Study of the contact problem of Bernoulli-Euler nano beams using gradient elasticity theory.

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Abstract: A mathematical model of the package for two nano Bernoulli-Euler beams connected through boundary conditions is constructed. Used gradient theory of deformations and the theory of the account of contact interaction of B. Ya. Kantor. The influence of the length dimension parameters on the modes of beam vibrations is studied.

1. Introduction
Micro- and nano-beams, plates, and shells are widely applied in micro- and nano-electromechanical systems such as vibration sensors [1], micro-cables [2], and micro-switches [3]. Dependence of elastic behaviour on the body dimensions in the micro scale has been experimentally observed in metals [4, 5] and alloys [6] as well as polymers [7] and crystals [8], biomembranes [9].

The classical mechanics of a rigid body does allow neither for interpretation nor prediction of the micron- and submicron-scale behaviour due to lack of a parameter controlling the scale effects. In recent years, many efforts have been devoted to propose numerous theories allowing for modelling the scale effects in continua, including the couple stress theory [10, 11], non-local theory of elasticity [12], gradient theory of elasticity [13] as well as theory of surface elasticity [14].

Beams and beam designs are widely used as elements of numerous micro devices in modern engineering. In many cases, the structural elements of the devices are under the influence of complex external dynamic excitations. Investigations of non-linear dynamics and contact interactions of the beam structures belong to important (but unsolved) challenging problems in the field of fabricating various sensors and amplifiers.

2. Problem formulation
The studied mechanical structure consists of two beams occupying a 2D part of the space $\mathbb{R}^2$ with the rectangular system of coordinates introduced in the following way: in the body of beams the so-called reference line $z=0$ is fixed, the axis OX goes from left to right along the middle beams line, whereas the axis OZ tends down and is perpendicular to OX. In the given system of coordinates the mentioned structure is defined as the 2D space: $\Omega = \left\{ x \in [0,a], \frac{-h}{2} \leq z \leq \frac{h}{2}, 0 \leq t < \infty \right\}$ (Figure 1).

The imposed fundamental hypotheses and assumptions are as follows:
(i) each beam consists of one layer;
(ii) beams are isotropic, elastic and they obey the Hooke’s law;
(iii) a beam longitudinal dimension essentially over creases its transverse dimension and its unit thickness;
(iv) the undeformed axis is a straight line;
(v) during the deformation of the beam, load acts along the OZ axis and the external forces do not change their directions;
(vi) contact pressure is within the Kantor model \[15\];
(vii) normal stresses on the surfaces are parallel to each other and they are negligibly small;

![Figure 1](image)

**Figure 1.** The computation scheme of two beams with a clearance.

Based on the higher order stress theory of Mindlin \[10\] and Lam et al. \[8\] proposed the theory of elasticity of the deformation gradient, in which, in addition to the classical equations of equilibrium of forces and moments, a new additional equilibrium equation is introduced, which determines the behavior of stresses of higher orders and the equation of equilibrium of moments. Three parameters of the material length scale are introduced for isotropic linear elastic materials \((l_0, l_1, l_2)\). According to this theory, the total strain energy density is a function of the symmetric strain tensor, the dilation gradient vector, the deviator tension gradient tensor, and the symmetric rotation gradient tensor.

In this paper, a mathematical model of Bernoulli-Euler nanobeams connected through boundary conditions under the action of transverse load is constructed. Three material length scale parameters are introduced to account for dimensional effects \((l_0, l_1, l_2)\). To account for the contact between the beams, a Winkler coupling between the compression and the contact pressure between the two beams is used \[15\]:

\[
\Psi = \frac{1}{2} \left[ 1 + \text{sign}(w_i - h_i - w_z) \right], \quad (1)
\]

where \(\Psi = 1\), if \(w_i > w_z + h_i\) to that is, there is contact between the plate and the beam, else \(\Psi = 0\), \(w_1, w_2, h_i\) - deflections of the first and second beams and the gap between them, respectively.

The mathematical model of contact between two nanobeams, based on the kinematic Bernoulli-Euler hypothesis, is described by a system of resolving equations:

\[
\begin{align*}
\frac{\partial^6 w}{\partial x^6} & \left( \lambda + 2\mu \right) + \frac{1}{12} l_i^2 \mu bh + \frac{96}{225} l_i^2 \frac{\mu bh}{l_i^2} - \\
& - \frac{\partial^6 w}{\partial x^6} \left( \frac{2 l_i^2 \mu bh}{12} + \frac{14}{225} \frac{l_i^2 \mu bh}{12} \right) - q_m +
\end{align*}
\]

\[
+ (-1)^m K (w_i - w_z - \hat{h}) \Psi = \frac{\partial^2 w}{\partial t^2} + \varepsilon \frac{\partial w}{\partial t}, \quad (2)
\]

where \(m\) - beam number \((m=1,2)\), \(\hat{h}\) - the gap between the beams. Boundary conditions are:

\[
w_m = 0; \quad \frac{\partial w_m}{\partial x} = 0. \quad (3)
\]

Initial conditions are:
\[ w_m(x,0) = 0; \quad \dot{w}_m(x,0) = 0. \quad (4) \]

The system (2-4) was reduced to the Cauchy problem using the finite difference method \( O(h^2) \). The Cauchy problem was solved by the Runge-Kutta method of 4 orders. A study of convergence by the method of finite differences, on the basis of which the optimal number of partitions was chosen, was carried out. The partitioning step for the Runge-Kutta method was determined according to the Runge principle.

The so far defined problem is solved with the help of known methods of non-linear dynamics as well the qualitative theory of differential equations. Namely, time histories (signals), phase portraits, Poincaré maps, FFT (Fast Fourier Transform), wavelet transformations, regarding to both beams are constructed and monitored while carrying out the numerical computations. Though various types of wavelets have been tested, the Morlet wavelet has been eventually applied as most suitable for our purpose. In addition, the use of Morlet wavelet has given a new approach to the study of chaotic phase synchronization, understood here as synchronization of phases of oscillations, for chaotic behaviour mechanical system. Note that the phase locking phenomenon implies locking of the frequencies. In the case of the wavelet transform application, the wavelet surface \( W(s,t_0) = \|W(s,t_0)\|\exp[j\phi_0(t_0)] \) presents the system behaviour regarding each time scale \( s \) at an arbitrary time instant \( t_0 \). The magnitude \( \|W(s,t_0)\| \) characterizes the occurrence and intensity of the counterpart time scale \( s \) at time instant \( t_0 \). In addition, the integral energy distribution of the wavelet spectrum versus time scale \( E(s) = \|W(s,t_0)\|^2 dt \) is introduced. The monitored phase \( \phi_0(t_0) = \text{arg}[W(s,t_0)] \) is defined for each of the time scale \( s \), i.e. we are able to follow dynamical behaviour of each time scale \( s \) with the help of the counterpart phases. The occurrence of phase synchronization implies phase locking effects on the synchronized time scales \( s \), i.e. \( \|\phi_0(t) - \phi_0(t)\| < \text{const} \).

For the considered tasks the following parameter values were used: \( a/h = 30, \quad q_l = 300 \sin(5.3t), \quad q_2 = 0, \quad \epsilon = 0.01, \quad \tilde{h} = 0.01. \)

The research results for two of the nine considered combinations of coefficients \((l_0, l_1, l_2)\) are shown in Table 1, 2. For the first case (Table 1.), coefficients \( l_0, l_1 \) are zero and \( l_2 = 0.3 \), i.e. considered beams on the couple stress theory. The second case (Table 2.) is characterized by the values \( l_0 = 0.3; \quad l_1 = 0.3; \quad l_2 = 0.3, \) i.e. all three dimensional factors are taken into account. The figures in the first row show the deflections of the first \( w_1 \) and second beams \( w_2 \). The second line shows the phase synchronization of oscillations \( \varphi_1 - \varphi_2 \), marked in dark color. The third and fourth lines show the Fourier spectrum \( S(\omega) \), the wavelet spectrum based on the Morlet wavelet for the first and second beams, respectively.

3. Conclusion.

Signals (deflections of beams) for values \( l_0 = 0.3, \quad l_1 = 0.3, \quad l_2 = 0.3 \) become mirror symmetric at \( t = 350 \), unlike the first case where symmetry occurs after \( t = 500 \). Thus, it can be said that taking into account all the dimensional parameters of the length of the material increases the stability of the vibrations of the interacting beams. Synchronization at the frequency of the external force for the second case is more clearly expressed in the entire time interval. Comparison of Fourier spectra and wavelet spectra for indicates a qualitative difference between the two cases. At \( l_0 = 0, \quad l_2 = 0, \quad l_1 = 0.3 \) the spectra of both beams are qualitatively similar. It speaks about the equality of the vibration energy. When \( l_0 = 0.3, \quad l_1 = 0.3, \quad l_2 = 0.3 \) spectra differ significantly the First Beam, which is affected by the load, has an energy maximum at the excitation frequency \( \omega_p = 5.3 \), which is absent from the second beam. Phase portraits of both cases for the first and second beams are qualitatively identical. Poincare sections are also similar.
Table 1. Vibration characteristics of beams for the case $l_0 = 0, l_1 = 0, l_2 = 0.3$.

Table 2. Vibration characteristics of beams for the case $l_0 = 0.3, l_1 = 0.3, l_2 = 0.3$.
4. Concluding remarks. The paper presents a mathematical model of contact interaction of nano-beams. The model is based on the kinematic hypothesis of Bernoulli – Euler and gradient elasticity theory. The study of the influence of the values of the dimensional parameters of the length of the material on the nature of vibrations. It is shown that taking into account all three parameters of the length of the material of the significantly changes the form of the signal, Fourier and wavelet spectra, as well as synchronization.

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