The small energy exchange via nucleon recoils in neutrino–nucleon scattering is now supposed to be one of the important factors for successful explosions of core-collapse supernovae (CCSNe), as they can change neutrino spectra through accumulation of a large number of scatterings. In deterministic methods employed for neutrino transport in CCSN simulations, we normally cannot afford to deploy a large enough number of energy bins needed to resolve this small energy exchange, and subgrid techniques are employed one way or another. In this paper, we study quantitatively with the Monte Carlo (MC) method how well such a treatment performs. We first investigate the effects of nucleon recoils on the neutrino spectra and confirm that the average energy is reduced by $\sim 15\%$ for heavy-lepton neutrinos and much smaller amounts for other types of neutrinos in a typical postbounce situation. It is also observed that the nucleon scattering dominates the electron scattering in the thermalization of neutrino spectra in all flavors. We then study possible artifacts that the coarse energy grid may produce in the deterministic methods. In order to mimic the latter calculation, we redistribute MC particles in each energy bin after a certain interval in a couple of ways and study how the results are affected, depending on the energy resolution. We also discuss the possible implications of our results for the deterministic methods.

**Abstract**

The small energy exchange via nucleon recoils in neutrino–nucleon scattering is now supposed to be one of the important factors for successful explosions of core-collapse supernovae (CCSNe), as they can change neutrino spectra through accumulation of a large number of scatterings. In deterministic methods employed for neutrino transport in CCSN simulations, we normally cannot afford to deploy a large enough number of energy bins needed to resolve this small energy exchange, and subgrid techniques are employed one way or another. In this paper, we study quantitatively with the Monte Carlo (MC) method how well such a treatment performs. We first investigate the effects of nucleon recoils on the neutrino spectra and confirm that the average energy is reduced by $\sim 15\%$ for heavy-lepton neutrinos and much smaller amounts for other types of neutrinos in a typical postbounce situation. It is also observed that the nucleon scattering dominates the electron scattering in the thermalization of neutrino spectra in all flavors. We then study possible artifacts that the coarse energy grid may produce in the deterministic methods. In order to mimic the latter calculation, we redistribute MC particles in each energy bin after a certain interval in a couple of ways and study how the results are affected, depending on the energy resolution. We also discuss the possible implications of our results for the deterministic methods.

Unified Astronomy Thesaurus concepts: Core-collapse supernovae (304); Supernova neutrinos (1666)

1. Introduction

Core-collapse supernovae (CCSNe) are violent explosions of massive stars with $M_{\text{ZAMS}} \gtrsim 8 M_\odot$. The explosion is instigated by the gravitational collapse of a central core, which is followed by the formation of a shock wave at core bounce. If the shock wave passes through the central core and propagates through the outer envelopes up to the stellar surface, these envelopes are ejected and a compact remnant is left behind at the center. In numerical simulations, the shock wave stagnates inside the core, and how to get the shock wave out of the core has been explored for a long time but has not been settled yet (Janka 2012; Kotake et al. 2012; Müller 2019, and references therein). One of the favored mechanisms for shock revival is heating by neutrinos emitted from a proto–neutron star (PNS); this is called the neutrino heating mechanism. In multidimensional simulations, nonspherical matter motions, such as convection or the standing accretion shock instability, push up the shock wave and enhance the neutrino heating behind it (Blondin et al. 2003; Iwakami et al. 2008), and shock revival has recently been obtained more often than not (Dolence et al. 2015; Lentz et al. 2015; Melson et al. 2015; Roberts et al. 2016; Skinner et al. 2016; Summa et al. 2016; Takiwaki et al. 2016; Müller et al. 2017; Radice et al. 2017; O’Connor & Couch 2018; Ott et al. 2018; Burrows et al. 2019, 2020; Vartanyan et al. 2019).

Neutrino reaction rates are certainly important for the SN explosion. Bruenn (1985) provided a comprehensive set of neutrino opacities, which have been widely incorporated in SN simulations. Possible corrections to these rates have been investigated for the last 30 yr. For example, the important updates are summarized in Kotake et al. (2018; see also references therein). They have been taken into account in numerical simulations of late (Buras et al. 2006; Lentz et al. 2012; Müller et al. 2012; Kotake et al. 2018; Burrows et al. 2020).

Nucleon recoils in neutrino–nucleon scattering are one of the important updates. Since the energy exchange by the nucleon recoil is only a few percent of the initial neutrino energy owing to the nucleon mass being much larger than the typical neutrino energy $\leq 100 \text{ MeV}$, they were considered to be less important in the spectral formation than electron scattering, in which the energy exchange is much more efficient, and ignored in past SN simulations. The cross section of the nucleon scattering is much larger than that of electron scattering, however, and it is possible that neutrino spectra are changed by the nucleon recoil, especially for heavy-lepton neutrinos, which interact with matter only via neutral current reactions. As a matter of fact, the effects of the nucleon recoil have already been investigated. For example, Keil et al. (2003) used their Monte Carlo (MC) code for the assessment and demonstrated that the average neutrino energy is indeed decreased by the nucleon recoil. The effects have also been studied by dynamical simulations of CCSNe (Rampp & Janka 2002; Buras et al. 2006; Marek & Janka 2009; Hüdepohl et al. 2010; Lentz et al. 2012, 2015; Müller et al. 2012; Pluimbi et al. 2015; Skinner et al. 2016; Radice et al. 2017; Bruenn et al. 2020; Kotake et al. 2018; Burrows et al. 2019; Glas et al. 2019; Rahman et al. 2019; Vartanyan et al. 2019). They found that the nucleon recoil reduces the opacity for neutrinos and accelerates the PNS cooling, which in turn increases neutrino luminosities, thus helping shock revival.

We revisit this issue from a bit different point of view. In most CCSN simulations, one employs a deterministic method for neutrino transport. In so doing, we normally cannot afford to deploy a sufficiently large number of energy bins to resolve...
the small energy exchange by the nucleon recoil. For example, only 20 energy bins are deployed to cover the range of 0–300 MeV in our CCSN simulations with a full Boltzmann neutrino transport (Nagakura et al. 2018, 2019b; Harada et al. 2019), and the widths of these energy bins are larger by an order than the typical energy exchange through the nucleon recoil. Note that although in those simulations, energy subgrids are normally employed to evaluate the transfer rate from an energy cell to the next one (Buras et al. 2006), the resolution problem still remains, since the neutrino distribution in the energy bin is not assumed one way or another. We are not certain how this issue is addressed by other groups in their multi-energy-group transfer computations; unfortunately, there is no mention of it in the literature. The problem may be common, though. We will hence try to quantify the effects of the coarse energy grid and present a possible improvement in this paper.

We perform neutrino transport calculations with our own MC code for a static hydrodynamical background derived from one of our dynamical SN simulations. Note that these MC simulations are free of the energy-resolution problem. It is also mentioned that in this study, we do not use the approximation given by Horowitz (2002) but rather employ the exact reaction rate for nucleon scattering. After validating our MC code, we look into the effects of the nucleon recoil on neutrino spectra, that is, how they are thermalized with radius, comparing their contributions with those from others, particularly electron scattering, in detail. We then assess the energy-resolution issue by introducing energy grids with different numbers of grid points: \( N_E = 10 \) and 20 in our MC calculations. Note that the latter energy grid is exactly the same as the one used in our CCSN simulations with the finite-difference Boltzmann solver. In order to mimic the situation in the deterministic methods, including our Boltzmann solver, we repeatedly redistribute the MC particles in each energy bin by hand in a couple of ways after some periods given by the typical time step of CCSN simulations and see their effects on neutrino spectra.

The organization of the paper is as follows. The new features in our MC code are briefly described in Section 2, particularly the treatment of neutrino–nucleon scattering; several numerical tests for the verification of our new code are presented in Section 3; the effects of the nucleon recoil on neutrino spectra are discussed in Section 4; the possible influence of energy resolution in the finite-difference methods is studied in Section 5; and finally, we give a summary and discussions in Section 6.

2. Numerical Methods of MC Transport

2.1. Stochastic Method versus Deterministic Method

There are two representative approaches to the numerical solution of the radiation transport equation: the deterministic method and the stochastic method. In the former method, such as the \( S_n \) method (see, e.g., Castor 2004), we finite-difference the transport equation in phase space and find solutions for the resultant algebraic equations with given initial and boundary conditions. In the MC method, as a representative of the latter method, we follow the tracks of “sample particles,” which represent a bundle of radiation particles interacting with matter. The interactions are treated probabilistically, and physical quantities, such as the distribution function of radiation, are obtained by collecting individual sample evolutions. Each method has its own advantages and drawbacks.

In the deterministic method, it is normally no problem to treat the entire system that typically encompasses both optically thick and thin regions. The time-dependent coupling with hydrodynamics is also straightforward. On the other hand, the numerical resolution is mainly determined by the number of mesh points one can afford, and, as repeatedly mentioned, the energy grid number cannot be very large, particularly in multispatial dimensions. This may be particularly critical for the treatment of the small energy exchanges in the nucleon scattering, and special care, such as the employment of subgrids, is normally taken (Buras et al. 2006; Bruenn et al. 2020). Recently, Suwa et al. (2019) showed that the Fokker–Planck approximation is also useful. It is noted that even if such a measure is taken, the coarse-resolution problem may remain, since the neutrino energy spectrum is still represented on the coarse energy grid point.

The MC method is mesh-free in principle and hence favorable for multidimensional simulations. Various reactions can be treated in a simple and direct way. In fact, the small energy exchanges in the nucleon scattering pose no problem in this approach. On the other hand, statistical errors inherent to the probabilistic description and slow convergence scaled as \( \sqrt{N} \) are big disadvantages for the MC method. It is normally counted as another demerit that it is difficult to treat the optically thick regime and/or couplings with hydrodynamics (but see Abdikamalov et al. 2012; Richers et al. 2017).

In this study, we employ the MC method for neutrino transport for two reasons. First, we focus on the nucleon recoil, which can be treated most accurately with the MC method, as explained above. Second, we are concerned with the thermalization of the neutrino spectrum via nucleon scattering; hence, we do not need to worry about the high-density region, where the MC method performs poorly. As a matter of fact, neutrinos are already thermalized by other processes well inside the neutrino sphere, and we have only to impose the thermal distribution functions as the inner boundary condition (but see Section 4.2 for more details of our treatment).

2.2. New Features in Our MC Code

Here we summarize some new features of our MC code worth particular mention. Other information on the code is provided in Appendices A–C.

The basics are essentially the same as in previous works (Tubbs 1978; Janka & Hillebrandt 1989; Keil et al. 2003). The main difference in the neutrino transport from the photon transport is the Fermi blocking at the final state. For example, neutrino scatterings are suppressed by the blocking factor \( 1 - f \), where the distribution function is denoted by \( f \). This makes the transport equation nonlinear, and we need to reconstruct the distribution function at an appropriate rate during the MC simulation (see Appendices B.2 and B.3). For that purpose, we employ 20 energy grid points in the range of

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5 Note, however, that we intentionally neglect the weak magnetism in order to focus on the effects of the nucleon recoil in this study. The incorporation of the weak magnetism in our MC code is straightforward, though.

6 We normally discretize matter distributions in the background for the MC simulation. The mesh is also introduced in phase space to reconstruct the distribution function of radiation, which is especially the case in the calculation of the neutrino transport, in which the Fermi blocking should be taken into account (see Section 2.2).
0–300 MeV uniformly in the logarithmic scale and deploy 10 angular grid points at the Gaussian quadrature points.

In our code, four emission and two scattering processes are implemented (see Table 1). Here we focus on the nucleon scattering, the key reaction in this paper. As mentioned earlier, we treat this process as precisely as possible. We do not adopt the approximate formula commonly used but rather employ the exact reaction rate, which is essentially the same as for the electron scattering. We store three quantities relevant for the calculation of the reaction rate, \( P_{\nu}, P_{\bar{\nu}}, \) and \( \sigma_N \), separately as tables for various combinations of density, temperature, and electron fraction. They are functions of \( E_{\nu}, E_{\bar{\nu}} \) and \( \psi \) (or some of them), which are the neutrino energies before and after scattering and the scattering angle, i.e., the angle that the incident and outgoing momenta make, respectively (see Equations (13)–(16) in Section 2.3). The numbers of grid points are 200 for two neutrino energies and 100 for \( \psi \). As for \( E_{\nu} \), we first uniformly discretize the energy range of 0–300 MeV into 20 cells in the logarithmic scale and then uniformly divide each energy cell further into 10 cells in the linear scale. Since \( R_{\nu}(E_{\nu}, E'_{\nu}, \psi) \) has a narrow peak around \( E'_{\nu} = E_{\nu} \), we set the energy range for the grid of \( E'_{\nu} \) as a function of \( E_{\nu} \) depending on \( E_{\nu} \). Finally, we uniformly divide \( [0, 2\pi] \) for \( \psi \) into 100 cells in \( \sqrt{1 - \cos \psi} \) for numerical convenience. The table actually captures the reaction rates only for \( E_{\nu} \lesssim E'_{\nu} \). Those for the opposite case, \( E_{\nu} > E'_{\nu} \), are derived from the former so that the detailed balance relation should be satisfied. The detailed procedure is given in Appendix A.

For a given incident energy \( E_{\nu} \), the scattering angle \( \psi(\theta'_\nu, \phi'_\nu) \) and the energy after scattering \( E'_{\nu} \) are determined probabilistically according to \( P_{\nu} \) and \( P_{E_{\nu}} \), which are derived from \( R_{\nu}(E_{\nu}, E'_{\nu}, \psi) \) as the (normalized) cumulative distributions (see Equations (15) and (16)). The azimuth of the scattering direction \( \Psi \) is determined randomly in the range of \( [0, 2\pi] \). Then, the propagation direction of the neutrinos after scattering in phase space specified by the zenith and azimuth angles measured from the local radial direction, \( (\theta'_{\nu}, \phi'_{\nu}) \), is given from the angles \( (\psi, \Psi) \) by an appropriate coordinate transformation.

Note that the distributions \( P_{\nu} \) and \( P_{E_{\nu}} \) do not include the blocking factor \( 1 - f \) (see Section 2.3). It is taken into account after \( E'_{\nu}, \theta'_{\nu}, \) and \( \phi'_{\nu} \) are determined in this way. We throw dice yet again to get a random number \( z \) in the range of \([0, 1]\). If the condition \( f(r, \nu, \nu') < z < 1 \) is satisfied, we accept this scattering; otherwise, it is “blocked,” and the energy and angles of neutrinos are not changed after all. Note that this procedure correctly reproduces the mean free path in the presence of Fermi blocking. It has the advantage that the reaction table can be independent of the neutrino distribution.

### 2.3. Reaction Rate of Neutrino–Nucleon Scattering

The reaction rate of the neutrino–nucleon scattering is given in essentially the same way as for the electron scattering (Mezzacappa & Bruenn 1993):

\[
R_{\text{rec}}(q, q') = \frac{G_F^2}{2\pi^2} \frac{1}{E_{\nu}E_{\nu'}()} |\beta_1 + \beta_2 + \beta_3 I_2|.
\]

In the above expression, \( G_F = 1.166364 \times 10^{-11} \text{MeV}^{-2} \) is the Fermi coupling constant and \( \beta \)'s are the following combinations of the coupling constants: \( \beta_1 = (C_V - C_A)^2 \), \( \beta_2 = (C_V + C_A)^2 \) and \( \beta_3 = C_A^2 - C_V^2 \). \( I_n \)'s are functions of the energies \( E_{\nu}, E_{\nu'} \) of the incident and outgoing neutrinos and the angle \( \psi \) between their momenta \( q \) and \( q' \):

\[
I_1 = \frac{2\pi T}{D_{\nu} E_{\nu} E_{\nu}^{'2}} (1 - \cos \psi)^2 \exp \left( \frac{E_{\nu} - E_{\nu}{'}}{T} \right) - 1
\]

\[
I_2 = \frac{2\pi T^2}{D_{\nu} E_{\nu} E_{\nu}^{'2}} (1 - \cos \psi)^2 \exp \left( \frac{E_{\nu} - E_{\nu}{'}}{T} \right) - 1
\]

\[
I_3 = \frac{2\pi T^2}{D_{\nu} E_{\nu} E_{\nu}^{'2}} (1 - \cos \psi)^2 \frac{G_0\{y_0\}}{\exp \left( \frac{E_{\nu} - E_{\nu}{'}}{T} \right) - 1}
\]

with

\[
D_{\nu} \equiv E_{\nu} + E_{\nu}^{'2} - 2E_{\nu}E_{\nu}^{'} \cos \psi
\]

\[
A \equiv E_{\nu} + E_{\nu}^{'2} + E_{\nu}E_{\nu}^{'} (3 + \cos \psi)
\]

\[
B \equiv E_{\nu}^{'2} E_{\nu} + E_{\nu}E_{\nu}^{'} (3 - \cos \psi)
\]

\[
C \equiv E_{\nu}^{'2} \left[ E_{\nu}^{'2} - E_{\nu} \cos \psi \right]^2 - \frac{E_{\nu}^{'2}}{2} (1 - \cos^2 \psi)
\]

\[
- \frac{1}{2} (1 + \cos \psi \frac{m_{\nu}^2}{2} \Delta^{2})
\]

Here, \( y_0 = E_{\nu}T, \eta = \mu_{\nu}T, \eta' = \eta + (E_{\nu} - E_{\nu}{'})/T \) and \( G_\eta(y) \equiv F_\eta(\eta' - y) - F_\eta(\eta - y) \), in which the Fermi integral

| Reactions | Base | rl1 | rl2 |
|-----------|------|-----|-----|
| Electron–positron pair annihilation | pair | \( e^- + e^+ \rightarrow \nu + \bar{\nu} \) | ✓ | ✓ | ✓ |
| Bragg scattering | brems | \( N + N \rightarrow N + N + \nu + \bar{\nu} \) | ✓ | ✓ | ✓ |
| Electron capture | ecp | \( p + e^- \rightarrow n + \nu \) | ✓ | ✓ | ✓ |
| Positron capture | psc | \( n + e^+ \rightarrow p + \bar{\nu} \) | ✓ | ✓ | ✓ |
| Nucleon scattering | nsc (Bruenn) | \( N + \nu \rightarrow N + \nu \) | ✓ | ✓ | ✓ |
| Nucleon scattering | nsc (rec) | | | | |
| Electron scattering | esc | \( e^- + \nu \rightarrow e^- + \nu \) | ✓ | ✓ | ✓ |
| Positron scattering | psc | \( e^+ + \nu \rightarrow e^+ + \nu \) | ✓ | ✓ | ✓ |

**Note:** The base model incorporates the subset of neutrino reactions normally considered in dynamical SN simulations. The nucleon recoil in the nucleon scattering is taken into account in model rl, whereas the electron/positron scattering is also included in model rl2.
$F_{\rho}(z)$ is defined as

$$F_{\rho}(z) = \int_{0}^{\infty} \frac{x^n}{e^{x-z} + 1} \, dx,$$

and $E_{N0}$ is expressed as

$$E_{N0} = \frac{E_{\nu} - E'_{\nu}}{2} + \frac{\Delta}{2} \sqrt{1 + \frac{2m_N^2}{E_{\nu}E'_{\nu}(1 - \cos \psi)}}. \quad (10)$$

Assuming that the energy exchange is much smaller than the neutrino energy before scattering, $\Delta E/E_{\nu} \ll 1$, and the nucleon mass is infinitely large, $m_N \rightarrow \infty$, one reproduces the reaction rate given by Bruenn (1985), which is commonly incorporated in SN simulations,

$$R_{\text{Bruenn}} = \frac{2\pi G_F^2}{\hbar c} \eta_{NN} \delta(E_{\nu} - E'_{\nu}) \times \left[ (h_{\nu}^N)^2 + 3(h_{\nu}^N)^2 + [(h_{\nu}^N)^2 - (h_{\nu}^\nu)^2] \cos \psi \right]. \quad (11)$$

$\eta_{NN}$ is defined as

$$\eta_{NN} \equiv \int \frac{2d^3p_N}{(2\pi)^3} F_N(\tilde{E})[1 - F_N(\tilde{E})], \quad (12)$$

where $F_N(\tilde{E}) = 1/[1 + \exp(\tilde{E} - \mu_N)/T]$ is the Fermi–Dirac distribution of nucleons with the nonrelativistic energy $\tilde{E} = p_N^2/2m_N$.

The exact and (Bruenn’s) approximate total cross sections are obtained by integrating the corresponding reaction rates $R_* = R_{\text{rec}}$, $R_{\text{Bruenn}}$,

$$\sigma_N = \int R_* d \cos \psi, \quad (13)$$

with

$$R_* = \frac{1}{(2\pi)^3} \int 2\pi E_{\nu}^4 R_{\nu} dE_{\nu}. \quad (14)$$

The quantities after scattering $E'_{\nu}$, $\theta'_{\nu}$, and $\phi'_{\nu}$ are determined as follows. We first determine the scattering angle $\psi$ according to the normalized cumulative distribution:

$$P_\psi(\cos \psi; E_{\nu}) = \int_{-1}^{\cos \psi} \int 2\pi E_{\nu}^4 R_{\nu} dE_{\nu} d \cos \psi.$$  

$$= \frac{1}{\int_{-1}^{1} \int 2\pi E_{\nu}^4 R_{\nu} dE_{\nu} d \cos \psi}. \quad (15)$$

For the derived $\psi$, the energy after scattering is determined in the same way according to the following normalized cumulative distribution:

$$P_{E'_{\nu}}(E'_{\nu}; \cos \psi_k, E_{\nu})$$

$$= \frac{\int_{E'_{\nu}^{\min}}^{E'_{\nu}^{\max}} 2\pi E_{\nu}^4 R_{\nu} dE_{\nu} d \cos \psi_k}{\int_{E'_{\nu}^{\min}}^{E'_{\nu}^{\max}} 2\pi E_{\nu}^4 R_{\nu} dE_{\nu} d \cos \psi_k}.$$  

The minimum and maximum energies $E'_{\nu}^{\min}$ and $E'_{\nu}^{\max}$ in the integration are determined so that the reaction rates there should be $10^{-5}$ times smaller than the maximum rate.

The treatments of other reactions are summarized in Appendices B and C.

### 3. Code Verification

In this section, we present some of the test calculations we conducted for the verification of our MC code. We first compare the results obtained with our MC code and those with another deterministic Boltzmann solver based on finite difference (Nagakura et al. 2014, 2017, 2019a) in Section 3.1. The numerical treatment of the detailed balance in the neutrino–nucleon scattering, a key ingredient in this paper, is then verified in the computation of the thermalization of the neutrino spectrum via this process in a single spatial zone in Section 3.2.

#### 3.1. Comparison with the Finite-difference Boltzmann Solver

We verify our MC code with another Boltzmann solver developed by Nagakura et al. (2014, 2017, 2019a), which is a finite-difference code based on the $S_N$ method. We take a similar strategy to that in Richers et al. (2017): we employ a snapshot at 100 ms after bounce taken from our realistic one-dimensional dynamical SN simulation with $M_{\text{AMS}} = 11.2 M_\odot$ (Nagakura et al. 2019a). Fixing the matter distribution so obtained, we run the two neutrino transport codes to obtain a steady neutrino distribution. Note that the same background model is used for the later studies. The top three panels in Figure 1 show the radial profiles of density, temperature, and electron fraction in this model. We focus on two regions: region I ($r = 20–25$ km) and region II ($r = 28–34$ km), shaded in yellow. In the former region, neutrinos are nearly in thermal equilibrium with matter, whereas in the latter region, they get gradually out of equilibrium as the density decreases and their distribution starts to become anisotropic.

The set of neutrino reactions employed in this comparison is referred to as “base” in Table 1. Note that the nucleon recoil is not included. We initially deploy $2 \times 10^6$ sample particles and determine the weight of the sample particles from the actual number of neutrinos. The weight is unchanged throughout the calculations.

We adopt a time step of $dt_l = 10^{-7}$ s, which is the same as that for updating the neutrino distribution function in this case (see Appendix B). We adopt exactly the same spatial grid as employed in the SN simulation and assume that hydrodynamical quantities are constant in each cell. In order to set the inner and outer boundary conditions, we introduce ghost cells at both sides of the active region and deploy sample particles uniformly there according to the distribution functions imposed at the boundaries. Turning off all interactions with matter, we follow the motions of these sample particles in the ghost cells to make the fluxes at the boundaries as close to the prescribed values as possible.

We follow the time evolution of neutrino distribution by MC simulations until the system settles down to a nearly steady state, in which the total number of sample particles does not change more than 0.5% from a certain value for the total number of sample particles. We then take the average over 8000 time steps ($8 \times 10^{-4}$ s) after the steady state is achieved to reduce the statistical error. Note that neutrinos with $E_{\nu} \gtrsim 5$ MeV experience scatterings with nucleons more than 10 times during this period. This may be understood from the total mean free path for the nucleon scattering in the bottom panel of Figure 1. This extra period is indeed longer than the timescale on which the above small fluctuations occur. After

Note that we use the exact reaction rate $R_{\text{rec}}$ for the cross sections of nucleon scattering $\sigma_N$ in the bottom panel of Figure 1.
the extra period, we evaluate the energy spectra of the neutrino number densities from the mean distribution function.

Figure 2 shows the comparison of the energy spectra $dN(r, E_{\nu})/dE_{\nu}$ for $\nu_e$ (top), $\bar{\nu}_e$ (middle), and $\nu_x$ (bottom) between the MC code and the finite-difference Boltzmann solver. The left panels exhibit the results in region I. Colored lines correspond to the results of the MC calculation for $\cos \theta_{\nu} = 0.973$ at different radii. As mentioned earlier, we use 20 energy grid points and 10 angular grid points for these plots. Counting the number of sample particles in each cell and dividing it with the cell volume, we assign the derived value to the midpoint of the cell. This procedure is identical to that employed by the finite-difference Boltzmann solver, a fact that facilitates comparison. Gray symbols in the figure present the results obtained with the finite-difference Boltzmann solver. We find a good agreement between the two methods.

In the right panels, on the other hand, we pick up the neutrino spectra at $r = 34$ km in region II. Different colors denote different values of $\cos \theta_{\nu}$. One can see that the angular distributions of neutrinos start to become forward-peaked, with $\nu_e$ being the most anisotropic, as expected. The neutrino spectra given by our MC code are again in excellent agreement with those by the finite-difference Boltzmann solver in this slightly outer region.

3.2. Thermalization by Nucleon Recoils

In this paper, we focus on the effects of the nucleon recoil, particularly the thermalization of neutrinos. In so doing, the detailed balance should be satisfied in the numerical simulations:

$$R_{\text{rec}}(E_{\nu}, E'_{\nu}, \cos \theta_{\nu}) f_{eq}(E_{\nu})(1 - f_{eq}(E'_{\nu})) = R_{\text{rec}}(E'_{\nu}, E_{\nu}, \cos \theta_{\nu})(1 - f_{eq}(E_{\nu})) f_{eq}(E'_{\nu}).$$

(18)

This is ensured simply by calculating the reaction rates for $E_{\nu} < E'_{\nu}$ and obtaining those for the other case, $E'_{\nu} > E_{\nu}$, from the former so that the detailed balance should be guaranteed. We tabulate the reaction rates so obtained for the thermodynamical conditions encountered in the matter background. The detailed procedure is described in Appendix A.

Ignoring the spatial dependence, this time we perform a one-zone calculation with $T = 9.96$ MeV and the chemical potential of neutrons, $\mu_n = 921$ MeV. We follow the thermalization of neutrino spectra only by neutrino-neutron scatterings in this test. We inject $10^6$ sample particles with the monochromatic energy, $E_{\nu} = 30$ MeV, as an initial condition. Figure 3 shows the time evolution of the neutrino spectrum. For this plot, we employ a uniform-energy mesh with 50 grid points instead of the 20 mesh points employed in the situation. Different colors correspond to different time steps. The expected thermal spectrum (red dotted) is obtained from the Fermi–Dirac distribution $f_{eq}$ as

$$\frac{dN(E_{\nu})}{dE_{\nu}} = \frac{1}{(2\pi\hbar c)^3} \frac{4\pi E_{\nu}^2}{1 + \exp\left(\frac{E_{\nu} - \mu_n}{T}\right)}.$$

(19)

Since the total number of neutrinos $N$ is conserved in this calculation, the chemical potential of neutrinos $\mu_\nu$ is determined by $N$ and $T$. In this test, we set $N = 10^{20}$, which leads to $\mu_\nu = -1.75$ MeV. We find that the neutrino spectrum indeed approaches this distribution, and they are in good agreement with each other at the end ($t = 9.95 \times 10^{-4}$ s; see the red dotted and black solid lines in Figure 3). This lends confidence to our treatment of the detailed balance in the nucleon scattering. In the bottom panel of the same figure, we also give the mean free time of neutrinos $t_{\text{mfp}}$ for the neutrino–neutron scattering:

$$t_{\text{mfp}} = \frac{\lambda_n}{c} = \frac{\lambda_n}{\sigma_{\nu n} c}.$$

(20)

The exact reaction rate $R_{\text{rec,n}}$ is used for the cross section $\sigma_{\nu n}$ in the evaluation. We find that the computation time is long...
enough to guarantee the thermalization, except at the lowest end of the energies, where the scattering occurs only rarely.

4. Impacts of Nucleon Recoils on Neutrino Spectra

We apply the MC code to the thermalization of energy spectra as neutrinos propagate outward in the postshock region. We pay particular attention to the relative importance of various processes, including the nucleon recoil for different neutrino flavors.

4.1. Isoenergy Limit of Nucleon Scattering

Before looking into the individual contributions of different processes to the thermalization of neutrino spectra in detail, it may be worthwhile to see the isoenergy limit of the nucleon scattering, which was derived by Bruenn (1985) and employed in most SN simulations in the past. The Bruenn rate (Equation (11)) can be derived from the generic expression for the nonisoenergetic scattering (Equations (1)–(10)) by taking a limit of $m_N \to \infty$ and $\Delta E/E_\nu \to 0$.

The top panel of Figure 4 shows the dependence on the proton mass of the reaction rate for the proton scattering. The vertical axis is the reaction rate $R_{\text{rec},p}$, and the horizontal axis is the ratio of the energy change to the initial energy, $\Delta E/E_\nu$. It is clear that as the proton mass increases, the energy exchange becomes smaller, making the reaction rate more sharply peaked at $\Delta E/E_\nu = 0$, the isoenergetic scattering limit. Note that in these calculations of $R_{\text{rec},p}$, we modify the chemical potential of protons so that the number density should be unchanged.

In addition to the energy redistribution, the effect of the proton recoil is the reduction of the reaction rate at high energies and/or backward scattering angles, as shown in the middle and bottom panels of Figure 4, respectively, for $T = 5.85$ MeV, $\rho = 10^{12}$ g cm$^{-3}$, and $\mu_p = 907$ MeV. We find that the latter effectively modifies the angular dependence of the nucleon scattering, making it less backward-peaked.
4.2. Sensitivity of Neutrino Spectra on Recoils in the Nucleon Scattering

We assess the impact of the nucleon recoil by comparing the energy spectra in MC simulations with/without the recoil on a realistic CCSN matter background. We run the MC code to obtain steady-state solutions of the neutrino transport on the static but nonuniform matter background given by the same progenitor model employed in the code verification (see Figure 1). The inner and outer boundaries are put at 20 and 100 km, respectively. The neutrino fluxes coming in from these boundaries are obtained automatically by setting the neutrino distribution functions on the ghost mesh points to the ones derived from the SN simulation.

As the first comparison, we adopt two sets of neutrino reactions: “base” and “r1,” given in Table 1. In the r1 set, the nucleon recoil is taken into account, in addition to the base set. For both cases, we use $2 \times 10^6$ sample particles and take $dt = 10^{-7}$ s for the time step of updating neutrino distribution functions.

Figure 5 shows the energy spectra of neutrino number densities obtained in the two calculations. For these plots, we take a volume average of the sample particles in each energy bin and assign the derived value to the midpoint of the energy bin. In order to reduce the statistical error, we take the average over the extra 8000 time steps after the steady state is achieved. Colors denote the radii at which the spectra are evaluated, and solid and dotted lines show the results for the r1 and base sets, respectively. The spectra of $\nu_e$ (top) and $\bar{\nu}_e$ (middle) do not change with the inclusion of the nucleon recoil, whereas high-energy $\nu_x$ is depleted and low-energy $\nu_x$ is increased due to downscattering by nucleons (bottom). As a result, the average energy of $\nu_x$ is reduced by $\sim15\%$ at the outer boundary, as shown in Figure 6. Note that the maximum difference is $\sim30\%$ at $r \sim 40$ km. The number density of $\nu_x$ is also decreased by $\sim7\%$ at the outer boundary. This is due to the opacity reduction caused by the nucleon recoil itself, as well as by the decrease of average energy.

In order to understand the different responses to the inclusion of the nucleon recoil among different flavors, we show the rates per volume for different reactions as a function of radius in the left panels of Figure 7. Line colors denote the different reactions. The vertical axis shows the number of neutrinos that experience each neutrino reaction per unit time and volume, denoted by $n_x$. We take the same average procedure here for $n_x$ as for the spectra in Figure 5. One finds in the top panel that the
electron capture dominates the other reactions for $\nu_e$. This is the reason why the spectrum is not changed by the inclusion of the nucleon recoil. Note that the number of nucleon scatterings itself is smaller than that of electron captures by a factor of $\sim 5$.

The dominant reaction for $\nu_x$, on the other hand, is the nucleon scattering in the absence of charged-current reactions (see the bottom panel). As a result, the spectrum is pinched by the inclusion of the nucleon recoil. For $\bar{\nu}_e$ (middle), the number of nucleon scatterings is larger than that of the other reactions. Although this seems at first glance to contradict the previous result that the spectrum of $\bar{\nu}_e$ is not affected by the nucleon recoil, this is simply due to the small energy exchange in the nucleon scattering.

The right panels of Figure 7 demonstrates this. They show the energies exchanged between neutrino and matter for different reactions. The vertical axis is the exchanged energy per unit time and volume and denoted by $E_s$. In the figure, the pair annihilation and bremsstrahlung are put together into "others." We find for $\nu_e$ (top) and $\nu_x$ (bottom) that the orders of lines in the right panels are unchanged from those in the corresponding left panels. For $\bar{\nu}_e$ (middle), on the other hand, the positron capture is dominant over the nucleon scattering in terms of the energy exchange, although the opposite is true for the reaction rates. This is, as mentioned above, due to the small energy exchange in the individual scattering on nucleons. As a result, the nucleon recoil affects the spectrum of $\nu_x$ but not $\bar{\nu}_e$. Note that our result is qualitatively consistent with the result in Keil et al. (2003).

We have so far omitted electron/positron scatterings on purpose. The energy exchange per scattering for electrons and positrons is much larger than that for nucleons because of the smaller mass of the former, $m = 0.511 \text{ MeV}$. In the top panel of Figure 8, we compare the energy exchanges between the two scatterings for the incident neutrino energy $E_\nu = 25 \text{ MeV}$ and the scattering angle $\cos \theta = -1.0$. Note that we show the case of $\bar{\nu}_e$ for the electron/positron scattering. The vertical and horizontal axes are the normalized reaction rate and the ratio of the energy exchange to the incident energy, respectively. It is clear that the peak of the reaction rate for the electron/positron scattering is dislocated from the isoenergy condition $\Delta E/E_\nu = 0$ by a large amount, which means that neutrinos give more energy to electrons/positrons than nucleons, on average. In the bottom panel of Figure 8, we show the total cross sections for the two scatterings as a function of the incident neutrino energy. For the electron/positron scattering, we give them separately for the three neutrino flavors. We calculate these cross sections at $T = 5.85 \text{ MeV}$, $\rho = 1.01 \times 10^{12} \text{ g cm}^{-3}$, $Y_e = 0.10$, $\mu_p = 907 \text{ MeV}$, $\mu_n = 924 \text{ MeV}$, and $\mu_e = 19.6 \text{ MeV}$. One finds that
the nucleon scattering has larger cross sections at $E_{\nu} \gtrsim$ a few MeV because of the different energy dependences of the total cross sections: $\sigma \propto E_{\nu}^2$ for the nucleon scattering but $\sigma \propto E_{\nu}$ for the electron/positron scattering.

We now rerun the MC code, this time with the e1 set of neutrino reactions given in Table 1, in which the electron/positron scattering is taken into account in addition to the r1 set. The number of sample particles and the time step $dt_f$ are the same as those in the previous calculations. This run is meant to see the relative importance of the two scatterings in thermalizing the neutrino spectra.

Figure 9 is the same as the right panels of Figure 7 except for the addition of the electron/positron scattering, shown in orange. We find that apart from the charged-current reactions for $\nu_e$ and $\bar{\nu}_e$, the accumulation of small recoils in the nucleon scattering is more important than a smaller number of large recoils in the electron/positron scattering in the thermalization of neutrinos, at least for this particular model. Indeed, we find that the energy spectra of neutrinos are almost identical to those without the electron/positron scattering$^8$ (see Figure 5). Note also that Thompson et al. (2000) calculated the thermalization of $\nu_x$ in uniform background matter with their own MC code and reached the same conclusion.

5. Implications for the Numerical Implementation of Nucleon Recoils in the Finite-difference Method

The nucleon recoil affects the neutrino luminosity and dynamics of explosions, as discussed in the literature (Rampp & Janka 2002; Buras et al. 2006; Marek & Janka 2009; Hüdepohl et al. 2010; Lentz et al. 2012; Müller et al. 2012; Lentz et al. 2015; Pllumbi et al. 2015; Skinner et al. 2016; Radice et al. 2017; Bruenn et al. 2020; Kotake et al. 2018; Burrows et al. 2019; Glas et al. 2019; Rahman et al. 2019; Vartanyan et al. 2019).

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$^8$ Note that the cross section of the electron/positron scattering for low-energy neutrinos ($\sim$ a few MeV) is higher than that of the nucleon scattering. On the other hand, those low-energy neutrinos have already decoupled from matter; hence, the energy spectra of neutrinos at low energy are less sensitive to the change of the cross sections.
Although one of the deterministic, finite-difference methods is normally employed for neutrino transport in the CCSN simulations, we cannot afford to deploy a sufficiently large number of energy bins to resolve the small energy exchange via the nucleon recoil in each scattering. Some subgrid technique is hence commonly adopted (Buras et al. 2006). Note that how to treat those neutrinos that have crossed the energy-cell boundary via the nucleon scattering in each energy bin is another concern, since it could lead to numerical diffusions in energy. Unfortunately, we cannot find any statement on this issue in the literature. Hence, in this section, we will conduct some experimental MC runs to investigate the possible consequences of some numerical implementations of the nucleon recoil in deterministic transport schemes such as the SN method. We quantify the effects of coarse energy grids on the energy spectrum of neutrinos and present a possible improvement.

When the cell width of the energy grid is much larger than the typical value of the energy exchange in the scattering, it is certainly inappropriate to use the cell-center values of energies and neutrino distribution functions to evaluate the rate of the scattering that transfers neutrinos in one energy cell to another adjacent to it. This is because only those neutrinos existing in the close vicinity of the energy-cell boundary can cross over it to the next cell. In the deterministic method adopting such an energy grid, it is necessary to somehow reconstruct the neutrino distribution inside the energy bin to estimate the neutrino populations near the cell boundary and calculate the scattering rate based on them; once the neutrinos enter the next energy cell, they are mixed with others in the same cell, and their individual energies are forgotten. This last mixing may result in numerical diffusions in the energy space, which is the focus in the following. In this study, we mimic the above situation in the deterministic method by periodically introducing artificial redistributions of sample particles in each energy bin in our MC code; for every $dt_f$ during the normal MC simulation, we reconstruct the neutrino distribution in each energy bin, as
explained in the next paragraph in detail, and then redistribute the sample particles in the bin according to the reconstructed distribution.

We adopt three prescriptions of the redistribution in each energy bin: “flat,” “linear+Ncons,” and “linear+NEcons.” The first one is the simplest but coarsest reconstruction, in which we homogenize the distribution of sample particles in each energy bin. In the second and third cases, we introduce a linear distribution as follows:

$$\frac{dN_{T,k}}{dE} = a_k E + b_k,$$

with the inclination, intercept, and total number of sample particles in the $k$th energy bin $a_k$, $b_k$, and $N_{T,k}$, respectively. In both cases, we impose the condition that the number of MC particles should be unchanged in each bin. For the second case, we determine $a_k$ as the weighted average of two inclinations, $a_1$ and $a_2$:

$$a_k = a_1 \frac{E_{v,k+1} - E_{sm,k}}{E_{v,k} - E_{v,k-1}} + a_2 \frac{E_{sm,k} - E_{v,k-1}}{E_{v,k} - E_{v,k-1}},$$

$$a_1 = \frac{N_{T,k}}{(E_{v,k+1} - E_{v,k})},$$

$$a_2 = \frac{N_{T,k}}{(E_{v,k} - E_{v,k-1})},$$

with the midpoint energy of the $k$th energy bin $E_{sm,k}$; $b_k$ is obtained from the number conservation:

$$N_{T,k} = \int_{E_{v,k-1}}^{E_{v,k}} (a_1 E + b_1) \, dE.$$  \hspace{1cm} (25)

For the third case, in addition to Equation (25), we impose the energy conservation in each energy bin; that is, the total energy of the $k$th energy bin $E_{T,k}$ should be unchanged by the redistribution,

$$E_{T,k} = \int_{E_{v,k-1}}^{E_{v,k}} (a_1 E + b_1) E \, dE,$$  \hspace{1cm} (26)

to determine $a_k$ and $b_k$ for the third case. Note that we derive $N_{T,k}$ and $E_{T,k}$ just by counting the number of particles and summing up their energies in the $k$th energy bin before redistribution, respectively.

We introduce two energy grids with different numbers of cells, $N_{E_v} = 10$ and $20$, to cover the energy range of 0–300 MeV. Note that the latter is exactly the same as the energy grid employed in the Boltzmann solver by Nagakura et al. (2019a). We focus on the spectra of $\nu_x$, which are affected most by the inclusion of the nucleon recoil, as shown in the previous sections. The artificial redistributions of sample particles in each energy bin are repeated on the timescale of a single time step in CCSN simulations to mimic the situation in the deterministic method. We adopt as a matter background the same hydrodynamical model as that employed in the previous sections (see Figure 1) but with the restricted range of $20 \leq r/\text{[km]} \leq 100$, and we deploy the same number of sample particles and use the same $dt_\nu$ as well. We run the MC code, starting with the spectrum obtained in the previous steady-state calculations but with the redistribution implemented. The r1 set of neutrino reactions is adopted. We take the average of the distribution function over the extra 8000 time steps after a new steady state is achieved.

Figure 10 demonstrates the three different reconstructions of the neutrino spectrum described above for the two energy grids with $N_{E_v} = 20$ (top panel) and 10 (bottom panel). The gray line is an original spectrum obtained by the normal MC calculation without the artificial redistribution of sample particles in each energy bin. The colored lines denote the spectra reconstructed as explained above. In the case of $N_{E_v} = 20$, the linear+Ncons and linear+NEcons models give similar distributions (see the green and red lines), whereas they are more deviated from each other for $N_{E_v} = 10$. This difference turns out to be important later.

Figure 11 shows the energy spectra obtained as a result of the new simulations with the artificial redistributions (upper half) and their deviations $\Delta$ from the original, supposedly correct ones (lower half) at $r = 20$ (top), 60 (middle), and 100 (bottom) km. The color coding is the same as before. In the case of $N_{E_v} = 20$, presented in the left panels, we find that the flat redistribution produces errors as large as $\sim 20\%$ near the average energy (see the orange lines). This is because a larger number of sample particles can get across the boundaries of the energy bins and move to the next cells thanks to the redistribution and may be regarded as the overestimation of the energy exchange via the nucleon recoil.
In the two linear redistribution models (green and red lines), the error is reduced to a few percent. We find smaller differences in the former model at lower and higher energies, whereas the latter model reproduces the peak of the neutrino spectra better. It is difficult to say which of the two is better from these results. If we reduce the number of energy grids to $N_{E} = 10$, however, their results differ more from each other. The error $\Delta$ in the linear+$N_{E}\text{cons}$ model increases but still stays within 10% for almost all energies, even at large radii. The spectra for the linear+$N_{E}\text{cons}$ model, on the other hand, deviate from the correct ones by $\sim 20\%$. This difference is a consequence of the difference in the redistributions, which we found becomes remarkable when the energy grid gets coarser. Note that $N_{E} = 10$ is not very low compared to that employed in current CCSN simulations.

We also investigate the effects of the energy redistribution on the first angular moment $F'$, defined as

$$\frac{dF'(r, E_{\nu})}{dE_{\nu}} = \frac{2\pi c}{(\hbar c)^3} \int E_{\nu}^3 \cos \theta_{\nu} f(r, E_{\nu}, \cos \theta_{\nu}) d\cos \theta_{\nu}. \quad (27)$$

Figure 12 shows the energy spectra of $F'$ at the same radii as those in Figure 11. The left and right panels exhibit the results with

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9 We conducted a resolution study on the initial number of MC sample particles for the linear+$N_{E}\text{cons}$ model and confirmed that the reconstructed distributions are not sensitive to MC noises for the current number of MC particles.
We again find that the linear + NEcons model reconstructs the original energy spectra of $F_{\nu}$ better than the other models. We had better hence impose, if possible, the energy conservation in reconstructing the neutrino distribution in each energy bin to incorporate the nucleon recoil in neutrino transport accurately, particularly when the energy resolution is not high. This will be possible if not only the number but also the energy in each energy bin is stored in the transport.

6. Summary and Discussions

The nucleon recoils in the neutrino–nucleon scattering are one of the important factors for the dynamics of SN explosions and neutrino observations, and their effects have already been investigated in the literature. In these studies, the deterministic methods are normally adopted for neutrino transport. In so doing, we cannot afford to deploy a sufficiently large number of energy bins to resolve the small energy exchange in the nucleon recoil. In this paper, we have performed neutrino transport calculations with our own MC code for a static matter background derived from a dynamical SN simulation to quantify the effects of the coarse energy grid and suggest a possible improvement in the subgrid modeling.

We first conducted two test calculations for the verification of our MC code. We compared steady-state solutions obtained with the MC code and those with our finite-difference Boltzmann solver, in which we employ a matter background computed from one of our recent CCSN simulations (Nagakura et al. 2019a). The nucleon recoil was ignored in this comparison. We demonstrated that the two results are in excellent agreement with each other. In order to confirm the detailed balance in our treatment of the nucleon recoil, we did a one-zone calculation of the thermalization of the neutrino spectrum via neutron scattering. This is ensured by calculating the reaction rates only for $\xi n n E$ and deriving those for $\xi n n E' E$ from them via the detailed balance relation. We confirmed that the neutrino spectrum approaches a thermal distribution, as expected.

We then ran the MC code to compute the thermalization of energy spectra as neutrinos propagate outward in the postshock region. We first studied the large proton mass limit of the proton scattering, in which it becomes isoenergetic, and made clear three important effects of the recoil on its reaction rate: the broadening of neutrino spectra, the reduction of the cross section, and the change of the angle dependence of the reaction rate. We then reapplied the MC code to the neutrino transport calculations on the same static matter background as that employed in the code verification but with the nucleon recoil being incorporated this time.
We found a significant change in the spectra of $\nu_e$ with the inclusion of the nucleon recoil. High-energy $\nu_e$ is depleted, while low-energy $\nu_e$ is increased due to downscattering, and their average energy is reduced by $\sim 15\%$. The spectra of $\nu_e$ and $\bar{\nu}_e$, on the other hand, do not change much with the inclusion of the nucleon recoil.

These different responses to the nucleon recoils among different flavors of neutrinos are explained as follows. The number of nucleon scatterings is smaller than that of electron captures by a factor of $\sim 5$ for $\nu_e$, whereas the dominant reaction for $\nu_e$ is the nucleon scattering. For $\bar{\nu}_e$, the number of nucleon scatterings is larger than that of other reactions, which seems to contradict the result that the spectrum of $\bar{\nu}_e$ is not changed by the nucleon recoil. The reason is simply that the energy exchange in the nucleon scattering is much smaller.

Next, we incorporated the electron/positron scattering in the MC code and compared the contributions to thermalization between the two scatterings. The energy exchange per scattering for the electron/positron scattering is much larger than that for the nucleon scattering because of the smaller mass of the former, $m_e = 0.511$ MeV, whereas the cross section of the latter is larger than that of the former at $E_e \gtrsim 1$ few MeV. We found that the accumulation of small recoils in the nucleon scattering is more important than a smaller number of large recoils in the electron/positron scattering in the thermalization of neutrinos, at least for this particular model.

We then conducted some experimental MC runs to investigate the implications for the numerical implementation of the nucleon recoils in the Boltzmann solver, in which they are included in our axisymmetric CCSN simulations with the Boltzmann solver (Nagakura et al. 2018, 2019b). We adopted two energy grids with different numbers of grid points: $N_{E_e} = 10$ and 20. Note that the latter grid is exactly the same as that employed in our asymmetric CCSN simulations with the Boltzmann solver (Nagakura et al. 2018, 2019b).

We ran the MC code with these redistribution schemes implemented for the same matter background as that in the previous calculations without the redistribution of sample particles. We found that the neutrino spectra in the flat model deviate from the correct one by $\sim 20\%$, even in the high energy resolution $N_{E_e} = 20$, whereas the difference is reduced to a few percent in the linear+NCons and linear+NECons models. Both of the latter two models can reconstruct the original spectra equally accurately for $N_{E_e} = 20$. If we reduce the number of energy grid points to $N_{E_e} = 10$, however, their results differ from each other. Although the errors in the linear+NECons model are still within $10\%$ at almost all energies, even in the outer region, they rise up to $\sim 20\%$ in the linear+NCons model. We also find the same conclusion for the energy spectra of the first angular moment. Since the energy resolution typically employed in the deterministic methods is rarely higher than the $N_{E_e} = 20$ case in this paper, it is recommended to somehow keep track of not only the number but also the energy in each energy bin and use the number and energy conservations to reconstruct the subgrid distributions of neutrinos when dealing with the small energy exchange in the nucleon recoil.

Our next task is to actually implement these subgrid modelings into the Boltzmann solver, in which they are currently employing the reaction rate of Bruenn (1985) for the nucleon scattering, and to perform CCSN simulations. This will enable us to quantitatively discuss the effects of the nucleon recoil, particularly its energy-resolution dependence, on the dynamics of explosion and PNS cooling. It should also be important from the observational point of view.

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Appendix A

Determination of Neutrino Energy after Scattering

In our code, we employ the reaction rate tables for the nucleon and electron/positron scatterings. In order to ensure the detailed balance between the direct and inverse reactions between the initial and final states with the neutrino energies $E_\nu$ and $E'_\nu$, respectively, we use the following method.

1. $E_\nu \leq E'_\nu \leq E_{\text{max}}$. The reaction rates for upscatterings $E_\nu < E'_\nu$ are included in the table, and we get $E'_\nu$ interpolating data in the table. We use the modified reaction rate $\mathcal{R}$ instead of $R_{\text{rec}}$ for convenience,

$$\mathcal{R}(E_\nu, \Delta E, \cos \psi) = R_{\text{rec}}(E_\nu, E'_\nu, \cos \psi) \exp \left( -\frac{E_\nu}{T} \right),$$

with the energy difference $\Delta E \equiv E'_\nu - E_\nu$. The modified reaction rates are described by the reaction rates in the table $R_{ij} \equiv \mathcal{R}(E_\nu, \Delta E, \cos \psi)$ with the on-grid neutrino energy $E_i$ employed in the table $E_1 \leq E_i \leq E_2$ and $E'_1 \leq E'_\nu \leq E'_2$ and the energy difference $\Delta E_{ij} \equiv E'_i - E_i$.

$$\mathcal{R}(E_\nu, \Delta E, \cos \psi) = q_1 k_1 R_{11} + q_2 k_2 R_{12} + q_3 k_3 R_{21} + q_4 k_4 R_{22},$$

where the coefficients are defined as follows:

$$q_1 = \frac{E_2 - E_\nu}{E_2 - E_1}, \quad q_2 = \frac{E_\nu - E_1}{E_2 - E_1},$$

$$k_1 = \frac{\Delta E_{12} - \Delta E}{\Delta E_{12} - \Delta E_{11}}, \quad k_2 = \frac{\Delta E - \Delta E_{11}}{\Delta E_{12} - \Delta E_{11}},$$

$$k'_1 = \frac{\Delta E_{22} - \Delta E}{\Delta E_{22} - \Delta E_{21}}, \quad k'_2 = \frac{\Delta E - \Delta E_{21}}{\Delta E_{22} - \Delta E_{21}}.$$
\( E_\nu \leq E_\nu' \) using the relation
\[
\bar{R}(E_\nu, E_\nu', \cos \psi) = \bar{R}(E_\nu', E_\nu, \cos \psi)
\]  
(A6)

based on the detailed balance. The modified reaction rate is described as
\[
\bar{R}(E_\nu, E_\nu', \cos \psi) = q_3 k_3 \bar{R}_3 + q_4 k_4 \bar{R}_4,
\]
(A7)

with the neutrino energy employed in the table \( E_3 \leq E_\nu' \leq E_4 \) and \( E_1 \leq E_\nu \leq E_4' \), the energy difference \( \Delta E' \equiv E_\nu - E_\nu' \), and the coefficients
\[
q_3 \equiv \frac{E_4 - E_\nu'}{E_4 - E_3}, \quad q_4 \equiv \frac{E_\nu' - E_3}{E_4 - E_3}, \quad (A8)
\]
\[
k_3 \equiv \frac{\Delta E_{34} - \Delta E'}{\Delta E_{34} - \Delta E_{33}}, \quad k_4 \equiv \frac{\Delta E' - \Delta E_{33}}{\Delta E_{44} - \Delta E_{43}}, \quad (A9)
\]

\[
k'_3 \equiv \frac{\Delta E_{44} - \Delta E'}{\Delta E_{44} - \Delta E_{43}}, \quad k'_4 \equiv \frac{\Delta E' - \Delta E_{43}}{\Delta E_{44} - \Delta E_{43}}, \quad (A10)
\]

The total rate integrated over \( E_\nu' \) is
\[
A \equiv \int_{E_{\nu min}}^{E_{\nu max}} \bar{R}(E_\nu, E_\nu', \cos \psi) 2\pi E_\nu'^2 dE_\nu' \]
\[
= \int_{E_{\nu min}}^{E_{\nu max}} \bar{R}(E_\nu, E_\nu', \cos \psi) \exp \left( \frac{E_\nu}{T} \right) 2\pi E_\nu'^2 dE_\nu' \]
\[
= 2\pi \exp \left( \frac{E_\nu}{T} \right) \left[ \int_{E_{\nu min}}^{E_{\nu}} \bar{R}(E_\nu, E_\nu', \cos \psi) E_\nu'^2 dE_\nu' \right. 
\]
\[
+ \int_{E_{\nu}}^{E_{\nu max}} \bar{R}(E_\nu, E_\nu', \cos \psi) E_\nu'^2 dE_\nu' \right]
\]
\[
= \frac{1}{4} (E_\nu^4 - E_{\nu min}^4) A_{11} + \frac{1}{3} (E_\nu^3 - E_{\nu min}^3) A_{12}
\]
\[
+ \frac{1}{5} (E_{\nu max}^5 - E_\nu^5) B_{11} + \frac{1}{4} (E_{\nu max}^4 - E_\nu^4) B_{12}
\]
\[
+ \frac{1}{3} (E_{\nu max}^3 - E_\nu^3) B_{13}, \quad (A11)
\]

with the minimum and maximum energies \( E_{\nu min} \), \( E_{\nu max} \), at which the reaction rates become \( 10^{-5} \) times less than the peak value, and the coefficients
\[
A_{11} = -\bar{R}_{11} + \frac{1}{\Delta E_{12} - \Delta E_{21}} q_1 - \frac{1}{\Delta E_{22} - \Delta E_{21}} q_2, \quad (A12)
\]
\[
A_{12} = \left( \frac{\Delta E_{12} + E_\nu}{E_\nu} \right) \bar{R}_{12} - \left( \frac{\Delta E_{11} + E_\nu}{E_\nu} \right) \bar{R}_{12} q_1
\]
\[
+ \left( \frac{\Delta E_{22} + E_\nu}{E_\nu} \right) \bar{R}_{22} - \left( \frac{\Delta E_{21} + E_\nu}{E_\nu} \right) \bar{R}_{22} q_2, \quad (A13)
\]
\[
B_{11} = \frac{1}{E_\nu - 3} \left( \frac{\bar{R}_{11}(E_\nu - \Delta E_{34} + E_\nu) - \bar{R}_{11}(E_\nu - \Delta E_{35} + E_\nu)}{\Delta E_{34} - \Delta E_{33}} + \frac{\bar{R}_{34} - \bar{R}_{34}}{\Delta E_{44} - \Delta E_{43}} \right), \quad (A14)
\]
\[
B_{12} = \frac{1}{E_\nu - 3} \left( \frac{\bar{R}_{12}(E_\nu - \Delta E_{34} + E_\nu) - \bar{R}_{12}(E_\nu - \Delta E_{35} + E_\nu)}{\Delta E_{34} - \Delta E_{33}} + \frac{\bar{R}_{34}(E_\nu - \Delta E_{44} + E_\nu) - \bar{R}_{34}(E_\nu - \Delta E_{44} + E_\nu)}{\Delta E_{44} - \Delta E_{43}} \right). \quad (A15)
\]

The neutrino energy after scattering \( E_\nu' \) is determined by the random number \( x \) in the range of \( [0, 1] \) and the normalized spectrum \( \int_{E_{\nu min}}^{E_{\nu max}} \bar{R}(E_\nu, E_\nu', \cos \psi) 2\pi E_\nu'^2 dE_\nu'/A \).

**Appendix B**

**Numerical Method of Our MC Code**

**B.1. Sample Particles**

In the MC method, we follow the tracks of sample particles, which represent a bundle of neutrinos, interacting with matter. The numbers of sample particles \( N_s \) and physical neutrinos \( N_p \) are related to the weight \( W_t \) as follows:
\[
W_t = \frac{N_s}{N_p}. \quad (B1)
\]

In our simulations, the weight is constant in all of the time and calculation domain.

**B.2. Treatments of the Transport of Sample Particles**

Each sample particle has six-dimensional information about a space \( (r, \theta, \phi) \) and a phase space \( (E_\nu, \theta_\nu, \phi_\nu) \), and we calculate their time evolutions by solving geometric equations. In order to calculate the transport of sample particles, we introduce three lengths: reaction length \( l_r \), background length \( l_b \), and distribution length \( l_d \).

1. Reaction length \( l_r \). We define a “reaction length,” which is a distance between the points where the sample particle interacts with matter, by the optical depth
\[
\tau(S, E_\nu) = \int_0^S \frac{1}{\lambda(r, E_\nu)} ds \quad (B2)
\]

using the local mean free path \( \lambda \),
\[
\lambda(r, E_\nu) = \frac{1}{\sigma_{tot}}, \quad (B3)
\]

with the total cross section \( \sigma_{tot} = \sum_\alpha \sigma_\alpha (r, E_\nu) \) using the cross section of the \( \alpha \)th type of reaction \( \sigma_\alpha \). The reaction occurs at \( \tau(l_r, E_\nu) \equiv \tau_{max} \), which is determined by the random number obeying the Poisson distribution whose average becomes 1.

2. Background length \( l_b \). We employ the results of the dynamical SN simulations as a background for the neutrino transport calculations. We assume that the hydrodynamical values, i.e., density, temperature, and chemical potential of matter, are uniform in each spatial zone. A “background length” is defined by the distance between the nearest spatial boundary of the hydrodynamical background and the current position of a sample particle.
Section B.3. Evaluation of the Neutrino Distribution Function

In this calculation, we employ the spherical symmetric background, and the neutrino distribution function is reduced to \( f(r, E_\nu, \theta_\nu) \). At every time step, we count the number of sample particles inside each volume element in a space and a phase space and calculate the \( i,j,k \) th discretized neutrino distribution function \( f_{ijk} \),

\[
    f_{ijk} = \frac{N_{ijk} W_i}{V_{r,i} N_{m,j,k}},
\]

where \( i,j,k \) describe the components of \( r, E_\nu, \) and \( \theta_\nu \), respectively; the total number of sample particles in the \( i,j,k \) th volume element \( N_{ijk} \); the \( i \) th spatial volume element \( V_{r,i} = 4\pi (r_i^3 - r_{i-1}^3)/3 \); and the \( j,k \) th phase space volume element \( N_{m,j,k} = 2\pi (\cos \theta_{i,k} - \cos \theta_{i,k-1})(E_\nu^{3,j} - E_\nu^{3,j-1})/3 \).

B.4. Treatments of Neutrino Reactions

Neutrinos interact with matter via several reactions inside stars (see Table 1). We divide the neutrino reactions into three processes—absorption, emission, and scattering—and adopt different treatments for them in our MC code.

B.4.1. Absorption and Scattering

Existing samples are absorbed or scattered by matter. After the subsequent reaction point is determined by the reaction rate, we stop following the track of this sample particle at this point.

B.4.2. Emission

The total number of neutrinos emitted during a time step \( dt \) in units of spatial volume is calculated by the reaction rate \( R_{\nu,\text{ems}} \), and we add the corresponding number of sample particles uniformly in that volume element at the beginning of each time step. The energies and angles of the sample particles are distributed following the distribution of the reaction rate. We put the distribution time into sample particles randomly in the range of \([0, dt]\) in order to get the constant emission rate and calculate their evolution in the same way as those for existing sample particles.

Appendix C

Neutrino Reactions

C.1. Electron/Positron Scatterings

The reaction rates of the electron/positron scattering are derived from a similar form as the nucleon scattering in Equations (1)-(10), if we change the coefficients \( \beta \) summarized in Table C1, the target mass \( m_N \rightarrow m_e \) and the chemical potential \( \mu_N \rightarrow \mu_e - \mu_e \), for electrons and positrons, respectively. In this paper, we denote the total reaction rates of electron and positron scatterings as \( R_{\text{esc}} \). Their cross section \( \sigma_{\text{esc}} \) and normalized spectra \( P_\nu \) and \( P_\nu' \) are defined in the same way as those for the nucleon scattering. Note that we should distinguish the reaction rates of \( \nu_\tau \) and \( \bar{\nu}_\tau \), but we adopt that of \( \nu_\tau \) in this study.

C.2. Electron Capture on Free Proton and Positron Capture on Free Neutron

The emission rates of electron and positron captures on free nucleons, \( R_{\nu,\text{ems}} \) and \( R_{\nu,\text{ems}} \), are calculated by Bruenn (1985),

\[
    R_{\nu,\text{ems}} = \frac{G_F^2}{\pi \hbar c} \eta_B (g_\nu^2 + 3g_e^2)(E_\nu + Q)^2 \times \sqrt{1 - \frac{m_e^2}{(E_\nu + Q)^2}} f_\nu(E_\nu + Q),
\]

\[
    R_{\nu,\text{ems}} = \frac{G_F^2}{\pi \hbar c} \eta_B (g_\nu^2 + 3g_e^2)(E_\nu - Q)^2 \times \sqrt{1 - \frac{m_e^2}{(E_\nu - Q)^2}} f_\nu(E_\nu - Q) \times \Theta(E_\nu - Q - m_e),
\]

in which nucleons are nonrelativistic, and they neglect nucleon recoils. The absorption rates are derived from the detailed balance relations, \( R_{\nu,\text{abs}}(1 - f_\nu) = R_{\nu,\text{ems}} \), using the Table C1

| Reaction | \( \beta_1 \) | \( \beta_2 \) | \( \beta_3 \) |
|----------|---------------|---------------|---------------|
| \( \nu_e^- / \bar{\nu}_e^+ \) | \( (C_{\nu_e} + C_{\bar{\nu}_e})^2 \) | \( (C_{\nu_e} - C_{\bar{\nu}_e})^2 \) | \( C_{\bar{\nu}_e}^2 - C_{\nu_e}^2 \) |
| \( \nu_e^+ / \bar{\nu}_e^- \) | \( (C_{\nu_e} - C_{\bar{\nu}_e})^2 \) | \( (C_{\nu_e} + C_{\bar{\nu}_e})^2 \) | \( C_{\bar{\nu}_e}^2 - C_{\nu_e}^2 \) |
| \( \nu_e^- \) | \( (C_{\nu_e} + C_{\bar{\nu}_e})^2 \) | \( (C_{\nu_e} - C_{\bar{\nu}_e})^2 \) | \( C_{\bar{\nu}_e}^2 - C_{\nu_e}^2 \) |
| \( \nu_e^+ \) | \( (C_{\nu_e} - C_{\bar{\nu}_e})^2 \) | \( (C_{\nu_e} + C_{\bar{\nu}_e})^2 \) | \( C_{\bar{\nu}_e}^2 - C_{\nu_e}^2 \) |

Note. In this expression, \( C_{\nu_e} = C_{\nu_e} + 1 \) and \( C_{\bar{\nu}_e} = C_{\bar{\nu}_e} + 1 \) with \( C_{\nu_e} = -1/2 + 2 \sin^2 \theta_\nu \) and \( C_{\bar{\nu}_e} = 1/2 \).
Fermi–Dirac distribution of electrons and positrons \( f_{\text{eq}} \) with the chemical potential \( \mu_e \) for electron captures and \( -\mu_p \) for positron captures; the reaction rates \( R_{\text{nuc}} = R_{\text{EC,abs}} + R_{\text{PC,abs}} \). The cross sections are calculated as \( \sigma = R_{\text{nuc}} \).

C.3. Electron–Positron Pair Annihilation

We use the reaction rate of the electron–positron pair annihilation \( R_{\text{pair}} \) described in Kato et al. 2017 (see Equations (1)–(9) in that paper). The emission rate and cross section for neutrinos are derived from the integrals of the reaction rate in a phase space for antineutrinos,

\[
R_{\text{pair,ems}} = \int \frac{1}{2E_p(2\pi)^3} \frac{2\pi E_p^2}{2E_p(2\pi)^3} \times R_{\text{pair}}(1 - f_p) d\cos\psi dE_p,
\]

\[
\sigma_{\text{pair}} = \int \frac{1}{2E_p(2\pi)^3} \frac{2\pi E_p^2}{2E_p(2\pi)^3} \times R_{\text{pair}} f_p d\cos\psi dE_p,
\]

with the energy of antineutrinos \( E_\nu \), the angle between four momenta of the neutrino pair \( \psi \), and the distribution for antineutrinos \( f_p \). For antineutrinos, we integrate the reaction rate over \( E_\nu \) instead of \( E_\nu \). In this calculation, we employ the distribution function for the other neutrinos derived from the background CCSN simulations.

C.4. Nucleon Bremsstrahlung

We calculate the reaction rate of the nucleon bremsstrahlung \( R_{\text{brem}} \) based on Friman & Maxwell (1979) and Maxwell (1987). The emission and absorption rates \( R_{\text{brem,ems}} \) and \( R_{\text{brem,abs}} \), and the cross section \( \sigma_{\text{brem}} \) are derived in the same way as those for pair annihilations.

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10 The reaction rate in Kato et al. (2017) is described in the natural unit \( e = h = 1 \). In this paper, \( R_{\text{pair}} \) is defined by multiplying a factor of \( 1/\hbar c \) to that in the previous paper.