Boundary regularity for quasilinear elliptic equations with general Dirichlet boundary data

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Abstract

We study global regularity for solutions of quasilinear elliptic equations of the form \( \text{div} \ A(x, u, \nabla u) = \text{div} \ F \) in rough domains \( \Omega \) in \( \mathbb{R}^n \) with nonhomogeneous Dirichlet boundary condition. The vector field \( A \) is assumed to be continuous in \( u \), and its growth in \( \nabla u \) is like that of the \( p \)-Laplace operator. We establish global gradient estimates in weighted Morrey spaces for weak solutions \( u \) to the equation under the Reifenberg flat condition for \( \Omega \), a small BMO condition in \( x \) for \( A \), and an optimal condition for the Dirichlet boundary data.

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1 Introduction

We investigate global gradient estimates for weak solutions to the Dirichlet problem

\[
\begin{aligned}
\text{div} \ A(x, u, \nabla u) &= \text{div} \ F & \text{in } & \Omega, \\
u &= \psi & \text{on } & \partial \Omega,
\end{aligned}
\]

(1.1)

when \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) (\( n \geq 2 \)) and the vector field \( A \) is only continuous in the \( u \) variable and possibly discontinuous in the \( x \) variable. Let \( K \subset \mathbb{R} \) be an open interval and consider the general vector field

\[ A = A(x, z, \xi) : \Omega \times \overline{K} \times \mathbb{R}^n \rightarrow \mathbb{R}^n \]

which is a Carathéodory map, that is, \( A(x, z, \xi) \) is measurable in \( x \) for every \( (z, \xi) \in \overline{K} \times \mathbb{R}^n \) and continuous in \( (z, \xi) \) for a.e. \( x \). We assume that \( \xi \mapsto A(x, z, \xi) \) is differentiable on \( \mathbb{R}^n \setminus \{0\} \) for a.e. \( x \) and all \( z \in \overline{K} \). Also, there exist constants \( \Lambda > 0, 1 < p < \infty \), and a nondecreasing and right continuous function \( \omega : [0, \infty) \rightarrow [0, \infty) \) with \( \omega(0) = 0 \) such that the following conditions are satisfied for a.e. \( x \in \Omega \) and all \( z \in \overline{K} \):

\[
\langle \partial_\xi A(x, z, \xi) \eta, \eta \rangle \geq \Lambda^{-1} |\xi|^{p-2} |\eta|^2 \quad \forall \xi \in \mathbb{R}^n \setminus \{0\} \text{ and } \forall \eta \in \mathbb{R}^n, \tag{1.2}
\]

\[
|A(x, z, \xi)| + |\xi| |\partial_\xi A(x, z, \xi)| \leq \Lambda |\xi|^{p-1} \quad \forall \xi \in \mathbb{R}^n, \tag{1.3}
\]

\[
|A(x, z_1, \xi) - A(x, z_2, \xi)| \leq \Lambda |\xi|^{p-1} \omega(|z_1 - z_2|) \quad \forall z_1, z_2 \in \overline{K} \text{ and } \forall \xi \in \mathbb{R}^n. \tag{1.4}
\]

Interior \( C^{1,\alpha} \) theory for the homogeneous equation associated to (1.1) was established by DiBenedetto [6] and Tolksdorf [22] extending the celebrated Hölder gradient estimates by Uraltseva [26] and Uhlenbeck [25].

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for the homogeneous \( p \)-Laplace equation. Furthermore, Lieberman \[13\] derived the global \( C^{1,\alpha} \) estimates for bounded weak solutions of the corresponding Dirichlet boundary problem when both the domain and the boundary data are of class \( C^{1,\alpha} \). On the other hand, interior \( W^{1,q} \) estimates for the nonhomogeneous quasilinear equation (1.1) was investigated in \[17\] using the perturbation method by Caffarelli-Peral \[5\] together with the two-parameter scaling technique introduced in \[10\] to deal with a specific parabolic equation. When \( A \) has sufficiently small BMO oscillation in \( x \) and is Lipschitz continuous in the \( z \) variable, it was established in \[17\] that: \( |F| \frac{\partial}{\partial z} \in L^q_{\text{loc}} \implies \nabla u \in L^q_{\text{loc}} \) for any \( q > p \). This result was extended in \[16,18,19\] to cover more general situations. In particular, the authors of \[2\] derived the corresponding global estimates for Reifenberg flat domains and for zero Dirichlet boundary data. Moreover, they were able to weaken the condition on \( A \) to cover more general situations. In a recent paper \[7\] with Di Fazio, we obtained interior gradient estimates in generalized weighted Morrey spaces for solutions of (1.1) when \( A \) is merely continuous in \( z \). This, in particular, extends the gradient estimates in the classical Morrey spaces obtained in \[15,20\] for the case \( A(x,z,\xi) = A(x,\xi) \).

Our purpose of the current work is two folds. On one hand, we extend the mentioned result in \[2\] to general and optimal Dirichlet boundary condition. On the other hand, we develop the boundary counterpart of the interior estimates in \[7\] by deriving global gradient estimates in generalized weighted Morrey spaces for bounded solutions of (1.1) in Reifenberg flat domains. These two goals are treated in a unified manner and our achieved results give a comprehensive picture of gradient estimates for equation (1.1). In what follows we consider a bounded domain with its boundary being flat in the following sense of Reifenberg \[21\].

**Definition 1.1.** Let \( \delta, R > 0 \). A bounded domain \( \Omega \subset \mathbb{R}^n \) (\( n \geq 2 \)) is said to be \((\delta,R)\)-Reifenberg flat if for every \( \bar{x} \in \partial \Omega \) and every \( \rho \in (0,R) \) there is a local coordinate system \( \{x_1, \ldots, x_n\} \) with origin at the point \( \bar{x} \) and such that

\[
B_\rho(\bar{x}) \cap \{x : x_n > \rho \delta\} \subset B_\rho(\bar{x}) \cap \Omega \subset B_\rho(\bar{x}) \cap \{x : x_n > -\rho \delta\}.
\]

The Reifenberg flatness means that the boundary \( \partial \Omega \) is well approximated by hyperplanes at every point and at every scale. We note that a domain \( \Omega \) is Reifenberg flat if its boundary is \( C^1 \) smooth or, more generally, its boundary is locally given as the graph of a Lipschitz continuous function with small Lipschitz constant. However, the class of Reifenberg flat domains is much larger and contains domains with rough fractal boundaries (see \[23\]). In order to state our main results, let us recall the so-called Muckenhoupt class of \( A \), weights. By definition, a weight is a nonnegative locally integrable function on \( \mathbb{R}^n \) that is positive almost everywhere. A weight \( w \) belongs to the class \( A_s \), \( 1 < s < \infty \), if

\[
[w]_{A_s} := \sup_{B} \left( \int_{B} w(x) \, dx \right) \left( \int_{B} w(x)^{-s} \, dx \right)^{s^{-1}} < \infty,
\]

where the supremum is taken over all balls \( B \) in \( \mathbb{R}^n \). We also say that \( w \) belongs to the class \( A_{\infty} \) if

\[
[w]_{A_{\infty}} := \sup_{B} \left( \int_{B} w(x) \, dx \right) \exp \left( \int_{B} \log w(x)^{-1} \, dx \right) < \infty.
\]

Now let \( U \subset \mathbb{R}^n \) be a bounded open set, \( w \) be a weight, \( 1 \leq q < \infty \), and \( \psi \) be a positive function on the set of nonempty open balls in \( \mathbb{R}^n \). A function \( g : U \rightarrow \mathbb{R} \) is said to belong to the weighted space \( L^q_w(U) \) if

\[
\|g\|_{L^q_w(U)} := \left( \int_{U} |g(x)|^q w(x) \, dx \right)^{\frac{1}{q}} < \infty.
\]
We define the weighted Morrey space $M_{w}^{q,p}(U)$ to be the set of all functions $g \in L_{w}^{q}(U)$ satisfying

$$
\|g\|_{M_{w}^{q,p}(U)} := \sup_{x \in U, 0 < r \leq \text{diam}(U)} \left( \frac{\varphi(B_{r}(x))}{w(B_{r}(x))} \int_{B_{r}(x) \cap U} |g(x)|^{q}w \, dx \right)^{\frac{1}{q}} < \infty.
$$

(1.5)

Notice that $M_{w}^{q,p}(U) = L_{w}^{q}(U)$ if $\varphi(B) = w(B)$, and we obtain the classical Morrey space $M^{q,p}(U)$ ($0 \leq A \leq n$) by taking $w = 1$ and $\varphi(B) = |B|$. For brevity, the Morrey space $M_{w}^{q,p}(U)$ with $w = 1$ will be denoted by $M^{q,p}(U)$. Let us next recall the class $\mathcal{B}_{\alpha}$ for $\varphi$ that was introduced in [7].

**Definition 1.2.** Let $\varphi$ be a positive function on the set of nonempty open balls in $\mathbb{R}^{n}$. We say that $\varphi$ belongs to the class $\mathcal{B}_{\alpha}$ with $\alpha \geq 0$ if there exists $C > 0$ such that

$$
\frac{\varphi(B_{r}(x))}{\varphi(B_{s}(x))} \leq C \left( \frac{r}{s} \right)^{\alpha}
$$

for every $x \in \mathbb{R}^{n}$ and every $0 < r \leq s$. We define $\mathcal{B}_{+} := \cup_{\alpha > 0} \mathcal{B}_{\alpha} \subset \mathcal{B}_{0}$.

Throughout the paper the conjugate exponent of a number $l \in (1, \infty)$ is denoted by $l'$. We will also adopt the following notation: $\Omega_{r}(y) := \Omega \cap B_{r}(y)$, $\Omega_{r} := \Omega_{r}(0)$, and $\langle \lambda \rangle_{\Omega_{r}(y)}(z, \xi) := \frac{1}{|B_{r}(y) \cap \Omega_{r}(y)|} \int_{B_{r}(y) \cap \Omega_{r}(y)} \lambda(x, z, \xi) \, dx$. On the other hand, $\bar{g}_{E} := \frac{1}{|E|} \int_{E} g(x) \, dx$ whenever $g \in L^{1}(E)$ with $E \subset \mathbb{R}^{n}$ being a measurable and bounded set. For a fixed number $M > 0$, we define

$$
\Theta_{\Omega_{r}(y)}(\Lambda) := \sup_{z \in \mathbb{R}^{n}, -M \leq M} \frac{1}{|B_{r}(y)|} \int_{\Omega_{r}(y)} \left[ \sup_{\xi \neq 0} \frac{|\Lambda(x, z, \xi) - \langle \lambda \rangle_{\Omega_{r}(y)}(z, \xi)|}{|\xi|^{p_{1}} - 1} \right] \, dx.
$$

(1.6)

Our first main result is:

**Theorem 1.3.** Let $\Lambda$ satisfy (1.2)–(1.4) with $p > 1$, and let $w$ be an $A_{\infty}$ weight for some $1 < s < \infty$. Then for any $q \geq p$, $M > 0$, and $\varphi \in \mathcal{B}_{+}$ with $\sup_{x \in \Omega} \varphi(B_{\text{diam}(\Omega)}(x)) < \infty$, there exists a constant $\delta = \delta(p, q, n, \omega, \Lambda, M, s, [w]_{A_{s}}) > 0$ such that: if $\Omega$ is $(\delta, R)$-Reifenberg flat,

$$
\sup_{0 < r < R} \sup_{y \in \Omega} \Theta_{\Omega_{r}(y)}(\Lambda) \leq \delta,
$$

(1.7)

and $u$ is a weak solution of (1.1) satisfying $\|u\|_{L^{\infty}(\Omega)} + \|\psi\|_{L^{\infty}(\Omega)} \leq M$, we have

$$
\|\nabla u\|_{M_{w}^{q,p}(\Omega)} \leq C \left( \|\nabla u\|_{L^{p}(\Omega)} + \|M_{\Omega}(|\psi|^{p} + |F|^{p'})\|_{M^{q,p}(\Omega)}^{\frac{1}{q}} + \|M_{\Omega}(|\psi|^{p} + |F|^{p'})\|_{M_{w}^{q,p}(\Omega)}^{\frac{1}{q}} \right)
$$

(1.8)

Here $M_{\Omega}$ denotes the centered Hardy–Littlewood maximal operator (see Definition 3.1 in [7]), and $C > 0$ is a constant depending only on $q$, $p$, $n$, $\omega$, $\Lambda$, $M$, $\varphi$, $s$, $R$, $\text{diam}(\Omega)$, and $[w]_{A_{s}}$.

The above theorem holds true for any weight $w$ in the class $A_{\infty}$. When $q > p$ and certain additional information about the weights and $\varphi, \phi$ is given, we can further estimate the two quantities in (1.8) involving the maximal function of $|F|^{p'}$ to obtain:
The organization of the paper is as follows. We recall some preliminary results in Subsections 2.1–2.2 and state two key regularity results (Theorems 2.9 and 2.10) in Subsection 2.3 about gradient estimates in Assumption 1.4. Let $A$ satisfy (1.2)–(1.4) with $p > 1$. Let $q > p$, $w \in A_\infty$, $v \in A_\infty^q$, $\varphi \in \mathcal{B}_q$ with $\sup_{x \in \Omega} \varphi(B_{\text{diam}(\Omega)}(x)) < \infty$, and $\phi \in \mathcal{B}_0$ satisfy

\[
[w, v^1-(\frac{p}{p})_\infty]^q \leq \sup_{B} \left( \int_B w \, dx \right) \left( \int_B v^{1-(\frac{p}{p})_\infty} \, dx \right)^{\frac{p}{p} - 1} < \infty, \tag{1.9}
\]

\[
\frac{v(2B)}{w(2B)} \frac{1}{\phi(2B)} \leq C \varphi(B)^{1-(\frac{p}{p})_\infty} \quad \text{for all balls } B \subset \mathbb{R}^n. \tag{1.10}
\]

Then for any $M > 0$, there exists a small constant $\delta = \delta(p, q, n, \omega, \Lambda, M, [w]_A) > 0$ such that: if $\Omega$ is $(\delta, R)$-Reifenberg flat, (1.7) holds, and $u$ is a weak solution of (1.1) satisfying $\|u\|_{L^\infty(\Omega)} + \|\varphi\|_{L^\infty(\Omega)} \leq M$, we have

\[
\|\nabla u\|_{A^{q,p}(\Omega)} \leq C\|\nabla \psi + |F|^{\frac{q}{p}}\|_{A^{q,p}(\Omega)} \tag{1.11}
\]

with $C$ depending only on $p$, $q$, $n$, $\omega$, $\Lambda$, $\varphi$, $\phi$, $C_\ast$, $R$, $\text{diam}(\Omega)$, $[w]_A$, $[v]_A$, and $[w, v^1-(\frac{p}{p})_\infty]^q$.

This result complements the global $C^{1,\alpha}$ regularity developed by Lieberman [13] for the corresponding homogeneous equation when $(x, z) \mapsto A(x, z, \cdot)$ is $C^\alpha$ and both $\partial \Omega$ and $\psi$ belong to the class $C^{1,\alpha}$. The boundedness assumption for $u$ in Theorems 1.3 and 1.4 is used to handle the dependence of the principal part on the solution itself. A simple inspection of our proofs reveals that this condition is not needed in the special case $A(x, z, \xi) = A(x, \xi)$. As far as we know all available global $W^{1,q}$ estimates for nonlinear equation of $p$-Laplacian type are only established for identically zero Dirichlet boundary data. Our boundary data is general and the imposed condition on $\psi$ is optimal in view of the linear case. We expect that our method of dealing with nonhomogeneous boundary data can be useful for other nonlinear elliptic equations.

The central point in proving the above main results is to be able to show that gradients of weak solutions $u$ to equation (1.1) can be approximated in an invariant way by bounded gradients in $L^p$ norm (see Proposition 3.2). In order to achieve this we encounter four main difficulties: the discontinuity of $A(x, z, \xi)$ in the $x$ variable and the dependence of $A$ on the $z$ variable, the nonhomogeneous Dirichlet boundary data, the roughness of the domain, and the fact that equations of the form (1.1) are not invariant with respect to dilations and rescaling of domains. Using the method in [4] and an argument in [12,17], we deal with the first two issues in Lemmas 3.3 and 3.4 by freezing the $x$ and $z$ variables and comparing solution $u$ of (1.1) to that of the corresponding frozen equation with zero Dirichlet boundary data. The third issue is handled by a compactness argument allowing us to extract information from the limiting equations whose domains have flat boundaries, see the proofs of Lemma 3.5 and Lemma 3.A. To overcome the last issue about the lack of the invariant structure, we use the key idea introduced in [10,17] by enlarging the class of equations under consideration in a suitable way. Precisely, we consider the associated quasilinear elliptic equations with two parameters, i.e. equation (2.4) below. The chief advantage of working with (2.4) is that equations of this form are invariant with respect to dilations and rescaling of domains. However, there arise new difficulties in dealing with the parameters. It is essential for the success of our analysis that all the obtained approximation estimates and involving constants must be independent of the two parameters. The large part of this paper is devoted to achieving this.
weighted $L^q$ spaces. Section 3 is devoted to proving Proposition 3.2 which shows that gradients of weak solutions to two-parameter equation (2.4) can be approximated by bounded gradients in a small neighborhood of any point in the domain. Using this crucial result, we establish in Subsection 4.1 some density estimates for gradients and then prove Theorems 2.9 and 2.10 in Subsection 4.2. Finally, the main results stated in Theorems 1.3 and 1.4 are derived in Subsection 4.3 as consequences of Theorem 2.9.

2 Preliminaries and key regularity results

2.1 Basic properties of $A_s$ weights and estimates for the maximal function

Given a weight $w$ and a measurable set $E \subset \mathbb{R}^n$, we use the notation $dw(x) = w(x) \, dx$ and $w(E) = \int_E w(x) \, dx$.

**Lemma 2.1.** Let $w \in A_s$ for some $1 < s < \infty$. Then there exist $0 < \beta \leq 1$ and $K > 0$ depending only on $n$ and $[w]_{A_s}$ such that

$$\frac{[w]_{A_s}^{-1}}{B} \left( \frac{|E|^s}{|B|} \right) \leq \frac{w(E)}{w(B)} \leq K \left( \frac{|E|^s}{|B|} \right)^\beta$$

(2.1)

for all balls $B$ and all measurable sets $E \subset B$. In particular, $w$ is doubling with $w(2B) \leq 2^{ns}[w]_{A_s} w(B)$.

**Lemma 2.2** (Characterizations of $A_\infty$ weights). Suppose that $w$ is a weight. Then $w$ is in $A_\infty$ if and only if there exist $0 < A, \nu < \infty$ such that for all balls $B$ and all measurable sets $E \subset B$ we have

$$\frac{w(E)}{w(B)} \leq A \left( \frac{|E|}{|B|} \right)^\nu.$$  

(2.2)

When $w \in A_\infty$, the above constants $A$ and $\nu$ depend only on $n$ and $[w]_{A_\infty}$. Conversely, given constants $A$ and $\nu$ satisfying (2.2), we have $[w]_{A_\infty} \leq C(n, A, \nu)$.

Let $\tilde{M}$ denotes the uncentered Hardy–Littlewood maximal operator (see Definition 3.1 in [7]). For two weights $w_1$ and $w_2$, let

$$[w_1, w_2]_{A_q} := \sup_B \left( \frac{\int_B w_1(x) \, dx}{\int_B w_2(x) \, dx} \right)^{q-1},$$

$$[w_1, w_2]_{S_q} := \sup_B \left( \frac{1}{w_2(B)} \int_B [\tilde{M}(w_2 \chi_B)]^q w_1 \, dx \right)^{\frac{1}{q}}.$$  

Then $[w_1, w_2]_{A_q} \leq [w, w_2]_{S_q}^q$. Moreover, we have the following estimates from [7] for the maximal function in weighted Morrey spaces.

**Lemma 2.3** (Corollary 3.6 in [7]). Let $w, v$ be two weights and $\varphi, \phi$ be two positive functions on the set of nonempty open balls in $\mathbb{R}^n$. Let $1 < q < \infty$ and $U \subset \mathbb{R}^n$ be a bounded open set. Assume that $w$ is doubling and there exists a constant $C_* > 0$ such that

$$\sup_{2r \leq s \leq 2\text{diam}(U)} \frac{v(B_r(y))}{w(B_r(y))} \frac{1}{\varphi(B_r(y))} \leq C_* \frac{1}{\phi(B_r(y))}$$

for all $y \in U$ and $0 < r \leq \text{diam}(U)$.

(2.3)

Assume in addition that one of the following two conditions is satisfied:
(A) There exists \( r > 1 \) such that
\[
\sup_B \left( \int_B w \, dx \right) \left( \int_B v^{(1-q')} \, dx \right)^{\frac{q}{1-q'}} < \infty.
\]

(B) \([w, v^{1-q'}]_{\lambda_q} < \infty\) and \(v^{1-q'} \in A_{\infty}\).

Then \([w, v^{1-q'}]_{\lambda_q} < \infty\) and there exists a constant \( C > 0 \) depending only on \( n, q, C_\alpha \), the doubling constant for \( w \), and \([w, v^{1-q'}]_{\lambda_q} \) such that
\[
\|\tilde{M}_U(f)\|_{\mathcal{M}^n_{\alpha}(U)} \leq C\|f\|_{\mathcal{M}^n_{\alpha}(U)} \quad \forall f \in L^1(U).
\]

**Lemma 2.4** (Corollary 3.7 in [7]). Let \( 1 < q < \infty \), \( w \in A_q \), and \( U \subset \mathbb{R}^n \) be a bounded open set. Assume that \( \varphi \) and \( \phi \) are two positive functions on the set of open balls in \( \mathbb{R}^n \) such that there exists \( C_\alpha > 0 \) satisfying
\[
\sup_{2r \leq s \leq \text{diam}(U)} \phi(B_s(y))^{-1} \leq C_\alpha \varphi(B_r(y))^{-1} \quad \text{for all } y \in U \text{ and } 0 < r \leq \text{diam}(U).
\]

Then there exists a constant \( C > 0 \) depending only on \( n, q, C_\alpha \), and \([w]_{\lambda_q} \) such that
\[
\|\tilde{M}_U(f)\|_{\mathcal{M}^n_{\alpha}(U)} \leq C\|f\|_{\mathcal{M}^n_{\alpha}(U)} \quad \text{for any } f \in L^1(U).
\]

### 2.2 An energy inequality and some classical regularity estimates

Let us consider the following equation
\[
\begin{cases}
\text{div} \left[ \frac{\mathbf{A}(x, \lambda \theta u, \lambda \nabla u)}{\lambda^{p-1}} \right] = \text{div} F & \text{in } \Omega, \\
u = \psi & \text{on } \partial \Omega,
\end{cases}
\tag{2.4}
\]

where \( \lambda, \theta > 0 \) are two parameters. We will use the fact (see the proof of Lemma 1) that condition \((1.2)\) implies that
\[
\langle \mathbf{A}(x, z, \xi) - \mathbf{A}(x, z, \eta), \xi - \eta \rangle \geq \begin{cases}
4^{1-p} \Lambda^{-1} |\xi - \eta|^p & \text{if } p \geq 2, \\
4^{-1} \Lambda^{-1} (|\xi| + |\eta|)^{p-2} |\xi - \eta|^2 & \text{if } 1 < p < 2
\end{cases}
\tag{2.5}
\]

for a.e. \( x \in \Omega \), all \( z \in \mathbb{R}^n \), and all \( \xi, \eta \in \mathbb{R}^n \).

**Proposition 2.5** (energy estimate). Let \( \psi \in W^{1,p}(\Omega) \) and \( u \) be a weak solution of \((2.4)\). Then we have
\[
\int_\Omega |\nabla u|^p \, dx \leq C(p, n, \Lambda) \int_\Omega (|\nabla \psi|^p + |F|^p) \, dx.
\tag{2.6}
\]

**Proof.** Let \( \tilde{\mathbf{A}}(x, z, \xi) := \frac{\mathbf{A}(x, \lambda \theta u, \lambda \nabla u)}{\lambda^{p-1}} \). Then by using \( u - \psi \) as a test function in equation \((2.4)\) we get
\[
\int_\Omega \langle \tilde{\mathbf{A}}(x, u, \nabla u), \nabla u \rangle \, dx = \int_\Omega \langle \tilde{\mathbf{A}}(x, u, \nabla u), \nabla \psi \rangle \, dx + \int_\Omega \langle F, \nabla u - \nabla \psi \rangle \, dx.
\]

But it follows from \((2.5)\) for \( \eta = 0 \) and the fact \( \mathbf{A}(x, z, 0) = 0 \) that \( \langle \mathbf{A}(x, z, \xi), \xi \rangle \geq C(p, \Lambda) |\xi|^p \). Therefore, we obtain
\[
\int_\Omega |\nabla u|^p \, dx \leq C \left[ \int_\Omega |\nabla u|^{p-1} |\nabla \psi| \, dx + \int_\Omega |F||\nabla u| \, dx + \int_\Omega |F||\nabla \psi| \, dx \right].
\]

We deduce from this and Young’s inequality that estimate \((2.6)\) holds true. \(\Box\)
Let \( \mathbf{a} = \mathbf{a}(x, \xi) : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) be measurable in \( x \) for every \( \xi \in \mathbb{R}^n \) and continuous in \( \xi \) for a.e. \( x \in \Omega \). In addition, we assume that there exist constants \( \Lambda > 0 \) and \( 1 < p < \infty \) such that the following structural conditions are satisfied for a.e. \( x \in \Omega \) and all \( \xi \in \mathbb{R}^n \):

\[
\langle \mathbf{a}(x, \xi), \xi \rangle \geq \Lambda^{-1} |\xi|^p \quad \text{and} \quad |\mathbf{a}(x, \xi)| \leq \Lambda |\xi|^{p-1}.
\]

The following interior Hölder estimate is classical, see for instance [3][14].

**Theorem 2.6** (interior Hölder estimate). Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain and \( \mathbf{a}(x, \xi) \) satisfy \((2.7)\). Suppose \( w \in W^{1,p}_\text{loc}(\Omega) \) is a weak solution of \( \text{div} \mathbf{a}(x, \nabla w) = 0 \) in \( \Omega \). Then \( u \) is continuous in \( \Omega \) and has the following bound on its modulus of continuity: if \( B_R(\bar{x}) \subset \Omega \) and \( r \in (0, R) \), then we have

\[
\text{osc}_{B_r(x)} w \leq C \left( \frac{r}{R} \right)^\beta \text{osc}_{B_R(\bar{x})} w,
\]

where \( C > 0 \) and \( \beta \in (0, 1) \) depend only on \( p \), \( n \), and \( \Lambda \).

**Proof.** This is a special case of Theorem 4.11 in [14]. Notice that due to a much more general equation in [14] their corresponding constants \( C \) and \( \beta \) depend also on \( \|w\|_{L^\infty(B_R(\bar{x}))} \). However, an inspection of their proof in page 196 reveals that with structural condition \((2.7)\) these constants can be chosen to depend only on \( p \), \( n \), and \( \Lambda \) as stated. \( \square \)

The next regularity result is well known and is a special case of [11] Theorem 1.1 and [14] Theorem 4.19 and Corollary 4.20] (see also [24] and [3] Theorem 6.8 and estimate (7.54)).

**Theorem 2.7** (global higher integrable and Hölder estimates). Let \( \Omega \subset \mathbb{R}^n \) (\( n \geq 2 \)) be a bounded domain, \( \mathbf{a}(x, \xi) \) satisfy \((2.7)\), \( x_0 \in \partial \Omega \), and \( r_0 > 0 \). Assume that there exist positive constants \( c_* \) and \( \rho_* \) satisfying

\[
|B_{\rho}(z) \setminus \Omega| \geq c_* |B_{\rho}(z)|
\]

for all \( z \in B_{r_0}(x_0) \cap \partial \Omega \) and all \( \rho \in (0, \rho_*) \). Suppose that \( w \in W^{1,p}(\Omega_{r_0}(x_0)) \) is a weak solution of

\[
\begin{cases}
\text{div} \mathbf{a}(x, \nabla w) = 0 & \text{in} \quad \Omega_{r_0}(x_0), \\
w = 0 & \text{on} \quad B_{r_0}(x_0) \cap \partial \Omega.
\end{cases}
\]

Then

(i) There exist constants \( p_0 \in (p, \infty) \) and \( C > 0 \) depending only on \( p \), \( n \), \( c_* \), and \( \Lambda \) such that: if \( 0 < r < s \leq r_0 \) and \( B_s(y) \subset \Omega_{r_0}(x_0) \), we have

\[
\left( \frac{1}{|B_s(y)|} \int_{\Omega_{r_0}(y)} |\nabla w|^p \, dx \right)^{\frac{1}{p}} \leq C \left( \frac{1}{|B_s(y)|} \int_{\Omega_{r_0}(y)} |\nabla w|^{p_0} \, dx \right)^{\frac{1}{p_0}}.
\]

(ii) For any \( z \in B_{r_0}(x_0) \cap \partial \Omega \) and any \( r \in (0, r_0] \), we have

\[
\text{osc}_{B_r(z)} w \leq C \left( \frac{r}{r_0} \right)^\beta \|w\|_{L^{p_0}(\Omega_{r_0}(z))},
\]

where \( C > 0 \) depends only on \( p \), \( n \), and \( \Lambda \), while \( \beta \in (0, 1) \) depends in addition on \( c_* \).
Proof. The result in (i) is from [11] Theorem 1.1 and the precise estimate for $\nabla w$ can be tracked from their proof and the use of Gehring’s lemma. On the other hand, the result in (ii) is a particular case of Theorem 4.19 in [14]. The estimate obtained in [14] is less explicit than ours due to their more general equation. However, our stated oscillation estimate follows from their proof in pages 200-201 and our simple structural condition (2.7). \hfill \Box

The next result is a special case of [13, Lemma 5] and plays an essential role in proving our main results.

**Theorem 2.8 (Boundary Lipschitz estimate).** Let $a : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vector field such that $\xi \mapsto a(\xi)$ is differentiable on $\mathbb{R}^n \setminus \{0\}$ and $a$ satisfies (1.2)–(1.3). Suppose that $w \in W^{1,p}(B_R^+)$ is a weak solution of $\text{div} a(\nabla w) = 0$ in $B_R^+$ and $w = 0$ on $B_R \cap \{x : x_n = 0\}$. Then we have

$$\sup_{B_R^+} |\nabla w|^p \leq C(p, n, \Lambda) \frac{1}{R^n} \int_{B_R^+} |\nabla w|^p \, dx.$$ 

### 2.3 Key regularity results

**Theorem 2.9.** Let $A$ satisfy (1.2)–(1.4) with $p > 1$, and let $w \in A_s$ for some $1 < s < \infty$. For any $q \geq p$ and $M > 0$, there exists a constant $\delta = \delta(p, q, n, \omega, \Lambda, M, s, [w]_{A_s}) > 0$ such that: if $\Omega$ is $(\delta, R)$-Reifenberg flat, $\lambda > 0$, $\theta > 0$, (1.7) holds, and $u$ is a weak solution of (2.4) satisfying $\|u\|_{L^\infty(\Omega)} + \|\psi\|_{L^\infty(\Omega)} \leq \frac{M}{\theta}$, then

$$\int_{\Omega} |\nabla u|^q \, dw \leq C \left( \|\nabla u\|^q_{L^\infty(\Omega)} + \int_{\Omega} M_{\Omega}(|\nabla \psi|^p + |F|^p)^{\frac{q}{p}} \, dw \right).$$

(2.9)

Here $C > 0$ is a constant depending only on $q, p, n, \omega, \Lambda, M, s, R, \text{diam}(\Omega)$, and $[w]_{A_s}$.

We also have the following localized version of Theorem 2.9.

**Theorem 2.10.** Let $A$ satisfy (1.2)–(1.4) with $p > 1$, and let $w \in A_s$ for some $1 < s < \infty$. For any $q \geq p$ and $M > 0$, there exists a constant $\delta = \delta(p, q, n, \omega, \Lambda, M, s, [w]_{A_s}) > 0$ such that: if $\Omega$ is $(\delta, R)$-Reifenberg flat, $\lambda > 0$, $\theta > 0$, (1.7) holds, and $u$ is a weak solution of (2.4) satisfying $\|u\|_{L^\infty(\Omega)} + \|\psi\|_{L^\infty(\Omega)} \leq \frac{M}{\theta}$, then

$$\frac{1}{w(B_r(y))} \int_{\Omega_{\lambda}(y)} |\nabla u|^q \, dw \leq C \left( \frac{1}{w(B_{2r}(y))} \int_{\Omega_{2\lambda}(y)} |\nabla u|^p \, dx \right)^{\frac{q}{p}} + \frac{1}{w(B_r(y))} \int_{\Omega_{\lambda}(y)} M_{\Omega_{2\lambda}(y)}(|\nabla \psi|^p + |F|^p)^{\frac{q}{p}} \, dw \right]$$

for every $y \in \Omega$ and $r > 0$. Here $C > 0$ depends only on $q, p, n, \omega, \Lambda, M, s, R, \text{diam}(\Omega)$, and $[w]_{A_s}$.

The above two results play a crucial role in proving our main theorems stated in Section 1 and their proofs will be given in Subsection 4.2. As a consequence of Theorem 2.9 and the maximal function estimate, we get:

**Corollary 2.11 (global weighted $L^q$ estimate).** Let $A$ satisfy (1.2)–(1.4) with $p > 1$. Then for any $q > p$, $M > 0$, and any weight $w \in A_{\frac{q}{p}}$, there exists a constant $\delta > 0$ such that: if $\Omega$ is $(\delta, R)$-Reifenberg flat, $\lambda > 0$, $\theta > 0$, (1.7) holds, and $u$ is a weak solution of (2.4) satisfying $\|u\|_{L^\infty(\Omega)} + \|\psi\|_{L^\infty(\Omega)} \leq \frac{M}{\theta}$, we have

$$\int_{\Omega} |\nabla u|^q \, dw \leq C \int_{\Omega} (|\nabla \psi|^q + |F|^q)^{\frac{p}{q}} \, dw.$$

Here $C, \delta$ are constants depending only on $q, p, n, \omega, \Lambda, M, R, \text{diam}(\Omega)$, and $[w]_{A_{\frac{q}{p}}}$. 

8
Proof. Since \( q > p \), we have from Theorem 2.9 and Muckenhoupt’s strong type weighted estimate for the maximal function that
\[
\int_{\Omega} |\nabla u|^q \, dw \leq C \left( \|\nabla u\|_{L^p(\Omega)}^q + \int_{\Omega} \left[ |\nabla \psi|^q + |F|^q \right] \, dw \right). \tag{2.10}
\]
From Proposition 2.5 and Hölder inequality, we also have
\[
\|\nabla u\|_{L^p(\Omega)}^q \leq C \left( \int_{\Omega} (|\nabla \psi|^p + |F|^p) \, dx \right)^{\frac{q}{p}} \leq C \left( \int_{\Omega} \frac{\varepsilon}{|\nabla \psi|} \, dx \right)^{\frac{q}{p}} \int_{\Omega} \left[ |\nabla \psi|^q + |F|^q \right] \, dw. \tag{2.11}
\]
Let \( \tilde{x} \in \mathbb{R}^n \) be such that \( \Omega \subset B_0 := B(\tilde{x}, \text{diam} (\Omega)) \). Then
\[
\left( \int_{\Omega} \frac{\varepsilon}{|\nabla \psi|} \, dx \right)^{\frac{q}{p}} \leq \left( \int_{B_0} \frac{\varepsilon}{|\nabla \psi|} \, dx \right)^{\frac{q}{p}} \leq [w]_{A} \frac{\|B_0\|^q}{\# w(B_0)} \leq [w]_{A} \frac{\|B_0\|^q}{\# w(\Omega)}.
\]
This together with (2.10)–(2.11) yields the desired conclusion. \( \square \)

3 Approximating gradients of solutions

In this section, we always suppose that \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) with \( n \geq 2 \). For the next result, we also assume that
\[
0 \in \partial \Omega \quad \text{and} \quad \left| B_{\rho}(z) \setminus \Omega \right| \geq \frac{1}{3} |B_{\rho}(z)| \quad \text{for all} \quad z \in B_4 \cap \partial \Omega \quad \text{and all} \quad 0 < \rho < 2. \tag{3.1}
\]

Lemma 3.1. Let \( A \) satisfy (1.2)–(1.4), and \( M > 0 \). For any \( \varepsilon \in (0, 1) \), there exist small positive constants \( \delta \) and \( \sigma \) depending only on \( \varepsilon, p, n, \omega, \Lambda, \) and \( M \) such that: if \( \lambda > 0, \theta > 0, \) \( \Omega \) satisfies (3.1),
\[
B_{3\sigma} \cap \{x_n > 3\sigma \delta\} \subset \Omega_{3\sigma} \subset B_{3\sigma} \cap \{x_n > -3\sigma \delta\}, \tag{3.2}
\]
y \( \in B_1 \) satisfies either \( y = 0 \) or \( B_{4\sigma}(y) \subset \Omega_2 \),
\[
\Theta_{\Omega_{3\sigma}(y)}(A) \leq \delta \quad \text{and} \quad \frac{1}{|B_4|} \int_{\Omega_4} (|\nabla \psi|^p + |F|^p) \, dx \leq \delta,
\]
and \( u \) is a weak solution of (2.4) satisfying
\[
\|u\|_{L^p(\Omega_4)} + \|\psi\|_{L^p(\Omega_4)} \leq \frac{M}{\lambda \theta}, \quad \frac{1}{|B_4|} \int_{\Omega_4} |\nabla u|^p \, dx \leq 1, \quad \text{and} \quad \frac{1}{|B_{4\sigma}(y)|} \int_{\Omega_{4\sigma}(y)} |\nabla u|^p \, dx \leq 1,
\]
then there exists a function \( v \in W^{1,p}(\Omega_{2\sigma}(y)) \) such that
\[
\|\nabla v\|_{L^p(\Omega_{2\sigma}(y))} \leq C(p, n, \Lambda) \quad \text{and} \quad \frac{1}{|B_{2\sigma}(y)|} \int_{\Omega_{2\sigma}(y)} |\nabla u - \nabla v|^p \, dx \leq \varepsilon^p.
\]
By translating and scaling, we obtain:
Proposition 3.2. Let $A$ satisfy (1.2)–(1.4), and $M > 0$. For any $e \in (0, 1]$, there exist small positive constants $\delta$ and $\sigma$ depending only on $e$, $p$, $n$, $\omega$, $\Lambda$, and $M$ such that: if $\Omega$ is $(\delta, R)$-Reifenberg flat, $\lambda > 0$, $\theta > 0$, $\bar{y} \in \partial \Omega$, $r \in (0, \frac{1}{2}]$, $y \in B_r(\bar{y})$ satisfies either $y = \bar{y}$ or $B_{4rr}(y) \subset \Omega_{2r}(\bar{y})$.

$$\Theta_{\Omega_{3rr}(y)}(A) \leq \delta \quad \text{and} \quad \frac{1}{|B_{4r}(\bar{y})|} \int_{\Omega_{4r}(\bar{y})} |(\nabla \psi |^{p} + |F|^{p'}) \, dx \leq \delta,$$

and $u$ is a weak solution of (2.4) satisfying

$$\|u\|_{L^{\infty}(\Omega_{4r}(\bar{y}))} + \|\psi\|_{L^{\infty}(\Omega_{4r}(\bar{y}))} \leq \frac{M}{\lambda^2}, \quad \frac{1}{|B_{4r}(\bar{y})|} \int_{\Omega_{4r}(\bar{y})} |\nabla u|^{p} \, dx \leq 1, \quad \text{and} \quad \frac{1}{|B_{4rr}(y)|} \int_{\Omega_{4rr}(y)} |\nabla u|^{p} \, dx \leq 1,$$

then there exists a function $v \in W^{1,p}(\Omega_{2rr}(\bar{y}))$ such that

$$\|\nabla v\|_{L^{\infty}(\Omega_{2rr}(\bar{y}))} \leq C(p, n, \Lambda) \quad \text{and} \quad \frac{1}{|B_{2rr}(y)|} \int_{\Omega_{2rr}(y)} |\nabla u - \nabla v|^{p} \, dx \leq \epsilon^{p}.$$

Proof. The result is obtained by translating and scaling, and then applying Lemma 3.1. Precisely, let $\tilde{\Omega} := \{r^{-1}(x - \bar{y}) : x \in \Omega\}$ and $\tilde{y} := r^{-1}(y - \bar{y}) \in B_1$. Then $0 \in \partial \tilde{\Omega}$ and $\tilde{\Omega}$ is $(\delta, R/r)$-Reifenberg flat with $R/r > 2$. Thus it follows from Definition 1.1 that $\tilde{\Omega}$ satisfies the so-called $(A)$-property: there exists a positive constant $K = K(\delta)$ such that

$$K|B_{\rho}(z)| \leq |B_{\rho}(z) \cap \tilde{\Omega}| \leq (1 - K)|B_{\rho}(z)|$$

for all $z \in \partial \tilde{\Omega}$ and all $\rho \in (0, R/r)$. Moreover, $K(\delta) \to 1/2$ when $\delta \to 0^+$. As $\delta > 0$ is small, we deduce that $\tilde{\Omega}$ satisfies condition (3.1). By rotating the standard coordinate system $(x_1, ..., x_n)$ if necessary, we also have from Definition 1.1 that

$$B_1 \cap \{x_n > \rho \delta\} \subset \tilde{\Omega}_\rho \subset B_1 \cap \{x_n > -\rho \delta\}$$

for any $\rho \leq 2$. In particular, condition (3.2) is verified for $\tilde{\Omega}$ as well. We next define

$$\tilde{A}(x, z, \xi) = A(rx + \bar{y}, z, \xi), \quad \tilde{F}(x) = F(rx + \bar{y}), \quad \tilde{u}(x) = r^{-1}u(rx + \bar{y}), \quad \tilde{\psi}(x) = r^{-1}\psi(rx + \bar{y}), \quad \text{and} \quad \tilde{\theta} = \theta r.$$

Then $\tilde{u}$ is a weak solution of $\text{div} \left( \frac{\tilde{A}(x, \tilde{u}, \nabla \tilde{u})}{\lambda^2} \right) = \text{div} \tilde{F}$ in $\tilde{\Omega}$ and $\tilde{u} = \tilde{\psi}$ on $\partial \tilde{\Omega}$. Moreover,

$$\|\tilde{u}\|_{L^{\infty}(\tilde{\Omega})} + \|\tilde{\psi}\|_{L^{\infty}(\tilde{\Omega})} \leq \frac{M}{\lambda^2}, \quad \frac{1}{|B_{4r}(\bar{y})|} \int_{\tilde{\Omega}_{4r}} |\nabla \tilde{u}|^{p} \, dx = \frac{1}{|B_{4r}(\bar{y})|} \int_{\Omega_{4r}(\bar{y})} |\nabla u|^{p} \, dz \leq 1,$$

$$\frac{1}{|B_{4rr}(y)|} \int_{\Omega_{4rr}(y)} |\nabla u|^{p} \, dz \leq 1, \quad \Theta_{\Omega_{3rr}(y)}(\tilde{A}) = \Theta_{\Omega_{3rr}(y)}(A) \leq \delta,$$

and

$$\frac{1}{|B_{4r}(\bar{y})|} \int_{\Omega_{4r}(\bar{y})} ((\nabla \tilde{\psi}|^{p} + |\tilde{F}|^{p'}) \, dx = \frac{1}{|B_{4r}(\bar{y})|} \int_{\Omega_{4r}(\bar{y})} ((\nabla \psi|^{p} + |F|^{p'}) \, dz \leq \delta.$$

Therefore, we can apply Lemma 3.1 to conclude that there exists a function $\tilde{v} \in W^{1,p}(\Omega_{2rr}(\bar{y}))$ such that

$$\|\nabla \tilde{v}\|_{L^{\infty}(\Omega_{2rr}(\bar{y}))} \leq C(p, n, \Lambda) \quad \text{and} \quad \frac{1}{|B_{2rr}(y)|} \int_{\Omega_{2rr}(y)} |\nabla \tilde{u} - \nabla \tilde{v}|^{p} \, dx \leq \epsilon^{p}.$$
Let \( v(x) := r\bar{v}(r^{-1}(x - \bar{y})) \). Then we infer that

\[
\|\nabla v\|_{L^\infty(\Omega_{5r}(y))} \leq C(p, n, \Lambda) \quad \text{and} \quad \frac{1}{|B_{2r}(y)|} \int_{\Omega_{2r}(y)} |\nabla u - \nabla v|^p \, dx \leq e^p.
\]

\[
\square
\]

The rest of this section is devoted to proving Lemma 3.1. The first step is:

**Lemma 3.3.** For any \( \varepsilon > 0 \), there exist small positive constants \( \delta \) and \( \sigma \) depending only on \( \varepsilon, p, n, \omega, \Lambda, \) and \( M \) such that: if \( \lambda > 0, \theta > 0, \Omega \) satisfies (3.1), \( y \in B_1 \) satisfies either \( y = 0 \) or \( B_{4r}(y) \subset \Omega_2 \),

\[
\int_{B_4} (|\nabla \psi|^p + |F|^p) \, dx \leq \delta,
\]

and \( u \) is a weak solution of (2.4) satisfying

\[
\|u\|_{L^\infty(\Omega_4)} + \|\psi\|_{L^\infty(\Omega_4)} \leq \frac{M}{\lambda^\theta} \quad \text{and} \quad \frac{1}{|B_4|} \int_{\Omega_4} |\nabla u|^p \, dx \leq 1,
\]

then

\[
\frac{1}{|B_{4r}(y)|} \int_{\Omega_{4r}(y)} |\nabla u - \nabla f|^p \, dx \leq e^p,
\]

where \( f \) is a weak solution of

\[
\begin{align*}
\text{div} \left( \frac{\mathbf{A}(x, \lambda \bar{h} \Omega_{4r}(y), \lambda \nabla f)}{\lambda^{p-1}} \right) &= 0 \quad \text{in} \quad \Omega_{4r}(y), \\
f &= h \quad \text{on} \quad \partial \Omega_{4r}(y)
\end{align*}
\]

with \( h \) being a weak solution of

\[
\begin{align*}
\text{div} \left( \frac{\mathbf{A}(x, \lambda \bar{u} \Omega_4, \lambda \nabla h)}{\lambda^{p-1}} \right) &= 0 \quad \text{in} \quad \Omega_4, \\
h &= u - \psi \quad \text{on} \quad \partial \Omega_4.
\end{align*}
\]

**Proof.** We only present the proof for \( p \geq 2 \) using an idea in [1][2]. The argument for the case \( 1 < p < 2 \) is similar with some slight adjustments which can be found in [2][7]. For convenience, let \( \tilde{\mathbf{A}}(x, z, \xi) := \frac{\mathbf{A}(x, \lambda \bar{u} \Omega_4, \lambda \nabla h)}{\lambda^{p-1}} \). We write

\[
\nabla u - \nabla f = \nabla (u - h) + \nabla (h - f)
\]

and will estimate \( \|\nabla (u - h)\|_{L^p(\Omega_{4r}(y))} \) and \( \|\nabla (h - f)\|_{L^p(\Omega_{4r}(y))} \). By using \( u - \psi - h \) as a test function in the equations for \( u \) and \( h \) we have

\[
\int_{\Omega_4} \langle \tilde{\mathbf{A}}(x, u, \nabla u) - \tilde{\mathbf{A}}(x, u, \nabla h), \nabla (u - \psi - h) \rangle \, dx = \int_{\Omega_4} \langle \mathbf{F}, \nabla (u - \psi - h) \rangle \, dx
\]

yielding

\[
\int_{\Omega_4} \langle \tilde{\mathbf{A}}(x, u, \nabla u) - \tilde{\mathbf{A}}(x, u, \nabla h), \nabla (u - h) \rangle \, dx \leq \Lambda \int_{\Omega_4} \|\nabla \psi\| \|\nabla u\|^{p-1} \, dx + \int_{\Omega_4} |\mathbf{F}|(|\nabla (u - h)| + |\nabla \psi|) \, dx.
\]
We then use \( (2.5) \) to bound the above left hand side from below. As a consequence, we obtain
\[
\int_{\Omega_4} |\nabla (u - h)|^p \, dx \leq C(p, \Lambda) \left[ \int_{\Omega_4} |\nabla \psi||\nabla u|^{p-1} + |\nabla h|^{p-1} \right] \, dx + \int_{\Omega_4} |F||(|\nabla (u - h)| + |\nabla \psi|) \, dx.
\]
Hence we infer from Young and Hölder inequalities, and the energy estimate in Proposition 2.5 that
\[
\int_{\Omega_4} |\nabla (u - h)|^p \, dx \leq C \left[ \|\nabla \psi\|_{L^p(\Omega_4)} \left(\|\nabla u\|_{L^p(\Omega_4)}^{p-1} + \|\nabla h\|_{L^p(\Omega_4)}^{p-1} \right) + \|F\|_{L^p(\Omega_4)} \right] + \int_{\Omega_4} |F|^\prime \, dx
\]
\[
\leq C \left[ \|\nabla \psi\|_{L^p(\Omega_4)} \left(\|\nabla (u - \psi)\|_{L^p(\Omega_4)}^{p-1} + \|F\|_{L^p(\Omega_4)} \right) \right] + \int_{\Omega_4} |F|^\prime \, dx
\]
\[
\leq C \left[ \|\nabla \psi\|_{L^p(\Omega_4)} \left(\|\nabla \psi\|_{L^p(\Omega_4)}^{p-1} + \|F\|_{L^p(\Omega_4)} \right) \right] + \int_{\Omega_4} |F|^\prime \, dx.
\]
Using the assumptions we then obtain
\[
\int_{\Omega_4} |\nabla (u - h)|^p \, dx \leq C \left[ \|\nabla \psi\|_{L^p(\Omega_4)} \right] + \int_{\Omega_4} |F|^\prime \, dx, \quad (3.5)
\]
which together with the fact that \( B_{4r}(y) \subset B_2 \) implies that
\[
\frac{1}{|B_{4r}(y)|} \int_{\Omega_{4r}(y)} |\nabla (u - h)|^p \, dx \leq \frac{C}{\sigma n} \left[ \|\nabla \psi\|_{L^p(\Omega_4)} \right] + \int_{\Omega_4} |F|^\prime \, dx. \quad (3.6)
\]
By letting \( m := \bar{u}_{\Omega_{4r}(y)} \) and using \( h - f \) as a test function in the equations for \( h \) and \( f \), we have
\[
\int_{\Omega_{4r}(y)} \langle \bar{A}(x, m, \nabla f), \nabla (h - f) \rangle \, dx = \int_{\Omega_{4r}(y)} \langle \bar{A}(x, u, \nabla h), \nabla (h - f) \rangle \, dx.
\]
This together with \( (2.5) \) gives
\[
\int_{\Omega_{4r}(y)} |\nabla (h - f)|^p \, dx \leq 4^{p-1} \Lambda \int_{\Omega_{4r}(y)} \langle \bar{A}(x, m, \nabla h) - \bar{A}(x, m, \nabla f), \nabla (h - f) \rangle \, dx
\]
\[
= 4^{p-1} \Lambda \int_{\Omega_{4r}(y)} \langle \bar{A}(x, m, \nabla h) - \bar{A}(x, m, \nabla h), \nabla (h - f) \rangle \, dx
\]
\[
\leq 4^{p-1} \Lambda \int_{\Omega_{4r}(y)} \min \{2\Lambda, \omega(\lambda \theta|u - m|)\} |\nabla h|^{p-1} |\nabla (h - f)| \, dx. \quad (3.7)
\]
As a consequence of \( (3.7) \) and Young’s inequality, we obtain
\[
\int_{\Omega_{4r}(y)} |\nabla (h - f)|^p \, dx \leq C(p, \Lambda) \int_{\Omega_{4r}(y)} |\nabla h|^p \, dx.
\]
Let \( \phi \in C_0^\infty(B_4) \) be the standard cutoff function satisfying \( 0 \leq \phi \leq 1, \, \phi = 1 \) in \( B_2 \), and \( |\nabla \phi| \leq 1 \). Then it follows by taking \( h \phi^\prime \) as a test function in equation \( (3.4) \) for \( h \) that
\[
\int_{\Omega_2} |\nabla h|^p \, dx \leq C \int_{\Omega_4} |h|^p \, dx.
\]
Thus by combining with the above estimate and the fact \( \Omega_{4r}(y) \subset \Omega_2 \) we conclude that
\[
\frac{1}{|B_{4r}(y)|} \int_{\Omega_{4r}(y)} |\nabla (h - f)|^p \, dx \leq \frac{C}{\sigma n} \int_{\Omega_4} |h|^p \, dx \leq \frac{C_0}{\sigma n} |h|_{L^\infty(\Omega_4)}^p \leq \frac{C_0}{\sigma n} |u - \psi|_{L^\infty(\Omega_4)}^p \leq \frac{C_0}{\sigma n} \left( \frac{M_0}{\Lambda \theta} \right)^p.
\]
This together with (3.6) gives the desired conclusion if \( \frac{C}{\sigma^n (M^p)^{\rho}} \leq \epsilon^p / 2 \). We hence only need to consider the case
\[
\frac{C}{\sigma^n (M^p)^{\rho}} > \frac{\epsilon^p}{2}.
\] (3.8)

For this, note first that (3.5) and the assumption yield
\[
\|\nabla h\|_{L^p(\Omega)} \leq \|\nabla (h - u)\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)} \leq C \left[ (\|\nabla \psi\|_{L^p(\Omega)} + \int_{\Omega} |F|^p \, dx)^{\frac{1}{p'}} + 1 \right] \leq C.
\]
As \( h = 0 \) on \( B_4 \cap \partial \Omega \) and \( \Omega \) satisfies (3.1), we can use this estimate together with the higher integrability for \( \nabla h \) given by Theorem 2.7 to conclude that
\[
\frac{1}{|B_{4r}(y)|} \int_{\Omega_{4r}(y)} \|\nabla h\|_{L^{p_0}(\Omega)}^\rho \, dx \leq C \left( \frac{1}{|B_4(y)|} \int_{\Omega_4(y)} \|\nabla h\|_{L^{p_0}(\Omega)}^\rho \, dx \right)^{\frac{1}{p'}} \leq C \quad (3.9)
\]
with \( p_0 > p \) and \( C > 0 \) depending only on \( p, n, \) and \( \Lambda \). We deduce from (3.7), Young and Hölder inequalities, and (3.9) that
\[
\frac{1}{|B_{4r}(y)|} \int_{\Omega_{4r}(y)} \|\nabla (h - f)\|_{L^{p_0}(\Omega)}^\rho \, dx \leq C \frac{1}{|B_{4r}(y)|} \int_{\Omega_{4r}(y)} \omega (A \theta |u - m|)^{\rho'} \|\nabla h\|_{L^{p_0}(\Omega)}^\rho \, dx \\
\leq C \left[ \frac{1}{|B_{4r}(y)|} \int_{\Omega_{4r}(y)} \omega (A \theta |u - m|)^{\rho'^{\frac{p_0}{p}}} \, dx \right]^{\frac{p_0}{p'}} \left[ \frac{1}{|B_{4r}(y)|} \int_{\Omega_{4r}(y)} \|\nabla h\|_{L^{p_0}(\Omega)}^\rho \, dx \right]^{\frac{\rho'}{p_0}} \\
\leq C \left[ \frac{1}{|B_{4r}(y)|} \int_{\Omega_{4r}(y)} \omega (A \theta |u - m|)^{\rho'^{\frac{p_0}{p}}} \, dx \right]^{\frac{p_0}{p'}}.
\]

But for any \( \gamma > 0 \), we have
\[
\int_{\Omega_{4r}(y)} \omega (A \theta |u - m|)^{\rho'^{\frac{p_0}{p}}} \, dx = \int_{\Omega_{4r}(y) : A \theta |u - m| \leq \gamma} \omega (A \theta |u - m|)^{\rho'^{\frac{p_0}{p}}} \, dx + \int_{\Omega_{4r}(y) : A \theta |u - m| > \gamma} \omega (A \theta |u - m|)^{\rho'^{\frac{p_0}{p}}} \, dx \\
\leq |\Omega_{4r}(y)| \omega (A \theta |u - m|)^{\rho'^{\frac{p_0}{p}}} + \omega (2M)^{\rho'^{\frac{p_0}{p}}} \gamma^\rho \int_{\Omega_{4r}(y)} (A \theta |u - m|)^{\rho} \, dx.
\]

Therefore, we infer that
\[
\frac{1}{|B_{4r}(y)|} \int_{\Omega_{4r}(y)} |\nabla (h - f)|^p \, dx \leq C \omega (A \gamma)^{\rho'} + C \omega (2M)^{\rho'} \left( \frac{(A \theta)^p}{|B_{4r}(y)|} \int_{\Omega_{4r}(y)} |u - m|^p \, dx \right)^{\frac{p_0 - p}{p_0}} \quad (3.10)
\]
for all \( \gamma > 0 \). Let us now estimate the last integral in (3.10). As
\[
\|u - m\|_{L^p(\Omega_{4r}(y))} \leq \|u -(h + \psi)\|_{L^p(\Omega_{4r}(y))} + \|(h + \psi)\|_{L^p(\Omega_{4r}(y))} + \|(h + \psi)\|_{L^p(\Omega_{4r}(y))} - \|\tilde{U}_{4r}(y)\|_{L^p(\Omega_{4r}(y))} \leq 2 \|u -(h + \psi)\|_{L^p(\Omega_{4r}(y)} + \|\psi - \tilde{\psi}_{4r}(y)\|_{L^p(\Omega_{4r}(y))} + \|h - \tilde{h}_{4r}(y)\|_{L^p(\Omega_{4r}(y))},
\]

it follows from Sobolev and Poincaré inequalities that
\[
\frac{1}{|B_{4r}(y)|} \int_{\Omega_{4r}(y)} |u - m|^p \, dx \leq C (\sigma^n \int_{\Omega_{4r}(y)} |\nabla (u - h - \psi)|^p \, dx + \frac{\sigma^p}{|B_{4r}(y)|} \int_{\Omega_{4r}(y)} |\nabla \psi|^p \, dx + (\text{osc } h)^p). \quad (3.11)
\]
We now use the fact $h = 0$ on $B_4 \cap \partial \Omega$ and $\Omega$ satisfies (3.1) to estimate the oscillation of $h$ when $y = 0$ or $B_{4r}(y) \subset \Omega_2$. In the first case, we can directly employ Theorem 2.7 to get

$$\text{osc}_{\Omega_{4r}(y)} h = \text{osc}_{\Omega_{4r}(y)} h \leq C \left( \frac{4\sigma}{r} \right) \|h\|_{L^\infty(\Omega_2)} \leq C \sigma \|h\|_{L^\infty(\Omega_2)}.$$  (3.12)

In the second case, let $r := \text{dist}(y, \partial \Omega) = |y - x_0| \leq |y| < 1$ for some $x_0 \in \partial \Omega \cap B_2$. Then as $B_r(y) \subset B_2(x_0)$, we have from Theorem 2.6 and Theorem 2.7 that

$$\text{osc}_{\Omega_{4r}(y)} h = \text{osc}_{\Omega_{4r}(y)} h \leq C \left( \frac{4\sigma}{r} \right) \text{osc}_{\Omega_2(x_0)} h \leq C \left( \frac{\sigma}{r} \right) \left( \frac{2r}{2} \right) \|h\|_{L^\infty(\Omega_2(x_0))} \leq C \sigma \|h\|_{L^\infty(\Omega_2)}.$$  (3.13)

From (3.11)–(3.13), the fact $\|h\|_{L^\infty(\Omega_2)} \leq M/\lambda \theta$, and (3.8), we obtain

$$\frac{(\lambda \theta)^p}{|B_{4r}(y)|} \int_{\Omega_{4r}(y)} |u - m|^p \, dx \leq C \int_{\Omega_4} |\nabla (u - h)|^p \, dx + \int_{\Omega_4} |\nabla \psi|^p \, dx + (M \sigma \beta)^p.$$  (3.14)

Plugging this estimate into (3.10) gives

$$\frac{1}{|B_{4r}(y)|} \int_{\Omega_{4r}(y)} |\nabla (h - f)|^p \, dx \leq C \omega(\gamma) \frac{\omega(2M)^p}{(\lambda \theta)^p} \left( \frac{\sigma^{-2n}}{\gamma \epsilon^2} \left( \int_{\Omega_4} |\nabla (u - h)|^p \, dx + \int_{\Omega_4} |\nabla \psi|^p \, dx + \left( \frac{\sigma \beta}{\gamma} \right)^p \right) \right).$$

By combining this with (3.6) and using (3.5) we obtain

$$\frac{1}{|B_{4r}(y)|} \int_{\Omega_{4r}(y)} |\nabla (u - f)|^p \, dx \leq C \omega(\gamma) \frac{\omega(2M)^p}{(\lambda \theta)^p} \left( \frac{\sigma^{-2n}}{\gamma \epsilon^2} \left( \int_{\Omega_4} |\nabla \psi|^p \, dx + \int_{\Omega_4} |\psi|^p \, dx \right) + \left( \frac{\sigma \beta}{\gamma} \right)^p \right)$$

for every $\gamma > 0$. From this, we get the desired conclusion by choosing $\gamma$ small first, then $\sigma$, and $\delta$ last. \hfill \Box

Our second step is to show that the gradient of the solution $f$ to (3.3) can be approximated by the gradient of a solution to a homogeneous equation with constant coefficient. Precisely, we have:

**Lemma 3.4.** Let $\epsilon \in (0, 1]$, and let $\sigma$ be its corresponding constant given by Lemma 3.3. Let $\Omega, \psi, F, u,$ and $h$ be as in Lemma 3.3 and assume in addition that $\frac{1}{|B_{4r}(y)|} \int_{\Omega_{4r}(y)} |\nabla \psi|^p \, dx \leq 1$. Suppose that $f$ is a weak solution of (3.3) and $w$ is a weak solution of

$$\begin{cases}
\text{div} \left[ \mathbf{A}(\lambda \theta \Omega_{3r}(y)) \left( \frac{\lambda \theta}{\epsilon} \frac{\lambda \theta}{\epsilon} \right) \nabla w \right] = 0 & \text{in } \Omega_{3r}(y), \\
w = f & \text{on } \partial \Omega_{3r}(y).
\end{cases}$$  (3.14)

There exist constants $p_0 \in (p, \infty)$ and $C > 0$ depending only on $p, n, and \mathbf{A}$ such that: if $p \geq 2$, then

$$\frac{1}{|B_{3r}(y)|} \int_{\Omega_{3r}(y)} |\nabla f - \nabla w|^p \, dx \leq C \Theta_{\Omega_{3r}(y)}(\mathbf{A}) \frac{p_0 - p}{p_0}.$$  (3.15)
and if $1 < p < 2$, then
\[
\frac{1}{|B_{2r}(y)|} \int_{\Omega_{2r}(y)} |\nabla f - \nabla u|^p \, dx \leq \tau + C \tau^{(1-\frac{2}{p})p'} \Theta_{\Omega_{2r}(y)}(A) \frac{p_0-p}{p_0} \quad \text{for every } \tau \in (0,1).
\]

Proof. For convenience, define $a(x, \xi) := \frac{\Lambda(x,\xi)h(x,\xi)}{\tilde{u}_{\Omega_{2r}(y)}}$ with $m := \tilde{u}_{\Omega_{2r}(y)}$. We first consider the case $p \geq 2$. Then using (2.5) we get
\[
\int_{\Omega_{3r}(y)} |\nabla (f - w)|^p \, dx \leq \Lambda \int_{\Omega_{3r}(y)} \langle (a)_{\Omega_{3r}(y)}(\nabla f) - (a)_{\Omega_{3r}(y)}(\nabla w), \nabla (f - w) \rangle \, dx := I. \tag{3.16}
\]

To estimate the term $I$, we use $f - w$ as a test function in equations (3.14) and (3.3) to obtain
\[
\int_{\Omega_{3r}(y)} \langle (a)_{\Omega_{3r}(y)}(\nabla w), \nabla (f - w) \rangle \, dx = \int_{\Omega_{3r}(y)} \langle a(x, \nabla f), \nabla (f - w) \rangle \, dx.
\]

Hence,
\[
I = \Lambda \int_{\Omega_{3r}(y)} \langle (a)_{\Omega_{3r}(y)}(\nabla f) - a(x, \nabla f), \nabla (f - w) \rangle \, dx
\leq \Lambda \int_{\Omega_{3r}(y)} \sup_{\xi \neq 0} \frac{|a(x, \xi) - a_{\Omega_{3r}(y)}(\xi)|}{|\xi|^{p-1}} |\nabla f|^{p-1} |\nabla (f - w)| \, dx. \tag{3.17}
\]

We now claim that there exist constants $p_0 \in (p, \infty)$ and $C > 0$ depending only on $p, n,$ and $\Lambda$ such that
\[
\left( \frac{1}{|B_{3r}(y)|} \int_{\Omega_{3r}(y)} |\nabla f|^{p_0} \, dx \right)^{\frac{1}{p_0}} \leq C \left( \frac{1}{|B_{4r}(y)|} \int_{\Omega_{4r}(y)} |\nabla f|^{p_0} \, dx \right)^{\frac{1}{p_0}}. \tag{3.18}
\]

Indeed, this follows from the classical interior higher integrability if $B_{4r}(y) \subset \Omega_2$. In the case $y = 0$, we obtain (3.18) from the boundary higher integrability in Theorem 2.7 by using the fact $f = h = 0$ on $B_{4r} \cap \partial \Omega$ and the assumption that $\Omega$ satisfies (3.1). Thanks to Lemma 3.3 we also have
\[
\frac{1}{|B_{4r}(y)|} \int_{\Omega_{4r}(y)} |\nabla f|^p \, dx \leq \frac{2^{p-1}}{|B_{4r}(y)|} \left[ \int_{\Omega_{4r}(y)} |\nabla f - \nabla u|^p \, dx + \int_{\Omega_{4r}(y)} |\nabla u|^p \, dx \right] \leq 2^{p-1} [c^p + 1] \leq 2^p. \tag{3.19}
\]

Therefore, we infer that
\[
\left( \frac{1}{|B_{3r}(y)|} \int_{\Omega_{3r}(y)} |\nabla f|^{p_0} \, dx \right)^{\frac{1}{p_0}} \leq C.
\]

This together with (3.16)–(3.17), Young and Hölder inequalities, and the fact $|a(x, \xi)| \leq \Lambda |\xi|^{p-1}$ gives
\[
\frac{1}{|B_{3r}(y)|} \int_{\Omega_{3r}(y)} |\nabla (f - w)|^p \, dx \leq C \frac{1}{|B_{3r}(y)|} \int_{\Omega_{3r}(y)} \left[ \sup_{\xi \neq 0} \frac{|a(x, \xi) - a_{\Omega_{3r}(y)}(\xi)|}{|\xi|^{p-1}} \right]^{\frac{p}{p_0}} |\nabla f|^{p_0} \, dx
\leq C \left( \frac{1}{|B_{3r}(y)|} \int_{\Omega_{3r}(y)} \left[ \sup_{\xi \neq 0} \frac{|a(x, \xi) - a_{\Omega_{3r}(y)}(\xi)|}{|\xi|^{p-1}} \right]^{\frac{p}{p_0}} \, dx \right)^{\frac{p_0}{p}} \left( \frac{1}{|B_{3r}(y)|} \int_{\Omega_{3r}(y)} |\nabla f|^{p_0} \, dx \right)^{\frac{p}{p_0}}
\leq C \left( \frac{1}{|B_{3r}(y)|} \int_{\Omega_{3r}(y)} \left[ \sup_{\xi \neq 0} \frac{|a(x, \xi) - a_{\Omega_{3r}(y)}(\xi)|}{|\xi|^{p-1}} \right] \, dx \right)^{\frac{p_0}{p}}.
\]
But we have from the definition of $a$ that
\[
\sup_{\xi \neq 0} \frac{|a(x, \xi) - (a)_{\Omega_{3\sigma}(y)}(\xi)|}{|\xi|^{p-1}} = \sup_{\eta \neq 0} \frac{|A(x, \lambda \theta m, \eta) - \langle A \rangle_{\Omega_{3\sigma}(y)}(\lambda \theta m, \eta)|}{|\eta|^{p-1}}.
\]
Thus we conclude that
\[
\frac{1}{|B_{3\sigma}(y)|} \int_{\Omega_{3\sigma}(y)} |\nabla (f-w)|^p \, dx \leq C \left( \frac{1}{|B_{3\sigma}(y)|} \int_{\Omega_{3\sigma}(y)} \sup_{\xi \neq 0} \frac{|A(x, \lambda \theta m, \xi) - \langle A \rangle_{\Omega_{3\sigma}(y)}(\lambda \theta m, \xi)|}{|\xi|^{p-1}} \, dx \right)^{\frac{p-2}{p}}.
\]
This together with the fact $\lambda \theta m \in \overline{\mathbb{R}} \cap [-M, M]$ and the definition of $\Theta_{\Omega_{3\sigma}}(A)$ given by (1.6) yields estimate (3.15). We next consider the case $1 < p < 2$. Then condition (1.2) in [17, Lemma 3.1] is satisfied thanks to (2.5). Therefore, instead of (3.16) we now have from [17, Lemma 3.1] that
\[
\int_{\Omega_{3\sigma}(y)} |\nabla (f-w)|^p \, dx \leq \tau \int_{\Omega_{3\sigma}(y)} |\nabla f|^p \, dx + C_p \tau^{1-\frac{2}{p}} I \quad \text{for all} \quad \tau \in (0, \frac{1}{2}).
\]
Then we deduce from estimate (3.17) for $I$ and Young’s inequality that
\[
\frac{1}{|B_{3\sigma}(y)|} \int_{\Omega_{3\sigma}(y)} |\nabla (f-w)|^p \, dx \leq 2 \tau \frac{1}{|B_{3\sigma}(y)|} \int_{\Omega_{3\sigma}(y)} |\nabla f|^p \, dx + C(p, \Lambda) \tau^{(1-\frac{2}{p})p'} \frac{1}{|B_{3\sigma}(y)|} \int_{\Omega_{3\sigma}(y)} \left[ \sup_{\xi \neq 0} \frac{|a(x, \xi) - (a)_{\Omega_{3\sigma}(y)}(\xi)|}{|\xi|^{p-1}} \right]^{p'} |\nabla f|^p \, dx.
\]
The first integral is estimated by (3.19) and the last integral can be estimated exactly as above. As a consequence, we obtain
\[
\frac{1}{|B_{3\sigma}(y)|} \int_{\Omega_{3\sigma}(y)} |\nabla (f-w)|^p \, dx \leq 2^{p+1} \left( \frac{4}{3} \right)^n \tau + C(p, n, \Lambda) \tau^{(1-\frac{2}{p})p'} \Theta_{\Omega_{3\sigma}(y)}(A)^{\frac{p-2}{p}} \quad \text{for all} \quad \tau \in (0, \frac{1}{2}).
\]

To obtain Lemma 3.1 our last step is to show that the gradient of the solution $w$ to (3.14) can be approximated by a bounded gradient. Precisely, we have:

**Lemma 3.5.** Let $a : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field as in Theorem 2.8. Let $\epsilon > 0$, and let $\sigma$ be its corresponding constant given by Lemma 3.3. Then there exists a constant $\delta > 0$ depending only on $\epsilon$, $p$, $\Lambda$, and $n$ satisfying: if $\Omega \subset \mathbb{R}^n$ is a bounded domain such that (3.2) holds, $y \in B_1$ satisfies either $y = 0$ or $B_{4\sigma}(y) \subset \Omega$, and $w \in W^{1,p}(\Omega_{3\sigma}(y))$ is a weak solution of
\[
\begin{cases}
\text{div } a(\nabla w) = 0 & \text{in } \Omega_{3\sigma}(y), \\
w = 0 & \text{on } B_{3\sigma}(y) \cap \partial \Omega,
\end{cases}
\]
with $\frac{1}{|B_{3\sigma}(y)|} \int_{\Omega_{3\sigma}(y)} |\nabla w|^p \, dx \leq 1$, then there exists a function $v \in W^{1,p}(\Omega_{2\sigma}(y))$ such that
\[
||\nabla v||_{L^{\infty}(\Omega_{2\sigma}(y))} \leq C(p, n, \Lambda) \quad \text{and} \quad \frac{1}{|B_{2\sigma}(y)|} \int_{\Omega_{2\sigma}(y)} |\nabla v - \nabla w|^p \, dx \leq \epsilon^p.
\]
Proof. If $B_{d}(y) \subset \Omega$, then the conclusion follows by simply taking $v = w$ and using the interior Lipschitz estimate for $w$. Notice that condition (3.2) is not used for this case. Thus it suffices to consider the case $y = 0$. If we let $\tilde{\Omega} := \{ \sigma^{-1} x : x \in \Omega \}$ and $\tilde{w}(x) := \sigma^{-1} w(\sigma x)$, then $\tilde{\Omega}$ satisfies

$$B_{3} \cap \{ x_{n} > 3 \delta \} \subset \tilde{\Omega} \subset B_{3} \cap \{ x_{n} > -3 \delta \}$$

and $\tilde{w}$ is a weak solution of $\text{div} \ a(\nabla \tilde{w}) = 0$ in $\tilde{\Omega}$ with $\tilde{w} = 0$ on $B_{3} \cap \partial \tilde{\Omega}$. Moreover, $\tilde{w}$ satisfies: $\tilde{w} \in W^{1, p}(\tilde{\Omega})$ and $\int_{\tilde{\Omega}} |\nabla \tilde{w}|^{p} \, dx \leq 1$. Therefore, by working with $\tilde{\Omega}$ and $\tilde{w}$ we can assume in addition that $\sigma = 1$, and only need to show that there exists a function $v \in W^{1, p}(\Omega_{2})$ such that

$$\| \nabla v \|_{L_{\infty}(\Omega_{2})} \leq C(p, n, \Lambda) \quad \text{and} \quad \int_{\Omega_{2}} |\nabla w - \nabla v|^{p} \, dx \leq \epsilon^{p}.$$  

For this, we first apply Lemma A.1 to conclude that there exists a weak solution $\tilde{v}$ to the equation

$$\begin{cases}
\text{div} \ a(\nabla \tilde{v}) = 0 & \text{in} \ B_{3}^{+}, \\
\tilde{v} = 0 & \text{on} \ B_{3} \cap \{ x_{n} = 0 \}
\end{cases}$$

with $\frac{1}{|B_{3}|} \int_{B_{3}} |\nabla \tilde{v}|^{p} \, dx \leq (4p \Lambda^{2})^{p}$ such that

$$\left( \int_{B_{3} \cap \{ x_{n} > 3 \delta \}} |w - \tilde{v}|^{p} \, dx \right)^{\frac{1}{p}} \leq \epsilon^{p}. \quad (3.20)$$

Then by the Lipschitz estimate in Theorem 2.8 we also have

$$\| \nabla \tilde{v} \|_{L_{\infty}(B_{3}^{+})}^{p} \leq C \int_{B_{3}^{+}} |\nabla \tilde{v}|^{p} \, dx \leq C. \quad (3.21)$$

Let us extend $\tilde{v}$ from $B_{3}^{+}$ to $B_{3}$ by the zero extension. Due to (3.21) the resulting function $\tilde{v}$ satisfies: $\tilde{v} \in W^{1, p}(B_{3})$ and

$$\| \nabla \tilde{v} \|_{L_{\infty}(B_{3})} \leq C. \quad (3.22)$$

Moreover, $\tilde{v}$ is a weak solution of $\text{div} \ a(\nabla \tilde{v}) = -\tilde{g}_{x}$ in $B_{3}$ with $\tilde{g}(x) := \chi_{\{ x_{n} < \delta \}}(x) a_{n}(\nabla \tilde{v}(x, 0))$ for $x = (x', x_{n}) \in B_{3}$. Notice that if we let $D_{0} := B_{3} \cap \{ x_{n} > 3 \delta \}$, then (3.20) gives

$$\int_{\Omega_{2}^{+}} |w - \tilde{v}|^{p} \, dx \leq \int_{\Omega_{2}^{+} \setminus D_{0}} |w - \tilde{v}|^{p} \, dx + \int_{D_{0}} |w - \tilde{v}|^{p} \, dx \leq 2^{p-1} \left[ \int_{\Omega_{2}^{+} \setminus D_{0}} |w|^{p} \, dx + \int_{\Omega_{2}^{+} \setminus D_{0}} |\tilde{v}|^{p} \, dx \right] + \epsilon^{p^{2}}.$$

If $p > n$, then we can bound $L_{\infty}$ norms of $w$ and $\tilde{v}$ since $\| w \|_{W^{1, p}(\Omega_{2}^{+})} + \| \tilde{v} \|_{W^{1, p}(\Omega_{2}^{+})} \leq C$. As a consequence, we deduce that

$$\int_{\Omega_{2}^{+}} |w - \tilde{v}|^{p} \, dx \leq C \left[ |\Omega_{2}^{+} \setminus D_{0}|^{\frac{p}{n}} + \epsilon^{p^{2}} \right] \leq C \delta^{\frac{p}{n}} + \epsilon^{p^{2}}. \quad (3.23)$$
In case $p < n$, we also have (3.25) as the above estimate together with Hölder and Poincaré inequalities implies that

$$
\int_{\Omega_2^{1\gamma}} |w - \tilde{v}|^p \, dx \leq 2^{p-1} \left[ \left( \int_{\Omega_2} |w|^{\frac{np}{n-p}} \, dx \right)^{\frac{n-p}{n}} + \left( \int_{B_2^{\gamma} \setminus D_\delta} |\tilde{v}|^{\frac{np}{n-p}} \, dx \right)^{\frac{n-p}{n}} \right]_{\Omega_2} \leq C \left( \int_{\Omega_2} |\nabla w|^p \, dx \right)^{\frac{n-p}{n}} + \left( \int_{B_2^{\gamma} \setminus D_\delta} |\nabla \tilde{v}|^p \, dx \right)^{\frac{n-p}{n}} + \epsilon^p.
$$

Let us define $v(x) := \tilde{v}(x, x_n - 3\delta) = \tilde{v}(x - 3\delta e_n)$. Then from the equation for $\tilde{v}$ we infer that $v$ is a weak solution of

$$
\begin{cases}
\text{div } a(\nabla v) = -g_{x_n} & \text{in } \Omega_3, \\
v = 0 & \text{on } B_3 \cap \partial \Omega,
\end{cases}
$$

where

$$
g(x) := \chi_{|x| < 3\delta}(x) a_n(\nabla v(x', 3\delta)) = \chi_{|x| < 3\delta}(x) a_n(\nabla \tilde{v}(x', 0)) \quad \text{for } x = (x', x_n) \in \Omega_3.
$$

Let $D_n^{\delta, \gamma} = \frac{|\nabla (v(x') - 3\delta e_n)|}{3\delta}$ denote the $n$th difference quotient. From (3.25) and $L^p$ estimate for $\nabla \tilde{v}$, we also have

$$
||w - v||_{L^p(\Omega_2)} \leq ||w - \tilde{v}||_{L^p(\Omega_2)} + \left( \int_{\Omega_2} |(v(x) - \tilde{v}(x - 3\delta e_n))|^p \, dx \right)^{\frac{1}{p}} \leq ||w - \tilde{v}||_{L^p(\Omega_2)} + 3\delta ||D_n^{\delta, \gamma} \tilde{v}||_{L^p(\Omega_2)} + \epsilon^p.
$$

Take $\phi \in C_0^\infty(B_2^{\gamma})$ be the standard cutoff function satisfying $\phi = 1$ in $B_2$. Then by using $\phi^p(w - v)$ as a test function in the equations for $w$ and (3.24), we obtain

$$
\int_{\Omega_3} \langle a(\nabla w), \nabla [\phi^p(w - v)] \rangle \, dx = \int_{\Omega_3} \langle a(\nabla v), \nabla [\phi^p(w - v)] \rangle \, dx + \int_{\Omega_3} g[\phi^p(w - v)]_n \, dx.
$$

We can now follow the proof of [3, Lemma 3.7] to get

$$
\int_{\Omega_2} |\nabla (w - v)|^p \, dx \leq \epsilon^p.
$$

For clarity, let us include the argument for the case $p \geq 2$. Indeed, we can rewrite (3.26) as

$$
\int_{\Omega_3} \langle a(\nabla w) - a(\nabla v), \nabla (w - v) \rangle \phi^p \, dx = p \int_{\Omega_3} \langle a(\nabla v) - a(\nabla w), \nabla \phi \rangle \phi^{p-1} (w - v) \, dx

+ \int_{\Omega_3 \setminus D_\delta} a_n(\nabla v(x', 3\delta)) ((w - v)_n \phi^p + p \phi_{x_n} \phi^{p-1}(w - v) \, dx.
$$

It follows that

$$
\int_{\Omega_3} |\nabla (w - v)|^p \, dx \leq C \int_{\Omega_3} |\nabla \phi|^{p-1} |\nabla v| \, dx + C \int_{\Omega_3 \setminus D_\delta} \nabla v(x', 3\delta))^{p-1} |\nabla (w - v)| \phi^p \, dx

+ C \int_{\Omega_3 \setminus D_\delta} |\nabla v(x', 3\delta)|^{p-1} \phi^{p-1} (w - v) \, dx.
$$
Then Young and Hölder inequalities yield
\[
\int_{\Omega_2} |\nabla (w - v)|^p \phi^p \, dx \leq C \left[ \|\nabla v\|_{L^p(\Omega_2)}^{p-1} + \|\nabla w\|_{L^p(\Omega_2)}^{p-1} + \left( \int_{\Omega_2 \setminus D_\delta} |\nabla v(x', 3\delta)|^p \, dx \right)^{\frac{p-1}{p}} \right] \|w - v\|_{L^p(\Omega_2)}
\]
\[ + C \int_{\Omega_2 \setminus D_\delta} |\nabla v(x', 3\delta)|^p \, dx.
\]
This together with the fact $\nabla v(x', 3\delta) = \nabla \tilde{v}(x', 0)$, (3.22), and (3.25) gives
\[
\int_{\Omega_2} |\nabla (w - v)|^p \, dx \leq C \left( \|w - v\|_{L^p(\Omega_2)} + \|\nabla v - \nabla \tilde{v}\|_{L^p(\Omega_{2r}(y))} + \|\nabla f - \nabla w\|_{L^p(\Omega_{2r}(y))} + \|\nabla f - \nabla \tilde{v}\|_{L^p(\Omega_{2r}(y))} \right).
\]

**Proof of Lemma 3.1** The conclusion follows from Lemmas 3.3–3.5 noting that
\[
\|\nabla u - \nabla v\|_{L^p(\Omega_{2r}(y))} \leq \|\nabla u - \nabla f\|_{L^p(\Omega_{2r}(y))} + \|\nabla f - \nabla w\|_{L^p(\Omega_{2r}(y))} + \|\nabla f - \nabla \tilde{v}\|_{L^p(\Omega_{2r}(y))}.
\]

\[
\square
\]

## 4 Density and gradient estimates

We derive global gradient estimates for weak solution $u$ of (2.4) by estimating the distribution functions of the maximal function of $|\nabla u|^p$. This is carried out in the next two subsections, while the last subsection (Subsection 4.3) is devoted to proving the main results stated in Section 1.

### 4.1 Density estimates

The next result gives a density estimate for the distribution of $M_{U_t}(|\nabla u|^p)$. It roughly says that if the maximal function $M_{U_t}(|\nabla u|^p)$ is bounded at one point in $B_{\sigma r}(y)$ then this property can be propagated for all points in $B_{\sigma r}(y)$ except on a set of small measure $w$.

**Lemma 4.1.** Assume that $A$ satisfies (1.2)–(1.4), $\psi \in W^{1,p}(\Omega)$, and $F \in L^{p'}(\Omega; \mathbb{R}^n)$. Let $M > 0$ and $w$ be an $A_\infty$ weight. There exists a constant $N = N(p, n, \Lambda) > 1$ satisfying for any $\epsilon > 0$, we can find small positive constants $\delta$ and $\sigma$ depending only on $\epsilon$, $p$, $n$, $\omega$, $\Lambda$, and $[\psi]_{A_\infty}$ such that: if $\Omega$ is $(\delta, R)$-Reifenberg flat, $\lambda > 0$, $\theta > 0$, $U \subset \Omega$ is an open set, $\bar{y} \in \partial \Omega$, and

\[
\sup_{0 < \rho < \frac{R}{2}} \sup_{y \in \Omega \cap B_{\rho}(\bar{y})} \Theta_{\Omega_{\delta r}(y)}(A) \leq \delta,
\]

then for any weak solution $u$ of (2.4) with $\|u\|_{L^\infty(\Omega)} + \|\psi\|_{L^\infty(\Omega)} \leq \frac{M}{\delta_0}$, for any $r \in (0, \frac{R}{2})$ satisfying $\Omega_{\delta r}(\bar{y}) \subset U$, for $y \in B_{\lambda}(\bar{y})$ satisfies either $y = \bar{y}$ or $A^{\lambda r}(y) \subset \Omega_{2\delta}(\bar{y})$, and

\[
B_{\sigma r}(y) \cap \{ U : M_{U_t}(|\nabla u|^p) \leq 1 \} \cap \{ U : M_{U_t}(|\nabla \psi|^p + |F|^p) \leq \delta \} \neq \emptyset,
\]

we have

\[
w\left( \{ U : M_{U_t}(|\nabla u|^p) > N \} \cap B_{\sigma r}(y) \right) < \epsilon w(B_{\sigma r}(y)),
\]

\[
\square
\]
Proof. By (4.1) there exists $x_0 \in B_{\sigma r}(y) \cap U$ such that
\begin{equation}
M_U(|\nabla u|^p)(x_0) \leq 1 \quad \text{and} \quad M_U(|\nabla \psi|^p + |F|^p)(x_0) \leq \delta.
\end{equation}
This together with the facts $B_{4r}(\tilde{y}) \subset B_{6r}(x_0)$ and $B_{4\sigma r}(y) \subset B_{5\sigma r}(x_0) \cap B_{2r}(\tilde{y})$ implies that
\begin{align*}
\frac{1}{|B_{4r}(\tilde{y})|} \int_{\Omega_{4r}(\tilde{y})} |\nabla u|^p \, dx &\leq \left( \frac{6}{4} \right)^{\rho} \frac{1}{|B_{6r}(x_0)|} \int_{B_{6r}(x_0) \cap U} |\nabla u|^p \, dx \leq \left( \frac{3}{2} \right)^{\rho}, \\
\frac{1}{|B_{4\sigma r}(y)|} \int_{\Omega_{4\sigma r}(y)} |\nabla u|^p \, dx &\leq \left( \frac{5}{4} \right)^{\rho} \frac{1}{|B_{5\sigma r}(x_0)|} \int_{B_{5\sigma r}(x_0) \cap U} |\nabla u|^p \, dx \leq \left( \frac{5}{4} \right)^{\rho}, \\
\frac{1}{|B_{4r}(\tilde{y})|} \int_{\Omega_{4r}(\tilde{y})} (|\nabla \psi|^p + |F|^p) \, dx &\leq \left( \frac{6}{4} \right)^{\rho} \frac{1}{|B_{6r}(x_0)|} \int_{B_{6r}(x_0) \cap U} (|\nabla \psi|^p + |F|^p) \, dx \leq \left( \frac{3}{2} \right)^{\rho} \delta.
\end{align*}
In addition, we have from the assumption that either $y = \tilde{y} \in \partial \Omega$ or $B_{4\sigma r}(y) \subset \Omega_{2r}(\tilde{y})$. Therefore, we can apply Proposition 3.2 for $\tilde{\varepsilon} \in (0, 1]$ that will be determined later. As a consequence, we obtain there exists $v \in W^{1,p}(\Omega_{2\sigma r}(y))$ such that
\begin{equation}
|\nabla v|^p \leq C \left( \rho, n, \Lambda \right) \quad \text{and} \quad \frac{1}{|B_{2\sigma r}(y)|} \int_{\Omega_{2\sigma r}(y)} |\nabla u - \nabla v|^p \, dx \leq \tilde{\varepsilon}^p.
\end{equation}
We claim that (4.2) and (4.3) yield
\begin{equation}
\{ \Omega_{\tau r}(y) : M_{\Omega_{2\sigma r}(y)}(|\nabla u - \nabla v|^p) \leq C \} \subset \{ \Omega_{\tau r}(y) : M_U(|\nabla u|^p) \leq N \}
\end{equation}
with $N := \max \{ 2^p C + 3^p \}$. Indeed, let $x$ be a point in the set on the left hand side of (4.4), and consider $B_p(x)$. If $\rho \leq \sigma r$, then $B_p(x) \subset B_{2\sigma r}(y)$ and hence
\begin{align*}
\frac{1}{|B_p(x)|} \int_{B_p(x) \cap U} |\nabla u|^p \, dy &\leq \frac{2^{p-1}}{|B_p(x)|} \int_{\Omega_p(x)} |\nabla u - \nabla v|^p \, dy + \int_{\Omega_p(x)} |\nabla v|^p \, dy \\
&\leq 2^{p-1} \left[ M_{\Omega_{2\sigma r}(y)}(|\nabla u - \nabla v|^p)(x) + |\nabla v|^p \right] \leq 2^p C.
\end{align*}
On the other hand, if $\rho > \sigma r$ then $B_p(x) \subset B_{3\rho}(x_0)$. This and the first inequality in (4.2) give
\begin{equation}
\frac{1}{|B_p(x)|} \int_{B_p(x) \cap U} |\nabla u|^p \, dy \leq \frac{3^a}{|B_{3\rho}(x_0)|} \int_{B_{3\rho}(x_0) \cap U} |\nabla u|^p \, dy \leq 3^a.
\end{equation}
Therefore, $M_U(|\nabla u|^p)(x) \leq N$ and claim (4.4) is proved. Notice that (4.4) is equivalent to
\begin{equation}
\{ \Omega_{\tau r}(y) : M_U(|\nabla u|^p) > N \} \subset \{ \Omega_{\tau r}(y) : M_{\Omega_{2\sigma r}(y)}(|\nabla u - \nabla v|^p) > C \}.
\end{equation}
It follows from this, the weak type $1 \,-\, 1$ estimate, and (4.3) that
\begin{equation}
\int_{\Omega_{\tau r}(y)} |\nabla u - \nabla v|^p \, dx \leq C \int_{\Omega_{2\sigma r}(y)} |\nabla u - \nabla v|^p \, dx \leq C_1 \tilde{\varepsilon}^p |B_{\tau r}(y)|.
\end{equation}
We then infer from property (2.2) that
\[ w\left( \Omega_{cr}(y) : M_U(\|\nabla u\|^p) > N \right) \leq A \left( \frac{\|\Omega_{cr}(y) : M_U(\|\nabla u\|^p) > N \|}{|B_{\sigma r}(y)|} \right)^{\nu} w(B_{\sigma r}(y)) \]
with \( A \) and \( \nu \) being the constants given by characterization (2.2) for \( w \). We choose \( \bar{e}^p := \min \{ C_1^{-1}(eA^{-1})^{\frac{1}{p}}, 1 \} \) to complete the proof.

**Lemma 4.2.** Assume that \( A \) satisfies (1.2)-(1.4), \( \psi \in W^{1,p}(\Omega) \), and \( F \in L^p(\Omega; \mathbb{R}^n) \). Let \( M > 0 \) and \( w \in A_s \) for some \( 1 < s < \infty \). There exists a constant \( N = N(p,n,\Lambda) > 1 \) satisfying for any \( e > 0 \), we can find small positive constants \( \delta \) and \( \sigma \) depending only on \( e, p, n, \omega, \Lambda, M, s, \) and \([w]_{A_s}\) such that: if \( \Omega \) is \((\delta, R)\)-Reifenberg flat, \( \lambda > 0, \theta > 0 \), \( U \subset \Omega \) is an open set, and
\[ \sup_{0 < r < \frac{\delta}{2}} \sup_{y \in \Omega} \Theta_{\Omega_{3\sigma r}(y)}(A) \leq \delta, \]
then for any weak solution \( u \) of (2.4) with \( ||u||_{L^\infty(\Omega)} + ||\psi||_{L^\infty(\Omega)} \leq M \), and for any \( y \in \Omega, 0 < r < R/10 \) with \( \Omega_{21r}(y) \subset U \) and
\[ B_{\sigma r}(y) \cap \{ U : M_U(\|\nabla u\|^p) \leq 1 \} \cap \{ U : M_U(\|\nabla \psi\|^p + |F|^p) \leq \delta \} \neq \emptyset, \]
we have
\[ w\left( \{| U : M_U(\|\nabla u\|^p) > N \} \cap B_{\sigma r}(y) \right) < e w(B_{\sigma r}(y)). \]

**Proof.** We consider the following possibilities:

**Case 1** (away from the boundary): \( B_r(y) \subset \Omega \). Then we are in the interior case and \( B_r(y) = \Omega_r(y) \subset U \). Hence, we obtain the conclusion by using Lemma 5.1 in [7].

**Case 2** (near boundary): \( B_{4r}(y) \subset \Omega \) but \( B_r(y) \cap \partial \Omega \neq \emptyset \). Let \( \bar{y} \in B_r(y) \cap \partial \Omega \). Then
\[ B_r(y) \subset B_2(\bar{y}) \subset B_5(\bar{y}). \]
In particular, we have \( B_{4r}(y) \subset \Omega_{21r}(\bar{y}) \). Moreover, \( \Omega_{4r}(y) \subset \Omega_{5r}(y) \subset U \) since \( B_{4r}(\bar{y}) \subset B_5(y) \). Therefore, we can use Lemma [4.4] to obtain the desired result.

**Case 3** (boundary): \( B_{4r}(y) \cap \partial \Omega \neq \emptyset \). Let \( \bar{y} \in B_{4r}(y) \cap \partial \Omega \). Then
\[ B_{\sigma r}(y) \subset B_{5\sigma r}(\bar{y}) \subset B_{9r}(y). \]
This together with assumption (4.6) yields
\[ B_{5\sigma r}(\bar{y}) \cap \{ U : M_U(\|\nabla u\|^p) \leq 1 \} \cap \{ U : M_U(\|\nabla \psi\|^p + |F|^p) \leq \delta \} \neq \emptyset. \]
We also have \( \Omega_{20r}(\bar{y}) \subset \Omega_{21r}(y) \subset U \) as \( B_{20r}(\bar{y}) \subset B_{21r}(y) \). Therefore, by applying Lemma [4.1] for \( y = \bar{y} \) and \( \varepsilon_1 := 9^{-s} \varepsilon/[w]_{A_s} \), we obtain
\[ w\left( \{| U : M_U(\|\nabla u\|^p) > N \} \cap B_{5\sigma r}(\bar{y}) \right) < \varepsilon_1 w(B_{5\sigma r}(\bar{y})). \]
It follows from this and (4.7) that
\[ w\left( \{| U : M_U(\|\nabla u\|^p) > N \} \cap B_{\sigma r}(y) \right) < \varepsilon_1 w(B_{9r}(y)) \leq \varepsilon_1 9^{ns} [w]_{A_s} w(B_{\sigma r}(y)) = e w(B_{\sigma r}(y)). \]

□
Let us fix $\bar{R} := R/100$ and a finite collection of points \( \{z_i\}_{i=1}^L \subset \Omega \) such that \( \overline{\Omega} \subset \bigcup_{i=1}^L B_{\sigma \bar{R}}(z_i) \). We also consider an open set \( V \subset \Omega \) satisfying: there exists a constant \( c_0 \in (0, 1) \) such that
\[
|B_\rho(z) \cap V| \geq c_0 |B_\rho(z)| \quad \text{for all } z \in \partial V \text{ and all } 0 < \rho < \sigma \bar{R}.
\] (4.8)

In view of Lemma 4.1 and as in [15, Lemma 3.8], we can apply Krylov-Safanov lemma, which is a variation of the Vitali covering lemma, to obtain:

**Lemma 4.3.** Assume that \( A \) satisfies (1.2)–(1.4), \( \psi \in W^{1, p}(\Omega) \), and \( F \in L^p(\Omega; \mathbb{R}^n) \). Let \( M > 0 \) and \( w \in A_s \) for some \( 1 < s < \infty \). There exists a constant \( N = N(p, n, \Lambda) > 1 \) satisfying for any \( \varepsilon > 0 \), we can find small positive constants \( \delta \) and \( \sigma \) depending only on \( e, p, n, \omega, \Lambda, M, s, \) and \( |w|_A \), such that: if \( \Omega \) is \((\delta, \bar{R})\)-Reifenberg flat, \( \eta > 0 \), \( \theta > 0 \), and \( V \subset U \subset \Omega \) are open sets satisfying (4.5) and \( \Omega_{\sigma \varepsilon}(y) \subset U \) for every \( y \in V \), then for any weak solution \( u \in W^{1, p}(\Omega) \) of (2.4) with \( ||u||_{L^\infty(\Omega)} + ||\psi||_{L^\infty(\Omega)} \leq \frac{M}{100} \) and
\[
w \left( \left\{ V : M_U(|\nabla u|^p) > N \right\} \right) < \varepsilon \ w(B_{\sigma \bar{R}}(z_i)) \quad \forall i = 1, 2, ..., L,
\] (4.9)
we have
\[
w \left( \left\{ V : M_U(|\nabla u|^p) > N \right\} \right) \leq \left( \frac{10^p}{c_0} \right)^{\varepsilon} |w|_A^\varepsilon \left[ w \left( \left\{ V : M_U(|\nabla u|^p) > 1 \right\} \right) + w \left( \left\{ V : M_U(|\nabla \psi|^p + |F|^p) > \delta \right\} \right) \right].
\]

**Proof.** For \( \varepsilon > 0 \), let \( N, \delta, \) and \( \sigma \) be the corresponding constants given by Lemma 4.2. Set
\[
C = \{ V : M_U(|\nabla u|^p) > N \} \quad \text{and} \quad D = \{ V : M_U(|\nabla u|^p) > 1 \} \cup \{ V : M_U(|\nabla \psi|^p + |F|^p) > \delta \}.
\]

Let \( y \) be any point in \( C \), and define
\[
m(r) := \frac{w(C \cap B_{\varepsilon r}(y))}{w(B_{\varepsilon r}(y))} \quad \text{for} \quad r > 0.
\]
The lower semicontinuity of \( M_U(|\nabla u|^p) \) implies that \( C \) is open, and hence \( \lim_{r \to 0^+} m(r) = 1 \). Moreover, as \( y \in B_{\varepsilon \bar{R}}(z_i) \) for some \( i \) we have from condition (4.9) that
\[
m(r) \leq \frac{w(C)}{w(B_{\varepsilon \bar{R}}(z_i))} < \varepsilon \quad \forall r \geq 2\bar{R}.
\]

Therefore, there exists \( r_y \in (0, 2\bar{R}) \) such that \( m(r_y) = \varepsilon \) and \( m(r) < \varepsilon \) for all \( r > r_y \). That is,
\[
w(C \cap B_{\varepsilon r_y}(y)) = \varepsilon w(B_{\varepsilon r_y}(y)) \quad \text{and} \quad w(C \cap B_{\varepsilon r_y}(y)) < \varepsilon w(B_{\varepsilon r_y}(y)) \quad \forall r > r_y.
\] (4.10)

Thus by Vitali’s covering lemma we can select a countable sequence \( \{y_i\}_{i=1}^\infty \) such that \( \{B_{\varepsilon r_y}(y_i)\} \) is a sequence of disjoint balls and
\[
C \subset \bigcup_{i=1}^\infty B_{5\varepsilon r_y}(y_i),
\]

22
where \( r_i := r_{y_i} \). Since \( w\left( \{ U : M_{U}((\nabla u)^p) > N \} \cap B_{r_{y_i}}(y_i) \} \) \( \geq w(C \cap B_{r_{y_i}}(y_i)) = \varepsilon \ w(B_{r_{y_i}}(y_i)) \) by (4.10) and \( r_i < 2\bar{R} = R/50 \), it follows from Lemma 4.2 that

\[
B_{r_{y_i}}(y_i) \cap V \subset D. \tag{4.11}
\]

We have

\[
w(C) \leq w\left( \bigcup_{i=1}^{\infty} B_{5r_{y_i}}(y_i) \cap C \right) \leq \sum_{i=1}^{\infty} w(B_{5r_{y_i}}(y_i) \cap C) \leq \varepsilon \sum_{i=1}^{\infty} w(B_{5r_{y_i}}(y_i)) \leq \varepsilon [w]_{A_s} 5^{n\alpha} \sum_{i=1}^{\infty} w(B_{r_{y_i}}(y_i)). \tag{4.12}
\]

Let \( y \in V \) and \( 0 < \rho < 2\sigma \bar{R} \). If \( B_{\rho}(y) \subset V \), then \( |B_{\rho}(y) \cap V| = |B_{\rho}(y)| = 2^{-n}|B_{\rho}(y)| \). Otherwise, there exists \( z \in B_{\rho}(y) \cap \partial V \) and hence it follows from assumption (4.8) that \( |B_{\rho}(y) \cap V| \geq |B_{\rho}(z) \cap V| = c_0 |B_{\rho}(z)| = c_0 2^{-n}|B_{\rho}(y)| \). Combining these, we conclude that

\[
\sup_{y \in V, 0 < \rho < 2\sigma \bar{R}} \frac{|B_{\rho}(y)|}{|B_{\rho}(y) \cap V|} \leq \frac{2^n}{c_0}.
\]

This together with property (2.1) gives

\[
w(B_{r_{y_i}}(y_i)) \leq [w]_{A_s} \left( \frac{|B_{r_{y_i}}(y_i)|}{|B_{r_{y_i}}(y_i) \cap V|} \right)^s w(B_{r_{y_i}}(y_i) \cap V) \leq [w]_{A_s} \left( \frac{2^n}{c_0} \right)^s \sum_{i=1}^{\infty} w(B_{r_{y_i}}(y_i)) \leq [w]_{A_s} \left( \frac{10^n}{c_0} \right)^s \sum_{i=1}^{\infty} \sum_{i=1}^{\infty} w(B_{r_{y_i}}(y_i) \cap V).
\]

We deduce from this and (4.11)–(4.12) that

\[
w(C) \leq \varepsilon [w]_{A_s} \left( \frac{10^n}{c_0} \right)^s \sum_{i=1}^{\infty} \sum_{i=1}^{\infty} w(B_{r_{y_i}}(y_i) \cap V) = \varepsilon [w]_{A_s} \left( \frac{10^n}{c_0} \right)^s \sum_{i=1}^{\infty} \sum_{i=1}^{\infty} w(B_{r_{y_i}}(y_i) \cap V) \leq \varepsilon [w]_{A_s} \left( \frac{10^n}{c_0} \right)^s w(D),
\]

which yields the desired estimate. \( \square \)

4.2 Global gradient estimates in weighted \( L^q \) spaces

We are now ready to prove Theorem 2.9 and Theorem 2.10.

**Proof of Theorem 2.9** Let \( N = N(p, n, \Lambda) > 1 \) be as in Lemma 4.3 and let \( l = q/p \geq 1 \). We choose \( \varepsilon = \varepsilon(p, q, n, \Lambda, s, [w]_{A_s}) > 0 \) be such that

\[
\varepsilon_1 \overset{\text{def}}{=} 20^n [w]_{A_s}^2 \varepsilon = \frac{1}{2N^l},
\]

and let \( \delta \) and \( \sigma \) (depending only on \( p, q, n, \omega, \Lambda, M, s, [w]_{A_s} \)) be the corresponding positive constants given by Lemma 4.3. Assume for the moment that \( u \) is a weak solution of (2.4) satisfying

\[
w(\{ \Omega : M_{\Omega}((\nabla u)^p) > N \}) < \varepsilon \ w(B_{\sigma \bar{R}}(z_i)) \quad \forall i = 1, 2, ..., L. \tag{4.13}
\]
Notice that $V = \Omega$ satisfies condition (4.8) with $c_0 = 1/2^n$ since for any $z \in \partial \Omega$ and $0 < \rho < \sigma \bar{R}$ we have from the Reifenberg flat condition that
\[
|B_{\rho}(z) \cap \Omega| \geq |B_{\rho}(z) \cap \{x : x_n > z_n + \rho \delta\}| \geq \frac{(1 - \delta)^n}{n} |B_{\rho}(z)| \geq \frac{1}{2^n} |B_{\rho}(z)|.
\]

Thus by applying Lemma 4.3 for $V = U = \Omega$ we obtain
\[
w(\Omega : M_{\Omega}(|\nabla u|^p) > N)) \leq \varepsilon_1 \left[w(\Omega : M_{\Omega}(|\nabla u|^p) > 1) + w(\Omega : M_{\Omega}(|\nabla \psi|^p + |F|^p) > \delta)\right]. \tag{4.14}
\]

Let us iterate this estimate by considering
\[
u(x) = \frac{u(x)}{N^2}, \quad \psi(x) = \frac{\psi(x)}{N^2}, \quad F(x) = \frac{F(x)}{N^2} \quad \text{and} \quad \lambda = \frac{1}{N^2}.
\]

It is clear that $\|\nu_1\|_{L^n(\Omega)} + \|\psi_1\|_{L^n(\Omega)} \leq \frac{M}{\lambda_1^p}$, and $u_1 \in W^{1,p}(\Omega)$ is a weak solution of $\text{div} \left[\frac{A(x,\lambda_1^p \nu_1,\lambda_1^p \nabla \nu_1)}{\lambda_1^p}\right] = \text{div} F_1$ in $\Omega$ and $u_1 = \psi_1$ on $\partial \Omega$. Moreover, thanks to (4.13) we have
\[
w(\Omega : M_{\Omega}(|\nabla u_1|^p) > N)) = w(\Omega : M_{\Omega}(|\nabla u_1|^p) > N^2)) < \varepsilon w(B_{\sigma \bar{R}}(z_i)) \quad \forall i = 1, 2, ..., L.
\]

Therefore, by applying Lemma 4.3 to $u_1$ we get
\[
w(\Omega : M_{\Omega}(|\nabla u_1|^p) > N)) \leq \varepsilon_1 \left[w(\Omega : M_{\Omega}(|\nabla u_1|^p) > 1) + w(\Omega : M_{\Omega}(|\nabla \psi_1|^p + |F_1|^p) > \delta)\right]
\]
\[
= \varepsilon_1 \left[w(\Omega : M_{\Omega}(|\nabla u_1|^p) > N)) + w(\Omega : M_{\Omega}(|\nabla \psi|^p + |F|^p) > \delta N))\right].
\]

We infer from this and (4.14) that
\[
w(\Omega : M_{\Omega}(|\nabla u|^p) > N^2)) \leq \varepsilon_1^2 w(\Omega : M_{\Omega}(|\nabla u|^p) > 1)) + \varepsilon_1 w(\Omega : M_{\Omega}(|\nabla \psi|^p + |F|^p) > \delta N)),
\]
where $G^p := |\nabla \psi|^p + |F|^p$. By repeating the iteration, we then conclude that
\[
w(\Omega : M_{\Omega}(|\nabla u|^p) > N^k)) \leq \varepsilon_1^k w(\Omega : M_{\Omega}(|\nabla u|^p) > 1)) + \sum_{i=1}^{k} \varepsilon_1^i w(\Omega : M_{\Omega}(|\nabla \psi|^p + |F|^p) > \delta N^{k-i}) \quad \forall k \geq 1.
\]

This together with
\[
\int_{\Omega} M_{\Omega}(|\nabla u|^p)^j \, dw = l \int_0^\infty t^{l-1} w(\Omega : M_{\Omega}(|\nabla u|^p) > t)) \, dt
\]
\[
= l \int_0^N t^{l-1} w(\Omega : M_{\Omega}(|\nabla u|^p) > t)) \, dt + l \sum_{k=1}^{N^{l+1}} \int_{N^k}^{N^{k+1}} t^{l-1} w(\Omega : M_{\Omega}(|\nabla u|^p) > t)) \, dt
\]
\[
\leq N^l w(\Omega) + (N^l - 1) \sum_{k=1}^{\infty} N^{lk} w(\Omega : M_{\Omega}(|\nabla u|^p) > N^k))
\]

24
We have from Lemma 8(iii) that
\[
\int_{\Omega} M_{\Omega}(|\nabla u|^p) \, dw \leq N^l w(\Omega) + (N^l - 1)w(\Omega) \sum_{k=1}^{\infty} (\varepsilon_1 N^l)^k + \sum_{k=1}^{\infty} \sum_{i=1}^{k} (N^l - 1)N^{l_k} \varepsilon_i w(\{|\Omega : M_{\Omega}(G^p) > \delta N^{k-i}\}).
\]

But we have
\[
\sum_{k=1}^{\infty} \sum_{i=1}^{k} (N^l - 1)N^{l_k} \varepsilon_i w(\{|\Omega : M_{\Omega}(G^p) > \delta N^{k-i}\})
\]
\[
= \left(\frac{N}{\delta}\right)^l \sum_{i=1}^{\infty} (\varepsilon_1 N^l)^i \left[ \sum_{k=i}^{\infty} (N^l - 1)\delta N^{l_k-i} w(\{|\Omega : M_{\Omega}(G^p) > \delta N^{k-i}\}) \right]
\]
\[
= \left(\frac{N}{\delta}\right)^l \sum_{i=1}^{\infty} (\varepsilon_1 N^l)^i \sum_{j=0}^{\infty} (N^l - 1)\delta N^{l_j} w(\{|\Omega : M_{\Omega}(G^p) > \delta N^{j}\}) \right] \leq \left(\frac{N}{\delta}\right)^l \left[ \int_{\Omega} M_{\Omega}(G^p)^j \, dw \right] \sum_{i=1}^{\infty} (\varepsilon_1 N^l)^j.
\]

Thus we infer that
\[
\int_{\Omega} M_{\Omega}(|\nabla u|^p) \, dw \leq N^l w(\Omega) + \left(\frac{N}{\delta}\right)^l \int_{\Omega} M_{\Omega}(G^p)^j \, dw \sum_{i=1}^{\infty} (\varepsilon_1 N^l)^j
\]
\[
= N^l w(\Omega) + \left(\frac{N}{\delta}\right)^l \int_{\Omega} M_{\Omega}(G^p)^j \, dw \sum_{k=1}^{\infty} 2^{-k} \leq C \left( w(\Omega) + \int_{\Omega} M_{\Omega}(G^p)^j \, dw \right)
\]

with the constant C depending only on p, q, n, \omega, \Lambda, M, s, and [w]_{A,s}. This together with the facts \( l = q/p \) and \( |\nabla u(x)|^p \leq M_{\Omega}(|\nabla u|^p)(x) \) for a.e. \( x \in \Omega \) yields
\[
\int_{\Omega} |\nabla u|^p \, dw \leq C \left( 1 + \int_{\Omega} M_{\Omega}(\psi)^p + |F|^p \right). \tag{4.16}
\]

We next remove the extra assumption on \( T > 0 \) for \( u \). Notice that for any constant \( T > 0 \), by using the weak type 1 – 1 estimate for the maximal function we get
\[
\left| \{|\Omega : M_{\Omega}(|\nabla u|^p) > N T^p \} \right| \leq C \frac{N T^p}{N^l} \int_{\Omega} |\nabla u|^p \, dx. \tag{4.17}
\]

We have from Lemma 8(iii) that \( R \leq 4d \) with \( d := \text{diam}(\Omega) \), which implies that \( \bigcup_{i=1}^{L} B_{\sigma R}(z_i) \subset \hat{B} := B(z_1, 2d) \). Let \( \hat{u}(x, t) = u(x, t)/T \), where \( \psi \in C_c^\infty(\mathbb{R}) \), and \( 
\]
\[
\gamma^p := \frac{2C|\nabla u|^p_{L^p(\Omega)}}{|\hat{u}|} \left( \frac{2d}{\sigma R} \right)^{\alpha s} \left[ w \right]_{A,K}^{1/\beta}
\]
\[
\text{with } K \text{ and } \beta \text{ being the constants given by Lemma 2.1. Then it follows from (4.17) that}
\]
\[
\left| \{|\Omega : M_{\Omega}(|\nabla \hat{u}|^p) > N \} \right| \leq 2^{-1} \left( \frac{\sigma R}{2d} \right)^{\alpha s} \left[ w \right]_{A,K}^{1/\beta} |\hat{B}|.
\]

25
This together with property (2.1) for $w$ gives
\[
 w((\Omega : M_\Omega((|\nabla u|^p) > N)) \leq \left(\frac{\sigma \bar{R}}{2d}\right)^{\frac{1}{p}} \frac{2^p}{\omega_n} w(B_{\sigma \bar{R}}(z_i)) \quad \forall i = 1, 2, \ldots, L.
\]
Hence we can apply (4.16) to $\bar{u}$ with $F$, $\psi$, and $\lambda$ being replaced by $\bar{F} = F/\gamma^{p-1}$, $\bar{\psi} = \psi/\gamma$, and $\bar{\lambda} = \lambda \gamma$. By reversing back to the functions $u$ and $F$, we then obtain
\[
\int_\Omega |\nabla u|^q \, dw \leq C\left(\gamma^q + \int_\Omega M_\Omega((|\nabla \psi|^p + |F|^p)^\frac{q}{r} \, dw)\right) \leq C\left(\|\nabla u\|_{L^p(\Omega)}^q + \int_\Omega M_\Omega((|\nabla \psi|^p + |F|^p)^\frac{q}{r} \, dw)\right)
\]
with $C$ now also depending on $\bar{R} = R/100$ and $\text{diam}(\Omega)$. This yields estimate (2.9) as desired. \hfill \Box

**Proof of Theorem 2.10.** Let $N = N(p, n, \Lambda) > 1$ be as in Lemma 4.3 and let $l = q/p \geq 1$. We choose $\epsilon = \epsilon(p, q, n, \Lambda, s, |w|_{A_s}) > 0$ be such that
\[
\epsilon_1 \overset{\text{def}}{=} 80^{q^2} |w|_{A_s}^2 \epsilon = \frac{1}{2n},
\]
and let $\delta$ and $\sigma$ (depending only on $p, q, n, \omega, \Lambda, M, s,$ and $|w|_{A_s}$) be the corresponding positive constants given by Lemma 4.3. Let $y_0 \in \overline{\Omega}$. We first consider the case $r \geq R/2$. Let $U = \Omega_{\frac{r}{2}}(y_0)$ and $V = \Omega_{\frac{r}{4}}(y_0)$. Since $r \geq R/2$, we get $\Omega_{\frac{r}{4}}(y) \subset U$ for every $y \in V$. We next verify condition (4.8) for $V$ in order to apply Lemma 4.3. For any $z \in \partial V$ and $0 < \rho < \sigma \bar{R}$, we consider the following two possibilities:

**Case 1:** $y_0 \in B_{\rho}(z)$. Then $B_{\rho}(z) \subset B_{\frac{r}{4}}(y_0) \subset B_{\rho}(y_0)$. In particular, $z \notin \partial B_{\rho}(y_0)$ and so we must have $z \in \partial \Omega$. These together with the Reifenberg flat condition for $\Omega$ give
\[
|B_{\rho}(z) \cap V| = |B_{\rho}(z) \cap \Omega| \geq |B_{\rho}(z) \cap \{x : x_n > z_n + \rho \delta\}| \geq \frac{(1 - \delta)^n}{n} |B_{\rho}(z)| \geq \frac{1}{2^n} |B_{\rho}(z)|.
\]

**Case 2:** $y_0 \notin B_{\rho}(z)$. Then the line passing through $z$ and $y_0$ intersects $\partial B_{\rho}(z)$ at two distinct points, say $a_1$ and $a_2$ with $a_1$ being the one on the same side as $y_0$ with respect to the point $z$. As $|a_1 - z| = \rho \leq |y_0 - z|$, we have in addition that $a_1$ belongs to the line segment $[z, y_0] \subset \Omega$ connecting $z$ and $y_0$. In particular, $a_1 \in B_{\rho}(y_0)$ since $|a_1 - y_0| < |z - y_0| \leq r$. By letting $w$ be the midpoint of $a_1$ and $z$ we obviously have $w \in \overline{\Omega}$ and $B_{\frac{\rho}{2}}(w) \subset B_{\rho}(z)$. Due to $y_0 \notin \{w, z\}$, we have $|w - y_0| = |z - y_0| - |z - w| \leq r - \frac{\rho}{4}$. Hence $B_{\frac{\rho}{2}}(w) \subset B_{\rho}(y_0)$ as $x \in B_{\frac{\rho}{2}}(w)$ implies that $|x - y_0| \leq |x - w| + |w - y_0| < \frac{\rho}{4} + r - \frac{\rho}{4} = r$. Therefore, we infer that $B_{\frac{\rho}{2}}(w) \subset B_{\rho}(z) \cap B_{\rho}(y_0)$ giving
\[
|B_{\rho}(z) \cap V| = |B_{\rho}(z) \cap B_{\rho}(y_0) \cap \Omega| \geq |B_{\frac{\rho}{2}}(w) \cap \Omega|.
\]
If $B_{\frac{\rho}{2}}(w) \subset \Omega$, then it follows from (4.18) that $|B_{\rho}(z) \cap V| \geq |B_{\frac{\rho}{2}}(w)| = 4^{-n}|B_{\rho}(z)|$. Otherwise, there exists $\tilde{w} \in B_{\frac{\rho}{2}}(w) \cap \partial \Omega$ implying that $B_{\frac{\rho}{2}}(\tilde{w}) \subset B_{\frac{\rho}{2}}(w)$. Then by combining with (4.18) and the Reifenberg flat condition we obtain
\[
|B_{\rho}(z) \cap V| \geq |B_{\frac{\rho}{2}}(\tilde{w}) \cap \Omega| \geq |B_{\frac{\rho}{2}}(\tilde{w}) \cap \{x : x_n > \tilde{w}_n + \frac{\rho}{4} \delta\}| \geq \frac{(1 - \delta)^n}{n} |B_{\frac{\rho}{2}}(\tilde{w})| \geq \frac{1}{8^n} |B_{\rho}(z)|.
\]
Therefore, we can repeat the same arguments as in the proof of Theorem 2.9 to obtain then we can apply Lemma 4.3 to get that

we can repeat the same arguments as in the proof of Theorem 2.9 to obtain then we can apply Lemma 4.3 to get that

\[ w(V : M_U(|\nabla u|^p) > N) \leq e \cdot w(B_{\sigma R}(z_i)) \quad \forall i = 1, 2, \ldots, L, \]

then we can apply Lemma 4.3 to get that

\[ w([v : M_U(|\nabla u|^p) > 1]) + w([v : M_U(|\nabla \psi|^p + |F|^p') > \delta]) \]

Therefore, we can repeat the same arguments as in the proof of Theorem 2.9 to obtain

\[ \int \nabla u dw \leq C \int \nabla u dw + \int \nabla u dw \]

with the constant C depending only on \( p, q, n, \omega, \Lambda, M, R, s, \) and \([w]_{A_r} \). Using the definitions of \( U \) and \( V \), we infer from this estimate that

\[ \frac{1}{w(B_r(y_0))} \int_{\Omega_r(y_0)} |\nabla u|^q dw \leq C \left( \frac{w(\Omega_r(y_0))}{w(B_r(y_0))} \int_{\Omega_r(y_0)} |\nabla u|^q dw + \int_{\Omega_r(y_0)} M_{\Omega_r(y_0)}(|\nabla \psi|^p + |F|^p')^\# dw \right) \]

\[ \leq C \left( \frac{1}{w(B_r(y_0))} \int_{\Omega_r(y_0)} M_{\Omega_r(y_0)}(|\nabla \psi|^p + |F|^p')^\# dw \right) \quad \forall r \geq \frac{R}{2}. \]

We next consider the case \( 0 < r < R/2 \). Let us rescale the problem by setting \( y_0 := r^{-1}y_0, \Omega := \{ r^{-1}x : x \in \Omega \} \), \( \tilde{w}(x) := w(rx) \), and

\[ \tilde{\mathbf{A}}(x, z, \xi) \equiv \mathbf{A}(x, z, \xi), \quad \tilde{\mathbf{F}}(x) = \mathbf{F}(x), \quad \tilde{\mathbf{u}}(x) = r^{-1}u(rx), \quad \tilde{\mathbf{\psi}}(x) = r^{-1}\psi(rx), \quad \tilde{\theta} = \theta r. \]

Then \( \tilde{u} \) is a weak solution of \( \text{div} \left[ \frac{\tilde{\mathbf{A}}(x, \tilde{\mathbf{u}}, \tilde{\mathbf{\psi}}, \tilde{\mathbf{\psi}})}{|\nabla \tilde{\mathbf{u}}|^p} \right] = \text{div} \tilde{\mathbf{F}} \) in \( \tilde{\Omega} \) and \( \tilde{\mathbf{u}} = \tilde{\mathbf{\psi}} \) on \( \partial \tilde{\Omega} \). We also have

\[ \left( \int_{B_{r}(z)} \tilde{w}(x) dx \right)^{-1} \left( \int_{B_{r}(z)} \tilde{w}(x)^{\frac{n}{n-p}} dx \right)^{-1} \left( \int_{B_{r}(z)} w(y)^{\frac{n}{n-p}} dy \right)^{-1} \]

for any ball \( B_{r}(z) \subset \mathbb{R}^n \), which implies that \([\tilde{w}]_{A_r} = [w]_{A_r} \). Moreover, \( \tilde{\Omega} \) is \((\delta, R/r)\)-Reifenberg flat with \( R/r > 2 \), and hence \( \tilde{\Omega} \) is \((\delta, 2)\)-Reifenberg flat. Therefore, we can apply estimate (4.20) for \( R = 2 \) and for \( \tilde{u} \) and weight \( \tilde{w}(x) \) to obtain

\[ \frac{1}{\tilde{w}(B_1(\tilde{y}_0))} \int_{\tilde{\Omega}_1(\tilde{y}_0)} |\nabla \tilde{u}|^p d\tilde{w} \leq C \left[ \int_{\tilde{\Omega}_1(\tilde{y}_0)} |\nabla \tilde{u}|^p d\tilde{w} \right] \]

Since

\[ \tilde{w}(B_1(\tilde{y}_0)) = \int_{B_1(y_0)} w(rx) dx = r^{-n} w(B_r(y_0)), \quad \int_{\tilde{\Omega}_1(\tilde{y}_0)} |\nabla \tilde{u}|^p d\tilde{w} = r^{-n} \int_{\Omega_1(y_0)} |\nabla u|^p dy, \]

\[ M_{\Omega_2(y_0)}(|\nabla \psi|^p + |F|^p')(x) = M_{\Omega_2(y_0)}(|\nabla \psi|^p + |F|^p')(rx), \]

by changing variables we see that (4.21) is equivalent to

\[ \frac{1}{w(B_r(y_0))} \int_{\Omega_r(y_0)} |\nabla u|^q dw \leq C \left[ \left( \int_{\Omega_2(y_0)} |\nabla u|^p dx \right)^\# + \frac{1}{w(B_r(y_0))} \int_{\Omega_r(y_0)} M_{\Omega_2(y_0)}(|\nabla \psi|^p + |F|^p')^\# dw \right] \]

for \( 0 < r < R/2 \). This estimate together with (4.20) gives the conclusion of the theorem. We note that unlike the situation in (4.20), the constant C is independent of R when \( r < R/2 \). \( \square \)
Remark 4.4. It is important to stress that in order to derive the above estimate for one particular region \( \Omega_e(y) \), we only need to assume \( u = \psi \) on the portion \( \partial \Omega \cap B_2(y) \).

### 4.3 Global gradient estimates in weighted Morrey spaces

In this subsection we present the proofs of Theorem 1.3 and Theorem 1.4.

**Proof of Theorem 1.3** Let \( G^p := |\phi|^p + |F|^p \). Let \( \bar{x} \in \Omega \) and \( 0 < r : = \text{diam}(\Omega) \). Then by applying the localized estimate in Theorem 2.10 we obtain

\[
\frac{1}{|B_r(\bar{x})|} \int_{B_r(\bar{x}) \cap \Omega} |\nabla u|^q dw \leq C \left( \frac{1}{|B_r(\bar{x})|} \int_{B_r(\bar{x}) \cap \Omega} |\nabla u|^p dx \right)^{\frac{q}{p}} + \frac{1}{|w(B_r(\bar{x}))|} \int_{B_r(\bar{x}) \cap \Omega} M_\Omega(G^p) \frac{dw}{w} \right). \tag{4.22}
\]

We next estimate the first term in the above right hand side. For this, let \( \varepsilon \in (0, n) \) to be determined later and use the trick in [15, Page 2506] to write

\[
\frac{1}{|B_{2r}(\bar{x})|} \int_{B_{2r}(\bar{x}) \cap \Omega} |\nabla u|^p dx = \omega_n^{-1} (2r)^{-\varepsilon} \int_{B_{2r}(\bar{x}) \cap \Omega} |\nabla u|^p \tilde{w} dx \leq \omega_n^{-1} (2r)^{-\varepsilon} \int_{\Omega} |\nabla u|^p \tilde{w} dx
\]

with \( \omega_n := |B_1| \) and \( \tilde{w} \) being the weight defined by

\[
\tilde{w}(x) := \min \{|x - \bar{x}|^{-\varepsilon}, (2r)^{-\varepsilon}\}.
\]

As \( \tilde{w} \in A_1 \) with \( [\tilde{w}]_{A_1} \leq C(t, \varepsilon, n) \) for any \( 1 < t < \infty \) (see [15, Lemma 3.2]), we can apply Theorem 2.9 with \( q = p \) to estimate the above last integral. As a consequence, we obtain

\[
\frac{1}{|B_{2r}(\bar{x})|} \int_{B_{2r}(\bar{x}) \cap \Omega} |\nabla u|^p dx \leq C (2r)^{-\varepsilon} \left( \tilde{w}(\Omega) \|\nabla u\|_{L^p(\Omega)} + \int_{\Omega} M_\Omega(G^p) \tilde{w} dx \right)
\]

\[
\leq C (2r)^{-\varepsilon} \left( \|\nabla u\|_{L^p(\Omega)} + \int_{\Omega} M_\Omega(G^p) \tilde{w} dx \right) \tag{4.23}
\]

with \( C > 0 \) depending only on \( p, n, \omega, \Lambda, M, R, \text{diam}(\Omega) \), and \( \varepsilon \). Notice that to obtain the last inequality we have used the fact

\[
\tilde{w}(\Omega) \leq \int_{B_d(\bar{x})} \|x - \bar{x}\|^{-\varepsilon} dx = \omega_n \int_0^d t^{\varepsilon-1} dt = \frac{\omega_n}{\varepsilon} d^\varepsilon.
\]

To bound the last integral in (4.23), we employ Fubini’s theorem to get

\[
\int_{\Omega} M_\Omega(G^p) \tilde{w} dx = \int_0^\infty \int_{\Omega \cap \tilde{w}(\Omega) > t} M_\Omega(G^p) dxdt \leq \int_0^{(2r)^{-\varepsilon}} \int_{B_d(\bar{x}) \cap \Omega} M_\Omega(G^p) dxdt
\]

\[
\leq \int_0^{(2r)^{-\varepsilon}} \int_{\Omega} M_\Omega(G^p) dxdt + \int_{d^{\varepsilon>1}}^{(2r)^{-\varepsilon}} \int_{B_{d^{\varepsilon>1}}(\bar{x}) \cap \Omega} M_\Omega(G^p) dxdt.
\]

Since \( \Omega = B_d(\bar{x}) \cap \Omega \), we then deduce that

\[
\int_{\Omega} M_\Omega(G^p) \tilde{w} dx \leq C \|[M_\Omega(G^p)]\|_{M^1\text{w}^p(\Omega)} \int_{d^{\varepsilon>1}}^{(2r)^{-\varepsilon}} t^{-\frac{n}{p}} \varphi(B_{d^{\varepsilon>1}}(\bar{x})) \frac{dt}{t} + \int_{d^{\varepsilon>1}}^{(2r)^{-\varepsilon}} t^{-\frac{n}{p}} \varphi(B_{d^{\varepsilon>1}}(\bar{x})) \frac{dt}{t}.
\]
where we recall that $M^{1, q}(U)$ denotes the Morrey space $M_w^{1, q}(U)$ with $w = 1$. As $\varphi \in B_+$ by the assumption and $\mathcal{B}_\alpha$ is decreasing in $\alpha$, there exists $\alpha \in (0, n)$ such that $\varphi \in \mathcal{B}_\alpha$. Then it follows if $\varepsilon < \alpha p / q$ that

$$
\int_\Omega M_\Omega(G^p) \, d\bar{x} \leq C(2r)^{\alpha \frac{d}{q}} \|M_\Omega(G^p)\|_{M^{1, q}(\bar{x})} \left[ \int_0^{(2r)^{\alpha \frac{d}{q}} \varphi(B_\varepsilon(\bar{x})) \, \frac{d\bar{x}}{\bar{x}} + \varphi(B_2r(\bar{x})) \, \frac{d\bar{x}}{\bar{x}} \right] 
\leq C r^\varepsilon \varphi(B_\varepsilon(\bar{x})) \|M_\Omega(G^p)\|_{M^{1, q}(\Omega)}.
$$

Combining this with (4.23), we arrive at:

$$
\frac{1}{|B_2r(\bar{x})|} \int_{B_2r(\bar{x}) \cap \Omega} \|\nabla u\|^q \, dw \leq C r^\varepsilon \|\nabla u\|^q_{L^p(\Omega)} + \|M_\Omega(G^p)\|_{M^{1, q}(\Omega)}.
$$

Therefore, we infer from (4.22) and the fact $\varphi \in \mathcal{B}_\alpha$ that

$$
\frac{\varphi(B_r(\bar{x}))}{|w(B_r(\bar{x}))|} \int_{B_r(\bar{x}) \cap \Omega} \|\nabla u\|^q \, w \leq C r^\varepsilon \left[ \|\nabla u\|^q_{L^p(\Omega)} + ||M_\Omega(G^p)\|_{M^{1, q}(\Omega)} \right]
\leq C r^\varepsilon \left[ \|\nabla u\|^q_{L^p(\Omega)} + ||M_\Omega(G^p)\|_{M^{1, q}(\Omega)} \right]
$$

for all $\bar{x} \in \Omega$ and $0 < r \leq d$. By taking $\varepsilon = \frac{\varepsilon}{2} q$ and as sup$_{\bar{x} \in \Omega} \varphi(B_r(\bar{x})) < \infty$, this gives estimate (1.8). □

**Proof of Theorem 1.4** Let $G^p := |\psi|^p + |F|^p$. Since $v \in A_{\frac{q}{p}}$, Lemma 2.1 in [7] gives $v^{1-\frac{q}{p}'} \in A_{\frac{q}{p}'} \subset A_{\infty}$. Thus our assumptions imply that condition (B) in Lemma 2.3 is satisfied. Also as $\varphi \in \mathcal{B}_0$, it is clear that $1.10$ yields (2.3). Indeed, for any $y \in \mathbb{R}^n$ and any $s \geq 2r > 0$ we have from (1.10) and $\varphi \in \mathcal{B}_0$ that

$$
\frac{\psi(B_s(y))}{\psi(B_s(y))} \leq C, \quad \frac{\psi(B_s(y))}{\psi(B_s(y))} \leq C, \quad \frac{1}{\psi(B_s(y))} \leq C
$$

yielding (2.3). Moreover, by [8] Theorem 9.3.3 there exist $s \in (1, \infty)$ and $C > 0$ depending only on $n$ and $[w]_{A_\infty}$ such that $[w]_{A_s} \leq C$. Therefore, it follows from Theorem 1.3 and Lemma 2.3 that

$$
\|\nabla u\|_{A_\infty^{s}(\Omega)} \leq C \left( \|\nabla u\|_{L^p(\Omega)} + ||M_\Omega(G^p)||_{M^{1, q}(\Omega)} \right)
$$

with $C > 0$ depending only on $q$, $p$, $n$, $\omega$, $A$, $M$, $R$, diam$(\Omega)$, $\varphi$, $C_\omega$, $[w]_{A_\omega}$, $[v]_{A_{\frac{q}{p}}}$, and $[w, v^{1-\frac{q}{p}'}]_{A_{\frac{q}{p}}'}$. Thus it remains to estimate the middle term on the right-hand side of (4.24). Let $l := q/p > 1$. Then for any nonnegative function $g \in L^1(\Omega)$, we obtain from Hölder inequality and assumption (1.9) that

$$
\frac{\varphi(B_r(\bar{x}))}{|B_r(\bar{x})|} \left( \int_{B_r(\bar{x}) \cap \Omega} g \, dx \right)^l \leq \frac{\varphi(B_r(\bar{x}))}{|B_r(\bar{x})|} \left( \int_{B_r(\bar{x}) \cap \Omega} g^l \, dx \right) \left( \int_{B_r(\bar{x}) \cap \Omega} v^{l-1} \, dx \right)^{-1}
\leq [w, v^{l-1}, \varphi(B_r(\bar{x}))]_{A_\infty^{s}(\Omega)} \frac{\varphi(B_r(\bar{x}))}{|B_r(\bar{x})|} \left( \int_{B_r(\bar{x}) \cap \Omega} g^l \, dx \right) \leq [w, v^{l-1}, l]_A \|g\|_{M^{1, q}(\Omega)}
$$

29
for all \( \bar{x} \in \Omega \) and all \( 0 < R \leq \text{diam}(\Omega) \), where
\[
\hat{\varphi}(B) := \frac{v(B)}{w(B)} \varphi(B).
\]

Hence we infer that
\[
\|M_{\Omega}(G^p)\|_{\mathcal{M}^{1/p,\hat{w}}(\Omega)} \leq [w, v^{1-p}]_{A_1}^{1/p} \|M_{\Omega}(G^p)\|_{\mathcal{M}^{1/p,\hat{\varphi}}(\Omega)}.
\]

Using \( \varphi \in B_0 \), condition (1.10), and the doubling property of \( w \) due to Lemma 2.1, we have
\[
\sup_{s \geq 2r} \frac{1}{\varphi(B_s(y))} \leq C \frac{1}{\varphi(B_2(y))} \leq CC \frac{w(B_2(y))}{v(B_2(y))} \frac{1}{\varphi(B_2(y))} \leq C' \frac{w(B_r(y))}{v(B_r(y))} \frac{1}{\varphi(B_r(y))} = C' \frac{1}{\hat{\varphi}(B_r(y))}
\]
for all \( y \in \mathbb{R}^n \) and \( r > 0 \). Thus as \( v \in A_1 \) we can use the strong type estimate for the Hardy–Littlewood maximal function given by Lemma 2.4 to estimate the right hand side of (4.25). As a result, we get
\[
\|M_{\Omega}(G^p)\|_{\mathcal{M}^{1/p,\hat{w}}(\Omega)} \leq C \|G^p\|_{\mathcal{M}^{1/p,\hat{\varphi}}(\Omega)} = C \|G\|_{\mathcal{M}^{1/p,\hat{\varphi}}(\Omega)}.
\]

This and (4.24) yield desired estimate (1.11).

\[\square\]

Appendix A  A compactness argument

**Lemma A.1.** Let \( a : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be a continuous vector field such that (2.5) holds and \( |a(\xi)| \leq \Lambda |\xi|^{p-1} \) for some constants \( p > 1 \) and \( \Lambda > 0 \). Then for any \( \varepsilon > 0 \), there exists a constant \( \delta > 0 \) depending only on \( \varepsilon, p, \Lambda, \) and \( n \) satisfying: if \( \Omega \subset \mathbb{R}^n \) is a bounded domain with
\[
B_3 \cap \{x_n > 3\delta\} \subset \Omega \subset B_3 \cap \{x_n > -3\delta\},
\]
\( w \in W^{1,p}(\Omega_3) \) is a weak solution of
\[
\begin{cases}
\text{div } a(\nabla w) = 0 & \text{in } \Omega_3,
\
w = 0 & \text{on } B_3 \cap \partial \Omega
\end{cases}
\]
with \( \frac{1}{|B_3|} \int_{\Omega_3} |\nabla w|^p \, dx \leq 1 \), then there exists a weak solution \( v \) of
\[
\begin{cases}
\text{div } a(\nabla v) = 0 & \text{in } B_3^+,
\
v = 0 & \text{on } B_3 \cap \{x_n = 0\}
\end{cases}
\]
satisfying \( \frac{1}{|B_3|} \int_{B_3} |\nabla v|^p \, dx \leq (4^p \Lambda^2)^p \) such that
\[
(\int_{B_3 \cap \{x_n > 3\delta\}} |w - v|^p \, dx)^{1/p} \leq \varepsilon^p.
\]
Proof. Assume to the contrary that the statement is false. Then there exist \( \varepsilon_0 > 0, n \in \mathbb{N}, p > 1, \Lambda > 0 \), and sequences \( \{a^k\}, \{\Omega^k\}, \{w^k\} \) such that for each \( k \in \mathbb{N} \) we have: \( a^k \) satisfies (2.5) and \( |a^k(\xi)| \leq \Lambda |\xi|^{p-1} \), \( B_3 \cap \{x_n > \frac{3}{k}\} \subset \Omega^k \subset B_3 \cap \{x_n > \frac{3}{k}\} \),

\[
B_3 \cap \{x_n > \frac{3}{k}\} \subset \Omega^k \subset B_3 \cap \{x_n > \frac{3}{k}\},
\]

\( w^k \in W^{1,p}(\Omega^k) \) is a weak solution of

\[
\begin{align*}
\text{div } a^k(\nabla w^k) &= 0 \quad \text{in } \Omega^k, \\
\ |w^k| &= 0 \quad \text{on } B_3 \cap \partial \Omega^k
\end{align*}
\]

with \( \int_{\Omega^k} |\nabla w^k|^p \, dx \leq |B_3| \), and

\[
(\int_{B_3 \cap \{x_n > \frac{3}{k}\}} |w^k - v|^p \, dx)^{\frac{1}{p}} > \varepsilon_0^p
\]

\[(A.1)\]

for every weak solution \( v \) of

\[
\begin{align*}
\text{div } a^k(\nabla v) &= 0 \quad \text{in } B_3^+, \\
\ v &= 0 \quad \text{on } B_3 \cap \{x_n = 0\}
\end{align*}
\]

satisfying \( \frac{1}{|B_3|} \int_{B_3} |\nabla v|^p \, dx \leq (4^p \Lambda^2)^p \). Notice that \( ||w^k||_{W^{1,p}(\Omega^k)} \leq C \) by using Pointcaré inequality. Then as in [3][7] we can show that there exist a continuous vector field \( a : \mathbb{R}^n \rightarrow \mathbb{R}^n \) and a function \( w \in W^{1,p}(B_3^+) \) with \( \int_{B_3^+} |\nabla w|^p \, dx \leq |B_3| \) such that up to a subsequence we have \( a^k(\xi) \rightarrow a(\xi) \) for all \( \xi \in \mathbb{R}^n \), \( w^k \rightarrow w \) strongly in \( L^p_{\text{loc}}(B_3^+) \), and \( \nabla w^k \rightarrow \nabla w \) weakly in \( L^p_{\text{loc}}(B_3^+) \). Consequently, \( a \) satisfies (2.5) and \( |a(\xi)| \leq \Lambda |\xi|^{p-1} \). We then infer by passing to the limit that \( w \in W^{1,p}(B_3^+) \) is a weak solution of the equation

\[
\begin{align*}
\text{div } a(\nabla w) &= 0 \quad \text{in } B_3^+, \\
\ w &= 0 \quad \text{on } B_3 \cap \{x_n = 0\}
\end{align*}
\]

\[(A.2)\]

Now for each \( k \in \mathbb{N} \), let \( v^k \) be a weak solution of

\[
\begin{align*}
\text{div } a^k(\nabla v^k) &= 0 \quad \text{in } B_3^+, \\
\ v^k &= w \quad \text{on } \partial B_3^+.
\end{align*}
\]

In particular, we have \( v^k = 0 \) on \( B_3 \cap \{x_n = 0\} \). Moreover, by using \( v^k - w \) as a test function and due to the structural conditions for \( a^k \) we get

\[
\int_{B_3^+} |\nabla v^k|^p \, dx \leq 4^p \Lambda^2 \int_{B_3^+} |\nabla v|^p \, dx \leq \frac{1}{p'} \int_{B_3^+} |\nabla v^k|^{p-1} |\nabla w| \, dx + \frac{1}{p'} \int_{B_3^+} |\nabla v|^p \, dx + \frac{1}{p'} (4^p \Lambda^2)^p \int_{B_3^+} |\nabla w|^p \, dx
\]

yielding \( \int_{B_3^+} |\nabla v^k|^p \, dx \leq (4^p \Lambda^2)^p |B_3| \). Hence it follows from (A.1) that

\[
(\int_{B_3 \cap \{x_n > \frac{3}{k}\}} |w^k - v^k|^p \, dx)^{\frac{1}{p}} > \varepsilon_0^p,
\]

\[(A.3)\]
Notice that Pointcaré inequality implies that the sequence \( \{v^k\} \) is bounded in \( W^{1,p}(B^+_3) \). Thus there exists a function \( v \in W^{1,p}(B^+_3) \) such that up to a subsequence we have \( v^k \to v \) strongly in \( L^p(B^+_3) \) and \( \nabla v^k \to \nabla v \) weakly in \( L^p(B^+_3) \). By passing to the limit we see that \( v \in W^{1,p}(B^+_3) \) is a weak solution of equation (A.2). But as \( (A.2) \) has a unique weak solution since \( a \) satisfies \( (2.5) \), we infer that \( v \equiv w \). Therefore, \( w^k - v^k \to w - v = 0 \) strongly in \( L^p_{loc}(B^+_3) \). This contradicts \( (A.3) \) and the proof is complete. \( \Box \)

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