DEFINABILITY IN THE SUBSTRUCTURE ORDERING OF FINITE DIRECTED GRAPHS

ÁDÁM KUNOS

Abstract. We deal with first-order definability in the substructure ordering \((\mathcal{D}; \sqsubseteq)\) of finite directed graphs. In two papers, the author has already investigated the first-order language of the embeddability ordering \((\mathcal{D}; \leq)\). The latter has turned out to be quite strong, e.g., it has been shown that, modulo edge-reversing (on the whole graphs), it can express the full second-order language of directed graphs. Now we show that, with finitely many directed graphs added as constants, the first order language of \((\mathcal{D}; \sqsubseteq)\) can express that of \((\mathcal{D}; \leq)\).

The limits of the expressive power of such languages are intimately related to the automorphism groups of the orderings. Previously, analogue investigations have found the concerning automorphism groups to be quite trivial, e.g., the automorphism group of \((\mathcal{D}; \leq)\) is isomorphic to \(\mathbb{Z}_2\). Here, unprecedentedly, this is not the case. Even though we conjecture that the automorphism group is isomorphic to \((\mathbb{Z}_4 \times \alpha \mathbb{Z}_2) \times \mathbb{Z}_2^2\), with a particular \(\alpha\) in the semidirect product, we only prove it is finite.

1. Introduction and formulation of our main theorems

In 2009–2010 J. Ježek and R. McKenzie published a series of papers \([1–4]\) in which they have examined (among other things) the first-order definability in the substructure orderings of finite mathematical structures with a given type, and determined the automorphism group of these orderings. They considered finite semilattices \([1]\), ordered sets \([4]\), distributive lattices \([2]\) and lattices \([3]\). Similar investigations \([5–9]\) have emerged since. The current paper is one of such, connected strongly to the author’s papers \([5, 6]\) that dealt with the embeddability ordering of finite directed graphs. Now, instead of embeddability, we are examining the substructure ordering of finite directed graphs.

Let us consider a nonempty set \(V\) and a binary relation \(E \subseteq V^2\). We call the pair \(G = (V, E)\) a directed graph or just digraph. Let \(\mathcal{D}\) denote the set of isomorphism types of finite digraphs. The elements of \(V = V(G)\) and \(E = E(G)\) are called the vertices and edges of \(G\), respectively. A digraph \(G\) is said to be embeddable into \(G'\), and we write \(G \leq G'\), if there exists an injective homomorphism \(\varphi : G \rightarrow G'\), i.e. an injective map for which \((v_1, v_2) \in E(G)\) implies \((\varphi(v_1), \varphi(v_2)) \in E(G')\). A digraph \(G\) is a substructure of \(G'\), and we write \(G \sqsubseteq G'\), if it is isomorphic to an induced substructure (on some subset of the vertices) of \(G'\). Every substructure is embeddable but the converse is not true. The names of these two concepts often mix both orally and on paper when it is clear from the context which notion

This research was supported by the UNKP-17-3 New National Excellence Program of the Ministry of Human Capacities.
we are using the whole time. In the present paper, however, we must be very cautious as both concepts are used alternately throughout the whole paper. It is easy to see that both $\leq$ and $\sqsubseteq$ are partial orders on $D$. Both partially ordered sets are naturally graded. The digraph $G$ is on the $n$th level of $(D; \leq)$ or $(D; \sqsubseteq)$ if $|V(G)| + |E(G)| = n$ or $|V(G)| = n$, respectively. See Figures 1 and 2 for the bottoms of the Hasse diagrams of the two partial orders.

**Figure 1.** The bottom part of the Hasse diagram of $(D; \leq)$.

**Figure 2.** The bottom part of the Hasse diagram of $(D; \sqsubseteq)$.

Let $(A; \leq)$ be an arbitrary poset. An $n$-ary relation $R$ is said to be (first-order) definable in $(A; \leq)$ if there exists a first-order formula $\Psi(x_1, x_2, \ldots, x_n)$ with free variables $x_1, x_2, \ldots, x_n$ in the language of partially ordered sets such that for any $a_1, a_2, \ldots, a_n \in A$, $\Psi(a_1, a_2, \ldots, a_n)$ holds in $(A; \leq)$ if and only if $(a_1, a_2, \ldots, a_n) \in R$. A subset of $A$ is definable if it is definable as a unary relation. An element $a \in A$ is said to be definable if the set $\{a\}$ is definable.

Our main result is the following.
Theorem 1. There exists a finite set of finite directed graphs \( \{C_1, \ldots, C_k\} \) such that the binary embeddability relation,
\[
\{(G, G') : G \leq G'\},
\]
is definable in the first-order language of \((D; \sqsubseteq, C_1, \ldots, C_k)\). Consequently, every relation definable in the first-order language of \((D; \leq)\) is definable in that of \((D; \sqsubseteq, C_1, \ldots, C_k)\).

In itself, this theorem is quite weightless, what fills it with content is that we already know \([5, 6]\) that the first-order language of \((D; \leq)\) is surprisingly strong. The paper \([6]\) has two parts. The first deals with definability in \((D; \leq)\), the second determines the automorphism group of \((D; \leq)\) (building on the first part, of course). The paper \([5]\) extends the main result of the first part of \([6]\), hence if one is only interested in definability, it is enough to read \([5]\). The main result there \([5, \text{Theorem 5}]\) is some kind of a characterization of the first-order definable relations in \((D; \leq)\). To even state the result precisely, there is a 3-page-long preparation which we don’t repeat here. We only provide some corollaries, demonstrating the power of definability in \((D; \leq)\). With Theorem \([5]\) these corollaries convert immediately to statements for the first-order language of \((D; \sqsubseteq, C_1, \ldots, C_k)\). As this paper is about the substructure ordering, we formulate these versions, rather than the versions talking about \((D; \leq)\).

Corollary 2. There exists a finite set of finite directed graphs \( \{C_1, \ldots, C_k\} \) such that in the first-order language of \((D; \sqsubseteq, C_1, \ldots, C_k)\)

- every single digraph \(G\) is definable,
- the set of weakly connected digraphs is definable, moreover,
- the full second-order language of digraphs becomes available.

Again, for the full scope of Theorem \([1]\), see \([3, \text{Section 2}]\).

The papers \([1-4, 6, 9]\), beyond dealing with definability, determined the automorphism groups of the orderings in question. In every case, the automorphisms came naturally and the automorphism groups were either trivial or isomorphic to \(\mathbb{Z}_2\). Despite all expectations, the partially ordered set \((D, \sqsubseteq)\) stands out in that aspect. There are automorphisms far from trivial. Unfortunately, we are not able to determine the automorphism group, we can only prove it is finite.

Theorem 3. The automorphism group of \((D, \sqsubseteq)\) is finite.

Even though we can not prove it, we formulate a conjecture for the automorphism group.

In Section 2, we prove Theorem \([3]\) and tell our conjecture on the automorphism group in detail. Section 3 contains the proof of Theorem \([1]\) without some technicalities. In Section 4, the reader finds the technicalities skipped in Section 3.

2. ON THE AUTOMORPHISM GROUP OF \((D, \sqsubseteq)\)

First, we prove Theorem \([3]\) using Theorem \([1]\).

Proof of Theorem \([3]\). It is clear that the orbits of the automorphism group are finite as an automorphism can only move a digraph inside its level in \((D, \sqsubseteq)\). Thus, it is enough to present a finite set of digraphs such that the only automorphism fixing them all is the identity. We claim that \(\{C_1, \ldots, C_k\}\) of Theorem \([1]\) suffices. Let \(\varphi\) be an automorphism that fixes all \(C_i\). Let \(G \in D\) be arbitrary. We need to show
that $\varphi(G) = G$. We know from Corollary 2 that there exists a formula $\phi_G(x)$ with one free variable, that defines $G$ in first order language of $(\mathcal{D}, \sqsubseteq, C_1, \ldots, C_k)$. If we change all occurrences of $C_i$ to $\varphi(C_i)$ in $\phi_G(x)$, then we get a formula $\phi_{\varphi(G)}(x)$ defining $\varphi(G)$. For $\varphi$ fixes all $C_i$s, $\phi_G(x) = \phi_{\varphi(G)}(x)$, implying $G = \varphi(G)$. □

In the remaining part of the section, we present the automorphisms that we know of. Here, no claim is proven rigorously, they are all rather conjectures. Our intention is just to offer some insight on how the author sees the automorphism group at the moment. All pairs of vertices fall into one of the following three categories in directed graphs, based on the number of loops they have. A pair of vertices is loop-free if there is no loop in it, it is loop-full if both vertices have loops, and it is mixed if one vertex does have a loop, while the other does not. Similarly, we categorize with regard to the number of non-loop edges. A pair of vertices is disconnected if there is no edge between the two vertices, strongly connected if there are edges in both directions, and weakly connected if there is edge only in one direction. We are ready to formulate some automorphisms. We do so by telling how to get $\varphi_i(G)$ from $G$.

$\varphi_1$: Put loops on loop-free vertices, and clear the loops from loop-full ones.

$\varphi_2$: On pairs of vertices that are weakly connected and loop-free, change the direction of the edges.

$\varphi_3$: On pairs of vertices that are weakly connected and loop-full, change the direction of the edges.

$\varphi_4$: On pairs of vertices that are weakly connected and mixed, change the direction of the edges.

$\varphi_5$: On loop-free pairs, change disconnected pairs to strongly connected ones and vica versa.

$\varphi_6$: On loop-full pairs, change disconnected pairs to strongly connected ones and vica versa.

$\varphi_7$: On mixed pairs, change disconnected pairs to strongly connected ones and vica versa (with the positions of the loops staying the same).

Obviously, arbitrary compositions of these are again automorphisms. All the listed automorphisms are of order two. Unfortunately, they do not commute, e.g., $\varphi_1 \varphi_2 \neq \varphi_2 \varphi_1$. Let $\langle \rangle$ stand for subgroup generation. For the subgroup $S := \langle \varphi_i : 1 \leq i \leq 7 \rangle$ of the automorphism group, the seven-element generator set, it is given by, is not even minimal as, for example, $\varphi_1 \varphi_2 \varphi_1 = \varphi_3$. The automorphism $\varphi_1 \varphi_2$ is of order 4. We have seen now that the automorphism group is far from $\mathbb{Z}_2^7$, which may be the first guess after seeing the seven automorphisms listed above. Still, we think that the automorphism group has $128(= 2^7)$ elements. It seems that $S$ is the internal direct product of its subgroups $\langle \varphi_1, \varphi_2, \varphi_3, \varphi_5, \varphi_6 \rangle$ and $\langle \varphi_4, \varphi_7 \rangle$. Furthermore, the factor $\langle \varphi_1, \varphi_2, \varphi_3, \varphi_5, \varphi_6 \rangle$ is the internal semidirect product of $\langle \varphi_1 \rangle$ acting on $\langle \varphi_2, \varphi_3, \varphi_5, \varphi_6 \rangle$:

$$\langle \varphi_1, \varphi_2, \varphi_3, \varphi_5, \varphi_6 \rangle = \langle \varphi_2, \varphi_3, \varphi_5, \varphi_6 \rangle \rtimes \langle \varphi_1 \rangle,$$

and $\langle \varphi_2, \varphi_3, \varphi_5, \varphi_6 \rangle$ factors into the internal direct product

$$\langle \varphi_2, \varphi_3, \varphi_5, \varphi_6 \rangle = \langle \varphi_2 \rangle \times \langle \varphi_3 \rangle \times \langle \varphi_5 \rangle \times \langle \varphi_6 \rangle.$$
These observations all need a proper checking, but they give rise to the conjecture that $S$ is isomorphic to

$$(\mathbb{Z}_4^2 \times \alpha \mathbb{Z}_2) \times \mathbb{Z}_2^2,$$

where $(a, b, c, d) \xrightarrow{\alpha(1)} (b, a, d, c)$.

It would be nice to see a digraph with a 128-element orbit under the action of $S$. We nominate the digraph of Fig. 3 for this.

Even though we cannot prove that there are no more automorphisms beyond the ones in $S$, we conjecture so.

Figure 3.

3. The proof of Theorem 1 without some technicalities

As long and technical as it may seem, the whole proof of Theorem 1 is based on a simple idea, which we outline here. We get substructures of a directed graph by leaving out vertices, while, to get embeddable digraphs, we can leave out vertices and edges both. We want to define the latter, so we need to be able to leave out edges somehow. Our main idea is the following. In a digraph $G$, if there is an edge $(u, v) \in E(G)$, then we add a vertex and two edges to “support” the edge $(u, v)$. Namely, we add $w$ to the set of vertices, and the edges $(u, w)$ and $(w, v)$ to the set of edges. After the addition, we say that the edge $(u, v)$ is “supported”. The idea is that the supportedness of an edge can be terminated by leaving out a vertex, in the previous example $w$, what we can do by taking substructures. Roughly, what we should do is: support all edges, take a substructure, and in one more step, leave only the supported edges in. Of course, there seems to be many problems with this (if told in such a simplified way). Firstly, how can we distinguish between the supporting vertices and the original ones? This appears to be an essential part of the plan. Secondly, the plan ended with “leave only the supported edges in” which just looks running into the original problem again: we cannot leave edges out. Even though the plan seems flawed for these reasons, it is manageable. The whole section is no more than building the apparatus and carrying it out.

Definition 4. In this section, we use three particular automorphism:

- the loop-exchange automorphism, denoted by $l$, which is $\varphi_1$ (of the previous section),
- the edge-reverse (transposition) automorphism, denoted by $t$, which is $\varphi_2\varphi_3\varphi_4$, and
- the complement automorphism, denoted by $c$, which is $\prod_{i=1}^{7}\varphi_i$.

The edge-reverse automorphism just reverses all edges in a digraph, while the complement automorphism replaces $E(G)$ with $V(G)^2 \setminus E(G)$.

Some basic definitions follow.
Definition 5. For digraphs $G, G' \in D$, let $G \cup G'$ denote their disjoint union, as usual.

Definition 6. Let $E_n (n = 1, 2, \ldots)$ denote the “empty” digraph with $n$ vertices and $F_n (n = 1, 2, \ldots)$ denote the “full” digraph with $n$ vertices:

$V(E_n) = \{v_1, v_2, \ldots, v_n\}$, $E(E_n) = \emptyset$,

$V(F_n) = \{v_1, v_2, \ldots, v_n\}$, $E(F_n) = V(F_n)^2$.

Definition 7. Let $I_n (n = 1, 2, \ldots)$, $O_n (n = 3, 4, \ldots)$, and $L_n (n = 1, 2, \ldots)$ be the following (Fig. 4.) digraphs:

$V(I_n) = V(O_n) = V(L_n) = \{v_1, v_2, \ldots, v_n\}$,

$E(I_n) = \{(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n)\}$,

$E(O_n) = \{(v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n), (v_n, v_1)\}$,

$E(L_n) = \{(v_1, v_1), (v_2, v_2), \ldots, (v_n, v_n)\}$.

The digraphs $I_n$ are called lines, and the digraphs $O_n$ are called circles.

Note $E_1 = I_1$.

![Diagram of I_n, O_n, and L_n](image)

Figure 4.

Definition 8. A directed graph is called an IO-graph if it satisfies the following conditions. The only one-element substructure of it is $E_1$. If $X$ is a two-element substructure then it is either $E_2$ or $I_2$. If $X$ is a three-element substructure then $X$ is $E_3$, or $I_2 \cup E_1$, or $I_3$, or $O_3$. Let the set of IO-graphs be denoted by $IO$.

Lemma 9. The set $IO$ is definable.

Proof. Observe that the set $IO$ is already given by a first-order definition, using the one, two, and three element digraphs as constants. □

Observe that the set $IO$ is closed under taking substructures. The following lemma motivates our notation $IO$.

Lemma 10. A directed graph is an IO-graph if and only if it is a disjoint union of lines and/or circles.

Proof. Straightforward induction on the number of vertices suffices, using the closedness mentioned prior to the lemma. □

Lemma 11. The set $\{O_n : n \geq 3\}$ is definable.
Proof. It is clear that all elements of the set are $IO$-graphs, we just need to choose which. It is easy to see that, in $IO$, those that have a unique lower-cover (within $IO$) are:

$$G \cup \cdots \cup G,$$

where $G \in \{E_1, I_2\} \cup \{O_n : n \geq 3\}$, for $k \geq 1$ except when $X = E_1$, then $k > 1$. In this set, the desired digraphs are exactly those that are minimal (in this particular set) and have $I_3$ or $O_3$ as a substructure.

**Definition 12.** A digraph is called loop-full if all vertices have loops on them, and loop-free if none. The loop-full part of a digraph is the maximal loop-full substructure of it, and the loop-free part is the maximal loop-free substructure.

**Lemma 13.** The relation

$$\{(G, F, G \cup F) : G, F \in \mathcal{D}, G \text{ is loop-full and } F \text{ is loop-free}\}$$

is definable.

**Proof.** The relation consists of those triples $(X, Y, Z)$ for which

- $X$ is the loop-full part of $Z$,
- $Y$ is the loop-free part of $Z$, and
- there is no two element substructure of $Z$ that consists exactly one loop and has a non-loop edge in it.

□

**Definition 14.** Let $L \rightarrow$ denote the digraph with

$$V(L \rightarrow) = \{v_1, v_2\}, \text{ and } E(G) = \{(v_1, v_1), (v_1, v_2)\}.$$

**Definition 15.** Let $G$ be a loop-full digraph with $V(G) = \{v_1, \ldots, v_n\}$. Then $l(G)$ is loop-free. Let the set of its vertices be $l(G) = \{v'_1, \ldots, v'_n\}$ with

$$\text{for } i \neq j : (v'_i, v'_j) \in E(l(G)) \iff (v_i, v_j) \in E(G).$$

Let $G \rightarrow l(G)$ denote the digraph for which

$$V(G \rightarrow l(G)) = V(G) \cup V(l(G)), \text{ and }$$

$$E(G \rightarrow l(G)) = E(G) \cup E(l(G)) \cup \{(v_i, v'_i) : 1 \leq i \leq n\}.$$  

**Lemma 16.** The relation

$$\{(G, l(G), G \rightarrow l(G)) : G \in \mathcal{D}, G \text{ is loop-full}\}$$

is definable.

**Proof.** Let us consider the triples $(X, Y, Z)$ for which

- $X$ is the loop-full part of $Z$, and $Y$ is the loop-free part of $Z$,
- $X \cup E_1 \not\subseteq Z$, and $Y \cup L_1 \not\subseteq Z$ (both are definable by Lemma 13),
- on two points, the only substructure having exactly one loop and at least one non-loop edge is $L \rightarrow$, and
- no digraph of the first two pictures of Fig. 5 is a substructure. We consider the dashed edges possibilities, either we draw them (individually) or not. In this way, there are 6 (isomorphism types) encoded into the first two pictures of Fig. 5. We exclude them all.
Now we have ensured that the edges $L \rightarrow$ constitute a bijection between the vertices of $X$ and $Y$ in $Z$. It only remains to force this bijection to be edge and non-edge preserving as well. This can be done by requiring the additional the property

- Consider the third picture of Figure 5 as before, the dashed edges are possibilities. We forbid those from being substructures in which the dashed edges are not symmetrically drawn on the two (loop-full and loop-free) sides.

\[\square\]

![Figure 5.](image)

We are going to need some basic arithmetic later. We define addition in the following lemma.

**Lemma 17.** The following relation is definable:

\[\{(E_n, E_m, E_{n+m}) : n, m \geq 1\}\].

**Proof.** The set \{E_n\} is definable as it consists of $E_1$ plus those digraphs which have only $E_2$ as a two-element substructure. $E_n \cup (L_m \rightarrow E_m)$ is the digraph $X$ for which

- $E_n \cup L_m \sqsubseteq X$ (using Lemma 13),
- $L_m \rightarrow E_m \sqsubseteq X$ (using Lemma 10),
- the second digraph of Fig. 5 without the dashed edges, is not a substructure,
- $E_{n+1} \cup L_m \not\sqsubseteq X$ (E_{n+1} is just the cover of $E_n$ in \{E_n\}),
- on two points, the only subgraph having a non-loop edge is $L \rightarrow$,
- the maximal loop-full subgraph of $X$ is $L_m$, and
- the maximal loop-free subgraph of $X$ is of the form $E_i$.

The $E_i$ of the last condition is $E_{n+m}$. \[\square\]

**Lemma 18.** The following relation is definable:

\[(1) \quad \{ (G, F) : G \text{ and } F \text{ have the same number of vertices} \}\].

**Proof.** We “determine” the number of vertices for the loop-full and the loop-free parts of the graphs separately and add them using Lemma 17. Let $G_1$ denote the loop-full part of $G$, and $G_2$ denote the loop-free part. Let $X$ denote the digraph with the following properties:

- The loop-full part of $X$ is $G_1$, and the loop-free part is $E_i$ for some $i$.
- On two points, the only substructure having exactly one loop and at least one non-loop edge is $L \rightarrow$.
- $G_1 \cup E_1 \not\sqsubseteq X$, and $E_i \cup L_1 \not\sqsubseteq X$. 

\[\square\]
• Just as in the proof of Lemma 16, no digraph of the 6 digraphs of the first two pictures of Fig. 5 is a substructure. (No matter, we wouldn’t even need all 6 in this case.)

Observe that in $X$, the edges $L \rightarrow$ constitute a bijection between $G_1$ and $E_i$, consequently $i$ in the first condition is $|V(G_1)|$.

Now we proceed analogously for the loop-free part, $G_2$. We do not write all the conditions down again, as they are just the ones above converted with the automorphism $l$. This way, we get $L_j$ with $j = |V(G_2)|$. To conclude, we use the relation of Lemma 16 to get $E_j$ and Lemma 17 to obtain the desired $E_{i+j}$, marking the number of vertices of $G$.

Finally, $(G, F) \in \mathbb{H}$ holds if and only if, by doing the same, we get the same $E_{i+j'}$ marking the number of vertices. □

We define some more arithmetic in the following lemma, namely multiplication.

Lemma 19. The following relation is definable:
$$\{(E_n, E_m, E_{nm}) : n, m \geq 1\}.$$  

Proof. The relation $\{(E_i, F_i) : i = 1, 2, \ldots\}$ is definable as, beyond $(E_1, F_1)$, for $i > 1$, $F_i$ is the only digraph having the same vertices as $E_i$ that has only $F_2$ as a two element substructure. Let $X$ be a digraph that is maximal with the following properties:

1. $E_1 \not\subseteq X$ to ensure that the relation $E(X)$ is reflexive.
2. $l(I_2) \not\subseteq X$ to ensure that the relation $E(X)$ is symmetric.
3. The digraph of Fig. 6 is not a substructure of $X$ to ensure that the relation $E(X)$ is transitive.
4. $L_n$ is the maximal $L_i$ subgraph.
5. $F_m$ is the maximal $F_i$ subgraph.

The conditions 1-3 force $E(X)$ to be an equivalence. Condition 4 tells the equivalence has at most $n$ classes and condition 5 requires the classes to have at most $m$ elements. It is easy to see that such an equivalence relation has a base set of at most $nm$ elements, hence $|V(X)| = nm$. Thus, we can finish with Lemma 18. □

**Figure 6.**

Lemma 20. Disjoint union of IO graphs is definable, i.e. the following relation is definable:
$$\{(G_1, G_2, G_1 \cup G_2) : G_1, G_2 \in IO\}.$$  

Proof. Using $G_1$ and $G_2$, we want to define

(2) $$G_1 \cup (l(G_2) \rightarrow G_2),$$

whose loop-free part is the sought $G_1 \cup G_2$. For this, let $X$ satisfy the following conditions.
\begin{itemize}
\item $|V(X)| = |V(G_1)| + 2|V(G_2)|$ (using Lemmas 18 and 17).
\item $G_1 \cup l(G_2) \subseteq X$.
\item $l(G_2) \rightarrow G_2 \subseteq X$.
\end{itemize}

It is easy to see that these three conditions ensure that (2) is embeddable (not substructure!) into $X$: there can be edges between the subgraphs $G_1$ and $G_2$ which we need to exclude. If there is an edge from $G_2$ to $G_1$ (in this particular direction), then the first graph of Fig. 7 is a substructure, without the dashed edges. Analogously, if an edge goes from $G_1$ to $G_2$, then the second digraph of Fig. 7 is a substructure, without the dashed edges. Thus we need to exclude these two subgraphs. Let $Y$ satisfy the following conditions.
\begin{itemize}
\item $|V(Y)| = |V(G_2)| + 2$, and $Y \supseteq l(G_2)$.
\item $I_2$ and $L_\rightarrow$ are substructures of $Y$.
\item The digraph of Fig. 8 is not a substructure of $Y$.
\end{itemize}

These three conditions do not define the two digraphs of Fig. 7 without the dashed edges, they rather define the set of those with the dashed edges meant as possibilities, as usual. However none of the dashed edges can actually appear in our $X$ so by excluding all such, we do not do more than by excluding only the two without the dashed edges. Finally, (2) is the loop-free part of $X$.\hfill $\square$

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{Figure7.png}
\caption{Figure 7.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{Figure8.png}
\caption{Figure 8.}
\end{figure}

**Lemma 21.** The following set is definable.
\[
\{ G : G \text{ is a disjoint union of circles of different sizes} \}.
\]

*Proof.* The set of digraphs that are disjoint unions of circles contains those $IO$ graphs that have unique upper-covers (in the set IO). In this set, the digraphs of the form $O_i \cup O_i$ are those that have a unique circle substructure $O_i$ and have twice as many vertices as $O_i$. We have defined two sets of digraphs, the set of the lemma is just the set of those digraphs of the first set that have no substructures from the second.\hfill $\square$
Lemma 22. The following relation is definable.

\[(3) \quad \{(O^*, G \cup O^*) : G \in D \text{ and } O^* \text{ is a disjoint union of } |V(G)| \text{-many circles of different sizes such that the smallest has at least } |V(G)| + 1 \text{ vertices}\} \]

Proof. First, we define a relation counting the number of circles in \(O^*\), actually we formulate it without the restriction on the sizes of the circles:

\[(4) \quad \{(E_i, O) : O \text{ is a disjoint union of } i \text{ circles}\}. \]

The set of \(O\)'s of this relation was defined in the first sentence of the proof of Lemma 21. Let \(O'\) denote such a substructure of \(O\) that has no circle in it and has a maximal number of vertices with this property. Then \(i + |V(O')| = |V(O)|\) holds for the \(i\) of (4), thus we can conclude with the addition relation defined earlier.

Let \(O^*\) be an element of the set defined in Lemma 21 and \(i\) be the number of its circles. Let \(X\) satisfy:

- \(|V(X)| = |V(O^*)| + i\).
- The smallest circle in \(O^*\) has at least \(i + 1\) vertices.
- \(O^* \subseteq X\).
- \(X\) does not have a substructure \(Y\) for which
  - \(|V(Y)| = |V(O^*)| + 1\), and \(Y \supseteq O^*\),
  - \(Y\) is loop-free, and
  - \(Y\) is not an \(IO\)-graph.
- \(X\) does not have a substructure \(Y\) for which
  - \(|V(Y)| = |V(O^*)| + 1\), and \(Y \supseteq O^*\),
  - \(Y\) has a loop in it, and
  - \(Y\) has one of \(L_{\rightarrow}\) or \(t(L_{\rightarrow})\) or \(c(L_1 \cup E_1)\) as a substructure.

With these properties, \(X\) is of the required form \(G \cup O^*\).

Remark 1. Let us remark here that for the smoothest continuation of the proof, we should have had \((G, G \cup O^*)\) instead of \((O^*, G \cup O^*)\) (with the same assumptions) in (4). The definability of this, however, seems to be out of reach (at least for the author) at this point. That is why we proceed in the following, somewhat inelegant, way.

Lemma 23. There exists a definable relation \(R\) for which

\[(5) \quad \{(G, O^*, G \cup O^*) : (O^*, G \cup O^*) \in (4) \} \subseteq R \subseteq \{(G, O^*, G_+ \cup O^*) : (O^*, G_+ \cup O^*) \in (4), |V(G)| = |V(G_+)|, G \leq G_+\}. \]

Observe that the last condition in the formula, \(G \leq G_+\), has embeddability (not substructureness) in it.

Proof. We define a sufficient \(R\) as a set of triples \((G, O^*, X)\) for which the following hold.

1. \((O^*, X) \in (4)\).
2. \(|V(X)| = |V(O^*)| + |V(G)|\).
3. \(G \subseteq X\).
4. Let \(G^{\text{max}}_{IO}\) be an \(IO\)-substructure of \(G\) that has a maximal number of vertices. Note that this implies \(\sqsubseteq\)-maximality as well. We require \(G^{\text{max}}_{IO} \cup O^* \subseteq X\) (with Lemma 20).
Lemma 25. \[ \text{with } V(G) = G_{IO} \text{, and let } G' \text{ denote “the rest” } (G = G_{IO} \cup G') \text{. At first glance, it might look like if condition } 3 \text{ was enough to force } X = G \cup O^*. \text{ Unfortunately, this is not the case though, as condition } 3 \text{ is not able to force } G_{IO} \text{ outside } O^*, \text{ because } G_{IO} \subseteq O^* \text{ is possible. On the other hand, } G' \cup O^* \subseteq X \text{ is ensured by condition } 3, \text{ as } O^* \text{ can only have } IO\text{-graph substructures. It is not hard to see that the last condition makes up for the deficiency we just saw, i.e. it “forces } G_{IO} \text{ out of } O^*”. \text{ However, } X = G \cup O^* \text{ is still not necessary as there can be “unwanted” edges between } G_{IO} \text{ and } G' \text{ in } X, \text{ but the right-side containment of } \Box \text{ lets this happen.} \]

Some technical tools follow. We introduce digraphs that we denote using the symbol $\sigma$. The motivation is the shape of the digraphs, as usual. Note, that the same notations were used in the papers [5, 6] in a slightly different way.

**Definition 24.** Let $V(O_n) = \{v_1, \ldots, v_n\}$ and let us define two digraphs with

\[ V(\sigma_n) := V(O_n) \cup \{u_1, u_2\}, \quad E(\sigma_n) := E(O_n) \cup \{(v_1, u_1), (u_1, u_2)\}, \quad \text{and} \]

\[ V(\sigma^L_n) := V(\sigma_n), \quad E(\sigma^L_n) := E(\sigma_n) \cup \{(u_2, u_2)\}. \]

Now let $m$ be a different positive integer from $n$ and define $\sigma_m$ and $\sigma^L_m$ analogously with $V(\sigma'_m) = V(\sigma'_m) = \{v'_1, \ldots, v'_m, u'_1, u'_2\}$.

Now we are going to deal with pairs of the digraphs just defined, which leaves us $4 = 2 \times 2$ cases with respect to the presence of the loops. To avoid the tiresomeness of listing all 4 possibilities all the time, we resort to the following notation. We say, let $(\Box, \forall) \in \{\emptyset, L\}^2$, and for example, in the case $(\Box, \forall) = (L, \emptyset)$, we mean $(\sigma^L_n, \sigma'_m)$ by $(\sigma^\Box_n, \sigma^\forall_m)$, naturally.

Let $(\Box, \forall) \in \{\emptyset, L\}^2$. We introduce two more types of digraphs with

\[ V(\sigma_n \rightarrow \sigma'_m) := V(\sigma_n) \cup V(\sigma'_m), \quad E(\sigma_n \rightarrow \sigma'_m) := E(\sigma_n) \cup E(\sigma'_m) \cup \{(u_2, u'_2)\}, \quad \text{and} \]

\[ V(\sigma_n \leftarrow \sigma'_m) := V(\sigma_n) \cup V(\sigma'_m), \quad E(\sigma_n \leftarrow \sigma'_m) := E(\sigma_n) \leftarrow E(\sigma'_m) \cup \{(u'_2, u_2)\}. \]

**Lemma 25.** The following relation is definable for all $(\Box, \forall) \in \{\emptyset, L\}^2$.

\[ \{(E_i, E_j, \sigma^\Box_i \rightarrow \sigma^\forall_j, \sigma^\Box_i \leftarrow \sigma^\forall_j, \sigma^\Box_i \rightarrow \sigma^\forall_j, \sigma^\Box_i \leftarrow \sigma^\forall_j) : i, j > 3, i \neq j\} \]

The proof is put in the last section for its technical nature.

The following definition is not a technicality any more as it is a construction of great importance in the remaining half of the proof.

**Definition 26.** Let $G$ be a digraph on $n$ vertices with $V(G) = \{v_1, \ldots, v_n\}$, and let $(O^*, G \cup O^*) \in \mathbb{B}$ with $V(O^*) = \{u_i : 1 \leq j \leq n, 1 \leq i \leq i_j\}$ such that the $m$th circle $O_{k_m}$ of $O^*$ consists of the vertices $\{u_i^m : 1 \leq i \leq i_m\}$. Let $C(O^*) = \{O_{k_1}, \ldots, O_{k_n}\}$ denote the set of the circles of $O^*$ and let $\alpha : C(O^*) \rightarrow V(G)$ be a bijective map. We introduce the notation $G \obar{\alpha} O^*$ for the digraph with

\[ V(G \obar{\alpha} O^*) = V(G \cup O^*) \cup \{w_1, \ldots, w_n\}, \quad \text{and} \]

\[ E(G \obar{\alpha} O^*) = E(G \cup O^*) \cup \{(u_i, w_j) : 1 \leq j \leq n\} \cup \{(w_j, \alpha(O_{k_j})) : 1 \leq j \leq n\}. \]

**Lemma 27.** The following relation is definable.

\[ \{(O^*, G \cup O^*, G \obar{\alpha} O^*) : (O^*, G \cup O^*) \in \mathbb{B}, \alpha : C(O^*) \rightarrow V(G)\}. \]
Proof. As we already defined (3), we only need to define the digraphs \( G \xrightarrow{\beta} O^* \) (using \( O^* \) and \( G \cup O^* \)). The relation of the lemma consists of those triples \((O^*, G \cup O^*, X)\) for which:

- Let \( V(G \cup O^*) = V(O^*) + n \). Then \( V(X) = V(G \cup O^*) + n \).
- \( G \cup O^* \subseteq X \).
- \( O_i \leq O^* \) implies \( \sigma_i \subseteq X \) or \( \sigma_i^L \subseteq X \).
- \( O_i, O_j \leq O^* \) (\( i \neq j \)) implies \( \sigma_i^D \cup \sigma_j^D \subseteq X \), or \( \sigma_i^D \rightarrow \sigma_j^D \subseteq X \), or \( \sigma_j^D \rightarrow \sigma_i^D \subseteq X \), or \( \sigma_i^D \leftrightarrow \sigma_j^D \subseteq X \) for some \((\Box, \bigtriangledown) \in \{{0, L}\}^2\).

\(\Box\)

In the following definition, we introduce the soul of our proof: the edge-supporting construction. Before starting to study the long definition, it is worth to read the simplified idea of it, back at the beginning of this section.

**Definition 28.** In this definition, we introduce the edge-supporting construction. Let \( G \) be a digraph with

\[
V(G) = \{v_1, \ldots, v_n\}, \quad \text{and} \quad E(G) = \{e_1, \ldots, e_k\}.
\]

Note that \( k \leq n^2 \) is necessary. Let \( p_1 \) and \( p_2 \) be two maps from \( E(G) \) to \( \{v_1, \ldots, v_n\} \) defined by the rule

\[
\forall e \in E(G) : e = (v_{p_1(e)}, v_{p_2(e)}).
\]

Let us introduce a digraph \( G_s \) with

\[
V(G_s) := V(G) \cup \{v_1^s, \ldots, v_n^s\}, \quad \text{and} \quad E(G_s) := E(G) \cup \bigcup_{i=1}^{k} \{\{v_{p_1(e)}, v_{p_2(e)}\}, (v_i^s, v_{p_2(e)})\}.
\]

Let

\[
O^* = O_{l_1} \cup O_{l_2} \cup \cdots \cup O_{l_n} \quad \text{such that} \quad n^2 + n < l_1 < l_2 < \cdots < l_n.
\]

Let \( D_s \) be a set of integers with

\[
|D_s| = k = |E(G)|, \quad \text{and} \quad x \in D_s \Rightarrow x > l_n.
\]

Let \( s \) be a bijective map from \( D_s \), satisfying (8), to \( \{v_1^s, \ldots, v_n^s\} \). Let

\[
O^*_s := O^* \cup \bigcup_{x \in D_s} O_x \quad \text{with} \quad V(O^*_s) = \{u_i^j : j \in \{l_1, \ldots, l_n\} \cup D_s, 1 \leq i \leq j\}.
\]

Let \( \alpha : C(O^*) \to V(G) \) be a bijective map. We define the digraph \((G \xrightarrow{\beta} O^*)_s \) by

\[
(G \xrightarrow{\beta} O^*)_s := G_s \xrightarrow{\beta} O^*_s, \quad \text{where} \quad \beta|_{C(O^*)} := \alpha, \quad \beta|_{O_x, x \in D_s} := \{(O_x, s(x)) : x \in D_s\},
\]

and say its an edge-supporting digraph for \( G \).

**Remark 2.** Note that the definition of the edge-supporting digraphs includes a condition for the size of the circles of \( O^* \). That condition is very important here, and was not present in (7). We need to be cautious about this later on.

**Lemma 29.** The following relation is definable.

\[
\{(O^*, G \xrightarrow{\alpha} O^*, (G \xrightarrow{\alpha} O^*)_s) : (G \xrightarrow{\alpha} O^*)_s \text{ is an edge-supporting digraph for } G\}
\]

**Proof.** The relation in question consists of those triples \((X, Y, Z)\) for which the highlighted conditions hold. There are explanations inserted between the conditions.
There exists a triple \((X, W, Y)\in \mathcal{J}\), meaning \((X, Y)\) is of the form \((O^*, G \xleftarrow{\beta} O^*)\).

Thus, instead of \((X, Y)\), we use \((O^*, G \xleftarrow{\beta} O^*)\) from now on in the proof. To ensure the structure of \(O^*\) (see Remark 2), first, we determine the number of vertices of \(G\) with

\[|V(O^*)| + 2n = |V(G \xleftarrow{\beta} O^*)|,\]

meaning \(G\) has \(n\) vertices. Now we are ready to shape \(O^*\).

- \(O_i \subseteq O^*\) implies \(i > n^2 + n\).

We turn to defining \(Z\) of the triple we started with.

There exists a triple \((W_1, W_2, Z)\in \mathcal{J}\), meaning \((W_1, Z)\) is of the form \((O^*_s, G_s \xleftarrow{\beta} O^*_s)\).

At this point, \(O^*_s, G_s\), and \(\beta\) are just notations yet, we need additional conditions to make them be like in Definition \[28\].

- \(O^* \subseteq O^*_s\),
- \(O_i \subseteq O^*_s\) implies \(i \geq l_1\), where \(l_1\) is the size of the smallest circle of \(O^*\), as before.
- \(G \xleftarrow{\beta} O^* \subseteq Z\).
- If \(O_i \subseteq O^*\) and \(\sigma_i^l \subseteq Z\), then there exists \(k > l_n\) for which \(\sigma_k \xrightarrow{i} \sigma_l \subseteq Z\) and \(\sigma_k \xrightarrow{\ell} \sigma_l \subseteq Z\) both hold. Additionally, if \(l\) is different from \(i, k\), and \(O_l \subseteq O^*_s\), then there exists \(\sigma \in \{\emptyset, L\}\) for which \(\sigma_k \cup \sigma_l^\sigma\) holds.
- If \(O_i, O_j \subseteq O^*, i \neq j\), and \(\sigma_i^\square \xrightarrow{\ell} \sigma_j^\square \subseteq Z\) with \(\{(\square, \ell)\in \{\emptyset, L\}^2\}\), then there exists \(k > l_n\) for which \(\sigma_k \xrightarrow{i} \sigma_k \subseteq Z\) and \(\sigma_k \xrightarrow{\ell} \sigma_k \subseteq Z\) both hold. Additionally, if \(l\) is different from \(i, j, k\), and \(O_l \subseteq O^*_s\), then there exists \(\sigma \in \{\emptyset, L\}\) for which \(\sigma_k \cup \sigma_l^\sigma\) holds.
- If \(O_i, O_j \subseteq O^*, i \neq j\), and \(\sigma_i^\square \leftrightarrow \sigma_j^\square \subseteq Z\) with some \(\{(\square, \ell)\in \{\emptyset, L\}^2\}\), then there exist two different \(k_1, k_2 > l_n\) for which all of

\[\sigma_i^\square \xrightarrow{\ell} \sigma_{k_1}, \sigma_{k_1} \xrightarrow{i} \sigma_j^\square, \sigma_j^\square \xrightarrow{\ell} \sigma_{k_2}, \text{ and } \sigma_{k_2} \xrightarrow{i} \sigma_j^\square\]

are substructures of \(Z\). Additionally, if \(l\) is different from \(i, j, k_i\), and \(O_l \subseteq O^*_s\), then there exists \(\sigma \in \{\emptyset, L\}\) for which \(\sigma_{k_1} \cup \sigma_l^\sigma\) holds for \(i = 1, 2\).

- If \(O_k \subseteq O^*_s\) and \(k > l_n\), then \(k\) is one of the \(k_s\) or \(k_0\)s of the previous three conditions.

It is not hard to see that these conditions provide the structure we need. \(\square\)

We can now handle the problem described in Remark 1. The next lemma does just that.

**Lemma 30.** The following relation is definable.

\[(10)\]

\[\{(G, O^*, G \xleftarrow{\beta} O^*) : \text{the circles of } O^* \text{ have more than } |V(G)|^2 + |V(G)| \text{ vertices.}\}\]

**Proof.** It is sufficient to define the relation

\[\{(G, O^*, G \cup O^*) : \text{the circles of } O^* \text{ have more than } |V(G)|^2 + |V(G)| \text{ vertices.}\}\]

as if we have this, we can easily finish the proof with \[7\]. We start with the \(R\) of Lemma \[28\]. For a pair \((G, O^*)\), we need to find the triple \((G, O^*, G_+ \cup O^*)\) of \(R\) such that \(G_+\) has the least possible number of edges. With \[7\] and \[9\], the relation

\[\{(G_+ \cup O^*, (G_+ \xleftarrow{\beta} O^*)), (G_+ \xleftarrow{\beta} O^*) \} \text{ is an edge-supporting digraph for } G_+\]
is defined easily. To conclude, pick a pair from this relation whose second component has a least number of vertices possible. The first element of this pair is $G \cup O^*$. □

We are finally ready to prove our main theorem.

**Proof of Theorem 4.** With (10) and (9) one can easily define the relation

$$\{(G, O^*), (G, O^*) : (G, O^*) \text{ is an edge-supporting digraph for } G\}.$$

Fix a triple $(G, O^*), (G, O^*)$ of this relation and let $n$ be the number of vertices of $G$. We need to show that the set of digraphs embeddable into $G$ is definable. Let $X \subseteq (G, O^*)$ and let $(G_X, O_X, G_X, O_X)$ be a triple of the relation (10) for which the following conditions hold. (We have to be careful (see Remark 2), the listed conditions do not contradict the assumption of (10).)

- $O_i \subseteq O^*_X$ holds if and only if both $O_i \subseteq O^*$, and $\sigma^\gamma_i \subseteq X$ for some $\Box \in \{\emptyset, L\}$ hold.
- If $O, O_j \subseteq O_X$, $i \neq j$, and $(\Box, \vee) \in \{\emptyset, L\}^2$, then
  - $\sigma^\gamma_i \cup \sigma^\gamma_j \subseteq X$ holds if and only if one of the following three holds:
    * $\sigma^\gamma_i \cup \sigma^\gamma_j \subseteq X$, or
    * $\sigma^\gamma_i \rightarrow \sigma^\gamma_j \subseteq X$, but the edge is not supported in $X$, i.e. there exists no $k > \ell_0$ (where $\ell_0$ is the size of the largest circle of $O^*$), as before for which $\sigma^\gamma_i \rightarrow \sigma^\gamma_k \subseteq X$ and $\sigma^\gamma_k \rightarrow \sigma^\gamma_j \subseteq X$ both hold, or
    * $\sigma^\gamma_i \leftrightarrow \sigma^\gamma_j \subseteq X$, but none of the two edges is supported in $X$.
  - $\sigma^\gamma_i \rightarrow \sigma^\gamma_j \subseteq X$ holds if and only if one of the following two holds
    * $\sigma^\gamma_i \rightarrow \sigma^\gamma_j \subseteq X$, and the edge is supported in $X$, or
    * $\sigma^\gamma_i \leftrightarrow \sigma^\gamma_j \subseteq X$, but only the “$i \rightarrow j$” edge is supported in $X$.
  - $\sigma^\gamma_i \leftrightarrow \sigma^\gamma_j \subseteq X$ holds if and only if $\sigma^\gamma_i \leftrightarrow \sigma^\gamma_j \subseteq X$ and both edges are supported in $X$.

It is clear that $G_X \leq G$ and all embeddable digraphs can be obtained this way. □

4. **The remaining technicalities**

**Definition 31.** The sum of the number of (both in- and out-)edges for a vertex, not counting the loops, is called the **loop-free degree** of the vertex.

**Lemma 32.** Let $0 \leq p$ and $1 \leq q$ be two fixed integers. We can define, with finitely many constants added to $(D, \subseteq)$, the set of digraphs that contain at most $p$ many vertices with loop-free degree at least $q$ each.

Before the easy proof, note that we can only use this lemma if we have a fixed constant, say $K = 4$, for the whole paper, such that all usage of the lemma restricts to $p, q \leq K$. Otherwise there would be no guarantee we are using finitely many constants at all. Fortunately, $K = 4$ will just do for the whole paper.

**Proof.** Observe that the digraph $G$ has more than $p$ many vertices with at least $q$ loop-free degree each, if and only if it has an at most $(p + 1)q$ element “certificate” substructure with the same property. Hence, by forbidding all those (finitely many) certificates, we define the set we need. □
Proof of Lemma 25. Let us consider $E_i$ and $E_j$ given. We define the other components of the relation.

We start with $\sigma_i|^O$ which is just the digraph $X$ for which

- $|V(X)| = i + 2$.
- $O_i \subseteq X$.
- We use Lemma 32 with $p = 1$, and $q = 3$, i.e. $X$ has at most one vertex with loop-free degree at least 3.
- We use Lemma 32 with $p = 0$, and $q = 4$ as well.
- The first digraph of Fig. 9 is a substructure. The $\Box$ symbol is understood naturally, if $\square = L$, then there is a loop there, if $\square = \emptyset$, then there is not.
- Depending on $\Box$,
  - if $\Box = \emptyset$, then $O_i \cup E_1 \subseteq X$, that is the only cover of $O_i$ among the $IO$-graphs,
  - if $\Box = L$, then $O_i \cup L_1 \subseteq X$, that is definable with Lemma 13.

We now start to deal with $\sigma_i|^O \cup \sigma_j|^O$. $O_i \cup O_j$ is the digraph with $i + j$ vertices that is a disjoint union of circles and both $O_i$ and $O_j$ are substructures. $\sigma_i|^O \cup \sigma_j|^O$ is the digraph $X$ for which

- $|V(X)| = |V(\sigma_i|^O)| + |V(\sigma_j|^O)|$.
- $\sigma_i|^O \subseteq X$, and $\sigma_j|^O \subseteq X$.
- We use Lemma 32 with $p = 2$, $q = 3$ and with $p = 0$, $q = 4$.
- Depending on $(\Box, \Box)$,
  - if $(\Box, \Box) = (\emptyset, \emptyset)$, then $O_i \cup O_j \cup E_2 \subseteq X$, which is just the digraph $Y$ for which
    - $|V(Y)| = i + j + 2$, and $O_i \cup O_j \subseteq X$,
    - $Y$ has the maximal substructure $E_k$ (among the ones with the previous property).
  - if $(\Box, \Box) = (L, \emptyset)$ or $(\emptyset, L)$, then $O_i \cup O_j \cup E_1 \cup L_1 \subseteq X$, which is just the digraph $Y$ for which
    - $|V(Y)| = i + j + 2$, and $O_i \cup O_j \subseteq X$,
    - $O_i \cup O_j \cup E_1$, which is the only $IO$-graph cover of $O_i \cup O_j$, is a subgraph,
    - on two elements, there is no subgraph with both a loop and a loop-free edge.
  - if $(\Box, \Box) = (L, L)$ then $O_i \cup O_j \cup L_2 \subseteq X$.

Now we turn to $\sigma_i|^\Box \rightarrow \sigma_j|^\Box$, which is just the digraph $X$ for which

- $|V(X)| = |V(\sigma_i|^\Box)| + |V(\sigma_j|^\Box)|$.
- $\sigma_i|^\Box \subseteq X$, and $\sigma_j|^\Box \subseteq X$.
- We use Lemma 32 with $p = 2$, $q = 3$ and with $p = 0$, $q = 4$.
- The second digraph of Fig. 9 is substructure of $X$.  

\begin{figure}[h]
\centering  
\includegraphics[width=\textwidth]{diagram}
\caption{Figure 9.}
\end{figure}
Finally, $\sigma_i^\triangledown \leftrightarrow \sigma_j^\triangledown$ is defined with the analogues of the conditions for $\sigma_i^\triangledown \rightarrow \sigma_j^\triangledown$.

References

[1] J. Ľeţek and R. McKenzie. Definability in substructure orderings, i: Finite semilattices. *Algebra universalis*, 61(1):59, 2009.
[2] J. Ľeţek and R. McKenzie. Definability in substructure orderings, iii: Finite distributive lattices. *Algebra universalis*, 61(3):283, 2009.
[3] J. Ľeţek and R. McKenzie. Definability in substructure orderings, iv: Finite lattices. *Algebra universalis*, 61(3):301, 2009.
[4] J. Ľeţek and R. McKenzie. Definability in substructure orderings, ii: Finite ordered sets. *Order*, 27(2):115–145, 2010.
[5] A. Kunos. Definability in the embeddability ordering of finite directed graphs, ii. *http://arxiv.org/abs/1806.07871*.
[6] A. Kunos. Definability in the embeddability ordering of finite directed graphs. *Order*, 32(1):117–133, 2015.
[7] R. Ramanujam and R. S. Thinniyam. *Definability in First Order Theories of Graph Orderings*, pages 331–348. Springer International Publishing, Cham, 2016.
[8] R. S. Thinniyam. *Definability of Recursive Predicates in the Induced Subgraph Order*, pages 211–223. Springer Berlin Heidelberg, Berlin, Heidelberg, 2017.
[9] A. Wires. Definability in the substructure ordering of simple graphs. *Annals of Combinatorics*, 20(1):139–176, 2016.