**Research Article**

**Computing Analysis for First Zagreb Connection Index and Coindex of Resultant Graphs**

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A numeric parameter which studies the behaviour, structural, toxicological, experimental, and physicochemical properties of chemical compounds under several graphs’ isomorphism is known as topological index. In 2018, Ali and Trinajstić studied the first Zagreb connection index \( ZC_1 \) to evaluate the value of a molecule. This concept was first studied by Gutman and Trinajstić in 1972 to find the solution of \( \pi \)-electron energy of alternant hydrocarbons. In this paper, the first Zagreb connection index and coindex are obtained in the form of exact formulae and upper bounds for the resultant graphs in terms of different indices of their factor graphs, where the resultant graphs are obtained by the product-related operations on graphs such as tensor product, strong product, symmetric difference, and disjunction. At the end, an analysis of the obtained results for the first Zagreb connection index and coindex on the aforesaid resultant graphs is interpreted with the help of numerical values and graphical depictions.

1. Introduction

Topological indices (TIs) are used in the study of cheminformatics which has key role in the formational structure of quantitative structures’ activity and property relationships to examine the chemical reactivity and experimental activity of a chemical compound in a molecular graph [1]. So, TIs predict both physical and chemical features that are defined in the molecular graphs such as surface tension, solubility, connectivity, freezing point, boiling point, melting point, critical temperature, polarizability, heat of evaporation, and formation [2]. In addition, the medical behaviours and a number of drug particles of different compounds have formed with the help of various TIs in the pharmaceutical networks (see [3]).

In particular, the TIs called by connection number-based Zagreb indices are used to compute the correlation values among various octane isomers such as heat of evaporation, acenitic factor, molecular weight, connectivity, critical temperature, stability, and density (see [4, 5]).

Product graphs play an essential part to develop new molecular graphs from simple graphs. For this purpose, Ashrafi et al. [6] defined the concept of coindices for several products on graphs. Das et al. [7] computed upper bounds for multiplicative Zagreb indices of operations such as join, corona, Cartesian, disjunction, and composition. The reformulated, multiplicative, hyper, first, second, and third Zagreb coindices with certain properties are defined in [8–13]. Relations between Zagreb coindices and some distance-based indices are computed in [14]. For more details, see [15–18]. For this purpose, we can define some operations such as tensor product, strong product, symmetric difference, and disjunction in the following definitions.

**Definition 1.** Tensor (or Kronecker or conjunctive or direct) product of two graphs \( G_1 \) and \( G_2 \) is a graph \( G_1 \otimes G_2 \) with vertex set \( V(G_1 \otimes G_2) = V(G_1) \otimes V(G_2) \) and edge set \( E(G_1 \otimes G_2) = \{(a_1, b_1)(a_2, b_2), \text{ where } (a_1, b_1), (a_2, b_2) \in V(G_1) \otimes V(G_2)\} \) and

\[
[a_1 a_2 \in E(G_1)] \land [b_1 b_2 \in E(G_2)].
\]
The connection number of a vertex \((a, b)\) of \(G_1 \otimes G_2\) is defined by
\[
\tau_{G_1 \otimes G_2}(a, b) = \tau_{G_1}(a) + \tau_{G_2}(b) + \tau_{G_1}(a)\tau_{G_2}(b).
\]
For more details, see Figure 1.

**Definition 2.** Strong product or normal product \((G_1 \otimes G_2)\) of two graphs \(G_1\) and \(G_2\) is obtained by taking the vertex set and edge set as \(V(Q_1 \otimes Q_2) = V(Q_1) \otimes V(Q_2)\) and \(E(Q_1 \otimes Q_2) = \{(a_1, b_1), (a_2, b_2)\}, \) where \((a_1, b_1), (a_2, b_2) \in V(Q_1) \otimes V(Q_2)\) with condition
\[
\begin{align*}
&\text{either } [a_1 = a_2 \in V(G_1) \land b_1 b_2 \in E(Q_2)] \\
&\text{or } [a_1 a_2 \in E(Q_1) \land b_1 b_2 \in E(Q_2)].
\end{align*}
\]
The connection number of a vertex \((a, b)\) of \(Q_1 \otimes Q_2\) is defined by
\[
\tau_{G_1 \otimes G_2}(a, b) = \tau_{G_1}(a) + \tau_{G_2}(b) + d_{G_1}(a)d_{G_2}(b)
+ d_{G_1}(a)d_{G_2}(b)^{\tau_{G_2}(a)}
+ \tau_{G_1}(a)d_{G_2}(b)
+ \tau_{G_1}(a)d_{G_2}(b)^{\tau_{G_2}(a)},
\]
if \(n_1, n_2 \geq 4\). For more details, see Figure 2.

**Definition 3.** Symmetric difference of two graphs \(Q_1\) and \(Q_2\) is a graph \(G_1 \oplus G_2\) with vertex set \(V(Q_1 \oplus Q_2) = V(Q_1) \oplus V(Q_2)\) and edge set \(E(Q_1 \oplus Q_2) = \{a_1, b_1\}, (a_2, b_2),\) where \((a_1, b_1), (a_2, b_2) \in V(Q_1) \oplus V(Q_2)\) and
\[
\begin{align*}
&[a_1 a_2 \in E(Q_1)] \\
&\text{or } [b_1 b_2 \in E(Q_2)],
\end{align*}
\]
but not both hold at the same time, respectively.
The connection number of a vertex \((a, b)\) of \(G_1 \oplus G_2\) is defined by
\[
\tau_{G_1 \oplus G_2}(a, b) = n_1 n_2 - 1 - n_1 d_{G_1}(a) - n_1 d_{G_2}(b)
+ 2d_{G_1}(a)d_{G_2}(b)
= n_2[n_1 d_{G_1}(a) + d_{G_2}(b)2d_{G_1}(a) - n_1 - 1].
\]
For more details, see Figure 3.

**Definition 4.** Disjunction of two graphs \(G_1\) and \(G_2\) is a graph \(G_1 \circ G_2\) with vertex set \(V(Q_1 \circ Q_2) = V(Q_1) \circ V(Q_2)\) and edge set \(E(Q_1 \circ Q_2) = \{a_1, b_1\}, (a_2, b_2),\) where \((a_1, b_1), (a_2, b_2) \in V(Q_1) \circ V(Q_2)\) and
\[
\begin{align*}
&[a_1 a_2 \in E(Q_1)] \\
&\text{or } [b_1 b_2 \in E(Q_2)].
\end{align*}
\]
The connection number of a vertex \((a, b)\) of \(G_1 \circ G_2\) is defined by
\[
\tau_{G_1 \circ G_2}(a, b) = n_1 n_2 - 1 - n_1 d_{G_1}(a) - n_1 d_{G_2}(b)
+ d_{G_1}(a)d_{G_2}(b)
= n_2[n_1 d_{G_1}(a) + d_{G_2}(b)[d_{G_1}(a) - n_1] - 1.
\]

For more details, see Figure 4.

The graph theory provides the significant tools in the field of modern chemistry that is exploited to develop several types of molecular graphs and also predicts their chemical properties. In 1972, Gutman and Trinajstić [19] defined the first degree-based (number of vertices at distance one) TI called the first Zagreb index to compute the total π-electron energy of the molecules in molecular graphs. There are several TIs in literature, but degree-based TIs are studied more than others (see [20–29]).

Ali and Trinajstić [30] restudied the concept of connection number-based (number of vertices at distance two) TIs that were also defined by Gutman and Trinajstić in the same paper in 1972 to compute the total electron energy of the alternant hydrocarbons. They recalled them as Zagreb connection indices and reported that the chemical capability of the Zagreb connection indices is better than the classical Zagreb indices for the thirteen physicochemical properties of octane isomers. After two years, a few works have delivered on these connection number-based descriptors. Ali et al. [31] computed the analysis of Zagreb connection indices and coindices for some chemical structures of operations on graphs. For further studies and properties of the Zagreb connection indices, we refer to [32–40]. Recently, Gong et al. [41, 42] developed blocking lot-streaming flow shop scheduling problems and dynamic interval multi-objective optimization problems. These problems have been considered in various studies which have a close relation with the topic considered in this paper. For more details, see [43, 44].

In this paper, we compute the analysis for the first Zagreb connection index and coindex of the resultant graphs in the shape of exact formulae and upper bounds in terms of their factor graphs, where resultant graphs are obtained by operations such as tensor product, strong product, symmetric difference, and disjunction. Moreover, at the end, an analysis of the first Zagreb connection index and coindex on the aforesaid operations is included with the help of their numerical values and graphical depictions.

The rest of the paper is as follows: Section 2 represents the preliminary definitions and key results which are used in the main results, Section 3 contains the main results of product on graphs, and Section 4 includes the analysis and conclusions.

### 2. Preliminaries

Let \(G = (V(G), E(G))\) be a simple graph such that the order and size of \(G\) are \(|V(G)| = n\) and \(|E(G)| = e\). The distance \(d(a, b)\) between any two vertices \(a\) and \(b\) of a graph \(G\) is equal to the length of a shortest path connecting them. For \(b \in V(G)\) and a positive integer \(p, N_p(b/G) = \)
\[ u \in V(G) : d(a, b) = p \] denotes the open \( p \)-neighborhood of \( b \) in a graph \( G \), where \( d_p(b/G) = |N_p(b/G)| \) is called \( p \)-distance degree of a vertex \( b \) in any graph \( G \). The degree of a vertex \( b \) in a graph \( G \) is the number of edges incident on it, and it is denoted by \( d_G(b) \). In particular:

If \( p = 1 \), \( d_1(b/G) = |N_1(b/G)| = d_G(b) = \text{degree of } b \) (number of vertices at distance one from \( b \))

If \( p = 2 \), \( d_2(b/G) = |N_2(b/G)| = \tau_G(b) = \text{connection number of } b \) (number of vertices at distance two from \( b \))

The complement of a graph \( G \) is denoted by \( \overline{G} \). It is also simple with the same vertex set as of \( G \), but edge set is defined as \( E(\overline{G}) = \{ab : a, b \in V(G) \text{ and } ab \notin E(G)\} \). Thus, \( E(G) \cup E(\overline{G}) = E(K_n) \), where \( K_n \) is a complete graph of
order $n$ and size $|E(K_n)| = \binom{n}{2}$. Moreover, if $|E(G)| = e$, then $|E(G_e)| = \binom{n}{2} - e = \mu$ and $d_G(b) = n - 1 - d_G(b)$, where $d_G(b)$ and $d_G(b)$ are the degrees of the vertex $b$ in $G$ and $G_e$, respectively. Now, throughout the paper, for two graphs $G_1$ and $G_2$, we assume that $|V(G_1)| = n_1$, $|V(G_2)| = n_2$, $|E(G_1)| = e_1$, and $|E(G_2)| = e_2$. Finally, it is important to note that Zagreb connection coindices of $G$ are not Zagreb connection indices of $G$ because the connection number works according to $G$. For any terminology or notion which are not mentioned here, we refer to [45, 46].

**Definition 5.** For a graph $G$, the first Zagreb index ($M_1(G)$) and second Zagreb index ($M_2(G)$) are defined as

$$
M_1(G) = \sum_{ab \in E(G)} [d_G(a) + d_G(b)],
$$

$$
M_2(G) = \sum_{ab \in E(G)} [d_G(a) \times d_G(b)].
$$

These degree-based indices are defined by Gutman, Trinajstić, and Ruscic (see [19, 47]) which are frequently used to predict better outcomes of the various parameters related to the molecular networks such as chirality, complexity, entropy, heat energy, ZE-isomerism, heat capacity, absolute value of correlation coefficient, chromatographic, retention times in chromatographic, pH, and molar ratio [19, 48]. Symmetrical to these degree-based TIs, the connection-based TIs are discussed in Definitions 6 and 7.

**Definition 6.** For a graph $G$, the first Zagreb connection index ($ZC_1(G)$) is defined as

$$
ZC_1(G) = \sum_{b \in V(G)} [\tau_G(b)]^2.
$$

**Definition 7.** For a graph $G$, the modified first Zagreb connection index ($ZC'_1(G)$) and second Zagreb connection index ($ZC_2(G)$) are defined as

$$
ZC_1'(G) = \sum_{abc \in E(G)} [\tau_G(a) + \tau_G(b)],
$$

$$
ZC_2(G) = \sum_{abc \in E(G)} [\tau_G(a) \times \tau_G(b)].
$$

These coindices associated with the degree-based classical Zagreb indices are defined by Ashrafi et al. (see [6]). The coindices associated with the connection-based Zagreb indices are defined in Definition 9.

**Definition 8.** For a graph $G$, the first Zagreb coindex ($\overline{M}_1(G)$) and second Zagreb coindex ($\overline{M}_2(G)$) are defined as

$$
\overline{M}_1(G) = \sum_{ab \notin E(G)} [d_G(a) + d_G(b)],
$$

$$
\overline{M}_2(G) = \sum_{ab \notin E(G)} [d_G(a) \times d_G(b)].
$$

The modified coindices associated with the connection number-based Zagreb indices are defined in Definition 10. These modified coindices associated with the connection number-based Zagreb indices are defined by Ali et al. (see [49]).

**Definition 9.** For a graph $G$, the first Zagreb connection coindex ($\overline{ZC}_1(G)$) and second Zagreb connection coindex ($\overline{ZC}_2(G)$) are defined as

$$
\overline{ZC}_1(G) = \sum_{abc \notin E(G)} [\tau_G(a) + \tau_G(b)],
$$

$$
\overline{ZC}_2(G) = \sum_{abc \notin E(G)} [\tau_G(a) \times \tau_G(b)].
$$

**Figure 4:** (a) $G_1 \cong C_3$, (b) $G_2 \cong P_2$, and (c) disjunction ($C_3 \cdot P_2$).
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The degree/connection-based coindices defined in Definitions 8–10 study the various physicochemical and isomer properties of molecules on the basis of the adjacency and nonadjacency pairs of vertices in the molecular networks. For more details, see [6, 30, 31, 36].

Now, we present some important results which are used in the main results.

**Lemma 1** (see [50]). Let $G$ be a connected graph and $\overline{G}$ be its complement with $n$ vertices and $e$ edges. Then,

1. $\sum_{b \in V(G)} d_G(b) = 2e$
2. $M_G(\overline{G}) \geq M_G(G) - 4$

**Lemma 2** (see [36]). Let $G$ be a connected and $[C_1, C_4]$-free graph with $n$ vertices and $e$ edges. Then,

1. $\sum_{b \in V(G)} \tau_G(b) = M_G(G) - 2e_1 = A$
2. $\sum_{b \in V(G)} \tau_G(b) = M_G(G) - 2e_2 = B$

**3. Main Results**

This section contains the main results for the first Zagreb connection index ($ZC_1$) and first Zagreb connection coindex ($ZC_1^\perp$) of the product on graphs obtained under the operations of tensor product, strong product, symmetric difference, and disjunction.

**Theorem 1.** Let $G_1$ and $G_2$ be two connected and $[C_1, C_4]$-free graphs. Then, $ZC_1$ and $ZC_1^\perp$ of the tensor product $(G_1 \otimes G_2)$ are

\[
ZC_1(G_1 \otimes G_2) = n_2 ZC_1(G_1) + n_1 ZC_1(G_2) + ZC_1(G_1) ZC_1(G_2) + 2AB + 2AZC_1(G_2) + 2BZC_1(G_1),
\]

\[
ZC_1^\perp(G_1 \otimes G_2) = 2\mu_2 ZC_1(G_1) + 2\mu_1 ZC_1(G_2) + 2 \sum_{a \in V(G_1)} \sum_{b \in V(G_2)} \tau_G(a) \tau_G(b) + \tau_G(a) \tau_G(b),
\]

where $a \in V(G_1), b \in V(G_2),$ and $(a, b) \in G_1 \otimes G_2$.

**Proof.** Since $\tau_G(b) = \tau_G(a) + \tau_G(b) + \tau_G(a) \tau_G(b),$ where $a \in V(G_1), b \in V(G_2),$ and $(a, b) \in G_1 \otimes G_2$.

\[
ZC_1(G_1 \otimes G_2) = \sum_{(a, b) \in V(G_1) \otimes V(G_2)} \left[ \tau_G(a, b) \right]^2
\]

\[
= \sum_{a \in V(G_1)} \sum_{b \in V(G_2)} \left[ \tau_G(a) + \tau_G(b) + \tau_G(a) \tau_G(b) \right]^2
\]

\[
= \sum_{a \in V(G_1)} \sum_{b \in V(G_2)} \left[ \tau_G(a)^2 + \tau_G(b)^2 + \tau_G(a)^2 \tau_G(b) + 2\tau_G(a) + \tau_G(b) + 2\tau_G(a) \tau_G(b) \right]
\]

\[
= \sum_{a \in V(G_1)} \left[ \tau_G(a)^2 + ZC_1(G_2) + \tau_G(a) \tau_G(1)(G_2) + 2\tau_G(a)(M_1(G_2) - 2e_2) \right]
\]

Using Lemma 2 (a) and (b),

\[
ZC_1(G_1 \otimes G_2) = n_2 ZC_1(G_1) + n_1 ZC_1(G_2) + ZC_1(G_1) ZC_1(G_2) + 2AB + 2AZC_1(G_2) + 2BZC_1(G_1),
\]

\[
ZC_1^\perp(G_1 \otimes G_2) = 2\mu_2 ZC_1(G_1) + 2\mu_1 ZC_1(G_2) + 2 \sum_{a \in V(G_1)} \sum_{b \in V(G_2)} \tau_G(a) \tau_G(b) + \tau_G(a) \tau_G(b).\]
Theorem 2. Let $G_1$ and $G_2$ be two connected and
{$C_3, C_4$}-free graphs. Then, $ZC_1$ and $\overline{ZC}_1$ of the strong
product $(G_1 \circ G_2)$ are

$$ZC_1(G_1 \circ G_2) = 2BZC_1(G_1) + 2AZC_1(G_2) + 2ZC_1(G_1)ZC_1(G_2) + (n_2 + 2B)ZC_1(G_1)$$

$$+ (n_1 + 2A)ZC_1(G_2) + ZC_1(G_1)ZC_1(G_2) + 2AB + 2(n_2 + B)\sum_{a \in V(G_1)} d_{G_1}(a) \tau_{G_1}^2(a) + 2(n_1 + A)$$

$$\cdot \sum_{b \in V(G_2)} d_{G_2}(b) \tau_{G_2}^2(b) + n_2 \sum_{a \in V(G_1)} d_{G_1}(a) \tau_{G_1}^2(a) + n_1 \sum_{b \in V(G_2)} d_{G_2}(b) \tau_{G_2}^2(b),$$

$$\overline{ZC}_1(G_1 \circ G_2) \leq (n_2 + 2\mu_2)MZC_2(G_1) + (n_1 + 2\mu_1)MZC_2(G_2) + (n_2 + 2\mu_2 + B)\overline{ZC}_1(G_1)$$

$$+ (n_1 + 2\mu_1 + A)\overline{ZC}_1(G_2) + 2[\mu_2 MZC_2(G_1) + \mu_1 MZC_2(G_2)] + 2[\mu_2 A + \mu_1 B]$$

$$+ 2 \sum_{a \in V(G_1), b \in V(G_2)} \sum_{a, a \neq b \in E(G_2), b \neq b \in E(G_1)} \tau_{G_1}(a) \tau_{G_1}(b_{1}) + \tau_{G_1}(a_{2}) \tau_{G_1}(b_{2}).$$

Proof. Since $\tau_{G_1 \circ G_2}(a, b) = \tau_{G_1}(a)[1 + d_{G_1}(a)] + \tau_{G_2}(b)[1 + d_{G_2}(b) + \tau_{G_1}(a)]$, where $a \in V(G_1), b \in V(G_2)$, and $(a, b) \in G_1 \circ G_2$.

$$ZC_1(G_1 \circ G_2) = \sum_{(a, b) \in V(G_1 \circ G_2)} \left[ \tau_{G_1 \circ G_2}(a, b) \right]^2$$

$$= \sum_{a \in V(G_1), b \in V(G_2)} \sum_{a \in V(G_1), b \in V(G_2)} \left[ \tau_{G_1}(a)[1 + d_{G_1}(a)] + \tau_{G_2}(b)[1 + d_{G_2}(b) + \tau_{G_1}(a)] \right]^2$$

$$= \sum_{a \in V(G_1), b \in V(G_2)} \sum_{a \in V(G_1), b \in V(G_2)} \left[ \tau_{G_1}^2(a)[1 + d_{G_1}(a)]^2 + \tau_{G_2}^2(b)[1 + d_{G_2}(b) + \tau_{G_1}(a)]^2 +$$

$$+ 2\tau_{G_1}(a)\tau_{G_2}(b)[1 + d_{G_1}(a)] \tau_{G_1}(a) \right]$$

$$+ 2\tau_{G_1}(a)\tau_{G_2}(b)[1 + d_{G_1}(a)] \tau_{G_1}(a) + d_{G_1}(a)d_{G_2}(b) + d_{G_1}(a)\tau_{G_1}(a)]$$

$$= \sum_{a \in V(G_1), b \in V(G_2)} \sum_{a \in V(G_1), b \in V(G_2)} \left[ \tau_{G_1}^2(a)[1 + d_{G_1}(a)]^2 + \tau_{G_2}^2(b)[1 + d_{G_2}(b) + \tau_{G_1}(a)]^2 +$$

$$+ \tau_{G_1}(a)\tau_{G_2}(b)[1 + d_{G_1}(a)] \tau_{G_1}(a) \right]$$

$$+ 2\tau_{G_1}(a)\tau_{G_2}(b)[1 + d_{G_1}(a)] \tau_{G_1}(a) + d_{G_1}(a)d_{G_2}(b) + d_{G_1}(a)\tau_{G_1}(a)]$$

$$+ 2\tau_{G_1}(a)\tau_{G_2}(b)[1 + d_{G_1}(a)] \tau_{G_1}(a) \tau_{G_1}(b) + 2d_{G_1}(a)\tau_{G_1}(a)d_{G_2}(b) \tau_{G_1}(b) + 2d_{G_1}(a)\tau_{G_1}(a)\tau_{G_1}(b) \tau_{G_1}(b).$$
Using Lemma 2 (a) and (b),

\[
\begin{align*}
ZC_1(G_1 \otimes G_2) & = 2BZC^*_1(G_1) + 2AZC^*_1(G_2) \\
& + 2ZC^*_1(G_1)ZC^*_1(G_2) + (n_2 + 2B)ZC_1(G_1) \\
& + (n_1 + 2A)ZC_1(G_2) + ZC_1(G_1)ZC_1(G_2) \\
& + 2AB + 2(n_2 + B) \sum_{a \in V(G_1)} d_{G_1}(a) r_{G_1}(a) \\
& + 2(n_1 + A) \sum_{b \in V(G_2)} d_{G_2}(b) r_{G_2}(b) \\
& + n_2 \sum_{a \in V(G_1)} d_{G_1}(a) r_{G_1}(a) + n_1 \sum_{b \in V(G_2)} d_{G_2}(b) r_{G_2}(b),
\end{align*}
\]

(22)

\[
\begin{align*}
ZC_1(G_1 \otimes G_2) & = \sum_{(a,b)(a',b') \notin E(G_1, G_2)} \left[ \tau_{G_1 \otimes G_2}(a_1, b_1) + \tau_{G_1 \otimes G_2}(a_2, b_2) \right] \\
& = \sum_{a \in V(G_1)} \sum_{b \in V(G_2)} \left[ \tau_{G_1 \otimes G_2}(a, b_1) + \tau_{G_1 \otimes G_2}(a, b_2) \right] \\
& + \sum_{b \in V(G_2)} \sum_{a \in V(G_1)} \left[ \tau_{G_1 \otimes G_2}(a_1, b) + \tau_{G_1 \otimes G_2}(a_2, b) \right] \\
& + \sum_{a_1, a_2 \notin E(G_1)} \sum_{b_1, b_2 \notin E(G_2)} \left[ \tau_{G_1 \otimes G_2}(a_1, b_1) + \tau_{G_1 \otimes G_2}(a_2, b_2) \right].
\end{align*}
\]

(23)

We take

\[
\begin{align*}
& \sum_{a \in V(G_1)} \sum_{b \in V(G_2)} \left[ \tau_{G_1 \otimes G_2}(a, b_1) + \tau_{G_1 \otimes G_2}(a, b_2) \right] \\
& \quad = \sum_{a \in V(G_1)} \sum_{b \in V(G_2)} \left[ \tau_{G_1}(a) + \tau_{G_2}(b_1) + d_{G_1}(a) \tau_{G_1}(a) + d_{G_2}(b_1) \tau_{G_2}(b_1) + \tau_{G_1}(a) \tau_{G_2}(b_1) \right] \\
& \quad + \left[ \tau_{G_1}(a) + d_{G_2}(b_1) \right] \left[ \tau_{G_1}(a) + d_{G_2}(b_2) \tau_{G_2}(b_2) + \tau_{G_1}(a) \tau_{G_2}(b_2) \right] \\
& \quad = \sum_{a \in V(G_1)} \sum_{b \in V(G_2)} \left[ 2\tau_{G_1}(a) + \left[ \tau_{G_2}(b_1) + \tau_{G_2}(b_2) \right] + 2d_{G_1}(a) \tau_{G_2}(a) + \left[ d_{G_2}(b_1) \tau_{G_2}(b_1) \right. \right. \\
& \left. \left. + d_{G_2}(b_2) \tau_{G_2}(b_2) \right] + \tau_{G_1}(a) \left[ \tau_{G_2}(b_1) + \tau_{G_2}(b_2) \right] \right].
\end{align*}
\]

(24)

We know that

\[
\sum_{b, b_2 \notin E(G_2)} = \binom{n_2}{2} - \epsilon_3 = \mu_2
\]

(25)

Also, we take

\[
\begin{align*}
& \sum_{a \in V(G_1)} \sum_{b \in V(G_2)} \left[ \tau_{G_1 \otimes G_2}(a_1, b) + \tau_{G_1 \otimes G_2}(a_2, b) \right] \\
& \quad \sum_{a \in V(G_1)} \sum_{b \in V(G_2)} \left[ \left[ \tau_{G_1}(a_1) + \tau_{G_2}(b_1) + d_{G_1}(a_1) \tau_{G_1}(a_1) + d_{G_2}(b_1) \tau_{G_2}(b_1) + \tau_{G_1}(a_1) \tau_{G_2}(b_1) \right] \\
& \quad + \left[ \tau_{G_1}(a_2) + \tau_{G_2}(b_2) + d_{G_1}(a_2) \tau_{G_1}(a_2) + d_{G_2}(b_2) \tau_{G_2}(b_2) + \tau_{G_1}(a_2) \tau_{G_2}(b_2) \right] \right] \\
& \quad = n_2ZC_1(G_1) + 2\mu_1B + n_2MZC_2(G_1) + 2\mu_1ZC^*_1(G_2) + BZC_1(G_1).
\]

(26)
Again, we take

\[
\sum_{a_1, a_2 \notin E(G_1)} \sum_{b_1, b_2 \notin E(G_2)} \left[ \tau_{G_1 \otimes G_2}(a_1, b_1) + \tau_{G_1 \otimes G_2}(a_2, b_2) \right]
\]

\[
\leq 2 \sum_{a_1, a_2 \notin E(G_1)} \sum_{b_1, b_2 \notin E(G_2)} \left[ \{ \tau_{G_1}(a_1) + \tau_{G_2}(b_1) + dG_1(a_1) + dG_2(b_1) + \tau_{G_1}(a_1) \tau_{G_2}(b_1) \right.

+ \left. \{ \tau_{G_1}(a_2) + \tau_{G_2}(b_2) + dG_1(a_2) + dG_2(b_2) + \tau_{G_1}(a_2) \tau_{G_2}(b_2) \} \right].
\] (27)

Similarly,

\[
2 \mu_2 ZC_1(G_1) + 2 \mu_1 ZC_1(G_2) + 2 \mu_3 MZC_2(G_1) + 2 \mu_4 MZC_2(G_2)
\]

\[
+ 2 \sum_{a_1, a_2 \notin E(G_1)} \sum_{b_1, b_2 \notin E(G_2)} \left[ \tau_{G_1}(a_1) \tau_{G_2}(b_1) + \tau_{G_1}(a_2) \tau_{G_2}(b_2) \right].
\]

Consequently,

\[
ZC_1(G_1 \otimes G_2) \leq (n_2 + 2 \mu_2) MZC_2(G_1) + (n_1 + 2 \mu_1) MZC_2(G_2) + (n_2 + 2 \mu_2 + B) ZC_1(G_1)
\]

\[
+ (n_1 + 2 \mu_1 + A) ZC_1(G_2) + 2 \mu_2 ZC_1(G_1) + \mu_1 ZC_1(G_2) + 2 \mu_2 A + 2 \mu_1 B
\]

\[
+ 2 \sum_{a_1, a_2 \notin E(G_1)} \sum_{b_1, b_2 \notin E(G_2)} \left[ \tau_{G_1}(a_1) \tau_{G_2}(b_1) + \tau_{G_1}(a_2) \tau_{G_2}(b_2) \right].
\] (28)

**Theorem 3.** Let \( G_1 \) and \( G_2 \) be two connected and \( \{C_3, C_4\} \)-free graphs. Then, \( ZC_1 \) and \( ZC_1 \) of the symmetric difference \((G_1 \oplus G_2)\) are

\[
ZC_1(G_1 \oplus G_2) = n_2(n^2_2 - 8e_2) M_1(G_1) + n_1(n^2_1 - 8e_1) M_1(G_2) + 4 M_1(G_1) M_1(G_2)
\]

\[
+ n_1 n_2(n^2_1 n^2_2 - 4n^2_1 e_1 - 4n^2_2 e_2 - 2n_1 n_2 + 24e_1 e_2 + 1) + 4(n^2_1 e_2 + n^2_2 e_1 - 4e_1 e_2),
\]

\[
ZC_1(G_1 \oplus G_2) = (4e_2 - n^2_2) \overline{M}_1(G_1) + (4e_1 - n^2_1) \overline{M}_1(G_2) + 2 \mu_1(n_1 n^2_2 - n_2 - 2n_1 e_2)
\]

\[
+ 2 \mu_2(n_1 n^2_2 - n_1 - 2n_1 e_1).
\] (29)

**Proof.** Since \( \tau_{G_1 \oplus G_2}(a, b) = n_1 [n_1 - d_{G_1}(a)] + d_{G_2}(b) \cdot [2d_{G_1}(a) - n_1 - 1], \) where \( a \in V(G_1), \) \( b \in V(G_2) \), and \( (a, b) \in G_1 \oplus G_2 \),
\[ ZC_1(G_1 \ominus G_2) = \sum_{(a,b) \in V(G_1 \ominus G_2)} \left[ \tau_{G_1 \ominus G_2}(a,b) \right]^2 \]
\[ = \sum_{a \in V(G_1)} \sum_{b \in V(G_2)} \left[ n_2 \{ n_1 - d_{G_1}(a) \} + \hat{d}_{G_2}(b) \{ 2d_{G_1}(a) - n_1 \} - 1 \right]^2 \]
\[ = \sum_{a \in V(G_1)} \sum_{b \in V(G_2)} \left[ n_2^2 \{ n_1 - d_{G_1}(a) \}^2 + \hat{d}_{G_2}(b) \{ 2d_{G_1}(a) - n_1 \}^2 + 1 + 2n_2 \hat{d}_{G_2}(b) \{ n_1 - d_{G_1}(a) \} \right] \]
\[ \{ 2d_{G_1}(a) - n_1 \} - 2d_{G_2}(b) \{ 2d_{G_1}(a) - n_1 \} - 2n_2 \{ n_1 - d_{G_1}(a) \} \} \]
\[ = n_1^3 d_3 + n_1^3 M_1(G_1) - 4n_1 n_2 e_1 + 4M_1(G_1) M_1(G_2) + n_1^3 M_1(G_2) - 8n_1 e_1 M_1(G_2) + n_1 n_2 \]
\[ + 16n_2 e_2 - 4n_1^2 e_1 - 8n_2 e_2 M_1(G_1) + 8n_2 e_1 e_2 - 16e_1 e_2 + 4n_1^2 e_2 - 4n_2^2 e_1, \]
\[ ZC_1(G_1 \ominus G_2) = n_2 \{ n_1^2 - 8e_2 \} M_1(G_1) + n_1 \{ n_1^2 - 8e_1 \} M_1(G_2) + 4M_1(G_1) M_1(G_2) + n_1 n_2 \]
\[ (n_1^2 n_2^2 - 4n_1^2 e_2 - 2n_2 e_2 + 24e_1 e_2 + 1) + 4(n_1^2 e_2 + n_1^2 e_2 - 4e_1 e_2). \]

We take

\[ \sum_{a \in V(G_1)} \sum_{b \in V(G_2)} \left[ \tau_{G_1 \ominus G_2}(a,b) \right] \]
\[ = \sum_{a \in V(G_1)} \sum_{b \in V(G_2)} \left[ n_1 n_2 - 1 - n_2 d_{G_1}(a) - n_1 d_{G_2}(b) + 2d_{G_1}(a)d_{G_2}(b) \right] \]
\[ + \left[ n_1 n_2 - 1 - n_2 d_{G_1}(a) - n_1 d_{G_2}(b) + 2d_{G_1}(a) d_{G_2}(b) \right]. \]
Similarly = \( 2n_1^2 n_2^2 - 2n_1 \mu_2 - 4n_2 e_1 \mu_2 - n_1^2 \bar{M}_1(G_2) + 4e_1 \bar{M}_1(G_2). \]

Also, we take

\[ \sum_{b \in V(G_2)} \sum_{a \in E(G_1)} \left[ \tau_{G_1 \ominus G_2}(a,b) \right]. \]
\[ Similarly = 2n_1 n_2^2 \mu_2 - 2n_2 \mu_1 - n_2^2 \bar{M}_1(G_1) - 4n_1 e_2 \mu_1 + 4e_2 \bar{M}_1(G_1). \]
Again, we take (null case)

\[
N = \sum_{a_1,a_2 \not\in E(G_1)} \sum_{b_1,b_2 \not\in E(G_2)} \left[ \tau_{G_1 \oplus G_2}(a_1,b_1) + \tau_{G_1 \oplus G_2}(a_2,b_2) \right] = 0. \tag{35}
\]

We further take (also null case)

\[
N = \sum_{b_1,b_2 \not\in E(G_1)} \sum_{a_1,a_2 \not\in E(G_2)} \left[ \tau_{G_1 \oplus G_2}(a_1,b_1) + \tau_{G_1 \oplus G_2}(a_2,b_2) \right] = 0. \tag{36}
\]

Consequently,

\[
\mathcal{ZC}_1(G_1 \oplus G_2) = (4e_2 - n_2^2)M_1(G_1) + (4e_1 - n_1^2)M_1(G_2) + 2\mu_1(n_1n_2^2 - n_2 - 2n_1e_2) + 2\mu_2(n_1^2n_2 - n_1 - 2n_2e_1). \tag{37}
\]

\[\blacksquare\]

\[
\mathcal{ZC}_1(G_1 \oplus G_2) = n_2(n_2^2 - 4e_2)M_1(G_1) + n_1(n_1^2 - 4e_1)M_1(G_2) + M_1(G_1)M_1(G_2) + n_1n_2(n_2^2 - 4n_2^2e_1 - 4n_1^2e_2 - 2n_1n_2 + 16n_1e_2 + 1) + 4(n_1^2e_2 + n_2^2e_1 - 2e_1e_2). \tag{38}
\]

\[
\mathcal{ZC}_1(G_1 \oplus G_2) = (2e_2 - n_2^2 - 2n_2^2\mu_2)M_1(G_1) + (2e_1 - n_1^2 - 2n_1\mu_1)M_1(G_2) + 2\mu_1(n_1n_2^2 - n_2 - 2n_1e_2) + 2\mu_2(n_1^2n_2 - n_1 - 2n_2e_1) + 4\mu_1\mu_2(n_1n_2 - 1) + 2\sum_{a_1,a_2 \not\in E(G_1)} \sum_{b_1,b_2 \not\in E(G_2)} \left[ d_{G_1}(a_1)d_{G_1}(b_1) + d_{G_2}(a_2)d_{G_2}(b_2) \right]. \tag{39}
\]

\[\blacksquare\]

Proof. Since \(\tau_{G_1 \oplus G_2}(a,b) = n_2[n_1 - d_{G_1}(a)] + d_{G_1}(b)[d_{G_1}(a) - n_1] - 1\), where \(a \in V(G_1), b \in V(G_2)\), and \((a,b) \in E(G_1 \oplus G_2)\),

\[
\mathcal{ZC}_1(G_1 \oplus G_2) = \sum_{(a,b) \in E(G_1 \oplus G_2)} \left[ \tau_{G_1 \oplus G_2}(a,b) \right]^2 \tag{40}
\]

\[= \sum_{a \in V(G_1)} \sum_{b \in V(G_2)} \left[ n_2[n_1 - d_{G_1}(a)] + d_{G_1}(b)[d_{G_1}(a) - n_1] - 1 \right]^2. \]
Similarly,

\[
ZC_1(G_1 \oplus G_2) = n_2(n_2^2 - 4e_2)M_1(G_1) + n_1(n_1^2 - 4e_1)M_1(G_2) + M_1(G_1)M_1(G_2) + n_1n_2
\cdot \left( n_2^2 - 4n_2^2e_1 - 4e_1^2 - 2n_1n_2 + 16e_1e_2 + 1 \right) + 4(n_1^2e_1 + n_2^2e_1 - 2e_1e_2).
\]

\[
ZC_1(G_1 \oplus G_2) = \sum_{(a_1,b_1)(a_2,b_2) \notin E(G_1 \oplus G_2)} \left[ r_{G_1 \oplus G_2}(a_1,b_1) + r_{G_1 \oplus G_2}(a_2,b_2) \right]
\]

\[
= \sum_{a \in V(G_1)} \sum_{b \in V(G_2)} \left[ r_{G_1 \oplus G_2}(a,b_1) + r_{G_1 \oplus G_2}(a,b_2) \right] + \sum_{b \in V(G_2)} \sum_{a \in V(G_1)} \left[ r_{G_1 \oplus G_2}(a_1,b) + r_{G_1 \oplus G_2}(a_2,b) \right]
\]

\[
= \sum_{a_1,a_2 \notin E(G_1)} \sum_{b_1,b_2 \notin E(G_2)} \left[ r_{G_1 \oplus G_2}(a_1,b_1) + r_{G_1 \oplus G_2}(a_2,b_2) \right] + \sum_{b_1 \notin E(G_2)} \sum_{a_1 \notin E(G_1)} \left[ r_{G_1 \oplus G_2}(a_1,b_1) + r_{G_1 \oplus G_2}(a_2,b_2) \right]
\]

\[
= \sum_{a_1,a_2 \notin E(G_1)} \sum_{b_1,b_2 \notin E(G_2)} \left[ r_{G_1 \oplus G_2}(a_1,b_1) + r_{G_1 \oplus G_2}(a_2,b_2) \right] + \sum_{b_1 \notin E(G_2)} \sum_{a_1 \notin E(G_1)} \left[ r_{G_1 \oplus G_2}(a_1,b_1) + r_{G_1 \oplus G_2}(a_2,b_2) \right]
\]

We take

\[
\sum_{a \in V(G_1)} \sum_{b \in V(G_2)} \left[ r_{G_1 \oplus G_2}(a,b_1) + r_{G_1 \oplus G_2}(a,b_2) \right]
\]

\[
= \sum_{a \in V(G_1)} \sum_{b \in V(G_2)} \left[ \{ n_1n_2 - 1 - n_2d_{G_1}(a) - n_1d_{G_2}(b) + d_{G_1}(a)d_{G_2}(b) \} \right]
\]

\[
+ \{ n_1n_2 - 1 - n_2d_{G_1}(a) - n_1d_{G_2}(b) + d_{G_1}(a)d_{G_2}(b) \} \right]
\]

\[
\text{Similarly} = 2n_2^2n_1\mu_2 - 2n_1\mu_2 - 4n_2\varepsilon_1\mu_2 - n_2^2M_1(G_2) + 2e_1M_1(G_2).
\]

Also, we take

\[
\sum_{a_1,a_2 \notin E(G_1)} \sum_{b_1,b_2 \notin E(G_2)} \left[ r_{G_1 \oplus G_2}(a_1,b_1) + r_{G_1 \oplus G_2}(a_2,b_2) \right]
\]

\[
= 2 \sum_{a_1,a_2 \notin E(G_1)} \sum_{b_1,b_2 \notin E(G_2)} \left[ \{ n_1n_2 - 1 - n_2d_{G_1}(a_1) - n_1d_{G_2}(b_1) + d_{G_1}(a_1)d_{G_2}(b_1) \} \right]
\]

\[
+ \{ n_1n_2 - 1 - n_2d_{G_1}(a_1) - n_1d_{G_2}(b_1) + d_{G_1}(a_1)d_{G_2}(b_1) \} \right]
\]

\[
\text{Similarly} = 4n_2n_1\mu_1\mu_2 - 4\mu_1\mu_2 - 2n_2\mu_2M_1(G_1) - 2n_1\mu_1M_1(G_2)
\]

\[
+ 2 \sum_{a_1,a_2 \notin E(G_1)} \sum_{b_1,b_2 \notin E(G_2)} \left[ d_{G_1}(a_1)d_{G_2}(b_1) + d_{G_1}(a_2)d_{G_2}(b_2) \right].
\]
Again, we take

\[ \sum_{a, a \notin E(G_1)} \sum_{b, b \notin E(G_2)} \left[ \tau_{G_1 \oplus G_2}(a_1, b) + \tau_{G_1 \oplus G_2}(a_2, b) \right]. \]

Similarly, \( 2n_1n_2^2\mu_1 - 2n_2\mu_1 - n_2^2n_1\lambda_1 = 4n_1n_2\mu_1 + 2n_2n_1\lambda_1 \)

We further take (null case)

\[ N = \sum_{a, a \notin E(G_1)} \sum_{b, b \notin E(G_2)} \left[ \tau_{G_1 \oplus G_2}(a_1, b_1) + \tau_{G_1 \oplus G_2}(a_2, b_2) \right] = 0. \]

Furthermore, we take (also null case)

\[ N = \sum_{a, a \notin E(G_1)} \sum_{b, b \notin E(G_2)} \left[ \tau_{G_1 \oplus G_2}(a_1, b_1) + \tau_{G_1 \oplus G_2}(a_2, b_2) \right] = 0. \]

Consequently,

\[ ZC_1(G_1 \oplus G_2) = (2e_2 - n_2^2 - 2n_2\mu_2)M_1(G_1) + (2e_1 - n_1^2 - 2n_1\mu_1)M_1(G_2) \]
\[ + 2 \mu_1(n_1n_2^2 - n_2 - 2n_1e_2) + 2 \mu_2(n_2^2n_1 - n_1 - 2n_2e_1) + 4 \mu_1\mu_2(n_1n_2 - 1) \]
\[ + 2 \sum_{a, a \notin E(G_1)} \sum_{b, b \notin E(G_2)} \left[ d_{G_1}(a_1)d_{G_2}(b_1) + d_{G_1}(a_2)d_{G_2}(b_2) \right]. \]

\[ \square \]

4. Analysis and Conclusions

In this section, we compute the analysis for the first Zagreb connection index \((ZC_1(\theta))\) and first Zagreb coindex \((ZC_1(\psi))\) of tensor product are obtained as follows:

(a) \( ZC_1(P_m \otimes P_n) \leq 64mn - 138m - 138n + 292 \)

(b) \( ZC_1(P_m \otimes P_n) \leq 4mn - 4m - 6n + 6 \)

The upper bounds for the first Zagreb connection index \((ZC_1(\lambda))\) of tensor product are obtained as follows [39]:

\[ ZC_1(P_m \otimes P_n) \leq 64mn - 56m - 124n + 100. \]

Table 1 and Figure 5 depict the numerical and graphical behaviours of the analysis between exact formulae and upper bounds for the first Zagreb connection index and coindex of tensor product by using values \( m = n \).

4.1. Tensor Product. Let \( P_m \) and \( P_n \) be two particular alkanes called by paths, then the tensor product \((P_m \otimes P_n)\) is obtained by the product of \( P_m \) and \( P_n \). For \( m = 3 \) and \( n = 3 \), see Figure 10.

4.2. Strong Product. Let \( P_m \) and \( P_n \) be two particular alkanes called by paths, then the strong product \((P_m \otimes P_n)\) is
Using Theorem 2, the exact formulae for the first Zagreb connection index \( ZC_1(\theta_1) \) and first Zagreb connection coindex \( ZC_1(\psi_1) \) of strong product are obtained as follows [39]:

\[
ZC_1(P_m \otimes P_n) \leq 102mn - 174m - 174n + 344
\]

The upper bounds for first Zagreb connection index \( ZC_1(\lambda_2) \) of strong product are obtained as follows [39]:

\[
ZC_1(P_m \otimes P_n) \leq 4mn^2 + 16m^2 - 5mn - 4m - 12n^3 - 40n^2 + 24n + 4. 
\]

For \( m = 4 \) and \( n = 4 \), see Figure 11.

Table 1: Analysis for index and coindex of exact formulae and upper bounds of \( \theta_1 = P_m \otimes P_n \), \( \lambda_1 = P_m \otimes P_{n'} \), and \( \psi_1 = P_n \otimes P_n \), respectively.

| (m = n) | ZC_1(\theta_1) | ZC_1(\lambda_1) | ZC_1(\psi_1) |
|---------|----------------|----------------|-------------|
| 1       | 80             | -16            | 0           |
| 2       | -4             | -4             | 2           |
| 3       | 40             | 136            | 12          |
| 4       | 212            | 404            | 30          |
| 5       | 512            | 800            | 56          |
| 6       | 940            | 1324           | 90          |
| 7       | 1496           | 1976           | 132         |
| 8       | 2180           | 2756           | 182         |
| 9       | 2992           | 3664           | 240         |
| 10      | 3932           | 4700           | 306         |
| 11      | 5000           | 5864           | 380         |
| 12      | 6196           | 7156           | 462         |
| 13      | 7520           | 8576           | 552         |
| 14      | 8972           | 10124          | 650         |
| 15      | 10552          | 11800          | 756         |

Table 2: Analysis for index and coindex of exact formulae and upper bounds of \( \theta_2 = P_m \otimes P_n \), \( \lambda_2 = P_m \otimes P_{n'} \), and \( \psi_2 = P_n \otimes P_{n'} \), respectively.

| (m = n) | ZC_1(\theta_2) | ZC_1(\lambda_2) | ZC_1(\psi_2) |
|---------|----------------|----------------|-------------|
| 1       | 98             | -13            | 22          |
| 2       | 56             | -40            | 88          |
| 3       | 218            | 91             | 234         |
| 4       | 584            | 644            | 460         |
| 5       | 1154           | 1979           | 766         |
| 6       | 1928           | 4552           | 1152        |
| 7       | 2906           | 8915           | 1618        |
| 8       | 4088           | 15716          | 2164        |
| 9       | 5474           | 25699          | 2790        |
| 10      | 7064           | 39704          | 3496        |
| 11      | 8858           | 58667          | 4282        |
| 12      | 10856          | 83620          | 5148        |
| 13      | 13058          | 115691         | 6094        |
| 14      | 15464          | 156104         | 7120        |
| 15      | 18074          | 206179         | 8226        |

Table 3: Analysis for index and coindex of exact formulae and upper bounds of \( \theta_3 = P_m \otimes P_{n'} \), \( \lambda_3 = P_m \otimes P_{n'} \), and \( \psi_3 = P_n \otimes P_n \), respectively.

| (m = n) | ZC_1(\theta_3) | ZC_1(\lambda_3) | ZC_1(\psi_3) |
|---------|----------------|----------------|-------------|
| 1       | 12             | 33             | 0           |
| 2       | 4              | 34             | 0           |
| 3       | 116            | 153            | 44          |
| 4       | 912            | 390            | 264         |
| 5       | 4348           | 745            | 888         |
| 6       | 15332          | 1218           | 2240        |
| 7       | 44004          | 1809           | 4740        |
| 8       | 108736         | 2518           | 8904        |
| 9       | 239852         | 3345           | 15344       |
| 10      | 484068         | 4290           | 24768       |
| 11      | 909652         | 5353           | 37980       |
| 12      | 1612304        | 6534           | 55880       |
| 13      | 2721756        | 7833           | 79464       |
| 14      | 4409092        | 9250           | 109824      |
| 15      | 6894788        | 10785          | 148148      |

Table 4: Analysis for index and coindex of exact formulae and upper bounds of \( \theta_4 = P_m \otimes P_n \), \( \lambda_4 = P_m \otimes P_{n'} \), and \( \psi_4 = P_n \otimes P_{n'} \), respectively.

| (m = n) | ZC_1(\theta_4) | ZC_1(\lambda_4) | ZC_1(\psi_4) |
|---------|----------------|----------------|-------------|
| 1       | 0              | 141            | 0           |
| 2       | 0              | 40             | -2          |
| 3       | 40             | 189            | 40          |
| 4       | 492            | 588            | 378         |
| 5       | 2928           | 1237           | 1456        |
| 6       | 11680          | 2136           | 3910        |
| 7       | 36120          | 3285           | 8568        |
| 8       | 93660          | 4684           | 16450       |
| 9       | 213472         | 6333           | 28768       |
| 10      | 440928         | 8232           | 46926       |
| 11      | 842760         | 10381          | 72520       |
| 12      | 1512940        | 12780          | 107338      |
| 13      | 2579280        | 15429          | 153360      |
| 14      | 4210752        | 18328          | 212758      |
| 15      | 6676153        | 21477          | 287896      |

Table 5: Particular numeric values of obtained results for index and coindex.

| Product graphs | ZC_1(\theta) | ZC_1(\lambda) | ZC_1(\psi) |
|----------------|--------------|---------------|------------|
| (P_m \otimes P_1) | 40           | 136           | 12         |
| (P_1 \otimes P_2) | 584          | 644           | 460        |
| (P_m \otimes P_2) | 116          | 153           | 44         |
| (P_1 \otimes P_1) | 40           | 189           | 40         |

obtained by the product of \( P_m \) and \( P_n \). For \( m = 4 \) and \( n = 4 \), see Figure 11.

4.3. Symmetric Difference. Let \( P_m \) and \( P_n \) be two particular alkanes called by paths, then the symmetric difference \( (P_m \triangle P_n) \) is obtained by the product of \( P_m \) and \( P_n \). For \( m = 3 \) and \( n = 3 \), see Figure 12.
Using Theorem 3, the exact formulae for the first Zagreb connection index \((ZC_1(\lambda_3))\) and first Zagreb connection coindex \((ZC_1(\psi_3))\) of symmetric difference are obtained as follows:

\[
ZC_1(P_m \ominus P_n) \leq m^3n^3 - 4m^3n^2 + 8m^3n - 6m^3 - 4m^2n^3 + 22m^2n^2 - 52m^2n + 44m^2 + 8mn^3 - 52mn^2
+ 137mn - 128m - 6n^3 + 44n^2 - 128n + 128, \tag{51}
\]

\[
ZC_1(P_m \ominus P_n) \leq 4m^2n^2 - 9m^2n + 5m^2 - 9mn^2 + 20mn - 12m + 5n^2 - 12n + 8. \tag{52}
\]

The upper bounds for first Zagreb connection index \((ZC_1(\lambda_3))\) of symmetric difference are obtained as follows [40]:

\[
ZC_1(P_m \ominus P_n) \leq 59mn - 88m - 88n + 150. \tag{53}
\]

Table 3 and Figure 7 depict the numerical and graphical behaviours of the analysis between exact formulae and upper bounds for first Zagreb connection index and coindex of symmetric difference by using values \(m = n\).

4.4. Disjunction. Let \(P_m\) and \(P_n\) be two particular alkanes called by paths, then the disjunction \((P_m \oplus P_n)\) is obtained by the product of \(P_m\) and \(P_n\). For \(m = 3\) and \(n = 3\), see Figure 13.

Using Theorem 4, the exact formulae for the first Zagreb connection index \((ZC_1(\lambda_4))\) and first Zagreb connection coindex \((ZC_1(\psi_4))\) of disjunction are obtained as follows:
Figure 9: Analysis of obtained results for product on graphs with respect to Table 5.

Figure 10: (a) $G_1 \cong P_3$ (b) $G_2 \cong P_3$, and (c) tensor product ($P_3 \circ P_3$).

Figure 11: (a) $G_1 \cong P_4$ (b) $G_2 \cong P_4$, and (c) strong product ($P_4 \otimes P_4$).
Figure 12: (a) $G_1 \equiv P_3$ (b) $G_2 \equiv P_3$, and (c) symmetric difference ($P_3 \oplus P_3$).

Figure 13: (a) $G_1 \equiv P_3$ (b) $G_2 \equiv P_3$, and (c) disjunction ($P_3 \oplus P_3$).

Figure 14: Analysis for $ZC_1 (\theta)$ of exact formula.
The upper bounds for first Zagreb connection index \((ZC_1(\lambda_4))\) of disjunction are obtained as follows \([39]\):

\[
ZC_1(P_m \oplus P_n) \leq 125mn - 238m - 238n + 492. \tag{56}
\]

Table 4 and Figure 8 depict the numerical and graphical behaviours of the analysis between exact formulae and upper bounds for the first Zagreb connection index and coindex of disjunction by using values \(m = n\).

Now, from Tables 1–5 and Figures 5–9 and 14–16, we close our discussion with the following conclusions:

The behaviours of all the connection number-based Zagreb index and coindex for operations on graphs such as tensor product, strong product, symmetric difference, and disjunction are increased in the following order, respectively, as \(ZC_1(\psi) \geq ZC_1(\lambda) \geq ZC_1(\theta)\).

For increasing values of \(m\) and \(n\), the upper bound for the first Zagreb connection index of products on graphs are working rapidly than all the exact formula for the first Zagreb connection index, respectively.

In certain intervals of the values of \(m\) and \(n\), all the first Zagreb connection coindices attain the maximum values on increasing values of \(m\) and \(n\). In Figures 5–8, we analyse that the first Zagreb connection coindex
attains more upper layer than other TIs in all the operations.

Table 5 and Figure 9 interpret the particular analysis of the obtained results for index and coindex on operations such as tensor product, strong product, symmetric difference, and disjunction. This particular analysis also concludes that the first Zagreb connection coindex attains more upper layer than other TIs in all the operations.

In particular, Figures 14–16 interpret the exact formula for the first Zagreb connection index, upper bound for the first Zagreb connection index, and exact formula for the first Zagreb connection coindex which are dominant on operations from tensor product to disjunction, respectively. In addition, we analyse that the first Zagreb connection coindex of operation disjunction has attained more upper layer than all the other operations for connection number-based index and coindex.

The investigation of these indices and coinindices for the resultant graphs obtained from other operations of graphs (subtraction, switching, zig-zag product, addition, rooted product, modular product etc.) is still open.

**Data Availability**

All data used to support the findings of this study are included within the article. However, additional data will be made available from the corresponding author upon request.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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