Non-abelian monopoles and vortices

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1 Introduction

In the short time since their discovery, the Seiberg–Witten equations have already proved to be a powerful tool in the study of smooth four-manifolds. Virtually all the hard-won gains that have been obtained using the heavy machinery of Donaldson invariants, can be recovered with a fraction of the effort if the (SU(2)) anti-self-duality equations are replaced by the (U(1)) Seiberg–Witten equations. In addition, the new equations probe deep features of symplectic structures. They have also been used to study geometric questions on Kähler surfaces. (See [D3] for a useful survey.)

The impressive success of the original equations has naturally led to speculation about possible generalizations and other related sets of equations. The original equations as proposed by Seiberg–Witten are associated with a Hermitian line bundle, and thus with the abelian group U(1). One way to generalize the equations is thus to look for versions based on larger, non-abelian groups. This means replacing the line bundle with a higher rank complex vector bundle. Indeed a number of authors have proposed versions of the equations along these lines. These include, among others, Okonek and Teleman [OT1, OT2], Pidstrigach and Tyurin [PT], Labastida and Mariño [LM], as well as the second author [G5]. Some of these (cf. [PT]) play a key role in attempts to prove the conjecture of Witten [W] concerning the equivalence of the old Donaldson and the new Seiberg–Witten invariants. (See also [D3] and [FL] for more recent progress in this direction.)

It is striking that no two of the above mentioned authors consider precisely the same set of equations. One conclusion to be drawn from this abundance of equations, is that there is apparently more than one natural way to write down non abelian versions of the Seiberg-Witten equations. This leads to the question: Are some versions more reasonable, or more natural, than others? The material in this paper gives one perspective on this question.

The main idea in our point of view is to exploit the special form of the Seiberg-Witten equations in the case where the four manifold is a Kähler surface. In this case the original Seiberg–Witten equations are known to reduce essentially to familiar equations in gauge theory known as the abelian vortex equations. Looked at from the opposite direction, the Seiberg–Witten equations serve as a “Riemannian version” of the vortex equations. The key point is that there are a number of well motivated, natural generalizations of such vortex equations. All of these are defined on complex vector bundles over Kähler...
manifolds, and thus in particular over Kähler surfaces. Our guiding principle is that the generalizations of the Seiberg-Witten equations should provide "Riemannian versions" of these vortex-type equations over Kähler surfaces.

In this paper we explore essentially two such non-abelian generalizations. We also make some remarks concerning a different aspect of the relation between the vortex and the Seiberg-Witten equations. This aspect has to do with the parameters which appear in the vortex equations. In their original form, these were taken to be real numbers, i.e. constant functions on the base manifolds. In the versions that emerge from the Seiberg-Witten equations, the analogous terms turn out to be non-constant functions (related to the scalar curvature). This has prompted a closer look at the affected terms in the vortex equation. We discuss various ways of incorporating — and interpreting — this level of generality in the analysis of the vortex equations.

In the interests of completeness, we have included a certain amount of standard background material on the Seiberg-Witten and vortex equations.

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2 The Seiberg–Witten monopole equations

In this section we briefly review the Seiberg–Witten equations and the analysis of these equations in the Kähler case. For more details, see the original papers by Witten [W] and Kronheimer and Mrowka [KM], or any recent survey on the subject (e.g. [D3, G5]).

Let \((X, g)\) be a compact, oriented, Riemannian four-manifold. To write the Seiberg–Witten equations one needs a Spin\(_c\)-structure on \(X\). This involves the choice of a Hermitian line bundle \(L\) on \(X\) satisfying that \(c_1(L) \equiv w_2(X) \mod 2\). A Spin\(_c\)-structure is then a lift of the fibre product of the \(\text{SO}(4)\)-bundle of orthonormal frames of \((X, g)\) with the \(U(1)\)-bundle defined by \(L\) to a Spin\(_c\)(4)-bundle, according to the short exact sequence

\[
0 \rightarrow \mathbb{Z}_2 \rightarrow \text{Spin}^c(4) \rightarrow \text{SO}(4) \times U(1) \rightarrow 1.
\]

Using the two fundamental irreducible 2-dimensional representations of Spin\(_c\)(4)—the so-called Spin representations—we can construct the associated vector bundles of positive and negative spinors \(S_L^\pm\). These are rank 2 Hermitian vector bundles whose determinant is \(L [1, \text{LaM}]\). The set of Spin\(_c\)-structures on \(X\) is thus parametrised, up to the finite group \(H^1(X, \mathbb{Z}_2)\), by

\[
\text{Spin}^c(X) = \{ c \in H^2(X, \mathbb{Z}) \mid c \equiv w_2(X) \mod 2 \}.
\]

Let us fix a Spin\(_c\)-structure \(c \in \text{Spin}^c(X)\), and let \(L = L_c\) be the corresponding Hermitian line bundle, and \(S_L^\pm\) the corresponding spinor bundles. The Seiberg–Witten monopole equations are equations for a pair \((A, \Psi)\) consisting of a unitary connection on \(L\) and a
smooth section of $S^+_L$. Using the connection $A$ one has the Dirac operator

$$D_A : \Gamma(S^+_L) \rightarrow \Gamma(S^-_L).$$

The first condition is that $\Psi$ must be in the kernel of the Dirac operator.

The curvature $F_A \in \Omega^2 = \Omega^+ \oplus \Omega^-$, can be decomposed in the self-dual and anti-selfdual parts

$$F_A = F_A^+ + F_A^-.$$

Using the spinor $\Psi$ we can consider another self-dual 2-form that we may couple to $F_A^+$ to obtain our second equation. Let $\text{ad}_0 S^+_L$ be the subbundle of the adjoint bundle of $S^+_L$ consisting of the traceless skew-Hermitian endomorphisms — its fibres are hence isomorphic to $\text{su}(2)$. We have a map

$$\Omega^0(S^+_L) \rightarrow \Omega^0(\text{ad}_0 S^+_L)$$

$$\Psi \mapsto i(\Psi \otimes \Psi^*)_0,$$

where $\Psi^*$ is the adjoint of $\Psi$, and the 0 subindex means that we are taking the trace-free part. This map is fibrewise modelled on the map $\mathbb{C}^2 \rightarrow \text{su}(2)$, given by $v \mapsto i(v\pi^*)_0$.

One of the basic ingredients that makes the Seiberg–Witten equations possible is the identification between the space of self-dual 2-forms and the skew-Hermitian automorphism of the positive spin representation [AHS]. This is a basic fact in Clifford algebras in dimension four, that takes place at each point of the manifold, and that can be carried out over the whole manifold precisely when one has a Spin$^c$-structure. More specifically, we have the isomorphism

$$\text{ad}_0 S^+_L \cong \Lambda^+.$$ (1)

We can now interpret $i(\Psi \otimes \Psi^*)_0$ as a section of $\Lambda^+$, i.e. as an element in $\Omega^+$. The monopole equations consist in the system of equations

$$\begin{align*}
D_A \Psi &= 0 \\
F_A^+ &= i(\Psi \otimes \Psi^*)_0
\end{align*}$$ (2)

In writing the second equation there is an abuse of notation, since we are not specifying what the isomorphism (1) is. Notice also that $F_A^+$ is a purely imaginary self-dual 2-form, and hence we are in fact identifying $\text{ad}_0 S^+_L$ with $i\Lambda^+$.

We shall analyse now the monopole equations in the case in which $(X, g)$ is Kähler. Recall that a Kähler manifold is Spin if and only if there exists a square root of the canonical bundle $K^{1/2}$ [A, H]. Moreover the spinor bundles are

$$S^+ = (\Lambda^0 \oplus \Lambda^{0,2}) \otimes K^{1/2} = K^{1/2} \oplus K^{-1/2}$$

$$S^- = \Lambda^{0,1} \otimes K^{1/2}.$$

In this situation the spinor bundles for the Spin$^c$-structure $c$ are given by $S^+_L = S^\pm \otimes L^{1/2}$ (notice that $L^{1/2}$ exists since $c_1(L) \equiv 0 \mod 2$). Even if $X$ is not Spin, i.e. even if $K^{1/2}$ and $L^{1/2}$ do not exist, the bundles $S^\pm_L$ do exist. In other words, there exists a square root of $K \otimes L$. Let us denote

$$\hat{L} = (K \otimes L)^{1/2}.$$
Then
\[ S^+_L = \hat{L} \oplus \Lambda^{0,2} \otimes \hat{L}, \quad S^-_L = \Lambda^{0,1} \otimes \hat{L} \]

and
\[ \Gamma(S^+_L) = \Omega^0(\hat{L}) \oplus \Omega^{0,2}(\hat{L}). \]

We can write \( \Psi \) according to this decomposition as a pair \( \Psi = (\phi, \beta) \). The Dirac operator can be written in this language as
\[ \overline{\partial}_A + \overline{\partial}_A^* : \Omega^0(\hat{L}) \oplus \Omega^{0,2}(\hat{L}) \rightarrow \Omega^{0,1}(\hat{L}), \]

where \( \overline{\partial}_A \) is the \( \overline{\partial} \) operator on \( \hat{L} \) corresponding to the connection \( A \) on \( \hat{L} \) defined by the connection \( A \) on \( L \) and the metric connection on \( K \) (cf. [H]).

On the other hand recall that
\[ \Lambda^+ \otimes C = \Lambda^0 \omega \oplus \Lambda^{2,0} \oplus \Lambda^{0,2}, \]

where \( \omega \) is the Kähler form. According to this decomposition, the isomorphism (4) (or rather \( \text{ad}_0 S^+_L \cong i\Lambda^+ \)) is explicitly given by
\[ i(\Psi \otimes \Psi^*)_0 \mapsto i(|\phi|^2 - |\beta|^2)\omega + \beta\overline{\phi} - \phi\overline{\beta}. \] (3)

We may thus write the monopole equations (4) as
\[ \begin{align*}
\overline{\partial}_A \phi + \overline{\partial}_A^* \beta &= 0 \\
\Lambda F_A &= i(|\phi|^2 - |\beta|^2) \\
F^{2,0}_A &= -\phi\overline{\beta} \\
F^{0,2}_A &= \beta\overline{\phi}
\end{align*} \] (4)

where \( \Lambda F_A \) is the contraction of the curvature with the Kähler form.

It is not difficult to see (cf. [W]) that the solutions to these equations are such that either \( \beta = 0 \) or \( \phi = 0 \), and moreover it is not possible to have irreducible solutions, i.e. solutions with \( \Psi \neq 0 \), of both types simultaneously for a fixed Spin\(^c\)-structure. We thus have one of the following two situations:

(i) \( \beta = 0 \) and the equations reduce to
\[ \begin{align*}
F^{0,2}_A &= 0 \\
\overline{\partial}_A \phi &= 0 \\
\Lambda F_A &= i|\phi|^2
\end{align*} \] (5)

(ii) \( \phi = 0 \) and then
\[ \begin{align*}
F^{0,2}_A &= 0 \\
\overline{\partial}_A^* \beta &= 0 \\
\Lambda F_A &= -i|\beta|^2.
\end{align*} \] (6)

Remark. We have omitted the equation \( F^{2,0}_A = 0 \), since by unitarity of the connection this is equivalent to \( F^{0,2}_A = 0 \).
Clearly if we have solutions to (5), from the third equation in (5) we obtain that \( \deg L \leq 0 \), while from (6) we have \( \deg L \geq 0 \), where \( \deg L \) is the degree of \( L \) with respect to the Kähler metric defined as in (11). Since we are interested only in irreducible solutions, obviously these two situations cannot occur simultaneously. The Hodge star operator interchanges these two cases, and we can thus concentrate on case (i).

Equations (5) are essentially the equations known as the vortex equations. These are generalisations of the vortex equations on the Euclidean plane studied by Jaffe and Taubes [T1, T2, JT], and have been extensively studied (e.g. in [B1, B2, G2, G3]) for compact Kähler manifolds of arbitrary dimension. In that setting, the equations are the following:

Let \((X, \omega)\) be a compact Kähler manifold of arbitrary dimension, and let \((L, h)\) be a Hermitian \(C^\infty\) line bundle over \(X\). Let \(\tau \in \mathbb{R}\). The \(\tau\)-vortex equations

\[
\begin{align*}
F^{0,2}_A &= 0, \\
\overline{\partial}_A \phi &= 0, \\
\Lambda F_A &= \frac{i}{2}(|\phi|^2 - \tau)
\end{align*}
\] (7)

are equations for a pair \((A, \phi)\) consisting of a connection on \((L, h)\) and a smooth section of \(L\). The first equation means that \(A\) defines a holomorphic structure on \(L\), while the second says that \(\phi\) must be holomorphic with respect to this holomorphic structure.

Coming back to the monopole equations, we first observe that (5) can be rewritten in terms of \(\hat{A}\) only, i.e. not involving simultaneously \(A\) and \(\hat{A}\). To do this we recall that \(\hat{A} = (A \otimes a_K)^{1/2}\), where \(a_K\) is the metric connection on \(K\). We thus have

\[
F_{\hat{A}} = \frac{1}{2}(F_A + F_{a_K}),
\] (8)

and hence \(F^{0,2}_A = 0\) is equivalent to \(F^{0,2}_{\hat{A}} = 0\) since \(F^{0,2}_{a_K} = 0\).

From (8), and using that \(s = -i\Lambda F_{a_K}\) is the scalar curvature, we obtain that

\[
\Lambda F_A = 2\Lambda F_{\hat{A}} - is,
\]

and hence (5) is equivalent to

\[
\begin{align*}
F^{0,2}_{\hat{A}} &= 0, \\
\overline{\partial}_{\hat{A}} \phi &= 0, \\
\Lambda F_{\hat{A}} &= \frac{i}{2}(|\phi|^2 + s)
\end{align*}
\] (9)

These are the vortex equations on \(\hat{L}\), but with the parameter \(\tau\) replaced by minus the scalar curvature. As we will explain in Section 6 the existence proofs for the vortex equations can be easily modified to give an existence theorem for the equations obtained by replacing the parameter \(\tau\) by a function \(t \in C^\infty(X, \mathbb{R})\) in (7). However, to compute the Seiberg–Witten invariants one can slightly perturb equations (2) in such a way that, when \(\beta = 0\), equations (2) reduce to the constant function vortex equations, i.e. to (7) (see e.g. [G5]).
3 Non-abelian vortex equations

As we have seen in the previous section, the Seiberg–Witten monopole equations can be considered as a four-dimensional Riemannian generalisation of the vortex equations. This suggests that we may find interesting Seiberg–Witten-type equations by considering the corresponding analogues of different equations of vortex-type existing in the literature. With this objective in mind, in this section we shall review three different non-abelian generalisations of the vortex equations described above. The first one consists in studying the vortex equations on a Hermitian vector bundle of arbitrary rank. The other two involve two vector bundles, one of which will actually be a line bundle in most cases.

Let \((E, H)\) be a Hermitian vector bundle over a compact Kähler manifold \((X, \omega)\) of complex dimension \(n\). Let \(\tau \in \mathbb{R}\). The \(\tau\)-vortex equations were generalised to this situation in [B2]. As in the line bundle case, one studies equations

\[
\begin{aligned}
F_A^{0,2} &= 0 \\
\overline{\partial}_A \phi &= 0 \\
\Lambda F_A &= i(\phi \otimes \phi^* - \tau I)
\end{aligned}
\]

for a pair \((A, \phi)\) consisting of a unitary connection on \((E, H)\) and a smooth section of \(E\). By \(\phi^*\) we denote the adjoint of \(\phi\) with respect to \(H\), and \(I \in \text{End} E\) is the identity. Notice that in the abelian equations (7) there is a \(1/2\) in the third equation, while in (10) there is not. This is not essential since by applying a constant complex gauge transformation we can introduce an arbitrary positive constant in front of \(i\phi \otimes \phi^*\).

These vortex equations appear naturally as the equations satisfied by the minima of the Yang–Mills–Higgs functional. This is a functional defined on the product of the space \(\mathcal{A}\) of unitary connections on \((E, H)\) and the space of smooth sections \(\Omega^0(E)\) by

\[
\text{YMH}_\tau(A, \phi) = \|F_A\|^2 + 2\|d_A \phi\|^2 + \|\phi \otimes \phi^* - \tau I\|^2,
\]

where \(\|\|\) denotes the \(L^2\)-metric.

This is easily seen by rewriting the Yang–Mills–Higgs functional — using the Kähler identities — as

\[
\text{YMH}_\tau(A, \phi) = 4\|F_A^{0,2}\|^2 + 4\|\overline{\partial}_A \phi\|^2 + \|i\Lambda F_A + \phi \otimes \phi^* - \tau I\|^2
\]

\[
+ 4\pi \tau \deg E - \frac{8\pi^2}{(n - 2)!} \int_X \text{ch}_2(E) \wedge \omega^{n-2},
\]

where \(\deg E\) is the degree of \(E\) defined as

\[
\deg E = \int_X c_1(E) \wedge \omega^{n-1},
\]

and \(\text{ch}_2(E)\) is the second Chern character of \(E\), which is represented in terms of the curvature by

\[
\text{ch}_2(E) = -\frac{1}{8\pi^2} \text{Tr}(F_A \wedge F_A).
\]
Clearly \( \text{YMH}_\tau \) achieves its minimum value

\[
4\pi \tau \deg E - \frac{8\pi^2}{(n-2)!} \int_X \text{ch}_2(E) \wedge \omega^{n-2}
\]

if and only if \((A, \phi)\) is a solution to equations (10) (see [B1] for details).

As we will explain in Section 6, the vortex equations also have a symplectic interpretation as moment map equations. The moment map in question is for a symplectic action of \( \mathcal{G}(E) \), i.e. the unitary gauge group of \( E \), on a certain infinite dimensional symplectic space.

A natural generalization of these equations is obtained if we regard the section \( \phi \) in (10) as a morphism from the trivial line bundle to \( E \). One can replace the trivial line bundle by a vector bundle of arbitrary rank and study equations for connections on both bundles and a morphism from one to the other. These are the \emph{coupled vortex equations} introduced in [G4].

Let \((E, H)\) and \((F, K)\) be smooth Hermitian vector bundles over \( X \). Let \( A \) and \( B \) be unitary connections on \((E, H)\) and \((F, K)\) resp., and let \( \phi \in \Omega^0(\text{Hom}(F, E)) \). Let \( \tau \) and \( \tau' \) be real parameters. The coupled vortex equations are

\[
\begin{align*}
F^0_A &= 0 \\
F^0_B &= 0 \\
\partial_{A,B} &\phi = 0 \\
i\Lambda F_A + \phi\phi^* &= \tau E \\
i\Lambda F_B - \phi^* \phi &= \tau' F
\end{align*}
\]

As in the case of the vortex equations described above, equations (12) correspond to the minima of a certain Yang–Mills–Higgs functional and are also moment map equations (cf. [G4]). The appropriate functional in this case is defined as

\[
\text{YMH}_{\tau, \tau'}(A, B, \phi) = \|F_A\|^2 + \|F_B\|^2 + 2\|d_{A\otimes B}\phi\|^2 + \|\phi\phi^* - \tau E\|^2 + \|\phi^* \phi - \tau' F\|^2.
\]

The moment map is now for a symplectic action of \( \mathcal{G}(E) \times \mathcal{G}(F) \), i.e. for the product of the unitary groups of \( E \) and \( F \).

In this paper we will be mostly interested in the case in which \( F = L \) is a line bundle. In this situation the equations can be written as

\[
\begin{align*}
F^0_A &= 0 \\
F^0_B &= 0 \\
\partial_{A,B} &\phi = 0 \\
i\Lambda F_A + \phi \otimes \phi^* &= \tau E \\
i\Lambda F_B - |\phi|^2 &= \tau'
\end{align*}
\]

It is clear, from taking the trace of the last two equations in (13) and integrating, that to solve (13) \( \tau \) and \( \tau' \) must be related by

\[
\tau \text{ rank } E + \tau' = \deg E + \deg L,
\]

(14)
hence there is only one free parameter.

We shall consider next the framed vortex equations. The situation is very similar to the previous one in that it also involves two vector bundles. In this case, however, the connection on one of the bundles is fixed. More specifically, let \((E, H)\) and \((F, K)\) be two Hermitian vector bundles. Let \(B\) a fixed Hermitian connection on \((F, K)\) such that \(F_B^{0,2} = 0\).

The equations are now for a unitary connection \(A\) on \((E, H)\), and \(\phi \in \Omega^0(\text{Hom}(F, E))\). As explained in [BDGW], the appropriate equations are

\[
\begin{align*}
F_A^{0,2} &= 0 \\
\overline{\partial}_{A,B} \phi &= 0 \\
i A F_A + \phi \phi^* &= \tau I_E \\
\end{align*}
\]

The relation between these equations and the full coupled vortex equations is perhaps best understood from the symplectic point of view. One sees that the effect of fixing the data on \(F\) is to reduce the symmetry group in the problem from \(G(E) \times G(F)\) to \(G(E)\). The new equations must thus correspond to the moment map for the subgroup \(G(E) \subset G(E) \times G(F)\). But the moment maps for \(G(E)\) and for \(G(E) \times G(F)\) are related by a projection from the Lie algebra of \(G(E) \times G(F)\) onto the summand corresponding to the Lie algebra of \(G(E)\). The effect of this projection is to eliminate the last equations in (13) (cf. [BDGW] for more details).

The appropriate moduli space problem corresponds to that of studying morphisms from a vector bundle with a fixed holomorphic structure to another vector bundle in which the holomorphic structure is varying. Such moduli spaces have been studied by Huybrecht and Lehn [HL1, HL2], who refer to these objects as framed modules.

As in the coupled vortex equations, we will be mostly interested in the case in which \(F = L\) is a line bundle.

All the vortex-type equations that we have considered so far involve one or two real parameters \(\tau\) and \(\tau'\). As we have seen in Section 2, the study of the Seiberg–Witten monopole equations leads to abelian vortex equations in which \(\tau\) is replaced by a function. The same will happen in the generalizations of the monopole equations that we are about to discuss. This will be analysed in detail in Section 6.

4 Non-abelian monopole equations

Let us go back to the set-up of Section 2 and let \((X, g)\) be a compact, oriented, Riemannian, four-dimensional manifold. Let \(c \in \text{Spin}^c(X)\) be a fixed \(\text{Spin}^c\)-structure, with corresponding Hermitian line bundle \(L\) and bundles of spinors \(S_L^\pm\). Let \((E, H)\) be a Hermitian vector bundle on \(X\). Let \(\Psi \in \Gamma(S_L^+ \otimes E)\). Using the metrics on \(S_L^+\) and \(E\) one has the antilinear identification

\[
S_L^+ \otimes E \rightarrow S_L^{++} \otimes E^* \\
\Psi \mapsto \Psi^*.
\]
and hence\[ \Psi \otimes \Psi^* \in \text{End}(S_L^+ \otimes E). \]

We shall also need the map\[ \text{End}(S_L^+ \otimes E) \longrightarrow \text{End}_0(S_L^+) \otimes \text{End} E \]
\[ \Psi \otimes \Psi^* \mapsto (\Psi \otimes \Psi^*)_0, \]
as well as the map\[ \text{End}_0(S_L^+) \otimes \text{End} E \xrightarrow{\text{Tr}} \text{End}_0(S_L^+) \]
\[ (\Psi \otimes \Psi^*)_0 \mapsto \text{Tr}(\Psi \otimes \Psi^*)_0 \]
obtained from the trace map\[ \text{End}_0 E \longrightarrow \text{End} E \xrightarrow{\text{Tr}} \mathbf{C} \longrightarrow 0. \]

The endomorphism \((\Psi \otimes \Psi^*)_0\) should not be confused with the completely traceless part of \(\Psi \otimes \Psi^*\) — here we are only removing the trace corresponding to \(S_L^+\).

In this paper we shall consider essentially two different non-abelian generalizations of the Seiberg–Witten equations. While in the first one we will fix a connection \(b\) on \(L\) and study equations for a pair \((A, \Psi)\), where \(A\) is a unitary connection on \((E, H)\) and \(\Psi \in \Gamma(S_L^+ \otimes E)\), in the second one we will allow \(b\) to vary as well, and hence our system of equations will be one for triples \((A, b, \Psi)\).

The first non-abelian version of the Seiberg–Witten equations is suggested by the framed vortex equations: Let \(b\) be a fixed unitary connection on \(L\) and \(A\) be a unitary connection on \(E\). Using these two connections and the Levi–Civita connection one can consider the coupled Dirac operator\[ D_{A,b} : \Gamma(S_L^+ \otimes E) \longrightarrow \Gamma(S_L^- \otimes E). \]
and study the equations\[ \begin{aligned}
D_{A,b} \Psi &= 0 \\
F_A^+ &= i(\Psi \otimes \Psi^*)_0
\end{aligned} \]
for the unknowns \(A\) and \(\Psi \in \Gamma(S_L^+ \otimes E)\).

The Bochner–Weitzenböck formula for \(D_{A,b}\) is given by\[ D_{A,b}^* D_{A,b} = \nabla_{A,b}^* \nabla_{A,b} + \frac{s}{4} + c(F_{A,b}), \]
where\[ F_{A,b} = F_A + \frac{1}{2} F_b \otimes I_E \]
and \(\nabla_{A,b}\) is the connection on \(E \otimes S_L\) determined by the connections \(A\) and \(b\). The term \(c(F_{A,b})\) in \((17)\) means Clifford multiplication by \(F_{A,b}\). In fact the action of \(F_{A,b}\) on \(\Psi \in \Gamma(S_L^+ \otimes E)\) coincides with the Clifford multiplication with \(F_{A,b}^+\) (see [LaM] for example).
It is thus natural to perturb the second equation in (16) by adding the constant self-dual 2-form \( \frac{1}{2} F_{b}^+ \), and consider instead equations

\[
\begin{align*}
D_{A,b} \Psi &= 0 \\
F_{A}^+ &= i(\Psi \otimes \Psi^*)_0
\end{align*}
\]

These are the equations studied in [OT1].

The abelian Seiberg–Witten equations (2) for a Spin\(^c\)-structure with line bundle \( \hat{L} \) can be recovered from (16) or (18) by considering the Hermitian bundle \( E = (\hat{L} \otimes L^*)^{1/2} \). Notice that this square root exists since \( c_1(\hat{L}) \equiv c_1(L) \mod 2 \).

Other non-abelian versions of the Seiberg–Witten equations that have been considered include replacing the \( U(r)\)-bundle \( (E, H) \) by an \( SU(r)\)-bundle, and study equations (16) in which \( (\Psi \otimes \Psi^*)_0 \) is replaced by the completely trace-free part of \( \Psi \otimes \Psi^* \). These are the equations studied in [LM] (see also [OT2]). In other versions, like the one considered in [PT], one fixes the connection of \( \text{det} E \otimes L \) instead of fixing that of \( L \).

In all the versions mentioned above one considers spinors coupled to a bundle associated the fundamental representation of \( U(r) \) or \( SU(r) \). Another direction in which the monopole equations can be generalized is by considering any compact Lie group \( G \) and/or an arbitrary representation. When the manifold is Kähler some of these correspond to the vortex-type equations described in [G1]. These more general equations will be dealt with somewhere else.

We shall consider next the case in which the connection on \( L \) is also varying. It is clear that we cannot consider the same equations as in the previous situation, since we would not obtain an elliptic complex in the linearization of the equations — we need an extra equation. As the coupled vortex equations (13) suggest, it is natural to consider the following set of equations for the triple \( (A, b, \Psi) \):

\[
\begin{align*}
D_{A,b} \Psi &= 0 \\
F_{A}^+ &= i(\Psi \otimes \Psi^*)_0 \\
F_{b}^+ &= 2i \text{Tr}(\Psi \otimes \Psi^*)_0
\end{align*}
\]

5 The Kähler case

Let now \( (X, \omega) \) be a compact Kähler surface. Let us fix a Spin\(^c\)-structure \( c \in \text{Spin}^c(X) \), and let \( L = L_c \) be the corresponding Hermitian line bundle. As mentioned in Section 2, the corresponding spinor bundles for the Spin\(^c\)-structure \( c \) are given by

\[
S_L^+ = \hat{L} \oplus \Lambda^{0,2} \otimes \hat{L} \quad \text{and} \quad S_L^- = \Lambda^{0,1} \otimes \hat{L},
\]

where

\[
\hat{L} = (K \otimes L)^{1/2}.
\]

Let \( (E, H) \) be a Hermitian vector bundle over \( X \).

\[
E \otimes S_L^+ = E \otimes \hat{L} \oplus \Lambda^{0,2} \otimes E \otimes \hat{L}
\]
and hence

$$\Gamma(E \otimes S^*_L) = \Omega^0(E \otimes \hat{L}) \oplus \Omega^{0,2}(E \otimes \hat{L}).$$

We can write $\Psi$ according to this decomposition as a pair $\Psi = (\phi, \beta)$. Let $b$ and $A$ be unitary connections on $L$ and $E$, respectively. Let $a_K$ be the metric connection on $K$. We shall denote by $\hat{b}$ the connection on $\hat{L}$ defined by $b$ and $a_K$. The Dirac operator $D_{A,b}$ can be written in this language as

$$\overline{\partial}_{A,b} + \overline{\partial}_{A,b} : \Omega^0(E \otimes \hat{L}) \oplus \Omega^{0,2}(E \otimes \hat{L}) \rightarrow \Omega^{0,1}(E \otimes \hat{L}),$$

where $\overline{\partial}_{A,b}$ is the $\overline{\partial}$ operator on $E \otimes \hat{L}$ corresponding to the connections $A$ and $\hat{b}$.

### 5.1 Fixed connection on $L$

We shall perturb equations (18) by a self-dual 2-form $\alpha$ and consider

$$D_{A,b} \Psi = 0 \quad \text{and} \quad F_{A,b}^+ = i((\Psi \otimes \Psi^*)_0 + \alpha I). \quad (21)$$

We will choose the perturbation to be of Kähler type, that is $\alpha = -f\omega$, where $f$ is a smooth real function.

Similarly to the abelian case, we can write (21) as

$$\overline{\partial}_{A,b}\phi + \overline{\partial}_{A,b}\beta = 0 \quad \Lambda F_{A,b} = i(\phi \otimes \phi^* - \Lambda^2 \beta \otimes \beta^* - fI)$$

$$F_{A,b}^{2,0} = -\phi \otimes \beta^* \quad F_{A,b}^{0,2} = \beta \otimes \phi^*.$$

By $\Lambda^2$ we denote the operation of contracting twice with the Kähler form.

As in the abelian case, one can see that the solutions to these equations are such that either $\beta = 0$ or $\phi = 0$. More precisely

**Proposition 5.1** Let $\mathcal{F} = \frac{1}{2\pi} \int_X f$. The only solutions to (22) satisfy either

(i) $\beta = 0$,

$$F_{A,b}^{0,2} = 0 \quad \overline{\partial}_{A,b}\phi = 0 \quad \Lambda F_{A,b} = i(\phi \otimes \phi^* - fI), \quad (23)$$

then $\mu(E) - 1/2 \deg L \leq \mathcal{F}$, or

(ii) $\phi = 0$,

$$F_{A,b}^{0,2} = 0 \quad \overline{\partial}_{A,b}\beta = 0 \quad \Lambda F_{A,b} = -i(\Lambda^2 \beta \otimes \beta^* + fI) \quad (24)$$

and then $\mu(E) - 1/2 \deg L \geq \mathcal{F}$. 

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Proof. One uses exactly the same method as the one used by Witten in the abelian case. Consider the transformation \((A, \phi, \beta) \mapsto (A, -\phi, \beta)\). Although this is not a symmetry of equations (22), if \((A, \phi, \beta)\) is a solution \((A, \phi, -\beta)\) must also be. This is easily seen by considering the functional

\[
SW(A, \Psi) = \|F_{A,b}^+ - i((\Psi \otimes \Psi^*)_0 + \alpha I)\|^2 + 2\|D_{A,b}\Psi\|^2.
\]

Using the Bochner–Weitzenböck formula (17) and the fact that on a Kähler manifold the decomposition (20) is parallel with respect to the connection \(\nabla_{A,b}\), we have

\[
SW(A, \Psi) = \|F_{A,b}^+\|^2 + 2\|\nabla_{A,b}\phi\|^2 + 2\|\nabla_{A,b}\beta\|^2 + \|i((\Psi \otimes \Psi^*)_0 + \alpha I)\|^2
\]

\[
+ \int_X \frac{\delta}{2}(|\phi|^2 + |\beta|^2) - 2\int_X \langle F_{A,b}, i\alpha I \rangle.
\]

Remark. Notice the analogy between this and the way of rewriting the Yang–Mills–Higgs functional in Section 3 using the Kähler identities. In fact in the Kähler case both things are essentially equivalent.

Of course the only way in which the two type of solutions can occur simultaneously is if \(\mu(E) = 1/2 \deg L + \bar{f}\), then \(\Psi = 0\) and the equations reduce essentially to the Hermitian–Einstein equations.

Since the Hodge operator interchanges the roles of \(\phi\) and \(\beta\) we may concentrate in the case \(\phi \neq 0\). We shall write equations (23) in a way that we can identify them as the vortex equations discussed in Section 3. To do this denote by

\[
\hat{E} = E \otimes \hat{L} \quad \text{and} \quad \hat{A} = A \otimes \hat{b},
\]

where recall that \(\hat{b} = b \otimes a_K\). We have that

\[
F_{\hat{A}} = F_{A,b} + \frac{1}{2} F_{a_K},
\]

and hence (23) is equivalent to

\[
\begin{align*}
F_{\hat{A}}^{0,2} &= 0 \\
\overline{\partial} A^\phi &= 0 \\
\Lambda F_{\hat{A}} &= i(\phi \otimes \phi^* + (s/2 - f)I)
\end{align*}
\]

where \(s = -i \Lambda F_{a_K}\) is the scalar curvature of \((X, \omega)\), and we have used that \(a_K\) is integrable, i.e. \(F_{a_K}^{0,2} = 0\). These are indeed the vortex equations (14) on the bundle \(\hat{E}\), with the parameter \(\tau\) replaced by the function \(f - s/2\).

Equations (23) can also be interpreted as the framed vortex equations (14). To see this we choose the fixed connection \(b\) to be integrable. We then have that \(F_{A,b}^{0,2} = F_{\hat{A}}^{0,2}\) and (23) is equivalent to

\[
\begin{align*}
F_{\hat{A}}^{0,2} &= 0 \\
\overline{\partial} A_{A,b^r}^\phi &= 0 \\
\Lambda F_{\hat{A}} &= i(\phi \otimes \phi^* - tI)
\end{align*}
\]
where \( t = f + \frac{i}{2} \Lambda F_b \). We thus obtain the framed vortex equations (13) on \((E, \hat{L}^*)\) with fixed connection \( \hat{b}^* \) on \( \hat{L}^* \), and parameter \( \tau \) replaced by the function \( t \).

These two slightly different points of view in relating (23) to the vortex equations reflect the close relation between the usual vortex equations and the framed vortex equations, as we will explain in Section 6.

### 5.2 Variable connection on \( L \)

We come now to equations (19). In the Kähler situation these equations can be written as

\[
\begin{align*}
\bar{\partial}_{A,b} \phi + \bar{\partial}_{A,b} \beta &= 0 \\
\Lambda F_A &= i(\phi \otimes \phi^* - \Lambda^2 \beta \otimes \beta^*) \\
\Lambda F_b^+ &= 2i(|\phi|^2 - |\beta|^2) \\
F_A^{0,0} &= i(\phi \otimes \phi^* - \Lambda^2 \beta \otimes \beta^*) \\
F^0_{0,2} &= \beta \otimes \phi^* \\
F_{b,2}^+ &= -2 \text{Tr}(\phi \otimes \beta^*) \\
F_{b,2}^0 &= 2 \text{Tr}(\beta \otimes \phi^*) .
\end{align*}
\]

By taking the third equation, subtracting twice the trace of the second equation in (27), and integrating we obtain that, in order to have solutions, we need

\[
\deg E = \frac{1}{2} \deg L.
\]

To avoid this restriction we can perturb, as in the previous case, the coupled monopole equations by adding fixed self-dual forms \( \alpha, \gamma \), i.e. by considering

\[
\begin{align*}
D_{A,b} \Psi &= 0 \\
F_A^+ &= i(\Psi \otimes \Psi^*)_0 + i\alpha \mathbb{I}_E \\
F_b^+ &= 2i \text{Tr}(\Psi \otimes \Psi^*)_0 + i\gamma
\end{align*}
\]

In the Kähler case we shall choose

\[
\alpha = -f \omega \quad \gamma = 2f' \omega,
\]

where \( f \) and \( f' \) are smooth real functions on \( X \).

With this choice of perturbation the second and third equations in (27) become respectively

\[
\begin{align*}
i\Lambda F_A + \phi \otimes \phi^* - \Lambda^2 \beta \otimes \beta^* &= f \mathbb{I}_E \\
i\Lambda F_b^+ + 2(|\phi|^2 - |\beta|^2) &= -2f'.
\end{align*}
\]

A necessary condition for existence of solutions is now

\[
\deg E - \frac{1}{2} \deg L = \int_X (rf + f'),
\]

where \( r = \text{rank } E \).

We shall study the more general monopole equations (28).
Proposition 5.2 Let $f, f' \in C^\infty(X, \mathbb{R})$ be related by (29) and denote
\[
\mathcal{F} = \frac{1}{2\pi} \int_X f \quad \text{and} \quad \overline{\mathcal{F}} = \frac{1}{2\pi} \int_X f'.
\]
The only solutions to (27) are such that either

(i) $\beta = 0$,

\[
\begin{align*}
F^0_{A, 2} & = 0 \\
F^0_{b, 2} & = 0 \\
\partial_{A, b}^* \phi & = 0 \\
i\Lambda F_A + \phi \otimes \phi^* & = f_I \\
i\Lambda F_b + 2|\phi|^2 & = -2f',
\end{align*}
\]

then $\mu(E) \leq \mathcal{F}$, $\deg L \geq 2\overline{\mathcal{F}}$ or

(ii) $\phi = 0$,

\[
\begin{align*}
F^0_{A, 2} & = 0 \\
F^0_{b, 2} & = 0 \\
\partial_{A, b} \beta & = 0 \\
i\Lambda F_A - \Lambda^2 \beta \otimes \beta^* & = f_I \\
i\Lambda F_b - 2|\beta|^2 & = -2f'
\end{align*}
\]

and then $\mu(E) \geq \mathcal{F}$, $\deg L \leq 2\overline{\mathcal{F}}$.

Proof. This is proved similarly to Proposition 5.1, by considering the functional

\[
\text{SW}(A, b, \Psi) = \|F^+_A - i(\Psi \otimes \Psi^*)_0 - i\alpha I_E\|^2 + 2\|F^+_b - 2i\text{Tr}(\Psi \otimes \Psi^*)_0 - i\gamma\|^2 + 2\|D_{A, b}\Psi\|^2,
\]

and observing that

\[
\langle F^+_A, i(\Psi \otimes \Psi^*)_0 \rangle + \frac{1}{2}\langle F^+_b, 2i\text{Tr}(\Psi \otimes \Psi^*)_0 \rangle = \langle F^+_A, b, i(\Psi \otimes \Psi^*)_0 \rangle.
\]

Again, we can focus on case (i), since by means of the Hodge operator we can interchange the roles of $\phi$ and $\beta$.

The system of equations (30) is equivalent to

\[
\begin{align*}
F^0_{A, 2} & = 0 \\
F^0_{b, 2} & = 0 \\
\partial_{A, b}^* \phi & = 0 \\
i\Lambda F_A + \phi \otimes \phi^* & = f_I \\
i\Lambda F_b - |\phi|^2 & = (f' - \frac{s^2}{2})
\end{align*}
\]

where $\hat{b}^*$ is the dual connection to $\hat{b}$ on $\hat{L}^*$, which satisfies that $F_{\hat{b}^*} = -F_b$.

We can now identify (32) as the coupled vortex equations (13) on $(E, \hat{L}^*)$, with the parameters $\tau$ and $\tau'$ replaced by functions. The existence of solutions to these equations as well as the description of the moduli space of all solutions will be dealt with in the next section.
6 Back to the vortex equations

We study now the existence of solutions to the monopole equations in the Kähler case — equations (25), (26) and (32) — or equivalently the vortex equations in which the parameters have been replaced by functions.

6.1 The $t$-vortex equations

In this section we will examine what happens if the parameter $\tau$, which appears on the $\tau$-vortex equations (10) is permitted to be a non-constant smooth function, say $t$. We will refer to the resulting equations as the $t$-vortex equations. The main result, namely that existence of solutions is governed entirely by the average value of $t$, has already been observed by Okonek and Teleman. Here we give a somewhat different proof than that in [OT1]. We also discuss some interpretations and implications of the result.

Let $E \to X$ be a rank $r$, smooth complex bundle over a closed Kähler manifold $(X, \omega)$. It is convenient to look at equations (10) from a different although equivalent point of view. Instead of fixing a Hermitian metric $H$ and solving for $(A, \phi)$ satisfying (10) we fix a $\overline{\partial}$-operator on $E, \overline{\partial}_E$ say, and a section $\phi \in H^0(X, E)$, where $E$ is the holomorphic bundle determined by $\overline{\partial}_E$, and solve for a metric $H$ satisfying

$$i\Lambda F_H + \phi \otimes \phi^* H = \tau I,$$

where $F_H$ is the curvature of the metric connection determined by $\overline{\partial}_E$ and $H$. It will be important to explicitly write $\phi^* H$ to denote the adjoint of $\phi$ with respect to the metric $H$.

Equation (33) can be regarded as the defining condition for a special metric on the holomorphic pair $(E, \phi)$. In [32] the first author showed that there is a Hitchin–Kobayashi correspondence between the existence of such metrics and a stability condition for holomorphic pairs. The appropriate notion of stability is as follows.

**Definition 6.1** Define the degree of any coherent sheaf $\mathcal{E}' \subset \mathcal{E}$ to be

$$\deg \mathcal{E}' = \int_X c_1(\mathcal{E}') \cdot \omega^{n-1},$$

and define the slope of $\mathcal{E}'$ by

$$\mu(\mathcal{E}') = \frac{\deg \mathcal{E}'}{\text{rank} \mathcal{E}'}.$$

Fix $\tau \in \mathbb{R}$. The holomorphic pair $(\mathcal{E}, \phi)$ is called $\tau$-stable if

1. $\mu(\mathcal{E}') < \tau$ for every coherent subsheaf $\mathcal{E}' \subset \mathcal{E}$, and
2. $\mu(\mathcal{E}/\mathcal{E}'') > \tau$ for every non-trivial coherent subsheaf $\mathcal{E}''$ such that $\phi \in H^0(X, \mathcal{E}'').$

The Hitchin–Kobayashi correspondence is expressed by the following two propositions.
Proposition 6.2 \([B2]\) Fix \(\tau \in \mathbb{R}\), let \((E, \phi)\) be a holomorphic pair, and suppose that there exists a metric, \(H\), satisfying the \(\tau\)-vortex equation \((33)\). Then either

1. the holomorphic pair \((E, \phi)\) is \(\tau\)-stable, or
2. the bundle \(E\) splits holomorphically as \(E = E_\phi \oplus E_{ps}\) with \(\phi \in H^0(X, E_\phi)\), and such that the holomorphic pair \((E_\phi, \phi)\) is \(\tau\)-stable, and \(E_{ps}\) is polystable with slope equal to \(\tau\).

Proposition 6.3 \([B2]\) Fix \(\tau \in \mathbb{R}\), let \((E, \phi)\) be a \(\tau\)-stable holomorphic pair. Then there is a unique smooth metric, \(H\), on \(E\) such that the \(\tau\)-vortex equation \((33)\) is satisfied.

Remark. We are assuming that the Kähler metric is normalized so that \(\operatorname{Vol}(X) = 2\pi\). Otherwise we need to introduce the factor \(\frac{2\pi}{\operatorname{Vol}(X)}\) in the right hand side of \((33)\).

Suppose now that we replace \(\tau\) in equation \((33)\) by a smooth real valued function \(t \in C^\infty(X, \mathbb{R})\) and study

\[
i\Lambda F_H + \phi \otimes \phi^* H = iI. \tag{34}\]

The question we wish to address is: how does this affect the Hitchin-Kobayashi correspondence?

One direction is clear: replacing the constant \(\tau\) by the smooth function \(t\) has absolutely no effect on the proof of Proposition 6.2 (Theorem 2.1.6 in \([B2]\)). The same proof thus yields

Theorem 6.4 (cf. also Theorem 3.3, \([OT1]\)) Fix a smooth function \(t \in C^\infty(X, \mathbb{R})\). Let \(\tau = \frac{1}{2\pi} \int t\). If a holomorphic pair \((E, \phi)\) supports a solution to the \(t\)-vortex equation \((33)\), then either

1. the holomorphic pair \((E, \phi)\) is \(\tau\)-stable, or
2. the bundle \(E\) splits holomorphically as \(E = E_\phi \oplus E_{ps}\) with \(\phi \in H^0(X, E_\phi)\), and such that the holomorphic pair \((E_\phi, \phi)\) is \(\tau\)-stable, and \(E_{ps}\) is polystable with slope equal to \(\tau\).

We now consider the converse result, i.e. the analogue of Proposition 6.3 (Theorem 3.1.1 in \([B2]\)). As shown in \([OT1]\), the \(t\)-vortex equation can be reformulated as an equation with a constant right hand side. Let \(\tau = \bar{t}\). Since \(\int_X (\tau - t) = 0\), we can find a smooth function \(u\) such that \(\Delta(u) = \tau - t\). Thus \((34)\) is equivalent to

\[
i\Lambda(F_H + \Delta(u)I) + \phi \otimes \phi^* H = \tau I. \tag{35}\]

If we define a new metric \(K = He^u\), then \((35)\) becomes

\[
i\Lambda F_K + e^{-u} \phi \otimes \phi^* K = \tau I. \tag{36}\]

This is almost the \(\tau\)-vortex equation, the only difference being the prefactor \(e^{-u}\) in the second term. In the analysis of this situation by Okonek and Teleman, they indicate how the proof (of Theorem 3.1.1) in \([B2]\) can be modified to accommodate this new wrinkle. (The
proof in [B2] employs a modified Donaldson functional on the space of Hermitian metrics on the bundle \( \mathcal{E} \). In [OT1], the authors generalize the functional so that it accommodates the extra factor of \( e^{-u} \), and argue that this has little effect on the proof.

It is interesting to observe that the same result can be achieved without any modification at all of the proof in [B2], if one enlarges the category in which the proof is applied. This can be seen as follows. If we set

\[
\phi_u = e^{-u/2} \phi ,
\]

then (36) becomes

\[
i \Lambda F_K + \phi_u \otimes \phi_u^* K = \tau I .
\]

We thus see that

**Lemma 6.5** Let \( u \) be given by \( \Delta(u) = \tau - t \). The pair \((\mathcal{E}, \phi)\) admits a metric \( H \) satisfying the \( t \)-vortex equation if and only if the pair \((\mathcal{E}, \phi_u)\) admits a metric \( K \) satisfying the \( \tau \)-vortex equation. The metrics \( H \) and \( K \) are related by \( K = H e^u \).

It is important to notice that, unless \( u \) is a constant function, \( \phi_u \) is not a holomorphic section of \( \mathcal{E} \). Indeed \( \overline{\partial}_E \phi_u = -\frac{1}{2} \overline{\partial}(u) \phi_u \). Our key observation is that in Theorem 3.1.1 in [B2], the holomorphicity of \( \phi \) is not required either in the statement or in the proof of the theorem. The proof, and thus the result remains unchanged if the holomorphic section \( \phi \) is replaced by a smooth section \( \phi_u \) related to \( \phi \) by \( \phi_u = e^{-u/2} \phi \). The basic reason can be traced back to the following simple fact:

**Lemma 6.6** Let \( \phi, u, \) and \( \phi_u \) be as above.

1. Let \( \mathcal{E}' \subset \mathcal{E} \) be any holomorphic subbundle of \( \mathcal{E} \). Then \( \phi \) is a section of \( \mathcal{E}' \) if and only if \( \phi_u \) is a section of \( \mathcal{E}' \).

2. Let \( s \) be any smooth endomorphism of \( \mathcal{E} \). Then \( s \phi = 0 \) if and only if \( s \phi_u = 0 \).

Notice, for instance, that if we define

\[
H_u^0(X, \mathcal{E}) = e^{-u/2} H^0(X, \mathcal{E})
= \{e^{-u/2} \phi \in \Omega^0(X, E) \mid \phi \in H^0(X, \mathcal{E})\} ,
\]

then the definition of \( \tau \)-stability can be applied to any pair \((\mathcal{E}, \phi_u)\) where \( \phi_u \in H_u^0(X, \mathcal{E}) \).

**Definition 6.7** A pair \((\mathcal{E}, \phi_u)\), where \( \phi \) is in \( H_u^0(X, \mathcal{E}) \), will be called a \( u \)-holomorphic pair.

In view of Lemma 6.6, it follows that the \( u \)-holomorphic pair \((\mathcal{E}, \phi_u)\) is \( \tau \)-stable if and only if the holomorphic pair \((\mathcal{E}, \phi)\) is \( \tau \)-stable (where \( \phi \) and \( \phi_u \) are related by (37)). Furthermore, without any alteration whatsoever, the proof of Theorem 3.1.1 in [B2] can be applied to a \( u \)-holomorphic pair to prove:
Proposition 6.8 Fix $u \in C^\infty(X, \mathbb{R})$ and $\tau \in \mathbb{R}$. Let $(\mathcal{E}, \phi_u)$ be a $\tau$-stable $u$-holomorphic pair. Then $E$ admits a unique smooth metric, say $K$, such that the $\tau$-vortex equation is satisfied, i.e. such that

$$i\Lambda F_K + \phi_u \otimes \phi_u^* = \tau I.$$ 

Taken together, Lemma 6.5 and Proposition 6.8 thus prove

Theorem 6.9 Fix $\tau = \bar{\tau}$ and suppose that $(\mathcal{E}, \phi)$ is a $\tau$-stable pair. Then $E$ supports a metric satisfying the $t$-vortex equation.

The above results describe the sense in which the vortex equation is insensitive to the precise form of the parameter $t$. This can be made precise by considering the moduli spaces. Let $C$ be the space of holomorphic structures (or, equivalently, integrable $\overline{\partial}_E$-operators) on $E$, and let $\mathcal{H} = \{(\overline{\partial}_E, \phi) \in C \times \Omega^0(X, E) \mid \overline{\partial}_E \phi = 0\}$ be the configuration space of holomorphic pairs on $E$. Let $\mathcal{V}_t \subset \mathcal{H}$ be the set of $t$-vortices, i.e.

$$\mathcal{V}_t = \{(\overline{\partial}_E, \phi) \in \mathcal{H} \mid \text{there is a metric satisfying the } t \text{-vortex equation}\}.$$ 

The above results can then be summarized by saying that

1. $\mathcal{V}_t = \mathcal{V}_\tau$ for all functions $t$ such that $\bar{t} = \tau$, and

2. for generic values of $\tau$, we can identify $\mathcal{V}_t = \mathcal{H}_\tau$ where $\mathcal{H}_\tau$ denotes the set of $\tau$-stable holomorphic pairs. In fact, as complex spaces, $\mathcal{V}_t/\mathcal{G} = \mathcal{H}_\tau$ where $\mathcal{G}$ is the moduli space of $\tau$-stable holomorphic pairs — which has the structure of a variety (cf. [Be, BD1, BD2, G4, HL1, HL2, L1]).

Remark. In the case in which $E = L$ is a line bundle the $\tau$-stability condition reduces to $\text{deg } L < \tau$, and the moduli space of $\tau$-stable pairs is nothing else but the space of non-negative divisors supported by $L$, where by a non-negative divisor we mean either an effective divisor or the zero divisor.

Nevertheless, the function $t$ does carry some information. For example, the metrics which satisfy the $t$-vortex equation (for fixed $\overline{\partial}_L$ and $\phi$) do depend on $t$. The following observations shed some light on the role played by $t$.

As described in [G4, BDG], the holomorphic pair $(\mathcal{E}, \phi)$ can be identified with the holomorphic triple $(\mathcal{E}, \mathcal{O}, \phi)$, where $\mathcal{O}$ is the structure sheaf and $\phi$ is a morphism $\mathcal{O} \rightarrow \mathcal{E}$. From this point of view, the natural equations to consider are the framed vortex equations (13). Coming back to the set-up of Section 3 we want to study equations (13) for a vector bundle $E$ of arbitrary rank and $F = L_0$, the trivial line bundle. As for the usual vortex equations, we will look at (13) as equations for a metric on $E$. To do this we fix the holomorphic structure $\overline{\partial}_{L_0}$ on $L_0$ to be the trivial one i.e. $(L_0, \overline{\partial}_{L_0}) = \mathcal{O}$, and consider a holomorphic structure $\overline{\partial}_E$ on $E$. Then we take $\phi : \mathcal{O} \rightarrow \mathcal{E}$ to be a holomorphic morphism, where $\mathcal{E} = (E, \overline{\partial}_E)$. In contrast with the coupled vortex equations (13) that we will analyse later, here we need to fix a metric $h$ on $L_0$. Then solving (13) is equivalent to solving for a metric $H$ on $E$ satisfying

$$i\Lambda F_H + \phi \otimes \phi^* = \tau I.$$ 

(39)
It is important to notice that now $\phi^*$ is the adjoint of $\phi$ with respect to both metrics $H$ and $h$. The identification between $(\mathcal{E}, \phi)$ and $(\mathcal{E}, \mathcal{O}, \phi)$ requires a choice of trivializing frame for $\mathcal{O}$, say $f$. If $h(f, f) = e^u$, then (39) becomes

$$i\Lambda F_H + e^{-u} \phi \otimes \phi^* H = \tau I .$$

Thus we recover the usual $\tau$-vortex equation when we select the metric on $L_0$ for which the holomorphic frame of $\mathcal{O}$ is also a unitary frame. For other choices of $h$ we see that we get essentially equation (38), or equivalently, the $t$-vortex equation.

From this point of view, the function $t$ is determined by the metric on $L_0$. The impact of non-constant $t$ can also be understood from the symplectic point of view. If we fix a metric, say $H$, on $E$, the induced inner products on $C$ and on $\Omega^0(X, E)$ can be combined to give a symplectic structure on the configuration space $\mathcal{H}$. Denoting the symplectic forms on $C$ and $\Omega^0(X, E)$ by $\omega_{H,C}$ and $\omega_{H,0}$ respectively, we take

$$\omega_{H,H} = \omega_{H,C} + \omega_{H,0}$$

as the symplectic form on $\mathcal{H}$. Let $\mathcal{G}_H$ be the unitary gauge group of $E$ determined by $H$, and let $\text{Lie} \mathcal{G}_H$ be its Lie algebra. A moment map $\mu: \mathcal{H} \to \text{Lie} \mathcal{G}_H^*$ for the action of $\mathcal{G}_H$ on the symplectic space $(\mathcal{H}, \omega_{H,H})$ is given by

$$\mu_{H,H}(\overline{\partial}_E, \phi) = \Lambda F_{\overline{\partial}_E,H} - i\phi \otimes \phi^* H ,$$

where we have written $F_{\overline{\partial}_E,H}$ instead of $F_H$ to emphasize that $F_H$ depends also on the holomorphic structure on $E$. The $\tau$-vortex equation is thus equivalent to the condition

$$\mu_{H,H}^{-1}(-i\tau I) = \mathcal{V}_\tau / \mathcal{G}_H .$$

Replacing $\tau$ by the non-constant function $t$ has no effect on the identification $\mathcal{V}_t / \mathcal{G}_H^c = \mu_{-H,H}^{-1}(-it I) / \mathcal{G}_H$: The element $it I$ is still a central element in $\text{Lie} \mathcal{G}_H^*$, so the symplectic quotient at this level is well defined. (The problem comes in proving that $\mathcal{V}_t / \mathcal{G}_H = \mathcal{B}_t$.) An alternative point of view makes use of the equivalence between the equations (34) and (38). We define

$$\mu_{H,K}(\overline{\partial}_E, \phi) = \Lambda F_{\overline{\partial}_E,H} - i\phi \otimes \phi^* K ,$$

where $H$ and $K$ are metrics on $E$. By the above results, $K = He^u$ where $\Delta(u) = \tau - t$, then

$$\mu_{H,K}^{-1}(-it I) = \mu_{-H,K}^{-1}(-i\tau I) .$$

The point is that $\mu_{H,K}$ is also a moment map for the action of $\mathcal{G}_H$. It arises when the symplectic structure on $\mathcal{H}$ is taken to be

$$\omega_{H,K} = \omega_{H,C} + \omega_{K,0} .$$

From this point of view, the function $t$ arises from a deformation of the symplectic structure on $\mathcal{H}$. 

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6.2 The coupled vortex equations

Let us consider the set-up in Section 3 for the coupled vortex equations (13). As in the previous situation, we want to look at (13) as equations for metrics . In order to do this let us fix holomorphic structures \( \partial_E \) and \( \partial_L \) on \( E \) and \( L \) respectively. Denote by \( E \) and \( L \) the corresponding holomorphic vector bundles. Let \( \phi \in H^0(E \otimes L^*) \). Equations (13) are then equivalent to solving

\[
\begin{align*}
\iota \Lambda F_H + \phi \otimes \phi^* &= \tau E \\
\iota \Lambda F_K - |\phi|^2 &= \tau'
\end{align*}
\]

for metrics \( H \) and \( K \) on \( E \) and \( L \) respectively.

A Hitchin–Kobayashi correspondence was proved in [G4]. The appropriate notion of stability for \((E, L, \phi)\) can be expressed in terms of the stability of a pair, namely

**Definition 6.10** The holomorphic triple \((E, L, \phi)\) is said to be \(\tau\)-stable if the holomorphic pair \((E \otimes L^*, \phi)\) is \((\tau - \deg L)\)-stable.

**Theorem 6.11 ([G4])** Let \( \tau \) and \( \tau' \) be real numbers satisfying (14). Let \((E, L, \phi)\) a holomorphic triple. Suppose that there exist metrics \( H \) and \( K \) satisfying (40), then either \((E, L, \phi)\) is \(\tau\)-stable or the bundle \( E \) splits holomorphically as \( E_{\phi} \oplus E_{ps} \) with \( \phi \in H^0(X, E_{\phi} \otimes L^*) \), and such that \((E_{\phi}, L, \phi)\) is \(\tau\)-stable and \( E_{ps} \) is polystable with slope equal to \(\tau\).

Conversely, let \((E, L, \phi)\) be a \(\tau\)-stable triple then there are unique smooth metrics \( H \) and \( K \) satisfying the coupled vortex equations (40).

Suppose now that we replace \(\tau\) and \(\tau'\) in (40) by smooth functions \( t, t' \in C^\infty(X, \mathbb{R}) \), i.e. we consider

\[
\begin{align*}
\iota \Lambda F_H + \phi \otimes \phi^* &= t E \\
\iota \Lambda F_K - |\phi|^2 &= t'
\end{align*}
\]

The first thing that we observe is that in order to have solutions \( t \) and \( t' \) must satisfy

\[
\int_X (rt + t') = \deg E + \deg L
\]

where \( r = \text{rank} E \).

Next, let us recall the main ideas in the proof of Theorem 6.11. The basic fact is that the coupled vortex equations (40) are a dimensional reduction of the Hermitian–Einstein equation for a metric on a certain vector bundle over \( X \times \mathbb{P}^1 \). This bundle \( F \), canonically associated to the holomorphic triple \((E, L, \phi)\), is an extension on \( X \times \mathbb{P}^1 \) of the form

\[
0 \longrightarrow p^*E \longrightarrow F \longrightarrow p^*L \otimes q^*\mathcal{O}(2) \longrightarrow 0,
\]

where \( p \) and \( q \) are the projections from \( X \times \mathbb{P}^1 \) to \( X \) and \( \mathbb{P}^1 \) respectively. This is simply because \( H^1(X \times \mathbb{P}^1, p^*(E \otimes L^*) \otimes q^*\mathcal{O}(2)) \approx H^0(X, E \otimes L^*) \)
Let $SU(2)$ act on $X \times \mathbb{P}^1$, trivially on $X$, and in the standard way on $\mathbb{P}^1 \cong SU(2)/U(1)$. This action can be lifted to an action on $\mathcal{F}$, trivial on $p^*\mathcal{E}$ and $p^*\mathcal{L}$, and standard on $q^*\mathcal{O}(2)$. The bundle $\mathcal{F}$ is in this way an $SU(2)$-equivariant holomorphic vector bundle.

Let $\tau$ and $\tau'$ be related by (14) and let $\sigma = 4\pi$ \(\tau' - \tau\) (44) be positive. Consider the $SU(2)$-invariant Kähler metric on $X \times \mathbb{P}^1$ whose Kähler form is

$$\omega_\sigma = p^*\omega + \sigma q^*\omega_{\mathbb{P}^1},$$

where $\omega$ is the Kähler form on $X$ (normalized such that $\text{Vol}(X) = 2\pi$), and $\omega_{\mathbb{P}^1}$ is the Fubini-Study metric with volume 1.

Theorem 6.11 is then a consequence of the following two propositions and the Hitchin–Kobayashi correspondence proved by Donaldson [D1, D2], and Uhlenbeck and Yau [UY]).

**Proposition 6.12 ([G4])** The triple $(\mathcal{E}, \mathcal{L}, \phi)$ admits a solution to (41) if and only if the vector bundle $F$ in (43) has a ($SU(2)$-invariant) Hermitian–Einstein metric with respect to $\omega_\sigma$.

**Proposition 6.13 ([G4])** Suppose that $\mathcal{E}$ is not isomorphic to $\mathcal{L}$. Then the triple $(\mathcal{E}, \mathcal{L}, \phi)$ is $\tau$-stable if and only if $\mathcal{F}$ is stable with respect to $\omega_\sigma$. If $\mathcal{E} \cong \mathcal{L}$, then $\mathcal{F} \cong p^*\mathcal{L} \otimes q^*\mathcal{O}(1) \oplus p^*\mathcal{L} \otimes q^*\mathcal{O}(1)$.

Suppose first that the functions $t$ and $t'$, in addition to satisfying (42), verify that there is a positive constant $\sigma$ so that

$$t - t' = \frac{4\pi}{\sigma}. \quad (45)$$

The proof of Proposition 6.12 then yields

**Proposition 6.14** The triple $(\mathcal{E}, \mathcal{L}, \phi)$ admits a solution to (41) if and only if the vector bundle $\mathcal{F}$ in (43) has a ($SU(2)$-invariant) metric satisfying the weak Hermitian–Einstein equation

$$i\Lambda_\sigma F_H = t \mathbf{I} \quad (46)$$

with respect to the Kähler form $\omega_\sigma = p^*\omega_X \oplus \sigma q^*\omega_{\mathbb{P}^1}$.

But the existence of a weak Hermitian–Einstein metric is in fact equivalent to the existence of a Hermitian–Einstein metric — as one can see simply by applying a conformal change to the metric — and hence equivalent to the stability of the bundle. We can then combine again Propositions 6.14 and 6.13 to prove the following.
Theorem 6.15 Fix smooth functions $t, t' \in C^\infty(X, \mathbb{R})$ satisfying (42) and (45). Let $\overline{t} = \frac{1}{2\pi} \int_X t$ and $\overline{t'} = \frac{1}{2\pi} \int_X t'$. Let $(\mathcal{E}, \mathcal{L}, \phi)$ a holomorphic triple. Suppose that there exist metrics $H$ and $K$ satisfying (41), then either $(\mathcal{E}, \mathcal{L}, \phi)$ is $\overline{t}$-stable or the bundle $\mathcal{E}$ splits holomorphically as $\mathcal{E}_\phi \oplus \mathcal{E}_{ps}$ with $\phi \in H^0(X, \mathcal{E}_\phi \otimes \mathcal{L}^*)$, and such that $(\mathcal{E}_\phi, \mathcal{L}, \phi)$ is $\overline{t}$-stable and $\mathcal{E}_{ps}$ is polystable with slope equal to $\overline{t}$.

Conversely, let $(\mathcal{E}, \mathcal{L}, \phi)$ be a $\overline{t}$-stable triple then there are unique smooth metrics $H$ and $K$ satisfying equations (44).

We will show now that the general coupled vortex equations (41) with $t$ and $t'$ satisfying simply (42) are also a dimensional reduction, but in this case of a metric on $\mathcal{F}$ satisfying a certain deformation of the Hermitian–Einstein condition. We set, as above,

$$\sigma = \frac{4\pi}{\overline{t} - \overline{t'}}$$

where $\overline{t}$ and $\overline{t'}$ denote the average values of $t$ and $t'$ respectively. Again the proof of Proposition 6.12 yields

Proposition 6.16 The triple $(\mathcal{E}, \mathcal{L}, \phi)$ admits a solution to (41) if and only if the vector bundle $\mathcal{F}$ in (43) has a $(SU(2)$-invariant) metric satisfying the deformed Hermitian–Einstein equation

$$i \Lambda_\sigma F_H = \overline{t} I + \begin{pmatrix} (t - \overline{t}) I_1 & 0 \\ 0 & (t' - \overline{t'}) I_2 \end{pmatrix},$$

with respect to the Kähler form

$$\omega_\sigma = p^* \omega_X \oplus \sigma q^* \omega_{\mathbb{P}^1}.$$
stability for the extension depending on the parameter $\alpha = \tau_1 - \tau_2$. To define this stability condition consider any coherent subsheaf $\mathcal{E}' \subset \mathcal{E}$ and write it as a subextension

$$0 \to \mathcal{E}'_1 \to \mathcal{E}' \to \mathcal{E}'_2 \to 0.$$ 

Define the $\alpha$-slope of $\mathcal{E}'$ as

$$\mu_\alpha(\mathcal{E}') = \mu(\mathcal{E}') + \alpha \frac{\text{rank} \mathcal{E}'_2}{\text{rank} \mathcal{E}'}. $$

Then we say that (48) is $\alpha$-stable if and only if for every non-trivial subsheaf $\mathcal{E}' \subset \mathcal{E}$

$$\mu_\alpha(\mathcal{E}') < \mu_\alpha(\mathcal{E}).$$

We proved

**Theorem 6.17** ([BG2]) Let $\alpha = \tau_1 - \tau_2 \leq 0$ and suppose that (48) is indecomposable (as an extension), then $\mathcal{E}$ admits a metric satisfying (49) if and only (48) is $\alpha$-stable.

**Remark.** If $\alpha = 0$ (49) reduces to the Hermitian–Einstein equation and the stability condition is the usual stability of the bundle $\mathcal{E}$.

The deformed Hermitian–Einstein equation in Proposition 6.16 differs from (49) only in that the constants $\tau_1$ and $\tau_2$ have been replaced by smooth functions, $t_1$ and $t_2$, satisfying

$$\int (r_1 t_1 + r_2 t_2) = \deg \mathcal{E}. $$

By the same methods used in [BG2] one can readily show one direction of the Hitchin-Kobayashi correspondence, namely

**Theorem 6.18** Let $t_1$ and $t_2$ be smooth real functions satisfying (50) and such that $\alpha = f(t_1 - t_2) \leq 0$. Then the existence of a metric $H$ on $\mathcal{E}$ satisfying

$$i \Lambda F_H = \begin{pmatrix} t_1 I_1 & 0 \\ 0 & t_2 I_2 \end{pmatrix},$$

implies the $\alpha$-stability of (48).

It should likewise be possible to adapt the proof of the other direction of the Hitchin-Kobayashi correspondence. This will then allow one (by taking $\alpha = 0$) to establish a more general version of Theorem 6.15 valid when $t$ and $t'$ are smooth functions satisfying just (12). We will discuss this in a future publication.

To describe the moduli space, let $\mathcal{C}_E$ and $\mathcal{C}_L$ the sets of holomorphic structures on $E$ and $L$ respectively. Consider the set

$$\mathcal{H}(E, L) = \{(\overline{\partial}_E, \overline{\partial}_L, \phi) \in \mathcal{C}_E \times \mathcal{C}_L \times \Omega^0(\text{Hom}(L, E)) \mid \phi \in H^0(X, \mathcal{E} \otimes \mathcal{L}^*) \} $$

(52)
of holomorphic triples on \((E, L)\), where \(\mathcal{E}\) and \(\mathcal{L}\) denote the holomorphic vector bundles defined by \(\overline{\partial}_E\) and \(\overline{\partial}_L\) respectively. Let \(\mathcal{H}_\tau(E, L) \subseteq \mathcal{H}(E, L)\) be the set of \(\tau\)-stable holomorphic triples. This set is invariant under the action of the complex gauge groups of \(E\) and \(L\), \(G^c_E\) and \(G^c_L\), say. The moduli space of \(\tau\)-stable triples is defined as

\[ B_\tau(E, L) = \mathcal{H}_\tau(E, L) / G^c_E \times G^c_L. \]

The set \(B_\tau(E, L)\), which has naturally the structure of a variety (cf. [G4]), is closely related to the moduli space of stable pairs—this is not surprising in view of the definition 6.10. More precisely, the map \((\mathcal{E}, \mathcal{L}, \phi) \mapsto (\mathcal{E} \otimes \mathcal{L}^*, \phi)\) exhibits \(B_\tau(E, L)\) as a Pic\(^0\)-principal bundle over the moduli space of \((\tau - \deg L)\)-stable pairs on \(E \otimes L^*\).

The study of the general equations (12), i.e. the case in which \(F\) is of arbitrary rank, requires the introduction of a new notion of stability. This was carried out in [BG1]. All the results explained above should extend appropriately to the higher rank case when one replaces \(\tau\) and \(\tau'\) in (12) by functions \(t\) and \(t'\).

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