The Hanna Neumann Conjecture and the rank of the join

Joshua E. Hunt*

October 22, 2015

Abstract

The Hanna Neumann conjecture gives a bound on the intersection of finitely generated subgroups of free groups. We explore a natural extension of this result, which turns out to be true only in the finite index case, and provide counterexamples for the general case. We also see that the graph-based method of generating random subgroups of free groups developed by Bassino, Nicaud and Weil is well-suited to generating subgroups with non-trivial intersections. The same method is then used to generate a counterexample to a similar conjecture of Guzman.

1 Introduction

Throughout, let $F$ be a finitely generated (non-trivial) free group, and $H,K$ be finitely generated subgroups of $F$. We define the reduced rank of $H$ to be

$$\tau(H) = \max(0, \text{rank}(H) - 1)$$

The Hanna Neumann conjecture states that

$$\tau(H \cap K) \leq \tau(H) \tau(K)$$

A strengthening of this, the Strengthened Hanna Neumann conjecture, was proposed by Walter Neumann [Neu]:

$$\sum_{HgK \text{ s.t. } H \cap gKg^{-1} \neq \{1\}} \tau(H \cap gKg^{-1}) \leq \tau(H) \tau(K)$$

This was proved independently by Joel Friedman [Fri] and Igor Mineyev [Min] in 2011.

The results above do not involve the join of $H$ and $K$ (i.e. the group generated by $H \cup K$, which we denote $H \vee K$). By way of analogy with the inclusion/exclusion principle, it seems natural to suppose that when $H \cap K$ is “large”, $H \vee K$ is “not much bigger” than $H$ or $K$. Accordingly, Henry Wilton conjectured

**Conjecture 1** (Wilton). Let $H, K$ be finitely generated subgroups of $F$. Then

$$\tau(H \vee K) \tau(H \cap K) \leq \tau(H) \tau(K)$$

We will refer to this as the Inclusion/Exclusion Hanna Neumann Conjecture (IEHNC). This turns out to be true if we also assume that $K$ is of finite index in $H \vee K$:

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*E-mail: joshuahunt94@gmail.com
The author was funded by a Trinity College Summer Research Studentship.

[1] In a private communication
Theorem 2. Let $H, K$ be finitely generated subgroups of $F$, with $K$ of finite index in $F$. Then

$$r(H \vee K) r(H \cap K) \leq r(H) r(K)$$

Unfortunately, the IEHNC is no longer true if we do not make this assumption, and so does not give a strengthening of the Hanna Neumann conjecture.

Note that the Strong Hanna Neumann conjecture is invariant under conjugation of $H$ or $K$ in $F$, while the IEHNC is not. As such, we would not expect to be able to combine the two results.

The IEHNC is slightly stronger than Rosemary Guzman’s “Group-theoretic conjecture” \cite{Guz}: Conjecture 3 (Guzman). Let $H, K$ be finitely generated subgroups of $F$ with

$$m = rk(H) = rk(K) \leq rk(H \cap K)$$

for some $m \geq 2$. Then

$$rk(H \vee K) \leq m$$

Note that the IEHNC implies Guzman’s conjecture. We give a counterexample to both Guzman’s conjecture and the IEHNC (without the finite index assumption) in section 4. Furthermore, example 1 in fact shows that there is no $c > 0$ such that

$$r(H \vee K) r(H \cap K) \leq c r(H) r(K)$$

for all $H, K$ finitely generated subgroups of $F$.

The counterexample given to Guzman’s conjecture has $m = 5$. Louder and McReynolds \cite{Lou} and Kent \cite{Ken} both proved conjecture 3 for the case $m = 2$, and Guzman herself proved it for $m = 3$ \cite{Guz}, so it remains to determine whether or not it holds for $m = 4$.

The first counterexamples to both the IEHNC and Guzman’s conjecture were generated using a computer search, with the software package GAP. In section 5 we discuss the methods used for this, and their suitability to investigating similar problems about free groups. We would be more than happy to share the source code on request.

Acknowledgements

The author gratefully acknowledges the support of a Trinity College Summer Research Studentship, and would like to thank Henry Wilton for supervising the project. He would also like to thank Trenton Schirmer for pointing out the connection between the IEHNC and Guzman’s conjecture.

2 Stallings graphs

2.1 The category of graphs

Finitely generated subgroups of free groups can be represented by immersions (locally injective maps) of finite graphs, as introduced by Stallings in \cite{Sta}. We consider a graph to be a pair $X = (V(X), E(X))$ of sets of vertices and directed edges, along with a function $E \to E$ and a function $E \to V$. A graph is finite if both its vertex set and edge set are finite. For each $e \in E$ we have an associated edge $\tau(e) \in E$, the reversal of $e$, and an associated vertex $\iota(e) \in V$, the initial vertex of $e$. We require that $\tau = e$ and $\tau \neq e$. We define the terminal vertex of $e$ to be $\tau(e) = \iota(\tau)$.

A map of finite graphs $f : X \to Y$ is a pair of functions $E(X) \to E(Y)$ and $V(X) \to V(Y)$ such that this structure is preserved, i.e. $f(\iota(e)) = \iota(f(e))$ and $f(\tau(e)) = f(\tau)$. This determines a category of finite graphs in which we may consider pullbacks, pushouts, products, and so on.

The pullback in particular is important (theorem 4 below). This always exists, and can be constructed explicitly. Given maps of finite graphs

$$f : X \to Y$$

and

$$g : Z \to Y$$

there is a unique map $h : Z \to X$ such that

$$f \circ h = g$$

This map is the pullback of $f$ and $g$. It can be constructed explicitly by considering the pullback of $E(X)$ and $E(Y)$, and $V(X)$ and $V(Y)$.
the pullback is given by
\[ V(Y \times_X Z) = \{ (v, v') \in V(Y) \times V(Z) : f(v) = g(v') \} \]
\[ E(Y \times_X Z) = \{ (e, e') \in E(Y) \times E(Z) : f(e) = g(e') \} \]

2.2 Immersions and coverings

We define the \textit{star of} \( v \) in \( X \) to be the set
\[ \text{St}_X(v) = \{ e \in E(X) : \iota(e) = v \} \]
Given a map of graphs \( f : X \to Y \) and a vertex \( v \in V(X) \), we get an induced map \( \text{St}_X(v) \to \text{St}_Y(f(v)) \); \( e \mapsto f(e) \). A map of graphs is said to be an \textit{immersion} if this induced map is injective for every \( v \in V(X) \), and a \textit{covering} if it is bijective for every \( v \in V(X) \).
(We often denote immersions as \( f : X \hookrightarrow Y \).)

Any map of based graphs \( f : (X, x_0) \to (Y, y_0) \) induces a homomorphism of fundamental groups \( f_\ast : \pi_1(X, x_0) \to \pi_1(Y, f(x_0)) \) for any \( x_0 \in V(X) \). Furthermore, if \( f \) is an immersion then this homomorphism is in fact injective. Using the technique of Stallings folding we can represent any finitely generated subgroup \( H \leq \pi_1(Y, y_0) \) (\( Y \) a finite graph) as a based immersion \( f : (X, x_0) \to (Y, y_0) \), where \( f_\ast \pi_1(X, x_0) = H \) and \( X \) is a finite graph. For more details, see section 5.4 of [Sta].

Remark 1. Note that \( r(H) - 1 = -\chi(X) \) (where \( \pi_1(X, x_0) \cong H \)). This is because if we pick a maximal spanning tree of \( X \), then each edge of \( X \) not in this tree determines a basis element of \( \pi_1(X, x_0) \) and reduces the Euler characteristic of the graph by 1, and the Euler characteristic of the maximal spanning tree itself is 1. Therefore questions about reduced rank can be reduced to questions about Euler characteristic, which is the technique used in the proofs below.

2.3 Useful results about subgroups and graphs

A key tool needed to investigate the IEHNC using graphs is the following theorem from [Sta]:

**Theorem 4.** Let
\[
\begin{array}{ccc}
Y_3 & \xrightarrow{g_1} & Y_1 \\
\downarrow{g_2} & & \downarrow{f_1} \\
Y_2 & \xrightarrow{f_2} & X
\end{array}
\]
be a pullback diagram of finite graphs. Suppose that \( f_1, f_2 \) are immersions. Let \( v_1 \in V(Y_1) \), \( v_2 \in V(Y_2) \), \( w \in V(X) \) such that
\[ f_1(v_1) = f_2(v_2) = w \]
Let \( v_3 = (v_1, v_2) \in V(Y_3) \). Define \( f_3 = f_1 g_1 = f_2 g_2 \). Let
\[ H_4 = (f_3)_\ast (\pi_1(Y_3, v_3)) \]
Then \( H_3 = H_1 \cap H_2 \).

**Proof.** See [Sta], theorem 5.5. \( \square \)

This means that we can explicitly construct the immersion representing the intersection
of any two subgroups.

Finally, when the subgroup is of finite index we get extra information about the immersion representing it:

**Lemma 5.** Let $H \leq F$ be represented by an immersion of finite graphs $f : Y \to X$, where $X$ is a rose (i.e. has only one vertex). Then $H$ is of finite index in $F$ iff $Y$ is a covering space of $X$.

**Proof.** See [Sta], remark 7.6 on page 562 \hfill \square

**Remark 2.** Note that this in particular implies that the number of vertices in $Y$ is equal to the index of $H$ in $F$. This is an instance of a more general result from the theory of covering spaces, which will be used again below: if $f : (Y, y_0) \to (X, x_0)$ is a based covering map then the index of $f, \pi_1(Y, y_0)$ in $\pi_1(X, x_0)$ is equal to the number of sheets in the covering.

## 3 Inclusion/Exclusion Hanna Neumann Conjecture

In order to prove theorem [2] we will make use of the following result:

**Lemma 6.** Let $H$ have finite index in $F$. Then

$$\tau(H) = \tau(F)|F : H|$$

Additionally, if $H$ is represented by a covering $g : Y \to Z$ where $\pi(Z, z_0) \cong F$, then

$$\tau(H) = \tau(F)|V(Y)|/|V(Z)|$$

**Proof.** By picking a free basis for $F$, we can represent $F$ by a rose with $rk(F)$ petals, say $X$, and $H$ by an immersion of finite graphs $f : W \to X$.

By lemma 3 $f$ is a covering and $W$ has $|F : H|$ vertices. $W$ has $rk(F)|F : H|$ edges (each vertex of $W$ has valence $2rk(F)$ since $W$ is a covering), so

$$\chi(W) = (1 - rk(F))|F : H|$$

Since $F$ is non-trivial, $\tau(F) = rk(F) - 1$ and $\tau(H) = -\chi(Y)$, which gives the first equality.

To get the second equality, we note that $|F : H| = |V(Y)|/|V(Z)|$ (since both sides are equal to the number of sheets of the covering $g : Y \to Z$). \hfill \square

We are now in a position to prove theorem [2]

**Theorem 2.** Let $H, K$ be finitely generated subgroups of $F$, with $K$ of finite index in $F$. Then

$$\tau(H \vee K) \tau(H \cap K) \leq \tau(H) \tau(K)$$

**Proof.** Firstly, we note that we can take $F$ to be $H \vee K$: we have $K \leq (H \vee K) \leq F$ and so $|F : K| = |F : H \vee K|/|H \vee K : K|$, hence $K$ is of finite index in $H \vee K$.

Identify $F$ with $\pi_1(X, x_0)$, where $X$ is a rose with $rk(F)$ petals. Let $H$ (resp. $K$) be represented by the immersion of finite graphs $g_1 : (Y, y_0) \to (X, x_0)$ (resp. $g_2 : (Z, z_0) \to (X, x_0)$), and construct the pullback $Y \times_X Z$. Let $W$ be the component of the pullback that contains the vertex $w_0 := (y_0, z_0)$. Then, by theorem [4] $\pi_1(W, w_0) \cong H \cap K$.

$$\begin{array}{ccc}
(W, w_0) & \xrightarrow{f_1} & (Y, y_0) \\
|f_2| \downarrow & & \downarrow g_1 \\
(Z, z_0) & \xrightarrow{g_2} & (X, x_0)
\end{array}
\quad
\begin{array}{ccc}
H \cap K & \xrightarrow{(f_1)_*} & H \\
|f_2|_* \downarrow & & \downarrow (g_1)_* \\
K & \xrightarrow{(g_2)_*} & F
\end{array}$$

\[4\]
The IEHNC does not necessarily hold when neither \( H \) nor \( K \) is of finite index in \( H \lor K \).
Indeed, we can show that there is no \( c > 0 \) such that
\[
\tau(H \cap K) \tau(H \lor K) \leq c \tau(H) \tau(K)
\]
holds for all \( H, K \) finitely generated subgroups of \( F \). An example demonstrating this is given below.

**Example 1.** Let
\[
H = \langle x_2 x_1^{-1}, x_1 x_2 x_1^{-2}, \ldots, x_1^{v-2} x_2 x_1^{-(v-1)}, x_3, x_4, \ldots, x_{\ell+2} \rangle
\]
and
\[
K = \langle x_1, x_2 \rangle
\]
We then get
\[
H \cap K = \langle x_2 x_1^{-1}, x_1 x_2 x_1^{-2}, \ldots, x_1^{v-2} x_2 x_1^{-(v-1)} \rangle
\]
\[
H \lor K = \langle x_1, \ldots, x_{\ell+2} \rangle
\]
Figure 2: Counterexample to conjecture \( \Box \) (Guzman)

These are illustrated in the pushout diagram shown in fig. 1 in which \( H \cong \pi_1(Y, y_0) \) and \( K \cong \pi_1(Z, z_0) \). For this choice of \( H \) and \( K \) we have

\[
\frac{\tau(H \cap K)}{\tau(H \vee K)} = \frac{(v - 2)(\ell + 1)}{v + \ell - 2}
\]

Setting \( v = \ell \), we obtain \( (v^2 - v - 2)/(2v - 2) \), so as \( v \to \infty \) the ratio gets arbitrarily large.

Using the graph generation algorithm detailed in section 5, we were able to find a counterexample to Guzman’s “group-theoretic conjecture” as well.

**Example 2.** Let \( F = F(a, b, c, d, x, y) \), and let

\[
H = \langle a, b, x, y^2, yxy^{-1} \rangle \\
K = \langle c, d, y, x^2, xyx^{-1} \rangle
\]

We then get

\[
H \cap K = \langle y^2, yx^2y^{-1}, x^2, yxy^{-1}, x, yxyx \rangle \\
H \vee K = \langle a, b, c, d, x, y \rangle
\]

and so disprove Guzman’s conjecture.

These are illustrated in the pushout diagram shown in fig. 2 (where red and blue correspond to \( x \) and \( y \)).

## 5 Graph-based generation algorithm

In order to investigate the above questions about subgroups of free groups, and to find the first counterexamples (though not the ones presented above), we used GAP to generate random subgroups of free groups.

Historically, random subgroups of free groups were usually generated by the “word-based distribution”, in which \( k \)-tuples of reduced words \( (g_1, \ldots, g_k) \) are chosen in \( F \), with each \( g_i \) having length less than some fixed \( n \). We then consider the subgroup \( H = \langle g_1, \ldots, g_k \rangle \).

Recently, Bassino, Nicaud and Weil proposed the “graph-based distribution”, which generates a random Stallings graph with a fixed number of vertices, and then computes its fundamental group (see section 3 of [Bas]). The algorithm first generates a random \( r(F) \)-tuple of partial injections (by a procedure given explicitly in [Bas]), with an \( a \)-labelled edge going from \( v \) to \( w \) in the Stallings graph if and only if the partial injection corresponding
Figure 3: The percentage of pairs of subgroups out of a sample of 10,000 randomly-generated subgroup pairs that fail to satisfy the IEHNC, against the parameter to the model (either vertices or maximal length of generating word). The sample contained only subgroups whose reduced ranks are strictly positive, as otherwise the IEHNC holds trivially. The subplots are (left-to-right, top-to-bottom): graph-based, word-based with 4 generators, word-based with 6 generators, and word-based with 8 generators. The different trendlines represent different ranks of the ambient group, as indicated in the legend.

to a sends v to w. If the generated graph is not connected, or has leaves other than the basepoint, then it is discarded and a new one is generated (a “rejection algorithm”). The fundamental group of this graph can be found by constructing a spanning tree, and then the GAP package FGA was used to calculate the intersection, join and rank of the subgroups generated.

Both the word- and graph-based distributions were able to generate counterexamples; however, the proportion of examples checked that were counterexamples is significantly higher in the graph-based distribution (fig. 3), as was the proportion of examples which had a non-trivial intersection (fig. 4). Interestingly, the dip in proportion of non-trivial intersections in the graph-based distribution coincides with the peak of the proportion of counterexamples generated, contrary to what might be naively expected; we are unable to explain this.

These figures suggest that the graph-based distribution is better-suited for investigating similar questions using computers.
Figure 4: The percentage of pairs of subgroups out of a sample of 10,000 randomly-generated subgroup pairs that have a non-trivial intersection, against the parameter to the model (either vertices or maximal length of generating word). The sample used was the same as for fig. 3. The subplots and trendlines are also the same as in that figure.

References

[Guz] Rosemary Guzman, Hyperbolic 3-manifolds with k-free fundamental group, Topology and its Applications, volume 173, pp. 142–156, 2014. (arXiv:1201.5911 [math.GT])

[Sta] John R. Stallings, Topology of Finite Graphs, Inventiones mathematicae, volume 71, issue 3, pp. 551–565, 1983

[Neu] Walter Neumann, On intersections of finitely generated subgroups of free groups, Groups—Canberra 1989, Lecture Notes in Mathematics, volume 1456, Springer, Berlin, pp. 161–170, 1990

[Fri] Joel Friedman, Sheaves on Graphs, Their Homological Invariants, and a Proof of the Hanna Neumann Conjecture, Memoirs of the AMS, volume 233, number 1100, 2015. (arXiv:1105.0129 [math.CO])

[Min] Igor Mineyev, Groups, graphs, and the Hanna Neumann Conjecture, Journal of Topology and Analysis, volume 4, issue 1, 2012

[Ken] Richard P. Kent IV, Intersections and joins of free groups, Algebraic & Geometric Topology, volume 9, issue 1, pp. 305–335, 2009. (arXiv:0901.3774 [math.GR])

[Lou] Larsen Louder and D. B. McReynolds, Graphs of subgroups of free groups, Algebraic & Geometric Topology, volume 9, issue 1, pp. 327–335, 2009. (arXiv:0901.3774 [math.GR])

[Bas] F. Bassino, C. Nicaud and P. Weil, Random generation of finitely generated subgroups of a free group, International Journal of Algebra and Computation, volume 18, issue 02, pp. 375–405, 2008. (arXiv:0707.3185 [math.GR])

[Bas2] F. Bassino, A. Martino, C. Nicaud, E. Ventura and P. Weil, Statistical properties of subgroups of free groups, Random Struct. Alg., volume 42, pp. 349–373, 2013. (arXiv:1001.4472 [math.GR])
[GAP] The GAP Group, *GAP – Groups, Algorithms, and Programming*, Version 4.7.8; 2015, http://www.gap-system.org.

[FGA] C. Sievers, *Free Group Algorithms - a GAP package*, Version 1.2.0; 2012, http://www.gap-system.org/Packages/fga.html