HEAT KERNEL ANALYSIS ON SEMI-INFINITE LIE GROUPS

TAI MELCHER

Abstract. This paper studies Brownian motion and heat kernel measure on a class of infinite dimensional Lie groups. We prove a Cameron-Martin type quasi-invariance theorem for the heat kernel measure and give estimates on the $L^p$ norms of the Radon-Nikodym derivatives. We also prove that a logarithmic Sobolev inequality holds in this setting.

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1. Introduction

We define Brownian motion on a class of infinite dimensional Lie algebras which we call semi-infinite Lie algebras. We then prove a Cameron-Martin type quasi-invariance result for the associated heat kernel measure, as well as a logarithmic Sobolev inequality. A particular example of these semi-infinite Lie algebras was treated in [10], and we build on the methods used there.

We briefly describe here the main results and give an outline of the paper; see Sections 2 and 3 for definitions. Let $(W, H, \mu)$ be an abstract Wiener space and $v$ be a finite dimensional Lie algebra equipped with an inner product. Let $g = W \oplus v$ be a Lie algebra extension of $W$ by $v$, and we will call $g_{CM} = H \oplus v$ the Cameron-Martin...
Lie subalgebra of $\mathfrak{g}$. If $\mathfrak{g}$ is nilpotent, we may define an explicit group operation on $\mathfrak{g}$ via the Baker-Campbell-Hausdorff-Dynkin formula, and $W \oplus v$ equipped with this group operation will be denoted by $G$. Similarly, $G_{CM} = H \oplus v$ with the same group operation is called the Cameron-Martin subgroup of $G$, and we equip $G_{CM}$ with the left invariant Riemannian metric which agrees with the inner product

$$\langle (A, a), (B, b) \rangle_{g_{CM}} = \langle A, B \rangle_H + \langle a, b \rangle_v$$
onumber

on $g_{CM} \cong T_eG_{CM}$.

In Section 2, we set the notation and give some standard facts needed about abstract Wiener spaces and extensions of Lie algebras. In Section 3, we construct the semi-infinite Lie algebras and give some examples. We make some additional requirements so that the Lie bracket on $\mathfrak{g}$ is continuous, making $\mathfrak{g}$ into a Banach Lie algebra. In Section 3.2, this gives bounded Hilbert-Schmidt norms for the Lie bracket, and, in Section 3.4, lower bounds on the Ricci curvature of $G$ and a uniform lower bound on certain finite dimensional approximations of $G$.

In Section 4, we define Brownian motion on $G$ as the solution to a stochastic differential equation with respect to a Wiener process on $\mathfrak{g}$. Let $B_t$ denote Brownian motion on $\mathfrak{g}$. Then, Brownian motion on $G$ is the solution to the Stratonovich stochastic differential equation

$$\delta g_t = g_t \delta B_t := L_{g_t} \delta B_t, \quad \text{with } g_0 = e = (0, 0).$$

For $t > 0$, let $\Delta_n(t)$ denote the simplex in $\mathbb{R}^n$ given by

$$\{ s = (s_1, \cdots, s_n) \in \mathbb{R}^n : 0 < s_1 < s_2 < \cdots < s_n < t \}.$$ 

Let $S_n$ denote the permutation group on $(1, \cdots, n)$, and, for each $\sigma \in S_n$, let $e(\sigma)$ denote the number of “errors” in the ordering $(\sigma(1), \sigma(2), \cdots, \sigma(n))$, that is,

$$e(\sigma) = \# \{ j < n : \sigma(j) > \sigma(j + 1) \}. $$

Then the Brownian motion on $G$ may be written as

$$g_t = \sum_{n=1}^{r-1} \sum_{\sigma \in S_n} \left( -1 \right)^{e(\sigma)} \frac{1}{n^2} \left[ \frac{n-1}{e(\sigma)} \right] \int_{\Delta_n(t)} \left[ \cdots \left[ \delta B_{s(1)}, \delta B_{s(2)}, \cdots, \delta B_{s(n)} \right] \right],$$

where this sum is finite since $\mathfrak{g}$ is assumed to be nilpotent. In Section 4, we show that these stochastic integrals are well-defined and each may be expressed as a sum of iterated Itô integrals. We also show that $g_t$ may be realized as a limit of Brownian motions living on the finite dimensional approximations to $G$. In particular, we show in Proposition 4.9 that this convergence holds in $L^p$, for all $p \in [1, \infty)$.

In Theorem 5.3, we apply the previous results and a theorem from [11] to prove that $\nu_t = \text{Law}(g_t)$ is invariant under (right or left) translation by elements of $G_{CM}$. Moreover, this theorem gives good bounds on the $L^p$-norms of the Radon-Nikodym derivatives. These results are important for future applications to spaces of holomorphic functions on $G$, as in [12]. We also show in Theorem 5.7 that a logarithmic Sobolev inequality holds for polynomial cylinder functions on $G$.

For heat kernel analysis, quasi-invariance results, and logarithmic Sobolev inequalities in related infinite dimensional settings, see [1, 17].
the sequel. For proofs of these results, see Section 2 of [10]. Also see [6, 19] for more on abstract Wiener spaces and some particular examples.

Suppose that $W$ is a real separable Banach space and $B_W$ is the Borel $\sigma$-algebra on $W$.

**Definition 2.1.** A measure $\mu$ on $(W, B_W)$ is called a (mean zero, non-degenerate) Gaussian measure provided that its characteristic functional is given by

$$\hat{\mu}(u) := \int_W e^{i u(x)} d\mu(x) = e^{-\frac{1}{2}q(u,u)} \quad \text{for all } u \in W^*,
$$

for $q = q_\mu : W^* \times W^* \to \mathbb{R}$ a symmetric, positive definite quadratic form. That is, $q$ is a real inner product on $W^*$.

**Theorem 2.2.** Let $\mu$ be a Gaussian measure on a real separable Banach space $W$. For $1 \leq p < \infty$, let

$$(2.2) \quad C_p := \int_W \|w\|_W^p d\mu(w).$$

For $w \in W$, let

$$\|w\|_H := \sup_{u \in W^* \setminus \{0\}} \frac{|u(w)|}{\sqrt{q(u,u)}}$$

and define the Cameron-Martin subspace $H \subset W$ by

$$H := \{h \in W : \|h\|_H < \infty\}.$$

Then

1. For all $1 \leq p < \infty$, $C_p < \infty$.
2. $H$ is a dense subspace of $W$.
3. There exists a unique inner product $\langle \cdot, \cdot \rangle_H$ on $H$ such that $\|h\|_H^2_H = \langle h, h \rangle_H$ for all $h \in H$, and $H$ is a separable Hilbert space with respect to this inner product.
4. For any $h \in H$, $\|h\|_W \leq \sqrt{C_2} \|h\|_H$.
5. If $\{k_j\}_{j=1}^\infty$ is an orthonormal basis of $H$ and $\varphi$ is a bounded linear map from $W$ to a real Hilbert space $C$, then

$$(2.3) \quad \|\varphi\|_{H^* \otimes C}^2 := \sum_{j=1}^\infty \|\varphi(k_j)\|_C^2 = \int_W \|\varphi(w)\|_C^2 d\mu(w) < \infty.$$

A simple consequence of (2.3) is that

$$(2.4) \quad \|\varphi\|_{H^* \otimes C}^2 \leq \|\varphi\|_{W^* \otimes C}^2 \int_W \|w\|_W^2 d\mu(w) = C_2 \|\varphi\|_{W^* \otimes C}^2.$$

**2.2. Extensions of Lie algebras.** Suppose $\mathfrak{v}$ is a Lie algebra and $\text{Der}(\mathfrak{v})$ is the set of derivations on $\mathfrak{v}$. That is, $\text{Der}(\mathfrak{v})$ consists of all linear maps $\rho : \mathfrak{v} \to \mathfrak{v}$ satisfying Leibniz's rule:

$$\rho([X,Y]_\mathfrak{v}) = [\rho(X), Y]_\mathfrak{v} + [X, \rho(Y)]_\mathfrak{v}.$$

$\text{Der}(\mathfrak{v})$ forms a Lie algebra with Lie bracket defined by the commutator:

$$[\rho_1, \rho_2] = \rho_1 \rho_2 - \rho_2 \rho_1, \quad \text{for } \rho_1, \rho_2 \in \text{Der}(\mathfrak{v}).$$

$\text{Der}(\mathfrak{v})$ is a subset of linear maps on $\mathfrak{v}$, so if $\mathfrak{v}$ is a normed vector space, one may equip $\text{Der}(\mathfrak{v})$ with the usual norm

$$(2.5) \quad \|\rho\|_0 = \sup\{\|\rho(X)\|_\mathfrak{v} : \|X\|_\mathfrak{v} = 1\}.$$
Now suppose that \( h \) and \( v \) are Lie algebras, and that there is a linear mapping
\[
\alpha : h \to \text{Der}(v)
\]
and a skew-symmetric bilinear mapping
\[
\omega : h \times h \to v,
\]
satisfying, for all \( X, Y, Z \in h \),
\[
(B1) \quad [\alpha_X, \alpha_Y] - \alpha_{[X,Y]} = \text{ad}_\omega(X,Y)
\]
and
\[
(B2) \quad \sum_{\text{cyclic}} (\alpha_X \omega(Y,Z) - \omega([X,Y],Z)) = 0.
\]
Then, one may verify that, for \( X_1 + V_1, X_2 + V_2 \in h \oplus v \),
\[
[X_1 + V_1, X_2 + V_2] := [X_1, X_2]_h + \omega(X_1, X_2) + \alpha_{X_1} V_2 - \alpha_{X_2} V_1 + [V_1, V_2]_v
\]
defines a Lie bracket on \( g := h \oplus v \), and we say \( g \) is an extension of \( h \) over \( v \). That is, \( g \) is the Lie algebra with ideal \( v \) and quotient algebra \( g/v = h \). The associated exact sequence is
\[
0 \to v \xrightarrow{\iota_1} g \xrightarrow{\pi_2} h \to 0,
\]
where \( \iota_1 \) is inclusion and \( \pi_2 \) is projection. In fact, the following theorem (see, for example, [2]) states that these are the only extensions of \( h \) over \( v \).

**Theorem 2.3.** Isomorphism classes of extensions of \( h \) over \( v \) (that is, short exact sequences of Lie algebras \( 0 \to v \to g \to h \to 0 \)) modulo the equivalence described by the commutative diagram of Lie algebra homomorphisms
\[
\begin{array}{cccccc}
0 & \to & v & \xrightarrow{\iota} & g & \xrightarrow{\pi} & h & \xrightarrow{\text{id}} & 0 \\
& & \downarrow{\text{id}} & & \downarrow{\varphi} & & \downarrow{\text{id}} & & \\
0 & \to & v & \xrightarrow{\iota} & g' & \xrightarrow{\pi} & h & \xrightarrow{0} & 0,
\end{array}
\]
correspond bijectively to equivalence classes of pairs of linear maps \( \alpha : h \to \text{Der}(v) \) and skew-symmetric bilinear maps \( \omega : h \times h \to v \) satisfying \((B1)\) and \((B2)\), where \((\alpha, \omega) \equiv (\alpha', \omega')\) if there exists a linear \( b : h \to v \) such that
\[
\alpha'_{X} = \alpha_{X} + \text{ad}_b(X),
\]
and
\[
\omega'(X,Y) = \omega(X,Y) + \alpha_X b(Y) - \alpha_Y b(X) - b([X,Y]) + [b(X), b(Y)]_v.
\]
The corresponding isomorphism \( \varphi : g \to g' \) is given by \( \varphi(X + V) = X - b(X) + V \).

When \( v = V \) is an abelian Lie algebra, these pairs consist of a Lie algebra homomorphism \( \alpha : h \to \text{gl}(V) \) and \( \omega \in H^2(h,V) \) is a Chevalley cohomology class with coefficients in the \( h \)-module \( V \) (see [16], Chapter 1, Sections 3.1 and 4.5). For definitions and details on extensions of Lie algebras, see Section XIV.5 of [7]. Reference [2] also gives a nice (although unpublished) summary. Reference [20] gives some conditions under which the extension of \( h \) over \( v \) is nilpotent (when \( h \) and \( v \) are nilpotent); [22] gives a characterization of extensions of a Lie algebra over a Heisenberg Lie algebra.

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Throughout the rest of this paper \((W, H, \mu)\) will denote a real abstract Wiener space, and \(v\) will denote a Lie algebra with \(\dim(v) = N < \infty\), equipped with an inner product \(\langle \cdot, \cdot \rangle_v\) and a continuous Lie bracket \([\cdot, \cdot]_v\). Note that this implies that there exists a constant \(c_0 \leq \infty\) such that
\[\|[X, Y]\|_v \leq c_0 \|[X]\|_v \|[Y]\|_v,\]
for all \(X, Y \in v\). For simplicity, we will assume that \(c_0 \equiv 1\). Also, \(\text{Der}(v)\) will denote the derivations of \(v\), equipped with the norm defined in (2.5).

**Definition 3.1.** Let \((W, H, \mu)\) be an abstract Wiener space and \(v\) a finite dimensional Lie algebra. Then \(g = W \oplus v\) endowed with a Lie bracket satisfying
1. \([g, g] \subset v\), and
2. \([\cdot, \cdot] : g \times g \to g\) is continuous,
will be called a *semi-infinite Lie algebra*.

Motivated by the discussion in Section 2.2, we may consider \(W\) as an abelian Lie algebra and construct extensions of \(W\) over \(v\). So suppose there is a skew-symmetric continuous bilinear mapping \(\omega : W \times W \to v\) and a continuous linear mapping \(\alpha : W \to \text{Der}(v)\) such that \(\alpha\) and \(\omega\) satisfy (B1) and (B2), which in this setting become
\[(C1) \quad [\alpha_X, \alpha_Y] = \text{ad}_{\omega(X,Y)}\]
and
\[(C2) \quad \alpha_X \omega(Y, Z) + \alpha_Y \omega(Z, X) + \alpha_Z \omega(X,Y) = 0,\]
for all \(X, Y, Z \in W\). Then we may define a Lie algebra structure on \(g := W \oplus v\) via the Lie bracket
\[([X_1, V_1], [X_2, V_2])_g := (0, \omega(X_1, X_2) + \alpha_{X_1} V_2 - \alpha_{X_2} V_1 + [V_1, V_2]_v).\]
The vector space \(g\) is also a Banach space in the norm
\[\|w, v\|_g := \|w\|_W + \|v\|_v,\]
and \(g_{CM} := H \oplus v\) is a Hilbert space with respect to the inner product
\[\langle (A, a), (B, b) \rangle_{g_{CM}} := \langle A, B \rangle_H + \langle a, b \rangle_v.\]
The associated Hilbertian norm on \(g_{CM}\) is given by
\[\|(A, a)\|_{g_{CM}} := \sqrt{\|A\|^2_H + \|a\|^2_v}.\]

**Notation 3.2.** Let
\[\|\omega\|_0 := \sup\{\|\omega(w_1, w_2)\|_v : \|w_1\|_W = \|w_2\|_W = 1\}\]
and
\[\|\alpha\|_0 := \sup\{\|\alpha w v\|_0 : \|w\|_W = \|v\|_v = 1\}\]
be the uniform norms of \(\omega\) and \(\alpha\), which are finite by their assumed continuity.
It will be useful to note that
\begin{equation}
(3.1) \quad \|\cdot\|_0 := \sup\{\|g_1, g_2\|_g : \|g_1\|_g = \|g_2\|_g = 1\} \leq \|\omega\|_0 + 2\|\alpha\|_0 + 1 < \infty,
\end{equation}
and similarly
\begin{equation}
(3.2) \quad C := C(\omega, \alpha) := \sup\{\|[h, k]\|_g : \|h\|_{g_{CM}} = \|k\|_{g_{CM}} = 1\} \leq \|\cdot\|_0 < \infty.
\end{equation}
Thus, for all \(\ell = 1, \cdots, r - 1\),
\[\|\text{ad}^\ell_k k\|_g \leq C^\ell \|h\|_{g_{CM}} \|k\|_{g_{CM}}.\]

If \(v\) is nilpotent, \(\omega\) and \(\alpha\) may be chosen so that \(g\) is a nilpotent Lie algebra (see Section 3.1 for some examples). For \(g\) nilpotent of step \(r\), the Baker-Campbell-Hausdorff-Dynkin formula implies that
\[\log(e^A e^B) = A + B + \sum_{k=1}^{r-1} \sum_{n,m \in I_k} a^k_{n,m} \text{ad}^n_A \text{ad}^m_B \text{ad}^n_A \text{ad}^m_B A,
\]
for all \(A, B \in g\), where
\begin{equation}
(3.3) \quad a^k_{n,m} := \frac{(-1)^k}{(k+1)m! n! (|n| + 1)}.
\end{equation}

\(I_k := \{(n, m) \in \mathbb{Z}^k \times \mathbb{Z}^k : n_i + m_i > 0 \text{ for all } 1 \leq i \leq k\}\), and for each multi-index \(n \in \mathbb{Z}^k\), \(n! = n_1! \cdots n_k!\) and \(|n| = n_1 + \cdots + n_k\), see, for example, [15]. Since \(g\) is nilpotent of step \(r\),
\[\text{ad}^n_A \text{ad}^m_B \text{ad}^n_A \text{ad}^m_B A = 0 \quad \text{if } |n| + |m| \geq r,
\]
for \(A, B \in g\). In particular, one may verify that
\begin{equation}
(3.4) \quad g \cdot h = g + h + \sum_{k=1}^{r-1} \sum_{n,m \in I_k} a^k_{n,m} \text{ad}^n_A \text{ad}^m_B \text{ad}^n_A \text{ad}^m_B g
\end{equation}
defines a group structure on \(g\). Note that \(g^{-1} = -g\) and the identity \(e = (0, 0)\).

**Definition 3.3.** When we wish to emphasize the group structure on \(g\), we will denote \(g\) by \(G\). Similarly, when we wish to view \(g_{CM}\) as a subgroup of \(G\), it will be denoted by \(G_{CM}\) and will be called the Cameron-Martin subgroup.

(Since \(g\) is simply connected and nilpotent, the exponential map is a global diffeomorphism (see, for example, Theorems 3.6.2 of [25] or 1.2.1 of [9]), and we may identify \(g\) and \(G\) under exponential coordinates. In particular, we may view \(g\) as both a Lie algebra and Lie group.)

**Lemma 3.4.** The Banach space topologies on \(g\) and \(g_{CM}\) make \(G\) and \(G_{CM}\) into topological groups.

**Proof.** Since \(g\) and \(g_{CM}\) are topological vector spaces, \(g \mapsto g^{-1} = -g\) and \((g_1, g_2) \mapsto g_1 + g_2\) are continuous by definition. The map \((g_1, g_2) \mapsto [g_1, g_2]\) is continuous in both the \(g\) and \(g_{CM}\) topologies by the estimates in equations (3.1) and (3.2). It then follows from (3.4) that \((g_1, g_2) \mapsto g_1 \cdot g_2\) is continuous as well. 

3.1. Examples. In this section, we give a few simple examples of semi-infinite Lie algebras.

**Example 3.5.** If \( v \) is a finite dimensional inner product space, we may consider \( v \) as an abelian Lie algebra, and taking \( \alpha \equiv 0 \) yields the infinite dimensional (step 2, stratified) Heisenberg like Lie algebras described in [10].

**Example 3.6.** Suppose \( v \) is an \( N \)-dimensional nilpotent Lie algebra. One standard way to construct Lie algebra extensions is as follows. Let \( \beta : W \to v \) be a continuous linear map, and define \( \alpha : W \to \text{Der}(v) \) as the inner derivation \( \alpha_X := \text{ad}_\beta(X) \). In this case, \( \Box 1 \) and \( \Box 2 \) are both satisfied if \( \omega : W \times W \to v \) is given by \( \omega(X,Y) := [\beta(X),\beta(Y)]_v \). Thus, \( g \) has Lie bracket

\[
[(X,V),(Y,U)]_\beta = (0,[\beta(X),\beta(Y)]_v + [\beta(X),U]_v - [\beta(Y),V]_v + [V,U]_v),
\]

and, if \( v \) is nilpotent, \( \alpha \) is a nilpotent Lie algebra of step \( r \), then \( g \) is nilpotent of step \( r \).

One should note for this construction that, since \( \beta \) is linear, we have the decomposition \( W = \text{Nul}(\beta) \oplus \text{Nul}(\beta)^\perp \), where \( \dim(\text{Nul}(\beta)^\perp) \leq \dim(v) = N \). Thus, for \( X = X_1 + X_2, Y = Y_1 + Y_2 \in W \),

\[
\omega(X_1 + X_2, Y_1 + Y_2) = [\beta(X_1 + X_2), \beta(Y_1 + Y_2)] = [\beta(X_2), \beta(Y_2)],
\]

and \( \omega \) is a map on \( \text{Nul}(\beta)^\perp \times \text{Nul}(\beta)^\perp \). Thus, \( \text{Nul}(\beta), \text{Nul}(\beta) = \{0\} \) and similarly \( \text{Nul}(\beta), v = \{0\} \). So

\[
g = W \oplus v = \text{Nul}(\beta) \oplus \text{Nul}(\beta)^\perp \oplus v
\]
is in a sense just an extension of the finite dimensional subspace \( \text{Nul}(\beta)^\perp \) by \( v \).

**Example 3.7.** One can generalize the previous example by taking a linear map \( \beta : W \to h \), where \( h \) is nilpotent Lie algebra, and constructing an extension of \( h \) by a nilpotent Lie algebra. For the sake of a concrete example, consider the following. Let

\[
W = W(\mathbb{R}^3) = \{\sigma : [0,1] \to \mathbb{R}^3 : \sigma \text{ is continuous and } \sigma(0) = 0\}
\]

and

\[
H = \left\{ \sigma \in W : \sigma \text{ is absolutely continuous and } \int_0^1 \|\dot{\sigma}(s)\|^2 \, ds < \infty \right\},
\]

so that \( (W,H) \) is standard Wiener space. Let \( v = \mathbb{R}^3 \) be an abelian Lie algebra. Let \( \sigma = \int_0^1 \sigma(s) \, ds = (\sigma_1, \sigma_2, \sigma_3) \), and define \( \omega : W \times W \to \mathbb{R}^3 \) by

\[
\omega(\sigma,\tau) = (\bar{\sigma}_1 \bar{\tau}_2 - \bar{\tau}_1 \bar{\sigma}_2, \bar{\sigma}_2 \bar{\tau}_3 - \bar{\tau}_2 \bar{\sigma}_3, 0)
\]

and \( \alpha_\sigma : \mathbb{R}^3 \to \mathbb{R}^3 \) by

\[
\alpha_\sigma(x,y,z) = (0,0,\bar{\sigma}_1 y - \bar{\sigma}_3 x).
\]

Then \( \alpha_\sigma \alpha_\tau = 0 \) and \( \Box 1 \) is trivially satisfied. Using that

\[
\alpha_\omega(\sigma,\tau) = (0,0,\bar{\sigma}_1 (\bar{\sigma}_2 \bar{\tau}_3 - \bar{\tau}_2 \bar{\sigma}_3) - \bar{\sigma}_3 (\bar{\sigma}_1 \bar{\tau}_2 - \bar{\tau}_1 \bar{\sigma}_2))
\]

one may verify that \( \Box 2 \) is satisfied. Thus, the Lie bracket for this extension \( g = W \oplus \mathbb{R}^3 \) is given by

\[
[(\sigma,v), (\tau,u)] = (0, \bar{\sigma}_1 \bar{\tau}_2 - \bar{\tau}_1 \bar{\sigma}_2, \bar{\sigma}_2 \bar{\tau}_3 - \bar{\tau}_2 \bar{\sigma}_3, \bar{\sigma}_1 u_2 - \bar{\sigma}_3 u_1 + \bar{\tau}_1 v_2 - \bar{\tau}_3 v_1),
\]

\[
[[\kappa,w], [(\sigma,v), (\tau,u)]] = (0,0,0, \bar{\kappa}_1 (\bar{\sigma}_2 \bar{\tau}_3 - \bar{\tau}_2 \bar{\sigma}_3) - \bar{\kappa}_3 (\bar{\sigma}_1 \bar{\tau}_2 - \bar{\tau}_1 \bar{\sigma}_2)),
\]

and all higher order brackets are 0.
Note that this construction corresponds to the extension $g = \mathbb{R}^3 \oplus \mathbb{R}^3$, the $4 \times 4$ upper triangular matrices. To see this, let $U = \mathbb{R}^3$ and $V = v = \mathbb{R}^3$, and define $\omega' : U \times U \to V$ by
\[ \omega((a, b, c), (a', b', c')) = \begin{pmatrix} 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & a' & 0 & 0 \\ 0 & 0 & b' & 0 \\ 0 & 0 & 0 & c' \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & ab' - ba' & 0 & 0 \\ 0 & 0 & be' - cb' & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \]
and $\alpha' : U \to gl(V)$ by
\[ \alpha_{(a,b,c)}(x, y, z) = \begin{pmatrix} 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & x & z & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & ay - cx & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \]
Then $\omega = \omega' \circ \beta$ and $\alpha = \alpha' \circ \beta$ where $\beta : W \to U$ is given by $\beta(\sigma) = (\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3)$.

**Example 3.8.** Consider $V = \mathbb{R}^n \oplus \mathbb{R}$ as an abelian Lie algebra. For $\omega : W \times W \to \mathbb{R}^n$, we may write $\omega = (\omega_1, \cdots, \omega_n)$, where $\omega_i : W \times W \to \mathbb{R}$ are bilinear, anti-symmetric, and continuous maps. Similarly, for $\alpha : W \times \mathbb{R}^n \to \mathbb{R}$, we have $\alpha_i(\cdot) = \alpha.e_i$, where $\{e_i\}_{i=1}^n$ is the standard basis for $\mathbb{R}^n$. Thus,
\[ \alpha_{\omega}(a_1, \ldots, a_n) = \sum_{i=1}^n a_i \alpha_i(\omega). \]
Then $\alpha$ and $\omega$ satisfy (C2) as long as
\[ \alpha_1 \wedge \omega_1 + \cdots + \alpha_n \wedge \omega_n = 0. \]

In the case $n = 1$, this is not very interesting, since $\alpha \wedge \omega = 0$ implies that $\omega = \alpha \wedge \beta$ for some $\beta \in W^*$. For $n = 2$, we have $V = \mathbb{R}^2 \oplus \mathbb{R}$. Let $\Omega : W \times W \to \mathbb{R}$ be bilinear, antisymmetric, and continuous, and $\gamma : W \to \mathbb{R}$ be linear and continuous. Then define $\omega : W \times W \to \mathbb{R}^2$ by $\omega = (\Omega, \Omega)$ and $\alpha : W \times \mathbb{R}^2 \to \mathbb{R}$ by $\alpha_1 = \gamma$ and $\alpha_2 = -\gamma$, so that for any $u, w \in W$ and $v = (v_1, v_2) \in \mathbb{R}^2$,
\[ \omega(w, u) = (\Omega(w, u), \Omega(w, u)) \quad \alpha \omega v = \gamma(w)(v_1 - v_2). \]
Note that, for any $w, u, h \in W$, $\omega$ and $\alpha$ satisfy
\[ \alpha_h \omega(w, u) = \alpha_h(\Omega(w, u), \Omega(w, u)) = \gamma(h)(\Omega(w, u) - \Omega(w, u)) = 0. \]
Thus, for any \((w, v, x), (w', v', x'), (w'', v'', x'')\) \(\in W \oplus v\),
\[
[(w, v, x), (w', v', x')] = (0, \omega(w, w'), \alpha_w v' - \alpha_{w'} v)
\]
\[
= (0, (\Omega(w, w'), \Omega(w, w')), \gamma(w)(v'_1 - v'_2) + \gamma(w')(v_1 - v_2)),
\]
\[
[(w'', v'', x''), [(w, v, x), (w', v', x')]] = (0, 0, \alpha_{w''} \omega(w, w')) = 0,
\]
and \(\mathfrak{g}\) is a step 2 Lie algebra. The group operation is given by
\[
(w, v) \cdot (w', v', x') = (w + w', v + v' + \frac{1}{2} \Omega(w, w'), \Omega(w, w')), \quad x + x' + \frac{1}{2} (\gamma(w)(v'_1 - v'_2) + \gamma(w')(v_1 - v_2)).
\]

As an example of a particular appropriate \(\Omega\) and \(\gamma\), again let \(W = W(\mathbb{R}^3)\) and \(H\) be as in Example [3.7]. Suppose \(\varphi\) is an anti-symmetric bilinear form on \(\mathbb{R}^3\), \(\rho : \mathbb{R}^3 \to \mathbb{R}\) is a linear map, and let \(\eta\) be finite measure on \([0,1]\). Then we may define
\[
\Omega(\sigma, \tau) = \int_0^1 \varphi(\sigma(s), \tau(s)) \, d\eta(s)
\]
and
\[
\gamma(\sigma) = \int_0^1 \rho(\sigma(s)) \, d\eta(s).
\]

**Example 3.9.** Here we make a slight modification on the previous example to construct a stratified step 3 Lie algebra. Let \(v = \mathbb{R}^6 = \mathbb{R}^3 \oplus \mathbb{R}^2 \oplus \mathbb{R}\) be an abelian Lie algebra. Let \(\Omega\) and \(\gamma\) be as in the previous example. Define \(\omega : W \times W \to \mathbb{R}^3\) by
\[
\omega(w, u) = (\Omega(w, u), \Omega(w, u), \Omega(w, u))
\]
and \(\alpha : W \times v \to v\) by
\[
\alpha_w((v_1, v_2, v_3), (x_1, x_2), y) = (0, (\gamma(w)(v_1 - v_2), \gamma(w)(v_2 - v_3)), \gamma(w)(x_1 - x_2))
\]
(so \(\alpha_w\) is a particular element of the \(6 \times 6\) strictly lower triangular matrices). Then
\[
\alpha_w \alpha_u = \alpha_u \alpha_w \quad \text{and so} \quad \alpha \text{ satisfies (C1)}, \quad \text{and also}
\]
\[
\alpha_{w'} \omega(w, u) = (0, (\gamma(v)(\Omega(w, u) - \Omega(w, u)), \gamma(v)(\Omega(w, u) - \Omega(w, u)), 0) = 0,
\]
so \(\alpha\) and \(\omega\) satisfy (C2) trivially. The Lie bracket is given by
\[
[(w, v, x, y), (w', v', x', y')] = (0, \omega(w, w'), \alpha_w v' - \alpha_{w'} v, \alpha_w x' - \alpha_{w'} x),
\]
or, more explicitly, this may be written componentwise as
\[
[(w, v, x, y), (w', v', x', y')]_2 = (\Omega(w, w'), \Omega(w, w'), \Omega(w, w')) \in \mathbb{R}^3,
\]
\[
[(w, v, x, y), (w', v', x', y')]_3
\]
\[
= (\gamma(w)(v'_1 - v'_2) - \gamma(w')(v_1 - v_2), \gamma(w)(v'_2 - v'_3) - \gamma(w')(v_2 - v_3)) \in \mathbb{R}^2,
\]
and
\[
[(w, v, x, y), (w', v', x', y')]_4 = \gamma(w)(x'_1 - x'_2) - \gamma(w')(x_1 - x_2) \in \mathbb{R}.
\]
Thus,
\[
[(w'', v'', x'', y''), ([w, v, x, y), (w', v', x', y')]]
= (0, 0, \alpha_w \omega(w, w'), \alpha_w \omega'(\alpha_v v' - \alpha_w v))
= (0, 0, \alpha_w \omega \alpha_v v' - \alpha_w \omega \alpha_w v)
= (0, 0, 0, \gamma(w'')(\gamma(v_1 - v_3)) - \gamma(w')\gamma(w')(v_1 - v_3)),
\]
and all higher order brackets are 0. So for \( g = (w, v, x, y) \) and \( g' = (w', v', x', y') \),
the group operation is given by
\[
(g \cdot g')_1 = w + w'
\]
\[
(g \cdot g')_2 = v + v' + \frac{1}{2} \omega(w, w')
\]
\[
(g \cdot g')_3 = x + x' + \frac{1}{2} (\alpha_w v' - \alpha_v v)
\]
\[
(g \cdot g')_4 = y + y' + \frac{1}{2} (\alpha_w x' - \alpha_v x) + \frac{1}{12} (\alpha_w^2 v' + \alpha_v^2 v - \alpha_w \alpha_v (v - v')).
\]

Clearly, this example may be further modified to make nilpotent Lie algebras of arbitrary step.

### 3.2. Hilbert-Schmidt norms

In this section, we will show that the assumed continuity of \( \omega \) and \( \alpha \) makes the Lie bracket into a Hilbert-Schmidt operator on \( \mathfrak{g}_{\mathcal{C}M} \). This result will be needed later in guaranteeing that our stochastic integrals are well-defined.

**Notation 3.10.** Let \( H_1, \ldots, H_n \) and \( V \) be Hilbert spaces, and let \( \{h_j^i\}_{j=1}^{\dim(H_i)} \) denote an orthonormal basis for each \( H_i \). If \( \rho : H_1 \times \cdots \times H_n \to V \) is a multilinear map, then the Hilbert-Schmidt norm of \( \rho \) is defined by
\[
\|\rho\|_2^2 := \|\rho|_{H_1^* \otimes \cdots \otimes H_n^* \otimes V} = \sum_{j_1, \ldots, j_n} \|\rho(h_{j_1}^1, \ldots, h_{j_n}^n)\|_V^2.
\]

In particular, for \( H \) an infinite dimensional Hilbert space with orthonormal basis \( \{h_i\}_{i=1}^{\infty} \), \( \rho : H^* \otimes \cdots \otimes V \) is Hilbert-Schmidt if
\[
\|\rho\|_2^2 = \|\rho|_{(H^*)^n \otimes V} = \sum_{j_1, \ldots, j_n=1}^{\infty} \|\rho(h_{j_1}, \ldots, h_{j_n})\|_V^2 < \infty.
\]

One may verify directly that these norms are independent of the chosen bases.

**Proposition 3.11.** For all \( w \in W \) and \( x \in \mathfrak{v} \),
\[
\|\alpha_w \cdot \|_{\mathfrak{v}^* \otimes W} \leq N\|\alpha\|_2^2 \|w\|_W^2 \quad \text{and} \quad \|\alpha \cdot x\|_{H^* \otimes \mathfrak{v}} \leq C_2\|\alpha\|_2^2 \|x\|_V^2,
\]
where \( C_2 \) is as in equation (2.24). Also,
\[
\|\omega(w, \cdot)\|_{H^* \otimes \mathfrak{v}} \leq C_2\|\omega\|_2 \|w\|_W^2.
\]

Furthermore,
\[
\|\alpha\|_2^2 \leq NC_2\|\alpha\|_0^2 < \infty \quad \text{and} \quad \|\omega\|_2^2 \leq C_2^2\|\omega\|_0^2 < \infty.
\]

**Proof.** Let \( \{e_i\}_{i=1}^{N} \) be an orthonormal basis of \( \mathfrak{v} \). Then, for any \( w \in W \),
\[
\|\alpha_w \cdot \|_{\mathfrak{v}^* \otimes W} = \sum_{i=1}^{N} \|\alpha_w e_i\|_w^2 \leq \sum_{i=1}^{N} \|\alpha\|_0^2 \|w\|_W^2 \|e_i\|_V^2 = N\|\alpha\|_0^2 \|w\|_W^2.
\]
For fixed \( x \in v, \alpha \cdot x : W \to v \) is a continuous linear map. Thus, equation (2.3) gives

\[
\|\alpha \cdot x\|_{H^s \otimes v}^2 = \int_W \|\alpha \cdot x\|_{v}^2 \, d\mu(w)
\]
\[
\leq \int_W \|\alpha\|_0^2 \|w\|_v^2 \|x\|_{v}^2 \, d\mu(w) = C_2 \|\alpha\|_0^2 \|x\|_v^2.
\]

Similarly, for fixed \( w \in W \) and \( \omega(w, \cdot) : W \to v \),

\[
\|\omega(w, \cdot)\|_{H^s \otimes v}^2 = \int_W \|\omega(w, w')\|_{v}^2 \, d\mu(w')
\]
\[
\leq \int_W \|\omega\|_0^2 \|w\|_v^2 \|w'\|_{v}^2 \, d\mu(w') = C_2 \|\omega\|_0^2 \|w\|_v^2.
\]

Since \( w \mapsto \alpha_w \) is a continuous linear map from \( W \) to \( v^* \otimes v \), it follows from equations (2.3) and (2.4) that

\[
\|\alpha\|_0^2 = \int_W \|\alpha_w \cdot w\|_{v^* \otimes v}^2 \, d\mu(w) \leq \int_W N \|\alpha\|_0^2 \|w\|_v^2 \, d\mu(w) = NC_2 \|\alpha\|_0^2,
\]

and since \( w \mapsto \omega(w, \cdot) \) is a continuous linear map from \( W \) to \( H^s \otimes v \),

\[
\|\omega\|_0^2 = \int_W \|\omega(h, \cdot)\|_{H^s \otimes v}^2 \, d\mu(h) \leq \int_W C_2 \|\omega\|_0^2 \|w\|_v^2 \, d\mu(w) = C_2^2 \|\omega\|_0^2.
\]

This proposition easily gives the following result.

**Corollary 3.12.** For all \( m \geq 2 \), \( [[[\cdot, \ldots, \cdot]]_{\otimes m} : \mathfrak{g}^M \to v \) is Hilbert-Schmidt.

**Proof.** For \( m = 2 \), this follows from the previous proposition and the continuity of the Lie bracket on \( v \), since taking \( \{h_i\}_{i=1}^\infty = \{k_i\}_{i=1}^\infty \cup \{e_j\}_{j=1}^N \), where \( \{k_i\}_{i=1}^\infty \) and \( \{e_j\}_{j=1}^N \) are orthonormal bases of \( H \) and \( v \), respectively, gives

\[
\|[\cdot, \cdot]\|_0^2 = \|[[\cdot, \ldots, \cdot]]_{\otimes 2} \mathfrak{g}^M \otimes \mathfrak{g}^M \otimes v = \sum_{i_1, i_2=1}^\infty \|[h_{i_1}, h_{i_2}]\|_0^2
\]
\[
= \sum_{i_1, i_2=1}^\infty \|\omega(k_{i_1}, k_{i_2})\|_0^2 + \sum_{i_1=1}^N \sum_{j_2=1}^N \|\alpha_{k_{i_1}} e_{j_2}\|_0^2
\]
\[
+ \sum_{i_2=1}^\infty \sum_{j_1=1}^N \|\alpha_{k_{i_2}} e_{j_1}\|_0^2 + \sum_{j_1, j_2=1}^N \|[e_{j_1}, e_{j_2}]\|_0^2
\]
\[
= \|\omega\|_0^2 + 2\|\alpha\|_0^2 + N < \infty.
\]

Now assume the statement is true for all \( m = 2, \ldots, \ell \). Consider \( m = \ell + 1 \). Writing \( [[[h_{i_1}, h_{i_2}], \ldots, h_{i_2}]] \in v \) in terms of the orthonormal basis \( \{e_j\}_{j=1}^N \) and using multiple
applications of the Cauchy-Schwarz inequality gives
\[
\|[[[\cdot], \ldots, \cdot]]\|_2^2 = \|[[[\cdot], \ldots, \cdot]](g_{CM}^{\otimes 2} \otimes v)
\]
\[
= \sum_{i_1, \ldots, i_{k+1}=1}^\infty \|[[[h_{i_1}, h_{i_2}], \ldots, h_{i_{k+1}}]]\|_2^2
\]
\[
= \sum_{i_1, \ldots, i_{k+1}=1}^\infty \left\| \sum_{j=1}^N \langle e_j, [h_{i_1}, h_{i_2}], \ldots, h_{i_{k+1}}] \rangle \right\|^2 \|e_j\|_2^2
\]
\[
\leq N \sum_{i_1, \ldots, i_{k+1}=1}^\infty \sum_{j=1}^N \|\langle e_j, [h_{i_1}, h_{i_2}], \ldots, h_{i_{k+1}}] \rangle\|^2 \|e_j\|_2^2 \|e_j\|_2^2
\]
\[
\leq N \left( \sum_{i_1, \ldots, i_{k+1}=1}^\infty \|\langle e_j, [h_{i_1}, h_{i_2}], \ldots, h_{i_{k+1}}] \rangle\|^2 \|e_j\|_2^2 \right) \left( \sum_{j=1}^N \sum_{i_1, \ldots, i_{k+1}=1}^\infty \|\langle e_j, [h_{i_1}, h_{i_2}], \ldots, h_{i_{k+1}}] \rangle\|^2 \right)
\]
\[
\leq N \left\| [[\cdot], \ldots, \cdot] \right\|^2_{g_{CM}^{\otimes 2} \otimes v} \left\| [[[\cdot], \ldots, \cdot]] \right\|^2_{g_{CM}^{\otimes 2} \otimes v},
\]
where in the penultimate inequality we have used that all terms in the sums are positive. The last line is finite by the induction hypothesis.

3.3. Length and distance. In this section, we define the Riemannian distance on $G_{CM}$ and show that the topology induced by this metric is equivalent to the Hilbert topology induced by $\| \cdot \|_{g_{CM}}$.

For $g \in G$, let $L_g : G \to G$ and $R_g : G \to G$ denote left and right multiplication by $g$, respectively. As $G$ is a vector space, to each $g \in G$ we can associate the tangent space $T_g G$ to $G$ at $g$, which is naturally isomorphic to $G$.

**Notation 3.13.** For $f : G \to \mathbb{R}$ a Fréchet smooth function and $v, x \in G$ and $h \in g$, let

\[
f'(x)_h := \partial_h f(x) = \frac{d}{dt} \bigg|_0 f(x + th),
\]

and let $v_x \in T_x G$ denote the tangent vector satisfying $v_x f = f'(x) v$. If $\sigma(t)$ is any smooth curve in $G$ such that $\sigma(0) = x$ and $\sigma(0) = v$ (for example, $\sigma(t) = x + tv$), then

\[
L_{gx} v_x = \frac{d}{dt} \bigg|_0 g \cdot \sigma(t).
\]

**Notation 3.14.** Let $T > 0$ and $C^1([0, T], G_{CM})$ denote the collection of $C^1$-paths $g : [0, T] \to G_{CM}$. The length of $g$ is defined as

\[
\ell_{CM}(g) := \int_0^T \| L_{g^{-1}(s)} g'(s) \|_{g_{CM}} ds.
\]

The Riemannian distance between $x, y \in G_{CM}$ then takes the usual form

\[
d_{CM}(x, y) := \inf \{ \ell_{CM}(g) : g \in C^1([0, T], G_{CM}) \text{ such that } g(0) = x \text{ and } g(T) = y \}.
\]

Note that the value of $T$ in the definition of $d_{CM}$ is irrelevant since the length functional is invariant under reparameterization.
Proposition 3.15. For \( g, x \in G \) and \( v_x \in T_xG \),

\[
L_g v_x = v + \sum_{k=1}^{r-1} \sum_{(n,m) \in I_k} a_{n,m}^k \times \sum_{j \in \{1, \ldots, k\}} \sum_{m_j > 0} \left( \text{ad} g \text{ad}_x \sum_{\ell=0}^{m_j-1} \text{ad}^\ell \text{ad}_x \cdots \text{ad}^n \text{ad}_x \right) g
\]

where \( a_{n,m}^k \) are the coefficients in the group multiplication given in equation (3.3).

**Proof.** The proof is a simple computation. Let \( x(t) = x + tv \), and first note that

\[
\frac{d}{dt} \bigg|_0 \text{ad}_g \text{ad}_x \sum_{\ell=0}^{m_j-1} \text{ad}^\ell \text{ad}_x \cdots \text{ad}^n \text{ad}_x g = \sum_{j \in \{1, \ldots, k\}} \sum_{m_j > 0} \left( \text{ad} g \text{ad}_x \sum_{\ell=0}^{m_j-1} \text{ad}^\ell \text{ad}_x \cdots \text{ad}^n \text{ad}_x \right) g.
\]

Then using (3.4) and plugging this into

\[
L_g v_x = \frac{d}{dt} \bigg|_0 g \cdot x(t)
\]

yields the desired result.

**Example 3.16 (The step 3 case).** When \( r = 3 \), the group operation is

\[ g \cdot h = g + h + \frac{1}{2} [g, h] + \frac{1}{12} ([g, [g, h]] + [h, [h, g]]). \]

Thus,

\[
L_g v_x = \frac{d}{dt} \bigg|_0 g \cdot x(t)
\]

\[
= \frac{d}{dt} \bigg|_0 \left( g + x(t) + \frac{1}{2} [g, x(t)] + \frac{1}{12} ([g, [g, x(t)]] + [x(t), [x(t), g]]) \right)
\]

\[
= v + \frac{1}{2} [g, v] + \frac{1}{12} ([g, [g, v]] + [v, [x, g]] + [x, [v, g]]).
\]

Proposition 3.17. There exists \( K_1 = K_1(a \land b) < \infty \) for \( a, b \geq 0 \) such that \( K_1(0) = 0 \) and, for all \( x, y \in G_{CM} \),

\[
d_{CM}(x, y) \leq (1 + K_1(\|x\|_{G_{CM}} \land \|y\|_{G_{CM}}))\|y - x\|_{G_{CM}} + o(\|y - x\|_{G_{CM}}^2).
\]
Proof. For notational simplicity, let $T = 1$. If $g(s)$ is a path in $C^1_{CM}$ for $0 \leq s \leq 1$, then, by equation (3.7), taking $g = g^{-1}(s)$, $x = g(s)$, and $v_{g(s)} = g'(s)$,

$$
\ell_{CM}(g) = \int_0^1 \left\| g'(s) + \sum_{k=1}^{r-1} \sum_{(n,m) \in \mathcal{I}_k} d_{n,m}^k \sum_{m_j > 0} \sum_{\ell = 0}^{m_j - 1} \text{ad}^{n_1}_g g^{-1}(s) \text{ad}^{m_1}_g \cdots \text{ad}^{n_j}_g g^{-1}(s) \text{ad}^{m_j}_g g^{-1}(s) \right\|_g ds 
$$

$$
= \int_0^1 \left\| g'(s) + \sum_{k=1}^{r-1} \sum_{(n,m) \in \mathcal{I}_k} (-1)^{n_1} 1_{\{m_k > 0\}} d_{n,m}^k \text{ad}^{n_1}_g g^{-1}(s) g'(s) \right\|_g ds 
$$

(3.8)

$$
= \int_0^1 \left\| g'(s) + \sum_{\ell=1}^{r-1} d_{\ell} \text{ad}^{\ell}_g g'(s) \right\|_g ds,
$$

where

$$
d_{\ell} := \sum_{k=1}^{\ell} \sum_{(n,m) \in \mathcal{I}_k} (-1)^{n_1} 1_{\{m_k > 0\}} d_{n,m}^k.
$$

Taking $g(s) = x + s(y - x)$ for $0 \leq s \leq 1$, this gives

$$
d_{CM}(x,y) \leq \ell_{CM}(g)
$$

$$
= \int_0^1 \left\| (y - x) + \sum_{\ell=1}^{r-1} d_{\ell} \text{ad}^{\ell}_x s(y - x) \right\|_{gCM} ds 
$$

$$
= \int_0^1 \left\| (y - x) + \sum_{\ell=1}^{r-1} d_{\ell} \sum_{(n,m) \in \mathcal{I}_k} s^{n_1} \text{ad}^{n_1}_x \cdots \text{ad}^{n_\ell}_x (y - x) \right\|_{gCM} ds.
$$

Splitting off all terms in the sum of order two or higher and evaluating the integral gives

$$
d_{CM}(x,y) \leq \left\| (y - x) + \sum_{\ell=1}^{r-1} d_{\ell} \text{ad}^{\ell}_x (y - x) \right\|_{gCM} 
$$

$$
+ \left\| \sum_{\ell=1}^{r-1} d_{\ell} \sum_{(n,m) \in \mathcal{I}_k} \frac{1_{\{n_1 > 0\}}}{|n| + 1} \text{ad}^{n_1}_x \cdots \text{ad}^{n_\ell}_x (y - x) \right\|_{gCM} 
$$

$$
\leq \left( 1 + \sum_{\ell=1}^{r-1} \sum_{(n,m) \in \mathcal{I}_k} \frac{1_{\{n_1 > 0\}}}{|n| + 1} \right) \|y - x\|_{gCM} + o \left( \|y - x\|_{gCM}^2 \right),
$$

where $C = C(\omega, \alpha)$ is as defined in (3.2). Interchanging the roles of $x$ and $y$ in $g(s)$, and thus in this inequality, completes the proof.

Notation 3.18. Let $\tau$ denote the norm topology on $G_{CM}$ and $\tau_d$ denote the topology induced by $d_{CM}$.
Proposition 3.19. For any \( y \in G \) and \( W \in \tau \) such that \( y \in W \), there exists \( U \in \tau_d \) such that \( y \in U \subset W \).

Proof. First we will show that, there exists \( \varepsilon_0 > 0 \) such that, for any \( x, y \in G_{CM} \) and \( \varepsilon \in (0, \varepsilon_0/2) \), if \( d_{CM}(x, y) < \varepsilon \), then \( \| x^{-1}y \|_{g_{CM}} < 2\varepsilon \). Then we will show that the continuity of the map \( x \mapsto \| x^{-1}y \|_{g_{CM}} \) (for fixed \( y \)) suffices to complete the proof.

Let \( d_\varepsilon \) be as in equation (3.9) and \( C = C(\omega, \alpha) \) be as in equation (3.2). Let

\[
\kappa := \sum_{\ell=1}^{r-1} |d_\varepsilon| C^\ell,
\]

and take \( \varepsilon := 1/2\kappa \wedge 1 \). Let \( B_{\varepsilon_0} := \{ x \in g_{CM} : \| x \|_{g_{CM}} \leq \varepsilon_0 \} \). Suppose \( y \in B_{\varepsilon_0} \), and let \( g : [0, 1] \to G_{CM} \) be a \( C^1 \)-path such that \( g(0) = e \) and \( g(1) = y \). Further, let \( T \in [0, 1] \) be the first time that \( g \) exits \( B_{\varepsilon_0} \), with the convention that \( T = 1 \) if \( g([0, 1]) \subset B_{\varepsilon_0} \). Then, by equation (3.8),

\[
\ell_{CM}(g2-) \geq \ell_{CM}(g|_{[0,T]})
\]

\[
\geq \int_0^T \| g'(s) \|_{g_{CM}} ds - \sum_{\ell=1}^{r-1} |d_\varepsilon| \left\| \text{ad}_{g(s)}^\ell g'(s) \right\|_{g_{CM}} ds
\]

\[
\geq \left( 1 - \sum_{\ell=1}^{r-1} |d_\varepsilon| C^\ell \epsilon \right) \int_0^T \| g'(s) \|_{g_{CM}} ds \geq (1 - \kappa \epsilon_0) \| g(T) \|_{g_{CM}} \geq \frac{\epsilon_0}{2} \| y \|_{g_{CM}}.
\]

Taking the infimum over \( g \) implies that

\[
d_{CM}(e, y) \geq \frac{\epsilon_0}{2} \| y \|_{g_{CM}}, \quad \text{for all } y \in B_{\varepsilon_0}.
\]

Now, if \( y \notin B_{\varepsilon_0} \), then the path \( g \) would have had to exit \( B_{\varepsilon_0} \) and \( \ell_{CM}(g) \geq \| g(T) \|_{g_{CM}} / 2 = \epsilon_0 / 2 \) implies that \( d_{CM}(e, y) \geq \epsilon_0 / 2 \). Thus,

\[
d_{CM}(e, y) \geq \frac{\epsilon_0}{2} \min(\epsilon_0, \| y \|_{g_{CM}}), \quad \text{for all } y \in G_{CM}.
\]

By the left invariance of \( d_{CM} \), this implies that, for any \( x, y \in G_{CM} \),

\[
d_{CM}(x, y) = d_{CM}(e, x^{-1}y) \geq \frac{\epsilon_0}{2} \min(\epsilon_0, \| x^{-1}y \|_{g_{CM}}).
\]

So if \( d_{CM}(x, y) < \epsilon_0 / 2 \), then \( \| x^{-1}y \|_{g_{CM}} \leq 2d_{CM}(x, y) \).

Now let \( W \in \tau \) (non-empty) and fix \( y \in W \). Recall that Lemma 3.4 implies that the map \( x \mapsto \| x^{-1}y \|_{g_{CM}} \) is \( \tau \)-continuous, and clearly \( \| x^{-1}y \|_{g_{CM}} = 0 \) if and only if \( x = y \). Thus,

\[
A_N(y) := \left\{ x : \| x^{-1}y \|_{g_{CM}} < \frac{1}{n} \right\} \downarrow \{ y \},
\]

and there exists \( N \) sufficiently large that \( 1/N < \epsilon_0 / 2 \) and \( A_N(y) \subset W \). Then

\[
B_N(y) := \left\{ x : d_{CM}(x, y) < \frac{1}{2N} \right\} \subset \tau_d
\]

satisfies \( B_N(y) \subset A_N(y) \), since \( x \in B_N(y) \) implies that \( d_{CM}(x, y) < 1/2N < \epsilon_0 / 4 \) and thus \( \| x^{-1}y \|_{g_{CM}} \leq 2d_{CM}(x, y) < 1/N \). Thus, \( B_N(y) \subset W \).

In particular, taking \( W = \{ x : \| y-x \|_{g_{CM}} < \delta \} \) for some \( \delta > 0 \) in this proposition, the proof implies that there exists \( N \) such that \( d_{CM}(x, y) < 1/2N \) implies that \( \| y-x \|_{g_{CM}} < \delta \). Propositions 3.17 and 3.19 give the following corollary.
Corollary 3.20. The topologies $\tau$ and $\tau_H$ are equivalent.

3.4. Ricci curvature. In this section, we compute the Ricci curvature of certain finite dimensional approximations of $G$ and show that it is bounded below uniformly. This result will be used in Section 5.1 to give $L^p$-bounds on Radon Nikodym derivatives of $\nu_t$. It will also be applied in Section 5.2 to prove a logarithmic Sobolev inequality for $\nu_t$.

Let $i : H \to W$ be the inclusion map, and $i^* : W^* \to H^*$ be its transpose. That is, $i^* \ell := \ell \circ i$ for all $\ell \in W^*$. Also, let $H_* := \{ h \in H : \langle \cdot, h \rangle_H \in \text{Range}(i^*) \subset H \}$. That is, for $h \in H$, $h \in H_*$ if and only if $\langle \cdot, h \rangle_H \in H^*$ extends to a continuous linear functional on $W$, which we will continue to denote by $\langle \cdot, h \rangle_H$. Because $H$ is a dense subspace of $W$, $i^*$ is injective and thus has a dense range. Since $h \mapsto \langle \cdot, h \rangle_H$ as a map from $H$ to $H^*$ is a conjugate linear isometric isomorphism, it follows that $H_* \ni h \mapsto \langle \cdot, h \rangle_H \in W^*$ is a conjugate linear isomorphism also, and so $H_*$ is a dense subspace of $H$.

Now suppose that $P : H \to H$ is a finite rank orthogonal projection such that $PH \subset H_*$. Let $\{ k_i \}_{i=1}^m$ be an orthonormal basis for $PH$. Then we may extend $P$ to a (unique) continuous operator from $W \to H$ (still denoted by $P$) by letting

$$Pw := \sum_{j=1}^m \langle w, k_j \rangle_H k_j \tag{3.10}$$

for all $w \in W$.

Notation 3.21. Let $\text{Proj}(W)$ denote the collection of finite rank projections on $W$ such that $PW \subset H_*$ and $P|_H : H \to H$ is an orthogonal projection, that is, $P$ has the form given in equation (3.10). Further, let $G_P := PW \oplus v$ (a subgroup of $G_{CM}$), and we equip $G_P$ with the left invariant Riemannian metric induced from the restriction of the inner product on $g_{CM} = H \oplus v$ to $\text{Lie}(G_P) = PH \oplus v =: g_{CM}^P$. Let $\text{Ric}^P$ denote the associated Ricci tensor at the identity in $G_P$.

Proposition 3.22. For $X = (A, a) \in g_{CM}^P$,

$$\langle \text{Ric}^P X, X \rangle_{g_{CM}^P} = \frac{1}{4} \| [a, [, ]] \|_{(g_{CM}^P)^* \otimes (g_{CM}^P)^*}^2 \left( \frac{1}{2} \| [, ] \|_{(g_{CM}^P)^* \otimes (g_{CM}^P)^*}^2 - \frac{1}{2} \| [a, X] \|_{(g_{CM}^P)^* \otimes (g_{CM}^P)^*}^2 \right),$$

where $(g_{CM}^P)^* = (PH)^* \oplus v^*$.

Proof. For $g$ any nilpotent Lie algebra with orthonormal basis $\Gamma$,

$$\langle \text{Ric} X, X \rangle = \frac{1}{4} \sum_{Y \in \Gamma} \| \text{ad}_Y^* X \|^2 - \frac{1}{2} \sum_{Y \in \Gamma} \| \text{ad}_Y X \|^2, \tag{3.11}$$

for all $X \in g$: see for example Theorem 7.30 and Corollary 7.33 of [5].

So let $\Gamma_m := \{ h_i \}_{i=1}^{m+N} = \{ (k_i, 0) \}_{i=1}^m \cup \{ (0, e_j) \}_{j=1}^N$ be an orthonormal basis of $g_{CM}^P = PH \oplus v$, where $\{ k_i \}_{i=1}^m$ and $\{ e_j \}_{j=1}^N$ are orthonormal bases of $PH$ and $v$, respectively. Then, for $Y \in g_{CM}^P$,

$$\text{ad}_Y X = \sum_{h_i \in \Gamma_m} \langle \text{ad}_Y^* X, h_i \rangle_{g_{CM}} h_i = \sum_{h_i \in \Gamma_m} \langle X, \text{ad}_Y h_i \rangle_{g_{CM}} h_i.$$
Thus,
\[
\sum_{h_i \in \Gamma_m} \| \text{ad} h_i X \|^2_{\mathfrak{g}_{CM}} = \sum_{h_i, h_j \in \Gamma_m} \langle X, \text{ad} h_i h_j \rangle_{\mathfrak{g}_{CM}}^2 = \sum_{h_i, h_j \in \Gamma_m} \langle X, [h_i, h_j] \rangle_{\mathfrak{g}_{CM}}^2.
\]
Plugging this into \(3.11\) gives
\[
\langle \text{Ric}^P X, X \rangle_{\mathfrak{g}^P_{CM}} = \frac{1}{4} \sum_{h_i, h_j \in \Gamma_m} \langle X, [h_i, h_j] \rangle_{\mathfrak{g}_{CM}}^2 - \frac{1}{2} \sum_{h_i \in \Gamma_m} \| [h_i, X] \|_{\mathfrak{g}_{CM}}^2
\]
\[
= \frac{1}{4} \sum_{h_i, h_j \in \Gamma_m} \langle a, [h_i, h_j] \rangle_{\mathfrak{g}}^2 - \frac{1}{2} \sum_{h_i \in \Gamma_m} \| [h_i, X] \|_{\mathfrak{g}}^2,
\]
\[\text{Corollary 3.23. Let } K := \frac{1}{2} \sup \{ \| [\cdot, \cdot] \|_{\mathfrak{g}_{CM} \otimes \mathfrak{v}}^2 : \| X \|_{\mathfrak{g}_{CM}} = 1 \}.\]
Then \(K > -\infty\) and \(K\) is the largest constant such that
\[
\langle \text{Ric}^P X, X \rangle_{\mathfrak{g}^P_{CM}} \geq K \| X \|_{\mathfrak{g}^P_{CM}}^2, \quad \text{for all } X \in \mathfrak{g}^P_{CM},
\]
holds uniformly for all \(P \in \text{Proj}(W)\).

\textbf{Proof.} The first assertion is simple, since
\[
K \geq \frac{1}{2} \| [\cdot, \cdot] \|_{\mathfrak{g}_{CM} \otimes \mathfrak{v}}^2 > -\infty,
\]
by Corollary 3.12. Now, for \(P \in \text{Proj}(W)\) as in Notation 3.21, Proposition 3.22 implies that
\[
\langle \text{Ric}^P X, X \rangle_{\mathfrak{g}^P_{CM}} \geq \frac{1}{2} \| [\cdot, X] \|_{\mathfrak{g}^P_{CM} \otimes \mathfrak{v}}^2.
\]
Thus,
\[
\frac{\langle \text{Ric}^P X, X \rangle_{\mathfrak{g}^P_{CM}}}{\| X \|_{\mathfrak{g}^P_{CM}}^2} \geq \frac{1}{2} \frac{\| [\cdot, X] \|_{\mathfrak{g}^P_{CM} \otimes \mathfrak{v}}^2}{\| X \|_{\mathfrak{g}^P_{CM}}^2}
\]
\[
\geq \frac{1}{2} \sup \{ \| [\cdot, \cdot] \|_{\mathfrak{g}^P_{CM} \otimes \mathfrak{v}}^2 : \| X \|_{\mathfrak{g}_{CM}} = 1 \} =: K_P.
\]
Noting that the infimum of \(K_P\) over all \(P \in \text{Proj}(W)\) is \(K\) completes the proof. \(\Box\)

\textbf{Remark 3.24.} Of course, one can compute the Ricci curvature for \(\mathfrak{g} = W \oplus \mathfrak{v}\) just as in Proposition 3.22. Choose an orthonormal basis \(\Gamma = \{ h_i \}_{i=1}^\infty = \{ (k_i, 0) \}_{i=1}^\infty \cup \{ (0, e_j) \}_{j=1}^N\) of \(\mathfrak{g}_{CM} = H \oplus \mathfrak{v}\), where \(\{ k_i \}_{i=1}^\infty\) is an orthonormal basis of \(H\), and \(\{ e_j \}_{j=1}^N\) is an orthonormal basis of \(\mathfrak{v}\). Then, for all \(X = (A, a) \in \mathfrak{g}_{CM}\),
\[
\langle \text{Ric} X, X \rangle_{\mathfrak{g}_{CM}} = \frac{1}{4} \sum_{i,j=1}^\infty \langle a, [h_i, h_j] \rangle_{\mathfrak{g}}^2 - \frac{1}{2} \sum_{i=1}^\infty \| [h_i, X] \|_{\mathfrak{g}}^2.
\]
\[
= \frac{1}{4} \| [a, [\cdot, \cdot]] \|_{\mathfrak{g}_{CM} \otimes \mathfrak{g}_{CM}}^2 - \frac{1}{2} \| [\cdot, X] \|_{\mathfrak{g}_{CM} \otimes \mathfrak{v}}^2 \geq K \| X \|_{\mathfrak{g}_{CM}}^2.
\]
4. Brownian motion

Suppose that $B_t$ is a smooth curve in $g_{CM}$ with $B_0 = 0$, and consider the differential equation

$$\dot{g}_t = L_{g_t^*} \dot{B}_t, \quad \text{with} \quad g_0 = e.$$  

The solution $g_t$ may be written as follows (see [24]): For $t > 0$, let $\Delta_n(t)$ denote the simplex in $\mathbb{R}^n$ given by

$$\{s = (s_1, \ldots, s_n) \in \mathbb{R}^n : 0 < s_1 < s_2 < \cdots < s_n < t\}.$$  

Let $S_n$ denote the permutation group on $(1, \cdots, n)$, and for each $\sigma \in S_n$, let $e(\sigma)$ denote the number of “errors” in the ordering $(\sigma(1), \sigma(2), \ldots, \sigma(n))$, that is, $e(\sigma) = \# \{j < n : \sigma(j) > \sigma(j + 1)\}$. Then

$$g_t = \sum_{n=1}^r \sum_{\sigma \in S_n} \left( (-1)^{e(\sigma)} / n! \right) n^2 \left[ n - 1 \choose e(\sigma) \right] \int_{\Delta_n(t)} [\cdots [\dot{B}_{\sigma(n)}], \ldots, \dot{B}_{\sigma(2)}], \dot{B}_{\sigma(1)}] ds.$$  

For $n \in \{1, \ldots, r\}$ and $\sigma \in S_n$, let $F_n^\sigma : g_{CM}^\otimes n \to \mathfrak{v}$ be the linear map given by

$$F_n^\sigma(k_1 \otimes \cdots \otimes k_n) := [\cdots [k_{\sigma(n)}], \cdots, k_{\sigma(2)}], k_{\sigma(1)}].$$  

Recall that $F_n^\sigma$ is Hilbert-Schmidt by Corollary 4.12. Then we may write

$$g_t = \sum_{n=1}^{r-1} \sum_{\sigma \in S_n} c_n^\sigma F_n^\sigma \left( \int_{\Delta_n(t)} \dot{B}_{s_1} \otimes \cdots \otimes \dot{B}_{s_n} ds \right).$$  

Using this as our motivation, we first explore stochastic integral analogues of equation (4.3) where the smooth curve $B$ is replaced by Brownian motion on $g$.

4.1. Multiple Itô integrals. Let $\langle \cdot, \cdot \rangle_{g_{CM}^\otimes n}$ denote the inner product on $g_{CM}^\otimes n$, arising from the inner product on $g_{CM}$. Also, let $\{k_i\}_{i=1}^\infty \subset H_*$ be an orthonormal basis of $H$, and define $P_m \in \text{Proj}(W)$ by

$$P_m(w) = \sum_{i=1}^m \langle w, k_i \rangle_H k_i, \quad \text{for all} \ w \in W,$$

as in equation (3.10), and define

$$\pi_m(w, x) := \pi_{P_m}(w, x) := (P_m(w), x) \in G_{P_m}.$$  

Of course, $\dim(G_{P_m}) = m + N$, but in a mild abuse of notation, we will use $\{h_i\}_{i=1}^m$ to denote an orthonormal basis of $G_{P_m}$, rather than the more cumbersome $\{h_i\}_{i=1}^{m+N} = \{(k_i, 0)\}_{i=1}^m \cup \{(0, e_i)\}_{i=1}^N$, where $\{e_i\}_{i=1}^N$ is a orthonormal basis of $\mathfrak{v}$.

Let $\{B_t\}_{t \geq 0} = \{ (\beta_t, \beta^g_t) \}_{t \geq 0}$ be a Brownian motion on $g = W \oplus \mathfrak{v}$ with variance determined by

$$\mathbb{E} \langle [B_s, h]_{g_{CM}}, [B_t, k]_{g_{CM}} \rangle = \langle h, k \rangle_{g_{CM}} \min(s, t),$$  

for all $s, t \geq 0$ and $h = (A, a)$ and $k = (C, c)$, such that $A, C \in H_*$ and $a, c \in \mathfrak{v}$. Then $\pi_m B = (P_m \beta, \beta^g)$ is a Brownian motion on $g^\otimes m = P_m W \oplus \mathfrak{v} \subset g_{CM}$. 
Proposition 4.1. For $\xi \in L^2(\Delta_n(t), \mathfrak{g}_{CM}^\otimes_n)$ a continuous mapping, let

$$J^m_n(\xi)_t := \int_{\Delta_n(t)} \langle \xi(s), d\pi_m B_{s_1} \otimes \cdots \otimes d\pi_m B_{s_n} \rangle_{\mathfrak{g}_{CM}^\otimes_n}.$$ 

Then $\{J^m_n(\xi)_t\}_{t \geq 0}$ is a continuous $L^2$-martingale such that, for all $m$,

$$\mathbb{E}|J^m_n(\xi)_t|^2 \leq \|\xi\|^2_{L^2(\Delta_n(t), \mathfrak{g}_{CM}^\otimes_n)},$$

and there exists a continuous $L^2$-martingale $\{J_n(\xi)_t\}_{t \geq 0}$ such that

$$\lim_{m \to \infty} \mathbb{E} \left[ \sup_{\tau \leq t} |J^m_n(\xi)_{\tau} - J_n(\xi)_{\tau}|^2 \right] = 0,$$

for all $t < \infty$. In particular,

$$J_n(\xi)_t := \int_{\Delta_n(t)} \langle \xi(s), dB_{s_1} \otimes \cdots \otimes dB_{s_n} \rangle_{\mathfrak{g}_{CM}^\otimes_n},$$

and $J_n(\xi)_t$ is well-defined independent of the choice of orthonormal basis $\{h_i\}_{i=1}^\infty$ in $\mathfrak{g}_{CM}$.

Proof. Note first that,

$$J^m_n(\xi)_t = \sum_{i_1, \ldots, i_n=1}^m \int_{\Delta_n(t)} \langle \xi(s), h_{i_1} \otimes \cdots \otimes h_{i_n} \rangle_{\mathfrak{g}_{CM}^\otimes_n} dB_{s_1}^i \cdots dB_{s_n}^i$$

where $\{B_i^j\}_{i=1}^m$ are independent real valued Brownian motions. Let $\xi_{i_1, \ldots, i_n} := \langle \xi, h_{i_1} \otimes \cdots \otimes h_{i_n} \rangle$. Then

$$|\xi_{i_1, \ldots, i_n}(s)|^2 \leq \|\xi(s)\|^2_{\mathfrak{g}_{CM}^\otimes_n}$$

and $\xi_{i_1, \ldots, i_n} \in L^2(\Delta_n(t))$. Thus, $J^m_n(\xi)_t$ is defined as a (finite dimensional) vector-valued multiple Wiener-Itô integral, see for example [18, 23].

Now note that

$$dJ^m_n(\xi)_t = \int_{\Delta_n-1(t)} \langle \xi(s_1, \ldots, s_{n-1}, t), d\pi_m B_{s_1} \otimes \cdots \otimes d\pi_m B_{s_{n-1}} \otimes d\pi_m B_t \rangle_{\mathfrak{g}_{CM}^\otimes_n}$$

$$=: \sum_{i=1}^m \int_{\Delta_n-1(t)} \langle \xi(s_1, \ldots, s_{n-1}, t), d\pi_m B_{s_1} \otimes \cdots \otimes d\pi_m B_{s_{n-1}} \otimes h_i \rangle_{\mathfrak{g}_{CM}^\otimes_n} dB_t^i.$$

Thus, the quadratic variation $\langle J^m(\xi) \rangle_t$ is given by

$$\sum_{i=1}^m \int_0^t \int_{\Delta_n-1(\tau)} \langle \xi(s_1, \ldots, s_{n-1}, \tau), d\pi_m B_{s_1} \otimes \cdots \otimes d\pi_m B_{s_{n-1}} \otimes h_i \rangle_{\mathfrak{g}_{CM}^\otimes_n}^2 d\tau,$$

and

$$\mathbb{E}|J^m_n(\xi)_t|^2 = \mathbb{E}\langle J^m_n(\xi) \rangle_t$$

$$= \sum_{i_1=1}^m \int_0^t \mathbb{E} \left[ \sum_{i_2=1}^m \int_0^{\tau_1} \int_{\Delta_n-2(\tau_2)} \langle \xi(s_1, \ldots, s_{n-2}, \tau_2, \tau_1), d\pi_m B_{s_1} \otimes \cdots \otimes d\pi_m B_{s_{n-2}} \otimes h_{i_2} \otimes h_{i_1} \rangle_{\mathfrak{g}_{CM}^\otimes_n} \right]^2 d\tau_1.$$
Iterating this procedure \( n \) times gives

\[
\mathbb{E}|J_n^m(\xi)_t|^2 = \sum_{i_1, \ldots, i_n=1}^{m} \int_{\Delta_n(t)} \left| \langle \xi(\tau_1, \ldots, \tau_n), h_{i_1} \otimes \cdots \otimes h_{i_n} \rangle_{\mathfrak{g}^\otimes M} \right|^2 d\tau_1 \cdots d\tau_n
\]

\[
= \int_{\Delta_n(t)} \| \pi_{\otimes n}^\otimes \xi(s) \|_{\mathfrak{g}^\otimes M}^2 \leq \| \xi \|_{L^2(\Delta_n(t), \mathfrak{g}^\otimes M)}^2,
\]

and thus, for each \( n \), \( J_n^m(\xi)_t \) is bounded uniformly in \( L^2 \) independent of \( m \).

A similar argument shows that the sequence \( \{ J_n^m(\xi)_t \}_{m=1}^{\infty} \) is Cauchy in \( L^2 \). For \( m \leq \ell \), consider

\[
J_n^\ell(\xi)_t - J_n^m(\xi)_t = \sum_{j=1}^{n} \int_{\Delta_n(t)} \langle \xi(s), d\pi_t B_{s_j} \otimes \cdots \otimes d\pi_t B_{s_{j-1}} \otimes d(\pi_t - \pi_m) B_{s_j} \otimes d\pi_m B_{s_{j+1}} \otimes \cdots \otimes d\pi_m B_{s_n} \rangle.
\]

Thus, applying Cauchy-Schwarz and computing as in equation (4.8),

\[
\mathbb{E}|J_n^\ell(\xi)_t - J_n^m(\xi)_t|^2 \leq n \sum_{j=1}^{n} \sum_{i_1, \ldots, i_{j-1}=1}^{\ell} \sum_{i_j=1}^{m+1} \sum_{i_{j+1, \ldots, i_n=1}^{\ell}} \int_{\Delta_n(t)} \left| \langle \xi(s), h_{i_1} \otimes \cdots \otimes h_{i_n} \rangle_{\mathfrak{g}^\otimes M} \right|^2 ds \rightarrow 0,
\]

as \( \ell, m \rightarrow \infty \), since

\[
\| \xi \|_{L^2(\Delta_n(t), \mathfrak{g}^\otimes M)}^2 = \int_{\Delta_n(t)} \| \xi(s) \|_{\mathfrak{g}^\otimes M}^2 ds
\]

\[
= \int_{\Delta_n(t)} \sum_{i_1, \ldots, i_n=1}^{\infty} \left| \langle \xi(s), h_{i_1} \otimes \cdots \otimes h_{i_n} \rangle_{\mathfrak{g}^\otimes M} \right|^2 ds < \infty.
\]

Since the space of continuous \( L^2 \)-martingales is complete in the norm \( M \rightarrow \mathbb{E}|M_t|^2 \), there exists a continuous martingale \( \{ X_t \}_{t \geq 0} \) such that

\[
\lim_{m \rightarrow \infty} \mathbb{E}|J_n^m(\xi)_t - X_t|^2 = 0.
\]

To see that \( X_t \) is independent of basis, suppose now that \( \{ h'_j \}_{j=1}^{\infty} \subset H_{\ast} \) is another orthonormal basis for \( H \) and \( P'_m : W \rightarrow H_{\ast} \) and \( \pi'_m : G \rightarrow G_{P'_m} \) are the corresponding orthogonal projections, that is,

\[
P'_m w := \sum_{i=1}^{m} \langle w, h'_i \rangle_W h'_i,
\]

and \( \pi'_m(w, x) = (P'_m w, x) \). Let

\[
J_n^{m'}(\xi)_t = \int_{\Delta_n(t)} \langle \xi(s), d\pi'_m B_{s_1} \otimes \cdots \otimes d\pi'_m B_{s_n} \rangle_{\mathfrak{g}^\otimes M'}.
\]
Then, using equation (4.9) with \( \pi_t \) replaced by \( \pi'_m \), applying Cauchy-Schwarz, and again computing as in (4.8), gives

\[
E \left| J^m_n (\xi_t) - J^{m'}_n (\xi_t) \right|^2 \leq n \sum_{j=1}^{\infty} \int_{\Delta_n(t)} \sum_{i_1, \ldots, i_n=1}^{\infty} \left| \langle \xi(s), \pi_m h_{i_1} \otimes \cdots \otimes \pi_m h_{i_j-1} \otimes (\pi_m - I) h_{i_j} \otimes \pi'_m h_{i_{j+1}} \otimes \cdots \otimes \pi'_m h_{i_n} \rangle \right|^2 ds.
\]

Writing \( \pi_m - \pi'_m = (\pi_m - I) + (I - \pi'_m) \), and considering terms for each fixed \( j \), we have

\[
\int_{\Delta_n(t)} \sum_{i_1, \ldots, i_n=1}^{\infty} \left| \langle \xi(s), \pi_m h_{i_1} \otimes \cdots \otimes \pi_m h_{i_{j-1}} \otimes (\pi_m - I) h_{i_j} \otimes \pi'_m h_{i_{j+1}} \otimes \cdots \otimes \pi'_m h_{i_n} \rangle \right|^2 ds
\]

\[
= \int_{\Delta_n(t)} \sum_{i_1, \ldots, i_j=1}^{m} \sum_{i_{j+1}=m+1}^{\infty} \sum_{i_{j+2}, \ldots, i_n=1}^{\infty} \left| \langle \xi(s), h_{i_1} \otimes \cdots \otimes h_{i_{j-1}} \otimes h_{i_j} \otimes \pi'_m h_{i_{j+1}} \otimes \cdots \otimes \pi'_m h_{i_n} \rangle \right|^2 ds
\]

\[
\leq \int_{\Delta_n(t)} \sum_{i_1, \ldots, i_j=1}^{m} \sum_{i_{j+1}=m+1}^{\infty} \sum_{i_{j+2}, \ldots, i_n=1}^{\infty} \left| \langle \xi(s), h_{i_1} \otimes \cdots \otimes h_{i_{j-1}} \otimes \pi'_m h_{i_{j+1}} \otimes \cdots \otimes \pi'_m h_{i_n} \rangle \right|^2 ds \to 0,
\]
as \( m \to \infty \). Similarly,

\[
\int_{\Delta_n(t)} \sum_{i_1, \ldots, i_n=1}^{\infty} \left| \langle \xi(s), \pi_m h_{i_1} \otimes \cdots \otimes \pi_m h_{i_{j-1}} \rangle \right|^2 ds
\]

\[
= \int_{\Delta_n(t)} \sum_{i_1, \ldots, i_j=1}^{m} \sum_{i_{j+1}=m+1}^{\infty} \sum_{i_{j+2}, \ldots, i_n=1}^{\infty} \left| \langle \xi(s), h_{i_1} \otimes \cdots \otimes h_{i_{j-1}} \otimes \pi'_m h_{i_{j+1}} \otimes \cdots \otimes \pi'_m h_{i_n} \rangle \right|^2 ds
\]

\[
\leq \int_{\Delta_n(t)} \sum_{i_1, \ldots, i_j=1}^{m} \sum_{i_{j+1}=m+1}^{\infty} \sum_{i_{j+2}, \ldots, i_n=1}^{\infty} \left| \langle \xi(s), h_{i_1} \otimes \cdots \otimes h_{i_{j-1}} \otimes \pi'_m h_{i_{j+1}} \otimes \cdots \otimes \pi'_m h_{i_n} \rangle \right|^2 ds \to 0,
\]
as \( m \to \infty \). Thus,

\[
\lim_{m \to \infty} E \left| J^m_n (\xi_t) - J^{m'}_n (\xi_t) \right|^2 = 0.
\]
and $X_t$ is independent of the choice of orthonormal basis. In particular, replacing $J_n(\xi)$ in (4.10) by $J_n(\xi)_t$ as given in equation (4.7), and taking the limit as $m \to \infty$, shows that $X_t = J_n(\xi)_t$ satisfies (4.11). Combining this with Doob’s maximal inequality proves equation (4.6).

A simple linearity argument extends the map $J_n$ to functions taking values in $(g_{CM})^\otimes n \otimes \mathfrak{v}$.

**Corollary 4.2.** Let $F \in L^2(\Delta_n(t), (g_{CM})^\otimes n \otimes \mathfrak{v})$ be a continuous map. That is, $F : \Delta_n(t) \times g_{CM}^\otimes n \to \mathfrak{v}$ is a map continuous in $s$ and linear on $g_{CM}^\otimes n$ such that

$$\int_{\Delta_n(t)} \|F(s)\|^2 ds = \int_{\Delta_n(t)} \sum_{j_1, \ldots, j_n=1}^\infty \|F(s)(h_{j_1} \otimes \cdots \otimes h_{j_n})\|^2_\mathfrak{v} ds < \infty.$$  

Then

$$J_n^m(F)_t := \int_{\Delta_n(t)} F(s)(d\pi_m B_{s_1} \otimes \cdots \otimes d\pi_m B_{s_n})$$  

is a continuous $L^2$-martingale, and there exists a continuous $\mathfrak{v}$-valued $L^2$-martingale \( \{J_n(F)_t\}_{t \geq 0} \) such that

$$\lim_{m \to \infty} \mathbb{E} \left[ \sup_{\tau \leq t} \|J_n^m(\xi)\_\tau - J_n(\xi)\_\tau\|^2_\mathfrak{v} \right] = 0,$$

for all $t < \infty$. The martingale $J_n(\xi)_t$ is well-defined independent of the choice of orthonormal basis \( \{h_i\}_{i=1}^\infty \) in (4.4), and will be denoted by

$$J_n(F)_t := \int_{\Delta_n(t)} F(s)(dB_{s_1} \otimes \cdots \otimes dB_{s_n}).$$

**Proof.** Let \( \{e_j\}_{j=1}^N \) be an orthonormal basis of $\mathfrak{v}$. Then for any $k_1, \ldots, k_n \in g_{CM}$,

$$F(s)(k_1 \otimes \cdots \otimes k_n) = \sum_{j=1}^N \langle F(s)(k_1 \otimes \cdots \otimes k_n), e_j \rangle e_j.$$  

Since \( \langle F(s)(\cdot), e_j \rangle \) is linear on $g_{CM}^\otimes n$, for each $s$ there exists $\xi_j(s) \in g_{CM}^\otimes n$ such that

$$\langle \xi_j(s), k_1 \otimes \cdots \otimes k_n \rangle = \langle F(s)(k_1 \otimes \cdots \otimes k_n), e_j \rangle.$$  

If $\xi_j : \Delta_n(t) \to g_{CM}^\otimes n$ is defined by equation (4.12), then

$$\|\xi_j\|_{L^2(\Delta_n(t), g_{CM}^\otimes n)} \leq \int_{\Delta_n(t)} \|F(s)\|^2_\mathfrak{v} ds < \infty.$$  

Thus,

$$J_n(F)_t = \sum_{j=1}^N \int_{\Delta_n(t)} \langle \xi_j(s), dB_{s_1} \otimes \cdots \otimes dB_{s_n} \rangle e_j = \sum_{j=1}^N J_n(\xi_j)e_j,$$

is well-defined, and, for each $j$, $J_n(\xi_j)$ is a martingale as defined in Proposition 4.1.
4.2. Brownian motion and finite dimensional approximations. Again let $B_t$ denote Brownian motion on $\mathfrak{g}$. By equation (4.1), the solution to the Stratonovich stochastic differential equation

$$\delta g_t = L_{g_t^*} \delta B_t, \quad \text{with } g_0 = e,$$

should be given by

$$g_t = \sum_{n=1}^{r-1} \sum_{\sigma \in S_n} c_n^\sigma \int_{\Delta_n(t)} \left[ \cdots [\delta B_{\sigma(1)}, \delta B_{\sigma(2)}], \cdots, \delta B_{\sigma(n)} \right],$$

for coefficients $c_n^\sigma$ determined by equation (4.1).

To see that this process is well-defined, consider the following. Let $\{M_n(t)\}_{t \geq 0}$ denote the process in $\mathfrak{g} \otimes^n$ defined by

$$M_n(t) := \int_{\Delta_n(t)} \delta B_{s_1} \otimes \cdots \otimes \delta B_{s_n}.$$ 

By repeatedly applying the definition of the Stratonovich integral, the iterated Stratonovich integral $M_n(t)$ may be realized as a linear combination of iterated Itô integrals:

$$M_n(t) = \sum_{m=\lfloor n/2 \rfloor}^{n} \frac{1}{2^{n-m}} \sum_{\alpha \in J_m^n} I_n^\alpha(\alpha),$$

where

$$J_m^n := \left\{ (\alpha_1, \ldots, \alpha_m) \in \{1, 2\}^m : \sum_{i=1}^{m} \alpha_i = n \right\},$$

and, for $\alpha \in J_m^n$, $I_n^\alpha(\alpha)$ is the iterated Itô integral

$$I_n^\alpha(\alpha) = \int_{\Delta_n(t)} dX_{s_1}^1 \otimes \cdots \otimes dX_{s_m}^m$$

with

$$dX_s^i = \begin{cases} dB_s & \text{if } \alpha_i = 1 \\ \sum_{j=1}^{\infty} h_j \otimes h_j ds & \text{if } \alpha_i = 2 \end{cases}.$$

compare with Proposition 1 of [4].

As in equation (4.2), letting

$$F_n^\sigma(k_1 \otimes \cdots \otimes k_n) := \left[ \cdots [k_{\sigma(1)}, k_{\sigma(2)}], \cdots, k_{\sigma(n)} \right],$$

we may write

$$g_t = \sum_{n=1}^{r-1} \sum_{\sigma \in S_n} c_n^\sigma F_n^\sigma(M_n(t))$$

$$= \sum_{n=1}^{r-1} \sum_{\sigma \in S_n} \sum_{m=\lfloor n/2 \rfloor}^{n} \frac{1}{2^{n-m}} \sum_{\alpha \in J_m^n} F_n^\sigma(I_n^\alpha(\alpha)),$$

presuming the integrals $F_n^\sigma(I_n^\alpha(\alpha))$ are defined.

For each $\alpha$, let $p_n = \# \{ i : \alpha_i = 1 \}$ and $q_n = \# \{ i : \alpha_i = 2 \}$ (so that $p_n + q_n = m$ when $\alpha \in J_m^n$), and let

$$J_n := \bigcup_{m=\lfloor n/2 \rfloor}^{n} J_m^n.$$
Then, for each \( \sigma \in \mathcal{S}_n \) and \( \alpha \in \mathcal{J}_n \),

\[
F_n^\sigma(I^n_\tau(\alpha)) = \int_{\Delta_{p_n}(t)} f_\alpha(s, t) \tilde{F}_n^{\sigma, \alpha}(dB_{s_1} \otimes \cdots \otimes dB_{s_{p_n}}),
\]

where \( \tilde{F}_n^{\sigma, \alpha} \) and \( f_\alpha \) are defined as follows. \( \tilde{F}_n^{\sigma, \alpha} : g^{\otimes p_n} \to g \) is defined by

\[
(4.13) \quad \tilde{F}_n^{\sigma, \alpha}(k_1 \otimes \cdots \otimes k_{p_n}) := \sum_{j_1, \ldots, j_{q_\alpha} = 1}^\infty F_n^{\sigma, \prime}(k_1 \otimes \cdots \otimes k_{p_n} \otimes h_{j_1} \otimes h_{j_1} \otimes \cdots h_{j_{q_\alpha}} \otimes h_{j_{q_\alpha}}),
\]

for \( \{h_j\}_{j=1}^\infty \) an orthonormal basis of \( g_{CM} \) and \( \sigma' = \sigma'(\alpha) \in \mathcal{S}_n \) given by \( \sigma' = \sigma \circ \tau^{-1} \), for any \( \tau \in \mathcal{S}_n \) such that

\[
\tau(dX_1 \otimes \cdots \otimes dX_m) = \sum_{j_1, \ldots, j_{q_\alpha} = 1}^\infty dB_{s_1} \otimes \cdots \otimes dB_{s_{p_n}} \otimes h_{j_1} \otimes h_{j_1} \otimes \cdots h_{j_{q_\alpha}} \otimes h_{j_{q_\alpha}} ds_1 \ldots ds_{q_\alpha}.
\]

The function \( f_\alpha \) is a polynomial of order \( q_\alpha \) in \( s = (s_1, \ldots, s_{p_n}) \) and \( t \). Thus, \( f_\alpha \) may be written as

\[
(4.14) \quad f_\alpha(s, t) = \sum_{a=0}^{q_\alpha} b_\alpha^a t^a \tilde{f}_{\alpha, a}(s),
\]

for some coefficients \( b_\alpha^a \in \mathbb{R} \) and polynomials \( \tilde{f}_{\alpha, a} \) of degree \( q_\alpha - a \) in \( s \). If \( \tilde{F}_n^{\sigma, \alpha} \) is Hilbert-Schmidt on \( g_{CM}^{\otimes p_n} \), then

\[
\int_{\Delta_{p_n}(t)} \left\| \tilde{f}_{\alpha, a}(s) \tilde{F}_n^{\sigma, \alpha} \right\|_2^2 ds = \left\| \tilde{f}_{\alpha, a} \right\|_{L^2(\Delta_{p_n}(t))} \left\| \tilde{F}_n^{\sigma, \alpha} \right\|_2^2 < \infty,
\]

and Corollary 4.2 implies that

\[
(4.15) \quad F_n^\sigma(I^n_\tau(\alpha)) = \sum_{a=0}^{q_\alpha} k_\alpha^a t^a J_n(\tilde{f}_{\alpha, a} \tilde{F}_n^{\sigma, \alpha})_t
\]

is well-defined. In particular, if \( \alpha_m = 1 \), then \( f_\alpha = f_\alpha(s) \) does not depend on \( t \), and Corollary 4.2 implies that \( F_n^\sigma(I^n_\tau(\alpha)) \) is a \( v \)-valued \( L^2 \)-martingale.

The next two results show that \( \tilde{F}_n^{\sigma, \alpha} \) is Hilbert-Schmidt as desired.

**Lemma 4.3.** Let \( n \in \{2, \ldots, r\} \), \( \sigma \in \mathcal{S}_n \), and \( \alpha \in \mathcal{J}_n \). For any \( v \in \mathfrak{v} \), \( \langle \tilde{F}_n^{\sigma, \alpha}, v \rangle \) is a Hilbert-Schmidt operator on \( g_{CM}^{\otimes p_n} \).

**Proof.** First consider the case \( n = 2 \). In this case, \( p_\alpha = 0 \) or \( p_\alpha = 2 \). If \( p_\alpha = 0 \), then \( \tilde{F}_2^{\sigma, \alpha} = \sum_{i=1}^\infty F_2^{\sigma'}(h_i \otimes h_i) = 0 \). If \( p_\alpha = 2 \), then \( \tilde{F}_2^{\sigma, \alpha}(k_1 \otimes k_2) = F_2^{\sigma'}(k_1 \otimes k_2) = [k_{\sigma(1)}, k_{\sigma(2)}] \) is Hilbert-Schmidt by Corollary 3.12 and thus \( \langle \tilde{F}_2^{\sigma, \alpha}, v \rangle \) is Hilbert-Schmidt. For \( n = 3 \), \( p_\alpha = 1 \) or \( p_\alpha = 3 \). If \( p_\alpha = 3 \), then \( \alpha = (1, 1, 1) \) and

\[
\tilde{F}_3^{\sigma, \alpha}(k_1 \otimes k_2 \otimes k_3) = F_3^{\sigma'}(k_1 \otimes k_2 \otimes k_3) = [[k_{\sigma(1)}, k_{\sigma(2)}], k_{\sigma(3)}]
\]

is Hilbert-Schmidt, again by Corollary 3.12. If \( p_\alpha = 1 \), then \( \alpha = (1, 2) \) or \( \alpha = (2, 1) \) and

\[
\tilde{F}_3^{\sigma, \alpha}(k) = \sum_{i=1}^{\infty} F_3^{\sigma'}(k \otimes h_i \otimes h_i),
\]

is Hilbert-Schmidt. For \( n = 4 \), \( p_\alpha = 2 \). If \( p_\alpha = 2 \), then \( \tilde{F}_4^{\sigma, \alpha}(k_1 \otimes k_2 \otimes k_3 \otimes k_4) = [k_{\sigma(1)}, k_{\sigma(2)}, k_{\sigma(3)}] \) is Hilbert-Schmidt, again by Corollary 3.12. If \( p_\alpha = 1 \), then \( \alpha = (1, 2, 1) \) or \( \alpha = (2, 1, 1) \) and

\[
\tilde{F}_4^{\sigma, \alpha}(k) = \sum_{i=1}^{\infty} F_4^{\sigma'}(k \otimes h_i \otimes h_i \otimes h_i),
\]

is Hilbert-Schmidt. For \( n = 5 \), \( p_\alpha = 3 \). If \( p_\alpha = 3 \), then \( \alpha = (1, 2, 2) \) or \( \alpha = (2, 1, 2) \) and

\[
\tilde{F}_5^{\sigma, \alpha}(k_1 \otimes k_2 \otimes k_3 \otimes k_4 \otimes k_5) = F_5^{\sigma'}(k_1 \otimes k_2 \otimes k_3 \otimes k_4 \otimes k_5) = [[k_{\sigma(1)}, k_{\sigma(2)}], k_{\sigma(3)}] \]

is Hilbert-Schmidt, again by Corollary 3.12. If \( p_\alpha = 2 \), then \( \alpha = (1, 2, 1, 1) \) or \( \alpha = (2, 1, 1, 1) \) and

\[
\tilde{F}_5^{\sigma, \alpha}(k) = \sum_{i=1}^{\infty} F_5^{\sigma'}(k \otimes h_i \otimes h_i \otimes h_i \otimes h_i),
\]

is Hilbert-Schmidt. For \( n = 6 \), \( p_\alpha = 4 \). If \( p_\alpha = 4 \), then \( \alpha = (1, 2, 2, 1) \) or \( \alpha = (2, 1, 2, 1) \) and

\[
\tilde{F}_6^{\sigma, \alpha}(k_1 \otimes k_2 \otimes k_3 \otimes k_4 \otimes k_5 \otimes k_6) = F_6^{\sigma'}(k_1 \otimes k_2 \otimes k_3 \otimes k_4 \otimes k_5 \otimes k_6) = [[k_{\sigma(1)}, k_{\sigma(2)}], k_{\sigma(3)}] \]

is Hilbert-Schmidt, again by Corollary 3.12. If \( p_\alpha = 3 \), then \( \alpha = (1, 2, 2, 1) \) or \( \alpha = (2, 1, 2, 1) \) and

\[
\tilde{F}_6^{\sigma, \alpha}(k) = \sum_{i=1}^{\infty} F_6^{\sigma'}(k \otimes h_i \otimes h_i \otimes h_i \otimes h_i \otimes h_i),
\]

is Hilbert-Schmidt. For \( n = 7 \), \( p_\alpha = 5 \). If \( p_\alpha = 5 \), then \( \alpha = (1, 2, 2, 2) \) or \( \alpha = (2, 1, 2, 2) \) and

\[
\tilde{F}_7^{\sigma, \alpha}(k) = \sum_{i=1}^{\infty} F_7^{\sigma'}(k \otimes h_i \otimes h_i \otimes h_i \otimes h_i \otimes h_i \otimes h_i),
\]

is well-defined. In particular, if \( \alpha_m = 1 \), then \( f_\alpha = f_\alpha(s) \) does not depend on \( t \), and Corollary 4.2 implies that \( F_n^\sigma(I^n_\tau(\alpha)) \) is a \( v \)-valued \( L^2 \)-martingale.

The next two results show that \( \tilde{F}_n^{\sigma, \alpha} \) is Hilbert-Schmidt as desired.
and we need only consider the case that 

\[ F_3^{\sigma'}(k \otimes h \otimes h) = \lbrack h, k \rbrack. \]

So let \( \{k_i\}_{i=1}^\infty \) be an orthonormal basis of \( \mathfrak{g}_{CM} \) and \( \{e_{\ell}\}_{\ell=1}^N \) be an orthonormal basis of \( \mathfrak{v} \). As in the proof of Corollary 3.12 expanding terms in an orthonormal basis of \( \mathfrak{v} \) and applying the Cauchy-Schwarz inequality gives

\[
\| (\hat{F}_3^{\sigma,\alpha}, v) \|_2^2 = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left| \langle \lbrack h_j, k_i \rbrack, h_j, v \rangle \right|^2 = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{k=1}^{N} \left| \langle e_{\ell}, \lbrack h_j, k_i \rbrack \rangle \right|^2 \\
\leq N \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{k=1}^{N} \left| \langle e_{\ell}, \lbrack h_j, k_i \rbrack \rangle \right|^2 \\
\leq N \left( \sum_{j=1}^{\infty} \sum_{\ell=1}^{N} \left| \langle e_{\ell}, h_j \rangle \langle e_{\ell}, \lbrack h_j, k_i \rbrack \rangle \right| \right) \left( \sum_{j=1}^{\infty} \sum_{i,j=1}^{N} \left| \langle \lbrack h_j, k_i \rbrack \rangle \right|^2 \right) \\
\leq N \left( \sum_{j=1}^{\infty} \sum_{\ell=1}^{N} \| \langle e_{\ell}, h_j \rangle \|_2 \left\| \langle \lbrack h_j, k_i \rbrack \rangle \right\|_2 \right) \\
\leq N \| v \|_2^2 < D - 1 > \| \cdot \|_2 \cdot \| \cdot \|_2.
\]

Now assume \( (\hat{F}_n^{\sigma,\alpha}, v) \) is Hilbert-Schmidt for all \( \sigma \in S_{n-1} \) and \( \alpha \in J_{n-1} \), and consider \( (\hat{F}_n^{\sigma,\alpha}, v) \) for some \( \sigma \in S_n \) and \( \alpha \in J_{n}' \). Let \( a = p_{\alpha} \) and \( b = q_{\alpha} \), and note that either \( a \geq 1 \) and

\[
\hat{F}_n^{\sigma,\alpha}(k_1 \otimes \cdots \otimes k_a) \\
= \sum_{j_1,\ldots,j_a} F_n' \left( k_1 \otimes \cdots \otimes k_a \otimes h_{j_1} \otimes h_{j_1} \otimes \cdots \otimes h_{j_b} \otimes h_{j_b} \right) \\
= \sum_{j_1,\ldots,j_a} [F_n'' \left( k_1 \otimes \cdots \otimes k_{d-1} \otimes k_{d+1} \otimes \cdots \otimes k_a \otimes h_{j_1} \otimes \cdots \otimes h_{j_b} \right), k_d] \\
(4.16) = [\hat{F}_n^{\sigma,\beta} \left( k_1 \otimes \cdots \otimes k_{d-1} \otimes k_{d+1} \otimes \cdots \otimes k_a \right), k_d],
\]

for some \( d \in \{1, \ldots, a\}, \sigma'', \tau \in S_{n-1} \), and \( \beta \in J_{n-1}' \) such that \( p_{\beta} = p_{\alpha} - 1 \) and \( q_{\beta} = q_{\alpha} \), or \( b \geq 1 \) and

\[
\hat{F}_n^{\sigma,\alpha}(k_1 \otimes \cdots \otimes k_a) \\
= \sum_{j_1,\ldots,j_a} [F_n'' \left( k_1 \otimes \cdots \otimes k_a \otimes h_{j_1} \otimes \cdots \otimes h_{j_{d-1}} \otimes h_{j_{d+1}} \otimes \cdots \otimes h_{j_b} \right), h_{j_d}] \\
(4.17) = \sum_{j_d} [\hat{F}_n^{\sigma,\beta} \left( k_1 \otimes \cdots \otimes k_a \otimes h_{j_d} \right), h_{j_d}],
\]
for some \( d \in \{1, \ldots, b\} \), \( \sigma'', \tau \in S_{n-1} \) and \( \beta \in J_{n-1}^m \) such that \( p_\beta = p_\alpha + 1 \) and \( q_\beta = q_\alpha - 1 \). In the first case, working as above for \( n = 3 \),

\[
\left\| \langle \hat{F}_n^\sigma, v \rangle \right\|_2^2 = \sum_{i_1, \ldots, i_a = 1}^\infty \sum_{j_1, \ldots, j_b = 1}^\infty \langle F_n^\sigma(k_{i_1} \otimes \cdots \otimes k_{i_a} \otimes h_{j_1} \otimes \cdots \otimes h_{j_b}), v \rangle^2
\]

\[
= \sum_{i_1, \ldots, i_a = 1}^\infty \sum_{j_1, \ldots, j_b = 1}^N \langle F_{n-1}^\sigma(k_{i_1} \otimes \cdots \otimes h_{j_b}), v \rangle^2
\]

\[
\leq N \sum_{i_1, \ldots, i_a = 1}^\infty \sum_{j_1, \ldots, j_b = 1}^N \langle F_{n-1}^\sigma(k_{i_1} \otimes \cdots \otimes h_{j_b}), e_\ell \rangle \langle [e_\ell, k_{i_a}], v \rangle^2
\]

\[
= N \sum_{i_1, \ldots, i_a = 1}^\infty \sum_{j_1, \ldots, j_b = 1}^N \langle [e_\ell, k_{i_a}], v \rangle^2 \sum_{j_1, \ldots, j_b = 1}^\infty \langle F_{n-1}^\sigma(k_{i_1} \otimes \cdots \otimes h_{j_b}), e_\ell \rangle
\]

which is finite by the induction hypothesis. Similarly, in the second case

\[
\left\| \langle \hat{F}_n^\sigma, v \rangle \right\|_2^2 = \sum_{i_1, \ldots, i_a = 1}^\infty \sum_{j_1, \ldots, j_b = 1}^\infty \langle F_n^\sigma(k_{i_1} \otimes \cdots \otimes h_{j_b}), [h_{j_a}], v \rangle^2
\]

\[
\leq N \sum_{i_1, \ldots, i_a = 1}^\infty \sum_{j_1, \ldots, j_b = 1}^\infty \langle F_{n-1}^\sigma(k_{i_1} \otimes \cdots \otimes h_{j_b}), e_\ell \rangle \langle [e_\ell, h_{j_a}], v \rangle^2
\]

\[
\leq N \left( \sum_{i_1, \ldots, i_a = 1}^\infty \sum_{j_1, \ldots, j_b = 1}^\infty \sum_{j_1, \ldots, j_b = 1}^{j_d+1} \langle F_{n-1}^\sigma(k_{i_1} \otimes \cdots \otimes h_{j_b}), e_\ell \rangle^2 \right)^2
\]

\[
\leq N \sum_{\ell = 1}^N \left\langle \hat{F}_{n-1}^{\tau, \beta}, e_\ell \right\rangle_2^2 \cdot \left\| v \right\|_2^2 \left\| [\cdot, \cdot] \right\|_2^2
\]

Proposition 4.4. Let \( n \in \{2, \ldots, r\} \), \( \sigma \in S_n \), and \( \alpha \in J_n \). Then \( \hat{F}_n^\sigma : g^{\otimes p_\alpha}_{CM} \rightarrow v \) is Hilbert-Schmidt.
**Proof.** This proof is analogous to that of Lemma 4.3. For \( \hat{F}_{n}^{\sigma, \alpha} \) as in equation (4.17), we have

\[
\| \hat{F}_{n}^{\sigma, \alpha} \|_{2}^{2} = \sum_{i_{1}, \ldots, i_{n} = 1}^{\infty} \left\| \sum_{j_{1}, \ldots, j_{n} = 1}^{\infty} [F_{n-1}^{\sigma, \alpha}(k_{i_{1}} \otimes \cdots \otimes h_{j_{n}}), h_{j_{n}}] \right\|^{2}
\]

\[
\leq N \left( \sum_{i_{1}, \ldots, i_{n} = 1}^{\infty} \left\| \sum_{j_{1}, \ldots, j_{n} = 1}^{\infty} \langle F_{n-1}^{\sigma, \alpha}(k_{i_{1}} \otimes \cdots \otimes h_{j_{n}}), e_{t} \rangle \right\|^{2} \right)
\]

\[
\times \left( \sum_{i_{1}, \ldots, i_{n} = 1}^{\infty} \sum_{j_{1}, \ldots, j_{n} = 1}^{\infty} \left\| \sum_{j_{1}, \ldots, j_{n} = 1}^{\infty} \langle F_{n-1}^{\sigma, \alpha}(k_{i_{1}} \otimes \cdots \otimes h_{j_{n}}), e_{t} \rangle \right\|^{2} \right)
\]

\[
\leq N \| \cdot \|_{2}^{2} \sum_{d=1}^{N} \left\| \langle \hat{F}^{\tau, \beta}_{n-1}, e_{t} \rangle \right\|_{2}^{2},
\]

which is finite by Corollary 3.12 and Lemma 4.3. In a similar way, one may show that, for \( \hat{F}^{\sigma, \alpha} \) as in equation (4.16),

\[
\| \hat{F}^{\sigma, \alpha} \|_{2}^{2} \leq N \| \cdot \|_{2}^{2} \sum_{d=1}^{N} \left\| \langle \hat{F}^{\tau, \beta}_{n-1}, e_{t} \rangle \right\|_{2}^{2}.
\]

**Remark 4.5.** The proofs of the previous propositions rely strongly on \( \mathfrak{v} \) being finite dimensional. Thus, if we were to extend the results of this paper to \( \mathfrak{v} \) an infinite dimensional Lie algebra, another proof would be required here, or more likely, some trace class requirements on the Lie bracket of \( \mathfrak{g} \).

Proposition 4.4 allows us to make the following definition.

**Definition 4.6.** A **Brownian motion** on \( G \) is the continuous \( G \)-valued process defined by

\[
g_{t} = \sum_{n=1}^{r} \sum_{\sigma \in S_{n}} \sum_{m=\lfloor n/2 \rfloor}^{n} \frac{c_{n}^{\sigma}}{2^{m}} \sum_{\alpha \in \mathcal{J}_{m}} \int_{\Delta_{\nu_{m}}(t)} f_{\alpha}(s, t) \hat{F}_{n}^{\sigma, \alpha}(dB_{s_{1}} \otimes \cdots \otimes dB_{s_{m}}),
\]

where

\[
c_{n}^{\sigma} = (-1)^{e(\sigma)} n^{2} \left[ \frac{n-1}{e(\sigma)} \right],
\]

\( \hat{F}^{\sigma, \alpha} \) is as defined in (4.13) and \( f_{\alpha} \) is a polynomial of degree \( \alpha \) in \( s = (s_{1}, \ldots, s_{m}) \) and \( t \) as described in (4.14). For \( t > 0 \), let \( \nu_{t} = \text{Law}(g_{t}) \) be the heat kernel measure at time \( t \), a probability measure on \( G \).
Example 4.7 (The step 3 case). Suppose that \( g \) is nilpotent of step 3. Then

\[
g_t = \sum_{n=1}^{3} \sum_{\sigma \in S_n} c_n^\sigma F_n^\sigma (M_n(t))
\]

\[
= \sum_{n=1}^{3} \sum_{\sigma \in S_n} \frac{n}{2n-m} \sum_{\alpha \in J_n^m} F_n^\sigma (I_n^m (\alpha))
\]

\[
= \sum_{n=1}^{3} \sum_{\sigma \in S_n} \frac{n}{2n-m} \sum_{\alpha \in J_n^m} \int_{\Delta_{kn}(t)} f_\alpha(s, t) \hat{F}_n^\sigma,\alpha (dB_{s_1} \otimes \cdots \otimes dB_{s_{kn}}).
\]

For \( n = 1 \), there is the single term given by

\[
M_1(t) = \int_0^t \delta B_s = B_t.
\]

For \( n = 2 \), \( J_2 = \{(1, 1), (2)\} \), and so

\[
M_2(t) = I_2^2 ((1, 1)) + \frac{1}{2} I_2^2 ((2))
\]

\[
= \int_{\Delta_2(t)} dB_{s_1} \otimes dB_{s_2} + \frac{1}{2} \int_0^t h_i \otimes h_i ds_2
\]

\[
= \int_{\Delta_2(t)} dB_{s_1} \otimes dB_{s_2} + \frac{1}{2} \sum_{i=1}^\infty h_i \otimes h_i.
\]

There are of course just two permutations: \( \sigma = (12) \) with \( e(\sigma) = 0 \) and \( c_2^\sigma = \frac{1}{4} \), and \( \tau = (21) \) with \( e(\tau) = 1 \) and \( c_2^\tau = -\frac{1}{4} \), and, by the antisymmetry of the Lie bracket,

\[
\sum_{\sigma \in S_2} c_2^\sigma F_2^\sigma (M_2(t)) = \frac{1}{4} [dB_{s_1}, dB_{s_2}] - \frac{1}{4} [dB_{s_2}, dB_{s_1}] = \frac{1}{2} [dB_{s_1}, dB_{s_2}].
\]

For \( n = 3 \), the permutations are (123) with \( e = 0 \), (213), (132), (312), (231) with \( e = 1 \), and (321) with \( e = 2 \). Thus,

\[
\sum_{\sigma \in S_3} c_3^\sigma F_3^\sigma (k_1 \otimes k_2 \otimes k_3) = \frac{1}{9} [[k_1, k_2], k_3] + \frac{1}{18} [[k_2, k_1], k_3] + \frac{1}{18} [[k_1, k_3], k_2]
\]

\[
- \frac{1}{18} [[k_3, k_1], k_2] - \frac{1}{18} [[k_2, k_3], k_1] + \frac{1}{9} [[k_3, k_2], k_1]
\]

\[
(4.18)
\]

\[
= \frac{1}{6} [[k_1, k_2], k_3] + \frac{1}{6} [[k_3, k_2], k_1].
\]
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Also, \( J_3 = \{(1,1,1), (1,2), (2,1)\} \), and so

\[
M_3(t) = I^3_3((1,1,1)) + \frac{1}{2} I^3_1((1,2)) + \frac{1}{2} I^3_1((2,1))
\]

\[
= \int_{\Delta_3(t)} dB_{s_1} \otimes dB_{s_2} \otimes dB_{s_3} + \frac{1}{2} \int_{\Delta_2(t)} \sum_{i=1}^{\infty} dB_{s_1} \otimes h_i \otimes h_i ds_3
\]

\[
+ \frac{1}{2} \int_0^t \sum_{i=1}^{\infty} s_3 h_i \otimes h_i \otimes dB_{s_3}
\]

\[
= \int_{\Delta_3(t)} dB_{s_1} \otimes dB_{s_2} \otimes dB_{s_3} + \frac{1}{2} \int_0^t \sum_{i=1}^{\infty} (t - s_1) dB_{s_1} \otimes h_i \otimes h_i
\]

\[
+ \frac{1}{2} \int_0^t \sum_{i=1}^{\infty} s_3 h_i \otimes h_i \otimes dB_{s_3}.
\]

Note that \( f_{(1,2)}(s,t) = t - s_1 \) and \( f_{(2,1)}(s,t) = s_3 \). Plugging this into equation (4.15) gives, for the \( \alpha = (1,1,1) \in J_3^3 \) term,

\[
\sum_{\sigma \in S_3} c_3^\sigma F_3^\sigma (I^3_3((1,1,1))) = \sum_{\sigma \in S_3} c_3^\sigma \int_{\Delta_3(t)} F_3^\sigma (dB_{s_1} \otimes dB_{s_2} \otimes dB_{s_3})
\]

\[
= \frac{1}{6} \int_{\Delta_3(t)} ([dB_{s_1}, dB_{s_2}], dB_{s_3}) + ([dB_{s_3}, dB_{s_2}], dB_{s_1})).
\]

For \( \alpha = (1,2) \in J_3^2 \),

\[
\sum_{\sigma \in S_3} c_3^\sigma F_3^\sigma (I_3((1,2))) = \frac{1}{6} \int_0^t \sum_{i=1}^{\infty} (t - s_1)[[dB_{s_1}, h_i], h_i],
\]

and

\[
\hat{F}_{3}^{\sigma,(1,2)}(k) = \sum_{i=1}^{\infty} F_3^\sigma (k \otimes h_i \otimes h_i)
\]

with \( \sigma' = \sigma \). For \( \alpha = (2,1) \in J_3^2 \),

\[
\sum_{\sigma \in S_3} c_3^\sigma F_3^\sigma (I_3((2,1))) = \frac{1}{6} \int_0^t \sum_{i=1}^{\infty} s_3 [[dB_{s_3}, h_i], h_i],
\]

and note that, in this case,

\[
\hat{F}_{3}^{\sigma,(2,1)}(k) = \sum_{i=1}^{\infty} F_3^\sigma (k \otimes h_i \otimes h_i) = \sum_{i=1}^{\infty} F_3^\sigma (h_i \otimes h_i \otimes k),
\]
and so $\sigma' = \sigma \circ (231)$ (or $\sigma' = \sigma \circ (321)$). Combining the above, Brownian motion on $G$ may be written as

$$g_t = B_t + \frac{1}{2} \int_{\Delta_2(t)} [dB_{s_1}, dB_{s_2}] + \frac{1}{12} \int_{\Delta_3(t)} \left( [[dB_{s_1}, dB_{s_2}], dB_{s_3}] + [[dB_{s_2}, dB_{s_3}], dB_{s_1}] \right) + \frac{1}{24} \sum_{i=1}^{\infty} \int_0^t (t-s) [[dB_{s_1}, h_i], h_i] + s [[dB_{s_2}, h_i], h_i] + \frac{1}{24} \sum_{i=1}^{\infty} t [[B_t, h_i], h_i].$$

Remark 4.8. In principle, the Brownian motion on $G$ has generator

$$\Delta = \sum_{i=1}^{\infty} \tilde{h}_i^2,$$

where $\{h_1\}_{i=1}^{\infty}$ is an orthonormal basis of $\mathfrak{g}_{CM} = H \oplus \mathfrak{v}$ and $\tilde{h}$ is the unique left invariant vector field on $G$ such that $\tilde{h}(e) = h$, and $\Delta$ is well-defined independent of the choice of orthonormal basis. Then the heat kernel measure $\{\nu_t\}_{t>0}$ has the standard characterization as the unique family of probability measures such that $\nu_t(f) := \int_G f \, d\nu_t$ is continuously differentiable in $t$ for all $f \in C^0_c(G)$ and satisfies

$$\frac{d}{dt} \nu_t(f) = \frac{1}{2} \nu_t(\Delta f) \quad \text{with} \lim_{t \downarrow 0} \nu_t(f) = f(c).$$

However, this realization of $\nu_t$ is not necessary for our results.

**Proposition 4.9** (Finite dimensional approximations). For $P \in \text{Proj}(W)$, let $g_t^P$ be the continuous process on $G_P$ defined by

$$g_t^P = \sum_{n=1}^{r} \sum_{\sigma \in S_n} \sum_{m=\lceil n/2 \rceil}^{n} \frac{c_n^\sigma}{2^{n-m}} \sum_{\alpha \in \mathcal{J}_m} \int_{\Delta_{\pi\alpha}(t)} f_{\alpha}(s, t) \tilde{F}_{n, \alpha}^{\pi, \alpha}(d\pi B_{s_1} \otimes \cdots \otimes d\pi B_{s_{m}}),$$

for $\pi(w, x) = (Pw, x)$. Then $g_t^P$ is Brownian motion on $G_P$. In particular, let $g_t^P = g_t^{P_1}$, for projections $\{P_t\}_{t=1}^{\infty} \subset \text{Proj}(W)$ as in equation (4.4). Then, for all $p \in [1, \infty)$ and $t < \infty$,

$$\lim_{t \to \infty} \mathbb{E} \left[ \sup_{\tau \leq t} \left\| g_{\tau}^t - g_{\tau}^P \right\|_g^p \right] = 0.$$

**Proof.** First note that $g_t^P$ solves the Stratonovich equation $\delta g_t^P = L_{g_t^P} \delta PB_t$ with $g_0^P = e$, see [4] [5] [3]. Thus, $g_t^P$ is a $G_P$-valued Brownian motion.

Now, if $\beta_t$ a Brownian motion on $W$, then, for all $p \in [1, \infty)$,

$$\lim_{t \to \infty} \mathbb{E} \left[ \sup_{\tau \leq t} \left\| P_t \beta_\tau - \beta_\tau \right\|_W^p \right] = 0;$$
see, for example, Proposition 4.6 of [10]. Thus,
\[
\lim_{t \to \infty} \mathbb{E} \left[ \sup_{\tau \leq t} \| \pi_t B_{\tau} - B_{\tau} \|_p^p \right] = 0.
\]

By equation (4.13) and its preceding discussion,
\[
g_t^\ell = \sum_{n=1}^{r} \sum_{\sigma \in S_n, m=\lfloor n/2 \rfloor} c_{\alpha} \cdot \frac{\mathcal{G}_n}{\sqrt{\alpha}} \sum_{\alpha \in J_n} \sum_{a=0}^{\eta_n} b_{n}^{a} t^{\alpha} J_n^{\ell}(\tilde{f}_n \tilde{F}_{\alpha})_{t},
\]
and thus, to verify (4.19), it suffices to show that, for all \( p \in [1, \infty) \),
\[
\lim_{t \to \infty} \mathbb{E} \left[ \sup_{\tau \leq t} \left\| J_n^{\ell}(\tilde{f}_n \tilde{F}_{\alpha})_{t} - J_n(\tilde{f}_n \tilde{F}_{\alpha})_{t} \right\|_p^p \right] = 0,
\]
for all \( n \in \{2, \ldots, r\} \), \( \sigma \in S_n \) and \( \alpha \in J_n \). By Proposition 4.13, \( \tilde{F}_{n,\sigma,\alpha} \) is Hilbert-Schmidt, and recall that \( \tilde{f}_n \) is a deterministic polynomial function in \( s \). Thus \( J_n^{\ell}(\tilde{f}_n \tilde{F}_{n,\sigma,\alpha}) \) and \( J_n(\tilde{f}_n \tilde{F}_{n,\sigma,\alpha}) \) are \( \nu \)-valued martingales as defined in Corollary 4.2. So, by Doob’s maximal inequality, it suffices to show that
\[
\lim_{t \to \infty} \mathbb{E} \left\| J_n^{\ell}(\tilde{f}_n \tilde{F}_{n,\sigma,\alpha})_{t} - J_n(\tilde{f}_n \tilde{F}_{n,\sigma,\alpha})_{t} \right\|_p^p = 0
\]
Corollary 4.2 gives the limit for \( p = 2 \). For \( p > 2 \), since each \( J_n^{\ell}(\tilde{f}_n \tilde{F}_{n,\sigma,\alpha}) \) and \( J_n(\tilde{f}_n \tilde{F}_{n,\sigma,\alpha}) \) has chaos expansion terminating at degree \( n \), a theorem of Nelson (see Lemma 2 of [21] and pp. 216-217 of [20]) implies that, for each \( j \in \mathbb{N} \), there exists \( c_j < \infty \) such that
\[
\mathbb{E} \left( J_n^{\ell}(\tilde{f}_n \tilde{F}_{n,\sigma,\alpha})_{t} - J_n(\tilde{f}_n \tilde{F}_{n,\sigma,\alpha})_{t} \right)^{2j} \leq c_j \left( \mathbb{E} \left( J_n^{\ell}(\tilde{f}_n \tilde{F}_{n,\sigma,\alpha})_{t} - J_n(\tilde{f}_n \tilde{F}_{n,\sigma,\alpha})_{t} \right)^{2} \right)^j.
\]

5. Heat kernel measure

We collect here some properties of the heat kernel measure on \( G \). The following results are completely analogous to Corollary 4.9 of [10] and Proposition 4.6 in [12]. The proofs are included here for the convenience of the reader.

**Proposition 5.1.** For any \( t > 0 \), the heat kernel measure \( \nu_t \) is invariant under the inversion map \( g \mapsto g^{-1} \) for any \( g \in G \).

**Proof.** The heat kernel measures \( \nu_t^{P_n} = \text{Law}(g_t^{n}) \) on the finite dimensional groups \( G_{P_n} \) are invariant under inversion (see, for example, [13]). Suppose that \( f : G \to \mathbb{R} \) is a bounded continuous function. By passing to a subsequence if necessary, we may assume that the sequence of \( G_{P_n} \)-valued random variables \( \{g_t^n\}_{n=1}^{\infty} \) in Proposition 4.9 converges almost surely to \( g_t \). Thus, by dominated convergence,
\[
\mathbb{E} \left[ f(g_t^{-1}) \right] = \lim_{n \to \infty} \mathbb{E} \left[ f(g_t^n) \right] = \mathbb{E} \left[ f(g_t) \right].
\]
Since \( \nu_t \) is the law of \( g_t \), this completes the proof.

**Proposition 5.2.** For all \( t > 0 \), \( \nu_t(G_{CM}) = 0 \).

**Proof.** Let \( \mu_t \) denote Wiener measure on \( W \) with variance \( t \). Then for a bounded measurable function \( f \) on \( G = W \oplus \nu \) such that \( f(w,x) = f(w) \),
\[
\int_{G} f(w) \, d\nu_t(w,x) = \mathbb{E}[f(\beta_t)] = \int_{W} f(w) \, d\mu_t(w).
\]
Let \( \pi : W \times v \to W \) be the projection \( \pi(w, x) = w \). Then \( \pi_* \nu_t = \mu_t \), and thus
\[
\nu_t(G_{CM}) = \nu_t(\pi^{-1}(H)) = \pi_* \nu_t(H) = \mu_t(H) = 0.
\]

This proposition gives some justification to our calling \( G_{CM} \) the Cameron-Martin subgroup of \( G \). In the next section, we further justify this by showing that a Cameron-Martin type quasi-invariance theorem holds for \( \nu_t \).

5.1. Quasi-invariance and Radon-Nikodym derivative estimates. The following theorem states that the heat kernel measure \( \nu_t = \text{Law}(g_t) \) is quasi-invariant under left and right translation by elements of \( G_{CM} \) and gives estimates for the Radon-Nikodym derivatives of the translated measures.

**Theorem 5.3.** For all \( h \in G_{CM} \) and \( t > 0 \), \( \nu_t \circ L^{-1}_h \) and \( \nu_t \circ R^{-1}_h \) are absolutely continuous with respect to \( \nu_t \). Let
\[
Z^l_h := \frac{d(\nu_t \circ L^{-1}_h)}{d\nu_t} \quad \text{and} \quad Z^r_h := \frac{d(\nu_t \circ R^{-1}_h)}{d\nu_t}
\]
be the Radon-Nikodym derivatives, \( K \) be lower bound on the Ricci curvature of \( G \) as in Corollary 3.23 and
\[
c(t) := \frac{t}{e^t - 1}, \quad \text{for all } t \in \mathbb{R},
\]
with the convention that \( c(0) = 1 \). Then, \( Z^l_h, Z^r_h \in L^p(\nu_t) \) for all \( p \in [1, \infty) \), and both satisfy the estimate
\[
\|Z^*_h\|_{L^p(\nu_t)} \leq \exp \left( \frac{c(Kt)(p - 1)}{2t} d^2_{CM}(e, h) \right),
\]
where \( * = l \) or \( * = r \).

**Proof.** As in [11], the proof of this theorem is an application of Theorem 7.3 and Corollary 7.4 in [11] on the quasi-invariance of heat kernel measures for inductive limits of finite dimensional Lie groups. In applying these results, the reader should take \( G_0 = G_{CM} \), \( A = \text{Proj}(W) \), \( s_P = \pi_P \), \( \nu_P = \text{Law}(g_P) \), and \( \nu = \nu_t = \text{Law}(g_t) \). We now verify that the hypotheses of Theorem 7.3 of [11] are satisfied.

By Corollary 3.23 the inductive limit group \( \cup_{P \in \text{Proj}(W)} G_P \) is a dense subgroup of \( G_{CM} \). By Proposition 4.10 for any \( \{P_n\}_{n=1}^\infty \subset \text{Proj}(W) \) with \( P_n|_H \uparrow I_H \) and \( f : G \to \mathbb{R} \) a bounded continuous function,
\[
(5.1) \int_G f \, d\nu = \lim_{n \to \infty} \int_{G_{P_n}} (f \circ i_{P_n}) \, d\nu_{P_n},
\]
and thus the heat kernel measure is consistent on finite dimensional projections of \( G_{CM} \). Corollary 3.23 says that \( K > -\infty \) and Ric\(^P\) \( \geq K g^P \), for all \( P \in \text{Proj}(W) \), and thus the Ricci curvature is uniformly bounded on these projections. Lastly, the length of a path in the inductive limit group can be approximated by the lengths of paths in the finite dimensional projections. That is, for any \( P_0 \in \text{Proj}(W) \) and \( \varphi \in C^1([0, 1], G_{CM}) \) with \( \varphi(0) = e \), there exists an increasing sequence \( \{P_n\}_{n=1}^\infty \subset \text{Proj}(W) \) such that \( P_0 \subset P_n, P_n|_H \uparrow I_H \), and
\[
\ell_{CM}(\varphi) = \lim_{n \to \infty} \ell_{G_{P_n}}(\pi_n \circ \varphi).
\]
To see this, let \( \varphi(t) = (A(t), a(t)) \) be a path in \( G_{CM} \), and recall that, by equation \( [3.8] \),

\[
I_{G_{F_n}}(\pi_n \circ \varphi) = \int_0^1 \left\| \pi_n \varphi'(s) + \sum_{\ell=1}^{r-1} d_\ell \text{ad}_{\pi_n \varphi(s)} \pi_n \varphi'(s) \right\|_{g_{CM}} ds
\]

\[
= \int_0^1 \left\| P_n A'(s) \right\|_H^2 + \left\| A'(s) + \sum_{\ell=1}^{r-1} d_\ell \text{ad}_{\pi_n \varphi(s)} \pi_n \varphi'(s) \right\|_\nu ds
\]

Applying dominated convergence to this equation shows that \( [5.1] \) holds for any such choice of \( P_n \mid_H \uparrow I_H \) such that \( P_0 \subset P_n \).

We also have the usual strong converse to quasi-invariance of \( \nu_t \) under translations by elements in \( G_{CM} \).

**Proposition 5.4.** For \( h \in G \setminus G_{CM} \) and \( t > 0, (\nu_t \circ L_{h^{-1}}) \) and \( \nu_t \) are singular and \( (\nu_t \circ R_{h^{-1}}) \) and \( \nu_t \) are singular.

**Proof.** Again, let \( \mu_t \) denote Wiener measure on \( W \) with variance \( t \). Let \( h = (A, a) \in G \setminus G_{CM} \) with \( A \in W \setminus H \) and \( a \in \mathfrak{v} \). Given a measurable subset \( U \subset W \),

\[
\nu_t(U \times \mathfrak{v}) = P(\beta_t \in U) = \mu_t(U).
\]

If \( A \in W \setminus H \), \( \mu_t(\cdot - A) \) and \( \mu_t \) are singular; for example, see Corollary 2.5.3 of [6]. Thus, there are disjoint subsets \( W_0 \) and \( W_1 \) of \( W \) such that \( \mu_t(W_0) = 1 = \mu_t(W_1 - A) \). Note that

\[
L_{k}^{-1}(U \times \mathfrak{v}) = R_{k}^{-1}(U \times \mathfrak{v}) = (U - A) \times \mathfrak{v}.
\]

Thus, for \( G_i := W_i \times \mathfrak{v} \) for \( i = 0, 1 \), \( G \) is the disjoint union of \( G_0 \) and \( G_1 \), and \( \nu_t(G_0) = \mu_t(W_0) = 1 \) while

\[
\nu_t \left( R_{k}^{-1}(G_1) \right) = \nu_t \left( L_{k}^{-1}(G_1) \right) = \nu_t((W_1 - A) \times \mathfrak{v}) = \mu_t(W_1 - A) = 1.
\]

**Proposition 5.5.** For all \( h \in G_{CM} \) and \( t > 0, Z_{h}^t(g) = Z_{h^{-1}}^t(g^{-1}) \).

**Proof.** By Proposition 5.1, \( \nu_t \) is invariant under inversions. Thus

\[
\int_G f(g \cdot h) \, d\nu_t(g) = \int_G f \left( (h^{-1} \cdot g)^{-1} \right) \, d\nu_t(g) = \int_G f \left( (h^{-1} \cdot g)^{-1} \right) \, d\nu_t(g)
\]

\[
= \int_G f(g) \, Z_{h^{-1}}^t(g) \, d\nu_t(g) = \int_G f(g)Z_{h^{-1}}^t(g) \, d\nu_t(g).
\]

5.2. Logarithmic Sobolev inequality.

**Definition 5.6.** A function \( f : G \to \mathbb{R} \) is said to be a (smooth) cylinder function if \( f = F \circ \pi_P \) for some \( P \in \text{Proj}(W) \) and some (smooth) function \( F : G_P \to \mathbb{R} \).

Also, \( f \) is a cylinder polynomial if \( f = F \circ \pi_P \) for \( F \) a polynomial function on \( G_P \).

**Theorem 5.7.** Given a cylinder polynomial \( f \) on \( G \), let \( \nabla f : G \to g_{CM} \) be the gradient of \( f \), the unique element of \( g_{CM} \) such that

\[
\langle \nabla f(g), h \rangle_{g_{CM}} = \hat{h} f(g) := f'(g)(L_{g}, h_g),
\]
for all \( h \in \mathfrak{g}_{CM} \). Then for \( K \) as in Corollary 5.3
\[
\int_G (f^2 \ln f^2) \, d\nu_t - \left( \int_G f^2 \, d\nu_t \right) \cdot \ln \left( \int_G f^2 \, d\nu_t \right) \leq 2 \frac{1 - e^{-Kt}}{K} \int_G \|\nabla f\|^2_{\mathfrak{g}_{CM}} \, d\nu_t.
\]

Proof. Following the method of Bakry and Ledoux applied to \( G_P \) (see Theorem 2.9 of [14] for the case needed here) shows that
\[
E \left[ (f^2 \ln f^2) \left( g^P_t \right) \right] - E \left[ f^2 \left( g^P_t \right) \right] \ln E \left[ f^2 \left( g^P_t \right) \right] \leq 2 \frac{1 - e^{-K_P t}}{K_P} E \| (\nabla^P f) \left( g^P_t \right) \|^2_{\mathfrak{g}_{CM}},
\]
for \( K_P \) as in equation (3.12). Since the function \( x \mapsto (1 - e^{-x})/x \) is decreasing and \( K \leq K_P \) for all \( P \in \text{Proj}(W) \), this estimate also holds with \( K_P \) replaced with \( K \).

Now applying Proposition 5.4 to pass to the limit as \( P \uparrow I \) gives the desired result.

\[ \blacksquare \]

Remark 5.8. It is desirable to state Theorem 5.7 for a larger class of functions in \( L^2(\nu_t) \). To do this, one must prove that the gradient operator \( \nabla : L^2(\nu_t) \to L^2(\nu_t) \otimes \mathfrak{g}_{CM} \) is closable. Unfortunately, Theorem 5.7 doesn’t give good information on the dependence of the Radon-Nikodym derivatives \( Z^I_t \) and \( Z^h_t \) on \( h \), and so at this point we can’t prove the necessary integration by parts formulae to show closability.

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Department of Mathematics, University of Virginia, Charlottesville, VA 22936

E-mail address: melcher@virginia.edu