Linear Dependence Between Hereditary Quasirandomness Conditions

Xiaoyu He

July 19, 2017

Abstract

Answering a question of Simonovits and Sós, Conlon, Fox, and Sudakov proved that for any fixed graph $H$, and any $\epsilon > 0$, there exists $\delta > 0$ polynomial in $\epsilon$, such that if $G$ is an $n$-vertex graph with the property that every $U \subseteq V(G)$ contains $p^e(H)|U|^v(H) \pm \delta n^v(H)$ labeled copies of $H$, then $G$ is $(p, \epsilon)$-quasirandom in the sense that every subset $U \subseteq G$ contains $\frac{1}{2}p|U|^2 \pm \epsilon n^2$ edges. They conjectured that $\delta$ may be taken to be linear in $\epsilon$ and proved this in the case that $H$ is a clique. We study a labelled version of this quasirandomness property proposed by Reiher and Schacht, and show that for any fixed graph $H$ on $r$ vertices $v_1, \ldots, v_r$, and any $\epsilon > 0$, there exists $\delta > 0$ linear in $\epsilon$, if $G$ is an $n$-vertex graph with the property that every sequence of $r$ subsets $U_1, \ldots, U_r \subseteq V(G)$, the number of copies of $H$ with each $v_i$ in $U_i$ is $p^e(H) \prod |U_i| \pm \delta n^v(H)$, then $G$ is $(p, \epsilon)$-quasirandom.

1 Introduction

Random-like objects, in particular quasirandom graphs, have become a central object of study in combinatorics and theoretical computer science (see for example the survey article of Krivelevich and Sudakov [6]). In this paper, we will prove a generalization of a result of Conlon, Fox, and Sudakov [3] on quasirandom graphs that ties in with a line of research motivated by two important principles of extremal graph theory. They are:

1. Many “natural” properties of random graphs are equivalent.

2. Many things can be proved by Szemerédi’s regularity lemma, but few things should be.

Although certain notions of quasirandom graphs were studied earlier earlier, such as in Thomason’s work on “jumbled” graphs [11], Principle 1 first appeared in the seminal work of Chung, Graham and Wilson [2].

The Erdős-Rényi random graph $G(n, p)$ is the random graph on $n$ vertices where each of the $\binom{n}{2}$ edges is drawn independently with probability $p$. A priori, any number of properties of the prototypical random graph $G(n, p)$ could be used to define quasirandomness, but Chung, Graham, and Wilson [2] discovered that many of these properties are qualitatively equivalent, leading to a canonical notion of quasirandomness for graphs. Write $V(G), E(G)$
to be the vertices (resp. edges) of a graph $G$, and $v(G) = |V(G)|$, $e(G) = |E(G)|$. Also, write $x = y \pm \Delta$ if $|x - y| \leq \Delta$.

**Theorem 1.** (Chung, Graham, and Wilson [2]). Let $p \in (0, 1)$. The following are equivalent, up to choice of $\epsilon > 0$:

1. For every graph $H$, the number of induced subgraphs of $G$ isomorphic to $H$ is
   \[ p^{e(H)}(1 - p)^{e(H)}v(G)^v(H) \pm \epsilon v(G)^v(H). \]

2. There exists a graph $H$ such that every induced subgraph $G[U]$ of $G$ contains $p^{e(H)}|U|^v(H) \pm \epsilon v(G)^v(H)$ (not necessarily induced) subgraphs isomorphic to $H$.

3. The number of 4-cycles in $G$ is at most $p^4v(G)^4 + \epsilon v(G)^4$.

4. The two largest (in absolute value) eigenvalues $\lambda_1, \lambda_2$ of the adjacency matrix of $G$ satisfy $\lambda_1 = (p \pm \epsilon)v(G)$, $|\lambda_2| \leq \epsilon v(G)$.

5. For every vertex subset $U$ of $G$, $e(U) = \frac{1}{2}p|U|^2 \pm \epsilon v(G)^2$.

Explicitly, we mean that for any properties $i, j \in \{1, 2, 3, 4, 5\}$ above, and any $\epsilon > 0$, there exists $\delta > 0$ such that if $G$ satisfies property $i$ for $\delta$, then it also satisfies property $j$ for $\epsilon$.

Although these conditions are equivalent, the relationships between the various $\epsilon$’s are not so well understood. For example, let $P_{H,p}(\epsilon)$ be property 2 above for some fixed $H, p$ and $\epsilon > 0$, the so-called “hereditary quasirandomness” condition because it is inherited by all induced subgraphs. Then Simonovits and Sós [9] were able to prove using Szemerédi’s regularity lemma [10] that for any two graphs $H, H'$, $P_{H,p}(\delta) \Rightarrow P_{H',p}(\epsilon)$ where $\delta^{-1}$ is growing as a tower function of $\epsilon^{-1}$. The Simonovits-Sós conjecture (Conjecture 3 below) is that this dependence can be made linear in $\epsilon$.

Roughly speaking, Szemerédi’s regularity lemma states that given any $\epsilon > 0$, every graph $G$ can be decomposed into $K = K(\epsilon)$ parts $V_1, \ldots, V_K$ (the “regularity partition”) such that the edges between most pairs $V_i, V_j$, are within $\epsilon$ of being random. Although the regularity lemma is an extraordinarily powerful tool, the quantative dependence of $K$ on $\epsilon$ is of tower-type growth. Thus, tower-type quantitative dependency is indicative of a straightforward application of the regularity lemma.

Principle 2 states that in practice the regularity lemma is unnecessary, and we can expect to improve quantitative bounds by avoiding its use. When this is possible, the tower-type bounds provided by the regularity lemma can usually be replaced by exponential or even polynomial growth. One of the most important examples of Principle 2 is the weak regularity lemma of Frieze-Kannan [5], which proves that a graph $G$ can be decomposed into a regularity partition with only exponentially many parts in $\epsilon$ if we replace the regularity condition by a weaker global version. Problems amenable to the regularity method sometimes only require the Frieze-Kannan weak regularity condition. For example, it is possible to prove Conjecture 3 with exponential $\delta$-$\epsilon$ dependence using this method.

Further progress on Conjecture 3 was made by Conlon, Fox, and Sudakov [3] by carefully optimizing the regularity proof, achieving polynomial dependency of $\delta^{-1}$ on $\epsilon^{-1}$. In this
paper we will study a variation of Conjecture 3 introduced by Reiher and Schacht [8], where instead of counting copies of an unlabelled graph $H$ in any induced subgraph $G[U]$ of $G$, we count labelled copies of $H$ with the $i$-th vertex $v_i$ lying in a prescribed subset $U_i \subset V(G)$. In this situation, we prove the optimal linear dependence by extending a counting argument from [3].

2 Background

We first offer some notation for counting copies of subgraphs in a given graph. If $G = (V, E)$ is a graph and $U \subseteq V(G)$, then write $G[U]$ for the induced subgraph of $G$ on vertex subset $U$.

Let $H, G$ be two labelled graphs where $H$ has $r$ vertices $v_1, \ldots, v_r$. Let $U_1, U_2, \ldots, U_r$ be vertex subsets of $G$. Then define $c(H, G[U]; U_1, \ldots, U_r)$ to be the number of (labelled graph) homomorphisms $\phi : H \to G$ with $\phi(v_i) \in U_i$. We abbreviate $c(H, G)$ for the total number of homomorphisms from $H$ to the induced subgraph on $U$. We think of $c(H, G[U]; U, \ldots, U)$ as the number of (non-induced) labelled copies of $H$ in $G$ with each vertex in a predetermined subset, counting degenerate ones.

Definition 2. If $H$ is a fixed graph, we say $G$ satisfies $P^*_H(p)$ if

$$c(H, G[U]) = p^{\epsilon(H)} |U|^{\nu(H)} \pm \epsilon \nu(G)^{\nu(H)}.$$

In other words, the condition $P^*_H(p)$ is that every induced subgraph contains the right number of copies of $H$. Note for any fixed $\epsilon > 0$ and $n$ sufficiently large, the random graph $G(n, p)$ satisfies $P^*_H(p)$ almost surely. Simonovits and Sós [9] proved using the Szemerédi’s regularity lemma that $P^*_H(p)$ are all equivalent, in the sense that for any $H, H', \epsilon$, there exists $\delta$ such that $P^*_H(\delta) \implies P^*_{H'}(\epsilon)$. Unfortunately, the dependence of $\delta$ on $\epsilon$ in their proof is of tower type because of the use of the regularity lemma. Note that it suffices to show that $P^*_H(\delta) \implies P^*_K(\epsilon)$ where $K$ is the graph with a single edge; the other direction is given by a straightforward counting lemma, which we state as Lemma 10 below.

Conlon, Fox, and Sudakov [3] were able to tailor the regularity method to this problem to prove the same result with polynomial dependence of the form $\delta = \Omega(\epsilon^{f(p, \nu(H))})$ where $f$ depends only on $p$ and $\nu(H)$. They conjectured that the dependence is in fact linear, and proved it for the clique case $H = K_n$.

Conjecture 3. For any graph $H$, and real numbers $p \in [0, 1]$, $\delta > 0$, we have

$$P^*_H(\delta) \implies P^*_K(\epsilon)$$

for some $\epsilon = O_H(p)(\delta)$.

Independently, Reiher and Schacht [8] showed a similar polynomial dependence for a stronger notion of quasirandomness which takes configurations into account.
Definition 4. If $H$ is a fixed graph, we say $G$ satisfies $R_{H,p}(\epsilon)$ if for every sequence of $v(H)$ disjoint vertex subsets $U_1, \ldots, U_{v(H)} \subseteq G$,

$$c(H, G; U_1, \ldots, U_{v(H)}) = p_{v(H)}^{\epsilon |U_i|} \pm \epsilon v(G)^{v(H)}.$$ 

In this paper we show the linear dependence in Conjecture 3 using the stronger condition $R_{H,p}(\epsilon)$.

Theorem 5. For any graph $H$, and real numbers $p \in [0, 1]$, $\delta > 0$, we have

$$R_{H,p}(\delta) \implies P_{K_2,p}^*(\epsilon)$$

for some $\epsilon = O_{H}(p^{-3e(H)}\delta)$.

The converse is a standard counting lemma; we show it in Lemma 10.

We will begin by proving that $R_{H,p}(\epsilon)$ is equivalent up to linear change of $\epsilon$ to $R'_{H,p}(\epsilon)$, which is the same condition with disjointness removed.

Definition 6. If $H$ is a fixed graph, we say $G$ satisfies $R'_{H,p}(\epsilon)$ if for every sequence of $v(H)$ (not necessarily disjoint) vertex subsets $U_1, \ldots, U_{v(H)} \subseteq G$,

$$c(H, G; U_1, \ldots, U_{v(H)}) = p_{v(H)}^{\epsilon |U_i|} \pm \epsilon v(G)^{v(H)}.$$ 

After this simple argument, we show that the argument of Conlon, Fox, and Sudakov which gives linear dependence in $P_{H,p}^*(\delta) \implies P_{K_2,p}^*(\epsilon)$ when $H$ is a clique extends naturally to all $H$ under the stronger condition $R'_{H,p}(\epsilon)$. Note that because cliques are completely transitive graphs, it is easily shown that $R'_{H,p}(\epsilon)$ is equivalent to $P_{H,p}^*(\epsilon)$ in this case, up to linear change in $\epsilon$.

The original conjecture of Conlon, Fox, Sudakov remains open. By Theorem 5, it suffices to show that $P_{H,p}^*(\epsilon)$ and $R_{H,p}(\epsilon)$ are equivalent up to linear change in $\epsilon$.

Conjecture 7. For any graph $H$, and real numbers $p \in [0, 1]$, $\delta > 0$, we have

$$P_{H,p}^*(\delta) \implies R_{H,p}(\epsilon)$$

for some $\epsilon = O_{H,p}(\delta)$.

The other direction is easy by inclusion-exclusion.

3 Preliminaries

Here we reduce $R_{H,p}(\epsilon)$ to $R'_{H,p}(\epsilon)$ and then collect some standard results that we need from graph theory.
Lemma 8. For any graph $H$, and real numbers $p \in [0, 1]$, $\delta > 0$, we have

$$R_{H,p}(\delta) \implies R'_{H,p}(\epsilon)$$

for some $\epsilon = O_H(\delta)$.

Proof. If $K$ is a graph on the integers in $[1, v(H)]$, we say a graph $G$ satisfies condition $R^K_{H,p}(\epsilon)$ if for every sequence of vertex subsets $U_1, \ldots, U_{v(H)} \subseteq G$ such that $U_i \cap U_j = \emptyset$ whenever $(i, j)$ is an edge of $K$,

$$c(H, G; U_1, \ldots, U_{v(H)}) = p^{\epsilon(H)} \prod_{i=1}^{v(H)} |U_i| \pm \epsilon v(G)^{v(H)}.$$

When $K$ is complete, $R^K_{H,p}(\epsilon)$ is exactly $R_{H,p}(\epsilon)$ and when $K$ is empty it is $R'_{H,p}(\epsilon)$.

To prove the lemma inductively, it suffices to show that if $K'$ has one more edge than $K$, $R^K_{H,p}(\delta) \implies R^{K'}_{H,p}(\epsilon)$ for some $\delta = O_H(\epsilon)$.

Let $G$ satisfy $R^{K'}_{H,p}(\delta)$. Without loss of generality $K' = K \cup (1, 2)$. Let $U_1, \ldots, U_{v(H)}$ be a sequence of vertex subsets of $G$ such that $U_i \cap U_j = \emptyset$ whenever $(i, j)$ is an edge of $K$. We will to show that

$$c(H, G; U_1, \ldots, U_{v(H)}) = p^{\epsilon(H)} \prod_{i=1}^{v(H)} |U_i| \pm (6 + o(1))\delta v(G)^{v(H)}.$$

If $U_1 \cap U_2 = \emptyset$ we are immediately done. Otherwise, write

$$c(H, G; U_1, U_2, \ldots) = c(H, G; U_1 \setminus U_2, U_2, \ldots) + c(H, G; U_1 \cap U_2, U_2 \setminus U_1, \ldots) + c(H, G; U_1 \cap U_2, U_1 \cap U_2, \ldots).$$

We pick up a $2\delta v(G)^{v(H)}$ error from directly applying $R^{K'}_{H,p}(\delta)$ to the first two terms. It remains to show that assuming $U_1 = U_2 = U$,

$$c(H, G; U, U, U_3, \ldots, U_{v(H)}) = p^{\epsilon(H)} \prod_{i=1}^{v(H)} |U_i| \pm (4 + o(1))\delta v(G)^{v(H)}.$$

For this, we may assume (by adding or removing a vertex if necessary) that $|U|$ is even, since a single vertex lies in a negligible $O(v(G)^{v(H) - 1})$ copies of $H$. Pick $U'_1, U'_2$ to be a random equitable bipartition of $U$, i.e. uniformly out of all the ways to evenly divide $U$ into two equal subsets. The number of homomorphisms $\phi : H \to G$ with $\phi(v_1) = \phi(v_2)$ is $O(v(G)^{v(H) - 1})$. Apart from these, each homomorphism in $c(H, G; U_1, U_2, U_3, \ldots)$ is counted in $c(H, G; U'_1, U'_2, U_3, \ldots)$ with probability $1/4$, and so by linearity of expectation,

$$E[c(H, G; U'_1, U'_2, U_3, \ldots)] = \frac{1}{4}c(H, G; U, U, U_3, \ldots) + O(v(G)^{v(H) - 1}).$$
On the other hand, because \( U'_1 \) and \( U'_2 \) are disjoint, the count on the left is controlled by \( R^*_{H,p}(\delta) \), and so

\[
c(H, G; U, U, U_3, \ldots) = p^{e(H)} \prod_{i=1}^{v(H)} |U_i| \pm (4 + o(1))\delta v(G)^{v(H)},
\]
as desired. In particular we have shown the result with

\[
\epsilon \leq \left(6^{v(H)} + o(1)\right)\delta.
\]

\[\square\]

The next lemma is needed to give a preliminary lower bound on the edge density of a graph satisfying \( R^*_{H,p}(\delta) \). It is a corollary of a stronger bound of Alon [1].

**Lemma 9.** If \( H \) is a graph with no isolated vertices, for any graph \( G \)

\[
c(H, G) = O(e(G)^{v(H)}).
\]

We also reiterate the standard counting lemma, see for example Section 10.5 of Lovász’ problem book [7]. It tells us how to count copies of \( H \) given quasirandomness. If \( A, B \subseteq V(G) \), let \( e(A, B) = c(K_2, G, A, B) \) be the number of edges between \( A \) and \( B \), with the caveat that if \( A \) and \( B \) intersect we count each edge within \( G[A \cap B] \) twice. In particular, \( e(A, A) = 2e(A) \), since the former counts labelled edges.

**Lemma 10.** If \( G, H \) are graphs and \( G \) satisfies \( P^*_{K_2,p}(\delta) \) then \( G \) satisfies \( R^*_{H,p}(4e(H)\delta) \).

**Proof.** Since

\[
e(A, B) = e(A \cup B) + e(A \cap B) - e(A \setminus B) - e(B \setminus A),
\]

\( P^*_{K_2,p}(\delta) \) implies that for every pair \( A, B \subseteq V(G) \),

\[
\left| e(A, B) - p|A||B| \right| \leq 4\delta v(G)^2.
\]

Let \( 1_G(u, v) \) be the indicator function of edges of \( G \). Another way of writing the above inequality is that for any \( f, g : V(G) \rightarrow \{0, 1\} \),

\[
\left| \sum_{u, v \in V(G)} f(u)g(v)(1_G(u, v) - p) \right| \leq 4\delta v(G)^2. \tag{3.1}
\]

We show the remaining calculation when \( H = K_3 \); it generalizes easily. The quantity \( c(H, G; A, B, C) \) in question satisfies

\[
c(H, G; A, B, C) - p^3|A||B||C| = \sum_{u, v, w \in V(G)} 1_A(u)1_B(v)1_C(w)\left(1_G(u, v)1_G(v, w)1_G(w, u) - p^3\right)
\]

\[
= \sum_{w \in V(G)} 1_C(w) \sum_{u, v \in V(G)} 1_A(u)1_B(v)(1_G(u, v) - p)1_G(v, w)1_G(w, u) + p \sum_{u \in V(G)} 1_A(u) \sum_{v, w \in V(G)} 1_B(v)1_C(w)(1_G(v, w) - p)1_G(w, u)
\]

\[
+ p^2 \sum_{v \in V(G)} 1_B(v) \sum_{w, u \in V(G)} 1_C(w)1_A(u)(1_G(w, u) - p),
\]

\[\]
and so applying (3.1) to each of the inner double sums, we obtain by the triangle inequality
\[
\left| c(H, G; A, B, C) - p^3|A||B||C| \right| \leq 4(1 + p + p^2)\delta v(G)^3 \\
\leq 12\delta v(G)^3,
\]
as desired. In general we expand into a telescoping sum with \(e(H)\) terms, whence the bound \(4e(H)\delta v(G)^3\).

Finally, we require a corollary of a lemma of Erdős, Goldberg, Pach, and Spencer [4].

**Lemma 11.** Let \(G\) be a graph with edge density \(q = c(K_2, G)\). If there is a subset \(S \subseteq V(G)\) for which \(|e(S) - q(|S|)| \geq D\), then there exists a set \(S' \subseteq V(G)\) of order \(\frac{1}{4}v(G)\) such that
\[
|e(S') - q\left(\frac{|S'|}{2}\right)| \geq \left(\frac{1}{4} + o(1)\right)D,
\]
where \(o(1)\) goes to zero as a function of \(D\).

**Corollary 12.** Let \(G\) be a graph with edge density \(q = c(K_2, G)\). If there is a subset \(S \subseteq V(G)\) for which \(|e(S) - q(|S|)| \geq D\), then there exist two disjoint subsets \(X, Y \subseteq V(G)\) of size \(\frac{1}{4}v(G)\) such that
\[
|e(X) - e(Y)| \geq \left(\frac{1}{16} + o(1)\right)D,
\]
where \(o(1)\) goes to zero as a function of \(D\).

**Proof.** Apply Lemma 11 to find a set \(S'\) with the stated properties. Now, pick a uniformly random subset \(A \subseteq V(G)\), pick \(X = A \cap S'\), and pick a uniformly random subset \(Y \subseteq V(G) \setminus A\). The marginal distribution of \(X\) is just the uniform distribution on subsets of \(S'\), and the marginal distribution of \(Y\) is just the distribution obtained by independently adding each \(v \in V(G)\) to \(Y\) with probability \(\frac{1}{4}\). Thus \(|X|, e(X)| should be tightly concentrated about \(\frac{1}{2}|S'|, \frac{1}{4}e(S')\) and \(|Y|, e(Y)| should be tightly concentrated about \(\frac{1}{4}v(G), \frac{1}{16}e(G)\).

Applying Azuma’s inequality we see that with high probability \(e(X) = (\frac{1}{4} + o(1))e(S')\) and \(e(Y) = (\frac{1}{16} + o(1))e(G)\). Meanwhile \(|X|, |Y|\) are binomially distributed with the same mean and variance, and thus are both within \(o(v(G))\) of \(\frac{1}{4}v(G)\) with high probability. Adding or removing \(o(v(G))\) edges and therefore \(o(v(G)^2)\) edges, we obtain \(|X| = |Y| = \frac{1}{4}v(G)\) without significantly changing their edge densities. The result follows. \(\square\)

### 4 The Main Lemma

With the stronger quasirandomness condition \(R_{H,p}^r(\delta)\), we can extend the proof for the clique case in Conlon, Fox, Sudakov [3]. Fix a vertex \(v_0 \in H\) with degree \(r\). We will want to control the average difference of \(|d^r(u) - d^r(v)|\) over all pairs \(u, v \in G\), where \(d^r(u)\) is the \(r\)-th power of the degree of \(u\). The arguments are essentially identical.

**Definition 13.** Let \(H' = H \setminus \{v_0\}\). Define \(c(u, v)\) to be the number of pairs of homomorphisms \(\phi, \psi : H \to G\) such that \(\phi(v_0) = u, \psi(v_0) = v\), and \(\phi|_{H'} = \psi|_{H'}\), i.e. the number of copies of \(H'\) in \(G\) that extend to a copy of \(H\) when we add either \(u\) or \(v\) for the first vertex.
Lemma 14. Let $H$ be a graph with a vertex $v_0$ of degree $r$, and let $G$ be a graph that satisfies $R_{H,p}(\delta)$. Then,
\[
\sum_{u,v \in V(G)} |d^r(u) - d^r(v)| = O(p^{-\epsilon(H)} \delta v(G)^{r+2}).
\]

Proof. We control the left hand side by estimating $c(u, v) \approx p^\epsilon(H) d^r(u) v(G)^{v(H) - r - 1}$. By the triangle inequality
\[
\sum_{u,v \in V(G)} |d^r(u) - d^r(v)| \leq 2p^{-\epsilon(H)} v(G)^{-v(H) + r + 1} \sum_{u,v \in V(G)} |p^{-\epsilon(H)} d^r(u) v(G)^{v(H) - r - 1} - c(u, v)|.
\]

Now for a fixed $u$, the sum over $v$ is
\[
\Sigma_u = \sum_{v \in V(G)} |c(u, v) - p^{-\epsilon(H)} d^r(u) v(G)^{v(H) - r - 1}|.
\]
Breaking up $V(G) = V^+ \cup V^-$ where $c(u, v) - p^{-\epsilon(H)} d^r(u) v(G)^{v(H) - r - 1}$ is nonnegative on $v \in V^+$ and negative on $v \in V^-$, the corresponding sums are
\[
\Sigma_u = \Sigma_u^+ - \Sigma_u^-.
\]
where
\[
\Sigma_u^* = \sum_{v \in V^*} \left( c(u, v) - p^{-\epsilon(H)} d^r(u) v(G)^{v(H) - r - 1} \right)
\]
\[
= \left( \sum_{v \in V^*} c(u, v) \right) - p^{-\epsilon(H)} d^r(u) |V^*| v(G)^{v(H) - r - 1},
\]
where $\ast$ is $+$ or $-$. On the other hand, for a fixed $u$, the sum of $c(u, v)$ over $v \in V^*$ is exactly
\[
\sum_{v \in V^*} c(u, v) = c(H, G; v^*, \underbrace{N(u), \ldots, N(u)}_{r}, \underbrace{V(G), \ldots, V(G)}_{v(H) - r - 1}).
\]
By $R_{H,p}(\delta)$ it follows that
\[
|\Sigma_u^*| \leq \delta v(G)^{v(H)},
\]
and so
\[
\sum_{u,v} |d^r(u) - d^r(v)| \leq 4p^{-\epsilon(H)} v(G)^{-v(H) + r + 1} \sum_u \delta v(G)^{v(H)} = O(p^{-\epsilon(H)} \delta v(G)^{r+2}). \]

\]

In the language of Reiher and Schacht [8], the total sum of $c(u, v)$ is the number of copies in $G$ of the graph $K$ obtained from $H$ by “doubling” the first vertex.
5 Proof of the Main Theorem 5

Conlon, Fox, and Sudakov [3] prove the following elementary inequality to fully exploit the above degree bound.

**Lemma 15.** (Corollary 2.2 from [3].) Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be two sets of $n$ non-negative integers. Then, for any $r \in \mathbb{N}$,

$$
\sum_{i,j=1}^{n} |b_j^r - a_i^r| \geq \sum_{i=1}^{n} b_j - \sum_{i=1}^{n} a_i .
$$

Now we finish the proof of Theorem 5 in the following form.

**Lemma 16.** If $H$ is a connected graph and $G$ is a graph satisfying $R_{H,p}^r(\delta)$, then $G$ satisfies $P_{K_2,p}^r(\epsilon)$ for some $\epsilon = O_H(p^{-3e(H)\delta})$.

**Proof.** Suppose $G$ satisfies $R_{H,p}^r(\delta)$. In particular, $c(H, G) \geq p^{e(H)v(G)v(H)} - \delta v(G)v(H)$. We can assume that $\delta$ is negligible compared to $p^{e(H)v(G)v(H)}$, so by Lemma 9,

$$
p^{e(H)v(G)v(H)} = O(e(G)v(H))
$$

and, we get a lower bound

$$
q = \Omega(p^{e(H)/v(H)})
$$

on the edge density $q = c(K_2, G)$ of $G$. Now we show that $G$ satisfies $P_{K_2,q}^r(\gamma)$ where $\gamma = O_H(q^{-r+1}p^{-e(H)\delta})$, and then that $|p - q|$ is small.

Let $r$ be the minimum degree of $H$. Suppose $G$ does not satisfy $P_{K_2,q}^r(\gamma)$ for some $\gamma = O_H(q^{-r+1}p^{-e(H)\delta})$, where we choose the implicit constant later. It must therefore contain a subset $S$ such that

$$
|e(S) - q\left(\frac{|S|}{2}\right)| > \gamma v(G)^2.
$$

Applying Corollary 12 and changing the implicit constant in $\gamma$ by a factor of $\frac{1}{16} + o(1)$, we may pick $X, Y$ of size $v(G)/4$ for which

$$
|e(X) - e(Y)| \geq \gamma v(G)^2,
$$

Without loss of generality, $e(X) > e(Y)$. Let $G' = G[X \cup Y]$, and apply Lemma 14 to this graph. Writing $d_{G'}(u)$ for the degree in $G'$ of $u$.

$$
\sum_{u,v \in V(G')} |d_{G'}^r(u) - d_{G'}^r(v)| = O(p^{-e(H)\delta}v(G')^{r+2}).
$$

On the other hand, this sum contains the differences $d_{G'}^r(u) - d_{G'}^r(v)$ where $(u, v) \in X \times Y$,
and these are large on average by Lemma 15:

\[
\sum_{u, v \in V(G')} |d'_{G'}(u) - d'_{G'}(v)| \geq \sum_{u \in X} \sum_{v \in Y} |d'_{G'}(u) - d'_{G'}(v)| \\
\geq \sum_{u \in X} \sum_{v \in Y} d'_{G'}^{-1}(u) d_{G'}(u) - \sum_{v \in Y} d_{G'}(v) \\
\geq |X| \left( \frac{1}{|X|} \sum_{u \in X} d_{G'}(u) \right)^{r-1} (e(X) - e(Y)) \\
= \Omega(q^{r-1} v(G)^{r+2}).
\]

Combining with the upper bound (5.2), it follows that

\[ q^{r-1} \gamma = O(p^{-e(H)\delta}), \]

which contradicts our choice of \( \gamma \) if the constant in \( \gamma \) is sufficiently large. We have shown that \( G \) must satisfy \( P_{K_2, q}^*(\gamma) \) for some \( \gamma = O(q^{-r+1} p^{-e(H)\delta}) \). But then the counting lemma tells us that \( G \) also satisfies \( R'_{H, q}(4e(H)\gamma) \), whereby the number of labelled density of \( H \) is close to both \( p^{e(H)} \) and \( q^{e(H)} \):

\[
|p^{e(H)} - q^{e(H)}| \leq 4e(H)\gamma + \delta, \\
|p - q| = O_H(p^{-e(H) + 1} \gamma)
\]

and so \( G \) also satisfies \( P_{K_2, p}^*(\epsilon) \) for \( \epsilon = O_H(p^{-3e(H)\delta}) \), as desired. \( \square \)

Combining Lemma 8 with Lemma 16, Theorem 5 is proved.

**Acknowledgements**

The author would like to thank Jacob Fox for calling attention to this problem and for many helpful discussions, and David Conlon and Mathias Schacht for stimulating conversations.

**References**

[1] N. Alon, On the number of subgraphs of prescribed type of graphs with a given number of edges, *Israel J. Math.* 38 (1981), 116–130.

[2] F. R. K. Chung, R. L. Graham and R. M. Wilson, Quasi-random graphs, *Combinatorica* 9 (1989), 345–362.

[3] D. Conlon, J. Fox, and B. Sudakov, Hereditary quasirandomness without regularity, arXiv:1611.02099 [math.CO].

[4] P. Erdős, M. Goldberg, J. Pach and J. Spencer, Cutting a graph into two dissimilar halves, *J. Graph Theory* 12 (1988), 121–131.
[5] A. Frieze and R. Kannan, Quick approximation to matrices and applications, *Combinatorica* 19 (1999), 175–220.

[6] M. Krivelevich and B. Sudakov, Pseudo-random graphs, in More sets, graphs and numbers, 199–262, Bolyai Soc. Math. Stud. 15, Springer, Berlin, 2006.

[7] L. Lovász, *Combinatorial problems and exercises*, 2nd edition. AMS Chelsea Publishing, Providence, RI, 2007.

[8] C. Reiher and M. Schacht, in preparation.

[9] M. Simonovits and V. T. Sós, Hereditarily extended properties, quasi-random graphs and not necessarily induced subgraphs, *Combinatorica* 17 (1997), 577–596.

[10] E. Szemerédi, Regular partitions of graphs, Problèmes combinatoires et théorie des graphes, *Colloq. Internat. CNRS* 260 (1978), 399–401.

[11] A. Thomason, Pseudo-random graphs, in: Proceedings of Random Graphs, Poznań 1985, M. Karoński, ed., *Annals of Discrete Math.* 33 (1987), 307–331.