CONTINUITY OF BALL PACKING DENSITY ON MODULI SPACES OF TORIC MANIFOLDS

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Abstract. In this note we characterize the continuity of the maximal toric ball packing density function $M_T \to [0,1]$ on the moduli space of $2n$-dimensional closed symplectic toric manifolds $M_T$ with respect to the natural metric structure on it. We will prove that the density function, which is a toric symplectic invariant, is everywhere discontinuous unless restricted to suitable (still large) moduli subspaces where it is everywhere continuous.

1. Introduction

In symplectic topology the ball packing problem asks how much of the volume of a symplectic manifold $(M, \omega)$ can be approximated by symplectically embedded disjoint balls. An optimal packing is one for which the sum of these disjoint volumes divided by $\text{vol}(M)$ is as large as it can be, taking into consideration all possible such packings.

In the theory of completely integrable toral actions, or completely integrable systems, this is also a natural question. But the embedded balls have to respect not only the symplectic form, but all the integrable system. The best way to think of this problem is about an approximation theorem for our integrable system by disjoint integrable systems on balls. This problem is more rigid, because fixed points of the system have to coincide with the origin of the ball, and the symplectic balls mapped this way have less flexibility.

In this paper, we will concentrate on the class of integrable systems coming from toral actions, because for this we have a good understanding [7, 6, 9], but the question is interesting for any integrable system. These particular systems are usually called toric, or symplectic toric, and a rich structure theory due to Kostant, Guillemin-Sternberg, Delzant among others led to a complete classification in the 1980s [2, 3, 5]. We refer to Section 2 for the precise definition of symplectic toric manifolds, and to Section 3 for the definition of the moduli space $M_T$ of such manifolds, where $T$ denotes the standard torus of dimension precisely half the dimension of the manifolds in $M_T$.

In dimension $2n = 4$ (so in the case of symplectic toric 4–manifolds) it is known [8] that $M_T$ is a neither a locally compact nor a complete metric space but its completion is well understood and describable in explicit terms.
Alessio Figalli and Álvaro Pelayo

It is natural to wonder how the optimal density function \( M_T \to [0,1] \) that assigns to a symplectic toric manifold its optimal density (see e.g. [8], and recall that by construction it is a symplectic toric invariant) behaves with respect to the topologies on the domain (induced by the metric) and on the target (with the standard topology). The following result answers this question:

**Theorem A.** Let \( T \) be an \( n \)-dimensional torus, let \( M_T \) be the moduli space of symplectic toric \( 2n \)-manifolds, and let

\[
\Omega : M_T \to [0,1]
\]

be the function which assigns to a manifold the density of its optimal toric ball packing. Let \( N \geq 1 \) be an integer and let \( M_T^N \) be the set of symplectic toric manifolds with precisely \( N \) points fixed by the \( T \)-action, which implies that \( M_T = \bigcup_{N \geq 1} M_T^N \). Then:

1. \( \Omega \) is discontinuous at every \( (M,\omega,T) \in M_T \), and the restriction \( \Omega|_{M_T^N} \) is continuous for each \( N \geq 1 \).
2. Given \( (M,\omega,T) \in M_T^N \subset M_T \), define \( \Omega_i(M,\omega,T) \), where \( 1 \leq i \leq N \), to be the maximal density computed along all packings avoiding balls with center at the \( i \)th fixed point of the \( T \)-action. Then \( M_T^N \) is the largest neighborhood of \( M \) in \( M_T \) where \( \Omega \) is continuous if and only if \( \Omega_i(M,\omega,T) < \Omega(M,\omega,T) \) for all \( i \) with \( 1 \leq i \leq N \).

The aforementioned classification of symplectic toric manifolds is in terms of a class of convex polytopes in \( \mathbb{R}^n \), called now Delzant polytopes. In practice this means that the a priori very complicated packing problem for these systems can be formulated and studied in terms of convex geometry, which is what was done in [7, 6, 9]. This is because symplectic toric manifolds have an associated momentum map

\[
\mu : M \to \mathbb{R}^n,
\]

which is a Morse-Bott function with a number of remarkable properties and which captures all the main features of the symplectic geometry of \( (M,\omega) \) and of the Hamiltonian action \( \psi \) of \( T \) on \( M \) (the proof of this is due to Atiyah, Guillemin, Sternberg, and Delzant). In fact \( \mu(M) \) is a convex polytope in \( \mathbb{R}^n \).

The strategy of this paper is to prove a slightly more general result about polytopes, Theorem 6.1, which implies Theorem A by virtue of a theorem proven in [6] (reviewed in the present paper, see Theorem 5.5).

**Structure of the paper.**

- in Section 2 we review what symplectic toric \( 2n \)-manifolds are;
- in Section 3 we review the construction of the moduli space \( M_T \) of symplectic toric manifolds;
- in Section 4 we review the definition of the density function on \( M_T \);
• in Section 5 we explain how to reduce the proof of Theorem A to the proof of a theorem in convex geometry (Theorem 6.1);
• in Section 6 we state and prove Theorem 6.1.

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2. Symplectic toric manifolds

Let \((M, \omega)\) denote a 2n-dimensional compact connected smooth manifold equipped with a symplectic form \(\omega\). We let \(T_k \cong (S^1)^k\) denote the \(k\)-dimensional torus, and throughout this paper we use \(T := T^n\) to denote the \(n\)-dimensional standard torus. Denote by \(\mathfrak{t}\) the Lie algebra \(\text{Lie}(T)\) of \(T\) and by \(\mathfrak{t}^*\) the dual of \(\mathfrak{t}\). A \(\omega\)-preserving action \(\psi : T^k \times M \to M\) of a \(k\)-dimensional torus is Hamiltonian if there exists a map \(\mu : M \to \mathfrak{t}^*\), called a momentum map, satisfying Hamilton’s equation

\[ i_{\xi_M} \omega = d\langle \mu, \xi \rangle, \]

for all \(\xi \in \mathfrak{t}\). Note that a momentum map is well defined up to translation by an element of \(\mathfrak{t}^*\). Nevertheless, we will ignore this ambiguity and refer to the momentum map. It is well known that if \((M, \omega)\) admits an effective and Hamiltonian action of \(T^k\), then \(k \leq n\). The maximal case is the object of study in this paper.

Definition 2.1. A symplectic-toric manifold or Delzant manifold, is a quadruple \((M, \omega, T, \mu)\) consisting of the 2n-dimensional compact connected symplectic manifold \((M, \omega)\) equipped with an effective Hamiltonian action \(\psi : T \times M \to M\) of an \(n\)-dimensional torus \(T\) with momentum map \(\mu\). When there is no confusion sometimes we write \((M, \omega, T)\).

Strictly speaking, the momentum map \(\mu\) of the Hamiltonian action of \(T\) on a manifold \(M\) is a map \(M \to \mathfrak{t}^*\). However, the presentation is simpler, if from the beginning we identify both \(\mathfrak{t}\) and \(\mathfrak{t}^*\) with \(\mathbb{R}^n\) and consider \(\mu\) as a map from \(M\) to \(\mathbb{R}^n\). The procedure to do this, that we now describe, is standard but not canonical.

Choose an epimorphism \(E : \mathbb{R} \to T^1\), for instance, \(x \mapsto e^{2\pi \sqrt{-1} x}\). This Lie group epimorphism has discrete center \(\mathbb{Z}\) and the inverse of the corresponding Lie algebra isomorphism is given by \(\text{Lie}(T^1) \ni \frac{\partial}{\partial x} \mapsto \frac{1}{2\pi} e_k \in \mathbb{R}\). Thus, for \(T\) we get the non-canonical isomorphism between the corresponding commutative Lie algebras

\[ \text{Lie}(T) = \mathfrak{t} \ni \frac{\partial}{\partial x_k} \mapsto \frac{1}{2\pi} e_k \in \mathbb{R}^n, \]

where \(e_k\) is the \(k^{th}\) element in the canonical basis of \(\mathbb{R}^n\). Choosing an inner product \(\langle \cdot, \cdot \rangle\) on \(\mathfrak{t}\), we obtain an isomorphism \(\mathfrak{t} \to \mathfrak{t}^*\), and hence taking its inverse and composing it with the isomorphism \(\mathfrak{t} \to \mathbb{R}^n\) described above, we
get an isomorphism $\mathcal{I} : t^* \to \mathbb{R}^n$. In this way, we obtain a momentum map $\mu = \mu_{\mathcal{I}} : M \to \mathbb{R}^n$.

**Example 2.2.** Equip the open radius $r$ ball $B_r \subset \mathbb{C}^n$ with the standard symplectic form $\omega_0 = \frac{i}{2} \sum_j dz_j \wedge d\overline{z}_j$. The action $\text{Rot} : \mathbb{T} \times B_r \to B_r$ of $\mathbb{T}$ given by $(\theta_1, \ldots, \theta_n) \cdot (z_1, \ldots, z_n) = (\theta_1 z_1, \ldots, \theta_n z_n)$ is Hamiltonian. Its momentum map $\mu^{B_r}$ has components $\mu_k^{B_r} = |z_k|^2$. Its image, which we shall denote by $\Delta^n(r)$, is given by

$$\Delta^n(r) = \text{ConvHull}(0, r^2 e_1, \ldots, r^2 e_n) \setminus \text{ConvHull}(r^2 e_1, \ldots, r^2 e_n),$$

where $\{e_i\}_{i=1}^n$ is the standard basis of $\mathbb{R}^n$.

For a symplectic-toric manifold $(M, \omega, T, \mu)$, the number of fixed points of the $T$-action is known to coincide with the Euler characteristic $\chi(M)$ (see eg. [4]). It follows that a toric ball packing $P$ consists of at most $\chi(M)$ disjoint equivariant balls. The images of momentum maps for symplectic-toric manifolds are a particular class of polytopes, called **Delzant polytopes**.

**Definition 2.3** ([6]). A simple $n$-dimensional convex polytope $\Delta \subset \mathbb{R}^n$ is said to be **Delzant (or smooth)** if for each vertex $v$, the edges meeting at $v$ are all of the form $v + t_i u_i$ where $t_i > 0$ and $\{u_1, \ldots, u_n\}$ define a basis of the $\mathbb{Z}$-module $\mathbb{Z}^n$.

A polytope is describable as the intersection of closed half-spaces $\Delta := \bigcap_{i=1}^F \{x \in \mathbb{R}^n \mid \langle x, u_i \rangle \geq \lambda_i \}$ where the vector $u_i$ is an inward pointing normal vector to the $i$th facet of $\Delta$ and each $\lambda_i$ is a real scalar. In this notation, the polytope $\Delta$ is Delzant if and only if there are precisely $n$ facets incident to each vertex of $\Delta$ and the inward pointing normals to these facets $u_1, \ldots, u_n$ can be chosen to be a $\mathbb{Z}$-basis of $\mathbb{Z}^n$. For each face $\Delta'$ of $\Delta$ of codimension $k$, there is a unique multi-index $I_{\Delta'}$ of length $k$, $I_{\Delta'} = \{i_1, \ldots, i_k\}$, $1 \leq i_1 < \ldots < i_k \leq n$, such that $\Delta' = \{x \in \mathbb{R}^n \mid \langle x, u_j \rangle = \lambda_j, \forall j \in I_{\Delta'}\}$.

**Definition 2.4** (eg. [4]). Letting $\sigma_{\Delta'}$ denote the cone in $\mathbb{R}^n$ generated by the vectors $\{u_j \mid j \in I_{\Delta'}\}$, the complete regular $n$-dimensional fan associated to $\Delta$ is given by $\text{Fan}(\Delta) := \{\sigma_{\Delta'} \mid \Delta' \text{ is a face of } \Delta\}$.

It is a well known fact that if two Delzant polytopes have the same associated fan, their associated symplectic-toric manifolds are equivariantly diffeomorphic.

### 3. Moduli spaces of toric manifolds

With the conventions above, where $T$ and the identification $\mathcal{I} : t^* \to \mathbb{R}^n$ are fixed, we next define the moduli space of toric manifolds.

#### 3.1. The moduli relation.

Let $(M, \omega, T, \mu : M \to \mathbb{R}^n)$ and $(M', \omega', T, \mu' : M \to \mathbb{R}^n)$ be symplectic toric manifolds. These two symplectic toric manifolds are **isomorphic** if there exists an equivariant symplectomorphism $\varphi : M \to M'$.
(i.e., \( \varphi \) is a diffeomorphism satisfying \( \varphi^*\omega' = \omega \) which intertwines the \( T \) actions) such that \( \mu' \circ \varphi = \mu \) (see also [1, Definition I.1.16]). We denote by \( \mathcal{M}_T := \mathcal{M}_T^L \)

the moduli space (the set of equivalence classes) of 2n-dimensional symplectic toric manifolds under this equivalence relation. The motivation for introducing this moduli space comes from the following seminal result, due to Delzant.

**Theorem 3.1.** (Delzant [3, Theorem 2.1]) Let \( (M, \omega, T, \mu) \), \( (M', \omega', T, \mu') \) be two toric symplectic manifolds. If \( \mu(M) = \mu'(M') \) then \( (M, \omega, T, \mu) \) and \( (M', \omega', T, \mu') \) are isomorphic.

The convexity theorem of Atiyah [2] and Guillemin-Sternberg [5] asserts that the image of the momentum map is a convex polytope. In addition, if the action is toric (the acting torus is precisely half the dimension of the manifold) the momentum image is a Delzant polytope. Let \( \mathcal{D}_T \) denote the set of Delzant polytopes. As a consequence of Theorem 3.1 the following map

\[
\begin{equation}
[(M, \omega, T, \mu)] \ni M_T \longmapsto \mu(M) \in \mathcal{D}_T
\end{equation}
\]

is an injection. However, Delzant also shows how from a Delzant polytope it is possible to reconstruct a symplectic toric manifold, thus implying that (3.1) is a bijection. To simplify the notation we usually write \( (M, \omega, T, \mu) \) identifying the representative with its equivalence class \( [(M, \omega, T, \mu)] \) in \( \mathcal{M}_T \).

3.2. **Metric on \( \mathcal{M}_T \).** We consider the space of Delzant polytopes \( \mathcal{D}_T \) and turn it into a metric space by endowing it with the distance function given by the volume of the symmetric difference \( (\Delta_1 \setminus \Delta_2) \cup (\Delta_2 \setminus \Delta_1) \) of any two polytopes. The map (3.1) allows us to define a metric \( d_T \) on \( \mathcal{M}_T \) as the pullback of the metric defined on \( \mathcal{D}_T \), thereby getting the metric space \( (\mathcal{M}_T, d_T) \). This metric induces a topology \( \nu \) on \( \mathcal{M}_T \) (so, by definition, it follows that \( (\mathcal{M}_T, \nu) \) is a metrizable topological space).

Let \( \mathcal{B}(\mathbb{R}^n) \) be the \( \sigma \)-algebra of Borel sets of \( \mathbb{R}^n \), \( \lambda: \mathcal{B}(\mathbb{R}^n) \to \mathbb{R}_{\geq 0} \cup \{\infty\} \) the Lebesgue measure on \( \mathbb{R}^n \), and \( \mathcal{B}'(\mathbb{R}^n) \subset \mathcal{B}(\mathbb{R}^n) \) the Borel sets with finite Lebesgue measure. Define

\[
\begin{equation}
d(A, B) := \|\chi_A - \chi_B\|_{L^1},
\end{equation}
\]

where \( \chi_C: \mathbb{R}^n \to \mathbb{R} \) denotes the characteristic function of \( C \in \mathcal{B}'(\mathbb{R}^n) \). This extends the distance function defined above on \( \mathcal{D}_T \), but it is not a metric on \( \mathcal{B}'(\mathbb{R}^n) \). Identifying the sets \( A, B \in \mathcal{B}'(\mathbb{R}^n) \) for which \( d(A, B) = 0 \), we obtain a metric on the resulting quotient space of \( \mathcal{B}'(\mathbb{R}^n) \). Let \( \hat{C} \) be the space of convex compact subsets of \( \mathbb{R}^n \) with positive Lebesgue measure, \( \emptyset \) the empty set, and \( \hat{C} := C \cup \{\emptyset\} \). Then \( \hat{C} \) equipped with the distance function \( d \) in (3.2) is a metric space.
3.3. **Topological structure in dimension 4.** The following theorem, which holds in dimension 4, describes properties of the topology of the aforementioned moduli spaces.

**Theorem 3.2** ([8]). Suppose that \( n = 2 \), that is, \( \mathbb{T} \) is a 2-dimensional torus. Then \( (\mathcal{M}_\mathbb{T}, d_{\mathbb{T}}) \) is neither locally compact nor a complete metric space. Its completion can be identified with the metric space \( (\hat{\mathcal{C}}, d) \) in the following sense: identifying \( (\mathcal{M}_\mathbb{T}, d_{\mathbb{T}}) \) with \( (\mathcal{D}_\mathbb{T}, d) \) via (3.1), the completion of \( (\mathcal{D}_\mathbb{T}, d) \) is \( (\hat{\mathcal{C}}, d) \).

4. **Density function**

Let \( (M, \omega, \mathbb{T}, \mu) \) be a symplectic-toric manifold, \( \Lambda \in \text{Aut}(\mathbb{T}) \), and \( r > 0 \). A subset \( B \subset M \) is said to be a \( \Lambda \)-equivariantly embedded symplectic ball of radius \( r \) if there exists a symplectic embedding \( f : \mathbb{B}_r \to M \) with image \( B \) and such that the following diagram commutes:

\[
\begin{array}{ccc}
\mathbb{T}^n \times \mathbb{B}_r & \xrightarrow{\Lambda \times f} & \mathbb{T}^n \times M \\
\downarrow \text{Rot} & & \downarrow \psi \\
\mathbb{B}_r & \xrightarrow{f} & M
\end{array}
\]

We say that the \( \Lambda \)-equivariantly embedded symplectic ball \( B \) has center \( f(0) \in M \). We say that a subset \( B' \subset M \) is an equivariantly embedded symplectic ball of radius \( r' \) if there exists \( \Lambda' \in \text{Aut}(\mathbb{T}) \) such that \( B' \) is a \( \Lambda' \)-equivariantly embedded symplectic ball of radius \( r' \).

We define the symplectic volume of a subset \( A \subset M \) by \( \text{vol}_\omega(A) := \int_A \omega^n \).

**Definition 4.1** (Definition 1.6 in [6]). A toric ball packing of \( M \) is a disjoint union \( \mathcal{P} := \bigsqcup_{\alpha \in A} B_\alpha \) of equivariantly embedded symplectic balls \( B_\alpha \) (of possibly varying radii) in \( M \). The density \( \Omega(\mathcal{P}) \) of a packing \( \mathcal{P} \) is defined by \( \Omega(\mathcal{P}) := \text{vol}_\omega(\mathcal{P}) / \text{vol}_\omega(M) \). The density \( \Omega(M, \omega, \psi) \) of a symplectic-toric manifold \( (M, \omega, \psi) \) is defined by

\[
\Omega(M, \omega, \psi) := \sup \{ \Omega(\mathcal{P}) \mid \mathcal{P} \text{ is a toric packing of } M \}.
\]

A packing achieving this density is said to be a maximal density (or optimal) packing.

The maximal density function (which assigns to a manifold the density of its optimal toric ball packing) is interesting because it is a symplectic invariant. In this paper we analyze the following problem, see Theorem A.

**Problem 4.2** (Problem 4 in [8]). The maximal density function \( \Omega : \mathcal{M}_\mathbb{T} / \simeq \to (0, 1] \) is most interesting when considered on \( \mathcal{M}_\mathbb{T} / \simeq \), where \( \simeq \) corresponds to rescaling the symplectic form (it does not change the density). With respect to the quotient topology on \( \mathcal{M}_\mathbb{T} / \simeq \), is \( \Omega \) continuous?
5. Convex geometry

The following definitions appeared in [6] in a slightly different but equivalent form.

Definition 5.1. Let $\Delta$ be a Delzant polytope. A subset $\Sigma \subset \Delta$ is said to be an admissible simplex of radius $r$ with center at the vertex $v \in \Delta$ if $\Sigma$ is the image of $\Delta(r^{1/2})$ by an element of $\text{AGL}(n, \mathbb{Z})$ which takes the origin to $v$ and the edges of $\Delta(r^{1/2})$ to the edges of $\Delta$ meeting $v$. For a vertex $v \in \Delta$, we put $r_v := \max\{r > 0 \mid \exists \text{an admissible simplex of radius } r \text{ with center } v\}$.

Remark 5.2. In view of (2.1), the simplex $\Delta(r^{1/2})$ may be identified with the set obtained by removing from $\text{ConvHull}(0, re_1, \ldots, re_n)$ the facet not containing the origin. For this reason we say that $\text{AGL}(n, \mathbb{Z})$ images of $\Delta(r^{1/2})$ as in the above definition have radius $r$ instead of radius $r^{1/2}$.

We denote the Euclidean volume of a subset $A \subset \Delta$ by $\text{vol}_{\text{euc}}(A)$.

Definition 5.3. Let $\Delta$ be a Delzant polytope. An admissible packing of $\Delta$ is a disjoint union $\mathcal{P} := \bigsqcup_{\alpha \in A} \Sigma_{\alpha}$ of admissible simplices (of possibly varying radii) in $\Delta$. The density $\Omega(\mathcal{P})$ of a packing $\mathcal{P}$ is $\Omega(\mathcal{P}) := \frac{\text{vol}_{\text{euc}}(\mathcal{P})}{\text{vol}_{\text{euc}}(\Delta)}$. The density $\Omega(\Delta)$ of a Delzant polytope $\Delta$ is defined by

$$\Omega(\Delta) := \sup\{\Omega(\mathcal{P}) \mid \mathcal{P} \text{ is an admissible packing of } \Delta\}.$$ 

A packing achieving this density is said to be a maximal density (or optimal) packing.

The next lemma shows that admissible simplices in $\Delta$ are parametrized by their centers and radii. The rational or SL$(n, \mathbb{Z})$-length of an interval $I \subset \mathbb{R}^n$ with rational slope is the unique number $l := \text{length}_{\mathbb{Q}}(I)$ such that $I$ is $\text{AGL}(n, \mathbb{Z})$-congruent to an interval of length $l$ on a coordinate axis. For a vertex $v$ in a Delzant polytope, we denote the $n$ edges leaving $v$ by $e^1_v, \ldots, e^n_v$. By Definition 2.3, each $e^i_v$ is of the form $v + t^i_v u^i_v$ with $t^i_v > 0$ and $\{u^i_v\}_{i=1}^n$ defining a $\mathbb{Z}$-basis of $\mathbb{Z}^n$. In this notation, we have that $\text{length}_{\mathbb{Q}}(e^1_v) = t^1_v$.

Lemma 5.4 ([6]). Let $\Delta$ be a Delzant polytope. Then, for each vertex $v \in \Delta$ it holds $r_v = \min\{\text{length}_{\mathbb{Q}}(e^1_v), \ldots, \text{length}_{\mathbb{Q}}(e^n_v)\}$. Also, there is an admissible simplex $\Sigma(v, r)$ of radius $r$ with center $v$ if and only if $0 \leq r \leq r_v$. Moreover this admissible simplex is unique and $\text{vol}_{\text{euc}}(\Sigma(v, r)) = r^n/n!$.

Theorem 5.5 ([6]). Let $(M, \omega, T, \mu)$ be a symplectic-toric manifold with associated Delzant polytope $\Delta := \mu(M)$.

(i) Let $B \subset M$ be an equivariantly embedded symplectic ball of radius $r$ and center $p \in M$. Then $\mu(B)$ is an admissible simplex of radius $r^2$ in $\Delta$ with center $\mu(p)$. Conversely, if $\Sigma \subset \Delta$ is an admissible simplex of radius $r$, then there exists an equivariantly embedded symplectic ball $B \subset M$ of radius $r^{1/2}$ satisfying $\mu(B) = \Sigma$. 

(ii) For each toric ball packing $\mathcal{P}$ of $M$, $\mu(\mathcal{P})$ is an admissible packing of $\Delta$ satisfying $\Omega(\mathcal{P}) = \Omega(\mu(\mathcal{P}))$. Moreover, given an admissible packing $\mathcal{Q}$ of $\Delta$, there exists a toric ball packing $\mathcal{P}$ of $M$ satisfying $\mu(\mathcal{P}) = \mathcal{Q}$.

6. The combinatorial convexity statement

In this section we state and prove a theorem in convex geometry, that in view of Theorem 5.5 implies Theorem A. Recall the notion of fan, eg. Definition 2.4.

**Theorem 6.1.** Let $N \geq 1$ be an integer and let $\mathcal{P}^N$ be the set of Delzant polytopes of $N$ vertices, and let $\mathcal{P}$ be the set of all Delzant polytopes, so that $\mathcal{P} = \bigcup_{N \geq 1} \mathcal{P}^N$. Then:

1. $\Omega$ is discontinuous at every $\Delta \in \mathcal{P}$, and the restriction $\Omega|_{\mathcal{P}^N}$ is continuous for each $N \geq 1$.

2. Given $\Delta \in \mathcal{P}^N \subset \mathcal{P}$, define $\{\Omega_i(\Delta)\}_{1 \leq i \leq N}$ to be the maximal density avoiding vertex $i$. Then $\mathcal{P}^N$ is the largest set containing $\Delta$ where $\Omega$ is continuous if and only if $\Omega_i(\Delta) < \Omega(\Delta)$ for all $1 \leq i \leq N$.

**Proof.** The proof of this result is based on geometric arguments that can be understood more easily in the case $n = 2$ (i.e., when dealing with Delzant polygons) but they work with minor modifications in any dimension (notice that we will rely only on results from [9] that hold in any dimension). Moreover, the case of symplectic toric 4-manifolds is currently the most interesting as the topology of the moduli space is understood only in that case (see Theorem 3.2). For this reason we detail the proof only in the 2-dimensional case, leaving the details of the general case to the interested reader.

In the proof of this result we will constantly make use of the following simple observation (see also the discussion below): given $\Delta \in \mathcal{P}$, a neighborhood of $\Delta$ is made up of polygons where either we translate the sizes in a parallel way, or we chop some small corners of $\Delta$. In particular the number of vertices can only increase.

We prove (1) first. To show that $\Omega$ is discontinuous at any $\Delta \in \mathcal{P}^N$, we fix $\epsilon > 0$ small and we chop all the corners, adding around each of them a small side of length $\leq \epsilon$. This gives us a polygon with $2N$ vertices with the following property: when considering the definition of density, any simplex will have at least one side of length at most $\epsilon$ while the others are universally bounded, so the total volume will be at most $CN\epsilon$. Since $\epsilon$ can be arbitrarily small, this proves that $\Omega$ cannot be continuous.

On the other hand, we show that $\Omega$ is continuous when restricted to $\mathcal{P}^N$. To see this, we call an angle $\alpha$ smooth if it can be obtained as the angle of a smooth simplex. Then, because

$$\frac{1}{2} = \frac{\ell_1 \ell_2 \sin \alpha}{2}, \quad \ell_1 \geq 1, \quad \ell_2 \geq 1,$$
we see that
\[ \ell_1, \ell_2 \leq C. \]
Now, fix \( \Delta \in \mathcal{P}_N \), and let \( \alpha_1, \ldots, \alpha_n \) be the angles of \( \Delta \).
Then, if \( \Delta' = (\alpha'_1, \ldots, \alpha'_N) \in \mathcal{P}^N \) with \( d(\Delta', \Delta) \ll 1 \), since the set of smooth angles \( \alpha \) is finite it follows that \( \alpha'_i = \alpha_i \). Hence, for every \( \Delta' \) close to \( \Delta \) we are just translating the sides \( \alpha_1, \ldots, \alpha_n \) in a parallel way, and we know that \( \Omega \) is continuous along this family of transformations [9], so \( \Omega \) is continuous on the whole \( \mathcal{P}^N \), proving (1).

Next we show (2). Assume first that
\[ \Omega(\Delta) = \Omega_i(\Delta) \text{ for some } i \in \{1, \ldots, N\}. \]
This implies that we can find an optimal family in the definition of \( \Omega \) with no simplex centered at \( i \) (simply take an optimal family in the definition of \( \Omega_i(\Delta) \)). Then, by slightly reducing the radius of each simplex we can keep the density arbitrarily close to \( \Omega(\Delta) \) making sure that no simplex touches the vertex \( i \). This gives us the possibility to chop a small corner around the vertex \( i \), obtaining Delzant polytope \( \Delta' \in \mathcal{P}^{N+1} \) keeping the density still arbitrarily close \( \Omega(\Delta) \). This procedure shows that we can find a family \( \{\Delta_k\}_{k \geq 1} \subset \mathcal{P}^{N+1} \) with
\[ d(\Delta_k, \Delta) \to 0 \text{ and } \Omega(\Delta_k) \to \Omega(\Delta), \]
proving that there is a set larger than \( \mathcal{P}^N \) where \( \Omega \) is continuous at \( \Delta \).

Viceversa, if there exists \( \eta > 0 \) such that
\[ \Omega_i(\Delta) \leq \Omega(\Delta) - \eta \text{ for all } i = 1, \ldots, N, \]
if we chop any corner around one of the vertices (say around \( i \)) then necessarily the density will be close to \( \Omega_i(\Delta) \), hence less than
\[ \Omega(\Delta) - \eta/2, \]
thus discontinuous. This concludes the proof of (2). \[ \square \]

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