Observational Constraints on the Tilted Spatially Flat and the Untilted Nonflat $\phi$CDM Dynamical Dark Energy Inflation Models

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Abstract

We constrain spatially flat tilted and nonflat untitled scalar field ($\phi$) dynamical dark energy inflation ($\phi$CDM) models by using Planck 2015 cosmic microwave background (CMB) anisotropy measurements and recent baryonic acoustic oscillation distance observations, Type Ia supernovae apparent magnitude data, Hubble parameter measurements, and growth rate data. We assume an inverse power-law scalar field potential energy density $V(\phi) = V_0 \phi^{-\alpha}$. We find that the combination of the CMB data with the four non-CMB data sets significantly improves parameter constraints and strengthens the evidence for nonflatness in the nonflat untitled $\phi$CDM case from 1.8$\sigma$ for the CMB measurements only to more than 3.1$\sigma$ for the combined data. In the nonflat untitled $\phi$CDM model, current observations favor a spatially closed universe with spatial curvature contributing about two-thirds of a percent of the present cosmological energy budget. The flat tilted $\phi$CDM model is a 0.4$\sigma$ better fit to the data than is the standard flat tilted $\Lambda$CDM model: current data allow for the possibility that dark energy is dynamical. The nonflat tilted $\phi$CDM model is in better accord with the Dark Energy Survey bounds on the rms amplitude of mass fluctuations now ($\sigma_8$) as a function of the nonrelativistic matter density parameter now ($\Omega_{m0}$) but it does not provide as good a fit to the larger-multipole Planck 2015 CMB anisotropy data as does the standard flat tilted $\Lambda$CDM model. A few cosmological parameter value measurements differ significantly when determined using the tilted flat and the untitled nonflat $\phi$CDM models, including the cold dark matter density parameter and the reionization optical depth.

Key words: cosmic background radiation – cosmological parameters – inflation – large-scale structure of universe – methods: statistical

1. Introduction

In the standard flat $\Lambda$CDM cosmogony (Peebles 1984) the cosmological energy budget is currently dominated by the cosmological constant $\Lambda$, which is responsible for powering the currently accelerated cosmological expansion.3 This standard $\Lambda$CDM model is consistent with most observational constraints, including cosmic microwave background (CMB) anisotropy measurements (Planck Collaboration et al. 2016), baryonic acoustic oscillation (BAO) distance observations (Alam et al. 2017; Ryan et al. 2018), Type Ia supernovae (SN Ia) apparent magnitude data (Scolnic et al. 2018), and Hubble parameter measurements (Farooq et al. 2017; Yu et al. 2018).

The standard flat $\Lambda$CDM inflation cosmogony is characterized by six cosmological parameters usually picked to be $\Omega_{c0} h^2$ and $\Omega_{b0} h^2$, the current values of the cold dark matter and baryonic matter density parameters multiplied by the square of the Hubble constant $H_0$ (in units of 100 km s$^{-1}$ Mpc$^{-1}$); $A_s$ and $n_s$, the amplitude and spectral index of the primordial fractional energy density inhomogeneity power-law power spectrum; $\theta_{MC}$, the angular size of the sound horizon at recombination; and $\tau$, the reionization optical depth.

While the standard $\Lambda$CDM model assumes flat spatial geometry, current observational data allow for slightly curved spatial hypersurfaces. Current measurements also allow a dark energy density that decreases slowly in time (and so also varies weakly spatially) and do not require a space- and time-independent $\Lambda$. Theoretically, it seems easier to accommodate dynamical dark energy than a $\Lambda$.

$\chi$CDM is a simple and widely used dynamical dark energy parameterization. Here the equation of state relating the dark energy fluid pressure and energy density is $p_X = w X$, where $w$ is the equation of state parameter and the additional seventh cosmological parameter. $\chi$CDM does not provide a consistent description of the evolution of energy density spatial inhomogeneities and so is not a physically consistent description of dark energy. The simplest physically consistent dynamical dark energy model is $\phi$CDM (Peebles & Ratra 1988; Ratra & Peebles 1988). In this model, the dynamical dark energy is a scalar field $\phi$ with potential energy density $V(\phi) \propto \phi^{-\alpha}$ and $\alpha > 0$ is the additional seventh cosmological parameter.4

There have been a number of suggestions that some measurements favor dynamical dark energy over a $\Lambda$ (Sahni et al. 2014; Ding et al. 2015; Solá et al. 2015, 2018a, 2017a, 2017b, 2017c, 2018b; Zheng et al. 2016; Gómez-Valent & Solá 2017, 2018; Zhang et al. 2017b; Zhao et al. 2017; Cao et al. 2018). These analyses made a number of simplifying assumptions, either ignoring CMB anisotropy data, or only

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3 For reviews of the standard model, see Ratra & Vogeley (2008), Martin (2012), Brax (2018), and Luković et al. (2018). In this model, cold dark matter (CDM) and baryonic matter, both nonrelativistic, are the second and third largest contributors to the current cosmological energy budget; earlier they dominated over $\Lambda$ and were responsible for decelerating the cosmological expansion.

4 Many cosmological data sets have been used to place constraints on the $\phi$CDM model (see, e.g., Yashar et al. 2009; Samushia et al. 2007; Samushia & Ratra 2010; Chen & Ratra 2011b; Campanelli et al. 2012; Avsajanishvili et al. 2015; Solá et al. 2018a, 2017b; Zhai et al. 2017; Sangwan et al. 2018, and references therein).
approximately accounting for it, or using it in the context of a generalized XCDM parameterization of dynamical dark energy. Some of these analyses also include a high $H_0$ value determined from the local expansion rate in the data collections they use to investigate dark energy dynamics.\footnote{We exclude this high local $H_0$ value from the data we use here to constrain cosmological model parameters, as it is inconsistent with the other data sets we utilize for this purpose, in the models we study.}

Ooba et al. (2018c) have more exactly analyzed the Planck CMB data (as well as a few BAO distance measurements) by using the seven parameter flat XCDM and $\phi$CDM dynamical dark energy tilted inflation models and discovered that both were weakly favored by the data, compared to the standard six parameter flat $\Lambda$CDM model, by 1.1$\sigma$ and 1.3$\sigma$ for the XCDM and $\phi$CDM cases.\footnote{Park & Ratra (2018b) used a much larger compilation of non-CMB data in an analysis of the tilted flat XCDM parameterization, confirming the Ooba et al. (2018c) findings, but at a lower level of significance, 0.3$\sigma$ instead of 1.1$\sigma$.} These are not significant improvements over the standard flat $\Lambda$CDM case, but current data allow for the possibility that dark energy is dynamical. Furthermore, both dynamical dark energy models decrease the tension between the Planck CMB and the weak lensing observational bounds on $\sigma_8$, the current value of rms fractional energy density inhomogeneity averaged over $8h^{-1}$ Mpc radius spheres.

Nonflat models have a characteristic length set by the nonvanishing spatial curvature and an energy density inhomogeneity power spectrum in a nonflat model that does not fully account for this spatial curvature length scale (as was done in the analyses of nonflat models by Planck Collaboration et al. 2016) is not physically consistent. Nonflat cosmological inflation models are the only known way of defining physically consistent fractional energy density inhomogeneity power spectra in nonflat models. For open geometries, the openbubble inflation model (Gott 1982) is used to derive the non-power-law power spectrum (Ratra & Peebles 1994, 1995). For closed geometries the Hawking prescription for the initial state of the universe (Hawking 1984; Ratra 1985) defines a closed inflation model that is used to compute the non-power-law power spectrum (Ratra 2017). Unlike in the flat inflation case, there is no simple way to also accommodate tilt in nonflat inflation models. In the nonflat case, $n_s$ is no longer a free parameter but is instead replaced by the current spatial curvature density parameter $\Omega_k$.

Ooba et al. (2018a) used this physically consistent nonflat untilted model non-power-law power spectrum of energy density spatial inhomogeneities in analyses of the Planck 2015 CMB anisotropy measurements (Planck Collaboration et al. 2016) and found that these data do not require flat spatial geometry in the six parameter nonflat untilted $\Lambda$CDM inflation model.\footnote{Non-CMB observations do not provide tight constraints on spatial curvature (Farooq et al. 2015, 2017; Chen et al. 2016; Li et al. 2016; Yu & Wang 2016; L’Huillier & Shafer 2017; Rana et al. 2017; Wei & Wu 2017; Mitra et al. 2018; Ryan et al. 2018; Yu et al. 2018), with the recent exception of a collection of all of the most recent BAO, Hubble parameter, and SN Ia data, which (weakly) favor closed spatial geometry (Park & Ratra 2018c), as well as a recent collection of deuterium abundances that favor flat spatial hypersurfaces (Penton et al. 2018).} Park & Ratra (2018a, 2018b) confirmed the results of Ooba et al. (2018a) by using the largest compilation of reliable observational data to study the nonflat untilted $\Lambda$CDM inflation model, and found stronger evidence for nonflatness, 5.2$\sigma$, favoring a very slightly closed model. The Planck 2015 CMB anisotropy data also do not require flat spatial surfaces in the seven parameter nonflat untilted XCDM dynamical dark energy inflation parameterization (Ooba et al. 2017). In the XCDM parameterization, $w$ is the seventh cosmological parameter with $n_s$ again replaced by $\Omega_k$. Using a much larger compilation of non-CMB data, Park & Ratra (2018b) confirmed the Ooba et al. (2017) results with higher significance: in the untilted nonflat XCDM case, the data favor a closed model at 3.4$\sigma$ significance and favor dynamical dark energy over a cosmological constant at 1.2$\sigma$ significance. In the seven parameter nonflat untilted $\phi$CDM dynamical dark energy inflation model (Pavlov et al. 2013)—with $\alpha$ as the seventh cosmological parameter—Ooba et al. (2018b) again discovered that Planck 2015 CMB anisotropy data do not demand flat spatial hypersurfaces. In both the XCDM and $\phi$CDM dynamical dark energy inflation cases, the data again favor a very mildly closed model. All three closed models are more compatible with weak lensing $\sigma_8$ constraints but do not fit the higher-$\ell$ $C_\ell$ data as well as the flat models do.

In this paper, we determine observational limits on parameters of the seven parameter flat tilted $\phi$CDM and the seven parameter nonflat untilted $\phi$CDM dynamical dark energy inflation models. For this purpose, we use the same observational data as Park & Ratra (2018b), the Planck CMB anisotropy, the Pantheon collection of 1048 SN Ia apparent magnitudes (Scolina et al. 2018), and a collection of BAO distances, Hubble parameters, and growth rates (see Park & Ratra 2018a, 2018b for the data compilation and update).

We find that the seven parameter flat tilted $\phi$CDM inflation model provides a better fit to these data than does the six parameter standard tilted $\Lambda$CDM model. However, for the larger compilation of data here the $\phi$CDM dynamical dark energy inflation model is only 0.40$\sigma$ better than the standard $\Lambda$CDM model (compared to the 1.3$\sigma$ Ooba et al. 2018c found with their smaller data collection). While not a significant improvement over the standard model, the $\phi$CDM model cannot be ruled out. In agreement with Ooba et al. (2018c), we also do not detect a deviation from $\alpha = 0$ (a cosmological constant) for the flat $\phi$CDM model.\footnote{These conclusions do not agree with those from earlier approximate analyses, based on less, as well as less reliable, data (Solà et al. 2018a, 2017b), that favor the flat $\phi$CDM model over the flat $\Lambda$CDM one by more than 3$\sigma$ and find $\alpha$ deviating from 0 by more than 2$\sigma$. As discussed elsewhere and below, the number of degrees of freedom of the Planck 2015 data are ambiguous and the nonflat untilted $\phi$CDM model and the flat tilted $\Lambda$CDM model are not nested, thus it is impossible to translate the $\Delta AIC$’s we compute here to quantitative goodness of fit probabilities, consequently a large number of our statements about goodness of fit are qualitative. See below and see Park & Ratra (2018a, 2018b) for more details about this issue.}

Our results for the nonflat untilted $\phi$CDM inflation model, derived using many more non-CMB observations, are consistent with and strengthen the Ooba et al. (2018b) conclusions. For the full data collection we use here, we find a more than 3.1$\sigma$ deviation from spatial flatness. The nonflat untilted $\phi$CDM model better fits the weak lensing $\sigma_8$-$\Omega_m$ bound. For the full data collection we use here (including CMB lensing data), the best-fit nonflat untilted $\phi$CDM model has a reduced low-$\ell$ CMB temperature anisotropy multipole number ($\ell$) power spectrum $C_\ell$ and is more compatible with the observations. However, overall, the standard tilted flat $\Lambda$CDM model better fits the CMB data.\footnote{As discussed elsewhere and below, the number of degrees of freedom of the Planck 2015 data are ambiguous and the nonflat untilted $\phi$CDM model and the flat tilted $\Lambda$CDM model are not nested, thus it is impossible to translate the $\Delta AIC$’s we compute here to quantitative goodness of fit probabilities, consequently a large number of our statements about goodness of fit are qualitative. See below and see Park & Ratra (2018a, 2018b) for more details about this issue.}

These data determine $H_0$ in an almost model independent way with a value that is compatible with most other estimates.

As found in Park & Ratra (2018a, 2018b), however, $\Omega_k h^2$ and $\tau$
differ significantly between the tilted flat and the untilted nonflat models and so care must be taken when utilizing cosmological measurements of such parameters.

In Section 2, we summarize the data sets we use in our analyses. In Section 3, we summarize the methods we use in our analyses here. Observational constraints following from these data for the flat tilted \( \phi \)CDM and the nonflat untilted \( \phi \)CDM inflation models are presented and discussed in Section 4. We summarize our main results in Section 5.

2. Data

Following Park & Ratra (2018a, 2018b), we utilize the Planck 2015 LT + lowP and LT + lowP + lensing CMB anisotropy measurements (Planck Collaboration et al. 2016) to set bounds on the parameters of the \( \phi \)CDM dynamical dark energy model. Here LT is the low-\( \ell \) (2 \( \leq \ell \) \leq 29) and high-\( \ell \) (30 \( \leq \ell \) \leq 2508; PlikTT) Planck temperature-only \( C_\ell \) angular power spectrum measurements and lowP is the low-\( \ell \) polarization \( C_\ell^{TT}, C_\ell^{EE}, \) and \( E_\ell^{BB} \) angular power spectra measurements at 2 \( \leq \ell \) \leq 29. The collection of low-\( \ell \) CMB temperature and polarization power spectra is called lowTEB. The CMB lensing data we use is the measured Planck lensing potential power spectrum, and the abbreviations LT + lowP and LT + lowP + lensing are used for the CMB data without and with CMB lensing data, respectively. The Planck collaboration recommends using the LT + lowP + lensing data combination as a conservative choice for parameter estimation (see the footnote of Table 4 of Planck Collaboration et al. 2016).

The SN Ia data set we use is the Pantheon set of 1048 SN Ia apparent magnitude observations over the wide redshift (\( z \)) range of 0.01 < \( z \) < 2.3 (Scolnic et al. 2018). This data set includes 276 SN Ia (0.03 < \( z \) < 0.65) from the Pan-STARRS1 Medium Deep Survey and SN Ia distance measurements from the SDSS, SNLS and low-\( z \) HST collections. We use the abbreviation SN to refer to the Pantheon sample.

We use the compilation of BAO data given in Table 1 of Park & Ratra (2018a). As in Park & Ratra (2018b), we use the updated BAO data point, \( D_v (\theta_{ad}/r_d) = 3843 \pm 147 \text{ Mpc of Ata et al. (2018)}, \) instead of the old value. See Section 2.3 of Park & Ratra (2018a) for more details. We note that the BAO data from BOSS DR12 (Alam et al. 2017) include growth rate \( f(\theta) \) and radial BAO \( H(z) \) data that are correlated with the other BOSS DR12 BAO measurements. We use the abbreviation BAO to refer to this BAO data compilation.

We also use the Hubble parameter, \( H(z) \) (with 31 data points in total),\(^{10}\) and growth rate (with 10 points in total), \( \sigma_8(z) \), observations of Tables 2 and 3 of Park & Ratra (2018a).

3. Methods

In the \( \phi \)CDM model we study here, the minimally coupled dark energy scalar field \( \phi \) has an inverse power-law potential energy density

\[
V(\phi) = V_0 \phi^{-\alpha},
\]

with \( \alpha > 0 \) being a constant parameter and \( V_0 \) being determined in terms of \( \alpha \) (Peebles & Ratra 1988). When \( \alpha \) goes to zero, the dark energy behaves like the cosmological constant \( \Lambda \).

We evolve a system of multiple components including radiation, neutrinos, matter, and the scalar field (that only directly couples to the gravitational field). The evolution equations for the spatially homogeneous background and the spatial inhomogeneity linear perturbations are summarized in Hwang & Noh (2001, 2002). For the homogeneous background scalar field, we use the initial conditions of Peebles & Ratra (1988) at scale factor \( a_i = 10^{-10} \). This places the homogeneous background scalar field on the attractor/tracker solution (Peebles & Ratra 1988; Ratra & Peebles 1988; Pavlov et al. 2013).\(^11\) As initial conditions of the spatially inhomogeneous scalar field perturbation and its time derivative, we take them to vanish in the CDM-comoving gauge (this is a synchronous gauge without gauge modes) at \( a = 10^{-10} \).\(^12\)

For background evolution, we numerically solve the equation of motion of the scalar field,

\[
\phi'' + \left( 3 \frac{\dot{H}}{H^2} \right) \phi' - \frac{\dot{V}_0 \alpha}{3} \frac{\dot{H}}{H} \phi^{-\alpha-1} = 0,
\]

where \( \phi' \equiv d\phi/d\ln a, H = \dot{a}/a, \dot{V}_0 \equiv V_0/H_0^2, \) and an overdot denotes the time derivative \( d/dt \). For the matter and dark energy dominated epochs, the normalized Hubble parameter \( H(a) \) can be written as

\[
\left( \frac{H}{H_0} \right)^2 = \frac{1}{1 - \frac{1}{6}(\phi')^2} \left[ \Omega_m a^{-3} + \Omega_\Lambda a^{-2} + \frac{1}{3} \frac{\dot{V}_0}{V_0} \phi^{-\alpha} \right].
\]

where \( \Omega_m \) and \( \Omega_{\Lambda} \) are present values of the matter and curvature density parameters, respectively, and we have chosen units such that 8\( \pi \)\( G \equiv 1 \). In actual calculations of Equation (3), we have taken into account the contribution of photons as well as massless and massive neutrinos. Given cosmological parameters and initial conditions for the scalar field, we adjust the value of \( V_0 \) to satisfy the condition \( H/H_0 = 1 \) at the present epoch \( (a_0 = 1) \) by applying the bisection method.

To estimate the likelihood distributions of \( \phi \)CDM model parameters, we use the CAMB/COSMOMC package (Nov. 2016 version) (Challinor & Lasenby 1999; Lewis et al. 2000; Lewis & Bridle 2002). CAMB is used to compute the theoretical CMB temperature anisotropy, polarization, and lensing potential power spectra, as well as the matter density power spectrum, by solving for the evolution of the cosmological spatial inhomogeneity linear perturbations. COSMOMC determines model parameter values that are favored by the observational data by using the Markov chain Monte Carlo (MCMC) method. Since the current version of the CAMB/COSMOMC package cannot be applied to scalar field dynamical dark energy models, we generalized CAMB by including the dynamical equations of motion for the spatially

\(^{10}\) Hubble parameters have been measured over a wide range of redshift, from the present epoch to well beyond the cosmological deceleration-acceleration transition redshift. They provide evidence that this transition occurred and they have been used to measure the redshift of this transition at roughly the value expected in standard \( \Lambda \)CDM and other dark energy models (Farooq & Ratra 2013; Farooq et al. 2013, 2017; Capozziello et al. 2014; Moresco et al. 2016; Haridasu et al. 2018a; Jesus et al. 2018; Yu et al. 2018).

\(^{11}\) As a consequence of there being an attractor/tracker solution of the background scalar field nonlinear equation of motion (coupled to the Friedmann equation), the long-term time evolution is independent of the chosen initial conditions. However, there can be differences caused by different approaches from different initial conditions to the attractor/tracker solution and future data might require a more careful study of initial conditions’ effects.

\(^{12}\) The evolution of spatially inhomogeneous scalar field quantities linearly perturbed about the background attractor solution also show tracking behavior and so are largely independent of the choice of initial conditions (Ratra & Peebles 1988; Brax et al. 2000).
homogeneous background and spatial inhomogeneity linear perturbation quantities for the scalar field inverse power-law potential energy density model. CAMB uses the RECFAST routine to compute the recombination history of the universe (Seager et al. 1999; Wong et al. 2008). We modified RECFAST to use the background evolution of the $\phi$CDM model. We also altered the COSMOMC parameter interface to use the scalar field potential energy density parameter $\alpha$ as a new free parameter, in place of the constant equation of state parameter $w$ of the $\Lambda$CDM model. Unlike in the $\Lambda$CDM and XCDM analyses of Park & Ratra (2018a, 2018b), here we use the Hubble constant $H_0$ as a new free parameter, instead of $\theta_{MC}$ (a default free parameter used in COSMOMC). There are two reasons for this change. First, $\theta_{MC}$, the approximate angular size of the sound horizon at the decoupling epoch, is based on the fitting formula of the sound horizon size given in Hu & Sugiyama (1996) and is appropriate for models with a negligible level of dark energy in the early universe. In general, however, scalar field dark energy can be non-negligible at early times, depending on the scalar field potential energy density parameters and the initial conditions (e.g., see Park et al. 2014 for episodic domination of scalar field dark energy in the early universe). In the $\phi$CDM model we study here, a large value of $\alpha$ can result in a significant amount of dark energy at early times. Thus, a more accurate model parameterization is needed. Second, using the angular size of the sound horizon ($\theta$) as a free parameter is less suitable in the presence of scalar field dark energy. Scalar field dark energy has its own dynamical equation that needs to be numerically evolved and so it is a matter of practical difficulty to adjust other cosmological parameter values along with the potential parameters of the scalar field to reproduce $\theta$, a quantity that is obtained from an integration of the spatially homogeneous background equations of motion. The drawback of choosing the Hubble constant as a free parameter is that this makes it difficult to achieve MCMC convergence as the Hubble constant has degeneracy with spatial curvature and with the dark energy parameter $\alpha$ resulting in likelihood distributions that are degenerate and non-Gaussian.

Figure 1. Top: Evolution of the dark energy scalar field density parameter ($\Omega_{f}$) and equation of state parameter ($w_{f}$) in the tilted flat $\phi$CDM model for integer values of $\alpha$ from 1 to 5. The black solid curve is for $\Lambda$CDM which corresponds to $\phi$CDM with $\alpha = 0$. For these illustrations all other cosmological parameters are fixed to the mean values of the $\Lambda$CDM model parameters constrained by using the Planck 2015 CMB (TT + lowP) and the four non-CMB data sets (see Table 5 bottom-right panel of Park & Ratra 2018a). Bottom: Theoretical predictions for matter density and CMB temperature anisotropy angular power spectra in the $\phi$CDM model at the corresponding $\alpha$ values. The ratios of the $\phi$CDM model power spectra relative to the $\Lambda$CDM one are shown in the lower panels.
Figure 1 shows the evolution of the scalar field dark energy density parameter \( \Omega_f \) and equation of state parameter \( w_f = p_f / \rho_f \) as well as theoretical predictions for matter density and CMB temperature anisotropy angular power spectra in the spatially flat \( \Lambda \)CDM model for some \( \alpha \) values. The other cosmological parameters are fixed to the mean \( \Lambda \)CDM model parameters obtained by using the Planck 2015 CMB (TT + lowP) and the four non-CMB data sets (see the bottom right panel of Table 5 in Park & Ratra 2018a). We can expect that the spatially flat \( \phi \)CDM model with large \( \alpha \) can be excluded by CMB data alone. However, we will see that the nonflat \( \phi \)CDM model with large values of \( \alpha \) can be consistent with Planck CMB data.13

Figure 2. Likelihood distributions of the tilted flat \( \phi \)CDM model parameters constrained by using Planck CMB TT + lowP, SN, BAO, \( H(z) \), and \( f_{\sigma 8} \) data. Two-dimensional marginalized likelihood constraint contours and one-dimensional likelihoods are plotted for when each set of non-CMB data is combined with the Planck TT + lowP measurements (left panel) and when the growth rate, Hubble parameter, and SN data, as well as their combination, are combined with the TT + lowP + BAO data (right panel). For viewing clarity, the cases of TT + lowP (left) and TT + lowP + BAO (right panel) are shown with solid black curves.

Figure 3. Same as Figure 2 but now also accounting for the Planck CMB lensing data.

However, from the bottom left panels of Figures 1 and 4, we see that matter power spectrum measurements over a wide range of wavenumbers, such as those shown in Figure 19 of Planck Collaboration et al. (2018), exclude large \( \alpha \) values.

13
The primordial fractional energy density spatial inhomogeneity power spectrum in the tilted flat \( \phi \)CDM inflation model (Lucchin & Matarrese 1985; Ratra 1992, 1989) is
\[
P(k) = A_s \left( \frac{k}{k_0} \right)^{n_s},
\]
where \( k \) is wavenumber and \( A_s \) is the amplitude of the power spectrum at the pivot scale wavenumber \( k_0 = 0.05 \) Mpc\(^{-1}\). The corresponding power spectrum in the nonflat untitled \( \phi \)CDM inflation model (Ratra & Peebles 1995; Ratra 2017) is
\[
P(q) \propto \frac{(q^2 - 4K^2)}{q(q^2 - K)},
\]
which becomes the \( n_s = 1 \) spectrum in the flat limit (when \( K = 0 \)). For scalar-type perturbations, \( q = \sqrt{k^2 + c^2} \) is the wavenumber where spatial curvature \( K = -(H_0^2/c^2) \Omega_k \) and \( c \) is the speed of light. For the negative \( \Omega_k \) closed model, normal modes are characterized by positive integers \( \nu = qK^{-1/2} = 3, 4, 5, \ldots \). For the nonflat model, we use \( P(q) \) as the initial perturbation power spectrum and normalize its amplitude at \( k_0 \) to \( A_s \).

Our analyses methods are those described in Section 3.2 of Park & Ratra (2018a) and Section 3 of Park & Ratra (2018b).

4. Observational Constraints

We constrain the tilted flat \( \phi \)CDM model with seven cosmological parameters \((\Omega_m h^2, \Omega_c h^2, H_0, \tau, A_s, n_s, \) and \( \alpha \)) and the untitled nonflat \( \phi \)CDM model with seven parameters \((\Omega_m h^2, \Omega_c h^2, H_0, \Omega_k, \tau, A_s, \) and \( \alpha \)). The calibration and foreground model parameters of the Planck data are also constrained as nuisance parameters by the COSMOMC program. In all parameter constraint tables presented in this work, we also list three derived parameters, \( \theta_{MC}, \Omega_m \) (present value of the nonrelativistic matter density parameter), and \( \sigma_8 \).

We use the COSMOMC settings adopted by the Planck team (Planck Collaboration et al. 2016) and the same priors on the model parameters as well as the same values of the present CMB temperature \( T_0 = 2.7255 \) K, the effective number of neutrino species \( (N_{\text{eff}} = 3.046) \), and one massive neutrino species (with mass \( m_{\nu} = 0.06 \) eV) as used in Park & Ratra (2018a, 2018b). We set tophat priors on the scalar field potential energy density parameter \( 0 < \alpha < 10 \) and on the Hubble constant \( 0.2 \lesssim h \lesssim 1.0 \). However, as detailed below,

### Table 1

| Parameter | TT+lowP | TT+lowP+SN | TT+lowP+BAO |
|-----------|---------|------------|-------------|
| \( \Omega_m h^2 \) | 0.02218 ± 0.00023 | 0.02231 ± 0.00023 | 0.02235 ± 0.00021 |
| \( \Omega_c h^2 \) | 0.1202 ± 0.0023 | 0.1184 ± 0.0019 | 0.1177 ± 0.0015 |
| \( H_0 \) [km s\(^{-1}\) Mpc\(^{-1}\)] | 63.3 ± 3.1 | 67.20 ± 0.91 | 66.97 ± 0.80 |
| \( \tau \) | 0.077 ± 0.019 | 0.083 ± 0.019 | 0.085 ± 0.018 |
| ln(10\(^{10}\)A\(_s\)) | 3.089 ± 0.037 | 3.097 ± 0.037 | 3.100 ± 0.035 |
| \( n_s \) | 0.9646 ± 0.00064 | 0.9686 ± 0.0058 | 0.9702 ± 0.0049 |
| \( \alpha \) [95.4% C.L.] | <1.49 | <0.19 | <0.32 |

1009\(_{MC}\) | 1.04063 ± 0.00049 | 1.04084 ± 0.00046 | 1.04095 ± 0.00043 |
| \( \Omega_m \) | 0.359 ± 0.039 | 0.313 ± 0.012 | 0.3139 ± 0.0085 |
| \( \sigma_8 \) | 0.791 ± 0.032 | 0.822 ± 0.016 | 0.816 ± 0.016 |

| Parameter | TT+lowP+H(\(z\)) | TT+lowP+SN+BAO | TT+lowP+SN+BAO+H(\(z\))+f\(_{\theta}\) |
|-----------|-----------------|----------------|----------------------------------|
| \( \Omega_m h^2 \) | 0.02226 ± 0.00023 | 0.02236 ± 0.00020 | 0.02237 ± 0.00020 |
| \( \Omega_c h^2 \) | 0.1193 ± 0.0021 | 0.1177 ± 0.0013 | 0.1177 ± 0.0013 |
| \( H_0 \) [km s\(^{-1}\) Mpc\(^{-1}\)] | 65.4 ± 1.9 | 64.44 ± 0.59 | 64.45 ± 0.61 |
| \( \tau \) | 0.080 ± 0.019 | 0.084 ± 0.017 | 0.084 ± 0.018 |
| ln(10\(^{10}\)A\(_s\)) | 3.094 ± 0.036 | 3.098 ± 0.034 | 3.098 ± 0.035 |
| \( n_s \) | 0.9666 ± 0.0061 | 0.9704 ± 0.0045 | 0.9704 ± 0.0046 |
| \( \alpha \) [95.4% C.L.] | <0.68 | <0.19 | <0.20 |

1009\(_{MC}\) | 1.04075 ± 0.00047 | 1.04093 ± 0.00043 | 1.04096 ± 0.00042 |
| \( \Omega_m \) | 0.334 ± 0.022 | 0.3094 ± 0.0069 | 0.3093 ± 0.0070 |
| \( \sigma_8 \) | 0.808 ± 0.022 | 0.820 ± 0.014 | 0.819 ± 0.015 |

| Parameter | TT+lowP+f\(_{\theta}\) | TT+lowP+BAO+f\(_{\theta}\) | TT+lowP+SN+BAO+H(\(z\))+f\(_{\theta}\) |
|-----------|-----------------|----------------|----------------------------------|
| \( \Omega_m h^2 \) | 0.02233 ± 0.00023 | 0.02239 ± 0.00020 | 0.02238 ± 0.00020 |
| \( \Omega_c h^2 \) | 0.1175 ± 0.0020 | 0.1168 ± 0.0014 | 0.1169 ± 0.0013 |
| \( H_0 \) [km s\(^{-1}\) Mpc\(^{-1}\)] | 65.6 ± 2.3 | 67.17 ± 0.83 | 67.61 ± 0.62 |
| \( \tau \) | 0.075 ± 0.019 | 0.079 ± 0.018 | 0.076 ± 0.018 |
| ln(10\(^{10}\)A\(_s\)) | 3.079 ± 0.037 | 3.084 ± 0.034 | 3.079 ± 0.035 |
| \( n_s \) | 0.9701 ± 0.0060 | 0.9721 ± 0.0048 | 0.9716 ± 0.0046 |
| \( \alpha \) [95.4% C.L.] | <0.85 | <0.33 | <0.21 |

1009\(_{MC}\) | 1.04091 ± 0.00047 | 1.04103 ± 0.00042 | 1.04101 ± 0.00042 |
| \( \Omega_m \) | 0.328 ± 0.026 | 0.3101 ± 0.0084 | 0.3062 ± 0.0069 |
| \( \sigma_8 \) | 0.792 ± 0.024 | 0.805 ± 0.015 | 0.808 ± 0.014 |
constraining the nonflat $\phi$CDM models using the Planck CMB data alone is a complicated task due to the highly degenerate and non-Gaussian likelihood distributions of $H_0$, $\Omega_b$, and $\sigma_8$, that make it difficult for the MCMC chains to converge. In this case (for only the CMB TT + lowP and TT + lowP + lensing data alone analyses), we apply a more restrictive tophat prior on the Hubble constant, 0.45 < $h$ < 1.0, to achieve convergence of the MCMC chains in a reasonable amount of time (given our computational resources).

Our results for the flat tilted $\phi$CDM model are given in Figures 2 and 3 and Tables 1 and 2. The likelihood distributions for the TT + lowP (+lensing) + SN + BAO data combination (ignoring or accounting for the CMB lensing data) are omitted in the figures since they are very similar to those for the TT + lowP (+lensing) + SN + BAO + $H(z)$ combination. The results for the flat tilted $\phi$CDM model in the TT + lowP panel of Table 1 and in the TT + lowP + lensing panel in Table 2 agree well with the corresponding entries in Table 2 of Ooba et al. (2018c), except for the 2$\sigma$ upper limit on $\alpha$ for the TT + lowP case where we find $\alpha < 1.49$ while Ooba et al. (2018c) give $\alpha < 1.1$. Ooba et al. (2018c) use CLASS (Blas et al. 2011) for computing the $C_l$’s and Monte Python (Audren et al. 2013) for the MCMC analyses, so it is comforting that both our results agree well.

Tables 1 and 2 show that, when added to the Planck anisotropy data, for the flat tilted $\phi$CDM cosmogony, the BAO distance observations are largely more constraining than the $f_{\sigma_8}$, SN, or $H(z)$ data, except for $\alpha$ when the SN measurements are more restrictive than the BAO ones and for $\sigma_8$ in the TT + lowP + lensing case when again the SN measurements are more restrictive than the BAO ones. This is very similar to the results of the XCDM analyses (Park & Ratra 2018b). As in the XCDM case, each of the four non-CMB data sets used with the CMB data provide approximately equally tight bounds on $\Omega_b h^2$, $\tau$, and $A_s$. We also note that the full combination of CMB and non-CMB data sets gives a somewhat worse constraint on the potential energy density parameter $\alpha$ than does the CMB + SN case, because the BAO, $H(z)$, and $f_{\sigma_8}$ data favor a wider range of $\alpha$ and so weaken the $\alpha$ constraint. For a similar reason, the combination of the CMB and SN + BAO + $H(z)$ data constrains $\alpha$ tighter than does the full data combination.

Next, we use the same observational data to explore and constrain the parameter space of untilted nonflat $\phi$CDM models. For these models the MCMC parameter search using only the Planck CMB data (either TT + lowP or TT + lowP + lensing).
lensing) is very slow because of the highly degenerate and non-Gaussian shape of the likelihood distributions of $H_0$, $\Omega_k$, and $\alpha$. The overall shape of the likelihood function in the three-dimensional space of these three parameters can be described as a sheet of bent paper. Thus the full likelihood distribution is not well approximated by a simple multivariate Gaussian function. In practical terms the problem is that the MCMC random walks in the parameter space that are usually determined by the square root of the covariance matrix of model parameters multiplied by a random number vector drawn from a Gaussian distribution does not properly propagate throughout the whole space but stays within a local maximum of the likelihood distribution.

The most dramatic feature of the nonflat $\phi$CDM analyses is that for this model the CMB data poorly constrains $\alpha$ and is also consistent with large values of $\alpha$, unlike in the spatially flat $\phi$CDM model. This phenomenon can be understood more easily by comparing CMB data parameter constraints for the nonflat $\phi$CDM and XCDM models. Figure 4 shows several examples of the $\phi$CDM model with large $\alpha$’s that are consistent with the Planck CMB observations. Here the parameters of the nonflat $\phi$CDM models were chosen from the unconverged MCMC chains and are consistent with Planck observations. Bottom: theoretical predictions for matter density and CMB temperature anisotropy angular power spectra for the $\phi$CDM models.

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14 $\alpha$ governs the dynamics of dark energy and when spatial curvature is allowed to vary, this weakens the constraints on dark energy dynamics from any data set (see, e.g., Farooq et al. 2017, and references therein).

15 For example, for the nonflat $\phi$CDM model with $\alpha \approx 10$, which corresponds to the red curves in Figure 4, the parameters are $\Omega_\Lambda h^2 = 0.029846$, $\Omega_\phi h^2 = 0.10990$, $\Omega_\nu h^2 = 6.4514 \times 10^{-4}$, $\Omega_Y = -0.2023$, $\alpha = 9.795$, $\tau = 0.1123$, and $A_s = 2.2921 \times 10^{-9}$ (at $k_0 = 0.05$ Mpc$^{-1}$).
untitled nonflat XCDM parameterizations with $w \approx -0.15$ are consistent with Planck TT + lowP data for small Hubble constant and highly negative $\Omega_k$. A similar thing happens in the nonflat $\phi$CDM model for large values of $\alpha$, and $\phi$CDM models with very large $\alpha$ around 10 still have CMB power spectra that are consistent with the $\Lambda$CDM prediction (black curve in Figure 4) within observational precision. However, as shown in the matter power spectra panel of Figure 4, these $\phi$CDM models are excluded by large-scale structure observations.

Figure 5 (top left panel) shows the likelihood distributions of the untitled nonflat $\phi$CDM model parameters constrained by using the Planck TT + lowP (red dots) and TT + lowP + lensing (blue dots) data. These are approximate estimates based on several sets of unconverged MCMC chains. For comparison, we present results for the untitled nonflat XCDM parameterization parameters in the top right panel. Here we use MCMC chain elements computed for dark energy equation of state parameter $w > -1$ to derive likelihood contours of nonflat XCDM model parameters. Thus the resulting likelihood distributions differ from those obtained from the full MCMC outputs presented in Park & Ratra (2018b). The bottom panels show three-dimensional views of some selected untitled nonflat $\phi$CDM model MCMC chain elements with $\Delta \chi^2 < 18$ relative to the minimum value (see Table 5 below), constrained using Planck TT + lowP (red dots) and TT + lowP + lensing (blue dots) data. The five green dots for the TT + lowP data indicate the position of the five untitled nonflat $\phi$CDM models presented in Figure 4.
the CMB data alone analyses for the more restrictive Hubble constant prior are shown in Figure 6. Note that for the Planck TT + lowP data the constraint on $\alpha$ seems tighter than for the case of the TT + lowP + lensing data. This is because the region of large $\alpha$ but small Hubble constant favored by TT + lowP data is excluded by the more restrictive Hubble constant prior.

Our results for the nonflat untilted $\phi$CDM model are presented in Figures 7 and 8 and Tables 3 and 4. As in the flat tilted $\phi$CDM models, the likelihood distributions for TT + lowP (+lensing) + SN + BAO data (ignoring or accounting for the CMB lensing data) are omitted in the figures since they are very similar to those for TT + lowP (+lensing) + SN + BAO + $H(z)$ data.

The entries for the nonflat untilted $\phi$CDM model in Table 3 (TT + lowP panel) and those in Table 4 (TT + lowP + lensing panel) are very consistent with the corresponding entries in Table 1 of Ooba et al. (2018b). Ooba et al. (2018b) computed $C_\ell$ by using CLASS (Blas et al. 2011) and used the Monte Python software package (Audren et al. 2013) for the MCMC analyses; it is reassuring that both sets of results agree well.

The parameter constraints are more interesting in the nonflat untilted $\phi$CDM model than in the flat tilted case. The general behavior of the cosmological parameter constraints are similar to those in the XCDM model (Park & Ratra 2018b). When CMB lensing data are accounted for, Table 4, Planck CMB data with either $H(z)$, BAO, SN, or $f\sigma_8$ data, provide roughly equally tight constraints on $\Omega_b h^2$, $\Omega_c h^2$, and $\theta_{MC}$, while CMB + BAO measurements provide the tightest limits on $H_0$, $\tau$, $A_s$, $\Omega_k$, $\Omega_m$, and $\sigma_8$, with CMB + SN setting the tightest limits on $\alpha$. We note that the full combination of CMB and non-CMB data results in somewhat weaker constraints on $\alpha$ (compared to the CMB + SN case) and on $\Omega_b h^2$ (compared to the CMB + $f\sigma_8$ case).

Let us focus on the results for CMB TT + lowP + lensing data, presented in Figures 3 and 8 and Tables 2 and 4, where we see that adding in turn each of the four sets of non-CMB measurements to the CMB measurements (left panels in the two figures) results in likelihood contours that are quite compatible with each other, as well as with the CMB alone contours, for both the flat tilted and the nonflat untilted $\phi$CDM model. It might be significant that the four sets of non-CMB observations do not pull the CMB only contours in very different directions. This is also true for the flat tilted $\phi$CDM model when CMB lensing data are ignored (left panel of Figure 2). However, in the nonflat untilted $\phi$CDM cosmogony without the lensing data each of the four sets of non-CMB data

Figure 6. Likelihood distributions of the untilted nonflat $\phi$CDM model parameters for the Planck CMB TT + lowP and TT + lowP + lensing data. Here we have used the more restrictive Hubble constant prior, $h \geq 0.45$. The MCMC processes have statistically converged.
added to the CMB data (left panel of Figure 7) push the results toward a smaller $|\Omega_k|$ (closer to flat space) and larger $H_0$ as well as slightly larger $\tau$ and $A_s$ and slightly smaller $\Omega_bh^2$ than is preferred by the CMB data alone, but all five constraint contour sets are largely mutually compatible, except for the $H_0$ and $\Omega_k$ constraints where the TT + lowP data alone results differ from those derived using TT + lowP in combination with any one of the four non-CMB data sets.

Although augmenting the CMB data with the BAO data typically results in the largest difference, each of the other three sets of non-CMB data also contribute. Considering the TT + lowP + BAO data, we see from Table 2 for the flat tilted...
Table 3

| Parameter | TT+lowP (h ≥ 0.45) | TT+lowP+SN | TT+lowP+BAO |
|-----------|--------------------|-------------|-------------|
| Ω_ch^2   | 0.02329 ± 0.00021 | 0.02313 ± 0.00020 | 0.02306 ± 0.00020 |
| Ω_bh^2   | 0.1095 ± 0.0011  | 0.1094 ± 0.0011  | 0.1096 ± 0.0011  |
| H_0 [km s^{-1} Mpc^{-1}] | 48.2 ± 2.5 | 64.2 ± 2.3 | 66.68 ± 0.91 |
| τ       | 0.108 ± 0.021  | 0.132 ± 0.018  | 0.138 ± 0.016  |
| ln(10^{10}A_s) | 3.126 ± 0.042 | 3.174 ± 0.036 | 3.185 ± 0.033 |
| Ω_k     | -0.074 ± 0.016 | -0.0162 ± 0.0064 | -0.0067 ± 0.0023 |
| α [95.4% C.L.] | <1.81 | <0.20 | <0.46 |

**1008MC**

|      | TT+lowP+H(z) | TT+lowP+SN+BAO | TT+lowP+SN+BAO+H(z) |
|------|-------------|----------------|----------------------|
| Ω_m  | 0.573 ± 0.057 | 0.324 ± 0.023 | 0.2999 ± 0.0086 |
| σ_s  | 0.733 ± 0.026 | 0.819 ± 0.017 | 0.815 ± 0.017 |

| Parameter | TT+lowP+σ_s | TT+lowP+BAO+σ_s | TT+lowP+SN+BAO+H(z)+σ_s |
|-----------|-------------|----------------|--------------------------|
| Ω_ch^2   | 0.02309 ± 0.00021 | 0.02306 ± 0.00019 | 0.02307 ± 0.00019 |
| Ω_bh^2   | 0.1098 ± 0.0011  | 0.1096 ± 0.0011  | 0.1097 ± 0.0011  |
| H_0 [km s^{-1} Mpc^{-1}] | 65.3 ± 2.2 | 67.16 ± 0.70 | 67.24 ± 0.72 |
| τ       | 0.136 ± 0.017  | 0.134 ± 0.016  | 0.136 ± 0.016  |
| ln(10^{10}A_s) | 3.182 ± 0.035 | 3.178 ± 0.033 | 3.182 ± 0.033 |
| Ω_k     | -0.0100 ± 0.0052 | -0.0073 ± 0.0020 | -0.0068 ± 0.0019 |
| α [95.4% C.L.] | <0.60 | <0.28 | <0.29 |

**1008MC**

|      | TT+lowP+σ_s | TT+lowP+BAO+σ_s | TT+lowP+SN+BAO+H(z)+σ_s |
|------|-------------|----------------|--------------------------|
| Ω_m  | 0.314 ± 0.022 | 0.2957 ± 0.0063 | 0.2952 ± 0.0066 |
| σ_s  | 0.813 ± 0.022 | 0.819 ± 0.016  | 0.820 ± 0.016  |

| Parameter | TT+lowP+σ_s | TT+lowP+BAO+σ_s | TT+lowP+SN+BAO+H(z)+σ_s |
|-----------|-------------|----------------|--------------------------|
| Ω_ch^2   | 0.02309 ± 0.00021 | 0.02304 ± 0.00020 | 0.02307 ± 0.00020 |
| Ω_bh^2   | 0.1091 ± 0.0010  | 0.1093 ± 0.0011  | 0.1092 ± 0.0011  |
| H_0 [km s^{-1} Mpc^{-1}] | 63.8 ± 4.0 | 66.91 ± 0.91 | 67.37 ± 0.71 |
| τ       | 0.122 ± 0.019  | 0.128 ± 0.017  | 0.126 ± 0.016  |
| ln(10^{10}A_s) | 3.152 ± 0.039 | 3.164 ± 0.033 | 3.161 ± 0.033 |
| Ω_k     | -0.0136 ± 0.0090 | -0.0063 ± 0.0023 | -0.0065 ± 0.0020 |
| α [95.4% C.L.] | <0.86 | <0.46 | <0.30 |

**1008MC**

|      | TT+lowP+σ_s | TT+lowP+BAO+σ_s | TT+lowP+SN+BAO+H(z)+σ_s |
|------|-------------|----------------|--------------------------|
| Ω_m  | 0.330 ± 0.042 | 0.2972 ± 0.0084 | 0.2930 ± 0.0062 |
| σ_s  | 0.789 ± 0.028 | 0.805 ± 0.016  | 0.809 ± 0.015  |

ϕCDM model that the H_0 error bar is reduced the most by the full compilation of measurements relative to the CMB + BAO observations compilation, followed by the Omega error bar decrease compared to the CMB + BAO data collection. For the nonflat tilted ϕCDM model, from Table 4, the error bars that reduce the most when CMB (accounting for lensing) data are used in combination with the four sets of non-CMB data are those on Ω_m and H_0 (relative to the CMB + BAO combination).

Focusing again on the TT + lowP + lensing data, Tables 2 and 4, for the flat tilted ϕCDM model, we see that augmenting the CMB data with the four non-CMB data sets most affects H_0, Ω_m, Ω_ch^2, and Ω_k, with the H_0 and σ_s central values moving up by 1.4σ and 0.98σ and the Ω_m and Ω_ch^2 central values moving down by 1.3σ and 1.1σ, all of the CMB data only error bars; ln(10^{10}A_s) is not much affected by including the four non-CMB sets of data, changing by only 0.065σ. The situation for the nonflat untilted ϕCDM model is a little more dramatic, with H_0 and σ_s central values moving up by 1.9σ and 1.7σ of the CMB data only error bars, Ω_m decreasing by 1.6σ, and Ω_k more closely approaching flatness by 1.5σ; in this case the Ω_ch^2 central value is not affected.

Figure 9 shows marginalized likelihood contours in the Ω_m–σ plane for the flat tilted ϕCDM model and in α–Ω_k plane for the nonflat untilted ϕCDM case. For CMB TT + lowP + lensing measurements combined with the non-CMB observations, the flat ϕCDM model prefers α = 0, favoring the cosmological constant over dynamical dark energy. However, the nonflat ϕCDM model, when constrained using all the data, prefers closed spatial hypersurfaces and also mildly prefers dynamical dark energy with scalar field potential energy density parameter α > 0. Estimating 68.3% and 95.4% confidence limits of α using the information on the right-hand side with respect to the peak value (mode) in the marginalized one-dimensional likelihood distribution, the mode ±1σ (2σ) values for α are 0.113 ± 0.094 (0.19).

More precisely, including the four sets of non-CMB data, we discover in the tilted flat ϕCDM model (bottom right panel of Table 2) that α < 0.22 (at 2σ), which is more tightly restricted to α = 0 and a cosmological constant than is the original
Ooba et al. (2018c) finding of \( \alpha < 0.28 \) (at 2\( \sigma \), the last column of their Table 2).\(^{17}\)

However, perhaps the most interesting consequence of adding in the four non-CMB data sets is the significant improvement of the evidence for nonflatness in the nonflat untitled \( \phi \)CDM model, with it increasing to \( \Omega_k = -0.0063 \pm 0.0020 \), a more than 3\( \sigma \) deviation from flatness now, for the total data compilation in the bottom right panel of Table 4, compared to the 1.8\( \sigma \) away from flatness for the CMB only case. This is now accompanied by very mild evidence favoring dynamical dark energy, see the right panel of Figure 9. This result is compatible with and strengthens that of Ooba et al. (2017) who found \( \Omega_k = -0.006 \pm 0.003 \) from Planck CMB data combined with a few BAO measurements. In favoring a closed geometry, the BAO measurements are the most important of the four non-CMB data sets.

From the total data combination (also accounting for CMB lensing data) in Tables 2 and 4, \( H_0 \) measured using the flat tilted and the nonflat untitled \( \phi \)CDM models, 67.63 \pm 0.62 and 67.36 \pm 0.72 km s\(^{-1}\) Mpc\(^{-1}\), are quite consistent with each other to within 0.28\( \sigma \) (of the quadrature sum of both the error bars). These values agrees with the median statistics measurement \( H_0 = 68 \pm 2.8 \) km s\(^{-1}\) Mpc\(^{-1}\) (Chen & Ratra 2011a), which agrees with earlier median statistics estimates (Gott et al. 2001; Chen et al. 2003). They are also compatible with many recent estimates of \( H_0 \) (Luković et al. 2016; Chen et al. 2017; DES Collaboration et al. 2018b; L’Huillier & Shafieloo 2017; Lin & Ishak 2017; Wang et al. 2017; da Silva & Cavalcanti 2018; Gómez-Valent & Amendola 2018; Haridasu et al. 2018a, 2018b; Yu et al. 2018; Zhang 2018; Zhang et al. 2018), but, as is well known, they are lower than the local expansion rate estimate of \( H_0 = 73.48 \pm 1.66 \) km s\(^{-1}\) Mpc\(^{-1}\) (Riess et al. 2018).\(^{18}\)

We find that \( H_0 \) and \( \sigma_8 \) (see discussion below) are the only measured parameters whose values are almost independent of

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\(^{17}\) These results do not agree with those of earlier approximate analyses that used less reliable, and less, data, and indicated evidence for \( \alpha \) deviating from 0 by more than 2\( \sigma \) (Solà et al. 2018a, 2017b).

\(^{18}\) This local expansion rate estimate is 3.3\( \sigma \) (3.4\( \sigma \)), of the quadrature sum of both the error bars, larger than the \( H_0 \) value measured here using the flat tilted \( \phi \)CDM (nonflat untilded \( \phi \)CDM) model. However, some other local expansion rate determinations finds somewhat lower \( H_0 \)’s with somewhat larger error bars (Rigault et al. 2015; Dhawan et al. 2017; Zhang et al. 2017a; Fernández Arenas et al. 2018); for related discussions see Roman et al. (2018), Kim et al. (2018), and Jones et al. (2018).
the cosmological model (spatial curvature and tilt) used in the analysis. Measurements of other parameters determined by using the two \(\phi\)CDM models differ more significantly. Specifically, measurements determined using the total data (also including CMB lensing) of \(\Omega_m\), \(\theta_{MC}\), \(\ln(10^{10}A_s)\), \(\Omega_b h^2\), \(\tau\), and \(\Omega_c h^2\), differ by 1.4\(\sigma\), 1.9\(\sigma\), 2.3\(\sigma\), 2.6\(\sigma\), and 4.6\(\sigma\) (of the quadrature sum of both the error bars). For some parameters, especially \(\Omega_b h^2\) as well as possibly \(\tau\), \(\Omega_c h^2\), and \(A_s\), the model dependence of the value results in a much larger uncertainty than that due to the statistical uncertainty in the given cosmological model. This was first detected when comparing measurements made using the flat tilted \(\Lambda\)CDM and the nonflat untitled \(\Lambda\)CDM model (Park & Ratra 2018a) and is also present in the XCDM case (Park & Ratra 2018b). From Tables 2 and 4, for the total data collection (also including CMB lensing), we find in the tilted flat \(\phi\)CDM (untitled nonflat \(\phi\)CDM) case 0.046 \(\leq\) \(\tau\) \(\leq\) 0.102 (0.098 \(\leq\) \(\tau\) \(\leq\) 0.146) and 0.02198 \(\leq\) \(\Omega_c h^2\) \(\leq\) 0.02278 (0.02264 \(\leq\) \(\Omega_c h^2\) \(\leq\) 0.02344) at 2\(\sigma\), which are almost disjoint, and 0.1142 \(\leq\) \(\Omega_b h^2\) \(\leq\) 0.1194 (0.1073 \(\leq\) \(\Omega_b h^2\) \(\leq\) 0.1113), which are completely separated from each other. Current cosmological data cannot be used to measure \(\Omega_b h^2\) or \(\tau\) (and possibly some of the other cosmological parameters also) in a model independent way.

From the total data combination (also including CMB lensing data), \(\sigma_8\), measured using the two \(\phi\)CDM models, Tables 2 and 4, agree to 0.034\(\sigma\) (of the quadrature sum of both the error bars). Figures 10 and 11 show the marginalized two-dimensional \(\Omega_m-\sigma_8\) likelihood contours for the flat tilted and nonflat untitled \(\phi\)CDM models constrained using the CMB and non-CMB data. In each panel, we also show the \(\Lambda\)CDM model constraints from a combined analysis of the first year galaxy clustering and weak lensing data of the Dark Energy Survey (DES Y1 All) (DES Collaboration et al. 2018a), whose 1\(\sigma\) confidence limits are \(\Omega_m = 0.264^{+0.032}_{-0.019}\) and \(\sigma_8 = 0.807^{+0.062}_{-0.041}\). The marginalized likelihood contours in the \(\Omega_m-\sigma_8\) plane derived by adding each of the four sets of non-CMB data in turn to the CMB data are consistent with each other. Here the BAO data provide the most stringent constraints among the four sets of non-CMB data.

Although the \(\sigma_8\) constraints from the flat tilted and nonflat untitled \(\phi\)CDM analyses (ignoring and accounting for CMB lensing data) are consistent with the DES Y1 All measurements, the \(\Omega_m\) bounds here prefer a larger value by about 1.3\(\sigma\) (of the quadrature sum of both the error bars) for the flat tilted \(\phi\)CDM case for the total data collection. We note that the best-fit point of the nonflat untitled \(\phi\)CDM model constrained by using the CMB data (also including lensing) combined with all non-CMB measurements enters inside the 1\(\sigma\) (68.3%) region of the DES Y1 All likelihood distribution (lower right panel of Figure 11), unlike the case for the flat tilted \(\phi\)CDM model (Figure 10 lower right panel).

Table 5 lists \(\chi^2\)'s for the best-fit flat tilted and nonflat untitled \(\phi\)CDM models. The best-fit position in parameter space is found using Powell’s minimization method, an efficient algorithm to locate the \(\chi^2\) minimum. We list the \(\chi^2\) contribution of each data set. The total \(\chi^2\) is the sum of the individual ones from the high-\(\ell\) CMB TT likelihood (\(\chi^2_{\text{Planck,TT}}\), the low-\(\ell\) CMB power spectra of temperature and polarization (\(\chi^2_{\text{lowTEB, lensing}}\), lensing (\(\chi^2_{\text{lensing}}\), SN (\(\chi^2_{\text{SN}}\), BAO (\(\chi^2_{\text{BAO}}\), \(f(z)\) (\(\chi^2_{f\Delta z}\), \(H(z)\) data (\(\chi^2_{H(z)}\), and from the foreground nuisance parameters (\(\chi^2_{\text{prize}}\). As a result of the nonstandard normalization of the Planck data likelihoods, the number of CMB degrees of freedom is ambiguous. Thus, the absolute value of \(\chi^2\) for the Planck CMB data is arbitrary, and only the relative difference between \(\chi^2\) of one model and another is meaningful.
for the Planck data. For the non-CMB data, the degrees of freedom are 10, 15, 31, and 1042\(^{19}\) for the \(f_{\sigma_8}\), BAO, \(H(z)\), and SN observations, respectively, resulting in 1098 degrees of freedom all together. The reduced \(\chi^2\) for the individual non-CMB data sets are \(\chi^2/\nu \approx 1\). There are 189 points in the Planck TT + lowP (binned) CMB data anisotropy angular power spectrum and 197 points when the CMB lensing measurements are included.

In the last column of Table 5, we list \(\Delta \chi^2\), the excess \(\chi^2\) of the best-fit seven parameter \(\phi\)CDM model relative to the \(\chi^2\) of the related six parameter \(\Lambda\)CDM model that is constrained by using the same data combination. The minimum \(\chi^2\) values for the \(\Lambda\)CDM and XCDM models are presented in Tables 7 and 8 of Park & Ratra (2018b). These models are nested; the seven parameter flat tilted \(\phi\)CDM (nonflat untitled \(\phi\)CDM) model reduces to the six parameter flat tilted \(\Lambda\)CDM (nonflat untitled \(\Lambda\)CDM) model when \(\alpha\) goes to zero.\(^{20}\) Here the ambiguity in the number of Planck CMB data degrees of freedom is no longer an obstacle to converting the \(\Delta \chi^2\) to a relative goodness of fit probability. From \(\sqrt{\Delta \chi^2}\), for the complete data (accounting for CMB lensing), for a single additional free

\(^{19}\) This is the number of degrees of freedom for the flat \(\Lambda\)CDM model, given by the number of data points (1048) minus the number of parameters such as the matter density (\(\Omega_m\)) and the five internal nuisance parameters.

\(^{20}\) This is also true of the XCDM parameterization when the equation of state parameter \(w\) goes to \(-1\).
parameter, we find that the flat tilted $\phi$CDM (nonflat untilted $\phi$CDM) model is a 0.40$\sigma$ (0.93$\sigma$) better fit to the data than is the flat tilted $\Lambda$CDM (nonflat untilted $\Lambda$CDM) model. These findings are compatible with those of Ooba et al. (2018c) and Ooba et al. (2017).

Of all three flat cases, tilted flat $\phi$CDM best fits the combined data (although there is no significant difference between all three cases), but at a lower level of significance than the 1.3$\sigma$ found by Ooba et al. (2018c) using a very small sample of non-CMB data compared to what we have used here, and far from the 3$\sigma$ or 4$\sigma$ result found in earlier approximate analyses by Solà et al. (2018a, 2017b). While the tilted flat $\phi$CDM and XCDM cases do not provide a much better fit to the data, available data allow for the possibility that dark energy is dynamical.

It is clear that relative to the flat models, in terms of $\Delta \chi^2$ values, the nonflat models do a worse job of fitting the higher-$\ell$ $C_\ell$'s than they do at fitting the lower-$\ell$ $C_\ell$'s. However, the models are not nested so it is not possible to turn these differences into relative goodness of fit probabilities (as the number of degrees of freedom of the Planck 2015 data is ambiguous). We note that there have been studies on systematic differences between constraints determined from the higher-$\ell$ and the lower-$\ell$ Planck 2015 data (Addison et al. 2016; Planck Collaboration et al. 2017). Also, in the tilted flat $\Lambda$CDM model, there seem

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21 XCDM does not do as well as $\phi$CDM, with the flat tilted XCDM (nonflat untilted XCDM) parameterization being a 0.28$\sigma$ (0.87$\sigma$) better fit to the data than the flat tilted $\Lambda$CDM (nonflat untilted XCDM) model (Park & Ratra 2018b). We emphasize that nonflat untilted $\Lambda$CDM does not fit as well as flat tilted $\Lambda$CDM, although as discussed in Ooba et al. (2017, 2018a, 2018b) and Park & Ratra (2018a, 2018b), it is not known how to transform this into a relative probability because the Planck 2015 CMB data number of degrees of freedom is unavailable and the two six parameter models are not nested.
to be inconsistencies between the higher-ℓ Planck and the South Pole Telescope CMB data (Aylor et al. 2017). Possibly, if these differences are real, when they are resolved this could result in a decrease of the Δχ² found here.

Figures 12 and 13 plot the CMB high-ℓ TT, and the low-ℓ TT, TE, and EE power spectra of the best-fit flat tilted and nonflat tilted ΩCDM dynamical dark energy inflation models, ignoring and accounting for the lensing data, respectively. The best-fit flat tilted ΩCDM models favored by the CMB and non-CMB data are in good agreement with the observed CMB power spectra at all ℓ (this is also the case for the best-fit flat tilted XCDM parameterization, Park & Ratra 2018b). However, similar to the nonflat ΩCDM and XCDM cases studied in Park & Ratra (2018a, 2018b), the nonflat tilted ΩCDM model constrained with the Planck 2015 CMB anisotropy data and each non-CMB data set generally provides a poorer fit to the low-ℓ EE power spectrum while it provides a better fit to the low-ℓ TT power spectrum (see the bottom left panel of Figures 12 and 13). The best-fit C₇ model power spectra shapes relative to the Planck CMB data points are compatible with the χ² values listed in Table 5. For example, the best-fit tilted nonflat ΩCDM model constrained by using the TT + lowP + H(2) data has an EE power spectrum shape at low-ℓ that is most deviant from the Planck data and a corresponding value of χ² |lowTT,TEB | < 10500.10 that is larger by 3.69 relative to the best-fit flat tilted ΩCDM model χ² |lowTT,TEB | 10496.41 for the TT + lowP data (see Table 5 here and Table 7 of Park & Ratra 2018b).

Figure 14 shows the best-fit primordial power spectra of fractional energy density spatial inhomogeneity perturbations for the nonflat tilted ΩCDM model constrained by using the Planck TT + lowP (left) and TT + lowP + lensing (right panel) data together with the other non-CMB data sets. The low q region reduction in power in the best-fit closed tilted ΩCDM inflation model power spectra in Figure 14 contributes to the TT power

| Data Sets | χ² | χ² | χ² | χ² | χ² | χ² | Total χ² | Δχ² |
|-----------|----|----|----|----|----|----|--------|----|
| Tilted flat ΩCDM model | 766.02 | 10496.47 | ... | ... | ... | ... | 0.24 | 11262.25 | +0.32 |
| +SN | 763.68 | 10496.29 | 1036.31 | 2.01 | 12298.30 | -0.01 |
| +BAO | 764.40 | 10495.97 | 12.85 | 1.95 | 11275.18 | -0.07 |
| +H(2) | 763.35 | 10496.70 | 14.93 | 1.88 | 11276.86 | -0.07 |
| +f₀s | 766.75 | 10494.91 | 12.13 | 1.96 | 11275.75 | -0.05 |
| +BAO+f₀s | 767.30 | 10494.94 | 12.04 | 1.99 | 11288.25 | -0.25 |
| +SN+BAO | 764.23 | 10496.03 | 1036.12 | 2.03 | 12311.36 | -0.05 |
| +SN+BAO+H(2) | 763.23 | 10494.04 | 1036.16 | 2.01 | 12326.21 | +0.00 |
| +SN+BAO+H(2)+f₀s | 766.85 | 10494.80 | 1036.07 | 2.11 | 12339.24 | -0.12 |

Note. Δχ² of tilted flat or untitled nonflat ΩCDM model represents the excess value relative to χ² of the tilted flat or untitled nonflat ΩCDM model estimated for the same combination of data sets (listed in Table 7 of Park & Ratra 2018b).
Figure 12. Best-fit CMB anisotropy angular power spectra of (a) flat tilted (top five panels) and (b) nonflat untilted $\phi$CDM models (bottom five panels) constrained by using the Planck 2015 CMB TT + lowP data (ignoring the lensing data) in conjunction with BAO, $H(z)$, SN, and $f_{\sigma 8}$ data. For comparison, the best-fit angular power spectra of the flat tilted $\Lambda$CDM model are shown as black curves. $\delta D_l$ residuals for the TT power spectra are shown with respect to the flat tilted $\Lambda$CDM power spectrum that best fits the TT + lowP data.
Figure 13. Same as Figure 12 but now accounting for the CMB lensing data. $\delta D_\ell$ residuals of the TT power spectra are shown with respect to the flat tilted $\Lambda$CDM power spectrum that best fits the TT + lowP + lensing data.
reduction at low-$\ell$ of the best-fit closed untilted model $C_\ell$ (see Figures 12 and 13 lower panels) relative to the best-fit flat tilted model $C_\ell$.\footnote{The usual and integrated Sachs–Wolfe effects, as well as other effects, also play a role in determining the shape of the low-$\ell$ $C_\ell$.} The most dramatic case is that of the best-fit untilted nonflat $\phi$CDM model for the TT + lowP data, consistent with the low-$\ell$ TT power reduction (Figures 12(b)).\footnote{Figure 24 (bottom-right panel) of Planck Collaboration et al. (2018) shows the primordial power spectrum derived from the Planck CMB data. (We note that their Figure 24 has been derived under the assumption that space is flat, and consequently ignores the effect of the spatial curvature Sachs–Wolfe and Integrated Sachs–Wolfe effects on the CMB power spectra that were used in the derivation of this figure.) This power spectrum is a power law over wavenumbers in the interval $5 \times 10^{-4} \lesssim k \ [\text{Mpc}^{-1}] \lesssim 2 \times 10^{-1}$ but at smaller wavenumbers their power spectrum amplitude errors are much larger and the Planck power spectrum is not inconsistent with our closed model power spectra plotted in Figure 14.}

5. Conclusion

We have used the flat tilted and the nonflat untilted $\phi$CDM dynamical dark energy inflation models to measure cosmological parameters from a reliable, large compilation of observational data.

Our main findings, in summary, are:

1. We confirm, but at a lower significance of $0.40\sigma$ than the result of Ooba et al. (2018c), that the flat tilted $\phi$CDM model better fits the data than the standard flat tilted $\Lambda$CDM model. While the improvement is not significant, it does mean that current data allow for the possibility that dark energy is dynamical.

2. In the nonflat untilted $\phi$CDM case, we confirm, with greater significance, the Ooba et al. (2018b) result that cosmological data does not require flat spatial hypersurfaces for this model, and that the nonflat untilted $\phi$CDM model better fits (at $0.93\sigma$) the data than does the nonflat untilted $\Lambda$CDM model (qualitatively the standard flat tilted $\Lambda$CDM model provides a better fit to the data than does the nonflat untilted $\Lambda$CDM model). In the nonflat untilted $\phi$CDM model, these data (including CMB lensing data) favor a closed model at more than $3.1\sigma$ significance, in which spatial curvature contributes a little less than two-thirds of a percent of the cosmological energy budget now.

3. $H_0$ is measured here in a manner that is almost model independent and is consistent with many other $H_0$ measurements. However, as is well known that an estimate of $H_0$ from the local expansion rate (Riess et al. 2018) is about $3.3\sigma$ larger.

4. $\sigma_8$ here is measured in an almost model independent manner and is consistent with the recent DES estimate (DES Collaboration et al. 2018a).

5. The value of $\Omega_{m}$ is more model dependent than the value of $\sigma_8$ and the $\Omega_{m}$ value measured using the nonflat untilted $\phi$CDM model is more consistent with the recent DES estimate (DES Collaboration et al. 2018a).

6. $\Omega_{b}h^2$, $\tau$, and a few of the other cosmological parameter values are quite model dependent.

These results are very similar to those for the XCDM dynamical dark energy parameterization presented in Park & Ratra (2018b).

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