ON THE APPLICATION OF THE MATRIX FORMALISM FOR THE HEAT KERNEL TO NUMBER THEORY

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Earlier, in the study of combinatorial properties of the heat kernel of the Laplace operator with covariant derivative, a diagram technique and matrix formalism were constructed. In particular, the obtained formalism allows one to control the coefficients of the heat kernel, which is useful for calculations. In this paper, we consider a simple case with an Abelian connection in the two-dimensional space. This model allows us to give a mathematical description of the operators and find a relation between these operators and generating functions of numbers. Bibliography: 22 titles.

Dedicated to M. A. Semenov-Tian-Shansky on the occasion of his 70th birthday

1. Introduction

This work is a consequence of the technique obtained in [1] when studying the combinatorial properties of the heat kernel for the Laplace operator with covariant derivative in the case of a gauge connection with smooth components. In short, the main result of [1] can be formulated as follows. The diagonal part of the heat kernel $K(x, x, \tau)$ for the $d$-dimensional ($d > 1$) Laplace operator with covariant derivative $\partial_\mu + \omega_\mu(x)$ can be represented as an asymptotic series $(4\pi\tau)^{-d/2} \sum_{n=0}^{\infty} \tau^n a_n(x, x)$, whose coefficients are called Seeley–DeWitt coefficients. The calculation of these coefficients is a time-consuming task, which has been studied for many years, see [2–8]. The paper proposes a technique that allows one to obtain the answer for an arbitrary coefficient (previously, similar methods have been constructed in [9–12], but they have fundamental differences with the one we suggest). The main elements of the technique are operators $A^\mu, B^\mu, S^\mu_l$, and $S^1_l$ which act on matrices with two rows and an arbitrary number of columns and are defined by the rules

$B^\mu \left( \begin{array}{cccc} \nu_1 & \nu_2 & \ldots & \nu_n \\ k_1 & k_2 & \ldots & k_n \end{array} \right) = \left( \begin{array}{cccc} \mu \nu_1 & \nu_2 & \ldots & \nu_n \\ k_1 + 1 & k_2 & \ldots & k_n \end{array} \right) + \ldots + \left( \begin{array}{cccc} \nu_1 & \nu_2 & \ldots & \mu \nu_n \\ k_1 + 1 & k_2 + 1 & \ldots & k_n + 1 \end{array} \right) - B^\mu \left( \begin{array}{c} 1 \\ k \end{array} \right) = 0,$

$A^\mu \left( \begin{array}{cccc} \nu_1 & \nu_2 & \ldots & \nu_n \\ k_1 & k_2 & \ldots & k_n \end{array} \right) = \left( \begin{array}{cccc} \mu & \nu_1 & \nu_2 & \ldots & \nu_n \\ k_1 + 1 & k_2 & \ldots & k_n \end{array} \right) + \ldots + \left( \begin{array}{cccc} \nu_1 & \nu_2 & \ldots & \mu \\ k_1 + 1 & k_2 + 1 & \ldots & k_n + 1 \end{array} \right),$

$S^\mu_l \left( \begin{array}{cccc} \nu_1 & \nu_2 & \ldots & \nu_n \\ k_1 & k_2 & \ldots & k_n \end{array} \right) = \left( \begin{array}{cccc} \mu & \nu_1 & \nu_2 & \ldots & \nu_n \\ l & k_1 & k_2 & \ldots & k_n \end{array} \right) - S^\mu_l \left( \begin{array}{c} 1 \\ k \end{array} \right) = 0.$

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where \( \mu \in \{1, \ldots, d\} \), \( \nu_1, \ldots, \nu_n \in \{1, \ldots, d\} \cup \{1\} \), and \( l, k_1, \ldots, k_n \in \mathbb{N} \). The element \( \mathbb{1} \) has a special meaning and will not be discussed here. Analogously, one can introduce an operator \( S_1^l \) and a map \( \Upsilon \) which sends a column to a tensor with some coefficient according to the rules

\[
\Upsilon \left( \frac{\mu_1, \nu}{k} \right) = \frac{1}{k} \sum_{k=1}^{n} \nabla_{\mu_1, \nu} F_{\mu k \nu}(x), \quad \Upsilon \left( \frac{1}{k} \right) = \frac{1}{k},
\]

where \( k \in \mathbb{N} \), \( I_n = \{1, \ldots, n\} \), \( \nu, \mu_1, \ldots, \mu_n \in \{1, \ldots, d\} \), \( \mu_{l_1} = \mu_1 \ldots \mu_n \), and \( F_{\mu \nu}(x) = \partial_{\mu} \omega_{\nu}(x) - \partial_{\nu} \omega_{\mu}(x) + [\omega_{\mu}(x), \omega_{\nu}(x)] \). It is assumed that a matrix is a set of columns. Thus, the coefficients have the form

\[
a_n(x, x) = \Upsilon \prod_{k=1}^{n} S_1^{s_1}(A^{s_1} + B^{s_1})(A^{s_2} + B^{s_2}) \mathbb{1},
\]

where the initial element \( \mathbb{1} \) is defined by the properties

\[
B^{s_1} \mathbb{1} = 0, \quad A^{s_1} \mathbb{1} = \left( \begin{array}{c} \mu_1 \\ 1 \end{array} \right), \quad S_1^{s_1} = \left( \begin{array}{c} \mu \\ l \end{array} \right), \quad S_1^{s_1} = \left( \begin{array}{c} 1 \\ l \end{array} \right), \quad \Upsilon \mathbb{1} = 1.
\]

The main objects of study in this paper are the operators \( A, B, \) and \( \Upsilon \) defined by formulas (1), (2), and (3). They are a simplified version of \( A^s, B^s, \) and \( \Upsilon \).

2. Problem statement

First of all, several simplifications of the definitions should be made. We will consider the case of the action by operators with the same top indices (without loss of generality, they can be \( A^1 \) and \( B^1 \)). It is seen from the definition that the action of the operator \( B^s \) on the first element of a column results in adding an additional index \( \mu \). Since the first components of columns consist of ones, we suggest to consider the special case where the top element of a column shows how many times the operator \( B^1 \) has been applied to this column. Thus, using the simplifications mentioned above, the definitions of the operators under study have the following form.

**Definition 1.**

\[
A \left( \begin{array}{cccc} s_1 & s_2 & \cdots & s_n \\ k_1 & k_2 & \cdots & k_n \end{array} \right) = \left( \begin{array}{cccc} 0 & s_1 & s_2 & \cdots & s_n \\ k_1 & k_1 & k_2 & \cdots & k_n \end{array} \right) + \cdots + \left( \begin{array}{cccc} s_1 & s_2 & \cdots & 0 \\ k_1 & k_1 + 1 & k_2 & \cdots & k_n + 1 \end{array} \right) + \left( \begin{array}{cccc} s_1 & s_2 & \cdots & s_n \\ k_1 + 1 & k_2 & \cdots & k_n + 1 \end{array} \right) + \left( \begin{array}{cccc} s_1 & s_2 & \cdots & s_n \\ k_1 & k_2 & \cdots & k_n \end{array} \right). (1)
\]

**Definition 2.**

\[
B \left( \begin{array}{cccc} s_1 & s_2 & \cdots & s_n \\ k_1 & k_2 & \cdots & k_n \end{array} \right) = \left( \begin{array}{cccc} s_1 + 1 & s_2 & \cdots & s_n \\ k_1 & k_1 + 1 & k_2 & \cdots & k_n \end{array} \right) + \cdots + \left( \begin{array}{cccc} s_1 & s_2 + 1 & \cdots & s_n \\ k_1 & k_1 + 1 & k_2 + 1 & \cdots & k_n \end{array} \right) + \left( \begin{array}{cccc} s_1 & s_2 & \cdots & s_n + 1 \\ k_1 + 1 & k_1 & k_2 + 1 & \cdots & k_n + 1 \end{array} \right). (2)
\]

**Definition 3.**

\[
\Upsilon \left( \begin{array}{cccc} s_1 & s_2 & \cdots & s_n \\ k_1 & k_2 & \cdots & k_n \end{array} \right) = \prod_{i=1}^{n} \left( \frac{s_i}{k_i} \right), (3)
\]

where \( k_1, \ldots, k_n \in \mathbb{N} \) and \( s_1, \ldots, s_n \in \mathbb{N} \cup \{0\} \). The initial element \( \mathbb{1} \) is defined by the following conditions:

\[
B \mathbb{1} = 0, \quad A \mathbb{1} = \left( \begin{array}{c} 0 \\ 1 \end{array} \right), \quad \Upsilon \mathbb{1} = 1. (4)
\]

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The main purpose of this paper is to give a mathematically correct definition (see (10)–(13)) of the operators $A$, $B$, and $\Upsilon$ and to find a relation (see (19) and (20)) between generating functions and numbers of the form 

$$\varpi_n = \Upsilon(A + B)^n 1, \quad n \geq 0.$$  

(5)

3. Auxilliary model

3.1. Motivation. When constructing the model, it is convenient to take into account several heuristic conditions which make the model simpler and clearer and do not interfere with the study of the operators $A$, $B$, and $\Upsilon$:

- connection components are scalar functions;
- the first connection component is zero ($\omega_1(x) = 0$);
- $d = 2$.

Also, it is useful to observe that we can use the property $\partial_x^n(xe^x)|_{x=0} = n$, since the number in the first row shows how many times the operator $B$ has been applied to the column.

3.2. The two-dimensional model. This set of conditions allows one to find the second connection component $\omega_2(x_1, x_2)$, where $x = (x_1, x_2)$. The construction of the diagram technique and the matrix formalism imply a relation between the tension component $F_{\mu\nu}(x)$ and the definition of the operator $B$ which can be formulated as

$$x_2 F_{21}(x_1, x_2) = x_1 e^{x_1}.$$  

(6)

Lemma 3.1. The classical solution to (6) with the condition $\omega_2(0, x_2) = 0$, where $x_2 \neq 0$, has the form

$$\omega_2(x_1, x_2) = \frac{1}{x_2} (e^{x_1} (1 - x_1) - 1).$$  

(7)

3.3. The path-ordered exponential. In the case of commutative connection components, the path-ordered exponential

$$\Phi(x, y) := 1 + \sum_{n=1}^{\infty} (-1)^n \int_0^1 \int_0^1 \ldots \int_0^1 ds_1 \ldots ds_n \frac{dz_1^{\nu_1}}{ds_1} \ldots \frac{dz_n^{\nu_n}}{ds_n} \omega_{\nu_1}(z_1) \ldots \omega_{\nu_n}(z_n),$$

where $z_\mu^{\nu} = y^{\nu} + (x^{\nu} - y^{\nu}) \prod_{k=1}^{n} s_k$, can be reduced to an exponential function.

Lemma 3.2. If $x = (x_1, x_2)$ and $y = (y_1, y_2)$, then

$$\Phi(x, y) = \exp \left\{ \int_0^{x_1} \frac{dt}{t} (e^t (t - 1) + 1) \left( \frac{t(x_2 - y_2)}{t(y_2(x_1 - y_1) + (x_2 - y_2)(t - y_1))} \right) \right\}.$$ 

Proof. One should use (7), the parametrization $z_\mu(s) = (1 - s)y_\mu + sx_\mu$, and a change of variables of the form $s \mapsto t = y_1 + s(x_1 - y_1)$:

$$- \int_0^1 ds \frac{dz_\mu}{ds} \omega_\mu(z(s)) = - \int_0^1 ds \frac{x_2 - y_2}{z_2(s)} \left( e^{z_1(s)} (1 - z_1(s)) - 1 \right).$$

□
3.4. The generating function

**Theorem 3.3.** For $n \in \mathbb{N} \cup \{0\}$,

$$\Upsilon(A + B)^n 1 = \partial^n_x \exp \left\{ \int_0^x \frac{dt}{t} (e^t(t - 1) + 1) \right\} \bigg|_{x=0}. \quad (8)$$

**Proof.** After choosing the initial point $y = (0,0)$, the formula (see [13])

$$(\partial + \omega(x))\Phi(x,y) = \int_0^1 ds \frac{dz(s)}{ds} \Phi(x,z(s)) F_{\nu\mu}(z(s)) \Phi(z(s),y)$$

in the case under consideration (6) reduces to

$$\partial_1 \Phi(x,0) = \int_0^1 ds \Phi(x,sx) sx_1 e^{sx_2} \Phi(sx,0).$$

To complete the proof, we must use the differentiation theorem for diagrams, since

- the numbers in the top row are obtained due to the action of the derivative on factors of the form $ze^z$, and the degrees of the parametrization parameters to the left of them increase by 1;
- the action of the derivative on the ordered exponential results in the appearance of a new column and a change of degrees. $\square$

4. The algebraic structure

The main motivation for the description of this combinatorics is the operator $A$, which maps a matrix with $n$ columns to a set of matrices with $n + 1$ columns. At the same time, matrices are equal (as elements of the space) if and only if all their components coincide. All this suggests to introduce a vector space, comultiplication, and other operations leading to bialgebras (see [14,15]).

4.1. Monoids. For further work, it is convenient to introduce two sets

$$G_1 = \mathbb{N} \cup \{0\}, \quad G_2 = \left\{ \frac{1}{q} : q \in \mathbb{N} \right\} \cup \{\infty\},$$

where the infinity sign has the meaning of the inverse element to zero with respect to the standard product of numbers. Also, we can introduce a binary operation $*$ on $G_2$ which maps arbitrary elements $p,q \in G_2$ according to the following rule:

$$*: (p,q) \mapsto \left( \frac{1}{p} + \frac{1}{q} \right)^{-1} \in G_2.$$

Since $*$ is associative and the element $\infty$ is a unit in $G_2$, we obtain the following lemma.

**Lemma 4.1.** The sets $G_1$ and $G_2$ are monoids with respect to the operations $+$ and $*$, respectively.

Using $G_1$ and $G_2$, we can define two vector spaces $\mathbb{R}^{G_i}$, $i = 1,2$, each consisting of the linear combinations of basis vectors $e^i_q$ where $q \in G_i$. 686
4.2. Bialgebras. In order to construct a bialgebra, we should extend the vector space to a unital associative algebra and a counital coassociative coalgebra satisfying the necessary properties, which can be represented in the form of four commutative diagrams. Thus, four operations on \( R^G \) can be defined:

(i) a multiplication \( \mu_1 : R^G \otimes R^G \mapsto R^G \):

\[
\mu_1(e^1_p \otimes e^1_q) = e^1_{p+q} \text{ for any } e^1_p, e^1_q \in R^G;
\]

(ii) a unit \( \eta_1 : R \mapsto R^G \):

\[
\eta_1(1) = e^1_0;
\]

(iii) a comultiplication \( \Delta_1 : R^G \mapsto R^G \otimes R^G \):

\[
\Delta_1(e^1_q) = e^1_q \otimes e^1_q \text{ for any } e^1_q \in R^G;
\]

(iv) a counit \( \varepsilon_1 : R^G \mapsto R \):

\[
\varepsilon_1(e^1_q) = 1 \text{ for any } e^1_q \in R^G.
\]

In a similar way, we construct operations \( \mu_2, \eta_2, \Delta_2, \varepsilon_2 \) for \( R^G \), replacing the upper index in basis vectors and substituting \( \ast \) instead of + and \( \infty \) instead of 0.

Lemma 4.2. \( (R^G, \mu_1, \eta_1, \Delta_1, \varepsilon_1) \) and \( (R^G, \mu_2, \eta_2, \Delta_2, \varepsilon_2) \) are bialgebras.

Corollary 4.2.1. \( (V, \mu, \eta, \Delta, \varepsilon) \) is a bialgebra, where

\[
V = \left( \bigoplus_{G \in G^1} R^{G} \right), \quad \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \quad \eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}, \quad \Delta = \begin{pmatrix} \Delta_1 \\ \Delta_2 \end{pmatrix}, \quad \varepsilon = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}.
\]

4.3. The tensor algebra. To construct a tensor algebra, we should introduce maps \( |1| \) and \( |2| \) from \( R^G \) to \( R \) according to the following rules (see [16]):

\[
|1| : R^G \mapsto \mathbb{R}, \quad \sum_{g \in G^1} \alpha_g e^1_g \bigg|_1 = \sum_{g \in G^1} |\alpha_g| g \text{ for every } \alpha_g \in \mathbb{R},
\]

\[
|2| : R^G \mapsto \mathbb{R}, \quad \sum_{g \in G^2 \setminus \{\infty\}} \beta_g e^2_g \bigg|_2 = \sum_{g \in G^2 \setminus \{\infty\}} |\beta_g| g \text{ for every } \beta_g \in \mathbb{R}.
\]

Lemma 4.3. \( |1| \) and \( |2| \) are semi norms on \( R^G \) and \( R^G \), respectively.

The objects introduced above can be extended to a seminorm \( | | : \mathcal{V} \mapsto R \) on \( V \) according to the following formula:

\[
\left| \sum_{g_1 \in G_1, g_2 \in G_2} \alpha_{g_1,g_2} \left( e^1_{g_1} \right) \right| = \sum_{g_1 \in G_1, g_2 \in G_2 \setminus \{\infty\}} |\alpha_{g_1,g_2}| |g_1 g_2| \text{ for every } \alpha_{g_1,g_2} \in \mathbb{R}.
\]

It is obvious from construction that \( | | \) is a seminorm on \( V \). Hence, after introducing the tensor algebra

\[
T(V) = \bigoplus_{n=0}^{\infty} V^\otimes n,
\]

the seminorm \( | | \) can be extended from \( V \) to \( T(V) \) similarly to \( (9) \), taking into account the fact that on \( \mathbb{R} \) it coincides with the absolute value. If \( H = T(V)/\mathcal{L} \) where \( \mathcal{L} = \{ v \in T(V) : |v| = 0 \} \), then \( | | \) is a norm on \( H \).
4.4. An algebraic definition of the operators. The above set of operations is not sufficient to define the operators, so we must introduce additional maps

\[ \theta_1 : \mathbb{R} \mapsto \mathbb{R}^{G_1}, \quad \theta_2 : \mathbb{R} \mapsto \mathbb{R}^{G_2} \]

acting according to the rules

\[ \theta_1(1) = e_1^1, \quad \theta_2(1) = e_1^1. \]

In this case, the operators \( A, B, \) and \( \Upsilon \) obtain quite a clear meaning \( A, B, \) and \(| | \), which can be formulated as follows.

**Theorem 4.4.** The operator \( A : T(V) \mapsto T(V) \) acts according to the rule

\[ A(1) = \left( \frac{\eta_1}{\theta_2} \right) \left( \begin{array}{c} 1 \\ e_1^1 \end{array} \right), \quad (10) \]

\[ A(v) = \sum_{j=0}^{k-1} \mu^{\otimes(j+1)} \otimes \text{id}^{\otimes(k-j)} \circ \left( \left( \frac{\eta_1}{\theta_2} \right) \otimes \text{id} \right)^{\otimes j} \otimes \left( \frac{\eta_1 \otimes \eta_1 \otimes \text{id}}{\theta_2 \otimes \Delta_2} \right)^{\otimes j} \otimes \text{id}^{\otimes(k-1-j)} v, \]

\[ + \mu^{\otimes k} \otimes \text{id} \circ \left( \left( \frac{\eta_1}{\theta_2} \otimes \text{id} \right)^{\otimes k} \otimes \left( \frac{\eta_1}{\theta_2} \right) \right) \otimes \text{id}^{\otimes(k-1-j)} v, \quad v \in V^{\otimes k}, \quad k \in \mathbb{N}. \quad (11) \]

The operator \( B : T(V) \mapsto T(V) \) acts according to the rule \( B(1) = 0, \)

\[ B(v) = \sum_{j=0}^{k-1} \mu^{\otimes(j+1)} \otimes \text{id}^{\otimes(k-j)} \circ \left( \left( \frac{\eta_1}{\theta_2} \otimes \text{id} \right)^{\otimes j} \otimes \left( \frac{\eta_1 \otimes \eta_1 \otimes \text{id}}{\theta_2 \otimes \Delta_2} \right)^{\otimes j} \otimes \text{id}^{\otimes(k-1-j)} v, \]

\[ v \in V^{\otimes k}, \quad k \in \mathbb{N}. \quad (12) \]

The operator \( \Upsilon : T(V) \mapsto \mathbb{R}_+ \cup \{0\} \) is given by

\[ \Upsilon(v) = |v|, \quad v \in T(V). \quad (13) \]

**Proof.** It suffices to check the action of (11) and (12) on the basis vectors from \( V \): for any \( g_1 \in G_1, \ g_2 \in G_2, \)

\[ \mu \circ \left( \frac{\eta_1}{\theta_2} \right) \otimes \text{id} \left( \begin{array}{c} e_1^{g_2} \\ e_1^{g_2} \end{array} \right) = \mu \left( \begin{array}{c} e_0^1 \\ e_0^1 \end{array} \right) = \left( \begin{array}{c} e_0^1 \\ e_0^1 \end{array} \right), \]

\[ \mu \otimes \text{id} \circ \left( \frac{\eta_1 \otimes \eta_1 \otimes \text{id}}{\theta_2 \otimes \Delta_2} \right) \left( \begin{array}{c} e_1^{g_2} \\ e_1^{g_2} \end{array} \right) = \mu \otimes \text{id} \left( \begin{array}{c} e_0^2 \\ e_0^2 \end{array} \right) = \left( \begin{array}{c} e_0^2 \\ e_0^2 \end{array} \right), \]

\[ \mu \circ \left( \frac{\theta_1 \otimes \text{id}}{\theta_2 \otimes \text{id}} \right) \left( \begin{array}{c} e_0^1 \\ e_0^1 \end{array} \right) = \mu \left( \begin{array}{c} e_0^1 \\ e_0^1 \end{array} \right) = \left( \begin{array}{c} e_0^1 \\ e_0^1 \end{array} \right). \quad \square \]

5. Generalization

The resulting combinatorics was based on the assumption \( x_2 F_2(x_1, x_2) = x_1 e^{x_1} \), which controls the action of the operator \( B \) by the rule (2). However, we can relax this assumption, thus widening the class of operators under consideration.

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5.1. The generating function. To this end, we should consider an equality of a more general form. It can be written as
\[ x_2 F_{21}(x_1, x_2) = f(x_1), \]
(14)
where the Taylor series expansion of \( f(x_1) \) in a neighborhood of zero is given by the formula
\[ f(x_1) = \sum_{k=0}^{\infty} \frac{b_k}{k!} x_1^k. \]
Under the new assumption, after repeating all the operations performed earlier, the generating function takes the form
\[ \Phi(x_1, 0) = \exp \left\{ \frac{x_1}{t} \int_0^t \int_0^s ds f(s) \right\}. \]
(15)

5.2. The algebraic structure. In the case (14), in order to describe the algebraic structure of the model, we should modify only the definition of the map \( | \cdot |_1 \), i.e., the kernel of the seminorm, and the normalization of the basis vectors. In the case where the function \( f(x) \) has a negative Taylor coefficient, the map \( | \cdot |_1 \) loses its meaning of a seminorm. Formally, these transformations look as follows:
\[ |e_g^1|_1 = g \rightarrow |e_g^1|_1 = b_g \quad \text{for} \quad g \in G_1, \]
(16)
\[ \ker(| \cdot |_1) = \{e_0^1\} \rightarrow \ker(| \cdot |_1) = \{e_g^1, \forall g \in G_1 : b_g = 0\}. \]
(17)

Theorem 5.1. The transition from the model with the condition \( x_2 F_{21}(x_1, x_2) = x_1 e^{x_1} \) to the model with the condition \( x_2 F_{21}(x_1, x_2) = f(x_1) \) can be performed by changing the generating function to (15) and the definition of \( | \cdot |_1 \) to (16) and (17).

5.3. The inverse problem. A natural question arises: given a generating function \( F(x) \), can one construct operators \( A, B, \) and \( \Upsilon \) such that the relation \( \Upsilon (A + B)^n 1 = F^{(n)}(0) \) holds for \( n \geq 0 \)? The answer is positive, and first one should observe that the function \( f(x_1) \) can be recovered from the ordered exponential \( \Phi(x_1, 0) \) according to the following lemma.

Lemma 5.2.
\[ f(x_1) = \frac{\Phi(x_1, 0) \Phi'(x_1, 0) + x_1 \Phi(x_1, 0) \Phi''(x_1, 0) - (\Phi'(x_1, 0))^2}{(\Phi(x_1, 0))^2}. \]
(18)

Proof. We have
\[ x_2 F_{21}(x_1, x_2) = -\frac{x_1}{x_2} \partial_{x_1} (\ln(\Phi(x_1, 0))), \]
\[ x_2 F_{21}(x_1, x_2) = -x_2 \partial_{x_2} \omega_2(x_1, x_2). \]

It should be noted that the generating function \( \Phi(x_1, 0) \) has a special property, namely, \( \Phi(0, 0) = 1 \). Clearly, not every generating function has this property, so formula (18) is not applicable for all generating functions, due to a problem observed in [22]. However, there are several obvious ways to modify the scheme.

Theorem 5.3. If \( F(x_1) \) is a generating function of numbers such that \( F(0) \neq 0 \), then the function \( f(x_1) \) can be constructed by formula (18) for
\[ \Phi(x_1, 0) = \frac{F(x_1)}{F(0)}, \]
which causes the following change in the initial data: \( 1 \mapsto F(0) \) in (5) and
\[ \omega_n = \Upsilon (A + B)^n F(0). \]
(19)
Theorem 5.4. If \( F(x_1) \) is an arbitrary generating function of numbers, then the function \( f(x_1) \) can be constructed by formula (18) for
\[
\Phi(x_1, 0) = F(x_1) - F(0) + 1,
\]
which gives the answer in the form
\[
\varpi_n = \Upsilon(A + B)^n 1 + (F(0) - 1)\delta_{n0}.
\]

5.4. Examples.

| Numbers                   | \( F(x) \)                      | \( f(x) \)                      |
|---------------------------|---------------------------------|---------------------------------|
| Catalan numbers [17]      | \( \frac{1 - \sqrt{1 - 4x}}{2x} \) | \( \frac{1}{(1 - 4x)^{3/2}} \) |
| Bell numbers [20, 21]     | \( e^{x-1} \)                   | \( e^x(x + 1) \)               |
| Binomial expansion        | \( (1 + x)^n, \alpha \in \mathbb{C} \) | \( \frac{\alpha}{(1 + x)^n}, \alpha \in \mathbb{C} \) |
| Exponential numbers [18, 19] | \( e^{\sin(x)} \)            | \( \cos(x) - x \sin(x) \)       |

6. Corollaries

6.1. The operator function. The special case \( \phi(A, B) = (A + B)^n \), where \( n \in \mathbb{N} \cup \{0\} \), was studied above. In this section, our main interest is to the numbers \( \Upsilon \phi(A, B)1 \), where the function under consideration has the form
\[
\Upsilon \prod_{k=1}^{n} \Lambda_{k}^{\sigma_k} 1,
\]
where \( n \in \mathbb{N} \), \( \Lambda_k \in \{A, B\} \), and \( \sigma_k \in \mathbb{N} \cup \{0\} \) for \( k \in \{1, \ldots, n\} \). Using the diagram technique, one can very easily see that the operator \( A \) acts only on ordered exponentials (lines), while the operator \( B \) differentiates all functions (circles) except exponentials. But, due to the commutativity of the connection components \( \omega_{\mu}(x) \), the ordered exponential can be factored out from the expression as a single factor \( \Phi(x, 0) \). In this case, \( A \) is the operator of multiplication by the function \( \partial_x \ln \Phi(x, 0) \). Therefore, the operator \( B \) acts as the derivative with connection \( -\ln'_x \Phi(x, 0) \).

Theorem 6.1. If \( n \in \mathbb{N} \), \( \Lambda_k \in \{A, B\} \), and \( \sigma_k \in \mathbb{N} \cup \{0\} \) for \( k \in \{1, \ldots, n\} \), and
\[
\phi(A, B) = \prod_{k=1}^{n} \Lambda_{k}^{\sigma_k},
\]
then
\[
\Upsilon \phi(A, B)1 = \phi(\ln'_x \Phi(x, 0), \partial_x - \ln'_x \Phi(x, 0))\Phi(x, 0)|_{x=0}.
\]

6.2. Estimates on the ordered exponential. The numbers \( \varpi_n \) obtained above can be used to derive local estimates on covariant derivatives of the ordered exponential. The result can be written as follows.

Theorem 6.2. If the dimension is equal to \( d \), \( \nabla_{\mu} = \partial_\mu + \omega_{\mu}(x) \), \( \Phi(x, y) \) is the ordered exponential constructed from the connection components \( \omega_{\mu}(x) \), and for some neighborhood \( U_{\delta}(x_0) \), where \( \delta > 0 \), of some point \( x_0 \in \mathbb{R}^d \) there exists \( C > 0 \) such that for \( x \in U_{\delta}(x_0) \) the inequalities
\[
|\nabla_{\mu_1} \cdots \nabla_{\mu_{k-2}} F_{\mu_{k-1}\mu_{k}}(x)| \leq C^k \text{ for any } k > 1, \mu_1, \ldots, \mu_k \in \{1, \ldots, d\}
\]
hold, then
\[
\left| \prod_{k=1}^{n} \nabla_{\mu_k} \Phi(x, y) \right|_{y=x} \leq d^n C^{2n} \varpi_{2n} \text{ for every } n \in \mathbb{N}.
\]
\[
\prod_{k=1}^{n} \int_0^1 dss^{k-1} (x-y)^{\rho} \partial_{\rho_k} \nabla \mu_k \Phi(x,y) \bigg|_{y=x} \leq d^n C^{2n} \frac{\varpi_n}{n!} \quad \text{for every } n \in \mathbb{N}.
\]

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REFERENCES

1. A. V. Ivanov, “Diagram technique for the heat kernel of the covariant Laplace operator,” Theoret. Math. Phys., 198, No. 1, 100–117 (2019).
2. H. P. McKean and I. M. Singer, J. Differential Geom., 1, 43–69 (1967).
3. P. Gilkey, J. Differential Geom., 10, 601–618 (1975).
4. P. Amsterdamski, A. L. Berkin, and D. J. O’Connor, Classical Quantum Gravity, 6, 1981–1991 (1989).
5. I. G. Avramidi, Phys. Lett. B, 238, 92–97 (1990).
6. I. G. Avramidi, Nucl. Phys. B, 355, 712–754 (1991).
7. A. E. M. van de Ven, Classical Quantum Gravity, 15, 2311–2344 (1998).
8. D. V. Vassilevich, Phys. Rep., 388, 279–360 (2003).
9. A. O. Barvinsky and G. A. Vilkovisky, Nucl. Phys. B, 282, 163–188 (1987).
10. A. O. Barvinsky and G. A. Vilkovisky, Nucl. Phys. B, 333, 471–511 (1990).
11. A. O. Barvinsky and G. A. Vilkovisky, Nucl. Phys. B, 333, 512–524 (1990).
12. I. G. Avramidi, Comm. Math. Phys., 288, 963–1006 (2009).
13. G. M. Shore, Ann. Phys., 137, 262–305 (1981).
14. S. Dascalescu, C. Nastasescu, and S. Raianu, Hopf Algebras: An Introduction, Marcel Dekker, New York (2000).
15. M. E. Sweedler, Hopf Algebras, W. A. Benjamin, New York (1969).
16. E. F. Krause, Taxicab Geometry, Courier Corporation (1975).
17. P. Hilton and J. Pedersen, Math. Intelligencer, 13, 64–75 (1991).
18. E. T. Bell, Amer. Math. Monthly, 41, 411–419 (1934).
19. J. Riordan, An Introduction to Combinatorial Analysis, Wiley, New York (1958).
20. G. T. Williams, Amer. Math. Monthly, 52, 323–327 (1945).
21. E. A. Enneking and J. C. Ahuja, Fibonacci Quart., 14, 67–73 (1976).
22. M. Pourahmadi, Amer. Math. Monthly, 91, 303–307 (1984).