1-Motives Associated to the Limiting Mixed Hodge Structures of Degenerations of Curves

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Abstract

In this article, we will give the Deligne 1-motives up to isogeny corresponding to the \( \mathbb{Q} \)-limiting mixed Hodge structures of semi-stable degenerations of curves, by using logarithmic structures and Steenbrink’s cohomological mixed Hodge complexes associated to semi-stable degenerations of curves.

Introduction

For a semi-stable degeneration of smooth projective curves, we have a limiting mixed Hodge structure over the nearby fiber. The goal of this paper is to give a geometric construction of the 1-motive associated to the limiting mixed Hodge structure.

Given a semi-stable degeneration of smooth projective curves \( f : X \to \Delta \) over a small disk \( \Delta \), with a chosen parameter \( t : \Delta \to \mathbb{C} \). The central fiber of \( f \), \( X_0 = f^{-1}(0) \), is a stable curve. Let \( X^* = X - X_0 \). Denote the inclusion \( X_0 \hookrightarrow X \) by \( i \) and the inclusion \( X^* \hookrightarrow X \) by \( j \). Then we have the following diagram:

\[
\begin{array}{ccc}
X_\infty & \xrightarrow{e} & X^* \\
\downarrow f^r & & \downarrow f^r \\
\mathbb{H} & \xrightarrow{\pi} & \Delta^* \\
\end{array}
\]

where \( \mathbb{H} \) is the upper half plane, \( \pi : \mathbb{H} \to \Delta^* \) is the universal cover, and the squares are Cartesian. Denote the composition map \( j \circ e \) by \( k \).

We call the fiber \( X_\infty \) a nearby fiber. It is homotopic to any fibre \( X_s \) of \( f \), for \( s \neq 0 \). \( X \) is homotopy equivalent to \( X_0 \) by a retraction \( r : X \to X_0 \). We call the composition map \( sp : X_s \xrightarrow{j} X \xrightarrow{r} X_0 \) a specialization. Following from [PeSt] and [SGA7], we have the construction of a complex of nearby cocycles \( \Psi_f \mathbb{C}_X := i^* Rk_* \mathbb{C}_X \) (respectively, \( \Psi_f \mathbb{Q}_X := i^* Rk_* \mathbb{Q}_X \)) in the derived category \( D^+(X_0, \mathbb{C}) \) (respectively, \( D^+(X_0, \mathbb{Q}) \)). According to Steenbrink [St1, Lemma 4.3], we have \( H^1(X_\infty, \mathbb{C}) \cong \).
$\mathbb{H}^1(\Psi_f \underline{L}_X)$ (respectively, $H^1(X_\infty, \mathbb{Q}) \cong \mathbb{H}^1(\Psi_f \underline{Q}_X)$). Schmid [Sch] and Steenbrink [St1] showed that the cohomology $H^1(X_\infty, \mathbb{C})$ admits a mixed Hodge structure, which is called the \textit{limiting mixed Hodge structure}, as the “limit” of pure Hodge structures $H^1(X_s, \mathbb{C})$ of general fibers when $s$ approaches to $0$ in the disk $\Delta$.

For the construction of the limiting mixed Hodge structure, note first that we have a de Rham complex over $X_0$ with logarithmic poles: $\Omega^\cdot_{X/\Delta}(\log X_0) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_0}$, which is isomorphic to $\Psi_f \underline{L}_X$ in the derived category $D^+(X_0, \mathbb{C})$. Steenbrink observed that the weight filtration of the limiting mixed Hodge structure cannot be constructed on $\Omega^\cdot_{X/\Delta}(\log X_0) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_0}$. To define the weight filtration of the limiting mixed Hodge structure, Steenbrink constructed the so called \textit{Steenbrink double complex} $A^*_t$ (refer to section 2.1), which depends on the parameter $t : \Delta \rightarrow \mathbb{C}$ we chose. The Steenbrink double complex gives a resolution of $\Omega^\cdot_{X/\Delta}(\log X_0) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_0}$, therefore there is an isomorphism $\mathbb{H}^1(X_0, \Omega^\cdot_{X/\Delta}(\log X_0) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_0}) \cong \mathbb{H}^1(\text{Tot}(A^*_t))$, which depends on the choice of $t$. Over the Steenbrink double complex, we can define the weight filtration and Hodge filtration, which give the limiting mixed Hodge structure after taking the hypercohomology. We will recall the Steenbrink double complex in section 2.

Deligne [De1] has built up an equivalence of two categories

$$
\begin{align*}
\begin{array}{c|c|c}
\text{Z-mixed Hodge structure} & \text{1-motive } [L \xrightarrow{\phi} G], & \text{with } L \text{ a free Abelian group of finite rank,} \\
(H_0, W, F^\cdot) \text{ of type } \{(-1, 1), & \text{and } 0 \rightarrow T \rightarrow G \rightarrow A \rightarrow 0 \text{ an extension of an Abelian variety } A \text{ by a torus } T. \\
(-1, 0), (0, -1), (0, 0)\} & \text{structure of weight } -1. \\
\text{with } & \\
G^H_{\text{1-motive}} H & \\
\end{array}
\end{align*}
$$

We will briefly recall the above equivalence of categories in section 3.

For elementary geometric examples of the above equivalence of categories, we consider a complex algebraic curve $X$, not necessarily smooth projective. We know that the first cohomology $H^1(X, \mathbb{C})(1)$ carries a Z-mixed Hodge structure of type $\{(-1, 1), (-1, 0), (0, -1), (0, 0)\}$ with $G^H_{\text{1-motive}} H$ a polarized pure Hodge structure of weight $-1$, where “(1)” is the Tate twist. When $X$ is a smooth projective curve, $H^1(X, \mathbb{C})(1)$ carries a pure Z-Hodge structure of weight $-1$. Then the corresponding 1-motive is $[0 \rightarrow J(X)]$, where $J(X)$ is the Jacobian of the curve $X$. For a nodal projective curve $X$, $H^1(X, \mathbb{C})(1)$ actually carries a Z-mixed Hodge structure of type $\{(-1, 1), (-1, 0), (0, 0)\}$. Then the corresponding 1-motive is actually $[0 \rightarrow G]$, where $G$ is the extension $0 \rightarrow T \rightarrow G \rightarrow J(\tilde{X}) \rightarrow 0$, with $J(\tilde{X})$ the Jacobian variety of the normalization of $X$, and $T$ the torus determined by the singularities of $X$. When $X$ is a smooth curve with $\tilde{X}$ the smooth
projective curve containing $X$ as an open subset, $H^1(X, \mathbb{C})(1)$ actually carries a $\mathbb{Z}$-mixed Hodge structure of type $\{(-1,0), (0,-1), (0,0)\}$. The corresponding 1-motive is actually $[L \to J(\bar{X})]$, where $J(\bar{X})$ is the Jacobian variety of $\bar{X}$ and $L$ is the free Abelian group of rank $n-1$ with generators $\{p_1 - p_0, ..., p_{n-1} - p_0\}$, where the set of points $\{p_0, p_1, ..., p_{n-1}\}$ is $\bar{X} - X$. Then the 1-motive map $\mu$ is actually the Abel-Jacobian map. For the algebraic construction of the 1-motive associated to a general curve, refer to Deligne [De1, section 10.3]. For the algebraic construction of 1-motives associated to the first cohomology of surfaces, refer to Carlson [Carl1].

Go back to the limiting mixed Hodge structure of a semi-stable degeneration of curves $f : X \to \Delta$. Once the parameter $t : \Delta \to \mathbb{C}$ is fixed, we have the Steenbrink double complex $A^\ast_t$ and the hypercohomology of its total complex $H^1(Tot(A^\ast_t))$. Steenbrink [St1] constructed an abstract bifiltered $\mathbb{Q}$-cohomological mixed Hodge complex $((A^\ast_t, W_\ast), (A^\ast_t, W_\ast, F^\ast))$, depending on $t$, which gives a $\mathbb{Q}$-mixed Hodge structure over $H^1(Tot(A^\ast_t))$. Later on, Steenbrink [St2] constructed the $\mathbb{Q}$-mixed Hodge structure over $H^1(Tot(A^\ast_t))$ using log geometry. By using the monodromy weight spectral sequence, it is not hard to show that the $\mathbb{Q}$-limiting mixed hodge structure $H^1(Tot(A^\ast_t))((1))$ is of type $\{(-1, -1), (-1, 0), (0, -1), (0,0)\}$ ([St1, Proposition 4.21]). By the correspondence of the two categories described above, it corresponds to a Deligne 1-motive up to isogeny $[L \overset{\mu}{\to} G \otimes \mathbb{Q}]$ (Deligne 1-motive tensor with $\mathbb{Q}$), where $L$ is a finite $\mathbb{Q}$ vector space and $G$ is a semi-Abelian variety. Actually $G$ is $Pic^0(X_0)$ and $L$ is given by nodal points in $X_0$, which will be shown in section 4.1. The main purpose of this article is to understand the 1-motive map $\mu_t$ geometrically, which will be discussed in section 4.2, 4.3, and 4.4. For the reason of calculation, we will use the $\mathbb{Q}$-cohomological mixed Hodge complex constructed by Steenbrink through log geometry.

We would like to mention the work by Jerome William Hoffman [Hoff]. He gives the 1-motives associated to $\mathbb{Z}$-limiting mixed Hodge structures for semi-stable degenerations of curves, which coincides with the 1-motive in this paper after tensoring with $\mathbb{Q}$. The method in [Hoff] is rather topological and analytical.

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1 Logarithmic Structure and Koszul Complex

In this section, we will briefly recall the logarithmic structures associated to the geometric setup in the introduction and some terminologies that will be used in the following sections.

1.1 Logarithmic structure associated to the semi-stable degeneration of curves

For the semi-stable degeneration of curves \( f : X \to \Delta \), we have the log structure for the embedding \( i : X_0 \hookrightarrow X \), which is the sheaf of monoids \( \mathcal{M}_X := \mathcal{O}_X \cap j_* \mathcal{O}_{X_0}^* \) with the natural structure morphism \( \alpha : \mathcal{M}_X \to \mathcal{O}_X \) such that \( \alpha^{-1}(\mathcal{O}_X^*) \cong \mathcal{O}_X^* \). Similarly we have the log structure \( \mathcal{M}_\Delta \) for the embedding \( 0 \hookrightarrow \Delta \). An analytic space with a log structure \((X, \mathcal{M}_X)\) is called a log space. The family \( f \) gives a morphism of log spaces \( f : (X, \mathcal{M}_X) \to (\Delta, \mathcal{M}_\Delta) \). We will omit the definition of morphisms of log spaces, since we will not use it in the following sections. However, for the definitions of a morphism of log spaces, refer to Steenbrink [St2, section 4] or Illusie [Ill2, section 1].

Restricting the log space \((X, \mathcal{M}_X)\) to the central fiber \( X_0 \), we get a log space \((X_0, \mathcal{M}_{X_0})\), where \( \mathcal{M}_{X_0} \) and the structure morphism \( \beta : \mathcal{M}_{X_0} \to \mathcal{O}_{X_0} \) are defined through the following pushout square in the category of sheaves of monoids:

\[
\begin{array}{c}
\gamma^{-1}(\mathcal{O}_{X_0}^*) & \xrightarrow{i^{-1}\mathcal{M}_X} & \mathcal{O}_{X_0}^* \\
\downarrow \gamma & & \downarrow \beta \\
\mathcal{O}_{X_0}^* & \xrightarrow{\gamma} & \mathcal{M}_{X_0} \\
\end{array}
\]

(1.1)

where \( \gamma : i^{-1}\mathcal{M}_X \to \mathcal{O}_{X_0} \) is the composition map \( i^{-1}\mathcal{M}_X \to i^{-1}\mathcal{O}_X \to \mathcal{O}_{X_0} \). Hence \( \mathcal{M}_{X_0} = \mathcal{O}_{X_0}^* \oplus i^{-1}\mathcal{M}_X / \sim \), where the equivalent relation is: \((f, m) \sim (f', m')\) if and only if there exists \( a, b \in \gamma^{-1}(\mathcal{O}_{X_0}^*)\) such that \( m/m' = b/a \) and \( f/f' = \gamma(a)/\gamma(b) \).

For the log space \((X, \mathcal{M}_X)\), there exists a short exact sequence

\[
0 \to \mathcal{O}_X^* \to \mathcal{M}_X \to (a_1)_* \mathcal{N}_{X_0[1]} \to 0,
\]

where \( a_1 : X_0[1] \to X_0 \) is the normalization of the nodal curve \( X_0 \). We also denote \( a_1 : X_0[1] \to X \) for the composition of the normalization morphism and the inclusion.
When we restrict the log space \((X, \mathcal{M}_X)\) to \(X_0\), we still have the following short exact sequence

\[
0 \longrightarrow \mathcal{O}_{X_0}^* \longrightarrow \mathcal{M}_{X_0} \longrightarrow (a_1)_* \mathcal{N}_{X_0[1]} \longrightarrow 0.
\]

Since the family \(f\) is semi-stable, there exists a global section \(\tilde{t} = t \circ f \in \Gamma(\mathcal{M}_X)\) being mapped to \((1, \ldots, 1)\), the generator of \((a_1)_* \mathcal{N}_{X_0[1]}\). Similarly, there is a global section \(\tilde{t} \in \Gamma(\mathcal{M}_{X_0})\) which is mapped to \((1, \ldots, 1)\).

Let’s end this subsection with the following notations that we will use later.

**Notation 1.1.1:**

(1) Denote \(X_0[2]\) the 2-fold intersection of the irreducible components of \(X_0\), i.e., the set of nodal points of \(X_0\), and \(a_2 : X_0[2] \to X\) be the natural inclusion map.

(2) Denote the number of the nodal points of \(X_0\) by \(d\) and the number of the irreducible components by \(n\). For convenience, we fix an order of the irreducible components \(\{X_0, i\}_{i=1}^n\) of \(X_0\).

### 1.2 Koszul complexes associated to homomorphisms of free Abelian groups

In this subsection we will briefly recall several constructions and results about the divided power envelop and the Koszul complex in Steenbrink [St2], Fujisawa [Fu2], and Illusie [Ill1]. Our discussion is for free Abelian groups, but the constructions and results can be generalized to sheaves of free Abelian groups.

Let \(E\) be a free Abelian group. Denote \(T(E), S(E), \text{ and } \bigwedge(F)\) to be the tensor algebra, the symmetric algebra, and the exterior algebra of \(E\), respectively. For \(x \in E\) and \(n \in \mathbb{N}\) denote \(\gamma_n(x) = x^n/n!\). Note that \(\gamma_n(x + y) = \sum_{i+j=n} \gamma_i(x)\gamma_j(y)\). Let \(\{e_\alpha\}\) be a basis of \(E\). Define \(\Gamma_n(E)\) to be a free subgroup of \(S_n(E \otimes \mathbb{Q})\) generated by \(\{\gamma_{n_1}(e_{\alpha_1}) \ldots \gamma_{n_m}(e_{\alpha_m})): \sum_{i=1}^m n_i = n\}\), then \(\Gamma(E) = \bigoplus_n \Gamma_n(E) \subset S(E_\mathbb{Q})\) is a subalgebra.

For any homomorphism of free Abelian groups \(\varepsilon : E \to F\) we have an induced homomorphism \(\Gamma(\varepsilon) : \Gamma(E) \to \Gamma(F)\) of graded \(\mathbb{Z}\)-algebras. Then we have the bigraded algebra \(Kos(\varepsilon) := \Gamma(E) \otimes \bigwedge(F)\). Note that \(Kos(\varepsilon)\) is commutative in the first degree and anti-commutative in the second degree.

**Lemma 1.2.1 ([Ill1, Prop 4.3.1.2]):** For any homomorphism of free Abelian groups \(\varepsilon : E \to F\), there exists a unique differential \(d\) of bidegree \((-1, 1)\) of \(Kos(\varepsilon)\) satisfying
the following axioms:

(i) $d(xx') = (dx)x' + (-1)^q x(dx')$ for $x$ a homogeneous element of bidegree $(p, q)$ in $\text{Kos}(u)$;

(ii) $d(x[k] \otimes 1) = x[k-1] \otimes \varepsilon (x)$ for $x \in E$, where $x[k] = \gamma_k (x)$;

(iii) $d(1 \otimes y) = 0$ for $y \in F$.

Denote $\text{Kos}^{p+q}(\varepsilon)^q = \text{Kos}^{p,q}(\varepsilon) := \Gamma_p(E) \otimes \Lambda^q(F)$. Then we have a cochain complex (w.r.t the index $q$) of free Abelian groups $\text{Kos}^{n}(\varepsilon)^*$, $p+q=n$, with the above differential $d$:

$$\cdots \longrightarrow \text{Kos}^{p+1,q-1}(\varepsilon) \overset{d^{p+1,q-1}}{\longrightarrow} \text{Kos}^{p,q}(\varepsilon) \overset{d^{p,q}}{\longrightarrow} \text{Kos}^{p-1,q+1}(\varepsilon) \overset{d^{p-1,q+1}}{\longrightarrow} \cdots$$

The above differential $d^{p,q}$ is explicitly given by:

$$d^{p,q}(\gamma_{n_1}(x_1) \ldots \gamma_{n_k}(x_k)) \otimes y = \sum_{i=1}^k \gamma_{n_i-1}(x_i) \gamma_{n_k}(x_k) \otimes \varepsilon (x_i) \wedge y,$$

where $x_i \in E$, $\Sigma_i n_i = p$, $y \in \Lambda^q(F)$.

**Lemma 1.2.2 ([Ill1, Prop 4.3.1.6]):** Suppose that $\varepsilon$ is a homomorphism as that in Lemma 1.2.1. Moreover, assume $\text{Coker}(\varepsilon)$ is a free Abelian group. Then we have the following isomorphism:

$$H^p(\text{Kos}^n(\varepsilon)^*) \cong \Gamma_{n-p}(\text{Ker}(\varepsilon)) \otimes \Lambda^p(\text{Coker}(\varepsilon)),$$

where $H^p(\text{Kos}^n(\varepsilon)^*)$ is the cohomology of the complex $\text{Kos}^n(\varepsilon)^*$.

**Remark 1.2.3 ([Fu1, (1.5)]):** For any free Abelian subgroup $G \subset F$ and any integer $0 \leq m \leq q$, we can define a subgroup $W(G)_m \text{Kos}^{p,q}(\varepsilon)$ of $\text{Kos}^{p,q}(\varepsilon)$ to be the image of the morphism $\Gamma_p(E) \otimes \Lambda^m G \otimes \Lambda^m F \to \Gamma_p(E) \otimes \Lambda^q F; x \otimes y \otimes z \mapsto x \otimes (y \wedge z)$. Moreover, if $\varepsilon(E) \subset G$, the above construction gives us a finite increasing filtration $W(G)$, on the complex $\text{Kos}^n(\varepsilon)^*$.

**Proposition 1.2.4 ([Fu1, Proposition (1.6)]):** Suppose that $E, G, F,$ and $\varepsilon : E \to F$ are as defined in Remark 1.2.3. Denote $\varepsilon_G : E \to G$ to be the induced map by $\varepsilon$. Then
we have the following isomorphism of complexes induced by the morphism in Remark 1.2.3:

$$G_r^{W(G)} K^s_n(\varepsilon)^r \cong K^s_n(\varepsilon)^{[-m]} \otimes \Lambda(F/G),$$

where $[-m]$ means shifting the complex by $m$.

Note that in Fujisawa [Fu1], the above proposition is stated for sheaves of free Abelian groups.

Remark 1.2.5: The construction of the Koszul complex above can be generalized to any flat $A$-module, where $A$ is a subalgebra of $\mathbb{C}$, e.g. $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{C}$.

2 Steenbrink’s Double Complex

In this section, we will recall the Steenbrink’s limiting cohomological mixed Hodge complex on the complex level and the rational level, associated to the geometric setting in the introduction.

2.1 The complex structure

Consider the sheaves of log differentials $\Omega^i_X(\log X_0)$, $i = 0, 1, 2$. For each $i$, we have the Deligne’s filtration:

$$W_p \Omega^i_X(\log X_0) := \begin{cases} 
0 & \text{for } p < 0, \\
\Omega^i_X(\log X_0) & \text{for } p \geq i, \\
\Omega^{i-p}_X \wedge \Omega^p_X(\log X_0) & \text{for } 0 \leq p \leq i.
\end{cases}$$

Fix a parameter $t : \Delta \to \mathbb{C}$. Define the Steenbrink double complex as follows.

$$A^{p,q} := \Omega^{p,q+1}_X(\log X_0)/W_p \Omega^{p+q+1}_X(\log X_0)$$

with two differentials:

$d' : A^{p,q} \to A^{p+1,q}$ by $d'(\omega) = \text{class of } \theta \wedge \omega$, where $\theta = f^*(dt/t)$, $t$ is the parameter chosen for $\Delta$;

$d'' : A^{p,q} \to A^{p,q+1}$ by $d''(\omega) = \text{class of } d\omega$.

Define the monodromy weight filtration $M$, and Hodge filtration $F^*$ of the double complex defined as follows.

$$M_r A^{p,q} := \text{The image of } W_{2p+r+1} \Omega^{p,q+1}_X(\log X_0) \text{ in } A^{p,q}.$$

The Hodge filtration $F^*$ is defined by the “stupid filtration” of the double complex.
Specifically, for the semi-stable degeneration $f$ of curves, we have the double complex $A_t^{**}$:

$$
\begin{align*}
\Omega_X^2(\log X_0)/\Omega^2_X \\
\downarrow d'' \\
\Omega_X^1(\log X_0)/\Omega^1_X \longrightarrow \Omega_X^2(\log X_0)/\Omega^1_X \wedge \Omega_X^1(\log X_0).
\end{align*}
$$

(2.1)

If we denote the total complex $Tot(A_t^{**})$ of the above double complex to be $A_t^{*}$, then we have the $C$-structure of a cohomological mixed Hodge complex $(A_t^{*}, M^{*}, F^{*})$, which gives the limiting mixed Hodge structure associated to the semi-stable degeneration of curves. From now on, for simplicity, we just denote $A_t^{*}$ by $A^{*}$, for the fixed parameter $t$.

2.2 The rational structure

In this section, we follow the terminologies and constructions in Steenbrink [St2] to give a rational structure of the limiting mixed Hodge structures of a semistable degeneration of curves.

Under the basic constructions in section 1.1, we consider the morphism $e : \mathcal{O}_X \to \mathcal{M}_X$, which is the composition of the exponential map $e^{2\pi \sqrt{-1} t} : \mathcal{O}_X \to \mathcal{O}_X$ and the inclusion map $\mathcal{O}_X^* \hookrightarrow \mathcal{M}_X := \mathcal{O}_X \cap j_* \mathcal{O}_U^*$. Then we have the following exact sequence of sheaves (Steenbrink [St2, Lemma 2.7]):

$$
\begin{align*}
0 \longrightarrow \mathbb{Z}_X \longrightarrow \mathcal{O}_X \overset{e}{\longrightarrow} \mathcal{M}_X^{gp} \longrightarrow (a_1)_* \mathbb{Z}_{X_0[1]} \longrightarrow 0,
\end{align*}
$$

where $\mathcal{M}_X^{gp}$ is the groupification of the sheaf of monoids.

As in section 1.1, restricting the log structure $\mathcal{M}_X$ along the closed immersion $i : X_0 \hookrightarrow X$ to $X_0$ provides a log structure $\mathcal{M}_{X_0}$ over $X_0$. Then we have the following exact sequence over $X_0$ [St2, (3.9)],

$$
\begin{align*}
0 \longrightarrow \mathbb{Z}_{X_0} \longrightarrow \mathcal{O}_{X_0} \overset{e}{\longrightarrow} \mathcal{M}_{X_0}^{gp} \longrightarrow (a_1)_* \mathbb{Z}_{X_0[1]} \longrightarrow 0.
\end{align*}
$$

After tensoring with $\mathbb{Q}$, we get an exact sequence

$$
\begin{align*}
0 \longrightarrow \mathbb{Q}_{X_0} \longrightarrow \mathcal{O}_{X_0} \overset{e}{\longrightarrow} \mathcal{M}_{X_0}^{gp} \otimes \mathbb{Q} \longrightarrow (a_1)_* \mathbb{Q}_{X_0[1]} \longrightarrow 0,
\end{align*}
$$

with the global section $\tilde{t} \in \Gamma(X_0, \mathcal{M}_{X_0}^{gp} \otimes \mathbb{Q})$ which is mapped to $(1,...,1)$. From now on, for simplicity, we will denote $\mathcal{M}_{X_0}^{gp} \otimes \mathbb{Q}$ by $\mathcal{M}$.

Consider the morphism $e : \mathcal{O}_{X_0} \to \mathcal{M}$ of sheaves of free Abelian groups in the above exact sequence. We have the following data in terms of terminologies in section 1.2:
1) \( \mathcal{K}_n^K := \text{Kos}^{n+2}(e)^q = \text{Sym}_q^{2+n-n}(\mathcal{O}_{X_0}) \otimes \bigwedge^m \mathcal{M}; \)

\hspace{1em} (2.2)

2) \( W_m K_n^K := \text{The image of Sym}_q^{2+n-q}(\mathcal{O}_{X_0}) \otimes \bigwedge^{q-m} e(\mathcal{O}_{X_0}) \otimes \bigwedge^m \mathcal{M} \) in \( K_n^K; \)

3) \( \theta : K_n^K \rightarrow K_{n+1}^{q+1}(1) \) is induced by the wedge product with \( 2\sqrt{-1}(i \otimes 1), \)

where \( \text{Sym}_q^{2+n}(\mathcal{L}) \) is the symmetric \( \mathbb{Q} \)-algebra of degree \( k \) and \( K_{n+1}^{q+1}(1) \) means \( (2\sqrt{-1})K_{n+1}^{q+1}. \)

**Theorem 2.2.1 (Steenbrink):** The morphism

\[
\phi_m : W_m K_n^K \rightarrow W_m \Omega_X^n(\log X_0), \text{ for } n \geq 0;
\]

\[
f_1 \ldots f_{2+n-q} \otimes e(g_1) \wedge ... \wedge e(g_{q-m}) \otimes y_1 \wedge ... \wedge y_m
\]

\[
\mapsto (2\sqrt{-1})^{-n-q}(\prod_{i=1}^{n-q} f_i)dg_1 \wedge ... \wedge dg_{q-m} \wedge \frac{dy_1}{y_1} \wedge ... \wedge \frac{dy_q}{y_q}
\]

induces a filtered quasi-isomorphism between \( (K_n^K \otimes \mathbb{C}, W_\ast) \) and \( (\Omega_X^n(\log X_0), W_\ast). \)

We will reproduce the proof here.

**Proof.** It suffices to show that \( G^W_m(\phi_\ast) : G^W_m K_n^K \otimes \mathbb{C} \rightarrow G^W_m \Omega_X^n(\log X_0) \) is a quasi-isomorphism for all \( m. \)

(1) \( m = 0 \)

\( \mathcal{H}^0(G^W_0 \Omega_X^n(\log X_0)) \cong \mathbb{C}_X \) and \( \mathcal{H}^i(G^W_0 \Omega_X^n(\log X_0)) = 0, \text{ for } i > 0, \) by Steenbrink [St1, Corollary (1.9)].

By Proposition 1.2.4, we have

\[
G^W_0 K_n^K \cong \text{Kos}^{n+2}(\mathcal{O}_{X_0} \xrightarrow{\text{e}^{2\sqrt{-1}}} \mathcal{O}_{X_0}^\ast \otimes \mathbb{Q}^\ast).
\]

Then by Lemma 1.2.2, we have \( \mathcal{H}^0(G^W_0 K_n^K) \cong \mathbb{Q}_X \) and \( \mathcal{H}^i(G^W_0 K_n^K) = 0, \text{ for } i > 0. \)

(2) \( m = 1 \)

\( \mathcal{H}^1(G^W_1 \Omega_X^n(\log X_0)) \cong (a_1) \mathcal{C} X_0[1] \) and \( \mathcal{H}^i(G^W_1 \Omega_X^n(\log X_0)) = 0, \text{ for } i \neq 1, \) by Steenbrink [St1, Corollary (1.9)].

By Proposition 1.2.4, we have

\[
G^W_1 K_n^K \cong \text{Kos}^{n+1}(\mathcal{O}_{X_0} \xrightarrow{\text{e}^{2\sqrt{-1}}} \mathcal{O}_{X_0}^\ast \otimes \mathbb{Q}[1] \otimes (a_1) \mathcal{Q} X_0[1].
\]

Then by Lemma 1.2.2, we have \( \mathcal{H}^0(G^W_1 K_n^K) \cong (a_1) \mathcal{Q} X_0[1] \) and \( \mathcal{H}^i(G^W_1 K_n^K) = 0, \text{ for } i \neq 1. \)

The argument for \( m \geq 2 \) is exactly the same.

\[ \blacksquare \]

To get the rational structure of the limiting mixed Hodge structure, let’s first recall the Steenbrink’s double complex in Steenbrink [St2]. The original double complex in
Steenbrink [St2] is constructed over \( \mathbb{Z} \) by using log geometry. Here we reconstruct it over \( \mathbb{Q} \) by using the data (2.2) and the results above.

**Notation 2.2.2:** Let \( K \) be a complex. Denote \( K^p(n)[m] := (2\pi i)^n K^{p+m} \).

**Construction 2.2.3 (\( \mathbb{Q} \)-Steenbrink double complex):**

\[
A^p_q = (K^p/W_p)(p+1) \text{ for } p, q \geq 0 \text{ with differentials}
\]

\[
\begin{align*}
&d' : A^p_q \to A^{p+1,q}, & d'(x \otimes y) = x \otimes (\theta \wedge y), \text{ where } \theta = (2\pi i)(1 \otimes \bar{t}) \\
&d'' : A^p_q \to A^{p,q+1}, & d''(x \otimes y) = d(x \otimes y), \text{ where } d'' \text{ is the differential of the kozul complex.}
\end{align*}
\]

We also have the monodromy weight filtration \( M \),

\[
M_r A^p_q := \text{the image of } W_{2p+r+1}K^p/p+1 \text{ in } A^p_q.
\]

Denote the total complex \( \text{Tot}(A^*; \mathbb{Q}) \) by \( A^*_{\mathbb{Q}} \).

The morphism in Theorem 2.2.1

\[
\phi_m : W_m K^q \to W_m \Omega^q_X(\log X_0)
\]

induces a morphism of filtered double complexes

\[
\phi : (A^*_{\mathbb{Q}}, M_\sigma) \to (A^*; M_\sigma).
\]

**Theorem 2.2.4 (Steenbrink):**

1. \( \text{Gr}_m^M A^q_{\mathbb{Q}} \cong \bigoplus_{p \geq 0, -m} (G^W_{m+2p+1}K^p)(p+1)[1] \).
2. \( \phi : (A^q_{\mathbb{Q}} \otimes \mathbb{C}, M_\sigma) \to (A^*; M_\sigma) \) is a filtered quasi-isomorphism.
3. We have the following commutative diagram

\[
\begin{array}{ccc}
i^* R_k k^* K^q_0 \otimes \mathbb{C} & \longrightarrow & i^* K^q_0 \otimes \mathbb{C} \\
\downarrow & & \downarrow \\
(A \text{Nearby Cycle}) & \longrightarrow & A^q_{\mathbb{Q}} \otimes \mathbb{C}
\end{array}
\]

\[
\begin{array}{ccc}
i^* Rk^* \Omega^q_X(\log X_0)[\log t] & \longrightarrow & A^* \\
\downarrow & & \downarrow \\
i^* R k^* C_X & \longrightarrow & i^* \Omega^q_X(\log X_0)[\log t]
\end{array}
\]

such that every morphism in the above diagram is a quasi-isomorphism. This means that the \( \mathbb{Q} \)-cohomological mixed Hodge complex \( (A^q_{\mathbb{Q}}, M_\sigma), (A^*; M_\sigma, F^*) \) coincides with the one in Steenbrink [St1, section 4].

Let’s reproduce the proof for the \( \mathbb{Q} \)-Steenbrink double complex in Construction 2.2.3.

**Proof.** (1) follows from direct computations.
\[ M_r A_{n}^{*} = \]

\[
\begin{array}{ccc}
0 & 0 \\
W_{r+1}K_{0}^{2} & \rightarrow & W_{r+1}K_{1}^{2} \\
W_{r+1}K_{0}^{3} & \rightarrow & W_{r+1}K_{1}^{3} \\
(1) & \rightarrow & (2) \\
(0,0) & \sim & (0,0) \\
0 & 0 \\
\end{array}
\]

Thus \( Gr_{r} M A_{n}^{*} \) equals to the following double complex

\[
\begin{array}{ccc}
0 & 0 \\
W_{r+1}K_{0}^{2} & \rightarrow & W_{r+1}K_{1}^{2} \\
W_{r+1}K_{0}^{3} & \rightarrow & W_{r+1}K_{1}^{3} \\
(1) & \rightarrow & (2) \\
(0,0) & \sim & (0,0) \\
0 & 0 \\
\end{array}
\]

Note that all the horizontal morphisms in diagram (2.5) are zero. Thus the total complex is a direct sum of vertical complexes with a shift by 1, i.e.,

\[
Gr_{r} M A_{n}^{*} \cong \bigoplus_{p \geq 0, -m} (Gr_{m+2p+1}^{W} K_{p}^{*})(p+1)[1].
\]

(2) First, claim that \( Gr_{m} M A_{n}^{*} \) is quasi-isomorphic to 0 for \( m \neq -1, 0, 1 \).

In fact, by Theorem 2.2.1, \( Gr_{m+2p+1}^{W} K_{p}^{*} \) is quasi-isomorphic to the complex \( Gr_{m+2p+1}^{W} \Omega_{X}(\log X_{0})[1] \). Since \( Gr_{r}^{W} \Omega_{X}(\log X_{0}) = 0 \), for \( r \neq 0, 1, 2 \), \( Gr_{m}^{W} A_{n} \) is quasi-isomorphic to 0 for \( m \neq -1, 0, 1 \) by Theorem 2.2.4 (1).

For \( r = -1 \), \( Gr_{-1}^{W} A_{Q} \otimes \mathbb{C} \cong Gr_{2}^{W} K_{1}[1] \otimes \mathbb{C} \cong Gr_{2}^{W} \Omega_{X}(\log X_{0})[1] \cong Gr_{-1}^{W} A_{n}^{*} \). The last isomorphism follows from Steenbrink [St1, Lemma (4.18)]. For \( r = 0, 1 \), the calculations are the same.

(3) Note that the bottom row of diagram (2.3) contains two quasi-isomorphisms constructed by N. Katz in Steenbrink [St1, section 2.6]. The last vertical arrow in diagram (2.3) is a quasi-isomorphism by Theorem 2.2.4 (2). The first and middle vertical arrows are quasi-isomorphisms by Theorem 2.2.1, for \( n = 0 \). Thus the top row contains two quasi-isomorphisms.
3 1-Motives Associated to Abstract Mixed Hodge Structures

In this section, we will briefly recall the theory of Deligne 1-motives. Following from [De1, (10.1.3)], we have an equivalence of categories as mentioned in the introduction:

\[
\begin{align*}
\text{Z-mixed Hodge structure} & \quad \text{1-motive} \quad [L \xrightarrow{\mu} G], \text{ with } L \text{ a free Abelian group of finite rank,} \\
(H_Z, W, F^\cdot) \text{ of type } \{(-1, -1), \quad & \text{and } 0 \to T \to G \to A \to 0 \text{ an extension of an Abelian variety } A \text{ by a torus } T. \\
(-1, 0), (0, -1), (0, 0) \} \quad \text{with} & \\
\text{Gr} W_1 H \text{ a polarized pure Hodge structure of weight } -1. &
\end{align*}
\]

(1) For an arbitrary 1-motive \( M \)

\[
\begin{array}{ccccccccc}
0 & \to & T & \to & G & \to & A & \to & 0,
\end{array}
\]

the corresponding mixed Hodge structure is constructed as follows

\[
\begin{array}{ccccccccc}
0 & \to & H_1(G) & \to & \text{Lie} G & \xrightarrow{\exp} & G & \to & 0 \\
& & & & \uparrow & \alpha & \downarrow & \mu & \\
& & 0 & \to & H_1(G) & \to & H_Z & \to & L & \to & 0,
\end{array}
\]

where the square on the right is the pullback of morphisms \( \mu \) and \( \exp \).

\( H_Z \) is the integral lattice of the corresponding mixed Hodge structure. Denote \( H := H_Z \otimes \mathbb{C} \). The Weight filtrations are \( W_{-1}H_Z := H_1(G), W_{-2}H_Z := \ker \{ H_1(G) \to H_1(A) \} \). The Hodge filtration is \( F^0H := \ker \{ \alpha \otimes \mathbb{C} : H \to \text{Lie} G \} \).

(2) For the other direction, we start from a mixed Hodge structure \((H_Z, W, F^\cdot)\) of the above type, to construct the corresponding 1-motive.

Firstly, we introduce a construction given by P. Deligne [De1].

Since \( \text{Gr} W_{-1} H \) is polarizable, the complex torus \( A = \text{Gr} W_{-1} H_{\mathbb{C}} / (F^0 \text{Gr} W_{-1} H_{\mathbb{C}} + \text{Gr} W_{-1} H_Z) \) is an Abelian variety. Let \( T \) be the torus of the character group of the dual of \( \text{Gr} W_{-1}^\vee (H_Z) \).

The complex analytic group \( G = W_{-1} H_{\mathbb{C}} / (F^0 W_{-1} H_{\mathbb{C}} + W_{-1} H_Z) \) is an extension of \( A \)
by $T$, as a semi-Abelian variety. Let $L = G_0^W H_Z$. Then the corresponding 1-motive is the map $\mu : L \to G$ making the following diagram commutative:

$$
\begin{array}{ccccccccc}
0 & \rightarrow & W_{-1}H_Z & \rightarrow & W_{-1}H_C/F^0W_{-1}H_C & \rightarrow & G & \rightarrow & 0 \\
& & \downarrow \wr & & \downarrow \sim & & \downarrow \mu & & \\
0 & \rightarrow & W_{-1}H_Z & \rightarrow & H_Z & \rightarrow & L & \rightarrow & 0.
\end{array}
$$

**Remark 3.0.1:** When we consider $\mathbb{Q}$-mixed Hodge structures, we get 1-motives up to isogeny. We just replace $H_Z$ by $H_\mathbb{Q}$ in the above diagram.

Next we give another construction by J. Carlson, which will not be use it in the following sections. Recall the following theorem about an extension of mixed Hodge structures.

**Theorem 3.0.2 ([PeSt, Theorem 3.31]):** Let $A$ and $B$ be $\mathbb{Q}$-mixed Hodge structures, then there is a canonical isomorphism

$$
\text{Ext}^1_{\text{MHS}}(A, B) \cong W_0\text{Hom}(A, B)_C/(F^0W_0\text{Hom}(A, B)_C + W_0\text{Hom}(A, B)_\mathbb{Q}).
$$

**Corollary 3.0.3:** Note that if $A$ and $B$ are separated mixed Hodge structures (i.e., the weights of $B$ are less than the weights of $A$), we have

$$
\text{Ext}^1_{\text{MHS}}(A, B) \cong \text{Hom}(A, B)_C/(F^0\text{Hom}(A, B)_C + \text{Hom}(A, B)_\mathbb{Q}).
$$

In fact, the isomorphism is given explicitly as follows. Let

$$
0 \rightarrow B_\mathbb{Q} \xrightarrow{\beta} H_\mathbb{Q} \xrightarrow{\delta} A_\mathbb{Q} \rightarrow 0
$$

be a separated extension of $\mathbb{Q}$-mixed Hodge structures. Choose any retraction $r : H_\mathbb{Q} \rightarrow B_\mathbb{Q}$, i.e., $r \circ \beta = id_B$ and a section $\sigma_F$ of $\alpha_C : H_C \rightarrow A_C$ preserving Hodge filtration, then the above extension corresponds to the element represented by $r_C \circ \sigma_F$ in Corollary 3.0.2.

If $B$ is a mixed Hodge structure of type $\{(-1, -1), (-1, 0), (0, -1)\}$ and $A$ is a mixed Hodge structure of type $\{(0, 0)\}$, the 1-motive up to isogeny of the above extension $H$ is given by the morphism $r$:

$$
A_\mathbb{Q} \xrightarrow{\sigma_F|_{A_\mathbb{Q}}} H_C \xrightarrow{r_C} B_C/(F^0B_C + B_\mathbb{Q})
$$

by Carlson [Carl2, Proposition 3, Lemma 4].
4 1-Motives upto Isogeny Associated to Semi-Stable Degenerations of Curves

4.1 Abstract construction for 1-motives

For the family \( f : X \to \Delta \) and parameter \( t \) as that in the introduction, the limiting mixed Hodge structure \((\mathbb{H}^1(A^1_\mathbb{Q}), W), (\mathbb{H}^1(A^t), W, F^t)\) constructed in section 2 is of type \(\{(0, 0), (0, 1), (1, 0), (1, 1)\}\). After taking a Tate twist, it is of type \(\{(-1, -1), (0, -1), (-1, 0), (0, 0)\}\). Also, by Lemma 4.1.2, \(Gr^W_1 \mathbb{H}^1(A^t)\) is polarized. Then we have an associated 1-motive upto isogeny as explained in section 3: \(L = Gr^W_1 \mathbb{H}^1(A^t)\), \(G = \mathbb{H}^1(A^t)/(F^1 + \mathbb{H}^1(A^t))\), \(\mu_t : L \to G\) which relies on parameter \(t\). In this subsection, we give an abstract description of the 1-motive map \(\mu_t\). From now on we will only write \(\mathbb{H}(A^t)\) for the \(\mathbb{Q}\)-limiting mixed Hodge structure and write \(\mathbb{H}(A^t)\) for its underling \(\mathbb{Q}\) structure.

**Lemma 4.1.1:** \(L = Gr^W_1 \mathbb{H}^1(A^t) \cong \ker\{H^0(X_0[2], \mathbb{Q}(1)) \to H^2(X_0[1], \mathbb{Q}(1))\}\), where \(\mathbb{Q}(1)\) means \(2\pi i\mathbb{Q}\).

**Proof.** Consider the monodromy weight spectral sequence

\[ E_{-m}^{1,m} = H^1(Gr^M_m A^t_\mathbb{Q}) \Rightarrow H^1(A^t_\mathbb{Q}) \]

which degenerates at \(E_2\), i.e., \(E_{-m}^{1,m} = \text{Cohomology of } E_{-m}^{1,m-1} \to E_{-m}^{1,m} \to E_{-m+1}^{1,m-1} \to \cdots\) \(E_{-m}^{1,m} = Gr^W_{m+1} \mathbb{H}^1(A^t)\). Thus \(Gr^W_2 \mathbb{H}^1(A^t)\) is Cohomology of \(\mathbb{H}^0(Gr^M_1 A^t) \to \mathbb{H}^1(Gr^M_1 A^t) \to H^2(Gr^M_0 A^t)\)\).

By Theorem 2.2.4, \(\mathbb{H}^0(Gr^M_2 A^t) = 0\); \(\mathbb{H}^1(Gr^M_1 A^t) = \mathbb{H}^1(Gr^W_0 K^0_0[1]) = H^0(X_0[2], \mathbb{Q}(1))\); \(\mathbb{H}^2(Gr^M_0 A^t) = H^2(X_0[1], \mathbb{Q}(1))\). Thus \(L \cong \ker\{H^0(X_0[2], \mathbb{Q}(1)) \to H^2(X_0[1], \mathbb{Q}(1))\}\).

**Lemma 4.1.2 ([St1], (4.27)):** The sub-\(\mathbb{Q}\)-mixed Hodge structure \(W_1 \mathbb{H}^1(A^t)\) is isomorphic to the canonical mixed Hodge structure \(H^1(X_0)\) of the singular curve \(X_0\).

**Proof.** Let’s sketch the idea of the proof here. Recall from Steenbrink [St1, (4.22)], \(\mathbb{H}^1(X_0, A^t)\) has an additional structure, the residue of Gauss-Manin connection \(N = \text{Res}_0 \nabla : \mathbb{H}^1(X_0, A^t) \to \mathbb{H}^1(X_0, A^t)(1)\), where “(1)” is the Tate twist of a mixed Hodge structure.
In fact, \( N \) is induced by an endomorphism \( v : A^{*} \rightarrow A^{*+1,*-1} \), the canonical projection \( \Omega^{p+q+1}(\log X_0)/W_p\Omega^{p+q+1}(\log X_0) \rightarrow \Omega^{p+q+1}(\log X_0)/W_{p+1}\Omega^{p+q+1}(\log X_0) \).

It is easy to see that \( \ker v = \Omega_X^{1} \wedge \Omega_X^{1}(\log X_0)/\Omega_X^{2} \)

\[
\begin{align*}
\Omega_X^{1}(\log X_0)/\Omega_X^{1} & \xrightarrow{d''} \Omega_X^{2}(\log X_0)/\Omega_X^{1} \\
& \xrightarrow{d'} \Omega_X^{2}(\log X_0)/\Omega_X^{1} \wedge \Omega_X^{1}(\log X_0).
\end{align*}
\]

Through Poincaré residue map, \( \ker v \) is isomorphic to

\[
(a_1)_*\Omega_{X_0[1]}^{1} \xrightarrow{d} (a_2)_* \bigoplus_{p \in X_0[2]} \mathbb{C}_p,
\]

where \( \theta \) is taking the difference of functions at \( p \), according to the order of components of \( X_0[1] \), i.e., \( f_i - f_j \) for \( i < j \), where \( f_i \) and \( f_j \) are local functions on \( X_{0,i} \) and \( X_{0,j} \), respectively, and \( p \in a_1(X_{0,i}) \cap a_1(X_{0,j}) \).

Note that the double complex (4.2) gives the mixed Hodge structure \( W_1H_1(X_0) \) and the double complex \( \ker v \) gives the mixed Hodge structure \( W_1H_1(A^*/squaresmall) \). The same argument works for \( \mathbb{Q} \)-structures. Hence \( H^1(X_0) \cong W_1H_1(A^*) \) as \( \mathbb{Q} \)-mixed Hodge structures.

\[
\text{Lemma 4.1.3: } H^1(A^*)/F_1H^1(A^*) \cong W_1H_1(A^*)/F_1W_1H_1(A^*).
\]

\[
\text{Proof. } \text{This follows immediately from Deligne Splitting of mixed Hodge structures [PeSt, Lemma 3.4]. In fact, for the mixed Hodge structure } (H, W_1, F_1) \text{ of type } \{(0, 0), (0, 1), (1, 0), (1, 1)\}, \text{ } W_1 \text{ and } F_1 \text{ generate } H \text{ as vector spaces.}
\]

\[\text{Lemma 4.1.4: } G = W_1H_1(A^*)/(F_1W_1H_1(A^*) + W_1H_1(A^*)) \text{ is isomorphic to } Pic^0(X_0) \otimes \mathbb{Q}.
\]

\[
\text{Proof. } \text{By Lemma 4.1.2, we have } W_1H_1(A^*)/(F_1W_1H_1(A^*) + W_1H_1(A^*)) \cong H^1(X_0, \mathbb{C})/(F_1H^1(X_0, \mathbb{C}) + H^1(X_0, \mathbb{Q})) \cong Pic^0(X_0) \otimes \mathbb{Q}.
\]
Before we state Theorem 4.1.6, let’s give the following ingredients.

I) The morphism
\[ \varphi : \mathbb{H}^1(A_\mathbb{Q}) \longrightarrow H^1(O_{X_0}) \]
is the composition of canonical morphisms of cohomologies:
\[ \mathbb{H}^1(A_\mathbb{Q}) \longrightarrow \mathbb{H}^1(A') \longrightarrow \mathbb{H}^1(A')/F^1\mathbb{H}^1(A') = \mathbb{H}^1(Gr_0^\mathbb{Q}A') \]
\[ \cong W_1\mathbb{H}^1(A')/F^1W_1\mathbb{H}^1(A') \cong H^1(X_0, \mathbb{C})/F^1H^1(X_0, \mathbb{C}) \cong H^1(O_{X_0}), \]

where \( \mathbb{H}^1(A')/F^1\mathbb{H}^1(A') = \mathbb{H}^1(Gr_0^\mathbb{Q}A') \) is from the degeneration of Hodge spectral sequence at \( E_1 \)-page, \( W_1\mathbb{H}^1(A')/F^1W_1\mathbb{H}^1(A') \cong H^1(X_0, \mathbb{C})/F^1H^1(X_0, \mathbb{C}) \) follows from Lemma 4.1.2, and \( H^1(X_0, \mathbb{C})/F^1H^1(X_0, \mathbb{C}) = H^1((a_1)_*O_{X_0[1]} \rightarrow (a_2)_* \bigoplus_{p \in X_0[2]} \mathbb{C}_p) \) as that in the diagram (4.2).

Thus \( \mathbb{H}^1(A_\mathbb{Q}) \rightarrow H^1(O_{X_0}) \) is induced by morphisms of complexes \( A_\mathbb{Q} \rightarrow A_\mathbb{Q} \rightarrow Gr_0^\mathbb{Q}A_\mathbb{Q} \overset{P.R}{\longrightarrow} \{(a_1)_*O_{X_0[1]} \rightarrow (a_2)_* \bigoplus_{p \in X_0[2]} \mathbb{C}_p\} \), where \( P.R \) is the Poincaré residue map.

II) We have canonical morphisms
\[ \mathbb{H}^1(A_\mathbb{Q}) \longrightarrow Gr_2^W\mathbb{H}^1(A_\mathbb{Q}) \longrightarrow \mathbb{H}^1(Gr_1^MA_\mathbb{Q}), \]
and the following lemma.

**Lemma 4.1.5:** The composition of two maps
\[ \mathbb{H}^1(A_\mathbb{Q}) \longrightarrow Gr_2^W\mathbb{H}^1(A_\mathbb{Q}) \longrightarrow \mathbb{H}^1(Gr_1^MA_\mathbb{Q}) \]
is induced by the canonical morphism of complexes \( A_\mathbb{Q} \rightarrow Gr_1^MA_\mathbb{Q} \).

**Proof.** Consider the monodromy weight spectral sequence
\[ E_{-m,1+m}^1 = \mathbb{H}^1(Gr_m^MA_\mathbb{Q}) \Rightarrow \mathbb{H}^1(A_\mathbb{Q}) \]
induced by the filtered complex \( M_{-1}A_\mathbb{Q} \subset M_0A_\mathbb{Q} \subset M_1A_\mathbb{Q} \). It degenerates at \( E_2 \).
\[ E_{-1,2}^\infty = im\{\mathbb{H}^1(A_\mathbb{Q}) \rightarrow \mathbb{H}^1(Gr_1^MA_\mathbb{Q})\} \]. Also, \( E_{-1,2}^\infty = E_2^{-1,2} = Gr_2^W\mathbb{H}^1(A_\mathbb{Q}). \)

III) The composite morphism
\[ H^1(O_{X_0}) \overset{exp}{\longrightarrow} Pic^0(X_0) \otimes \mathbb{Q} \longrightarrow Pic(X_0) \otimes \mathbb{Q} \]
is induced by the morphism of complexes

$$
\begin{array}{cccccc}
0 & \longrightarrow & (a_1)_* \mathcal{O}_{X_0[1]} & \theta & (a_2)_* & \bigoplus_{p \in X_0[2]} \mathbb{C}_p & \longrightarrow & 0 \\
\downarrow{\exp(2\pi i)} & & \downarrow{\exp(2\pi i)} & & & & \\
1 & \longrightarrow & (a_1)_* \mathcal{O}_{X_0[1]} \otimes_{\mathbb{Z}} \mathbb{Q} & \theta^* & (a_2)_* & \bigoplus_{p \in X_0[2]} \mathbb{C}_p^* \otimes_{\mathbb{Z}} \mathbb{Q} & \longrightarrow & 1,
\end{array}
$$

where $\theta^*$ is taking the ratio of functions at $p$, according to the order of components of $X_0[1]$, i.e., $f_i/f_j$ for $i < j$, where $f_i$ and $f_j$ are local functions on $X_{0,i}$ and $X_{0,j}$, respectively, and $p \in a_1(X_{0,i}) \cap a_1(X_{0,j})$.

**Theorem 4.1.6:** The 1-motive map $\mu_t$ of the extension of $\mathbb{Q}$-mixed Hodge structures $0 \to W_1^1 \mathbb{H}^1(A^*) \to \mathbb{H}^1(A^*) \to Gr_2^W \mathbb{H}^1(A^*) \to 0$ is the morphism making the following diagram commutative

$$
\begin{array}{cccccc}
H^1(\mathcal{O}_{X_0}) & \xrightarrow{\exp} & Pic^0(X_0) \otimes \mathbb{Q} & \\
\varphi & | & \mu_t & | \\
\mathbb{H}^1(A_{\mathbb{Q}}^1) & \xrightarrow{pr} & L & \xrightarrow{\mu_t} & Gr_2^W \mathbb{H}^1(A_{\mathbb{Q}}^1),
\end{array}
$$

where $pr : \mathbb{H}^1(A_{\mathbb{Q}}^1) \to L$ is the usual projection map. The morphisms in the diagram (4.3), except $\mu_t$, are discussed in I), II), III) previously.

**Proof.** By Lemma 4.1.2, we have $\ker(exp) = H^1(X_0, \mathbb{Q})$. Also, $\ker(pr) = W_1^1 \mathbb{H}^1(A_{\mathbb{Q}}^1)$. Then by the construction of $\varphi$ we have the following commutative diagram:

$$
\begin{array}{cccccc}
0 & \longrightarrow & H^1(X_0, \mathbb{Q}) & \xrightarrow{\varphi} & H^1(\mathcal{O}_{X_0}) & \xrightarrow{exp} & Pic^0(X_0) \otimes \mathbb{Q} & \longrightarrow & 0 \\
\uparrow{\cong} & & \uparrow{\varphi} & & \uparrow{\exp} & & \uparrow{\mu_t} & & \uparrow{0} \\
0 & \longrightarrow & W_1^1 \mathbb{H}^1(A_{\mathbb{Q}}^1) & \xrightarrow{pr} & \mathbb{H}^1(A_{\mathbb{Q}}^1) & \xrightarrow{\mu_t} & L & \longrightarrow & 0.
\end{array}
$$

Thus there exist a unique map $\mu_t : L \to Pic^0(X_0) \otimes \mathbb{Q}$ making the above diagram commutative. Then by construction (2) of section 3, $\mu_t$ is the 1-motive of the extension $0 \to W_1^1 \mathbb{H}^1(A^*) \to \mathbb{H}^1(A^*) \to Gr_2^W \mathbb{H}^1(A^*) \to 0$.

What’s more, by diagram chasing, $\mathbb{H}^1(A_{\mathbb{Q}}^1)$ is the fiber product of morphisms $exp$ and $\mu_t$.

4.2 Geometric description of 1-motives I

By Lemma 4.1.1, we have $L \cong \ker\{H^0(X_0[2], \mathbb{Q}(1)) \longrightarrow H^2(X_0[1], \mathbb{Q}(1))\} \xrightarrow{\cong} \ker$
The local defining equation of $X$ along a fiber $X_0$ we have $H^0(X_0[2], Z) \rightarrow H^2(X_0[1], Z) \otimes \mathbb{Q} \cong \ker \{ H_0(X_0[2], Z) \rightarrow H_0(X_0[1], Z) \} \otimes \mathbb{Q}$, where the last isomorphism is from Poincaré duality. In the last three subsections, we want to find out a nice geometric description of the morphism $\mu_t : L \rightarrow \text{Pic}^0(X_0) \otimes \mathbb{Q}$ in Theorem 4.1.6.

Note that we have the following short exact sequence:

$$1 \rightarrow \mathbb{C}^{*d+1-n} \rightarrow \text{Pic}^0(X_0) \rightarrow \text{Pic}^0(X_0[1]) \rightarrow 1,$$

where $d$ is the number of nodal points of $X_0$, and $n$ is the number of irreducible components of $X_0$. Geometrically, any line bundle $L$ over $X_0$ is coming from the pullback line bundle $\tilde{L} := a_1^*L$ over $X_0[1]$ glued along the preimage of nodal points. The gluing data is encoded in $\mathbb{C}^{*d+1-n}$. Based on the above discussion, to understand $\mu_t$, we first calculate $a_1^* \circ \mu_t$ in the following commutative diagram

$$
\begin{array}{cccc}
H^1(O_{X_0}) & \xrightarrow{\exp} & \text{Pic}^0(X_0) \otimes \mathbb{Q} & \xrightarrow{a_1^*} \text{Pic}^0(X_0[1]) \otimes \mathbb{Q} \\
\downarrow & & & & \\
\mathbb{H}(A_0^1) & \xrightarrow{pr} & Gr^W_1 \mathbb{H}(A_0^1).
\end{array}
$$

Before we formulate next theorem, let’s make the following notation.

**Notation 4.2.1:** For any nodal point $p \in X_0[2]$, we denote $a_1^{-1}(p)$ by $\{p', p''\}$, where $p'$ is contained in $X_{0,i'}$ and $p''$ is contained in $X_{0,i''}$, for $i' < i''$.

**Theorem 4.2.2:** The morphism $a_1^* \circ \mu_t : \ker \{ H_0(X_0[2], Z) \rightarrow H_0(X_0[1], Z) \} \otimes \mathbb{Q} \rightarrow \text{Pic}(X_0[1] \otimes \mathbb{Q})$ is given by $D = \sum_{p \in X_0[2]} n_p p \mapsto O_{X_0[1]}(\sum_{p \in X_0[2]} n_p(p' - p''))$, where $a_1^*D$ is a degree zero divisor on each component of $X_0[1]$.

Before the proof of Theorem 4.2.2, we prove the following lemma first. Denote the set of nodal points of $X_0$ by $S$. Consider the normalization map $a_1 : X_0[1] \rightarrow X_0$. We have the open smooth curve $X_0[1] - a_1^{-1}(S)$, which we denote by $Y$.

**Lemma 4.2.3:** There is a canonical injective morphism of $\mathbb{Q}$-mixed Hodge structures $\mathbb{H}^1(A^1)/W_0 \mathbb{H}^1(A^1) \hookrightarrow H^1(Y)$.

**Proof.** In Hain [Hain, section 5.3], we have another construction of the nearby fiber $X_\infty$ through real oriented blowup. Let $X_0'$ be the real oriented blowup of $X_0[1]$ along $a_1^{-1}(S)$. Then $X_\infty$ is obtained from $X_0'$ by gluing the gluing data for which is given by the local defining equation of $X_0$ and the parameter $t$ we chose. Topologically, we have $Y \hookrightarrow X_0' \rightarrow X_\infty$.

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The induced map $\mathbb{H}^1(A^*) \cong H^1(X_\infty) \to H^1(Y)$ is a morphism of $\mathbb{Q}$-mixed Hodge structures by [Hain, Theorem 14]. Then we have the following commutative diagram of $\mathbb{Q}$-mixed Hodge structures

\[
\begin{array}{c}
0 \\
0 \to W_0 H^1(X_0) \to H^1(X_0) \to H^1(Y) \to Gr^W_2 H^1(Y) \to 0 \\
0 \to W_0 \mathbb{H}^1(A^*) \to \mathbb{H}^1(A^*) \\
Gr^W_2 H^1(A^*) \\
0 \\
\end{array}
\]

The first row is an exact sequence of $\mathbb{Q}$-mixed Hodge structures, which is obtained from the inclusion of curves $Y \hookrightarrow X_0$. The isomorphism in the first column is obtained from Lemma 4.1.2. Also, by Lemma 4.1.2, the second column is a short exact sequence of $\mathbb{Q}$-mixed Hodge structures. The inclusion map $Gr^W_2 H^1(A^*) \hookrightarrow Gr^W_2 H^1(Y)$ is obtained from Lemma 4.1.1. Thus, from the above diagram, we get an exact sequence of $\mathbb{Q}$-mixed Hodge structures

\[0 \to W_0 \mathbb{H}^1(A^*) \to \mathbb{H}^1(A^*) \to H^1(Y).\]

**Proof of Theorem 4.2.2:** Consider the $\mathbb{Q}$-mixed Hodge structure for the open curve $Y$. By construction (2) in section 3, we have the following commutative diagram

\[
\begin{array}{c}
H^1(\mathcal{O}_{X_0[1]}) \xrightarrow{\exp} Pic^0(X_0[1]) \otimes \mathbb{Q} \\
\mu_Y \downarrow \\
H^1(Y, \mathbb{Q}) \xrightarrow{pr} Gr^W_2 H^1(Y, \mathbb{Q}) \\
\end{array}
\]

where $\mu_Y$ is the 1-motive associated to the $\mathbb{Q}$-mixed Hodge structure $H^1(Y)$. Then by Lemma 4.2.3, we have the following commutative diagram

\[
\begin{array}{c}
H^1(\mathcal{O}_{X_0}) \xrightarrow{\exp} Pic^0(X_0[1]) \otimes \mathbb{Q} \\
\phi \downarrow \\
\mathbb{H}^1(A_\mathbb{Q}) \xrightarrow{pr} Gr^W_2 \mathbb{H}^1(A_\mathbb{Q}) \\
\end{array}
\]

Thus we have $a_1^* \circ \mu_\mathbb{Q} \cong \mu_Y |_{Gr^W_2 \mathbb{H}^1(A_\mathbb{Q})}$. By Deligne [De1, (10.3.8)], the 1-motive map $\mu_Y |_{Gr^W_2 \mathbb{H}^1(A_\mathbb{Q})} : ker \{H_0(X_0[2], \mathbb{Z}) \to H_0(X_0[1], \mathbb{Z})\} \otimes \mathbb{Q} \to Pic^0(X_0[1] \otimes \mathbb{Q})$ is just the
map \( D = \sum_{p \in X_0[2]} n_p p \mapsto \mathcal{O}_{X_0[1]}(\sum_{p \in X_0[2]} n_p (p' - p'')) \), where \( a_1^* D \) is a degree zero divisor on each component of \( X_0[1] \). Hence Theorem 4.2.2 holds.

4.3 Geometric description of 1-motives II

In subsection 4.2, we have already known that if \( D = \sum_{p \in X_0[2]} n_p p \otimes \mathbb{1} \in \mathcal{L} \), the 1-motive \( \mu_t(D) \) can be obtained from a line bundle \( \mathcal{O}_{X_0[1]}(\sum_{p \in X_0[2]} n_p (p' - p'')) \) with gluing data along the pairs \((p', p'')\). In this subsection, let’s describe an educated guess of 1-motives \( \mu_t \) associated to the limiting mixed Hodge structure of degeneration of curves in Deligne [De2].

Recall that in the introduction, we have \( f : X \to \Delta \), with a chosen parameter \( t : \Delta \to \mathbb{C} \). It gives a global section \( \tilde{t} := t \circ f \in \Gamma(X, \mathcal{M}_X) \), where \( \mathcal{M}_X = \mathcal{O}_X \cap j_1^* \mathcal{O}_{T^1} \). \( \tilde{t} \) has a zero along each component \( \{X_{0,i}\} \) of order 1, since the monodromy is unipotent.

For any nodal point \( p \in X_0 \subset X \), we choose an open neighborhood \( V_p \) of \( p \) in analytic topology of \( X \), such that \( f : V_p \to \Delta \) can be defined to be \( \{(u_1, u_2, \tilde{t}) \in \mathbb{C}^3 \mid u_1 \cdot u_2 = \tilde{t}\} \), i.e., the following diagram commutes:

\[
\begin{array}{ccc}
W' & \xrightarrow{\text{closed immersion}} & \mathbb{C}^2 \times \mathbb{C} \\
V_p & \xrightarrow{t} & \mathbb{C} \times \mathbb{C} \\
\downarrow & & \downarrow \text{pr}_2 \\
\mathbb{C} & \xrightarrow{p} & \mathbb{C}
\end{array}
\]

(4.8)

where \( W \) is defined to be \( \{(u_1, u_2, \tilde{t}) \in \mathbb{C}^3 \mid u_1 \cdot u_2 = \tilde{t}\} \). Note that \( \{u_2 = 0\} \) and \( \{u_1 = 0\} \) give the two components of \( V_p \cap X_0 \). Thus we can assume that \( u_1 \) is a local coordinate function at \( p' \) and \( u_2 \) can be regarded as a coordinate function at \( p'' \) in terms of Notation 4.2.1.

**Notation 4.3.1:** Pick a set of open complex polydisks \( \{V_\alpha\}_{\alpha \in I} \) in \( X \), s.t., \( \mathcal{U} = \{U_\alpha := V_\alpha \cap X_0\}_{\alpha \in I} \) is a covering of \( X_0 \). We also assume that \( U_\alpha \) is simply connected, connected and containing at most one nodal point, for all \( \alpha \). Also assume that every nodal point is contained in at most one \( U_\alpha \).

For the covering \( \mathcal{U} = \bigcup_{\alpha \in I} U_\alpha \) of \( X_0 \) in the above notation, as is described in the diagram (4.8), we can choose \( u_{1,\alpha}, u_{2,\alpha} \) for any \( V_\alpha \) containing some nodal point \( p \) in \( X_0 \). We always assume that \( u_{1,\alpha} \) is the coordinate function at \( p' \) and \( u_{2,\alpha} \) is the coordinate
function at $p''$, in terms of Notation 4.2.1.

**Construction 4.3.2:** Define $\nu_t : L = \ker \{ H_0(X_0[2], \mathbb{Z}) \to H_0(X_0[1], \mathbb{Z}) \} \otimes \mathbb{Q} \longrightarrow \text{Pic}^0(X_0) \otimes \mathbb{Q}$ as follows.

For any element $D = \sum_{p \in X_0[2]} n_p p^1 \otimes 1 \in L$, attach it with the line bundle $L(D) := \mathcal{O}_{X_0[1]}(\sum_{p \in X_0[2]} n_p (p' - p'')) \in \text{Pic}^0(X_0[1])$. For each point $p \in U_\alpha$, we have gluing data:

$$\rho_p : \text{Fiber of } L(D) \text{ at } p' \sim \text{Fiber of } L(D) \text{ at } p'' \quad (4.9)$$

induced by the commutative diagram:

$$\begin{array}{ccc}
L(D) \otimes (\mathcal{O}_{X_0[1],p'} / m_p) & \sim & L(D) \otimes (\mathcal{O}_{X_0[1],p''} / m_p) \\
\cong (\mathcal{O}_{X_0[1],p''} / m_p) & \cong (\mathcal{O}_{X_0[1],p''} / m_p)
\end{array} \quad (4.10)$$

Through this gluing data, we get a line bundle in $\text{Pic}^0(X_0) \otimes \mathbb{Q}$, which is defined to be the image $\nu_t(D)$.

Next we want to describe the $\nu_t(D)$ in terms of Čech double complex.

We already have the resolution $1 \to \mathcal{O}_{X_0}^{\alpha_2} \to (a_1)_* \mathcal{O}_{X_0[1]}^{\alpha_2} \to (a_2)_* \bigoplus_{p \in X_0[2]} C_p^* \to 1$ of $\mathcal{O}_{X_0}^*$. The Čech double complex of $(a_1)_* \mathcal{O}_{X_0[1]}^{\alpha_2} \to (a_2)_* \bigoplus_{p \in X_0[2]} C_p^*$ is

$$\begin{array}{ccc}
1 & \to & \mathcal{O}_{X_0}^{\alpha_2} \to (a_2)_* \bigoplus_{p \in X_0[2]} C_p^* \otimes_{\mathbb{Z}} \mathbb{Q} \to 1 \\
\delta & \delta & \delta \\
(0,0) \sim \mathcal{C}(U,(a_1)_* \mathcal{O}_{X_0[1]}^{\alpha_2} \otimes_{\mathbb{Z}} \mathbb{Q}) \to \mathcal{C}^1(U,(a_2)_* \mathcal{O}_{X_0[1]}^{\alpha_2} \otimes_{\mathbb{Z}} \mathbb{Q}) \to \cdots \\
1 & \to & 1
\end{array} \quad (4.11)$$

Denote the above Čech double complex by $E^{\alpha}$ and its total complex by $E^\alpha$. Then
we have the canonical isomorphisms

$$\tilde{H}^1(E) \cong \tilde{H}^1((a_1)_{\ast}\mathcal{O}^\ast_{X_0[1]} \to (a_2)_{\ast}\mathcal{O}^\ast_{X_0}), \bigoplus_{p \in X_0[2]} C^p \cong H^1(X_0, \mathcal{O}^\ast_{X_0}) \cong Pic(X_0),$$

so we get a canonical morphism $\Psi : C^0(U,(a_2)_{\ast}\bigoplus_{p \in X_0[2]} C^p \otimes \mathbb{Q}) \oplus Z^1(U,(a_1)_{\ast}\mathcal{O}^\ast_{X_0[1]} \otimes \mathbb{Q}) \to Pic(X_0)$, where $Z^1(U,(a_1)_{\ast}\mathcal{O}^\ast_{X_0[1]} \otimes \mathbb{Q})$ is the group of 1-cocycles. It is also clear that $C^0(U,(a_2)_{\ast}\bigoplus_{p \in X_0[2]} C^p \otimes \mathbb{Q}) = \bigoplus_{p \in X_0[2]} C^p \otimes \mathbb{Q}$. We want to find a representative of $\nu_1(D)$ in $(\bigoplus_{p \in X_0[2]} C^p \otimes \mathbb{Q}) \oplus Z^1(U,(a_1)_{\ast}\mathcal{O}^\ast_{X_0[1]} \otimes \mathbb{Q})$, for $D = \Sigma n_p p \otimes 1 \in L$.

**Notation 4.3.3:** For any node $p \in U_\alpha$, where $U_\alpha \subset U$, we have local coordinate functions $u_{1,\alpha}, u_{2,\alpha}$ near $p'$ and $p''$ as is discussed previously. We denote the function $w^\alpha$ over $a_1^{-1}U_\alpha$ to be $u_{1,\alpha}$ over the connected component containing $p'$, and $1/u_{2,\alpha}$ over the other component containing $p''$. For convenience, if $U_\beta$ does not contain nodal point, we denote $w^\beta = 1$.

**Theorem 4.3.4:** Let $(g^\alpha \otimes 1) \in C^0(U,(a_1)_{\ast}\mathcal{O}^\ast_{X_0[1]} \otimes \mathbb{Q})$, where $g^\alpha \in \Gamma(\mathcal{O}^\ast_{X_0[1]}, a_1^{-1}U_\alpha)$. Take any $D = \Sigma n_p p \in L$, define $\tilde{g}^\alpha = g^\alpha(w^\alpha)^{n_\alpha}$, where

$$n_\alpha = \begin{cases} n_p & \text{for } p \in U_\alpha \\ 1 & \text{for } U_\alpha \text{ containing no nodal point} \end{cases}$$

Then the representative $(\theta^\ast(g^\alpha \otimes 1), (\tilde{g}^\alpha/\tilde{g}^\beta|_{U_\alpha}))$ gives $\nu_1(D)$ in $Pic^0(X_0)$.

**Proof.** It is clear that the representative $(\theta^\ast(g^\alpha \otimes 1), (\tilde{g}^\alpha/\tilde{g}^\beta|_{U_\alpha}))$ is independent of the choice of $(g^\alpha \otimes 1)$. For simplicity, we can assume $\theta^\ast(g^\alpha \otimes 1) = (1, 1, \ldots, 1) \otimes 1$. For this propose, we can choose $g^\alpha = 1$, for all $\alpha$. Then the representative is $((1,\ldots,1) \otimes 1, ((w^\alpha)^{n_\alpha}/(w^\beta)^{n_\beta}|_{U_\beta})).$

Note that the quasi-isomorphism of complexes

$$\begin{array}{c}
1 \longrightarrow \mathcal{O}^\ast_{X_0} \longrightarrow 1 \\
\downarrow \quad a_1 \\
1 \longrightarrow (a_1)_{\ast}\mathcal{O}^\ast_{X_0[1]} \longrightarrow \bigoplus_{p \in X_0[2]} C^p \longrightarrow 1
\end{array}$$
gives the morphism of two Čech double complexes as below:

\[
\begin{array}{ccc}
1 & \rightarrow & 1 \\
\theta' & & \theta' \\
C^0(U, (a_1)_*, \mathcal{O}_{X_0}^* \otimes_\mathbb{Z} \mathbb{Q}) & \rightarrow & C^1(U, (a_1)_*, \mathcal{O}_{X_0}^* \otimes_\mathbb{Z} \mathbb{Q}) \\
\downarrow & & \downarrow \\
C^0(U, (a_2)_* \bigoplus_{p \in X_0[2]} C_p^* \otimes_\mathbb{Z} \mathbb{Q}) & \rightarrow & 1 \\
\theta' & & \theta' \\
C^0(U, (a_1)_*, \mathcal{O}_{X_0[1]}^* \otimes_\mathbb{Z} \mathbb{Q}) & \rightarrow & C^1(U, (a_1)_*, \mathcal{O}_{X_0[1]}^* \otimes_\mathbb{Z} \mathbb{Q}) \\
\downarrow & & \downarrow \\
\cdots & & \cdots \\
\end{array}
\]

The Čech 1-cocycle \(((u^\alpha)^{n_\alpha}/(u^\beta)^{n_\beta})_{U_{\alpha\beta}} \in Z^1(U, (a_1)_*, \mathcal{O}_{X_0}^* \otimes_\mathbb{Z} \mathbb{Q})\) maps to the representative \(((1, ..., 1) \otimes 1, ((u^\alpha)^{n_\alpha}/(u^\beta)^{n_\beta})_{U_{\alpha\beta}})\). Also, through the diagram (4.10), it is clear that the representative \(((u^\alpha)^{n_\alpha}/(u^\beta)^{n_\beta})_{U_{\alpha\beta}} \in Z^1(U, (a_1)_*, \mathcal{O}_{X_0}^* \otimes_\mathbb{Z} \mathbb{Q})\) gives the line bundle \(\nu_t(D) \in \text{Pic}^0(X_0)\).

\[\blacksquare\]

### 4.4 Geometric description of 1-motives III

Recall that our ultimate goal is to describe the geometric meaning of the morphism \(\mu_t : L \rightarrow \text{Pic}^0(X_0) \otimes \mathbb{Q}\). In this subsection, we will prove that the 1-motive \(\mu_t\) coincides with \(\nu_t\) in subsection 4.3. In order to achieve this goal, let’s consider the rational structure \((A_0^\ast, M)\) given in section 2.2.

We have double complexes \(A_0^\ast, M_0 A_0^\ast, Gr_1^M A_0^\ast\) as follows:

1. \(A_0^\ast =

\[
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\wedge^2 M_{\wedge \mathcal{O}_{X_0}^*} \rightarrow & \wedge^3 M_{\wedge \mathcal{O}_{X_0}^*} \rightarrow & \cdots \\
\downarrow d & & \downarrow d \\
\wedge^2 M_{\wedge \mathcal{O}_{X_0}^*} \rightarrow & \wedge^3 M_{\wedge \mathcal{O}_{X_0}^*} \rightarrow & \cdots \\
(0, 0) \rightarrow & (0, 0) \rightarrow & (0, 0) \rightarrow \\
\wedge^2 M_{\wedge \mathcal{O}_{X_0}^*} \rightarrow & \wedge^3 M_{\wedge \mathcal{O}_{X_0}^*} \rightarrow & \cdots \\
0 & 0 & 0 \\
\end{array}
\]

(4.12)
Lemma 4.4.1:

1. \( \Lambda^2 M_{e(O_X)} \cong (a_2)_* Q_{X_0}[2]; \)
2. \( \mathcal{O}_{X_0} \otimes \Lambda^2 M_{e(O_X)} \cong (a_2)_* Q_{X_0}[2]; \)
3. \( \mathcal{O}_{X_0} \otimes \Lambda^3 M_{e(O_X)} \cong 0. \)

Proof. (1) Consider the morphism of sheaves of free Abelian groups \( e : \mathcal{O}_{X_0} \to M. \) Denote \( f : \mathcal{O}_{X_0} \to e(\mathcal{O}_{X_0}). \) Note that \( \Lambda^2 M_{e(O_X)} = Gr_2^{W(e(O_X))} K\text{os}^2(e)^2. \) By Proposition 1.2.4, \( Gr_2^{W(e(O_X))} K\text{os}^2(e)^2 \cong K\text{os}^0(f)^2[-2] \otimes [\text{coker}(e)] \cong \Lambda((a_1)_* Q_{X_0}[1]) \cong (a_2)_* Q_{X_0}[2]. \) The arguments for (2) and (3) are similar to the one that for (1).

Consider the diagram (4.3)
We want to work on the complexes, so we extend the above square to be the following commutative diagram:

$$
\begin{array}{ccc}
H^1(O_{X_0}) & \xrightarrow{\text{exp}} & \text{Pic}^0(X_0) \otimes \mathbb{Q} \\
\downarrow \quad \mu_t & & \downarrow \text{inclusion} \\
H^1(A_Q) & \xrightarrow{\text{pr}} & \text{Gr}^W_2 H^1(A_Q) \cong H^1(\text{Gr}_1^M A_Q) \\
\ker\{H_0(X_0[2], \mathbb{Z}) \to H_0(X_0[1], \mathbb{Z})\} \otimes \mathbb{Q} & \cong & H_0(X_0[2], \mathbb{Z}) \otimes \mathbb{Q}.
\end{array}
$$

(4.15)

To compute $H^1(A_Q)$, $H^1(\text{Gr}_1^M A_Q)$, $H^1(O_{X_0})$, and $\text{Pic}(X_0) \otimes \mathbb{Q}$, we use Čech cohomology. Taking into account Lemma 4.4.1 and the choice of the covering $\mathcal{U}$, we have the following Čech double complexes of $A_Q$, $\text{Gr}_1^M A_Q$, and $O_{X_0} \otimes \mathbb{Z} \otimes \mathbb{Q}$ which is denoted to be $C^{\cdot, \cdot}$, $D^{\cdot, \cdot}$, and $E^{\cdot, \cdot}$, respectively.

(1) $C^{\cdot, \cdot}$

$$
\begin{array}{ccccc}
0 & & & & \\
\uparrow & & & & \\
0 & & & & \\
\uparrow & & & & \\
C^0(\mathcal{U}, O_{X_0} \otimes M(2)) & \xrightarrow{\delta} & \cdots & & \\
\downarrow D & & & & \\
C^0(\mathcal{U}, O_{X_0} \otimes \wedge^2 M(1)) & \xrightarrow{\delta} & C^1(\mathcal{U}, O_{X_0} \otimes \wedge^2 M(1)) & \xrightarrow{\delta} & \cdots \\
\downarrow D & & & & \\
(0, 0) & \xrightarrow{\delta} & C^0(\mathcal{U}, O_{X_0} \otimes M(1)) & \xrightarrow{\delta} & C^1(\mathcal{U}, O_{X_0} \otimes M(1)) & \xrightarrow{\delta} & \cdots
\end{array}
$$

(4.16)

(2) $D^{\cdot, \cdot}$

$$
\begin{array}{cccc}
0 & & & \\
\uparrow D & & & \\
0 & & & \\
\uparrow D & & & \\
C^0(\mathcal{U}, O_{X_0} \otimes M(1)) & \rightarrow & 0 & \\
(0, 0) & \rightarrow & 0 & \\
\end{array}
$$

(4.17)
(3) $E^{\psi}$

\[
\begin{align*}
C^0(\mathcal{U}, (a_2)_*) & \oplus \mathbb{C}_p^* \otimes \mathbb{Q} & \rightarrow & 1 \\
\delta' & & \delta & \\
(0,0) & \rightarrow C^0(\mathcal{U}, (a_1)_* \mathcal{O}_{X_0[1]}^* \otimes \mathbb{Q}) & \rightarrow & C^1(\mathcal{U}, (a_1)_* \mathcal{O}_{X_0[1]}^* \otimes \mathbb{Q}) \\
1 & \rightarrow & 1 & \rightarrow \cdots
\end{align*}
\]

(4.18)

Also, we have morphisms $\Psi_1 : C^{\psi} \rightarrow D^{\psi}$ induced by natural projection, and $\Psi_2 : C^{\psi} \rightarrow E^{\psi}$ induced by canonical morphisms $\mathcal{A}_q^* \rightarrow \{(a_1)_* \mathcal{O}_{X_0[1]} \rightarrow (a_2)_* \bigoplus \mathbb{C}_p \exp\}_{p \in X_0[2]} \{[a_1], \mathcal{O}_{X_0[1]}^* \otimes \mathbb{Q}\}$, which is described in I) section 4.1.

**Theorem 4.4.2:** The abstract 1-motive $\mu_t$ in Theorem 4.1.6 is isomorphic to the geometric 1-motive $\nu_t$ constructed in section 4.3.

**Proof.** Take any element $D = \sum_{p \in X_0[2]} n_p p \otimes 1 \in L = \text{Gr}^W_2 \mathbb{H}^1(C^{\psi})$. Consider the line bundle $L(D) := \mathcal{O}_{X_0[1]}(\sum_{p \in X_0[2]} n_p (p' - p'')) \otimes \text{Pic}^0(X_0[1])$, where $a_1^{-1}(p) = \{p', p''\}$. We take local functions $(\tilde{g}^\alpha, U_\alpha)$ of $L(D)$ as in Theorem 4.3.4, with $\theta^* (g^\alpha \otimes 1) = (1, 1, ..., 1) \otimes 1$. Thus we have $\{\tilde{g}^\alpha / g^\beta|_{U_{\alpha\beta}}\}$ in $\mathbb{Z}^1(\mathcal{U}, (a_1)_* \mathcal{O}_{X_0[1]}^* \otimes \mathbb{Q})$ which gives the line bundle $L(D)$.

By Theorem 4.2.2, we can find an element in $\mathbb{H}^1(\mathcal{A}_q\mathcal{O})$ that is mapped to $L(D)$. Through the morphism $\Psi_2$ from $C^{\psi}$ to $E^{\psi}$, we want to lift the 1-cocycle $\kappa$ to $C^0(\mathcal{U}, \mathcal{O}_{X_0[1]} \otimes \mathbb{Q})$.

Let $G^\alpha_{\alpha\beta}$ be an extension of $\tilde{g}^\alpha|_{U_{\alpha\beta}}$ in $V_{\alpha\beta}$, where $V_\alpha$ is introduced in Notation 4.3.1 ($G^\alpha_{\alpha\beta} = G^\alpha_{\beta\alpha}$).

1) When $U_\alpha$ contains a node, $G^\alpha_{\alpha\beta}$ can be extended to be a holomorphic function $g^\alpha(u_{1,\alpha})^{n_\alpha}$ or a meromorphic function $g^\alpha(1/u_{2,\alpha})^{n_\alpha}$ in $V_\alpha$, which depends on $\beta$. Here we refer to the notation in Theorem 4.3.4. Also note that $g^\alpha(u_{1,\alpha})^{n_\alpha} \wedge \tilde{l} = (g^\alpha(u_{1,\alpha})^{n_\alpha} / \tilde{l}^{n_\alpha}) \wedge \tilde{l} = g^\alpha(1/u_{2,\alpha})^{n_\alpha} \wedge \tilde{l}$. Thus we have a well-defined extension, denoted by $G^\alpha \wedge \tilde{l}$, over $V_\alpha$, which is independent of $\beta$.

2) When $U_\alpha$ does not contain any node, $G^\alpha_{\alpha\beta}$ can be extended to some holomorphic function $G^\alpha$ over $V_\alpha$. Then we also have a well-defined element $G^\alpha \wedge \tilde{l}$ over $V_\alpha$.  

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Then we get \( \{G^\alpha \wedge \tilde{t}\} \in C^0(\mathcal{U}, \frac{\wedge^2 \mathcal{M}}{e(O_{X_0}) \wedge \mathcal{M}}(1)) \), which represents the element \( D = \sum_{p \in X_0[^2]} n_p p \otimes 1 \in L \) through the Čech double complex \( D^{**} \). In the Čech double complex \( C^\bullet \), we take an element \( \kappa = ((f^\alpha \otimes (m_1^\alpha \wedge m_2^\alpha), G^\alpha \wedge \tilde{t}), (\log G^\alpha_{\alpha \beta} - \log G^\beta_{\alpha \beta}) \otimes \tilde{t}) \in C^0(\mathcal{U}, \frac{O_{X_0} \wedge \wedge^2 \mathcal{M}}{e(O_{X_0}) \wedge \mathcal{M}}(2) \oplus \frac{\wedge^2 \mathcal{M}}{e(O_{X_0}) \wedge \mathcal{M}}(1)) \oplus C^1(\mathcal{U}, \frac{O_{X_0} \wedge \wedge^2 \mathcal{M}}{e(O_{X_0}) \wedge \mathcal{M}}(1)) \) with \( (f^\alpha \otimes (m_1^\alpha \wedge m_2^\alpha) \otimes \tilde{t}) \) to be determined. Note first that in the diagram (4.16), since \( d''(\log G^\alpha_{\alpha \beta} - \log G^\beta_{\alpha \beta}) = G^\alpha_{\alpha \beta} - G^\beta_{\alpha \beta} \), where \( d'' \) is the differential in Construction 2.2.3, we have \( \delta((1 \otimes (G^\alpha \wedge \tilde{t}), G^\alpha \wedge \tilde{t})) = D((\log G^\alpha_{\alpha \beta} - \log G^\beta_{\alpha \beta}) \otimes \tilde{t}) \). Also, \( \delta((\log G^\alpha_{\alpha \beta} - \log G^\beta_{\alpha \beta}) \otimes \tilde{t}) = 0 \). In order for \( \kappa \) to be a 1-cocycle, we just need to let \( d(f^\alpha \otimes (m_1^\alpha \wedge m_2^\alpha)) = \theta(G^\alpha \wedge \tilde{t}) = \tilde{t} \wedge G^\alpha \wedge \tilde{t} = 1 \wedge G^\alpha \wedge \tilde{t} \). Therefore we can take \( f^\alpha = 1, m_1^\alpha = G^\alpha, m_2^\alpha = \tilde{t}|_{V_a} \), and then \( ((1 \otimes (G^\alpha \wedge \tilde{t}), G^\alpha \wedge \tilde{t}), (\log G^\alpha_{\alpha \beta} - \log G^\beta_{\alpha \beta}) \otimes \tilde{t}) \) is a 1-cocycle.

Through morphism \( \Psi_1 \), the image of \( ((1 \otimes (G^\alpha \wedge \tilde{t}), G^\alpha \wedge \tilde{t}), (\log G^\alpha_{\alpha \beta} - \log G^\beta_{\alpha \beta}) \otimes \tilde{t}) \) is \( G^\alpha \wedge \tilde{t} \), which represents \( D = \sum_{p \in X_0[^2]} n_p p \otimes 1 \in L \). Through morphism \( \Psi_2 \), the image of \( ((1 \otimes (G^\alpha \wedge \tilde{t}), G^\alpha \wedge \tilde{t}), (\log G^\alpha_{\alpha \beta} - \log G^\beta_{\alpha \beta}) \otimes \tilde{t}) \) is \((1, 1, ..., 1), (\tilde{g}^\alpha/\tilde{g}^\beta|_{V_{\alpha \beta}})\), which represents \( \nu_1(D) \) by Theorem 4.3.4. Therefore we proved that \( \mu_1 \) is isomorphic to \( \nu_1 \).
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