Diffeomorphisms and spin foam models

Laurent Freidel*

Perimeter Institute for Theoretical Physics
35 King street North,
Waterloo N2J-2G9, Ontario, Canada

and

Laboratoire de Physique, École Normale Supérieure de Lyon
46 allée d’Italie, 69364 Lyon Cedex 07, France

David Louapre†

Laboratoire de Physique, École Normale Supérieure de Lyon
46 allée d’Italie, 69364 Lyon Cedex 07, France‡

(Dated: January 4, 2022)

Abstract

We study the action of diffeomorphisms on spin foam models. We prove that in 3 dimensions, there is a residual action of the diffeomorphisms that explains the naive divergences of state sum models. We present the gauge fixing of this symmetry and show that it explains the original renormalization of Ponzano-Regge model. We discuss the implication this action of diffeomorphisms has on higher dimensional spin foam models and especially the finite ones.

* Electronic address: lfreidel@perimeterinstitute.ca
† Electronic address: dlouapre@ens-lyon.fr
‡ UMR 5672 du CNRS
I. INTRODUCTION

Spin foam models are an attempt to describe the geometry of spacetime at the quantum level. They give a construction of transition amplitudes between initial and final spatial geometries labeled by spin network states. A spin foam is a 2-dimensional cell complex $F$ with polygonal faces labeled by representations $j_f$ and edges labeled by intertwining operators $i_e$. Given a spin foam we associate an amplitude $A_f, A_e$ and $A_v$ to the faces, edges and vertices of the spin foam respectively. They depend only locally on the spins $j_f$ and intertwiners $i_e$, e.g the vertex amplitude is computed using the spins labeling the faces incident to that vertex and intertwiners on incident edges. The amplitude of a spin foam is then computed as the product of these local weights:

$$Z_F(j_f, i_e) = \prod_{f \in F_2} A_f(j_f) \prod_{e \in F_1} A_e(j_e, i_e) \prod_{v \in F_0} A_v(j_v, i_v).$$

The corresponding partition function associated to the two dimensional cell complex $F$ is given by summation over all admissible spins and intertwiners:

$$Z(F) = \sum_{\{j_f\}, \{i_e\}} Z(j_f, i_e).$$

Spin foams can be understood to be dual to triangulations. The vertex of a spin foam is dual to a codimension 0 simplex. We can therefore think about the amplitude $A_v$ as the ‘bulk’ term, interpreted as corresponding to the discrete version of the amplitude $e^{iS}$. The edges and faces amplitudes are dual to codimension 1 and 2 simplices and we will refer to them as the discrete measure of integration. There is so far a general consensus on the form of the vertex amplitudes, in 3d they are given by the 6-j symbol [1] and by the 10-j symbol in 4d [2].

However, there are some open debates on the possible measures one should use in the state sum. A measure for 4d euclidean gravity was first proposed in [3] by DePietri et al. using group field theory technics. This measure is such that the summation (2) diverges if we do not have a positive cosmological constant. Another proposal was made later in [4] by Perez and Rovelli and it was shown that this measure leads to convergent amplitudes [5]. Numerical studies of these models and some others were made by Baez et al. in [6], we refer the reader to this paper for a more detail account of this issue. It is a key issue since the infinite versus finite models have very different properties. Let us remark that the divergence of the partition function is not a good reason to discard the corresponding models since we are interested in correlation functions or transition amplitudes which are ratio of divergent amplitudes and could be well defined. In this paper we will address this issue first in the context of three dimensional gravity.

In retrospect, the very first spin foam model was the Ponzano–Regge model of 3-dimensional riemannian quantum gravity [1]. In this model, the vertex amplitude is given by the 6-j symbol, one of its key property is the fact that its semi-classical asymptotics is governed by the exponential of the Regge action. The measure was uniquely determined in order to obtain a partition function invariant under refinement of the triangulation. This choice of measure is such that the partition function is divergent and it is only after a regularization and the division by an ad hoc divergent factor sometimes called the anomaly that the state sum is formally independent of the choice of triangulation. In the original paper
of Ponzano-Regge the division by this infinite factor was required in order to have a well defined continuum limit of the partition function. In all subsequent papers on this model the same reason for the overall factor was always advocated.

Our aim in this paper is to explain this divergence as the infinite gauge volume of a remaining gauge symmetry. This symmetry is the translational part of the local Poincaré symmetry and is classically equivalent to the diffeomorphism symmetry. It is clear that the choice of a triangulation (or a spin foam) breaks the full covariance of the theory, however this does not mean that there is no residual action of the diffeomorphism group on a fixed spin foam. We show in three dimensions that indeed there is a residual action of the diffeomorphism group which acts at the vertices of the fixed triangulation. This result is in fact known to specialist of Regge calculus [7]. We also prove that the infinite anomaly factor is necessary in order to divide out the volume of this residual symmetry group.

The plan of the paper is as follows. In section II, we recall the classical gauge symmetries of 3d gravity and show how they are related to each other. In section III, we present the construction of the Ponzano-Regge model, using a discretization of the partition function. We insist on the way the gauge symmetries are implemented at this level. In particular we describe how to implement the translational symmetry. In section IV, we relate the divergence of the Ponzano-Regge model with the volume of the translational symmetry and carry a gauge fixing procedure. In section V, we conclude with a discussion on the general implications of these results for higher dimensional models. We argue that we can generally expect a residual action of diffeomorphisms on spin foams, leading to a physical interpretation of the divergences. We discuss the consequences of this observation for the finite spin foam models.

II. CLASSICAL GAUGE SYMMETRIES IN 3D GRAVITY

In this part we recall the different gauge symmetries of the classical action of 3d gravity, and how they are related together. We consider the first order formalism for 3d gravity. The field variables are the triad frame field $e^i_\mu$ ($i = 1, 2, 3$) and the spin connection $\omega^i_\mu$. The metric is reconstructed as usual from the triad $g_{\mu\nu} = e^i_\mu \eta_{ij} e^j_\nu$ where $\eta = (+, +, +)$ for euclidean gravity and $\eta = (-, +, +)$ for lorentzian gravity. In the following, we will denote by $e^i, \omega^i$ the one forms $e^i_\mu dx^\mu, \omega^i_\mu dx^\mu$. We also introduce the $SU(2)$ Lie algebra generator $J_i$, taken to be $-i/2$ times the Pauli matrices, satisfying $[J_i, J_j] = \epsilon_{ijk} \eta^{kl} J_l$, where $\epsilon_{ijk}$ is the antisymmetric tensor. The trace is such that $\text{tr}(J_i J_j) = -\frac{1}{2} \delta_{ij}$. One can thus define the Lie algebra valued one-forms $e = e^i J_i$ and $\omega = \omega^i J_i$. The action is

$$S[e, \omega] = -\frac{1}{2} \int_M \epsilon_{ijk} e^i \wedge F^{jk}(\omega) = \int_M \text{tr}(e \wedge F(\omega)), \quad (3)$$

where $\wedge$ is the antisymmetric product of forms and $F(\omega) = d\omega + \omega \wedge \omega$ is the curvature of $\omega$. The equations of motion of this theory are

$$d_\omega e = 0, \quad (4)$$

$$F(\omega) = 0, \quad (5)$$

where $d_\omega = d + \omega$ denotes the covariant derivative.
Classically, this theory has three kind of symmetries. First, it is invariant under local Lorentz gauge symmetry:
\[
\begin{align*}
\delta^L_X \omega &= d \omega X, \\
\delta^L_X e &= [e, X],
\end{align*}
\]
parametrized by a Lie algebra element \( X \). It is also invariant under diffeomorphism, for a vector field \( \xi^\mu \), the action is
\[
\begin{align*}
\delta^D_\xi e &= d(\iota_\xi e) + \iota_\xi (d e), \\
\delta^D_\xi \omega &= d(\iota_\xi \omega) + \iota_\xi (d \omega),
\end{align*}
\]
where \( \iota_\xi \) denotes the interior product. These are the usual gauge symmetries of gravity. However this theory admits another symmetry that we call *translational symmetry*, given by
\[
\begin{align*}
\delta^T_\phi \omega &= 0, \\
\delta^T_\phi e &= d \omega \phi,
\end{align*}
\]
for \( \phi \) in the Lie algebra. As seen by inserting this transformation in the action (3), this symmetry is due to the Bianchi identity \( d \omega F = 0 \). These three types of symmetries are not all independent, one can see that
\[
\begin{align*}
\delta^D_\xi e &= \delta^L_{(i \iota_\xi \omega)} e + \delta^T_{(i \iota_\xi e)} e + i \iota_\xi (d \omega e), \\
\delta^D_\xi \omega &= \delta^L_{(i \iota_\xi \omega)} \omega + \delta^T_{(i \iota_\xi e)} \omega + i \iota_\xi (F(\omega)).
\end{align*}
\]
If one uses the equations of motion (4) and (5), one clearly sees that on-shell we have
\[
\delta^D_\xi = \delta^L_{(i \iota_\xi \omega)} + \delta^T_{(i \iota_\xi e)}. \tag{11}
\]
This shows that on-shell, the diffeomorphism symmetry is recovered as a combination of Lorentz and translational symmetry, for field dependent parameters of transformation \( X = \iota_\xi \omega, \phi = \iota_\xi e \). Note that the combination \( \delta^L + \delta^T \) is in fact a local Poincaré gauge symmetry. Let’s denote by \( P_i \) the generators of the translations, they commute between themselves and satisfy \([J_i, P_j] = \epsilon_{ijk} \eta^{kl} P_k\). We can introduce a Poincaré connection \( A = \omega^i J_i + e^i P_i \). The symmetry \( \delta^L + \delta^T \) is just the local Poincaré gauge symmetry for this connection.

### III. GAUGE SYMMETRIES IN DISCRETE QUANTUM GRAVITY

In this section we recall briefly the discretization procedure leading to the Ponzano-Regge model for 3d euclidian quantum gravity. We emphasize the implementation of gauge symmetries and explain how the translational symmetry arises at the level of this model. We are interested in the computation of the partition function of 3 dimensional gravity, i-e formally
\[
Z = \int D e D \omega \ e^{i S(e, \omega)} . \tag{12}
\]
In order to do this computation we will first choose a triangulation \( \Delta \), discretize the action with respect to this triangulation and then compute the discretized path integral. The
continuum limit is then obtained by refining the triangulation. This methodology and the resulting computation are not new, they have been considered and refined several times in the literature [8–10], our approach will be very close to the treatment in [10]. The new point of our approach is to insist on the fact that when we compute the partition function (12), and more generally transition amplitudes, one integrates over all \( e \) and \( A \) modulo gauge transformations. In the continuum one usually uses the Fadeev-Popov procedure. In the discrete approach what one should do is identify the residual gauge symmetries that are left after choosing the triangulation, and divide out the volume of this residual discrete gauge symmetry. In the following we do the proper analysis and find that there is a residual Lorentz gauge symmetry acting at the vertices of the dual triangulation and a residual translational symmetry associated with the vertices of the triangulation.

A. Discretized model

The action (3) can be discretized in order to formulate a discretized version of 3d quantum gravity. One considers a triangulation \( \Delta \) of the manifold where all the edges are oriented. The triad \( e \) is a 1-form and as such it is naturally integrated on one dimensional structures. We integrate it along the edges \( e \in \Delta \) of the triangulation, and replace it by the collection of Lie-algebra elements \( X_e \) obtained in this way. To discretize the connection field, one considers the dual 2-complex \( \Delta^* \). One can naturally consider the holonomy of the connection (which is a 1-form) along the edges \( e^* \) of the dual two complex. This assigns a group element \( g_{e^*} \) to each dual edge \( e^* \). The discretized curvature is obtained as the holonomy of the connection around a whole dual face \( f^* \), i.e as the ordered product of corresponding group elements living on the dual edges

\[
 g_{f^*} = \prod_{e^* \in f^*} g_{e^*}.
\]  

To be more precise, in order to define this holonomy we chose a vertex of the dual triangulation \( v^* \) and chose a path \( P_{v^*,f^*} \) in the dual triangulation which starts at \( v^* \), goes to a point on \( f^* \), goes around the face and come back by the same initial path. The path should be oriented in a way compatible with the orientation of the edge \( e \) dual to \( f^* \). Different choices of initial points \( v^* \) or different choices of pathes \( P_{v^*,f^*} \) are equivalent, since the corresponding group elements we obtain are related by transformations \( g_{f^*} \to g_0 g_{f^*} g_0^{-1} \), which are gauge transformations, as will be proved in the following. If we take the logarithm of this group element (13), we get a Lie algebra element\(^1 \) \( Z_{f^*} \) such that \( g_{f^*} = e^{Z_{f^*}} \). We denote this Lie algebra element by \( Z_e \) for simplicity, this is valid since there is a one to one correspondence between edges of \( \Delta \) and dual faces. So within this discretization the dynamical variables are \((X_e, g_{e^*})\), and the action is simply expressed as

\[
 S[X_e, g_{e^*}] = \sum_e \text{tr}(X_e Z_e).
\]  

The partition function is

\[
 Z(\Delta) = \int_{G^{E^*}} \prod_e dX_e \prod_{e^*} dg_{e^*} \ e^{i \sum_e \text{tr}(X_e Z_e)},
\]  

\(^1\) We restrict the Lie algebra element \( Z \) to be in the region around 0 in which \( \exp \) is an isomorphism between this region and the group.
where \( g \) denotes the Lie algebra, \( E \) and \( E^* \) denote the number of edges in \( \Delta \) and \( \Delta^* \). It is clear that this partition function does not depend on the choice of the orientation, since both \( X_e \) and \( Z_e \) change sign if we change the orientation of \( e \). Before calculating this discretized partition function, we are going to show how the gauge symmetries of the classical action are reflected in this discretization.

### B. Discrete gauge symmetries

Since \( g_{e^*} \) is the holonomy of the connection along the dual edge \( e^* \), the gauge group acts by left and right multiplication at the vertices of the dual lattice

\[
\begin{align*}
g_{e^*} &\rightarrow k_{s(e^*)}^{-1} g_{e^*} k_{t(e^*)}, \\
\end{align*}
\]

where \( s(e^*) \) (resp. \( t(e^*) \)) denotes the source (resp. the target) of the edge \( e^* \). Since \( g_{f^*} \) is a product of edges group elements (see eq.13) starting and ending at the same vertex \( v^* \), the gauge group acts by conjugation on this variables. The discrete action (14) is invariant under the transformation

\[
\begin{align*}
Z_e &\rightarrow g_v^{-1} Z_e g_{v^*}, \\
X_e &\rightarrow g_v^{-1} X_e g_{v^*}.
\end{align*}
\]

This is the implementation of the classical local Lorentz transformation (6) at the level of the discrete model.

The continuum translational transformation is

\[
\delta e = d\phi + [\omega, \phi]
\]

where \( \phi \) is a zero-form valued in the Lie algebra. We have seen that \( e \) is naturally integrated on the edges of the triangulation, \( X_e = \int e. \) The 0-form \( \phi \) is naturally discretized at the vertices of the triangulation in terms of a collection of Lie algebra elements \( \Phi_v \). We therefore expect the discrete transformation to be

\[
\delta X_e = \Phi_{t(e)} - [\Omega_e^{(e)}, \Phi_{t(e)}] - \Phi_{s(e)} + [\Omega_e^{s(e)}, \Phi_{s(e)}].
\]

Recall that we choose an orientation of the edges, so it is clear that the discretization of \( d\Phi \) leads to \( \Phi_{t(e)} - \Phi_{s(e)} \). We define \( \Omega_e^v \) to be an integrated version of \( \omega \) on \( e \) starting from \( v \)

\[
\Omega_e^v \sim \int_{v \rightarrow e} \omega + ...
\]

in the limit where all edges lengths are small. This explains why we have a plus sign for \( [\Omega_e^{s(e)}, \Phi_{s(e)}] \) and a minus sign for \( [\Omega_e^{t(e)}, \Phi_{t(e)}] \). One can isolate the action at a vertex

\[
\delta X_e = \Phi_v - [\Omega_e^v, \Phi_v].
\]

The expression (22) corresponds to the integrated version of the classical translational symmetry (8). The problem in this expression is that the connection \( \omega \) has originally been discretized by integrating on the dual edges \( e^* \), and there is therefore no natural discrete expression for the integration \( \Omega_e^v \) of \( \omega \) on the original edge \( e \). To see how to deal with this
problem, we examine the Bianchi identity which is at the origin of this symmetry, and show how this identity holds at the discretized level.

Classically, the Bianchi identity is

$$dF + [\omega, F] = 0. \quad (23)$$

This is an identity about 3-forms. Integrated on a 3d volume $V$, this leads to

$$0 = \int_{\partial V} F + \int_V [\omega, F] \quad (24)$$

One can consider a vertex of the original triangulation and the surface of dual 2-faces surrounding it. If the triangulation is regular, this surface $S$ has the topology of a 2-sphere and one can consider the integrated Bianchi identity (24) on the interior $B$ of this 2-sphere. Decomposing $B$ as the disjoint union of cones obtained by linking each dual face to the central vertex, one can rewrite it

$$\int_S F + \int_B [\omega, F] = \sum_{e \supset v} Z_e + \sum_{e \supset v} [\Omega^v_e, Z_e] \quad (25)$$

where $\Omega^v_e$ is an integrated version of $\omega$ on the edge $e$ surrounded by the corresponding cone. Note that we also choose to define $\Omega^v_e$ as the integration of $\omega$ along $e$ starting from $v$, which means along an edge outgoing from the interior of $B$, which is necessary to integrate in this way the Bianchi identity. From this argument we expect that the element $\Omega^v_e$ appearing in the translational symmetry (22) can be given by the understanding of the discrete Bianchi identity. Our analysis is closed to the one already given in [9] by Kawamoto et al. In our computation their statements are made precise by a careful analysis of the origin of the discrete Bianchi identity.

To do so, we begin by observing that if one considers the dual faces surrounding the vertex, it exists an order on the faces $f_1...f_n$ and a collection of group elements $g_{f_i}$ representing their curvature such that

$$\prod_{i=1..n} g_{f_i} = 1 \quad (26)$$

One can now take the logarithm of this expression

$$\ln \left( \prod_{e \supset v} e^{Z_e} \right) = 0 \quad (27)$$

In the case of an abelian group, this expression is just $\sum Z_e = 0$, which corresponds to the discretized version of the abelian Bianchi identity $dF = 0$. However in the general case of a non-abelian group, taking the logarithm leads to a more complicated result which has to be expressed using the Baker-Campbell-Hausdorff formula. In the appendix A, we show (see eq. (A15)) that the logarithm (27) can be rewritten as

$$\sum_{e \supset v} (Z_e + [\Omega^v_e, Z_e]) = 0, \quad (28)$$

2 Remember that we had some freedom to chose a vertex in the dual triangulation and a path connecting this vertex to the dual face. We need to use this freedom in order for the identity (26) to be true. We don’t spell out the details of the construction as it is standard, and more or less obvious from a drawing.
where $\Omega^v_e$ is a Lie algebra element explicitly given in terms of the $Z_e$ on the edges meeting at the vertex $v$

$$\Omega^v_e = \sum_{e' \supset v, e' \neq e} Z_{e'} + \sum_{e', e'' \supset v} c_{ee'e''} [Z_{e'}, Z_{e''}] + \cdots , \quad (29)$$

the dots stand for higher commutator terms constructed in terms of $Z_e$ and $c_{ee'e''}$ are explicit coefficients given in (A16). See (A12) for the complete formal expression of $\Omega^v_e$ including all higher commutators. The expression (28) is the Bianchi identity at the discretized level, and its construction provides an expression for the elements $\Omega^v_e$.

It is now clear that due to the discretized Bianchi identity, the discretized action is invariant under the symmetry

$$\delta X_e = \Phi_v - [\Omega^v_e, \Phi_v] \quad \text{if} \quad v \subset e \quad \delta X_e = 0 \quad \text{if} \quad v \notin e \quad (30)$$

for $\Phi_v$ a Lie algebra element associated to the vertex $v$. The variation of the discretized action under this transform is

$$\delta S = \sum_{e \supset v} \text{tr}(\Phi_v Z_e - [\Omega^v_e, \Phi_v] Z_e) \quad (31)$$

$$= \text{tr} \left[ \Phi_v \left( \sum_{e \supset v} Z_e + [\Omega^v_e, Z_e] \right) \right] = 0 \quad (32)$$

In the first equality we use cyclicity of the trace and the last equality is due to the discretized Bianchi identity (28). The action is invariant under the transform (30) which is the discrete action of the translational symmetry. We have only considered the action at one vertex but this can be extended for all the vertices of the triangulation, the transformation is parametrized by a Lie algebra element $\Phi_v$ for each vertex.

**IV. GAUGE SYMMETRIES AND DIVERGENCES**

In this part we complete the construction of the Ponzano-Regge model, taking into account the infinite volume of the translational gauge symmetry.

**A. Division by the gauge volume**

We have seen that the local Lorentz gauge symmetry is parameterized by group elements acting at the vertices of the dual $\Delta^*$, while the translational symmetry is parameterized by Lie algebra elements acting at the vertices of $\Delta$. If we denote by $V$ (resp. $V^*$) the number of vertices of the triangulation (resp. the dual complex), one has to divide the naive (non gauge fixed) discretized partition function by the total gauge volume $Vol(G)^V \times Vol(g)^V$. $Vol(G)$ denotes the volume of the Lorentz group, it is finite for euclidean gravity$^3$ $G = SU(2)$.

---

$^3$ It is infinite in the case of lorentzian gravity $G = SL(2, R)$ but we can take care of it by appropriate regularization, see [11]. In this paper we restrict ourselves to euclidean gravity for simplicity, but considering the lorentzian case will not really change our discussion.
\( V_0(g) \) denotes the infinite volume of the Lie algebra. Dividing by the gauge volume, the partition function (15) is rewritten

\[
Z(\Delta) = \frac{1}{Vol(G)^V \times Vol(g)^V} \int_{G^e} dX_e \prod_{e} dX_e e^{i \sum_{e} \text{tr}(X_e Z_e)}.
\]  

(34)

One can find a choice of measures on the Lie algebra and the group

\[
dX = \frac{1}{4\pi} r^2 dr \sin \phi d\phi d\psi,
\]

(35)

\[
dg = \frac{1}{2\pi} \sin^2 \theta d\theta \sin \phi d\phi d\psi.
\]

(36)

such that \( V(G) = 1 \), and

\[
\int_{G} dX e^{i \text{tr}(XZ)} = \delta(e^Z),
\]

(37)

where \( \delta \) is the delta function on the group for the measure (36). It is expanded on the characters of the representations of \( G \) using Plancherel decomposition

\[
\delta(g) = \sum_j d_j \chi^j(g),
\]

(38)

where \( \chi^j \) designs the character of the spin \( j \) representation. The integrals over group elements in the partition function (34) are performed using the relations on the integrations of product matrices of representations. The resulting contributions are given by Wigner \( 3j \)-symbols on the dual edges, which recombine into \( 6j \) symbols associated to tetrahedra. One thus get the Ponzano-Regge model

\[
Z(\Delta) = \frac{1}{[Vol(g)]^V} \sum_{\{J_e\}} \prod_{e} d_{J_e} \prod_{t} \{6j\}.
\]

(39)

This expression was interpreted originally by Ponzano and Regge as a partition function for discrete 3d gravity. In terms of simplicial geometry, an edge carrying a spin \( j \) is interpreted as an edge of length \( j + 1/2 \), from the fact that the asymptotic of the \( 6j \)-symbol leads to the Regge action for the tetrahedron with lengths \( j + 1/2 \), and the original observation due to Wigner that the asymptotic of the square of the \( 6j \)-symbol is interpreted as the classical probability to have a tetrahedron with lengths \( j + 1/2 \). The equation (39) is purely formal since both the spin state sum and the volume of the Lie algebra are infinite. We therefore need to introduce a cut-off in order to regularize this expression. Our prescription is to restrict the Lie algebra elements to be \( |X_e| < k \). This has for consequence to restrict the summation in the state sum over a finite number of spins. This can be seen from the Kirillov correspondence between representations and coadjoint orbits in the Lie algebra. The character of the spin \( j \) representation is expressed by the Kirillov formula

\[
\chi^j(e^Z) = \frac{\int_{\mathcal{O}_j} e^{i \text{tr}(XZ)} d_j \tilde{X}}{\int_{\mathcal{O}_0} e^{i \text{tr}(XZ)} d_0 \tilde{X}}
\]

(40)

\&

Strictly speaking, the equation (37) is true only when \( Z \) is around 0.
where $\mathcal{O}_j$ is the sphere of radius $2j + 1$ and $d_j \vec{X}$ the measure such that the volume of $\mathcal{O}_j$ is equal to $2j + 1$. This shows that restricting the Lie algebra elements by $|X| < k$ restricts the representations to $2j + 1 < k$ i.e. $j \leq \frac{k-2}{2}$. With this cut, the volume of the Lie algebra becomes finite and equal to $V(k) = \frac{k^3}{3}$ with our choice of measure (35). This can be also obtained from the cut-off on the representations in the Plancherel formula (38), since for large $k$

$$V(k) = \int_{|X|<k} dX e^{\text{tr}(X \cdot 0)} \sim \sum_{d_j<k} d_j^2 \sim \frac{k^3}{3}. \quad (41)$$

The renormalized partition function is thus obtain by sending the cut-off to infinity

$$Z(\Delta) = \lim_{k \to \infty} \sum_{\{d_j<k\}} \prod_v \frac{1}{V(k)} \prod_e d_j \prod_t \{6j\}. \quad (42)$$

The partition function for another triangulation $\Delta'$ obtained by refinement of $\Delta$ is formally such that $Z(\Delta') = Z(\Delta)$. In the original paper of Ponzano-Regge, the renormalization by $V(k)^{-V}$ was motivated by the requirement to have invariance under refinement. Our argumentation shows that this is not just an ad-hoc renormalization, but it arises from the division of the volume of the translational symmetry acting at the vertices.

**B. Gauge fixing procedure**

In this part, we carry a precise Fadeev-Popov gauge fixing procedure for the translational symmetry. To do so, we choose a maximal tree $T$ in the 1-skeleton of the triangulation $\Delta$. A maximal tree is a graph touching every vertex of the triangulation, without forming a closed loop. The main property of a maximal tree is that it exists an unique path in $T$ between any two vertices. A natural way to gauge fix the symmetry is to impose that all $X_e$ on $T$ are zero. Since a maximal tree with $V$ vertices contains $V-1$ edges, this gauge fixing procedure will fix all the symmetry except a global translation. We isolate one of the vertices in the tree (called the root). This induces an order in the tree and an orientation on the edges, each edge being oriented along the direction of the path from the root to each vertex. Each edge has thus a source $s(e)$ and a target $t(e)$. Moreover each vertex $v$ can be unambiguously associated to the edge $e$ such that $v = t(e)$. The (inverse of the) Fadeev-Popov determinant for this procedure is

$$\Delta^{-1} = \int_{g \in \mathcal{G}} [\prod_e d\Phi_{t(e)}] \prod_{e \in T} \delta(\Phi_{t(e)} - [\Omega_{e}^{t(e)}, \Phi_{t(e)}] - \Phi_{s(e)} + [\Omega_{e}^{s(e)}, \Phi_{s(e)}]). \quad (43)$$

To compute the F-P determinant, we have to compute the Jacobian of the transformation, or equivalently the wedge product of all the $d\Phi_{t(e)}$ where

$$\tilde{\Phi}_{t(e)} = \Phi_{t(e)} - [\Omega_{e}^{t(e)}, \Phi_{t(e)}] - \Phi_{s(e)} + [\Omega_{e}^{s(e)}, \Phi_{s(e)}]. \quad (44)$$

One can see that the variable $\Phi_{t(e)}$ for an external edge $e$ appears only in $\tilde{\Phi}_{t(e)}$, thus we have

$$d\tilde{\Phi}_{t(e)} = d(\Phi_{t(e)} - [\Omega_{e}^{t(e)}, \Phi_{t(e)}]) + \cdots \quad (45)$$
where the dots are for terms that will give zero when taking the wedge product with others \(d\Phi\), since this product does not contain \(d\Phi_t\). This reasoning apply for all external edges, and can be continued for other edges. In particular we have

\[
d \left( \Phi_t(e) - [\Omega_t(e), \Phi_t(e)] \right) = (1 + |\Omega_t(e)|^2)d\Phi_t(e)
\]

where \(|\Omega_t(e)|^2\) denotes the square of the norm of \(\Omega_t(e)\) as Lie algebra element. Therefore one gets for the F-P determinant

\[
\Delta = \prod_{e \in T} (1 + |\Omega_t(e)|^2)
\]

From now we denote \(\Omega_e = \Omega_t(e)\). The partition function on the triangulation \(\Delta\) for a gauge fixing along the tree \(T\) becomes

\[
Z(\Delta, T) = \prod_{e \in \Delta/T} \int dX_e \prod_{e^* \in \Delta^*} \int dg_{e^*} \left( \prod_{e \in T} (1 + |\Omega_e|^2) \right) e^{i\text{tr}(\sum_{e \in \Delta/T} X_e Z_e)},
\]

where the \(Z_e\) and \(\Omega_e\) depend on the group elements. The integration on the remaining \(X_e\) variables can be performed, leading to

\[
Z(\Delta, T) = \prod_{e^* \in \Delta^*} \int dg_{e^*} \left( \prod_{e \in T} (1 + |\Omega_e|^2) \right) \prod_{e \in \Delta/T} \delta(e^{Z_e})
\]

Now one can prove that if the \(Z_e\) are zero on \(\Delta/T\), as imposed by the delta functions, then the \(\Omega_e\) are zero on the maximal tree \(T\) and the F-P determinant reduces to 1. Consider an edge \(e\) in \(T\), one of its vertices \(v\) and the corresponding \(\Omega_e\). \(\Omega_e\) is defined (see (A12)) in terms of the \(Z\) variables of all edges meeting at \(v\). Suppose first that this vertex is an external vertex of \(T\), which means that \(e\) is the only edge of \(T\) meeting it. All other edges \(e'\) meeting it are in \(\Delta/T\) and the \(\delta\) functions impose \(Z_{e'} = 0\) for them. In that case, the only non-zero variable in the explicit expression (A12) of \(\Omega_e\) is \(Z_e\), it is thus clear that all the commutators vanish. So, for this external vertices, \(\Omega_e\) is zero. Moreover if we consider the Bianchi identity (28) at this vertex, separating, in the sum, \(e\) from the other edges meeting, it says

\[
Z_e + [\Omega_e, Z_e] + \sum_{e' \neq e} (Z_{e'} + [\Omega_{e'}, Z_{e'}]) = 0
\]

As \(Z_{e'} = 0\) and \(\Omega_e = 0\), this identity gives \(Z_e = 0\). Thus for all external vertices and corresponding edges of \(T\), \(\Omega_e = 0\) and \(Z_e = 0\). If one removes all these edges from the tree, one is left with new external vertices, for which the same reasoning apply. The procedure can be extended to the whole tree and all the \(\Omega_e\) and \(Z_e\) are zero on the tree. The F-P determinant is thus one and the partition function is rewritten

\[
Z(\Delta, T) = \prod_{e^* \in \Delta^*} \int dg_{e^*} \prod_{e \in \Delta/T} \delta(e^{Z_e}).
\]

At the level of the discretized partition function, the gauge fixing of the \(X_e\) variables to zero for \(e \in T\) is translated into a projection on the spins \(j_e = 0\) for the edges of the tree \(T\)

\[
Z(\Delta, T) = \sum_{\{j_e\}} \left( \prod_{e \in T} \delta_{j_e, 0} \right) \prod_{e} d_{j_e} \prod_{t} \{6j\}.
\]
In the triangulation, this has the following consequences: if we set \( j_e = 0 \) for an edge \( e \) belonging to \( n \) triangular faces, each of these faces is suppressed, and the two other edges of each face are identified. We call this move the collapse of the edge \( e \). One can see that this is a topological move, so if we perform the collapse of \( \Delta \) along any tree \( T \) and we denote the resulting triangulation \( \Delta_T \), this triangulation is equivalent to the original one. The relevance of this definition lies in the fact that

\[
Z(\Delta, T) = Z(\Delta_T).
\]

Note that \( \Delta_T \) is well-defined\(^5\) since the tree structure prevents us to try to collapse an edge previously collapsed due to the identification arising in the process. Note also that it still remains a global translational symmetry acting at the vertex which was the root of the tree, and which has to be gauge fixed. Finally, the result is formally independent of the choice of a maximal tree: \( Z(\Delta_T) = Z(\Delta_{T'}). \) This is clear since on one hand different choices of trees amounts to different choices of gauge and it is usually expected that after gauge fixing the theory is BRST invariant, expressing the fact that all the gauge fixings are equivalent. On the other hand, we have seen that the collapsed triangulations along different trees are topologically equivalent, leading to the same conclusion.

C. Positive cosmological constant case

We consider now the case of gravity with positive cosmological constant. The euclidian space with positive cosmological constant \( \Lambda \) is a 3-sphere of radius \( 1/\sqrt{\Lambda} \). This is a space of finite volume \( 1/\Lambda^{3/2} \) (computed for the normalized Haar measure for \( S^3 \)). In such a space, the maximum geodesic length is obtained for the half-perimeter of a great circle, namely \( L_{\text{max}} = \pi/\sqrt{\Lambda} \). Since a maximum length exists in such space, we expect that to be translated at the level of the quantum model as a cut-off in the allowed spins. Also in the discrete models, the translational symmetry acts at the vertices by translation in the spacetime. In a positive cosmological constant space, the volume accessible by translation is finite and equal to the volume of the space. As the space is of finite volume, it acts as a cut-off for the translational symmetry, since it forbids arbitrary large translations of the vertices. We therefore expect a finite gauge volume corresponding to the volume of the space \( 1/\Lambda^{3/2} \).

These two expectations concerning the cut-off on the lengths and the finite gauge volume are precisely what happens in the Turaev-Viro model\([12]\). The Turaev-Viro model is obtained as a deformation of the Ponzano-Regge model using the quantum group \( SU_q(2) \) instead of \( SU(2) \). It depends on an additional parameter \( k \) such that \( q = e^{2\pi i/k} \). This parameter can be linked in several ways with the cosmological constant \( \Lambda \). First it is well-known that the spins of the representations of \( SU_q(2) \) are strictly smaller than \( k - 1 \). This is clear since the quantum dimension of the spin \( j \) representation

\[
[d_j]_k = \frac{\sin(\pi(2j + 1)/k)}{\sin(\pi/k)}
\]

is zero when \( j = k - 1 \). We have seen that the length for an edge carrying a spin \( j \) is given by \( j + 1/2 \), so from the point of view of the Turaev-Viro model, the maximum length is

\(^5\) as a generalized triangulation
$L_{\text{max}} = \frac{k}{2}$. If we identify it with the maximum geodesic length in the 3-sphere of radius $\frac{1}{\Lambda}$ we get $k = \frac{2\pi}{\sqrt{\Lambda}}$. This interpretation of the level $k$ can also be done using the link between Turaev-Viro for $q = e^{2\pi i/k}$ and Chern-Simons theory at level $k$ [13]. The same interpretation is also manifest in the asymptotic of the quantum 6j-symbol [14] which leads to the exponential of the Regge action with cosmological constant $\Lambda = 4\pi^2/k^2$.

It is well known that the triangulation independence of the Turaev-Viro model requires the multiplication of the state sum by the factor $Vol(SU(2)_q)^{-V}$ where

$$Vol(SU(2)_q) = \sum_{j=1}^{(k-2)/2} [d_j^2] = \frac{k}{2\sin^2(\pi/k)}$$

(55)

For a small cosmological constant (large $k$), it behaves as

$$Vol(SU_q(2)) \sim \frac{k^3}{2\pi^2} \sim \frac{4\pi}{\Lambda^{3/2}}.$$  

(56)

which scales as the volume of the 3-sphere for cosmological constant $\Lambda$. This is not a surprise if we interpret $Vol(SU(2)_q)$ as the volume of the translational group acting at the vertex.

One can notice that despite the fact that the quantum dimension $[2j + 1]_k$ for large $k$ is asymptotic to the dimension of the classical representation $2j + 1$, the large $k$ behavior of the gauge volume is different in the Turaev-Viro model and in the Ponzano-Regge regularization

$$\sum_{j=1}^{(k-2)/2} [d_j^2]_k \not\sim \sum_{j=1}^{(k-2)/2} d_j^2$$

(57)

This fact seems puzzling but can be understood geometrically. We have seen that the LHS is interpreted as the volume of a 3-sphere while the RHS is the volume of a 3-ball.

V. DISCUSSION

In this paper we have proven that a carefully discretized spin foam model for 3-dimensional gravity possess not only the usual Lorentz gauge symmetry, but also a local translational symmetry, acting at the vertices of the triangulation, and parameterized by Lie algebra elements. At the classical level, this symmetry is related to the diffeomorphism symmetry, so we can interpret this result as the existence of a residual action of the diffeomorphism group on 3-dimensional triangulated space-time.

We have shown that the volume of this symmetry is infinite and scales with the number of vertices of the 3-dimensional triangulation. If we correctly quantize the theory via path integral we have to divide out this infinite gauge volume. The prescription we obtain for the regularized partition function after the division by the volume of the gauge group is the same as the one proposed a long time ago by Ponzano and Regge. However, in their paper and in all subsequent paper on the subject, the infinite volume factor was required in order to implement the continuum limit. Our result prove that the requirement of gauge invariance is enough to explain this factor and the fact that the partition function possess the correct continuum limit is a consequence of this requirement without any further renormalisation. We also show that gauge fixing this symmetry amounts to compute the partition function on a collapsed triangulation.
So what does this example teach us on higher dimensional spin foam models? We can think of spin foam models as discretized models of gravity where the vertices of the spin foam are dual to higher dimensional simplices, the edges of the spin foam are dual to codimension 1 simplices and the faces of the spin foam are dual to codimension 2 simplices. The representation labels $j$ are carried by the codimension 2 simplices. They are therefore interpreted as a length ($l \sim l_p j$) in 3 dimensions and an area ($A \sim l_p^2 j$) in 4 dimensions. The first lesson from 3 dimensions, is that we can expect a non trivial residual action of the diffeomorphisms on the spin foam. The corresponding remaining diffeomorphisms will not change the connectivity of the spin foam but will act on its representation labels. This is not a new idea, it was advocated a long time ago in the context of Regge calculus by Rocek and Williams [7]. In the context of Regge triangulation, the labels are carried by the edges of the triangulation and they proved that around the flat solution there is an action of the diffeomorphism at the vertices of the triangulation. This can be easily understood as follows. Suppose we have a Regge triangulation of flat space time. To obtain such a triangulation, one can first triangulate $\mathbb{R}^4$ and then put as Regge labels on the edges the lengths of the edges measured with the flat metric. Now by action of a diffeomorphism on $\mathbb{R}^4$ we can translate one vertex of the triangulation without moving the others. This will gives us an other Regge triangulation which differs from the first one by a relabeling of the edges surrounding the vertex. This triangulation described the same piecewise linear geometry. This indicates that Regge triangulation still carry a residual action of the diffeomorphism group. It was proven by Rocek and Williams that such transformations leave the Regge action invariant.

In the 4d case, the situation in spin foam models is different from Regge calculus. First, the labels are carried by the faces of the triangulation, instead of the edges. Second, if we interpret spin foams as evolving spin networks as was first done in the literature [15], there is no obvious relation with piecewise linear geometry. However one can still make a link between the discretized partition function and spin foam models [10], and try to interpret translational symmetry as in 3d, in terms of variation of the spin labels. At the classical level, one can consider the bivector field $B^{IJ} = e^I \wedge e^J$, where $e^I$ is the frame field. This is a 2-form and its natural discretization on a triangulation is on faces, and is given by $B^f_{IJ} = E_1^I \wedge E_2^J$, where $E_1$ and $E_2$ are the discretization of the frame field $e$ along two edges of the face $f$. Now at the discretized level, the spin $j_f$ carried by the face $f$ can be interpreted as the norm of the simple bivector $B_f$. In 3d we interpreted the remaining diffeomorphism symmetry as changing the lengths of the edges, keeping the geometry fixed. We want to argue that a similar interpretation survive in 4d. It has been shown by Horowitz [16] that in the first order formalism of gravity, we can trade off diffeomorphism for translational symmetry in any dimension. Around flat space, this translational symmetry acts only on the continuous frame field as $\delta e^I = d_\omega \Phi^I$ where $\Phi^I$ is a 0-form field. The resulting transformation on the bivector field $B^{IJ} = e^I \wedge e^J$ is $\delta B^{IJ} = d_\omega \Psi^{IJ}$ where $\Psi^{IJ} = \Phi^I [e^J]$ is a frame dependent parameter of transformation. This is a 1-form, naturally discretized on the edges of the triangulation. One therefore expect the translational symmetry to act on the discretized $B$ field at edges of the triangulation. This leads to an action of this transformation on the 3-cells of the spin foam. This is consistent with the fact that the Bianchi identity responsible for this symmetry is a 3-form discretized along the 3-cells of the spin foam. In the light of what happens in 3d, one expects a divergence for each 3-cell of the spin foam if the diffeomorphism symmetry is not broken and if the geometry we are integrating contains flat space. That’s exactly the type of divergences which is occurring in some of the spin foam
models first proposed in [3] and analyzed in [4]. This open the possibility to give a physical interpretation of these divergences as coming from a residual action of the diffeomorphisms. Of course in 4 dimensions this is only a plausibility argument so far and it deserves more study.

This argumentation therefore raise the question of the meaning of finite spin foam models. There are two different types of spin foam model which have a convergent sum over spins. The first type of model is obtained when we consider euclidean gravity with a positive cosmological constant, in that case the sum over spin is restricted to be finite (the spins cannot be bigger than the cosmological constant in Planck units). There is still an action of the diffeomorphism group in 3-dimensions, but since the 3-sphere has a finite volume the volume of this group is finite. This is consistent with the interpretation that a positive cosmological constant suppress spacetimes with large volumes. This is extensively used in the context of dynamical triangulation for instance, where a positive cosmological constant is needed to make the summation convergent. In 4 dimension the inclusion of a positive cosmological constant in the euclidean theory will also lead to convergent spin foam models. We can argue in the same way that this doesn’t mean that the diffeomorphism group do not act on the spin foams but only that the volume of the symmetry group is finite in that case. In this case we therefore have a physical interpretation of the finiteness of the model. It is important to note that adding a cosmological constant help us only if its positive and if the spacetime is euclidean. If we consider lorentzian gravity with any cosmological constant or euclidean gravity with a negative cosmological constant we cannot expect any convergence of the models since the volume of the corresponding homogeneous spaces are all infinite in these cases and so will be the volume of the residual translational group.

The second kind of convergent models were first considered in [4] and correspond to a different choices of edge amplitudes. This is interpreted as a different choice of measure in spin foams, as we discussed in the introduction. Since we expect an action of diffeomorphisms on spin foam we have to understand why this action does not translate in the amplitude by an infinite factor. In these models there is no parameter we can vary which can be interpreted as a cosmological constant so we cannot really argue that the convergence possess a clear physical interpretation. An interpretation of the finiteness of the amplitude is that diffeomorphism symmetry is broken in these model. This means that we expect these models to assign different amplitudes to gauge equivalent solutions. This can be explicitly checked in three dimensions where a modification of the measure of spin foam models similar to the one done in 4-dimension can be achieved [17]. One possible interpretation for that could be that these models are in fact gauged fixed models and should not carry a representation of the diffeomorphism group. This is however unlikely, since a Fadeev-Popov gauge fixing procedure generally involves a highly non-local determinant. The amplitude of finite spin foam models are expressed as a product of local weights (cf eq. 1) which cannot be interpreted as a Fadeev-Popov determinant. In [6] numerical simulation showed that the amplitude of the finite model are dominated by the spins 0 and 1/2 and for all practical purpose the summation over spins can be restricted these ones. In the spin foam model, the amplitude associated with a spin foam where a given face is carrying the spin 0 is equal to the amplitude of the spin foam where this face as been collapsed. So we can interpret the amplitude $A(S) = \sum_{j=0,1/2} A(j)$ of a given spin foam $S$ as a sum over all spin foams included in $S$

$$A(S) = \sum_{S' \subset S} \alpha^{N_0} \beta^{N_1} \gamma^{N_2},$$

where $N_0, N_1, N_2$ denotes the number of 0, 1, 2 cells of the spin foams and $\alpha = A_v(j = 1/2)$,
\( \beta = A_e(j = 1/2), \gamma = A_f(j = 1/2) \) in the notation of (1). As such, the finite spin foam models are similar to dynamical triangulations models with a fixed edge lengths \( j = 1/2 \). In dynamical triangulations, \( \beta = 1 \) and the parameters \( \alpha, \gamma \) can be freely tuned, they depend on the Newton coupling constant, the cosmological constant and the scale of the lattice spacing [18]. It is known that dynamical triangulations models do not carry a representation of the diffeomorphisms. However, in the limit where \( S \) becomes infinite we can tune the parameters of the theory to recover a continuum limit in some region of phase space and restore in this limit the action of diffeomorphisms.

So the finite spin foam models have features very similar to dynamical triangulations models, namely they break the action of diffeomorphism. It does not go along the line of the spin foam philosophy which is to keep most of the symmetry of the continuum theory in the intermediate triangulation dependant levels. So far this requirement has been focused on local Lorentz symmetry but we have argued in this paper that there is the possibility to even fulfill this requirement for the local Poincaré symmetry or diffeomorphism. One reason for this requirement is the hope that it would help to get a final theory with the correct symmetries when we get rid of the triangulation dependance. This can be of course disputed and one can argue that the symmetry is restored in the coarse grain limit even if it is not implemented at the microscopic level. There is no very good reason yet to prefer one mechanism over the other. However, we still think that finite models have undesirable features. First, they behave much alike dynamical triangulation models which is not bad by itself but then it seems hard to expect spin foam to do better than what has been done in this context. The second point is that in dynamical triangulations models the parameters are freely tuned and this is important to restore diffeomorphism symmetry in the coarse grain limit. This is not a feature of spin foam models so far.

In order to conclude, we have showed in this paper that there is a residual action of translational symmetry on an individual spin foam and that this explains the infinity or ‘anomaly’ of the Ponzano-Regge model. We have argued that such a residual action of translational symmetry should be expected for higher dimensional models. This would give a physical meaning to the divergences observed in the 4D model proposed in [3]. On the contrary we have disputed the relevance of the finite spin foam models proposed in [4]. Our argumentation is by no mean conclusive since it contains hypothesis and unresolved issues but we hope that it will launch a fruitful and successful debate on this issues.

Acknowledgments : We thank A. Ashtekar for relevant questions on the gauge fixing procedure. We thank H. Pfeiffer and A. Perez for discussions. D. L. is supported by a MENRT grant and Eurodoc program from Région Rhône-Alpes. L. F. is supported by CNRS and an ACI-Blanche grant.

APPENDIX A: BAKER-CAMPBELL-HAUSDORFF FORMULA

In this section we recall how to formally obtain the Baker-Campbell-Hausdorff formula. In particular we obtain the expression (28) used in the text. We are interested in the computation of the expression

\[
X = \ln \left( e^{Z_1} e^{Z_2} ... e^{Z_N} \right)
\]  

(A1)
for $Z_1, \ldots, Z_N$ Lie algebra elements. One can expand the exponentials

$$X = \ln \left[ \left( \sum_{p_1} Z_1^{p_1} \right) \left( \sum_{p_2} Z_2^{p_2} \right) \cdots \left( \sum_{p_N} Z_N^{p_N} \right) \right] \quad (A2)$$

Isolating the term $p_1 = p_2 = \ldots = p_N = 0$ one can rewrite it

$$X = \ln \left[ 1 + \sum_{p_1 + \ldots + p_N \geq 1} \frac{Z_1^{p_1} Z_2^{p_2} \cdots Z_N^{p_N}}{p_1! p_2! \cdots p_N!} \right] \quad (A3)$$

Expanding now the logarithm one gets

$$X = \sum_{k=1}^{+\infty} \frac{(-1)^{k-1}}{k} \left( \sum_{p_1 + \ldots + p_N \geq 1} \frac{Z_1^{p_1} Z_2^{p_2} \cdots Z_N^{p_N}}{p_1! p_2! \cdots p_N!} \right)^k \quad (A4)$$

Expanding the term to the power $k$, one gets a sum over all the ways to take the orderer product of $k$ terms in the sum, namely

$$X = \sum_{k=1}^{+\infty} \frac{(-1)^{k-1}}{k} \sum_{\mathcal{I}^k} \left[ \frac{Z_1^{p_1}}{p_1!} \frac{Z_2^{p_2}}{p_2!} \cdots \frac{Z_N^{p_N}}{p_N!} \right] \left[ \frac{Z_1^{q_1}}{p_1!} \frac{Z_2^{q_2}}{p_2!} \cdots \frac{Z_N^{q_N}}{p_N!} \right] \cdots \left[ \frac{Z_1^{r_k}}{p_1!} \frac{Z_2^{r_k}}{p_2!} \cdots \frac{Z_N^{r_k}}{p_N!} \right] \quad (A5)$$

where the sum is over the set $\mathcal{I}^k$

$$\mathcal{I}^k = \left\{ p_i^j \in \mathbb{N}, i = 1 \ldots N, j = 1 \ldots k \mid \sum_i p_i^j \geq 1 \ \forall j \right\} \quad (A6)$$

One thus obtains an expression of $X$ as a polynome in the $Z_i$. Each monome is obtained as a product of $k$ subterms, the condition $\sum_i p_i^j \geq 1 \ \forall j$ states that each of these subterms is non-empty. $X$ is a Lie algebra element, while each monome in the sum at the LHS is generically an element of the universal enveloping algebra $U(\mathfrak{g})$. The Lie algebra element of $U(\mathfrak{g})$ are characterized by the fact one can give $U(\mathfrak{g})$ a coproduct such that the Lie algebra elements are the primitive elements of this coproduct. One can apply at both sides of the equation a projector $\pi : U(\mathfrak{g}) \to \mathfrak{g}$ which projects on the Lie algebra elements of $U(\mathfrak{g})$. It exists different projectors of this type, we will use the most common namely the Dynkin projector. It is defined in the following way, consider a monome which is written as $x_1 x_2 \ldots x_M$ for $x_i \in \mathfrak{g}$ the projector is defined

$$\pi(x_1 \ldots x_M) = \frac{1}{M} [[[[x_1, x_2], x_3] \ldots x_{M-1}], x_M] \quad (A7)$$

In particular, we have for $M \geq 2$

$$\pi(x_1 \ldots x_M) = \frac{M - 1}{M} [\pi(x_1 \ldots x_{M-1}), x_M] \quad (A8)$$
This is the projector which allows to express the Baker-Campbell-Hausdorff formula in terms of commutators of the originals $Z_i$. Consider one of the monome, i.e., fix $k$ and a set of indices $p_i^j$. We consider the action of this projector

$$
\pi \left[ \left( Z_{p_1^1}^1 Z_{p_2^2}^2 ... Z_{N_i}^{p_N^N} \right) \left( Z_{p_1^1}^2 Z_{p_2^2}^2 ... Z_{N_i}^{p_N^N} \right) ... \left( Z_{p_1^k}^k Z_{p_2^k}^k ... Z_{N_i}^{p_N^N} \right) \right]
$$

(A9)

The size of the monome is denoted $M = \sum_{i,j} p_i^j$. First, if the monome is of size $M = 1$, this means that there is only one subterm (recall the subterms are asked to be non-empty), hence $k = 1$, and the projector acts as the identity. Now we consider the general case $M \geq 2$. To use the expression (A8), we need to understand what is the last term of the monome. We know that the $k$-th subterm is non-empty since $\sum_{i} p_{i}^{k} \geq 1$. We pick up the last non-zero element in the sequence $p_{k}^1 ... p_{N}^k$ and denote it $p_{i_0}^{k}$. This means that the monome ends with the element $Z_{i_0}$ and that the action of $\pi$ on it is expressed as a commutator

$$
\left[ \frac{M-1}{M} \pi \left( Z_{1}^{p_{1}^1} ... Z_{N}^{p_{N}^N} \right) \right], Z_{i_0}
$$

(A10)

The total projection of all the monomes of size $M \geq 2$ of (A5) can be rewritten as a sum of commutators of a Lie algebra element with a single $Z_{i_0}$

$$
\sum_{i_0} [\Omega_{i_0}, Z_{i_0}]
$$

(A11)

where $\Omega_{i_0}$ is defined by

$$
\Omega_{i_0} = \sum_{k=1}^{+\infty} \frac{(-1)^{k-1}}{k} \sum_{I_{k}^{i_0}} \left( \prod_{i,j} \frac{1}{p_{i}^{j}} \right) \left( \sum_{i,j} p_{i}^{j} \right) \pi \left[ Z_{1}^{p_{1}^1} Z_{2}^{p_{2}^2} ... Z_{N}^{p_{N}^N} \right] \pi \left[ Z_{1}^{p_{1}^1} Z_{2}^{p_{2}^2} ... Z_{N}^{p_{N}^N} \right]
$$

(A12)

where the set of indices $I_{k}^{i_0}$ is

$$
p_{i}^{j} \in \mathbb{N} \mid \sum_{i} p_{i}^{j} \geq 1, i = 1..N, j = 1..k - 1
$$

(A13)

$$
p_{i}^{k} \in \mathbb{N} \mid \sum_{i} p_{i}^{k} \geq 0, i = 1..i_0
$$

(A14)

This set $I_{k}^{i_0}$ means that we have taken the set $I^{k}$ but cut the indices in the last subterm $k$ at order $i_0$ and drop the condition $\sum_{i} p_{i} \geq 1$. Putting together the results for $M = 1$ and $M \geq 2$ we get

$$
X = \ln \left( e^{Z_1} ... e^{Z_N} \right) = \sum_{i_0} Z_{i_0} + [\Omega_{i_0}, Z_{i_0}]
$$

(A15)

One can estimate the first terms in the complicated expression of the $\Omega_k$

$$
\Omega_{k} = \frac{1}{2} \sum_{k<i} Z_{i}
$$

$$
+ \frac{1}{6} \sum_{i<j<k} [Z_{i}, Z_{j}] - \frac{1}{6} \sum_{k<i<j} [Z_{i}, Z_{j}] + \frac{1}{12} \sum_{i} [Z_{i}, Z_{k}]
$$

+ ... 

(A16)
[1] G. Ponzano and T. Regge, *Semiclassical limit of Racah coefficients*, in Spectroscopic and Group Theoretical Methods in Physics, ed. F. Bloch, North-Holland, New York, 1968.

[2] J. W. Barrett, L. Crane, *Relativistic spin networks and quantum gravity*, J.Math.Phys. 39 (1998) 3296-3302, gr-qc/9709028.

[3] R. De Pietri, L. Freidel, K. Krasnov, C. Rovelli, *Barrett-Crane model from a Boulatov-Ooguri field theory over a homogeneous space*, Nucl. Phys. B574 (2000) 785-806, hep-th/9907154.

[4] A. Perez, C. Rovelli, *A spin foam model without bubble divergences*, Nucl. Phys. B599 (2001) 255-282, gr-qc/0006107.

[5] A. Perez, *Finiteness of a spinfoam model for euclidean quantum general relativity*, Nucl.Phys. B599 (2001) 427-434, gr-qc/0011058.

[6] J. C. Baez, J.D. Christensen, T. R. Halford, D. C. Tsang, *Spin foam models of riemannian quantum gravity*, Class. Quant. Grav. 19 (2002) 4627-4648, gr-qc/0202017.

[7] M. Rocek, R.M. Williams, *The quantization of Regge calculus*, Z. Phys. C21:371 (1984).

[8] H. Ooguri, *Partition functions and topology-changing amplitudes in the 3D lattice gravity of Ponzano and Regge*, Nucl. Phys. B382 (1992) 276-304, hep-th/9112072.

[9] N. Kawamoto, H.B. Nielsen, N. Sato, *Lattice Chern-Simons gravity via Ponzano-Regge model*, Nucl. Phys. B555 (1999) 629-649, hep-th/9902165.

[10] L. Freidel, K. Krasnov, *Spin foam models and the classical action principle*, Adv. Theor. Math. Phys. 2 (1999) 1183-1247, hep-th/9907092.

[11] L. Freidel, D. Louapre, *3-dimensional Lorentzian gravity*, to be published.

[12] V. Turaev, O. Viro, *State sum invariants of 3-manifolds and quantum 6j-symbols*, Topology 31 (1992), 865-902.

[13] L. Freidel, K. Krasnov, *Discrete space-time volume for 3-dimensional BF theory and quantum gravity*, Class.Quant.Grav. 16 (1999) 351-362, hep-th/9804185.

[14] S. Mizoguchi, T. Tada, *3-dimensional gravity from the Turaev-Viro invariant*, Phys.Rev.Lett. 68 (1992) 1795-1798, hep-th/9110057.

[15] M. P. Reisenberger, C. Rovelli, *“Sum over surfaces” form of loop quantum gravity*, Phys. Rev. D56 (1997) 3490-3508, gr-qc/9612035.

[16] G. T. Horowitz, *Topology change in general relativity*, 6th Marcel Grossman meeting, Kyoto, Japan (1991), hep-th/9109030.

[17] L. Freidel, D. Louapre, *Non-perturbative summation over 3D discrete topologies*, hep-th/0211026.

[18] J. Ambjorn, J. Jurkiewicz, Y. Watabiki, *Dynamical triangulations, a gateway to quantum gravity?*, J. Math. Phys. 36 (1995) 6299-6339, hep-th/9503108.