On a Class Almost Contact Manifolds with Norden Metric

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1. PRELIMINARIES

Let \((M, \varphi, \xi, \eta, g)\) be a \((2n + 1)\)-dimensional almost contact manifold with Norden metric \((B\text{-metric})\), i.e. \((\varphi, \xi, \eta)\) is an almost contact structure, and \(g\) is a pseudo-Riemannian metric, called a Norden metric \((B\text{-metric})\) such that [1]

\[
\varphi^2 x = -x + \eta(x)\xi, \quad \eta(\xi) = 1,
\]

\[
g(\varphi x, \varphi y) = -g(x, y) + \eta(x)\eta(y).
\]

The associated metric \(\tilde{g}\) of \(g\) is defined by

\[
\tilde{g}(x, y) = g(x, \varphi y) + \eta(x)\eta(y)
\]

and is a Norden metric, too. Both metrics are necessarily of signature \((n + 1, n)\).
Let $\nabla$ be the Levi-Civita connection of $g$. The fundamental tensor $F$ is defined by

$$F(x, y, z) = g((\nabla_x \varphi)y, z)$$

and has the following properties

$$F(x, y, z) = F(x, z, y),$$
$$F(x, \varphi y, \varphi z) = F(x, y, z) - F(x, \xi, z)\eta(y) - F(x, y, \xi)\eta(z).$$

The following 1-forms are associated with $F$:

$$\theta(x) = g^{ij} F(e_i, e_j, x), \quad \theta^*(x) = g^{ij} F(e_i, \varphi e_j, x),$$
$$\omega(x) = F(\xi, \xi, x), \quad \omega^* = \omega \circ \varphi.$$

The corresponding vector field to $\omega$ is denoted by $\Omega$, i.e. $\omega(x) = g(x, \Omega)$. 
The Nijenhuis tensor $N$ of the almost contact structure $(\varphi, \xi, \eta)$ is defined by [6]

\[
N(x, y) = \varphi^2 [x, y] + [\varphi x, \varphi y] - \varphi [\varphi x, y] - \varphi [x, \varphi y]
+ (\nabla_x \eta) y. \xi - (\nabla_y \eta) x. \xi
\]

The almost contact structure is said to be integrable if $N = 0$. In this case the almost contact manifold is called \textit{normal} [6].
A classification of the almost contact manifolds with Norden metric is introduced in [1]. Eleven basic classes $\mathcal{F}_i$ ($i = 1, 2, ..., 11$) are characterized there according to the properties of $F$.

The classes for which $F$ is expressed explicitly by the other structural tensors are called main classes.

In the present work we focus our attention on the class $\mathcal{F}_{11}$ given by

$$\mathcal{F}_{11} : \quad F(x, y, z) = \eta(x)\{\eta(y)\omega(z) + \eta(z)\omega(y)\}.$$
The curvature tensor $R$ of $\nabla$ is defined as usually by

$$R(x, y)z = \nabla_x \nabla_y z - \nabla_y \nabla_x z - \nabla_{[x, y]} z,$$

and its corresponding tensor of type (0,4) is given by

$$R(x, y, z, u) = g(R(x, y)z, u).$$

The Ricci tensor $\rho$ and the scalar curvatures $\tau$ and $\tau^*$ are defined by, respectively

$$\rho(y, z) = g^{ij} R(e_i, y, z, e_j), \quad \tau = g^{ij} \rho(e_i, e_j), \quad \tau^* = g^{ij} \rho(e_i, \varphi e_j).$$

$R$ is called a $\varphi$-Kähler-type tensor if

$$R(x, y, \varphi z, \varphi u) = -R(x, y, z, u).$$
Let $\alpha = \{x, y\}$ be a non-degenerate 2-section spanned by the vectors $x, y \in T_p M$, $p \in M$. The sectional curvature of $\alpha$ is defined by

$$k(\alpha; p) = \frac{R(x, y, y, x)}{\pi_1(x, y, y, x)},$$

where $\pi_1(x, y, z, u) = g(y, z)g(x, u) - g(x, z)g(y, u)$.

In [5] there are introduced the following special sections in $T_p M$:

- a $\xi$-section if $\alpha = \{x, \xi\}$;
- a $\varphi$-holomorphic section if $\varphi \alpha = \alpha$;
- a totally real section if $\varphi \alpha \perp \alpha$ with respect to $g$. 
The square norms of $\nabla \varphi$, $\nabla \eta$ and $\nabla \xi$ are defined by, respectively [3]:

$$||\nabla \varphi||^2 = g^{ij} g^{ks} g \left((\nabla e_i \varphi) e_k, (\nabla e_j \varphi) e_s\right),$$

$$||\nabla \eta||^2 = ||\nabla \xi||^2 = g^{ij} g^{ks} (\nabla e_i \eta) e_k (\nabla e_j \eta) e_s.$$

**Definition 1.1.** An almost contact manifold with Norden metric is called *isotropic Kählerian* if

$$||\nabla \varphi||^2 = ||\nabla \eta||^2 = 0 \quad (||\nabla \varphi||^2 = ||\nabla \xi||^2 = 0).$$
2. CURVATURE PROPERTIES OF $\mathcal{F}_{11}$-MANIFOLDS

**Proposition 2.1.** On a $\mathcal{F}_{11}$-manifold it is valid

$$||\nabla \varphi||^2 = -||N||^2 = -2||\nabla \eta||^2 = 2\omega(\Omega).$$

**Corollary 2.1.** On a $\mathcal{F}_{11}$-manifold the following conditions are equivalent:

(i) the manifold is isotropic Kählerian;

(ii) the vector $\Omega$ is isotopic, i.e. $\omega(\Omega) = 0$;

(iii) the Nijenhuis tensor $N$ is isotropic.
Proposition 2.2. On a $\mathcal{F}_{11}$-manifold we have

$$\tau + \tau^{**} = 2\text{div}(\varphi \Omega) = 2\rho(\xi, \xi),$$

where $\tau^{**} = g^{is}g^{jk}R(e_i, e_j, \varphi e_k, \varphi e_s)$.

Proposition 2.3 The curvature tensor of a $\mathcal{F}_{11}$-manifold with Nor-dan metric satisfies

$$R(x, y, \varphi z, \varphi u) = -R(x, y, z, u) + \psi_4(S)(x, y, z, u),$$

where the tensor $\psi_4(S)$ is defined by [4]

$$\psi_4(S)(x, y, z, u) = \eta(y)\eta(z)S(x, u) - \eta(x)\eta(z)S(y, u)$$
$$+ \eta(x)\eta(u)S(y, z) - \eta(y)\eta(u)S(x, z).$$

$$S(x, y) = (\nabla_x \omega)\varphi y - \omega(\varphi x)\omega(\varphi y).$$
Proposition 2.4. The curvature tensor of a $\mathcal{F}_{11}$-manifold with Norden metric is $\varphi$-K"{a}herian iff

$$(\nabla_x \omega^*)y = \eta(x)\eta(y)\omega(\Omega) + \omega^*(x)\omega^*(y),$$

where $\omega^* = \omega \circ \varphi.$
3. AN EXAMPLE

Let $G$ be a $(2n + 1)$-dimensional real connected Lie group, and $\mathfrak{g}$ be its corresponding Lie algebra. If $\{x_0, x_1, \ldots, x_{2n}\}$ is a basis of left-invariant vector fields on $G$, we define a left-invariant almost contact structure $(\varphi, \xi, \eta)$ by

$$
\varphi x_i = x_{i+n}, \quad \varphi x_{i+n} = -x_i, \quad \varphi x_0 = 0, \quad i = 1, 2, \ldots, n,
$$

$$
\xi = x_0, \quad \eta(x_0) = 1, \quad \eta(x_j) = 0, \quad j = 1, 2, \ldots, 2n.
$$

We also define a left-invariant pseudo-Riemannian metric $g$ on $G$ by

$$
g(x_0, x_0) = g(x_i, x_i) = -g(x_{i+n}, x_{i+n}) = 1, \quad i = 1, 2, \ldots, n,
$$

$$
g(x_j, x_k) = 0, \quad j \neq k, \quad j, k = 0, 1, \ldots, 2n.
$$

Then, $(G, \varphi, \xi, \eta, g)$ is an almost contact manifold with Norden metric.
Let the Lie algebra $\mathfrak{g}$ of $G$ be given by the following non-zero commutators

$$[x_i, x_0] = \lambda_i x_0, \quad i = 1, 2, ..., 2n, \quad (3.1)$$

where $\lambda_i \in \mathbb{R}$.

Equalities (3.1) determine a $2n$-parametric family of solvable Lie algebras.

Further, we study the manifold $(G, \varphi, \xi, \eta, g)$ with Lie algebra $\mathfrak{g}$ defined by (3.1).
The components of the Levi-Civita connection:

\[
\nabla_{x_i}x_j = \nabla_{x_i}\xi = 0, \quad \nabla_\xi x_i = -\lambda_i \xi, \quad i, j = 1, 2, \ldots, 2n,
\]

\[
\nabla_\xi \xi = \sum_{k=1}^{n}(\lambda_k x_k - \lambda_{k+n} x_{k+n}).
\]

The essential non-zero components of $F$:

\[
F(\xi, \xi, x_i) = \omega(x_i) = -\lambda_{i+n}, \quad F(\xi, \xi, x_{i+n}) = \omega(x_{i+n}) = \lambda_{i},
\]

$i = 1, 2, \ldots, n$.

**Proposition 3.1.** The almost contact manifold with Norden metric $(G, \varphi, \xi, \eta, g)$ defined by (3.1) belongs to the class $\mathcal{F}_{11}$.

**Remark 3.1.** The considered manifold has closed 1-forms $\omega$ and $\omega^*$. 
Curvature properties

- The components of the curvature tensor $R$:

$$R(x_i, \xi, \xi, x_j) = -\lambda_i \lambda_j, \quad i, j = 1, 2, ..., 2n.$$ 

Because of $R(x_i, x_j, \varphi x_k, \varphi x_s) = 0$ for all $i, j, k, s = 0, 1, ..., 2n$ and Proposition 2.3 we obtain

**Proposition 3.2.** The curvature tensor and the Ricci tensor of the $\mathcal{F}_{11}$-manifold $(G, \varphi, \xi, \eta, g)$ defined by (3.1) have the form, respectively

$$R = \psi_4(S), \quad \rho(x, y) = \eta(x)\eta(y)\text{tr}S + S(x, y),$$

where $S(x, y) = (\nabla_x \omega)\varphi y - \omega(\varphi x)\omega(\varphi y)$ and $\text{tr}S = \text{div}(\varphi \Omega)$. 
• The essential components of the Ricci tensor $\rho$:

\[ \rho(x_i, x_j) = -\lambda_i \lambda_j, \quad i = 1, 2, \ldots, 2n, \]

\[ \rho(\xi, \xi) = -\sum_{k=1}^{n} \left( \lambda_k^2 - \lambda_{k+n}^2 \right). \]

• The scalar curvatures $\tau$ and $\tau^*$:

\[ \tau = -2 \sum_{k=1}^{n} \left( \lambda_k^2 - \lambda_{k+n}^2 \right), \quad \tau^* = -2 \sum_{k=1}^{n} \lambda_k \lambda_{k+n}. \]
• Sectional curvatures:

The characteristic 2-sections $\alpha_{ij}$ spanned by the vectors $\{x_i, x_j\}$ are the following:

$\xi$-sections $\alpha_{0,i} \ (i = 1, 2, ..., 2n)$

$\varphi$-holomorphic sections $\alpha_{i,i+n} \ (i = 1, 2, ..., n)$

the rest $\alpha_{ij}$ are totally real sections.

**Proposition 3.3.** The $\mathcal{F}_{11}$-manifold with Norden metric $(G, \varphi, \xi, \eta, g)$ defined by (3.1) has zero totally real and $\varphi$-holomorphic sectional curvatures, and its $\xi$-sectional curvatures are given by

$$k(\alpha_{0,i}) = -\frac{\chi_i^2}{g(x_i, x_i)}, \quad i = 1, 2, ..., 2n.$$
Isotropic Kähler properties

• The vector field $\Omega$ corresponding to the 1-form $\omega$:

$$\Omega = - \sum_{k=1}^{n} (\lambda_{k+n}x_k + \lambda_kx_{k+n}), \quad \omega(\Omega) = - \sum_{k=1}^{n} (\lambda_k^2 - \lambda_{k+n}^2).$$

**Proposition 3.4.** The $\mathcal{F}_{11}$-manifold with Norden metric $(G, \varphi, \xi, \eta, g)$ defined by (3.1) is isotropic Kählerian iff the condition

$$\sum_{k=1}^{n} \left( \lambda_k^2 - \lambda_{k+n}^2 \right) = 0$$

holds.
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