Selfgravitating nonlinear scalar fields

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Abstract

We investigate the Cauchy problem for the Einstein - scalar field equations in asymptotically flat spherically symmetric spacetimes, in the standard 1 + 3 formulation. We prove the local existence and uniqueness of solutions for initial data given on a space-like hypersurface in the Sobolev $H_1 \cap H_{1,4}$ space. Solutions exist globally if a central (integral) singularity does not form and/or outside an outgoing null hypersurface. An explicit example demonstrates that there exists a local evolution with a naked initial curvature singularity at the symmetry centre.

I. Introduction.

The local Cauchy problem in General Relativity has been solved long ago by Y. Choquet-Bruhat [1] in the so-called harmonic gauge but the global Cauchy problem still remains unsolved, despite progress made in recent years. The list of known results concerning evolution in asymptotically flat spacetimes includes the global existence of almost Minkowskian geometries [2] and two special cases of spherically symmetric systems - massless scalar fields with characteristic initial data [3] and the Vlasov - Einstein equations [4]. On the other hand the validity of the main open question of gravitational physics, the cosmic censorship hypothesis [5], would demand the existence of global Cauchy solutions. More radically, the cosmic censorship question can be identified with the global Cauchy problem [6].

In this paper we consider the Cauchy problem for Einstein equations coupled to a class of nonlinear scalar fields. We specialize to spherically symmetric and asymptotically flat systems, with initial data prescribed on a space-like hypersurface. Our interest is in finding
the weakest possible solutions; that is motivated by the existence of an $L_2$ apriori estimate induced (in the absence of black holes) by the conservation of the asymptotic mass in asymptotically flat spacetimes. An ultimate reduction to the $L_2$ class would mean that the global evolution exists in the absence of black holes. We did not achieve that aim although the differentiability of solutions considered here is $H_1 \cap H_{1,4}$, i.e., weaker than of classical $C^1$ solutions. We show elsewhere \cite{7} that the breakdown of the evolution, i.e., the lapse collapse at the symmetry center, must be associated with the infinite value of the $H_1$ norm of a solution and with the conical singularity.

The plan of this paper is the following. Section II presents the Einstein - scalar field equations. They can be reduced to a system of two first order characteristic equations on $[0, \infty) \times [0, T]$. Further analysis of metric coefficients of the equations allows one to reduce that to a single ”symmetrized” equation on $(-\infty, \infty) \times [0, T]$. Section III comprises a number of estimates that will be used in further sections. Section IV consists of the main local result, Theorem 6. The local existence is proved in a standard way, by a combination of the ”viscosity” and compactness methods. The global uniqueness is shown in Section V. Section VI presents a proof of a related global Stefan problem, i.e., that a global Cauchy evolution exists outside an event horizon and, in particular, the Schwarzschild radius $R = 2m$. Section VII shows the global existence, assuming that a ”central integral singularity” does not exist. Section VIII presents an example of initial configuration for the scalar field with a singularity at the symmetry center that can be seen by external observers, i.e., it is naked. That demonstrates, in our opinion, that the concept of pointwise singularities shall be replaced by a smaller class of quasilocal (integral) singularities and that the concept of the cosmic censorship shall be accordingly reformulated.

II. Equations.

In spherically symmetric spacetimes one can always choose a diagonal line element
\[ ds^2 = -N^2(r,t)dt^2 + a(r,t)dr^2 + R^2(r,t)d\Omega^2, \]  

(1)

where \( t \) is a time coordinate, \( r \) - a radial coordinate, \( R \) - an areal radius and \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \) - the line element on the unit sphere, \( 0 \leq \phi < 2\pi \) and \( 0 \leq \theta \leq \pi \). At spatial infinity \( N = 1 \) and \( a = 1 \), for asymptotically flat spacetimes. Below we adopt the standard convention that Greek letters change from 0 to 3 while Latin indices range from 1 to 3.

Einstein equations \( R_{\mu\nu} - \frac{g_{\mu\nu}}{2}R = 8\pi T_{\mu\nu} \) can be written as a \( 1 + 3 \) system of initial constraints and evolution equations \[8\]. Let \( \Sigma_t \) be a foliation of Cauchy hypersurfaces, with \( g_{ij} \) the intrinsic metric and \( K_{ij} \) the extrinsic curvature. We adopt the convention of Wald \[8\] so that the metric signature is \((-+++\)) , \( K_{ij} = \frac{\partial g_{ij}}{\partial t} \) and \( trK = g^{ij}K_{ij} \). Let \( T_{\mu\nu} \) be the energy - momentum tensor of a matter field. The matter energy density is \( \rho = -T_{00} \) and the matter current density reads \( j_i = NT_{i0} \).

Then the Einstein constraints read:

\[(3) \quad R - K_{ij}K^{ij} + (trK)^2 = 16\pi \rho \tag{2} \]

\[ \nabla_iK^{ij} - \nabla^jtrK = -8\pi j^j \tag{3} \]

Above \( (3) \) \( R \) is the scalar curvature of the intrinsic metric of \( \Sigma_t \). It is useful to express the Einstein equations in terms of the mean curvature of a two-dimensional sphere centered around the symmetry center of \( \Sigma_t \), \( p = \frac{2hR}{\sqrt{a}R} \), and the following components of the extrinsic curvature:

\[ trK - K = 2K^0_\phi = 2K^\phi_r, \quad K = K^r_r. \tag{4} \]

The constraints in terms of \( K \) and \( p \) read

\[ \frac{\partial_r(pR)}{\sqrt{a}} = -8\pi R\rho - \frac{3R}{4}(K)^2 + \frac{R}{4}(trK)^2 + \frac{R}{2}KtrK - \frac{Rp^2}{4} + \frac{1}{R} \tag{5} \]

\[ \frac{\partial_r(R^3(K - trK))}{\sqrt{a}} = -8\pi R^3 \frac{j_r}{\sqrt{a}} - ptrKR^3 \tag{6} \]

The two remaining equations are the evolution Einstein equation:
\[ \partial_0 (K - trK) = \frac{3N}{2} (K)^2 + \frac{N}{2} (trK)^2 - 2NKtrK - \frac{p^2 R}{\sqrt{a}} \partial_r \frac{N}{pR} + 8\pi N (T^r_r + \rho) \quad (7) \]

and the lapse equation:

\[ \nabla_i \partial^i N = N \left( \frac{3}{2} (K)^2 + \frac{(trK)^2}{2} - KtrK + 4\pi (\rho + T^i_i) \right) + \partial_0 trK. \quad (8) \]

The above equations yield (via the Bianchi identity) the energy-momentum conservation equations:

\[ \partial_0 \frac{j_r}{\sqrt{a}} + N(trK + K) \frac{j_r}{\sqrt{a}} + \frac{N}{\sqrt{a}} \partial_r T^r_r + \frac{\partial_r N}{\sqrt{a}} (T^r_r + \rho) + Np(T^r_r - T^\phi_\phi) = 0 \quad (9) \]

\[ - \partial_0 \rho - \frac{Np}{\sqrt{a}} j_r - \frac{N}{\sqrt{a}} \partial_r (\frac{j_r}{\sqrt{a}}) - \frac{2\partial_r N}{a} j_r - NK(T^r_r - T^\phi_\phi) - NtrK(\rho + T^\phi_\phi) = 0. \quad (10) \]

The above equations can be converted to a system of nonlinear integral equations. They are particularly simple in the so-called polar gauge \( trK = K \). By solving the hamiltonian constraint one obtains

\[ pR = 2\sqrt{1 - \frac{2m}{R} + \frac{2m(R)}{R}}, \quad (11) \]

where \( m \) is the asymptotic mass and \( m(R) = 4\pi \int_R^\infty drr^2 \rho \) and from the evolution equation

\[ N = \frac{pR}{2} \beta(R), \quad (12) \]

\[ \beta(r) = e^{16\pi \int_r^\infty (-T^r_r - \rho) \frac{1}{pR} ds}. \quad (13) \]

The line element reads

\[ ds^2 = -dt^2 N^2 + \frac{4}{(pR)^2} dR^2 + R^2 d\Omega^2. \quad (14) \]

The above equations mean that metric functions can be expressed as some integrals of matter fields and that spherically symmetric Einstein equations do not exhibit any dynamical meaning. The whole dynamics of a selfgravitating spherical system is contained in the evolution of the material field.
The nonlinear scalar field equation is given by the second order equation

\[ \nabla_\mu \nabla^\mu \phi - W'(\phi) = 0, \]  

(15)

where \( W(\phi) \) is the scalar field potential and \( W'(x) = \frac{d}{dx}W(x) \). They can be written in the characteristic form, as a system of two first-order equations,

\[ (\partial_0 + \frac{NpR}{2} \partial_R)V = \frac{8\pi N}{p} V(j - T) - \frac{Np}{4} V - \frac{NUp}{2} - \frac{NV}{pR^2} + NW', \]  

(16)

and

\[ (\partial_0 - \frac{NpR}{2} \partial_R)U = \frac{8\pi N}{p} U(j + T) + \frac{Np}{4} U + \frac{NVp}{2} + \frac{NU}{pR^2} - NW'; \]  

(17)

above

\[ V = D\phi \]

\[ U = D\phi \]

\[ j = N \frac{T^0}{\sqrt{a}} = \frac{1}{4}(V^2 - U^2) \]

\[ T = T^r = \frac{1}{4}(V^2 + U^2) - W(\phi). \]

\[ \rho = \frac{1}{4}(V^2 + U^2) + W(\phi), \]  

(18)

where

\[ D = -\frac{1}{N} \partial_0 + \frac{pR}{2} \partial_R \]

\[ D = \frac{1}{N} \partial_0 + \frac{pR}{2} \partial_R \]  

(19)

Equations (17) and (18) are hyperbolic in a ”strict sense” if \( NpR > 0 \) i.e. \( p > 0 \).

Define

\[ \phi = \frac{\tilde{\phi}}{R} \]

\[ h_+ = \frac{D\tilde{\phi}}{pR} \]

\[ h_- = \frac{D\tilde{\phi}}{pR} \]

\[ \hat{h} = \frac{1}{2R} \int_0^R dr(h_+ + h_-) \]  

(20)
Notice that $\hat{\phi} = \int_0^R dr (h_+ + h_-)$, since by continuity one has to impose $\hat{\phi}(0) = 0$.

Define also

$$\delta(R) = \frac{N p R}{2} \quad (21)$$

Differentiation of (24) gives, with the help of the hamiltonian constraint (3), the relation

$$\partial_R \delta = \frac{3-\delta}{R} - 8\pi W R \beta.$$

That allows one to express $\delta$ in the following useful form

$$\delta(R) = \frac{1}{R} \int_0^R \beta dr - \frac{8\pi}{R} \int_0^R drr^2 \beta W(\hat{h}). \quad (22)$$

One can show, after some calculations, that

$$\beta(R) = e^{-8\pi \int_R^\infty \frac{d}{r} (h_+ - \hat{h})^2 + (h_- - \hat{h})^2} \quad (23)$$

**Remark.** One can easily show, by analyzing the Einstein constraints, that if a collapsing system possess apparent horizons, then at least one of them (the innermost apparent horizon) must be immersed inside matter. In the polar gauge apparent horizons coincide with minimal surfaces, i.e., surfaces at which the trace $p$ of the second fundamental form vanishes. Formulae (22 and 23) imply now that derivatives of metric functions $\beta, \gamma$ and $\delta$ must be singular at the location of the innermost minimal surface. That means that smooth solutions can exist only if minimal surfaces are absent, that is, the system of equations is strictly hyperbolic.

The scalar field equations reduce now to two first order equations

$$(\partial_0 + \delta \partial_R) h_+ = (h_+ - \hat{h})(8\pi \beta RW + \frac{\gamma}{R}) + \frac{\beta R}{2} W'$$

$$(\partial_0 - \delta \partial_R) h_- = -(h_- - \hat{h})(8\pi \beta RW + \frac{\gamma}{R}) - \frac{\beta R}{2} W',$$ \quad (24)

where $\gamma(R) = \delta(R) - \beta(R)$. Let us define a function by $h(R) = h_+(R)$ for $R > 0$ and $h(R) = h_-(-R)$ for $R \leq 0$. Then one can write down functions $\hat{h}$ and $\beta$ as follows

$$\hat{h} = \frac{1}{2R} \int_{-R}^R dr h(r)$$

$$\beta(R) = e^{-8\pi (\int_R^\infty + \int_{-R}^{-\infty} \frac{d}{r} (h - \hat{h})^2}. \quad (25)$$
¿From that follows \( \hat{h}(R) = \hat{h}(-R), \beta(R) = \beta(-R) \) and \( W(R) = W(-R). \) \( \beta \) infer that \( \delta(R) = \delta(-R) \) and that implies \( \gamma(R) = \gamma(-R). \) Therefore the system of two first order equations (24) can be written as a single first order equation on a "symmetrized" domain \(-\infty \leq R \leq \infty,\)

\[
(\partial_0 + \delta \partial_R) h = (h - \hat{h})(8\pi \beta RW + \frac{\gamma}{R}) + \frac{\beta R}{2} W'.
\] (26)

That is the central equation of this paper; together with definitions of \( h, \hat{h}, \beta, \delta \) and \( \gamma \) it encodes the all information carried by Einstein equations coupled to the scalar field. Notice that \( \int_{-\infty}^{\infty} dr h(r) = \int_{0}^{\infty} dr (h(r) + h(-r)) = \int_{0}^{\infty} dr \partial_r \tilde{\phi} = 0; \) therefore initial data \( h_0 \) of compact support must satisfy the condition

\[
\int_{-\infty}^{\infty} dr h_0 = 0.
\] (27)

One can easily show, using relations between metric functions and their symmetry properties, that if (27) holds true then \( \int_{-\infty}^{\infty} dr h(r, t) = 0 \) in the existence interval of a solution.

In Theorem 6 of Section 4 we formulate and then prove the local existence of solutions of (24). Its uniqueness is shown in Theorem 8. Let us point out that the above equation incorporates some of sigma models (but let us point out that the local existence result of Theorem 6 does not apply to them). The description of selfgravitating Yang - Mills \( SU(2) \) potentials reduces also to a single equation of a similar form.

### III. Estimates of metric functions and of \( \hat{h}(R). \)

We define Sobolev space \( H_1(V) \) - as a completion of \( C^1 \)-functions in the norm

\[
||f||_{H_1(-\infty, \infty)} = \left( \int_{-\infty}^{\infty} dR (\partial_R f)^2 \right)^{1/2}.
\] This Section contains a number of estimates that will be used later in order to prove the local existence and uniqueness of solutions.

Define

\[
\hat{h}(R) = \frac{1}{2R} \int_{-R}^{R} h(r) dr.
\] (28)
Lemma 1. Let $h(R) \in H_1(-\infty, \infty)$, $h(r)$ be of compact support with $h(r) = 0$ for any $|r| > R_0$ and let $1 > \delta > 0$. Then $|\hat{h}| \leq C||h||_{H_1(-\infty, \infty)}$, $\hat{h}(R) \in L_2(0, \infty)$ and
\[
||r^\delta \partial_r \hat{h}||_{L_2(-\infty, \infty)} \leq \frac{2R^2}{\delta}||h||_{H_1(-\infty, \infty)}.
\]

Proof.

Step 1.
\[
|h(R) - h(0)| \leq ||h||_{H_1(-\infty, \infty)} \frac{R^1/2}{\sqrt{2}},
\]
\[
|h(0)| \leq ||h||_{H_1(-\infty, \infty)} \frac{R^1/2}{\sqrt{2}}.
\]

Proof of Step 1.

$h(r)$ is of class $H_1(-\infty, \infty)$, that is $h \in C^{1/2}$. Now we have
\[
|h(R) - h(0)| = \left| \int_{-R}^{R} dr \partial_r h(r) \right| \leq ||\partial_r h||_{L_2(-\infty, \infty)} \frac{R^{1/2}}{\sqrt{2}} \leq ||h||_{H_1(-\infty, \infty)} \frac{R^{1/2}}{\sqrt{2}},
\]
where the first inequality follows from the Schwartz inequality. For a function $h$ vanishing outside a region $|R| \leq R_0$ one obtains $h(0) \leq ||h||_{H_1(-\infty, \infty)} \frac{R^{1/2}}{\sqrt{2}}$. That proves Step 1.

Step 2.
\[
|\hat{h}(R) - h(0)| \leq R^{1/2}||h||_{H_1(-\infty, \infty)};
\]

Proof of Step 2.

Notice the identity
\[
|\hat{h}(R) - h(0)| = \left| \frac{1}{2R} \int_{-R}^{R} (h(r) - h(0)) dr \right|
\]
and use the estimation of Step 1. That immediately yields $|\hat{h}(R) - h(0)| \leq \frac{R^{1/2}}{3\sqrt{2}} ||h||_{H_1(-\infty, \infty)}$.

Step 3.

\[
|\partial_R \hat{h}(R)| \leq 2R^{-1/2}||h||_{H_1(-\infty, \infty)}.
\]

Proof of Step 3. From (28) one gets
\[
\partial_R \hat{h}(R) = -\frac{1}{2R^2} \int_{-R}^{R} dr (h(r) - h(0)) + \frac{1}{2R}(h(R) + h(-R) - 2h(0)).
\]
Using Step 1 and performing simple integrations, one arrives at
\[ |\partial_R \hat{h}(R)| \leq \frac{2}{\sqrt{R}} \|h\|_{H_1(-\infty,\infty)}. \]

Proof of Lemma 1. Estimations follow directly from definitions of corresponding norms and from Steps 1 - 3. One has to use the assumption that a support of \( h(R) \) is finite, which gives \( \hat{h}(R) = \frac{C}{R} \) outside the support of \( h \); that ensures the \( L_2 \) integrability of \( \hat{h} \).

Define
\[ < h >= h - \hat{h}. \quad (29) \]

**Lemma 2.** Let \( h \) satisfies conditions of Lemma 1 and \( 1 > \eta > 0 \). Then
\[ |< h >| \leq |CR^{1/2}||h||_{H_1(-\infty,\infty)} \]
\[ ||R^n\partial_R < h >||_{L_2(-\infty,\infty)} \leq C||h||_{H_1(-\infty,\infty)} \quad (30) \]

Proof of Lemma 2.

Notice that \( < h > (R) = h(R) - h(0) - \frac{1}{2\pi} \int_{-R}^{R} dr (h(r) - h(0)) \) where \( |h(r) - h(0)| \) is bounded by Step 1. That gives the first estimate of Lemma 2.

The second, integral, bound on \( \partial_R < h > = \partial_R h - \partial_R \hat{h} \)
follows immediately from Lemma 1.

Define
\[ \beta(R) = e^{-8\pi(\int_{-R}^{\infty} + \int_{-\infty}^{-R}) dr \frac{1}{\pi} <h>^2} \]

**Lemma 3.** Let \( h \) satisfies conditions of Lemma 1. Then
i) \( e^{-C||h||_{H_1(-\infty,\infty)}^2} \leq \beta(R) \leq 1; \)
ii) \( |\partial_R \beta(R)| \leq C||h||_{H_1(-\infty,\infty)}^2 \) and \( \partial_R \beta(R)|_{R=0} = 0, \)

with \( C \)'s being some constants depending on the support of \( h(R) \).

Proof.
i) Obviously $\beta(R) \leq 1$. The lower bound of i) follows from the first estimate of Lemma 2 on $< h >$. Invoking to the finiteness of the support of $h(r)$ and $\hat{h}(r)$, one arrives at the sought inequality.

ii) Direct differentiation of $\beta$ with respect $R$ yields

$$\frac{d}{dR} \beta(R) = \frac{8\pi}{R} (\frac{< h >^2}{R} + \frac{< h >^2}{R}) \beta(R).$$

The first estimate of Lemma 2 yields $|\partial_R \beta(R)| \leq C||h||^2_{H^1(\mathbb{R})}$. From Lemma 2 we have $|< h(R) >| \leq R^{1/2} ||h||_{H^1(\mathbb{R})}$; $\partial_R \beta(R)$ is continuous for $R \neq 0$ and, being an antisymmetric function of $R$, must vanish at the origin. That gives ii).

Define

$$\gamma(R) = \frac{1}{R} \int_0^R \beta dr - \beta(0) - \frac{8\pi}{R} \int_0^R dr r^2 \beta W(\hat{h}).$$

**Lemma 4.** Let $h$ satisfies conditions of Lemma 1. Assume that $|W(x)|$ can be bounded from above by a polynomial of k-th order in $x$ with constant coefficients, and $W(x) \geq 0$. Then

i) $|\gamma(R)| \leq CR(||h||^2_{H^1(\mathbb{R})} ||h||^k_{H^1(\mathbb{R})})$,

ii) $|\partial_R \gamma(R)| \leq C(||h||^2_{H^1(\mathbb{R})} ||h||^k_{H^1(\mathbb{R})})$,

where $C$ changes from a line to line, but it depends only on $R_0$ and coefficients of $W$.

Moreover, $\partial_R \gamma(0) = 0$.

Proof of Lemma 4 is straightforward and consists in applying hitherto proven estimates in order to bound the derivatives of $\gamma$ in question.

i) Notice that

$$|\gamma(R)| = \left| \frac{1}{R} \int_0^R (\beta(r) - \beta(0)) dr - \beta(0) - \frac{8\pi}{R} \int_0^R dr r^2 \beta W(\hat{h}) \right| =$$

$$\left| \frac{1}{R} \int_0^R dr \int_0^r ds \partial_s \beta(s) - \frac{8\pi}{R} \int_0^R dr r^2 \beta W(\hat{h}) \right| \leq$$

$$CR(||h||^2_{H^1(\mathbb{R})} ||h||^k_{H^1(\mathbb{R})}),$$

where in the last line we used the estimation ii) of Lemma 3.
Using the mean value theorem, one can write the second line of the preceding equation as

\[- \int_{\theta R}^{R} \partial_r \beta dr - \frac{8\pi}{R} \int_{0}^{R} drr^2 \beta W(h),\]  

(33)

where \(1 > \theta > 0\). From that and from the estimation ii) of Lemma 3 one arrives at the second estimate of Lemma 4. By antisymmetry and continuity of \(\partial R \beta\) we have also \(\partial R \gamma(0) = 0\). That accomplishes the proof of Lemma 4.

Define

\[\delta(R) = \gamma(R) + \beta(R);\]

estimates of derivatives of \(\delta\) up to first order follow immediately from those of \(\gamma\) and \(\beta\). Thus

**Lemma 5.** Let \(h\) satisfies conditions of Lemma 1. Assume that \(|W(x)|\) is bounded by a polynomial of \(k\)-th order in \(x\), and \(W(x) \geq 0\). Then

i) \(\delta(R) \leq CR(||h||_{H^1_{[-\infty,\infty]}}^2 + ||h||_{H^1_{[-\infty,\infty]}}^k),\)

ii) \(|\partial R \delta(R)| \leq C(||h||_{H^1_{[-\infty,\infty]}}^2 + ||h||_{H^1_{[-\infty,\infty]}}^{2k}),\)

and \(\partial R \delta|_{R=0} = 0\). Above \(C\) changes from a line to line, but it depends only on \(R_0\) and coefficients of \(W\).

IV. The existence of local Cauchy solutions.

**Definition.** We define \(H^1_{1,4}(a, b)\) as a completion of classical \(C^1\) functions \(f\) of compact support in the \(||\partial_r f||_{L^4(a,b)}\) norm.

In the case of \(a, b < \infty\) we have the inclusion \(H^1_{1,4}(a, b) \subset H^1_1(a, b)\).

We will frequently use the following technical result.

**Proposition A.** Let \(f\) be a continuous function of a compact support \(\Omega = [-R_0, R_0] \times (0, T)\) and \(A, B\) some constants (depending on \(R_0\) and \(T\)) such that for all \(0 \leq t \leq T\)

i) \(||f||_{H^1_{[-R_0, R_0]}} < A.\)

ii) \(||\partial_0 f||_{L^4_{[-R_0, R_0]}} < B.\)
Then there exists a constant $C$ depending only on $R_0$ and $T$ such that
\[
|f(R_1, t_1) - f(R_2, t_2)| \leq C(|R_1 - R_2|^{1/2} + |t_1 - t_2|^{1/2}).
\]

For the proof see [9]. Below we shall outline its main points. A part of the above statement, the "equal time inequality", can be proven in a way similar to that employed in Step 1. Similarly as before one shows that
\[
|f(R_1, t_1) - f(R_2, t_2)| \leq \|f\|_{H_1} \sqrt{|R_1 - R_2|}. \tag{34}
\]

we use the Schwartz inequality and the assumption ii).

By continuity of $f$ there exists a point $R_3$ lying between $R_1$ and $R_2$ such that $|f(R_3, t_1) - f(R_3, t_2)| \leq B(t_2 - t_1)\sqrt{(R_2 - R_1)}$. Notice that $|f(R_1, t_1) - f(R_1, t_2)| \leq |f(R_1, t_1) - f(R_3, t_1)| + |f(R_3, t_2) - f(R_1, t_2)| + |f(R_3, t_1) - f(R_3, t_2)|$; employing the "equal time" inequality for the function $f$ at fixed times $t_1$ and $t_2$ and choosing $\frac{t_2 - t_1}{T} = \frac{R_2 - R_1}{2R_0}$ one arrives at $|f(R, t_1) - f(R, t_2)| \leq C\sqrt{t_2 - t_1}$ for any $R \in (R_1, R_2)$. Combining that result with the "equal time" inequality one accomplishes the proof of Proposition A.

Theorem 6. Let the initial data of equation (24) on an initial slice $\Sigma_0$ be of compact support, $\inf_{\Sigma_0} \delta > 0$ and

i) $h_0 \in H_{1,4}(-\infty, \infty)$; assume also that

ii) $\int_{-\infty}^{\infty} dr h_0(R, t) = 0$. Let $0 \leq W(x)$ and $|W'(x)|$ be bounded by a polynomial with constant coefficients of order $k$. Then there exists a local Cauchy solution of (24).

Remark. Theorem 6 implies the existence of a foliation $\Sigma_t$ for some $T(0 \leq t < T)$, with $h_+, h_- \in H_1(0, \infty)$, and with no minimal surfaces on any leaf $\Sigma_t$. Indeed, having $h(R, t)$, one determines all metric functions and the scalar field itself - see Section II for corresponding formulæ. That minimal surfaces are absent in a local evolution follows from the proof,
where the positivity of \( \delta \) is proven. The existence of a local evolution of (26) can be proven without the assumption ii); the latter is needed to make the identification with the Einstein - scalar field equations (see a remark at the end of Section 2).

**Proof.**

Let us notice that Eq. (26) is nonlocal and integro-differential. We prove its solvability from first principles.

In the part A of the proof we consider a regularized equation in \( H_1 \). The existence of a local in time solution is proven in a standard way, using a method of successive approximations and then standard compactness method.

In the part B we show that if initial data are in \( H_{1,4} \), then the regularization can be removed. Once again the compactness method ensures the existence of a weakly convergent subsequence, whose limit is the sought (local in time) solution of the reduced equations (26).

**Part A.**

Let us define a regularized equation,

\[
(\partial_0 + \delta_\partial_R)h = < h > \left( 8\pi \beta R W + \frac{\gamma |R|}{R} \right) + \frac{\beta R}{2} W',
\]

where all coefficients are defined as in section II. (The introduction of the parameter \( \epsilon \) is reminiscent of the viscosity method known in the Navier - Stokes equation.) Denote a solution of (35) by \( h_\epsilon \). Define a sequence of functions \( h_{n\epsilon}(t, R) \) as follows:

\[
h_{0\epsilon}(t, R) = h(t = 0, R)
\]

and \( h_{n\epsilon} \) is a solution of

\[
(\partial_0 + \delta_{n-1}\partial_R)h_{n\epsilon} = < h_{n-1} > \left( 8\pi \beta_{n-1} R W_{n-1} + \frac{\gamma_{n-1} |R|}{R} \right) + \frac{\beta_{n-1} R}{2} W_{n-1}',
\]

where

\[
\beta_n(R) = e^{-8\pi(\int_R^\infty + \int_{-\infty}^{-R} dR') \frac{dR'}{R} <h_n>^2}
\]
\[\delta_n(R) = \frac{1}{R} \int_0^R \beta_n dr - \frac{8\pi}{R} \int_0^R r^2 \beta_n W_n(\hat{h}_n)\]
\[\gamma_n(R) = \delta_n(R) - \beta_n(R)\]
\[\hat{h}_n = \frac{1}{2R} \int_0^R dr (h_{ne}(r) + h_{ne}(-r))\]
\[< h_n > = h_{ne} - \hat{h}_n\] (37)

We use the method of induction to show the existence of a sequence of functions for a small but nonzero interval of time, such that
\[||h_{ne}||_{H_1(-\infty, \infty)} \leq \frac{1}{(C^* - (4k' - 1)\bar{C}t)^{\frac{1}{4k'-1}}}\],
(38)
where \(\bar{C}\) is the same constant that appears in Eq. (42), \(k' = \text{sup}(1, k)\) and \((C^*)^{-1/(4k'-1)} = ||h(t = 0)||_{H_1(-\infty, \infty)}\). Thus, \(\bar{C}\) and \(C^*\) are some constants that depend only on initial data, \(k\) and coefficients of the polynomial \(W\).

i) step \(n = 0\) is trivial. \(h_0\) is at least \(C^{1/2}\) as a function of \(R\) and \(t\) and it obviously satisfies the bound. Coefficients of (7.3) are \(C^{3/2}\), thence there exists a solution \(h_{1\epsilon} \in C^{1/2}\), by a standard result for linear equations [10] and Proposition A.

ii) let there exists a solution \(h_{ne} \in H_1(-\infty, \infty)\) for some \(n\). One easily infers that \(h_{ne}\) satisfies the conditions of the preceding Proposition, so that \(h_n\) is \(C^{1/2}\) as a function of \(R\) and \(t\). Notice that \(\delta_n(t, R) \leq 1\). That means that the support of \(h_n\) at a time \(t\) must be placed within \(-R^0 - t, R_0 + t\), that is, it remains bounded.

There exists also a short interval of time such that \(\delta_n(t, R)\) is positive, since initially \(\delta_n(0, R) = \delta(0, R) > 0\). We prove that using the induction hypothesis. By direct computation one shows that
\[\partial_0 \delta_n = -\frac{8\pi}{R} \int_0^R dr r^2 [W_n \partial_0 \beta_n + \beta_n W_n \partial_0 \hat{h}_n] + \frac{1}{R} \int_0^R dr \partial_0 \beta_n\]
and, from the definition of \(\beta_n\) and \(W_n\) and the approximating equation (36),
\[\partial_0 \beta_n = -16\pi \beta_n (\int_r^\infty + \int_r^{-\infty}) \frac{dr}{r} < h_n > [A_n(r) - \frac{1}{2r} \int_r^\infty d\tilde{r} A_n(\tilde{r})]\]
where
\[A_n = -\delta_{n-1} \partial_R h_n + < h_n > \left(8\pi \beta_{n-1} RW_{n-1} + \frac{|R|^2 \gamma_{n-1}}{R} + \beta_{n-1} RW'_{n-1}/2\right).\]
One can bound $\hat{A}_n$ by $CR^{1/2}||h_n\epsilon||_{H^1}$ and $|\partial_0\beta_n(R)|$ and $|\partial_0\delta_n|$ by $C||h_n\epsilon||_{H^1}$, using the estimates of Lemmata 1 -5. (C is a constant that depends only on initial data and may change from line to line and $x$ is a number depending only on $W_n$.) That shows, using the induction hypothesis on the behaviour of Sobolev norms of $h_n\epsilon$, that $\beta_n$ and $\delta_n$ are nonzero and finite for a sufficiently small time $t$, if their initial values are nonzero.

Then various differentiability properties of $<h_n\epsilon, h_n\epsilon, \beta_n, \gamma_n$ and $\delta_n$ follow immediately from Lemmata 1-5 and Steps 1-3 of Lemma 1. In particular, the coefficient $\delta_n(t, R)$ is easily shown to be $C^{3/2}$ while the right hand side of the approximating equation is certainly at least $C^0$; that guarantees the existence of $h_{n+1\epsilon}$, thanks to a standard existence theorem for linear equations as formulated by, for instance, Petrovsky. The boundedness of $W'(\hat{h}_{n\epsilon})$ is controlled due to estimates of $\hat{h}_{n\epsilon}$ and the assumption that $W'(x)$ is bounded by a polynomial in $x$ with bounded coefficients. We shall show that the $H_1$ norm of $h_{n+1\epsilon}$ is bounded by a number that depends only on initial data and $W$; that would mean also that the interval of the existence of $h_{n\epsilon}$ is bounded from below by a number that does not depend on the index $n$. In order to do so, let differentiate the equation (35) with respect $R$. That gives an equation of the form

$$((\partial_0 + \delta_n \partial_R)\partial_R h_{n+1\epsilon} = \frac{d}{dR} <h_n\epsilon>(8\pi \beta_n R W_n + \frac{\gamma_n|R|\epsilon}{R}) + \frac{\beta_n R^2}{2} W'_n) -$$

$$\frac{1}{2}\frac{d}{dR} h_{n+1\epsilon} \frac{d}{dR} \delta_n.$$  \hspace{1cm} (39)

Multiplying that equation by $\frac{d}{dR} h_{n+1\epsilon}$, integrating over the whole real line and integrating by parts, one arrives at

$$\partial_0 \frac{1}{2} ||\partial_R h_{n+1\epsilon}||_{L^2(-\infty, \infty)}^2 = \int_{-\infty}^{\infty} \frac{d}{dR} h_{n+1\epsilon} dR \left[ \frac{d}{dR} <h_n\epsilon>(8\pi \beta_n R W_n + \frac{\gamma_n|R|\epsilon}{R}) + \frac{\beta_n R^2}{2} W'_n +

< h_n\epsilon > \frac{d}{dR} (8\pi \beta_n R W_n + \frac{\gamma_n|R|\epsilon}{R}) - \right.

\left. \frac{1}{2}\frac{d}{dR} h_{n+1\epsilon} \frac{d}{dR} \delta_n \right]$$ \hspace{1cm} (40)

One can use the estimates of Lemmata 1 - 5 and Steps 1-3 and eventually arrive at the inequality

15
\[
\frac{d}{dt}||h_{n+1,\epsilon}||_{H_1(\mathbb{R})} \leq C||h_{n+1,\epsilon}||_{H_1(\mathbb{R})} \left(||h_n||_{H_1(\mathbb{R})}^{4k} + ||h_{ne}||_{H_1(\mathbb{R})}^4\right),
\]

where \(C\) depends only on \(k\) and initial data. Introducing a new constant \(\tilde{C}\) and \(k' = \sup(4k, 4)\), one gets the following inequality

\[
\frac{d}{dt}||h_{n+1,\epsilon}||_{H_1(\mathbb{R})} \leq \tilde{C}||h_{ne}||_{H_1(\mathbb{R})}^{4k'}.
\]

Using the induction hypothesis and integrating (42), one arrives at

\[
||h_{n+1,\epsilon}||_{H_1(\mathbb{R})} \leq \frac{1}{(C^* - (4k' - 1)\tilde{C}t)^{4k' - 1}}
\]

which concludes the proof of the induction hypothesis. (43) shows that the Sobolev norm of each function \(h_{ne}\) is bounded by \(n\) - independent number and, that the interval \(T\) of the existence of solutions of all approximating equations is bounded away from zero by a number that is \(n\) - independent, \(0 < T < \frac{C^*}{C(4k' - 1)}\). From the approximating equation and (43) one deduces that \(\int_{-\infty}^{\infty} dr (\partial_0 h_n(r, t))^2 \leq C\), where \(C\) is \(t\) - and \(n\) - independent. Therefore \(h_{ne}\) satisfies conditions of Proposition A, which implies that the sequence \(h_{ne}\) is equicontinuous and equibounded.

Obviously, also \(\int_0^T dt \int_{-\infty}^{\infty} dr ||h_{ne}||^2 + \int_0^T dt \int_{-\infty}^{\infty} dr |\partial_t h_{ne}|^2 \leq C\) for some \(C\). Now, the standard compactness argument shows the existence of a subsequence \(h_{n,\epsilon}\) weakly convergent to \(h_\epsilon\) in \(H_1([0, T] \times \mathbb{R})\). \(h_{ne}\) is equicontinuous and equibounded, therefore by the Arzela-Ascoli theorem it contains a subsequence convergent pointwise to a limit \(h_\epsilon\). \(h_\epsilon\) in turn, being a limit of functions satisfying conditions of Proposition A, must be of class \(C^{1/2}\). The pointwise convergence to \(h_\epsilon\) and \(C^{1/2}\) continuity of \(h_\epsilon\) implies the pointwise convergence of \(\delta_n, \gamma_n, \beta_n\) and \(\hat{h}_n\) to functionals depending on the limiting solution \(h_\epsilon\). Thus the right hand side of (33) tends pointwise to an expression depending on the weak limit \(h_\epsilon\). We can conclude that \(h_{n,\epsilon}\) tends to a weak (distributional) solution \(h_\epsilon\) of the equation (35).

The norm of \(h_\epsilon\) is bounded by a constant that depends on \(\frac{1}{\epsilon}\), so it would become infinite when removing regularization, that is if \(\epsilon \to 0\). We will show, however, that there
exists a subset of initial data which gives rise to an evolution that survives the removal of regularization.

**Part B.**

Let initial data be of of compact support and \( \partial_R h \epsilon L_4(-\infty, \infty) \); that implies also that \( h \epsilon H_1(-\infty, \infty) \). Thus, by the result proven in Part A, there exists a local evolution. Now one can show that \( \partial_R h \epsilon L_4(-\infty, \infty) \) for some time \( 0 < t < T \).

One easily shows that in such a case all estimates of Lemmae 1-5 improve by a factor \( R^{1/4} \). We have, in particular,

\[
|\partial_R < h_\epsilon > | \leq C \frac{||\partial_R h_0||_{L_4(-\infty, \infty)}}{R^{1/4}}
\]

for any \( t < T \) and with \( C \) being (possibly) \( \epsilon \)-dependent.

In such a case we can improve, however, the statement of Lemma 2, to get

**Lemma 7.** Let \( \partial_r h \epsilon L_4(-\infty, \infty) \). Then a solution of the regularized equation satisfies the following estimates

\[
| < h_\epsilon(t) > | \leq CR^{3/4}||h_\epsilon(t)||_{L_4(-\infty, \infty)}
\]

\[
||R^{\eta-1/2}\partial_R < h_\epsilon(t) > ||_{L_2(-\infty, \infty)} \leq C||h_\epsilon||_{H_1(-\infty, \infty)},
\]

where \( C \) is \( \epsilon \)-independent.

With this new estimate one can show that \( H_{1,4} \) and \( H_1 \) norms of \( h_\epsilon \) remain uniformly bounded for \( \epsilon \to 0 \). Take a sequence of \( \epsilon_i \) tending to 0 as \( i \to \infty \); there exists a subsequence of \( h_{\epsilon_i} \) that is weakly convergent in \( H_1 \) to a limit \( h \); that is the sought solution of the equation \([33]\), as can be shown by repeating arguments used in the final part of Part A. Also, \( ||\partial_R h||_{L_4} < C \). That accomplishes the proof of Theorem 6.

**V. Uniqueness of solutions.**

**Theorem 8.** Under conditions of Theorem 6, if \( W \) and \( W' \) are Lipschitz continuous, there exists a unique Cauchy solution of the reduced equation \([24]\).
Proof.

Let $h_1$ and $h_2$ be two solutions satisfying given initial data of class $H_1 \cap H_{1,4}$. We have $h_1(t = 0, R) = h_2(t = 0, R)$.

Let the suffix "1" or "2" means that a function in question $\beta, \gamma, \delta, \hat{h}, < h >$ depends on $h_1$ or $h_2$, respectively. Notice that $< f > + < g >= < f + g >$. We have

$$\beta_1(R, t) = e^{-8\pi(f^\infty_R + f^{-\infty}_R) \frac{<h_1^2>}{r}} = \beta_2 e^{-8\pi(f^\infty_R + f^{-\infty}_R) \frac{<h_1 + h_2 > < -h_1 + h_2>}{r}}.$$  \(46\)

We can prove

Lemma 9. Under conditions of Theorem 8,

$$\left| \left( \int_R^\infty + \int_{-R}^{-\infty} dr < h_1 + h_2 > < -h_1 + h_2> \right) \right| \leq C \left| h_1 - h_2 \right|_{L_4(-\infty, \infty)}.$$  \(47\)

Indeed, using several times the Schwarz inequality, the inequality $(a - b)^2 \leq 2a^2 + 2b^2$ and the improved (for $h_i \in H_{1,4}$) estimate of Lemma 7

$$\left| < h > (R) \right| \leq CR^{3/4} \| \partial_t h \|_{L_4(-\infty, \infty)}^2,$$

one bounds the integral of \(17\) by

$$\left[ \int_{-\infty}^\infty \frac{dr < h_1 + h_2 >^2}{r^2} \right]^{1/2} \left[ \int_{-\infty}^\infty \frac{dr < h_1 - h_2 >^2}{r} \right]^{1/2} \leq C \left[ \int_{-\infty}^\infty dR \left( (h_1 - h_2)^2 + \frac{1}{4R^2} \left[ \int_{-R}^R (h_1 - h_2) dr \right]^2 \right) \right]^{1/2} \leq C \left| h_1 - h_2 \right|_{L_4(-\infty, \infty)}.$$  \(48\)

In the above calculation we used the finiteness of the support of initial data; the constant $C$ depends on the support of initial data and on $H_{1,4}$ norms of $h_1$ and $h_2$.

The above lemma yields, for small values of $\| h_1 - h_2 \|_{L_4(-\infty, \infty)}$ the following estimation

$$\left| \beta_1(R, t) - \beta_2(R, t) \right| \leq C \left| h_1 - h_2 \right|_{L_4(-\infty, \infty)}.$$  \(49\)

In a similar way one shows that

$$\| (\hat{h}_1 - \hat{h}_2) \| \leq \frac{C}{R^{1/4}} \| h_1 - h_2 \|_{L_4(-\infty, \infty)}.$$  \(50\)
and (using Lipschitz continuity)
\[ |W(\hat{h}_1) - W(\hat{h}_2)| \leq \frac{C}{R^{1/4}} \|h_1 - h_2\|_{L_4(-\infty, \infty)}. \] (51)

An analogous relation holds for the difference \(|W'(\hat{h}_1) - W'(\hat{h}_2)|\).

From (49), (22) and the \(W\) and \(W'\) estimates, one shows that
\[ |\delta_1(R, t) - \delta_2(R, t)| \leq C \|h_1 - h_2\|_{L_4(-\infty, \infty)}. \] (52)

Similarly one arrives at
\[ |\gamma_1(R, t) - \gamma_2(R, t)| \leq C \|h_1 - h_2\|_{L_4(-\infty, \infty)}. \] (53)

Above and below \(C\) is a certain constant that changes from line to line, independent of \(t\) and \(R\).

Subtracting the reduced equations for \(h_2\) from that for \(h_1\) and using the above estimates on the right hand side of the substracted equations, one gets (below \(\Delta h = h_1 - h_2\))
\[ (\partial_0 + \delta_1 \partial_R) \Delta h + (\delta_1 - \delta_2) \partial_R h_2 \leq C |\Delta h| + \left( |h_1| + |h_2| \right) \|\Delta h\|_{L_4(-\infty, \infty)}; \] (54)
multiplying (54) by \((\Delta h)^3\), once again estimating the difference \(|\delta_1 - \delta_2|\) by \(\|\Delta h\|_{L_4(-\infty, \infty)}\) and integrating by parts, one eventually arrives at the inequality
\[ \partial_0 \|\Delta h\|_{L_4(-\infty, \infty)}^4 \leq C \|\Delta h\|_{L_4(-\infty, \infty)}^4; \] (55)
that implies \(\|\Delta h\|_{L_4(-\infty, \infty)} = 0\), since at \(t = 0\) \(\Delta h = 0\). The last inequality holds true for sufficiently small \(t\). \(h_1\) and \(h_2\) are continuous functions, therefore \(h_1 = h_2\) at least for sufficiently small intervals of time. Iteration of that reasoning leads to the conclusion that if there exists a solution of the reduced equation, then it is unique in the \(L_\infty\) norm.

That means, in turn, that the possible nonuniqueness can be seen on the level of first derivatives of \(h\) and, even if there exist two solutions with different derivatives, then still \(\gamma_1 = \gamma_2\), \(\hat{h}_1 = \hat{h}_2\), \(\delta_1 = \delta_2\) and \(\gamma_1 = \gamma_2\) up to their first derivatives.

Using that one can easily show that also the \(H_1\) norm of the difference \(\Delta h\) must vanish. In fact, let \(dh = \partial_R \Delta h\); from the reduced equation one gets
\[ \partial_0 dh = -\partial_R(\delta dh) + dhF \] (56)

where \( F \) denotes terms which do not involve \( dh \). Integrating (56) over \( R \) one gets, after employing various estimates proven in the first part of this paper

\[ \partial_0 ||dh||_{L_2(-\infty, \infty)} \leq C ||dh||_{L_2(-\infty, \infty)}; \] (57)

that yields \( ||dh||_{L_2(-\infty, \infty)} = 0 \), from Grönewall inequality, since at \( t = 0 \) we have \( ||dh||_{L_2(-\infty, \infty)} = 0 \). Combining that with the already proven fact, we conclude that the solution of the reduced equation is unique in the sense of \( H_1 \). A similar reasoning gives uniqueness in \( H_{1,4} \). That ends the proof of Theorem 8.

VI. External Cauchy problem.

It occurs that that there are two main problems in proving the global existence.

One is due to difficulties in estimating needed quantities at the origin. That we omit by considering a sort of an external Cauchy evolution; we will investigate whether initial data of Einstein - scalar field equations give rise to an evolution that exists globally outside any outgoing null hypersurface originating at an initial spacelike hypersurface.

The other is the possible emergence of minimal surfaces during an evolution; that would mean that equations become singular. We will show that this does not happen; that fact is well known and it shows that polar gauges are deficient in the sense that they do avoid regions of spacetime with minimal surfaces; if initially minimal surfaces are absent, then they cannot develop in Cauchy slices satisfying the condition \( trK = K^r_r \) during a finite evolution, assuming that a dominant energy condition is satisfied. The proof of this claim goes as follows. Let \( \Omega^\text{out}_{R'} = [(R, t) : |R| \geq R_{in}, \frac{dR_{in}}{dt} = \delta, R_{in}(t = 0) = R' > 0] \) be a patch of hypersurfaces that evolve from an initial slice \( \Sigma_{R'_{in}}^\text{out} \) (which is free of minimal surfaces) and let \( \Sigma_{R_{out}}^\text{out} \) be the first slice with a minimal surface located at an areal radius \( R_m \). \( R_{in}(t) \) describes the location of the free inner boundary. By the regularity of the evolution, the four-metric
is at least $C^1$ on that piece of the space-time; thus there exists a null ingoing geodesic joining the four-point $(R_m, t)$ with a point $(R' > R_m, 0)$ lying on the initial slice. Along that geodesic the mean curvature $p$ decreases from an initial nonzero value $p_0$ to 0. The change of the mean curvature along the ingoing null geodesic is given, however, by one of the Raychaudhuri equations (that can be obtained, in that case, by manipulating the evolution Einstein equation and the Hamiltonian constraint). We use the geodesic coordinates with the line element $ds^2 = -N^2 dt^2 + dl^2 + R^2 d\Omega^2$ One can find, after some calculations, that
\begin{equation}
(\partial_t - N \partial_l)(pR) = 8\pi NR(j + \rho) + \frac{N}{4R}(p^2 R^2 - 4);
\end{equation}
using now the energy condition $|j| \leq \rho$, definition of the lapse $N$ and the fact that $R$ is lowering along the ingoing null ray, one gets the inequality
\begin{equation}
(\partial_t - N \partial_l R)(pR) \geq -\frac{pR\beta(R)}{2R_m} \geq -\frac{pR}{2R_m}.
\end{equation}
In the last line I used the estimation $\beta(R) \leq 1$. Equation (59) yields
\begin{equation}
pR \geq \sup_{\Sigma_{R'}^\text{out}} (pR)e^{-t/2R_m},
\end{equation}
which must be nonzero for $t < \infty$. Thus we obtained a contradiction, that enforces us to accept that polar gauge slicings cannot penetrate regions with minimal surfaces. Notice also that (60) gives a lower bound for the minimal value of mean curvature on subsequent Cauchy slices; that will used later.

Now we state the main result of this section.

**Theorem 10.** Take a part $\Sigma_{R'} = [(R, t = 0) : |R| \geq R', ]$ (with $R' \geq 0$) of the initial hypersurface $\Sigma_0$ . Let initial data of equation (26) on an initial slice be of compact support, $\inf_{\Sigma_{R'}} \delta > 0$, the mass function $m(R') \leq m$ (with $h(R') = h(-R') = 0$ if $m(R') = m$) and
i) $h_0 \in H_{1,4}(\Sigma_{R'})$; assume also that
ii) $(\int_{-\infty}^{-R'} + \int_{R'}^{\infty})dr h_0(R, t) = 0$. Let $0 \leq W(x)$ and $|W'(x)|$ be bounded by a polynomial with constant coefficients of order $k$. Then there exists a global Cauchy solution in $\Omega_{R'}^\text{out}$. Theorem 10 has been proven in [11], under stronger differentiability conditions, $h \in H_2$. Below we present a modified proof that bases on the results of preceding sections.
First of all, we have to write down the reduced problem in a modified (but equivalent) form. While keeping $\beta$ in the form (25), we choose the representation (21) of $\delta$, namely
\[
\delta(R) = \frac{(pR)^2}{4} \beta(R) = \left(1 - \frac{2m}{|R|} + \frac{2m(R)}{|R|} \right) \beta(R), \tag{61}
\]
where $m$ is the asymptotic mass and
\[
m(R) = 4\pi \int_{-R}^{\infty} dr r^2 \rho = 4\pi \left( \int_{-R}^{\infty} dr - \int_{-\infty}^{-R} dr \right) \left( \frac{\delta(r)}{\beta(r)} < \hat{h} >^2 + \frac{r^2}{2} W(\hat{h}) \right), \tag{62}
\]
It is convenient to deal with $\hat{h}$ expressed as follows
\[
\hat{h}(R) = -\frac{1}{2R} \left( \int_{-\infty}^{-R} + \int_{R}^{\infty} \right) h(r) dr \tag{63}
\]
which is equivalent to the expression (23) used before, if $\int_{-\infty}^{\infty} dr h(r) = 0$. With those forms of $\beta, \delta, h$ and $\gamma$ it is obvious that solutions of the reduced equation (26) outside any given outgoing null cone $\delta_{R'}$ do not depend on its interior. We use that fact in proving Theorem 10.

Namely, we smoothly extend initial data across $R' > 0$ to vacuum, keeping conditions $\int_{-\infty}^{\infty} h(r) dr = 0$, $m = m(0)$ and $\delta > 0$, to get $h(r) = 0$ for $|r| < R' - \eta$ for some $\eta > 0$; it is easy to see that there exist extensions which do not change significantly the required $H_1$ and $H_{1,4}$ Sobolev norms. Therefore we may use the local result of Theorem 6 to infer the existence of a local solution; notice that according to the preceding remark, outside an outgoing null cone $\delta_{H_{R'}}$ (including the cone itself) defined by $\frac{dR}{dt} = \delta, R(t = 0) = R'$, the solution is independent of the extension.

There is a number of useful local estimates; obviously $0 < \beta \leq 1$, $0 < \delta \leq 1$, $\gamma \leq 2$ and $m(R) \leq m$. We need a bound on $\hat{h}$. That is proven in the following

**Lemma 11.** Under conditions of Theorem 6,
\[
|\hat{h}(R)| \leq \frac{\sqrt{m}}{\sqrt{4\pi R(\inf_{R' \geq R} \frac{\delta}{\beta})^{1/2}}} \tag{64}
\]
Proof of Lemma 11. (Assume, for simplicity, $R > 0$).
We have \( \hat{h}(R) = - \int_R^\infty dr \partial_r \hat{h} dr = (\int_R^\infty + \int_{-\infty}^{-R}) \frac{\epsilon \hat{h} \hat{r}}{r} dr; \) using the Schwartz inequality, we get
\[
|\hat{h}(R)| \leq \left[ \left( \int_R^\infty dr + \int_{-\infty}^{-R} \right) \frac{\delta(r)}{\beta(r)} \left( < h >^2 + \frac{1}{2} W(\hat{h}) \right) \right]^{1/2}
\]
which is bounded by
\[
\frac{1}{R^{1/2} \left( \inf_{R' \geq R} \frac{\delta}{\beta} \right)^{1/2}} \left[ \left( \int_R^\infty dr + \int_{-\infty}^{-R} \delta(r) \beta(r) \left( < h >^2 + \frac{1}{2} W(\hat{h}) \right) \right) \right]^{1/2}.
\]
The integral term is bounded from above by \( \frac{m(R)}{4\pi} \) which in turn is not bigger than \( \frac{m}{4\pi} \). That ends the proof of Lemma 11.

Take now the patch of slices of constant \( t \) of \( \Omega_{\text{out}}^R \). It is easy to find, manipulating with the reduced equation (26) that the rate of change of the \( L_p \) norm of \( h \) (for any even value of \( p \)) along the external foliation is bounded,
\[
\frac{d}{dt} ||h||_{L_p(\Sigma_{\text{out}}^R)} \leq C ||h||_{L_p(\Sigma_{\text{out}}^R)},
\]
where \( C \) depends on the above local estimates. \( C \) can be infinite if minimal surfaces appear, but that cannot happen for \( t < \infty \), as proven at the beginning of this section. Therefore the growth of \( L_p \) norm is controlled. A similar reasoning gives a control also of \( H_1 \) and \( H_{1,4} \) norms.

The bootstrap argument yields now immediately the global existence. Indeed, let \( T \) be the maximal existence interval of a solution in the exterior region: thus at any \( T - \eta \) all norms are finite. By using the above reasoning one shows that relevant norms must be finite at \( t = T \), which leads to contradiction. That ends the proof of Theorem 10.

**Remark on smoothness.** In the external region one can reduce the smoothness requirements on \( h \) from \( H_{1,4} \) to \( H_1 \). Indeed, if initial data vanish on a compact neigbourhood of the symmetry center on the initial hypersurface, then there exists a compact domain \([-R(T), R(T)] \times [0, T] \) with null data. In such a case the regularization procedure of Section IV is not necessary and one can show the existence of local solutions in \( H_1 \). In the globalization part presented above one uses only those local extensions that are null close to the origin \( R = 0 \); therefore also the global existence extends to \( H_1 \). That reasoning allows
one to conclude that Theorem 10 holds true also for matter with selfinteraction $W$ that is singular at the origin but satisfies the remaining boundedness conditions. Thus there exists a global evolution in an external region for Yang Mills $SU(2)$ fields (with $W = \frac{(1-\phi^2)^2}{2R^2}$) and skyrmionic $SU(2)$ fields (with $W = \frac{\sin^2(\phi)}{2R^2}$).

**Remarks on the free boundary $\delta H_{R'}$.** It is easy to notice (see (12) that outside the Schwarzschild region, $R' \geq 2m$, minimal surfaces must be absent ($\delta$ or $pR$ must be strictly positive). In that case the inner boundary $\delta H_{R'}$ of $\Omega^\text{out}_{R'}$ escapes to spatial infinity, $|R_{in}(t)|$ increases without bound. Therefore initial data posed outside the Schwarzschild radius always give rise to global external solutions.

In the alternative case, with the initial hypersurface entering the interior of the Schwarzschild sphere, we may consider two situations:

i) a rather trivial case, when the inner boundary $\delta_{R'}$ of some of the future Cauchy slices crosses (at some finite $t'$) through the sphere located at the areal radius $2m$; in that case we have the global existence, with the conclusion, that $\delta H_{R'}$ escapes to spatial infinity.

ii) the inner boundary "freezes" close to a sphere of an areal radius $R < 2m$.

We will investigate the second point in more detail. One can easily show that the area of an outermost apparent horizon cannot decrease (see, e.g. [12]); in fact it has to increase whenever matter (satisfying the strong energy condition) crosses through the horizon; that has to move acausally outwards. Asymptotically the areal radius of the apparent horizon becomes equal to $2m_B$, where $m_B$ is the Bondi mass of the black hole. Take a part $\Sigma^\text{out}_{r_0}$ of the initial hypersurface that does not include minimal surfaces. Then data on $\Sigma^\text{out}_{r_0}$ give rise to a local evolution, according to the local Theorem 6. The global evolution prolongs until the free inner boundary freezes at some areal radius $R < 2m$, close to the (anticipated) minimal surface. In such a case one can take a slightly smaller initial open end $\Sigma^\text{out}_{r'} \subset \Sigma^\text{out}_{r_0}$; that evolves to a spacetime that freezes at a later time than the previous one. Continuing that procedure ad infinitum one finds finally a smallest open end such that the area $dH$ of a null inner boundary $\delta_{R'}$ a corresponding spacetime $H$ still stabilizes at a value $4\pi R^2_{B'}$.

$\delta_{R'}$ is an event horizon and half of $R_B$ is the Bondi mass. Thus there exists a solution
for that exterior region $\Sigma_{R,t}^{\text{out}}$ whose inner boundary coincides with an event horizon that is asymptotic to a minimal surface located somewhere at $R \leq 2m$. That solution is global in the sense that it does exist for arbitrarily large $t$, but on the other hand it does not cover a part of the physical spacetime which is hidden behind an apparent horizon.

VII. Global existence and central integral singularities.

**Theorem 12.** Assume conditions of Theorem 6. Assume that there exists a small cylinder $\Omega^{R_0} = [(R, t) : R \leq R_0]$ such that a contribution to $H_1 \cap H_{1,4}$ norm of $h$ from a spatial section $t = \text{const}, \Omega^{R_0}_t$, of $\Omega^{R_0}$ is uniformly bounded, $||h||_{H_{1,4}(\Omega^{R_0}_t)} < C$, where $C$ is $t$-independent. Then the Cauchy evolution of the Einstein-scalar field system exists globally.

**Proof.** In the first part of the proof we will use the global existence of solutions of the related Stefan problem. Using Theorem 10, the proof of Theorem 12 proceeds as follows. Take $\Omega^{\text{out}}_{R_0/2}$; by the proof of Theorem 10, the norm of $h$ in $\Omega^{\text{out}}_{R_0/2}$ is bounded by at most exponentially increasing function of time, $||h||_{H_1(\Omega^{\text{out}}_{R_0/2}, t)} < C_0 e^{ct}$. Thus, taking into account the uniform bound in $\Omega^{R_0}_t$, we have $||h||_{H_1(\Omega^{\text{out}}_{R_0/2}, t)} < \infty$ at a time $t = R_0/2$. Now, take a portion $\Sigma_{t,R_0/2}$ of the Cauchy slice at a time $t$; using the same reasoning as before, we can extend the existence period from $R_0/2$ into $R_0$. Iterating that reasoning we infer the global existence.

If we assume a condition stronger than in Theorem 12, namely that (keeping the same notation)

$$
\sup_{0 < R_0 < 2m} \left[ \frac{1}{R_0^{1/2}} \frac{||h||^2_{H_{1,4}(\Omega^{R_0}_t)}}{||h||^2_{H_{1,4}(\Omega^{R_0}_t)}} \right] < C,
$$

where $C$ is small enough then the spacetime is geodesically complete. For definiteness, consider the massless scalar fields; then $C = \frac{1}{48\pi}$. Indeed, from (11) and (12) one obtains

$$
\delta(R) = \beta(R) \left( 1 - \frac{8\pi}{R} \int_0^R dr \left( \frac{\delta(r)}{\beta(r)} \left( h(R) > 2 + h(-R) > 2 \right) \right) \right) \geq
$$

$$
\beta(R) \left( 1 - \frac{1}{48R^{1/2} \pi} ||h||^2_{L_4(\Omega^{R_0}_t)} \right) \geq
$$
\[ \beta(R) \left(1 - \frac{1}{48 R^{1/2} \pi} \| h \|_{H^{1,4}(\mathbb{R}^3_0)}^2 \right) > 0. \]  

(66)

The first inequality follows from \( < h >^2 \leq 2 h^2 + 2 \dot{h}^2 \), \( \int_R^R dr \dot{h}^2 dr \leq 2 R^{1/2} \| h \|_{L^4(-R,R)} \) and \( \int_R^R dr \hat{h}^2 dr \leq R^{1/2} \| h \|_{L^4(-R,R)} \). (66) means that all time-like and null-like geodesics have infinite proper or affine length.

**Remark.** Theorem 12 essentially states that if there is no central singularity (understood as a portion of spacetime that gives infinite contribution to the \( H^{1,4} \) norm of \( h \)), then there is no singularity at all. That is an accordance with a corresponding result proven by Rein, Rendall and Schaeffer [4] in the case of the Vlasov - Einstein system.

That conclusion is consistent with an analogous result in [13], proven entirely in the framework of initial data formalism, which shows the absence of \( L^2 \) singularities on any Cauchy slice with a regular trace of extrinsic curvature and with (at most ) a conical singularity in the symmetry center.

VIII. Naked singularities.

We will give an example of an initial configuration that gives rise to \( H^{1,4} \) evolution and that is characterized by a pointwise curvature singularity at the symmetry center. That singularity can be seen by external observers placed at spatial infinity. Other examples of naked singularities in various material systems can be found in [14].

As a material model we choose a scalar field with the nonlinear selfinteraction potential

\[ W(\phi) = \sin^2(\phi). \] (67)

Assume that \( h(R) = 0 \) outside some \( |R_0|, h(R) = \lambda \epsilon(R)|R|^{\alpha} \) for \( |R| < R_0 \) and \( \frac{3}{4} < \alpha < 1 \) (where \( \epsilon(x) = 1 \) for \( x > 0 \) and \(-1 \) for \( x < 0 \)), with a smooth transition in between. Then \( \hat{h} \) vanishes identically in the initial hypersurface, \( \beta(R, t = 0) = e^{-\frac{2\alpha \lambda^2}{\alpha} (R_0)^{2\alpha} - R^{2\alpha}} \) \( e^{-\frac{8\alpha \lambda^2}{\alpha} (R_0)^{2\alpha}} \) and \( \delta(R) = \frac{1}{R} \int_0^R \beta(r) dr \geq e^{-\frac{8\alpha \lambda^2}{\alpha} (R_0)^{2\alpha}} \). The energy density \( \rho \) at \( t = 0 \) is equal to \( 4\pi \frac{\dot{h}^2}{R^2} \) and it is divergent like \( R^{-2+2\alpha} \) at the origin. The hamiltonian constraint (2)
yields now that the three-dimensional Ricci scalar $R$ is also divergent like $R^{−2+2α}$.

The $H_{1,4}$ norm of $h$ is finite and it is merely proportional to $\frac{λ}{4α−3}$. Therefore there exists an evolution in an interval $T$. From the local analysis of Section IV one obtains

$$||h||_{H_{1,4}(t)} \leq \frac{1}{(C^* − (4k' − 1)λCt)^{\frac{1}{2k'−1}}}$$

(68)

where $k' ≥ 1, (C^*)^{\frac{1}{4k'-1}} = \frac{1}{||h||_{H_{1,4}(t=0)}}$ and $C$ is a certain constant. Therefore the smaller is $λ$, the bigger is the existence interval $T$.

By differentiation of the metric function $δ$ one obtains

$$\partial_0 δ = -\frac{8π}{R} \int_0^R drr^2 [W \partial_0 β + βW' \partial_0 \dot{h}] + \frac{1}{R} \int_0^R dr \partial_0 β$$

and, from the definition of $β$ and $W$ and the reduced equation (66),

$$\partial_0 β = -16πβ(\int_R^∞ + \int_{−∞}^R \frac{dr}{r} < h > [A(r) − \frac{1}{2r} \int_r^∞ d\tilde{r} A(\tilde{r})])$$

where

$$A = −δrRh+ < h > (8πβRW + \frac{γ}{R}) + βRW'/2.$$  

One can bound $|\partial_0 β(R)|$ by $CR||h||_{H_{1,4}}^{k'}$ and then also $|\partial_0 δ|$ by $C||h||_{H_{1}}^{k'}$, using estimates analogous to those of Lemme 1 -5. Combining that with (68) yields

$$κ ≡ \inf_R δ(R, t) ≥ e^{-\frac{8πλγ}{α}(R_0)^{2α}} − C(λ)T,$$  

(69)

where $C(λ) → 0$ for $λ → 0$ and $t ≤ T$.

Let $λ$ be so small as to have $\frac{2m}{κ} < T$. Then (since $δ$ is at least $C^1$) there exists a solution $R(t)$ of the null geodesic equation

$$\frac{dR}{dt} = δ$$

(70)

with the initial value $R(0) = 0$ such that $R(T) > 2m$. The exterior of the cylinder $R = 2m$ is geodesically complete, therefore the null geodesic $R(t)$ reaches any asymptotic observer.

Hence we established that the central curvature singularity is naked.
A closer investigation shall reveal that this singularity is not strong in the sense of Tipler [15].

That example suggests that the concept of pointwise singularities shall be replaced by a class of quasilocal (integral) singularities. The latter are understood as those local singularities that are characterized by infinite values of some quasilocal (integral) quantities (e.g., some Sobolev norms). That notion of a singularity seems to be more natural than the idea of strong singularities, since quasilocal quantities can be directly related to a quantitative description of the Cauchy evolution. Consequently, the concept of the cosmic censorship shall be accordingly reformulated.

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