Inverse Nodal Problems for Dirac-Type Integro-Differential Operators with Linear Functions in the Boundary Condition

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ABSTRACT
In this article, Dirac-type integro-differential operator with linear functions in the boundary condition is considered. We obtain asymptotic expressions for the solution of the differential system and derive the large eigenvalues and nodal points. We also give a constructive procedure for solving an inverse nodal problem. We prove that a dense subset of the nodes determines the coefficients of the differential part of the operator and gives partial information for the integral part of it.

1. Introduction
The article aims to solve the inverse nodal problem for the boundary value problem (BVP) generated by the following Dirac-type integro-differential system

\[ BY'(x) + \Omega(x)Y(x) + \int_0^x M(x,t)Y(t)dt = \mu Y(x), \quad x \in [0, \pi], \] (1)

with the boundary conditions

\[ (a_1 + \mu \cos \phi)y_1(0) + (a_2 + \mu \sin \phi)y_2(0) = 0, \] (2)

\[ (b_1 + \mu \cos \rho)y_1(\pi) + (b_2 + \mu \sin \rho)y_2(\pi) = 0. \] (3)

Here \( \phi, \rho, a_1, a_2, b_1, \) and \( b_2 \) are real constants, \( 0 \leq \phi, \rho < \pi, \) and \( \mu \) are the spectral parameters,
\[
B = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}, \quad \Omega(x) = \begin{pmatrix}
p(x) & 0 \\
0 & q(x)
\end{pmatrix}, \quad M(x, t) = \\
\begin{pmatrix}
M_{11}(x, t) & M_{12}(x, t) \\
M_{21}(x, t) & M_{22}(x, t)
\end{pmatrix}, \quad Y(x) = \begin{pmatrix}
y_1(x) \\
y_2(x)
\end{pmatrix}, \quad p(x), q(x), \text{ and } M_{ij}(x, t),
\]

\((i, j = 1, 2)\), are real-valued functions in \(W^1_2[0, \pi]\) with respect to \(t\) and in \(L^2[0, \pi]\) with respect to \(x\).

Inverse spectral problems consist in recovering operators from their spectral characteristics. One of the important spectral characteristics is the zeros of eigenfunctions called nodes or nodal points. The inverse nodal problems for Sturm–Liouville operator was first handled and resolved by McLaughlin in 1988 [1]. In this study, it has been shown that a dense subset of zeros of eigenfunctions uniquely determines the potential of the Sturm Liouville operator up to the mean value. In 1989, Hald and McLaughlin gave a numerical method for reconstructing potential function from nodes for more general boundary conditions [2]. In 1997, Yang suggested a constructive procedure for reconstructing the potential and the boundary condition of the Sturm–Liouville problem from nodes of its eigenfunctions [3]. In this work, the author also gave the uniqueness theorem for general boundary conditions using the same method as McLaughlin. Inverse nodal problems have been addressed by various researchers in several papers for different operators [4–14]. The inverse nodal problems for Dirac operators with various boundary conditions have been studied and shown that a dense subset of nodes is enough to determine the coefficients of the operators in [15–17].

In recent years, integro-differential operators attracted much attention of mathematicians. Such operators have important applications in many fields of science (see monographs [18, 19] and references therein). Therefore, many researchers are currently working on inverse problems for these operators [20–35]. The inverse nodal problem for Dirac-type integro-differential operators with Robin boundary conditions was first studied by [36]. This operator with parameter-dependent boundary conditions linearly and nonlinearly were studied by [37] and [38], respectively. In their studies, the authors considered \(p(x)\) and \(q(x)\), which are the components of the potential function \(\Omega(x)\), as a special case such that \(p(x)−q(x) = \text{const}\). In this study, we consider \(p(x)\) and \(q(x)\) as two independent functions and investigate a more general case. In this way, we have the opportunity to determine \(p(x)\) and \(q(x)\) separately. We deal with an inverse nodal problem of reconstructing the Dirac-type integro-differential operators with the spectral parameter contained linearly in the both boundary conditions. It is shown that the coefficients of the differential part of the operator can be determined using a dense subset of the nodes. For this problem, we also give a
constructive procedure for determining the coefficients as well as the useful asymptotics regarding the solution, eigenvalues, and nodes.

2. Asymptotics of the solutions

In this section, we will first obtain the integral equations for the solutions of the Dirac-type integro-differential system (1). Using these equations, we will calculate the asymptotics of solutions of (1) with the help of the successive approximation method.

Denote by \( \varphi(x, \mu) = \begin{pmatrix} \varphi_1(x, \mu) \\ \varphi_2(x, \mu) \end{pmatrix} \), the solution of (1), satisfying the initial condition \( \varphi(0, \mu) = \left( \begin{array}{c} \mu \sin \phi + a_2 \\ -\mu \cos \phi - a_1 \end{array} \right) \). For each fixed \( x \) and \( t \), this solution is entire with respect to \( \mu \), and satisfy

\[
\begin{align*}
\varphi_1(x, \mu) &= \mu \sin (\mu x + \phi) + a_1 \sin \mu x + a_2 \cos \mu x \\
&+ \int_0^x p(t) \varphi_1(t, \mu) \sin \mu(x-t) dt + \int_0^x r(t) \varphi_2(t, \mu) \cos \mu(t-x) dt \\
&+ \int_0^x \left\{ M_{11}(t, \xi) \varphi_1(\xi, \mu) + M_{12}(t, \xi) \varphi_2(\xi, \mu) \right\} \sin \mu(x-t) d\xi dt \\
&+ \int_0^x \left\{ M_{21}(t, \xi) \varphi_1(\xi, \mu) + M_{22}(t, \xi) \varphi_2(\xi, \mu) \right\} \cos \mu(x-t) d\xi dt,
\end{align*}
\]

(4)

\[
\begin{align*}
\varphi_2(x, \mu) &= -\mu \cos (\mu x + \phi) - a_1 \cos \mu x + a_2 \sin \mu x \\
&- \int_0^x p(t) \varphi_1(t, \mu) \cos \mu(x-t) dt + \int_0^x r(t) \varphi_2(t, \mu) \sin \mu(t-x) dt \\
&- \int_0^x \left\{ M_{11}(t, \xi) \varphi_1(\xi, \mu) + M_{12}(t, \xi) \varphi_2(\xi, \mu) \right\} \cos \mu(x-t) d\xi dt \\
&+ \int_0^x \left\{ M_{21}(t, \xi) \varphi_1(\xi, \mu) + M_{22}(t, \xi) \varphi_2(\xi, \mu) \right\} \sin \mu(x-t) d\xi dt.
\end{align*}
\]

(5)

Theorem 1. The components \( \varphi_1(x, \mu) \) and \( \varphi_2(x, \mu) \) of the solutions have the following asymptotic expansions:
\[ \varphi_1(x, \mu) = \mu \sin [\mu x + \phi - \omega(x)] + a_1 \sin [\mu x - \omega(x)] \\
+ a_2 \cos [\mu x - \omega(x)] + \frac{v(x)}{4} \sin [\mu x + \phi - \omega(x)] \\
+ \frac{a_1}{4\mu} v(x) \sin [\mu x - \omega(x)] + \frac{a_2}{4\mu} v(x) \cos [\mu x - \omega(x)] \\
- \frac{1}{8} \psi(x) \cos [\mu x + \phi - \omega(x)] - \frac{a_1}{8\mu} \psi(x) \cos [\mu x - \omega(x)] \\
+ \frac{a_2}{8\mu} \psi(x) \sin [\mu x - \omega(x)] - \frac{v(0)}{4} \sin [\phi - \mu x + \omega(x)] \\
+ \frac{a_1}{4\mu} v(0) \sin [\mu x - \omega(x)] - \frac{a_2}{4\mu} v(0) \cos [\mu x - \omega(x)] \]  \tag{6}

\[ \varphi_2(x, \mu) = -\mu \cos [\mu x - \omega(x) + \phi] - a_1 \cos [\mu x - \omega(x)] \\
+ a_2 \sin [\mu x - \omega(x)] + \frac{v(x)}{4} \cos [\mu x - \omega(x) + \phi] \\
+ \frac{a_1}{4\mu} v(x) \cos [\mu x - \omega(x)] - \frac{a_2}{4\mu} v(x) \sin [\mu x - \omega(x)] \\
- \frac{1}{8} \psi(x) \sin [\mu x + \phi - \omega(x)] - \frac{a_1}{8\mu} \psi(x) \sin [\mu x - \omega(x)] \\
- \frac{a_2}{8\mu} \psi(x) \cos [\mu x - \omega(x)] - \frac{v(0)}{4} \cos [\phi - \mu x + \omega(x)] \\
- \frac{a_1}{4\mu} v(0) \cos [\mu x - \omega(x)] - \frac{a_2}{4\mu} v(0) \sin [\mu x - \omega(x)] \]  \tag{7}

for sufficiently large $|\mu|$, uniformly in $x$. Here, $\omega(x) = \frac{1}{2} \int_0^x (p(t) + r(t)) dt$, $v(x) = p(x) - q(x)$, $\psi(x) = \int_0^x v^2(t) dt$, $K(x) = \int_0^x (M_{11}(t,t) + M_{22}(t,t)) dt$, $L(x) = \int_0^x (M_{12}(t, t) - M_{21}(t, t)) dt$ and $\tau = \text{Im} \mu$. 
Proof. Use the expansion equations (4) and (5) and put

\[
\varphi_{1,0}(x, \mu) = \mu \sin(\mu x + \phi) + a_1 \sin \mu x + a_2 \cos \mu x,
\]

\[
\varphi_{1,k+1}(x, \mu) = \int_0^x p(t) \varphi_{1,k}(t, \mu) \sin \mu(x-t) dt + \int_0^x r(t) \varphi_{2,k}(t, \mu) \cos \mu(t-x) dt
\]

\[
+ \int_0^x \int_0^x \left\{ M_{11}(t, \zeta) \varphi_{1,k}(\zeta, \mu) + M_{12}(t, \zeta) \varphi_{2,k}(\zeta, \mu) \right\} \sin \mu(x-t) d\zeta dt
\]

\[
+ \int_0^x \int_0^x \left\{ M_{21}(t, \zeta) \varphi_{1,k}(\zeta, \mu) + M_{22}(t, \zeta) \varphi_{2,k}(\zeta, \mu) \right\} \cos \mu(x-t) d\zeta dt,
\]

and

\[
\varphi_{2,0}(x, \mu) = -\mu \cos(\mu x + \phi) - a_1 \cos \mu x + a_2 \sin \mu x,
\]

\[
\varphi_{2,k+1}(x, \mu) = -\int_0^x p(t) \varphi_{1,k}(t, \mu) \cos \mu(x-t) dt + \int_0^x r(t) \varphi_{2,k}(t, \mu) \sin \mu(t-x) dt
\]

\[
- \int_0^x \int_0^x \left\{ M_{11}(t, \zeta) \varphi_{1,k}(\zeta, \mu) + M_{12}(t, \zeta) \varphi_{2,k}(\zeta, \mu) \right\} \cos \mu(x-t) d\zeta dt
\]

\[
+ \int_0^x \int_0^x \left\{ M_{21}(t, \zeta) \varphi_{1,k}(\zeta, \mu) + M_{22}(t, \zeta) \varphi_{2,k}(\zeta, \mu) \right\} \sin \mu(x-t) d\zeta dt.
\]

Apply the successive approximations method to have

\[
\varphi_{1,1}(x, \mu) = -\omega(x) \mu \cos(\mu x + \phi) - \omega(x) a_1 \cos \mu x + \omega(x) a_2 \sin \mu x
\]

\[
+ \frac{v(x)}{4} \sin(\mu x + \phi) + \frac{a_1}{4 \mu} v(x) \sin \mu x + \frac{a_2}{4 \mu} v(x) \cos \mu x
\]

\[
+ \frac{v(0)}{4} \sin(\mu x - \phi) + \frac{a_1}{4 \mu} v(0) \sin \mu x - \frac{a_2}{4 \mu} v(0) \cos \mu x
\]

\[
- \frac{1}{2} K(x) \sin(\mu x + \phi) - \frac{a_1}{2 \mu} K(x) \sin \mu x - \frac{a_2}{2 \mu} K(x) \cos \mu x
\]

\[
+ \frac{1}{2} L(x) \cos(\mu x + \phi) + \frac{a_1}{2 \mu} L(x) \cos \mu x - \frac{a_2}{2 \mu} L(x) \sin \mu x + o \left( \frac{e^{\gamma|x|}}{\mu} \right),
\]
\[ \varphi_{2,1}(x, \mu) = -\omega(x) \mu \sin (\mu x + \phi) - \omega(x) a_1 \sin \mu x + \omega(x) a_2 \cos \mu x + \frac{\nu(x)}{4} \cos (\mu x + \phi) + \frac{a_1}{4\mu} \nu(x) \cos \mu x - \frac{a_2}{4\mu} \nu(x) \sin \mu x \]

\[ - \frac{\nu(0)}{4} \cos (\mu x - \phi) - \frac{a_1}{4\mu} \nu(0) \cos \mu x - \frac{a_2}{4\mu} \nu(0) \sin \mu x \]

\[ + \frac{1}{2} K(x) \cos (\mu x + \phi) + \frac{a_1}{2\mu} K(x) \cos \mu x - \frac{a_2}{2\mu} K(x) \sin \mu x \]

\[ + \frac{1}{2} L(x) \sin (\mu x + \phi) + \frac{a_1}{2\mu} L(x) \sin \mu x + \frac{a_2}{2\mu} L(x) \cos \mu x + o \left( \frac{e|\mu|^{x}}{\mu} \right), \]

and for \( \kappa \geq 1 \)

\[ \varphi_{1,2\kappa+1}(x, \mu) = (-1)^{\kappa+1} \frac{\omega^{2\kappa+1}(x)}{(2\kappa + 1)!} \mu \cos (\mu x + \phi) + (-1)^{\kappa+1} a_1 \frac{\omega^{2\kappa+1}(x)}{(2\kappa + 1)!} \cos \mu x \]

\[ + (-1)^{\kappa} a_2 \frac{\omega^{2\kappa+1}(x)}{(2\kappa + 1)!} \sin \mu x + (-1)^{\kappa} \frac{\nu(x)}{4} \frac{\omega^{2\kappa}(x)}{(2\kappa)!} \sin \mu x + \frac{\nu(0)}{4} \frac{\omega^{2\kappa}(x)}{(2\kappa)!} \cos \mu x \]

\[ - (-1)^{\kappa+1} \frac{\nu(0)}{4} \frac{\omega^{2\kappa}(x)}{(2\kappa)!} \sin (\mu x - \phi) + (-1)^{\kappa} a_1 \frac{\nu(0)}{4\mu} \frac{\omega^{2\kappa}(x)}{(2\kappa)!} \sin \mu x \]

\[ + (-1)^{\kappa} a_2 \frac{\nu(0)}{4\mu} \frac{\omega^{2\kappa}(x)}{(2\kappa)!} \cos \mu x + (-1)^{\kappa} \frac{\psi(x)}{8} \frac{\omega^{2\kappa-1}(x)}{(2\kappa - 1)!} \sin (\mu x + \phi) \]

\[ + (-1)^{\kappa} a_1 \frac{\psi(x)}{8\mu} \frac{\omega^{2\kappa-1}(x)}{(2\kappa - 1)!} \sin \mu x + (-1)^{\kappa} \frac{\psi(x)}{8\mu} \frac{\omega^{2\kappa-1}(x)}{(2\kappa - 1)!} \cos \mu x \]

\[ + (-1)^{\kappa} \frac{1}{2} \frac{\omega^{2\kappa}(x)}{(2\kappa)!} K(x) \sin (\mu x + \phi) + (-1)^{\kappa} \frac{1}{2} \frac{\omega^{2\kappa}(x)}{(2\kappa)!} K(x) \sin \mu x \]

\[ + (-1)^{\kappa} \frac{1}{2} \frac{\omega^{2\kappa}(x)}{(2\kappa)!} K(x) \cos \mu x + (-1)^{\kappa} \frac{1}{2} \frac{\omega^{2\kappa}(x)}{(2\kappa)!} L(x) \cos (\mu x + \phi) \]

\[ + (-1)^{\kappa} \frac{1}{2} \frac{\omega^{2\kappa}(x)}{(2\kappa)!} L(x) \cos \mu x + (-1)^{\kappa+1} \frac{1}{2} \frac{\omega^{2\kappa}(x)}{(2\kappa)!} \frac{L(x)}{(2\kappa)!} \sin \mu x \]

\[ + o \left( \frac{e|\mu|^{x}}{\mu} \right), \]

for sufficiently large \( |\mu| \), uniformly in \( x \). Other iterations can be calculated in a similar way. Hence, the Proof of the Theorem 1 is completed. \qed

3. Asymptotics of the eigenvalues

Define the characteristic function of the considered problem by

\[ \Delta(\mu) = \varphi_1(\pi, \mu)(\mu \cos \rho + b_1) + \varphi_2(\pi, \mu)(\mu \sin \rho + b_2), \quad (8) \]

the zeros of this entire function coincide with the eigenvalues. Substituting (6) and (7) into the (8), the asymptotic expression of \( \Delta(\mu) \) can be written as follows,
\[ \Delta(\mu) = \mu^2 \left\{ \begin{array}{l}
\sin [\mu\pi - \omega(\pi) + \phi - \rho] \\
+ \frac{a_1}{\mu} \sin [\mu\pi - \omega(\pi) + \phi - \rho] \cos \phi - \frac{a_1}{\mu} \cos [\mu\pi - \omega(\pi) + \phi - \rho] \sin \phi \\
+ \frac{a_2}{\mu} \cos [\mu\pi - \omega(\pi) + \phi - \rho] \cos \phi + \frac{a_2}{\mu} \sin [\mu\pi - \omega(\pi) + \phi - \rho] \sin \phi \\
+ \frac{v(\pi)}{4\mu} \sin [\mu\pi - \omega(\pi) + \phi - \rho] \cos 2\rho + \frac{v(\pi)}{4\mu} \cos [\mu\pi - \omega(\pi) + \phi - \rho] \sin 2\rho \\
- \frac{\psi(\pi)}{8\mu} \cos [\mu\pi - \omega(\pi) + \phi - \rho] + \frac{\psi(0)}{4\mu} \sin [\mu\pi - \omega(\pi) + \phi - \rho] \\
- \frac{K(\pi)}{2\mu} \sin [\mu\pi - \omega(\pi) + \phi - \rho] + \frac{L(\pi)}{2\mu} \cos [\mu\pi - \omega(\pi) + \phi - \rho] \\
+ \frac{b_1}{\mu} \sin [\mu\pi - \omega(\pi) + \phi - \rho] \cos \rho + \frac{b_1}{\mu} \cos [\mu\pi - \omega(\pi) + \phi - \rho] \sin \rho \\
- \frac{b_2}{\mu} \cos [\mu\pi - \omega(\pi) + \phi - \rho] \cos \rho + \frac{b_2}{\mu} \sin [\mu\pi - \omega(\pi) + \phi - \rho] \\
\sin \rho + O\left(\frac{e^{|\pi|}}{\mu}\right) \end{array} \right\} \]

from this expression, for sufficiently large \(|n|\), we get

\[ \mu_n = n - 2 + \frac{\omega(\pi) + \rho - \phi}{\pi} + \frac{D}{(n - 2)\pi} + O\left(\frac{1}{n^2}\right), \quad n \geq 3, \quad (9) \]

and similarly

\[ \mu_n = n + 2 + \frac{\omega(\pi) + \rho - \phi}{\pi} + \frac{D}{(n + 2)\pi} + O\left(\frac{1}{n^2}\right), \quad n \leq -3 \quad (10) \]

where \(D = a_1 \sin \phi - a_2 \cos \phi - \frac{v(\pi)}{4} \sin 2\rho + \frac{\psi(\pi)}{8} - \frac{L(\pi)}{2} - b_1 \sin \rho + b_2 \cos \rho \)

4. Asymptotics of the nodal points

We call the zeros \(x \in (0, \pi)\) of the function \(\varphi_1(x, \mu_n)\), the nodes or the nodal points. The following Lemma establishes the existence of the nodes for sufficiently large \(n > 0\) and also describes their asymptotic behavior.

**Lemma 1.** The function \(\varphi_1(x, \mu_n)\) has exactly \(n - 2\) nodes

\[ \left\{ x_j^n : j = 0, 1, \ldots, n - 3 \right\} \text{ in } (0, \pi) : \]

\[ 0 < x^0_n < x^1_n < \ldots < x^{n-3}_n < \pi. \]
Moreover, \( \{ x_n^i \} \) satisfy the following relations:

\[
x_n^i = \frac{j\pi}{n-2} - \frac{j\pi}{n-2} \frac{\omega(\pi) + \rho - \phi}{(n-2)\pi} + \frac{\omega(x_n^i) - \phi}{n-2}
\]

\[
+ \frac{j\pi}{n-2} \frac{(\omega(\pi) + \rho - \phi)^2 - D\pi}{(n-2)^2 \pi^2} - \left( \omega(x_n^i) - \phi \right) \left( \frac{\omega(\pi) + \rho - \phi}{(n-2)^2 \pi} \right)
\]

\[
+ \frac{1}{8(n-2)^2} \left\{ 8a_1 \sin \phi - 8a_2 \cos \phi + \psi(x_n^i) + 2\nu(0) \sin 2\phi - 4L(x_n^i) \right\}
\]

\[+ O \left( \frac{1}{n^3} \right) \quad (11)\]

**Proof.** Consider the equation \( \varphi_1(x_n^i, \mu_n) = 0 \) on \((0, \pi)\), which is equivalent to

\[
\mu_n \sin \left[ \mu_n x_n^i - \omega(x_n^i) + \phi \right] + a_1 \sin \left[ \mu_n x_n^i - \omega(x_n^i) + \phi \right] \cos \phi
\]

\[- a_1 \cos \left[ \mu_n x_n^i - \omega(x_n^i) + \phi \right] \sin \phi + a_2 \cos \left[ \mu_n x_n^i - \omega(x_n^i) + \phi \right] \sin \phi
\]

\[+ a_2 \sin \left[ \mu_n x_n^i - \omega(x_n^i) + \phi \right] \sin \phi + \frac{v(x_n^i)}{4} \sin \left[ \mu_n x_n^i - \omega(x_n^i) + \phi \right] \sin \phi
\]

\[+ \frac{a_1}{4\mu_n} v(x_n^i) \sin \left[ \mu_n x_n^i - \omega(x_n^i) + \phi \right] \cos \phi - \frac{a_1}{4\mu_n} v(x_n^i) \cos \left[ \mu_n x_n^i - \omega(x_n^i) + \phi \right] \sin \phi
\]

\[+ a_2 \frac{v(x_n^i)}{4\mu_n} \cos \left[ \mu_n x_n^i - \omega(x_n^i) + \phi \right] \cos \phi + \frac{v(0)}{4} \sin \left[ \mu_n x_n^i - \omega(x_n^i) + \phi \right] \cos 2\phi
\]

\[+ \frac{v(0)}{4} \cos \left[ \mu_n x_n^i - \omega(x_n^i) + \phi \right] \sin 2\phi + \frac{a_1}{4\mu_n} v(0) \sin \left[ \mu_n x_n^i - \omega(x_n^i) + \phi \right] \cos \phi
\]

\[- \frac{a_1}{4\mu_n} v(0) \cos \left[ \mu_n x_n^i - \omega(x_n^i) + \phi \right] \sin \phi \sin \frac{1}{2} K(x) \sin \left[ \mu_n x - \omega(x) + \phi \right]
\]

\[- \frac{a_1}{2\mu_n} K(x_n^i) \sin \left[ \mu_n x_n^i - \omega(x_n^i) + \phi \right] \cos \phi + \frac{a_1}{2\mu_n} K(x_n^i) \cos \left[ \mu_n x_n^i - \omega(x_n^i) + \phi \right] \sin \phi
\]

\[- \frac{a_2}{2\mu_n} K(x_n^i) \cos \left[ \mu_n x_n^i - \omega(x_n^i) + \phi \right] \cos \phi - \frac{a_2}{2\mu_n} K(x_n^i) \sin \left[ \mu_n x_n^i - \omega(x_n^i) + \phi \right] \sin \phi
\]

\[+ \frac{1}{2} L(x_n^i) \cos \left[ \mu_n x_n^i - \omega(x_n^i) + \phi \right] + \frac{a_1}{2\mu_n} L(x_n^i) \cos \left[ \mu_n x_n^i - \omega(x_n^i) + \phi \right] \cos \phi
\]
\[ + \frac{a_1}{2\mu_n} L(x_n') \sin \left[ \mu_n x_n' - \omega(x_n') + \phi \right] \sin \phi - \frac{a_2}{2\mu_n} L(x_n') \sin \left[ \mu_n x_n' - \omega(x_n') + \phi \right] \cos \phi \]

\[ + \frac{a_2}{2\mu_n} L(x_n') \cos \left[ \mu_n x_n' - \omega(x_n') + \phi \right] \sin \phi + o(\frac{e^{[\pi]} n}{\mu_n}) = 0 \]

this implies
\[
\tan \left[ \mu_n x_n' - \omega(x_n') + \phi \right] = \\
\left\{ 1 + \frac{1}{4\mu_n} (4a_1 \cos \phi + 4a_2 \sin \phi + v(x_n') + v(0) \cos 2\phi - 2K(x_n')) + O\left( \frac{1}{\mu_n^2} \right) \right\}^{-1} \times \\
\left\{ \frac{1}{8\mu_n} (8a_1 \sin \phi - 8a_2 \cos \phi + \psi(x_n') + 2v(0) \sin 2\phi - 4L(x_n')) + O\left( \frac{1}{\mu_n} \right) \right\} \times \\
\left\{ \frac{1}{8\mu_n} (8a_1 \sin \phi - 8a_2 \cos \phi + \psi(x_n') + 2v(0) \sin 2\phi - 4L(x_n')) + O\left( \frac{1}{\mu_n} \right) \right\} \times \\
\frac{1}{8\mu_n} (8a_1 \sin \phi - 8a_2 \cos \phi + \psi(x_n') + 2v(0) \sin 2\phi - 4L(x_n')) + O\left( \frac{1}{\mu_n^2} \right). \]

If we solve the last equation by using Taylor’s expansion formula for arctangent yields
\[
x_n^j = \mu_n^{-1} \left\{ \omega(x_n^j) - \phi + j\pi \right. \\
+ \frac{1}{8\mu_n} (8a_1 \sin \phi - 8a_2 \cos \phi + \psi(x_n^j) + 2v(0) \sin 2\phi - 4L(x_n^j)) \} + O\left( \frac{1}{\mu_n} \right) \}
\]
then substituting the relation
\[
\mu_n^{-1} = \frac{1}{n-2} \left\{ 1 - \frac{\omega(\pi) - \phi + \rho}{(n-2)\pi} - \frac{D\pi - [\omega(\pi) - \phi + \rho]^2}{(n-2)^2\pi^2} + O \left( \frac{1}{n^3} \right) \right\}
\]
into the last equation, we arrive at
\[
x_n^j = \frac{j\pi}{n-2} - \frac{j\pi}{n-2} \frac{\omega(\pi) + \rho - \phi}{(n-2)\pi} + \frac{\omega(x_n^j) - \phi}{n-2} \\
+ \frac{j\pi}{n-2} \frac{((\omega(\pi) + \rho - \phi)^2 - D\pi)}{(n-2)^2\pi^2} - (\omega(x_n^j) - \phi) \frac{(\omega(\pi) + \rho - \phi)}{(n-2)^2\pi} \\
+ \frac{1}{8(n-2)^2} \left\{ 8a_1 \sin \phi - 8a_2 \cos \phi + \psi(x_n^j) + 2v(0) \sin 2\phi - 4L(x_n^j) \right\} \\
+ O\left( \frac{1}{n^3} \right)
\]
for sufficiently large \( n > 0 \).
5. Inverse nodal problem

In this section, we prove a uniqueness theorem and develop a constructive procedure for solving the inverse nodal problem.

Let denote the set of nodes by $\Psi$, then a sequence $(x_n^j) \subset \Psi$ can be chosen such that $\lim x_n^j = x$. In this case, the following limits are exist and finite:

$$h(x) := \lim_{n \to \infty} \left(x_n^j - \frac{j\pi}{n}\right)(n - 2) = -x\left(\frac{\omega(\pi) + \rho - \phi}{\pi}\right) + \omega(x) - \phi$$

(12)

and

$$g(x) := \lim_{n \to \infty} \left(x_n^j - \frac{j\pi}{n} + \frac{j\pi \omega(\pi) - \phi + \rho - \omega(x) - \phi}{n}\right)8(n - 2)^2$$

$$= x\left((\frac{\omega(\pi) + \rho - \phi}{\pi})^2 - D\pi\right) - (\omega(x) - \phi)\left(\frac{\omega(\pi) + \rho - \phi}{8\pi}\right)$$

$$+ 8a_1 \sin \phi - 8a_2 \cos \phi + \psi(x) + 2\nu(0) \sin 2\phi - 4L(x)$$

(13)

Now, we can give the following theorem and formulate a constructive procedure for reconstructing the potential of the considered inverse problem. Without loss of generality, we assume $\omega(\pi) = 0$.

**Theorem 2.** The given dense subset of the set $\Psi$ uniquely determines $p(x)$ and $q(x)$ a.e. on $(0, \pi)$, and the coefficients $\phi$ and $\rho$. Moreover, $p(x)$, $q(x)$, $\phi$, and $\rho$ can be reconstructed by the following algorithm:

1. for each $x \in (0, \pi)$, choose a sequence $(x_n^{j(n)}) \subset \Psi$, such that $\lim_{n \to \infty} x_n^{j(n)} = x$;
2. find the function $h(x)$ via (12) and calculate
   $$\phi = -h(0)$$
   $$\rho = -h(\pi)$$
   $$\omega'(x) = \frac{1}{2}(p(x) + q(x)) = h'(x) + \frac{\rho - \phi}{\pi}$$
3. If $L'(x)$ is known, find the function $g(x)$ via (13) and calculate
   $$p(x) = h'(x) + \frac{\rho - \phi}{\pi} + \xi(x)$$
   $$q(x) = h'(x) + \frac{\rho - \phi}{\pi} - \xi(x)$$
where, \( \zeta^2(x) = \frac{1}{4} \left( g'(x) + \left( h'(x) + \frac{\rho-\phi}{\pi} \right) \frac{\rho-\phi}{8\pi} + 4L'(x) \right) \)

(4) If one of \( a_1 \) and \( a_2 \) is known, calculate the other by the formula

\[
a_1 \sin \phi - a_2 \cos \phi = \frac{g(0)}{8} - \phi \frac{\rho-\phi}{64\pi} - \frac{(p(0)-q(0)) \sin 2\phi}{4}
\]

(5) If \( p(x) \) and \( q(x) \) are known, calculate

\[
L'(x) = -\frac{g'(x)}{4} - \frac{(p(x) + q(x))(\rho-\phi)}{8\pi} + \frac{(p(x) - q(x))^2}{4}
\]

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