Abstract. Let $\pi$ be a group. The aim of this paper is to construct the category of Yetter-Drinfeld modules over the quasi-Turaev group coalgebra $H = (\{H_\alpha\}_{\alpha \in \pi}, \Delta, \varepsilon, S, \Phi)$, and prove that this category is isomorphic to the center of the representation category of $H$. Therefore a new Turaev braided group category is constructed.

Keywords: Yetter-Drinfeld module; Quasi-Hopf group coalgebra; Turaev braided group category; Center construction.

Mathematics Subject Classification: 16W30.

Introduction

Given a group $\pi$, Turaev in [6] introduced the notion of a braided $\pi$-monoidal category which is called Turaev braided group category in this paper, and showed that such a category gives rise to a 3-dimensional homotopy quantum field theory. Meanwhile such a category plays a key role in the construction of Hennings-type invariants of flat group-bundles over complements of link in the 3-sphere, see [8].

For the above reasons, it becomes very important to construct Turaev braided group category. Based on the work of [4], more results have been obtained in [2] and [9], where the method used in [4] was applied to weak Hopf algebras and regular multiplier Hopf algebras. It is well-known that there is another approach to the construction, for instance, in [1] the authors introduced the notion of quasi-Hopf group coalgebras and proved that the representation category of quasitriangular quasi-Hopf group coalgebras is exactly a Turaev braided group category.

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M. Zunino in [11] constructed the Yetter-Drinfeld category of crossed Hopf group coalgebra and showed that it is a Turaev braided group category. Motivated by this construction, in this paper, we will generalize this result to quasi-Turaev Hopf group coalgebra defined in [1]. The notion of Yetter-Drinfeld category of quasi-Turaev Hopf group coalgebra will be given, and the isomorphism between Yetter-Drinfeld category and the category of the center of representation category of quasi-Turaev Hopf group coalgebra will be established. Moreover, both of the categories are Turaev braided group categories.

This paper is organized as follows: In section 1, we will recall the notions of crossed $T$-category and its center and quasi-Turaev group coalgebra. In section 2, we will construct the Yetter-Drinfeld module over the quasi-Turaev group coalgebra and prove that the Yetter-Drinfeld category is isomorphic to the center of the representation category.

Throughout this article, let $k$ be a fixed field. All the algebras and linear spaces are over $k$; unadorned $\otimes$ means $\otimes_k$.

1 Preliminary

In this section, we will recall the definitions and notations relevant to Turaev braided group categories.

1.1 Crossed $T$-category

A tensor category $\mathcal{C} = (\mathcal{C}, \otimes, a, l, r)$ is a category $\mathcal{C}$ endowed with a functor $\otimes: \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ (the tensor product), an object $I \in \mathcal{C}$ (the tensor unit), and natural isomorphisms $a = a_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$ for all $U, V, W \in \mathcal{C}$ (the associativity constraint), $l = l_U : I \otimes U \to U$ (the left unit constraint) and $r = r_U : U \otimes I \to U$ (the right unit constraint) for all $U \in \mathcal{C}$ such that for all $U, V, W, X \in \mathcal{C}$, the associativity pentagon

$$a_{U,V,W} \circ a_{U \otimes V,W,X} = (U \otimes a_{V,W,X}) \circ a_{U,V \otimes W,X} \circ (a_{U,V,W} \otimes X),$$

and the triangle

$$(U \otimes l_V) \circ (r_U \otimes V) = a_{U,V} ,$$

are satisfied. A tensor category $\mathcal{C}$ is strict when all the constraints are identities.

Let $\pi$ be a group with the unit 1. Recall from [11] that a crossed category $\mathcal{C}$ (over $\pi$) is given by the following data:

- $\mathcal{C}$ is a tensor category.
- A family of subcategory $\{\mathcal{C}_\alpha\}_{\alpha \in \pi}$ such that $\mathcal{C}$ is a disjont union of this family and that $U \otimes V \in \mathcal{C}_{\alpha \beta}$ for any $\alpha, \beta \in \pi$, $U \in \mathcal{C}_\alpha$ and $V \in \mathcal{C}_\beta$.
- A group homomorphism $\varphi : \pi \to aut(\mathcal{C}), \beta \mapsto \varphi_\beta$, the conjugation, where $aut(\mathcal{C})$ is the group of the invertible strict tensor functors from $\mathcal{C}$ to itself, such that $\varphi_\beta(\mathcal{C}_\alpha) = \mathcal{C}_{\beta \alpha \beta^{-1}}$ for any $\alpha, \beta \in \pi$.
We will use the left index notation in Turaev: Given $\beta \in \pi$ and an object $V \in C$, the functor $\varphi_\beta$ will be denoted by $\beta(\cdot)$ or $V(\cdot)$ and $\beta^{-1}(\cdot)$ will be denoted by $\overline{V}(\cdot)$. Since $V(\cdot)$ is a functor, for any object $U \in C$ and any composition of morphism $g \circ f$ in $C$, we obtain $V(id_U) = id_{V(U)}$ and $V(g \circ f) = Vg \circ Vf$. Since the conjugation $\varphi : \pi \to aut(C)$ is a group homomorphism, for any $V, W \in C$, we have $V \otimes W(\cdot) = V(W(\cdot))$ and $1(\cdot) = V(\overline{V}(\cdot)) = id_C$. Since for any $V \in C$, the functor $V(\cdot)$ is strict, we have $V(f \otimes g) = Vf \otimes Vg$ for any morphism $f$ and $g$ in $C$, and $V(1) = 1$.

A Turaev braided $\pi$-category is a crossed $T$-category $C$ endowed with a braiding, i.e., a family of isomorphisms

$$c = \{c_{U,V} : U \otimes V \to V(U \otimes V)\}_{U,V \in C}$$

obeying the following conditions:

- For any morphism $f \in Hom_{C_\alpha}(U,U')$ and $g \in Hom_{C_\beta}(V,V')$, we have
  $$(\alpha g \otimes f) \circ c_{U,V} = c_{U',V'} \circ (f \otimes g),$$

- For all $U, V, W \in C$, we have
  $$c_{U,V \otimes W} = a_{U,V,W,U}^{-1} \circ (U \otimes c_{U,W} \circ (c_{U,V} \otimes W)) \circ a_{U,V,W}^{-1}, \quad (1.1)$$
  $$c_{U \otimes V, W} = a_{U \otimes V, W,U} \circ (c_{U,V \otimes W} \circ (U \otimes c_{V,W} \circ (c_{V,W} \otimes V))) \circ a_{U,V,W}^{-1}. \quad (1.2)$$

- For any $U, V \in C$ and $\alpha \in \pi$, $\varphi_\alpha(c_{U,V}) = \varphi_\alpha(U) \circ \alpha V$.

1.2 The center of a crossed $T$-category

Let $C$ be a crossed $T$-category. The center of $C$ is the braided crossed $T$-category $\mathcal{Z}(C)$ defined as follows:

1. The objects of $\mathcal{Z}(C)$ are the pairs $(U, c_{U,\cdot})$ satisfying the following conditions:
   - $U$ is an object of $C$.
   - $c_{U,\cdot}$ is a natural isomorphism from the functor $U \otimes \cdot$ to the functor $U(\cdot) \otimes U$ such that for any objects $V, W \in C$, the identity (1.1) is satisfied.

2. The morphism in $\mathcal{Z}(C)$ from $(U, c_{U,\cdot})$ to $(V, c'_{V,\cdot})$ is a morphism $f : U \to V$ such that for any object $X \in C$,
   $$(U X \otimes f) \circ c_{U,X} = c'_{V,X} \circ (f \otimes X). \quad (1.3)$$

The composition of two morphisms in $\mathcal{Z}(C)$ is given by the composition in $C$. 

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3. Given $Z_1 = (U, c_{U, -})$ and $Z_2 = (V, c_{V, -})$ in $\mathcal{Z}(\mathcal{C})$, the tensor product $Z_1 \otimes Z_2$ in $\mathcal{Z}(\mathcal{C})$ is the couple $(U \otimes V, (c \otimes c')_{U \otimes V, -})$, where for any object $W \in \mathcal{C}$, $(c \otimes c')_{U \otimes V, -}$ is obtained by

$$(c \otimes c')_{U \otimes V, W} = a_{U \otimes V, W} \circ (c_{U, V} \otimes V) \circ a_{U, V, W}^{-1} \circ (U \otimes c'_{V, W}) \circ a_{U, V, W}. \quad (1.4)$$

4. The unit of $\mathcal{Z}(\mathcal{C})$ is the couple $(I, id_{-})$, where $I$ is the unit of $\mathcal{C}$.

5. For any $\alpha \in \pi$, the $\alpha$th component of $\mathcal{Z}(\mathcal{C})$, denoted $Z_\alpha(\mathcal{C})$, is the full subcategory of $\mathcal{Z}(\mathcal{C})$ whose objects are the pairs $(U, c_{U, -})$, where $U \in \mathcal{C}_\alpha$.

6. For any $\beta \in \pi$, the automorphism $\varphi_{Z, \beta}$ is given by, for any $(U, c_{U, -}) \in \mathcal{Z}(\mathcal{C})$,

$$\varphi_{Z, \beta}(U, c_{U, -}) = (\varphi_{\beta}(U), \varphi_{Z, \beta}(c_{U, -})), \quad (1.5)$$

where $\varphi_{Z, \beta}(c_{U, -}) \varphi_{\beta}(U, c_{U, -}) = \varphi_{\beta}(c_{U, \varphi_{\beta}^{-1}(U)})$ for any $X \in \mathcal{C}$.

7. The braiding $\varepsilon$ in $\mathcal{Z}(\mathcal{C})$ is obtained by setting $c_{Z_1, Z_2} = c_{U, V}$ for any $Z_1 = (U, c_{U, -}), Z_2 = (V, c_{V, -}) \in \mathcal{Z}(\mathcal{C})$.

### 1.3 Quasi-Turaev group coalgebras

Recall from [1], a family of algebras $H = \{H_\alpha\}_{\alpha \in \pi}$ is a quasi-semi-T-coalgebra if there exist a family of morphisms of algebra $\Delta = \{\Delta_{\alpha, \beta} : H_\alpha \otimes H_\beta \to H_\gamma\}_{\alpha, \beta \in \pi}$, a morphism of algebra $\varepsilon : H_1 \to k$ and a family of invertible elements $\{\Phi_{\alpha, \beta, \gamma} \in H_\alpha \otimes H_\beta \otimes H_\gamma\}_{\alpha, \beta, \gamma \in \pi}$ such that

$$(H_\alpha \otimes \Delta_{\beta, \gamma}) \Delta_{\alpha, \beta, \gamma}(h) \Phi_{\alpha, \beta, \gamma} = \Phi_{\alpha, \beta, \gamma} \Delta_{\alpha, \beta} \otimes H_\gamma \Delta_{\alpha, \beta, \gamma}(h), \quad (1.6)$$

$$(H_\alpha \otimes \varepsilon)(\Delta_{\alpha, 1}(a)) = a, \quad (\varepsilon \otimes H_\alpha)(\Delta_{1, \alpha}(a)) = a, \quad (1.7)$$

$$(1_\alpha \otimes \Phi_{\beta, \gamma, \lambda})(H_\alpha \otimes \Delta_{\beta, \gamma} \otimes H_\lambda)(\Phi_{\alpha, \beta, \gamma, \lambda})(\Phi_{\alpha, \beta, \gamma} \otimes 1_\lambda) = (H_\alpha \otimes H_\beta \otimes \Delta_{\lambda, \gamma})(\Phi_{\alpha, \beta, \gamma, \lambda})(\Delta_{\alpha, \beta} \otimes H_\gamma \otimes H_\lambda)(\Phi_{\alpha, \beta, \gamma, \lambda}), \quad (1.8)$$

$$(H_\alpha \otimes \varepsilon \otimes H_\beta)(1_\alpha \otimes 1_\beta \otimes 1_\gamma) = 1_\alpha \otimes 1_\beta \otimes 1_\gamma \quad (1.9)$$

for all $h \in H_{\alpha, \beta, \gamma}$ and $a \in H_\alpha$. $\Delta$ is called comultiplication, and $\varepsilon$ the counit.

In our computations, we will use the Sweedler-Heyneman notation $\Delta_{a, b}(b) = b_{(1, a)} \otimes b_{(2, b)}$ for all $b \in H_{a, \beta}$ (summation implicitly understood). Since $\Delta$ is only quasi-coassociative, we adopt further convention

$$(id_\alpha \otimes \Delta_{\beta, \gamma}) \Delta_{\alpha, \beta, \gamma}(h) = h_{(1, \alpha)} \otimes h_{(2, \beta, \gamma)}(1, \beta) \otimes h_{(2, \beta, \gamma)}(2, \gamma),$$

$$(\Delta_{a, b} \otimes id_\gamma) \Delta_{\alpha, \beta, \gamma}(h) = h_{(1, a, \beta)}(1, \alpha) \otimes h_{(1, a, \beta)}(2, \beta) \otimes h_{(2, \gamma)},$$
for all $h \in H_{\alpha\beta\gamma}$. We will denote the components of $\Phi$ by capital letters, and the ones of $\Phi^{-1}$ by small letters, namely,

$$\Phi_{\alpha,\beta,\gamma} = Y_1^\alpha \otimes Y_2^\beta \otimes Y_3^\gamma = T_1^\alpha \otimes T_2^\beta \otimes T_3^\gamma = \cdots$$

$$\Phi_{\alpha,\beta,\gamma}^{-1} = y_1^\alpha \otimes y_2^\beta \otimes y_3^\gamma = t_1^\alpha \otimes t_2^\beta \otimes t_3^\gamma = \cdots$$

A quasi-Hopf group coalgebra is a quasi-semi-T-coalgebra $H = \{H_{\alpha}\}_{\alpha \in \pi}, \Delta, \varepsilon, S$ endowed with a family of invertible anti-automorphisms of algebra $S = \{S_{\alpha} : H_{\alpha} \to H_{\alpha}^{-1}\}_{\alpha \in \pi}$ (the antipode) and elements $\{p_{\alpha}, q_{\alpha} \in H_{\alpha}\}_{\alpha \in \pi}$ such that the following conditions hold:

$$S_{\alpha}(h_{(1,\alpha)})p_{\alpha^{-1}}h_{(2,\alpha^{-1})} = \varepsilon(h)p_{\alpha^{-1}}, \quad h_{(1,\alpha)}q_{\alpha}S_{\alpha^{-1}}(h_{(2,\alpha^{-1})}) = \varepsilon(h)q_{\alpha}, \quad (1.10)$$

$$Y_1^\alpha q_{\alpha}S_{\alpha^{-1}}(Y_2^\alpha p_{\alpha}Y_3^\alpha) = 1_{\alpha}, \quad \varepsilon \circ \Phi_{\alpha,\beta,\gamma} = \varepsilon \circ \Phi_{\alpha,\beta,\gamma}. \quad (1.11)$$

A quasi-Turaev $\pi$-coalgebra is a quasi-Hopf $\pi$-coalgebra $H = \{H_{\alpha}\}_{\alpha \in \pi}, \Delta, \varepsilon, \Phi$ with a family of $k$-linear maps $\varphi = \{\varphi_{\beta} : H_{\alpha} \to H_{\beta\alpha^{-1}}\}_{\alpha,\beta \in \pi}$ (the crossing) such that the following conditions hold:

• For any $\beta \in \pi$, $\varphi_{\beta}$ is an algebra isomorphism.

• $\varphi_{\beta}$ preserves the comultiplication and the counit, i.e., for any $\alpha, \beta, \gamma \in \pi$,

$$\varphi_{\beta} \circ \Delta_{\alpha,\gamma} = \Delta_{\beta\alpha^{-1},\beta\gamma^{-1}} \circ \varphi_{\beta},$$

$$\varepsilon \circ \varphi_{\beta} = \varepsilon.$$  

• $\varphi$ is multiplicative in the sense that $\varphi_{\beta} \varphi_{\beta'} = \varphi_{\beta\beta'}$ for all $\beta, \beta' \in \pi$.

• The family $\Phi$ is invariant under the crossing, i.e., for any $\Phi_{\alpha,\beta,\gamma}$,

$$(\varphi_{\eta} \otimes \varphi_{\theta} \otimes \varphi_{\theta})\Phi_{\alpha,\beta,\gamma} = \Phi_{\eta\alpha^{-1},\theta\beta^{-1},\theta\gamma^{-1}}.$$  

## 2 Main results

In this section, we will give the main result of this paper. First of all, we need some preparations. For any Hopf group coalgebra $H = \{H_{\alpha}\}, \Delta, \varepsilon, S$, we obviously have the following identity

$$h_{(1,\alpha)} \otimes h_{(2,\beta)}S_{\beta^{-1}}(h_{(3,\beta^{-1})}) = h \otimes 1_{\beta},$$

for all $\alpha, \beta \in \pi$ and $h \in H_{\alpha}$. We will need the generalization of this formula to the quasi-Hopf group coalgebra setting. The following lemma will be given without proof.
Lemma 2.1. Let $H = (\{H_{\alpha}\}, \Delta, \epsilon, S)$ be a quasi-Hopf group coalgebra. Set

\begin{align*}
I_{\alpha,\beta}^R &= I_{\alpha}^1 \otimes I_{\beta}^2 = y_{\alpha}^1 \otimes y_{\beta}^2 q_{\beta^{-1}}(y_{\beta^{-1}}^3), \\
J_{\alpha,\beta}^R &= J_{\alpha}^1 \otimes J_{\beta}^2 = Y_{\alpha}^1 \otimes S_{\beta^{-1}}^{-1}(p_{\beta^{-1}} Y_{\beta^{-1}}^3) Y_{\beta}^2, \\
I_{\alpha,\beta}^L &= I_{\alpha}^1 \otimes I_{\beta}^2 = Y_{\alpha}^2 \Delta^{-1}(Y_{\alpha^{-1}}^1 q_{\alpha^{-1}}) \otimes Y_{\beta}^3, \\
J_{\alpha,\beta}^L &= J_{\alpha}^1 \otimes J_{\beta}^2 = S_{\alpha^{-1}}(y_{\alpha^{-1}}^1 p_{\alpha} y_{\alpha}^2 \otimes y_{\beta}^3).
\end{align*}

Then for all $h \in H_{\alpha}$ and $a \in H_{\beta}$, we have

\begin{align*}
\Delta_{\alpha,\beta}(h_{(1,\alpha\beta)}) I_{\alpha,\beta}^R [1 \otimes S_{\beta^{-1}}^{-1}(h_{(2,\beta^{-1})})] = I_{\alpha,\beta}^R [h \otimes 1], \\
[1 \otimes S_{\beta^{-1}}^{-1}(h_{(2,\beta^{-1})})] J_{\alpha,\beta}^R \Delta_{\alpha,\beta}(h_{(1,\alpha\beta)}) = [h \otimes 1] J_{\alpha,\beta}^R, \\
\Delta_{\alpha,\beta}(a_{(2,\alpha\beta)}) I_{\alpha,\beta}^L [S_{\alpha^{-1}}^{-1}(a_{(1,\alpha^{-1})}) \otimes 1] = I_{\alpha,\beta}^L [1 \otimes a], \\
[S_{\alpha^{-1}}^{-1}(a_{(1,\alpha^{-1})}) \otimes 1] J_{\alpha,\beta}^L \Delta_{\alpha,\beta}(a_{(2,\alpha\beta)}) = J_{\alpha,\beta}^L [1 \otimes a].
\end{align*}

And the following relations hold:

\begin{align*}
\Delta_{\alpha,\beta}(J_{\alpha,\beta}^1) I_{\alpha,\beta}^R [1 \otimes S_{\beta^{-1}}^{-1}(J_{\beta^{-1}}^2)] &= 1_{\alpha} \otimes 1_{\beta}, \\
[1 \otimes S_{\beta^{-1}}^{-1}(J_{\beta^{-1}}^2)] J_{\alpha,\beta}^R \Delta_{\alpha,\beta}(J_{\alpha,\beta}^1) &= 1_{\alpha} \otimes 1_{\beta}, \\
\Delta_{\alpha,\beta}(J_{\alpha,\beta}^2) I_{\alpha,\beta}^L [S_{\alpha^{-1}}^{-1}(J_{\alpha^{-1}}^1) \otimes 1_{\beta}] &= 1_{\alpha} \otimes 1_{\beta}, \\
[S_{\alpha^{-1}}^{-1}(J_{\alpha^{-1}}^1) \otimes 1_{\beta}] J_{\alpha,\beta}^L \Delta_{\alpha,\beta}(J_{\alpha,\beta}^2) &= 1_{\alpha} \otimes 1_{\beta}.
\end{align*}

In [11], M. Zunino defined the Yetter-Drinfeld module over the crossed group coalgebra, and S. Majid in [3] ingeniously constructed the Yetter-Drinfeld module over quasi-Hopf algebra. With these help, we have the following definition.

Definition 2.2. Fix an element $\alpha \in \pi$. An $\alpha$-Yetter-Drinfeld module or $YD_{\alpha}$-module is a couple $V = \{V, \rho_V = \{\rho_{V,\lambda}\}_{\lambda \in \pi}\}$, where $\rho_{V,\lambda} : V \to V \otimes H_\lambda$, $v \mapsto v_{(0)} \otimes v_{(1,\lambda)}$ is a $k$-linear morphism such that the following conditions are satisfied:

1. $V$ is a left $H_\alpha$-module,
2. $V$ is counitary in the sense that
   \[(id \otimes \epsilon) \circ \rho_{V,1} = id.\]
3. For all $v \in V$,
   \[
   (y_{\alpha}^3 \cdot v_{(0)})_{(0,0)} \otimes (y_{\alpha}^3 \cdot v_{(0,0)})_{(1,\lambda_1)} y_{\lambda_1}^1 \otimes y_{\lambda_2}^3 v_{(1,\lambda_2)}
   = \Phi_{\alpha,\lambda_1,\lambda_2}^{-1} \cdot [(y_{\alpha}^3 \cdot v)_{(0)} \otimes (y_{\alpha}^3 \cdot v)_{(1,\lambda_1\lambda_2)} y_{\lambda_1}^1 \otimes (y_{\alpha}^3 \cdot v)_{(1,\lambda_1\lambda_2)(2,\lambda_2)} y_{\lambda_2}^3].
   \]
4. For all $h \in H_{\alpha\beta}$ and $v \in V$,
   \[
   h_{(1,\alpha)} \cdot v_{(0,0)} \otimes h_{(2,\beta)} v_{(1,\beta)} = (h_{(2,\alpha)} \cdot v)_{(0)} \otimes (h_{(2,\alpha)} \cdot v)_{(1,\beta)} \varphi_{\alpha^{-1}}(h_{(1,\alpha\beta^{-1})}).
   \]
Remark 2.3. Note that in the above definition, when the quasi-Hopf group coalgebra is trivial, i.e., \( \varphi_{\alpha, \beta, \lambda} = 1_\alpha \otimes 1_\beta \otimes 1_\lambda \) for any \( \alpha, \beta, \lambda \in \pi \), then we have a \( \text{YD}_\alpha \)-module over Hopf group coalgebra introduced in [11].

Given two \( \text{YD}_\alpha \)-modules \((U, \rho_U)\) and \((V, \rho_V)\), a linear map \( f : U \rightarrow V \) is said to be a morphism of \( \text{YD}_\alpha \)-module if \( f \) is \( H_\alpha \)-linear and for any \( \lambda \in \pi \),

\[
\rho_{V, \lambda} \circ f = (f \otimes H_\lambda) \circ \rho_{U, \lambda}.
\]

Let \( \text{YD}(H) \) be the disjoint union of the categories \( \text{YD}_\alpha(H) \) for all \( \alpha \in \pi \). The category \( \text{YD}(H) \) admits the structure of a braided \( T \)-category as follows:

- The tensor product of a \( \text{YD}_\alpha \)-module \((V, \rho_V)\) and a \( \text{YD}_\beta \)-module \((W, \rho_W)\) is a \( \text{YD}_{\alpha \beta} \)-module \((V \otimes W, \rho_{V \otimes W})\), where for any \( v \in V, w \in W \) and \( \lambda \in \pi \),

\[
\rho_{V \otimes W}(v \otimes w) = t_{\lambda}^1 Y_{\lambda}^1 (y_{\alpha}^2 \cdot v)(0,0) \otimes t_{\lambda}^2 (Y_{\beta}^2 y_{\beta}^3 \cdot w)(0,0) \otimes t_{\lambda}^3 (Y_{\lambda}^2 (y_{\alpha}^2 \cdot v)(1,\lambda) Y_{\lambda}^2 (y_{\alpha}^2 \cdot v)(1,\beta \lambda - 1)) y_{\lambda}^3. \tag{2.16}
\]

The unit of \( \text{YD}(H) \) is the pair \((k, \rho_k)\), where for any \( \lambda \in \pi \), \( \rho_{\lambda}(1) = 1 \otimes 1_\lambda \). Then the tensor product of arrows is given by the tensor product of \( k \)-linear maps.

- For any \( \beta \in \pi \), the conjugation functor \( \beta(\cdot) \) is given as follows. Let \((V, \rho_V)\) be a \( \text{YD}_\alpha \)-module and we set \( \beta(V, \rho_V) = (\beta V, \rho_{\beta V}) \), where for any \( \lambda \in \pi \) and \( v \in V \),

\[
\rho_{\beta V}(v) = \beta((\beta^{-1} v)(0,0)) \otimes \varphi_{\beta}(\beta^{-1} v)(1,\beta^{-1} \lambda) \lambda). \tag{2.17}
\]

For any morphism \( f : (V, \rho_V) \rightarrow (W, \rho_W) \) of \( \text{YD} \)-module and any \( v \in V \), we set \( \beta(f)(\beta v) = \beta(f(v)) \).

- For any \( \text{YD}_\alpha \)-module \((V, \rho_V)\) and any \( \text{YD}_\beta \)-module \((W, \rho_W)\), the braiding \( c \) is given by

\[
c_{V,W}(v \otimes w) = \alpha[J_{(1, \beta)}^1 y_{\beta}^1 S_{\beta - 1} (J_{(2, \beta)}^2 y_{\beta}^3 - 1) (I_{(1, \beta)}^2 v)(1, \beta - 1) I_{(1, \beta - 1)}^1 v)(1, \xi - 1)] y_{\lambda}^1 \otimes J_{(2, \alpha)}^1 y_{\alpha}^2 (I_{(2, \alpha)}^2 v)(0,0). \tag{2.18}
\]

Lemma 2.4. For a fixed element \( \alpha \in \pi \), let \((V, c_{V,-})\) be any object in \( Z_{\alpha}(\text{Rep}(H)) \). For any \( \lambda \in \pi \), define the linear map \( \rho_{V, \lambda} : V \rightarrow V \otimes H_\lambda \) by

\[
\rho_{V, \lambda}(v) = c_{V, H_\lambda}^1 (a_1 \lambda \otimes v). \tag{2.19}
\]

Then the pair \( V = (V, \rho_V = \{\rho_{V, \lambda}\}_{\lambda \in \pi}) \) is a \( \text{YD}_\alpha \)-module. Hence we have a functor \( F_1 : Z(\text{Rep}(H)) \rightarrow \text{YD}(H) \) given by \( F_1(V, c_{V,-}) = (V, \rho_V) \) and \( F_1(f) = f \), where \( f \) is a morphism in \( Z(\text{Rep}(H)) \).

Proof. We just need to verify that \((V, \rho_V)\) satisfies the axioms of \( \text{YD}_\alpha \)-modules.

First of all, for any \( \lambda_1, \lambda_2 \in \pi \), consider \( H_{\lambda_1} \) and \( H_{\lambda_2} \) as the modules over themselves. By [1,1], we have

\[
a_{V,H_{\lambda_1},H_{\lambda_2}}^{-1} c_{V,H_{\lambda_1} \otimes H_{\lambda_2}}^{-1} \circ a_{V,H_{\lambda_1},V,H_{\lambda_2}}^{-1} = (c_{V,H_{\lambda_1} \otimes H_{\lambda_2}}^{-1} \circ a_{V,H_{\lambda_1},V,H_{\lambda_2}}^{-1} \circ (V \otimes H_{\lambda_1} \otimes c_{V,H_{\lambda_2}}^{-1}).
\]

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For all \( v \in V \), both of the sides evaluating at \( \alpha \lambda_1 \otimes \alpha \lambda_2 \otimes v \), we have

\[
(y^2_\alpha \cdot v(0,0))(0,0) \otimes (y^2_\alpha \cdot v(0,0))(1,\lambda_1) y^1_\lambda \otimes y^3_\lambda v(1,\lambda_2) \\
= y_{\alpha,\lambda_1,\lambda_2} \cdot [(y^3_\alpha \cdot v)(0,0) \otimes (y^3_\alpha \cdot v)(1,\lambda_1,\lambda_2)(1,\lambda_1) y^1_\lambda \otimes (y^3_\alpha \cdot v)(1,\lambda_1,\lambda_2)(2,\lambda_2) y^2_\lambda].
\]

The counitarity of \( V \) is obvious.

Secondly for all \( v \in V \) and \( h \in H_{\alpha,\lambda} \), we have on one hand,

\[
h \cdot c^{-1}_{V,H}(\alpha \lambda \otimes v) = h \cdot (v(0,0) \otimes v(1,\lambda)) = h_{(1,\alpha)} \cdot v(0,0) \otimes h_{(2,\lambda)} v(1,\lambda),
\]

and on the other hand,

\[
c^{-1}_{V,H}(h \cdot (\alpha \lambda \otimes v)) = c_{V,H}(h \cdot (1,\alpha,\lambda_{\alpha^{-1}}) \cdot \alpha \lambda \otimes h_{(2,\alpha)} \cdot v))
\]

\[
= c_{V,H}(\alpha (\varphi_{\alpha^{-1}}(h_{(1,\alpha,\lambda_{\alpha^{-1}})})) \otimes h_{(2,\alpha)} \cdot v))
\]

\[
= (h_{(2,\alpha)} \cdot v)(0,0) \otimes (h_{(2,\alpha)} \cdot v)(1,\lambda) \varphi_{\alpha^{-1}}(h_{(1,\alpha,\lambda_{\alpha^{-1}})}).
\]

Since the braiding \( c_{V,H} \) is \( H \)-linear, we obtain

\[
h_{(1,\alpha)} \cdot v(0,0) \otimes h_{(2,\lambda)} v(1,\lambda) = (h_{(1,\alpha)} \cdot v)(0,0) \otimes (h_{(2,\alpha)} \cdot v)(1,\lambda) \varphi_{\alpha^{-1}}(h_{(1,\alpha,\lambda_{\alpha^{-1}})}).
\]

Finally, let \( f : (V, c_{V,-}) \to (W, c_{W,-}) \) is a morphism in \( Z_\alpha(Rep(H)) \), then as the case of Hopf group coalgebra, \( f \) gives rise to a morphism of \( YD_\alpha \)-module. It is easy to see that \( F_1 \) is a functor. This completes the proof. \( \Box \)

Assume that \((V, \rho_V)\) is an object in the category \( YD_\alpha(H) \). For any \( \lambda \in \pi \) and left \( H_\lambda \)-module \( X \), give the linear map \( c_{V,X} : V \otimes X \to \alpha X \otimes V \) by

\[
c_{V,X}(v \otimes x) = \alpha_[(1,\lambda)] J_{(1,\lambda)}^1 y_{\lambda_{\alpha^{-1}}} \cdot (J_{\lambda_{\alpha^{-1}}}^2 y_{\lambda_{\alpha^{-1}}} (\tilde{F}_\alpha \cdot v)(1,\lambda_{\alpha^{-1}}) \tilde{J}_{\lambda_{\alpha^{-1}}}^1 \cdot x) \otimes J_{(2,\alpha)}^1 y_{\alpha} \cdot (\tilde{F}_\alpha \cdot v)(0,0),
\]

for all \( v \in V \) and \( x \in X \).

**Lemma 2.5.** The couple \((V, c_{V,-})\) is an object in \( Z(Rep(H)) \). Hence we have a functor \( F_2 : YD(H) \to Z(Rep(H)) \) given by \( F_2(V, \rho_V) = (V, c_{V,-}) \) and \( F_2(f) = f \), where \( f \) is a morphism in \( YD(H) \). The functors \( F_1 \) and \( F_2 \) are inverses.

**Proof.** Firstly for any \( \lambda \in \pi \) and left \( H_\lambda \)-module \( X \), we set a morphism \( \hat{c}_{V,X} : \alpha X \otimes V \to V \otimes X \) by

\[
\hat{c}_{V,X}(\alpha x \otimes v) = v(0,0) \otimes v(1,\lambda) \cdot x.
\]
Then
\[ \hat{c}_{V,X} \circ c_{V,X} (v \otimes x) = \hat{c}_{V,X}(\ alpha J^1_{(\lambda_1)} y^{1}_{\lambda_1} S_{\lambda_1}^{-1} (J^2_{\lambda_1} - y^{3}_{\lambda_1} (\tilde{t}^2_{\lambda_1} \cdot v)(1,\lambda_1^{-1} \tilde{t}^1_{\lambda_1} \cdot x)) \otimes J^1_{(\alpha)} y^{2}_{\alpha} \cdot (\tilde{t}^2_{\alpha} \cdot v)(0,0)) = [J^1_{(\alpha)} y^{2}_{\alpha} \cdot (\tilde{t}^2_{\alpha} \cdot v)(0,0)](0,0) \otimes [J^1_{(\alpha)} y^{2}_{\alpha} \cdot (\tilde{t}^2_{\alpha} \cdot v)(0,0)](1,\lambda_1) \varphi_{\alpha}^{-1}(J^1_{(1,\alpha \lambda_1^{-1})}) \varphi_{\alpha}^{-1}(y^{1}_{\alpha \lambda_1^{-1}}) \cdot [S_{\lambda_1}^{-1} (J^2_{\lambda_1} - y^{3}_{\lambda_1} (\tilde{t}^2_{\lambda_1} \cdot v)(1,\lambda_1^{-1} \tilde{t}^1_{\lambda_1} \cdot x)] \]

(ii) \[ \hat{c}_{V,X}(\alpha x \otimes v) = \hat{c}_{V,X}(\alpha (y^{2}_{\alpha} \cdot (\tilde{t}^2_{\alpha} \cdot v)(0,0)) \otimes \varphi_{\alpha}^{-1}(J^1_{(1,\lambda_1^{-1})}) \varphi_{\alpha}^{-1}(y^{1}_{\lambda_1^{-1}}) \cdot [S_{\lambda_1}^{-1} (J^2_{\lambda_1} - y^{3}_{\lambda_1} (\tilde{t}^2_{\lambda_1} \cdot v)(1,\lambda_1^{-1} \tilde{t}^1_{\lambda_1} \cdot x)] \]

\[ \hat{c}_{V,X}(\alpha x \otimes v) = \hat{c}_{V,X}(\alpha (y^{2}_{\alpha} \cdot (\tilde{t}^2_{\alpha} \cdot v)(0,0)) \otimes \varphi_{\alpha}^{-1}(J^1_{(1,\lambda_1^{-1})}) \varphi_{\alpha}^{-1}(y^{1}_{\lambda_1^{-1}}) \cdot [S_{\lambda_1}^{-1} (J^2_{\lambda_1} - y^{3}_{\lambda_1} (\tilde{t}^2_{\lambda_1} \cdot v)(1,\lambda_1^{-1} \tilde{t}^1_{\lambda_1} \cdot x)] \]

Hence \( \hat{c}_{V,X} \circ c_{V,X} = id_{V \otimes X} \). Similarly \( \hat{c}_{V,X} \circ c_{V,X} = id_{X \otimes V} \). Therefore \( \hat{c}_{V,X} \) and \( c_{V,X} \) are inverses.

Secondly for any \( h \in H_{\alpha \lambda} \),
\[ h \cdot c^{-1}_{V,X}(\alpha x \otimes v) = h_{(1,\alpha)} \cdot v(0,0) \otimes h_{(2,\alpha)} v(1,\lambda) \cdot x \]

That is, \( c^{-1}_{V,X} \) is \( H_{\alpha \lambda} \)-linear, so is \( c_{V,X} \). The naturality of \( c_{V,X} \) is straightforward to verify.

Next suppose that \( X_1 \) is an \( H_{\lambda_1} \)-module and \( X_2 \) an \( H_{\lambda_2} \)-module for all \( \lambda_1, \lambda_2 \in \pi \), and for any \( x_1 \in X_1, x_2 \in X_2 \),
\[ a_{V,X_1, X_2} \circ (c^{-1}_{V,X_1} \otimes X_2) \circ a_{\alpha X_1, V,X_2} \circ (\alpha X_1 \otimes c^{-1}_{V,X_2} \circ a_{\alpha X_1, \alpha X_2, V}(\alpha x_1 \otimes \alpha x_2 \otimes v) \]

\[ = T^1_{\lambda_1} \cdot [y^{2}_{\alpha} \cdot (Y^{3}_{\alpha} \cdot v)(0,0)](0,0) \otimes T^2_{\lambda_2} \cdot [y^{2}_{\alpha} \cdot (Y^{3}_{\alpha} \cdot v)(0,0)](1,\lambda_1) Y_{\lambda_1}^{-1} Y_{\lambda_1}^{1} \cdot x_1 \]

\[ \otimes T^3_{\lambda_2} y^{2}_{\lambda_2}(Y^{3}_{\alpha} \cdot v)(1,\lambda_2) Y_{\lambda_2}^{1} \cdot x_2 \]

\[ = (y^{3}_{\alpha} Y^{3}_{\alpha} \cdot v)(0,0) \otimes (y^{3}_{\alpha} Y^{3}_{\alpha} \cdot v)(1,\lambda_1 \lambda_2) (1,\lambda_1) Y_{\lambda_1}^{1} \cdot x_1 \otimes (y^{3}_{\alpha} Y^{3}_{\alpha} \cdot v)(1,\lambda_1 \lambda_2) (2,\lambda_2) Y_{\lambda_2}^{2} \cdot x_2 \]

\[ = v(0,0) \otimes v(1,\lambda_1 \lambda_2)(1,\lambda_1) \cdot x_1 \otimes v(1,\lambda_1 \lambda_2)(2,\lambda_2) Y_{\lambda_2}^{2} \cdot x_2 \]

\[ = c^{-1}_{V,X_1 \otimes X_2}(\alpha x_1 \otimes \alpha x_2 \otimes v) \].

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Let $V, W$ be YD$_\alpha$-modules, $f : V \to W$ be any morphism of YD$_\alpha$-module. For any $H_\lambda$-module $X$ and $x \in X$,

$$c_{W,X} \circ (f \otimes \text{id})(v \otimes x) = c_{W,X}(f(v) \otimes x)$$

$$= \alpha[J^1_{(1,\lambda)} y_\lambda^1 S_{\lambda^{-1}}(J^2_{\lambda^{-1}} y_\lambda^2 (\tilde{I}_\alpha \cdot f(v)) (1,\lambda^{-1}) \tilde{I}_\lambda^{-1}) \cdot x] \otimes J^1_{(2,\alpha)} y_\alpha^2 \cdot (\tilde{I}_\alpha \cdot f(v))(0,0)$$

$$= \alpha[J^1_{(1,\lambda)} y_\lambda^1 S_{\lambda^{-1}}(J^2_{\lambda^{-1}} y_\lambda^2 (f(\tilde{I}_\alpha \cdot v)) (1,\lambda^{-1}) \tilde{I}_\lambda^{-1}) \cdot x] \otimes J^1_{(2,\alpha)} y_\alpha^2 \cdot (f(\tilde{I}_\alpha \cdot v))(0,0)$$

$$= \alpha[J^1_{(1,\lambda)} y_\lambda^1 S_{\lambda^{-1}}(J^2_{\lambda^{-1}} y_\lambda^2 (\tilde{I}_\alpha \cdot v)(1,\lambda^{-1}) \tilde{I}_\lambda^{-1}) \cdot x] \otimes J^1_{(2,\alpha)} y_\alpha^2 \cdot f((\tilde{I}_\alpha \cdot v))(0,0)$$

$$= \alpha(\text{id} \otimes f) c_{V,X}(v \otimes x).$$

That is, $f$ is a morphism in $\mathcal{Z}(\text{Rep}(H))$. Finally by similar arguments in [11], we know that $F_1$ and $F_2$ are inverses. This completes the proof.

**Theorem 2.6.** The category YD($H$) is isomorphic to the category $\mathcal{Z}(\text{Rep}(H))$. This isomorphism induces the structure of braided $T$-category on YD($H$).

**Proof.** This isomorphism holds via the functors $F_1$ and $F_2$.

Let $(V, \rho_V)$ be a YD$_\alpha$-module and $(W, \rho_W)$ be a YD$_\beta$-module. Suppose that $(V, c_{V,-}) = F_2(V, \rho_V)$ and $(W, c'_{W,-}) = F_2(W, \rho_W)$ and set

$$(V, \rho_V) \otimes (W, \rho_W) = F_1(F_2(V, \rho_V) \otimes F_2(W, \rho_W)) = F_1(V \otimes W, (c \otimes c')_{V \otimes W,-}).$$

For any $v \in V, w \in W$, we have

$$\rho_{V \otimes W,-}(v \otimes w) = ((c \otimes c')_{V \otimes W,H}\lambda^{-1}(\alpha \beta 1 \otimes v \otimes w)$$

$$= a_{V,W,H}^{-1}(V \otimes c_{W,H}^{-1}) \circ a_{V,\beta H,W} \circ (c_{V,\beta H}^{-1} \otimes W) \circ a_{H,H,W}^{-1}(\alpha \beta 1 \otimes v \otimes w)$$

$$= y_a^1 Y_a^1 \cdot (t_{\alpha} a \cdot v)(0,0) \otimes y_\beta^2 \cdot (Y_\beta^2 t_{\beta} \cdot w)(0,0)$$

$$\otimes y_\lambda^3 (Y_\lambda^3 t_{\lambda} \cdot w)(1,\lambda) Y_\lambda^1 \nu_{\beta^{-1}}(t_{\alpha} a \cdot v)(1,\beta \lambda^{-1} t_{\lambda}^1)$$

where

$$(c_{V,\beta H}^{-1} \otimes W)(\alpha \beta t_{\lambda}^1 \otimes t_{\alpha} a \cdot v \otimes t_{\beta}^3 \cdot w)$$

$$= (t_{\alpha} a \cdot v)(0,0) \otimes \beta [\nu_{\beta^{-1}}(t_{\alpha} a \cdot v)(1,\beta \lambda^{-1} t_{\lambda}^1) \otimes t_{\beta}^3 \cdot w].$$

The part concerning the tensor unit of YD($H$) is trivial. By similar arguments in [11], we can verify the condition (2.16)–(2.18). This completes the proof.

**Acknowledgement**

This work was supported by the NSF of China (No. 11371088) and the NSF of Jiangsu Province (No. BK2012736).

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