Potential relations for nonrelativistic and relativistic two-body Schrödinger equations

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Relations between nonrelativistic and relativistic two-body equations, also allowing for different masses, are studied and explicit expressions are given. One example is the Blankenbecler Sugar equation. The corresponding expressions for the boosted two-body potentials are provided.

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The modern high precision NN potentials CD-Bonn [1], Nijm I, II [2] and AV18 [3] appear in a nonrelativistic Schrödinger equation. Turning to a relativistic version thereof which replaces the nonrelativistic kinetic energy by the relativistic one requires a modified NN potential to describe the same NN phase shifts. In case of AV18 such modifications have been worked out in [4]. One can also use an analytical momentum change transformation [5], which might be sufficient for a first orientation, but which cannot replace the necessary changes in the potential introduced physics-wise (see also [6] for more formal aspects).

In case of CD-Bonn and the Nijmegen potentials relativistic kinematics has been applied relating the relative momentum $\vec{k}$ and the energy $E$ for two equal mass particles in the c.m. system.

$$E = 2\sqrt{m^2 + \vec{k}^2}. \quad \text{(1)}$$

For such a choice one can switch between the Schrödinger equation in the relativistic and the nonrelativistic form by a simple trick [7, 8]. Put $\omega = \sqrt{m^2 + \vec{k}^2}$ then the relativistic NN Schrödinger equation reads

$$(2\omega + v)\psi = 2\omega_0\psi. \quad \text{(2)}$$

Applying $(2\omega + v)$ again on both sides yields

$$(4\omega^2 + 2\omega v + 2v^2 + v^2)\psi = 4\omega_0^2\psi. \quad \text{(3)}$$

After a simple algebra one arrives at the nonrelativistic Schrödinger equation

$$\left(\frac{k^2}{m} + V_{nr}\right)\psi = \frac{k_0^2}{m}\psi. \quad \text{(4)}$$

This means that solving Eq. (4) together with the relativistic kinematic relation (1) is in fact a relativistic treatment. One can easily invert the steps from Eq. (2) to (5) and finds

$$\omega = \sqrt{4mV_{nr} + 4\omega^2 - 2v} \quad \text{(6)}$$

which relates the potential $v$ in the relativistic NN Schrödinger equation to the potential $V_{nr}$ in the nonrelativistic Schrödinger equation. Important thereby is of course the relation (1). This connection is also valid for the Nijmegen potentials, since also there the relativistic relation (1) has been used.

In previous work (see for instance [11, 12]) it is stated that the bound state eigenvalue to Eq. (4) is $\omega_b$, which is $M_b^2/4m - \epsilon_b^2$ and differs from $E_b \equiv M_b - 2m$ by the small amount $E_b^2/4m$. Here $M_b$ is the mass of the bound two-body system. Though this is correct one can use a different definition of the binding energy, which replaces $E_b$ by $\epsilon_b$ defined as

$$M_b \equiv \sqrt{4m^2 + 4\epsilon_b m} \quad \text{(7)}$$

Then in Eq. (4) $\epsilon_b$ occurs directly without correction term and moreover it can naturally be written as $\epsilon_b = -\kappa^2/m$ in agreement with the form of the energy eigenvalue in Eq. (1). Expanding Eq. (7) one obtains $M_b = 2m + \epsilon_b - \epsilon_b^2/4m + \ldots$ the lowest order of which agrees with the usual definition of the binding energy.

For the practical application of (6) one can proceed in close analogy to the representation shown in [3] for the boosted NN potentials. We obtain (for details see Appendix)

$$\langle \vec{k} | v | \vec{k}' \rangle = \Psi_b(\vec{k}) M_b \Psi_b(\vec{k}')$$

$$+ \frac{m}{k^2 - k'^2} \left\{ 2 \omega_0 \Re\left[T(\vec{k}, \vec{k}; k^2/m)\right] - 2\omega_0' \Re\left[T(\vec{k}, \vec{k}'; k'^2/m)\right] \right\}$$

$$+ \frac{m^2}{k^2 - k'^2} \times$$

$$\left\{ \mathcal{P} \int d^3 k'' \frac{2\omega''}{k''^2 - k'^2} T(\vec{k}, \vec{k}''; k''^2/m) T^* (\vec{k}, \vec{k}''; k'^2/m) \right. \right.$$

$$\left. - \mathcal{P} \int d^3 k'' \frac{2\omega''}{k''^2 - k'^2} T(\vec{k}, \vec{k}''; k''^2/m) T^* (\vec{k}, \vec{k}''; k'^2/m) \right\} \quad \text{(8)}$$

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where $\Psi_b(k)$ is the nonrelativistic deuteron wave function of Eq.(4), $M_b$ the mass of the deuteron, $T$ the standard NN $T$-matrix related to $V_{nr}$ via the nonrelativistic Lippmann Schwinger equation and $\omega'' = \sqrt{m^2 + k'^2}$.

Like in [3] one can also obtain the boosted potential $v_p$ in a frame, where the total NN momentum is different from zero:

$$v_p = \sqrt{(2\omega + v)^2 + p^2} - \sqrt{4\omega^2 + p^2}. \quad (9)$$

In this case $v_p$ is connected to $V_{nr}$ as

$$v_p = \sqrt{4mV_{nr} + 4\omega^2 + p^2} - \sqrt{4\omega^2 + p^2}. \quad (10)$$

The explicit representation is

$$\langle \tilde{k}|v_p|\tilde{k}'\rangle = \Psi_b(\tilde{k})\sqrt{M_b^2 + p^2}\Psi_b(\tilde{k}')$$

$$+ \frac{m}{k^2 - k'^2} \left\{ \sqrt{4\omega^2 + p^2}^{2} \Re[T(\tilde{k}, \tilde{k}'; \frac{k'^2}{m})] \right\}$$

$$\times \sqrt{4\omega^2 + p^2}^{2} \Re[T(\tilde{k}, \tilde{k}'; \frac{k'^2}{m})]$$

$$\{ \mathcal{P} \int d^3k'' \frac{\sqrt{4\omega''^2 + p^2}^{2}}{k''^2 - k^2} T(\tilde{k}, \tilde{k}''; \frac{k''^2}{m}) T^*(\tilde{k}, \tilde{k}''; \frac{k''^2}{m}) \}$$

$$- \mathcal{P} \int d^3k'' \frac{\sqrt{4\omega''^2 + p^2}^{2}}{k''^2 - k^2} T(\tilde{k}, \tilde{k}''; \frac{k''^2}{m}) T^*(\tilde{k}, \tilde{k}''; \frac{k''^2}{m}). \quad (11)$$

This is to be used together with the kinetic energy $h_0 = \sqrt{4\omega^2 + p^2}$.

In [3] we did not use the correct relation [10] for CD-Bonn and the Nijmegen potentials to arrive at $v$ but the analytical momentum change transformation [5]. Using [10] might change the outcome for the relativistic energy shifts in the triton binding energy shown in [3]. This is left to a forthcoming calculation.

Now we ask a more general question. Assume as a starting point a relativistic three-dimensional equation for a two-body system with general masses in the total momentum zero frame, which defines the wave functions for bound and scattering states. It is assumed that they fulfill the completeness relation

$$\sum_b |\Psi_b><\Psi_b| + \int d\tilde{k}|\Psi_{\tilde{k}}><\Psi_{\tilde{k}}| = 1 \quad (12)$$

and obey the homogeneous and inhomogeneous equations

$$|\Psi_b> = G_0 V |\Psi_b> \quad (13)$$

$$|\Psi_{\tilde{k}}> = |\tilde{k}> + G_0 T |\tilde{k}>. \quad (14)$$

Here $V$, $G_0$ and $T$ are the corresponding potential, free propagator (with the correct kinematical cut) and $T$-operator, respectively. (Note, this $T$ is different from the previous one). Then the energies related to $|\Psi_b>$ are $M_b$

and for the scattering states $|\Psi_{\tilde{k}}>$ they are $\omega_1 + \omega_2$ with $\omega_i = \sqrt{m_i^2 + k_i^2}$. We want to rewrite these equations into the form of a standard relativistic two-body equation in the system with total momentum zero:

$$h \equiv \tilde{\omega}_1 + \tilde{\omega}_2 + v \quad (15)$$

where $\tilde{\omega}_i$ are the corresponding operators related to $\omega_i$. This means that $h$ should have the same bound and scattering states. Therefore the operator $h$ can be represented in terms of the bound and scattering states of the underlying relativistic two-body equation as

$$h = \sum_b |\Psi_b><M_b|$$

$$+ \int d\tilde{k}|\Psi_{\tilde{k}}>(\omega_1(\tilde{k}) + \omega_2(\tilde{k})) <\Psi_{\tilde{k}}|. \quad (16)$$

Using that form one obviously gets

$$v \equiv h - \tilde{\omega}_1 - \tilde{\omega}_2 = \sum_b |\Psi_b><M_b|$$

$$+ \int d\tilde{k}|\Psi_{\tilde{k}}>(\omega_1(\tilde{k}) + \omega_2(\tilde{k})) <\Psi_{\tilde{k}}|$$

$$- \int d\tilde{k}|\tilde{k}>(\omega_1(\tilde{k}) + \omega_2(\tilde{k})) <\tilde{k}|. \quad (17)$$

This leads to the momentum representation

$$<\tilde{k}|v|\tilde{k}'> = \sum_b \langle \Psi_b(\tilde{k})M_b \Psi_b(\tilde{k}'):$$

$$+ \int d\tilde{k}'<\tilde{k}|\Psi_{\tilde{k}}'> (\omega_1(\tilde{k}') + \omega_2(\tilde{k}')) <\Psi_{\tilde{k}}'|\tilde{k}' >$$

$$- \delta(\tilde{k} - \tilde{k}') (\omega_1(\tilde{k}) + \omega_2(\tilde{k}))) \rangle. \quad (18)$$

We insert now the underlying form of the assumed relativistic equation [14] and get for the part with the scattering states

$$<\tilde{k}|v|\tilde{k}>_{scatt}$$

$$= \int d\tilde{k}' \delta(\tilde{k} - \tilde{k}') G_0(k, k') <\tilde{k}|T|\tilde{k}' >$$

$$\langle \omega_1(\tilde{k}') + \omega_2(\tilde{k}')(\tilde{k}'|T|\tilde{k}') \delta(k - k') \rangle$$

$$- \delta(\tilde{k} - \tilde{k}') (\omega_1(\tilde{k}) + \omega_2(\tilde{k}))). \quad (19)$$

As an example we regard the Blankenbecler-Sugar (BBS) equation resulting from the Bethe-Salpeter (BS) equation by replacing the propagator in the BS equation with the propagator [10]

$$g_{BBS} \equiv \frac{1}{2\pi i} \int_{(m_1 + m_2)^2}^{\infty} ds' \frac{1}{(2\pi)^2} \frac{\delta^+(k_0^2 - m_1^2) \delta^+(k_1^2 - m_2^2)}{s' - s - i\epsilon}$$

Putting the four-momenta $k_{1,2} = \frac{1}{2} P \pm k$ and $P_0(P'_0) = \sqrt{s} (\sqrt{s'})$ one obtains [10]

$$g_{BBS} = \frac{s P_0}{\omega_1 \omega_2 (\omega_1 + \omega_2)^2} \left( \frac{\omega_1 + \omega_2}{s - i\epsilon} \right). \quad (21)$$
This yields the following equation of the type (14):
\[ |\Psi_{\tilde{k}^I} > = |\tilde{k}^I > + \frac{1}{(\omega_1 + \omega_2)^2 - (\omega_1' + \omega_2')^2 - \i \epsilon} T|\tilde{k}^I > \] (22)
In arriving at (22) we have redefined the matrix elements of \( T \) and \( V \) occurring in the Bethe Salpeter equation after inserting (21) by multiplying from both sides \( \sqrt{\pi (\omega_1 + \omega_2)/\omega_1 \omega_2} \) and \( \sqrt{\pi (\omega_1' + \omega_2')/\omega_1' \omega_2'} \), respectively. We denote \( \omega_i' = \sqrt{m_i^2 + k^2} \).

The bound states are defined by an equation corresponding to (13) with the same propagator as in (22) and the redefined \( V \). We assume the completeness relation (12) to be valid and obtain the two-body interaction \( v \) in Eq. (15) as
\[
<k|v|\tilde{k}^I> = \sum_b \Psi_b(\tilde{k}) M_b \Psi_b(\tilde{k}^I) + \frac{1}{(\omega_1 + \omega_2)^2 - (\omega_1' + \omega_2')^2} \times
\{(\omega_1 + \omega_2)\Re[T(\tilde{k}^I, \tilde{k}; \omega_1 + \omega_2)] - (\omega_1' + \omega_2')\Re[T(\tilde{k}^I, \tilde{k}^I; \omega_1' + \omega_2')]\}
\]

\[
+ 1 \times \frac{1}{(\omega_1 + \omega_2)^2 - (\omega_1' + \omega_2')^2} \times
\{ \mathcal{P} \int d^3k' (\omega_1'' + \omega_2'')^2 - (\omega_1 + \omega_2)^2 \times
T(\tilde{k}, \tilde{k}'; \omega_1'' + \omega_2'') T^*(\tilde{k}^I, \tilde{k}'; \omega_1' + \omega_2') - \mathcal{P} \int d^3k' (\omega_1'' + \omega_2'')^2 - (\omega_1' + \omega_2')^2 \times
T(\tilde{k}, \tilde{k}'; \omega_1'' + \omega_2'') T^*(\tilde{k}^I, \tilde{k}'; \omega_1'' + \omega_2') \}. \tag{23}
\]

To arrive at (23) the steps are analogous to the steps laid out in the Appendix.

The boosted potential which is to be taken together with the kinetic energy \( \sqrt{(\omega_1 + \omega_2)^2 + p^2} \) results simply by replacing \( M_b \) by \( \sqrt{M_b^2 + p^2} \) and the \( (\omega_1 + \omega_2) \)'s in the numerators by \( \sqrt{(\omega_1 + \omega_2)^2 + p^2} \).

Finally we address the question whether the relativistic two-body Schrödinger equation corresponding to \( h \) from Eq. (19) can be cast into the form of a nonrelativistic two-body Schrödinger equation
\[
h_{nr} \Psi = \frac{\hat{\mathcal{E}}^2}{2\mu} + v_{nr} \Psi = \frac{\hat{\mathcal{E}}^2}{2\mu} \Psi \tag{24}
\]
where \( \mu \) is the standard reduced mass (\( \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \)). We require again that \( h_{nr} \) has the same bound and scattering states as \( h \). Therefore
\[
v_{nr} \equiv h_{nr} - \frac{\hat{\mathcal{E}}^2}{2\mu} = \sum_b \{ |\Psi_b > \epsilon_b < \Psi_b| \}
\]
\[
+ \int d\tilde{k}^I |\Psi_{\tilde{k}^I} > \frac{\hat{\mathcal{E}}^2}{2\mu} < \Psi_{\tilde{k}^I}| \tilde{k}^I > - \frac{\hat{\mathcal{E}}^2}{2\mu} . \tag{25}
\]
In momentum representation this yields
\[
<k|v_{nr}|\tilde{k}^I> = \sum_b \Psi_b(\tilde{k}) \epsilon_b \Psi_b(\tilde{k}^I)
\]
\[
+ \int d\tilde{k}'' <\tilde{k}'|\Psi_{\tilde{k}''} > \sqrt{\frac{\hat{\mathcal{E}}^2}{2\mu}} <\Psi_{\tilde{k}''}|\tilde{k}'' > - 2\omega <\tilde{k} - \tilde{k}' >
\]
\[
+ \int d\tilde{k}'' <\tilde{k}'|\Psi_{\tilde{k}''} > \sqrt{\frac{\hat{\mathcal{E}}^2}{2\mu}} m <\Psi_{\tilde{k}''}|\tilde{k}'' > . \tag{27}
\]

In summary we regarded relations between relativistic and nonrelativistic two-body equations. For equal mass particles the relativistic two-body Schrödinger equation can be converted identically into a nonrelativistic Schrödinger equation and vice versa if the relativistic connection between energy and momentum is kept. An explicit expression for the relativistic potential in terms of the nonrelativistic T-matrix and bound state is provided. Also the corresponding relativistic potential in a moving frame has been worked out. Further we regarded a more general question. Often relativistic three-dimensional equations are proposed, which do not have the standard form of a relativistic two-body Schrödinger equation. Can they be rewritten into such a form? We addressed that question and exemplified a solution for the case of the Blankenbecler Sugar equation for two different mass particles. Finally we provided an explicit expression for the nonrelativistic potential occurring together with the standard nonrelativistic kinetic energy of two different mass particles if bound and scattering state information is available from the underlying relativistic two-body Schrödinger equation. In this case because of two square root expressions in the kinetic energy the simple trick possible for two equal mass particles is not applicable and we are left with a more complicated form.

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**Derivation of Eq. (3):**
We insert the complete basis of eigenstates related to Eq. (Assuming one bound state). This leads to
\[
<k|v|\tilde{k}^I> = \sum_b <\Psi_b| M_b <\Psi_b|\tilde{k}^I > - 2\omega <\tilde{k} - \tilde{k}' > + \int d\tilde{k}'' <\tilde{k}'|\Psi_{\tilde{k}''} > \sqrt{\frac{\hat{\mathcal{E}}^2}{2\mu}} m <\Psi_{\tilde{k}''}|\tilde{k}'' > . \tag{27}
\]
Next we use the well known decomposition

\[ \langle \vec{k} | \Psi_{k'} \rangle = \delta(\vec{k} - \vec{k'}) + \frac{T(\vec{k}, \vec{k'}; \frac{k'^2}{m})}{k^2 + i\epsilon - k'^2/m} \]  \hspace{1cm} (28)

and arrive at

\[ \langle \vec{k} | v | \vec{k'} \rangle = \Psi_b(\vec{k}) M_b \Psi_b(\vec{k'}) + \frac{T^*(\vec{k}, \vec{k'}; \frac{k'^2}{m})}{k^2 + i\epsilon - k'^2/m} 2\omega(k') \]
\[ + \int d\vec{k''} T(\vec{k}, \vec{k''}; \frac{k'^2}{m}) T^*(\vec{k}, \vec{k''}; \frac{k'^2}{m}) 2\omega(\vec{k''}) \]  \hspace{1cm} (29)

The integral requires some care and we keep the limiting processes for the two scattering states separately by putting

\[ \frac{1}{k'^2 - k^2 + i\epsilon} \frac{1}{k'^2 - k'^2 - i\epsilon} \]
\[ \xrightarrow{\mathcal{P}} \frac{1}{k'^2 - k^2 + i\epsilon} \frac{1}{k'^2 - k'^2 - i\epsilon} \]
\[ \xrightarrow{\mathcal{P}} \frac{1}{k'^2 - k^2 + i\epsilon} \frac{1}{k'^2 - k'^2 - i\epsilon} \]  \hspace{1cm} (30)

This allows us to perform one limit firstly with the result

\[ \int d\vec{k''} f(\vec{k''}) \]
\[ = \lim_{\epsilon \rightarrow +0} \frac{1}{k'^2 - k^2 + i\epsilon} \frac{1}{k'^2 - k'^2 - i\epsilon} \times \mathcal{P} \int d\vec{k''} f(\vec{k''}) \]
\[ -i\pi \lim_{\epsilon \rightarrow +0} \frac{1}{k'^2 - k^2 + i\epsilon} \times \left( \int d\vec{k''} f(\vec{k''}) \delta(k'^2 - k^2) + \int d\vec{k''} f(\vec{k''}) \delta(k'^2 - k'^2) \right) \]  \hspace{1cm} (32)

The principal value prescription is denoted via “\( \mathcal{P} \)”. In our case

\[ f(\vec{k''}) = 2m^2\omega(\vec{k''})T(\vec{k}, \vec{k''}; \frac{k'^2}{m}) T^*(\vec{k}, \vec{k''}; \frac{k'^2}{m}) \]  \hspace{1cm} (33)

and therefore

\[ \int d\vec{k''} f(\vec{k''}) \delta(k'^2 - k^2) \]
\[ = 2m^2 \int d\vec{k''} \omega(\vec{k''}) T(\vec{k}, \vec{k''}; \frac{k'^2}{m}) T^*(\vec{k}, \vec{k''}; \frac{k'^2}{m}) \delta(q^2 - k'^2) \]  \hspace{1cm} (34)

This is part of the unitary relation

\[ T(\vec{k}, \vec{k''}; \frac{q^2}{m}) - T^*(\vec{k'}, \vec{k''}; \frac{q^2}{m}) \]
\[ = -2i\pi m \int d\vec{k''} T(\vec{k}, \vec{k''}; \frac{q^2}{m}) T^*(\vec{k'}, \vec{k''}; \frac{q^2}{m}) \delta(q^2 - k'^2) \]  \hspace{1cm} (35)

Consequently

\[ 2m^2 \int d\vec{k''} \omega(\vec{k''}) T(\vec{k}, \vec{k''}; \frac{k'^2}{m}) T^*(\vec{k}, \vec{k''}; \frac{k'^2}{m}) \delta(k'^2 - k^2) \]
\[ = -\frac{2\omega(k)m}{\pi} \Im[T(\vec{k}, \vec{k}; \frac{k^2}{m})] \]  \hspace{1cm} (36)

and

\[ -i\pi \lim_{\epsilon \rightarrow +0} \frac{1}{k'^2 - k^2 - i\epsilon} \times \left( \int d\vec{k''} f(\vec{k''}) \delta(k''^2 - k^2) + \int d\vec{k''} f(\vec{k''}) \delta(k''^2 - k'^2) \right) \]
\[ = \lim_{\epsilon \rightarrow +0} \frac{2m}{k'^2 - k^2 - i\epsilon} \times \left( \omega(k)\Im[T(\vec{k}, \vec{k}; \frac{k^2}{m})] + \omega(k')\Im[T(\vec{k}, \vec{k}; \frac{k'^2}{m})] \right) \]  \hspace{1cm} (37)

Combined with Eq. 29 certain terms cancel and one arrives at Eq. 8.