Elementary parabolic twist

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Abstract

The twist deformations for simple Lie algebras $U(g)$ whose twisting elements $F$ are known explicitly are usually defined on the carrier subspace injected in the Borel subalgebra $B^+(g)$. We solve the problem of creating the parabolic twist $F_P$ whose carrier algebra $P$ not only covers $B^+(g)$ but also intersects nontrivially with $B^-(g)$. This algebra $P$ is the parabolic subalgebra in $\mathfrak{sl}(3)$ and has the structure of the algebra of two-dimensional motions. The parabolic twist is explicitly constructed as a composition of the extended jordanian twist $F_{EJ}$ and the new factor $F_{D}$. The latter can be considered as a special version of the jordanian twist. The twisted costructure is found for $U(L)$ and the corresponding universal $R$-matrix is presented.

1 Introduction

There are two main sets of constant classical quasitriangular $r$-matrices for semisimple Lie algebras $g$: nonskewsymmetric and skewsymmetric. The first set was classified by Belavin and Drinfeld. The corresponding $r$-matrices were associated with the subsets $T_{1,2}$ of the root system $\Lambda(g)$ and the morphisms $T$ connecting them (Belavin-Drinfeld triples). The classification scheme for skewsymmetric $r$-matrices was developed by Stolin. In this scheme listing of skewsymmetric solutions of the classical Yang-Baxter equation (CYBE) was reduced to the classification of quasi-Frobenius subalgebras in $g$, their normalizers and the evaluation of the cohomology group $H^2(g)$.

The existence of quantizations was proved by Drinfeld for all Lie bialgebras. But that proof does not allow one to compute the quantization explicitly. For a long period of time the universal $R$-matrices were known only for some classical $r$-matrices (standard or Drinfeld-Jimbo, and Cremmer-Gervais for nonskewsymmetric and jordanian, Reshetikhin and GGS for skewsymmetric solutions and some other).

It was also known that in nonskewsymmetric case some solutions of quantum YBE can be transformed into the standard one by a special kind of twist. 

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The corresponding twisting element connecting Cremmer-Gervais and standard \( R \)-matrices was constructed by Kulish and Mudrov \[10\]. Recently Etingof, Schedler and Schiffmann \[11\] had proved that such twists exist for any quantized Belavin-Drinfeld triple and thus had solved the problem of explicit quantization for nonskewsymmetric \( r \)-matrices. The alternative expressions for the twisting elements mentioned above where proposed in \[12\] by Isaev and Ogievetsky.

The quantization problem for quasitriangular Lie bialgebras with skewsymmetric \( r \)-matrices was reduced by Drinfeld \[13\] to the solution of the twist equations

\[
\begin{align*}
F_{12}(\Delta \otimes id)F &= F_{23}(id \otimes \Delta)F, \\
(\epsilon \otimes id)F &= (id \otimes \epsilon)F.
\end{align*}
\]

Such solutions are called twisting elements \( F \in U(\mathfrak{L}) \otimes U(\mathfrak{L}) \) and the minimal \( \mathfrak{L} \subset \mathfrak{g} \) on which \( F \) is defined is the carrier subalgebra of the twist. The well known triangular universal \( R \)-matrices were found when the corresponding solutions of \( \mathfrak{L} \) were constructed (jordanian \[7\], Reshetikhin \[8\] and GGS-twists \[9\]). Starting with the construction of the extended jordanian twists (EJT’s) deeper understanding of the peculiarities of skewsymmetric class of solutions was achieved \[14\]. The essential part of the variety of skew \( r \)-matrices (with carriers in the Borel subalgebras \( \mathfrak{B}(\mathfrak{g}) \subset \mathfrak{g} \)) was explicitly quantized. The corresponding twisting elements were found to be the compositions ("chains") of EJT’s. It becomes clear that EJT’s play the fundamental role in quantizing the skewsymmetric \( r \)-matrices. On the other hand in many cases the canonical forms of EJT’s are insufficient and the precedent twisting factors of a chain can induce deformations of the consequent ones \[15\].

Let \( \Lambda = \Lambda^+ \cup \Lambda^- \) be the root system of \( \mathfrak{g} \) and \( V_\theta = V^- \oplus V_\theta \oplus V^+ \) – the corresponding triangular decomposition. While constructing the twisting elements explicitly the most difficult is the situation when the carrier \( \mathfrak{L} \) subalgebra intersects nontrivially both \( V^+ \) and \( V^- \). (The intersection must be considered trivial when there exist an automorphism bringing \( \mathfrak{L} \) into \( V^+ \).) In that case a chain of twists starting in \( V^+ \) deforms the costructure of the space \( V_\mathfrak{L} \cap V^- \) so that the corresponding basic elements lose their quasiprimitivity. This deformed costructure of \( \mathfrak{L} \) must be used in the twist equation (1) to find the twisting factors that could enlarge the initial chain. This leads to severe difficulties.

In this paper we consider the simplest example of the carrier algebra \( \mathfrak{L} \) containing the Borel subalgebra \( \mathfrak{B}^+(\mathfrak{g}) \) and a one-dimensional subspace from \( \mathfrak{B}^-(\mathfrak{g}) \). This is the maximal parabolic subalgebra \( \mathfrak{P} \) in \( \mathfrak{g} = \mathfrak{sl}(3) \). As was shown by Gerstenhaber and Giaquinto \[13\] the \( r_\mathfrak{P} \)-matrices for the parabolic subalgebras lay in the boundaries of the smooth varieties of Cremmer-Gervais solutions \( r_{CG} \) of the modified CYBE. In particular there exists such an element \( x \in \mathfrak{sl}(3) \) that the following expression

\[
\exp(-t \text{ad}(x)) \circ r_{CG} = r_{CG} + t \cdot r_{\mathfrak{P}}\]

is a solution of CYBE with an arbitrary parameter \( t \).
Our aim is to construct the twisting element \( F_\nu \) with the carrier subalgebra 
\[ \mathfrak{P} \equiv \mathfrak{P}_1 = \mathfrak{B}^+(\mathfrak{sl}(3)) \supset V_{-\alpha_2} = \mathfrak{B}^+(\mathfrak{sl}(3)) \supset V(E_{32}), \]  
(2)
where the generators \( E_{ik} \) are the usual matrix units. The problem is solved in two steps (see Section 2). First we perform the peripheric extended twisting \( F_P \) with the carrier in \( \mathfrak{B}^+(\mathfrak{sl}(3)) \). This gives the first two factors \( \Phi_J \) and \( \Phi_E \) of the twisting element \( F_\nu \). Studying the obtained costructure of the deformed algebra \( U_P(\mathfrak{P}) \) we can predict the form of the last factor \( F_D \) whose carrier contains \( E_{32} \). Our main statement is that this \( F_D \) is a solution of the twist equations for \( \mathfrak{P} \) with the costructure deformed by the peripheric twist \( F_P = \Phi_E \Phi_J \). The proof demonstrates the connection between the generalized Verma’s identity and the factorization of Drinfeld equation for \( F_D \). It follows that the composition \( F_\nu = F_D F_E \) forms the twist with the parabolic carrier subalgebra \( \mathfrak{P} \). We call it the elementary parabolic twist. In Section 3 the corresponding \( R_\nu \) matrix is evaluated in the defining representation. The applications of this result and possible extensions of the construction are discussed.

2 Twisting the universal enveloping algebra \( U(\mathfrak{P}) \)

The algebra \( U(\mathfrak{sl}(3)) \) will be considered in the canonical \( \mathfrak{gl}(3) \)-basis \( \{E_{ij}; \ i, j = 1, 2, 3\} \) with the relations
\[ [E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{kj}. \]
For the Cartan subalgebra we shall use the following generators
\[ H_{13}^+ = \frac{1}{3} E_{11} - \frac{2}{3} E_{22} + \frac{1}{3} E_{33}, \]
\[ H_{23}^+ = \frac{2}{3} E_{11} - \frac{1}{3} E_{22} - \frac{1}{3} E_{33}, \]
\[ H_{23}^- = E_{22} - E_{33} = H_{23}^+ - 2 H_{13}^+, \]
Notice that the dual vectors, \( H_{13}^{\perp} \) and \( H_{23}^{\perp} \), are orthogonal to the roots \( \alpha_{13} \) and \( \alpha_{23} \) respectively.

Our aim is to construct the twist whose carrier subalgebra \( \mathfrak{P} \) is generated by the Borel subalgebra \( \mathfrak{B}^+(\mathfrak{sl}(3)) \) and the element \( E_{32} \), i.e. by the parabolic subalgebra of \( \mathfrak{sl}(3) \). The Cartan subalgebra acts on the root generators of \( \mathfrak{P} \) as follows:
\[ [H_{13}^+, E_{12}] = E_{12}, \quad [H_{23}^+, E_{12}] = E_{12}, \]
\[ [H_{13}^+, E_{13}] = 0, \quad [H_{23}^+, E_{13}] = E_{13}, \]
\[ [H_{13}^+, E_{23}] = -E_{23}, \quad [H_{23}^+, E_{23}] = 0, \]
\[ [H_{13}^+, E_{32}] = E_{32}, \quad [H_{23}^+, E_{32}] = 0. \]
For the algebra \( \mathfrak{P} \) the twisting can be started by applying the extended jordanian twist \( F_{EJ} \) based on the Borel subalgebra \( \mathfrak{B}_1 \) generated by some Cartan
generator $H_1$ and a generator that can be chosen to be $E_{13}$:

$$F_{\xi, J} = \exp(-E_{23} \otimes E_{12} e^{\beta \sigma_{13}}) \exp(H_1 \otimes \sigma_{13})$$

(3)

with $\sigma_{13} = \ln(1 + E_{13})$ and $[H_1, E_{13}] \neq 0$. Our aim is to enlarge the canonical extended jordanian twist to incorporate the additional Cartan generator $H_2$ and the element $E_{32}$ into the carrier subalgebra $\mathfrak{P}$ of the final twisting element. In the general case $H_2$ forms Borel subalgebras (say $\mathfrak{B}_2$) with both $E_{13}$ and $E_{32}$. Evidently one can suppose that some kind of jordanian twist (based on the subalgebra $\mathfrak{B}_2$) might be applicable to the algebra $U_{\xi, J}(\mathfrak{sl}(3))$ deformed by $F_{\xi, J}$. Thus it is natural to consider the following supposition: the twisting element with the parabolic carrier in $U(\mathfrak{sl}(3))$ has the form

$$F_{\nu} = F_{D} F_{\xi} F_{J} = F_{D} F_{\xi} J =$$

$$= \exp(H_2 \otimes (b\sigma_{13} + \sigma_{32})) \exp(-E_{23} \otimes E_{12} e^{-\beta \sigma_{13}}) \exp(H_1 \otimes \sigma_{13}).$$

(4)

Here $[H_1, E_{12}] = \beta E_{12}$ and $\sigma_{32} = \ln(1 + E_{32})$. We have two free parameters ($b$ and $\beta$) and two yet undetermined Cartan generators: $H_1$ and $H_2$. The ordering of tensor factors in the extension $F_{\xi}$ isn’t essential and similar considerations work for the alternative ordering $[17]$.

In our notations the $r$-matrix studied in $[18]$ looks like

$$r_{\nu} = H_{23} \wedge E_{13} + E_{12} \wedge E_{23} + H_{13} \wedge E_{32}.$$

(5)

This $r$-matrix imposes the following conditions on the generators:

$$H_1 + b H_2 = H_{23}^+, \quad H_2 = H_{13}^+. $$

The factorization property $F_{\nu} = F_{D} F_{\xi} F_{J}$ leads to the additional restriction: the extended jordanian factor $F_{\xi, J}$ in $F_{\nu}$ must survive after the generator $E_{32}$ is scaled and the scale parameter is sent to zero. Thus we get the additional condition: $\beta + b = 1$. Moreover the explicit form (3) of the $r$-matrix clearly indicates that it is natural to rearrange the factors of $F_{\nu}$:

$$F_{\nu} = F_{D} F_{\xi} F_{J} =$$

$$= \exp(H_{13}^+ \otimes (b\sigma_{13} + \sigma_{32})) \times$$

$$\times \exp(-E_{23} \otimes E_{12} e^{-(1-b)\sigma_{13}}) \exp((H_{23}^+ - b H_{13}^-) \otimes \sigma_{13})$$

$$= F_{D} F_{R} F_{P} = F_{D} F_{R} F_{\xi} F_{J} =$$

$$= \exp(H_{13}^+ \otimes (b\sigma_{13} + \sigma_{32})) \exp(-b H_{13}^- \otimes \sigma_{13}) \times$$

$$\times \exp(-E_{23} \otimes E_{12} e^{-\sigma_{13}}) \exp(H_{23}^+ \otimes \sigma_{13}).$$

We shall use this form to justify the proposed expression for the parabolic twisting element and to fix the value of $b$. Let us impose on $F_{\nu}$ the Drinfeld condition with the costructure of the algebra $U_{\nu}(\mathfrak{sl}(3))$. The latter is the result of the twist deformation performed by the so called peripheric extended jordanian twist $F_{P}$ $[17]$:

$$F_{P} = F_{\xi} F_{J} = \exp(-E_{23} \otimes E_{12} e^{-\sigma_{13}}) \exp(H_{23}^+ \otimes \sigma_{13}).$$

(6)
The twisting element $F_P$ has the 4-dimensional carrier subalgebra $L \subset \mathfrak{p}$ generated by the set $\{H_{23}, E_{12}, E_{23}, E_{13}\}$. The costructure of the twisted algebra $U_P(\mathfrak{p})$ is defined by the following coproducts:

$$
\Delta_P(H_{13}^+) = H_{13}^+ \otimes 1 + 1 \otimes H_{13}^+,
\Delta_P(E_{12}) = E_{12} \otimes e^{\sigma_{13}} + e^{\sigma_{13}} \otimes E_{12},
\Delta_P(E_{13}) = E_{13} \otimes e^{\sigma_{13}} + 1 \otimes E_{13},
\Delta_P(E_{23}) = E_{23} \otimes e^{-\sigma_{13}} + 1 \otimes E_{23},
\Delta_P(H_{23}^+) = H_{23}^+ \otimes 1 + 1 \otimes H_{23}^+ + E_{23} \otimes E_{12} e^{-2\sigma_{13}}.
\Delta_P(E_{32}) = E_{32} \otimes 1 + 1 \otimes E_{32} + 2H_{13}^+ \otimes E_{12} e^{-\sigma_{13}}.
$$

Now we are ready to evaluate the constant $b$. Let us expand the element $F_{DR} = F_D F_R$ with respect to $b$:

$$
F_{DR} = \exp(H_{13}^+ \otimes \sigma_{32} + \frac{b}{2}(H_{13}^+)^2 \otimes [\sigma_{13}, \sigma_{32}] + \cdots) \pmod{b^2}
$$

In the Drinfeld equation for $F_{DR}$ in the form (8) let us collect the coefficients for the first two powers of $b$. Taking into account the relation $[\sigma_{13}, E_{32}] = E_{12} e^{-\sigma_{13}}$ and the form of the last term in $\Delta_P(E_{32})$ we immediately come to the conclusion that $b = 2$.

We want to demonstrate that the Hopf algebra $U_P(\mathfrak{p})$ (with the costructure (7)) can be additionally twisted by the factor $F_{DR}$ that depends on $E_{32}$ and $H_{13}^+$. As a result the composition of the four factors $\Phi_J, \Phi_E, F_R$ and $F_D$ will form the twisting element $F_\nu = F_{DR} F_P = F_{DR} F_P \Phi_E \Phi_J$ with the carrier algebra $\mathfrak{p}$.

**Theorem**

Algebra $U_P(\mathfrak{p})$ admits the twist with the element

$$
F_{DR} = \exp(H_{13}^+ \otimes (2\sigma_{13} + \sigma_{32})) \exp(-2H_{13}^+ \otimes \sigma_{13})
$$

**Proof**

Consider the generalized Verma identity presented in [18] in the following form:

$$
e^{x \ln(1+ta)} e^{(x+y) \ln(1+sb)} e^{y \ln(1+ta)} = e^{y \ln(1+sb)} e^{(x+y) \ln(1+ta)} e^{x \ln(1+sb)}.
$$

The operators $a$ and $b$ are subject to the relations:

$$
[a, [a, b]] = 0
$$

and

$$
[b, [b, a]] = 0;
$$

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The coproducts for the six generators of $P$ and twisted parabolic algebra:

$$\Delta \sigma \cdot e^{\sigma_{13}} = (e^{\sigma_{13}} e^{\sigma_{32}} e^{\sigma_{13}}) \xi.$$ 

Now we need only to differentiate this expression and to put $\xi = 0$. This results in the following formula:

$$\ln(e^{\sigma_{13}} e^{\sigma_{32}} e^{\sigma_{13}}) = (2\sigma_{13} + \sigma_{32}).$$

As a consequence the element $F$ satisfies the factorized Drinfeld equations:

$$F = e^{\Delta \sigma \cdot e^{\sigma_{13}} \cdots \Delta \sigma \cdot e^{\sigma_{13}}}(\Delta \sigma \cdot e^{\sigma_{13}}) = e^{2\sigma_{13} + \sigma_{32}}.$$ 

Consider the deformed coproduct $F \Delta \sigma = (F \Delta \sigma)_{13}(F \Delta \sigma)_{23}$.

As a consequence the element $F \Delta \sigma$ acquires the form

$$F \Delta \sigma = \exp(H_{13} \otimes \ln(e^{\sigma_{13}} e^{\sigma_{32}} e^{\sigma_{13}})) \exp(-2H_{13} \otimes \sigma_{13}).$$

Consider the deformed coproduct $F \Delta \sigma = (F \Delta \sigma)_{13}(F \Delta \sigma)_{23}$. This is easy to verify that it is group-like. Together with the primitivity of $H_{13}$ in $U_P(\mathfrak{g})$ and the special form of the last factor in $F \Delta \sigma$ this brings us to the conclusion that $F \Delta \sigma$ satisfies the factorized Drinfeld equations:

$$(\Delta \sigma \otimes id)(F \Delta \sigma) = (F \Delta \sigma)_{13}(F \Delta \sigma)_{23}$$

and

$$(id \otimes \Delta \sigma)(F \Delta \sigma) = (F \Delta \sigma)_{12}(F \Delta \sigma)_{13}$$

and therefore satisfies also the initial equation (1). This proves the theorem. \[\square\]

Applying $F \Delta \sigma$ to the algebra $U_P(\mathfrak{g})$ we find the final costruction of the twisted parabolic algebra:

$$U_P(\mathfrak{g}) \xrightarrow{\Phi} U_P(\mathfrak{g}).$$

The coproducts for the six generators of $\mathfrak{g}$ obtain the following form:

$$\Delta_{\phi}(H_{13}^+) = 1 \otimes H_{13}^+ + (H_{13}^+ \otimes 1)(1 \otimes 1 + C)^{-1},$$

$$\Delta_{\phi}(H_{23}^+) = 1 \otimes H_{23}^+ + H_{23}^+ \otimes e^{-\sigma_{13}} + (E_{23} \otimes E_{12} e^{-\sigma_{13}} +$$

$$+(H_{23}^+ - H_{13}^+ \otimes 1)(1 \otimes 1 + C))(1 \otimes e^{\sigma_{13}} e^{\sigma_{32}})^{-1},$$

$$\Delta_{\phi}(E_{12}) = E_{12} \otimes e^{\sigma_{32}} e^{\sigma_{13}} + e^{\sigma_{13}} \otimes E_{12} + H_{13}^+ E_{12} \otimes E_{12},$$

$$\Delta_{\phi}(E_{13}) = (e^{\sigma_{13}} \otimes e^{\sigma_{13}} e^{\sigma_{32}})(1 \otimes 1 + C)^{-1} - 1 \otimes 1,$$

$$\Delta_{\phi}(E_{23}) = (E_{23} \otimes e^{-\sigma_{13}} + H_{13}^+ \otimes (2H_{13} - H_{23}) -$$

$$-(H_{13}^+)^2 \otimes e^{-\sigma_{13}} + H_{13}^+ \otimes 1)(1 \otimes 1 + C)^{-1} +$$

$$+(H_{13}^+ H_{13}^+ - 1 \otimes 1)(1 \otimes 1 + C)^{-2} + 1 \otimes E_{23},$$

$$\Delta_{\phi}(E_{32}) = E_{32} \otimes e^{\sigma_{32}} + 1 \otimes E_{32} +$$

$$+(E_{32} + 2e^{\sigma_{32}} H_{13}^+) \otimes E_{12} e^{-\sigma_{13}} +$$

$$+(E_{32} + e^{\sigma_{32}} H_{13}^+) H_{13}^+ \otimes (E_{12})^2 (e^{\sigma_{13}} e^{\sigma_{32}} e^{\sigma_{13}})^{-1}.$$ 

where

$$C = 1 \otimes E_{32} + H_{13}^+ \otimes E_{12} e^{-\sigma_{13}}.$$
We can consider the Hopf algebra $U_P(\mathcal{P})$ as a result of the integral twist deformation

$$U(\mathcal{P}) \xrightarrow{\mathcal{F}_\psi} U_P(\mathcal{P}),$$

where the element $\mathcal{F}_\psi$ can be written in the form

$$\mathcal{F}_\psi = \mathcal{F}_D \mathcal{F}_E \mathcal{J} = \exp(H^\perp_{13} \otimes (2\sigma_{13} + \sigma_{32})) \exp(-E_{23} \otimes E_{12} e^{\sigma_{13}}) \exp(H_{23} \otimes \sigma_{13})$$

and is called the elementary parabolic twisting element.

### 3 Universal element and $R$-matrix

The parabolic twist $\mathcal{F}_\psi$ can be supplied with two natural parameters corresponding to two jordanian-like deformations. The following rescaling of the generators is an automorphism of the algebra $\mathcal{P}$:

$$E_{13} \rightarrow \xi E_{13}, \quad E_{32} \rightarrow \zeta E_{32}, \quad E_{12} \rightarrow \xi \zeta E_{12}, \quad E_{23} \rightarrow \frac{1}{\zeta} E_{23}. $$

It induces the parametrization of the twisting element

$$\mathcal{F}_\psi(\xi, \zeta) = \exp(H^\perp_{13} \otimes (2\sigma_{13}(\xi) + \sigma_{32}(\zeta))) \exp(-\xi E_{23} \otimes E_{12} e^{\sigma_{13}(\xi)}) \exp(H_{23} \otimes \sigma_{13}(\xi)), \quad (12)$$

where $\sigma_{ij}(\xi) = \ln(1 + \xi E_{ij})$.

The result of the parabolic twisting with the element (12) is the 2-dimensional smooth variety of Hopf algebras $U_P(\mathcal{P}; \xi, \zeta)$. The parameters are independent. In the limit points we get the ordinary twists:

$$\mathcal{F}_\psi(\xi, \zeta) \xrightarrow{\xi \rightarrow 0} \mathcal{F}_P(\xi), \quad \mathcal{F}_\psi(\xi, \zeta) \xrightarrow{\zeta \rightarrow 0} \mathcal{F}_J(\zeta).$$

Here the first limit is the parametrized peripheric extended twist (13), the second is the jordanian twist for the Borel subalgebra generated by $\{H^\perp_{13}, E_{32}\}$ with the twisting element $\mathcal{F}_J(\zeta) = e^{H^\perp_{13} \otimes \sigma_{32}(\zeta)}$.

The universal $R$-matrix for $U_P(\mathcal{P}; \xi, \zeta)$ is defined by the standard expression:

$$R_P(\xi, \zeta) = (\mathcal{F}_\psi(\xi, \zeta))_{21} (\mathcal{F}_\psi(\xi, \zeta))^{-1} = \exp((2\sigma_{13}(\xi) + \sigma_{32}(\zeta)) \otimes H^\perp_{13}) \exp(-\xi E_{12} e^{\sigma_{13}(\xi)} \otimes E_{23}) \exp(\sigma_{13}(\xi) \otimes H_{23}) \times \exp(-H_{23} \otimes \sigma_{13}(\xi)) \exp(\xi E_{23} \otimes E_{12} e^{\sigma_{13}(\xi)}) \exp(-H^\perp_{13} \otimes (2\sigma_{13}(\xi) + \sigma_{32}(\zeta))). \quad (13)$$

If we choose the parameters to be proportional ($\zeta = \eta \xi$) then the expression (13) can be considered as a quantized version of the classical $r$-matrix

$$r_P(\eta) = H^\perp_{23} \wedge E_{13} + E_{12} \wedge E_{23} + \eta H^\perp_{13} \wedge E_{32}.$$
In the fundamental representation the $R$-matrix has the form:

$$R_\nu = 1 \otimes 1 +$$

$$(E_{13} \wedge (\frac{2}{3}E_{11} - \frac{1}{3}E_{22} - \frac{1}{3}E_{33}) + E_{21} \wedge E_{12}) \xi +$$

$$+ \frac{4}{9} (E_{13} \otimes E_{13}) \xi^2 +$$

$$(E_{32} \wedge (\frac{1}{3}E_{11} - \frac{1}{3}E_{22} + \frac{1}{3}E_{33})) \zeta +$$

$$+ \frac{4}{9} (E_{32} \otimes E_{32}) \zeta^2 +$$

$$+ \frac{1}{3} (E_{12} \otimes (\frac{1}{3}E_{11} - \frac{2}{3}E_{22} + \frac{1}{3}E_{33}) + (\frac{1}{3}E_{11} - \frac{2}{3}E_{22} + \frac{1}{3}E_{33}) \otimes E_{12} +$$

$$+ \frac{1}{3} (E_{13} \otimes E_{32} + E_{32} \otimes E_{13}) \zeta \xi -$$

$$- \frac{2}{81} (E_{12} \otimes E_{12}) \xi^2 \zeta^2 + \frac{2}{27} (E_{12} \wedge E_{32}) \xi \zeta^2 +$$

$$+ \frac{1}{27} (E_{12} \wedge E_{13}) \xi^2 \zeta.$$

4 Conclusions

We have demonstrated that with the help of the peripheric EJT’s and the jordanian-like factors the carrier subalgebra can be enlarged so that the negative sector of the Cartan decomposition is partially occupied. The elementary parabolic twist $\mathcal{F}_\nu$ can be applied to any Lie algebra that contains the parabolic carrier subalgebra $\mathfrak{P}$. The algebras $\mathfrak{sl}(3)$ and $\mathfrak{G}_2$ are the only ones with this property among the simple Lie algebras with rank($\mathfrak{g}$) = 2. Any simple algebra whose rank is greater than 2 contains $\mathfrak{P}$ and consequently can be twisted by $\mathcal{F}_\nu$.

There are substantial reasons to suppose that constructions similar to the composition $\mathcal{F}_\nu = \mathcal{F}_P \mathcal{F}_{EJT} = \mathcal{F}_P \mathcal{F}_R \mathcal{F}_P$ exist for the parabolic subalgebras of larger dimensions. So the elementary parabolic twist can be considered as the first step towards the invasion of the negative sector of the Cartan decomposition.

There exist other twists with 6-dimensional carriers, for example the one constructed in [15]. In the latter case the twist can be presented as a chain containing the extended jordanian twist and the jordanian twist on the deformed carrier space. It is important to know whether the similar interpretation can be used in the case of the parabolic twist.

Any maximal parabolic subalgebra $\mathfrak{P}_i$ with the missing negative simple root $\lambda_i$ can be considered as a semidirect product of the maximal simple subalgebra in $\mathfrak{P}_i$ and the ideal generated by the basic elements $E_{\alpha_k}$ whose positive roots $\alpha_k$ contain the simple component $\lambda_i$. This means that the corresponding Hopf algebras $U_{\nu_\nu}(\mathfrak{P}_i)$ are the examples of the algebras of motion $\mathfrak{g}$ quantized by twists whose carriers coincide with $\mathfrak{g}$. In the case considered here the algebra $U(\mathfrak{P})$ is the universal enveloping of the semidirect product $\mathfrak{P} \cong \mathfrak{gl}(2) \dagger (2)$, that is the algebra of two-dimensional motions.

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