THE $K$-THEORY OF TORIC SCHEMES OVER REGULAR RINGS OF MIXED CHARACTERISTIC

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Abstract. We show that if $X$ is a toric scheme over a regular commutative ring $k$ then the direct limit of the $K$-groups of $X$ taken over any infinite sequence of nontrivial dilations is homotopy invariant. This theorem was previously known for regular commutative rings containing a field. The affine case of our result was conjectured by Gubeladze. We prove analogous results when $k$ is replaced by an appropriate $K$-regular, not necessarily commutative $k$-algebra.

Introduction

Let $A = (A, ·)$ be a commutative monoid. For each integer $c \geq 2$, the $c$-th power map $\theta_c : A \to A$, $\theta_c(a) = a^c$, is an endomorphism of $A$; it is called a dilation. For any ring $k$, $\theta_e$ induces an endomorphism of the monoid ring $k[A]$, its $K$-theory $K_*(k[A])$, and its homotopy $K$-theory $KH_*(k[A])$.

The affine Dilation Theorem says that the monoid of dilations acts nilpotently on the reduced $K$-theory of $k[A]$, at least when $A$ is a submonoid of a torsionfree abelian group, $A$ has no nontrivial units and $k$ is an appropriately regular ring.

Case (a) of the following theorem verifies a conjecture of Gubeladze ([20, 1.1]).

Dilation Theorem 0.1. Let $A$ be a submonoid of a torsionfree abelian group. Assume that $A$ has no non-trivial units. Let $e = (c_1, c_2, \ldots)$ be a sequence of integers with $c_i \geq 2$ for all $i$. If $\Lambda$ is any of the rings listed below, then the following canonical map is an isomorphism:

\begin{equation}
K_*(\Lambda) \xrightarrow{\cong} \varinjlim_{\theta_e} K_*(\Lambda[A]).
\end{equation}

a) a regular commutative ring;

b) a commutative $C^*$-algebra;

c) an associative regular ring that admits the structure of a flat algebra over a regular commutative ring $k_0$ of finite Krull dimension;

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d) an associative ring that is \( K \)-regular and that admits the structure of a flat algebra over a regular commutative ring \( k_0 \) of finite Krull dimension having infinite residue fields; or

e) an associative ring that is \( K \)-regular and that admits the structure of a flat algebra over a regular commutative ring \( k_0 \) of finite Krull dimension such that for every prime \( p \subset k_0 \) having finite residue field, the ring \( \Lambda \otimes_{k_0} k' \) is \( K \)-regular for every étale \((k_0)_p\)-algebra \( k'\).

Recall that an associative ring \( \Lambda \) is called (right) \( K \)-regular if it is (right) Noetherian and every finitely generated (right) \( \Lambda \) module has finite projective dimension. It is called \( K \)-regular if the canonical map \( K_*(\Lambda) \to K_*(\Lambda[x_1, \ldots, x_n]) \) is an isomorphism for all \( n \geq 0 \). Also recall that, up to canonical isomorphism, a (unital) commutative \( C^* \)-algebra is the ring \( C(T) \) of complex-valued continuous functions on a compact Hausdorff space \( T \). All rings considered in this article are unital; in particular, nonunital \( C^* \)-algebras are not considered.

Gubeladze proved the Dilation Theorem for regular \( \mathbb{Q} \)-algebras in [20] and [21]; it was established for regular \( k \) of positive characteristic by the authors in [9]. Gubeladze also verified the theorem when \( A \) is a “simplicial” monoid for all commutative regular rings \( k \) in [19]. Further, his result [18, Theorem 3.2.2] implies that if \( A \) is simplicial and \( k \) is a not necessarily commutative \( K \)-regular ring, then the map \((0.2)\) is an isomorphism whenever \( \epsilon \) is a constant prime sequence \( \epsilon = (p, p, \ldots) \).

**Remark.** Nilpotence for the constant prime sequences \( \epsilon = (p, p, \ldots) \) implies nilpotence for all constant sequences \( \epsilon = (n, n, \ldots) \) because \((c_1)\ast (c_2)\ast = (c_2)\ast (c_1)\ast \ast \). However, it doesn’t imply nilpotence for sequences containing infinitely many primes.

As we showed in [8], it is useful to pass from abelian monoids to pointed abelian monoids (adding an element ‘0’), and to generalize even further to monoid schemes, associating the affine monoid scheme \( \text{MSpec}(A) \) to a pointed monoid \( A \). A monoid scheme \( X \) has a \( k \)-realization \( X_k \) over any commutative ring \( k \); if \( X = \text{MSpec}(A) \) then \( X_k = \text{Spec}(k[A]) \). Again there are dilation maps \( \theta_\epsilon : X \to X \), defined locally as the \( \epsilon \)-th power map on affine open subschemes, and \( \theta_\epsilon \) induces dilations of both \( X_k \) and its \( K \)-theory.

Even if \( \Lambda \) is a noncommutative ring, we can still make sense out of \( K(X_\Lambda) \), although the scheme \( X_\Lambda \) is not defined. If \( X = \text{MSpec}(A) \) then \( K(X_\Lambda) \) is \( K(\Lambda[A]) \); in general, the spectrum \( K(X_\Lambda) \) may either be defined using Zariski descent on \( X \), or equivalently as the \( K \)-theory of the dg category \( \text{perf}(X_k) \otimes_k^L \Lambda \); see Example 4.3 for details.

Theorem 0.3 below is the **Dilation Theorem** for monoid schemes; the hypotheses on \( X \) are satisfied whenever it is a toric monoid scheme or, more generally, a partially cancellative torsionfree (pctf) monoid scheme of finite type. (The definitions of all these terms are recalled below, at the end of this introduction.) They are also satisfied by \( X = \text{MSpec}(A) \) when \( A \) is a monoid of finite type satisfying the assumptions in Theorem 0.1. Observe also that since every abelian monoid is the filtered colimit of its finitely generated submonoids, and since \( K \)-theory commutes with such colimits, the finitely generated case of Theorem 0.1 implies the general case. We remark also that for \( A \) and \( \Lambda \) as in Theorem 0.1, \( K_*(\Lambda) = KH_*(\Lambda) = KH_*(\Lambda[A]) \). Thus Theorem 0.1 follows from Theorem 0.3.

**Theorem 0.3.** Let \( X \) be a separated, partially cancellative, torsionfree monoid scheme of finite type, let \( \epsilon = (c_1, c_2, \ldots) \) be a sequence of integers with \( c_i \geq 2 \) for
all $i$ and let $\Lambda$ be an associative ring satisfying one of (a) through (e) in Theorem 0.1. Then the canonical map is an isomorphism:

\[
\lim_{\theta} K_{\ast}(X_{\Lambda}) \cong \lim_{\theta} KH_{\ast}(X_{\Lambda}).
\]

The particular case of this theorem when $\Lambda$ is a commutative regular ring containing a field was proven by the authors in [7] and [9], and was used to verify Gubeladze’s conjecture for these rings.

We remark that for commutative noetherian rings $k$, Theorem 0.3 is equivalent to the affine Dilation Theorem 0.1, by Zariski descent. However, the affine case seems to be no easier to prove. Our proof of the Dilation Theorem follows the strategy used in [9], using Theorem 0.3 to deduce 0.1. The basic outline is as follows:

**Step 1.** We show that the singularities of any nice monoid scheme may be resolved by a finite sequence of blow-ups along smooth normally flat centers. This is based upon a theorem of Bierstone-Milman (see [2]). Even if we begin with a monoid, the blow-ups of its monoid scheme take us out of the realm of (affine) monoids.

This step is carried out in Section 2, and is preceded by a short calculation with monoids in Section 1. This step is independent of $\Lambda$.

**Step 2.** In [8], we introduced the notion of cdh descent for presheaves on pctf monoid schemes, and showed that the functor $X \mapsto KH(X_{\Lambda})$ from pctf monoid schemes to spectra satisfies cdh descent when $\Lambda$ is a commutative regular ring of finite Krull dimension that contains a field. In Section 3, we review and use Step 1 to modify this proof, replacing the assumptions on $\Lambda$ given above by the assumptions that $\Lambda$ is a flat associative algebra over a commutative regular ring of finite Krull dimension all of whose residue fields are infinite; see Corollaries 3.7 and 4.4. Also in this section, we apply the arguments of [9] to prove that both the fiber of the Jones-Goodwillie Chern character $K(X_{\Lambda}) \otimes \mathbb{Q} \to HN(X_{\Lambda} \otimes \mathbb{Q})$ and the fiber of the $p$-local cyclotomic trace with $\mathbb{Z}/p^n$-coefficients to the pro-spectrum $\{TC^\nu(X_{\Lambda}; p)\}_\nu$ satisfy cdh descent whenever $\Lambda$ is as in Theorem 0.1 (see Propositions 4.7 and 4.11).

**Step 3.** For a presheaf of spectra $E$ on monoid schemes, write $F_E$ for the homotopy fiber of the map from $E$ to its cdh-fibrant replacement. From Step 2 we conclude that if $\Lambda$ is a flat $K$-regular algebra over acommutative regular ring $k_0$ of finite Krull dimension all of whose residue fields are infinite, then $F_K$ is equivalent to the fiber of the map $K \to KH$, its rationalization is equivalent to $F_{HN(- \otimes \mathbb{Q})}$, and for each $n$ the cofiber $F_K/p^n$ of multiplication by $p^n$ is equivalent to the pro-spectrum $\{F_{E_{\nu}}\}$ for $E_{\nu} = TC^\nu(-; p)/p^n$. Making use of the constructions developed in [9, Section 6], we prove that for an arbitrary ring $\Lambda$, monoid scheme $X$ and $\nu \geq 1$, both $F_{HN(- \otimes \mathbb{Q})}(X)$ and $F_{TC^\nu(-; p)}(X)$ become contractible upon taking the colimit over any infinite sequence of non-trivial dilations. The case of $TC^\nu(-; p)$ is considered in Section 5 (see Theorem 5.3) and that of $HN(- \otimes \mathbb{Q})$ in Section 6 (see Corollary 6.3).

**Step 4.** Theorem 0.3 is finally obtained in Section 7: Theorem 7.5 gives parts (c) and (e) and the other parts are consequences. The particular case when $\Lambda$ is commutative and contains a field is [9, Theorem 8.3]. The case when $\Lambda$ is flat over a commutative regular ring $k_0$ of finite Krull dimension all of whose residue fields are infinite follows using Steps 2-3 and the argument of the proof of [9, Theorem 8.3]. We show further that, if $k_0$ has finite Krull dimension and $\Lambda \supset k_0$ satisfies the hypothesis of the theorem, then the map (0.4) is an isomorphism if and only if this happens for every local ring of $k_0$ (Lemma 7.1). Then we show that for $k_0$ a
regular local ring, the case when the residue field is infinite implies the case of finite residue field. This establishes parts (c) and (e) of the theorem. Part (d) is a special case of part (e) and part (b) follows from the fact that commutative $C^*$-algebras are $K$-regular ([11, Theorem 8.2]). Finally we observe that every commutative regular ring $\Lambda$ with Spec($\Lambda$) connected is a flat extension of either a finite field or a localization of the ring of integers; this proves (a).

Some notation: Given a presheaf $E$ from some full subcategory of the category of monoid schemes to spectra (or spaces, or chain complexes, or equivariant spectra or spaces) and a sequence $c = (c_1, c_2, c_3, \ldots)$ of integers larger than 1, we write $E^c$ for the (sequential) homotopy colimit $\varinjlim_n E_{a_{c_n}}$, again a presheaf of spectra (or spaces, or ...) on the same category of monoid schemes. If our presheaf is obtained by sending the monoid scheme $X$ to the spectrum (or space, etc.) $E(X_\Lambda)$ for a ring $\Lambda$ we will sometimes (when no confusion is possible) abuse notation and write $E^c$ for the presheaf $X \mapsto \varinjlim_n E(X_\Lambda)$.

Monoid terminology: Unless otherwise stated, a monoid is a pointed abelian monoid written multiplicatively, i.e., it is an abelian monoid with unit 1 and a basepoint $0$ satisfying $a \cdot 0 = 0$ for all $a \in A$. If $B$ is an unpointed monoid, adjoining an element $0$ yields a pointed monoid $B_+$. If $A$ is pointed monoid, we write $A'$ for the subset $A \setminus \{0\}$. We say that $A$ is cancellative if $ab = ac$ implies $b = c$ or $a = 0$ for any $a, b, c \in A$. In particular, if $A$ is cancellative, then $A'$ is an unpointed submonoid of $A$. We say $A$ is torsionfree if whenever $a^n = b^n$ for $a, b \in A$ and some $n \geq 1$, we have $a = b$. A monoid is cancellative and torsionfree if and only if it is a submonoid of the pointed monoid $T_+$ associated to a torsionfree abelian group $T$.

An ideal $I$ of a monoid $A$ is a subset containing 0 and satisfying $AI \subseteq I$. In this case, the quotient monoid $A/I$ is defined by collapsing $I$ to 0; the product in $A/I$ is the unique one making the canonical surjection $A \rightarrow A/I$ a morphism. A monoid $A$ is said to be partially cancellative if it is a quotient $C/I$ of a cancellative monoid $C$ by some ideal $I$; if $C$ is both cancellative and torsionfree, we say that $A$ is pcf.

The prime ideals in $A$ form a space $\text{MSpec}(A)$, with a sheaf of monoids. A monoid scheme is a space $X$ with a sheaf of monoids $\mathcal{A}$ that is locally isomorphic to $\text{MSpec}(A)$ for some $A$. A closed subscheme $Z$ is equivariant if its structure sheaf is $\mathcal{A}/\mathcal{I}$ for a sheaf of ideals $\mathcal{I}$.

1. Free $A$-sets

Given a monoid $A$, an $A$-set is a pointed set $X$, with basepoint 0, together with a function $A \times X \rightarrow X$, written $(a, x) \mapsto a \cdot x$, such that $a \cdot (a' \cdot x) = a a' \cdot x$ for all $a, a' \in A$ and $x \in X$. The $k$-realization $k[X]$ of an $A$-set is a $k[A]$-module. The monoid $A$ is an $A$-set in an obvious way, and given any collection of $A$-sets, their wedge sum is again an $A$-set. A subset $B$ of an $A$-set $X$ is a basis if $X = \bigvee_{b \in B} Ab$ and if $ab = a'b'$, for $a, a' \in A'$ and $b, b' \in B$, implies that either $b = b'$ and $a = a'$ or $a = a' = 0$. We say that an $A$-set $X$ is free if it has a basis, or equivalently if it is isomorphic to a wedge sum of copies of $A$. $\bigvee_{i \in I} A$.

The goal of this section is to prove the following result.

Theorem 1.1. Let $A$ be a cancellative monoid, $X$ a finitely generated $A$-set and $k$ a commutative ring. If $k[X]$ is a free $k[A]$-module, then $X$ is a free $A$-set.
Example 1.2. Suppose that $A = G_+$ for an abelian group $G$. Then $X \setminus \{0\}$ is a disjoint union of orbits $G/H_i$, and $k[X]$ is the direct sum of the $k[G/H_i]$. If $k[X]$ is a free $k[A]$-module, then each $H_i$ is trivial, so $X$ is a free $A$-set.

The proof requires a sequence of preliminary results.

Let $A^\times$ denote the group of units of $A$ and define the quotient monoid $\overline{A} = A/\sim$ where $a \sim b$ if and only if $a = ub$ for some $u \in A^\times$. Similarly, $\overline{X} = X \times_A \overline{A}$ is the $\overline{A}$-set $X/\sim$, where $x \sim y$ if and only if $x = uy$ for some $u \in A^\times$.

Lemma 1.3. For any monoid $A$ and any commutative ring $k$, $k[\overline{A}] \cong k[A] \otimes_{k[A^\times]} k$. If $k[X]$ is free as a $k[A]$-module, then $k[\overline{X}]$ is free as a $k[\overline{A}]$-module.

Proof. The kernel $I$ of $k[A^\times] \to k$ is generated by $\{u - 1 : u \in A^\times\}$; we have to show that the surjection $k[A]/Ik[A] \to k[\overline{A}]$ is an isomorphism. Choose a set of representatives $\{a_i\}$ for $\sim$, so that $A$ is the disjoint union of 0 and the sets $A^\times a_i$. Then $k[A]$ is the direct sum of the $k[A^\times a_i]$, and $k[A]/Ik[A]$ is the direct sum of the $k[a_i]$, i.e., $k[\overline{A}]$. The result for free modules follows from the equation below (see [8, paragraph below diagram 1.8])

$$k[X] = k[X \times_A \overline{A}] \cong k[X] \otimes_{k[A]} k[\overline{A}].$$

We will say that an $A$-set $X$ is cancellative if $ax = bx$ implies $a = b$ for any nonzero $x \in X$ and $a, b \in A$.

Lemma 1.4. Assume $A$ is a cancellative monoid and $X$ is a cancellative $A$-set. Then $X$ is free as an $A$-set if and only if $\overline{X}$ is free as an $\overline{A}$-set.

Proof. The forward direction holds since $\bigvee_I \overline{A} = \bigvee_I \overline{A}$. Conversely, suppose that $\overline{X}$ is free as an $\overline{A}$-set with basis $\overline{B}$. Let $B$ be a subset of $X$ given by choosing one representative for each element of $\overline{B}$. It is clear that $B$ generates $X$, and is an $A$-basis of $X$ because $X$ is cancellative: if $ab = a'b$ in $X$ then $a = a'$.

Remark 1.5. In Lemma 1.4, the hypothesis that $X$ be cancellative is needed, as the example $A = \{0, 1, -1\}$ and $X = \{0, 1\}$ shows; in this case $\overline{A} = \overline{X}$.

Remark 1.6. If $A$ is cancellative and has no non-trivial units, we define a partial ordering on $A$ by $b \leq c$ if $c = ab$ for some $a \in A$. This partial ordering respects the group operation on $A$: if $a \leq b$ then $ac \leq bc$ for all $c \in A$.

The group completion $G_+$ of such a monoid $A$ is also partially ordered: $h \leq g$ if $g = ah$ for some $a \in A$. The inclusion $A \subset G_+$ respects the partial ordering.

We say that a ring $R$ is $A$-graded if $R = \bigoplus_{a \in A} R_a$ with $R_aR_b \subseteq R_{ab}$ and $R_0 = \{0\}$. If $A$ has no non-trivial units, then $R_{\neq 1} = \bigoplus_{a \neq 1} R_a$ is an ideal of $R$, with $R_1 = R/R_{\neq 1}$.

Example 1.7. For any monoid $A$ and ring $k$, the monoid ring $k[A]$ is $A$-graded with $k[A]_a := k \cdot a$. If $A$ has no non-trivial units, $k[A]_{\neq 1} = k[m]$ where $m = m_A$ is the unique maximal ideal of $A$.

Lemma 1.8. Let $A$ be a cancellative monoid with no non-trivial units. Suppose $R$ is an $A$-graded ring. If $T$ is a finitely generated nonzero $A$-graded $R$-module such that $T_0 = 0$, then $R_{\neq 1}T \neq T$. 
Proof. Set $S = \{a \in A' : T_a \neq 0\}$, and suppose that $T$ is generated by homogeneous elements $x_1, \ldots, x_l$ of degrees $s_1, \ldots, s_l \ (s_i \in S)$. Let $S_0$ denote the set of minimal elements in $\{s_1, \ldots, s_l\}$ with respect to the partial ordering in Remark 1.6. Then every element of $S_0$ is minimal in $S$. If $s \in S$, there is a non-zero homogeneous element $t_0$ in $T_0$. Write $t = \sum r_i x_i$. Grouping the sum by common degrees, we may assume the $r_i$ are homogeneous and that $r_i = 0$ unless $s = |r_i x_i|$. Since $t \neq 0$, there is an $i$ so that $s = |r_i x_i| \geq s_i$. This shows that $s \geq s_0$ for some $s_0 \in S_0$.

Fix $s \in S_0$ and let $t$ be a non-zero element of $T_s$. If $t \in R_{\neq 1} T$ then $t = \sum_i r_i x_i$ with $r_i \in R_{\neq 1}$. As before, we may assume all the $r_i$ are homogenous and nonzero, and that $r_i x_i \in R_s$ for all $i$. Since $r_i \in R_{\neq 1}$, we have $|r_i| > 1$ and so $|r_i x_i| > 1|x_i| = s_i$. This contradicts the minimality of $s$, showing that $T \neq R_{\neq 1} T$. \hfill \Box

**Corollary 1.9.** Let $A$ be a cancellative monoid with no nontrivial units. Suppose $R$ is an $A$-graded ring and $T$ is a finitely generated, free graded $R$-module. If $Y$ is any set of homogeneous elements in $T$ whose image in $T/R_{\neq 1} T$ is a basis of the free $R_1$-module $T/R_{\neq 1} T$, then $Y$ is a basis of $T$ as an $R$-module.

**Proof.** The classical proof applies; here are the details:

First note that $Y$ must be finite, since it maps mod $R_{\neq 1}$ to a basis of a free, finitely generated $R_1$-module. Let $L$ denote the free graded $R$-module on the set $Y$. The canonical map $\pi : L \to T$ is graded and $L_0 = T_0 = 0$; hence its kernel $K$ and cokernel $C$ are both graded modules, and $K_0 = C_0 = 0$. By applying $- \otimes_R R/R_{\neq 1}$ to this map results in an isomorphism by assumption, $C/R_{\neq 1} C = 0$. By Lemma 1.8, $C = 0$ and hence $\pi$ is surjective. Since $T$ is free graded, $\pi$ splits (in the graded category). In particular, $K$ is a direct summand of a finitely generated free module and hence finitely generated. It follows that $K/R_{\neq 1} K = 0$ as well, whence $K = 0$ using the Lemma again. \hfill \Box

**Lemma 1.10.** Suppose $A$ is a monoid, $X$ is an $A$-set and $S \subseteq A$ is a multiplicatively closed subset. Then the natural homomorphism of $k[A]$-modules $S^{-1} k[X] \to k[S^{-1} X]$ is an isomorphism.

**Proof.** The inverse homomorphism is obtained from the natural map of $A$-sets $S^{-1} X \to S^{-1} k[X]$ by adjunction. \hfill \Box

**Lemma 1.11.** If $A$ is a cancellative monoid and $X$ an $A$-set so that $k[X]$ is a free $k[A]$-module, then $X$ is a cancellative.

**Proof.** Consider the group completion of $A$, which is isomorphic to $G_+$ for some abelian group $G$. Since $k[A] \subseteq k[G]$ and $k[X]$ is free, we have $k[X] \subseteq k[X] \otimes_{k[A]} k[G]$. By Lemma 1.10 and [8, Lemma 5.5], $k[X] \otimes_{k[A]} k[G] = k[X_G]$, where $X_G$ is the localization of $X$ at $A'$. It follows that $X \subseteq X_G$, and the latter is a free $G_+$-set by Example 1.2. Thus, $X$ is a cancellative $A$-set as asserted. \hfill \Box

**Proof of Theorem 1.1.** By Lemma 1.11, $X$ is a cancellative $A$-set. Thus Lemmas 1.3 and 1.4 imply that we may assume that $A$ has no non-trivial units.

Define a partial ordering on $X$ by $x \leq y$ if and only if $y = ax$ for some $a \in A$. The reflexive and transitive properties are clear. Since $X$ is cancellative, it is also symmetric: if $x \leq y$ and $y \leq x$ there are $a, b \in A$ with $y = ax, x = by$ and hence $x = abx$. This implies that $ab = 1$; since $A$ has no nontrivial units, we have $a = b = 1$ and hence $x = y$. (When $X = A$, this is the partial ordering of Remark 1.6.) As $X$ is finitely generated, there is a finite set $B$ of minimal elements of $X$
with respect to the partial ordering. In fact, $B$ is the unique smallest collection of generators of $X$. We will prove that $B$ forms a basis of $X$.

Let $F$ denote the free graded $A$-set on $B$. We need to show that the canonical map $F \xrightarrow{\pi} X$ sending $a[b]$ to $ab$ is a bijection. The map is surjective since $B$ generates $X$ by construction. Passing to $k$-realization gives a surjection of graded $k[A]$-modules, $k[F] \rightarrow k[X]$. By Corollary 1.9, with $R = k[A]$ and $Y = B$, it suffices to prove that this map becomes an isomorphism after applying $\sim \otimes_{k[A]} k$, where $k = k[A]/k[A]_{\neq 1}$; this will show that $k[F] \rightarrow k[X]$ and hence $F \rightarrow X$, are injective. By construction, there is a canonical isomorphism $k[F] \otimes_{k[A]} k \cong k[B_+]$, and, by Example 1.7,

$$k[X] \otimes_{k[A]} k = k[X] \otimes_{k[A]} k[A/m_A] = k[X/m_A X].$$

Since the natural map $B_+ \rightarrow X/m_A X$ induced by $\pi$ is an isomorphism, the induced map $k[B_+] \rightarrow k[X/A_+ X]$ is also an isomorphism. Thus $B$ is a basis of $X$. \hfill $\square$

2. Normal flatness for monoid schemes

Recall from [8, 6.4] that a finitely generated pointed monoid $A$ is smooth if $A$ is the wedge product of a finitely generated free pointed monoid $T$ by $\Gamma_+$, where $\Gamma$ is a finitely generated free abelian group. That is, for some $d$ and $r$:

$$A = \langle t_1, \ldots, t_d, s_1, 1/s_1, \ldots, s_r, 1/s_r \rangle.$$

Note that $k[A]$ is the Laurent polynomial ring $k[t_1, \ldots, t_d, s_1, 1/s_1, \ldots, s_r, 1/s_r]$ for every ring $k$. More generally, we say that a monoid scheme $X$ is smooth if each stalk monoid $A_x$ is smooth.

Recall that a commutative ring $R$ is said to be normally flat along an ideal $J$ if each quotient $J^n/J^{n+1}$ is a projective module over the ring $R/J$. A scheme is said to be normally flat along a closed subscheme if locally the coordinate ring is normally flat along the ideal defining it.

**Definition 2.1.** A monoid $A$ is said to be normally flat along an ideal $I$ if each quotient $I^n/I^{n+1}$ is a free $A/I$-set. Let $X$ be a monoid scheme of finite type with structure sheaf $A$; we say that $X$ is normally flat along an equivariant closed subscheme $Z$ if for every $x \in X$ the monoid $A_x$ is normally flat along the stalk of the ideal defining $Z$.

**Proposition 2.2.** Let $I$ be a finitely generated ideal in a monoid $A$, with $A/I$ smooth. The following are equivalent:

1) $A$ is normally flat along $I$.
2) $k[A]$ is normally flat along the ideal $k[I]$ for every commutative ring $k$.
3) $\mathbb{Q}[A]$ is normally flat along the ideal $\mathbb{Q}[I]$.

**Proof.** The $k$-realization functor commutes with colimits, since it is left adjoint to the forgetful functor, so $k[I^n/k[I]^{n+1} \cong k[I^n/I^{n+1}]$. Hence (1) implies (2). It is clear that (2) implies (3). Suppose that (3) holds, and write $J$ for $\mathbb{Q}[I]$. Then each quotient $J^n/J^{n+1} \cong \mathbb{Q}[J^n/J^{n+1}]$ is a projective module over the ring $\mathbb{Q}[A]/J = \mathbb{Q}[A/I]$, and, since all projective $\mathbb{Q}[A/I]$-modules are free [31], Theorem 1.1 implies that each $A/I$-set $I^n/I^{n+1}$ is free. Hence (1) holds. \hfill $\square$
Corollary 2.3. Let $Z$ be a smooth equivariant closed subscheme of $X$, a monoid scheme of finite type. The following are equivalent:

1) $X$ is normally flat along $Z$.
2) $X_k$ is normally flat along $Z_k$ for every commutative ring $k$.
3) $X_\mathbb{Q}$ is normally flat along $Z_\mathbb{Q}$.

Proof. Again, it suffices to prove that (3) implies (1). As the conditions are local, it suffices to pick $x \in X$ and assume that $X = \text{MSpec}(A)$ and $Z = \text{MSpec}(A/I)$ for $A = A_x$, with $A/I$ smooth. This case is covered by Proposition 2.2. □

Theorem 2.4. Let $X$ be a separated cancellative torsionfree monoid scheme of finite type, embedded as an equivariant closed subscheme in a smooth toric scheme. Then there is a sequence of blow-ups along smooth equivariant centers $Z_i \subset X_i$, $0 \leq i \leq n - 1$, $Y = X_n \to \cdots \to X_0 = X$ such that $Y$ is smooth and each $X_i$ is normally flat along $Z_i$. In addition, for any commutative ring $k$, each $(X_i)_k$ is normally flat along $(Z_i)_k$.

Proof. The case $k = \mathbb{Q}$ of Theorem 14.1 in our paper [8] states that there is a sequence of blowups along smooth equivariant centers $Z_i \subset X_i$, such that each $(X_i)_\mathbb{Q}$ is normally flat along $(Z_i)_\mathbb{Q}$. By Corollary 2.3, each $X_i$ is normally flat along $Z_i$, and each $(X_i)_k$ is normally flat along $(Z_i)_k$. □

3. A descent theorem for functors via realizations

In this section, we recall the notion of cdh descent, establish a technical result generalizing [8, Theorem 14.3] to the present context, and use it to promote several results in op. cit. to our situation.

In more detail, let $\mathcal{M}_\text{pctf}$ denote the category of separated pctf monoid schemes of finite type. Fix a commutative regular ring $k_0$ of finite Krull dimension all of whose residue fields are infinite, and suppose $k_0 \subseteq k$ is a flat extension of commutative rings. We will show that if $F$ is a presheaf of spectra on the category $\text{Sch}/k$ of separated $k$-schemes essentially of finite presentation that satisfies a weak version of cdh descent (see Definition 3.4), then the presheaf $F$ on $\mathcal{M}_\text{pctf}$ defined by $F(X) = F(X_k)$ satisfies cdh descent; see Theorem 3.6.

We first recall the necessary definitions from [8] and [36]. By a cd structure on a category, we mean a family of distinguished commutative squares

\[
\begin{array}{ccc}
D & \longrightarrow & Y \\
\downarrow & & \downarrow p \\
C & \leftarrow & X.
\end{array}
\]

Definition 3.2. A cartesian square (3.1) in $\mathcal{M}_\text{pctf}$ is called

1) an abstract blow-up square if $p$ is proper, $e$ is an equivariant closed immersion, and $p$ maps the open complement $Y \setminus D$ isomorphically onto $X \setminus C$;
2) a Zariski square if $p$ and $e$ form an open cover of $X$;
3) a cdh square if it is either a Zariski or an abstract blow-up square.

The cdh topology on $\mathcal{M}_\text{pctf}$ is the topology generated by the cdh squares. It is a bounded, complete and regular cd structure by [8, Theorems 12.7 and 12.8].
A presheaf of spectra $F$ on $\mathcal{M}_{\text{petf}}$ satisfies the Mayer-Vietoris property for some family $\mathcal{C}$ of cartesian squares if $F(\emptyset) = \ast$ and the application of $F$ to each member of the family gives a homotopy cartesian square of spectra.

**Definition 3.3.** Let $F$ be a presheaf of spectra on $\mathcal{M}_{\text{petf}}$. We say that $F$ satisfies cdh descent on $\mathcal{M}_{\text{petf}}$ if the canonical map $F(X) \to \mathbb{H}_{\text{cdh}}(X, F)$ is a weak equivalence of spectra for all $X$. Here $\mathbb{H}_{\text{cdh}}(\ast, F)$ is the fibrant replacement of $F$ in the model structure of $[24]$ and $[25]$. By $[8$, Proposition 12.10$,]$ this is equivalent to $F$ satisfying the Mayer-Vietoris property for the cdh structure; i.e., for both the family of Zariski squares and the family of abstract blow-up squares.

Here is a restatement of $[8$, Definition 13.8$.]$

**Definition 3.4.** Let $k$ be a commutative ring and let $\text{Sch}/k$ be the category of separated $k$-schemes, essentially of finite presentation. A presheaf of spectra $F$ on $\text{Sch}/k$ is said to satisfy weak cdh descent if it satisfies the Mayer-Vietoris property for all open covers, finite abstract blow-ups, and blow-ups along regularly embedded subschemes.

**Example 3.4.1.** The presheaf $K_1$ satisfies weak cdh descent. This was observed in $[8$, Example 13.11$]$, the hypothesis there that the ring $k$ be Noetherian is not needed. Indeed, the Mayer-Vietoris property for finite abstract blow-ups (excision for ideals and invariance under nilpotent extensions) is proved in $[37]$ for general rings, and Mayer-Vietoris for open covers as well as Mayer-Vietoris for blow-ups along regular sequences applies to the category of quasi-separated and quasi-compact schemes.

We will see other examples in Theorems 3.6 and 4.2 below.

**Theorem 3.5.** Let $k$ be a commutative regular ring of finite Krull dimension, with infinite residue fields, and $F$ a presheaf of spectra on $\text{Sch}/k$. If $F$ satisfies weak cdh descent on $\text{Sch}/k$, then the presheaf $F(X) = F(X_k)$ satisfies cdh descent on $\mathcal{M}_{\text{petf}}$.

**Proof.** The proof is almost identical to the proof of Theorem 14.3 in our earlier paper $[8]$, which assumed that $k$ contains a field.

We merely point out the adjustments necessary for the proof to work for $k$. Although the proof in loc. cit. makes no direct use of the hypothesis that $k$ contains a field, this hypothesis is buried in the references to Lemma 13.9 and Theorem 14.2 of $[8]$. (Theorem 13.3 and Proposition 13.6 are referred to, but apply as stated).

Although the hypothesis of Lemma 13.9 is that $k$ is a commutative regular domain containing an infinite field, the proof goes through if we only assume that $k$ is a commutative regular ring of finite Krull dimension such that every residue field of $k$ is infinite. The hypothesis that $k$ is regular is needed so that $X_k$ is a Cohen-Macaulay scheme for every toric monoid scheme $X$ (by $[23]$). The finite Krull dimension hypothesis is needed in Lemma 13.9 in order that the local-global spectral sequence converges (and so the proof of Lemma 13.9 is flawed as written; see the remark below for the correction).

The last hypothesis, that every residue field of $k$ is infinite, is needed to use Lemma 13.7 of $[8]$. Without this hypothesis, we would have to pass to finite extensions of each local ring of $k$ to get a minimal reduction generated by a regular sequence.

Finally, the hypothesis in Theorem 14.2 that $k$ contains a field is required to make use of the Bierstone-Milman theorem $[8$, Theorem 14.1$.]$. Replacing it by the more general Theorem 2.4 in this paper makes the proof go through. □
Remark 3.5.1. As explained in the proof above, the hypothesis that $k$ have finite Krull dimension is missing in Lemma 13.9 and Theorem 14.3 of [8]. This however does not affect the main results of [8], since $K$-theory commutes with filtering colimits and, by Popescu’s theorem [32], every regular ring containing a field is a filtering colimit of regular rings of finite Krull dimension.

Theorem 3.6. Let $k_0$ be a commutative regular ring of finite Krull dimension all of whose residue fields are infinite, let $i : k_0 \subseteq k$ be a flat extension of commutative rings. If $F$ is a presheaf of spectra on $\text{Sch}/k$ satisfying weak cdh descent on $\text{Sch}/k_0$, then the presheaf $F_k$ on $\mathcal{M}_{\text{pctf}}$ defined by $F_k(X) = F(X_k)$ satisfies cdh descent.

Proof. Let $i_* F$ denote the direct image presheaf on $\text{Sch}/k_0$, defined by $i_* F(S) = F(S_k)$. Since the flat basechange $i_*$ preserves open immersions, closed immersions, surjective morphisms, finite morphisms, regular closed immersions, and blow-ups, it also preserves open covers, finite abstract blow-ups, and blow-ups along regular closed immersions. Because $F$ satisfies weak cdh descent on $\text{Sch}/k$, $i_* F$ satisfies weak cdh descent on $\text{Sch}/k_0$. The result is now immediate from Theorem 3.5.

Given a commutative ring $k$, let $\mathcal{K}_k$ and $KH_k$ denote the presheaves of spectra on $\mathcal{M}_{\text{pctf}}$ sending $X$ to $K(X_k)$ and $KH(X_k)$, respectively. Here is the analogue of [8, Corollary 14.5].

Corollary 3.7. Let $k_0$ be a commutative regular ring of finite Krull dimension all of whose residue fields are infinite, and let $k$ be a commutative flat $K$-regular $k_0$-algebra. Then the presheaf of spectra $KH_k$ satisfies cdh descent on $\mathcal{M}_{\text{pctf}}$, and the maps

$$KH_k(X) \to H_{\text{cdh}}(X, KH_k) \leftarrow H_{\text{cdh}}(X, \mathcal{K}_k)$$

are weak equivalences for all $X$ in $\mathcal{M}_{\text{pctf}}$.

Proof. As observed in Example 3.4.1, the presheaf $KH$ on $\text{Sch}/k$ satisfies weak cdh descent. Thus the first statement follows from Theorem 3.6, and therefore the left arrow above is an equivalence.

For the other arrow, we mimic the argument we used in [8, Example 13.11]. By [8, Theorem 11.1], every cdh covering in $\mathcal{M}_{\text{pctf}}$ admits a refinement consisting of smooth monoid schemes. If $X$ is smooth, then $X_k$ is locally the spectrum of a Laurent polynomial ring over $k$. Since $k$ is assumed to be $K$-regular, we conclude that for smooth $X$, $\mathcal{K}_k(X) \to KH_k(X)$ is an equivalence, using Mayer-Vietoris for open covers and the Fundamental Theorem of $K$-theory. The result for general $X$ in $\mathcal{M}_{\text{pctf}}$ now follows from the Mayer-Vietoris property for $H_{\text{cdh}}(X, \mathcal{K}_k)$ and resolution of singularities for monoid schemes.

Remark 3.7.1. If $k$ is commutative and noetherian, the result follows from [3]; see [27].

4. Presheaves of spectra and dg categories

Let $k_0$ be a commutative ring and let $E$ be a functor from small dg $k_0$-categories to spectra. We may use $E$ in two different ways to obtain a functor on $\text{Sch}/k_0$. First, by regarding a $k_0$-algebra as a dg category with just one object, we may restrict $E$ to a functor of commutative $k_0$-algebras. This restriction induces a presheaf $\mathcal{E}$ on $\text{Sch}/k_0$ by mapping $S \mapsto E(\mathcal{O}(S))$ on $\text{Sch}/k_0$, and its fibrant replacement is $\mathbb{H}_{\text{zar}}(-, \mathcal{E})$. On the other hand, we may simply compose $E$ with the functor
that sends a scheme $S$ to the dg $k_0$-category $\text{perf}(S)$ of perfect complexes on $S$, obtaining the functor $E(\text{perf}(\blank))$; see [6, Example 2.7] or [30, Section 2.4] for a precise definition.

Following Sections 3 and 5 of [26], we say $E$ is dg Morita invariant if it sends dg Morita equivalences to weak equivalences; we say that $E$ localizing if it sends short exact sequences of dg categories to fibration sequences.

**Lemma 4.1.** If $E$ is a functor from small dg $k_0$-categories to spectra that is dg Morita invariant and localizing, then the functors $H_{\text{zar}}(\blank, E)$ and $E(\text{perf}(\blank))$ are equivalent.

**Proof.** For a commutative ring $R$, the functor $R \to \text{perf}(R)$ is a dg Morita equivalence and thus $E(R) \xrightarrow{\sim} E(\text{perf}(R))$ is an equivalence. It follows that the natural transformation of presheaves on $\text{Sch}/k_0$ from $E(\mathcal{O}(\blank))$ to $E(\text{perf}(\mathcal{O}(\blank)))$ is an equivalence Zariski locally. The induced functor on fibrant replacements is thus also an equivalence. Since $E$ is localizing, $S \mapsto E(\text{perf}(S))$ satisfies Zariski descent (and even Nisnevich descent) by [33, Theorem 3.1]. We thus get a pair of natural equivalences

$$H_{\text{zar}}(S, E) \xrightarrow{\sim} H_{\text{zar}}(S, E(\text{perf}(\blank))) \xleftarrow{\sim} E(\text{perf}(S))$$

for all $S \in \text{Sch}/k_0$. □

From now on, if $S \in \text{Sch}/k_0$ and $E$ is a functor from small dg $k_0$-categories that is dg Morita invariant and localizing, by $E(S)$ we shall always mean $E(\text{perf}(S))$.

Given two small dg $k_0$-categories $\mathcal{A}$ and $\mathcal{B}$, we write $\mathcal{A} \otimes_{k_0} \mathcal{B}$ for their dg tensor product (as defined in [26, Sec. 2.3]).

**Theorem 4.2.** Let $k_0$ be a commutative ring, $E$ a functor from small dg $k_0$-categories to spectra and $\Lambda$ a flat (not necessarily commutative) $k_0$-algebra. Assume

1. $E$ is dg Morita invariant and
2. $E$ is localizing.

Then the presheaf $E_\Lambda$ of spectra on $\text{Sch}/k_0$, sending $S$ to

$$E_\Lambda(S) = E(\text{perf}(S) \otimes_{k_0} \Lambda),$$

satisfies the Mayer-Vietoris property with respect to open covers (Zariski descent) and blow-ups of regularly embedded subschemes.

Assume furthermore that

3. the restriction of $E$ to $k_0$-algebras satisfies excision for ideals
   and is invariant under nilpotent ring extensions.

Then $E_\Lambda$ satisfies weak cdh descent on $\text{Sch}/k_0$. In particular, by Theorem 3.5, the functor $E_\Lambda(X) = E_\Lambda(X_{k_0})$ satisfies cdh descent on $\mathcal{M}_{\text{perf}}$.

**Proof.** Because $\Lambda$ is flat, tensoring with $\Lambda$ preserves dg Morita equivalences and short exact sequences of dg categories. Hence $E_\Lambda$ is both dg Morita invariant and localizing whenever $E$ is. In particular it satisfies Zariski descent, by the discussion above. The Mayer-Vietoris property for blow-ups along regularly embedded subschemes also follows from hypotheses (1) and (2) using [6, Lemma 1.5], as pointed out in [10, Theorem 3.2] and implicitly in [6, Theorem 2.10]. To finish the proof we must show that if $E$ also satisfies (3) then it has the Mayer-Vietoris property for finite abstract blow-ups. As observed in the proof of [6, Theorem 3.12], the latter property follows from excision for ideals, invariance under nilpotent ring extensions
and Zariski descent. Tensoring with \( \Lambda \) preserves Milnor squares of \( k_0 \)-algebras; because \( E \) satisfies excision for ideals, so does \( E_\Lambda \). If \( I \) is a nilpotent ideal, so is \( I \otimes_{k_0} \Lambda \), so hypothesis (3) implies that \( E_\Lambda \) is invariant under nilpotent ring extensions. \( \square \)

**Remark 4.2.1.** If \( E \) is a dg Morita invariant, localizing functor from small dg \( k_0 \)-categories to spectra, and \( k \) is a flat commutative \( k_0 \)-algebra, then \( E(S_k) \) is naturally equivalent to the \( E_k(S) \) of Theorem 4.2. Indeed, the result holds when \( S = \text{Spec}(R) \) is an affine scheme, since \( R_k \to \text{perf}(R_k) \) and \( \text{perf}(R) \otimes_{k_0} k \to \text{perf}(R_k) \) are dg Morita equivalences; the result for arbitrary schemes follows from the fact that, by Theorem 4.2, both \( E_k \) and \( E(-k) \) satisfy Zariski descent. In particular, it follows that the functors \( E(X) = E(X_k) \) and \( E_\Lambda(X) = E_\Lambda(X_k) \) on \( M_{\text{pctf}} \) are naturally equivalent.

Motivated by all this, we shall often write \( E(X_\Lambda) \) for \( E_\Lambda(X) \) when \( \Lambda \) is a flat but not necessarily commutative \( k_0 \)-algebra, even if no scheme \( X_\Lambda \) is defined. By Lemma 4.1, \( E(X_\Lambda) \) is equivalent to the functor defined by Zariski descent from the spectra \( E(A[A]) \) on the affine opens \( \text{MSpec}(A) \) of \( X \).

**Example 4.3.** For any dg category \( A \), abusing notation a bit, we also write \( \text{perf}(A) \) for the (unenriched) category whose morphisms are \( A \)-module homomorphisms; i.e., the category whose hom sets are the zero cycles of underlying dg category. Then \( \text{perf}(A) \) may be regarded as a Waldhausen category; the weak equivalences are quasi-isomorphisms and the cofibrations are \( A \)-module homomorphisms which admit retractions as homomorphisms of graded \( A \)-modules; see [26, Sec. 5.2] for example.

We define \( K(A) \) to be the (Waldhausen) \( K \)-theory of \( \text{perf}(A) \). As pointed out in loc. cit., this functor is localizing and Morita invariant, and its restriction to \( k_0 \)-algebras is naturally equivalent to the usual \( K \)-theory of algebras. Theorem 4.2 applies to \( KH \), which satisfies (3), showing that \( KH_\Lambda(S) \) satisfies cdh descent on \( \text{Sch}/k_0 \), and \( KH_\Lambda(X) \) satisfies cdh descent on \( M_{\text{pctf}} \).

**Corollary 4.4.** Let \( k_0 \) be a commutative regular ring of finite Krull dimension all of whose residue fields are infinite, and let \( \Lambda \) be a flat \( K \)-regular associative \( k_0 \)-algebra. Then the presheaf of spectra \( KH_\Lambda \) satisfies cdh descent on \( M_{\text{pctf}} \), and the maps

\[
KH_\Lambda(X) \to H_{\text{cdh}}(X,KH_\Lambda) \leftarrow H_{\text{cdh}}(X,K_\Lambda)
\]

are weak equivalences for all \( X \) in \( M_{\text{pctf}} \).

**Proof.** The left arrow is a weak equivalence by Example 4.3. The proof that the other arrow is an equivalence is exactly like the corresponding proof of Corollary 3.7. \( \square \)

**Example 4.5.** The Hochschild homology, cyclic homology and negative cyclic homology of a dg category \( A \) are defined using the mixed complex of \( \text{perf}(A) \); see [26, Sec. 5.3]. Moreover, the restrictions of these functors to flat algebras (resp., schemes) are isomorphic to the usual homology theories of algebras (resp., of schemes).

**Notation 4.6.** Given a presheaf of spectra \( \mathcal{F} \) on some category we will write \( \mathcal{F} \otimes \mathbb{Q} \) for the rationalization of \( \mathcal{F} \) and \( \mathcal{F}/n \) for the cofiber of multiplication by \( n \) on \( \mathcal{F} \). We make an exception for the presheaf of spectra \( HN(-_\Lambda) \) on \( M_{\text{pctf}} \) sending \( X \) to negative cyclic homology \( HN(X_\Lambda) \). By \( HN_{\Lambda\mathbb{Q}}(X) \) we will mean \( HN(X_\Lambda\otimes\mathbb{Q}) \),
which is defined as in Remark 4.2.1. The Jones-Goodwillie Chern character is a natural morphism on algebras, from $K(\Lambda) \otimes \mathbb{Q}$ to $HN(\Lambda \otimes \mathbb{Q})$; see [16, p. 351]. It extends to a natural transformation of functors on dg categories, and hence a natural morphism $K_\Lambda(X) \otimes \mathbb{Q} \to HN_{\Lambda\mathbb{Q}}(X)$ on monoid schemes (see [4, Example 9.10], [28, Section 4.4] for the Chern character for dg categories).

**Proposition 4.7.** Let $k_0$ be commutative regular ring of finite Krull dimension all of whose residue fields are infinite, and let $\Lambda$ be a flat, associative $k_0$-algebra that is $K$-regular. For any monoid scheme $X$ in $\mathcal{M}_{\text{pctf}}$, the following square of spectra is homotopy cartesian:

$$
\begin{array}{ccc}
K_\Lambda(X) \otimes \mathbb{Q} & \longrightarrow & KH_\Lambda(X) \otimes \mathbb{Q} \\
\downarrow & & \downarrow \\
HN_{\Lambda\mathbb{Q}}(X) & \longrightarrow & \mathbb{H}_{\text{cdh}}(X, HN_{\Lambda\mathbb{Q}}).
\end{array}
$$

**Proof.** By Corollary 4.4 the presheaves in the right column satisfy cdh-descent on $\mathcal{M}_{\text{pctf}}$. Thus to prove the assertion, we need to show that the homotopy fiber $F$ of the left vertical map satisfies cdh descent. Since $K_\Lambda \otimes \mathbb{Q}$ and $HN_{\Lambda\mathbb{Q}}$ satisfy Zariski descent, so does $F$. Given Examples 4.3 and 4.5, Theorem 4.2 shows that it suffices to show that $F$ (the corresponding homotopy fiber on $\text{Sch}/k_0$) satisfies excision for ideals and invariance under nilpotent extensions. The first of these properties is the main theorem of [5]; the second is the scheme version of Goodwillie’s [16].

The analogue of the Jones-Goodwillie Chern character in characteristic $p$ is the cyclotomic trace; it is a compatible family of morphisms $\operatorname{tr}^\nu : K(\Lambda) \to TC^\nu(\Lambda; p)$, where $\Lambda$ is an associative ring and the pro-spectrum $\{TC^\nu(\Lambda; p)\}_\nu$ is $p$-local topological cyclic homology. For each $n \geq 1$, the cyclotomic trace induces a map $K(\Lambda)/p^n \to TC^\nu(\Lambda; p)/p^n$. Letting $\nu$ vary, we get a strict map of pro-spectra. Blumberg and Mandell showed in [1, 1.1, 1.2, 7.1, 7.3] that $TC^\nu$ extends to a functor on dg categories that is localizing, Morita invariant and satisfies Zariski descent on schemes.

As shown in [1] the cyclotomic trace extends to a natural transformation of functors on dg categories; in particular, if $X$ is a monoid scheme and $\Lambda$ is an associative, flat algebra over some commutative ring $k_0$, we obtain a natural map $\operatorname{tr}^\nu : K(X_\Lambda)/p^n \to TC^\nu(X_\Lambda;p)$.

Recall from [14] (or [8, Section 14]) that a strict map $\{E^\nu\} \to \{F^\nu\}$ of pro-spectra is said to be a weak equivalence if for every $q$ the induced map $\{\pi_q(E^\nu)\} \to \{\pi_q(F^\nu)\}$ is an isomorphism of pro-abelian groups. Let

$$
\begin{array}{ccc}
\{E^\nu\} & \longrightarrow & \{F^\nu\} \\
\downarrow & & \downarrow \\
\{G^\nu\} & \longrightarrow & \{H^\nu\}
\end{array}
$$

be a square diagram of strict maps of pro-spectra. We say that (4.8) is homotopy cartesian if the canonical map from the upper left pro-spectrum to the level-wise homotopy limit of the other terms is a weak equivalence.
The presheaves of pro-spectra we shall consider come from pro-presheaves of spectra, that is, from inverse systems of presheaves of spectra. Let \( \{ F^\nu \} \) be a pro-presheaf of spectra on \( \text{Sch}/k \); for \( X \) in \( \mathcal{M}_{\text{pctf}} \), write \( F^\nu(X) \) for \( F^\nu(X_k) \).

The pro-analogue of \( \{ F^\nu \} \) (or \( \{ F^\nu \} \)) having the Mayer-Vietoris property for a family of squares is the obvious one, as are the notions 3.3, 3.4 of \( \{ F^\nu \} \) having cdh descent or weak cdh descent. Here is the pro-analogue of Theorem 3.5.

**Theorem 4.9.** Let \( \{ F^\nu \} \) be a pro-presheaf of spectra on \( \text{Sch}/k \), where \( k \) is a commutative regular ring of finite Krull dimension all of whose residue fields are infinite. If \( \{ F^\nu \} \), regarded as a presheaf of pro-spectra, satisfies weak cdh descent on \( \text{Sch}/k \), then the presheaf \( \{ F^\nu \} \) defined by \( F^\nu(X) = F^\nu(X_k) \) satisfies cdh descent on \( \mathcal{M}_{\text{pctf}} \).

**Proof.** We modify the proof of Theorem 3.5 (which is in turn a modification of the proof of [8, 14.3]). By construction, each \( H_{\text{cdh}}(−, F^\nu) \) satisfies cdh descent. As noted in the proof of Theorem 3.5, the proof of Theorem 13.9 in [8] goes through to show that the presheaf of pro-spectra \( \{ F^\nu \} \) has the Mayer-Vietoris property for “nice” blow-up squares (Definition 13.5 in [8]). Now each \( F^\nu \) has the Mayer-Vietoris property for Zariski squares and smooth blow-up squares, so by [8, 12.13] each \( F^\nu \) satisfies scdh descent (terminology of [8, 12.12]). Hence each \( \{ F^\nu(X) \} \rightarrow \{ H_{\text{cdh}}(X, F^\nu) \} \) is a weak equivalence of pro-spectra by Theorem 14.2 of [8].

**Corollary 4.10.** Assume that \( k_0 \) is a commutative regular ring of finite Krull dimension all of whose residue fields are infinite, and that \( \Lambda \) is a flat unital associative \( k_0 \)-algebra. Let \( \{ E^\nu \} \) be an inverse system of functors from the category of small dg \( k_0 \)-categories to spectra such that each \( E^\nu \) is localizing and dg Morita invariant.

If the restriction of \( \{ E^\nu \} \) to \( k_0 \)-algebras, regarded as a functor to pro-spectra, satisfies excision for ideals and is invariant under nilpotent ring extensions, then the presheaf of pro-spectra \( \{ E^\nu_\Lambda \} \), defined by

\[
E^\nu_\Lambda(S) = E^\nu(\text{perf}(S) \otimes_{k_0} \Lambda),
\]

satisfies weak cdh descent on \( \text{Sch}/k_0 \).

By Theorem 4.9, the functor \( \{ E^\nu_\Lambda \} \) satisfies cdh descent on \( \mathcal{M}_{\text{pctf}} \).

**Proof.** The proof of Theorem 4.2 applies, using Theorem 4.9 in place of Theorem 3.5.

**Proposition 4.11.** Let \( k_0 \) be a commutative regular ring of finite Krull dimension all of whose residue fields are infinite, and let \( \Lambda \) be a flat associative \( k_0 \)-algebra that is \( K \)-regular. For any monoid scheme \( X \) in \( \mathcal{M}_{\text{pctf}} \), any prime \( p \) and all \( n > 0 \), the following square of strict maps of pro-spectra is homotopy cartesian.

\[
\begin{array}{ccc}
K_\Lambda/p^n(X) & \longrightarrow & KH_\Lambda/p^n(X) \\
\downarrow & & \downarrow \\
\{ TC^\nu(X_\Lambda; p)/p^n \}_\nu & \longrightarrow & \{ H_{\text{cdh}}(X_\Lambda, TC^\nu(−; p)/p^n) \}_\nu.
\end{array}
\]

The vertical maps are induced by the cyclotomic trace.

**Proof.** As in Proposition 4.7, it suffices to prove that the homotopy fiber \( \{ F^\nu(X) \} \) of the left vertical map satisfies cdh descent as a functor on \( \mathcal{M}_{\text{pctf}} \), in the sense
that \( \{F^\nu\} \to \{ \mathbb{P}_{cdh}(\_; F^\nu) \} \) is a weak equivalence of pro-spectra. (This uses the fact that pro-abelian groups form an abelian category.)

For each dg category \( A \), let \( F^\nu(A) \) denote the homotopy fiber of \( K/p^n(A) \to TC^\nu(A; p)/p^n \). Each \( F^\nu \) is localizing and dg Morita invariant; these properties are inherited from \( K/p^n \) and \( TC^\nu(-; p)/p^n \). To see that \( \{F^\nu\} \) satisfies weak descent on \( \text{Sch}/k \), and hence that \( \{F^\nu\} \) satisfies cdh descent on \( \mathcal{M}_{pctf} \), it suffices by Corollary 4.10 to show that the restriction of the pro-presheaf \( \{F^\nu\} \) to algebras, regarded as a functor to pro-spectra, satisfies excision for ideals and invariance under nilpotent ring extensions. Excision for ideals is [13, Theorem 1]: if \( I \lhd R \) is an ideal and \( f : R \to S \) is a ring homomorphism mapping \( I \) bijectively to an ideal of \( S \), then the map \( \{F^\nu(R, I)\} \to \{F^\nu(S, f(I))\} \) is a weak equivalence. Invariance under nilpotent ring extensions follows from McCarthy’s theorem [29] as strengthened in [12, Theorem 2.1.1]: if \( I \) is nilpotent then \( \{F^\nu(R, I)\} \) is weakly equivalent to a point. \( \square \)

5. \( \tilde{\Omega} \) and Dilated Cyclic Homologies.

Here we briefly recall the results of Section 6 of our paper [9]. These results do not require the existence of a base field.

The cyclic bar construction \( N^\nu(A) \) of a pointed monoid \( A \) is the cyclic set whose underlying set of \( n \)-simplices is \( A \cap \cdots \cap A \) \((n + 1) \) factors), with \( t_n \) being rotation of the entries to the left. It is \( A \)-graded: for \( a \in A \), a simplex \((a_0, \ldots, a_n)\) is in \( N^\nu(A; a) \) if \( \prod a_i = a \). In [9, 3.1], we introduced the dilated cyclic bar construction

\[
\tilde{N}^\nu(A) = \bigvee_{a \in A} N^\nu(A[\frac{1}{a}]; a),
\]

where the element \( a \in N^\nu(A[\frac{1}{a}]; a) \) refers to the element \( \frac{a}{1} \) of \( A[\frac{1}{a}] \). Thus an \( n \)-simplex of \( \tilde{N}^\nu(A) \) is given by \((a; a_0, \ldots, a_n)\) where \( a \in A \), \( a_0, \ldots, a_n \in A[\frac{1}{a}] \), and the equation \( a_0 \cdots a_n = \frac{a}{1} \) holds in \( A[\frac{1}{a}] \).

The geometric realizations of both \( N^\nu(A) \) and \( \tilde{N}^\nu(A) \) are \( S^1 \)-spaces, meaning they have a continuous action of the circle group \( S^1 \).

We now assume that \( A \) is a quotient \( \tilde{A}/I \) of a cancellative monoid \( \tilde{A} \) (i.e., \( A \) is partially cancellative), and that \( A \) is reduced in the sense that \( a^n = 0 \) implies \( a = 0 \) (this is equivalent to the assertion that if \( a, b \in A \) satisfy \( a^n = b^n \) for all \( n > 1 \) then \( a = b \); see [9, 1.6]). Such a monoid is naturally contained in a monoid \( A_{sn} \), that is seminormal, meaning that whenever \( x, y \in A_{sn} \) satisfy \( x^3 = y^2 \), then there is a (unique) \( z \in A_{sn} \) such that \( x = z^2 \) and \( y = z^3 \). In fact, \( A \to A_{sn} \) is universal with respect to maps from \( A \) to seminormal monoids; see [9, Proposition 1.15].

**Definition 5.1.** ([9, 4.1]) For a pc monoid \( A \), we define \( \tilde{\Omega}_A \) to be the \( S^1 \)-space

\[
\tilde{\Omega}_A = \{\tilde{N}^\nu(A_{sn})\}.
\]

Since \( A \to A_{sn} \) is a functor, the assignment \((X, A) \to \tilde{\Omega}_A = \tilde{\Omega}_A(X) \) yields a presheaf on \( \mathcal{M}_{pctf} \). Moreover, there is a natural map \( |N^\nu(A)| \to \tilde{\Omega}_A \) of \( S^1 \)-spaces.

Given a sequence \( c = (c_1, c_2, \ldots) \) of integers with \( c_i \geq 2 \) for all \( i \), it is shown in [9, 3.6] that \( |N^\nu(A)|^{\geq c_1} \tilde{\Omega}_A^{c_1} \) is an \( S^1 \)-homotopy equivalence.

The \( S^1 \)-equivariant smash product \( \tilde{\Omega}_A \wedge T \) of \( \tilde{\Omega}_A \) with any \( S^1 \)-spectrum \( T \) is a presheaf of \( S^1 \)-spectra on \( \mathcal{M}_{pctf} \). For every integer \( r \geq 1 \), we write \( \tilde{\Omega}^{T, r} \) for the
presheaf of fixed-point spectra $X \mapsto (\tilde{\Omega}_{A(X)} \wedge T)^{C_r}$ on $\mathcal{M}_{pctf}$ (where $C_r$ is the cyclic subgroup of $S^1$ having $r$ elements), and $\mathbb{H}_{zar}(-, \tilde{\Omega}^{T,r}_{T})$ for its fibrant replacement for the Zariski topology. The following result was proven in [9, Theorem 6.2].

**Theorem 5.2.** For any $S^1$-spectrum $T$ and integer $r \geq 1$, the presheaf of spectra $\mathbb{H}_{zar}(-, \tilde{\Omega}^{T,r}_{T})$ satisfies cdh descent on $\mathcal{M}_{pctf}$.

Hesselholt and Madsen proved in [22, Theorem 7.1] that for any associative ring $\Lambda$ and monoid $A$, there is a natural equivalence of cyclotomic spectra,

$$TH(\Lambda) \wedge |N^{cy}(A)| \xrightarrow{\approx} TH(\Lambda[A]),$$

where $TH$ is topological Hochschild homology. Combining this with Theorem 5.2, we showed in [9, Corollary 6.6] that the dilated topological cyclic homology $X \mapsto TC^\nu(X_{k};p)^{c}$ satisfies cdh-descent on $\mathcal{M}_{pctf}$. Here is a more general statement.

**Theorem 5.3.** ([9, 6.6]) Let $c = (c_1, c_2, \ldots)$ be a sequence of integers with $c_i \geq 2$ for all $i$. For any associative ring $\Lambda$ and integer $\nu \geq 1$, the spectrum-valued functor $X \mapsto TC^\nu(X_{\Lambda};p)^{c}$ satisfies cdh descent on $\mathcal{M}_{pctf}$.

**Proof.** The proofs of Theorem 6.5 and Corollary 6.6 in [9] go through with $TH(k)$ replaced by $TH(\Lambda)$. $\square$

6. Descent for Hochschild and Cyclic Homology

We write $HH(\Lambda)$ for the Eilenberg-MacLane spectrum associated to the absolute Hochschild homology of a ring $\Lambda$ (i.e., relative to the base ring $\mathbb{Z}$) and similarly for cyclic and negative cyclic homology. Since the Hochschild complex of $\Lambda$ is a cyclic complex, $HH(\Lambda)$ is an $S^1$-spectrum.

**Remark 6.1.** The Hochschild homology we are using in this section is the classical version defined by the bar complex of an algebra. There is also a derived version (see [15, IV]), and this is the one that coincides with the Hochschild homology of dg categories used in the rest of this paper. The results of this section we need going forward (Corollaries 6.3 and 6.4) deal with $\mathbb{Q}$-algebras; for such algebras, the derived and classical definitions agree.

**Theorem 6.2.** Let $c = (c_1, c_2, \ldots)$ be a sequence of integers with $c_i \geq 2$. For any associative ring $\Lambda$, the spectrum-valued functors

$$X \mapsto HH(X_{\Lambda})^{c} \quad \text{and} \quad X \mapsto HC(X_{\Lambda})^{c}$$

satisfy cdh descent on $\mathcal{M}_{pctf}$.

**Proof.** By Theorem 5.2 with $T = HH(\Lambda)$ and $r = 1$, $\mathbb{H}_{zar}(-, HH(\Lambda) \wedge \tilde{\Omega})$ satisfies cdh descent on $\mathcal{M}_{pctf}$. Since $|N^{cy}|^{c} \xrightarrow{\approx} |\tilde{\Omega}|^{c}$, we see that $\mathbb{H}_{zar}(-, HH(\Lambda) \wedge |N^{cy}|)^{c}$ also satisfies cdh descent. This is equivalent to $HH(-)^{c} \simeq \mathbb{H}_{zar}(-, HH(-)^{c})$ because there is a natural weak equivalence of spectra

$$HH(\Lambda) \wedge |N^{cy}(A)| \simeq HH(\Lambda[A]).$$

The $HH$ assertion now follows. The $HC$ assertion follows from this, together with the SBI sequence connecting cyclic homology and Hochschild homology (see [38, 9.6.11]). In more detail, let $\mathbb{HH}$ and $\mathbb{H}$ denote the presheaves $\mathbb{HH}(X) = \mathbb{H}(X)$ for $X \mapsto \mathbb{HH}(X)^{c}$.
follows that

Proof. Combine Proposition 4.7 and Corollary 6.3 (the latter applied using the Corollary 6.4. Let \( \Lambda \) be a sequence of integers with each \( c_i \geq 2 \). Then \( X \mapsto HN(X_\Lambda)^i \) satisfies cdh descent on \( \mathcal{M}_{pcft} \).

Proof. As in the proof of Corollary 3.13 in [6], the un-dilated presheaf \( HP \) satisfies weak cdh descent on \( Sch/k \). By Theorem 3.6, \( HP(-\Lambda) \) satisfies cdh descent on \( \mathcal{M}_{pcft} \). Therefore the dilated presheaf \( X \mapsto HP(X_\Lambda)^i \) also satisfies cdh descent on \( \mathcal{M}_{pcft} \). Using the SBI sequence for \( HN \) and \( HP \), together with Theorem 6.2, it follows that \( X \mapsto HN(X_\Lambda)^i \) satisfies cdh descent on \( \mathcal{M}_{pcft} \).

Corollary 6.4. Let \( k_0 \) be commutative regular ring of finite Krull dimension all of whose residue fields are infinite, and let \( \Lambda \) be a flat \( K \)-regular \( k_0 \)-algebra. Then

\[ K(X_\Lambda)^i \otimes \mathbb{Q} \cong K H(X_\Lambda)^i \otimes \mathbb{Q}. \]

Proof. Combine Proposition 4.7 and Corollary 6.3 (the latter applied using the \( \mathbb{Q} \)-algebra \( \Lambda \otimes \mathbb{Q} \)).

7. Main theorem

Lemma 7.1. Let \( X \) be a monoid scheme and \( \xi = (c_1, c_2, \ldots) \) a sequence of integers with \( c_i \geq 2 \) for all \( i \). Let \( k \) be a commutative Noetherian ring of finite Krull dimension, and let \( \Lambda \) be any associative \( k \)-algebra. Suppose that for every prime ideal \( \wp \) of \( k \), the natural map

\[ \phi_{\wp} : K^i(X_{\Lambda_{\wp}}) \to K H^i(X_{\Lambda_{\wp}}) \]

is a weak equivalence. Then \( \phi_k \) is a weak equivalence.

Proof. Write \( \mathcal{F} \) for the presheaf on \( S = \text{Spec}(k) \) sending the open \( U \subset S \) to the homotopy fiber of \( \phi_{\mathcal{O}(U)} \). The hypothesis implies that the Zariski sheaves \( a_{zar} \pi_{\wp} \mathcal{F} \) associated to \( \pi_{\wp} \mathcal{F} \) vanish on \( S \). Because the source and target, viewed as functors of \( U \), both satisfy Zariski descent [34, 8.1], so does \( \mathcal{F} \). Since \( k \) is Noetherian and of finite Krull dimension, there is a spectral sequence converging to \( \pi_* \mathcal{F}(k) \) with \( E_2^{p,q} = H^n_{zar}(\text{Spec } k, a_{zar} \pi_{-q} \mathcal{F}) = 0 \); see [39, Theorem V.10.11]. Hence \( \mathcal{F}(k) \simeq * \) and \( \phi_k \) is a weak homotopy equivalence.

Remark 7.2. An argument similar to that in the proof of Lemma 7.1 shows that if \( k \) is Noetherian of finite Krull dimension and \( \phi_{kh} \) is a weak equivalence for the henselization of every local ring of \( k \), then \( \phi_k \) is a weak equivalence. One simply has to substitute the Nisnevich for the Zariski topology and use [39, Remark V.10.11.1].
Lemma 7.3. Let \((R, \mathfrak{m})\) be a commutative local ring with finite residue field. For each integer \(l > 0\), there is a finite étale extension \(R \to R'\) of rank \(l\) with \((R', \mathfrak{m}R')\) local.

Proof. Let \(\mathfrak{m}\) be the maximal ideal of \(R\) and set \(k = R/\mathfrak{m}\). Because \(k\) is finite, there is separable field extension \(k'/k\) of degree \(l\). Pick a primitive element \(\alpha \in k'\) for this field extension, so that \(k' = k(\alpha)\), and let \(\bar{f} \in k[x]\) be the minimum polynomial of \(\alpha\). Let \(f \in R[x]\) be a monic lift of \(\bar{f}\) and set \(R' = R[x]/\langle f \rangle\). Then \(R \to R'\) is a finite flat extension of rank \(l\) and \(R'\) is local because \(R'/\mathfrak{m}R' = k'\) is a field. Since the field extension is separable, \(f'(\alpha) \neq 0\) and hence \(f'(x)\) is a unit in \(R'\). This proves \(R \to R'\) is an étale extension. \(\square\)

Recall from the introduction that an associative ring \(\Gamma\) is (right) regular if it is (right) Noetherian and every finitely generated (right) module has finite projective dimension.

Lemma 7.4. Let \(\Gamma\) be an associative algebra over a commutative ring \(k\) and \(f \in k[x]\) a monic polynomial such that the derivative \(df/dx\) is invertible in \(k' = k[x]/\langle f \rangle\). If \(\Gamma\) is regular, then so is \(\Gamma' = \Gamma \otimes_k k'\).

Proof. First of all, \(\Gamma'\) is Noetherian since it is finite as a module over \(\Gamma\), which is Noetherian by assumption. We must show every finitely generated right \(\Gamma'\)-module \(M\) has finite projective dimension. This is clear for \(M\) of the form \(N \otimes_k k'\) for some finitely generated \(\Gamma\)-module \(N\), since \(k \to k'\) is flat and we are assuming that \(\Gamma\) is regular. If \(M\) is any \(\Gamma'\)-module, write \(M' = M \otimes_k k'\); the multiplication map \(\mu : M' \to M\) is a surjective homomorphism of \(\Gamma'\)-modules. On the other hand, because \(k'\) is a separable \(k\)-algebra, there exists an idempotent \(e = \sum_i x_i \otimes y_i \in k' \otimes_k k'\) such that \(\sum_i x_i y_i = 1 \in k'\) and such that \((x \otimes 1 - 1 \otimes x)e = 0\) for all \(x \in k'\). One checks that the map \(s : M \to M'\), \(m \mapsto me\) is \(\Gamma'\)-linear and that \(\mu s = 1_M\). Thus \(M\) is a direct summand of \(M'\), and a direct summand of a module of finite projective dimension is itself of finite projective dimension. \(\square\)

We are now ready to prove our main theorem.

Theorem 7.5. Let \(k_0\) be a commutative regular ring of finite Krull dimension, \(\Lambda\) a flat (not necessarily commutative) \(k_0\)-algebra, and assume that one of the following three conditions holds:

1. \(\Lambda\) is regular,
2. \(\Lambda\) is \(K\)-regular and all of the residue fields of \(k_0\) are infinite, or
3. \(\Lambda\) is \(K\)-regular and for every prime \(p \subseteq k_0\) whose residue field is finite and every étale \((k_0)_p\)-algebra \(k'\), the ring \(\Lambda \otimes_{k_0} k'\) is \(K\)-regular.

Let \(X\) be a pcf monoid scheme of finite type, and \(\zeta = (c_1, c_2, \ldots)\) any sequence of integers with \(c_i \geq 2\) for all \(i\). Then the natural map

\[K^\zeta(X_\Lambda) \to KH^\zeta(X_\Lambda)\]

is a weak equivalence.

Proof. Both regularity and \(K\)-regularity localize under central nonzero divisors \([39, \text{Lemma V.8.5}].\) It follows from this and from Lemma 7.1 that we may assume that \(k_0\) is local.

Suppose first that the residue fields of \(k_0\) are infinite. Let \(F_K\) be the homotopy fiber of the map \(K \to KH\). We must show that the homotopy groups of
If \( \pi_f(X_\Lambda) \) vanish. By Corollary 6.4 we have \( \pi_*(\mathcal{F}_k(X_\Lambda)) \otimes \mathbb{Q} = 0 \). Hence the groups \( \pi_*(\mathcal{F}_k(X_\Lambda)) \) are torsion. We will be done if we show that multiplication by any prime \( p \) induces an isomorphism on \( \pi_*(\mathcal{F}_k(X_\Lambda)) \), or equivalently, that \( \pi_*(\mathcal{F}_k(X_\Lambda)/p) = 0 \). This follows from Proposition 4.11 and Theorem 5.3, using [9, Lemma 8.2], as in the proof of [9, Theorem 8.3]. This proves the result if (2) holds.

Now assume that the residue field \( k_0/m \) of \( k_0 \) is finite and that either (1) or (3) hold. Let \( \mathcal{F}^c(k_0) \) be the fiber of the map \( K^c(X_\Lambda) \to KH^c(X_\Lambda) \). We must show that the homotopy groups of \( \mathcal{F}^c(k_0) \) vanish. Let \( l \) be a prime number. By Lemma 7.3 there exists a tower of finite, étale extensions \( k_0 \subset k_1 \subset \cdots \) of the form \( k_{i+1} = k_i[x]/(f_i) \) with \( df_i/dx \) invertible in \( k_{i+1} \), such that \( (k_n, \mathfrak{m}k_n) \) is local and has rank \( l^n \) over \( k_0 \), for all \( n \).

Set \( k' = \bigcup_n k_n \); it is Noetherian by [17, 0.1f, 10.3.1.3]. Then \( (k', \mathfrak{m}k') \) is regular local, with infinite residue field, and \( k' \subset \Lambda \otimes_{k_0} k' \) is \( K \)-regular by hypothesis. If (1) holds, then by Lemma 7.4, \( \Lambda_n = \Lambda \otimes_{k_0} k_n \) is regular and thus \( K \)-regular for each \( n \), whence again we conclude that \( \Lambda \otimes_{k_0} k' \) is \( K \)-regular. Hence

\[
0 = \pi_*(\mathcal{F}^c(k')) = \text{colim} \pi_*(\mathcal{F}^c(k_n)).
\]

Since \( \Lambda \to \Lambda \otimes_{k_0} k_n \) is finite and flat, there is a natural transfer map

\[
K(X_\Lambda, k_n) \to K(X_\Lambda) \text{ such that the composition } K(X_\Lambda) \to K(X_\Lambda, k_n) \to K(X_\Lambda) \text{ induces multiplication by } l^n \text{ on homotopy groups, for all } n.
\]

By naturality, \( K^c \) also admits such a transfer map. Likewise, \( KH \)-theory and hence \( KH^c \)-theory have such transfer maps and they are compatible with the map \( K^c \to KH^c \). We obtain a map \( \mathcal{F}^c(k_n) \to \mathcal{F}^c(k_0) \) such that the composition \( \mathcal{F}^c(k_0) \to \mathcal{F}^c(k_n) \to \mathcal{F}^c(k_0) \) induces multiplication by \( l^n \) on homotopy groups. It follows that the kernel of \( \pi_*(\mathcal{F}^c(k_0)) \to \pi_*(\mathcal{F}^c(k')) \) is an \( l \)-primary torsion group. Since this occurs for every prime \( l \), we must have \( \pi_*(\mathcal{F}^c(k_0)) = 0 \). \( \square \)

Theorem 7.5 implies Theorem 0.3:

**Proof of Theorem 0.3.** Cases (c) and (e) of Theorem 0.3 are cases (1) and (3) of Theorem 7.5, and case (d) is a special case of (c). If \( k \) is a regular commutative ring and \( \text{Spec}(k) \) is connected, then either \( k \) is flat over a field \( k_0 \) or it is flat over the ring of integers. Hence it satisfies the hypothesis of Theorem 7.5. If \( k \) is an arbitrary commutative regular ring, then it is a finite product of regular rings with connected Spec. Thus case (a) of Theorem 0.3 is proved.

If \( \Lambda \) is a commutative \( C^* \)-algebra, then it is flat over the field \( k_0 = \mathbb{C} \) and satisfies the \( K \)-regularity hypothesis by [11, Theorem 8.1]. Thus case (b) of Theorem 0.3 also follows from Theorem 7.5. \( \square \)

**Remark 7.6.** If in Theorem 7.5 we assume that \( \Lambda \) is commutative Noetherian and \( K \)-regular then every étale extension of \( \Lambda \) is \( K \)-regular by van der Kallen’s theorem [35, Theorem 3.2]. In particular the assumption that \( \Lambda \otimes_{k_0} k' \) is \( K \)-regular for every étale \( k_0 \)-algebra \( k' \) is superfluous in this case.

**Remark 7.7.** Theorems 0.1 and 0.3 for commutative \( C^* \)-algebras can alternatively be derived from Gubeladze’s main result of [21] using [11, Theorem 7.7].
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