Corrections to Sirlin’s Theorem in $\mathcal{O}(p^6)$ Chiral Perturbation Theory

P. Post and K. Schilcher

Institut für Physik, Johannes Gutenberg Universität, Staudinger Weg 7, D-55099 Mainz, Germany

Abstract

We present the results of the first two-loop calculation of a form factor in full $SU(3) \times SU(3)$ Chiral Perturbation Theory. We choose a specific linear combination of $\pi^+, K^+, K^0$ and $K\pi$ form factors (the one appearing in Sirlin’s theorem) which does not get contributions from order $p^6$ operators with unknown constants. For the charge radii, the correction to the previous one-loop result turns out to be significant, but still there is no agreement with the present data due to large experimental uncertainties in the kaon charge radii.

1E-mail: post@vipmzw.physik.uni-mainz.de, schilcher@vipmzw.physik.uni-mainz.de
Chiral perturbation theory (ChPT) has been applied with great success to low energy hadronic phenomena \[1\, 3, 4\]. Presently there is an emerging effort to extend the calculations to two-loop order so as to allow quantitative comparison with experiments and test the convergence properties of ChPT. Up to now most of the two-loops results have been obtained in the chiral $SU(2)\times SU(2)$-limit which is obviously a serious limitation as $K$-mesons are excluded from loops. To our knowledge a complete $SU(3)\times SU(3)$ calculation exists so far only for two-point-functions of current-correlators \[3\]. In this note we will present the results of the first full $SU(3)\times SU(3)$ form factor calculation. We choose a specific combination of weak and electromagnetic meson form factors which does not involve arbitrary renormalization constants of new operators.

ChPT is formulated in terms of an effective Lagrangian involving an increasing number of covariant derivatives, external fields (including quark mass terms) and field strength tensors,

$$\mathcal{L}_{\text{eff}} = \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \mathcal{L}^{(6)} + \ldots .$$

The lowest-order term is \[1\]

$$\mathcal{L}^{(2)} = \frac{F^2}{4} \text{Tr}(D_\mu U D^\mu U^\dagger) + \frac{F^2}{4} \text{Tr}(\chi U^\dagger + U\chi^\dagger)$$

with $U(x) = \exp[i\Phi(x)/F]$ where $\Phi$ is the $3 \times 3$ matrix made up of the Goldstone fields $(\pi, K, \eta)$, $F$ is the pion decay constant in the chiral limit, and $\chi$ is related to the quark mass matrix (for details see \[3\]). The term $\mathcal{L}^{(4)}$ is of order $p^4$ and involves 10 new operators which are to be renormalized by imposing the same number of independent experimental input data. The renormalization constants $L_1$ to $L_{10}$ are commonly defined in dimensional regularization. The operators of $\mathcal{L}^{(6)}$ have been exhaustively analysed in \[3\] and found to be 143 in number. The corresponding number of free constants seems to be prohibitive. There are, however, subsets of experiments like the weak and electromagnetic form factors of mesons which involve only a small number of new renormalization constants. ChPT to order $p^6$ then leads to relations between and predictions of specifics of their $t$-dependence. In one combination of these form factors the $\mathcal{L}^{(6)}$-constants all cancel. This is the combination entering Sirlin’s relation \[6\] and which should vanish in the chiral limit.

The relevant vector current form factors are defined as follows:

$$\langle \pi^+, p' \mid J_\mu \mid \pi^+, p \rangle = (p + p')_\mu F^\pi(t)$$

$$\langle K, p' \mid J_\mu \mid K, p \rangle = (p + p')_\mu F^K(t)$$

$$\langle \pi^-, p' \mid \bar{u}\gamma_\mu s \mid K^0, p \rangle = (p + p')_\mu f^K_{\pi^-}(t) + (p - p')_\mu f^K_{\pi^+}(t),$$

where $t = (p' - p)^2$ and $J_\mu$ is the electromagnetic current carried by the light quarks, $J_\mu = \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d - \frac{1}{3}\bar{s}\gamma_\mu s$. Sirlin’s low-energy theorem then states that up to second order in the quark mass difference $m_s - m, \hat{m} = \frac{1}{2}(m_u + m_d)$, the linear combination

$$\Delta(t) = \frac{1}{2} F^{\pi^+}(t) + \frac{1}{2} F^K_{\pi^+}(t) + F^K_{\pi^-}(t) - f^K_{\pi^-}(t)$$

vanishes. The effect of heavy quarks in the electromagnetic current is neglected. Sirlin’s relation generalizes the Ademollo Gatto theorem \[3\] to $t \neq 0$. For $t = 0$ eqn. (\[6\]) yields no prediction as $f^K_{\pi^-}(0)$ still depends on unknown constants of $\mathcal{L}^{(6)}$. In the relation for the charge radii (and higher Taylor coefficients), however, all arbitrary constants cancel and an unambiguous prediction remains.

The diagrams contributing to the $t$-dependence of the form factors at order $p^6$ are represented symbolically in fig. (1). There are further graphs which do not depend on $t$ and are omitted here. The $\mathcal{O}(p^6)$ contributions arise from the lowest order diagrams (a),(b) owing to wave function, mass and decay constant renormalization, from the one-loop-diagrams (c)–(f) with one vertex from
\[ \Delta(t) = \delta_4(t) + \delta_6(t) + \ldots \]

where \( \delta_4 \) is the \( \mathcal{O}(p^4) \)-result,

\[
\delta_4(t) = \frac{1}{8F^2} \left\{ -3A(m_2^2) + 4A(m_K^2) - A(m_{\eta}^2) + 12B_21(q^2; m_1^2, m_2^2) \\
- 20B_21(q^2; m_K^2, m_1^2) + 12B_21(q^2; m_K^2, m_2^2) - 4B_21(q^2; m_{\eta}^2, m_1^2) \right\},
\]

\[ \mathcal{L}^{(4)} \text{, from the reducible two-loop diagrams (g)–(j), from the irreducible two-loop diagram (k) and from the tree graph (l) with one vertex from } \mathcal{L}^{(6)}, \text{ which yields no contributions to Sirlin’s linear combination } \Delta(t). \text{ Diagrams (b), (f) and (l) are polynomial in } t \text{ due to the derivative couplings in the vertices.} \]

All reducible diagrams (a)–(j) involve only well-known one-loop integrals \[ (7) \] (calculated to order \( \varepsilon = 2 - D/2 \), i.e. the massive one-loop tadpole

\[ A(m^2) = \mu^{4-D} \int \frac{d^Dk}{i(2\pi)^D} \frac{1}{k^2 - m^2} \]

and the one-loop two-point-functions \( B_{21}(q^2; m_1^2, m_2^2) \) and \( B_{31}(q^2; m_1^2, m_2^2) \) defined by the tensor decompositions

\[
\mu^{4-D} \int \frac{d^Dk}{i(2\pi)^D} \left[ \frac{k^\mu k^\nu}{[(k + q)^2 - m_1^2] [k^2 - m_2^2]} \right] = q^\mu q^\nu B_{20}(q^2; m_1^2, m_2^2) + g^{\mu\nu} B_{21}(q^2; m_1^2, m_2^2) \]
\[
\mu^{4-D} \int \frac{d^Dk}{i(2\pi)^D} \left[ \frac{k^\mu k^\nu k^\rho}{[(k + q)^2 - m_1^2] [k^2 - m_2^2]} \right] = q^\mu q^\nu q^\rho B_{30} + (g^{\mu\nu} q^\rho + g^{\mu\rho} q^\nu + g^{\nu\rho} q^\mu) B_{31}.
\]

In terms of these integrals Sirlin’s relation reads

\[ \Delta(t) = \delta_4(t) + \delta_6(t) + \ldots \]
and $\delta_6$ the $O(p^6)$-contribution. Up to a constant which is irrelevant for the $t$-dependence of $\Delta(t)$, $\delta_6$ is given as a sum

$$
\delta_6(t) = \text{Red}_1(t) + \text{Red}_2(t) + \text{Irr}(t) + \text{const},
$$

where the reducible one-loop part $\text{Red}_1(t)$ collects all terms involving $\mathcal{L}^{(4)}$-parameters $L_1, \ldots, L_{10}$,

$$
\text{Red}_1(t) = \frac{1}{4F^3} \left\{ -3A(m_\eta^2)q^2L_9 + 4A(m_\pi^2)q^2L_9 - A(m_\pi^2)q^2L_9 \\
+ 16B_31(q^2;m_\eta^2;2m_K^2)L_3(m_K^2 - m_\pi^2) \\
- 16B_31(q^2;m_\pi^2;m_\eta^2)(8L_1 + 4L_2 + L_3)(m_K^2 - m_\pi^2) \\
+ 4B_21(q^2;m_\eta^2,m_\pi^2)[2L_3(m_K^2 - m_\eta^2 - 3q^2) + 12L_5m_\pi^2 + 3q^2L_9] \\
+ 4B_21(q^2;m_\eta^2,m_\pi^2)[8q^2L_1 - 4q^2L_2 + 10q^2L_3 - 16L_4m_K^2 - 20L_5m_\pi^2 - 5q^2L_9] \\
+ 4B_21(q^2;m_\pi^2,m_\eta^2)[8q^2L_1 - 4q^2L_2 + 2q^2L_3 - 16L_4m_\pi^2 - 4L_5m_\eta^2 - q^2L_9] \\
+ 4B_21(q^2;m_\eta^2,m_\pi^2)[16L_1(m_K^2 - m_\pi^2 + q^2) - 8L_2(m_K^2 - m_\pi^2 - q^2) \\
- 2L_3(m_K^2 - m_\pi^2 + 3q^2) + 16L_4(m_\pi^2 + m_\eta^2) + 12L_5m_\pi^2 + 3q^2L_9] \right\},
$$

$\text{Red}_2(t)$ denotes the reducible two-loop parts,

$$
\text{Red}_2(t) = \frac{1}{144F^3} \left\{ 99A(m_\eta^2)B_21(q^2;m_\eta^2,m_K^2) + 102A(m_\eta^2)B_21(q^2;m_\eta^2,m_\pi^2) \\
+ 75A(m_\eta^2)B_21(q^2;m_\eta^2,m_\eta^2) - 6A(m_\eta^2)B_21(q^2;m_\eta^2,m_\eta^2) \\
+ 318A(m_K^2)B_21(q^2;m_\eta^2,m_K^2) - 924A(m_K^2)B_21(q^2;m_\eta^2,m_K^2) \\
+ 350A(m_\pi^2)B_21(q^2;m_\eta^2,m_\pi^2) - 104A(m_\pi^2)B_21(q^2;m_\eta^2,m_\pi^2) \\
+ 483A(m_\pi^2)B_21(q^2;m_\pi^2,m_K^2) - 678A(m_\pi^2)B_21(q^2;m_\pi^2,m_K^2) \\
+ 475A(m_\pi^2)B_21(q^2;m_\pi^2,m_\pi^2) - 190A(m_\pi^2)B_21(q^2;m_\pi^2,m_\pi^2) \\
- 324B_21(q^2;m_\eta^2,m_K^2)^2 + 1008B_21(q^2;m_\pi^2,m_K^2)^2 \\
+ 144B_21(q^2;m_\pi^2,m_\pi^2)^2 - 648B_21(q^2;m_\pi^2,m_K^2)B_21(q^2;m_\pi^2,m_\pi^2) \\
- 324B_21(q^2;m_\pi^2,m_K^2)^2 + 144B_21(q^2;m_\pi^2,m_K^2)B_21(q^2;m_\eta^2,m_\eta^2) \right\},
$$

and $\text{Irr}(t)$ the irreducible two-loop contribution from diagram (k). Diagram (k) cannot be calculated analytically, unless all masses are equal. Instead, for arbitrary masses and tensor numerators, it can be reduced via dispersion techniques to a one-dimensional integral which is done numerically. Its contribution to the charge radius

$$
\langle r^2 \rangle_{\text{Sirlin}} := 6 \frac{d\Delta}{dt} \bigg|_{t=0}
$$

turns out to be small if one uses the generalized Gasser-Leutwyler renormalization scheme, i.e. multiplication of each $O(p^6)$-contribution with the $\overline{\text{MS}}$-type factor

$$
\left[ \varepsilon (4\pi)^{\varepsilon} \Gamma(-1 + \varepsilon) \right]^2 = \exp \left[ 2\varepsilon (\gamma - 1 - \log 4\pi) - \varepsilon^2 \left( \frac{\pi^2}{6} + 1 \right) \right] + O(\varepsilon^3).
$$

For $\mu = 770 \text{ MeV}$, $F_\pi = 92.4 \text{ MeV}$, $m_K = 495 \text{ MeV}$, $m_\pi = 135 \text{ MeV}$ and $m_\eta = 548.8 \text{ MeV}$ we find

$$
\langle r^2 \rangle_{\text{Sirlin}} = \left[ 0.006 \text{ fm}^2 \right] O(p^4) + \left[ 0.017(3) \text{ fm}^2 \right] \text{red. } O(p^6) + \left[ - 0.002 \text{ fm}^2 \right] \text{irr. } O(p^6)
$$

$$
= (0.021 \pm 0.003) \text{ fm}^2
$$

(15)
where the error is due to uncertainties in the $L^4$-parameters $L_i$ involved. This is to be compared with the experimental point $\langle r^2 \rangle_{\text{exp}} = -(0.025 \pm 0.041) \text{ fm}^2$ which is based on the data

\begin{align*}
\langle r^2 \rangle_{\pi^+} &= (0.439 \pm 0.008) \text{ fm}^2 \quad [11] \\
\langle r^2 \rangle_{K^+} &= (0.34 \pm 0.05) \text{ fm}^2 \quad [12] \\
\langle r^2 \rangle_{K^0} &= -(0.054 \pm 0.026) \text{ fm}^2 \quad [13] \\
\langle r^2 \rangle_{K\pi} &= (0.36 \pm 0.02) \text{ fm}^2 \quad [14].
\end{align*}

We have actually calculated the complete $t$-dependence of $\Delta(t)$, but we find only slight deviations from linearity [10]. From figure 2 we see that Sirlin’s relation is poorly satisfied. Actually the $O(p^6)$-contribution tends to increase the disagreement. The main experimental uncertainty lies in the kaon form factors which ought to be remeasured with higher accuracy. In particular it may be argued [3] that the $K^+$ charge radius $\langle r^2 \rangle_{K^+}$ should be larger than $\langle r^2 \rangle_{K\pi}$ which would bring prediction and experiment into better agreement.

We have checked our calculations in several ways:

- In the special case of all masses equal, the irreducible two-loop integrals were compared to known analytic results [15].

- The electromagnetic form factors have to satisfy the Ward identity. This holds separately for the group of reducible and the group of irreducible diagrams.

- Non-polynomial divergences have to disappear in the sum of all loop-diagrams.

In this note we have reported the results of the first $O(p^6)$ calculation of a form factor in full chiral $SU(3) \times SU(3)$ perturbation theory. We chose a particular combination of weak and electromagnetic form factors due to Sirlin which is independent of the new arbitrary renormalization constants of $L^6$ (except at $t = 0$). The correction to the previous one-loop result [3] turns out to be significant. Comparison of Sirlin’s linear combination of charge radii with data is inconclusive due to large experimental uncertainties of the kaon charge radii. A significant improvement in the precision of the charged kaon form factor should be feasible in the future COMPASS experiment [16].

P.P. was supported by the ”Studienstiftung des deutschen Volkes”.

References

[1] S. Weinberg, Physica 96A, 327 (1979)
[2] J. Gasser, H. Leutwyler, Ann. Phys. 158, 142 (1984); Nucl. Phys. B250, 465 (1985)
[3] J. Gasser, H. Leutwyler, Nucl. Phys. B250, 517 (1985)
[4] E. Golowich, J. Kambor, Nucl. Phys. B447, 373 (1995)
[5] H.W. Fearing, S. Scherer, Phys. Rev. D53, 315 (1996)
[6] A. Sirlin, Ann. of Phys. 61, 294 (1970); Phys. Rev. Lett. 43, 904 (1979)
[7] M. Ademollo, R. Gatto, Phys. Rev. Lett. 13, 264 (1964)
[8] G. t’Hooft, M. Veltman, Nucl. Phys. B153, 365, (1979)
[9] P. Post, J.B. Tausk, Mod. Phys. Lett. A11, 2115 (1996)
[10] P. Post, K. Schilcher, to be published
[11] NA7 Collaboration, S.R. Amendolia et al., Nucl. Phys. B277, 168 (1986)
[12] S.R. Amendolia et al., Phys. Lett. B178, 435 (1986)
[13] W.R. Molzon et al., Phys. Rev. Lett. 41, 1213 (1978)
[14] Particle Data Group, Phys. Rev. D54, 420 (1996)
[15] D. Bessis and M. Pusterla, Nuovo Cim. 54A, 243 (1968)
[16] D. von Harrach, private communication