Research article

Fractional inequalities of the Hermite–Hadamard type for \( m \)-polynomial convex and harmonically convex functions

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Abstract: In this paper, it is our purpose to establish some new fractional inequalities of the Hermite–Hadamard type for the \( m \)-polynomial convex and harmonically convex functions. Our results involve the Caputo–Fabrizio and \( \zeta \)-Riemann–Liouville fractional integral operators. They generalize, complement and extend existing results in the literature. By taking \( m \geq 2 \), we deduce loads of new and interesting inequalities. We expect that the thought laid out in this work will provoke advance examinations in this course.

Keywords: Hermite–Hadamard; \( m \)-polynomial convex; \( m \)-polynomial harmonically convex; Riemann–Liouville; Caputo–Fabrizio

Mathematics Subject Classification: 26D15, 26A51, 26D10

1. Introduction

The sets \( T \) and \( S \subseteq \mathbb{R} \setminus \{0\} \) are called convex and harmonically convex, respectively if

\[
\begin{align*}
\varsigma q + (1 - \varsigma)z & \in T \quad \text{for all } q, z \in T \quad \text{and } \varsigma \in [0, 1]; \\
\frac{qz}{\varsigma q + (1 - \varsigma)z} & \in S \quad \text{for all } q, z \in S \quad \text{and } \varsigma \in [0, 1].
\end{align*}
\]
Whenever used, we shall always consider $T$ as a convex set and $S$ as a harmonically convex set. Let $m \in \mathbb{N}$. Recall that a function $\varphi : T \to \mathbb{R}$ is said to be $m$-polynomial convex [31] on $T$ if

$$\varphi(\varsigma q + (1 - \varsigma)z) \leq \frac{1}{m} \sum_{\theta = 1}^{m} \left[ 1 - (1 - \varsigma)^{\theta} \right] \varphi(q) + \frac{1}{m} \sum_{\theta = 1}^{m} \left[ 1 - \varsigma^{\theta} \right] \varphi(z)$$

for all $q, z \in S$ and $\varsigma \in [0, 1]$. For this class of functions, Toplu et al. established the following double inequality of the Hermite–Hadamard type.

**Theorem 1** ([31]). Let $\varphi : T \to \mathbb{R}$ be an $m$-polynomial convex function. If $\xi, \delta \in T$ with $\xi < \delta$, and $\varphi$ is Lebesgue integrable on $[\xi, \delta]$, then the following Hermite–Hadamard type inequality holds:

$$\frac{2^{-1}m}{m + 2^{-m} - 1} \varphi \left( \frac{\xi + \delta}{2} \right) \leq \frac{1}{\delta - \xi} \int_{\xi}^{\delta} \varphi(r) \, dr \leq \frac{\varphi(\xi) + \varphi(\delta)}{m} \sum_{\theta = 1}^{m} \frac{\theta}{\theta + 1}. \tag{1.1}$$

The inequality (1.1) boils down to the classical Hermite–Hadamard inequality for convex functions if we take $m = 1$. Recently, Awan et al. [2] introduced the notion of $m$-polynomial harmonically convex functions as follows: a real valued function $\varphi : S \to \mathbb{R}^+ := [0, \infty)$ is $m$-harmonically convex if

$$\varphi \left( \frac{qz}{\varsigma q + (1 - \varsigma)z} \right) \leq \frac{1}{m} \sum_{\theta = 1}^{m} \left[ 1 - (1 - \varsigma)^{\theta} \right] \varphi(q) + \frac{1}{m} \sum_{\theta = 1}^{m} \left[ 1 - \varsigma^{\theta} \right] \varphi(z) \tag{1.2}$$

for all $q, z \in S$ and $\varsigma \in [0, 1]$. In the same paper, the authors established the following Hermite–Hadamard type inequality for this class of functions:

**Theorem 2** ([2]). Let $\varphi : S \to \mathbb{R}^+$ be an $m$-polynomial harmonically convex function. If $\xi, \delta \in S$ with $0 < \xi < \delta$, and $\varphi$ is Lebesgue integrable on $[\xi, \delta]$, then the following Hermite–Hadamard type inequality holds:

$$\frac{2^{-1}m}{m + 2^{-m} - 1} \varphi \left( \frac{2\xi\delta}{\xi + \delta} \right) \leq \frac{\xi\delta}{\delta - \xi} \int_{\xi}^{\delta} \varphi(r) \, dr \leq \frac{\varphi(\xi) + \varphi(\delta)}{m} \sum_{\theta = 1}^{m} \frac{\theta}{\theta + 1}.$$

In the sequel, we will denote the sets of all $m$-polynomial convex and $m$-polynomial harmonically convex functions from $A$ into $B$ by $\textbf{XP}_m(A, B)$ and $\textbf{HXP}_m(A, B)$, respectively. The classical Hermite–Hadamard inequality has generated load of generalizations and extensions to other class of convexity. There are dozens of articles in this direction. We invite the interested reader to see the following articles [3–6, 8, 10–20, 22–30, 32–34] and the references cited therein.

Now, recall that the left- and right-sided $\xi$-Riemann–Liouville fractional integral operators $\mathcal{I}_{\xi}^{\mathcal{F}_\xi}$ and $\mathcal{I}_{\mathcal{F}_\xi}^{\xi}$ of order $\epsilon > 0$, for a real valued continuous function $\varphi(r)$, are defined as ([21]):

$$\mathcal{I}_{\xi}^{\mathcal{F}_\xi} \varphi(r) = \frac{1}{\Gamma(\epsilon)} \int_{\xi}^{r} (r - \varsigma)^{\varsigma-1} \varphi(\varsigma) \, d\varsigma, \quad r > \xi,$$

and

$$\mathcal{I}_{\mathcal{F}_\xi}^{\xi} \varphi(r) = \frac{1}{\Gamma(\epsilon)} \int_{\varsigma}^{\delta} (\varsigma - r)^{\varsigma-1} \varphi(\varsigma) \, d\varsigma, \quad r < \delta,$$
where $\zeta > 0$, and $\Gamma_{\zeta}$ is the $\zeta$-gamma function given by

$$\Gamma_{\zeta}(r) := \int_0^\infty s^{r-1} e^{-s} d\zeta, \quad Re(r) > 0,$$

with the properties $\Gamma_{\zeta}(r + \zeta) = r\Gamma_{\zeta}(r)$ and $\Gamma_{\zeta}(1) = 1$. If $\zeta = 1$, we simply write

$$\mathcal{I}_{\zeta}^r \varphi = \mathcal{J}_{\zeta}^r \varphi \quad \text{and} \quad \mathcal{J}_{\delta}^r \varphi = \mathcal{J}_{\delta}^r \varphi.$$

The beta function $\mathcal{B}$ is defined by

$$\mathcal{B}(u, v) = \int_0^1 s^{u-1}(1-s)^{v-1} \, ds \quad \text{for} \quad Re(u) > 0, Re(v) > 0. \quad (1.3)$$

Another fractional integral operators of interest is the Caputo–Fabrizio operators [1]: let $L^2(\xi, \delta)$ be the space of square integrable functions on the interval $(\xi, \delta)$ and $H^1(\xi, \delta) := \{g \mid g \in L^2(\xi, \delta) \quad \text{and} \quad g' \in L^2(\xi, \delta)\}$. If $\varphi \in H^1(\xi, \delta)$, $\xi < \delta$ and $\mu \in [0, 1]$, then the left- and right-sided Caputo–Fabrizio fractional integral operators $\mathcal{C}_\xi^\mu \varphi$ and $\mathcal{C}_\delta^\mu \varphi$ are defined by

$$\mathcal{C}_\xi^\mu \varphi(s) = \frac{1 - \mu}{B(\mu)} \varphi(s) + \frac{\mu}{B(\mu)} \int_\xi^s \varphi(r) \, dr \quad (1.4)$$

and

$$\mathcal{C}_\delta^\mu \varphi(s) = \frac{1 - \mu}{B(\mu)} \varphi(s) + \frac{\mu}{B(\mu)} \int_s^\delta \varphi(r) \, dr, \quad (1.5)$$

where $B : [0, 1] \to (0, \infty)$ is a normalization function satisfying $B(0) = B(1) = 1$.

Using these fractional integral operators in (1.4) and (1.5), Gürbüz et al. established the following fractional version of the Hermite–Hadamard inequality:

**Theorem 3 ([7]).** Let $\varphi : T \to \mathbb{R}$ be a convex function on $T$. If $\xi, \delta \in T$ with $\xi < \delta$, and $\varphi$ is Lebesgue integrable on $[\xi, \delta]$, then the following double inequality holds:

$$\varphi\left(\frac{\xi + \delta}{2}\right) \leq \frac{B(\mu)}{\mu(\delta - \xi)} \left[ \mathcal{C}_\xi^\mu \varphi(s) + \mathcal{C}_\delta^\mu \varphi(s) - \frac{2(1 - \mu)}{B(\mu)} \varphi(s) \right] \leq \frac{\varphi(\xi) + \varphi(\delta)}{2},$$

where $\mu \in [0, 1]$, $s \in [\xi, \delta]$ and $B(\mu) > 0$ is a normalization function.

Since the classes of convexity introduced here are new, much work have not been done in this sense. This work is geared towards further development around inequalities for these classes. In view of this, we aim to achieve the following objectives:

1. To establish new Hermite–Hadamard type inequalities for the class of $m$-polynomial convex functions involving the Caputo–Fabrizio integral operators. Our first result in this direction generalizes and extends Theorem 3.

2. To obtain inequalities of the Hermite–Hadamard type for functions that are $m$-polynomial harmonically convex functions via the $\zeta$-Riemann–Liouville fractional integral operators. This, in turn, also complement and generalize some existing results in the literature.
2. Main results

2.1. Inequalities for $m$-polynomial convex functions

Inequalities of the Hermite–Hadamard type, for $m$-polynomial convex functions, are hereby presented. The results, presented herein, involve the Caputo–Fabrizio operators.

**Theorem 4.** Let $\varphi : T \to \mathbb{R}$ be a Lebesgue integrable function on $[\xi, \delta]$ with $\xi < \delta$ and $\xi, \delta \in T$. If $\varphi \in \text{XP}_m(T, \mathbb{R})$, then

$$
\frac{2^{1-m}}{m + 2^{m-1}} \varphi\left(\frac{\xi + \delta}{2}\right) \leq \frac{B(\mu)}{\mu(\delta - \xi)} \left[c_f I^\mu_\xi \varphi(s) + c_f I^\mu_\delta \varphi(s) - \frac{2(1 - \mu)}{B(\mu)} \varphi(s)\right]
$$

$$
\leq \frac{\varphi(\xi) + \varphi(\delta)}{m} \sum_{\vartheta=1}^{m} \frac{\vartheta}{\vartheta + 1},
$$

where $\mu \in (0, 1]$, $s \in [\xi, \delta]$ and $B(\mu) > 0$ is a normalization function.

*Proof.* Given that $\varphi \in \text{XP}_m(T, \mathbb{R})$, it follows from (1.1) that

$$
\frac{m}{m + 2^{m-1}} \varphi\left(\frac{\xi + \delta}{2}\right) \leq \frac{2}{\delta - \xi} \int_\xi^\delta \varphi(r) \, dr
$$

$$
= \frac{2}{\delta - \xi} \left[\int_\xi^s \varphi(r) \, dr + \int_s^\delta \varphi(r) \, dr\right].
$$

(2.1)

Multiplying both sides of (2.1) by $\frac{\mu(\delta - \xi)}{2B(\mu)}$ gives:

$$
\frac{\mu(\delta - \xi)}{2B(\mu)} \frac{m}{m + 2^{m-1}} \varphi\left(\frac{\xi + \delta}{2}\right) \leq \frac{\mu}{B(\mu)} \left[\int_\xi^s \varphi(r) \, dr + \int_s^\delta \varphi(r) \, dr\right].
$$

(2.2)

By adding $\frac{2(1 - \mu)}{B(\mu)} \varphi(s)$ to both sides of (2.2), we get:

$$
\frac{2(1 - \mu)}{B(\mu)} \varphi(s) + \frac{\mu(\delta - \xi)}{2B(\mu)} \frac{m}{m + 2^{m-1}} \varphi\left(\frac{\xi + \delta}{2}\right)
\leq \frac{2(1 - \mu)}{B(\mu)} \varphi(s) + \frac{\mu}{B(\mu)} \left[\int_\xi^s \varphi(r) \, dr + \int_s^\delta \varphi(r) \, dr\right]
$$

$$
= \left[\frac{(1 - \mu)}{B(\mu)} \varphi(s) + \frac{\mu}{B(\mu)} \int_\xi^s \varphi(r) \, dr\right]
$$

$$
+ \left[\frac{(1 - \mu)}{B(\mu)} \varphi(s) + \frac{\mu}{B(\mu)} \int_s^\delta \varphi(r) \, dr\right]
$$

$$
c_f I^\mu_\xi \varphi(s) + c_f I^\mu_\delta \varphi(s).
$$

This implies that

$$
\frac{2(1 - \mu)}{B(\mu)} \varphi(s) + \frac{\mu(\delta - \xi)}{2B(\mu)} \frac{m}{m + 2^{m-1}} \varphi\left(\frac{\xi + \delta}{2}\right)
\leq c_f I^\mu_\xi \varphi(s) + c_f I^\mu_\delta \varphi(s).
$$

(2.3)
On the other hand, we also get from (1.1) the following inequality:

$$\frac{2}{\delta - \xi} \int_\xi^\delta \varphi(r) \, dr \leq \frac{\varphi(\xi) + \varphi(\delta)}{m} \sum_{\theta = 1}^m \frac{2\theta}{\theta + 1}. \tag{2.4}$$

If we multiply (2.4) by $\frac{\mu(\delta - \xi)}{B(\mu)}$ and then add $\frac{2(1 - \mu)}{B(\mu)} \varphi(s)$ to the resulting inequality, we obtain:

$$c_f I_\xi^\mu \varphi(s) + c_f I_\delta^\mu \varphi(s) \leq \frac{\mu(\delta - \xi) \varphi(\xi) + \varphi(\delta)}{B(\mu)} \sum_{\theta = 1}^m \frac{\theta}{\theta + 1} + \frac{2(1 - \mu)}{B(\mu)} \varphi(s). \tag{2.5}$$

Hence, the desired result is obtained by combining (2.3) and (2.5). \qed

**Remark 1.** By taking $m = 1$, Theorem 4 becomes Theorem 3.

**Theorem 5.** Let $\varphi, \psi : T \to \mathbb{R}$ be two functions such that $\varphi \psi$ is Lebesgue integrable function on $[\xi, \delta]$ with $\xi < \delta$ and $\xi, \delta \in T$. If $\varphi \in \text{XP}_{m_1}(S, \mathbb{R})$, $\psi \in \text{XP}_{m_2}(T, \mathbb{R})$, then

$$\frac{B(\mu)}{\mu(\delta - \xi)} \left[ c_f I_\xi^\mu \varphi(s) \psi(s) + c_f I_\delta^\mu \varphi(s) \psi(s) - \frac{2(1 - \mu)}{B(\mu)} \varphi(s) \psi(s) \right]$$

$$\leq \int_0^1 \left[ \Delta_1(\varsigma) \varphi(\varsigma) \psi(\varsigma) + \Delta_2(\varsigma) \varphi(\varsigma) \psi(\varsigma) - \Delta_3(\varsigma) \varphi(\varsigma) \psi(\varsigma) + \Delta_4(\varsigma) \varphi(\varsigma) \psi(\varsigma) \right] d\varsigma,$$

where $\mu \in (0, 1]$, $s \in [\xi, \delta]$ and $B(\mu) > 0$ is a normalization function, and

$$\Delta_1(\varsigma) := \frac{1}{m_1 m_2} \sum_{\theta = 1}^{m_1} \sum_{\theta = 1}^{m_2} \left[ 1 - (1 - \varsigma)^\theta \right] \sum_{\theta = 1}^{m_1} \left[ 1 - (1 - \varsigma)^\theta \right];$$

$$\Delta_2(\varsigma) := \frac{1}{m_1 m_2} \sum_{\theta = 1}^{m_1} \sum_{\theta = 1}^{m_2} \left[ 1 - (1 - \varsigma)^\theta \right] \sum_{\theta = 1}^{m_1} \left[ 1 - (1 - \varsigma)^\theta \right];$$

$$\Delta_3(\varsigma) := \frac{1}{m_1 m_2} \sum_{\theta = 1}^{m_1} \sum_{\theta = 1}^{m_2} \left[ 1 - (1 - \varsigma)^\theta \right] \sum_{\theta = 1}^{m_1} \left[ 1 - (1 - \varsigma)^\theta \right];$$

$$\Delta_4(\varsigma) := \frac{1}{m_1 m_2} \sum_{\theta = 1}^{m_1} \sum_{\theta = 1}^{m_2} \left[ 1 - (1 - \varsigma)^\theta \right] \sum_{\theta = 1}^{m_1} \left[ 1 - (1 - \varsigma)^\theta \right].$$

**Proof.** Let $\varphi \in \text{XP}_{m_1}(T, \mathbb{R})$ and $\psi \in \text{XP}_{m_2}(T, \mathbb{R})$. Then for $\varsigma \in [0, 1]$, we have:

$$\varphi(\varsigma \xi + (1 - \varsigma)\delta) \leq \frac{1}{m_1} \sum_{\theta = 1}^{m_1} \left[ 1 - (1 - \varsigma)^\theta \right] \varphi(\xi) + \frac{1}{m_1} \sum_{\theta = 1}^{m_1} \left[ 1 - (1 - \varsigma)^\theta \right] \varphi(\delta) \tag{2.6}$$

and

$$\psi(\varsigma \xi + (1 - \varsigma)\delta) \leq \frac{1}{m_1} \sum_{\theta = 1}^{m_1} \left[ 1 - (1 - \varsigma)^\theta \right] \psi(\xi) + \frac{1}{m_1} \sum_{\theta = 1}^{m_1} \left[ 1 - (1 - \varsigma)^\theta \right] \psi(\delta). \tag{2.7}$$
Multiplying (2.6) and (2.7) gives:
\[
\varphi(\xi + (1 - \zeta)\delta) \nu(\xi + (1 - \zeta)\delta) \\
\leq \frac{1}{m_1} \frac{1}{m_2} \sum_{\theta=1}^{m_1} \left[ 1 - (1 - \zeta)^\theta \right] \sum_{\theta=1}^{m_2} \left[ 1 - (1 - \zeta)^\theta \right] \varphi(\xi) \nu(\xi) \\
+ \frac{1}{m_1} \frac{1}{m_2} \sum_{\theta=1}^{m_1} \left[ 1 - (1 - \zeta)^\theta \right] \sum_{\theta=1}^{m_2} \left[ 1 - \zeta^\theta \right] \varphi(\xi) \nu(\xi) \\
+ \frac{1}{m_1} \frac{1}{m_2} \sum_{\theta=1}^{m_1} \left[ 1 - \zeta^\theta \right] \sum_{\theta=1}^{m_2} \left[ 1 - (1 - \zeta)^\theta \right] \varphi(\delta) \nu(\xi) \\
+ \frac{1}{m_1} \frac{1}{m_2} \sum_{\theta=1}^{m_1} \left[ 1 - \zeta^\theta \right] \sum_{\theta=1}^{m_2} \left[ 1 - \zeta^\theta \right] \varphi(\delta) \nu(\delta) \\
:= \Delta_1(\zeta)\varphi(\xi) \nu(\xi) + \Delta_2(\zeta)\varphi(\xi) \nu(\delta) + \Delta_3(\zeta)\varphi(\delta) \nu(\xi) + \Delta_4(\zeta)\varphi(\delta) \nu(\delta).
\]

This implies that
\[
\varphi(\xi + (1 - \zeta)\delta) \nu(\xi + (1 - \zeta)\delta) \\
\leq \Delta_1(\zeta)\varphi(\xi) \nu(\xi) + \Delta_2(\zeta)\varphi(\xi) \nu(\delta) + \Delta_3(\zeta)\varphi(\delta) \nu(\xi) + \Delta_4(\zeta)\varphi(\delta) \nu(\delta). \tag{2.8}
\]

Integrating both sides of (2.8) with respect to \(\zeta\) over \([0, 1]\) results to:
\[
\frac{2}{\delta - \xi} \int_\xi^0 \varphi(r) \nu(r) \, dr \\
\leq 2 \int_0^1 \left[ \Delta_1(\zeta)\varphi(\xi) \nu(\xi) + \Delta_2(\zeta)\varphi(\xi) \nu(\delta) + \Delta_3(\zeta)\varphi(\delta) \nu(\xi) + \Delta_4(\zeta)\varphi(\delta) \nu(\delta) \right] d\zeta \\
:= N(\xi, \delta).
\]

That is,
\[
\frac{2}{\delta - \xi} \left[ \int_\xi^0 \varphi(r) \nu(r) \, dr + \int_s^\delta \varphi(r) \nu(r) \, dr \right] \\
\leq N(\xi, \delta). \tag{2.9}
\]

Now, multiplying (2.9) by \(\frac{\mu(\delta - \xi)}{2B(\mu)}\) and then adding \(\frac{2(1 - \mu)}{B(\mu)}\varphi(s) \nu(s)\) to the result to obtain:
\[
\frac{\mu}{B(\mu)} \left[ \int_\xi^s \varphi(r) \nu(r) \, dr + \int_s^\delta \varphi(r) \nu(r) \, dr \right] + \frac{2(1 - \mu)}{B(\mu)} \varphi(s) \nu(s) \\
\leq \frac{\mu(\delta - \xi)}{2B(\mu)} N(\xi, \delta) + \frac{2(1 - \mu)}{B(\mu)} \varphi(s) \nu(s).
\]

Hence,
\[
c^f \int_\xi^\mu \varphi(s) \nu(s) + c^f \int_\delta^\mu \varphi(s) \nu(s) \\
\leq \frac{\mu(\delta - \xi)}{2B(\mu)} N(\xi, \delta) + \frac{2(1 - \mu)}{B(\mu)} \varphi(s) \nu(s),
\]
from which we get the intended inequality. \(\square\)

**Remark 2.** Set \(m_1 = m_2 = 1\) in Theorem 5. Then we recover [7, Theorem 3].
2.2. Inequalities for m-polynomial harmonically convex functions

In this subsection, we present some new Hermite–Hadamard type results involving the $\zeta$-Riemann–Liouville fractional integral operators.

**Theorem 6.** Let $\varphi : S \to \mathbb{R}^+$ be a Lebesgue integrable function on $[\xi, \delta]$ with $0 < \xi < \delta$ and $\xi, \delta \in S$. If $\varphi \in HXP_m(S, \mathbb{R}^+)$ and $\xi, \epsilon > 0$, then

\[
\frac{1}{m + 2^{-m} - 1} \varphi \left( \frac{2\xi \delta}{\xi + \delta} \right) 
\leq \frac{\Gamma(\epsilon + \zeta)}{m} \left[ \zeta J_{\epsilon, \xi}^\zeta (\varphi \circ \tilde{\varphi}) \left( \frac{1}{\xi} \right) + \zeta J_{\epsilon, \delta}^\zeta (\varphi \circ \tilde{\varphi}) \left( \frac{1}{\delta} \right) \right]
\leq \frac{\varphi(\xi) + \varphi(\delta)}{m^2} \sum_{\vartheta=1}^m \left[ 2 - \frac{\epsilon}{\epsilon + \zeta \vartheta} - \frac{\epsilon}{\zeta} B \left( \frac{\epsilon}{\zeta}, \vartheta + 1 \right) \right],
\]

where $\tilde{\varphi}(r) = \frac{1}{r}$ and $B$ is the beta function defined by (1.3).

**Proof.** Given that $\varphi \in HXP_m(S, \mathbb{R}^+)$, we get the following relation:

\[
\varphi \left( \frac{q z}{q \varphi + z} \right) \leq \frac{1}{m} \sum_{\vartheta=1}^m \left[ 1 - \frac{1}{2^\vartheta} \right] \varphi(q) + \frac{1}{m} \sum_{\vartheta=1}^m \left[ 1 - \frac{1}{2^\vartheta} \right] \varphi(z).
\]

This implies that for all $q, z \in S$:

\[
\varphi \left( \frac{2q z}{q + z} \right) \leq \frac{1}{m} \sum_{\vartheta=1}^m \left[ 1 - \frac{1}{2^\vartheta} \right] \left( \varphi(q) + \varphi(z) \right).
\]

(2.10)

Now, let $q = \frac{\xi \delta}{\xi \vartheta + (1 - \vartheta) \xi}$ and $z = \frac{\xi \delta}{\xi \vartheta + (1 - \vartheta) \xi}$. Then (2.10) becomes:

\[
\varphi \left( \frac{2\xi \delta}{\xi + \delta} \right) \leq \frac{1}{m} \sum_{\vartheta=1}^m \left[ 1 - \frac{1}{2^\vartheta} \right] \left\{ \varphi \left( \frac{\xi \delta}{\xi \vartheta + (1 - \vartheta) \xi} \right) + \varphi \left( \frac{\xi \delta}{\xi \vartheta + (1 - \vartheta) \xi} \right) \right\}. \tag{2.11}
\]
Multiplying both sides of (2.11) by $\varsigma^{z-1}$ and integrating with respect to $\varsigma$ over $[0, 1]$, we get:

\[
\int_0^1 \varsigma^{z-1} \varphi \left( \frac{2\xi\delta}{\xi + \delta} \right) d\varsigma 
\leq \frac{1}{m} \sum_{\sigma=1}^{m} \left( 1 - \frac{1}{2^\sigma} \right) \int_0^1 \varsigma^{z-1} \left\{ \varphi \left( \frac{\xi\delta}{\varsigma\xi + (1 - \varsigma)\delta} \right) + \varphi \left( \frac{\xi\delta}{\varsigma\delta + (1 - \varsigma)\xi} \right) \right\} d\varsigma 
\]

\[
= \frac{1}{m} \sum_{\sigma=1}^{m} \left( 1 - \frac{1}{2^\sigma} \right) \int_0^1 \varsigma^{z-1} \varphi \left( \frac{\xi\delta}{\varsigma\xi + (1 - \varsigma)\delta} \right) d\varsigma 
+ \int_0^1 \varsigma^{z-1} \varphi \left( \frac{\xi\delta}{\varsigma\delta + (1 - \varsigma)\xi} \right) d\varsigma 
\]

\[
= \frac{1}{m} \sum_{\sigma=1}^{m} \left( 1 - \frac{1}{2^\sigma} \right) \left( \frac{\xi\delta}{\delta - \xi} \right) \int_{\frac{1}{2}}^1 \left( \frac{1}{\xi} - r \right)^{z-1} \varphi \left( \frac{1}{r} \right) dr 
+ \left( \frac{\xi\delta}{\delta - \xi} \right) \int_{\frac{1}{2}}^1 \left( r - \frac{1}{\delta} \right)^{z-1} \varphi \left( \frac{1}{r} \right) dr 
\]

\[
= \frac{\zeta \Gamma_{\zeta} (\epsilon) \delta}{m} \sum_{\sigma=1}^{m} \left( 1 - \frac{1}{2^\sigma} \right) \left( \frac{\xi\delta}{\delta - \xi} \right) \left\{ \frac{1}{\zeta \Gamma_{\zeta} (\epsilon)} \int_{\frac{1}{2}}^1 \left( \frac{1}{\xi} - r \right)^{z-1} \varphi \left( \frac{1}{r} \right) dr 
\right. 
+ \left. \frac{1}{\zeta \Gamma_{\zeta} (\epsilon)} \int_{\frac{1}{2}}^1 \left( r - \frac{1}{\delta} \right)^{z-1} \varphi \left( \frac{1}{r} \right) dr \right\} ,
\]

where $\varphi(r) = \frac{1}{r}$. This implies that

\[
\frac{1}{m + 2^{-m} - 1} \varphi \left( \frac{2\xi\delta}{\xi + \delta} \right) \leq \frac{\zeta \Gamma_{\zeta} (\epsilon + \zeta)}{m} \left( \frac{\xi\delta}{\delta - \xi} \right) \left\{ \zeta \int_{\frac{1}{2}}^1 \varphi \left( \frac{1}{\xi} \right) + \zeta \int_{\frac{1}{2}}^1 \varphi \left( \frac{1}{\delta} \right) \right\} .
\]

(2.12)

Next, substituting $q = \xi$ and $z = \delta$ in (1.2) gives

\[
\varphi \left( \frac{\xi\delta}{\varsigma\xi + (1 - \varsigma)\delta} \right) \leq \frac{1}{m} \sum_{\sigma=1}^{m} \left[ 1 - (1 - \varsigma)^{\sigma} \right] \varphi(\xi) + \frac{1}{m} \sum_{\sigma=1}^{m} \left[ 1 - \varsigma^{\sigma} \right] \varphi(\delta).
\]

(2.13)

Reversing the role of $\xi$ and $\delta$ in (2.13) produces:

\[
\varphi \left( \frac{\xi\delta}{\varsigma\delta + (1 - \varsigma)\xi} \right) \leq \frac{1}{m} \sum_{\sigma=1}^{m} \left[ 1 - (1 - \varsigma)^{\sigma} \right] \varphi(\delta) + \frac{1}{m} \sum_{\sigma=1}^{m} \left[ 1 - \varsigma^{\sigma} \right] \varphi(\xi).
\]

(2.14)

If we now add (2.13) and (2.15), multiply the resulting inequality by $\varsigma^{z-1}$ and integrate with respect to
\( \zeta \in [0, 1] \), then we obtain:

\[
\int_0^1 \frac{1}{s^{\zeta - 1}} \left\{ \varphi \left( \frac{\xi \delta}{s^{\xi} + (1 - \zeta)\delta} \right) + \varphi \left( \frac{\xi \delta}{s^{\xi} + (1 - \zeta)\xi} \right) \right\} \, ds \\
\leq \frac{\varphi(\xi) + \varphi(\delta)}{m} \sum_{\vartheta = 1}^{m} \int_0^1 \left[ 2s^{\zeta - 1} - s^{\zeta - 1}(1 - \zeta)^{\vartheta} - s^{\zeta + \vartheta - 1} \right] \, ds \quad (2.15)
\]

From (2.15), we get:

\[
\frac{\Gamma_\epsilon(\epsilon + \zeta)}{m} \left( \frac{\xi \delta}{\delta - \xi} \right)^\zeta \left[ \zeta J_\zeta^\epsilon (\varphi \circ \tilde{\varphi}) \left( \frac{1}{\xi} \right) + \zeta J_\zeta^\epsilon (\varphi \circ \tilde{\varphi}) \left( \frac{1}{\delta} \right) \right] \\
\leq \frac{\varphi(\xi) + \varphi(\delta)}{m^2} \sum_{\vartheta = 1}^{m} \left[ 2 - \frac{\epsilon}{\epsilon + \zeta} - \frac{\epsilon}{\zeta} B \left( \frac{\epsilon}{\zeta}, \vartheta + 1 \right) \right] \quad (2.16)
\]

Combining (2.12) and (2.16), we get the desired result.

\[ \square \]

**Remark 3.** If we take \( \epsilon = \zeta = 1 \), then Theorem 6 reduces to Theorem 2. If, on the other hand, we let \( m = 1 \), then we get from Theorem 6 the following corollary:

**Corollary 1.** Let \( \varphi : S \to \mathbb{R}^+ \) be a Lebesgue integrable function on \([\xi, \delta]\) with \( \xi < \delta \) and \( \xi, \delta \in S \). If \( \varphi \) is harmonically convex and \( \zeta, \epsilon > 0 \), then

\[
\varphi \left( \frac{2\xi \delta}{\xi + \delta} \right) \\
\leq \frac{\Gamma_\epsilon(\epsilon + \zeta)}{2} \left( \frac{\xi \delta}{\delta - \xi} \right)^\zeta \left[ \zeta J_\zeta^\epsilon (\varphi \circ \tilde{\varphi}) \left( \frac{1}{\xi} \right) + \zeta J_\zeta^\epsilon (\varphi \circ \tilde{\varphi}) \left( \frac{1}{\delta} \right) \right] \\
\leq \frac{\varphi(\xi) + \varphi(\delta)}{2}.
\]

**Theorem 7.** Let \( \varphi, \nu : S \to \mathbb{R}^+ \) be two functions such that \( \varphi \nu \) is Lebesgue integrable function on \([\xi, \delta]\) with \( 0 < \xi < \delta \) and \( \xi, \delta \in S \). If \( \varphi \in HXP_{m_1} (S, \mathbb{R}^+) \), \( \nu \in HXP_{m_2} (S, \mathbb{R}^+) \) and \( \zeta, \epsilon > 0 \), then

\[
\left( \frac{\xi \delta}{\delta - \xi} \right)^\zeta \left[ \zeta J_\zeta^\epsilon (\varphi \circ \tilde{\varphi}) \left( \frac{1}{\xi} \right) + \zeta J_\zeta^\epsilon (\varphi \circ \tilde{\varphi}) \left( \frac{1}{\delta} \right) \right] \\
\leq \frac{D(\xi, \delta)}{\zeta \Gamma_\epsilon(\epsilon)} \int_0^1 s^{\zeta - 1} \left[ \zeta \Delta_1(\xi) + \Delta_1(\xi) \right] \, ds + \frac{F(\xi, \delta)}{\zeta \Gamma_\epsilon(\epsilon)} \int_0^1 s^{\zeta - 1} \left[ \zeta \Delta_2(\xi) + \Delta_3(\xi) \right] \, ds,
\]

where \( D(\xi, \delta) := \varphi(\xi) \nu(\xi) + \varphi(\delta) \nu(\delta), F(\xi, \delta) := \varphi(\xi) \nu(\delta) + \varphi(\delta) \nu(\xi), \) \( \tilde{\varphi} \) is as defined in Theorem 6, and \( \Delta_j(\cdot), j = 1, 4, \) as defined in Theorem 5.

**Proof.** Given that \( \varphi \in HXP_{m_1} (S, \mathbb{R}^+) \) and \( \nu \in HXP_{m_2} (S, \mathbb{R}^+) \), we get:

\[
\varphi \left( \frac{\xi \delta}{s^{\xi} + (1 - \zeta)\delta} \right) \leq \frac{1}{m_1} \sum_{\vartheta = 1}^{m_1} \left[ 1 - (1 - \zeta)^{\vartheta} \right] \varphi(\xi) + \frac{1}{m_1} \sum_{\vartheta = 1}^{m_1} \left[ 1 - \zeta^{\vartheta} \right] \varphi(\delta)
\]
and
\[
\psi \left( \frac{\xi \delta}{\zeta \xi + (1 - \zeta)\delta} \right) \leq \frac{1}{m_1} \sum_{\theta=1}^{m_1} \left[ 1 - (1 - \zeta)^\theta \right] \psi(\xi) + \frac{1}{m_1} \sum_{\theta=1}^{m_1} \left[ 1 - \zeta^\theta \right] \psi(\delta).
\]

This implies:
\[
\psi \left( \frac{\xi \delta}{\zeta \xi + (1 - \zeta)\delta} \right) \psi \left( \frac{\xi \delta}{\zeta \xi + (1 - \zeta)\delta} \right) \leq \frac{1}{m_1} \frac{1}{m_2} \sum_{\theta=1}^{m_1} \left[ 1 - (1 - \zeta)^\theta \right] \sum_{m_2} \left[ 1 - (1 - \zeta)^\theta \right] \phi(\xi) \psi(\delta)
\]
\[
+ \frac{1}{m_1} \frac{1}{m_2} \sum_{\theta=1}^{m_1} \left[ 1 - \zeta^\theta \right] \sum_{m_2} \left[ 1 - (1 - \zeta)^\theta \right] \phi(\xi) \psi(\delta)
\]
\[
+ \frac{1}{m_1} \frac{1}{m_2} \sum_{\theta=1}^{m_1} \left[ 1 - \zeta^\theta \right] \sum_{\theta=1}^{m_2} \left[ 1 - (1 - \zeta)^\theta \right] \phi(\xi) \psi(\delta)
\]
\[
+ \frac{1}{m_1} \frac{1}{m_2} \sum_{\theta=1}^{m_1} \left[ 1 - (1 - \zeta)^\theta \right] \sum_{\theta=1}^{m_2} \left[ 1 - \zeta^\theta \right] \phi(\xi) \psi(\delta)
\]
\[
:= \Delta_1(\zeta) \phi(\xi) \psi(\delta) + \Delta_2(\zeta) \phi(\xi) \psi(\delta) + \Delta_3(\zeta) \phi(\xi) \psi(\delta) + \Delta_4(\zeta) \phi(\xi) \psi(\delta).
\]

This gives:
\[
\psi \left( \frac{\xi \delta}{\zeta \xi + (1 - \zeta)\delta} \right) \psi \left( \frac{\xi \delta}{\zeta \xi + (1 - \zeta)\delta} \right) \leq \Delta_1(\zeta) \phi(\xi) \psi(\delta) + \Delta_2(\zeta) \phi(\xi) \psi(\delta) + \Delta_3(\zeta) \phi(\xi) \psi(\delta) + \Delta_4(\zeta) \phi(\xi) \psi(\delta).
\]

Similarly, we also have
\[
\psi \left( \frac{\xi \delta}{\zeta \delta + (1 - \zeta)\xi} \right) \psi \left( \frac{\xi \delta}{\zeta \delta + (1 - \zeta)\xi} \right) \leq \Delta_1(\zeta) \phi(\xi) \psi(\delta) + \Delta_2(\zeta) \phi(\xi) \psi(\delta) + \Delta_3(\zeta) \phi(\xi) \psi(\delta) + \Delta_4(\zeta) \phi(\xi) \psi(\delta).
\]

Adding (2.17) and (2.18), we get
\[
\phi \left( \frac{\xi \delta}{\zeta \xi + (1 - \zeta)\delta} \right) \phi \left( \frac{\xi \delta}{\zeta \xi + (1 - \zeta)\delta} \right) \leq \left( \phi(\xi) \psi(\delta) + \phi(\delta) \psi(\xi) \right) \left[ \Delta_1(\zeta) + \Delta_4(\zeta) \right]
\]
\[
+ \left( \phi(\xi) \psi(\delta) + \phi(\delta) \psi(\xi) \right) \left[ \Delta_2(\zeta) + \Delta_3(\zeta) \right].
\]
Now, multiplying both sides of (2.19) by $z^{i-1}$ and integrating with respect to $z$ over $[0, 1]$, gives:

\[
\begin{align*}
\zeta \Gamma(z) \left( \frac{\xi \delta}{\delta - \xi} \right)^{\xi} & \left[ \zeta T_{\xi}^{\varepsilon} (\varphi \nu \circ \tilde{\varphi}) \left( \frac{1}{\xi} \right) + \zeta T_{\xi}^{\varepsilon} ( \varphi \nu \circ \tilde{\varphi}) \left( \frac{1}{\delta} \right) \right] \\
= & \int_{0}^{1} \zeta^{\xi-1} \varphi \left( \frac{\xi \delta}{\zeta \xi + (1 - z) \delta} \right) \nu \left( \frac{\xi \delta}{\zeta \xi + (1 - z) \xi} \right) dz \\
+ & \int_{0}^{1} \zeta^{\xi-1} \varphi \left( \frac{\xi \delta}{\zeta \delta + (1 - z) \xi} \right) \nu \left( \frac{\xi \delta}{\zeta \delta + (1 - z) \delta} \right) dz \\
\leq & \left( \varphi(z) \nu(z) + \varphi(\delta) \nu(\delta) \right) \int_{0}^{1} \zeta^{\xi-1} \left[ \Delta_1(z) + \Delta_4(z) \right] dz \\
& + \left( \varphi(z) \nu(z) + \varphi(\delta) \nu(\delta) \right) \int_{0}^{1} \zeta^{\xi-1} \left[ \Delta_2(z) + \Delta_3(z) \right] dz \\
:= & D(\xi, \delta) \int_{0}^{1} \zeta^{\xi-1} \left[ \Delta_1(z) + \Delta_4(z) \right] dz + F(\xi, \delta) \int_{0}^{1} \zeta^{\xi-1} \left[ \Delta_2(z) + \Delta_3(z) \right].
\end{align*}
\]

Hence, this completes the proof.

\[\square\]

**Corollary 2.** Let $\varphi, \nu : S \to \mathbb{R}^+$ be two functions such that $\varphi \nu$ is Lebesgue integrable function on $[\xi, \delta]$ with $0 < \xi < \delta$ and $\xi, \delta \in S$. If $\varphi$ and $\nu$ are harmonically convex and $\xi, \epsilon > 0$, then

\[
\left( \frac{\xi \delta}{\delta - \xi} \right)^{\xi} \left[ \zeta T_{\xi}^{\varepsilon} (\varphi \nu \circ \tilde{\varphi}) \left( \frac{1}{\xi} \right) + \zeta T_{\xi}^{\varepsilon} ( \varphi \nu \circ \tilde{\varphi}) \left( \frac{1}{\delta} \right) \right] \\
\leq \frac{D(\xi, \delta)}{\Gamma(z)} \left[ \frac{1}{\epsilon} \frac{2}{\epsilon + 2z} - \frac{2}{\epsilon + \xi} \right] + F(\xi, \delta) \left[ \frac{2}{\epsilon + \xi} - \frac{2}{\epsilon + 2z} \right].
\]

**Proof.** Let $m_1 = m_2 = 1$. Then, $\Delta_1(z) = z^2$, $\Delta_2(z) = \Delta_3(z) = z - z^2$ and $\Delta_4(z) = 1 - 2z + z^2$. The intended result follows by using Theorem 7. \[\square\]

**Theorem 8.** Let $\varphi, \nu : S \to \mathbb{R}^+$ be two functions such that $\varphi \nu$ is Lebesgue integrable function on $[\xi, \delta]$ with $0 < \xi < \delta$ and $\xi, \delta \in S$. If $\varphi \in HXP_{m_1}(S, \mathbb{R}^+)$, $\nu \in HXP_{m_2}(S, \mathbb{R}^+)$ and $\xi, \epsilon > 0$, then

\[
\frac{m_1 m_2}{(m_1 + 2 - m_1 - 1)(m_2 + 2 - m_2 - 1)} \varphi \left( \frac{2 \xi \delta}{\xi + \delta} \right) \nu \left( \frac{2 \xi \delta}{\xi + \delta} \right) \\
\leq \Gamma(z) \left( \frac{\xi \delta}{\delta - \xi} \right)^{\xi} \left[ \zeta T_{\xi}^{\varepsilon} (\varphi \nu \circ \tilde{\varphi}) \left( \frac{1}{\xi} \right) + \zeta T_{\xi}^{\varepsilon} ( \varphi \nu \circ \tilde{\varphi}) \left( \frac{1}{\delta} \right) \right] \\
+ \frac{\xi}{\zeta} \int_{0}^{1} \zeta^{\xi-1} \left[ \Lambda_{m_1}(z) \Lambda_{m_2}(z) + \Lambda_{m_1}(z) \Lambda_{m_2}(z) \right] D(\xi, \delta) \\
+ \left[ \Lambda_{m_1}(z) \Lambda_{m_2}(z) + \Lambda_{m_1}(z) \Lambda_{m_2}(z) \right] F(\xi, \delta) dz,
\]

where $\tilde{\varphi}$ is defined in Theorem 6, $\Lambda_m(z) = \frac{1}{m} \sum_{\theta=1}^{m} \left[ 1 - (1 - z)^\theta \right]$ and $\tilde{\Lambda}_m(z) = \frac{1}{m} \sum_{\theta=1}^{m} \left[ 1 - z^\theta \right]$. 

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Proof. We start by noticing that:

$$\tilde{\Lambda}_m \left( \frac{1}{2} \right) = \Lambda_m \left( \frac{1}{2} \right) := E_m := \frac{m + 2^{-m} - 1}{m}.$$  

Now, let $\varsigma \in [0, 1]$. Hence, from (2.11), one gets:

$$\varphi \left( \frac{2\xi\delta}{\xi + \delta} \right) \leq E_{m_1} \left\{ \varphi \left( \frac{\xi\delta}{\varsigma\xi + (1 - \varsigma)\delta} \right) + \varphi \left( \frac{\xi\delta}{\varsigma\delta + (1 - \varsigma)\xi} \right) \right\}$$

and

$$\nu \left( \frac{2\xi\delta}{\xi + \delta} \right) \leq E_{m_1} \left\{ \nu \left( \frac{\xi\delta}{\varsigma\xi + (1 - \varsigma)\delta} \right) + \nu \left( \frac{\xi\delta}{\varsigma\delta + (1 - \varsigma)\xi} \right) \right\}.$$  

Now,

$$\varphi \left( \frac{2\xi\delta}{\xi + \delta} \right) \nu \left( \frac{2\xi\delta}{\xi + \delta} \right) \leq E_{m_1} E_{m_2} \left[ \varphi \left( \frac{\xi\delta}{\varsigma\xi + (1 - \varsigma)\delta} \right) \nu \left( \frac{\xi\delta}{\varsigma\delta + (1 - \varsigma)\xi} \right) + \varphi \left( \frac{\xi\delta}{\varsigma\delta + (1 - \varsigma)\xi} \right) \nu \left( \frac{\xi\delta}{\varsigma\delta + (1 - \varsigma)\xi} \right) \right]$$

$$\quad + E_{m_1} E_{m_2} \left[ \varphi \left( \frac{\xi\delta}{\varsigma\xi + (1 - \varsigma)\delta} \right) \nu \left( \frac{\xi\delta}{\varsigma\delta + (1 - \varsigma)\xi} \right) + \varphi \left( \frac{\xi\delta}{\varsigma\delta + (1 - \varsigma)\xi} \right) \nu \left( \frac{\xi\delta}{\varsigma\delta + (1 - \varsigma)\xi} \right) \right]$$

$$\quad + \Lambda_{m_1} (\varsigma) \varphi(\xi) + \Lambda_{m_1} (\varsigma) \varphi(\delta) \] \left[ \Lambda_{m_2} (\varsigma) \nu(\delta) + \Lambda_{m_2} (\varsigma) \nu(\xi) \right]$$

$$\quad + \Lambda_{m_1} (\varsigma) \varphi(\delta) + \Lambda_{m_1} (\varsigma) \varphi(\xi) \] \left[ \Lambda_{m_2} (\varsigma) \nu(\xi) + \Lambda_{m_2} (\varsigma) \nu(\delta) \right]$$

$$\quad = E_{m_1} E_{m_2} \left[ \varphi \left( \frac{\xi\delta}{\varsigma\xi + (1 - \varsigma)\delta} \right) \nu \left( \frac{\xi\delta}{\varsigma\delta + (1 - \varsigma)\xi} \right) + \varphi \left( \frac{\xi\delta}{\varsigma\delta + (1 - \varsigma)\xi} \right) \nu \left( \frac{\xi\delta}{\varsigma\delta + (1 - \varsigma)\xi} \right) \right]$$

$$\quad + E_{m_1} E_{m_2} \left[ \Lambda_{m_1} (\varsigma) \Lambda_{m_2} (\varsigma) + \Lambda_{m_1} (\varsigma) \Lambda_{m_2} (\varsigma) \right] \left[ \varphi(\xi) \nu(\xi) + \varphi(\delta) \nu(\delta) \right]$$

$$\quad + \Lambda_{m_1} (\varsigma) \Lambda_{m_2} (\varsigma) + \Lambda_{m_1} (\varsigma) \Lambda_{m_2} (\varsigma) \right] \left[ \varphi(\xi) \nu(\delta) + \varphi(\delta) \nu(\xi) \right]$$

$$\quad = E_{m_1} E_{m_2} \left[ \varphi \left( \frac{\xi\delta}{\varsigma\xi + (1 - \varsigma)\delta} \right) \nu \left( \frac{\xi\delta}{\varsigma\delta + (1 - \varsigma)\xi} \right) + \varphi \left( \frac{\xi\delta}{\varsigma\delta + (1 - \varsigma)\xi} \right) \nu \left( \frac{\xi\delta}{\varsigma\delta + (1 - \varsigma)\xi} \right) \right]$$

$$\quad + E_{m_1} E_{m_2} \left[ \Lambda_{m_1} (\varsigma) \Lambda_{m_2} (\varsigma) + \Lambda_{m_1} (\varsigma) \Lambda_{m_2} (\varsigma) \right] \mathcal{D}(\xi, \delta)$$

$$\quad + \Lambda_{m_1} (\varsigma) \Lambda_{m_2} (\varsigma) + \Lambda_{m_1} (\varsigma) \Lambda_{m_2} (\varsigma) \] \mathcal{F}(\xi, \delta).$$
This implies that

\[
\varphi \left( \frac{2\xi\delta}{\xi + \delta} \right) \nu \left( \frac{2\xi\delta}{\xi + \delta} \right) \leq E_m E_m \left[ \varphi \left( \frac{\xi \delta}{\xi + (1 - \zeta) \delta} \right) \nu \left( \frac{\xi \delta}{\xi + (1 - \zeta) \delta} \right) + \varphi \left( \frac{\xi \delta}{\xi + (1 - \zeta) \delta} \right) \nu \left( \frac{\xi \delta}{\xi + (1 - \zeta) \delta} \right) \right] 
\]

\[+ E_m E_m \left[ \left\{ \Lambda_m (\zeta) \Lambda_m (\delta) + \tilde{\Lambda}_m (\zeta) \Lambda_m (\delta) \right\} D(\xi, \delta) \right. \]

\[+ \left[ \Lambda_m (\zeta) \Lambda_m (\delta) + \tilde{\Lambda}_m (\zeta) \tilde{\Lambda}_m (\delta) \right] \mathcal{F}(\xi, \delta) \right] \] .

Multiplying both sides of (2.20) by \( \zeta^{\zeta - 1} \) and integrating with respect to \( \zeta \) over \([0, 1]\) to get:

\[
\int_0^1 \zeta^{\zeta - 1} \varphi \left( \frac{2\xi\delta}{\xi + \delta} \right) \nu \left( \frac{2\xi\delta}{\xi + \delta} \right) d\zeta
\]

\[
\leq E_m E_m \int_0^1 \zeta^{\zeta - 1} \left[ \varphi \left( \frac{\xi \delta}{\xi + (1 - \zeta) \delta} \right) \nu \left( \frac{\xi \delta}{\xi + (1 - \zeta) \delta} \right) + \varphi \left( \frac{\xi \delta}{\xi + (1 - \zeta) \delta} \right) \nu \left( \frac{\xi \delta}{\xi + (1 - \zeta) \delta} \right) \right] d\zeta
\]

\[+ E_m E_m \int_0^1 \zeta^{\zeta - 1} \left[ \Lambda_m (\zeta) \Lambda_m (\delta) + \tilde{\Lambda}_m (\zeta) \Lambda_m (\delta) \right] D(\xi, \delta)
\]

\[+ \left[ \Lambda_m (\zeta) \Lambda_m (\delta) + \tilde{\Lambda}_m (\zeta) \tilde{\Lambda}_m (\delta) \right] \mathcal{F}(\xi, \delta) \right] d\zeta.
\]

The required result follows.

\[\square\]

**Corollary 3.** Let \( \varphi, \nu : S \to \mathbb{R}^+ \) be two functions such that \( \varphi \nu \) is Lebesgue integrable function on \([\xi, \delta]\) with \( 0 < \xi < \delta \) and \( \xi, \delta \in S \). If \( \varphi \) and \( \nu \) are harmonically convex and \( \zeta, \epsilon > 0 \), then

\[
\varphi \left( \frac{2\xi\delta}{\xi + \delta} \right) \nu \left( \frac{2\xi\delta}{\xi + \delta} \right) \leq \frac{\Gamma_1 (\epsilon + \zeta)}{4} \left[ \frac{\xi \delta}{\delta - \xi} \right] \int \left[ \xi J^\epsilon_{\xi} (\varphi \nu \circ \tilde{\varphi}) \left( \frac{1}{\xi} \right) + \xi J^\epsilon_{\xi} (\varphi \nu \circ \tilde{\varphi}) \left( \frac{1}{\delta} \right) \right] d\xi
\]

\[+ \frac{1}{2} \left[ \frac{\epsilon}{\epsilon + \zeta} - \frac{\epsilon}{\epsilon + 2\zeta} \right] D(\xi, \delta) + \frac{1}{4} \left[ 1 + \frac{2\epsilon}{\epsilon + 2\zeta} - \frac{2\epsilon}{\epsilon + \zeta} \right] \mathcal{F}(\xi, \delta).
\]
Proof. Let \( m_1 = m_2 = 1 \). Then, \( \Lambda_{m_1}(\varsigma) = \Lambda_{m_2}(\varsigma) = \varsigma \) and \( \tilde{\Lambda}_{m_1}(\varsigma) = \tilde{\Lambda}_{m_2}(\varsigma) = 1 - \varsigma \). The intended result follows by using Theorem 8. \( \square \)

3. Conclusion

Utilizing the Caputo–Fabrizio and generalized Riemann–Liouville fractional integral operators, we proved some inequalities of the Hermite–Hadamard kinds for \( m \)-polynomial convex and harmonically convex functions. Our results generalize, extend and complement results in [7, 9, 31].

Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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