On Complete Convergence in Mean for Double Sums of Independent Random Elements in Banach Spaces

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Abstract
In this work, conditions are provided under which a normed double sum of independent random elements in a real separable Rademacher type $p$ Banach space converges completely to 0 in mean of order $p$. These conditions for the complete convergence in mean of order $p$ are shown to provide an exact characterization of Rademacher type $p$ Banach spaces. In case the Banach space is not of Rademacher type $p$, it is proved that the complete convergence in mean of order $p$ of a normed double sum implies a strong law of large numbers.

Key Words and Phrases: Double array; Complete convergence in mean; Strong law of large numbers; Real separable Banach space; Rademacher type $p$ Banach space.

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1 Introduction
Let $\{V_{mn}, m \geq 1, n \geq 1\}$ be a double array of random elements in a real separable Banach space $X$ with norm $\|\cdot\|$. Throughout this paper, we write

$$S(m, n) = S_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} V_{ij}, m \geq 1, n \geq 1.$$  

For $a, b \in \mathbb{R}$, $\max\{a, b\}$ will be denoted by $a \vee b$. The symbol $C$ will denote a generic constant ($0 < C < \infty$) which is not necessarily the same one in each appearance.

We recall that $V_{mn}$ is said to converge completely to 0 (denoted $V_{mn} \xrightarrow{c} 0$) if

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} P(\|V_{mn}\| > \varepsilon) < \infty \text{ for all } \varepsilon > 0$$

and that for $p > 0$, $V_{mn}$ is said to converge to 0 in mean of order $p$ as $m \vee n \to \infty$ (denoted $V_{mn} \xrightarrow{L_p} 0$ as $m \vee n \to \infty$) if $$E\|V_{mn}\|^p \to 0 \text{ as } m \vee n \to \infty.$$  

By the Borel-Cantelli lemma, $V_{mn} \xrightarrow{c} 0$ ensure that $V_{mn} \to 0$ almost surely (a.s.) as $m \vee n \to \infty$ (see, e.g., [17]). But the modes of convergence $V_{mn} \xrightarrow{c} 0$ and $V_{mn} \xrightarrow{L_p} 0$ are not comparable in general. The double array $V_{mn}$ is said to converge completely to 0 in mean of order $p$ (denoted $V_{mn} \xrightarrow{c,L_p} 0$) if

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} E\|V_{mn}\|^p < \infty.$$  

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It is easy to see that $V_{mn} \overset{c,L}{\rightarrow} 0$ ensure both $V_{mn} \overset{c}{\rightarrow} 0$ and $V_{mn} \overset{L}{\rightarrow} 0$ as $m \lor n \to \infty$. However, as we will see later in Example 4.1 that the converse is not true.

The notion of complete convergence in mean of order $p$ ($p > 0$) was apparently first investigated by Chow [1] in the (real-valued) random variables case. Rosalsky, Thanh and Volodin [16] studied the complete convergence in mean of order $p$ for sequences of independent random elements in Banach spaces and provided through this mode of convergence a new characterization of Rademacher type $p$ Banach spaces. In this paper, we establish the double sum versions for the main results in [16]. This is done by using recent results by Rosalsky, Thanh and Thuy in [18]. The main results are Theorems 3.1 and 3.3. Theorem 3.1 provides conditions under which the normed double sum $S_{mn}/(mn)^{(p+1)/p}$ converges completely to 0 in mean of order $p$, $1 \leq p \leq 2$. Moreover, these conditions for $S_{mn}/(mn)^{(p+1)/p}$ converging completely to 0 in mean of order $p$ are shown to provide an exact characterization of Rademacher type $p$ Banach spaces. Theorem 3.3 shows that in general Banach spaces, the condition $S_{mn}/(mn)^{(p+1)/p} \overset{c,L}{\rightarrow} 0$ for some $p \geq 1$ implies the strong law of large numbers (SLLN) $S_{mn}/(mn) \to 0$ a.s. as $m \lor n \to \infty$.

The reader may refer to Gut [6], Gut and Stadtmüller [7, 8], Móricz [11], Móricz, Su and Taylor [12], Móricz, Stadtmüller and Thalmaier [13], Smythe [19] and references therein for SLLN and other limit theorems for double arrays of random variables. Rosalsky and Thanh [17] gave a brief discussion of a historical nature concerning double sums and on their importance in the field of statistical physics. In a major survey article [14], Pyke discussed fluctuation theory, the limiting Brownian sheet, the SLLN, and the law of the iterated logarithm for double arrays of independent identically distributed real-valued random variables. Recently, Klesov [10] published a comprehensive book on multiple sums.

The plan of the paper is as follows. Notation, technical definitions, and six known propositions and lemmas which are used in proving the main results are consolidated into Section 2. The main results are established in Section 3. In Section 4, two illustrating examples concerning the sharpness of Theorems 3.1 and 3.3 are presented.

## 2 Preliminaries

In this section, notation, lemmas and propositions which are needed in connection with the main results will be presented.

The expected value or mean of a Banach space $X$-valued random element $V$, denoted $EV$, is defined to be the Pettis integral provided it exists. If $E\|V\| < \infty$, then (see, e.g., Taylor [21, p. 40]) $V$ has an expected value. But the expected value can exist when $E\|V\| = \infty$. For an example, see Taylor [21, p. 41].

The reader may refer to Hoffmann-Jørgensen and Pisier [9] for definition, properties and examples of Rademacher type $p$ Banach spaces. Hoffmann-Jørgensen and Pisier [9] proved for $1 \leq p \leq 2$ that a real separable Banach space is of Rademacher type $p$ if and only if there exists a constant $C$ depending only on $p$ such that
\[ E \left\| \sum_{j=1}^{n} V_j \right\|^p \leq C \sum_{j=1}^{n} E\|V_j\|^p \]  \hspace{1cm} (2.1)

for every finite collection \( \{V_1, \ldots, V_n\} \) of independent mean 0 random elements.

The proof of the following simple lemma can be found in [16].

**Lemma 2.1.** Let \( \{V_n, n \geq 1\} \) be a sequence of independent mean 0 random elements in a real separable Banach space. Then for all \( p \geq 1 \), the sequence \( \{E\|\sum_{j=1}^{n} V_j\|^p, n \geq 1\} \) is nondecreasing.

Proposition 2.2 is a double sum analogue of the classical Kolmogorov SLLN in Banach spaces.

**Proposition 2.2** (Rosalsky and Thanh [15]). Let \( 1 \leq p \leq 2 \) and let \( X \) be real separable Banach space. Then the following two statements are equivalent:

(i) The Banach space \( X \) is of Rademacher type \( p \).

(ii) For every double array \( \{V_{mn}, m \geq 1, n \geq 1\} \) of independent mean 0 random elements in \( X \) and every choice of constants \( \alpha > 0 \) and \( \beta > 0 \), the condition

\[ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} E\|V_{mn}\|^p \frac{1}{m^\alpha n^\beta} < \infty \]  \hspace{1cm} (2.2)

implies that the SLLN

\[ \frac{S_{mn}}{m^\alpha n^\beta} \to 0 \text{ a.s. as } m \vee n \to \infty \]  \hspace{1cm} (2.3)

obtains.

If the Banach space is not of Rademacher type \( p \), condition (2.2) alone does not ensure the SLLN (2.3) (see [18 Example 5.1]). The next proposition is a recent result of Rosalsky, Thanh and Thuy [18] which considers the law of large numbers for double sums in a general real separable Banach space. It shows that if (2.2) holds, the the SLLN (2.3) and the weak law of large numbers (WLLN) (2.4) are equivalent.

**Proposition 2.3** (Rosalsky, Thanh and Thuy [18]). Let \( \alpha > 0, \beta > 0 \) and let \( \{V_{mn}, m \geq 1, n \geq 1\} \) be a double array of independent random elements in a real separable Banach space. Assume that (2.2) holds for some \( 1 \leq p \leq 2 \), then the SLLN (2.3) holds if and only if

\[ \frac{S_{mn}}{m^\alpha n^\beta} \overset{P}{\to} 0 \text{ as } m \vee n \to \infty. \]  \hspace{1cm} (2.4)

The next lemma is Lemma 3.2 in [18] which enables to study the SLLN through the symmetrization procedures.

**Lemma 2.4** (Rosalsky, Thanh and Thuy [18]). Let \( \alpha > 0, \beta > 0 \) and let \( V = \{V_{mn}, m \geq 1, n \geq 1\} \) and \( V' = \{V'_{mn}, m \geq 1, n \geq 1\} \) be two double arrays of independent random elements in a real separable Banach space such that \( V \) and \( V' \) are independent copies of each other. Let \( S_{mn}^* = \sum_{i=1}^{m} \sum_{j=1}^{n} (V_{ij} - V'_{ij}) \). Then

\[ \frac{S_{mn}}{m^\alpha n^\beta} \to 0 \text{ a.s. as } m \vee n \to \infty \]
if and only if
\[
\frac{S_{mn}^*}{m^\alpha n^\beta} \to 0 \text{ a.s. as } m \lor n \to \infty
\]
and
\[
\frac{S_{mn}}{m^\alpha n^\beta} \overset{P}{\to} 0 \text{ as } m \lor n \to \infty.
\]

Lemma 2.5 considers the SLLN for double arrays of symmetric independent random elements.

Lemma 2.5 (Rosalsky, Thanh and Thuy [18]). Let \( \{V_{mn}, m \geq 1, n \geq 1\} \) be a double array of independent symmetric random elements in a real separable Banach space. Then
\[
\frac{S_{mn}}{mn} \to 0 \text{ a.s. as } m \lor n \to \infty
\]
if and only if
\[
\sum_{i=2^m+1}^{2^{m+1}} \sum_{j=2^n+1}^{2^{n+1}} V_{ij}^2 \to 0 \text{ a.s. as } m \lor n \to \infty.
\]

The last lemma is a simple consequence of Lemma 1 of Etemadi [4].

Lemma 2.6. Let \( X \) and \( Y \) be two independent symmetric random elements in a real separable Banach space. Then for all \( t > 0 \),
\[
P(\|X\| > t) \leq 2P(\|X + Y\| > t).
\]

3 Main Results

With the preliminaries accounted for, the first main result may be established. Theorem 3.1 provides a new characterization of Rademacher type \( p \) Banach spaces through the complete convergence in mean of order \( p \) for normed double sums.

Theorem 3.1. Let \( 1 \leq p \leq 2 \) and let \( X \) be a real separable Banach space. Then the following statements are equivalent:

(i) \( X \) is of Rademacher type \( p \).

(ii) For every double array \( \{V_{mn}, m \geq 1, n \geq 1\} \) of independent mean 0 random elements in \( X \), the condition
\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{E\|V_{mn}\|^p}{m^p n^p} < \infty
\]
implies
\[
\frac{S_{mn}}{(mn)^{(p+1)/p}} \overset{c.L}{\to} 0.
\]

Proof. Note that
\[
\sum_{m=1}^{\infty} \sum_{n=j}^{\infty} \frac{1}{(mn)^{p+1}} \approx \frac{1}{p^2} \frac{1}{(ij)^p}.
\]
Assume that (i) holds. Let \( \{V_{mn}, m \geq 1, n \geq 1\} \) be a double array of independent mean 0 random elements in \( X \) satisfying (3.1). Then

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} E \left\| \frac{S_{mn}}{(mn)^{(p+1)/p}} \right\|^p \leq C \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} E \|V_{ij}\|^p \frac{1}{(mn)^{p+1}} \quad \text{(by (2.1))}
\]

\[
= C \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{m=i}^{\infty} \sum_{n=j}^{\infty} E \|V_{ij}\|^p \frac{1}{(mn)^{p+1}}
\]

\[
= C \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} E \|V_{ij}\|^p \sum_{m=i}^{\infty} \sum_{n=j}^{\infty} \frac{1}{(mn)^{p+1}}
\]

\[
\leq C \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{i^p j^p} = \sum_{i=1}^{\infty} \frac{1}{i^p} E \|S_{mn}\|^p \quad \text{(by (3.3))}
\]

\[
< \infty \quad \text{(by (3.1)).}
\]

So (3.2) is proved. This ends the proof of the implication \((i) \Rightarrow (ii)\).

Now, assume that (ii) holds. Let \( \{V_{mn}, m \geq 1, n \geq 1\} \) be a double array of independent mean 0 random elements in \( X \) such that (3.1) holds. In view of Proposition 2.2, it suffices to verify that

\[
\frac{S_{mn}}{mn} \to 0 \text{ a.s. as } m \lor n \to \infty. \quad (3.4)
\]

Now (3.2) holds by (3.1) and (ii) and so

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} E \|S_{mn}\|^p \frac{1}{(mn)^{p+1}} < \infty. \quad (3.5)
\]

Thus

\[
E \left\| \frac{S_{mn}}{mn} \right\|^p = \frac{1}{(mn)^p} E \|S_{mn}\|^p
\]

\[
\leq C \sum_{k=m}^{\infty} \sum_{l=n}^{\infty} \frac{1}{(kl)^{p+1}} E \|S_{mn}\|^p \quad \text{(by (3.3))}
\]

\[
\leq C \sum_{k=m}^{\infty} \sum_{l=n}^{\infty} \frac{1}{(kl)^{p+1}} E \|S_{kl}\|^p \quad \text{(by Lemma 2.1)}
\]

\[
\to 0 \text{ as } m \lor n \to \infty \quad \text{(by (3.5)).}
\]

Then by Markov’s inequality \( \frac{S_{mn}}{mn} \to 0 \text{ as } m \lor n \to \infty \) and so (3.4) holds by Proposition 2.3. The proof of the implication \((ii) \Rightarrow (i)\) is completed.

\[\square\]

**Remark 3.2.** From the proof of Theorem 3.1 we see that if (3.5) holds for some \( p \geq 1 \), we obtain \( \frac{S_{mn}}{mn} \to 0 \) in \( L_p \) as \( m \lor n \to \infty \).

This remark will be used in the proof of Theorem 3.3.
In the following theorem, we show that $S_{mn}/(mn)^{(p+1)/p} \overset{c.L}{\to} 0$ for some $p \geq 1$ implies $S_{mn}/(mn) \to 0$ a.s. as $m \vee n \to \infty$. We emphasize that we are not assuming that the Banach space is of Rademacher type $p$.

**Theorem 3.3.** Let $\{V_{mn}, m \geq 1, n \geq 1\}$ be a double array of independent random elements in a real separable Banach space. If

$$\frac{S_{mn}}{(mn)^{(p+1)/p}} \overset{c.L}{\to} 0 \text{ for some } p \geq 1,$$

then

$$\frac{S_{mn}}{mn} \to 0 \text{ a.s. as } m \vee n \to \infty.$$  

**Remark 3.4.** (i) In [16], Rosalsky, Thanh and Volodin established Theorem 3.3 for 1-dimensional case with $1 \leq p \leq 2$. The proof we presented here for the double sum version is much more complicated. As we will see in the proof that the condition $p \leq 2$ is not needed.

(ii) Recently, Son, Thang and Dung [20] proved a result on complete convergence in mean of order $p$ without assuming that the summands are independent. More precise, they proved that for arbitrary double array $\{V_{mn}, m \geq 1, n \geq 1\}$ in a real separable Banach space, the condition

$$\frac{1}{(mn)^{(p+1)/p}} \max_{k \leq m, l \leq n} \|S_{kl}\| \overset{c.L}{\to} 0 \text{ for some } 1 \leq p \leq 2$$

implies

$$\frac{1}{mn} \max_{k \leq m, l \leq n} \|S_{kl}\| \to 0 \text{ a.s. as } m \vee n \to \infty.$$  

Their result and ours are not comparable and do not imply each other, and our proof is completely different from theirs. Moreover, we will show in Example 4.2 that in our Theorem 3.3, the independence assumption cannot be weakened to the assumption that the random elements are pairwise independent.

The proof of Theorem 3.3 has several steps so we will break it up into two lemmas. These lemmas may be of independent interest. The first lemma provides a necessary and sufficient condition for SLLN $S_{mn}/(mn) \to 0$ a.s. as $m \vee n \to \infty$ when $\{V_{mn}, m \geq 1, n \geq 1\}$ is comprised of independent symmetric random elements. Lemma 3.5 is a double sum analogue of Theorem 1 of Etemadi [3].

**Lemma 3.5.** Let $\{V_{mn}, m \geq 1, n \geq 1\}$ be a double array of independent symmetric random elements in a real separable Banach space. Then

$$\frac{S_{mn}}{mn} \to 0 \text{ a.s. as } m \vee n \to \infty$$

if and only if

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn} P \left( \left\| \sum_{i=m+1}^{2m} \sum_{j=n+1}^{2n} V_{ij} \right\| > \varepsilon mn \right) < \infty \text{ for all } \varepsilon > 0.$$  

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Proof. Assume that (3.8) holds and let \( \varepsilon > 0 \) be arbitrary. It is easy to see that (3.8) implies

\[
\sum_{i=2^k+1}^{2^{k+1}} \sum_{j=2^l+1}^{2^{l+1}} V_{ij} \to 0 \text{ a.s. as } k \lor l \to \infty.
\]

By the Borel-Cantelli lemma, it implies

\[
\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} P \left( \left\| \sum_{i=2^k+1}^{2^{k+1}} \sum_{j=2^l+1}^{2^{l+1}} V_{ij} \right\| > \frac{\varepsilon}{4} 2^k 2^l \right) < \infty. \tag{3.10}
\]

Similarly, we have

\[
\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} P \left( \left\| \sum_{i=2^k+1}^{2^{k+1}} \sum_{j=2^l+1}^{2^{l+1}} V_{ij} \right\| > \frac{\varepsilon}{4} 2^k 2^l \right) < \infty, \tag{3.11}
\]

\[
\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} P \left( \left\| \sum_{i=2^k+1}^{2^{k+1}} \sum_{j=2^l+1}^{2^{l+1}} V_{ij} \right\| > \frac{\varepsilon}{4} 2^k 2^l \right) < \infty, \tag{3.12}
\]

and

\[
\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} P \left( \left\| \sum_{i=2^k+1}^{2^{k+1}} \sum_{j=2^l+1}^{2^{l+1}} V_{ij} \right\| > \frac{\varepsilon}{4} 2^k 2^l \right) < \infty. \tag{3.13}
\]

Now, we have

\[
\sum_{m=2^k+1}^{2^{k+1}} \sum_{n=2^l+1}^{2^{l+1}} \sum_{i=m+1}^{2m} \sum_{j=n+1}^{2n} V_{ij} > \varepsilon m n
\]

\[
\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{1}{mn} P \left( \left\| \sum_{i=m+1}^{2m} \sum_{j=n+1}^{2n} V_{ij} \right\| > \varepsilon m n \right)
\]

\[
= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{1}{mn} \sum_{m=2^k+1}^{2^{k+1}} \sum_{n=2^l+1}^{2^{l+1}} P \left( \left\| \sum_{i=m+1}^{2m} \sum_{j=n+1}^{2n} V_{ij} \right\| > \varepsilon m n \right)
\]

\[
\leq 2 \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{1}{2^{k+l}} \sum_{m=2^k+1}^{2^{k+1}} \sum_{n=2^l+1}^{2^{l+1}} P \left( \left\| \sum_{i=2^k+1}^{2^{k+1}} \sum_{j=2^l+1}^{2^{l+1}} V_{ij} \right\| > \varepsilon 2^k 2^l \right) \tag{by Lemma 2.6}
\]

\[
= 2 \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} P \left( \left\| \sum_{i=2^k+1}^{2^{k+1}} \sum_{j=2^l+1}^{2^{l+1}} V_{ij} \right\| > \varepsilon 2^k 2^l \right)
\]

\[
\leq 2 \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} P \left( \left\| \sum_{i=2^k+1}^{2^{k+1}} \sum_{j=2^l+1}^{2^{l+1}} V_{ij} \right\| > \frac{\varepsilon}{4} 2^k 2^l \right)
\]

\[
+ 2 \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} P \left( \left\| \sum_{i=2^k+1}^{2^{k+1}} \sum_{j=2^l+1}^{2^{l+1}} V_{ij} \right\| > \frac{\varepsilon}{4} 2^k 2^l \right)
\]

\[
+ 2 \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} P \left( \left\| \sum_{i=2^k+1}^{2^{k+1}} \sum_{j=2^l+1}^{2^{l+1}} V_{ij} \right\| > \frac{\varepsilon}{4} 2^k 2^l \right)
\]

\[
+ 2 \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} P \left( \left\| \sum_{i=2^k+1}^{2^{k+1}} \sum_{j=2^l+1}^{2^{l+1}} V_{ij} \right\| > \frac{\varepsilon}{4} 2^k 2^l \right)
\]

\[
< \infty \tag{by (3.10)-(3.13)}.
\]

The proof of the implication \((3.8) \Rightarrow (3.9)\) is thus completed.
Now, we assume (3.9) holds. Then for arbitrary $\varepsilon > 0$,

$$
\infty > \sum_{m=4}^{\infty} \sum_{n=4}^{\infty} \frac{1}{mn} P \left( \max_{i=m+1}^{2m} \max_{j=n+1}^{2n} |V_{ij}| > \varepsilon mn \right)
$$

$$
= \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{1}{mn} P \left( \max_{i=m+1}^{2m} \max_{j=n+1}^{2n} |V_{ij}| > \varepsilon mn \right)
$$

$$
= \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{1}{mn} P \left( \max_{i=m+1}^{2m} \max_{j=n+1}^{2n} |V_{ij}| > \varepsilon mn \right)
$$

$$
+ \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{1}{mn} P \left( \max_{i=m+1}^{2m} \max_{j=n+1}^{2n} |V_{ij}| > \varepsilon mn \right)
$$

$$
+ \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{1}{mn} P \left( \max_{i=m+1}^{2m} \max_{j=n+1}^{2n} |V_{ij}| > \varepsilon mn \right)
$$

$$
+ \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{1}{mn} P \left( \max_{i=m+1}^{2m} \max_{j=n+1}^{2n} |V_{ij}| > \varepsilon mn \right)
$$

$$
\geq \frac{1}{32} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} P \left( \sum_{i=2k+1}^{2^{k+2}} \sum_{j=2^{l+1}}^{2^{l+2}} |V_{ij}| > \varepsilon 2^{k+2} 2^{l+2} \right)
$$

$$
+ \frac{1}{24} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} P \left( \sum_{i=2^{l+1}+2k+1}^{2^{l+2}} \sum_{j=2^{l+1}+2k+1}^{2^{l+2}} |V_{ij}| > \varepsilon 2^{k+2} 2^{l+2} \right)
$$

$$
+ \frac{1}{24} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} P \left( \sum_{i=2^{l+1}+2k+1}^{2^{l+2}} \sum_{j=2^{l+1}+2k+1}^{2^{l+2}} |V_{ij}| > \varepsilon 2^{k+2} 2^{l+2} \right)
$$

$$
+ \frac{1}{18} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} P \left( \sum_{i=2^{l+1}+2k+1}^{2^{l+2}} \sum_{j=2^{l+1}+2k+1}^{2^{l+2}} |V_{ij}| > \varepsilon 2^{k+2} 2^{l+2} \right)
$$

(by Lemma 2.6)

$$
\geq \frac{1}{32} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} P \left( \sum_{i=2k+1}^{2^{k+2}} \sum_{j=2^{l+1}+2k+1}^{2^{l+2}} |V_{ij}| > 4\varepsilon 2^{k+2} 2^{l+2} \right)
$$

By the Borel-Cantelli lemma, it follows that

$$
\sum_{i=2^{l+1}+2k+1}^{2^{l+2}} \sum_{j=2^{l+1}+2k+1}^{2^{l+2}} |V_{ij}| \to 0 \text{ a.s. as } k \lor l \to \infty.
$$

(3.14)

Applying Lemma 2.6 (3.9) follows from (3.14).

The following lemma is similar to Lemma 3.5 but the random elements $\{V_{mn}, m \geq 1, n \geq 1\}$ are not assumed to be symmetric. It is a double sum analogue of Theorem 2 of Etemadi [3].

**Lemma 3.6.** Let $\{V_{mn}, m \geq 1, n \geq 1\}$ be a double array of independent random elements in a real separable Banach space. Then

$$
\frac{S_{mn}}{mn} \to 0 \text{ a.s. as } m \lor n \to \infty
$$

(3.15)
if and only if

$$\frac{S_{mn}}{mn} \to 0 \text{ a.s. as } m \lor n \to \infty$$  \hspace{1cm} (3.16)

and

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn} P \left( \left\| \sum_{i=m+1}^{2m} \sum_{j=n+1}^{2n} V_{ij} \right\| > \varepsilon mn \right) < \infty \text{ for all } \varepsilon > 0. \hspace{1cm} (3.17)$$

Proof. Let $$V' = \{V_{mn}', m \geq 1, n \geq 1\}$$ and $$S'_{mn}$$ be as in Lemma 2.4 and set

$$Y_{mn} = 2 \sum_{i=m+1}^{2m} \sum_{j=n+1}^{2n} V_{ij}, \ m \geq 1, n \geq 1,$$

and

$$Y_{mn}' = 2 \sum_{i=m+1}^{2m} \sum_{j=n+1}^{2n} V_{ij}', \ m \geq 1, n \geq 1.$$

Proof of the implication ((3.15) ⇒ (3.16) and (3.17)): Assume (3.15) holds and let $$\varepsilon > 0$$ be arbitrary. By using Lemma 2.4, we get

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn} \left( \sum_{i=m+1}^{2m} \sum_{j=n+1}^{2n} (V_{ij} - V_{ij}') \right) \to 0 \text{ a.s. as } m \lor n \to \infty.$$

It follows from Lemma 3.5 that

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn} \left( \sum_{i=m+1}^{2m} \sum_{j=n+1}^{2n} (V_{ij} - V_{ij}') \right) > \varepsilon 2 mn \hspace{1cm} (3.18)$$

Let $$\mu_{mn} = \text{median of } \|Y_{mn}\|$$. Then it is clear that (3.15) implies

$$\frac{\mu_{mn}}{mn} \to 0 \text{ as } m \lor n \to \infty.$$

Thus, for $$k \lor l$$ large enough,

$$\sum_{m=k}^{\infty} \sum_{n=l}^{\infty} \frac{1}{mn} P \left( \|Y_{mn}\| > \varepsilon mn \right) \leq \sum_{m=k}^{\infty} \sum_{n=l}^{\infty} \frac{1}{mn} P \left( \|Y_{mn}\| - \mu_{mn} > \frac{\varepsilon}{2 mn} \right)$$

$$\leq 2 \sum_{m=k}^{\infty} \sum_{n=l}^{\infty} \frac{1}{mn} P \left( \|Y_{mn}\| - \|Y_{mn}'\| > \frac{\varepsilon}{2 mn} \right)$$

(by the weak symmetrization inequality [5, p.134])

$$\leq 2 \sum_{m=k}^{\infty} \sum_{n=l}^{\infty} \frac{1}{mn} P \left( \|Y_{mn}' - Y_{mn}\| > \frac{\varepsilon}{2 mn} \right)$$

$$< \infty \text{ (by (3.18))}$$

thereby proving (3.17). Of course (3.15) immediately implies (3.16).

Proof of the implication ((3.16) and (3.17) ⇒ (3.15)): Assume that (3.16) and (3.17) hold. Again, let $$\varepsilon > 0$$ be arbitrary, then

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn} P(\|Y_{mn} - Y_{mn}'\| > \varepsilon mn) \leq 2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn} P(\|Y_{mn}\| > \frac{\varepsilon}{2 mn})$$

$$< \infty \text{ (by (3.17)).}$$
It thus follows from Lemma 3.5 that
\[ \frac{S_{mn}}{mn} \to 0 \text{ a.s. as } m \lor n \to \infty. \tag{3.19} \]

By applying Lemma 2.4, (3.15) follows from (3.16) and (3.19).

**Proof of Theorem 3.3.** From (3.6), we have
\[ \sum_{m=1}^\infty \sum_{n=1}^\infty E\left\| S_{mn} \right\|^p (mn)^{p+1} < \infty \tag{3.20} \]

Using Remark 3.2 we get from (3.20) that
\[ \frac{S_{mn}}{mn} \to 0 \text{ in } L^p \text{ as } m \lor n \to \infty. \tag{3.21} \]

It thus follows from (3.21) and Markov’s inequality that
\[ \frac{S_{mn}}{mn} \to 0 \text{ as } m \lor n \to \infty. \tag{3.22} \]

On the other hand,
\[ \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{1}{mn} P\left( \frac{1}{2m} \sum_{i=m+1}^{2m} \sum_{j=n+1}^{2n} V_{ij} > \varepsilon mn \right) \]
\[ \leq \sum_{m=1}^\infty \sum_{n=1}^\infty \varepsilon^p (mn)^{p+1} E\left\| \frac{1}{2m} \sum_{i=m+1}^{2m} \sum_{j=n+1}^{2n} V_{ij} \right\|^p \text{ (by Markov’s inequality)} \]
\[ = \sum_{m=1}^\infty \sum_{n=1}^\infty \varepsilon^p (mn)^{p+1} E\left( \left\| S_{(2m,2n)} - S_{(2m,n)} - S_{(m,2n)} + S_{(m,n)} \right\|^p \right) \]
\[ \leq \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{C}{(mn)^{p+1}} E\left( \left\| S_{(2m,2n)} \right\|^p + \left\| S_{(2m,n)} \right\|^p + \left\| S_{(m,2n)} \right\|^p + \left\| S_{(m,n)} \right\|^p \right) \]
\[ < \infty \text{ (by (3.20))}. \]

Combining this and (3.22), we see that the conclusion (3.7) follows from Lemma 3.6.

**4 Illustrating Examples**

By Theorem 3.1 if a real separable Banach space is not of Rademacher type \( p \) where \( 1 < p \leq 2 \), then there exists a double array of independent mean 0 random elements for which (3.11) holds but (3.12) fails. The following example, which was inspired by an example of Rosalsky and Thanh [17], exhibits such a double array of random elements in the Banach space \( \ell_1 \). This example will also demonstrate that, there exists a double array of random elements \( \{T_{mn}, m \geq 1, n \geq 1\} \) satisfying \( T_{mn} \overset{c}{\to} 0 \) and \( T_{mn} \overset{L^p}{\to} 0 \) as \( m \lor n \to \infty \), but \( T_{mn} \overset{c,L^p}{\not\to} 0 \).

**Example 4.1.** Let \( 1 < p \leq 2 \) and consider the Banach space \( \ell_1 \) (which is not of Rademacher type \( p \)). Let \( \psi^{(k)} \) denote the element of \( \ell_1 \) having 1 in its \( k \)th position and 0 elsewhere, \( k \geq 1 \). Let \( \varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) be a
one-to-one and onto mapping. Let \( \{V_{mn}, m \geq 1, n \geq 1\} \) be a double array of independent random elements in \( \ell_1 \) by requiring the \( \{V_{mn}, m \geq 1, n \geq 1\} \) to be independent with

\[
P \left( V_{mn} = v^{(\varphi(m, n))} \right) = P \left( V_{mn} = -v^{(\varphi(m, n))} \right) = \frac{1}{2}, \quad m \geq 1, n \geq 1.
\]

We have

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} E \left\| V_{mn} \right\|^p \left( mn \right)^p = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{(mn)^p} < \infty.
\]

Hence (3.1) holds but

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} E \left\| S_{mn} \right\|^p \left( mn \right)^{(p+1)/p} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn} = \infty
\]

and so (3.2) fails. Moreover, since for all \( \varepsilon > 0 \) and all large \( m \lor n \)

\[
P \left( \frac{\left\| S_{mn} \right\|}{(mn)^{(p+1)/p}} > \varepsilon \right) = P \left( \frac{1}{(mn)^{1/p}} > \varepsilon \right) = 0,
\]

it follows that

\[
\frac{S_{mn}}{(mn)^{(p+1)/p}} \xrightarrow{c} 0.
\]

Now by the computation in (4.1), we have

\[
E \left\| \frac{S_{mn}}{(mn)^{(p+1)/p}} \right\|^p = \frac{1}{mn} \rightarrow 0 \quad \text{as} \quad m \lor n \rightarrow \infty
\]

and so

\[
\frac{S_{mn}}{(mn)^{(p+1)/p}} \xrightarrow{L_p} 0 \quad \text{as} \quad m \lor n \rightarrow \infty.
\]

Consequently

\[
\frac{S_{mn}}{(mn)^{(p+1)/p}} \xrightarrow{c} 0 \quad \text{and} \quad \frac{S_{mn}}{(mn)^{(p+1)/p}} \xrightarrow{L_p} 0 \quad \text{as} \quad m \lor n \rightarrow \infty
\]

but

\[
\frac{S_{mn}}{(mn)^{(p+1)/p}} \xrightarrow{c,L_p} 0.
\]

The following example shows that in general, the independence assumption in Theorem 3.3 cannot be weakened to the assumption that the summands are pairwise independent. The example is based on Theorem 3 in Csörge, Tandori and Totik [2].

**Example 4.2.** Csörge, Tandori and Totik [2, Theorem 3] constructed a sequence of pairwise independent real-valued random variables \( \{X_m, m \geq 1\} \) satisfying \( EX_m = 0, EX_m^2 < \infty \), and

\[
\sum_{m=2}^{\infty} \frac{EX_m^2 (\log \log m)^{1-\varepsilon}}{m^2} < \infty, \quad \varepsilon > 0,
\]

\[
P \left( \limsup_{m \rightarrow \infty} \frac{\sum_{i=1}^{m} X_i}{m} = \infty \right) > 0.
\]
For $m \geq 1$ we set $V_{mn} = X_m$ if $n = 1$ and $V_{mn} = 0$ if $n \geq 2$. In Theorem 3.3 let $p = 2$, then
\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} E \left\| \frac{S_{mn}}{(mn)^{(p+1)/p}} \right\|^p = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sum_{i=1}^{m} \sum_{j=1}^{n} EV_{ij}^2}{(mn)^3} = \sum_{n=1}^{\infty} \frac{\sum_{i=1}^{n} EX_{i}^2}{m^3} \leq C \sum_{i=1}^{\infty} \frac{EX_{i}^2}{i^2} < \infty \text{ (by (4.2)).}
\]
So (3.6) holds. However, it follows from (4.3) that (3.7) fails.

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