Higher associate class partially balanced incomplete block designs using graphs for agricultural experiments

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Abstract
Some new association schemes and construction methods of 3- and 4-associate class partially balanced incomplete block (PBIB) designs based on these schemes using chosen graphs have been given. A catalogue is included of efficient PBIB designs with number of treatments \( v \leq 100 \).

Keywords: Pappus hexagon association scheme, partially balanced incomplete block design, star polygon association scheme, extended G6 graph association scheme

1. Introduction
Partially balanced incomplete block (PBIB) designs are formally introduced by Bose and Nair (1939) \[^{[3]}\]. Two class PBIB designs have been extensively studied in the literature and an exhaustive catalogue of these designs found in Clatworthy (1973) \[^{[4]}\]. A lot of literature is available on PBIB designs based on 3- or higher class association schemes. PBIB (3) designs based on rectangular association scheme are an important class of block designs with factorial structure for experiments with two factors were studied by Vartak (1955) \[^{[34]}\], Sharma and Das (1985) \[^{[24]}\], Suen (1989) \[^{[29]}\], Srivastava et al. (2000) \[^{[28]}\], and Parsad et al. (2007a, 2007b) \[^{[18, 19]}\]. The nested group divisible designs, a class of PBIB (3) designs, useful for 3-factor experiments was introduced by Roy (1953) \[^{[22]}\] were subsequently studied by Raghavarao (1960) \[^{[20]}\], Miao et al. (1996) \[^{[15]}\], and Kageyama and Singh (2002) \[^{[10]}\]. More generalized association scheme called extended group divisible association scheme and designs based on this scheme are known as extended group divisible (EGD) designs was discovered by Hinkelmann (1964) \[^{[9]}\]. Many useful applications of these designs and their catalogue are given in Parsad et al. (2007a, 2007b) \[^{[18, 19]}\]. PBIB (3) designs using circular lattices were developed by Rao (1956) \[^{[21]}\] for \( v = 2n^2 \) treatments, where \( n \geq 2 \) and these were further generalized by Varghese and Sharma (2004) \[^{[33]}\] to accommodate \( 2sn^2 \) treatments form \( s \geq 2 \). Also, Varghese et al. (2004) \[^{[25]}\] reported exhaustive work on 3-class PBIB designs with a comprehensive catalogue for \( v, b \leq 100, k \leq 20 \) and their applications to partial diallel crosses. Sharma et al. (2010) \[^{[23]}\] extended the work on PBIB (3) designs further by proposing tetrahedral and cubical association schemes and methods of constructions of PBIB (3) designs based on these schemes. Garg et al. (2011) \[^{[7]}\] developed some new triangular and four associate class PBIB designs in two replications. Furthermore, Sharma et al. (2013) \[^{[23]}\] presented web solutions for PBIB designs. Garg and Farooq (2014) \[^{[6]}\] introduced 2- and 3-associate class PBIB designs through chosen lines and graphs. Kipkemoi et al. (2013) \[^{[13]}\] and Kipkemoi et al. (2015) \[^{[14]}\] also gave some PBIB (3) designs in two replications. Recently, Singh and Garg (2020) \[^{[26]}\] constructed some 2- and 3-associate class PBIB designs through edges and paths of graphs. Investigations of association schemes for 4-associate class PBIB designs have been curbed mainly to Nair (1951) \[^{[10]}\], Tharthare (1963, 1965) \[^{[31, 30]}\], and Garg et al. (2011) \[^{[7]}\]. In the present investigation, three new association schemes with three and four associate classes and methods for constructing PBIB designs based on these schemes using some graphs are given in Section 2. Conclusions and future scope are covered in Section 3. Catalogues of PBIB designs for \( v \leq 100 \) along with the coefficient of error variance in the average variance \( (V\bar{V}) \) and efficiency (E) as compared to a randomized complete block design have been obtained from these construction methods and are presented in the Appendix.

2. Graphical Association schemes and Constructions of PBIB designs
We present 3- and 4-class graphical association schemes which may be considered as generalizations of Theorem 3.1, Theorem 3.2, Theorem 3.3, and Theorem 3.6 of Garg and Farooq (2014) \[^{[6]}\] and methods of constructing PBIB designs based on these schemes in the sequel.
2.1 Star Polygon Association scheme

Let \( v = 10m \) \((m \geq 2)\) be the number of treatments. Arrange these treatments on the vertices of a Star Polygon graph such that each vertex contains exactly \( m \) distinct treatments. Now we define the association scheme on these \( v \) treatments as follows: Treatment \( \beta \) is the first associate of \( \alpha \), if \( \beta \) lies on the same vertex of \( \alpha \); the second associate, if \( \beta \) lies on any of the quadruplets (where each quadruplet has precisely four equidistant vertices) that intersect the vertex of \( \alpha \) and third associates, otherwise. The parameters of first kind and association matrices (called as parameters of second of kind) of the association scheme are given respectively: \( v = 10m \), \( n_1 = m - 1 \), \( n_2 = 6m \), \( n_3 = 3m \), and

\[
P_1 = \begin{bmatrix} m-2 & 0 & 0 \\ 0 & 6m & 0 \\ 0 & 0 & m-1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & m-1 & 0 \\ m-1 & 3m & 2m \\ 0 & 2m & m \end{bmatrix}
\]

and \( P_3 = \begin{bmatrix} 0 & 0 & m-1 \\ 0 & 4m & 2m \\ m-1 & 2m & 0 \end{bmatrix} \).

Illustration 2.1: Let \( v = 30 \) \((= 10 \times 3)\) treatments are arranged on the vertices of a Star Polygon graph such that each vertex contains exactly three distinct treatments as shown in Fig. 1.

Here, \( n_1 = 2 \), \( n_2 = 18 \), \( n_3 = 9 \) and the three associates of treatments, say 1, 2, 7 and 22 are as given in Table 1. Association matrices of this illustration are as follows:

\[
P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 9 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 2 & 0 & 9 \\ 2 & 6 & 6 \\ 0 & 6 & 3 \end{bmatrix} \text{ and } P_3 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 12 & 6 \\ 2 & 6 & 0 \end{bmatrix}
\]

| Treatment | 1\(^{st}\) associates | 2\(^{nd}\) associates | 3\(^{rd}\) associates |
|-----------|-----------------------|-----------------------|-----------------------|
| 1         | 2, 3                  | 7, 8, 9, 10, 11, 12, 16, 17, 18, 19, 20, 21, 25, 26, 27, 28, 29, 30 | 4, 5, 6, 13, 14, 15, 22, 23, 24 |
| 2         | 1, 3                  | 7, 8, 9, 10, 11, 12, 16, 17, 18, 19, 20, 21, 25, 26, 27, 28, 29, 30 | 4, 5, 6, 13, 14, 15, 22, 23, 24 |
| 7         | 8, 9                  | 1, 2, 3, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27 | 4, 5, 6, 10, 11, 12, 28, 29, 30 |
| 22        | 23, 24                | 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 19, 20, 21, 25, 26, 27 | 1, 2, 3, 16, 17, 18, 28, 29, 30 |

2.1 Methods of construction of PBIB (3) designs Using Star Polygon graph

In this section, we give two construction methods of PBIB designs based on Star Polygon association scheme with three associate classes in two and three replications.

Method 2.1.1: Let \( v = 10m \) \((m \geq 2)\) treatments are arranged on the vertices of a Star Polygon graph as indicated in the association scheme. By taking all possible triangles as blocks, this process implies a PBIB(3) design based on Star Polygon association scheme with parameters \( v = 10m \), \( b = 10 \), \( r = 3 \), \( k = 3m \), \( \lambda_1 = 3 \), \( \lambda_2 = 1 \), \( \lambda_3 = 0 \).

Example 2.1.1: Let \( v = 30 \) treatments are arranged on the vertices of a Star Polygon graph as given in Fig. 1. By following the procedure of Method 2.1.1, we can get a PBIB(3) design based on Star Polygon association scheme with block contents as

\[(1, 2, 3, 16, 17, 18, 28, 29, 30), (4, 5, 6, 16, 17, 18, 19, 20, 21), (7, 8, 9, 19, 20, 21, 22, 23, 24), (10, 11, 12, 22, 23, 24, 25, 26, 27), (13, 14, 15, 25, 26, 27, 28, 29, 30), (1, 2, 3, 7, 8, 9, 25, 26, 27)\]

Method 2.1.2: Let \( v = 10m \) \((m \geq 2)\) treatments are arranged on the vertices of a Star Polygon graph as defined in the association scheme. Form the blocks of the design each one corresponding to a quadruplets by taking together the treatments that lie on the four vertices (points) of each quadruplet as the block, which results the PBIB(3) design based on Star Polygon association scheme with parameters \( v = 10m \), \( b = 5 \), \( r = 2 \), \( k = 4m \), \( \lambda_1 = 1 \), \( \lambda_2 = 1 \), \( \lambda_3 = 0 \).

Example 2.1.2: Let \( v = 30 \) treatments are defined on the Star Polygon association scheme. Then by using the procedure of Method 2.1.2, we can get a PBIB(3) design based on Star Polygon association scheme with block contents as...
(1, 2, 3, 10, 11, 12, 25, 26, 27, 28, 29, 30)  
(1, 2, 3, 7, 8, 9, 16, 17, 18, 19, 20, 21)  
(4, 5, 6, 10, 11, 12, 19, 20, 21, 22, 23, 24)  
(4, 5, 6, 13, 14, 15, 16, 17, 18, 28, 29, 30)  
(7, 8, 9, 13, 14, 15, 22, 23, 24, 25, 26, 27)  

The parameters of this design are \( v = 30, b = 5, r = 2, k = 12, \lambda_1 = 2, \lambda_2 = 1, \lambda_3 = 0 \).

A total of 9 PBIB(3) designs generated by Method 2.1.2 are catalogued in Table 2.2 of the Appendix for \( v \leq 100 \). In Table 2.2, efficiency of the designs constructed using this method ranges from 0.8878 to 0.9763.

**Remark 2.1:** For \( m = 1 \), the Method 2.1.1 and Method 2.1.2 are reduced to 2-class PBIB design from Theorem 3.1 and Theorem 3.2 respectively, further details on this, reader may refer to Garg and Farooq (2014) [6].

### 2.2 Pappus Hexagon Association scheme

Let \( v = 9m (m \geq 2) \) be the number of treatments. Arrange these treatments on a Pappus Hexagon graph’s vertices so that each vertex has exactly \( m \) distinct treatments. The association scheme is now defined as follows: Two treatments are first associates if they lie on same vertex. Second associates, if they lie on two different vertices of same triangle that intersect the vertex of first associates and third associates, otherwise. The parameters and association matrices of the association scheme are given respectively: \( v = 9m, n_1 = m - 1, n_2 = 6m, n_3 = 2m, \) and

\[
P_1 = \begin{bmatrix} m-2 & 0 & 0 \\ 0 & 6m & 0 \\ 0 & 0 & 2m \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & m-1 & 0 \\ m-1 & 3m & 2m \\ 0 & 2m & 0 \end{bmatrix}
\]

### Illustration 2.2:

Let \( v = 18 \) treatments are arranged on the vertices of a Pappus Hexagon graph such that each vertex contains exactly two distinct treatments is given in Fig. 2.

![Fig 2: Arrangement of 20 treatments on vertices of a Pappus Hexagon](image)

Here, \( n_1 = 1, n_2 = 12, n_3 = 4 \) and the three associates of treatments, say 1, 2, 3 and 7 are as given in Table 2. Association matrices of above illustration are:

\[
P_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 & 6 & 4 \\ 0 & 4 & 0 \\ 0 & 6 & 4 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 12 & 0 \\ 1 & 0 & 2 \end{bmatrix}
\]

| Treatment | 1st associates | 2nd associates | 3rd associates |
|-----------|----------------|----------------|---------------|
| 1         | 2              | 3, 4, 5, 6, 9, 10, 11, 12, 13, 14, 15, 16 | 7, 8, 17, 18 |
| 2         | 1              | 3, 4, 5, 6, 9, 10, 11, 12, 13, 14, 15, 16 | 7, 8, 17, 18 |
| 3         | 4              | 1, 2, 5, 6, 7, 8, 11, 12, 13, 14, 17, 18 | 9, 10, 15, 16 |
| 7         | 8              | 3, 4, 5, 6, 9, 10, 11, 12, 13, 14, 15, 16 | 1, 2, 17, 18 |

### 2.2.2 Method of constructing PBIB (3) design Using Pappus Hexagon graph

In this section, we give a construction method of PBIB design based on Pappus Hexagon association scheme with three associate classes in six replications.

**Method 2.2.1:** Let \( v = 9m (m \geq 2) \) treatments are defined on the Pappus Hexagon association scheme. By taking together the treatments that lies on all the three vertices of each triangle taken as the block of size \( 3m \). Repeat this process to all possible triangles exist in the Pappus Hexagon association scheme. By taking the Pappus Hexagon scheme is now defined as follows: Two treatments are first associates and third associates, if they lie on same vertex. Second associates, if they lie on two different vertices of same triangle that intersect the vertex of first associates and third associates, otherwise. The parameters and association matrices of the association scheme are given respectively: \( v = 9m, n_1 = m - 1, n_2 = 6m, n_3 = 2m, \) and

\[
P_1 = \begin{bmatrix} 0 & m-1 & 0 \\ m-1 & 3m & 2m \\ 0 & 2m & 0 \end{bmatrix}
\]

The parameters of above design are \( v = 18, b = 18, r = 6, k = 6, \lambda_1 = 6, \lambda_2 = 2, \lambda_3 = 0. \)

**Example 2.2.1:** Let \( v = 18 \) treatments are defined on the Pappus Hexagon association scheme. Now as per the procedure of Method 2.2.1, one can get a PBIB(3) design based on Pappus Hexagon association scheme with block structure as

\[
(1, 2, 5, 6, 15, 16); \quad (3, 4, 7, 8, 11, 12) \\
(1, 2, 3, 4, 13, 14); \quad (5, 6, 15, 16, 17, 18)
\]

The parameters of above design are \( v = 18, b = 18, r = 6, k = 6, \lambda_1 = 6, \lambda_2 = 2, \lambda_3 = 0. \)

The PBIB (3) designs generated by Method 2.2.1 are catalogued in Table 2.3 for \( v \leq 100 \) in the Appendix. This method yielded a total of 10 designs, each with efficiency in the range of 0.8500 to 0.9708, as shown in Table 2.3.

**Remark 2.2:** For \( m = 1 \), the Method 2.2.1 is reduced to Theorem 3.3 which yields two class PBIB designs found in Garg and Farooq (2014) [6].

Next higher association scheme is named as Extended G6 graph association scheme. A construction method of 4-associate class PBIB design based on this scheme is constructed with the help of G6 graph quoted in Garg and Farooq (2014) [6] and the scheme is defined in sequel.
2.3 Extended G₆ Graph Association scheme

Let the number of treatments be \( v = 6m \) \( \forall \) \( m \geq 2 \). Arrange these \( v \) treatments on the six vertices (points) of a G₆ graph such that each vertex contains exactly \( m \) distinct treatments. Now we define the association scheme as follows: Treatment \( \theta_2 \) is the first associate of \( \theta_1 \), if \( \theta_2 \) lies on the same vertex of \( \theta_1 \); the second associate, if \( \theta_2 \) lies on any of the two vertices of same triangle that intersect the vertex of \( \theta_1 \); the third associate, if \( \theta_2 \) lies on the collinear points (the points that lie on the same straight line or in a single line) and fourth associates, otherwise. The parameters of the association scheme are given by \( v = 6m \), \( n_1 = m − 1 \), \( n_2 = 2m \), \( n_3 = 2m \), \( n_4 = m \). Association matrices of the association scheme are given below:

\[
P_1 = \begin{bmatrix}
(m−2) & 0 & 0 & 0 \\
0 & 2m & 0 & 0 \\
0 & 0 & 2m & 0 \\
0 & 0 & 0 & m
\end{bmatrix},
P_2 = \begin{bmatrix}
0 & (m−1) & 0 & 0 \\
(m−1) & 0 & m & 0 \\
0 & m & 0 & m \\
0 & 0 & m & 0
\end{bmatrix},
P_3 = \begin{bmatrix}
0 & 0 & (m−1) & 0 \\
0 & m & 0 & m \\
(m−1) & 0 & m & 0 \\
0 & m & 0 & 0
\end{bmatrix},
\]

and \( P_4 = \begin{bmatrix}
0 & 0 & 0 & (m−1) \\
0 & 2m & 0 & 0 \\
0 & 0 & 2m & 0 \\
(m−1) & 0 & 0 & 0
\end{bmatrix} \).

Illustration 2.3: Let \( v = 18 \) \( (= 6 \times 3) \) treatments are arranged on the vertices of G₆ graph such that each vertex contains exactly three distinct treatments is given in Fig. 3.

![Fig 3: Arrangement of 18 treatments on vertices of G₆ graph](image)

Here, \( n_1 = 2 \), \( n_2 = 6 \), \( n_3 = 6 \), \( n_4 = 3 \) and the four associates of treatments, say 1, 2, 4, 10 and 7 are as given in Table 3.

Also, association matrices of this scheme are as follows;

\[
P_1 = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 6 & 0 & 0 \\
0 & 0 & 6 & 0 \\
0 & 0 & 0 & 3
\end{bmatrix},
P_2 = \begin{bmatrix}
0 & 2 & 0 & 0 \\
2 & 0 & 3 & 0 \\
0 & 3 & 0 & 3 \\
0 & 0 & 3 & 0
\end{bmatrix},
P_3 = \begin{bmatrix}
0 & 0 & 2 & 0 \\
0 & 3 & 0 & 3 \\
2 & 0 & 3 & 0 \\
0 & 3 & 0 & 0
\end{bmatrix}\]

and \( P_4 = \begin{bmatrix}
0 & 0 & 0 & 2 \\
0 & 0 & 6 & 0 \\
0 & 6 & 0 & 0 \\
2 & 0 & 0 & 0
\end{bmatrix} \).

Table 3: Different associates of treatments 1, 2, 4, 10 and 7

| Treatment | 1st associates | 2nd associates | 3rd associates | 4th associates |
|-----------|--------------|---------------|--------------|--------------|
| 1          | 2, 3         | 4, 5, 6, 16, 17, 18 | 10, 11, 12, 13, 14, 15 | 7, 8, 9 |
| 2          | 1, 3         | 4, 5, 6, 16, 17, 18 | 10, 11, 12, 13, 14, 15 | 7, 8, 9 |
| 4          | 5, 6         | 1, 2, 3, 13, 14, 15 | 7, 8, 9, 16, 17, 18 | 10, 11, 12 |
| 10         | 11, 12       | 7, 8, 9, 16, 17, 18 | 1, 2, 3, 13, 14, 15 | 4, 5, 6 |
| 7          | 8, 9         | 10, 11, 12, 13, 14, 15 | 4, 5, 6, 16, 17, 18 | 1, 2, 3 |

(4, 5, 6, 7, 8, 9, 13, 14, 15)
(7, 8, 9, 10, 11, 12, 13, 14, 15)

The parameters of the design are \( v = 18 \), \( b = 6 \), \( r = 3 \), \( k = 9 \), \( \lambda_1 = 3 \), \( \lambda_2 = 2 \), \( \lambda_3 = 1 \), \( \lambda_4 = 0 \).

A total of 15 PBIB (4) designs generated by Method 2.3.1 for \( v \leq 100 \) are catalogued in Table 2.4 of the Appendix. Efficiency of the designs from this method lies in the range of 0.8644 to 0.9821, as mentioned in Table 2.4.

Remark 2.3: For \( m = 1 \), the Method 2.3.1 is coincides with 3-class PBIB designs of Theorem 3.6 by Garg and Farooq (2014) [6].

2.3 Method of constructing PBIB (4) design Using G₆ graph

In this section, we give a construction method of PBIB design based on Extended G₆ graph association scheme with four associate classes in three replications.

Method 2.3.1: Let \( v = 6m \) \( (m \geq 2) \) treatments are defined on Extended G₆ graph association scheme. By taking together the treatments that lies on all the three vertices of each triangle taken as the block of size 3m. Repeat this process to all possible triangles exist in the G₆ graph then resultant design is the PBIB(4) design based on Extended G₆ graph association scheme with parameters \( v = 6m \), \( b = 6 \), \( r = 3 \), \( k = 3m \), \( \lambda_1 = 3 \), \( \lambda_2 = 2 \), \( \lambda_3 = 1 \), \( \lambda_4 = 0 \).

Example 2.3.1: Let \( v = 18 \) treatments are defined on the Extended G₆ graph association scheme. By using the procedure of Method 2.3.1, we can get a PBIB(4) design based on Extended G₆ graph association scheme with block layout as

\[
(1, 2, 3, 4, 5, 6, 16, 17, 18) \\
(1, 2, 3, 4, 5, 6, 13, 14, 15) \\
(1, 2, 3, 10, 11, 12, 16, 17, 18) \\
(7, 8, 9, 10, 11, 12, 16, 17, 18)
\]
IARI, New Delhi, and ICAR-IASRI, New Delhi for providing

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5. References

1. Agrawal HC. A four-class cyclic association scheme and related partial diallel crosses. Sankhyā: The Indian Journal of Statistics, Series B. 1985;47:78-90.

2. Arya AS, Narain P. Partial diallel crosses based on some association schemes with three and four associate classes. Sankhyā: The Indian Journal of Statistics, Series B, 1978, 394-9.

3. Bose RC, Nair KR. Partially balanced incomplete block designs. Sankhyā: The Indian Journal of Statistics. 1939;4:337-72.

4. Clatworthy WH. Tables of two-associate-class partially balanced designs. US National Bureau of Standards. 1973.

5. Fyle JL, Gilbert N. Partial diallel crosses. Biometrics. 1963;19(2):278-86.

6. Garg DK, Farooq SA. Construction of PBIB designs through chosen lines and triangles of graphs. International Journal of Mathematics Trends and Technology. 2014;8(1):25-32.

7. Garg DK, Jhaji HS, Mishra G. Construction of Some New Triangular and Four Associate Class PBIB Designs with Two Replicates. International Journal of Mathematical Sciences and Applications. 2011;1(2):808-21.

8. Hinkelmann K, Kempthorne O. Two classes of group divisible partial diallel crosses. Biometrika. 1963;50(3/4):281-91.

9. Hinkelmann K. Extended group divisible partially balanced incomplete block designs. The Annals of Mathematical Statistics, 1964, 681-95.

10. Kageyama S, Singh MK. Constructions of group divisible and nested group divisible designs. Utilitas Mathematica. 2002;61:167-74.

11. Kaushik LS, Puri PD. Partial diallel crosses based on generalized right angled association scheme. Communications in Statistics-Theory and Methods. 1989;18(7):2501-10.

12. Kaushik LS. Partial diallel crosses based on three associate class association schemes. Journal of Applied Statistics. 1999;26(2):195-201.

13. Kipkemoi EC, Koske JK, Mutiso JM. Construction of three-associate class partially balanced incomplete block designs in two replicates. American Journal of Mathematical Science and Applications. 2013;1(1):61-65.

14. Kipkemoi EC, Mutiso JM, Bekolle D. Construction of some new three associate class partially balanced incomplete block designs in two replicates. International Journal of Academic Research in Progressive Education and Development. 2015;1:188-192.

15. Miao Y, Kageyama S, Duan X. Further constructions of nested group divisible designs. Journal of the Japan Statistical Society. 1996;26(2):231-9.

16. Nair KR. Rectangular lattices and partially balanced incomplete block designs. Biometrics. 1951;7(2):145-54.

17. Narain P, Rao CS, Nigam AK. Partial diallel crosses based on extended triangular association scheme (India). Indian Journal of Genetics and Plant Breeding. 1974.

18. Parsad R, Gupta VK, Srivastava R. Designs for cropping systems research. Journal of Statistical Planning and Inference. 2007;137(5):1687-703.

19. Parsad R, Kageyama S, Gupta VK. Use of complementary property of block designs in PBIB designs. Ars Combinatoria. 2007b;85:173-182.

20. Raghavarao D. A generalization of group divisible designs. The Annals of Mathematical Statistics. 1960, 756-71.

21. Rao CR. A general class of quasifactorial and related designs. Sankhyā: The Indian Journal of Statistics (1933-1960). 1956;17(2):165-74.

22. Roy PM. Hierarchical group divisible incomplete block designs with m associate classes. Science and Culture. 1953;19:210-1.

23. Sharma A, Varghese C, Jaggi S. Web solutions for Partially Balanced Incomplete Block Designs. I.A.S.R.I./B.-02/2012, IASRI, New Delhi, 2013.

24. Sharma VK, Das MN. On resolvable incomplete block designs 1. Australian Journal of Statistics. 1985;27(3):298-302.

25. Sharma VK, Varghese C, Jaggi S. Tetrahedral and cubical association schemes with related PBIB (3) designs. Model assisted statistics and applications. 2010;5(2):93-9.

26. Singh GP, Garg DK. PBIB designs constructed through edges and paths of graphs. International Journal of Mathematics Trends and Technology. 2020;66(5):70-74.

27. Singh M, Hinkelmann K. Partial diallel crosses in incomplete blocks. Biometrics. 1995;51:1302-14.

28. Srivastava R, Parsad R, Gupta VK. Structure resistant factorial designs. Sankhyā: The Indian Journal of Statistics, Series B. 2000;62(2):257-65.

29. Suen CY. Some rectangular designs constructed by the method of differences. Journal of Statistical Planning and Inference. 1989;21(2):273-6.

30. Tharthare SK. Generalized right angular designs. The Annals of Mathematical Statistics. 1965;36(5):1535-53.

31. Tharthare SK. Right angular designs. The Annals of Mathematical Statistics. 1963;34(3):1057-67.

32. Varghese C, Sharma VK, Jaggi S, Sharma A. Three-associate class partially balanced incomplete block designs and their application to partial diallel crosses. Project report, IASRI publication, New Delhi. 2004.

33. Varghese C, Sharma VK. A series of resolvable PBIB (3) designs with two replicates. Metrika. 2004;60(2):251-4.

34. Vartak MN. On an application of Kronecker product of smatrices to statistical designs. The Annals of Mathematical Statistics. 1964, 756-71.

35. Williams E, Piepho HP, Whitaker D. Augmented p-rep designs. Biometrical Journal. 2011;53(1):19-27.
## Appendix

### Table 2.1: PBIB (3) Designs based on Star Polygon Association Scheme with $v \leq 100$ Using Method 2.1.1

| Sl. No. | $v$ | $b$ | $r$ | $k$ | $n_1$ | $n_2$ | $n_3$ | $\overline{V}$ | $E$   |
|---------|-----|-----|-----|-----|-------|-------|-------|------------|------|
| 1       | 13  | 10  | 3   | 6   | 3     | 1     | 1     | 12         | 6    |
| 2       | 10  | 5   | 2   | 8   | 2     | 1     | 0     | 12         | 6    |
| 3       | 7   | 5   | 2   | 10  | 5     | 5     | 5     | 10         | 5    |
| 4       | 5   | 5   | 2   | 10  | 5     | 5     | 5     | 10         | 5    |
| 5       | 3   | 5   | 2   | 10  | 5     | 5     | 5     | 10         | 5    |
| 6       | 2   | 5   | 2   | 10  | 5     | 5     | 5     | 10         | 5    |
| 7       | 2   | 5   | 2   | 10  | 5     | 5     | 5     | 10         | 5    |
| 8       | 2   | 5   | 2   | 10  | 5     | 5     | 5     | 10         | 5    |
| 9       | 10  | 3   | 3   | 9   | 3     | 3     | 3     | 9          | 3    |

### Table 2.2: PBIB (3) Designs based on Star Polygon Association Scheme with $v \leq 100$ Using Method 2.1.2

| Sl. No. | $v$ | $b$ | $r$ | $k$ | $n_1$ | $n_2$ | $n_3$ | $\overline{V}$ | $E$   |
|---------|-----|-----|-----|-----|-------|-------|-------|------------|------|
| 1       | 13  | 10  | 3   | 6   | 3     | 1     | 1     | 12         | 6    |
| 2       | 10  | 5   | 2   | 8   | 2     | 1     | 0     | 12         | 6    |
| 3       | 7   | 5   | 2   | 10  | 5     | 5     | 5     | 10         | 5    |
| 4       | 5   | 5   | 2   | 10  | 5     | 5     | 5     | 10         | 5    |
| 5       | 3   | 5   | 2   | 10  | 5     | 5     | 5     | 10         | 5    |
| 6       | 2   | 5   | 2   | 10  | 5     | 5     | 5     | 10         | 5    |
| 7       | 2   | 5   | 2   | 10  | 5     | 5     | 5     | 10         | 5    |
| 8       | 2   | 5   | 2   | 10  | 5     | 5     | 5     | 10         | 5    |
| 9       | 10  | 3   | 3   | 9   | 3     | 3     | 3     | 9          | 3    |

### Table 2.3: PBIB (3) Designs based on Pappus Hexagon Association Scheme with $v \leq 100$ Using Method 2.2.1

| Sl. No. | $v$ | $b$ | $r$ | $k$ | $n_1$ | $n_2$ | $n_3$ | $\overline{V}$ | $E$   |
|---------|-----|-----|-----|-----|-------|-------|-------|------------|------|
| 1       | 18  | 18  | 6   | 6   | 6     | 2     | 0     | 12         | 4    |
| 2       | 18  | 18  | 6   | 6   | 6     | 2     | 0     | 12         | 4    |
| 3       | 18  | 18  | 6   | 6   | 6     | 2     | 0     | 12         | 4    |
| 4       | 18  | 18  | 6   | 6   | 6     | 2     | 0     | 12         | 4    |
| 5       | 18  | 18  | 6   | 6   | 6     | 2     | 0     | 12         | 4    |
| 6       | 18  | 18  | 6   | 6   | 6     | 2     | 0     | 12         | 4    |
| 7       | 18  | 18  | 6   | 6   | 6     | 2     | 0     | 12         | 4    |
| 8       | 18  | 18  | 6   | 6   | 6     | 2     | 0     | 12         | 4    |
| 9       | 18  | 18  | 6   | 6   | 6     | 2     | 0     | 12         | 4    |

### Table 2.4: PBIB (4) Designs based on Extended G6 graph Association Scheme with $v \leq 100$ Using Method 2.3.1

| Sl. No. | $v$ | $b$ | $r$ | $k$ | $n_1$ | $n_2$ | $n_3$ | $n_4$ | $\overline{V}$ | $E$   |
|---------|-----|-----|-----|-----|-------|-------|-------|-------|------------|------|
| 1       | 12  | 6   | 3   | 6   | 3     | 2     | 1     | 1     | 4           | 4    | 4         | 2    | 0.7712 | 0.8644 |
| 2       | 12  | 6   | 3   | 6   | 3     | 2     | 1     | 1     | 4           | 4    | 4         | 2    | 0.7343 | 0.9078 |
| 3       | 12  | 6   | 3   | 6   | 3     | 2     | 1     | 1     | 4           | 4    | 4         | 2    | 0.7166 | 0.9302 |
| 4       | 12  | 6   | 3   | 6   | 3     | 2     | 1     | 1     | 4           | 4    | 4         | 2    | 0.7063 | 0.9438 |
| 5       | 12  | 6   | 3   | 6   | 3     | 2     | 1     | 1     | 4           | 4    | 4         | 2    | 0.6995 | 0.9530 |
| 6       | 12  | 6   | 3   | 6   | 3     | 2     | 1     | 1     | 4           | 4    | 4         | 2    | 0.6947 | 0.9596 |
| 7       | 12  | 6   | 3   | 6   | 3     | 2     | 1     | 1     | 4           | 4    | 4         | 2    | 0.6911 | 0.9645 |
| 8       | 12  | 6   | 3   | 6   | 3     | 2     | 1     | 1     | 4           | 4    | 4         | 2    | 0.6883 | 0.9684 |
| 9       | 12  | 6   | 3   | 6   | 3     | 2     | 1     | 1     | 4           | 4    | 4         | 2    | 0.6861 | 0.9716 |
| 10      | 12  | 6   | 3   | 6   | 3     | 2     | 1     | 1     | 4           | 4    | 4         | 2    | 0.6843 | 0.9741 |
| 11      | 12  | 6   | 3   | 6   | 3     | 2     | 1     | 1     | 4           | 4    | 4         | 2    | 0.6828 | 0.9762 |
| 12      | 12  | 6   | 3   | 6   | 3     | 2     | 1     | 1     | 4           | 4    | 4         | 2    | 0.6816 | 0.9780 |
| 13      | 12  | 6   | 3   | 6   | 3     | 2     | 1     | 1     | 4           | 4    | 4         | 2    | 0.6805 | 0.9796 |
| 14      | 12  | 6   | 3   | 6   | 3     | 2     | 1     | 1     | 4           | 4    | 4         | 2    | 0.6795 | 0.9809 |
| 15      | 12  | 6   | 3   | 6   | 3     | 2     | 1     | 1     | 4           | 4    | 4         | 2    | 0.6787 | 0.9821 |