Bifurcation analysis and chaos of a discrete-time Kolmogorov model

A. Q. Khan a, S. Khaliq a, O. Tunç b, A. Khaliq c, M. Javaid a and I. Ahmed d

aDepartment of Mathematics, University of Azad Jammu & Kashmir, Muzaffarabad, Pakistan; bDepartment of Computer Programing Baskale Vocational School, Van Yuzuncu Yil University, Van, Turkey; cDepartment of Mathematics, Riphah International University, Lahore, Pakistan; dDepartment of Mathematics, Mirpur University of Science and Technology (MUST), Mirpur, Pakistan

ABSTRACT

In this paper, we explore local dynamical characteristics with different topological classifications at fixed points, bifurcations and chaos in the discrete Kolmogorov model. More precisely, we investigate the existence of trivial, boundary and interior fixed points of the discrete Kolmogorov model by algebraic techniques. We prove that for all involved parameters, the discrete Kolmogorov model has trivial and two boundary fixed points, and the interior fixed point under specific parametric condition. Further we explore the local dynamics with topological classifications at fixed points and existence of periodic points of the discrete Kolmogorov model simultaneously. We also explore the occurrence of bifurcation at fixed points and prove that at boundary points there exists no flip bifurcation but it occurs at the interior fixed point. Moreover, we utilize feedback control method to stabilize chaos appears in the Kolmogorov model. Finally, we present numerical simulations to verify corresponding theoretical results and also reveal some new dynamics.

1. Introduction

In mathematics, diffeomorphism is an invertible function that maps one differentiable manifold to another such that both the function and its inverse are smooth. In nature this property can be examined in different states of ecosystem. In ecosystem, different components interact with each other and also with the environment such as structure of trophic, relation among diversity production and flow of energy emerging from the variations of components that are interacted with each other [1]. These interactions affect the time taken by the ecosystem in its growth and therefore describing in cross scale-lay out solves the basic issues of services related to ecosystem [2–4]. An ecosystem develops over time in a well competitive adaptive manner. During its development competition exists between different components both at inter-specific and intra-specific levels, and is the basic class of dynamics [5,6].

In natural frameworks, time can be considered as a continuous function for many models [7–9]. In population’s model, a permanent arrangement of deaths and births can be seen but in numerous life situations this standard is not appropriate (e.g. fish’s genesis plan). In discrete type of population, it is not appropriate to take time as standard of an unbroken function or continuous function [10]. Biological systems are the main components of ecosystem. In recent years, number of scientists examined different aspects of a biological system, for example a predator–prey model, host parasite model, food chain model, hyperchaotic system and many others [11–24]. These are all related to a discrete type of population and numerous authors analysed the stability, bifurcation, chaotic behaviour and global dynamics along with biological interpretation of obtained results [25–33]. In a biological system, mutualism is a beneficial relationship exists between two equivalent species. Mutualism exists in large number and this can also be modified. The Lotka–Volterra model provided an opportunity for several scientific models to analyse mutualistic relationships [34–37]. For instance, May [38] suggested the two-species continuous-time Kolmogorov model represented by the following system of differential equations:

\[
\begin{align*}
\frac{dx}{dt} &= r_1 x \left(1 - \frac{x}{\beta_1 + \alpha_1 y}\right), \\
\frac{dy}{dt} &= r_2 y \left(1 - \frac{y}{\beta_2 + \alpha_2 x}\right),
\end{align*}
\]

where \(x(t)\) and \(y(t)\) denote the densities of species, and the parameters \(r_1, r_2, \alpha_1, \alpha_2, \beta_1\) and \(\beta_2\) are the positive numbers. It is important here to mention that the discrete-time models governed by difference equations are more appropriate than the continuous ones in the case where populations have non-overlapping
generations, and discrete models can also provide efficient computational results for numerical simulations. So, in this study we will explore the bifurcation analysis and chaos of the discrete model corresponding to (1). By applying the forward Euler scheme, (1) becomes of the following form:

\[
\begin{align*}
    x_{t+1} &= (1 + h r_1)x_t - \frac{h r_1 x_t^2}{\beta_1 + \alpha_1 y_t}, \\
y_{t+1} &= (1 + h r_2)y_t - \frac{h r_2 y_t^2}{\beta_2 + \alpha_2 x_t},
\end{align*}
\]

(2)

The subsequent section purely dedicated for the dynamical characteristics at fixed points of the discrete Kolmogorov model (2). In Section 3, periodic points of period-1, 2, 3, ..., K of the discrete Kolmogorov model (2) are explored, whereas comprehensive analysis of the bifurcation at fixed points is explored in Section 4. In Section 5, chaos control is explored by feedback control method whereas Section 6 is about the presentation of numerical simulations to verify obtained results. The summary of present study is given in Section 7.

2. Dynamical characteristics at fixed points of discrete Kolmogorov model (2)

The present section is purely devoted for the exploration of dynamical characteristics at equilibrium points of the discrete Kolmogorov model (2) in \( \mathbb{R}^2 = \{(x,y) : x, y \geq 0\} \). For this, first we explore the existence of equilibrium points along with variational matrix.

2.1. Exploration of fixed points along with variational matrix

The summarized results regarding the exploration of fixed points can be stated as a following Lemma.

**Lemma 2.1:** In \( \mathbb{R}^2 = \{(x,y) : x, y \geq 0\} \), discrete Kolmogorov model (2) has at most four fixed points. More specifically,

(1) \( \forall \ r_1, r_2, \alpha_1, \alpha_2, \beta_1, \beta_2, h, \) Kolmogorov model (2) has trivial and boundary fixed points \( P = (0, 0), Q = (0, \beta_2) \) and \( R = (\beta_1, 0) \) respectively;

(2) If \( \alpha_1 < \frac{1}{\alpha_2} \) then Kolmogorov model (2) has interior fixed point \( S = \left( \frac{\beta_1 + \alpha_1 \beta_2}{1 - \alpha_1 \alpha_2}, \frac{\beta_2 + \alpha_2 \beta_1}{1 - \alpha_1 \alpha_2} \right) \).

Additionally, the linearized form of (2) at fixed point \( X = (x, y) \) under the map

\[
(f_1, f_2) \mapsto (x_{t+1}, y_{t+1}),
\]

is

\[
\Omega_{t+1} = V_X \Omega_t,
\]

where

\[
\Omega_t = \begin{pmatrix} x_t \\ y_t \end{pmatrix},
\]

and

\[
V_X = \begin{pmatrix} 1 + h r_1 - \frac{2 h r_1 x_t}{\beta_1 + \alpha_1 y_t} & \frac{h \alpha_1 r_1 x_t^2}{(\beta_1 + \alpha_1 y_t)^2} \\ \frac{h r_2 y_t^2}{(\beta_2 + \alpha_2 x_t)^2} & 1 + h r_2 - \frac{2 h r_2 y_t}{\beta_2 + \alpha_2 x_t} \end{pmatrix},
\]

and

\[
f_1 := (1 + r_1) x - \frac{r_1 h x_t^2}{\beta_1 + \alpha_1 y_t},
\]

\[
f_2 := (1 + r_2) y - \frac{r_2 h y_t^2}{\beta_2 + \alpha_2 x_t}.
\]

2.2. Exploration of topological classifications at fixed points \( P, Q \) and \( R \)

The local dynamical characteristics with different topological classifications at fixed points \( P, Q \) and \( R \) of the discrete Kolmogorov model (2) are summarized in Table 1.

2.3. Exploration of topological classifications at fixed point \( S \)

The variational matrix at fixed point \( S \) is

\[
V_S = \begin{pmatrix} 1 - h r_1 & h \alpha_1 r_1 \\ h \beta_2 r_2 & 1 - h r_2 \end{pmatrix},
\]

with corresponding characteristic equation is of the form

\[
\lambda^2 + \varrho_1 \lambda + \varrho_2 = 0,
\]

where

\[
\varrho_1 = h (r_1 + r_2) - 2,
\]

\[
\varrho_2 = 1 - h (r_1 + r_2) + h^2 r_1 r_2 (1 - \alpha_1 \alpha_2).
\]

Finally, roots of (9) become

\[
\lambda_{1,2} = \frac{-\varrho_1 \pm \sqrt{\Delta}}{2},
\]

where

\[
\Delta = \varrho_1^2 - 4 \varrho_2,
\]

\[= h^2 (r_1 - r_2)^2 + 4 r_1 r_2 \alpha_1 \alpha_2 > 0.\]

Table 1. Topological classifications at fixed points \( P, Q \) and \( R \).

| Fixed points | Corresponding behaviour |
|--------------|------------------------|
| \( P \)      | Source but never sink, saddle and non-hyperbolic |
| \( Q \)      | Never sink; source if \( h > \frac{2}{r_2} \); saddle if \( 0 < h < \frac{2}{r_2} \); non-hyperbolic if \( h = \frac{2}{r_2} \); |
| \( R \)      | Never sink; source if \( h > \frac{2}{r_1} \); saddle if \( 0 < h < \frac{2}{r_1} \); non-hyperbolic if \( h = \frac{2}{r_1} \); |

Since \( \Delta > 0 \), therefore it is important here to note that fixed point \( S \) is never stable focus, unstable focus and non-hyperbolic. So we will summarize the behaviour of the discrete Kolmogorov model (2) at \( S \) in Table 2.
Table 2. Topological classifications at fixed point $S$.

| Properties          | Respective parametric conditions                                      |
|---------------------|---------------------------------------------------------------------|
| Stable node         | $0 < h < \min\left\{ \frac{r_1 + r_2 - \sqrt{(r_1 - r_2)^2 + 4r_1r_2\alpha_2}}{r_2(1 - \alpha_1\alpha_2)}, \frac{r_1 + r_2 - \sqrt{(r_1 - r_2)^2 + 4r_1r_2\alpha_2}}{r_2(1 - \alpha_1\alpha_2)} \right\}$ |
| Unstable node       | $h > \max\left\{ \frac{r_1 + r_2 - \sqrt{(r_1 - r_2)^2 + 4r_1r_2\alpha_2}}{r_2(1 - \alpha_1\alpha_2)}, \frac{r_1 + r_2 - \sqrt{(r_1 - r_2)^2 + 4r_1r_2\alpha_2}}{r_2(1 - \alpha_1\alpha_2)} \right\}$ |
| Non-hyperbolic      | $h = \frac{r_1 + r_2 - \sqrt{(r_1 - r_2)^2 + 4r_1r_2\alpha_2}}{r_2(1 - \alpha_1\alpha_2)}$ or $h = \frac{r_1 + r_2 - \sqrt{(r_1 - r_2)^2 + 4r_1r_2\alpha_2}}{r_2(1 - \alpha_1\alpha_2)}$ |

3. Exploration of periodic points having period-1, 2, 3, ..., $n$ of the discrete Kolmogorov model (2)

The periodic points having period-1, 2, 3, ..., $n$ of the discrete Kolmogorov model (2) are explored in this section by utilizing Definition 1.1.2 of [39]. More precisely, these results are summarized as a following proposition.

**Proposition 3.1:** For exploration of periodic points having period-1, 2, 3, ..., $n$ following results hold:

1. Fixed point $P$ of the Kolmogorov model (2) is a periodic point having period-1, 2, 3, ..., $n$.
2. Fixed point $Q$ of the Kolmogorov model (2) is a periodic point having period-1, 2, 3, ..., $n$.
3. Fixed point $R$ of the Kolmogorov model (2) is a periodic point having period-1, 2, 3, ..., $n$.
4. Fixed point $S$ of the Kolmogorov model (2) is a periodic point having period-1, 2, 3, ..., $n$.

**Proof:** (1) From (2), one can define

$$ F(x, y) := (f_1, f_2), \quad (13) $$

where $f_1$ and $f_2$ are defined in (7). Now after some computation from (13), one gets

$$ F = (f_1, f_2), \Rightarrow F_{P=0,0} = P, $$

$$ F^2 = \left(1 + r_2h\right) f_1 + \frac{r_1h^2}{\alpha_1 + \beta_1f_2}, $$

$$ (1 + r_2h) f_2 + \frac{r_2h^2}{\alpha_2 + \beta_2f_1} \Rightarrow F_{P=0,0}^2 = P, $$

$$ F^3 = \left(1 + r_1h\right) f_2 + \frac{r_1h(f_2)^2}{\alpha_1 + \beta_1f_1}, $$

$$ (1 + r_2h) f_2 + \frac{r_2h(f_2)^2}{\alpha_2 + \beta_2f_1} \Rightarrow F_{P=0,0}^3 = P, $$

$$ \vdots $$

$$ F^n = \left(1 + r_1h\right) f_1^{n-1} + \frac{r_1h(f_1)^{n-1}}{\alpha_1 + \beta_1f_1^n}, $$

which shows that $S$ of Kolmogorov model (2) is a periodic point having period-1, 2, 3, ..., $n$. 

From (14), one can conclude that fixed point $P$ of the Kolmogorov model (2) is a periodic point having period-1, 2, 3, ..., $n$.

(2) From (14), we have

$$ F_{Q=0,0} = Q, $$

$$ F_{Q=0,0}^2 = Q, $$

$$ F_{Q=0,0}^3 = Q, $$

$$ \vdots $$

$$ F_{Q=0,0}^n = Q, $$

which shows that $Q$ of the Kolmogorov model (2) is a periodic point having period-1, 2, 3, ..., $n$.

(3) Similarly from (14), the computation yields

$$ F_{R=0,0} = R, $$

$$ F_{R=0,0}^2 = R, $$

$$ F_{R=0,0}^3 = R, $$

$$ \vdots $$

$$ F_{R=0,0}^n = R, $$

which shows that $R$ of the Kolmogorov model (2) is a periodic point having period-1, 2, 3, ..., $n$.

(4) Finally from (14), we have

$$ F_{S=0,0} = S, $$

$$ F_{S=0,0}^2 = S, $$

$$ F_{S=0,0}^3 = S, $$

$$ \vdots $$

$$ F_{S=0,0}^n = S, $$

which shows that $S$ of Kolmogorov model (2) is a periodic point having period-1, 2, 3, ..., $n$. 

\[\square\]
4. Bifurcation analysis of the discrete-time Kolmogorov model (2) at Q, R and S

The full possible bifurcation analysis of the discrete-time Kolmogorov model (2) at Q, R and S by bifurcation theory \([40,41]\) is given in this section. Our investigation in this section reveals that at boundary fixed points Q, R there exist no flip bifurcation but flip bifurcation occurs at interior fixed point S and no other bifurcation occurs at it. So, for the completeness of this section first we will give definition of flip bifurcation as follows:

**Definition 4.1:** Bifurcation related to the existence of \(\lambda_1 = -1\) is known as flip bifurcation.

4.1. Bifurcation analysis at Q

From obtained results which are depicted in Table 1, it is noted that Q is non-hyperbolic if \(h = \frac{2}{r_2}\). Therefore eigenvalues of \(\mathcal{V}_Q\) at \(h = \frac{2}{r_2}\) are \(\lambda_2|_{h=\frac{2}{r_2}} = -1\) but \(\lambda_1|_{h=\frac{2}{r_2}} = 1 + r_1\frac{r_2}{r_1} \neq 1\) or \(-1\), which conclude that at Q Kolmogorov model (2) may undergo flip bifurcation if involved parameters \((h, \alpha_1, \alpha_2, r_1, r_2, \beta_1, \beta_2)\) are located in the following set:

\[
\mathcal{F}|_Q = \{(h, \alpha_1, \alpha_2, r_1, r_2, \beta_1, \beta_2), h = \frac{2}{r_2}\}. \tag{18}
\]

The following theorem shows that at Q no flip bifurcation exists if \((h, \alpha_1, \alpha_2, r_1, r_2, \beta_1, \beta_2) \in \mathcal{F}|_Q\).

**Theorem 4.2:** If \((h, \alpha_1, \alpha_2, r_1, r_2, \beta_1, \beta_2) \in \mathcal{F}|_Q\) then discrete Kolmogorov model (2) cannot undergo flip bifurcation.

**Proof:** It is noted that discrete Kolmogorov model (2) is invariant with respect to \(x = 0\) and hence for exploring mentioned bifurcation one restrict it to the line \(x = 0\), where (2) takes the form

\[
y_{t+1} = (1 + r_2 h) y_t - \frac{r_2 h}{\beta_2} y_t^2. \tag{19}
\]

Now from (19), one denote the map

\[
f_2 := (1 + r_2 h) y - \frac{r_2 h}{\beta_2} y^2. \tag{20}
\]

From (20), the computation yields

\[
\frac{\partial f_2}{\partial y} |_{h=h^*, y=y^*} = -1, \tag{21}
\]

\[
\frac{\partial^2 f_2}{\partial y^2} |_{h=h^*, y=y^*} = -\frac{4}{\beta_2} \neq 0, \tag{22}
\]

and

\[
\frac{\partial^2 f_2}{\partial h} |_{h=h^*, y=y^*} = 0, \tag{23}
\]

where \(h^* = \frac{2}{r_1}\) and \(y^* = \beta_2\). From (23), one can conclude that discrete Kolmogorov model (2) cannot undergo flip bifurcation if \((h, \alpha_1, \alpha_2, r_1, r_2, \beta_1, \beta_2) \in \mathcal{F}|_Q\).

4.2. Bifurcation analysis at R

From summarized results, which are depicted in Table 1, it is noted that fixed point R is non-hyperbolic if \(h = \frac{2}{r_1}\). Therefore, eigenvalues of \(\mathcal{V}_R\) at \(h = \frac{2}{r_1}\) are \(\lambda_1|_{h=\frac{2}{r_1}} = -1\) but \(\lambda_2|_{h=\frac{2}{r_1}} = 1 + r_2\frac{r_1}{r_2} \neq 1\) or \(-1\), which conclude that at R Kolmogorov model (2) may undergo flip bifurcation if involved parameters \((h, \alpha_1, \alpha_2, r_1, r_2, \beta_1, \beta_2)\) are located in the following set:

\[
\mathcal{F}|_R = \{(h, \alpha_1, \alpha_2, r_1, r_2, \beta_1, \beta_2), h = \frac{2}{r_1}\}. \tag{24}
\]

But by computation, the following theorem shows that it cannot occur if \((h, \alpha_1, \alpha_2, r_1, r_2, \beta_1, \beta_2) \in \mathcal{F}|_R\).

**Theorem 4.3:** If \((h, \alpha_1, \alpha_2, r_1, r_2, \beta_1, \beta_2) \in \mathcal{F}|_R\), then discrete Kolmogorov model (2) cannot undergo flip bifurcation.

**Proof:** Since discrete Kolmogorov model (2) is invariant with respect to \(y = 0\) and hence for exploring the flip bifurcation one restrict it to the line \(y = 0\), where (2) takes the form

\[
x_{t+1} = (1 + r_1 h) x_t - \frac{r_1 h}{\beta_1} x_t^2. \tag{25}
\]

Now from (25), one denote the map

\[
f_1 := (1 + r_1 h) x - \frac{r_1 h}{\beta_1} x^2. \tag{26}
\]

From (26), the computation yields

\[
\frac{\partial f_1}{\partial x} |_{h=h^*, y=y^*} = -1, \tag{27}
\]

\[
\frac{\partial^2 f_1}{\partial x^2} |_{h=h^*, y=y^*} = -\frac{4}{\beta_1} \neq 0, \tag{28}
\]

and

\[
\frac{\partial^2 f_1}{\partial h} |_{h=h^*, y=y^*} = 0, \tag{29}
\]

where \(h^* = \frac{2}{r_1}\) and \(y^* = \beta_1\). From (29), one can conclude that discrete Kolmogorov model (2) cannot undergo flip bifurcation if \((h, \alpha_1, \alpha_2, r_1, r_2, \beta_1, \beta_2) \in \mathcal{F}|_R\).

4.3. Bifurcation analysis at S

From Table 2, it is noted that fixed point S of the discrete-time Kolmogorov model (2) is non-hyperbolic if \(h = \frac{2}{r_2}\). Therefore eigenvalues of variational matrix \(\mathcal{V}_S\) at non-hyperbolic condition \(h = \frac{2}{r_2}\) are computed and one gets

\[
\lambda_1|_{h=\frac{2}{r_2}} = -1 \tag{30}
\]

\[
\lambda_2|_{h=\frac{2}{r_2}} = 1 + r_1\frac{r_2}{r_1} \neq 1 \tag{31}
\]
\begin{align*}
= 3 - \frac{(r_1 + r_2)^2}{r_1 r_2 (1 - \alpha_1 \alpha_2)} & \quad - \frac{(r_1 + r_2)\sqrt{(r_1 - r_2)^2 + 4r_1 r_2 \alpha_1 \alpha_2}}{r_1 r_2 (1 - \alpha_1 \alpha_2)} \neq 1 \text{ or } -1,
\end{align*}

which conclude that at S, Kolmogorov model (2) may undergo flip bifurcation if \((h, \alpha_1, \alpha_2, r_1, r_2, \beta_1, \beta_2)\) are located in the following set:

\[ F_{15} = \left\{ \begin{array}{l}
(h, \alpha_1, \alpha_2, r_1, r_2, \beta_1, \beta_2), \\
h = \frac{r_1 + r_2 + \sqrt{(r_1 - r_2)^2 + 4r_1 r_2 \alpha_1 \alpha_2}}{r_1 r_2 (1 - \alpha_1 \alpha_2)} \end{array} \right\} \quad \text{(30)} \]

Hereafter in the following, we will present comprehensive flip bifurcation analysis at S of the Kolmogorov model (2).

**Theorem 4.4:** If \((h, \alpha_1, \alpha_2, r_1, r_2, \beta_1, \beta_2) \in F_{15}\), then discrete Kolmogorov model (2) undergoes the flip bifurcation.

**Proof:** From Table 2, it is recalled that S is non-hyperbolic if \(h = \frac{r_1 + r_2 + \sqrt{(r_1 - r_2)^2 + 4r_1 r_2 \alpha_1 \alpha_2}}{r_1 r_2 (1 - \alpha_1 \alpha_2)}\). Moreover, at parametric condition \(h = \frac{r_1 + r_2 + \sqrt{(r_1 - r_2)^2 + 4r_1 r_2 \alpha_1 \alpha_2}}{r_1 r_2 (1 - \alpha_1 \alpha_2)}\), \(\lambda_1|_{h = \frac{r_1 + r_2 + \sqrt{(r_1 - r_2)^2 + 4r_1 r_2 \alpha_1 \alpha_2}}{r_1 r_2 (1 - \alpha_1 \alpha_2)}} = -1 \) whereas

\[ \lambda_2|_{h = \frac{r_1 + r_2 + \sqrt{(r_1 - r_2)^2 + 4r_1 r_2 \alpha_1 \alpha_2}}{r_1 r_2 (1 - \alpha_1 \alpha_2)}} = 3 - \frac{(r_1 + r_2)^2}{r_1 r_2 (1 - \alpha_1 \alpha_2)} & \quad - \frac{(r_1 + r_2)\sqrt{(r_1 - r_2)^2 + 4r_1 r_2 \alpha_1 \alpha_2}}{r_1 r_2 (1 - \alpha_1 \alpha_2)} \neq 1 \text{ or } -1,
\]

which gives the existence of flip bifurcation at S by choosing h as a bifurcation parameter. So if h varies in a neighbourhood of \(h^*\) then model (2) takes the following form:

\[
x_{t+1} = (1 + r_1(h^* + \epsilon))x_t - \frac{(h^* + \epsilon) r_1 x_t^2}{\beta_1 + \alpha_1 y_t}, \\
y_{t+1} = (1 + r_2(h^* + \epsilon))y_t - \frac{(h^* + \epsilon) r_2 y_t^2}{\beta_2 + \alpha_2 x_t}.
\]

Now using the following transformation in order to transform S into \(P = O(0,0)\)

\[
u_t = x_t - x^*, \quad v_t = y_t - y^*.
\]

In view of (32), (31) takes the following form:

\[
\begin{align*}
u_{t+1} &= D_{11} u_t + D_{12} v_t + D_{13} u_t^2 + D_{14} u_t v_t + D_{15} v_t^2 \\
&+ \Gamma_{01} u_t v_t + \Gamma_{02} v_t v_t + \Gamma_{03} u_t^2 v_t \\
&+ \Gamma_{04} u_t v_t + \Gamma_{05} v_t^2,
\end{align*}
\[
\begin{align*}
u_{t+1} &= D_{21} u_t + D_{22} v_t + D_{23} u_t^2 + D_{24} u_t v_t + D_{25} v_t^2 \\
&+ \Gamma_{06} u_t v_t + \Gamma_{07} v_t v_t + \Gamma_{08} u_t^2 v_t \\
&+ \Gamma_{09} u_t v_t + \Gamma_{10} v_t^2,
\end{align*}
\]

\[
\begin{align*}
D_{11} &= 1 + r_1 h^* - \frac{2h^* r_1 x^*}{\beta_1 + \alpha_1 y^*}, \quad D_{12} = \frac{h^* r_1 x^*}{(\beta_1 + \alpha_1 y^*)^2}, \\
D_{13} &= -\frac{h^* r_1}{\beta_1 + \alpha_1 y^*}, \\
D_{14} &= \frac{2\alpha_1 h^* (r_1 x^*)}{(\beta_1 + \alpha_1 y^*)^2}, \\
D_{15} &= -\frac{h^* \alpha_1^2 r_1 x^2}{(\beta_1 + \alpha_1 y^*)^3}, \\
\Gamma_{01} &= r_1 - \frac{2r_1 x^*}{\beta_1 + \alpha_1 y^*}, \\
\Gamma_{02} &= \frac{\alpha_1 r_1 x^*}{(\beta_1 + \alpha_1 y^*)^2}, \quad \Gamma_{03} = -\frac{r_1}{\beta_1 + \alpha_1 y^*}, \\
\Gamma_{04} &= \frac{2\alpha_1 r_1 x^*}{(\beta_1 + \alpha_1 y^*)^2}, \\
\Gamma_{05} &= -\frac{\alpha_1^2 r_1 x^2}{(\beta_1 + \alpha_1 y^*)^3}, \\
D_{21} &= \frac{h^* \alpha_2 r_2 y^*}{(\beta_2 + \alpha_2 x^*)^2}, \\
D_{22} &= 1 + r_2 h^* - \frac{2h^* r_2 y^*}{\beta_2 + \alpha_2 x^*}, \\
D_{23} &= \frac{h^* \alpha_2^2 r_2 y^2}{(\beta_2 + \alpha_2 x^*)^3}, \\
D_{24} &= \frac{2h^* \alpha_2 r_2 y^*}{(\beta_2 + \alpha_2 x^*)^2}, \\
D_{25} &= \frac{h^* r_2}{(\beta_2 + \alpha_2 x^*)^2}, \\
D_{26} &= \frac{\alpha_2 r_2 y^*}{(\beta_2 + \alpha_2 x^*)^2}, \\
\Gamma_{06} &= \frac{\alpha_2^2 r_2 y^2}{(\beta_2 + \alpha_2 x^*)^3}, \\
\Gamma_{07} &= \frac{r_2}{(\beta_2 + \alpha_2 x^*)^2}, \\
\Gamma_{08} &= \frac{2\alpha_2^2 r_2 y^2}{(\beta_2 + \alpha_2 x^*)^3}, \\
\Gamma_{09} &= \frac{2\alpha_2 r_2 y^*}{(\beta_2 + \alpha_2 x^*)^2}, \quad \Gamma_{10} = -\frac{r_2}{(\beta_2 + \alpha_2 x^*)^2}.
\end{align*}
\]

Now using the following translation:

\[
\begin{pmatrix}
u_t \\
v_{t+1}
\end{pmatrix} = \begin{pmatrix} D_{12} & D_{11} \\ -1 - D_{11} & \lambda_2 - D_{11} \end{pmatrix} \begin{pmatrix} X_t \\
Y_t
\end{pmatrix} + \begin{pmatrix} P(\epsilon) \\
Q(\epsilon)
\end{pmatrix},
\]

(35)

(33) becomes

\[
\begin{pmatrix} X_{t+1} \\
Y_{t+1}
\end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} X_t \\
Y_t
\end{pmatrix} + \begin{pmatrix} P(\epsilon) \\
Q(\epsilon)
\end{pmatrix},
\]

(36)

where

\[
P(\epsilon) = \frac{D_{13} (\lambda_2 - D_{11}) - D_{12} D_{23} u_t^2}{D_{12} (1 + \lambda_2)} + \frac{D_{14} (\lambda_2 - D_{11}) - D_{12} D_{24} u_t v_t}{D_{12} (1 + \lambda_2)} + \frac{D_{15} (\lambda_2 - D_{11}) - D_{12} D_{25} v_t^2}{D_{12} (1 + \lambda_2)} + \frac{\Gamma_{01} (\lambda_2 - D_{11}) - D_{12} \Gamma_{06} u_t v_t}{D_{12} (1 + \lambda_2)} + \frac{\Gamma_{02} (\lambda_2 - D_{11}) - D_{12} \Gamma_{07} v_t^2}{D_{12} (1 + \lambda_2)}.
\]
Hereafter for the map (36), centre manifold
The computation yields
\[ Q(\epsilon) = \frac{D_{13}(1 + D_{11}) + D_{12}D_{23}u_t}{D_{12}(1 + \lambda_2)} + \frac{D_{14}(1 + D_{11}) + D_{12}D_{24}u_t}{D_{12}(1 + \lambda_2)} + \frac{D_{15}(1 + D_{11}) + D_{12}D_{25}v_t}{D_{12}(1 + \lambda_2)} + \frac{\Gamma_{01}(1 + D_{11}) + D_{12}\Gamma_{06}u_t}{D_{12}(1 + \lambda_2)} + \frac{\Gamma_{02}(1 + D_{11}) + D_{12}\Gamma_{07}v_t}{D_{12}(1 + \lambda_2)} + \frac{\Gamma_{03}(1 + D_{11}) + D_{12}\Gamma_{08}u_t}{D_{12}(1 + \lambda_2)} + \frac{\Gamma_{04}(1 + D_{11}) + D_{12}\Gamma_{09}u_t}{D_{12}(1 + \lambda_2)} + \frac{\Gamma_{05}(1 + D_{11}) + D_{12}\Gamma_{06}v_t}{D_{12}(1 + \lambda_2)} + \frac{\Gamma_{06}(1 + D_{11}) + D_{12}\Gamma_{07}v_t}{D_{12}(1 + \lambda_2)} + \frac{\Gamma_{07}(1 + D_{11}) + D_{12}\Gamma_{08}v_t}{D_{12}(1 + \lambda_2)} + \frac{\Gamma_{08}(1 + D_{11}) + D_{12}\Gamma_{09}v_t}{D_{12}(1 + \lambda_2)} + \frac{\Gamma_{09}(1 + D_{11}) + D_{12}\Gamma_{10}v_t}{D_{12}(1 + \lambda_2)} \]
\[ u_t^2 = D_{12}^2(X_t^2 + 2X_tY_t + Y_t^2), \]
\[ u_tv_t = -D_{12}(1 + D_{11})X_t^2 + (D_{12}(\lambda_2 - D_{11}) - D_{12}(1 + D_{11}))X_tY_t + D_{12}(\lambda_2 - D_{11})Y_t^2, \]
\[ v_t^2 = (1 + D_{11})^2X_t^2 - 2(1 + D_{11})(\lambda_2 - D_{11})X_tY_t + (\lambda_2 - D_{11})^2Y_t^2, \]
\[ u_t = D_{12}X_t^2 + D_{12}Y_t^2, \]
\[ v_t = (-1 - D_{11})X_t^2 + (\lambda_2 - D_{11})Y_t^2, \]
\[ u_t^2 = D_{12}^2(X_t^2 + 2X_tY_t + Y_t^2), \]
\[ u_tv_t = -D_{12}(1 + D_{11})X_t^2 + (D_{12}(\lambda_2 - D_{11}) - D_{12}(1 + D_{11}))X_tY_t + D_{12}(\lambda_2 - D_{11})Y_t^2, \]
\[ v_t^2 = (1 + D_{11})^2X_t^2 - 2(1 + D_{11})(\lambda_2 - D_{11})X_tY_t + (\lambda_2 - D_{11})^2Y_t^2. \]
\[ h_2 = \frac{(1 + D_{11})^2[D_{13}(1 + D_{11}) + D_{12}D_{25}]}{D_{12}(1 - \lambda_2^2)}, \]
\[ h_3 = 0. \] (39)

Finally the map (36) is restricted to \( M^P \) as follows:
\[ f(x_t) = -x_t + v_tX_t^2 + v_2x_t^2 + v_3x_t^2 + v_4x_t^2 + v_5x_t^2 + O((|X_t| + |\epsilon|)^4), \] (40)

where
\[ v_1 = \frac{1}{1 + \lambda_2} \left[ \frac{D_{12}D_{13}(\lambda_2 - D_{11}) - (1 + D_{11})}{D_{12}(1 + \lambda_2)} \times (D_{14}(\lambda_2 - D_{11}) - D_{12}D_{24}) - D_{15}^2D_{23} - D_{25}(1 + D_{11})^2 \frac{D_{15}(\lambda_2 - D_{11})(1 + D_{11})^2}{D_{12}} \right], \]
\[ v_2 = \frac{1}{1 + \lambda_2} \left[ \Gamma_{01}(\lambda_2 - D_{11}) - \Gamma_{06}D_{12} + \Gamma_{07}(1 + D_{11}) - \frac{\Gamma_{02}(\lambda_2 - D_{11})(1 + D_{11})}{D_{12}} \right], \]
\[ v_3 = \frac{1}{1 + \lambda_2} \left[ 2h_2D_{12}D_{13}(\lambda_2 - D_{11}) - 2h_2D_{12}D_{23} + (h_2D_{12} - D_{12}^2D_{23}) + (h_2D_{12} - D_{12})(1 + D_{11})) \times (D_{14}(\lambda_2 - D_{11}) - D_{12}D_{24}) + 2h_2D_{12}(1 + D_{11})(\lambda_2 - D_{11}) - \frac{D_{15}(\lambda_2 - D_{11})(1 + D_{11})}{D_{12}} \right], \]
\[ h_4 = \frac{1}{1 + \lambda_2} \left[ \frac{h_1(\lambda_2) - D_{11}}{D_{12}}(\Gamma_{02}(\lambda_2 - D_{11}) - D_{12}\Gamma_{07}) + (\Gamma_{03}(\lambda_2 - D_{11}) - D_{12}\Gamma_{08})D_{12} - (1 + D_{11})(\Gamma_{04}(\lambda_2 - D_{11}) - D_{12}\Gamma_{09}) + (1 + D_{11})^2(\Gamma_{05}(\lambda_2 - D_{11}) - D_{12}\Gamma_{10}) \right], \]
\[ v_4 = \frac{1}{1 + \lambda_2} \left[ \frac{2D_{12}(1 + D_{11})(\lambda_2 - D_{11}) - D_{12}D_{23}}{D_{12}} + (h_1(\lambda_2 - D_{11}) - h_1(1 + D_{11})) \right], \]
\[
\begin{align*}
\times (D_{14} (\lambda_2 - D_{11}) - D_{12} D_{24}) \\
= 2h_1 (\lambda_2 - D_{11}) (1 + D_{11}) \\
\times (D_{15} (\lambda_2 - D_{11}) - D_{12} D_{25}) \right]. 
\end{align*}
\]

(41)

In order to ensure flip bifurcation for map (40), the discriminatory quantities should be non-zero \([40, 41]\)

\[
\begin{align*}
Q_1 &= \left( \frac{\partial^2 f}{\partial x_1^2} + \frac{1}{2} \frac{\partial f}{\partial x_1} \frac{\partial^2 f}{\partial x_1^2} \right)_{p} \\
Q_2 &= \left( \frac{1}{6} \frac{\partial^3 f}{\partial x_1^3} + \frac{1}{2} \frac{\partial^2 f}{\partial x_1^2} \right)_{p}.
\end{align*}
\]

(42)

After calculating, one gets

\[
\begin{align*}
Q_1 &= \frac{r_1 r_2}{4r_1 r_2 (1 - \alpha_1 \alpha_2) - (r_1 + r_2)^2 - (r_1 + r_2) A} \\
&\times \left[ -\alpha_1 \alpha_2 (r_1 + r_2) + \alpha_1 \alpha_2 A + \frac{r_1 (1 - \alpha_1 \alpha_2)}{r_1 + r_2 + A} \\
&\times (r_1 + r_2 + A - 2r_1 (1 - \alpha_1 \alpha_2)) \right] \neq 0,
\end{align*}
\]

(43)

\[
\begin{align*}
Q_2 &= \frac{h_1 r_1 r_2 (1 - \alpha_1 \alpha_2)}{4r_1 r_2 (1 - \alpha_1 \alpha_2) - (r_1 + r_2)^2 - (r_1 + r_2) A} \\
&\times \left[ \frac{2 (r_1 + r_2 + A)^2}{r_1 r_2 (1 - \alpha_1 \alpha_2)^2} \left( \alpha_1 (r_1 + r_2) \\
&+ \alpha_1 A - 2 \alpha_1 r_1 (1 - \alpha_1 \alpha_2) \right) + 2 \alpha_1^2 \alpha_2^2 A^2 \\
&\times \frac{r_1 + r_2 + A}{r_1 (1 - \alpha_1 \alpha_2)} \\
&\times \left( \frac{r_1^2 - r_2^2}{r_1 r_2 (1 - \alpha_1 \alpha_2)} + \frac{(r_1 - r_2) A}{r_1 r_2 (1 - \alpha_1 \alpha_2)} \right) \\
&\times \left( 4 \alpha_1 - 2 \alpha_1 (r_1 + r_2) \right) \frac{r_1 + r_2 + A}{r_1 (1 - \alpha_1 \alpha_2)} \\
&\times \left( - \frac{r_1 + r_2 + A}{r_1 (1 - \alpha_1 \alpha_2)} - \frac{2 \alpha_1 (1 - \alpha_1 \alpha_2)}{r_1 (1 - \alpha_1 \alpha_2)} \right) \\
&\times \left( - \frac{r_1 + r_2 + A}{r_1 (1 - \alpha_1 \alpha_2)} - \frac{2 \alpha_1 (1 - \alpha_1 \alpha_2)}{r_1 (1 - \alpha_1 \alpha_2)} \right) \\
&\times \left( \frac{r_1 + r_2 + A}{r_1 (1 - \alpha_1 \alpha_2)} + \frac{\alpha_1 A}{r_1 (1 - \alpha_1 \alpha_2)} \right) \\
&\times \left( \frac{4 (r_1 + r_2)^2}{r_1 r_2 (1 - \alpha_1 \alpha_2)^2} + \frac{4 \alpha_1 (r_1 + r_2)}{r_1 r_2 (1 - \alpha_1 \alpha_2)} \right) \\
&\times \left( - \frac{4 (r_1 + r_2)^2}{r_1 r_2 (1 - \alpha_1 \alpha_2)^2} - \frac{8 \alpha_1 a_2}{r_1 r_2 (1 - \alpha_1 \alpha_2)} \right) \\
&\times \left( - \frac{4 (r_1 + r_2) A}{r_1 r_2 (1 - \alpha_1 \alpha_2)} \right) \right].
\end{align*}
\]

(44)

where

\[
\begin{align*}
h_1 &= \frac{r_1 r_2 (1 - \alpha_1 \alpha_2)}{r_1^2 r_2^2 (1 - \alpha_1 \alpha_2)^2} \\
&\times \left[ -\frac{\alpha_1^2 \alpha_2^2 (r_1 + r_2)^3}{r_1 (1 - \alpha_1 \alpha_2) (r_1 + r_2 + A)^3} \\
&\times \left( 2 \alpha_1 (r_1 + r_2 + A) \right) \\
&\times \left( \frac{r_1 r_2 (1 - \alpha_1 \alpha_2) (r_1 + r_2 + A)}{r_1 (1 - \alpha_1 \alpha_2) (r_1 + r_2 + A) \beta_1 + \alpha_1 \beta_2} \right) \\
&\times \left( 4 \alpha_1^2 (r_1 + r_2 + A) \beta_2 + \alpha_1 \beta_1 \right) \\
&\times \left( 2 \alpha_1^2 (r_1 + r_2 + A) \beta_1 + \alpha_1 \beta_2 \right) \left( 2 \alpha_1 (r_1 + r_2) \left( \beta_2 - \alpha_2 \beta_1 \right) \right) \\
&\times \left( 2 \alpha_1 (r_1 + r_2 + A) \beta_1 + \alpha_1 \beta_2 \right) \\
&\times \left( 2 \alpha_1 (r_1 + r_2) \beta_2 + \alpha_1 \beta_1 \right) \right].
\end{align*}
\]

(45)

and

\[
A = \sqrt{(r_1 - r_2)^2 + 4r_1 r_2 \alpha_1 \alpha_2}.
\]

(46)

Now in view of (45) and (46), from (44) if one gets \(Q_2 \neq 0\) as \((h, \alpha_1, \alpha_2, r_1, r_2, \beta_1, \beta_2) \in \mathcal{F}\) then discrete Kolmogorov model (2) undergo the flip bifurcation. Additionally period-2 points from \(S\) are stable (respectively unstable) if \(Q_2 > 0\) (respectively \(Q_2 < 0\).
**Remark 4.1:** Since \( \Delta = h^2((r_1 - r_2)^2 + 4r_1r_2\alpha_1\alpha_2) > 0 \), and hence fixed point \( S \) of the discrete-time Kolmogorov model (2) is neither stable focus, unstable focus and non-hyperbolic. Therefore, one can conclude that at fixed point \( S \) of the discrete-time Kolmogorov model (2) no hopf bifurcation occurs.

5. Chaos control

This section is purely devoted for the exploration of chaos control in the sense of state feedback control method [10,32,42–45]. For the completeness of this section, first we will give the definition of marginal stability.

**Definition 5.1 ([46,47]):** System is marginally stable if it is neither asymptotically stable nor unstable. Moreover we can also say that system is marginally stable if its impulse response is bounded.

Now discrete Kolmogorov model (2) takes the following form:

\[
x_{t+1} = (1 + r_1 h)x_t - \frac{r_1 h x^2_t}{\beta_1 + \alpha_1 y_t} + u_t
\]

\[
y_{t+1} = (1 + r_2 h)y_t - \frac{r_2 h y^2_t}{\beta_2 + \alpha_2 x_t},
\]

by adding \( u_t = -l_1(x_t - \frac{\beta_1 + \alpha_1 \beta_2}{1 - \alpha_1 \alpha_2}) - l_2(y_t - \frac{\beta_2 + \alpha_2 \beta_1}{1 - \alpha_1 \alpha_2}) \) as a control force, where \( l_1, l_2 \) denotes feedback gains. Moreover, the variational matrix \( V^C_S \) for the controlled system (47) evaluated at interior fixed point \( S \) under the map

\[
(f^C_1, f^C_2) \mapsto (x_{t+1}, y_{t+1}),
\]

where

\[
f^C_1 : = (1 + r_1 h)x_t - \frac{r_1 h x^2_t}{\beta_1 + \alpha_1 y_t} - l_1 \left( x_t - \frac{\beta_1 + \alpha_1 \beta_2}{1 - \alpha_1 \alpha_2} \right)
\]

\[
- l_2 \left( y_t - \frac{\beta_2 + \alpha_2 \beta_1}{1 - \alpha_1 \alpha_2} \right),
\]

\[
f^C_2 : = (1 + r_2 h)y_t - \frac{r_2 h y^2_t}{\beta_2 + \alpha_2 x_t},
\]

is

\[
V^C_S = \begin{pmatrix}
1 - r_1 h - l_1 & hr_1 \alpha_1 - l_2 \\
hr_2 \alpha_2 & 1 - r_2 h
\end{pmatrix}.
\]

**Figure 1.** Stable node of the Kolmogorov model (2): (a) \( h = 0.43 \) with \((0.94, 0.85)\), (b) \( h = 0.432 \) with \((0.94, 0.85)\), (c) \( h = 0.436 \) with \((0.94, 0.85)\) and (d) \( h = 0.4353 \) with \((0.94, 0.85)\).
If $\lambda_{1,2}$ denotes the characteristic roots of $V_s^C$ at $S$, then

$$\lambda_1 + \lambda_2 = -l_1 - (r_1 + r_2) h + 2, \quad (51)$$

$$\lambda_1 \lambda_2 = -(1 - r_2 h) l_1 + h r_2 \alpha_2 l_2 + h^2 r_1 r_2 (1 - \alpha_1 \alpha_2) - h (r_1 + r_2) + 1. \quad (52)$$

The lines of marginal stability are determined by solving the equations $\lambda_1 = \pm 1$ and $\lambda_1 \lambda_2 = 1$. These conditions guarantee that the eigenvalues $\lambda_1$ and $\lambda_2$ have modulus less than unity. If $\lambda_1 \lambda_2 = 1$, then from (52), one gets

$$L_1 : -(1 - r_2 h) l_1 + h r_2 \alpha_2 l_2 + h^2 r_1 r_2 (1 - \alpha_1 \alpha_2) - h (r_1 + r_2) + 1 = 0. \quad (53)$$

If $\lambda_1 = 1$, then from (51) and (52) one gets

$$L_2 : r_2 h l_1 + h r_2 \alpha_2 l_2 + h^2 r_1 r_2 (1 - \alpha_1 \alpha_2) + 2 = 0. \quad (54)$$

Finally if $\lambda_1 = -1$ then from (51) and (52) one gets

$$L_3 : (2 - r_2 h) l_1 - r_2 h \alpha_2 l_2 + 2 (r_1 + r_2) h - r_1 r_2 h^2 (1 - \alpha_1 \alpha_2) - 4 = 0. \quad (55)$$

Therefore from (53), (54) and (55), lines $L_1, L_2$ and $L_3$ in $(l_1, l_2)$-plane give the triangular region, which further give the fact that $|\lambda_{1,2}| < 1$.

6. Numerical simulations

We present numerical simulations to validate the obtained results in this section. For this, first we will give the definition of Lyapunov exponent.

**Definition 6.1 ([39]):** The Lyapunov exponent for the map

$$f : \mathbb{R} \mapsto \mathbb{R} \quad (56)$$

is defined by

$$L = \lim_{n \to \infty} \ln \left| \frac{d^{\frac{1}{n}} f^k(x) |_{x_0}}{dx} \right|^\frac{1}{n}. \quad (57)$$

The following two cases are to be considered for the completeness of this section.

**Case 1.** If $r_1 = 3.25$, $r_2 = 2.3$, $\alpha_1 = 0.13$, $\alpha_2 = 0.24$, $\beta_1 = 3.9$, $\beta_2 = 2.9$, then from non-hyperbolic condition

**Figure 2.** (a,b) Flip bifurcation diagram of the Kolmogorov model (2) with $h \in [0.001, 2.99]$; (c) maximum Lyapunov exponents corresponding with $(0.94, 0.85)$.
one gets $h = r_1 + r_2 + \sqrt{(r_1 - r_2)^2 + 4r_1r_2\alpha_1\alpha_2} / r_1r_2(1 - \alpha_1\alpha_2) = 0.953461990244541$. Moreover for $r_1 = 3.25$, $r_2 = 2.3$, $\alpha_1 = 0.13$, $\alpha_2 = 0.24$, $\beta_1 = 3.9$, $\beta_2 = 2.9$, we also get

$$h = r_1 + r_2 - \sqrt{(r_1 - r_2)^2 + 4r_1r_2\alpha_1\alpha_2} / r_1r_2(1 - \alpha_1\alpha_2) = 0.5793103392103719.$$  

From theoretical discussion, $S$ of the discrete Kolmogorov model (2) is stable node if $0 < h < \min(0.953461990244541, 0.5793103392103719)$. So, if one chose the bifurcation value $0 < h = 0.43 < \min(0.953461990244541, 0.5793103392103719)$, then it is clear from Figure 1(a) that fixed point $S = (4.414739884393064, 3.9595375722543347)$ of Kolmogorov model (2) is stable node. Moreover, if $h = 0.432, 0.436, 0.435 < $
min\{0.953461990244541, 0.5793103392103719\}, then Figure 1(b–d) also indicates that the fixed point $S = (4.414739884393064, 3.9595375722543347)$ of Kolmogorov model (2) is also stable node. Further, if $h > 0.953461990244541$, then fixed point $S$ becomes unstable and meanwhile flip bifurcation occurs, i.e. if $h = 0.976 > 0.953461990244541$ then by Mathematica computation from (43) one get $Q_1 = -2.1005368459391356 \neq 0$. Moreover, from (44) one gets $Q_2 = 0.22483096824141$, which indicates the fact that stable period-2 points bifurcate from $S$, and hence flip bifurcation diagram with maximum Lyapunov exponents are drawn and presented in Figure 2. Finally, Figure 3 has been drawn which shows the complex dynamics of the Kolmogorov model (2) with orbits of period-6, 7, 9, 10, 14, 24.

Case 2. Now we will prove the validity of obtained results in Section 5. For instance, if $h = 0.987, r_1 = 3.25, r_2 = 2.3, \alpha_1 = 0.13, \alpha_2 = 0.24, \beta_1 = 3.9, \beta_2 = 2.9$, then from (53), (54) and (55) one gets

$L_1 : -1.024/l_1 + 0.688160000000001/l_2 + 0.25515459200000024 = 0,$ (58)

$L_2 : 2.024/l_1 + 0.6881600000000001/l_2 + 7.139154592 = 0,$ (59)

$L_3 : -0.02400000000000002/l_1 - 0.688160000000001/l_2 - 0.6288454079999992 = 0.$ (60)
Moreover, (58), (59) and (60) determine triangular region that gives $|\lambda_{1,2}| < 1$ (see Figure 4). Moreover, $t$ vs $x_1$ and $y_1$ have been plotted for system (47) with $h_1 = -3.8839999999999995, h_2 = 1.0492638456172975$, which shows that unstable trajectories are stabilized (see Figure 5). Finally, the strange attractors for the Kolmogorov model (2) are also plotted and drawn in Figure 6.

7. Conclusion

The presented work is about the local dynamics, bifurcations and chaos in the Kolmogorov model (2) in $\mathbb{R}_+^2 = \{(x, y) : x, y \geq 0\}$. Algebraically, it is proved that for all parametric values $r_1, r_2, \alpha_1, \alpha_2, \beta_1, \beta_2, h$, Kolmogorov model (2) has trivial solution $P = (0, 0)$, two boundary fixed points $Q = (0, \beta_2), R = (\beta_1, 0)$, and interior fixed point $S$ if $\alpha_1 < \frac{2}{r_1}$. By linear stability theory, it is proved that fixed point $P$ is never sink, saddle, non-hyperbolic but it is source; $Q$ is source if $h > \frac{2}{r_1}$, saddle if $0 < h < \frac{2}{r_1}$, non-hyperbolic if $h = \frac{2}{r_1}$ but it is never sink; $R$ is source if $h > \frac{2}{r_2}$, saddle if $0 < h < \frac{2}{r_2}$, non-hyperbolic if $h = \frac{2}{r_2}$ but it is never sink; and $S$ is stable node if $0 < h < \min\left\{\frac{r_1+r_2+\sqrt{(r_1-r_2)^2+4r_1r_2\alpha_1\alpha_2}}{r_1r_2(1-\alpha_1\alpha_2)}, \frac{r_1+r_2-\sqrt{(r_1-r_2)^2+4r_1r_2\alpha_1\alpha_2}}{r_1r_2(1-\alpha_1\alpha_2)}\right\}$.

unstable node if $h > \max\left\{\frac{r_1+r_2+\sqrt{(r_1-r_2)^2+4r_1r_2\alpha_1\alpha_2}}{r_1r_2(1-\alpha_1\alpha_2)}, \frac{r_1+r_2-\sqrt{(r_1-r_2)^2+4r_1r_2\alpha_1\alpha_2}}{r_1r_2(1-\alpha_1\alpha_2)}\right\}$, non-hyperbolic if $h = \frac{r_1+r_2+\sqrt{(r_1-r_2)^2+4r_1r_2\alpha_1\alpha_2}}{r_1r_2(1-\alpha_1\alpha_2)}$ or $h = \frac{r_1+r_2-\sqrt{(r_1-r_2)^2+4r_1r_2\alpha_1\alpha_2}}{r_1r_2(1-\alpha_1\alpha_2)}$, but it is never stable focus, unstable focus and non-hyperbolic. Further, it is proved that fixed points $P, Q, R \text{ and } S$ are periodic having period-1, 2, 3, …, $n$. Moreover to understand the system under consideration deeply, we also explored the existence of possible bifurcations at fixed points $Q, R \text{ and } S$. It is proved that at fixed points $Q, R$, Kolmogorov model (2) does not undergo flip bifurcation if involved parameters respectively go through the curves $\mathcal{F}_Q = \{(h, \alpha_1, \alpha_2, r_1, r_2, \beta_1, \beta_2), h = \frac{\beta_1}{2}\}, \mathcal{F}_R = \{(h, \alpha_1, \alpha_2, r_1, r_2, \beta_1, \beta_2), h = \frac{\beta_2}{2}\}$ but at interior fixed point $S$ model undergoes flip bifurcation if involved parameters respectively go through the curve $\mathcal{F}_S = \{(h, \alpha_1, \alpha_2, r_1, r_2, \beta_1, \beta_2), h = \frac{r_1 + r_2 + \sqrt{(r_1 - r_2)^2 + 4r_1r_2\alpha_1\alpha_2}}{r_1r_2(1 - \alpha_1\alpha_2)}\}$. 

Figure 6. Strange attractors for the discrete Kolmogorov model (2): (a) $h = 0.982$ with $(0.74, 0.55)$, (b) $h = 0.9854$ with $(0.54, 0.55)$ and (c) $h = 0.994$ with $(0.2254, 0.2255)$. 

\[\text{Figure 6. Strange attractors for the discrete Kolmogorov model (2): (a) $h = 0.982$ with $(0.74, 0.55)$, (b) $h = 0.9854$ with $(0.54, 0.55)$ and (c) $h = 0.994$ with $(0.2254, 0.2255)$.}\]
Further state feedback control method is utilized to stabilize chaos existing in discrete Kolmogorov model (2). Finally, corresponding theoretical results have been verified numerically. This research can provide a framework for theoretical basis and help for the research in different aspects of biology specifically in the field of ecology.

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