ON THE DANILOV-GIZATULLIN ISOMORPHISM THEOREM
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Abstract. A Danilov-Gizatullin surface is a normal affine surface $V = \Sigma_d \setminus S$ which is a complement to an ample section $S$ in a Hirzebruch surface $\Sigma_d$. By a surprising result due to Danilov and Gizatullin [DaGi] $V$ depends only on $n = S^2$ and neither on $d$ nor on $S$. In this note we provide a new and simple proof of this Isomorphism Theorem.

1. The Danilov-Gizatullin theorem

By definition, a Danilov-Gizatullin surface is the complement $V = \Sigma_d \setminus S$ of an ample section $S$ in a Hirzebruch surface $\Sigma_d$, $d \geq 0$. In particular $n := S^2 > d$. The purpose of this note is to give a short proof of the following result of Danilov and Gizatullin [DaGi, Theorem 5.8.1].

Theorem 1.1. The isomorphism type of $V_n = \Sigma_d \setminus S$ only depends on $n$. In particular, it neither depends on $d$ nor on the choice of the section $S$.

For other proofs we refer the reader to [DaGi] and [CNR, Corollary 4.8]. In the forthcoming paper [FKZ2, Theorem 1.0.5] we extend the Isomorphism Theorem 1.1 to a larger class of affine surfaces. However, the proof of this latter result is much harder.

2. Proof of the Danilov-Gizatullin theorem

2.1. Extended divisors of Danilov-Gizatullin surfaces. Let as before $V = \Sigma_d \setminus S$ be a Danilov-Gizatullin surface, where $S$ is an ample section in a Hirzebruch surface $\Sigma_d$, $d \geq 0$ with $n := S^2 > d$. Picking a point, say, $A \in S$ and performing a sequence of $n$ blowups at $A$ and its infinitesimally near points on $S$ leads to a new SNC completion $(\overline{V}, D)$ of $V$. The new boundary $D = C_0 + C_1 + \ldots + C_n$ forms a zigzag i.e., a linear chain of rational curves with weights $C_0^2 = 0$, $C_1^2 = -1$ and $C_i^2 = -2$ for $i = 2, \ldots, n$. Here $C_0 \cong S$ is the proper transform of $S$. The linear system $|C_0|$ on $\overline{V}$ defines a $\mathbb{P}^1$-fibration $\Phi_0 : \overline{V} \to \mathbb{P}^1$ for which $C_0$ is a fiber and $C_1$ is a section. Choosing an appropriate affine coordinate on $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$ we may suppose that $\Phi_0^{-1}(\infty) = C_0$ and $\Phi_0^{-1}(0)$ contains the subchain $C_2 + \ldots + C_n$ of $D$. The reduced curve $D_{\text{ext}} = \Phi_0^{-1}(0) \cup C_0 \cup C_1$ is called the extended divisor of the completion $(\overline{V}, D)$ of $V$. The following lemma appeared implicitly in the proof of Proposition 1 in [Gi] (cf. also [FKZ1]). To make this note self-contained we provide a short argument.

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Lemma 2.1. (a) For every $a \neq 0$ the fiber $\Phi_0^{-1}(a)$ is reduced and isomorphic to $\mathbb{P}^1$.  
(b) $D_{\text{ext}} = \Phi_0^{-1}(0) \cup C_0 \cup C_1$ is an SNC divisor with dual graph

\[
D_{\text{ext}} : \begin{array}{c}
\begin{array}{c}
0 \\
C_0 \\
-1 \\
C_1 \\
-2 \\
C_2 \\
\vdots \\
-2 \\
C_s \\
\vdots \\
-2 \\
C_n
\end{array}
\end{array}
\begin{array}{c}
1 - s \\
F_1 \\
-1 \\
F_0
\end{array}
\]

for some $s$ with $2 \leq s \leq n$.

Proof. (a) follows easily from the fact that the affine surface $V = \bar{V} \setminus D$ does not contain complete curves.

To deduce (b), we note first that $\bar{V}$ has Picard number $n + 2$, since $\bar{V}$ is obtained from $\Sigma_d$ by a sequence of $n$ blowups. Since $C_1 \cdot C_2 = 1$, the part $\Phi_0^{-1}(0) - C_2$ of the fiber $\Phi_0^{-1}(0)$ can be blown down to a smooth point. Since $C_1^2 = -1$, after this contraction we arrive at the Hirzebruch surface $\Sigma_1$, which has Picard number 2. Hence the fiber $\Phi_0^{-1}(0)$ consists of $n + 1$ components. In other words, $\Phi_0^{-1}(0)$ contains, besides the chain $C_2 + \ldots + C_n$, exactly 2 further components $F_0$ and $F_1$ called feathers [FKZ]. These are disjoint smooth rational curves, which meet the chain $C_2 + \ldots + C_n$ transversally at two distinct smooth points. Indeed, $\Phi_0^{-1}(0)$ is an SNC divisor without cycles and the affine surface $V$ does not contain complete curves. In particular, $(F_0 \cup F_1) \setminus D$ is a union of two disjoint smooth curves on $V$ isomorphic to $\mathbb{A}^1$.

Since $\Phi_0^{-1}(0) - C_2$ can be blown down to a smooth point and $C_i^2 = -2$ for $i \geq 2$, at least one of these feathers, call it $F_0$, must be a $(-1)$-curve. We claim that $F_0$ cannot meet a component $C_i$ with $3 \leq i \leq n - 1$. Indeed, otherwise the contraction of $F_0 + C_r + C_{r+1}$ would result in $C_{r-1}^2 = 0$ without the total fiber over 0 being irreducible, which is impossible. Hence $F_0$ meets either $C_2$ or $C_n$.

If $F_0$ meets $C_2$ then $F_0 + C_2 + \ldots + C_n$ is contractible to a smooth point. Thus the image of $F_1$ will become a smooth fiber of the contracted surface. This is only possible if $F_1$ is a $(-1)$-curve attached to $C_n$. Hence after interchanging $F_0$ and $F_1$ the divisor $D_{\text{ext}}$ is as in $(\text{I})$ with $s = 2$.

Therefore we may assume for the rest of the proof that $F_0$ is attached at $C_n$ and $F_1$ at $C_s$, where $2 \leq s \leq n$. Contracting the chain $F_0 + C_2 + \ldots + C_n$ within the fiber $\Phi_0^{-1}(0)$ yields an irreducible fiber $F'_1$ with $(F'_1)^2 = 0$. This determines the index $s$ in a unique way, namely, $s = 1 - F'_1^2$. \hfill \Box

2.2. Jumping feathers. The construction in [2.1] depends on the initial choice of the point $A \in S$. In particular, the extended divisor $D_{\text{ext}} = D_{\text{ext}}(A)$ and the integer $s = s(A)$ depend on $A$. The aim of this subsection is to show that $s(A) = 2$ for a general choice of $A \in S$.

2.2. Let $F_0 = F_0(A)$ and $F_1 = F_1(A)$ denote the images of the feathers $F_0 = F_0(A)$ and $F_1 = F_1(A)$, respectively, in the Hirzebruch surface $\Sigma_d$ under the blowdown $\sigma : \bar{V} \to \Sigma_d$ of the chain $C_1 + \ldots + C_n$. These images meet each other and the original section $S = \sigma(C_0)$ at the point $A$ and satisfy

\[
(2) \quad \bar{F}_0^2 = 0, \quad \bar{F}_0 \cdot \bar{F}_1 = \bar{F}_0 \cdot S = 1, \quad \bar{F}_1^2 = n - 2s + 2, \quad \bar{F}_1 \cdot S = n - s + 1,
\]

where $s = s(A)$. Hence $F_0 = F_0(A)$ is the fiber through $A$ of the canonical projection $\pi : \Sigma_d \to \mathbb{P}^1$ and $F_1 = F_1(A)$ is a section of $\pi$. The sections $S$ and $F_1$ meet only at $A$, where they can be tangent (osculating).
We let below

$$s_0 = s(A_0) = \min_{A \in S} \{ s(A) \}, \quad l = \bar{F}_1(A_0)^2 + 1 \quad \text{and} \quad m = \bar{F}_1(A_0) \cdot S.$$  

For the next proposition see e.g., Lemma 7 and the following Remark in [Gi], or Proposition 4.8.11 in [DaGi, II]. Our proof is based essentially on the same idea.

**Proposition 2.3.** (a) $s(A) = s_0$ for a general point $A \in S$, and

(b) $s_0 = 2$.

*Proof.* For a general point $A \in S$ and an arbitrary point $A' \in S$ we have $\bar{F}_1(A) \sim \bar{F}_1(A') + k\bar{F}_0$ for some $k \geq 0$. Hence $\bar{F}_1(A)^2 = \bar{F}_1(A')^2 + 2k \geq \bar{F}_1(A')^2$. Using (2) it follows that

$$s(A) = 1 - F_1(A)^2 \leq s(A') = 1 - F_1(A')^2.$$ 

Thus $s(A) = s_0$ for all points $A$ in a Zariski open subset $S_0 \subseteq S$, which implies (a).

To deduce (b) we note that by (3)

$$l = n - 2s_0 + 3 \leq n - s_0 + 1 = m$$

with equality if and only if $s_0 = 2$. Thus it is enough to show that $l \geq m$. Restriction to $S$ yields

$$\bar{F}_1(A)|S = m[A] \in \text{Div}(S) \quad \forall A \in S_0.$$ 

Consider the linear systems

$$|\bar{F}_1(A_0)| \cong \mathbb{P}^l \quad \text{and} \quad |\mathcal{O}_S(m)| \cong \mathbb{P}^m$$

on $\Sigma_d$ and $S \cong \mathbb{P}^1$, respectively, and the linear map

$$\rho : \mathbb{P}^l \longrightarrow \mathbb{P}^m, \quad F \longmapsto F|S.$$ 

The set of divisors

$$\Gamma_m = \{ m[A] \}_{A \in S}$$

represents a rational normal curve of degree $m$ in $\mathbb{P}^m = |\mathcal{O}_S(m)|$. In view of (4) the linear subspace $\overline{\rho(\mathbb{P}^l)}$ contains $\Gamma_m$. Since the curve $\Gamma_m$ is linearly non-degenerate we have $\rho(\mathbb{P}^l) = \mathbb{P}^m$ and so $l \geq m$, as desired. \qed

### 2.3. Elementary shifts

We consider as before a completion $V = \bar{V} \setminus D$ of a Danilov-Gizatullin surface $V$ as in [2.1]

#### 2.4. Choosing $A$ generically, according to Proposition 2.3 we may suppose in the sequel that $s = s(A) = 2$ and $F_0^2 = F_1^2 = -1$.

By (1) in Lemma 2.1 blowing down in $\bar{V}$ the feathers $F_0, F_1$ and then the chain $C_3 + \ldots + C_n$ yields the Hirzebruch surface $\Sigma_1$, in which $C_0$ and $C_2$ become fibers and $C_1$ a section. Reversing this contraction, the above completion $\bar{V}$ can be obtained from $\Sigma_1$ by a sequence of blowups as follows. The sequence starts by the blowup with center at a point $P_3 \in C_2 \setminus C_1$ to create the next component $C_3$ of the zigzag $D$. Then we perform subsequent blowups with centers at points $P_4, \ldots, P_{n+1}$ infinitesimally near to $P_3$, where for each $i = 4, \ldots, n$ the blowup of $P_i \in C_{i-1} \setminus C_{i-2}$ creates the next component $C_i$ of the zigzag. The blowup with center at $P_{n+1} \in C_n \setminus C_{n-1}$ creates the feather $F_0$. Finally we blow up at a point $Q \in C_2 \setminus C_1$ different from $P_3$ to create the feather $F_1$. In this way we recover the given completion $\bar{V}$ with extended divisor $D_{\text{ext}}$ as in (1), where $s = 2$. 

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[1] DaGi: Daigle, C. (1993) *On the Danilov-Gizatullin isomorphism theorem*. 
[2] Gi: Gizatullin, R. (1989) *Finite generation of the Picard group of certain surfaces*. 
[3] P: Pukhlikov, A. (1997) *Algebraic Geometry*.
We observe that the sequence $P_3, \ldots, P_{n+1}, Q$ depends on the original triplet $(\Sigma_d, S, A)$. It follows that, varying the points $P_3, \ldots, P_{n+1}, Q$ and then contracting the chain $C_1 + \ldots + C_n = D - C_0$ on the resulting surface $\hat{V}$, we can obtain all possible Danilov-Gizatullin surfaces

$$V = \hat{V} \setminus D \cong \Sigma_d \setminus S \quad \text{with} \quad S^2 = n \quad \text{and} \quad 0 \leq d \leq n - 1.$$ 

Thus to deduce the Danilov-Gizatullin Isomorphism Theorem it suffices to establish the following fact.

**Proposition 2.5.** The isomorphism type of the affine surface $V = \hat{V} \setminus D$ does not depend on the choice of the blowup centers $P_3, \ldots, P_{n+1}$ and $Q$ as above.

The proof proceeds in several steps.

**2.6.** First we note that in our construction it suffices to keep track only of some partial completions rather than of the whole complete surfaces. We can choose affine coordinates $(x, y)$ in $\Sigma_1 \setminus (C_0 \cup C_1) \cong \mathbb{A}^2$ so that $C_2 \setminus C_1 = \{x = 0\}$, $P = P_3 = (0, 0)$ and $Q = (0, 1)$. The affine surface $V$ can be obtained from the affine plane $\mathbb{A}^2$ by performing subsequent blowups with centers at the points $P_3, \ldots, P_{n+1}$ and $Q$ as in 2.4

and then deleting the curve $C_2 \cup \ldots \cup C_n = D \setminus (C_0 \cup C_1)$.

With $X_2 = \mathbb{A}^2$, for every $i = 3, \ldots, n+1$ we let $X_i$ denote the result of the subsequent blowups of $\mathbb{A}^2$ with centers $P_3, \ldots, P_i$. This gives a tower of blowups

$$\hat{V} \setminus (C_0 \cup C_1) =: X_{n+2} \to X_{n+1} \to X_n \to \ldots \to X_2 = \mathbb{A}^2,$$

where in the last step the point $Q$ is blown up to create $F_1$.

**2.7.** Let us exhibit a special case of this construction. Consider the standard action

$$(\lambda_1, \lambda_2) : (x, y) \mapsto (\lambda_1 x, \lambda_2 y)$$

of the 2-torus $T = (\mathbb{C}^*)^2$ on the affine plane $X_2 = \mathbb{A}^2$. We claim that there is a unique sequence of points $(0, 0) = P_3 = P^o_3, \ldots, P_{n+1} = P^o_{n+1}$ such that the torus action can be lifted to $X_i$ for $i = 3, \ldots, n+1$. Indeed, if by induction the $T$-action is lifted already to $X_i$ with $i \geq 2$, then on $C_i \setminus C_{i-1} \cong \mathbb{A}^1$ the induced $T$-action has a unique fixed point $P^o_{i+1}$. Blowing up this point the $T$-action can be lifted further to $X_{i+1}$. Blowing up finally $Q = (0, 1) \in C_2 \setminus C_1$ and deleting $C_2 \cup \ldots \cup C_n$ we arrive at a unique *standard* Danilov-Gizatullin surface $V_{st} = V_{st}(n)$.

Let us note that $T$ acts transitively on $(C_2 \setminus C_1) \setminus \{(0, 0)\}$. Thus up to isomorphism, the resulting affine surface $V_{st}$ does not depend on the choice of $Q$.

**2.8.** Consider now an automorphism $h$ of $\mathbb{A}^2$ fixing the $y$-axis pointwise. It moves the blowup centers $P_3, \ldots, P_{n+1}$ to new positions $P'_3, \ldots, P'_{n+1}$, while $P_3$ and $Q$ remain unchanged. It is easily seen that $h$ induces an isomorphism between $V$ and the resulting new affine surface $V'$. We show in Lemma 2.9 below that applying a suitable automorphism $h$, we can choose $V'$ to be the standard surface $V_{st}$ as in 2.7. This implies immediately Proposition 2.5 and as well Theorem 1.1 More precisely, our $h$ will be composed of *elementary shifts*

$$h_{a,t} : (x, y) \mapsto (x, y + ax^t), \quad \text{where} \quad a \in \mathbb{C} \quad \text{and} \quad t \geq 0.$$ 

**Lemma 2.9.** By a sequence of elementary shifts as in (6) we can move the blowup centers $P_3, \ldots, P_n$ into the points $P^o_3, \ldots, P^o_n$ so that $V$ is isomorphic to $V_{st}$.
Proof. Since $X_2 = \mathbb{A}^2$ the assertion is obviously true for $i = 2$. The point $P_3 = (0, 0)$ being fixed by $T$, the torus action can be lifted to $X_3$. The blowup with center at $P_3$ has a coordinate presentation

$$(x_3, y_3) = (x, y/x), \quad \text{or, equivalently,} \quad (x, y) = (x_3, x_3y_3),$$

where the exceptional curve $C_3$ is given by $x_3 = 0$ and the proper transform of $C_2$ by $y_3 = \infty$. The action of $T$ in these coordinates is

$$(\lambda_1, \lambda_2)(x_3, y_3) = (\lambda_1x_3, \lambda_1^{-1}\lambda_2y_3),$$

while the elementary shift $h_{a,t}$ can be written as

$$h_{a,t} : (x_3, y_3) \mapsto (x_3, y_3 + ax_3^{t-1}).$$

Thus in $(x_3, y_3)$-coordinates $P_3^t = (0, 0)$. Furthermore for $t = 1$, the shift $h_{a,1}$ yields a translation on the axis $C_3 \setminus C_2 = \{x_3 = 0\}$, while $h_{a,t}$ with $t \geq 2$ is the identity on this axis. Applying $h_{a,1}$ for a suitable $a$ we can move the point $P_4 \in C_3 \setminus C_2$ to $P_3^t$. Repeating the argument recursively, we can achieve that $P_i = P_i^t$ for $i \leq n + 1$, as required. \hfill \Box

Remarks 2.10. 1. The surface $X_{n+1}$ as in [2.7] is toric, and the $T$-action on $X_{n+1}$ stabilizes the chain $C_2 \cup \ldots \cup C_n \cup F_0$. There is a 1-parameter subgroup $G$ of the torus (namely, the stationary subgroup of the point $Q = (0, 1)$), which lifts to $X_{n+2}$ and then restricts to $V_{st} = X_{n+2} \setminus (C_2 \cup \ldots \cup C_n)$. Fixing an isomorphism $G \cong \mathbb{C}^*$ gives a $\mathbb{C}^*$-action on $V_{st}$. As follows from [FKZ2, 1.0.6], $V_{st} = V_{st}(n)$ is the normalization of the surface $W_n \subseteq \mathbb{A}^3$ with equation

$$x^{n-1}y = (z - 1)(z + 1)^{n-1}.$$
Λ_s, such that \( \bar{V} \) is its equivariant standard completion. Note that the isomorphism class of \((\bar{V}, D)\) is independent on the choice of the point \( P \in C_s \setminus (F_1 \cup C_{s-1}) \). Indeed this point can be moved by the \( T \)-action yielding conjugated \( \mathbb{C}^* \)-actions on \( V_n \).

Contracting the chain \( C_1 + \ldots + C_n \) leads to a Hirzebruch surface \( \Sigma_d \) such that the image of \( F_0 \) is a fiber of the ruling \( \Sigma_d \to \mathbb{P}^1 \). Moreover, the image \( S \) of \( C_0 \) is an ample section with \( S^2 = n \) so that \( V_n = \Sigma_d \setminus S \). The image of \( F_1 \) is another section with \( F_1^2 = n + 2 - 2s \). In particular, if this number is negative then \( d = 2s - 2 - n \).

One can show that the \( \Lambda_s, s = 2, \ldots, n \) represent all conjugacy classes of \( \mathbb{C}^* \)-actions on \( V_n \). Moreover, inverting the action \( \Lambda_s \) with respect to the isomorphism \( t \mapsto t^{-1} \) of \( \mathbb{C}^* \) yields the action \( \Lambda_{n-s+2} \). Thus after inversion, if necessary, we may suppose that \( 2s - 2 \geq n \) so that \( V_n \cong \Sigma_d \setminus S \) as above with \( d = 2s - 2 - n \).

As was remarked by Peter Russell, with the exception of Proposition 2.3 our proof is also valid for Danilov-Gizatullin surfaces over an algebraically closed field of any characteristic \( p \). Moreover Proposition 2.3 holds as soon as \( p = 0 \) or \( p \) and \( m \) are coprime. In particular it follows that the Isomorphism Theorem holds in the cases \( p = 0 \) and \( p \geq n - 2 \). This latter result was shown already in [DaGi]. However for \( p = 2 \) and \( n = 56 \) there is an infinite number of isomorphism types of Danilov-Gizatullin surfaces; see [DaGi] §9.

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