LOCALLY TRIANGULAR GRAPHS
AND RECTAGRAPHS WITH SYMMETRY

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Abstract. Locally triangular graphs are known to be halved graphs of bipartite rectagraphs, which are connected triangle-free graphs in which every 2-arc lies in a unique quadrangle. A graph Γ is locally rank 3 if there exists $G \leqslant \text{Aut}(\Gamma)$ such that for each vertex $u$, the permutation group induced by the vertex stabiliser $G_u$ on the neighbourhood $\Gamma(u)$ is transitive of rank 3. One natural place to seek locally rank 3 graphs is among the locally triangular graphs, where every induced neighbourhood graph is isomorphic to a triangular graph $T_n$. This is because the graph $T_n$, which has vertex set the 2-subsets of $\{1, \ldots, n\}$ and edge set the pairs of 2-subsets intersecting at one point, admits a rank 3 group of automorphisms. In this paper, we classify the locally 4-homogeneous rectagraphs under some additional structural assumptions. We then use this result to classify the connected locally triangular graphs that are also locally rank 3.

1. Introduction

A finite simple undirected graph Γ is locally $\Delta$ (or $\mathcal{C}$) for some graph $\Delta$ (or class of graphs $\mathcal{C}$) if for every vertex $u \in V\Gamma$, the graph induced by the neighbourhood $\Gamma(u)$ is isomorphic to $\Delta$ (or some graph in $\mathcal{C}$). There is a well-established tradition of classifying such graphs. Extending this concept, there is a trend of studying graphs $\Gamma$ for which some set of vertices related to $u$ is highly symmetric for each $u \in V\Gamma$, such as locally projective graphs [16], or locally $s$-arc transitive graphs [11, 21, 31].

A graph $\Gamma$ is locally 2-arc transitive if $\Gamma$ contains a 2-arc and there exists $G \leqslant \text{Aut}(\Gamma)$ such that, for every $u \in V\Gamma$, the stabiliser $G_u$ acts transitively on the 2-arcs starting at $u$, where a 2-arc is a tuple of vertices $(u, v, w)$ such that $u \neq w$ and $u, w \in \Gamma(v)$. A 2-arc $(u, v, w)$ is either a triangle when $u$ and $w$ are adjacent, or a 2-geodesic when $u$ and $w$ are at distance 2 apart. Connected non-complete graphs with girth 3 contain triangles and 2-geodesics with the same initial vertex and are therefore never locally 2-arc transitive, but for the most symmetrical of these graphs, the 2-arcs with any given initial vertex $u$ fall into two $\text{Aut}(\Gamma)_u$-orbits, namely the triangles and the 2-geodesics. This is equivalent to the permutation group induced by $\text{Aut}(\Gamma)_u$ on $\Gamma(u)$ being transitive of rank 3 for all $u \in V\Gamma$ (cf. Proposition 5.4).

Motivated by this observation, we introduce the following definition: a graph $\Gamma$ is locally rank 3 with respect to $G$ if $\Gamma$ has no vertices with valency 0 and $G \leqslant \text{Aut}(\Gamma)$ such that, for all $u \in V\Gamma$, the permutation group induced by $G_u$ on $\Gamma_u$ is transitive of rank 3 (cf. §5 for the definition of rank). We also say that $\Gamma$ is locally rank 3 if it is locally rank 3 with respect to some $G$. Note that there exist connected graphs that are both locally rank 3 and locally 2-arc transitive (cf. §5); necessarily, these graphs are either complete or have girth at least 4.

In this paper, we study and classify a family of locally rank 3 graphs associated with the class of rectagraphs (defined below). Note that locally disconnected locally rank 3 graphs were recently analysed in a more general setting by Devillers et al. [9], but the graphs we are interested in are locally connected.

Locally triangular and locally rank 3 graphs. The most basic example of a rank 3 permutation group is the action of the symmetric group $S_n$ on the set $\binom{\{1, \ldots, n\}}{2}$.
There are two graphs corresponding to this action. One is the \textit{triangular graph} $T_n$, where a pair of 2-subsets are adjacent whenever they intersect at exactly one point, and the other is the complement $\overline{T}_n$ of $T_n$.

Remarkably, connected locally $T_n$ graphs were completely classified by Hall and Shult \cite{13}. These graphs are very well behaved, for $\overline{T}_{n+2}$ is itself locally $T_n$, and for $n \geq 7$, it is the only such connected graph. In particular, locally $T_n$ graphs have bounded diameter, and we will see that they are all locally rank 3 (cf. Corollary 1.2). Note that $\overline{T}_5$ is the Petersen graph, so this classification includes locally Petersen graphs, first classified by Hall \cite{12}.

However, the behaviour of locally $T_n$ graphs is much wilder. For example, the graph $T_n$ is not locally $T_m$ for any $m$, nor is it ever locally rank 3. Moreover, the diameter of locally triangular graphs is unbounded, for the halved $n$-cube is locally $T_n$ and has diameter $\lfloor n/2 \rfloor$. In fact, every connected component of the distance 2 graph of a coset graph of a linear code over $\mathbb{F}_2^n$ with minimum distance at least seven is locally $T_n$ (cf. Lemma 4.4), but this graph is rarely locally rank 3 (cf. Theorem 5.6 and Proposition 5.7).

We say that a graph is \textit{locally triangular} if it is locally $C_5$, where $C_5$ is the class of all triangular graphs. It turns out that a connected locally triangular graph is always locally $T_n$ for some $n$ \cite[Proposition 4.3.9]{5}. Strongly regular locally $T_n$ graphs were classified in \cite{23}, and 1-homogeneous locally triangular graphs were classified in \cite[Theorem 4.4]{18}. The graphs in these classifications all appear in our first main result Theorem 1.1, where we completely classify the connected graphs that are locally triangular and locally rank 3.

\textbf{Theorem 1.1.} A connected graph $\Gamma$ is locally rank 3 and locally triangular if and only if $\Gamma$ is the halved graph of one of the following bipartite graphs.

\begin{itemize}
  \item[(i)] The n-cube $Q_n$ where $n \geq 3$.
  \item[(ii)] The folded n-cube $\overline{Q}_n$ where $n$ is even and $n \geq 8$.
  \item[(iii)] The bipartite double of the coset graph of the binary Golay code $C_{23}$.
  \item[(iv)] The coset graph of the extended binary Golay code $C_{24}$.
\end{itemize}

Moreover, $\Gamma$ is locally rank 3 with respect to $G \leq \text{Aut}(\Gamma)$ if and only if $G$ is listed in Table 1.

\begin{table}[h]
\begin{tabular}{|c|c|c|}
\hline
$\Gamma$ & $n$ & $G$ \\
\hline
$\frac{1}{2}Q_n$ & $n \geq 5$ & $2^{n-1} \rtimes S_n$, $2^{n-1} \rtimes A_n$ \\
& 3 & $A_4$ \\
& 4 & $2^4 \rtimes S_4$, $2^3 \rtimes S_4$, $(2^3 \rtimes A_4).2$ \\
& 9 & $2^8 \rtimes \text{PGL}_2(8)$ \\
& 11, 12, 23, 24 & $2^{n-1} \rtimes M_n$ \\
$\frac{1}{2}\overline{Q}_n$ & $n \geq 8$ even & $2^{n-2} \rtimes S_n$, $2^{n-2} \rtimes A_n$ \\
& 12, 24 & $2^{n-2} \rtimes M_n$ \\
$\frac{1}{2}\Gamma(C_{23}).2$ & 23 & $2^{11} \rtimes M_{23}$ \\
$\frac{1}{2}\Gamma(C_{24}).2$ & 24 & $2^{11} \rtimes M_{24}$ \\
\hline
\end{tabular}
\caption{\label{table1} $G \leq \text{Aut}(\Gamma)$ for which $\Gamma$ is locally rank 3}
\end{table}

The graphs of Theorem 1.1 are described in §2.5 (see also §2.1). They are all distance-transitive graphs of valency $\left(\begin{array}{c}n \\ 2\end{array}\right)$ where $n = 23$ in (iii) and $n = 24$ in (iv). As a corollary of Theorem 1.1 and \cite[Theorem 2]{13}, we obtain a classification of graphs that are locally rank 3 where the local action is that of some subgroup of $S_n$ on the set $\left(\begin{array}{c}n \\ 2\end{array}\right)$ of 2-subsets of $\{1, \ldots, n\}$. In particular, this classification includes the locally $T_n$ graphs for $n \geq 5$.

\textbf{Corollary 1.2.} Let $\Gamma$ be a connected non-complete graph with girth 3. For $n \geq 5$, let $\mathcal{C}_n$ denote the class of groups $H \leq S_n$ such that $H$ is transitive of rank 3 on $\left(\begin{array}{c}n \\ 2\end{array}\right)$. Then the following are equivalent.
The exceptional locally triangular graphs are closely related to rectagraphs, which are connected triangle-free graphs in which every 2-arc lies in a unique quadrangle. Indeed, if Π is a rectagraph with \( a_2(Π) = 0 \) and \( c_3(Π) = 3 \), then every connected component of the distance 2 graph of Π is locally triangular. (The parameters \( a_i \) and \( c_i \) are the same as those defined for distance-regular graphs; see §2.1.) Conversely, by [5, Proposition 4.3.9], every connected locally triangular graph is a halved graph of some bipartite rectagraph Π with \( c_3(Π) = 3 \).

Rectagraphs were first named by Neumaier [27], though they were studied before this [7]. More generally, rectagraphs are examples of \((0, 2)\)-graphs, which have been classified for small valency [4, 6]. Bipartite rectagraphs also have links to geometry, for such graphs are precisely the incidence graphs of semibiplanes, which are connected point-block incidence structures for which any two distinct points lie in exactly 0 or 2 blocks and any two distinct blocks intersect in exactly 0 or 2 points. These structures were first examined in [15, 32] and their study is an active area of research.

Theorem 1.1 is largely a consequence of our second main result, which concerns rectagraphs. We say that a group \( H \) acting on a set \( Ω \) with \( |Ω| \geq 4 \) is 4-homogeneous if \( H \) acts transitively on the set of 4-subsets of \( Ω \).

**Theorem 1.3.** Let \( Π \) be a rectagraph with \( a_2(Π) = 0 \) and \( c_3(Π) = 3 \). There exists \( u \in VII \) such that \( |Π(u)| \geq 4 \) and \( \text{Aut}(Π)_u \) is 4-homogeneous on \( Π(u) \) if and only if \( Π \) is one of the following.

(i) The \( n \)-cube \( Q_n \) where \( n \geq 4 \).
(ii) The folded \( n \)-cube \( □_n \) where \( n \geq 7 \).
(iii) The bipartite double of the coset graph of the binary Golay code \( C_{23} \).
(iv) The coset graph of the binary Golay code \( C_{23} \).
(v) The coset graph of the extended binary Golay code \( C_{24} \).

We remark that no assumptions of vertex-transitivity or distance-regularity are made in Theorems 1.1 or 1.3, though vertex-transitivity is a consequence of the assumptions of Theorem 1.1 (cf. Lemma 5.1).

The graphs appearing in Theorem 1.3 are described in §2.5. They are all distance-transitive graphs of valency \( n \), where \( n = 23 \) in (iii)-(iv) and \( n = 24 \) in (v). Moreover, all of these graphs are coset graphs of linear codes in \( F_2^n \). Indeed, the \( n \)-cube is the coset graph of the zero code, the folded \( n \)-cube is the coset graph of the repetition code, and the bipartite double of the coset graph of the binary Golay code \( C_{23} \) is isomorphic to the coset graph of the even weight vectors in \( C_{23} \). The graphs of Theorem 1.3 are also examples of affine 2-arc transitive graphs. (Note that a 2-arc transitive graph is precisely a vertex-transitive locally 2-arc transitive graph.) The primitive and bi-primitive affine 2-arc transitive graphs were classified in [17]. The non-bipartite graphs of Theorem 1.3 are all primitive, while the bipartite graphs of Theorem 1.3 are bi-primitive except when \( Π \) is the \( n \)-cube and \( n \) is even.
A result of Cameron [7, Theorem 4.5] implies that if \( II \) is a vertex-transitive rectagraph with \( a_2(II) = 0 \) and \( c_3(II) = 3 \) such that the action of \( \text{Aut}(II)_u \) on \( II(u) \) is permutation isomorphic to the natural action of \( S_n \) or \( A_n \) on \( [n] := \{1, \ldots, n\} \), then either \( II \) is the \( n \)-cube, or \( n \geq 7 \) and \( II \) is the folded \( n \)-cube. We obtain this result as a corollary of Theorem 1.3 without the assumption of vertex-transitivity.

**Corollary 1.4.** Let \( II \) be a rectagraph with \( a_2(II) = 0 \) and \( c_3(II) = 3 \). There exists \( u \in V II \) such that the action of \( \text{Aut}(II)_u \) on \( II(u) \) is permutation isomorphic to the natural action of \( S_n \) or \( A_n \) on \( [n] \) if and only if \( II \) is the \( n \)-cube, or \( n \geq 7 \) and \( II \) is the folded \( n \)-cube.

In addition, Brouwer proved in [3] that any distance-regular bipartite graph \( II \) with parameters \( c_i(II) = i \) for all \( i \) (and some additional assumptions) is the \( n \)-cube, the halved \( n \)-cube or the coset graph of the extended binary Golay code. Replacing the distance-regularity condition with a local symmetry condition, we obtain a similar result.

**Corollary 1.5.** Let \( II \) be a connected bipartite graph with \( c_2(II) = 2 \) and \( c_3(II) = 3 \). There exists \( u \in V II \) such that \( |II(u)| \geq 5 \) and \( \text{Aut}(II)_u \) is \( 5 \)-transitive on \( II(u) \) if and only if \( II \) is the \( n \)-cube where \( n \geq 5 \), or the folded \( n \)-cube where \( n \geq 8 \) and \( n \) is even, or the coset graph of the extended binary Golay code.

The proof of Theorem 1.3 proceeds as follows. If \( II \) is any rectagraph with \( a_2(II) = 0 \) and \( c_3(II) = 3 \), then for some \( n \) there is a map \( \pi : Q_n \rightarrow II \) (called a covering) that preserves the local structure of the \( n \)-cube [5, §4.3B]. By some unpublished observations of Matsumoto [24], proved here in a slightly more general context, there is a group \( K^\pi \) of automorphisms of \( Q_n \) associated to the covering \( \pi \) that completely determines the structure of \( II \) as a quotient of \( Q_n \) (cf. Proposition 3.4). Often, this group \( K^\pi \) turns out to be a linear code in \( F_2^\pi \) as well as an \( F_2 N^0_0 \)-module, where \( N^0_0 \) is some subgroup of \( S_n \) normalising \( K^\pi \), in which case we can use coding theory and representation theory to determine the group \( K^\pi \) and therefore \( II \) itself.

**Open problems.** One natural extension of our work would be to generalise Theorem 1.1 to the \( q \)-analogue of the triangular graph, the so-called Grassmann graph, whose vertices are the 2-subspaces of an \( F_q \)-vector space, with two 2-subspaces adjacent whenever their intersection has dimension one. Examples of locally Grassmann graphs include the graph of alternating forms over \( F_2 \) and the graph of quadratic forms over \( F_2 \) [26]. Surprisingly, it is claimed in [19, Theorem 3] that locally Grassmann graphs only exist when \( q = 2 \). Unfortunately, no proof or reference to one is given.

Moreover, since all of the graphs of Theorem 1.1, Corollary 1.2 and Theorem 1.3 turn out to be distance-transitive, it would be interesting to have a direct proof of this fact.

**Outline.** In §2, we define some notation and give some information concerning coverings, quotient graphs and binary linear codes, including the definitions of the graphs of Theorems 1.1 and 1.3 and Corollary 1.2. In §3, we explore the structure of rectagraphs covered by \( n \)-cubes and use this information to prove Theorem 1.3 and Corollaries 1.4 and 1.5. In §4, we give some properties of locally triangular graphs, and in §5, we examine locally rank 3 graphs and then prove Theorem 1.1 and Corollary 1.2. Note that with the exception of Corollary 1.4, all of the main results of this paper depend on the classification of the finite simple groups, as their proofs use the classification of the multiply transitive permutation groups.

## 2. Preliminaries

Unless otherwise specified, all graphs in this paper are finite, undirected and simple (no multiple edges or loops), all groups are finite, and all functions and actions are written on the right. Basic graph theoretical terminology may be found in [5], and basic group theoretical terminology may be found in [8, 33]. The notation used to denote the finite simple groups is consistent with that of [33].
2.1. Notation and basic definitions. Let $\mathbb{F}_2^n$ be the vector space of $n$-tuples over the field $\mathbb{F}_2 = \{0, 1\}$. The weight $|u|$ of a vector $u \in \mathbb{F}_2^n$ is the number of non-zero coordinates in $u$, and the Hamming distance of $u, v \in \mathbb{F}_2^n$ is the number of coordinates at which $u$ and $v$ differ, or equivalently, $|u + v|$. For $1 \leq i_1 < \cdots < i_m \leq n$, let $e_{i_1, \ldots, i_m}$ denote the vector of weight $m$ in $\mathbb{F}_2^n$ whose $i_j$-th coordinate is 1 for $1 \leq j \leq m$. Also, let $E_n$ denote the set of vectors in $\mathbb{F}_2^n$ with even weight. Context permitting, we will write 0 for the $n$-tuple $(0, \ldots, 0)$ and 1 for the $n$-tuple $(1, \ldots, 1)$. We write $\binom{n}{i}$ for the set of $i$-subsets of $[n] := \{1, \ldots, n\}$.

Let $G$ and $H$ be groups. Then $G \rtimes H$ denotes a semidirect product with normal subgroup $G$ and subgroup $H$, and $G.H$ denotes a group with normal subgroup $G$ and quotient $H$. Moreover, $G \wr H$ denotes the wreath product $G^n \rtimes H$, where $H$ acts on $[n]$. If $G$ acts on $\Omega$ and $\Delta := \{\omega_1, \ldots, \omega_m\} \subseteq \Omega$, then we write $G_\Delta$ for the setwise stabiliser of $\Delta$ in $G$ and $G_{\omega_1, \ldots, \omega_m}$ for the pointwise stabiliser of $\Delta$ in $G$. We also write $\omega G$ for the orbit of $G$ containing $\omega_1$. The induced permutation group $G^\Omega$ of $G$ is defined to be the image of the permutation representation $G \to \text{Sym}(\Omega)$ and is isomorphic to $G/K$, where $K$ is the kernel of the action of $G$ on $\Omega$. Hence $G^\Omega$ is a subgroup of $\text{Sym}(\Omega)$. The symmetric group and alternating group on $n$ points are denoted by $S_n$ and $A_n$ respectively. For a field $F$, we denote the group algebra of $G$ over $F$ by $FG$, and if $G \leq S_n$, then the permutation module of $G$ over $F$ is the $FG$-module $F^n$ where $G$ acts by permuting coordinates.

If $G$ and $H$ are groups acting on $\Omega$ and $\Delta$ respectively, then $G$ and $H$ are permutation isomorphic if the action of $G$ or $H$ is faithful and there exists a group isomorphism $\psi : G \to H$ and a bijection $\varphi : \Omega \to \Delta$ for which $(\omega g)\varphi = (\varphi g)\psi$ for all $\omega \in \Omega$ and $g \in G$. Note that if $G$ and $H$ are permutation isomorphic, then both groups must act faithfully. In particular, the definition given is equivalent to the more standard definition of permutation isomorphism, which requires that both $G$ and $H$ act faithfully.

Let $\Gamma$ be a graph. We write $\mathcal{V}T$ for the vertex set of $\Gamma$, $\mathcal{E}T$ for the edge set of $\Gamma$, and $\text{Aut}(\Gamma)$ for the automorphism group of $\Gamma$. We say that $\Gamma$ is $G$-vertex-transitive (respectively $G$-edge-transitive) if $G \leq \text{Aut}(\Gamma)$ and $G$ acts transitively on $\mathcal{V}T$ (respectively $\mathcal{E}T$). If $X \subseteq \mathcal{V}T$, then $[X]$ denotes the subgraph of $\Gamma$ induced by $X$. The distance between $u, v \in \mathcal{V}T$ is denoted by $d_\Gamma(u, v)$, and for $u \in \mathcal{V}T$ and any integer $i \geq 0$, we define $\Gamma_i(u) := \{v \in \mathcal{V}T : d_\Gamma(u, v) = i\}$. In particular, we write $\Gamma(u)$ for the neighbourhood $\Gamma_1(u)$. For $u, v \in \mathcal{V}T$ such that $d_\Gamma(u, v) = i$, let

$$c_i(u, v) := |\Gamma_{i-1}(u) \cap \Gamma(v)|,$$
$$a_i(u, v) := |\Gamma_i(u) \cap \Gamma(v)|.$$

We write $c_i(\Gamma)$ (respectively $a_i(\Gamma)$) whenever $c_i(u, v)$ (respectively $a_i(u, v)$) does not depend on the choice of $u$ and $v$, and we omit the $\Gamma$ when context permits. Note that if $\Gamma$ is bipartite, then $a_i = 0$ for all $i$. We write $\overline{\Gamma}$ for the complement of $\Gamma$. The complete graph on $n$ vertices is denoted by $K_n$, and the complete multipartite graph with $n$ parts of size $m$ is denoted by $K_n[m]$. The distance 2 graph $\Gamma_2$ of $\Gamma$ has vertex set $\mathcal{V}T$, where two vertices are adjacent whenever their distance in $\Gamma$ is 2. If $\Gamma$ is connected but not bipartite, then $\Gamma_2$ is connected, and if $\Gamma$ is connected and bipartite, then $\Gamma_2$ has exactly two connected components; these are called the halved graphs of $\Gamma$. We write $\frac{1}{2}\Gamma$ for a halved graph of $\Gamma$ whenever the halved graphs of $\Gamma$ are isomorphic.

The bipartite double $\Gamma.2$ of $\Gamma$ has vertex set $\mathcal{V}T \times \mathbb{F}_2$, where vertices $(u, x)$ and $(v, y)$ are adjacent whenever $u$ and $v$ are adjacent in $\Gamma$ and $x \neq y$. The graph $\Gamma.2$ is bipartite, and it is connected if and only if $\Gamma$ is connected but not bipartite.

2.2. Covering maps. Let $\Gamma$ and $\Pi$ be graphs. A map $\pi : \Gamma \to \Pi$ is a local bijection if $\pi$ induces a bijection from $\Gamma(x)$ onto $\Pi(x, \pi)$ for all $x \in \mathcal{V}T$. Note that a local bijection is also called a local isomorphism, but we prefer the former term since the induced neighbourhood graphs need not be isomorphic in general. A surjective local bijection is a covering. Whenever a covering $\pi : \Gamma \to \Pi$ exists, we say that $\Pi$ is covered by $\Gamma$.

Here are some basic but important properties of local bijections and coverings.
Lemma 2.1. Let $\Gamma$ and $\Pi$ be graphs. If $\Pi$ is connected, then any local bijection $\pi : \Gamma \to \Pi$ is a covering.

Proof. If $u \in \Pi$ is adjacent to $x\pi$ for some $x \in \Gamma$, then $u \in \Pi(x\pi) = \Gamma(x)\pi$, and so $u = y\pi$ for some $y \in \Gamma(x)$. Since $\Pi$ is connected, $\pi$ is a covering. $\square$

Note that there exist local bijections that are not coverings. For example, there is a local bijection from $K_2 \times 2$ to itself whose image is $K_2$.

Lemma 2.2. Let $\Gamma$ and $\Pi$ be graphs, and let $\pi : \Gamma \to \Pi$ be a covering. Then the following hold.

(i) If $u_1, u_2 \in \Pi$ are adjacent, then for every $x_1 \in u_1\pi^{-1}$, there exists a unique $x_2 \in u_2\pi^{-1}$ such that $x_1$ and $x_2$ are adjacent.

(ii) If $\Pi$ is connected, then $|u\pi^{-1}| = |v\pi^{-1}|$ for all $u, v \in \Pi$.

Proof. (i) Let $x_1 \in u_1\pi^{-1}$. Then $x_2 \in \Pi(u_1\pi)$. Since $\pi$ is a local bijection, there exists a unique $x_2 \in \Pi(x_1\pi)$ such that $u_2 = x_2\pi$ (and hence $x_2 \in u_2\pi^{-1}$).

(ii) Follows from (i). $\square$

Let $\Gamma$ and $\Pi$ be graphs, and let $\pi : \Gamma \to \Pi$ be a covering. If $g \in \text{Aut}(\Gamma)$, then $g : \Gamma \to \Gamma$, so we may compose the functions $g$ and $\pi$ to obtain a new covering $g\pi : \Gamma \to \Pi$. Since $g\pi = \pi$ if and only if $\pi = g^{-1}\pi$, it follows that $\{g \in \text{Aut}(\Gamma) : g\pi = \pi\}$ forms a subgroup of $\text{Aut}(\Gamma)$. We denote this subgroup by $K^\pi$. We will see in §3 that $K^\pi$ is fundamental to the study of many rectagraphs. Here is one useful property of this group.

Lemma 2.3. Let $\Gamma$ and $\Pi$ be graphs where $\Gamma$ is connected, and let $\pi : \Gamma \to \Pi$ be a covering. Then $K^\pi_x = 1$ for all $x \in \Gamma^{\Pi}$.

Proof. Let $x \in \Gamma^\Pi$ and $y \in \Gamma(x)$. Set $u := x\pi$ and $v := y\pi$. Then $v \in \Pi(u)$. Let $g \in K^\pi_x$. Then $y\beta\pi = y(g\pi) = y\pi$, so $y\beta \in v\pi^{-1}$, but $x\beta = x$, so $y$ and $y\beta$ are both neighbours of $x$ in $v\pi^{-1}$. Hence $y = y\beta$ by Lemma 2.2 (i). Since $\Gamma$ is connected, it follows that $g$ fixes $\Gamma^\Pi$ pointwise, and so $g = 1$. $\square$

2.3. Quotient graphs. For a graph $\Gamma$ and a partition $B$ of $\Gamma^\Pi$, the quotient graph $\Gamma_B$ is the graph with vertex set $B$, where $B_1, B_2 \in B$ are adjacent whenever there exists $x_1 \in B_1$ and $x_2 \in B_2$ such that $x_1$ and $x_2$ are adjacent in $\Gamma$. Note that $\Gamma_B$ can contain loops.

Any graph covered by $\Gamma$ is naturally isomorphic to a quotient graph of $\Gamma$, for if $\Pi$ is a graph and $\pi : \Gamma \to \Pi$ is a covering, then $\Pi \simeq \Gamma_B$ where $B$ is the set of preimages of $\pi$ in $\Gamma^\Pi$. On the other hand, given a graph $\Gamma$ and a partition $B$ of $\Gamma^\Pi$, there is a natural surjective map $\pi : \Gamma \to \Gamma_B$ sending a vertex in $\Gamma$ to the part in $B$ containing it, and this map is a covering precisely when the following two conditions hold: (i) no $B \in B$ contains an edge, and (ii) if $B_1, B_2 \in \Gamma^\Pi$ are adjacent, then for every $x_1 \in B_1$, there exists a unique $x_2 \in B_2$ such that $x_1$ and $x_2$ are adjacent (cf. Lemma 2.2).

One useful way of defining partitions for graph quotients is to use the orbits of a normal subgroup of a group of automorphisms, for we retain some control of the automorphism group, the valency and the local action of the quotient.

Let $M \leq \text{Aut}(\Gamma)$ (where $M$ is not necessarily normal in $\text{Aut}(\Gamma)$), and let $B$ be the set of orbits of $M$ on $\Gamma^\Pi$. We say that $\Gamma_B$ is a normal quotient of $\Gamma$ and write $\Gamma_M$ for $\Gamma_B$. If $M \leq G \leq \text{Aut}(\Gamma)$, then $G$ acts naturally on $B$ by $(x^M)^g := (x^g)^M$ for all $x \in \Gamma^\Pi$ and $g \in G$. Moreover, this action preserves adjacency, so $G/K \leq \text{Aut}(\Gamma_M)$, where $K$ is the kernel of the action of $G$ on $B$. Note that $M \leq K$, but $M \neq K$ in general.

Recall from §2.2 that if $\pi : \Gamma \to \Pi$ is a covering, then $K^\pi = \{g \in \text{Aut}(\Gamma) : g\pi = \pi\}$, and let $N^\pi$ denote the normaliser of $K^\pi$ in $\text{Aut}(\Gamma)$. In the following, we begin to see the importance of the group $K^\pi$.

Lemma 2.4. Let $\Gamma$ be a connected graph, and let $M \leq \text{Aut}(\Gamma)$. If the natural map $\pi : \Gamma \to \Gamma_M$ is a covering, then the following hold.
(i) $K^\pi = M$, and $K^\pi$ is the kernel of the action of $N^\pi$ on $VT_M$.

(ii) $N^\pi/K^\pi \leq \text{Aut}(\Gamma_M)$.

(iii) $N^\pi_x \simeq (N^\pi/K^\pi)_x$ for all $x \in VT$.

Proof. Clearly $M \leq K^\pi$. If $g \in K^\pi$ and $x \in VT$, then $(x^g)^M = x(g\pi) = x\pi = x^M$, so $x^g = x^m$ for some $m \in M$. Then $gm^{-1} \in K^\pi_x$, but $K^\pi_x = 1$ by Lemma 2.3, so $g \in M$. Thus $M = K^\pi$. In particular, $N^\pi$ acts on $VT_M$, and $K^\pi$ is contained in the kernel of this action. If $g \in N^\pi$ and $(x^g)^M = x^m$ for all $x \in VT$, then $x(g\pi) = x\pi = x\pi$ for all $x \in VT$, so $g \in K^\pi$. Thus $K^\pi$ is the kernel of the action of $N^\pi$ on $VT_M$, proving (i) and (ii).

Let $x \in VT$. There is a group homomorphism $\psi: N^\pi_x \rightarrow (N^\pi/K^\pi)_x$ defined by $g \mapsto K^\pi g$ for all $g \in N^\pi_x$, and it is injective since $N^\pi_x \cap K^\pi = 1$ by Lemma 2.3. To see that $\psi$ is surjective, let $K^\pi g \in (N^\pi/K^\pi)_x$. Then $x^g$ and $x$ are in the same $K^\pi$-orbit, so there exists $k \in K^\pi$ such that $x^gk = x$. Thus $gk \in N^\pi_x$ and $(gk)\psi = K^\pi gk = K^\pi (gkg^{-1})g = K^\pi g$, as desired. $\square$

2.4. Coset graphs of binary linear codes. The $n$-cube $Q_n$ is defined to be the graph with vertex set $\mathbb{F}_2^n$, where two $n$-tuples are adjacent whenever their Hamming distance is 1. The $n$-cube is a connected regular bipartite graph of valency $n$ with parts the sets of vectors of even and odd weight. Its automorphism group is $\mathbb{F}_2^n \rtimes S_n$, where $\mathbb{F}_2^n$ acts on $VQ_n = \mathbb{F}_2^n$ by translation, and $S_n$ acts by permuting coordinates.

In this paper, we will be interested in normal quotients of $Q_n$, particularly those that are formed using additive subgroups of $\mathbb{F}_2^n$. If $C$ is such a group, we define the coset graph of $C$, denoted by $\Gamma(C)$, to be the normal quotient $(Q_n)/C$. This graph is so named because the orbits of $C$ on $\mathbb{F}_2^n$ are the cosets of $C$ in $\mathbb{F}_2^n$. Note that $\Gamma(C)$ is loop-free if and only if $e_i \notin C$ for all $i \leq n$.

The additive subgroups of $\mathbb{F}_2^n$ are precisely the binary linear codes, for which we now give some basic terminology. A binary linear code $C$ is defined to be a subspace of $\mathbb{F}_2^n$. The vectors in $C$ are called codewords. The minimum distance of $C$ is defined to be $\infty$ when $C = \{0\}$, and the minimum Hamming distance between distinct codewords in $C$ otherwise. Note that the minimum distance of a non-zero code is precisely the minimum weight of the non-zero codewords.

We say that $C$ is an $[n,r,d]$-code, where $r$ is the dimension of $C$ and $d$ is the minimum distance of $C$. If every codeword in $C$ has even weight, then $C$ is an even code. The automorphism group of $C$, denoted by $\text{Aut}(C)$, is defined to be the stabiliser of $C$ in $S_n$, where $S_n$ acts on $\mathbb{F}_2^n$ by permuting coordinates.

The following describes some elementary but important properties of binary linear codes and the coset graphs formed from them.

**Lemma 2.5.** Let $C$ be a binary linear $[n,r,d]$-code where $d \geq 2$. Let $\Pi := \Gamma(C)$. Then the following hold.

(i) The graph $\Pi$ is bipartite if and only if $C$ is an even code.

(ii) If $\Pi$ is bipartite, then the halved graphs of $\Pi$ are isomorphic.

(iii) If $C$ is not an even code, then $\Pi_2 \simeq \Gamma(C \cap E_n)$.

(iv) If $d \geq 3$, then the natural map $\pi: Q_n \rightarrow \Pi$ is a covering.

(v) If $d \geq 5$, then $a_1(\Pi) = 0$ and $a_2(\Pi) = 2$.

(vi) If $d \geq 7$, then $a_2(\Pi) = 0$ and $a_3(\Pi) = 3$.

Proof. (i) If $C$ is even, then the weights of vectors in a coset of $C$ have the same parity. Edges in $Q_n$ only occur between vectors whose weights have different parity, so $\Pi$ is bipartite.

If $C$ is not even, then we may choose $c \in C$ of minimal odd weight. There exists a path $(x_0,\ldots,x_m)$ in $Q_n$ where $x_0 = 0$, $x_m = c$, and $m = |c|$. The minimality of $c$ and $m \geq d \geq 2$ then imply that $(x_0+c,\ldots,x_m+C)$ is an odd cycle, so $\Pi$ is not bipartite.

(ii) Let $\Gamma$ and $\Sigma$ be the halved graphs of $\Pi$ (if they exist). By the proof of (i), we may define a map $\varphi: \Gamma \rightarrow \Sigma$ by $x+C \mapsto x+e_1+C$ for all $x+C \in VT$. It is routine to verify that $\varphi$ is a graph isomorphism.
(iii) Define a map \( \varphi : \Gamma(C \cap E_n) \to \Pi.2 \) by \( x + C \cap E_n \mapsto (x + C, |x| \mod 2) \) for all \( x \in VQ_n \). It is routine to verify that \( \varphi \) is a well-defined graph isomorphism.

(iv) This follows from the fact that \( \Pi(x + C) = \{ x + e_i + C : i \in [n] \} \) for all \( x \in VQ_n \).

(v) This follows from the structure of \( \Pi(x + C) \) observed in (iv) together with \( \Pi_2(x + C) = \{ x + e_{i,j} + C : \{i, j\} \in \binom{n}{2} \} \) for all \( x \in VQ_n \).

(vi) This follows from the observations in (iv) and (v) together with \( \Pi_3(x + C) = \{ x + e_{i,j,k} + C : \{i, j, k\} \in \binom{n}{3} \} \) for all \( x \in VQ_n \).

Thus whenever \( \Gamma(C) \) is bipartite, its halved graphs are isomorphic, and so we may refer to the halved graph of \( \Gamma(C) \). In particular, \( C \subseteq E_n \), and we may assume that the halved graph of \( \Gamma(C) \) has vertex set \( E_n/C \). Note that if \( C \) is not an even code, then there is a natural graph isomorphism between \( \frac{1}{2} \Gamma(C \cap E_n) \) and the distance 2 graph \( \Gamma(C) \).

2.5. Graphs of Theorems 1.1 and 1.3. The graphs of Theorem 1.3 are all coset graphs of binary linear codes, and the graphs of Theorem 1.1 are all halved graphs of coset graphs of even binary linear codes. There are five relevant binary linear codes for these theorems, two of which are elementary.

The most basic code is the zero code \( \{0\} \), whose coset graph is simply \( Q_n \). It is an even binary linear \([n, 0, \infty]-\)code with automorphism group \( S_n \). The \( n \)-cube is distance-transitive with automorphism group \( 2^n \rtimes S_n \) and vertex stabiliser \( S_n \). The automorphism group of the halved \( n \)-cube is \( 2^{n-1} \rtimes S_n \) for \( n \geq 5 \), \( 2^4 \rtimes S_4 \) for \( n = 4 \), and \( S_4 \) for \( n = 3 \) [5, p. 265]. It has vertex stabiliser \( S_n \) when \( n \neq 4 \), and vertex stabiliser \( S_4 \times C_2 \) when \( n = 4 \). The halved \( n \)-cube is also distance-transitive, as is any halved graph of a bipartite distance-transitive graph [5, Theorem 4.1.10].

The other basic code is the repetition code \( \{0, 1\} \), whose coset graph \( \square_n \) is called the folded \( n \)-cube. It is a binary linear \([n, 1, n]-\)code with automorphism group \( S_n \), and it is an even code if and only if \( n \) is even. The folded \( n \)-cube is distance-transitive with automorphism group \( 2^{n-1} \rtimes S_n \) and vertex stabiliser \( S_n \) for \( n \geq 5 \). When \( n \) is even and \( n \geq 8 \), the halved folded \( n \)-cube is distance-transitive with automorphism group \( 2^{n-2} \rtimes S_{n} \) and vertex stabiliser \( S_n \) [5, p. 265].

The remaining codes are Golay codes. The extended binary Golay code \( C_{24} \) is a binary linear [24, 12, 8]-code. There are various methods for constructing this code; see [33], for example. Its automorphism group is the Mathieu group \( M_{24} \), and it is an even code. The graph \( \Gamma(C_{24}) \) is distance-transitive with automorphism group \( 2^{12} \rtimes M_{24} \) and vertex stabiliser \( M_{24} \) [5, Theorem 11.3.2]. The halved graph of \( \Gamma(C_{24}) \) is distance-transitive with automorphism group \( 2^{11} \rtimes M_{24} \) and vertex stabiliser \( M_{24} \) (cf. Lemma 5.5).

If we remove one fixed coordinate from every codeword in the extended binary Golay code, then we obtain the binary Golay code \( C_{23} \), which is a binary linear [23, 12, 7]-code. Its automorphism group is the Mathieu group \( M_{23} \), the stabiliser in \( M_{24} \) of a point, and it is not an even code. The graph \( \Gamma(C_{23}) \) is distance-transitive with automorphism group \( 2^{11} \rtimes M_{23} \) and vertex stabiliser \( M_{23} \) [5, Theorem 11.3.4].

Lastly, the set of vectors in \( C_{23} \) with even weight forms a binary linear [23, 11, 8]-code. Its automorphism group is \( M_{23} \) since automorphisms of codes preserve weight, and its coset graph is isomorphic to the bipartite double \( \Gamma(C_{23}) \) by Lemma 2.5 (iii). The graph \( \Gamma(C_{23}) \) is distance-transitive with automorphism group \( 2^{12} \rtimes M_{23} \) and vertex stabiliser \( M_{23} \) [5, p. 362]. The halved graph of \( \Gamma(C_{23}) \) is distance-transitive with automorphism group \( 2^{11} \rtimes M_{23} \) and vertex stabiliser \( M_{23} \) (cf. Lemma 5.5).

Note that if \( C \) is one of the linear codes in \( F_2^n \) defined above, or any binary linear code with \( d \geq 5 \), then \( \text{Aut}(\Gamma(C)) \) can be described uniformly as \( (F_2^n/C) \rtimes \text{Aut}(C) \) (cf. Lemma 3.9), where the vertex set \( F_2^n/C \) acts by translation and \( \text{Aut}(C) \) acts by permuting coordinates. Moreover, when \( C \) is an even code and \( d \geq 5 \), the automorphism group of the halved graph of \( \Gamma(C) \) always
contains \((E_n/C) \rtimes \text{Aut}(C)\) (cf. Lemma 3.10), and is in fact equal to this group for \(d \geq 7\) except when \(C\) is the zero code and \(n = 4\) (cf. Lemma 5.5).

2.6. Graphs of Corollary 1.2. The graphs of Corollary 1.2 that are not described in Theorem 1.1 are precisely the locally \(T_n\) graphs for \(n \geq 5\), classified by Hall and Shult [13, Theorem 2]. The graph \(T_{n+2}\) is itself locally \(T_n\) for \(n \geq 5\). This graph is strongly regular and distance-transitive with automorphism group \(S_n \times C_2\) and vertex stabiliser \(S_n \times C_2\), where \(C_2\) fixes the neighbourhood of the vertex pointwise.

The remaining locally \(T_n\) graphs only occur when \(n = 5\) or \(6\). When \(n = 5\), the graph \(T_5\) is isomorphic to the Petersen graph. Graphs that are locally Petersen were classified by Hall [12]. Besides \(T_5\), there are two such graphs, both of which are commuting involutions graphs. Given a group \(G\) and a conjugacy class \(\mathcal{C}\) of involutions in \(G\), the commuting involutions graph of \(\mathcal{C}\) in \(G\) has vertex set \(\mathcal{C}\), where two (distinct) involutions are adjacent whenever they commute. For example, \(T_5\) is the commuting involutions graph of the class of transpositions in \(S_n\).

One of the exceptional locally Petersen graphs is the Conway-Smith graph, which is the commuting graph of transposition preimages in the group 3. This graph has 32 vertices and diameter 3. Using GAP [1, 10, 30], we determined that it is distance-transitive and has automorphism group \(P\Omega^+(2,5)\times 2\) and vertex stabiliser \(S_6 \times C_2\), where \(C_2\) fixes the neighbourhood of the vertex pointwise. See [5, Theorem 12.2.2] for more details.

The other exceptional locally Petersen graph is the commuting involutions graph of the class of involutory Galois field automorphism in the group \(P\Sigma L_2(25)\). It is distance-transitive and has automorphism group \(P\Sigma L_2(25)\) and vertex stabiliser \(S_6 \times C_2\), where \(C_2\) fixes the neighbourhood of the vertex pointwise. See [5, Theorem 13.2.3] for more details.

When \(n = 6\), there are again two locally \(T_6\) graphs besides \(T_8\). Let \(Sp_{2n}(2)\) denote the graph whose vertices are the non-zero vectors of a \(2n\)-dimensional \(F_2\)-vector space, with two vectors adjacent whenever they are perpendicular with respect to a given non-degenerate symplectic form. The locally \(Sp_{2n}(2)\) graphs were classified in [13, Theorem 5], and since \(T_6 \simeq Sp_4(2)\), this classification determines the locally \(T_6\) graphs. These graphs all arise as subgraphs of \(Sp_6(2)\), including \(T_8\), which is isomorphic to the complement of a hyperbolic quadric in \(Sp_6(2)\).

One of the exceptional locally \(T_6\) graphs is the complement of an elliptic quadric in \(Sp_6(2)\). This graph has 36 vertices and diameter 2. Using GAP [1, 10, 30], we determined that it is distance-transitive with the following distance distribution diagram.

\[
\begin{array}{cccccccc}
1 & 15 & 1 & 1 & 8 & 6 & 20 & 9 \\
5 & 6 & 8 & 6 & 1 & 15 & 15 & 1
\end{array}
\]

We also determined that it has automorphism group \(P\Omega^+(2,6)\times 2\) and vertex stabiliser \(S_6 \times C_2\), where \(C_2\) fixes the neighbourhood of the vertex pointwise.

The other exceptional locally \(T_6\) graph is the complement of a hyperplane in \(Sp_6(2)\). This graph has 32 vertices and diameter 3. Using GAP [1, 10, 30], we determined that it is distance-transitive with the following distance distribution diagram.

\[
\begin{array}{cccccccc}
1 & 15 & 1 & 1 & 8 & 8 & 15 & 1 \\
5 & 6 & 6 & 1 & 15 & 1 & 15 & 1
\end{array}
\]

We also determined that it has automorphism group \(2^5 \times S_6\) and vertex stabiliser \(S_6\). Interestingly, this graph is isomorphic to a connected component in the distance 4 graph of \(Q_6\) (where \(x, y \in VQ_6\) are adjacent whenever \(dQ_6(x, y) = 4\)). It is also isomorphic to the subgraph of the complement of \(\frac{1}{2}Q_6\) where the edge between \(x\) and \(x + 1\) is removed for all \(x \in V(\frac{1}{2}Q_6)\).

3. Rectagraphs

Recall that a rectagraph is a connected triangle-free graph in which every 2-arc determines a unique quadrangle. Equivalently, a rectagraph is a connected graph with \(a_1 = 0\) and \(a_2 = 2\). By [5, Proposition 1.1.2], every rectagraph is regular.
The most basic example of a rectagraph is also the most important one: the $n$-cube $Q_n$. The $n$-cubes are extremal rectagraphs, for a rectagraph $\Pi$ of valency $n$ has at most $2^n$ vertices with equality if and only if $\Pi$ is the $n$-cube [5, Proposition 1.13.1].

Other examples of rectagraphs include coset graphs of binary linear codes with minimum distance at least five, all of which are covered by the $n$-cube (cf. Lemma 2.5 (iv)-(v)). We begin with a fundamental generalisation of this result for a certain class of rectagraphs. Note that this result can be proved under more general assumptions than $a_2 = 0$ and $c_3 = 3$, but these suffice for our purposes.

**Lemma 3.1** ([5]). Let $\Pi$ be a rectagraph of valency $n$ where $a_2 = 0$ and $c_3 = 3$. For any $u \in V\Pi$ with neighbours $u_1, \ldots, u_n$, there exists a covering $\pi : Q_n \rightarrow \Pi$ such that $0\pi = u$ and $e_i\pi = u_i$ for all $i \leq n$.

**Proof.** By [5, Lemma 4.3.5] and the proof of [5, Proposition 4.3.6], there exists a map $\pi : Q_n \rightarrow \Pi$ for which $0\pi = u$ and $e_i\pi = u_i$ for all $i \leq n$, and also if $d_{Q_n}(x,y) \leq 2$ for $x, y \in VQ_n$, then $d_{Q_n}(x,y) = d_{\Pi}(x\pi, y\pi)$. Since $Q_n$ and $\Pi$ have valency $n$, it follows that $\pi$ is a local bijection, and so $\pi$ is a covering by Lemma 2.1. \hfill $\square$

In fact, whenever a covering of a rectagraph by an $n$-cube exists, it is essentially unique.

**Lemma 3.2.** Let $\Pi$ be a rectagraph. Let $\pi : Q_n \rightarrow \Pi$ and $\theta : Q_n \rightarrow \Pi$ be coverings. If $0\pi = 0\theta$ and $e_i\pi = e_i\theta$ for all $i \leq n$, then $\pi = \theta$.

**Proof.** We prove that $x\pi = x\theta$ for all $x \in VQ_n$ by induction on $|x|$. If $|x| \leq 1$, then $x\pi = x\theta$ by assumption, so we may assume that $|x| \geq 2$. Let $i$ and $j$ be non-zero coordinates of $x$. By induction, $(x + e_i)\pi = (x + e_i)\theta$, $(x + e_j)\pi = (x + e_j)\theta$ and $(x + e_i + e_j)\pi = (x + e_i + e_j)\theta$. But any covering maps quadrangles to quadrangles, so $x\pi$ and $x\theta$ are both vertices of $\Pi$ that are adjacent to $(x + e_i)\pi$ and $(x + e_j)\pi$ but distinct from $(x + e_i + e_j)\pi$. Thus $x\pi = x\theta$. \hfill $\square$

The next two results are slight generalisations of some unpublished work by Matsumoto [24], who observed that a certain type of rectagraph covered by $Q_n$ must also be a normal quotient of $Q_n$ [24, Proposition 2] and then determined the automorphism group of this normal quotient [24, Lemma 5].

First we recall some notation. Let $\Gamma$ and $\Pi$ be graphs, and let $\pi : \Gamma \rightarrow \Pi$ be a covering. Recall from §2.2 that $K^\pi = \{g \in \text{Aut}(\Gamma) : g|_\pi = \pi\} \leq \text{Aut}(\Gamma)$, and recall from §2.3 that $N^\pi$ denotes the normaliser of $K^\pi$ in $\text{Aut}(\Gamma)$. The groups $K^\pi$ and $N^\pi$ play a vital role in the study of rectagraphs, as we now see. Note that we write elements of $\text{Aut}(Q_n) = F_2 \wr S_n$ in the form $(x, \sigma)$ where $x \in F_2^n$ and $\sigma \in S_n$.

**Lemma 3.3.** Let $\Pi$ be a rectagraph and $\pi : Q_n \rightarrow \Pi$ a covering. Then $v\pi^{-1}$ is a regular $K^\pi$-orbit for all $v \in V\Pi$.

**Proof.** Let $u := 0\pi$. For $x \in v\pi^{-1}$ and $g \in K^\pi$, we have $v = x\pi = x(g\pi) = x\theta\pi$, so $v\pi^{-1}$ is a $K^\pi$-invariant set. Since $|v\pi^{-1}| = |u\pi^{-1}|$ by Lemma 2.2 (ii) and $K^\pi = 1$ for all $x \in VQ_n$ by Lemma 2.3, it suffices to prove that $u\pi^{-1}$ is a $K^\pi$-orbit. If $\{0\} = u\pi^{-1}$, then $K^\pi = K_0^\pi = 1$, so $u\pi^{-1}$ is a $K^\pi$-orbit, as desired. Otherwise, let $0 \neq y \in u\pi^{-1}$. Then $\{e_i\pi : 1 \leq i \leq n\} = \Pi(0\pi) = \Pi(y\pi) = \{(y + e_i)\pi : 1 \leq i \leq n\}$, so there exists $\sigma \in S_n$ for which $e_i\pi = (y + e_i)\sigma\pi$ for all $i \leq n$. Let $g := (y\sigma^{-1}, \sigma) \in F_2^n \times S_n = \text{Aut}(Q_n)$. Then $0\pi y = y$ and $e_i \pi = y + e_i_e \sigma\pi$ for all $i \leq n$. This implies that $y\pi$ and $\pi$ are coverings that agree on $\{0\} \cup Q_n(0)$, so $y\pi = \pi$ by Lemma 3.2. Hence $g \in K^\pi$. Since $0\pi = y$, it follows that $u\pi^{-1}$ is a $K^\pi$-orbit. \hfill $\square$

Thus any rectagraph covered by an $n$-cube is a normal quotient of $Q_n$. We record this fact in Proposition 3.4 below. Moreover, using Lemma 3.3, we can describe the automorphism group of a rectagraph covered by an $n$-cube in terms of groups related to the covering.

**Proposition 3.4.** Let $\Pi$ be a rectagraph and $\pi : Q_n \rightarrow \Pi$ a covering. Then $\Pi \simeq (Q_n)_{K^\pi}$ and $\text{Aut}(\Pi) \simeq N^\pi/K^\pi$. 
Lemma 3.5. Let $\Pi$ be a rectagraph and $\pi : Q_n \to \Pi$ a covering. Let $0 \pi := u$. Then the actions of $\text{Aut}(\Pi)_u$ on $\Pi(u)$ and $N^\pi_0$ on $\binom{\sigma}{0}$ are permutation isomorphic for $i = 1, 2$.

Proof. As in the proof of Proposition 3.4, there is no loss of generality in assuming that $\Pi = (Q_n)_{K^\pi}$ and $\pi$ is the natural map. By Lemma 2.4 and Proposition 3.4, there is a group isomorphism $\psi : N^\pi_0 \to \text{Aut}(\Pi)_u$ defined by $(x\pi)^{\sigma'} := (x\pi)^{\pi'}$ for all $x \in VQ_n$ and $\sigma \in N^\pi_0$. Since $\pi$ is a covering, there is a bijection $\varphi_1 : [n] \to \Pi(u)$ defined by $i \mapsto e_i\pi$ for all $i \in [n]$, and since $\Pi$ is a rectagraph of valency $n$, there is a bijection $\varphi_2 : \binom{\sigma}{0} \to \Pi_2(u)$ defined by $\{i, j\} \mapsto e_{i,j}\pi$ for all $\{i, j\} \in \binom{\sigma}{0}$. It is routine to verify that $\psi$ is a permutation isomorphism with respect to each of these bijections. □

Note that since $N^\pi_0 \leq S_n$, the vector space $\mathbb{F}_2^3$ is naturally an $\mathbb{F}_2N^\pi_0$-module; indeed, it is the permutation module of $N^\pi_0$ over $\mathbb{F}_2$. Then the set $E_n$ of vectors in $\mathbb{F}_2^3$ with even weight is an $\mathbb{F}_2N^\pi_0$-submodule of $\mathbb{F}_2^3$. Moreover, we have the following important observation.

Lemma 3.6. Let $\Pi$ be a rectagraph and $\pi : Q_n \to \Pi$ a covering. Then $K^\pi \cap \mathbb{F}_2^3$ is an $\mathbb{F}_2N^\pi_0$-submodule of the permutation module $\mathbb{F}_2^3$ that does not contain $E_n$.

Proof. If $x \in K^\pi \cap \mathbb{F}_2^3$ and $\sigma \in N^\pi_0$, then $(x\pi, 1) = (0, \sigma)^{-1}(x, 1)(0, \sigma) \in K^\pi$ since $N^\pi_0$ normalises $K^\pi$, and so $x\pi \in K^\pi \cap \mathbb{F}_2^3$. Thus $K^\pi \cap \mathbb{F}_2^3$ is an $F_2N^\pi_0$-submodule of $\mathbb{F}_2^3$. If $K^\pi \cap \mathbb{F}_2^3 \geq E_n$, then $e_1, e_2 \in K^\pi \cap \mathbb{F}_2^3$, and so $0 \pi = 0^{e_1, e_2} \pi = e_1, e_2 \pi$, but this is impossible since $\pi$ is a covering and the vertices 0 and $e_1, e_2$ are both neighbours of $e_1$ in $Q_n$. □

Here is one useful application of Lemma 3.6, for which we require the following definition. Let $G \leq S_n$. Note that $E_n$ and $\{0, 1\}$ are both $\mathbb{F}_2G$-submodules of the permutation module $\mathbb{F}_2^3$. Then we may define the heart of $G$ over $\mathbb{F}_2$ to be the $\mathbb{F}_2G$-module $E_n/(E_n \cap \{0, 1\})$.

Lemma 3.7. Let $\Pi$ be a rectagraph and $\pi : Q_n \to \Pi$ a covering. Suppose that $K^\pi \leq \mathbb{F}_2^3$ and $N^\pi_0$ is transitive on $[n]$. If the heart of $N^\pi_0$ over $\mathbb{F}_2$ is irreducible, then $\Pi \simeq Q_n$ or $\square_n$.

Proof. Lemma 3.6 implies that $K^\pi$ is an $\mathbb{F}_2N^\pi_0$-submodule of $\mathbb{F}_2^3$ that does not contain $E_n$. Since $N^\pi_0$ is transitive on $[n]$ and the heart of $N^\pi_0$ over $\mathbb{F}_2$ is irreducible, the only $\mathbb{F}_2N^\pi_0$-submodules of $\mathbb{F}_2^3$ are $\{0\}$, $\{0, 1\}$, $E_n$ and $\mathbb{F}_2^3$ by [25, Lemma 2]. Thus $K^\pi = \{0\}$ or $\{0, 1\}$. Since $\Pi \simeq (Q_n)_{K^\pi}$ by Proposition 3.4, it follows that $\Pi \simeq Q_n$ or $\square_n$. □

Here are some sufficient conditions for $K^\pi$ to be contained in $\mathbb{F}_2^3$.

Lemma 3.8. Let $\Pi$ be a rectagraph and $\pi : Q_n \to \Pi$ a covering. Then the following hold.

(i) If $N^\pi_0$ is $S_n$ or $A_n$ where $n \geq 5$, then $K^\pi \leq \mathbb{F}_2^3$. 

(ii) If $N_0^\pi$ is 2-transitive on $[n]$ and $n$ is not a power of 2, then $K^\pi \leq F_2^n$.

**Proof.** Let $\phi : F_2^n \times S_n \to S_n$ be the natural projection map. Note that $K^\pi \rho$ is a 2-group, for $K^\pi \rho$ is a quotient of $K^\pi$, and the preimages of $\pi$ partition $V\{Q_n = F_2^n$ and have size $|K^\pi|$ by Lemma 3.3.

Since $N_0^\pi$ normalises $K^\pi$ in $Aut(Q_n)$, it follows that $N_0^\pi$ normalises $K^\pi$ in $S_n$. In particular, if $N_0^\pi$ is $S_n$ or $A_n$ where $n \geq 5$, then since $K^\pi \rho$ is a 2-group, we must have $K^\pi \leq F_2^n$, proving (i).

Now suppose that $N_0^\pi$ is 2-transitive on $[n]$ and that there exists $\sigma \in K^\pi \rho$ where $i^\sigma = j$ and $i \neq j$. If $k \in [n] \setminus \{i\}$, then there exists $\tau \in N_0^\pi$ with $i^\tau = i$ and $j^\tau = k$, so $\tau^{-1}\sigma\tau$ maps $i$ to $k$ and lies in $K^\pi \rho$. Thus $K^\pi \rho$ is a transitive subgroup of $S_n$. In particular, $n$ divides $|K^\pi \rho|$, which is a power of 2, proving (ii).

Next we see that Proposition 3.4 provides a uniform description of the automorphism groups of many coset graphs of binary linear codes, including the graphs of Theorem 1.3.

**Lemma 3.9.** Let $C$ be a binary linear $[n, r, d]$-code where $d \geq 5$. Then $Aut(\Gamma(C)) = (F_2^n / C) \times Aut(C)$.

**Proof.** By Lemma 2.5, the natural map $\pi : Q_n \to \Gamma(C)$ is a covering and $\Gamma(C)$ is a rectagraph. Then $K^\pi = C$ by Lemma 2.4, so $N^\pi = F_2^n / C \times Aut(C)$. Thus $Aut(\Gamma(C)) \simeq N^\pi / K^\pi \simeq (F_2^n / C) \times Aut(C)$ by Proposition 3.4. □

We are now in a position to prove Theorem 1.3.

**Proof of Theorem 1.3.** Suppose that $\Pi$ is a rectagraph with $a_2 = 0$ and $c_3 = 3$, and suppose that there exists $u \in VII$ such that $|\Pi(u)| \geq 4$ and $Aut(\Pi)_u$ acts 4-homogeneously on $\Pi(u)$. By [5, Proposition 1.1.2], $\Pi$ is regular of valency $n$ for some $n \geq 4$, and by Lemma 3.1, there exists a covering $\pi : Q_n \to \Pi$ such that $0\pi = u$. Then $\Pi \simeq (Q_n)K^\pi$ by Proposition 3.4 and $N^\pi_0$ is a 4-homogeneous subgroup of $S_n$ by Lemma 3.5.

Since $\Pi$ is a rectagraph with $a_2 = 0$ and $c_3 = 3$, we obtain $|\Pi_2(u)| = \binom{n}{2}$ and $|\Pi_3(u)| = \binom{n}{3}$ by a counting argument. Thus $1 + n + \binom{n}{2} + \binom{n}{3} \leq |VII|$. But $|VII|$ divides $2^n$ by Lemma 2.2 (ii), so if $n \leq 6$, then $|VII| = 2^n$, in which case $K^\pi = \{0\}$ and $\Pi \simeq Q_n$.

Similarly, if $n = 7$, then $|VII| = 2^6$ or $2^7$, and $\Pi \simeq Q_7$ in the latter case, so we may assume that $|VII| = 2^6$. Then $|K^\pi| = 2$. Since $N^\pi_0$ is 4-homogeneous, it is also 3-homogeneous, and so it is 2-transitive by [22, Theorem 2]. Then $K^\pi \leq F_2^n$ by Lemma 3.8 (ii), so $K^\pi = \{0, x\}$ for some $0 \neq x \in F_2^n$. Then $x^2 = x$ for all $\sigma \in N_0^\pi$ by Lemma 3.6. Since $N_0^\pi$ is transitive, it follows that $x = 1$, and so $\Pi \simeq \square_7$.

Thus we may assume that $n \geq 8$. Then $N_0^\pi$ is 3-transitive by [22, Theorem 2]. First suppose that $N_0^\pi$ is not 4-transitive. Then $(N_0^\pi, n)$ is one of $(PSL_2(8), 9)$, $(PG_2(8), 9)$ or $(PG_2(32), 33)$ by [20], in which case $K^\pi \leq F_2^n$ by Lemma 3.8 (ii). Since the heart of $N_0^\pi$ over $F_2$ is irreducible by [25], it follows from Lemma 3.7 that $\Pi \simeq Q_n$ or $\square_n$.

Hence we may assume that $n \geq 8$ and $N_0^\pi$ is 4-transitive. By the classification of the finite simple groups, it follows that $(N_0^\pi, n)$ is one of $(S_n, n)$ or $(A_n, n)$ for $n \geq 8$, or $(M_n, n)$ for $n = 11, 12, 23$ or 24 (cf. [8, Theorem 4.11]). Again, we have $K^\pi \leq F_2^n$ by Lemma 3.8.

If $\Pi \simeq Q_n$ or $\square_n$, then we are done, so we may assume that $\Pi \not\simeq Q_n, 0, 1$. By Lemma 3.7, the heart of $N_0^\pi$ over $F_2$ is therefore reducible, so $N_0^\pi$ is $M_{23}$ or $M_{24}$ by [25]. Note that $K^\pi$ is an $F_2$-module $N_0^\pi$-submodule of $F_2^n$ distinct from $E_9$ or $F_3^n$ by Lemma 3.6. Using MAGMA [2], it can be checked that if $N_0^\pi = M_{23}$, then $K^\pi$ is either the binary Golay code $C_{23}$, in which case $\Pi \simeq \Gamma(C_{23})$, or $K^\pi$ is $C_{23} \cap E_{23}$, in which case $\Pi \simeq \Gamma(C_{23}, 2)$ by Lemma 2.5 (iii). Similarly, if $N_0^\pi = M_{24}$, then $K^\pi$ is the extended binary Golay code $C_{24}$, in which case $\Pi \simeq \Gamma(C_{24})$.

Conversely, suppose that $\Pi$ is one of the graphs described in (i)-(v) of the statement of the theorem. Then $\Pi = \Gamma(C)$ where $C$ is a binary linear $[n, r, d]$-code with $d \geq 7$ (this holds for (iii) by Lemma 2.5 (iii)), so $\Pi$ is a rectagraph with $a_2 = 0$ and $c_3 = 3$ by Lemma 2.5. Moreover, by
Lemma 3.9. \(\text{Aut}(\Pi)_{0+C}\) is \(S_n\) in cases (i)-(ii), \(M_{23}\) in cases (iii)-(iv), and \(M_{24}\) in case (v), and so \(\text{Aut}(\Pi)_{0+C}\) is 4-homogeneous on \(\Pi(0+C)\).

Though Corollary 1.4 follows immediately from Theorem 1.3 since the \(n\)-cube is the only rectagraph of valency \(n\) for \(n \leq 3\), we provide a direct proof of this result that does not use the classification of the finite simple groups.

**Proof of Corollary 1.4.** Let \(\Pi\) be a rectagraph with \(a_2 = 0\) and \(c_3 = 3\), where for some \(u \in V\Gamma\), the action of \(\text{Aut}(\Pi)_{\gamma}\) on \(\Pi(u)\) is permutation isomorphic to the natural action of \(S_n\) or \(A_n\) on \([n]\). By [5, Proposition 1.1.2], \(\Pi\) is regular of valency \(n\), and by Lemma 3.1, there is a covering \(\pi : Q_n \to \Pi\) such that \(0\pi = u\). Then \(\Pi \cong (Q_n)K^n\) by Proposition 3.4 and \(N_3^n\) is \(S_n\) or \(A_n\) by Lemma 3.5. For \(n \leq 3\), we have \(\Pi \cong Q_n\), and for \(n = 4\), we have \(\Pi \cong Q_4\) since \(a_2 = 0\) and \(c_3 = 3\), so we may assume that \(n \geq 5\). Then \(K^n \leq Q_4^n\) by Lemma 3.8 (i), and it is routine to verify that the heart of \(N_3^n\) over \(\mathbb{F}_2\) is irreducible. Thus \(\Pi \cong Q_n\) or \(\sqcap_n\) by Lemma 3.7, and \(n \geq 7\) when \(\Pi \cong \sqcap_n\) since \(a_2 = 0\) and \(c_3 = 3\). The converse is straightforward.

Next we see that Corollary 1.5 is a natural consequence of Theorem 1.3.

**Proof of Corollary 1.5.** Note that \(\Pi\) is a rectagraph. Suppose that there exists \(u \in \Pi\) such that \(|\Pi(u)| \geq 5\) and \(\text{Aut}(\Pi)_{\gamma}\) is 5-transitive on \(\Pi(u)\). Any 5-transitive group is 4-homogeneous, so \(\Pi\) is one of the bipartite graphs listed in Theorem 1.3. Since \(M_{23}\) is not 5-transitive on 23 points, it follows that \(\Pi\) is \(Q_n\) where \(n \geq 5\), or \(\sqcap_n\) where \(n\) is even (by Lemma 2.5) and \(n \geq 8\), or \(\Gamma(C_{42})\). The converse is straightforward.

To finish this section, we see that the automorphism group of a rectagraph \(\Pi\) is closely related to that of a connected component in the distance 2 graph \(\Pi_2\) of \(\Pi\).

**Lemma 3.10.** Let \(\Pi\) be a rectagraph of valency \(n \geq 3\), and let \(\Gamma\) be a connected component of \(\Pi_2\). Then the map \(\varphi : \text{Aut}(\Pi)_{\gamma} \to \text{Aut}(\Gamma)\) defined by \(g \mapsto g|_{V\Gamma}\) for all \(g \in \text{Aut}(\Pi)_{\gamma}\) is an injective group homomorphism.

**Proof.** The map \(\varphi\) is well-defined since automorphisms preserve distance, so \(\varphi\) is a group homomorphism. If \(\varphi\) is not bipartite, then \(V\Gamma = V\Pi\), in which case \(\varphi\) is injective, so we may assume that \(\Pi\) is bipartite. Let \(X := V\Gamma\) and \(Y := V\Pi \setminus X\), and suppose that \(g \in \text{Aut}(\Pi)\) fixes \(X\) pointwise. Let \(y_1 \in Y\). Choose \(y_2 \in \Pi_2(y_1)\), and let \(x_1\) and \(x_2\) be the two vertices of \(X\) lying in the quadrangle determined by \(y_1\) and \(y_2\). Then \(g\) either fixes or interchanges \(y_1\) and \(y_2\). Let \(y_3 \in Y\) be adjacent to \(x_1\) but distinct from \(y_1\) and \(y_2\), and let \(x_3\) be the vertex in \(X\) distinct from \(x_1\) lying in the unique quadrangle determined by \(y_1\) and \(y_3\). Again, \(g\) either fixes or interchanges \(y_1\) and \(y_3\). Hence \(g\) fixes \(y_1\). As \(y_1\) was arbitrary, it follows that \(g\) fixes \(X\) pointwise, and so \(g = 1\), as desired.

We remark that \(\text{Aut}(\Pi)_{\gamma}\) and \(\text{Aut}(\Gamma)\) need not be isomorphic in general: if \(\Pi\) is the 4-cube, then \(\text{Aut}(\Pi)_{\gamma} = 2^4 \times S_4\) while \(\text{Aut}(\Gamma) = 2^4 \times S_4\).

4. **Locally Triangular Graphs**

Recall that for \(n \geq 2\), the **triangular graph** \(T_n\) is the graph whose vertices are the 2-subsets of \(\{1, \ldots, n\}\), where a pair of 2-subsets are adjacent whenever they intersect at exactly one point. It is well-known that \(\text{Aut}(T_n) = S_n\) for \(n \geq 3\) and \(n \neq 4\), while \(\text{Aut}(T_4) = S_4 \times C_2\) (cf. [5, Proposition 9.1.2]).

Furthermore, recall that a graph \(\Gamma\) is **locally triangular** if \([\Gamma(u)]\) is isomorphic to a triangular graph for all \(u \in V\Gamma\) and locally \(T_n\) if \([\Gamma(u)] \cong T_n\) for all \(u \in V\Gamma\). By [5, Proposition 4.3.9], every connected locally triangular graph is locally \(T_n\) for some \(n\). Thus there is no loss of generality in focusing on graphs that are locally \(T_n\). Note that if \(n\) is 2, 3 or 4, then \(T_n = K_1, K_3\) or \(K_{3^3}\) respectively, and so the only connected locally \(T_n\) graph is \(K_2, K_4\) or \(K_{4^2}\) respectively.
Let \( \Gamma \) be a graph that is locally \( T_n \). If \( n \geq 4 \), then \( \Gamma \) is not complete and has two families of maximal cliques, corresponding to the two families of maximal cliques in \( T_n \). A maximal clique in \( T_n \) consists either of the 2-subsets of \([n]\) containing some fixed \( i \in [n] \), or the 2-subsets of \([n]\) contained in some 3-subset of \([n]\). In particular, the maximal cliques in \( \Gamma \) either have size 4 or \( n \), and so they are easily distinguished for \( n \geq 5 \).

By [5, Proposition 4.3.9], every connected graph \( \Gamma \) that is locally \( T_n \) is a halved graph of some bipartite rectagraph \( \Pi \) where \( c_3(\Pi) = 3 \). If \( n \leq 4 \), then we may take \( \Pi \) to be the \( n \)-cube, and if \( n \geq 5 \), then \( \Pi \) is defined to be the bipartite graph with parts \( \Delta \) and \( \Delta \), where \( \Delta \) is the set of \( n \)-cliques of \( \Gamma \), and vertices \( u \in \Delta \) and \( B \in \Delta \) are adjacent whenever \( u \in B \). This was first observed in [28, Proposition 3]. Moreover, we have the following.

**Lemma 4.1.** Let \( \Pi \) be a bipartite rectagraph with halved graphs \( \Delta \) and \( \Delta \). If \( \Gamma \) is locally \( T_n \) where \( n \geq 5 \) and \( B \) is the set of \( n \)-cliques of \( \Gamma \), then there exists a bijection \( \varphi : \Delta \to B \) such that \( u \in \Delta \) is adjacent to \( v \in \Delta \) if and only if \( u \in v \varphi \). Furthermore, we have \( c_3(\Pi) = 3 \).

**Proof.** Recall that \( \Pi \) is regular by [5, Proposition 1.1.2]. Since \( |\Pi_2(u)| = |\Gamma(u)| = \binom{n}{2} \) for any \( u \in \Delta \), a counting argument shows that \( \Pi \) has valency \( n \). In particular, if \( v \in \Delta \), then \( \Pi(v) \) is an \( n \)-clique in \( \Gamma \), and so \( \Pi(v) \in \Delta \). Define \( \varphi : V \Gamma \to B \) by \( v \mapsto \Pi(v) \) for all \( v \in \Delta \). Then \( u \in \Delta \) is adjacent to \( v \in \Delta \) if and only if \( u \in v \varphi \). Moreover, \( \varphi \) is injective since \( \Pi \) is a rectagraph and \( n \geq 3 \). Since the parts of a regular bipartite graph must have the same size, and since \( \Delta \) and \( \Delta \) are the parts of a regular bipartite graph by the proof of [5, Proposition 4.3.9], it follows that \( |\Delta| = |\Delta| = |\Delta| \). Thus \( \varphi \) is surjective. Since \( \Pi \) is then isomorphic to the bipartite rectagraph constructed in the proof of [5, Proposition 4.3.9], we have \( c_3(\Pi) = 3 \). \( \Box \)

Thus a connected graph that is locally \( T_n \) is a halved graph of a unique bipartite rectagraph (this is also the case for \( n \leq 4 \)). Note, however, that such a graph may also be the distance 2 graph of a non-bipartite rectagraph.

**Lemma 4.1** has various applications. For example, it follows from Lemma 4.1 and a straightforward exercise that if \( \Pi \) is a bipartite rectagraph and \( \Gamma \) is a halved graph of \( \Pi \), then \( \Gamma \) is locally triangular if and only if \( c_3(\Pi) = 3 \). In addition, Lemma 4.1 enables us to determine the relationship between the automorphism group of a bipartite rectagraph and its locally triangular halved graph.

**Lemma 4.2.** Let \( \Pi \) be a bipartite rectagraph, and let \( \Gamma \) be a halved graph of \( \Pi \) that is locally \( T_n \) where \( n \geq 5 \). Then the following hold.

(i) \( Aut(\Gamma) \cong Aut(\Pi)_{\Delta} \).

(ii) The actions of \( Aut(\Gamma)_{\Delta} \) and \( Aut(\Pi)_{\Delta} \) on \( \Gamma(u) = \Pi(u) \) are faithful and permutation isomorphic for all \( u \in \Delta \).

**Proof.** By Lemma 4.1, we may identify \( \Pi \setminus \Delta \) with the set \( \Delta \) of \( n \)-cliques of \( \Gamma \) in such a way that \( u \in \Delta \) is adjacent to \( B \in \Delta \) whenever \( u \in B \). Note that \( \Pi \) has valency \( n \geq 5 \). Then the map \( \varphi : Aut(\Pi)_{\Delta} \to Aut(\Gamma)_{\Delta} \) defined by \( g \mapsto g|_{\Delta} \) for all \( g \in Aut(\Pi)_{\Delta} \) is an injective group homomorphism by Lemma 3.10. Since \( Aut(\Gamma) \) acts naturally on \( \Delta \), it follows that \( Aut(\Gamma) \) acts on \( \Pi\Delta = \Delta \cup \Delta \). This action is faithful and preserves adjacency in \( \Pi \), and so \( \varphi \) is surjective. Thus (i) holds.

Let \( u \in \Delta \). Since \( c_3(\Pi) = 3 \) by Lemma 4.1, there exists a covering \( \pi : Q_4 \to \Pi \) such that \( 0 \pi = u \) by Lemma 3.1. Then \( Aut(\Pi)_{\Delta} \) acts faithfully on \( \Pi(u) \) by Lemma 3.5. Observe that \( Aut(\Pi)_{\Delta} \leq Aut(\Pi)_{\Delta} \), for if \( g \in Aut(\Pi)_{\Delta} \) and \( v \in \Delta \), then \( d_{\Pi}(u, v^g) = d_{\Pi}(u, v) \), so \( v^g \in \Delta \). Hence \( Aut(\Pi)_{\Delta} \simeq \{g|_{\Delta} : g \in Aut(\Pi)_{\Delta}\} = Aut(\Gamma)_{\Delta} \). Then \( Aut(\Gamma)_{\Delta} \) acts faithfully on \( \Gamma(u) \), and the actions of \( Aut(\Gamma)_{\Delta} \) and \( Aut(\Pi)_{\Delta} \) on \( \Gamma(u) = \Pi(u) \) are permutation isomorphic.

Note that Lemma 4.2 does not hold when \( n = 4 \), for then \( \Pi \cong Q_4 \), in which case \( Aut(\Pi)_{\Delta} \neq Aut(\Gamma)_{\Delta} \), as observed after the proof of Lemma 3.10, and \( Aut(\Pi)_{\Delta} \neq Aut(\Gamma)_{\Delta} \), for \( Aut(\Pi)_{\Delta} = S_4 \) while \( Aut(\Gamma)_{\Delta} = S_4 \times C_2 \).

Lemma 4.2 has the following interesting consequence.
**Proposition 4.3.** Let $\Gamma$ be a connected locally triangular graph. Then $\text{Aut}(\Gamma)_u$ acts faithfully on $\Gamma(u)$ for all $u \in V_T$.

**Proof.** By [5, Proposition 4.3.9], there exists an integer $n \geq 2$ such that $\Gamma$ is locally $T_n$, and $\Gamma$ is a halved graph of some bipartite rectagraph $\Pi$. If $n \leq 4$, then the result is trivial since $\Gamma$ is $K_2$, $K_4$ or $K_4[2]$, and if $n \geq 5$, then the result follows from Lemma 4.2.

To finish this section, we see that there are many examples of graphs that are locally $T_n$.

**Lemma 4.4.** Let $C$ be a binary linear $[n,r,d]$-code where $n \geq 2$ and $d \geq 7$. Then any connected component of $\Gamma(C)_2$ is locally $T_n$.

**Proof.** Let $G$ be a connected component of $\Gamma(C)_2$, and let $x + C \in V_T$. Define a map $\varphi : T_n \to [\Gamma(x + C)]$ by $\{i,j\} \mapsto x + e_{i,j} + C$ for all $\{i,j\} \in \binom{[n]}{2}$. It is routine to verify that $\varphi$ is a graph isomorphism.

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5. Locally rank 3 graphs

In this section, we prove Theorem 1.1 and Corollary 1.2. We begin with some preliminary observations about locally rank 3 graphs.

Let $G$ be a transitive permutation group on $\Omega$, and let $\omega \in \Omega$. A suborbit of $G$ is an orbit of the stabiliser $G_\omega$ on $\Omega$, and the rank of $G$ is the number of suborbits of $G$ (which is independent of the choice of $\omega$ by transitivity). The rank is also the number of orbits of $G$ on $\Omega \times \Omega$.

Recall that a graph $\Gamma$ is locally rank 3 with respect to $G$ if $\Gamma$ has no vertices with valency 0 and $G \leq \text{Aut}(\Gamma)$ such that $G_u(\Gamma)$ is transitive of rank 3 on $\Gamma(u)$ for every $u \in V_T$. We also say that $\Gamma$ is locally rank 3 when $\Gamma$ is locally rank 3 with respect to some $G$. Since $G_u$ acts transitively on $\Gamma(u)$ for all $u \in V_T$, we obtain the following.

**Lemma 5.1.** Let $\Gamma$ be a connected graph that is locally rank 3 with respect to $G$. Then $\Gamma$ is $G$-edge-transitive. If $\Gamma$ is not bipartite, then $\Gamma$ is also $G$-vertex-transitive.

**Proof.** Observe that any pair of vertices with a common neighbour lie in the same $G$-orbit. Then connectivity implies that $\Gamma$ is $G$-edge-transitive. If $\Gamma$ is not bipartite, then $\Gamma$ contains an odd length cycle, so there exists an edge whose ends are in the same $G$-orbit. There is a path of even length from every vertex in $\Gamma$ to one of these ends, so $\Gamma$ is $G$-vertex transitive.

If a graph $\Gamma$ is complete or has girth at least 4, then the induced neighbourhood graphs are complete or have no edges, in which case the suborbits of $G_u(\Gamma)$ will depend entirely on the automorphism group $G$. However, when $\Gamma$ is a locally rank 3 non-complete graph with girth 3, these suborbits can be described combinatorially.

**Lemma 5.2.** Let $\Gamma$ be a connected non-complete graph with girth 3 that is locally rank 3 with respect to $G$. Then for every $u \in V_T$ and $v \in \Gamma(u)$, the orbits of $G_{u,v}$ on $\Gamma(u)$ are $\{v\}$, $\Gamma(u) \cap \Gamma(v)$ and $\Gamma(u) \cap \Gamma_2(v)$.

**Proof.** Fix $u \in V_T$ and $v \in \Gamma(u)$. Clearly the sets $\{v\}$, $\Gamma(u) \cap \Gamma(v)$ and $\Gamma(u) \cap \Gamma_2(v)$ are $G_{u,v}$-invariant and partition $\Gamma(u)$, so it suffices to show that the latter two are non-empty. Observe that since $\Gamma$ has girth 3, it is not bipartite, and so it is $G$-vertex-transitive by Lemma 5.1. In particular, the induced neighbourhood graphs of $\Gamma$ are all isomorphic. If $\Gamma(u) \cap \Gamma(v) = \emptyset$, then $\Gamma(v)$ has no edges, and so $\Gamma$ has no triangles, a contradiction. Similarly, if $\Gamma(u) \cap \Gamma_2(v) = \emptyset$, then $\Gamma(u)$ is complete, and so $\Gamma$ is complete, a contradiction.

Using this result, we see that it is usually sufficient to consider the full automorphism group in order to classify a locally rank 3 graph with girth 3.

**Lemma 5.3.** Let $\Gamma$ be a connected non-complete graph with girth 3 that is locally rank 3 with respect to $G$. If $G \leq H \leq \text{Aut}(\Gamma)$, then $\Gamma$ is locally rank 3 with respect to $H$. 
Proof. Let \( u \in \Gamma (u) \) and \( v \in \Gamma (u) \). By Lemma 5.2, the orbits of \( G_{u,v} \) on \( \Gamma (u) \) are \( \{v\}, \Gamma (u) \cap \Gamma (v) \) and \( \Gamma (u) \cap \Gamma (v) \). These are \( H_{u,v} \)-invariant sets, so they are the orbits of \( H_{u,v} \) on \( \Gamma (u) \). □

Note that Lemma 5.3 does not hold in general. For example, the complete graph \( K_4 \) is locally rank 3 with respect to \( A_4 \) but not \( \text{Aut}(K_4) = S_4 \). Moreover, let \( n := \binom{n}{2} \) where \( m \geq 4 \), and view \( S_m \) as a subgroup of \( S_n \) via the action of \( S_m \) on \( \binom{m}{2} \). Then \( Q_n \) has girth 4 and is locally rank 3 with respect to \( \mathbb{F}_3^2 \times S_m \), but not \( \text{Aut}(Q_n) = \mathbb{F}_3^2 \times S_m \). Thus \( K_4 \) and \( Q_n \) (for \( n = \binom{n}{2} \) and \( m \geq 4 \)) are examples of graphs that are locally rank 3 and locally 2-arc transitive.

Recall that a 2-arc \( (u,v,w) \) is either a triangle when \( u \) and \( w \) are adjacent, or a 2-geodesic when \( u \) and \( w \) are distance 2 apart. For any \( u \in \Gamma(T) \), the (possibly empty) sets of triangles and 2-geodesics with initial vertex \( u \) are \( G_u \)-invariant. It turns out that for a connected non-complete graph with girth 3, these two sets are \( G_u \)-orbits precisely when \( \Gamma \) is locally rank 3 with respect to \( G \). This result, which we now prove, is somewhat surprising, for the assumption that these two sets form orbits does not, at first glance, seem likely to imply that \( G_u \) is transitive on \( \Gamma(u) \) for all \( u \in \Gamma(T) \).

**Proposition 5.4.** Let \( \Gamma \) be a connected non-complete graph with girth 3, and let \( G \leq \text{Aut}(\Gamma) \). Then the following are equivalent.

(i) \( \Gamma \) is locally rank 3 with respect to \( G \).

(ii) For each \( u \in \Gamma(T) \), there are two orbits of \( G_u \) on the 2-arcs starting at \( u \), namely the set of triangles starting at \( u \) and the set of 2-geodesics starting at \( u \).

Proof. Suppose that for each \( u \in \Gamma(T) \), there are two orbits of \( G_u \) on the 2-arcs starting at \( u \), namely the sets of triangles and 2-geodesics. Let \( u \in \Gamma(T) \). First we claim that \( G_u \) acts transitively on \( \Gamma(u) \). Since \( u \) lies in a triangle, there exists \( v \in \Gamma(u) \) and \( w \in \Gamma(u) \cap \Gamma(v) \). Let \( x \in \Gamma(u) \), and suppose for a contradiction that \( \Gamma(u) \cap \Gamma(x) = \emptyset \). Since \( x \) lies in a triangle and therefore does not have valency 1, there exists \( y \in \Gamma_2(u) \cap \Gamma(x) \). Moreover, since \( (w,v,u) \) and \( (w,v,u) \) are triangles, there is some \( g \in G_w \) that interchanges \( v \) and \( x \). In particular, since \( v \) and \( x \) are not adjacent, \( x^y \) lies in \( \Gamma_2(u) \cap \Gamma(v) \). Then \( (u,v,x^y) \) and \( (u,x,y) \) are 2-geodesics, so \( v^h = x \) for some \( h \in G_u \), in which case \( w^h \in \Gamma(u) \cap \Gamma(x) \), a contradiction. Thus there exists \( y \in \Gamma(u) \cap \Gamma(x) \), and since \( (u,v,w) \) and \( (u,x,y) \) are triangles, we have \( x \in \Gamma(u) \), as desired.

Now we determine the orbits of \( G_{u,v} \) on \( \Gamma(u) \), where \( v \in \Gamma(u) \). If \( x,y \in \Gamma(u) \cap \Gamma(v) \), then \( (v,u,x) \) and \( (v,u,y) \) are triangles, so there exists \( g \in G_{u,v} \) such that \( x^g = y \). Similarly, if \( x,y \in \Gamma(u) \cap \Gamma_2(v) \), then \( (u,v,x) \) and \( (u,y) \) are 2-geodesics, so there exists \( g \in G_{u,v} \) such that \( x^g = y \). Thus the orbits of \( G_{u,v} \) on \( \Gamma(u) \) are \( \{v\}, \Gamma(u) \cap \Gamma(v) \) and \( \Gamma(u) \cap \Gamma_2(v) \), and so \( \Gamma \) is locally rank 3 with respect to \( G \).

Conversely, suppose that \( \Gamma \) is locally rank 3 with respect to \( G \). Let \( u \in \Gamma(T) \). Recall that the sets of triangles and 2-geodesics starting at \( u \) are \( G_u \)-invariant. They are also non-empty, for there exists \( v \in \Gamma(u) \) (since \( G_u \) has no vertices of valency 0 by assumption), and so there exists \( w \in \Gamma(v) \cap \Gamma(u) \) and \( x \in \Gamma(v) \cap \Gamma_2(u) \) by Lemma 5.2, in which case \( (u,v,u) \) is a triangle and \( (u,v,x) \) is a 2-geodesic.

Now let \( (u,v,w) \) and \( (u,x,y) \) be 2-arc. Since \( G_u \) is transitive on \( \Gamma(u) \), there exists \( g \in G_u \) such that \( v^g = x \). If \( (u,v,w) \) and \( (u,x,y) \) are triangles, then \( (u,x,w^g) \) is a triangle, so \( w^g, y \in \Gamma(x) \cap \Gamma(u) \), and by Lemma 5.2 there exists \( h \in G_{x,u} \) such that \( w^{gh} = y \). Thus \( (u,v,w)^{gh} = (u,x,y) \), so the set of triangles starting at \( u \) is a \( G_u \)-orbit. Similarly, if \( (u,v,w) \) and \( (u,x,y) \) are 2-geodesics, then \( (u,x,w^g) \) is a 2-geodesic, so \( w^g, y \in \Gamma(x) \cap \Gamma_2(u) \), and by Lemma 5.2 there exists \( h \in G_{x,u} \) such that \( w^{gh} = y \). Thus \( (u,v,w)^{gh} = (u,x,y) \), so the set of 2-geodesics starting at \( u \) is a \( G_u \)-orbit. □

Next we provide some results that will be used in the proof of Theorem 1.1. The first gives us information about the automorphism groups of halved coset graphs of binary linear codes. Recall that if \( C \) is a binary linear \([n,r,d]\)-code, then \( \text{Aut}(C) \) is the subgroup of \( S_n \) that preserves \( C \).
Lemma 5.5. Let C be an even binary linear \([n, r, d]\)-code where \(n \geq 5\) and \(d \geq 7\). Let \(\Gamma\) be a halved graph of \(\Gamma(C)\). Then \(\Aut(\Gamma) \cong (E_n/C) \rtimes \Aut(C)\). Moreover, let \(\rho : \Aut(\Gamma) \to \Aut(C)\) be the natural projection map, and let \(G \leq \Aut(\Gamma)\). Then the following hold.

(i) The actions of \(G_{x+C}\) on \(\Gamma(x + C)\) and \((G_{x+C})\rho \text{ on } (\frac{n}{2})\) are permutation isomorphic for all \(x \in E_n\).

(ii) If \(G = (E_n/C) \rtimes H\) where \(H \leq \Aut(C)\) is transitive of rank 3 on \((\frac{n}{2})\), then \(\Gamma\) is locally rank 3 with respect to \(G\).

(iii) If \(H := (G_{0+C})\rho = (G_{x+C})\rho\) for all \(x \in E_n\) and \(H\) is transitive on \((\frac{n}{2})\), then \(G = (E_n/C) \rtimes H\).

Proof. By Lemma 2.5, we may assume that \(VT = E_n/C\). By Lemma 4.4 and its proof, the graph \(\Gamma\) is locally \(T_n\), and for each \(x \in E_n\), there is a bijection \(\varphi_x : \Gamma(x + C) \to (\frac{n}{2})\) defined by \(x + e_{i,j} + C \mapsto \{i, j\}\) for all \(i < j \leq n\). Recall from Lemma 2.5 (v) that \(\Gamma(C)\) is a rectagraph and recall from Lemma 3.9 that \(\Aut(\Gamma(C)) = (\mathbb{F}_2^n/C) \rtimes \Aut(C)\). Then \(\Aut(\Gamma) = (E_n/C) \rtimes \Aut(C)\) by Lemma 4.2.

Let \(x \in E_n\). Since \(G_{x+C} = G \cap \{(x + x^\sigma + C, \sigma^{-1}) : \sigma \in \Aut(C)\}\), the actions of \(G_{x+C}\) on \(\Gamma(x + C)\) and \((G_{x+C})\rho \text{ on } (\frac{n}{2})\) are permutation isomorphic under the group isomorphism \(\rho|_{G_{x+C}}\) and the bijection \(\varphi_x\). This proves (i).

If \(G = (E_n/C) \rtimes H\) where \(H \leq \Aut(C)\) is transitive of rank 3 on \((\frac{n}{2})\), then \(\Gamma\) is \(G\)-vertex-transitive and \((G_{0+C})\rho = (G_{x+C})\rho\) for all \(x \in E_n\), and suppose that \(H\) is transitive on \((\frac{n}{2})\). Since \(G_{0+C} = \{(0 + C, \sigma) : \sigma \in H\}\), it follows that \(\Gamma(x + x^\sigma + C, 1) \in G\) for all \(x \in E_n\) and \(\sigma \in H\). Let \(i, j, k \in [n]\) be pairwise distinct. There exists \(\sigma \in H\) such that \(\{i, k\}^\sigma = \{k, j\}\), so \((e_{i,j} + C, 1) = (e_{i,k} + e_{i,k}^\sigma + C, 1) \in G\). Hence \(E_n/C \leq G\), and so \(G = (E_n/C) \rtimes H\), proving (ii).

In the following, we invoke the classification of the finite simple groups to determine which subgroups of \(S_n\) are transitive on \(\frac{n}{2}\).

Theorem 5.6. Let \(H \leq S_n\) where \(n \geq 5\). Then \(H\) is transitive on \(\frac{n}{2}\) if and only if \((H, n)\) is one of \((S_n, n)\) or \((A_n, n)\) for \(n \geq 5\), \((\text{P}T\text{L}_2(8), 9)\), or \((M_n, n)\) for \(n = 11, 12, 23\) or 24.

Proof. If \(H\) is transitive on \(\frac{n}{2}\), then by [14, Lemma 5], either \(H\) is 4-transitive or \((H, n)\) is one of \((A_5, 5)\) or \((\text{P}T\text{L}_2(8), 9)\). By the classification of the finite simple groups, the 4-transitive subgroups of \(S_n\) are \(S_n\) for \(n \geq 5\), \(A_n\) for \(n \geq 6\), and \(M_n\) for \(n = 11, 12, 23\) or 24 (cf. [8, Theorem 4.11]). Conversely, all of these groups are transitive on \(\frac{n}{2}\).

Surprisingly, using Theorem 5.6, we can prove that the converse to Lemma 5.5 (ii) holds.

Proposition 5.7. Let \(C\) be an even binary linear \([n, r, d]\)-code where \(n \geq 5\) and \(d \geq 7\). A halved graph of \(\Gamma(C)\) is locally rank 3 with respect to \(G\) if and only if \(G = (E_n/C) \rtimes H\) where \(H \leq \Aut(C)\) is transitive on rank 3 on \((\frac{n}{2})\).

Proof. Let \(\Gamma\) be a halved graph of \(\Gamma(C)\), and suppose that \(\Gamma\) is locally rank 3 with respect to \(G\). Then \(\Aut(\Gamma) = (E_n/C) \rtimes \Aut(C)\) by Lemma 5.5. Again, we may assume that \(VT = E_n/C\) by Lemma 2.5. Let \(\rho : \Aut(\Gamma) \to \Aut(C)\) be the natural projection map and \(H := (G_{0+C})\rho\). Then by Lemma 5.5 (i), \(H\) is transitive on rank 3 on \((\frac{n}{2})\), and so it suffices to show that \(H = (G_{x+C})\rho\) for all \(x \in E_n\), for then \(G = (E_n/C) \rtimes H\) by Lemma 5.5 (iii), as desired.

Since \(H\) is transitive on rank 3 on \((\frac{n}{2})\) and \(H \leq G\rho \leq S_n\) where \(n \geq 5\), the group \(G\rho\) is also transitive on rank 3 on \((\frac{n}{2})\). Moreover, \((G_{x+C})\rho\) is transitive on rank 3 on \((\frac{n}{2})\) for all \(x \in E_n\) by Lemma 5.5 (i). Thus \((G\rho, n)\) and \(((G_{x+C})\rho, n)\) for all \(x \in E_n\) are one of \((S_n, n)\) or \((A_n, n)\) for \(n \geq 5\), \((\text{P}T\text{L}_2(8), 9)\), or \((M_n, n)\) for \(n = 11, 12, 23\) or 24 by Theorem 5.6.

If \((G\rho, n)\) is one of \((\text{P}T\text{L}_2(8), 9)\) or \((M_n, n)\) where \(n = 11, 12, 23\) or 24, then \((G_{x+C})\rho = G\rho\) for all \(x \in E_n\), as desired, so we may assume that \(A_n \leq G\rho\). Fix \(x \in E_n\). Since \(|(G_{x+C})\rho| = |G_{x+C}|\) by Lemma 5.5 (i), the index \(|G\rho : (G_{x+C})\rho|\) divides \(|G : G_{x+C}|\), but \(G\) acts transitively on...
VT by Lemma 5.1 since \( \Gamma \) contains triangles, and \(|V| = 2^n\), so \(|G_{\rho} : (G_{x+C})\rho| = 2^r\) for some \( r \geq 0 \). Since \((G_{x+C})\rho\) is one of \( S_n, A_n, PTL_2(8)\), or \( M_n\) for \( n = 11, 12, 23 \) or \( 24\), we must have \( A_n \leq (G_{x+C})\rho\). Since \( G \) acts transitively on \( VT\), the groups \( G_{0+C} \) and \( G_{x+C} \) are conjugate in \( G\), and so the groups \( H \) and \((G_{x+C})\rho\) are conjugate in \( G_{\rho}\). Thus \( H = (G_{x+C})\rho\).

The converse is Lemma 5.5 (ii), so the proof is complete. \(\square\)

Recall from §2.4 that if \( C \) is a linear code in \( \mathbb{F}_2^n \) that is not even, then \( \Gamma(C) \) is transitive on \( \Pi(C) \). Thus Proposition 5.7 can be used for codes that are not even as well.

Next we deal with small \( n \). When \( n = 3 \), the only locally \( T_3 \simeq K_3 \) graph is \( K_4 \), which has automorphism group \( S_4 \).

**Lemma 5.8.** \( K_4 \) is locally rank 3 with respect to \( G \) if and only if \( G = A_4 \).

**Proof.** Suppose that \( \Gamma := K_4 \) is locally rank 3 with respect to \( G \). Clearly \( G_u \) acts faithfully on \( \Gamma(u) \) for all \( u \in VT \). Since \( S_3 \) has rank 2 on \( \binom{3}{2} \), it follows that \( G_u \simeq A_3 \) for all \( u \in VT \), and so \( G \) contains all of the 3-cycles of \( A_4 \). Thus \( G = A_4 \), and \( \Gamma \) is indeed locally rank 3 with respect to \( G \).

Similarly, when \( n = 4 \), the only locally \( T_4 \simeq K_{3[2]} \) graph is \( K_{4[2]} \), which has automorphism group \( S_2 \wr S_4 \). In the following, we write elements of \( S_2 \wr S_4 \) in the form \( (\sigma_1, \sigma_2, \sigma_3, \sigma_4) \in S_2^4 \) and \( \sigma \in S_4 \). We also write \( (12) \) for the transposition in \( S_2 \) and \( S_4 \) that interchanges 1 and 2.

**Lemma 5.9.** The graph \( K_{4[2]} \) is locally rank 3 with respect to \( G \) if and only if \( G \) is one of \( S_2 \wr S_4, E \rtimes S_4 \) or \((E \rtimes A_4)(\tau)\), where \( E := \{(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \in S_2^4 : |\{i : \sigma_i \neq 1\}| \equiv 0 \mod 2\} \) and \( \tau := ((12), 1, 1, 1)(12) \).

Note that if we identify \( S_2 \) with \( \mathbb{F}_2 \), then the subgroup \( E \) of \( S_2^4 \) defined in Lemma 5.9 corresponds to the subgroup \( E_4 \) of \( \mathbb{F}_2^4 \).

**Proof.** Suppose that \( \Gamma := K_{4[2]} \) is locally rank 3 with respect to \( G \). Then \( \text{Aut}(\Gamma) = S_2 \wr S_4 \). Clearly \( G_u \) acts faithfully on \( \Gamma(u) \) for all \( u \in VT \). If \( G = S_2 \wr S_4 \), then we are done, so we may assume that \( G < S_2 \wr S_4 \). Fix \( u \in VT \), and let \( v, w \in \Gamma(u) \) correspond to \( \{1, 2\} \) and \( \{1, 3\} \) respectively. Then \( G_{u,v} \) has an orbit of size 4 containing \( w \). Since \( G \) is transitive on \( VT \) by Lemma 5.1, and since \( G_u \) is transitive on \( \Gamma(u) \), we have that \( |G| = |G_u| = |G_{u,v}| = 48 = |(u,v,w)\{2\} \wr 23 \cdot |23| \), and so \( G_{u,v,w} = 1 \) and \( G \) has index 2 in \( S_2 \wr S_4 \). Let \( \rho : S_2^4 \rtimes S_4 \to S_4 \) be the natural projection map. Then \( G \leq S_2^4 \rtimes G_{\rho} \), so \( 12 \leq |G_{\rho}| \). Note that \( G_{\rho} \neq A_4 \), or else \( G = S_2 \rtimes A_4 \) and \( G_{u,v,w} = S_2 \), a contradiction. Hence \( G_{\rho} = S_4 \), so \( G \) is either \( E \rtimes S_4 \) or \((E \rtimes A_4)(\tau)\). Conversely, \( \Gamma \) is indeed locally rank 3 with respect to \( S_2 \wr S_4, E \rtimes S_4 \) and \((E \rtimes A_4)(\tau)\).

Now we prove Theorem 1.1.

**Proof of Theorem 1.1.** For this proof, recall that the bipartite double \( \Gamma(C_{23}) \) is isomorphic to \( \Gamma(C_{23} \cap E_{23}) \) by Lemma 2.5 (iii).

Let \( \Gamma \) be a connected graph that is locally triangular and locally rank 3 with respect to \( G \). We wish to show that \( \Gamma \) is one of the graphs in (i)-(iv) and \( G \) is one of the groups in Table 1.

The graph \( \Gamma \) is locally \( T_n \) for some integer \( n \geq 2 \) by [5, Proposition 4.3.9]. If \( n = 2 \), then \( G_{\Gamma(u)} \) has rank 1, a contradiction. If \( n = 3 \), then \( \Gamma \) is locally \( K_3 \), and so \( \Gamma \simeq K_3 \simeq Q_3 \) and \( G \) is one of the groups in Table 1 by Lemma 5.8. If \( n = 4 \), then \( \Gamma \) is locally \( K_{3[2]} \), and so \( \Gamma \simeq K_{4[2]} \simeq Q_4 \) and \( G \) is one of the groups in Table 1 by Lemma 5.9. Thus we may assume that \( n \geq 5 \).

By [5, Propositions 1.1.2 and 4.3.9], \( \Gamma \) is a halved graph of a bipartite rectagraph \( \Pi \) of valency \( n \) with \( c_3(\Pi) = 3 \). Let \( u \in VT \). By Lemma 3.1, there exists a covering \( \pi : Q_n \to \Pi \) such that \( 0 \pi = u \). Since \( n \geq 5 \), the group \( N_0^n \) is transitive of rank 3 on \( \binom{n}{2} \) by Lemmas 3.5, 4.2 (ii) and 5.3. It is then routine to verify that \( N_0^n \) acts 4-homogeneously on \( [n] \), and so \( \text{Aut}(\Pi)u \) acts 4-homogeneously on \( \Pi(u) \) by Lemma 3.5.
Since Π is bipartite, it follows from Theorem 1.3 and Lemma 2.5 (i) that Π is isomorphic to the coset graph $Γ(C)$ where one of the following occurs: $C = \{0\}$ and $n ≥ 5$; $C = \{0, 1\}$, $n$ is even and $n ≥ 8$; $C = C_{23} \cap E_{23}$ and $n = 23$; or $C = C_{24}$ and $n = 24$. Since the halved graphs of Π are then isomorphic by Lemma 2.5 (ii), the graph $Γ$ is the halved graph of Π. Thus $Γ$ is one of the graphs described in (i)-(iv) where $n ≥ 5$, and so $G$ is one of the groups in Table 1 by Theorem 5.6 and Proposition 5.7.

Conversely, let $Γ$ be one of the graphs in (i)-(iv) and $G$ one of the groups in Table 1. Then $Γ$ is locally triangular by Lemma 4.4. If $n = 3$ or 4, then $Γ$ is isomorphic to $K_4$ or $K_{4[2]}$ respectively, so $Γ$ is locally rank $3$ with respect to $G$ by Lemmas 5.8 and 5.9. If $n ≥ 5$, then $Γ$ is locally rank $3$ with respect to $G$ by Theorem 5.6 and Proposition 5.7.

Lastly, we prove Corollary 1.2.

Proof of Corollary 1.2. Suppose that there exists $G ≤ \text{Aut}(Γ)$ such that, for all $u ∈ VΓ$, the action of $G_u$ on $Γ(u)$ is permutation isomorphic to the action of $H$ on $\binom{u}{2}$, where $H ≤ S_n$ is transitive of rank 3 on $\binom{u}{2}$ and $n ≥ 5$. Let $u ∈ VΓ$ and $v ∈ Γ(u)$. Lemma 5.2 implies that the orbits of $G_u,v$ on $Γ(u)$ are $\{v\}$, $Γ(u) \cap Γ(v)$ and $Γ(u) \cap Γ_2(v)$. Without loss of generality, we may assume that $v$ corresponds to $\{1, 2\}$. Note that the orbits of $H_{1,2}$ on $\binom{u}{2} \setminus \{\{1, 2\}\}$ consist of the set $X$ of 2-subsets containing either 1 or 2, and the set $Y$ of 2-subsets containing neither 1 nor 2. If $X = Γ(u) \cap Γ(v)$, then $[Γ(u)] ≃ T_n$, and if $X = Γ(u) \cap Γ_2(v)$, then $[Γ(u)] ≃ T_n$. Since $Γ$ has girth 3, it is $G$-vertex transitive by Lemma 5.1, and so $Γ$ is either locally $T_n$ or locally $T_n$. In the former case, Theorem 1.1 applies, and in the latter case, [13, Theorem 2] applies.

Conversely, if $Γ$ is a graph from Theorem 1.1, then $Γ$ is locally rank 3 and locally $T_n$, so the claim holds. If $Γ = T_{n+2}$ where $n ≥ 5$, then $\text{Aut}(Γ)_u = S_n × C_2$ and $\text{Aut}(Γ)_u = S_8$ for all $u ∈ VΓ$, in which case the claim holds with $G = \text{Aut}(Γ)$. We obtain the same result when $n = 5$ and $Γ$ is the Conway-Smith graph by [5, Theorem 13.2.3], or when $n = 5$ and $Γ$ is the commuting involutions graph of the conjugacy class of the involutory Galois field automorphism in $ΨL_2(25)$ by [5, Proposition 12.2.2], or when $n = 6$ and $Γ$ is the complement of an elliptic quadric in the graph $Sp_6(2)$ by GAP [1, 10, 30]. Lastly, if $n = 6$ and $Γ$ is the complement of a hyperplane in the graph $Sp_6(2)$, then $\text{Aut}(Γ)_u = S_8$ for all $u ∈ VΓ$ by GAP [1, 10, 30], in which case the claim holds with $G = \text{Aut}(Γ)$.

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References

[1] Bamberg, J., Beuten, A., De Beule, J., Cara, P., Lavrauw, M., and Neunhoeffer, M. FinInG – a GAP package, Version 1.0, 2013. http://cage.ugent.be/geometry/fining.php.
[2] Bosma, W., Cannon, J., and Playoust, C. The Magma algebra system. I. The user language. J. Symbolic Comput. 24 (1997), 235–265.
[3] Brouwer, A. E. On the uniqueness of a certain thin near octagon (or partial 2-geometry, or parallelism) derived from the binary Golay code. IEEE Transactions on Information Theory 29 (1983), 370–371.
[4] Brouwer, A. E. Classification of small $(0,2)$-graphs. J. Combinatorial Theory, Series A 113 (2006), 1636–1645.
[5] Brouwer, A. E., Cohen, A. M., and Neumaier, A. Distance-regular graphs. Springer-Verlag, Berlin, 1989.
[6] Brouwer, A. E., and Östergård, P. R. J. Classification of the (0, 2)-graphs of valency 8. *Discrete Math.* 309 (2009), 532–547.

[7] Cameron, P. J. Suborbits in transitive permutation groups. In *Combinatorics: Proceedings of the NATO Advanced Study Institute held at Nijenrode Castle, Breukelen, The Netherlands, 8-20 July 1974*, Springer, 1975, pp. 419–450.

[8] Cameron, P. J. *Permutation groups*. Cambridge University Press, New York, 1999.

[9] Devillers, A., Jin, W., Li, C. H., and Praeger, C. E. Local 2-geodesic transitivity and clique graphs. *J. Combinatorial Theory, Series A* 120 (2013), 500–508.

[10] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.6.3*, 2013. [http://www.gap-system.org](http://www.gap-system.org).

[11] Giudici, M., Li, C. H., and Praeger, C. E. Analysing finite locally s-arc transitive graphs. *Trans. Amer. Math. Soc.* 356 (2004), 291–317.

[12] Hall, J. I. Locally Petersen graphs. *J. Graph Th.* 4 (1980), 173–187.

[13] Hall, J. I., and Shult, E. E. Locally cotriangular graphs. *Geom. Dedicata* 18 (1985), 113–159.

[14] Higman, D. G. Characterization of families of rank 3 permutation groups by the subdegrees I. *Arch. Math.* 21 (1970), 151–156.

[15] Hughes, D. R. Biplanes and semi-biplanes. In *Combinatorial Mathematics, Proc. Internat. Conf. Combinatorial Theory, Australian Nat. Univ., Canberra, 1977*, in Lecture Notes in Math., vol. 686. Springer-Verlag, Berlin, 1978, pp. 55–58.

[16] Ivanov, A. A. *Geometry of Sporadic Groups I, Petersen and Tilde Geometries*. Cambridge University Press, Cambridge, 1999.

[17] Ivanov, A. A., and Praeger, C. E. On finite affine 2-arc transitive graphs. *Europ. J. Combinatorics* 14 (1993), 421–444.

[18] Jurišić, A., and Koolen, J. 1-homogeneous graphs with cocktail party μ-graphs. *J. Algebraic Combinatorics* 18 (2003), 79–98.

[19] Kabanov, V. V., Makhnev, A. A., and Paduchikh, D. V. Characterization of certain distance-regular graphs by forbidden subgraphs. *Doklady Math.* 75 (2007), 420–423.

[20] Kantor, W. M. k-Homogeneous groups. *Math. Z.* 124 (1972), 261–265.

[21] Leemans, D. Locally s-arc-transitive graphs related to sporadic simple groups. *J. Algebra* 322 (2009), 882–892.

[22] Livingstone, D., and Wagner, A. Transitivity of finite permutation groups on unordered sets. *Math. Z.* 90 (1965), 393–403.

[23] Makhnev, A. A. On the graphs with μ-subgraphs isomorphic to K_{u×2}. In *Proc. Steklov Inst. Math.* (2001), vol. 2, pp. S169–S178.

[24] Matsumoto, M. On the classification of locally Hamming distance-regular graphs. *RIMS Kôkyûroku* 768 (1991), 50–61. [http://www.kurims.kyoto-u.ac.jp/~kyodo/kokyuroku/contents/pdf/0768-07.pdf](http://www.kurims.kyoto-u.ac.jp/~kyodo/kokyuroku/contents/pdf/0768-07.pdf).

[25] Mortimer, B. The modular permutation representations of the known doubly transitive groups. *Proc. London Math. Soc.* 41 (1980), 1–20.

[26] Munemasa, A., Pasechnik, D. V., and Shpectorov, S. V. A local characterization of the graphs of alternating forms and the graphs of quadratic forms over GF(2). In *Finite Geometries and Combinatorics* (1993), Cambridge University Press, Cambridge, pp. 303–318.

[27] Neumaier, A. Rectagraphs, diagrams, and Suzuki’s sporadic simple group. *Ann. Discrete Math.* 15 (1982), 305–318.

[28] Neumaier, A. Characterization of a class of distance regular graphs. *J. Reine Angew. Math.* 357 (1985), 182–192.

[29] Rifà, J., and Huguet, L. Classification of a class of distance-regular graphs via completely regular codes. *Discrete Applied Math.* 26 (1990), 289–300.

[30] Soicher, L. H. *Grape – a GAP package*, Version 4.3, 2012. [http://www.maths.qmul.ac](http://www.maths.qmul.ac).
[31] van Bon, J. On locally s-arc transitive graphs with trivial edge kernel. Bull. London Math. Soc. 43 (2011), 799–804.

[32] Wild, P. R. On semibiplanes. PhD thesis, University of London, 1980.

[33] Wilson, R. A. The finite simple groups. Springer, London, 2009.

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