A SINGULAR YAMABE PROBLEM ON MANIFOLDS WITH SOLID CONES

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ABSTRACT. We study the existence of conformal metrics on noncompact Riemannian manifolds with noncompact boundary, which are complete as metric spaces and have negative constant scalar curvature in the interior and negative constant mean curvature on the boundary. These metrics are constructed on smooth manifolds obtained by removing \( d \)-dimensional submanifolds from certain \( n \)-dimensional compact spaces locally modelled on generalized solid cones. We prove the existence of such metrics if and only if \( d > (n - 2)/2 \). Our main theorem is inspired by the classical results by Aviles-McOwen and Loewner-Nirenberg known in the literature as the “singular Yamabe problem”.

1. INTRODUCTION

The singular Yamabe problem is an extension, to noncompact Riemannian manifolds, of the classical Yamabe problem. While the classical problem [30] is formulated for closed manifolds (i.e., compact and without boundary), its singular version concerns manifolds obtained by removing a closed subset from a closed manifold. Precisely, the following problem is considered:

**Problem 1.1.** Let \((M, g)\) be a closed smooth Riemannian manifold of dimension \( n \geq 3 \) and let \( F \subset M \) be a closed subset (which we call a “singular set”). Is there a complete metric on \( M \setminus F \) which is conformal to \( g \) and has scalar curvature \( R = \text{constant} \)?

As the following example suggest, the dimension of the singular set is closely related to the sign of \( R \):

**Example 1.2.** Let \( S^n \) be the \( n \)-dimensional unit sphere and let \( S^d \) be a sphere of dimension \( 0 \leq d \leq n - 1 \), totally geodesic in \( S^n \). The stereographic projection \( \psi : S^n \setminus \{p\} \to \mathbb{R}^n \) gives a conformal equivalence between \( S^n \setminus S^d \) and \( \mathbb{R}^n \setminus \mathbb{R}^d \), where \( p \) is the north pole of \( S^n \) and \( \mathbb{R}^d = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_{d+1} = \ldots = x_n = 0 \} \).

On the other hand, the conformal change

\[
dx_1^2 + \ldots + dx_n^2 \mapsto \frac{dx_1^2}{x_{d+1}^2} + \ldots + \frac{dx_n^2}{x_{d+1}^2}
\]

transforms \( \mathbb{R}^n \setminus \mathbb{R}^d \) into \( \mathbb{H}^{d+1} \times S^{n-d-1} \), where \( \mathbb{H}^k \) represents the hyperbolic space of dimension \( k \). This product manifold has constant scalar curvature

\[
R = (n - d - 1)(n - d - 2) - d(d + 1) = (n - 1)(n - 2 - 2d),
\]

whose signal is determined by \( \frac{n-2}{2} - d \).

Historically, the Singular Yamabe problem originated from the seminal paper [19], by Loewner and Nirenberg, which handled the case \( R < 0 \) on spheres. The case of a general compact Riemannian manifolds was studied by Aviles and McOwen:

**Theorem 1.3.** Suppose that \((M, g)\) is a closed Riemannian manifold of dimension \( n \). Let \( F \) be a closed smooth submanifold of \( M \), with dimension \( 0 \leq d \leq n - 1 \). Then the Problem 1.1 has a solution with \( R = \text{constant} < 0 \) if and only if \( d > \frac{n-2}{2} \).

An immediate consequence of the particular case \( d = n - 1 \) is stated as follows:

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Corollary 1.4. If \((M, g)\) is a compact Riemannian manifold with non-empty smooth boundary \(\partial M\), then \(M \setminus \partial M\) admits a complete metric, conformal to \(g\), with constant negative scalar curvature.

Remark 1.5. We refer the reader to [11, 12, 17] for later developments considering more general singular sets. The corresponding fully nonlinear version of the singular Yamabe problem was studied in [14].

It is natural to seek for an extension, to noncompact manifolds, of the Escobar-Yamabe problem for compact manifolds with boundary introduced in [8], where one searches for a conformal metric with constant scalar curvature in the interior and constant mean curvature on the boundary. The first challenge we face is the quest for a compact space from which one should remove a singular subset. It turns out that the intersection angle between the boundary and the singular set plays an important role in the analysis as the next example suggests.

Example 1.6. For \(h > 0\) and \(d \in \{1, \ldots, n−1\}\), consider the generalized solid cone
\[
C_{d,h} = \left\{ x \in \mathbb{R}^n \mid x_1 \geq h \left( x_{d+1}^2 + \cdots + x_n^2 \right)^{\frac{1}{2}} \right\},
\]
and the singular set
\[
\mathbb{R}^d_+ = \{ x \in \mathbb{R}^n \mid x_1 \geq 0, x_{d+1} = \ldots = x_n = 0 \},
\]
as in Figure 1.

![Figure 1. The generalized solid cone \(C_{d,h}\).](image)

Observe that \(\mathbb{R}^d_+\) intersects the boundary cone
\[
\partial C_{d,h} = \left\{ x \in \mathbb{R}^n \mid x_1 = h \left( x_{d+1}^2 + \cdots + x_n^2 \right)^{\frac{1}{2}} \right\},
\]
along the “vertex”
\[
VC_{d,h} = \{ x \in \mathbb{R}^n \mid x_1 = 0 = x_{d+1} = \ldots = x_n \},
\]
with an angle \(\theta \in (0, \pi/2)\) determined by \(h = \cot \theta\). Now, \(C_{d,h} \setminus \mathbb{R}^d_+ \subset \mathbb{R}^n \setminus \mathbb{R}^d\) is a manifold with smooth noncompact boundary
\[
\partial (C_{d,h} \setminus \mathbb{R}^d_+) = \left\{ x \in \mathbb{R}^n \mid x_1 = h \left( x_{d+1}^2 + \cdots + x_n^2 \right)^{\frac{1}{2}}, x_1 > 0 \right\}
\]
in the classical sense. Hence, \(C_{d,h} \setminus \mathbb{R}^d_+\), with the metric
\[
\frac{dx_1^2 + \cdots + dx_n^2}{x_{d+1}^2 + \cdots + x_n^2} \quad (1.1)
\]
is a region of \(\mathbb{S}^{d+1} \times \mathbb{S}^{n-d-1}\), so that it has scalar curvature \((n−1)(n−2−2d)\).

On the other hand, a direct calculation carried out in Section 2 shows that \(\partial (C_{d,h} \setminus \mathbb{R}^d_+)\) has constant mean curvature
\[
-\frac{dh}{\sqrt{h^2 + 1}} = -d \cos \theta.
\]
Observe that, while the sign of the scalar curvature is determined by \( \frac{n-2}{2} - d \), the sign of the boundary mean curvature is determined by \(- \cos \theta\).

Adding the infinity of \( \mathbb{R}^n \) to that structure, one obtains the compact topological space
\[ M = C_{d,h} \cup \{ \infty \} . \]
If we set \( \Gamma = \mathbb{R}^d \cup \{ \infty \} \) we obtain the noncompact manifold \( M \setminus \Gamma = C_{d,h} \setminus \mathbb{R}^d \) with noncompact boundary \( \partial (C_{d,h} \setminus \mathbb{R}^d) \) which will be our model in this paper. So, in analogy with Problem 1.1, the pair \( (M, \Gamma) \) plays the same role as \( (S^n, S^d) \) does in Example 1.2. The reader may find interesting to see \( M \) as a compact subset of \( S^n \) by means of the stereographic projection; see Figure 2.

The above construction motivates us to define a “solid conical manifold” as a topological space \( M \) with a differential structure locally modeled on \( C_{d,h} \); see Section 2 for the precise definitions. This produces a nonsmooth boundary \( \partial M \) corresponding to the points of \( M \) taken to \( \partial C_{d,h} \) by the coordinate charts, and a vertex \( VM \) corresponding to the ones taken to \( VC_{d,h} \). From one such solid conical manifold, we remove a closed singular set \( \Gamma \) which is given by a finite union \( \bigcup \Gamma_m \) of \( d_m \)-dimensional submanifolds \( \Gamma_m \) with boundary \( \partial \Gamma_m \), in such a way that \( VM = \bigcup \partial \Gamma_m \).

To that structure we add a smooth Riemannian metric \( g \) on \( M \). This metric defines the intersection angle between \( \Gamma \) and \( \partial M \) along the vertex \( VM \), which is assumed to coincide with the angle \( \theta \) coming from the solid cone modeling it; see Definition 2.3.

We raise the following question:

**Problem 1.7.** Is there a Riemannian metric on \( M \setminus \Gamma \), complete as a metric space and conformal to \( g \), that has constant scalar curvature in \( M \setminus \Gamma \) and constant mean curvature on \( \partial (M \setminus \Gamma) = \partial M \setminus VM \)?

One important feature of this problem is that, although \( M \) is not a manifold in the classical sense, \( M \setminus \Gamma \) is a noncompact manifold with smooth noncompact boundary \( \partial (M \setminus \Gamma) = \partial M \setminus VM \) in the classical sense. So, this can be viewed as an existence problem of conformal metrics on noncompact Riemannian manifolds with constant scalar curvature and constant boundary mean curvature, while \( M \) is viewed as the compactification of \( M \setminus \Gamma \).

Our approach to Problem 1.7 is a modification of the arguments in [4, 5] which depend essentially on maximum principles, variational technics and elliptical estimates adapted to our settings. In analytical terms, we are searching for a solution \( u > 0 \) of the problem
\[
\begin{align*}
-D_\theta u + \frac{(n-2)}{4(n-1)} R_\theta u + c_0 u^{\frac{n+2}{n-2}} &= 0 & \text{in } M \setminus \Gamma, \\
\frac{\partial u}{\partial \nu_\theta} + \frac{n-2}{2(n-1)} H_\theta u + c_1 u^{\frac{n}{n-2}} &= 0 & \text{on } \partial M \setminus VM, \\
\liminf_{p \to \Gamma} u(p) \, \text{dist}_g(p, \Gamma)^{\frac{2}{n-2}} &> 0.
\end{align*}
\]
Here, \( c_0 > 0 \) and \( c_1 > 0 \) are constant on each connected component of \( M \setminus \Gamma \) and \( \partial M \setminus VM \) respectively. Besides, \( R_\theta \) stands for the scalar curvature, \( \Delta_\theta = \text{div}_g \nabla_\theta \) is the Laplace operator and \( \nu_\theta \) is the outward unit normal vector to \( \partial M \setminus VM \), so that \( H_\theta = \text{div}_g \nu_\theta \) is its mean curvature.
Our main result, see Theorem 3.1 below, implies in particular that, given \( c_0, c_1 > 0 \) as above, one can find a solution \( \tilde{g} \) to the Problem 1.7 with
\[
\begin{align*}
R_{\tilde{g}} &= -c_0, \quad M \setminus \Gamma, \\
H_{\tilde{g}} &= -c_1, \quad \partial M \setminus \nu M,
\end{align*}
\]
if and only if \( \dim \Gamma_m > \frac{n-2}{2} \) for all \( m \). In fact, our result is more general as it allows for \( c_0 \) and \( c_1 \) to be smooth functions on \( M \setminus \Gamma \) and \( \partial M \setminus \nu M \) respectively, bounded above and below by positive constants.

Remark 1.8. The case \( d = n-1 \) has special interest because of its close relationship with cornered manifolds (see Appendix A.1). In our model, Example 1.6, the singular set
\[
\mathbb{R}^{n-1}_+ = \{ x \in \mathbb{R}^n \mid x_1 \geq 0 = x_n \},
\]
leads to a decomposition of \( M = C_{n-1, h} \cup \{ \infty \} \) into two connected components, each one being a cornered manifold itself. In this case, the metric (1.1) is the hyperbolic metric
\[
x_n^{-2}(dx_1^2 + \ldots + dx_n^2),
\]
so that each connected component of \( \mathbb{R}^n \setminus \mathbb{R}^{n-1}_+ \) is a copy of \( \mathbb{H}^n \). Further, the two connected components of the boundary \( x_1 = h(x_n) \neq 0 \) are hyperspheres of \( \mathbb{H}^n \), i.e., hypersurfaces equidistant from the totally geodesic hypersurface \( x_1 = 0 \); see Figure 3.

More generally, let \( M \) be a cornered manifold \( M \) with boundary \( \partial M = \Gamma \cup \Sigma \), where \( \Gamma \) and \( \Sigma \) are smooth hypersurfaces of \( M \) with the common boundary \( \partial \Gamma = \partial \Sigma = \Gamma \cap \Sigma \) being a codimension two corner of \( M \). Let \( g \) be a Riemannian metric on \( M \). If we assume that \( \Gamma \) and \( \Sigma \) make intersection angles \( \theta_m \) along each connected component of \( \Gamma \cap \Sigma \), then doubling \( M \) along \( \Gamma \) produces a solid conical manifold with \( d_m = n-1 \) and \( h_m = \cot \theta_m \).

The same proof of Theorem 3.1 also leads to the following analogue of Corollary 1.4.

Corollary 1.9. Let \( M \) be a cornered manifold as in Remark 1.8 and let \( g \) be a Riemannian metric on \( M \). Assume that the angle between \( \Gamma \) and \( \Sigma \) along each connected component of the corner \( \Gamma \cap \Sigma \) is a constant in the interval \((0, \pi/2)\). Given smooth functions \( c_0 \) and \( c_1 \) on \( M \setminus \Gamma \) and \( \Sigma \) respectively, both being bounded above and below by positive constants, there exists a smooth metric \( \tilde{g} \in \lceil g \rceil \) on \( M \setminus \Gamma \), complete as a metric space, satisfying
\[
\begin{align*}
R_{\tilde{g}} &= -c_0, \quad M \setminus \Gamma, \\
H_{\tilde{g}} &= -c_1, \quad \Sigma.
\end{align*}
\]

We conclude this section by discussing a couple of questions arising naturally from our results. The first one is the uniqueness of the metrics obtained in Corollary 1.9. In the case of Corollary 1.4, uniqueness was proved in [24] and [20], where regularity of solutions up to the singular set is studied. We believe that similar results should hold in our settings, where an asymptotic behavior is expected with respect to the boundary component \( \Gamma \). We also believe that a hyperboloidal initial data problem for Einstein’s field equations could be formulated in the presence of a noncompact
boundary. This motivates one to pursue results similar to [2]. In that case, the boundary mean curvature would be coupled with the scalar in the interior to compose the hyperboloidal initial data set. This is strongly motivated by the results in [1] where the boundary mean curvature plays central role in the definitions of dominant energy conditions and total mass invariants.

The second question is the case when $c_1$ and $c_2$ are not both negative. Although the case $c_1 = 0$ and $c_2 < 0$ should be similar to the one when $c_1, c_2 < 0$, handled here, the situation when $c_1$ is positive should require more refined techniques as the behavior near the singular set is not a priori determined. We refer the reader to [6, 21, 22, 23, 24, 27, 29] for interesting results that could be extended to our setting.

This paper is organized as follows. In Section 2, we give precise definitions of the objects involved and discuss further our model, namely, Example 1. In Section 3, we state and prove our main result, Theorem 3.1. The Appendix contains some technical tools used in this paper.

2. Preliminaries and formal definitions

In this section, we define the type of manifolds appearing in our main theorem stated and proved in Section 3 below. For $n \geq 2$, they are modeled on sets of the form

$$C_{d,h} = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 \geq h \left( x_{d+1}^2 + \cdots + x_n^2 \right)^{\frac{1}{2}} \right\},$$

where $h > 0$ and $d \in \{1, \ldots, n-1\}$, as described in the Introduction. We now provide the precise definitions.

**Definition 2.1.** A smooth solid conical manifold of dimension $n$ is a paracompact Hausdorff topological space $M$ and a family of homeomorphisms (called the charts)

$$\varphi_{\alpha} : U_{\alpha} \to \varphi_{\alpha}(U_{\alpha}) \subset M,$$

where each $U_{\alpha} \subset C_{d_\alpha, h_\alpha}$ is a relative open subset, satisfying the following three conditions:

(i) $\bigcup_{\alpha} \varphi_{\alpha}(U_{\alpha}) = M$.
(ii) For any pair $\alpha, \beta$, with $W := \varphi_{\alpha}(U_{\alpha}) \cap \varphi_{\beta}(U_{\beta}) \neq \emptyset$, the map

$$\varphi_{\beta}^{-1} \circ \varphi_{\alpha} : \varphi_{\alpha}^{-1}(W) \to \varphi_{\beta}^{-1}(W)$$

is smooth \footnote{Suppose $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$, we say that a map $f : A \to B$ is smooth if, for any $p \in A$, $f$ has a smooth extension in a neighborhood of $p$, i.e., there exist an open subset $V \subset \mathbb{R}^n$ with $p \in V$ and a smooth map $\tilde{f} : V \to \mathbb{R}^m$ with $f|_{\partial V \cap A} \equiv f|_{V \cap A}$.}

(iii) The family $\{(U_{\alpha}, \varphi_{\alpha})\}$ is maximal relative to the conditions (i) and (ii).

We define the vertex

$$VM = \{ p \in M \mid \exists (U_{\alpha}, \varphi_{\alpha}) \text{ and } x \in U_{\alpha} \text{ such that } x_1 = 0 \text{ and } \varphi_{\alpha}(x) = p \},$$

the interior

$$\text{int}(M) = \{ p \in M \mid \exists (U_{\alpha}, \varphi_{\alpha}) \text{ such that } p \in \varphi_{\alpha}(U_{\alpha}) \text{ and } U_{\alpha} \text{ is an open subset of } \mathbb{R}^n \},$$

and the boundary $\partial M = M \setminus \text{int}(M)$ of $M$. Observe that if $p \in VM$ then for any chart $(U_{\alpha}, \varphi_{\alpha})$ we have that $x_1 = 0$ for $x = \varphi^{-1}_{\alpha}(p)$. This easily follows from the property that an immersion of an open subset of $\mathbb{R}^n$ into $\mathbb{R}^m$ takes smooth curves into smooth curves.

Denote by $C^\infty(M)$ the $\mathbb{R}$-algebra of smooth functions $f : M \to \mathbb{R}$. As for a classical manifold, for $p \in M$ we define its tangent space by

$$T_pM := \{ v : C^\infty(M) \to \mathbb{R} \mid v \text{ is a linear map and } v(fg) = v(f)g(p) + f(p)v(g) \forall f, g \in C^\infty(M) \}.$$  

For a chart $(U_{\alpha}, \varphi_{\alpha})$ of $M$ and $p \in \varphi_{\alpha}(U_{\alpha})$, define

$$\partial_i(p) : f \in C^\infty(M) \mapsto \frac{\partial f \circ \varphi_{\alpha}}{\partial x_i} \left( \varphi^{-1}_{\alpha}(p) \right), \quad i = 1, \ldots, n.$$  

Then the set $\{\partial_1(p), \ldots, \partial_n(p)\}$ is a coordinate frame at $T_pM$.

**Definition 2.2.** Let $M^n$ be a compact smooth solid conical manifold and let $\{\Gamma_m^n\}$ be a finite disjoint family of $d_m$-dimensional compact submanifolds of $M$ with boundary $\partial \Gamma_m = \Gamma_m \cap \partial M$ such that

(i) $\bigcup_m \partial \Gamma_m = VM$. 

The family $\{(U_{\alpha}, \varphi_{\alpha})\}$ is maximal relative to the conditions (i) and (ii).
If $p \in \partial \Gamma_m$ and $(U_{\alpha}, \varphi_{\alpha})$ is a chart of $M$ with $p \in \varphi_{\alpha}(U_{\alpha})$, then
\[
\varphi_{\alpha}|_{U_{\alpha} \cap \{x_{d_m+1} = \cdots = x_n = 0\}}
\]
is a chart of $\Gamma_m$; see Figure 4.

We define the singular set of $M$ as $\Gamma = \bigcup_m \Gamma_m$ and call the pair $(M, \Gamma)$ a solid conical singular space (s.c.s.s.).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4.png}
\caption{A local chart of a s.c.s.s.}
\end{figure}

A Riemannian metric on a smooth solid conical manifold $M$ is a correspondence which associates to each point $p$ of $M$ an inner product $g(\cdot, \cdot)_p$ (that is, a symmetric, bilinear, positive definite form) on the tangent space $T_pM$, which varies smoothly in the following sense: if $(U_{\alpha}, \varphi_{\alpha})$ is a chart of $M$ at $p$ then $x \mapsto g(\partial_i, \partial_j)_{\varphi_{\alpha}(x)}$ is a smooth function on $U_{\alpha}$.

Definition 2.3. A Riemannian s.c.s.s. is a s.c.s.s. $(M, \Gamma)$ endowed with a Riemannian metric $g$, satisfying the following conformal condition along $\nabla M$: for all $m$ and $p \in \partial \Gamma_m$ there is $(U_{\alpha}, \varphi_{\alpha})$, where $U_{\alpha}$ is open in $C_{d_m, h_m}$, such that for some positive function $\varrho$,
\[
\varphi_{\alpha}^* g(x) = \varrho(x) \delta_{R^n} \quad \text{for all } x \in U_{\alpha} \cap \{x_1 = x_{d_m+1} = \cdots = x_n = 0\},
\]
where $\delta_{R^n}$ is the Euclidean metric.

This definition simply ensures that, $h_m = \cot \theta_m$, where $\theta_m$ is the angle between $\Gamma_m$ and $\partial M$ along $\nabla M$, calculated with respect to the metric $g$.

The following technical lemma will be used later:

Lemma 2.4. Let $(M, \Gamma)$ be a s.c.s.s. Then we have:

(i) There exists $\varepsilon_0 > 0$ such that $\rho(\cdot) = \text{dist}_g(\cdot, \Gamma)$ is smooth in $\{p \in M \mid 0 < \rho(p) < \varepsilon_0\}$,

\[
\rho(p) = \text{dist}_g(p, \Gamma_i) \quad \text{if } \text{dist}_g(p, \Gamma_i) < \varepsilon_0
\]

and

\[
\{p \in M \mid \text{dist}_g(p, \Gamma_i) < \varepsilon_0\} \cap \{p \in M \mid \text{dist}_g(p, \Gamma_j) < \varepsilon_0\} = \emptyset \quad \text{if } i \neq j.
\]

(ii) In a coordinate neighborhood $\varphi_{\alpha}(U_{\alpha})$, with $U_{\alpha} \subset C_{d_m, h_m}$, satisfying (2.1), the following limits hold:
\[
g(\nabla \rho, \nu)_p \to \frac{h_m}{\sqrt{1 + h_m^2}} \quad \text{and} \quad \rho(p) H_g(p) \to \frac{(n - d_m - 1) h_m}{\sqrt{1 + h_m^2}},
\]
as $p \to \nabla M$ along $\partial M \setminus \nabla M$, where $\nu$ is the outward unit normal vector to the boundary $\partial M \setminus \nabla M$.

(iii) For all $\varepsilon > 0$ there exists $\delta \in (0, \varepsilon_0)$ such that
\[
|(\rho \Delta \rho)(p) - (n - d_i - 1)| < \varepsilon \quad \text{if } 0 < \text{dist}_g(p, \Gamma_i) < \delta.
\]
Proof. Set

\[ C_{d,m,h_m} = \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 > h_m \left(x_{d+m+1}^2 + \ldots + x_n^2\right)^{\frac{1}{2}} \right\}. \]

We first construct local extensions of \( M \) across \( VM \). If \( p \in VM \cap \Gamma_m \) we can assume that there is a chart \((U_\alpha, \varphi_\alpha)\) at \( p \), with \( U_\alpha = C_{d,m,h_m} \cap B_x \), \( B_x = \{ x \in \mathbb{R}^n \mid \| x \| < \varepsilon \} \), such that there exist smooth functions \( g_{ij} : B_x \to \mathbb{R} \) satisfying

\[ g_{ij}|_{U_\alpha} = g(\partial_i, \partial_j) \circ \varphi_\alpha \]

and \([\hat{g}_{ij}]\) is symmetric and positive definite. Let \( \hat{U}_\alpha \) be the set obtained from \((B_x \setminus C_{d,m,h_m}) \cup \varphi_\alpha(U_\alpha)\) by identifying the points \( x \) and \( \varphi_\alpha(x) \) whenever \( x \in \partial C_{d,m,h_m} \cap B_x \). Set

\[ \hat{\varphi}_\alpha(x) := \begin{cases} \varphi_\alpha(x) & \text{if } x \in U_\alpha, \\ x & \text{if } x \in B_x \setminus C_{d,m,h_m}. \end{cases} \tag{2.3} \]

Observe that \( \hat{U}_\alpha \) is a manifold without boundary with a Riemannian metric defined by

\[ \hat{g} \left( \partial_i, \partial_j \right) := \hat{g}_{ij} \circ \varphi_\alpha^{-1}. \]

where \( \{ \hat{\partial}_i \} \) is the coordinate frame associated to \( \hat{\varphi}_\alpha \). Similarly, if \( \mathbb{R}^{d_m} := \mathbb{R}^n \cap \{ x_{d+m+1} = \ldots = x_n = 0 \} \), we can extend \( \hat{\varphi}_\alpha(U_\alpha \cap \mathbb{R}^{d_m}) \subset \Gamma \) to a smooth manifold without boundary denoted by \( U_\alpha \cap \mathbb{R}^{d_m} \).

Choosing \( \varepsilon \) smaller if necessary, we can prove (ii) and also that \( \hat{\rho}|_{\varphi_\alpha(U_\alpha)} = \rho|_{\varphi_\alpha(U_\alpha)} \). The item (i) easily follows from this.

The proof of (iii) can be found in [19, pp. 257], and concludes the proof of the lemma. \( \blacksquare \)

We end this section returning to our model, Example 1.6 which is given by the compact set \( C_{d,h} \cup \{ \infty \} \) endowed with the metric

\[ g_0(x) := \rho(x)^{-2} \delta_{\mathbb{R}^n}, \]

where \( \rho(x) = \text{dist}_{\delta_{\mathbb{R}^n}}(x, \Gamma) = \sqrt{x_{d+1}^2 + \ldots + x_n^2}. \)

Observe that the inversion \( x \mapsto x/|x|^2 \) provides a chart on a neighborhood of \( \infty \) and it is also an isometry of \( g_0 \). We finally determine the mean curvature \( H_{g_0} \) of \( \partial C_{d,h} \setminus \{ \infty \} \).

**Lemma 2.5.** We have

\[ H_{g_0} = - \frac{dh}{\sqrt{1 + h^2}}. \tag{2.4} \]

**Proof.** Let \((\mathcal{O}, \Theta)\) be a chart of \( \mathbb{S}^{n-d-1} \) and let \( \{ \partial_i \} \) be its associated coordinate frame. Set

\[ U = \{(r, y, z, s) \in \mathbb{R} \times \mathbb{R}^{d-1} \times \mathbb{R}^{n-d-1} \times \mathbb{R} \mid z \in \mathcal{O}, \ r > 0 \ \text{and} \ -r/h < s \}, \]

and define

\[ \varphi : (r, y, z, s) \in U \mapsto (hr - s, y, (hs + r)\Theta(z)) \in \mathbb{R}^n. \]

We see that \( \varphi \) is a chart of \( \mathbb{R}^n \setminus \mathbb{R}^d \) and \( \varphi(r, y, z) := \varphi(r, y, z, 0) \) is a chart of \( \partial C_{d,h} \setminus \{ \infty \} \). Let \( \{ \partial_i \}_{i=1}^d \) be the associated coordinate frame to \( \varphi \). We can see \( v_{g_0}(r, y, z) = (|\partial_i|_{g_0}^{-1}\partial_i) (r, y, z, 0) \) is the outward unit normal vector to the boundary \( \partial C_{d,h} \setminus \{ \infty \} \). We have

\[ g_0(\partial_1, \partial_1)(r, y, z, s) = \frac{1 + h^2}{(hs + r)^2}, \quad g_0(\partial_i, \partial_i)(r, y, z, s) = \frac{1}{(hs + r)^2}, \]

\[ g_0(\partial_j, \partial_k)(r, y, z, s) = \delta_{jk} \left( \partial_{\theta_{j+d}}, \partial_{\theta_{k+d}} \right)(z), \quad g_0(\partial_n, \partial_n)(r, y, z, 0) = \frac{1 + h^2}{r^2}, \]

where \( i = 2, \ldots, d \) and \( j, k = d + 1, \ldots, n - 1 \). Observe also that \( \{ \partial_i \}_{i=1}^d \) is orthogonal.

Finally, the mean curvature is given by

\[ H_{g_0} = \frac{1}{2} \sum_{i,j=1}^{n-1} g_0^{ij} v_{g_0}(g_0, ij) = \frac{1}{2} \sum_{i=1}^d \frac{r}{\sqrt{1 + h^2}} \frac{\partial g_0, ii}{\partial s} = - \frac{dh}{\sqrt{1 + h^2}}. \]

This proves (2.4).
As a final remark, notice that
\[ H_{\delta_u}(r, y, z) = \frac{(n - d - 1)h}{\sqrt{1 + h^2}} r, \]
so that the mean curvature, when calculated in terms of \( \delta_{\delta_u} \), blows up as we approach the singular set.

3. The main theorem

In this section, we state and prove our main result:

**Theorem 3.1.** Let \((M, \Gamma, g)\) be a Riemannian s.c.s.s. with \( n \geq 3 \) and \( \Gamma = \cup_m \Gamma_m \). Let \( c_0 \in L^\infty(M \setminus \Gamma) \) and \( c_1 \in L^\infty(\partial M \setminus \Omega M) \) be smooth functions bounded below by a positive constant. There exists a metric \( \tilde{g} \) on \( M \setminus \Gamma \), conformal to \( g \) and complete as a metric space, with scalar curvature \( R_{\tilde{g}} = -c_0 \) in \( M \setminus \Gamma \) and mean curvature \( H_{\tilde{g}} = -c_1 \) on \( \partial M \setminus \Omega M \) if and only if \( \dim \Gamma_m > \frac{n-2}{2} \) for all \( m \).

To that end, we will prove the existence of a positive smooth solution \( u \) of the following problem:

\[
\begin{cases}
-\Delta_g u + \frac{(n - 2)}{4(n - 1)} R_g u + c_0 u^{\frac{n+2}{n-2}} = 0 & \text{in } M \setminus \Gamma, \\
\frac{\partial u}{\partial \nu_g} + \frac{n - 2}{2(n - 1)} H_g u + c_1 u^{\frac{n-2}{n+2}} = 0 & \text{on } \partial M \setminus \Omega M, \\
\lim_{p \to \Gamma} u(p) (\text{dist}_g(p, \Gamma)^{\frac{n-2}{2}} > 0.
\end{cases}
\] (3.1)

Observe that the inequality above ensures that \( \tilde{g} = u^{-\frac{2}{n-2}} g \) is complete as a metric space on \( M \setminus \Gamma \).

Our proof goes along the same lines as [4]. We also provide the necessary modifications of some results in [5]. For any open set \( U \subset M \setminus \Gamma \) such that \( U \cap \partial M \neq \emptyset \), we set
\[ D_U = \partial U \cap \text{int}(M) \quad \text{and} \quad N_U = U \cap \partial M \]
and define
\[
\lambda_g(U) = \inf_{\zeta \in C^\infty_0(U)} \frac{\int_U \left( |\nabla \zeta|^2_g + \frac{n-2}{2(n-1)} R_g \zeta^2 \right) \, dv_g + \frac{n-2}{2(n-1)} \int_{N_U} H_g \zeta^2 \, d\sigma_g}{\int_U \zeta^2 \, dv_g},
\] (3.2)
where \( dv_g \) and \( d\sigma_g \) are the volume and the area elements respectively. We divide the proof of the existence part in two cases, according to the sign of \( \lambda_g(M \setminus \Gamma) \). The non-negative and negative cases are handled in Subsections 3.1 and 3.2 respectively. In Subsection 3.3 we prove the converse statement, i.e., that the existence of a positive solution to (3.1) implies that \( \dim \Gamma_m > \frac{n-2}{2} \) for all \( m \).

3.1. The case \( \lambda_g(M \setminus \Gamma) \geq 0 \). Unless otherwise stated, \( \Omega \) will denote an open subset of \( M \setminus \Gamma \) satisfying the following:

**Assumption 3.1.** The closure \( \overline{\Omega} \subset M \setminus \Gamma \) is a compact \( n \)-submanifold with corners locally modeled on \( \mathbb{R}^{n-2} \times [0, \infty)^2 \) (see the Appendix A.1 for the precise definition) and \( \overline{D}_\Omega \) and \( \overline{N}_\Omega \) are \((n-1)\)-submanifolds with smooth boundaries \( \partial \overline{D}_\Omega := \overline{D}_\Omega \setminus D_\Omega \) and \( \partial \overline{N}_\Omega := \overline{N}_\Omega \setminus N_\Omega \), respectively (observe that \( \partial \overline{D}_\Omega \cap \partial \overline{N}_\Omega \subset \partial M \setminus \Omega M \)).

Recall that \( \rho(\cdot) = \text{dist}_g(\cdot, \Gamma) \) and set
\[ M_j = \{ p \in M \mid \rho(p) > j^{-1} \}, \quad j = 1, 2, \ldots. \]
Since \( \Gamma \) is compact and \( 0 \leq g(\nabla \rho, \nu) < 1 \) near \( \Omega M \) (see the Lemma 2.4(ii)), for large \( j \), \( M_j \) is an open set in \( M \) satisfying the Assumption 3.1.
For simplicity we write,

\[ D_j = D_{M_j}, \quad N_j = N_{M_j}, \quad D = D_{\Omega} \quad \text{and} \quad N = N_{\Omega}; \]

see Figure 5. The following maximum principle is proved as in [25, Chapter 2, Section 5] (see also [26]):

**Proposition 3.2.** Let \( U \) be an open bounded set of \( \mathbb{R}^n \) satisfying \( \partial U = \partial D \cup \partial N, \partial D \cap \partial N = \emptyset \) and \( \partial D = \partial N \), where \( N \) is a hypersurface (without boundary) in \( \mathbb{R}^n \), and \( D \neq \emptyset \). Let \( \{ a^{ij} \} \) be symmetric and uniformly elliptic coefficients with \( a^{ij} \in C_0^{0,1} - (U), i, j \in \{1, \ldots, n\}, q > n \). Suppose that \( d \in L^\infty(U), \hat{c}_0 \in L^2(U), \hat{c}_0 \geq 0, \hat{c}_1, \hat{c}_2 : N \to \mathbb{R}, \hat{c}_1 \geq 0 \). Consider the operators

\[
Lu := -\left( a^{ij} u_{x_i} \right)_{x_j} - du \quad \text{and} \quad Bu := \frac{\partial u}{\partial \nu} + \hat{c}_2 u,
\]

and let \( f_0, f_1 : \mathbb{R} \to \mathbb{R} \) be non decreasing functions. Assume that

\[
u, v, w \in C^1(U \cup N) \cap C(\overline{U}),
\]

\[
f_0(u), f_0(v) \in L^2(U), \quad \text{and}
\]

\[
\begin{cases}
Lu + \hat{c}_0 f_0(u) \leq Lv + \hat{c}_0 f_0(v) & \text{in } U, \\
u \leq v & \text{on } D, \\
Bu + \hat{c}_1 f_1(u) \leq Bv + \hat{c}_1 f_1(v) & \text{on } N,
\end{cases}
\]

\[
\begin{cases}
Lw \geq 0 & \text{in } U, \\
Bw \geq 0 & \text{on } N, \\
w > 0 & \text{in } \overline{U},
\end{cases}
\]

are satisfied in the weak sense in \( U \), and in the classical sense on \( N \). Then

\[
u \leq v \quad \text{in } U.
\]

Set

\[
\mathcal{L}_g u := -\Delta_g u + \frac{(n-2)}{4(n-1)} R_g u \quad \text{and} \quad B_g u := \frac{\partial u}{\partial \nu_g} + \frac{n-2}{2(n-1)} H_g u.
\]

**Lemma 3.3.** If \( u, v \in C^2(\Omega) \cap C(\overline{\Omega}) \) are non-negative functions such that

\[
\begin{cases}
\mathcal{L}_g u + c_0 u^{\frac{n+2}{n-2}} \leq \mathcal{L}_g v + c_0 v^{\frac{n+2}{n-2}} & \text{in } \Omega, \\
u \leq v & \text{on } D, \\
B_g u + c_1 u^{\frac{n}{n-2}} \leq B_g v + c_1 v^{\frac{n}{n-2}} & \text{on } N,
\end{cases}
\]

then

\[
u \leq v.
\]
Proof. Choose \( j \) large enough such that \( \overline{\Omega} \subset M_j \). The condition \( \lambda_{\delta}(M\setminus\Gamma) \geq 0 \) implies \( \lambda_{\delta}(M_j) \geq 0 \). Using standard variational arguments (see for example the proof of Proposition A.11 in the appendix), we can find \( \phi_j \in C^\infty(M_j) \cap C(\overline{M_j}) \) a minimum of the variational problem

\[
\lambda_{\delta}(M_j) := \inf_{\zeta \in C^\infty_0(M_j)} \frac{\int_{M_j} \left( |\nabla \zeta|^2 + \frac{n+2}{4(n-1)} R_g \zeta^2 \right) \, dv_g + \frac{n-2}{2(n-1)} \int_{\partial M_j} H_g \zeta^2 \, d\sigma_g}{\int_{M_j} \zeta^2 \, dv_g},
\]

so that

\[
\begin{aligned}
\mathcal{L}_g \phi_j &= \lambda_{\delta}(M_j) \phi_j & \text{in } M_j, \\
\phi_j &= 0 & \text{on } \mathcal{D}_j, \\
\mathcal{B}_g \phi_j &= 0 & \text{on } \mathcal{N}_j, \\
\phi_j &> 0 & \text{in } M_j.
\end{aligned}
\]

(3.3)

In particular, the conditions of Proposition 3.2 are fulfilled with \( w = \phi_j \) and the result follows.

Lemma 3.4. There exists \( u \in C^\infty(\Omega) \) such that

\[
\begin{aligned}
\mathcal{L}_g u + c_0 u^{\frac{n+2}{n-2}} &= 0 & \text{in } \Omega, \\
u(x) &\to \infty & \text{as } x \to \overline{\Omega}, \\
\mathcal{B}_g u + c_1 u^{\frac{n}{n-2}} &= 0 & \text{on } \mathcal{N}, \\
u &> 0 & \text{in } \Omega.
\end{aligned}
\]

(3.4)

Proof. As in the last proof, we choose \( j \) large such that \( \overline{\Omega} \subset M_j \) and \( \phi_{j+1} \) satisfying (3.3). Then \( \tilde{g} = \phi_{j+1}^{-\frac{2}{n-2}} g \) satisfies

\[
\begin{aligned}
R_{\tilde{g}} &= \frac{4(n-1)}{(n-2)} \lambda_{\delta}(M_{j+1}) \phi_{j+1}^{-\frac{2}{n-2}} \geq 0 & \text{in } M_j, \\
H_{\tilde{g}} &= \phi_{j+1}^{-\frac{2}{n-2}} \left( \frac{2(n-1)}{n-2} \frac{\partial \phi_{j+1}}{\partial v_{\tilde{g}}} + \phi_{j+1} H_g \right) = 0 & \text{on } \mathcal{N}_j.
\end{aligned}
\]

It follows from Proposition A.11 that, for every positive integer \( m \), there is \( w_m \in C^{2,\alpha}(\Omega) \cap C^{0,\alpha}(\overline{\Omega}) \) such that

\[
\begin{aligned}
\mathcal{L}_{\tilde{g}} w_m + c_0 w_m^{\frac{n+2}{n-2}} &= 0 & \text{in } \Omega, \\
w_m &= m & \text{on } \overline{\Omega}, \\
\mathcal{B}_{\tilde{g}} w_m + c_1 w_m^{\frac{n}{n-2}} &= 0 & \text{on } \mathcal{N}, \\
w_m &> 0 & \text{in } \Omega.
\end{aligned}
\]

By the Lemma 3.3 the \( w_m \) are monotonic increasing and by Theorem A.15 the \( w_m \) are uniformly bounded on any open subset \( V \subset \Omega \) with \( \overline{V} \subset \Omega \). Using elliptic estimates as in the last paragraph of the proof of Proposition A.11 we can assume that \( w_m \) converges \( C^2 \) in compact subsets of \( \Omega \) to a solution \( w \) of

\[
\begin{aligned}
\mathcal{L}_g w + c_0 w^{\frac{n+2}{n-2}} &= 0 & \text{in } \Omega, \\
w(x) &\to \infty & \text{as } x \to \overline{\Omega}, \\
\mathcal{B}_g w + c_1 w^{\frac{n}{n-2}} &= 0 & \text{on } \mathcal{N}, \\
w &> 0 & \text{in } \Omega.
\end{aligned}
\]

(3.5)

Hence, \( u = \phi_{j+1} w \) solves (3.4), and the regularity of \( u \) follows from standard elliptic arguments.
By Lemma 3.3, the \( u \) are monotonically decreasing, i.e., \( u_j \geq u_{j+1} \) in \( M_j \). Similarly as we did in the proof of Lemma 3.4, for \( u(x) := \inf_{m \geq j} u_m (x) \), \( x \in M_j \), we have that \( u \in C^\infty (M \setminus \Gamma) \) and

\[
\begin{align*}
\mathcal{L}_g u_j + c_0 u_j^{\frac{n+2}{n-2}} &= 0 \quad \text{in} \ M_j, \\
u_j(x) &\to \infty \quad \text{as} \ x \to \partial M_j, \\
\mathcal{B}_g u_j + c_1 u_j^{\frac{n}{n-2}} &= 0 \quad \text{on} \ N_j, \\
u_j > 0 &\quad \text{in} \ M_j.
\end{align*}
\]

By Lemma 3.4, the \( u_j \) are monotonically decreasing, i.e., \( u_j \geq u_{j+1} \) in \( M_j \). Similarly as we did in the proof of Lemma 3.4, for \( u(x) := \inf_{m \geq j} u_m (x) \), \( x \in M_j \), we have that \( u \in C^\infty (M \setminus \Gamma) \) and

\[
\begin{align*}
\mathcal{L}_g u + c_0 u^{\frac{n+2}{n-2}} &= 0 \quad \text{in} \ M \setminus \Gamma, \\
\mathcal{B}_g u + c_1 u^{\frac{n}{n-2}} &= 0 \quad \text{on} \ \partial M \setminus \partial \Gamma M,
\end{align*}
\]

By Lemma 3.4, there exists \( C_* \) a positive constant independent of \( C_* \). Indeed, the inequality (3.6) is a consequence of Lemma 2.4 (ii), that implies \( (\rho(p) \Delta \rho(p) - \frac{n}{2}) \to (\frac{n}{2} - d_j) \) as \( p \to \Gamma_j \). The inequality (3.7) is a consequence of Lemma 2.4 (ii).

On \( M_m \setminus M_{m_0} = \{ p \in M | m^{-1} < \rho(p) < m_0 \} \), \( \varepsilon^{-1} < m_0 < m \), we have

\[
\begin{align*}
\mathcal{L}_g \psi + c_0 \psi^{\frac{n+2}{n-2}} &= \frac{n-2}{2} \left( \rho \Delta \rho - \frac{n}{2} \rho \right) + c_0 \| \nabla \psi \|^2 \leq C_* \frac{n-2}{2} \left( \rho \Delta \rho - \frac{n}{2} \rho \right) + C_* \rho^{\frac{n}{n-2}} \rho H_2 \leq C_* \rho^{\frac{n}{n-2} - \frac{n}{2}} \left( \rho \Delta \rho - \frac{n}{2} \rho \right) + C_* \rho^{\frac{n}{n-2} - \frac{n}{2}} \rho H_2 \leq C_* \rho^{\frac{n}{n-2} - \frac{n}{2}} \left( \rho \Delta \rho - \frac{n}{2} \rho \right) + C_* \rho^{\frac{n}{n-2} - \frac{n}{2}} \rho H_2.
\end{align*}
\]

by (3.6). Moreover,

\[
\begin{align*}
\mathcal{B}_g \psi + c_1 \psi^{\frac{n}{n-2}} &= \frac{n-2}{2} \left( \rho \Delta \rho - \frac{n}{2} \rho \right) + \rho H_2 \leq \frac{n-2}{2} \left( \rho \Delta \rho - \frac{n}{2} \rho \right) + \rho H_2 \leq \frac{n-2}{2} \left( \rho \Delta \rho - \frac{n}{2} \rho \right) + \rho H_2 \leq \frac{n-2}{2} \left( \rho \Delta \rho - \frac{n}{2} \rho \right) + \rho H_2.
\end{align*}
\]

above we have used Lemma 2.4 (ii) and (3.7). Then, for \( m_0 \) large enough and very small \( C_* \), we have that

\[
\begin{align*}
\mathcal{L}_g \psi + c_0 \psi^{\frac{n+2}{n-2}} &= \frac{n-2}{2} \left( \rho \Delta \rho - \frac{n}{2} \rho \right) \leq 0 \quad \text{in} \ M_m \setminus M_{m_0}, \\
\mathcal{B}_g \psi + c_1 \psi^{\frac{n}{n-2}} &= \frac{n-2}{2} \left( \rho \Delta \rho - \frac{n}{2} \rho \right) \leq 0 \quad \text{on} \ N_m \setminus N_{m_0}.
\end{align*}
\]

Applying the Lemma 3.3

\[
\psi \geq C_0 m_0^{\frac{2-n}{2}} \quad \text{on} \ M_m \setminus M_{m_0}.
\]
In particular, $$u \geq \psi - C_{\star}m_{0}^{-\frac{2-n}{n}}$$ on \( \{ p \in M \mid 0 < \rho(p) \leq m_{0}^{-1} \} \).

Therefore, $$\liminf_{p \to \Gamma} u(p)\rho^{\frac{n-2}{2}}(p) > 0,$$

and, by maximum principle arguments, \( u > 0 \) on \( M \setminus \Gamma \).

3.2. The case \( \lambda_{g}(M \setminus \Gamma) < 0 \). Let \( \varepsilon_{0} > 0 \) be given by the Lemma 2.4(i), and let \( m_{0} > \varepsilon_{0}^{-1} \) be such that $$\rho^{2}R_{g} + 2(n-1)\left( \rho \Delta_{g} \rho - \frac{n}{2} \right) < -C_{2}$$

and $$\rho H_{g} - (n-1)g(\nabla \rho, \nu) < -C_{2},$$

on \( \{ p \in M \mid 0 < \rho(p) < m_{0}^{-1} \} \), where \( C_{2} \) is a positive constant.

Let \( \zeta \in C_{c}^{\infty}(M \setminus \overline{M_{m_{0}+1}}) \) be such that

\[
\begin{cases}
0 \leq \zeta \leq 1 & \text{in } M, \\
\zeta = 1 & \text{on } M \setminus \overline{M_{m_{0}+2}}.
\end{cases}
\]

Set $$\tilde{\rho}(p) = \begin{cases} 
(\rho(p) - 1)\zeta(p) + 1, & \text{if } p \in M \setminus \overline{(M_{m_{0}+1} \cup \Gamma)} , \\
1, & \text{if } p \in M_{m_{0}+1}.
\end{cases}$$

and $$\tilde{g} := \tilde{\rho}^{-2}g.$$ On \( M \setminus \overline{(M_{m_{0}+2} \cup \Gamma)} \) we have $$R_{\tilde{g}} = \rho^{2}R_{g} + 2(n-1)\left( \rho \Delta_{g} \rho - \frac{n}{2} \right)$$

and $$H_{\tilde{g}} = \rho H_{g} - (n-1)g(\nabla \rho, \nu).$$

By Lemma 2.4(ii), observe that \( H_{\tilde{g}} \in L^{\infty} \). For small \( \varepsilon > 0 \) we have

\[
\begin{cases}
R_{\tilde{g}}\varepsilon + c_{0}\varepsilon^{\frac{n-2}{2}} \leq 0 & \text{in } M \setminus \overline{(M_{m_{0}+2} \cup \Gamma)}, \\
H_{\tilde{g}}\varepsilon + c_{1}\varepsilon^{\frac{n-2}{2}} \leq 0 & \text{on } \partial M \setminus \overline{(N_{m_{0}+2} \cup VM)}.
\end{cases}
\]

Step 1. It is easy to see that the property \( \lambda_{g}(M \setminus \Gamma) < 0 \) is preserved under conformal changes of the metric. Choose \( j_{1} > m_{0} + 2 \) such that \( \lambda_{g}(M_{j_{1}}) < 0 \). As in the positive case, we choose \( \phi_{j_{1}} \in C^{2,\alpha}(M_{j_{1}}) \cap C^{0,\alpha}(\overline{M_{j_{1}}}) \) to be a minimum of the variational problem

$$\hat{\lambda}_{\tilde{g}}(M_{j_{1}}) := \inf_{\zeta \in C_{c}^{\infty}(M_{j_{1}})} \int_{M_{j_{1}}} \left( \nabla \zeta \right)^{2} + \frac{n-2}{4(n-1)}R_{\tilde{g}} \zeta^{2} \right) dv_{\tilde{g}} + \frac{n-2}{4(n-1)} \int_{\partial M_{j_{1}}} H_{\tilde{g}} \zeta^{2} d\sigma_{\tilde{g}}

\int_{M_{j_{1}}} \zeta^{2} d\sigma_{\tilde{g}} + \int_{N_{j_{1}}} \zeta^{2} d\sigma_{\tilde{g}}.$$

Then \( \hat{\lambda}_{\tilde{g}}(M_{j_{1}}) < 0 \) and

\[
\begin{cases}
\mathcal{L}_{\tilde{g}}\phi_{j_{1}} = \hat{\lambda}_{\tilde{g}}(M_{j_{1}}) \phi_{j_{1}} & \text{in } M_{j_{1}}, \\
\phi_{j_{1}} = 0 & \text{on } D_{j_{1}}, \\
\mathcal{B}_{\tilde{g}}\phi_{j_{1}} = \hat{\lambda}_{\tilde{g}}(M_{j_{1}}) \phi_{j_{1}} & \text{on } N_{j_{1}}, \\
\phi_{j_{1}} > 0 & \text{in } M_{j_{1}}.
\end{cases}
\]

Choose \( \delta > 0 \) such that

\[
\begin{cases}
\| c_{0} \|_{L^{\infty}} (\delta \phi_{j_{1}})^{\frac{n-2}{2}} \leq -\hat{\lambda}_{\tilde{g}}(M_{j_{1}}) & \text{in } M_{j_{1}}, \\
\| c_{1} \|_{L^{\infty}} (\delta \phi_{j_{1}})^{\frac{n-2}{2}} \leq -\hat{\lambda}_{\tilde{g}}(M_{j_{1}}) & \text{on } N_{j_{1}}.
\end{cases}
\]

If \( \psi_{j_{1}} := \delta \phi_{j_{1}} \),

\[
\begin{cases}
\mathcal{L}_{\tilde{g}}\psi_{j_{1}} + c_{0}\psi_{j_{1}}^{\frac{n-2}{2}} \leq 0 & \text{in } M_{j_{1}}, \\
\psi_{j_{1}} = 0 & \text{on } D_{j_{1}}, \\
\mathcal{B}_{\tilde{g}}\psi_{j_{1}} + c_{1}\psi_{j_{1}}^{\frac{n-2}{2}} \leq 0 & \text{on } N_{j_{1}}, \\
\psi_{j_{1}} > 0 & \text{in } M_{j_{1}}.
\end{cases}
\]
Using standard elliptic estimates, we see that the sequences
\[ u_j = u_0^- \leq u_1^- \leq \cdots \leq u_m^- \leq \cdots \leq u_1^+ \leq \cdots \leq u_0^+ = S \]
and
\[
\begin{cases}
-\Delta \tilde{g} u_m^{\pm} + S^{\frac{n-2}{n-2}} u_m^{\pm} = F_m & \text{in } M_j, \\
\frac{\partial u_m^{\pm}}{\partial \tilde{g}} + S^{\frac{n-2}{n-2}} u_m^{\pm} = G_m & \text{on } N_j,
\end{cases}
\]
where \( u_0^- = \psi_j, u_0^+ = S \),
\[
F_m := \left(S^{\frac{n-2}{n-2}} - R_{\tilde{g}}\right) u_m^{\pm} - c_0 |u_m^{\pm}|^{\frac{n+2}{n-2}} \quad \text{and} \quad G_m := \left(S^{\frac{n-2}{n-2}} - H_{\tilde{g}}\right) u_m^{\pm} - c_1 |u_m^{\pm}|^{\frac{n}{n-2}}.
\]
Indeed, we can assume that
\[
t \in [0, S] \mapsto \left(S^{\frac{n-2}{n-2}} - R_{\tilde{g}}(p)\right) t - c_0(p)t^{\frac{n+2}{n-2}} \quad \text{and} \quad t \in [0, S] \mapsto \left(S^{\frac{n-2}{n-2}} - H_{\tilde{g}}(q)\right) t - c_1(q)t^{\frac{n}{n-2}}
\]
are non-decreasing functions. So, the inequalities (3.8) are consequences of the following of the maximum principle, which is proved using \([13, \text{Theorem 8.16}]\) and \([26, \text{Hopf boundary point lemma}]\):

**Proposition 3.5.** Assume the hypotheses of Proposition 3.2 and \( d \in L^\infty(U) \) with \( d \leq 0 \). Let \( \dot{c}_2 : N \to [0, \infty) \) be a function and \( u \in H^1(U) \cap C(\overline{U}) \). Suppose
\[
\lim_{t \to 0^-} \frac{u(x + tv(x)) - u(x)}{t}
\]
exists for all \( x \in N \) and
\[
\begin{cases}
- (a^{ij} u_{x_i x_j}) - d u \leq 0 & \text{in } U, \\
u \leq 0 & \text{on } D, \\
\lim_{t \to 0^-} \frac{u(x + tv(x)) - u(x)}{t} + \dot{c}_2(x)u(x) \leq 0 & \text{on } N,
\end{cases}
\]
is satisfied in the weak sense in \( U \), and in the classical sense on \( N \), where \( \nu \) is the outward unit normal vector to \( N \). Then
\[
u \leq 0 \quad \text{in } U.
\]
By (3.8), both sequences \( \{u_m^+\} \) and \( \{u_m^-\} \) converge pointwise, so we set
\[
u^- := \lim_{m \to \infty} u_m^- \quad \text{and} \quad \nu^+ := \lim_{m \to \infty} u_m^+.
\]
Using standard elliptic estimates, we see that the sequences \( \{u_m^\pm\} \) are uniformly bounded in \( C^{2,\alpha}(M_j) \cap C^{0,\alpha'}(\overline{M_j}) \). It follows that \( u^\pm \in C^{2,\alpha'}(M_j) \cap C^{0,\alpha'}(\overline{M_j}) \) for \( \alpha' < \alpha \) and
\[
\begin{cases}
L_{\tilde{g}} u^\pm + c_0 \left(u^\pm\right)^{\frac{n-2}{n-2}} = 0 & \text{in } M_j, \\
u^\pm = \varepsilon & \text{on } D_j, \\
B_{\tilde{g}} u^\pm + c_1 \left(u^\pm\right)^{\frac{n-2}{n-2}} = 0 & \text{on } N_j, \\
u^\pm > 0 & \text{in } M_j.
\end{cases}
\]

**Step 2.** For \( k > j_1 \), we define
\[
u_k^-(p) = \begin{cases}
u^-(p), & \text{if } p \in M_j, \\
\varepsilon, & \text{if } p \in M_k \setminus M_j.
\end{cases}
\]
Proceeding as in Step 1 (replacing \(\psi_j\) by \(u_{k0}\)), there exist \(u^\pm_k \in C^{2,\alpha}(M_k) \cap C^{0,\alpha}(\overline{M_k})\) such that
\[
\begin{align*}
\mathcal{L}_g u_k^+ + c_0 \left( u_k^+ \right)^{\frac{n+2}{n-2}} &= 0 \quad \text{in } M_k, \\
\mathcal{L}_g u_k^- + c_0 \left( u_k^- \right)^{\frac{n+2}{n-2}} &= 0 \quad \text{in } M_k \setminus M_j,
\end{align*}
\]
\[
\begin{align*}
B_g u_k^+ + c_1 \left( u_k^+ \right)^{\frac{n+2}{n-2}} &= 0 \quad \text{on } N_k, \\
B_g u_k^- + c_1 \left( u_k^- \right)^{\frac{n+2}{n-2}} &= 0 \quad \text{on } M_k \setminus \overline{M_j}.
\end{align*}
\]

Using Theorem A.15 and standard elliptic arguments, we see that there is \(u \in C^\infty(M \setminus \Gamma)\) such that
\[
\begin{align*}
\mathcal{L}_g u + c_0 u^{\frac{n+2}{n-2}} &= 0 \quad \text{in } M \setminus \Gamma, \\
\mathcal{L}_g u + c_1 u^{\frac{n+2}{n-2}} &= 0 \quad \text{on } \partial M \setminus \overline{\nu M},
\end{align*}
\]
\[
\begin{align*}
u \geq \epsilon \quad \text{in } M \setminus (M_j \cup \Gamma), \\
u > 0 \quad \text{in } M \setminus \Gamma.
\end{align*}
\]

Recalling that we are working with the metric \(\tilde{g} = \rho^{-2} g\), we have that \(w = u \rho^{\frac{n+2}{2n}}\) satisfies
\[
\begin{align*}
-\Delta_g w + \left( \frac{n-2}{4(n-1)} \right) R_g w + c_0 w^{\frac{n+2}{n-2}} &= 0 \quad \text{in } M \setminus \Gamma, \\
\frac{\partial w}{\partial \nu_g} + \frac{n-2}{2(n-1)} H_g w + c_1 w^{\frac{n+2}{n-2}} &= 0 \quad \text{on } \partial M \setminus \overline{\nu M},
\end{align*}
\]
\[
\liminf_{p \to \Gamma} w(p) \operatorname{dist}_g(p, \Gamma) \geq \epsilon.
\]

This completes the proof of the case \(\lambda_g(M \setminus \Gamma) < 0\).

3.3. The “only if” part. Following the lines of [41] we suppose that \(d_r \leq (n-2)/2\), for some \(r\), and \(\tilde{g} = u^{-\frac{2}{n-2}} g\) has scalar curvature \(R_{\tilde{g}} = -c_0\) on \(M \setminus \Gamma\) and mean curvature \(H_{\tilde{g}} = -c_1\) on \(\partial M \setminus \overline{\nu M}\). Then \(u\) satisfies
\[
\begin{align*}
\nabla^2 u + \left( \frac{n-2}{4(n-1)} \right) R_g u + c_0 u^{\frac{n+2}{n-2}} &= 0 \quad \text{in } M \setminus \Gamma, \\
\frac{\partial u}{\partial \nu_g} - \frac{n-2}{2(n-1)} H_g u + c_1 u^{\frac{n+2}{n-2}} &= 0 \quad \text{on } \partial M \setminus \overline{\nu M},
\end{align*}
\]
and
\[
\begin{align*}
-\int_M g(\nabla u, \nabla \zeta) d\nu_g &= \int_M \left( \frac{n-2}{4(n-1)} \right) R_g u \zeta d\nu_g + \int_M \frac{n-2}{2(n-1)} c_0 u^{\frac{n+2}{n-2}} \zeta d\nu_g \\
+ \int_{\partial M} \frac{n-2}{2(n-1)} \left( H_g + c_1 u^{\frac{n-2}{n+2}} \right) u \zeta d\sigma_g.
\end{align*}
\]

for all \(\zeta \in C_0^1(M \setminus \Gamma)\).

Assume by contradiction that \(\tilde{g}\) is complete as a metric space. In particular, \(u(p) \to \infty\) as \(\rho(p) = \operatorname{dist}_g(p, \Gamma) \to 0\). By Lemma 2.4(ii), we can assume that
\[
H_{\tilde{g}} \geq 0 \quad \text{near } \Gamma_r.
\]

From (3.10), for \(\delta > 0\) small enough, \(u\) satisfies
\[
\begin{align*}
\Delta_g u \geq C_0 u^{\frac{n+2}{n-2}}, \\
\frac{\partial u}{\partial \nu_g} \leq -C_0 u^{\frac{n+2}{n-2}},
\end{align*}
\]
in \(\{0 < \rho < \delta\}\), where \(C_0 = C_0(n, c_0, c_1)\) is a positive constant.

Let \((U_\alpha, \varphi_\alpha)\) be a chart with the property (2.1) of Definition 2.3 satisfying \(U_\alpha \subset C_{d_r, h_r}\). Let \(V \subset U_\alpha \cap \{0 \leq \rho_{R^*_\alpha} < \delta\}\) be an open subset. There exists a small constant \(k = k(h_r, V) > 0\) that verifies
\[
B(x_0, k \rho_{R^*_\alpha}(x_0)) \cap C_{d_r, h_r} \subset U_\alpha \setminus \{x_{d_r+1} = \cdots = x_n = 0\}
\]
for all \(x_0 \in V \setminus \{x_{d_r+1} = \cdots = x_n = 0\}\), where \(\rho_{R^*_\alpha}(x_0) = \operatorname{dist}_{R^*_\alpha}(x_0, \{x_{d_r+1} = \cdots = x_n = 0\})\).
Let $x_0 \in V \setminus \{x_{d+1} = \cdots = x_n = 0\}$ be fixed. For a suitable constant $C_1 = C_1(n, c_0, c_1, g) > 0$ the function

$$w(x) = \frac{C_1 (k_{R^n}(x_0))^{\frac{n+2}{2n}}}{(k_{R^n}(x_0)^2 - |x-x_0|^2)^{\frac{n+2}{2n}}}$$

satisfies

$$\begin{cases}
\Delta_g (w \circ \varphi_\alpha^{-1}) & \leq C_0 (w \circ \varphi_\alpha^{-1})^{\frac{n+2}{n-2}}, \\
\frac{\partial (w \circ \varphi_\alpha^{-1})}{\partial \nu_g} & \geq -C_0 (w \circ \varphi_\alpha^{-1})^{\frac{n+2}{n-2}},
\end{cases}$$

on $\varphi_\alpha (B (x_0, k_{R^n}(x_0)) \cap C_{d, h_0})$.

We will show that $u \leq w \circ \varphi_\alpha^{-1}$ on $\varphi_\alpha (B (x_0, k_{R^n}(x_0)) \cap C_{d, h_0})$. Suppose by contradiction this inequality does not hold. Define

$$B^+ = \{p \in \varphi_\alpha (B (x_0, k_{R^n}(x_0)) \cap C_{d, h_0}) \mid u(p) > w \circ \varphi_\alpha^{-1}(p)\} \neq \emptyset,$$

and observe that $\overline{B^+} \subset \varphi_\alpha (B (x_0, k_{R^n}(x_0)) \cap C_{d, h_0}) \subset M \setminus \Gamma$.

We consider the case $B^+ \cap \partial M \neq \emptyset$; the other case follows the same way. By (3.13) and (3.15),

$$\begin{cases}
-\Delta_g (w \circ \varphi_\alpha^{-1} - u) & \geq 0 \quad \text{in } B^+ , \\
\frac{\partial (w \circ \varphi_\alpha^{-1} - u)}{\partial \nu_g} & \geq 0 \quad \text{on } \partial B^+ \cap \text{int}(M) , \\
\frac{\partial (w \circ \varphi_\alpha^{-1} - u)}{\partial \nu_g} & \geq 0 \quad \text{on } B^+ \cap \partial M .
\end{cases}$$

It follows from the maximum principle that $u \leq w \circ \varphi_\alpha^{-1}$ on $B^+$. Thus, we have a contradiction. So,

$$u \leq w \circ \varphi_\alpha^{-1} \text{ on } \varphi_\alpha (B (x_0, k_{R^n}(x_0)) \cap C_{d, h_0}).$$

In particular,

$$u(\varphi_\alpha(x_0)) \leq w(x_0) = C_1 (k_{R^n}(x_0))^{\frac{n+2}{2n}}.$$

We can assume that there is an appropriate constant $C_2 > 0$ such that $\rho \circ \varphi_\alpha \leq C_2 k_{R^n}$ in $U_\alpha$. Then, since $M$ is compact, there is $\delta_1 > 0$ small enough so that

$$u(p) \leq C_3 \rho(p)^{\frac{2n+2}{n}} \quad \text{if } 0 < \rho(p) < \delta_1 ,$$

where $C_3$ is a positive constant independent of $u$.

Next we follow the argument in [3, pp.628-630] to show that $u$ is bounded near $\Gamma_*$. This is where the assumption $d_r \leq \frac{4n}{n-2}$ will be used. By (3.11) and (3.12), there is $\delta > 0$ small enough such that $u$ satisfies

$$- \int_M g(\nabla u, \nabla \zeta) dv_g \geq C_4(n, c_0) \int_M u^{\frac{n+2}{2}} \zeta dv_g ,$$

for $\zeta \in C^1_c (\{0 < \rho(p) < \delta\})$ with $\zeta \geq 0$.

Let $\kappa : \mathbb{R} \to \mathbb{R} \subset C^\infty$ be increasing, bounded such that $\kappa(t) \equiv 0$ if $t < 0$. Additionally, for each positive integer $m$, let $\xi_m : \mathbb{R} \to \mathbb{R}$ be a smooth function such that

$$\xi_m(t) = \begin{cases}
1 & \text{if } t > \frac{1}{m} , \\
0 & \text{if } t < \frac{1}{2m} .
\end{cases}$$

$0 \leq \xi_m \leq 1$ and $|\xi_m'| \leq C_5 m$, for a suitable positive constant $C_5$. Let $\beta > \max(\rho = \rho_0)$ where $\rho_0 \in (0, \delta_1)$. By (3.17), if $m > 3/\rho_0$,

$$\begin{align*}
C_4 \int_M u^{\frac{n+2}{2n}} (\kappa(u) - \xi_m \circ \rho) dv_g & \leq - \int_M g(\nabla u, \nabla (\kappa(u - \beta) \xi_m \circ \rho)) dv_g \\
& = - \int_M \kappa'(u - \beta) \xi_m \circ \rho |\nabla (u - \beta)|^2 dv_g - \int_M (\kappa(u - \beta) \xi_m' \circ \rho g (\nabla (u - \beta)^+, \nabla \rho)) dv_g \\
& \leq C_6 m |\nabla (u - \beta)^+|_{L^2(\{\frac{1}{m} \leq \rho \leq \frac{1}{2m}\})} \left( \rho \leq \frac{1}{m} \right) .
\end{align*}$$

(3.18)
Next we will estimate the r.h.s. of (3.18). Let \( z_m : \mathbb{R} \to \mathbb{R} \) be a smooth function such that
\[
z_m(t) = \begin{cases} 
1 & \text{if } t \in \left( \frac{1}{2m}, \frac{1}{m} \right), \\
0 & \text{if } t \in \left( -\infty, \frac{1}{4m} \right) \cup \left( \frac{5}{4m}, \infty \right), 
\end{cases}
\]

\( 0 \leq z_m \leq 1 \) and \( |z'_m| \leq C_7 m \), where \( C_7 \) is a positive constant. Since \( (u - \beta)^+ (z_m \circ \rho)^2 \in H^1(M) \) is non-negative and has support in \( M \backslash \Gamma \), by (3.17),
\[
\int_M (z_m \circ \rho)^2 |\nabla (u - \beta)^+|^2 \, dv_g + \int_M 2(u - \beta)^+ (z_m \circ \rho) g (\nabla (u - \beta)^+, z_m \nabla \rho) \, dv_g \leq 0.
\]
Thus,
\[
\|\nabla (u - \beta)^+\|_{L^2\left(\left\{ \frac{1}{m} \leq \rho \leq \frac{1}{m} \right\}\right)} \leq C_7 m\|u - \beta)^+\|_{L^2\left(\left\{ \frac{1}{m} \leq \rho \leq \frac{1}{m} \right\}\right)}.
\]

By (3.15), (3.18) and (3.19),
\[
\int_M u^{n-2} \kappa (u - \beta) \xi \circ \rho \, dv_g \leq C_{10} m^{2+ \frac{n-2}{2} + d_r - n} < C_{11},
\]
where \( C_{11} \) is a positive constant independent of \( u \) and \( m \). So \( \int_M u^{n-2} < \infty \). By (3.18) and (3.16),
\[
\int_M (u - \beta)^+ \left( \frac{1}{m} \leq \rho \leq \frac{1}{m} \right) \left| \frac{(u - \beta)^+}{\kappa} \right|^2 \, dv_g \leq C_{12} m^{\frac{2}{n-2} + \frac{4}{n-2}} \left( \frac{1}{m} \right) \leq C_{13} m^{\frac{3}{n-2}} \leq C_{14} m^{\frac{3}{n-2} - \frac{4}{n-2}} \leq C_{15} m^{-4}.
\]

Making \( m \to \infty \) we see that \( \int_M u^{n-2} \kappa (u - \beta) \, dv_g = 0 \). Therefore, \( u \) is bounded near \( \Gamma_r \) and we have a contradiction. Hence, \( \tilde{g} \) is not complete as a metric space.

**Appendix**

**A.1. Analysis on cornered manifolds.** In this appendix, we will work with compact cornered \( n \)-manifolds locally modeled by \( \mathbb{R}^{n-2} \times [0, \infty)^2 \). More precisely (see [16] for a general definition):

**Definition A.1.** A smooth cornered manifold of dimension \( n \) is a paracompact Hausdorff topological space \( \overline{M} \) and a family of homeomorphisms
\[
\varphi_a : U_a \to \varphi_a(U_a) \subset \overline{M},
\]
where \( U_a \subset \mathbb{R}^{n-2} \times [0, \infty)^2 \) is a relative open subset, satisfying the following three conditions:

(i) \( \bigcup_a \varphi_a(U_a) = \overline{M} \),
(ii) For any pair \( \alpha, \beta \), with \( W := \varphi_a(U_a) \cap \varphi_\beta(U_\beta) \neq \emptyset \), the mappings \( \varphi_\beta^{-1} \circ \varphi_a : \varphi_a^{-1}(W) \to \varphi_\beta^{-1}(W) \) is smooth.
(iii) The family \( \{(U_a, \varphi_a)\} \) is maximal relative to the conditions (i) and (ii).

The pair \( (U_a, \varphi_a) \) (or the mapping \( \varphi_a \)) with \( p \in \varphi_a(U_a) \) is called a chart (or system of coordinates) of \( \overline{M} \) at \( p \). A family \( \{(U_a, \varphi_a)\} \) satisfying (i) and (ii) is called a smooth structure on \( \overline{M} \).

We define the corner by
\[
\partial \overline{M} := \{ q \in \overline{M} | q = \varphi(y, 0, 0) \text{ for some chart } (U, \varphi) \text{ of } \overline{M} \text{ and } y \in \mathbb{R}^{n-2} \},
\]
Moreover, for every \( p < \infty \) and \( \alpha + \beta \leq p \), we have the following Sobolev embedding theorems:

\[
\|u\|_{W^{\alpha, p}(\mathcal{M})} = \left( \int_{\mathcal{M}} |u|^p d\nu + \int_{\mathcal{N}} |\nabla u|^p d\nu \right)^{\frac{1}{p}}.
\]

For \( 1 \leq p < \infty \) we define the Sobolev space \( W^{1, p}(\mathcal{M}) \) by

\[
W^{1, p}(\mathcal{M}) := \left\{ u \in L^p_{\text{loc}}(\mathcal{M}) \mid u \in L^p(\mathcal{M}), \exists \nabla u \text{ and } \nabla u \in W^{1, p}(\mathcal{M}) \right\},
\]
equipped with the norm

\[
\|u\|_{W^{1, p}(\mathcal{M})} := \left( \int_{\mathcal{M}} |u|^p d\nu + \int_{\mathcal{N}} |\nabla u|^p d\nu \right)^{\frac{1}{p}}.
\]

Similarly to the Euclidean case (see [9, 10] when \( \mathcal{M} \) is an open subset of \( \mathbb{R}^n \) with Lipschitz boundary) we have the following Sobolev embedding theorems:

**Theorem A.3.** Assume \( 1 \leq p < n \), and \( u \in W^{1, p}(\mathcal{M}) \). Then \( u \in L^{p^*}(\mathcal{M}) \), \( p^* = \frac{np}{n-p} \), with the estimate

\[
\|u\|_{L^{p^*}(\mathcal{M})} \leq C(p, n, \mathcal{M}) \|u\|_{W^{1, p}(\mathcal{M})}.
\]

**Theorem A.4.** (Trace theorem). Assume \( 1 \leq p < \infty \). Then there is a continuous boundary trace embedding

\[
\mathcal{T} : W^{1, p}(\mathcal{M}) \to L^{\frac{(n-1)p}{n-p}}(\partial \mathcal{M})
\]
such that

\[
\mathcal{T} u = u|_{\partial \mathcal{M}} \quad \text{if } u \in W^{1, p}(\mathcal{M}) \cap C(\mathcal{M}).
\]

Moreover, for every \( q \in \left[ 1, \frac{(n-1)p}{n-p} \right) \) the trace embedding \( W^{1, p}(\mathcal{M}) \to L^q(\partial \mathcal{M}) \) is compact.

Set

\[
H^1_\mathcal{D}(\mathcal{M}) := \left\{ u \in H^1(\mathcal{M}) \mid u|_{\mathcal{D}} = 0 \right\}.
\]

We define the bilinear form

\[
\mathcal{B}[u, v] := \int_{\mathcal{M}} (g(\nabla u, \nabla v) + cuv) d\nu + \int_{\mathcal{N}} c_2 uv d\sigma_g \quad (u, v \in H^1_\mathcal{D}(\mathcal{M})),
\]

where \( c \in L^\infty(\mathcal{M}) \) and \( c_2 \in L^\infty(\mathcal{N}) \), and we set

\[
\mathcal{B}_{\mu_1, \mu_2}[u, v] := \mathcal{B}[u, v] + \mu_1 \int_{\mathcal{M}} uv d\nu + \mu_2 \int_{\mathcal{N}} uv d\sigma_g \quad (u, v \in H^1_\mathcal{D}(\mathcal{M})),
\]

for \( \mu_1, \mu_2 \in \mathbb{R} \).

**Proposition A.5.** There is a number \( \gamma \geq 0 \) such that for each

\[
\mu_1 \geq \gamma, \quad \mu_2 \geq 0,
\]
and each function

\[
f_1 \in L^2(\mathcal{M}) \quad \text{and} \quad f_2 \in L^2(\mathcal{N}),
\]
there exists a unique weak solution \( u \in H^1_\mathcal{D}(\mathcal{M}) \) of the mixed-boundary-value problem:

\[
\mathcal{B}_{\mu_1, \mu_2}[u, v] = \int_{\mathcal{M}} f_1 v d\nu + \int_{\mathcal{N}} f_2 v d\sigma_g \quad (v \in H^1_\mathcal{D}(\mathcal{M})).
\]
Proof. Choose \( \gamma \geq 0 \) and \( \alpha, \beta > 0 \) such that
\[
|B[u,v]| \leq \alpha \|u\|_{H^1(\mathcal{M})} \|v\|_{H^1(\mathcal{M})}
\]
and
\[
\beta \|u\|_{H^1(\mathcal{M})} \leq B[u,u] + \gamma \|u\|^2_{L^2(\mathcal{M})} + \mu_2 \|u\|_{L^2(\mathcal{M})}^2,
\]
hold for all \( \mu_2 \geq 0 \) and \( u, v \in H^1_0(\mathcal{M}) \). Then \( B_{\mu_1,\mu_2}[\cdot,\cdot] \), with \( \mu_1 \geq \gamma \), satisfies the hypotheses of the Lax-Milgram Theorem.

Now fix \( f_1 \in L^2(\mathcal{M}) \) and \( f_2 \in L^2(\mathcal{N}) \). Set \( Q(v) := \int_{\mathcal{M}} f_1 v \mathrm{d}\nu_{\gamma} + \int_{\mathcal{N}} f_2 v \mathrm{d}\sigma_{\gamma} \), which is a bounded linear functional on \( H^1_0(\mathcal{M}) \). By the Lax-Milgram theorem, there exists a unique \( u \in H^1_0(\mathcal{M}) \) satisfying
\[
B_{\mu_1,\mu_2}[u,v] = Q(v),
\]
for all \( v \in H^1_0(\mathcal{M}) \).

\[\blacksquare\]

### A.2 Elliptic estimates

In this subsection, we adapt the proof of the next proposition to estimate (in a coordinate neighborhood) our solutions on the boundary. Our main results are Lemmas A.9 and A.10. Denote by \( B \), the ball \( \{ x \in \mathbb{R}^n \mid |x| < r \} \).

**Proposition A.6.** (See [15, Theorem 4.1].) Suppose \( n \geq 2 \), \( a^{ij} \in L^\infty(B_1) \) and \( c \in L^q(B_1) \) for some \( q > n/2 \) satisfy
\[
a^{ij}(x)y_iy_j \geq \lambda |y|^2 \text{ for any } x \in B_1, \quad y \in \mathbb{R}^n,
\]
and
\[
\|a^{ij}\|_{L^\infty(B_1)} + \|c\|_{L^q(B_1)} \leq \Lambda,
\]
for some positive constants \( \lambda \) and \( \Lambda \). Suppose that \( u \in H^1(B_1) \) is a subsolution in the sense that the inequality
\[
\int_{B_1} a^{ij} u_{x_i} \zeta_{x_j} + cu \zeta \mathrm{d}x \leq \int_{B_1} f \zeta \mathrm{d}x
\]
holds for any \( \zeta \in H^1_0(B_1) \) and \( \zeta \geq 0 \) in \( B_1 \).

If \( f \in L^p(B_1) \), then \( u^+ \in L^\infty_{\text{loc}}(B_1) \). Moreover, there holds for any \( r \in (0,1) \) and \( p > 0 \)
\[
\sup_{B_r} u^+ \leq C \left[ \frac{1}{(1-r)^n} \|u^+\|_{L^p(B_1)} + \|f\|_{L^p(B_1)} \right]
\]
where \( C = C(\lambda, \Lambda, p, q, n, B_1) > 0 \).

For \( r > 0 \), we write
\[
K_r = \left\{ x \in \mathbb{R}^n \mid \max_{i \in \{1, \ldots, n\}} |x_i| < r \right\}.
\]

Set
\[
D = K_1 \cap \{ x_n > 0, \ x_{n-1} > 0 \}, \quad \mathcal{N} = K_1 \cap \{ x_{n-1} = 0, \ x_n > 0 \}.
\]

Let \( n \geq 3 \) and \( V_0 \subset \mathbb{R}^n \) be open bounded sets with \( V_0 \subseteq V_1 \) and \( \zeta \in C^\infty_c(V_1) \) satisfying \( 1 \geq \zeta \geq 0 \) and \( \zeta|_{V_0} \equiv 1 \). For the next lemma, we assume that there exist \( u \in H^1(B) \) and constants \( S_* \geq 0, \ C_* > 0 \) such that
\[
\|(u-k)^+ \zeta\|_{L^{2^*}(B)} \leq C_* \|\nabla [(u-k)^+ \zeta]\|_{L^{2^*}(B)} \text{ for all } k \geq S_*,
\]
where \( 2^* = \frac{2n}{n-2} \). Write \( s_1 = \sup_{B\subset \mathbb{R}^n} |\nabla \zeta| \)
\[
v(k,i) = \{ x \in V_i \cap B \mid u(x) > k \} \quad \text{and} \quad v_N(k,i) = \{ x \in V_i \cap \mathcal{N} \mid u(x) > k \},
\]
i = 0, 1. For \( k \in \mathbb{R} \), set \( u_k = (u-k)^+ \).

We also assume that \( a^{ij} \in L^\infty(B) \) is uniformly elliptic respect to \( \lambda \) (i.e., \( \text{(A.3)} \) holds), \( c \in L^q(B) \) and \( c_1 \in L^{q_1}(\mathcal{N}) \) for some \( q, q_1 > n - 1 \), with
\[
\sum_{i,j} \|a^{ij}\|_{L^\infty(B)} + \|c\|_{L^q(B)} + \|c_1\|_{L^{q_1}(\mathcal{N})} \leq \Lambda,
\]
for some positive constant \( \Lambda \).
Lemma A.7. Suppose $f \in L^q(B)$, $f_1 \in L^{q_1}(\mathcal{N})$ and
\[
\int_B (a^j u_x, v_x + cuv) \, dx + \int_{\mathcal{N}} c_1 u v d\sigma \leq \int_B f_1 v \, dx + \int_{\mathcal{N}} f_1 v \, d\sigma \quad \forall k \geq \mathcal{S}_r, v = u_k \zeta^2. \tag{A.5}
\]
Set
\[
\Psi(k, i) := \sqrt{\int_{\mathcal{V}(k, i)} u_k^2 \, dx + \int_{\mathcal{N}(k, i)} u_k^2 \, d\sigma}, \quad i = 0, 1.
\]
There exist $\varepsilon = \varepsilon(q, q_1, n) > 0$ and $\mathcal{N} = N(\alpha, \lambda, n, B, c, c_1) > 0$ such that if
\[
h > k > N \max \{\|u^+\|_{L^2(B)}, \|u^+\|_{L^2(\mathcal{N})}, \mathcal{S}_r\},
\]
then
\[
\Psi(h, 0) \leq C \left[ s_1 \frac{1}{(h - k)^\varepsilon} + \frac{\|f\|_{L^q(B)} + \|f_1\|_{L^{q_1}(\mathcal{N})} + \mathcal{S}_r}{(h - k)^{1+\varepsilon}} \right] \psi^{1+\varepsilon}(k, 1),
\]
where $C = C(\alpha, \lambda, n, B, c, c_1, q_1) > 0$.
Proof. We will follow the proof in [15, Theorem 4.1].

Claim 1. We have
\[
\int_B |\nabla (u_k \zeta)|^2 \, dx \leq C_1(\alpha, \lambda, n) \left( s_1^2 \int_{\mathcal{V}(h, 1)} u_k^2 \, dx + \int_B a^j u_x (u_k \zeta)^2 \, dx \right).
\]

The inequality (A.5) and the Claim\[\square\] implies
\[
\int_B |\nabla (u_k \zeta)|^2 \, dx \leq C_1 \left( s_1^2 \int_{\mathcal{V}(h, 1)} u_k^2 \, dx - \int_B cuv \, dx - \int_{\mathcal{N}} c_1 u v d\sigma + \int_B f v \, dx + \int_{\mathcal{N}} f_1 v \, d\sigma \right), \tag{A.6}
\]
where $v = u_k \zeta^2$. Next we will estimate the terms on the right side of (A.6).

Claim 2. The following estimates hold:

(i) \[\int_{\mathcal{N}} c_1 u (u_k \zeta^2) \, d\sigma \leq C_2(n, B) \|c_1\|_{L^q(\mathcal{N})} \{u_k \zeta \neq 0\} \cap \mathcal{N}^\frac{1}{\sqrt{n-1}} \frac{1}{2} \int_B |\nabla (u_k \zeta)|^2 \, dx + h^2 \|c_1\|_{L^q(\mathcal{N})} \{u_k \zeta \neq 0\} \cap \mathcal{N}^{1-\frac{1}{n}}.\]

(ii) \[\int_{\mathcal{N}} f_1 u_k \zeta^2 \, d\sigma \leq C_3(n, B) \left( \delta^{-1} \{u_k \zeta \neq 0\} \cap \mathcal{N}^\frac{1}{\sqrt{n-1}} \frac{1}{2} \|f_1\|_{L^{q_1}(\mathcal{N})} + \delta \right) \int_B |\nabla (u_k \zeta)|^2 \, dx,\]
for all $\delta > 0$.

(iii) \[\int_B cuv \zeta^2 \, dx \leq C_4(n, B) \|c\|_{L^q(B)} \{u_k \zeta \neq 0\}^{\frac{3}{2}-\frac{1}{2}} \int_B |\nabla (u_k \zeta)|^2 \, dx + h^2 \|c\|_{L^q(B)} \{u_k \zeta \neq 0\}^{-\frac{1}{2}}.\]

(iv) \[\int_B f \zeta^2 \, dx \leq C_5(n, B) \left( \delta^{-1} \{u_k \zeta \neq 0\}^{1+\frac{3}{2}-\frac{3}{2}} \|f\|_{L^{q_1}(B)} + \delta \right) \int_B |\nabla (u_k \zeta)|^2 \, dx,\]
for all $\delta > 0$.

Proof. We will start with the following three inequalities that will be used later. The Hölder’s inequality and the Trace theorem imply
\[
\|u_k \zeta\|_{L^2(\mathcal{N})} \leq \{u_k \zeta \neq 0\} \cap \mathcal{N}^\frac{1}{\sqrt{n-1}} \|u_k \zeta\|_{L^{2(n-1)}}^{\frac{1}{n-1}} \leq C_6(n, B) \{u_k \zeta \neq 0\} \cap \mathcal{N}^\frac{1}{\sqrt{n-1}} \|u_k \zeta\|_{H^1(B)}. \tag{A.7}
\]
On the other hand, by (A.4),
\[ \int_B (u_h \zeta)^2 \, dx \leq \left( \| u_h \zeta \|_2 \right)^2 \| u_h \zeta \|_2^2 (B) \leq C^2 \| \{ u_h \zeta \neq 0 \} \|_2^2 \| \nabla (u_h \zeta) \|_2^2 \, dx, \]  
(A.8)
for \( h \geq S_* \), where \( \frac{2q}{2q-2} = \frac{q}{h} \). Further, from (A.7) and (A.8),
\[ \int_{\mathcal{N}} (u_h \zeta)^2 \, d\sigma \leq C_7 (n, B) \| \{ u_h \zeta \neq 0 \} \cap \mathcal{N} \|^{\frac{n-3}{n-2}} \int_B \| \nabla (u_h \zeta) \|_2^2 \, dx \]  
(A.9)

(i) Since \( \frac{n-2}{n-1} + \frac{1}{q_1} < 1 \),
\[ \begin{aligned}
\int_{\mathcal{N}} c_1 u (u_h \zeta^2) \, d\sigma &= - \int_{\{ u_h \zeta \neq 0 \} \cap \mathcal{N}} c_1 \left( u_h^2 + h u_h \right) \zeta^2 \, d\sigma \\
&\leq 2 \int_{\{ u_h \zeta \neq 0 \} \cap \mathcal{N}} |c_1| u_h^2 \zeta^2 \, d\sigma + h^2 \int_{\{ u_h \zeta \neq 0 \} \cap \mathcal{N}} |c_1| \zeta^2 \, d\sigma \\
&\leq 2 \left( \int_{\mathcal{N}} |c_1| q_1 \, d\sigma \right)^{\frac{1}{q_1}} \left( \int_{\{ u_h \zeta \neq 0 \} \cap \mathcal{N}} |u_h \zeta|^{\frac{2(n-1)}{2q-2}} \, d\sigma \right)^{\frac{n-2}{n-1}} \| \{ u_h \zeta \neq 0 \} \cap \mathcal{N} \|^{1-\frac{n-2}{n-1}} \\
&+ h^2 \| c_1 \|_{L^2(\mathcal{N})} \| \{ u_h \zeta \neq 0 \} \cap \mathcal{N} \|^{1-\frac{1}{n}} \left( \int_B (u_h \zeta)^2 \, dx + \int_B \| \nabla (u_h \zeta) \|_2^2 \, dx \right) \\
&\leq C_8 (n, B) \| c_1 \|_{L^2(\mathcal{N})} \| \{ u_h \zeta \neq 0 \} \cap \mathcal{N} \|^{1-\frac{1}{n}} \left( \int_B (u_h \zeta)^2 \, dx + \int_B \| \nabla (u_h \zeta) \|_2^2 \, dx \right) \\
&+ h^2 \| c_1 \|_{L^2(\mathcal{N})} \| \{ u_h \zeta \neq 0 \} \cap \mathcal{N} \|^{1-\frac{1}{n}} ,
\end{aligned} \]

by (A.7). Using (A.8) we conclude the proof of (i).

(ii) Since \( q_1 > n - 1 - \frac{1}{q_1} + \frac{n-2}{2(n-1)} < 1 \) and \( \zeta \leq 1 \),
\[ \int_{\mathcal{N}} f_1 u_h \zeta^2 \, d\sigma \leq \left( \int_{\mathcal{N}} |f_1|^{q_1} \, d\sigma \right)^{\frac{1}{q_1}} \left( \int_{\mathcal{N}} |u_h \zeta|^{\frac{2(n-1)}{2q-2}} \, d\sigma \right)^{\frac{n-2}{n-1}} \| \{ u_h \zeta \neq 0 \} \cap \mathcal{N} \|^{1-\frac{1}{n}} \left( \int_B (u_h \zeta)^2 \, dx + \int_B \| \nabla (u_h \zeta) \|_2^2 \, dx \right) . \]

By (A.7) and (A.8),
\[ \begin{aligned}
\int_{\mathcal{N}} |f_1| u_h \zeta^2 \, d\sigma &\leq C_9 \| f_1 \|_{L^{q_1}(\mathcal{N})} \left( \int_B \| \nabla (u_h \zeta) \|_2^2 \, dx \right)^{\frac{1}{2}} \| \{ u_h \zeta \neq 0 \} \cap \mathcal{N} \|^{\frac{n-2}{2q} - \frac{1}{n}} \| f_1 \|_{L^{q_1}(\mathcal{N})} + C_9 \delta \int_B \| \nabla (u_h \zeta) \|_2^2 \, dx .
\end{aligned} \]

This proves the item (ii). The proofs of (iii) and (iv) are the same as in [15].

Let us observe that \( q > \frac{n}{2} \), \( q_1 > n - 1 \), \( \{ u_h \zeta \neq 0 \} \subset V(h, 1) \), \( \{ u_h \zeta \neq 0 \} \cap \mathcal{N} \subset V(h, 1) \), \( |V(h, 1)| \leq h^{-1} \int V(h, 1) u^+ \) and \( |\nabla V(h, 1)| \leq h^{-1} \int V(h, 1) u^+ \). By (A.6) and Claim 2 there exists a constant \( N = N (\lambda, \Lambda, n, B, c, c_1) > 0 \) such that if \( h > N \max \{ \| u^+ \|_{L^2(B)}, \| u^+ \|_{L^2(\mathcal{N})}, \mathcal{S}_v \} \) then
\[ |V(h, 1)|, |\nabla V(h, 1)| < 1. \]

(A.10)

and
\[ \begin{aligned}
\int_B \| \nabla (u_h \zeta) \|_2^2 \, dx &\leq C_{10} \left[ s_1 \int_{V(h, 1)} u_h^2 \, dx + \left( \| f_1 \|_{L^{q_1}(B)}^2 + h^2 \right) \| \{ u_h \zeta \neq 0 \} \|^{1-\frac{1}{4}} + \left( \| f_1 \|_{L^{q_1}(\mathcal{N})} + h^2 \right) \| \{ u_h \zeta \neq 0 \} \cap \mathcal{N} \|^{1-\frac{1}{n}} \right] .
\end{aligned} \]

(A.11)

where \( C_{10} = C_{10}(\lambda, \Lambda, n, B, c, c_1) > 0 \).
From (A.7) - (A.9) and (A.11), we have

\[
\int_B (u_h \zeta)^2 dx \leq C_{11} \left[ s^2 \|u_h \zeta \neq 0\| + \int_{\mathcal{V}(h, 1)} u_h^2 dx + \int_{\mathcal{V}(h, 1)} u_h^2 d\sigma \right]
\]

\[
+ \left( \|f\|_L^5(B) + \|f_1\|_L^5(\mathcal{N}) + h \right) \left( \|u_h \zeta \neq 0\| + \|u_h \zeta \neq 0\| \right)^{1 + \frac{5}{2}} + \left( \|u_h \zeta \neq 0\| \right)^{1 + \frac{5}{2}} \right] .
\]

(A.12)

and

\[
\int_{\mathcal{N}} (u_h \zeta)^2 d\sigma \leq C_{12} \left[ s^2 \|u_h \zeta \neq 0\| \cap \mathcal{N} \right] \left( \int_{\mathcal{V}(h, 1)} u_h^2 dx + \int_{\mathcal{V}(h, 1)} u_h^2 d\sigma \right)
\]

\[
+ \left( \|f\|_L^5(B) + \|f_1\|_L^5(\mathcal{N}) + h \right) \left( \|u_h \zeta \neq 0\| \cap \mathcal{N} \right) \left( \|u_h \zeta \neq 0\| \right)^{1 + \frac{5}{2}} \right] ,
\]

(A.13)

where \( C_i = C_i(\lambda, \Lambda, n, B, c, c_1) > 0, \) \( i = 11, 12. \)

On the other hand. Set \( \epsilon = \frac{1}{n+1} - \frac{1}{\min_{q \in \mathcal{Q}^1}}, \) by Young’s inequality,

\[
\|u_h \zeta \neq 0\| \cap \mathcal{N} \right) \leq \frac{1}{C_{13}} \left( \|u_h \zeta \neq 0\| \right)^{1 + \epsilon} + \frac{C_{13} - 1}{C_{13}} \left( \|u_h \zeta \neq 0\| \right)^{1 + \frac{1}{\epsilon}},
\]

(A.14)

\[
\|u_h \zeta \neq 0\| \cap \mathcal{N} \right) \leq \frac{1}{C_{14}} \left( \|u_h \zeta \neq 0\| \right)^{1 + \frac{1}{\epsilon}} + \frac{C_{14} - 1}{C_{14}} \left( \|u_h \zeta \neq 0\| \right)^{1 + \frac{1}{\epsilon}},
\]

(A.15)

where \( C_{13} = \frac{n}{2} (1 + \epsilon) \) and \( C_{14} = (n - 1)(1 + \epsilon). \) Observe that

\[
\left( 1 - \frac{1}{q_1} \right) \frac{C_{13}}{C_{13} - 1} \geq 1 + \epsilon \quad \text{and} \quad \left( 1 - \frac{1}{q} \right) \frac{C_{14}}{C_{14} - 1} \geq 1 + \epsilon.
\]

The inequalities (A.10), (A.12) - (A.15) imply

\[
\Psi(h, 0)^2 \leq C_{15} \left[ s^2 \left( \|u_h \zeta \neq 0\| \right) - \|u_h \zeta \neq 0\| \cap \mathcal{N} \right] \left( \Psi(h, 1)^2 \right.
\]

\[
+ \left( \|f\|_L^5(B) + \|f_1\|_L^5(\mathcal{N}) + h \right) \left( \|u_h \zeta \neq 0\| \right)^{1 + \frac{1}{\epsilon}} \right],
\]

(A.16)

where \( C_{15} = C_{15}(\lambda, \Lambda, n, B, c, c_1, q, q_1) > 0. \) Consider (A.16) and the following claim, which is proved as in [13]:

**Claim 3.** If \( h > k, \) then

\[
\|u_h \zeta \neq 0\| \leq \frac{1}{(h - k)^2} \int_{\mathcal{V}(h, 1)} u_h^2 dx, \quad \|u_h \zeta \neq 0\| \cap \mathcal{N} \leq \frac{1}{(h - k)^2} \int_{\mathcal{V}(h, 1)} u_h^2 d\sigma,
\]

\[
\int_{\mathcal{V}(h, 1)} u_h^2 dx \leq \int_{\mathcal{V}(h, 1)} u_h^2 dx \quad \text{and} \quad \int_{\mathcal{V}(h, 1)} u_h^2 d\sigma \leq \int_{\mathcal{V}(h, 1)} u_h^2 d\sigma.
\]

Therefore,

\[
\Psi(h, 0)^2 \leq C_{16} \left[ s^2 \left( \frac{1}{(h - k)^2} \right)^\frac{1}{2} + \left( \|f\|_L^5(B) + \|f_1\|_L^5(\mathcal{N}) + h \right) \left( \|u_h \zeta \neq 0\| \right)^{1 + \epsilon} \right],
\]

(A.17)

where \( C_{16} = C_{16}(\lambda, \Lambda, n, B, c, c_1, q, q_1) > 0. \) This proves the Lemma [A.7]

Observe that if \( u \in H^1(B) \) and \( u|_D \in L^\infty(D) \) then \((u - k)^+|_D = 0 \forall k \geq \|u\|_L^\infty(D). \) By Sobolev embedding inequalities and Lemma [A.12] we have that

\[
\|u - k\|^\frac{1}{2} \|L^\infty(B) \leq C(n, B) \|\nabla \|u - k\|\|^\frac{1}{2} \|L^2(B)
\]

(A.17)

for all \( k \geq \|u\|_L^\infty(D) \) and \( \zeta \in C_c^\infty(\mathbb{R}^n). \) Therefore the condition (A.4) is satisfied.

For a set \( A \subset \mathbb{R}^n \) and \( t \in \mathbb{R}, \) we write

\[
tA := \{tx \in \mathbb{R}^n \mid x \in A\}.
\]
Lemma A.8. Suppose that \( u \in H^1(B) \), \( u|_D \in L^\infty(D) \), \( S_+ \geq \|u\|_{L^\infty(D)} \), and
\[
\int_B \left( a^{ij} u_{x_i} v_{x_j} + cuv \right) \, dx + \int_N c_1 \, v \, d\sigma \leq \int_B f \, v \, dx + \int_N f_1 \, v \, d\sigma, \quad v = (u - k)^+ \zeta^2,
\]
for all \( k \geq S_+ \) and \( \zeta \in C^\infty_{\text{c}}(K_1) \) with \( \zeta \geq 0 \). Assume \( f \in L^p(B) \) and \( f_1 \in L^{q_1}(N) \), then if \( p, p_1 \geq 2 \)
\[
\sup_{2^{-1}B} u^+ + \sup_{2^{-1}N} u^+ \leq C \left( \|u^+\|_{L^p(B)} + \|u^+\|_{L^{q_1}(N)} + S_+ + \|f_1\|_{L^{q_1}(N)} + \|f\|_{L^p(B)} \right),
\]
where \( C = C(\lambda, \Lambda, p, p_1, q, q_1, n, B) > 0 \).

Proof. We again follow the lines of [15, Theorem 4.1]. As observed above, the condition \( \mathbf{A.4} \) is satisfied. Then, by Lemma A.7, we have \( u^+ \in L^\infty(2^{-1}B) \cap L^\infty(2^{-1}N) \) and
\[
\sup_{2^{-1}B} u^+ + \sup_{2^{-1}N} u^+ \leq C \left( \|u^+\|_{L^2(B)} + \|u^+\|_{L^2(N)} + S_+ + \|f_1\|_{L^\infty(N)} + \|f\|_{L^2(B)} \right),
\]
where \( C = C(\lambda, \Lambda, q, q_1, n, B) > 0 \). Using the Hölder’s inequality we can conclude the proof.

The proof of the next two lemmas are similar to the one of Lemma A.8.

Lemma A.9. Suppose that \( u \in H^1(B) \), \( u|_D \in L^\infty(D) \), \( S_+ \geq \|u\|_{L^\infty}, \) and
\[
\int_B \left( a^{ij} u_{x_i} v_{x_j} + cuv \right) \, dx + \int_N c_1 \, v \, d\sigma \leq \int_B f \, v \, dx + \int_N f_1 \, v \, d\sigma, \quad v = (u - k)^+ \zeta^2,
\]
for all \( k \geq S_+ \) and \( \zeta \in C^\infty_{\text{c}}(K_1 \cap \{x_n > 0\}) \) with \( \zeta \geq 0 \). Assume \( f \in L^p(B) \), then if \( p \geq 2 \) and \( X \) is a compact set with \( X \subset B \cup D \) we have that
\[
\sup_{X \cap B} u^+ \leq C \left( \|u^+\|_{L^p(B)} + S_+ + \|f\|_{L^p(B)} \right),
\]
where \( C = C(\lambda, \Lambda, p, q, n, X, B) > 0 \).

We have if \( \zeta \in C^\infty_{\text{c}}(K_1 \cap \{x_n > 0\}) \), then \( \zeta|_D = 0 \) and
\[
\| (u - k)^+ \zeta \|_{L^\infty(B)} \leq C \| \nabla ((u - k)^+ \zeta) \|_{L^2(B)} \forall k \geq 0,
\]
where \( C = C(n, B) > 0 \), which implies that the condition \( \mathbf{A.4} \) is satisfied.

The following lemma will be important in the proof of Theorem A.15 below.

Lemma A.10. Suppose that \( u \in H^1(B) \) and
\[
\int_B \left( a^{ij} u_{x_i} v_{x_j} + cuv \right) \, dx + \int_N c_1 \, v \, d\sigma \leq \int_B f \, v \, dx + \int_N f_1 \, v \, d\sigma, \quad v = (u - k)^+ \zeta^2,
\]
for all \( k \geq 0 \) and \( \zeta \in C^\infty_{\text{c}}(K_1 \cap \{x_n > 0\}) \) with \( \zeta \geq 0 \). Assume \( f \in L^p(B) \) and \( f_1 \in L^{q_1}(N) \), then if \( p, p_1 \geq 2 \) and \( X \) is a compact set with \( X \subset B \cup N \) we have that
\[
\sup_{X \cap B} u^+ + \sup_{X \cap N} u^+ \leq C \left( \|u^+\|_{L^p(B)} + \|u^+\|_{L^{q_1}(N)} + \|f_1\|_{L^{q_1}(N)} + \|f\|_{L^p(B)} \right),
\]
where \( C = C(\lambda, \Lambda, p, p_1, q, q_1, n, X, B) > 0 \).

A.3. Nonlinear solutions and estimates. Recall that \( \overline{M} = M \cup \overline{D} \cup \overline{N} \) denotes a cornered manifold satisfying \( \mathbf{A.1} \). Let \( c, c_0 \in L^\infty(M), \) \( c_1 \in L^\infty(N) \) be smooth functions such that \( c, c_0, c_1 \geq 0 \). For \( f \in H^1(M) \), set
\[
H^1_1(M) = \{ u \in H^1(M) \mid u|_D = f|_D \}.
\]

Our first goal now is to prove the following:

Proposition A.11. Suppose that \( f|_D \in L^\infty(D) \) is non-negative, non-trivial, and Hölder continuous. Then there exists a solution \( u \in C^{2,\alpha}(M \cup N) \cap C(\overline{M}) \) of the mixed boundary problem
\[
\begin{cases}
-\Delta u + cu + c_0 u^{\frac{n+2}{n-2}} = 0 & \text{in } M, \\
u
u \text{ on } \partial M, \\
\partial u
\end{cases}
\]
\[
satisfying
\begin{cases}
\frac{\partial u}{\partial \nu} + c_1 u^{\frac{n-2}{n+2}} = 0 & \text{on } N, \\
u
\text{ in } M \cup N.
\end{cases}
\]

(A.18)

(A.19)
Proof. Suppose by contradiction that there exists a sequence where further, by Theorems A.3 and A.4, which is coercive and closed (the latter being a consequence of Theorem A.4), then also, the following Poincaré inequality holds: 

\[ \int_{\mathcal{M}} u M^2 \, dv_g = 1 \quad \text{and} \quad \int_{\mathcal{M}} |\nabla u_m|^2 \, dv_g \to 0 \quad \text{as} \quad m \to \infty. \]

Hence we may assume that strongly in \( L^2(\mathcal{M}) \) and weakly in \( H^1(\mathcal{M}) \). Thus, 

\[ \nabla u_m \to 0 \quad \text{and} \quad \|u_0\|_{L^2(\mathcal{M})} = 1. \]

As \( H^1_0(\mathcal{M}) \) is convex and closed (the latter being a consequence of Theorem A.4), then 

\[ u_0 \in H^1_0(\mathcal{M}). \]

Therefore, \( u_0 = 0 \). This is a contradiction.

Set 

\[ F(t) := \frac{n - 2}{2n} \left( t^2 \right)^{\frac{n-2}{4}} \quad \text{and} \quad G(t) := \frac{n - 2}{2(n - 1)} \left( t^2 \right)^{\frac{n-1}{4}}. \]

Observe that \( F, G \) are convex functions and \( F(u) \in L^1(\mathcal{M}), G(u) \in L^1(\partial \mathcal{M}) \) for all \( u \in H^1(\mathcal{M}) \). We define the functional \( \mathcal{I} : H^1_0(\mathcal{M}) \to \mathbb{R}, \) by 

\[ \mathcal{I}(u) = \frac{1}{2} \left( \int_{\mathcal{M}} |\nabla u|^2 \, dv_g + \int_{\mathcal{M}} cu^2 \, dv_g \right) + \int_{\mathcal{M}} c_0 F(u) \, dv_g + \int_{\mathcal{N}} c_1 G(u) \, d\sigma_g. \]

Then \( u \in H^1_0(\mathcal{M}) \) is a critical point of \( \mathcal{I} \) if and only if it satisfies 

\[ \int_{\mathcal{M}} \left( g(\nabla u, \nabla v) + cuv + c_0 |u|^{\frac{n-2}{4}} uv \right) \, dv_g + \int_{\mathcal{N}} c_1 |u|^{\frac{n-1}{4}} u v \, d\sigma_g = 0 \quad \forall v \in H^1_0(\mathcal{M}), \quad (A.20) \]

i.e., \( u \) is a weak solution of (A.18).

Lemma A.13. The functional \( \mathcal{I} \) is w.l.s.c. and weakly coercive.

Proof. By Lemma A.12

\[ C_1 \|u\|_{H^1(\mathcal{M})}^2 - C_2 \|f\|_{H^1(\mathcal{M})}^2 \leq \mathcal{I}(u) \quad \forall u \in H^1_0(\mathcal{M}), \quad (A.21) \]

and further, by Theorems A.3 and A.4

\[ \mathcal{I}(u) \leq C_3 \left( \|u\|_{H^1(\mathcal{M})}^2 + \int_{\mathcal{M}} F(u) \, dv_g + \int_{\mathcal{N}} G(u) \, d\sigma_g \right) \leq C_4 \left( \|u\|_{H^1(\mathcal{M})}^2 + \|u\|_{H^1(\mathcal{M})}^2 + \|u\|_{H^1(\mathcal{M})}^{\frac{2(n-1)}{n-4}} \right), \]

where \( C_i = C_i(n, \mathcal{M}), i = 1, 2, C_j = C_j(c, c_0, c_1, n, \mathcal{M}), j = 3, 4, \) are positive constants. By (A.21), \( \mathcal{I} \) is weakly coercive. That \( \mathcal{I} \) is w.l.s.c. follows from the following result:

Proposition A.14. [31 Theorem 1.41] Consider the functional \( \mathcal{I} : C \subset X \to \mathbb{R}, \) where \( X \) is a real Banach space. Suppose \( C \) is closed and convex, \( \mathcal{I} \) is convex and continuous. Then \( \mathcal{I} \) is w.l.s.c.

This completes the proof of Lemma A.13.
Proof of Proposition A.11. Set
\[ \ell = \inf_{u \in H^1_0(M)} I(u). \]  
(A.22)

It follows from (A.21) that \( \ell > -\infty \). So, there exists a sequence \( \{u_m\} \subset H^1_0(M) \) such that
\[ I(u_m) \to \ell, \]
(A.23)

Hence, using (A.21) again, \( \{u_m\} \) is uniformly bounded in \( H^1(M) \) and so we may assume that this sequence converges weakly in \( H^1(M) \) to some \( u_0 \in H^1(M) \). Since \( H^1_0(M) \) is weakly closed, \( u_0 \in H^1_0(M) \). Then
\[ I(u_0) \leq \ell, \]
(A.24)

because \( I \) is w.l.s.c. So,
\[ \ell = I(u_0). \]
(A.25)

Therefore, we conclude that there exists \( u \in H^1_0(M) \) such that (A.20) holds.

Now assume that
\[ f|_{\mathcal{D}} \geq 0, \]
i.e., \( f^- := -\min\{f, 0\} \) satisfies \( f^-|_{\mathcal{D}} = 0 \). Observe that \( |(v|_{\mathcal{D}}) = (|v|)|_{\mathcal{D}} \) for any \( v \in W^{1, p}(M) \).

Since \( u^-|_{\mathcal{D}} = -\min\{f, 0\} = 0 \),
\[ I(u^+ + u^-) = I(u^+ - u^-) \quad \text{and} \quad (u^+ + u^-)|_{\mathcal{D}} = f, \]
we can assume that
\[ u \geq 0. \]

In particular, by (A.20),
\[ \int_\mathcal{M} g(\nabla u, \nabla v) \, dv_g \leq 0 \]
(A.26)

for all \( v \in H^1_0(M) \) with \( v \geq 0 \). Hence, Proposition A.6 together with (A.26) gives us \( u \in L^{\infty}(\mathcal{M}) \). Then, the Lemmas A.8–A.10 imply
\[ u \in L^{\infty}(\mathcal{M}). \]
(A.27)

By [7] Proposition 2.4, \( u \) is Hölder continuous on \( \mathcal{M} \). Using [15] Corollary 4.23 and Theorem 4.24 and [18] Theorem 2 we see that \( u \in C^{1, \alpha}(\mathcal{M} \cup \mathcal{N}) \). Hence, it follows from standard elliptic estimates that \( u \in C^{2, \alpha}(\mathcal{M} \cup \mathcal{N}) \). In particular, (A.18) holds in the classical sense. Finally, the strong maximum principle and the Hopf’s lemma gives (A.19).

The next result is used in Section 3 to estimate in compact sets the solutions obtained above.

**Theorem A.15.** Let \( X \subset \mathcal{M} \cup \mathcal{N} \) be a compact set. Suppose that \( c, c_0 \in L^\infty(\mathcal{M}), c_1, c_2 \in L^\infty(\mathcal{N}) \) and \( \|c\|_{L^\infty} + \|c_0\|_{L^\infty} \leq \Lambda \). Let \( 1 < \alpha < \frac{n+2}{n-2} \) and \( 1 < \alpha_1 < \frac{n}{n-2} \). Assume that \( u \in H^1(\mathcal{M}), u \geq 0 \) and
\[ \int_\mathcal{M} (g(\nabla u, \nabla v) + cuv + c_0 u^n v) \, dv_g + \int_{\mathcal{N}} (e_2 u^v + c_1 u^{n_1} v) \, d\sigma_g \leq 0, \]
(A.28)

for all \( v \in H^1_0(\mathcal{M}) \) with \( v \geq 0 \).

If \( c \geq -S_1, c_2 \geq -S_2, \) with \( S_1 \geq 0, \) and \( c_0 \geq S_0, \) \( c_1 \geq S_1, \) with \( S_0, S_1 > 0 \), then
\[ \sup_X u \leq C_\Lambda \alpha S_0 S_1 \alpha_1 a_1 n_1 g \]
(A.29)

**Proof.** We follow the steps of the proof of [5] Theorem 1.1. By the compactness of \( X \) we can find \( \varepsilon > 0 \) and a finite number of charts \( (B_{3\varepsilon}, \varphi_i), (K_{3\varepsilon} \cap \{x_n \geq 0\}, \psi_j) \) such that
\[ X \subset (\cup_{i} \varphi_i(B_{\varepsilon})) \cup (\cup_j \psi_j (K_{\varepsilon} \cap \{x_n \geq 0\})), \]

\( \psi_j(K_{3\varepsilon} \cap \{x_n = 0\}) \subset \mathcal{N} \) and \( \varphi_i(B_{3\varepsilon}) \subset \mathcal{M} \).

We have the following two cases:
(i) \( \sup_X u = \sup_{\varphi_i(B_{\varepsilon})} u \), for some \( i \).
(ii) \( \sup_X u = \sup_{\psi_j(K_{\varepsilon}^+)} u \), for some \( j \), where \( K_{\varepsilon}^+ := K_{\varepsilon} \cap \{x_n > 0\} \).
If the first case holds, the proof of \((A.29)\) is the same as in \([5]\). For the second case, observe that, by \((A.28)\),
\[
\int_M g(\nabla u, \nabla v) \, dv_g - S \int_M uv \, dv_g - S_2 \int_N uv \, d\sigma_g \leq 0, \tag{A.30}
\]
for all \(v \in H^1_0(M)\) with \(v \geq 0\).

Set \(K^\circ \equiv K_\varepsilon \cap \{x_n = 0\}\) and let \(p, p_1 \geq 2\) be constants to be determined below. Applying Lemma \((A.10)\) to inequality \((A.30)\) we have
\[
\sup_{K^\circ} u \circ \psi_j + \sup_{K^\circ} u \circ \psi_j \leq C \left( S, S_2, p, p_1, n, K^\circ_\varepsilon \right) \left( \|u \circ \psi_j\|_{L^p(K^\circ_\varepsilon)} + \|u \circ \psi\|_{L^{p_1}(K^\circ_\varepsilon)} \right). \tag{A.31}
\]

Now, let \(\xi \in C^\infty_\alpha(K_\varepsilon \cap \{x_n \geq 0\})\), \(\xi \geq 0\), be such that \(\xi \equiv 1\) on \(\psi_j(K_\varepsilon \cap \{x_n \geq 0\})\). Setting \(\beta = \frac{2(\alpha+1)}{\alpha-1} > 2\) and replacing \(u \xi^\beta\) into \((A.28)\) we have
\[
S_0 \int_M u^\alpha + 1 \xi^\beta \, dv_g + S_1 \int_N u^\alpha + 1 \xi^\beta \, d\sigma_g \leq - \int_M \left[ \xi^\beta |\nabla u|^2 + g(\nabla u, \beta u \xi^\beta - 1 \nabla \xi) \right] \, dv_g + \int_M S u^2 \xi^\beta \, dv_g + \int_N S_2 u^2 \xi^\beta \, d\sigma_g.
\]
By Cauchy-Schwarz,
\[
- \beta u \xi^\beta - 1 \, g(\nabla u, \nabla \xi) \leq \xi^\beta |\nabla u|^2 + \beta^2 u^2 \xi^\beta - 2 |\nabla \xi|^2,
\]
so we obtain
\[
S_0 \int_M u^\alpha + 1 \xi^\beta \, dv_g + S_1 \int_N u^\alpha + 1 \xi^\beta \, d\sigma_g \leq \beta^2 \int_M u^2 \xi^\beta - 2 |\nabla \xi|^2 \, dv_g + S \int_M u^2 \xi^\beta \, dv_g + S_2 \int_N u^2 \xi^\beta \, d\sigma_g,
\]
\[
\leq \beta^2 \left( \int_M u^\alpha + 1 \xi^\beta \, dv_g \right)^{\frac{1}{\alpha+1}} \left( \int_M |\nabla \xi|^2 \, dv_g \right)^{\frac{\alpha-1}{\alpha+1}} + S \left( \int_M u^\alpha + 1 \xi^\beta \, dv_g \right)^{\frac{1}{\alpha+1}} + S_2 \left( \int_N \xi^\beta \, d\sigma_g \right)^{\frac{\alpha-1}{\alpha+1}}.
\]
Then, by Young’s Inequality,
\[
\int_M u^\alpha + 1 \xi^\beta \, dv_g + \int_N u^\alpha + 1 \xi^\beta \, d\sigma_g \leq C_1 \left( \int_M |\nabla \xi|^2 \, dv_g + \int_M \xi^\beta \, dv_g + \int_N \xi^\beta \, d\sigma_g \right),
\]
where \(C_1 = C_1 \left( S, S_0, S_1, S_2, \alpha, \alpha_1 \right) > 0\). Then
\[
\|u \circ \psi_j\|_{L^{p+1}(K^\circ_\varepsilon)} + \|u \circ \psi\|_{L^{p+1}(K^\circ_\varepsilon)} \leq C_2 \left( S, S_0, S_1, S_2, \alpha, \alpha_1, n, X \right).
\]
Finally, if \(p = \alpha + 1\) and \(p_1 = \alpha_1 + 1\), by \((A.31)\) we have
\[
\sup_X u \leq C \left( A, S, S_0, S_1, S_2, \alpha, \alpha_1, n, g, X \right).
\]

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