A restricted Magnus property for profinite surface groups

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August 26, 2014

Abstract

Magnus [16] proved that, given two elements $x$ and $y$ of a finitely generated free group $F$ with equal normal closures $\langle x \rangle^F = \langle y \rangle^F$, then $x$ is conjugated either to $y$ or $y^{-1}$. More recently ([7] and [8]), this property, called the Magnus property, has been generalized to oriented surface groups.

In this paper, we consider an analogue property for profinite surface groups. While Magnus property, in general, does not hold in the profinite setting, it does hold in some restricted form. In particular, for $\mathcal{S}$ a class of finite groups, we prove that, if $x$ and $y$ are elements of the pro-$\mathcal{S}$ completion $\hat{\Pi}^{\mathcal{S}}$ of an orientable surface group $\Pi$, such that, for all $n \in \mathbb{N}$, there holds $\langle x^n \rangle^{\hat{\Pi}^{\mathcal{S}}} = \langle y^n \rangle^{\hat{\Pi}^{\mathcal{S}}}$, then $x$ is conjugated to $y^s$ for some $s \in (\hat{\mathbb{Z}}^{\mathcal{S}})^*$. As a matter of fact, a much more general property is proved and further extended to a wider class of profinite completions.

The most important application of the above results is to provide a kind of linearization for the complex of profinite curves, introduced by the first author in [4]. This is applied to extend to profinite multi-twists the results of [3] on centralizers of profinite Dehn twists in the congruence completion of the Teichmüller group.

1 Introduction

Let $\Pi$ be an oriented surface group, that is to say the fundamental group of an oriented Riemann surface of finite type.

**Definition 1.1.** A class of finite groups (cf. Definition 3.1 in [11]) is a full subcategory $\mathcal{S}$ of the category of finite groups which is closed under taking subgroups, homomorphic images and extensions (meaning that a short exact sequence of finite groups is in $\mathcal{S}$ whenever its exterior terms are). We always assume that $\mathcal{S}$ contains a nontrivial group.

For $\mathcal{S}$ a class of finite groups, the pro-$\mathcal{S}$ completion $\hat{\Pi}^{\mathcal{S}}$ of $\Pi$ is the inverse limit of the finite quotients of $\Pi$ which belong to $\mathcal{S}$. The profinite group $\hat{\Pi}^{\mathcal{S}}$ is also called a pro-$\mathcal{S}$ surface group.

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Let us give some examples. Fixed a non-empty set of primes \( \Lambda \), let then \( \mathcal{S} \) be the category of finite \( \Lambda \)-groups, i.e. finite groups whose orders are product of primes in \( \Lambda \). In this case, the corresponding profinite completion is denoted by \( \hat{\Pi}^{\Lambda} \) and called the pro-\( \Lambda \) completion of \( \Pi \). The two cases of interest are usually when \( \Lambda \) is the set of all primes, in which case \( \hat{\Pi}^{\Lambda} \) is just the profinite completion \( \hat{\Pi} \) of \( \Pi \), and when \( \Lambda \) consists of only one prime \( p \), in which case \( \hat{\Pi}^{\Lambda} \) is the pro-\( p \) completion of \( \Pi \) and is denoted by \( \hat{\Pi}^{(p)} \). Another important example of class of finite groups is the class of finite solvable groups.

The main purpose of this paper is to prove for the pro-\( \mathcal{S} \) surface group \( \hat{\Pi}^{\mathcal{S}} \) some properties which are analogous to the Magnus property proved for \( \Pi \) in \cite{2} and \cite{3}.

For a given element \( x \) of a (profinite) group \( G \), let us denote by \( \langle x \rangle \) and \( \langle x \rangle^{G} \), respectively, the (closed) subgroup and the (closed) normal subgroup generated (topologically) by \( x \) in \( G \). The Magnus property for the discrete group \( \Pi \) says that if, for two given elements \( x, y \in \Pi \), the normal subgroup \( \langle x \rangle^{\Pi} \) equals the normal subgroup \( \langle y \rangle^{\Pi} \), then \( x \) is conjugated either to \( y \) or \( y^{-1} \). This property cannot be transported literally to the profinite case since \( \hat{\Z} \) has more units than just \( \{\pm 1\} \) and so the property would fail already for \( \hat{\Z} \). Moreover, even if we take this into account, there are counterexamples to the analogue property which can be formulated for the profinite completion \( \hat{\Pi}^{\mathcal{S}} \).

A counterexample for a free profinite group of finite rank is the following. Let us denote by \( M(G) \) the intersection of all maximal normal subgroups of a group \( G \). Let then \( U \) be a normal subgroup of a finitely generated free profinite group \( F \) such that \( U/M(U) \) is a direct product of non-abelian simple groups (for instance, let \( U \) be the kernel of the natural epimorphism of \( F \) onto the maximal prosolvable quotient of \( F \)).

By Proposition 8.3.6 in Chapter 8 of \cite{20}, the subgroup \( U \) is the normal closure of an element \( u \in U \) if and only if \( U/M(U) \) is the normal closure of \( u \cdot M(U) \). Now, in an infinite direct product of non-abelian simple groups, there are plenty of elements and groups generated by them which are non-conjugate but normally generate \( U/M(U) \). For instance, any element with non-trivial projection to every direct simple factor of \( U/M(U) \) has this property. However, having different order in some of the projections, such elements are not conjugate.

A counterexample for the pro-\( p \) case is instead the following. Let \( F = F(x, y) \) be the free pro-\( p \) group in two generators. Then, the normal closure of \( x \) has generators which are not conjugated to powers of \( x \). Indeed, this follows from the fact that \( \langle x \rangle^{F} \), modulo its Frattini subgroup, identifies with the completed group ring \( \mathbb{F}_{p}[\langle \langle y \rangle \rangle] \) and this has more units than just the powers of \( y \).

For this reason, the profinite analogue of Magnus property should be rather formulated saying that, if \( x, y \in \hat{\Pi}^{\mathcal{S}} \) satisfy the stronger condition \( \langle x \rangle^{\hat{\Pi}^{\mathcal{S}}} = \langle y \rangle^{\hat{\Pi}^{\mathcal{S}}} \), for all \( n \) which are a product of primes in \( \Lambda^{\mathcal{S}} \), then \( x \) is conjugated to a power \( y^{s} \), where \( s \) is a unit of the standard pro-\( \mathcal{S} \) cyclic group \( \hat{\Z}^{\mathcal{S}} \), i.e. the pro-\( \mathcal{S} \) completion of \( \Z \). The latter group is more explicitly described as follows. Let \( \Lambda^{\mathcal{S}} \) be the set of primes which occur as orders of groups in \( \mathcal{S} \), there is then a natural isomorphism \( \hat{\Z}^{\mathcal{S}} \cong \prod_{p \in \Lambda^{\mathcal{S}}} \Z_{p} \).

A first instance of the profinite analogue of Magnus property for a free profinite group of finite rank follows from a deep theorem of Wise (cf. \cite{22}, but with the further restriction that one of the two elements be abstract:
Theorem 1.2. Let \( \hat{F} \) be a free profinite group of finite rank and \( y \in F \) be an element of an abstract dense free subgroup \( F \). If, for some \( x \in \hat{F} \) and all \( n \in \mathbb{N}^+ \), there holds that \( x^{k_n} \in \langle y^n \rangle^{\hat{F}} \), for some \( k_n \in \mathbb{N}^+ \), then \( x \) is conjugated to \( y^s \) for some \( s \in \hat{\mathbb{Z}} \).

We do not know in which generality the Magnus property holds for profinite surface groups. In what follows, we will restrict to the case where one of the two elements satisfies some geometric conditions similar to those considered in [7].

Definition 1.3. (i) Let us fix a presentation of \( \Pi \) as the fundamental group of a Riemann surface \( S \). A subset of non-trivial elements \( \sigma = \{ \gamma_1, \ldots, \gamma_h \} \subset \Pi \) is simple if there is a set of disjoint simple closed curves (briefly, s.c.c.’s) \( \tilde{\sigma} = \{ \tilde{\gamma}_1, \ldots, \tilde{\gamma}_h \} \) on \( S \), such that they are two by two non-isotopic and \( \tilde{\gamma}_i \) belongs to the free homotopy class of \( \gamma_i \) for \( i = 1, \ldots, h \). A s.c.c. on \( S \) is peripheral if it bounds a 1-punctured disc.

(ii) Let \( \mathcal{S} \) be a class of finite groups. A pro-\( \mathcal{S} \) surface group \( \hat{\Pi}^{\mathcal{S}} \) is the pro-\( \mathcal{S} \) completion of a surface group \( \Pi \). It is endowed with a natural monomorphism with dense image \( \Pi \to \hat{\Pi}^{\mathcal{S}} \). Let \( \Gamma \) be the group of mapping classes of self-homeomorphisms of \( S \) fixing the base point of \( \Pi \) and let \( \hat{\Gamma}^{\mathcal{S}} \) be its closure in \( \text{Aut}(\hat{\Pi}^{\mathcal{S}}) \). Then, a subset of elements \( \sigma = \{ \gamma_1, \ldots, \gamma_h \} \subset \hat{\Pi}^{\mathcal{S}} \) is simple if it is in the orbit of the image of a simple set \( \sigma' \subset \Pi \) for the action of \( \hat{\Gamma}^{\mathcal{S}} \).

(iii) A subset of elements \( \sigma = \{ \gamma_1, \ldots, \gamma_h \} \subset \hat{\Pi}^{\mathcal{S}} \) is absolutely simple if it is in the \( \text{Aut}(\hat{\Pi}^{\mathcal{S}}) \)-orbit of the image of a set \( \sigma' \subset \Pi \) which is simple for some presentation of \( \Pi \) as the fundamental group of a Riemann surface.

Remark 1.4. Let \( \Pi \) be a free group of rank \( n \) and \( \hat{\Pi}^{\mathcal{S}} \) either its pro-\( \Lambda \) completion, for some non-empty set of primes \( \Lambda \), or its pro-solvable completion. Then, given a minimal set \( \{ \alpha_1, \ldots, \alpha_n \} \) of topological generators for \( \hat{\Pi}^{\Lambda} \), any element of this set and any product of commutators \( \prod_{i=1}^{k} [\alpha_i, \alpha_{n-i}] \) and of commutators and generators of the form \( \prod_{i=1}^{k} [\alpha_i, \alpha_{n-i}] \alpha_{i+1} \alpha_{i+2} \cdots \alpha_j \), for \( 1 \leq k \leq \lfloor n/2 \rfloor \) and \( i+1 \leq j \leq n-k-1 \), is absolutely simple.

Part (i) of the above definition can be rephrased, group-theoretically, saying that there is a graph of groups \( \mathcal{G} \), whose vertex groups are finitely generated free groups of rank at least 2, together with an isomorphism \( \pi_1(\mathcal{G}) \cong \Pi \) which identifies the set of edge groups of \( \mathcal{G} \) with the set of cyclic subgroups of \( \Pi \) generated by non-peripheral elements of \( \sigma \).

Part (iii) just says that, modulo automorphisms, the profinite group \( \hat{\Pi}^{\mathcal{S}} \) can be realized as a profinite completion of a discrete group \( \Pi \) of the above type.

In this setting, we are actually going to prove (cf. [3]) the following stronger statement:

Theorem 1.5. Let \( \sigma = \{ \gamma_1, \ldots, \gamma_h \} \subset \hat{\Pi}^{\mathcal{S}} \) be an absolutely simple subset and let us denote by \( \sigma^*_n \) for \( n \in \mathbb{N}^+ \), the closed normal subgroup of \( \hat{\Pi}^{\mathcal{S}} \) generated by \( n \)-th powers of elements of \( \sigma \). Let \( y \in \hat{\Pi}^{\mathcal{S}} \) be an element such that, for all \( n \) a product of primes in \( \Lambda^{\mathcal{S}} \), there exists a \( k_n \in \mathbb{N}^+ \) with the property that \( y^{k_n} \in \sigma^*_n \). Then, for some \( s \in \hat{\mathbb{Z}}^{\mathcal{S}} \) and \( i \in \{ 1, \ldots, h \} \), the element \( y \) is conjugated to the element \( \gamma_i^s \).
An immediate corollary of Theorem 1.5 is the restricted Magnus property for profinite surface groups, mentioned above:

**Corollary 1.6.** Let \( x \in \hat{\Pi}^\forall \) be an absolutely simple element and \( y \in \hat{\Pi}^\forall \) an arbitrary element. Let us denote by \( \langle x^n \rangle_{\hat{\Pi}^\forall} \) and \( \langle y^n \rangle_{\hat{\Pi}^\forall} \) the closed normal subgroups of \( \hat{\Pi}^\forall \) generated respectively by \( x^n \) and \( y^n \), for \( n \in \mathbb{N}^+ \). If, for all \( n \) a product of primes in \( \Lambda^\forall \), there holds \( \langle x^n \rangle_{\hat{\Pi}^\forall} = \langle y^n \rangle_{\hat{\Pi}^\forall} \), then, for some \( s \in (\hat{\mathbb{Z}}^\forall)^* \), the element \( y \) is conjugated to \( x^s \).

# 2 The proof of Theorem 1.2

In order to prove Theorem 1.2 we need a preliminary result of independent interest, which follows from the work of Wise [24]:

**Theorem 2.1.** Let \( G = \langle F \mid r^n \rangle \) be a one-relator group with torsion and let \( y \in \hat{G} \) be a torsion element of its profinite completion. Then, \( y \) is conjugate in \( \hat{G} \) to a power of the image \( \bar{r} \) of \( r \) in \( G \).

**Proof.** The result is well-known for the discrete group \( G \). Hence it is enough to show that a torsion element \( y \in \hat{G} \) is conjugate to an element of \( G \). Every one-relator group \( G \) embeds naturally into a free product \( G' = G \ast \mathbb{Z} \) which is an HNN extension \( HNN(H,M,t) \) of a simpler one-relator group \( H \), where \( M \) is a free subgroup generated by subsets of the generators of the presentation of \( G \) (cf. the Magnus-Moldavanskii construction in Section 18.b [24]). The hierarchy is finite and terminates at a virtually free group of the form \( \mathbb{Z}/n * F \), where \( F \) is free. Let us use induction on the length of the hierarchy to prove that \( y \) is conjugate to an element of \( G \). If the length is zero, this means that \( r = 1 \) and \( G \) is free and then \( \hat{G} \) is torsion free. In this case, the claim trivially holds.

According to Theorem 18.1 [24], the Magnus-Moldavanskii hierarchy is quasi-convex for any one-relator group with torsion, i.e. for one-relator groups with torsion, the subgroups \( H,M,M' \) are quasi-convex at each level of the hierarchy. Wise showed that \( G \) has a finite index subgroup \( G_0 \) that embeds as a quasi-convex subgroup of a right-angled Artin group. It follows that every quasi-convex subgroup of \( G \) is a virtual retract and is hence separable (cf. Theorem 7.3 [12]).

Thus the hierarchy is separable (including finite index subgroups of the groups of the hierarchy) and so the profinite completion functor extends the hierarchy on \( G \) to a hierarchy on \( \hat{G} \). In particular \( \hat{G}' = \hat{G} \amalg \hat{\mathbb{Z}} = HNN(\hat{H},\hat{M},t) \). By Theorem 3.10 [25], any torsion element of a profinite HNN-extension is conjugate to an element of the base group. We may then assume that our torsion element \( y \) is in \( \hat{H} \) and use the induction hypothesis to conclude that it is conjugated to an element of \( H \) and so of \( G' \). Since \( \hat{G} \) is a free factor of \( \hat{G}' \) and \( y \in \hat{G} \), it follows that \( y \) is actually conjugated to an element of \( G \).

Let us recall that an abstract group \( G \) is **good**, if the natural homomorphism \( G \to \hat{G} \) of the group to its profinite completion induces an isomorphism on cohomology with finite coefficients (cf. Exercises §2.6 [21]). From the proof of Theorem 2.1 and Theorem 1.4 [11], it then follows:
Theorem 2.2. One-relator groups with torsion are good.

Proof of Theorem 2.2. By Theorem 2.1, the element \( x^k \langle y^n \rangle^{\hat{F}} / \langle y^n \rangle^{\hat{F}} \) of the quotient group \( \hat{F} / \langle y^n \rangle^{\hat{F}} \) is conjugated to a power of the element \( y \langle y^n \rangle^{\hat{F}} / \langle y^n \rangle^{\hat{F}} \) for every \( n \in \mathbb{N}^+ \). The result then follows taking the inverse limit of all these quotients for \( n \in \mathbb{N}^+ \). \( \square \)

3 A geometric proof of Theorem 1.5.

An (orientable) Fuchsian group \( \Pi \) is a group which admits a presentation of the form:

\[
\Pi = \langle \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, x_1, \ldots, x_d, y_1, \ldots, y_s | \prod_{i=1}^g [\alpha_i, \beta_i]; x_1^{m_1}, \ldots, x_d^{m_d}, \text{ for } m_1, \ldots, m_d \in \mathbb{N}^+ \rangle.
\]

We say that a Fuchsian group with the above presentation \( \Pi \) is hyperbolic if there holds \( 2g - 2 + d + s > 0 \). Group-theoretically, this may be reformulated by saying that a Fuchsian group \( \Pi \) is hyperbolic if it is not abelian. Hyperbolic Fuchsian groups arise as topological fundamental groups of complex hyperbolic orbicurves (see, for instance, [18] for a definition).

The integer \( g \) is the genus of an orbifold Riemann surface \( S \) whose fundamental group has the standard presentation given to the Fuchsian group \( \Pi \). Let us then observe that the same Fuchsian group as an abstract group can be given distinct presentations corresponding to orbifold Riemann surfaces of different genuses.

Following [18], the order of the Fuchsian group \( \Pi \) is the least common multiple of the integers \( m_1, \ldots, m_d \), i.e. of the orders of the cyclic subgroups generated by \( x_1, \ldots, x_d \) in \( \Pi \). A finite subgroup of \( \Pi \) in the conjugacy class of the cyclic subgroup \( \langle x_i \rangle \), for \( i = 1, \ldots, d \), is called a decomposition group.

Let \( S \) be an orbifold Riemann surface whose fundamental group can be identified with the Fuchsian group \( \Pi \) described above. Then, to a finite index subgroup \( K \) of \( \Pi \), is associated an unramified covering \( S_K \to S \). If \( K \) is a normal subgroup, the covering is normal with covering transformation group the quotient \( G_K := \Pi/K \). The orbifold Riemann surface \( S_K \) is representable, i.e. the decomposition groups of its points are all trivial, if and only if, \( \langle x_i \rangle^a \cap K = \{1\} \), for all \( i = 1, \ldots, n \) and all elements \( a \in \Pi \). If \( K \) is a normal subgroup, then it is enough to ask that \( \langle x_i \rangle \cap K = \{1\} \), for all \( i = 1, \ldots, n \).

The following is a reformulation of Lemma 2.11 [18] in our terminology, of which, however, we prefer to give an independent proof:

Lemma 3.1. Let \( \Pi \) be a Fuchsian group with the presentation given above and let \( \mathcal{S} \) be a class of finite groups. Let us assume that \( \Pi \) contains a torsion free normal subgroup \( H \) such that \( \Pi/H \in \mathcal{S} \). Let \( \hat{\Pi}^{\mathcal{S}} \) be the pro-\( \mathcal{S} \) completion of \( \Pi \). Then, there hold:

(i) \( \langle x_i \rangle \cap \langle x_j \rangle^h \neq \{1\} \), if and only if \( i = j \) and \( h \in \langle x_i \rangle \). In particular, the subgroup \( \langle x_i \rangle \) is self-normalizing in the profinite group \( \hat{\Pi}^{\mathcal{S}} \), for \( i = 1, \ldots, n \).
(ii) Every finite non-trivial subgroup \( C \) of \( \hat{\Pi}^\mathcal{S} \) is contained in a decomposition group, i.e., in a subgroup \( \langle x_i \rangle^h \) for some \( i \in \{1, \ldots, n\} \) and \( h \in \hat{\Pi}^\mathcal{S} \).

Proof. Let \( S \) be an orbifold Riemann surface such that \( \Pi = \pi_1(S) \) and let \( S_H \to S \) be the covering associated to the subgroup \( H \) of \( \Pi \). By hypothesis, \( S_H \) is a Riemann surface. The \( \mathcal{S} \)-solenoid \( S^\mathcal{S} \) (cf. [6], for more details on this construction) is defined to be the inverse limit space \( S^\mathcal{S} := \lim_{\mathcal{S}} S_K \) of the coverings \( S_K \to S \) associated to normal subgroups \( K \) of \( \Pi \) such that \( \Pi/K \in \mathcal{S} \). Let us observe that \( S^\mathcal{S} \cong S_\hat{H}^\mathcal{S} \), where the latter space is the inverse limit of the coverings \( S_K \to S_H \) associated to normal subgroups \( K \) of \( H \) such that \( H/K \in \mathcal{S} \). There is then a series of natural isomorphisms:

\[
H^k(S^\mathcal{S}, \mathbb{Z}/p) \cong H^k(S_\hat{H}^\mathcal{S}, \mathbb{Z}/p) := \lim_{\mathcal{S}} H^k(S_K, \mathbb{Z}/p) \cong \lim_{\mathcal{S}} H^k(K, \mathbb{Z}/p).
\]

It is well known that a surface group is \( p \)-good for all primes \( p \) (cf. and Definition 8.2 [5]). Thus (see, for instance, Theorem 8.3 [5]), there holds \( \lim_{\mathcal{S}} H^k(K, \mathbb{Z}/p) = 0 \), for \( k > 0 \) and all primes \( p \in \Lambda^\mathcal{S} \). It follows that, for all \( p \in \Lambda^\mathcal{S} \), there hold \( H^k(S^\mathcal{S}, \mathbb{Z}/p) = \{0\} \), for \( k > 0 \), and \( H^0(S^\mathcal{S}, \mathbb{Z}/p) = \mathbb{Z}/p \).

There is a natural continuous action of \( \hat{\Pi}^\mathcal{S} \) on the \( \mathcal{S} \)-solenoid \( S^\mathcal{S} \) and the decompositions groups of \( \hat{\Pi}^\mathcal{S} \) naturally identify with the stabilizers of points of \( S^\mathcal{S} \) in the inverse image of points of the orbifold Riemann surface \( S \) which have non-trivial isotropy groups.

In order to prove (i), it is enough to show that the intersection of the stabilizers of two points \( P_1 \) and \( P_2 \) contains a cyclic subgroup \( C_p \) of prime order \( p \in \Lambda^\mathcal{S} \) if and only if there holds \( P_1 = P_2 \).

Let us observe that the solenoid \( S^\mathcal{S} \) can be triangulated by a simplicial profinite set in such a way that the inverse images of the orbifold points of \( S \) with non-trivial isotropy group are realized inside the set of 0-simplices. Therefore, it is possible to apply to the action of \( C_p \) on the solenoid \( S^\mathcal{S} \) the results of [22].

By the results of §5 in [22], item (b) of Theorem 10.5, Chap. VII [9] generalizes to profinite spaces. Therefore, since the profinite space \( S^\mathcal{S} \) is \( p \)-acyclic, for all \( p \in \Lambda^\mathcal{S} \), it follows that \( (S^\mathcal{S})^C \) is also \( p \)-acyclic, where \( (S^\mathcal{S})^C \) is the fixed point set of the action of the \( p \)-group \( C_p \) on the \( \mathcal{S} \)-solenoid \( S^\mathcal{S} \). In particular, \( (S^\mathcal{S})^C \) is connected and thus consists of only one point. In particular, there holds \( P_1 = P_2 \).

Let us now prove (ii). Here, we basically proceed like in the proof of Lemma 2.11 [18]. So, let \( C \) be a finite subgroup of \( \hat{\Pi}^\mathcal{S} \). It then holds \( C \in \mathcal{S} \). Let us assume moreover that \( C \) is solvable. Then, by induction on the order of \( C \), we can further assume that either:

a) \( C \) is of prime order \( p \in \Lambda^\mathcal{S} \);

b) \( C \) is an extension of a group of prime order \( p \in \Lambda^\mathcal{S} \) by a non-trivial subgroup \( C_1 \subseteq C \) which is contained in the decomposition group \( A \).

If a) is satisfied, then, since the space \( S^\mathcal{S} \) is \( p \)-acyclic, there holds \( (S^\mathcal{S})^C \neq \emptyset \), i.e. the subgroup \( C \) is contained in a decomposition group of \( \hat{\Pi}^\mathcal{S} \).
If b) holds, by replacing the profinite group $\hat{\Pi}^S$ with its open subgroup $C_1 \cdot \hat{H}^S$, we can actually assume that $C_1$ is a decomposition group. But then, by (i), it is also self-normalizing and there holds $C_1 = C$.

For $C$ any finite subgroup of $\hat{\Pi}^S$, the above arguments show that the Sylow subgroups of $C$ are cyclic. By a classical result of group theory, the group $C$ is then solvable and we are reduced to the case already treated.

The next step is to generalize Lemma 3.1 to the fundamental group of a graph of hyperbolic Fuchsian groups. More precisely, let $(G, Y)$ be a graph of groups such that the vertex groups $G_v$, for every vertex $v \in v(Y)$, are Fuchsian groups and the edge groups $G_e$ identify with maximal finite subgroups of the vertex groups, for every edge $e \in e(Y)$. Then, we say that $\pi_1(G, Y)$ is an orbifold nodal Riemann surface, i.e. nodally degenerate orbifold Riemann surfaces.

As above, if $X$ is an orbifold nodal Riemann surface, then its fundamental group is virtually torsion free, if and only if $X$ admits a finite étale (eq. finite étale Galois) covering $Y \to X$, where $Y$ is a connected nodal Riemann surface.

**Definition 3.2.** (i) Let $\mathcal{S}$ be a class of finite groups, i.e. closed by taking subgroups, homomorphic images and extensions. A pro-$\mathcal{S}$ nodal Fuchsian group $\hat{\Pi}^\mathcal{S}$ is the pro-$\mathcal{S}$ completion of a nodal Fuchsian group $\Pi$.

(ii) We say that a finite subgroup $D$ of a nodal Fuchsian group $\pi_1(G, Y)$ or of its pro-$\mathcal{S}$ completion $\hat{\pi}_1^\mathcal{S}(G, Y)$ is a decomposition group of type $I$ if it is in the conjugacy class of a decomposition subgroup $I$ of a vertex group of $(G, Y)$.

The following result is a result of independent interest we need in order to generalize Lemma 3.1 to pro-$\mathcal{S}$ nodal Fuchsian groups:

**Lemma 3.3.** Let $\mathcal{S}$ be a class of finite groups and let $(\mathcal{G}, Y)$ be a finite graph of (discrete) groups, with finite edge groups, such that $G = \pi_1(\mathcal{G}, Y)$ is residually-$\mathcal{S}$. Then, the pro-$\mathcal{S}$ completion $\hat{G}^\mathcal{S}$ of $G$ is isomorphic to the pro-$\mathcal{S}$ fundamental group $\hat{\pi}_1^\mathcal{S}(\mathcal{G}^\mathcal{S}, Y)$ of the finite graph of pro-$\mathcal{S}$ groups $(\mathcal{G}^\mathcal{S}, Y)$ obtained from $(\mathcal{G}, Y)$ by taking the pro-$\mathcal{S}$ completion of each vertex group and the vertex groups of $(\mathcal{G}^\mathcal{S}, Y)$ embed in $\hat{G}^\mathcal{S}$.

**Proof.** Since $G$ is residually $\mathcal{S}$ one can find an open (in the pro-$\mathcal{S}$ topology) subgroup $H$ of $G$ that intersects trivially all the edge groups. Then it suffices to show that the pro-$\mathcal{S}$ topology of $G$ or equivalently of $H$ induces the full pro-$\mathcal{S}$ topology on $H \cap \mathcal{G}(v)$. But $H \cap \mathcal{G}(v)$ is a free factor of $H$ so this statement is just Corollary 3.1.6 in [20].

Let us then extend Lemma 3.1 to pro-$\mathcal{S}$ nodal Fuchsian groups:

**Lemma 3.4.** Let $\Pi := \pi_1(\mathcal{G}, Y)$ be a nodal Fuchsian group and let $\mathcal{S}$ be a class of finite groups. Let us assume that $\Pi$ contains a torsion free normal subgroup $H$ such that $\Pi/H \in \mathcal{S}$. Let then $\hat{\Pi}^\mathcal{S}$ be the pro-$\mathcal{S}$ completion of $\Pi$. Let $D_1$ and $D_2$ be decomposition groups of $\hat{\Pi}^\mathcal{S}$ of type $I_1$ and $I_2$, respectively. Then, there hold:
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(i) The profinite group $\hat{\Pi}$ is virtually torsion free.

(ii) $\hat{\Pi} = \hat{\pi}_1(\hat{\mathcal{G}}, Y)$ and the vertex groups of $(\hat{\mathcal{G}}, Y)$ embed in $\hat{\Pi}$.

(iii) $D_1 \cap D_2 \neq \{1\}$, if and only if $D_1 = D_2$ and $I_1, I_2$ are contained and conjugated in a vertex group $G_v$ for some vertex $v \in v(Y)$.

(iv) The decomposition groups of $\hat{\Pi}$ are self-normalizing.

(v) Every finite non-trivial subgroup $C$ of $\hat{\Pi}$ is contained in a decomposition group.

Proof. Since the subgroup $H$ is torsion free, it is a free product of surface groups. Moreover, since $\Pi/H \in \mathcal{S}$, the closure of the subgroup $H$ in $\hat{\Pi}$ coincides with its pro-$\mathcal{S}$ completion $\hat{H}$ and so is a free pro-$\mathcal{S}$ product of pro-$\mathcal{S}$ surface groups. By Theorem 3.10 and Remark 3.18 [25], any finite subgroup of the profinite group $\hat{H}$ is contained in a free factor of $\hat{H}$. As observed in the proof of Lemma 3.3, such a free factor is the pro-$\mathcal{S}$ completion of the corresponding free factor of the group $H$, hence it is a pro-$\mathcal{S}$ surface group, which is torsion free. It follows that $\hat{H}$ is torsion free. This proves (i).

In particular, by the above proof, the group $\Pi = \pi_1(\mathcal{G}, Y)$ is residually-$\mathcal{S}$. Item (ii) then follows from Lemma 3.3.

By Theorem 3.10 and Remark 3.18 [25], any finite subgroup of the profinite group $\hat{\Pi}$ is contained in a vertex group of $(\hat{\mathcal{G}}, Y)$. Moreover, by Theorem 3.12 [25], the intersection of the conjugates of two distinct vertex groups is conjugate to a subgroup of an edge group.

Since the edge groups of $(\hat{\mathcal{G}}, Y)$ identify with decompositions groups of the vertex groups, if $D_1 \cap D_2 \neq \{1\}$, then they are both conjugated to the same decomposition group of some vertex group of $(\hat{\mathcal{G}}, Y)$. Thus, items (iii), (iv), (v) follows from Lemma 3.1.

Proof of Theorem 1.5. It is not restrictive to assume that $\sigma = \{\gamma_1, \ldots, \gamma_h\}$ is a simple subset of $\Pi$. Let $\sigma_n^\Pi$, for $n \in \mathbb{N}^+$, the closed normal subgroup of $\Pi$ generated by $n$-th powers of elements of $\sigma$.

Let us denote by $S_n$ the orbifold nodal Riemann surface obtained topologically from $S$ glueing a disc to each s.c.c. in $\sigma$ with an attaching map of degree $n$. Then, there is a natural isomorphism $\pi_1(S_n) \cong \Pi/\sigma_n^\Pi$.

Therefore, the quotient group $\Pi/\sigma_n^\Pi$ is a nodal Fuchsian group whose decomposition groups are the conjugacy classes of the subgroups generated by the images of the elements in $\sigma$ and the quotient group $\hat{\Pi}/\sigma_n^\Pi$ is the pro-$\mathcal{S}$ completion of $\Pi/\sigma_n^\Pi$.

Let us prove that, if $n$ is a product of primes in $\Lambda_{\mathcal{S}}$, then the nodal Fuchsian group $\Pi/\sigma_n^\Pi$ satisfies the hypotheses of Lemma 3.4. It is enough to show that there is a torsion free, normal subgroup $H$ of the quotient group $\Pi/\sigma_n^\Pi$ of index a product of powers of primes in $\Lambda_{\mathcal{S}}$ and such that the quotient of $\Pi/\sigma_n^\Pi$ by $H$ is a metabelian group, or, equivalently, that there exists a normal, metabelian $\Lambda_{\mathcal{S}}$-covering $S''_n \rightarrow S_n$ such that $S''_n$ is representable.

Let $S' \rightarrow S$ be the abelian $\Lambda_{\mathcal{S}}$-covering associated to the characteristic subgroup $[\Pi, \Pi]^{\Pi n}$ of $\Pi$. This covering has the property that its restriction to every s.c.c. of $S'$,
which covers either a non-separating s.c.c. or, in case there is more than one puncture on $S$, a peripheral s.c.c. on $S$, has degree $n$.

Since $\sigma_n^p < [\Pi, \Pi]\Pi^n$, there is an induced $\Lambda_{\gamma'}$-covering $S'_n \to S_n$ which ramifies with order $n$ over the orbifold points of $S_n$ corresponding to the non-separating and, in case there is more than one puncture on $S$, the peripheral s.c.c.'s in $\sigma$. Therefore the orbifold Riemann surface $S'_n$ is representable over those points.

From the same argument used in the proof of Lemma 3.10 [3], it follows that a s.c.c. contained in the inverse image in $S'$ of a non-peripheral separating s.c.c. $\gamma$ in the set $\sigma$ is non-separating and its image in $S'_n$ is homologically non-trivial. In case there is only one peripheral s.c.c. in $\sigma$, a s.c.c. contained in its inverse image in $S'$ has also homologically non-trivial image in $S'_n$.

Let $\Pi' := [\Pi, \Pi]\Pi^n$. The metabelian $\Lambda_{\gamma'}$-covering $S'' \to S$ associated to the normal subgroup $\sigma_n^p[\Pi', \Pi'](\Pi')^n$ of $\Pi$ has then the property that its restriction to every s.c.c. of $S''$, lying above a s.c.c. of $\sigma$, has degree $n$. Therefore, the induced metabelian $\Lambda_{\gamma'}$-covering $S''_n \to S_n$ is such that $S''_n$ is representable.

For an element $a \in \hat{\Pi}_{\gamma'}$, let us denote by $\tilde{a}$ its image in the quotient group $\hat{\Pi}_{\gamma'}/\sigma_n^p_{\gamma'}$. The image $\tilde{y}$ of the given $y$ then has finite order.

From Lemma 3.4 it follows that $\tilde{g} \in \langle \gamma_i \rangle^x$, for some $i \in \{1, \ldots, h\}$ and $\tilde{x} \in \hat{\Pi}_{\gamma'}/\sigma_n^p_{\gamma'}$. Since this holds for all $n$ which are a product of primes in $\Lambda_{\gamma'}$, by an inverse limit argument, it actually holds $y \in \langle \gamma_i \rangle^x$, for some $i \in \{1, \ldots, h\}$ and $x \in \hat{\Pi}_{\gamma'}$.

We say that an element $x$ of a profinite group $G$ is full if the $p$-component $\langle x \rangle^p$ of the pro-cyclic group generated by $x$ is non-trivial for every prime $p$ dividing the order of $G$.

By cohomological methods, it is possible to show that normalizers of full elements in a non-abelian pro-$\mathcal{F}$ surface groups are pro-cyclic. For absolutely simple elements in the pro-$\mathcal{F}$ completion of a surface group, this also follows from an argument similar to the one given in the proof of Theorem 1.5.

**Proposition 3.5.** Let $\mathcal{F}$ be a class of finite groups and let $\hat{\Pi}_{\gamma'}$ be the pro-$\mathcal{F}$ completion of a non-abelian surface group $\Pi$. Let $x$ be an absolutely simple element of $\hat{\Pi}_{\gamma'}$. Then, for all $n \in \hat{\mathbb{Z}}_{\gamma'} \setminus \{0\}$, there holds:

$$N_{\hat{\Pi}_{\gamma'}}(\langle x^n \rangle) = N_{\hat{\Pi}_{\gamma'}}(\langle x \rangle) = \langle x \rangle.$$  

**Proof.** We can assume that $x$ is a simple element of $\Pi$. The quotient group $\Xi_h := \Pi/(x^h)\Pi$ is a Fuchsian group which satisfies the hypotheses of Lemma 3.3. Let $x_h$ be the image of $x$ in $\Xi_h$. Then, the cyclic group $C_h$ generated by $x_h$ is a decomposition group of $\Xi_h$.

From Lemma 3.3 it follows that, in the pro-$\mathcal{F}$ completion $\Xi'_h$ of $\Xi_h$, for $n \in \mathbb{N}^+$ and $h > n$, there holds: $N_{\Xi'_h}(C_h) = N_{\Xi'_h}(C'^n_h) = C_h$. The conclusion of the proposition then follows taking the inverse limit for $h \to \infty$. \hfill $\square$

A multi-curve (cf. Definition 5.2) $\sigma$ on a Riemann surface $S$ is a set $\{\gamma_0, \ldots, \gamma_k\}$ of disjoint, non-trivial, non-peripheral s.c.c.'s on $S$, such that they are two by two non-isotopic. The complement $S \setminus \sigma$ is then a disjoint union of hyperbolic Riemann surfaces
\[ \prod_{i=0}^{h} S_i \] and, for some choices of base points and a path between them, the fundamental group \( \Pi_i := \pi_1(S_i) \), for \( i = 0, \ldots, h \), identifies with a subgroup of \( \Pi := \pi_1(S) \).

Let \( Y_\sigma \) be the graph which has for vertices the connected components of \( S \setminus \sigma \) and for edges the elements of \( \sigma \), where two vertices \( S_i \) and \( S_j \) are joined by the edge \( \gamma_i \) if \( \gamma_i \) is a boundary component of both \( S_i \) and \( S_j \). Then, the group \( \Pi \) is naturally isomorphic to the fundamental group of the graph of groups \( (\mathcal{G},Y_\sigma) \) with vertex groups \( \mathcal{G}_i = \Pi_i \), for \( i = 0, \ldots, h \), and with edge groups the cyclic groups \( C_j \), for \( j = 0, \ldots, k \), generated by representatives in \( \Pi \) of the s.c.c.’s in \( \sigma \).

From (ii) Lemma 3.14 and an argument similar to that of the proof of Theorem 1.5, it follows a description of the pro-\( \mathcal{G} \) completion of \( \Pi \) in terms of the above graph of groups:

**Theorem 3.6.** Let \( \sigma \) be a multi-curve on a Riemann surface \( S \), let \( S \setminus \sigma = \bigcup_{i=0}^{h} S_i \) be the decomposition in connected components and let \( \Pi_i \) be the fundamental group of \( S_i \), for \( i = 0, \ldots, h \). The fundamental group \( \Pi \) of \( S \) is naturally isomorphic to the fundamental group of the graph of groups \( (\mathcal{G},\mathcal{Y}_\sigma) \), described above, with vertex groups the groups \( \Pi_i \), for \( i = 0, \ldots, h \). Let \( \mathcal{G} \) be a class of finite groups. Then, the pro-\( \mathcal{G} \) completion \( \hat{\Pi}^\mathcal{G} \) of \( \Pi \) is the pro-\( \mathcal{G} \) fundamental group of the graph of profinite groups \( (\hat{\mathcal{G}},\mathcal{Y}_\sigma) \) whose vertex and edge groups are the pro-\( \mathcal{G} \) completions of vertex and edge groups of the graph of groups \( (\mathcal{G},\mathcal{Y}_\sigma) \). Moreover, the vertex groups \( \hat{\Pi}_i^\mathcal{G} \), for \( i = 0, \ldots, h \), embed in \( \hat{\Pi}^\mathcal{G} \) and are their own normalizers in this group.

**Proof.** Let us also denote by \( \sigma \) a set of representatives in \( \Pi \) of the s.c.c.’s in \( \sigma \). As in the proof of Theorem 1.5 let us consider the quotient group \( \Pi/\sigma_n^\Pi \), which is a nodal Fuchsian group satisfying the hypotheses of Lemma 3.14. This group is the fundamental group of the graph of groups \( (\mathcal{G}_n,\mathcal{Y}_\sigma) \) whose vertex groups are the fundamental groups of the orbifolds obtained, from the connected components of the manifold with boundary obtained cutting \( S \) in \( \sigma \), attaching discs to their boundary components with an attaching map of degree \( n \).

By (ii) Lemma 3.14, its pro-\( \mathcal{G} \) completion \( \hat{\Pi}^\mathcal{G}/\sigma_n^\hat{\Pi}^\mathcal{G} \) is the pro-\( \mathcal{G} \) fundamental group \( \hat{\Pi}_i^\mathcal{G} (\mathcal{G},\mathcal{Y}_\sigma) \) of the finite graph of pro-\( \mathcal{G} \) groups \( (\hat{\mathcal{G}}_n,\mathcal{Y}_\sigma) \) obtained from \( (\mathcal{G},\mathcal{Y}) \) by taking the pro-\( \mathcal{G} \) completion of each vertex group of \( (\mathcal{G}_n,\mathcal{Y}_\sigma) \) and the vertex groups of \( (\hat{\mathcal{G}}_n,\mathcal{Y}_\sigma) \) embed in \( \hat{\Pi}^\mathcal{G}/\sigma_n^\hat{\Pi}^\mathcal{G} \).

The inverse limit, for \( n \to \infty \), of the graphs of pro-\( \mathcal{G} \) groups \( (\hat{\mathcal{G}}_n,\mathcal{Y}_\sigma) \) is the graph of groups \( (\hat{\mathcal{G}},\mathcal{Y}_\sigma) \), whose vertex groups are the pro-\( \mathcal{G} \) completions of the fundamental groups of the connected components of \( S \setminus \sigma \). The pro-\( \mathcal{G} \) fundamental group of \( (\hat{\mathcal{G}},\mathcal{Y}_\sigma) \) is the inverse limit, for \( n \to \infty \), of the groups \( \hat{\Pi}^\mathcal{G}/\sigma_n^\hat{\Pi}^\mathcal{G} \), i.e. the pro-\( \mathcal{G} \) completion \( \hat{\Pi}^\mathcal{G} \) of \( \Pi \). Therefore, the vertex groups of \( (\hat{\mathcal{G}},\mathcal{Y}_\sigma) \) embed in \( \hat{\Pi}^\mathcal{G} \). The last statement in the theorem then follows from Corollary 3.13 and Remark 3.18 [25].  \(\square\)
4 Relative versions of the restricted Magnus property.

For the applications given in Section 5 we need a finer result than Theorem 1.5. Let us give the following definition:

Definition 4.1. (i) For a given profinite group $G$ (possibly finite) and a given prime $p \geq 2$, let us denote respectively by $G^{(p)}$ and $G^{nil}$ its maximal pro-$p$ and pro-nilpotent quotients. It is clear that, if the group $G$ is pro-nilpotent, there holds $G = \prod G^{(p)}$ and the pro-$p$ group $G^{(p)}$ is naturally identified with the $p$-Sylow subgroup of $G$.

(ii) For $K$ a normal open subgroup of a profinite group $G$ and a given prime $p \geq 2$, let $G^{(p)}_K$ and $G^{nil}_K$ be, respectively, the quotients of $G$ by the kernels of the natural epimorphisms $K \to K^{(p)}$ and $K \to K^{nil}$. Let us call them, respectively, the relative maximal pro-$p$ and pro-nilpotent quotients of $G$ with respect to the subgroup $K$.

(iii) For $K$ a normal finite index subgroup of a discrete group $G$ and a given prime $p \geq 2$, let us denote by $\hat{G}^{(p)}_K$ and $\hat{G}^{nil}_K$, respectively, the quotients of the profinite completion $\hat{G}$ by the kernels of the natural epimorphisms $\hat{K} \to \hat{K}^{(p)}$ and $\hat{K} \to \hat{K}^{nil}$. Let us call $\hat{G}^{(p)}_K$ and $\hat{G}^{nil}_K$, respectively, the relative pro-$p$ and pro-nilpotent completion of $G$ with respect to the subgroup $K$.

Mochizuki’s Lemma 3.1 admits the following generalization to relative pro-$p$ completions of Fuchsian groups:

Lemma 4.2. Let $\Pi$ be a Fuchsian group with the presentation given in $\S 3$. Let $K$ be a finite index normal subgroup of $\Pi$ which contains a torsion free normal subgroup $H$ of index a power of $p$, for a prime $p \geq 2$. Let then $\hat{\Pi}^{(p)}_K$ be the relative pro-$p$ completion of $\Pi$ with respect to the subgroup $K$. Let $D_i$ be a decomposition group of $\hat{\Pi}^{(p)}_K$ in the conjugacy class of the cyclic subgroup $\langle x_i \rangle$, for $i = 1, \ldots, n$. Then, there hold:

(i) For all $x \in \hat{\Pi}^{(p)}_K$, there holds $D_i^{(p)} \cap (D_j^{(p)})^x \neq \{1\}$, if and only if, $i = j$ and $D_i = (D_i)^x$.

In particular, for all $i = 1, \ldots, n$ such that $D_i^{(p)} \neq \{1\}$, there is a series of identities:

$$N_{\hat{\Pi}^{(p)}_K}(D_i) = N_{\hat{\Pi}^{(p)}_K}(D_i^{(p)}) = D_i.$$

(ii) Every finite nilpotent subgroup $C$ of $\hat{\Pi}^{(p)}_K$, such that $C^{(p)} \neq \{1\}$, is contained in a decomposition group of $\hat{\Pi}^{(p)}_K$.

(iii) Therefore, the decomposition groups $D$ of $\Pi^{(p)}_K$, such that $D^{(p)} \neq \{1\}$, may be characterized as the maximal finite nilpotent subgroups $M$ of the profinite group $\Pi^{(p)}_K$ such that $M^{(p)} \neq \{1\}$. 
Proof. The non-hyperbolic case is trivial. So let us assume that \( \Pi \) is a hyperbolic Fuchsian group. Let \( S \) be a hyperbolic orbifold Riemann surface such that \( \Pi \cong \pi_1(S) \) and let \( S_K \to S \) be the Galois unramified covering associated to the normal finite index subgroup \( K \) of \( \Pi \). Let us also denote by \( S_H \to S_K \) the normal unramified \( p \)-covering associated to the subgroup \( H \) of \( K \). By hypothesis, \( S_H \) is a hyperbolic Riemann surface and the \( p \)-adic solenoid \( \mathcal{S}_K^{(p)} \) of \( S_K \), i.e. the inverse limit of all unramified \( p \)-coverings of \( S_K \), identifies with the \( p \)-adic solenoid \( \mathcal{S}_H^{(p)} \) of \( S_H \). Let \( \{L \triangleleft H \} \) be the set of subgroups of \( H \) open for the \( p \)-topology. There is then a series of natural isomorphisms:

\[
H^k(\mathcal{S}_K^{(p)}, \mathbb{Z}/p) \cong H^k(\mathcal{S}_H^{(p)}, \mathbb{Z}/p) := \lim_{L \triangleleft_o H} H^k(S_L, \mathbb{Z}/p) \cong \lim_{L \triangleleft_o H} H^k(L, \mathbb{Z}/p).
\]

Since \( H \) is \( p \)-good, there hold \( H^k(\mathcal{S}_K^{(p)}, \mathbb{Z}/p) = \{0\} \), for \( k > 0 \), and \( H^0(\mathcal{S}_K^{(p)}, \mathbb{Z}/p) = \mathbb{Z}/p \).

There is a natural continuous action of \( \Pi_K^{(p)} \) on the \( p \)-adic solenoid \( \mathcal{S}_K^{(p)} \) and the decompositions groups of \( \Pi_K^{(p)} \) identify with the stabilizers of points of \( \mathcal{S}_K^{(p)} \) in the inverse image of points of the orbifold Riemann surface \( S \) which have non-trivial isotropy groups.

The proof then proceeds exactly like the proof of Lemma 3.1. In order to prove (i), it is enough to show that the intersection of the stabilizers of two such points \( P_1 \) and \( P_2 \) contains a cyclic subgroup \( C_p \) of order \( p \) if and only if \( P_1 = P_2 \).

Since the profinite space \( \mathcal{S}_K^{(p)} \) is \( p \)-acyclic, it follows that \( (\mathcal{S}_K^{(p)})^C_{p} \) is also \( p \)-acyclic, where \( (\mathcal{S}_K^{(p)})^C_{p} \) is the fixed point set of the action of the \( p \)-group \( C_p \) on the \( p \)-adic solenoid \( \mathcal{S}_K^{(p)} \). In particular, \((\mathcal{S}_K^{(p)})^C_{p} \) is connected and thus consists of only one point. In particular, there holds \( P_1 = P_2 \).

The proof of item (ii) essentially follows from the same arguments of the proof of item (ii) of Lemma 3.1. Since \( C \) is nilpotent, it has a unique normal and, by hypothesis, non-trivial \( p \)-Sylow subgroup \( C^{(p)} \). Proceeding by induction on the order of \( C \), we can assume either that \( C \) is of order \( p \) or that \( C \) is an extension of a group of prime order by a non-trivial subgroup \( C_1 \leq C \), which is contained in a decomposition group \( D \) and such that \( C^{(p)} \leq C_1 \).

In the first case, since the profinite space \( \mathcal{S}_K^{(p)} \) is \( p \)-acyclic and of finite dimension, it follows that \((\mathcal{S}_K^{(p)})^C \) is non-empty, i.e. \( C \) is contained in a decomposition group.

In the second case, replacing \( \hat{\Pi}_K^{(p)} \) by its open subgroup \( \hat{H}^{(p)} \cdot C \), which is also the relative pro-\( p \) completion of a Fuchsian group with respect to some normal subgroup, we may assume that actually \( C_1 = D \leq C \). Since \( C_1 \) is normal in \( C \) and \( D \) is self-normalizing, it follows \( C = D \).

Item (iii) is just a reformulation of (ii). \( \square \)

The next step is to generalize Lemma 4.2 to nodal Fuchsian groups.

**Definition 4.3.** (i) A virtual pro-\( p \) nodal Fuchsian group \( \hat{\Pi}_K^{(p)} \) is the relative pro-\( p \) completion of a nodal Fuchsian group \( \Pi \) with respect to a given finite index normal subgroup \( K \).
(ii) We say that a finite subgroup $D$ of a virtual pro-$p$ nodal Fuchsian group $\hat{\pi}_1(\mathcal{G}, Y)^{(p)}_K$ is a decomposition group of type I if it is in the conjugacy class of a decomposition subgroup $I$ of a vertex group of $(\mathcal{G}, Y)$.

(iii) Sets of simple and absolutely simple elements of $\hat{\Pi}^{(p)}_K$ are defined as in Definition 1.3. Thus, a subset of elements $\sigma = \{\gamma_1, \ldots, \gamma_h\} \subset \hat{\Pi}^{(p)}_K$ is absolutely simple if it is in the $\text{Aut}(\hat{\Pi}^{(p)}_K)$-orbit of the image of a set $\sigma' \subset \Pi$ which is simple for some presentation of $\Pi$ as the fundamental group of a Riemann surface.

**Lemma 4.4.** Let $\Pi := \pi_1(G, Y)$ be a nodal Fuchsian group. Let $K$ be a finite index normal subgroup of $\Pi$ and $p \geq 2$ a prime such that, for some normal subgroup $H$ of $K$ of index a power of $p$ and for all vertices $v$ of $Y$, the subgroups $H \cap G_v$ are torsion free. Let then $\hat{\Pi}^{(p)}_K$ be the relative pro-$p$ completion of $\Pi$ with respect to the subgroup $K$. Let $D_1$ and $D_2$ be decomposition groups of $\hat{\Pi}^{(p)}_K$ of type $I_1$ and $I_2$, respectively, such that their $p$-Sylow subgroups are non-trivial. Then, there hold:

(i) The profinite group $\hat{\Pi}^{(p)}_K$ is virtually torsion free.

(ii) It holds $D_1^{(p)} \cap D_2^{(p)} \neq \{1\}$, if and only if $D_1 = D_2$ and $I_1, I_2$ are contained and conjugated in a vertex group $G_v$, for some vertex $v \in v(Y)$. In particular, if $D$ is a decomposition group of $\hat{\Pi}^{(p)}_K$ such that $D^{(p)} \neq \{1\}$, there is a series of identities:

$$N_{\hat{\Pi}^{(p)}_K}(D) = N_{\hat{\Pi}^{(p)}_K}(D^{(p)}) = D.$$

(iii) Every finite nilpotent subgroup $C$ of $\hat{\Pi}^{(p)}_K$, such that $C^{(p)} \neq \{1\}$, is contained in a decomposition group.

(iv) Therefore, the decomposition groups of $\hat{\Pi}^{(p)}_K$ with a nontrivial $p$-component may be characterized as the maximal finite nilpotent subgroups of $\hat{\Pi}^{(p)}_K$ with a nontrivial $p$-component.

**Proof.** From the same argument used to prove item (i) of Lemma 3.4, it follows that $\hat{H}^{(p)}$ is a torsion free group.

In order to prove (ii), let us consider the Galois unramified covering $S_K \rightarrow S$ associated to the normal finite index subgroup $K$ of $\Pi$ and let $S_H \rightarrow S_K$ be the $p$-covering associated to the subgroup $H$ of $K$. As in the proof of (i) Lemma 4.2, the $p$-adic solenoid $S_{(p)}^{(p)}$ of $S_K$, i.e. the inverse limit of all unramified $p$-coverings of $S_K$, identifies with the $p$-adic solenoid $S_H^{(p)}$ of $S_H$. Since $H$ is a free product of surface groups, it is $p$-good. Therefore, as in the proof of Lemma 4.2 there hold $H^k(S_{(p)}^{(p)}, \mathbb{Z}/p) = H^k(S_H^{(p)}, \mathbb{Z}/p) = \{0\}$, for $k > 0$, and $H^0(S_{(p)}^{(p)}, \mathbb{Z}/p) = \mathbb{Z}/p$. Then, the proof proceeds exactly as for Lemma 4.2.

**Definition 4.5.** For an element $a \in \hat{\Pi}^{(p)}_K$, let $\nu_K(a)$ be the minimal natural number such that there holds $a^{\nu_K(a)} \in \hat{\Pi}^{(p)}_K$. For $\sigma = \{\gamma_1, \ldots, \gamma_h\} \subset \hat{\Pi}^{(p)}_K$ an absolutely simple subset, let then $\sigma_{K, m}$, for $n \in \mathbb{N}^+$, be the closed normal subgroup of $\hat{\Pi}^{(p)}_K$ generated by the elements $\gamma_1^{\nu_K(\gamma_1)}, \ldots, \gamma_h^{\nu_K(\gamma_h)}$. 


We then have the following generalization of Theorem 1.5

**Theorem 4.6.** Given an absolutely simple subset $\sigma = \{\gamma_1, \ldots, \gamma_h\} \subset \hat{\Pi}_K^{(p)}$, let $y \in \hat{\Pi}_K^{(p)}$ be an element such that $y^{\nu_K(y)}$ is a generator of the pro-$p$ group $K^{(p)}$ and, for every $n = p^t$, with $t \in \mathbb{N}^+$, there exists a $k_n \in \mathbb{N}^+$ with the property that there holds $y^{k_n} \in \sigma_{K,n}$. Then, for some $s \in \hat{\mathbb{Z}} \setminus \{0\}$ and $i \in \{1, \ldots, h\}$, the element $y$ is conjugated to $\gamma_i^s$.

**Proof.** It is not restrictive to assume that $\sigma = \{\gamma_1, \ldots, \gamma_h\}$ is a simple subset of $\Pi$. Let us then denote by $\sigma_{K,n}$ the normal subgroup of $\Pi$ generated by the elements:

$$\gamma_1^{n\nu_K(\gamma_1)}, \ldots, \gamma_h^{n\nu_K(\gamma_h)}.$$

The quotient group $\Pi/\sigma_{K,n}^\Pi$ is a nodal Fuchsian group whose decomposition groups are the conjugacy classes of the subgroups generated by the images of the elements in $\sigma$.

By the same argument used in the proof of Theorem 1.5, the group $\Pi/\sigma_{K,n}^\Pi$ contains a normal, torsion free subgroup, which is contained as a subgroup of index a power of $p$ in the image of the subgroup $\hat{K}$. Its relative pro-$p$ completion with respect to the image of $K$ is then exactly the quotient group $\hat{\Pi}_K^{(p)}/\sigma_{K,n}^{\hat{\Pi}_K^{(p)}}$.

For an element $a \in \Pi_K^{(p)}$, let us denote by $\bar{a}$ its image in the quotient group $\hat{\Pi}_K^{(p)}/\sigma_{K,n}^{\hat{\Pi}_K^{(p)}}$.

Since the natural epimorphism $\hat{K}^{(p)} \twoheadrightarrow H_1(K, \mathbb{Z}/p)$ factors through the quotient group $\hat{K}^{(p)}/\sigma_{K,n}^{\hat{\Pi}_K^{(p)}}$ and, by hypotheses, the lift $y^{\nu_K(y)}$ is a generator of the pro-$p$ group $\hat{K}^{(p)}$, the image $\bar{y}$ of $y$ then has finite order divisible by $p$.

From Lemma 4.1 it then follows that, for all $n = p^t$, there holds $\bar{y} \in \langle \gamma_i \rangle^{\bar{x}}$, for some $i \in \{1, \ldots, h\}$ and some $\bar{x} \in \hat{\Pi}_K^{(p)}/\sigma_{K,n}^{\hat{\Pi}_K^{(p)}}$. By an inverse limit argument, we conclude that there holds $y \in \langle \gamma_i \rangle^x$, for some $i \in \{1, \ldots, h\}$ and some $x \in \hat{\Pi}_K^{(p)}$. \hfill $\square$

**Corollary 4.7.** Let $K$ be a normal finite index subgroup of $\Pi$ such that, for every absolutely simple element $x \in \hat{\Pi}_K^{(p)}$, the lift $x^{\nu_K(y)} \in \hat{K}^{(p)}$ is a generator. Given an absolutely simple subset $\sigma = \{\gamma_1, \ldots, \gamma_h\} \subset \hat{\Pi}_K^{(p)}$, let $y \in \hat{\Pi}_K^{(p)}$ be an absolutely simple element such that, for every $n = p^t$, with $t \in \mathbb{N}^+$, there exists a $k_n \in \mathbb{N}^+$ with the property that there holds $y^{k_n} \in \sigma_{K,n}^{\hat{\Pi}_K^{(p)}}$. Then, for some $s \in \hat{\mathbb{Z}}^+$ and $i \in \{1, \ldots, h\}$, the element $y$ is conjugated to $\gamma_i^s$.

**Remark 4.8.** In Lemma 3.10 [3], it is defined a characteristic finite index subgroup $K_\ell$ of $\Pi$ such that, for every simple element $x \in \Pi$, the lift $x^{\nu_K(x)} \in K_\ell$ is a generator. The definition of this group can be rephrased, group theoretically as follows. Let $K$ be a finite index characteristic subgroup of $\Pi$ such that, for all simple (and so for all absolutely simple) elements $\gamma \in \Pi$, there holds $\gamma \notin K$. For a given integer $\ell > 1$, we let then $K_\ell := [K, K][K, K]^\ell$. For any absolutely simple element $x \in \hat{\Pi}_K^{(p)}$, the lift $x^{\nu_K(x)} \in \hat{K}_\ell^{(p)}$ is then a generator. It follows that any normal finite index subgroup $N$ of $\Pi$, contained in $K_\ell$, satisfies the hypothesis of Corollary 4.7.

Corollary 4.7 and Remark 4.8 yield a substantial refinement of Theorem 1.5. We need to fix some more notations.
Definition 4.9. For an open normal subgroup $K$ of $\hat{\Pi}$ and $a \in \hat{\Pi}$, let $\nu_K(a)$ be the minimal natural number such that $a^{\nu_K(a)} \in K$. For an absolutely simple subset $\sigma = \{\gamma_1, \ldots, \gamma_h\} \subset \hat{\Pi}$, let then $\sigma_{K,n}^\Pi$, for $n \in \mathbb{N}^+$, be the closed normal subgroup of $\hat{\Pi}$ generated by the elements $\gamma_1^{\nu_K(\gamma_1)}, \ldots, \gamma_h^{\nu_K(\gamma_h)}$.

Let $\hat{\Pi}$ be the profinite completion of a hyperbolic surface group $\Pi$ and $\hat{\Pi}_K^{(p)}$ its relative pro-$p$ completion with respect to some normal finite index subgroup $K$. There is then a natural epimorphism:

$$\psi_K^{(p)} : \hat{\Pi} \to \hat{\Pi}_K^{(p)}.$$ 

From an inverse limit argument, Corollary 4.10 and Remark 4.8 it follows:

**Corollary 4.10.** Let $\sigma = \{\gamma_1, \ldots, \gamma_h\} \subset \hat{\Pi}$ be an absolutely simple subset and $p$ a fixed prime. Let $y \in \hat{\Pi}$ be an absolutely simple element such that, for a cofinal system of open normal subgroups $\{K\}$ of $\hat{\Pi}$ and every $n = p^t$, with $t \in \mathbb{N}^+$, there exists a $\mu_{K,n} \in \mathbb{N}^+$ with the property that there holds $\psi_K^{(p)}(y)^{\mu_{K,n}} \in \psi_K^{(p)}(\sigma_{K,n}^\Pi)$. Then, for some $s \in \hat{\mathbb{Z}}^*$ and $i \in \{1, \ldots, h\}$, the element $y$ is conjugated to $\gamma_i^s$.

The assumption that the element $y$ is absolutely simple can be dropped reformulating Corollary 4.10 for relative pro-nilpotent, instead of relative pro-$p$, completions. Let us denote by $\psi_K^{nil} : \hat{\Pi} \to \hat{\Pi}_K^{nil}$ the natural epimorphism. It holds:

**Theorem 4.11.** Let $\sigma = \{\gamma_1, \ldots, \gamma_h\} \subset \hat{\Pi}$ be an absolutely simple subset. Let $y \in \hat{\Pi}$ be an element such that, for a cofinal system of open normal subgroups $\{K\}$ of $\hat{\Pi}$ and every $n \in \mathbb{N}^+$, there exists a $\mu_{K,n} \in \mathbb{N}^+$ with the property that $\psi_K^{nil}(y)^{\mu_{K,n}} \in \psi_K^{nil}(\sigma_{K,n}^\Pi)$. Then, for some $s \in \hat{\mathbb{Z}}$ and $i \in \{1, \ldots, h\}$, the element $y$ is conjugated to $\gamma_i^s$.

**Proof.** Let us assume that $y \neq 1$. Since $y$ has infinite order a Sylow $p$-subgroup $Y_p$ of $\langle y \rangle$ is infinite for some prime $p$. Let $y_p$ be a generator of $Y_p$ and $U$ an open subgroup of $\hat{\Pi}$ such that $y_p$ is a generator of the maximal pro-$p$ quotient $U^{(p)}$ of $U$. Such a $U$ exists because $Y_p$ is the intersection of all open subgroups containing it. Let $K_U$ be a subgroup in the cofinal system $\{K\}$ contained in $U$. Then its image $\hat{K}_U$ in $U^{(p)}$ intersects $\langle y_p \rangle$ non-trivially and is such that $\hat{K}_U \cap \langle y_p \rangle \not\subseteq \Phi(\hat{K}_U)$. Therefore, since $\hat{K}_U$ is a quotient of $K_U^{(p)}$, the image of $Y_p \cap K_U$ in $K_U^{(p)}$ is not contained in $\Phi(K_U^{(p)})$. This means that $\psi_{K_U^{(p)}}(y)^{\nu_{K,U}(y)}$ is a generator of $K_U^{(p)}$. Then, the conclusion follows from Theorem 4.6 and the usual inverse limit argument.

Proceeding as in the proof of Theorem 4.6 we can also describe normalizers of absolutely simple elements in the relative pro-$p$ completion of a surface group:

**Proposition 4.12.** Let $\hat{\Pi}_K^{(p)}$ be the relative pro-$p$ completion of a non-abelian surface group $\Pi$ with respect to some normal finite index subgroup $K$ and let $x$ be an absolutely simple element of $\hat{\Pi}_K^{(p)}$. Then, for all $n \in \hat{\mathbb{Z}} \setminus \{0\}$, there holds:

$$N_{\hat{\Pi}_K^{(p)}}(\langle x^n \rangle) = N_{\hat{\Pi}_K^{(p)}}(\langle x \rangle) = \langle x \rangle.$$
Proof. We can assume that $x$ is a simple element of $\Pi$. Let then $k > 0$ the smallest integer such that $x^k \in K$. The quotient group $\Xi := \Pi/\langle x^{hk} \rangle \Pi$ is a Fuchsian group which satisfies the hypotheses of Lemma 4.4 with respect to the normal subgroup $K_h$, image of $K$ in the quotient group $\Xi$. Let $x_h$ be the image of $x$ in $\Xi$. Then, the cyclic group $C_h$ generated by $x_h$ is a decomposition group of $\Xi$ with non-trivial $p$-component for $h > 0$.

From Lemma 4.4, it follows that in the relative pro-$p$ completion $\Xi^{(p)}_{K_h}$ of $\Xi$ with respect to the subgroup $K_h$, for $h > 0$ and $p^h \nmid n$, there holds:

$$N_{\Xi^{(p)}_{K_h}}(C_h) = N_{\Xi^{(p)}_h}(C^n_h) = C_h.$$ 

The conclusion of the proposition then follows taking the inverse limit for $h \to \infty$. 

5  A linearization of the complex of profinite curves.

Let $S_g$ be a closed orientable Riemann surface of genus $g$ and let $\{P_1, \ldots, P_n\}$ be a set of distinct points on $S_g$. The Teichmüller modular group $\Gamma_{g,n}$, for $2g - 2 + n > 0$, is defined to be the group of isotopy classes of diffeomorphisms or, equivalently, of homeomorphisms of the surface $S_g$ which preserve the orientation and the given ordered set $\{P_1, \ldots, P_n\}$ of marked points:

$$\Gamma_{g,n} := \text{Diff}^+(S_g, n)/\text{Diff}_0(S_g, n) \cong \text{Hom}^+(S_g, n)/\text{Hom}_0(S_g, n),$$

where $\text{Diff}(S_g, n)$ and $\text{Hom}(S_g, n)$ denote the connected components of the identity in the respective topological groups.

Forgetting the last marked point $P_{n+1}$ induces an epimorphism of Teichmüller modular groups $\Gamma_{g,n+1} \to \Gamma_{g,n}$, called the point-pushing map relative to $P_{n+1}$.

Let $S_{g,n}$ be the differentiable surface obtained removing the points $P_1, \ldots, P_n$ from $S_g$ and let $\Pi_{g,n} := \pi_1(S_{g,n}, P_{n+1})$. The point-pushing homomorphism induces a short exact sequence of Teichmüller modular groups, called the Birman exact sequence:

$$1 \to \Pi_{g,n} \to \Gamma_{g,n+1} \to \Gamma_{g,n} \to 1.$$ 

The monomorphism $i: \Pi_{g,n} \hookrightarrow \Gamma_{g,n+1}$ sends the isotopy class of a $P_{n+1}$-pointed oriented closed curve $\gamma$ to the isotopy class of the homeomorphism $i(\gamma)$ defined pushing the base point $P_{n+1}$ all along the path $\gamma$ in the direction given by the orientation of $\gamma$, from which the name point-pushing map. It is clear that $i(\gamma)$ is isotopic to the identity for isotopies which are allowed to move the base point $P_{n+1}$.

There are natural faithful representations, induced by the action of homeomorphisms on the fundamental group of the Riemann surface $S_{g,n}$:

$$\rho_{g,n}: \Gamma_{g,n} \hookrightarrow \text{Out}(\Pi_{g,n}) \quad \text{and} \quad \rho'_{g,n+1}: \Gamma_{g,n+1} \hookrightarrow \text{Aut}(\Pi_{g,n}).$$

Since the automorphism $\rho'_{g,n+1}(i(\gamma))$ is the inner automorphism $\text{inn}\gamma$, induced by $\gamma$, for all elements $\gamma \in \Pi_{g,n}$, the exactness of the Birman sequence then follows from the
exactness of the standard group-theoretical short exact sequence:

\[ 1 \to \Pi_{g,n} \xrightarrow{\text{inn}} \text{Aut}(\Pi_{g,n}) \to \text{Out}(\Pi_{g,n}) \to 1. \]

It also follows that the representations \( \rho_{g,n} \) and \( \rho'_{g,n+1} \) can be recovered, algebraically, from the Birman exact sequence and the action by restriction of inner automorphisms of \( \Gamma_{g,n+1} \) on its normal subgroup \( \Pi_{g,n} \).

Let us now switch to the profinite setting. Let, as usual, \( \hat{\Pi}_{g,n} \) be the profinite completion of the fundamental group \( \Pi_{g,n} \). Since the profinite group \( \hat{\Pi}_{g,n} \) is center-free, there is also a natural short exact sequence:

\[ 1 \to \hat{\Pi}_{g,n} \xrightarrow{\text{inn}} \text{Aut}(\hat{\Pi}_{g,n}) \to \text{Out}(\hat{\Pi}_{g,n}) \to 1. \]

Let us mention here a fundamental result of Nikolov and Segal [19] which asserts that any finite index subgroup of any topologically finitely generated profinite group \( G \) is open. Since such a profinite group \( G \) has also a basis of neighborhoods of the identity consisting of open characteristic subgroups, it follows that all automorphisms of \( G \) are continuous and that \( \text{Aut}(G) \) is a profinite group as well.

Let \( \hat{\Gamma}_{g,n} \), for \( 2g - 2 + n > 0 \), be the profinite completion of the Teichmüller modular group. From the universal property of the profinite completion, it follows that there are natural representations:

\[ \hat{\rho}_{g,n} : \hat{\Gamma}_{g,n} \to \text{Out}(\hat{\Pi}_{g,n}) \quad \text{and} \quad \hat{\rho}'_{g,n+1} : \hat{\Gamma}_{g,n+1} \to \text{Aut}(\hat{\Pi}_{g,n}). \]

Let us then recall a few definitions from [4].

**Definition 5.1.** For \( 2g - 2 + n > 0 \), let the profinite groups \( \hat{\Gamma}_{g,n+1} \) and \( \hat{\Gamma}_{g,n} \) be, respectively, the image of \( \hat{\rho}'_{g,n+1} \) in \( \text{Aut}(\hat{\Pi}_{g,n}) \) and of \( \hat{\rho}_{g,n} \) in \( \text{Out}(\hat{\Pi}_{g,n}) \). For all \( n \geq 0 \), there is a natural isomorphism \( \hat{\Gamma}_{g,n+1} \cong \hat{\Gamma}_{g,n+1} \) (cf. [2] and [13]). We then call \( \hat{\Gamma}_{g,n} \) the congruence completion of the Teichmüller group or, more simply, the procongruence Teichmüller group.

One of the most important objects in Teichmüller theory is the complex of curves:

**Definition 5.2.** A simple closed curve (s.c.c.) \( \gamma \) on the Riemann surface \( S_{g,n} \) is non-peripheral if it does not bound a disc with less than two punctures. A multi-curve \( \sigma \) on \( S_{g,n} \) is a set of disjoint, non-trivial, non-peripheral s.c.c.’s on \( S_{g,n} \), such that they are two by two non-isotopic. The complex of curves \( C(S_{g,n}) \) is the abstract simplicial complex whose simplices are isotopy classes of multi-curves on \( S_{g,n} \).

It is easy to check that the combinatorial dimension of \( C(S_{g,n}) \) is \( n - 4 \) for \( g = 0 \) and \( 3g - 4 + n \) for \( g \geq 1 \). There is a natural simplicial action of \( \Gamma_{g,n} \) on \( C(S_{g,n}) \).

In order to construct a profinite version of the complex of curves, we need to reformulate its definition in more algebraic terms.

Let \( \mathcal{L}_{g,n} = C(S_{g,n})_0 \), for \( 2g - 2 + n > 0 \), be the set of isotopy classes of non-peripheral simple closed curves on \( S_{g,n} \). Let \( \Pi_{g,n}/\sim \) be the set of conjugacy classes of elements of \( \Pi_{g,n} \) and let \( \mathcal{P}_2(\Pi_{g,n}/\sim) \) be the set of unordered pairs of elements of \( \Pi_{g,n}/\sim \).
For a given $\gamma \in \Pi_{g,n}$, let us denote by $\gamma^{\pm 1}$ the set $\{\gamma, \gamma^{-1}\}$ and by $[\gamma^{\pm 1}]$ its equivalence class in $\mathcal{P}_2(\Pi_{g,n}/\sim)$. Let us then define the natural embedding $\iota: \mathcal{L}_{g,n} \hookrightarrow \mathcal{P}_2(\Pi_{g,n}/\sim)$, choosing, for an element $\gamma \in \mathcal{L}_{g,n}$, an element $\tilde{\gamma}_s \in \Pi_{g,n}$ whose free homotopy class contains $\gamma$ and letting $\iota(\gamma) := [\tilde{\gamma}_s^{\pm 1}]$.

Let $\hat{\Pi}_{g,n}/\sim$ be the set of conjugacy classes of elements of $\hat{\Pi}_{g,n}$ and $\mathcal{P}_2(\hat{\Pi}_{g,n}/\sim)$ the profinite set of unordered pairs of elements of $\hat{\Pi}_{g,n}/\sim$. Since $\Pi_{g,n}$ is conjugacy separable (cf. [23]) the set $\Pi_{g,n}/\sim$ embeds in the profinite set $\hat{\Pi}_{g,n}/\sim$. So, let us define the set of non-peripheral profinite s.c.c.’s $\hat{\mathcal{L}}_{g,n}$ on $S_{g,n}$ to be the closure of the set $\iota(\mathcal{L}_{g,n})$ inside the profinite set $\mathcal{P}_2(\hat{\Pi}_{g,n}/\sim)$. When it is clear from the context, we omit the subscripts and denote these sets simply by $\mathcal{L}$ and $\hat{\mathcal{L}}$.

An ordering of the set $\{\alpha, \alpha^{-1}\}$ is preserved by the conjugacy action and defines an orientation for the associated equivalence class $[\alpha^{\pm 1}] \in \hat{\mathcal{L}}$.

For all $k \geq 0$, there is a natural embedding of the set $C(S_{g,n})_{k-1}$ of isotopy classes of multi-curves on $S_{g,n}$ of cardinality $k$ into the profinite set $\mathcal{P}_k(\hat{\mathcal{L}})$ of unordered subsets of $k$ elements of $\hat{\mathcal{L}}$. Let us then define the set of profinite multi-curves on $S_{g,n}$ as the union of the closures of the sets $C(S_{g,n})_{k-1}$ inside the profinite sets $\mathcal{P}_k(\hat{\mathcal{L}})$, for all $k > 0$.

Let us observe that the sets of elements of $\hat{\Pi}_{g,n}$ in the class of a profinite multi-curve on $S_{g,n}$ are simple in the sense of Definition 1.3. However, in general, the class in $\mathcal{P}_k(\hat{\mathcal{L}})$ of a simple subset of $k$ elements of $\hat{\Pi}_{g,n}$ is not a profinite multi-curve because it may contain peripheral elements.

**Definition 5.3.** (cf. [4]) Let $L(\hat{\Pi}_{g,n})$, for $2g-2+n > 0$, be the abstract simplicial profinite complex whose simplices are the profinite multi-curves on $S_{g,n}$. The abstract simplicial profinite complex $L(\hat{\Pi}_{g,n})$ is called the complex of profinite curves on $S_{g,n}$.

For $2g-2+n > 0$, there is a natural continuous action of the procongruence Teichmüller group $\hat{\Gamma}_{g,n}$ on the complex of profinite curves $L(\hat{\Pi}_{g,n})$. There are finitely many orbits of $\hat{\Gamma}_{g,n}$ in $L(\hat{\Pi}_{g,n})_k$ each containing an element of $C(S_{g,n})_k$, for $k \geq 0$, and, by the results of Section 4 [4], these orbits correspond to the possible topological types of a surface $S_{g,n} \setminus \sigma$, for $\sigma$ a multi-curve on $S_{g,n}$.

The main result of §4 [4] (cf. Theorem 4.2) was that, for all $k \geq 0$, the profinite set $L(\hat{\Pi}_{g,n})_k$ is the $\hat{\Gamma}_{g,n}$-completion of the discrete $\Gamma_{g,n}$-set $C(S_{g,n})_k$, i.e. there is a natural continuous isomorphism of $\hat{\Gamma}_{g,n}$-sets:

$$L(\hat{\Pi}_{g,n})_k \cong \lim_{\leftarrow} \Gamma_{\lambda, k} / C(S_{g,n})_k / \Gamma_{\lambda},$$

where $\{\Gamma_{\lambda}\}_{\lambda \in \Lambda}$ is a tower of finite index normal subgroup of $\Gamma_{g,n}$ which forms a fundamental system of neighborhoods of the identity for the congruence topology.

From classical Teichmüller theory, we know that the set $\mathcal{L}$ of isotopy classes of non-peripheral s.c.c.’s on $S_{g,n}$ parametrizes the set of Dehn twists of $\Gamma_{g,n}$, which is the standard set of generators for this group. In other words, the assignment $\gamma \mapsto \tau_{\gamma}$, for $\gamma \in \mathcal{L}$, defines an embedding $d: \mathcal{L} \hookrightarrow \Gamma_{g,n}$, for $2g-2+n > 0$. 


Theorem 5.4. Of the procongruence Teichmüller group: (cf. Theorem 5.1) is that this provides a parametrization of the set of profinite Dehn twists. This linearization result can be “linearized” and then extract, as a consequence of this process, both a new proof and definitions. Let \( K \). The purpose of this section is to show how the complex of profinite curves \( \hat{\Gamma}_{g,n} \) can be obtained from \( \Gamma_{g,n} \) filling its punctures and, for a commutative unitary ring of coefficients \( A \), let \( H_1(\Delta K, A) \) be its first homology group. There is a natural map \( \psi_K: K \to H_1(\Delta K, A) \).

**Definition 5.5.** For a given \( \gamma \in \hat{\Delta} \), let us denote by the same letter an element of the profinite group \( \hat{\Pi}_{g,n} \) in the class of the given profinite s.c.c.. Let \( \nu_K(\gamma) \) be the smallest positive integer such that \( \gamma^{\nu_K(\gamma)} \in K \). For a profinite multi-curve \( \sigma \in L(\hat{\Pi}_{g,n}) \), let us also denote by \( \sigma \) a simple subset of \( \hat{\Pi}_{g,n} \) in the class of the given multi-curve. Let then \( V_{K,\sigma} \) be the primitive \( A \)-submodule of \( H_1(\Delta K, A) \) generated by the \( G_K \)-orbit of the subset \( \psi_K(\gamma^{\nu_K(\gamma)}) \). For \( \sigma = \{\gamma_1, \ldots, \gamma_h\} \subset \hat{\Pi}_{g,n} \) a simple subset, this is the same as the \( A \)-submodule generated by the image \( \psi_K(\sigma_{\hat{\Pi}_{g,n}}) \in H_1(\Delta K, A) \) (cf. Theorem 4.6). Let then \( \text{Gr}(H_1(\Delta K, A)) \) be the absolute Grassmanian of primitive \( A \)-submodules of the homology group \( H_1(\Delta K, A) \), that is to say the disjoint union of the Grassmanians of primitive, \( k \)-dimensional, \( A \)-submodules of the homology group \( H_1(\Delta K, A) \), for all \( 1 \leq k \leq \text{rank} H_1(\Delta K, A) \).
For $A_p$ equal to the ring of $p$-adic integers $\mathbb{Z}_p$ or the finite field $\mathbb{F}_p$, the absolute Grassmannian $\text{Gr}(H_1(\widehat{S}_K, A_p))$ has a natural structure of profinite space, while, for $A_p = \mathbb{Q}_p$, it is a locally compact totally disconnected Hausdorff space. In all cases, for $\sigma \in L(\widehat{\Pi}_{g,n})$, the assignment $\sigma \mapsto V_{K,\sigma}$ defines a natural continuous $\Gamma_{g,n}$-equivariant map:

$$\Psi_{K,p} : L(\widehat{\Pi}_{g,n}) \longrightarrow \text{Gr}(H_1(\widehat{S}_K, A_p)).$$

**Theorem 5.6.** For $p > 0$ a prime number, let $A_p = \mathbb{F}_p$, $\mathbb{Z}_p$ or $\mathbb{Q}_p$. For $2g - 2 + n > 0$, there is a natural continuous $\Gamma_{g,n}$-equivariant injective map:

$$\Psi_p := \prod_{K \in \Pi_{g,n}} \Psi_{K,p} : L(\widehat{\Pi}_{g,n}) \hookrightarrow \prod_{K \in \Pi_{g,n}} \text{Gr}(H_1(\widehat{S}_K, A_p)),$$

where $\{K\}$ is a cofinal system of open normal subgroups of the profinite group $\widehat{\Pi}_{g,n}$.

**Proof.** For $A_p = \mathbb{Z}_p$, the submodule $V_{K,\sigma}$ of $H_1(\widehat{S}_K, \mathbb{Z}_p)$ is primitive. Therefore, it is enough to prove the theorem for $A_p = \mathbb{F}_p$.

Let $u_1, \ldots, u_n \in \Pi_{g,n}$ be simple loops around the punctures on the surface $S_{g,n}$ labeled by the points $P_1, \ldots, P_n$, respectively.

For $K$ an open normal subgroup of $\widehat{\Pi}_{g,n}$, a simple set of elements $\sigma = \{\gamma_1, \ldots, \gamma_h\}$ and $s \in \mathbb{N}^+$, let us denote by $\sigma^{\Pi_{g,n}}$ the closed normal subgroup generated by the set of elements:

$$\gamma_1^{s\nu_K(\gamma_1)}, \ldots, \gamma_h^{s\nu_K(\gamma_h)}, u_1^{s\nu_K(u_1)}, \ldots, u_n^{s\nu_K(u_n)}.$$

For $\sigma = \{\gamma_1, \ldots, \gamma_h\} \in L(\widehat{\Pi}_{g,n})$, let $\hat{V}_{K,\sigma}$ be the subspace of the $\mathbb{F}_p$-vector space $K_p^{ab} := H_1(K, \mathbb{F}_p)$ which is the image of the normal subgroup $\hat{\sigma}^{\Pi_{g,n}}$ of $K$ by the natural epimorphism $K \to K_p^{ab}$. The assignment $\gamma \mapsto \hat{V}_{K,\sigma}$ then defines a natural continuous $\Gamma_{g,n}$-equivariant map:

$$\hat{\Psi}_{K,p} : L(\widehat{\Pi}_{g,n}) \longrightarrow \text{Gr}(K_p^{ab}).$$

For $\xi = \{\delta_1, \ldots, \delta_h\} \in L(\widehat{\Pi}_{g,n})$, let us also denote by $\xi^{\Pi_{g,n}}$ the closed normal subgroup generated by the set of elements:

$$\delta_1^{s\nu_K(\gamma_1)}, \ldots, \delta_h^{s\nu_K(\gamma_h)}, u_1^{s\nu_K(u_1)}, \ldots, u_n^{s\nu_K(u_n)}.$$

Let us show that, if $\sigma \neq \xi \in L(\widehat{\Pi}_{g,n})$, there is an open normal subgroup $K$ of $\widehat{\Pi}_{g,n}$ such that there holds $\hat{\Psi}_{K,p}(\sigma) \neq \hat{\Psi}_{K,p}(\xi)$. This obviously implies that there holds as well $\Psi_{K,p}(\sigma) \neq \Psi_{K,p}(\xi)$, proving Theorem 5.6.

A consequence of Theorem 4.2 [4] is also that not every element of $\sigma$ is conjugated to a $\widehat{\mathbb{Z}}$-power of an element in $\xi$. Since no element of $\sigma$ is conjugated to $u_k$, for $k = 1, \ldots, n$, from Corollary [4.10] it then follows that there is an open normal subgroup $H$ of $\widehat{\Pi}_{g,n}$ and an $s = p^t$, for $t \in \mathbb{N}$, such that the images of the subgroups $\hat{\sigma}^{\Pi_{g,n}}$ and $\xi^{\Pi_{g,n}}$ in the maximal
pro-p quotient $H^{(p)}$ of the profinite group $H$ are distinct. Moreover, by Remark 4.8, we can assume that all elements $\gamma_i^{p_{K}(\psi)}$ and $\delta_j^{p_{K}(\psi)}$, for $i, j = 1, \ldots, h$, are generators in $H$.

Let then $L$ be an open normal subgroup of $\tilde{\Pi}_{g,n}$ contained in $H$ and of index a power of $p$ in $H$ such that there holds $\nu_L(x) = s\nu_H(x)$, for $x = \gamma_i, \delta_j$ or $u_k$, for $i, j = 1, \ldots, h$ and $k = 1, \ldots, n$. Then, the images of the subgroups $\tilde{\sigma}_{L,1} = \tilde{\sigma}_{H,s}$ and $\tilde{\xi}_{L,1} = \tilde{\xi}_{H,s}$ in the maximal pro-p quotient $L^{(p)}$ of $L$ are also distinct. We need one more lemma:

**Lemma 5.7.** Let $L^{(p)}$ be a pro-p group and $N_1 \neq N_2$ normal subgroups of $L^{(p)}$ invariant for the action of a finite subgroup $G$ of $\text{Out}(L^{(p)})$. Then, there exists an open normal subgroup $U$ of $L^{(p)}$, containing $N_1, N_2$ and invariant for the action of $G$, such that $N_1\Phi(U) \neq N_2\Phi(U)$, where, for a given group $H$, we denote by $\Phi(H)$ its Frattini subgroup.

**Proof.** Note that $N_1 \subseteq N_1 N_2$. Hence, it suffices to prove the existence of an open normal subgroup $U$ of $L^{(p)}$ containing $N_1 N_2$ and invariant for the action of $G$ such that there holds $N_1\Phi(U) \neq N_1 N_2\Phi(U)$.

Let $\{V_v\}_{v \in \Upsilon}$ be the set of all open normal subgroups of $L^{(p)}$ containing $N_1 N_2$ and invariant for the action of $G$, it then holds $\bigcap_{v \in \Upsilon} V_v = N_1 N_2$ and $\bigcap_{v \in \Upsilon} \Phi(V_v) = \Phi(N_1 N_2)$. If $N_1\Phi(V_v) = N_1 N_2\Phi(V_v)$, for all $v \in \Upsilon$, then there holds as well:

$$N_1 = N_1\Phi(N_1) = \bigcap_{v \in \Upsilon} N_1\Phi(V_v) = \bigcap_{v \in \Upsilon} N_1 N_2\Phi(V_v) = N_1 N_2\Phi(N_1 N_2) = N_1 N_2.$$ 

Therefore, since $N_1 \neq N_1 N_2$, there holds $N_1\Phi(V_v) \neq N_1 N_2\Phi(V_v)$, for some $v \in \Upsilon$. \qed

In order to complete the proof, we apply Lemma 5.7 with $G = \tilde{\Pi}_{g,n}/L$, $N_1 = \psi^{(p)}(\tilde{\sigma}_{L,1}^{n,n})$ and $N_2 = \psi^{(p)}(\tilde{\xi}_{L,1}^{n,n})$. Then, let $U$ be an open normal subgroup of $L^{(p)}$ as in the statement of Lemma 5.7 let $U'$ be its inverse image in $\tilde{\Pi}_{g,n}$, which is also a normal subgroup, and let $K := L \cap U'$. Since $K$ contains both $\tilde{\sigma}_{L,1}^{n,n}$ and $\tilde{\xi}_{L,1}^{n,n}$, there holds $\tilde{\psi}_{K,p}(\sigma) \neq \tilde{\psi}_{K,p}(\xi)$.

Thanks to Theorem 5.6, it is now possible to give a partial parametrization also to profinite multi-twists, i.e. products of powers of commuting profinite Dehn twists. Let us observe that, for $\sigma = \{\gamma_0, \ldots, \gamma_k\}$ a profinite multi-curve on $S_{g,n}$, the set $\{\tau_{\gamma_0}, \ldots, \tau_{\gamma_k}\}$ is a set of commuting profinite Dehn twists.

**Theorem 5.8.** For $2g - 2 + n > 0$, let $\sigma = \{\gamma_1, \ldots, \gamma_s\}$ and $\sigma' = \{\delta_1, \ldots, \delta_t\}$ be two profinite multi-curves on $S_{g,n}$. Suppose that, there is an identity in $\Gamma_{g,n}$:

$$\tau_{\gamma_i}^{h_{i}} \tau_{\gamma_j}^{k_{j}} \cdots \tau_{\gamma_s}^{h_{s}} = \tau_{\delta_1}^{k_{1}} \tau_{\delta_2}^{k_{2}} \cdots \tau_{\delta_t}^{k_{t}},$$

for $h_i \in m_{\sigma} \cdot \mathbb{N}^+$ and $k_j \in m_{\sigma'} \cdot \mathbb{N}^+$, with $m_{\sigma}, m_{\sigma'} \in \mathbb{Z}^*$. Then, there hold:

(i) $t = s$;

(ii) there is a permutation $\phi \in \Sigma_s$ such that $\delta_i = \gamma_{\phi(i)}$ and $k_i = h_{\phi(i)}$, for $i = 1, \ldots, s$. 

Proof. If $\sigma \neq \sigma'$, by Theorem 5.6 there is an open characteristic subgroup $K$ of $\hat{\Pi}_{g,n}$ such that there holds $\Psi_{K,p}(\sigma) \neq \Psi_{K,p}(\sigma')$ in the $\mathbb{Q}_p$-vector space $H_1(S_K, \mathbb{Q}_p)$.

Let $\Gamma^K$ be the geometric level associated to $K$, i.e., the kernel of the natural representation $\Gamma_{g,n} \to \text{Out}(\hat{\Pi}_{g,n}/K)$. Then (cf. §2 [4]), there is a natural representation $\rho_{K,p}: \Gamma^K \to Z_{\text{Sp}(H_1(S_K, \mathbb{Q}_p))(G_K)}$.

Let $r \in \mathbb{N}^+$ be such that, for every profinite Dehn twist $\tau_\gamma \in \hat{\Gamma}_{g,n}$, there holds $\tau_\gamma^r \in \hat{\Gamma}^K$. From the results of §5 in [4], it follows that it is possible to recover the subspaces $\Psi_{K,p}(\sigma)$ and $\Psi_{K,p}(\sigma')$ as the cores (cf. remarks preceding Lemma 5.11 [4]) of the symmetric bilinear forms on $H_1(S_K, \mathbb{Q}_p)$ associated to the multi-transvections $\rho_{K,p}(\tau_{f,1}^r \cdots \tau_{f,s}^r)$ and $\rho_{K,p}(\tau_{1}^r \cdots \tau_{k}^r)$, respectively. Therefore, the hypotheses of the theorem imply $\sigma = \sigma'$, but then items (i) and (ii) follow immediately.

An immediate consequence of Theorem 5.8 is a description of centralizers of profinite multi-twists of the procongruence Teichmüller group (for maximal multi-curves, this result also appeared in [14], cf. Theorems D and E).

Corollary 5.9. (i) For $2g-2+n > 0$, let $\sigma = \{\gamma_1, \ldots, \gamma_s\}$ be a profinite multi-curves on $S_{g,n}$ and $(h_1, \ldots, h_k) \in (m_\sigma \cdot \mathbb{N}^+)^k$ a multi-index, with $m_\sigma \in \hat{\mathbb{Z}}$. Then, there holds:

$$Z_{\Gamma_{g,n}}(\tau_{\gamma_1}^{h_1} \cdots \tau_{\gamma_k}^{h_k}) = N_{\Gamma_{g,n}}((\tau_{\gamma_1}^{h_1} \cdots \tau_{\gamma_k}^{h_k})) = N_{\Gamma_{g,n}}((\tau_{\gamma_1}, \ldots, \tau_{\gamma_k})).$$

(ii) Let us assume that $\sigma = \{\gamma_1, \ldots, \gamma_s\}$ is a multi-curve on $S_{g,n}$ such that there holds $S_{g,n} \setminus \{\gamma_1, \ldots, \gamma_k\} \cong S_{g_1,n_1} \amalg \cdots \amalg S_{g_h,n_h}$. Then, the centralizer in the procongruence Teichmüller modular group $\hat{\Gamma}_{g,n}$ of the multi-twist $\tau_{\gamma_1}^{h_1} \cdots \tau_{\gamma_k}^{h_k}$ is the closure, inside $\hat{\Gamma}_{g,n}$, of the stabilizer $\Gamma_\sigma < \Gamma_{g,n}$. Therefore, it is described by the exact sequences:

$$1 \to \hat{\Gamma}_\sigma \to Z_{\hat{\Gamma}_{g,n}}(\tau_{\gamma_1}^{h_1} \cdots \tau_{\gamma_k}^{h_k}) \to \text{Sym}^+(\sigma),$$

$$1 \to \bigoplus_{i=1}^k \hat{\mathbb{Z}} \cdot \tau_{\gamma_i} \to \hat{\Gamma}_\sigma \to \hat{\Gamma}_{g_1,n_1} \times \cdots \times \hat{\Gamma}_{g_h,n_h} \to 1,$$

where $\text{Sym}^+(\sigma)$ is the group of signed permutations on the set $\sigma$.

Proof. As we already observed, for $f \in \hat{\Gamma}_{g,n}$, there holds the identity:

$$f \cdot (\tau_{\gamma_1}^{h_1} \cdots \tau_{\gamma_k}^{h_k}) \cdot f^{-1} = \tau_{f(\gamma_1)}^{h_1} \cdots \tau_{f(\gamma_k)}^{h_k}.$$ 

The conclusion then follows from Theorem 5.8 Corollary 6.4 and Theorem 6.6 in [4].

6 Galois actions on hyperbolic curves.

Let $C$ be a hyperbolic curve defined over a number field $\mathbb{k}$. Let us fix an embedding $\mathbb{k} \subset \bar{\mathbb{Q}}$ and a $\bar{\mathbb{Q}}$-valued point $\hat{\xi} \in C$. The structural morphism $C \to \text{Spec}(\mathbb{k})$ induces a short exact sequence of algebraic fundamental groups:

$$1 \to \pi_1(C \times_\mathbb{k} \bar{\mathbb{Q}}, \hat{\xi}) \to \pi_1(C, \hat{\xi}) \to G_\mathbb{k} \to 1,$$
where $G_k$ is the absolute Galois group and the group $\pi_1(C \times_k \overline{\mathbb{Q}})$ is isomorphic to the profinite completion of a hyperbolic surface group. Associated to the above short exact sequence, is the outer Galois representation:

$$\rho_C: G_k \to \text{Out}(\pi_1(C \times_k \overline{\mathbb{Q}}, \xi)).$$

By Theorem 2.2 [17], Corollary 6.3 [13] and Theorem 7.7 [4], the representation $\rho_C$ is faithful. In this section, as an application of the restricted Magnus property for profinite surface groups, we are going to prove some refinements of these results.

**Theorem 6.1.** Let $C$ be a smooth $n$-punctured, genus $g$ curve, defined over a number field $k$. For $2g - 2 + n > 0$ and $3g - 3 + n > 0$, the faithful outer Galois representation $\rho_C$ induces a faithful representation:

$$\omega_{g,n}: G_k \hookrightarrow \text{Aut}(L(\widehat{\Pi}_{g,n})).$$

**Proof.** By the definition of profinite simple closed curves, it is clear that the representation $\rho_C$ induces a natural representation $G_k \to \text{Aut}(\widehat{L}_{g,n})$, which induces the natural representation $\omega_{g,n}$. The faithfulness of the representation $\omega_{g,n}$ then follows from Theorem 7.2 and Corollary 7.6 of [4].

Let $\{C^\lambda\}_{\lambda \in \Lambda}$ be the tower of Galois étale connected covering of $C$ associated to characteristic subgroups of $\widehat{\Pi} := \pi_1(C \times_k \overline{\mathbb{Q}}, \xi)$ and let us denote by $G^\lambda$ the covering transformation group of the covering $C^\lambda \to C$. For $\lambda \in \Lambda$, let us also denote by $C^\lambda$ the smooth projective curve obtained from $C$ filling in its punctures.

For a $G$-vector space $V$, let us denote by $\text{Gr}_G(V)$ the absolute Grassmanian of $G$-invariant subspaces of $V$. The outer Galois representation $\rho_C$ then induces, for every $\lambda \in \Lambda$, a natural representation $G_k \to \text{Aut}(\text{Gr}_{\widehat{\Pi}}(H^1_{\text{ét}}(C^\lambda, \mathbb{Q}_\ell)))$.

Let $H^1_{\text{ét}}(C^\lambda, \mathbb{Q}_\ell) := \lim_{\to \lambda \in \Lambda} H^1_{\text{ét}}(C^\lambda, \mathbb{Q}_\ell)$, which is endowed with a natural continuous action of the profinite group $\widehat{\Pi}$. Let then $\text{Gr}_{\widehat{\Pi}}(H^1_{\text{ét}}(C^\lambda, \mathbb{Q}_\ell))$ be the absolute Grassmanian of $\widehat{\Pi}$-invariant subspaces of $H^1_{\text{ét}}(C^\lambda, \mathbb{Q}_\ell)$.

From Theorem 5.1 and Theorem 5.6, it follows:

**Corollary 6.2.** Let $C$ be a hyperbolic curve defined over a number field $k$ with non-trivial moduli space. The associated outer Galois representation $\rho_C$ then induces a natural faithful representation:

$$G_k \hookrightarrow \text{Aut}(\text{Gr}_{\widehat{\Pi}}(H^1_{\text{ét}}(C^\lambda, \mathbb{Q}_\ell))).$$

**References**

[1] M. Artin, B. Mazur. *Etale Homotopy*. Springer Lecture Notes in Mathematics 100 (1969).
[2] M. Boggi. The congruence subgroup property for the hyperelliptic Teichmüller modular group: the closed surface case. Pre-print.

[3] M. Boggi. Galois covers of moduli spaces of curves and loci of curves with symmetries. Geom. Dedicata n. 168 (2014), 113–142.

[4] M. Boggi. On the procongruence completion of the Teichmüller modular group. Trans. Amer. Math. Soc. 366 (2014), no. 10, 5185–5221.

[5] M. Boggi. Continuous cohomology and homology of profinite groups. arXiv:0306381v2 (2014).

[6] M. Boggi, P.A. Zalesskii. Characterizing closed curves on Riemann surfaces via homology groups of coverings. arXiv:1111.2373v3 (2014).

[7] O. Bogopolski, E. Kudryavtseva, H. Zieschang Simple curves on surfaces and an analog of a theorem of Magnus for surface groups. Math. Zeitschrift n. 247 (2004), 595–609.

[8] O. Bogopolski. A surface groups analogue of a theorem of Magnus. From Geometric methods in group theory. Contemp. Math. n. 372, Amer. Math. Soc., Providence, RI (2005), 59–69.

[9] K.S. Brown. Cohomology of groups. Graduate Texts in Mathematics 87, Springer-Verlag (1982).

[10] E.K. Grossman. On residual finiteness of certain mapping class groups. J. London Math. Soc. n. 2, vol. 9 (1974), 160–164.

[11] F. Grunewald, A. Jaikin-Zapirain, P.A. Zalesskii. Cohomological goodness and the profinite completion of Bianchi groups. Duke Math. J. 144 (2008), no. 1, 53–72.

[12] F. Haglund, D.T. Wise. Special cube complexes. Geom. Funct. Anal. 17 (2008), no. 5, 1551–1620.

[13] Y. Hoshi, S. Mochizuki. On the combinatorial anabelian geometry of nodally nondegenerate outer representations. RIMS pre-print 1677 (2009).

[14] Y. Hoshi, S. Mochizuki. Topics surrounding the combinatorial anabelian geometry of hyperbolic curves II: tripods and combinatorial cuspidalization. RIMS pre-print 1762 (2014).

[15] W. Magnus, A. Karrass, D. Solitar Combinatorial group theory. Second Revised Edition. Dover Publications, Inc. New York (1976).

[16] W. Magnus. Über diskontinuierliche Gruppen mit einer definierenden Relation. (Der Freiheitssatz.) J. Reine Angew. Math. n. 163 (1930), 141–165.
REFERENCES

[17] M. Matsumoto. Galois representations on profinite braid groups on curves. J. Reine Angew. Math. n. 474 (1996), 169–219.

[18] S. Mochizuki. Absolute anabelian cuspidalizations of proper hyperbolic curves. J. of Math. Kyoto Univ. n. 47 (2007), 451–539.

[19] N. Nikolov, D. Segal. Finite index subgroups in profinite groups. C. R. Math. Acad. Sci. Paris n. 337 (2003), 303–308.

[20] L. Ribes, P.A. Zalesskii. Profinite groups. Erg. der Math. und ihrer Grenz. 3. Folge 40, Springer-Verlag (2000).

[21] J.P. Serre. Cohomologie galoisienne. Cinquième édition, révisée et complétée. Lectures Notes in Mathematics n.5, Springer-Verlag (1997).

[22] C. Scheiderer. Farrell cohomology and Brown theorems for profinite groups. Manuscripta Math. n. 91 (1996), 247–281.

[23] P.F. Stebe. Conjugacy separability of certain free products with amalgamation. Trans. Amer. Math. Soc. 156 (1971), 119–129.

[24] D.T. Wise. The structure of groups with a quasiconvex hierarchy. Pre-print.

[25] P.A. Zalesskii, O. Melnikov. Subgroups of profinite groups acting on trees. Math. USSR Sbornik, 63 (1989), 405–424.

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