GENERALIZED CHEBYSHEV I-BERNSTEIN BASES TRANSFORMATION

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Abstract. We provide a closed form for the matrix transformation of the generalized Chebyshev of the first kind (Chebyshev-I) polynomial basis into Bernstein polynomial basis, and the matrix transformation of Bernstein polynomial basis into generalized Chebyshev-I polynomial basis. Also, we provide an explicit closed form of the generalized Chebyshev-I polynomials of degree \( r \leq n \) in terms of the Bernstein basis of fixed degree \( n \).

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1. Introduction

Approximation is important to many numerical methods, since it is possible to approximate arbitrary continuous function by a polynomial. On the other hand, polynomials can be characterized in many different bases such as power and Bernstein basis. Every type of polynomial basis has its strength, where by suitable choice of the basis numerous problems can be solved and many complications can be removed.

1.1. Bernstein polynomials. The Bernstein polynomials have been studied comprehensively and there exist many great enduring works on theses polynomials.

Definition 1.1. The \( n+1 \) Bernstein polynomials \( B^n_v(x) \) of degree \( n \), \( x \in [0,1] \), \( v = 0,1,\ldots,n \), are defined by:

\[
B^n_v(x) = \begin{cases} 
{n \choose v} x^v (1-x)^{n-v} & v = 0,1,\ldots,n \欺
0 & \text{else} 
\end{cases},
\]

where \( {n \choose v} \), \( v = 0,\ldots,n \) are the binomial coefficients.

Bernstein polynomials are known for their geometric and analytic properties [3][11], where the basis are known to be stable. They are all non-negative, \( B^n_v(x) \geq 0, x \in [0,1] \), and satisfy symmetry relation \( B^n_v(x) = B^n_{n-v}(1-x) \). The Bernstein polynomials of degree \( n \) can be defined by combining two Bernstein polynomials of degree \( n-1 \). The \( k \)th \( n \)th degree Bernstein polynomial can be defined by the following relation

\[
B^n_k(x) = (1-x)B^n_{k-1}(x) + xB^n_{k-1}(x), \quad k = 0,\ldots,n; n \geq 1
\]

where \( B^n_0(x) = 0 \) and \( B^n_k(x) = 0 \) for \( k < 0 \) or \( k > n \).

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In addition, it is possible to write Bernstein polynomials of degree \( r \) where \( r \leq n \) in terms of the Bernstein polynomials of degree \( n \) using the degree elevation defined by [2]:

\[
B^r_k(x) = \sum_{i=k}^{n-r+k} \binom{n}{i-k} \binom{n-r}{i-k} B^n_i(x), \quad k = 0, 1, \ldots, r.
\]

These remarkable properties make the Bernstein polynomials significant for the development of Bézier curves and surfaces in Computer Aided Geometric Design. The Bernstein polynomials are the standard basis for the Bézier representations of curves and surfaces in Computer Aided Geometric Design. However, the Bernstein polynomials are not orthogonal and could not be used effectively in the least-squares approximation [9], and thus the calculations performed in finding the least-square approximation polynomial of degree \( m \) do not decrease the calculations to obtain the least-squares approximation polynomial of degree \( m + 1 \). Since then a theory of approximation has been developed and many approximation techniques have been presented and examined. The method of least square approximation accompanied by orthogonality polynomials is one of these methods.

**Definition 1.2.** For a function \( f(x) \), continuous on \([0,1]\) the least square approximation requires finding a polynomial (Least-Squares Polynomial)

\[
p_n(x) = a_0 \varphi_0(x) + a_1 \varphi_1(x) + \cdots + a_n \varphi_n(x)
\]

that minimizes the error

\[
E(a_0, a_1, \ldots, a_n) = \int_0^1 (f(x) - p_n(x))^2\,dx.
\]

Such computations can be made effective by using a special type of polynomials, called orthogonal polynomials. Choosing \( \{\varphi_0(x), \varphi_1(x), \ldots, \varphi_n(x)\} \) to be orthogonal simplifies the least-squares approximation problem. The matrix of the normal equations is diagonalized, which simplifies calculations and gives a closed form for \( a_i, i = 0, 1, \ldots, n \). Moreover, once \( p_n(x) \) is known, it is only needed to compute \( a_{n+1} \) to get \( p_{n+1}(x) \). See [9] for more details on the least squares approximations.

1.2. **Factorials and Semi-Factorials.** We present some results concerning factorials, semi-factorials (double factorials), and combinatorial identities. The semi-factorial of an integer \( m \) is given by

\[
(2m - 1)!! = (2m - 1)(2m - 3)(2m - 5)\ldots(3)(1) \quad \text{if } m \text{ is odd}
\]

\[
m!! = (m)(m-2)(m-4)\ldots(4)(2) \quad \text{if } m \text{ is even},
\]

where \( 0!! = (-1)!! = 1 \). From (1.3), we have the following definition.

**Definition 1.3.** For an integer \( n \), the double factorial is defined as

\[
n!! = \begin{cases} 
2^\frac{n}{2} \frac{(\frac{n}{2})!}{\sqrt{\pi}}, & \text{if } n \text{ is even} \\
2^{\frac{n-1}{2}} \frac{(\frac{n-1}{2})!}{\sqrt{\pi}}, & \text{if } n \text{ is odd}
\end{cases}
\]

From the definition, we can derive the factorial of an integer minus half as

\[
\left( n - \frac{1}{2} \right)! = \frac{n!(2n-1)!!\sqrt{\pi}}{(2n)!!}.
\]
1.3. Chebyshev-I polynomials. The Chebyshev-I polynomials of degree \( n \) are a set of orthogonal polynomials defined as the solutions to the Chebyshev-I differential equation and denoted \( T_n(x) \). They are the orthogonal polynomials, except for a constant factor, with respect to the weight function

\[
W(x) = \frac{1}{\sqrt{1-x^2}}.
\]

They are used as an approximation to "a least squares fit". They are a special case of the Jacobi polynomials with \( \alpha = \beta = -\frac{1}{2} \), and related to the Jacobi polynomials by the relation

\[
(1.6) \quad P^{(-\frac{1}{2},-\frac{1}{2})}_n(1)T_n(x) = P^{(-\frac{1}{2},-\frac{1}{2})}_n(x).
\]

The univariate classical orthogonal polynomials are traditionally defined on \([-1, 1]\), however, it is more convenient to use \([0, 1]\). Since authors are not uniform on using the notations, and for the convenience we recall the following expression for univariate Chebyshev-I polynomials of degree \( n \) in \( x \) [5],

\[
(1.7) \quad T_n(x) := \frac{(2n)!!}{(2n-1)!!} \sum_{k=0}^{n} \binom{n - \frac{1}{2}}{n - k} \frac{(n - \frac{1}{2})}{k} \left( \frac{x - 1}{2} \right)^{k} \left( \frac{x + 1}{2} \right)^{n-k},
\]

which it can be transformed in terms of Bernstein basis on \( x \in [0, 1] \) as

\[
(1.8) \quad T_n(2x - 1) := \frac{2^{2n}(n!)^2}{(2n)!} \sum_{k=0}^{n} (-1)^{n+1} \binom{n - \frac{1}{2}}{k} \binom{n - \frac{1}{2}}{n-k} B^n_k(x).
\]

Using combinatorial representation gives more compact and clear formulas, these have also been used by Szegö [10]. By expanding the right-hand side and using (1.5) with some simplifications, we have

\[
\left( \frac{n - \frac{1}{2}}{n - k} \right) \left( \frac{n - \frac{1}{2}}{k} \right) = \frac{(2n-1)!!}{2^n(n-k)!} \frac{2^k}{(2k-1)!!} \frac{(2n-1)!!}{2^n k!} \frac{2^{n-k}}{(2n-2k-1)!!} = \frac{1}{2^n(n-k)!} \frac{(2n-1)!!}{(2k-1)!!} \frac{2^n}{(2(n-k)-1)!!}.
\]

Using the fact \((2n)! = (2n-1)!! 2^{n} n!\) we get

\[
(1.9) \quad T_n(2x - 1) := \frac{2^{2n}(n!)^2}{(2n)!} \sum_{k=0}^{n} (-1)^{n+1} \binom{2n}{2k} \binom{2n}{n-k} B^n_k(x).
\]

Also, Chebyshev-I polynomials may be represented in terms of hypergeometric series as follows [7],

\[
(1.10) \quad T_n(x) := 2F_1 \left( -n, n; \frac{1-x}{2} \right).
\]

Moreover, the Chebyshev-I polynomials satisfy the orthogonality relation

\[
(1.11) \quad \int_0^1 (1 - x)^{-\frac{1}{2}} x^{-\frac{1}{2}} T_n(x) T_m(x) dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{\pi}{2} & \text{if } m = n = 0 \\ \frac{n\pi}{2} & \text{if } m = n = 1, 2, \ldots \end{cases}.
\]
1.4. The generalized Chebyshev-I polynomials. For \( M, N \geq 0 \) the generalized Chebyshev-I polynomials \( \{ T_{n}^{(M,N)}(x) \}_{n=0}^{\infty} \) of degree \( n \) are orthogonal on the interval \([-1, 1] \) with respect to the weight function

\[
\frac{1}{\pi}(1-x)^{-\frac{1}{2}}(1+x)^{-\frac{1}{2}} + M\delta(x+1) + N\delta(x-1),
\]

and can be written \[1\] as

\[
T_{n}^{(M,N)}(x) = \frac{(2n-1)!!}{(2n)!!} T_{n}(x) + \sum_{k=0}^{n} \frac{(2k)!\lambda_{k}}{2^{2k}(k!)^2} T_{k}(x),
\]

where

\[
\lambda_{k} = \frac{4M}{(2k-3)(k-1)!} + \frac{4N}{(2k-3)(k-1)!} + \frac{4MN}{(k-1)!(k-2)!}.
\]

1.4.1. Generalized Chebyshev-I Polynomials using Bernstein basis. The following theorem \[1\] shows how the generalized Chebyshev-I polynomial \( r \)

\[
\mathcal{T}_{r}^{(M,N)}(x) = \frac{(2r-1)!!}{(2r)!!} T_{r}(x) + \sum_{k=0}^{r} \frac{(2k)!\lambda_{k}}{2^{2k}(k!)^2} \sum_{j=0}^{k} (-1)^{k-j} \eta_{j,k} B_{j}^{(r)}(x),
\]

where \( \lambda_{k} \) defined in \[1.14\], \( \eta_{i,r} = \frac{(2r-1-i)!!}{2^{i}(i!)!!} \), \( i = 0, 1, \ldots, r \), and \( \eta_{0,r} = \frac{1}{2^{r}} \frac{(2r)!}{(2r-1)!} \).

Moreover, the coefficients \( \eta_{i,r} \) satisfy the recurrence relation

\[
\eta_{i,r} = \frac{2r-2i+1}{2i-1} \eta_{i-1,r}, \quad i = 1, \ldots, r.
\]

2. Main Results

In this section we provide a closed form for the matrix transformation of the generalized Chebyshev-I polynomial basis into Bernstein polynomial basis, and for Bernstein polynomial basis into generalized Chebyshev-I polynomial basis.

2.1. Generalized Chebyshev-I to Bernstein transformation. Rababah \[8\] provided some results concerning the univariate Chebyshev case. In the following theorem we mimicking the procedure in \[8\] to generalize the results for the generalized case. The theorem will be used to combine the superior performance of the least-squares of the generalized Chebyshev-I polynomials with the geometric insights of the Bernstein polynomials basis.

**Theorem 2.1.** The entries \( M_{i,r}, i, r = 0, 1, \ldots, n \) of the matrix transformation of the generalized Chebyshev-I polynomial basis into Bernstein polynomial basis of
degree $n$ are given by

\begin{equation}
M_{i,r}^n = \binom{n}{i}^{-1} \frac{(2r)!}{2^{2r}(r!)^2} \sum_{k=\max(0,i+r-n)}^{\min(i,r)} (-1)^r \frac{r-k}{i-k} \frac{(r-\frac{1}{2})}{k} \frac{(r-\frac{1}{2})}{r-k} \sum_{j=\max(0,i+k-n)}^r \lambda_k \left( \frac{n}{i} \right) \frac{(2k)!}{2^{2k}(k!)^2} \sum_{j=\max(0,i+j-n)}^{\min(i,k)} (-1)^{k-j} \frac{(n-k)}{i-j} \frac{(k-\frac{1}{2})}{j} \frac{(k-\frac{1}{2})}{k-j}.
\end{equation}

Proof. Polynomials can be written as a linear combination of certain elementary polynomials. Any polynomial $p_n(x), x \in [0,1]$ of degree $n$, can be uniquely written in this way. Write $p_n(x)$ as a linear combination of the Bernstein polynomial basis as

$$p_n(x) = \sum_{r=0}^{n} c_r B_r^n(x)$$

and as a linear combination of the generalized Chebyshev-I polynomials as follows

$$p_n(x) = \sum_{i=0}^{n} d_i \mathcal{G}_i^{(M,N)}(x).$$

We are interested in the transformation matrix $M$ where $c = M.d$ which transforms the generalized Chebyshev-I coefficients $\{d_i\}_{i=0}^{n}$ into the Bernstein coefficients $\{c_r\}_{r=0}^{n}$.

\begin{equation}
c_i = \sum_{r=0}^{n} M_{i,r}^n d_r,
\end{equation}

which can be written in matrix format as

\begin{equation}
\begin{bmatrix}
c_0 \\
c_1 \\
\vdots \\
c_n
\end{bmatrix} =
\begin{bmatrix}
M_{0,0}^n & M_{0,1}^n & \cdots & M_{0,n}^n \\
M_{1,0}^n & M_{1,1}^n & \cdots & M_{1,n}^n \\
\vdots & \vdots & \ddots & \vdots \\
M_{n,0}^n & M_{n,1}^n & \cdots & M_{n,n}^n
\end{bmatrix}
\begin{bmatrix}
d_0 \\
d_1 \\
\vdots \\
d_n
\end{bmatrix}.
\end{equation}

Furthermore, use Theorem 1.1 to write the generalized Chebyshev-I polynomials $\mathcal{G}_r^{(M,N)}(x)$ of degree $r \leq n$ in terms of Bernstein polynomial basis of degree $n$ as

\begin{equation}
\mathcal{G}_r^{(M,N)}(x) = \sum_{i=0}^{n} N_{r,i}^n B_i^n(x), \quad r = 0, 1, \ldots, n,
\end{equation}

where $N_{r,i}^n$ the entries of the $(n+1) \times (n+1)$ basis conversion matrix $N$. Thus, the elements of $c$ can be written in the form

\begin{equation}
c_i = \sum_{r=0}^{n} d_r N_{r,i}^n.
\end{equation}

By comparing equations (2.2) and (2.5), it is clear that $M = N^T$.

Now, we need to write each Bernstein polynomial of degree $r$ where $r \leq n$ in terms of Bernstein polynomials of degree $n$. By substituting the degree elevation defined by [2]

\begin{equation}
B_k^n(x) = \sum_{i=k}^{n-r+k} \binom{r}{k} \binom{n-r}{i-k} B_i^n(x), \quad k = 0, 1, \ldots, r,
\end{equation}

we obtain
into (1.15) and rearrange the order of summations, we find that the entries of the matrix $N$ for $r = 0, 1, \ldots, n$ are given by

$$
N_{r,i}^n = \binom{n}{i}^{-1} \frac{(2r)!}{2^{2r}(r!)^2} \sum_{k=\min(0,i+r-n)}^{\min(i,r)} (-1)^{r-k} \binom{n-r}{i-k} \left( r - \frac{1}{2} \right) \left( r - \frac{k}{2} \right) + \sum_{k=0}^{r} \lambda_k \binom{n}{i}^{-1} \frac{(2k)!}{2^{2k}(k!)^2} \sum_{j=\max(0,i+k-n)}^{\min(i,k)} (-1)^{k-j} \binom{n-k}{i-j} \left( k - \frac{1}{2} \right) \left( k - \frac{j}{2} \right).
$$

Thus, the matrix $M$ can be obtained by transposing the entries of the matrix $N$. □

In the following Corollary, we express the generalized Chebyshev-I polynomials $\mathcal{T}_r^{(M,N)}(x)$ of degree $r \leq n$ in terms of the Bernstein basis of fixed degree $n$.

**Corollary 2.1.** The generalized Chebyshev-I polynomials $\mathcal{T}_r^{(M,N)}(x)$ of degree less than or equal to $n$ can be expressed in the Bernstein basis of fixed degree $n$ by the following formula

$$
\mathcal{T}_r^{(M,N)}(x) = \sum_{i=0}^{n} N_{r,i}^n B_i^n(x), \quad r = 0, 1, \ldots, n,
$$

where $N_{r,i}^n = \mu_{r,i}^n + \sum_{k=0}^{r} \lambda_k \mu_{k,i}^n$ and $\mu_{r,i}^n$ defined as

$$
\mu_{r,i}^n = \frac{(2r)!}{2^{2r}(r!)^2} \sum_{k=\max(0,i+r-n)}^{\min(i,r)} (-1)^{r-k} \binom{n-r}{i-k} \left( r - \frac{1}{2} \right) \left( r - \frac{k}{2} \right).
$$

**Proof.** Using (2.4) from the proof of Theorem 2.1, any generalized Chebyshev-I polynomials $\mathcal{T}_r^{(M,N)}(x)$ of degree $r \leq n$ can be expressed in terms of Bernstein basis of fixed degree $n$ as $\mathcal{T}_r^{(M,N)}(x) = \sum_{i=0}^{n} N_{r,i}^n B_i^n(x), \quad r = 0, 1, \ldots, n$, where the matrix $N$ can be obtained by transposing the entries of the matrix $M$ defined in (2.1). But,

$$
\left( r - \frac{1}{2} \right) \left( r - \frac{k}{2} \right) = \frac{(2r-1)!!}{2^r(r-k)!(2k-1)!!} \frac{(2r-1)!!}{2^r(2r-1)!!} \frac{2r-1}{2^r(2r-1)!!} = \frac{1}{2^r(r-k)!k!(2k-1)!!(2r-k-1)!!}.
$$

Using the fact $(2r)! = (2r-1)!!2^r r!$ we get

$$
\left( \frac{n-r}{i-k} \right) \left( \frac{r-\frac{1}{2}}{r-\frac{k}{2}} \right) = \frac{(n-r)\binom{\frac{r}{2}}{\frac{k}{2}}}{\binom{n}{i}} = \frac{(n-r)\binom{\frac{r}{2}}{\frac{k}{2}}}{\binom{n}{i}}.
$$

So, the entries $N_{r,i}^n$ can be rewritten as

$$
N_{r,i}^n = \mu_{r,i}^n + \sum_{k=0}^{r} \lambda_k \mu_{k,i}^n,
$$

where

$$
\mu_{r,i}^n = \frac{(2r)!}{2^{2r}(r!)^2} \sum_{k=\max(0,i+r-n)}^{\min(i,r)} (-1)^{r-k} \binom{n-r}{i-k} \left( r - \frac{1}{2} \right) \left( r - \frac{k}{2} \right).
$$

□
2.2. Transformation of Bernstein Basis into the Generalized Chebyshev-I Polynomial Basis. In the first section, we discussed some analytic and geometric properties for Bernstein polynomials. It is worth mentioning that Bernstein polynomials can be integrated easily as
\[
\int_0^1 B_k^n(x)dx = \frac{1}{n+1}, \quad k = 0, 1, \ldots, n.
\]
Moreover, the product of two Bernstein polynomials is a Bernstein polynomial and given by
\[
B_i^n(x)B_j^m(x) = \binom{n}{i}\binom{m}{j}B_{i+j}^{n+m}(x).
\]
But when integrating the product of Bernstein polynomials with the weighted Generalized Chebyshev-I Polynomials, we the following interesting theorem.

**Theorem 2.2.** \cite{1} (M. AlQudah) Let \( B_i^n(x) \) be the Bernstein polynomial of degree \( n \) and \( \mathcal{T}_i^{(M,N)}(x) \) be the generalized Chebyshev-I polynomial of degree \( i \), then for \( i, r = 0, 1, \ldots, n \) we have
\[
\int_0^1 (1-x)^{-\frac{1}{2}} x^{-\frac{1}{2}} B_i^n(x)\mathcal{T}_i^{(M,N)}(x)dx = \left(\frac{n}{r}\right)\frac{(2i)!}{2^{2i}(i)!^2} \sum_{k=0}^{i} \frac{(-1)^{i-k}}{k} \frac{(i - \frac{1}{2})}{i - k} \left(\frac{i}{2}\right) B(r + k + \frac{1}{2}, n + i - r - k + \frac{1}{2})
\]
\[
+ \sum_{d=0}^{i} \sum_{j=0}^{d} \frac{\lambda_d(n)}{2^{2d}(d)!^2} \sum_{k=0}^{i} \frac{(-1)^{i-k}}{2^{2k}k!^2} \left(\frac{2}{k}\right) \left(\frac{2i}{k}\right) \frac{1}{2^{2j}j!^2} B(r + j + \frac{1}{2}, n + d - r - j + \frac{1}{2})
\]
where \( B(x, y) \) is the Beta function.

**Proof.** See \cite{1} for the proof. \( \square \)

**Theorem 2.3.** The entries \( M_{i,r}^{-1}, i, r = 0, 1, \ldots, n \) of the matrix of transformation of the Bernstein polynomial basis into the generalized Chebyshev-I polynomial basis of degree \( n \) are given by
\[
(2.8) \quad M_{i,r}^{-1} = \Phi_i \frac{(n)}{r} \left[ 2^{2i}(i)!^2 \Psi_{k,i}^{n,r} + \frac{2^{2i}(i)!^2}{(2i)!} \sum_{d=0}^{i} \frac{(2d)!\lambda_d}{2^{2d}(d)!^2} \Psi_{j,d}^{n,r} \right],
\]
where \( \lambda_i \) defined in (1.14), \( \Phi_i = \begin{cases} 2/\pi & \text{if } i = 0 \\ 1/\pi & \text{if } i \neq 0 \end{cases} \),
\[
\Psi_{k,i}^{n,r} = \sum_{k=0}^{i} \frac{(-1)^{i-k}}{2^{2k}} \left(\frac{2i}{k}\right) \left(\frac{2i}{k}\right) B(r + k + \frac{1}{2}, n + i - r - k + \frac{1}{2}),
\]
and \( B(x, y) \) is the Beta function.

**Proof.** To write the Bernstein polynomial basis into generalized Chebyshev-I polynomial basis of degree \( n \), we invert the transformation formula in (2.3) to get
\[
(2.9) \quad d = M^{-1}c.
\]
Let $M^{-1}_{n,i}, N^{-1}_{r,i}, i, r = 0, \ldots, n$ be the entries of $M^{-1}$ and $N^{-1}$ respectively. The transformation of Bernstein polynomial into generalized Chebyshev-I polynomial basis of degree $n$ can then be written as

$$B^*_r(x) = \sum_{i=0}^{n} N^{-1}_{r,i} \mathcal{F}^{(M,N)}_i(x). \tag{2.10}$$

The entries $N^{-1}_{r,i}, i, r = 0, 1, \ldots, n$ can be found by we multiply (2.10) by $(x - x^2)^{-\frac{1}{2}} \mathcal{F}^{(M,N)}_i(x)$ and integrate over $[0, 1]$ to get

$$\int_0^1 (x - x^2)^{-\frac{1}{2}} B^*_r(x) \mathcal{F}^{(M,N)}_i(x) dx = \sum_{i=0}^{n} N^{-1}_{r,i} \int_0^1 (x - x^2)^{-\frac{1}{2}} \mathcal{F}^{(M,N)}_i(x) \mathcal{F}^{(M,N)}_i(x) dx. \tag{2.11}$$

By using the orthogonality relation (1.11) we get

$$\int_0^1 (x - x^2)^{-\frac{1}{2}} B^*_r(x) \mathcal{F}^{(M,N)}_i(x) dx = \left\{ \begin{array}{ll}
\frac{\pi}{2} N^{-1}_{r,i} \left( \frac{(2i)!}{2^{2i}(i)!} \right) (1 + \lambda_i)^2 & \text{if } i = 0 \\
\pi N^{-1}_{r,i} \left( \frac{(2i)!}{2^{2i}(i)!} \right) (1 + \lambda_i)^2 & \text{if } i \neq 0.
\end{array} \right. \tag{2.12}$$

Taking into account Theorem 2.2, we obtain

$$N^{-1}_{n,i} = \frac{\Phi_i(n)}{(1 + \lambda_i)^2} \left[ \frac{2i^n}{(2i)!} \right] \sum_{k=0}^{i} \frac{(-1)^{i-k}}{2^{2i}} \binom{2i}{i} \binom{2i}{2k} B(r + k + 1/2, n + i - r - k + 1/2)$$

$$+ \left( \frac{2^{2i}(i)!^2}{(2i)!} \right)^2 \sum_{d=0}^{i} \frac{(2d)!\lambda_d}{2^{2d}(d)!^2} \sum_{j=0}^{d} \frac{(-1)^{d-j}}{2^{2d}} \binom{2d}{d} \binom{2d}{2j} B(r + j + 1/2, n + d - r - j + 1/2). \tag{2.13}$$

By reordering the terms we have

$$N^{-1}_{r,i} = \frac{\Phi_i(n)}{(1 + \lambda_i)^2} \left[ \frac{2^{2i}(i)!^2}{(2i)!} \right] \Psi^{n,r}_{k,i} + \left( \frac{2^{2i}(i)!^2}{(2i)!} \right)^2 \sum_{d=0}^{i} \frac{(2d)!\lambda_d}{2^{2d}(d)!^2} \Psi^{n,r}_{j,d}, \tag{2.14}$$

where $\Phi_i = \left\{ \begin{array}{ll}
2/\pi & \text{if } i = 0 \\
1/\pi & \text{if } i \neq 0
\end{array} \right.$, $\lambda_i$ defined in (1.14),

$$\Psi^{n,r}_{k,i} = \sum_{k=0}^{i} \frac{(-1)^{i-k}}{2^{2i}} \binom{2i}{i} \binom{2i}{2k} B(r + k + 1/2, n + i - r - k + 1/2),$$

and $B(x, y)$ is the Beta function. The entries of $M^{-1}$ are obtained by transposition of $N^{-1}$. \hfill \Box

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