A TREE-ARROWING GRAPH

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Dedicated to the memory of Eric Milner

Abstract. We answer a variant of a question of Rödl and Voigt by showing that, for a given infinite cardinal \( \lambda \), there is a graph \( G \) of cardinality \( \kappa = (2^\lambda)^+ \) such that for any colouring of the edges of \( G \) with \( \lambda \) colours, there is an induced copy of the \( \kappa \)-tree in \( G \) in the set theoretic sense with all edges having the same colour.

Keywords: Partition relation, graph, tree, cardinal number, stationary set, normal filter.

AMS Subject Classification (1991): 03, 04

1. Introduction

\( G = (V, E) \) is a graph with vertex set \( V \) and edge set \( E \), where \( E \subseteq [V]^2 \). The graph \( \mathcal{H} = (W, F) \) is a subgraph of \( G \) if \( W \subseteq V \) and \( F \subseteq E \), it is an induced subgraph if \( F = E \cap [W]^2 \). If \( \lambda \) is a cardinal, the partition relation

\[ G \rightarrow (\mathcal{H})^2_\lambda, \]

(1)

means that if \( c : E \rightarrow \lambda \) is any colouring of the edges of \( G \) with \( \lambda \) colours, then there is an induced copy of \( \mathcal{H} \) in \( G \) in which all the edges have the same colour. There is a related notion \( G \rightarrow (\mathcal{H})^1_\lambda \), for vertex colourings of graphs. However, there is an essential difference since, for any given graph \( \mathcal{H} \) and any \( \lambda \), there is

\(^1\)Paper Sh 578 in Shelah’s publication list. Research supported by “The Israel Science Foundation” administered by The Israel Academy of Sciences and Humanities.

\(^2\)Research supported by NSERC grant #69-0982.
some \( G \) such that \( G \to (H)^1_{\lambda} \) holds. This is not true for edge-colourings; Hajnal and Komjath \([2]\) proved the consistency of a negative answer, and Shelah \([5]\) proved that a positive answer is also consistent. It is therefore of some interest to have instances of graphs \( H \) such that (1) holds for some \( G \), and then, of course, one can ask for the smallest such \( G \).

Rödl and Voigt \([4]\) (see also \([3]\)) proved a result of this kind by showing that for any infinite cardinal \( \lambda \) and a suitably large \( \kappa \), there is a graph \( G_\kappa \) of cardinality \( \kappa \) such that \( G_\kappa \to (T_{\kappa})^2_{\lambda} \) holds, where \( T_{\kappa} \) is the tree in which every vertex has degree \( \kappa \) (see below). More precisely, ‘suitably large’ means that the ordinary partition relation

\[
\text{cf}(\kappa) \to (\omega)^3_{\lambda}
\]

holds so that, by \([1]\), \( \kappa \geq (2^\lambda)^+ \); in fact, they showed in this case that the ubiquitous shift-graph on \( \kappa \) works. Rödl and Voigt \([4]\) then asked, what is the smallest cardinal \( \kappa \) such that (2) holds? It is easily seen that (2) is false if \( \kappa \leq 2^\lambda \), and they conjectured that it holds (for some suitable graph \( G_\kappa \)) if \( \kappa = (2^\lambda)^+ \). In this paper we prove that (2) holds with \( T_\kappa \) replaced by \( T(\kappa) \), a related graph which we call the transitive \( \kappa \)-tree defined in the next section.

2. Preliminaries

For an infinite cardinal \( \kappa \) we denote by \( <^\omega \kappa \) the set of all increasing finite sequences of ordinals in \( \kappa \). The length of an element \( s = \langle s_0, \ldots, s_{n-1} \rangle \in <^\omega \kappa \) is denoted by \( \ell n(s) = n \). Also, we define

\[
\max(s) = \begin{cases} 
-1 & \text{if } s = \langle \rangle, \text{ the empty sequence}, \\
 s_{\ell n(s) - 1} & \text{if } \ell n(s) > 0.
\end{cases}
\]

If \( s = \langle s_0, \ldots, s_{n-1} \rangle \) and \( t = \langle t_0, \ldots, t_{m-1} \rangle \) are two elements of \( <^\omega \kappa \), we write \( s \triangleleft t \) to denote the fact that \( s \) is a proper initial segment of \( t \), that is \( n < m \) and \( s_i = t_i \) for \( i < n \), and in this case we write \( s = t|n \). We also write \( s = t_* \) if \( m = n + 1 \) and \( s \triangleleft t \). If \( s, t \) are distinct and \( \triangleleft \)-incomparable we write \( s \perp t \). The \( \kappa \)-tree of height \( \omega \) is the graph \( T_\kappa \) on \( <^\omega \kappa \) with edge set

\[
E_\kappa = \{ \{ s, t \} : s, t \in <^\omega \kappa \land s = t_* \}.
\]

We shall also consider a related graph, the transitive \( \kappa \)-tree of height \( \omega \), which is the graph \( T(\kappa) \) on \( <^\omega \kappa \) with edge set

\[
F_\kappa = \{ \{ s, t \} : s, t \in <^\omega \kappa \land s \triangleleft t \}.
\]

We shall prove the following theorem.
Theorem 2.1. Let \( \lambda \) be an infinite cardinal, and let \( \kappa = (2^\lambda)^+ \). Then there is a graph \( G_\kappa \) of cardinality \( \kappa \) such that

\[
G_\kappa \to (T)^2_\kappa,
\]

where \( T \) is \( T(\kappa) \).

Remark. Instead of \( \kappa = (2^\lambda)^+ \), it is enough that \( \kappa \) be any regular cardinal such that \( |\alpha|^\lambda < \kappa \) holds for all \( \alpha < \kappa \). The same proof works.

The construction of a suitable \( G_\kappa \) depends upon the following (slightly weaker version of a) theorem of Shelah [7] (or more [8, 3.5]):

(●) Let \( \lambda \) be an infinite cardinal, \( \kappa = (2^\lambda)^+ \), \( S = \{ \alpha < \kappa : \text{cf}(\alpha) = \lambda^+ \} \). Then there are a sequence \( C = \langle C_\delta : \delta \in S \rangle \) and a sequence \( h^*_\delta : C_\delta \to 2 \) and such that, for any club \( K \) in \( \kappa \), there is a stationary subset \( B_K \) of \( S \cap K \) such that for each \( \delta \in B_K \) and each \( i < \lambda \), \( \min(C_\delta \setminus (\alpha + 1)) \in K \) is cofinal in \( \delta \).

Remarks. 1. The result is also true if 2, the range of each \( h^*_\delta \), is replaced by \( \lambda \); also, if \( \kappa = \lambda^{++} \), we can also require that \( D_K(\delta, i) \) be a stationary subset of \( \delta \) for each \( \delta \in B_K \) and each \( i < \lambda \) (see [8]).

2. If \( 2^\lambda > \lambda^+ \), then the following stronger assertion is true (see Shelah [6]):

(●●) There is a sequence \( C = \langle C_\delta : \delta \in S \rangle \) such that \( C_\delta \) is a club in \( \delta \) having order type \( \lambda^+ \) and, for any club \( K \) in \( \kappa \) and any stationary subset \( S' \subseteq S \), there is a stationary subset \( B_K \subseteq S' \cap K \) such that \( C_\delta \subseteq K \) for each \( \delta \in B_K \). Using this result instead of (●), the proof of Theorem 2.1 for the case when \( 2^\lambda > \lambda^+ \) may be slightly simplified.

We will prove that Theorem 2.1 holds with the graph \( G_\kappa = (\kappa, E) \), where

\[
E = \{ \{ \alpha, \beta \} : \beta \in S \land \min(C_\beta) < \alpha < \beta \land h^*_\beta(\sup(\alpha \cap C_\beta)) = 0 \},
\]

and the \( C_\beta \) and \( h^*_\beta \) are as described in (●).

3. The case \( T = T(\kappa) \)

We prove the result for the case of the transitive tree \( T(\kappa) \).

Proof: Let \( c : E \to \lambda \) be any \( \lambda \)-colouring of the edges of \( G_\kappa \). For each \( \zeta \in \lambda \) consider the following two-person game \( G_\zeta \). The game has \( \omega \) moves. At the \( n \)-th stage the first player \( P_1 \) chooses ordinals \( \alpha_n, \beta_n \), and then the second player \( P_2 \) chooses two ordinals \( \gamma_n, \delta_n \) so that

\[
\alpha_n < \beta_n < \gamma_n < \delta_n < \kappa, \tag{3}
\]

\[
\delta_m < \alpha_n \quad (m < n). \tag{4}
\]
The player $P_2$ is declared the winner in a play of the game if he succeeds in choosing the $\gamma_n$ so that

$$\{\gamma_m, \gamma_n\} \in E, \quad c(\{\gamma_m, \gamma_n\}) = \zeta \quad (m < n < \omega), \quad \text{(5)}$$

and

$$\{\xi, \gamma_n\} \notin E \quad \text{for} \quad \xi \in (\alpha_m, \beta_m) \quad \text{and} \quad m \leq n < \omega. \quad \text{(6)}$$

(As usual, $(\alpha, \beta)$ denotes the open interval $\{\xi : \alpha < \xi < \beta\}$ and $[\alpha, \beta]$ is the corresponding closed interval.)

The proof of the theorem depends upon the following two facts:

**Fact A:** For some $\zeta < \lambda$, $P_2$ has a winning strategy for the game $G_\zeta$.

**Fact B:** If $P_2$ can win $G_\zeta$, then the graph $G_\kappa$ contains an induced copy of $T(\kappa)$ with all edges coloured $\zeta$.

**Proof of Fact B.** We assume that $\zeta < \lambda$ and that the second player $P_2$ has a winning strategy $\sigma_\zeta$ for the game $G_\zeta$. We shall define ordinals $\alpha_s, \beta_s, \gamma_s, \delta_s$ for $s$ a vertex of $T(\kappa)$ so that the following conditions are satisfied:

(a) For each $s$ the sequence

$$\langle (\alpha_{s|i}, \beta_{s|i}, \gamma_{s|i}, \delta_{s|i}) : i < \ell_n(s) \rangle$$

consists of the first $2\ell_n(s)$ moves in a proper play of the game $G_\zeta$ in which $P_2$ uses the winning strategy $\sigma_\zeta$.

(b) $\gamma_s \neq \gamma_t$ if $s \neq t$.

(c) If $s \perp t$, then $\{\gamma_s, \gamma_t\} \notin E$.

Since (5) holds, these conditions imply that the map $s \mapsto \gamma_s$ is an embedding of the tree $T(\kappa)$ into the graph $G_\kappa$ and all the edges of the image have colour $\zeta$.

In fact, we shall choose the $\alpha_s, \beta_s, \gamma_s, \delta_s$ so that (a) holds and so that the following condition is satisfied:

(d) For any vertices $s, t$ of $T(\kappa)$, if $s \perp t$, then

EITHER

(i) $[\gamma_s, \delta_s] \subset \bigcup_{i \leq \ell_n(t)} (\alpha_{t|i}, \beta_{t|i})$,

OR

(ii) $[\gamma_t, \delta_t] \subset \bigcup_{i \leq \ell_n(s)} (\alpha_{s|i}, \beta_{s|i})$.

The conditions (a) and (d), and the fact that $P_2$ is using the winning strategy $\sigma_\zeta$, ensure that (b) and (c) also hold.

We define $\alpha_s, \beta_s, \gamma_s, \delta_s$ by induction on $\max(s)$. Let $\alpha_0 = 0, \beta_0 = 1$, and then let $(\gamma_0, \delta_0)$ be $P_2$’s response in the game $G_\zeta$ using his winning strategy $\sigma_\zeta$. Now let $0 \leq \xi < \kappa$, and suppose that we have suitably defined $\alpha_s, \beta_s, \gamma_s, \delta_s$ for all vertices $s$ of $T(\kappa)$ such that $\max(s) < \xi$. We need to define these when $\max(s) = \xi$. 

Let \((t_i : i < \theta(\xi))\) be an enumeration of all the nodes \(s\) of \(T(\kappa)\) with \(\max(s) = \xi\). Then \(1 \leq \theta(\xi) \leq 2^\lambda < \kappa\). Now inductively choose the \(\alpha_{t_i}, \beta_{t_i}, \gamma_{t_i}, \delta_{t_i}\) for \(i < \theta(\xi)\) so that
\[
\alpha_{t_i} = \delta_{t_i} + 1,
\]
and if \(i = 0\), \(\beta_{t_0} = \alpha_{t_0} + 1\) and if \(i > 0\)
\[
\beta_{t_i} = \sup\{\delta_s + 2 : \max(s) < \xi \text{ or } s = t_j \text{ for some } j < i\}.
\]
The corresponding pairs \((\gamma_{t_i}, \delta_{t_i})\) are determined by the strategy \(\sigma_{t_i}\). With these choices it is easily seen that (a) continues to hold; we have to check that (d) also holds when \(s \perp t\) and \(\max(s) = \xi\) or \(\max(t) = \xi\).

If \(\max(s) = \max(t) = \xi\), then \(s = t_i\) and \(t = t_j\), where say \(i < j\). Then
\[
\alpha_t = \delta_{t_*} + 1 < \beta_s < \gamma_s < \delta_s < \beta_t,
\]
and so (d)(i) holds.

Suppose \(\max(s) < \xi = \max(t)\). Then by the induction hypothesis, either (i) or (ii) of (d) holds when we replace \(t\) by \(t_*\). Suppose first that (d)(i) holds. Then for some \(m \leq \ell n(t_*\rangle\) we have that
\[
\alpha_{t_*\mid m} < \gamma_s < \delta_s < \beta_{t_*\mid m}.
\]
It follows that (d)(i) also holds for \(s\) and \(t\) since \(t\mid m = t_*\mid m\). Now suppose that (d)(ii) holds so that, for some \(m \leq \ell n(s)\),
\[
\alpha_{s\mid m} < \gamma_{t_*} < \delta_{t_*} < \beta_{s\mid m}.
\]
Then, by the definitions of \(\alpha_t\) and \(\beta_t\), it follows that
\[
\alpha_t = \delta_{t_*} + 1 \leq \beta_s < \gamma_s < \delta_s < \beta_t,
\]
so that again (d)(i) holds for \(s\) and \(t\). Similarly, if \(\max(t) < \xi = \max(s)\).

**Proof of Fact A.** We have to show that \(P_2\) wins the game \(G_{t_i}\) for some \(\xi < \lambda\). Suppose for a contradiction that this is false. Since the games are open and hence determined, it follows that \(P_1\) has a winning strategy, say \(\tau_{t_i}\), for the game \(G_{t_i}\) for every \(\xi < \lambda\).

For convenience we write \(c(\{\alpha, \beta\}) = -1\) if \(\{\alpha, \beta\} \notin E\), so that \(c\) is defined on all pairs \(\{\alpha, \beta\} \in [\kappa]^2\). For each bounded subset \(X \subseteq \kappa\) define an equivalence relation \(\varepsilon_X\) on \(S \setminus (\sup(X) + 1)\) so that \(\beta \varepsilon_X \gamma\) holds if and only if
(i) \(\beta, \gamma \in S\) and \(\sup(X) < \beta, \gamma < \kappa\);
(ii) \(c(\{\alpha, \beta\}) = c(\{\alpha, \gamma\})\) for all \(\alpha \in X\);
(iii) \(X \cap C_{\beta} = X \cap C_{\gamma}\), (iv) for \(\alpha \in X\), \(\alpha \leq \min(C_{\beta}) \iff \alpha \leq \min(C_{\gamma})\),
\[
\text{tp}(\alpha \cap C_{\beta}) = \text{tp}(\alpha \cap C_{\gamma}) \text{ and } \text{h}_{\beta}(\sup(\alpha \cap C_{\beta})) = \text{h}_{\gamma}(\sup(\alpha \cap C_{\gamma}))\] (for \(\alpha > \min(C_{\beta})\)).
Note that the equivalence relation $e_X$ has at most $(\lambda^+)^{|X|} \leq 2^{|X|}$ classes. Also, if $Y \subseteq X$, then $\beta e_X \gamma \Rightarrow \beta e_Y \gamma$.

Since $\kappa = (2^\lambda)^+$, there is a continuous increasing sequence of ordinals $(\rho_\eta : \eta < \kappa)$ in $\kappa$ such that the following two conditions hold:

(o) If $X \subseteq \rho_\eta$, $|X| \leq \lambda$ and $\rho_\eta < \beta < \kappa$, then there is some $\gamma \in (\rho_\eta, \rho_{\eta+1})$ such that $\beta e_X \gamma$

(oo) $\rho_\eta$ is closed under $\tau_\zeta$ for all $\zeta < \lambda$. In other words, if at the $n$-th stage of a play in the game $G_\zeta$, player $P_2$ chooses $\gamma_0 < \delta_0 < \rho_\eta$, then $P_1$’s response using $\tau_\zeta$ is to choose $\alpha_{n+1}, \beta_{n+1}$ so that $\delta_n < \alpha_{n+1} < \beta_{n+1} < \rho_\eta$.

Since $K = \{\rho_\eta : \eta < \kappa\}$ is a club in $\kappa$, there is some $\delta \in S$ such that $\min(C_\delta) \in K$ and, for $\varepsilon \in \{0, 1\}$,

$$A_\varepsilon = \{\alpha \in C_\delta \cap K : h_\delta^\varepsilon(\alpha) = \varepsilon \land \min(C_\delta \setminus (\alpha + 1)) \in K\}$$

is an unbounded subset of $\delta$. Let $C_\delta = \{i_\sigma : \sigma < \lambda^+\}$, where $i_0 < i_1 < \cdots$.

We claim that the following assertion holds for some $\zeta < \lambda$.

(*) $\zeta$: If $X \subseteq \delta$, $|X| \leq \lambda$, then there are $\sigma < \lambda^+$ and $\gamma$ such that (a) $\sup(X) < i_\sigma < \gamma < i_{\sigma+1}$, (b) $i_\sigma \in A_0$, (c) $\gamma e_\delta \delta$, and (d) $c(\gamma, \delta) = \zeta$.

For suppose the claim is false. Then, for each $\zeta < \lambda$ there is a counter-example $X_\zeta$. Let $X = \bigcup \{X_\zeta : \zeta < \lambda\}$. Then $X \subseteq \delta$ and $|X| \leq \lambda$ and so, for some $\alpha \in A_0$, $\sup(X) < \alpha < \delta$. There are $\eta < \kappa$ and $\sigma < \lambda^+$ such that $\alpha = \rho_\eta = i_\sigma$, and therefore, by the choice of $\rho_{\eta+1}$, there is $\gamma$ such that $\rho_\eta < \gamma < \rho_{\eta+1}$ and $\gamma e_\delta \delta$.

Since $\alpha = i_\sigma \in A_0$, $i_{\sigma+1} = \min(C_\delta \setminus (\alpha + 1)) \in K$. So $\rho_{\eta+1} \leq i_{\sigma+1}$. Therefore, $\sup(C_\delta \cap \gamma) = i_\sigma$, and since $\alpha = i_\sigma \in A_0$, we have that $h_\delta^\varepsilon(\sup(C_\delta \cap \gamma)) = 0$. Therefore, $\{\gamma, \delta\}$ is an edge of $G$ and there is some $\zeta \in \lambda$ such that $c(\gamma, \delta) = \zeta$.

But this contradicts the choice of $X_\zeta \subseteq X$, and hence (**) holds for some $\zeta < \lambda$.

By induction on $n < \omega$ we now choose ordinals $\alpha_n, \beta_n, \gamma_n, \delta_n$ in $\delta$ and $\sigma(n) < \lambda^+$ so that the following conditions are satisfied:

A: $\{(|\alpha_m, \beta_m, \gamma_m, \delta_m| : m \leq n\}$ is an initial segment of a play in the game $G_\zeta$ in which $P_1$ uses the winning strategy $\tau_\zeta$.

B: $\alpha_0, \beta_0 < \min(C_\delta)$.

C: $\gamma_n = \min\{\gamma : \gamma > i_{\sigma(2n)} \land \gamma e_\delta \delta \land c(\gamma, \delta) = \zeta\}$, where

$$X_n = \bigcup \{\{\alpha_\ell, \beta_\ell, \gamma_\ell, \delta_\ell\} : \ell < n\} \cup \{\alpha_n, \beta_n\} \cup \bigcup \{\{i_{\sigma(\ell)}, i_{\sigma(\ell)+1}\} : \ell < 2n\}.$$

D: $\delta_n = i_{\sigma(2n+1)}$.

E: For $n > 0$, $[\alpha_n, \beta_n] \subseteq (\delta_{n-1}, i_{\sigma(2n-1)+1})$.

F: $i_{\sigma(n)}$ belongs to $A_0$ or $A_1$ according as $n$ is even or odd and $\sigma(n)+1 < \sigma(n+1)$.

We have to prove that it is possible to choose the $\alpha_n$ etc., so that these conditions are satisfied. Clearly (B) holds since, by (oo), the first moves by $P_1$ using the statey $\tau_\zeta$ are $\alpha_0 < \beta_0 < \rho_0$ and $\rho_0 \leq \min(C_\delta) \in K$. By (**) there are $\sigma(0) < \lambda^+$
and $\gamma$ such that $i_{\sigma(0)} \in A_0$, $i_{\sigma(0)} < \gamma < i_{\sigma(0)+1}$, $\gamma \in X_0 \delta$, where $X_0 = \{\alpha_0, \beta_0\}$ and $c(\gamma, \delta) = \zeta$; let $\gamma_0$ be the least such $\gamma$. Now let $\sigma(1) > \sigma(0) + 1$ be minimal so that $i_{\sigma(1)} \in A_1$, and put $\delta_0 = i_{\sigma(1)}$. Now suppose that $n > 0$ and that the $\alpha_m, \beta_m, \gamma_m, \delta_m, \sigma(2m)$ and $\sigma(2m + 1)$ have been suitably defined for all $m < n$. Let $\rho \in K$ be minimal such that $\rho > \delta_{n-1}$. $P_1$ chooses $\alpha_n, \beta_n$ using the strategy $\tau_{\zeta}$ so that $\delta_{n-1} < \alpha_n < \beta_n < \rho$. Since $\delta_{n-1} = i_{\sigma(2n-1)} \in A_1$, it follows that $i_{\sigma(2n-1)+1} \in K$ and hence $\rho \leq i_{\sigma(2n-1)+1}$. Now by (*)$\zeta$, there are $\sigma(2n)$ and $\gamma$ so that $i_{\sigma(2n)} \in A_0$, $i_{\sigma(2n)} < \gamma < i_{\sigma(2n)+1}$, $\gamma \in X_n \delta$ (where $X_n$ is as described in (C)), and $c(\gamma, \delta) = \zeta$; let $\gamma_n$ be the least such $\gamma$. Note that, since $i_{\sigma(2n)} \in A_0$, $i_{\sigma(2n)+1} = \min(C_{\delta} \setminus (i_{\sigma(2n)+1} + 1)) \in K$. Finally, choose a minimal ordinal $\sigma(2n+1) > \sigma(2n) + 1$ so that $\delta_n = i_{\sigma(2n+1)} \in A_1$. This completes the definition of the $\alpha_n$ etc., so that (A)-(F) hold.

By (C) it follows that $c(\gamma_n, \delta) = \zeta$ for all $n < \omega$, and hence $c(\gamma_m, \gamma_n) = \zeta$ holds for all $m < n < \omega$ since $\gamma_m \in X_n$ and $\gamma_n \in X_n \delta$. There is no edge of $G_\kappa$ from $\delta$ to $(\alpha_0, \beta_0)$ since $\beta_0 < \min(C_{\delta})$. Since $\gamma_n \in X_n \delta$ and $\beta_0 \in X_n$, it follows that $\beta_0 < \min(C_{\gamma_n})$ also, and so there is no edge from $\gamma_n$ to $(\alpha_0, \beta_0)$ either. By the construction, for $0 < m < \omega$, $i_{\sigma(2m-1)} < \alpha_m < \beta_m < i_{\sigma(2m-1)+1}$, and hence $C_{\delta} \cap (\alpha_m, \beta_m) = \emptyset$. Therefore, for any $\xi \in (\alpha_m, \beta_m)$, $h^*_{\gamma_n}(\sup(\xi \cap C_\delta)) = h^*_{\gamma_n}(i_{\sigma(2m-1)+1}) = 1$ by (F), and so there is no edge of $G$ from $\delta$ to $(\alpha_m, \beta_m)$. If $0 < m < n < \omega$, then $\gamma_n \notin X_n \delta$ and therefore,

$$\text{tp}(\alpha_m \cap C_{\gamma_n}) = \text{tp}(\alpha_m \cap C_{\delta}) = \text{tp}(\beta_m \cap C_{\delta}) = \text{tp}(\beta_m \cap C_{\gamma_n}).$$

Therefore, for any $\xi \in (\alpha_m, \beta_m)$, it follows that

$$h^*_{\gamma_n}(\sup(\xi \cap C_{\gamma_n})) = h^*_{\gamma_n}(\sup(\alpha_m \cap C_{\gamma_n})) = h^*_{\gamma_n}(\sup(\alpha_m \cap C_{\delta})) = 1$$

and so there are no edges of $G$ from $\gamma_n$ to $(\alpha_m, \beta_m)$ either.

Thus we have produced a play in the game $G_\zeta$ in which $P_1$ uses the strategy $\tau_{\zeta}$ but the second player $P_2$ wins! This contradicts the assumption that $\sigma_{\zeta}$ is a winning strategy for the first player, and completes the proof. \(\square\)

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