i$, where each $\Lambda_i = \Lambda^* \mathbb{Z}_{(-1)}$, the total exterior algebra on the free graded $\mathbb{Z}$-module on a single generator of degree $-1$. The latter generator arises from a generator of $H_1(S^1 \times S^1)/\text{tor}$. The following subalgebra of $A^*(Y_M)$ will be useful later:

$$A^*(M) = A^*(M) \otimes \bigotimes_{i \in \Lambda} \Lambda_i$$

with respect to the aforementioned factorization of $A^*(Y_M)$.

It is shown in paper I of [KLT] that one may define a variant of (filtered) $ECH$ on $Y_M$ with the special stable Hamiltonian structure $(a, w)$. This $ECH$ comes in four flavors, and is denoted as a whole as $ech^\phi$. Like the Heegaard Floer homology,
the superscript ◦ stands for −, ∞, +, ∧, and they fit into fundamental long exact sequences in exact analogy with the Heegaard Floer homology. They also come equipped with $A^\dagger(M)$-module structures. The origin of the four flavors and the $U$-action are in complete parallel to those in $HF$.

By taking the cylinders $H_i$ in $Y_M$ to be long and thin, the decomposition $Y_M$ as a union of $M_\delta$ ($M$ with small balls near critical points removed) and all the $H_i$’s, a gluing argument (cf. papers II and III of [KLT]) allows one to compute $ech^\circ$ by combining arguments from prior work of Lipschitz and Taubes: In [Lip], Lipshitz reinterpreted the Heegaard Floer homology as a certain variant of ECH (with Lagrangian boundary condition) on $I \times \Sigma$, where $\Sigma$ is the Heegaard surface, and the Lagrangian boundary condition given by curves in the Heegaard diagram. On the other hand, Taubes’s work in [Tz] and its sequels give explicit description of holomorphic curves in the symplectization of $S^1 \times S^2$ equipped with the contact structure previously mentioned. The result of this computation is summarized as follows:

Let $\hat{V} = \mathbb{Z}[y]$ with an odd generator $y$ of degree 1. Regard this as a free $\wedge^\bullet \mathbb{Z}_{(-1)}$-module with the degree $-1$ generator of the graded algebra $\wedge^\bullet \mathbb{Z}_{(-1)}$ acting as $\partial_y$.

5.1 Theorem There exists a system of graded $\hat{A}^\dagger(M)$-module isomorphisms from $ech^\circ$ to $HF^0(M,s) \otimes_\mathbb{Z} \hat{V} \otimes G$. These isomorphisms are natural with respect to the fundamental sequences of $HF^\circ$.

This is a direct consequence of Theorem 1.1 in paper III of [KLT], and constitutes the first system of isomorphisms for the proof of Theorem 2.1.

The second system of isomorphisms is the stable Hamiltonian analog of Theorems 2.2 and 2.5 mentioned in Section 1, and is summarized in the next theorem. There, $H_{SW}^0(Y)$ denotes a filtered version of large perturbation Seiberg-Witten Floer homology on the special stable Hamiltonian 3-manifold $Y$.

5.2 Theorem (Theorem 3.4, paper I of [KLT]) There exists a system of isomorphisms from $ech^\circ$ to $H_{SW}^0(Y)$ as $\hat{A}^\dagger(Y)$-modules that is natural with respect to the fundamental exact sequences on both sides.

See paper IV of [KLT] for the definition of $H_{SW}^0(Y)$ as well as a complete proof.
The third isomorphism theorem used for the proof of Theorem 2.1 relates the filtered large perturbation Seiberg-Witten-Floer homologies $H^\circ_{SW}(Y)$ with the balanced Seiberg-Witten-Floer homologies of $M$. This is summarized in the next theorem. The notion $HMT^\circ$ below refers to a filtered version of large-perturbation Seiberg-Witten-Floer homology on $\hat{M} \cong M \# \gamma_0(S^1 \times S^2)$, originally introduced in [L].

5.3 Theorem (Theorem 1.1 of paper V of [KLT])  
(1) There exists a system of isomorphisms of $\hat{A}_\beta(M)$-modules
\[
H^\circ_{SW}(Y) \cong HMT^\circ \otimes \hat{V}^\otimes G, \quad \circ = -, \infty, +, \wedge,
\]
that preserves the relative gradings and is natural with respect to the fundamental long exact sequences on both sides.

(2) There exists a system of canonical isomorphisms of $\hat{A}_\beta(M)$-modules from
\[
HMT^\circ, \circ = -, \infty, +, \wedge \text{ respectively to } H \hat{M}(M, s, c\beta), \circ = \wedge, -, \vee, \sim,
\]
that preserves the relative gradings and is natural with respect to the fundamental long exact sequences on both sides.

An ingredient of the proof of part (2) above involves some homological algebra related to the so-called “Koszul duality” in $S^1$-equivariant homology theories. This mechanism converts the algebraic structures of $HF^\circ$ and $HMT^\circ$, resulting from filtration, to the algebraic structures on $H \hat{M}$, resulting from $S^1$-equivariance.

A central part of the proof consists of certain filtered versions of connected sum formula for perturbed Seiberg-Witten-Floer homologies. The filtered cobordism formula makes use of cobordism maps between Floer homologies of two stable Hamiltonian 3-manifolds $Y_-, Y_+$ induced from a cobordism $W$ corresponding to attaching a 4-dimensional 3-handle along a separating 2-sphere in $Y_-$ and their “time-reversal” $\bar{W}$ (corresponding to attaching 4-dimensional 1-handles along two points lying on different connected components of the $Y_+$). The filtration on the perturbed Seiberg-Witten Floer complex of $Y_\pm$ is induced by a particular Reeb orbit $\gamma_\pm$ of the stable Hamiltonian structure. For example, for $Y_\pm = Y_M$, $\gamma_\pm$ is the curve $\gamma_0$ through $\mathcal{H}_0$ described before. It is essential to show that the aforementioned cobordism maps preserve the filtration, and this constitutes the technical core and occupies the bulk of paper V of [KLT]. (In fact, a special case
is required to show that $H_{SW}^0$ and $HM^0$ are well-defined filtered Floer homologies, and this occupies a substantial part of paper IV of [KLT]. To this end, a particular holomorphic cylinder $\mathbb{R} \times S^1$ in $W$ ending in $\gamma_-$ and $\gamma_+$ is introduced, and the key is a positivity result of certain curvature integral over this cylinder. Cf. Proposition 3.4 in paper V of [KLT].

It is important to note that the aforementioned positivity result applies only to very special cobordisms. For example, $W$ must contains an open set diffeomorphic to $\mathbb{R} \times S^2$. A set of very stringent conditions on the asymptotic behavior of the geometry of $W$ is also required. Cf. Sections 3.2-3.3 of paper V of [KLT]. As a result, a rather lengthy portion of the latter article is spent on constructing cobordisms satisfying these conditions. (Section 9 of [KLT], paper V). These special cobordisms will be useful for comparison the 4-dimensional aspects of the Seiberg-Witten and Heegaard Floer theories. More about this will be said in the ensuing sections.

It is highly desirable to generalize the above positivity result to more general cobordisms. For general 4-dimensional cobordisms equipped with a nontrivial self-dual harmonic form, this is carried out in [L2].

6 4-manifolds and TQFT

According to Atiyah [A], 3- and 4-manifold gauge invariants should fit into a certain “Topological Quantum Field Theory” (TQFT), which assigns (Floer) homology groups to closed 3-manifolds, and for 4-dimensional cobordisms, homomorphisms between the Floer groups of the 3-manifolds at the beginning and the end of the cobordism. Aside from some minor glitches (to be explained later), the TQFT structures on both Heegaard Floer theory and Seiberg-Witten theory have been fairly well developed. In contrast, due to technical difficulties, there is little progress on constructing TQFT for ECH.

6.1 Question Given a symplectic cobordism $(W, \omega)$ between two 3-manifolds with contact forms $(Y_0, \alpha_0)$, $(Y_1, \alpha_1)$, are there appropriate ECH maps associated to $(W, \omega)$ (defined by counting $p$-holomorphic curves in $(W, \omega)$)? Show that these are equivalent to the Seiberg-Witten cobordism map $\hat{HM}(W)$.

Regarding the second question, there are partial results towards the $HM \to ECH$
direction of the proposed equivalence for exact symplectic cobordisms in [HT2].

A parallel question may be asked for PFH: Consider the category \(
\mathcal{C}
\) whose objects are pairs \((Y, \theta)\), where \(\theta\) is a nowhere vanishing harmonic 1-form on the 3-manifold \(Y\), and whose morphisms between \((Y_0, \theta_0), (Y_1, \theta_1)\) are pairs \((W, w)\) consisting of a 4-dimensional cobordism \(W\) and a nowhere vanishing harmonic 2-form \(w\) on \(W\) satisfying the following conditions: \((W, w)\) are asymptotic to pairs \((Y_0, w_0), (Y_1, w_1)\) in the ends, where \(w_0 = *\theta_0, w_1 = *\theta_1\) are nowhere vanishing harmonic 2-forms on the 3-manifolds \(Y_0, Y_1\) respectively. By adapting the works by Donaldson and Gompf relating Lefschetz fibrations and symplectic structures (cf. e.g. [D]), it should not be difficult to demonstrate that this category is equivalent to the category called “FCOB” in [U].

**6.2 Question** Construct a TQFT from the category \(\mathcal{C}\) that associates to each object \((Y, \theta)\) its PFH, and to each morphism \((W, w)\) a map between the PFH of the ends of the cobordism. Show that this TQFT is isomorphic to a restriction of the (large perturbation) \(HM\) TQFT.

An application of the convergence theorem in [L] provides the \(HM \rightarrow PFH\) part of the solution to the second question above. As for the first question, an analog of the desired \(PFH\) TQFT is constructed by Usher in [U]; in fact, they are expected to be equivalent.

We now provide a little more details about the 4-manifold invariants in \(HM\) and \(HF\) theories. Let \(Y_0, Y_1\) be nonempty closed, connected, oriented 3-manifolds, and let \(W\) be a connected oriented cobordism from the former to the latter. Fix a spin-c structure \(s\) on \(W\); let \(t_0 := s|_{Y_0}, t_1 := s|_{Y_1}\) respectively. In [OS2], Ozsvath-Szabo defined maps \(F^\circ(W, s) : HF^\circ(Y_0, t_0) \rightarrow HF^\circ(Y_1, t_1)\) for \(\circ = -, \infty, +, \land\) along the following lines: First, decompose \(W\) into a sequence of elementary cobordisms, each corresponding to either a 1-, 2-, or 3-handle attachment. We call these respectively the index 1, 2, or 3 elementary cobordisms below. Explicit formulae are given for cobordism maps associated to each type of elementary cobordisms. The cobordism map \(F^\circ(W)\) is then defined as the composition of cobordism maps associated with the each step of the aforementioned handlebody decomposition of \(W\).

Let \((X, s)\) be a closed connected spin-c 4-manifold with \(b_2^+ > 1\). Ozsvath-Szabo also introduced an invariant \(\Phi_{X, s}\) taking values in \(\mathbb{Z}/\pm\) from a mixture
of the aforementioned cobordism maps in different flavors: Take out two balls in $X$ and view the latter as a cobordism $(W_X, s_W)$ from $S^3$ to $S^3$. It is shown (cf. [OS2] Lemma 8.2) that $F^\infty(W, s) = 0$ for any connected oriented cobordism $(W, s)$ between connected oriented closed 3-manifolds $(Y_0, t_0), (Y_1, t_1)$, if $b_2^+(W) > 0$. Thus, in this case the map $F^-(W, s)$ has a lift, denoted $F^{\text{mix}}(W, s) : HF^-(Y_0, t_0) \to HF^+(Y_1, t_1)$. Moreover, when $b_2^+(W) > 1$, this lift is canonical. Noting that both the Heegaard Floer homologies $HF^-(S^3)$ and $HF^+(S^3)$ are isomorphic to $\mathbb{Z}$, denote a generator of the former by $\Theta_+$, and a generator of the latter by $\Theta_-$. Set $\Phi_{X,s}$ to be the coefficient of $\Theta_+$ in $F(W_X, s_W)$.

The 4-manifold story on the Seiberg-Witten side is completely parallel. For each flavor $\odot = \wedge, -, \vee, \sim$, [KM] defines a homomorphism

$$HM(W, s) : \hat{HM}^\bullet(Y_0, s_0) \to \hat{HM}^\bullet(Y_1, s_1),$$

and for $W$ with $b_2^+ > 1$, there is also a mixed invariant $\hat{HM}(W, s)$. Let $1, \tilde{1}$ respectively denote the standard generators of $\hat{HM}^\bullet(S^3) \simeq \mathbb{Z}$ and $\hat{HM}^\bullet(S^3) \simeq \mathbb{Z}$, it is shown in Proposition 27.4.1 (cf. also Propositions 3.6.1 and 3.8.2) of [KM] that the coefficient of the mixed invariant, $\langle \hat{HM}(W_X, s_W)1, \tilde{1} \rangle$ is equal to the closed 4-manifold Seiberg-Witten invariant $m(X, s)$ introduced in late 1990’s. The latter is defined by counting solutions of the Seiberg-Witten equations on $X$; cf. also [KM] Definitions 27.1.6-1.7.

With these invariants defined, Corollary 23.1.7 in [KM] asserts that the Seiberg-Witten theory is a weak version of TQFT. It is not a TQFT in the strictest sense, as instead of the category of all oriented 3-manifolds and 4-dimensional cobordisms, the Seiberg-Witten functor is defined on the smaller category, denoted by COB in [KM], whose objects consist of nonempty, connected oriented 3-manifolds. In this TQFT the Floer homologies come in four flavors, and duality takes Floer homologies in one flavor to that of another. As a consequence, the additional condition $b_2^+ > 1$ is required to recover the closed 4-manifold invariant from the Seiberg-Witten TQFT. The Heegaard Floer theory has a parallel (weak) TQFT structure, established recently in [JT]. It is natural to expect:

**6.3 Conjecture** The isomorphisms between $HF$ and $HM$ in Theorem 2.1 are natural with respect to the TQFT structures on both sides, and that their closed 4-manifold invariants agree.
If established, this would imply the equivalence of their respective contact invariants, and that the \( HF = HM \) isomorphism is natural with respect to the surgery exact sequences on both sides.

Unlike Heegaard Floer theory, the sign ambiguity in the Seiberg-Witten 4-manifold invariants \( \hat{H}M(W, s) \) and \( m(X, s) \) can be eliminated by fixing “homological orientations”. This leads one to ask:

**6.4 Question** Can the Heegaard Floer invariants \( F^0_W \) and \( \Phi_X \) be refined to obtain \( \mathbb{Z} \)-valued invariants for fixed homological orientations?

There is brief discussion in [KM] generalizing the Seiberg-Witten 4-manifold invariants above to non-exact perturbations and local coefficients. It would be interesting to explore the corresponding “perturbed Heegaard Floer 4-manifold invariants” extending the “perturbed Heegaard Floer homology” mentioned in Section 2 above to a TQFT.

Meanwhile, one may try to remove the connectedness assumption in the definition of cobordism maps in either theory:

**6.5 Question** What would be an appropriate general formulation for invariants of 4-manifold cobordisms between possibly disconnected 3-manifolds, in either Seiberg-Witten or Heegaard Floer theory?

Cobordism maps between possibly disconnected 3-manifolds has been defined in some simple special cases, such as those used for the connected sum formula mentioned in Section 4.

7 Sketching a proof of Conjecture 6.3

In an article under preparation, the author will give a proof of Conjecture 6.3. Some salient features of this proof, especially those pertaining to the prior discussion, are summarized below.

First, in view of the construction of the 4-manifold invariants \( F^0_{W,s} \) and \( \Phi_{X,s} \) from compositions of elementary cobordisms and the composition theorems of their counterparts in Seiberg-Witten theory (cf. [KM]), it suffices to compare the cobordism maps on both sides for elementary cobordisms.
It is verified in [KLT] that the isomorphisms in Theorem 2.1 map the aforementioned generators \( \Theta_- \in HF^- (S^3) \), \( \Theta_+ \in HF^+ (S^3) \) respectively to the generators \( 1 \in \hat{HM} (S^3) \), \( \tilde{1} \in \hat{HM} (S^3) \) modulo signs.

Index 1 and index 3 elementary cobordisms are “time-reversals” of each other; so we consider only one of them. Take an index 1 cobordism \( W \) from \( M \) to \( M' \cong M \# (S^1 \times S^2) \). Choose a Heegaard diagram \((\Sigma, \alpha, \beta)\) for \( M \) and let \( f \) be a self-indexing Morse function on \( M \) associated to this Heegaard diagram. The chosen Heegaard diagram for \( M \) induces one for \( M' \), \((\Sigma', \alpha', \beta') = (\Sigma, \alpha, \beta) \# (E, \alpha_0, \beta_0)\), where \((E, \alpha_0, \beta_0)\) denotes the so-called standard Heegaard diagram of \( S^1 \times S^2 \), with the Heegaard surface \( E \) being a torus, and \( \alpha_0, \beta_0 \) are embedded circles on \( E \) intersecting transversely at two points. Correspondingly, this Heegaard diagram is associated to a self-indexing Morse function \( f' \) on \( M' \) that has a single pair of index 3 and index 0 critical points, and \( G + 1 \) pairs of index 2 and index 1 critical points. Here, the first \( G \) descending cycles from index 2 critical points are the \( G \) mutually disjoint circles \( \alpha \) on the \( \Sigma \)-summand of \( \Sigma' \), and the first \( G \) ascending cycles from the index 1 critical points are the \( G \) mutually disjoint circles in \( \beta \) on the \( \Sigma \)-summand of \( \Sigma' \). The descending and ascending cycles of the last pair of index 2-index 1 critical points lie on the \( E \)-summand of \( \Sigma' \) and are respectively \( \alpha_0 \) and \( \beta_0 \). By expressing the Heegaard diagram of \( M' \) as a connected sum in this way, the chain group \( CF^0(M') \) may be expressed as a tensor product \( CF^0(M) \otimes \hat{V} \), and in [OS2], the map \( F_W: HF^0(M) \to HF^0(M') \) is defined as the map induced by \( CF^0(M) \to CF^0(M) \otimes \hat{V}: \xi \mapsto \xi \otimes y \).

Recall from Section 4 the construction of the auxiliary manifold \( Y \) with a stable Hamiltonian structure from a 3-manifold \( M \) and a self-indexing Morse function \( f \) on it. As this construction will be applied to different pairs of 3-manifolds and Morse functions, to be specific we denote the stable Hamiltonian structure constructed from the pair \((M, f)\) by \( Y(M, f) \). In paper V of [KLT], as intermediate steps for the application of filtered connected sum formula, we also constructed a family of related auxiliary 3-manifolds \( Y_i, i = 0, \ldots, G \), so that \( Y_0 = Y \) and \( Y_G = M \). These will be denoted by \( Y_i(M, f) \). Topologically, \( Y_i \) is diffeomorphic to \( M \) connected summing with \( G + 1 - i \) copies of \( S^1 \times S^2 \). The geometric structures on \( Y'_i \)'s for \( i > 0 \) are not stable Hamiltonian, but like the pair \((a, w)\) of 1- and 2-forms characterizing a stable Hamiltonian structure, they are associated with certain pairs of 1- and 2-forms \((a_i, w_i)\) that may vanish somewhere on \( Y_i \).
Each of these $Y_i$’s contains a circle used to define a filtration on the associated large-perturbation Seiberg-Witten Floer complex. This circle lies away from the zeros of $(a_i, w_i)$, and abusing notation, we denote it by $\gamma_0$ for all $i = 0, \ldots, G$.

Perturb the Morse function $f$ inside a small ball $B^3_q$ to create a pair of cancelling index 1 and index 2 critical points $q$, and denote the resulting Morse function $f_s$. By a simple modification of the construction in Section 9 of paper V of [KLT], we construct an index 1 elementary cobordism $W_i$ between $Y_{i-1}(M, f)$ and $Y_i(M', f')$, with $Y_{-1}(M, f) := Y(M, f_s)$. It is endowed with a geometric structure so that the associated cobordism map preserves filtration. By composing these index 1 elementary cobordisms with the appropriate index 1 cobordisms in the proof of the filtered connected sum formula, and noting that the order of the composition may be permuted, by induction the computation of $\hat{HM}(W)$ may be reduced to the computation of the filtered cobordism map associated to $W_0$. The computation of the latter makes further use of composition results: Compose the index 1 elementary cobordism $W_0$ with an index 3 elementary cobordism $W'$ from the proof of the filtered connected sum formula, with the attaching sphere being $\partial B^3_q$, the right end of this cobordism is a disjoint union of $Y = Y(M, f_s)$ with $(S^1 \times S^2)$, the latter being equipped with the round metric and an exact perturbation. Attach a 2-handle along the core circle in $S^1 \times S^2$ to form a final cobordism which is diffeomorphic to the product cobordism $\mathbb{R} \times Y$ with a 4-ball removed. Evaluating the associated cobordism map on the generator $\hat{1}$ of $\hat{HM}(S^3)$ then yields the identity map on the Seiberg-Witten-Floer homology of $Y$. With this, the cobordism maps associated to $W_0$ may be computed from those of $W'$, which is known from the filtered connected sum formula.

Next, we compare maps associated to index 2 elementary cobordisms. Let $W_K$ now be an elementary cobordisms from $M$ to $M_K$, obtained by attaching a 4-dimensional 2-handle along a framed knot $K \subset M$. In [OS2], the cobordism map $F_{W_K}$ is defined by counting “holomorphic triangles”. Take a Heegaard diagram $(\Sigma, \alpha, \beta)$ adapted to the pair $(M, K)$. This means that $M$ is obtained by gluing two $G$-handlebodies along the sets $\alpha = \{\alpha_1, \ldots, \alpha_G\}$, $\beta = \{\beta_1, \ldots, \beta_G\}$ of disjoint $G$-circles, where $\beta_G = m_K$, the meridian of the knot $K$. Meanwhile, the framing of $K$ is represented by a circle $\gamma_G$ on $\Sigma$. Let $\gamma = \{\gamma_1, \ldots, \gamma_G\}$, where for $i = 1, \ldots, G - 1$, $\gamma_i$ is a circle on $\Sigma$ obtained by perturbing $\beta_i$ so that it intersects $\beta_i$ transversely at 2 points. As described in [OS2], the “Heegaard triple”
$(\Sigma, \alpha, \beta, \gamma)$ defines a 4-dimensional cobordism $X_{\alpha, \beta, \gamma}$ from $M \sqcup \#^{G-1}(S^1 \times S^2)$ to $M_K$. Associated to this are holomorphic triangle maps $F^\circ_{\alpha, \beta, \gamma} : HF^\circ(M) \otimes HF^-(\#^{G-1}(S^1 \times S^2)) \to HF^\circ(M_K)$ (Cf. e.g. [OS2] Section 2.3). The elementary cobordism $W_K$ may be recovered from $X_{\alpha, \beta, \gamma}$ by filling in the boundary component $\#^{G-1}(S^1 \times S^2)$ with the boundary connected sum $\#^{G-1}(S^1 \times B^3)$. The cobordism map $F^\circ_{W_K} : HF^\circ(M) \to HF^\circ(M_K)$ is defined in terms of the holomorphic triangle maps $F^\circ_{\alpha, \beta, \gamma}$ by evaluating the latter on the highest degree generator (unique modulo signs) of $HF^-(\#^{G-1}(S^1 \times S^2)) \simeq \bigotimes^{G-1} \hat{V}$.

Let $\Delta$ denote a triangle. Extending the aforementioned re-interpretation of the Heegaard Floer chain maps, in [Lip] the Heegaard triple maps have an alternative definition as counting invariants of holomorphic curves in the 4-manifold $\Delta \times \Sigma$, with boundary conditions specified by the triple $(\alpha, \beta, \gamma)$. This interpretation is not directly compatible with our proof in [KLT] of $HF = HM$ through intermediate Floer homologies. Instead, we reinterpret $F_{W_K}$ as a counting invariant of holomorphic sections in a 4-dimensional Lefschetz fibration over $\mathbb{R} \times I$ that contains one single singular points. The boundary conditions are specified by $(\alpha, \beta)$, and in this picture, $\gamma_G$ arises as the Dehn twist of $\beta_G$ along the vanishing cycle of the Lefschetz singularity. The relation between this interpretation and Lipshitz’s has antecedent analogs, see e.g. [S].

With this 4-dimensional interpretation of $F^\circ_{W_K}$, we may now modify the cobordism $W_K$ in a straightforward manner to get an auxiliary cobordism $W_Y$ between $Y_M$ and $Y_{M_K}$. This cobordism induces maps between the intermediate Floer homologies $H^0_{SW}$ of $M$ and $M_K$, and Theorems 5.1, 5.2 have direct analogs that compute this map in terms of the Heegaard Floer map $F^\circ_{W_K}$. Meanwhile, an application of the filtered connected sum formula, Theorem 5.3 allows one to relate this map with the Seiberg-Witten cobordism maps $HM(W_K)$.

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