Universal correlations in spectra of the lattice QCD Dirac operator
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Recently, Kalkreuter obtained complete Dirac spectra for SU(2) lattice gauge theory both for staggered fermions and for Wilson fermions. The lattice size was as large as $12^4$. We performed a statistical analysis of these data and found that the eigenvalue correlations can be described by the Gaussian Symplectic Ensemble for staggered fermions and by the Gaussian Orthogonal Ensemble for Wilson fermions. In both cases long range spectral fluctuations are strongly suppressed: the variance of a sequence of levels containing $n$ eigenvalues on average is given by $\Sigma^2(n) \sim (2 \log n)/\beta \pi^2$ ($\beta$ is equal to 4 and 1, respectively) instead of $\Sigma^2(n) = n$ for a random sequence of levels. Our findings are in agreement with the anti-unitary symmetry of the lattice Dirac operator for $N_c = 2$ with staggered fermions which differs from Wilson fermions (with the continuum anti-unitary symmetry). For $N_c = 3$, we predict that the eigenvalue correlations are given by the Gaussian Unitary Ensemble.

1. INTRODUCTION

The study of spectra of the QCD Dirac operator is of great importance for the understanding of the mechanism of chiral symmetry breaking and its restoration above the critical temperature $T_c$. Moreover, the fluctuations of the eigenvalues provide information on the distribution of the fermion determinants, and ultimately, on the validity of the quenched approximation. In this work, we study the Dirac spectrum for SU(2) color both with Kogut-Susskind (KS) and Wilson fermions.

We consider the eigenvalue problems

\begin{equation}
\begin{aligned}
i D^{KS} \psi = \lambda \psi, \\
Q^W_5 \psi \equiv \frac{1}{8 + m} \gamma_5 (D^W + m) \psi = \lambda \psi,
\end{aligned}
\end{equation}

where $D^{KS}$ and $D^W$ are the usual KS and Wilson lattice QCD Dirac operators. The Dirac matrix is tri-diagonalized by Cullum’s and Willoughby’s Lanczos procedure\textsuperscript{3}. The eigenvalues are then obtained by a standard QL algorithm. This improved Lanczos algorithm makes it possible to obtain all eigenvalues. The accuracy of the eigenvalues can then be checked by means of sum rules for the sum of the squares of the eigenvalues of the lattice Dirac operator\textsuperscript{4}.

2. SPECTRA OF COMPLEX SYSTEMS AND RANDOM MATRIX THEORY

A basic assumption in the analysis of spectra of complex systems is that the variations of the average spectral density, $\bar{\rho}(\lambda)$, and the spectral fluctuations about this average separate. This allows us to unfold the spectrum $\{\lambda_k\}$ into $\{\lambda'_k\}$, with average spectral density equal to one, according to $\lambda'_k = \int_{-\infty}^{\lambda_k} d\lambda \bar{\rho}(\lambda)$. We study the distribution of the number of unfolded eigenvalues, $\{n_k(n)\}$, in a stretch of length $n$, by means of its variance, $\Sigma^2(n)$, and its first two cumulants, $\gamma_1(n)$ and $\gamma_2(n)$.

These statistics can be obtained analytically for the invariant random matrix ensembles of hermitean matrices with probability distribution given by $P(H) \sim \exp(-\text{Tr} H^\dagger H)$. The matrix elements can be either real (Gaussian Orthogonal Ensemble (GOE)), complex (Gaussian Unitary Ensemble (GUE)) or quaternion real (Gaussian Symplectic Ensemble (GSE)). They are characterized by the Dyson index $\beta$, which is equal to 1, 2 and 4, respectively. The most notable property of random matrix ensembles is the stiffness of the spectrum, i.e. $\Sigma^2(n) \sim (2/\pi^2 \beta) \log(n)$, instead of $n$ for independently distributed eigenvalues.

In the past decade, spectra of many complex systems have been studied in great detail. The
main conclusion is that, if the corresponding classical system is chaotic, the spectral correlations are given by the random matrix theory (RMT) with the same (anti-unitary) symmetries as the original Hamiltonian. We also note that the average spectral density is generically not given by RMT.

3. SPECTRAL CORRELATIONS OF THE DIRAC OPERATOR

For a lattice size of $6^3 \times 12$ and Wilson fermions with $\beta = 2.12$ and $\kappa = 0.15$, all eigenvalues were obtained for 8 independent gauge field configurations. A histogram of the spectral density (normalized to 1) in the region $[-0.1, 0.1]$ of each of these configurations is shown in figure 1. The fluctuations of the Dirac spectrum over the ensemble and the statistical fluctuations from bin to bin seem to coincide. This so called ‘spectral ergodicity’, the equality of ensemble averages and spectral averages, is a well known property of RMT. We have also calculated the ensemble average of the number variance of these spectra unfolded with the ensemble averaged spectral density. Within the limited statistics we found complete agreement with the GOE prediction. For long level sequences much better statistics can be obtained from a spectral average rather than an ensemble average which we will use in the remainder of the paper. Below we will show that the spectral average of $\Sigma_2(n)$ for Wilson fermions is given by the GOE as well.

For $SU(2)$, the anti-unitary symmetry of the KS Dirac operator and the Wilson Dirac operator is different. We have $[\tau_2 K, D^{KS}] = 0$ and $[\gamma_5 C \tau_2 K, Q_5^W] = 0$, where $K$ is the complex conjugation operator and $C$ the charge conjugation matrix. Because $(\tau_2 K)^2 = -1$, whereas $(\gamma_5 C \tau_2 K)^2 = 1$ the KS Dirac matrix can be organized into real quaternions, whereas the Wilson Dirac matrix is real in an appropriate basis. Consequently, we expect that the spectral correlations of the eigenvalues of $D^{KS}$ are given by the GSE, and those of $Q_5^W$ are given by the GOE. In Figs. 3 and 4 we show the number variance, $\Sigma_2(n)$, and the first two cumulants, $\gamma_1(n)$ and $\gamma_2(n)$, versus $n$. The points are obtained from the unfolded lattice spectra, with the average integrated spectral density obtained by fitting a second order polynomial to a stretch of 500-1000 eigenvalues. Analytical results are represented by the full (GOE) and dotted curves (GSE). The KS results are for a $12^4$ lattice with 4 dynamical flavors with a mass of $ma = 0.05$. The results for dynamical Wilson fermions were obtained on a $8^3 \times 12$ lattice. The values of $\beta$ and $\kappa$ are shown in the label of the figure. Clearly, perfect agreement with the random matrix prediction is observed. We also studied spectral correlations for weaker coupling $\beta = 2.8$, and stronger coupling $\beta = 1.8$, and both quenched and unquenched field configurations. The number variance was calculated up to $n = 100$. The stationarity of the spectral correlations was investigated by studying correlations of stretches of 200 eigenvalues starting from zero. In all cases no deviations from the random matrix predictions were found.

4. DISCUSSION AND CONCLUSIONS

We have seen that the spectral correlations of the KS Dirac operator are given by the GSE, contrary to those of the Wilson Dirac operator, with the anti-unitary symmetries of the continuum theory which are given by the GOE. One might wonder what happens to the spectral correlations of $D^{KS}$ in the continuum limit. In a suitable basis, we can write $D^{KS} = D^{cont} + a^2 B$. The operator $a^2 B$ is relevant for the spectral fluctuations if its norm is larger than the level spacing of $D^{cont}$. At fixed volume, the level spacing of $\lambda a$ in the bulk of the spectrum is of order $1/N = a^4/V$.
which is much less than the norm of the perturbing operator. Only for the low-lying spectrum with $\lambda \sim 1/V$ for $a \to 0$, a transition to the GOE is possible. Apparently, much larger lattices are required to study this transition.

Finally, for $SU(3)$ color, the Dirac operator does not have any additional anti-unitary symmetries, and we predict that the spectral correlations for both KS fermions and Wilson fermions are given by the GUE. In all cases the spectral fluctuations are strongly suppressed with respect to uncorrelated eigenvalues. To some extent, QCD is selfquenching.

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