Abstract. We consider the degree 4 L-function associated to an automorphic cuspidal representation \( \pi \) of the symplectic group \( \text{GSp}_4 \). Starting with Beilinson’s Eisenstein symbol, we construct some motivic cohomology classes over the universal abelian scheme over the Shimura variety of \( \text{GSp}_4 \). We show that the image of these classes under the absolute Hodge regulator vanishes on the boundary of the Baily-Borel compactification of the Shimura variety. This allows to relate these classes to the product of an archimedean integral, Harris’ occult period invariant, a Deligne period and the special value of this L-function predicted by Beilinson’s conjecture. The representation \( \pi \) is assumed to be stable, of multiplicity one and to have a Bessel associated to an isotropic symmetric matrix.

2000 Mathematics Subject Classification: Primary 11F46; Secondary 19F27.

Keywords and Phrases: Shimura varieties, mixed Hodge modules, higher regulators, Rankin-Selberg integral.

Contents

1 Preliminaries
1.1 Conventions and notations ............................................. 6
1.2 Mixed Hodge modules and absolute Hodge cohomology ............ 11
1.3 The Shimura varieties .................................................. 13
1.4 Hodge structures and discrete series L-packets ....................... 15

2 Eisenstein classes for \( \text{GSp}_4 \)
2.1 Higher direct images of variations in the Baily-Borel compactification .... 20
2.2 Computations of higher direct images ............................... 22
2.3 Interior cohomology and boundary cohomology .................... 25
2.4 The vanishing on the boundary ...................................... 26
2.5 The Eisenstein classes ................................................. 30
3 Introduction

Given a pure motive $M$ over $\mathbb{Q}$, Beilinson’s conjectures relate the first non-zero term of the Taylor expansion at 0 (the special value at 0) of the L-function of $M$ to the Betti realization of mixed motives:

$$\operatorname{Ext}^1(\mathbb{Q}(0), M) \to \operatorname{Ext}^1_R(\mathbb{R}(0), M_B).$$

Here the left hand term is the space of 1-extensions between the trivial motive and $M$, in the abelian category $\operatorname{MM}(\mathbb{Q})$ of mixed motives over $\mathbb{Q}$. This category has not been discovered yet, but the triangulated category $\operatorname{DM}(\mathbb{Q})$ of Voevodsky ([51] Ch.5) has several properties of the bounded derived category $\mathcal{D}^b\operatorname{MM}(\mathbb{Q})$: for example it contains the category of Chow motives over $\mathbb{Q}$ and is the source of a contravariant triangulated functor

$$\operatorname{DM}(\mathbb{Q}) \to \mathcal{D}^b\operatorname{MHS}^+_R$$

in the bounded derived category of $\mathbb{R}$-mixed Hodge structures with infinite Frobenius ([25], [26]).

Conjecture. (Beilinson) Let $E$ be an extension of $\mathbb{Q}$ and $M$ a Chow motive over $\mathbb{Q}$, with coefficients in $E$ and of weight $\leq -3$.

(i) Then the image of the Betti realization functor

$$\operatorname{Hom}_{\operatorname{DM}(\mathbb{Q})}(M, \mathbb{Q}(0)[1]) \to \operatorname{Ext}^1_{\operatorname{MHS}^+_R}(\mathbb{R}(0), M_{B_R})$$

is an $E$-structure of the left hand term.

(ii) Let $\mathcal{D}(M)$ be the Deligne $E$-structure on the highest exterior power $\det \operatorname{Ext}^1_{\operatorname{MHS}^+_R}(\mathbb{R}(0), M_{B_R})$ and $L(s, M) = \prod_{v<\infty} L_v(s, M) \in E \otimes \mathbb{C}$ be the L-function of $M$. Then the highest exterior power $\mathcal{R}(M)$ of the $E$-structure given by (i) satisfies

$$\mathcal{R}(M) = L(0, M)\mathcal{D}(M).$$
Higher regulators, periods and special values...

The assumption on the weight implies (conjecturally) that 0 is in the convergence region of $L(s, M)$. Originally, the conjecture has been stated for the motive $H^1(X)(n)$ associated to a proper and smooth $\mathbb{Q}$-scheme $X$ (6 8.4.2 and 38 6.1). It relates the special value $L^*(0, H^4(X)(n))$ to the image under Beilinson’s regulator of the motivic cohomology space $H^{k+4}_{\mathcal{M}}(X, \mathbb{Q}(n))$. But as the weight $i - 2n$ is assumed to be $\leq -3$, hence non zero, this space is conjecturally isomorphic to the space of 1-extensions $\text{Ext}_{\mathcal{M}(\mathbb{Q})}^1(\mathbb{Q}(0), H^4(X)(n))$ (see 30 Rem. 4.12 (c)). This formulation of the conjecture, well known to specialists, has the advantage to make transparent the connection with Selmer groups associated to $p$-adic representations, objects far more familiar in number theory.

All but one successful attempt to prove the conjecture involved three main ideas, already present in Beilinson’s work on elliptic modular forms [5]: construction of motivic cohomology classes starting with Beilinson’s Eisenstein symbol, calculation of their image under the regulator map and comparison of the result with the L-value via the Rankin-Selberg method. Building on these ideas, Kings proved a weak form of the conjecture for some Hilbert modular forms over a real quadratic field [31]. We adress the case of the group of similitudes $\text{GSp}_4 = \{ g \in \text{GL}(V_4) \mid \exists \nu(g) \in \mathbb{G}_m, \forall v, w \in V_4, \psi(gv, gw) = \nu(g) \psi(v, w) \}$. of a four-dimensionnal symplectic space $(V_4, \psi)$ over $\mathbb{Q}$. Fix two integers $k \geq k' \geq 0$ and an imbedding

$$\text{GL}_2 \times_{\mathbb{G}_m} \text{GL}_2 \to \text{GSp}_4$$

of the group of couples of matrices with the same determinant into $\text{GSp}_4$. On the level of Shimura varieties, this gives rise to closed imbeddings of codimension one

$$(E \times E')^{k+k'} \to \mathbb{A}^{k+k'}$$

of the product of two modular curves, with their universal elliptic curves, into the Siegel modular variety of dimension 3, with it’s universal abelian surface. The Eisenstein symbol [5], [32], [17] provides motivic cohomology classes in $H^{k+k'}_{\mathcal{M}}(\mathbb{Q}(n+1))$ for every integer $n$. Then the image under the Gysin morphism $\iota_*$ of cup product of Eisenstein symbols for $k$ and $k'$ defines a subspace $\mathcal{E}^{k,k'}$ of the motivic cohomology space $H^{k+k'+4}_{\mathcal{M}}(\mathbb{A}^{k+k'}, \mathbb{Q}(k+k'+3))$ (see 2.5.2). Let $V_2$ be the standard representation of $\text{GL}_2$ and $\det$ be the determinant. Up to isomorphism, there exists a unique representation $W^{k,k'}$ of $\text{GSp}_4$ whose restriction $\iota^*W^{k,k'}$ contains the irreducible representation $(\text{Sym}^k V_2 \otimes \text{Sym}^{k'} V_2) \otimes \det \otimes 3$. One associates to $W^{k,k'}$ a variation of $\mathbb{Q}$-Hodge structures on $S$ of weight $-c(W^{k,k'})$, where $c(W^{k,k'}) = k + k' + 6$ is the central character of $W^{k,k'}$. Hence the interior cohomology

$$H^3(S, W^{k,k'}) = \text{Im} (H^3_b(S, W^{k,k'}) \to H^3(S, W^{k,k'}))$$

is a $\mathbb{Q}$-Hodge structure pure of weight $w = -c(W^{k,k'}) + 3$. Our first result is the following

**Theorem.** Assume $k > k' > 0$ and $k \neq k' + 3$. Then the space $\mathcal{E}^{k,k'}$ is mapped under Beilinson’s regulator

$$r : H^{k+k'+4}_{\mathcal{M}}(\mathbb{A}^{k+k'}, \mathbb{Q}(k+k'+3)) \to H^{k+k'+4}(\mathbb{A}^{k+k'}, \mathbb{R}(k+k'+3)) \otimes \overline{\mathbb{Q}}$$
into the space of 1-extensions \( \text{Ext}_{\text{MHS}}^1 (\mathbb{R}(0), H^3(S, W^{k,k'})_\mathbb{R}) \otimes \mathbb{Q} \).

In fact by the very definition of \( \mathcal{E}^{k,k'} \), one has \( r(\mathcal{E}^{k,k'}) \subset \text{Ext}_{\text{MHS}}^1 (\mathbb{R}(0), H^3(S, W^{k,k'})_\mathbb{R}) \otimes \mathbb{Q} \). The theorem follows from the vanishing of \( r(\mathcal{E}^{k,k'}) \) on the boundary of the Baily-Borel compactification of \( S \) (propositions 2.9 and 2.10).

Let \( \pi = \bigotimes_v \pi_v \) be an irreducible cuspidal automorphic representation of \( \text{GSp}_4 \) and \( S \) be the finite set of finite places where \( \pi \) is ramified together with the infinite place. The degree four \( L \)-function of \( \pi \) is the Langlands Euler product

\[
L_S(s, \pi) = \prod_{v \in S} L(s, \pi_v)
\]

associated to the standard representation of the Langlands dual \( L^{\text{GSp}_4} = \text{GSp}_4(\mathbb{C}) \). Assume that the archimedean component \( \pi_\infty \) of \( \pi \) is in the discrete series \( L \)-packet \( P(W^{k,k'}) = \{ \pi_H, \pi_W, \pi_{W'}, \pi_{H'} \} \) associated to \( W^{k,k'} \). Then by [21], it’s non-archimedean part \( \pi_f \) is defined over a number field \( E(\pi_f) \).

Hence the space

\[
M_B(\pi_f, W^{k,k'}) = \text{Hom}_{\mathbb{Q}[G(\mathbb{A}_f)]} (\text{Res}_{E(\pi_f)/\mathbb{Q}} \pi_f, H^3(S, W^{k,k'}))
\]

is a pure \( \mathbb{Q} \)-Hodge structure with coefficients in \( E(\pi_f) \), of weight \( w \). Let us also assume that \( \pi \) is stable and of multiplicity one, which means that the multiplicities of representations with the same non-archimedean part than \( \pi \) and with archimedean part in \( P(W^{k,k'}) \) satisfy

\[
m(\pi_\infty^H \otimes \pi_f) = m(\pi_\infty^W \otimes \pi_f) = m(\pi_{\infty}^W \otimes \pi_f) = m(\pi_{\infty}^H \otimes \pi_f) = 1.
\]

Such representations represent the majority of representations with archimedean component in \( P(W^{k,k'}) \) (see [2] (a) p. 78). An example is given by the theta lift to \( \text{GSp}_4 \) of an Hilbert modular form over a real quadratic field ([31] Th. 8.6 (1)). By the work of Taylor [50], Laumon [35] and Weissauer [74] one knows that for every prime number \( l \), the local \( L \)-factor at a place \( v \notin S \cup \{ l \} \) of the l-adic Galois representation \( M_l(\pi_f, W^{k,k'}) \) corresponding to \( M_B(\pi_f, W^{k,k'}) \) is related to the Langlands local \( L \)-factor by

\[
L_v(s, M_l(\pi_f, W^{k,k'})) = L(s - \frac{3}{2}, \tilde{\pi}_v),
\]

where \( \tilde{\pi} \) is the dual of \( \pi \). This equality is compatible with functional equations on both sides (section 3.7). Fix an embedding of \( E(\pi_f) \) in \( \mathbb{Q} \). Stability and multiplicity one imply that the \( \mathbb{R} \otimes \mathbb{Q} \)-module \( \text{Ext}_{\text{MHS}}^1 (\mathbb{R}(0), M_B(\pi_f, W^{k,k'})_\mathbb{R}) \otimes_{E(\pi_f)} \mathbb{Q} \) is of rank one. Let \( \mathcal{D}(\pi_f, W^{k,k'}) \) be its Deligne \( \mathbb{Q} \)-structure (definition 3.4 (ii)). The preceding theorem gives another sub \( \mathbb{Q} \)-space \( \mathcal{R}(\pi_f, W^{k,k'}) \) of the space of 1-extensions \( \text{Ext}_{\text{MHS}}^1 (\mathbb{R}(0), M_B(\pi_f, W^{k,k'})_\mathbb{R}) \otimes_{E(\pi_f)} \mathbb{Q} \) that is to be compared to \( \mathcal{D}(\pi_f, W^{k,k'}) \). One compares the two by comparing their cup-products by the cohomology class associated to cusp form \( \phi \) on \( \text{GSp}_4 \) (lemma 3.9). The cup-product with \( \mathcal{R}(\pi_f, W^{k,k'}) \) turns out to be a global adelic integral of the product of \( \phi \) by an Eisenstein series (proposition 3.22); and when the cohomology class associated to \( \phi \) is assumed to be rational in de Rham cohomology, the cup-product with \( \mathcal{D}(\pi_f, W^{k,k'}) \) is a Deligne period (proposition 3.10). In the following, we’ll denote by \( \pi' \) the representation \( \tilde{\pi} \mid^{k-k'-3} \). Unfortunately, we also denote by \( \pi \) the length of a circle of radius \( \frac{1}{2} \).
Theorem. Let $Z$ be the center of $GL_2 \times \mathbb{G}_m \times GL_2$. Assume that $\pi$ is stable, of multiplicity one, with archimedean part belonging to the subset of generic elements $\{\pi_{W, \infty}, \pi_{\infty}^W\}$ of $P(W^{k', k})$ and with central character of sign $(-1)^{k+k'}$. Let $\alpha_1$ and $\alpha_2$ be two Hecke characters of $k^\times$ of respective norms $-k'$ and $-k$ and of respective signs $(-1)^k$ and $(-1)^{k'}$. Let $c(-1)^{k+k'+1}(\pi_f, W^{k', k})$ be the Deligne period of sign $(-1)^{k+k'+1}$ associated to $\pi_f$ and $W^{k, k'}$. Then there is an Eisenstein series $E^\Phi(x, \alpha_1, k + k' + 3 - \frac{3}{2})$ such that

$$\frac{R(\pi_f, W^{k', k})}{D(\pi_f, W^{k', k})} \equiv c(-1)^{k+k'+1}(\pi_f, W^{k', k}) \int \left| Z(\mathbb{A}) (GL_2 \times \mathbb{G}_m \times GL_2)(\mathbb{Q}) \right| (GL_2 \times \mathbb{G}_m \times GL_2)(\mathbb{A}) \phi(x) E^\Phi(x, \alpha_1, k + k' + 3 - \frac{3}{2}) dx$$

up to multiplication by a non-zero algebraic number and a power of $i$ and of $\pi$.

The integral appearing in the theorem was defined and studied by Piatetski-Shapiro in order to prove the analytic continuation and the functional equation of $L_S(s, \pi)$. The non vanishing of the integral, i.e. of $R(\pi_f, W^{k, k})$, depends on the Fourier expansion of the form $\phi$ along the Siegel maximal parabolic of $GSp_4$, whose unipotent radical $W^0$ is the group of symmetric matrices of size 2. More precisely, it is equivalent to the existence of a Bessel model of $\phi$ relative to $(\beta, \alpha_1, \alpha_2)$ where $\beta \in W^0(\mathbb{Q})$ is an isotropic matrix (lemma 3.31). As the Fourier expansion of cuspidal Siegel modular forms is indexed by definite positive symmetric matrices, the integral would be zero if $\phi$ were such a form. So it is remarkable that mixed Hodge theory forces to choose a cusp form $\phi$ which is a generic member of a discrete series L-packet (remark following lemma 3.31), in other terms a form that is neither a holomorphic nor an anti-holomorphic Siegel modular form. Assuming the existence of the global Bessel model, the integral is expanded in an Euler product of local integrals, that can be assumed to be non zero algebraic numbers if the local Bessel model is algebraic (lemma 3.32). For holomorphic forms, the q-expansion principle says that the rationality in de Rham cohomology is equivalent to the one of the Fourier expansion. Here, it was noticed by Harris that the rational structure on $\pi'$ given by de Rham cohomology and the one given by the Bessel model differ by a number, the occult period invariant $a(\pi', \beta, \alpha_1, \alpha_2)$ (definition 3.33). Relying on Furusawa’s computations (proposition 3.32) of the unramified non-archimedean integrals we deduce our main result.

Theorem. Let $S'$ be a finite set of ramified places containing $S$. We keep assumptions and notations of the preceding theorem. Let $\beta$ be an isotropic symmetric matrix in $GL_2(\mathbb{Q})$ and $Z_{\infty}$ be the archimedean integral appearing in the factorization (3.6.1). Assume that $\pi'$ has a Bessel model relative to $(\beta, \alpha_1, \alpha_2)$. Then

$$R(\pi_f, W^{k', k}) \equiv Z_{\infty} a(\pi', \beta, \alpha_1, \alpha_2) c(-1)^{k+k'+1}(\pi_f, W^{k', k}) L_{S'}(-\frac{3}{2}, \pi_f) D(\pi_f, W^{k', k})$$

up to multiplication by a non-zero algebraic number and a power of $i$ and of $\pi$.

Remarks.

- Powers of $i$ and of $\pi$ appearing in the above theorem are computable but this seems worthless while the archimedean integral $Z_{\infty}$ hasn’t been computed. We expect this integral to be the value
at zero of the Gamma factor predicted by the rule of Serre (see section 3.7), up to integral powers of $i$ and of $\pi$. A similar integral has been computed in [36] Th. 12.2.

- The theorem above shows that the non vanishing of the regulator, which is predicted by Beilinson's conjecture, implies the existence of a Bessel model for the stable automorphic representation $\pi$, relative to $(\beta, \alpha_1\alpha_2)$.

- Our theorem deals with the value of a partial $L$-function whereas the conjecture is about the complete $L$-function. It is a consequence of the conjecture $C_6$ of Serre [48] that at the places $v$ of bad reduction the local $L$-factor $L_v(s, M)$ is a well defined algebraic number at 0. Hence assuming $C_6$, one can replace the complete $L$-function by the partial $L$-function in the statement of the conjecture for motives with coefficients in $\mathbb{Q}$.

- In the paper [24], the occult period invariant is related to the critical values of the degree four $L$-function. Without going into the details of the careful sign analysis made in [loc. cit.], let us mention that the main results of this paper and Deligne's conjecture [15] show that this invariant looks like the inverse of a Deligne period $c(\pi_f, W^{kk})$. This is compatible with the above result and Beilinson's conjecture.

- Except the major technical complexity, the main new phenomenon appearing in the study of Beilinson's conjecture for stable forms on $\text{GSp}_4$ is the occurrence of the generic forms in the Rankin-Selberg integral. This leads to the presence of the occult period and of the Deligne period in the final result.

1 Preliminaries

1.1 Conventions and notations

1.1.1 Given a ring $A$, a $A$-algebra $B$ and a $A$-module $M$, resp. a $A$-scheme $X$, we’ll denote by $M_B$, resp. $X_B$, the $B$-module $M \otimes_A B$, resp. the $B$-scheme $X \otimes_A B$. We fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$. For $x, y \in \mathbb{C} \otimes \overline{\mathbb{Q}}$, we write $x \equiv y$ if $x$ and $y$ coincide up to left multiplication by a power of $i$ and of $\pi$ and right multiplication by an element of $\mathbb{Q}^\times$.

1.1.2 Let $G_2$, resp. $G_4$, the group $\text{GL}_2$, resp. $\text{GSp}_4$. The multiplicative group $\mathbb{G}_m$ will sometimes be denoted by $G_0$. We have the similitude factor $\nu_4 : G_4 \to G_0$ and the determinant $\nu_2 : G_2 \to G_0$. We write as usual $\text{Sp}_4 = \text{Ker} \nu_4$. Let $\Pi_2$ be the group $G_2 \times_{G_0} G_2$ of couples of invertible matrices with the same determinant. We choose a symplectic basis $(e_1, e_2, e_3, e_4)$ of $(V_4, \psi)$ such that

$$\psi = \begin{pmatrix} 0 & \mathbf{I}_2 \\ -\mathbf{I}_2 & 0 \end{pmatrix},$$

where $\mathbf{I}_2$ denotes the identity matrix of size 2. Hence we have a symplectic isomorphism $V_2 \oplus V_2 \simeq V_4$ and the imbedding $\iota : \Pi_2 \to G_4$ is given by

$$\iota \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a' & 0 & b' \\ c & 0 & d & 0 \\ 0 & c' & 0 & d' \end{pmatrix}.$$  \hspace{1cm} (1.1.1)
The symbol $n$ will denote either $2$ or $4$. Let $T_n$, $B_n$ and $Z_n$ be the diagonal maximal torus, the standard Borel and the center of $G_n$. We have

$$\begin{align*}
T_2 &= \left\{ \text{diag}(\alpha, \alpha^{-1} \nu) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \nu \end{pmatrix}, (\alpha, \nu) \in \mathbb{G}_m^2 \right\}, \\
T_4 &= \left\{ \text{diag}(\alpha_1, \alpha_2, \alpha_1^{-1} \nu, \alpha_2^{-1} \nu) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & \alpha_1^{-1} \nu & 0 \\ 0 & 0 & 0 & \alpha_2^{-1} \nu \end{pmatrix}, (\alpha_1, \alpha_2, \nu) \in \mathbb{G}_m^3 \right\}.
\end{align*}$$

The group of characters $X^*(T_n)$ is identified to $\mathbb{Z}^{n+1}$ via $\lambda(k, t) : \text{diag}(\alpha, \alpha^{-1} \nu) \mapsto \alpha_1^k \nu$ and $\lambda(k, k', t) : \text{diag}(\alpha_1, \alpha_2, \alpha_1^{-1} \nu, \alpha_2^{-1} \nu) \mapsto \alpha_1^k \alpha_2^{k'} \nu$. Write $\rho_1 = \lambda(1, -1, 0)$ and $\rho_2 = \lambda(0, 2, 0)$. Then the roots of $T_4$ in $G_4$ are $R = \{ \pm \rho_1, \pm \rho_2, \pm (\rho_1 + \rho_2), \pm (2 \rho_1 + \rho_2) \}$ and the positive roots with respect to $B_4$ are $R^+ = \{ \rho_1, \rho_2, \rho_1 + \rho_2, 2 \rho_1 + \rho_2 \}$ (Gr § 16.1). The Weyl group $W_4$ is the dihedral group generated by reflections of $\mathbb{R}^2$ with axis $\rho_1$ and $\rho_2$. The reflection with axis $\rho$ will be denoted by $s_\rho$. For $G_4$, dominants, resp. regular weights are the $\lambda(k, k', t)$ with $k > k' > 0$, resp. $k > k'$ > 0. Given a representation $E_n$ of $G_n$, denote by $E_n(t)$ the representation $E_n \otimes \nu_n^{\otimes t}$, resp. $E_n \otimes \nu_n^{\otimes -t}$, if $t > 0$, resp. if $t < 0$. For any dominant weight $\lambda(k, k', t)$, there exists a unique, up to isomorphism, irreducible subrepresentation $E_4 \subset V_4^{\otimes k + k'}(t)$ of highest weight $\lambda(k, k', t)$ (loc. cit. Th. 17.5) and all irreducible representations are obtained in this way. Note that for weight reasons, the restriction $\iota^*E$ to $\Pi_2$ contains the irreducible representation $(\text{Sym}^k V_2 \boxtimes \text{Sym}^k V_2)(t)$. The symplectic form induces an isomorphism $V_4 \cong V_4(1)$. As a consequence, if $E$ is as above we have $E \cong E(k + k' + 2t)$. Dominants, resp. regular, weights of $T_2$ are the $\lambda(k, t)$ with $k \geq 0$, resp. $k > 0$. The irreducible representation of $G_2$ of highest weight $\lambda(k, k')$ is the twisted symmetric product $\text{Sym}^k V_2(t)$. We fix once and for all an irreducible representation $W^k k'$ of $G_4$ of highest weight $\lambda(k, k', 3)$.

1.1.3 The unitary group $U(n/2) = \{ X \in \text{GL}_n(\mathbb{C}) |^t X X = I_2 \}$ is identified to a maximal compact subgroup $K'_n$ of $\text{Sp}_n(\mathbb{R})$ by the map $X = A + i B \mapsto \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$. Then $K_{n, \infty} = Z(\mathbb{R}) K'_n$ is a subgroup of $G_n(\mathbb{R})$ maximal compact modulo the center and $T'_4 = K_{2, \infty} \times G_m K_{2, \infty}$ is a maximal elliptic torus of $G_4(\mathbb{R})$. Let $\mathbb{R}^\pm$ be the non zero real numbers of sign $\pm 1$ and $G_n(\mathbb{R})^\pm = \nu_n^{-1}(\mathbb{R}^\pm)$. Write $g_n$, $t'_n$ and $t_n$ for the complex Lie algebras of $G_n$, $K'_n$ and $K_n$ respectively. We have

$$t'_n = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} | A, B \in M_2(\mathbb{C}) \right\}.$$

The Cartan decomposition is $g_n = t_n \oplus p_n$ with $p_n = p_n^+ \oplus p_n^-$ and

$$p_n^\pm = \left\{ M = \begin{pmatrix} Z & \pm i Z \\ \pm i Z & -Z \end{pmatrix} | Z \in M_2(\mathbb{R}), \; ^t Z = Z \right\}.$$

Let $J_2 = \begin{pmatrix} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \end{pmatrix} \in G_2(\mathbb{C})$ and $J_4 = \iota(J_2 \times J_2) \in G_4(\mathbb{C})$. Then $\text{Ad} J_n : g \mapsto J_n^{-1} g J_n$ induce an isomorphism $K_{2, \infty} \cong T_2$, resp. $T'_4(\mathbb{C}) \cong T_4$, for $n = 2$, resp. $n = 4$. For $c \equiv n \mod 2$ and
$c' \equiv n + n' \mod 2$ denote by $\lambda'(n, c)$ and $\lambda'(n, n', c')$ respectively the characters

$$\lambda(n, \frac{c-n}{2}) \circ \text{Ad} J_2 : \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \mapsto (x + iy)^n(x^2 + y^2)^{\frac{c-n}{2}},$$

$$\lambda(n, n', \frac{c-n-n'}{2}) \circ \text{Ad} J_4 : \begin{pmatrix} x & 0 & y & 0 \\ 0 & x' & 0 & y' \\ -y & 0 & x & 0 \\ 0 & -y' & 0 & x' \end{pmatrix} \mapsto (x + iy)^n(x' + iy')^{n'}(x^2 + y^2)^{\frac{c-n-n'}{2}}. \quad (1.1.3)$$

**Lemma 1.1.** Let $v^\pm \in \mathfrak{p}_2^\pm$ corresponding to $Z = \pm \frac{1}{2}$ in $\mathfrak{t}_1, \mathfrak{t}_2$. (i) The weight of $v^\pm$ is $\lambda'(\pm 2, 0)$.

(ii) The vectors $\vartheta = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 & i \\ -1 & 0 & i & 0 \\ 0 & -i & 0 & 1 \\ -i & 0 & -1 & 0 \end{pmatrix}$ and $\overline{\vartheta} = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 & -i \\ -1 & 0 & -i & 0 \\ 0 & i & 0 & 1 \\ i & 0 & -1 & 0 \end{pmatrix}$ in $\mathfrak{t}_4$ are of weight $\lambda'(1, -1, 0)$ and $\lambda'(-1, 1, 0)$ respectively.

**Proof.** Easy computation. □

The lemma shows that the compact roots are $\Delta_{K_{4, \infty}} = \{ \pm \rho_i \}$, the roots in the Lie algebra $\mathfrak{p}_4^\pm$ are $\{ \pm \rho_i, \pm (\rho_i + \rho_j), \pm (2\rho_i + \rho_j) \}$, the positive compact root is $\rho_i$ and the dominant weights of $K_{4, \infty}$ are the $\lambda'(n, n', c)$ with $n \geq n'$. The compact Weyl group $W_{K_{4, \infty}}$ is the subgroup of $W_4$ generated by the reflexion of axis $\rho_1 + \rho_2$.

**Lemma 1.2.** Let $m \geq 0$ be an integer. The choice of a symplectic basis $(X, Y)$ of $V_2$ gives an identification of $\text{Sym}^m V_2$ with the space of homogeneous polynomials of degree $m$ in $X$ and $Y$.

(a) For $0 \leq j \leq m$ let $b_j^{m}(X, Y) = (iX - Y)^j(iX + Y)^{m-j}$. Then $b_j^m$ if of weight $\lambda'(2j - m, m)$.

(b) Denote by $[ \cdot, \cdot ]_2$ the pairing $\text{Sym}^m \tilde{V}_2 \otimes \text{Sym}^m \tilde{V}_2 \rightarrow \mathbb{Q}(-m)$. There is a basis $(a_j^m)_{0 \leq j \leq m}$ of $\text{Sym}^m \tilde{V}_2$ such that

(i) the weight of $a_j^m$ is $\lambda'(m - 2j, -m)$,

(ii) one has $v^+ a_j^m = \begin{cases} -j a_{j-1}^m & \text{if } j \geq 1 \\ 0 & \text{else.} \end{cases}$ and $v^- a_j^m = \begin{cases} -(m - j) a_{j+1}^m & \text{if } j \leq m - 1 \\ 0 & \text{else.} \end{cases}$

(iii) $a_j^m = (-1)^m a_{m-j}^m$.

(iv) $[a_i^m, a_j^m]_2 = \begin{cases} 1 & \text{if } i + j = m \\ 0 & \text{else.} \end{cases}$

**Proof.** (a) Easy computation. (b) Write $b_{m+1}^m = b_{-1}^m = 0$. Then we have $v^+ b_j^m = (m - j)b_{j+1}^m$, $v^- b_j^m = jb_{j-1}^m$. By (a), in the dual basis $(\tilde{b}_j^m)_j$, the vector $\tilde{b}_j^m$ is of weight $\lambda'(m - 2j, -m)$ and by the Leibniz rule we have $v^+ \tilde{b}_j^m = -(m - j + 1)\tilde{b}_{j-1}^m$ and $v^- \tilde{b}_j^m = -(j + 1)\tilde{b}_{j+1}^m$. Define inductively $\alpha_0 = 1, \alpha_j = j/(m - j + 1)\alpha_{j-1}$. Then one checks easily that the basis $(a_j^m = \alpha_j \tilde{b}_j^m)_j$ has the properties (i), (ii) and (iii). Property (iv) follows from the consideration of weights. □
1.1.4 Let $S = \text{Res}_{\mathbb{C}/\mathbb{R}} G_m$ be the Deligne torus. We follow the convention of $[10]$ (1.1.1) and $[40]$ 1.3 associating to a representation $\rho : S \to \text{GL}(V)$ in a $\mathbb{R}$-vector space correspond the decompositions $V^* = \bigoplus_{p,q} V^{p,q}$ where $V^{p,q}$ is the factor where $z \in S(\mathbb{R}) = \mathbb{C}^\times$ acts by $z^{p}z^{-q}$. Note that $\mathbb{R}^\times \subset \mathbb{C}^\times$ acts on the factor $V^{p,q}$ by the character $x \mapsto x^{-t}$ where $t = p + q$.

1.1.5 Fix a split reductive algebraic linear group $G$ over $\mathbb{Q}$, with center $Z$ and let $K_{\infty}$ be a maximal compact modulo the center subgroup of $G(\mathbb{R})$. In the following $G$ will be $G_4$ or $H_2$. Denote by $C^\infty(G(\mathbb{Q})\backslash G(\mathbb{A}))$, resp. $C^\infty_c(G(\mathbb{Q})\backslash G(\mathbb{A}))$, the space of functions $G(\mathbb{Q})\backslash G(\mathbb{A}) \to \mathbb{C}$ whose restriction to $G(\mathbb{R})$ is $K_{\infty}$-finite, of class $C^\infty$, resp. of class $C^\infty$ and with compact support modulo the center, and whose restriction to $G(\mathfrak{a}_f)$ is locally constant with compact support. Denoting by $C^\infty_c(G(\mathbb{Q})\backslash G(\mathbb{A}))$ the spaces of rapidly decreasing and slowly increasing functions (see $[22]$ § 1 for the definition) respectively, we have the sequence of inclusions

$$C^\infty_c(G(\mathbb{Q})\backslash G(\mathbb{A})) \subset C^\infty_c(G(\mathbb{Q})\backslash G(\mathbb{A})) \subset C^\infty_c(G(\mathbb{Q})\backslash G(\mathbb{A})) \subset C^\infty_c(G(\mathbb{Q})\backslash G(\mathbb{A})).$$

Let $\mathfrak{g} = (\text{Lie} G)_{\mathbb{C}}$ and $\mathfrak{t} = (\text{Lie} K_{\infty})_{\mathbb{C}}$. The choice of a generator 1 of the highest exterior power \( \bigwedge^{\max} \mathfrak{g}/\mathfrak{t} \) determines a left translation invariant measure $dg_{\infty}$ on $G(\mathbb{R})/K_{\infty}$. We choose a Haar measure $dk$ on $Z(\mathbb{R}) \backslash K_{\infty}$ with total mass equal to one. So $dg_{\infty}dk$ is a Haar measure on $Z(\mathbb{R}) \backslash G(\mathbb{R})$. On $G(\mathfrak{a}_f) = \prod_p G(\mathbb{Q}_p)$ take the Haar measure $dg_f = \prod_p dg_p$ where $dg_p$ is the Haar measure on $G(\mathbb{Q}_p)$ for which $G(\mathbb{Z}_p)$ has volume one. Hence the choice of 1 determines a Haar measure $dg = dg_{\infty}dg_f$ on $Z(\mathbb{R}) \backslash G(\mathbb{A})$. Given a Hecke character $\omega : \mathbb{Q}^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$ let $\chi_\omega = \omega \circ \nu$, where $\nu$ is $\nu_2$ if $G = G_2$, $G_2 \times_{G_m} G_2$ and is $\nu_4$ if $G = G_4$. Let

$$L^2(Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)$$

be the Hilbert space of square integrable functions modulo the center, with the action of $G(\mathbb{A})$ by right translation. Let $C^\infty_c(G(\mathbb{Q})\backslash G(\mathbb{A})) = \bigcup \omega L^2(Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A}), \omega) \cap C^\infty_c(G(\mathbb{Q})\backslash G(\mathbb{A}))$ be the Hilbert sum of its closed irreducible subspaces $\omega L^2(Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A}), \omega) = \bigoplus_{m=1}^\infty m(\pi)^m$ with finite multiplicities ($[9]$ 4.6). The $\pi$’s appearing in the decomposition are the automorphic cuspidal representations of $G$ with central character $\omega$. Unfortunately, we will use the same letter $\pi$ to denote automorphic representations and the length of a circle of radius 1/2. Let $T \subset B \subset G$ be a maximal torus and a Borel. Given a Hecke character $\chi : T(\mathbb{Q})\backslash T(\mathbb{A}) \to \mathbb{C}^\times$ we denote by $\text{ind}_{B(\mathbb{A})}^{G(\mathbb{A})}\chi$ the space of functions $f : G(\mathbb{A}) \to \mathbb{C}$ that are $K_{\infty}$-finite, locally constant on $G(\mathfrak{a}_f)$ and satisfy $f(tx) = \chi(t)f(x)$. The space $\text{ind}_{B(\mathbb{A})}^{G(\mathbb{A})}\chi$ is a $(\mathfrak{g},K_{\infty}) \times G(\mathfrak{a}_f)$ module by right translation.

1.1.6 Given a $(\mathfrak{g},K_{\infty})$-module $V$, we’ll consider the complex $C^\ast(\mathfrak{g},K_{\infty},V) = \text{Hom}_{K_{\infty}}(\bigwedge^\ast \mathfrak{g}/\mathfrak{t}, V)$ ($[10]$ 1) and we’ll denote by $H^\ast(\mathfrak{g},K_{\infty},V)$ it’s cohomology. A congruence subgroup of $G(\mathbb{Q})$ is a subgroup of the shape $G(\mathbb{Q}) \cap K$ for a compact open subgroup $K \subset G(\mathfrak{a}_f)$. An arithmetic subgroup of $G(\mathbb{Q})$ is a subgroup commensurable with a (hence with every) congruence subgroup of $G(\mathbb{Q})$. Every compact open subgroup $K \subset G(\mathfrak{a}_f)$ will be assumed to be neat and we will freely use that the subgroup and the image of a neat group are neat, that neatness is invariant under inner automorphisms of $G(\mathfrak{a}_f)$ and that a group that is finite and neat is trivial (loc. cit. 0.6).
1.1.7 Let $T$ be a torus defined over $\mathbb{Q}$ and $\lambda_0$ be an algebraic character $T \to \mathbb{G}_m$. Denote by $T(\mathbb{R})^+$ the neutral component of $T(\mathbb{R})$. An algebraic character of $T$ of type $\lambda$ is a continuous character $\lambda : T(\mathbb{Q}) \backslash T(\mathbb{A}) \to \mathbb{C}^\times$ whose restriction to $T(\mathbb{R})^+$ coincides the one of with $\lambda(\mathbb{R})^{-1}$. For $T = \mathbb{G}_m$, we will say "algebraic Hecke character of type $r$" instead of "algebraic Hecke character of type $x \mapsto x^r". The sign of such a character $\lambda$ is defined as $\text{sgn}(\lambda) = \lambda|_{\mathbb{R}^\times}(-1)$.

1.1.8 Let us now recall a definition and some basic properties of the motivic cohomology functor. Let $k$ be a field of characteristic zero, let $\text{Sm}/k$ be the category of smooth $k$-schemes and let $R$ be a commutative ring. Then we have the tensor triangulated category $\text{DM}_{gm}^\text{eff}(k)$ of effective geometric motives over $k$ and the rigid tensor triangulated category $\text{DM}_{gm}(k)$ of geometric motives over $k$ ([51] Def. 2.1.1 p. 192 and (4.3.7)). Denote by $\text{DM}_{gm}^\text{eff}(k)_R$ and $\text{DM}_{gm}(k)_R$ the categories deduced by extending the scalars to $R$. There is a fully faithful covariant functor

$$\text{Sm}/k \xrightarrow{M_{gm}} \text{DM}_{gm}^\text{eff}(k)_R$$

and a tensor triangulated covariant functor

$$\text{DM}_{gm}^\text{eff}(k)_R \xrightarrow{\text{DM}_{gm}(k)_R}$$

We denote again by $M_{gm}$ the composition $\text{Sm}/k \longrightarrow \text{DM}_{gm}^\text{eff}(k)_R \longrightarrow \text{DM}_{gm}(k)_R$. For $X \in \text{Sm}/k$ and two integers $m$ and $n$, the $n$-th motivic cohomology group of $X$ with coefficients in $R(m)$ is the $R$-module

$$H^n_{\text{DM}}(X, R(m)) = \text{Hom}_{\text{DM}_{gm}(k)_R}(M_{gm}(X), R(m)[n]).$$

We see $H^n_{\text{DM}}(X, R(m))$ as a substitute for the group of $n$-extensions $\text{Ext}^n_{\text{MM}(X)}(R(0), R(m))$ in the category $\text{MM}(X)$ of mixed motivic sheaves over $X$. Given a morphism $f : X \longrightarrow Y \in \text{Sm}/k$, we write

$$f^* = \text{Hom}_{\text{DM}_{gm}(k)_R}(M_{gm}(f), R(m)[n]) : H^n_{\text{DM}}(Y, R(m)) \longrightarrow H^n_{\text{DM}}(X, R(m)).$$

For $X, X' \in \text{Sm}/k$, we have $M_{gm}(X) \otimes M_{gm}(X') = M_{gm}(X \times X')$. Let $p_X : X \times X' \longrightarrow X$ and $p_{X'} : X \times X' \longrightarrow X'$ be the projections. The external cup-product

$$H^n_{\text{DM}}(X, R(m)) \otimes H^{n'}_{\text{DM}}(X', R(m')) \xrightarrow{\cup} H^{n+n'}_{\text{DM}}(X \times X', R(m + m'))$$

is defined by $f \otimes g \mapsto p_X^*(f) \otimes p_{X'}^*(g)$. By [loc.cit.] Ch.4 p. 186 et Ch.5 Prop. 4.2.9 there is a functorial isomorphism

$$H^n_{\text{DM}}(X, R(m)) \simeq \text{CH}^m(X, 2m - n) \otimes R$$

where the right hand term denotes Bloch’s higher Chow group [8]. By [loc.cit.] Th. 9.1, when $R$ is a $\mathbb{Q}$-algebra, the Chern character is a functorial isomorphism

$$H^n_{\text{DM}}(X, R(m)) \simeq K^{(n)}_{2m-n}(X) \otimes R$$

where the right hand term denotes Quillen’s algebraic $K$-theory of $X$. Note that in [5], this equality is taken as a definition of the left hand term. In this work we will only consider the case $k = \mathbb{Q}$ and $R = \overline{\mathbb{Q}}$. The properties of motivic cohomology that we use don’t rely on Voevodsky’s construction.
1.2 Mixed Hodge modules and absolute Hodge cohomology

Mixed Hodge modules are the relative version of mixed Hodge structures, and are the analogues of mixed l-adic perverse sheaves. On the derived categories of mixed Hodge modules, we have the formalism of Grothendieck’s 6 functors, Verdier duality and a nice functorial behaviour of weights. Absolute Hodge cohomology is the absolute cohomology associated to Mixed Hodge modules. In this section we fix some conventions and recall some basic properties of mixed Hodge structures, mixed Hodge modules and absolute Hodge cohomology.

Let $A$ be a subfield of $\mathbb{R}$. A pure $A$-Hodge structure of weight $n$ is a finite dimensional $A$-vector space $M$ whose complexified space $M_C$ has a finite and exhaustive decreasing filtration $F^*M_C$ that is $n$-opposed to the conjugate filtration $F^*M_C$ ([12] Def. 1.2.3). Write $M^{p,q} = F^p M_C \cap F^q M_C$. Then one has a decomposition $M_C = \bigoplus_{p+q=n} M^{p,q}$ and $F^p M_C = \bigoplus_{p' \geq p} M^{p',q}$ ([loc. cit.] Prop. 1.2.5 (ii)).

**Definition 1.3.** (i) A mixed $A$-Hodge structure $(M, W, M, F^*M_C)$ is a finite dimensional $A$-vector space endowed with a finite increasing (weight) filtration $W, M$, and whose complexification $M_C$ has a finite decreasing (Hodge) filtration $F^*M_C$ inducing a pure polarisable ([loc. cit.] 2.3) Hodge structure of weight $n$ on the $n$-th graded piece $Gr^n M$. (ii) A mixed real $A$-Hodge structure is a mixed $A$-Hodge structure whose underlying vector space has a $A$-linear involution $F_\infty$ stabilizing the weight filtration and whose $\mathbb{C}$-antilinear complexification $F_\infty \otimes \tau$ stabilizes the Hodge filtration.

The involution $F_\infty \otimes \tau$ is called the de Rham involution. We denote by $\text{MHS}^+_A$ the category of mixed real $A$-Hodge structures, which is an abelian category by [loc. cit.] 2.3.5 (i).

**Definition 1.4.** Let $E$ be a ring. A mixed real $A$-Hodge structure with coefficients in $E$ is a couple $(M, a)$ where $M \in \text{MHS}^+_A$ and $a : E \to \text{End}_{\text{MHS}^+_A}(M)$ is a ring homomorphism.

Let $\text{Sch}(\mathbb{Q})$ be the category of quasi-projective $\mathbb{Q}$-schemes. For $s : X \to \text{Spec} \mathbb{Q} \in \text{Sch}(\mathbb{Q})$ we consider the abelian category $\text{MHM}_A(X/\mathbb{R})$ of real algebraic mixed $A$-Hodge modules ([27] Def. A.2.4). Objects $M \in \text{MHM}_A(X/\mathbb{R})$ have an increasing weight filtration $W, M$ and we say that $M$ is of weight $\geq n$, resp. $\leq n$, if $Gr^i M = 0$ for $i < n$, resp. $i > n$. Let $D^b(C(X), A)$ be the bounded derived category of sheaves of $A$ modules with constructible cohomology objects and consider $\text{Perv}_A(X_C) \subset D^b(C(X), A)$ the subcategory of perverse sheaves for the autodual perversity on $X_C$. By [3] the natural functor $D^b \text{Perv}_A(X_C) \to D^b(X_C, A)$ is an equivalence. According to [43] Th. 0.1 there is a functor $\text{rat} : \text{MHM}_A(X/\mathbb{R}) \to \text{Perv}_A(X_C)$ which is faithful and exact. We'll denote again by $\text{rat} : D^b \text{MHM}_A(X/\mathbb{R}) \to D^b(C(X), A)$ the derived functor. For $M \in D^b \text{MHM}_A(X/\mathbb{R})$, the cohomology objects $H^i M$ satisfy $\text{rat}(H^i M) = \check{H}^i \text{rat}(M)$ where $\check{H}^i$ is the perverse cohomology functor. When $X$ is smooth and purely of dimension $d$, for any local system $V$ of $A$-vector sheaves on $X_C$, the complex $V[d]$ concentrated in degree $-d$ is an object of $\text{Perv}_A(X_C)$. We identify in this way the category of local systems on $X_C$ to an abelian subcategory of $\text{Perv}_A(X_C)$. Let $\text{MHM}_A(X/\mathbb{R})^s$ be the full subcategory of $\text{MHM}_A(X/\mathbb{R})$ whose underlying perverse sheaf is a local system and $\text{Var}_A(X/\mathbb{R})$ the category of real admissible polarizable variations of mixed $A$-Hodge structures over $X$ ([27] Def. A.2.1 b). There is an equivalence $\text{Var}_A(X/\mathbb{R}) \simeq \text{MHM}_A(X/\mathbb{R})^s$ ([loc. cit.] Def A.2.4 b) that we denote by $V \mapsto V^t$. As a consequence we have an equivalence $\text{MHM}_A(\text{Spec} \mathbb{Q}/\mathbb{R}) = \text{MHM}_A^+(\mathbb{Q})$ ([loc. cit.] Lem. A.2.2). In the following, we'll simply say ”variation” instead of ”real admissible polarizable variations of mixed $A$-Hodge structures”.
Theorem 1.5. \cite{[42], [43], 0.1, [27]} Th. A.2.5. The derived categories $D_b^{\text{MHM}}A(\text{/}\mathbb{R})$ have the formalism of Grothendieck's 6 functors $(f^*, f_!, f_*, f^!, \text{Hom}, \hat{\otimes})$ and Verdier duality $\mathbb{D}$. These functors commute with $\text{rat}$.

One should be warned that the perverse $t$-structure gives rise to unusual shifts in the exact sequence relating the cohomology of a closed subscheme

$$ \cdots \rightarrow H^i_c(U, M) \rightarrow \mathcal{H}^i(U, M) \rightarrow H^{i-\epsilon}(Y, i^* j_* M) \rightarrow H^{i+1}_c(U, M) \rightarrow \ldots,$$

and to the restriction map in absolute Hodge cohomology $H^i_h(X, M) \rightarrow H^{i-c}_h(Y, i_* j^* M)$. Furthermore by \cite{[1.2.1]} we have $1 = s^* 1[d]$, hence

$$ H^i_h(X, M) = \text{Hom}_{D^b^{\text{MHM}}A(X/\mathbb{R})}(1, s_* M[i - d]). \quad (1.2.2) $$

For any $M \in \text{MHS}_\mathbb{R}^+$ and $i > 1$, we have $\text{Ext}_i^{\text{MHS}_\mathbb{R}^+}(1, M) = 0$ \cite{[6]} Cor. 1.10). As a consequence, for $A = \mathbb{R}$, the Leray spectral sequence computing the right hand side of \cite{[1.2.2]} reduces to an exact sequence

$$ 0 \rightarrow \text{Ext}_i^{1, \text{MHS}_\mathbb{R}^+}(1, H^{i-1}(X, M)) \rightarrow H^i_h(X, M) \rightarrow \text{Hom}_{\text{MHS}_\mathbb{R}^+}(1, H^i(X, M)) \rightarrow 0. \quad (1.2.3) $$
Furthermore, for $M \in \text{MHS}_k^+$ one has a canonical isomorphism

$$\text{Ext}^1_{\text{MHS}_k^+}(1, M) \simeq \left( \frac{W_0 M_C}{W_0 M_C \cap W_0 M + W_0 F^0 M_C} \right)^+ \tag{1.2.4}$$

(see [35] 2.5.3). In the following, we only consider absolute Hodge cohomology with real coefficients.

**Proposition 1.7.** [39] (4.5.1.) Let $f : X \to Y \in \text{Sch}(\mathbb{Q})$ purely of relative dimension $d$ and $V \in \text{Var}(X/\mathbb{R})$ be pure of weight $p$. Then $H_{i-d} f_* V$ is of weight $\geq i + p$ and $H_{i-d} f_* V$ is of weight $\leq i + p$.

### 1.3 The Shimura varieties

We work in the context of mixed Shimura data and mixed Shimura varieties, for which we refer to [40]. The semi-direct product $V_n \rtimes G_n$ is imbedded in $\text{GL}_{n+1}$ by

$$(v, g) \mapsto \begin{pmatrix} 1 & 0 \\ v & g \end{pmatrix}$$

and will be viewed as a linear algebraic group over $\mathbb{Q}$. Let $m \geq 0$ be an integer and denote by $P_2^m$, $P_4^m$ and $P_4^m$ the groups $V_{2}^{\oplus m} \rtimes G_2$, $(V_2 \oplus V_2)^m \rtimes H_2$ and $V_{2}^{\oplus m} \rtimes G_4$ respectively. The symplectic isomorphism $V_2 \oplus V_2 \simeq V_4$ together with the imbedding $\iota$ induces a commutative diagram

$$\begin{array}{ccc}
P_2^m & \rightarrow & P_4^m \\
\downarrow & & \downarrow \\
\Pi_2 & \rightarrow & G_4.
\end{array}$$

Let $h_2^m : \mathbb{S}(\mathbb{R}) \longrightarrow P_2^m(\mathbb{R})$ be the morphism

$$z = x + iy \longmapsto \begin{pmatrix} x \\ -y \\ x \end{pmatrix}.$$ 

Write $h_2^{2m} = h_2^m \rtimes h_2^m : \mathbb{S}(\mathbb{R}) \longrightarrow P_2^{2m}(\mathbb{R})$ and $h_4^m = \iota' \circ h_2^{2m} : \mathbb{S}(\mathbb{R}) \longrightarrow P_4^m(\mathbb{R})$. The conjugacy classes $H_2^{2m}$, $H_2^m$ and $H_4^m$ under $P_2^{2m}(\mathbb{R})$, $P_4^m(\mathbb{R})$ and $P_4^m(\mathbb{R})$ of $h_2^{2m}$, $h_2^m$ and $h_4^m$ respectively define mixed Shimura data $(H_2^{2m}, P_2^{2m})$, $(H_2^m, P_2^m)$ and $(H_4^m, P_4^m)$ whose reflex field is $\mathbb{Q}$ ([loc. cit.] 2.2.5). Finally $h_0^m = h_0^0 = \det \circ h_2^m : \mathbb{S} \to G_0$, which is part of a pure Shimura datum $(G_0, H_0^0)$.

**Lemma 1.8.** For $n \in \{0, 2, 4\}$, let $(\rho, E)$ be an algebraic representation of $G_n$. Then for every $x \in H_0^n$ the morphism $\rho_x : \mathbb{S} \to \text{GL}(E_\mathbb{R})$ induces a mixed $\mathbb{R}$-Hodge structure on $E_\mathbb{R}$ whose weight filtration doesn’t depend on $x$. The $k$-th graded part $Gr_k^W E_\mathbb{R}$ is the direct factor of $E_\mathbb{R}$ where $t \in \mathbb{R}^\times \subset \mathbb{C}^\times = \mathbb{S}(\mathbb{R})$ acts by multiplication by $t^{-k}$.

**Proof.** This is a reformulation of our convention 1.1.4. The independence on $x$ of the weight filtration follows from the fact that the restriction of $h_0^n(\mathbb{R}) : \mathbb{C}^\times \to G_n(\mathbb{R})$ to $\mathbb{R}^\times$ is central.

**Proposition 1.9.** Let $K \subset P_4^m(\mathbb{A}_f)$ be a compact open subgroup.
The space \( \mathbb{P}^m_n(Q) \backslash \mathcal{H}^m_n \times \mathbb{P}^m_n(\mathbb{A}_f)/K \) is the set of complex points of a smooth quasi-projective \( \mathbb{Q} \)-scheme \( A^m_{n,K} \).

For every \( g \in \mathbb{P}^m_n(\mathbb{A}_f) \) and every compact open subgroup \( L \subset \mathbb{P}^m_n(\mathbb{A}_f) \) such that \( K \subset gLg^{-1} \), right multiplication by \( g \)

\[
P^m_n(Q) \backslash \mathcal{H}^m_n \times \mathbb{P}^m_n(\mathbb{A}_f)/K \xrightarrow{gK} P^m_n(Q) \backslash \mathcal{H}^m_n \times \mathbb{P}^m_n(\mathbb{A}_f)/L
\]
descends to an étale cover \( A^m_{n,K} \xrightarrow{g} A^m_{n,L} \).

The projection \( \pi^m_n : \mathbb{P}^m_n \rightarrow \mathbb{G}_n \) induces a morphism of Shimura data \( \lambda^m_n : (\mathcal{H}^m_n, \mathbb{P}^m_n) \rightarrow (\mathcal{H}^0_n, \mathbb{G}_n) \) hence a continuous map

\[
\lambda^m_n : P^m_n(Q) \backslash \mathcal{H}^m_n \times \mathbb{P}^m_n(\mathbb{A}_f)/K \longrightarrow G_n(Q) \backslash \mathcal{H}^n_n \times G_n(\mathbb{A}_f)/\pi^m_n(K)
\]
that descends to a morphism of \( \mathbb{Q} \)-schemes \( \lambda^m_n : A^m_{n,K} \longrightarrow A^0_n \pi^m_n(K) \). Furthermore \( \lambda^m_n \) is an abelian scheme of relative dimension \( \frac{n(n+2)}{2} \).

If \( K \) is of the shape \( K_{\psi}^m \times K' \) for compact open subgroups \( K_V \) of \( V_n(\mathbb{A}_f) \) and \( K' \) of \( G_n(\mathbb{A}_f) \) stabilizing \( K_V \), then we have an isomorphism of \( A^0_{n,K'} \)-schemes

\[
A^m_{n,K} \cong A^1_{n,K_V \times K'} \times \cdots \times A^0_{n,K'} A^{1}_{n,K_V \times K'}.
\]

The \( \mathbb{Q} \)-scheme \( A^0_{n,K} \) is of dimension \( \frac{n(n+2)}{2} \).

The statement (v) follows from the identification of \( A^0_{n,K}(\mathbb{C}) \) with a finite disjoint union of quotients of the Siegel upper half-plane by arithmetic subgroups of \( \text{Sp}_4(\mathbb{Z}) \). Analogous statements of (i) and (ii) hold for the group \( \mathbb{P}^m_n \). Given a compact open subgroup \( K \subset \mathbb{P}^m_n(\mathbb{A}_f) \), we’ll denote by \( A^m_{n,K} \) the associated \( \mathbb{Q} \)-scheme.

**Proposition 11.10.** [loc. cit.] 3.10, 3.11, Prop. 11.10. Let \( K_0 \) be a compact open subgroup of \( \mathbb{G}_0(\mathbb{A}_f) \) and let \( F_{K_0} \) be the abelian extension of \( \mathbb{Q} \) associated to \( K_0 \) by class field theory. Let \( K \) and \( K' \) be two compact open subgroups of \( \mathbb{P}^m_2(\mathbb{A}_f) \) whose images under \( \pi^m_2(\mathbb{A}_f) \xrightarrow{\pi^m_2} \mathbb{G}_2(\mathbb{A}_f) \xrightarrow{\text{det}} \mathbb{G}_0(\mathbb{A}_f) \) coincide with \( K_0 \). Then \( K \times K' \) is a compact open subgroup of \( \mathbb{P}^m_2(\mathbb{A}_f) \) and we have an isomorphism of \( \mathbb{Q} \)-schemes

\[
A^m_{2,K \times K'} \cong A^m_{2,K} \times \text{Spec } F_{K_0} A^m_{2,K'}.
\]
These isomorphisms are compatible with the morphisms given by proposition \( \text{Loc. Cit.} \) (ii).

The diagram extends to a commutative diagram of Shimura data

\[
\begin{array}{ccc}
(\mathcal{H}^m_2, \mathbb{P}^m_2) & \xrightarrow{\lambda^m_2} & (\mathcal{H}^m_4, \mathbb{P}^m_4) \\
\downarrow & & \downarrow \\
(\mathcal{H}^m_2, \Pi_2) & \xrightarrow{\lambda^m_4} & (\mathcal{H}^0_4, \mathbb{G}_4).
\end{array}
\]
Proposition 1.11. [loc.cit.] 3.8 b. Keep the assumptions and notations of the previous proposition. Then there exists a compact open subgroup $L \subset \text{P}^n_\mathbb{A}_f$ containing $i(K \times K')$ and a commutative diagram

\[
\begin{array}{ccc}
A_{2K}^m \times \text{Spec } \mathbb{F}_{K_0} & \longrightarrow & A_{4L}^m \\
\lambda_m^2 \times \text{Spec } \mathbb{F}_{K_0} & \downarrow & \lambda_4^m \\
A_{2 \pi^m(K)}^0 \times \text{Spec } \mathbb{F}_{K_0} & \longrightarrow & A_{4 \pi^m(L)}^0
\end{array}
\]

whose horizontal arrows are closed imbeddings. These diagrams are compatible with the morphisms given by proposition 7.3 (ii).

1.4 Hodge structures and discrete series L-packets

In this section, we recall how the Hodge decomposition of the middle degree interior cohomology of the Siegel variety is related to archimedean discrete series L-packets.

Fix an irreducible algebraic representation $(\rho_n, E_n)$ of $G_n$. For every compact open subgroup $K \subset G_n(\mathbb{A}_f)$, we denote by $\mu_K(E_n)$ the real variation of mixed $\mathbb{Q}$-Hodge structures on $A^0_{nK}$ associated to $E_n$ (12 2).

Lemma 1.12. Let $\lambda(k, k', t)$ be the highest weight of $E_4$. Let $K_V \subset V_4(\mathbb{A}_f)$ be a compact open subgroup stable under $K$ and $\lambda_{4+\kappa}^4: A_{4+\kappa}^{k+k'} \times K \to A_{4+\kappa}^0$ the corresponding abelian scheme. Then $\mu_K(E_4)$ is direct factor of the complex $\lambda_{4+\kappa}^4 \mathbb{Q}(k+k'+t)[-k-k']$ in $D^b \text{HM}_{\mathbb{Q}}(A^0_4)$.

Proof. By (12.1), for any smooth $\mathbb{Q}$-scheme $Y$ and any smooth morphism $f: X \to Y$ of relative dimension $d$, we have $\text{rat}(\mathcal{H}^f, 1) = R^d f_* 1$. According to [55] Ch. 2, remark above lemma 2.6, the variation $\mu_K(V_4)$ is isomorphic to $\mathcal{H}^{-1} \lambda_{4+\kappa}^4 \mathbb{Q}(1)$. As a consequence $\mu_K(E_4)$ is direct factor of $(\mathcal{H}^{-1} \lambda_{4+\kappa}^4 \mathbb{Q}(1))^{\oplus k+k'}(t)$, which is a direct factor of

\[
(\bigwedge^{k+k'} \mathcal{H}^{1-2(k+k')} \lambda_{4+\kappa}^4 \mathbb{Q})(k+k'+t) \simeq (\mathcal{H}^{-k-k'} \lambda_{4+\kappa}^4 \mathbb{Q})(k+k'+t)
\]

because $\lambda_{4+\kappa}^4$ is an abelian scheme. Now as $\lambda_{4+\kappa}^4$ is proper, proposition 1.7 shows that $\lambda_{4+\kappa}^4 \mathbb{Q}$ is pure. So we have an isomorphism $\lambda_{4+\kappa}^4 \mathbb{Q} \cong \bigoplus_i \mathcal{H}^{k+k'} \mathbb{Q}[-i]$ by the decomposition theorem 1.5 (4.5.4), and the claim is proved.

Let $A^*(A^0_{nK}, E_n)$ be the de Rham complex of differential forms on $A^0_{nK}$ with values in the local system $\text{rat} \, \mu_K(E_n)$ and $A^*_{(2)}(A^0_{nK}, E_n)$ be the subcomplex of square integrable forms (16 1). Let $A^*(A^0_{n}, E_n) = \lim_{\longrightarrow} A^*(A^0_{nK}, E_n)$ and $A^*_{(2)}(A^0_{n}, E_n) = \lim_{\longrightarrow} A^*_{(2)}(A^0_{nK}, E_n)$. For every compact open subgroup $K_{V_4} \subset V_4(\mathbb{A}_f)$ stabilized by $K_n$ let $A^*(A^0_{nK_{V_4} \times K'}, \mathbb{C})$ be the de Rham complex of complex valued differential forms on $A^0_{nK_{V_4} \times K'}$ and let $A^*(A^0_{nK_{V_4} \times K'}, \mathbb{C}) = \lim_{\longrightarrow} A^*(A^0_{nK_{V_4} \times K'}, \mathbb{C})$.

For $g \in G_n(\mathbb{A}_f)$ and any compact open subgroup $L \subset G_n(\mathbb{A}_f)$ such that $L \subset gKg^{-1}$ we have a canonical isomorphism $[g]^\ast \mu_K(E_n) \cong \mu_L(E_n)$ (55 II.2). Hence there is an action of $G_n(\mathbb{A}_f)$ on $A^*(A^0_{n}, E_n), A^*_{(2)}(A^0_{n}, E_n)$ and $A^*(A^0_{nK}, E_n)$. [55]
Lemma 1.13. \cite{10} VII Cor. 2.7. There is a functorial $G_n(\mathbb{A}_f)$-equivariant commutative diagram of complexes

$$A^*_\sigma(A_{nK}, E_n) \sim \text{Hom}_{K_n}(\Lambda^* \mathfrak{g}_n/\mathfrak{k}_n, E_n \otimes C^\infty_{(2)}(G_n(\mathbb{Q}) \backslash G_n(\mathbb{A})))$$

whose horizontal arrows are isomorphisms. There is a functorial $G_n(\mathbb{A}_f)$-equivariant isomorphism of complexes

$$A^*(A_{nK}^k, C) \sim \text{Hom}_{K_n}(\Lambda^* (V_{n}^{\mathbb{C}} \mathfrak{g}_n/\mathfrak{k}_n, C^\infty (P^k_n(\mathbb{Q}) \backslash P^k_n(\mathbb{A}))).$$

Let $s_{nL}: A_{nL}^0 \to \text{Spec} \mathbb{Q}$ and $s_{nK}: A_{nK}^0 \to \text{Spec} \mathbb{Q}$ be the structural morphisms. Theorem 1.5 gives a commutative diagram of functors

$$s_{nK}^! \quad \sim \quad s_{nK}^*$$

where the isomorphism of the lower left hand corner follows from the fact that $[g]$ is proper. As the diagrams for $[g]$, $[h]$ and $[gh]$ are compatible, we have an action of the group $G_n(\mathbb{A}_f)$ on the inductive limits of singular cohomology spaces $H^*(A_{nL}^0, E_n) = \lim H^*(A_{nK}^0, \mu_K(E_n)))$ and cohomology with compact support $H^*(A_{nL}^0, E_n) = \lim H^*(A_{nK}^0, \mu_K(E_n)))$ with rational coefficients, for which the morphism $H^*(A_{nL}^0, E_n) \to H^*(A_{nK}^0, E_n)$ is equivariant. Denote by $H^*(A_{nL}^0, E_n)$ its image. The comparison theorem between interior and $L^2$-cohomology \cite{37} Prop. 1 is a $G_n(\mathbb{A}_f)$-equivariant isomorphism

$$H^*_c(A_{nL}^0, E_n)_C \simeq H^*_c(\mathfrak{g}_n, K_n, E_n \otimes C^\infty_{(2)}(G_n(\mathbb{Q}) \backslash G_n(\mathbb{A}))).$$

We want to identify the irreducible $G_n(\mathbb{A}_f)$-modules occurring in $H^*_c(A_{nL}^0, E_n)_C$. To this end recall that the discrete series of $G_n(\mathbb{R})$ are $(\mathfrak{g}_n, K_n, E_n)$-modules occurring in the space of square integrable functions on $G_n(\mathbb{R})/Z_n(\mathbb{R})$ (see \cite{33} Th. 8.51 for a precise definition).

Lemma 1.14. Let $\pi_\infty$ be a discrete series of $G_n(\mathbb{R})$. Then the following are equivalent:

(i) the complex $C^*_c(\mathfrak{g}_n, K_n, E_n \otimes \pi_\infty)$ is non zero and its differential is zero,

(ii) the $(\mathfrak{g}_n, K_n, E_n)$-module $\pi_\infty$ has the same central and infinitesimal characters than the dual of $E_n$.

Proof. Assume (i). As the adjoint action of the center on $\Lambda^* \mathfrak{g}_n/\mathfrak{k}_n$ is trivial, the non vanishing of $\text{Hom}_{K_n}(\Lambda^* \mathfrak{g}_n/\mathfrak{k}_n, E_n \otimes \pi_\infty)$ implies that $\pi_\infty$ and $E_n$ have opposite central characters. The statement on infinitesimal characters follows from \cite{10} § 5 Th. 5.3 (b). The fact that (ii) implies (i) follows from [loc. cit.] II. § 3 Prop. 3.1 (b) and II. § 5 Th. 5.3 (b).

By a theorem of Harish-Chandra \cite{33} Th. 9.20, the set $P(E_4)$ of isomorphism classes of discrete series of $G_4(\mathbb{R})$ whose central and infinitesimal characters coincide with the ones of the dual $E_4$ has
|\text{W}_4/\text{W}_{K_4}\rangle|= 4 \text{ elements } \pi_{H,\infty}^W, \pi_{Q,\infty}^W, \tilde{\pi}_{Q,\infty}^W \text{ and } \tilde{\pi}_{H,\infty}^W. \text{ If } \lambda(k, k', t) \text{ is the highest weight of } E_4, \text{ their minimal } K_4\text{-types have respective highest weight }

\lambda'(k + 3, k' + 3, -c_4), \lambda'(k + 3, -k' - 1, -c_4), \lambda'(k' + 1, -k - 3, -c_4), \lambda'(-k' - 3, -k - 3, -c_4)

\text{ with } c_4 = k + k' + 2t. \text{ We call } \pi_{H,\infty}^W, \text{ resp. } \tilde{\pi}_{H,\infty}^W, \text{ the holomorphic, resp. antiholomorphic, member of } P(E_4) \text{ and } \pi_{Q,\infty}^W, \tilde{\pi}_{Q,\infty}^W \text{ the generic members . A holomorphic Siegel modular forms give rise to an automorphic representation whose archimedean component is the holomorphic member of a discrete series L-packet [3].}

**Proposition 1.15.** Let } \pi_f \text{ a smooth irreducible } G_4(\lambda_f)-\text{module. Then }

\text{Hom}_{\mathbb{C}[G_4(\lambda_f)]}(\pi_f, H_1^0(A_4^0, E_4)\mathbb{C}) \neq 0

\text{if and only if } \pi_f \text{ is the non archimedean part of a cuspidal automorphic representation of } G_4 \text{ whose archimedean part belongs to } P(E_4).

**Proof.** The "if" part follows from the preceding lemma and the comparaison isomorphism [1.4,1]. Conversely assume } H^*(\mathfrak{g}_4, K_4\text{-}E_4, E_4 \otimes \pi_\infty) \neq 0. \text{ By [33] Th. 5.6 (b) this implies that } \pi_\infty \text{ is a \text{ cohomological induction } A_4(\lambda).} \text{ As } E_4 \text{ has regular highest weight, [loc. cit.] Th. 5.6 (a) implies that } q \text{ is the Lie algebra of the Borel. Hence } A_4(\lambda) \text{ is a discrete series ([33] Th. 11.178).} \blacklozenge

**Lemma 1.16.** Let } \lambda(k, k', t) \text{ be the highest weight of } E_4 \text{ and } p_4 = p_4^+ \oplus p_4^- \text{ the decomposition (1.1.3). For every } 0 \leq p, q \leq 3 \text{ choose a highest weight vector } \lambda_{p,q} \text{ of } \bigwedge^p p_4^+ \otimes \bigwedge^q p_4^- \text{. Let } v_{k,k',v_{-k,-k'},v_{-k',k}} \text{ and } v_{k,k',v_{-k,-k'}} \text{ be vectors of } E_4 \text{ of respective weights } \lambda(k, k', t), \lambda(-k,-k', t), \lambda(-k', k, t) \text{ and } \lambda(k', k, t) \text{ and } g^H, g^W, g^W, g^H \text{ be cuspidal factorizable forms whose non archimedean part is in a fixed } \pi_f \text{ and whose archimedean part is a highest weight vector of the minimal } K_4\text{-type of } \pi_{H,\infty}^W, \pi_{Q,\infty}^W, \tilde{\pi}_{Q,\infty}^W \text{ and } \tilde{\pi}_{H,\infty}^W \text{ respectively. Then we have well defined morphisms }

(\omega^H: \lambda_{3,0} \mapsto v_{-k,-k'} \otimes g^H) \in \text{Hom}_{K_4\text{-}E_4}(\bigwedge^3 p_4^+, E_4 \otimes \pi_{H,\infty}^W) \otimes \pi_f

(\omega^W: \lambda_{2,1} \mapsto v_{-k,-k'} \otimes g^W) \in \text{Hom}_{K_4\text{-}E_4}(\bigwedge^2 p_4^+ \otimes \bigwedge^1 p_4^-, E_4 \otimes \pi_{Q,\infty}^W) \otimes \pi_f,

(\omega^W: \lambda_{1,2} \mapsto v_{k',k} \otimes g^W) \in \text{Hom}_{K_4\text{-}E_4}(\bigwedge^1 p_4^+ \otimes \bigwedge^2 p_4^-, E_4 \otimes \tilde{\pi}_{Q,\infty}^W) \otimes \pi_f

(\omega^H: \lambda_{0,3} \mapsto v_{k',k} \otimes g^H) \in \text{Hom}_{K_4\text{-}E_4}(\bigwedge^3 p_4^-, E_4 \otimes \tilde{\pi}_{H,\infty}^W) \otimes \pi_f.

**Proof.** The list of weights of } p_4^+ \text{ given above the lemma [1.2] shows that for } 0 \leq p, q \leq 3 \text{ the representation } \bigwedge^p p_4^+ \otimes \bigwedge^q p_4^- \text{ is irreducible. Then the statement follows from the fact that weight of } \lambda_{p,q} \text{ and of the corresponding vector in the list above are the same.} \blacklozenge

The fiber of } \mu_K(E_n) \text{ at every point } x = (g_n h_n g_n^{-1}, g_f K) \text{ of } A_n^0 K_4(\mathbb{C}) \text{ is the mixed Hodge structure on } E_n \text{ given by lemma [1.8]. As a consequence, the variation } \mu_K(E_n) \text{ is pure, of weight } w = -c_n, \text{ where } x \mapsto x^{c_n} \text{ is the central character of } \rho_n. \text{ Let } \pi \text{ be a cuspidal automorphic representation of } G_4 \text{ whose archimedean part belongs to } P(E_4). \text{ Then by [21] it’s archimedean part } \pi_f \text{ is defined over its rationality field, a number field } E(\pi_f). \text{ We fix an imbedding } E(\pi_f) \subset \mathbb{Q} \text{ and we still denote by } \pi_f \text{ the model of } \pi_f \text{ over } E(\pi_f). \text{ Hence } M_B(\pi_f, E_4) = \text{Hom}_{\mathbb{Q}[G_4(\lambda_f)]}(\pi_f, H_1^0(A_4^0, E_4)) \text{ is an object of } \text{MHS}_{\mathbb{Q}}^+ E(\pi_f), \text{ that is pure of weight } w = -c_4 + 3 \text{ (proposition [1.7]).}
Proposition 1.17. The Hodge decomposition of $M_B(\pi_f, E_4)$ is

$$M_B(\pi_f, E_4)_C = M^{3-t,-k-k'-t}_B \oplus M^{2-k'-t,1-k-t}_B \oplus M^{1-k-t,2-k'-t}_B \oplus M^{k-k'-t,3-t}_B$$

with

$$M_B(\pi_f, E_4)^{3-t,-k-k'-t} = m(\pi^H_\infty \otimes \pi_f) \text{Hom}_{K_{4,\infty}}(\Lambda^3 p^+_4, E_4 \otimes \pi^H_\infty),$$

$$M_B(\pi_f, E_4)^{2-k'-t,1-k-t} = m(\pi^W_\infty \otimes \pi_f) \text{Hom}_{K_{4,\infty}}(\Lambda^2 p^+_4 \otimes p^-_4, E_4 \otimes \pi^W_\infty),$$

$$M_B(\pi_f, E_4)^{1-k-t,2-k'-t} = m(\pi^W_\infty \otimes \pi_f) \text{Hom}_{K_{4,\infty}}(p^+_4 \otimes \Lambda^2 p^-_4, E_4 \otimes \pi^W_\infty),$$

$$M_B(\pi_f, E_4)^{k-k'-t,3-t} = m(\pi^H_\infty \otimes \pi_f) \text{Hom}_{K_{4,\infty}}(\Lambda^3 p^-_4, E_4 \otimes \pi^H_\infty).$$

and for every $(p, q)$ and every $\pi_\infty \in P(E_4)$ we have $\dim_C \text{Hom}_{K_{4,\infty}}(\Lambda^p p^+_4 \otimes \Lambda^q p^-_4, E_4 \otimes \pi_\infty) = 1$.

Proof. By (1.4.1), lemma 1.14 and proposition 1.15 we have

$$M_B(\pi_f, E_4)_C = \bigoplus_{\pi_\infty \otimes \pi_f, \pi_\infty \in P(E_4)} m(\pi_\infty \otimes \pi_f) \text{Hom}_{K_{4,\infty}}(\Lambda^3 g_4/t_4, E_4 \otimes \pi_\infty).$$

Each $\text{Hom}_{K_{4,\infty}}(\Lambda^3 g_4/t_4, E_4 \otimes \pi_\infty)$ is one dimensional by [10] II. § 5 Th. 5.3. Then lemma 1.16 shows that

$$\text{Hom}_{K_{4,\infty}}(\Lambda^3 g_4/t_4, E_4 \otimes \pi^H_\infty) = \text{Hom}_{K_{4,\infty}}(\Lambda^3 p^+_4, E_4 \otimes \pi^H_\infty),$$

$$\text{Hom}_{K_{4,\infty}}(\Lambda^3 g_4/t_4, E_4 \otimes \pi^W_\infty) = \text{Hom}_{K_{4,\infty}}(\Lambda^2 p^+_4 \otimes p^-_4, E_4 \otimes \pi^W_\infty),$$

$$\text{Hom}_{K_{4,\infty}}(\Lambda^3 g_4/t_4, E_4 \otimes \pi^W_\infty) = \text{Hom}_{K_{4,\infty}}(p^+_4 \otimes \Lambda^2 p^-_4, E_4 \otimes \pi^W_\infty),$$

$$\text{Hom}_{K_{4,\infty}}(\Lambda^3 g_4/t_4, E_4 \otimes \pi^H_\infty) = \text{Hom}_{K_{4,\infty}}(\Lambda^3 p^-_4, E_4 \otimes \pi^H_\infty),$$

which proves the last statement of the proposition. Let us now compute the Hodge types. The morphism $h_1^0 : S \to G_{4R}$ gives rise to the Hodge decomposition

$$V_{4C} \oplus g_4/t_4 = V_0^{-1,0} \oplus V_1^{0,-1} \oplus (g_4/t_4)^{-1,1} \oplus (g_4/t_4)^{1,-1}$$

with $(g_4/t_4)^{-1,1} = p^+_4$ and $(g_4/t_4)^{1,-1} = p^-_4$. Furthermore $(V_0^{-1,0})^{k+k'} \oplus (g_4/t_4)^{-1,1}$ is the holomorphic tangent space of $H_{k+k'}^4$ at the origin (see [10] 1.7). Write

$$\Lambda^\cdot g_4/t_4 = \Lambda^\cdot p^+_4 \otimes \Lambda^\cdot p^-_4,$$

$$\Lambda^r s' V_4^{k+k'} = \Lambda^r (V_0^{-1,0})^{k+k'} \otimes \Lambda^s (V_0^{-1,0})^{k+k'},$$

$$\Lambda^* V_4^{k+k'} = \bigoplus_n \Lambda^n V_4^{k+k'}$$

so that $\bigwedge^{r',s'} V_4^{k+k'} = (\Lambda^* V_4^{k+k'})^{-r',-s'}$. The subspace of $A^\cdot (A_4^{k+k'}, C)$ of forms of type $(p, q)$ is
Let $E' = E_4(-k - k' - t)$ so that $E' \subset V_4^{\otimes k + k'} \subset \wedge^k V_4^{\otimes k + k'}$. Then the subspace of $A^*(A_4^0, E')$ of forms of type $(p, q)$ is $\bigoplus_{r, s} \text{Hom}_{K_4}((\Lambda^{r-s} g_4/\mathfrak{k}_4, (\Lambda^r V_4^{\otimes k + k'})^{p-r, q-s} \otimes \mathbb{C})$. The formula (1.1.3) gives

$$\chi(n, n', e) \circ h_4(\mathbb{R}) : x + iy \mapsto (x + iy)^{n+n'}\frac{(x - iy)^{e-n-n'}}{2}. $$

As a consequence the vector $v_{-k, k'}$ of the lemma [1.16] is of type $(k + k', 0)$. Hence the corresponding differential form $\omega^H$ is of type $(k + k' + 3, 0)$. Similarly one computes that $\omega^W$, $\bar{\omega}^W$ and $\bar{\omega}^H$ are of types $(k + 2, k' + 1)$, $(k' + 1, k + 2)$ and $(0, k + k' + 3)$. The types for the initial representation $E = E'(k + k' + t)$ are obtained by extensions of scalars to $C$.

**Proposition 1.18.** Let $\tau$ be the complex conjugation and $F_\infty$ the involution on $M_B(\pi_f, E_4)$ induced by the complex conjugation on $A^0_4(C)$. The complexified space $M_B(\pi_f, E_4)_C$ has a rational structure $M_{dR}(\pi_f, E_4)$ such that the following diagram commutes

$$
\begin{array}{ccc}
M_{dR}(\pi_f, E_4)_C & \xrightarrow{1 \otimes \tau} & M_{dR}(\pi_f, E_4)_C \\
\sim & & \sim \\
M_B(\pi_f, E_4)_C & \xrightarrow{F_\infty \otimes \tau} & M_B(\pi_f, E_4).
\end{array}
$$

Furthermore the Hodge filtration $F^* M_B(\pi_f, E_4)_C$ is obtained by extensions of scalars to $\mathbb{C}$ from a rational filtration $F^* M_{dR}(\pi_f, E_4) \subset M_{dR}(\pi_f, E_4)$.

**Proof.** See [23] Cor. 2.3.1 and above Th 2.2.7 for the compatibility with the theory of mixed Hodge modules.

We call $M_{dR}(\pi_f, E_4)$ the de Rham rational structure of $M_B(\pi_f, E_4)_C$.

### 2 Eisenstein classes for $\text{GSp}_4$

Let $K_n \subset G_n(\mathbb{A}_f)$ and $K_{V_n} \subset V_n(\mathbb{A}_f)$ be compact open subgroups such that $K_{V_n}$ is stable under $K_n$. From now on, we'll denote by $A_{K_{V_n} \times K_4} \to S_{K_4}$, resp. $E_{K_{V_2} \times K_2}$, the Siegel modular variety $A^1_{K_{V_4} \times K_4} \to A^0_{K_4}$, resp. the modular curve $A^1_{K_{V_2} \times K_2} \to A^0_{K_2}$ (proposition 1.9(ii)). We write $A^m_{K_{V_4} \times K_4}$, resp. $E^m_{K_{V_2} \times K_2}$, for the $m$-th fiber product $A_{K_{V_4} \times K_4} \times S_{K_4} \ldots \times S_{K_4} A_{K_{V_4} \times K_4}$, resp. $E_{K_{V_2} \times K_2} \times M_{K_2} \ldots \times M_{K_2} E_{K_{V_2} \times K_2}$.

In this section, we show the vanishing of the restriction to the boundary of a Gysin morphism associated to the imbedding of the product of two modular curves into the Siegel variety. This
will imply that some motivic cohomology classes in \(H^4_{M}(\mathbb{K}_{k} \times \mathbb{K}_{k'}, \mathbb{Q}(k + k' + 3))\), constructed via the Eisenstein symbol, are mapped into \(\text{Ext}^1_{\text{MHS}^+}(\mathbb{R}(0), H^0_{\text{M}}(\mathbb{S}_{k}, W^{k,k}_k)) \otimes \mathbb{Q}\) under Beilinson’s regulator. The proof of the vanishing result relies on the computation of higher direct images of variations in the Baily-Borel compactification.

### 2.1 Higher direct images of variations in the Baily-Borel compactification

The boundary of the Baily-Borel compactification of a Shimura variety associated to a group \(G\) is stratified by Shimura varieties associated to the Levi subgroups of \(G\). Furthermore the closure in the boundary of such a Shimura variety is its own Baily-Borel compactification. The main result of [12], that is stated at the end of this section, describes the restriction to a stratum of the higher direct image in the Baily-Borel compactification of the variation associated to an algebraic representation of \(G\).

Let \((G, \mathcal{H})\) one of the pure Shimura data \((G_2, \mathcal{H}_2^0)\), \((\Pi_2, \mathcal{H}_2^0)\) or \((G_4, \mathcal{H}_4^0)\) (2.2). For every maximal parabolic subgroup \(Q \subset G\), there exists a normal subgroup \(P_1 \subset Q\) underlying a mixed Shimura datum \((P_1, \mathcal{X}_1)\) ([40] 4.11), called a rational boundary component of \((G, \mathcal{H})\). Let \(W_1\) be the unipotent radical of \(P_1\) and let \(q : P_1 \to G_1 = P_1 / W_1\) be the projection on the Levi. Denote by \((G_1, \mathcal{H}_1)\) the quotient pure Shimura datum \((P_1, \mathcal{X}_1)/W_1\) ([40] prop. 2.9). Consider

\[
\mathbb{G}_a \times G_0 = \left\{ \begin{pmatrix} \alpha & m \\ 0 & 1 \end{pmatrix}, \alpha \in G_0, m \in \mathbb{G}_a \right\}, \tag{2.1.1}
\]

\[
Q^0 = W^0 \times (G_0 \times \text{GL}_2) = \left\{ \begin{pmatrix} \alpha A & A.M \\ 0 & t_A^{-1} \end{pmatrix}, \alpha \in G_0, A \in \text{GL}_2, tM = M \right\}, \tag{2.1.2}
\]

\[
Q^1 = W^1 \times (G_2 \times \mathbb{G}_m) = \left\{ \begin{pmatrix} \alpha & * & * \\ 0 & a & b \\ 0 & 0 & \beta \\ 0 & c & d \end{pmatrix} \in G_4, \alpha \beta = ad - bc \right\}. \tag{2.1.3}
\]

Then we have ([loc. cit.] 4.25)
The parabolic \( Q^0 \) \((2.1.2)\) is called the Siegel parabolic. The correspondence between maximal parabolics and the simple roots \([19] \S 23.2\) shows that \( \dim W^0 = \dim W^1 = 3 \).

For every \( g \in G(\mathbb{A}_f) \) let \( K_1 = gKg^{-1} \cap P_1(\mathbb{A}_f) \) and

\[
M(G_1, \mathcal{H}_1)_{q(K_1)}(\mathbb{C}) = G_1(\mathbb{Q}) \backslash \mathcal{H}_1 \times G_1(\mathbb{A}_f)/q(K_1).
\]

By [loc. cit.] 12.3 (b), the map \( M(G_1, \mathcal{H}_1)_{q(K_1)}(\mathbb{C}) \to M(G, \mathcal{H})_K^*(\mathbb{C}) \) descends to

\[
i_1 : M(G_1, \mathcal{H}_1)_{q(K_1)} \to M(G, \mathcal{H})_K^*.
\]

Furthermore, varying \((G_1, \mathcal{H}_1)\) and \( g \), we obtain a stratification of \( M(G, \mathcal{H})_K^* \) by locally closed subschemes \((12) 1\).

Now let \( E \) be an algebraic representation of \( G \). As \( G_1 \) acts on \( W_1 \) and on \( E \), it acts on the cohomology \( H^*(W_1, E) \). So the latter is endowed with a mixed \( \mathbb{R} \)-Hodge structure (lemma \((12) 2\)).

We denote by \( \mu \) the canonical construction of variations on a Shimura variety, associated to algebraic representations of it’s underlying group (see \((12) 2\)).

| \( G \)       | \( Q \)       | \( P_1 \)         | \( Q/P_1 \)       | \( G_1 \)       | \( W_1 \)       |
|-------------|--------------|------------------|------------------|----------------|----------------|
| \( G_2 \)   | \( B_2 \)    | \( G_a \times G_0 \) | \( G_m \)        | \( G_0 \)       | \( G_a \)       |
| \( G_2 \times G_0 \) \( G_2 \) | \( Q^0 = B_2 \times G_0 \) \( G_2 \) | \( (G_a \times G_0) \times (G_a \times G_0) \) | \( G_m^2 \) | \( G_0 \)       | \( G_0 + G_a \) |
| \( \mathcal{Q}^1 = G_2 \times G_0 \) \( B_2 \) | \( G_2 \times G_0 \) \( G_a \times G_0 \) | \( G_m \)        | \( G_2 \)       | \( G_a \)       |
| \( \mathcal{Q}^1 = B_2 \times G_0 \) \( G_2 \) | \( (G_a \times G_0) \times G_0 \) \( G_2 \) | \( G_m \)        | \( G_2 \)       | \( G_a \)       |
| \( G_4 \)   | \( Q^0 = \text{Stab}_{G_4}(b_1^1, b_2^1) \) | \( W^0 \times G_0 \) | \( \text{GL}_2 \) | \( G_0 \)       | \( W^0 \)       |
| \( Q^1 = \text{Stab}_{G_4}(b_1^1) \) | \( W^1 \times G_2 \) | \( G_m \)        | \( G_2 \)       | \( W^1 \)       |
Theorem 2.1. [loc. cit.] Th. 2.6, 2.9. We have an isomorphism

\[ i_1^\ast j_\ast \mu_K(E) = \bigoplus_n \mathcal{H}^n(i_1^\ast j_\ast \mu_K(E))[-n] \]

in \( D^b\text{MHM}_\mathbb{R}(M(G_1, H_1)_{q(K_1)}/\mathbb{R}) \). Let \( c \) be the codimension of the immersion \( i_1 \). There exists an arithmetic subgroup \( H_C \subset \mathbb{Q}/P_1(\mathbb{Q}) \) such that

\[ \mathcal{H}^n i_1^\ast j_\ast \mu_K(E) = \bigoplus_{p+q=n+c} \mu_{q(K_1)} H^p(H_C, H^q(W_1, E)) \]

in \( \text{MHM}_\mathbb{R}(M(G_1, H_1)_{q(K_1)}/\mathbb{R}) \). There is an isomorphism of local systems over \( M(G_1, H_1)_{q(K_1)} \)

\[ \text{rat} \ G_k^W \mathcal{H}^n i_1^\ast j_\ast \mu_K(E) = \text{rat} \bigoplus_{p+q=n+c} \mu_{q(K_1)} H^p(H_C, G_k^W H^q(W_1, E)). \]

The functor \( \mu \) will often be suppressed from the notation.

The following result of Kostant, together with the last statement of the above theorem, will allow us to compute the weights of \( \mathcal{H}^n i_1^\ast j_\ast \mu_K(E) \). Let \( q = \text{Lie} \mathbb{Q} \) with Levi decomposition \( q = u_1 \oplus t \). Let \( h \) be the diagonal Cartan subalgebra of \( g \), let \( \Delta^+(g, h) \) be the set of positive roots with respect to the standard Borel, let \( \Delta(u_1, h) \subset \Delta^+(g, h) \) be the roots of \( u_1 \), let \( \rho \) be the half sum of positive roots and let \( W(g, h) \) be the Weyl group. For \( w \in W(g, h) \) consider

\[ \Delta^+(w) = \{ \alpha \in \Delta^+(g, h) | w^{-1} \alpha \not\in \Delta^+(g, h) \}, \]

\[ l(w) = |\Delta^+(w)|, \]

\[ W' = \{ w \in W | \Delta^+(w) \subset \Delta(u_1, h) \}. \]

We also denote by \( F_\mu \) the irreducible representation of \( t \) of highest weight \( \mu \).

Theorem 2.2. [52] Th. 3.2.3 Let \( E_\lambda \) be the irreducible representation of \( g \) of highest weight \( \lambda \). Then

\[ H^i(u_1, E_\lambda) = \bigoplus_{\{ w \in W' | l(w) = i \}} F_{w(\lambda + \rho) - \rho}. \]

2.2 Computations of higher direct images

In this section, we rely on the previous theorems to carry out some explicit computations. We use freely the notations of theorems 2.1 and 2.2. The reader should consult 2.1.4 to follow the computations. The field of coefficients of mixed Hodge modules is the field \( \mathbb{R} \).

Let \( K \subset G_2(A_f) \) be a compact open subgroup. We simply denote by \( M \) the modular curve \( M_K \). Let

\[ M \longrightarrow \text{M} \leftarrow \partial \text{M} \]

be the Baily-Borel compactification of \( M \) with the complementary reduced closed imbedding of the boundary.
Lemma 2.3. Let $k \geq 0$ and $t$ be two integers. In the category $\text{MHM}(\partial \mathbb{M}/\mathbb{R})$ we have the identities

$$
\mathcal{H}^{-1} i''_* j''_* \text{Sym}^k \mathbb{V}_2(t) = 1(k + t),
$$

$$
\mathcal{H}^0 i''_* j''_* \text{Sym}^k \mathbb{V}_2(t) = 1(t - 1),
$$

$$
\mathcal{H}^m i''_* j''_* \text{Sym}^k \mathbb{V}_2(t) = 0 \text{ for } m > 0.
$$

Proof. This is the case $G = G_2$. The group $H_C$ is a neat arithmetic subgroup of $\mathbb{G}_m(\mathbb{Q})$, hence is trivial. Theorem \ref{thm:higher-regulators} yields

$$
\mathcal{H}^m i''_* j''_* \text{Sym}^k \mathbb{V}_2(t) = \mu \mathcal{H}^{m+1}(\text{Lie} \mathbb{G}_a, \text{Sym}^k \mathbb{V}_2(t))
$$

for every integer $m$. As $\text{Lie} \mathbb{G}_a$ is of dimension one, we have $\mathcal{H}^m i''_* j''_* \text{Sym}^k \mathbb{V}_2(t) = 0$ for $m > 0$. Let us compute the action of $G_0$ over $\mathcal{H}^m i''_* j''_* \text{Sym}^k \mathbb{V}_2(t)$ for $m = -1, 0$. Look at $\text{Sym}^k \mathbb{V}_2$ as the space of homogeneous polynomials of degree $k$ in $X$ and $Y$. Then $\mathcal{H}^0(\text{Lie} \mathbb{G}_a, \text{Sym}^k \mathbb{V}_2(t))$ is $(X^k)(t)$, which is of weight $k + t$. The Lie algebra $\text{Lie} \mathbb{G}_a$ is of weight $\lambda(-2, 1)^{-1}$ for the adjoint action of the maximal torus $T_2$. Hence the choice of a generator $u$ of $\text{Lie} \mathbb{G}_a$ induces an isomorphism $\mathcal{H}^1(\text{Lie} \mathbb{G}_a, \text{Sym}^k \mathbb{V}_2(t)) = \lambda(-2, 1) \oplus (Y^k)(t)$, which is of weight $t - 1$. The statement now follows from lemma \ref{lem:weyl-group-action} and the definition of $\mu$. $\square$

Let $K' \subset G_2(\mathbb{A}_f)$ be another compact open subgroup and let $M'$, $M'^*$ and $\partial M'$ be the modular curve $M_{K'}$, its compactification and it’s boundary. The compact open subgroup $\det K'$ of $G_0(\mathbb{A}_f)$ is assumed to coincide with $\det K$ and is denoted by $K_0$. Let $M \times M'$ be the fiber product $M \times \text{Spec} F_{K_0} M'$ (proposition \ref{prop:representation-theory}) and let

$$
M \times M' \xrightarrow{j'} (M \times M')^* \xleftarrow{i'} \partial(M \times M')
$$

be the Baily-Borel compactification of $M \times M'$ with the complementary reduced closed imbedding of the boundary. Denote by $\partial(M \times M')_i$ the stratum of dimension $i$ of $\partial(M \times M')$. Then $\partial(M \times M')_0$ is $\partial M \times \partial M'$ and $\partial(M \times M')_1$ is $\partial M \times M' \cup M \times \partial M'$. As a consequence we have

$$
\mathcal{D}^b_{\text{MHM}}(\partial(M \times M')_1/\mathbb{R}) = \mathcal{D}^b_{\text{MHM}}(\partial M \times M'/\mathbb{R}) \oplus \mathcal{D}^b_{\text{MHM}}(M \times \partial M'/\mathbb{R})
$$

in an obvious sense and in a compatible way with Grothendieck’s functors. We have an open imbedding and the complementary reduced closed imbedding

$$
\partial(M \times M')_1 \xrightarrow{i'_1} \partial(M \times M') \xleftarrow{i'_0} \partial(M \times M')_0.
$$

Lemma 2.4. Fix integers $k, k' \geq 0$ and $t$. Then in $\text{MHM}(\partial(M \times M')_1/\mathbb{R})$ we have the identities

$$
\mathcal{H}^{-1} i'_1 i''_* j'_1 \left( \text{Sym}^k \mathbb{V}_2 \boxtimes \text{Sym}^{k'} \mathbb{V}_2 \right)(t) = \text{Sym}^{k'} \mathbb{V}_2(k + 2t) \oplus \text{Sym}^k \mathbb{V}_2(k' + 2t),
$$

$$
\mathcal{H}^0 i'_1 i''_* j'_1 \left( \text{Sym}^k \mathbb{V}_2 \boxtimes \text{Sym}^{k'} \mathbb{V}_2 \right)(t) = \text{Sym}^{k'} \mathbb{V}_2(2t - 1) \oplus \text{Sym}^k \mathbb{V}_2(2t - 1),
$$

$$
\mathcal{H}^m i'_1 i''_* j'_1 \left( \text{Sym}^k \mathbb{V}_2 \boxtimes \text{Sym}^{k'} \mathbb{V}_2 \right)(t) = 0 \text{ for } m > 0.
$$
Proof. This is the case $G = \Pi_2$ and $G_1 = G_2$. Write $S = (\text{Sym}^k V_2 \boxtimes \text{Sym}^{k'} V_2)(2t)$. As in the proof of the preceding lemma, it follows from theorem 2.1 on $\partial M \times M'$ and the Künneth formula for Lie algebra cohomology (10) I § 1 that

$$\mathcal{H}^{-1}i_1^*i^*j_*^*(\text{Sym}^k V_2 \boxtimes \text{Sym}^{k'} V_2)(t) = \mu H^0(\text{Lie} G_2, \text{Sym}^k V_2(t)) \otimes \mu H^0(0, \text{Sym}^{k'} V_2(t)) = 1(k + t) \otimes \text{Sym}^{k'} V_2(t) = \text{Sym}^{k} V_2(k + 2t),$$

similarly $\mathcal{H}^0i_1^*i^*j_*^*(\text{Sym}^k V_2 \boxtimes \text{Sym}^{k'} V_2)(t) = \text{Sym}^{k'} V_2(2t - 1)$. Furthermore, for dimension reasons $\mathcal{H}^m i_1^*i^*j_*^*S = 0$ for $m > 0$. The result on the stratum $M \times \delta M'$ is obtained by exchanging $k$ and $k'$.

Let $L$ be a compact open subgroup of $G_4(\kappa_f)$. Denote by $S$ the Siegel threefold $S_{2,k}$ and by

$$S \xrightarrow{j} S^* \xleftarrow{i} \partial S$$

it’s Baily-Borel compactification. Let $\partial S_i$ be the stratum of dimension $i$ of $\partial S$. We have an open imbedding and the complementary closed reduced imbedding

$$\partial S_1 \xrightarrow{i_1} \partial S \xleftarrow{i_0} \partial S_0.$$  

**Lemma 2.5.** Let $E$ be the irreducible representation of $G_4$ of highest weight $\lambda = \lambda(k, k', t)$. If $t \leq 3$ then the mixed Hodge modules $\mathcal{H}^0 \rho \otimes E$ and $\mathcal{H}^1 \rho \otimes j_* E$ on $\partial S_0$ have weights $> 0$.

**Proof.** This is the case $G = G_4$ and $G_1 = G_0$. Let $\mathfrak{h} = \text{Lie} T_4$ and let $\mathfrak{q}^0 = \text{Lie} Q^0 = \mathfrak{u}^0 \oplus \mathfrak{l}^0$ with $\mathfrak{u}^0 = \text{Lie} W^0$ and $\mathfrak{l}^0 = \text{Lie} (G_0 \times \text{GL}_2)$ (see 2.1.2). Note that as $W^0$ is abelian the set $\Delta(\mathfrak{u}^0, \mathfrak{h})$ contains two long roots, hence $\Delta(\mathfrak{u}^0, \mathfrak{h}) = \{\rho_2, \rho_1 + \rho_2, 2\rho_1 + \rho_2\}$. With the notations of theorem 2.2 we have $W' = \{w_0 = \text{Id}, w_1 = s_{2\rho_1 + \rho_2}, w_2 = s_\rho_1 \circ s_{2\rho_1 + \rho_2}, w_3 = s_{\rho_1}\}$ with $l(w_i) = i$. We have $w_2(\lambda + \rho) - \rho = \lambda(k' - 1, -k - 3, t)$ and $w_3(\lambda + \rho) - \rho = \lambda(-k' - 3, -k - 3, t)$. The irreducible representation of $\mathfrak{l}^0$ of highest weight $\mu$ is the tensor product of the representation of $\text{Lie} G_0$ of highest weight $\mu|_{\text{Lie} G_0}$ and of a representation of $\text{Lie} \text{GL}_2$. One sees from 2.1.2 that the restriction of $\lambda(k, k', t)$ to $G_0 \subset W^0 \times (G_0 \times \text{GL}_2)$ is $x \mapsto x^{k+k'+t}$. Hence theorem 2.2 shows that $x \in G_0$ acts on $H^2(\mathfrak{u}^0, E)$ by multiplication by $x^{-(k-k'+4-t)}$ and on $H^3(\mathfrak{u}^0, W)$ by multiplication by $x^{-(k+k'+6-t)}$. So lemma 1.8 implies that $H^2(\mathfrak{u}^0, E)$ and $H^3(\mathfrak{u}^0, W)$ have weights $0 > 0$. The group $H_C$ is an arithmetic subgroup of $P_1(\mathbb{Q}) = \text{GL}_2(\mathbb{Q})$, hence it acts properly and discontinuously on $\mathcal{H}^0_{\mathfrak{h}}$ (12 Lem. 1.2). In particular stabilizers of points in $\mathcal{H}^0_{\mathfrak{h}}$ are finite. By our running neatness assumption $H_C$ is torsionfree, hence the space $\mathcal{H}^1_{\mathfrak{h}}$ is a universal cover of $\mathcal{H}^0_{\mathfrak{h}}/H_C$. Furthermore $H_C$ is commensurable to $\text{SL}_2(\mathbb{Z})$ and the space $\mathcal{H}^0_{\mathfrak{h}}/\text{SL}_2(\mathbb{Z})$ is known to be not compact. Hence $\mathcal{H}^0_{\mathfrak{h}}/H_C$ is not compact. This implies that $H_C$ has cohomological dimension 1. As $\mathfrak{u}^0$ has dimension 3 one has $H^m(\mathfrak{u}^0, E) = 0$ for $m > 3$, and theorem 2.1 yields

$$\mathcal{H}^0i_0^*j_* E = \mu H^0(H_C, H^3(\mathfrak{u}^0, E))) \oplus \mu H^1(H_C, H^2(\mathfrak{u}^0, E))),$$

$$\mathcal{H}^1i_0^*j_* E = \mu H^1(H_C, H^3(\mathfrak{u}^0, E)).$$

The conclusion now follows from the last statement of theorem 2.1.
Lemma 2.6. Let $E$ be the irreducible representation of $G_4$ of highest weight $\lambda = \lambda(k, k', t)$. Then in $\text{MHM}(\partial S_1/\mathbb{R})$ we have the identities

\[
\mathcal{H}^{-2}i^*_1i^*_jE = \mathcal{H}^{1}i^*_1i^*_jE = \text{Sym}^{k'}V_2(t),
\]

\[
\mathcal{H}^{-1}i^*_1i^*_jE = \mathcal{H}^{0}i^*_1i^*_jE = \text{Sym}^{k+1}V_2(t),
\]

\[
\mathcal{H}^{m}i^*_1i^*_jE = 0 \quad \text{for } m > 1.
\]

Proof. Let $q^1 = \text{Lie}Q^1 = u^1 \oplus t^1$ with $u^1 = \text{Lie}W^1$ and $t^1 = \text{Lie}(G_2 \times \mathbb{G}_m)$ (see 2.1.3). We have $\Delta(u^1, h) = \{\rho_1, \rho_1 + \rho_2, 2\rho_1 + \rho_2\}$ and $W' = \{w_0 = \text{Id}, w_1 = s_{\rho_1 + \rho_2}, w_2 = s_{\rho_2} \circ s_{\rho_1 + \rho_2}, w_3 = s_{\rho_2}\}$ with $\{(w_i) = i$. Furthermore

\[
w_1(\lambda + \rho) - \rho = \lambda(k' - 1, k + 1, t),
\]

\[
w_2(\lambda + \rho) - \rho = \lambda(-k' - 3, k + 1, t),
\]

\[
w_3(\lambda + \rho) - \rho = \lambda(-k - 4, k', t).
\]

As the restriction of the character $\lambda(k, k', t)$ to the maximal torus $T_2$ of $G_2 \subset W^1 \times (G_2 \times \mathbb{G}_m)$ is $\lambda(k', t)$, theorem 2.2 shows $H^0(u_1, E) \simeq H^0(u_1, E) \simeq \text{Sym}^{k'}V_2(t)$ and $H^1(u_1, E) \simeq H^2(u_1, E) \simeq \text{Sym}^{k+1}V_2(t)$. As $u^1$ has dimension 3, the cohomology $H^m(u^1, E)$ vanishes for $m > 3$. The group $H_C$ is a neat arithmetic subgroup of $\mathbb{G}_m(Q)$, hence is trivial. So theorem 2.1 yields

\[
H^0i^*_1i^*_jE = \mu H^{0+2}(u^1, E).
\]

\[\square\]

2.3 Interior cohomology and boundary cohomology

Let $E$ be the irreducible representation of $G_4$ of regular highest weight $\lambda(k, k', t)$. We show that when $t \leq 3$, the kernel of the restriction to the boundary in absolute Hodge cohomology $H^q(S, E) \rightarrow H^q_h(\partial S, i^*_jE)$ is the space of 1-extensions $\text{Ext}^1_{\text{MHM}}(1, H^q(S, E))$. We rely on the computations of section 3.2 and on a theorem of Saper, valid for any Shimura variety, that we apply for the symplectic groups $G_n$. For the statement fix $K \subset G_n(k_f)$ a compact open subgroup and denote again by $A^0_nK$ the Shimura variety of level $K$, a $\mathbb{Q}$-scheme of dimension $d_n = n(n + 2)/8$.

Theorem 2.7. [43], [47]. Th. 5. Let $V$ be an representation of $G_n$ of regular highest weight. Then the singular cohomology $H^i(A^0_nK, \mu_K(V))$ vanishes for $i < d_n$.

Note that by Poincaré duality, this implies the vanishing of the cohomology with compact supports $H^i(A^0_nK, V)$ for $i > d_n$. Let us now come back to the notations of the previous section 3.2. In the following, we will repeatedly use the fact that the variation associated to the representation $\text{Sym}^kV_2(t)$ is pure of weight $-k - 2t$ (see the remark following lemma 1.16)

Lemma 2.8. If $t \leq 3$ then

\[
\text{Hom}_{\text{MHS}}(1, H^q(S, E)) = \text{Hom}_{\text{MHS}}(1, H^2(\partial S, i^*_jE)) = \text{Hom}_{\text{MHS}}(1, H^1(\partial S, i^*_jE)) = 0.
\]
Proof. By the remark following theorem 2.7 we have $H^i_c(S, E) = H^2(S, E) = 0$. Hence we have an isomorphism $H^4(S, E) \simeq H^2(\partial S, i^*j_*E)$. The exact triangle
\[
i_1i_1^*j_*E \to i^*j_*E \to i_0^*i_0^*j_*E + \to
\]
gives rise to the exact sequence $H^2(\partial S_1, i_1^*j_*E) \to H^2(\partial S, i^*j_*E) \to H^1(\partial S_0, i_0^*j_*E)$. As $\partial S_0$ has dimension 0, theorem 2.1 shows that $H^1(\partial S_0, i_0^*j_*E) = H^0(\partial S_0, \mathcal{H}^1_i j_*E)$, which has no weight zero according to lemma 2.8. Theorem 2.4 and lemma 2.6 show that $H_c^2(\partial S_1, i_1^*j_*E)$ is the direct sum of cohomology with compact supports of variations of negative weight. Then proposition 1.7 implies that $H^2_c(\partial S_1, i_1^*j_*E)$ has negative weight. As a consequence
\[
H_{MHS}^1(1, H^4(S, E)) = H_{MHS}^1(1, H^2(\partial S, i^*j_*E)) = 0.
\]
One shows in the same way that $H_{MHS}^1(1, H^1(\partial S, i^*j_*E)) = 0$.

Proposition 2.9. If $t \leq 3$. Then we have an exact sequence
\[
0 \to \text{Ext}^1_{MHS}(1, H^3(S, E)) \to H^4(S, E) \to H^2(\partial S, i^*j_*E).
\]

Proof. By the remark following theorem 2.7 we have $H^4_c(S, E) = 0$. Hence there is an exact sequence
\[
0 \to H^3_c(S, E) \to H^3(S, E) \to H^1(\partial S, i^*j_*E) \to 0.
\]
Applying the functor $X \to \text{Hom}_{MHS}(1, X)$ and using the last equality of the previous lemma we get an exact sequence
\[
0 \to \text{Ext}^1(1, H^3_c(S, E)) \to \text{Ext}^1(1, H^3(S, E)) \to \text{Ext}^1(1, H^1(\partial S, i^*j_*E)).
\]
By the first two equalities of lemma 2.8 and the exact sequence 1.2.3 we have
\[
\text{Ext}^1(1, H^3_c(S, E)) = H^1_c(S, E),
\]
\[
\text{Ext}^1(1, H^1(\partial S, i^*j_*E)) = H^2_c(\partial S, i^*j_*E).
\]

2.4 The vanishing on the boundary

Assume now that the compact open subgroup $L$ of $G_4(\mathbb{A}_f)$ is given by the proposition 1111 so that we have a closed immersion of codimension one $\iota : M \times M' \to S$. By functoriality of the Baily-Borel compactification [40] 4.16 and 12.3 (b), we have commutative diagrams with cartesian squares
\[
M \times M' \xrightarrow{j'} (M \times M')^* \xleftarrow{i'} \partial(M \times M') \xrightarrow{\partial} \partial S. \tag{2.4.1}
\]
Let $F$ be the irreducible representation $(\text{Sym}^k V_2 \otimes \text{Sym}^{k'} V_2)(t - 1)$. We denote again by $E$ the irreducible representation of $G_4$ of highest weight $\lambda(k, k', t)$ but we drop the assumption that $\lambda(k, k', t)$ is regular.

**Lemma 2.10.** In $D^b \text{MHM}(S^* / \mathbb{R})$ we have a commutative diagram

$$
\begin{aligned}
\partial(M \times M')_0 & \xrightarrow{i'_0} \partial(M \times M') \xleftarrow{i'_1} \partial(M \times M')_1 \\
\partial S_0 & \xrightarrow{i_0} \partial S \xleftarrow{i_1} \partial S_1.
\end{aligned}
$$

(2.4.2)

Proof. Recall the fixed imbedding $\iota : \Pi_2 \rightarrow G_4$. As $S$ and $M \times M'$ are smooth, purely of dimension 3 and 2 respectively, we have $\mathbb{D}(i_* \mu(\overline{E}(3))) = i'_* \mu(E)$ and $\mathbb{D}(\mu(F)(2)) = \mu(F)$ respectively. In the category of representations of $\Pi_2$ there is a projection $i^* \overline{E}(3) \rightarrow \overline{F}(2)$ in . Applying $\mu$ we get $i^*_\mu(\overline{E}(3)) \rightarrow \mu(\overline{F})(2)$, so by 1.2.1, we obtain a morphism $i^*_\mu(\overline{E})(3)[-1] \rightarrow \mu(\overline{F})(2)$ in $D^b \text{MHM}(M \times M' / \mathbb{R})$. Applying $\mathbb{D}$ we get $\mu(F) \rightarrow i'_* \mu(E)[1]$, then by adjunction $i_\mu(F) \rightarrow \mu(E)[1]$. Because $\iota$ is proper we have $i_\iota = i_*$, so we have a morphism $\iota : i_\iota \mu(F) \rightarrow \mu(E)[1]$. By functoriality and the proper base change theorem 13 4.4.3 we have $i_* i^* j_* \iota = i_* i^* p_* j'_\mu \mu(F) = i_* q_* i''_* j'_\mu \mu(F) \rightarrow i_* i^* j_* \mu(E)[1]$. In the same way we have the morphism $i_* i_1 i_1^* i^* j_* \iota : i_* i_1 i_1^* i_* q_* i^* j'_\mu \mu(F) \rightarrow i_* i_1 i_1^* i_* i^* j_* \mu(E)[1]$. Then the sought for diagram is $j_* \iota : i_* i^* j_* \iota \rightarrow i_* i_1 i_1^* i^* j_* \mu(E)$.

From now on, we suppress the functor $\mu$ from the notation.

**Corollary 2.11.** We have a commutative diagram

$$
\begin{aligned}
H^*_\mathbb{Z}(M \times M, F) & \xrightarrow{} H^*_\mathbb{H}(\partial(M \times M), i^* j'_* F) \xrightarrow{} H^*_\mathbb{H}(\partial(M \times M)_1, i'_1 j'_* F) \\
H^*_\mathbb{H}(S, E) & \xrightarrow{} H^*_\mathbb{H}(\partial S, i^* j_* E) \xrightarrow{} H^*_\mathbb{H}(\partial S_1, i'_1 j_* E).
\end{aligned}
$$

Proof. This follows by applying the functor $X \mapsto \text{Hom}_{D^b \text{MHM}}(S_{K_4}^* / \mathbb{R})(s^1, X)$ to the diagram of lemma 2.10 where $s : S_{K_4}^* \rightarrow \text{Spec} \mathbb{Q}$ is the structural morphism.

We are going to show that the diagonal of the left hand square of the above diagram is the zero map, under some assumptions on the weight of $E$.

**Lemma 2.12.** Let $H \subset G_2(k_f)$ be a compact open subgroup and $M_H$ be the modular curve of level $H$. If the real mixed Hodge structure $H^1(M_H, \text{Sym}^k V_2(t))$ has weight zero then either $t = 1$ or $2t = 1 - k$. 

Proof. We have an exact sequence of real mixed Hodge structures

\[ 0 \to H^1(M, \text{Sym}^k V_2(t)) \to H^1(M, \text{Sym}^k V_2(t)) \to H^0(\partial M, i''^* j''_* \text{Sym}^k V_2(t)) \to H^0(\partial M, i''^* j''_* \text{Sym}^k V_2(t)) \to 0. \]

As \( \partial M \) is of dimension zero, theorem 2.1 and lemma 2.3 show

\[ H^0(\partial M, i''^* j''_* \text{Sym}^k V_2(t)) = H^0(\partial M, H^0 i''^* j'_* \text{Sym}^k V_2(t)) = 1(t - 1). \]

The assertion now follows from the fact that \( H^1(M, \text{Sym}^k V_2(t)) \) is pure of weight \( 1 - k - 2t \).

Lemma 2.13. (i) If \( t \leq 3 \) then \( H^1_h(\partial S_0, i''_0^* j'_* E) = 0. \)

(ii) If in the highest weight \( \lambda(k, k', t) \) of \( E \) we have \( k' > 0 \), then

\[ H^1_h(\partial S_1, i''_1^* j'_* E) = H^1_h(\partial S_1, H^0 i''_1^* j'_* E). \]

Proof. (i) As \( \partial S_0 \) is of dimension zero, theorem 2.1 yields \( H^0(\partial S_0, i''_0^* j'_* E) = H^0(\partial S_0, H^0 i''_0^* j'_* E) \).

As a consequence \( H^0(\partial S_0, H^0 i''_0^* j'_* E) \) and \( H^0(\partial S_0, H^1 i''_1^* j'_* E) \) have positive weights by lemma 2.3. Hence the extremal terms in the exact sequence (1.2.3) for \( H^1_h(\partial S_0, i''_0^* j'_* E) \) are zero, so \( H^1_h(\partial S_0, i''_0^* j'_* E) = 0 \). (ii) By theorem 2.1 and lemma 2.6 we have

\[ H^1_h(\partial S_1, i''_1^* j'_* E) = \bigoplus_{p=-2}^{1} H^{2+p}_h(\partial S_1, H^p i''_1^* j'_* E). \]

By assumption all the \( H^p i''_1^* j'_* E \) are variations associated to a representation of regular highest weight, hence have no cohomology in degree zero by theorem 2.7. Furthermore, as \( \partial S_1 \) is a non proper curve, singular cohomology of local systems vanishes in degrees \( \geq 2 \). Then the exact sequence (1.2.3) shows

\[ H^1_h(\partial S_1, H^{-2} i''_1^* j'_* E) = 0, \]

\[ H^1_h(\partial S_1, H^{-1} i''_1^* j'_* E) = H_{\text{MHS}}(1, H^1(\partial S_1, \text{Sym}^{k+1} V_2(t))), \]

\[ H^1_h(\partial S_1, H^0 i''_1^* j'_* E) = 0. \]

To conclude, note that \( H_{\text{MHS}}(1, H^1(\partial S_1, \text{Sym}^{k+1} V_2(t))) = 0 \) by lemma 2.12.

Lemma 2.14. The right hand morphism \( q_1 \) of the diagram (2.4.2) is étale.

Proof. As \( \partial(M \times M')_1 = \partial M \times M' \cap \partial M' \), it is sufficient to consider the restriction of \( q_1 \) to \( \partial M \times M' \). According to 2.14, the corresponding boundary component is a mixed Shimura datum associated to \( P_i^2 = (G_a \times G_0) \times \mathbb{G}_a \), \( G_2 \) be the group in the last line of (2.1.3). Denote by \( q^n : P^n_i \to G_2 \) the projection on the Levi. The description of the Baily-Borel compactification given in subsection 3.1 shows that \( \partial M \times M' \) is a sum of the modular curves \( \hat{M}_{n^2(g(K \times K')g^{-1} \cap P^n_2(\mathcal{A}_j))} \) for \( g \in \Pi_2(\mathcal{A}_j) \) and that \( S_1 \) is the sum of the \( \hat{M}_{n^2(g_0 K, g^{-1} \cap P^n_2(\mathcal{A}_j))} \) for \( g \in P^n_2(\mathcal{A}_j) \). By proposition 1.11 \( \iota(K \times K') \subset L \). The restriction of \( q_1 \) to a \( \hat{M}_{n^2(g(K \times K')g^{-1} \cap P^n_2(\mathcal{A}_j))} \) is then the projection corresponding to the inclusion \( q^2(g(K \times K')g^{-1} \cap P^n_2(\mathcal{A}_j)) \subset q^2(\iota(g) L_0(g)^{-1} \cap P^n_2(\mathcal{A}_j)) \), which is étale (3.6).
Lemma 2.15. If $k - k' \neq 2t - 3$ then the vertical right hand arrow of the diagram of corollary 2.11 is the zero map.

Proof. By lemma 2.13 (ii) it is enough to show that $\text{Hom}_{\text{DM}(\mathcal{O}_{S_1}/\mathbb{R})}}(q_1, \mathcal{H}^{-1}i_1^*i^*j_*F, \mathcal{H}^0i_1^*i^*j_*E)$ vanishes. Theorem 2.1 and lemma 2.4 give $q_1i_1^*i^*j_*F = q_1\mathcal{H}^{-1}i_1^*i^*j_*F[1] \oplus q_1\mathcal{H}^{-1}i_1^*i^*j_*F$ with

$$
\mathcal{H}^{-1}i_1^*i^*j_*F = \text{Sym}^k V_2(k + 2t) \oplus \text{Sym}^k V_2(k' + 2t),
$$

$$
\mathcal{H}^0i_1^*i^*j_*F = \text{Sym}^k V_2(2t - 1) \oplus \text{Sym}^k V_2(2t - 1).
$$

As $q_1$ is proper and étale we have

$$
\text{Hom}_{\text{DM}(\mathcal{O}_{S_1}/\mathbb{R})}}(q_1, \mathcal{H}^{-1}i_1^*i^*j_*F[1], \mathcal{H}^0i_1^*i^*j_*E)
$$

$$
= \text{Hom}_{\text{DM}(\mathcal{O}_{S_1}/\mathbb{R})}}(q_1, \mathcal{H}^{-1}i_1^*i^*j_*F, \mathcal{H}^0i_1^*i^*j_*E[-1])
$$

$$
= \text{Hom}_{\text{DM}(\text{M}(\text{M} \times \text{M}),/\mathbb{R})}}(\mathcal{H}^{-1}i_1^*i^*j_*F, q_1^*\mathcal{H}^0i_1^*i^*j_*E[-1])
$$

$$
= \text{Hom}_{\text{DM}(\text{M}(\text{M} \times \text{M}),/\mathbb{R})}}(\mathcal{H}^{-1}i_1^*i^*j_*F, q_1^*\mathcal{H}^0i_1^*i^*j_*E[-1])
$$

$$
= \text{Hom}_{\text{DM}(\text{M}(\text{M} \times \text{M}),/\mathbb{R})}}(\mathcal{H}^{-1}i_1^*i^*j_*F, q_1^*\mathcal{H}^0i_1^*i^*j_*E[-1])
$$

$$
= \text{Hom}_{\text{DM}(\text{M}(\text{M} \times \text{M}),/\mathbb{R})}}(\mathcal{H}^{-1}i_1^*i^*j_*F, q_1^*\mathcal{H}^0i_1^*i^*j_*E[-1])
$$

$$
= \text{Hom}_{\text{DM}(\text{M}(\text{M} \times \text{M}),/\mathbb{R})}}(\mathcal{H}^{-1}i_1^*i^*j_*F, q_1^*\mathcal{H}^0i_1^*i^*j_*E[-1])
$$

$$
= \text{Hom}_{\text{DM}(\text{M}(\text{M} \times \text{M}),/\mathbb{R})}}(\mathcal{H}^{-1}i_1^*i^*j_*F, q_1^*\mathcal{H}^0i_1^*i^*j_*E[-1])
$$

$$
= \text{Hom}_{\text{DM}(\text{M}(\text{M} \times \text{M}),/\mathbb{R})}}(\mathcal{H}^{-1}i_1^*i^*j_*F, q_1^*\mathcal{H}^0i_1^*i^*j_*E[-1])
$$

where in the last equality, which follows from 2.2 Lem. 2.6, we denote by the superscript $\vee$ the dual variation. The last term of the sequence of equalities above is the absolute Hodge cohomology in degree $-1$ with coefficients in a variation. The exact sequence 12.23 shows that it is zero.

On the other hand

$$
\text{Hom}_{\text{DM}(\mathcal{O}_{S_1}/\mathbb{R})}}(q_1, \mathcal{H}^0i_1^*i^*j_*F, \mathcal{H}^0i_1^*i^*j_*E)
$$

$$
= \text{Hom}_{\text{DM}(\text{M}(\text{M} \times \text{M}),/\mathbb{R})}}(\mathcal{H}^0i_1^*i^*j_*F, \mathcal{H}^0i_1^*i^*j_*E)
$$

The variation on the source has weights $-k' - 4t + 2$ and $-k - 4t + 2$ and the variation on the target has weights $-k - 1 - 2t$. Our assumption is that the weights on the source and on the target are different, hence $\text{Hom}_{\text{DM}(\mathcal{O}_{S_1}/\mathbb{R})}}(q_1, \mathcal{H}^0i_1^*i^*j_*F, \mathcal{H}^0i_1^*i^*j_*E) = 0$. 

Proposition 2.16. If $t \in \{2, 3\}$ and $k - k' \neq 2t - 3$ then the diagonal morphism of the left hand square in the diagram of corollary 2.11 is the zero map.
Proof. The lower line of the commutative diagram (2.4.2) gives rise to a commutative diagram of exact triangles

\[
\begin{array}{ccc}
    i_1 i_1^* j_* E & \rightarrow & i_* j_* E & \rightarrow & i_0 i_0^* j_* E
\
    \downarrow & & \downarrow & & \downarrow
\
    i_1 i_1^* j_* E & \rightarrow & i_1 i_1^* j_* E & \rightarrow & i_0 i_0^* i_1 i_1^* j_* E
\end{array}
\]

in \(\text{D}^b\text{MHM}(\partial S^*/\mathbb{R})\) (see [7] 1.4.7.1). Hence we have a commutative diagram with exact lines

\[
\begin{array}{ccc}
    H^2_h(\partial S, i_1 i_1^* j_* E) & \rightarrow & H^2_h(\partial S, i_* j_* E) & \rightarrow & H^1_h(\partial S_0, i_0^* j_* E)
\
    \downarrow & & \downarrow & & \downarrow
\
    H^2_h(\partial S_0, i_0^* i_1 i_1^* j_* E) & \rightarrow & H^2_h(\partial S_1, i_1^* j_* E) & \rightarrow & H^1_h(\partial S_0, i_0^* i_1 i_1^* j_* E).
\end{array}
\]

According to the lemma 2.13 (i), the upper right hand corner is zero and according to lemma 2.15 the composite map \(H^2_h(\text{M} \times M', F) \xrightarrow{b} H^2_h(\partial S, i^* j_* E) \rightarrow H^2_h(\partial S_1, i_1^* j_* E)\) in the diagram of corollary 2.11 is zero. Hence \(b\) factors through \(H^2_h(\partial S_0, i_0^* i_1 i_1^* j_* E) \rightarrow H^2_h(\partial S_1, i_1^* j_* E)\).

Let us show that the source of this morphism is zero. By theorem 2.1 and lemma 2.6 on has \(H^2_h(\partial S_0, i_0^* i_1 i_1^* j_* E) = \bigoplus_{q=-2}^{1} H^0 h_{-q-p}(\partial S_0, i_0^* i_1 i_1^* j_* E)\). Let \(j_1 : \partial S_1 \rightarrow \partial S_1\) be the Baily-Borel compactification and \(i_1 : \partial S_1 \rightarrow \partial S_1\) be the complementary reduced closed immersion. By [40] 7.6 there is a commutative diagram with cartesian squares

\[
\begin{array}{ccc}
    \partial S_1 & \xrightarrow{j_1} & \partial S_1
\
    \downarrow & & \downarrow
\
    \partial S_1 & \xrightarrow{i_1} & \partial S_0
\end{array}
\]

where \(r\) is an étale cover. Functoriality and proper base change give

\[
i_0^* i_1 \mathcal{H}^1 i_1^* j_* E = i_0^* r_* j_* \mathcal{H}^1 i_1^* j_* E = i_1^* i_1^* j_* E.
\]

Then using first theorem 2.1 and lemma 2.3 and second that \(\partial \partial S_1\) has dimension zero we get

\[
H^0_h(\partial S_0, i_0^* i_1 i_1^* j_* E) = \bigoplus_{q=-2}^{1} H^0 h_{-q-p}(\partial S_1, r_* \mathcal{H}^1 i_1^* j_* E) = H^0_h(\partial \partial S_1, \mathcal{H}^{-1} i^* j_1, \mathcal{H}^1 i_1^* j_* E) \oplus H^0_h(\partial S_1, \mathcal{H}^0 i_1^* j_1, \mathcal{H}^0 i_1^* j_* E).
\]

By lemmas 2.6 and 2.3 we get \(\mathcal{H}^{-1} i^* j_1, \mathcal{H}^1 i_1^* j_* E = 1(k + t)\) and \(\mathcal{H}^0 i_1^* j_1, \mathcal{H}^0 i_1^* j_* E = 1(t - 1)\). By assumption these variations have no weight zero. So the exact sequence (1.2.3) shows that \(H^0_h(\partial S_0, i_0^* i_1 i_1^* j_* E) = 0\) and the proof is complete. \(\square\)

2.5 The Eisenstein classes

From now on, when we write a cohomology space of a Shimura variety without mentioning the level, we mean the inductive limit of the cohomology spaces of the corresponding finite level Shimura
varieties. Such a space is endowed with the action of the \( \mathbb{A}_f \)-valued points of the group underlying the Shimura variety, which is induced by right multiplication on the complex points (see section 1.4). For example \( H_{\mathcal{M}}^m(\mathbb{Q}(s)) \) denotes the inductive limit \( \lim_{\leftarrow} H_{\mathcal{M}}^m(\mathbb{Q}(s)) \) indexed by compact open subgroups \( K_{\mathfrak{v}} \times K \subset \mathbb{P}_2^1(\mathbb{A}_f) \), endowed with the action of \( \mathbb{P}_2(\mathbb{A}_f) \).

Let \( m > 0 \) be an integer. Denote by \( \mathbb{T}_2(\mathbb{R})^+ \) the neutral component of \( \mathbb{T}_2(\mathbb{R}) \) and consider the \( G_2(\mathbb{A}_f) \)-module \( \mathbb{B}_m = \bigoplus \text{ind}_{\mathbb{T}_2(\mathfrak{b}_2)} G_2(\mathbb{A}_f) \eta_f \), where the sum runs over all algebraic Hecke characters \( \eta_f : \mathbb{T}_2(\mathbb{Q}) \setminus \mathbb{T}_2(\mathbb{A}) \rightarrow \mathbb{C}^\times \) of type \( \lambda(-m - 2, -1) \). We denote \( \mathbb{B}_m \subset \mathbb{B}_m \) the sub \( G_2(\mathbb{A}_f) \)-module of \( \mathbb{Q} \)-valued functions. The Eisenstein symbol [5] § 3 and 4.3 (see also [32], [47]) is a \( G_2(\mathbb{A}_f) \)-equivariant morphism

\[
\text{Eis}^m : \mathbb{B}_m \mathbb{B}_m^0 \rightarrow H_{\mathcal{M}}^{m+1}(\mathbb{Q}(m + 1)).
\]  

Fix two integers \( k \geq k' \geq 0 \). Let \( K_{\mathfrak{v}} \times K \) and \( K_{\mathfrak{v}}' \times K' \) be compact open subgroups of \( \mathbb{P}_2^1(\mathbb{A}_f) \) such that \( \det K = \det K' = K_0 \) and let \( L \) be a compact open subgroup of \( G_4(\mathbb{A}_f) \) given by proposition 1.11. There is a commutative diagram

\[
\begin{array}{ccc}
E_{K_{\mathfrak{v}} \times K}^{k+k'} & \xrightarrow{\iota'} & A_{(K_{\mathfrak{v}}\oplus K_{\mathfrak{v}}') \times L}^{k+k'} \\
\downarrow & & \downarrow \\
M_{K} \times F_{K_0} & \xrightarrow{\iota} & S_L
\end{array}
\]

whose horizontal arrows are closed imbeddings of codimension one. Denote by

\[
\iota' : H_{\mathcal{M}}^{k+k'+2}(E_{K_{\mathfrak{v}} \times K}^{k+k'} \times F_{K_0}, E_{K_{\mathfrak{v}}' \times K'}^{k+k'}) \rightarrow H_{\mathcal{M}}^{k+k'+4}(A_{(K_{\mathfrak{v}}\oplus K_{\mathfrak{v}}') \times L}, \mathbb{Q}(k + k' + 3))
\]

the Gysin morphism associated to \( \iota' \) ([51] § 5 (3.5.4)). The projections \( p : E_{K_{\mathfrak{v}} \times K}^{k+k'} \rightarrow E_{K_{\mathfrak{v}}'}^{k+k'} \) and \( p' : E_{K_{\mathfrak{v}}' \times K'}^{k+k'} \rightarrow E_{K_{\mathfrak{v}}' \times K'}^{k+k'} \) on the first and \( k' \) last factors respectively obviously extend to morphisms of mixed Shimura data which yield a projective system of projections \( p : E_{K_{\mathfrak{v}} \times K}^{k+k'} \rightarrow E_{K_{\mathfrak{v}} \times K}^{k+k'} \) and \( p' : E_{K_{\mathfrak{v}}' \times K'}^{k+k'} \rightarrow E_{K_{\mathfrak{v}}' \times K'}^{k+k'} \), indexed by compact open subgroups of \( \mathbb{P}_2^1(\mathbb{A}_f) \). Denoting by \( \sqcup \) the external cup-product, we get on the inductive limit

\[
\text{Eis}^{k,k'} = \iota'_*(p^* \text{Eis}^k \sqcup p'^* \text{Eis}^{k'}) : \mathbb{B}_m^0 \otimes \mathbb{B}_m^0 \rightarrow H_{\mathcal{M}}^{k+k'+4}(A_{k+k'+3}, \mathbb{Q}(k + k' + 3)).
\]  

We call the \( \text{Eis}^{k,k'}(\phi_f \circ \phi'_f) \) the Eisenstein classes.

**Theorem 2.17.** Let \( \mathcal{E}^{k,k'} \) be the sub \( G_4(\mathbb{A}_f) \)-module of \( H_{\mathcal{M}}^{k+k'+4}(A_{k+k'+3}, \mathbb{Q}(k + k' + 3)) \) generated by the image of \( \text{Eis}^{k,k'} \). If \( k > k' > 0 \) and \( k \neq k' + 3 \) then the regulator

\[
r_h : H_{\mathcal{M}}^{k+k'+4}(A_{k+k'+3}, \mathbb{Q}(k + k' + 3)) \rightarrow H_{\mathcal{M}}^{k+k'+4}(A_{k+k'+3}, \mathbb{R}(k + k' + 3))
\]

maps \( \mathcal{E}^{k,k'} \) to \( \text{Ext}_{\text{MHS}}^1(1, H^1_\mathbb{Q}(S, \omega^{k,k'})) \).
Lemma invariant under the de Rham involution. The projection 
implies together with for 3.1 Deligne’s rational structure 
cohomology. By construction functoriality of for 
real and imaginary part induce an isomorphism of 
Proof. Let \( r_h : H^m_\mathbb{M}(E^m, \overline{\mathbb{Q}}(m+1)) \to H^{m+1}_\mathbb{M}(E^m, \mathbb{R}(m+1)) \) be the regulator in absolute Hodge 
cohomology. By construction \( r \circ \text{Eis}^m \) factors through \( H^1_\mathbb{M}(\text{Sym}^m V_2(1)) \) (see \[3.1\]). Then by 
functionality of \( r \), the composite map \( r \circ \text{Eis}^{k,k'} \) factors through

\[
\begin{array}{ccc}
H^1_\mathbb{M}(\text{Sym}^k V_2(1)) \otimes H^1_\mathbb{M}(\text{Sym}^{k'} V_2(1)) & \to & H^2_\mathbb{M}(M \times M', (\text{Sym}^k V_2 \boxtimes \text{Sym}^{k'} V_2)(2)) \\
& & \downarrow \circlearrowright \\
& & H^3_\mathbb{M}(\text{Sym}^{k,k'})
\end{array}
\]

where the first map is the external cup-product and the second is the left hand vertical map 
appearing in the diagram of corollary \[2.11\]. By propositions \[2.9\] and \[2.16\] the second map factors 
through \( \text{Ext}^1_{\text{MHS}_\mathbb{R}}(1, \text{H}^3_\mathbb{M}(\text{Sym}^{k,k'})) \). 

\[\Box\]

Notation. For \( \phi_f \otimes \phi_f' \in \mathcal{B}^0_k \otimes \mathcal{B}^0_k \) we write \( \text{Eis}_h^{k,k'}(\phi_f \otimes \phi_f') = (r_h \circ \text{Eis}^{k,k'})(\phi_f \otimes \phi_f') \).

3 Connection with the special value

Let \( \pi = \pi_\infty \otimes \pi_f \) be an irreducible cuspidal automorphic representation of \( G_1 \), whose archimedean component \( \pi_\infty \) is in the discrete series \( L \)-packet \( \text{P}(W^{kk'}) \). Let \( \text{M}_B(\pi_f, W^{kk'}) \subset \text{MHS}_\mathbb{Q}^+ \text{E}(\pi_f) \) the 
associated \( \mathbb{Q} \)-Hodge structure with infinite Frobenius and coefficients in \( E(\pi_f) \). It is pure, of weight 
\(-k - k' - 3 \). We also have the de Rham rational structure \( \text{M}_{dR}(\pi_f, W^{kk'}) \subset \text{M}_B(\pi_f, W^{kk'}) \). 

3.1 Deligne’s rational structure

For \( M \in \text{MHS}_\mathbb{R}^+ \), we denote by \( M^+ \subset M \subset \text{M}_C \), resp. \( M^+_C \subset \text{M}_C \), the subspaces of vectors that are invariant under the de Rham involution.

**Lemma 3.1.** Let \( M \in \text{MHS}_\mathbb{R}^+ \) of weight \( w < 0 \). There is an exact sequence of \( \mathbb{R} \)-vector spaces

\[
0 \to F^0 M_{dR} \to M^+ \to (M(-1))^+ \to \text{Ext}^1_{\text{MHS}_\mathbb{R}^+}(1, M) \to 0.
\]

**Proof.** Real and imaginary part induce an isomorphism of \( \mathbb{R} \)-vector spaces \( M_C = M \oplus M(-1) \) stable under the de Rham involution. The projection \( \text{Im} : M_C \to M(-1) \) is given by \( v \mapsto \frac{1}{2}(v - \bar{v}) \). As 
complex conjugation exchanges \( M^{p,q} \) and \( M^{q,p} \), for \( p \neq q \) we have \( \text{Ker} \text{Im} \cap M_{C}^{p,q} = 0 \). Then it 
follows from the fact that \( M \) has negative weight that

\[
(\text{Ker} \text{Im}) \cap F^0 M_C = (\text{Ker} \text{Im}) \cap \bigoplus_{p \geq 0, p+q = w} M^{p,q} = (\text{Ker} \text{Im}) \cap \bigoplus_{p \geq 0, 2p = w} M^{p,p} = 0.
\]

Together with \( F^0 M_{dR} \subset M^+_C \) this shows that \( F^0 M_{dR} \subset M(-1)^+ \). The hypothesis on the weight 
implies \( W_0 M = M \). As a consequence \([2.3]\) yields

\[
\text{Ext}^1_{\text{MHS}_\mathbb{R}^+}(1, M) = M^+_C / (M + F^0 M_C)^+ = M(-1)^+ / F^0 M_{dR}.
\]

\[\Box\]
Corollary 3.2. (i) There is an exact sequence of $\mathbb{R} \otimes E(\pi_f)$-modules of finite rank

$$0 \longrightarrow F^0 M_{dR}(\pi_f, W^{k_k'})_R \longrightarrow M_B(\pi_f, W^{k_k'})_R(-1)^+ \longrightarrow \text{Ext}^1_{\text{MHS}_B^+}(1, M_{\text{R}(\pi_f, W^{k_k'})_R}) \longrightarrow 0.$$ 

(ii) We have $\text{Ext}^1_{\text{MHS}_B^+}(1, M_B(\pi_f, W^{k_k'})_R) \subset \text{Hom}_{\mathbb{Q}[G_4(\lambda_f)]}(\pi_f, \text{Ext}^1_{\text{MHS}_B^+}(1, H^3(S, W^{k_k'})_R)).$

Proof. (i) follows immediately from the preceding lemma. (ii) As filtering inductive limits are exact, the preceding lemma gives an exact sequence of $\mathbb{R}$-vector spaces

$$0 \longrightarrow F^0 H^3_{dR}(S, W^{k_k'})_R \longrightarrow H^3(S, W^{k_k'})_R(-1)^+ \longrightarrow \text{Ext}^1_{\text{MHS}_B^+}(1, H^3(S, W^{k_k'})_R) \longrightarrow 0.$$ 

The statement follows by applying the functor $X \mapsto \text{Hom}_{\mathbb{Q}[G_4(\lambda_f)]}(\pi_f, X).$

Lemma 3.3. (i) Let $M_B(\pi_f, W^{k_k'})_{C_{\infty}=\pm 1} \subset M_B(\pi_f, W^{k_k'})_C$ be the subspace where the involution $F_{\infty}$ induced by the complex conjugation on $S(\mathbb{C})$ acts by $\pm 1$. We have

$$\text{rk}_{C \otimes E(\pi_f)} M(\pi_f, W^{k_k'})_{C_{\infty}=\pm 1} = m(\pi^H_{\infty} \otimes \pi_f) + m(\pi^W_{\infty} \otimes \pi_f).$$

(ii) We have

$$\text{rk}_{R \otimes E(\pi_f)} F^0 M_{dR}(\pi_f, W^{k_k'})_R = m(\pi^H_{\infty} \otimes \pi_f),$$

$$\text{rk}_{R \otimes E(\pi_f)} M_B(\pi_f, W^{k_k'})_R(-1)_{R^+} = m(\pi^H_{\infty} \otimes \pi_f) + m(\pi^W_{\infty} \otimes \pi_f),$$

$$\text{rk}_{R \otimes E(\pi_f)} \text{Ext}^1_{\text{MHS}_B^+}(1, M_B(\pi_f, W^{k_k'})_R) = m(\pi^W_{\infty} \otimes \pi_f).$$

Proof. (i) By proposition 1.17, the Hodge decomposition is $M_{B \mathbb{C}} = M_{0,w} \oplus M_{p,q} \oplus M_{q,p} \oplus M_{w,0}$ with $0 > p > q > w$. Furthermore we have $\text{rk}_{C \otimes E(\pi_f)} M_{0,w} = \text{rk}_{C \otimes E(\pi_f)} M_{w,0} = m(\pi^H_{\infty} \otimes \pi_f)$ and $\text{rk}_{C \otimes E(\pi_f)} M_{p,q} = \text{rk}_{C \otimes E(\pi_f)} M_{q,p} = m(\pi^W_{\infty} \otimes \pi_f)$. Let $(e^H_i)$ and $(e^W_i)$ be basis of $M_{0,w}$ and $M_{p,q}$. Then the statement follows from the fact that $(1/2(e^H_i \pm F_{\infty} e^H_i)1/2(e^W_i \pm F_{\infty} e^W_i))$ is a basis of $M_{B \mathbb{C}}$. (ii) The first equality follows from $F^0 M_{dR}(\pi_f, W^{k_k'})_C = M_{0,w}$ and the second part the fact that the complexification of $M_{B \mathbb{R}}(-1)^+$ is one of the $M_{B \mathbb{C}}$ isomorphic to $M_{B \mathbb{C}}$. Then the exact sequence of corollary 3.2 (i) gives the last equality.

Denote by $F^0 M_{dR}(\pi_f, W^{k_k'})^\vee_R$ the dual of $F^0 M_{dR}(\pi_f, W^{k_k'})_R$. Then the exact sequence of $\mathbb{R} \otimes E(\pi_f)$-modules of corollary 3.2 (i) induces an isomorphism between the highest exterior powers

$$\det \text{Ext}^1_{\text{MHS}_B^+}(1, M_B(\pi_f, W^{k_k'})_R) \cong \det F^0 M_{dR}(\pi_f, W^{k_k'})^\vee_R \otimes \det M_B(\pi_f, W^{k_k'})_R(-1)^+.$$ 

Definition 3.4. (29) Con. 4.6 (c), (38) 2 (i) Beilinson’s $E(\pi_f)$-structure is

$$B(\pi_f, W^{k_k'}) = \det F^0 M_{dR}(\pi_f, W^{k_k'})^\vee_R \otimes \det M_B(\pi_f, W^{k_k'})_R(-1)^+.$$ 

(ii) Let $\delta(\pi_f) \in (\mathbb{C} \otimes E(\pi_f))^\vee$ be the determinant of the comparaison isomorphism between singular and de Rham cohomology $M_B(\pi_f, W^{k_k'})_\mathbb{C} \rightarrow M_{dR}(\pi_f, W^{k_k'})_\mathbb{C}$ computed in basis defined over $E(\pi_f)$ on both sides. Then Deligne’s $E(\pi_f)$-structure is $D(\pi_f, W^{k_k'}) = \delta(\pi_f)^{-1}B(\pi_f, W^{k_k'})$. 

Hypothesis 1. From now on, we assume that the automorphic representation \( \pi \) is stable and of multiplicity one, which means that

\[
m(\pi_\infty^H \otimes \pi_f) = m(\pi_\infty^W \otimes \pi_f) = m(\pi_\infty^W \otimes \pi_f) = m(\pi_\infty^H \otimes \pi_f) = 1.
\]

We compute \( \delta(\pi_f) \) thanks to the following result of Weissauer: fix a prime number \( l \) and denote by \( H^q_{\et}(S \otimes \overline{Q}, W_{k'}^k) \) the \( l \)-adic interior cohomology. There is a continuous action of \( \text{Gal}(\overline{Q}/Q) \) on the \( \mathbf{Q}_l \otimes E(\pi_f) \)-module \( M_l(\pi_f, W_{k'}^k) = \text{Hom}_{\mathbf{Q}[G_4]}(\pi_f, H^q_{\et}(S \otimes \overline{Q}, W_{k'}^k)) \), which is of rank four by the hypothesis on \( \pi \), proposition 3.6 and the comparison isomorphism between singular and étale cohomology. For every prime \( \lambda \) of \( E(\pi_f) \) over \( l \), denote by \( E(\pi_f)_\lambda \) the \( \lambda \)-adic completion of \( E(\pi_f) \) and \( M_l(\pi_f, W_{k'}^k)_\lambda \) the associated \( \text{GL}_4(E(\pi_f)_\lambda) \)-valued Galois representation. The following reflects the fact that \( G_4 \) is its own Langlands dual group.

**Theorem 3.5.** ([54] Th. IV) Choose a compatible system of primitive \( l \)-th roots of unity in \( \overline{Q} \) and denote by \( \mu_l : \text{Gal}(\overline{Q}/Q) \to \mathbb{Z}_l^* \) the \( l \)-adic cyclotomic character. Let \( \omega_{\pi_f}^0 \) be the finite order part of the central character of \( \pi_f \) and let \( \lambda \) be a prime of \( E(\pi_f) \) above \( l \). Then the four dimensional Galois representation \( M_l(\pi_f, W_{k'}^k)_\lambda \) has values in \( G_4(E(\pi_f)_\lambda) \) and for every \( g \in \text{Gal}(\overline{Q}/Q) \) we have

\[
\nu_4(g) = \omega_{\pi_f}^0(g)^{-1} \mu_l(g)^{-k-k'-3}.
\]

Let \( N \) be the conductor of \( \omega_{\pi_f}^0 \) and see \( \omega_{\pi_f}^0 \), as a Dirichlet character \( (\mathbb{Z}/N)^* \to \mathbb{C}^* \). We denote by \( \epsilon(\omega_{\pi_f}^0) \) the associated Gauss sum \( \epsilon(\omega_{\pi_f}^0) = \sum_{n \in (\mathbb{Z}/N)^*} \omega_{\pi_f}^0(n) \otimes e^{2\pi i n/N} \).

**Corollary 3.6.** With the notation of definition 3.4 (ii) we have \( \delta(\pi_f) \equiv \epsilon(\omega_{\pi_f}^0)^2 \).

**Proof.** As the determinant of a matrix of \( G_4 \) is the square of its similitude factor, theorem 3.4 shows that \( \det M_l(\pi_f, W_{k'}^k) = M_l(\omega_{\pi_f}^0)^{-2}(-2w) \). Then the conclusion follows from [115] Th. 5.6, (5.1.9) and Prop. 6.5.

Let us finally give an explicit description of Beilinson’s \( E(\pi_f) \)-structure. Chose a basis \((a, b)\) of \( F \) and let \( \theta \) be a generator of \( F^{\text{der}}/F \). Via the inclusion of the exact sequence of corollary 3.2 (i), we get \( \lambda, \mu \in \mathbb{R} \otimes E(\pi_f) \) such that \( \theta = \lambda a + \mu b \).

**Lemma 3.7.** Let \( v = xa + yb \in M_B(\pi_f, W_{k'}^k)_R(-1) \). Then the projection

\[
M_B(\pi_f, W_{k'}^k)_R(-1) \xrightarrow{\text{Ext}^1_{\text{MHS}_k}(1, M_B(\pi_f, W_{k'}^k)_R)}
\]

maps \( v \) into \( B(\pi_f, W_{k'}^k) \) if and only if \( x \mu - \lambda y \in E(\pi_f) \).

**Proof.** The choice of \( \theta \) induces an isomorphism \( B(\pi_f, W_{k'}^k) \simeq \text{det}_{E(\pi_f)} M_B(\pi_f, W_{k'}^k)(-1)^+ \) on one side and the map

\[
\wedge \theta : M_B(\pi_f, W_{k'}^k)_R(-1) \xrightarrow{\text{det}_{E(\pi_f)} M_B(\pi_f, W_{k'}^k)(-1)^+} \text{Ext}^1_{\text{MHS}_k}(1, M_B(\pi_f, W_{k'}^k)_R)
\]

which is part of the commutative diagram with exact lines

\[
\begin{align*}
0 \xrightarrow{} F^0 M_{dR} \xrightarrow{} & M_B(\pi_f, W_{k'}^k)_R(-1)^+ \xrightarrow{\wedge \theta} \text{det}_{E(\pi_f)} M_B(\pi_f, W_{k'}^k)(-1)_R^+ \xrightarrow{} 0 \\
0 \xrightarrow{} F^0 M_{dR} \xrightarrow{} & M_B(\pi_f, W_{k'}^k)_R(-1)^+ \xrightarrow{} \text{Ext}^1_{\text{MHS}_k}(1, M_B(\pi_f, W_{k'}^k)_R) \xrightarrow{} 0
\end{align*}
\]
on the other side. This shows that \( v \) is mapped into \( \mathcal{B}(\pi_f, W^{k,k'}) \) if and only if \( v \wedge \theta = \rho a \wedge b \) for some \( \rho \in E(\pi_f) \), which means that \( x\mu - y\lambda \in E(\pi_f) \).

It follows from theorem 2.17 that the space \( \text{Hom}_{\mathbb{Q}[G_4(\mathcal{A}_f)]}(\pi_f, r(\mathcal{E}^{k,k'})) \) is a sub \( \mathbb{Q} \)-vector space of the \( \mathbb{R} \otimes \mathbb{Q} \)-module \( \text{Hom}_{\mathbb{Q}[G_4(\mathcal{A}_f)]}(\pi_f, \text{Ext}^1_{\text{MHS}^+}(1, H^3_1(S, W^{k,k'}))) \). Corollary 3.2 (ii) gives an inclusion of \( \mathbb{R} \otimes E(\pi_f) \)-modules \( \text{Ext}^1_{\text{MHS}^+}(1, M_B(\pi_f, W^{k,k'})_R) \subset \text{Hom}_{\mathbb{Q}[G_4(\mathcal{A}_f)]}(\pi_f, \text{Ext}^1_{\text{MHS}^+}(1, H^3_1(S, W^{k,k'}))) \). The \( \mathbb{Q} \)-subspace \( \text{Hom}_{\mathbb{Q}[G_4(\mathcal{A}_f)]}(\pi_f, r(\mathcal{E}^{k,k'})) \cap \text{Ext}^1_{\text{MHS}^+}(1, M_B(\pi_f, W^{k,k'})_R) \) of the \( \mathbb{R} \otimes \mathbb{Q} \)-module \( \text{Ext}^1_{\text{MHS}^+}(1, M_B(\pi_f, W^{k,k'})_R) \) is denoted \( \mathcal{R}(\pi_f, W^{k,k'}) \). The \( \mathbb{Q} \)-structures obtained by those of definition 3.4 by extensions of scalars to \( \mathbb{Q} \) are again denoted by \( \mathcal{B}(\pi_f, W^{k,k'}) \) and \( \mathcal{D}(\pi_f, W^{k,k'}) \). Our task is now to compare the two \( \mathbb{Q} \)-structures \( \mathcal{R}(\pi_f, W^{k,k'}) \) and \( \mathcal{D}(\pi_f, W^{k,k'}) \). As in [5] and [31], Poincaré duality reduces the problem to the computation of two cup-products.

3.2 Poincaré duality and 1-extensions of mixed Hodge structures

Fix a generator \( 1 \) of the highest exterior power \( \Lambda^6 \mathfrak{g}_4/\mathfrak{t}_4 \). This gives rise to an orientation on the manifold \( S(C) \) (lemma 1.13). Then consider the perfect pairing \( [\ , \ ] : W^{k,k'} \otimes W^{k,k'}(-3) \to \mathbb{Q}(k+k') \) and endow the \( \mathbb{Q} \)-Hodge structure \( \mathbb{Q}(k+k') \) with the action of \( G_4(\mathcal{A}_f) \) via \( \nu_4^{-k-k'} \beta \). By 50 p. 294-295 and the isomorphism between interior and intersection cohomology (37 Prop. 1), Poincaré duality

\[
H^3_1(S, W^{k,k'}) \otimes H^3_1(S, W^{k,k'}(-3)) \to \mathbb{Q}(k+k')
\]

is \( G_4(\mathcal{A}_f) \)-equivariant. Hence we have a perfect pairing of \( \mathbb{Q} \)-Hodge structures with coefficients in \( E(\pi_f) \)

\[
M_B(\pi_f, W^{k,k'}) \otimes M_B(\pi_f|\nu_4|^{k-k'-3}, W^{k,k'}(-3)) \xrightarrow{\cdot B} E(\pi_f)(k+k')_B. \tag{3.2.1}
\]

In de Rham cohomology, we have a perfect pairing of filtered \( E(\pi_f) \)-vector spaces

\[
M_{dR}(\pi_f, W^{k,k'}) \otimes M_{dR}(\pi_f|\nu_4|^{k-k'-3}, W^{k,k'}(-3)) \xrightarrow{\cdot dR} E(\pi_f)(k+k')_{dR} \tag{3.2.2}
\]

whose complexification is induced by the pairing between rapidly decreasing and slowly increasing differential forms (22 Prop. 2.5, Prop. 1.4.4)

\[
\text{Hom}(\Lambda^3 \mathfrak{g}_4/\mathfrak{t}_4, W^{k,k'} \otimes C^\infty_{rd}(G_4(\mathbb{Q}) \backslash G_4(\mathcal{A})) \otimes \text{Hom}(\Lambda^3 \mathfrak{g}_4/\mathfrak{t}_4, W^{k,k'}(-3) \otimes C^\infty_{rd}(G_4(\mathbb{Q}) \backslash G_4(\mathcal{A}))) \to \mathbb{C}
\]

\[
\omega \otimes \omega' \to \int_{Z_4(\mathcal{A})G_4(\mathbb{Q})/G_4(\mathcal{A})} \frac{1}{(2\pi i)^{k+k'}} |\nu_4(x)|^{k+k'+3} \omega \wedge \omega'(1)(x)dx.
\]

Extending scalars to \( \mathbb{C} \), these pairings give a commutative diagram

\[
\begin{array}{ccc}
M_B(\pi_f, W^{k,k'})_C \otimes M_B(\pi_f|\nu_4|^{k-k'-3}, W^{k,k'}(-3))_C & \xrightarrow{\cdot B} & \mathbb{C} \otimes E(\pi_f)(k+k')_B \\
\sim & & \sim \\
M_{dR}(\pi_f, W^{k,k'})_C \otimes M_{dR}(\pi_f|\nu_4|^{k-k'-3}, W^{k,k'}(-3))_C & \xrightarrow{\cdot dR} & \mathbb{C} \otimes E(\pi_f)(k+k')_{dR}
\end{array}
\tag{3.2.3}
\]

whose vertical lines are isomorphisms.
Lemma 3.8. Denote by $F_\infty$ be the involution on $M_B(\pi_f, W^{k, k'})$ and $M_B(\pi_f | u_4)^{-k - k' - 3, W^{k, k'}} (-3)_C$ induced by complex conjugation on $S(\mathbb{C})$ and $M_B(\pi_f, W^{k, k'})_{C F_\infty = (-1)}$ the direct factor of the vector space $M_B(\pi_f, W^{k, k'})_C$ where $F_\infty$ acts by multiplication by $(-1)^s$.

(i) We have $M_B(\pi_f, W^{k, k'})(-1)^{\frac{1}{2}} = M_B(\pi_f, W^{k, k'})_{C F_\infty = (-1)^{k + k'}}$.

(ii) The dual of $M_B(\pi_f, W^{k, k'})(-1)^{\frac{1}{2}}$ under the pairing $\langle \cdot, \cdot \rangle$ is $M_B(\pi_f | u_4)^{-k - k' - 3, W^{k, k'}} (-3)^{F_\infty = 1}_{C}$.

Proof. (i) Lemma 1.12 implies that $H^3(S, W^{k, k'})$ is a direct factor of $H^{3+k+k'}(\mathbb{A}^{k+k'}, \mathbb{Q}(k + k' + 3))$. So $M_B(\pi_f, W^{k, k'})_{\mathbb{R}}(-1)^{\frac{1}{2}}$ is a direct factor of $\text{Hom}_{\mathbb{Q}(\mathbb{G}_a(k_f))}(\pi_f, H^{3+k+k'}(\mathbb{A}^{k+k'}, \mathbb{R}(k + k' + 2)^{+})$. The symbol $^+$ denotes the invariants under the $\mathbb{C}$-antilinear complexification of $F_\infty$. Hence $F_\infty$ acts by multiplication by $(-1)^{k+k'}$ on $H^{3+k+k'}(\mathbb{A}^{k+k'}, \mathbb{R}(k + k' + 2)^{+})$. (ii) follows from the identities $(F_\infty u, F_\infty v)_B = F_\infty (u, v)_B = (-1)^{k+k'} (u, v)_B$.

By proposition 3.8, we have the Hodge decomposition

$$M_B(\pi_f | u_4)^{-k - k' - 3, W^{k, k'}} (-3)_C = M_B^{-k, k'} + M_B^{-k, k'} + M_B^{-k, k'} + M_B^{-k, k'}$$

Lemma 3.9. Let $\sigma \in M_{dR}(\pi_f | u_4)^{-k - k' - 3, W^{k, k'}} (-3)_C$ be a de Rham cohomology class mapped into $M_B^{-k, k'} + M_B^{-k, k'}$ by the comparison isomorphism and let $\sigma^+ = \frac{1}{2}(\sigma + F_\infty \sigma)$ its projection on $M_B(\pi_f | u_4)^{-k - k' - 3, W^{k, k'}} (-3)^{F_\infty = 1}_{C}$. Then the cup-product

$$M_B(\pi_f, W^{k, k'})_{\mathbb{R}}(-1)^{+} \subset M_B(\pi_f, W^{k, k'})_{C (\sigma^+, \cdot)_B} \mathbb{C} \otimes E(\pi_f)$$

induces a commutative diagram

$$\begin{array}{ccc}
M_B(\pi_f, W^{k, k'})_{\mathbb{R}}(-1)^+ & \longrightarrow & \text{Ext}_\text{MHS}_k^1(1, M_B(\pi_f, W^{k, k'})_{\mathbb{R}}) \\
\langle \sigma^+, \cdot \rangle_B & \downarrow & \langle \sigma^+, \cdot \rangle_B \\
\mathbb{C} \otimes E(\pi_f) & \longleftarrow & \\
\end{array}$$

Proof. By the exact sequence of corollary 3.24 (i), we have to show that $\langle \sigma^+, \cdot \rangle_B$ vanishes on $F^0 M_B(\pi_f, W^{k, k'})_{\mathbb{R}}$. As the inclusion $F^0 M_B(\pi_f, W^{k, k'})_{\mathbb{R}} \subset M_B(\pi_f, W^{k, k'})_{\mathbb{R}}(-1)^{+}$ is induced by $v \mapsto \frac{1}{2}(v - \bar{v})$, it maps $F^0 M_B(\pi_f, W^{k, k'}) \subset F^0 M_B(\pi_f, W^{k, k'})_C = M_B^{k - k' - 3, 0}$ into the direct sum $M_B^{-k - k' - 3} + M_B^{-k - k' - 3, 0}$. So the vanishing follows from the choice of the Hodge types of $\sigma$.

Remark. Note that the lemma would be false if $\sigma$ had a component in $M_B^{-k - k' - 3} + M_B^{-k - k' - 3, 0}$.

As $\text{Ext}_\text{MHS}_k^1(1, M_B(\pi_f, W^{k, k'})_{\mathbb{R}}$ is of rank one, we have

$$\langle \sigma^+, D(\pi_f, W^{k, k'}) \rangle_B R(\pi_f, W^{k, k'}) = \langle \sigma^+, R(\pi_f, W^{k, k'}) \rangle_B D(\pi_f, W^{k, k'})$$

so we are reduced to compute the cup-products $\langle \sigma^+, D(\pi_f, W^{k, k'}) \rangle_B$ and $\langle \sigma^+, R(\pi_f, W^{k, k'}) \rangle_B$. 
3.3 The period computation

Let us recall the definition of the Deligne periods ([15] 1.7) of $M(\pi_f, W^{k'k})$. We use the notation of the lemma [3.3] (i). We have the Hodge decomposition

$$M_B(\pi_f, W^{k'k})_C = M_B^{0,-k-k'-3} \oplus M_B^{-1-k',-2-k} \oplus M_B^{-2-k,-1-k'} \oplus M_B^{-k-k'-3,0}$$

(proposition [1.17]). The subspaces $F^\pm \subset M_{dR}(\pi_f, W^{k'k})$ of [loc. cit.] are given by

$$F^+(\pi_f, W^{k'k})_C = F^-(\pi_f, W^{k'k})_C = M_B^{0,-k-k'-3} \oplus M_B^{-1-k',-2-k}.$$ 

The map

$$F^\pm : M_B(\pi_f, W^{k'k})_C \subset M_B(\pi_f, W^{k'k})_C \simeq M_{dR}(\pi_f, W^{k'k})_C \longrightarrow (M_{dR}(\pi_f, W^{k'k})/F^\mp)_C$$

is easily seen to be an isomorphism. The determinant of this isomorphism computed in rational bases on both sides is the Deligne period $c^\pm(\pi_f, W^{k'k})$. It is an element of $(\mathbb{C} \otimes \mathbb{Q})^\times$ whose class for the relation $\equiv$ doesn’t depend on the choice of the basis. Similarly for $M(\tilde{\pi}_f|\nu_4|^{-k-k'-3}, W^{k'k'}(-3))$ we have

$$F^+(\tilde{\pi}_f|\nu_4|^{-k-k'-3}, W^{k'k'}(-3))_C = F^-(\tilde{\pi}_f|\nu_4|^{-k-k'-3}, W^{k'k'}(-3))_C = M_B^{1-k-k'} \oplus M_B^{2-k,1-k}$$

The Deligne periods $c^\pm(\tilde{\pi}_f|\nu_4|^{-k-k'-3}, W^{k'k'}(-3))$ of $M(\tilde{\pi}_f|\nu_4|^{-k-k'-3}, W^{k'k'}(-3))$ are defined as above. We have an isomorphism

$$M_B(\tilde{\pi}_f|\nu_4|^{-k-k'-3}, W^{k'k'}(-3))^{1-k,2-k'} \simeq [F^{1-k}M_{dR}(\tilde{\pi}_f|\nu_4|^{-k-k'-3}, W^{k'k'}(-3))/F^{\mp}M_{dR}]_C$$

making $F^{1-k}M_{dR}(\tilde{\pi}_f|\nu_4|^{-k-k'-3}, W^{k'k'}(-3))/F^{\mp}M_{dR}$ an $E(\pi_f)$-structure of the left hand term.

**Proposition 3.10.** Let $\omega^0_{\pi_f}$ be the finite order part of the central character of $\pi$. Denote by $\text{sgn}(\omega^0_{\pi_f})$ its sign and by $\epsilon(\omega^0_{\pi_f})$ be the associated Gauss sum. If $\sigma$ is in the above $E(\pi_f)$-structure then

$$\langle \sigma^+, B(\pi_f, W^{k'k}) \rangle_B \equiv \frac{\epsilon(\omega^0_{\pi_f})^2}{c^{(-1)^{k+k'+1}}(\pi_f, W^{k'k})}.$$ 

**Proof.** Let $B' = xa + yb \in M_B(\pi_f, W^{k'k})_R(-1)^+$, with $x\mu - y\lambda = 1$. By lemma [3.7] this implies that $B'$ is mapped to $B(\pi_f, W^{k'k})$ by $M_B(\pi_f, W^{k'k})_R(-1)^+ \rightarrow \text{Ext}^1_{\text{MHS}_3}(1, M_B(\pi_f, W^{k'k})_R)$, hence that

$$\langle \sigma^+, B(\pi_f, W^{k'k}) \rangle_B \equiv \langle \sigma^+, B' \rangle_B.$$ 

Fix generators $1_{dR}$ and $1_B$ of $E(\pi_f)(k + k')_{dR}$ and $E(\pi_f)(k + k')_B$ respectively. The right hand vertical isomorphism in the diagram ([3.2,3]) maps $1_B$ to $(2i\pi)^{k+k'}1_{dR} \equiv 1_{dR}$. Let

$$\theta' \in M_{dR}(\tilde{\pi}_f|\nu_4|^{-k-k'-3}, W^{k'k'}(-3))$$

such that $\langle \theta, \theta' \rangle_{dR} = 1_{dR}$. Recall that the Hodge types of the class $\sigma$ have been chosen to belong to $\{(2-k',1-k),(1-k,2-k')\}$. As $\theta \in F^0M_{dR}(\pi_f, W^{k'k})_C = M_B^{0,-k-k'-3}$, the dual $\theta'$ has a non zero component on $M_B^{k-k',-3}$. So the couple $(\sigma, \theta')$ is a rational basis of
The vectors \(2i\pi a\) and \(2i\pi b\) are rational vectors of \(M_B(\pi_f, W^{kk'})_C\), hence \((\frac{1}{2i\pi}a', \frac{1}{2i\pi}b')\) is a rational basis of the vector space \(M_B(\pi_f|_{V_4})^{-k-k'-3}, W^{kk'}(-3))_C\), with values in \(R\). Consider the isomorphism

\[I_\infty : M_B(\pi_f|_{V_4})^{-k-k'-3}, W^{kk'}(-3))_C \rightarrow (M_{\text{dR}}(\pi_f|_{V_4})^{-k-k'-3}, W^{kk'}(-3))/F^\pm_C.\]

Write \(I_\infty(\frac{1}{2i\pi}a') = u\sigma + v\theta'\) and \(I_\infty(\frac{1}{2i\pi}b') = u'\omega + v'\theta'\). Then the definition of the Deligne period is \(c^+(\pi_f|_{V_4})^{-k-k'-3}, W^{kk'}(-3)) = uv' - u'v\). We have \(\lambda_1 B \equiv (\frac{1}{2i\pi}a', \theta)_B \equiv (u\omega + v\theta', \theta)_{\text{dR}} \equiv v_1 \text{dR}\), the second identity following from the fact that \(\langle \cdot, \cdot \rangle_{\text{dR}}\) vanishes on \(F^\pm\). So \(\lambda \equiv v\). Similarly we have \(\mu \equiv v'\). Then we have

\[\langle \sigma^+, B' \rangle \equiv \langle I^{-1}_\infty \sigma, B' \rangle \equiv c^+(\pi_f|_{V_4})^{-k-k'-3}, W^{kk'}(-3))^{-1}(v'a' - vb', xa + yb) \equiv c^+(\pi_f|_{V_4})^{-k-k'-3}, W^{kk'}(-3))^{-1}(\mu a - \lambda b, xa + yb) \equiv c^+(\pi_f|_{V_4})^{-k-k'-3}, W^{kk'}(-3))^{-1}(\mu x - \lambda y) \equiv c^+(\pi_f|_{V_4})^{-k-k'-3}, W^{kk'}(-3)^{-1}.\]

Now it follows from [13] Prop. 5.1 and (5.1.8) and corollary 3.3.4 that

\[c^+(\pi_f|_{V_4})^{-k-k'-3}, W^{kk'}(-3)) \equiv \delta(\pi_f)^{1-1}(-1)^{k+k'+1}(\pi_f, W^{kk'}) \equiv \epsilon(\omega_2^0 |_{\pi_f}^{-2}(-1)^{k+k'+1}(\pi_f, W^{kk'}).\]

The computation of \(\langle \sigma^+, R(\pi_f, W^{kk'})_B \rangle\) is based on the explicit description of \(R(\pi_f, W^{kk'})\) in Deligne cohomology. Let us recall some of it’s basic properties.

### 3.4 Deligne Cohomology

A definition of Deligne cohomology and real Deligne cohomology can be found in [38] 7. It allows to give an explicit description of absolute Hodge cohomology classes by means of currents and differential forms.

We denote by \(Sm(Q)\) the category of smooth quasi-projective \(Q\)-schemes. Let \(X \in Sm(Q)\).

**Proposition 3.11.** ([loc. cit.] 7.1. There is a functorial morphism from absolute Hodge cohomology to real Deligne cohomology \(H^n_{\text{dR}B}(X, 1(t)) \rightarrow H^n_B(X/R, \mathbb{R}(t))\) which is an isomorphism for \(A = \mathbb{R}\) and \(m \leq t\).

**Remark.** The definition of absolute Hodge cohomology used in [loc. cit.] is not the one we use here. The compatibility between the two is proved in [27] Th. A.2.7.

Let \(S^m(X(C), \mathbb{R}(t))\) and \(S^m_{\text{dR}}(X(C), \mathbb{R}(t))\) be the space of \(C^\infty\) differential forms and \(C^\infty\) differential forms with compact support on \(X(C)\) of degree \(m\) with values in \(\mathbb{R}(t)\). Fix a smooth compactification

\[X \overset{j}{\longrightarrow} X^* \overset{i}{\longleftarrow} Y\]
whose boundary is a normal crossing divisor and let $\Omega^m_Y(X)$ be the space of degree $m$ holomorphic differential forms with logarithmic poles along $Y$, together with its Hodge filtration $F^*\Omega^m_Y(X)$ (3.1 and 3.2.2).

**Proposition 3.12.** Let $\pi_k : \mathbb{C} \to \mathbb{R}(k)$ be the projection $z \mapsto \frac{1}{2}(z + (-1)^k \overline{z})$. For $m \geq 1$ we have an isomorphism

$$H^m_D(X, \mathbb{R}(m))$$

$$\simeq \{(\phi, \omega) \in \mathcal{S}^{m-1}(X, \mathbb{R}(m - 1)) \oplus \Omega^m_Y(X) \mid d\phi = \pi_{m-1}(\omega)/\{(d\phi, 0) \mid \phi \in \mathcal{S}^{m-2}(X, \mathbb{R}(m - 1))\}.$$ 

Now denote by $T^m(X(\mathbb{C}), \mathbb{R}(t))$ the space of linear forms on $\mathcal{S}^m(X(\mathbb{C}), \mathbb{R}(t))$ that are continuous for the $C^\infty$ topology. Elements of $T^m(X(\mathbb{C}), \mathbb{R}(t))$ are called currents of degree $m$ on $X(\mathbb{C})$ with values in $\mathbb{R}(t)$. Differential of forms induces a differential $T^m(X(\mathbb{C}), \mathbb{R}(t)) \to T^{m-1}(X(\mathbb{C}), \mathbb{R}(t))$ and the cohomology of the complex defined in this way is the singular cohomology $H^m(X(\mathbb{C}), \mathbb{R}(t))$.

**Proposition 3.13.** Let $H^m_D(X, \mathbb{R}(t))$ be the $m$-th Deligne homology space of $X$ with values in $\mathbb{R}(t)$ ([loc. cit.] Def. 1.9). Then we have a natural isomorphism $H^{m+2d}_D(X, \mathbb{R}(t + d)) \simeq H^m_D(X, \mathbb{R}(t))$ and a natural isomorphism

$$H^m_D(X, \mathbb{R}(t)) \simeq \{(S, T) \in T^{m-1}(X(\mathbb{C}), \mathbb{R}(t - 1)) \oplus F^tT^m_Y(X) \mid dS = \pi_{t-1}T\}/\{(\tilde S, \tilde T)\}.$$ 

**Proof.** The first isomorphism is proven in [loc. cit.] Th. 1.15 and the second in [31] Lem. 6.3.9. □

As a consequence we have an explicit description of the Gysin morphism in Deligne cohomology.

**Lemma 3.14.** Let $\iota : Z \to X \in Sm(Q)$ a closed imbedding purely of codimension $c$. Represent Deligne cohomology classes by currents via proposition 3.13. Then the Gysin morphism $H^m_D(X, \mathbb{R}(t)) \to H^{m+2c}_D(X, \mathbb{R}(t + c))$ is given by $(S, T) \mapsto (\iota_!S, \iota_!T)$.

Choose an orientation of the complex manifold $X(\mathbb{C})$. For $\eta \in \mathcal{S}^m(X(\mathbb{C}), \mathbb{R}(t))$ we denote by $T_\eta \in T^{m-2d}(X(\mathbb{C}), \mathbb{R}(t - d))$ the current defined by $\alpha \mapsto 1/(2\pi)^d f \alpha \wedge \eta$. Similarly we denote by $T : (\Omega^I_v(X), F^\circ) \to (T^m_Y(X(\mathbb{C})), [-2d], F^\circ - d)$ the natural inclusion ([29] Lem. 1.2) of filtered complexes. For $t = m + d$, the isomorphism (i) of proposition 3.13 is then given via the explicit description of each member by

$$(\phi, \omega) \in \mathcal{S}^{m+2d-1}(X, \mathbb{R}(t + d - 1)) \oplus \Omega^{m+2d}(X) \mapsto (T_\phi, T_\omega) \in T^{m-1}(X(\mathbb{C}), \mathbb{R}(t - 1)) \oplus F^tT^m_Y(X(\mathbb{C})).$$

**Lemma 3.15.** Represent Deligne cohomology classes by currents as in proposition 3.13 (i) and (ii). The application mapping a closed form $\eta \in \mathcal{S}^m(X, \mathbb{R}(t))$ to the class $(T_\eta, 0)$ induces a morphism

$$H^m_D(X, \mathbb{R}(t)) \longrightarrow H^{m+1}_D(X, \mathbb{R}(t + 1)).$$

**Proof.** The current $T_\eta \in T^{m-2d}(X(\mathbb{C}), \mathbb{R}(t - d))$ satisfies $dT_\eta = 0$ hence defines a deligne homology class $(T_\eta, 0) \in H^m_D(X, \mathbb{R}(m + 1 - d)) = H^{m+1}_D(X, \mathbb{R}(m + 1))$ (proposition 3.13), which is zero when $\eta$ is exact. □

Higher regulators, periods and special values...
Finally we give the explicit formula for the external cup-product in Deligne cohomology.

**Proposition 3.16.** Let \( X_1, X_2 \in \text{Sm}(\mathbb{Q}) \) and let \( p_i : X_1 \times X_2 \to X_i \) be the projections. Then the external cup-product \( H^m_D(X_1, \mathbb{R}(t)) \otimes H^m_D(X_2, \mathbb{R}(t')) \to H^{m+m'}_D(X_1 \times X_2, \mathbb{R}(t+t')) \) is given via the isomorphism of proposition 3.12 by

\[
(\phi, \omega) \otimes (\phi', \omega') \mapsto (p_{X_1}^* \phi \wedge p_{X_2}^* \pi_m \omega' + (-1)^mp_{X_1}^* \pi_m \omega \wedge p_{X_2}^* \phi')^m \wedge p_{X_1}^* \omega' \wedge p_{X_2}^* \omega').
\]

**Proof.** This follows from the formula for the internal cup-product [17] 2.5.1.

### 3.5 The Rankin-Selberg integral

#### 3.5.1 The Eisenstein classes in Deligne cohomology

We begin by writing down the real analytic Eisenstein series describing the Eisenstein symbol in Deligne cohomology. Then we obtain a description of the image under the regulator in Deligne cohomology of the Eisenstein classes (2.5.2) by means of currents.

**Proposition 3.17.** [11] Prop. 2.5.4. Let \( n \geq 0 \) be an integer. Denote by \( v^\pm \in g_2 \) the vector corresponding to \( Z = \frac{1}{t} \) in (1.1.2). There exists a family \( (\phi^n_r)_{r=n \mod 2} \) of functions \( \phi^n_r \) in \( \text{ind}^{G_2(\mathbb{R})}(\lambda(n,-1)) \) such that

(i) the function \( \phi^n_r \) is of weight \( \lambda(r, n - 2) \),

(ii) we have \( v^+ \phi^n_r = \begin{cases} \frac{n+r}{2} \phi^n_{r+2} & \text{if } r \neq -n \\ 0 & \text{if } r = -n. \end{cases} \)

and \( v^- \phi^n_r = \begin{cases} \frac{n-r}{2} \phi^n_{r-2} & \text{if } r \neq n \\ 0 & \text{if } r = n. \end{cases} \)

(iii) we have \( \tilde{\phi}^n_r = \phi^n_{-r} \).

Let \( \eta_\infty \otimes \eta_f \) be an algebraic Hecke character of \( T_2 \) of type \( \lambda(m - 2, -1) \). Let \( \phi = \phi_\infty \otimes \phi_f \) be a function in \( \text{ind}^{G_2(A)}(\lambda) \) with \( \phi_\infty = \phi^{m+2}_{m+2} \) and such that \( \phi_f \) has values in \( \overline{Q} \). Then the map

\[
\omega^\pm_m(\phi_f) : v^\pm \mapsto (2i\pi)^{m+1} a^m_{\frac{1}{2}(m \pm m)} \otimes \phi^{m+2}_{m \pm m} \otimes \phi_f
\]

where \( a^m_{\frac{1}{2}(m \pm m)} \) are the vectors of \( \text{Sym}^m \overline{V}_2 \otimes \mathbb{C} \) defined in lemma [12] is a vector valued differential form \( \omega^\pm_m \in \text{Hom}_{\mathbb{K}_2}(g_2/t_2, \text{Sym}^m \overline{V}_2 \otimes \text{ind}^{G_2(A)}(\lambda)) \) and we have \( \overline{\omega^\pm_m(\phi_f)} = (-1)^m \omega^\pm_m(\phi_f) \).

Furthermore the function

\[
\Theta_m(\phi_f) = \frac{(2i\pi)^{m+1}}{2(m+1)} \sum_{j=0}^m a^m_j \otimes \phi^{m+2}_{2j-m} \otimes \phi_f \in \text{Hom}_{\mathbb{K}_2}(\Lambda^0 g_2/t_2, \text{Sym}^m \overline{V}_2 \otimes \text{ind}^{G_2(A)}(\lambda)).
\]

satisfies \( \Theta_m(\phi_f) = (-1)^m \Theta_m(\phi_f) \) and \( d\Theta_m(\phi_f) = \pi_m \omega^\pm_m(\phi_f) \) where \( \pi_m : \mathbb{C} \to \mathbb{R}(m) = (2i\pi)^m \mathbb{R} \) denotes the projection \( z \mapsto \frac{1}{2}(z + (-1)^m z) \) ([31] Lem. 6.3.3).

We define the Eisenstein series

\[
E_m(\phi_f) = \sum_{\gamma \in B_2(\mathbb{Q}) \backslash G_2(\mathbb{Q})} \gamma^* \Theta_m(\phi_f) \in \mathcal{A}^0(\mathbb{M}, (\text{Sym}^m \overline{V}_2 \otimes \mathbb{R})(m))
\]

\[
E'_m(\phi_f) = \sum_{\gamma \in B_2(\mathbb{Q}) \backslash G_2(\mathbb{Q})} \gamma^* \omega^\pm_m(\phi_f) \in \mathcal{A}^1(\mathbb{M}, \text{Sym}^m \overline{V}_2 \otimes \mathbb{C}).
\]
As $m > 0$, they are absolutely convergent ([28] Prop. 19.3), hence satisfy $dE_m(\phi_f) = \pi_m E'_m(\phi_f)$. Identify $A^0(M, \text{Sym}^m V_{2\mathbb{R}})\mathbb{C})$ and $A^1(M, \text{Sym}^m V_{2\mathbb{C}})$ with direct factors of $A^m(E^m, \mathbb{R}(m))$ and $A^{m+1}(E^m, \mathbb{C})$ respectively, as in the proof of proposition [31] (6.3.5) the form $E_m(\phi_f)$ has logarithmic poles along a toroidal compactification of $E^m$. So via the isomorphism of proposition [3.12] the data $\text{Eis}_m^D(\phi_f) = (E_m(\phi_f), E'_m(\phi_f))$ defines a class in $H^{m+1}_D(E^m, \mathbb{R}(m + 1))$.

**Proposition 3.18.** [37] 6.3. Let $r_D : H^{m+1}(E^m, \mathbb{Q}(m + 1)) \to H^{m+1}_D(E^m, \mathbb{R}(m + 1))\mathbb{Q}$ be the regulator in Deligne cohomology and $\text{Eis}_m^D : B^0_m \to H^{m+1}_D(E^m, \mathbb{R}(m + 1))$ be the Eisenstein symbol. Then for every $\phi_f \in B^0_m$ we have $(r_D \circ \text{Eis}_m^D)(\phi_f) = \text{Eis}_m^D(\phi_f)$.

With this proposition and the explicit formulas for the external cup-product and the Gysin morphism given in section 3.4, we have the following description of the image of $\text{Eis}^{k,k'}$ ([25.2]) in Deligne cohomology.

**Proposition 3.19.** (i) Denote by $(P_k, k(\phi_f \otimes \phi_f'), P'_k, k(\phi_f \otimes \phi_f')) \in H^{k+k'+2}(E^k \times E^{k'}, \mathbb{R}(k+k'+2))\mathbb{Q}$ the class of the image of $\phi_f \otimes \phi_f'$ by the external cup-product

$$\text{Eis}_D^k \cup \text{Eis}_D^{k'} : B^0_k \otimes B^0_{k'} \to H^{k+k'+2}(E^k \times E^{k'}, \mathbb{R}(k+k'+2)).$$

Then

$$P_k(\phi_f \otimes \phi_f') = p^* E_k(\phi_f) \otimes p'^* \pi_{k-E'_k}(\phi_f) + (-1)^k p^* \pi_{k-E'_k}(\phi_f) \otimes p'^* E_{k'}(\phi_f'),$$

$$P'_k(\phi_f \otimes \phi_f') = p^* E'_k(\phi_f) \otimes p'^* E_{k'}(\phi_f'),$$

where $p$ and $p'$ denote the projections of ([25.3]).

(ii) Let $r_D : H^{k+k'}_{\mathbb{A}}(A^{k+k'}, \mathbb{Q}(k+k'+3)) \to H^{k+k'+4}(A^{k+k'}, \mathbb{R}(k+k'+3))\mathbb{Q}$ be the regulator in Deligne cohomology. For $\phi_f \otimes \phi_f' \in B^0_k \otimes B^0_{k'}$ let

$$\text{Eis}^{k,k'}_D(\phi_f \otimes \phi_f') = (r_D \circ \text{Eis}^{k,k'})(\phi_f \otimes \phi_f') \in H^{k+k'+4}(A^{k+k'}, \mathbb{R}(k+k'+3)).$$

Then via the isomorphism of proposition [3.14] we have

$$\text{Eis}^{k,k'}_D(\phi_f \otimes \phi_f') = (\tau, T_{p'} \circ P_k, (\phi_f \otimes \phi_f'), \tau \circ T_{p'} \circ P'_k(\phi_f \otimes \phi_f')).$$

**Proof.** This follows from lemmas [3.14] [3.16] and the functoriality of the regulator.

### 3.5.2 Computation of the regulator

In concrete terms, the class $\sigma$ of lemma [3.19] is a harmonic vector valued differential forms associated to an automorphic form whose archimedean component is a non-holomorphic member of the discrete series $L$-packet $P(W^{k,k'}(-3))$. This association depends on choices that we have to make precise. In the following, we denote by $\pi'$ the representation $|\cdot|^{-k-k'-3}$.

Let $\pi_2$ and $I_2$ be the complex Lie algebras of $\Pi_2$ and $L_{2\infty}$ respectively. We have the decomposition $\pi_2/I_2 = (\mathfrak{g}_2^+ \oplus \mathfrak{g}_2^-) \oplus (\mathfrak{p}_2^+ \oplus \mathfrak{p}_2^-)$. Denote by $(x, y)$ the components of vectors of $\pi_2/I_2$ in this decomposition. Then $((v^+, 0), (0, v^+), (v^-, 0), (0, v^-))$ is a basis of $\pi_2/I_2$. Let $\mathfrak{q}_2 = \mathfrak{p}_2^+ \oplus \mathfrak{p}_2^- \subset \pi_2/I_2$. Then the imbedding $\iota : \Pi_2 \to G_4$ ([1.1.1]) induces an imbedding

$$\iota : \pi_2/I_2 = \mathfrak{q}_2^+ \oplus \mathfrak{q}_2^- \to \mathfrak{g}_4/t_4 = \mathfrak{p}_4^+ \oplus \mathfrak{p}_4^-$$
mapping \( q_2^+ \) to \( p_4^+ \).

**Lemma 3.20.** Consider the vector \( \overline{\partial} \in \mathfrak{t}_4 \) defined in Lemma 1.17 (ii). Then there is a unique highest weight vector \( \lambda_{2,1} \in \bigwedge^2 p_1^+ \otimes p_4^- \) such that \( \overline{\partial} \lambda_{2,1} = (0, v^+) \wedge (v^+, 0) \wedge (0, v^-) \). Furthermore we have \( \overline{\partial} \lambda_{2,1} = t[(0, v^+) \wedge (v^+, 0) \wedge (v^-, 0)] \) for some non zero complex number \( t \).

**Proof.** The representation \( \bigwedge^2 p_1^+ \otimes p_4^- \otimes \mathfrak{t}_4 \) is irreducible with highest weight \( \lambda'(3, -1, 0) \). By Lemma 1.11 (i), the vector \((0, v^+) \wedge (v^+, 0) \wedge (0, v^-)\) is of weight \( \lambda'(2, 0, 0) = \lambda'(3, -1, 0) + \lambda'(-1, 1, 0) \). As \( \lambda'(-1, 1, 0) \) can’t be written as the sum of two roots of \( \Delta - \Delta^+ \), the character \( \lambda'(2, 0, 0) \) has multiplicity one in \( \bigwedge^2 p_1^+ \otimes p_4^- \). Then the first statement follows from the fact that \( \overline{\partial} \) has weight \( \lambda'(-1, 1, 0) \) (Lemma 1.11 (iii)). As the weight of \((0, v^+) \wedge (v^+, 0) \wedge (v^-, 0)\) is

\[
\lambda'(0, 2, 0) = \lambda'(3, -1, 0) + 3\lambda'(-1, 1, 0),
\]

the second statement follows by the same argument. \( \square \)

**Remark.** It’s not necessary to specify \( t \) because it will disappear in the following computations.

**Lemma 3.21.** Let \( \lambda_{2,1} \in \bigwedge^2 p_1^+ \otimes p_4^- \) be the vector given by the previous lemma. Let

\[
v_{-k, k'} = a_{k}^l \otimes a_{k'}^l \otimes 1 \in \text{Sym}^k \mathcal{V}_2(k) \otimes \text{Sym}^k \mathcal{V}_2(k') \cong \text{Sym}^k \mathcal{V}_2 \otimes \text{Sym}^k \mathcal{V}_2 \subset t^* W^{kk'}(-3).
\]

Assume that the archimedean component \( \pi_{\infty}^{\prime} \) is the element of \( P(W^{kk'}(-3)) \) whose minimal \( \mathbb{K}_4^{\infty} \)-type has highest weight \( \lambda'(k + 3, -k' - 1, -k - k') \). Denote by \( \pi'(k + 3, -k' - 1, -k - k') \subset \pi' \) the sub \( \mathbb{G}_4(\mathbb{A}_f) \)-module of forms whose weight is \( \lambda'(k + 3, -k' - 1, -k - k') \). For every automorphic form \( g \) in \( \pi'(k + 3, -k' - 1, -k - k') \) let \( \omega(g) \in \text{Hom}_{\mathbb{K}_4^{\infty}}(\bigwedge^2 p_1^+ \otimes p_4^-, W^{kk'}(-3) \otimes \pi_{\infty}^{\prime}) \) be the map defined by \( \omega(g) : \lambda_{2,1} \mapsto v_{-k, k'} \otimes g \). This defines a \( \mathbb{G}_4(\mathbb{A}_f) \)-equivariant embedding

\[
\omega : \pi'(k + 3, -k' - 1, -k - k') \longrightarrow H^3_{\text{dR}}(S, W^{kk'}(-3))
\]

mapped into \( H^3_{\mathbb{B}^1}(S, W^{kk'}(-3)) \) by the comparaison isomorphism.

**Proof.** Note that \( v_{-k, k'} \) has weight \( \lambda'(-k, k', k + k') \) (Lemma 1.2 (i)) hence that \( \omega(g) \) is well defined (Lemma 1.10). Then the statement is a reformulation of proposition 1.17. \( \square \)

Fix now a form \( g \in \pi' \) as in the previous lemma.

**Lemma 3.22.** (i) The only non zero values taken by \( \omega(g) \) on the basis of \( \bigwedge^3 \pi_2/\mathfrak{l}_2 \) deduced from the basis \( ((v^+, 0), (0, v^+), (v^-, 0), (0, v^-)) \) of \( \pi_2/\mathfrak{l}_2 \) are

\[
\omega(g)[(0, v^+)] = v_{-k, k'} \otimes \overline{\partial} g,
\]

\[
\omega(g)[(0, v^+)] = \frac{1}{t} v_{-k, k'} \otimes \overline{\partial} g.
\]

(ii) The only non zero values taken by \( F_{\infty} \omega(g) \) on the basis of \( \bigwedge^3 \pi_2/\mathfrak{l}_2 \) deduced from the basis \( ((v^+, 0), (0, v^+), (v^-, 0), (0, v^-)) \) of \( \pi_2/\mathfrak{l}_2 \) are

\[
F_{\infty} \omega(g)[(0, v^+)] = v_{-k', k} \otimes \overline{\partial} g,
\]

\[
F_{\infty} \omega(g)[(v^+, 0)] = w_{-k', k} \otimes \overline{\partial} g.
\]
where $v_{-k', k}, w_{-k', k} \in W^{k, k'}(-3)$ is a vector of weight $\lambda'(-k', k, k')$ and $\overline{g}$ and $\overline{g}'$ automorphic forms with archimedean component in the element of $P(W^{k, k'}(-3))$ of minimal $K_{4\infty}$-type of highest weight $\lambda'(k' + 1, -k - 3, -k - k')$.

Proof. (i) By definition (lemma 1.16) the form $\omega(g)$ vanishes outside the direct factor $\Lambda^3(p_+^* p_+^*)$ of

$$\Lambda^3(p_+^* p_+^*) = \Lambda^3 p_+^* \Lambda^2 p_+^* \Lambda^1 p_+^* = \Lambda^3 p_+^* \Lambda^3 p_+^* .$$

Hence it’s restriction to $\Lambda^3 \pi_2/\mathbb{Z}$ vanishes outside the direct factor $\Lambda^3 q_2^* q_2^*$ of $\Lambda^3 \pi_2/\mathbb{Z}$. We have $\omega(g)(\{0, v^+\} \cap (v^+, 0) \cap (0, v^-)) = \omega(g)[\overline{g} + v_{-k', k'} \otimes \overline{g}]$. As $v_{-k', k'}$ has weight $\lambda'(-k', k, k')$ and the weights of $W^{k, k'}(-3)$ are inside the convex hull of the orbit under $W_4$ of the dominant weight $\lambda(k, k', k')$ we have $\overline{g} = 0$. So $\omega(g)(\{0, v^+\} \cap (v^+, 0) \cap (0, v^-)) = v_{-k', k'} \otimes \overline{g}$. The identity $\omega(g)(\{0, v^+\} \cap (v^+, 0) \cap (0, v^-)) = v_{-k', k'} \otimes \overline{g}$ is proven in the same way. The last two identities follow from the first two by applying complex conjugation. (ii) As above we have $\overline{g} = 0$ so the second statement is proven in the same way than the first.

Remark. We don’t need to be more precise in the statement (ii) of the previous lemma because the terms $\overline{g}(\{0, v^+\} \cap (v^+, 0) \cap (0, v^-))$ will disappear in the following computation.

Proposition 3.23. Take $\sigma = \omega(g)$. Then for $\phi_f \otimes \phi_f' \in B^0 \otimes B^{0'}_k$ we have

$$\langle \sigma^+, \text{Eis}^{k, k'}_n(\phi_f \otimes \phi_f') \rangle \equiv \int_{\overline{g}(z) \in B^0_2(\mathbb{Q}) \otimes B^{0'}_k(\mathbb{Q})} \text{Eis}^{k, k'}_n(\phi_f \otimes \phi_f')(x) \, dx.$$

Proof. Represent the Deligne cohomology classes in $H^3_{k'^{+} + k'^{+}}(A^{k + k'}; \mathbb{R}(k + k' + 3))$ by couples of currents as in proposition 3.13. Consider the composite map

$$\int_{\overline{g}(\zeta) \in B^0_2(\mathbb{Q}) \otimes B^{0'}_k(\mathbb{Q})} \Gamma^*(\phi_f \otimes \phi_f')(x) \, dx.$$

By lemma 3.13 it maps a closed differential form $\alpha$ to the class $(T_\alpha, 0)$. Let $\eta(\phi_f \otimes \phi_f')$ be a closed form in $S^3(\mathcal{W}^{k, k'})$ lifting $\text{Eis}^{k, k'}_n(\phi_f \otimes \phi_f')$. With the notation of proposition 3.19 this means that $$(T_{\eta(\phi_f \otimes \phi_f')}, 0) = (\iota_* T_{\phi_f \otimes \phi_f'})(\phi_f \otimes \phi_f').$$

So there exists a current $S'$ such that $T_{\eta(\phi_f \otimes \phi_f')} = \iota_* T_{\phi_f \otimes \phi_f'}(\phi_f \otimes \phi_f') + dS'$. Hence

$$\langle \sigma^+, \text{Eis}^{k, k'}_n(\phi_f \otimes \phi_f') \rangle = \langle \iota^*(\omega(g) + F_{\infty}(g)), \eta(\phi_f \otimes \phi_f') \rangle$$

$$= \langle \iota^*(\omega(g) + F_{\infty}(g)), \iota_* T_{\phi_f \otimes \phi_f'} \rangle$$

$$= \frac{1}{2} \langle \iota^*(\omega(g)), p^* \phi_f \otimes \phi_f' \rangle + \frac{1}{2} \langle \iota^* F_{\infty}(g), p^* \phi_f \otimes \phi_f' \rangle.$$
Let $1 \in A^4 \pi_2/I_2$ be the generator $1 = (0, v^+) \land (v^+, 0) \land (v^-, 0) \land (0, v^-)$. By the explicit description of the pairing (3.2.2) we have
\[
\langle t^* \omega(g), p^* P_{k,k'}(\phi_f \otimes \phi_f') \rangle = \int_{\mathbb{Z}_2^2(A)\mathbb{H}_2(Q)/\mathbb{H}_2(A)} [t^* \omega(g) \land p^* P_{k,k'}(\phi_f \otimes \phi_f')(1)] dx
\]
where $[\ , ] : t^* W^{k,k'}(-3) \otimes t^* W^{k,k'}(-3) \to 1(-k-k')$ is the contraction. Denote by $S_4$ the symmetric group on the set of four elements and $\epsilon(\sigma)$ the signature of $\sigma \in S_4$. Write $e_1 = (0, v^+), e_2 = (v^+, 0), e_3 = (v^-, 0)$ and $e_4 = (v^-, v^-)$. Then
\[
(t^* \omega(g) \land p^* P_{k,k'}(\phi_f \otimes \phi_f'))(1) = \sum_{\sigma \in S_4} \epsilon(\sigma) \epsilon^* \omega(g)[e_{\sigma(1)} \land e_{\sigma(2)} \land e_{\sigma(3)}] \otimes p^* P_{k,k'}(\phi_f \otimes \phi_f')(e_{\sigma(4)})
\]
By lemma 3.22 (i) the sum reduces to
\[
t^* \omega(g)[(0, v^+) \land (v^+, 0) \land (v^-, 0)] \otimes p^* P_{k,k'}(\phi_f \otimes \phi_f')(v^-, 0)
\]
\[-t^* \omega(g)[(0, v^+) \land (v^+, 0) \land (v^-, 0)] \otimes p^* P_{k,k'}(\phi_f \otimes \phi_f')(0, v^-).
\]
By proposition 3.19 we have
\[
p^* P_{k,k'}(\phi_f \otimes \phi_f')(v^-, 0)
\]
\[
= \sum_{\gamma} \sum_{\gamma'} \gamma^* \Theta_k(\phi_f)(1) \otimes \gamma'^* (\omega^+_k(\phi_f') \land +(-1)^k \gamma^* (\omega^+_k(\phi_f) + \omega^-_k(\phi_f))(v^-) \otimes \gamma'^* \Theta_{k'}(\phi_f')(1)
\]
\[
= \frac{(2\pi)^{k+k'+2}}{2(k'+1)} \sum_{j=0}^{k'} \sum_{\gamma \in B_2(Q) \land G_2(Q)} \sum_{\gamma' \in B_2(Q) \land G_2(Q)} a_{j}^k \otimes a_{j}^{k'} \otimes \Gamma^* \phi^{k'_{-2} \otimes \phi_f} \gamma'^* [\phi^{k'_{-2} \otimes \phi_f}]
\]
Similarly
\[
p^* P_{k,k'}(\phi_f \otimes \phi_f')(0, v^-)
\]
\[
= \sum_{\gamma} \sum_{\gamma'} \gamma^* \Theta_k(\phi_f)(1) \otimes \gamma'^* (\omega^-_k(\phi_f') \land +(-1)^k \gamma^* (\omega^-_k(\phi_f) + \omega^-_k(\phi_f))(0) \otimes \gamma'^* \Theta_{k'}(\phi_f')(1)
\]
\[
= \frac{(2\pi)^{k+k'+2}}{2(k'+1)} \sum_{j=0}^{k'} \sum_{\gamma \in B_2(Q) \land G_2(Q)} \sum_{\gamma' \in B_2(Q) \land G_2(Q)} a_{j}^k \otimes a_{j}^{k'} \otimes \Gamma^* \phi^{k'_{-2} \otimes \phi_f} \gamma'^* [\phi^{k'_{-2} \otimes \phi_f}]
\]

By lemma 1.2 (iv) we have

\[
[a_k^l \otimes a_0^j, a_k^l \otimes a_j^k] \equiv \begin{cases} 1 & \text{if } l = 0 \text{ and } j = k' \\
0 & \text{else}. \end{cases}
\]

Hence it follows from lemma 3.22 (i) that

\[
[t^* \omega(g)](0, v^+) \wedge (v^+, 0) \wedge (0, v^-) \otimes p^* P_{k', k'} (\phi_f \otimes \phi'_f)(v^-, 0) \equiv \overline{\partial g} \sum_{\gamma \in B_2'(\Gamma)} \Gamma^*(\phi_{k-2}^j \phi_{k'}^k \otimes \phi_f \phi'_f),
\]

\[
[t^* \omega(g)](0, v^+) \wedge (v^+, 0) \wedge (v^-, 0) \otimes p^* P_{k', k'} (\phi_f \otimes \phi'_f)(0, v^-) \equiv 0.
\]

It remains to compute the term \([t^* F_{\infty}(\omega(g)) \wedge p^* P_{k', k'} (\phi_f \otimes \phi'_f)(\mathbf{1})]\). We claim that this term is zero. Indeed, it follows as above from lemma 3.22 (ii) that

\[
\lambda^* F_{\infty}(\omega(g)) \wedge p^* P_{k', k'} (\phi_f \otimes \phi'_f)(\mathbf{1})
\]

\[
= F_{\infty}(\omega(g))[(0, v^+) \wedge (0, v^-) \wedge (0, v^-)] \otimes p^* P_{k', k'} (\phi_f \otimes \phi'_f)(v^+, 0)
\]

\[
- F_{\infty}(\omega(g))[v^+, 0) \wedge (0, v^-) \wedge (v^-, 0)] \otimes p^* P_{k', k'} (\phi_f \otimes \phi'_f)(0, v^+).
\]

As above \(p^* P_{k', k'} (\phi_f \otimes \phi'_f)(v^+, 0)\) and \(p^* P_{k', k'} (\phi_f \otimes \phi'_f)(0, v^+)\) are vectors of

\[
\text{Sym}^k \bar{V}_2 \otimes \text{Sym}^{k'} \bar{V}_2 \otimes C^\infty(\Pi_2(\Gamma) \setminus \Pi_2(A)).
\]

As the vectors \(v_{k', k}\) and \(w_{-k', k}\) appearing in lemma 3.22 (ii) have weight \(\lambda'(-k', k, k + k')\) and as \(k > k'\), these vectors can’t be paired non trivially with vectors of \(\text{Sym}^k \bar{V}_2 \otimes \text{Sym}^{k'} \bar{V}_2\). So

\[
\langle \omega(g)^+, \text{Eis}_h^{k, k'}(\phi_f \otimes \phi'_f) \rangle = \frac{1}{2} \langle t^* \omega(g), p^* P_{k', k'} (\phi_f \otimes \phi'_f) \rangle
\]

\[= \int_{\mathcal{Z}_2(A) \setminus \Pi_2(\Gamma) / \Pi_2(\Gamma)} \overline{\partial g}(x) \sum_{\gamma \in B_2'(\Gamma) \setminus B_2(\Gamma)} \Gamma^*(\phi_{k-2}^j \phi_{k'}^k \otimes \phi_f \phi'_f)(x) \, dx.
\]

We are going to show that the Eisenstein series appearing in the above integral coincides with an half-integral value of the Eisenstein series defined and studied by Piatetski-Shapiro in [39], for some choice of data. To this end, let us recall some facts of [loc. cit.].

### 3.5.3 Piatetski-Shapiro’s integral

Let \(W^0 = \{ M \in M_2(\mathbb{Q}) \mid \tau M = M \} \) be the unipotent radical of the Siegel parabolic \(Q^0\) (2.1.4) and

\[
M^0 = Q^0 / W^0 = \mathbb{G}_m \times \text{GL}_2 = \left\{ \begin{pmatrix} \alpha A & 0 \\ 0 & \alpha A^{-1} \end{pmatrix} \mid \alpha \in \mathbb{G}_m, A \in \text{GL}_2 \right\}
\]

be the Levi. It acts on \(W^0\) by \((\alpha, A) \cdot \gamma = \alpha^t A \gamma A\). We say that a symmetric matrix \(\beta \in W^0\) is isotropic, resp. anisotropic, if the associated quadratic form is isotropic, resp. anisotropic, over \(\mathbb{Q}\). Fix an invertible matrix \(\beta \in W^0\).
Lemma 3.24. Let $D_\beta$ be the stabilizer of $\beta$ in $M^0$. There exists a 2 dimensional $\mathbb{Q}$-algebra $K_\beta$ such that $D_\beta \simeq \text{Res}_{K_\beta/\mathbb{Q}} \mathbb{G}_m$. Furthermore

$$K_\beta = \begin{cases} \text{a quadratic extension of } \mathbb{Q} \text{ if } \beta \text{ is anisotropic}, \\ \mathbb{Q} \times \mathbb{Q} \text{ else.} \end{cases}$$

Proof. For anisotropic $\beta$, one is reduced to $\beta = \begin{pmatrix} 1 & 0 \\ 0 & -\rho \end{pmatrix}$ where $\rho \in \mathbb{Q}^\times$ is not a square. Then $x + \sqrt{\rho} y \longrightarrow \left( (x^2 - \rho y^2)^{-1}, \begin{pmatrix} x & \rho y \\ y & x \end{pmatrix} \right)$ gives an isomorphism $\mathbb{Q}(\sqrt{\rho})^\times \simeq D_\beta$. For isotropic $\beta$, one is reduced to $\beta = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$. Then $(x, y) \longrightarrow \left( (xy)^{-1}, \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \right)$ gives an isomorphism $(\mathbb{Q} \times \mathbb{Q})^\times \simeq D_\beta$. \qed

Notation. We denote by $t \mapsto \overline{t}$ the natural involution of $K_\beta$ and by $l_\beta : W^0 \to \mathbb{G}_a$ the morphism $x \longrightarrow \text{tr}(\beta x)$. Let $R_\beta = W^0 \times D_\beta$, let $C_\beta$ be the center of $R_\beta$, let $N_\beta = \text{Ker} l_\beta \subset W^0$ and let $G_\beta$ be the fiber product

$$
\begin{array}{ccc}
G_\beta & \longrightarrow & \text{Res}_{K_\beta/\mathbb{Q}} \text{GL}_2 \\
\downarrow & & \downarrow \text{det} \\
\mathbb{G}_m & \longrightarrow & \text{Res}_{K_\beta/\mathbb{Q}} \mathbb{G}_m.
\end{array}
$$

Denote by $B'_\beta$ the Borel subgroup of upper triangular matrices and let $V_\beta = \text{Res}_{K_\beta/\mathbb{Q}} K^2$. Note that if $\beta$ is isotropic then $G_\beta$ is the group $\Pi_2 = G_2 \times \mathbb{G}_a \times G_2$.

Lemma 3.25. [39] Prop. 2.1 There is a closed imbedding $\iota_\beta : G_\beta \to G$ such that we have the identity $\iota_\beta(G_\beta) \cap R_\beta = N_\beta \times D_\beta$. If $\beta$ is isotropic we can choose $\iota_\beta = t$ (1.1.1).

Now let $\mu : \mathbb{Q}^\times \backslash \mathbb{A}^\times \longrightarrow \mathbb{C}^\times$ and $\nu : D_\beta(\mathbb{Q}) \backslash D_\beta(\mathbb{A}) \longrightarrow \mathbb{C}^\times$ be continuous characters. Consider the character $\chi_{\mu, \nu, s}$ given by

$$
\chi_{\mu, \nu, s} \left( \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{t} & 0 \\ 0 & t \end{pmatrix} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \right) = \mu(x)|x|^s \frac{1}{2} \nu(t)^{-1}(t). \tag{3.5.1}
$$

Then for any Schwartz-Bruhat function $\Phi$ on $V_\beta(\mathbb{A})$, the function on $G_\beta(\mathbb{A})$ defined by

$$
g \longrightarrow f^\Phi(g, \mu, \nu, s) = \mu(\text{det}g)|\text{det}g|^{s + \frac{1}{2}} \int_{K_\beta} \Phi((0, t)g)|t\tilde{t}|^{s + \frac{1}{2}} \mu(t)\nu(t) d^{\times} t
$$

belongs to the induced representation $\text{ind}_{B'_\beta(\mathbb{Q}) \backslash G_\beta(\mathbb{Q})}^{G_\beta(\mathbb{A})} \chi_{\mu, \nu, s}$, i.e.

$$
f^\Phi \left( \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{t} & 0 \\ 0 & t \end{pmatrix} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \mu, \nu, s \right) = \mu(x)|x|^s \frac{1}{2} \nu(t)^{-1}(t)f^\Phi(g, \mu, \nu, s).$$

Proposition 3.26. [39] Th. 5.1 For $s \in \mathbb{C}$ the Eisenstein series on $G_\beta(\mathbb{A})$ given by

$$E^\Phi(x, \mu, \nu, s) = \sum_{\Gamma \in B'_\beta(\mathbb{Q}) \backslash G_\beta(\mathbb{Q})} f^\Phi(\Gamma x, \mu, \nu, s)$$
is absolutely convergent for Res big enough and has a meromorphic continuation to \( \mathbb{C} \) satisfying the functional equation

\[
E^\Phi(x, \mu, \nu, s) = E^{\hat{\Phi}}(x, \mu^{-1} \nu^{-1}, \nu, 1 - s)
\]

where \( \hat{\Phi} \) is the Fourier transform of \( \Phi \) and \( \nu(a) = \nu(\bar{a}) \).

**Remark.** Note that the central character of \( E^\Phi(x, \mu, \nu, s) \) is \( \nu^{-1} \).

In order to relate the Eisenstein series appearing in proposition 3.23 to the preceding one, we assume from now on that \( \beta \) is the isotropic matrix \( \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} \).

**Lemma 3.27.** Denote by \( ((x_1, y_1), (x_2, y_2)) \) the coordinates of vectors in \( V_\delta(\mathbb{C}) = (\mathbb{C} \oplus \mathbb{C})^{\otimes 2} \). Let \( m, m' \) be two positive integers. For \( 0 \leq j \leq m \) and \( 0 \leq j' \leq m' \) let \( b_j^m(x_1, y_1) \) and \( b_{j'}^{m'}(x_2, y_2) \) be the polynomials of lemma 1.2 (a). Given a Schwartz-Bruhat function \( \Phi = \Phi \) where \( \Phi \) is the Fourier transform of \( \phi \), we have

\[
\Phi = \Phi \cdot \chi \cdot \phi
\]

where \( \chi \) is algebraic Hecke characters of \( \mathbb{G}_m \) of respective signs \( (-1)^k \) and \( (-1)^k \). Then

(i) The character \( \chi_{1, \alpha_1 \alpha_2, k + k' + 3 - 3/2} \) of \( B_\delta(\mathbb{A}) = B_2(\mathbb{A}) \times G_0(\mathbb{A}) B_2(\mathbb{A}) \) can be written

\[
\chi_{1, \alpha_1 \alpha_2, k + k' + 3 - 3/2} = \lambda \times \lambda'
\]

where \( \lambda \) and \( \lambda' \) are algebraic Hecke characters of \( T_2 \) of types \( \lambda(-k - 2, 1) \) and \( \lambda(-k' - 2, 1) \).

(ii) There exists a Schwartz-Bruhat function \( \Phi = \Phi_\infty \otimes \Phi_f \) on \( V_\beta(\mathbb{A}) \) and \( \phi \otimes \phi_f \in \mathcal{B}_k \otimes \mathcal{B}_f \) such that

\[
f^\Phi(1, 1, \alpha_1 \alpha_2, k + k' + 3 - 3/2) = \phi_{-k-2} \phi_{k'} \otimes \phi_f \phi_f.
\]

**Proof.** (i) We have \( \alpha_1 = \left| \right|^{-k} \alpha_1^0 \) and \( \alpha_2 = \left| \right|^k \alpha_2^0 \) where \( \alpha_1^0 \) and \( \alpha_2^0 \) are of finite order. Write

\[
\begin{pmatrix}
( 1 & b_1 \\
0 & d_1 \\
\end{pmatrix},
( 0 & b_2 \\
\end{pmatrix} =
\begin{pmatrix}
( a_1 d_2, a_2/d_1 \\
0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
( d_1, d_2 \\
0 & (d_1, d_2) \\
\end{pmatrix}
\begin{pmatrix}
( 1 & a_1 b_2 + b_1 d_2 \\
0 & 1 \\
\end{pmatrix}.
\]

By definition 3.5.1 we have
Consider the restriction $\lambda|_{T_2(\mathbb{R})}$. It is given by $\lambda|_{T_2(\mathbb{R})} \left( \begin{array}{c} a_1 \\ 0 \\ d_1 \end{array} \right) = |a_1|^{k+2}a_1d_1^{-1}sgn(d_1)^k$ where $sgn(d_1)$ is the sign of $d_1$. For $\left( \begin{array}{c} a_1 \\ 0 \\ d_1 \end{array} \right) \in T_2(\mathbb{R})^+$ we have $a_1d_1 > 0$, so we have the identity $\lambda \left( \begin{array}{c} a_1 \\ 0 \\ d_1 \end{array} \right) = a_1^{k+2}(a_1d_1)^{-1}$. This means that $\lambda$ is an algebraic Hecke character of type $\lambda(-k-2,1)$. Similarly $\lambda'$ is an algebraic Hecke characters of type $\lambda(-k'-2,1)$. (ii) For every $\Phi$ and every $\phi_f \otimes \phi_f'$ the functions $f^\Phi(-,1,\alpha_1\alpha_2,k+k'+3-3/2)$ and $\phi_{k-k'2}\phi_{k'}' \otimes \phi_f \phi_f'$ belong to $\text{ind}\mathcal{H}_2(\mathfrak{a})(\lambda \times \lambda')$. By the Iwasawa decomposition $\Pi_2(\mathbb{R}) = B_2(\mathbb{R})L_2^\infty$, the archimedean component of elements of $\text{ind}\mathcal{H}_2(\mathfrak{a})(\lambda \times \lambda')$ are characterized by their restriction to the torus $L_2^\infty$. Assume that $\Phi = \Phi^{\omega} \otimes \Phi_f$, then $f^{\Phi} = f^{\Phi^{\omega}} \otimes f^{\Phi_f}$. Lemma 3.27 shows that there exists $\Phi^{\omega}$, such that the archimedean component of $f^{\Phi^{\omega}}(-,1,\alpha_1\alpha_2,k+k'+3-3/2)$ is $\phi_{k-k'2}\phi_{k'}'$. If $\Phi_f$ has values in $\mathbb{Q}$, the function $f^{\Phi_f}(-,1,\alpha_1\alpha_2,k+k'+3-3/2)$ belongs to $\mathcal{B}_k^0 \otimes \mathcal{B}_k'$, and the conclusion follows.

We can now deduce our second main result.

**Theorem 3.29.** Assume that the de Rham cohomology class $\omega(g)$ associated to $g$ is rational in the sense of proposition 3.10 and the central character $\omega_{\pi'}$ of $\pi'$ satisfies $\omega_{\pi'} = \alpha_1\alpha_2$. Then there is a Schwartz-Bruhat function $\Phi = \Phi^{\omega} \otimes \Phi_f$ on $(V_2 \otimes V_2)(\mathfrak{a})$ whose non archimedean component $\Phi_f$ has values in $\mathbb{Q}$, and such that

$$\frac{\mathcal{R}(\pi_f,W^{kk'})}{\mathcal{D}(\pi_f,W^{kk'})} \equiv c^{(-1)^{k+k'+1}}(\pi_f,W^{kk'}) \int_{[Z_2(\mathfrak{a})(\Pi_2(\mathbb{Q})]/(\Pi_2(\mathfrak{a}))} \mathcal{D}g(x) E^{\Phi}(x,\alpha_1\alpha_2,k+k'+3-\frac{3}{2})dx.$$

**Proof.** Direct consequence of the identity 3.24(i), corollary 3.6, propositions 3.10, 3.24 and lemma 3.28 (ii).
In the following, we denote by $\phi$ the cusp-form $\overline{dg}$ and by $Z(\phi, \alpha_1 \alpha_2, s)$ the global zeta integral

$$Z(\phi, \alpha_1 \alpha_2, s) = \int_{[Z_2^0(\mathbb{A}) \backslash H_2(\mathbb{Q})] \backslash [H_2(\mathbb{A})]} \phi(x) E^\Phi(x, \alpha_1 \alpha_2, s) dx.$$ 

### 3.6 Bessel models, occult period invariants and proof of the main theorem

Beilinson’s conjecture predicts that the regulator is non zero. The previous theorem shows that the non-vanishing of $\mathcal{R}(\pi_f, W^{kk'})$ is equivalent to the the non vanishing of $Z(\phi, \alpha_1 \alpha_2, k + k' + 3 - \frac{3}{2})$. As we will see, this depends on the existence of a Fourier coefficient, more precisely of a Bessel model, relative to the isotropic matrix $\beta$, of the integrated form $\phi$. Assuming the existence of such a Bessel model, we expand the global integral $Z(\phi, \alpha_1 \alpha_2, k + k' + 3 - \frac{3}{2})$ in an Euler product of local integrals.

Fix a non trivial continuous character $\psi : \mathbb{Q} \backslash \mathbb{A} \to \mathbb{C}^\times$. For $\xi \in W^0(\mathbb{Q})$ let $\psi_\xi$ be the character $W^0(\mathbb{Q}) \backslash W^0(\mathbb{A}) \to \mathbb{C}^\times$ defined by $s \mapsto \psi_\xi tr(\xi s)$. Then the map $\xi \mapsto \psi_\xi$ identifies the discrete group $W^0(\mathbb{Q})$ with the Pontrjagin dual of the locally compact abelian group $W^0(\mathbb{Q}) \backslash W^0(\mathbb{A})$. By Fourier theory, the restriction to $W^0(\mathbb{A})$ of a form $\phi$ on $G_A$ has a Fourier expansion

$$\phi|_{W^0(\mathbb{A})} = \sum_{\xi \in W^0(\mathbb{Q})} \phi_\xi \psi_\xi$$

where $\phi_\xi$ is the Fourier coefficient

$$\phi_\xi = \int_{W^0(\mathbb{Q}) \backslash W^0(\mathbb{A})} \phi(n) \psi_\xi^{-1}(n) dn.$$ 

Now let $(\alpha_1, \alpha_2) : D_\beta(\mathbb{Q}) \backslash D_\beta(\mathbb{A}) \to \mathbb{C}^\times$ be the character $(x, y) \mapsto \alpha_1(x) \alpha_2(y)$ where $\alpha_1$ and $\alpha_2$ are the characters defined in lemma 3.28.

**Lemma 3.30.** [22] Proof of Th. 5.2, [20]. Let $W_\phi$ be the function on $G_A(\mathbb{A})$ defined by

$$x \mapsto W_\phi(x) = \int_{C_\beta(\mathbb{A}) D_\beta(\mathbb{Q}) \backslash D_\beta(\mathbb{A})} (\alpha_1, \alpha_2)(d) \int_{W^0(\mathbb{Q}) \backslash W^0(\mathbb{A})} \phi(n dx) \psi_\beta^{-1}(n) dn.$$ 

Then we have

$$Z(\phi, \alpha_1 \alpha_2, s) = \int_{D_\beta(\mathbb{A}) N_\beta(\mathbb{Q}) \backslash G_\beta(\mathbb{A})} W_\phi(x) f^\Phi(x, 1, \alpha_1 \alpha_2, s) dx.$$ 

We say that the representation $\pi'$ generated by $\phi$ has a Bessel model relative to $(\beta, \alpha_1 \alpha_2)$ if the function $W_\phi$ is non zero. The lemma shows that the existence of a Bessel model of $\pi'$ relative to $(\beta, \alpha_1 \alpha_2)$ is necessary for the non vanishing of $Z(\phi, \alpha_1 \alpha_2)$, i.e. of $\mathcal{R}(\pi_f, W^{kk'})$ (theorem 3.29), which is predicted by Beilinson’s conjecture. It is a classical fact that cuspidal Siegel modular forms don’t have such a Bessel model (II Th. 2.3.12 and [3]). Then it is remarkable that because of mixed Hodge theory, we were forced to choose a form $\phi$ whose archimedean part is generic (remark following lemma 3.30).
Hypothesis 2. From now on we assume that the representation $\pi'$ has a Bessel model relative to $(\beta, \alpha_1, \alpha_2)$. If we denote by $B_{\pi'}$ the $(g_4, K_{1, \infty}) \times G_4(\mathbb{A}_f)$-module generated by the function $W_\phi$, the existence of the Bessel model is equivalent to the fact that the $(g_4, K_{1, \infty}) \times G_4(\mathbb{A}_f)$-equivariant functional $l : \pi' \to B_{\pi'}$ mapping $\phi$ to $W_\phi$ is an isomorphism.

Now assume that the form $\phi$ and the Schwartz-Bruhat function $\Phi$ are factorizable; write $\phi = \bigotimes_v \phi_v$ and $\Phi = \bigotimes_v \Phi_v$. Then we have a corresponding factorization $f^\phi = \prod_v f^{\phi_v}$. By the unicity of local Bessel models ([39] Th. 3.1), the integral $Z(\phi, \alpha_1, \alpha_2, s)$ expands in a product of local integrals

$$Z(\phi, \alpha_1, \alpha_2, s) = \prod_v Z_v(\phi_v, \alpha_1, \alpha_2, s)$$

(3.6.1)

where

$$Z_v(\phi_v, \alpha_1, \alpha_2, s) = \int_{(D_\beta(Q_v)N_\beta(Q_v)) / G_\beta(Q_v)} W_{\phi_v}(x_v) f^{\phi_v}(x_v, 1, \alpha_1, \alpha_2, s) dx_v.$$

Proposition 3.31. Let $p$ be a finite place where $\pi'_p$, $\alpha_1_p$ and $\alpha_2_p$ are unramified and let $\Phi_p$ be the characteristic function of $V_\beta(\mathbb{Z}_p)$. Assume that $\phi_p \in \pi'_p$ is invariant under $G_4(\mathbb{Z}_p)$. Then

$$Z_p(\phi_p, \alpha_1_p, \alpha_2_p, s) = L(s, \pi'_p).$$

Proof. The representation $\pi'_p$ it is the unique irreducible subquotient of an unramified principal series representation $\text{ind}_{B_4(Q_p)}^{G_4(Q_p)} \chi_p$ ([13] Prop. 2.6), where $\chi_p$ is an unramified character of $T_4(\mathbb{Q}_p)$. Write

$$\xi_1 \xi_2 = \chi_p \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \xi_1 \xi_3 = \chi_p \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & p \end{pmatrix},$$

$$\xi_3 \xi_4 = \chi_p \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}, \quad \xi_2 \xi_4 = \chi_p \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then by [3] Lem. 4, the local L-factor at $p$ is

$$L(s, \pi'_p) = \frac{1}{(1 - \xi_1 \xi_2 p^{-s})(1 - \xi_1 \xi_3 p^{-s})(1 - \xi_2 \xi_4 p^{-s})(1 - \xi_3 \xi_4 p^{-s})}.$$
We have

$$D_\beta = \left\{ \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & x \end{pmatrix} \mid x, y \in \mathbb{G}_m \right\},$$

$$N_\beta = \left\{ \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \mid a, c \in \mathbb{G}_a \right\}$$

and $G_\beta = \Pi_2$. The function $f_{\beta_p}(x, 1, \alpha_{1p}, \alpha_{2p}, s)$ belongs to \text{ind}^{G_\beta(\mathbb{Q}_p)}_{B_\beta'(\mathbb{Q}_p)} \chi_{1, \alpha_{1p}, \alpha_{2p}, s}$. Hence it follows from the Iwasawa decomposition $G_\beta(\mathbb{Q}_p) = B_\beta'(\mathbb{Q}_p)G_\beta(\mathbb{Z}_p)$ that

$$Z_p(\phi_\beta, \alpha_{1p}, \alpha_{2p}, s) = f_{\beta_p}(1, 1, \alpha_{1p}, \alpha_{2p}, s) \int_{\mathbb{Q}_p^2} |x|^{-(s+\frac{1}{2})} W_{\phi_\beta} \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} d^4 x,$$

where

$$f_{\beta_p}(1, 1, \alpha_{1p}, \alpha_{2p}, s) = \int_{\mathbb{Q}_p^2} 1_{\mathbb{Z}_p \times \mathbb{Z}_p} (t_1, t_2)|t_1 t_2|^{s+\frac{1}{2}} \alpha_{1p}(t_1) \alpha_{2p}(t_2) d^4 t_1 d^4 t_2 = L(s + 1/2, \alpha_{1p}) L(s + 1/2, \alpha_{2p}).$$

For all $k \in K_p$ and $n \in N_\beta(\mathbb{Q}_p)$ the Bessel functionnal satisfies $W_{\phi_\beta}(n x k) = \psi_{\beta_p}(n) W_{\phi_\beta}(x)$. Write

$$u(x_1, x_2, x_3, x_4) = \begin{pmatrix} x_1 x_2 & 0 & 0 & 0 \\ 0 & x_1 x_3 & 0 & 0 \\ 0 & 0 & x_3 x_4 & 0 \\ 0 & 0 & 0 & x_2 x_4 \end{pmatrix}.$$

For every integer $m$ and every $n \in N_\beta(\mathbb{Q}_p)$ we have

$$W_{\phi_\beta}[u(p^{-m}, 1, 1, 1)] = W_{\phi_\beta}[u(p^{-m}, 1, 1, 1) n u(p^{-m}, 1, 1, 1)^{-1} u(p^{-m}, 1, 1, 1)] = \psi_{\beta_p}(u(p^{-m}, 1, 1, 1) n u(p^{-m}, 1, 1, 1)^{-1}) W_{\phi_\beta}[u(p^{-m}, 1, 1, 1)].$$

Now assume $m > 0$. In the additive group $N_\beta(\mathbb{Q}_p)$ we have

$$p^m u(p^{-m}, 1, 1, 1) n u(p^{-m}, 1, 1, 1)^{-1} = n.$$

Hence $\psi_{\beta_p}(u(p^{-m}, 1, 1, 1) n u(p^{-m}, 1, 1, 1)^{-1}) \neq 1$ for some $n$ and $W_{\phi_\beta}[u(p^{-m}, 1, 1, 1)] = 0$. So the integral to be computed becomes

$$Z_p(\phi_\beta, \alpha_{1p}, \alpha_{2p}, s) = L(s + 1/2, \alpha_{1p}) L(s + 1/2, \alpha_{2p}) \sum_{m \geq 0} p^{-m(s+1/2)} W_{\phi_\beta} \begin{pmatrix} p^m & 0 & 0 & 0 \\ 0 & p^m & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$
Let \( B = \sum_{w \in W} (-1)^{l(w)} w \) in the group algebra \( \mathbb{Z}[W] \), where \( l(w) \) denotes the length of \( w \) (see above theorem \[2.2\]). Write \( \beta_i = \alpha_{i,p}(p) \). By \[18\] Cor. 1.9 (2) we have

\[
W_{\phi_p} \left( \begin{pmatrix} p^m & 0 & 0 & 0 \\
0 & p^m & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \end{pmatrix} \right) = p^{\frac{1}{2}B[\zeta_1^{n+3}, \zeta_2, \zeta_3] \prod_{i=1,2} (1 - \zeta_i^4 \beta_p p^{-\frac{1}{2}})(1 - \zeta_i^4 \beta_p p^{-\frac{1}{2}})}
\]

(1 - p^{-1})B[\zeta_1^2, \zeta_2, \zeta_3].

Denote by \( \rho_n \) the irreducible representation \( \text{Sym}^n V_4 \) of \( G \). Then

\[
L(s, \pi'_p) = \sum_{n \geq 0} p^{-n} \text{tr} \rho_n(u(\xi_1, \xi_2, \xi_3, \xi_4)) = \sum_{n \geq 0} p^{-n} \frac{B[\zeta_1^{n+3}, \zeta_2, \zeta_3]}{B[\zeta_1^2, \zeta_2, \zeta_3]}
\]

where the second identity follows from the Weyl character formula (\[19\] Th. 24.2). According to \[20\], an elementary computation shows that

\[
Z_p(\phi_p, \alpha_{1,p} \alpha_{2,p}, s) = L(s, \pi'_p).
\]

In order to get rid of the non archimedean ramified integrals we refer to the following

**Lemma 3.32.** [\[21\]] Lem. 3.5.4. If \( W_{\phi_p} \) has algebraic values then \( Z_p(\phi_p, \alpha_{1,p} \alpha_{2,p}, k + k' + 3 - \frac{3}{2}) \in \overline{\mathbb{Q}} \).

We can chose \( \phi_p \) and \( \Phi_p \) such that \( Z_p(\phi_p, \alpha_{1,p} \alpha_{2,p}) \in \overline{\mathbb{Q}} \).

Recall that the cusp form \( \phi \) is \( \phi_\infty = g \) where \( g \in \pi' \) is a form whose archimedean part \( g_\infty \) is a vector of highest weight \( \lambda'(k + 3, -k' - 1, -k - k') \) in the minimal \( K_{4,M} \)-type of \( \pi'_\infty \) (lemma \[3.21\]), and \( \overline{\theta} \in \mathfrak{t}_1 \).

**Proposition 3.33.** [loc. cit.] Prop. 3.5.2. Let \( B_{\pi'}^{\text{arith}} \subset B_{\pi'} \) be the sub \( G_4(\mathbb{A}_f) \)-module of functions of the shape \( W_{g_{\infty}} W_f \) where \( W_f \) has values in \( \overline{\mathbb{Q}} \). Let

\[
\omega: \pi'(k + 3, -k' - 1, -k - k') \rightarrow H^3_{dR, f}(S, W^{k+k'}(-3))_{\mathbb{C}}
\]

be the \( G_4(\mathbb{A}_f) \)-equivariant imbedding of lemma \[3.27\]. Then there exists an element \( a(\pi', \beta, \alpha \alpha_2) \) of \( \mathbb{C} \otimes \overline{\mathbb{Q}} \) well defined up \( \equiv \) and such that for every \( g \in \pi'(k + 3, -k' - 1, -k - k') \) whose image \( \omega(g) \) is defined over \( \overline{\mathbb{Q}} \), we have \( l(g) \in a(\pi', \beta, \alpha \alpha_2) B_{\pi'}^{\text{arith}} \subset B_{\pi'} \).

We can now deduce our main result.

**Theorem 3.34.** Denote by \( Z_\infty \) the archimedean integral \( Z_\infty(\phi_\infty, \alpha_1, \alpha_2, k + k' + 3 - \frac{3}{2}) \). Let \( S \) be a finite set of ramified non-archimedean primes together with the infinite prime. Assume that the de Rham cohomology class \( \omega(g) \) associated to \( g \) is rational in the sense of proposition \[3.10\] and that the central character \( \omega_{\pi'} \) of \( \pi' \) satisfies \( \omega_{\pi'} = \alpha_1 \alpha_2 \). Then we have

\[
\mathcal{R}(\pi_f, W^{k+k'}) \equiv Z_\infty a(\pi', \beta, \alpha \alpha_2) e^{(-1)^{k+k'+1}(\pi_f, W^{k+k'})} L_S(-\frac{3}{2}, \pi_f) D(\pi_f, W^{k+k'}).
\]
Proof. By theorem 3.23 and the factorisation \(3.6.1\) we have

\[
\mathcal{R}(\pi_f, W^{k,k'}) \equiv c^{-1+k+k'+1}(\pi_f, W^{k,k'}) \prod_v Z_v(\phi_v, \alpha_1, \alpha_2, v, k + k' + 3 - \frac{3}{2}) \mathcal{D}(\pi_f, W^{k,k'}). \]

As \(\omega(g)\) is rational, it follows from proposition 3.33 that \(l(g) = a(\alpha_1, \alpha_2)W_{p,\infty} \prod_p W_{g_p}\) where each \(W_{g_p}\) has values in \(\mathbb{Q}\). Hence by proposition 3.31 and lemma 3.32 we have

\[
\mathcal{R}(\pi_f, W^{k,k'}) \equiv Z_\infty a(\alpha_1, \alpha_2) c^{-1+k+k'+1}(\pi_f, W^{k,k'}) L_S(k + k' + 3 - \frac{3}{2}, \pi') \mathcal{D}(\pi_f, W^{k,k'})
\]

and as \(\pi' = \tilde{\pi} \mid -k-k'-3\), we have \(L_S(k + k' + 3 - \frac{3}{2}, \pi') = L_S(-\frac{3}{2}, \tilde{\pi})\). \(\square\)

3.7 Gamma factor and compatibility of functional equations

3.7.1 Gamma factor

Let \(\Gamma(s)\) be the usual Gamma function and let \(\Gamma_C(s) = (2\pi)^{-s}\Gamma(s)\). By proposition 1.2.2 the Hodge decomposition of \(M_B(\pi_f, W^{k,k'})\) is

\[
M_B(\pi_f, W^{k,k'})_C = M_B(\pi_f, W^{k,k'}) \cong M_B^{0,-k-k'-3} \oplus M_B^{-1-k',-2-k} \oplus M_B^{-2-k,-1-k'} \oplus M_B^{-k-k'-3,0}.
\]

Our stability and multiplicity one hypothesis on \(\pi\) implies that \(h(p, q) = \dim M^\omega_{p,q} = 1\) for every \((p, q)\). By the rule of Serre [49] (25), the Gamma factor associated to \(M_B(\pi_f, W^{k,k'})\) is then

\[
L_\infty(s) = \prod_{p < q} \Gamma_C(s-p) h(p, q) = \Gamma_C(s + k + k' + 3) \Gamma_C(s + k + 2).
\]

3.7.2 Compatibility of functional equations

Theorem 3.34 relates vectors in the space of 1-extensions \(\text{Ext}^1_{MHS}(1, M_B(\pi_f, W^{k,k'}))\) to the L-value \(L_S(-\frac{3}{2}, \tilde{\pi}_f)\). To see that this corresponds to the L-value \(L_S(0, M_i(\pi_f, W^{k,k'}))\) predicted by Beilinson’s conjecture, we check the compatibility of automorphic and motivic functional equations.

For every prime number \(l\) the completed L-function

\[
\Lambda(s, M(\pi_f, W^{k,k'})) = L_\infty(s, M_B(\pi_f, W^{k,k'})) L(s, M_i(\pi_f, W^{k,k'}))
\]

conjecturally satisfies a functional equation

\[
\Lambda(s, M(\pi_f, W^{k,k'})) = \epsilon(s) \Lambda(1 - s, M(\pi_f, W^{k,k'})^\vee),
\]

where \(M(\pi_f, W^{k,k'})^\vee\) denotes the dual of \(M(\pi_f, W^{k,k'})\) ([38] 1.5). By theorem 3.35 we have an isomorphism \(M_i(\pi_f, W^{k,k'})^\vee \cong M_i(\pi_f, W^{k,k'}) \otimes M_i(\omega^0_{\pi_f})(w)\). Hence the functional equation becomes

\[
L(s, M_i(\pi_f, W^{k,k'})) = \epsilon(s) L(w + 1 - s, M_i(\pi_f, W^{k,k'}) \otimes M_i(\omega^0_{\pi_f})).
\]

As the archimedean component of \(\pi\) belongs to \(P(W^{k,k'})\), the central character of \(\pi\) is of the shape \(\omega_\pi = |\omega_\pi^0|\). The automorphic functional equation [39] Th. 5.3 gives

\[
L(s - \frac{3}{2}, \tilde{\pi}) = \epsilon(s) L(1 - s + \frac{3}{2}, \pi).
\]
As \( \pi \simeq \hat{\pi} \otimes \omega_{\pi} \) (\cite[Prop. 2.3]{Beilinson}) we have

\[
L(s - \frac{3}{2}, \hat{\pi}) = \epsilon(s) L(1 - s + \frac{3}{2}, \hat{\pi} \otimes \omega_{\pi}) \\
= \epsilon(s) L(-c + 1 - s + \frac{3}{2}, \hat{\pi} \otimes \omega_{\pi}^0) \\
= \epsilon(s) L(w + 1 - s - \frac{3}{2}, \hat{\pi} \otimes \omega_{\pi}^0).
\]

Hence the L-function \( L(s - \frac{3}{2}, \hat{\pi}) \) satisfies the conjectural functional equation of \( L(s, M_l(\pi_f, W^{k,k'})) \) and the L-value \( L(-\frac{3}{2}, \hat{\pi}) \) appearing in theorem \(
\ref{thm:main}
\) corresponds to the L-value predicted by Beilinson’s conjecture. We expect the archimedean integral \( Z_\infty \) of theorem \(
\ref{thm:main}
\) to coincide up to multiplication by powers of \( i \) and \( \pi \) with \( L_\infty(0) \), i.e. a non zero integer up to an integral power of \( 2\pi \).

Acknowledgments. The results of this paper were obtained during the preparation of my phd thesis under supervision of Jörg Wildeshaus. It’s a pleasure to thank him for his constant support. I would also like to thank Michael Harris for stressing the importance of the occult period invariant, and Masaaki Furusawa for sending me his unramified computations and allowing me to include them in this work. Parts of this work have been realized in the Centre de Recerca Matematica de Bellaterra during a stay financed by the European arithmetic geometry network, and in the Mathematisches Institut der Universität Bonn with a financement of the SFB "Periods, moduli spaces and geometry of algebraic varieties." I thank both institutions for their support.

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