Rotating Charged Black Strings in General Relativity

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Abstract

Einstein-Maxwell equations with a cosmological constant are analyzed in a four dimensional stationary spacetime admitting in addition a two dimensional group $G_2$ of spatial isometries. We find charged rotating black string solutions. For open black strings the mass ($M$), angular momentum ($J$) and charge ($Q$) line densities can be defined using the Hamiltonian formalism of Brown and York. It is shown through dimensional reduction that $M$, $J$ and $Q$ are respectively the mass, angular momentum and charge of a related three dimensional black hole. For closed black strings one can define the total mass, charge and angular momentum of the solution. These closed black string solutions have flat torus topology. The black string solutions are classified according to the mass, charge and angular momentum parameters. The causal structure is studied and some Penrose diagrams are shown. There are similarities between the charged rotating black string and the Kerr-Newman spacetime. The solution has Cauchy and event horizons, ergosphere, timelike singularities, closed timelike curves, and extremal cases. Both the similarities and differences of these black strings and Kerr-Newman black holes are explored. We comment on the implications these solutions might have on the hoop conjecture.

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I. Introduction

The theory of gravitational collapse and the theory of black holes are two distinct but linked subjects. From the work of Oppenheimer and Snyder [1] and Penrose’s theorem [2] we know that if General Relativity is correct, then realistic, slightly non-spherical, complete collapse leads to the formation of a black hole and a singularity. There are also studies hinting that the introduction of a cosmological constant does not alter this picture [3].

On the other hand, highly non-spherical collapse is not so well understood. The collapse of prolate spheroids, i.e. spindles, is not only astrophysically interesting but also important to a better understanding of both the cosmic censorship [4] and hoop [5] conjectures. Prolate collapse has been studied in some detail [5, 6, 7, 8] and it was shown that fully relativistic effects, totally different from the spherical case, come into play.

Collapse of cylindrical systems and other idealized models was used by Thorne to mimic prolate collapse [5]. This study led to the formulation of the hoop conjecture which states that horizons form when and only when a mass gets compacted into a region whose circumference in every direction is less than its Schwarzschild circumference, $4\pi GM$ (the velocity of light is equal to one in this paper). Thus, cylindrical collapsing matter will not form a
black hole. However, the hoop conjecture was given for spacetimes with zero cosmological constant. In the presence of a negative cosmological constant one can expect the occurrence of major changes. Indeed, we show in the present work that there are black hole solutions with cylindrical symmetry if a negative cosmological constant is present (a fact that does not happen for zero cosmological constant). These cylindrical black holes are also called black strings. We study the charged rotating black string and show that apart from spacetime being asymptotically anti-de Sitter in the radial direction (and not asymptotically flat) the black string solution has many similarities with the Kerr-Newman black hole. The existence of black strings suggests that they can form from the collapse of matter with cylindrical symmetry. This hint can indeed be confirmed as we next show when we take into account the three dimensional (3D) black hole of Bañados, Teitelboim and Zanelli (BTZ) [9].

In pure (zero cosmological constant) 3D General Relativity the theory of gravitational collapse has some features, which like cylindrical collapse in four dimensions (4D), do not resemble at all 4D spherical collapse. For instance, the radius of a static fluid 1-sphere does not depend on the pressure, it depends solely on the energy density of the fluid. In fact, in pure 3D Einstein’s gravity there is no gravitational attraction, and so there is no gravitational collapse [10, 11], essentially because the theory has no local
degrees of freedom and no Newtonian limit. Without cosmological constant there is no analogue in 3D of the Kerr and Schwarzschild black holes spacetimes. However, with a negative cosmological constant the situation changes. It has been shown that a black hole exists \cite{9, 12}. This is a surprising result, since the black hole spacetime has constant curvature. However, the solution is only locally anti-de Sitter, but globally has the topology of a black hole. The rotating case resembles the Kerr metric and the non-rotating case the Schwarzschild solution, although there is no polynomial singularity, only a causal singularity. It has also been shown \cite{13} that gravitational collapse in 3D with $\Lambda < 0$ leads to a black hole. This result connects the theory of gravitational collapse with the theory of black holes in 3D.

It is interesting to discuss further the common points between 4D cylindrical General Relativity and other 3D theories \cite{14, 15}. It is well known that straight infinite cosmic string dynamics and the dynamics of point particles in 3D General Relativity coincide. Thus, results in 3D point particle General Relativity can be directly translated into results in straight cosmic string theory (a special case of cylindrical symmetry in General Relativity \cite{16}). The 3D point particle is a solution of the the 3D theory given by Einstein’s action,

$$ S = \frac{1}{16\pi G} \int d^3x \sqrt{-g}(R - 2\Lambda) + S_f , $$

(1.1)

where $S_f$ is the action for other fields that might be present, $R$ is the curvature
scalar, $g$ is the determinant of the metric, $\Lambda$ is the cosmological constant and in the 3D case $G$ is usual taken as $G = \frac{1}{8}$, although for the moment we leave it free. The usual point particle solution appears for zero $\Lambda$ with appropriate matter stress-energy density. The action (1.1) with $\Lambda < 0$ yields the 3D BTZ black hole. Now the question that we can ask is how can we relate a 3D theory, in particular 3D Einstein’s gravity, with 4D cylindrical General Relativity. By the well known dimensional reduction procedure one is able to connect both theories. A 4D metric, $g_{\mu\nu}$ ($\mu\nu = 0, 1, 2, 3$), with one Killing vector can be written (in a particular instance) as,

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = g_{mn}dx^m dx^n + e^{-4\phi}dz^2,$$

(1.2)

where $g_{mn}$ and $\phi$ are metric functions, $m, n = 0, 1, 2$ and $z$ is the Killing coordinate. Equation (1.2) is invariant under $z \rightarrow -z$. A cylindrical symmetric metric can then be taken from (1.2) by imposing that the azimuthal coordinate, $\varphi$, also yields a Killing direction. Einstein-Hilbert action in 4D

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g}(R - 2\Lambda),$$

(1.3)

plus metric (1.2) give through dimensional reduction the following 3D theory:

$$S = \frac{1}{16\pi G} \int d^3x \sqrt{-g}e^{-2\phi}(R - 2\Lambda).$$

(1.4)

In equations (1.3) and (1.4) $R$ refers to the 4D and 3D Ricci scalars, respectively. In general $\phi$ is not zero (or a constant). However, the special case
\( \phi = 0 \) yields 3D General Relativity, given in equation (1.1). Thus, cylindrical symmetry in which the Killing direction, \( dz \), has got a constant as associated metric function, is related directly with 3D General Relativity. If the energy-momentum tensors in the 4D and 3D theories are also appropriately related then 4D cosmic strings and 3D point particles have the same dynamics. In the same way, by a careful choice of an appropriate energy-momentum tensor in 4D, the 3D BTZ black hole can be associated with a black string in 4D [17, 18]. Thus, one can now translate the results in 3D gravitational collapse with negative cosmological constant [11, 13] to 4D cylindrical symmetry and conclude that cylindrical collapse in a negative cosmological constant background produces black strings.

We consider here in this paper, the case in which the dilaton \( \phi \) is not a constant. We show that the theory also has black holes similar to the Kerr-Newman black holes, with a polynomial timelike singularity hidden behind the event and Cauchy horizons. When the charge is zero, the rotating solution does not resemble so much the Kerr solution, the singularity is spacelike hidden behind a single event horizon. In addition, in the non-rotating uncharged case, apart from the topology and asymptotics, the solution is identical to the Schwarzschild solution. The 4D black string metric has a corresponding 3D black hole solution. It is likely, in view of the arguments given above, that cylindrical collapse with appropriate matter in a \( \Lambda < 0 \) background will
produce the black strings discussed in this paper. We note that these black holes have a cylindrical or toroidal event horizon, in contrast with Hawking’s theorem [19] which states that the topology of the event horizon is spherical. Again, the presence of a negative cosmological constant alters the situation [20].

Cylindrical symmetry, as emphasized by Thorne [5], is an idealized situation. It is possible that the Universe we live in contains an infinite cosmic string. It is also possible, however less likely, that the Universe is crossed by an infinite black string. Yet, one can always argue that close enough to a loop string, spacetime resembles the spacetime of an infinite cosmic string. In the same way, one could argue that close enough to a toroidal finite black hole, spacetime resembles the spacetime of the infinite black string.

In section II we give the equations and the solutions. In section III we discuss and find the mass and angular momentum of the solutions. In section IV we study the causal structure of the charged rotating black string. In section V we study the causal structure of the uncharged rotating black string. In section VI we discuss other solutions. In section VII we study the geodesics. In section VIII we relate the 4D black strings with 3D black holes.
and in section IX we present the conclusions.

II. Equations and Solutions

We consider Einstein-Hilbert action in four dimensions with a cosmological term in the presence of an electromagnetic field. The total action is

\[ S + S_{em} = \frac{1}{16\pi G} \int d^4x \sqrt{-g}(R - 2\Lambda) - \frac{1}{16\pi} \int d^4x \sqrt{-g} F_{\mu\nu} F^{\mu\nu}, \tag{2.1} \]

where \( S \) was defined in (1.3), and \( R \) and \( g \) have also been defined in the previous section. The Maxwell tensor is

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{2.2} \]

\( A_\mu \) being the vector potential. We study solutions of the Einstein-Maxwell equations with cylindrical symmetry. By this we mean spacetimes admitting a commutative two dimensional Lie group \( G_2 \) of isometries. The topology of the two dimensional space generated by \( G_2 \) can be (i) \( R \times S^1 \), the standard cylindrically symmetric model, with orbits diffeomorphic either to cylinders or to \( R \) (i.e, \( G_2 = R \times U(1) \)), (ii) \( S^1 \times S^1 \) the flat torus \( T^2 \) model \( (G_2 = U(1) \times U(1)) \), and (iii) \( R^2 \). We will focus upon (i) and (ii). We then choose a cylindrical coordinate system \((x^0, x^1, x^2, x^3) = (t, r, \varphi, z)\) with \(-\infty < t < +\infty, 0 \leq r < +\infty, -\infty < z < +\infty, 0 \leq \varphi < 2\pi\). In the toroidal model (ii) the range of the coordinate \( z \) is \( 0 \leq \alpha z < 2\pi \). The electromagnetic four
potential is given by \( A_\mu = -h(r)\delta_\mu^0 \), where \( h(r) \) is an arbitrary function of the radial coordinate \( r \). Solving the Einstein-Maxwell equations yielded by (2.1) for a static cylindrically symmetric spacetime we find,

\[
ds^2 = -\left(\alpha^2 r^2 - \frac{b}{\alpha r} + \frac{c^2}{\alpha^2 r^2}\right) dt^2 - \frac{dr^2}{\alpha^2 r^2 - \frac{b}{\alpha r} + \frac{c^2}{\alpha^2 r^2}} + r^2 d\varphi^2 + \alpha^2 r^2 dz^2
\]

(2.3)

\[
h(r) = \frac{2\lambda}{\alpha r} + \text{const.}
\]

(2.4)

where \( \alpha^2 \equiv -\frac{1}{3} \Lambda, c^2 \equiv 4G\lambda^2 \) and \( b \) and \( \lambda \) are integration constants. It is easy to show, for instance using Gauss’s law, that \( \lambda \) is the linear charge density of the \( z \)-line, and \( b = 4GM \) with \( M \) being the mass per unit length of the \( z \)-line as we will show in the next section. Depending on the relative values of \( b \) and \( c \), metric (2.3) can represent a static black string. In the case there is a black string, an analysis of the Einstein-Rosen bridge (see e.g. [22]) of the solution (2.3) shows that spacetime is not simply connected which here implies that the first Betti number of the manifold is one, i.e., closed curves encircling the horizon cannot be shrunk to a point.

There is also a stationary solution that follows from equations (2.4) given by

\[
ds^2 = -\left[(\gamma^2 - \frac{\omega^2}{\alpha^2})\alpha^2 r^2 - \frac{\omega^2 b}{\alpha r} + \frac{\omega^2 c^2}{\alpha^2 r^2}\right] dt^2 - \frac{dr^2}{\alpha^2 r^2 - \frac{\omega^2 b}{\alpha r} + \frac{\omega^2 c^2}{\alpha^2 r^2}} + \gamma^2 r^2 d\varphi^2 + \omega^2 r^2 dz^2
\]

(2.5)

\[
A_\mu = -\gamma h(r)\delta_\mu^0 + \frac{\omega}{\alpha^2} h(r)\delta_\mu^2,
\]

(2.6)
where $\omega$ and $\gamma$ are constants, $h(r) = \frac{2\lambda}{\alpha r}$, and the coordinates have the same range as in the static case. Solution (2.5) can represent a stationary black string. If one compactifies the $z$ coordinate ($0 \leq \alpha z < 2\pi$) one has a closed black string. In this case, one can also put the coordinate $z$ to rotate. However, this simply represents a bad choice of coordinates. One can always find principal directions in which spacetime rotates only along one of these ($\varphi$, say) as in (2.5). This also follows from the fact that the first Betti number is one.

For an observer at radial infinity, the standard cylindrical spacetime model (with $R \times S^1$ topology) given by the metric (2.5) extends uniformly over the infinite $z$–line. Thus one expects that, as $r \to \infty$, the total energy as well as the total charge is infinite. The quantities that can be interpreted physically are the mass and charge densities, i.e., mass and charge per unit length of the string. In fact we have already found above the finite and well defined line charge density (of the $z$-line) as an integration constant in Einstein-Maxwell equations. For the close black string (the flat torus model with $S^1 \times S^1$) topology) the total energy and total charge are well defined quantities. In order to properly define such quantities we use the Hamiltonian formalism and the prescription of Brown and York [23].

Let us recall that spacetime of (2.5) is also asymptotically anti-de Sitter
whose metric we write here for later reference.

\[ ds^2 = -\left(\gamma^2 - \frac{\omega^2}{\alpha^2}\right)\alpha^2 r^2 dt^2 + \frac{dr^2}{\alpha^2 r^2} + \left(\gamma^2 - \frac{\omega^2}{\alpha^2}\right) r^2 d\varphi^2 + \alpha^2 r^2 dz^2. \]  

(2.7)

To have the usual form of the anti-de Sitter metric we choose \( \gamma^2 - \frac{\omega^2}{\alpha^2} = 1 \). This is also the background reference space, since metric (2.5) reduces to (2.7) if the black hole is not present.

**III. Mass, angular momentum and charge of the black string**

To use Brown and York formalism let us write the metric (2.5) in the suitable canonical form

\[ ds^2 = -N^0^2 dt^2 + R^2 (N^\varphi dt + d\varphi)^2 + \frac{dR^2}{f^2} + e^{-4\phi} dz^2, \]  

(3.1)

where

\[ N^0^2 = \left(\gamma^2 - \frac{\omega^2}{\alpha^2}\right)^2 \left(\alpha^2 r^2 - \frac{b}{\alpha r} + \frac{c^2}{\alpha^2 r^2}\right) \frac{r^2}{R}, \quad N^\varphi = -\frac{\omega^\varphi}{\alpha^2 R^2} \left(\frac{b}{\alpha r} - \frac{c^2}{\alpha^2 r^2}\right), \]

\[ R^2 = \gamma^2 r^2 - \frac{\omega^2}{\alpha^2} \left(\alpha^2 r^2 - \frac{b}{\alpha r} + \frac{c^2}{\alpha^2 r^2}\right), \quad f^2 = \left(\alpha^2 r^2 - \frac{b}{\alpha r} + \frac{c^2}{\alpha^2 r^2}\right) \left(\frac{dR}{dr}\right)^2, \]

\[ e^{-4\phi} = \alpha^2 r^2. \]  

(3.2)

In metric (3.1) \( N^0 \) and \( N^\varphi \) are respectively the lapse and shift functions. We then choose a region \( \mathcal{M} \) of spacetime bounded by \( R = \) constant and two space-like hypersurfaces \( t = t_1 \) and \( t = t_2 \). The surface \( t = \) constant, \( R = \) constant is the two-boundary \( B \) of the three-space \( \Sigma \). The three-boundary of
\( \mathcal{M} \) (\( ^3B \) according to Brown and York symbology) is in the present case the product of \( B \) with timelike lines \((R = \text{constant}, \varphi = \text{constant}, z = \text{constant})\). Then, \( B \) can be thought also as the intersection of \( \Sigma \) with \( ^3B \). Notice that, since we are now treating the standard cylindrical symmetric model with \( R \times S^1 \) topology, \( B \) is an infinite cylindrical surface with radius \( R \) and infinite total area. Then, in order to avoid infinite quantities during calculations we select just a finite region of such a surface, which we call \( B_z \). We choose \( B_z \) to be between \( z = z_1 \) and \( z = z_2 \). In the present case the metric \( \sigma_{ab} \) can be obtained from (3.1) by putting \( dt = 0 \) and \( dR = 0 \) (thus, \( a, b = 2, 3 \)), while the three-space metric \( h_{ij} \) is obtained by putting \( dt = 0 \) (thus, \( i, j = 1, 2, 3 \)). The two-metric on \( B \), \( \sigma_{ab} \), can also be viewed as a spacetime tensor \( \sigma_{\mu\nu} \) as well as a tensor on the three-space, \( \sigma_{ij} \).

Now we see that the metric given in equation (3.1) admits the two Killing vectors needed in order to define mass and angular momentum: a timelike Killing vector \( \xi^\mu_t = (\frac{\partial}{\partial t})^\mu \) and a spacelike (axial) Killing vector \( \xi^\mu_\varphi = (\frac{\partial}{\partial \varphi})^\mu \). Brown and York arrived at the following definition of a global charge \( Q_\xi \) related to any given Killing vector \( \xi^\mu \):

\[
Q_\xi = \int_B d^2x \sqrt{\sigma} (\epsilon u^\mu + j^\mu) \xi_\mu ,
\]

where \( u^\mu \) is the timelike future pointing normal to \( \Sigma \), \( \sigma \) is the determinant of \( \sigma_{ab} \), \( \epsilon \) and \( j^\mu = (0, j^i) \) are respectively the energy and momentum surface
density on $B$, and are given by:

\[
\epsilon = \frac{k}{8\pi G}, \quad j^i = \frac{\sigma^i n_k \Pi^{jk}}{16\pi G \sqrt{h}},
\]

(3.4)

where $n^k$ is the unit normal to the two-boundary $B$ on the three-space $\Sigma$, $k$ is the trace of the extrinsic curvature of $B$, $\Pi^{ij}$ is the conjugate momentum in $\Sigma$ and $h$ is the determinant of $h_{ij}$.

In (3.3) $\epsilon$ and $j^i$ are defined in such a way that they vanish for the background (reference) spacetime. The meaning of such reference spacetime can be seen as follows. Suppose one is interested in the energy (mass) of a segment of spacetime and tries to calculate it using Hamiltonian formalism with the gravitational action $S_1$ obtained from (3.1). The result contains also the contribution from the background spacetime which in our case is the anti-de Sitter spacetime whose metric is given by (2.7), or equivalently, by (3.1) with $b = 0$ and $c = 0$. Such a “background” spacetime energy does not vanish even in the absence of the black string, and it diverges at radial infinity. Then, in order to properly define the energy of the black string one must get rid of the contribution of the anti-de Sitter background spacetime to the total energy of the black string spacetime. Let us then define $S_0$ as the action related to the reference spacetime, while $S_1$ is the full gravitational action of the black string spacetime obtained from metric (3.1). Both, $S_1$ and $S_0$ are functionals of the lapse and shift functions suitable defined on the
boundary $^3B$ \cite{23, 24}. Thus, from $S_1$ we get the total surface energy density at a given point $P$ on the boundary $B$ as $\epsilon_1 = k_1/8\pi G$ where $k_1$ is the trace of the extrinsic curvature of $B$, obtained from (3.1), when the black string is present. In a similar way, from $S_0$ we get $\epsilon_0 = k_0/8\pi G$, the background surface energy density at the same point $P$ on $B$. $k_0$ is the trace of the extrinsic curvature obtained from (3.1) with $b = 0$ and $c = 0$.

Now, we see that the surface energy density $\epsilon = \epsilon_1 - \epsilon_0$ defined at each point on the boundary $B$ is in fact the contribution to energy density due only to the presence of the black string, since we are subtracting the contribution of the background reference space from the total energy density of the black string spacetime. Notice also that this “true” energy density (of the black string) $\epsilon$ may be obtained from the “true” gravitational action $S = S_1 - S_0$ defined on the boundary $^3B$.

It is all the same regarding to, for instance, the angular momentum of the black string spacetime. In order to properly define the angular momentum, the background surface momentum density $j^i_0$ must be subtracted from the “total” surface density of the black string spacetime $j^i_1$. Thus, in (3.3) we have to take, $\epsilon = (k_1 - k_0)/8\pi G$ and $j^i = j^i_1 - j^i_0$.

Then, equation (3.3) yields the following expressions for the total energy and angular momentum of a segment $\Delta z$ of the two-boundary, $B_z$, at radial infinity ($R \to \infty$), of the black string spacetime: $M_t = \frac{\Delta z}{8G} b \left( 2\gamma^2 + \frac{\omega^2}{\alpha^2} \right)$,
\( J_\varphi = \frac{3\Delta z}{8G} \frac{\gamma \omega}{\alpha^2} \). From these we define the mass and angular momentum line densities of the spacetime as,

\[
M \equiv \frac{M_t}{\Delta z} = \frac{b}{8G} \left( 2\gamma^2 + \frac{\omega^2}{\alpha^2} \right), \quad (3.5)
\]

\[
J \equiv \frac{J_\varphi}{\Delta z} = \frac{3b}{8G} \frac{\gamma \omega}{\alpha^2}. \quad (3.6)
\]

Then one can solve a quadratic equation for \( \gamma^2 \) and \( \frac{\omega^2}{\alpha^2} \). It gives two distinct solutions

\[
\gamma^2 = \frac{2GM}{b} + \frac{2G}{b} \sqrt{M^2 - \frac{8J^2\alpha^2}{9}} \quad ; \quad \frac{\omega^2}{\alpha^2} = \frac{4GM}{b} - \frac{4G}{b} \sqrt{M^2 - \frac{8J^2\alpha^2}{9}} \quad (3.7)
\]

\[
\gamma^2 = \frac{2GM}{b} - \frac{2G}{b} \sqrt{M^2 - \frac{8J^2\alpha^2}{9}} \quad ; \quad \frac{\omega^2}{\alpha^2} = \frac{4GM}{b} + \frac{4G}{b} \sqrt{M^2 - \frac{8J^2\alpha^2}{9}} \quad (3.8)
\]

We will concentrate on (3.7).

One can also work out the total energy \( M_c \) and angular momentum \( J_c \) for the closed black string. In terms of \( M \) and \( J \) they are given by \( M_c = \frac{2\pi}{\alpha} M \) and \( J_c = \frac{2\pi}{\alpha} J \).

Now we turn our attention to the electric charge of the rotating black hole. In the Hamiltonian formalism the canonical field variables are the spatial components of the vector potential (described by the 1-form \( A_\mu dx^\mu \)) and the conjugate momentum \( E^i \), while the component \( A_0 \) is the Lagrange multiplier for the Gauss-Law constraint. Notice that \( E^i \) can be viewed as a tensor in the three-space \( \Sigma \). The electromagnetic field gives rise to boundary
terms in the action so that the electric charge $Q_z$ is defined from them as the conjugate quantity to $A_0$ in the action [26]:

$$Q_z = \frac{1}{4\pi} \int_{B_z} d^2x \frac{n_i \mathcal{E}^i \sqrt{\sigma}}{\sqrt{h}}, \quad \mathcal{E}^i = \frac{R e^{-2\phi}}{N} \left( A_{0,R} - N^\sigma A_{2,R} \right), \quad (3.9)$$

where we have suppressed the term related to the reference space because it vanishes since $\mathcal{E}_0^i$ is identically zero. The quantity $n_i \mathcal{E}^i$ can also be viewed as a surface charge density on the two boundary $B$.

Then, using (3.9) and the vector potential given in (2.6) we find (recall that $B_z$ is the surface of a cylinder whose radius is $R \to \infty$ and height is $\Delta z$) $Q_z = \gamma \lambda \Delta z$. Again we see that the total charge is in fact infinite, but the line charge density of the string, which is the quantity that appears in the metric,

$$Q \equiv \frac{Q_z}{\Delta z} = \gamma \lambda, \quad (3.10)$$

is finite and well defined. The electric charge of the closed black string is $Q_c = \frac{2\pi}{\alpha} Q$.

Finally, using (3.7) and (3.10), the metric (2.5) assumes the form

$$ds^2 = -\left( \alpha^2 r^2 - \frac{2G(M+\Omega)}{\alpha r} + \frac{4GQ^2}{\alpha^4 r^2} \right) dt^2 - \frac{16GJ}{3\alpha r} \left( 1 - \frac{2Q^2}{(M+\Omega)\alpha r} \right) dt d\varphi + \left[ r^2 + \frac{4G(M-\Omega)}{\alpha^4 r} \left( 1 - \frac{2}{(M+\Omega)\alpha r} \right) \right] d\varphi^2 + \frac{dr^2}{\alpha^2 r^2} - \frac{2G(3\Omega-M)}{\alpha r} + \frac{4GQ^2}{\alpha^4 r^2} + \alpha^2 r^2 dz^2, \quad (3.11)$$
where

\[ \Omega = \sqrt{M^2 - \frac{8J^2\alpha^2}{9}}. \quad (3.12) \]

The asymptotic group of this metric, as \( r \to \infty \), is \( R \times \) conformal group in two dimensions [27].

IV. Causal Structure of the Charged Rotating Black String Spacetime

In order to study the metric and its causal structure it is useful to define the parameter \( a \) (with units of angular momentum per unit mass),

\[ a^2\alpha^2 \equiv 1 - \frac{\Omega}{M} \quad (4.1) \]

such that

\[ 1 + \frac{\Omega}{M} = 2\left(1 - \frac{a^2\alpha^2}{2}\right), \quad 3\frac{\Omega}{M} - 1 = 2\left(1 - \frac{3}{2}a^2\alpha^2\right). \quad (4.2) \]

The relation between \( J \) and \( a \) is given by

\[ J = \frac{3}{2}aM\sqrt{1 - \frac{a^2\alpha^2}{2}}. \quad (4.3) \]

The range of \( a \) is \( 0 \leq a\alpha \leq 1 \). From now on we put \( G = 1 \) (note that in [14, 15] we have normalized differently, with \( G = \frac{1}{8} \)). With these definitions the metric (3.11) assumes the form

\[ ds^2 = -\left(\alpha^2r^2 - \frac{4M(1-a^2\alpha^2)}{\alpha r^2} + \frac{4Q^2}{\alpha^2 r^2}\right)dt^2 + \]
\[-\frac{4aM}{\alpha r} \sqrt{1-\frac{\alpha^2}{r^2}} \left(1 - \frac{Q^2}{M(1-\frac{\alpha^2}{r^2})} \right) 2dt d\varphi + \]
\[+ \left(\alpha^2 r^2 - \frac{4M(1-\frac{\alpha^2}{r^2})}{\alpha r} + \frac{4Q^2}{\alpha r^2} \frac{(1-\frac{\alpha^2}{r^2})}{(1-\frac{\alpha^2}{r^2})} \right)^{-1} dr^2 + \]
\[+ \left[r^2 + \frac{4Ma^2}{\alpha r} \left(1 - \frac{Q^2}{(1-\frac{\alpha^2}{r^2})} \right) \right] d\varphi^2 + \alpha^2 r^2 dz^2. \quad (4.4)\]

In order to compare metric (4.4) with the well-known Kerr-Newman metric we write explicitly here the Kerr-Newman metric on the equatorial plane

\[ds^2 = -(1 - \frac{2m}{r} + \frac{e^2}{r^2})dt^2 + \]
\[- \frac{2ma}{r} (1 - \frac{e^2}{2mr}) 2dt d\varphi + \]
\[+ (1 - \frac{2m}{r} + \frac{a^2 + e^2}{r^2})^{-1} dr^2 + \]
\[+ \left[r^2 + a^2 \left(1 + \frac{2m}{r} - \frac{e^2}{r^2} \right) \right] d\varphi^2 + r^2 d\theta^2, \quad (4.5)\]

where \((m, a, e)\) are the Kerr-Newman parameters, mass, specific angular momentum and charge, respectively. We can now see that the metric for a rotating cylindrical symmetric spacetime asymptotically anti-de Sitter, given in (4.4), has many similarities with the metric on the equatorial plane for an axisymmetric rotating spacetime asymptotically flat given by the Kerr-Newman metric in (4.5). Of course, there are differences, and we will explore both the differences and similarities along this paper.

Metric (4.4) has a singularity at \(r = 0\). The Kretschmann scalar \(K\) is

\[K = 24\alpha^4 \left(1 + \frac{b^2}{2\alpha^6 r^6}\right) - \frac{48c^2}{\alpha^3 r^7} \left(b - \frac{7c^2}{6\alpha r}\right) \quad (4.6)\]
where $b$ and $c$ can be picked up from (2.5) and (4.4). Thus $K$ diverges at $r = 0$. The solution has totally different character depending on whether $r > 0$ or $r < 0$. The important black hole solution exists for $r > 0$ which case we analyze first.

**IV.1 $r \geq 0$ (or $M > 0$)**

To analyze the causal structure and follow the procedure of Boyer and Lindquist [28] and Carter [29] we put metric (1.4) in the form,

$$ds^2 = -\Delta \left( \gamma dt - \frac{\omega}{\alpha^2} d\varphi \right)^2 + r^2 \left( \gamma d\varphi - \omega dt \right)^2 + \frac{dr^2}{\Delta} + \alpha^2 r^2 dz^2,$$

(4.7)

where now,

$$\Delta = \alpha^2 r^2 - \frac{b}{\alpha r} + \frac{c^2}{\alpha^2 r^2},$$

(4.8)

$$b = 4M \left( 1 - \frac{3}{2} a^2 \alpha^2 \right),$$

(4.9)

$$c^2 = 4Q^2 \left( \frac{1 - \frac{3}{2} a^2 \alpha^2}{1 - \frac{3}{2} a^2 \alpha^2} \right),$$

(4.10)

$$\gamma = \sqrt{\frac{1 - \frac{3}{2} a^2 \alpha^2}{1 - \frac{3}{2} a^2 \alpha^2}},$$

(4.11)

$$\omega = \frac{a \alpha^2}{\sqrt{1 - \frac{3}{2} a^2 \alpha^2}}.$$  

(4.12)

There are horizons whenever

$$\Delta = 0,$$  

(4.13)
i.e., at the roots of $\Delta$. One knows that the non-extremal situations in the Kerr-Newman metric are given by $0 \leq \frac{a^2}{m^2} \leq 1 - \frac{e^2}{m^2}$. Here, to have horizons one needs either one of the two conditions:

$$0 \leq a^2 \alpha^2 \leq \frac{2}{3} - \frac{128}{81} \frac{Q^6}{M^4(1 - \frac{1}{2}a^2 \alpha^2)^3},$$

or

$$\frac{2}{3} < a^2 \alpha^2 \leq 1. \quad (4.15)$$

Thus there are five distinct cases depending on the value of the charge and angular momentum: (i) $0 \leq a^2 \alpha^2 \leq \frac{2}{3} - \frac{128}{81} \frac{Q^6}{M^4(1 - \frac{1}{2}a^2 \alpha^2)^3}$, which yields the black hole solution with event and Cauchy horizons. (ii) $a^2 \alpha^2 = \frac{2}{3} - \frac{128}{81} \frac{Q^6}{M^4(1 - \frac{1}{2}a^2 \alpha^2)^3}$, which corresponds to the extreme case, where the two horizons merge. (iii) $\frac{2}{3} - \frac{128}{81} \frac{Q^6}{M^4(1 - \frac{1}{2}a^2 \alpha^2)^3} < a^2 \alpha^2 < \frac{2}{3}$, corresponding to naked singularities solutions. (iv) $a^2 \alpha^2 = \frac{2}{3}$, which gives a null singularity. (v) $\frac{2}{3} < a^2 \alpha^2 < 1$, which gives a black hole solution with one horizon. The most interesting solutions are given in items (i) and (ii). Solutions (iv) and (v) do not have partners in the Kerr-Newman family. In figure 4.1, we show the black hole and naked singularity regions, and the extremal black hole line dividing those two regions, as well as the other solutions in the upper part of the figure. We now analyze each item in turn.

(i) $0 \leq a^2 \alpha^2 \leq \frac{2}{3} - \frac{128}{81} \frac{Q^6}{M^4(1 - \frac{1}{2}a^2 \alpha^2)^3}$

This is the charged-rotating black string spacetime. As we will see this
one is indeed very similar to the Kerr-Newman black hole. The structure has
event and Cauchy horizons, timelike singularities, and closed timelike curves.

Now, following Boyer and Lindquist we choose a new angular coordinate
which straightens out the helicoidal null geodesics that pile up around the
event horizon. A good choice is

$$\varphi = \gamma \varphi - \omega t. \quad (4.16)$$

In this case the metric reads,

$$ds^2 = - \left( \alpha^2 r^2 - \frac{b}{\alpha r} + \frac{c^2}{\alpha^2 r^2} \right) \left( \frac{dr}{\gamma} - \frac{\omega}{\alpha^2 \gamma} d\varphi \right)^2 + r^2 d\varphi^2 +$$

$$+ \left( \alpha^2 r^2 - \frac{b}{\alpha r} + \frac{c^2}{\alpha^2 r^2} \right)^{-1} dr^2 + \alpha^2 r^2 dz^2. \quad (4.17)$$

The horizons are given for the zeros of the lapse function, i.e. when $\Delta = 0$.

We find that $\Delta$ has two roots $r_+$ and $r_-$ which are given by,

$$r_+ = b^2 \sqrt{s + \sqrt{2s^2 - 4q^2}} \frac{s}{2\alpha} \quad (4.18)$$

and

$$r_- = b^2 \sqrt{s - \sqrt{2s^2 - 4q^2}} \frac{s}{2\alpha} \quad (4.19)$$

where,

$$s = \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 - 4 \left( \frac{4q^2}{3} \right)^3} \right)^{\frac{1}{3}} + \left( \frac{1}{2} - \frac{1}{2} \sqrt{1 - 4 \left( \frac{4q^2}{3} \right)^3} \right)^{\frac{1}{3}}, \quad (4.20)$$

$$q^2 = \frac{c^2}{b^4}. \quad (4.21)$$
and $b$ and $c$ are given in equations (4.9) and (4.10).

We now introduce a Kruskal coordinate patch around each of the roots of $\Delta$, $r_+$ and $r_-$. The first patch constructed around $r_+$ is valid for $r_- < r < \infty$.

In the region $r_- < r \leq r_+$ the null Kruskal coordinates $U$ and $V$ are given by,

$$
U = \left( \frac{a(r_+ - r)}{b^{1/4}} \right)^{1/2} \left( \frac{a(r_+ - r_-)}{b^{1/4}} \right)^{-\frac{C r_-^2}{B 2 r^2}} F(r) \exp \left( -\frac{A r_+ - r_-}{2 r^2} \alpha \gamma \right),
$$

$$
V = \left( \frac{a(r_+ - r)}{b^{1/4}} \right)^{1/2} \left( \frac{a(r_+ - r_-)}{b^{1/4}} \right)^{-\frac{C r_-^2}{B 2 r^2}} F(r) \exp \left( \frac{A r_+ - r_-}{2 r^2} \alpha \gamma \right), \quad (4.22)
$$

For $r_+ \leq r < \infty$ we put,

$$
U = -\left( \frac{a(r_+ - r)}{b^{1/4}} \right)^{1/2} \left( \frac{a(r_+ - r_-)}{b^{1/4}} \right)^{-\frac{C r_-^2}{B 2 r^2}} F(r) \exp \left( -\frac{A r_+ - r_-}{2 r^2} \alpha \gamma \right),
$$

$$
V = \left( \frac{a(r_+ - r)}{b^{1/4}} \right)^{1/2} \left( \frac{a(r_+ - r_-)}{b^{1/4}} \right)^{-\frac{C r_-^2}{B 2 r^2}} F(r) \exp \left( \frac{A r_+ - r_-}{2 r^2} \alpha \gamma \right). \quad (4.23)
$$

The following definitions have been introduced in order to facilitate the notation,

$$
A \equiv \left( r_+^2 + r_-^2 \right)^2 + 2 \left( r_+ + r_- \right)^4, \quad (4.24)
$$

$$
B \equiv \frac{1}{\alpha} \left( r_+^2 + r_-^2 \right)^2 + 2 \left( r_+ + r_- \right)^2, \quad (4.25)
$$

$$
C \equiv \frac{1}{\alpha} \left( r_+ + r_- \right)^2 + 2 r_+^2, \quad (4.26)
$$

$$
D \equiv \frac{1}{2\alpha} \left( r_+ + r_- \right)^3, \quad (4.27)
$$

$$
E \equiv \frac{1}{\alpha} \frac{\left( r_+^2 + r_-^2 \right)^2 + 2 \left( r_+ + r_- \right)^2 \left( r_+^2 + r_-^2 + r_+ r_- \right)}{\sqrt{(r_+ + r_-)^2 + 2 \left( r_+^2 + r_-^2 \right)}}, \quad (4.28)
$$
and finally,
\[
F (r) \equiv \left( \frac{\alpha^2}{b^8} [r^2 + (r_+ + r_-) r + (r_+^2 + r_-^2 + r_+ r_-)] \right)^{-\frac{B r_+ - r_-}{2r_+^2}} \exp \left( \frac{E r_+ - r_-}{2r_+^2} \arctan \frac{2r_+ (r_+ + r_-)}{\sqrt{(r_+ + r_-)^2 + 2(r_+^2 + r_-^2)}} \right). \tag{4.29}
\]

In this first coordinate patch, \( r_- < r \leq \infty \), the metric can be written as,
\[
\begin{align*}
ds^2 &= -\frac{b^2 \left[ \frac{\alpha(r-r_+)b^0}{\alpha b^1} \right]^{1+G(r)_{+}}}{\alpha k_+ r^2} G_+ (r) dU dV + \\
&\quad \frac{\alpha}{\sqrt{1 - \frac{a^2}{2r_+^2}}} \left[ \frac{\alpha(r-r_-)b^0}{\alpha b^1} \right]^{1+G(r)_{+}} G_+ (r) (VdU - UdV) d\varphi + \\
&\quad \left( r^2 - \Delta \frac{a^2}{2r_+^2} \right) d\varphi^2 + \alpha^2 r^2 dz^2, \tag{4.30}
\end{align*}
\]

where,
\[
G_+(r) \equiv \frac{r^2 + (r_+ + r_-) r + (r_+^2 + r_-^2 + r_+ r_-)}{F^2(r)}, \tag{4.31}
\]

and
\[
k_+ = \frac{A r_+ - r_-}{B r_+^2}. \tag{4.32}
\]

We see that the metric given in (4.30) is regular in this patch, and in particular is regular at \( r_+ \). It is however singular at \( r_- \). To have a metric non-singular at \( r_- \) one has to define new Kruskal coordinates for the patch \( 0 < r < r_+ \).

For \( 0 < r \leq r_- \) we have,
\[
\begin{align*}
U &= -\left( \frac{\alpha(r_+ - r_-)}{b^0} \right) \frac{B r_+}{2r_+^2} \left( \frac{\alpha(r_+ - r_-)}{b^0} \right) \frac{1}{2} H (r) \exp \left( \frac{A r_+ - r_-}{2r_+^2} \alpha \frac{1}{4} \right), \\
V &= \left( \frac{\alpha(r_+ - r_-)}{b^0} \right) \frac{B r_+}{2r_+^2} \left( \frac{\alpha(r_+ - r_-)}{b^0} \right) \frac{1}{2} H (r) \exp \left( -\frac{A r_+ - r_-}{2r_+^2} \alpha \frac{1}{4} \right), \tag{4.33}
\end{align*}
\]

For $r_- \leq r < r_+$ we have,

$$U = \left( \frac{\alpha(r_+-r)}{b^+} \right) \frac{H(r)}{2} \exp \left( \frac{A r_+-r_-}{2r_-} - \frac{\alpha^2}{2} \right),$$

$$V = \left( \frac{\alpha(r+r_--r)}{b^-} \right) \frac{H(r)}{2} \exp \left( -\frac{A r_+-r_-}{2r_-} - \frac{\alpha^2}{2} \right),$$

(4.34)

where,

$$H(r) = \left( \frac{\alpha^2}{b^2} \right) \left[ r^2 + (r_+ + r_-) r + (r_+^2 + r_-^2 + r_+ r_-) \right] \frac{H^2(r)}{2},$$

$$\exp \left( -\frac{E r_+-r_-}{2r_-} \arctan \frac{2r+(r_+ r_-)}{\sqrt{(r_+ r_-)^2+2(r_+^2+r_-^2)}} \right).$$

(4.35)

The metric for this second patch can be written as

$$ds^2 = -2 \frac{b^{\frac{3}{2}} \alpha(r+r_-)/b^+}{\alpha^2 k_- r^2} \sqrt{G_- (r)} dU dV +$$

$$- \frac{a}{\alpha \sqrt{1-a^2 r^2}} \frac{b^{\frac{3}{2}} \alpha(r+r_-)/b^+}{k_- r^2} \sqrt{G_- (r)} (V dU - U dV) d\psi +$$

$$+ \left( r^2 - \Delta \frac{a^2}{1-a^2 r^2} \right) d\varphi^2 + \alpha^2 r^2 dz^2,$$

(4.36)

where,

$$G_- (r) \equiv \frac{r^2 + (r_+ + r_-) r + (r_+^2 + r_-^2 + r_+ r_-)}{H^2(r)}$$

(4.37)

and

$$k_- = \frac{A r_+ - r_-}{B 2r_-^2}.$$  

(4.38)

The metric is regular at $r_-$ and is singular at $r = 0$. To construct the Penrose diagram we have to define the Penrose coordinates, $\psi$, $\xi$ by the
usual arctangent functions of $U$ and $V$,

$$U = \tan \frac{1}{2}(\psi - \xi) \quad V = \tan \frac{1}{2}(\psi + \xi) \quad (4.39)$$

From (4.39), (4.22) and (4.23) we have that in the first patch (i) the line $r = \infty$ is mapped into two symmetrical curved timelike lines (ii) the line $r = r_+$ is mapped into two mutual perpendicular straight lines at $45^0$. From (4.33) and (4.34) we see that (i) $r = 0$ is mapped into a curved timelike line and (ii) $r = r_-$ is mapped into two mutual perpendicular straight null lines at $45^0$. One has to join these two different patches (see [30, 12]) and then repeat them over in the vertical. The result is the Penrose diagram shown in figure 4.2. The lines $r = 0$ and $r = \infty$ are drawn as vertical lines, although in the coordinates $\psi$ and $\xi$ they should be curved outwards, bulged. It is always possible to change coordinates so that the lines are indeed vertical.

To study closed timelike curves (CTCs) we first note that the angular Killing vector $\frac{\partial}{\partial \varphi}$ has norm given by

$$\frac{\partial}{\partial \varphi} \cdot \frac{\partial}{\partial \varphi} = g_{\varphi \varphi} = r^2 + \frac{4M^2}{\alpha r} - \frac{4a^2 Q^2}{(1 - \frac{a^2}{\alpha^2})^2 r^2}. \quad (4.40)$$

There are CTCs for $g_{\varphi \varphi} < 0$. One can show that the radius for which $g_{\varphi \varphi} = 0$ is

$$r_{\text{CTC}} = \left(\frac{a^2 \alpha^2 b}{1 - \frac{3}{2}a^2 \alpha^2}\right)\frac{1}{2} \sqrt{2\pi^2 + 4\pi^2 - \pi - \sqrt{\pi}} \quad (4.41)$$
where,
\[
\bar{s} = \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 + 4 \left( \frac{47^2}{3} \right)^3} \right)^{\frac{1}{3}} + \left( \frac{1}{2} - \frac{1}{2} \sqrt{1 + 4 \left( \frac{47^2}{3} \right)^3} \right)^{\frac{1}{3}},
\]
(4.42)
\[
\bar{q}^2 = \frac{a^2 \alpha^2 c^2}{1 - \frac{a^2 \alpha^2}{2} b^2}
\]
(4.43)

and \(b\) and \(c\) are given in equations (4.9) (4.10). One can also show that \(r_+ > r_{CTC}\) always, as in figure 4.2. Following Carter [29] the region \(0 < r < r_+\) is a vicious region, i.e., there are CTCs through every point of it.

The singularity is at \(r = 0\) and timelike. Going to Kerr-Schild like coordinates near the singularity one can show that the singularity has a ring structure, like the Kerr-Newman black hole. Indeed, near \(r = 0\) the metric takes the form
\[
ds^2 \simeq -\frac{4Q^2}{\alpha^2 r^2} (dt - \frac{a}{\sqrt{1 - \frac{a^2 \alpha^2}{2}}} d\varphi)^2.
\]
(4.44)

If one now changes coordinates to \(x = r \cos \varphi - a(1 - \frac{a^2 \alpha^2}{2})^{-\frac{1}{2}} \sin \varphi\) and \(y = r \sin \varphi + a(1 - \frac{a^2 \alpha^2}{2})^{-\frac{1}{2}} \cos \varphi\) one finds that at \(r \rightarrow 0\) the metric is given by
\[
ds^2 \simeq -\frac{4Q^2}{\alpha^2 r^2} \left[ dt - \frac{\sqrt{1 - \frac{a^2 \alpha^2}{2}}}{a} (xy - ydx) \right]^2.
\]
(4.45)

Now, \(x^2 + y^2 = \frac{a^2}{1 - \frac{a^2 \alpha^2}{2}} + r^2 \simeq \frac{a^2}{1 - \frac{a^2 \alpha^2}{2}}\). Thus the singularity at \(r = 0\), like the Kerr-Newman singularity, has a ring structure. However, unlike Kerr-Newman, one cannot penetrate to the inside of the singularity.
From (4.4) one can see that there is an infinite redshift surface given by the zero of the coefficient in front of $dt^2$. It is also a quartic equation and one can easily find $r_{rs}$. It is always outside the event horizon, unless $a = 0$ in which case it coincides with the event horizon. We also note that if there is no rotation $a = 0$ then the Penrose diagram in figure 4.2 is identical. However there are no CTCs and the singularity loses the ring structure.

(ii) $a^2\alpha^2 = \frac{2}{3} - \frac{128}{81} \frac{Q^6}{M^4(1-\frac{1}{2}a^2\alpha^2)^3}$

The extreme case is given when $Q$ is connected to $M$ and $a$ through the relation,

$$Q^6 = \frac{27}{64} M^2 \left( 1 - \frac{3}{2} a^2 \alpha^2 \right) \left( 1 - \frac{1}{2} a^2 \alpha^2 \right)^3,$$  \hspace{1cm} (4.46)

which can also be put in the form $a^2\alpha^2 = \frac{2}{3} - \frac{128}{81} \frac{Q^6}{M^4(1-\frac{1}{2}a^2\alpha^2)^3}$ as above. In figure 4.1 we have drawn the line which gives the values of $Q$ and $a$ (in suitable $M$ units) compatible with this case. The event and Cauchy horizons join together in one single horizon $r_+$ given by

$$r_+ = \frac{4Q^2}{3M\alpha(1 - \frac{a^2\alpha^2}{2})}.$$ \hspace{1cm} (4.47)

The function $\Delta$ is now,

$$\Delta = \frac{a^2(r - r_+)^2(r^2 + 2r_+r + 3r_+^2)}{r^2},$$ \hspace{1cm} (4.48)

so the metric (4.7) turns to

$$ds^2 = \frac{a^2(r-r_+)^2(r^2+2r_+r+3r_+^2)}{r^2} (\gamma dt - \frac{\omega}{\alpha^2} d\varphi)^2 + \frac{r^2 dr^2}{a^2(r-r_+)^2(r^2+2r_+r+3r_+^2)} + \cdots$$
where $\gamma$ and $\omega$ are defined in (4.11) and (4.12) respectively. There are no Kruskal coordinates. To draw the Penrose diagram we resort first to the double null coordinates $u$ and $v$,

$$u = \alpha \left( \frac{t}{\gamma} - r_\ast \right) \quad \text{and} \quad v = \alpha \left( \frac{t}{\gamma} + r_\ast \right)$$

(4.50)

where $r_\ast$ is the tortoise coordinate given by

$$r_\ast = \frac{2}{9 \alpha r_+} \ln \left[ \frac{\alpha (r-r_\ast)}{b^\frac{1}{3}} \right] - \frac{1}{6 \alpha (r-r_\ast)} - \frac{1}{9 \alpha r_+} \ln \left[ \frac{\alpha^2 (r^2+2r_\ast r+3r_\ast^2)}{b^\frac{2}{3}} \right] +$$

$$+ \frac{7 \sqrt{2}}{18 \sqrt{2} r_+} \arctan \frac{r+r_\ast}{\sqrt{2} r_\ast}.$$  

(4.51)

Defining the new angular coordinate as before $\phi = \gamma \varphi - \omega t$, the metric (4.49) is now

$$ds^2 = -\frac{\alpha^2 (r-r_\ast)^2 (r^2+2r_\ast r+3r_\ast^2)}{r^2} dt^2 + \frac{r^2 dz^2}{\gamma^2} +$$

$$+ \frac{\Delta \omega}{\alpha^2 \gamma^2} 2 dtd\phi + (r^2 - \Delta \frac{\omega^2}{\alpha^2 \gamma^2}) d\phi^2 + \alpha^2 r^2 dz^2.$$  

(4.52)

Now define the Penrose coordinates $\psi$ and $\xi$ via the relations

$$u = \tan \frac{1}{2} (\psi - \xi) \quad \text{and} \quad v = \tan \frac{1}{2} (\psi + \xi)$$

(4.53)

Then the metric (4.52) turns into

$$ds^2 = -\frac{(r-r_\ast)^2 (r^2+2r_\ast r+3r_\ast^2)}{r^2} d\psi^2 - d\xi^2 +$$

$$+ \frac{\Delta \omega}{\alpha^2 \gamma^2} 2 dtd\phi + (r^2 - \Delta \frac{\omega^2}{\alpha^2 \gamma^2}) d\phi^2 + \alpha^2 r^2 dz^2.$$  

(4.54)

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where \( t \) is given implicitly in terms of \( \psi \) and \( \xi \). From the defining equations (4.50) and (4.53) we have

\[
\frac{\sin \xi}{\cos \psi + \cos \xi} = \alpha r^* \tag{4.55}
\]

Then one can draw the Penrose diagram (see figure 4.3). The lines \( r = r_+ \) are given by the equation \( \psi = \pm \xi + n\pi \) with \( n \) any integer, and therefore are lines at 45°. The lines \( r = 0 \) and \( r = \infty \) are timelike lines given by an equation of the form \( \frac{\sin \xi}{\cos \psi + \cos \xi} = \) constant, where the constant is easily found from \( r^* \). These are not straight vertical lines. However by a further coordinate transformation it is possible to straighten them out as it is shown in the figure. The metric (4.54) is regular at \( r = r_+ \) because the zeros of the denominator and numerator cancel each other.

The CTC radius is defined in (4.41), where now we take the values for \( Q \) and \( a \) appropriate for the extreme case. This line is also shown in the diagram. There is also an infinite redshift line, \( r_{rs} \). If \( a = 0 \) there are no CTCs and \( r_{rs} \) joins the event horizon.

\[
(iii) \quad \frac{2}{3} - \frac{128}{81} \frac{Q^6}{M^4(1 - \frac{1}{2} a^2 \alpha^2)^3} < a^2 \alpha^2 < \frac{2}{3}
\]

In this case there are no roots for \( \Delta \) as defined in (4.8). Therefore there are no horizons. The singularity is timelike and naked. Infinity is also timelike. There is an infinite redshift surface if the following inequality is satisfied

\[
Q^6 \leq \frac{27}{64} (1 - \frac{1}{2} a^2 \alpha^2)^4 M^4. \tag{4.56}
\]
There are CTCs if \( a \neq 0 \). The Penrose diagram is sketched in figure 4.4.

(iv) \( a^2 \alpha^2 = \frac{2}{3} \)

For \( a^2 \alpha^2 = \frac{2}{3} \) the form of the metric (4.7) is not valid, one has to go back to the form (4.4). The Kretschmann scalar is a constant. However, since the first Betti number of the manifold is one, there is a topological singularity at \( r = 0 \), which is a null surface. There is an infinite redshift surface and a closed timelike radius. The Penrose diagram is sketched in figure 4.5.

(v) \( \frac{2}{3} < a^2 \alpha^2 \leq 1 \)

Here \( \Delta \) has one root, and therefore there is one horizon. Thus, it represents a black hole. The singularity at \( r = 0 \) is polynomial and spacelike. The CTC radius is outside the event horizon. If \( Q \) is sufficiently small there is an ergosphere, see figure 4.6.

IV.2 \( r \leq 0 \) (or \( M < 0 \))

In order to be complete we continue to analyze the solution found in (4.4). The solutions displayed below are rather exotic, although not without interest.

When we say \( r \leq 0 \) it amounts to do \( r \to -r \) in the metric given in equation (4.4), or equivalently \( M \to -M \). Thus the following solution is a
negative mass solution. The metric is then,

$$ds^2 = -\left(\alpha^2 r^2 + \frac{4M(1-a^2\alpha^2)}{\alpha r} + \frac{4Q^2}{\alpha^2 r^2}\right)dt^2 +$$

$$-\frac{4aM\sqrt{1-\frac{a^2\alpha^2}{r^2}}}{\alpha r^2} \left(1 + \frac{Q^2}{M(1-a^2\alpha^2)}\right) 2dtd\varphi +$$

$$+ \left(\alpha^2 r^2 + \frac{4M(1-\frac{3}{2}a^2\alpha^2)}{\alpha r} + \frac{4Q^2}{\alpha^2 r^2(1-\frac{3}{2}a^2\alpha^2)}\right)^{-1} dr^2 +$$

$$+ \left[r^2 - \frac{4Ma^2}{\alpha r} \left(1 + \frac{2Q^2}{(1-\frac{3}{2}a^2\alpha^2)M\alpha r}\right)\right] d\varphi^2 + \alpha^2 r^2 dz^2,$$  \hspace{1cm} (4.57)

where we have done $J \rightarrow -J$ (or $a \rightarrow -a$) since this does not change the character of the metric. Now, the function $\Delta$ is

$$\Delta = \alpha^2 r^2 + \frac{b}{\alpha r} + \frac{c^2}{\alpha^2 r^2},$$  \hspace{1cm} (4.58)

with $b = 4M \left(1 - \frac{3}{2}a^2\alpha^2\right)$ and $c^2 = 4Q^2 \left(\frac{1}{1-\frac{3}{2}a^2\alpha^2}\right)$ as in equations (4.9) and (4.10). However, the parameter $M$ still positive, represents now negative mass. Note then that whether $\Delta$ has roots or not depends only on the value of $a$. Recall that $0 \leq a^2\alpha^2 \leq 1$. Thus, looking at the $g_{00}$ term of (4.57), one sees that there are no infinite redshift surfaces. There are three distinct cases.

(i) $0 < a^2\alpha^2 < \frac{2}{3}$

Here $\Delta > 0$ always, therefore there are no roots, no horizons. There is a timelike singularity at $r = 0$. At $r \rightarrow \infty$ spacetime is anti-de Sitter. There is a $r_{\text{CTC}}$. Supressing the infinite redshift surface $r_{\text{rs}}$, figure 4.4 serves as a representation of the Penrose diagram. If $a = 0$ there are no CTCs.
There is a topological (non-polynomial) null singularity at $r = 0$. The singularity is topological because the first Betti number of the manifold is one. See figure 4.5. If $a = 0$ there are no CTCs. See figure 4.5 (with $r_{rs}$ supressed) for a representation of the Penrose diagram.

(iii) $\frac{2}{3} < a^2 \alpha^2 < 1$

In this case there is one root in $\Delta$ for positive $r$, therefore there is one horizon located at

$$r_h = |b|^{\frac{1}{2}} \frac{\sqrt{\sigma} + \sqrt{2\sigma^2 + 4q^2 - \sigma}}{2\alpha},$$

where,

$$\sigma = \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 + 4 \left( \frac{4q^2}{3} \right)^3} \right)^{\frac{1}{3}} + \left( \frac{1}{2} - \frac{1}{2} \sqrt{1 - 4 \left( \frac{4q^2}{3} \right)^3} \right)^{\frac{1}{3}},$$

and here $q = \frac{|c^2|}{|\omega|^2}$. There are also CTCs. It is easy to check that $r_{CTC} > r_h$. See figure 4.6 (with $r_{rs}$ supressed) for the Penrose diagram. For $a = 0$ there are no CTCs.

V. Causal Structure of the Uncharged Rotating Black String Spacetime

This case has been proposed in [14, 15]. Since the solution and its causal
structure are totally different in character from the charged solution we present it here in detail.

For $Q = 0$ the metric (4.4) simplifies to

$$ds^2 = -\left(\alpha^2 r^2 - \frac{4M(1 - \frac{a^2 \alpha^2}{\alpha r})}{\alpha r}\right) dt^2 +$$

$$-\frac{4aM}{\sqrt{1 - \frac{a^2 \alpha^2}{\alpha r}}} 2dt d\varphi +$$

$$+ \left(\alpha^2 r^2 - \frac{4M(1 - \frac{3a^2 \alpha^2}{\alpha r})}{\alpha r}\right)^{-1} dr^2 +$$

$$+ \left(r^2 + \frac{4Ma^2}{\alpha r}\right) d\varphi^2 + \alpha^2 r^2 dz^2.$$  (5.1)

It can also be put in a form like (4.7)

$$ds^2 = -(\alpha^2 r^2 - \frac{b}{\alpha r}) \left(\gamma dt - \frac{\omega}{\alpha r} d\varphi\right)^2 +$$

$$+ r^2 (\gamma d\varphi - \omega dt)^2 + \frac{dr^2}{\alpha^2 r^2 - \frac{b}{\alpha r}} + \alpha^2 r^2 dz^2.$$  (5.2)

and now we have

$$\Delta = \alpha^2 r^2 - \frac{b}{\alpha r}$$  (5.3)

with $b = 4M \left(1 - \frac{3a^2 \alpha^2}{2}\right)$ as in (4.9) and $\gamma$ and $\omega$ given as in (4.11) and (4.12) respectively. The solution changes character depending on whether $r \leq 0$ or $r \geq 0$ as in the charged case. We study first $r \geq 0$ which contains the most interesting rotating black hole solution.

**V.1** $r \geq 0$
From (5.1) we see that to have horizons one needs
\[ 0 \leq a^2 \alpha^2 < \frac{2}{3} \] (5.4)

One has therefore three distinct cases which depend on the value of \( a \).

(i) \( 0 < a^2 \alpha^2 < \frac{2}{3} \)

In this case \( \Delta \) has one root, which locates the horizon at the radius
\[ r_+ = \frac{b^\frac{1}{2}}{\alpha} = \left[ 4M(1 - \frac{3}{2} a^2 \alpha^2) \right]^{\frac{1}{3}} \] (5.5)

To find the Kruskal coordinates we first define the tortoise coordinate \( r_* = \int \frac{dr}{\Delta} \). Using (5.3) we find
\[ r_* = \frac{1}{\alpha^2 r_+} \left[ \frac{1}{6} \ln \frac{(r - r_+)^2}{r^2 + r_+ r + r_+^2} + \frac{1}{\sqrt{3}} \arctan \frac{2r + r_+}{\sqrt{3}r_+} \right] \] (5.6)

Then the Kruskal coordinates are for \( r \geq r_+ \),
\[ U = -e^{-\frac{3}{2} a^2 \alpha^2 (r_+ - r_+)} = -(r - r_+)^{\frac{1}{2}} \left[ \frac{\sqrt{3} \arctan \frac{2r + r_+}{\sqrt{3}r_+}}{e} \right]^{\frac{1}{2}} \frac{1}{(r^2 + r_+ r + r_+^2)^{\frac{1}{2}}} e^{-\frac{3}{2} a^2 \alpha^2 t}, \]
\[ V = e^{-\frac{3}{2} a^2 \alpha^2 (\frac{1}{\gamma} + r_+)} = (r - r_+)^{\frac{1}{2}} \left[ \frac{\sqrt{3} \arctan \frac{2r + r_+}{\sqrt{3}r_+}}{e} \right]^{\frac{1}{2}} \frac{1}{(r^2 + r_+ r + r_+^2)^{\frac{1}{2}}} e^{\frac{3}{2} a^2 r_+ t}. \] (5.7)

And for \( 0 < r < r_+ \) we choose the following coordinates,
\[ U = (r_+ - r)^{\frac{1}{2}} \left[ \frac{\sqrt{3} \arctan \frac{2r + r_+}{\sqrt{3}r_+}}{e} \right]^{\frac{1}{2}} \frac{1}{(r^2 + r_+ r + r_+^2)^{\frac{1}{2}}} e^{-\frac{3}{2} a^2 \alpha^2 t}, \]
\[ V = (r_+ - r)^{\frac{1}{2}} \left[ \frac{\sqrt{3} \arctan \frac{2r + r_+}{\sqrt{3}r_+}}{e} \right]^{\frac{1}{2}} \frac{1}{(r^2 + r_+ r + r_+^2)^{\frac{1}{2}}} e^{\frac{3}{2} a^2 r_+ t}. \] (5.8)
Then defining the usual new angular coordinate $\varphi = \gamma d\varphi - \omega dt$ one finds that metric (5.1) is written in Kruskal coordinates as

$$ds^2 = -\frac{4}{9\alpha^2 r_+^2} \frac{(r^2 + r_+^2)^2}{\sqrt{3} \arctan \frac{2\sqrt{2} r_+}{\sqrt{3} r_+^2}} dUdV +$$

$$+ \frac{2}{3r_+} \frac{(r^2 + r_+^2)^2}{\sqrt{3} \arctan \frac{2\sqrt{2} r_+}{\sqrt{3} r_+^2}} \sqrt{\frac{1}{1 - \frac{a^2}{r_+^2}}} (VdU - UdV)d\varphi +$$

$$+(r^2 - \Delta \frac{a^2}{1 - \frac{a^2}{r_+^2}})d\varphi^2 + \alpha^2 r^2 dz^2.$$  (5.9)

The metric is regular at $r = r_+$. To draw the Kruskal diagram it is a simple matter. One has only to take the product $UV$ appropriately from equations (5.7) and (5.8). Then one has the usual hyperbolas in the Kruskal diagram. To revert to the Penrose diagram one takes as usual the arctangents of $U$ and $V$. The diagram is represented in figure 5.1.

There are no CTCs. The infinite redshift surface $r_{rs}$ is given when $g_{00} = 0$ in (5.1), i.e., $r_{rs} = \frac{1}{\alpha} \left[ 4M \left( 1 - \frac{3}{2} a^2 \alpha^2 \right) \right]^\frac{1}{2}$. If the black hole has no rotation, $a = 0$, then the Penrose diagram looks the same. However, in this case the infinite redshift surface coincides with the event horizon.

(ii) $a^2 \alpha^2 = \frac{2}{3}$

In this case $b = 0$ The Kretschmann scalar is constant and there is no polynomial singularity. However there is a topological singularity at $r = 0$. The singularity is a null line. There is an infinite redshift surface at $r_{rs} = (\frac{2M\alpha^2}{a})^\frac{1}{2}$. This can be considered the extremal uncharged black hole. See figure 5.2.
(iii) $\frac{2}{3} < a^2 \alpha^2 < 1$

In this case $\Delta$ is positive for all $r$. There are no horizons. There is still an infinite redshift surface. The singularity at $r = 0$ is timelike. At $r \to \infty$ spacetime is anti-de Sitter. The Penrose diagram is represented in figure 5.3.

V.2 $r \leq 0$

When we do $r \to -r$ the metric (5.1) turns into (4.57) with $c^2 = 0$, i.e., $Q = 0$. For $r \geq 0$ we have just seen that the character of the solution is totally different depending on whether it has charge or not. For $r \leq 0$ the character of the solution does not change at all. Thus there are the same three cases as in section IV.2 and the Penrose diagrams are simply the same.

VI. Causal Structure of Other Solutions

Now, when solving equation (4.4) for $\gamma$ and $\omega$ we obtained quadratic equations. The solutions we have discussed so far in sections IV and V are all solutions with positive sign in front of the square root of the discriminant, see equation (3.7). We now discuss solutions with negative sign, given by (3.8). This amounts to do $\Omega \to -\Omega$ in (3.11). Then with the definitions
we obtain the following metric,

$$ds^2 = -\left(\alpha^2 r^2 - \frac{2Ma^2\alpha^2}{ar} + \frac{4Q^2 a^2\alpha^2}{a^2 r^2}\right)dt^2 +$$

$$-\frac{4aM}{\sqrt{1 - a^2}} \left(1 - \frac{2Q}{Ma}\right) 2dtd\varphi +$$

$$+ \left(\alpha^2 r^2 + \frac{8M(1 - a^2\alpha^2)}{ar} - \frac{16Q^2(1 - a^2\alpha^2)}{a^2 r^2}\right)^{-1} dr^2 +$$

$$+ \left[r^2 + \frac{8M(1 - a^2\alpha^2)}{a^3 r} \left(1 - \frac{2Q}{Ma}\right)\right] d\varphi^2 + \alpha^2 r^2 dz^2, \quad (6.1)$$

where we have defined

$$\overline{Q}^2 = \frac{Q^2}{a^2\alpha^2}. \quad (6.2)$$

Then to study the causal structure one can subdivide it again in \( r \geq 0 \) and \( r \leq 0 \).

**VI.1 \( r \geq 0 \)**

These are the positive mass solutions. We have two distinct cases. Indeed, if \( \overline{Q} \neq 0 \) there is one horizon only, \( r_h \). There are CTCs and \( r_{CTC} > r_h \) always. There is an infinite redshift surface for certain values of \( a \) and \( Q \). If \( \frac{512Q^6}{a^6\alpha^6} > 27M^4 \) there are no infinite redshift surfaces (see figure 4.6 for the Penrose diagram). On the other hand if \( \overline{Q} = 0 \) then there are no horizons, the singularity is naked. There is an infinite redshift surface and there are no CTCs. The Penrose diagram looks like figure 4.4.

**VI.2 \( r \leq 0 \)**

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In this case the metric reads,

\[ ds^2 = -\left(\alpha^2 r^2 + \frac{2Ma^2\alpha^2}{ar} + \frac{4Q^2a^2\alpha^2}{a^2r^2}\right) dt^2 + \\
-\frac{4aM\sqrt{1 - \frac{a^2\alpha^2}{ar} + 2}}{\alpha r} \left(1 + \frac{2Q}{Mar}\right) 2 dt d\varphi + \\
+ \left(\alpha^2 r^2 - \frac{8M(1 - \frac{a^2\alpha^2}{ar})}{\alpha r} - \frac{16Q^2(1 - \frac{a^2\alpha^2}{a^2r^2})}{\alpha^2 r^2}\right)^{-1} dr^2 + \\
+ \left[r^2 - \frac{8M(1 - \frac{a^2\alpha^2}{a^2 r})}{a^3 r} \left(1 + \frac{2Q}{Mar}\right)\right] d\varphi^2 + \alpha^2 r^2 dz^2, \quad (6.3) \]

Now, this is a solution with negative mass. \( M \) represents negative mass. From the term in \( dr^2 \) we see that there is a horizon always. From the \( g_{00} \) component we see that there are no infinite redshift surfaces. There are CTCs and one can check that \( r_{CTC} > r_H \) always (see figure 4.6).

**VII. Geodesic structure**

We study the geodesic equations with emphasis in the null geodesics. We find the equations and their first integrals. We study the effective potential and show the possible turning points. We comment on the timelike geodesics. The results, very interesting, underline the similarities and non-similarities with the Kerr and Schwarzschild black hole.

As we have seen in the previous sections, there are several particular cases whose geodesic structure should be studied separately. However, we are going to fix attention on the two most relevant cases. Namely, the black
string with two horizons considered in section IV.1(i) and the extremal black
string case of section IV.1(ii).

The situation we are interested here is given in equation (4.14). Then,
the horizons \( r = r_+ \) (event horizon) and \( r = r_- \) (Cauchy horizon) are given
respectively by equations (4.18) and (4.19). There is also an infinite redshift
surface outside the horizon at \( r = r_{rs} \) as shown in figure 4.2. For \( r < r_{rs} \)
all observers with fixed \( r \) and \( z \) must orbit the black hole in the direction it
rotates.

In order to be more specific we must study frame dragging in metric (4.4).
Thus, the angular velocity \( \Omega = d\varphi/dt \) of stationary observers is constrained
by

\[
\Omega_{\text{min}} = \omega - \sqrt{\omega^2 - g_{tt}/g_{\varphi\varphi}} \quad \text{< } \Omega \text{ < } \Omega_{\text{max}} = \omega + \sqrt{\omega^2 - g_{tt}/g_{\varphi\varphi}}
\]

\[
\omega = \frac{4aM\sqrt{1 - \frac{a^2\alpha^2}{2}} \left(1 - \frac{Q^2}{M(1 - \frac{a^2\alpha^2}{2})\alpha r}\right)}{\alpha r^3 + 4Ma^2 \left(1 - \frac{Q^2}{(1 - \frac{a^2\alpha^2}{2})^{3/2}M\alpha r}\right)}
\] (7.1)

From (7.1) we see that \( \Omega_{\text{min}} \) is null at \( r = r_{rs} \) and is positive for \( r < r_{rs} \),
which means that there is no static observers inside such a surface. It is also
worth to note that for \( r \to \infty \), \( \omega \) falls as \( r^{-3} \) just like in the case of the
Kerr metric and in constrast with BTZ [9] black hole where \( \omega = J/2r^2 \). Let
us now turn our attention to the geodesic equations.

In the metric (4.4) there are three Killing vectors: one of them \( \varepsilon^\mu \equiv \xi_t^\mu =
(\partial_t)^\mu \) is timelike, associated to the time translation invariance, and the two
others $\phi^\mu \equiv \xi^\mu = \left( \frac{\partial}{\partial x} \right)^\mu$ and $\zeta^\mu \equiv \xi^\mu = \left( \frac{\partial}{\partial z} \right)^\mu$ are both spacelike associated to the invariance of the metric under rotations around the symmetry axis and $z$-translations, respectively (we are considering the standard cylindrically symmetric model, $G_2 = R \times U(1)$). Then, considering the geodesic motion in such a spacetime, the (three) constants related to these Killing vectors are:

$$E = -g_{\mu\nu}\varepsilon^\mu u^\nu; \quad L = g_{\mu\nu}\phi^\mu u^\nu; \quad P = g_{\mu\nu}\zeta^\mu u^\nu,$$

(7.2)

where $u^\mu = \frac{dx^\mu}{d\tau}$ with $\tau$ being a (affine) parameter on the geodesic curve. As $u^\mu$ is the tangent vector to the curve it may be normalized as

$$u^\mu u_\mu = -\epsilon^2,$$

(7.3)

with $\epsilon^2 = 1(0)$ for timelike (null) geodesics.

Using (7.2), (7.3) and metric (4.4) we may write the geodesic equations in the form

$$\dot t = \frac{1}{Br^2} \left( E \alpha^2 r^4 + 4Mc_0 \alpha r - \frac{4Q^2c_0}{1 - \frac{\alpha^2a^2}{2}} \right),$$

$$\dot \varphi = \frac{1}{Br^2} \left( L \alpha^4 r^4 + 4Mc_0 \sqrt{1 - \frac{\alpha^2a^2}{2}} \alpha r - \frac{4Q^2c_0}{\sqrt{1 - \frac{\alpha^2a^2}{2}}} \right),$$

$$\dot z = \frac{P}{\alpha^2 r^2},$$

$$\alpha^4 r^4 \dot r^2 = -\left( \epsilon^2 \alpha^2 r^2 + P^2 + \frac{\alpha^2c_0^2}{1 - \frac{3\alpha^2a^2}{2}} \right) B + \alpha^4 r^4 \frac{c_0^2}{1 - \frac{3\alpha^2a^2}{2}},$$

(7.4)

where $\dot t = \frac{dt}{d\tau}$, etc. In the above equations we have introduced the definitions

$$c_0 = Ea - L \sqrt{1 - \frac{\alpha^2a^2}{2}},$$

(7.5)
\[ c_1 = E \sqrt{1 - \frac{\alpha^2 a^2}{2} - \alpha^2 La}, \]  
(7.6)

\[ B = \alpha^4 r^4 - 4M \left(1 - \frac{3a^2 \alpha^2}{2}\right) \alpha r + 4Q^2 \alpha \left(1 - \frac{a^2 \alpha^2}{2}\right). \]  
(7.7)

It is worth to note that since the spacetime is not asymptotically flat the constants \( E \) and \( L \) cannot be interpreted as the local energy and angular momentum at infinity.

In order to study the behaviour of geodesic lines in the black string spacetime, the important equation to analyze is the one that dictates the behaviour along the radial coordinate. It is not easy to give a complete description of the motion of the \( r \) coordinate, because the governing function, i.e., the right hand side of (7.4), is in fact a polynomial of sixth degree and the equation cannot be exactly integrated. Nevertheless, it is possible after some algebraic and numerical manipulations to reach interesting conclusions.

To simplify discussions let us study first the static charged solution where \( a^2 = 0 \) and then the general case with rotation.

**VII.1 Static Case**

In the static case \( J = 0 \), or equivalently \( a = 0 \), and the equation for \( \dot{r} \) may be used to find the main properties of timelike and null geodesics. Then by putting \( a = 0 \) in (7.4) we get

\[ \dot{t} = \frac{E \alpha^2 r^2}{B}, \quad \dot{\phi} = \frac{L}{r^2}, \quad \dot{z} = \frac{P}{\alpha^2 r^2}, \]  
(7.8)
\[ r^2 = E^2 - V_{eff}^2, \quad V_{eff}^2 = \left( \epsilon^2 \alpha^2 r^2 + P^2 + \alpha^2 L^2 \right) \frac{B}{\alpha^4 r^4}, \quad (7.9) \]

where

\[ B = \alpha^4 r^4 - 4 M \alpha r + 4 Q^2. \quad (7.10) \]

Notice that the equation for the radial coordinate has the same polynomial form as (7.4). In the static charged black string case, condition (4.14) reduces to

\[ Q^6 \leq \frac{27}{64} M^4. \quad (7.11) \]

Let us consider each case of (7.11) in turn.

(a) \( Q^6 < \frac{27}{64} M^4 \)

Then, \( B \) has two roots \( r_\pm \) give respectively by equations (4.18)–(4.21) with \( a = 0 \).

\[ r_+ = (4M)^{\frac{1}{2}} \sqrt{\frac{\sqrt{s} + \sqrt{2 \sqrt{s^2 - Q^2 \left( \frac{2}{M} \right)^3} - s}}{2\alpha}} \quad (7.12) \]

and

\[ r_- = (4M)^{\frac{1}{2}} \sqrt{\frac{\sqrt{s} - \sqrt{2 \sqrt{s^2 - Q^2 \left( \frac{2}{M} \right)^3} - s}}{2\alpha}} \quad (7.13) \]

where,

\[ s = \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{64 Q^6}{27 M^4}} \right)^{\frac{1}{4}} + \left( \frac{1}{2} - \frac{1}{2} \sqrt{1 - \frac{64 Q^6}{27 M^4}} \right)^{\frac{1}{4}}, \quad (7.14) \]
From the explicit form of the effective potential (7.9) we see that there are two turning points, where \( r^2 = 0, \ r_2 \) and \( r_1 \), with \( r_2 > r_+ \) and \( r_1 < r_- \), for any finite values of the parameters of motion \( E, L, \) and \( P \). The allowed region for geodesic particles is between \( r_1 \) and \( r_2 \) where the right hand side of (7.9) is a positive number. For null geodesics with \( E^2 - \alpha^2 L^2 - P^2 \geq 0 \) the turning point \( r_2 \) is at infinity. Notice also that for motion with \( E = 0 \), the two turning points coincide with the horizons and such geodesic particles must be confined to this region.

Thus, due to the simplicity of the function \( V_{\text{eff}} \), it is easy to see from (7.9) and (7.10) how the radial coordinate varies during the geodesic motion, (even without integrating such equation exactly). For null geodesics the effective potential is given by

\[
V_{\text{eff}}^2 = (P^2 + \alpha^2 L^2) \left( 1 - \frac{4M}{\alpha^2 r^3} + \frac{4Q^2}{\alpha^4 r^4} \right). \tag{7.15}
\]

Then, from (7.9) and (7.15) we can draw the following conclusions.

(i) If \( E^2 - \alpha^2 L^2 - P^2 > 0 \) and \( \alpha^2 L^2 + P^2 \neq 0 \) null particles produced near the horizon may escape to \( r = +\infty \), while particles coming in from infinity reach a minimum distance from the black string where they are then scattered by the potential barrier and spiral back to infinity.

(ii) Null geodesic particles with \( E^2 - \alpha^2 L^2 - P^2 < 0 \) produced near the horizon reach a maximum distance from the black string at \( r = r_2 \) and are
scattered to \( r = r_1 \). Then, the radial coordinate \( r \) starts to increase reaching \( r_2 \) again, and so on. The geodesic oscillates indefinitely between \( r_1 \) and \( r_2 \) crossing the horizons an infinite number of times.

(iii) Just radial null geodesics can reach the singularity \( r = 0 \). To see it notice that \( V_{\text{eff}} = 0 \) \( (P^2 = 0, L^2 = 0) \) and then the radial equation yields

\[
r = r_0 \pm E\tau, \tag{7.16}
\]

where \( r_0 \) is an integration constant. Such relation shows that this kind of line extends either from infinity to the singularity or from the singularity to infinity.

(iv) If \( E^2 = \alpha^2 L^2 + P^2 \) the equation for \( \frac{dr}{d\tau} \) may be integrated exactly producing

\[
\pm 15 \sqrt{\alpha M(P^2 + \alpha^2 L^2)} \tau + \text{const.} = \left( 3r^2 + \frac{4Q^2}{\alpha M} r + \frac{8Q^2}{\alpha^2 M^2} \right) \sqrt{r - \frac{Q^2}{\alpha M}},
\]

or,

\[
r = \frac{Q^2}{\alpha M} + \frac{ME^2}{\alpha^3 L^2} (\varphi - \varphi_0)^2, \tag{7.17}
\]

where \( \varphi_0 \) is a constant. Then we can see that in such a case null geodesics are spirals that start at \( r_1 = \frac{Q^2}{\alpha M} \) and may reach infinite radial distances. They can also start at \( r = +\infty \) spiralling to \( r_1 \) and then getting back to infinity.

In the same way, some conclusions may be found for timelike geodesics.

(v) From the asymptotic form of \( V_{\text{eff}} \) it is easy to see that timelike geodesics cannot reach the singularity \( r = 0 \) nor escape to \( r \to +\infty \). The
trajectory of timelike geodesic particles are bounded between $r_1$ and $r_2$. The behaviour is the same as described in (ii) for null geodesics.

(vi) The fact that there is a local minimum of the effective potential for both null and timelike geodesics ensures that there are stable circular orbits with radius between $r_2$ and $r_1$.

(b) $Q^6 = \frac{27}{64} M^4$

Here, equations (7.11)–(7.13) reduce to $r_+ = r_- = \frac{4Q^2}{3M\alpha}$. We then see that for the extremal static charged black string the general features of the geodesics are the same as for the non-extremal case discussed above. All the properties listed in paragraph (i)–(vi) still hold. There are, however, some peculiarities.

All geodesics with $E^2 = 0$ must have $r = \frac{4Q^2}{3M\alpha}$. In fact, the right hand side of (7.9) is null for $r = \frac{4Q^2}{3M\alpha}$ and is negative for all other values of $r$.

A geodesic like this can represent a helical line ($P^2 \neq 0$, $L^2 \neq 0$), a circle ($P^2 = 0$, $L^2 \neq 0$), a strait line along the $z$ direction ($P^2 \neq 0$, $L^2 = 0$), or even a “radial” geodesic with $r = r_+$ ($P^2 = 0$, $L^2 = 0$). It is worth to note that even though $E = 0$, it follows from (7.7) that $\dot{t}$ may be different from zero for particles on the horizon, since $B = 0$ there. Such geodesic orbits are stable against small radial perturbations.

VII.2 The rotating case
As we have mentioned before, we constrain ourselves to the rotating black string case, where \( a \) is restricted by \( 0 < a^2 \alpha^2 \leq \frac{2}{3} - \frac{128}{81} \frac{Q^6}{(1-\frac{2}{9} \alpha^2 a^2)^3 M^4} \). Due to the similarity among equation (7.4) and (7.8) the behaviour of radial geodesic motion in both static and rotating black string spacetimes are also very similar, as we shall see below.

(a) \( 0 < a^2 \alpha^2 < \frac{2}{3} - \frac{128}{81} \frac{Q^6}{(1-\frac{2}{9} \alpha^2 a)^3 M^4} \).

This condition represents the non-extremal black string where polynomial \( B \) has two real positive roots, the two horizons \( r_+ \) and \( r_- \), given respectively by (4.18) and (4.19). It is negative in the region \( r_- < r < r_+ \), otherwise it is positive.

Then, we see from (7.4) that as in the static case there are two turning points \( r_2 \) and \( r_1 \), with \( r_2 > r_+ \) and \( r_1 < r_- \), and the allowed region for the radial coordinate of geodesic particles is between \( r_1 \) and \( r_2 \). Again, for null geodesics with \( E^2 - \alpha^2 L^2 - P^2 \geq 0 \) the turning point \( r_2 \) is at infinity. Here, however, the turning points coincide with the horizons for timelike and null geodesics whose motions are such that \( c_1 = 0 \), i.e., the radial coordinate of the geodesics must be between the horizons if the energy and angular momentum of the particle satisfy \( E \sqrt{1 - \frac{a^2 \alpha^2}{2}} - \alpha^2 La = 0 \).

We then start studying the motion of null particles. Once more the im-
Important equation to analyze is (7.4) which we rewrite here (with \( \varepsilon^2 = 0 \))

\[
\alpha^4 r^4 \dot{r}^2 = (E^2 - P^2 - \alpha^2 L^2)\alpha^4 r^4 + 4M \left( P^2 \left( 1 - \frac{3\alpha^2 a^2}{2} \right) + \alpha^2 c_0^2 \right) \alpha r - \frac{4Q^2}{1 - \frac{\alpha^2 a^2}{2}} \left( P^2 \left( 1 - \frac{3\alpha^2 a^2}{2} \right) + \alpha^2 c_0^2 \right),
\]

(7.18)

where \( c_0^2 \) is given by (7.5). Then, comparing (7.18) with (7.9) and (7.15) we can take the following conclusions.

(i) All what was said in VII.1 (i) also holds for the rotating black string. Null particles can escape to infinity only if they have more than the escape energy, i.e., if \( E^2 \geq \alpha^2 L^2 + P^2 \).

(ii) The conclusions in VII.1 (ii) are also valid here. Thus, null geodesics with \( E^2 < \alpha^2 L^2 + P^2 \) are bounded between a maximum and a minimum radial distance and cross the horizons an infinite number of times. These kind of geodesics do not reach the singularity, since in this case \( c_0^2 \neq 0 \) (see (iii) below).

(iii) Only null geodesics with \( P^2 = 0 \) and \( c_0^2 = 0 \) can reach the singularity. These conditions imply \( E^2 - \alpha^2 L^2 > 0 \). Such geodesics describe the motion of particles which do not travel in the \( z \) direction and whose angular momentum and energy are related by \( \frac{L}{E} = a/\sqrt{1 - \frac{\alpha^2 a^2}{2}} = \frac{\omega}{\gamma} \), where \( \gamma \) and \( \omega \) are given respectively by (4.11) and (4.12). Such geodesic motion satisfies the relation \( \frac{d\phi}{dt} = \frac{\alpha^2 L}{E} = \frac{\omega}{\gamma} \), which means that this kind of geodesics rotates with the same angular velocity of the black string. They are radial geodesics (with respect
to the rotating string) in the sense that the angular coordinate $\bar{\phi}$ defined in equation (4.16) is constant ($\frac{d\bar{\phi}}{d\tau} = 0$). In this case, the radial equation can be integrated producing the simple relation (see also (7.16))

$$r = r_0 \pm \frac{E}{\sqrt{1 - \frac{\alpha^2 a^2}{2}}} \tau = r_0 \pm \frac{L}{a} \tau, \quad (7.19)$$

which describes either a geodesic coming from large values of $r$ and spiralling to the singularity in a finite proper time or a spiral line starting at $r = 0$ and extending to infinity.

(iv) There is also a particular case similar to VII.1 (iv). If $E^2 = \alpha^2 L^2 + P^2$ the equation for $\frac{dr}{d\tau}$ yields (see also (7.17))

$$\pm 15A \sqrt{\alpha M \tau + \text{const.}} = \left(3r^2 + 4r_1 r + 8r_1^2\right) \sqrt{r - r_1}, \quad (7.20)$$

where $\alpha r_1 = \frac{Q^2}{M \left(1 - \frac{\alpha^2 a^2}{2}\right)}$ and $A = a \alpha^2 L - \sqrt{(P^2 + \alpha^2 L^2) \left(1 - \frac{\alpha^2 a^2}{2}\right)}$. Such equation describes either spiralling null geodesics that start at $r_1$ and reach infinite radial distances, or geodesics that start at $r = +\infty$ and spiral towards $r_1$ and then return to infinity.

(v) Regarding timelike geodesics we have to consider the full radial equation (7.4). The general features are much the same as for null particles with $E^2 - \alpha^2 - P^2 < 0$ discussed in VII.1 (ii) above. That is to say, the motion of any timelike geodesic is bounded within the region $r_1 \leq r \leq r_2$. Then, an uncharged timelike particle emitted by the black string reaches the maximum distance at $r = r_2$. Then is pulled back crosses inwards the two the horizons
$r_+$ and $r_-$ and reaches a minimum distance at $r = r_1$. There, it is scattered by a potential barrier to $r_2$, after crossing $r_+$ and $r_-$ of a new universe in the Penrose diagram (see figure 4.2). Thus, the geodesic has an infinite length, it crosses an infinite number of times the two horizons.

(vi) From the form of the effective potential it also follows that there are stable circular geodesics.

\[(b) \ a^2 \alpha^2 = \frac{2}{3} - \frac{128}{81} \frac{Q^6}{(1 - \frac{1}{2} \alpha^2 a^2)^3 M^4}.\]

This corresponds to the extremal rotating black string. The particularities of geodesic motion here are similar to the extremal static case studied in VII.1(b). As we have seen, all particles whith $c_1^2 = 0$, have their motion restricted to the region bounded by the two horizons. Then, since the two horizons coincide here, the only possible geodesics with $E \sqrt{1 - \frac{a^2 \alpha^2}{2}} = \alpha^2 La$ are the ones that have constant radial coordinate $r = \frac{4Q^2}{3Ma(1 - \frac{a^2 \alpha^2}{2})}$. For $c_1^2 \neq 0$ the behaviour of the radial coordinate of the geodesic motion is the same as in the non-extremal case.

Finally, let us recall that there are other three special cases considered in section IV.1. Namely, (1) the naked singularity case (cf. IV.1(iii)); (2) the topological singularity case (cf. IV.1(iv)); and (3) the other black string case (cf. IV.1(v)). In all these cases the asymptotic behaviour of geodesics for large values of $r$ is the same as for the rotating black string spacetime. The most important particularity is that in (2) and (3) some particular timelike
(in addition to the null) geodesics can also reach the singularity.

VIII. Dimensional reduction and the 3D black hole

In this section we discuss the connection between the 4D black string and a 3D charged and rotating black hole through the dimensional reduction of actions (2.1) under suitable conditions as seen bellow. Such a procedure was used in ref. [14] in order to define the parameters of the 3D black hole and it was then possible to postulate mass and angular momentum per unit length for the 4D black string by simply taking the values of the mass and angular momentum from the 3D spacetime. Repeating the dimensional reduction formalism, but now for the charged case, we are now able to show explicitly that the mass, angular momentum and electric charge of the 3D black hole are exactly the line density quantities of the 4D black string defined in section III.

Consider a 4D spacetime metric admitting one spacelike Killing vector, $\frac{\partial}{\partial z}$. In such a case the 4D metric may be written in the form

$$ds^2 = (g_{mn} - X_m X_n) dx^m dx^n + 2g_{mn} e^{-4\phi} X^m dx^n dz + e^{-4\phi} dz^2,$$

(8.1)

where the 3D metric $g_{mn}$ ($m, n = 0, 1, 2$) and $\phi$ and $X^m$ are metric functions independent of $z$. The spacetime we are considering here (see e.g. equations (2.1) and (3.2)) is a particular case of (8.1) where $X^m = 0$. After the splitting
and taking $X^m = 0$, the electromagnetic action can be put in the following suitable form

$$S_{em} = \frac{1}{8\pi} \int d^4x \sqrt{-g} \left( \frac{H^{mn}H_{mn}}{2} + e^{4\phi} B^m B_n \right),$$  \hspace{1cm} (8.2)$$

where we have introduced the definitions

$$H_{mn} \equiv F_{mn}, \quad B_m \equiv F_{3m} = -F_{m3},$$  \hspace{1cm} (8.3)$$

$$F_{\mu\nu} \equiv \begin{pmatrix} H_{mn} & -B_n \\ B_m & 0 \end{pmatrix},$$  \hspace{1cm} (8.4)$$

with $H^{mn} \equiv g^{mp}g^{nq}H_{pq}$, etc. In the above equations, greek indices run from 0 to 3 and roman indices from 0 to 2.

For our purposes in this paper we may also put $B^m = 0$ since it vanishes identically for the charged stationary black string. For instance, according to (2.6), the electromagnetic potential assumes the form $A_{\mu} = -\gamma h(r)\delta^0_{\mu} + \frac{\omega}{ar} h(r)\delta^2_{\mu}$, where $h(r) = \frac{2A}{ar}$, from which follows that $B_m \equiv F_{3m} = 0$.

Then after dimensional reduction we obtain the following 3D actions

$$S_{em} = -\frac{1}{16\pi} \int d^3x \sqrt{-g} e^{-2\phi} (H^{mn}H_{mn}),$$  \hspace{1cm} (8.5)$$

$$S = \frac{1}{16\pi G} \int d^3x \sqrt{-g} e^{-2\phi} (R - 2\Lambda).$$  \hspace{1cm} (8.6)$$

By varying the action $S + S_{em}$ with respect to $g_{mn}, \phi$ and the electromagnetic potential $A_m$ one obtains equations of motions identical to 4D Einstein-Maxwell equations with one Killing vector $\frac{\partial}{\partial z}$. Imposing stationarity and
axial symmetry in the 3D theory one finds a 3D charged black hole solution which generalizes the stationary 3D black hole of [14] and whose metric can be written in the form:

\[ ds^2 = - \left( \alpha^2 r^2 - \frac{4M(1-a^2)}{\alpha} + \frac{4Q^2}{\alpha^2 r^2} \right) dt^2 + \]

\[ - \frac{4aM}{\alpha r} \sqrt{1 - \frac{a^2}{\alpha^2 r^2}} \left( 1 - \frac{Q^2}{M(1-a^2/\alpha^2)} \right) 2dd\phi + \]

\[ + \left( \alpha^2 r^2 - \frac{4M(1-\frac{3}{2}a^2)}{\alpha} + \frac{4Q^2}{\alpha^2 r^2} \left( 1-\frac{3}{2} \frac{a^2}{\alpha^2} \right) \right)^{-1} dr^2 + \]

\[ + \left[ r^2 + \frac{4Ma^2}{\alpha r} \left( 1-\frac{Q^2}{M(1-3/2a^2)} \right) \right] d\phi^2, \quad e^{-2\phi} = d_0\alpha r, \quad \text{(8.7)} \]

where \( d_0 \) is a constant which here we take equal to unity, \( d_0 = 1 \). To find the mass and angular momentum in 3D, and how it relates to the mass and angular momentum per unit length in 4D, we write equation (8.7) in the canonical form,

\[ ds^2 = -N^0 dt^2 + R^2 (N^\phi dt + d\phi)^2 + \frac{dR^2}{f^2}, \]

\[ e^{-2\phi} = \alpha r, \quad \text{(8.8)} \]

where \( N^0, N^\phi, R, f^2 \) are defined in equation (3.2), or else can be taken directly from (8.7). From (8.5), (8.6) and (8.2) we get an action written in the Hamiltonian formalism,

\[ S = -\frac{1}{8\pi} \int dt \left\{ N \left[ \frac{1}{2} e^{-2\phi} R^2 \left( \frac{N^\phi, R}{N^2} \right)^2 + e^{-2\phi} (f^2) R \left( 1 - 2R \frac{d\phi}{dR} \right) \right] - \right\} \]

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\[-4 f^2 \left( e^{-2\phi} R \frac{d\phi}{dR} \right)_{,R} + 2 e^{-2\phi} RA \right] + N^\phi \left( e^{-2\phi} R^3 \frac{N\phi R}{N} \right)_{,R} \right) dR \\
+ S_{em} + B;\]

\[S_{em} \equiv \int dt \left\{ N \left[ \frac{e^{-2\phi}}{4R} \left( e^{4\phi} E^2 + (A_{2,R})^2 \right) \right] + N^\phi \left( \frac{E A_{2,R}}{2} + \frac{A_0 E_R}{2} \right) \right\} dR \]

where \(N \equiv \frac{N_0}{f}\), \(N^\phi\) and \(A_0\) are Lagrange multipliers, and \(B\) is a surface term.

The next step is analyzing each surface term that follows by varying the action (8.9) with respect to \(\phi\), \(f^2\) and \(A_2\) and their conjugate momenta \[24, 4, 12\]. It follows that the surface terms that comes from the gravitational action for the charged 3D black hole do not change with respect to the uncharged 3D black hole. Moreover, apart from the term \(A_0 \frac{E}{2}\), the new surface terms in the total action coming from the electromagnetic field also vanish when \(R \to \infty\). That is to say, the conjugate quantities to \(N\) and \(N^\phi\) which correspond respectively to (ADM) mass and angular momentum of the charged 3D black hole are the same as if the black hole was not charged and are given respectively by the right hand side of (3.5) and (3.6).

The only new surface term which survive at infinity is \(A_0 \delta \left( \frac{E}{2} \right)\) from which we find the electric charge of the black hole (as the conjugate quantity to the Lagrange multiplier \(A_0\)):

\[Q \equiv \frac{1}{2} \left( \delta E \right)_{R=\infty} = \gamma \lambda, \quad (8.10)\]

which is exactly the same as (3.10).

From these results we see that the mass, angular momentum and charge
of the 3D black hole are exactly the mass $M$, angular momentum $J$ and charge $Q$ per unit length of the (4D) black string. However, this is valid only for $d_0 > 0$ in (8.7). Notice also that $M$ and $J$ given in (3.3) and (3.4) are identical to the mass and angular momentum of [14], apart from the normalization factor $G = \frac{1}{8}$.

Also note that whenever $\phi = 0$ in (8.6) one obtains the Einstein’s 3D gravity and the BTZ 3D black hole follows. This black hole can therefore be related to a 4D cylindrical black hole in a simple manner, although the corresponding 4D black string has a non-zero energy-momentum tensor [18].

IX. Conclusions

We have found a solution for a charged rotating black string in General Relativity. The maximal analytical extension of the charged solutions, studied in sections IV, have shown some similarities with the Kerr-Newman family of black holes. There is an ergosphere, there are event and Cauchy horizons, closed timelike curves inside the Cauchy horizon and timelike singularities. There are regions, such as the region behind the past event horizon (also appearing in the Kerr-Newman solutions), which will be covered when one performs complete gravitational collapse of some cylindrical matter in an appropriate background.
The maximal analytical extensions of the uncharged solutions do not resemble so much the Kerr solution. The rotating uncharged case shows similarities with the Schwarzschild black hole. For instance, the singularity is spacelike.

It was also shown that this black string solution corresponds to a 3D black hole. Thus, one has a framework in which it is possible to relate lower dimensional results with 4D General Relativity (see also [14, 15, 32, 33, 34]).

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Bibliography

[1] J. R. Oppenheimer, H. Snyder, *Phys. Rev.* **56** (1939) 455.

[2] R. Penrose, *Phys. Rev. Lett.* **14** (1965) 57.

[3] W. A. Hiscock, *J. Math. Phys.* **29** (1988) 443.

[4] R. Penrose, *Riv. Nuovo Cimento* **1** (Numero Special) (1969) 252.

[5] K. S. Thorne, *in Magic without Magic*, ed. J. R. Klauder, Freeman and Company (1972), p. 231.

[6] T. A. Apostolatos, K. S. Thorne, *Phys. Rev D* **46** (1992) 2435.

[7] F. Echeverria, *Phys. Rev. D* **47** (1993) 2271.

[8] S. L. Shapiro, S. A. Teukolky, *Phys. Rev. Lett.* **66** (1991) 994.

[9] M. Bañados, C. Teitelboim and J. Zanelli, *Phys. Rev. Lett.* **69**, (1992) 1849.

[10] S. Giddings, J. Abbot, K. Kuchar, *Gen. Rel. Grav.* **16** (1984) 751.
[11] N. Cruz, J. Zanelli, *Class. Quantum Grav.* **12** (1995) 975.

[12] M. Bañados, M. Henneaux, C. Teitelboim and J. Zanelli, *Phys. Rev. D* **48**, (1993) 1506.

[13] R. B. Mann, S. F. Ross, *Phys. Rev. D* **47** (1993) 3319.

[14] J.P.S. Lemos, *Phys. Lett. B* **353** (1995) 46.

[15] J.P.S. Lemos, “Cylindrical Black Hole in General Relativity”, gr-qc/9404041 (1994).

[16] A. Vilenkin, *Phys. Rep.* **121** (1985) 263.

[17] N. Kaloper, *Phys. Rev. D* **48** (1993) 4658.

[18] J. P. S. Lemos, V. Zanchin, *Phys. Rev. D* **53** (1996) 4684.

[19] S. W. Hawking, *Comm. Math. Phys.* **25** (1972) 152.

[20] T. Jacobson, S. Venkataramani, *Class. Quantum Grav.* **12** (1995) 1055.

[21] B. K. Berger, P. T. Chruściel, V. Moncrief, *Annals Phys.* **237** (1995) 322.

[22] C. W. Misner, K. S. Thorne, J. A. Wheeler, *Gravitation*, Freeman and Company (1973).

[23] J.D. Brown and J.W. York, Jr., *Phys. Rev. D* **47** (1993) 1407.

56
[24] T. Regge and C. Teitelboim, *Ann. Phys.* (NY) **88**, (1974) 286.

[25] J.D. Brown and J.W. York, Jr., *Cont. Math.* **132** (1992) 129.

[26] J.D. Brown, E.A. Martinez, and J.W. York, Jr., *Phys. Rev. Lett.* **66** (1991) 2281.

[27] J. D. Brown, M. Henneux, *Comm. Math. Phys.* **104**, 207 (1986).

[28] R. H. Boyer, R. W. Lindquist, *J. Math. Phys.* **8** (1967) 265.

[29] B. Carter, *Phys. Rev.* **174** (1968) 1559.

[30] S. Chandrasekhar, *The Mathematical Theory of Black Holes*, Oxford University Press (1983).

[31] B. Carter, *Phy. Lett.* **21** (1966) 423.

[32] P. M. Sá, A. Kleber, J. P. S. Lemos, *Class. Quantum Grav.* **13** (1996) 125.

[33] J. P. S. Lemos, *Class. Quantum Gravity* **12** (1995) 1081.

[34] J. P. S. Lemos, Paulo Sá, *Phys. Rev. D.* **49** (1994) 2897.
Figure Captions

Figure 4.1 – The five regions and lines which yield solutions of different nature are shown.

Figure 4.2 – The Penrose diagram representing the non-extreme charged rotating cylindrical black hole. The double line at $r = 0$ indicates a scalar polynomial singularity.

Figure 4.3 – The Penrose diagram for the extremal charged rotating cylindrical black hole.

Figure 4.4 – The Penrose diagram for the charged rotating naked singularity.

Figure 4.5 – The Penrose diagram for the charged rotating null singularity.

Figure 4.6 – The Penrose diagram for the charged rotating solution with one horizon.

Figure 5.1 – The Penrose diagram for the rotating uncharged black hole.

Figure 5.2 – The Penrose diagram for the extremal rotating uncharged black hole. The singularity at $r = 0$ is non-polynomial.

Figure 5.3 – The Penrose diagram for the rotating uncharged naked singularity.
null singularity line

black holes (with one horizon)

extremal singularities

diamond pattern

black holes

\( \frac{64 Q^6}{27 M^4} \)
Figure 4.2
Figure 4.4

\[ r = 0 \quad r = \infty \]

\[ r_{CTC} \quad r_{RS} \]
Figure 4.5
Figure 4.6
Figure 5.1
Figure 5.2

\[ r = 0 \]
\[ r = 0 \]
\[ r_{rs} \]
\[ r = \infty \]
Figure 5.3

$r=0$  $r_{rs}$  $r=\infty$