ON MINIMAL EDGE VERSION OF DOUBLY RESOLVING SETS OF A GRAPH

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Abstract. In this paper, we introduce the edge version of doubly resolving set of a graph which is based on the edge distances of the graph. As a main result, we computed the minimum cardinality $\psi_E$ of edge version of doubly resolving sets of family of $n$-sunlet graph $S_n$ and prism graph $Y_n$.

1. Introduction and Preliminaries

Let us take a graph $G = (V(G), E(G))$, which is simple, connected and undirected, where its vertex set is $V(G)$ and edge set is $E(G)$. The order of a graph $G$ is $|V(G)|$ and the size of a graph $G$ is $|E(G)|$. The distance $d(a, b)$ between the vertices $a, b \in V(G)$ is the length of a shortest path between them. If $d(c, a) \neq d(c, b)$, then the vertex $c \in V(G)$ is said to resolve two vertices $a$ and $b$ of $V(G)$. Suppose that $N = \{n_1, n_2, \ldots, n_k\} \subseteq V(G)$ is an ordered set and $m$ is a vertex of $V(G)$, then the representation $r(m, N)$ of $m$ with respect to $N$ is the k-tuple $(d(m, n_1), d(m, n_2), \ldots, d(m, n_k))$. If different vertices of $G$ have different representations with respect to $N$, then the set $N$ is said to be a resolving set of $G$. The metric basis of $G$ is basically a resolving set having minimum cardinality. The cardinality of metric basis is represented by $\text{dim}(G)$, and is called metric dimension of $G$.

In [19], Slater introduced the idea of resolving sets and also in [10], Harary and Melter introduced this concept individually. Different applications of this idea has been introduced in the fields like network discovery and verification [1], robot navigation [15] and

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The introduction of doubly resolving sets is given by Caceres et al. (see [3]) by presenting its connection with metric dimension of the cartesian product $G \square G$ of the graph $G$.

The doubly resolving sets create a valuable means for finding upper bounds on the metric dimension of graphs. The vertices $a$ and $b$ of the graph $G$ with order $|V(G)| \geq 2$ are supposed to doubly resolve vertices $u_1$ and $v_1$ of the graph $G$ if $d(u_1, a) - d(u_1, b) \neq d(v_1, a) - d(v_1, b)$. A subset $D$ of vertices doubly resolves $G$ if every two vertices in $G$ are doubly resolved by some two vertices of $D$. Precisely, in $G$ there do not exist any two different vertices having the same difference between their corresponding metric coordinates with respect to $D$. A doubly resolving set with minimum cardinality is called the minimal doubly resolving set. The minimum cardinality of a doubly resolving set for $G$ is represented by $\psi(G)$. In case of some convex polytopes, hamming and prism graphs, the minimal doubly resolving sets has been obtained in [13], [14] and [4] respectively.

Since, the line graph $L(G)$ of a graph $G$ is defined as, the graph whose vertices are the edges of $G$, with two adjacent vertices if the corresponding edges have one vertex common in $G$. In mathematics, the metric properties of line graph have been studied to a great extent (see [2, 5, 6, 17, 18]) and in chemistry literature, its significant applications have been proved (see [7, 8, 9]). In [16], the edge version of metric dimension have been introduced, which is defined as:

**Definitions 1.1.**

1. The edge distance $d_E(f, g)$ between two edges $f, g \in E(G)$ is the length of a shortest path between vertices $f$ and $g$ in the line graph $L(G)$.

2. If $d_E(e, f) \neq d_E(e, g)$, then the edge $e \in E(G)$ is said to edge resolve two edges $f$ and $g$ of $E(G)$. 
(3) Suppose that \( N_E = \{f_1, f_2, \ldots, f_k\} \subseteq E(G) \) is an ordered set and \( e \) is an edge of \( E(G) \), then the edge version of representation \( r_E(e, N_E) \) of \( e \) with respect to \( N_E \) is the \( k \)-tuple \( (d_E(e, f_1), d_E(e, f_2), \ldots, d_E(e, f_k)) \).

(4) If different edges of \( G \) have different edge version of representations with respect to \( N_E \), then the set \( N_E \) is said to be an edge version of resolving set of \( G \).

(5) The edge version of metric basis of \( G \) is basically an edge version of resolving set having minimum cardinality. The cardinality of edge version of metric basis is represented by \( \dim_E(G) \), and is called edge version of metric dimension of \( G \).

The following theorems in [16] are important for us.

**Theorem 1.2.** Let \( S_n \) be the family of \( n \)-sunlet graph then

\[
\dim_E(S_n) = \begin{cases} 
2, & \text{if } n \text{ is even;} \\
3, & \text{if } n \text{ is odd.}
\end{cases}
\]

**Theorem 1.3.** Let \( Y_n \) be the family of prism graph then \( \dim_E(Y_n) = 3 \) for \( n \geq 3 \).

In this article, we proposed minimal edge version of doubly resolving sets of a graph \( G \), based on edge distances of graph \( G \) as follows:

**Definitions 1.4.**

1. The edges \( f \) and \( g \) of the graph \( G \) with size \( |E(G)| \geq 2 \) are supposed to edge doubly resolve edges \( f_1 \) and \( f_2 \) of the graph \( G \) if \( d_E(f_1, f) - d_E(f_1, g) \neq d_E(f_2, f) - d_E(f_2, g) \).

2. Let \( D_E = \{f_1, f_2, \ldots, f_k\} \) be an ordered set of the edges of \( G \) then if any two edges \( e \neq f \in E(G) \) are edge doubly resolved by some two edges of set \( D_E \) then the set \( D_E \subseteq E(G) \) is said to be an edge version of doubly resolving set of \( G \). The minimum cardinality of an edge version of doubly resolving set of \( G \) is represented by \( \psi_E(G) \).
Note that every edge version of doubly resolving set is an edge version of resolving set, which implies \( \dim_E(G) \leq \psi_E(G) \) for all graphs \( G \).

2. The edge version of doubly resolving sets for family of \( n \)-sunlet graph \( S_n \).

The family of \( n \)-sunlet graph \( S_n \) is obtained by joining \( n \) pendant edges to a cycle graph \( C_n \) (see Figure 1).

![Figure 1. \( n \)-sunlet graph \( S_n \)](image)

For our purpose, we label the inner edges of \( S_n \) by \( \{e_i : \forall 0 \leq i \leq n - 1\} \) and the pendent edges by \( \{f_i : \forall 0 \leq i \leq n - 1\} \) as shown in Figure 1.

![Figure 2. \( L(S_n) \) of \( n \)-sunlet graph \( S_n \)](image)

As motivated by the Theorem 1.2, we obtain

\[
\psi_E(S_n) \geq \begin{cases} 
2, & \text{if } n \text{ is even;} \\
3, & \text{if } n \text{ is odd.} 
\end{cases}
\]

Furthermore, we will show that \( \psi_E(S_n) = 3 \) for \( n \geq 4 \).
In order to calculate the edge distances for family of \(n\)-sunlet graphs \(S_n\), consider the line graph \(L(S_n)\) as shown in Figure 2. Define \(S_i(e_0) = \{e \in E(S_n) : d(e_0, e) = i\}\). For \(\psi_E(S_n)\) with \(n \geq 4\), we can locate the sets \(S_i(e_0)\) that are represented in the Table 1. It is clearly observed from Figure 2 that \(S_i(e_0) = \emptyset\) when \(i \geq k + 1\) for \(n = 2k\), and \(S_i(e_0) = \emptyset\) when \(i \geq k + 2\) for \(n = 2k + 1\).

From the above mentioned sets \(S_i(e_0)\), it is clear that they can be utilized to define the edge distances between two arbitrary edges of \(E(S_n)\) in the subsequent way.

| \(n\) | \(i\) | \(S_i(e_0)\) |
|------|------|------------|
| \(2k (k \geq 2)\) | \(k\) | \(\{f_{k-1}, f_k\}\) |
| \(2k + 1 (k \geq 2)\) | \(k\) | \(\{f_{k-1}, f_k, f_{k+1}\}\) |
| \(k + 1\) | | \(\{f_k\}\) |

The symmetry in Figure 2 shows that \(d_E(e_i, e_j) = d_E(e_0, e_{|j-i|})\) for \(0 \leq |j-i| \leq n - 1\).

If \(n = 2k\), where \(k \geq 2\), we have

\[
d_E(f_i, f_j) = \begin{cases} 
  d_E(e_0, f_{|j-i|}) - 1, & \text{if } |j-i| = 0; \\
  d_E(e_0, f_{|j-i|}), & \text{if } 1 \leq |j-i| < k; \\
  d_E(e_0, f_{|j-i|}) + 1, & \text{if } k \leq |j-i| \leq n - 1,
\end{cases}
\]

If \(n = 2k + 1\) where \(k \geq 2\), we have

\[
d_E(e_i, f_j) = \begin{cases} 
  d_E(e_0, f_{|j-i|}), & \text{if } 0 \leq |j-i| \leq n - 1 \text{ for } i \leq j; \\
  d_E(e_0, f_{|j-i|}) - 1, & \text{if } 1 \leq |j-i| < k \text{ for } i > j; \\
  d_E(e_0, f_{|j-i|}), & \text{if } |j-i| = k \text{ for } i > j; \\
  d_E(e_0, f_{|j-i|}) + 1, & \text{if } k < |j-i| \leq n - 1 \text{ for } i > j.
\end{cases}
\]
recreate the edge distances between any two edges from $\psi$ Lemma 2.1.

As we know that for resolving set $S$ of them the resultant non-edge doubly resolved pair of edges from edge set $S$ edge version of doubly resolving set of $\{two edges of the set $e$ To verify, let us take an example, the edges $d$ $\psi$ Lemma 2.2.

As a result, if we know the edge distance $d_E(e_0, e)$ for any $e \in E(S_n)$, then one can recreate the edge distances between any two edges from $E(S_n)$.

**Lemma 2.1.** $\psi_E(S_n) > 2$, for $n = 2k$, $k \geq 2$.

**Proof.** As we know that for $n = 2k$, $\psi_E(S_n) \geq 2$. So it is necessary to prove that each of the subset $D_E$ of edge set $E(S_n)$ such that $|D_E| = 2$ is not an edge version of doubly resolving set for $S_n$. In Table 2 seven possible types of set $D_E$ are presented and for each of them the resultant non-edge doubly resolved pair of edges from edge set $E(S_n)$ is found. To verify, let us take an example, the edges $e_k, e_{k+1}$ are not edge doubly resolved by any two edges of the set $\{e_0, e_i; k < i \leq n - 1\}$. Obviously, for $k < i \leq n - 1$, we have

$$d_E(e_0, e_k) = d_E(e_0, e_{|k-0|}) = k, \quad d_E(e_0, e_{k+1}) = d_E(e_0, e_{|k+1-0|}) = k - 1, \quad d_E(e_i, e_k) = d_E(e_i, e_{|k-i|}) = i - k \quad \text{and} \quad d_E(e_i, e_{k+1}) = d_E(e_i, e_{|k+1-i|}) = i - k - 1.$$  

So, $d_E(e_0, e_k) - d_E(e_0, e_{k+1}) = d_E(e_i, e_k) - d_E(e_i, e_{k+1}) = 1$, that is, $\{e_0, e_i; k < i \leq n - 1\}$ is not an edge version of doubly resolving set of $S_n$. Using this procedure we can verify all other non-edge doubly resolved pairs of edges for all other possible types of $D_E$ from Table 2.

**Lemma 2.2.** $\psi_E(S_n) = 3$, for $n = 2k$, $k \geq 2$. 

$\Box$
Table 2. Non-edge doubly resolved pairs of $S_n$ for $n = 2k$, $k \geq 2$

| $D_E$ | Non-edge doubly resolved pairs |
|-------|--------------------------------|
| $\{e_0, e_i\}, 0 < i < k$ | $\{e_0, e_{n-1}\}$ |
| $\{e_0, e_i\}, k < i \leq n-1$ | $\{e_k, e_{k+1}\}$ |
| $\{e_0, f_i\}, 0 \leq i < k$ | $\{e_0, f_{n-1}\}$ |
| $\{e_0, f_i\}, k \leq i \leq n-1$ | $\{e_0, f_0\}$ |
| $\{f_0, f_i\}, 1 \leq i < k$ | $\{e_k, f_k\}$ |
| $\{f_0, f_k\}$ | $\{e_0, e_1\}$ |
| $\{f_0, f_i\}, k < i \leq n-1$ | $\{e_1, f_1\}$ |

Proof. The Table 3 demonstrate that edge version of representations of $S_n$ in relation to the set $D^*_E = \{e_0, e_1, e_k\}$ in a different manner.

Table 3. Vectors of edge metric coordinates for $S_n$, $n = 2k$, $k \geq 2$

| $i$ | $S_i(e_0)$ | $D^*_E = \{e_0, e_1, e_k\}$ |
|-----|------------|-----------------------------|
| 0   | $e_0$      | $(0, 1, k)$                  |
| $1 \leq i < k$ | $f_{i-1}, e_i, f_{n-i}, e_{n-i}$ | $(i, i-1, k+1 - i)$ |
| i = k | $f_{k-1}, f_k, e_k$ | $(k, k-1, 1)$ |

Now from Table 3 as $e_0 \in D^*_E$, so the first edge version of metric coordinate of the vector of $e_0 \in S_i(e_0)$ is equal to 0. For each $i \in \{1, 2, 3, \ldots, k\}$, one can easily check that there are no two edges $h_1, h_2 \in S_i(e_0)$ such that $r_E(h_1, D^*_E) - r_E(h_2, D^*_E) = 0$. Also, for each $i, j \in \{1, 2, 3, \ldots, k\}, i \neq j$, there are no two edges $h_1 \in S_i(e_0)$ and $h_2 \in S_j(e_0)$ such that $r_E(h_1, D^*_E) - r_E(h_2, D^*_E) = i - j$. In this manner, the set $D^*_E = \{e_0, e_1, e_k\}$ is the minimal edge version of doubly resolving set for $S_n$ with $n = 2k$, $k \geq 2$ and hence Lemma 2.2 holds.

Lemma 2.3. $\psi_E(S_n) = 3$, for $n = 2k + 1$, $k \geq 2$. 

Table 4. Vectors of edge metric coordinates for $S_n$, $n = 2k + 1$, $k \geq 2$

| $i$  | $S_i(e_0)$ | $D^*_E = \{e_0, e_1, e_{k+1}\}$ |
|------|------------|----------------------------------|
| 0    | $e_0$      | $(0, 1, k)$                      |
| $1 \leq i < k$ | $f_{i-1}$ | $(i, i - 1, k + 2 - i)$          |
|      | $e_i$      | $(i, i - 1, k + 1 - i)$          |
|      | $f_{n-i}$  | $(i, i + 1, k + 1 - i)$          |
|      | $e_{n-1}$  | $(i, i + 1, k - i)$              |
| $i = k$ | $f_{k-1}$ | $(k, k - 1, 2)$                  |
|      | $e_k$      | $(k, k - 1, 1)$                  |
|      | $f_{k+1}$  | $(k, k + 1, 1)$                  |
|      | $e_{k+1}$  | $(k, k, 0)$                      |
| $i = k + 1$ | $f_k$     | $(k + 1, k, 1)$                  |

Proof. The Table 4 demonstrate that the edge version of representations of $S_n$ in relation to the set $D^*_E = \{e_0, e_1, e_{k+1}\}$ in a different way.

Now from Table 4 as $e_0 \in D^*_E$, so the first edge version of metric coordinate of the vector of $e_0 \in S_i(e_0)$ is equal to 0. Similarly for each $i \in \{1, 2, 3, \ldots, k+1\}$, one can easily find that there are no two edges $h_1, h_2 \in S_i(e_0)$ such that $r_E(h_1, D^*_E) - r_E(h_2, D^*_E) = 0$. Likewise, for every $i, j \in \{1, 2, 3, \ldots, k+1\}$, $i \neq j$, there are no two edges $h_1 \in S_i(e_0)$ and $h_2 \in S_j(e_0)$ such that $r_E(h_1, D^*_E) - r_E(h_2, D^*_E) = i - j$. Like so, the set $D^*_E = \{e_0, e_1, e_{k+1}\}$ is the minimal edge version of doubly resolving set for $S_n$ with $n = 2k + 1$, $k \geq 2$ and consequently Lemma 2.3 holds.

\[ \square \]

It is displayed from the whole technique that $\psi_E(S_n) = 3$, for $n \geq 4$. We state the resulting main theorem by using Lemma 2.2 and Lemma 2.3 as mentioned below;

**Theorem 2.4.** Let $S_n$ be the $n$-sunlet graph for $n \geq 4$. Then $\psi_E(S_n) = 3$. 
3. The edge version of doubly resolving sets for family of prism graph $Y_n$.

A family of prism graph $Y_n$ is cartesian product graph $C_n \times P_2$, where $C_n$ is cycle graph of order $n$ and $P_2$ is a path of order 2 (see Figure 3).

![Figure 3. Prism graph $Y_n$](image)

The family of prism graph $Y_n$ consists of 4-sided faces and $n$-sided faces. For our purpose, we label the inner cycle edges of $Y_n$ by \( \{e_i : 0 \leq i \leq n - 1\} \), middle edges by \( \{f_i : 0 \leq i \leq n - 1\} \) and the outer cycle edges by \( \{g_i : 1 \leq i \leq n - 1\} \) as shown in Figure 3.

![Figure 4. $L(Y_n)$ of prism graph $Y_n$](image)

As motivated by the Theorem 1.3, we obtain $\psi_{E}(Y_n) \geq 3$. Furthermore, we will show that $\psi_{E}(Y_n) = 3$ for $n \geq 6$.

In order to calculate the edge distances for family of prism graphs $Y_n$, consider the line graph $L(Y_n)$ as shown in Figure 4.
Define $S_i(f_0) = \{ f \in E(Y_n) : d_E(f_0, f) = i \}$. For $\psi_E(Y_n)$ with $n \geq 6$, we can locate the sets $S_i(f_0)$ that are represented in the Table 5. It is clearly observed from Figure 4 that $S_i(f_0) = \emptyset$ for $i \geq k + 2$. From the above mentioned sets $S_i(f_0)$, it is clear that they can be utilized to define the edge distance between two arbitrary edges of $E(Y_n)$ in the subsequent way.

**Table 5.** $S_i(f_0)$ for $Y_n$

| $n$    | $i$    | $S_i(f_0)$                             |
|--------|--------|----------------------------------------|
| 1      | 1      | $\{e_0, g_0, e_{n-1}, g_{n-1}\}$      |
| $2 \leq i \leq k$ | $k + 1$ | $\{f_i\}$                             |
| $2k(k \geq 3)$ | $k + 1$ | $\{f_k\}$                             |
| $2k + 1(k \geq 3)$ | $k + 1$ | $\{f_k, e_k, g_k, f_{k+1}\}$          |

The symmetry in Figure 4 shows that $d_E(f_i, f_j) = d_E(f_0, f_{j-i})$ for $0 \leq |j - i| \leq n - 1$.

If $n = 2k$, where $k \geq 3$, we have

$$d_E(e_i, e_j) = d_E(g_i, g_j) = \begin{cases} 
    d_E(f_0, e_{j-i}) - 1, & \text{if } 0 \leq |j - i| < k; \\
    d_E(f_0, e_{j-i}), & \text{if } k \leq |j - i| \leq n - 1,
\end{cases}$$

$$d_E(f_i, e_j) = d_E(f_i, g_j) = \begin{cases} 
    d_E(f_0, e_{j-i}), & \text{if } 0 \leq |j - i| \leq n - 1, \text{ for } i \leq j; \\
    d_E(f_0, e_{j-i}) - 1, & \text{if } 1 \leq |j - i| < k, \text{ for } i > j; \\
    d_E(f_0, e_{j-i}), & \text{if } |j - i| = k, \text{ for } i > j; \\
    d_E(f_0, e_{j-i}) + 1, & \text{if } k < |j - i| \leq n - 1, \text{ for } i > j,
\end{cases}$$

$$d_E(e_i, g_j) = \begin{cases} 
    d_E(f_0, e_{j-i}) + 1, & \text{if } |j - i| = 0; \\
    d_E(f_0, e_{j-i}), & \text{if } 1 \leq |j - i| < k; \\
    d_E(f_0, e_{j-i}) + 1, & \text{if } k \leq |j - i| \leq n - 1.
\end{cases}$$

If $n = 2k + 1$ where $k \geq 3$, we have
\[ d_E(e_i, e_j) = d_E(g_i, g_j) = \begin{cases} 
    d_E(f_0, e_{|j-i|}) - 1, & \text{if } 0 \leq |j-i| \leq k; \\
    d_E(f_0, e_{|j-i|}), & \text{if } k < |j-i| \leq n-1, 
\end{cases} \]

\[ d_E(f_i, e_j) = d_E(f_i, g_j) = \begin{cases} 
    d_E(f_0, e_{|j-i|}), & \text{if } 0 \leq |j-i| \leq n-1 \text{ for } i \leq j; \\
    d_E(f_0, e_{|j-i|}) - 1, & \text{if } 1 \leq |j-i| \leq k \text{ for } i > j; \\
    d_E(f_0, e_{|j-i|}) + 1, & \text{if } k < |j-i| \leq n-1 \text{ for } i > j, 
\end{cases} \]

\[ d_E(e_i, g_j) = \begin{cases} 
    d_E(f_0, e_{|j-i|}) + 1, & \text{if } |j-i| = 0; \\
    d_E(f_0, e_{|j-i|}), & \text{if } 1 \leq |j-i| \leq k; \\
    d_E(f_0, e_{|j-i|}) + 1, & \text{if } k < |j-i| \leq n-1. 
\end{cases} \]

As a result, if we know the edge distance \( d_E(f_0, f) \) for any \( f \in E(Y_n) \) then one can recreate the edge distances between any two edges from \( E(Y_n) \).

**Lemma 3.1.** \( \psi_E(Y_n) = 3 \), for \( n = 2k, k \geq 3 \).

**Proof.** The Table 6 demonstrate that edge version of representations of \( Y_n \) in relation to the set \( D^*_E = \{ e_0, e_{k-1}, f_{k+1} \} \) in a different manner.

Now from Table 6 as \( e_0 \in D^*_E \), so the first edge version of metric coordinate of the vector of \( f_0 \in S_i(f_0) \) is equal to 1. For each \( i \in \{1, 2, 3, \ldots, k+1\} \), one can easily check that there are no two edges \( h_1, h_2 \in S_i(f_0) \) such that \( r_E(h_1, D^*_E) - r_E(h_2, D^*_E) = 0 \). Also, for each \( i, j \in \{1, 2, 3, \ldots, k+1\}, i \neq j \), there are no two edges \( h_1 \in S_i(f_0) \) and \( h_2 \in S_j(f_0) \) such that \( r_E(h_1, D^*_E) - r_E(h_2, D^*_E) = i - j \). In this manner, the set \( D^*_E = \{ e_0, e_{k-1}, f_{k+1} \} \) is the minimal edge version of doubly resolving set for \( Y_n \) with \( n = 2k, k \geq 3 \) and hence Lemma 3.1 holds.

**Lemma 3.2.** \( \psi_E(Y_n) = 3 \), for \( n = 2k+1, k \geq 3 \).
Table 6. Vectors of edge metric coordinates for $Y_n$, $n = 2k$, $k \geq 3$

| $i$   | $S_i(f_0)$ | $D_E^* = \{e_0, e_k, f_{k+1}\}$ |
|-------|------------|----------------------------------|
| 0     | $f_0$      | $(1, k, k)$                      |
| 1     | $e_0$      | $(0, k - 1, k)$                  |
|       | $g_0$      | $(2, k, k)$                      |
|       | $e_{n-1}$  | $(1, k, k - 1)$                  |
|       | $g_{n-1}$  | $(2, k + 1, k - 1)$              |
| 2     | $f_1$      | $(1, k - 1, k + 1)$              |
|       | $e_1$      | $(1, k - 2, k)$                  |
|       | $g_1$      | $(2, k - 1, k)$                  |
|       | $f_{n-1}$  | $(2, k, k - 1)$                  |
|       | $e_{n-2}$  | $(2, k - 1, k - 2)$              |
|       | $g_{n-2}$  | $(3, k, k - 2)$                  |
| $3 \leq i \leq k$ | $f_{i-1}$ | $(i - 1, k + 1 - i, k + 3 - i)$ |
|       | $e_{i-1}$  | $(i - 1, k - i, k + 2 - i)$      |
|       | $g_{i-1}$  | $\begin{cases} (k, 2, 2), & \text{if } i = k; \\ (i, k + 1 - i, k + 2 - i), & \text{if } i < k. \end{cases}$ |
|       | $f_{n+1-i}$ | $\begin{cases} (i, k + 2 - i, k + 1 - i), & \text{if } i < k \end{cases}$ |
|       | $e_{n-i}$  | $\begin{cases} (i, k + 1 - i, k - i), & \text{if } i < k \end{cases}$ |
|       | $g_{n-i}$  | $\begin{cases} (k + 1, 2, 1), & \text{if } i = k; \\ (i, k + 2 - i, k - i), & \text{if } i < k \end{cases}$ |

Proof. The Table 7 demonstrate that the edge version of representations of $Y_n$ in relation to the set $D_E^* = \{e_0, e_k, g_{k+2}\}$ in a different way.
Table 7. Vectors of edge metric coordinates for $Y_n$, $n = 2k + 1$, $k \geq 3$

| $i$ | $S_i(f_0)$ | $D_E = \{e_0, e_k, g_{k+2}\}$ |
|-----|------------|-------------------------------|
| 0   | $f_0$     | $(1, k+1, k-1)$               |
| 1   | $e_0$     | $(0, k, k)$                   |
|     | $g_0$     | $(2, k+1, k-1)$               |
|     | $e_{n-1}$ | $(1, k, k-1)$                 |
|     | $g_{n-1}$ | $(2, k+1, k-2)$               |
| 2   | $f_1$     | $(1, k, k)$                   |
|     | $e_1$     | $(1, k-1, k+1)$               |
|     | $g_1$     | $(2, k, k)$                   |
|     | $f_{n-1}$ | $(2, k, k-2)$                 |
|     | $e_{n-2}$ | \(\begin{cases} (2, 2, 2), & \text{if } k = 3; \\ (2, k-i, k-2), & \text{if } k < 3. \end{cases}\) |
|     | $g_{n-2}$ | $(3, k, k-3)$                 |
| 3 \leq i \leq k | $f_{i-1}$ | $(i-1, k+2-i, k+4-i)$         |
|     | $e_{i-1}$ | $(i-1, k+1-i, k+4-i)$         |
|     | $g_{i-1}$ | $(i, k+2-i, k+3-i)$           |
|     | $f_{n+1-i}$ | \(\begin{cases} (i, k+2-i, k-i), & \text{if } i+1 \leq k \\ (k, 1, 2), & \text{if } i = k; \end{cases}\) |
|     | $e_{n-i}$ | \(\begin{cases} (i, 2, 2), & \text{if } i + 1 = k; \\ (i, k+1-i, k-i), & \text{if } i + 1 < k \end{cases}\) |
|     | $g_{n-i}$ | \(\begin{cases} (k+1, 2, 1), & \text{if } i = k; \\ (i+1, k+2-i, k-1-i), & \text{if } i + 1 \leq k \end{cases}\) |
| $i = k + 1$ | $f_k$     | $(k, 1, 3)$                   |
|     | $e_k$     | $(k, 0, 3)$                   |
|     | $g_k$     | $(k+1, 2, 2)$                 |
|     | $f_{k+1}$ | $(k+1, 1, 2)$                 |
Now from Table 7, as $e_0 \in D_E^*$, so the first edge version of metric coordinate of the vector of $f_0 \in S_i(f_0)$ is equal to 1. Similarly for each $i \in \{1, 2, 3, \ldots, k+1\}$, one can easily find that here are no two edges $h_1, h_2 \in S_i(f_0)$ such that $r_E(h_1, D_E^*) - r_E(h_2, D_E^*) = 0$. Likewise, for every $i, j \in \{1, 2, 3, \ldots, k+1\}, i \neq j$, there are no two edges $h_1 \in S_i(f_0)$ and $h_2 \in S_j(f_0)$ such that $r_E(h_1, D_E^*) - r_E(h_2, D_E^*) = i - j$. Like so, the set $D_E^* = \{e_0, e_k, g_{k+2}\}$ is the minimal edge version of doubly resolving set for $Y_n$ with $n = 2k + 1$, $k \geq 3$ and consequently Lemma 3.2 holds.

\[\square\]

It is displayed from the whole technique that $\psi_E(Y_n) = 3$, for $n \geq 6$. We state the resulting main theorem by using Lemma 3.1 and Lemma 3.2 as mentioned below;

**Theorem 3.3.** Let $Y_n$ be the prism graph for $n \geq 6$. Then $\psi_E(Y_n) = 3$.

4. **Conclusion**

In this article, we computed the minimal edge version of doubly resolving sets and its cardinality $\psi_E(G)$ by considering $G$ as a family of $n$-sunlet graph $S_n$ and prism graph $Y_n$. In case of $n$-sunlet graphs, the graph is interesting to consider in the sense that its edge version of metric dimension $\dim_E(S_n)$ is dependent on the parity of $n$ for both even and odd cases. The cardinality $\psi_E(S_n)$ of minimal edge version of doubly resolving set of $n$-sunlet graph $S_n$ is independent from the parity of $n$. In the case of prism graph $Y_n$, the edge version of metric dimension $\dim_E(Y_n)$ and the cardinality $\psi_E(Y_n)$ of its minimal edge version of doubly resolving set are same for every $n \geq 6$.

**Open Problem 4.1.** Compute edge version of doubly resolving sets for some generalized Petersen graphs.
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