Momentum-based Accelerated Mirror Descent Stochastic Approximation for Robust Topology Optimization under Stochastic Loads

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Abstract

Robust topology optimization (RTO) improves the robustness of designs with respect to random sources in real-world structures, yet an accurate sensitivity analysis requires the solution of many systems of equations at each optimization step, leading to a high computational cost. To open up the full potential of RTO under a variety of random sources, this paper presents a momentum-based accelerated mirror descent stochastic approximation (AC-MDSA) approach to efficiently solve RTO problems involving various types of load uncertainties. The proposed framework can perform high-quality design updates with highly noisy stochastic gradients. We reduce the sample size to two (minimum for unbiased variance estimation) and show only two samples are sufficient for evaluating stochastic gradients to obtain robust designs, thus drastically reducing the computational cost. We derive the AC-MDSA update formula based on $\ell_1$-norm with entropy function, which is tailored to the geometry of the feasible domain. To accelerate and stabilize the algorithm, we integrate a momentum-based acceleration scheme, which also alleviates the step size sensitivity. Several 2D and 3D examples with various sizes are presented to demonstrate the effectiveness and efficiency of the proposed AC-MDSA framework to handle RTO involving various types of loading uncertainties.

Keywords: Robust topology optimization, stochastic approximation, load uncertainty, mirror descent stochastic approximation, acceleration scheme, step size strategies

1. Introduction

Topology optimization has been widely used in many disciplines, such as aerospace engineering \cite{1,2}, biomedical engineering \cite{3,4}, and architectural design \cite{5}. The main goal of topology optimization is to find the distribution of material to achieve optimized performance \cite{6,7}. While the classical setting of topology optimization assumes problem-related parameters that are deterministic, real-world structures are subjected to various sources of randomness, such as load, material

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property, and geometry, which can influence the layout of optimized designs. Thus, robust topology optimization (RTO) has been employed to improve the robustness of designs concerning random sources [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19].

One common random source comes from the loading, which includes load magnitudes, directions, locations, and distributions. Many studies have contributed to the RTO with load randomness using various approaches, such as semidefinite programming [20], conversion to many load cases [21, 22], first-order reliability method approximation [12], Karhunen-Loeve expansion to model stochastic load fields [13, 14], perturbation techniques [23], stochastic collocation [15], univariate dimension reduction [16], polynomial chaos expansion [17], game theory [24], linear elastic theory [25], and non-probabilistic interval uncertainty [18]. These approaches successfully produce robust optimized designs. For large-scale problems, particularly in three dimensions (3D), some may require a relatively high computational cost as they typically solve multiple systems of equations at each optimization step in order to accurately estimate the sensitivity information. In this work, we aim to reduce the computational cost associated with RTO problems using stochastic approximation.

Stochastic approximation (SA) [26] is a family of stochastic optimization methods known for its low computational cost and effectiveness [27]. In the standard setting, SA methods solve stochastic optimization problems with the objective function in the form of the expectation of a stochastic function [28]. Instead of computing the exact gradient, the classic SA method uses a stochastic one as the gradient descent direction. Thus, SA is also known as stochastic gradient descent (SGD). The SGD was initially developed by Robbins and Monro [26] and improved in [28, 27, 29]. In [27, 28], the classic SA (or SGD) is generalized to the mirror descent stochastic approximation (MDSA) by replacing the traditional $\ell_2$-norm definition of distance in SGD with a more general definition. With the general setting, MDSA adapts its update to the underlying geometry of the feasible space and obtain improvements in the convergence performance [28, 30, 31]. One of the most popular versions of MDSA is the entropic MDSA, which is based on the $\ell_1$-norm setting [28]. In a related area, smooth convex programming, accelerated methods (also known as momentum methods) were first developed by Polyak [32] and significantly improved by Nesterov [33, 34]. These methods are referred to as the accelerated gradient descent and proved to possess an unimprovable rate of convergence for convex problems as a linear Krylov subspace method [34]. The accelerated methods are incorporated into SA and MDSA to speed up the convergence of stochastic optimization [35, 36, 37, 38]. Inspired by a popular version of accelerated SA methods, accelerated mirror descent stochastic approximation (AC-MDSA) [35], this paper derives an AC-MDSA framework tailored for the topology optimization accounting for stochastic loads.

In the field of topology optimization, the idea of integrating stochastic optimization algorithms has been recently explored in a few studies. For instance, Zhang et al. [39] proposed a stochastic sampling algorithm that requires 5 to 6 samples to estimate the gradient and solve deterministic topology optimization problems with hundreds of load cases. De et al. [40] applied SGD algorithms
to compliance minimization of RTO problems with load uncertainty and shows improvements over GCMMA [11]. Pflug et al. [42] developed a continuous stochastic gradient method (CSG) that shows superiority over traditional SGD methods when applied to the expected compliance minimization (without the variance term). Both [40] and [42] treat the volume constraint as a penalization term in the objective function, thereby converting the constrained optimization to an unconstrained problem. The volume constraint represents a feasible domain bounded a plane in which $\ell_1$-norm-based entropic MDSA performs better than the $\ell_2$-norm-based SGD (and its variants) [28]. Recently, the $\ell_1$-norm-based entropic MDSA has been proposed and tailored for topology optimization with many deterministic load cases [31] and requires only a single sample at each optimization step, thereby significantly reducing the computational cost compared to the standard weighted average formulation [31]. Theoretical and numerical comparisons of the entropic MDSA and SGD are also carried out therein, and show better performance (objective function values and computational time) of the $\ell_1$-norm entropic MDSA than SGD ($\ell_2$-norm-based) for compliance minimization with a volume constraint [31]. The advantage of entropic MDSA comes from the use of $\ell_1$-norm and entropic distance function to mimic the underlying geometry of the feasible design space represented by the linear volume constraint [31]. Therefore, we focus on the entropic MDSA with the $\ell_1$-norm setting in this study.

In this work, we propose a novel momentum-based AC-MDSA algorithm to solve RTO problems with the volume constraint involving various types of loading uncertainties. The proposed AC-MDSA approach can perform high-quality design variable updates with noisy stochastic gradients. As a result, we demonstrate that only two samples are sufficient for computing the stochastic gradients at each optimization step, which is the minimum number of samples for unbiased variance estimation. Second, in order to adapt to the underlying geometry of the feasible set defined by the volume constraint, we derive the explicit update formula in the $\ell_1$-norm setting by introducing the entropy function as the distance-generating function in the AC-MDSA method. Third, we present adaptive step-size recalibration and damping schemes which, in conjunction with the momentum-based acceleration mechanism, to improve the convergence performance of AC-MDSA with significantly reduced sensitivity to various step size choices. Through numerical examples in both 2D and 3D, we showcase that the proposed AC-MDSA approach can efficiently produce robust designs with respect to different types of loading uncertainties and exhibits scalable performance for RTO problems of various problem sizes and geometries.

The remainder of this paper is organized as follows. Section 2 reviews the RTO formulation for compliance minimization problem considering various load uncertainties. Section 3 introduces the theoretical background of AC-MDSA and derives a momentum-based entropic AC-MDSA update algorithm for the RTO problem. Section 4 proposes algorithmic techniques for improving convergence performance, including adaptive step size recalibration and damping schemes. Section 5 presents four numerical examples illustrating the effectiveness and efficiency of the proposed entropic AC-MDSA algorithm in producing robust optimized designs under various loading uncer-
tainties. Finally, Section 6 provides concluding remarks.

2. Robust topology optimization formulation

In this section, the RTO formulation of compliance minimization problem considering loading uncertainties is introduced, and the unbiased estimations of the objective function and gradient of the RTO formulation are presented using a finite number of samples. In this work, we focus on the density-based approach [6].

For a given mesh consisting of \( n \) finite elements, the RTO aims to minimize the weighted sum of the mean and variance of the compliance under load randomness. More specifically, the RTO formulation is introduced as follows:

\[
\min_{\mathbf{x}} J(\mathbf{x}) = \frac{\kappa}{w} \mathbb{E}[C(\mathbf{x}, \xi)] + \frac{1 - \kappa}{w^2} \text{Var}[C(\mathbf{x}, \xi)]
\]

s.t. \( \frac{V(\mathbf{x})}{V_0} - V_f = 0 \)

\( x^{(i)} \in [0, 1], \quad i = 1, 2, ..., n \)

with \( K(E(\mathbf{x})) u(\mathbf{x}, \xi) = f(\xi) \),

where \( \mathbf{x} \) is the design variable vector; \( f(\xi) \) is the random load vector with \( \xi \) being a random vector representing various types of load uncertainty; \( K \) and \( u \) are the global stiffness matrix and displacement vector, respectively; \( V_0 \) is the total volume of the design domain; and \( V_f \) is the prescribed volume fraction. For a given structure with design variable \( \mathbf{x} \), \( V(\mathbf{x}) \) stands for the total volume of that structure as follows,

\[
V(\mathbf{x}) = \sum_{i=1}^{n} v^{(i)} \bar{x}^{(i)} = v^T \bar{x} = v^T H x,
\]

where \( v^{(i)} \) and \( \bar{x}^{(i)} \) is the volume and the filtered/physical density of the \( i \)th element, respectively; and \( H \) is the matrix representation of density filter [43, 2], such that \( \bar{x} = H x \), which is used to prevent the checkerboard pattern and achieve mesh-independent designs [44, 45, 46]. In addition, the modified simplified isotropic material with penalization (SIMP) [6, 47, 48] is adopted, which interpolates the Young’s modulus of each element as

\[
E^{(i)}(\bar{x}^{(i)}(\mathbf{x})) = E_{\text{min}} + (\bar{x}^{(i)}(\mathbf{x}))^p (E_0 - E_{\text{min}}), \quad i = 1, ..., n,
\]

where \( E_0 \) is the Young’s modulus of the solid material; \( E_{\text{min}} \) is the Ersatz stiffness which is taken to be \( 10^{-4} \); and \( p \) is the SIMP penalization parameters, which is taken to be 3 [49] in this study.

\[1\] A similar approach used by many studies is the weighted sum of mean and standard deviation as the objective function, this work focuses on the weighted sum of mean and variance.
The objective function of the RTO formulation (1) is a weighted sum of the expectation and the variance of the compliance,
\[ C(x, \xi) = f^T(\xi)u(x, \xi), \]
where \( E[\cdot] \) and \( \text{Var}[\cdot] \) stand for expectation and variance operators, respectively; and \( \kappa \in [0, 1] \) is a prescribed coefficient representing the relative importance of the expectation over the variance in the objective function. Because the expectation and the variance of the compliance have different units, we normalize their relative weights by \( w \) and \( w^2 \), respectively, where \( w = \bar{f}^T \bar{f}/E_0 \) with \( \bar{f} = E[f(\xi)] \) being the expectation of the load vector \( f(\xi) \).

The stochastic gradient of the objective function \( J \) in formulation (1) is given by
\[ g(x) \doteq \nabla_x J(x) = \mathbb{E}[G(x, \xi)] = \mathbb{E}\left[ \frac{\kappa}{w} G^\mu(x, \xi) + \frac{1 - \kappa}{w^2} G^{\text{Var}}(x, \xi) \right], \]
where
\[ G^\mu(x, \xi) \doteq \nabla_x C(x, \xi) \quad \text{and} \quad G^{\text{Var}}(x, \xi) \doteq 2 \left( C(x, \xi) - \mathbb{E}[C(x, \xi)] \right) \nabla_x C(x, \xi), \]
respectively. In the above expressions, the stochastic gradient of the compliance with respect to the design variable, \( \nabla_x C(x, \xi) \), is obtained through the chain rule as
\[ \nabla_x C(x, \xi) = H^T \nabla_{\bar{x}} C(x, \xi), \]
where \( \nabla_{\bar{x}} C(x, \xi) \) is the stochastic gradient of the compliance with respect to the filtered design variable \( \bar{x} \), whose \( i \)th component is given by
\[ \frac{\partial C(x, \xi)}{\partial \bar{x}^{(i)}} = -p(\bar{x}^{(i)})^{p-1} \left( \mathbf{u}^{(i)}(x, \xi) \right)^T \mathbf{k}_0^{(i)} \mathbf{u}^{(i)}(x, \xi) \]
Symbols \( \mathbf{u}^{(i)} \) and \( \mathbf{k}_0^{(i)} \) are the nodal displacement vector and the element stiffness matrix (corresponds to solid material) of the \( i \)th element, respectively.

In this work, we employ unbiased estimators of the objective function and its stochastic gradient. The unbiased estimators of \( \mathbb{E}[C(x, \xi)] \) and \( \text{Var}[C(x, \xi)] \) using \( m \) samples are denoted by \( \mu_m(x) \) and \( \text{Var}_m(x) \) with \( \text{Var}_m(x) = (\sigma_m(x))^2 \), where \( \sigma_m(x) \) is the estimate of the standard deviation. The estimators \( \mu_m(x) \) and \( \text{Var}_m(x) \) are given by
\[ \mu_m(x) = \frac{1}{m} \sum_{j=1}^{m} C(x, \xi_j) \quad \text{and} \]
\[ \text{Var}_m(x) = \frac{1}{m-1} \sum_{j=1}^{m} \left( C(x, \xi_j) - \mu_m(x) \right)^2, \]
respectively, where \( \xi_j, j = 1, \ldots, m \) are independent and identically distributed (i.i.d.) samples of the random vector \( \xi \). Notice that, for the variance estimator \( \text{Var}_m(x) \), it requires \( m \geq 2 \).
Accordingly, the unbiased estimator of objective function $J(x)$, denoted by $J_m(x)$, is given by

$$J_m(x) = \frac{\kappa}{w} \mu_m(x) + \frac{1 - \kappa}{w^2} \text{Var}_m(x)$$  \hspace{1cm} (9)$$

Similarly, the unbiased estimators of $\nabla_x \left( \mathbb{E}[C(x, \xi)] \right)$ and $\nabla_x \left( \text{Var}[C(x, \xi)] \right)$ using $m$ samples, which are respectively denoted by $G_{m}^\mu(x)$ and $G_{m}^{\text{Var}}(x)$, take the forms of

$$G_{m}^\mu(x) = \frac{1}{m} \sum_{j=1}^{m} \nabla_x C(x, \xi_j) \quad \text{and}$$

$$G_{m}^{\text{Var}}(x) = \frac{2}{m-1} \left\{ \sum_{j=1}^{m} \left( C(x, \xi_j) \nabla_x C(x, \xi_j) \right) - \mu_m^C(x) G_{m}^\mu(x) \right\}$$  \hspace{1cm} (10)$$

Accordingly, the corresponding unbiased estimator of the stochastic gradient of the objective function $g(x)$ using $m$ samples, which is later denoted as $G_m$, takes the form of

$$G_{m}(x) = \frac{\kappa}{w} G_{m}^\mu(x) + \frac{1 - \kappa}{w^2} G_{m}^{\text{Var}}(x)$$  \hspace{1cm} (11)$$

We note that, as required by $G_{m}^{\text{Var}}(x)$, at least two i.i.d. samples are needed to evaluate the above unbiased gradient estimator, namely $m \geq 2$.

We remark that, if we use the unbiased gradient estimator (11) together with the commonly used design update schemes in topology optimization, a large sample size $m$ is needed. This is because those update algorithms typically require higher accuracy in the estimation of gradient (11) to perform high-quality updates [31], which leads to a large sample size $m$ and the solution of $m$ linear systems (in the limit of $m \to \infty$, we have $G_m(x) \to g(x)$). Thus, the associated computational cost can be prohibitive, particularly for large-scale problems.

To address this challenge, we propose an accelerated MDSA algorithm in Section 3 tailored for the RTO formulation (1). Compared with the standard optimization algorithms in topology optimization, AC-MDSA is a stochastic optimization method, which can perform high-quality design variable update with highly noisy gradient estimations. As we demonstrate in the design examples, with the tailored AC-MDSA method proposed in this work, we can efficiently and accurately solve RTO problems with only two samples (i.e., $m = 2$) at every optimization step, where 2 is the minimum sample size for the unbiased gradient estimator.

3. Accelerated Mirror Descent Stochastic Approximation: theory and algorithm

This section introduces the background of AC-MDSA and derives the update algorithm when applied to the RTO problem. We first review the general framework of the MDSA [27, 28] and introduce an accelerated MDSA using momentum-based techniques. One major advantage of the MDSA is that, through its general definition of the distance-generating function, the design variable
update can be adapted according to the underlying geometry of the feasible set (see [28, 30] for
detailed discussions). Exploiting this advantage, this section then derives the update formula of the
AC-MDSA in the \( \ell_1 \)-norm setting with the entropy function as the distance-generating function.

3.1. Mirror descent stochastic approximation (MDSA)

The MDSA is introduced in [28] to solve stochastic optimization problems of the form

\[
\min_{x \in X} \{ \phi(x) = \mathbb{E} [ \Phi(x, \xi) ] \},
\]

where \( X \subset \mathbb{R}^n \) is the feasible set of \( x \) (typically assumed to be a nonempty bounded convex set),
and \( \xi \) is a random vector with a given probability distribution. The gradient of the objective
function is given by:

\[
\nabla \phi(x) = \mathbb{E} [ \nabla_x \Phi(x, \xi) ] = \mathbb{E} [ G(x, \xi) ],
\]

where \( G(x, \xi) = \nabla_x \Phi(x, \xi) \) is the stochastic gradient. We note that, although we assume the
differentiability of \( \Phi(x, \xi) \) with respect to \( x \), the above setting is applicable to the non-smooth
case [28].

Before we introduce the general framework of MDSA, let us first introduce the relevant nota-
tions [28]. We denote \( \| \cdot \| \) as a generalized norm defined on \( \mathbb{R}^n \) with \( \|x\|_* = \sup_{\|y\| \leq 1} y^T x \)
being its dual norm. We define \( \omega(\cdot) : X \to \mathbb{R} \) as a distance-generating function with modulus \( \alpha > 0 \) with
respect to norm \( \| \cdot \| \), such that \( \omega(\cdot) \) is convex and continuous on \( X \), continuously differentiable,
and strongly convex with parameter \( \alpha \) with respect to \( \| \cdot \| \), namely,

\[
(x' - x)^T (\nabla \omega(x') - \nabla \omega(x)) \geq \alpha \|x' - x\|^2 \quad \forall \ x', x \in X
\]

Based on the distance-generating function \( \omega(\cdot) \), we then introduce a prox-function (also known
as the Bregman divergence [50]) \( B : X \times X \to \mathbb{R}_+ \) as:

\[
B(x, z) = \omega(z) - [\omega(x) + \nabla \omega(x)^T (z - x)]
\]

Notice that, due to the convexity of \( \omega(\cdot) \), we can show that \( B(x, \cdot) \) is non-negative. Associated
with the distance-generating function \( \omega(\cdot) \), a prox-mapping \( P_x : \mathbb{R}^n \to X \) can be defined as:

\[
P_x(y) = \arg \min_{z \in X} \{ y^T (z - x) + B(x, z) \}
\]

Notice that because of the strong convexity of \( B(x, \cdot) \), the above prox-mapping is well defined
and has a unique value. Making use of the prox-mapping, the MDSA update the design variable
according to the following formula,

\[ x_{k+1} = P_{x_k} \left( \eta_k G_m(x_k) \right) = \arg \min_{z \in X} \left\{ \left( G_m(x_k) \right)^T (z - x_k) + \frac{1}{\eta_k} B(x_k, z) \right\}, \quad (17) \]

where \( x_k, \eta_k > 0 \), and \( G_m(x_k) \) are the design variable, step size, and unbiased gradient estimator (using \( m \) samples) at optimization step \( k \), respectively.

To gain a better understanding, let us take a closer look at the above update formula. The first term in the right bracket of (17) is a linear approximation of the objective function at \( x_k \) using the gradient estimator \( G_m(x_k) \), and the second term is a strongly convex function scaled by \( 1/\eta_k \). We can show that, at \( z = x_k \), the gradient of \( B(x_k, z) \) vanishes and, as a result, the gradient of the entire expression in the bracket with respect to \( z \) equals to \( G_m(x_k) \). In addition, the Hessian of the expression in the bracket equals to \( \nabla^2 \omega(z) \) scaled by \( 1/\eta_k \). This indicates that the expression in the bracket can be deemed as a convex approximation of the original objective function using the stochastic gradient \( G_m(x_k) \), and the local curvature of the expression can be controlled through \( \eta_k \).

We conclude this subsection with several remarks on the MDSA framework. First, same as the classical SA methods, the MDSA method can work with highly noisy gradient estimators. To ensure the convergence of MDSA update when applied to general stochastic optimization problems, only a single sample is required with a properly chosen step size policy [28]. This is firstly demonstrated in topology optimization by [31] for a randomized formulation to optimize structures under many deterministic load cases and the reduction to one sample load case. Alternatively, one can evaluate the gradient estimator \( G_m(x_k) \) using multiple samples [39] and integrate with a commonly used update scheme (e.g., Optimality Criteria). Second, as compared to the classical SA approaches, the MDSA framework allows for a more general setup mainly because of the general definition of distance generating-function \( \omega(\cdot) \). In fact, if we choose distance-generating function \( \omega(x) = 1/2||x||^2 \) with \( || \cdot || \) being the Euclidean norm, the MDSA update becomes the classic SA (or equivalently SGD) method [26, 28]. As demonstrated theoretically and numerically in [28] for general stochastic optimization problems, by choosing a proper distance-generating function, MDSA can adapt the update to the geometry of the problem, which leads to better performance in accuracy and convergence. This advantage is exploited in [31] for a deterministic topology optimization problem.

### 3.2. Accelerated Mirror Descent Stochastic Approximation (AC-MDSA)

In general, the performance of the MDSA is sensitive to the choice of the step size policy. Large step size in MDSA could potentially lead to divergence, while too small step size may result in slow convergence. To alleviate this sensitivity, we introduce an accelerated version of MDSA [35], referred to as AC-MDSA, which makes use of momentum-based acceleration techniques. The AC-MDSA algorithm is proposed in [35] for general stochastic optimization problems and is shown
to achieve the optimal convergence rate for convex problems. The general update algorithm is presented in Algorithm 1. Compared to the classic MDSA, which updates the $x_k$ sequence, the AC-MDSA includes updating two additional sequences, namely a “middle variable” $x_{md}^k$ and an “aggregated variable” $x_{ag}^k$.

Algorithm 1

1: **Initialize:** $x_1, x_{ag}^1 = x_1$, step sizes $\beta_1$ and $\eta_1$.
2: **Set:** $x_{md}^k = \beta_k^{-1} x_k + (1 - \beta_k^{-1}) x_{ag}^k$, with $\beta_k$ computed by (32).
3: **Get $x_{k+1}$ using MDSA update (17):**
   $x_{k+1} = P_{x_k} \left( \eta_k G_m(x_{md}^k) \right)$, with $\eta_k$ computed by (29).
4: **Set:** $x_{ag}^{k+1} = \beta_k^{-1} x_{k+1} + (1 - \beta_k^{-1}) x_{ag}^k$

The modifications from the standard MDSA update (17) in the AC-MDSA algorithm mainly lie in three aspects. First, in addition to the sequence of $x_k$, the algorithm updates the sequences $x_{md}^k$ and $x_{ag}^k$. We note that the converged sequence $x_{ag}^k$ represents the final solution. As we show in the next subsection, $x_{ag}^N$ represents the history weighted average of $x_k$ from step 1 to step $N$ with a linear weight [35]. The introduction of two additional sequences adds a negligible computational cost as they are vector additions. Second, the AC-MDSA algorithm performs update $x_{k+1}$ using the gradient estimator evaluated at $x_{md}^k$ instead of $x_k$. Third, compared with MDSA, AC-MDSA requires the specification of $\beta_k$, which acts as a weight factor to compute $x_{md}^k$ and $x_{ag}^k$.

3.3. **An Entropic AC-MDSA tailored for robust topology optimization**

Having presented the general frameworks of MDSA and AC-MDSA, we now derive an AC-MDSA algorithm with the $\ell_1$-norm tailored for the RTO problem [1] and propose the explicit update formula. With the volume (linear) and box constraints, the feasible set $X$ of the RTO formulation (1) is given by

$$X = \{ x \in \mathbb{R}^n : \frac{V(x)}{V_0} - V_f \leq 0, x^{(i)} \in [0, 1], \quad i = 1, \ldots, n \} \quad (18)$$

We define a scaled design variable vector $\tilde{x}$ such that $\tilde{x}^{(i)} := \tilde{v}^{(i)} x^{(i)}$ with $\tilde{v}^{(i)}$ being $\tilde{v}^{(i)} := (H^T v)^{(i)}/(V_0 V_f)$. The corresponding feasible set of the scaled variable is:

$$\tilde{X} = \{ \tilde{x} \in \mathbb{R}^n : \sum_{i=1}^n \tilde{x}^{(i)} - 1 \leq 0, \tilde{x}^{(i)} \in [0, \tilde{v}^{(i)}], \quad i = 1, \ldots, n \} \quad (19)$$

Accordingly, the gradient estimator of the objective function in (1) with respect to $\tilde{x}$ is obtained as:

$$\tilde{G}_m(\tilde{x}) = \text{diag} \left( \frac{1}{\tilde{v}^{(i)}} \right) G_m(x) \quad (20)$$
The feasible set $\tilde{X}$ is similar to a standard simplex set. Thus, we choose the $\ell_1$-norm with entropy function as $\omega$ in the proposed AC-MDSA algorithm, denoted as entropic AC-MDSA, because this setup leads to an improved convergence performance over the $\ell_2$-norm setting (which leads to the classical SA/SGD method) for a simplex set [31]. The distance-generating function $\omega$ of the entropic AC-MDSA takes the following form,

$$
\omega(\tilde{x}) = \sum_{i=1}^{n} \tilde{x}^{(i)} \ln \tilde{x}^{(i)},
$$

and the corresponding prox-function becomes

$$
B(\tilde{x}, z) = \sum_{i=1}^{n} \left( z^{(i)} \ln \frac{z^{(i)}}{\tilde{x}^{(i)}} - z^{(i)} + \tilde{x}^{(i)} \right)
$$

By plugging (22) into the prox-mapping (16) and dropping the constant terms, the update formula (17) becomes

$$
\tilde{x}_{k+1} = P_{\tilde{x}_k} \left( \eta_k \tilde{G}_m (\tilde{x}_k) \right) = \arg\min_{z \in \tilde{X}} \left\{ \left( \tilde{G}_m (\tilde{x}_k) \right)^T z + \frac{1}{\eta_k} \sum_{i=1}^{n} \left( z_i \ln \frac{z^{(i)}}{\tilde{x}^{(i)}} - z^{(i)} \right) \right\}
$$

The above formula is given as a minimization problem, where $\tilde{x}_{k+1}$ is its unique minimizer. Next, we derive an explicit update formula for (23). The Lagrangian of (23) with respect to the (scaled) volume constraint is given by

$$
L(z, \lambda) = \tilde{G}_m^T (\tilde{x}_k) z + \frac{1}{\eta_k} \sum_{i=1}^{n} \left( z^{(i)} \ln \frac{z^{(i)}}{\tilde{x}^{(i)}} - z^{(i)} \right) + \lambda (l^T z - 1),
$$

where $\lambda \in \mathbb{R}_+$ is the Lagrange multiplier associated with the constraint and $l$ is a constant vector whose components are all 1. Imposing the gradient condition $\nabla_z L(z, \lambda) = 0$ gives:

$$
\frac{\partial L(z, \lambda)}{\partial z_i} = \tilde{G}_m^{(i)} (\tilde{x}_k) + \frac{1}{\eta_k} \ln \frac{z^{(i)}}{\tilde{x}^{(i)}} + \lambda = 0,
$$

which can be recast as

$$
z^{(i)} (\lambda) = \tilde{x}^{(i)}_k \exp \left( -\eta_k \left( \tilde{G}_m^{(i)} (\tilde{x}_k) + \lambda \right) \right)
$$

Incorporating the box constraints, we then obtain the update formula:

$$
\tilde{x}^{(i)}_{k+1} (\lambda^*) = \begin{cases} 
  z^{(i)} (\lambda^*), & \text{if } \tilde{x}^{(i)}_{k+1} \leq z^{(i)} (\lambda^*) \leq \tilde{x}^{(i)}_{k+1}, \\
  \tilde{x}^{(i)}_{k+1}, & \text{if } z^{(i)} (\lambda^*) > \tilde{x}^{(i)}_{k+1}, \\
  \tilde{x}^{(i)}_{k+1}, & \text{if } z^{(i)} (\lambda^*) < \tilde{x}^{(i)}_{k+1},
\end{cases}
$$

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where \( \tilde{x}^{(i)}_{k+1} \) is the upper bound of updated design variables and \( \bar{x}^{(i)}_{k+1} \) is the lower bound, and \( \lambda^* \) solves equation \( \sum_{i=1}^n \bar{x}^{(i)}(\lambda) = 1 \). In practice, \( \lambda^* \) is obtained using the bi-section method. Notice that, in (27), we introduce a move limit denoted as \( \text{move} \), which will be adaptively adjusted by a damping scheme (see Section 4.3). After obtaining \( \bar{x}_{k+1} \), we map it back to the original feasible space as

\[
x^{(i)}_{k+1} := \frac{1}{\tilde{v}(i)} \tilde{x}^{(i)}_{k+1}
\]  

(28)

Finally, the proposed entropic AC-MDSA for RTO problem (1) is obtained by replacing (17) in the step 3 of Algorithm (1) with (26), (27), and (28). Several remarks can be made regarding the above entropic AC-MDSA update. First, the derived update (26) and (27) can handle both positive and negative stochastic gradients, thus it is also applicable to RTO problems with other objective functions, such as the compliance mechanism design [12, 19]. Second, as long as we start from a feasible initial guess, the \( \bar{x}_k \) (and \( x_k \)) always stays positive. Thus, the lower bound \( x_{\min} = 0 \) of the design variables is typically not active. By tailoring AC-MDSA algorithm to RTO, we find that, compared with the MDSA method, the AC-MDSA method can lead to accelerated convergence performance without an increase in computational cost. We also observe that the AC-MDSA method is considerably less sensitive to the choice of step sizes, which allows us to use larger step sizes. These advantages will be demonstrated in the numerical examples in Section 5.

4. Algorithmic parameters and implementation details of the entropic AC-MDSA for RTO

This section discusses the algorithmic and implementation details of the proposed entropic AC-MDSA algorithm for RTO problems. In particular, we present the step size policy and introduce related techniques (i.e., step size recalibration and adaptive damping) to accelerate the convergence performance and reduce the step-size sensitivity of the AC-MDSA algorithm.

4.1. Step size policy

Typically, the step size policy is critical for stochastic optimization algorithms. The step size policy adopted in this work is based on [35] and involves two sequences \( \eta_k \) and \( \beta_k \). The policy for \( \eta_k \) is given by

\[
\eta_k = \theta \eta \frac{k + 1}{2}, \quad k = 1, 2, ..., N,
\]

(29)

Because compliance problems have active volume constraints in practice, the proposed update formula assumes that mapped volume constraint is active throughout the optimization process, namely, \( \sum_{i=1}^n \tilde{x}^{(i)} = 1 \).
where $\theta$ is a user-defined scaling factor that adjusts the step size, and $\bar{\eta}$ is computed according to

$$
\bar{\eta} = \sqrt{6\alpha D_{\omega,\bar{X}}} \frac{1}{(N + 2)^{\frac{3}{2}} (4M^2 + \Sigma^2)^{\frac{1}{2}}},
$$

(30)

where $\alpha = 1$, $D_{\omega,\bar{X}}$ is the diameter of set $\bar{X}$ measured by the distance-generating function $\omega$, which is taken to be $\sqrt{\ln(n)}$ [35, 28]. Parameters $M$ and $\Sigma$ are estimates of the upper bounds of the stochastic gradients and its variance. In this work, they are estimated using sampling-based techniques,

$$
\mathcal{M} = \sqrt{\frac{1}{N_M} \sum_{i=1}^{N_M} \left\| \tilde{G}_m^{(i)}(\bar{x}_k) \right\|_\infty^2}
$$

and

$$
\Sigma = \sqrt{\frac{1}{N_\Sigma} \sum_{i=1}^{N_\Sigma} \left\| \tilde{G}_m^{(i)}(\bar{x}_k) - \mathcal{Q} \right\|_\infty^2},
$$

(31)

where $\tilde{G}_m^{(i)}$ denotes the $i$th evaluation of $\tilde{G}_m$ using an independent set of $m$ samples, $N_M$ and $N_\Sigma$ are the numbers of evaluations to estimate the upper bounds of the stochastic gradients and its variance, respectively, and $\mathcal{Q} = 1/N_M \sum_{i=1}^{N_M} \tilde{G}_m^{(i)}(\bar{x}_k)$. The policy for $\beta_k$ is given by

$$
\beta_k = \frac{k + 1}{2}, \quad k = 1, 2, ..., N
$$

(32)

If we plug (32) into the expression of aggregated variable $\bar{x}_{k+1}^{ag}$ in Algorithm (1), $\bar{x}_{k+1}^{ag}$ can be recast as [35]:

$$
\bar{x}_{k+1}^{ag} = \frac{2}{k + 1} \bar{x}_{k+1} + \frac{k - 1}{k + 1} \bar{x}_{k}^{ag} = \frac{\sum_{t=1}^{k} (t \bar{x}_{t+1})}{\sum_{t=1}^{k} t}
$$

(33)

The above expression indicates that the aggregated variable $\bar{x}_{k+1}^{ag}$, obtained by adopting $\beta$ policy [32], is the weighted average of the history of variable $\bar{x}_{t+1}$ with linearly varying weights. We note that history-averaging techniques are widely used in SA methods to suppress noise and accelerate convergence [28, 29]. Such a noise-suppressing strategy is different from Monte Carlo based methods [51], in which the noise is reduced through estimations using many samples within one optimization step.

4.2. Adaptive step recalibration scheme

The history-averaging technique can lead to a small change of designs as the optimization proceeds, leading to slow convergence. To address this, we propose a step size recalibration scheme to speed up the evolution of the design and convergence.

The basic idea of the recalibration scheme is to adaptively re-initialize the acceleration method throughout the optimization by tracking the changes of design variables. Specifically, we monitor the $\ell_2$-norm of the change of $\bar{x}_k^{ag}$, namely,

$$
\| \Delta x_k^{ag} \|_2 = \| x_{k+1}^{ag} - x_k^{ag} \|_2
$$

(34)
If $\|\Delta x_{sa}^{ag}\|_2$ becomes smaller than a tolerance $\epsilon_{rst}$, then we recompute $\bar{\eta}$, $\mathcal{M}$, and $\Sigma$ based on (30) and (31), and set $k = 1$ in evaluating $\eta_k$ and $\beta_k$. To avoid frequent recalibration, we require the number of optimization steps between two consecutive recalibrations to be larger than a prescribed minimum step $\Delta_{rst}$, and monitoring of $\|\Delta x_{sa}^{ag}\|_2$ starts after the first $N_{rst}$ steps. We note that recalibration schemes of similar forms are shown to be effective for accelerated gradient descent methods in other applications, for example, see [31].

4.3. Adaptive damping scheme

Because of the stochastic nature of the entropic AC-MDSA algorithm, optimization terminates at the maximum step unless a decaying step size policy is adopted. Thus, this work adopts the adaptive damping scheme proposed in [39] to effectively terminate the optimization after the design has converged. Inspired by the simulated annealing [52, 53], the adaptive damping scheme monitors the average progress of the design at each step and reduces the move limit when small progress is detected. The average progress of the design at $k^{th}$ step is characterized by the effective step ratio, $R_k$, which is defined as:

$$R_k := \frac{1}{N_D} \frac{\|E_k - E_{k-N_D+1}\|_2}{\|E - E_{k-1}\|_2},$$

(35)

where $E$ is the vector of elemental Young’s moduli, $N_D$ is the history window size.

The effective step ratio, $R_k$, represents the relative magnitude of the average design change over the past $N_D$ steps to the current design change. A small $R_k$ indicates slow progress over the previous $N_D$ steps, the move limit is then reduced. Specifically, if $R_k$ is lower than a tolerance $\epsilon_{damp}$, the move limit is scaled down by a factor $\tau$, i.e. $move = move/\tau$. Here, we use $\tau = 2$. The adaptive damping scheme is activated after a prescribed minimum number of steps $N_{damp}$.

4.4. Algorithm summary

To conclude this section, we summarize the proposed entropic AC-MDSA algorithm and its parameters in Algorithm 2. The objective function value and design quality are generally insensitive to most of the algorithm parameters, i.e., $N_{rst}$, $\Delta_{rst}$, $\epsilon_{rst}$, $N_{damp}$, $\epsilon_{damp}$, $\tau$, $N_D$, $N_{max}$, $N_{min}$, and $\epsilon$. We have investigated various parameter values and summarized the value ranges used in this study in Table 1 which are generally recommended. The step size scaling factor, $\theta$, has more influence on the results, as it directly adjusts the magnitude of the step size $\eta_k$. In general, a larger $\theta$ (and therefore larger $\eta_k$) leads to faster convergence and design evolution, but $\theta$ should not be too large as it may result in instability. The proper range of $\theta$ needs to be calibrated with a few pilot runs, but in general, the range that produces a stable and steady convergence is wide.
Table 1: Range of parameter values for AC-MDSA

| Parameter  | Value          | Usage                              |
|------------|----------------|------------------------------------|
| $N_{rst}$  | 100 or 300     |                                    |
| $\Delta_{rst}$ | 100            | Adaptive step recalibration        |
| $\epsilon_{rst}$ | 0.025         |                                    |
| $N_{damp}$ | 400 $\sim$ 450 |                                    |
| $\epsilon_{damp}$ | 0.05 or 0.075 | Adaptive damping scheme            |
| $\tau$     | 2              |                                    |
| $N_D$      | 100            |                                    |
| $N_{max}$  | 450 $\sim$ 600 | Termination of optimization        |
| $N_{min}$  | 400 $\sim$ 450 |                                    |
| $\epsilon$ | 0.01           |                                    |
| $N_M$      | 6              | Estimation of $\mathcal{M}$ and $\Sigma$ \(^{[31]}\) |
| $N_\Sigma$ | 6              |                                    |
Algorithm 2 Entropic AC-MDSA algorithm for robust topology optimization

1: Initialize: $x_1, \theta$
2: Set $\tilde{x}_1 = \text{diag}(\tilde{v}(i)) x_1, \tilde{x}_1^{ag} = \tilde{x}_1$; compute $\bar{\eta}$ using (30); and set $k_{in} = 1$.
3: for $k = 1, ..., N_{\max}$ do
4: \hspace{1em} if $k \geq N_{\text{rst}}$ and $k_{in} \geq \Delta_{\text{rst}}$ and $\|x_{k+1}^{ag} - x_k^{ag}\|_2 < \epsilon_{\text{rst}}$ then
5: \hspace{2em} Set $k_{in} = 1$ and $\tilde{x}_k = \tilde{x}_k^{ag}$
6: \hspace{2em} Compute $\bar{\eta}$ using (30)
7: \hspace{1em} end if
8: \hspace{1em} Set $\tilde{x}_k^{md} = \beta_{k_{in}}^{-1} \tilde{x}_k^{ag} + (1 - \beta_{k_{in}}^{-1}) \tilde{x}_k^{ag}$ with $\beta_{k_{in}}$ defined in (32).
9: \hspace{1em} Compute gradient estimator $\tilde{G}_m(\tilde{x}_k^{md})$ according to (20) and (11) using $m$ i.i.d samples.
10: \hspace{1em} Update $\tilde{x}_{k+1}$ using entropic MDSA (26)–(27)
11: \hspace{1em} Set $\tilde{x}_{k+1}^{ag} = \beta_{k_{in}}^{-1} \tilde{x}_{k+1}^{ag} + (1 - \beta_{k_{in}}^{-1}) \tilde{x}_k^{ag}$ with $\beta_{k_{in}}$ defined in (32).
12: \hspace{1em} Compute $x_{k+1}^{ag}$ using (28)
13: \hspace{1em} if $k \geq N_{\text{min}}$ and $\|x_{k+1}^{ag} - x_k^{ag}\|_\infty < \epsilon$ then
14: \hspace{2em} break
15: \hspace{1em} end if
16: \hspace{1em} Evaluate effective step ratio $R_k$ using (35)
17: \hspace{1em} if $k \geq N_{\text{damp}}$ and $R_k \leq \epsilon_{\text{damp}}$ then
18: \hspace{2em} $\text{move} = \text{move}/\tau$
19: \hspace{1em} end if
20: \hspace{1em} $k_{in} = k_{in} + 1$
21: end for
22: Output: $x^* = x_k^{ag}$

5. Numerical examples

This section presents four examples to demonstrate the effectiveness and efficiency of the entropic AC-MDSA algorithm. First, to verify the results by AC-MDSA, we compare the final designs, objective function values, and computational cost of the AC-MDSA with those from the Monte Carlo (MC) method. The MC method evaluates the sensitivity using $m = 1,000$ samples at each optimization step to get sufficiently accurate gradients and uses a popular optimization update algorithm, MMA [54], to update the design variables with the estimated sensitivity. The second example shows that the AC-MDSA, although using two samples, effectively reflects the influence of $\kappa$ (relative weight of mean and variance) through both designs and objective function values. Example 3 demonstrates the AC-MDSA using problems with different domain geometries, multiple random loads, and various mesh sizes. Finally, in Example 4, we solve a three-dimensional (3D) problem to show the applicability of the entropic AC-MDSA with an iterative linear solver.
The key information of the four examples is summarized in Table 3. The investigated $\kappa$ values for the robust designs are $\kappa = 0.8284, 0.618, 0.2824$, and they are chosen such that the equivalent ratio in terms of mean and standard deviation in the objective function with $w = 1$, i.e. $\kappa : \sqrt{1 - \kappa}$, is $2, 1, \frac{1}{3}$, which are commonly used values in the RTO literature. The $\kappa$ values are summarized in Table 2.

Table 2: Investigated $\kappa$ values and their equivalent mean-to-s.t.d. ratios

| $\kappa$ value | Equivalent ratio of mean : s.t.d. $(\kappa : \sqrt{1 - \kappa})$ with $w = 1$ |
|----------------|--------------------------------------------------------------------------------|
| 1             | -                                                                               |
| 0.828         | $1 : 0.5$                                                                      |
| 0.618         | $1 : 1$                                                                         |
| 0.282         | $1 : 3$                                                                         |

We implement the proposed AC-MDSA algorithm in the PolyTop code [55]. To comprehensively and fairly evaluate the algorithm’s performance, we carry out 50 consecutive and independent runs for each $\kappa$ studied in every 2D example and present the statistical data related to the algorithm’s performance. Notice the 50 trials are only for evaluating statistical consistency and are not required for practical use of the algorithm. The presented design for each $\kappa$ is a representative design chosen from the 50 trials and has an objective function value close to the mean value of the 50 objective function values. At the end of the optimization, denoting $x^*$ as the optimized solution, we use $m = 10,000$ samples to obtain accurate estimates of the objective function value, the mean, and the standard deviation of the compliance for the final design, denoted as $\hat{J}(x^*)$, $\hat{\mu}(x^*)$, and $\hat{\sigma}(x^*)$, respectively. For comparison, we also include the deterministic designs with the objective function being the compliance under deterministic loads that take the mean values of the random loads. The $\hat{\mu}(x^*)$ and $\hat{\sigma}(x^*)$ of the optimized deterministic design is evaluated using the same random load corresponding to the stochastic cases. The total wall-clock time and the number of optimization steps are reported. All the examples are performed on a machine with an Intel(R) Xeon(R) Silver 4116 CPU, 2.10GHz processor and 64 GB of RAM, running MATLAB R2018b. In this work, the state equation is solved using the sparse direct solver and preconditioned conjugate gradient solver for 2D and 3D problems, respectively. For most two-dimensional (2D) examples, we enforce the design symmetry about the vertical axis, and we study a 2D example without symmetry constraint. For the 3D example, we enforce design symmetry about the two vertical planes.
Table 3: Brief description of the numerical examples.

| Ex. | Dim. | Name                  | Load uncertainty          | Feature                                                                 |
|-----|------|-----------------------|---------------------------|-------------------------------------------------------------------------|
| 1   | 2D   | Simple column         | Random direction          | - Verification of entropic AC-MDSA with MC                               |
|     |      | benchmark             | $\sim U(\frac{11}{24}\pi, \frac{13}{24}\pi)$ | - Study of sample size, $m = 2, 10, 100$                                  |
|     |      |                       |                           | - Comparison of MDSA algorithms with and without acceleration             |
| 2   | 2D   | Half circle           | Deterministic vertical & random horizontal components $\sim N(0, 0.15^2)$ | - Study of $\kappa$ values, $\kappa = 1, 0.618, 0.282$                   |
| 3   | 2D   | Double hook & Torsion disk | Deterministic vertical/normal & multiple random horizontal/tangential components $\sim N(0, 0.1^2)$ | - Problem size study, $n = 114k, 51k, 13k$                               |
|     |      |                       |                           | - Complex design geometries and multiple independent random components    |
|     |      |                       |                           | - Comparison with MC                                                     |
| 4   | 3D   | Crane                 | Deterministic z-direction, multiple random x- & y-directions $\sim N(0, 0.1^2)$ | - Combination of entropic AC-MDSA with iterative linear solver           |

5.1. Example 1: Simple column benchmark

The first example is the simple column involving randomness in the load direction, which is commonly studied in the literature of RTO. We first verify the proposed entropic AC-MDSA (using two samples) by comparing its results with the ones obtained by the MC method (using 1000 samples). Then, we demonstrate the robustness of the entropic AC-MDSA algorithm with respect to different sample sizes $m$ (thus different accuracy levels) for computing the gradient estimator. Finally, we compare the performance of the entropic AC-MDSA algorithm with the entropic MDSA (without acceleration).

Figure 1a shows the design domain and boundary conditions of the simple column problem. The domain is fixed at the bottom and is subjected to a load $f$ with a deterministic magnitude of 1 and a random direction, defined by $\alpha \sim U(\frac{11}{24}\pi, \frac{13}{24}\pi)$ with the standard deviation being $\frac{1}{12\sqrt{12}}\pi$, which is in the common range used in the literature [22][13]. We consider three cases: a deterministic design ($\alpha \sim U(\frac{1}{2}\pi, \frac{5}{2}\pi)$), a robust design with $\kappa = 1$, and a robust design with $\kappa = 0.618$. The mesh size
For the entropic AC-MDSA algorithm, we use the sample size $m = 2$, $\theta = 600n$, $N_{rst} = 100$, $N_{damp} = 400$, $\epsilon_{damp} = 0.05$, $N_{max} = 500$, and $N_{max} = 400$. The filter radius begins to reduce to $R = 1.2$ with an interval of 30 steps. For the MC method, we use $m = 1000$ and $N_{max} = 100$, and the filter radius starts to decrease at the 60th step, which is at the same stage relative to the $N_{max}$ ($60/100 = 0.6$) as the one in AC-MDSA ($300/500 = 0.6$). We chose the relatively small $N_{max} = 100$ for the MC because the computational cost for MC with $m = 1000$ samples is excessive.

Figure 1: Geometry and boundary conditions of (a) Example 1: simple column, $H = W = 100$, point load $f$ has a deterministic magnitude of 1 and a random load direction $\alpha \sim U(\frac{11}{24}\pi, \frac{13}{24}\pi)$; (b) Example 2: half circle, $D = 1$, point load has a deterministic vertical component $f_y = 1$ and a random horizontal component $f_x \sim N(0, 0.15^2)$.

Figure 2: Final designs of (a) AC-MDSA, deterministic; (b) AC-MDSA, $\kappa = 1$; (c) AC-MDSA, $\kappa = 0.618$; (d) MC, deterministic; (e) MC, $\kappa = 1$; (f) MC, $\kappa = 0.618$. The design in (b) and (C), respectively, is a representative design chosen from the 50 trials.
5.1.1. Verification of AC-MDSA with MC

Here, we verify the entropic AC-MDSA algorithm with MC method by comparing the representative final designs and objective function values as shown in Figure 2 and statistics in Table 4. For each $\kappa$, the representative design of the AC-MDSA in Figure 2 has an objective function value close to the mean of the objective function values of the 50 trials. For the deterministic cases (Figures 2a and 2d), the entropic AC-MDSA and MC methods produce similar designs with comparable objective values, demonstrating that the entropic AC-MDSA can also be used to solve deterministic problems. Notice that in the deterministic case, even though the $\hat{\mu}$ and $\hat{\sigma}$ are identical for the AC-MDSA and MC, the $\hat{J}$ are different. This is because the $\hat{\mu}$ and $\hat{\sigma}$ are evaluated with the random load, and $\hat{J}$ is obtained with the deterministic load, which is not computed based on $\hat{\mu}$ and $\hat{\sigma}$. In the robust designs with $\kappa = 1$ (Figures 2b and 2e), both methods produce similar designs with two split legs, and the design by AC-MDSA has slightly wider distances between the two legs and a slightly lower $\hat{J}(x^*)$ (and lower $\hat{\mu}(x^*)$ and $\hat{\sigma}(x^*)$). In the robust designs with $\kappa = 0.618$ (Figures 2c and 2f), both AC-MDSA and MC methods produce similar designs, and the design by AC-MDSA has a smaller distance between the two legs and a lower $\hat{J}(x^*)$. This comparison verifies that, with only two samples in each optimization step, the entropic AC-MDSA produces similar designs and objective function values as the MC method with 1,000 samples. We note that even though the MC achieves slightly higher objective function values, MC’s solution can be potentially improved with more optimization steps and more computational time. Comparing designs with various $\kappa$ values, the design with higher weight in variance ($\kappa = 0.618$) has wider legs and smaller $\hat{\sigma}(x^*)$. In terms of computational efficiency, AC-MDSA generally has low computational costs as indicated in Table 4 due to its use of two samples.

![Figure 3: Performance comparison of the proposed AC-MDSA algorithm and the MC method: $\hat{\mu}(x^*)$ versus $\hat{\sigma}(x^*)$ for $\kappa = 1$ and $\kappa = 0.618$. AC-MDSA includes 50 trials for each $\kappa$ value. (Representative designs from each case is shown next to the highlighted markers.)](image-url)
To evaluate the overall performance and consistency of the entropic AC-MDSA, the $\hat{\mu}(x^*)$ versus $\hat{\sigma}(x^*)$ of the 50 independent trials (one trial is one run of the numerical experiment) are plotted in Figure 3. We observe that 50 independent trials with $\kappa = 1$ lead to similar $\hat{\mu}(x^*)$, and those 50 trials with $\kappa = 0.618$ (higher weight on Var) have similar $\hat{\sigma}(x^*)$, indicating the AC-MDSA algorithm produces consistent designs. Also, the ones with $\kappa = 0.618$ have considerably lower $\hat{\sigma}(x^*)$ and higher $\hat{\mu}(x^*)$ than those with $\kappa = 1$, demonstrating the algorithm can effectively reflect the impact of $\kappa$ with only two samples. In the $\kappa = 0.618$ case, although MC method produces a design with the lowest $\hat{\sigma}(x^*)$, its $\hat{\mu}(x^*)$ is considerably higher than the designs produced by AC-MDSA, resulting in an overall higher objective function value.

Figure 4 shows the history of the estimated objective values of AC-MDSA and MC methods for $\kappa = 1$ and $\kappa = 0.618$. Note that the objective history of AC-MDSA is more oscillatory than the one of the MC method because the objective function in the entropic AC-MDSA is estimated with $m = 2$ samples per step, and the one in the MC is estimated with $m = 1000$ samples. However, the true objective of AC-MDSA evaluated at the end of the optimization with 10000 samples has a similar value to that obtained by MC as indicated in Figure 2.
5.1.2. Study of sample size

Next, we study the influence of various sample sizes \( m \), which is used to compute the stochastic gradient, on the performance of the proposed AC-MDSA. We consider \( m = 2, m = 10, m = 100 \) samples. Figure 5 shows the history of the error (norm) of stochastic gradients estimated using the three \( m \) values for \( \kappa = 1 \) (Figures 5 a and b) and \( \kappa = 0.618 \) (Figures 5 c and d). The error is defined as the difference between the estimated gradient using \( m \) samples and the reference estimated gradient using 1000 samples. Several observations can be made. First, as we expect, a larger \( m \) leads to a smaller difference between the estimated gradient and the reference estimated gradient. Second, the cosine of the angle between the stochastic and the reference estimated gradient vectors for both \( \kappa = 1 \) and \( \kappa = 0.618 \) are close to 1 after the first few steps, indicating the estimated gradient with a small sample size has fairly accurate directions, but this observation can be problem-dependent.

Various sample sizes \( m \) produce gradient estimators with different accuracy levels; thus, we study the sensitivity of the AC-MDSA performance to \( m \). Table 4 summarizes the performance and the associated computational cost of the entropic AC-MDSA with \( m = 2, 10, \) and 100 samples and compares with the ones from the MC method. The statistics in for the entropic AC-MDSA in Table 4 are averaged over the 50 independent trials (for evaluating statistical consistency and are not needed in practice). The computational time shown in Table 4 is for reference. To have
a more comprehensive comparison of computational cost, more investigation is needed. For the entropic AC-MDSA, a larger $m$ leads to small differences in the final objective function values and convergence steps. This showcases that the proposed entropic AC-MDSA can perform high-quality updates with highly noisy gradient estimators (i.e., $m = 2$). Therefore, we use $m = 2$ for the remaining studies.

Table 4: Performance of AC-MDSA (averaged over 50 trials) and MC methods: Simple column example

| Algorithm | $\kappa$ | $J(x^*)$ (avg.) | $\hat{\mu}(x^*)$ (avg.) | $\hat{\sigma}(x^*)$ (avg.) | $N_{\text{step}}$ (avg.) | $N_{\text{solve}}$ (sec.) | WC time (sec.) | WC time $\frac{N_{\text{step}}}{N_{\text{step}}}$ (sec.) |
|-----------|---------|----------------|-------------------------|--------------------------|-----------------|-----------------|----------------|----------------|
| AC-MDSA   | 1       | 9.02           | 9.02                    | 1.13                     | 411.3           | 862.6           | 81.1           | 0.2            |
|           | $m = 2$ | 0.618          | 5.93                    | 9.41                     | 0.55            | 420.9           | 881.8          | 82.8           |
| AC-MDSA   | 1       | 9.00           | 9.00                    | 1.07                     | 407.6           | 855.3           | 136.0          | 0.3            |
|           | $m = 10$| 0.618          | 5.96                    | 9.48                     | 0.52            | 414.0           | 868.1          | 138.5          |
| AC-MDSA   | 1       | 8.99           | 8.99                    | 1.07                     | 403.4           | 846.7           | 807.3          | 2.0            |
|           | $m = 100$| 0.618         | 5.97                    | 9.49                     | 0.52            | 410.6           | 861.2          | 824.4          |
| MC        | 1       | 9.23           | 9.23                    | 1.47                     | 100             | 100000.0        | 1985.0         | 19.9           |
|           | $m = 1000$| 0.618       | 6.28                    | 10.05                    | 0.41            | 100             | 100000.0       | 1972.7         |

5.1.3. Comparison of AC-MDSA and MDSA algorithms (with and without acceleration)

We compare the performance of the entropic AC-MDSA algorithm with the entropic MDSA algorithm (without acceleration) to demonstrate the advantage of the acceleration technique. In particular, we aim to demonstrate that, with the acceleration scheme, the AC-MDSA is less sensitive to various step sizes. We consider the case of $\kappa = 0.618$ and use the same step size recalibration, damping, and filter radius reduction setup for the MDSA algorithm. The symmetry of the designs is not imposed in this comparison. For the entropic MDSA, the step size formula is adopted from [28, 31]. Figure 6 shows the final designs of AC-MDSA and MDSA with three values of step size scaling factor $\theta$. Notice $\theta$ is set to a smaller value than previous cases, and this is because when the symmetry constraint is absent, the algorithm needs a smaller step size to guarantee stable and steady convergence. Each design is a representative one selected from the results of 20 independent trials. The range of $\theta$ value for the entropic MDSA is determined based on pilot runs. As shown in Figure 6, the entropic MDSA is more sensitive to different choices of $\theta$ (i.e., different step sizes) than the entropic AC-MDSA. For various $\theta$ values considered, the entropic AC-MDSA yields similar results (which are also similar to Figures 2c and f) with comparable performance, whereas the entropic MDSA yields less consistent results. Besides, although the design symmetry is not imposed, the entropic AC-MDSA produces nearly-symmetric designs while the entropic MDSA yields asymmetric ones, indicating the entropic AC-MDSA is more robust and stable than
the entropic MDSA (without acceleration). Thus, the remaining of the study uses the entropic AC-MDSA algorithm.

\[
\begin{align*}
\text{a, AC-MDSA } & \kappa = 0.618 \ m = 2 \\
& \theta = 2000 \\
& \mu = 6.69 \\
& \hat{\mu} = 10.69 \\
& \sigma = 0.48 \\
\text{b, AC-MDSA } & \kappa = 0.618 \ m = 2 \\
& \theta = 6000 \\
& \mu = 6.54 \\
& \hat{\mu} = 10.46 \\
& \sigma = 0.46 \\
\text{c, AC-MDSA } & \kappa = 0.618 \ m = 2 \\
& \theta = 10000 \\
& \mu = 6.49 \\
& \hat{\mu} = 10.37 \\
& \sigma = 0.47 \\
\text{d, MDSA } & \kappa = 0.618 \ m = 2 \\
& \theta = 10 \\
& \mu = 6.99 \\
& \hat{\mu} = 10.78 \\
& \sigma = 0.93 \\
\text{e, MDSA } & \kappa = 0.618 \ m = 2 \\
& \theta = 30 \\
& \mu = 7.32 \\
& \hat{\mu} = 10.83 \\
& \sigma = 1.28 \\
\text{f, MDSA } & \kappa = 0.618 \ m = 2 \\
& \theta = 50 \\
& \mu = 7.07 \\
& \hat{\mu} = 10.62 \\
& \sigma = 1.16
\end{align*}
\]

Figure 6: Final design of (a) AC-MDSA: \(\theta = 2000\); (b) AC-MDSA: \(\theta = 6000\); (c) AC-MDSA: \(\theta = 10,000\); (d) MDSA: \(\theta = 10\); (e) MDSA: \(\theta = 30\); (f) MDSA: \(\theta = 50\). The design in each case, respectively, is a representative design chosen from the 20 trials.

5.2. Example 2: Half circle

The second example demonstrates that the entropic AC-MDSA effectively captures the influence of various \(\kappa\) values (relative weight of mean and variance for compliance) on the designs. Figure 1b shows the design domain and boundary conditions. The domain (discretized by \(n = 40,000\) polygonal elements [55]) is fixed on the outer perimeter and subjected to a point load that has a deterministic vertical component with magnitude 1 and random horizontal component \(\sim \mathcal{N}(0, 0.15^2)\).

We consider three cases: \(\kappa = 1\), \(\kappa = 0.618\), and \(\kappa = 0.282\). The filter radius \(R\) is initialized as 0.03 and reduced to 0.004 after 300 steps with an interval of 30 steps. We choose \(\theta = 8000n\), \(\theta = 100n\), and \(\theta = 10n\) for \(\kappa = 1\), \(\kappa = 0.618\), and \(\kappa = 0.282\), respectively; \(N_{\text{rest}} = 300\), \(N_{\text{damp}} = 450\), \(\epsilon_{\text{damp}} = 0.05\), \(N_{\text{max}} = 600\), and \(N_{\text{min}} = 450\).

Figure 7 shows the designs obtained by the entropic AC-MDSA for the deterministic case (i.e., the horizontal load is 0) and three stochastic cases with a wide range of \(\kappa\). For the stochastic cases, each design is a representative one from 50 independent trials with the objective function values close to the mean of the 50 objective function values. The three stochastic designs show the impact of various \(\kappa\) values: as \(\kappa\) decreases (more weight on the variance), the angle between the two arms increases, improving the robustness in resisting the random horizontal load.
The impact of varying $\kappa$ is shown in Figure 8, which plots $\hat{\mu}(x^*)$ versus $\hat{\sigma}(x^*)$ of a total of 150 independent trials (50 for each $\kappa$) with representative designs. Several observations can be made. First, as $\kappa$ decreases, $\hat{\sigma}(x^*)$ decreases (indicating improved robustness) and $\hat{\mu}(x^*)$ increases. The AC-MDSA produces consistent designs for each $\kappa$ case. Second, the designs for larger $\kappa$ typically have similar $\hat{\mu}(x^*)$ but widely distributed $\hat{\sigma}(x^*)$, while the designs for smaller $\kappa$ typically have similar $\hat{\sigma}(x^*)$ but widely distributed $\hat{\mu}(x^*)$. This observation is consistent with the definition of the objective function in (9).

Figure 8: $\hat{\mu}(x^*)$ versus $\hat{\sigma}(x^*)$ of the 50 trials by the entropic AC-MDSA. (Highlighted markers correspond to the presented designs.)

### 5.3. Example 3: Robust designs with multiple random loads

The third example, which includes the double hook and the disk problem, is designed to show that the AC-MDSA algorithm can tackle problems with various problem sizes, geometries, and multiple independent random loads. Additionally, using the double hook example, we show that the parameters of the AC-MDSA algorithm are insensitive to various mesh sizes. Figure 9 shows the design domains and boundary conditions of the double hook and the disk problems.
Figure 9: Geometry and boundary conditions of (a) double hook, \( W = 4, W_1 = 1, H_1 = 1, H_2 = 1.5 \), two point loads have deterministic vertical components \( f_{1y} = f_{2y} = 1 \) and random horizontal components \( f_{1x}, f_{2x} \sim N(0, 0.1^2) \); (b) disk, \( D_{out} = 2, D_{in} = 0.6 \), five point loads have deterministic normal components \( f_{1N} = f_{2N} = f_{3N} = f_{4N} = f_{5N} = 1 \) and random tangential components \( f_{1T}, f_{2T}, f_{3T}, f_{4T}, f_{5T} \sim N(0, 0.1^2) \).

Figure 10: Double hook: deterministic and robust designs (\( \kappa = 1 \) and \( \kappa = 0.618 \)) obtained by the AC-MDSA algorithm: (a)-(c) \( n = 114,048 \), (d)-(f) \( n = 50,688 \), (g)-(i) \( n = 12,672 \). The design of (b), (c), (e), (f), (h), and (i), respectively, is a representative design chosen from the 50 trials.

5.3.1. Double hook

In the double hook problem, the two point loads have deterministic vertical components with magnitudes 1 and random horizontal components \( \sim N(0, 0.1^2) \). We use \( \theta = n \) and \( \theta = 0.03n \) for
κ = 1 and κ = 0.618, respectively, and N_{rst} = 100, N_{damp} = 450, \epsilon_{damp} = 0.075, N_{max} = 600, and N_{min} = 450. We first evaluate the sensitivity of the AC-MDSA algorithmic parameters (e.g., step size factor θ and initial step to monitor recalibration N_{rst}) to various mesh sizes, n = 114,048, n = 50,688 and n = 12,672. For comparison, we also solve the problem using MC method with 1000 samples.

Comparing (vertically) the designs with three problem sizes, as shown in Figure 10, they have consistent geometric features and similar objective function values for each case of κ. This observation demonstrates that the proposed AC-MDSA algorithm and associated parameters can lead to mesh-insensitive designs. Consistent observations can be made among designs from various problem sizes. The deterministic and robust designs differ in the upper domain. The deterministic design forms a single connection to the support, resulting in less resistance to moments and horizontal loads. The robust design with κ = 1 has two separated arms without braces, which can carry moment but is weak in resisting horizontal shear forces. The robust design with κ = 0.618 forms a brace with separated arms, indicating an improved strength to resist the stochastic lateral load. The increase in robustness is also revealed in the decrease in ˆσ(x*) of the three designs from left to right.

The final designs and objective function values obtained by the MC method with 1000 samples are shown in Figure 11. The main geometric features are similar to the designs from the AC-MDSA with two samples, but with more small branches. For the objective function values, AC-MDSA achieves a slightly lower value in the κ = 1 design and an identical value in the κ = 0.618 design as compared to the MC method. Figure 12 (a) shows ˆμ(x*) and ˆσ(x*) of the 50 trials from the AC-MDSA and one trial from the MC method. We can observe that a lower κ value produces designs with lower ˆσ(x*). The statistics, including computational cost, is shown in Table 5, and the data related to AC-MDSA are averaged values over the 50 trials (for evaluating statistical consistency and are not needed in practice). The AC-MDSA algorithm solves approximately 1240 linear systems with an average wall-clock time of approximately 2.4 seconds per step.
5.3.2. Disk

In the disk problem, five loads are equally distributed on the outer perimeter, and each has a deterministic normal component 1 and a random tangential component $\sim \mathcal{N}(0, 0.1^2)$. We use $n = 72,000$ elements, $\theta = 3000n$ and $\theta = 0.1n$ for $\kappa = 1$ and $\kappa = 0.618$, respectively, $N_{rst} = 300$, $N_{damp} = 450$, $\epsilon_{damp} = 0.05$, $N_{max} = 600$, and $N_{min} = 450$. Figure 13 shows the representative optimized designs for deterministic, $\kappa = 1$, and $\kappa = 0.618$ cases. The deterministic design contains rods with uniform widths, resisting normal load components, whereas the robust design with $\kappa = 1$ leads to a structure with two branches that resist each random tangential loads. In the robust design with $\kappa = 0.618$, lateral braces are formed to further enhance the resistance of random components.
tangential loads. The $\sigma(x^*)$ values of the three designs confirm that the robustness is effectively improved when $\kappa$ drops. Table 6 and Figure 12 (b) show the statistics of the 50 trials from AC-MDSA, which solves two linear systems per step with an average wall-clock time of 1.7 seconds.

Both the double hook and disk examples show that the two-sample AC-MDSA can solve problems with various mesh sizes, complex geometries, and multiple independent random loads. As $\kappa$ changes, we observe apparent changes in both the designs and $\sigma(x^*)$ values. The optimized designs and algorithmic parameters are insensitive to the change of mesh sizes. Finally, we show that the AC-MDSA algorithm requires a low computational cost to handle RTO problems as it needs only two linear solves per step.

5.4. Example 4: three-dimensional crane

The last example, which solves a 3D crane problem, demonstrates the applicability and efficiency of the entropic AC-MDSA. Figure 14 shows the domain and boundary conditions. The domain is fixed on the top and subjected to two point loads that have deterministic $z$ components with magnitudes 1 and random $x$ and $y$ components $\sim \mathcal{N}(0, 0.1^2)$. The FE mesh consists of $n = 352,000$ hexahedral elements. The filter radius $R$ is initialized as 0.15 and starts to decrease after 320 steps by $1/30$ every 25 steps until 0.05. We use $\theta = 2000$ and impose symmetry constraints with respect to the $x$ and $y$ planes, and $N_{rst} = 100$, $N_{damp} = 430$, $\epsilon_{damp} = 0.05$, $N_{max} = 450$, and $N_{min} = 430$. The objective function values of the final designs are evaluated using 1,000 samples.
We use the GPU-accelerated preconditioned conjugate gradient (PCG) built-in solver from Matlab with the Jacobi preconditioner and choose a relatively high tolerance of $10^{-4}$ for convergence as the entropic AC-MDSA does not require accurate evaluation of sensitivity.

![Figure 14: Geometry and boundary conditions of Example 4: 3D crane, $W = 4, W_1 = 1, H_1 = 1.5, H_2 = 1, B = 1$, two point loads have deterministic $z$ components $f_{1z} = f_{2z} = 1$ and random $x$ and $y$ components $f_{1x}, f_{1y}, f_{2x}, f_{2y} \sim N(0, 0.1^2)$.](image)

![Figure 15: Optimized designs and objective function values of 3D crane from AC-MDSA: (a) deterministic; (b) $\kappa = 1$; (c) $\kappa = 0.828$. The design in (b) and (c), respectively, is a representative design chosen from the 10 trials.](image)

The optimized designs and objective function values for deterministic, $\kappa = 1$, and $\kappa = 0.828$ cases are shown in Figure 15. We observe that the entropic AC-MDSA algorithm captures the influence of different $\kappa$ values on the final designs, both qualitatively and quantitatively. Qual-
itatively, in the deterministic design (Figure 15a), no lateral braces are formed among the four columns on the upper part, resulting in poor resistance to shear in the $x$-direction and torque in the $x - y$ plane. In addition, the material in the lower part is mostly distributed within the $x - z$ plane, which also leads to poor resistance to loads in the $y$-direction. The robust design with $\kappa = 1$ (Figure 15b), on the other hand, forms pairs of braces in the $x - z$ planes between the four columns, which improves the resistance to the random load components in the $x$- and $y$-directions that potentially impose shear and torsion. However, no braces appear in the $y - z$ planes. In the lower part, two branches are formed in the upper and middle chords of the beam. These branches can increase the stiffness of resisting the random loads in the $y$-direction. Finally, the robust design with $\kappa = 0.828$ (Figure 15c) forms four braces in both the $x - z$ and $y - z$ planes between the four columns, leading to the highest resistance to the shear and torsion imposed by the random load components in the $x$- and $y$-directions. In the lower part of the design, the upper chord branches are further split to enhance the resistance to loads in the $y$-direction. The middle chord becomes two independent members, and the lower chord splits into two branches. In addition, two members connecting the two lower chords are formed. These features clearly indicate the increase in the structural robustness when $\kappa$ decreases. Quantitatively, the influence of $\kappa$ is also revealed by the values of $\hat{\mu}(x^*)$ and $\hat{\sigma}(x^*)$ of the optimized designs. For the deterministic case, the design has both the highest $\hat{\mu}(x^*)$ and $\hat{\sigma}(x^*)$ because the load randomness is not considered in the optimization. For the robust designs, as $\kappa$ becomes smaller, $\hat{\mu}(x^*)$ increases while $\hat{\sigma}(x^*)$ decreases considerably, which is consistent with the corresponding importance in the objective function of the RTO formulation (1).

This 3D example shows that the proposed AC-MDSA algorithm effectively produces designs with various levels of robustness. The AC-MDSA uses a relatively high tolerance for the iterative linear solve, which may suggest high tolerance can be used to reduce computational cost further as AC-MDSA does not require accurate evaluation of gradients. However, more investigation is needed to verify this potential.

6. Concluding remarks

In this work, we introduce a momentum-based accelerated mirror descent stochastic approximation algorithm to solve RTO problems involving various load randomness efficiently and effectively. Built upon MDSA, the proposed AC-MDSA framework is capable of performing high-quality design variable updates with highly noisy stochastic gradients. We show that stochastic gradients evaluated using only two samples (two being the smallest sample size for unbiased gradient estimators) are sufficient to obtain robust designs in RTO. We derive the AC-MDSA update in the $\ell_1$-norm setting using the entropy function as the distance-generating function. The AC-MDSA algorithm is shown to exhibit stable convergence performance insensitive to various step size choices. In addition, several techniques, including an adaptive step-size recalibration scheme
and an adaptive damping scheme, are developed to improve the convergence performance. Several 2D and 3D numerical examples involving various geometries, problem sizes, uncertainties are presented, demonstrating that the proposed AC-MDSA algorithm with only two samples effectively and efficiently handles RTO problems involving various types of load uncertainties.

In the simple column benchmark, the AC-MDSA with two samples produces designs with no worse objective function values than the MC method with 1000 samples for both robust designs with a low computational cost. The study on sample size shows that, although a larger number of samples results in higher accuracy in sensitivity, two samples are sufficient to produce designs with similar objective function values. In addition, the AC-MDSA shows superior stability than the standard MDSA for a wide range of step sizes. The half circle example demonstrates that AC-MDSA effectively reflects various levels of robustness through geometric features and standard deviation of compliance $\hat{\sigma}(x^*)$ of the final designs. As $\kappa$ (relative weight of mean and variance of compliance in the objective function) decreases, $\hat{\sigma}(x^*)$ become smaller consistently. The double hook and disk examples show that the AC-MDSA can tackle various geometries and multiple independent random loads. The mesh size study further demonstrates the consistency of the AC-MDSA and insensitivity of algorithm parameters to various problem sizes. For the larger problem size ($n = 114,048$), the AC-MDSA algorithm obtains similar optimized designs and objective function values compared to those from the MC method with 1000 samples with a small computational cost. As the problem size increases, this difference in computational cost magnifies because the total computational cost becomes dominated by the procedure of solving state equations. The 3D crane example demonstrates the effectiveness and applicability of the proposed AC-MDSA algorithm.

We note that the AC-MDSA has the potential to use loose tolerance for the iterative linear solver due to its low accuracy requirement for the gradient, which could further save computational cost. However, further study is needed to verify and make use of this potential advantage.

This work has investigated design cases with and without the symmetry constraint. While the designs without symmetry constraint show a certain level of asymmetry, the asymmetry appears to be mild, as indicated in Figure 6. Also, although the proposed AC-MDSA requires several pilot runs to calibrate the appropriate range of step size scaling factor, the range is generally wide, and different values in the range provide similar performance and final designs. Further studies about calculating the step size are desired. Last but not least, this work focuses on load uncertainty in compliance minimization RTO problems, and extension of the proposed AC-MDSA algorithm to other uncertainties or problems is valuable for future studies.

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