Stable Pairs and Log Flips

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1. Introduction.

Let $C$ be a Riemann surface of genus $g$, let $K_C$ be a canonical divisor on $C$, and let $D$ be an effective divisor on $C$ of degree $d$. Two basic rational maps arise in this context, namely the linear series map:

$$\phi_{|K_C + D|} : C \to |K_C + D|^* \cong \mathbb{P}^{d+g-2}$$

and the Serre correspondence:

$$\psi_{|K_C + D|} : |K_C + D|^* \to M_C(2, D)$$

where $M_C(2, D)$ is the moduli space for semistable bundles $E$ on $C$ satisfying $\text{rk}(E) = 2$ and $\text{det}(E) \cong \mathcal{O}_C(D)$. The latter arises when we consider hyperplanes in $|K_C + D|$ (via Serre duality) as lines in $\text{Ext}^1(\mathcal{O}_C(D), \mathcal{O}_C)$ hence as extensions:

$$\epsilon : 0 \to \mathcal{O}_C \to E \to \mathcal{O}_C(D) \to 0 \quad \text{(modulo equivalence and scalars)}.$$

The following is an ancient result:

**Theorem 1.1.** If $d > 2$, then $\phi_{|K_C + D|}$ is an embedding.

In case $d > 2$ and $d + g - 2 = \dim(|K_C + D|) > 1$, we will let

$$X := \text{bl}(|K_C + D|^*, C),$$

the blow-up of $|K_C + D|^*$ along the image of $C$ (which we will also denote by $C$).

In contrast, the Serre correspondence is a morphism only when $d \leq 2$, as it is undefined at points of $C$ (and certain secant varieties) for larger values of $d$. The structure of the Serre correspondence for large $d$ (particularly $d > 2g - 2$, in which case it is dominant) is, however, well understood now, due largely to recent work of the author and Michael Thaddeus. The first goal of this paper (§2) is to explain Thaddeus’ interpretation of the Serre correspondence as a sequence of simple birational maps of moduli spaces of stable pairs followed by a contraction. We will discuss the birational maps at length, but let’s start with an example of a contraction. This is another ancient result:

**Theorem 1.2.** If $d > 3$, then $C \subset |K_C + D|^*$ is cut out scheme-theoretically by quadric hypersurfaces.

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If we let $H, E \subset X$ be the pull-back of a hyperplane divisor and an exceptional divisor, respectively, then Theorem 1.2 can be reinterpreted as the statement that the linear series $|2H - E|$ on $X$ is base-point free. As with any base-point free linear series, one obtains a well-defined contraction:

$$\gamma : X \to Y$$

(i.e. $\gamma$ satisfies $\gamma_*\mathcal{O}_X = \mathcal{O}_Y$).

For example, if $g = 2$ and $d = 4$ then $\gamma : X \to Y$ is generically a conic bundle over $Y \cong MC(2, \mathcal{O}_C(D)) \cong \mathbb{P}^3$, and the blow-up followed by the contraction resolve the Serre correspondence:

$$\xymatrix{ X \ar[r]_{\psi|_{K_C+D}} & \mathbb{P}^4 \ar[r] & \mathbb{P}^3 }$$

In §2, we will show (following Thaddeus) how the moduli spaces of stable pairs produce varieties $Y_1 = |K_C + D|^*, X_1 = X, Y_2 = Y, X_2, Y_3, ..., X_{\frac{d-2}{2}}$ and a diagram of contractions beginning with the blow-down and $\gamma$ and ending with a morphism to $MC(2, D)$ (the last morphism is a contraction iff $d > 2g - 2$):

$$\xymatrix{ X \ar[r]_{\gamma} & X_2 \ar[r] & \cdots \ar[r] & X_{\frac{d-2}{2}} \ar[r] & MC(2, D) \ar[d]|{K_C + D}^* \ar@/_/[ll]_Y \ar@/_/[lld]_Y \ar@/_/[llld]_Y \ar@/_/[lllld]_Y }$$

Just as the curve $C$ sits in $|K_C + D|^*$ via the linear series map, there are natural inclusions of the $k$-th symmetric products $C_k$ of $C$ in each $Y_k$. (For $k > 1$, the $Y_k$ are singular along $C_k$). And just as the blow-down $X \to |K_C + D|^*$ contracts a projective bundle to $C$ and is an isomorphism elsewhere, so too is each of the contractions to a $Y_k$ an isomorphism off a projective bundle, which is contracted to $C_k$. Thus the diagram above decomposes the Serre correspondence as a sequence of “irreducible” birational maps $X_k \to X_{k+1}$ followed by a morphism to $MC(2, D)$.

In the second part of the paper, we bring in the log minimal model program. The idea is to explain the diagram above as a sequence of log flips starting from $X$. This will not provide a new proof of the results of §2, since the existence of log flips is only known in dimension 3. Nevertheless, the idea is to find an explanation for the existence of the diagram without recourse to moduli spaces. (I will use vector bundle techniques, however, to prove the required properties of certain divisors on $X$). The justification is the observation that from this point of view, it is not unreasonable to expect more log flips than just those involving moduli of stable pairs in case $d < 2g - 2$. As motivation, recall the analogues of Theorems 1.1 and 1.3 for canonical curves which are not explained by Thaddeus’ stable pair construction.

**Theorem 1.3 (Noether).** If $C$ is not hyperelliptic, then $\phi|_{K_C}| : C \to |K_C|^*$ is an embedding.

**Theorem 1.4 (Petri).** If $C$ is neither hyperelliptic nor trigonal nor a smooth plane quintic, then the image curve $C \subset |K_C|^*$ is cut out scheme-theoretically by quadric hypersurfaces.
Section 2 begins with a general discussion of slope functions and stability. The categories of triples (following Bradlow and Garcia-Prada \[\text{BG}\]) and Bradlow pairs (following Thaddeus \[\text{T1}\]) are investigated, together with their natural families of slope functions and stability conditions. (A triple consists of two bundles plus a morphism. Triples are introduced here because the appearance of a parameter in the slope function has a very natural explanation in this context. Also, Bradlow pairs, which consist of a bundle plus a section, can be viewed as a special case of triples. See also \[\text{B2}, \text{RV}\] and in greatest generality, \[\text{KN}\], where moduli are constructed for pairs consisting of a vector bundle and a subspace of its space of sections —“Brill-Noether pairs”. A variant of this, where the subspace comes along with a basis, is also considered in \[\text{BDW}\].) Thaddeus’ theorem on the existence of moduli spaces of stable pairs in rank two is given, as well as the properties of these moduli spaces which are relevant to our discussion.

Section 3 starts with some of the definitions and expectations of the log minimal model program. Suitable \(\mathbb{Q}\)-divisors \(F\) on \(X\) are produced determinantally, and are proven to be log canonical, via a sequence of blow-ups of \(X\) along smooth centers first considered by the author in \[\text{B1}\]. Using these \(\mathbb{Q}\)-divisors, we are able to prove that all the simple birational maps \(X_k \rightarrow X_{k+1}\) produced by the diagram above are log flips, in the sense of the log minimal model program. In contrast to §2, several of the results of this section have not previously appeared elsewhere, and are due to the author.

**Connections and Problems:** Three “schools” of algebraic geometry are represented in this paper – curves in projective space, moduli spaces and the minimal model program – and my hope in writing it is to stimulate more interaction among them. Concretely, I think there are some interesting areas where a combination of ideas from the various schools may yield real progress. For example:

**Higher Dimension:** Stable pairs on surfaces have been studied (see \[\text{HL}\] and LePotier’s “coherent systems” \[\text{I}\]), and the general problem of studying the behavior of GIT quotients (such as these) as the values of some parameters pass across a “critical” wall has been studied by \[\text{DH}\] and \[\text{T2}\]. However, to the best of my knowledge, no family of moduli spaces has been constructed which addresses the following systematically:

**Question:** Does there exist an analogous diagram of contractions and simple birational transformations for “sufficiently positive” embeddings of higher dimensional varieties? (And how positive is sufficiently positive?)

**Vanishing:** The existence of “special” log canonical divisors on \(X\) would have some practical applications, even in the absence of a proof for the log minimal model program. There are general versions of Kodaira vanishing (see \[\text{K}\]) which would apply in the presence of such divisors to yield vanishings for the higher cohomologies of sheaves on \(|K_C + D|^*\) of the form \(I_k^*(n)\) (see \[\text{B4}\]). From this point of view, the following would yield concrete applications:

**Question:** Given \(d < 2g - 2\), what is the minimum value for a ratio \(\frac{a}{b}\) of positive rational numbers such that there exists an effective divisor \(F \equiv aH - bE\) on \(X\)? And how large may \(a\) be taken given \(\frac{a}{b}\) so that \(F\) is log canonical?

**(Variant):** Same question if \(C\) and/or \(D\) are general in moduli.
2. Stable Bradlow Pairs.

Let \( \mathcal{A} \) be a category with a zero object in which kernels and cokernels exist, as well as direct sums. Let \( S \subset \text{Ob}(\mathcal{A}) \) be a subset which is closed under direct sums.

**Definition 2.1.** A function \( \mu : S \to \mathbb{R} \) is called a slope function if for all short exact sequences \( 0 \to B \to A \to C \to 0 \) of elements of \( S \),

\[
\mu(B) < \mu(A) \Leftrightarrow \mu(A) < \mu(C) \quad \text{and} \quad \mu(B) = \mu(A) \Leftrightarrow \mu(A) = \mu(C)
\]

Given a slope function \( \mu : S \to \mathbb{R} \):

**Definition 2.2.** An object \( A \in S \) is called stable if \( \mu(B) < \mu(A) \) whenever \( B \in S \) and there exists an injection \( B \hookrightarrow A \) other than the identity. \( A \) is called semistable if \( \mu(B) \leq \mu(A) \) above, strictly semistable if it is semistable but not stable, and unstable if it is not semistable.

**Definition 2.3.**

(a) If \( A \in S \) is unstable, then a filtration:

\[
0 \hookrightarrow A_1 \hookrightarrow A_2 \hookrightarrow \cdots \hookrightarrow A_n = A
\]

by elements of \( S \) is called a Harder-Narasimhan filtration if the \( A_i/A_{i-1} \) are all in \( S \) and semistable and \( \mu(A_1) > \mu(A_2/A_1) > \cdots > \mu(A_n/A_{n-1}) \).

(b) If \( A \in S \) is semistable, then a filtration

\[
0 \hookrightarrow A_1 \hookrightarrow A_2 \hookrightarrow \cdots \hookrightarrow A_n = A
\]

is called a Jordan-Hölder filtration if the \( A_i/A_{i-1} \) are all in \( S \) and stable. Given a Jordan-Hölder filtration of \( A \), the object \( \text{gr}(A) := \oplus_{i=1}^n A_i/A_{i-1} \in S \) is called the associated graded of the filtration. Two Jordan-Hölder filtrations are called \( s \)-equivalent if their associated graded objects are isomorphic.

**Example 2.4.** (Vector Bundles on \( C \)):

- \( \mathcal{A} \) is the category of isomorphism classes of vector bundles on \( C \).
- \( S = \text{Ob}(\mathcal{A}) - \{0\} \).
- \( \mu : S \to \mathbb{Q} \) is the usual slope function \( \mu(E) = \deg(E)/\text{rk}(E) \). Then:
  
  (a) Harder-Narasimhan filtrations always exist (and up to isomorphism only depend upon the isomorphism class of \( E \)).
  
  (b) Jordan-Hölder filtrations always exist, producing associated graded objects which only depend upon the isomorphism class of \( E \). In particular, \( s \)-equivalence becomes an equivalence relation on isomorphism classes of semistable bundles.
  
  (c) For fixed invariants \( r \) (the rank) and either \( d \) (the degree) or \( \mathcal{O}_C(D) \) (the isomorphism class of the determinant), there are projective coarse moduli spaces \( M_C(r,d) \) (respectively, \( M_C(r,D) \)) for the functors “families of semistable vector bundles modulo \( s \)-equivalence with the given invariants”. (See [S] for details.)

The next example is due to Bradlow and Garcia-Prada ([BG]).

**Example 2.5.** (Triples on \( C \)):

- \( \mathcal{A} \) is the category of isomorphism classes of triples \((E,f,F)\), where \( E,F \) are vector bundles on \( C \) and \( f : E \to F \) is a homomorphism. A triple is called nontrivial if \( F \neq 0 \) and nondegenerate if \( f \) has maximal rank at some point. A morphism in this category is a pair \((\alpha,\beta)\) consisting of morphisms \( \alpha : E \to E' \) and \( \beta : F \to F' \) in the category of vector bundles, such that the following diagram commutes:
Both $\alpha$ and $\beta$ need to be injective to make $(\alpha, \beta) : (E, f, F) \to (E', f', F')$ injective as a morphism of triples. Direct sums obviously exist as the “free” sum of triples: 

$$(E, f, F) \oplus (E', f', F') = (E \oplus E', f \oplus f', F \oplus F').$$

- $S$ is the set of nontrivial triples.
- For each $\sigma \in \mathbb{R}$ and $(E, f, F) \in S$, let:

$$\mu_\sigma(E, f, F) = \frac{\deg(E) + \deg(F) + \sigma(rk(E) + \text{rk}(F))}{\text{rk}(F)}.$$

Note the asymmetry in the slope function! One says $(E, f, F)$ is $\sigma$-stable if it is stable with respect to the slope function $\mu_\sigma$.

**Theorem 2.6.** (Theorem 6.1) For fixed invariants: 

$$r_1 = \text{rk}(E), r_2 = \text{rk}(F), d_1 = \deg(E), d_2 = \deg(F) \text{ and } \sigma \in \mathbb{R}$$

a coarse moduli space exists for the functor “families of non-degenerate $\sigma$-stable triples with the given invariants”, which is moreover projective if $r_1 + r_2$ is relatively prime to $d_1 + d_2$ and $\sigma$ is “generic” (see [BG]).

**Explanation of the Parameter:** The idea is to relate stable triples $(E, f, F)$ on $C$ to (equivariantly) stable equivariant bundles $G$ on $C \times \mathbb{P}^1$. (The action is the automorphism group of $\mathbb{P}^1$ acting on the second factor.) This is a consequence of Künneth, which gives an isomorphism

$$\text{Hom}_{\mathcal{O}_C}(E, F) \cong \text{Ext}^1_{\mathcal{O}_{C \times \mathbb{P}^1}}(p^*E \otimes q^*\mathcal{O}_{\mathbb{P}^1}(2), p^*F),$$

telling us to look for $G$ in the corresponding extension:

$$0 \to p^*F \to G \to p^*E \otimes q^*\mathcal{O}_{\mathbb{P}^1}(2) \to 0.$$ 

(this technique is often called “dimensional reduction”.)

The main point is now that ample line bundles on $C \times \mathbb{P}^1$ are of the form $p^*L \otimes q^*M$, and (an equivariant version of) Gieseker stability for bundles on $C \times \mathbb{P}^1$ depends upon a parameter, namely the ratio $\deg(L)/\deg(M)$. Stability of $G$ with respect to a given ratio translates into $\sigma$-stability for triples for a fixed value of $\sigma$.

We are most concerned with the following, first considered by Bradlow in [Br].

**Example 2.7.** (Pairs on $C$):

- Restrict the category of triples to the objects: $(\mathcal{O}_C, f, E)$ and $(0, 0, E)$. These objects we will call pairs, following the literature. This full subcategory is closed under kernels and cokernels. It is not closed under arbitrary direct sums, but if a set of pairs is given, with the property that at most one of them is of the form $(\mathcal{O}_C, f, E)$, then their direct sum does lie in the subcategory. (This will be enough to construct associated gradeds for Jordan-Hölder filtrations!)

- $S$ is the set of nontrivial pairs (i.e. $E \neq 0$) and a pair of the form $(\mathcal{O}_C, f, E)$ is nondegenerate if and only if $f \neq 0$.

- The slope functions $\mu_\sigma$ are the same as for triples.
With respect to this slope function, observe that a pair \((\mathcal{O}_C, f, E)\) is \(\sigma\)-stable if and only if:

(i) \(\mu(F) < \mu(E) - \sigma(\frac{1}{\text{rk}(F)} - \frac{1}{\text{rk}(E)})\) for each \((\mathcal{O}_C, g, F) \hookrightarrow (\mathcal{O}_C, f, E)\), and

(ii) \(\mu(F) < \mu(E) + \sigma(\frac{1}{\text{rk}(E)})\) for each \((0, 0, F) \hookrightarrow (\mathcal{O}_C, f, E)\).

**Theorem 2.8.** ([1],[1.1]-[1.19]) Fix invariants:

\[ \text{rk}(E) = 2 \quad \text{and det}(E) \cong \mathcal{O}_C(D), \quad \text{with deg}(D) = d. \]

Then:

(a) Harder-Narasimhan and Jordan-Hölder filtrations exist if \(\sigma > 0\) and yield a well-defined \(s\)-equivalence for \(\sigma\)-semi-stable pairs.

(b) For each \(\sigma > 0\), a projective variety \(M_C(2, D, \sigma)\) (abbreviated \(M_\sigma\)) coarsely represents the functor: “families of nondegenerate \(\sigma\)-semistable pairs \((\mathcal{O}_C, f, E)\) modulo \(s\)-equivalence”. There is a universal family over the open locus parametrising stable pairs, which is smooth and irreducible.

Full proofs of the properties listed below can be found in [1].

( Please note that our \(\sigma\) differs from the \(\sigma\) in [1] by a factor of 2.)

**Properties of Stable Pairs:** Fix \(\text{rk}(E) = 2\) and \(\text{det}(E) \cong \mathcal{O}_C(D)\). Also assume that \(g \geq 2\) (but see the note at the end of this section). Then:

(a) There are no \(\sigma\)-semi-stable pairs if \(\sigma < 0\) or \(\sigma > d\).

(b) There are always 0-semi-stable pairs, though no 0-stable pairs.

(c) \(M_d\) is a point.

**Proof:** If \(\sigma < 0\), then by (ii) above, \((0, 0, E) \hookrightarrow (\mathcal{O}_C, f, E)\) destabilizes any pair. If \(\sigma > d\) and \((\mathcal{O}_C, f, E)\) is given, let \(L\) be the line-bundle image of \(\mathcal{O}_C\) in \(E\) with induced map \(s : \mathcal{O}_C \to L\). Then using (i) above, the pair is destabilized by the natural inclusion \((\mathcal{O}_C, s, L) \hookrightarrow (\mathcal{O}_C, f, E)\).

If \(\sigma = 0\), then conditions (i) and (ii) coincide, telling us that \((\mathcal{O}_C, f, E)\) is semistable if and only if \(E\) is semistable, and that no pair is 0-stable. Note that there is no Jordan-Hölder filtration.

If \(\sigma = d\), then by the analysis in the proof of (a), \((\mathcal{O}_C, f, E)\) is \(\sigma\)-semi-stable if and only if \(f : \mathcal{O}_C \to E\) has no zeroes, and all such \((\mathcal{O}_C, f, E)\) are \(s\)-equivalent, with associated graded \((\mathcal{O}_C, \text{id}, \mathcal{O}_C) \oplus (0, 0, \mathcal{O}_C(D))\). So there are no \(\sigma\)-stable pairs, and the moduli space is a point.

In contrast to the boundary cases presented here, the stable locus in \(M_\sigma\) will be nonempty if \(0 < \sigma < d\).

**Critical Points and Local Triviality:**

Let \(\Gamma = \{0 < c < d \mid c \equiv d \pmod{2}\}\).

For each \(\sigma\), let \(Z_\sigma \subset M_\sigma\) be the locus of \(\sigma\)-strictly-semistable pairs.

(a) If \(\sigma \notin \Gamma\), then \(Z_\sigma = \emptyset\) (i.e. \(\sigma\)-semistable \(\Rightarrow\) \(\sigma\)-stable).

(e) If \(c = d - 2n \in \Gamma\), then \(Z_c \cong C_n\), the \(n\)-th symmetric product of \(C\).

(f) If \(I \subset (0, d) - \Gamma\) is an interval and \(\sigma, \sigma' \in I\), then \(M_\sigma \cong M_{\sigma'}\).
(g) Suppose that $c \in \Gamma$ and $c^- < c < c^+$ are real numbers in the neighboring intervals of $(0, d) - \Gamma$. Then there are surjective morphisms:

$$
\begin{array}{c}
M_{c^-} \\
\downarrow f^- \\
M_c \\
\uparrow f^+
\end{array}
\quad \begin{array}{c}
M_{c^+} \\
\downarrow f^+
\end{array}
$$

**Key Point:** $f^-$ and $f^+$ are isomorphisms away from $Z_c \subset M_c$ and projective bundles over $Z_c$. (The projective bundles are identified in the proof).

**Proof:** If $\sigma > 0$ and if $F$ is the bundle in a destabilizing subpair of $(\mathcal{O}_C, f, E)$, then it is easy to see that $F$ is a line bundle. But if $\sigma \not\in \Gamma$, then the right side of (i) and (ii) are not integers, whereas $\mu(F)$ is an integer. So we cannot have equality. This proves (d).

Suppose $c = d - 2n \in \Gamma$. Then a $c$-strictly semistable pair $(\mathcal{O}_C, f, E)$ has a subpair which is either isomorphic to $(\mathcal{O}_C, s, \mathcal{O}_C(A))$ (where $s$ is the tautological section, $\deg(A) = n$, and $\mathcal{O}_C(A)$ is the image of $\mathcal{O}_C$ in $E$) or else it is isomorphic to $(0, 0, L)$, where $\deg(L) = d - n$. But either possibility forces the associated graded for the pair $(\mathcal{O}_C, f, E)$ to be of the form $(\mathcal{O}_C, s, \mathcal{O}_C(A)) \oplus (0, 0, \mathcal{O}_C(D - A))$, and these are parametrized by $C_n$.

The stability conditions do not change when $\sigma$ moves within an interval $I \subset (0, d) - \Gamma$ (again because $\mu(F) \in \mathbb{Z}$) so the moduli spaces are isomorphic by the universal property of a coarse moduli space.

If $c \in \Gamma$, then apart from $Z_c$, the stability conditions do not change when $c$ is replaced by $c^-$ or $c^+$, so the first part of the key point follows as in the previous paragraph.

Let $c = d - 2n$, and consider $(\mathcal{O}_C, s, \mathcal{O}_C(A)) \oplus (0, 0, \mathcal{O}_C(D - A)) \in Z_c$. Then it follows that among all pairs with this associated graded, exactly those pairs of the form:

$$
\begin{array}{c}
\mathcal{O}_C \\
\downarrow 0 \rightarrow \mathcal{O}_C(A) \rightarrow E \rightarrow \mathcal{O}_C(D - A) \rightarrow 0
\end{array}
$$

are $c^-$-stable, and these are parametrized by $|K_C + D - 2A|$, which has dimension $d - 2n + g - 2$ (independent of $A$) since $d - 2n > 0$.

On the other hand, among all pairs with this associated graded, exactly those pairs of the form:

$$
\begin{array}{c}
\mathcal{O}_C \\
\downarrow 0 \rightarrow \mathcal{O}_C(D - A) \rightarrow E \rightarrow \mathcal{O}_C(A) \rightarrow 0
\end{array}
$$

are $c^+$-stable. Such pairs are parametrized by $\mathbb{P}(V)$, where $V$ sits in the long exact sequence:

$$
H^0(C, \mathcal{O}_C(D - A)) \rightarrow V \rightarrow H^1(C, \mathcal{O}_C(D - 2A)) \rightarrow H^1(C, \mathcal{O}_C(D - A)).
$$

(in fact, $\mathbb{P}(V)$ is naturally isomorphic to $\mathbb{P}(H^0(C, \mathcal{O}_C(D - A) \otimes \mathcal{O}_A)^*))$.

Thus the dimension of $\mathbb{P}(V)$ is $n - 1$, independent of $A$. 

So there are a finite number of moduli spaces $M_{\sigma}$, linked by morphisms as in
the following diagram:

$$
\cdots \xrightarrow{\cdots} X_2 \xrightarrow{\cdots} X_1 \xleftarrow{\cdots} X_0 \xleftarrow{\cdots} M_{d-4} \xrightarrow{\cdots} M_{d-2} \xleftarrow{\cdots} M_{d}
$$

where each $X_n \cong M_{(d-2n)} \cong M_{(d-2n)-2}$.

Theorems 1.1 and 1.2 are embedded in this diagram because of:

**Large Values of $\sigma$:**

(h) If $d > 0$, then $X_0 \cong |K_C + D|^*$. If $d > 2$, then $M_{d-2} \cong |K_C + D|^*$ and $X_1 \cong X = bl(|K_C + D|^*, C)$. Moreover, the morphism $f^- : X_1 \to M_{d-2}$ is the blow-down.

If $d > 4$, then $f^+ : X_1 \to M_{d-4}$ is the contraction $\gamma : X \to Y$.

**Proof:** A special case of the proof of (g) shows that $X_0 \cong |K_C + D|^*$.

Another special case of the proof of (g) shows that $f^+ : X_0 \to M_{d-2}$ is an isomorphism, because the “exceptional” part of the map is a $\mathbb{P}^1$-bundle(!)

When (g) is applied to the map $f^- : X_1 \to M_{d-2} = |K_C + D|^*$, one discovers that the exceptional locus is a divisor, which is a projective bundle over $C$, hence $f^-$ is the blow-down.

Finally, when (g) is applied to the map $f^+ : X_1 \to M_{d-4}$, the exceptional set consists of lines spanned by two points of $C$ (i.e. the secant lines) which are contracted to points. This means that the linear series which realizes $f^+$ must be a multiple of $|2H - E|$, so the fact that $f^+$ has connected fibers implies $f^+$ is equal to $\gamma$.

So we’ve got Theorem 1.1 and (a very precise) Theorem 1.2 when $d > 4$. To see what happens for $d = 4$ from this point of view (for example, in the case $g = 2$ and $d = 4$ considered in the introduction), we need to analyze:

**All Values of $\sigma$:**

(i) If $n > 1$, then $X_n$ (if defined) is isomorphic to $X_1$ off codimension 2.

The maps $f^+ : X_n \to M_{d-2n-2}$ are multiples of $|(n + 1)H - nE|$.

The maps $f^- : X_n \to M_{d-2n}$ are multiples of $|nH - (n - 1)E|$.

(j) If $\sigma$ is in the first interval of $(0, d) - \Gamma$, then there is a morphism:

$$f : M_{\sigma} = X_{\left\lceil \frac{d}{2} \right\rceil} \to M_C(2, D)$$

which is the contraction determined by high multiples of $|dH - (d - 2)E|$.

**Proof:** The first part of (i) is a dimension count using (g), which allows us to transfer linear series from $X_1 = X$ over to each $X_n$. The reader is referred to [17], where the ample cone is constructed for each $X_n$, the boundary of which gives properties (i) and (j). Notice in particular, that each $f^+$ and $f^-$ is a very simple contraction by property (g), but that the final map $f$ in (j) can have rather more complicated behavior, as in the example of the introduction.
Thus the mirror image of the diagram following property (g) gives us:

\[
\begin{array}{ccc}
X_1 & \rightarrow & X_2 \\
\downarrow & & \downarrow \\
|K_C + D|^* & \rightarrow & Y = M_{d-4} \\
\end{array}
\]

which is the advertised generalization of Theorems 1.1 and 1.2.

Note: Most of this analysis also applies to curves of genus 0 and 1.

**genus 1:** All properties (a)-(j) apply. The only difference between this and the general case occurs when \( d \) is even, in which case \( M_C(2, D) \) is isomorphic to \( \mathbb{P}^1 \), rather than a point, as one would expect by a dimension count. For example, if \( d = 4 \), then \( \gamma : X = \text{bl}(\mathbb{P}^3, C) \rightarrow Y = \mathbb{P}^1 \) is the contraction determined by the pencil of quadrics vanishing along \( C \).

**genus 0:** Properties (a)-(j) apply if \( d \) is even and \( M_C(2, D) \) is a point, corresponding to the vector bundle \( \mathcal{O}_{\mathbb{P}^1}(\frac{d}{2}) \oplus \mathcal{O}_{\mathbb{P}^1}(\frac{d}{2}) \). On the other hand, if \( d = 2n + 1 \), then \( M_\sigma = \emptyset \) if \( \sigma < 1 \), because all bundles are unstable. Other than this, which forces obvious changes to properties (b),(c),(g) and (j), everything is as in the general case. Notice that in this case, \( M_1 \) is isomorphic to \( \mathbb{P}^n = (\mathbb{P}^1)^n \), by property (e). For example, when \( d = 5 \), then \( \gamma : X = \text{bl}(\mathbb{P}^3, C) \rightarrow Y = M_1 = \mathbb{P}^2 \) is the contraction determined by the web of quadrics vanishing along the twisted cubic. This contraction is a \( \mathbb{P}^1 \) bundle, a special case of the key point of property (g).

### 3. Log Flips.

The goal of this section is to interpret the birational maps:

\[
X = X_1 \rightarrow X_2 \rightarrow ... \rightarrow X_{\left\lfloor \frac{d-1}{2} \right\rfloor}
\]

as flips, in the sense of the minimal model program. In fact, they are not flips, but rather their inverses are flips (at least initially), in the traditional sense. While this is an interesting observation, it is not the one I want to pursue, because the inverses point in the wrong direction, from the point of view of Theorems 1.1 and 1.2. For example, with this interpretation, the first contraction \( \gamma : X \rightarrow Y \) is not a flipping contraction, but rather contracts curves whose intersection with \( K_X \) are positive. Such contractions are hard to understand, in general. Fortunately, the theory of log minimal models provides a means for turning the flips around, provided we can find suitable divisors on the \( X_k \). We begin with a quick tour of the parts of the log minimal model program relevant to our discussion.

Let \( X \) be a smooth projective variety.

**Definition 3.1.** A \( \mathbb{Q} \)-divisor on \( X \) is a finite sum of distinct prime divisors with rational coefficients. It is effective if all the coefficients are non-negative. Intersections with curves, self intersections and numerical equivalence are all defined as with ordinary divisors.

Let \( F \) be an effective \( \mathbb{Q} \)-divisor on \( X \). If \( F = \sum \alpha_i F_i \), then the support of \( F \), denoted \( \text{Supp}(F) \), is the union of the prime divisors \( F_i \) which appear in \( F \) with positive coefficients.
Definition 3.2. If $F$ is an effective divisor on $X$, then a log resolution of $(X,F)$ is a morphism $f : \tilde{X} \to X$ with the property that $\tilde{X}$ is smooth, and $\sum f$-exceptional divisors $+ f^{-1}_*(\text{Supp}(F))$ is a normal crossings divisor.

Note: If $f : Y \to X$ is any birational morphism of smooth varieties, and if $D$ is a Q-divisor on $X$, then $E_f$ will denote the sum of the $f$-exceptional divisors, and $f^*(D)$ and $f^{-1}_*(D)$ will denote the total transform and the strict transform of $D$ on $Y$, respectively. (They are well-defined by linearity.)

Let $F = \sum \alpha_i F_i$ be an effective Q-divisor.

Definition 3.3. $F$ is log canonical if each coefficient $\alpha_i \leq 1$, and there is a log resolution of $(X,F)$ with the property that all coefficients of the components of $E_f$ are at least $-1$ in the Q-divisor:

$$(K_{\tilde{X}} - f^* K_X) + (f^{-1}_*(F) - f^*(F))$$

Note: This property is independent of the log resolution.

Definition 3.4. Suppose that $B \subset X$ is a curve (which we also identify with its image in $H_2(X,\mathbb{R})$). Then $B$ spans an extremal ray of the cone of effective curves on $X$ if there is an element $\lambda \in H^2(X,\mathbb{R})$ such that:

(i) $\lambda(B) = 0$ and

(ii) if $\beta \in H_2(X,\mathbb{R})$ is a limit of sums $\sum c_i B_i$ of curves with positive (real) coefficients, then $\lambda(\beta) \geq 0$ with equality if and only if $\beta$ is a multiple of $B$.

Definition 3.5. Suppose that $B$ spans an extremal ray and $f : X \to Y$ is a morphism satisfying $f_* (\mathcal{O}_X) = \mathcal{O}_Y$. If $B$ is contained in a fiber of $f$, and if moreover every curve contained in every fiber of $f$ is homologous to a (rational) multiple of $B$, then $f$ (which is uniquely determined if it exists) is called the extremal contraction associated to $B$.

Suppose that $F$ is a log-canonical divisor on $X$. A basic result of the log minimal model program is the following (see [CKM] and [Ketal]):

Contraction Theorem: If $B \subset X$ spans an extremal ray and $B.(K_X + F) < 0$, then there is an extremal contraction $\gamma : X \to Y$ associated to $B$.

A central question of the minimal model program is:

Do Log Flips Exist?: Suppose the contraction $f : X \to Y$ of the theorem is an isomorphism off codimension 2 in $X$. Then does there exist a morphism $f^+ : X^+ \to Y$ with the following properties:

(a) $f^+$ is an isomorphism off codimension 2 in $X^+$. Let $(K_X + F)^+$ be the strict transform of $K_X + F$ in $X^+$.

(b) If $B^+ \subset X^+$ is a curve lying in a fiber of $f^+$, then $B^+.((K_X + F)^+) > 0$.

(c) The singularities of $X^+$ (or rather, of $(X^+, F^+)$) are not too bad (for example, so that we can even define the intersections $B^+.((K_X + F)^+$ in (b)).

When $F = 0$ and the dimension of $X$ is 3, then the affirmative answer to this question is a deep theorem of Mori (together with a definition of “not too bad”, of course). The answer is also known to be yes for arbitrary $F$ and dimension 3.
The interested reader is urged to consult [CKM] and [Ketal], as well as Kollár’s notes in this Proceedings for an introduction to the minimal and log minimal model programs and other applications.

Next, we construct a morphism which will eventually be a log resolution.

Let $M$ be a line bundle on $C$, let $C_k$ be the $k$-th symmetric product of $C$, and let $V = H^0(C, M)$. If $M$ has the following property:

$$(*)_k : \text{For all } D \in C_k, \dim(H^0(C, M(-D))) = \dim(V) - k$$

then each such divisor $D$ determines a $P^{k-1} \subset P(V)$, which is called the span of $D$. Given that property $(*)_k$ holds, the $k$-th secant variety is:

$$\Sigma_k(C) = \bigcup_{D \in C_k} \text{span}(D) \subset P(V).$$

If $M$ satisfies $(*)_2$ (i.e. $M$ is very ample), let $X = \text{bl}(P(V), C)$ (as in §1).

**Observation:** $O_C(K_C + D)$ satisfies $(*)_d$ (Riemann-Roch!)

The following construction blows up the secant varieties of $C$.

**Theorem 3.6.** ([B1] Theorem 1) (a) Suppose $n \geq 1$ and $M$ is a line bundle with property $(*)_2n$. Then there is a birational morphism $f : \bar{X} \to X$ which is a composition of the following blow-ups:

$$f^{(1)} : X^{(1)} = X \text{ blows up the strict transform of } \Sigma_2(C),$$

$$f^{(2)} : X^{(2)} \to X^{(1)} \text{ blows up the strict transform of } \Sigma_3(C),$$

$$\vdots$$

$$f^{(n)} : \bar{X} = X^{(n)} \to X^{(n-1)} \text{ blows up the strict transform of } \Sigma_n(C)$$

Moreover, the strict transform of each $\Sigma_k(C)$ in $X^{(k-1)}$ is smooth and irreducible of dimension $2k - 1$, transverse to all exceptional divisors, so in particular $\bar{X}$ is smooth.

For consistency, let $f^{(1)} : X \to P(V)$ be the blow-down. Let $E^{(k)}$ be the strict transform in $\bar{X}$ of each $f^{(k)}$-exceptional divisor. Then $E^{(1)} + \ldots + E^{(n)}$ is a normal crossings divisor on $\bar{X}$ with $n$ smooth components.

If $M$ is a line bundle that does not satisfy $(*)_2$, let $\bar{X} = P(V)$. Then:

(b) (Terracini recursiveness) Suppose $k \leq n$ and $x \in \Sigma_k(C) - \Sigma_{k-1}(C)$. Then the fiber

$$(f^{(k)})^{-1}(x) \subset X^{(k)}$$

is naturally isomorphic to $P(H^0(C, M(-2A)))$, where $A$ is the unique divisor of degree $k$ whose span contains $x$. Moreover, the fiber

$$f^{-1}(x) \subset E^{(k)} \subset \bar{X}$$

is isomorphic to $\bar{X}_A$, the variety obtained by applying (a) of the Theorem to the line bundle $M(-2A)$.

(c) If $g \geq 2$ and if $M = O_C(K_C + D)$, then there is a **morphism**

$$\bar{\psi}|_{K_C+D} : \bar{X} \to MC(2, D)$$
which extends $\psi_{[K_C+D]}$. When restricted to a fiber $f^{-1}(x)$ of part (b), $\tilde{\psi}_{[K_C+D]}$ agrees with $\psi_{[K_C+D-2A]}$ (and this property determines $\tilde{\psi}_{[K_C+D]}$ uniquely!)

For the proof, see [31]. Notice that parts (a) and (b) make no reference to moduli, hence generalize to, for example, canonical embeddings, where condition $(*)_g$ is equivalent to the nonexistence of $g^1_0$'s. As for part (c), the idea is to construct a vector bundle on $C \times \tilde{X}$ by a sequence of elementary modifications of the bundle (constructed from the universal extension) on $C \times [K_C+D]^*$ along the exceptional divisors for each $f^{(k)}$, and to use this bundle to get the map to moduli.

In fact, though, the proof really constructs families of nondegenerate pairs $(\mathcal{O}_C,f,E)$ parametrized by the $\tilde{X}^{(k)}$ (in all genera) with the following property. For every $y \in \tilde{X}$ and every $\sigma \in [0,d]$ (or $[1,d]$ if $g = 0$ and $d$ is odd) there is an $X^{(k)}$ such that the image of $y$ in $X^{(k)}$ parametrizes a $\sigma$-semistable pair. Thus, for each $\sigma$, there is a natural morphism:

$$\psi_{\sigma} : \tilde{X} \rightarrow M_{\sigma}.$$ 

Now we construct log-canonical divisors on $X = \text{bl}([K_C+D]^*,C)$.

**Linear Algebra Construction:** Given any vector bundle $F$ on $C$, the cup product gives rise to a linear map:

$$c : \text{Ext}^1(F(D),F) \rightarrow \text{Hom}(H^0(C,F(D)),H^1(C,F))$$

Also, the summand $\mathcal{O}_C \rightarrow F \otimes F^*$ produces an inclusion of vector spaces:

$$i : \text{Ext}^1(\mathcal{O}_C(D),\mathcal{O}_C) \hookrightarrow \text{Ext}^1(F(D),F)$$

One can think of the composition $c \circ i$ pointwise as follows. Given

$$c : 0 \rightarrow \mathcal{O}_C \rightarrow E \rightarrow \mathcal{O}_C(D) \rightarrow 0$$

one tensors each term by $F$, and $c(i(\epsilon))$ is the connecting homomorphism:

$$c(i(\epsilon)) = \delta : H^0(C,F(D)) \rightarrow H^1(C,F).$$

When we lift $c \circ i$ to a map of trivial bundles on $[K_C+D]^*$, it determines a matrix $M(F)$ of linear forms on $[K_C+D]^*$ via:

$$\mathcal{O}_{[K_C+D]^*}(-1) \rightarrow \text{Hom}(H^0(C,F(D)),H^1(C,F)) \otimes \mathcal{O}_{[K_C+D]^*}.$$ 

**Proposition 3.7.** (a) For each $0 < k \leq \frac{d}{2}$, there is a nonempty open subset $U \subset \text{Pic}^{k-(g-1)}(C)$ such that

$$(*) \quad L \in U \Rightarrow h^0(C,L^{-1}(D)) = d - k \text{ and } h^1(C,L^{-1}) = k.$$ 

If $L \in U$, choose a basis for $H^0(C,L^{-1}(D))$, and let $I = (i_1,\ldots,i_k)$ be a multi-index with $1 \leq i_1 < \ldots < i_k \leq d - k$. Then $I$ determines a $k \times k$ minor $M_I(L^{-1})$ (choosing columns $i_1,\ldots,i_k$ from the matrix $M(L^{-1})$) yielding a divisor on $\tilde{X}$:

$$D_{L,I} \in [kH - (k-1)E^{(1)} - (k-2)E^{(2)} - \ldots - E^{(k-1)}],$$

(i.e. the generic multiplicity of $\text{det}(M_I(L^{-1}))$ along $\Sigma_i(C)$ is at least $k - i$).

Finally, if we let $V_k$ be the sub-linear-series spanned by the $D_{L,I}$, then $V_k$ is base-point-free (and independent of choices of basis).
(b) Suppose \( g > 0 \). Then for each \( 0 < l \leq d \), there is a nonempty open subset \( U \subset M_C(2, l - (2g - 2)) \) such that

\[
(*) \quad F \in U \Rightarrow h^0(C, F^{-1}(D)) = 2d - l \text{ and } h^1(C, F^{-1}) = l.
\]

If \( F \in U \), choose \( J = (j_1, ..., j_l) \) such that \( 1 \leq j_1 < ... < j_l \leq d - l \) and the minor \( M_J(F^{-1}) \) as in (a). Then \( \det(M_J(F^{-1})) \) determines a divisor:

\[
D_{F,J} \in \langle H - (l - 2)E^{(1)} - (l - 4)E^{(2)} - ... \rangle,
\]

and the sub-linear-series \( W_i \) spanned by the \( D_{F,J} \) is base-point-free.

**Proof:** The values for \( h^0 \) and \( h^1 \) in (*) are generic in \( \text{Pic} \) and \( M_C(2, *) \) respectively. \( U \) is an intersection of two nonempty open subsets.

For the next part, the following observation is crucial. Given an effective divisor \( A \) on \( C \), an extension \( e \in \text{Ext}^1(\mathcal{O}_C(D), \mathcal{O}_C) \) determines a point \( \tau \in \text{span}(A) \subset |K_C + D| \) if and only if the extension splits when pushed forward:

\[
0 \to \mathcal{O}_C \to E \to \mathcal{O}_C(D) \to 0
\]

Now suppose \( \tau \in \Sigma_i(C) \), so is in the span of some divisor \( A \) of degree \( i < k \). It follows (tensoring the inclusion \( \mathcal{O}_C(D-A) \hookrightarrow E \) by \( L^{-1} \)) that \( H^0(C, L^{-1}(D-A)) \subset \ker(e(v(e))) \), which by Riemann-Roch has dimension at least \( (d - k) - i \). Thus the rank of each \( M_i(L^{-1}) \) is at most \( i \), and therefore its determinant has multiplicity at least \( k - i \) at \( \tau \), from which the linear series computation in (a) follows. The linear series in (b) is computed similarly.

We prove base-point-freeness first when \( g = 0 \) and \( d = 2n + 1 \). Given \( k \), if \( y \in \bar{X} - (E^{(1)} \cup ... \cup E^{(k-1)}) \), then the bundle \( E \) associated to \( \tau = y \) is isomorphic to \( \mathcal{O}_{P^1}(m) \oplus \mathcal{O}_{P^1}(d - m) \) where \( d - m > m \geq k \). This is because of the crucial observation. It follows that \( h^0(C, E(1 - k)) = d - 2k \), so some \( M_I(\mathcal{O}_{P^1}(k - 1)) \) has full rank at \( \tau \), and thus \( y \) is not a base point. If \( y \in E^{(i)} \) for some (minimal) \( i < k \), then Theorem 3.6 (b) allows us to place \( y \) in a fiber over \( \Sigma_i(C) \) isomorphic to \( \bar{X}_A \) for some divisor \( A \) satisfying \( \deg(A) = i \). The restriction to this fiber of \( V_k \) is identified with the linear series \( V_{k-i} \) under the isomorphism with \( \bar{X}_A \), and so we can conclude base-point-freeness by induction.

The proof of base-point-freeness is similar in general. Suppose \( d = 2n + 1 \) or \( 2n + 2 \), so \( \mathcal{O}_C(K_C + D) \) satisfies (*)\(2n\). Given \( k \) (or \( l = 2k \) or \( 2k - 1 \)), first consider the points \( y \in \bar{X} - (E^{(1)} \cup ... \cup E^{(k-1)}) \). If \( y \in E^{(k)} \cup ... \cup E^{(n)} \), then we can find an \( L \subset U \) (or \( F \subset U \)) such that \( h^0(C, E \otimes L^{-1}) = d - 2k \) (or \( h^0(C, E \otimes F^{-1}) = 2d - 2l \)) because the bundle \( E \) associated to \( \tau = y \) fits in an exact sequence:

\[
0 \to \mathcal{O}_C(D - A) \to E \to \mathcal{O}_C(A) \to 0 \quad (k \leq \deg(A) \leq n).
\]

(This is a consequence of the crucial observation.) If \( y \) does not lie in an exceptional divisor, then the bundle \( E \) associated to the extension \( \tau = y \) is semistable (Theorem 3.6 (c) or the crucial observation) in which case the same fact about \( h^0(C, E \otimes L^{-1}) \) (or \( h^0(C, E \otimes F^{-1}) \)) is a standard result, for example, see [3.33], Lemma 3.6. Once this is achieved, one has base-point-freeness off the exceptional divisors \( E^{(1)} \cup ... \cup E^{(k-1)} \) and the points of these exceptional divisors are treated by induction using Theorem 3.4 (b) and the same identification of \( V_k \) with \( V_{k-l} \) (or \( W_l \) with \( W_{l-2l} \)) as above.
There are two exceptional cases (given in detail below) where the linear series \((V_k \text{ in case (a) and } W_l \text{ in case (b)})\) are trivial, which occur when \(\Sigma_n(C)\) is a divisor and \(k\) (or \(l\)) is maximal. In all other cases, we can use Bertini to find smooth members of the linear series which meet all the exceptional divisors \(E^{(1)}, \ldots, E^{(n)}\) transversally.

**Exceptional Cases:** (a) Suppose \(g = 0\) and \(d = 2n + 2 = 2k\). Then there is only one \(k \times k\) matrix \(M_f(O_{\mathbb{P}(}(-k - 1))\), and \(V_k\) has only one element. (So since it is base-point-free, it has to be trivial!) With a suitable choice of basis, \(M_f(O_{\mathbb{P}(}(-k - 1))\) is the standard square matrix:

\[
\begin{pmatrix}
    z_1 & z_2 & z_3 & \cdots & z_k \\
    z_2 & z_3 & z_4 & \cdots & z_{k+1} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    z_k & z_{k+1} & z_{k+2} & \cdots & z_{d-1}
\end{pmatrix}
\]

whose \(2 \times 2\) minors cut out the rational normal curve \(C \subset \mathbb{P}^{d-2} = |K_C + D|^*\) (see Proposition 9.7 in [ACGH]). Its determinant cuts out \(\Sigma_n(C) \subset \mathbb{P}^{d-2}\). The linear series \(V_k\) is trivial because here \(E^{(n)} \equiv nH - (n - 1)E^{(1)} - \ldots - 2E^{(n-1)}\) on \(\tilde{X}\). Notice that although we cannot use Bertini to find smooth divisors in \(V_k\), we can use Theorem 3.6 (a) to conclude that \(E^{(n)}\) itself is smooth and meets the other exceptional divisors transversally.

(b) Suppose \(g = 1\), and \(d = l = 2n + 1\). Then there is one \(l \times l\) matrix \(M_f(F^{-1})\) for each stable bundle \(F\) of rank 2 and degree \(d = d - (2g - 2)\). There is no reason a priori why this determines a trivial linear series, however, as in Exception (a), one computes that \(\Sigma_n(C) \subset |K_C + D|^*\) is a divisor, of degree \(d\), which must therefore be the zero locus of each determinant, and \(E^{(n)} \equiv dH - (d - 2)E^{(1)} - \ldots - 3E^{(n-1)}\). (The degree can be computed using Lemma 2.5 (Chapter VIII) from [ACGH].) Again, we will use the fact that \(E^{(n)}\) is smooth, intersecting the other exceptional divisors transversally.

**Remarks:** If \(l\) is even, the conditions \(h^0(C, F^{-1}(D)) = 2d - l\) and \(h^0(C, F^{-1}) = l\) may not be independent of the choice of a representative \(F\) for a semistable point in \(M_C(2, l - (2g - 2))\). However, if these properties are true for the associated graded, then they hold for all representatives, as easily checked. Moreover, if we let \(l = 2k\), then the split bundles determine an inclusion of linear series: \(V_k \cdot V_k \subseteq W_l\). (In genus 0 and 1, this is an equality!)

Next, we use the linear series to find:

**Some Log Canonical Divisors on \(X\):** Let \(F = f_*A\), where:

**Genus 0:** (a) If \(d = 2n + 1\), then \(A \in W_{2n}\) is a general member.

(b) If \(d = 2n + 2\), then \(A = E^{(n)} + A'\), where \(A' \in V_n\) is a general member.

**Genus 1:** (a) If \(d = 2n + 1\), then \(A = E^{(n)}\).

(b) If \(d = 2n + 2\), then \(A \in W_d\) is a general member.

**Genus \(\geq 2\):** \(A \in W_d\) is general.
Alternatively, one can think of $F$ in each case as the strict transform in $X$ of a hypersurface in $|K_C + D|^*$ (highly singular along the secant varieties). However, when we think of $F$ as the push-forward of a divisor $A$ on $\tilde{X}$, then the following becomes almost immediate.

**Claim:** In all the cases above, $f : \tilde{X} \to X$ is a log resolution of $(X, F)$ and $F$ is a log canonical divisor.

**Proof:** By Theorem 3.6 (a), all the $f$-exceptional divisors are smooth with normal crossings. In each case, the strict transform of the support of $F$ is the support of $A$, which is a sum of smooth divisors which intersect all others with normal crossings, either by Bertini or Theorem 3.6(a) again. So $f$ is a log resolution of $(X, F)$.

Since each blow-up $f^{(k)}$ was along a smooth center transverse to all exceptional divisors, it follows that the coefficient of $E^{(k)}$ in $K_{\tilde{X}} - f^*K_X$ is the codimension of $\Sigma_k(C)$ in $|K_C + D|^*$ minus 1, a consequence of Riemann-Hurwitz. It also follows that since we constructed $F$ as $f_A$, the coefficient of $E^{(k)}$ in $f^{-1}F - f^*F = A - f^*F$ is the (negative of the) generic multiplicity of $F$ along the strict transform of $\Sigma_k(C)$ in $X$, which is computed directly from the linear series in which $A$ lies. This is the information we need to check that $F$ is log canonical. The computations in genus 0 are left to the reader. Here is the data for genus $\geq 1$:

- codimension of $\Sigma_k(C)$ in $|K_C + D|^*$: $d + g - 2k - 1$.
- multiplicity of $F$ along $E^{(k)}$: $d - 2k$.

Since it follows that the coefficient of each $E^{(k)}$ in $(K_{\tilde{X}} - f^*K_X) + (A - f^*F)$ is $g - 2$, we see that $F$ is log canonical.

We can (and need to!) do a little better when $g \geq 2$ if we use $\mathbb{Q}$-divisors. If $p, q$ are positive integers, let $(W_d)^p$ be the linear series spanned by products of $p$ elements of $W_d$, and given a smooth element $G \in (W_d)^p$ (this linear series is base-point free), consider $F' = \frac{1}{q}f_*G$. This is not only numerically equivalent to $\frac{p}{q}F$ (as is easy to see), but all coefficients of the $E^{(k)}$ in the expression $f^{-1}F' - f^*F'$ are $\frac{p}{q}$ times the corresponding coefficients for $F$. We will abuse notation and say that this divisor is a member of $\frac{p}{q}F$, keeping in mind that if $p > q$, then the literal $\mathbb{Q}$-divisor $\frac{p}{q}F$ cannot be log canonical, by definition, while a member constructed in this way might be log canonical. In fact, if $d > 4$, then

\[
\left(\frac{d + g - 5}{d - 4}\right) F \text{ has a log canonical member}
\]

by the data above (keep in mind that $E^{(1)}$ is not $f$-exceptional).

Now we will relate these log canonical divisors to the diagram at the end of §2 constructed by stable pairs. Namely, recall that whenever $d > 2k$, there was a diagram:

\[
\begin{array}{ccc}
X_{k-1} & \xrightarrow{f^+} & X_k \\
& \swarrow f^- & \\
M_{d-2k} & & \\
\end{array}
\]

Moreover, $f^-$ and $f^+$ are obviously extremal ray contractions since each contracts a projective bundle over $C_k$ and $\dim(H_2(X, \mathbb{R})) = 2$. (Take any curve in
a projective-space fiber to span the extremal ray.) Finally, each contraction is an isomorphism off of codimension 2.

**Proposition 3.8.** If \( k = 2 \) or \( d > 2g - 2 \) and \( k \) is arbitrary, then the diagram above is a log flip for (the strict transform on \( X_{k-1} \) of) \( K_X + \left( \frac{d + g - 5}{d - 4} \right) F \).

**Proof:** We need to show: (a) the member of \( \left( \frac{d + g - 5}{d - 4} \right) F \) constructed as above is log canonical on each \( X_k \) (this will certainly suffice for the condition “not too bad” in the “definition” of log flips), and (b) If \( B \subset X_{k-1} \) and \( B^+ \subset X_k \) are curves spanning extremal rays, then \( B . (K_X + \left( \frac{d + g - 5}{d - 4} \right) F) < 0 \) and \( B^+. (K_X + \left( \frac{d + g - 5}{d - 4} \right) F) > 0 \).

We prove (b) first. Recall (property (i) from §2) that the map \( f^+ \) is a multiple of \( |kH - (k-1)E| \). Thus, if \( B \subset X_{k-1} \) is an extremal ray, then \( B . (H - \frac{k-1}{k} E) = 0 \). Moreover, \( |(k-1)H - (k-2)E| \) is nef on \( X_{k-1} \), and \( B \) is not contracted in this linear series, so it follows that \( B . (H - \frac{k-1}{k} E) > 0 \), and \( B . E > 0 \).

From the data:

\[
K_X \equiv -(d + g - 1)H + (d + g - 4)E,
\]

\[
F \equiv dH - (d - 2)E,
\]

we get \( K_X + \left( \frac{d + g - 5}{d - 4} \right) F \equiv \frac{4g - 1}{d - 4} \left( H - \frac{d + 2g - 6}{4g - 4} E \right) \), from which it follows that its intersection with \( B \) is negative when \( k = 2 \) or \( d > 2g - 2 \). Moreover, the mirror image of this argument shows that if \( B^+ \subset X_k \) is an extremal ray for \( f^- \), then its intersection with \( K_X + \left( \frac{d + g - 5}{d - 4} \right) F \) is positive in the same cases.

So the only thing left to see is the fact that the member of \( \left( \frac{d + g - 5}{d - 4} \right) F \) we constructed is log canonical on all \( X_k \), not just \( X = X_1 \), as was shown earlier. In fact, I claim a stronger result, which will explain all the maps as log flips:

**Lemma 3.9.** If \( d > 2k \), then a general member of \( \left( \frac{d + g - 2k - 1}{d - 2k} \right) F \) is log canonical on \( X_{k-1} \).

**Proof:** The construction is as before, pushing down a general element of \( (W_d)^{d^d + g - 2k - 1} \) and dividing by \( d - 2k \). After \( k - 1 \) elementary modifications, the proof of Theorem \( 5.6(c) \) (see [31]) produces a family of \( \sigma = n - 2k + 1 \)-stable pairs on \( C \) parametrized by \( \psi(k-1) \), hence a morphism \( \psi(k-1) : X(k-1) \to X_{k-1} \) since \( X_{k-1} = M_\sigma \). Moreover, the morphism \( \psi_\sigma : \tilde{X} \to X_{k-1} \) factors through \( \psi(k-1) \) via the composition of blow-downs \( f_{k-1} : \tilde{X} \to X^{(k-1)} \) from Theorem \( 3.5(a) \).

One checks that if \( G \in (W_d)^p \) (for any \( p > 0 \)), then \( (f_{k-1})_*G \) descends to a divisor on \( X_{k-1} \). Thus when we log resolve \( (X_{k-1}, \frac{E}{k} F) \) by the map \( \psi_\sigma \), then only the exceptional divisors \( E^{(k)} \) and above appear with a nonzero coefficient in \( \psi_\sigma^* \frac{E}{k} F - (\psi_\sigma)_*^{-1} \frac{F}{k} F \), and those, it is easy to see, appear with the same coefficients as in the earlier computation. The lemma immediately follows.

**Corollary 3.10.** Each rational map \( X_{k-1} \to X_k \) is a log flip.
Proof: Using the log canonical divisor on $X_{k-1}$ from the Lemma:

$$K_X + \left(\frac{d + g - 2k - 1}{d - 2k}\right) F \equiv k(2g - 2) \left( H - \frac{(2k - 1)(2g - 4) + d - 2}{k(2g - 2)} E \right)$$

has negative intersection with $B$ and positive intersection with $B^+$ (as in the proof of Proposition 3.8, keeping in mind the fact that $d > 2k$).

Final Remark: I have split off Proposition 3.8 from Corollary 3.10 (which is in a sense more powerful) to point out a curious fact. Namely, if $d > 2g - 2$, which is precisely when $\psi|_{K_C + D} : [K_C + D]^* \to M_C(2, D)$ is dominant, then we can construct a single $\mathbb{Q}$-divisor on $X$ for which all the maps $X_{k-1} \to X_k$ simultaneously become log flips. When $d \leq 2g - 2$, however, one needs to tailor the divisor to the variety $X_{k-1}$ and in fact it seems that no single $\mathbb{Q}$-divisor on $X$ will be log canonical and have the desired intersection properties with all the extremal rays. (At least the linear algebra construction does not produce such a divisor.)

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