Change of Base for Commutative Algebras

U. Ege Arslan, Z. Arvasi and Ö. Gürmen

Abstract

In this paper we examine on changing the base which induces a pair of functors for a subcategory of a category of crossed modules over commutative algebras. We give some examples and results on induced crossed modules.

Keywords: Induced modules, induced crossed modules, base change functors.

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Introduction

Let $S$ be a ring. Then there is a category $\text{Mod}/S$ of modules over $S$. If $\phi : S \to R$ is a ring homomorphism, then there is a functor $\phi^*$ from $\text{Mod}/R$ to $\text{Mod}/S$ where $S$ acts on an $R$-module via $\phi$. This functor has a left adjoint $\phi_*$ to $\phi^*$ giving the well known induced module via tensor product. This construction can be known as a “change of base” in a general module theory setting. Brown and Higgins [4] generalised that to higher dimension for the group theoretical case, that is, a morphism $\phi : P \to Q$ of groups determines a pullback functor $\phi^* : \text{XMod}/Q \to \text{XMod}/P$, where $\text{XMod}/Q$ denotes a subcategory of objects of a category $\text{XMod}$ of crossed modules over $Q$. The left adjoint $\phi_*$ to pullback gives the induces crossed modules. This is also given by pushouts of crossed modules.

In this work we will consider the appropriate analogue of that in the theory of crossed modules in commutative algebras. Although this construction has already been worked by Porter [11] and Shammu [13], we will reconsider and develop that in the light of the works of Brown and Wensley [6, 7]. The purpose of this paper is to give some new examples and results on crossed modules induced by a morphism of algebras $\phi : S \to R$ in the case when $\phi$ is the inclusion of an ideal. In the applications to commutative algebras, the induced crossed modules play an important role since the free crossed modules which are related to Koszul complexes given by Porter [12] are the special case of induced crossed modules. We believe that the induced crossed modules of commutative algebras give useful informations on Koszul-like constructions.

Conventions: Throughout this paper $k$ is a fixed commutative ring, $R$ a $k$-algebra with identity. All $k$-algebras will be assumed commutative and associative but there will not be requiring algebras to have unit elements unless stated otherwise.
1 Crossed Modules of Commutative Algebras

A crossed module of algebras, \((C, R, \partial)\), consists of an \(R\)-algebra \(C\) and a \(k\)-algebra \(R\) with an action of \(R\) on \(C\), \((r, c) \mapsto r \cdot c\) for \(c \in C, r \in R\), and an \(R\)-algebra morphism \(\partial : C \rightarrow R\) satisfying the following condition for all \(c, c' \in C\)

\[ \partial(c) \cdot c' = cc'. \]

This condition is called the Peiffer identity. We call \(R\), the base algebra and \(C\), the top algebra. When we wish to emphasise the base algebra \(R\), we call \((C, R, \partial)\), a crossed \(R\)-module.

A morphism of crossed modules from \((C, R, \partial)\) to \((C', R', \partial')\) is a pair \((f, \phi)\) of \(k\)-algebra morphisms \(f : C \rightarrow C', \phi : R \rightarrow R'\) such that

(i) \(\partial' f = \phi \partial\)

and

(ii) \(f(r \cdot c) = \phi(r) \cdot f(c)\)

for all \(c \in C, r \in R\). Thus one can obtain the category \(\text{XMod}\) of crossed modules of algebras. In the case of a morphism \((f, \phi)\) between crossed modules with the same base \(R\), say, where \(\phi\) is the identity on \(R\),

\[
\begin{array}{ccc}
C & \xrightarrow{f} & C' \\
\downarrow{\partial} & & \downarrow{\partial'} \\
R & & R
\end{array}
\]

then we say that \(f\) is a morphism of crossed \(R\)-modules. This gives a subcategory \(\text{XMod}/R\) of \(\text{XMod}\).

1.1 Examples of Crossed Modules

(i) Any ideal, \(I\), in \(R\) gives an inclusion map \(I \rightarrow R\), which is a crossed module then we will say \((I, R, i)\) is an ideal pair. In this case, of course, \(R\) acts on \(I\) by multiplication and the inclusion homomorphism \(i\) makes \((I, R, i)\) into a crossed module, an “inclusion crossed modules”. Conversely,

**Lemma 1** If \((C, R, \partial)\) is a crossed module, \(\partial(C)\) is an ideal of \(R\). \(\square\)

(ii) Any \(R\)-module \(M\) can be considered as an \(R\)-algebra with zero multiplication and hence the zero morphism \(0 : M \rightarrow R\) sending everything in \(M\) to the zero element of \(R\) is a crossed module. Again conversely:

**Lemma 2** If \((C, R, \partial)\) is a crossed module, \(\ker \partial\) is an ideal in \(C\) and inherits a natural \(R\)-module structure from \(R\)-action on \(C\). Moreover, \(\partial(C)\) acts trivially on \(\ker \partial\), hence \(\ker \partial\) has a natural \(R/\partial(C)\)-module structure. \(\square\)

As these two examples suggest, general crossed modules lie between the two extremes of ideal and modules. Both aspects are important.

(iii) In the category of algebras, the appropriate replacement for automorphism groups is the multiplication algebra defined by Mac Lane [10]. Then automorphism crossed module correspond to the multiplication crossed module \((R, M(R), \mu)\).
To see this crossed module, we need to assume $\text{Ann}(R) = 0$ or $R^2 = R$ and let $M(R)$ be the set of all multipliers $\delta : R \to R$ such that for all $c, c' \in C$, $\delta(rr') = \delta(r)r'$. $M(R)$ acts on $R$ by

$$M(R) \times R \rightarrow R \quad (\delta, r) \rightarrow \delta(r)$$

and there is a morphism $\mu : R \to M(R)$ defined by $\mu(r) = \delta_r$ with $\delta_r(r') = rr'$ for all $r, r' \in R$. (See [2] for details).

1.2 Free Crossed Modules

Let $(C, R, \partial)$ be a crossed module, let $Y$ be a set and let $\upsilon : Y \to C$ be a function, then $(C, R, \partial)$ is said to be a free crossed module with basis $\upsilon$ or alternatively, on the function $\partial \upsilon : Y \to R$ if for any crossed $R$-module $(A, R, \delta)$ and a function $w : Y \to A$ such that $\delta w = \partial \upsilon$, there is a unique morphism

$$\phi : (C, R, \partial) \longrightarrow (A, R, \delta)$$

such that the diagram

$$\begin{array}{ccc} C & \xrightarrow{\phi} & A \\ \partial \downarrow & & \downarrow \delta \\ R & & \end{array}$$

is commutative.

For our purpose, an important standard construction of free crossed $R$-modules is as follows:

Suppose given $f : Y \to R$. Let $E = R^+[Y]$, the positively graded part of the polynomial ring on $Y$. $f$ induces a morphism of $R$-algebras,

$$\theta : E \to R$$

defined on generators by

$$\theta(y) = f(y).$$

We define an ideal $P$ in $E$ (sometimes called by analogy with the group theoretical case, the Peiffer ideal relative to $f$) generated by the elements

$$\{pq - \theta(p)q : p, q \in E\}$$

clearly $\theta(P) = 0$, so putting $C = E/P$, one obtains an induced morphism

$$\delta : C \to R$$

which is the required free crossed $R$-module on $f$ [12].

This construction will be seen later as a special case of an induced crossed module.
2 Pullback Crossed Modules

Within the theory of modules and more generally of Abelian categories, there is a very important set of results known as Morita Theory describing between categories of modules. The idea is that let \( \phi : S \to R \) be a ring homomorphism and let \( M \) be a \( R \)-module, then we can obtain \( S \)-module \( \phi^*(M) \) by means of \( \phi \) for which the action is given by \( s \cdot m = \phi(s)m \), for \( s \in S, m \in M \). Then there is a functor

\[
\phi^* : \text{Mod}/R \to \text{Mod}/S.
\]

This functor has a left adjoint

\[
\phi_* : \text{Mod}/S \to \text{Mod}/R.
\]

Then each \( S \)-module \( N \) defines a \( R \)-module \( \phi_*(N) = S \otimes_S N \). This construction can be also known as a “change of base” in a module theory. In this section we will see the corresponding idea with crossed modules. We call these structures a pullback crossed module and induced crossed module, respectively. These functors had already been done by Porter, \[11\], under different names. Also Shammu \[13\], had considered in his thesis for non-commutative case. But we will deeply analyse these constructions by using the work of Brown-Wensley and Brown,Villanueva and Higgins \[6, 7, 5\]. Similar results are known for crossed modules of groups \[4, 6\], and Lie algebras \[8\].

We will define a pullback crossed module which is due to \[5\].

**Definition 3** Given a crossed module \( \partial : C \to R \) and a morphism of \( k \)-algebras \( \phi : S \to R \), the pullback crossed module can be given by

(i) a crossed \( S \)-module \( \phi^*(C, R, \partial) = (\partial^* : \phi^*(C) \to S) \)

(ii) given

\( (f, \phi) : (B, S, \mu) \to (C, R, \partial) \)

crossed module morphism, then there is a unique \( (f^*, \text{id}_S) \) crossed \( S \)-module morphism that commutes the following diagram:

\[
\begin{array}{ccc}
(B, S, \mu) & \xrightarrow{(f^*, \text{id}_S)} & (\phi^*(C), S, \partial^*) \\
\downarrow{(f, \phi)} & & \downarrow{(\phi', \phi)} \\
(C, R, \partial) & & (C, R, \partial)
\end{array}
\]

or more simply as

\[
\begin{array}{ccc}
B & \xrightarrow{f} & C \\
\downarrow{\mu} & & \downarrow{\partial} \\
S & \xrightarrow{\phi} & R
\end{array}
\]

\[
\begin{array}{ccc}
\phi^*(C) & \xrightarrow{\phi} & \phi'^*(C) \\
\downarrow{\phi^*} & & \downarrow{\phi'^*} \\
S & \xrightarrow{\phi} & R
\end{array}
\]

\[
\begin{array}{ccc}
\partial & \xrightarrow{\phi} & \partial' \\
\downarrow{\phi^*} & & \downarrow{\phi'^*} \\
S & \xrightarrow{\phi} & R
\end{array}
\]
2.1 Construction of Pullback Crossed Module

Let \((C, R, \partial)\) be a crossed \(R\)-module and let \(\phi : S \longrightarrow R\) be a morphism of \(k\)-algebras.

We define \(A = \{(c, s) \mid \phi(s) = \partial(c), \ s \in S, c \in C\} \subseteq C \times S\). \(A\) has the structure of a \(S\)-algebra by

\[
s \cdot (c, s') = (\phi(s) \cdot c, ss').
\]

If we take \(\phi^*(C) = A\), then \(\phi^*(C, R, \partial) = (\phi^*(C), S, \partial^*)\) is a pullback crossed module. We now show this as follows:

i) \(\partial^* : \phi^*(C) \rightarrow S, \partial^*((c, s)) = s\) is a crossed \(S\)-module. Since,

\[
\partial^*(c, s) \cdot (c', s') = s \cdot (c', s') = (\phi(s) \cdot c', ss') = (\partial(c) \cdot c', ss') = (cc', ss') = (c, s) (c', s')
\]

ii) \((\phi', \phi) : (\phi^*(C), S, \partial^*) \longrightarrow (C, R, \partial)\)

is a morphism of crossed module where \(\phi'(c, s) = c\). Since

\[
\phi'(s' \cdot (c, s)) = \phi'(\phi(s') \cdot c, s's) = \phi(s') \cdot c = \phi(s') \cdot \phi(c, s)
\]

and clearly \(\partial \phi' = \phi \partial^\ast\).

Suppose that

\((f, \phi) : (B, S, \mu) \longrightarrow (C, R, \partial)\)

is any crossed module morphism such that \(\partial f = \phi \mu\), then there is a unique morphism

\[
f^* : B \longrightarrow \phi^*(C)
\]

\[
x \mapsto (f(x), \mu(x))
\]

since \(\partial f(x) = \phi \mu(x)\) for all \(x \in B\). Now, let us show that \((f^*, \text{id}_S)\), is a crossed \(S\)-module morphism. For \(x \in B, s \in S\)

\[
f^*(s \cdot x) = (f(s \cdot x), \mu(s \cdot x)) = (\phi(s) \cdot f(x), s \mu(x)) = s \cdot (f(x), \mu(x)) = s \cdot f^*(x) = \text{id}_S(s) \cdot f^*(x),
\]

so \((f^*, \text{id}_S)\) is a crossed \(S\)-module morphism.

Finally; for all \(x \in \phi^*(C)\),

\[
(\partial^* f^*) (x) = \partial^* (f^*(x)) = \partial^* (f(x), \mu(x)) = \mu(x)
\]
so, $\partial^* f^* = \mu$ and

$$\phi^* f^*(x) = \phi'(f(x), \mu(x)) = f(x)$$

so $\phi^* f^* = f$. Thus, we get a functor

$$\phi^* : \text{XMod}/R \rightarrow \text{XMod}/S$$

which gives our pullback crossed module.

This pullback crossed module can be given by a pullback diagram.

**Corollary 4** Given a crossed module $(C, R, \partial)$ and a morphism $\phi : S \rightarrow R$ of $k$-algebras, there is a pullback diagram

![Pullback Diagram](image)

**Proof:** It is straight forward from a direct calculation. \( \square \)

### 2.2 Examples of Pullback Crossed Modules

1. Given crossed module $i = \partial : I \hookrightarrow R$ where $i$ is an inclusion of an ideal. The pullback crossed module is

$$\phi^* (I, R, \partial) = (\phi^* (I), S, \partial^*)$$

as,

$$\phi^* (I) = \{(i, s) \mid \phi(s) = \partial(i) = i, \; s \in S, \; i \in I\}$$

$$\cong \{s \in S \mid \phi(s) = i \in I\} = \phi^{-1}(I) \subseteq S.$$  

The pullback diagram is

![Pullback Diagram](image)

Particularly if $I = \{0\}$, then

$$\phi^*(\{0\}) \cong \{s \in S \mid \phi(s) = 0\} = \ker \phi$$

and so $(\ker \phi, S, \partial^*)$ is a pullback crossed modules. Kernels are thus particular cases of pullbacks. Also if $\phi$ is onto and $I = R$, then $\phi^*(R) = R \times S$
2. Given a crossed module $0 : M \rightarrow R$, $0(m) = 0$ where $M$ is any $R$-module, so it is also an $R$-algebra with zero multiplication. Then

$$\phi^* (M, R, 0) = (\phi^* (M), S, \partial^*)$$

where

$$\phi^* (M) = \{(m, s) \in M \times S \mid \phi(s) = \partial(m) = 0\}$$

$$\cong M \times \text{Ker} \phi$$

The corresponding pullback diagram is

\[
\begin{array}{ccc}
M \times \text{Ker} \phi & \xrightarrow{\delta^*} & M \\
\downarrow & & \downarrow 0 \\
S & \xrightarrow{\phi} & R.
\end{array}
\]

So if $\phi$ is injective ($\text{Ker} \phi = 0$), then $M \cong \phi^* (M)$. If $M = \{0\}$, then $\phi^* (M) \cong \text{Ker} \phi$.

3. If $\phi : S \rightarrow R$ is a morphism of algebras, then there may, or may not be a morphism $M(\phi) : M(S) \rightarrow M(R)$ such that

\[
\begin{array}{ccc}
S & \xrightarrow{\phi} & R \\
\downarrow & & \downarrow \\
M(S) & \xrightarrow{M(\phi)} & M(R)
\end{array}
\]

is a morphism of crossed modules. Using the following commutative diagram

\[
\begin{array}{ccc}
\phi^* (R) & \xrightarrow{\partial^*} & R \\
\downarrow & & \downarrow \\
S & \xrightarrow{\phi} & M(R) \\
\downarrow & & \downarrow M(\phi) \\
M(S) & \xrightarrow{M(\phi)} & M(R)
\end{array}
\]

we get the pullback diagram

\[
\begin{array}{ccc}
\phi^* (R) & \xrightarrow{\partial^*} & R \\
\downarrow & & \downarrow \\
M(S) & \xrightarrow{M(\phi)} & M(R)
\end{array}
\]
where
\[ \phi^\ast (R) = \{ (\gamma, r) : M(\phi)(\gamma) = \partial (r), \; \gamma \in M(S), \; r \in R \}. \]

3 Induced Crossed Modules

We will consider a functor \( \phi^\ast : \text{XMod}/S \to \text{XMod}/R \leftarrow \) left adjoint defined to the pullback \( \phi^\ast \) of the previous section. This functor has already been defined by Porter [11] for which he call it “extension along a morphism”. But we defined this functor by the universal property and analysed this construction deeply.

**Definition 5** For any crossed \( S \)-module \( \partial : D \to S \) and \( k \)-algebra morphism \( \phi : S \to R \), the induced crossed module can be given by
i) a crossed \( R \)-module \( \phi^\ast (D, S, \partial) = (\partial^\ast : \phi^\ast (D) \to R) \)
ii) Given \( (f, \phi) : (D, S, \partial) \to (B, R, \eta) \) crossed module morphism, then there is a unique \( (f^\ast, \text{id}_R) \) crossed \( R \)-module morphism such that commutes the following diagram

\[
\begin{array}{ccc}
(D, S, \partial) & \xrightarrow{(f, \phi)} & (B, R, \eta) \\
\downarrow (f, \phi) & \downarrow (\phi^\ast, \text{id}_R) & \\
(f^\ast, \text{id}_R) & \longleftarrow (\phi^\ast (D), R, \partial^\ast)
\end{array}
\]

or more simply as

\[
\begin{array}{ccc}
D & \xrightarrow{f} & B \\
\downarrow \phi^\ast & \downarrow \partial^\ast & \\
S & \xrightarrow{\phi} & R
\end{array}
\]

3.1 Construction of Induced Crossed Module

We will construct the induced crossed module as follows. Given a \( k \)-algebra morphism \( \phi : S \to R \) and a crossed module \( \partial : D \to S \), and let the set

\[ F(D \times R) \]

be a free algebra generated by the elements of \( D \times R \). Let \( P \) be the ideal generated by all the relations of the three following types:

\[
\begin{align*}
(d_1, r) + (d_2, r) &= (d_1 + d_2, r) \\
(s \cdot d, r) &= (d, \phi(s)r) \\
(d_1, r_1)(d_2, r_2) &= (d_2, r_1(\phi \partial_1)r_2)
\end{align*}
\]
for any \(d, d_1, d_2 \in D\), and \(r \in R\), \(s \in S\)
We define
\[ D \otimes_S R = F(D \times R)/P. \]

This is an \(R\)-algebra with
\[ r' \cdot (d \otimes r) = d \otimes r' r \]
for \(d \in D\), \(r, r' \in R\). If we take \(\phi_*(D) = D \otimes_S R\), then
\[ \phi_*(D, S, \partial) = (\phi_*(D), R, \partial_*) \]
is a induced crossed module. We will see it as follows:

i)
\[ \partial_* : D \otimes_S R \rightarrow R \]
\[
\begin{array}{c}
\partial_*(d \otimes r) \cdot (d_1 \otimes r_1) = ((\phi_*(d)) \cdot (d_1 \otimes r_1)) \\
= (d_1 \otimes \phi_*(d_1 \otimes r_1)) \\
= (\partial_*(d) \cdot d_1 \otimes r_1) \\
= (d_1 \otimes d_2 \otimes r_1) \\
= (d \otimes d_1 \otimes r_1)
\end{array}
\]

so \(\partial_*\) is a crossed \(R\)-module.

ii) Since \(R\) has a unit, \(\phi'_* : D \rightarrow \phi_*(D)\) is defined by \(\phi'(d) = (d \otimes 1)\), then
\[
\phi'_*(s \cdot d) = (s \cdot d \otimes 1) = (d \otimes \phi_*(s)) = (\phi_*(s) \cdot d \otimes 1) = \phi_*(s) \cdot \phi'_*(d)
\]
for \(d \in D\), \(s \in S\). So \((\phi'_*, \phi) : (D, S, \partial) \rightarrow (\phi_*(D), R, \partial_*)\) is a crossed module morphism.

Let
\[(f, \phi) : (D, S, \partial) \rightarrow (B, R, \eta)\]
be any crossed module morphism. Then there is a morphism \(f_*\) given by
\[
f_* : \phi_*(D) \rightarrow B
\]
\[
\begin{array}{c}
f_*(d \otimes r) = r \cdot f(d)
\end{array}
\]
Also, for \(x \in \phi_*(D)\), \(r \in R\)
\[
f_*(r \cdot (d_1 \otimes r_1)) = f_*(d_1 \otimes r_1) = r \cdot (d_1 \otimes r_1) = r \cdot f_*(d_1 \otimes r_1) = id_R(r) \cdot f_*(d_1 \otimes r_1)
\]
so \((f_*, id_R)\) is crossed \(R\)-module morphism.
Finally,
\[
(\eta f_\ast)((d \otimes r)) = \eta (f_\ast(d \otimes r)) \\
= \eta(r \cdot f(d)) \\
= r\eta(f(d)) \\
= r\phi(\partial(d)) \\
= \partial_\ast(d \otimes r) \\
= \text{id}_R \partial_\ast(d \otimes r)
\]
for each \((d \otimes r) \in \phi_\ast(D)\) and
\[
(f_\ast \phi')(d) = f_\ast(\phi'(d)) \\
= f_\ast((d \otimes 1)) \\
= f(d)
\]
so \(f_\ast \phi' = f\). Thus, we get a functor
\[
\phi_\ast : \text{XMod}/S \rightarrow \text{XMod}/R
\]
which gives our induced crossed module.

This induced crossed modules can be interpret in terms of pushout diagram.

**Corollary 6** Let \(\partial : D \rightarrow S\) be a crossed \(S\)-module and \(\phi : S \rightarrow R\), \(k\)-algebra morphism. Then there is a induced diagram

\[
\begin{array}{ccc}
D & \xrightarrow{\phi'} & \phi_\ast(D) \\
\downarrow{\partial} & & \downarrow{\partial_\ast} \\
S & \xrightarrow{\phi} & R.
\end{array}
\]

\(\square\)

**Theorem 7** For any \(k\)-algebra morphism \(\phi : S \rightarrow R\), there is adjoint functor pair \((\phi^*, \phi_\ast)\).

**Proof:** The proof can be easily shown using the universal property of the pushout. \(\square\)

### 3.2 Examples of Induced Crossed Modules

1. Let \(D = S\) and \(\partial = \text{id}_S : S \rightarrow S\) be identity crossed \(S\)-modules. The induced crossed module diagram is

\[
\begin{array}{ccc}
S & \xrightarrow{\psi} & \phi_\ast(S) \\
\downarrow{i=\partial} & & \downarrow{\partial_\ast} \\
S & \xrightarrow{\phi} & R
\end{array}
\]

where \(\phi_\ast(S) = S \otimes_S R\).
(Remark: S has not unit, otherwise S ⊗ S R ∼= R). When we take S = k+[X], the positively graded part of the polynomial algebra over k on the set of generators, X we have the induced crossed module constructed in section 1.2. ∂∗ : k+[X] ⊗k+[X] R → R which is the free R-module on f : X → R. Thus the free crossed modules is the special case of the induced of the induced crossed modules. We will examine this with respect to section 3.1.

Considering the free crossed module construction given in the first section, we have a diagram

\[
\begin{array}{ccc}
k^+[X] \times R & \xrightarrow{\theta} & R \\
\downarrow & & \downarrow \\
(k^+[X] \times R)/P & = & k^+[X] \otimes_{k+[X]} R
\end{array}
\]

where P is an ideal generated by all the relations given in section 3.1. Thus for all p ∈ k+[X], r ∈ R

\[
\theta((p_1, r) + (p_2, r) - (p_1 + p_2, r)) = \theta(p_1, r) + \theta(p_2, r) - \theta(p_1 + p_2, r) = \phi(p_1) r + \phi(p_2) r - (\phi(p_1) + \phi(p_2)) r = 0
\]

\[
\theta((p \cdot q) - (q, \phi(p) r)) = \theta(pq, r) - \theta(q, \phi(p) r) = \phi(pq) r + \phi(q) \phi(p) r = 0
\]

\[
\theta((p_1, r_1)(p_2, r_2) - (p_2, r_1 \phi(p_1) r_2)) = \theta(p_1, r_1) \theta(p_2, r_2) - \theta(p_2, r_1 \phi(p_1) r_2) = \phi(p_1) r_1 \phi(p_2) r_2 - \phi(p_2) r_1 \phi(p_1) r_2 = 0
\]

so \( \theta(P) = 0 \), and we have the pushout diagram

\[
\begin{array}{ccc}
X & \xrightarrow{id} & k^+[X] \\
\downarrow \phi & & \downarrow \phi_* \\
X & \xrightarrow{\partial} & k^+[X] \\
\downarrow & & \downarrow \\
X & \xrightarrow{\phi} & R
\end{array}
\]

where \( \phi_* (k^+[X]) = k^+[X] \otimes_{k+[X]} R \)

\[
\partial_* : k^+[X] \otimes_{k+[X]} R \to R, \quad p \otimes r \mapsto \phi(p) r
\]

with \( p \in k+[X], \partial(p) = p \).
2. Let $D$ be $S$-module and $0 = \partial : D \to S$ be zero morphism. The pushout diagram is

\[
\begin{array}{ccc}
D & \xrightarrow{\psi} & \phi_s(D) \\
\downarrow{\partial=0} & & \downarrow{\phi_s} \\
S & \xrightarrow{\phi} & R
\end{array}
\]

where
\[
\partial_s (d \otimes r) = \phi_s (\partial(d)) r = \phi_s(0) r = 0 r = 0
\]
so $\partial_s = 0$ and $P = 0$. Thus,
\[
\phi_s (D) = F(D \times R)
\]
Then, the induced crossed module is a free $S$-module on $D \times R$.

3. Given crossed module $i = \partial : I \hookrightarrow S$ where $i$ inclusion of an ideal. Using any surjective homomorphism $\phi : S \to S/I$ the induced diagram is

\[
\begin{array}{ccc}
I & \xrightarrow{\psi} & \phi_s(I) \\
\downarrow{i=\partial} & & \downarrow{\phi_s} \\
S & \xrightarrow{\phi} & S/I
\end{array}
\]
Thus we get $\phi_s (I) = I \otimes (S/I) \cong I/I^2$ which is an $S/I$-module. So $\phi_s$ does not preserve ideals.

4. Given crossed module $\partial : S \to M(S)$ with $\partial(s) = \delta_s$ and $\delta_s(s') = ss'$ for all $s, s' \in S$. If $M(\phi) : M(S) \to M(R)$ is a morphism where $\phi : S \to R$ is a morphism of algebras such that $(\phi, M(\phi))$ is a morphism of crossed modules. We get the induced diagram

\[
\begin{array}{ccc}
S & \xrightarrow{\phi_s} & \phi_s(S) \\
\downarrow{\partial} & & \downarrow{\phi_s} \\
M(S) & \xrightarrow{M(\phi)} & M(R)
\end{array}
\]
where $\phi_s (S) = S \otimes M(R)$.

3.3 Properties of Induced Crossed Module

$\phi_s (D)$ induced crossed $S$-module can be expressed more simply for the case when $\phi : S \to R$, $k$-algebra morphism, is an epimorphism or monomorphism.
3.3.1 Epimorphism case:

Proposition 8 Let $\partial : D \rightarrow S$ be a crossed $S$-module and $\phi : S \rightarrow R$ epimorphism with $\text{Ker}\phi = K$. Then

$$\phi_\ast(D) \cong D/KD$$

where $KD$ is an ideal of $D$ generated by $KD = \{k \cdot d \mid d \in D, k \in K\}$.

Proof: Because $S$ and $K$ acts trivially on $D/KD$, $R \cong S/K$ acts on $D/KD$. Indeed, because of

$$S \times D/KD \longrightarrow D/KD$$

$$(s, d + KD) \longmapsto s \cdot (d + KD) = sd + KD$$

and

$$K \times D/KD \longrightarrow D/KD$$

$$(k, d + KD) \longmapsto k \cdot (d + KD) = kd + KD$$

$S/K$ acts on $D/KD$ as following

As $\phi : S \rightarrow R$ is surjective, we get the following

$$R \cong S/K$$

and $R$ acts on $D/KD$.

$\beta : D/KD \rightarrow R$ given by $\beta(d + KD) = \partial(d) + K$ is a crossed $R$-module. Indeed,

$$\beta(d + KD) \cdot (d' + KD) = (\partial d + K) \cdot (d' + KD)$$

$$= \partial (d) \cdot d' + KD$$

$$= dd' + KD$$

$$= (d + KD)(d' + KD)$$

$(\rho, \phi) : (D, S, \partial) \longrightarrow (D/KD, R, \beta)$ is a crossed module morphism where $\rho : D \longrightarrow D/KD, \rho(d) = d + KD$ since $\rho(s \cdot d) = \phi(s) \cdot \rho(d)$.
Suppose that the following diagram of crossed module is commutative.

\[
\begin{array}{ccc}
D & \xrightarrow{\rho'} & D' \\
\downarrow & & \downarrow \\
S & \xrightarrow{\phi} & R
\end{array}
\]

Since \( \rho'(s \cdot d) = \phi(s) \cdot \rho'(d) \) for any \( d \in D, s \in S \), we have
\[
\rho'(k \cdot d) = \phi(k) \cdot \rho'(d) = 0 \cdot \rho'(d) = 0
\]
so \( \rho'(K D) = 0 \). Then, there is a unique morphism \( \mu : (D/K D) \rightarrow D' \) given by \( \mu(d + K D) = \rho'(d) \) such that \( \mu \rho = \rho' \) and \( \mu \) is well defined, because of \( \rho'(K D) = 0 \). Finally, the diagram

\[
\begin{array}{ccc}
D & \xrightarrow{\rho} & D/K D & \xrightarrow{\mu} & D' \\
\downarrow & & \downarrow & & \downarrow \\
S & \xrightarrow{\phi} & R & \xrightarrow{\rho'} & R
\end{array}
\]

commutes, since for all \( d \in D \)
\[
\beta(d + K D) = \phi \partial d = \partial d + K = \beta' \rho'(d) = \beta' \mu(d + K D)
\]
and
\[
\mu(r \cdot (d + K D)) = \mu((s \cdot d) + K D) = \mu(\rho(s \cdot d)) = \rho'(s \cdot d) = \phi(s) \cdot \rho'(d) = r \cdot \mu(d + K D)
\]
so \( \mu \) preserves the actions. \( \Box \)

### 3.3.2 Inducing Crossed Modules by an ideal inclusion

#### Monomorphism case:

In this subsection we consider the crossed modules induced by a morphism \( \phi : S \rightarrow R \) of \( k \)-algebras, the particular case when \( S \) is an ideal of \( R \).

If \( d \in D \), then the class of \( d \) in \( D/D^2 \) is written as \([d]\). If \( M \) is an algebra, then \( I(M) \) denotes the augmentation ideal of \( M \). Then the augmentation ideal of \( I(R/S) \) of a quotient algebra \( R/S \) has the basis \( \{ \bar{e}_{i_1} \bar{e}_{i_2} \cdots \bar{e}_{i_p}, i_1 \leq i_2 \leq \cdots \leq i_p, i_j \in I \}_{(i) \neq \emptyset} \), where \( \bar{e}_{i_j} \) is the projection of the basic element \( e_{i_j} \in I(R) \) on \( R/S \).

**Theorem 9** Let \( D \subseteq S \) be ideals of \( R \) so that \( R \) acts on \( S \) and \( D \) by multiplication. Let \( \partial : D \rightarrow S \), \( \phi : S \rightarrow R \) be the inclusions and let \( D \) denote
the crossed module \((D, S, \partial)\) with the multiplication action. Then the induced crossed \(R\)-module \(\phi_* (D)\) is isomorphic as a crossed \(R\)-module to

\[
\zeta : D \times (D/D^2 \otimes I(R/S)) \rightarrow R
\]

\[
(d, [t] \otimes \bar{x}) \mapsto d.
\]

The action is given by

\[
r \cdot (d, [t] \otimes \bar{r}) = (r \cdot d, [d] \otimes \bar{r} + [t] \otimes \bar{r} \bar{x} - [x \cdot t] \otimes \bar{r})
\]

for \(d, t \in D; \bar{x} \in I(R/S)\) where \(\bar{r}, \bar{r} \bar{r}\) denote the image of \(r, x\) in \(R/S\), respectively.

**Proof:** First we will show that \(T = (\zeta : T = (D \times (D/D^2 \otimes I(R/S))) \rightarrow R), \zeta (d, [t] \otimes \bar{x}) = d\) is a crossed module with the given action:

\[
\zeta (d', [t'] \otimes \bar{x}') \cdot (d, [t] \otimes \bar{x}) = d' \cdot (d, [t] \otimes \bar{x})
\]

\[
= (d' \cdot d, [d] \otimes \bar{r} + [t] \otimes \bar{r} \bar{x} - [x \cdot t] \otimes \bar{r})
\]

\[
= (d'd, 0)
\]

\[
= (d'd, [t't] \otimes \bar{x}' \bar{x})
\]

\[
= (d', [t'] \otimes \bar{x}') (d, [t] \otimes \bar{x})
\]

Consider \(i : D \rightarrow D \times (D/D^2 \otimes I(R/S)), i(d) = (d, 0)\). We have the following diagram.

\[
\begin{array}{ccc}
D & \xrightarrow{i} & T \\
\downarrow{\phi} & & \downarrow{\alpha} \\
S & \xrightarrow{\zeta} & R
\end{array}
\]

\[
C
\]

\[
\begin{array}{ccc}
D & \xrightarrow{i} & T \\
\downarrow{\phi} & & \downarrow{\alpha} \\
S & \xrightarrow{\zeta} & R
\end{array}
\]

Clearly we have a morphism of crossed modules \((i, \phi) : D \rightarrow T\). We just verify that this morphism satisfies condition ii) of Definition 2.3. That is, when a morphism of crossed module \((\beta, \phi) : (D, S, \partial) \rightarrow (C, R, \alpha)\) is given we prove that there is a unique morphism \(\bar{\phi} : T = D \times (D/D^2 \otimes I(R/S)) \rightarrow C\) such that \(\bar{\phi} i = \beta\) and \(\alpha \bar{\phi} = \zeta\). Since \(\bar{\phi}\) has to be a homomorphism and preserve the action we have

\[
\bar{\phi} (d, [t] \otimes \bar{e}_{(i)}) = \bar{\phi} (d(0), + (0, [t] \otimes \bar{e}_{(i)})
\]

\[
= \bar{\phi} (d(0), + (e_{(i)} \cdot [t] \otimes \bar{e}_{(i)}) + (-e_{(i)} \cdot t, 0))
\]

\[
= \bar{\phi} (d(0), + (e_{(i)} \cdot (t, 0) + (-e_{(i)} \cdot t, 0))
\]

\[
= \bar{\phi} (d(0), + \bar{\phi} (e_{(i)} \cdot (t, 0)) + \bar{\phi} ((-e_{(i)} \cdot t, 0))
\]

\[
= \bar{\phi} i(d) + e_{(i)} \cdot \beta(t) - \bar{\phi} (e_{(i)} \cdot t)
\]

\[
= \beta (d) + e_{(i)} \cdot \beta(t) - \beta (e_{(i)} \cdot t)
\]
for any \(d \in D, \left([t] \otimes e_{(i)}\right) \in D/D^2 \otimes I(R/S)\), This proves uniqueness of any such a \(\tilde{\phi}\). We now prove that this formula gives a well-defined morphism.

It is immediate from the formula that \(\phi : D \times (D/D^2 \otimes I(R/S)) \to C\) has to be \(\beta\) on the first factor and is defined on the second one by the map

\[
[t] \otimes e_{(i)} \mapsto e_{(i)} : \beta(t) - \beta(e_{(i)} \cdot t).
\]

We have to check that this latter map is well defined homomorphism. We define the function

\[
\gamma_r : D \rightarrow C
\]

\[
d \mapsto r \cdot \beta(d) - \beta(r \cdot d)
\]

and prove in turn the following statements.

3.5 \(\gamma_r(D)\) is contained in the annihilator \(\text{Ann}(C)\) of \(C\).

**Proof of 3.5** We use the fact that if \(d \in D\), then

\[
\alpha \gamma_r(d) = \alpha (r \cdot \beta(d) - \beta(r \cdot d)) = \alpha (r \cdot \beta(d)) - \alpha (r \cdot d)
\]

\[
= r \alpha (\beta(d)) - \phi \partial (r \cdot d)
\]

\[
= r \phi \partial (d) - \phi (r \partial (d))
\]

\[
= r \partial (d) - r \partial (d) = 0
\]

and that \((C, R, \alpha)\) is a crossed module. So \(\gamma_r(D) \subseteq \ker \alpha = \text{Ann}(C)\)

3.6 \(\gamma_r\) is a morphism which factors through \(D/D^2\).

**Proof of 3.6** Let \(d, d' \in D\) be, then

\[
\gamma_r(dd') = r \cdot \beta(dd') - \beta(r \cdot (dd'))
\]

\[
= r \cdot (\beta(d) \beta(d')) - \beta(r \cdot d) \beta(d')
\]

\[
= r \cdot \beta(d) \beta(d') - \beta(r \cdot d) \beta(d')
\]

\[
= \gamma_r(d) \beta(d')
\]

\[
= 0
\]

\[
= \gamma_r(d) \gamma_r(d')
\]

Consequently \(\gamma_r\) is a homomorphism of commutative algebras that factors through \(D/D^2\).

3.7 We define a morphism

\[
\gamma : D/D^2 \otimes I(R/S) \to C
\]

\[
[d] \otimes e_{(i)} \mapsto \gamma e_{(i)}(d)
\]

**Proof of 3.7** Since \(\beta\) is a \(S\)-equivariant homomorphism, the morphisms \(\gamma_r\) depend only on the classes \(\tau\) of \(r\) in \(R/S\). Thus, we define the morphism \(\gamma\) as mentioned.

3.8 The function \(\tilde{\phi}\) defined in the theorem satisfies \(\tilde{\phi}i = \beta\) and is a well defined morphism of \(R\)-crossed modules.

**Proof of 3.8** The function \(\tilde{\phi}\) is clearly a well-defined morphism of commutative algebras, since it is of the form \(\tilde{\phi}(d, u) = \beta(d) + \gamma(u)\), where \(\beta\) and \(\gamma\) are the morphisms of commutative algebras and \(\gamma(u)\) belongs to the annihilator of
Thus every element of $C$ is that $X \in p, q$ module the 2-boundaries where $d$ constructed in section 1 we can discuss the free crossed module

$$\phi((d, u)(d', u')) = \phi(dd', uu') = \beta(dd') + \gamma(uu') = \beta(d)\beta(d') + \gamma(u)\gamma(u') = \beta(d)\beta(d') + \beta(d)\gamma(u') + \gamma(u)\beta(d') + \gamma(u)\gamma(u') = (\beta(d) + \gamma(u))(\beta(d') + \gamma(u'))$$

$$= \phi(d, u)\phi(d', u')$$

Further, $\phi i = \beta$ and $\alpha\phi = \zeta$, as $\alpha\gamma$ is trivial:

$$\alpha\gamma([t] \otimes e_{(i)}) = \alpha(e_{(i)} \cdot \beta(t) - \beta(e_{(i)} \cdot t)) = e_{(i)} \cdot \alpha(\beta(t)) - \alpha(\beta(e_{(i)} \cdot t)) = e_{(i)} \cdot \phi \partial(t) - \phi \partial(e_{(i)} \cdot t) = e_{(i)} \cdot \partial(t) - \partial(e_{(i)} \cdot t) = 0.$$

Finally, we prove that $\tilde{\phi}$ preserves the action. This is the crucial part of the argument. Let $d, t \in D, r \in R$ and $r_{(i)}$ be an element in the basis of $I(R/S)$, then

$$\tilde{\phi}(r \cdot (d, [t] \otimes e_{(i)})) = \alpha(e_{(i)} \cdot \beta(t) - \beta(e_{(i)} \cdot t)) = e_{(i)} \cdot \phi \partial(t) - \phi \partial(e_{(i)} \cdot t) = e_{(i)} \cdot \partial(t) - \partial(e_{(i)} \cdot t) = 0.$$

$\square$

## 4 Application: Koszul Complex

As the free crossed modules are the special case of the induced crossed modules we can discuss the free crossed module $C \rightarrow R$ of commutative algebras was shown in [12] to have $C \cong R^n/d(\Lambda^2 R^n)$, i.e. the 2nd Koszul complex term module the 2-boundaries where $d : \Lambda^2 R^n \rightarrow R^n$ the Koszul differential. The idea is that $X$ is a finite set and given a function $f : X \rightarrow R$, then one can get

$$C = R^+[X]/P$$

constructed in section 1.2 where $P$ is generated by the elements $pq - \theta(p)q$ for $p, q \in R^+[X]$. Clearly $P$ is also generated by the elements

$$\{x_i x_j - f(x_i)y_j \mid 1 \leq i, j \leq n\}.$$

Thus every elements of $C$ can be represented by a linear form in $R^+[X]

$$c = \sum r_i x_i + P$$

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Also
\[ d(c) = \partial(\sum r_i x_i + P) = \sum r_i \partial(x_i) + \partial(P) = \sum r_i f(x_i) \]

Since this representation is unique (see [12]) there is thus an epimorphism
\[ R^n \rightarrow C \]
\[ e_i \mapsto x_i + P \]

where \( e_i \) is the \( i \)th basis element of \( R^n \). The kernel \( f \) this epimorphism is the image of the Koszul differential
\[ d : \Lambda^2 R^n \rightarrow R^n. \]

So we have
\[ R^n / d(\Lambda^2 R^n) \cong C. \]

When we take \( S = k^+[X] \) in a morphism \( \phi : S \rightarrow R \), there is the induced crossed module
\[ \partial : k^+[X] \otimes_{k^+[X]} R \rightarrow R \]

and thus
\[ k^+[X] \otimes_{k^+[X]} R \cong R^n / d(\Lambda^2 R^n) \]

This connection with the Koszul complex is taken in [11, 12]. (see also [1])

References

[1] Z. Arvasi, T. Porter, Simplicial and Crossed Resolutions of Commutative Algebras, *Journal of Algebras*, 181, 426-448, (1996).

[2] Z. Arvasi, U. Ege, Annihilators, Multipliers and Crossed Modules, *Applied Categorical Structures*, Vol. 11, 487-506, (2003).

[3] R. Brown, Coproducts of Crossed P-modules: Applications to Second Homotopy Groups and to the Homology of Groups, *Topology*, 23, 337-345, (1984).

[4] R. Brown, P. Higgins, On the Connection between the Second Relative Homotopy Groups of Some Related Spaces, *Proc. London Math. Soc.*, 3, 36, 193-212, (1978).

[5] R. Brown, R.S. Villanueva, P. Higgins, Nonabelian Algebraic topology, Preprint, (2004).

[6] R. Brown, C. D. Wensley, Computing Crossed Modules Induced by an Inclusion of a Normal Subgroup, with Applications to Homotopy 2-types, *Theory Appl. Categ.*, 2, 1, 3-16, (1996).

[7] R. Brown, C. D. Wensley, Computation and Homotopical Applications of Induced Crossed Modules, *J. Symbolic Comput.*, 35, 59-72, (2003).
[8] J.M. Casas, M. Ladra, Colimits in the Crossed Modules Category in Lie Algebras, *Georgian Mathematical Journal*, 7, 3, 461-474, (2000).

[9] J.L. Loday, Spaces with Finitely many non-trivial Homotopy Groups, *J. Pure and Applied Algebra*, Vol. 24, 179-202, (1982).

[10] S. Mac Lane, Extension and Obstructions for Rings, *Illinois Journal of Mathematics*, 121, 316-345, (1958).

[11] T. Porter, Some Categorical Results in the Category of Crossed Modules in Commutative Algebra, *J. Algebra*, 109, 415-429, (1978).

[12] T. Porter, Homology of Commutative Algebras and an Invariant of Simis and Vasconceles, *J. Algebra*, 99, 458-465, (1986).

[13] N.M. Shammu, Algebraic and an Categorical Structure of Category of Crossed Modules of Algebras, *Ph.D. Thesis, U.C.N.W*, (1992).

[14] J. H. C. Whitehead, Combinatorial Homotopy I and II, *Bull. Amer. Math. Soc.*, 55, 231-245 and 453-456, (1949).
Ummahan Ege Arslan
uege@ogu.edu.tr
Osmangazi University
Department of Mathematics and Computer Sciences
Art and Science Faculty
Eskişehir/Turkey

Zekeriya Arvasi
zarvasi@ogu.edu.tr
Osmangazi University
Department of Mathematics and Computer Sciences
Art and Science Faculty
Eskişehir/Turkey

Özgün Gürmen
ogurmen@dpu.edu.tr
Dumlupınar University
Department of Mathematics
Art and Science Faculty
Kütahya/Turkey