A sharp Rogers & Shephard type inequality for the $p$-difference body of planar convex bodies.

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Abstract

We prove a sharp Rogers & Shephard type inequality for the $p$-difference body of a convex body in the two-dimensional case, for every $p \geq 1$.

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1 Introduction

A convex body is a non-empty convex compact subset of $\mathbb{R}^n$; let us indicate the set of convex bodies in $\mathbb{R}^n$ with $\mathcal{K}^n$. To each convex body $K$ we can associate in a biuniquely way its support function $h_K$:

$$h_K(u) = \sup\{\langle x,u \rangle \mid x \in K \},$$

for all $u \in \mathbb{R}^n$, where $\langle \cdot, \cdot \rangle$ is the standard scalar product. The support function is a fundamental tool since the main properties of the body can be deduced from it.

One of the most interesting aspects of convex geometry, i.e. the theory of convex bodies, are geometric inequalities. An important family of inequalities are those leading to estimate the volume of a special body associated with a convex body (for example the difference body or the reflection body) in terms of the volume of the body itself.

A remarkable inequality of this type is the classical Rogers & Shephard inequality (see [1]) which asserts that for all $K \in \mathcal{K}^n$

$$V_n\left(K + (-K)\right) \leq \binom{2n}{n} V_n(K),$$

and equality holds if and only if $K$ is a simplex. Here $V_n(K)$ denotes the $n$-dimensional volume of $K$ (i.e. the $n$-dimensional Lebesgue measure). The body $K + (-K)$ is called difference body of $K$ and it is the Minkowski sum of $K$ and its reflected body with respect to the origin, $-K$. We recall more generally that the Minkowski sum of $K$ and $L \in \mathcal{K}^n$ is

$$K + L = \{z \in \mathbb{R}^n \mid z = k + l, \ k \in K, \ l \in L\}.$$
Another inequality due to Rogers & Shephard (12) concerns the convex hull (here denoted by conv) of $K$ and $-K$, under the assumption that the origin $o$ belongs to $K$:

$$V_n \left( \text{conv}(K \cup (-K)) \right) \leq 2^n V_n(K), \quad (2)$$

where equality holds if and only if $K$ is a simplex with one vertex at the origin.

In [6] Firey introduced a new operation for convex bodies, called $p$-sum, which depends on the parameter $p \geq 1$ and extends the Minkowski sum. An account on the theory of convex bodies based on the $p$-sum, the so called Brunn-Minkowski-Firey theory, can be found in the papers [8], [9] by Lutwak. Let us fix $K, L \in \mathcal{K}^n$ both containing the origin; the $p$-sum of $K$ and $L$, $K + p L$, is defined by its support function in the following way:

$$h_{K + p L}(u) = \left( h_K^p(u) + h_L^p(u) \right)^{\frac{1}{p}}, \quad u \in \mathbb{R}^n.$$ 

This definition admits a natural extension to the case $p = \infty$:

$$h_{K + \infty L}(u) = \lim_{p \to \infty} h_{K + p L}(u) = \max\{h_K(u), h_L(u)\}, \quad u \in \mathbb{R}^n.$$ 

Note that the extremal values $p = 1$ and $p = \infty$, correspond to the Minkowski sum and the convex hull of the union respectively. Indeed one has

$$h_{K + 1 L}(u) = h_K(u) + h_L(u) = h_{K + L}(u),$$

and

$$h_{K + \infty L}(u) = \max\{h_K(u), h_L(u)\} = h_{\text{conv}(K \cup L)}.$$ 

As proved by Firey [6], the $p$-sum is monotone with respect to the parameter $p$: for all $K, L \in \mathcal{K}^n$ such that $o \in K, L$, if $p \leq q$ then

$$K + q L \subseteq K + p L.$$ 

This implies that for all $p \geq 1$,

$$\text{conv}(K \cup L) \subseteq K + p L \subseteq K + L.$$ 

Another simple inclusion is

$$K + p L \subseteq 2^{\frac{1}{p}} \text{conv}(K \cup L).$$ 

In particular choosing $L = -K$ and using inequalities (1) and (2), we have:

$$V_n \left( K + p (-K) \right) \leq \min \left\{ \left( \frac{2n}{n} \right), \ 2^{\frac{n(1+p)}{p}} \right\} V_n(K).$$ 

A natural problem is then to find the best constant $c = c_{n,p}$, depending on $n$ and $p$, such that

$$V_n \left( K + p (-K) \right) \leq c_{n,p} V_n(K), \quad \text{for all } K \in \mathcal{K}^n, \ o \in K. \quad (3)$$ 

In this paper we solve this problem in the planar case $n = 2$, for every $p \geq 1$. 

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Theorem 1.1. For every $p \geq 1$ there exists a constant $c_p$ such that

$$V_2\left(K + p (-K)\right) \leq c_p V_2(K),$$

(4)

for all $K \in \mathcal{K}^2$. In particular if $K$ is a triangle with one vertex at the origin, then equality holds.

An explicit expression of $c_p$ will be presented in Section 3.

We will show the $p$-Rogers & Shephard inequality (4) as a consequence of a theorem about the $p$-sum of the so called parallel chord movements of convex bodies.

A parallel chord movement is a special one-parameter family of convex bodies which can be seen as continuous deformations of a fixed convex body. More precisely, fix $K \in \mathcal{K}^n$ and a direction $v \in \mathbb{R}^n$ which is the direction of the movement. We move each chord of $K$ parallel to $v$ in that direction with a certain speed and we consider the union of these chords as the time parameter varies. If the speed function is suitably chosen, namely if the union of the chords is convex for all values of the parameter, then the family of the resulting convex bodies is a parallel chord movement.

Parallel chord movements are special cases of a wider class of movements of convex bodies introduced by Rogers and Shephard in [13], which have been recently applied in the proof of several inequalities in convex geometry (see, for examples, [1], [2]-[5], [10]).

The importance of these movements is due principally to the behaviour of several geometric functionals with respect to the parameter of the movement. Indeed many of them, and the volume is the main example, are convex function of the time parameter of the movement.

In particular in this paper we prove that if $K_t$ is a parallel chord movement, then the volume of its $p$-difference body $V_n\left(K_t + p (-K_t)\right)$ is a convex function of $t$, for all $p \geq 1$. This result, together with a technique used in [1], leads to the proof of Theorem 1.1. As noted in [1] this technique is successful only in the planar case, so our method can not be used to prove inequality (3) in the general case $n \geq 2$.

The paper is organized as follows. In Section 2 we introduce several kinds of movements of convex bodies and we show some of their properties. Next to basic results we present a theorem about the $p$-sum of a particular type of movements. In section 3 we prove Theorem 1.1 as an application of the results concerning movements of convex bodies.

2 Shadow systems and linear parameter systems

A shadow system is a family of $n$-dimensional convex bodies $\{K(u)\}$ obtained as the projection of a fixed convex body $\tilde{K} \subseteq \mathbb{R}^{n+1}$ onto the hyperplane $\{e_{n+1}^+\}$, which we identify with $\mathbb{R}^n$, along the direction $e_{n+1} + u$. Here $u$ varies in $\{e_{n+1}^+\}$. The shadow system is said to be originated from the $(n+1)$-dimensional body $\tilde{K}$.
A linear parameter system is a family of convex bodies \( \{K_t\} \) that can be written in the form
\[
K_t = \text{conv}\{x_i + \lambda_i tv : i \in I\}, \quad t \in \mathcal{I};
\] (5)
where \( I \) is an arbitrary index set, \( \{x_i\}_{i \in I} \) and \( \{\lambda_i\}_{i \in I} \) are bounded subsets of \( \mathbb{R}^n \) and of \( \mathbb{R} \) respectively, \( \mathcal{I} \) is an interval of \( \mathbb{R} \) and \( v \in \mathbb{R}^n \) is the direction of the linear parameter system.

Linear parameter systems are shadow systems in which \( u \) lies on a line, indeed we have the following result.

**Proposition 2.1.** \( \{K_t\}_{t \in \mathcal{I}} \) is a linear parameter system in \( \mathbb{R}^n \) if and only if there exists a convex body \( \tilde{K} \) in \( \mathbb{R}^{n+1} \) such that for every \( t \in \mathcal{I}, K_t \) is the projection of \( \tilde{K} \) onto the hyperplane \( \{e_{n+1}^\perp\} \) along the direction \( e_{n+1} - tv \).

The idea to view linear parameter systems as projections of higher dimensional convex bodies is contained in the original papers by Rogers and Shephard ([13], [15]) and was largely used by Campi and Gronchi ([2]–[5]). For the sake of completeness here we present the proof of Proposition 2.1.

**Proof.** Let \( K_t \) be of the form (5) and let us define the body \( \tilde{K} \) as follows:
\[
\tilde{K} = \text{conv}\left(\{x_i + \lambda_i e_{n+1} : i \in I\}\right).
\]
For all \( t \in \mathcal{I} \) let us call \( L_t \) the projection of \( \tilde{K} \) onto \( \{e_{n+1}^\perp\} \) along \( e_{n+1} - tv \). For all \( y \in L_t \) there exists \( z \in \tilde{K} \) such that \( y = z - \langle z, e_{n+1}\rangle (e_{n+1} - tv) \). Furthermore there exist \( a_i \in e_{n+1}^\perp, \lambda_i \in \mathbb{R}, \) and \( \sigma_i \geq 0, i = 1, \ldots, n + 1, \) such that \( \sum_{i=1}^{n+1} \sigma_i = 1 \) and
\[
z = \sum_{i=1}^{n+1} \sigma_i (a_i + \lambda_i e_{n+1}).
\]
Therefore
\[
y = \sum_{i=1}^{n+1} \sigma_i (a_i + \lambda_i tv).
\]
This implies that \( L_t \) is contained in \( K_t \). To prove the reverse inclusion one can observe that the previous implications are true in both directions.

Conversely, let \( \tilde{K} \) be any \( (n+1) \)-dimensional convex body and fix \( t \in \mathcal{I} \); its projection onto \( \{e_{n+1}^\perp\} \) along \( e_{n+1} - tv \) is the set:
\[
L_t = \{e_{n+1}^\perp\} \cap \{z \in \mathbb{R}^{n+1} \mid z = x + t \xi, \; x \in \tilde{K}, \; t \in \mathbb{R}\}.
\]
This is equivalent to write
\[
L_t = \{x - \langle x, e_{n+1}\rangle e_{n+1} + t(-x, e_{n+1})w : x \in \tilde{K}\},
\]
and, by the convexity of \( \tilde{K} \), \( \{L_t\}_{t \in \mathcal{I}} \) is a linear parameter system as defined in (4). \( \square \)
From the previous proof it follows that the body $\tilde{K}$ which generates a linear parameter system of the form (4) can be explicitly written as:

$$\tilde{K} = \text{conv}\{x_i + \lambda_i e_{n+1} : i \in I\}. \quad (6)$$

Campi and Gronchi showed in [4] the following formula which relates the support functions of $K_t$ and $\tilde{K}$:

$$h_{K_t}(u) = h_{\tilde{K}}(u + t(u,v)e_{n+1}), \quad u \in \mathbb{R}^n, \ t \in I. \quad (7)$$

We can give a cinematic interpretation of a linear parameter system viewing the numbers $\lambda_i$ as the speeds of the points $x_i$ along the direction $v$ and $t$ as the time parameter.

If the index set $I$ is a convex body $K \in \mathcal{K}^n$ and the speed is a function of the point, then the linear parameter system is called continuos movement:

$$K_t = \text{conv}\{x + \alpha(x)tv : x \in K\}, \quad t \in I,$$

where $\alpha(\cdot)$ is a bounded function on $K$.

Assume that the speed function is constant on each chord parallel to $v$, i.e. $\alpha(x) = \beta(x|v^\perp)$ where $x|v^\perp$ is the projection of $x$ onto $\{v^\perp\}$ and $\beta$ is a function defined on the orthogonal projection of $K$ onto $\{v^\perp\}$. Moreover, if $\beta$ is such that convexity is preserved for any $t$, namely

$$\{x + \beta(x|v^\perp)tv : x \in K\} = \text{conv}\{x + \beta(x|v^\perp)tv : x \in K\},$$

then the continuos movement is called parallel chord movement.

In other words a parallel chord movement is obtained assigning to each chord parallel to the direction $v$ a speed vector $\beta(x|v^\perp)v$ and considering for each fixed time $t$ the union of these chords. Such union has to be convex. Notice that if $\{K_t\}_{t \in I}$ is a parallel chord movement, then the volume of $K_t$ is independent of $t$.

The following theorem is due to Rogers and Shephard (see [13]) and it is one of the main motivations for the use of linear parameter systems in the theory of convex bodies.

**Theorem 2.2.** The volume $V_n(K_t)$ of a linear parameter system is a convex function of the parameter $t$.

In [4] it is proved that the Minkowski sum of linear parameter systems is a linear parameter system. Here we extend this result to the $p$-sum. This fact is one of the main ingredients in the proof of the $p$-Rogers & Shephard inequality.

**Theorem 2.3.** Let $\{K_t\}_{t \in I}$ and $\{L_t\}_{t \in I}$ be linear parameter systems along the direction $v$ and let $p \geq 1$, then $\{K_t + L_t\}_{t \in I}$ is also a linear parameter system along the direction $v$.

The proof is a straightforward consequence of Proposition 2.1 and the following lemma.
Lemma 2.4. Let \( \{K_t\}_{t \in I} \) and \( \{L_t\}_{t \in I} \) be linear parameter systems along the same direction \( v \) and let \( K_t \) and \( L_t \) be the \((n+1)\)-dimensional convex bodies which generate \( K_t \) and \( L_t \) respectively, defined as in (7). Hence for all \( t \in I \), \( K_t + p \cdot L_t \) is the projection of \( \tilde{K} + p \cdot \tilde{L} \) onto the hyperplane \( \{e_{n+1} \} \) along the direction \( e_{n+1} - tv \).

Proof. Using (7) one has:

\[
h_p \left( K_t \right) = V_n \left( K_t + p \cdot \left( -K_t \right) \right) / V_n (K).
\]

This implies that \( K_t + p \cdot L_t \) is the projection of the body \( \tilde{K} + p \cdot \tilde{L} \) onto the hyperplane \( \{e_{n+1} \} \) along the direction \( e_{n+1} - tv \), which means, by Proposition (2.1), that \( K_t + p \cdot L_t \) is a linear parameter system along \( v \).

3 The proof of the \( p \)-Rogers & Shephard inequality

Let us call \( \mathcal{K}_0^n \) the set of convex bodies with non-empty interior and containing the origin and let us consider the functional \( F_p \) defined on \( \mathcal{K}_0^n \):

\[
F_p(K) = \frac{V_n \left( K + p \cdot (-K) \right)}{V_n(K)}.
\]

It is clear that the best constant \( c_{n,p} \) such that (3) holds is the supremum of \( F_p \) in \( \mathcal{K}_0^n \).

We will use linear parameter systems to find a maximum for the functional \( F_p \) in the planar-case. The starting point is the next proposition which follows from Theorem 2.2 and Theorem 2.3.

Proposition 3.1. If \( K_t \) is any parallel chord movement such that \( K_t \in \mathcal{K}_0^n \) for all \( t \in I \), then \( F_p(K_t) \) is a convex function of the parameter \( t \).

In [1] the following fact is proved: if \( P \) is a planar convex polygon with \( m \) vertices, \( m > 3 \), then there exists a parallel chord movement \( \{P_t\}_{t \in [t_0,t_1]} \), with \( t_0 < 0 < t_1 \), such that \( P = P_0 \) and \( P_t \) and \( P_{t_1} \) have at most \((m-1)\) vertices. By Proposition 3.1 it follows that:

\[
F_p(P) \leq \max \{ F_p(P_0), F_p(P_{t_1}) \}.
\]

Using recursively this fact we deduce that:

\[
\sup_{\mathcal{P}} F_p = \sup_{\mathcal{P}} F_p,
\]

where \( \mathcal{P} = \{ K \in \mathcal{K}_0^2 \mid K \text{ is a polygon} \} \) and \( \mathcal{T} = \{ K \in \mathcal{K}_0^2 \mid K \text{ is a triangle} \} \).

Moreover, by the continuity of \( F_p(\cdot) \) and a standard density argument, one has:

\[
\sup_{\mathcal{K}_0^2} F_p = \sup_{\mathcal{T}} F_p.
\]
In particular we are going to show that triangles with one vertex at the origin are maximizers for $F_p$. In order to do this, let $T \in \mathcal{T}$ and assume $o \in \text{int}(T)$ (int denotes the interior). Then there exists a parallel chord movement (whose elements are translates of $T$), $\{T_t\}_{t \in [t_0, t_1]}$ with $t_0 < 0 < t_1$, such that $T_0 = T$ and $o \in \text{bd}(T_{t_0})$, $o \in \text{bd}(T_{t_1})$, (bd denotes the boundary). Similarly, if $o \in \text{bd}(T)$, then there exists a parallel chord movement containing $T$, whose endpoints are triangles with one vertex at $o$. Using again Proposition 3.1 we have proved that

$$\sup_{\mathcal{K}_0^2} F_p = \sup_{\mathcal{K}_0} F_p,$$

where $\mathcal{K}_0$ the set of triangles with one vertex at the origin.

Note that $F_p$ is invariant under non-singular linear transformations. This implies that $F_p$ is constant on $\mathcal{K}_0$.

This argument proves the following result.

**Theorem 3.2.** If $T$ is a triangle in $\mathcal{K}_0^2$ with one vertex at the origin, then $T$ is a maximizer for $F_p$.

To compute the best constant $c_{2,p}$, we can choose as a maximizer the triangle with vertices at the origin, at $(1, 0)$ and $(0, 1)$; let us indicate it with $K$. Namely

$$c_{2,p} = \frac{V_2(K + p(-K))}{V_2(K)}.$$ 

Then to express the value of $c_{2,p}$ it is necessary to know how the $p$-difference body $K + p(-K)$ looks like. Here we use the parametrization of the boundary of a convex body in terms of its support function (see [14] Corollary 1.7.3).

The support function of $K + p(-K)$ is:

$$h_{K + p(-K)}(w) = \begin{cases} 
\cos \theta & \text{if } 0 \leq \theta < \frac{\pi}{4}, \\
\sin \theta & \text{if } \frac{\pi}{4} \leq \theta < \frac{\pi}{2}, \\
\left(\sin^p \theta + (-\cos \theta)^p\right)^{\frac{1}{p}} & \text{if } \frac{\pi}{2} \leq \theta \leq \pi,
\end{cases}$$

where $w = e^{i\theta} \in S^1$. Furthermore, by the symmetry of $K + p(-K)$,

$$h_{K + p(-K)}(e^{i\theta}) = h_{K + p(-K)}(e^{i(\theta-\pi)})$$

for all $\pi \leq \theta \leq 2\pi$. Then a parametrization for the boundary of $K + p(-K)$, for $1 < p < +\infty$, is $\zeta(\theta) = (x(\theta), y(\theta))$, where

$$x(\theta) = \begin{cases} 
1 - \frac{2}{\pi} \theta & \text{for } \theta \in [0, \frac{\pi}{2}], \\
(-\sin^p \theta + (-\cos \theta)^p)^{\frac{1-p}{p}} (-\cos \theta)^{p-1} & \text{for } \theta \in (\frac{\pi}{2}, \pi);
\end{cases}$$

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\[ g(\theta) = \begin{cases} \frac{2}{\pi} \theta & \text{for } \theta \in [0, \frac{\pi}{2}], \\ \left( \sin^p \theta + (-\cos \theta)^p \right)^{-\frac{1}{p}} \sin^{p-1} \theta & \text{for } \theta \in \left(\frac{\pi}{2}, \pi\right); \end{cases} \]

and the remaining part of the boundary can be found using the symmetry of the body.

A picture can perhaps better show the geometry of the body. In the following one \( K +_p (-K) \) is represented for the the values 1, 1.5, 2, 15, \( \infty \) of the parameter \( p \).

Using the above parametrization and Gauss-Green’s formulas we can express the area of \( K +_p (-K) \) and then the value of the best constant \( c_{2,p} \):

\[ c_{2,p} = 2 \left( 1 + (p - 1) \int_0^{\frac{\pi}{2}} \frac{\sin^{p-2} t \cos^{p-2} t}{\left( \sin^p t + \cos^p t \right)^{\frac{3(p-1)}{2p}}} dt \right), \quad 1 < p < +\infty. \]

References

[1] S. Campi, A. Colesanti, P. Gronchi, A note on Sylvester’s problem for random polytopes in a convex body, Rend. Ist. Mat. Univ. Trieste 31 (1999), 79-94.

[2] S. Campi, P. Gronchi, The \( L^p \)-Busemann-Petty centroid inequality, Adv. Math. 167 (2002), 128-141.

[3] S. Campi, P. Gronchi, On the reverse \( L^p \)-Busemann-Petty centroid inequality, Mathematika 49 (2002), 1-11.

[4] S. Campi, P. Gronchi, On volume product inequalities for convex sets, Proc. Amer. Math. Soc. 134, 8 (2006), 2393-2402.

[5] S. Campi, P. Gronchi, Volume inequalities for \( L^p \)-zonotopes (to appear on Mathematika).

[6] W.M.J. Firey, \( p \)-Means of convex bodies, Math. Scand. 10 (1962), 17-24.

[7] F. John, Extremum problems with inequalities as subsidiary conditions, Courant Anniversary Volume, Interscience, New York, 1948, 187-204.
[8] E. Lutwak, *The Brunn-Minkowski-Firey theory I. Mixed volumes and the Minkowski problem*, J. Differential Geom. **38** (1993), 131-150.

[9] E. Lutwak, *The Brunn-Minkowski-Firey theory II. Affine and geominimal surface area*, Adv. Math. **118**, 2 (1996), 244-294.

[10] M. Meyer, S. Reisner, *Shadow systems and volumes of polar convex bodies*, preprint (2006), [arXiv:math.MG/0606305](http://arxiv.org/abs/math.MG/0606305).

[11] C.A. Rogers, G.C. Shephard, *The difference body of a convex body*, Arch. Math. **8** (1957), 220-233.

[12] C.A. Rogers, G.C. Shephard, *Convex bodies associated with a given convex body*, J. Lond. Math. Soc. **33** (1958), 270-281.

[13] C.A. Rogers, G.C. Shephard, *Some extremal problems for convex bodies*, Mathematika **5** (1958), 93-102.

[14] R. Schneider, *Convex bodies: the Brunn-Minkowski theory*, Encyclopedia of Mathematics and its applications **44**, Cambridge University Press, Cambridge, 19993.

[15] G.C. Shephard, *Shadow system of convex sets*, Israel J. Math. **2** (1964), 229-236.

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