Abstract—Let \( \mathbb{Z}_n \) be the ring of Gaussian integers modulo a positive integer \( n \). Very recently, Camarero and Martínez [IEEE Trans. Inform. Theory, 62 (2016), 1183–1192], showed that for every prime number \( p > 5 \) such that \( p \equiv \pm 5 \pmod{12} \), the Cayley graph \( G_p = \text{Cay}(\mathbb{Z}_p[i], S_2) \), where \( S_2 \) is the set of units of \( \mathbb{Z}_p[i] \), induces a 2-quasi-perfect Lee code over \( \mathbb{Z}_p \). In this paper, we solve this conjecture. Our main tools are Deligne’s bound from 1977 for estimating a particular kind of trigonometric sum and a result of Lovász from 1975 (or of Babai from 1979) which gives the eigenvalues of Cayley graphs of finite Abelian groups. Our proof techniques may motivate more work in the interactions between spectral graph theory, character theory, and coding theory, and may provide new ideas towards the famous Golomb–Welch conjecture on the existence of perfect Lee codes.

I. INTRODUCTION

The long-standing Golomb–Welch conjecture [7] states that there are no perfect Lee codes for spheres of radius greater than 1 and dimension greater than 2. Resolving this conjecture has been one of the main motivations for studying perfect and quasi-perfect Lee codes. Very recently, Camarero and Martínez [2], showed that for every prime number \( p > 5 \) such that \( p \equiv \pm 5 \pmod{12} \), the Cayley graph \( G_p = \text{Cay}(\mathbb{Z}_p[i], S_2) \), where \( S_2 \) is the set of units of \( \mathbb{Z}_p[i] \), induces a 2-quasi-perfect Lee code over \( \mathbb{Z}_p \), where \( m = 2 \left\lfloor \frac{1}{2} \right\rfloor \). They also conjectured that \( G_p \) is a Ramanujan graph for every prime \( p \) such that \( p \equiv 3 \pmod{4} \). In this paper, we solve this conjecture. Our main tools are Deligne’s bound from 1977 for estimating a particular kind of trigonometric sum and a result of Lovász from 1975 (or of Babai from 1979) which gives the eigenvalues of Cayley graphs of finite Abelian groups. Our proof techniques may motivate more work in the interactions between spectral graph theory, character theory, and coding theory, and may provide new ideas towards the famous Golomb–Welch conjecture on the existence of perfect Lee codes.

II. THE CAYLEY GRAPHS ASSOCIATED WITH SOME QUASI-PERFECT LEE CODES ARE RAMANUJAN GRAPHS

Let \( \mathbb{Z}_n \) be the ring of Gaussian integers modulo a positive integer \( n \). Very recently, Camarero and Martínez [IEEE Trans. Inform. Theory, 62 (2016), 1183–1192], showed that for every prime number \( p > 5 \) such that \( p \equiv \pm 5 \pmod{12} \), the Cayley graph \( G_p = \text{Cay}(\mathbb{Z}_p[i], S_2) \), where \( S_2 \) is the set of units of \( \mathbb{Z}_p[i] \), induces a 2-quasi-perfect Lee code over \( \mathbb{Z}_p \). In this paper, we solve this conjecture. Our main tools are Deligne’s bound from 1977 for estimating a particular kind of trigonometric sum and a result of Lovász from 1975 (or of Babai from 1979) which gives the eigenvalues of Cayley graphs of finite Abelian groups. Our proof techniques may motivate more work in the interactions between spectral graph theory, character theory, and coding theory, and may provide new ideas towards the famous Golomb–Welch conjecture on the existence of perfect Lee codes.

In other words, Gaussian integers are the lattice points in the Euclidean plane. The norm of a Gaussian integer \( w = x + yi \) is \( N(w) = |w|^2 = x^2 + y^2 \). The elements of \( \mathbb{Z}[i] \) with norm 1 are called the units of \( \mathbb{Z}[i] \); so, the units of \( \mathbb{Z}[i] \) are just \( \pm 1 \) and \( \pm i \). Similarly, the ring of Gaussian integers modulo a positive integer \( n \) is defined as

\[ \mathbb{Z}[i]/n\mathbb{Z}[i] = \{ a + bi : a, b \in \mathbb{Z}_n, i = \sqrt{-1} \}. \]

Note that the definition of norm (and so unit) in the ring \( \mathbb{Z}_n[i] \) is the same as that of \( \mathbb{Z}[i] \) except that we need to evaluate the norm modulo \( n \). That is, the norm of \( z = a + bi \in \mathbb{Z}_n[i] \) is \( N(z) = a^2 + b^2 \pmod{n} \), and \( z = a + bi \in \mathbb{Z}_n[i] \) is a unit of \( \mathbb{Z}_n[i] \) if and only if

\[ a^2 + b^2 \equiv 1 \pmod{n}. \]

The following classical result gives necessary and sufficient conditions under which the ring \( \mathbb{Z}_n[i] \) is a field; see, e.g., [6, Fact 3].

Proposition I.1. Let \( n > 1 \) be an integer. The ring \( \mathbb{Z}_n[i] \) is a field if and only if \( n \) is a prime and \( n \equiv 3 \pmod{4} \).

Let \( \Gamma \) be a group written in additive notation. A non-empty subset \( S \subseteq \Gamma \) is said to be symmetric if \( S = -S \), where \( -S = \{-x : x \in S\} \). In other words, \( S \) is symmetric if \( -x \in S \) whenever \( x \in S \). Now, we define Cayley graphs:

Definition I.2. Let \( \Gamma \) be a group, written additively, and \( S \) be a finite symmetric subset of \( \Gamma \) which does not contain the identity element of \( \Gamma \). The Cayley graph of \( \Gamma \) with respect to \( S \), denoted by \( G = \text{Cay}(\Gamma, S) \), is the graph whose vertex set is \( \Gamma \), and such that \( u \sim v \) if and only if \( v - u \in S \). Note that the Cayley graph \( G = \text{Cay}(\Gamma, S) \) is undirected, simple, \( |S| \)-regular, and vertex-transitive. Also, \( G \) is connected if and only if \( S \) generates \( \Gamma \).

Roughly speaking, an expander is a highly connected sparse graph, that is, every subset of its vertices has a large set of neighbours. An important special case, namely, Ramanujan graphs are also of great interest. These graphs are actually ‘optimal’ expanders, from the spectral point of view. Roughly speaking, a Ramanujan graph is a connected regular graph whose second largest eigenvalue in absolute value is ‘asymptotically’ the smallest possible (or, equivalently, whose spectral gap is ‘asymptotically’ the largest possible). Formally, a finite, connected, \( k \)-regular graph \( G \) is called a Ramanujan graph if every eigenvalue \( \lambda \neq \pm k \) of \( G \) satisfies the bound

\[ |\lambda| \leq 2\sqrt{k-1}. \]

To this date, there are only a few explicit constructions (which are useful for applications) of expanders and Ramanujan graphs, all given using several strong (and seemingly
unrelated!) mathematical tools; mainly from number theory. These graphs have a great deal of seminal applications in many disciplines such as computer science, cryptography, coding theory, and even in pure mathematics! See [4], [6], [12] for detailed discussions and surveys on expanders and Ramanujan graphs, their interactions with other areas like number theory and group theory, and their many wide-ranging applications.

Now, we review some basic facts about group characters; see, e.g., [9], [13] for more details. A character of a group \( \Gamma \) is a group homomorphism from \( \Gamma \) to the unit circle \( S^1 = \{ z \in \mathbb{C} : |z| = 1 \} \). So, if \( \Gamma \) is a finite group then a character of \( \Gamma \) can be defined as a group homomorphism from \( \Gamma \) to \( \mathbb{C}^* \), the multiplicative group of non-zero complex numbers. A group character is common, by an eigenvalue (resp., eigenvector) of a graph is real and symmetric, all these eigenvalues have the number of connected components of \( \Gamma \). Furthermore, if \( \Gamma \) is connected then \( \chi(\Gamma) = -\chi^{-1}(\Gamma) \) if \( \chi \) is a character of \( \Gamma \).

### II. Proof Ingredients and Techniques

In this section, we prove the conjecture proposed in [2], by showing that the Cayley graph \( G_p = \text{Cay}(\mathbb{Z}_p[i], S_2) \) is a \((p+1)\)-regular Ramanujan graph. First, we mention the proof ingredients. The following proposition lists some classical facts from spectral graph theory; see, e.g., [3]. As it is common, by an eigenvalue (resp., eigenvector) of a graph we mean an eigenvalue (resp., eigenvector) of the adjacency matrix of that graph.

**Proposition II.1.** Let \( G \) be a simple graph (i.e., without loops or multiple edges) of order \( n \), with the adjacency matrix \( A(G) \), and with the maximum degree \( \Delta(G) \). Also, let \( \lambda_{\min}(G) \) and \( \lambda_{\max}(G) \) denote, respectively, the smallest and the largest eigenvalues of \( G \). The following facts hold:

- The graph \( G \) has \( n \) eigenvalues (including multiplicities), and since \( A(G) \) is real and symmetric, all these eigenvalues are real.
- We have \( \lambda_{\max}(G) \leq \Delta(G) \). Furthermore, if \( G \) is \( k \)-regular then \( \lambda_{\max}(G) = k \), and for every eigenvalue \( \lambda \) of \( G \), \( |\lambda| \leq k \).
- If \( G \) is \( k \)-regular then the multiplicity of the eigenvalue \( k \) equals the number of connected components of \( G \). So, if \( G \) is \( k \)-regular then \( G \) is connected if and only if the eigenvalue \( k \) has multiplicity one.
- The graph \( G \) is bipartite if and only if its spectrum is symmetric about 0. Also, if \( G \) is connected then \( G \) is bipartite if and only if \( \lambda_{\min}(G) = -\lambda_{\max}(G) \).

It is well-known that the spectra of Cayley graphs of finite groups can be expressed in terms of characters of the underlying group ([1], [11]). The following result determines the eigenvalues and eigenvectors of Cayley graphs of finite Abelian groups. The theorem follows from a more general result of Lovász [11] from 1975 (or of Babai [1] from 1979).

**Theorem II.2.** Let \( \Gamma \) be a finite Abelian group, \( \chi : \Gamma \to \mathbb{C}^* \) be a character of \( \Gamma \), and \( S \) be a symmetric subset of \( \Gamma \) which does not contain the identity element of \( \Gamma \). Then the vector \( v_\chi = (\chi(g))_{g \in \Gamma} \) is an eigenvector of the Cayley graph \( G = \text{Cay}(\Gamma, S) \), with the corresponding eigenvalue being

\[
\lambda_\chi = \sum_{s \in S} \chi(s).
\]

In order to find the degree of the Cayley graph \( G_p = \text{Cay}(\mathbb{Z}_p[i], S_2) \), we need to evaluate the number of solutions of certain quadratic congruences. The problem of counting the number of solutions of quadratic congruences in several variables has been investigated, in a general form, in [14], where a general formula is proved. Specifically, Tóth [14] considered the quadratic congruence

\[
a_1 x_1^2 + \cdots + a_k x_k^2 \equiv b \pmod{n}, \tag{II.1}
\]

where \( b \in \mathbb{Z} \), \( a = (a_1, \ldots, a_k) \in \mathbb{Z}^k \), and proved an explicit formula (see Theorem II.3 below) for the number \( N_k(b, n, a) \) of solutions \((x_1, \ldots, x_k) \in \mathbb{Z}_n^k\) of (II.1), when \( n \) is odd. The formula involves a special kind of trigonometric sums, namely, quadratic Gauss sums that we now define. Let \( e(x) = \exp(2\pi i x) \) be the complex exponential with period 1. For positive integers \( m \) and \( n \) with \( \gcd(m, n) = 1 \), the quantity

\[
S(m, n) = \sum_{j=1}^{n} e \left( \frac{mj^2}{n} \right). \tag{II.2}
\]

is called a quadratic Gauss sum.

**Theorem II.3.** Let \( k, b, n \) be integers \((k, n \geq 1)\), and \( a = (a_1, \ldots, a_k) \in \mathbb{Z}_n^k \). We have

\[
N_k(b, n, a) = n^{k-1} \sum_{d \mid n} \frac{1}{d} \sum_{m=1}^{d} e \left( \frac{-bm}{d} \right) S(ma_1, d) \cdots S(ma_k, d).
\]

Putting \( k = 2, a_1 = a_2 = 1, b = 1, \) and \( n = p^r \) (a power of a prime) in Theorem II.3 the following special case is obtained (see [14]):

**Lemma II.4.** Let \( p \) be a prime and \( r \) be a positive integer. The number \( N_2(1, p^r) \) of solutions of the quadratic congruence \( x^2 + y^2 \equiv 1 \pmod{p^r} \) is

\[
N_2(1, p^r) = \begin{cases} 
p^r (1 - \frac{1}{p}), & \text{if } p \equiv 1 \pmod{4}, r \geq 1; 
p^r (1 + \frac{1}{p}), & \text{if } p \equiv 3 \pmod{4}, r \geq 1; 
2, & \text{if } p = 2, r = 1; 
2^{r+1}, & \text{if } p = 2, r \geq 2. 
\end{cases}
\]

If \( \mathbb{F} \) and \( \mathbb{E} \) are fields and \( \mathbb{F} \subseteq \mathbb{E} \), then \( \mathbb{E} \) is said to be an extension of \( \mathbb{F} \), denoted by \( \mathbb{E}/\mathbb{F} \). The degree of a field extension \( \mathbb{E}/\mathbb{F} \), denoted by \( [\mathbb{E} : \mathbb{F}] \), is the dimension of \( \mathbb{E} \) as a vector space over \( \mathbb{F} \). A field extension \( \mathbb{E}/\mathbb{F} \) is called a finite extension if \( [\mathbb{E} : \mathbb{F}] < \infty \). Let \( F_{p^n} \) be a finite extension field of the finite field \( \mathbb{F}_p \). For \( \alpha \in F_{p^n} \), the field norm of \( \alpha \) is defined by (see, e.g., [10], Def. 2.27)

\[
N_{F_{p^n}/\mathbb{F}_p}(\alpha) = \alpha^{(q^n-1)/(q-1)}.
\]
The elements of \( F_{q^n} \) with field norm 1 are called the units of \( F_{q^n} \).

**Lemma II.5.** Let \( p \) be a prime such that \( p \equiv 3 \pmod{4} \). Then for every \( z \in \mathbb{Z}_p[i] \) the field norm of \( z \) coincides with the norm of \( z \) in the usual sense, that is, as the norm of a Gaussian integer modulo \( p \).

**Proof.** Let \( z = a + bi \in \mathbb{Z}_p[i] \), where \( p \) is a prime and \( p \equiv 3 \pmod{4} \). Then, by the above definition, the field norm of \( z \) equals

\[
N_{\mathbb{Z}_p[i]/\mathbb{Z}_p}(a + bi) = (a + bi)^{p+1} = (a + bi)^2 = (a^2 + b^2) \pmod{p},
\]

where we have used Fermat’s little theorem and also the binomial theorem for commutative rings of characteristic \( p \) (see, e.g., [10, Th. 1.46]) which says that in a commutative ring \( R \) of prime characteristic \( p \), we have

\[(x + y)^p = x^p + y^p,
\]

for every \( x, y \in R \) and every positive integer \( n \). Note that the value \( a^2 + b^2 \pmod{p} \) is just the norm of \( z \) as a Gaussian integer modulo \( p \).

Deligne [5] using tools from algebraic geometry and cohomology proved the following crucial bound.

**Theorem II.6.** Suppose that \( F_{q^n}/F_q \) is the field extension of degree \( n \) of the finite field \( F_q \). \( S_n \) is the set of units of \( F_{q^n} \), and \( \chi \) is a nontrivial character of the additive group of \( F_{q^n} \). Then

\[
\left| \sum_{s \in S_n} \chi(s) \right| \leq nq^{\frac{n-1}{2}}.
\]

Now, we are ready to prove our main result. This problem has been mentioned as Conjecture 31 in [1].

**Theorem II.7.** Let \( p \) be a prime, \( p \equiv 3 \pmod{4} \), and \( S_2 \) be the set of units of \( \mathbb{Z}_p[i] \). Then the Cayley graph \( G_p = \text{Cay}(\mathbb{Z}_p[i], S_2) \) is a \((p+1)\)-regular Ramanujan graph.

**Proof.** By Proposition II.1 the ring \( \mathbb{Z}_n[i] \) is a field if and only if \( n \) is a prime and \( n \equiv 3 \pmod{4} \). Thus, for a prime \( p \) with \( p \equiv 3 \pmod{4} \) we have \( \mathbb{Z}_p[i] \cong \mathbb{F}_p^2 \). Also, we know that for a prime \( p \) with \( p \equiv 3 \pmod{4} \), \( \mathbb{Z}_p[i] \) as an extension field of the finite field \( \mathbb{F}_p \) has degree 2 (because \( \{1, i\} \) can serve as a basis), that is, \( \mathbb{Z}_p[i] : \mathbb{F}_p = 2 \).

Note that \( S_2 \) is a symmetric subset of \( \mathbb{Z}_p[i] \) and does not contain the identity element of \( \mathbb{Z}_p[i] \). Since the Cayley graph \( G_p = \text{Cay}(\mathbb{Z}_p[i], S_2) \) is of order \( p^2 \), it has \( p^2 \) real eigenvalues. Also, by Lemma II.4 the number of solutions of the quadratic congruence \( x^2 + y^2 \equiv 1 \pmod{p} \) is \( p + 1 \), so, \( |S_2| = p + 1 \) which means that \( G_p \) is \((p+1)\)-regular. By Theorem II.2 the eigenvalues of \( G_p \) are determined by

\[
\lambda_n = \sum_{s \in S_2} \chi(s),
\]

where \( \chi \) runs over all characters of \( \mathbb{Z}_p[i] \); note that since \( \mathbb{Z}_p[i] \), as an additive group, is a finite Abelian group, it has \( p^2 \) distinct characters. The eigenvalue corresponding to the trivial character \( \chi_0 \) of \( \mathbb{Z}_p[i] \) equals

\[
\lambda_0 = \sum_{s \in S_2} \chi_0(s) = \sum_{s \in S_2} 1 = |S_2| = p + 1.
\]

Of course, as \( G_p \) is \((p+1)\)-regular, we already knew, by Proposition II.1 that \( p + 1 \) is an eigenvalue of \( G_p \) (in fact, the largest one).

Note that since \( p \) is a prime and \( p \equiv 3 \pmod{4} \), by Lemma II.5 for every \( z \in \mathbb{Z}_p[i] \) the field norm of \( z \) coincides with the norm of \( z \) as a Gaussian integer modulo \( p \), thus, the ‘field norm’ (and so unit) in Theorem II.6 is in fact the ‘norm’ (and so unit) we already have. Now, by Theorem II.6 the absolute values of the eigenvalues corresponding to the nontrivial characters \( \chi \neq \chi_0 \) of \( \mathbb{Z}_p[i] \) satisfy the bound

\[
|\lambda_\chi| = \left| \sum_{s \in S_2} \chi(s) \right| \leq 2\sqrt{p}.
\]

Therefore, \( G_p \) is a \((p+1)\)-regular Ramanujan graph. We remark that since \( G_p \) is \((p+1)\)-regular and the eigenvalue \( p + 1 \) has multiplicity one, by Proposition II.1 \( G_p \) is connected. This in turn implies that \( S_2 \) generates \( \mathbb{Z}_p[i] \).

Since by the above argument, \((-p+1)\) is not an eigenvalue of \( G_p \), by Proposition II.1 we get:

**Corollary II.8.** The Cayley graph \( G_p = \text{Cay}(\mathbb{Z}_p[i], S_2) \) is not bipartite. This implies that \( G_p \) has at least one odd cycle.

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