Fractional Negative Binomial and Polya Processes

P. VELLAISAMY AND A. MAHESHWARI

Department of Mathematics,
Indian Institute of Technology Bombay, Powai, Mumbai 400076, INDIA.

Abstract. In this paper, we define a fractional negative binomial process (FNBP) by replacing the Poisson process by a fractional Poisson process (FPP) in the gamma subordinated form of the negative binomial process. The long-range dependence of the FNBP and the infinite divisibility of the FPP and the FNBP are investigated. Also, the space fractional Polya process (SFPP) is defined by replacing the rate parameter $\lambda$ by a gamma random variable in the space fractional Poisson process. The properties of the FNBP and the SFPP and the connections to pde’s governing the density of FNBP and SFPP are also investigated.

1. Introduction

The fractional generalizations of classical stochastic processes have received considerable attention by researchers in the recent years. These generalizations have found applications in several disciplines such as control theory, quantum physics, option pricing, actuarial science and reliability. For example, fractional Poisson process (FPP) has been used recently in [15] to define a new family of quantum coherent states and also fractional generalization of Bell polynomials, Bell numbers and Stirling’s numbers of second kind. Also, a new renewal risk model, which is non-stationary and has long-range dependence property, is defined using the FPP in [5]. In this paper, we define a fractional generalization of negative binomial process and a space fractional version of Polya process. Quite recently, a fractional generalization of the negative binomial process is defined in [3] and [4]. We introduce here a different, but a natural generalization of the negative binomial process. It is known that negative binomial process can be viewed as subordinated Poisson process via a gamma subordinator. Let $\alpha > 0$, $p > 0$ and $\{\Gamma(t)\}_{t \geq 0}$ be a gamma process, where $\Gamma(t) \sim G(\alpha, pt)$, the gamma distribution with scale parameter $\alpha^{-1}$ and shape parameter $pt$. Let

$$Q(t, \lambda) = N(\Gamma(t), \lambda),$$

where $\{N(t, \lambda)\}_{t \geq 0}$ is the Poisson process with intensity $\lambda > 0$. Then $\{Q(t, \lambda)\}_{t \geq 0}$ is called a negative binomial process and $Q(t, \lambda) \sim NB(t|pt, \eta)$, the negative binomial distribution with parameters $\eta = \lambda/(\alpha + \lambda)$ and $pt$. For $0 < \beta < 1$, let $\{D_{\beta}(t)\}_{t \geq 0}$ be a $\beta$-stable
subordinator with index $0 < \beta < 1$, and $\{E_\beta(t)\}_{t \geq 0}$ be its (right-continuous) inverse stable subordinator defined by

\begin{equation}
E_\beta(t) = \inf\{s > 0 : D_\beta(s) > t\}, \quad t > 0.
\end{equation}

A natural generalization of $Q(t, \lambda)$ is to consider $Q_\beta(t, \lambda) = N_\beta(\Gamma(t), \lambda)$, where $\{N_\beta(t, \lambda)\}_{t \geq 0}$ is the FPP (see [14, 21]), and we call $\{Q_\beta(t, \lambda)\}_{t \geq 0}$ the fractional negative binomial process (FNBP). We will show that this process is different from the FNBP discussed in [3] and [4]. It is known that the Polya process is obtained by replacing parameter $\lambda$ by a gamma random variable in the definition of the Poisson process $\{N(t, \lambda)\}$. Let $\Gamma \sim G(\alpha, p)$ and $W_\Gamma(t) = N(t, \Gamma)$, where $\Gamma$ is independent of $N$. Then $\{W_\Gamma(t)\}_{t \geq 0}$ is called the Polya process. However, a fractional version of the Polya process has not been addressed in the literature. Recently, in [22], a space fractional Poisson process $\{\tilde{N}_\beta(t, \lambda)\}_{t \geq 0}$, where $\tilde{N}_\beta(t, \lambda) = N(D_\beta(t), \lambda)$, is introduced and its properties are investigated. We here introduce space fractional Polya process (SFPP), as a fractional generalization of Polya process, defined by the process $\{\tilde{W}_\Gamma(t)\}_{t \geq 0}$, where $\tilde{W}_\Gamma(t) = \tilde{N}_\beta(t, \Gamma)$. The time fractional version of the Polya process does not exist.

The paper is organized as follows. In Section 2 some preliminary notation and results are stated. In Section 3 we discuss the infinite divisibility of the FPP $\{N_\beta(t, \lambda)\}_{t \geq 0}$ and also that of $\{N(E_\beta^{\ast n}(t, \lambda))\}_{t \geq 0}$, where $E_\beta^{\ast n}(t)$ is the $n$-fold convolution of inverse stable subordinator $E_\beta(t)$. In Section 4 we define the FNBP, compute its one-dimensional distributions and discuss their properties. It is shown in particular that their one-dimensional distributions are not infinitely divisible, and they solve certain fractional pde’s. It is also shown that the FNBP exhibits long-range dependence property. In Section 5 we define the SFPP and show that it has the stationary increments and is stochastically continuous. However, it does not have independent increments and hence is not a Lévy process. The fractional pde’s governed by the SFPP with respect to both the variables $t$ and $p$ are also discussed.

2. Preliminaries

In this section, we introduce the preliminary notations and the results that will be used later. Let $\mathbb{Z}_+ = \{0, 1, \ldots, \}$ be the set of nonnegative integers.

2.1. Some special functions. We start with some special functions that will be required later.

The Mittag-Leffler function $L_\beta(z)$ is defined as (see [8])

\begin{equation}
L_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1 + \beta k)}, \quad \beta, z \in \mathbb{C} \text{ and } \Re(\beta) > 0.
\end{equation}

The M-Wright function $M_\beta(z)$ (see [10, 17]) is defined as

\begin{equation}
M_\beta(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n! \Gamma(-\beta n + (1 - \beta))} = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-z)^{n-1}}{(n-1)!} \Gamma(\beta n) \sin(\pi \beta n), \quad z \in \mathbb{C}, \quad 0 < \beta < 1.
\end{equation}
Let \( p, q \in \mathbb{Z}_+ \setminus \{0\} \). Also, for \( 1 \leq i \leq p, 1 \leq j \leq q \), let \( a_i, b_j, z \in \mathbb{C} \) and \( A_i, B_j \) are positive reals.

(i) The generalized Wright function (see [28, 7, 11]) is defined as

\[
(2.2) \quad \psi_p^q \left[ z \right] = \left( a_1, A_1 \right)_{1,p} \left( b_1, B_1 \right)_{1,q} \sum_{k=0}^{\infty} \prod_{i=1}^{p} \frac{\Gamma(a_i + A_i k)}{k!} \left( \frac{z^k}{k!} \right).
\]

(ii) An \( H \)-function [10, Section 1.2] is defined in terms of the Mellin-Barnes type integral as

\[
(2.3) \quad H_{m,n}^{m,n}(z) = \frac{z}{2\pi i} \int_L \chi_{m,n}^{m,n}(s)z^{-s}ds,
\]

where \( z \neq 0 \) and \( z^{-s} = \exp[-s\{\ln|z| + i\arg(z)\}] \). Here, \( \ln|z| \) represents the natural logarithm of \(|z|\) and \( \arg(z) \) is not necessarily the principal value. Also, an empty product is interpreted as unity and

\[
\chi_{m,n}^{m,n}(s) = \prod_{j=1}^{m} \Gamma(b_j + B_j s) \prod_{j=1}^{n} \Gamma(1 - a_j - A_j s) \prod_{j=m+1}^{q} \Gamma(1 - b_j - B_j s) \prod_{j=n+1}^{p} \Gamma(a_j + A_j s),
\]

where \( m, n, p, q \) are nonnegative integers such that \( 1 \leq n \leq p, 1 \leq m \leq q \) and

\( A_i(b_j + v) \neq B_j(a_i - k - 1) \)

for \( v, k \in \mathbb{Z}_+, j = 1, \ldots, m \) and \( i = 1, \ldots, n \). The contour \( L \) in (2.3) separate the points \( s = -\left( \frac{b_j + v}{B_j} \right) \) which are the poles of \( \Gamma(b_j + B_j s) \) from the points \( s = \left( \frac{1 - a_i + k}{A_i} \right) \) which are the poles of \( \Gamma(1 - a_i - A_i s) \), where \( 1 \leq j \leq m, 1 \leq i \leq n, \) and \( \nu \in \mathbb{Z}_+ \).

It is known that the generalized Wright function \( \psi_p^q \) given in (2.2) satisfies

\[
(2.4) \quad \psi_p^q \left[ z \left( \alpha_1, A_1 \right), \ldots, \left( \alpha_p, A_p \right) \right] = H_{1,p}^{1,p+1} \left[ z \left( 1 - \alpha_1, A_1 \right), \ldots, \left( 1 - \alpha_p, A_p \right) \right],
\]

(see [11, eq. (5.2)]).

2.2. Some elementary distributions. Let \( \{N(t, \lambda)\}_{t \geq 0} \) be the Poisson process with rate \( \lambda > 0 \), so that

\[
p(n|t, \lambda) = \mathbb{P}[N(t, \lambda) = n] = \frac{(\lambda t)^n e^{-\lambda t}}{n!}, \quad n \in \mathbb{Z}_+.
\]

For \( \alpha > 0, p > 0 \), let \( \{\Gamma(t)\}_{t \geq 0} \) be the gamma process, where \( \Gamma(t) \sim G(\alpha, pt) \) with density

\[
g(y|\alpha, pt) = \frac{\alpha^{pt}}{\Gamma(pt)} y^{pt-1} e^{-\alpha y}, \quad y > 0.
\]

\[
(2.5)
\]
We say a random variable $X$ follows a negative binomial distribution with parameters $\alpha > 0$ and $0 < \eta < 1$, denoted by $NB(\alpha, \eta)$, if

\begin{equation}
\mathbb{P}(X = n) = \binom{n + \alpha - 1}{n} \eta^n (1 - \eta)^\alpha, \quad n \in \mathbb{Z}_+.
\end{equation}

When $\alpha$ is a natural number, then $X$ denotes the number of successes before $\alpha$-th failure, in a sequence of Bernoulli trials with success probability $\eta$.

We say $X$ follows a logarithmic series distribution with parameter $\eta$, denoted by $LS(\eta)$ if

\begin{equation}
\mathbb{P}(X = n) = \frac{-(\eta)^n}{n \ln(1 - \eta)}, \quad n \in \mathbb{Z}_+ \setminus \{0\}.
\end{equation}

Let $\{D_\beta(t)\}_{t \geq 0}$ be the $\beta$-stable subordinator. Then the density of $D_\beta(t)$ is (see [10, eq. (4.7)])

\begin{equation}
g_\beta(x; t) = \beta t x^{-(\beta+1)} M_\beta(tx^{-\beta}), \quad x > 0.
\end{equation}

Let $\{E_\beta(t)\}_{t \geq 0}$ be the inverse $\beta$-stable subordinator defined in (1.1). Then the density of $E_\beta(t)$ is (see [10, eq. (5.7)])

\begin{equation}
h_\beta(x; t) = t^{-\beta} M_\beta(t^{-\beta} x), \quad x > 0.
\end{equation}

2.3. Some fractional derivatives. Let $AC[a, b]$ be the space of functions $f$ which are absolutely continuous on $[a, b]$ and

\begin{equation}
AC^m[a, b] = \left\{ f : [a, b] \rightarrow \mathbb{R}; \frac{d^{m-1}}{dt^{m-1}} f(t) \in AC[a, b] \right\},
\end{equation}

where $AC^1[a, b] = AC[a, b]$.

**Definition 2.1** (Riemann-Liouville fractional derivative). Let $m \in \mathbb{Z}_+ \setminus \{0\}$ and $\nu \geq 0$. If $f(t) \in AC^{m-1}[0, T], 0 \leq t \leq T$, then the (left-hand) Riemann-Liouville (R-L) fractional derivative $\partial^\nu_\tau (\cdot)$ (see [23, p. 37]) is defined by

\begin{equation}
\partial^\nu_\tau f(t) :=
\begin{cases}
\frac{1}{\Gamma(m - \nu)} \frac{d^m}{dt^m} \int_0^t \frac{f(s)}{(t-s)^{\nu+1-m}} ds, & m - 1 < \nu < m, \\
\frac{d^m}{dt^m} f(t), & \nu = m.
\end{cases}
\end{equation}

**Definition 2.2** (Caputo fractional derivative). Let $m \in \mathbb{Z}_+ \setminus \{0\}$ and $\nu \geq 0$. If $f(t) \in AC^{m}[0, T], 0 \leq t \leq T$, then the (left-hand) Caputo fractional derivative (see [12, Theorem 2.1]) $D^\nu_\tau (\cdot)$ is defined by

\begin{equation}
D^\nu_\tau f(t) :=
\begin{cases}
\frac{1}{\Gamma(m - \nu)} \int_0^t \frac{f^{(m)}(s)}{(t-s)^{\nu+1-m}} ds, & m - 1 < \nu < m, \\
\frac{d^m}{dt^m} f(t), & \nu = m.
\end{cases}
\end{equation}
The relation between R-L fractional derivative and Caputo fractional derivative is (see [12, eq. (2.4.6)])

$$\partial_\nu^\nu t f(t) = D_\nu^\nu t f(t) + \sum_{k=0}^{m-1} \frac{t^{k-\nu}}{\Gamma(k-\nu+1)} f^{(k)}(0^+),$$

where $f^{(k)}$ denotes the $k$-th derivative of $f$.

3. Fractional Poisson process

Let $0 < \beta \leq 1$. The fractional Poisson process (FPP) $\{N_\beta(t, \lambda)\}_{t \geq 0}$, which is a generalization of the Poisson process $\{N(t, \lambda)\}_{t \geq 0}$, solves the following fractional difference-differential equation (see [14, 18, 21]):

$$D_\beta^\beta t p_\beta(n|t, \lambda) = -\lambda p_\beta(n|t, \lambda) + \lambda p_\beta(n-1|t, \lambda),$$

for $n \geq 1$, (3.1)

$$D_\beta^\beta t p_\beta(0|t, \lambda) = -\lambda p_\beta(0|t, \lambda),$$

with $p_\beta(n|0, \lambda) = 1$, if $n = 0$, and is zero if $n \geq 1$. Here, $p_\beta(n|t, \lambda) = \mathbb{P}\{N_\beta(t, \lambda) = n\}$ and $D_\beta^\beta$ denotes the Caputo fractional derivative defined in (2.11). The pmf $p_\beta(n|t, \lambda)$ for the FPP is given by (see [14, 21])

$$p_\beta(n|t, \lambda) = \frac{(\lambda t^\beta)^n}{n!} \sum_{k=0}^{\infty} \frac{(n+k)!}{k!} \frac{(-\lambda t^\beta)^k}{\Gamma(\beta(k+n)+1)}.$$  (3.2)

Note that equation (3.2) can also be represented as

$$p_\beta(n|t, \lambda) = \frac{(\lambda t^\beta)^n}{n!} \psi_1\left[ -\lambda t^\beta \left| \begin{array}{c} (n+1,1) \\ (n+\beta+1, \beta) \end{array} \right| \right],$$

using the generalized Wright function defined in [22].

The mean and variance of FPP are given by (see [14])

$$\mathbb{E}N_\beta(t, \lambda) = \frac{\lambda t^\beta}{\Gamma(\beta + 1)};$$

(3.3)

$$\text{Var}(N_\beta(t, \lambda)) = \frac{\lambda t^\beta}{\Gamma(\beta + 1)} \left(1 + \frac{\lambda t^\beta}{\Gamma(\beta + 1)} \left(\frac{\beta B(\beta, 1/2)}{2^{\beta-1}} - 1\right)\right),$$

(3.4)

where $B(a, b)$ denotes the beta function. An alternative form for $\text{Var}(N_\beta(t, \lambda))$ is given in [2, eq. (2.8)] as

$$\text{Var}(N_\beta(t, \lambda)) = \frac{\lambda t^\beta}{\Gamma(\beta + 1)} + \frac{(\lambda t^\beta)^2}{\beta} \left(\frac{1}{\Gamma(2\beta)} - \frac{1}{\beta \Gamma^2(\beta)}\right).$$

(3.5)

It is also known that (see [21]) when $0 < \beta < 1$,

$$N_\beta(t, \lambda) = N(E_\beta(t), \lambda),$$

(3.6)

where $\{E_\beta(t)\}_{t \geq 0}$ is the inverse $\beta$-stable subordinator.

First we establish an important property of the FPP.

**Theorem 3.1.** Let $0 < \beta < 1$. The one-dimensional distributions of the FPP $\{N_\beta(t, \lambda)\}_{t \geq 0}$ are not infinitely divisible (i.d.).
Proof. Since the sample paths of \( \{D_\beta(t)\}_{t \geq 0} \) are strictly increasing, the process \( \{E_\beta(t)\}_{t \geq 0} \) has continuous sample paths. Further,

\[
P(E_\beta(t) \leq x) = P(D_\beta(x) \geq t).
\]

Let \( c > 0 \). It is well known that if \( D_\beta(t) \) is a \( \beta \)-stable process, then it is also self-similar with index \( 1/\beta \), that is,

\[
D_\beta(ct) \overset{d}{=} c^{1/\beta} D_\beta(t).
\]

Hence,

\[
P(E_\beta(ct) \leq x) = P(D_\beta(x) \geq ct) = P\left( \frac{1}{c} D_\beta(x) \geq t \right) = P\left( D_\beta\left( \frac{x}{ct} \right) \geq t \right)
\]

\[
= P\left( E_\beta(t) \leq \frac{x}{ct} \right) = P(c^\beta E_\beta(t) \leq x).
\]

That is,

\[
(3.7) \quad E_\beta(ct) \overset{d}{=} c^\beta E_\beta(t),
\]

showing that \( E_\beta(t) \) is also self-similar with index \( \beta \).

Observe now that

\[
N_\beta(t, \lambda) = N(E_\beta(t), \lambda) \overset{d}{=} N(t^\beta E_\beta(1), \lambda).
\]

By renewal theorem for the Poisson process,

\[
\lim_{t \to \infty} \frac{N(t, \lambda)}{t} = \frac{1}{\lambda}, \quad \text{a.s.}
\]

This implies, when \( E_\beta(t) \) is independent of \( N \),

\[
(3.8) \quad \lim_{t \to \infty} \frac{N(t^\beta E_\beta(1), \lambda)}{t^\beta} = E_\beta(1) \lim_{t \to \infty} \frac{N(t^\beta E_\beta(1), \lambda)}{t^\beta E_\beta(1)} = \frac{E_\beta(1)}{\lambda} \quad \text{a.s.}
\]

since \( E_\beta(1) > 0 \) a.s. Hence, for \( 0 < \beta < 1 \),

\[
\frac{N_\beta(t, \lambda)}{t^\beta} \overset{\mathcal{L}}{\to} \frac{E_\beta(1)}{\lambda},
\]

where \( \mathcal{L} \) denote the convergence in law. Assume now that \( N_\beta(t, \lambda) \) is i.d., then \( N_\beta(t, \lambda)/t^\beta \) is also i.d. for each \( t \). Since the limit of a sequence of i.d. random variable is also i.d. (see [24, Lemma 7.8, p. 34]), it follows that \( \frac{E_\beta(1)}{\lambda} \) or equivalently \( E_\beta(1) \) is i.d., which is a contradiction since, \( E_\beta(t) \) is not i.d. for \( t > 0 \) (see [26]). Hence, the result follows. \( \square \)

Let \( \{E_{\beta_1}(t)\}, \{E_{\beta_2}(t)\}, \ldots, \{E_{\beta_n}(t)\} \) be the corresponding inverse stable processes. Consider the process \( \{E_{\beta_n}^*(t)\} \), where \( E_{\beta_n}^*(t) = E_{\beta_1} \circ E_{\beta_2} \circ \ldots \circ E_{\beta_n}(t) \). By [26, Remark 2.5], we have that \( E_{\beta_n}^*(t) \) is not i.d. We have the following result for the Poisson process with time change \( E_{\beta_n}^*(t) \).

**Theorem 3.2.** The one-dimensional distributions of the subordinated Poisson process \( \{N(E_{\beta_n}^*(t), \lambda)\}_{t \geq 0} \) are not i.d.

**Proof.** For some \( c > 0 \) and using (3.7), we have

\[
E_{\beta_1}(E_{\beta_2}(ct)) \overset{d}{=} E_{\beta_1}(c^{\beta_2} E_{\beta_2}(t)) \overset{d}{=} c^{\beta_1 \beta_2} E_{\beta_1}(E_{\beta_2}(t)).
\]
Thus, in general, we have \( E^{*n}_{\beta}(ct) = c^\beta E^{*n}_{\beta}(t) \) and hence
\[
\frac{N(E^{*n}_{\beta}(t), \lambda)}{t^\beta} \xrightarrow{\text{L}} E^{*n}_{\beta}(1) \lambda.
\]
which is not i.d. and hence the result follows.

4. Fractional Negative Binomial Process

4.1. Definition and Properties. Let \( \{\Gamma(t)\}_{t \geq 0} \) be the gamma process, where \( \Gamma(t) \sim G(\alpha, pt) \) given in (2.5). The negative binomial process \( \{Q(t, \lambda)\}_{t \geq 0} = \{N(\Gamma(t), \lambda)\}_{t \geq 0} \) is a subordinated Poisson process (see [3, 13]) with
\[
\mathbb{P}[Q(t, \lambda) = n] = \delta(n|\alpha, pt, \lambda) = \frac{\alpha^pt^n}{n!\Gamma(pt)} \int_0^{\infty} y^{n+pt-1} e^{-y(\alpha+\lambda)} dy
\]

\[
= \left(\frac{pt + n - 1}{n}\right) \left(\frac{\alpha}{\alpha + \lambda}\right)^pt \left(\frac{\lambda}{\alpha + \lambda}\right)^n = \left(\frac{pt + n - 1}{n}\right) \eta^n(1 - \eta)^{pt},
\]
where \( \eta = \lambda/(\alpha + \lambda) \). That is, \( Q(t, \lambda) \sim NB(pt, \eta) \), for \( t > 0 \), defined in (2.6).

Let \( t \geq 0 \). We define the fractional negative binomial process (FNBP) as \( \{Q_\beta(t, \lambda)\} = \{N_\beta(\Gamma(t), \lambda)\} \), where \( \{N_\beta(t, \lambda)\} \) is the FPP. Let \( g(y|\alpha, pt) \) denotes the pdf of \( \Gamma(t) \) given in (2.5). Then,
\[
\mathbb{P}[Q_\beta(t, \lambda) = n] = \delta_\beta(n|\alpha, pt, \lambda) = \int_0^{\infty} p_\beta(n|y, \lambda)g(y|\alpha, pt) dy
\]

\[
= \frac{\lambda^n}{n!} \sum_{k=0}^{\infty} (-\lambda)^k \frac{(n+k)!}{k!} \frac{1}{\Gamma(\beta(n+k)+1)} \frac{\alpha^pt}{\Gamma(pt)} \int_0^{\infty} e^{-\alpha y^{(n+k)\beta+pt-1} dy}
\]

\[
= \frac{\lambda^n}{n!} \sum_{k=0}^{\infty} (-\lambda)^k \frac{(n+k)!}{k!} \frac{1}{\Gamma(\beta(n+k)+1)} \frac{\alpha^pt}{\Gamma(pt)} \frac{\Gamma((n+k)\beta+pt)}{\alpha^{(n+k)\beta+pt}}
\]

\[
= \left(\frac{\lambda}{\alpha^\beta}\right)^n \frac{1}{n!} \sum_{k=0}^{\infty} \frac{(n+k)!}{k!} \frac{1}{\Gamma(\beta(n+k)+1)} \frac{\Gamma((n+k)\beta+pt)}{\alpha^{(n+k)\beta+pt}} \left(\frac{-\lambda}{\alpha^\beta}\right)^k
\]

\[
(4.1)
\]

Assume \( |\frac{\lambda}{\alpha^\beta}| < 1 \). Then, by [11, Theorem 1(b)] and with \( \delta = 1^{-1}\beta^{-\beta}\beta^\beta = 1, \Delta = \beta - \beta - 1 = -1 \), the associated series of \( _2\psi_1 \) function in (4.1) converges. Thus, we have proved the following result.

Theorem 4.1. Let \( 0 < \beta \leq 1 \) and \( 0 < \lambda < \alpha^\beta \), where \( \alpha > 0 \). Then the FNBP \( \{Q_\beta(t, \lambda)\}_{t \geq 0} \) has the one-dimensional distributions, for \( n \in \mathbb{Z}_+ \), as
\[
\delta_\beta(n|\alpha, pt, \lambda) = \frac{1}{\Gamma(pt)n!} \left(\frac{\lambda}{\alpha^\beta}\right)^n _2\psi_1 \left[ \frac{-\lambda}{\alpha^\beta} (n+1, 1), \frac{\lambda}{\alpha^\beta} (n\beta+pt, \beta) \right] (n+1, 1, (n\beta+pt, \beta))
\]

\[
= \frac{1}{\Gamma(pt)n!} \left(\frac{\lambda}{\alpha^\beta}\right)^n H_{2,2}^{1,2} \left[ \frac{\lambda}{\alpha^\beta} (0, 1), \frac{\lambda}{\alpha^\beta} (-n, 1), \frac{\lambda}{\alpha^\beta} (1-n\beta-pt, \beta) \right] (-n, 1, \beta, (1-n\beta-pt, \beta)),
\]
which follows from (2.4).

When \( \beta = 1 \), we can see that \( \delta_{\beta}(n|\alpha, \beta, \lambda) \) reduces to the pmf of NB\((p, \eta)\) distribution.

We next check that \( \delta_{\beta}(n|\alpha, \beta, \lambda) \) is indeed a pmf for \( 0 < \beta < 1 \) also. Note that

\[
\sum_{n=0}^{\infty} \delta_{\beta}(n|\alpha, \beta, \lambda) = \sum_{n=0}^{\infty} \left( \frac{\lambda}{\alpha^\beta} \right)^n \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{\Gamma((n+k)\beta + \alpha \beta)}{\Gamma((n+k)\beta + \alpha \beta + 1)} \left( -\frac{\lambda}{\alpha^\beta} \right)^k
\]

\[
= \sum_{n=0}^{\infty} \left( \frac{\lambda/\alpha^\beta}{n! \Gamma(pt)} \right)^n \sum_{k=n}^{\infty} \frac{(k)!}{(k-n)! \Gamma(k\beta + 1)} \left( -\frac{\lambda}{\alpha^\beta} \right)^{k-n}
\]

\[
= \frac{1}{\Gamma(pt)} \sum_{n=0}^{\infty} \frac{\Gamma(k\beta + pt)}{\Gamma(k\beta + 1)} \sum_{k=0}^{\infty} \binom{k}{n} \left( \frac{\lambda}{\alpha^\beta} \right)^n \left( -\frac{\lambda}{\alpha^\beta} \right)^{k-n} = 1,
\]

since only the polynomial term corresponding to \( k = 0 \) remains.

**Remark 4.1.** Let \( 0 < \alpha^\beta < \lambda \). Then, using the representation given in (3.6) and from (2.9), we obtain also

\[
\delta_{\beta}(n|\alpha, \beta, \lambda) = \int_0^\infty \int_0^\infty p(n|x, \lambda)h_{\beta}(x, y)g(y|\alpha, pt)dx\,dy
\]

\[
= \frac{\alpha^\beta}{\lambda \Gamma(pt)} \sum_{k=0}^{\infty} \binom{n+k}{n} \frac{\Gamma(pt - \beta - \beta k)}{\Gamma(-\beta k + (1 - \beta))} \left( -\frac{\lambda}{\alpha^\beta} \right)^k
\]

\[
= \frac{\alpha^\beta}{n! \lambda \Gamma(pt)^2} \psi_1\left[ \frac{-\alpha^\beta}{\lambda}, \binom{n+1, 1}{1 - \beta, -\beta} \right].
\]

**Remark 4.2.** Using a conditioning argument and using (3.3), we get

\[
\mathbb{E}Q_{\beta}(t, \lambda) = \frac{\lambda}{\alpha^\beta \Gamma(\beta + 1)} \frac{\Gamma(pt + \beta)}{\Gamma(pt)} = \frac{\lambda}{\alpha^\beta} \frac{1}{\beta B(\beta, pt)}.
\]

By Stirling’s formula, \( \Gamma(pt + \beta)/\Gamma(pt) \sim (pt/e)^\beta \), for large \( t \), we get \( \mathbb{E}N_{\beta}(\Gamma(t), \lambda) \sim \left( \frac{\lambda}{\alpha^\beta} \right)^\beta \mathbb{E}N_{\beta}(t, \lambda) \).

Also, using (3.4),

\[
\text{Var}(Q_{\beta}(t, \lambda)) = \frac{\lambda}{\Gamma(\beta + 1)} \frac{\Gamma(pt + \beta)}{\alpha^\beta \Gamma(pt)} + \left( \frac{\alpha^{-\beta} \lambda}{\Gamma(\beta + 1)} \right)^2
\]

\[
\times \left[ \frac{\beta B(\beta, 1/2) \Gamma(pt + 2\beta)}{2^{1-\beta} \Gamma(pt)} - \left( \frac{\Gamma(pt + \beta)}{\Gamma(pt)} \right)^2 \right].
\]

Again, using (3.5), we get

\[
\text{Var}(Q_{\beta}(t, \lambda)) = \frac{\lambda}{\Gamma(\beta + 1)} \frac{\Gamma(pt + \beta)}{\alpha^\beta \Gamma(pt)} + \frac{\lambda^2 \Gamma(pt + 2\beta)}{\beta \alpha^2 \Gamma(pt)} \left( \frac{1}{\Gamma(2\beta)} - \frac{1}{\beta \Gamma^2(\beta)} \right).
\]

Note now

\[
\text{Var}(Q_{\beta}(t, \lambda)) - \mathbb{E}[Q_{\beta}(t, \lambda)] = \frac{\lambda^2}{\Gamma(2\beta + 1)} \frac{\Gamma(pt + 2\beta)}{\Gamma(pt)} - \left( \frac{\lambda}{\Gamma(\beta + 1) \alpha^\beta \Gamma(pt)} \right)^2.
\]
\[ = \left( \frac{\lambda}{\alpha^\beta} \right)^2 \frac{1}{\Gamma(pt)} \left[ \frac{\Gamma(pt + 2\beta)}{\Gamma(2\beta + 1)} - \left( \frac{\Gamma(pt + \beta)}{\Gamma(\beta + 1)} \right)^2 \frac{1}{\Gamma(pt)} \right], \]

which is computed in Mathematica, for \(\lambda = 1, \alpha = 1, \beta = 0.5, p = 1\) and \(t \in [0, 3]\), using the following code

\[
\text{Plot} \left[ \frac{1}{\Gamma(t)} \left( \frac{\Gamma(t + 2 \times 0.5)}{\Gamma(2 \times 0.5 + 1)} - \frac{1}{\Gamma(t)} \left( \frac{\Gamma(t + 0.5)}{\Gamma(1 + 0.5)} \right)^2 \right), \{t, 0, 3\}, \text{PlotRange} \rightarrow 0.2 \right].
\]

The following graph (see Figure 1) is obtained, which clearly shows that FNBP is not overdispersed.

Theorem 4.2. The one-dimensional distributions of the FNBP \(\{Q_\beta(t, \lambda)\}_{t \geq 0}\) are not i.d.

Proof. Since \(E_\beta(t) \overset{d}{=} t^\beta E_\beta(1)\), we have

\[ Q_\beta(t, \lambda) = N(E_\beta(\Gamma(t)), \lambda) \overset{d}{=} N((\Gamma(t))^\beta E_\beta(1), \lambda). \]

Using \(\text{Binomial}^2\),

\[
\lim_{t \to \infty} \frac{N((\Gamma(t))^\beta E_\beta(1), \lambda)}{t^\beta} = \lim_{t \to \infty} \frac{N((\Gamma(t))^\beta E_\beta(1), \lambda)}{(\Gamma(t))^\beta} \left( \frac{\Gamma(t)}{t} \right)^\beta = \frac{E_\beta(1)}{\lambda} (\mathbb{E}(\Gamma(1))^\beta = \frac{E_\beta(1)}{\lambda} \left( \frac{p}{\alpha} \right)^\beta, \ a.s.,
\]

since \(\Gamma(t) \to \infty\) and \(\Gamma(t)/t \to \mathbb{E}(\Gamma(1))\ a.s., \) as \(t \to \infty\). The result follow by contradiction, since \(E_\beta(1)\) is not i.d. \(\Box\)

Remark 4.3. (i) In fact, this can be generalized for any subordinator \(T(t)\), with \(\mathbb{E}(T(1)) < \infty\). For a subordinator, SLLN for Lévy processes yields \(\lim_{t \to \infty} \frac{T(t)}{t} = \mathbb{E} T(1)\ a.s.\) Thus,
Hence, substituting in (4.8), we get

\[
\frac{N_{\beta}(\Gamma(t), \lambda)}{t^{\beta}} \xrightarrow{\mathcal{L}} \frac{E(1)}{\lambda} (\mathbb{E}T(1))^\beta \text{ which is not i.d.}
\]

(ii) Since,

\[
\frac{N_{\beta}(\Gamma(t), \lambda)}{t^{\beta_1}t^{\beta_2} \cdots t^{\beta_n}} \xrightarrow{\mathcal{L}} \frac{E(1)}{\lambda^i} (\mathbb{E}T(1))^\frac{1}{\beta_1 + \beta_2 + \cdots + \beta_n}, \quad \text{as } t \to \infty,
\]

it follows that the distributions of \( \{N_{\beta}(\Gamma(t), \lambda)\}_{t \geq 0} \) are also not i.d.

**Theorem 4.3.** The FNBP \( \{Q_{\beta}(t, \lambda)\}_{t \geq 0} \) has long-range dependence (LRD) property.

**Proof.** For simplicity, let \( \{N_{\beta}(t)\} = \{N_{\beta}(t, \lambda)\} \) be the fractional Poisson process. Then from [16, eq. (14)],

\[
\text{Cov}(N_{\beta}(s), N_{\beta}(t)) = q s^\beta + q^2 [s^2 s \beta B(\beta, 1 + \beta) + \beta t \beta^2 B(\beta, 1 + \beta; s/t) - (st)^\beta], \quad 0 < s \leq t,
\]

where \( q = \frac{\lambda}{\Gamma(1 + \beta)} \) and \( B(a, b; x) = \int_0^x t^{a-1}(1 - t)^{b-1}dt, \ 0 < x < 1, \) is the incomplete beta function. Hence, from (3.3),

\[
E[N_{\beta}(s)N_{\beta}(t)] = q s^\beta + q^2 [s^2 s \beta B(\beta, 1 + \beta) + t \beta^2 B(\beta, 1 + \beta; s/t)].
\]

Let \( \{Q_{\beta}(t)\} = \{Q_{\beta}(t, \lambda)\} \) be the FNBP. Then from (4.7),

\[
E[Q_{\beta}(s)Q_{\beta}(t)] = E[E[N_{\beta}(\Gamma(s))N_{\beta}(\Gamma(t))]|\Gamma(s)\Gamma(t)]
\]

\[
= qE[\Gamma(s)^\beta] + q^2 E[\Gamma(s)^{2\beta}] \beta B(\beta, 1 + \beta) + q^2 \beta \int_0^\infty \int_0^\infty v^{\beta} B(\beta, 1 + \beta; \frac{u}{v}) l(u, v) du dv,
\]

where

\[
l(u, v) = \frac{\partial^2}{\partial u \partial v} \mathbb{P}(\Gamma(s) \leq u, \Gamma(t) \leq v)
\]

is the joint density of \((\Gamma(s), \Gamma(t))\). Since \( \{\Gamma(t)\} \) is a Lévy process and using (2.5),

\[
\mathbb{P}[\Gamma(s) \leq u, \Gamma(t) \leq v] = \mathbb{P}[\Gamma(s) \leq u, (\Gamma(t) - \Gamma(s)) \leq v - \Gamma(s)],
\]

\[
= \int_{x=0}^{u} g(x|\alpha, ps) \int_{y=0}^{v-x} g(y|\alpha, p(t-s)) dy dx.
\]

Hence,

\[
l(u, v) = g(u|\alpha, ps) g(v - u|\alpha, p(t-s)).
\]

Substituting in (4.8), we get

\[
E[Q_{\beta}(s)Q_{\beta}(t)] = qE[\Gamma(s)^\beta] + q^2 E[\Gamma(s)^{2\beta}] \beta B(\beta, 1 + \beta)
\]

\[
+ q^2 \beta \int_0^\infty \int_0^\infty v^{\beta} B(\beta, 1 + \beta; \frac{u}{v}) g(u|\alpha, ps) g(v - u|\alpha, p(t-s)) du dv.
\]

Hence, from (4.4)

\[
\text{Cov}(Q_{\beta}(s), Q_{\beta}(t)) = qE[\Gamma(s)^\beta] + q^2 E[\Gamma(s)^{2\beta}] \beta B(\beta, 1 + \beta) - \beta^2 E[\Gamma(s)^{3\beta}] E[\Gamma(t)^\beta]
\]

\[
+ q^2 \beta \int_0^\infty \int_0^\infty v^{\beta} B(\beta, 1 + \beta; \frac{u}{v}) g(u|\alpha, ps) g(v - u|\alpha, p(t-s)) du dv,
\]

(4.9)
which is the autocovariance function of the FNBP. It can be seen that, by Stirling’s approximation,
\[
\mathbb{E}(\Gamma(t)^\beta) = \frac{1}{\alpha^\beta} \frac{\Gamma(pt + \beta)}{\Gamma(pt)} \approx \left( \frac{pt}{e\alpha} \right)^\beta, \quad \text{for large } t.
\]
Since \(\lim_{t \to \infty} g(x|\alpha, t) = 0\), we have \(\lim_{t \to \infty} g(v - u|\alpha, p(t - s)) = 0\), for \(p > 0\) and fixed \(s > 0\). Therefore, (4.9) becomes, for large \(t\),
\[
\text{(4.10)} \quad \text{Cov}(Q_\beta(s), Q_\beta(t)) \approx q\mathbb{E}[\Gamma(s)^\beta] + q^2\mathbb{E}[\Gamma(s)^{2\beta}]B(\beta, 1 + \beta) - q^2\mathbb{E}[\Gamma(s)^\beta] \left( \frac{pt}{e\alpha} \right)^\beta.
\]
Similarly, for large \(t\),
\[
\text{Var}(Q_\beta(t)) \approx q \left( \frac{pt}{e\alpha} \right)^\beta + \frac{\lambda^2}{\Gamma(2\beta + 1)} \left( \frac{pt}{e\alpha} \right)^{2\beta} - q^2 \left( \frac{pt}{e\alpha} \right)^{2\beta} \\
= t^{2\beta} \left\{ q \left( \frac{p}{te\alpha} \right)^\beta + \frac{\lambda^2}{\Gamma(2\beta + 1)} \left( \frac{p}{e\alpha} \right)^{2\beta} - q^2 \left( \frac{p}{e\alpha} \right)^{2\beta} \right\} \\
\approx t^{2\beta} \left( \frac{p}{e\alpha} \right)^{2\beta} \left\{ \frac{\lambda^2}{\Gamma(2\beta + 1)} - q^2 \right\}.
\]
(4.11)
Thus, from (4.10) and (4.11), the correlation between \(Q_\beta(s)\) and \(Q_\beta(t)\), for large \(t > s\), is
\[
\text{Corr}(Q_\beta(s), Q_\beta(t)) \approx \frac{q\mathbb{E}[\Gamma(s)^\beta] + q^2\mathbb{E}[\Gamma(s)^{2\beta}]B(\beta, 1 + \beta) - q^2\mathbb{E}[\Gamma(s)^\beta] \left( \frac{pt}{e\alpha} \right)^\beta}{\sqrt{t^{2\beta} \left\{ \frac{\lambda^2}{\Gamma(2\beta + 1)} - q^2 \right\} \sqrt{\text{Var}(Q_\beta(s))}}} \\
= t^{-\beta} \left( \frac{q\mathbb{E}[\Gamma(s)^\beta] + q^2\mathbb{E}[\Gamma(s)^{2\beta}]B(\beta, 1 + \beta)}{\sqrt{\frac{\lambda^2}{\Gamma(2\beta + 1)} - q^2} \sqrt{\text{Var}(Q_\beta(s))}} \right) \\
- \frac{q^2\mathbb{E}[\Gamma(s)^\beta]}{\sqrt{\frac{\lambda^2}{\Gamma(2\beta + 1)} - q^2} \sqrt{\text{Var}(Q_\beta(s))}}
\]
which decays like the power law \(t^{-\beta}\). Hence, the FNBP exhibits LRD property. \(\square\)

Recently, Beghin [3] and Beghin and Macci [4] also studied the FNBP. For \(0 < \beta < 1\) and \(0 < \eta < 1\), they define FNBP as
\[
X_1(t) = \sum_{i=1}^{N_\beta(t,-\ln(1-\eta))} Y_i \quad \text{and} \quad X_2(t) = N(\Gamma_\beta^*(t), 1),
\]
in [4] and [3] respectively, where \(Y_i\)’s are logarithmic series \(LS(\eta)\) random variables, \(\{\Gamma_\beta^*(t)\}_{\geq 0} = \{\Gamma^*(E_\beta(t))\}_{\geq 0}\) is the fractional gamma process (see [3]) and \(\Gamma^* \sim G(\alpha, t)\). Note when \(\beta = 1\),
\[
X_1(t) \sim NB(t, \eta) \quad \text{and} \quad X_2 \sim \left(t, \frac{1}{1+\alpha}\right).
\]
Observe that, our definition of the FNBP is
\[
\text{(4.12)} \quad Q_\beta(t, \lambda) := N_\beta(\Gamma(t), \lambda) \overset{d}{=} N(E_\beta(\Gamma(t)), \lambda),
\]
which is a more natural extension of the negative binomial process. Note that subordinating the FPP to a gamma subordinator is more natural than subordinating the Poisson process to a fractional gamma subordinator. Also, our definition of the FNBP allows us to compute the one-dimensional distributions which exhibit LRD property. When $\beta = 1$, $Q_1(t, \lambda) = N(\Gamma(t), \lambda) \sim NB(pt, \frac{1}{\lambda+\alpha}), t > 0$ (see e.g. [27]). Let, for $i \geq 1$, $Y_i^* \sim LS \left( \frac{1}{\alpha+\lambda} \right)$. Then it can be seen that

$$Q_1(t, \lambda) = \sum_{i=1}^{N(p,t,-\ln\left(\frac{t}{\alpha+\lambda}\right))} Y_i^*.$$ 

The following result shows that our process is different from theirs.

**Lemma 4.1.** Let $0 < \beta < 1$. Then the process $\{N(\Gamma_\beta(t), \lambda)\}_{t \geq 0}$ and $\{Q_\beta(t, \lambda)\}_{t \geq 0}$ are different.

**Proof.** It is sufficient to prove that the one-dimensional distributions of the processes $\{E_\beta(\Gamma(t))\}_{t \geq 0}$ and $\{\Gamma(E_\beta(t))\}_{t \geq 0}$ are different. To see this, let $p(x, t)$ be pdf of $\Gamma(E_\beta(t))$, $q(x, t)$ be the pdf of $E_\beta(\Gamma(t))$ and $h_\beta(x, t)$ be the pdf of $E_\beta(t)$. Now, the Laplace transform of $p(x, t)$ in the space variable $x$ is

$$\tilde{p}(s, t) = \int_0^\infty e^{-sx} p(x, t) dx = \int_0^\infty e^{-sx} \int_0^\infty g(x|\alpha, py) h_\beta(y, t) dy dx$$

(4.13)

$$= \int_0^\infty \left( \frac{\alpha}{\alpha + s} \right)^{py} h_\beta(y, t) dy = \mathbb{E} \left[ \left( \frac{\alpha}{\alpha + s} \right)^{pE_\beta(t)} \right].$$

It is known (see [20, 6]) that $\mathbb{E}[e^{-sE_\beta(t)}] = L_\beta[-st^\beta]$ so that, with $u = e^{-s}$, $\tilde{p}(s, t) = \mathbb{E}[u^{E_\beta(t)}] = L_\beta[t^\beta \log u]$, where $L_\beta(z)$ is the Mittag-Leffler function defined in (2.1). So, (4.13) simplifies to

$$L_\beta[t^\beta \log \left( \frac{\alpha}{\alpha + s} \right)] = \sum_{k=0}^{\infty} \frac{(t^\beta p \log(\frac{\alpha}{\alpha + s}))^k}{\Gamma(\beta k + 1)}.$$

Similarly, the Laplace transform of $q(x, t)$ w.r.t. variable $x$ is

$$\tilde{q}(s, t) = \int_0^\infty e^{-sx} q(x, t) dx = \int_0^\infty \int_0^\infty e^{-sx} h_\beta(x, y) g(y|\alpha, pt) dy dx$$

$$= \int_0^\infty L_\beta(-sy^\beta) g(y|\alpha, pt) dy = \sum_{k=0}^{\infty} \frac{\Gamma(\beta k + pt)}{\Gamma(pt)\Gamma(1 + \beta k)} \left( \frac{-s}{\alpha^\beta} \right)^k$$

(4.15)

$$= \frac{1}{\Gamma(pt)}^2 \psi_1 \left[ \frac{-s}{\alpha^\beta}, (1, 1), (1, 1) \right].$$

It can be seen that (4.14) and (4.15) are different. For example, taking $\beta = 1/2, \alpha = 2, \lambda = 1, p = 1, s = 1$ and $t = 1$, the series in the right-hand side of (4.14) reduces to

$$\sum_{k=0}^{\infty} \frac{(\log 2/3)^k}{\Gamma(k/2 + 1)} = -e^{2\log(3/2)}(-1 + \text{Erf}[\log(3/2)])$$

(4.16)

where $\text{Erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$. But, the right-hand side of (4.15) reduces to

$$\sum_{k=0}^{\infty} \left( -\frac{1}{\sqrt{2}} \right)^k = 2 - \sqrt{2}.$$
Clearly, \([4.16]\) and \([4.17]\) are different, which proves the result.

\[\square\]

### 4.2. Connections to pde's.

In this section, we discuss pde's governed by the one-dimensional distributions of the FNBP.

**Theorem 4.4.** Let \(r \in \mathbb{Z}_+ \setminus \{0\}\). The pmf \([4.2]\) of FNBP solves the following pde:

\[
\frac{\partial^r}{\partial \lambda^r} \delta_r(n|\alpha, pt, \lambda) = \frac{1}{\alpha^\beta \Gamma(pt)n!} \sum_{i=0}^{\infty} \binom{\lambda}{\alpha^\beta} \binom{\lambda}{\alpha^\beta} \left[ (n, 1), (1 - n\beta - pt, \beta) \right]
\]

with

\[
\delta_r(n|\alpha, 0, \lambda) = \begin{cases} 1, & n = 0, \\ 0, & n \geq 1. \end{cases} \quad \text{and} \quad \delta_r(n|\alpha, pt, \lambda) = 0, \forall n < 0.
\]

**Proof.** Note first the \(H\)-function satisfies (see \([19, \text{eq. (1.4.1)}]\)), for \(r \in \mathbb{Z}_+ \setminus \{0\}\),

\[
\frac{\partial^r}{\partial z^r} \left\{ z^{-(\beta_1/B_1)} H_{p,q}^{m,n} \left[ \binom{\alpha_1, A_1, \ldots, (\alpha_p, A_p)}{\beta_1, B_1, \ldots, (\beta_q, B_q)} \right] \right\}
= \left( -\frac{\gamma}{B_1} \right) z^{-r-(\beta_1/B_1)} H_{p,q}^{m,n} \left[ \binom{\alpha_1, A_1, \ldots, (\alpha_p, A_p)}{\beta_1, B_1, \ldots, (\beta_q, B_q)} \right].
\]

Taking \(p = 2, q = 2, m = 1, n = 2, \alpha_1 = -n, \alpha_2 = 1 - n\beta - pt, \beta_1 = 0, \beta_2 = -n\beta, A_1 = B_1 = 1, A_2 = B_2 = 2\) and \(\gamma = 1\) we get

\[
\frac{\partial^r}{\partial z^r} H_{2,2}^{1,2} \left[ \binom{(-n, 1), (1 - n\beta - pt, \beta)}{(0, 1), (-n\beta, \beta)} \right] = (-1)^r z^{-r} H_{2,2}^{1,2} \left[ \binom{(-n, 1), (1 - n\beta - pt, \beta)}{(r, 1), (-n\beta, \beta)} \right]
\]

Now, differentiate \(r\) times the rhs of \([4.3]\) with respect to \(\lambda\), use \([4.20]\) and the Leibniz rule

\[
\frac{d^r}{dx^r} [u(x)v(x)] = \sum_{i=0}^{r} \binom{r}{i} \frac{d^i}{dx^i} [u(x)] \frac{d^{r-i}}{dx^{r-i}} [v(x)],
\]

to obtain the result in \([4.18]\).

\[\square\]

**Remark 4.4.** When \(r = 1\), we get

\[
\frac{\partial}{\partial \lambda} \delta_1(n|\alpha, pt, \lambda) = \frac{n}{\lambda} \delta_1(n|\alpha, pt, \lambda) - \frac{1}{\lambda \Gamma(pt)n!} \left[ \binom{\lambda}{\alpha^\beta} \right]^n H_{2,2}^{1,2} \left[ \binom{-n, 1, (1 - n\beta - pt, \beta)}{(1, 1), (1 - n\beta, \beta)} \right],
\]

with the initial condition given in \([4.19]\).

Next, we obtain the fractional pde in time variable \(t\) solved by FNBP distributions.

**Lemma 4.2.** The density \([2.5]\) of gamma process \(\Gamma(t) \sim G(\alpha, pt)\) satisfies the following fractional differential equation

\[
\partial_t^\gamma g(y|\alpha, pt) = p\partial_t^{-1} (\log(\alpha y) - \psi(pt)) g(y|\alpha, pt), \quad y > 0,
\]

\[
g(y|\alpha, 0) = 0,
\]

where \(\psi(x) := \Gamma'(x)/\Gamma(x)\) is the digamma function and \(\partial_t^\gamma (\cdot)\) is R-L derivative defined in \([2.10]\).
Proof. Note first that (see [25, eq. (3.6)])

\[ \frac{1}{\Gamma(t)} = \frac{1}{2\pi i} \int_{C} e^{z}e^{-t}dz, \]

where \( C \) is the Hankel contour given below:

![Fig. Hankel Contour](image)

Also,

\[ \int_{0}^{t} \frac{\left(\alpha y/z\right)^{ps}}{(t-s)^{\nu+1-m}}ds = \left(\frac{\alpha y}{z}\right)^{pt}\left(p \log \frac{\alpha y}{z}\right)^{\nu-m} \{ \Gamma(m-\nu) - \Gamma(m-\nu, pt \log \frac{\alpha y}{z}) \} \]

and

\[ \frac{d}{dt} \Gamma(m-\nu, pt \log \frac{\alpha y}{z}) = -p \left(\frac{\alpha y}{z}\right)^{pt} \log \frac{\alpha y}{z} \left(p \log \frac{\alpha y}{z}\right)^{m-1-\nu}, \]

which can be checked using Mathematica 8.0. Now, by definition,

\[
\partial_n g(y|\alpha, pt) = \frac{1}{\Gamma(m-\nu)} \frac{d^{m-1}}{dt^{m-1}} \left[ \int_{0}^{t} \frac{\left(\alpha y/z\right)^{ps}}{(t-s)^{\nu+1-m}}ds \right] \cdot \frac{e^{z}e^{-ay}}{\Gamma(ps)(t-s)^{\nu+1-m}}dz ds \\
= \frac{(ye^{ay})^{-1}}{\Gamma(m-\nu)} \frac{d^{m-1}}{dt^{m-1}} \left[ \int_{0}^{t} \frac{\left(\alpha y/z\right)^{ps}}{(t-s)^{\nu+1-m}} \frac{1}{2\pi i} \int_{C} e^{z}e^{-ps}dz ds \right] \\
= \frac{(ye^{ay})^{-1}}{\Gamma(m-\nu)} \frac{d^{m-1}}{dt^{m-1}} \left[ \frac{1}{2\pi i} \int_{C} e^{z} \left(\frac{\alpha y}{z}\right)^{pt} \left(p \log \frac{\alpha y}{z}\right)^{\nu-m} \right. \\
\left. \times \{ \Gamma(m-\nu) - \Gamma(m-\nu, pt \log \frac{\alpha y}{z}) \} \right] dz \\
= p \frac{(ye^{ay})^{-1}}{\Gamma(m-\nu)} \frac{d^{m-1}}{dt^{m-1}} \left[ \frac{1}{2\pi i} \int_{C} e^{z} \left(\frac{\alpha y}{z}\right)^{pt} \log \frac{\alpha y}{z} \left(p \log \frac{\alpha y}{z}\right)^{\nu-m} \right] \]
Corollary 4.1. The density (2.5) of gamma process $\Gamma(\nu, \beta)$ in time variable $t > \delta$, where

\[ \frac{p}{\Gamma(m - \nu)} \frac{d^{m-1}}{dt^{m-1}} \frac{1}{2\pi i} \int_C e^{\alpha y} \left( \frac{\alpha y}{z} \right)^{pt} \left( \log \left( \frac{\alpha y}{z} \right) \right)^{m-1-\nu} \]

\[ \times \left\{ -p \left( \frac{\alpha y}{z} \right)^{-\nu} \log \frac{\alpha y}{z} \left( \log \left( \frac{\alpha y}{z} \right) \right)^{m-1-\nu} \right\} \, dz \]  

(from (4.25))

\[ = \frac{p}{\Gamma(m - \nu)} \frac{d^{m-1}}{dt^{m-1}} \frac{1}{2\pi i} \int_C e^{\alpha y} \left( \frac{\alpha y}{z} \right)^{pt} \left( \log \left( \frac{\alpha y}{z} \right) \right)^{m-1-\nu} \]

\[ \times \left\{ \Gamma(m - \nu) - \Gamma \left( m - \nu, pt \log \frac{\alpha y}{z} \right) \right\} \, dz + \frac{p}{\Gamma(m - \nu)} \frac{d^{m-1}}{dt^{m-1}} \frac{1}{2\pi i} \int_C e^{\alpha y} \, dz \]

\[ = p \log(\alpha y) \frac{d^{m-1}}{dt^{m-1}} \frac{1}{2\pi i} \int_C e^{\alpha y} \left( \frac{\alpha y}{z} \right)^{pt} \left( \log \left( \frac{\alpha y}{z} \right) \right)^{m-1-\nu} \]

\[ \times \left\{ \Gamma(m - \nu) - \Gamma \left( m - \nu, pt \log \frac{\alpha y}{z} \right) \right\} \, dz \]

\[ = p \log(\alpha y) \frac{d^{m-1}}{dt^{m-1}} \frac{1}{2\pi i} \int_C e^{\alpha y} \left( \frac{\alpha y}{z} \right)^{pt} \log(z) \left( \log \left( \frac{\alpha y}{z} \right) \right)^{m-1-\nu} \]

\[ \times \left\{ \Gamma(m - \nu) - \Gamma \left( m - \nu, pt \log \frac{\alpha y}{z} \right) \right\} \, dz \quad \therefore \int_C e^{\alpha y} \, dz = 0 \]

\[ = p \log(\alpha y) \frac{d^{m-1}}{dt^{m-1}} \frac{1}{2\pi i} \int_C e^{\alpha y} \left( \frac{\alpha y}{z} \right)^{pt} \log(z) \left( \log \left( \frac{\alpha y}{z} \right) \right)^{m-1-\nu} \]

\[ \times \left\{ \Gamma(m - \nu) - \Gamma \left( m - \nu, pt \log \frac{\alpha y}{z} \right) \right\} \, dz \]

which proves the result.

The following corollary corresponds to the case $\nu = 1$.

**Corollary 4.1.** The density (2.5) of gamma process $\Gamma(t) \sim G(\alpha, pt)$ solves the following pde, in time variable $t > 0$ (with $g(y|\alpha,0) = 0$ ),

\[ \frac{\partial}{\partial t} g(y|\alpha, pt) = p \left( \log(\alpha y) - \psi(pt) \right) g(y|\alpha, pt), \; y > 0. \]

**Theorem 4.5.** The pmf (1.2) of FNBP solves the following fractional pde in time variable $t > 0$:

\[ \frac{1}{p} \partial_t^\nu \delta_\nu(n|\alpha, pt, \lambda) = \partial_t^{\nu-1} \left( \log(\alpha) - \psi(pt) \right) \delta_\nu(n|\alpha, pt, \lambda) + \int_0^\infty p_s(n|y, \lambda) \log(y) \partial_t^{\nu-1} g(y|\alpha, pt) \, dy, \]

where $\delta_\nu(n|\alpha, 0, \lambda) = 1$ if $n = 0$ and zero otherwise.

**Proof.** Let $m - 1 < \nu < m$, where $m$ is a positive integer. Then

\[ \delta_\nu(n|\alpha, pt, \lambda) = \partial_t^\nu \int_0^\infty p_s(n|y, \lambda) g(y|\alpha, pt) \, dy \]
we have as follows: The change in order of integration in (4.26) can be justified, using Fubini-Tonelli theorem,

Further, using (5.1), we have

for \( n \geq 0 \), with \( \eta \). Since the polynomial process satisfies by an independent gamma random variable \( \Gamma \sim G(\alpha, p) \) with density \( g(x|\alpha, p) \) given in (2.5). Then the pmf \( \eta(n|\alpha, p) = \mathbb{P}(W^T(t) = n) \) of the Polya process is given by

\[
\eta(n|\alpha, p) = \int_0^{\infty} p(n|t, x) g(x|\alpha, p) dx = \frac{t^n \Gamma(n + p)}{n! \Gamma(p)} \frac{\alpha^p}{(t + \alpha)^{p+n}},
\]

which is the pmf of \( \text{NB}(p, \frac{t}{\alpha + t}) \). Since the pmf \( p(n|t, \lambda) \) of the Poisson process satisfies

we have

Further, using (5.1), we have

\[
\int_0^{\infty} xp(n|t, x) g(x|\alpha, p) dx = \frac{n + p}{t + \alpha} \eta(n|t, \alpha, p) \quad ; \quad n \in \mathbb{Z}_+.
\]

Substituting (5.3) in (5.2), we obtain

for \( n \geq 0 \), with \( \eta(n|t, \alpha, p) = 0 \) for \( n < 0 \), is the underlying difference-differential equation satisfied by the Polya process.

Observe that the negative binomial process \( \{Q(t, \lambda)\}_{t \geq 0} \) is a Lévy process (see [13, 9]) so
that it has independent increments. However, the Polya process \( \{W^T(t)\}_{t \geq 0} \) is not a Lévy process, as it does not have independent increments (see Remark 5.2).

5.2. **Space fractional Polya process.** The space fractional Poisson process \( \{\tilde{N}_\beta(t, \lambda)\}_{t \geq 0} \), which is a generalization of the Poisson process \( \{N(t, \lambda)\}_{t \geq 0} \), defined in [22], can be viewed as (see [22, Remark 2.3])

\[
\tilde{N}_\beta(t, \lambda) \overset{d}{=} N(D_\beta(t), \lambda),
\]

where \( D_\beta(t) \) is the \( \beta \)-stable subordinator \( 0 < \beta < 1 \). The pmf of space fractional Poisson process is given by (see [22, Theorem 2.2])

\[
(5.4) \quad \tilde{p}_\beta(n|t, \lambda) = \mathbb{P}[\tilde{N}_\beta(t, \lambda) = n] = \frac{(-1)^n}{n!} \sum_{k=0}^{\infty} \frac{(-\lambda^\beta t)^k}{k!} \frac{\Gamma(\beta k + 1)}{\Gamma(\beta k + 1 - n)}, \quad t \geq 0, \ k \geq 0.
\]

It solves the following fractional difference-differential equation ([22, eq. (2.4)]).

\[
(5.5) \quad \frac{\partial}{\partial t} \tilde{p}_\beta(n|t, \lambda) = -\lambda^\beta (1 - B_n)^{\beta} \tilde{p}_\beta(n|t, \lambda), \quad \beta \in (0, 1],
\]

\[
\tilde{p}_\beta(n|0, \lambda, \beta) = \begin{cases} 
1, & \text{for } n = 0, \\
0, & \text{for } n > 0,
\end{cases}
\]

where \( B_x \) is the backward shift operator \( B_x u(x, t) = u(x - 1, t) \).

We replace \( \lambda \) in \( \tilde{N}_\beta(t, \lambda) \) by a gamma random variable \( \Gamma \sim G(\alpha, p) \), independent of \( \{D_\beta(t)\}_{t \geq 0} \) and \( N \), to obtain

\[
\tilde{W}_\beta^T(t) := \tilde{N}_\beta(t, \Gamma) = N(D_\beta(t), \Gamma),
\]

and call the process \( \{\tilde{W}_\beta^T(t)\}_{t \geq 0} \) the space fractional Polya process (SFPP).

**Theorem 5.1.** Let \( 0 < \beta \leq 1 \). The one-dimensional distributions of SFPP are

\[
(5.6) \quad \tilde{\eta}_\beta(n|t, \alpha, p) = \mathbb{P}[\tilde{W}_\beta^T(t) = n] = \frac{1}{\Gamma(p)} \frac{(-1)^n}{n!} \psi_1 \left[ \frac{t}{\alpha^\beta} \right] \left( 1, \beta \right), \quad (p, \beta) \quad \text{for } n \in \mathbb{Z}_+.
\]

**Proof.** Observe that

\[
(5.7) \quad \tilde{\eta}_\beta(n|t, \alpha, p) = \mathbb{E}[\mathbb{P}[\tilde{W}_\beta^T(t) = n]|\Gamma] = \int_0^\infty \tilde{p}_\beta(n|t, y) g(y|\alpha, p) dy
\]

\[
= \int_0^\infty \frac{(-1)^n}{n!} \sum_{k=0}^{\infty} \frac{(-y^\beta)^k}{k!} \frac{\Gamma(\beta k + 1)}{\Gamma(\beta k + 1 - n)} \frac{\alpha^p}{\Gamma(p)} y^{p-1} e^{-\alpha y} dy
\]

\[
= \frac{(-1)^n}{n!} \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \frac{\Gamma(\beta k + 1)}{\Gamma(\beta k + 1 - n)} \frac{\alpha^p}{\Gamma(p)} \int_0^\infty y^{\beta k+p-1} e^{-\alpha y} dy
\]

\[
= \frac{1}{\Gamma(p)} \frac{(-1)^n}{n!} \sum_{k=0}^{\infty} \frac{\Gamma(\beta k + 1)\Gamma(\beta k + p)}{\Gamma(\beta k + 1 - n)} \frac{(-t/\alpha^\beta)^k}{k!}
\]
\[
= \frac{1}{\Gamma(p)} \frac{(-1)^n}{n!} z^{\psi_1} \left[ -\frac{t}{\alpha^\beta} \Gamma(1-n, \beta) (1-n, \beta) \right].
\]

When \( \beta = 1 \), we get
\[
\overline{\eta}_1(n|t, \alpha, p) = \frac{1}{\Gamma(p)} \frac{(-1)^n}{n!} \sum_{k=0}^{\infty} \frac{\Gamma(k+1)\Gamma(k+p)(-t/\alpha)^k}{(k-n)! k!}
\]
\[
= \frac{1}{\Gamma(p)} \frac{(-1)^n}{n!} \sum_{k=0}^{\infty} \frac{\Gamma(k+p)}{(k-n)!} \left( \frac{-t}{\alpha} \right)^k \text{ (put } j = k - n) 
\]
\[
= \frac{1}{\Gamma(p)} \frac{(-1)^n}{n!} \sum_{n+j=0}^{\infty} \frac{\Gamma(n+j+p)}{j!} \left( \frac{-t}{\alpha} \right)^j 
= \frac{\Gamma(n+p)}{\Gamma(p)\alpha^n n!} \sum_{j=0}^{\infty} \frac{\Gamma(n+j+p)}{\Gamma(j+1)\Gamma(n+p)} \left( \frac{-t}{\alpha} \right)^j 
= \frac{t^n \Gamma(n+p)}{n! \Gamma(p)} \frac{\alpha^p}{(t+\alpha)^{p+n}},
\]
which is the pmf of NB\((p, \frac{t}{\alpha+t})\), as expected.

**Remark 5.1.** The time fractional generalization of Polya process, namely \( \{N(E_{\beta}(t), \Gamma)\}_{t \geq 0} \) seems not possible. For, the pmf of time fractional Polya process is given by
\[
P(N_{\beta}(t, \Gamma) = n) = \int_0^{\infty} p_{\beta}(n|t, x) g(x|\alpha, p) dx 
= \frac{t^\beta n}{\Gamma(p)n!\alpha^n} \sum_{k=0}^{\infty} \frac{\Gamma(n+1+k)\Gamma(n+p+k)}{\Gamma(\beta n+1+\beta k)\Gamma(1+k)} \left( \frac{-t^\beta}{\alpha} \right)^k 
\]
\[
= \frac{t^\beta n}{\Gamma(p)n!\alpha^n} \sum_{k=0}^{\infty} c_k,
\]
where
\[
c_k = \frac{\Gamma(n+1+k)\Gamma(n+p+k)}{\Gamma(\beta n+1+\beta k)\Gamma(1+k)} \left( \frac{-t^\beta}{\alpha} \right)^k.
\]
Using the Stirling approximation formula (for large \( z \)),
\[
\Gamma(z) \sim (2\pi)^{1/2} z^{1/2} e^{-z},
\]
we get, for large \( k \),
\[
c_k \sim \frac{\sqrt{2\pi} e^{-(n+1)(k/e)^k k^{n+1/2}} \sqrt{2\pi} e^{-(n+p)(k/e)^k k^{n+p-1/2}}}{2\pi^{1/2} \beta^{n+1/2+\beta k} e^{-(1+\beta n)(k/e)^{\beta k} k^{\beta n+1/2}} 2\pi^{1/2} (k/e)^{k^{1/2}} \left( \frac{-t^\beta}{\alpha} \right)^k 
= Ak^{(2n+p+\beta n-1/2)(k/e)^k (1-\beta) \beta - \beta k} \left( \frac{-t^\beta}{\alpha} \right)^k ,
\]
where \( A = e^{-(2n+p+\beta n)} \beta^{-(\beta n+1/2)} \). Applying the root test for the series in (5.8), we get
\[
\lim_{k \to \infty} \sqrt[k]{|c_k|} = \lim_{k \to \infty} |A^{1/k}(k^{1/k})^{(2n+p+\beta n-1/2)} \beta^{-\beta}(k/e)^{(1-\beta)(t/\alpha)}| = \infty,
\]
since \( 1 - \beta > 0 \). Hence, the series in (5.8) diverges and so the pmf of \( N(E_{\beta}(t), \Gamma) \) does not exist.
Theorem 5.2. The SFPP $\{\tilde{W}_\beta^\Gamma(t)\}_{t \geq 0}$ has stationary increments and is stochastically continuous.

Proof. Consider first the Polya process $\{W^\Gamma(t)\}_{t \geq 0}$.

(i) Stationary increments: Let $B$ be a Borel set. Then for $t \geq 0, s > 0$,
\[
\mathbb{P}[W^\Gamma(t + s) - W^\Gamma(s) \in B] = \mathbb{E}[\mathbb{P}[N(t + s, \Gamma) - N(s, \Gamma) \in B] | \Gamma] = \mathbb{E}[\mathbb{P}[N(t, \Gamma) \in B] | \Gamma] = \mathbb{P}[W^\Gamma(t) \in B],
\]
showing that $\{W^\Gamma(t)\}$ has stationary increments. It is known that the time change of a process with stationary increments by a process with stationary increments has stationary increments (see [1, Theorem 1.3.25]). Since $\{D_\beta(t)\}_{t \geq 0}$ has stationary increments, $\{\tilde{W}_\beta^\Gamma(t)\}_{t \geq 0}$ also has stationary increments.

(ii) Stochastic continuity: Note first that for any stationary processes $\{X(t)\}_{t \geq 0}$,
\[
\lim_{t \to s} \mathbb{P}[\|X(t) - X(s)\| > a] = 0 \Rightarrow \lim_{t \to 0} \mathbb{P}[\|X(t)\| > a] = 0, \quad \text{for } a > 0.
\]
Since the Poisson process $\{N(t, \lambda)\}_{t \geq 0}$ is stochastically continuous, we have for $a > 0$ and given $\epsilon > 0$, $\exists a \delta > 0$ such that $\mathbb{P}[\|N(t, \lambda)\| > a] < \epsilon, \forall t \in (0, \delta)$. Now, suppose $W^\Gamma(t) = N(t, \Gamma)$ is not stochastically continuous. Then $\exists a t_0 \in (0, \delta)$ such that $\mathbb{P}[\|W^\Gamma(t_0)\| > a] \geq \epsilon$. Again,
\[
\mathbb{P}[\|W^\Gamma(t_0)\| > a] = \mathbb{E}[\mathbb{P}[\|N(t_0, \Gamma)\| > a] | \Gamma], \quad \text{for } t_0 \in (0, \delta)
\]
\[
= \int_0^\infty \mathbb{P}[\|N(t_0, \lambda)\| > a] g(\lambda | \alpha, \delta) d\lambda, \quad \text{for } t_0 \in (0, \delta)
\]
\[
< \epsilon \int_0^\infty g(\lambda | \alpha, \delta) d\lambda \leq \epsilon.
\]
which is a contradiction. Hence, $\{W^\Gamma(t)\}$ is stochastically continuous. Also, by the similar conditioning arguments, it follows that $\{\tilde{W}_\beta^\Gamma(t)\}_{t \geq 0}$ is stochastically continuous. □

Remark 5.2. The Polya process $\{W^\Gamma(t)\}_{t \geq 0}$ and the SFPP $\{\tilde{W}_\beta^\Gamma(t)\}_{t \geq 0}$ are not Lévy processes, since they do not have independent increments. To see this, let $0 \leq t_1 < t_2 < t_3 < \infty$ and $B_1, B_2$ be Borel sets. Then
\[
\mathbb{P}[W^\Gamma(t_2) - W^\Gamma(t_1) \in B_1, W^\Gamma(t_3) - W^\Gamma(t_2) \in B_2] = \mathbb{E}[\mathbb{P}[N(t_3, \Gamma) - N(t_2, \Gamma) \in B_1, N(t_2, \Gamma) - N(t_1, \Gamma) \in B_2] | \Gamma]
\]
(5.9)
\[
\neq \mathbb{E}[\mathbb{P}[N(t_3, \Gamma) - N(t_2, \Gamma) \in B_1] \mathbb{E}[\mathbb{P}[N(t_2, \Gamma) - N(t_1, \Gamma) \in B_2] | \Gamma] \mathbb{E}[\mathbb{P}[N(t_2, \Gamma) - N(t_1, \Gamma) \in B_1] | \Gamma].
\]
(5.10)
Also, it is straightforward to check that the rhs of (5.9) $\neq$ rhs of (5.10). For example, when $t_1 = 1, t_2 = 2, t_3 = 3$, we get the rhs of (5.9) and (5.10) as
\[
\frac{1}{n!m! \Gamma(p)} \frac{\alpha^p}{\Gamma(n + m + p)} \Gamma(n + m + p) (\alpha + 2)^{n+m+p} \quad \text{and}
\]
Now repeating above computation \(k\) times, we get the desired result.

**Remark 5.3.** The mean \(\mathbb{E}(\tilde{W}_\beta^T(t))\) is infinite, which can be seen as follows. Consider the pgf of \(\tilde{W}_\beta^T(t)\) given by

\[
\mathbb{E}[u^{\tilde{W}_\beta^T(t)}] = \int_0^\infty G_\beta(u, t, \lambda) g(\lambda|\alpha, p) d\lambda,
\]

where \(G_\beta(u, t, \lambda) = \mathbb{E}[u^{\tilde{N}_\beta(t, \lambda)}] = e^{-\lambda^\beta t(1-u)}\), \(|u| \leq 1\) (see [22, eq. (2.12)]). Now differentiate with respect to \(u\), and let \(u \rightarrow 1\) to obtain infinity.

### 5.3. Connections to pde’s

We here discuss some pde connections associated with the distributions of the SFPP.

First, we establish a result for the process \(\{\tilde{W}_\beta^T(t)\}_{t \geq 0}\), similar to (5.5).

**Theorem 5.3.** The pmf \((5.6)\) satisfies the following pde in time variable \(t\):

\[
\frac{\partial^k}{\partial t^k} \tilde{\eta}_\beta(n|t, \alpha, p) = \left(1 - \frac{1 - B_n}{\alpha^\beta \Gamma(p)}\right)^k \tilde{\eta}_\beta(n|t, \alpha, p + k\beta)
\]

with \(\tilde{\eta}_\beta(n|0, \alpha, p) = 1\) if \(n = 0\) and zero otherwise.

**Proof.** Note from (5.7),

\[
\frac{\partial}{\partial t} \tilde{\eta}_\beta(n|t, \alpha, p) = \frac{\partial}{\partial t} \int_0^\infty \tilde{p}_\beta(n|t, y) g(y|\alpha, p) dy = \int_0^\infty \frac{\partial}{\partial t} \tilde{p}_\beta(n|t, y) g(y|\alpha, p) dy
\]

\[= \int_0^\infty -y^\beta (1 - B_n)^\beta \tilde{p}_\beta(n|t, y) g(y|\alpha, p) dy \text{ (Using (5.5))}
\]

\[= -(1 - B_n)^\beta \int_0^\infty y^\beta \tilde{p}_\beta(n|t, y) g(y|\alpha, p) dy \text{ (Using (5.13))}
\]

\[
(5.12)
\]

The last step is due to the fact

\[y^\beta g(y|\alpha, p) = \frac{\Gamma(p+\beta)}{\alpha^\beta \Gamma(p)} g(y|\alpha, p+\beta).
\]

Now repeating above computation \(k\)-times, we get the desired result. \(\square\)

**Corollary 5.1.** The pgf \(G_\beta(u, t, \alpha, p) = \mathbb{E}[u^{\tilde{W}_\beta^T(t)}], \ |u| \leq 1,\) satisfies the following \(k\)-th order pde:

\[
\frac{\partial^k}{\partial t^k} G_\beta(u|t, \alpha, p) = \left(1 - \frac{1 - u}{\alpha^\beta \Gamma(p)}\right)^k G_\beta(u|t, \alpha, p + k\beta),
\]

where \(G_\beta(u, 0, \alpha, p) = 1\), and \(k \in \mathbb{Z}_+ \setminus \{0\}\).
\textbf{Proof.} Note that

\[(1 - B_n)^\beta = \sum_{r=0}^{\infty} \frac{\Gamma(\beta + 1)}{\Gamma(j + 1)\Gamma(\beta - j + 1)} (-1)^r B_n^r.\]

From (5.11),

\[
\frac{\partial}{\partial t} G_\beta(u, t, \alpha, p) = \frac{\partial}{\partial t} \sum_{n=0}^{\infty} u^n \tilde{\eta}_\beta(n|t, \alpha, p) = \sum_{n=0}^{\infty} u^n \left( - (1 - B_n)^\beta \frac{\Gamma(p + \beta)}{\alpha^\beta \Gamma(p)} \right) \tilde{\eta}_\beta(n|t, \alpha, p + \beta)
\]

\[
= - \frac{\Gamma(p + \beta)}{\alpha^\beta \Gamma(p)} \sum_{n=0}^{\infty} u^n (1 - B_n)^\beta \tilde{\eta}_\beta(n|t, \alpha, p + \beta)
\]

\[
= - \frac{\Gamma(p + \beta)}{\alpha^\beta \Gamma(p)} \sum_{n=0}^{\infty} u^n \sum_{r=0}^{\infty} \frac{\Gamma(\beta + 1)}{\Gamma(j + 1)\Gamma(\beta - j + 1)} (-1)^r B_n^r \tilde{\eta}_\beta(n|t, \alpha, p + \beta)
\]

\[
= - \frac{\Gamma(p + \beta)}{\alpha^\beta \Gamma(p)} \sum_{r=0}^{\infty} \frac{\Gamma(\beta + 1)}{\Gamma(j + 1)\Gamma(\beta - j + 1)} (-1)^r \sum_{n=r}^{\infty} u^n \tilde{\eta}_\beta(n - r|t, \alpha, p + \beta)
\]

\[
= - (1 - u)^\beta \frac{\Gamma(p + \beta)}{\alpha^\beta \Gamma(p)} G_\beta(u|t, \alpha, p + \beta).
\]

Taking the derivative \(k\)-times, we get the result. \(\Box\)

Finally, we obtain the following result in variable \(p\).

\textbf{Theorem 5.4.} The pmf (5.6) satisfies the following fractional pde:

\[(5.15)\]

\[\partial_p^\nu \tilde{\eta}_\beta(n|t, \alpha, p) = \partial_p^{\nu - 1}(\log(\alpha) - \psi(p))\tilde{\eta}_\beta(n|t, \alpha, p) + \int_0^\infty \tilde{\eta}_\beta(n|t, \lambda) \log(\lambda) \partial_p^{\nu - 1} g(\lambda|\alpha, p) d\lambda,
\]

where \(\tilde{\eta}_\beta(n|0, \alpha, p) = 1\) if \(n = 0\) and zero otherwise.

\textbf{Proof.} Note that

\[\partial_p^\nu \tilde{\eta}_\beta(n|t, \alpha, p) = \partial_p^\nu \int_0^\infty \tilde{\eta}_\beta(n|t, \lambda) g(\lambda|\alpha, p) d\lambda = \int_0^\infty \tilde{\eta}_\beta(n|t, \lambda) \partial_p^\nu g(\lambda|\alpha, p) d\lambda,
\]

and the proof now follows from Lemma 4.2. \(\Box\)

\textbf{Acknowledgements.} The authors are grateful to the referees for their detailed report and numerous critical comments which improved the paper significantly, both in the content and the quality of the paper.
REFERENCES

[1] D. Applebaum. *Lévy Processes and Stochastic Calculus*. Second. Vol. 116. Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press, 2009, pp. xxx+460. ISBN: 978-0-521-73865-1.

[2] L. Beghin and E. Orsingher. “Fractional Poisson processes and related planar random motions.” In: *Electron. J. Probab.* 14 (2009), no. 61, 1790–1827.

[3] L. Beghin. “Fractional gamma processes and fractional gamma-subordinated processes.” In: *Submitted. arXiv:1305.1753 [math.PR]* (2013).

[4] L. Beghin and C. Macci. “Fractional discrete processes: compound and mixed Poisson representations.” In: *J. Appl. Probab.* 51.1 (Mar. 2014), pp. 9–36.

[5] R. Biard and B. Saussereau. “Fractional Poisson process: long-range dependence, applications in ruin theory.” In: *J. Appl. Probab. forthcoming* (2014).

[6] N. H. Bingham. “Limit theorems for occupation times of Markov processes.” In: *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* 17 (1971), pp. 1–22.

[7] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi. *Higher Transcendental Functions. Vol. I*. Based on notes left by Harry Bateman, With a preface by Mina Rees, With a foreword by E. C. Watson, Reprint of the 1953 original. Melbourne, Fla.: Robert E. Krieger Publishing Co. Inc., 1981, pp. xiii+302. ISBN: 0-89874-069-X.

[8] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi. *Higher Transcendental Functions. Vol. III*. Based, in part, on notes left by Harry Bateman. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955, pp. xvii+292.

[9] W. Feller. *An Introduction to Probability Theory and its Applications. Vol. II.* Second edition. New York: John Wiley & Sons Inc., 1971, pp. xxiv+669.

[10] R. Gorenflo and F. Mainardi. “On the fractional Poisson process and the discretized stable subordinator.” In: *Submitted. arXiv:1305.3074 [math.PR]* (2013).

[11] A. A. Kilbas, M. Saigo, and J. J. Trujillo. “On the generalized Wright function.” In: *Fract. Calc. Appl. Anal. 5.4* (2002). Dedicated to the 60th anniversary of Prof. Francesco Mainardi, pp. 437–460.

[12] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo. *Theory and applications of fractional differential equations*. Vol. 204. North-Holland Mathematics Studies. Elsevier Science B.V., Amsterdam, 2006, pp. xvi+523. ISBN: 978-0-444-51832-3; 0-444-51832-0.

[13] T. J. Kozubowski and K. Podgórski. “Distributional properties of the negative binomial Lévy process.” In: *Probab. Math. Statist.* 29.1 (2009), pp. 43–71.

[14] N. Laskin. “Fractional Poisson process.” In: *Commun. Nonlinear Sci. Numer. Simul. 8.3-4* (2003). Chaotic transport and complexity in classical and quantum dynamics, pp. 201–213.

[15] N. Laskin. “Some applications of the fractional Poisson probability distribution.” In: *J. Math. Phys.* 50.11 (2009), pp. 113513, 12.

[16] N. N. Leonenko, M. M. Meerschaert, R. L. Schilling, and A. Sikorskii. “Correlation structure of time-changed Lévy processes.” In: *Commun. Appl. Ind. Math.* (2014).

[17] F. Mainardi. *Fractional Calculus and Waves in Linear Viscoelasticity*. An introduction to mathematical models. London: Imperial College Press, 2010, pp. xx+347. ISBN: 978-1-84816-329-4; 1-84816-329-0.

[18] F. Mainardi, R. Gorenflo, and E. Scalas. “A fractional generalization of the Poisson processes.” In: *Vietnam J. Math. 32.* Special Issue (2004), pp. 53–64.

[19] A. M. Mathai, R. K. Saxena, and H. J. Haubold. *The H-function*. Theory and applications. New York: Springer, 2010, pp. xiv+266. ISBN: 978-1-4419-0915-2.

[20] M. M. Meerschaert and P. Straka. “Inverse stable subordinators.” In: *Math. Model. Nat. Phenom. 8.2* (2013), pp. 1–16.

[21] M. M. Meerschaert, E. Nane, and P. Vellaisamy. “The fractional Poisson process and the inverse stable subordinator.” In: *Electron. J. Probab.* 16 (2011), no. 59, 1600–1620.

[22] E. Orsingher and F. Polito. “The space-fractional Poisson process.” In: *Statist. Probab. Lett.* 82.4 (2012), pp. 852–858.

[23] S. G. Samko, A. A. Kilbas, and O. I. Marichev. *Fractional integrals and derivatives*. Theory and applications, Edited and with a foreword by S. M. Nikol’skii, Translated from the 1987 Russian original, Revised by the authors. Gordon and Breach Science Publishers, Yverdon, 1993, pp. xxvi+976. ISBN: 2-88124-864-0.

[24] K.-i. Sato. *Lévy Processes and Infinitely Divisible Distributions*. Vol. 68. Cambridge Studies in Advanced Mathematics. Translated from the 1990 Japanese original, Revised by the author. Cambridge: Cambridge University Press, 1999, pp. xii+486. ISBN: 0-521-55302-4.

[25] N. M. Temme. *Special functions*. A Wiley-Interscience Publication. An introduction to the classical functions of mathematical physics. New York: John Wiley & Sons Inc., 1996, pp. xiv+374. ISBN: 0-471-11313-1.
[26] P. Vellaisamy and A. Kumar. “Hitting times of an inverse Gaussian process.” In: Submitted. arXiv:1105.1468 [math.PR] (2013).

[27] P. Vellaisamy and M. Sreehari. “Some intrinsic properties of the gamma distribution.” In: J. Japan Statist. Soc. 40.1 (2010), pp. 133–144.

[28] E. M. Wright. “The asymptotic expansion of the generalized hypergeometric function.” In: Proc. London Math. Soc. (2) 46 (1940), pp. 389–408.