Bound state solutions of a Dirac particle undergoing a tensor interaction potentials via asymptotic iteration method

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ABSTRACT
Bound state solutions of a quantum Dirac particle with a tensor interactions terms are investigated by Asymptotic Iteration Method. The eigenvalues are calculated for tensor potentials and formulated into closed forms. Similarity Transformation is used to convert the Dirac equation into a generalized useful form. The s-wave analytical solutions for the relativistic problems of the Dirac-oscillator, Scarf, Rosen-Morse II, and Pöschl-Teller tensor potentials are obtained with an excellent agreement with previous works.

1. Introduction
Exact solutions and analytical solutions always attract the attention of researchers because of their benefits in capturing the entire solution cases of the problem. Furthermore, the solutions of Dirac equation also attract researchers in quantum physics due to their importance in relativistic region. For a particle with velocity closely to light velocity, Dirac equation is the suitable one. As a consequence of that, trends of using mathematical methods that produced exact solutions arise from time to time. Examples of mathematical physics that give exact solutions to Dirac equation are the factorization method [1], Nikiforov–Uvarof (NU) method [2], SUSY QM [3], Romanovski polynomials [4], shape invariance [5] and Asymptotic iteration method (AIM) [6].

Beside the previous method for Dirac equation, there are a new method to obtain solutions for Schrödinger equation like the Residual power series method and the kernel algorithm for first order differential equations [7–10].

On the other hand, tensor potential introduced into the Dirac equation with the substitution

\[ p \rightarrow p - i\beta \alpha \cdot \hat{T}(r) \]  

where \( \hat{T}(r) \) is the tensor interaction potential, \( \alpha \) and \( \beta \) are the Dirac matrices. The tensor potential is one of the important elements of the nucleon–nucleon interaction and in nuclear binding. The introduction of the tensor potentials appears in 1940s shortly after the appearance of nuclear physics [11,12]. Research show that the tensor potentials are connected to the spin-orbit coupling in the Dirac equation [11,12]. Spin-orbit coupling plays a significant role in strong interaction and in the modern fields in physics. Tensor potentials used to investigate nuclear properties and the degeneracy of the problem [13]. In specific, some types of the relativistic physical interaction can be treated as a tensor terms that add to Dirac equation. An example of these tensor potentials, is the interaction of a quantum particle with an anomalous magnetic moment [14].

Recently, the exact solutions of Dirac equation with a tensor interaction potential are considered and gained more attention. Pure Coulomb tensor potential [15], Morse potential [16], Pseudo harmonic Potential [17], Rosen-Morse [18], Quadratic Exponential-Type Potential Plus Eckart Potential Tensor [19], Manning-Rosen Plus Shifted Deng-Fan Potential [20] and Pöschl–Teller double-ring shaped harmonic potential [21] are examples of these investigations.

In this work, we attempt to find the bound states of four types of tensor potentials. The first tensor potential is introduced in 1991 by Moshinsky and Szczepaniak as Dirac-Oscillator [7–10]. The eigenvalues and eigenfunctions of the problem have been investigated by relativistic shape invariant approach [22]. The second one is the trigonometric Scarf tensor potential which has been solved by Romanovski polynomials method [23] and by shape invariance approach [22]. The third potential is Morse-Rosen II which is a well-known interaction

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potential which has been used to probe the vibrations in diatomic molecules [18]. Bound stated of this potential in Dirac equation studied in the relativistic shape invariant framework [22]. The fourth one is the Pöschl-Teller potential, which introduced to explore the vibrational excitations in molecule dynamics [24]. This potential had been investigated for Dirac equation in framework of supersymmetric quantum mechanics [25,26] and within the spin symmetry framework [27].

In this paper, we centered our work in the application of Asymptotic Iteration Method (AIM) for the four previous tensor potentials. AIM is a successful method to find the analytical and numerical solutions for Schrödinger, K-G and Dirac equations [28–33]. Although many years have been passed on this method, it still provides a powerful and favored technique for finding solutions of Dirac, Schrödinger and K-G equations. Researchers still dig on its solutions and improve the method. This is an example of very recent researches this year [34–36]. I think that the beauty of using AIM is their eigenvalues which arises neutrally from the iteration mechanism of the method (we will see that later). AIM has been used in second-order ordinary differential so that Dirac equation which is a first order equations is not used directly by AIM. When Dirac equation transforms to second-order differential equations complicated terms appeared. However, with similar transformation, these complicated terms can be reduced to simpler ones [16,18,37]. Similarity transformation is a linear change of coordinates. Similarity transformations are effective technique to convert partial differential equations to a set of ordinary differential equations. It is a transformation by which an n-independent variable partial differential system can be converted to a system with n-1 independent variable. When n = 2, applying the similarity transformation we come to deal with ordinary differential equations. In linear algebra, two matrices A and B are similar if they obey the following relation

\[ B = M^{-1}AM \]  

(2)

Similar matrices represent the same linear transformation under two different bases, with \( F \) being the change of basis matrix. The matrix \( M \) is called a similarity transformation.

In this paper, we will investigate the s-wave relativistic exact solution of Dirac equation for the Oscillator, Scarf, Rosen-Morse II, and Pöschl-Teller tensor potentials. We try to find its solutions directly by the systematic approach of AIM. We concentrate on s-wave cases because its solutions was available previously by different methods in order to be able to compare our results.

This manuscript is organized as follows: In Section 2, we will shortly remind the reader of Asymmetric Iteration Method (AIM). In Section 3, we will present an overview of the effect of similarity transformation to Dirac equation to get a general formula with any potentials (scalar, vector and tensor potentials). In Section 4, The application of AIM to Oscillator, Scarf, Rosen-Morse II, and Pöschl-Teller tensor potential problems for Dirac equations is presented and the results are discussed in each subsection. Finally, we conclude our results in Section 5.

2. The asymptotic iteration method (AIM)

The formalism of AIM is well-known which can be founded in details in Ref. [6]. To remind the reader, we will presented a short summary of AIM as follows: Any equation of the form

\[ \frac{d^2}{dx^2} f(x) = \lambda_0(x) \frac{d}{dx} f(x) + s_0(x) f(x) \]  

(3)

can be derivative up to \((i + 1)\) and \((i + 2)\) orders, When one re-arranging terms in the form

\[ f^{(i+1)}(x) = \lambda_{i-1}(x) f^{(i)}(x) + s_{i-1}(x) f(x), \]  

(4)

\[ f^{(i+2)}(x) = \lambda_i(x) f^{(i)}(x) + s_i(x) f(x), \]  

(5)

where

\[ \lambda_i(x) = \lambda_{i-1}^0(x) + s_{i-1}(x) + \lambda_0(x) \lambda_{i-1}(x), \]  

and

\[ s_i(x) = s_{i-1}(x) + s_0(x) \lambda_{i-1}(x) \]

then the ratio of \((i + 2)^{th}\) and \((i + 1)^{th}\) derivatives becomes

\[ \frac{f^{(i+2)}}{f^{(i+1)}} = \frac{d}{dx} \ln(f^{(i+1)}) = \frac{\left(f^{(i)} + \frac{s_i}{\lambda_i} f\right) \lambda_i}{\left(f^{(i+1)} + \frac{s_{i-1}}{\lambda_{i-1}} f\right) \lambda_{i-1}} \]  

(6)

The asymptotic aspect of the method arises when one is iterated up to large number of derivative iterations \(i\). This allows one to gain a great simplification in the previous equation by assuming

\[ \frac{s_i}{\lambda_i} = \frac{s_{i-1}}{\lambda_{i-1}} \equiv \alpha \]  

(7)

which yields

\[ \frac{d}{dx} \ln(f^{(i+1)}) = \frac{\lambda_i}{\lambda_{i-1}} \]  

(8)

Do Integration, the solution arise as

\[ f(x) = e^{-\int \alpha dx} \]  

(9)

The energy eigenvalues which can be obtained from the relation in Equation (7) by establishing the termination condition as

\[ \delta_i(x) = \lambda_i(x) s_{i-1}(x) - s_i(x) \lambda_{i-1}(x) = 0 \]  

(10)
3. Dirac equation and similarity transformation

A quantum mechanical fundamental particle of mass \( m \) under the influence of scalar potential \( S(r) \), vector potential \( V(r) \) and tensor potential \( \bar{\beta}\bar{\alpha}T(r) \) can be studied in relativistic motion regime by Dirac equation (in units of \( \hbar = c = 1 \)):

\[
H_D(\vec{r})\psi(\vec{r}) = E\psi(\vec{r})
\]  

with the Dirac Hamiltonian

\[
H_D(\vec{r}) = \bar{\alpha}\vec{p} + \beta(m + S(r)) + V(\vec{r}) - i\beta\bar{\alpha}\vec{T}(\vec{r})
\]  

where \( \bar{\alpha} \) and \( \beta \) are Dirac matrices:

\[
\bar{\alpha} = \begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{pmatrix}, \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

Guided by the standard steps of applying a similarity transformation to Equation (7), one finds [16]:

\[
H'_D\psi'(r) = E\psi'(r)
\]

with \( H'_D = MH_DM^{-1} \), \( \psi'(r) = M\psi(r) \) and \( M = \alpha + ib\bar{\alpha}\vec{T} \), where \( \vec{T} \) is the unit vector \( \vec{r}/r \) and \( \alpha \) and \( b \) are real constants to be determined later. The transformed wave function is usually taken to be

\[
\psi'_{nk}(\vec{r}) = \begin{pmatrix} f_{nk}(\vec{r}) \\ g_{nk}(\vec{r}) \end{pmatrix} = \begin{pmatrix} iR_{nk}(r)\Phi_{jm}^l(\theta, \varphi) \\ Q_{nk}(r)\vec{r}\Phi_{jm}^l(\theta, \varphi) \end{pmatrix}
\]

where \( R_{nk}(r) \), \( Q_{nk}(r) \) and \( \Phi_{jm}^l \) are the usual wave functions.

One can then calculate \( H'_D\psi' = E\psi' \) to obtain two coupled equations for \( R_{nk}(r) \) and \( Q_{nk}(r) \):

\[
\begin{pmatrix} H_1 & H_2 \\ H_3 & H_4 \end{pmatrix} \begin{pmatrix} R_{nk}(r) \\ Q_{nk}(r) \end{pmatrix} = E \begin{pmatrix} R_{nk}(r) \\ Q_{nk}(r) \end{pmatrix}
\]

with

\[
H_1 = (m + S(r))\cosh\vartheta + \sinh\vartheta \left( \frac{d}{\tilde{r}} + \frac{1}{\tilde{r}} \right) + V(r)
\]

\[
H_2 = \left( m + S(r) \right) \sinh\vartheta + \cosh\vartheta \left( \frac{d}{\tilde{r}} + \frac{1}{\tilde{r}} \right) - \frac{\tilde{r}}{4} + T(r)
\]

\[
H_3 = (m + S(r))\sinh\vartheta + \cosh\vartheta \left( \frac{d}{\tilde{r}} + \frac{1}{\tilde{r}} \right) + \frac{\tilde{r}}{4} - T(r)
\]

\[
H_4 = -\left( m + S(r) \right) \cosh\vartheta + \sinh\vartheta \left( \frac{d}{\tilde{r}} + \frac{1}{\tilde{r}} \right) - V(r)
\]

where the quantum number \( k = \pm(j + 1/2) \) for states with \( l = j \pm 1 \), \( \cosh\vartheta = (a^2 + b^2)/(a^2 - b^2) \), \( \sinh\vartheta = 2ab/(a^2 - b^2) \), and \( n \) denotes the radial quantum number.

The coupled equations for \( R(r) \) and \( Q(r) \) are given by

\[
H_1R_{nk}(r) + H_2Q_{nk}(r) = ER_{nk}(r)
\]

\[
H_3R_{nk}(r) + H_4Q_{nk}(r) = EQ_{nk}(r)
\]

Multiplying Equation (16) by \( \sinh\vartheta \), Equation (17) by \( \cosh\vartheta \) and then subtracting the results we obtain

\[
Q_{nk}(r) = \frac{1}{Z_1(r)}D_1(r)R_{nk}(r)
\]

where

\[
D_1(r) = \tilde{r} + X_1(r)
\]

\[
X_1(r) = \frac{1}{r} + \frac{k}{r}\cosh\vartheta - T(r)\cosh\vartheta - V(r)\sinh\vartheta + E\sinh\vartheta
\]

and

\[
Z_1(r) = m + S(r) - V(r)\cosh\vartheta + \frac{k}{r}\sinh\vartheta - T(r)\sinh\vartheta + E\cosh\vartheta
\]

Similarly, multiplying Equation (16) by \( \cosh\vartheta \), Equation (17) by \( \sinh\vartheta \) and then subtracting the results we obtain

\[
R_{nk}(r) = \frac{1}{Z_2(r)}D_2(r)Q_{nk}(r)
\]

where

\[
D_2(r) = \tilde{r} + X_2(r)
\]

\[
X_2(r) = \frac{1}{r} - \frac{k}{r}\cosh\vartheta + T(r)\cosh\vartheta + V(r)\sinh\vartheta - E\sinh\vartheta
\]

and

\[
Z_2(r) = m + S(r) + V(r)\cosh\vartheta - \frac{k}{r}\sinh\vartheta + T(r)\sinh\vartheta - E\cosh\vartheta
\]

By substituting from Equation (18) in (22) and solving the resulting equation for \( R_{nk}(r) \) we imply

\[
R_{nk}(r) = \frac{1}{Z_2(r)}D_2(r)\frac{1}{Z_1(r)}D_1(r)R_{nk}(r)
\]

Equation (26) can be simplified by using the identity: \( \cosh^2 - \sinh^2 = 1 \), to obtain

\[
R_{nk}(r) + \left( \frac{2}{r} - \frac{Z_2(r)}{Z_1(r)} \right) R_{nk}(r)
\]

\[
+ \left( U(r) - \frac{X_1(r)Z_1(r)}{Z_1(r)} \right) R_{nk}(r) = 0
\]
where
\[ U(r) = -\frac{k \cosh \theta + k^2}{r^2} + \frac{2kT(r)}{r} - m^2 - S^2 - T^2(r) \]
\[ - 2mS(r) - \sinh \theta V(r) - \cosh \theta T'(r) \]
\[ - 2E V(r) + V^2(r) + E^2 \]
(28)

The prime notation denotes differentiation with respect to \( r \).

Equation (27) is like a 2nd order Dirac equation but with a complicated term. Similar transformation here may play a significant simplification to Equation (27), by appropriate choice of the constants \( \cosh \theta \) and \( \sinh \theta \) one can attain a great simplification [37].

We can now introduce potentials in to Equation (27) and then by using the appropriate choice of the constants \( \cosh \theta \) and \( \sinh \theta \) one can simplify the equation to arrive to nearly a 2nd order Dirac equation then solutions can be found by applying AIM. The procedure of finding solutions will become clearer in the following sections.

4. Applications, results and discussion

In this section we will present the applications of AIM to a tensor potentials for the general Dirac equation Equation (27) as following:

4.1. Dirac-Oscillator tensor potential:

\[ (T = -A r, S = 0 \text{ and } V = 0) \]

where \( A \) is a positive constant, \( S \) and \( V \) are the scalar and vector potentials.

Substituting these potentials into Equation (13) yields an equation which need a more simplification. The choice \( \cosh \theta = 1 \) and \( \sinh \theta = 0 \), let \( Z_1'(r) = 0 \) and the remaining term; \( \frac{1}{2} R''_n(r) \) in Equation (27) can be removed by redefined \( R_n(r) \) as \( R_n(r) = r^{-1} \phi_{nk}(r) \), Equation (27) becomes;

\[ \phi_{nk}(r) + \left[ \frac{\varepsilon}{2} - \frac{k(k + 1)}{r^2} - (Ar)^2 \right] \phi_{nk}(r) = 0 \]
(29)

where
\[ \varepsilon = E^2_{nk} - m^2 + A - 2Ak \]
(30)

Hence the asymptotic behavior of the radial wave function \( \phi_{nk}(r) \) suggests ansatz;

\[ \phi_{nk}(r) = r^{k+1} e^{-(A/2)r^2} f_{nk}(r) \]
(31)

Substitutes this suggestion into Equation (22), We can get an equation which is suitable for the application of AIM as in Equation (10).

\[ f''_{nk}(r) + \lambda_0 f'_{nk}(r) + s_0 f_{nk}(r) = 0 \]
(32)

with
\[ \lambda_0 = 2Ar - \frac{2(k + 1)}{r}, \]
\[ s_0 = A(3 + 2k) - \varepsilon \]
(33)

Now, AIM is ready to apply to Equation (32) find solutions. The relativistic energy eigenvalues of the problem can be calculated by means of the termination condition as in Equation (10). By following the standard iterations of AIM and imposing the termination condition, the following results can be obtained;

for \( n = 0, \text{ gives } \varepsilon = A(3 + 2k) \)
\( n = 1 \rightarrow \varepsilon = A(7 + 2k), \)
\( n = 2 \rightarrow \varepsilon = A(11 + 2k) \)
\( \ldots, \text{ etc.} \)

From these sequences, one can predict a rule of \( \varepsilon \) for any \( n \) as
\[ \varepsilon = A(4n + 3 + 2k), \text{ for } n \geq 0 \]
(34)

Now the eigenvalues \( E_{nk} \) of the problem can be deduced directly from Equation (30);
\[ E_{nk} = \sqrt{m^2 + 2A(1 + 2k + 2n)} \text{ where } k = l, -l - 1 \]
(35)

One may notice that Equation (27) is satisfied by the Kummer confluents hypergeometric function \( _1F_1(a; b; z) \) [38], hence the corresponding general solutions Equation (27) of the problem can be obtained as
\[ \psi_{nk}(r) = Nk^{l+1} e^{-(A/2)r^2} _1F_1(a; b; -Ar^2) \]
(36)

where
\[ a = \frac{3}{4} - \frac{\varepsilon}{4A} + \frac{k}{2}, b = -\frac{1}{2} - k \]

and \( N \) is the normalization constant. These results are in exact agreement that founded in the previous work [22].

4.2. Scarf potential

The Scarf and it’s larger family of hyperbolic potentials play a fundamental role in atomic and molecular physics because it is used as tool to model the inter-atomic and inter-molecular interactions. The use of this potential extended to many fields of physics like solid state physics especially in modeling the periodic potential in one-dimensional crystal [39].

We will consider the following tensor potential type:
\[ T = -ATanh(\lambda r) - BSech(\lambda r) - \frac{k}{r}, S = 0 \text{ and } V = 0 \]
where \( A \) and \( B \) are positive constants.
Substituting these potentials into Equation (13), following the same procedures as in 5.1, removing the first order derivative, arranging terms, then the s-wave equation ($k = 0$) becomes;

$$\phi''_{nk}(r) + [\varepsilon + g \text{Sech}^2(\lambda r)] \phi_{nk}(r) + G \text{Sech}(\lambda r) \text{Tanh}(\lambda r) \phi_{nk}(r) = 0$$  \hspace{1cm} (37)

where

$$\varepsilon = (E_{nk}^2 - m^2 + A),$$

$$F = (A^2 - B^2 + A\lambda),$$ and

$$G = -B(\lambda + 2A).$$

Using the corresponding hyperbolic trigonometric identities for $\text{Tanh}$ and $\text{Sech}$ allows us to rewrite the previous equation in terms of $\text{Cosh}$ and $\text{Sinh}$. Then by using the identity; $\text{Cosh}^2(x) = 1 + \text{Sinh}^2(x)$, one can get an equation of one hyperbolic trigonometric variable in $\text{Sinh}(x)$. By change of variables as; $z = \text{Sinh}(\lambda r)$, one can arrive to

$$\lambda^2 (1 + z^2) \phi''(z) + \lambda^2 z \phi'(z) + \left[ \varepsilon + \frac{g z}{1 + z^2} + \frac{F}{1 + z^2} \right] \phi(z) = 0$$  \hspace{1cm} (39)

The asymptotic behavior of the radial wave function $\phi_{nk}$ suggests the ansatz;

$$\phi_{nk}(z) = (1 + z^2)^{\gamma/2} e^{-\lambda z} \text{ArcTanh}(r)f_{nk}(z)$$  \hspace{1cm} (40)

The values of $\gamma$, $\mu$ will be determined in a way that we were able to apply AIM. Inserting this ansatz into Equation (39),

$$f''_{nk}(z) + \lambda_1(z)f'_{nk}(z) + s_0(z)f_{nk}(z) = 0$$  \hspace{1cm} (41)

with

$$\lambda_1(z) = -z(1 + 2\gamma) + \mu,$$

$$s_0(z) = -\frac{(\varepsilon + \gamma^2)}{1 + z^2} + \frac{\rho z + \sigma}{(1 + z^2)^2}$$  \hspace{1cm} (42)

where

$$\rho = G - \frac{\mu}{2} + \gamma \mu$$

and

$$\sigma = \gamma^2 - \gamma - \frac{\mu^2}{4} - F.$$

To apply AIM, we must do more simplification to $s_0(z)$. We notice that the last terms in $s_0(z)$ can be canceled by letting $\rho = 0$ and $\sigma = 0$ from the free choices of $\mu$ and $\gamma$, which allow us to have

$$s_0(z) = -\frac{(\varepsilon + \gamma^2)}{1 + z^2}$$  \hspace{1cm} (43)

Now, we can apply AIM procedure to find the eigenvalues as in 5.1, the results as follows:

For $n = 0$ gives $\rightarrow \varepsilon = 0$,

$$n = 1 \rightarrow \varepsilon = -1 - 2\gamma - \gamma^2$$

$$n = 2 \rightarrow \varepsilon = -4 - 2\gamma - \gamma^2$$

$$n = 3 \rightarrow \varepsilon = -9 - 4\gamma - \gamma^2$$

$$n = 4 \rightarrow \varepsilon = -16 - 8\gamma - \gamma^2$$

... etc.

From these iterations, one can suggest that for any $n$;

$$\varepsilon = n^2 - 2n\gamma - \gamma^2$$ for $n \geq 0$  \hspace{1cm} (44)

Now, the eigenvalues $E_{nk}$ of the problem can be obtained from Equation (38). Our result in Equation (44) is in exact agreement with the earlier studies for this potential [22].

### 4.3. Rosen–Morse II

The trigonometric Rosen–Morse II potential, introduced after the investigation of the physicists Nathan Rosen and Philip Morse for the exactly solvable quantum potentials. This potential gain its importance in the fields of atomic and molecular spectroscopy.

This potential interested in quark physics (QCD) when researchers investigate the principal of the quark-gluon dynamics.

We consider the following Rosen–Morse II tensor potential;

$$T = -A \text{Cosh}(\lambda r) - B \text{Csch}(\lambda r) - \frac{k}{r}, \quad S = 0 \text{ and } V = 0$$

where $A$ and $B$ are a positive constants.

Substituting these potentials into our master Equation (27), following the same procedures as in 5.1, and using the corresponding hyperbolic trigonometric identities for $\text{Cosh}$ and $\text{Csch}$, removing the first order derivative as done in 5.1, arranging terms, the s-wave equation ($k = 0$) becomes;

$$\phi''_{nk}(r) + \left[ \varepsilon - \frac{F + G \text{Cosh}(\lambda r)}{\text{Sinh}(\lambda r)^2} \right] \phi_{nk}(r) = 0$$  \hspace{1cm} (45)

with

$$\varepsilon = E_{nk}^2 - m^2 + A,$$

$$F = (B^2 - A(A + \lambda))$$

and

$$G = B(\lambda + 2A).$$

By taking; $iz = \text{cosh}(\lambda r)$, we can put the previous equation into the following form:

$$(1 + z^2)\phi''_{nk}(z) + z\phi'_{nk}(z) + \left[ \varepsilon - \frac{Gz - F}{1 + z^2} \right] \phi_{nk}(z) = 0$$  \hspace{1cm} (47)

with

$$\varepsilon = (E_{nk}^2 - m^2 + A)/\lambda^2,$$

$$F = (B^2 - A(A + \lambda))/\lambda^2,$$

and

$$G = iB(\lambda + 2A)/\lambda^2.$$

We notice that Equation (47) is nearly the same as Equation (39), the difference in $G$ parameter as in
Applying AIM procedure to find the eigenvalues as in Equation (48), we will assume the same suggestion for asymmetric behavior as
\[
\phi_{nk}(z) = (1 + z^2)^{\gamma/2} e^{-\left(\frac{z}{2}\right) ArcTan(\gamma)} f_{nk}(z)
\]
(49)
The values of \( \gamma, \mu \) will be determined in a way that allows us to be able to apply AIM. Inserting this ansatz into Equation (47),
\[
f''_{nk}(z) + \lambda_0(z)f'_{nk}(z) + s_0(z)f_{nk}(z) = 0
\]
(50)
with
\[
\lambda_0(z) = \frac{-z(1 + 2\gamma) + \mu}{1 + z^2},
\]
\[
s_0(z) = -\frac{(\epsilon + \gamma^2)}{1 + z^2} + \frac{\rho z + \sigma}{(1 + z^2)^2}
\]
(51)
where
\[
\rho = G - \frac{\mu}{2} + \gamma \mu \quad \text{and} \quad \sigma = \gamma^2 - \gamma - \frac{\mu^2}{4} - F.
\]
To apply AIM, we must do more s to \( s_0(z) \). The last terms in \( s_0(z) \) can be canceled by letting \( \rho = 0 \) and \( \sigma = 0 \) from the free choices of \( \mu \) and \( \gamma \), which allow us to get
\[
s_0(z) = -\frac{(\epsilon + \gamma^2)}{1 + z^2}
\]
(52)
Applying AIM procedure to find the eigenvalues as in 5.2, the results as following;
\[
n = 0 \rightarrow \epsilon = 0
\]
\[
n = 1 \rightarrow \epsilon = -1 - 2\gamma - \gamma^2
\]
\[
n = 2 \rightarrow \epsilon = -4 - 4\gamma - \gamma^2
\]
\ldots \text{ etc.}

From these iterations, one can suggest for any \( n \);
\[
\epsilon = n^2 - 2n\gamma - \gamma^2, \text{ for } n \geq 0
\]
(53)
Now, the eigenvalues \( E_{nk} \) of the problem can be obtained from Equation (48).
Again, our result is in exact agreement with the earlier studies for this potential [22].

4.4. Pöschl-Teller tensor

Considering the following tensor potential;
\[
T = -A \ Tanh(\lambda r) - B \ Coth(\lambda r) - \frac{k}{r^2} S = 0 \quad \text{and} \quad V = 0
\]
where \( A \) and \( B \) are a positive constants.
Substituting these potentials into our master Equation (27), following the same procedures as in 5.3, and using the corresponding hyperbolic trigonometric identities for \( \text{Coth} \) and \( \text{Csch} \), removing the first order derivative as done in 5.1, arranging terms, the s-wave equation \((k = 0)\) becomes;
\[
\psi''_{nk}(r) + \left[ \epsilon + \frac{A(A + \lambda)}{\text{Cosh}^2(\lambda r)} - \frac{B(\lambda - \mu)}{\text{Sinh}^2(\lambda r)} \right] \psi_{nk}(r) = 0
\]
(54)
with
\[
\epsilon = (E_{nk}^2 - m^2 - (A + B)^2)
\]
(55)
Taking \( z = \sinh(\lambda r) \), we can put Equation (54) into the following form:
\[
(1 + z^2)\psi''_{nk}(z) + z\psi'_{nk}(z) + \left[ \epsilon + \frac{a(a + 1)}{1 + z^2} - \frac{b(b - 1)}{z^2} \right] \psi_{nk}(r) = 0
\]
(56)
where \( a \) and \( b \) defined as
\[
a(a + 1) = \frac{A(A + \lambda)}{\lambda^2}, \quad b(b - 1) = \frac{B(\lambda - \mu)}{\lambda^2} \quad \text{and} \quad \epsilon = \frac{\lambda^2}{\lambda^2}
\]
(57)
We will assume the same suggestion for asymmetric behavior as
\[
\psi_{nk}(z) = (1 + z^2)^{-a/2} \rho f_{nk}(z)
\]
(58)
Inserting this ansatz into Equation (56), one can arrive to this form
\[
f''_{nk}(z) + \lambda_0(z)f'_{nk}(z) + s_0(z)f_{nk}(z) = 0
\]
(59)
with
\[
\lambda_0(z) = -\frac{2b + z^2(1 - 2a + 2b)}{z + z^2},
\]
\[
s_0(z) = -\frac{\epsilon}{1 + z^2} - \frac{(b - a)^2}{(1 + z^2)^2}
\]
(60)
Applying AIM procedure to find the eigenvalues as in 5.3 leads to the following results;
\[
n = 0 \rightarrow \epsilon = -(0 - a + b)^2
\]
\[
n = 1 \rightarrow \epsilon = -(2 - a + b)^2
\]
\[
n = 2 \rightarrow \epsilon = -(4 - a + b)^2
\]
\ldots \text{ etc}
\[
\epsilon = -(2n - a + b)^2, \text{ for } n \geq 0
\]
(61)
The eigenvalues \( E_{nk} \) of the problem can be obtained easily from Equation (61) and Equation (57). Our result is in exact agreement with the earlier studies for this potential [22].
5. Conclusion

The relativistic energy eigenvalues of Dirac equation with a tensor potentials is investigated by using the Asymptotically Iteration Method (AIM). Similarity transformation is successfully used and converted Dirac equation into a general form that is suitable for AIM applications. The bound state solutions of the Dirac-oscillator, Scarf, Rosen-Morse II, and Pöschl-Teller tensor potentials are obtained with an excellent agreement with the previous works.

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