Cubic fourfolds containing a plane and $K3$
surfaces of Picard rank two

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Abstract

We present some new examples of families of cubic hypersurfaces in $\mathbb{P}^5(\mathbb{C})$ containing a plane whose associated quadric bundle does not have a rational section.

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1 Introduction

Let $X$ be a smooth cubic hypersurface in $\mathbb{P}^5(\mathbb{C})$. Investigating the rationality of $X$ is a classical problem in algebraic geometry. The general $X$ is conjectured to be not rational but not a single example of non rational cubic fourfold is known.

Cubic fourfolds containing a quartic scroll or a quintic del Pezzo surface are rational (see [F], [Mo]). Idem for those fourfolds containing a plane and a Veronese surface (see [Tr]). Beauville and Donagi showed in [BD] that also pfaffian cubic fourfolds are rational.

The closure of the locus of pfaffian cubic fourfolds is a divisor $C_{14}$ in the moduli space $C$ of all cubic fourfolds, while the fourfolds containing a plane form a divisor $C_8$ (see [H2]). The general fourfold containing a plane is also expected to be non rational. Nevertheless, Hassett showed in [H1] that there exists a countable infinite collection of divisors in $C_8$ which parameterize rational cubic fourfolds. The fourfolds containing a plane are birational to the total space of a quadric surface bundle by projecting from the plane: Hassett’s examples are rational since the associated quadric bundle has a rational section. We call these hypersurfaces trivially rational.

Auel et al. (see [ABBV]) have described a divisor in $C_8$ whose very general member parameterizes rational but not trivially rational cubic fourfolds. They are all pfaffian, so rational. In a recent paper, Bolognesi and Russo proved that every cubic hypersurface belonging to $C_{14}$ is rational [BR].

Using results on the Hodge structure of cubic fourfolds and $K3$ surfaces, we present a family of cubic fourfolds containing a plane which are not trivially rational...
rational. We don’t know if these fourfolds are rational. The rational example in [ABBV] is in our family.

The paper is organized as follows. In Sections 2 and 3 we recall some basic notions on lattices and $K3$ surfaces. We focus on $K3$ surfaces of Picard rank two recalling the fundamental work of Nikulin in [N]. Then in 3.1 we present the $K3$ surfaces of Picard rank two which are double covers of the plane ramified over a sextic curve. In 3.1.1 we construct a family $S(b,c)$ of double planes with Picard rank two. In Section 4 we recall how these surfaces are related to cubic 4–folds containing a plane. Such a cubic $X$ is birational to a quadric bundle $Y \longrightarrow \mathbb{P}^2$ which, in the general case, ramifies over a smooth sextic curve $C$. The Hodge structure of $X$ is strictly related to the Hodge structure of the $K3$ surface $S$ obtained as a double cover of the plane ramified over $C$ and parameterizing the rulings of the quadrics in the fibration $Y \longrightarrow \mathbb{P}^2$ (see [V, §1]). We use the following fact: the lattice $A(X)$ of 2–cycles modulo numerical equivalence on $X$ has rank three and even discriminant if $S$ has Picard rank two and even Néron-Severi discriminant (see [V, §1 Proposition 2]). In case of $\text{rk}(A(X)) = 3$ it is known that the quadric bundle $Y \longrightarrow \mathbb{P}^2$ does not have a rational section if and only if the discriminant of $A(X)$ is even (see Proposition 4.0.4).

We prove that if $X$ is not trivially rational, the discriminant $d(A(X))$ is even, without restrictions on the rank of $A(X)$ (see Proposition 4.0.6).

In 4.1 we recover the cubic hypersurfaces associated to the double planes $S(b,c)$ using the additional datum of an odd theta characteristic on the discriminant sextic (see [B, V]).

In Theorem 4.1.2 we prove that the fourfolds corresponding to $S(b,c)$ with $d$ even are not trivially rational. The rational example in [ABBV, Theorem 11] correspond to fourfolds associated to $S(2, -1)$. Theorem 4.1.2 gives only a sufficient condition for the existence of not trivially rational 4–folds: there are cubic fourfolds containing a plane associated to double planes $S(b,c)$ with $b$ odd which are not trivially rational (see Proposition 4.1.4).

## 2 Lattices

A lattice is a free $\mathbb{Z}$-module $L$ of finite rank with a $\mathbb{Z}$-valued symmetric bilinear form $b_L(x, y)$. A lattice is called even if the quadratic form $q_L$ associated to the bilinear form has only even values, odd otherwise. The discriminant $d(L)$ of a lattice is the determinant of the matrix of its bilinear form. A lattice is called non-degenerate if the discriminant is non-zero and unimodular if the discriminant is ±1. If the lattice $L$ is non-degenerate, the pair $(s_+, s_-)$, where $s_\pm$ denotes the multiplicity of the eigenvalue $\pm 1$ for the quadratic form associated to $L \otimes \mathbb{R}$, is called signature of $L$. Finally, we call $s_+ + s_-$ the rank of $L$ and $L$ is said indefinite if the associate quadratic form has both positive and negative values.

Given a lattice $L$, the lattice $L(m)$ is the $\mathbb{Z}$-module $L$ with bilinear form $b_{L(m)}(x, y) = mb_L(x, y)$. An isometry of lattices is an isomorphism preserving
the bilinear form. Given a sublattice $L' \subset L$, the embedding is primitive if $\frac{L}{L'}$ is free.

Let $L^* = \text{Hom}_\mathbb{Z}(L, \mathbb{Z}) = \{ x \in L \otimes \mathbb{Q} : b_L(x, l) \in \mathbb{Z}, \forall l \in L \}$ be the dual of the lattice $L$. There is a natural embedding $L \hookrightarrow L^*$ given by $l \mapsto b_L(l, -)$. There is the following

**Lemma 2.0.1.** [BPV, I, Lemma 2.1.] Let $L$ be a non-degenerate lattice. Then

1. $[L^* : L] = |d(L)|$

2. $[L : L']^2 = \frac{d(L')}{d(L)}$, where $L' \subset L$ is a sublattice with $\text{rk}(L') = \text{rk}(L)$.

Denote by $L$ a non-degenerate even lattice. The bilinear form $b_L$ induces a $\mathbb{Q}$-valued bilinear form on $L^*$ and so a finite quadratic form $q_{A_L} : L^*/L \to \mathbb{Q}/2\mathbb{Z}$ called the discriminant form of $L$. The group $L^*/L := A_L$ is the discriminant group of $L$.

### 2.1 Examples.

i) The lattice $\langle n \rangle$ is a free $\mathbb{Z}$-module of rank one, $\mathbb{Z}\langle e \rangle$, with bilinear form $b(e, e) = n$.

ii) The hyperbolic lattice is the even, unimodular, indefinite lattice with $\mathbb{Z}$-module $\mathbb{Z}\langle e_1, e_2 \rangle$ and bilinear form given by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We write

$$U = \left\{ \mathbb{Z}^2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

iii) The lattice $E_8$ has $\mathbb{Z}^8$ as $\mathbb{Z}$-module and the matrix of the bilinear form is the Cartan matrix of the root system of $E_8$. It is an even, unimodular and positive definite lattice.

### 3 K3 surfaces of rank two

A K3 surface is a smooth projective surface $S$ with trivial canonical class and $H^1(S, \mathcal{O}_S) = 0$.

It is well known that $H^2(S, \mathbb{Z})$ is an even, unimodular, indefinite lattice, with respect to the intersection form on $S$. It has rank 22, signature $(3, 19)$ and it is isomorphic to

$$\Lambda := U^\oplus 3 \oplus E_8(-1)^\oplus 2.$$
The lattice $\Lambda$ will be called the $K3$ lattice. The Hodge numbers are $(1, 20, 1)$, (see [BPV, VIII]). Denote by

$$NS(S) \cong H^2(S, \mathbb{Z}) \cap H^{1,1}(S)$$

the Néron-Severi lattice of $S$, it is a primitive sublattice of $H^2(S, \mathbb{Z})$. Rational, algebraic and homological equivalence coincide on a $K3$ surface.

The orthogonal complement $T(S)$ of $NS(S)$ in $H^2(S, \mathbb{Z})$ is the transcendental lattice of $S$.

The rank of $S$, $\rho(S)$, is the rank of $NS(S)$. The Hodge Index Theorem implies that $NS(S)$ has signature $(1, \rho(S) - 1)$ and that $T(S)$ has signature $(2, 20 - \rho(S))$. Let $l \in NS(S)$ be a class with $l^2 > 0$. The primitive cohomology $H^2(S, \mathbb{Z})^0$ is the orthogonal complement of the lattice $<l>$. 

Main tools for the study of $K3$ surfaces are the Torelli Theorem (see [LP] and [PSS]) and the Surjectivity of the Period Map (see [T]). The period of $S$ is given by $[\omega_S] = P(H^2,0(S))$ in the period domain

$$\Omega = \{x \in \mathbb{P}(\Lambda \otimes \mathbb{C})| x \cdot x = 0, x \cdot \bar{x} > 0\} \subset \mathbb{P}(\Lambda \otimes \mathbb{C}).$$

By the Torelli Theorem and the Surjectivity of the Period Map, an element $\omega$ in the period domain determines the $K3$ surface: given $\omega \in \Omega$ there exists a $K3$ surface $S_\omega$ (unique up to isomorphism) with period $\omega$ such that $H^2(S_\omega, \mathbb{Z})$ is isometric to $\Lambda$.

Nikulin in [N] made a deep study of lattice theory and integral quadratic forms with applications to the study of $K3$ surfaces. We recall the following which is crucial for our purposes

**Theorem 3.0.1.** [N, Theorem 1.14.4] [M, Corollary 2.9] If $\rho(S) \leq 10$, then every even lattice $M$ of signature $(1, \rho - 1)$ occurs as the Néron-Severi group of some $K3$ surface and the primitive embedding $M \hookrightarrow \Lambda$ is unique.

**Corollary 3.0.2.** All even lattices of rank $2$ and signature $(1, 1)$ occur as the Néron-Severi lattice $NS(S)$ of some $K3$ surface $S$ of rank two and the primitive embedding $NS(S) \hookrightarrow \Lambda$ is unique. Any such lattice has the form

$$M = \left\{ \mathbb{Z}^2, \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix} \right\}$$

with $a \geq 0$ and $b^2 - 4ac > 0$.

### 3.1 $K3$ surfaces double planes of rank two

A double covering of the projective plane $\varphi : S \rightarrow \mathbb{P}^2$ branched along a smooth sextic $C$ is a $K3$ surface: $\varphi_* (\mathcal{O}_S) \cong \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}(3)$, so $H^1(S, \mathcal{O}_S) = 0$ and $\omega_S \cong \varphi^*(\omega_{\mathbb{P}^2} \otimes \mathcal{O}(3)) \cong \mathcal{O}_S$. The $K3$ surface $S$ in this case is called a double plane. For general references on double planes, see [En] and [S]. An ample class $l \in NS(S)$
with \( l^2 = 2 \) is the pull-back of the class of a line in \( \mathbb{P}^2 \). If \( S \) has rank two the Néron-Severi lattice has the form

\[
L(b,c) = \left\{ \mathbb{Z}^2, \begin{pmatrix} 2 & b \\ b & 2c \end{pmatrix} \right\}.
\]

### 3.1.1 Examples.

i) Consider \( S \) a K3 surface double plane ramified over a smooth sextic with Néron-Severi lattice of the form

\[
L(1,-1) = \left\{ \mathbb{Z}^2, \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \right\}.
\]

This can be realized by taking a double cover of the plane ramified over a sextic curve having a tritangent line \( l \). The pull-back of \( l \) to \( S \) is a divisor splitting into two irreducible components \( l_1, l_2 \). The corresponding divisor classes are linearly independent. Both curves are isomorphic to \( l \) and \( l_1^2 = l_2^2 = -2 \).

ii) Analogously, if the Néron-Severi lattice has the form

\[
L(2,-1) = \left\{ \mathbb{Z}^2, \begin{pmatrix} 2 & 2 \\ 2 & -2 \end{pmatrix} \right\}
\]

the corresponding double plane \( S \) can be realized with a ramification sextic \( C \) which is tangent to a conic \( D \) in 6 points with multiplicity two. As before, \( \varphi^*(D) = D_1 + D_2 \), with \( D_1, D_2 \) isomorphic to \( D \) and \( D_1^2 = D_2^2 = -2 \).

The previous examples can be generalized as follows.

**Lemma 3.1.1.** If \( b > 0 \) and \( b^2 - 4c > 0 \), then the lattice

\[
L(b,c) = \left\{ \mathbb{Z}^2, \begin{pmatrix} 2 & b \\ b & 2c \end{pmatrix} \right\}
\]

is the Néron-Severi lattice of a double plane \( S_{(b,c)} \) with a smooth ramification sextic.

**Proof.** The lattice \( L(b,c) \) is even and it has signature \((1,1)\). By Theorem 3.0.1 and Corollary 3.0.2, \( L(b,c) \) occurs as the Picard group of a K3 surface: denote by \( S(b,c) = S_\alpha \) the K3 surface defined by \( \alpha \in \Omega \) with \( \alpha^\perp = L(b,c) \) and, moreover, generic with this property, hence \( L(b,c) = NS(S_{(b,c)}) \). Let \( H, A \) be the classes \((1,0)\) and \((0,1)\) in \( NS(S_{(b,c)}) \), respectively. For each divisor \( \Gamma \) with \( \Gamma^2 = -2 \) we have the Picard–Lefschetz reflection \( \pi_\Gamma \) of \( NS(S_{(b,c)}) \) defined by \( D \mapsto D + (D\Gamma)\Gamma \). If \( D' \) is another divisor on \( S_{(b,c)} \), then \( \pi_\Gamma(D)\pi_\Gamma(D') = DD' \), because \( \Gamma^2 = -2 \). The cone of big and nef divisors is a fundamental domain for the group generated by the above reflections (see for example [Huy1, Chapter 8,
Corollary 2.11]). In particular, we can find divisors $\Gamma_i$ with $\Gamma_i \Gamma_j = -2\delta_{i,j}$, $i = 1, \ldots, l$, such that

$$H' := H + \sum_{i=1}^l (H\Gamma_i)\Gamma_i$$

is nef. Let

$$A' := A + \sum_{i=1}^l (A\Gamma_i)\Gamma_i.$$

Thus $\text{NS}(S_{(b,c)}) = \langle H, A \rangle = \langle H', A' \rangle$. Omitting the prime in the superscript we can thus assume that $H$ is nef.

Let $H = F + M$ be its decomposition in the fixed part $F$ and the mobile part $M$, then $M$ is nef too. Observe that $M^2 = H^2 = 2$ (see for example [Huy1, Chapter 2, Remark 3.3]). Since, moreover, $M$ is without fixed part by definition, it defines a double cover $\varphi : S_{(b,c)} \to \mathbb{P}^2$. The ramification curve $C$ is smooth since a point $x \in S$ is singular iff $\varphi(x)$ is a singular point of $C$ (see for example [S, p.8]).

4 Cubic 4–folds containing a plane

Let $X$ be a smooth cubic hypersurface in $\mathbb{P}^5(\mathbb{C})$. Consider the cohomology group $H^4(X, \mathbb{Z})$ and denote with

$$A(X) = H^4(X, \mathbb{Z}) \cap H^{2,2}(X)$$

the lattice of the middle integral cohomology Hodge classes. Those classes are algebraic since $X$ verifies the integral Hodge conjecture (see [Mu] and [Zu]). The transcendentnal lattice $T(X)$ is the orthogonal complement of $A(X)$ (with respect to the intersection form on $X$).

From now on $X$ will indicate a cubic hypersurface in $\mathbb{P}^5$ containing a plane. Consider the projection from the plane $P$ onto a plane in $\mathbb{P}^5$ disjoint from $P$. Blowing up $X$ along $P$ one obtains a quadric bundle $\pi : Y \to \mathbb{P}^2$ branched over $C$, the discriminant sextic. If $X$ does not contain a second plane intersecting $P$, the curve $C$ is smooth and this means that the quadrics of the bundle have rank $\geq 3$ (see [V, §1 Lemme 2]).

Denote by $Q$ the class of such a quadric. One has $P + Q = H^2$, where $H$ is the hyperplane class associated to the embedding $X \hookrightarrow \mathbb{P}^5(\mathbb{C})$. The hypersurface $X$ is said to be very general if $A(X) = \langle H^2, P \rangle$ ($= \langle H^2, Q \rangle$). Denote $L := \langle H^2, P \rangle^1$.

$X$ is rational iff $Y$ is rational and a sufficient condition for the rationality of $Y$ is the existence of a rational section.

**Definition 4.0.2.** We call a cubic hypersurface $X \subset \mathbb{P}^5$ containing a plane trivially rational if the associated quadric bundle has a rational section.
This fact may be translated in a condition on the parity of the intersection of some 2-cycle on \( X \). More precisely, for a 2-cycle \( T \) in \( X \) consider the intersection index
\[
\delta(T) = T \cdot Q.
\]
Note that \( \delta(P) = -2 \) and \( \delta(H^2) = 2 \). So, if \( X \) is very general the index \( \delta \) takes only even values. There is the following result (see [H2, Theorem 3.1.], [ABBV, Proposition 2], [H1, Lemma 4.4.]).

**Theorem 4.0.3.** A cubic fourfold \( X \) containing a plane is trivially rational if and only if there exists a cycle \( T \) in \( A(X) \) with \( \delta(T) \) odd.

Using this Theorem it is easy to give (lattice-theoretic) hints to construct cubic fourfolds with \( \text{rk}(A(X)) > 2 \) and not trivially rational (see [H1, Lemma 4.4.] and [ABBV, Proposition 2]).

**Proposition 4.0.4.** Let \( X \) be a cubic fourfold containing a plane with \( \text{rk}(A(X)) = 3 \). Thus \( X \) is trivially rational if and only if \( d(A(X)) \) is odd.

**Proof.** The quadric bundle \( \pi : Y \rightarrow \mathbb{P}^2 \) has a rational section if and only if there exists a cycle \( T \in A(X) \) such that \( \delta(T) \) is odd (by Theorem 4.0.3). Since \( A(X) \) has rank 3, the sublattice \( \langle H^2, Q, T \rangle \) has finite index, hence Lemma 2.0.1 implies that, if \( \langle H^2, Q, T \rangle \) has odd discriminant, then \( d(A(X)) \) is odd as well.

Our aim now is to build some geometric examples. To do this, we need to better understand the links between Hodge theory and the geometry on a cubic 4-fold containing a plane. Here we follow Voisin [V, §1].

Let \( \varphi : S \rightarrow \mathbb{P}^2 \) be the double cover branched over \( C \), the discriminant sextic of the quadric bundle \( Y \rightarrow \mathbb{P}^2 \). The surface \( S \) parameterizes the rulings of the quadrics of the fibration. Let \( F \) be the Fano variety of lines in \( X \), the subvariety of the Grassmannian \( \text{Gr}(1,5) \) parameterizing lines contained in \( X \). The divisor \( D \subset F \) consisting of lines meeting \( P \) is identified with
\[
D = \{(l,s) \in F \times S : l \text{ is in the ruling of the quadric parameterized by } \varphi(s)\}.
\]
giving a \( \mathbb{P}^1 \)-bundle
\[
f : D \rightarrow S. \tag{1}
\]
The incidence graph restricted to \( D \)
\[
D \times X \supset Z_D \xrightarrow{p} D \xrightarrow{q} X
\]
defines the Abel-Jacobi map:
\[
\alpha_D = p_* q^* : H^4(X, \mathbb{Q}) \rightarrow H^2(D, \mathbb{Q})
\]
which induces an isomorphism of Hodge structures, see [V, §1 Proposition 1]. Before stating the next result, we recall that we denote by $L$ the orthogonal complement of the lattice $<H^2, P>$ in $H^4(X, \mathbb{Z})$, where $H$ is the hyperplane class and $P$ is the class of a plane contained in $X$.

**Proposition 4.0.5.** ([V, §1 Proposition 2], [ABBV, Proposition 1]) Let $X$ be a smooth cubic fourfold containing a plane. Then $\alpha_D(L) \subset f^*(H^2(S, \mathbb{Z})_0(-1))$ is a polarized Hodge substructure of index 2. Moreover, $\alpha_D(T(X)) \subset f^*T(S)(-1)$ is a sublattice of index $\epsilon$ dividing 2. In particular, $\text{rk } A(X) = \text{rk } (NS(S)) + 1$ and $d(A(X)) = (-1)^{\epsilon(S)-1}2^{\epsilon-1}d(NS(S))$.

We can also derive the following result, which amplifies Proposition 4.0.4.

**Proposition 4.0.6.** Let $X$ be a cubic fourfold containing a plane. If $X$ is not trivially rational, then $\alpha_D(T(X)) \subset f^*T(S)(-1)$ is a sublattice of index 2 and $d(A(X))$ is even.

**Proof.** The $\mathbb{P}^1$-bundle $f : D \longrightarrow S$ in (1) produces an element of order two in the Brauer group $\text{Br}(S)$ of $S$. The quadric bundle associated to $X$ does not have a rational section if and only if this element is not trivial in $\text{Br}(S)$ (see [Ku, Proposition 4.7.]). Recall that, if $S$ is a $K3$ surface, then

$$\text{Br}(S) \cong T(S)^* \otimes \mathbb{Q}/\mathbb{Z} \cong \text{Hom}(T(S), \mathbb{Q}/\mathbb{Z})$$

(see for example [vG, §2.1.]). An element of order 2 in $\text{Br}(S)$ defines a surjective homomorphism

$$\alpha : T(S) \longrightarrow \mathbb{Z}/2\mathbb{Z}$$

(2)

and thus a sublattice $T_{\alpha}$ of index 2 in $T(S)$. Voisin [V, §1] and van Geemen [vG, §9] give a geometric realization for this element $\alpha$ (see also [HVV11, §2]). More precisely, there exists $k \in H^2(S, \mathbb{Z})$ such that

$$\alpha_D(L) \cong \{v \in H^2(S, \mathbb{Z})^0 : <v, k>_S \equiv 0(\text{mod } 2)\}$$

and $k$ induces an element $\varphi$ in $\text{Hom}(H^2(S, \mathbb{Z}), \mathbb{Z}/2\mathbb{Z})$ which restricts to $\alpha$ in $T(S)$. By definition, $\ker \varphi \cong \alpha_D(L)$ and, since $\alpha_D(T(X)) \subseteq \alpha_D(L)$, we have $\alpha_D(T(X)) \subseteq f^*(T_{\alpha})(-1)$. Thus $\alpha_D(T(X)) \subset f^*T(S)(-1)$ is a sublattice of index 2 and $d(A(X))$ is even by Proposition 4.0.5. \qed

**Remark 4.0.7.** The lattice $T_{\alpha}$ is isometric to the transcendental lattice $T(S, \alpha)$ of the $\alpha$-twisted Hodge structure of $S$ (see [Huy3, Proposition 4.7] and [Huy2, Lemma 2.15]). If $u, v \in L$ one has that $<u, v>_X = -<\alpha_D(u), \alpha_D(v)>_S$ (see [V, Proposition 2 ii]). Thus Proposition 4.0.6 implies that, if $X$ is not trivially rational, then $\alpha_D(T(X))$ is isometric to $T(S, \alpha)(-1)$. 

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4.1 Theta-characteristics on the ramification curve $C$

A theta-characteristic on a smooth curve $C$ is a line bundle $\kappa$ such that $\kappa^{\otimes 2} = K_C$. We write $h^0(\kappa) := \dim H^0(C, \kappa)$.

Denote with $Q_x$ a quadric of the bundle $Y \rightarrow \mathbb{P}^2$. The map $x \mapsto Q_x \cap P$ gives a net of conics whose discriminant curve is a plane cubic $C_1$. The curve $C_1$ cuts a divisor $2D$ on the sextic $C$ and thus it determines an effective theta-characteristic on $C$ (see [V, §1 Lemme 7]). Conversely, the cubic hypersurface $X$ is determined by the curve $C$ plus an odd theta-characteristic (see [V, §1 Proposition 4]). The same result is implied by the following Proposition 4.1.1.

**Proposition 4.1.1.** ([B, Proposition 4.2.]) Let $C$ be a smooth plane curve of degree $d$, defined by an equation $F = 0$ and $\kappa$ an odd theta-characteristic on $C$ with $h^0(\kappa) = 1$. Thus, $\kappa$ admits a minimal resolution

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-2)^{d-3} \oplus \mathcal{O}_{\mathbb{P}^2}(-3) \xrightarrow{M} \mathcal{O}_{\mathbb{P}^2}(-1)^{d-3} \oplus \mathcal{O}_{\mathbb{P}^2} \rightarrow \kappa \rightarrow 0$$

with a symmetric matrix $M \in M_{(d-2) \times (d-2)}(\mathbb{C}[X_0, X_1, X_2])$ satisfying $\det M = F$, and of the form

$$M = \begin{pmatrix}
L_{1,1} & \cdots & L_{1,d-3} & Q_1 \\
\vdots & & \vdots & \vdots \\
L_{1,d-3} & \cdots & L_{d-3,d-3} & Q_{d-3} \\
Q_1 & \cdots & Q_{d-3} & H
\end{pmatrix}
$$

where the forms $L_{i,j}, Q_i, H$ are linear, quadratic and cubic respectively.

Conversely, the cokernel of a symmetric matrix $M$ as above is an odd theta-characteristic $\kappa$ on $C$ with $h^0(\kappa) = 1$.

We can now prove our main result.

**Theorem 4.1.2.** Consider the couple $(S_{(b,c)}, \kappa)$ where $S_{(b,c)}$ is a double plane defined as in Lemma 3.1.1 and $\kappa$ is a theta characteristic on the ramification curve $C$ with $h^0(\kappa) = 1$. If $b$ is even, then $(S_{(b,c)}, \kappa)$ determines a cubic fourfold which is not trivially rational.

**Proof.** Let $C$ be the ramification curve of $S := S_{(b,c)}$ and take a theta characteristic $\kappa$ on $C$ with $h^0(\kappa) = 1$. Proposition 4.1.1 says that the curve $C$ has a determinantal representation $F = \det M = 0$ with

$$M = \begin{pmatrix}
L_{1,1} & L_{1,2} & L_{1,3} & Q_1 \\
L_{1,2} & L_{2,2} & L_{2,3} & Q_2 \\
L_{1,3} & L_{2,3} & L_{3,3} & Q_3 \\
Q_1 & Q_2 & Q_3 & H
\end{pmatrix}.$$ 

The geometric interpretation is the following. Choose projective coordinates $[Z_1, Z_2, Z_3, X_0, X_1, X_2]$ in $\mathbb{P}^5(\mathbb{C})$ and define the cubic fourfold $X = X(S, \kappa)$ as
the zero set
\[ \sum_{i,j=1}^{3} Z_iZ_jL_{i,j}(X_0, X_1, X_2) + \sum_{i=1}^{3} 2Z_iQ_i(X_0, X_1, X_2) + H(X_0, X_1, X_2) = 0. \]

The cubic $X$ is smooth and it contains the plane $P := \{X_0 = X_1 = X_2 = 0\}$. The curve $C$ is the discriminant of the quadric bundle obtained by projecting the hypersurface $X$ from $P$.

The $K3$ surface $S$ has rank two and $b$ is even, so the discriminant of $NS(S)$ is even. This means that $A(X)$ has rank three and even discriminant by Proposition 4.0.5. That $X$ is not trivially rational follows now from Proposition 4.0.4.

**Remark 4.1.3.** Auel et al. in [ABBV] (see Theorem 11) show an explicit example of a pfaffian (hence rational) cubic fourfold associated to a $K3$ surface of type $S_{(2,-1)}$.

Theorem 4.1.2 gives only a sufficient condition for the existence of not trivially rational 4-folds.

**Proposition 4.1.4.** There exist double planes $S_{(b,c)}$ with $b$ odd determining cubic fourfolds containing a plane which are not trivially rational.

**Proof.** In [ABBV, Theorem 4] it is proved that the general fourfold $X$ in one of the irreducible components of $\mathcal{C}_8 \cap \mathcal{C}_{14}$ has $A(X)$ with intersection matrix given by

|       | $H^2$ | $P$ | $T$ |
|-------|-------|-----|-----|
| $H^2$ | 3     | 1   | 4   |
| $P$   | 1     | 3   | 2   |
| $T$   | 4     | 2   | 10  |

(4)

The discriminant sextic $C$ of the quadric bundle associated to $X$ is smooth and let $S = S_{(b,c)}$ the double plane branched on $C$. Since $d(A(X) = 36$, $X$ is not trivially rational by Proposition 4.0.4. Thus, $d(NS(S)) = -9$ by Proposition 4.0.5 and Proposition 4.0.6. We conclude that $b$ is odd.

**Remark 4.1.5.** Actually, the cubic in the previous example is already known to be rational since it is a pfaffian.

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