Complete synchronization of two Chen-Lee systems

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Abstract. This study demonstrates that complete synchronization of two Chen-Lee chaotic systems can be easily achieved. The upper bound of the Chen-Lee chaotic system is estimated numerically. A controller is designed to synchronize two chaotic systems. Sufficient conditions for synchronization are obtained using Lyapunov’s direct method. Two numerical examples are presented to verify the proposed synchronization approach.

1. Introduction
In recent years, the study of chaos synchronization has received increasing attention due to its predicted potentials in technological applications. Chaos synchronization was first described by Fujisaka and Yamada [1] in 1983, but the phenomenon did not attract much attention until Pecora and Carroll [2] introduced a method (called PC method) to synchronize two identical chaotic systems with different initial conditions. From that time, chaos synchronization has seen a flurry of research activities for over two decades. A wide variety of approaches have been successfully applied to the synchronization of chaotic systems which include adaptive control [3, 4], backstepping design [5-7], active control [8, 9], and nonlinear control [10, 11]. Using these methods, numerous works for the synchronization problems of well-known chaotic and hyperchaotic systems have been carried out by many researchers [12-14], to name just a few recent examples.

Control methods based on a single variable have been proposed and successfully applied to the synchronization of chaotic systems [15-17]. The above-mentioned methods are simple, efficient, and...
easy to implement in practical applications. However, not all possible selections of driving signals for a given chaotic system can lead to a synchronized state. For example, Liao and Lin [15] studied chaos synchronization of Lorenz systems, which can be achieved based on a single variable, x or y; whereas synchronization cannot be achieved based on the single variable, z.

Recently, while studying the anti-control of chaos, Chen and Lee [18] introduced a new chaotic attractor, called the Chen-Lee system [19]. This system displays comprehensive dynamic behaviors. A chaotic system is well-known to be bounded, and the estimation of its bounds is important in chaos control, chaos synchronization, and their applications. Therefore, the bounds of Chen-Lee chaotic systems were estimated by a numerical simulation. The results were then adopted to design a controller to synchronize two chaotic systems.

In this paper, the synchronization of two Chen-Lee chaotic systems is addressed by employing a linear feedback controller based on partial variables. When Lyapunov’s direct method was used to study chaos synchronization, the proposed method was simpler, and more efficient and powerful in finding a positive definite function. Finally, two numerical examples are presented to demonstrate the effectiveness of the proposed control scheme.

2. System description and estimating its bounds

For the anti-control of chaos in rigid-body motion, Chen and Lee introduced a new chaotic system, called the Chen-Lee system [19]. Chen and his co-workers [8, 11, 20] continued their studies of this system including chaos synchronization and the dynamics with fractional orders. The Chen-Lee system has also received considerable attention from different researchers [21-23]. The system is described by the following nonlinear differential equations denoted as system (1)

\[
\begin{align*}
\dot{x} &= -yz + ax \\
\dot{y} &= xz + by \\
\dot{z} &= (1/3)x + cy + e
\end{align*}
\]

where \(x, y,\) and \(z\) are state variables and \(a, b,\) and \(c\) are three system parameters. System (1) acts as a chaotic attractor when \((a, b, c) = (5, -10, -3.8)\).

A relatively accurate estimation of the upper bound of a chaotic system is very important but unfortunately quite difficult to achieve technically. Therefore, the upper bound of the Chen-Lee chaotic system is estimated by a numerical simulation in this section. Suppose that \(M_1, M_2,\) and \(M_3\) are the respective upper bounds of the absolute values of variables, \(x, y,\) and \(z.\) Since a chaotic system has bounded trajectories, the constants \(M_1, M_2,\) and \(M_3\) must exist. The approximate values of \(M_1, M_2,\) and \(M_3\) can be obtained through numerical simulations and their values are \(\approx 26.9, \approx 23.2,\) and \(\approx 17.6\) respectively. The results are also used to design a controller to synchronize two chaotic systems.

3. Synchronization of Chen-Lee systems based on partial variables

Consider two chaotic systems given by

\[
\begin{align*}
\dot{x} &= f(x, t) \\
\dot{y} &= g(y, t) + u(x, y, t)
\end{align*}
\]

where \(x, y \in R^n, f, g \in C^r [R \times R^n, R^n], u \in C^r [R \times R^n \times R^n, R^n],\) and \(r \geq 1.\) \(R^n\) comprises the set of non-negative real numbers. Assume that equation (2) is the drive system, equation (3) is the response system, and \(u(x, y, t)\) is the control vector. The response and drive systems are said to be synchronized if \(\forall x(t_0, y(t_0) \in R^n, \|x(t) - y(t)\| \to 0\) as \(t \to \infty.\)

**Theorem:** For Chen-Lee chaotic systems, the drive and response systems are defined as follows:

**Drive system**
where is the controller, which is introduced in system (5). Let , be the synchronization errors between the drive and response systems. Systems (4) and (5) can be synchronized by a linear feedback controller based on partial variables. Two different cases are considered here.

**Case 1:** The response system and drive systems are globally asymptotically synchronized under the control law: , , where , .

**Case 2:** The response system and drive systems are globally asymptotically synchronized under the control law: , , where , .

**Proof for Case 1:** By subtracting system (4) from system (5), the following error system is obtained:

\[
\begin{aligned}
\dot{e}_1 &= -e_2z_1 - e_3y_1 + (a - k_1)e_1 \\
\dot{e}_2 &= e_1e_3 + e_2z_1 + e_3x_1 + (b - k_2)e_2 \\
\dot{e}_3 &= 1/3(e_1e_2 + e_1y_1 + e_2x_1) + ce_3
\end{aligned}
\]  

Choose the following Lyapunov function, , as follows:

\[
V = \frac{1}{2}e_1^2 + \frac{1}{2}e_2^2 + \frac{3}{2}e_3^2
\]

It is clear that the Lyapunov function is a positive definite function. The time derivative of the Lyapunov function is

\[
\dot{V} = e_1\dot{e}_1 + e_2\dot{e}_2 + 3e_3\dot{e}_3 = -(k_1 - a)e_1^2 - (k_2 - b)e_2^2 + 3ce_3^2 + (2x_1 + e_1)e_2e_3 \\
\leq -(k_1 - a)e_1^2 - (k_2 - b)e_2^2 + 3ce_3^2 + 4M_1e_2e_3 = -e^TPe
\]

where is the error vector, and

\[
P = \begin{bmatrix}
(k_1 - a) & 0 & 0 \\
0 & (k_2 - b) & -2M_1 \\
0 & -2M_1 & -3c
\end{bmatrix}
\]

Obviously, to ensure that the origin of error system (6) is asymptotically stable, the matrix should be a positive definite matrix, which implies that is negative definite. This case is satisfied if and only if the following inequalities hold:

\[
k_1 > a \text{ and } k_2 > b - \frac{4M_1^2}{3c}
\]

Hence, the origin of error system (6) is asymptotically stable. Therefore, response system (5) synchronizes with drive system (4) as time tends to . Hence, the proof for case 1 is completed.

**Proof for Case 2:** Similarly, the error system is obtained:
\[
\begin{aligned}
\dot{e}_1 &= -e_2e_3 - e_2z_1 - e_3y_1 + (a - k_1)e_1 \\
\dot{e}_2 &= e_1e_3 + e_1z_1 + e_2x_1 + be_2 \\
\dot{e}_3 &= 1/3(e_1e_2 + e_1y_1 + e_2x_1) + (c - k_3)e_3
\end{aligned}
\] (11)

Similar to the above case, the Lyapunov function, \(V\), is defined as
\[
V = \frac{1}{2}e_1^2 + \frac{1}{2}e_2^2 + \frac{3}{2}e_3^2
\] (12)

Again, it is evident that the Lyapunov function is a positive definite function. The differential of the Lyapunov function (12) along the trajectory of error system (11) is
\[
\dot{V} = e_1\dot{e}_1 + e_2\dot{e}_2 + 3e_3\dot{e}_3 = -(k_1 - a)e_1^2 + be_2^2 - 3(k_3 - c)e_3^2 + (2x_1 + e_1)e_2e_3 \\
\leq -(k_1 - a)e_1^2 + be_2^2 - 3(k_3 - c)e_3^2 + 2M_1|e_2e_3| = -e^TPe
\] (13)

where \(e = [e_1, e_2, e_3]^T\) is the error vector, and
\[
P = \begin{bmatrix}
(k_1 - a) & 0 & 0 \\
0 & -b & -2M_1 \\
0 & -2M_1 & 3(k_3 - c)
\end{bmatrix}
\] (14)

Obviously, to ensure that the origin of error system (11) is asymptotically stable, matrix \(P\) should be positive definite, which implies that \(\dot{V}\) is negative definite. Based on the above analysis, the following inequalities must be satisfied:
\[
k_1 > a \quad \text{and} \quad k_3 > c - \frac{4M_1^2}{3b}.
\] (15)

According to the Lyapunov stability theory, error system (11) is asymptotically stable about the origin, i.e., \(e_1 = e_2 = e_3 = 0\). It can be concluded that systems (4) and (5) will achieve complete synchronization. Hence, the proof for case 2 is completed.

4. Numerical simulations

In this section, two numerical examples are used to demonstrate the effectiveness of the proposed method. In the numerical simulations, the fourth-order Runge-Kutta method is used to solve the systems with a time step equal to 0.001s.

**Example 1.** The constants of the Chen-Lee system are chosen as \(a = 5\), \(b = -10\), and \(c = -3.8\) in all simulations so that it exhibits chaotic behavior. The initial states of the drive and response systems are taken as \(x_1(0) = 0.2\), \(y_1(0) = 0.2\), and \(z_1(0) = 0.2\) and \(x_2(0) = 5.0\), \(y_2(0) = 5.0\), and \(z_2(0) = 5.0\), respectively. Thus the initial states of error system (6) are \(e_1(0) = 4.8\), \(e_2(0) = 4.8\), and \(e_3(0) = 4.8\). According to the theorem described above, the lower bound of the feedback control coefficients, \(k_1\) and \(k_2\), can easily be obtained. Through simulations, \(M_1 \approx 26.9\), \(M_2 \approx 23.2\), and \(M_3 \approx 17.6\) are obtained. Hence, the minimum lower bound for \(k_1\) and \(k_2\) can easily be set. The feedback coefficients, \(k_1\) and \(k_2\), are chosen as 5.1 and 244, respectively. Moreover, the control scheme is activated after 100 s. The simulation results are illustrated in figure 1. As we expect, one can observe that the response system begins to trace the drive system, and they finally become synchronized.

**Example 2.** Similarly, the initial states of the drive and response systems are chosen as \(x_1(0) = 0.2\), \(y_1(0) = 0.2\), and \(z_1(0) = 0.2\) and \(x_2(0) = 30\), \(y_2(0) = 30\), and \(z_2(0) = 30\), respectively. Thus the initial states of error system (11) are \(e_1(0) = 29.8\), \(e_2(0) = 29.8\), and \(e_3(0) = 29.8\). Based on the above-described theorem, the lower bound of the feedback control coefficients, \(k_1\) and \(k_2\), can also be
obtained. We may simply choose $k_1 = 5.1$ and $k_3 = 93$. Again, the control scheme is also activated after 100 s. The simulation results are displayed in figure 2. From these figures, it can be seen that the synchronization errors converge to zero and two identical systems indeed achieve chaos synchronization.

**Figure 1.** Synchronization errors, $e_1$, $e_2$, and $e_3$ versus time ($t$). The control law, $u_1 = -5.1(x_2 - x_1)$ and $u_2 = -244(y_2 - y_1)$, is activated at $t = 100$ s.
Figure 2. Synchronization errors, $e_1$, $e_2$, and $e_3$ versus time ($t$).

The control law, $u_1 = -5.1(x_2 - x_1)$ and $u_3 = -93(z_2 - z_1)$, is activated at $t = 100$ s.

5. Conclusions

This study demonstrates that complete synchronization of two Chen-Lee chaotic systems can be easily achieved based on partial variables. The bounds of Chen-Lee chaotic systems were also estimated by a numerical simulation. $M_1$, $M_2$, and $M_3$ are the upper bounds of the absolute values for variables $x$, $y$, and $z$, respectively. Through simulations, the results were obtained as $M_1 \approx 26.9$, $M_2 \approx 23.2$, and $M_3 \approx 17.6$. The above results were also adopted to design a controller to synchronize two chaotic systems. Sufficient conditions for synchronization were obtained by Lyapunov’s direct method. When Lyapunov’s direct method is used to study chaos synchronization, the proposed method is simpler, and more efficient and powerful in finding a positive definite function. Numerical simulations were also conducted to validate the proposed synchronization approaches.

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