Time-reversibility and integrability of $p : -q$ resonant vector fields

Jaume Giné$^1$, Valery G. Romanovski$^{2,3,4}$, Joan Torregrosa$^5$

$^1$Departament de Matemàtica, Universitat de Lleida, Av. Jaume II, 69, 25001 Lleida, Catalonia, Spain

$^2$Faculty of Electrical Engineering and Computer Science, University of Maribor, Koroška cesta 46, SI-2000 Maribor, Slovenia

$^3$Center for Applied Mathematics and Theoretical Physics, Mladinska 3, SI-2000 Maribor, Slovenia

$^4$Faculty of Natural Science and Mathematics, University of Maribor, Koroška cesta 160, SI-2000 Maribor, Slovenia

$^5$Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

Abstract

We study local analytical integrability in a neighborhood of $p : -q$ resonant singular point of a two-dimensional vector field and its connection to time-reversibility with respect to the non-smooth involution $\varphi(x, y) = (y^{p/q}, x^{q/p})$. Some generalizations of the theory developed by K. S. Sibirsky for $1 : -1$ resonant case to the $p : -q$ resonant case are presented.

1 Introduction

Consider an $n$-dimensional system of ordinary differential equations

$$\dot{x} = F(x)$$

(1.1)

where $F(x)$ is an $n$-dimensional vector functions defined on some domain $D$ of $\mathbb{R}^n$ or $\mathbb{C}^n$. It is said (see e.g. [2,15]) that system (1.3) is time-reversible on $D$ if there exists an involution $\psi$ defined on $D$ such that

$$D_\psi^{-1} \cdot F \circ \psi = -F.$$  

(1.2)

We say that system (1.3) is completely analytically integrable on $D$ if it admits $n-1$ functionally independent analytic first integrals on $D$.

A time-reversal symmetry is one of the fundamental symmetries that appears in nature, in particular, important both for classical and quantum mechanics. Various properties of systems exhibiting such symmetries have been studying by many authors, see e.g. [1,2,12,13,27,29] and the references therein.
Our paper is devoted to the investigation of the interconnection of time-reversibility and local integrability in a neighborhood of a singular point of systems of the form
\[ \dot{x} = Ax + X(x), \]
where \( A \) is an \( n \times n \) matrix with entries in \( \mathbb{R} \) or \( \mathbb{C} \), \( x = (x_1, \ldots, x_n) \), \( X(x) \) is a vector-function without constant and linear terms defined on some domain \( D \) of \( \mathbb{R}^n \) or \( \mathbb{C}^n \).

One of the first results in such studies is due to Poincaré. It follows from his results that if in the two-dimensional case the eigenvalue of \( A \) are pure imaginary and the system has an axis of symmetry passing through the origin, then it admits an analytic first integral in a neighborhood of the origin.

A generalization of this result is presented in [3], where it is shown that if system (1.3) is time-reversible with respect to a certain linear involution and two eigenvalues of the matrix \( A \) are pure imaginary, then under some assumptions the system has at least one analytic first integral in a neighborhood of the origin.

A detail study of the interconnection of time-reversibility and local integrability for systems (1.3) was presented in [17]. In [17] and [28] the notion of time reversibility was generalized to the case when on the right hand side of (1.2) ”−1" is replaced by a primitive root of unity.

In this paper we limit our consideration to the two-dimensional systems (1.3) with non-degenerate matrix \( A \). In the case when a 2-dim system (1.3) is real and the eigenvalues of \( A \) are pure imaginary, and the vector field is symmetric with respect to a curve passing through the origin, the origin of (1.3) is a center, and, therefore, has an analytic local integral in a neighborhood of the origin. This geometric argument was used in [30] in order to find some integrable systems in the family of real cubic systems (see also [2] for recent developments in this direction). The symmetry axis is, in the general case, an analytic curve passing through the origin. However, from the work of Montgomery and Zippin [18], any analytic involution \( \psi \) associated with a time-reversal symmetry can be linearized in such a way that the symmetry axis becomes a straight line. From this result in [1] the normal form theory is used to establish an algorithm to determine if a two-dimensional systems (1.3) is orbitally reversible.

A detail study of real polynomial systems which are time-reversible under reflection with respect to a line was performed by Sibirsky [24,25]. In particular, he showed that in the polynomial case the set of such systems in the space of parameters is the variety of a binomial ideal defined by invariants of the rotation group of the system (some similar results were obtained also in [5,16]). Later on the results obtained by Sibirsky were generalized to the case of complex systems (1.3) with 1 : −1 resonant singular point at the origin in [14,19,20].

In this paper we consider system (1.3) having a \( p : -q \) resonant singular point at the origin, which we write in the form
\[ \dot{x} = px - \sum_{j+k \geq 1, j \geq -1}^{\infty} a_{jk} x^{j+1} y^{k} = px(1 - \sum_{j+k \geq 1, j \geq -1}^{\infty} \frac{1}{p} a_{jk} x^{j} y^{k}) \]
\[ \dot{y} = -qy + \sum_{j+k \geq 1, j \geq -1}^{\infty} b_{kj} x^{j} y^{j+1} = -qy(1 - \sum_{j+k \geq 1, j \geq -1}^{\infty} \frac{1}{q} b_{kj} x^{j} y^{j}), \]
(1.4)
where \( p, q \in \mathbb{N}, \text{GCD}(p,q) = 1 \). Both vector field (1.4) and the associated differential operator are denoted by \( \mathcal{X} \).
Unless \( p = q = 1 \) this system is not time-reversible under a linear transformation. Our study deals with the time-reversibility of system (1.4) with respect to the involution

\[
\varphi(x, y) = (y^{p/q}, x^{q/p}).
\]  

(1.5)

We will prove that if a system (1.4) is time-reversible with respect to (1.5), then it admits an analytic first integral on a neighborhood of the origin. We will also give an extension of the results of [24, 25] and their generalizations obtained in [14, 19, 20] presenting an algorithm for finding subsets of polynomial systems (1.4) which are time-reversible with respect to (1.5) and generalizing the notion of Sibirsky ideal to polynomial systems (1.4). It will be shown that, in fact, the theory developed for 1 : −1 resonant case can be extended to systems (1.4), however not to the whole family, but only to a certain subfamily of (1.4).

2 Integrability of time-reversible systems

For system (1.4) it is always possible to find a series of the form (2.1)

\[
\Psi(x, y) = x^{q}y^{p} + \sum_{j+k \geq p+q; j, k \in \mathbb{N}_{0}} v_{j-k,p-q}x^{j}y^{k}
\]  

(2.1)

for which

\[
\mathcal{X}\Psi = g_{q,p}(x^{q}y^{p})^{2} + g_{2q,2p}(x^{q}y^{p})^{3} + g_{3q,3p}(x^{q}y^{p})^{4} + \cdots,
\]  

(2.2)

where \( g_{kq,kp} \), \( k = 1, 2, \ldots \), are polynomials in the parameters \( a_{jk}, b_{kj} \) of system (1.4). Polynomials \( g_{kq,kp} \) are called the saddle quantities of system (1.4) (sometimes also the focus quantities). System (1.4) corresponding to some fixed values \( a_{jk}^{*}, b_{kj}^{*} \) of the parameters has a local analytical first integral in a neighborhood of the origin if and only if \( g_{kq,kp}(a_{jk}^{*}, b_{kj}^{*}) = 0 \) \( \forall k \in \mathbb{N} \) (see e.g. [20, 23]).

The following theorem shows that if system (1.4) is time-reversible with respect to (1.5) then it has an analytic first integral of the form (2.1). Observe, that if \( p = q = 1 \) then the map (1.5) is just a permutation of the variables, so the statement presents a generalization of known results of [3, 17, 19] to the case of \( p : -q \) resonant systems.

**Theorem 1.** Assume that system (1.4) is time-reversible with respect to the involution (1.5), that is,

\[
D^{-1}_{\varphi} \circ \mathcal{X} = -\mathcal{X}.
\]  

(2.3)

Then it admits an analytic first integral of the form (2.1) in a neighborhood of the origin.

To prove the theorem we will need the following results.

**Lemma 1.** System (1.4) with \( p \) or \( q \) different from 1 is time-reversible with respect to (2.3) if and only if

\[
b_{qu,up} = \frac{q}{p} a_{qu,pv}, \quad a_{qu,pv} = \frac{p}{q} b_{qu,pv},
\]  

(2.4)

where \( u, v = 0, 1, 2, \ldots \) and the other coefficients in (1.4) are equal to zero.

**Proof.** Using involution (1.5), that is, performing the substitution

\[
x_{1} = y^{p/q}, \quad y_{1} = x^{q/p},
\]  

(2.5)
after straightforward calculations we obtain
\[ \dot{x}_1 = -px_1(1 - \sum_{j_1 + k_1\geq 1, j_1\geq -1} \frac{1}{q} b_{k_1j_1} x_1^{q_{k_1}} y_1^{q_{j_1}}) \]
\[ \dot{y}_1 = qy_1(1 - \sum_{j_1 + k_1\geq 1, j_1\geq -1} \frac{1}{p} a_{j_1k_1} x_1^{p_{j_1}} y_1^{p_{k_1}}). \] (2.6)

In view of (2.3) it should hold
\[ \frac{1}{p} a_{jk} x^j y^k = \frac{1}{q} b_{kj} x^{q_j} y^{q_k} \] (2.7)
where the exponents on the right hand side should be non-negative integers or
\[ \frac{qj_1}{p} = \frac{pj_1}{q} = -1. \]

However the latter equality is impossible unless \( p = q = 1 \). Thus, (2.7) can take place if we set \( j_1 = pu, k_1 = qv \), where \( u, v = 0, 1, 2, \ldots \). This yields formulae (2.4).

The following statement follows directly from (2.4).

**Corollary 1.** If system (1.4) with \( p \) or \( q \) different from 1 is time-reversible with respect to (2.3) then \( x = 0 \) and \( y = 0 \) are the separatrices of the system.

**Remark.** Formulas (2.4) and the results obtained below remain valid also in the case of l : −1 resonant singular points. But since the results in the l : −1 resonant case are known, below we work under the assumption \( p/q \neq 1 \) taking advantage from the fact that in such case the subscripts of the parameters \( a_{jk}, b_{kj} \) of (1.4) are non-negative.

We augment the set of coefficients in (2.1) with the collection
\[ J = \{v_{-q+s,q-s} : s = 0, \ldots, p+q\}, \]
where, in agreement with formula (2.1), we set \( v_{00} = 1 \) and \( v_{mn} = 0 \) for all other elements of \( J \), so that elements of \( J \) are the coefficients of the terms of degree \( p+q \) in \( \Psi(x,y) \) of the form (2.1).

By [22, p. 117] the coefficients \( v_{k_1,k_2} \) of series (2.1) can be computed recursively using the formula
\[ v_{k_1,k_2} = \begin{cases} \frac{1}{pk_1-qk_2} \sum_{s_1+s_2=0, s_1\geq-q, s_2\geq-p}^{k_1+k_2-1} [(s_1+q) a_{k_1-s_1,k_2-s_2} - (s_2+p) b_{k_1-s_1,k_2-s_2}] v_{s_1,s_2} & \text{if } pk_1 \neq qk_2 \\ 0 & \text{if } pk_1 = qk_2, \end{cases} \]
(2.8)

(in [22] formula (2.8) was obtained for the case of polynomial system (1.4) but, obviously, it remains valid also in the case when the right hand sides of (1.4) are series).

We order the index set of parameters \( a_{jk} \) in the first equation of (1.4) in some manner, say by degree lexicographic order from least to greatest, and write the ordered set as \( S = \{(1,0), (0,1), (-1,2), (2,0), \ldots \} \). Consistently with this we then order the parameters as \((a_{10}, a_{01}, a_{-1,2}, a_{20}, \ldots, b_{02}, b_{2,-1}, b_{10}, b_{01})\) so that any monomial appearing in \( v_{ij} \) has the
form $a_{10}^\nu a_{t_1}^\nu \cdots a_s^\nu f_{t_1}^{\nu t_1} \cdots b_{10}^{\nu t_1} b_0^{\nu t_1}$ for some $\nu = (\nu_1, \ldots, \nu_2)$ and $\ell = 1, 2, \ldots$. To simplify the notation, for $\nu \in \mathbb{N}_+^2$ we write

$$[\nu] = a_{10}^\nu a_{t_1}^\nu \cdots a_s^\nu f_{t_1}^{\nu t_1} \cdots b_{10}^{\nu t_1} b_0^{\nu t_1},$$

(2.9)

so if the $k$-th variable in the product is $a_{pq}$, then $2\ell - k + 1$-st variable is $b_{qp}$.

For each $m \in \mathbb{N}$ we consider the finite subset $S_m$ of the set $S$ which corresponds to the case when system (1.4) is a polynomial system of degree $\nu$ in the polynomial ring $\mathbb{C}[a,b]$ defined by

$$L^\nu = \text{Supp}(f),$$

(2.8)

Let $k[a,b]$ be the ring of polynomials in parameters $a_{jk}, b_{jk}$ of system (1.4) over the field $k$ and for $f \in k[a,b]$ we write $f = \sum_{\nu \in \text{Supp}(f)} f^{\nu}[\nu]$, where $\text{Supp}(f)$ denotes those $\nu \in \mathbb{N}_+^2$, $m = 1, 2, \ldots$, that the coefficient of $[\nu]$ in the polynomial $f$ is nonzero.

**Definition 1.** For $(j, k) \in \mathbb{N}_- \times \mathbb{N}_-$, a polynomial $f$ in the polynomial ring $\mathbb{C}[a,b]$ is a $(j, k)$-polynomial if, for every $\nu \in \text{Supp}(f)$, $L^\nu = (j, k)$ for all sufficiently large $\ell$.

The reader can consult [22, Section 3.4] for more details about $(j, k)$-polynomials in the case of polynomial system (1.4).

From now on we will limit our consideration to the systems of the form

$$\dot{x} = x(p - \sum_{t_1 + t_2 = 1}^{\infty} a_{qt_1, pt_2} x^{qt_1} y^{pt_2}),$$

$$\dot{y} = -y(q - \sum_{t_1 + t_2 = 1}^{\infty} b_{qt_2, pt_1} x^{qt_2} y^{pt_1}).$$

(2.11)

By Lemma 1 in the case when $p/q \neq 1$ systems (1.4), which are time-reversible with respect to involution (1.5), form a subfamily of systems (2.11), so we do not lose generality working with family (2.11) if we are interesting in time-reversibility with respect to (1.5).

**Lemma 2.** For system (2.11) if $v_{k_1, k_2}$ is a non-zero coefficient of series (2.1) computed by (2.8), then

$$k_1 = t_1 q, \quad k_2 = t_2 p$$

(2.12)

for some non-negative integer $t_1, t_2$. Moreover, under involution (2.3) the term

$$v_{qt_1, pt_2} x^{qt_1} y^{pt_2}$$

(2.13)

of (2.1) is changed to the term

$$v_{qt_2, pt_1} x^{qt_2} y^{pt_1}$$

(2.14)

and vice versa.
Proof. For system (2.11) the linear map (2.10) can be written in the form

\[ L^m(\nu) = (qL^m_1(\nu), pL^m_2(\nu)) \]

where \((L^m_1(\nu), L^m_2(\nu))\) is defined by (2.10) and \(L^{m-q+1}(\nu) = \ldots = L^{m-1}(\nu) = L^m(\nu)\). By Theorem 4 of [21] \(v_{k_1,k_2}\) is a \((k_1,k_2)\) polynomial. Therefore for each monomial \([\nu]\) of \(v_{k_1,k_2}\) it holds that

\[ (qL^m_1(\nu), pL^m_2(\nu)) = (k_1, k_2) \]

for all sufficiently large \(m\). It means that \(q\) divides \(k_1\) and \(p\) divides \(k_2\), that is, (2.12) holds.

Performing in (2.13) substitution (2.5) we see that (2.14) holds. \(\square\)

Remark. Theorem 4 of [21] mentioned above was formulated in [21] for the case of polynomial systems (1.4) but it remains correct also in the case when the right hand sides of (1.4) are series.

**Theorem 2.** The formal series (2.1) computed according to (2.8) is unchanged under involution (2.3), that is, in view of Lemma 2

\[ v_{qt_1,pt_2} = v_{qt_2,pt_1}. \]  

**Proof.** We prove the claim using induction on \(t_1 + t_2\). When \(t_1 = t_2 = 0\) we have by the definition \(v_{00} = 1\), so the claim holds.

Using Lemma 2 we can write formula (2.8) as

\[
\begin{aligned}
v_{qt_1,pt_2} &= \begin{cases} 
\frac{1}{pq(t_1-t_2)} \sum_{s_1+s_2=0}^{qt_1+pt_2-1} \left[(s_1+q)a_{qt_1-s_1,pt_2-s_2} - (s_2+p)b_{qt_1-s_1,pt_2-s_2}\right]v_{s_1,s_2} & \text{if } t_1 \neq t_2 \\
0 & \text{if } t_1 = t_2.
\end{cases}
\end{aligned}
\]  

(2.15)

In view of (2.4) and taking into account that \(v_{j,k}\) are \((j,k)\)-polynomials for \(t_1 \neq t_2\) we can change the rule of summation obtaining from (2.10)

\[
\begin{aligned}
v_{qt_1,pt_2} &= \frac{1}{pq(t_1-t_2)} \sum_{s_1=0}^{qt_1} \sum_{s_2=0}^{pt_2} [(s_1+q)a_{qt_1-s_1,pt_2-s_2} - (s_2+p)b_{qt_1-s_1,pt_2-s_2}]v_{s_1,s_2} \\
&= \frac{1}{pq(t_1-t_2)} \sum_{s_1=0}^{qt_1} \sum_{s_2=0}^{pt_2} [\tilde{s}_1 + 1)a_{q(t_1-\tilde{s}_1),p(t_2-\tilde{s}_2)} - (\tilde{s}_2 + 1)p\tilde{b}_{q(t_1-\tilde{s}_1),p(t_2-\tilde{s}_2)}]v_{\tilde{s}_1,\tilde{s}_2},
\end{aligned}
\]  

(2.16)

where \(s_1 = q\tilde{s}_1, s_2 = p\tilde{s}_2\).

Performing similar computations we have

\[
\begin{aligned}
v_{qt_2,pt_1} &= \frac{1}{pq(t_2-t_1)} \sum_{s_1=0}^{qt_2} \sum_{s_2=0}^{pt_1} [(s_1+q)a_{qt_2-s_1,pt_1-s_2} - (s_2+p)b_{qt_2-s_1,pt_1-s_2}]v_{s_1,s_2} \\
&= \frac{1}{pq(t_2-t_1)} \sum_{s_2=0}^{qt_2} \sum_{s_1=0}^{pt_1} [(\tilde{s}_2 + 1)a_{q(t_2-\tilde{s}_2),p(t_1-\tilde{s}_1)} - (\tilde{s}_1 + 1)p\tilde{b}_{q(t_2-\tilde{s}_2),p(t_1-\tilde{s}_1)}]v_{\tilde{s}_2,\tilde{s}_1},
\end{aligned}
\]  

(2.17)

where \(s_1 = q\tilde{s}_2, s_2 = p\tilde{s}_1\).
Using (2.4) we further obtain from (2.19)

\[ v_{q\tilde{s}_2,p\tilde{s}_1} = \frac{1}{pq(t_2 - t_1)} \sum_{\tilde{s}_2=0}^{t_2} \sum_{\tilde{s}_1=0}^{t_1} \left[ (\tilde{s}_2 + 1)p(b_{q(t_1-\tilde{s}_1)}(t_2-\tilde{s}_2) - (\tilde{s}_1 + 1)q(a_{q(t_1-\tilde{s}_1)}(t_2-\tilde{s}_2) \right] v_{q\tilde{s}_2,p\tilde{s}_1} = \]

\[ \frac{1}{pq(t_1 - t_2)} \sum_{\tilde{s}_1=0}^{t_1} \sum_{\tilde{s}_2=0}^{t_2} \left[ (\tilde{s}_1 + 1)q(a_{q(t_1-\tilde{s}_1)}(t_2-\tilde{s}_2) - (\tilde{s}_2 + 1)p(b_{q(t_1-\tilde{s}_1)}(t_2-\tilde{s}_2) \right] v_{q\tilde{s}_1,p\tilde{s}_2}, \tag{2.20} \]

where we have changed \( v_{q\tilde{s}_2,p\tilde{s}_1} \) to \( v_{q\tilde{s}_1,p\tilde{s}_2} \) using the induction hypothesis. Comparing the expressions for (2.18) and (2.20) we conclude that (2.15) holds, that is, the series \( \Psi(x, y) \) computed by (2.8) is unchanged under involution (2.3).

Using the obtained results we prove Theorem 1 as follows.

**Proof of Theorem 1.** Denote by \( X \) the vector field of system (2.11). By Theorem 2 the series \( \Psi(x, y) \) computed by (2.8) is unchanged under involution (1.5).

Assume that \( X \Psi = \alpha(x, y) \).

Since by our assumption the system is time-reversible, it also holds that

\[ X \Psi = -\alpha(x, y), \]

yielding \( \alpha(x, y) \equiv 0 \). That means, \( \Psi(x, y) \) is a formal first integral of (1.4). But then also there exists an analytic first integral of the form (2.1) (see e.g. [21]). □

**Corollary 2.** If system (1.4) is time-reversible with respect to involution (1.5), then the system in the distinguished Poincare-Dulac normal form is also time-reversible with respect to the same involution.

**Proof.** Since by Theorem 1 any time-reversible system (1.4) is locally analytically integrable, its distinguished normal form can be written as

\[ \dot{x} = px(1 + \sum_{k=1}^{\infty} g_k(x^q y^p)^k), \quad \dot{y} = -qy(1 + \sum_{k=1}^{\infty} g_k(x^q y^p)^k). \tag{2.21} \]

(see e.g. [21][30]). Clearly, the latter system is time-reversible with respect to (1.5). □

The normal form of any locally analytically integrable system (1.4) is given by (2.21). System (2.21) is time-reversible with respect to the involution (1.5). Therefore any locally analytically integrable system (1.4) is conjugate to a time-reversible system, in the sense that there exists a change of variables \( \phi \) that transforms the original system to the normal form (2.21) and consequently the original system is time-reversible with respect to the involution \( \bar{\psi} = \phi^{-1} \circ \psi \circ \phi \), where \( \psi \) is involution (1.5). Hence the analytically integrability of system (1.4) is always associated with a time-reversal symmetry. In fact, all the nondegenerate centers are conjugate to a time-reversible system, and all the nilpotent centers are orbitally time-reversible. This does not happens for systems with null linear part, see [12].

The problem with the map \( \bar{\psi} \) is that we have no idea about the form of \( \bar{\psi} \) not even the leading terms of such involution. Therefore from the found results we cannot deduce an algorithm based on the computation of the involution of the original system. However
several methods to compute the saddle or focus quantities are known, see for instance \[10, 11, 22\] and references therein. Nevertheless, always there exists a change \( \phi \) such that any differential system (1.4) is transformed to its normal form

\[
\dot{y}_1 = p y_1(1 + Y_1(y_1^q y_2^p)), \quad \dot{y}_2 = -q y_2(1 + Y_2(y_1^q y_2^p)). \tag{2.22}
\]

Next going through the change of variables \( z_1 = y_1^q \) and \( z_2 = y_2^p \) the normal form is transformed to the normal form of the resonance 1 : \(-1\) and the results known for such resonance can be applied to the \( p : -q \) resonance.

## 3 Conditions of time-reversibility and the Sibirsky ideal

In this section we propose an algorithmic approach which allows for a given polynomial family (2.11) to find the set of systems which are time-reversible with respect to (1.5). We also give a description of the set using the so-called Sibirsky ideal obtaining some generalizations of the results of [14, 19].

We will limit our consideration to polynomial systems of the form (2.11), that is, systems of the form

\[
\begin{align*}
\dot{x} &= x(p - \sum_{u+v=1}^{n} a_{qu,pv} x^{qu} y^{pv}), \\
\dot{y} &= -y(q - \sum_{u+v=1}^{n} b_{qv,pu} x^{qv} y^{pu}),
\end{align*} \tag{3.1}
\]

assuming that \( p/q \neq 1 \).

Denote by \( \ell \) the number of parameters in the first equation of (3.1). For \( k = 1, \ldots, \ell \) let

\[
\zeta_k = u_k - v_k \tag{3.2}
\]

and consider the ideal

\[
H = \langle 1 - w^\gamma, a_{qu,pv_k} - t_k, b_{qv,pu_k} - \frac{q}{p} \gamma \zeta_k t_k : k = 1, \ldots, \ell \rangle. \tag{3.3}
\]

**Proposition 1.** The following statements hold:

1) The Zariski closure of the set of systems in family (3.1), which are time-reversible with respect to involution (1.5) after the transformation

\[
x \to \alpha x, \quad y \to \alpha^{-1} y \tag{3.4}
\]

with \( \alpha \in \mathbb{C} \setminus \{0\} \), is the variety \( V(\mathcal{I}) \) of the ideal

\[
\mathcal{I} = H \cap \mathbb{C}[a, b]. \tag{3.5}
\]

2) If the parameters \( a_{qu,pv}, b_{qv,pu} \) of system (3.1) belong to the variety \( V(\mathcal{I}) \), then the system admits a local analytic first integral of the form (2.1).
Proof. Performing in system (3.1) transformation (3.4) we obtain the system of the same shape with the parameters \(a_{qu,pv}, b_{qv,pu}\) changed according to the rule

\[
a_{qu,pv} \mapsto \alpha^{pu-qa}a_{qu,pv}, \quad b_{qv,pu} \mapsto \alpha^{qu-qv}b_{qv,pu},
\]

where \(u + v = 1, \ldots, n\).

By Lemma 1 the system obtained after transformation (3.4) is time-reversible with respect to involution (1.5) if and only if for some \(\alpha \neq 0\)

\[
\alpha^{pu-qa}a_{qu,pv} = \frac{p}{q} \alpha^{qu-qv}b_{qv,pu},
\]

where \(u + v = 1, \ldots, n\). Equivalently, we can rewrite (3.6) as

\[
a_{qu_k,pv_k} = t_k, \quad b_{qu_k,pv_k} - \frac{q}{p} \gamma^{\alpha_k} t_k,
\]

where \(\gamma = \alpha^{-(p+q)}\), \(k = 1, \ldots, \ell\) and \(\alpha\) are defined by (3.2).

From (3.7) using the Implicitization Theorem (see e.g. [7]) we conclude that the first statement of the proposition holds.

2) By the construction \(V(\mathcal{I})\) is the Zariski closure of systems which are time-reversible with respect to (1.5) after a linear transformation (3.4), so, in view of Theorem 1 it is the Zariski closure of systems which admit a first integral of the form (2.1). However the set of systems in the space of parameters of (3.1) having an analytic first integral integral of the form (2.1) is an algebraic set (see e.g. Theorem 3.2.5 of [22]). Therefore all systems from \(V(\mathcal{I})\) admit an analytic first integral of the form (2.1).

Remark. Obviously, in generic case the set of time-reversible systems is a proper subset of \(V(\mathcal{I})\).

As an example we consider the 1 : −2 resonant system of the form (3.1) of degree five,

\[
\dot{x} = x(1 - \sum_{u+v=1}^{2} a_{2u,v} x^{2u} y^{v}) = x - a_{01} x y - a_{20} x^3 y - a_{21} x^2 y^2 - a_{40} x^5
\]

\[
\dot{y} = -y(2 - \sum_{u+v=1}^{2} b_{2u,v} x^{2u} y^{v}) = -2y + b_{01} y^2 + b_{20} x^2 y^2 + b_{21} x^2 y^2 + b_{40} x^4 y.
\]

Proposition 2. System (3.8) admits an analytic first integral of the form (2.1) if the 10-tuple \((a_{01}, \ldots, a_{40}, b_{40}, \ldots, b_{01})\) of its coefficients belong to the variety of the ideal \(\mathcal{I}\),

\[
\mathcal{I} = \langle 2a_{21} - b_{21}, -a_{40} b_{01} + 2a_{20} b_{02}, 4a_{02} a_{40} - b_{02} b_{40}, 8a_{02} a_{20}^2 - b_{02}^2 b_{40}, -2a_{02} a_{20} b_{20} + a_{01} b_{40}, 2a_{01} a_{40} b_{01} - a_{20} b_{02} b_{20}, 4a_{01} a_{20} - b_{01} b_{20}, -a_{02} b_{20}^2 + 2a_{01}^2 b_{40}, 8a_{01}^2 a_{40} - b_{02} b_{20}^2 \rangle.
\]

Proof. In the case of system (3.8) the ideal \(H\) of Proposition 1 is

\[
\langle a_{01} - t_1, a_{20} - t_2, a_{02} - t_3, a_{40} - t_4, a_{21} - t_5, b_{20} - 2t_1 \gamma^{-1}, b_{01} - 2t_2 \gamma, b_{40} - 2t_3 \gamma^{-2}, b_{02} - 2t_4 \gamma^2, b_{21} - 2t_5, 1 - w \gamma \rangle.
\]

Computing the reduced Groebner basis of this ideal with respect to the lexicographic ordering with \(\gamma > w > t_1 > t_2 > t_3 > t_4 > a_{01} > a_{02} > a_{20} > a_{21} > a_{40} > b_{01} > b_{02} > b_{20} > b_{21} > b_{40}\) we obtain the set of polynomials

\[
\{2a_{21} - b_{21}, 4a_{01} a_{20} - b_{01} b_{02}, a_{02} b_{20}^2 - 2a_{01}^2 b_{40}, 2a_{02} a_{20} b_{20} - a_{01} b_{01} b_{40}, 8a_{02} a_{20}^2 - b_{01} b_{40}, a_{40} b_{01}^2 - 2a_{20}^2 b_{42}, 2a_{01} a_{40} b_{01} - a_{20} b_{02} b_{20}, 8a_{01} a_{40} - b_{02} b_{20}, 4a_{02} a_{40} - b_{02} b_{40}, \ldots \}.
\]
where the dots stand for the polynomials which depend ob $\gamma, w, t_1, t_2, t_3, t_4, t_5$.

The polynomials of the Groebner basis which do not depend on $\gamma, w, t_1, t_2, t_3, t_4$ form a basis of the ideal $\mathcal{I}$ of Proposition 1, and they are exactly the polynomials defining the ideal in the statement of the present proposition. \hfill $\square$

**Remark.** By 1) of Proposition 1 the variety $\mathbf{V}(\mathcal{I})$ is the Zariski closure of the set of time-reversible systems in family $\{3.5\}$. We denote by $S$ the ordered set of subscripts of the coefficients of the nonlinear terms of the first equation in (3.1). Letting $\ell$ the number of elements of $S$, $S$ can be written as

$$S = \{(qu_1, pv_1), \ldots, (qu_\ell, pv_\ell)\} = \{\bar{i}_1, \ldots, \bar{i}_\ell\}.$$ 

For $\bar{i}_s = (qu_s, pv_s)$, let $\bar{j}_s = (qv_s, pu_s)$. We call $\bar{i}_s$ and $\bar{j}_s$ conjugate vectors (or conjugate indexes). Any monomial appearing in the coefficient $v_{k_1, k_2}$ of (2.1) has the form $a^{\nu_1}_{1_1} \cdots a^{\nu_\ell}_{\ell_1} b^{\nu_{\ell+1}}_{j_1} \cdots b^{\nu_{2\ell}}_{j_{2\ell}}$ for some $\nu = (\nu_1, \ldots, \nu_{2\ell})$. We use notation (2.10) adapted to the case of system (3.1), so now

$$[\nu] \overset{\text{def}}{=} a^{\nu_1}_{1_1} \cdots a^{\nu_\ell}_{\ell_1} b^{\nu_{\ell+1}}_{j_1} \cdots b^{\nu_{2\ell}}_{j_{2\ell}}.$$ 

For a given field $k$ we will write just $k[a, b]$ in place of $k[a_{1_1}, \ldots, a_{\ell_1}, b_{j_1}, \ldots, b_{j_{2\ell}}]$, and for $f \in k[a, b]$ write $f = \sum_{\nu \in \text{Supp}(f)} f^{(\nu)} [\nu]$, where $\text{Supp}(f)$ denotes those $\nu \in \mathbb{N}_{+}^{2\ell}$ such that the coefficient of $[\nu]$ in the polynomial $f$ is non-zero.

**Definition 2.** Let

$$f = \sum_{\nu \in \text{Supp}(f)} f^{(\nu)} a^{\nu_1}_{1_1} \cdots a^{\nu_\ell}_{\ell_1} b^{\nu_{\ell+1}}_{j_1} \cdots b^{\nu_{2\ell}}_{j_{2\ell}} \in \mathbb{C}[a, b].$$ 

The conjugate $\widehat{f}$ of $f$ is the polynomial obtained from $f$ by the involution

$$f^{(\nu)} \rightarrow \bar{f}^{(\nu)} : a_{q_i, p_j} \rightarrow b_{q_j, p_i}, b_{q_j, p_i} \rightarrow a_{q_i, p_j};$$

that is,

$$\widehat{f} = \sum_{\nu \in \text{Supp}(f)} \bar{f}^{(\nu)} a^{\nu_{\ell+1}}_{q_{j_1}} \cdots a^{\nu_{2\ell}}_{q_{j_{2\ell}}} b^{\nu_{\ell+1}}_{q_{j_1}} \cdots b^{\nu_{2\ell}}_{q_{j_{2\ell}}} \in \mathbb{C}[a, b].$$ 

Since $[\nu] = a^{\nu_1}_{1_1} \cdots a^{\nu_\ell}_{\ell_1} b^{\nu_{\ell+1}}_{j_1} \cdots b^{\nu_{2\ell}}_{j_{2\ell}}$, we have

$$\widehat{[\nu]} = a^{\nu_{\ell+1}}_{j_1} \cdots a^{\nu_{2\ell}}_{j_{2\ell}} b^{\nu_{\ell+1}}_{1_1} \cdots b^{\nu_{2\ell}}_{\ell_1},$$

so that

$$\widehat{[\nu_1, \ldots, \nu_{2\ell}]} = [\nu_{2\ell}, \ldots, \nu_1]. \quad (3.10)$$

For this reason we will also write, for $\nu = (\nu_1, \ldots, \nu_{2\ell})$, $\widehat{\nu} = (\nu_{2\ell}, \ldots, \nu_1)$.

Once the $\ell$–element set $S$ has been specified and ordered we let $L : \mathbb{N}_{+}^{2\ell} \rightarrow \mathbb{N}_{+}^2$ be the map defined by

$$L(\nu) = (L_1(\nu), L_2(\nu)) = \nu_1 \bar{i}_1 + \cdots + \nu_\ell \bar{i}_\ell + \nu_{\ell+1} \bar{j}_\ell + \cdots + \nu_{2\ell} \bar{i}_1. \quad (3.11)$$

which is similar to the map (2.10).

Let

$$\mathcal{M} = \{\nu \in \mathbb{N}_{+}^{2\ell} : L(\nu) = (qk, pk), \ k = 0, 1, 2, \ldots \}.$$ 

Clearly, $\mathcal{M}$ is an Abelian monoid.

Let $\mathcal{X}$ be the vector field of system (3.1). The following result was obtained in (21) (where speaking about $(s, t)$-polynomials we mean $(s, t)$-polynomials with respect to map $\{3.11\}$).
Theorem 3. Let family (3.1) be given. There exists a formal series \( \Psi(x,y) \) of the form (2.1) and polynomials \( g_{q,p}, g_{2q,2p}, \ldots \) in \( \mathbb{Q}[a,b] \) such that

(a) \[ X\Psi = \sum_{k=1}^{\infty} g_{q,k}x^{qk}y^{pk}; \] (3.12)

(b) for every pair \((i,j)\) \( \in \mathbb{N}_{-q} \times \mathbb{N}_{-p}, i + j \geq 0, v_{ij} \in \mathbb{Q}[a,b], \) and \( v_{ij} \) is an \((i,j)-\) polynomial;

(c) for every \( k \geq 1, v_{qk,pk} = 0; \) and

(d) for every \( k \geq 1, g_{qk,pk} \in \mathbb{Q}[a,b], \) and \( g_{qk,pk} \) is a \((qk,pk)-\)polynomial.

For a given family (3.1) and ordered set \( S \) of indices for any \( \nu \in \mathbb{N}^{2\ell} \) define \( V(\nu) \in \mathbb{Q} \) recursively, with respect to \( |\nu| = \nu_1 + \cdots + \nu_{2\ell} \), as follows:

\[ V((0,\ldots,0)) = 1; \]

for \( \nu \neq (0,\ldots,0) \)

\[ V(\nu) = 0 \quad \text{if} \quad pL_1(\nu) = qL_2(\nu); \]

and when \( pL_1(\nu) \neq qL_2(\nu) \),

\[ V(\nu) = \frac{1}{pL_1(\nu) - qL_2(\nu)} \times \]

\[ \left[ \sum_{j=1}^{\ell} V(\nu_1,\ldots,\nu_j-1,\ldots,\nu_{2\ell})(L_1(\nu_1,\ldots,\nu_j-1,\ldots,\nu_{2\ell}) + q) \right. \]

\[ - \sum_{j=\ell+1}^{2\ell} V(\nu_1,\ldots,\nu_j-1,\ldots,\nu_{2\ell})(L_2(\nu_1,\ldots,\nu_j-1,\ldots,\nu_{2\ell}) + p) \left. \right], \] (3.13)

where \( L(\nu) \) is defined by (3.11).

Theorem 4. For a family of systems of the form (3.1) let \( \Psi \) be the formal series of the form (2.1) computed by (2.8), \( \{g_{q,k} : k \in \mathbb{N}\} \) be the polynomials in \( \mathbb{C}[a,b] \) satisfying (3.12). Then

(a) for \( \nu \in \text{Supp}(v_{k_1,k_2}) \), the coefficient \( v_{k_1,k_2}^{(\nu)} \) of \([\nu]\) in \( v_{k_1,k_2} \) is \( V(\nu) \),

(b) for \( \nu \in \text{Supp}(g_{q,k,p}) \), the coefficient \( g_{q,k,p}^{(\nu)} \) of \([\nu]\) in \( g_{q,k,p} \) is

\[ g_{q,k,p}^{(\nu)} = - \left[ \sum_{j=1}^{\ell} V(\nu_1,\ldots,\nu_j-1,\ldots,\nu_{2\ell})(L_1(\nu_1,\ldots,\nu_j-1,\ldots,\nu_{2\ell}) + q) \right. \]

\[ - \sum_{j=\ell+1}^{2\ell} V(\nu_1,\ldots,\nu_j-1,\ldots,\nu_{2\ell})(L_2(\nu_1,\ldots,\nu_j-1,\ldots,\nu_{2\ell}) + p) \left. \right], \] (3.14)

and
(c) the following identities hold:

\[ \kappa V(\hat{\nu}) = V(\nu) \quad \text{and} \quad \kappa g^\nu_{q_k,p_k} = -g^\nu_{q_k,p_k} \quad \text{for all} \quad \nu \in \mathbb{N}_+^{2\ell} \]  
\[ V(\nu) = g^\nu_{q_k,p_k} = 0 \quad \text{if} \quad \hat{\nu} = \nu \neq (0, \ldots, 0), \] where

\[ \kappa = \left( \frac{q}{p} \right)^{(\nu_1+\cdots+\nu_\ell)-(\nu_{l+1}+\cdots+\nu_2)} . \]

Statements (a) and (b) of the theorem are proved in [21], statement (c) can be proved similarly as statement 3) of Theorem 3.4.5 of [22].

The next statements follows immediately from c) of Theorem 4.

Corollary 3. The saddle quantities \( g_{q_k,p_k} \) of system (3.1) have the form

\[ g_{q_k,p_k} = \frac{1}{2} \sum_{v: L(\nu) = (q_k,p_k)} g^\nu_{q_k,p_k}(\kappa[\nu] - [\hat{\nu}]). \] (3.16)

By the analogy with 1 : −1 resonant case we call the ideal

\[ I_{\text{Sib}} = \langle \kappa[\nu] - [\hat{\nu}] : \nu \in \mathcal{M} \rangle \] (3.17)

the Sibirsky ideal of system (3.1). Obviously, transformations (3.4) form a group. It is easy to see that for any \( \nu \in \mathcal{M} \) \( [\nu] \) is an invariant of group (3.4). Sibirsky studied such invariants for the case of 1 : −1 resonant system (1.4) and used the ideal (3.17) to describe the basis of the invariants and the number of axis of symmetry of the corresponding real systems [24, 25].

Theorem 5. If the \( 2\ell \)-tuple of parameters of (3.1) belong to \( V(I_{\text{Sib}}) \) then the corresponding system admits an analytic first integral of the form (2.1).

Proof. The conclusion follows from formula (3.16). \( \square \)

The following statement shows that the variety of the Sibirsky ideal is the Zariski closure of the set of systems, which are time-reversible with respect to (1.5). The proof is based on an adaption of the ideas of [26].

Theorem 6. Let \( \mathcal{I} \) be the ideal defined by (3.5). Then

\[ I_{\text{Sib}} = \mathcal{I}. \] (3.18)

Proof. For \( k = 1, \ldots, \ell \) let, as above, \( \zeta_k = u_k - v_k \) and consider the ring homomorphism

\[ \theta : \mathbb{Q}(a,b,t_1,\ldots,t_\ell,\gamma,w) \longrightarrow \mathbb{Q}(\gamma,t_1,\ldots,t_\ell) \]
defined by

\[ a_{q_k,u_k} \mapsto t_k, \quad b_{q_k,u_k} \mapsto \frac{q}{p} \gamma^k t_k, \quad w \mapsto 1/\gamma, \quad (k = 1, \ldots, \ell). \] (3.19)

Let \( H \) be the ideal (3.3). Clearly,

\[ H = \ker(\theta). \] (3.20)

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A reduced Groebner basis $G$ of $\mathbb{Q}[a, b] \cap H$ can be found computing a reduced Groebner basis of $H$ using an elimination ordering with $\{a_{qj, pj}, b_{qj, pj}\} < \{w, \gamma, \tau\}$ for all $j = 1, \ldots, \ell$, and then intersecting it with $\mathbb{Q}[a, b]$. Since $H$ is binomial, any reduced Groebner basis $G$ of $H$ also consists of binomials. This means that $I = H \cap \mathbb{Q}[a, b]$ is a binomial ideal.

We show that $I_{Sib} \subset I$. Taking into account that $\zeta_k = -\zeta_{2\ell-k}$ by (3.19) for any $\alpha = (\alpha_1, \ldots, \alpha_{2\ell}) \in \mathcal{M}$ we have

$$\theta([\alpha]) = t_1^{\alpha_1} \cdots t_\ell^{\alpha_\ell} t_{\ell+1}^{\alpha_{\ell+1}} (\frac{q}{p})^{\alpha_{\ell+1} \gamma} \cdots t_1^{\alpha_{2\ell}} (\frac{q}{p})^{\alpha_{2\ell} \gamma} \zeta_{\alpha_{2\ell}} =$$

$$= (\frac{q}{p})^{\alpha_{\ell+1} + \cdots + \alpha_{2\ell}} t_1^{\alpha_1} \cdots t_\ell^{\alpha_\ell} t_{\ell+1}^{\alpha_{\ell+1}} \cdots t_1^{\alpha_{2\ell}} \gamma - (\zeta_{\ell+1} \alpha_{\ell+1} + \cdots + \zeta_{2\ell} \alpha_{2\ell})$$

and

$$\theta([\hat{\alpha}]) = t_1^{\alpha_1} \cdots t_\ell^{\alpha_\ell} t_{\ell+1}^{\alpha_{\ell+1}} (\frac{q}{p})^{\alpha_1 \gamma} \cdots t_1^{\alpha_{2\ell}} (\frac{q}{p})^{\alpha_{2\ell} \gamma} \zeta_{\alpha_{2\ell}} =$$

$$= (\frac{q}{p})^{\alpha_1 + \cdots + \alpha_{\ell}} t_1^{\alpha_1} \cdots t_\ell^{\alpha_\ell} t_{\ell+1}^{\alpha_{\ell+1}} \cdots t_1^{\alpha_{2\ell}} \gamma - (\zeta_1 \alpha_1 + \cdots + \zeta_\ell \alpha_\ell)$$

Since

$$-(\zeta_{\ell+1} \alpha_{\ell+1} + \cdots + \zeta_{2\ell} \alpha_{2\ell}) = \zeta_1 \alpha_1 + \cdots + \zeta_\ell \alpha_\ell,$$

we obtain $\theta(\kappa[\alpha] - [\hat{\alpha}]) = 0$. Thus, $\kappa[\alpha] - [\hat{\alpha}] \in \ker(\theta)$ yielding $\kappa[\alpha] - [\hat{\alpha}] \in I$. The proof of the inclusion $I \subset I_{Sib}$ is similar as the proof in Theorem 5.2.2 of [22].

As an immediate corollary of Theorem 5.18 and Proposition 1 we have

**Theorem 7.** The variety of the Sibirsky ideal $I_{Sib}$ is the Zariski closure of the set $\mathcal{R}$ of all time-reversible systems in the family (3.1).

The above studies shows that the theory regarding to the computation and the structure of the saddle quantities for $1 : -1$ resonant systems has a counterpart also in the family of $p : -q$ resonant systems, however not in the whole family, but just in subfamilies of the form (2.11) and (3.1). It is in agreement with the known fact that the study of local integrability of $p : -q$ resonant systems is much more difficult that the studies in $1 : -1$ case, which can be observed already in the quadratic and the cubic case [1] [9].

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