Relativistic static fluid spheres with a linear equation of state

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It is shown that almost all known solutions of the kind mentioned in the title are easily derived in a unified manner when a simple ansatz is imposed on the metric. The Whittaker solution is an exception, replaced by a new solution with the same equation of state.

04.20.Jb

I. INTRODUCTION

Einstein’s equations for static and spherically symmetric perfect fluids have been studied by many authors [1-3]. The unknown functions are more than the equations, therefore, either an equation of state should be prescribed or some simplifying ansatz assumed. Schwarzschild found the first solution [4] in 1916. It was a degenerate case \((n = 0)\) of the linear equation of state

\[
\rho = np + \rho_0
\]  

where \(\rho\) and \(p\) are the density and the pressure of the fluid and \(n, \rho_0\) are constant parameters. A year later, two other solutions satisfying Eq. (1) were found, describing static cosmological models [5,6]. Tolman [7] used eight simplifying ansatze for the metric components to find new solutions. Some limiting cases of these satisfy Eq. (1) too, leading to a simple, but singular solution for arbitrary \(n\) and \(\rho_0 = 0\) [3]. Klein rediscovered this solution first for the case of incoherent radiation \(n = 3\) [8] and then for general \(n\) [9] by studying systematically the equation of state (1) with vanishing \(\rho_0\) (the so-called \(\gamma\)-law). This Klein-Tolman (KT) solution was rediscovered later three times more [10-12]. Meanwhile, two new solutions with \(\rho_0 \neq 0\) appeared in 1968, developing further the idea of Schwarzschild’s incompressible sphere. Whittaker argued that the gravitational mass density \(\rho + 3p\) should be constant [13], while Buchdahl and Land demanded that the speed of sound should be the largest possible, namely, equal to the speed of light [14].

The main stream of papers, however, utilized simplifying assumptions, which often lead to complicated equations of state, if any. Already Tolman used the ansatz

\[
e^{-\lambda} = a + br^2
\]

where \(a\) and \(b\) are constants, \(0 \leq a \leq 1\), and \(e^\lambda\) is the \(g_{rr}\) component of the metric. A thorough study of this ansatz was performed by Kuchowicz [15,16] who found seven different cases in total, many of them having rather involved expressions for the pressure and the density. He didn’t study whether his solutions satisfy Eq. (1).

What happens when we impose on the field equations both Eqs. (1) and (2)? An overdetermined system of differential equations is obtained which may be devoid of any non-trivial solutions. The real answer, derived in this paper, is somewhat surprising. All known explicit solutions with a linear equation of state satisfy also Eq. (2), except for the Whittaker solution. We find an additional solution with \(n = -3\) instead, which appears to be new.

It is much easier to impose both requirements directly on the field equations, than to study case by case the results of Ref. [16]. In this way we derive in a unified and simple manner almost all known solutions with linear equation of state, including a new hybrid between the Einstein static universe (ESU) and the KT solution. This programme is executed in the following section. The last section contains some discussion.

II. FIELD EQUATIONS AND THEIR SOLUTIONS

The metric of a static spherically symmetric spacetime reads

\[
ds^2 = e^{\nu} dt^2 - e^{\lambda} dr^2 - r^2 d\Omega^2
\]

where \(d\Omega^2\) is the metric on the two-sphere and \(\nu, \lambda\) depend on \(r\). The system of equations is
\[
\rho = \frac{\lambda'}{r} e^{-\lambda} + \frac{1}{r^2} (1 - e^{-\lambda})
\]

(4)

\[
p = \frac{\nu'}{r} e^{-\lambda} - \frac{1}{r^2} (1 - e^{-\lambda})
\]

(5)

\[
p' + \frac{1}{2} (\rho + p) \nu' = 0
\]

(6)

where the prime means a \( r \)-derivative and one of the field equations has been replaced by the contracted Bianchi identity (6). Let us impose Eq. (2). Then Eq. (4) gives a simple expression for the density

\[
\rho = -3b + \frac{1 - a}{r^2}
\]

(7)

which is regular at the centre if \( a = 1 \). When Eq. (1) holds, Eq. (6) may be integrated. Two different cases arise, \( n = -1 \) and \( n \neq -1 \).

Let us discuss first the case \( n = -1 \). Then Eq. (6) yields

\[
p = -\frac{1}{2} \rho_0 \nu + h
\]

(8)

with \( h \) being an integration constant. Plugging Eq. (8) into Eq. (1) and the resulting expression for \( \rho \) into Eq. (7) gives

\[
\frac{1}{2} \rho_0 \nu = h - \rho_0 - 3b + \frac{1 - a}{r^2}
\]

(9)

There are two possibilities. If \( \rho_0 = 0 \) then from Eq. (8) \( p = h \), from Eq. (1) \( \rho = -p \) and from Eq. (9) \( a = 1, b = h/3 \). Therefore

\[
e^{-\lambda} = 1 + \frac{h}{3} r^2
\]

(10)

while Eq. (5) gives \( \nu \) after a simple integration: \( \nu = -\lambda \). This is the de Sitter solution.

If \( \rho_0 \neq 0 \) then the insertion of Eqs. (2,8,9) into Eq. (5) yields the relation

\[
\rho_0 (\rho_0 + 2b) r^4 = -4 (1 - a) \left( a + b r^2 \right)
\]

(11)

Obviously, \( a = 1 \) and \( b = -\rho_0/2 \). From Eq. (7) \( \rho = 3\rho_0/2 \) and from Eq. (1) \( p = -\rho_0/2 \). Eq. (8) gives \( \nu = 1 + 2h/\rho_0 = \text{const} \), while Eq. (2) becomes

\[
e^{-\lambda} = 1 - \frac{\rho_0}{2} r^2
\]

(12)

The pressure and the density satisfy, in fact, the equation of state \( \rho + 3p = 0 \). Hence, we obtain ESU, although initially \( \rho = -p + \rho_0 \).

Let us continue with the generic case \( n \neq -1 \). Now the integration of Eq. (6) brings a different answer

\[
p = p_1 e^{-\frac{n+1}{2} \nu} - p_0
\]

(13)

where \( p_1 \) is a constant of integration, while \( p_0 = \rho_0 / (n + 1) \). Proceeding like we have done with Eq. (8) the analog of Eq. (9) is obtained

\[
np_1 e^{-\frac{n+1}{2} \nu} - np_0 + \rho_0 = -3b + \frac{1 - a}{r^2}
\]

(14)

Again, there are two possibilities. If \( n = 0 \) we have \( a = 1 \), \( b = -\rho_0/3 \) and from Eq. (1) \( \rho = \rho_0 = \text{const} \). The metric component \( \nu \) is obtained not from Eq. (14) but from Eq. (5), the last equation to be satisfied. For a general \( n \) it becomes a linear equation for \( y = e^{\frac{n+1}{2} \nu} \):
\[
\frac{2}{n+1}r (a + br^2) y' = \left[1 - a - (p_0 + b) r^2 \right] y + p_1 r^2
\]  \hspace{1cm} (15)

In our case it reduces to
\[
2 \left(1 - \frac{\rho_0}{3}r^2 \right) y' = -\frac{2}{3}r\rho_0 y + p_1 r
\]  \hspace{1cm} (16)

and is easily integrated
\[
e^\nu = \left(\frac{3p_1}{2\rho_0} + Ce^{-\lambda/2}\right)^2
\]  \hspace{1cm} (17)

\[
e^{-\lambda} = 1 - \frac{\rho_0}{3}r^2
\]  \hspace{1cm} (18)

which, after some change of notation, coincides with solution III of Tolman. This is the Schwarzschild interior solution.

If \( n \neq 0 \) both Eqs. (14) and (15) determine \( \nu \) and we must study their compatibility. Eq. (14) may be written as
\[
\frac{1}{y} = A + \frac{B}{r^2}
\]  \hspace{1cm} (19)

\[
A = -\frac{1}{n(n+1)p_1} \left[ \rho_0 + 3(n+1)b \right]
\]  \hspace{1cm} (20)

\[
B = \frac{1-a}{np_1}
\]  \hspace{1cm} (21)

When these relations are put into Eq. (15), a polynomial of second degree in \( r^2 \) arises, which must vanish identically. Three compatibility conditions result
\[
(1 - a) \left[ \frac{4n + (n+1)^2}{(n+1)^2} a - 1 \right] = 0
\]  \hspace{1cm} (22)

\[
A \left[ \rho_0 + (n+3)b \right] = 0
\]  \hspace{1cm} (23)

\[
B \left( (n + 1)\rho_0 + 4nb + (n + 1)(n + 3)b \right) - (n + 1)^2(1 - a)A = 0
\]  \hspace{1cm} (24)

Eq. (22) has two solutions for \( a \). Let \( a = 1 \). Then \( B = 0 \), \( \nu \) is a constant and \( \rho = -3b \). Eq. (24) is also satisfied, while Eq. (23) provides two subcases. If \( A = 0 \) then from Eq. (19) \( y = \infty \) which is unacceptable. If \( \rho_0 = -(n+3)b \) then Eq. (20) shows that \( A \neq 0 \) when \( b \neq 0 \). Eq. (1) gives \( p = b \), hence the equation of state is \( \rho + 3p = 0 \) for any \( n \).

We again obtain the ESU with constant \( \nu \) and
\[
e^{-\lambda} = 1 + br^2
\]  \hspace{1cm} (25)

This is the same as Eq. (12), since both can be written as \( e^{-\lambda} = 1 + pr^2 \).

The last case which remains is
\[
a = \frac{(n+1)^2}{4n + (n+1)^2}
\]  \hspace{1cm} (26)

where \( a \neq 1 \) since \( n \neq 0 \). Eq. (23), like before, offers two choices.

a) \( A = 0 (\rho_0 = -3(n + 1)b) \). Then Eq. (24) simplifies to \((n - 1)b = 0\). There are two subcases:

a1) \( n = 1, b \neq 0 \). Then \( \rho_0 = -6b \), Eq. (26) gives \( a = 1/2 \), Eq. (21) gives \( B = 1/2p_1 \) and Eqs. (2,19,7,1) give respectively
\[
e^{-\lambda} = \frac{1}{2} \left(1 - \frac{\rho_0}{3}r^2 \right)
\]  \hspace{1cm} (27)
\[ e^\nu = 2p_1 r^2 \] (28)

\[ \rho = \frac{1}{2} \left( \rho_0 + \frac{1}{r^2} \right) \] (29)

\[ \rho = p + \rho_0 \] (30)

This is precisely the solution of Buchdahl and Land [14] after \( e^\nu \) is rescaled.

a2) \( b = 0 \). Then \( \rho_0 = 0 \) and we easily find

\[ e^{-\lambda} = \frac{(n+1)^2}{4n+(n+1)^2} \] (31)

\[ e^{-\frac{n+1}{2} \nu} = \frac{4}{\left[ 4n+(n+1)^2 \right]^2} p_1 r^2 \] (32)

\[ \rho = np = \frac{4n}{\left[ 4n+(n+1)^2 \right]^2} r^2 \] (33)

This is the KT solution in curvature coordinates [3,8-12]. Eq. (31) shows that the solution exists when \( 4n+(n+1)^2 > 0 \). This condition is fulfilled for \( n \) outside the interval \( (-5.83 = -3 - 2\sqrt{2}, -3 + 2\sqrt{2} = -0.17) \). When \( n < -5.83 \) or \(-0.17 < n < 0 \) the solution has \( p \) and \( \rho \) of different signs. The physically realistic range is \( 1 \leq n \leq \infty \), which includes the important cases of stiff fluid \( (n = 1) \) and incoherent radiation \( (n = 3) \). The KT solution is singular at \( r = 0 \) but it can serve as an interior solution for \( r > r_0 > 0 \) in multi-layered perfect fluid models. It also represents a focal point for a regular solution, which has been studied numerically [8,17,18,19].

b) \( \rho_0 = -(n+3) b \). Since \( a \neq 1 \), Eq. (24) reduces to \( (n+3) b = 0 \). The two subcases are

b1) \( b = 0 \). Then \( A = 0 \) and we go back to case a).

b2) \( n = -3 \). Then \( \rho_0 = 0 \), \( a = -1/2 \), \( \rho + 3p = 0 \) and

\[ e^{-\lambda} = -\frac{1}{2} + br^2 \] (34)

\[ p_1 e^\nu = b - \frac{1}{2r^2} \] (35)

\[ \rho = -3b + \frac{3}{2r^2} \] (36)

The constant \( p_1 \) may be set to one by a time rescaling. This appears to be a new solution - a hybrid between ESU with its constant pressure and density and the KT solution. Its pressure, density and metric component \( g_{tt} \) are linear combinations of those two solutions, although Eqs. (4-6) form a highly non-linear system. When \( b = 0 \) we obtain the KT solution for \( n = -3 \) which, in fact, does not exist. When \( b \neq 0 \), \( e^\lambda \) and \( e^\nu \) are negative for small \( r \). However, if we choose \( r_0^2 = 1/2b \) and \( b > 0 \), then the solution exists for \( r > r_0 \), has positive pressure, negative density and represents a regular companion of ESU.

Whittaker has solved the general case \( n = -3 \) when \( \rho_0 \) may be different from zero and the ansatz (2) is not imposed [13]. However, the above solution has been missed. The reason is the following. Eqs. (1,4,5) lead to the relation

\[ (e^{\lambda+\nu})' = k r e^{2(\lambda+\nu)} \] (37)

where \( k \) is a constant. The integration of this equation yields

\[ e^{\lambda+\nu} = \frac{2}{c - kr^2} \] (38)

c being a constant. Whittaker assumes that \( k = ca \), where \( \alpha \) is yet another constant. This choice excludes the above solution which has \( c = 0 \), \( k = -2p_1 \neq 0 \) and in principle may be regained as a limiting case. However, when one sets \( \rho_0 = 0 \) in the results of Ref. [13] only ESU is obtained. It should be noticed that the singular term in Eq. (35) is different from the universal Schwarzschild term \( m/r \) which was correctly discarded from the interior solution.
III. DISCUSSION

In this paper we have studied the gravitation of static spherically symmetric perfect fluid solutions with a linear equation of state. We imposed the additional requirement (2), making the system of equations overdetermined. Amazingly, it gives six non-trivial solutions: the Schwarzschild interior solution for an incompressible sphere, the de Sitter cosmological solution, the Einstein static universe, the stiff fluid solution of Buchdahl and Land, the Klein-Tolman singular solution and a new solution with $\rho + 3p = 0$. Among the known solutions with linear equation of state only the Whittaker solution does not satisfy Eq. (2). The reason is that it satisfies Eq. (38) and if Eq (2) is required too, $\nu$ has the only option to remain a constant, which is too restrictive. These solutions have been found originally starting from a variety of physical considerations and following quite different mathematical approaches. Here they are derived in a unified way as a result of straightforward calculations and a new solution is added. Roughly speaking, it is a linear combination between ESU and the KT solution, which is unusual for a non-linear system of differential equations. It closes the list of solutions satisfying both Eqs. (1) and (2). The equation of state $\rho + 3p = \text{const}$ is unrealistic because the speed of sound is negative. However, models which satisfy it have interesting properties, are often integrable and occupy important places in different classification schemes. This concerns one whole class of algebraically special perfect fluids of Petrov type II or D [1] (Eq.(29.20)), a rigidly rotating solution of type D [20], the well-known Wahlquist solution [21-22] which has the Whittaker solution as a static limit and the Kerr metric as a vacuum subcase, and the recently found Wahlquist-Newman solution [23] which reduces to a number of well-known solutions in appropriate limits. An interesting phenomenon is the effective change of the equation of state by which ESU is obtained first from $\rho = -p + \rho_0$ and then from $\rho = np + \rho_0$. When $\rho_0 = 0$ a one-parameter sequence of KT solutions exists which serve as focal points for regular numerical solutions. When $\rho_0 \neq 0$ only two discrete cases satisfy Eq. (2) for $n = 0$ and $n = 1$, the third known one ($n = -3$) drops out of this scheme. Presumably when one replaces the ansatz (2) by a different one, other integrable cases will appear, although the system is overdetermined.

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