The impact of a random metric upon a diffusing particle

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October 10, 2024

Abstract

We show that if the singularity of the covariance of the random metric is \(|x|^{-2\gamma}\) then the mean value of the fourth power of the distance achieved in time \(t\) by a diffusing particle behaves as \(t^{2(1-\gamma)}\) for a small \(t\).

1 Introduction

An interest in a random diffusion appears in a theory of complex systems as well as in quantum gravity. In the first case one considers fluctuating diffusivity (see [1][2][3]) or random walks with modified jump probabilities (an overview in ref. [4]). The random metric in quantum field theories is studied in an attempt to construct a quantum field theory of all interactions. There is an old suggestion associated with the names of Landau and Pauli that the quantization of gravity can change the short distance behaviour of quantum field propagators allowing to ease the quantum field theory ultraviolet problems (for some recent investigations, see [5][6]). The propagator is defined as the kernel of the inverse of the second order differential operator. In the proper time representation the propagator is expressed by the heat kernel (as an integral over time). Then, a change of the short time behaviour of the heat kernel in a quantized metric leads to a modification of the short distance behaviour of the propagators. The problem has been studied numerically in causal triangulation approximation to quantum gravity [7][8][9], in Horava’s gravity at Lifshitz point [10][11] and in Liouville gravity [12]. These studies support the suggestion that quantum field theory in a quantized metric behaves like a field theory in lower dimensions (for recent reviews see [13][14] and references cited there). In a diffusion theory a random metric in the generator of diffusion can be interpreted as a random diffusion matrix. One can consider the additional randomness as a result of a diffusion on irregular (e.g., fractal) structures [13][15][16][17]. It is known that during such a diffusion the short time behaviour of the transition function is changing. In particular, the mean distance achieved by a diffusing particle at
small time is increasing faster than in the regular case. On the basis of this time behaviour one can define the random walker dimension which is related to the Hausdorff dimension and to the spectral dimension of the given fractal structure \cite{15,13}. In relation to quantum gravity it has been suggested already by J.A. Wheeler (see a discussion in \cite{14,18}) that a "foamy" structure of space-time at short distances can lead to an anomalous (more regular) behaviour of matter field propagators at short distances. As the field propagator is an integral over the time of the heat kernel the modified short time behaviour of the heat kernel implies a change of the short distance behaviour of the propagator. In particular, the results on a diffusion on fractals \cite{17} imply that the field propagators on fractal spaces are more regular.

In this paper we discuss a space-dependent random diffusion matrix mimicking the random metric of quantum (Euclidean) gravity. As long as the random metric has a regular covariance there is no effect upon the mean distance of a diffusing particle achieved for a small time $t$. We show that the mean fourth power of the distance achieved by a diffusing particle behaves as $t^{2(1-\gamma)}$ if the covariance of the metric has the singularity $|x|^{-2\gamma}$ (this short time behaviour would be $t^2$ for a diffusion on regular structures; we can treat rigorously the singularity with $0 \leq \gamma < \frac{1}{2}$). A transition function for a random singular metric requires a renormalization. After the renormalization the small time behaviour is changed as can already be seen by a scaling argument. We define the metric as a square of a tetrad (vierbein) in order to achieve positivity of the regular metric. Then, we consider the tetrad as a Gaussian process with a singular covariance. We define the singular metric as a Wick square of the tetrad. In such a case a renormalization of the transition function is needed when we go from a regular to a singular metric. This renormalization is the origin of the modified short time behaviour. The final effect is the same as the diffusion on the fractal structures \cite{17,15,13} indirectly confirming that a manifold equipped with a singular random metric may play the same role as a fractal space-time.

In \cite{19} and \cite{20} we have discussed a modified small time and short distance behaviour by means of a scaling transformation in the functional integral. The argument using the scaling transformation does not take into account that a definition of correlations of quantum fields requires normal ordering of fields and coupling constants renormalization. The eventual scaling should be used in conjunction with the renormalization group methods to establish the scale invariance at the renormalization group fixed point. In this paper we show by means of explicit calculations that the singular metric can indeed change the small time behaviour of the diffusing particle.

2 Diffusion in a random metric

In Euclidean quantum field theory correlation functions of the Euclidean field $\phi$ defined on a multidimensional Riemannian manifold (of $D = n + d$ dimensions)
are determined by a formal Gaussian measure

\[ d\phi \exp(-\frac{\sigma^2}{2} \int d^D \xi \mathcal{L}), \]

with

\[ \mathcal{L} = \sqrt{g} g^{AB} \partial^A \phi \partial^B \phi, \]

where \( g_{AB} \) is the Riemannian metric, \( g = \det g_{AB} \) and \( \partial^A = \frac{\partial}{\partial \xi^A} \). The two-point correlation function (the propagator) is the kernel of an inverse of the Laplace-Beltrami operator \( \tilde{A} \)

\[ \tilde{A} = \frac{\sigma^2}{2} \left( g^{AB} \frac{\partial^2}{\partial \xi^A \partial \xi^B} + \Gamma^A \frac{\partial}{\partial \xi^A} \right) \]

where \( \Gamma^A \) is the Christoffel symbol (expressed by derivatives of the metric) and we introduced a diffusion constant \( \sigma^2 \). In a generator of a diffusion the metric plays the role of the space-dependent diffusion matrix and \( \Gamma \) is a drift. In quantum gravity we still need to average the scalar field correlations with respect to a measure over the metric. In a perturbative expansion the first approximation to this measure is a Gaussian one with the covariance \((-\triangle)^{-1}\) where \( \triangle \) is the Laplacian on \( \mathbb{R}^{n+d} \).

We consider a simplified version of the Laplace-Beltrami operator \( \tilde{A} \) on \( \mathbb{R}^{n+d} \) with coordinates \( \xi = (x, X) \) where \( x \in \mathbb{R}^n \) and \( X \in \mathbb{R}^d \). We assume that the metric \( g^{AB} = (\delta^{ij}, g^{\mu\nu}(x)) \) has a block form. It is flat on a submanifold described by the coordinates \( x \) and depends only on \( x \) when restricted to the (physical) submanifold described by the coordinates \( X \). A partial justification of such a choice could come from Horava gravity \([10][11]\) (motivated by condensed state theory) where the \( x \) and \( X \) coordinates can scale in a different way so that finally the \( X \) dependence of the metric is negligible. Under this assumption the Laplace-Beltrami operator takes the form

\[ \tilde{A} = \frac{\sigma^2}{2} \left( \nabla_x^2 + \nabla_x \ln \det g \nabla_x + g^{\mu\nu}(x) \partial_\mu \partial_\nu \right), \]

(1)

where \( \det g = \det(g_{\mu\nu}) \) is the determinant of the matrix \( g_{AB} \) and

\[ \partial_\mu = \frac{\partial}{\partial X^\mu}. \]

We further simplify the operator \( \tilde{A} \) neglecting the drift term \( \nabla_x \ln \det g \). We arrive to the diffusion generator

\[ A = \frac{\sigma^2}{2} \left( \nabla_x^2 + g^{\mu\nu}(x) \partial_\mu \partial_\nu \right), \]

(2)

We study the short time behaviour of the diffusion generated by \( A \) (2) with a singular random diffusion metric \( g \). The effect of a drift has been studied in \([22][23]\) (and in references cited there) for a regular random drift. It has been
shown that a random drift can lead to a superdiffusion at large time but it does not change the short time behaviour. We shall give some arguments further on that this is the singular diffusion matrix $g$ (rather than a drift) that leads to an anomalous diffusion at small time.

In order to achieve the positive definiteness of the metric let us represent it by the tetrads (vierbeins) $e^\mu_a$:

$$ g^{\mu\nu}(x) = e^\mu_a(x)e^\nu_a(x), $$

where initially we assume that $e^\mu_a(x)$ are regular functions of $x$. We define the stochastic process (as in [20])

$$ x_t(x) = x + \sigma b_t, \tag{3} $$

$$ X^\mu_t(X) = X^\mu + \sigma \int_0^t e^\mu_a(x_s)dB^a_s, \tag{4} $$

where $(b_t, B_t)$ is the Brownian motion on $\mathbb{R}^{n+d}$, i.e., the Gaussian process with mean zero and the covariance

$$ \mathbb{E}[b_t^j b_s^l] = \min(t, s)\delta^{jl} $$

(and similarly for $B_t$).

If there is a drift (as in eq.(1)) then $x_t(x)$ satisfies an integral equation

$$ x_t(x) = x + \int_0^t \nabla_x \ln \det g(x_s)ds + \sigma b_t. $$

There is an additional time integral on the rhs of this equation suggesting that for a small time the drift can be neglected.

It is well-known [21] that the transition function $P_t(x, X, y, Y)$ of the diffusion process defines a semi-group $\exp(tA)$. So acting on a function $\psi$

$$ \left( \exp(tA)\psi \right)(x, X) = \int dYdY' P_t(x, X; y, Y)\psi(y, Y) = \mathbb{E}\left[\psi(x_t, X_t)\right]. \tag{5} $$

It follows from equation (5) that $P_t(x, X; y, Y)$ is the kernel $K$ of the operator $\exp(tA)$. This kernel can be expressed as

$$ K_t(x, X; y, Y) = \mathbb{E}\left[\delta(y - x_t(x))\delta(Y - X_t(X))\right] \tag{6} $$

or after the Fourier representation of $\delta$-functions

$$ K_t(x, X; y, Y) = (2\pi)^{-n-d} \int d\mathbf{p}d\mathbf{P} \mathbb{E}\left[\exp(i\mathbf{p}(y - x_t(x)))\exp(i\mathbf{P}(Y - X_t(X)))\right]. \tag{7} $$

We wish to calculate an average $< K_t >$ of the diffusion kernel $K_t$ over the metric. For a Gaussian random field $F$

$$ < \exp F >= \exp(< F > + \frac{1}{2} < F^2 >) $$
For non-Gaussian fields these are the first terms of the cumulant expansion.

We assume that \( e_\mu^a = \delta_\mu^a + \epsilon_\mu^a \) is the Gaussian random field with mean \( \delta_\mu^a \) and the covariance
\[
\left\langle e_\mu^a(x)e_\nu^b(y) \right\rangle = \kappa \delta^{a\nu}\epsilon^{bc} G(x-y),
\]  
(8)

(for simplicity of calculations we choose the matrix on the rhs of eq.(8) in the form \( \delta^{a\nu}\epsilon^{bc} = \delta^{a\nu}\delta_{ac} \)). We obtain a random perturbation of the Euclidean metric. In perturbative quantum gravity \( G(x-y) \) is the two-point correlation function of the graviton, \( G(x-y) \simeq \|x-y\|^{-2\gamma} \), where \( 2\gamma = n-2 \) if the graviton moves in the \( \mathbb{R}^n \) space and \( 2\gamma = n-d-2 \) if the graviton lives in the \( \mathbb{R}^{n+d} \) space. In a rigorous formulation we must restrict ourselves to \( 2\gamma < 1 \).

There are some results based on computer simulations \([7,8,9]\) (for \( n + d = 4 \)) suggesting that because of the graviton self-interaction \( \gamma < 1 \). For the semigroup \( \exp(t\epsilon A) \) generated by the operator \( A \) (2) \( x_i (3) \) does not depend on the metric whereas \( X_i \) being linear in \( e \) is Gaussian. Then, the mean value of \( K_\epsilon \) in the Gaussian field \( e \) is
\[
\left\langle K_\epsilon(x,X;y,Y) \right\rangle = (2\pi)^{-d} \int dP \exp(iP(Y-X))
\]  
\[
\mathbb{E} \left[ \delta(y-x_i(x)) \exp \left( -i\sigma P^a B_t^a - \frac{\sigma^2}{2} \left( \int_0^t P^c(x_s)dB_s^c \right)^2 \right) \right].
\]  
(9)

For the general Riemannian model (1) the process \( x_t \) with a non-linear drift \( \nabla \ln \det g \) is non-Gaussian. A calculation of expectation values of \( \exp(t\epsilon A) \) could be performed only in an approximate way, e.g., in cumulant expansion. As discussed at eq.(1) we rely on the assumption that a process with a constant diffusion and a random drift has the same short time behaviour as the Brownian motion \( b_t \). In such a case the estimates obtained for \( \left\langle \exp(t\epsilon A) \right\rangle \) will be valid also for \( \left\langle \exp(t\epsilon A) \right\rangle \).

The last term in eq.(9) is
\[
\exp(-\frac{\sigma^2}{2} \left( P_\mu Q^\mu_t \right)^2)
\]  
(10)

where
\[
P_\mu Q^\mu_t \equiv P_\mu \int_0^t \epsilon_\mu^a(x_s)dB_s^a.
\]  
(11)

From the Ito formula \([21,22]\)
\[
d(PQ)_t = 2P_\mu Q^\mu_t P_\nu dQ^\nu_t + P_\mu dQ^\mu_t P_\nu dQ^\nu_t
\]  
\[
= 2P_\mu \int_0^t \epsilon_\mu^a(x_s)dB_s^a P_\nu \epsilon_\nu^c(x_t)dB_t^c + P_\mu P_\nu \epsilon_\mu^a(x_t)\epsilon_\nu^c(x_t)dt.
\]  
(12)

Hence, integrating eq.(12)
\[
(PQ)_t = 2P_\mu \int_0^t \epsilon_\mu^a(x_s)dB_s^a \int_0^t P_\nu \epsilon_\nu^c(x_t)dB_t^c + P_\mu P_\nu \int_0^t \epsilon_\mu^a(x_s)\epsilon_\nu^c(x_t)ds.
\]  
(13)

The formula (13) is sometimes considered as a definition of the double stochastic integral \([25]\); it appears in quantum electrodynamics \([26,27]\).
At the beginning we treat $\epsilon^a_\mu$ as a regularized random field. In such a case the correlation function $G$ is a regular function. Now, we remove the regularization admitting singular $G$. In order to make $A\psi$ a well-defined random field we need the normal ordering of $A$:

$$A := \frac{1}{2} \left( \nabla^2 + \epsilon^a_\mu : (x) \partial_\mu \partial_\nu \right),$$

where

$$\epsilon^a_\mu := \epsilon^a_\mu(x) := \epsilon^a_\mu(x) - <\epsilon^a_\mu(x)\epsilon^a_\mu(x)>. \quad (14)$$

After the normal ordering in eq.(10)

$$< (PQ_t) > \rightarrow < (PQ_t) > - P^2 t \kappa G(0) d \equiv L_t. \quad (15)$$

We treat the renormalized kernel $K^R_t$ in eqs.(9)-(10) as a generalized function acting on regular functions $\psi$ (then: $A : \psi$ is a well-defined random field).

After averaging over the translation invariant random field $\epsilon^a_\mu(x)$ and the renormalization (15) we can write $< \exp(t : A : ) \psi >$ in terms of Fourier transforms as

$$\int dy dY \langle K^R_t(x,X,y,Y) \psi(y,Y) \rangle = (2\pi)^{-d} \int dP dK dkdY \exp(iP(Y - X)) \tilde{\psi}(k,K) \mathbb{E} \left[ \exp \left( - \frac{\sigma^2}{2} P^2 L_t \right) \exp(-iKY - i\sigma P^a B^a_t - i\kappa x_t(x)) \right], \quad (16)$$

where

$$L_t = 2 \int_0^t dB^a_s \int_0^s G(\sigma b_s - \sigma b_s') dB^a_s. \quad (17)$$

We expect that the effect of the random metric will be seen by an observation of the mean distance $|X - Y|^{2k}$ (as the $X$ coordinates are coupled to the metric).

We are interested in calculation of the average

$$\int dy dY \langle P_t(x,X,y,Y) |X - Y|^{2k} \rangle \quad (18)$$

over the metric.

We use the representation ($k$ is a natural number)

$$|X - Y|^{2k} \exp(iP(X - Y)) = (-1)^k \left( \frac{\partial^2}{\partial P^a \partial P^a} \right)^k \exp(iP(X - Y)) \quad (19)$$

in order to insert $|X - Y|^{2k}$ in eq.(16). Then, integrating by parts in eq.(16) (with $\psi$ depending only on $Y$) we obtain the result

$$\int dy dY \left( K^R_t(x,X;y,Y) \right)|X - Y|^{2k} \psi(Y) = \int dP \exp(-iPX) \tilde{\psi}(P)(-1)^k \left( \frac{\partial^2}{\partial P^a \partial P^a} \right)^k \mathbb{E} \left[ \exp \left( - i\sigma P^a B^a_t - \frac{\sigma^2}{2} P^2 L_t \right) \right]. \quad (20)$$
As \( L_t \) is not positive definite the integral (20) may not exist if \( \tilde{\psi}(\mathbf{P}) \) does not decay fast (faster than \( \exp(-\alpha \mathbf{P}^2) \)). We are interested in the limit \( \psi \to 1 \) (with a properly chosen topology in the space of functions \( \psi \))

\[
\int d\mathbf{d}Y \left[ K^R_t(x, \mathbf{X}, y, Y) \right] |\mathbf{X} - \mathbf{Y}|^{2k} \equiv \lim_{\psi \to 1} \int d\mathbf{d}Y \left[ K^R_t(x, \mathbf{X}, y, Y) \right] |\mathbf{X} - \mathbf{Y}|^{2k} \psi(Y) = \int d\mathbf{d}\delta(\mathbf{P}) \mathbb{E} \left[ (-1)^k \left( \frac{\partial^2}{\partial \mathbf{P} \partial \mathbf{P}'} \right)^k \exp \left( -i \sigma \mathbf{P}^a B^a_t - \frac{\sigma^2}{2} \mathbf{P}^2 L_t \right) \right].
\]

(21)

For the limit \( \tilde{\psi}(\mathbf{P}) \to \delta(\mathbf{P}) \) in eq.(21) we may apply the sequence (with \( \delta \to 0 \))

\[
\tilde{\psi} = \left( \int d\mathbf{P} \exp(-\frac{1}{2} |\mathbf{P}|^4) \right)^{-1} \exp(-\frac{1}{2} |\mathbf{P}|^4).
\]

Let us note the identity for \( \mathbf{B}_s \) and \( \mathbf{b}_s \) (in the sense that both sides have the same probability law)

\[
\mathbf{B}_s = \sqrt{t} \mathbf{B}_s.
\]

(22)

We assume a scale invariant correlation function for the metric

\[
G(x) = |x|^{-2\gamma}.
\]

(23)

As discussed at eq.(8) such a scale invariant correlation function appears in quantum gravity. Using the scale invariant \( G \) we obtain an exact dependence of \( L_t \) on \( t \)

\[
L_t = 2t^{1-\gamma} \int_0^t dB^a_s \int_0^s G(\sigma \mathbf{b}_s - \sigma \mathbf{b}_{s'}) dB^a_{s'} \equiv t^{1-\gamma} L_1.
\]

(24)

We are interested only in the short time behaviour of the process \( \mathbf{X}_t \). For such estimates it would be sufficient to assume that the behaviour (23) holds true at small distances in order to show that the formula (24) holds true for a small time with a negligible remainder. For the operator \( \hat{A} \) in the kernel of eq.(16) \( \mathbf{x}_t(x) \) would be a non-Gaussian stochastic process with the drift \( \nabla \ln \det g(\mathbf{x}_s) \) discussed at the beginning of this section (eq.(1)). Then, in eq.(17) we would have \( \sigma \mathbf{x}_s - \sigma \mathbf{x}_{s'} \) instead of \( \sigma \mathbf{b}_s - \sigma \mathbf{b}_{s'} \). In order to obtain the estimate (24) with a small remainder (required for our final result) we would need to show that the process \( \mathbf{x}_t(x) - x \) scales as the Brownian motion (22) for a small time. \( L_t \) in eq.(24) plays the role of a new time. In [21](chapter 3, sec.5) the diffusion matrix is applied in general for a random time change in order to replace \( \mathbf{e}(\mathbf{X}_t) dB \) by \( dB(\tau) \), where \( \tau \) is a random time. A random time change is discussed in [2][4] in order to generate processes with anomalous diffusion.

By means of the normal ordering (15) we have removed from eq.(9) the infinite term \( \mathbf{P}^2 G(0) t \) which would describe the standard behaviour of the Brownian motion. We obtain \( L_t \) as the new time variable (a random time change). According to eq.(24) this new time variable behaves as \( t^{1-\gamma} \). For a small time this random time is dominating the one resulting from \( \mathbf{B}_t^2 \sim t \). This is an intuitive explanation of the behaviour of the fourth moment of the process (4) which leads to \( t^{2(1-\gamma)} \) replacing \( t^2 \) of \( \mathbf{B}_t^2 \).

For the second moment we still have (as follows from eqs.(20) and (22)) that

\[
\int d\mathbf{d}Y \left[ K^R_t(x, \mathbf{X}, y, Y) \right] |\mathbf{X} - \mathbf{Y}|^2 = \sigma^2 t d,
\]

(25)
because \( \mathbb{E}[B_t^2] = td \) plays the role of time as a consequence of the vanishing of the expectation value of the random time

\[
\mathbb{E}[L_t] = 0.
\]

However, for \( k = 2 \) the random time \( t^{1-\gamma}L_t \) will be dominating for a small \( t \). We have

\[
\int d\mathbf{y}d\mathbf{Y} \left\langle k_r^R(\mathbf{x}, \mathbf{x}; \mathbf{y}, \mathbf{Y}) \right\rangle |\mathbf{Y} - \mathbf{X}|^4
= \int d\mathbf{P} \delta(\mathbf{P}) \left( \frac{\partial^2}{\partial P^j \partial P^k} \right) \mathbb{E} \left[ \exp \left( -i\mathbf{P} \mathbf{B}_t - \frac{\sigma^2}{2} \mathbf{P}^2 L_t \right) \right].
\]

According to eq.(24) to estimate the rhs of eq.(27) it is sufficient to estimate \( \mathbb{E}[B^2_tL_t] \) and \( \mathbb{E}[L^2_t] \). The first expectation value is bounded by the second one and by \( \mathbb{E}[B^4_t] \) on the basis of the Schwarz inequality. Using \( \mathbb{E}[\left( \int f dB_s \right)^2] = \mathbb{E}[\int f^2 ds] \) we obtain

\[
\mathbb{E}[L^2_t] = 4\sigma^{-4\gamma}\kappa^2 \mathbb{E} \left[ \int_0^1 ds \left( \int_0^s dB_{s'} |b_s - b_{s'}|^{-2\gamma} \right)^2 \right]
= 4\sigma^{-4\gamma}\kappa^2 \int_0^1 ds \int_0^s ds' \int d\mathbf{x}(2\pi)^{-\frac{n}{2}} \exp \left( -\frac{x^2}{2(s-s')} \right) |\mathbf{x}|^{-4\gamma}(s-s')^{-\frac{n}{2}}.
\]

We have

\[
\int d\mathbf{x}(2\pi)^{-\frac{n}{2}} \exp \left( -\frac{x^2}{2(s-s')} \right) |\mathbf{x}|^{-4\gamma} = C(s-s')^{-2\gamma+\frac{n}{2}}
\]

with a certain numerical constant \( C \) (for convergence of eq.(29) we need \( 4\gamma < n \)). Inserting (29) in eq.(28) we can see that the integral (28) is convergent if \( 2\gamma < 1 \) (so \( n \geq 2 \)). Hence, for \( 2\gamma < 1 \) we have

\[
\int d\mathbf{y}d\mathbf{Y} \left\langle k_r^R(\mathbf{x}, \mathbf{x}; \mathbf{y}, \mathbf{Y}) \right\rangle |\mathbf{Y} - \mathbf{X}|^4
= C_1\sigma^4 t^2 + C_2\kappa \sigma^4 t^{2-2\gamma} + C_3\sigma^4 \kappa^2 t^{2(1-\gamma)},
\]

where \( t^2 \) comes from \( \mathbb{E}[B^4_t] \), \( t^{2-\gamma} \) from \( \mathbb{E}[B^2_tL_t] \) and \( t^{2(1-\gamma)} \) from \( \mathbb{E}[L^2_t] \). \( C_j \) are numerical constants independent of \( \sigma \) and \( \kappa \). We can see that for a small time the last term in eq.(30) (which depends on the metric covariance) is dominating.

We could estimate the \( 2k \)-th power of the distance using the estimate [21]

\[
\mathbb{E}[\left( \int_0^t f dB_s \right)^{2k}] \leq c_k \mathbb{E}[\int_0^t f^{2k} ds]
\]

with certain constants \( c_k \). It follows that (18) is finite if \( k\gamma < 1 \).

3 Discussion

We have shown that the mean fourth power of the distance achieved by a diffusing particle in a random singular metric behaves as \( t^{2(1-\gamma)} \) for a small time.
(depending on the random metric singularity). Then, the index $1 - \gamma$ in the heat kernel appears in the kernel of $A^{-1} = \int_0^\infty dt \exp(tA)$ in $n + d$ dimensions. The behaviour of this kernel at short distances can be related to the reduced dimensionality $d(1 - \gamma)$ of $\mathbb{R}^d$. Concerning the diffusion on irregular structures [15][16], the result shows that a singular random diffusivity may have a similar effect as a fractal nature of the medium in which the diffusion takes place. In the literature it is the anomalous behaviour of diffusion at large time which is of the most interest [1][2]. It can happen in models with a random drift [22][23]. The short time behaviour is more stable. It can appear in random walk models with a modified random waiting time [4] and in scale invariant models of a diffusion on fractals. It seems that in general models with a random diffusion generator the anomalous diffusion at small time is possible if the diffusion matrix is a singular random field.

We think that the result could be generalized to an arbitrary Riemannian manifold (in particular for the generator $\tilde{A}$, eq.(1)). We were concerned with a small time $t$. In such a case only local coordinate neighborhood is relevant in an estimate of the heat kernel. Hopefully, the argument could be extended to an arbitrary Riemannian metric. In principle, the mean $2k$-th power of the distance could be measured in experiments as a confirmation of the randomness of the metric.

Concerning the spectral dimension of refs.[7][10][15][13], we could formally integrate in eqs.(7)-(9) over momenta with the result

$$\langle K^R_t(x, X; x, X) \rangle = (2\pi\sigma^2)^{-\frac{d}{2}} t^{-\frac{d}{2}(1 - \gamma)} E[(L_1)^{-\frac{d}{2}}] \quad (31)$$

The spectral dimension $\nu$ of the diffusion kernel defined in [7][10] when applied to eq.(31) gives

$$\nu = -2\frac{d}{\sigma^2} \ln \langle K^R_t(x, X; x, X) \rangle = n + d(1 - \gamma) \quad (32)$$

where $\tau = \ln t$. The final result (32) does not depend on $L_1$. However, the formula (31) for the diagonal of the kernel (16) makes sense only if the last term in eq.(31) (the expectation value) is finite what does not seem to be true because $L_1$ takes arbitrarily small values. We may expect that some numerical schemes with a proper regularization (discretization) can give a finite result for the spectral dimension (32). However, it is unlikely that the diagonal of the averaged heat kernel is finite with a singular random metric.

We would encounter a difficulty if we wished to define the quantum field theory propagator by an integration over the (proper) time in eq.(16). For the definition of the propagator we could apply the evolution operator $\exp(it : A : )$. Then, we would have no problem with the non-positivity of $(- : A : )$. We could also use the Gaussian metric $g^{\mu\nu}$ without any positivity property. Instead of the rigorous Euclidean functional integral we would have to apply the Feynman integral. The scaling properties applied in the derivation of the modified short distance behaviour (like in eqs.(27)-(28)) apply to the Feynman integral as well.
The difficulty consists in establishing the argument with the Feynman integral as a rigorous proof.

**Acknowledgement:** Co-financed from the research and research-commercialization fund of the University of Wroclaw

**Data availability statement:** data are available on request from the author Zbigniew Haba at zbigniew.haba@uwr.edu.pl

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