IMPROVED LOWER BOUND FOR ANALYTIC SCHRÖDINGER EIGENFUNCTIONS IN FORBIDDEN REGIONS

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Abstract. The point of this paper is to improve the reverse Agmon estimate discussed in [TW20] with assuming that the Schrödinger operator \( P(h) = -h^2 \Delta_g + V - E(h) \), \( E(h) \to E \) as \( h \to 0^+ \), is analytic on a compact, real-analytic Riemannian manifold \((M, g)\). In this paper, by considering a Neumann problem with applying Poisson representation and exterior mass estimates on hypersurfaces, we can prove an improved reverse Agmon estimate on a hypersurface.

1. Introduction

Let \((\mathcal{M}, g)\) be a compact, real-analytic \(n\)-dimensional Riemannian manifold and \( V(x) \in C^\omega(\mathcal{M}; \mathbb{R}) \) be a real-analytic potential. Assume that \( E \) is a regular value of \( V \) so that \( dV|_{V=E} \neq 0 \). The corresponding classically forbidden region is denoted as
\[
\Omega_E := \{ x \in \mathcal{M}; V(x) > E \},
\]
and the allowed region is the complement \( \Omega_E^c = \{ x \in \mathcal{M}; V(x) \leq E \} \) with boundary \( C^\omega \) hypersurface (ie. boundary caustic)
\[
\Lambda_E := \{ x \in \mathcal{M}; V(x) = E \}.
\]

Consider the Schrödinger equation
\[
P(h)u_h = 0,
\]
where \( P(h) := -h^2 \Delta_g + V(x) - E(h) \) and \( \{ u_h \} \) are \( L^2 \)-normalized eigenfunctions with eigenvalues \( E(h) \to E \) as \( h \to 0^+ \). The Agmon metric is given by
\[
g_E(x) := (V(x) - E)_+ g(x).
\]
The degenerate metric \( g_E \) is supported in the forbidden region \( \Omega_E \) and we denote the corresponding Riemannian distance function by \( d_E : \Omega_E \times \Omega_E \to \mathbb{R}^+ \). By a slight abuse of notation, we define the associated distance function to \( \Lambda_E \) by
\[
d_E(x) := d_E(x, \Lambda_E) = \inf_{y \in \Lambda_E} d_E(x, y), \quad x \in \Omega_E.
\]

In [TW20] ((\(\mathcal{M}, g, V\) is only required to be smooth), under the control and monotonicity assumptions (see [TW20, Definitions 1 and 2]), by applying Carleman estimate to pass across the caustic hypersurface [TW20 Theorem 1] authors prove that for any \( \varepsilon > 0 \) and \( h \in (0, h_0(\varepsilon)] \),
\[
\|e^{\gamma d_E/h} u_h\|_{H^1(\Lambda(\delta_1, \delta_2))} \geq C(\varepsilon, \delta_1, \delta_2) e^{-\beta(\varepsilon)/h},
\]
where \( \beta(\varepsilon) = O(\varepsilon) \) as \( \varepsilon \to 0^+ \). \( A(\delta_1, \delta_2) \subset \Omega_E \) is an annular domain near the boundary (precise definitions refer to [TW20]) and constant \( \tau_0 \geq 1 \).

As authors pointed out in [TW20], (1.5) is a partial reverse Agmon estimate, our objective in this paper is to get an improved result in the case where \((\mathcal{M}, g, V)\) is analytic. This is precisely the point of inequality (1.7). We note that the assumption that \((\mathcal{M}, g, V)\) is \( C^\infty \) is necessary in this article since real-analyticity allows for accuracy up to exponential errors in \( h \) in the pseudodifferential calculus, whereas in the \( C^\infty \) case, one can only work to \( O(h^\infty) \)-error. Since the eigenfunctions decay exponentially in \( h \) in forbidden regions, the usual \( C^\infty \) semiclassical calculus of operators is not accurate enough to deal with these functions in a rigorous fashion. In order to get an improved lower bound of eigenfunctions in forbidden regions, we first consider a Neumann problem.

Let \((\Omega_\Gamma, g)\) be a real-analytic \( n \)-dimensional Riemannian manifold with smooth boundary \( \Gamma \). Now consider following Neumann problem

\[
(-h^2 \Delta_g + V(x) - E(h)) u_h = 0 \quad \text{in} \ \Omega_\Gamma, \\
\partial_\nu u_h = 0 \quad \text{on} \ \Gamma,
\]

(1.6)

here \( \partial_\nu \) is the exterior normal derivative, \( V(x) > E \) in \( \Omega_\Gamma \) and \( \|u_h\|_{L^2(\Omega_\Gamma)} \leq 1 \).

In the following we fix a small constant \( r_0 \in (0, \text{in } (\Omega_\Gamma, g_E)) \) and let \( U_\Gamma(r_0) \) be a collar neighbourhood of \( \Gamma \) where we have Fermi coordinates \((x', x_n)\) with respect to the Agmon metric \( g_E \). The defining function \( x_n : \Omega_\Gamma \to \mathbb{R} \) is the distance to the boundary, with the property that \( 0 \leq x_n \leq r_0 \) in \( \Omega_\Gamma \) and \( x_n = 0 \) on \( \Gamma \). And \( x' \) is constant on geodesics normal to the boundary.

Denote level set \( \Gamma_\rho = \{ x = (x', x_n); \ x_n = \rho, 0 \leq \rho \leq r_0 \} \) and let \( \gamma_\rho : C^\infty(\Omega_\Gamma) \to C^\infty(\Gamma_\rho) \) be the restriction operator.

Motivated by [SU16] and [GT19], we use the Poisson representation for problem (1.6), whose parametrix is a Fourier integral operator with complex phase. As the upper half plane model (Sect. 2.1) reveals, the main difficulty of improving the decay rate is to show that the restricted eigenfunctions \( u_h \) do concentrate near the zero section, indeed we can prove that it localized to frequencies \( \geq \lambda^{1/2} h^{1/2} \) is \( O(\lambda^{-1/2}) \) (Proposition 3.4). Hence, with establishing a proper lower bound estimate (Proposition 2.7), we can get following main theorem of this paper (see Sect. 2.2 for the proof).

**Theorem 1.** If \( u_h \) solves (1.6), for \( \alpha \in \mathbb{N}^n \), there exist constants \( h_0(\alpha) > 0 \), \( \rho_0 > 0 \) and \( C(\alpha) > 0 \) such that for \( h \in (0, h_0(\alpha)] \)

\[
\| \partial^\alpha u_h \|_{L^2(\Gamma_\rho)} \geq C(\alpha) e^{-\rho/h} \| u_h \|_{L^2(\Gamma)} \quad \text{if } 0 < \rho < \rho_0.
\]

(1.7)

**Remark 1.** Notice that on compact set \( \Gamma_\rho \), the \( L^2 \) norm in ambient metric \( g \) and the one in Agmon metric \( g_E \) are comparable. In the following, we shall use \( \| \cdot \|_{L^2(\cdot, g)} \) to emphasize the \( L^2 \) norm in ambient metric \( g \) instead \( \| \cdot \|_{L^2(\cdot)} \) in the conformal Agmon metric \( g_E \). Notice

\[
\sqrt{\det g_E} = (V(x) - E)^{n/2} \sqrt{\det g}.
\]
We also use $x = (x', x_n)$ to denote Fermi coordinates in a neighborhood of the boundary $\Gamma$ in ambient metric $g$. For any $U \subset \subset \Gamma_\rho$ in one coordinate patch, by definition

$$\| \cdot \|^2_{L^2(U, g)} = \int_U | \cdot |^2 \sqrt{| \det g(x', \rho) |} \, dx' = \int_U | \cdot |^2 (V(x', \rho) - E)^{\frac{n}{2}} \sqrt{| \det g_E(x', \rho) |} \, dx'.$$

So by partition of unity, for any $W \subset \subset \Gamma_\rho$, $\| \cdot \|_{L^2(W)}$ and $\| \cdot \|_{L^2(W, g_E)}$ are comparable, which is equivalent to say, there exist $c, C > 0$ which are independent of $h$ such that

$$c \| \cdot \|_{L^2(W, g)} \leq \| \cdot \|_{L^2(W)} \leq C \| \cdot \|_{L^2(W, g)}.$$  \hspace{1cm} (1.8)

Hence we can write (1.7) in ambient metric

$$\| \partial^\alpha u_h \|_{L^2(\Gamma_\rho, g)} \geq C(\alpha) e^{-\rho/h} \| u_h \|_{L^2(\Gamma_\rho, g)} \quad \text{if } 0 < \rho < \rho_0,$$  \hspace{1cm} (1.9)

$\rho$ is the distance from hypersurface $\Gamma_\rho$ to hypersurface $\Gamma$ in Agmon metric $g_E$.

**Remark 2.** Back to our problem (1.3), if there exists a smooth separating hypersurface $\Gamma$ which is isotopic in classically forbidden region $\Omega_E$ to boundary caustic $\Lambda_E$ and $\Gamma$ bounds a domain $\Omega_\Gamma$ satisfying $\Omega_\Gamma \subset \Omega_E$ (see Figure 1) such that a sequence eigenfunctions $\{u_h\}$ satisfying $\partial_{x_n} u_h |_{\Gamma} = 0$, then Theorem 1 exactly shows that the sequence eigenfunctions $\{u_h\}$ exponentially decay in Agmon distance from the hypersurface $\Gamma$.

At present, we are unable to prove that (1.7) holds in the general setting without assuming Neumann condition, but we hope to return to this point elsewhere.

**1.1. Outline of the paper.** In Sect. 2.1 we discuss the exponential $L^2$ lower bound for the eigenfunctions $\{u_h\}$ in the upper half plane model. And in Sect. 2.2, we give the proof of Theorem 1. Finally the key ingredient of the proof, exterior mass estimate (Prop. 3.4), will be discussed in Sect. 3.

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2. Exponential lower bound of Analytic Schrödinger Eigenfunctions

In local coordinates, the Laplace-Beltrami operator $\Delta_g$ has following form,

$$
\Delta_g = \sum_{i,j} \frac{1}{\sqrt{|\det g|}} \frac{\partial}{\partial x_j} \left( \sqrt{|\det g|} g^{ij} \frac{\partial}{\partial x_i} \right),
$$

where $\det g$ is the determinant of $g$.

In local coordinates where $\{V(x) > E\}$, for sufficiently small $h$, the conformal Laplacian under Agmon metric $g_E$ is of the form

$$
\Delta_{g_E} = \frac{1}{V-E} \Delta_g + \sum_{i,j} \frac{n-2}{2} \frac{g^{ij}_E}{V-E} \partial_{x_j} V \partial_{x_i},
$$

here $n \geq 2$ is the dimension of $M$, correspondingly

$$
\frac{P(h)}{V-E} = -h^2 \Delta_{g_E} + h^2 \frac{n-2}{2} \sum_{i,j} \frac{g^{ij}_E}{V-E(h)} \partial_{x_j} V \partial_{x_i} + 1 + f(x;h),
$$

here $f(x;h) = \frac{E-E(h)}{V(x)-E(h)} \to 0$ as $h \to 0^+$

In terms of the Fermi coordinates $x = (x', x_n)$,

$$
\frac{P(h)}{V-E} = h^2 D_{x_n}^2 + R_h(x, D_{x'}) + 1 + f(x;h),
$$

where $R_h(x, D_{x'})$ is a second order elliptic differential operator in $x'$ with positive principal symbol $r(x, \xi')$. Indeed $R_h(x, D_{x'}) = -h^2 \Delta_{g_E} + h^2 \frac{\partial_{x_n} |\det g_E|}{2\det g_E} \partial_{x_n} + h^2 \frac{n-2}{2} \sum_{i,j} \frac{g^{ij}_E}{V-E} \partial_{x_j} V \partial_{x_i}$,

here $\Delta_{g_E} |_{x_n}$ is the induced tangential Laplacian, $\Delta_{g_E}'$, on $\Gamma_{x_n}$.

Set a conjugated operator

$$
Q(h) = e^{x_n/h} \frac{P(h)}{V-E} e^{-x_n/h} = h^2 D_{x_n}^2 + \tilde{R}_h(x, D_{x'}) + 2ihD_{x_n} + f(x;h),
$$

where $\tilde{R}_h(x, D_{x'}) = e^{x_n/h} R_h(x, D_{x'}) e^{-x_n/h}$ is a second order elliptic differential operator in $x'$ with same positive principal symbol $r(x, \xi')$.

Setting $v_h = e^{\frac{\varphi}{h}} u_h$, then

$$
Q(h)v_h = 0 \quad \text{in} \ \Omega_\Gamma, \tag{2.1}
$$

and $v_h(x',0) = u_h(x',0) := \varphi_h(x')$.

Let $U \subset T^*M$ be open. Following [Sj96], we define the notion of a classical analytic symbol (cl. a.s) of order $k$ and write $a \in \mathcal{S}^{m,k}_{cl}(U)$ provided $a \sim h^{-m}(a_0 + ha_1 + \ldots)$ in the sense that

$$
\left| a - h^{-m} \sum_{0 \leq j \leq (\xi)/C_0h} \partial^k_x \partial^j_\xi \varphi(x,\xi)a \right| = O(1)e^{-(\xi)/C_1h}, \quad |a_j| \leq C_0 C^j \xi^{k-j}, \quad (x, \xi) \in U.
$$
To keep track of powers of $h$ in the remainders, we use a special class of symbols than one used in [DJ18].

Fix parameter $0 \leq \rho < 1$, we say that an $h$-dependent symbol $a$ lies in the class $S^\text{comp}_\rho(U)$ if

1. $a(x, \xi; h)$ is smooth in $(x, \xi)$ in $U$, defined for $0 < h \leq 1$, and supported in an $h$-independent compact subset of $U$;
2. $a$ satisfies the derivative bounds
   \[
   \sup_{x, \xi} |\partial_x^\alpha \partial_\xi^\beta a(x, \xi; h)| \leq C_{\alpha \beta} h^{-\rho |\beta|}.
   \]  

**Remark 3.** $S^\text{comp}_\rho(U)$ is a special class of $S^\text{comp}_{L, \rho, \rho'}(U)$ used in [DJ18, Appendix]. As a model case, one can take $U = T^*\mathbb{R}^n$, $L = L_0 = \text{span}\{\partial_{x_1}, \ldots, \partial_{x_n}\}$ and $\rho' = 0$.

The corresponding standard semiclassical pseudodifferential operators have Schwartz kernels that are sums of the local integrals of the form

\[
O_p h(a)(x, y) := \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} e^{i(x-y, \xi)} a(x, \xi; h)d\xi.
\]

### 2.1. Upper half plane model.

Let us first explain heuristically where the proper $h$-exponential decay comes from. Consider $L^2$-normalized solutions $\{u_h\}$ on the upper half plane $\mathbb{R}^n_+ = \{(x', x_n) \in \mathbb{R}^n; x_n > 0\}$,

\[
(-h^2\Delta + 1) u_h(x) = 0 \quad \text{on } \mathbb{R}^n_+,
\]

\[
u_h(x', 0) = \phi_h(x') \quad \text{on } \partial \mathbb{R}^n_+.
\]

We assume that $\phi_h(x')$ concentrates near zero section $\xi' = 0$. More explicitly,

\[
\|O_p h(1 - \chi_\delta(\xi'))\phi_h\|_{L^2(\partial \mathbb{R}^n_+)} \leq \varepsilon\|\phi_h\|_{L^2(\partial \mathbb{R}^n_+)},
\]

here $\varepsilon$ is a positively small constant and $\chi_\delta \in C^\infty_0(T^*\mathbb{R}^n_+; [0, 1])$ is a cutoff supported near the zero section, with $\chi_\delta(x', \xi') = 1$ for $\{0 \leq |\xi'| \leq \delta/2\}$ and $\chi_\delta(x', \xi') = 0$ for $|\xi'| > \delta$. Here $\delta > 0$ is some arbitrarily small but fixed constant.

It’s straightforward to check that the restricted Poisson operator of (2.3) is of the form

\[
\gamma_p K w(x', \rho) = \frac{1}{(2\pi h)^{n-1}} \int \int e^{i(x'-y', \xi') - \frac{\rho}{2} |\xi'|^2 + 1} w(y') dy'd\xi'.
\]

Define the semiclassical Fourier transform for $h > 0$

\[
\mathcal{F}_h u(\xi) := \int_{\mathbb{R}^n} e^{-\frac{i}{h}(x, \xi)} u(x)dx
\]

and its inverse

\[
\mathcal{F}_h^{-1} v(x) := \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}(x, \xi)} v(\xi)d\xi.
\]
Denote $\partial \mathbb{R}^n_{\rho} = \{(x', \rho) \in \mathbb{R}^n; \rho \text{ is a fixed positive constant}\}$ and apply Plancherel formula

$$
\| (\gamma_\rho K) \varphi_h \|_{L^2(\partial \mathbb{R}^n_{\rho})} = \| F^{-1}_h \left( e^{-\frac{i}{\pi} \sqrt{|\xi'|^2 + 1}} (F_h \varphi_h) \right) \|_{L^2(\mathbb{R}^{n-1})}
= \frac{1}{(2\pi h)^{(n-1)/2}} \| e^{-\frac{i}{\pi} \sqrt{|\xi'|^2 + 1}} (F_h \varphi_h) \|_{L^2(\mathbb{R}^{n-1})}
\geq \frac{1}{(2\pi h)^{(n-1)/2}} \| e^{-\frac{i}{\pi} \sqrt{|\xi'|^2 + 1}} \chi_\delta(\xi') (F_h \varphi_h) \|_{L^2(\mathbb{R}^{n-1})}
\geq e^{-\frac{i}{\pi} \sqrt{|\xi'|^2 + 1}} \frac{1}{(2\pi h)^{(n-1)/2}} \| \chi_\delta(\xi') (F_h \varphi_h) \|_{L^2(\mathbb{R}^{n-1})}
= e^{-\frac{i}{\pi} \sqrt{|\xi'|^2 + 1}} \| O_{P_h}(\chi_\delta \varphi_h) \|_{L^2(\partial \mathbb{R}^n_{\rho})}.
$$

(2.5)

With the help of (2.4), for small $h$, the RHS of (2.5) is

$$
\geq e^{-\frac{i}{\pi} \sqrt{|\xi'|^2 + 1}} \left( \| \varphi_h \|_{L^2(\partial \mathbb{R}^n_{\rho})} - \| O_{P_h}(1 - \chi_\delta(\xi')) \varphi_h \|_{L^2(\partial \mathbb{R}^n_{\rho})} \right)
\geq \frac{1}{2} e^{-\frac{i}{\pi} \sqrt{|\xi'|^2 + 1}} \| \varphi_h \|_{L^2(\partial \mathbb{R}^n_{\rho})}.
$$

Consequently, under assumption (2.4)

$$
\| u_h \|_{L^2(\partial \mathbb{R}^n_{\rho})} = \langle \gamma_\rho K \varphi_h, \gamma_\rho K \varphi_h \rangle_{\partial \mathbb{R}^n_{\rho}}^{1/2} \geq \frac{1}{2} e^{-\frac{i}{\pi} \sqrt{|\xi'|^2 + 1}} \| \varphi_h \|_{L^2(\partial \mathbb{R}^n_{\rho})}.
$$

(2.6)

for sufficiently small $h > 0$.

Remark 4. If the boundary data $\varphi_h(x')$ is independent of $h$, we have similar inequality (2.6) with directly applying analytic stationary phase theorem without assumption (2.4).

From above argument, inequality (2.4) is a key ingredient. Intuitively, in order to get an improved reverse Agmon estimate, we need to establish similar inequality (Proposition 3.4).

2.2. Lower bound of the Poisson parametrix: proof of Theorem II. Throughout this section, we always work in the collar neighbourhood $U_{\Gamma}(r_0)$. First choose a covering of $\Gamma_\rho$ by finitely many coordinate charts $\{W_j \times \{\rho\} \}_{j \in J}$ and a corresponding smooth, locally finite partition of unity $\{\chi_j\}_{j \in J}$, with

$$
\sum_{j \in J} \chi_j(x') = 1, (x', \rho) \in W_j \times \{\rho\} \subset \Gamma_\rho, \chi_j(x') \in C_0^\infty(W_j).
$$

For convenience set $W = \bigcup_j W_j$, so $\Gamma_\rho = W \times \{\rho\}$.

It follows from [GT19] (or [SU16]) the Poisson kernel $K(x', x_n, y', \xi')$ of problem (2.1) has the following form

$$
K(x', x_n, y', h) = \frac{1}{(2\pi h)^{n-1}} \sum_{j,k} \chi_j(x') \int_{\mathbb{R}^{n-1}} e^{i \pi \left( |\varphi(x', x_n, \xi') - y' \cdot \xi'|^2 - a(x', x_n, \xi'; h) \chi_k(y') \right)} d\xi'
+ O(e^{-C_0/h}),
$$

(2.7)
here $a(x', x_n, \xi'; h)$ on $U_\Gamma(r_0) \times \mathbb{R}^{n-1}$ is a c.l.a.s of order 0 and the summation is for all $\chi_j$ and $\chi_k$ such that $\text{supp}(\chi_j) \cap \text{supp}(\chi_k) \neq \emptyset$. Also $a(x', 0, \xi'; h) = 1 + O(h)$, and there exist constants $\rho_0, C_1, C_2 > 0$, such that

$$C_1 \leq a(x', x_n, \xi'; h) \leq C_2 \quad \text{for } \forall 0 \leq x_n \leq \rho_0. \quad (2.8)$$

Here $\varphi$ solves following Hamilton-Jacobi equation

$$(\partial_{x_n} \varphi)^2 + 2i(\partial_{x_n} \varphi) + r(x', x_n, \partial_{x'} \varphi) = 0, \quad \varphi(x', 0, \xi') = (x', \xi'), \quad (2.9)$$

with $r(x', x_n, \xi') = \sum_{i,j} g_{ij}^{x_n}(x', x_n) \xi_i \xi_j$, the semiclassical principle symbol of $\tilde{R}_h(x', x_n, D_{x'})$. For simplicity, we write $r(x', x_n, \xi') = |\xi'|_{(x', x_n)}^2$. Such $\varphi$ in (2.9) exists by applying Cauchy-Kowalevski theorem since polynomial $r(x', x_n, \xi')$ is analytic.

By the positivity of the metric $g_E$ for $x_n \geq 0$,

$$\exists C, \tilde{C} > 0 \quad \text{such that } C|\xi'|^2 \leq |\xi'|_{(x', x_n)}^2 \leq \tilde{C}|\xi'|^2, \quad \text{for } \forall (x', x_n) \in U_\Gamma(r_0), \ x_n \geq 0. \quad (2.10)$$

With the natural branch of $r^{1/2}$ with a cut along the real negative axis, it follows by Taylor expansion in $\xi'$ near 0 that

$$\varphi = (x', \xi') + i \varphi_1(x', x_n, \xi') \quad (2.11)$$

where

$$\varphi_1(x', x_n, \xi') = \sum_{0 \leq |\alpha| \leq k-1} \frac{\partial^\alpha \varphi_1(x', x_n, 0)}{\alpha!} (\xi')^\alpha + R_k, \quad \varphi_1(x', 0, \xi') = 0. \quad (2.12)$$

For $\alpha = 0$, by (2.9) $\varphi_1(x', x_n, 0)$ solves following equation

$$(\partial_{x_n} \varphi_1)^2 + 2\partial_{x_n} \varphi_1 - \sum_{i,j} g_{ij}^{x_n}(x', x_n) \partial_{x_i} \varphi_1 \partial_{x_j} \varphi_1 = 0, \quad \varphi_1(x', 0, 0) = 0. \quad (2.13)$$

By the uniqueness of Cauchy-Kowalevski theorem, $\varphi_1(x', x_n, 0) = 0$ is the unique solution. Similarly, we have $\partial_{\xi_i} \varphi_1(x', x_n, 0) = 0$ for $1 \leq j \leq n - 1$. Hence,

$$\varphi_1(x', x_n, \xi') = O_{x', x_n}(|\xi'|^2) \quad \text{for small } |\xi'|. \quad (2.14)$$

By Taylor expansion near $x_n = 0$, we also have

$$\varphi_1(x', x_n, \xi') = x_n(\sqrt{|\xi'|_{(x', 0)}^2} + 1 - 1) + O(x_n^2 |\xi'|_{(x', 0)}) \quad \text{for large } |\xi'|. \quad (2.14)$$

Modulo $O(e^{-C_0/h})$ term, we write

$$\gamma_\rho K = O_{h}(\sigma_{\gamma_\rho K}),$$

where $\sigma_{\gamma_\rho K}(x', \xi') := \sum \chi_i e^{-\varphi_1(x', \rho, \xi')/h} a(x', \rho, \xi'; h)$.

Recall that $\varphi_h(x') := v_h \Gamma = v_h K$. One has

$$\gamma_\rho K \varphi_h := e^{\rho/h} u_h(x', \rho) = \frac{1}{(2\pi h)^{n-1}} \sum_{j,k} \chi_j(x') \int_{W_j} \int_{\mathbb{R}^{n-1}} e^{\frac{i}{h} |\varphi(x', \rho, \xi') - y'\xi'|-1} a(x', x_n, \xi', h) \chi_k(y') \varphi_h(y') d\xi' d\rho' + O(e^{-C_0/h}) \varphi_h(x'). \quad (2.15)$$
Choose a decreasing smooth function $b(x) \geq e^{-x}$ for $x \geq 0$ (see Figure 2) satisfying that

$$b(x) = e^{-x}, \quad 0 \leq x \leq \frac{M}{2},$$

$$b(x) = \zeta, \quad x \geq M,$$

and $|\frac{db}{dx}(x)| \leq C_n b(x)$, here $M$ is a large constant which will be determined later and $e^{-M} < \zeta < e^{-\frac{M}{2}}$ is a constant.

Now set

$$b_j(x', \rho, \xi'; h) = \chi_j(x') b(\varphi_1(x', \rho, \xi')/h).$$

By the definition of $b$ and the positivity (2.10) of the metric, one has

$$b_j(x', \rho, \xi'; h) \equiv \chi_j e^{-\varphi_1(x', \rho, \xi')/h} \quad \text{if} \quad |\xi'|^2 \leq c M h,$$

and

$$b_j(x', \rho, \xi'; h) \equiv \chi_j \zeta \quad \text{if} \quad |\xi'|^2 \geq c' M h,$$

here $c' > c$ are two positive constants.

One also has

$$b_j(x', \rho, \xi'; h) \geq \chi_j \zeta, \quad \text{for} \quad 0 \leq \rho < \rho_0.$$  

Using (2.13), this is straightforward to check that for $|\xi'|^2 \leq c' M h$

$$|\partial_{x'}^\alpha \varphi_1| \leq C_{n} M h \quad \text{if} \quad |\alpha| \geq 1,$$

$$|\partial_{\xi'}^\beta \varphi_1| \leq C_{n} M^{1/2} h^{1/2} \quad \text{if} \quad |\beta| = 1, \quad |\partial_{\xi'}^\beta \varphi_1| \leq C_{n} \quad \text{if} \quad |\beta| \geq 2. \quad (2.19)$$

By chain rule, this is straightforward to check that

$$|\partial_{x} x_i b_j| \leq C_M b_j, \quad |\partial_{\xi} \xi_i b_j| \leq C_M h^{-1/2} b_j,$$

$$|\partial_{x} x_i x_i b_j| \leq C_M b_j, \quad |\partial_{\xi} \xi_i \xi_i b_j| \leq C_M h^{-1} b_j. \quad (2.20)$$

Furthermore by Leibniz rule, one has

$$|\partial_{x}^\alpha \partial_{\xi}^\beta b_j| \leq C_{n} h^{-|\beta|} b_j, \quad (2.21)$$

which means that $b_j(x', \rho, \xi'; h) - \chi_j \zeta \in S^{\text{comp}}_T(T^*W_j)$.

For convenience, set $f_\rho(x', \xi'; h) := \sum_j b_j(x', \rho, \xi'; h)$. From (2.21), one has

$$|\partial_{x}^\alpha \partial_{\xi}^\beta f_\rho| \leq C_{n} h^{-|\beta|} f_\rho, \quad (2.22)$$

which means that $f_\rho(x', \xi'; h) - \zeta \in S^{\text{comp}}_T(T^*W)$. 

**Figure 2.**
Proposition 2.1. There exist $h_0 > 0$ and $C_3 > 0$ such that
\[
\|O_{\rho h}(f_{\rho a})\varphi_h\|_{L^2(\Gamma_\rho)} \geq C_3 \|\varphi_h\|_{L^2(W)}.
\]
(2.23)
for $0 < h < h_0$.

Proof. The proof essentially follows \[Zwo12\] Theorem 4.29. From the definition of $f_\rho$ and (2.22), we know $f_\rho^{-1} - \zeta^{-1} \in f_\rho^{-1}S_{1/2}^{\text{comp}}(T^*W)$.

By composition formula (see \[Zwo12\] (4.3.16) and \[DJ18\] (A.23)) and (2.22), one can get
\[
f_\rho \# f_\rho^{-1}(x', \zeta'; h) = 1 + r_1
\]
(2.24)
with
\[
r_1 \in h_{1/2}S_{1/2}^{\text{comp}}(T^*W)
\]
Likewise
\[
f_\rho^{-1} \# f_\rho(x', \zeta'; h) = 1 + r_2 \quad \text{with } r_2 \in h_{1/2}S_{1/2}^{\text{comp}}(T^*W).
\]
(2.25)
Hence if $R_1 := O_{\rho h}(r_1)$ and $R_2 := O_{\rho h}(r_2)$, we have
\[
O_{\rho h}(f_\rho) O_{\rho h}(f_\rho^{-1}) = I + R_1,
\]
\[
O_{\rho h}(f_\rho^{-1}) O_{\rho h}(f_\rho) = I + R_2,
\]
with $\|R_1\|_{L^2 \rightarrow L^2}, \|R_2\|_{L^2 \rightarrow L^2} = O(h_{1/2}^2) \leq 1/2$.

It remains to apply triangle inequality and \[DJ18\] Lemma A.5 to get that
\[
\|O_{\rho h}(f_\rho^{-1})\|_{L^2 \rightarrow L^2} \leq \|O_{\rho h}(f_\rho^{-1} - \zeta^{-1})\|_{L^2 \rightarrow L^2} + \zeta^{-1} \leq C\zeta^{-1},
\]
(2.26)
with noticing that $1 \leq f_\rho^{-1} \leq \zeta^{-1}$ and $f_\rho^{-1} - \zeta^{-1} \in f_\rho^{-1}S_{1/2}^{\text{comp}}(T^*W)$.

Then with the help of (2.8),
\[
\frac{C_1}{2} \|\varphi_h\|_{L^2(W)} \leq \|O_{\rho h}(a)\varphi_h\|_{L^2(\Gamma_\rho)}
\]
\[
\leq \|(I + R_2)^{-1}O_{\rho h}(f_\rho^{-1})O_{\rho h}(f_\rho)O_{\rho h}(a)\varphi_h\|_{L^2(\Gamma_\rho)}
\]
\[
\leq C\zeta^{-1}\|O_{\rho h}(f_\rho)O_{\rho h}(a)\varphi_h\|_{L^2(\Gamma_\rho)}
\]
\[
\leq \frac{C}{2} \zeta^{-1}\|O_{\rho h}(f_\rho a)\varphi_h\|_{L^2(\Gamma_\rho)}
\]
implies the result. The last line follows by applying composition formula. \[\square\]
there exists a constant $C$ in the 2nd last line, we applied the composition formula of operators with noticing that $\xi$ only depends on $\xi'$ (Figure 3).

Then by $L^2$ boundedness, one can conclude that

$$\|Op_h(f_\rho a - \sigma_{\gamma_p K})\|_{L^2(\Gamma)} \leq C_4 \zeta.$$  

Taking radial cutoffs $\chi_{in}, \chi_{out} \in C^\infty(\mathbb{R}; [0, 1])$ with $\chi_{in}(\xi') = \chi_{in}(|\xi'|; h, M)$ and $\chi_{out}(\xi') = \chi_{out}(|\xi'|; h, M)$ such that

$$\chi_{in}(\xi') + \chi_{out}(\xi') = 1$$

and

$$\chi_{in}(\xi') = 1, \quad |\xi'|^2 \leq \frac{c}{2} Mh,$$

$$\chi_{in}(\xi') = 0, \quad |\xi'|^2 \geq cMh,$$

here $c$ is a constant in (2.16).

According to (2.16), one has

$$(f_\rho a - \sigma_{\gamma_p K}) \chi_{out} = f_\rho a - \sigma_{\gamma_p K}.$$  

By Proposition 2.2

$$\|Op_h(f_\rho a - \sigma_{\gamma_p K}) \chi_{out} \varphi_h\|_{L^2(\Gamma^e)} \leq \|Op_h(f_\rho a - \sigma_{\gamma_p K}) \chi_{out} \varphi_h\|_{L^2(\Gamma^e)} \leq C_4 \zeta \|Op_h(\chi_{out}) \varphi_h\|_{L^2(W)},$$  

in the 2nd line, we applied the composition formula of operators with noticing that $\chi_{out}$ only depends on $\xi'$.

Thereafter, by triangle inequality and along with (2.23) and (2.27), from (2.15) one can conclude that

$$\|\rho^h \rho f(x', \rho)|_{L^2(\Gamma^e)} = \|Op_h(f_\rho a - \sigma_{\gamma_p K}) \varphi_h\|_{L^2(\Gamma^e)} - O(e^{-C_0/\rho}) \varphi_h\|_{L^2(W)}$$

$$\geq \|Op_h(f_\rho a - \sigma_{\gamma_p K}) \varphi_h\|_{L^2(\Gamma^e)} - \|Op_h(f_\rho a - \sigma_{\gamma_p K}) \varphi_h\|_{L^2(\Gamma^e)}$$

$$O(e^{-C_0/\rho}) \varphi_h\|_{L^2(W)}$$

$$\geq C_3 \|\varphi_h\|_{L^2(W)} - C_4 \zeta \|Op_h(\chi_{out}) \varphi_h\|_{L^2(W)}$$

$$O(e^{-C_0/\rho}) \varphi_h\|_{L^2(W)}.$$  

From Proposition 3.1 with taking $h$ sufficiently small and $M$ sufficiently large, one has

$$C_4 \zeta \|Op_h(\chi_{out}(|\xi'|; h, M)) \varphi_h\|_{L^2(W)} \ll C_3 \|\varphi_h\|_{L^2(W)}.$$  

Consequently, from (2.28) and (2.29), one has

$$\|u_h\|_{L^2(\Gamma^e)} \geq C e^{-\rho/h} \|\varphi_h\|_{L^2(W)} = C e^{-\rho/h} \|u_h\|_{L^2(\Gamma)} \quad \text{if } \rho \leq \rho_0.$$  

**Proposition 2.2.** $Op_h(f_\rho a - \sigma_{\gamma_p K}) : L^2(\Gamma) \rightarrow L^2(\Gamma)$ is a bounded operator. More explicitly, there exists a constant $C_4 > 0$ such that

$$\|Op_h(f_\rho a - \sigma_{\gamma_p K})\|_{L^2(\Gamma) \rightarrow L^2(\Gamma)} \leq C_4 \zeta.$$  

**Proof.** From the definition of $f_\rho$, one has $f_\rho - \chi e^{-\varphi_1(x', \rho, \xi')/h} \leq \zeta$ which is a Schwartz function in $\xi'$ (Figure 3).

Then by $L^2$ boundedness, one can conclude that

$$\|Op_h(f_\rho a - \sigma_{\gamma_p K})\|_{L^2(\Gamma) \rightarrow L^2(\Gamma)} = \|Op_h(f_\rho a - \sum \chi_i e^{-\varphi_1(x', \rho, \xi')/h})\|_{L^2(\Gamma) \rightarrow L^2(\Gamma)} \leq C_4 \zeta.$$  

□
The same argument works for $\|\partial^\alpha u_h\|_{L^2(\Gamma_\rho)}$ with each differentiation creating a power of $h^{-1}$.

3. Exterior mass estimates on $\Gamma$

Before presenting our main result in this section, we introduce following lemma.

**Lemma 3.1.** For small $\rho > 0$ one has

$$\|u_h\|_{L^2(\Gamma_\rho)} \leq C(\rho)e^{-\rho/h}\|u_h(x)\|_{L^2(\Gamma)}.$$  \hfill (3.1)

**Proof.** From equation (2.1) and (2.7), we know for any $(x', \rho) \in \Gamma_\rho$

$$e^{\rho/h}u_h(x', \rho)$$

$$= \frac{1}{(2\pi h)^{n-1}} \int_W \int_{\mathbb{R}^{n-1}} e^{\frac{i}{h}[\varphi(x', \rho, \xi') - \varphi(y', \xi')]a(x', \rho, \xi'; h)} \chi(x' - y') \varphi_h(y') d\xi' dy'$$

$$+ O(e^{-C_0/h}) \varphi_h(x')$$

$$= \frac{1}{(2\pi h)^{n-1}} \int_W \int_{\mathbb{R}^{n-1}} e^{\frac{i}{h}[\varphi(x', \rho, \xi') - \varphi(y', \xi')]a(x', \rho, \xi'; h)} \chi(x' - y') \varphi_h(y') d\xi' dy'$$

$$+ \frac{1}{(2\pi h)^{n-1}} \int_W \int_{\mathbb{R}^{n-1}} e^{\frac{i}{h}[\varphi(x', \rho, \xi') - \varphi(y', \xi')]a(x', \rho, \xi'; h)} \chi(x' - y') \varphi_h(y') d\xi' dy'$$

$$+ O(e^{-C_0/h}) \varphi_h(x'),$$  \hfill (3.2)

where $\chi(x' - y')$ is a near-diagonal cutoff. With help of equation (2.14), one can apply Cauchy-Schwartz inequality to get that the integrations in (3.2) are bounded by $C(\rho)\|\varphi_h\|_{L^2}$. Then we can complete the proof.

**Remark 5.** Like Agmon estimate, this lemma shows how fast the restricted eigenfunctions decay from a hypersurface in Forbidden regions. From (3.1), we know that (2.30) is sharp.

From now on we fix a small constant $r_{0,g} \in (0, \text{inj}(\Omega_\Gamma, g))$ and let $U_{\Gamma(r_{0,g})}$ be a collar neighbourhood of $\Gamma$ where we have Fermi coordinates $(x', x_n)$ with respect to the ambient metric $g$. We also set

$$\Gamma_{\rho,g} = \{x = (x', x_n); x_n = \rho, 0 \leq \rho \leq r_{0,g}\}.$$

**Lemma 3.2.** For small $\rho > 0$ one has

$$\|\partial_{x_n} u_h\|_{L^2(\Gamma_{\rho,g})} \leq C(\rho)h^{-1}\|u_h\|_{L^2(\Gamma)}.$$  \hfill (3.3)

**Proof.** Like we did in section 2.2 in ambient metric $g$ one has

$$u_h(x', x_n) = \tilde{K}\varphi_h + O(e^{-C_0/h}\varphi_h),$$  \hfill (3.4)

where

$$\tilde{K}(x', x_n, y', h) = \frac{1}{(2\pi h)^{n-1}} \sum_j \chi_j(x') \int_{\mathbb{R}^{n-1}} e^{\frac{i}{h}[\varphi(x', x_n, \xi') - \varphi(y', \xi')]a(x', x_n, \xi'; h)} \chi_k(y') d\xi'$$

$$+ O(e^{-C_0/h}),$$  \hfill (3.5)
here \(\tilde{a}(x', x_n, \xi'; h)\) on \(U_{1}(r_{0, 0}) \times \mathbb{R}^{n-1}\) is a c.l.a.s of order 0 and \(\tilde{a}(x', 0, \xi'; h) = 1 + O(h)\).

\(\tilde{\varphi}\) solves following Hamilton-Jacobi equation

\[
(\partial_{x'}\tilde{\varphi})^2 + V - E + \tilde{r}(x', x_n, \partial_{x'} \tilde{\varphi}) = 0, \quad \tilde{\varphi}(x', 0, \xi') = \langle x', \xi' \rangle, 
\]

with \(\tilde{r}(x', x_n, \xi') := \sum_{ij} g^{ij}(x', x_n) \xi_i \xi_j\). For simplicity, we write \(\tilde{r}(x', x_n, \xi') = |\xi'|_{(x', x_n)}^2\).

With the natural branch of \(\tilde{r}^{1/2}\) with a cut along the real negative axis, it follows by Taylor expansion in \(x_n\) near 0 that

\[
\tilde{\varphi} = \langle x', \xi' \rangle + i\tilde{\varphi}_1(x', x_n, \xi'),
\]

where

\[
\tilde{\varphi}_1(x', x_n, \xi') = x_n \left( V(x', 0) - E + |\xi'|_{(x', 0)}^2 \right) + O \left( x_n^2 |\xi'|_{(x', 0)} \right) \quad \text{for large } |\xi'|. 
\]

Then we can finish the proof following the steps in Lemma 3.1 with noticing that each differentiation creating a power of \(h^{-1}\).

Following results essentially follow [BHT18]. To emphasize the normal direction \(x_n\), let us write

\[
r = x_n, \tag{3.8}
\]

and set \(P(r) = -h^2 \Delta_d g_r\).

Let the smooth function \(\chi : \mathbb{R} \to [0, 1]\) satisfy \(\chi(t) = 0\) for \(t \leq 1\) and \(\chi(t) = 1\) for \(t \geq 2\).

Define \(f_\lambda(t)\) by

\[
f_\lambda(t) = \chi \left( \frac{t}{\lambda h} \right),
\]

here \(\lambda\) is a large parameter.

**Lemma 3.3.** One has

\[
\|(d/dr)^k f_\lambda(P)\|_{L^2(\Gamma, g)} \leq C_k \lambda^{-k} h^{-k}
\]

for \(k = 0, 1, 2,\) and \(h \leq 1\).

For the proof, one can refer [BHT18] Lemma 3.3 and Lemma 5.1.

**Proof.** We select an almost analytic extension of \(f_\lambda\), denoted by \(F_\lambda\). We extend the smooth function \(\chi(t)\) to an almost analytic extension \(\tilde{\chi}(z)\), satisfying

\[
|\partial_z^k \tilde{\chi}(z)| \leq C_{k,N} |Im z|^N \quad \forall z \in \mathbb{C}, \forall N \in \mathbb{N},
\]

here \(\partial\) is the usual \(d\)-bar operator, \(\partial_x + i\partial_y\). We then define

\[
F_\lambda(z) = \tilde{\chi} \left( \frac{z}{\lambda h} \right).
\]

Note that \(F_\lambda(t) = f_\lambda(t)\) for real \(t\), and since \(P(r) \geq 0\) as an operator, it follows that \(F_\lambda(P) = f_\lambda(P)\). Due to the scaling in the definition of \(h\), we have

\[
|\partial_z^k \tilde{\chi}(z)| \leq C_{k,N} \lambda^{-k(4+N+1)} h^{-k(4+N+1)} |Im z|^N \quad \forall z \in \mathbb{C}, \forall N \in \mathbb{N}. \tag{3.9}
\]

In addition, we can assume that \(\partial \tilde{\chi}\) is supported in the set \([1, 2] \times i[-1, 1]\). Consequently, \(\partial F_\lambda\) is supported in \([\lambda h, 2\lambda h] \times i[-\lambda h, \lambda h]\), which is a set of measure \(O(\lambda^2 h^2)\).
We can express \( f_\lambda(P) \) in terms of \( F_\lambda \) using the standard Helffer-Sjöstrand formula (see Theorem 8.1 [Zwo12])

\[
f_\lambda(P) = \frac{1}{2\pi} \int_{\mathbb{C}} \bar{\partial} F_\lambda(z)(P - z)^{-1} dL(z).
\] (3.10)

Here the integral is over the entire complex plane \( \mathbb{C} \), \( L(z) \) is Lebesgue measure on \( \mathbb{C} \). Using this formula we can easily express the \( r \)-derivatives of \( f_\lambda(P) \). For example, using

\[
(P - z)^{-1} = -(P - z)^{-1} \dot{P}(P - z)^{-1},
\]

with dots indicating differentiation with respect to \( r \). We have

\[
f_\lambda(P) = -\frac{1}{2\pi} \int_{\mathbb{C}} \bar{\partial} F_\lambda(z)(P - z)^{-1} \dot{P}(P - z)^{-1} dL(z),
\]

\[
f_\lambda(\ddot{P}) = \frac{2}{2\pi} \int_{\mathbb{C}} \bar{\partial} F_\lambda(z)(P - z)^{-1} \ddot{P}(P - z)^{-1} dL(z) - \frac{1}{2\pi} \int_{\mathbb{C}} \bar{\partial} F_\lambda(z)(P - z)^{-1} \dddot{P}(P - z)^{-1} dL(z). \] (3.11)

Note that \( 1 + P \) is invertible, so we have

\[
(P - z)^{-1} \dot{P}(P - z)^{-1} = (P - z)^{-1}(1 + P)(1 + P)^{-1} \dot{P}(1 + P)^{-1}(1 + P)(P - z)^{-1}.
\]

Note that \( (1 + P)^{-1} \dot{P}(1 + P)^{-1} \) has an \( O(1) \) operator norm bound on \( L^2(\Gamma_{r,g}) \) uniform in \( r \) and \( h \). On the other hand, using spectral theory, the operator norms of \( (1 + P)(P - z)^{-1} \) and \( (P - z)^{-1}(1 + P) \) are, for \( z \in \text{supp} \bar{\partial} F_\lambda \), bounded by

\[
\sup_{t \in [0, +\infty)} |(1 + t)(t - z)^{-1}| \leq C|\text{Im} \, z|^{-1}.
\]

Hence \( (P - z)^{-1} \dot{P}(P - z)^{-1} \) is bounded by \( C|\text{Im} \, z|^{-2} \) for \( z \in \text{supp} \bar{\partial} F_\lambda \).

Together with the bound on \( \bar{\partial} F_\lambda(z) \)

\[
|\bar{\partial} F_\lambda(z)| \leq C_N \lambda^{-(N+1)} h^{-(N+1)} |\text{Im} \, z|^{N} \quad \forall z \in \mathbb{C}, \forall N \in \mathbb{N},
\]

along with the \( O(\lambda^2 h^2) \) estimate on the area of the support of \( \bar{\partial} F_\lambda \), we can get

\[
\|f_\lambda(\dot{P})\|_{L^2(\Gamma_{r,g})} \leq C \lambda^{-1} h^{-1},
\]

with choosing \( N = 2 \).

Similarly the operator \( f_\lambda(\ddot{P}) \) has an operator norm bound of \( CM^{-4} h^{-4/3} \).

Then we can state our main result in this section,

**Proposition 3.4.** One has

\[
\langle f_\lambda(P) u_h, u_h \rangle_\Gamma \leq C \lambda^{-1} \|u_h\|_{L^2(\Gamma)}^2. \] (3.12)

**Remark 6.** The main idea of proof is inspired by [BHT18, Proposition 5.2]. Since we work on the eigenfunctions in forbidden regions, which have exponentially small upper bound in \( h \) rather than the ones in allowed regions, which have polynomial upper bound in \( h \), each quantity in our proof should be treated differently.
Also note that the above estimate is different from \[ \text{[BHT18, Proposition 5.2]} \] since the right hand side quantity is restricted on a hypersurface.

Proof. Let
\[
L(r) = h^2 \langle f_\lambda(P)u_h, u_h \rangle_{\Gamma_{r,g}}.
\]

With dots indicating differentiation with respect to \( r \), and notice \( \partial_r = \partial_{x_n} \)
\[
\dot{L}(r) = h^2 \langle f_\lambda(P)u_h, u_h \rangle_{\Gamma_{r,g}} + 2h^2 \langle f_\lambda(P)u_h, \partial_{x_n} u_h \rangle_{\Gamma_{r,g}}
\]
and
\[
\ddot{L}(r) = h^2 \langle f_\lambda(P)u_h, u_h \rangle_{\Gamma_{r,g}} + 4h \langle f_\lambda(P)u_h, h\partial_{x_n} u_h \rangle_{\Gamma_{r,g}} + 2 \langle f_\lambda(P)h\partial_{x_n} u_h, h\partial_{x_n} u_h \rangle_{\Gamma_{r,g}} + 2 \langle f_\lambda(P)u_h, h^2 \partial_{x_n}^2 u_h \rangle_{\Gamma_{r,g}}.
\]

By Taylor expansion we write
\[
L(r) = L(0) + r \dot{L}(0) + \int_0^r ds \int_0^s \ddot{L}(t) dt.
\]

The key observation is that the term \( \langle f_\lambda(P)u_h, h^2 \partial_{x_n}^2 u_h \rangle_{\Gamma_{r,g}} \) in (3.13) is strongly positive, since
\[
h^2 \partial_{x_n}^2 u_h = Pu_h + h^2 \frac{\partial_{x_n} |\det g|}{2|\det g|} \partial_{x_n} u_h + (V - E(h)) u_h
\]
and
\[
P \geq \lambda h
\]
on the support of \( f_\lambda \). We will show that, unless \( L(0) \) is very small, this term drives the polynomial growth in \( h \) of \( L \) which will contradict the value by integrating \( L \) on the interval \([0, \lambda h]\).

Using Lemma 3.3, we estimate
\[
\dot{L}(0) \geq -C\lambda^{-1} h \| u_h \|^2_{L^2(\Gamma)}.
\]

Similarly, for the terms on the right hand side of (3.13), we have
\[
\left| \langle f_\lambda(P)u_h, u_h \rangle_{\Gamma_{r,g}} \right| \leq C\lambda^{-2} h^{-2} \| u_h \|^2_{L^2(\Gamma_{r,g})},
\]

\[
4h \langle f_\lambda(P)u_h, h\partial_{x_n} u_h \rangle_{\Gamma_{r,g}} = 2h^2 \frac{d}{dx_n} \langle f_\lambda(P)u_h, u_h \rangle_{\Gamma_{r,g}} - 2h^2 \langle f_\lambda(P)u_h, u_h \rangle_{\Gamma_{r,g}}
\]
\[
\geq 2h^2 \frac{d}{dx_n} \langle f_\lambda(P)u_h, u_h \rangle_{\Gamma_{r,g}} - C\lambda^{-2} \| u_h \|^2_{L^2(\Gamma_{r,g})},
\]
and induce the key inequality from (3.15) and (3.16)

\[
2 \left\langle f_\lambda(P) u_h, h^2 \partial^2_{x_n} u_h \right\rangle_{\Gamma_{r,g}} = 2 \left\langle f_\lambda(P) u_h, P u_h + h^2 \partial^2_{x_n} \left| \operatorname{det} g \right| \frac{\partial}{\partial x_n} u_h + (V - E(h)) u_h \right\rangle_{\Gamma_{r,g}} \\
\geq 2T h^{-2} L(r) + \left\langle f_\lambda(P) u_h, h^2 \partial^2_{x_n} \left| \operatorname{det} g \right| \frac{\partial}{\partial x_n} u_h \right\rangle_{\Gamma_{r,g}} \\
\geq 2T h^{-2} L(r) - C \lambda h^2 \|u_h\|^2_{L^2(\Gamma_{r,g})} \left\| \partial_{x_n} u_h \right\|_{L^2(\Gamma_{r,g})} \\
\geq 2T h^{-2} L(r) - C \lambda h^2 \|u_h\|^2_{L^2(\Gamma)} \\
\geq 2T h^{-2} L(r) - C \lambda h^2 \|u_h\|^2_{L^2(\Gamma)} \\
\geq 2T h^{-2} L(r) - C h^{-1} \|u_h\|^2_{L^2(\Gamma)} \\
(3.18)
\]

in the second inequality notice that \( P + V - E(h) \geq T > 0 \) on the support of \( f_\lambda \), here \( T \) is a small constant independent of \( h \) and \( r \). Also we applied Cauchy-Schwartz inequality in the last inequality.

From Lemma 3.1, for any fixed small \( r \), taking \( h \) sufficiently small, one has \( \|u_h\|^2_{L^2(\Gamma_{r,g})} < \|u_h\|^2_{L^2(\Gamma)} \). Setting \( \tilde{L}(r) = \|u_h\|^2_{L^2(\Gamma_{r,g})} \), one also has \( \frac{d}{dr} \tilde{L}(0) = 2 \left\langle u_h, \dot{u}_h \right\rangle_{\Gamma} = 0 \). Hence \( \frac{d}{dr} \tilde{L}(r) < 0 \) for any small \( r \), which is equivalent to say that for any small \( r \),

\[
\|u_h\|^2_{L^2(\Gamma_{r,g})} < \|u_h\|^2_{L^2(\Gamma)}. \\
(3.19)
\]

With Lemma 3.2 and the boundary condition \( \|u_h\|_{L^2(\Gamma)} = 0 \), one can deduce that for any small \( r \),

\[
\|\partial_{x_n} u_h\|^2_{L^2(\Gamma_{r,g})} < C h^{-1} \|u_h\|^2_{L^2(\Gamma)}. \\
(3.20)
\]

Putting these together, we obtain

\[
\tilde{L}(r) \geq -C \lambda^{-2} \|u_h\|^2_{L^2(\Gamma)} + 2 h^2 \frac{d}{dx_n} \left\langle f_\lambda(P) u_h, u_h \right\rangle_{\Gamma_{r,g}} + 2 T h^{-2} L(r). \\
(3.21)
\]

Using this in (3.14), we get an inequality

\[
L(r) \geq L(0) - r C \lambda^{-1} h \|u_h\|^2_{L^2(\Gamma)} + \int_0^r ds \int_0^s dt \left( -C \lambda^{-2} \|u_h\|^2_{L^2(\Gamma)} + 2 h^2 \frac{d}{dt} \left\langle f_\lambda(P) u_h, u_h \right\rangle_{\Gamma_{r,g}} + 2 T h^{-2} L(t) \right). \\
\]

For \( r \in [0, \lambda h] \), by (3.19) and Lemma 3.3, the first two terms in the big bracket can be absorbed by the \( r C \lambda^{-1} h \|u_h\|^2_{L^2(\Gamma)} \) term if \( h \) is small enough. We get

\[
L(r) \geq L(0) - r C \lambda^{-1} h \|u_h\|^2_{L^2(\Gamma)} + 2 T \int_0^r ds \int_0^s h^{-2} L(t) dt. \\
\]

Then one has following comparison: \( L(r) \geq Z(r) \) where \( Z(r) \) satisfies the corresponding equality

\[
Z(r) = L(0) - r C \lambda^{-1} h \|u_h\|^2_{L^2(\Gamma)} + T \int_0^r ds \int_0^s h^{-2} Z(t) dt, \quad Z(0) = L(0). \\
\]

This we can solve exactly: differentiating twice gives us

\[
\ddot{Z}(r) = T h^{-2} Z(r), \quad Z(0) = L(0), \quad \dot{Z}(0) = -C \lambda^{-1} h \|u_h\|^2_{L^2(\Gamma)}. \\
(3.22)
\]
The solution is
\[ Z(r) = L(0)cosh(\sqrt{Th^{-1}}r) - \frac{1}{\sqrt{T}}C\lambda^{-1}h^2\|u_h\|_{L^2(\Gamma)}^2 sinh(\sqrt{Th^{-1}}r) \]
\[ \geq \left( L(0) - \frac{1}{\sqrt{T}}C\lambda^{-1}h^2\|u_h\|_{L^2(\Gamma)}^2 \right) e^{1/\sqrt{T}} \]

Now suppose, for a contradiction, that \( L(0) \) was bigger than \( \frac{1}{\sqrt{T}}C\lambda^{-1}h^2\|u_h\|_{L^2(\Gamma)}^2 \). This would tell us that
\[ L(r) \geq Z(r) \geq C\lambda^{-1}h^2\|u_h\|_{L^2(\Gamma)}^2 e^{1/\sqrt{T}} \quad r \in [0, \lambda h]. \] (3.23)

Integrating this on \([0, \lambda h]\) gives
\[ \int_0^{\lambda h} L(r)dr \geq C e^{1/\sqrt{T}} \lambda^{-1}h^3\|u_h\|_{L^2(\Gamma)}^2. \] (3.24)

On the other hand by the definition of \( L(r) \), one has
\[ \int_0^{\lambda h} L(r)dr \leq C\lambda h^3\|u_h\|_{L^2(\Gamma)}^2, \]
which contradicts with (3.24) for sufficiently large \( \lambda \). We conclude that
\[ L(0) \leq C\lambda^{-1}h^2\|u_h\|_{L^2(\Gamma)}^2, \]
proving this proposition.

\[ \square \]

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