Gauge transformation and symmetries of the commutative multi-component BKP hierarchy

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Abstract

In this paper, we defined a new multi-component B type Kadomtsev-Petviashvili (BKP) hierarchy that takes values in a commutative subalgebra of $\mathfrak{gl}(N, C)$. After this, we give the gauge transformation of this commutative multi-component BKP (CMBKP) hierarchy. Meanwhile, we construct a new constrained CMBKP hierarchy that contains some new integrable systems, including coupled KdV equations under a certain reduction. After this, the quantum torus symmetry and quantum torus constraint on the tau function of the commutative multi-component BKP hierarchy will be constructed.

Keywords: commutative multi-component BKP hierarchy, constrained commutative multi-component BKP hierarchy, gauge transformation, quantum torus symmetry

(Some figures may appear in colour only in the online journal)

1. Introduction

The KP and Toda lattice hierarchies as completely integrable systems have many important applications in mathematics and physics, including the theory of Lie algebra’s representation, orthogonal, polynomials, and a random matrix model [1–6]. The Kadomtsev-Petviashvili (KP) hierarchy has many kinds of reduction or extension, for example the B type Kadomtsev-Petviashvili (BKP), C type Kadomtsev-Petviashvili (CKP) hierarchies and so on. As important sub-hierarchies of the KP hierarchy, the constrained KP (cKP) hierarchy, the constrained BKP (cBKP) hierarchy, and the constrained CKP (cCKP) hierarchy play an important role in commutative integrable systems.

In [7], the Virasoro symmetry and ASvM formula of the BKP hierarchy were given. In [8, 9], the gauge transformations of the BKP, CKP, constrained BKP, and constrained CKP
hierarchies were constructed. In a paper [10], we constructed the generalized additional symmetries of the two-component BKP hierarchy and identified its algebraic structure. As a reduction of the two-component BKP hierarchy, the D-type Drinfeld–Sokolov hierarchy was found to be a good differential model to derive a complete Block-type infinite dimensional Lie algebra (also called torus Lie algebra). About the Block algebra and its quantization (quantum torus algebra) related to integrable systems, we did a series of works in [11]-[14]. In paper [15], we constructed the additional symmetries of the supersymmetric BKP hierarchy, which constitute a B-type SW_{1+\infty} Lie algebra. Further we generalized the SBKP hierarchy to a supersymmetric two-component BKP (S2BKP) hierarchy and a new supersymmetric Drinfeld–Sokolov hierarchy of type D, which has a super Block-type additional symmetry.

There is another kind of generalization of KP and Toda systems called a multi-component KP [16, 17] or a multicomponent Toda system, which is attracting more and more attention because of its wide use in many fields such as the fields of multiple orthogonal polynomials and non-intersecting Brownian motions. In [18], they considered a generalized multicomponent KP hierarchy, which contains N-independent generalized scalar KP hierarchies in particular by considering a commutative subalgebra of diagonal matrices. In [19], a formalism of multicomponent BKP hierarchies using the elementary geometry of spinors was developed by Kac and van de Leur. In [20], M. Mañas and Luis Martínez Alonso construct a relation between a multicomponent BKP hierarchy and Lame equations from Ramond fermions. The \( \tau \) functions of a 2N-multicomponent KP hierarchy provide solutions of the N-multicomponent, two-dimensional Toda hierarchy [4], which was considered from the point of view of the theory of multiple matrix orthogonal polynomials, non-intersecting Brownian motions, and the matrix Riemann–Hilber problem [21, 22]. The multicomponent Toda hierarchy in [21] is a periodic reduction of the bi-infinite, matrix-formed Toda hierarchy, which contains, matrix-formed Toda equation as the first flow equation. In [23], we defined the extended multicomponent Toda hierarchy and its Sato theory.

In [24], a new hierarchy called the \( \mathbb{Z}_m \)-KP hierarchy, which take values in a maximal commutative subalgebra of \( gl(m, \mathbb{C}) \), was constructed; meanwhile, the relation between the Frobenius manifold and dispersionless reduced \( \mathbb{Z}_m \)-KP hierarchy was discussed. This inspired us to consider the Hirota quadratic equation of the commutative version of the extended multicomponent Toda hierarchy in [25], which might be useful in Frobenius manifold theory.

This paper is arranged as follows. In the next section we recall the factorization problem and construct the multicomponent \( \mathbb{Z}_n \)-BKP hierarchy. In section 2, we will give the Lax equations of the commutative multicomponent BKP hierarchy. In section 3, multifold transformations of the CMBKP hierarchy will be constructed using the determinant technique in [26, 27]. We construct a new constrained CMBKP hierarchy that contains some new integrable systems, including a coupled commutative matrix KdV equation in section 4. In section 5, the quantum torus symmetry and quantum torus constraint on the tau function of the commutative multicomponent BKP hierarchy will be constructed. Section 6 will be devoted to short and discussions.

2. Lax equations of CMBKP hierarchy

In this section we will use the factorization problem to derive Lax equations. We will consider the linear space of the complex \( N \times N \) matrix-valued function \( g : \mathbb{R} \to M_N(\mathbb{C}) \) with the derivative operator \( \partial \). Then the set \( g \) of the Laurent series in \( \partial \) as an associative algebra is a Lie algebra under the standard commutator. This Lie algebra has the following important splitting.
The splitting (1) leads us to consider the following factorization of $g \in G$

$$g = g_+^{-1} \circ g_+,$$  \hspace{1cm} (2)

where $G_\pm$ have $g_\pm$ as their Lie algebras. $G_+$ is the set of invertible linear operators of the form $$\sum_{j \geq 0} g_j(x) \partial^j,$$ while $G_-$ is the set of invertible linear operators of the form $1 + \sum_{j < 0} g_j(x) \partial^j$.

This algebra has a maximal commutative subalgebra $Z_N = \mathbb{C}[\Gamma]/(\Gamma^N)$ and $\Gamma = (\delta_{i,j+1})_j \in gl(N, \mathbb{C})$. Denote $Z_N(\partial) \rightleftharpoons g_+$; then we have the following splitting

$$g_c = g_+ \oplus g_-.$$  \hspace{1cm} (3)

We denote $\cdot^*$ as a formal adjoint operation defined by $p^* = \sum (-1)^j \partial^j \circ p_j$ for an arbitrary $Z_N$-valued pseudo-differential operator $p = \sum p_j \partial^j$, and $(fg)^* = g^* f^*$ for two operators $f, g$. Here $\circ$ means the multiplication of two operators.

Before the work, we list some identities, which will be used in the following sections:

$$A^* = A,$$  \hspace{1cm} (4)

$$(AB)^* = BA,$$ \hspace{1cm} (5)

$$(A \circ \partial \circ B)^* = -B \circ \partial \circ A,$$ \hspace{1cm} (6)

where $A$ and $B$ are $N \times N$ $Z_N$-valued matrix functions. The Lax operator of the CMBKP hierarchy has the form

$$L = \partial + \sum_{j \geq 1} u_j \partial^{-1},$$  \hspace{1cm} (7)

where $u_i$ takes values in the commutative subalgebra $Z_N$. The operator must satisfy the following so-called B-type condition

$$L^* = -\partial \circ L \circ \partial^{-1}.$$ \hspace{1cm} (8)

The CMBKP hierarchy is defined by the following Lax equations:

$$\partial_{2k-1} L = \left[ (B_{2k-1})_+, L \right], \quad B_{2k-1} = L^{2k-1}, \quad k \geq 1.$$ \hspace{1cm} (9)

One can write the operators $L$ in a dressing form as

$$L = \Phi \circ \partial \circ \Phi^{-1}.$$ \hspace{1cm} (10)
where

$$\Phi = 1 + \sum_{i \geq 1} a_i \partial^{-i},$$

satisfying

$$\Phi^* = \partial \circ \Phi^{-1} \circ \partial^{-1}. \quad (12)$$

We call equation (12) the B-type condition of the CMBKP hierarchy. Given $L$, the dressing operators $\Phi$ are determined uniquely up to a multiplication to the right by operators with constant coefficients. The dressing operator $\Phi$ takes values in a B-type commutative Volterra group in $G$. The CMBKP hierarchy (23) can also be redefined as

$$\frac{\partial \Phi}{\partial t_{2k-1}} = -(L^{2k-1}) \circ \Phi,$$

with $k \geq 1$. In the the CMBKP hierarchy, we can derive an equation as the following

$$9w_{x,h} - 5w_{h,h} + \left( v_{xxxxx} - 5v_{xx,0} - 15v_{h}v_{x} + 15v_{vxxx} + 15v_{x}^{3} \right)^{x}_{x} = 0, \quad (14)$$

where $v = \int u_{t} dx$ is in the $Z_{N}$ algebra. We will call the equation (14) the CMBKP equation. When $N = 2$, we can derive the following two-component CMBKP equation as

$$9w_{x,t} - 5w_{h,t} + \left( w_{xxxx} - w_{xx,0} - 15w_{w}w_{x} + 15w_{wxxx} + 15w_{x}^{3} \right)^{x}_{x} = 0,$$

where $v = w + z \Gamma$. After freezing the $t_{3}$ flow, the CMBKP equation will be reduced to the commutative two-component Sawada–Kotera(CMSK) equation as

$$9w_{x,t} + \left( w_{xxxx} + 15w_{w}w_{xxx} + 15w_{x}^{3} \right)^{x}_{x} = 0,$$

$$9z_{x,t} + \left( z_{xxxx} + 15w_{z}w_{xxx} + 15z_{x}w_{xxx} + 45z_{x}w_{x}^{2} \right)^{x}_{x} = 0. \quad (18)$$

With the above preparation, it is time to construct gauge transformations for the CMBKP hierarchy in the next section.

### 3. Gauge transformations of the CMBKP hierarchy

In this section, we will consider the gauge transformation of the CMBKP hierarchy on the Lax operator

$$L^{[1]} = \partial + \sum_{i \geq 1} U_{[i]}^{[1]} \partial^{-i} = W \circ L \circ W^{-1}, \quad (19)$$

where $W$ is the gauge transformation operator. $L^{[1]}$ should satisfy the B-type condition

$$\left( L^{[1]} \right)^* = -\partial \circ L^{[1]} \circ \partial^{-1}, \quad (20)$$

which further implies

$$W^* = \partial \circ W^{-1} \circ \partial^{-1}. \quad (21)$$

This means that after the gauge transformation, the spectral problem about the $N \times N$ spectral matrix $\phi$ taking values in the commutative subalgebra $Z_{N}$ will preserve its form as.
To keep the Lax pair of the CMBKP hierarchy invariant, i.e.,

$$L \cdot \phi = \lambda \phi, \quad \frac{\partial \phi}{\partial t_n} = B_n \cdot \phi. \quad (22)$$

The dressing operator $W$ should satisfy the following dressing equation

$$W_n = -W \circ (B_n)_+ + \left(W \circ B_n \circ W^{-1}\right)_+ \circ W, \quad n = 1, 3, 5, \ldots, \quad (24)$$

where $W_+$ means the derivative of $W$ by $t_n$.

The evolutions of the eigenfunction $\phi$ and the adjoint eigenfunction $\psi$ of the CMBKP hierarchy are defined, respectively, by

$$\frac{\partial \phi}{\partial t_n} = B_n \cdot \phi, \quad \frac{\partial \psi}{\partial t_n} = -\left(B_n\right)^* \cdot \psi, \quad (25)$$

where $\phi = \phi(\lambda; t)$, $\psi = \psi(\lambda; t)$, and $t = (t_1, t_3, t_5, \ldots)$. To give the gauge transformation, we need the following lemma.

**Lemma 1.** The operator $B := \sum_{n=0}^{\infty} b_n \partial^n (B := \sum_{n=0}^{\infty} \partial^n \circ a_n)$ is a $Z_N$-valued differential operator, and $f, g$ (short for $f(x), g(x)$) are two matrix functions taking values in the commutative subalgebra $Z_N$. The following identities hold

$$\left(B \circ f \partial^{-1} \circ g\right)_- = (B \cdot f) \circ \partial^{-1} \circ g, \quad \left(f \partial^{-1} \circ g \circ B\right)_- = f \partial^{-1} \circ (B^* \cdot g). \quad (26)$$

**Proof.** Here we only give the proof of the second equation of (26) by direct calculation based on the first equation of (26)

$$\left(f \partial^{-1} \circ g \circ B\right)_- = \left(-B^* \circ g \circ \partial^{-1} \circ f\right)_-^* = \left((B^* \cdot g) \circ \partial^{-1} \circ f\right)_-^* = \sum_{m=0}^{\infty} \left(-a_m \left((- \partial)^m \cdot g\right) \partial^{-1} \circ f\right)_-^* = \sum_{m=0}^{\infty} (-1)^m f \partial^{-1} \circ \left(\partial^m \cdot g\right)_-^* \circ a_m = \sum_{m=0}^{\infty} f \partial^{-1} (-1)^m \circ a_m \left(\partial^m \cdot g\right) = f \partial^{-1} \circ \left(B^* \cdot g\right). \quad (27)$$

**Lemma 2.** The operators $T_D = \phi \circ \partial \circ \phi^{-1}$ and $T_I = \psi^{-1} \circ \partial^{-1} \circ \psi$ satisfy equation (24), which implies $T_D T_I = \phi \circ \partial \circ \phi^{-1} \circ \psi^{-1} \circ \partial^{-1} \circ \psi$ can also satisfy equation (24).

Now, we will find out the gauge transformation operator $W$ of the CMBKP hierarchy. Firstly, we consider the two operators

$$T_D(\phi) = \phi \circ \partial \circ \phi^{-1}, \quad T_I(\psi) = \psi^{-1} \circ \partial^{-1} \circ \psi, \quad (28)$$
where \( \phi \) and \( \psi \) are \( N \times N \) matrix-valued eigenfunctions taking values in the commutative subalgebra \( Z_N \). Then we have

\[
\left( T_D^{-1}(\phi) \right)^* = -T_I(\phi), \left( T_I^{-1}(\psi) \right)^* = -T_D(\psi).
\]  

(29)

We can easily get

\[
T_D(\phi) \cdot \phi = 0, \left( T_I^{-1}(\psi) \right)^* \cdot \psi = 0.
\]

(30)

Similarly to reference [26], we can consider two sets of matrix functions \( \{ \phi_i^{(0)}, i = 1, 2, \ldots; \phi_i^{(1)} \} \) and \( \{ \psi_i^{(0)}, i = 1, 2 \cdots; \psi_i^{(1)} \} \). For \( T_D(\phi) = \phi \circ \partial \circ \phi^{-1} \), we do iterations by the following two steps. For the first step, we consider:

\[
T_D^{(1)} = T_D^{(1)} \left( \phi_i^{(0)} \right) = \phi_i^{(0)} \circ \partial \circ \left( \phi_i^{(0)} \right)^{-1};
\]

we define the rule of transformation under \( T_D^{(1)} \) as

\[
\phi_i^{(1)} = T_D^{(1)} \left( \phi_i^{(0)} \right) = \phi_i^{(0)} \cdot \phi^{(1)} = \left( T_D^{(1)} \left( \phi_i^{(0)} \right) \right)^{-1} \cdot \phi^{(0)} = -T_I \left( \phi_i^{(0)} \right) \cdot \phi^{(0)},
\]

(32)

\[
\phi_i^{(1)} = T_D^{(1)} \left( \phi_i^{(0)} \right) = \phi_i^{(0)} \cdot \phi^{(1)} = \left( T_D^{(1)} \left( \phi_i^{(0)} \right) \right)^{-1} \cdot \phi^{(0)} = -T_I \left( \phi_i^{(0)} \right) \cdot \phi^{(0)},
\]

(33)

where \( i \geq 2 \) for \( \phi_i^{(1)} \) and

\[
\psi_i^{(1)} = -T_I \left( \phi_i^{(0)} \right) \cdot \left( \psi_i^{(0)} \right).
\]

(34)

For the second step, we consider:

\[
T_D^{(2)} = T_D^{(2)} \left( \phi_i^{(1)} \right) = \phi_i^{(1)} \circ \partial \circ \left( \phi_i^{(1)} \right)^{-1};
\]

we define the rule of transformation under \( T_D^{(2)} \) as

\[
\phi_i^{(2)} = T_D^{(2)} \left( \phi_i^{(1)} \right) = \phi_i^{(1)} \cdot \phi^{(2)} = \left( T_D^{(2)} \left( \phi_i^{(1)} \right) \right)^{-1} \cdot \phi^{(1)} = -T_I \left( \phi_i^{(1)} \right) \cdot \phi^{(1)},
\]

(36)

\[
\phi_i^{(2)} = T_D^{(2)} \left( \phi_i^{(1)} \right) = \phi_i^{(1)} \cdot \phi^{(2)} = \left( T_D^{(2)} \left( \phi_i^{(1)} \right) \right)^{-1} \cdot \phi^{(1)} = -T_I \left( \phi_i^{(1)} \right) \cdot \phi^{(1)},
\]

(37)

where \( i \geq 3 \) for \( \phi_i^{(2)} \) and

\[
\psi_i^{(2)} = -T_I \left( \phi_i^{(1)} \right) \cdot \left( \psi_i^{(1)} \right).
\]

(38)

For \( T_I(\psi) = \psi^{-1} \circ \partial^{-1} \circ \psi \), it obeys the following iterated rule:

For the first step, we consider:

\[
T_I^{(1)} = T_I^{(1)} \left( \psi_i^{(0)} \right) = \left( \psi_i^{(0)} \right)^{-1} \circ \partial^{-1} \circ \left( \psi_i^{(0)} \right),
\]

(39)

\[
\phi_i^{(1)} = T_I^{(1)} \left( \psi_i^{(0)} \right) \cdot \phi^{(0)} = \left( T_I^{(1)} \left( \psi_i^{(0)} \right) \right)^{-1} \cdot \phi^{(0)} = -T_D \left( \psi_i^{(0)} \right) \cdot \phi^{(0)},
\]

(40)

\[
\phi_i^{(1)} = T_I^{(1)} \left( \psi_i^{(0)} \right) \cdot \phi^{(0)} = \left( T_I^{(1)} \left( \psi_i^{(0)} \right) \right)^{-1} \cdot \phi^{(0)} = -T_D \left( \psi_i^{(0)} \right) \cdot \phi^{(0)},
\]

(41)

where \( i \geq 2 \) for \( \phi_i^{(1)} \) and

\[
\psi_i^{(1)} = -T_D \left( \psi_i^{(0)} \right) \cdot \left( \psi_i^{(0)} \right).
\]

(42)
For the second step, we consider:

\[ T_i^{(2)} = T_i^{(2)}(\psi_2^{(1)}) = (\psi_2^{(1)})^{-1} \circ \partial^{-1} \circ (\psi_2^{(1)}), \] (43)

\[ \phi_1^{(2)} = T_i^{(2)}(\psi_2^{(1)}) \cdot \phi_1^{(1)} = (T_i^{(2)}(\psi_2^{(1)}))^{-1} \cdot \psi_1^{(1)} = -T_D(\psi_2^{(1)}) \cdot \psi_1^{(1)}, \] (44)

\[ \phi_i^{(2)} = T_i^{(2)}(\psi_2^{(1)}) \cdot \phi_i^{(1)} = (T_i^{(2)}(\psi_2^{(1)}))^{-1} \cdot \psi_i^{(1)} = -T_D(\psi_2^{(1)}) \cdot \psi_i^{(1)}, \] (45)

where \( i \geq 3 \) for \( \psi_1^{(1)} \) and

\[ \psi_i^{(2)} = -T_D(\psi_2^{(1)}) \cdot (\psi_i^{(1)}). \] (46)

It is obvious that a single step of the operator \( T_D \) or \( I_i \) cannot keep the restriction of the B-type condition. We use

\[ W_i = T_{i+1} = T_i(\psi_1^{(1)}) \circ T_D(\phi_1^{(0)}) \] (47)

as the gauge transformation operator, and we have \( L_i^{(1)} = W_i L W_i^{-1} \). Let us check whether it satisfies the required constraint

\[ (L_i^{(1)})^* = -\partial L_i^{(1)} \partial^{-1}. \] (48)

We can calculate

\[ (L_i^{(1)})^* = \left( (\psi_1^{(1)})^{-1} \circ \partial^{-1} \circ \psi_1^{(1)} \circ \phi_1^{(0)} \circ \partial \circ (\psi_1^{(1)})^{-1} \circ \right. \]

\[ \times L \circ \phi_1^{(0)} \circ \partial^{-1} \circ (\phi_1^{(0)})^{-1} \circ (\psi_1^{(1)})^{-1} \circ \partial \circ (\psi_1^{(1)}) \] \]

\[ = -\left( (\psi_1^{(1)}) \circ \partial \circ (\psi_1^{(1)})^{-1} \circ \phi_1^{(0)} \circ \partial \circ (\psi_1^{(1)})^{-1} \circ \partial \circ (\psi_1^{(1)}) \right). \] (49)

and

\[ -\partial L_i^{(1)} \partial^{-1} = -\partial \circ \left( (\psi_1^{(1)})^{-1} \circ \partial^{-1} \circ \psi_1^{(1)} \circ \phi_1^{(0)} \circ \partial \circ (\phi_1^{(0)})^{-1} \right. \]

\[ \times L \circ \phi_1^{(0)} \circ \partial^{-1} \circ (\phi_1^{(0)})^{-1} \circ (\psi_1^{(1)})^{-1} \circ \partial \circ (\psi_1^{(1)}) \circ \partial^{-1} \] \]

\[ = \left( (\psi_1^{(1)}) \circ \partial \circ (\psi_1^{(1)})^{-1} \circ \phi_1^{(0)} \circ \partial \circ (\phi_1^{(0)})^{-1} \right), \] (50)

which means in order to keep the constraint \((L_i^{(1)})^* = -\partial L_i^{(1)} \partial^{-1}, T \) should satisfy the following equation:

\[ T_D(\psi_1^{(1)}) T_i(\phi_1^{(0)}) \circ \partial = \partial \circ T_i(\psi_1^{(1)}) T_D(\phi_1^{(0)}). \] (51)

where \( \psi_1^{(1)} = -\phi_1^{(0)} \int (\phi_1^{(0)}) \psi_1^{(0)} \) and \( \int \) means the integral about spatial variable \( x \).

Then we can acquire the following theorem, because the CMBKP hierarchy takes values in a commutative subalgebra, just like the case when \( N = 1 \), i.e., the case of the original BKP hierarchy.

**Theorem 1.** The B-type condition of the CMBKP hierarchy implies \( \psi_1^{(0)} \) and \( \phi_1^{(0)} \) have the following relation:

\[ \psi_1^{(0)} = \psi_{1,x}. \] (52)
The B-type reduction of \( L \) guarantees that there exists at least one solution \((\phi; \psi)\) that satisfies equation (52). In fact, the above theorem can be generalized to the case of the \( gl(N, \mathbb{C}) \)-valued multicomponent BKP hierarchy, which is not commutative.

The B-type condition of the \( gl(N, \mathbb{C}) \)-valued multicomponent BKP hierarchy implies that the noncommutative matrices \( \psi_1^{(0)} \) and \( \phi_1^{(0)} \) have the following relation:

\[
\left( (\phi_1^{(0)})^T \right)^{-2} \int_x \left( (\phi_1^{(0)})^T \psi_1^{(0)} - \psi_1^{(0)} \right) - \left( (\phi_1^{(0)})^T \right)^{-1} \left( \int_x (\phi_1^{(0)})^T \psi_1^{(0)} \right) \phi_1^{(0)} (\phi_1^{(0)})^{-1} + (\phi_1^{(0)})^T x = 0, \tag{53}
\]

where \( T \) means the transposition of matrices.

The proof of equation (53) will be skipped here, because the focus of this paper is about the CMBKP hierarchy. A thorough study on the \( gl(N, \mathbb{C}) \)-valued multicomponent BKP hierarchy will be contained in another work of ours in the future.

**Remark.** From equation (52) to equation (53), one can see clearly the difference of the BKP systems from \( Z_N \) to \( gl(N, \mathbb{C}) \).

In order to keep the B-type restriction of the Lax operator of the CMBKP hierarchy, we do iterations of the gauge transformation \( W_n = T_{n+1} \). In particular,

\[
W_2 = T_{2+2} = T_1 (\psi^{(3)}) \circ T_0 (\phi^{(2)}) \circ T_1 (\psi^{(1)}) \circ T_0 (\phi^{(0)}), \tag{54}
\]

\[
W_n = T_{n+n} = T_1 (\psi^{(2n-1)}) \circ T_0 (\phi^{(2n-2)}) \circ \cdots \circ T_1 (\psi^{(1)}) \circ T_0 (\phi^{(0)}), \tag{55}
\]

where \( \psi^{(2n-1)} = -T_1 ((\phi^{(2n-2)}) \cdot (\psi^{(2n-2)}), \psi^{(i)} = (\phi^{(i)})_i \). It can be easily checked that \( W_n \cdot \phi^{(0)} \big|_{n \in n} = 0, (W_n^{-1})^y \cdot (\psi^{(0)}) \big|_{n \in n} = 0 \). The relations \( \psi^{(i)} = (\phi^{(i)})_i, n = 1, 2, \ldots \) can keep the dressing procedures \( W_n = T_{n+1}, n = 1, 2, \ldots \) always preserving the B-type condition of the new Lax operators \( L^{(n)} \). This is similar as the case of the BKP hierarchy in [8].

We denote \( t = (t_1, t_2, t_3, \ldots) \) and introduce the \( Z_N \)-valued wave function as

\[
w(t; z) = \Phi \cdot e^{\xi(t; z)}, \tag{56}
\]

where the function \( \xi \) is defined as \( \xi(t; z) = \sum_{k \in 2^{n+1}} t_k z^k \). It is easy to see

\[
L w(t; z) = zw(t; z), \quad \frac{\partial w}{\partial t_{2n+1}} = L_{2n+1} w. \tag{57}
\]

The \( Z_N \)-valued tau function \( \tau \) of the CMBKP hierarchy can be defined in the form of the wave functions as

\[
w(t; z) = \frac{\tau(t - 2[z^{-1}])}{\tau(t)} e^{\xi(t; z)}, \tag{58}
\]

where \([z] = (z^3/3, z^5/5, \ldots)\).

The generating functions of n-step \( T_n \) and n-step \( T_i \) are denoted as \( (\phi_1, \ldots, \phi_{n-1}, \phi_n) \) and \( (\psi_1, \ldots, \psi_{n-1}, \psi_n) \) in order, respectively. The generating functions have the following B-type constraint

\[
\psi_1 = (\phi_1), \tag{59}
\]

Using the above gauge transformation, we can derive the gauge transformation on the tau function of the CMBKP hierarchy as
\[-(n+a) = G_{W,n}(\psi_0, \psi_{h-1}, \ldots, \psi_i, \phi_1, \ldots, \phi_{n-1}, \phi_n) \tau, \quad (60)\]

where the generalized Wronskian $G_{W,n}$ is defined as [9]

\[
G_{W,n}(g_k, g_{k-1}, \ldots, g_i; f_1, f_2, \ldots, f_n) =
\begin{vmatrix}
\int g_k f_1 & \int g_{k-1} f_1 & \ldots & \int g_{i+1} f_1 \\
\int g_k f_2 & \int g_{k-1} f_2 & \ldots & \int g_{i+1} f_2 \\
\vdots & \vdots & \ddots & \vdots \\
\int g_k f_n & \int g_{k-1} f_n & \ldots & \int g_{i+1} f_n \\

f_1 & f_2 & \ldots & f_n \\
\vdots & \vdots & \ddots & \vdots \\
(\Delta)^{(n-k-1)}(f_1) & (\Delta)^{(n-k-1)}(f_2) & \ldots & (\Delta)^{(n-k-1)}(f_n)
\end{vmatrix}.
\quad (62)\]

When $k = 0$, the generalized Wronskian $G_{W,n}$ will be reduced to the ordinary Wronskian. Now, we will only give the first gauge transformation of the CMBKP hierarchy included in the following proposition.

**Proposition 1.** If the eigenfunction $\phi$ and the adjoint eigenfunction $\psi$ satisfy equation (25), the one-fold gauge transformation operator of the CMBKP hierarchy

\[
W_i = (\psi_1^{(1)})^{-1} \circ \partial^{-1} \circ \psi_1^{(1)} \circ \phi_1^{(0)} \circ \partial \circ (\psi_1^{(0)})^{-1},
\]

satisfies $W_i \phi_1^{(0)} = 0$ and $(W_i^{-1})^* (\psi_1^{(0)}) = 0$. $W_i$ will generate new solutions $U_i^{[1]}$ from seed solutions $U_i$. To see it clearly, here we only give the transformations of the first two dynamic functions

\[
U_i^{[1]} = U_i + 2(\ln \phi_1^{(0)})_{xx}, \quad (64)
\]

\[
U_2^{[1]} = U_2 + 4(\ln \phi_1^{(0)})_{xx} (\ln \phi_1^{(0)})_x - 2 \left( \frac{\psi_1^{(0)}}{\phi_1^{(0)}} \right)_x. \quad (65)
\]

If we suppose $U_i = \begin{bmatrix} \alpha & 0 \\ \beta & \alpha \end{bmatrix}$, $U_2 = \begin{bmatrix} \gamma & 0 \\ \eta & \gamma \end{bmatrix}$, $\phi_1^{(0)} = \begin{bmatrix} \phi_0 & 0 \\ \phi_1 & \phi_0 \end{bmatrix}$, then we can derive the explicit transformation as

\[
\alpha^{[1]} = \alpha + 2(\ln \phi_0)_{xx}, \quad (66)
\]

\[
\beta^{[1]} = \beta + 2 \left( \frac{\phi_1}{\phi_0} \right)_{xx}, \quad (67)
\]

\[
\gamma^{[1]} = \gamma + 4(\ln \phi_0)_{xx} (\ln \phi_0)_x - 2 \left( \frac{\phi_0}{\phi_0} \right)_x. \quad (68)
\]
In the calculation, the identity

\[
\ln \phi_0 = \frac{\phi_1}{\phi_0} \ln \phi_0 = \frac{\phi_1}{\phi_0} \ln \phi_0
\]

is used.

For the case \( N = 1 \), \( W_1 \) will generate new solutions of the BKP hierarchy from the seed solutions.

4. Constrained CMBKP hierarchy

In this section, we will consider the operator of the constrained CMBKP(cCMBKP) hierarchy as

\[
L = \partial + u\partial^{-1}v - v\partial^{-1}u,
\]

where \( u \) and \( v \) are \( N \times N \) matrix functions taking values in \( \mathbb{Z}_N \). Here, \( u, v \) should satisfy the following Sato equation

\[
u_{2n-1} = B_{2n-1} \cdot u, \ n \in \mathbb{N},
\]

Because of the B-type condition equation (8), one can prove that the \( \partial^0 \) term does not exist in \( B_{2n-1} \), as mentioned in [1]. That means that \( u = v = 1 \) is a trivial solution.

Suppose \( u = q + p\Gamma, \ v = r + s\Gamma \), and consider the case when \( N = 2 \), i.e.,

\[
u = \begin{bmatrix} q & 0 \\ p & q \end{bmatrix}, \ u = \begin{bmatrix} r & 0 \\ s & r \end{bmatrix}
\]

Then we can further derive the following coupled equations

\[
q_t = q_{3x} + 3(qr_s - rq_s)q_x,
\]

\[
p_t = p_{3x} + 3 \left( (qr_s - rq_s)p_x + (pr_x + qs_x - sq_x - rp_x)q_x \right),
\]

\[
r_t = r_{3x} + 3(qr_s - rq_s)r_x,
\]

\[
s_t = s_{3x} + 3 \left( (qr_s - rq_s)s_x + (pr_x + qs_x - sq_x - rp_x)r_x \right).
\]

If \( q = r, p = s \), we can derive the following trivial equations

\[
q_t = q_{3x},
\]

\[
p_t = p_{3x}.
\]

If \( r = s = 1 \), we can derive the following coupled matrix KdV-like equation

\[
q_t = q_{3x} - 3q_x^2,
\]

\[
p_t = p_{3x} - 6q_x p_x - 3q_x^2.
\]

Similarly to [9], we can derive the new solutions generated from the seed solution \( q, r \)

\[
u^{(n+n)} = \frac{GW_{2n+1}(\psi^{(n-1)}, \psi^{(n-2)}, ..., \psi^{(1)}, u_x; u, \phi^{(1)}, ..., \phi^{(n-1)}, \phi^{(n)})}{GW_{2n}(\psi^{(n-1)}, \psi^{(n-2)}, ..., \psi^{(1)}, u_x; u, \phi^{(1)}, ..., \phi^{(n-2)}, \phi^{(n-1)})},
\]

(73)
\[ i^{(n+n)} = \frac{(-1)^nGW_{n-1,n} \left( \psi^{(n-2)}, \psi^{(n-3)}, \ldots, \psi^{(1)}, u, \phi^{(1)}, \ldots, \phi^{(n-2)}, \phi^{(n-1)} \right)}{GW_{n,n} \left( \psi^{(n-1)}, \psi^{(n-2)}, \ldots, \psi^{(1)}, u, \phi^{(1)}, \ldots, \phi^{(n-2)}, \phi^{(n-1)} \right)}, \]

(74)

where \( \phi^{(i)} = L/u \) and \( (\phi^{(i)}, \psi^{(i)}) \) have the same relation as equation (59). Also, the iteration on the constrained tau functions \( \tau_c \) of the constrained CMBKP hierarchy is derived as

\[ \tau^{(n+n)} = GW_{n,n} \left( \psi^{(n-1)}, \psi^{(n-2)}, \ldots, \psi^{(1)}, u, \phi^{(1)}, \ldots, \phi^{(n-2)}, \phi^{(n-1)} \right) \tau_c. \]

(75)

In the above process of calculations, all the elements in the above Wronskians must keep being always written in terms of \( \Gamma \). In this way, one can keep the new solutions \( u^{(n+n)}, \psi^{(n+n)} \) taking values in the algebra \( Z_N \).

5. Quantum torus constraint of CMBKP hierarchy

In this section, we will focus on the quantum torus symmetry of the CMBKP hierarchy. Firstly we define the operator \( \Gamma_B \) and the \( Z_N \)-valued Orlov–Shulman’s operator \( M \) as

\[ \Gamma_B = \sum_{i \in \mathbb{Z}_N^0} i u_i \partial^{-1}, \quad M = \Phi \Gamma_B \Phi^{-1}. \]

(76)

The Lax operator \( L \) and the \( Z_N \)-valued Orlov–Shulman’s \( M \) operator satisfy the following canonical relation

\[ [L, M] = 1. \]

(77)

With the above preparation, it is time to construct additional symmetries for the CMBKP hierarchy in the next part. Then it is easy to get that the operator \( M \) satisfies

\[ [L, M] = 1, \quad Mw(z) = \partial w(z); \]

\[ \frac{\partial M}{\partial t_k} = \left[ \left( L^k \right)_+, M \right], \quad k \in \mathbb{Z}_N^{\text{odd}}. \]

(79)

Given any pair of integers \((m,n)\) with \( m, n \geq 0 \), we will introduce the following \( Z_N \)-valued operator \( B_{mn} \)

\[ B_{mn} = M^m L^n - (-1)^m L^{n-1} M^m L. \]

(80)

For any \( Z_N \)-valued operator \( B_{mn} \) in (80), one has

\[ \frac{\partial B_{mn}}{\partial t_k} = \left[ \left( L^k \right)_+, B_{mn} \right], \quad k \in \mathbb{Z}_N^{\text{odd}}. \]

(81)

Using

\[ \Phi^* = \partial \Phi^{-1} \partial^{-1}, \quad \Gamma_B^* = \Gamma_B, \]

(82)

the \( Z_N \)-valued operator \( M \) satisfies the following identity,

\[ M^* = \partial L^{-1} M L \partial^{-1}. \]

(83)

It is easy to check that the \( Z_N \)-valued operator \( B_{mn} \) satisfies the B-type condition, namely,

\[ B_{mn}^* = - \partial B_{mn} \partial^{-1}. \]

(84)

Now we will denote the operator \( D_{mn} \) as

\[ D_{mn} := c^{mn} L^{-1} q^n - L^{-1} q^{-n} c^{mn} L. \]

(85)
Using equation (84), the B-type property of $D_{mn}$ can be derived as

$$D_{mn}^* = -\partial D_{mn} \partial^{-1}.$$ 

Therefore, we get the following important B-type condition, which the $\mathbb{Z}_N$-valued operator $D_{mn}$ satisfies

$$D_{mn}^* = -\partial D_{mn} \partial^{-1}. \quad (86)$$

Then, based on a quantum parameter $q$, the additional flows for the time variable $t_{m,n}$, $t_{m,n}^*$ are defined as follows

$$\frac{\partial \Phi}{\partial t_{m,n}} = -(B_{mn})_\Phi, \quad \frac{\partial \Phi}{\partial t_{m,n}^*} = -(D_{mn})_\Phi,$$ 

or equivalently rewritten as

$$\frac{\partial L}{\partial t_{m,n}} = -[(B_{mn}), L], \quad \frac{\partial M}{\partial t_{m,n}^*} = -[(D_{mn}), M]. \quad (88)$$

Generally, one can also derive

$$\partial t_{m,n}^*(D_{mn}) = -[(D_{lk}), D_{mn}]. \quad (89)$$

Using the similar proof as the BKP hierarchy in [14], the additional flows of $\partial t_{m,n}$ can be proven to be symmetries of the CMBKP hierarchy; i.e., they commute with all $\partial t_{n}$ flows of the CMBKP hierarchy.

The additional flows $\partial t_{m,n}$ of the CMBKP hierarchy form the $W_\infty$ algebra similarly as [7], which is about the BKP hierarchy.

Now it is time to identify the algebraic structure of the additional $t_{m,n}^*$ flows of the CMBKP hierarchy.

**Theorem 2.** The additional flows $\partial t_{m,n}^*$ of the CMBKP hierarchy form the positive half of the quantum torus algebra, i.e.,

$$\left[ \partial t_{m,n}^*, \partial t_{l,k}^* \right] = \left( q^{nl} - q^{lk} \right) \partial t_{n+l,m+k}^*, \quad n, m, l, k \geq 0. \quad (90)$$

**Remark.** The $t_{m,n}^*$ additional flows constitute a nice quantum torus algebra, because it is based on a commutative algebra. This is different from the multicomponent BKP, whose additional symmetry constitutes the multifold quantum torus algebra [28].

Next, similarly to the KP and BKP hierarchy [14], we will consider the quantum torus constraint on the $\mathbb{Z}_N$-valued tau function of the CMBKP hierarchy.

Similarly as [14], one has shown that

$$\partial_{t,m} \log w = \left( e^\theta - 1 \right) \frac{Z_{Z_N}^{(p+1)}(\tau)}{p + 1} \frac{1}{\tau}, \quad (91)$$
where
\[ \hat{\eta} = \sum_{i \in \mathbb{Z}^{d_{\text{rel}}}} \lambda^i \frac{\partial}{\partial \eta_i}, \]
and \( Z^{(p+1)}_q \) is the generator of the \( \mathbb{W}_\infty \) algebra. Then with the help of rewriting the quantum torus flow \( \hat{\eta}^* \) in terms of the \( \hat{\eta}_{\text{ps}} \) flows
\[ \hat{\eta}_{\text{ps}}^* = \sum_{p,s=0}^{\infty} \frac{t^p (\log q)^s}{p! s!} \hat{\eta}_{\text{ps}}^*, \]
and denoting
\[ L_{\text{ps}}^\psi := \sum_{p,s=0}^{\infty} \frac{t^p (\log q)^s Z^{(p+1)}_q}{p! s!} \]
the quantum torus constraint on the \( Z_\infty \)-valued wave function \( w \), i.e.,
\[ \hat{\eta}_{\text{ps}}^* w = 0, \]
will lead to the quantum torus constraint on the \( Z_\infty \)-valued tau function of the CMBKP hierarchy
\[ L_{\text{ps}}^\psi = c, \]
where \( c \) is a constant.

6. Conclusions and discussions

In this paper, we define a new multicomponent BKP hierarchy that takes values in a commutative subalgebra of \( gl(N, \mathbb{C}) \). After this, we give the gauge transformation of the commutative multicomponent BKP hierarchy. Meanwhile, we construct a new constrained CMBKP hierarchy that contains some integrable systems, including coupled matrix KdV equations under a certain reduction. After this, the quantum torus symmetry and quantum torus constraint of the commutative multicomponent BKP hierarchy are constructed. We are looking forward to the possible application of the quantum torus constraint in topological field theory and enumerating geometry. For the importance of the BKP hierarchy in representation theory and mathematical physics, an interesting question is the application of the commutative multicomponent BKP hierarchy in other theories such as the Frobenius manifold.

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