Regular flow emergence at inclined surface in a stratified turbulent medium

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Abstract. The authors focus on the following effect: near an inclined surface in a temperature (density) stratified turbulent medium in a gravity field, a regular flow along the slope should appear. This is due to the existence of a region of weakened turbulent exchange near the solid surface, in other words, to the appearance of spatial inhomogeneities of the effective coefficient of turbulent heat transfer. In this case, horizontal components of the gradients of temperature, density and, consequently, hydrostatic pressure arise near an inclined surface. This, in turn, leads to the emergence of an average (non-turbulent) slope current. The mentioned problem is just an example of a much wider range of phenomena – stratified flows in the gravity field, arising due to the horizontal inhomogeneity of the transfer coefficients. In conclusion, another example of such flow is given.

1. Introduction
In the present paper, attention is drawn to the following effect: in the stratified turbulent medium over a solid inclined surface, an ordered flow should arise. This is due to the existence of a region of weakened turbulent exchange near the solid surface, in other words, to the appearance of horizontal inhomogeneities of the effective coefficient of turbulent heat transfer.

2. Preliminary qualitative considerations
Let a stably stratified fluid (gaseous) medium be in the field of gravity (with height, the density decreases, and the temperature increases). In spite of a stable stratification, a medium is assumed to be turbulized (this is possible, for example, at the presence of a sufficiently intense horizontal flow with a vertical shear). Such situations are typical, for example, for the atmosphere and large basins. The stationary vertical profile of the mean (non-turbulent) temperature $T(z)$ is assumed to be a smooth function of the vertical coordinate $z$. The vertical heat flux caused by turbulent diffusion, under stationary conditions in the absence of volumetric sources and regular flows that transfer heat, does not depend on the height; in terms of the effective coefficient of turbulent exchange $K(z)$, this means product $K(z)\gamma(z)$ independence from the height, where $\gamma = dT/dz$. With a constant exchange coefficient, it is obviously, $\gamma = \text{const}$, $T$ is a linear function $z$. 
Now let the medium be bounded from below by a solid inclined surface, the temperature of which everywhere coincides with the temperature of the medium. Near the solid surface turbulence is weakened usually (this is an important circumstance, for example, for substances transfer in the lower atmosphere). If we assume a static (on average) state and a constant diffusion heat flow, then a decrease in the effective coefficient of turbulent exchange near the surface means an increase in the temperature vertical gradient $\gamma$ in this area. Therefore, other things being equal, the vertical temperature profile near the solid surface is deformed (dashed curves 1, 2 in Fig. 1), in comparison with the linear profile $T(z)$ at a constant coefficient of exchange.

**Figure 1.** Flow diagram (indicated by an arrow) near an inclined surface. Turbulent exchange is weakened near the solid lower boundary. Therefore, the temperature and density of the medium near the boundary are deflected from the "background". This leads to a horizontal dependence of the weight of the column of the medium, and to the appearance of a horizontal pressure gradient and flow along the slope.

If we compare the weight of the columns of the medium on vertical lines 3 and 4, then on the same horizontal line 5 (indicated by the chain-line in Fig. 1) this weight will differ: vertical line 4 includes the near-surface layer in which the temperature profile is curved. In the example considered in Fig. 1, the near-surface layer temperature is lowered due to the surface influence (in a general case, the sign of the temperature deviation depends on the boundary conditions). Consequently, in this example, the density of the medium at the surface is increased, and the hydrostatic pressure on the mentioned horizontal line 5 is inhomogeneous: it decreases to the left of the vertical 4. Already from these qualitative considerations, it can be seen that the presence of an inclined boundary leads to the appearance of horizontal gradients of temperature and pressure and, consequently, a pressure gradient force, which has a component along the slope and a regular flow directed there. In this example, due to the spatial inhomogeneity of turbulent heat exchange, a relatively cold (dense) layer of the medium is formed above the inclined surface; this layer flows down the slope under its own weight.

3. Common inaccuracy in slope current literature
The considerations given above are elementary, and it seems to be strange that this effect, evidently, was apparently not discussed earlier (with the exception of the author's mention of this idea in [1]), the more so as an extensive literature is devoted to the slope flows in geophysics (especially in atmosphere dynamics) (see, for example, [1–8] and the bibliography therein).

Back in the forties, the Prandtl model was proposed [2–4], which is still often considered as the basic one in the description of geophysical slope currents. But this model assumes constancy of the
exchange coefficients, so that the effect considered above is absent in that model. The slope flows in this model (and in many subsequent works) are caused by external thermal influences – temperature deviations or heat (buoyancy) sources/sinks, set on an inclined surface or near it. Later, in many works, the problem was analyzed taking into account the dependence of the exchange coefficients on the distance to the inclined surface (this bibliography is contained, for example, in [8]).

However, as far as can be understood, in these works, starting from the middle of the last century, the following systematic error is widespread.

A stationary one-dimensional system of equations relating the temperature disturbance $\theta(n)$ caused by the influence of the lower boundary ($n$ is the coordinate in the direction normal to the slope, $\varphi$ is the slope angle to the horizon) with the velocity component along the slope $u(n)$, accurate to notation, usually is written in the form [3, 5, 6, 8]:

$$0 = \frac{d}{dn} \left( \nu \left( \frac{du}{dn} \right) \right) + \alpha \theta \sin \varphi,$$

$$\nu \sin \varphi = \frac{d}{dn} \left( \kappa \frac{d\theta}{dn} \right)$$

(1)

Here $\nu$, $\kappa$ are the exchange coefficients which are assumed to be the known functions of the coordinate $n$ (in the case of turbulent exchange these coefficients are not differ usually and in the present work are often denoted by $K$); $\alpha$ is the thermal coefficient of the medium expansion, $g$ is the acceleration of gravity. The medium temperature (in the case of atmospheric air, the so-called potential temperature is considered [3]) is represented as the sum of the background stratification $\overline{T}$ and the deviation $\theta$, due to the presence of a given deviation of the inclined surface from the background.

In paper [5] it is rightly mentioned that in reality the total potential temperature $\overline{T} + \theta$ (and not the deviation of the potential temperature $\theta$) should appear in the second equation of (1). It follows that, instead of $d\theta/dn$, the above equation should contain the sum $d\theta/dn + d\overline{T}/dn = d\theta/dn + \gamma \cos \varphi$. Then in the work [5], considerations were made about the relative smallness of the contribution of the second term in the boundary and surface layers of the atmosphere, and it is not taken into account in the future. In many subsequent works, this term is not considered without comment.

The neglect of the term $\gamma \cos \varphi$ was not given importance for a long time, since at a constant exchange coefficient this does not affect the result, so that Prandtl's model leads in any case to the correct result. However, at $\kappa = K(n)$, the second equation (1) after correction becomes inhomogeneous, and this leads to the qualitatively new results.

4. The corrected equation and example of the numerical solution

At $\kappa = K(n)$, the second equation (1) after correction has the form:

$$- \nu \sin \varphi + K(n) \frac{d^2\theta}{dn^2} + \frac{dK(n)}{dn} \frac{d\theta}{dn} = -\gamma \frac{dK(n)}{dn} \cos \varphi.$$

(2)

Therefore, the static solution $\theta = 0, u = 0$ becomes impossible at any boundary conditions – the weight of a column of medium with a curved temperature (density) profile can not at all levels coincide with the weight of the "straight" profile. Thus, due to the dependence $K$ on $n$, an ordered slope current arises.

On the right side of the work (2), an efficient heat source/sink appeared which had not been studied previously. Let estimate its intensity. The total effective heat dissipation per unit area between the levels $n_1$ and $n_2$ ($n_2 > n_1$) is obviously

$$c_p \rho \gamma [K(n_2) - K(n_1)] \cos \varphi,$$

(3)
where \( c_p, \bar{\rho} \) - the heat capacity and average density of the medium, respectively. Let us take the values typical for the surface layer of the atmosphere: \( c_p = 10^3 \) J/(kgK), \( \bar{\rho} = 1 \) kg/m\(^3\), \( \gamma = 3 \cdot 10^{-3} \) K/m, \( K(n_2) - K(n_1) = 5 \) m\(^2\)/s, \( \cos \varphi = 1 \). In this case, expression (3) gives a noticeable effective heat source of 15 W/m\(^2\).

This result has a clear physical meaning. A decrease in the thermal conductivity coefficient near the lower boundary leads to the accumulation of heat, coming from above due to diffusion, in the lower layer of the medium.

Let us give a specific example of a numerical solution for the profile of the turbulent exchange coefficient considered in [3] (pp. 268, 269) ("Dorodnitsyn's relations"):

\[
K(n) = K_0 + (K_1 - K_0) \left( 1 - e^{-n/h} \right).
\]

Let us take the parameter values close to the numerical example considered in [3]: \( \varphi = 3 \cdot 10^{-2}, \gamma = 3 \cdot 10^{-3} \) K/m, \( K_1 = 5 \) m\(^2\)/s, \( K_0 = 10^{-2} \) m\(^2\)/s, \( \alpha = 3.4 \cdot 10^{-3} \) K\(^{-1}\), \( h = 50 \) m. At the inclined surface, homogeneous boundary conditions \( \theta = 0, u = 0 \) are set. Fig. 2 shows the results of the numerical solution of the system consisting of the first equation (1) and (2). The flow arising in this case is relatively weak. But the commonly used system (1) in this case gives a zero solution – a static state, which, as shown above, does not agree with obvious physical considerations.

From the above, the following conclusions can be drawn. The dependence of the thermal conductivity coefficient on the distance to the inclined lower boundary makes impossible the static state of the stratified medium. This effect is lost in the commonly considered generalizations of Prandtl's model.

![Figure 2](image.png)

**Figure 2.** Profiles of deviations of potential temperature (dashed line) and velocity along the slope (solid line) for the considered numerical example

It is important in the case of turbulent exchange, the intensity of which substantially depends on the distance to the solid boundary. Along the inclined lower boundary in a turbulent medium an ordered flow arises.

5. A similar example
It should be mentioned that the problem considered above is just an example of a much wider range of phenomena – stratified flows in the gravity field, arising due to the horizontal inhomogeneity of the transfer coefficients. We give one more example of such flow.

Let infinite medium be stably stratified:
\[ \Theta(z) = \Theta_0 + \gamma z = \Theta_0 + \gamma \cos \varphi. \]

Here \( \Theta(z) \) is the background profile of temperature (for the air in the atmosphere – the potential temperature [9]), \( \Theta_0 \) is the referent value of \( \Theta \), \( \gamma > 0 \) is the constant vertical gradient of temperature. We assume firstly that the thermal diffusivity coefficient \( K_u = \text{const} \).

In static regime, in this medium there is a uniform downward diffusive heat flow proportional to \( \gamma K_u \). Suppose that at some moment of time in the region below a certain inclined interface \( n = 0 \), the thermal diffusivity coefficient decreases to a constant value \( K_d < K_u \). At the inclined boundary the heat flow would be disrupted: over boundary it was proportional to \( \gamma K_u \), but below at the first time it would be equal to the smaller value \( \gamma K_d \). Therefore, a heat would accumulate near the inclined boundary. Due to buoyancy deviation, an upward movement has to occur along this boundary. This movement brings the colder volumes of medium from below that leads to the compensation of the excess heat accumulating due to the difference in the exchange coefficients. One can assume that, as a result, a stationary flow arises along the inclined boundary and ensures a heat balance (the geometry of the problem is shown schematically in Figure 3).

**Figure 3.** The downward vertical arrows schematically depict the diffusive heat flows above and below boundary (for different values of the heat conductivity coefficient).

We seek the corresponding stationary solution for the two-layer one-dimensional problem in the coordinate system \((s, n)\). For simplicity, we restrict ourselves to the case when the values of heat transfer and momentum coefficients coincide (the value of the Prandtl number is everywhere equal to unity – a common hypothesis in the description of turbulent exchange).

In the considered two-layer geometry of the problem with a jump in the exchange coefficient, the system consisting of the first equation (1) and (2) has the form
\[ \frac{d}{dn} \left[ K(n) \frac{du}{dn} \right] + \alpha g \theta \sin \phi = 0, \quad \frac{d}{dn} \left[ K(n) \frac{du}{dn} \right] - \gamma \mu \sin \phi = -\gamma \cos \phi (K_u - K_d) \delta(n), \]

where \( \delta \) is the symbol of the Dirac delta function.

Above the interface, this system has the form:
\[ 0 = K_u \frac{d^2 u}{dn^2} + \alpha g \theta \sin \phi, \quad \gamma \mu \sin \phi = K_u \frac{d^2 \theta}{dn^2}, \]

By eliminating one of the unknowns, get the equation
\[
\frac{d^4 u}{dn^4} + \frac{u}{h_u} = 0,
\]

(6)

where the spatial scale is

\[
h_u = \left( \frac{2K_u}{N \sin \varphi} \right)^{1/2},
\]

\[
N = (\alpha g \gamma)^{1/2}
\]

- a buoyancy frequency. The general solution of equation (6) represents a linear combination of four exponentials with complex indexes. Taking into account the damping condition of disturbances at \( n \to \infty \),

\[
u_u = \left( C_1 \sin \frac{n}{h_u} + C_2 \cos \frac{n}{h_u} \right) \exp \left( -\frac{n}{h_u} \right),
\]

where \( C_i \) are the integration constants.

Below boundary, the solution seems similar, but with the replacement of the sign of exponent and index « \( u \) » on « \( d \) »:

\[
u_d = \left( C_3 \sin \frac{n}{h_d} + C_4 \cos \frac{n}{h_d} \right) \exp \left( \frac{n}{h_d} \right), \quad h_d = \left( \frac{2K_d}{N \sin \varphi} \right)^{1/2}.
\]

From the original system of equations it is not difficult to obtain expressions for temperature deviations. Solutions on both sides of the interface are closed at \( n = 0 \), where the conditions have to be met:

\[
u_d = \nu_u, \quad \theta_u = \theta_d, \quad K_d \frac{d\theta_u}{dn} = K_u \frac{d\theta_u}{dn},
\]

\[
K_d \left( \frac{d\theta_u}{dn} + \gamma \cos \varphi \right) = K_u \left( \frac{d\theta_u}{dn} + \gamma \cos \varphi \right).
\]

(the last condition represents the equality of the diffusive heat flows on both sides of the interface and follows from the second equation (4)).

From these conditions, it is not difficult to obtain the expressions for the integration constants:

\[
C_1 = C_2 = C_4 = -C_3 = U \equiv \left( \frac{N}{2 \sin \varphi} \right)^{1/2} \left( K_u^{1/2} - K_d^{1/2} \right) \cos \varphi.
\]

Thus, a solution has the form:

\[
u = U \left( \pm \sin \frac{n}{h_{u,d}} + \cos \frac{n}{h_{u,d}} \right) \exp \left( \mp \frac{n}{h_{u,d}} \right),
\]

\[
\theta = \frac{NU}{\alpha g} \left( \mp \sin \frac{n}{h_{u,d}} + \cos \frac{n}{h_{u,d}} \right) \exp \left( \mp \frac{n}{h_{u,d}} \right),
\]

where upper sign and index « \( u \) » corresponds to the area above the interface.

It is seen that if the intensity of exchange in the lower area is more weaken than in the upper one \( K_d < K_u \), then \( U > 0 \); near the interface \( n = 0 \), as it should be expected, there is a positive temperature deviation and an upward movement along this boundary. It is interesting to note that with a decrease of slope angle \( \varphi \) this effect of inhomogeneity of the exchange coefficients increases: a thickness of the layers \( h_{u,d} \) increases; slightly inclined flows transfer little heat, and their sufficiently high velocity is required to maintain a heat balance near the interface. But it applies to the stationary
regime under investigation, the establishment time of which should increase with a decrease of the slope angle, since spatial scales $h_u, d$ are growing.

Let the parameters values typical for turbulent exchange in the atmosphere:

$N = 10^{-2} s^{-1}$, $K_u = 5$ m$^2$/s, $K_d = 1$ m$^2$/s. Then at $\varphi = 0.05$ an arising flow velocity is about 0.4 m/s, the temperature deviation is about 0.1 K.

6. Conclusion
The problems considered here are just examples of a much wider range of phenomena – stratified flows in the gravity field, arising due to the horizontal inhomogeneity of the transfer coefficients. A physical mechanism of flows arising seems quite clear. A background vertical diffusion heat flux passes through the medium stratified by temperature. Such a non-equilibrium medium can be at rest only when the density gradients are parallel to the direction of gravity. In the situations under consideration, for example, at the presence of a solid inclined surface, which affects the transfer coefficients, the horizontal symmetry of the background heat flux is broken, and this leads to the appearance of horizontal thermal inhomogeneities and, consequently, horizontal gradients of hydrostatic pressure.

References
[1] Ingel L Kh 2006 *Russ. Meteorol. and Hydrol.* **12** 75–77
[2] Prandtl L 1944 *Führer durch die Strömungslehre* (Goettingen)
[3] Gutman L N 1969 *Introduction to the Nonlinear Theory of Mesometeorological Processes* (Gidrometeoizdat, Leningrad) [in Russian]
[4] Chow F K, DeWekker S F J and Snyder B 2013 (Eds.) *Mountain Weather Research and Forecasting, Recent Progress and Current Challenges* (Springer, Berlin)
[5] Gutman L N 1983 *Tellus* **35A** 213–218
[6] Grisogono B, Oerlemans J, Atmos J 2001 *Sci.* **58** 3349–3354
[7] Ingel L Kh 2011 Fluid Dynamics **46** 505–513
[8] Giometto M G, Grandi R, Fang J, Monkewitz P A, Parlange M B 2017 *Bound.-Layer Meteorol.* **162** 307–317
[9] Gill A E 1982 *Atmosphere-Ocean Dynamics* (Academic Press, London)