Construction and analysis of the quadratic finite volume methods on tetrahedral meshes

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Abstract We construct and analyze a family of quadratic finite volume method (FVM) schemes over tetrahedral meshes. In order to prove the stability and the error estimate, we propose the minimum V-angle condition on tetrahedral meshes, and the surface and volume orthogonal conditions on dual meshes. Through the technique of element analysis, the local stability is equivalent to a positive definiteness of a $9 \times 9$ element matrix, which is difficult to analyze directly or even numerically. With the help of the surface orthogonal condition and the congruent transformation, this element matrix is reduced into a block diagonal matrix, and then we carry out the stability result under the minimum V-angle condition. It is worth mentioning that the minimum V-angle condition of the tetrahedral case is very different from a simple extension of the minimum angle condition for triangular meshes, while it is also convenient to use in practice. Based on the stability, we prove the optimal $H^1$ and $L^2$ error estimates, respectively, where the orthogonal conditions play an important role in ensuring the optimal $L^2$ convergence rate. Numerical experiments are presented to illustrate our theoretical results.

Keywords finite volume method, tetrahedral mesh, orthogonal condition, minimum V-angle condition, stability and convergence

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1 Introduction

The finite volume method (FVM) (see [1–3,6,7,11,14,20] and [22,27–30]), which is known for preserving the local conservation property, is one of the main numerical methods for solving partial differential equations. Till now, much progress has been made on the stability, error estimate and superconvergence of the FVMs. The linear FVM schemes have been well studied on any spatial dimension [5,10,13,15,21,25,33], and complete results of arbitrary $k$-order FVM schemes on one dimension (1D) are given by [4,32]. For higher-order FVMs on triangular meshes, Chen et al. [8,9] presented a unified analysis of the stability under the assumption of the minimum angle condition, which restricts the minimum interior angle of the triangular elements; while for some quadratic FVM schemes, the angle value of this condition is improved by [33,39–41], where [39–41] are based on a new trial-to-test mapping. In addition, by proposing the
orthogonal conditions, Wang and Li [31] carried out optimal $L^2$ error estimates for arbitrary $k$-order FVM schemes on triangular meshes. For the FVMs on quadrilateral meshes, Lv and Li [26] proved the stability and the optimal $L^2$ error estimate of the biquadratic FVM schemes, and Lin et al. [23], Yang et al. [36] and Zhang and Zou [37,38] presented the stability and the optimal $L^2$ estimates of arbitrary higher-order FVM schemes by considering the bilinear form of the FVMs as the Gaussian quadrature of the bilinear form of the finite element methods (FEMs). The dual meshes of the schemes in [23,26,36,38] are based on the Gauss points. Compared with the big progress in 1D and two dimensions (2D), the FVMs on three dimensions (3D), which have more applications in practice, are mainly concentrated on the linear schemes [5,12,19,34], and there are few results of higher-order FVMs in 3D (see, e.g., [35]). Especially for the quadratic FVMs on tetrahedral meshes, no result has been published yet.

In this paper, we construct and analyze a family of quadratic FVM schemes on tetrahedral meshes for the following elliptic boundary value problem:

$$
\begin{align*}
-\nabla \cdot (\kappa \nabla u) &= f \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \Gamma,
\end{align*}
$$

where $\Omega \subset \mathbb{R}^3$ is a bounded convex polyhedron with boundary $\Gamma = \partial \Omega$, $f \in L^2(\Omega)$ and the diffusion coefficient $\kappa(x_1, x_2, x_3)$ is a piecewise smooth function and bounded almost everywhere with positive lower and upper bounds $c_\ast$ and $c^\ast$, respectively. Some recent works about the FEMs on higher dimensions are referred to [16–18].

We introduce three parameters $(\alpha, \beta, \gamma)$ to construct the dual mesh, and a mapping $\Pi^*_{\lambda}$ from the trial space to the test space for theoretical analysis. For the stability and error analysis, we propose two key restrictions: (1) the orthogonal conditions on the surface and the volume for the dual mesh, which control the construction of the dual elements; (2) the minimum V-angle condition for the primary tetrahedral mesh, which restricts the local shape around each vertex of the tetrahedral elements. Under the orthogonal condition on the surface and the minimum V-angle condition, we prove the local stability by element analysis and obtain the optimal $H^1$ error estimate. Based on an equivalent discrete norm, the local stability is converted to a positive definiteness of a $9 \times 9$ symbolic matrix. With the help of the orthogonal condition on the surface and the congruent transformation, the $9 \times 9$ symbolic matrix is reduced to a block diagonal matrix containing a $3 \times 3$ matrix and a $6 \times 6$ matrix, where the $3 \times 3$ matrix is proved to be unconditionally positive definite, while the analysis of the $6 \times 6$ matrix is a challenge. Fortunately, we derive that for fixed parameters $(\alpha, \beta, \gamma, \lambda)$, the $6 \times 6$ symbolic matrix only relies on five certain plane angles of a tetrahedral element. Then under the minimum V-angle condition, the positive definiteness of the $6 \times 6$ matrix is guaranteed numerically, and the stability is obtained. On the other hand, under the orthogonal conditions on the surface and the volume, we prove the optimal $L^2$ error estimate by the Aubin-Nitsche technique.

The contributions of this paper are summarized as follows: (1) We first use three parameters $(\alpha, \beta, \gamma)$ to construct a family of quadratic FVM schemes on tetrahedral meshes. (2) Under the proposed minimum V-angle condition and the orthogonal condition on the surface, we prove the stability by the technique of element analysis. (3) Under the proposed orthogonal conditions on the surface and the volume, we obtain the optimal $L^2$ error estimate.

The rest of this paper is organized as follows. In Section 2, we construct a family of quadratic FVM schemes on tetrahedral meshes. In Section 3, we prove the local stability by the technique of element analysis. Then we give optimal $H^1$ and $L^2$ error estimates in Section 4. We provide numerical experiments to confirm our theoretical results in Section 5. Finally, we draw the conclusion in Section 6. Some symbolic matrices are put in Appendix A, and some relations in a tetrahedron and two proofs based on them are included in Appendix B.

In this paper, “$A \lesssim B$” means that $A$ can be bounded by $B$ multiplied by a constant that is independent of the parameters on which $A$ and $B$ may depend. “$A \sim B$” means both “$A \lesssim B$” and “$B \lesssim A$.”
2 Preliminaries

2.1 The quadratic finite volume method schemes

Primary mesh and trial function space. Let the primary mesh $\mathcal{T}_h = \{K\}$ be a conforming tetrahedral partition of $\Omega$, where $h = \max_{K \in \mathcal{T}_h} h_K$ and $h_K$ is the length of the largest edge of the tetrahedral element $K$. Assume that $\mathcal{T}_h$ is a regular partition, i.e., there exists a positive constant $\sigma$, independent of $h$, satisfying

$$\frac{h_K}{\rho_K} \leq \sigma, \quad \forall \ K \in \mathcal{T}_h,$$

where $\rho_K$ is the diameter of the inscribed sphere of $K$.

Denote the set of the four vertices and the six midpoints of $K$ by $\mathcal{N}_K$, and let $\mathcal{N}_h = \bigcup_{K \in \mathcal{T}_h} \mathcal{N}_K$. Then define the trial function space over $\mathcal{T}_h$ as

$$U_h = \{u_h : u_h \in C(\Omega), u_h |_{K} \in P^2(K), \forall K \in \mathcal{T}_h, u_h |_{\partial \Omega} = 0\},$$

where $P^2(K)$ is the quadratic polynomial space on $K$, and $u_h |_{K}$ is determined by its ten node values on $\mathcal{N}_K$.

Dual mesh and test function space. Define two sets

$$\mathcal{Z}_n^{(1)} = \{1, 2, \ldots, n\}, \quad \mathcal{Z}_4^{(2)} = \{(j, k) \mid j, k \in \mathcal{Z}_4^{(1)}, j < k\}.$$

For a tetrahedral element $K = \Delta^4P_1P_2P_3P_4$ (see Figure 1), $T_i$ is the triangular face of $K$ opposite to the vertex $P_i$ ($i \in \mathcal{Z}_4^{(1)}$), $F_i$ is the barycenter of $T_i$ ($i \in \mathcal{Z}_4^{(1)}$), and $M_{jk}$ is the midpoint of the edge $P_jP_k$ ($\{j, k\} \in \mathcal{Z}_4^{(2)}$). It is well known that the four central lines $\{P_iF_i\}_{i \in \mathcal{Z}_4^{(1)}}$ intersect at $Q$, the barycenter of $K$, and

$$\frac{|P_iF_i|}{|P_iM_{jk}|} = \frac{2}{3}, \quad (j, k) \in \mathcal{Z}_4^{(2)}, \quad \{i_1, i_2, j, k\} = \mathcal{Z}_4^{(1)},$$

$$\frac{|P_iQ|}{|P_iF_i|} = \frac{3}{4}, \quad i \in \mathcal{Z}_4^{(1)}.$$

![Figure 1](https://example.com/figure1.png) (Color online) A tetrahedral element $K = \Delta^4P_1P_2P_3P_4$
We introduce three parameters $\alpha$, $\beta$ and $\gamma$ to locate the dual nodes in $K$ such that

$$
\frac{P_{i_1}P_{i_2}}{P_iP_{i_2}} = \alpha \in \left(0, \frac{1}{2}\right), \ i_1, i_2 \in Z_4^{(1)}, \ i_1 \neq i_2,
$$

$$
\frac{P_iP_{j,k}}{P_iM_{j,k}} = \beta \in \left(0, \frac{2}{3}\right), \ i \in Z_4^{(1)}, \ (j,k) \in Z_4^{(2)}, \ i \notin \{j,k\},
$$

$$
\frac{P_iQ_i}{F_i} = \gamma \in \left(0, \frac{3}{4}\right), \ i \in Z_4^{(1)},
$$

(2.2)

where $P_{i_1,i_2}$, $P_{i,j,k}^\beta$ and $Q_i^\gamma$ are the dual nodes on the edges $P_{i_1}P_{i_2}$, the midlines $P_{i}M_{j,k}$ and the central lines $P_iF_i$, respectively (see connecting lines on Figure 1). For every dual node $P_{i,j,k}^\beta$ on the triangular face $T_i \ (\{i,j,k,l\} = Z_4^{(1)})$, connect the following three line segments:

$$
P_{i,j,k}^\beta P_{i,j,k}^\alpha, \ P_{i,j,k}^\beta P_{i,k}, \ P_{i,j,k}^\beta F_i
$$

(see connecting lines on $T_4$ in Figure 1). For every dual node $Q_i^\gamma$ in the interior of $K$, connect the following seven line segments:

$$
Q_i^\gamma P_{i,i_1}^\alpha \ (i_1 \in Z_4^{(1)} \setminus \{i\}), \ Q_i^\gamma P_{i,j,k}^\beta \ (j,k) \in Z_4^{(2)}, \ i \notin \{j,k\}, \ Q_i^\gamma F_i.
$$

Then we obtain a polyhedral partition $\{D^K_P, P \in \mathcal{N}_K\}$ of $K$, where $D^K_P$ is a subdomain of $K$ surrounding $P$ (see $D^K_{P_1}$ surrounding $P_1$ and $D^K_{M_{23}}$ surrounding $M_{23}$ in Figure 2). Denote by

$$
K^\ast_P = \bigcup_{K \in \mathcal{T}_h} D^K_P
$$

the dual element associated with $P \in \mathcal{N}_h$, and

$$
\mathcal{T}_h^* = \{K^\ast_P, P \in \mathcal{N}_h\}
$$

the dual mesh.

The test function space over $\mathcal{T}_h^*$ is defined as

$$
V_h = \{v_h \in L^2(\Omega) : v_h |_{K^\ast_P} = \text{constant}, \ \forall K^\ast_P \in \mathcal{T}_h^*; \ v_h |_{K^\ast_P} = 0, \ \forall P \in \partial \Omega \cap \mathcal{N}_h\}.
$$

Figure 2 (Color online) Two subdomains $D^K_{P_1}$ and $D^K_{M_{23}}$ of the polyhedral partition
The quadratic FVM schemes. The quadratic FVM for (1.1) is to find $u_h \in U_h$ such that

$$a_h(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h,$$

(2.3)

where

$$a_h(u_h, v_h) = - \sum_{K^* \in T_h^*} \int_{\partial K^*} (\kappa \nabla u_h) \cdot nv_h dS, \quad (f, v_h) = \sum_{K^* \in T_h^*} \int_{K^*} f v_h dx_1 dx_2 dx_3.$$

(2.4)

Here, $n$ is the unit outer normal vector of $\partial K^*$.

Remark 2.1. The dual mesh $T_h^*$ depends on the three parameters $(\alpha, \beta, \gamma)$, so the equation (2.3) actually leads to a family of quadratic FVM schemes.

2.2 Volume coordinates

We present the volume coordinates related with a tetrahedron. Let $K = \Delta^4 P_1 P_2 P_3 P_4$ be a tetrahedron with vertices $P_i = (x_{i1}, x_{i2}, x_{i3}, x_{i4})$ for $i \in \mathbb{Z}_4^{(1)}$ (see Figure 1). If these vertices are not coplanar, $K$ has the positive volume

$$|K| = \frac{1}{6} \begin{vmatrix} 1 & x_{11} & x_{21} & x_{31} \\ 1 & x_{12} & x_{22} & x_{32} \\ 1 & x_{13} & x_{23} & x_{33} \\ 1 & x_{14} & x_{24} & x_{34} \end{vmatrix}.$$  

(2.5)

Let $K_i$ ($i \in \mathbb{Z}_4^{(1)}$) be the tetrahedrons subtended at $P = (x_1, x_2, x_3)$ by the faces of $K$. Their volumes are $|K_i|$ ($i \in \mathbb{Z}_4^{(1)}$), where $|K_i|$ is obtained from $|K|$ by replacing the elements $1, x_{i1}, x_{i2}, x_{i3}$ by $1, x_1, x_2, x_3$, respectively. The volume coordinates $L_i$ ($i \in \mathbb{Z}_4^{(1)}$) are defined by the volume-ratios

$$L_i = \frac{|K_i|}{|K|}, \quad i \in \mathbb{Z}_4^{(1)}.$$  

(2.6)

We have relations among $(L_1, L_2, L_3, L_4)$ and $(x_1, x_2, x_3)$ as follows:

$$\begin{align*}
x_1 &= x_{11} L_1 + x_{12} L_2 + x_{13} L_3 + x_{14} L_4, \\
x_2 &= x_{21} L_1 + x_{22} L_2 + x_{23} L_3 + x_{24} L_4, \\
x_3 &= x_{31} L_1 + x_{32} L_2 + x_{33} L_3 + x_{34} L_4, \\
L_1 + L_2 + L_3 + L_4 &= 1,
\end{align*}$$

(2.7)

which transform the reference element $\hat{K} = \{(L_1, L_2, L_3) \mid L_1 \geq 0, L_2 \geq 0, L_3 \geq 0, L_1 + L_2 + L_3 \leq 1\}$ into any tetrahedral element $K = \Delta^4 P_1 P_2 P_3 P_4$.

A direct calculation of (2.5) yields

$$\nabla L_i = \left(\frac{\partial L_i}{\partial x_1}, \frac{\partial L_i}{\partial x_2}, \frac{\partial L_i}{\partial x_3}\right)^\top = -\frac{|T_i|}{3|K|} n_i, \quad i \in \mathbb{Z}_4^{(1)},$$

(2.8)

where $n_i$ and $|T_i|$ are the unit outer normal vector and the area of $T_i$, respectively. Let $\theta_{jk}$ be the dihedral angle associated with the edge $P_j P_k$ ($(j, k) \in \mathbb{Z}_4^{(2)}$) in $K$, and

$$r_{jk} = |\overrightarrow{P_j P_k}| \cot \theta_{jk}, \quad \forall (j, k) \in \mathbb{Z}_4^{(2)}, \quad R_i = \sum_{(j, k) \in \mathbb{Z}_4^{(2)}, j \neq \{j, k\}} r_{jk}, \quad \forall i \in \mathbb{Z}_4^{(1)}.$$  

(2.9)

Then we have Lemma 2.2.
Lemma 2.2.  For the volume coordinates $L_i$ ($i \in \mathbb{Z}_4^{(1)}$) given by (2.5), it holds that
\[
\begin{aligned}
6|K| (\nabla L_{j_1} \cdot \nabla L_{k_1}) &= -r_{j_2 k_2}, \\
6|K| (\nabla L_{j_2} \cdot \nabla L_{k_2}) &= 0,
\end{aligned}
\]
where $\{j_1, k_1, j_2, k_2\} \in \mathbb{Z}_4^{(2)}$.

Proof. By (2.8) and $n_{j_1} \cdot n_{k_1} = -\cos \theta_{j_2 k_2}$, we have
\[
6|K| (\nabla L_{j_1} \cdot \nabla L_{k_1}) = -\frac{2|T_j| |T_k|}{3|K|} \cos \theta_{j_2 k_2}.
\]
Then the first relation follows from the volume formula that
\[
3|K| = (2|T_j| |T_k| \sin \theta_{j_2 k_2}) / |P_{j_2 k_2}|.
\]
Combining the fact that $\nabla (L_1 + L_2 + L_3 + L_4) = 0$, we have the second relation. \hfill \Box

2.3 The mapping from the trial space to the test space

We define a transform operator from the trial space to the test space, which is meaningful in the theoretical analysis of the quadratic FVM schemes, especially for stability analysis.

Definition 2.3. For $\lambda \in \mathbb{R}$ and $\lambda \neq 0$, we define a transform operator $\Pi^*_\lambda$ from the trial space $U_h$ to the test space $V_h$ such that for any $u_h \in U_h$,
\[
\begin{aligned}
(\Pi^*_\lambda u_h)(P) &= u_h(P), \\
(\Pi^*_\lambda u_h)(M) &= \frac{1 - \lambda}{2} (u_h(P_M^1) + u_h(P_M^2)) + \lambda u_h(M),
\end{aligned}
\]
where $P \in \mathcal{N}_h$ is the vertex, and $M \in \mathcal{N}_h$ is the midpoint of the edge $P_M^1 P_M^2$.

Remark 2.4. The mapping $\Pi^*_\lambda$ is proposed only for the theoretical analysis of the quadratic FVM schemes, and it has no effect on practical computations of these schemes, while taking $\lambda = 1$, $\Pi^*_1$ is the traditional mapping $\Pi^*_0$ (see [20]).

For simplicity, we write the midpoints of the six edges in $K = \Delta^4 P_1 P_2 P_3 P_4$ (see Figure 1) as
\[
P_5 = M_{23}, \quad P_6 = M_{13}, \quad P_7 = M_{12}, \quad P_8 = M_{14}, \quad P_9 = M_{24}, \quad P_{10} = M_{34}.
\]
For each nodal point $P_i$ ($i \in \mathbb{Z}_4^{(1)}$), denote by $\phi_{P_i}$ or $\phi_i$ the corresponding local quadratic Lagrange basis function, and $\chi_i$ the corresponding local characteristic function of the dual element $K^*_P$. Let
\[
\Phi = (\phi_1, \phi_2, \ldots, \phi_{10})^T, \quad A = (\chi_1, \chi_2, \ldots, \chi_{10})^T
\]
and
\[
\mathbf{S} = \begin{pmatrix}
1 & 0 & 0 & 0 & \frac{1-\lambda}{2} & \frac{1-\lambda}{2} & 0 & 0 \\
0 & 1 & 0 & \frac{1-\lambda}{2} & 0 & \frac{1-\lambda}{2} & 0 & 0 \\
0 & 0 & 1 & \frac{1-\lambda}{2} & 0 & 0 & 0 & \frac{1-\lambda}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda
\end{pmatrix}.
\]
(2.10)

According to Definition 2.3, Lemma 2.5 shows a relation between $\Pi^*_\lambda$ and $\Pi^*_1$ associated with $\mathbf{S}$. 
Lemma 2.7. For the mapping $\Pi_h^*$ restricted on any tetrahedral element $K$, it holds that

$$\Pi_h^* \Phi = SA = S \Pi_h^* \Phi.$$  \hfill (2.11)

Proof. We have the following forms of the piecewise quadratic function $u_h$ and the piecewise constant function $\Pi_h^*u_h$ on $K$, i.e.,

$$u_h = u_K^T \Phi, \quad \Pi_h^* u_h = \tilde{u}_K^T A,$$

where $u_K = (u_1, u_2, \ldots, u_{10})^T$ with $u_i = u_h(P_i)$, and $\tilde{u}_K = (\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_{10})^T$ with $\tilde{u}_i = (\Pi_h^* u_h)(P_i)$ for $i \in \mathcal{Z}_{10}$. By Definition 2.3, it is observed that

$$\tilde{u}_K^T = u_K^T S.$$

Then we have

$$u_K^T \Pi_h^* \Phi = \Pi_h^* u_h = u_K^T S A,$$

which indicates $\Pi_h^* \Phi = S A$. Taking $\lambda = 1$, we obtain $\Pi_h^* \Phi = A$, and this completes the proof. \hfill $\square$

2.4 The orthogonal conditions

The orthogonal conditions proposed for the FVM schemes on triangular meshes [31] are used to prove the optimal $L^2$ error estimate. Here, we propose the orthogonal conditions on the surface and the volume for the quadratic FVM schemes on tetrahedral meshes. The orthogonal condition on the surface is also helpful to stability analysis.

Definition 2.6 (Orthogonal conditions). A quadratic FVM scheme or the corresponding dual mesh $\mathcal{T}^*_h$ is called to satisfy the orthogonal condition on the surface if the following equation associated with the mapping $\Pi_h^*$ holds:

$$\int_{T_i} g_1(v_1 - \Pi_h^* v_1) dS = 0, \quad \forall g_1, v_1 \in P^1(T_i), \quad T_i \in \partial K, \quad K \in \mathcal{T}_h,$$  \hfill (2.12)

and it is called to satisfy the orthogonal condition on the volume if the following equation associated with $\Pi_h^*$ holds:

$$\int_{K} g_2(v_2 - \Pi_h^* v_2) dx_1 dx_2 dx_3 = 0, \quad \forall g_2, v_2 \in P^1(K), \quad K \in \mathcal{T}_h.$$  \hfill (2.13)

Here, $P^1$ is the linear function space.

Lemma 2.7. The orthogonal condition on the surface (2.12) is equivalent to the parameter equation

$$\alpha \beta \left( -\frac{1}{2} + \frac{1}{3} \alpha + \frac{1}{4} \beta \right) + \frac{1}{54} = 0,$$  \hfill (2.14)

and the orthogonal condition on the volume (2.13) is equivalent to the parameter equation

$$\alpha \beta \gamma \left( -\frac{1}{2} \alpha + \frac{3}{8} \beta + \frac{1}{3} \gamma \right) + \frac{1}{480} = 0.$$  \hfill (2.15)

Proof. Firstly, consider the equation (2.12) on the reference element $\hat{K}$ (see (2.7)). It is equivalent to solve

$$\int_{\hat{T}_4} L_1^n (1 - \Pi_h^* 1) dL_1 dL_2 = 0, \quad n = 0, 1.$$  \hfill (2.16)

Since $1 = \hat{\phi}_1 + \hat{\phi}_2 + \hat{\phi}_3 + \hat{\phi}_5 + \hat{\phi}_6 + \hat{\phi}_7$ and $L_1 = \hat{\phi}_1 + (\hat{\phi}_6 + \hat{\phi}_7)/2$ holds on $\hat{T}_4$, by Lemma 2.5, it is easy to verify that

$$\int_{\hat{T}_4} (1 - \Pi_h^* 1) dL_1 dL_2 = 0.$$
and
\[ \iint_{T_4} (L_1 - \Pi^*_1 L_1) dL_1 dL_2 = 0. \]

In addition, we have the following integral results:
\[
\begin{align*}
t_1 &:= \iint_{T_4} L_1 \chi_1 dL_1 dL_2 = \frac{\alpha \beta}{2} \left( 1 - \frac{\alpha + \beta}{3} \right), \\
t_2 &:= \iint_{T_4} L_1 \chi_{2,3} dL_1 dL_2 = \frac{\alpha \beta (\alpha + \beta)}{12}, \\
t_3 &:= \iint_{T_4} L_1 \chi_{6,7} dL_1 dL_2 = \frac{2}{27} - \frac{\alpha \beta}{4} \left( 1 - \frac{\beta}{6} \right), \\
t_4 &:= \iint_{T_4} L_1 \chi_5 dL_1 dL_2 = \frac{1}{54} - \frac{\alpha \beta^2}{12},
\end{align*}
\]
(2.17)

where \( \chi_i \) is the local characteristic function of the dual element \( K^*_i \), and \( \chi_i, j \) means \( \chi_i \) or \( \chi_j \). Then using the above integral results yields
\[ \iint_{T_4} L_1 (1 - \Pi^*_1 L_1) dL_1 dL_2 = 0 \]
and
\[ \iint_{T_4} L_1 (L_1 - \Pi^*_1 L_1) dL_1 dL_2 = \iint_{T_4} L_1^2 dL_1 dL_2 - \iint_{T_4} L_1 \left( \chi_1 + \frac{\chi_6}{2} + \frac{\chi_7}{2} \right) dL_1 dL_2 = \frac{1}{12} - t_1 - t_3. \]

Thus solving (2.16) yields \( 1/12 - t_1 - t_3 = 0 \), which leads to (2.14). Noticing
\[ t_1 + 2t_2 + 2t_3 + t_4 = \iint_{T_4} L_1 (\chi_1 + \chi_2 + \chi_3 + \chi_5 + \chi_6 + \chi_7) dL_1 dL_2 = \frac{1}{6}, \]
we see that the parameter equation (2.14) can also be written as
\[ -t_1 + 2t_2 + t_4 = 0. \]
(2.18)

On the other hand, the equation (2.13) is equivalent to
\[
\begin{align*}
\iiint_{\hat{K}} L_1^n (1 - \Pi^*_1 L_1) dL_1 dL_2 dL_3 &= 0, \\
\iiint_{\hat{K}} L_1^n (L_1 - \Pi^*_1 L_1) dL_1 dL_2 dL_3 &= 0,
\end{align*}
\]
\( n = 0, 1. \)

Similarly, (2.15) can be derived.

**Remark 2.8.** Note that \( \lambda \) does not appear in the parameter equations (2.14) and (2.15). This means that the orthogonal conditions associated with \( \Pi^*_1 \) and \( \Pi^*_7 \) lead to the same parameter equations. In fact, \( \Pi^*_1 L_1 = \Pi^*_7 L_1 (i \in Z^1_4) \) for any given \( \lambda \in \mathbb{R} \) and \( \lambda \neq 0 \).

**Lemma 2.9.** The parameter equations (2.14) and (2.15) have infinite solutions. In addition, for any given \( \alpha \) in the range
\[ \frac{1}{2} - \frac{\sqrt{6}}{6} \approx 0.091752 < \alpha < \frac{1}{2}, \]
(2.19)
there is a unique solution for (2.14) and (2.15) as follows:
\[
\begin{align*}
\beta &= \left( 1 - \frac{2}{3} \alpha \right) - \sqrt{\left( 1 - \frac{2}{3} \alpha \right)^2 - \frac{2}{27 \alpha}}, \\
\gamma &= \left( \frac{3}{8} + \frac{1}{24 \alpha \beta} \right) - \sqrt{\left( \frac{3}{8} + \frac{1}{24 \alpha \beta} \right)^2 - \frac{1}{160 \alpha \beta}}.
\end{align*}
\]
Proof. We start by analyzing the parameter equation (2.14) for \( \alpha \in (0, 1/2) \) and \( \beta \in (0, 2/3) \) separately. Considering (2.14) as an equation for \( \beta \), we see that the two roots \( \beta_1 \) and \( \beta_2 \) \((\beta_1 \leq \beta_2)\) satisfy \( \beta_1 + \beta_2 = 2 - (4\alpha)/3 > 4/3 \). Thus, the reasonable root is

\[
\beta = \beta_1(\alpha) = \frac{1}{2} \alpha - \frac{1}{3} \alpha^2 - \sqrt{\left(\frac{1}{2} \alpha - \frac{1}{3} \alpha^2\right)^2 - \frac{1}{2} \alpha^3}.
\]  

(2.20)

Since \( 0 < \beta_1(\alpha) < 2/3 \), a direct calculation of this inequality about \( \alpha \) yields the range (2.19) of \( \alpha \).

Similarly, considering (2.14) as an equation for \( \alpha \), we have

\[
\frac{2}{3} - \frac{2\sqrt{6}}{9} (\approx 0.122336) < \beta < \frac{2}{3}.
\]  

(2.21)

Actually, every reasonable solution of (2.14) exactly meets (2.19) and (2.21). Moreover, the equation (2.20) indicates that \( \alpha \beta \) can be represented by \( \alpha \) as follows:

\[
\alpha \beta = \left(\alpha - \frac{2}{3} \alpha^2\right) - \sqrt{\left(\alpha - \frac{2}{3} \alpha^2\right)^2 - \frac{2}{27} \alpha^3}.
\]

(2.22)

By (2.19), we compute to arrive at the range of \( \alpha \beta \) that

\[
0.049913491374002 \leq \alpha \beta < \frac{1}{3} - \frac{\sqrt{6}}{9} (\approx 0.061168).
\]  

(2.23)

Then considering the parameter equation (2.15) as an equation for \( \gamma \in (0, 3/4) \), we have that the two roots \( \gamma_1 \) and \( \gamma_2 \) \((\gamma_1 \leq \gamma_2)\) satisfy \( \gamma_1 + \gamma_2 = 3 - (3\alpha)/2 - (9\beta)/8 > 3/2 \). Using (2.14) to simplify (2.15), we obtain

\[
\left(\frac{1}{3} \alpha \beta\right) \gamma^2 - \left(\frac{1}{4} \alpha \beta + \frac{1}{36}\right) \gamma + \frac{1}{480} = 0.
\]

Thus, the reasonable root is

\[
\gamma = \gamma_1(\alpha \beta) = \frac{\left(\frac{1}{4} \alpha \beta + \frac{1}{36}\right) - \sqrt{\left(\frac{1}{4} \alpha \beta + \frac{1}{36}\right)^2 - \frac{1}{2\alpha \beta} \alpha \beta}}{\frac{1}{2} \alpha \beta}.
\]  

(2.24)

Combining the range (2.22) of \( \alpha \beta \), we have

\[
\frac{30 + 5\sqrt{6} - \sqrt{960 + 270\sqrt{6}}}{40} (\approx 0.049533) < \gamma \leq 0.052908895445995,
\]

which lies in \((0, 3/4)\).

The above discussions show that for any given \( \alpha \) in (2.19), the parameters \( \beta \) (see (2.20)) and \( \gamma \) (see (2.23)) are uniquely obtained from the equations (2.14) and (2.15). Simplifying (2.20) and (2.23) completes the proof. \( \square \)

Remark 2.10. The quadratic FVM schemes satisfying the orthogonal conditions on tetrahedral meshes are infinite, and this is different from the case of the unique one on the triangular meshes [31].

3 Stability analysis

This section is devoted to the analysis for the stability of the quadratic FVM schemes (2.3). Assume the diffusion coefficient \( \kappa = 1 \). Our goal is to prove the local stability that for any \( K \in \mathcal{T}_h \),

\[
a^K_h(u_h, \Pi^*_h u_h) := a_h(u_h, \Pi^*_h u_h)|_K \geq |u_h|^2_{1, K}, \quad \forall u_h \in U_h.
\]

(3.1)

In Subsection 3.1, the local stability (3.1) is converted to a positive definiteness of a \( 9 \times 9 \) symbolic matrix based on an equivalent discrete norm. In Subsection 3.2, under the orthogonal condition on the
surface (2.18), firstly, for any regular tetrahedron \( K \in T_h \), it is proved that the \( 9 \times 9 \) symbolic matrix is positive definite for the given parameter \( \lambda \) in a certain range; secondly, for any general tetrahedron \( K \in T_h \), by the congruent transformation, the \( 9 \times 9 \) matrix is reduced to a block diagonal matrix containing a \( 3 \times 3 \) matrix and a \( 6 \times 6 \) matrix, where the \( 3 \times 3 \) matrix is proved to be unconditionally positive definite. In Subsection 3.3, we derive that for fixed parameters \((\alpha, \beta, \gamma, \lambda)\), the \( 6 \times 6 \) symbolic matrix only relies on five certain plane angles of a tetrahedral element. Then the minimum V-angle condition (3.25) is proposed to ensure the positive definiteness of the \( 6 \times 6 \) matrix numerically. Under the two restrictions (2.18) and (3.25), the stability is presented at the end of this section.

3.1 The element matrices

The \( 10 \times 10 \) element stiffness matrix and an equivalent discrete norm of \(|u_h|_{1,h}^2\) are presented. Then the local stability (3.1) is converted to a positive definiteness of a \( 9 \times 9 \) symbolic matrix.

According to the equation (2.4), we have the bilinear form on any element \( K \in T_h \) that

\[
a_h^K(u_h, \Pi_h^*u_h) = - \sum_{K^* \in T_h} \int_{\partial K^* \cap K} \nabla u_h \cdot n \Pi_h^*u_h dS.
\]

By Lemma 2.5, we have

\[
a_h^K(u_h, \Pi_h^*u_h) = a_h^K(u_h^\top \Phi, \Pi_h^*u_h^\top \Phi) = u_h^\top \mathcal{A}_{K, \lambda} u_K,
\]

where the element stiffness matrix \( \mathcal{A}_{K, \lambda} \) equals \( (a_{mn, \lambda})_{m,n \in \mathbb{Z}_{10}^1} \) with \( a_{mn, \lambda} = a_h^K(\phi_n, \Pi_h^*\phi_m) \), and

\[
\mathcal{A}_{K, \lambda} = S \mathcal{A}_{K, 1}, \tag{3.2}
\]

where the element matrix \( \mathcal{A}_{K, 1} \) equals \( (a_{mn}^1)_{m,n \in \mathbb{Z}_{10}^1} \) with

\[
a_{mn}^1 = a_h^K(\phi_n, \chi_m) = - \int_{\partial K_{mn}^* \cap K} \nabla \phi_n \cdot ndS.
\]

By Green’s formula, one obtains

\[
- \int_{\partial K_{mn}^* \cap K} \nabla \phi_n \cdot ndS = \int_{\partial K \cap K_{mn}^*} \nabla \phi_n \cdot ndS - \int_{K_{mn}^* \cap K} \Delta \phi_n dx_1 dx_2 dx_3, \quad \forall m,n \in \mathbb{Z}_{10}^1. \tag{3.3}
\]

Since \( \Delta \phi_n \ (n \in \mathbb{Z}_{10}^1 \) are constants, we split the element matrix \( \mathcal{A}_{K, 1} \) into two parts

\[
\mathcal{A}_{K, 1} = A - v_1 v_2^\top, \tag{3.4}
\]

where \( A \) is a \( 10 \times 10 \) matrix, and \( v_1 \) and \( v_2 \) are two column vectors. They are given by

\[
A = \left( \frac{1}{[K]} \int_{K_{mn}^* \cap K} dx_1 dx_2 dx_3 \right)_{m,n \in \mathbb{Z}_{10}^1},
\]

\[
v_1 = \left( \frac{1}{6[K]} \int_{K_{mn}^* \cap K} dx_1 dx_2 dx_3 \right)_{m,n \in \mathbb{Z}_{10}^1},
\]

\[
v_2 = \frac{1}{6[K]} \Delta \phi_n_{n \in \mathbb{Z}_{10}^1}.\]

By (2.2) of the parameters \((\alpha, \beta, \gamma)\), we have the explicit form of \( v_1 \) as follows:

\[
v_1 = \left( \begin{array}{cccccccc}
\alpha^2 & \alpha^2 & \alpha^2 & \alpha^2 & 1 - 4\alpha\beta & 1 - 4\alpha\beta & 1 - 4\alpha\beta & 1 - 4\alpha\beta \\
6 & 6 & 6 & 6 & 36 & 36 & 36 & 36
\end{array} \right) \top.
\]

By Lemma 2.2 and the expressions of ten local quadratic Lagrange basis functions

\[
\phi_{P_i} = L_i(2L_i - 1), \quad \forall i \in \mathbb{Z}_{4}^1, \quad \phi_{M_{jk}} = 4L_j L_k, \quad \forall (j,k) \in \mathbb{Z}_{4}^2, \tag{3.5}
\]

\[
\phi_{P_i} = L_i(2L_i - 1), \quad \forall i \in \mathbb{Z}_{4}^1, \quad \phi_{M_{jk}} = 4L_j L_k, \quad \forall (j,k) \in \mathbb{Z}_{4}^2, \tag{3.5}
\]
we have the explicit form of $\mathbf{v}_2$ as follows:

$$\mathbf{v}_2 = 4(R_1, R_2, R_3, R_4, -2r_{14}, -2r_{24}, -2r_{34}, -2r_{23}, -2r_{13}, -2r_{12})^T.$$  

From (3.5) and $L_1 + L_2 + L_3 + L_4 = 1$, we have

$$\nabla \phi_i = \sum_{i_1, i_2 \in \mathcal{Z}(1)} c_{i_1, i_2} L_{i_1} \nabla L_{i_2}$$

with some constants $c_{i_1, i_2}$. In view of this, Lemma 3.1 presents some important equations for deriving $A$.

**Lemma 3.1.** For the vertices $P_i$ ($i \in \mathcal{Z}(1)$) in the element $K$ and $L_{i_1} \nabla L_{i_2}$ ($i_1, i_2 \in \mathcal{Z}(1)$), we have the following integral results:

$$\iint_{\partial K \cap \hat{K}_{i_1}} L_{i_1} \nabla L_{i_2} \cdot n dS = \begin{cases} 6|K|(|\nabla L_{i_1} \cdot \nabla L_{i_2}|) t_1, & i_1 = i, \\ 6|K| \left( \sum_{l \in \{i, i_1\}} \nabla L_l \cdot \nabla L_{i_2} \right) t_2, & i_1 \neq i. \end{cases}$$

For the midpoints $M_{jk}$ ($(j, k) \in \mathcal{Z}(1)$) in the element $K$ and $L_{i_1} \nabla L_{i_2}$ ($i_1, i_2 \in \mathcal{Z}(1)$), we have the following integral results:

$$\iint_{\partial K \cap \hat{K}_{M_{jk}}} L_{i_1} \nabla L_{i_2} \cdot n dS = \begin{cases} 6|K| \left( \sum_{l \in \{j, k\}} \nabla L_l \cdot \nabla L_{i_2} \right) t_3, & i_1 \in \{j, k\}, \\ 6|K| \left( \sum_{l \in \{j, k, i_1\}} \nabla L_l \cdot \nabla L_{i_2} \right) t_4, & i_1 \notin \{j, k\}. \end{cases}$$

Here, $t_i, i \in \mathcal{Z}(1)$ are constants given by (2.17).

**Proof.** Noticing $\partial K \cap K_{P_i}^* = \bigcup_{i_4 \in \mathcal{Z}(1)} (T_{i_4} \cap K_{P_i}^*)$ ($i_3 \in \mathcal{Z}(1)$), we have

$$\iint_{\partial K \cap K_{P_i}^*} L_{i_1} \nabla L_{i_2} \cdot n dS = \sum_{i_4 \in \mathcal{Z}(1)} \iint_{T_{i_4}} L_{i_1} \chi_{i_3} \nabla L_{i_2} \cdot n_{i_4} ds, \quad \forall i_3 \in \mathcal{Z}(1),$$

where $\chi_{i_3}$ is the local characteristic function of the dual element $K_{P_i}^*$. By (2.8), replacing the vector $n_{i_4}$ by $-6|K| / (2|T_{i_4}|) \nabla L_{i_4}$ yields

$$\sum_{i_4 \in \mathcal{Z}(1)} \iint_{T_{i_4}} L_{i_1} \chi_{i_3} \nabla L_{i_2} \cdot n_{i_4} ds = -6|K| \sum_{i_4 \in \mathcal{Z}(1)} \iint_{T_{i_4}} \frac{1}{2|T_{i_4}|} L_{i_1} \chi_{i_3} \nabla L_{i_4} \cdot \nabla L_{i_2} ds$$

$$= -6|K|(|\nabla L_{i_4} \cdot \nabla L_{i_2}|) \sum_{i_4 \in \mathcal{Z}(1)} \iint_{T_{i_4}} L_{i_1} \chi_{i_3} ds.$$  

Combining the facts that $L_4 = 0$ holds on $T_i$ ($i \in \mathcal{Z}(1)$), $\nabla (L_1 + L_2 + L_3 + L_4) = 0$, and the integral results (2.17), one can reach the conclusion of Lemma 3.1. \hfill \Box

By Lemma 3.1, taking $A(1, 1)$ as an example, we have

$$A(1, 1) = \iint_{\partial K \cap \hat{K}_{P_1}} \nabla \phi_{P_1} \cdot n dS$$

$$= \iint_{\partial K \cap \hat{K}_{P_1}} 3L_1 \nabla L_1 \cdot n dS - \iint_{\partial K \cap \hat{K}_{P_1}} (L_2 + L_3 + L_4) \nabla L_1 \cdot n dS$$

$$= 6|K|(|\nabla L_1 \cdot \nabla L_1|) t_1 - 6|K|(|3 \nabla L_1 + \nabla L_2 + \nabla L_3 + \nabla L_4|) \cdot \nabla L_1 t_2$$

$$= (3t_1 - 2t_2) R_1.$$
Other entries are derived similarly, and the explicit form of $A$ is put in Appendix A.1.

Then consider $|u_h|_{1,K}^2$ with $u_h|_K = \sum_{i \in \mathbb{Z}^{(1)}} u_i \phi_i$ and $u_i = u_h(P_i)$ ($i \in \mathbb{Z}_{10}$). By (3.5) and $L_4 = 1 - L_1 - L_2 - L_3$, we have

$$u_h|_K = u_4 + L_1(-u_1 - 3u_4 + 4u_9) + L_2(-u_2 - 3u_4 + 4u_9) + L_3(-u_3 - 3u_4 + 4u_{10})$$
$$+ 2L_1^2(u_1 + u_4 - 2u_8) + 2L_2^2(u_2 + u_4 - 2u_8) + 2L_3^2(u_3 + u_4 - 2u_{10})$$
$$+ 4L_1L_2(u_4 + u_5 - u_9 - u_{10}) + 4L_1L_3(u_4 + u_6 - u_8 - u_{10}) + 4L_2L_3(u_4 + u_7 - u_8 - u_9)$$
$$= u_4 + (L_1, L_2, L_3, L_1^2, L_2^2, L_3^2, L_1L_2, L_1L_3, L_2L_3)Gu_K,$$

where $u_K = (u_1, u_2, \ldots, u_{10})^\top$, and $G$ is a $9 \times 10$ matrix given by

$$G = \begin{pmatrix}
-1 & 0 & 0 & -3 & 0 & 0 & 0 & 4 & 0 & 0 \\
0 & -1 & 0 & -3 & 0 & 0 & 0 & 4 & 0 & 0 \\
0 & 0 & -1 & -3 & 0 & 0 & 0 & 0 & 4 & 0 \\
2 & 0 & 0 & 2 & 0 & 0 & 0 & -4 & 0 & 0 \\
0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & -4 & 0 \\
0 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & -4 & 0 \\
0 & 0 & 0 & 4 & 0 & 0 & 0 & -4 & -4 & 0 \\
0 & 0 & 0 & 4 & 0 & 0 & 0 & -4 & -4 & 0 \\
0 & 0 & 0 & 4 & 0 & 0 & 0 & -4 & -4 & 0
\end{pmatrix}. \tag{3.7}$$

Note that the constant term of $u_h|_K$ vanishes in the derivative of $u_h|_K$. In view of this, Lemma 3.2 presents an equivalent discrete norm of $|u_h|_{1,K}^2$ associated with $G$.

**Lemma 3.2 (Discrete norm).** If $\mathcal{T}_h$ is a regular partition (2.1), then for each $K \in \mathcal{T}_h$, we have

$$|u_h|_{1,K}^2 \sim h_K \|Gu_K\|^2,$$  \tag{3.8}

where $\| \cdot \|$ is the Euclidean norm, and $G$ is given by (3.7).

**Proof.** Under the translation (2.6), let $\hat{u}_h|_K$ be the interpolation function on the reference element $\hat{K}$ (see (2.7)). Since $\mathcal{T}_h$ is a regular partition, the following relation for Sobolev semi-norms holds [20]:

$$|u_h|_{1,K}^2 \sim h_K|\hat{u}_h|_{1,K}^2.$$  \tag{3.9}

By (3.6), a direct calculation of $|\hat{u}_h|_{1,K}^2$ yields

$$|\hat{u}_h|_{1,K}^2 = \iint_K \sum_{i \in \mathbb{Z}^{(1)}} \left(\frac{\partial \hat{u}_h}{\partial L_i}\right)^2 dL_1dL_2dL_3$$
$$= \iint_K [(w_1Gu_K)^\top (w_1Gu_K) + (w_2Gu_K)^\top (w_2Gu_K) + (w_3Gu_K)^\top (w_3Gu_K)]dL_1dL_2dL_3$$
$$= (Gu_K)^\top W(Gu_K),$$

where

$$w_1 = (1, 0, 0, 2L_1, 0, 0, 0, L_3, L_2), \quad w_2 = (0, 1, 0, 0, 2L_2, 0, L_3, 0, L_1), \quad w_3 = (0, 0, 1, 0, 0, 2L_3, L_2, L_1, 0)$$
and

\[
W = \int \int \int_K (w_1^T w_1 + w_2^T w_2 + w_3^T w_3) dL_1 dL_2 dL_3 = \frac{1}{120} \begin{pmatrix}
20 & 0 & 0 & 0 & 0 & 0 & 5 & 5 \\
0 & 20 & 0 & 0 & 0 & 0 & 5 & 0 \\
0 & 0 & 20 & 0 & 0 & 0 & 5 & 0 \\
10 & 0 & 0 & 8 & 0 & 0 & 2 & 2 \\
0 & 10 & 0 & 0 & 8 & 0 & 2 & 0 \\
0 & 0 & 10 & 0 & 0 & 8 & 2 & 0 \\
0 & 5 & 5 & 0 & 2 & 2 & 4 & 1 \\
5 & 0 & 5 & 2 & 0 & 2 & 1 & 4 \\
5 & 5 & 0 & 2 & 2 & 0 & 1 & 1
\end{pmatrix}.
\]

It is verified that the real symmetric matrix \( W \) is positive definite. Therefore, \( |\hat{u}_{h_{1,K}}|^2 \sim \|G u_K\|^2 \), which together with (3.9) completes the proof.

Let \( T \) be the unique Moore-Penrose inverse of \( G \) such that \( T G T = T, G T G = G, (T G)^T = T G \) and \( (G T)^T = G T \). By computing in MATLAB, we obtain the \( 10 \times 9 \) matrix

\[
T = \frac{1}{40} \begin{pmatrix}
30 & -10 & -10 & 33 & -7 & -7 & -1 & -1 \\
-10 & 30 & -10 & -7 & 33 & -7 & -1 & -1 \\
-10 & -10 & 30 & -7 & -7 & 33 & -1 & -1 \\
-10 & -10 & -10 & -7 & -7 & -7 & -1 & -1 \\
-10 & 10 & 10 & -7 & 3 & 3 & 9 & -1 \\
10 & -10 & 10 & 3 & -7 & 3 & -1 & 9 \\
10 & 10 & -10 & 3 & 3 & -7 & -1 & 9 \\
10 & -10 & -10 & 3 & -7 & -7 & -1 & -1 \\
-10 & 10 & -10 & -7 & 3 & -7 & -1 & -1
\end{pmatrix}, \quad (3.10)
\]

and

\[
T G = E_{10} - \frac{1}{10} \mathbb{1}, \quad (3.11)
\]

where \( E_n \) is the \( n \times n \) identity matrix, and \( \mathbb{1} \) is the \( 10 \times 10 \) matrix in which all the entries are 1. The following Lemma 3.3 helps us to simplify the local stability (3.1).

**Lemma 3.3.** For the element stiffness matrix \( A_{K,\lambda} \) in (3.2), we have

\[
A_{K,\lambda} = G^T T^T A_{K,\lambda} T G,
\]

where \( G \) and \( T \) are given by (3.7) and (3.10), respectively.

**Proof.** Obviously, \( \mathbb{1} S = \mathbb{1} \) and \( A_{K,\lambda} \mathbb{1} = A_{K,\lambda} = 0, \) where \( 0 \) is the \( n \times n \) zero matrix. Then we derive \( A_{K,\lambda} \mathbb{1} = A_{K,\lambda} = 0 \) from (3.2). By (3.11), we obtain

\[
(T G)^T A_{K,\lambda} T G = \left(E_{10} - \frac{1}{10} \mathbb{1}\right)^T A_{K,\lambda} \left(E_{10} - \frac{1}{10} \mathbb{1}\right) = A_{K,\lambda} \left(E_{10} - \frac{1}{10} \mathbb{1}\right) = A_{K,\lambda}.
\]

This completes the proof.

Let \( E_{K,\lambda} \) be a \( 9 \times 9 \) symbolic matrix, given by

\[
E_{K,\lambda} = T^T A_{K,\lambda} T, \quad (3.12)
\]

By Lemma 3.3, we have

\[
\alpha_{h_{1,K}}(u_h, \Pi^h_{\lambda} u_h) = \frac{1}{2} u_{K}^T (A_{K,\lambda} + A_{K,\lambda}^T) u_K = \frac{1}{2} u_{K}^T (G^T E_{K,\lambda} G + G^T E_{K,\lambda}^T G) u_K = (G u_K)^T E_{K,\lambda} (G u_K),
\]
where $\mathbb{F}_{K,\lambda}$ is the symmetrization of $\mathbb{B}_{K,\lambda}$, i.e.,
\[ \mathbb{F}_{K,\lambda} = \frac{\mathbb{B}_{K,\lambda} + \mathbb{B}_{K,\lambda}^T}{2}. \tag{3.13} \]

Recalling the equivalent discrete norm (3.8), we see that the local stability (3.1) is equivalent to
\[ (G_K)^\top \mathbb{F}_{K,\lambda} (G_K) \geq h_K (G_K)^\top (G_K). \]

Therefore, the local stability (3.1) is converted into a positive definiteness of the matrix $\frac{1}{h_K} \mathbb{F}_{K,\lambda}$, which is proved in the following subsections.

### 3.2 A restriction on $T_h^+$ and some results

In this subsection, under the orthogonal condition on the surface (2.18), we prove a positive definiteness of $\frac{1}{h_K} \mathbb{F}_{K,\lambda}$ for any regular tetrahedron $K \in T_h$, and reduce $\frac{1}{h_K} \mathbb{F}_{K,\lambda}$ for any general tetrahedron $K \in T_h$.

Firstly, we try to deal with $\mathbb{B}_{K,\lambda}$ in (3.12). By (3.2) and (3.4), one obtains
\[ \mathbb{B}_{K,\lambda} = T^\top s_{K,\lambda} T = T^\top S_{K,\lambda} T = T^\top SAT - (T^\top s_T)(v_T^\top T). \tag{3.14} \]

By computing in MATLAB, we have
\[ T^\top s_T = s_0 (0, 0, 0, 3, 3, -1, -1, -1)^\top, \]
\[ v_T^\top T = (0, 0, 0, 2R_1, 2R_2, 2R_3, -2r_{14}, -2r_{24}, -2r_{34}), \tag{3.15} \]
and organize $T^\top SAT$ into a $3 \times 3$ block matrix as follows:
\[ T^\top SAT = \begin{pmatrix} s_0 M_K^{3 \times 3} & \frac{s_0}{2} M_K^{3 \times 3} + \frac{s_0}{2} Q_K^{(1)} & \frac{s_0}{4} (M_K^{3 \times 3} C_{3 \times 3} + \frac{s_0}{4} Q_K^{(2)}) \\ \frac{8s_0}{144} (C_{3 \times 3} M_K^{3 \times 3}) & L_{3 \times 3}^{(1)} & L_{3 \times 3}^{(2)} \\ (2s_0 + s_1) (C_{3 \times 3} M_K^{3 \times 3}) & L_{3 \times 3}^{(3)} & L_{3 \times 3}^{(4)} \end{pmatrix}. \tag{3.16} \]

Here,
\[ s_0 = \frac{1}{240} + \frac{4 \alpha \beta \gamma - 1}{144}, \quad s_1 = t_1 + 2t_2 + 2t_3 + t_4 = \frac{1}{6}, \quad s_2 = t_3 \lambda, \quad s_3 = t_4 \lambda, \]
\[ s^* = -t_1 + 2t_2 + t_4 \text{ and } s^* = 0 \text{ coincides with the orthogonal condition on the surface (2.18), and} \]
\[ M_K^{3 \times 3} = \begin{pmatrix} R_1 & -r_{24} & -r_{24} \\ -r_{34} & R_2 & -r_{14} \\ -r_{24} & -r_{14} & R_3 \end{pmatrix}, \quad C_{3 \times 3} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}. \tag{3.18} \]

We put those symbolic matrices $Q_K^{(1)}, Q_K^{(2)}, L_{3 \times 3}^{(1)}, L_{3 \times 3}^{(2)}, L_{3 \times 3}^{(3)}$ and $L_{3 \times 3}^{(4)}$ in Appendix A.2. Lemma 3.4 shows a property of $M_K^{3 \times 3}$, and the proof is included in Appendix B.2.

**Lemma 3.4.** If $T_h$ is a regular partition, then $\frac{1}{h_K} M_K^{3 \times 3}$ is unconditionally positive definite for any tetrahedral element $K \in T_h$.

Now, we consider $\frac{1}{h_K} \mathbb{F}_{K,\lambda}$ for any regular tetrahedron $K \in T_h$. The following result shows that $\frac{1}{h_K} \mathbb{F}_{K,\lambda}$ is positive definite for a given $\lambda$ in a certain range.

**Lemma 3.5.** Assume that $K$ is a regular tetrahedron. If the orthogonal condition on the surface (2.18) (i.e., $s^* = 0$) holds, then the element matrix $\frac{1}{h_K} \mathbb{F}_{K,\lambda}$ in (3.13) is positive definite if and only if $\lambda$ satisfies
\[ \frac{(2t_3 + 5t_4) - 2 \sqrt{2t_1(2t_3 + 3t_4)}}{6(2t_3 + t_4)^2} < \lambda < \frac{(2t_3 + 5t_4) + 2 \sqrt{2t_1(2t_3 + 3t_4)}}{6(2t_3 + t_4)^2}, \tag{3.19} \]
where $t_3$ and $t_4$ are constants in (2.17).
Proof. It is observed from (3.13) and (3.14) that every entry of \( \frac{1}{h_K} \mathcal{E}_{K, \lambda} \) is a linear combination of \( \frac{\nu_{jk}}{h_K} \) for \((j, k) \in \mathcal{Z}_4^{(2)}\). By (2.9), since \(K\) is a regular tetrahedron, we obtain
\[
\frac{r_{jk}}{h_K} = \frac{P_j P_k}{h_K} \cot \theta_{jk} = c, \quad \forall (j, k) \in \mathcal{Z}_4^{(2)},
\]
where \(c\) is a positive constant.

Choose an invertible \(9 \times 9\) matrix
\[
C_1 = \begin{pmatrix}
E_3 & 0_3 & 0_3 \\
0_3 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & -2 & 2 & -1 & 1 & 0 \\
0 & 2 & 0 & 0 & 0 & -1 \end{pmatrix}
\]
such that the first eight entries of \(C_1^\top T^\top S v_1\) and \(v_2^\top T C_1\) vanish, and then the parameter \(\gamma\) only appears in the corner of \(C_1^\top T^\top \mathcal{E}_{K, \lambda} C_1\). According to Lemma 3.4, substituting \(s_1 = 1/6\) and \(s^* = 0\) into (3.16), we compute the determinants that
\[
det \left( C_1^\top \frac{1}{h_K} \mathcal{E}_{K, \lambda} C_1 (1 : 4, 1 : 4) \right) = c^4 \varphi_1(s_2, s_3),
\]
\[
det \left( C_1^\top \frac{1}{h_K} \mathcal{E}_{K, \lambda} C_1 (1 : 5, 1 : 5) \right) = 8c^5 (s_2 - s_3) \varphi_1(s_2, s_3),
\]
\[
det \left( C_1^\top \frac{1}{h_K} \mathcal{E}_{K, \lambda} C_1 (1 : 6, 1 : 6) \right) = 81 \frac{c^6}{4} (s_2 - s_3)^2 \varphi_1(s_2, s_3),
\]
\[
det \left( C_1^\top \frac{1}{h_K} \mathcal{E}_{K, \lambda} C_1 (1 : 7, 1 : 7) \right) = \frac{243c^7}{8} (s_2 - s_3)^2 \varphi_1^2(s_2, s_3),
\]
\[
det \left( C_1^\top \frac{1}{h_K} \mathcal{E}_{K, \lambda} C_1 (1 : 8, 1 : 8) \right) = \frac{2187c^8}{32} (s_2 - s_3)^2 \varphi_1^3(s_2, s_3),
\]
\[
det \left( C_1^\top \frac{1}{h_K} \mathcal{E}_{K, \lambda} C_1 (1 : 9, 1 : 9) \right) = \frac{729c^9}{1280} (s_2 - s_3)^2 \varphi_1^3(s_2, s_3) \varphi_2(s_0, s_2, s_3),
\]
where \(s_0, s_2\) and \(s_3\) are given by (3.17), and
\[
\varphi_1(s_2, s_3) = \frac{2}{27} \left( -3(2s_2 + s_3)^2 + (2s_2 + 5s_3) - \frac{1}{12} \right),
\]
\[
\varphi_2(s_0, s_2, s_3) = -240 s_0 - 20 s_2 - 10 s_3 + 1 = \frac{5}{3} \alpha \beta (3 - 4 \gamma \lambda).
\]

It is clear that \(s_2 - s_3 = (t_3 - t_4) \lambda = (\alpha \beta^2 / 8 - \alpha \beta / 4 + 1/18) \lambda \) and \(\varphi_2(s_0, s_2, s_3)\) have the same signs with \(\lambda\) for any \(\alpha \in (0, 1/2), \beta \in (0, 2/3)\) and \(\gamma \in (0, 3/4)\). Therefore, \(\frac{1}{h_K} \mathcal{E}_{K, \lambda}\) is positive definite if and only if both \(\varphi_1(s_2, s_3)\) and \(\lambda\) are positive. A straight calculation of \(\varphi_1(s_2, s_3) \geq 0\) yields (3.19), which implies \(\lambda > 0\). In fact, since \(2t_4, 2t_3 + 3t_4\) and \(2t_3 + t_4\) are all positive for \(\alpha \in (0, 1/2)\) and \(\beta \in (0, 2/3)\), we have \(2t_4 + 3t_3 > 0\) by considering \(2t_4\) and \(2t_3 + 3t_4\) as values of \(x\) and \(y\) in \(x + y > 2 \sqrt{\lambda} > 0\) (\(y > x > 0\)).

Remark 3.6. \(\lambda = 1/(12t_4 + 6t_3) = 1/(1 - 3\alpha \beta)\) and \(\lambda = 1\) are always in the range (3.19).

For any general tetrahedron \(K \in \mathcal{T}_h\), we complete a reduction of \(\frac{1}{h_K} \mathcal{E}_{K, \lambda}\) under the orthogonal condition on the surface (2.18). Our initial idea is to construct a \(3 \times 3\) block transformation matrix \(C_2\) for the
3 × 3 block target matrix $B_{K,\lambda}$ such that the first block row and the column of $B_{K,\lambda}$ act on the other two block rows and columns. We mainly focus on these changes of the elements in the first block row and the column of $B_{K,\lambda}$. Consider a transformation matrix depending on two parameters $\eta_1$ and $\eta_2$ as follows:

$$C_2 = \begin{pmatrix} E_3 & 0_3 \\ \eta_1 E_3 & E_3 \\ \eta_2 C_{3\times 3} & \end{pmatrix}. $$

By (3.14), the corresponding congruent matrix of $B_{K,\lambda}$ is

$$C_2 B_{K,\lambda} C_2^T = C_2 T^T S A T C_2^T - C_2 T^T S v_1 v_2^T T C_2^T. $$

Noticing the first three elements of $T^T S v_1$ and $v_2^T T$ in (3.15) are all zeros, we have

$$C_2 T^T S v_1 v_2^T T C_2^T = (T^T S v_1)(v_2^T T) = s_0 \begin{pmatrix} 0_3 \\ 0_3 \\ 0_3 \end{pmatrix},$$

where the symbolic matrices $J_{3\times 3}^{(1)}$ and $J_{3\times 3}^{(2)}$ can be found in Appendix A.2. By (3.16), we have

$$C_2 T^T S A T C_2^T = \begin{pmatrix} s_0 M_{3\times 3}^K \\ 0_3 \\ 0_3 \\ \end{pmatrix},$$

$$\begin{pmatrix} 0_3 \\ (1 + \eta_1)s_1 - \frac{2 s_2 + s_3}{3} \end{pmatrix} M_{3\times 3}^K,$$

$$\begin{pmatrix} 0_3 \\ \eta_2 s_2 + \frac{2 s_4 + s_3}{3} \end{pmatrix} (C_{3\times 3} M_{3\times 3}^K).$$

Then by (3.20), we obtain

$$C_2 \frac{1}{h_K} B_{K,\lambda} C_2^T = \frac{1}{2 h_K} C_2 (B_{K,\lambda} + B_{K,\lambda}^T) C_2^T = \frac{1}{h_K} \begin{pmatrix} s_0 M_{3\times 3}^K \\ (1/4) \end{pmatrix},$$

where

$$(*)_1 = \begin{pmatrix} (3/4 + \eta_1) s_1 - \frac{2 s_2 + s_3}{4} M_{3\times 3}^K + \frac{s^*}{4} Q_{3\times 3}^{(1)} \end{pmatrix},$$

$$(*)_2 = \begin{pmatrix} (1/8 + \eta_2) s_2 + \frac{2 s_4 + s_3}{8} (M_{3\times 3} C_{3\times 3} + \frac{s^*}{8} Q_{3\times 3}^{(2)} \end{pmatrix},$$

and

$$N_{6\times 6}^{K,\lambda} = \begin{pmatrix} (\tilde{L}_{3\times 3}^{(1)} - 6 s_0 J_{3\times 3}^{(1)}) + (\tilde{L}_{3\times 3}^{(1)} - 6 s_0 J_{3\times 3}^{(1)})^T \\ (\tilde{L}_{3\times 3}^{(2)} + 6 s_0 J_{3\times 3}^{(2)}) + (\tilde{L}_{3\times 3}^{(2)} + 6 s_0 J_{3\times 3}^{(2)})^T \\ (\tilde{L}_{3\times 3}^{(4)} - 2 s_0 J_{3\times 3}^{(2)}) + (\tilde{L}_{3\times 3}^{(4)} - 2 s_0 J_{3\times 3}^{(2)})^T \end{pmatrix}. $$

(3.21)
Thanks to the orthogonal condition on the surface (2.18) which means \( s^* = 0 \), and meanwhile, choosing parameters \( \eta_1 \) and \( \eta_2 \) as follows:

\[
\eta_1 = \frac{2s_a + s_s}{4s_s^2} - \frac{3}{4}, \quad \eta_2 = -\frac{2s_a + s_s}{8s_s^2} - \frac{1}{8},
\]

then we have \((*)_1 = 0\) and \((*)_2 = 0\). Thus, \( \frac{1}{\eta_1} K_{N_{K,\lambda}} \) is reduced to a block diagonal matrix. According to Lemma 3.4, a positive definiteness of \( \frac{1}{\eta_1} K_{N_{K,\lambda}} \) is reducible to a positive definiteness of \( \frac{1}{\eta_1} N_{K,\lambda} \), which is proved in the next subsection.

### 3.3 A restriction on \( T_h \) and the stability

In this subsection, the minimum V-angle condition on \( T_h \) is proposed to ensure the positive definiteness of \( \frac{1}{\eta_1} N_{K,\lambda} \) numerically, and furthermore, the stability of the quadratic FVM schemes is presented.

For simplicity, write all the plane angles in \( K = \Delta^4 P_1 P_2 P_3 P_4 \) (see Figure 1) as

\[
\begin{aligned}
\theta_{1,P_1} &= \angle P_2 P_1 P_4, \quad \theta_{2,P_1} = \angle P_2 P_1 P_3, \quad \theta_{3,P_1} = \angle P_3 P_1 P_4, \\
\theta_{1,P_2} &= \angle P_1 P_2 P_4, \quad \theta_{2,P_2} = \angle P_1 P_2 P_3, \quad \theta_{3,P_2} = \angle P_3 P_1 P_4, \\
\theta_{1,P_3} &= \angle P_1 P_3 P_4, \quad \theta_{2,P_3} = \angle P_1 P_3 P_2, \quad \theta_{3,P_3} = \angle P_2 P_3 P_4, \\
\theta_{1,P_4} &= \angle P_1 P_4 P_2, \quad \theta_{2,P_4} = \angle P_1 P_4 P_3, \quad \theta_{3,P_4} = \angle P_2 P_4 P_3.
\end{aligned}
\]

(3.22)

Firstly, we define the “V-angle” at the vertex \( P_i \) of \( K \) as

\[
\theta_i = \theta_{1,P_i} + \theta_{2,P_i} + \theta_{3,P_i} - 2 \max \{ \theta_{1,P_i}, \theta_{2,P_i}, \theta_{3,P_i} \}, \quad i \in \mathbb{Z}_4^{(1)}.
\]

(3.23)

Let \( \theta_K := \min \{ \theta_i, i \in \mathbb{Z}_4^{(1)} \} \). The following Lemma 3.7 shows the properties of \( \theta_K \).

**Lemma 3.7.** The minimum V-angle \( \theta_K \) has the following properties:

(i) \( \theta_{1,P_i} + \theta_{2,P_i} + \theta_{3,P_i} \geq \theta_K, \quad \forall i \in \mathbb{Z}_3^{(1)}, \quad i_2 \in \mathbb{Z}_4^{(1)} \);

(ii) if \( \theta_K \leq 60^\circ \), and \( \theta_K = 60^\circ \) means that \( K \) is a regular tetrahedron.

**Proof.** Combining \( \theta_{1,P_i} + \theta_{2,P_i} + \theta_{3,P_i} \leq 2 \max \{ \theta_{1,P_i}, \theta_{2,P_i}, \theta_{3,P_i} \} + \min \{ \theta_{1,P_i}, \theta_{2,P_i}, \theta_{3,P_i} \} \) \( (i \in \mathbb{Z}_4^{(1)} \) and (3.23), we have \( \theta_{1,P_i} \leq \min \{ \theta_{1,P_i}, \theta_{2,P_i}, \theta_{3,P_i} \} \ (i \in \mathbb{Z}_4^{(1)} \), and then the first property follows from

\[
\theta_K = \min \{ \theta_i, i \in \mathbb{Z}_4^{(1)} \}.
\]

It is clear that \( \theta_K > 0^\circ \), and then we prove \( \theta_K \leq 60^\circ \). If the statement is not true, every plane angle of \( K \) is larger than \( 60^\circ \), which contradicts the fact that the sum of the three interior angles of a triangle is \( 180^\circ \). Moreover, if \( \theta_K = 60^\circ \), then all the plane angles in \( K \) are equal to \( 60^\circ \), and this means that \( K \) is a regular tetrahedron.

**Remark 3.8.** The poorly-shaped tetrahedrons classified in [24] possess at least one of the following two types of local shapes around a vertex: (i) In Figure 3(a), the three plane angles around a vertex are all small, the local shape of which is performed as “sharp”. (ii) In Figure 3(b), there exists an edge that is close to the angle (not small) formed by the other two edges around a vertex, the local shape of which is performed as “flat”. The minimum V-angle \( \theta_K \) bounded below ensures the shape regularity of the tetrahedron \( K \) by controlling the local shapes at each vertex. However, if we only restrict the minimum plane angle for each triangular face of a tetrahedron \( K \), similar to [8,33,39,40] for triangular meshes in 2D, then we cannot guarantee the shape regularity of the tetrahedron \( K \). The minimum plane angle of a degenerated tetrahedron \( K \) is \( 45^\circ \), which is big enough in \( (0^\circ, 60^\circ) \) (see Figure 4 as an example).

Let \( \Theta_5 := \{ \theta_{1,P_1}, \theta_{2,P_1}, \theta_{1,P_2}, \theta_{2,P_2}, \theta_{3,P_1}, \theta_{3,P_2} \} \) be the five plane angles of \( K \). Lemma 3.9 implies that a tetrahedron \( K \) is determined by its circumradius \( R_K \) and \( \Theta_5 \). The proof is included in Appendix B.3.

**Lemma 3.9.** All the plane angles \( \theta_{1,P_1}, \theta_{2,P_1}, \theta_{1,P_2}, \theta_{2,P_2}, \theta_{3,P_1}, \theta_{3,P_2} \) be the five plane angles \( \Theta_5 \).
The local shape around a vertex is "sharp".

(b) The local shape around a vertex is "flat"

Figure 3  Illustration of two types of local shapes

Figure 4  A degenerated tetrahedron $K$ whose four vertices coincide with the vertices of a square

According to Lemma B.1, Lemma B.2 listed in Appendix B.1 and Lemma 3.9, each of $r_{jk}$ ($(j,k) \in \mathbb{Z}^{(2)}_{4}$) defined in (2.9) can be represented by $R_K$ multiplying by a continuous function of $\Theta_5$. Taking $r_{12}$ as an example, we have

$$r_{12} = R_K \frac{2 \cos \theta_{12}}{\sqrt{(1 - \cos^2 \theta_{12}) + \cot^2 \theta_{2,P_3} + \cot^2 \theta_{1,P_4} - 2 \cot \theta_{2,P_3} \cot \theta_{1,P_4} \cos \theta_{12}}}$$

and

$$\cos \theta_{12} = \frac{\cos \theta_{1,P_4} - \cos \theta_{1,P_3} \cos \theta_{2,P_2}}{\sin \theta_{1,P_2} \sin \theta_{2,P_2}}.$$

For simplicity, the similar representations of other $r_{jk}$ ($(j,k) \in \mathbb{Z}^{(2)}_{4}$) are omitted here.

Obviously, every entry of $N_{6 \times 6}^{K,\lambda}$ in (3.21) is a linear combination of $r_{jk}$ ($(j,k) \in \mathbb{Z}^{(2)}_{4}$). Thus, $\frac{1}{h_K} S_{6 \times 6}^{K,\lambda}$ can be rewritten as $\frac{1}{h_K} N_{6 \times 6}^{6 \times 6}(\Theta_5, \alpha, \beta, \gamma, \lambda)$. We turn to discuss the positive definiteness of $N_{6 \times 6}(\Theta_5, \alpha, \beta, \gamma, \lambda)$ under a regular partition $T_h$. Subsequently, for any fixed group of parameters $(\alpha, \beta, \gamma, \lambda)$, we restrict the lower bound of the minimum V-angle $\theta_K$ to ensure the positiveness of $\det(N_{6 \times 6}(\Theta_5, \alpha, \beta, \gamma, \lambda))$.

Denote the reasonable range of $\Theta_5$ for a tetrahedron $K$ satisfying $\theta_K \geq v$ by $Q_v$ as follows:

$Q_v = \{\Theta_5 > 0^\circ \mid \theta_{1,r_1} + \theta_{1,r_2} < 180^\circ; \theta_{2,r_1} + \theta_{2,r_2} < 180^\circ; \theta_{1,r_2} + \theta_{2,r_2} + \theta_{3,r_2} < 360^\circ; \theta_{r_i} \geq v, i \in \mathbb{Z}^{(1)}_{4}\}.$

By Lemma 3.9, $\theta_{r_i} \geq v$ $(i \in \mathbb{Z}^{(1)}_{4})$ means some relations between $\Theta_5$ and $v$. In addition, Lemma 3.7 indicates that each angle of $\Theta_5 \in Q_v$ lies in $[v, 180^\circ - 2v]$. For parameters $(\alpha, \beta, \gamma, \lambda)$ fixed by (2.2), (2.14) and (3.19), we define an angle set as

$$V^*(\alpha, \beta, \gamma, \lambda) = \{v \in (0^\circ, 60^\circ) \mid \det(N_{6 \times 6}(\Theta_5, \alpha, \beta, \gamma, \lambda)) > 0, \forall \Theta_5 \in Q_v\}.$$

(3.24)

Note that $V^*(\alpha, \beta, \gamma, \lambda)$ is nonempty since Lemma 3.5 shows $60^\circ \in V^*(\alpha, \beta, \gamma, \lambda)$. Let $v^*(\alpha, \beta, \gamma, \lambda) = \inf V^*(\alpha, \beta, \gamma, \lambda)$. The following restriction on the primary mesh $T_h$ plays an important role for the stability analysis.
Definition 3.10 (Minimum V-angle condition). A quadratic FVM scheme or the corresponding primary mesh $T_h$ is called to satisfy the minimum V-angle condition, if there exist $\varepsilon^* > 0$ and $\lambda$ in the range (3.19) such that

$$\theta_K \geq v^*(\alpha, \beta, \gamma, \lambda) + \varepsilon^*, \quad \forall K \in T_h.$$  \hfill (3.25)

Remark 3.11. Under the traditional mapping $\Pi^1$, the parameter $\lambda$ in the minimum V-angle condition (3.25) is fixed to be 1. The mapping $\Pi^1$ gives us more chances to find a better $\lambda$ in (3.19) such that $v^*(\alpha, \beta, \gamma, \lambda)$ is smaller, which leads (3.25) to be a weaker restriction. Actually, for a given scheme (fixed $\alpha, \beta, \gamma$), we care about when $v^*(\alpha, \beta, \gamma, \lambda)$ reaches its minimum value for $\lambda$.

Remark 3.12. The minimum V-angle condition (3.25) for tetrahedral meshes is as convenient as the minimum angle condition [8, 33, 39, 40] for 2D triangular meshes in application. Other restrictions on tetrahedral meshes are referred to [24].

In Algorithm 1, we show a way to compute $v^*(\alpha, \beta, \gamma, \lambda)$ numerically. The basic idea is to find the minimum $v \in (0^\circ, 60^\circ]$ by the bisection process such that $\det(\overline{N}_6 \times 6(\Theta_5, \alpha, \beta, \gamma, \lambda))$ is positive for $\Theta_5 \in Q_v$.

In this process, for each $v$, we compute to check whether the positiveness of $\det(\overline{N}_6 \times 6(\Theta_5, \alpha, \beta, \gamma, \lambda))$ is satisfied for $\Theta_5$ in $P_v(1) \cap Q_v, \ldots, P_v(q) \cap Q_v$, where $P_v(n) (n \in Z_q(1))$ are discrete point sets. Here, the points in $P_v^{(n)}$ are evenly selected in $[v, 180^\circ - 2v]^5$, i.e.,

$$P_v^{(n)} = \left\{ (v, v, v, v, v) + \frac{180^\circ - 3v}{N_n} (i_1, i_2, i_3, i_4, i_5), \forall i_1, \ldots, i_5 \in \{0\} \cup Z_n^{(1)} \right\},$$

where $N_n$ is a division number of $[v, 180^\circ - 2v]$. We take $N_n (n \in Z_q^{(1)})$ as a group of increasing prime numbers to avoid repeating calculations of $\det(\overline{N}_6 \times 6(\Theta_5, \alpha, \beta, \gamma, \lambda))$. In Section 5, we will show the numerical performances of $v^*(\alpha, \beta, \gamma, \lambda)$ for four given quadratic FVM schemes.

Algorithm 1. Searching $v^*(\alpha, \beta, \gamma, \lambda)$

Require: $\alpha \in (0, 1/2), \beta \in (0, 2/3), \gamma \in (0, 3/4)$, a group of increasing prime numbers $N_n (n \in Z_q^{(1)})$, precision $\varepsilon > 0^\circ$;  
Ensure: $\alpha$ and $\beta$ satisfy (2.14), and $\lambda$ satisfies (3.19);  
1: $v_0 \leftarrow 0^\circ, v_1 \leftarrow 60^\circ$;  
2: while $v_1 - v_0 > \varepsilon$ do  
3: $v_* \leftarrow \frac{v_0 + v_1}{2}, n \leftarrow 1$;  
4: while $\det(\overline{N}_6 \times 6(\Theta_5, \alpha, \beta, \gamma, \lambda)) > 0, \forall \Theta_5 \in P_v(n) \cap Q_v$, and $n \leq q$ do  
5: $n \leftarrow n + 1$;  
6: end while  
7: if $n = q + 1$ then  
8: $v_1 \leftarrow v_*$;  
9: else  
10: $v_0 \leftarrow v_*$;  
11: end if  
12: end while  
13: $v^*(\alpha, \beta, \gamma, \lambda) \leftarrow v_*$.

Lemma 3.13. Assume that $F_{m \times m}(X) (X \in S_0)$ is an $m \times m$ real symmetric matrix and elementwisely continuous in a connected region $S_0$ of $\mathbb{R}^n$. If there exists an $X_1 \in S_0$ such that $F_{m \times m}(X_1)$ is positive definite, and the determinant of $F_{m \times m}(X)$ is always positive in $S_0$, then $F_{m \times m}(X)$ is positive definite for every $X \in S_0$.

Proof. If the statement is not true, there exist even numbers of eigenvalues of $F_{m \times m}(X)$ for a certain $X_1 \in S_0$ that are negative. Let one of these negative eigenvalues be $\mu_0(X_1) (< 0)$. Since $\mu_0(X_1) > 0$, by a simple continuity argument, there is a point $X_2 \in S_0$ such that $\mu_0(X_2) = 0$. This contradicts that the determinant of $F_{m \times m}(X)$ is positive for every $X \in S_0$. \hfill \Box

Under the orthogonal condition on the surface (2.14), Lemma 3.5 implies that $\overline{N}_6 \times 6(\Theta_5, \alpha, \beta, \gamma, \lambda)$ with $\Theta_5 = (60^\circ, 60^\circ, 60^\circ, 60^\circ)$ is positive definite for a given $\lambda$ in (3.19). On the other hand, if $T_h$ satisfies
the minimum V-angle condition (3.25), then det(\(\widehat{N}_b\times J(\Theta_5, \alpha, \beta, \gamma, \lambda)\)) > 0 for \(\Theta_5 \in Q_v\) holds for every \(K \in T_h\). Thus, under the two restrictions (2.14) and (3.25), Lemma 3.13 ensures the local stability (3.1), in which the hidden constants have a common lower bound. According to the above discussion, we present the stability as follows.

**Theorem 3.14 (Stability).** Assume that the diffusion coefficient \(\kappa\) is piecewise constant over \(T_h\). If a quadratic FVM scheme (2.3) satisfies the orthogonal condition on the surface (2.14) and the minimum V-angle condition (3.25), then the local stability (3.1) holds. Furthermore, the bilinear form \(a_h(\cdot, \Pi_h)\) is uniformly elliptic, i.e.,

\[ a_h(u_h, \Pi_h u_h) \geq |u_h|^2, \quad \forall u_h \in U_h. \]

### 4 Error analysis

In this section, we present optimal \(H^1\) and \(L^2\) error estimates of the quadratic FVM schemes. The \(H^1\) error estimate is based on the continuity and the stability, and the \(L^2\) error estimate follows the \(H^1\) result and the orthogonal conditions (2.12) and (2.13).

Define the piecewise \(H^2\) space over \(T_h\) as

\[ H_h^2(\Omega) = \{ u \in C(\Omega) : u|_K \in H^2(K), \forall K \in T_h \}. \]

Then we have the continuity of the quadratic FVM schemes (2.3).

**Lemma 4.1.** For the bilinear form \(a_h(\cdot, \Pi_h)\) in (2.4), we have

\[ |a_h(u, \Pi_h u)| \lesssim (|u|_1 + h|u|_2)|u|_1, \quad \forall u \in H_h^1(\Omega) \cap H_h^2(\Omega), \quad u_h \in U_h, \]

where \(|u|_2.h = (\sum_{K \in T_h} |u|^2_{L, K})^{1/2}\).

**Proof.** For \(u \in H_h^1(\Omega) \cap H_h^2(\Omega)\) and \(u_h \in U_h\), we rewrite

\[ a_h(u, \Pi_h u) = - \sum_{K \in T_h} \sum_{s^* \in S^*_K} \int_{s^*} (\kappa \nabla u) \cdot n [\Pi_h u]_{s^*} ds, \]

where \(S^*_K\) is the set of all the common faces of the dual elements contained in the interior of \(K\). For all the polygonal faces \(s^*\) in \(S^*_K\), this means \(\bigcup_{s^* \in S^*_K} s^* = \bigcup_{K^* \in T_h}(\partial K^* \cap K)\). In addition, \(n\) is the unit normal vector on \(s^*\) from a dual element \(K^*_1\) to its neighboring dual element \(K^*_2\), and

\[ [\Pi_h u]_{s^*} := \Pi_h u|_{K^*_1} - \Pi_h u|_{K^*_2} \]

is the jump of \(\Pi_h u\) on \(s^*\).

Then by the Cauchy-Schwarz inequality,

\[ |a_h(u, \Pi_h u)| \lesssim \left( \sum_{K \in T_h} \sum_{s^* \in S^*_K} |s^*|^{-1} \int_{s^*} [\Pi_h u]^2 ds \right)^{1/2} \left( \sum_{K \in T_h} \sum_{s^* \in S^*_K} |s^*| \int_{s^*} ((\kappa \nabla u) \cdot n)^2 ds \right)^{1/2} \]

\[ \lesssim \left( \sum_{K \in T_h} \sum_{s^* \in S^*_K} h_K |\Pi_h u|^2_{s^*} \right)^{1/2} \left( \sum_{K \in T_h} \sum_{s^* \in S^*_K} |s^*| h_K \int_{s^*} |\nabla u|^2 ds \right)^{1/2}. \]

We start with the first term on the right-hand side of (4.2). Definition 2.3 indicates that \([\Pi_h u]_{s^*}\) (\(s^* \in S^*_K\)) are linear combinations of \(u_h(P_i)\) (\(i \in Z^{(1)}_{10}\)). Thus, there exists a matrix \(K\) with 10 columns such that

\[ \sum_{s^* \in S^*_K} [\Pi_h u]^2_{s^*} = (K u_K)(K u_K)^T, \]

where \(u_K = (u_{11}, u_{12}, \ldots, u_{10})^T\) and \(u_i = u_h(P_i)\) (\(i \in Z^{(1)}_{10}\)). Obviously, \(u_{i1} = u_{i2}\) for \(i_1, i_2 \in Z^{(1)}_{10}\) yields \([\Pi_h u]_{s^*} = \Pi_h u|_{K^*_1} - \Pi_h u|_{K^*_2} = 0\) for \(s^* \in S^*_K\). Thus, taking \(u_K = u_1(1, 1, \ldots, 1)^T\), one obtains
Recalling the equivalent discrete norm (3.8), we have

$$\sum_{s^* \in S^K} [\Pi^*_h u_h|_{s^*}]^2 = 0,$$

which indicates $K u_h$ being a zero vector. It means that the row sum of $K$ is zero. By the relation (3.11), one arrives at

$$\langle K u_h \rangle^T (K u_h) = \langle K T G u_h \rangle^T (K T G u_h) = \langle G u_h \rangle^T (K T) (G u_h) \lesssim \langle G u_h \rangle^T (G u_h).$$

Recalling the equivalent discrete norm (3.8), we have

$$\left( \sum_{K \in T_h} \sum_{s^* \in S^K} h_K [\Pi^*_h u_h|_{s^*}]^2 \right)^{1/2} \lesssim \left( \sum_{K \in T_h} h_K \| G u_h \|^2 \right)^{1/2} \lesssim \left( \sum_{K \in T_h} |u_h|_{1,K}^2 \right)^{1/2} = |u_h|_1. \quad (4.3)$$

For the second term on the right-hand side of (4.2), let $\varphi = \nabla u_h$. Since $T_h$ is a regular partition (2.1), it is obvious that

$$\iint_{s^*} |\varphi|^2 dS \lesssim h_K^2 \iint_{s^*} |\varphi|^2 dS.$$ 

According to the trace theorem, we have

$$\sum_{s^* \in S^K} \iint_{s^*} |\varphi|^2 dS \lesssim \| \varphi \|_{1,K}^2.$$ 

The Sobolev norm and the semi-norm of $\varphi$ satisfy (see [20])

$$\| \varphi \|_{0,K}^2 \lesssim h_K^{-2} \| \varphi \|_{0,K}^2, \quad |\varphi|_{1,K}^2 \lesssim h_K^{-1} |\varphi|_{1,K}^2,$$

which lead to

$$\sum_{s^* \in S^K} \iint_{s^*} \varphi^2 dS \lesssim h_K^{-1} \| \varphi \|_{0,K}^2 + h_K |\varphi|_{1,K}^2.$$ 

Then

$$\left( \sum_{K \in T_h} \sum_{s^* \in S^K} \frac{|s^*|}{h_K} \iint_{s^*} (\nabla u)^2 dS \right)^{1/2} \lesssim \left( \sum_{K \in T_h} h_K (h_K^{-1} \| \nabla u \|_{0,K}^2 + h_K |\nabla u|_{1,K}^2) \right)^{1/2} \lesssim \left( \sum_{K \in T_h} (|u|_{1,K}^2 + h_K |u|_{2,K}^2) \right)^{1/2},$$

which together with (4.2) and (4.3) completes the proof. \qed

Based on Theorem 3.14, we give the stability for the variable $\kappa(x_1, x_2, x_3)$.

**Lemma 4.2.** Under the same conditions of Theorem 3.14 and more generally, assuming that $\kappa$ is piecewise $W^{1, \infty}$ over $T_h$, then we can see that the bilinear form $a_h(\cdot, \Pi^*_h \cdot)$ is uniformly elliptic for sufficiently small $h > 0$, i.e.,

$$a_h(u_h, \Pi^*_h u_h) \gtrsim |u_h|_1^2, \quad \forall u_h \in U_h. \quad (4.4)$$

**Proof.** Let

$$\overline{\pi}_h(u_h, \Pi^*_h u_h) = - \sum_{K \in T_h} \sum_{s^* \in S^K} \iint_{\partial K \cap K} \overline{\pi} \nabla u_h \cdot n \Pi^*_h u_h dS,$$

where $\overline{\pi}$ is a piecewise constant function that

$$\overline{\pi}|_K = |K|^{-1} \iint_{K} \kappa dx_1 dx_2 dx_3, \quad \forall K \in T_h.$$ 

Theorem 3.14 indicates that

$$\overline{\pi}_h(u_h, \Pi^*_h u_h) \gtrsim |u_h|_1^2, \quad \forall u_h \in U_h.$$
Similar to the proof of the continuity (4.1) and by the inverse estimate, we obtain

$$|a_h(u_h, \Pi_h^*u_h) - \pi_h(u_h, \Pi_h^*u_h)| = \left| \sum_{K \in \mathcal{T}_h} \sum_{S^* \in \mathcal{S}_K} \int_{S^*} (k - \pi) \nabla u_h \cdot n \left[ \Pi_h^*u_h \right]_{S^*} \right| \lesssim \|k - \pi\|_{0, \infty} \|u_h\|_1 + h \|u_h\|_{2, h} \|u_h\|_1 \lesssim h \|u_h\|_1^2, \quad \forall u_h \in U_h.$$  

Therefore,

$$a_h(u_h, \Pi_h^*u_h) \geq \pi_h(u_h, \Pi_h^*u_h) - |a_h(u_h, \Pi_h^*u_h) - \pi_h(u_h, \Pi_h^*u_h)| \gtrsim \|u_h\|_1^2, \quad \forall u_h \in U_h,$n

when $h$ is small enough.  

Then the $H^1$ error estimate follows the stability.

**Theorem 4.3** ($H^1$ error estimate). Suppose that $u \in H^1_0(\Omega) \cap H^3(\Omega)$ is the solution of (1.1). If the conditions of Lemma 4.2 are satisfied, then (2.3) has a unique solution $u_h \in U_h$, and

$$|u - u_h|_1 \lesssim h^2 |u|_3.$$  

**Proof.** Lemma 4.2 indicates

$$a_h(u_h, \Pi_h^*u_h) \geq 0, \quad \forall u_h \in U_h,$n

and the equality holds if and only if $u_h = 0$ which verifies the existence and uniqueness of $u_h$.

Apparently, $a_h(u, v_h) = (f, v_h), \quad \forall v_h \in V_h$, which together with (2.3) leads to the orthogonality

$$a_h(u_h - u, v_h) = 0, \quad \forall v_h \in V_h.$$  

(4.6)

Let $u_f \in U_h$ be the standard Lagrange quadratic interpolation of $u$ over $\mathcal{T}_h$. Then from (4.1), (4.4) and (4.6), we have

$$|u_h - u_f|_1^2 \lesssim a_h(u_h - u_f, \Pi_h^*(u_h - u_f)) = a_h(u - u_f, \Pi_h^*(u_h - u_f)) \lesssim \|u - u_f\|_1 + h \|u - u_f\|_{2, h} \|u_h - u_f\|_1.$$  

Eliminating $|u_h - u_f|_1$ and by the standard interpolation error estimate, we obtain

$$|u_h - u_f|_1 \lesssim |u - u_f|_1 + h \|u - u_f\|_{2, h} \lesssim h^2 |u|_3.$$  

Together with $|u - u_h|_1 \lesssim |u - u_f|_1 + |u_h - u_f|_1$, we obtain (4.5).  

We present the optimal $L^2$ error estimate in the following Theorem 4.4, which benefits from [31].

**Theorem 4.4** ($L^2$ error estimate). Suppose that $u \in H^1_0(\Omega) \cap H^3(\Omega)$ is the solution of (1.1). If the conditions of Lemma 4.2 and the orthogonal condition on the volume (2.13) are satisfied, then

$$\|u - u_h\|_0 \lesssim h^3 |u|_4.$$  

**Proof.** Consider an auxiliary problem: given $g \in L^2(\Omega)$, find $\omega_y \in H^1_0(\Omega)$ such that

$$a(v, \omega_y) = (g, v), \quad \forall v \in H^1_0(\Omega),$$

where $a(v, \omega_y) = \iint_{\Omega} (k\nabla v) \cdot \nabla \omega_y dx_1 dx_2 dx_3$. It is well known that this problem is regular, i.e., it attains a unique solution $\omega_y \in H^3(\Omega) \cap H^2(\Omega)$ satisfying $\|\omega_y\|_2 \lesssim \|g\|_0$.

Let $v = u - u_h$ in (4.8). By the orthogonality (4.6) and Green’s formula, we have

$$\begin{align*}
(g, u - u_h) &= a(u - u_h, w_g) = a(u - u_h, w_g - \Pi_h w_g) + a(u - u_h, \Pi_h w_g) - a_h(u - u_h, \Pi_h(\Pi_h w_g)) \\
&=: E_1 + E_2 + E_3
\end{align*}$$  

(4.9)

and

$$\begin{align*}
E_1 &= a(u - u_h, w_g - \Pi_h w_g), \\
E_2 &= \sum_{K \in \mathcal{T}_h} \iint_K (\nabla \cdot (k\nabla(u - u_h))) \Pi_h w_g - \Pi_h^*w_g dx_1 dx_2 dx_3, \\
E_3 &= \sum_{K \in \mathcal{T}_h} \int_{\partial K} (k\nabla(u - u_h)) \cdot n(\Pi_h w_g - \Pi_h^*w_g) dS,
\end{align*}$$

where

$$\begin{align*}
\Pi_h &= \Pi_h^* \Pi_h, \\
\Pi_h^* &= \Pi_h^* \Pi_h^*.
\end{align*}$$  

(4.10)
where $\Pi_h^1$ is the piecewise linear interpolation projection over $\mathcal{T}_h$. For convenience of writing, let $M^*_1 w_g = \Pi_h^1 w_g - \Pi_h^1 (\Pi_h^1 w_g)$. Consider $E_2 = E_{21} + E_{22}$ and $E_3 = E_{31} + E_{32}$ with

$$E_{21} = \sum_{K \in \mathcal{T}_h} \int_\Gamma \left( - \nabla \cdot \left( ((\kappa - \bar{\kappa}) \nabla (u - u_h)) M^*_1 w_g \right) dx_1 dx_2 dx_3, \right.$$

$$E_{22} = \sum_{K \in \mathcal{T}_h} \int_\Gamma \left( - \nabla \cdot \left( (\bar{\kappa} \nabla (u - u_h)) M^*_1 w_g \right) dx_1 dx_2 dx_3, \right.$$

$$E_{31} = \sum_{K \in \mathcal{T}_h} \int_{\partial K} ((\kappa - \bar{\kappa}) \nabla (u - u_h)) \cdot n M^*_1 w_g dS,$$

$$E_{32} = \sum_{K \in \mathcal{T}_h} \int_{\partial K} (\bar{\kappa} \nabla (u - u_h)) \cdot n M^*_1 w_g dS,$$

where $\bar{\kappa}$ and $\kappa$ are two piecewise constant functions that for $K \in \mathcal{T}_h$, $\bar{\kappa} |_K = |K|^{-1} \int_{K} \kappa dx_1 dx_2 dx_3$, and $\bar{\kappa} |_{T_i} = |T_i|^{-1} \int_{T_i} \kappa dS$, $\forall T_i \in \partial K$.

It follows from the proof of [31, Theorem 5.3] that

$$|E_{22}| \lesssim h^3 |u|_3 |w_g|_2. \quad |E_{21}| \lesssim h^3 |u|_3 |w_g|_1. \quad |E_{31}| \lesssim h^3 |u|_3 |w_g|_1. \quad (4.10)$$

The orthogonal condition on the volume (2.13) is used to estimate $E_{22}$. Noticing that $\frac{\partial^2 u}{\partial x_i^2} - \Pi_h^1 \frac{\partial^2 u}{\partial x_i^2} (i \in Z_3^{(1)})$ restricted on $K$ are linear functions, we obtain

$$|E_{22}| \lesssim \sum_{K \in \mathcal{T}_h} \sum_{i \in Z_3^{(1)}} \bar{\kappa} \int_\Gamma \left| \frac{\partial^2 (u - u_h)}{\partial x_i^2} M^*_1 w_g dx_1 dx_2 dx_3 \right|$$

$$\lesssim \sum_{K \in \mathcal{T}_h} \left\| \frac{\partial^2 u}{\partial x_i^2} - \Pi_h^1 \frac{\partial^2 u}{\partial x_i^2} \right\|_{0, K} \left\| M^*_1 w_g \right\|_{0, K} \lesssim h^3 |u|_3 |w_g|_1. \quad (4.11)$$

The orthogonal condition on the surface (2.12) is used to estimate $E_{32}$. By the boundary condition $w_g |_{\partial \Omega} = 0$, we have

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} (\bar{\kappa} \nabla u) \cdot n M^*_1 w_g dS = \int_{\partial \Omega} (\bar{\kappa} \nabla u) \cdot n M^*_1 w_g dS = 0,$$

which together with (2.12) yields

$$E_{32} = \sum_{K \in \mathcal{T}_h} \int_{\partial K} (\bar{\kappa} \nabla u) \cdot n M^*_1 w_g dS - \sum_{K \in \mathcal{T}_h} \int_{\partial K} (\bar{\kappa} \nabla u_h) \cdot n M^*_1 w_g dS = 0. \quad (4.12)$$

Let $g = u - u_h$ in (4.9). With the estimates (4.10)–(4.12) and the regularity $\|\omega_g\|_2 \lesssim \|g\|_0$, we obtain the $L^2$ error estimate (4.7).

The above optimal $L^2$ error estimate strongly depends on the orthogonal conditions on the surface (2.12) and the volume (2.13). We point out that the orthogonal conditions are sufficient conditions in the proof of the $L^2$ error estimate, however, a large number of experiments indicates that they are necessary to achieve the optimal convergence rate in the $L^2$ norm, which are shown in Section 5.

5 Numerical experiments

In Table 1, we present four quadratic FVM schemes, which are used in the numerical experiments. The first three schemes satisfy the orthogonal conditions on the surface (2.14) and the volume (2.15), while the last scheme only satisfies the orthogonal condition on the surface (2.14).
Example 5.1. According to Algorithm 1, we show in Figure 5 the relationships between $v^*(\alpha, \beta, \gamma, \lambda)$ and $\lambda$ for the schemes in Table 1. From the four figures, $v^*(\alpha, \beta, \gamma, \lambda)$ reaches its minimum value at $\lambda = 1/(1 - 3\alpha\beta)$. The values of $v^*(\alpha, \beta, \gamma, 1/(1 - 3\alpha\beta))$ are given in the last column of Table 1.

Example 5.2. Consider the model problem (1.1) with $\Omega = [0, 1]^3$. The primary mesh $T_h$ is constructed by first dividing $\Omega$ into $N^3$ cubes, and then dividing each cube into six tetrahedrons (see Figure 6). By (3.23), the uniform minimum V-angle is

$$\min\{\theta_K, K \in T_h\} = \arctan\left(\frac{\sqrt{2}}{2}\right) + 45^\circ - \arctan\sqrt{2} \approx 25.529^\circ.$$

Then for the four schemes in Table 1, it is observed from the last column of Table 1 that the primary mesh $T_h$ satisfies the minimum V-angle condition (3.25).

We apply the four schemes in Table 1 to the equation (1.1) with the coefficient $\kappa(x_1, x_2, x_3) = e^{x_1+2x_2+3x_3}$ and $f$ is chosen so that the exact solution is $u = \sin(\pi x_1)\sin(\pi x_2)\sin(\pi x_3)$. In Table 2, all of the four schemes possess the optimal $H^1$ convergence rate. Table 3 shows that the first three schemes have the optimal $L^2$ convergence rate, while the last scheme does not. These numerical results demonstrate our theoretical results.

### Table 1 Four quadratic FVM schemes

| Scheme   | $\alpha$ | $\beta$    | $\gamma$  | $v^*(\alpha, \beta, \gamma, 1/(1 - 3\alpha\beta))$ |
|----------|----------|------------|-----------|--------------------------------------------------|
| QFVS-1   | $1/10$   | $14/15 - 2\sqrt{66}/45$ | $0.050667311760225$ | $20.5^\circ$ |
| QFVS-2   | $1/2 - \sqrt{3}/6$ | $2/3 + \sqrt{3}/9 - \sqrt{21 + 6\sqrt{3}/9}$ | $0.052883496779577$ | $17.0^\circ$ |
| QFVS-3   | $2/5$    | $11/15 - 2\sqrt{774}/45$ | $0.05108569487555$ | $16.7^\circ$ |
| QFVS-4   | $1/2 - \sqrt{3}/6$ | $2/3 + \sqrt{3}/9 - \sqrt{21 + 6\sqrt{3}/9}$ | $1/4$ | $18.2^\circ$ |

![Figure 5](image_url) (Color online) The relationships between $v^*(\alpha, \beta, \gamma, \lambda)$ and $\lambda$.
Figure 6  A division of a cube

Table 2  $H^1$ convergence rates of Example 5.2

| $N$ | QFVS-1 | QFVS-2 | QFVS-3 | QFVS-4 |
|-----|--------|--------|--------|--------|
|     | $|u - u_h|_1$ Order | $|u - u_h|_1$ Order | $|u - u_h|_1$ Order | $|u - u_h|_1$ Order |
| 5   | 1.1526E−01 \  \ | 1.1454E−01 \  \ | 1.1478E−01 \  \ | 1.1294E−01 \  \ |
| 15  | 1.3110E−02 \  | 1.3098E−02 \  | 1.3102E−02 \  | 1.3105E−02 \  |
| 25  | 4.7300E−03 \  | 4.7285E−03 \  | 4.7289E−03 \  | 4.7380E−03 \  |
| 35  | 1.4612E−03 \  | 1.4611E−03 \  | 1.4611E−03 \  | 1.4649E−03 \  |

Table 3  $L^2$ convergence rates of Example 5.2

| $N$ | QFVS-1 | QFVS-2 | QFVS-3 | QFVS-4 |
|-----|--------|--------|--------|--------|
|     | $\|u - u_h\|_0$ Order | $\|u - u_h\|_0$ Order | $\|u - u_h\|_0$ Order | $\|u - u_h\|_0$ Order |
| 5   | 2.7360E−03 \  | 2.7238E−03 \  | 2.7202E−03 \  | 2.4588E−03 \  |
| 15  | 9.7046E−05 \  | 9.6983E−05 \  | 9.6969E−05 \  | 8.766E−05 \  |
| 25  | 2.0875E−05 \  | 2.0870E−05 \  | 2.0869E−05 \  | 1.552E−05 \  |
| 35  | 7.5988E−06 \  | 7.5978E−06 \  | 7.5976E−06 \  | 3.157E−05 \  |
| 45  | 3.573E−06 \  | 3.5733E−06 \  | 3.5733E−06 \  | 1.999E−05 \  |

Example 5.3 (Random mesh).  To further illustrate the performance of the proposed quadratic FVM schemes, we disturb the vertices of the tetrahedrons in Example 5.2 with the random rate $0.2/N$ in three directions, in which

(i) the eight vertices of $\Omega = [0, 1]^3$ are fixed;
(ii) the vertices on the twelve edges of $\Omega = [0, 1]^3$ are repositioned along the edges;
(iii) the vertices on the six faces of $\Omega = [0, 1]^3$ are repositioned on the faces in two directions.

For the four schemes in Table 1, we show in Tables 4 and 5 the numerical performances, which are consistent with the numerical performances in Example 5.2.

Remark 5.4.  Consider the same model problem as Example 5.2. Then for fixed $(\alpha, \beta)$, Figure 7 shows how convergence rates in $H^1$ and $L^2$ norms between $N = 10$ and $N = 20$ change with $\gamma$, where the first three pairs of $(\alpha, \beta)$ are the same as those in QFVS-1, QFVS-2 and QFVS-3, respectively, and the last pair is taken as

$$\alpha = 0.3, \quad \beta = 0.4.$$ 

In fact, a large number of experiments indicate that the orthogonal conditions are not only sufficient but also necessary to achieve the optimal convergence rate in the $L^2$ norm.
Table 4 $H^1$ convergence rates of Example 5.3

| N  | QFVS-1 $|u - u_h|_1$ Order | QFVS-2 $|u - u_h|_1$ Order | QFVS-3 $|u - u_h|_1$ Order | QFVS-4 $|u - u_h|_1$ Order |
|----|-----------------|-----------------|-----------------|-----------------|
| 5  | 1.2602E−01 \ 1.2429E−01 | 1.2090E−01 \ 2.0129 | 1.2429E−01 \ 1.9924 | 1.2061E−01 \ 1.9555 |
| 15 | 1.4108E−02 \ 1.9937 | 1.9931 \ 5.0786E−03 | 1.9549 \ 5.0767E−03 | 2.0129 \ 1.9958 |
| 25 | 5.0786E−03 \ 2.5947E−03 | 1.9988 \ 2.5977E−03 | 1.9924 \ 2.5959E−03 | 2.0129 \ 1.9934 |
| 35 | 1.5700E−03 \ 2.5967E−03 | 1.9958 \ 2.5977E−03 | 2.0129 \ 2.5959E−03 | 2.0129 \ 1.9977 |

Table 5 $L^2$ convergence rates of Example 5.3

| N  | QFVS-1 $\|u - u_h\|_0$ Order | QFVS-2 $\|u - u_h\|_0$ Order | QFVS-3 $\|u - u_h\|_0$ Order | QFVS-4 $\|u - u_h\|_0$ Order |
|----|-----------------|-----------------|-----------------|-----------------|
| 5  | 3.7870E−03 \ 3.4369E−03 | 3.3785E−03 \ 2.9683 | 3.4378E−03 \ 2.9747 | 3.1009E−03 \ 2.4652 |
| 15 | 1.2821E−04 \ 3.0508 | 1.2956E−04 \ 3.0409 | 2.9683E−05 \ 2.9857 | 6.4989E−05 \ 2.1339 |
| 25 | 2.6984E−05 \ 3.1588 | 2.7407E−05 \ 3.1588 | 2.9857 \ 2.1339 | 6.4989E−05 \ 2.1339 |
| 35 | 9.8073E−05 \ 3.0480 | 9.8279E−05 \ 2.9857 | 2.9857 \ 2.1339 | 6.4989E−05 \ 2.1339 |
| 45 | 4.6334E−06 \ 2.9766 | 4.6514E−06 \ 2.9766 | 2.9857 \ 2.1339 | 6.4989E−05 \ 2.1339 |

Figure 7 (Color online) The relationships between the convergence rate and $\gamma$

6 Conclusion

In this paper, we have constructed a family of quadratic FVM schemes on tetrahedral meshes by introducing three parameters ($\alpha$, $\beta$, $\gamma$) on the dual mesh. Under the proposed orthogonal conditions and the minimum V-angle condition, we derive the theoretical analysis. The theoretical results include:

- stability, which is the most important result in this paper;
- optimal $H^1$ and $L^2$ error estimates, where the $L^2$ convergence rate strongly depends on the
orthogonal conditions. These theoretical results are confirmed by some numerical experiments.

For higher $r$-order ($r \geq 3$) FVMs on tetrahedral meshes, the stability analysis is much more complex and it needs further study.

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Appendix A  Some symbolic matrices

Appendix A.1  The symbolic matrix A in (3.4)

It holds that

\[ A = \begin{pmatrix}
A^{(1)}_{4 \times 4} & 4(A^{(2)}_{6 \times 4})^\top \\
A^{(3)}_{6 \times 4} & 4A^{(4)}_{6 \times 6}
\end{pmatrix},
\]

where

\[ A^{(1)}_{4 \times 4} = \begin{pmatrix}
(3t_1 - 2t_2)R_1 + 4t_2R_2 + (t_1 - 2t_2)r_{24} & 4t_2R_3 + (t_1 - 2t_2)r_{24} & 4t_2R_4 + (t_1 - 2t_2)r_{23} \\
4t_2R_1 + (t_1 - 2t_2)r_{34} & (3t_1 - 2t_2)R_2 + 4t_2R_3 + (t_1 - 2t_2)r_{14} & 4t_2R_4 + (t_1 - 2t_2)r_{13} \\
4t_2R_1 + (t_1 - 2t_2)r_{24} & 4t_2R_2 + (t_1 - 2t_2)r_{14} & (3t_1 - 2t_2)R_3 + 4t_2R_4 + (t_1 - 2t_2)r_{12} \\
4t_2R_1 + (t_1 - 2t_2)r_{33} & 4t_2R_2 + (t_1 - 2t_2)r_{23} & 4t_2R_3 + (t_1 - 2t_2)r_{22} \\
(3t_1 - 2t_2)R_1 & (t_1 - 2t_2)R_2 & (t_1 - 2t_2)R_3 & (t_1 - 2t_2)R_4
\end{pmatrix},
\]

\[ A^{(2)}_{6 \times 4} = \begin{pmatrix}
-t_2(R_2 + R_3 - r_{12} - r_{13}) & t_2R_2 - (t_1 + t_2)r_{14} & t_2R_3 - (t_1 + t_2)r_{14} & t_2R_4 - (t_1 + t_2)r_{14} & -t_2(R_2 + R_3 - r_{24} - r_{34}) \\
t_2R_1 - (t_1 + t_2)r_{24} & -t_2(R_1 + R_3 - r_{12} - r_{23}) & t_2R_3 - (t_1 + t_2)r_{24} & t_2R_4 - (t_1 + t_2)r_{24} & -t_2(R_1 + R_3 - r_{14} - r_{34}) \\
t_2R_1 - (t_1 + t_2)r_{34} & t_2R_2 - (t_1 + t_2)r_{34} & -t_2(R_1 + R_2 - r_{13} - r_{23}) & t_2R_4 - (t_1 + t_2)r_{34} & -t_2(R_1 + R_2 - r_{14} - r_{24}) \\
t_2(R_2 + R_4 - r_{12} - r_{14}) & t_2R_2 - (t_1 + t_2)r_{13} & -t_2(R_2 + R_4 - r_{23} - r_{34}) & t_2R_4 - (t_1 + t_2)r_{13} & -t_2(R_2 + R_4 - r_{23} - r_{34}) \\
t_2(R_2 + R_3 - r_{13} - r_{14}) & -t_2(R_2 + R_3 - r_{23} - r_{24}) & t_2R_3 - (t_1 + t_2)r_{12} & t_2R_4 - (t_1 + t_2)r_{12} & -t_2(R_2 + R_3 - r_{23} - r_{24})
\end{pmatrix},
\]

\[ A^{(3)}_{6 \times 4} = \begin{pmatrix}
(t_5 + t_4)R_1 - (2t_5 - 3t_4)r_{23} & (2t_5 - t_4)(R_3 - r_{14}) & (2t_5 - t_4)(R_3 - r_{14}) & (2t_5 + t_4)R_4 - (2t_5 - 3t_4)r_{23} \\
(2t_5 + t_4)(R_1 - r_{14}) & (2t_3 + t_4)R_2 - (2t_3 - 3t_4)r_{13} & (2t_3 - t_4)(R_3 - r_{14}) & (2t_5 + t_4)R_4 - (2t_5 - 3t_4)r_{13} \\
(2t_3 + t_4)(R_1 - r_{13}) & (2t_5 - t_4)(R_2 - r_{23}) & (2t_3 + t_4)R_3 - (2t_3 - 3t_4)r_{12} & (2t_5 + t_4)R_4 - (2t_5 - 3t_4)r_{12} \\
(2t_5 + t_4)R_1 - (2t_5 - 3t_4)r_{24} & (2t_5 - t_4)(R_2 - r_{12}) & (2t_5 + t_4)R_3 - (2t_5 - 3t_4)r_{24} & (2t_5 + t_4)R_4 - (2t_5 - 3t_4)r_{12}
\end{pmatrix},
\]

\[ A^{(4)}_{6 \times 6} = \begin{pmatrix}
R_1 & R_2 & R_3 & R_4 & R_1 & R_2 \\
R_2 & R_3 & R_4 & R_1 & R_2 & R_3 \\
R_3 & R_4 & R_1 & R_2 & R_3 & R_4 \\
R_4 & R_1 & R_2 & R_3 & R_4 & R_1 \\
R_1 & R_2 & R_3 & R_4 & R_1 & R_2 \\
R_2 & R_3 & R_4 & R_1 & R_2 & R_3
\end{pmatrix}.
\]
Appendix A.2 The symbolic matrices $Q_{3x3}^{(1)}, Q_{3x3}^{(2)}, L_{3x3}^{(1)}, L_{3x3}^{(2)}, L_{3x3}^{(3)}$ and $L_{3x3}^{(4)}$ in (3.16)

$$
A^{(4)}_{\cos} = 
\begin{pmatrix}
  t_1(R_1 + R_2 + 2r_3) & t_1 r_4 - t_2 R_3 & t_2 R_2 - 2r_3 & t_2 R_2 + 2r_3 & t_2 R_1 - R_4 & t_2 R_1 + R_4 & t_2 R_1 - R_4 & t_2 R_1 + R_4 & t_2 R_1 - R_4 & t_2 R_1 + R_4 \\
  t_1 r_4 - t_2 R_3 & t_1 R_2 + 2r_3 & t_1 R_2 - 2r_3 & t_1 R_2 + 2r_3 & t_1 R_1 - R_4 & t_1 R_1 + R_4 & t_1 R_1 - R_4 & t_1 R_1 + R_4 & t_1 R_1 - R_4 & t_1 R_1 + R_4 \\
  t_1 R_2 - R_3 & t_1 R_2 - 2r_3 & t_1 R_2 + 2r_3 & t_1 R_2 - 2r_3 & t_1 R_1 - R_4 & t_1 R_1 + R_4 & t_1 R_1 - R_4 & t_1 R_1 + R_4 & t_1 R_1 - R_4 & t_1 R_1 + R_4 \\
  t_1 R_1 - R_3 & t_1 R_1 - 2r_3 & t_1 R_1 + 2r_3 & t_1 R_1 - 2r_3 & t_1 R_1 - R_4 & t_1 R_1 + R_4 & t_1 R_1 - R_4 & t_1 R_1 + R_4 & t_1 R_1 - R_4 & t_1 R_1 + R_4 \\
  -t_1 r_4 + t_2 R_3 & -t_1 R_2 + 2r_3 & -t_1 R_2 - 2r_3 & -t_1 R_2 + 2r_3 & -t_1 R_1 + R_4 & -t_1 R_1 - R_4 & -t_1 R_1 + R_4 & -t_1 R_1 - R_4 & -t_1 R_1 + R_4 & -t_1 R_1 - R_4 \\
  -t_1 R_2 + R_3 & -t_1 R_2 - 2r_3 & -t_1 R_2 + 2r_3 & -t_1 R_2 - 2r_3 & -t_1 R_1 + R_4 & -t_1 R_1 - R_4 & -t_1 R_1 + R_4 & -t_1 R_1 - R_4 & -t_1 R_1 + R_4 & -t_1 R_1 - R_4 \\
  -t_1 R_1 + R_3 & -t_1 R_1 - 2r_3 & -t_1 R_1 + 2r_3 & -t_1 R_1 - 2r_3 & -t_1 R_1 + R_4 & -t_1 R_1 - R_4 & -t_1 R_1 + R_4 & -t_1 R_1 - R_4 & -t_1 R_1 + R_4 & -t_1 R_1 - R_4 \\
  -t_1 R_1 - R_3 & -t_1 R_1 - 2r_3 & -t_1 R_1 + 2r_3 & -t_1 R_1 - 2r_3 & -t_1 R_1 + R_4 & -t_1 R_1 - R_4 & -t_1 R_1 + R_4 & -t_1 R_1 - R_4 & -t_1 R_1 + R_4 & -t_1 R_1 - R_4 \\
  -t_1 r_4 - t_2 R_3 & -t_1 R_2 - 2r_3 & -t_1 R_2 + 2r_3 & -t_1 R_2 + 2r_3 & -t_1 R_1 - R_4 & -t_1 R_1 + R_4 & -t_1 R_1 - R_4 & -t_1 R_1 + R_4 & -t_1 R_1 - R_4 & -t_1 R_1 + R_4 \\
  -t_1 R_2 - R_3 & -t_1 R_2 - 2r_3 & -t_1 R_2 + 2r_3 & -t_1 R_2 + 2r_3 & -t_1 R_1 - R_4 & -t_1 R_1 + R_4 & -t_1 R_1 - R_4 & -t_1 R_1 + R_4 & -t_1 R_1 - R_4 & -t_1 R_1 + R_4 \\
  -t_1 R_1 + R_3 & -t_1 R_1 - 2r_3 & -t_1 R_1 + 2r_3 & -t_1 R_1 - 2r_3 & -t_1 R_1 + R_4 & -t_1 R_1 - R_4 & -t_1 R_1 + R_4 & -t_1 R_1 - R_4 & -t_1 R_1 + R_4 & -t_1 R_1 - R_4 \\
  -t_1 R_1 - R_3 & -t_1 R_1 - 2r_3 & -t_1 R_1 + 2r_3 & -t_1 R_1 - 2r_3 & -t_1 R_1 + R_4 & -t_1 R_1 - R_4 & -t_1 R_1 + R_4 & -t_1 R_1 - R_4 & -t_1 R_1 + R_4 & -t_1 R_1 - R_4 \\
  -t_1 r_4 + t_2 R_3 & -t_1 R_2 - 2r_3 & -t_1 R_2 + 2r_3 & -t_1 R_2 + 2r_3 & -t_1 R_1 - R_4 & -t_1 R_1 + R_4 & -t_1 R_1 - R_4 & -t_1 R_1 + R_4 & -t_1 R_1 - R_4 & -t_1 R_1 + R_4 \\
  -t_1 R_2 + R_3 & -t_1 R_2 - 2r_3 & -t_1 R_2 + 2r_3 & -t_1 R_2 + 2r_3 & -t_1 R_1 - R_4 & -t_1 R_1 + R_4 & -t_1 R_1 - R_4 & -t_1 R_1 + R_4 & -t_1 R_1 - R_4 & -t_1 R_1 + R_4 \\
  -t_1 R_1 + R_3 & -t_1 R_1 - 2r_3 & -t_1 R_1 + 2r_3 & -t_1 R_1 - 2r_3 & -t_1 R_1 + R_4 & -t_1 R_1 - R_4 & -t_1 R_1 + R_4 & -t_1 R_1 - R_4 & -t_1 R_1 + R_4 & -t_1 R_1 - R_4 \\
  -t_1 R_1 - R_3 & -t_1 R_1 - 2r_3 & -t_1 R_1 + 2r_3 & -t_1 R_1 - 2r_3 & -t_1 R_1 + R_4 & -t_1 R_1 - R_4 & -t_1 R_1 + R_4 & -t_1 R_1 - R_4 & -t_1 R_1 + R_4 & -t_1 R_1 - R_4 \\
\end{pmatrix}.
$$

Appendix B Some relations in a tetrahedron and two related proofs

Appendix B.1 Some relations in a tetrahedron

In this subsection, we discuss some relations in a tetrahedron $K = \Delta^4 P_1 P_2 P_3 P_4$ about its plane angles $\theta_{i_1, i_2}$ $(i_1 \in Z_3^{(1)}, i_2 \in Z_4^{(1)})$ in (3.22), dihedral angles $\theta_{j k}$ ($(j, k) \in Z_4^{(2)})$, edge lengths $|P_j P_k| ((j, k) \in Z_4^{(2)})$ and circumradius $R_K$.

In Figure 8, we take a positively oriented orthogonal coordinate system $(x_1, x_2, x_3)$ such that $P_1$ is the origin, $\mathcal{P}_1 \mathcal{P}_2$ on the $x_1$ axis, and the $x_1 P_1 x_2$ plane coincides with the plane spanned by $\mathcal{P}_1 \mathcal{P}_2$ and $\mathcal{P}_1 \mathcal{P}_3$. Here, $O_1$ and $O_2$ are the circumcenters of $\Delta^4 P_1 P_2 P_3 P_4, \Delta^4 P_1 P_2 P_4$ and $\Delta^4 P_2 P_3$, respectively, and $M_{12}$ is the midpoint of $\mathcal{P}_1 \mathcal{P}_2$.

**Lemma B.1.** For a given $K = \Delta^4 P_1 P_2 P_3 P_4$ (see Figure 8), its plane angles and dihedral angles satisfy

$$
\cos \theta_{1_{12}} = \frac{\cos \theta_{1, r_1} - \cos \theta_{1, r_1} \cos \theta_{2, r_1}}{\sin \theta_{1, r_1} \sin \theta_{2, r_1}}, \quad (B.1)
$$

$$
\cos \theta_{1_{12}} = \frac{\cos \theta_{2, r_1} \sin \theta_{12} \cos \theta_{13} + \cos \theta_{12} \sin \theta_{13}}{\sqrt{(\cos \theta_{2, r_1} \sin \theta_{12} \cos \theta_{13} + \cos \theta_{12} \sin \theta_{13})^2 + \sin^2 \theta_{2, r_1} \sin^2 \theta_{12}}}, \quad (B.2)
$$
Let \( \mathbf{n}_i = (n_{i,1}, n_{i,2}, n_{i,3})^\top \) be the unit outer normal vector of the triangular face \( T_i \), and \( \mathbf{n}_{kl} = \frac{\mathbf{P}_k - \mathbf{P}_l}{|\mathbf{P}_k - \mathbf{P}_l|} = (n_{kl,1}, n_{kl,2}, n_{kl,3})^\top \) be the unit direction vector from \( P_k \) to \( P_l \). From Figure 8, it is easy to find that

\[
\mathbf{n}_{12} = (1, 0, 0)^\top, \quad \mathbf{n}_4 = (0, 0, -1)^\top, \quad \mathbf{n}_3 = (0, -\sin \theta_{12}, \cos \theta_{12})^\top, \\
\mathbf{n}_{13} = (\cos \theta_{2,3}, \sin \theta_{2,3}, 0)^\top, \quad \mathbf{n}_{14} = (\cos \theta_{1,4}, n_{14,2}, n_{14,3})^\top, \quad \mathbf{n}_2 = (n_{2,1}, n_{2,2}, \cos \theta_{13})^\top.
\]

Since \( \theta_{12} \) equals the angle between \( -\mathbf{n}_4 \) and \( \mathbf{n}_3 \), where \( \mathbf{n}_3 \) has the same direction as \( \mathbf{n}_{12} \times \mathbf{n}_{14} \), we have

\[
\cos \theta_{12} = \mathbf{n}_3 \cdot \mathbf{n}_4 = -\frac{\mathbf{n}_{12} \times \mathbf{n}_{14}}{|\mathbf{n}_{12} \times \mathbf{n}_{14}|} \cdot \mathbf{n}_4 = \frac{(0, n_{14,2}, -n_{14,3})}{\sin \theta_{1,4}} \cdot (0, 0, -1) = \frac{n_{14,2}}{\sin \theta_{1,4}}. \tag{B.3}
\]

Noticing that \( \theta_{3,1} \) is the angle between \( \mathbf{n}_{13} \) and \( \mathbf{n}_{14} \) satisfying \( \cos \theta_{3,1} = \mathbf{n}_{13} \cdot \mathbf{n}_{14} = \cos \theta_{1,4} \cos \theta_{2,3} + \sin \theta_{2,3} n_{14,2} \), we have

\[
n_{14,2} = \frac{\cos \theta_{3,1} - \cos \theta_{1,4} \cos \theta_{2,3}}{\sin \theta_{2,3}},
\]

which together with (B.3) leads to (B.1).

On the other hand, since \( \theta_{3,1} \) is the angle between \( \mathbf{n}_{13} \) and \( \mathbf{n}_{14} \), where \( \mathbf{n}_{14} \) has the same direction as \( \mathbf{n}_2 \times \mathbf{n}_3 \), we have

\[
\cos \theta_{3,1} = \mathbf{n}_{13} \cdot \mathbf{n}_{14} = \mathbf{n}_{13} \cdot \frac{\mathbf{n}_2 \times \mathbf{n}_3}{|\mathbf{n}_2 \times \mathbf{n}_3|}
\]

\[
= (\cos \theta_{2,3}, \sin \theta_{2,3}, 0)^\top \cdot \frac{(n_{2,2} \cos \theta_{12} + \cos \theta_{13} \sin \theta_{12}, -n_{2,1}, \cos \theta_{12}, -n_{2,1}, \sin \theta_{12})^\top}{\sqrt{(n_{2,2} \cos \theta_{12} + \cos \theta_{13} \sin \theta_{12})^2 + n_{2,1}^2}}
\]

\[
= \frac{\cos \theta_{2,3} n_{2,2} \cos \theta_{12} + \cos \theta_{13} \sin \theta_{12} - \sin \theta_{2,3} n_{2,1} \cos \theta_{12}}{\sqrt{(n_{2,2} \cos \theta_{12} + \cos \theta_{13} \sin \theta_{12})^2 + n_{2,1}^2}}. \tag{B.4}
\]

Noticing that \( \mathbf{n}_2 \cdot \mathbf{n}_{13} = \cos \theta_{2,3} n_{2,1} + \sin \theta_{2,3} n_{2,2} = 0 \) and \( |\mathbf{n}_2|^2 = n_{2,1}^2 + n_{2,2}^2 + \cos \theta_{12}^2 = 1 \), one obtains

\[
n_{2,1} = -\sin \theta_{2,3} \sin \theta_{12}, \quad n_{2,2} = \cos \theta_{2,3} \sin \theta_{12},
\]

which together with (B.4) leads to (B.2).
Lemma B.2. For a given $K = \triangle^4 P_1P_2P_3P_4$ (see Figure 8), the edge length $|P_1P_2|$ can be represented as follows:

$$|P_1P_2| = 2R_K \cdot \frac{\sin \theta_{12}}{\sqrt{\sin^2 \theta_{12} + \cos^2 \theta_{1, P_3} + \cos^2 \theta_{1,P_4} - 2 \cot \theta_{1,P_3} \cot \theta_{1,P_4} \cos \theta_{12}}}. \quad (B.5)$$

Proof. The relationship between the diameter and the chord length in the circumcircles of $\triangle P_1P_2P_4$ and $\triangle P_1P_2P_3$ yields

$$2\frac{|O_1M_{12}|}{|P_1P_2|} = \cot \theta_{1,P_4}, \quad 2\frac{|O_2M_{12}|}{|P_1P_2|} = \cot \theta_{2,P_3}.$$

Besides, the circumcenters $O_1$, $O_1$ and $O_2$ are on a common plane which passes through $M_{12}$ and is perpendicular to $O_1P_2$, and $\angle OO_1M_{12} = \angle OO_2M_{12} = \pi/2$. Thus, the four points $O_1$, $O_1$, $O_2$ and $M_{12}$ are on a common circle with the diameter $|O_1O_2|$. Then

$$\frac{|O_1O_2|}{|O_1M_{12}|} = \sin \theta_{12}.$$

Substituting the above equations and $|O_1O_2|^2 + \frac{|P_1P_2|^2}{4} = R_K^2$ into the law of cosines on $\triangle O_1M_{12}O_2$, i.e.,

$$|O_1O_2|^2 = |O_1M_{12}|^2 + |O_2M_{12}|^2 - 2|O_1M_{12}||O_2M_{12}| \cos \theta_{12},$$

one arrives at (B.5). \qed

Other relations of angles and length in $K = \triangle^4 P_1P_2P_3P_4$ similar to those given by Lemmas B.1 and B.2 can be easily derived. Without causing confusion, we use (B.1), (B.2) and (B.5) to represent themselves and other similar relations.

Appendix B.2 A proof of Lemma 3.4

We give a proof of Lemma 3.4. It suffices to verify that the three leading principal minors of $\frac{1}{h_K} M^K_{3 \times 3}$ are positive. By Lemma 2.2, we have $\frac{1}{h_K} R_i = \frac{2T_i^2}{3|K|K}$ and

$$R_1 R_2 = \left(\frac{2|T_1||T_2|}{3|K|K}\right)^2 \left(\frac{2|T_1||T_2|}{3|K|K} \cdot n_1 \cdot n_2\right)^2 (1 + \tan^2 \theta_{34}) = r_{34}^2 + \frac{|P_3P_4|^2}{h_K^2},$$

which leads to

$$\frac{1}{h_K^2} \begin{vmatrix} R_1 & -r_{34} \\ -r_{34} & R_2 \end{vmatrix} = \frac{|P_3P_4|^2}{h_K^2}.$$

Moreover, by $2S_{\triangle ABC} = |AB||AC| \sin \angle A$, we can write the determinant of $\frac{1}{h_K} M^K_{3 \times 3}$ as

$$\det \left(\frac{1}{h_K} M^K_{3 \times 3}\right) = \frac{1}{h_K^2} \begin{vmatrix} \frac{2T_1^2}{3|K|K} \cos \theta_{34} & -\frac{2T_1||T_3|}{3|K|K} \cos \theta_{34} & -\frac{2T_1||T_3|}{3|K|K} \cos \theta_{24} \\ -\frac{2T_2||T_3|}{3|K|K} \cos \theta_{24} & \frac{2T_2^2}{3|K|K} \cos \theta_{34} & -\frac{2T_2||T_3|}{3|K|K} \cos \theta_{14} \\ \frac{2T_3^2}{3|K|K} \cos \theta_{14} & -\frac{2T_3||T_3|}{3|K|K} \cos \theta_{14} & \frac{2T_3^2}{3|K|K} \cos \theta_{24} \end{vmatrix} = \frac{c_L^2}{h_K^2 (6|K|)^3} c_d, \quad (B.6)$$

where $c_L = |P_1P_2||P_2P_3||P_3P_4|$ and

$$c_d = \begin{vmatrix} \sin^2 \theta_{1,P_3} & -\sin \theta_{1,P_3} \sin \theta_{3,P_4} \cos \theta_{34} & -\sin \theta_{1,P_3} \sin \theta_{3,P_4} \cos \theta_{24} \\ -\sin \theta_{2,P_4} \sin \theta_{3,P_4} \cos \theta_{34} & \sin^2 \theta_{3,P_4} & -\sin \theta_{2,P_4} \sin \theta_{3,P_4} \cos \theta_{14} \\ -\sin \theta_{1,P_4} \sin \theta_{2,P_4} \cos \theta_{34} & -\sin \theta_{1,P_4} \sin \theta_{2,P_4} \cos \theta_{14} & \sin^2 \theta_{2,P_4} \end{vmatrix}.$$

The relation (B.1) and $\sin^2 \theta + \cos^2 \theta = 1$ yield

$$c_d = \begin{vmatrix} 1 - \cos^2 \theta_{2,P_4} & \cos \theta_{2,P_4} \cos \theta_{3,P_4} - \cos \theta_{1,P_4} \cos \theta_{3,P_4} & \cos \theta_{2,P_4} \cos \theta_{3,P_4} - \cos \theta_{1,P_4} \cos \theta_{3,P_4} \\ \cos \theta_{2,P_4} \cos \theta_{3,P_4} - \cos \theta_{1,P_4} \cos \theta_{3,P_4} & 1 - \cos^2 \theta_{3,P_4} & \cos \theta_{3,P_4} \cos \theta_{2,P_4} - \cos \theta_{1,P_4} \cos \theta_{2,P_4} \\ \cos \theta_{1,P_4} \cos \theta_{2,P_4} - \cos \theta_{1,P_4} \cos \theta_{2,P_4} & \cos \theta_{1,P_4} \cos \theta_{2,P_4} - \cos \theta_{1,P_4} \cos \theta_{2,P_4} & 1 - \cos^2 \theta_{1,P_4} \end{vmatrix}. \quad (B.7)$$
For $c_L$ in (B.6), there is the volume formula that
\[
6|K| = |T_1| |P_1 P_2| \sin \theta_{2, r_4} \sin \theta_{3, r_4} = c_L \sin \theta_{2, r_4} \sin \theta_{3, r_4} \sqrt{1 - \cos^2 \theta_{3, r_4}}
\]
\[
= c_L \sin \theta_{2, r_4} \sin \theta_{3, r_4} \sqrt{1 - \left( \frac{\cos \theta_{1, r_4} - \cos \theta_{2, r_4} \cos \theta_{3, r_4}}{\sin \theta_{2, r_4} \sin \theta_{3, r_4}} \right)^2} = c_L \sqrt{c_K}
\]  
(B.8)
with $c_K = 1 - \cos^2 \theta_{1, r_4} - \cos^2 \theta_{2, r_4} - \cos^2 \theta_{3, r_4} + 2 \cos \theta_{1, r_4} \cos \theta_{2, r_4} \cos \theta_{3, r_4}$. For $c_d$ in (B.6), multiplying (B.7) by a determinant whose value is 1, one obtains
\[
\begin{vmatrix}
1 & 0 & 0 & c_K \cos \theta_{2, r_4} \cos \theta_{3, r_4} - \cos \theta_{1, r_4} \cos \theta_{2, r_4} - \cos \theta_{3, r_4} \\
\cos \theta_{1, r_4} & 1 & 0 & 0 & 1 - \cos^2 \theta_{3, r_4} \\
\cos \theta_{3, r_4} & 0 & 1 & 0 & \cos \theta_{2, r_4} \cos \theta_{3, r_4} - \cos \theta_{2, r_4} \\
\end{vmatrix} = c_K^2. 
\]  
(B.9)
Then substituting (B.8) and (B.9) into (B.6) yields
\[
\det \left( \frac{1}{h_K} M_{3 \times 3}^K \right) = \frac{6|K|}{h_K^3}
\]
Since $T_6$ is a regular partition, the proof is completed.

**Appendix B.3 A proof of Lemma 3.9**

We give a proof of Lemma 3.9. By the representations (B.1) and (B.2), the remaining seven of the twelve plane angles $\theta_{i_1, r_{i_2}}$ ($i_1 \in Z_3^{(1)}, i_2 \in Z_4^{(1)}$) can be represented by $\Theta_5$ in the following order:
\[
\Theta_5 \rightarrow \begin{cases}
\theta_{1, r_4} = 180^\circ - \theta_{1, r_1} - \theta_{1, r_2}, \\
\theta_{2, r_3} = 180^\circ - \theta_{2, r_1} - \theta_{2, r_2}.
\end{cases}
\]
\[
\begin{align*}
\theta_{12} &= \arccos \left( \frac{\cos \theta_{3, r_2} - \cos \theta_{1, r_2} \cos \theta_{2, r_2}}{\sin \theta_{2, r_2} \sin \theta_{3, r_2}} \right), \\
\theta_{24} &= \arccos \left( \frac{\cos \theta_{1, r_2} - \cos \theta_{2, r_2} \cos \theta_{3, r_2}}{\sin \theta_{2, r_2} \sin \theta_{3, r_2}} \right).
\end{align*}
\]
\[
\theta_{3, r_1} = \arccos \left( \cos \theta_{1, r_2} \sin \theta_{1, r_1} \sin \theta_{2, r_1} + \cos \theta_{1, r_1} \cos \theta_{2, r_1} \right), \\
\theta_{14} = \arccos \left( \cos \theta_{2, r_3} - \cos \theta_{1, r_2} \cos \theta_{3, r_2} \right) \cdot \frac{\sin \theta_{1, r_2} \sin \theta_{3, r_2}}{\sin \theta_{2, r_2} \sin \theta_{3, r_2}}
\]
\[
\theta_{24} = \arccos \left( \frac{\cos \theta_{24} \sin \theta_{14} + \cos \theta_{1, r_2} \sin \theta_{24} \cos \theta_{14}}{\sqrt{\cos \theta_{24} \sin \theta_{14} + \cos \theta_{1, r_2} \sin \theta_{24} \cos \theta_{14}}^2 + \sin^2 \theta_{1, r_2} \sin^2 \theta_{24}} \right),
\]
\[
\theta_{3, r_4} = \arccos \left( \cos \theta_{24} \sin \theta_{14} \sin \theta_{2, r_4} + \cos \theta_{1, r_2} \cos \theta_{2, r_4} \right) \rightarrow \begin{cases}
\theta_{1, r_4} = 180^\circ - \theta_{3, r_1} - \theta_{2, r_4}, \\
\theta_{3, r_4} = 180^\circ - \theta_{3, r_3} - \theta_{2, r_4}.
\end{cases}
\]
Then by (B.1), the remaining three dihedral angles $\theta_{13}$, $\theta_{23}$ and $\theta_{34}$ can be derived similarly.