Eigenfunction expansions associated with 1d periodic differential operators of order $2n$

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Abstract

We prove an explicit formula for the spectral expansions in $L^2(\mathbb{R})$ generated by selfadjoint differential operators

$$(-1)^n \frac{d^{2n}}{dx^{2n}} + \sum_{j=0}^{n-1} \frac{d^j}{dx^j} p_j(x) \frac{d^j}{dx^j}, \quad p_j(x + \pi) = p_j(x), \quad x \in \mathbb{R}.$$ 

1 Statement of results

It is well known [1], see also [2], that for every Hill operator

$$H = -\frac{d^2}{dx^2} + q(x), \quad q(x) = q(x + \pi), \quad x \in \mathbb{R} \quad (1.1)$$

with a real-valued potential function $q(x)$ there exists a sequence of real numbers

$$\mu_0 = \mu^-_0 = \mu^+_0 < \mu^-_1 < \mu^+_1 < ... < \mu^-_k < \mu^+_k < ...$$

such that the spectrum of $L$ in the space $L^2(\mathbb{R})$ has the form

$$\sigma(L) = \bigcup_{k=0}^{\infty} [\mu^+_k, \mu^-_{k+1}].$$

In 1950 Titchmarsh proved ([1], see also [2]) that every real-valued function $f \in L^2(\mathbb{R})$ may be represented in the form

$$f(x) = \frac{1}{\pi} \sum_{n=0}^{\infty} (-1)^n \int_{\mu^-_n}^{\mu^+_n} d\mu \, p(\mu) \{ \phi(\pi, \mu) \theta(x, \mu) g(\mu) - \theta'(\pi, \mu) \phi(x, \mu) h(\mu) 
+ \frac{1}{2} (\phi'(\pi, \mu) - \theta(\pi, \mu)) \theta(x, \mu) h(\mu) + \frac{1}{2} (\phi'(\pi, \mu) - \theta(\pi, \mu)) \phi(x, \mu) g(\mu) \},$$

(1.2)
where $\theta(x, \mu)$ and $\phi(x, \mu)$ are solutions of the equation $(H - \mu I) = 0$ satisfying the initial conditions $\theta(0, \mu) = \phi'(0, \mu) = 1$, $\theta'(0, \mu) = \phi(0, \mu) = 0$, with $p(\mu) = (4 - (\theta(\pi, \mu) + \phi'(\pi, \mu))^2)^{-1/2}$ and

$$g(\mu) = \int_{\mathbb{R}} dx \, \theta(x, \mu) f(x), \quad h(\mu) = \int_{\mathbb{R}} dx \, \phi(x, \mu) f(x).$$

The expansion (1.2) determines explicitly the spectral matrix of operator $H$ and, in particular, shows that the multiplicity of its spectrum equals 2.

We consider arbitrary periodic self-adjoint differential operators

$$L = (-1)^n \frac{d^2}{dx^2} + \sum_{j=0}^{n-1} \frac{d^j}{dx^j} p_j(x) \frac{d^j}{dx^j}, \quad p_j(x + \pi) = p_j(x), \quad x \in \mathbb{R}, \quad (1.3)$$

with real-valued functions $p_j(x)$, $j = 0, 1, ..., n - 1$, such that

$$P(L) = \sum_{j=0}^{n-1} \int_0^\pi |p_j(x)| \, dx < \infty. \quad (1.4)$$

Similar to Hill operators, the spectrum of every such operator in the complex space $L^2(\mathbb{R})$ has the band structure, but in contrast to these operators its multiplicity may vary inside one spectral band. Using a general resolvent method due to Kodaira and Spencer, Dunford and Schwartz (cf., [3], Chap. XIII) proved a formula for the spectral matrix of operator (1.3) on an interval of a constant multiplicity of the spectrum.

Our aim is to obtain an expansion formula similar to (1.2) for operator (1.3), to derive from it an explicit formula for the spectral matrix of operator (1.3) and to prove that this matrix determines its coefficients uniquely. Our approach is based on a version of the Fourier transform proposed by Gel’fand [4] for a study of periodic differential operators.

To state our main result, let $\{u_k(x, \mu)\}_{k=1}^{2n}$ be the fundamental system of solutions of equation

$$Ly - \mu y = 0 \quad (1.5)$$

normalized by the initial conditions

$$u_k^{(j)}(0, \mu) = \delta_{k-1,j}, \quad k = 1, ..., 2n; \quad j = 0, ..., 2n - 1, \quad (1.6)$$

and let

$$U(\mu) = ||u_k^{(j-1)}(\pi, \mu)||_{k,j=1}^{2n} \quad (1.7)$$

be the monodromy matrix of $L$. The eigenvalues of $U(\mu)$ are solutions of the characteristic equation

$$\Delta(\mu, \rho) = 0 \quad (1.8)$$

where $\Delta(\mu, \rho) = \det(U(\mu) - \rho I)$ and are called the Floquet multipliers of $L$. If $\{v_1, ..., v_{2n}\}^\perp$ is an eigenvector corresponding to $\rho$, then the solution of (1.5) uniquely determined by the initial conditions

$$y^{(j)}(0) = v_{j+1}, \quad j = 0, ..., 2n - 1,$$
has a “quasi-periodic” property
\[ y(x + k\pi) = \rho^k y(x), \quad x \in \mathbb{R}, \quad k \in \mathbb{Z}, \]
and is called a Floquet solution of (1.5).

The following proposition is well known in the theory of ordinary differential operators with periodic coefficients, cf. [3].

**Theorem 1.1.** The spectrum \( \sigma(L) \) of an operator (1.3) is absolutely continuous and coincides with the set of all those \( \mu \) for which there exists a solution \( \rho \) of (1.8) with \( |\rho| = 1 \).

Let us define the functions
\[
E(x; \mu, \rho) = \begin{vmatrix}
  u_1(x, \mu) & \cdots & u_j(x, \mu) & \cdots & u_{2n}(x, \mu) \\
  u_1(\pi, \mu) - \rho & \cdots & u_j(\pi, \mu) & \cdots & u_{2n}(\pi, \mu) \\
  \cdots & \cdots & \cdots & \cdots & \cdots \\
  u_1^{(j-1)}(\pi, \mu) & \cdots & u_j^{(j-1)}(\pi, \mu) - \rho & \cdots & u_{2n}^{(j-1)}(\pi, \mu) \\
  \cdots & \cdots & \cdots & \cdots & \cdots \\
  u_1^{(2n-2)}(\pi, \mu) & \cdots & u_j^{(2n-2)}(\pi, \mu) & \cdots & u_{2n}^{(2n-2)}(\pi, \mu)
\end{vmatrix}
\]
and
\[
p(\mu, \rho) = |2\pi E(0; \mu, \rho^{-1})\Delta'(\mu, \rho)|^{-1}.
\]

We will see later on that with a proper choice of \( t \) and \( \mu \) the functions \( E(x; \mu, e^{\mu t}) \) are the Floquet solutions participating in the spectral expansion associated with \( L \). As to the function (1.10), it supplies the normalizing factors in such expansion.

**Theorem 1.2.** If \( \omega_k, k = 1, \ldots, 2n, \) are all values of \( \sqrt[2n]{(-1)^n} \) then for every \( k = 1, \ldots, 2n \) there exists a solution \( \rho_k(\mu) \) of (1.8), continuous on the real line and satisfying the asymptotic relation
\[
|\rho_k(\mu)| = e^{Re\omega_k \lambda n}(1 + o(1)), \quad \mu = \lambda^{2n}, \quad \lambda \to +\infty.
\]
Moreover, solutions \( \rho_k(\mu) \) are pair-wise distinct and analytic at points of \( \mathbb{R} \), except maybe points of a discrete set grouped in pairs asymptotically close to the set
\[
\mathcal{N} = \bigcup_{j=1}^{2n} \mathcal{N}_j, \quad \mathcal{N}_j = \{(-1)^j m^{2n} (\text{Im} \omega_j)^{-2n}\}_{m=1}^{\infty}
\]
where some of them coincide and their analyticity may fail.

If we set
\[
\sigma_k(L) = \{ \mu \in \mathbb{R} : |\rho_k(\mu)| = 1 \}, \quad k = 1, \ldots, 2n,
\]
then we obtain
\[
\sigma(L) = \bigcup_{k=1}^{2n} \sigma_k(L).
\]
Denote by $H_{2n}^2(L)$ the Hilbert space of complex $2n$-vector functions $\Phi = \{\phi_k(\mu)\}_{k=1}^{2n}$ on the sets $\{\sigma_k(L)\}_{k=1}^{2n}$ with the scalar product

$$(\Phi, \Psi) = \sum_{k=1}^{2n} \int_{\sigma_k(L)} d\mu \chi_k(\mu) p(\mu, \rho_k(\mu)) \phi_k(\mu) \overline{\psi_k(\mu)},$$

where $\chi_k(\mu)$ is the indicator function of the set $\sigma_k(L)$.

For the simplest operator

$$(-1)^n \frac{d^{2n}}{dx^{2n}}$$

we have $\rho_k(\mu) = \exp(\omega_k \mu^{1/2n} \pi)$ and the sets $\sigma_k(L)$ are reduced to the point 0 for all $k$'s with $\omega_k \neq \pm i$. Therefore a situation where some intervals $\sigma_k(L)$ degenerate to a point cannot be ruled out. As a result, some functions $\phi_k(\mu)$ may be trivial for all $\Phi \in H_{2n}^2(L)$.

**Theorem 1.3.** The relations

$$\Phi(\mu, \rho_k(\mu); f) = \int_{\mathbb{R}} dy \ f(y) \ E(y; \mu, (\rho_k(\mu))^{-1}), \quad k = 1, \ldots, 2n, \quad (1.15)$$

and

$$f(x) = \int_{\sigma(L)} d\mu \ \sum_{k=1}^{2n} \chi_k(\mu) p(\mu, \rho_k(\mu)) \phi_k(\mu) E(x; \mu, \rho_k(\mu)) \quad (1.16)$$

define one-to-one mapping of $L^2(\mathbb{R})$ onto $H_{2n}^2(L)$ and its inverse conjugating operator $L$ on the former space with the scalar operator $\mu I$ on the latter space.

The integrals (1.15) and (1.16) converge in the norms of the corresponding spaces and for every function $f \in L^2(\mathbb{R})$ the Parseval identity

$$\int_{\mathbb{R}} dx |f(x)|^2 = \int_{\sigma(L)} d\mu \ \sum_{k=1}^{2n} \chi_k(\mu) p(\mu, \rho_k(\mu)) |\Phi(\mu, \rho_k(\mu); f)|^2 \quad (1.17)$$

holds.

It is easy to see that in the case of Hill operators Theorem 1.3 states that for every function $f \in L^2(\mathbb{R})$ the representation

$$f(x) = \frac{1}{4\pi} \int_{\sigma(L)} d\mu \ \frac{|\phi(\pi, \mu)|}{\sqrt{1 - u_+^{(\mu)}|^2}} \{Y_+(x, \mu) F_- (\mu; f) + Y_-(x, \mu) F_+ (\mu; f)\}$$

is valid where

$$Y_{\pm}(x, \mu) = \theta(x, \mu) - \frac{u_{\pm}(\mu) \pm i \sqrt{1 - u_+^{(\mu)}|^2}}{\phi(\pi, \mu)} \phi(x, \mu)$$
are the Floquet solutions,
\[ F_{\pm}(\mu; f) = \int dy f(y) Y_{\pm}(y, \mu), \]
and
\[ u_{\pm}(\mu) = \frac{\theta(\pi, \mu) \pm \phi'(\pi, \mu)}{2}. \]
This is a complex version of expansion (1.2) obtained first in [11] and later in [12].

2 Solutions \( \rho_k(\mu) \) and Riemann surface \( \mathcal{R}(L) \)

Let us start from the proof of Theorem 1.2. Following [5] we enumarate the numbers \( \omega_k = \sqrt[2n]{-1} \) in such an order that
\[ 1 = \omega_1 > \Re \omega_2 = \Re \omega_3 > \ldots > \Re \omega_{2k} = \Re \omega_{2k+1} > \ldots > \omega_{2n} = -1, \quad \Im \omega_{2k} \geq 0, \tag{2.1} \]
if \( n \) is even, and
\[ \Re \omega_1 = \Re \omega_2 > \ldots > \Re \omega_{2k+1} = \Re \omega_{2k} > \ldots > \Re \omega_{2n-1} = \Re \omega_{2n}, \quad \Im \omega_{2k} \leq 0, \tag{2.2} \]
if \( n \) is odd. With such enumeration we have \( \omega_n = i \) both for odd and even values of \( n \). In what follows we restrict ourselves to the case \( n = 2p, p \in \mathbb{N} \). The alternative case differs from it by non-essential technical details.

Let
\[ T(r) = \{ \lambda : \lambda = z - e^{-i\pi/4n}, \quad |z| \geq r, \quad -\frac{1}{2n} \leq \arg z \leq 0 \}. \]
According to [5] there exists a fundamental matrix \( Y(x, \mu) \) of solutions to (1.5) representable in the form
\[ Y(x, \mu) = D(\lambda)\Omega(x, \lambda)E(\lambda x), \quad \mu = \lambda^{2n}, \quad x \in [0, \pi], \quad \lambda \in T(r), \tag{2.3} \]
with matrices
\[ D(\lambda) = ||\lambda^j \delta_{jk}||_{j,k=1}^{2n}, \quad E(x) = ||e^{\omega_k x} \delta_{jk}||_{j,k=1}^{2n}, \]
\[ \Omega(x, \lambda) = \Omega + \lambda^{-1} \Omega_1(x, \lambda), \quad \Omega = ||\omega_k^{j-1}||_{j,k=1}^{2n}, \]
where
\[ \sup_{x \in [0, \pi], \lambda \in T(r)} ||\Omega_1(x, \lambda)|| < \infty. \]
Since \( U(\pi, \mu) = Y(x, \mu)Y(0, \mu)^{-1} \), we can represent (1.8) in the form
\[ \det((I + o(1))E(\pi \lambda) - \rho I) = 0. \tag{2.4} \]
To investigate the latter equation we define the entire functions

\[ f_k(\lambda) = 1 - e^{(\omega_{k+1} - \omega_k) \lambda \pi}, \quad k = 1, \ldots, 2n - 1, \]

and denote by \( Z_k \) their zero sets. For every \( \delta > 0 \) we denote by \( U_k(\delta) \) the \( \delta \)-neighborhood of \( Z_k \). If

\[
T_k(r, \delta) = \begin{cases} 
T(r) \setminus U_1(\delta), & k = 1, \\
T(r) \setminus (U_k(\delta) \cup U_{k-1}(\delta)), & 1 < k < 2n - 1, \\
T(r) \setminus U_{2n-1}(\delta), & k = 2n,
\end{cases}
\]

then

\[
C^{-1} \delta \leq |f_k(\lambda)| \leq C \delta, \quad \lambda \in \partial U_k(\delta) \cap T_k(r, \delta),
\]

\[
C^{-1} \delta \leq |f_k(\lambda)| \leq C, \quad \lambda \in T_k(r, \delta),
\]

with a constant \( C > 1 \) not depending on \( \delta \). In what follows we fix \( \delta \) such that \( C \delta < 10^{-1} \).

First, we the substitute \( \rho = \sigma e^{\omega_1 \lambda \pi} \) in (2.4), divide the resulting equation by \( \rho = e^{2n \omega_1 \lambda \pi} \) and obtain the equivalent equation

\[
\sigma^{2n-2} (\sigma - 1) (\sigma - 1 - f_2(\lambda)) = Q_1(\sigma, \lambda),
\]

where \( Q_1(\sigma, \lambda) \) is a polynomial in \( \sigma \) with coefficients analytic in \( T_1(r, \delta) \) and vanishing as \( |\lambda| \to +\infty \). For sufficiently small \( \epsilon > 0 \) we find large \( r \) such that the function

\[
|Q_1(\sigma, \lambda) \sigma^{-(2n-2)} (\sigma - 1 - f_2(\lambda))^{-1}| \leq \frac{\epsilon}{2}, \quad \lambda \in T_1(r, \delta), \quad |\sigma - 1| \leq \epsilon,
\]

and using the Rouchet Theorem conclude that there exists the unique solution \( \sigma_1(\lambda) \) of (2.6) analytic in \( T_1(r, \delta) \) and such that \( |\sigma_1(\lambda) - 1| \leq \epsilon \). Since \((r, \infty) \in T_1(r, \delta)\), Equation (2.6) implies that

\[
\rho_1(\mu) = e^{\omega_1 \lambda \pi} \sigma_1(\lambda), \quad \mu = \lambda^{2n}, \quad \lambda \in T_1(r, \delta),
\]

is a solution of (1.8) satisfying (1.11) with \( k = 1 \).

In the same way the substitution \( \rho = \sigma e^{2n \omega_1 \lambda \pi} \) leads us to the equation

\[
(\sigma - 1) (\sigma - 1 - \sigma f_{2n-1}(\lambda)) = Q_{2n}(\sigma, \lambda), \quad \lambda \in T_{2n}(r, \delta),
\]

with its unique solution \( \sigma_{2n}(\lambda) \) analytic in \( T_{2n}(r, \delta) \) and such that

\[
\rho_{2n}(\mu) = e^{2n \omega_1 \lambda \pi} \sigma_{2n}(\lambda), \quad \mu = \lambda^{2n}, \quad \lambda \in T_{2n}(r, \delta),
\]

is a solution of (2.4) satisfying (1.11) with \( k = 2n \).

If now \( 1 < k = 2p < 2n \) and \( \rho = \sigma e^{\omega_k \lambda \pi} \) then Equation (2.4) takes on the form

\[
\sigma^{2n-k-1} (\sigma - 1 - \sigma f_{k-1}(\lambda)) (\sigma - 1) (\sigma - 1 + f_k(\lambda)) = Q_k(\sigma, \lambda), \quad \lambda \in T_k(r, \delta),
\]

(2.8)
where $Q_k(\sigma, \lambda)$ is a polynomial in $\sigma$ with coefficients analytic in $T_k(r, \delta)$ and vanishing as $|\lambda| \to +\infty$. As before we find that Equation (2.8) has the unique solution $\sigma_k(\lambda)$ analytic in $T_k(r, \delta)$ and satisfying the relation

$$\lim_{\lambda \in T_k(r, \delta); |\lambda| \to \infty} \sigma_k(\lambda) = 1 \quad (2.9)$$

but now, contrary to the case $k = 1$ and $k = 2n$, the set $T_k(r, \delta)$ does not contain a part of the ray $(r, +\infty)$ belonging to the set $U_k(\delta)$.

To define $\sigma_k(\lambda)$ inside the exceptional set $U_k(\delta)$ we assume, to be definite, that $\alpha \in N_k \cap T(r)$, define $D_\alpha = \{\lambda : |\lambda - \alpha| \leq \delta\}$ and represent (2.8) in the form

$$(\sigma - 1)(\sigma - 1 + f_k(\lambda)) = q_k(\sigma, \lambda), \quad \lambda \in D_\alpha, \quad (2.10)$$

where

$$q_k(\sigma, \lambda) = Q_k(\sigma, \lambda)\sigma^{-2n+k+1}(\sigma - 1 - \sigma f_k(\lambda))^{-1}.$$ 

If necessary we increase $r$ to satisfy the estimate

$$|q_k(\sigma, \lambda)| \leq 2C\delta^2, \quad |\sigma - 1| \leq C^2, \quad \lambda \in D_\alpha,$$

where $C$ is the constant from (2.5). According to the Rouchet Theorem for every $\lambda \in D_\alpha$ there exist two solutions of (2.10) satisfying $|\sigma - 1| \leq 2C\delta$. If these solutions coincide at some point $\lambda \in D_\alpha$ then $f_k^2(\lambda) = 4q_k(\sigma, \lambda)$. Once again we use the Rouchet Theorem and find that there are at most two such points $\lambda$. The analyticity of $q_k(\sigma, \lambda)$ in the domain $\{\lambda, \sigma : |\sigma - 1| \leq 2C\delta, \lambda \in D_\alpha\}$ implies that there exists two-valued analytic solution $\tilde{\sigma}_k(\lambda)$ of (2.10) in $D_\alpha$ satisfying the estimate

$$|\tilde{\sigma}_k(\lambda) - 1| \leq 2C\delta, \quad \lambda \in D_\alpha. \quad (2.11)$$

The function $\sigma_k(\lambda)$, $\lambda \in T_k(r, \delta)$, is a single-valued solution of Eq. (2.8) which is equivalent to Eq. (2.10) in $D_\alpha$ and because of (2.9) it coincides with a branch of $\tilde{\sigma}_k(\lambda)$. In other words, $\sigma_k(\lambda)$ is extended as a two-valued analytic function inside $D_\alpha$ with at most two ramification points.

Furthermore, with the same value of even $k = 2p$ we substitute $\rho = \sigma e^{\omega_{k+1}\lambda\pi}$ in (2.4) and obtain an equation of the same type as (2.8). As before we prove that there exists its unique solution $\sigma_{k+1}(\lambda)$ analytic in $T_{k+1}(r, \delta)$ and satisfying

$$\lim_{\lambda \in T_{k+1}(r, \delta); |\lambda| \to \infty} \sigma_{k+1}(\lambda) = 1. \quad (2.12)$$

Similar to $\tilde{\sigma}_k(\lambda)$ its two-valued analytic extension $\tilde{\sigma}_{k+1}(\lambda)$ inside exceptional discs $D_\alpha$ satisfies condition

$$|\tilde{\sigma}_{k+1}(\lambda) - 1| \leq 2C\delta, \quad \lambda \in D_\alpha. \quad (2.13)$$

Let us now consider two-valued solutions

$$\tilde{\rho}_k(\mu) = \tilde{\sigma}_k(\lambda)e^{\omega_k\lambda\pi}, \quad \tilde{\rho}_{k+1}(\mu) = \tilde{\sigma}_{k+1}(\lambda)e^{\omega_{k+1}\lambda\pi}, \quad \mu = \lambda^{2n}, \quad \lambda \in D_\alpha,$$
of (1.16). The coefficients of Equation (1.8) are real on the real line and therefore the function
\[ \rho_k^*(\mu) = \rho_k(\mu^*), \quad \mu = \lambda^{2^n}, \quad \lambda \in \partial D_\alpha, \]
is its solution as well. Since \( \omega_k = \omega_{k+1} \), we have
\[ \rho_k^*(\mu) = \sigma_k^*(\lambda)e^{\omega_k+1+\lambda\pi} = \rho_{k+1}(\mu), \quad \mu = \lambda^{2^n}, \quad \lambda \in \partial D_\alpha. \quad (2.14) \]
On the other hand,
\[ \rho_k^*(\mu) = \theta_k(\lambda)e^{\omega_k\lambda\pi}, \quad \theta_k(\lambda) = \sigma_k^*(\lambda)e^{(\omega_k+1-\omega_k)\lambda\pi}, \quad \mu = \lambda^{2^n}, \quad \lambda \in \partial D_\alpha, \]
and the estimate
\[ |\theta_k(\lambda) - 1| \leq |\sigma_k(\lambda)f_k(\lambda)| + |\sigma_k(\lambda) - 1| \leq 2C\delta, \quad \lambda \in \partial D_\alpha, \]
shows that \( \rho_k^*(\mu) \) is a single-valued analytic solution of (2.10) in \( \partial D_\alpha \). If \( \rho_k(\mu) = \rho_k^*(\mu) \) for \( \mu = \lambda^{2^n}, \lambda \in \partial D_\alpha \) then \( |f_k(\lambda)| = o(1) \) which is impossible for large \( \lambda \). Therefore for \( \mu = \lambda^{2^n}, \lambda \in \partial D_\alpha \), the functions \( \rho_k(\mu) \) and \( \rho_k^*(\mu) \) are different branches of the two-valued function \( \hat{\rho}_k(\mu) \).
To estimate the latter functions inside \( D_\alpha \) we set
\[ A_k(\lambda) = \frac{\hat{\rho}_k(\lambda^{2^n}) + \hat{\rho}_k^*(\lambda^{2^n})}{2}e^{-\lambda\pi\Re \omega_k}, \quad \lambda \in D_\alpha, \]
and
\[ B_k(\lambda) = \hat{\rho}_k(\lambda^{2^n})\hat{\rho}_k^*(\lambda^{2^n})e^{-2\lambda\pi\Re \omega_k}, \quad \lambda \in D_\alpha. \]
According to their definition both functions are single-valued and analytic inside \( D_\alpha \) and, as it follows from (2.9), (2.12) and (2.14), satisfy conditions
\[ A_k(\lambda) = \cos(\lambda\pi\Im \omega_k) + o(1), \quad B_k(\lambda) = 1 + o(1), \quad \lambda \in \partial D_\alpha. \]
According to the Maximum Principle the same representations are valid in the domain \( D_\alpha \). It means that \( \hat{\rho}_k(\mu)e^{-\lambda\pi\Re \omega_k} \) and \( \hat{\rho}_{k+1}(\mu)e^{-\lambda\pi\Re \omega_{k+1}} \) are solutions of the equation \( w^2 - 2(\cos(\lambda\pi\Im \omega_k) + o(1))w + 1 + o(1) = 0, \quad \lambda \in D_\alpha \), and therefore \( \hat{\rho}_k(\mu) = e^{\omega_k\lambda^\sigma}(1 + o(1)), \hat{\rho}_{k+1}(\mu) = e^{\omega_{k+1}\lambda^\sigma}(1 + o(1)), \quad \lambda \in D_\alpha \). To complete the proof of Theorem 1.2 we fix a point \( \mu = \lambda^{2^n} \) with \( \lambda \in T_k(r, \delta) \), a system \( \rho_k(\mu) = e^{\omega_k\lambda^\sigma}\sigma_k(\lambda), k = 1, \ldots, 2n, \) of solutions of (1.8) and extend all of them to \( \mathbb{R} \) as single-valued continuous functions, pair-wise distinct outside the discriminant set
\[ \mathcal{Z}(L) = \{ \mu : \Delta(\mu, \rho) = \Delta'_\rho(\mu, \rho) = 0 \}. \]
The latter is a zero set of the resultant \( R[\Delta, \Delta'_\rho] \), see [6]. Since there exist points \( \mu \) at which Equation (1.8) has \( 2n \) distinct roots, the resultant is a non-trivial function and the set \( \mathcal{Z}(L) \) has no finite accumulation points. As a result, the extended solutions are analytic outside \( \mathcal{Z}(L) \) and, according to the above estimates of \( \hat{\rho}_k(\mu) \) and \( \hat{\rho}_k^*(\mu) \), satisfy (1.11), which completes the proof of Theorem 1.2.
The entries of the monodromy matrix are entire functions of order $1/2n$, and Equation (1.8) defines the $2n$-sheeted Riemann surface

\[ \mathcal{R}(L) = \{ (\mu, \rho) : \Delta(\mu, \rho) = 0 \} \]

with the analytic function $\rho(\mu)$ on it.

**Lemma 2.1.** The Riemann surface $\mathcal{R}(L)$ is simply-connected.

**Proof.** Let \( \{ \rho_k(\mu) \}_{k=1}^{2n} \) be the system of unique solutions of (1.8) obtained in the proof of Theorem 1.2. As we have seen, these solutions are single-valued and analytic in the domains

\[ \Pi_k^-(r, \delta) = \{ \mu : \mu = \lambda^{2n}, \lambda \in T_k(r, \delta) \} \cap \{ \mu : \Im \mu \leq 0 \}, \]

and satisfy the asymptotic relations

\[ \rho_k(\mu) = e^{\omega_k \lambda^2 (1 + o(1)), \mu = \lambda^{2n}, |\lambda| \to \infty, \lambda \in T_k(r, \delta)}. \]

We will use their analytic extensions to describe the surface $\mathcal{R}(L)$.

First, we set

\[ \Pi_k^+(r, \delta) = \{ \mu : \mu = \tau, \tau \in \Pi_k^- (r, \delta) \} \]

and note that the function \( \rho_k^*(\mu) = \overline{\rho_k(\overline{\mu})} \) are single-valued and analytic solutions of (1.8) in the domain \( \Pi_k^+(r, \delta) \).

Furthermore, let

\[ c_k^+ = \partial\Pi_k^-(r, \delta) \cap \{ \mu \in \mathbb{R} : \mu \geq 0 \}, c_k^- = \partial\Pi_k^+(r, \delta) \cap \{ \mu \in \mathbb{R} : \mu \leq 0 \}. \]

Since the number $\omega_1$ is real, the solutions $\rho_1(\mu)$ and $\rho_k^*(\mu)$ have the same asymptotic behavior in $c_1^+$ and hence they coincide in $c_1^+$. By the same reason the solutions $\rho_{2n}(\mu)$ and $\rho_{2n}^*(\mu)$ coincide in $c_{2n}^+$.

For $k = 2p, p = 1, ..., n - 1$, the function $\rho_k(\mu)$ coincides with $\rho_{k+1}^*(\mu)$ on the set $c_k^+ = c_{k+1}^+$ and with $\rho_{k-1}^*(\mu)$ on the set $c_k^- = c_{k-1}^-$, while $\rho_{k+1}(\mu)$ coincides with $\rho_k^*(\mu)$ on the set $c_{k+1}^- = c_{k+2}^+$ and with $\rho_{k+2}^*(\mu)$ on the set $c_{k+2}^- = c_{k+3}^+$. If we glue together the pairs of the corresponding sets belonging to $\Pi_k^- (r, \delta)$ and $\Pi_k^+(r, \delta)$, then we obtain a Riemann surface $\mathcal{R}(r, \delta)$ with the single-valued analytic function $\rho(\mu)$ on it. The surface $\mathcal{R}(r, \delta)$ is the same for all operators of the form (1.3) and may be obtained from the surface $\mathcal{R}_0 = \{ (\mu, \lambda) : \mu = \lambda^{2n} \}$ corresponding to the simplest operator (1.14) after removing from it the disc \( \{ \mu : |\mu| \leq r \} \) and small neighborhoods of points projecting into the set $Z$ from (1.12). It is evident that the surface $\mathcal{R}_0$ is simply-connected. The surface $\mathcal{R}(L)$ results from the analytic extension of all functions $\rho_k(\mu)$ inside all exceptional sets.

To prove Lemma 2.1 let us assume that $(\mu_0, \rho(\mu_0))$ is an arbitrary non-ramified point of $\mathcal{R}(L)$ and $(\nu_0, \rho(\nu_0))$ is a point of the surface $\mathcal{R}(r, \delta)$. Denote by $l$ a simple smooth curve in the complex plain connecting $\mu_0$ with $\nu_0$ and
not containing points of the discriminant set $Z(L)$. According to the Monodromy Theorem [7] there exists the unique analytic continuation of $\rho(\mu)$ from a neighborhood of $\mu_0$ along $l$. The regular element of this continuation at a neighborhood of $\nu_0$ is a locally single-valued analytic solution of (1.8) and since the system $\{\rho_k(\mu)\}_{k=1}^{2n}$ contains all local solutions analytic at $\nu_0$, this element coincides with some function $\rho_k(\mu)$. It means that the lifting of $l$ to the surface $\mathcal{R}(L)$ connects the points $(\mu_0, \rho(\mu_0))$ and $(\nu_0, \rho_k(\nu_0))$ which proves the lemma.

We can give now a geometric description of the surface $\mathcal{R}(L)$. To this aim denote by $\mathcal{R}_k$ a copy of the complex plane cut along the following sets:

1. A simple smooth curve inside the disc $\{\mu : |\mu| \leq r\}$ containing the point $\mu = -r$ and all points of the discriminant set $Z(L)$ lying inside it;
2. The ray $\{\mu : \mu < -r\}$;
3. All real segments $[\alpha, \beta]$ where $\alpha$ and $\beta$ are neighboring ramification points of $\rho_k(\mu)$ with Re $\mu \geq r$;
4. All segments $[\alpha, \alpha]$ where $\alpha$ is a ramification point of either $\rho^*_{k-1}(\mu)$ with Re $\mu \leq -r$ if $k$ is even or $\rho^*_{k+1}(\mu)$ with Re $\mu \leq -r$ if $k$ is odd.

The function $\rho_k(\mu)$ is extended from $\Pi_k(r, \delta)$ to $\mathcal{R}_k$ as a single-valued analytic solution of (1.8) and by glueing the sheets $\mathcal{R}_j, j = 1, \ldots, 2n$, according to boundary values of functions $\rho_j(\mu)$ we obtain the surface $\mathcal{R}(L)$ with a single valued analytic function $\rho(\mu)$ on it.

**Corollary.** The transformation $J(\mu, \rho(\mu)) = (\mu, (\rho(\mu))^{-1})$ is an analytic involution in $\mathcal{R}(L)$.

Indeed, if we have $|\rho_n(\mu)| = 1$ with a sufficiently large real $\mu$ then $(\rho_n(\mu))^{-1} = \rho_n(\mu) = \rho_{n+1}(\mu)$. It means that $(\rho(\mu))^{-1}$ is a solution of (1.8). Since $\rho(\mu)$ is an analytic function in a simply-connected Riemann surface $\mathcal{R}(L)$, the function $(\rho(\mu))^{-1}$ is extended as a solution to entire surface and $(\mu, (\rho(\mu))^{-1})$ is its point, which proves Corollary.

To conclude the present section, we note that the characteristic polynomial of operator $L$ is of the form

$$\Delta(\mu, \rho) = \rho^{2n} + \sum_{k=1}^{2n-1} A_k(\mu)\rho^k + 1$$

where $A_k(\mu)$ are entire functions. It follows from Corollary that $A_k(\mu) = A_{2n-k}(\mu)$ for $k = 1, \ldots, 2n - 1$. A statement of such a type for canonical Hamiltonian systems is known as the Lyapunov-Poincaré Theorem, cf., [13, 14].

### 3 Band structure of the spectrum of operator $L$

To describe the band structure of the spectrum $\sigma(L)$ let us introduce the set

$$\mathcal{E}(L) = \{\mu \in \mathbb{R} : \text{there exists } \rho \in \mathbb{C}, |\rho| = 1, \text{ such that } \Delta(\mu, \rho) = \Delta(\mu, \rho(\mu)) = 0\}.$$

If

$$\mathcal{E}_k(L) = \{\mu \in \sigma(L) : \Delta(\mu, \rho_k(\mu)) = \Delta(\mu, \rho_k(\mu)) = 0\}$$

(3.1)
then
\[ \mathcal{E}(L) = \bigcup_{k=1}^{2n} \mathcal{E}_k(L). \]

The set \( \mathcal{E}(L) \) is a part of the discriminant set \( \mathcal{Z}(L) \) and hence it is countable with the unique accumulation point at \( +\infty \).

**Lemma 3.1.** If \( \rho_k(\mu_0) = e^{it_0} \) for some real \( \mu_0 \notin \mathcal{E}_k(L) \), \( t_0 \in [0, \pi] \) and some integer \( k, 1 \leq k \leq 2n \), then there exist the maximal closed interval \( S \subset [0, \pi] \) containing \( t_0 \) and the continuous monotonic function \( \mu(t) \) in \( S \) such that

(i) The relations
\[ \Delta(\mu(t), e^{it}) = 0, \quad t \in S; \quad \mu(t_0) = \mu_0; \quad \mu'(t) \neq 0, \quad t \in \text{int } S, \quad (3.2) \]
are valid;

(ii) The function \( \mu(t) \) maps \( S \) one-to-one onto the compact interval \( \ell \subset \sigma_k(L) \) with end-points in the set \( \mathcal{E}_k(L) \).

Proof. Since \( \mu_0 \notin \mathcal{E}_k(L) \), the surface \( R(L) \) is not ramified at the point \((\mu_0, \rho_k(\mu_0))\) and \( \rho_k(\mu) \) is a single-valued branch of \( \rho(\mu) \) analytic in a small complex neighborhood \( V_0 = \{ \mu : |\mu - \mu_0| < \epsilon \} \subset R_k \) of \( \mu_0 \). Its Taylor expansion at \( \mu_0 \) has the form
\[ \rho_k(\mu) = e^{it_0} + c_k(\mu - \mu_0)^p + \sum_{m=p+1}^{\infty} c_km(\mu - \mu_0)^m, \quad c_k \neq 0, \quad p \geq 1. \quad (3.3) \]

If here \( p \geq 2 \), then the pre-image with respect to \( \rho_k(\mu) \) of a small neighborhood of the point \( e^{it_0} \) on the unit circle \( U_0 = \{ z : |z| = 1 \} \) contains non-real points. According to Theorem 1.1 such points belong to the spectrum \( \sigma(L) \) which is impossible, since \( L \) is a selfadjoint operator. Therefore \( p = 1 \) in (3.3), \( \rho_k'(\mu_0) = c_k \neq 0 \) and the function \( \rho_k(\mu) \) maps \( V_0 \) one-to-one onto a small complex neighborhood \( \Theta_0 \) of the point \( e^{it_0} \in U_0 \).

According to Theorem 1.1 the pre-image of the arc \( \Theta_0 \cap U_0 \) consists of the points belonging to the spectrum \( \sigma(L) \) and hence coincides with the interval \( W_0 = V_0 \cap \mathbb{R}, \mu_0 \) being its inner point. The function
\[ \rho_k^*(\mu) = \overline{\rho_k(\mu)}, \quad \mu \in V_0, \quad (3.4) \]
is a single-valued branch of \( \rho(\mu) \) in \( V_0 \) different from \( \rho_k(\mu) \) and if \( \rho_k(\mu_0) = \pm 1 \) then \( \rho_k(\mu_0) = \rho_k^*(\mu_0) \) contradicting the assumption \( \mu_0 \notin \mathcal{E}_k(L) \). Therefore \( t_0 \notin \{ 0, \pi, 2\pi \} \).

Denote by \( \ell \) the largest closed interval in \( \sigma_k(L) \) which contains \( \mu_0 \), but does not contain points of the set \( \mathcal{E}_k(L) \) in its interior \( \ell^{(o)} \), and by \( \mu^- \) and \( \mu^+ \) the end-points of \( \ell \). The function \( \rho_k(\mu) \) maps \( \ell^{(o)} \) into the unit circle \( U_0 \). As before, we use the self-adjointness arguments and find \( \rho_k'(\mu_0) \neq 0, \mu \in \ell^{(o)}. \) Therefore for every \( \mu \in \ell^{(o)} \) there exists \( t \in [0, 2\pi) \) such that \( \rho(\mu) = e^{it} \) and

\[ \text{These arguments are similar to those used in [3], Ch.XIII.} \]
the local correspondence \( t = t(\mu) \) is one-to-one and analytic at points of \( \ell^{(0)} \). The function \( \rho_\mu(\mu) \) defined by (3.4) is a single-valued solution of (1.8) analytic in some neighborhood of \( \ell^{(0)} \) and hence it coincides with a single-valued branch \( \rho_q(\mu) \) of \( \rho(\mu) \). Since \( \rho_\mu(\mu) \neq \rho_\mu^*(\mu) \) in a neighborhood of \( \mu_0 \), we have \( q \neq k \).

Let now a point \( \mu_0 \) starts moving monotonically in \( \mathbb{R} \) towards one of end-points of \( \ell \). Then the corresponding point \( t = t(\mu) \) moves monotonically in \( \mathbb{R} \) and simultaneously the points \( \rho_k(\mu) \) and \( \rho_q(\mu) \) move in the opposite directions on the unit circle towards each other. These points may meet for the first time only if \( \rho_k(\mu) = e^{it} = e^{-it} = \rho_q(\mu) \), i.e., if either \( t = 0 \) or \( t = \pi \). For \( \mu \) corresponding to the meeting point we have \( \rho_k(\mu) = \rho_q(\mu) = \pm 1 \) which means that \( (\mu, \rho_k(\mu)) = (\mu, \rho_q(\mu)) \) is a ramification point of \( \mathcal{R}(L) \). Therefore \( \mu \) is a point of the set \( \mathcal{E}_k(L) \) and since \( \ell^{(0)} \) is the maximal open interval containing the point \( \mu_0 \) and no points of \( \mathcal{E}_k(L) \), we find that \( \mu^- \in \mathcal{E}_k(L) \) and \( \mu^+ \in \mathcal{E}_k(L) \).

The same claim is true if the points \( \rho_k(\mu) = e^{it} \) and \( \rho_q(\mu) = e^{-it} \) do not meet at all. Indeed, in this situation \( t \in (0, \pi) \) and because the monotonicity of the function \( t(\mu) \) there exist the limits

\[
    t_\pm = \lim_{\mu \to \mu_\pm} t(\mu) \in (0, \pi).
\]

If either \( \mu^- \) or \( \mu^+ \) does not belong to the set \( \mathcal{E}_k(L) \), we replace \( \mu_0 \) and \( t_0 \) in the assumptions of the lemma by either \( \mu^- \) and \( t_- \) or \( \mu^+ \) and \( t_+ \), respectively, and use the above arguments to find that the interval \( \ell \) fails its maximal property.

The contradiction proves that the end-points of \( \ell \) belong to \( \mathcal{E}_k(L) \). Differentiating the identity \( \rho_k(\mu(t)) = e^{it} \) we obtain \( \mu'(t) \neq 0 \) for \( t \in \text{int } S \) where \( S \) is the segment with end-points \( t_- \) and \( t_+ \), completing the proof of the lemma.

The following statement is a detailed version of of Theorem 1.1.

**Theorem 3.1.** For every fixed integer \( k = 1, \ldots, 2n \) there exist a system of non-overlapping compact intervals \( \ell_j^{(k)} \subset \sigma_k(L), j = 1, \ldots, N(k) \), with end-points in the set \( \mathcal{E}_k(L) \), the system of closed intervals \( S_j^{(k)} \), \( j = 1, \ldots, N(k) \), each contained in either \( [0, \pi] \) or \( [\pi, 2\pi] \) and the system of continuous monotonic functions \( \mu_j(t), t \in S_j^{(k)} \), \( j = 1, \ldots, N(k) \), such that

(i) The relations

\[
    \Delta(\mu_jk(t), \rho_k(\mu_jk(t))) = 0, \quad \rho_k(\mu_jk(t)) = e^{it}, \quad t \in S_j^{(k)}, \quad (3.5)
\]

\[
    \mu_j'{k}(t) \neq 0, \quad t \in \text{int } S_j^{(k)},
\]

are valid;

(ii) The functions \( \mu_jk(t) \) map \( S_j^{(k)} \) one-to-one onto \( \ell_j^{(k)} \);

(iii) The representation

\[
    \sigma_k(L) = \bigcup_{j=1}^{N(k)} \ell_j^{(k)}
\]

holds.
(iv) The spectrum \( \sigma(L) \) of operator (1.3) has the form

\[
\sigma(L) = \bigcup_{k=1}^{2n} \left( \bigcup_{j=1}^{N(k)} \ell_j^{(k)} \right).
\]

Proof. Let \( \mu_0 \) be a non-isolated point in the set \( \sigma_k(L) \). If \( \rho_k(\mu_0) = e^{it_0} \), \( t_0 \in [0, \pi] \) and \( \mu_0 \notin \mathcal{E}_k(L) \), then according to Lemma 3.1 there exist the closed intervals \( \ell \in \sigma_k(L) \) and \( S \in [0, \pi] \) and the function \( \mu(t) \) with properties stated in (i) and (ii).

If \( \rho_k(\mu_0) = e^{it_0} \), \( t_0 \in [\pi, 2\pi] \) and \( \mu_0 \notin \mathcal{E}_k(L) \) we apply Lemma 3.1 to the function \( \rho_k(\mu_0) \) defined by (3.4) and to \( t_0^* = 2\pi - t_0 \in [0, \pi] \) in the capacity of \( \rho_k(\mu_0) \) and \( t_0 \in [0, \pi] \) and, in addition to the interval \( \ell \), find the interval \( S^* \in [0, \pi] \) and the function \( \mu^*(t) \) with properties stated in Lemma 3.1. Now the intervals \( \ell \) and \( S = \{t \in [\pi, 2\pi] : t = 2\pi - t^*, \ t^* \in S^* \} \) and the function \( \mu(t) = \mu^*(2\pi - t) \) possess properties (i) and (ii).

If as before \( \mu_0 \) is a non-isolated point in the set \( \sigma_k(L) \) but \( \mu_0 \in \mathcal{E}_k(L) \), then we fix an interval \( V_0 = \{\mu \in \mathbb{R} : |\mu - \mu_0| < \epsilon \} \) without points of the set \( \mathcal{E}_k(L) \) distinct from \( \mu_0 \). We choose an arbitrary point \( \nu \in V_0 \cap \sigma_k(L), \nu \neq \mu_0 \), and again using Lemma 3.1 find the interval \( \ell \) containing \( \nu \), the interval \( S \) and the function \( \mu(t) \) with properties (i) and (ii). The end-points of \( \ell \) must be located in the set \( \mathcal{E}_k(L) \) and hence \( \mu_0 \) is one of them.

The remaining opportunity for \( \mu_0 \) is to be an isolated point in \( \sigma_k(L) \). If such a point does not belong to \( \mathcal{E}_k(L) \), then \( \rho_k(\mu) \) is a single-valued analytic function in a complex neighborhood of \( \mu_0 \). As shown in the proof of Lemma

3.1, it maps one-to-one a small interval \( \{\mu \in \mathbb{R} : |\mu - \mu_0| < \epsilon \} \) onto a small arc of the unit circle \( U_0 \) centered at \( \rho_k(\mu_0) \), and \( \mu_0 \) cannot be a non-isolated point in \( \sigma_k(L) \). Therefore \( \mu_0 \in \mathcal{E}_k(L) \) and if \( \rho_k(\mu_0) = e^{it_0} \) the conditions (i) and (ii) are satisfied with \( \ell = \{\mu_0\}, \ S = \{t_0\} \) and \( \mu(t) \equiv \mu_0 \) completing the proof of Theorem 3.1.

The previous analysis shows that the set \( \sigma_k(L) \) is formed by a system of compact intervals with end-points in the set \( \mathcal{E}_k(L) \). We denote these intervals by \( \ell_j^{(k)}, j = 1, \ldots, N(k) \), according to the ordering of their end-points on the real axis and by \( \mu_j^{(k)} \) and \( \mu_{jk}(t) \) the corresponding intervals in \([0, 2\pi]\) and functions defined on them, respectively.

**Remark 3.2.** Every point of the set \( \mathcal{E}(L) \) belongs to the spectrum \( \sigma(L) \) and is not isolated in it. Therefore such a point belongs to some non-trivial interval \( \ell_j^{(k)} \) and according to Theorem 3.1 must be its end-point.

4 Operators \( L_t \)

For every \( t \in [0, 2\pi] \) we denote by \( L_t \) the selfadjoint operator in the space \( L^2([0, \pi]) \) generated by the expression 1.3 and boundary conditions

\[
y^{(j)}(\pi) = e^{it}y^{(j)}(0), \quad 0 \leq j \leq 2n - 1, \quad (4.1)
\]
The spectrum $\sigma(L_t)$ of $L_t$ in the space $L^2[0, \pi]$ coincides with the set of all $\mu$'s satisfying Equation (1.8) with $\rho = e^{it}$ and Theorem 1.1 states

$$\sigma(L) = \bigcup_{t \in [0, 2\pi]} \sigma(L_t).$$

In this section we show that the spectrum $\sigma(L_t)$ is simple for all $t \in [0, 2\pi]$ except maybe finitely many points and describe eigen-functions for non-exceptional values of $t$.

**Lemma 4.1.** If $\Delta(\mu_0, e^{it}) = 0$ and $E(x; \mu, \rho)$ is defined by (1.9), then

$$\|E(\cdot; \mu_0, e^{it})\|^2_{L^2([0, \pi])} = (-1)^{n+1} e^{-it} \Delta(\mu_0, e^{it}) E(0; \mu_0, e^{-it}).$$

Proof. For every $\mu \in \mathbb{C}$ the definition (1.9) of $E(x; \mu, \rho)$ yields relations

$$E^{(j)}(\pi; \mu, e^{it}) - e^{it} E^{(j)}(0; \mu, e^{it}) = 0, \quad j = 0, \ldots, 2n - 2,$$

$$E^{(2n-1)}(\pi; \mu, e^{it}) - e^{it} E^{(2n-1)}(0; \mu, e^{it}) = -\Delta(\mu, e^{it}).$$

Using the Lagrange formula we find

$$(\mu - \mu_0)(E(\cdot; \mu, e^{it}), E(\cdot; \mu_0, e^{it})) = (-1)^{n+1} e^{-it} \Delta(\mu, e^{it}) E(0; \mu_0, e^{-it})$$

and (4.2) follows as $\mu \to \mu_0$.

**Lemma 4.2.** The set

$$\mathcal{T}(L) = \{t \in [0, 2\pi] : \Delta(\mu_0, e^{it}) E(0; \mu_0, e^{-it}) = 0 \text{ for some } \mu_0 \in \sigma(L_t)\}$$

is finite and contains the points $0, \pi, 2\pi$.

Proof. It is evident that the set $\mathcal{T}$ is the union of (maybe intersecting) sets

$$\mathcal{T}_1(L) = \{t \in [0, 2\pi] : \text{there exists } \mu_0 \in \sigma(L_t) \text{ such that } \Delta(\mu_0, e^{-it}) = 0\}$$

and

$$\mathcal{T}_2(L) = \{t \in [0, 2\pi] : \text{there exists } \mu_0 \in \sigma(L_t) \text{ such that } E(0; \mu_0, e^{it}) = 0\}.$$

If $\mu_0 \in \sigma(L_{t_0})$ then there exist the intervals $e^{(k)}_j$ and $S^{(k)}_j$ and the function $\mu_j(t), \ t \in S^{(k)}_j$ with properties described in Theorem 3.1 such that $\mu_0 \in \sigma_k(L), \ \rho_0(\mu_0) = e^{it_0}, \ t_0 \in S^{(k)}_j$. If $\mu_0 \in \mathcal{T}_1(L)$ then $\mu_0 \in \mathcal{E}_k(L)$ and $\mu_0$ is an end-point of $e^{(k)}_j$ implying that $t_0$ is an end-point of $S^{(k)}_j$. Let us show that the set of all $t$'s which are the end-points of intervals

$$S^{(k)}_j, \quad j = 1, \ldots, N(k); \quad k = 1, \ldots, 2n,$$

is finite.
For every $R > 0$ the number of intervals $\ell_j^{(k)}$ either partly or completely located in the disc $\{ \mu : |\mu| \leq R \}$ is finite and therefore the same is the number of intervals $S_j^{(k)}$ with the corresponding indices $k$ and $j$.

It follows from the asymptotic representation (1.11) that there are only two values of $k$ for which $N(k) = \infty$, and they are $n$ and $n + 1$. We choose $R$ large enough for end-points $\alpha_j^{(n)}$ and $\beta_j^{(n)}$ of intervals $\ell_j^{(n)}$ lying in the domain $\{ \mu : |\mu| \geq R \}$ to be close to the set $\{ m^{2n} \}_{m \in \mathbb{N}}$. In addition, we can assume that ramification points $(\mu, \rho_n(\mu))$ of the surface $\mathcal{R}(L)$ with $|\mu| \geq R$ are close to $(\alpha_j^{(n)}, \rho_n(\alpha_j^{(n)}))$ and $(\beta_j^{(n)}, \rho_n(\beta_j^{(n)}))$.

Let $\ell_j^{(n)}$ and $\ell_{j+1}^{(n)}$ be two adjacent intervals from the set $\sigma_n(L) \cap \{ \mu : \mu \geq R \}$.

If $\gamma_j = (j + 1/2)2^n$ then it follows from (1.11) that one of the numbers $\rho_n(\gamma_j)$ and $\rho_n(\gamma_j)$ is contained in the half-plane $\{ \mu : \Im \mu > 0 \}$ and another in $\{ \mu : \Im \mu < 0 \}$. By virtue of Lemma 3.1 one of the numbers $\rho_n^a$ and $\rho_n^b$ belongs to the segment $[0, \pi]$ and another to $[\pi, 2\pi]$.

Suppose that the gap $[\beta_j^{(n)}, \alpha_j^{(n)}]$ collapses to a point. If $\mu$ approaches this point then the continuity of $\rho_n(\mu)$ yields

$$
\lim_{\mu \to \beta_j^{(n)} - 0} \rho_n(\mu) = \lim_{\mu \to \alpha_j^{(n)} + 0} \rho_n(\mu).
$$

Both numbers here are of the form $e^{it}$, one with $t \in [0, \pi]$ and another with $t \in [\pi, 2\pi]$, and we conclude that either $t = 0$, or $t = \pi$ or $t = 2\pi$.

If the points $\beta_j^{(n)}$ and $\alpha_j^{(n)}$ are distinct, then they are separated by a non-degenerate open gap not containing points from the set $\sigma_n(L)$. In such a case the surface $\mathcal{R}(L)$ is ramified at both points $(\alpha_j^{(n)}, \rho_n(\alpha_j^{(n)}))$ and $(\beta_j^{(n)}, \rho_n(\beta_j^{(n)}))$: otherwise $\rho_n(\mu)$ is a single-valued analytic function in a small neighborhood of each such point, representation (3.3) is valid with $p = 1$ and with either $\mu_0 = \alpha_j^{(n)}$, or $\mu_0 = \beta_j^{(n)}$ and hence the gap $[\beta_j^{(n)}, \alpha_j^{(n)}]$ contains points of $\sigma_n(L)$ leading us to a contradiction. According to the choice of the number $R$ there are only two branches of $\rho(\mu)$ in small neighborhoods of points $(\alpha_j^{(n)}, \rho_n(\alpha_j^{(n)}))$ and $(\beta_j^{(n)}, \rho_n(\beta_j^{(n)}))$, and these are $\rho_n(\mu)$ and $\rho_n^+(\mu) = \rho_{n+1}(\mu)$. Since

$$
\lim_{\mu \to \beta_j^{(n)} - 0} \rho_n(\mu) = \lim_{\mu \to \beta_j^{(n)} - 0} \rho_n^+(\mu), \quad \lim_{\mu \to \alpha_j^{(n)} + 0} \rho_n(\mu) = \lim_{\mu \to \alpha_j^{(n)} + 0} \rho_n^+(\mu),
$$

$\Im \rho_n(\mu)>0$ and all four numbers here are of the form $e^{it}$: they are either $-1$ or $1$ and we again find that either $t = 0$ or $t = \pi$ or $t = 2\pi$. Thus the set $\mathcal{T}_l(L)$ is finite.

Let us prove now that the function $E(0; \mu, \rho(\mu))$ does not vanish identically on the surface $\mathcal{R}(L)$. First we note that this function is one of $(2n)^2$ minors of the matrix $U(\mu) - \rho(\mu)I$ and hence it is an entry of the matrix

$$
- \lim_{z \to \rho(\mu)} \Delta(\mu, z)(U(\mu) - zI)^{-1}. \quad (4.4)
$$
If \( \rho(\mu) \) is a simple root of Equation (1.8) then the previous expression is equal to
\[
\Delta'_\rho(\mu, \rho(\mu)) \ res\{(U(\mu) - zI)^{-1}; z = \rho(\mu)\}. \tag{4.5}
\]
Let \( \delta > 0 \) be fixed and let
\[
S_n(r, \delta) = \{ \mu : \mu = \lambda^{2n}, \lambda \in T_n(r, \delta) \} \cap \{ \mu : \text{Im} \mu \geq -1 \}
\]
where \( T_n(r, \delta) \) is defined in Section 2. Then relations (2.9) are fulfilled and the estimates
\[
|e^{\pi i \lambda} - e^{\pi \omega_k \lambda}| \geq d, \quad \omega_k \neq i, \quad \lambda \in T_n(r, \delta)
\]
are valid with a constant \( d > 0 \) not depending on \( \lambda \). Therefore for every \( \lambda \in T_n(r, \delta) \) and every \( z \) satisfying \(|z - e^{\pi i \lambda}| = d/4\) we have
\[
|z - e^{\pi \omega_k \lambda}| \geq |e^{\pi i \lambda} - e^{\pi \omega_k \lambda}| - |z - e^{\pi i \lambda}| \geq 3d/4, \quad \omega_k \neq i.
\]
As a result
\[
\|(E(\pi \lambda) - zI)^{-1}\| \leq C, \quad \lambda \in T_n(r, \delta), \quad |z - e^{\pi i \lambda}| = d/4,
\]
with a constant \( C \) not depending on either \( \mu \) or \( z \).

To calculate the residue in (4.5) we again use (2.9). Under the same restrictions on \( \mu \) and \( z \) as in the latter relation we have
\[
(U(\mu) - zI)^{-1} = D(\lambda)(\Omega + o(1))(E(\pi \lambda) - z(I + o(1))(\Omega^{-1} + o(1))(D(\lambda))^{-1}
\]
with \( \mu = \lambda^{2n}, \lambda \in T_n(r, \delta) \) and
\[
\|(E(\pi \lambda) - z(I + o(1)))^{-1}\| = \|(E(\pi \lambda) - zI)^{-1}(I + o(1))\| \leq C.
\]
If the number \( r \) is sufficiently large we have the estimates
\[
|\rho_n(\mu) - e^{\pi i \lambda}| \leq d/4, \quad |\rho_{n+1}(\mu) - e^{-\pi i \lambda}| \leq d/4, \quad \mu = \lambda^{2n}, \quad \lambda \in T_n(r, \delta).
\]
Therefore for the same \( \mu \)'s we obtain
\[
|\rho_k(\mu) - e^{\pi i \lambda}| \geq \begin{cases} 
\frac{1}{2}e^{\pi \text{Re}(\omega_k \lambda)} - |e^{\pi i \lambda}| & \geq d, \quad 1 \leq k \leq n - 1, \\
|e^{\pi i \lambda}| - \frac{1}{2}e^{\pi \text{Re}(\omega_k \lambda)} & \geq \frac{d}{2}, \quad n + 2 \leq k \leq 2n, \\
d - |\rho_{n+1}(\mu) - e^{-\pi i \lambda}| & \geq \frac{3d}{4}, \quad k = n + 1,
\end{cases}
\]
and hence the eigenvalue \( \rho_n(\mu) \) is inside the circle \( C(\mu, d) = \{ z \in \mathbb{C} : |z - e^{\pi i \lambda}| = d/4, \mu = \lambda^{2n} \} \), while all other eigenvalues \( \rho_k(\mu) \) are outside it. Since
\[
(E(\pi \lambda) - z(I + o(1)))^{-1} = (E(\pi \lambda) - zI)^{-1} + (E(\pi \lambda) - z(I + o(1)))^{-1}o(1)(E(\pi \lambda) - zI)^{-1},
\]
16
we obtain
\[
\text{res}\{(U(\mu) - zI)^{-1}; z = \rho_\mu(\mu)\} = \frac{1}{2\pi i} \oint_{C(\mu, d)} dz (U(\mu) - zI)^{-1} = D(\lambda)(\Omega P \Omega^{-1} + o(1))(D(\lambda))^{-1}
\]
where \(P = \|\delta_{p_\mu} \delta_{j,n} \|^2_{2n} \). Thus
\[
E(0; \mu, \rho_\mu(\mu)) = \mu \frac{2n-1}{2} \Delta^{(\mu)}(\mu, \rho_\mu(\mu))(\mu + o(1)), \quad \mu = \lambda^{2n}, \quad \lambda \in T_n(r, \delta), \quad (4.6)
\]
with a constant \(C \neq 0\) and we conclude that \(E(0; \mu, \rho(\mu))\) is a non-trivial analytic function on the surface \(R(L)\). Lemma 2.1 implies that zeros of this function may accumulate only at the point in infinity, and for every \(r > 0\) there exist finitely many solutions of the equation \(E(0; \mu, \rho(\mu)) = 0\) with \(|\mu| \leq r\). Since \(\rho(\mu)\) is not more than \(2n\)-valued function, for every such solution \(\mu\) there exists not more than \(2n\) values \(t \in (0, 2\pi)\) satisfying \(\rho(\mu) = e^{it}\).

Furthermore, it follows from (4.6) that the function \(E(0; \mu, \rho_\mu(\mu))\) may vanish for sufficiently large values of \(\text{Re} \mu\) inside exceptional sets \(D_k = \{\mu = \lambda^{2n}: |\lambda - k| \leq \delta, k \in \mathbb{N}\}\) only. If \(E(0; \mu, \rho_\mu(\mu)) = 0\) for \(\mu \in D_k \cap \sigma_n(L)\) then \(E(0; \mu, \rho_{\mu+1}(\mu)) = 0\) as well. In the case \(\rho_\mu(\mu) = e^{it} \neq \pm 1\) we have \(\rho_\mu(\mu) \neq \rho_{\mu+1}(\mu)\) and both \(\Delta_\mu'(\mu, \rho_\mu(\mu))\) and \(\Delta_{\mu+1}'(\mu, \rho_{\mu+1}(\mu))\) do not vanish. Therefore we can differentiate (3.5) at the point \(\mu\). The resulting identity
\[
\Delta_{\mu}^\prime(\mu, e^{it})\mu_{j,k}^\prime(t) + i e^{it} \Delta_{\mu}^\prime(\mu, e^{it}) = 0
\]
shows that \(\Delta_{\mu}^\prime(\mu, \rho_\mu(\mu))\) and \(\Delta_{\mu+1}^\prime(\mu, \rho_{\mu+1}(\mu))\) also do not vanish. According to (4.2) the number \(\mu\) is at least a double root of both \(E(0; \mu, \rho_\mu(\mu))\) and \(E(0; \mu, \rho_{\mu+1}(\mu))\). As a result \(\mu\) is a root of multiplicity at least 4 of the function \(E(0; \mu, \rho_\mu(\mu))E(0; \mu, \rho_{\mu+1}(\mu))\) which is single-valued and analytic inside \(D_k\). On the other hand, we have seen in the proof of Theorem 1.2 that for \(\mu \in D_k\) and \(\rho\) sufficiently close to \(\pm 1\) the representation \(\Delta(\mu, \rho) = (\rho^2 - 2)(\cos(\lambda \pi) + o(1))\rho + 1 + o(1)\phi(\mu, \rho)\) holds where \(\phi(\mu, \rho)\) is a polynomial in \(\rho\) whose coefficients, as well as all terms \(o(1)\), are single-valued analytic functions of \(\mu \in D_k\). Hence \(\omega(\mu) = \Delta(\mu, \rho_\mu(\mu))\Delta(\mu, \rho_{\mu+1}(\mu)) = (-4(\sin \pi \lambda)^2 + o(1))\psi(\mu)\) is a non-vanishing single-valued analytic function of \(\mu \in D_k\). We conclude that \(\omega(\mu)\) has only two roots in \(D_k\) while the Rouche Theorem claims, according to (4.6), that there are at least 4 such roots. Therefore \(E(0; \mu, \rho_\mu(\mu)) = 0\) is possible for \(\rho_{\mu}(\mu) = e^{it} = \pm 1\) only, proving that the set \(T_2(L)\) is finite.

**Theorem 4.1.** If \(t \in [0, 2\pi] \setminus T(L)\), then the spectrum \(\sigma(L_t)\) of the operator \(L_t\) is simple and every function \(f \in L^2([0, \pi])\) is representable by the \(L^2([0, \pi])\)-convergent orthogonal series
\[
f(x) = \sum_{\Delta(\mu, e^{it}) = 0} w(\mu, e^{it}) E(x; \mu, e^{it}) \int_0^\pi dy E(y; \mu, e^{-it}) f(y)
\]
with the weight function
\[
w(\mu, \rho) = |\Delta_{\mu}^\prime(\mu, \rho)| E(0; \mu, \rho^{-1})^{-1}.
\]

(4.8)
Proof. For every \( \mu \in \sigma(L) \) there exists an integer \( k, 1 \leq k \leq 2n \), such that \( \mu \) is a point of the set \( \sigma_k(L) \) defined by (1.13). We apply Theorem 3.1 and find an interval \( \ell_j^{(k)} \) containing \( \mu \), an interval \( \mathcal{S}_j^{(k)} \subset [0, 2\pi] \) containing \( t \) such that \( \rho_k(\mu) = e^{it} \) and a function \( \mu_{jk}(s) \) satisfying (3.5). If \( t \not\in \mathcal{T}(L) \) then \( \mu \) does not belong to the set \( \mathcal{E}_k(L) \) defined by (3.1). Therefore \( \mu \) is an inner point of \( \ell_j^{(k)} \), \( t \) is an inner point of \( \mathcal{S}_j^{(k)} \) and (4.7) shows that \( \Delta'_\mu(\mu, e^{it}) \neq 0 \). Hence \( \mu \) is a simple eigenvalue of \( L_t \), see [5]. For \( t \not\in \mathcal{T}(L) \) we have \( E(0; \mu, e^{it}) \neq 0 \), and it follows from Equation (4.2) that \( E(x; \mu, e^{it}) \) is an eigenfunction corresponding to \( \mu \). Theorem 4.1 states a well-known property of selfadjoint operators in Hilbert spaces.

5 Spectral expansions generated by \( L \)

In the present section we prove Theorem 1.3. The main tool in the proof is a version of the Fourier transform proposed by Gel’fand [4] for a study of differential operators with periodic coefficients.

Given a function \( f \in L^2(\mathbb{R}) \) we set, following [4],

\[
F(x, t) = (\mathcal{G}f)(x, t) = \sum_{r = -\infty}^{\infty} e^{-irt} f(x + \pi r), \quad t \in [0, 2\pi],
\]

and obtain a function \( F(x, t) \in L^2(Q) \) where \( Q = \{(x, t) : x \in [0, \pi], t \in [0, 2\pi]\} \). The inverse transform is given by

\[
f(x + \pi r) = \frac{1}{2\pi} \int_{0}^{2\pi} dt \, e^{ir t} \, F(x, t), \quad r \in \mathbb{Z}, \quad x \in [0, \pi].
\]

Since

\[
\|F\|_{L^2(Q)}^2 = \frac{1}{2\pi} \int_{Q} \, dx \, dt \, |F(x, t)|^2 = \int_{\mathbb{R}} \, dx \, |f(x)|^2 = \|f\|_{L^2(\mathbb{R})}^2,
\]

the Gel’fand transform \( \mathcal{G} \) is an isomorphic linear mapping of \( L^2(\mathbb{R}) \) onto \( L^2(Q) \).

For \( t \in [0, 2\pi] \) and \( k = 1, ..., 2n \) we denote by \( J_k(t) \) the set of all integers \( j \in \mathbb{N} \) such that \( \ell_j^{(k)} \cap \sigma(L_t) \neq \emptyset \).

Let \( 0 = t_0 < t_1 < ... < t_{M-1} < t_M = 2\pi \) be all points of the exceptional set \( \mathcal{T}(L) \) defined in Lemma 4.2 and let \( t \in (t_m, t_{m+1}) \) for some integer \( m, 0 \leq m \leq M - 1 \). Then to every \( j \in J_k(t) \) there corresponds the spectral band \( \ell_j^{(k)} \), the interval \( \mathcal{S}_j^{(k)} \subset [0, 2\pi] \) and the function \( \mu_{jk}(t), t \in \mathcal{S}_j^{(k)} \) with properties described in Theorem 4.1. The end-points \( m_{jk} \) and of \( M_{jk} \) of \( \mathcal{S}_j^{(k)} \) belong to the set \( \mathcal{T}(L) \) and

\[
\mathcal{S}_j^{(k)} = \bigcup_{p=m_{jk}}^{M_{jk}-1} [t_p, t_{p+1}].
\]
Of course, \( m_{jk} \leq m \leq M_{jk} \) and with \( s \) moving from \( t_m \) to \( t_{m+1} \) the point \( \mu_{jk}(s) \) runs over the interval

\[
\ell_{jm}^{(k)} = \{ \mu : \mu = \mu_{jk}(s), s \in (t_m, t_{m+1}) \} \subset \ell_j^{(k)}
\]

with end-points satisfying either \( \mu_{jk}(t_m) < \mu_{jk}(t_{m+1}) \) if \( \mu'_{jk}(t) > 0 \) or \( \mu_{jk}(t_m) > \mu_{jk}(t_{m+1}) \) if \( \mu'_{jk}(t) < 0 \) for \( t \in S_j^{(k)} \). In any case, the set \( J_k(t) \) does not change with \( t \) varying in \( (t_m, t_{m+1}) \) and

\[
\ell_j^{(k)} = \bigcup_{p=m_{jk}}^{M_{jk}-1} \ell_{jm}^{(k)}. \tag{5.1}
\]

Given an arbitrary function \( F \in L^2([0, \pi]) \) we set

\[
\phi(\mu, \rho; F) = \int_0^\pi dy \ F(y) \ E(y; \mu, \rho^{-1}).
\]

If here

\[
\mu = \mu_{jk}(t), \quad \rho = \rho_k(\mu_{jk}(t)) = e^{it}, \quad t \in (t_m, t_{m+1}); \quad F = Gf, \quad f \in L^2(\mathbb{R}),
\]

then

\[
e^{irt} E(y; \mu_{jk}(t), e^{it}) = E(y + \pi r; \mu_{jk}(t), e^{it}),
\]

and hence

\[
\phi(\mu_{jk}(t), e^{it}; F(., t)) = \int_0^\pi dy \ \sum_{r=-\infty}^{\infty} e^{-irt} f(y + \pi r) \ E(y; \mu_{jk}(t), e^{-it})
\]

\[
= \int_0^\pi dy \ \sum_{r=-\infty}^{\infty} f(y + \pi r) \ E(y + \pi r; \mu_{jk}(t), e^{-it})
\]

\[
= \int_\mathbb{R} dy \ f(y) \ E(y; \mu_{jk}(t), e^{-it})
\]

\[
= \Phi(\mu_{jk}(t), \rho_k(\mu_{jk}(t)); f)
\]

where \( \Phi(\mu, \rho_k(\mu); f) \) is defined by (1.15).

To prove Theorem 1.3 assume that \( f \) is an arbitrary function from the space \( L^2(\mathbb{R}) \) with the Gel’fand transform \( F = Gf \). According to the Fubini Theorem the function \( F(., t) \) belongs to the space \( L^2([0, \pi]) \) for almost all \( t \in [0, 2\pi] \), and for every such \( t \notin T(L) \) Theorem 4.1 permits us to represent it by the \( L^2([0, \pi]) \)-convergent orthogonal series

\[
F(x, t) = \sum_{k=1}^{2n} \sum_{j \in J_k(t)} \Psi(\mu_{jk}(t), \rho_k(\mu_{jk}(t)); F(., t)) E(x; \mu_{jk}(t), \rho_k(\mu_{jk}(t))) \tag{5.3}
\]
with \( \Psi(\mu, \rho; f) = w(\mu, \rho)\Phi(\mu, \rho; f) \) and \( w(\mu, \rho) \) defined by (4.8).

To prove that the series (5.3) converges in the space \( L^2(Q) \) let us introduce the partial sums

\[
F_q(x, t) = \sum_{k=1}^{2n} \sum_{j \in J_k(t), 1 \leq j \leq q} \Psi(\mu_{jk}(t), \rho_k(\mu_{jk}(t)); F(.,t))E(x; \mu_{jk}(t), \rho_k(\mu_{jk}(t)))
\]

and set \( \Theta(\mu, \rho; f) = w(\mu, \rho)|\phi(\mu, \rho; f)|^2 \). Because the orthogonality we have

\[
\|F_q(., t)\|_{L^2([0,\pi])}^2 = \sum_{k=1}^{2n} \sum_{j \in J_k(t), 1 \leq j \leq q} \Theta(\mu_{jk}(t), \rho_k(\mu_{jk}(t)); F(., t)).
\]

Since the set \( J_k(t) \) does not depend on \( t \) in \( (t_m, t_{m+1}) \), we can integrate the latter sum term by term in every such interval. After substituting \( \mu = \mu_{jk}(t) \) into the \( j \)-th integrated summand we take into account (5.1) and obtain

\[
\|F_q\|_{L^2(Q)}^2 = \frac{1}{2\pi} \sum_{m=1}^{M-1} \sum_{k=1}^{t_{m+1}-t_m} \int dt \|F_q(., t)\|_{L^2([0,\pi])}^2
\]

\[
= \sum_{m=1}^{M-1} \sum_{k=1}^{2n} \sum_{1 \leq j \leq q} \int_{\ell^j_{km}} \mu_2 \rho(\mu, \rho_k(\mu))|\Phi(\mu, \rho_k(\mu); f)|^2
\]

\[
= \sum_{k=1}^{2n} \sum_{1 \leq j \leq q} \int_{\ell^j_{km}} \mu_2 \rho(\mu, \rho_k(\mu))|\Phi(\mu, \rho_k(\mu); f)|^2
\]

According to the Bessel inequality applied to the series (5.3) we have \( \|F_q\|_{L^2(Q)}^2 = \|f\|_R^2 \) implying

\[
\sum_{k=1}^{2n} \sum_{j=1}^{N(k)} \int_{\ell^j_{km}} \mu_2 \rho(\mu, \rho_k(\mu))|\Phi(\mu, \rho_k(\mu); f)|^2 < \infty.
\]

Since

\[
\|F_{q'} - F_{q''}\|_{L^2(Q)}^2 = \sum_{k=1}^{2n} \sum_{q' < j \leq q''} \int_{\ell^j_{km}} \mu_2 \rho(\mu, \rho_k(\mu))|\Phi(\mu, \rho_k(\mu); f)|^2,
\]

the series (5.3) converges to \( F(x, t) \) in \( L^2(Q) \)-norm and we can apply to it the inverse Gel’fand transform term by term. The resulting identity

\[
f(x) = \sum_{k=1}^{2n} \sum_{j=1}^{N(k)} \int_{\ell^j_{km}} \mu_2 \rho(\mu, \rho_k(\mu))\Phi(\mu, \rho_k(\mu); f) E(x; \mu, \rho_k(\mu))
\]
is Equation (1.16) with $\phi_k(\mu) = \Phi(\mu, \rho_k(\mu); f)$.

Assume now that $\Phi = \{\phi_k(\mu)\}_{k=1}^{2n} \in \mathcal{H}_{2n}^2(L)$. Then for $k = 1, \ldots, 2n$ the sequences

$$\phi_k(\mu) = \{\phi_{jk}(\mu), \mu \in \ell_j^{(k)}, j = 1, 2, \ldots\}$$

satisfy the condition

$$\sum_{k=1}^{2n} \sum_{j=1}^{N(k)} \int_{\ell_j^{(k)}} d\mu p(\mu, \rho_k(\mu)) |\phi_{jk}(\mu)|^2 < \infty. \quad (5.5)$$

For $t \in (t_m, t_{m+1})$, $m = 0, \ldots, M - 1$, we define

$$F_q(x, t) = \sum_{k=1}^{2n} \sum_{j \in J_k(t), 1 \leq j \leq q} w(\mu_{jk}(t), e^{it}) \phi_{jk}(\mu_{jk}(t)) E(x; \mu_{jk}(t), e^{it})$$

and obtain

$$\|F_q(t, t) - F_{q''}(t, t)\|_{L^2([0,\pi])}^2 = \sum_{k=1}^{2n} \sum_{j \in J_k(t), q < j \leq q''} w(\mu_{jk}(t), e^{it}) |\phi_{jk}(\mu_{jk}(t))|^2.$$  

Similar to (5.4) we find

$$\|F_q' - F_{q''}\|_{L^2(Q)}^2 = \sum_{k=1}^{2n} \sum_{j \in J_k(t), q' < j \leq q''} \int_{\ell_j^{(k)}} d\mu p(\mu, \rho_k(\mu)) |\phi_{jk}(\mu)|^2.$$  

It follows from (5.5) that $\{F_q\}_{q=1}^\infty$ is a Cauchy sequence in the space $\mathcal{H}_{2n}^2(L)$ and hence there exists the function

$$F(x, t) = \lim_{q \to \infty} F_q(x, t) \in L^2(Q).$$

Its inverse Gel'fand transform $f = G^{-1} F$ has the form

$$f(x + \pi r) = \frac{1}{2\pi} \int_0^{2\pi} dt e^{i\pi r} F(x, t) = \frac{1}{2\pi} \int_0^{2\pi} dt e^{i\pi r} \lim_{q \to \infty} F_q(x, t)$$

$$= \lim_{q \to \infty} \sum_{k=1}^{2n} \sum_{m=1}^{M-1} \int_{t_m}^{t_{m+1}} \sum_{j \in J_k(t), 1 \leq j \leq q} p(\mu_{jk}(t), e^{it}) \phi_{jk}(\mu_{jk}(t)) E(x + \pi r; \mu_{jk}(t), e^{it})$$

$$= \sum_{k=1}^{2n} \sum_{j=1}^{N(k)} \int_{\ell_j^{(k)}} d\mu p(\mu, \rho_k(\mu)) \phi_{jk}(\mu) E(x + \pi r; \mu, \rho_k(\mu)).$$
Thus for every $\Phi \in H^2_2(L)$ there exists a function $f \in L^2(\mathbb{R})$ such that representation (1.16) is valid. To complete the proof of Theorem 1.3 let us show that in this representation

$$
\Phi(\mu, \rho_k(\mu); f) = \phi_k(\mu), \quad \mu \in \sigma_k(L), \quad k = 1, \ldots, 2n.
$$

It is sufficient to prove the above relation for an arbitrary step-like function

$$
\Phi_{\sigma} = \{\phi_k(\mu)\}_{k=1}^{2n}, \quad \phi_k(\mu) = \begin{cases} 
1, & k = k', \quad \mu \in \sigma \subset \ell_j^{(k')} \\
0, & \text{otherwise}
\end{cases}
$$

where $\sigma$ is a closed segment in the interior of a fixed band $\ell_j^{(k')}$ such that the corresponding interval $S_j^{(k')}$ does not contain points of the exceptional set $T(L)$ and hence the function $w(\mu_{j'k'}(t), e^{it})$ is continuous in it.

First we note that the function $f_{\sigma} \in L^2(\mathbb{R})$ defined by (1.16) with $\Phi = \Phi_{\sigma}$ has the form

$$
\Phi_{\sigma}(x) = \int_{\sigma} \mu \, p(\mu, \rho_k(\mu)) \, E(x; \mu, \rho_k(\mu))
$$

$$
= \frac{1}{2\pi} \int_{s_j^{(k')}} dt \, w(\mu_{j'k'}(t), e^{it}) \, E(x; \mu_{j'k'}(t), e^{it})
$$

where $s_j^{(k')} \subset S_j^{(k')}$ is the pre-image of $\sigma \subset \ell_j^{(k')}$ with respect to the function $\mu_{j'k'}(t)$.

Furthermore, for every integer $k$, $1 \leq k \leq 2n$, and real number $\mu \in \sigma_k(L) \setminus \mathcal{E}_k(L)$ there exist the unique integer $j$, $1 \leq j < N(k)$, and real number $\tau \in \text{int} S_j^{(k)}$ such that $\mu = \mu_{jk}(\tau)$. Therefore according to (1.15) we obtain

$$
\Phi(\mu, \rho_k(\mu); f_{\sigma}) = \int_{\mathbb{R}} dy \, E(y; \mu, \rho_k(\mu)) f_{\sigma}(y)
$$

$$
= \lim_{N \to \infty} \sum_{r=-N}^{N} \frac{1}{N} \int_{0}^{N} dy \, E(y + \pi r; \mu, \rho_k(\mu)) f_{\sigma}(y + \pi r)
$$

$$
= \lim_{N \to \infty} \int_{s_j^{(k')}} dt \, \delta_N(t-\tau) w(\mu_{j'k'}(t), e^{it}) \int_{0}^{N} dy \, E(y; \mu_{jk}(\tau), e^{-it}) E(y; \mu_{j'k'}(t), e^{it})
$$

where

$$
\delta_N(t) = \frac{1}{2\pi} \sum_{r=-N}^{N} e^{irt}
$$

is the Dirichlet kernel. For $\mu$ not being an end-point of $\ell_j^{(k')}$ we find

$$
\Phi(\mu, \rho_k(\mu); f_{\sigma})
$$
its inverse, respectively, conjugating operators $L$ define a one-to-one mapping of the space with \( L \) holds.

For \( k \neq k' \) and \( \mu_{jk}(\tau) = \mu_{j'k'}(\tau) = \mu \), then \( \rho_k(\mu) = \rho_{k'}(\mu) \) contrary to the restriction \( \mu \notin E_k(L) \) imposed on \( \mu \). It means that \( \mu_{j'k'}(\tau) \neq \mu_{jk}(\tau) \) for every \( k \neq k' \) and the eigen-functions \( E(y; \mu_{jk}(\tau), e^{i\tau}) \) and \( E(y; \mu_{j'k'}(\tau), e^{i\tau}) \) are orthogonal for all natural numbers \( j \) and \( j' \), yielding

\[
\Phi(\mu, \rho_k(\mu); f_\sigma) = 0, \quad k \neq k', \quad k = 1, \ldots, 2n.
\]

For \( k = k' \) we use the orthogonality of \( E(y; \mu_{jk}(\tau), e^{i\tau}) \) and \( E(y; \mu_{j'k}(\tau), e^{i\tau}) \) for \( j \neq j' \) and (4.2) and obtain \( \Phi(\mu, \rho_k(\mu); f_\sigma) = \delta_{kk'} \delta_{jj'} \chi_\sigma(\mu), \ \mu \in \ell^{(k')}. \)

Therefore \( \{\Phi(\mu, \rho_k(\mu); f_\sigma)\}_{k=1}^{2n} = \Phi_\sigma \) which completes the proof of Theorem 1.3.

6 Spectral matrix and uniqueness theorem

Let \( M(\mu) \) be a non-negative Hermitian \( 2n \times 2n \)-matrix-valued function defined on the spectrum \( \sigma(L) \). Denote by \( L^{2n}_\mathbb{C}(M) \) the space of complex \( 2n \)-vector functions \( F = \{F_q(\mu)\}_{q=1}^{2n} \) with the scalar product

\[
(F, G) = \int_{\sigma(L)} d\mu \ (M(\mu)F(\mu), G(\mu)).
\]

According to a general definition (cf., [5]), \( M(\mu) \) is called a spectral matrix of \( L \) if the relations

\[
F_q(\mu) = \int_{\mathbb{R}} dx \ f(x)u_q(\mu, x), \quad q = 1, \ldots, 2n, \tag{6.1}
\]

and

\[
f(x) = \int_{\sigma(L)} d\mu \ (M(\mu)F(\mu), Y(x, \mu)) \tag{6.2}
\]

with \( Y(x, \mu) = \text{col}\{u_1(x, \mu), \ldots, u_{2n}(x, \mu)\} \) define a one-to-one mapping of the space \( L^2(\mathbb{R}) \) onto the space \( L^{2n}_\mathbb{C}(M) \) and its inverse, respectively, conjugating operators \( L \) and \( \mu I \), with integrals in (6.1) and (6.2) converging in the norms of the corresponding spaces. As a result, the Parseval identity

\[
\int_{\mathbb{R}} dx \ |f(x)|^2 = \int_{\sigma(L)} d\mu \ (M(\mu)F(\mu), F(\mu)) \tag{6.3}
\]

holds.
Theorem 6.1. The spectral matrix $\mathcal{M}(\mu)$ of operator $L$ has the form

$$
\mathcal{M}(\mu) = \sum_{k=1}^{2n} \chi_k(\mu)\mathcal{M}(\mu, \rho_k(\mu)), \quad \mathcal{M}(\mu, \rho) = p(\mu, \rho)\|v_q(\mu, \rho)v_{q'}(\mu, \rho^{-1})\|_{q, q'=1}^{2n}
$$

where the function $p(\mu, \rho)$ is defined by (1.10) and the numbers $v_q(\mu, \rho)$, $q = 1, ..., 2n$ are uniquely defined by the representation

$$
E(x; \mu, \rho) = \sum_{q=1}^{2n} v_q(\mu, \rho) u_q(x, \mu).
$$

If two operators of the form (1.3) have the same spectral matrix $\mathcal{M}$, then their coefficients coincide.

Proof. Let $f(x)$ be an arbitrary function from the space $L^2(\mathbb{R})$ and let $F_q(\mu)$ be defined by (6.1). Then (1.15) reads

$$
\Phi(\mu, \rho_k(\mu); f) = \sum_{q=1}^{2n} v_q(\mu, (\rho_k(\mu))^{-1}) F_q(\mu)
$$

and Theorem 1.3 states that the representation (6.2) and identity (6.3) hold for functions from the space $L^2(\mathbb{R})$.

On the other hand, let $F = \{F_q(\mu)\}_{q=1}^{2n}$ be an arbitrary element of the space $L^2_{2n}(\mathcal{M})$. Then

$$
\int_{\sigma(L)} d\mu \left( \mathcal{M}(\mu) F(\mu), F(\mu) \right) < \infty
$$

and if

$$
\phi_k(\mu) = \sum_{q=1}^{2n} v_q(\mu, (\rho_k(\mu))^{-1}) F_q(\mu)
$$

then the vector $\Phi = \{\phi_k(\mu)\}_{k=1}^{2n}$ belongs to the space $\mathcal{H}_{2n}^2(L)$. We again use Theorem 1.3 and find that the function $f(x)$ defined by (1.16) belongs to the space $L^2(\mathbb{R})$ and the representation itself coincides with (6.2). As we have just seen, the function $f(x)$ is representable also by (6.2) with

$$
\tilde{F}_q(\mu) = \int_{\mathbb{R}} dx f(x)u_q(\mu, x), \quad q = 1, ..., 2n.
$$

The Parseval identity (6.3) implies $F = \tilde{F}$ which proves that the mapping (6.1) is from the space $L^2(\mathbb{R})$ onto the space $L^2_{2n}(\mathcal{M})$ and that $\mathcal{M}(\mu)$ is the spectral matrix of $L$.

To prove the second part of Theorem 6.1 we assume that the characteristic polynomial $\Delta(\mu, \rho)$ and the spectral matrix $\mathcal{M}(\mu)$ are known. It is easy to see that if $\Delta(\mu, \rho) = 0, p(\mu, \rho) \neq \infty$, and if, for a fixed $q'$, the vector

$$
V(\mu, \rho) = p(\mu, \rho)v_{q'}(\mu, \rho^{-1}) \{v_q(\mu, \rho)\}_{q=1}^{2n}.
$$

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does not vanish then it is an eigen-vector of the monodromy matrix $U(\mu)$ corresponding to its eigenvalue $\rho$. Now we note that the vector-function $V(\mu, \rho(\mu))$ is the $q'$-th line of the matrix $M(\mu, \rho(\mu))$ and hence it is also known.

Since $v_1(\mu, \rho) = E(0; \mu, \rho)$, we find that $V(\mu, \rho(\mu))$ is a non-trivial meromorphic vector-function on the surface $\mathcal{R}(L)$ and hence its zeros and poles are projected on a discrete set $Z_V(L)$ of the complex plane accumulating at the point in infinity.

Assume now $(\mu, \rho(\mu)) \in \mathcal{R}(L)$ to be such a point that $\mu \notin Z(L) \cup Z_V(L)$ where $Z(L)$ is a discriminant set of $\Delta(\mu, \rho)$. Then

- The eigenvalues $\rho_1(\mu), ..., \rho_{2n}(\mu)$ of the monodromy matrix $U(\mu)$ are pairwise distinct;
- The vectors $V(\mu, \rho_1(\mu)), ..., V(\mu, \rho_{2n}(\mu))$ are well-defined, do not vanish and hence form a linear independent system in the space $\mathbb{R}^{2n}$;
- The $2n \times 2n$ matrix $C(\mu, \rho(\mu))$ whose columns are vectors from the above system does not degenerate;
- The identity

$$U(\mu) = C(\mu, \rho(\mu)) \text{ diag}\{\rho_1(\mu), ..., \rho_{2n}(\mu)\} \{C(\mu, \rho(\mu))\}^{-1}, \mu \notin Z(L) \cup Z_V(L),$$

holds.

The factors in the above product are neither single-valued no analytic in $\mathbb{C}$: every rotation of a point $\mu$ around the projection of a ramification point results in a permutation of columns and at every point from the set $Z_V(L)$ the matrix $C(\mu, \rho(\mu))^{-1}$ may have a pole. Nevertheless, the product is single-valued in $\mathbb{C} \setminus (Z(L) \cup Z_V(L))$ and coincides with the entire matrix-function $U(\mu)$ outside the set of all its singular points. Hence all these points are removable singularities of the product which means that the spectral matrix $M(\mu)$ permits us to reconstruct completely the monodromy matrix. According to a uniqueness theorem proved by Leibenzon [8] the latter uniquely determines all coefficients of operator (1.3) which completes the proof of Theorem 6.1.

It follows from Theorem 6.1 that the spectral matrix $M(\mu)$ determines the coefficients of operator (1.3) uniquely. For Hill’s operators (1.1) on the entire real axis this statement is well-known and is a version of the uniqueness theorem proved by Marchenko [15] for Sturm-Liouville operators on a semi-axis and extended by Rofe-Beketov [16] to such operators on the entire real axis.

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