A BOUNDARY QUOTIENT DIAGRAM FOR RIGHT LCM SEMIGROUPS

NICOLAI STAMMEIER

Abstract. We propose a boundary quotient diagram for right LCM semigroups with property (AR) that generalizes the boundary quotient diagram for $\mathbb{N} \times \mathbb{N}^\times$. Our approach focuses on two important subsemigroups: the core subsemigroup and the semigroup of core irreducible elements. The diagram is then employed to unify several case studies on KMS-states, and we end with a discussion on $K$-theoretical aspects of the diagram motivated by recent findings for integral dynamics.

1. Introduction

A countable discrete semigroup $S$ is called right LCM if it is left cancellative and the intersection of two principal right ideals in $S$ is either empty or another principal right ideal. The terminology alludes to the existence of right least common multiples given the existence of any common right multiple. It is known that a left cancellative semigroup $S$ is right LCM if and only if Li's family of constructible right ideals $\mathcal{J}(S)$ is given by $\emptyset$ and the principal right ideals in $S$, see [Li12, BLS17]. We shall assume that the semigroup $S$ is not only right LCM and unital, but also has property (AR), which is explained in Section 2. But we remark here that no example of a right LCM semigroup without this property is known so far, see [BS16].

A classical example of a right LCM semigroup $S$ is $\mathbb{N} \times \mathbb{N}^\times$. Within the operator-algebraic context, this example is treated in great detail in the celebrated work [LR10], where Laca and Raeburn studied the Toeplitz algebra for the quasi-lattice ordered pair $(\mathbb{Q} \times \mathbb{Q}^\times_+, \mathbb{N} \times \mathbb{N}^\times)$, in particular with regards to its KMS-state structure for a natural dynamics. They also considered the boundary quotient $Q(\mathbb{N} \times \mathbb{N}^\times)$ of $T(\mathbb{N} \times \mathbb{N}^\times)$ in the sense of Crisp and Laca [CL07]. Using a suitable presentation for $T(\mathbb{N} \times \mathbb{N}^\times)$ by generators and relations, they show that $Q(\mathbb{N} \times \mathbb{N}^\times)$ is obtained from $T(\mathbb{N} \times \mathbb{N}^\times)$ by imposing two extra relations:

(a) The isometry corresponding to $(1, 1) \in \mathbb{N} \times \mathbb{N}^\times$ is a unitary.

(m) For every prime $p \in \mathbb{N}^\times$, the isometries for $\{(m, p) \mid 0 \leq m \leq p - 1\}$ form a Cuntz family, that is, their range projections sum up to one.

As a straightforward consequence, $Q(\mathbb{N} \times \mathbb{N}^\times)$ coincides with Cuntz’s $Q_n$ from [Cun08].

The results of [LR10] were extended in [BaHLR12] by introducing and analysing two complementary intermediate quotients between $T(\mathbb{N} \times \mathbb{N}^\times)$ and $Q(\mathbb{N} \times \mathbb{N}^\times)$: the additive boundary quotient $T_{add}(\mathbb{N} \times \mathbb{N}^\times)$, obtained by imposing (a), and the multiplicative

\textit{2010 Mathematics Subject Classification.} 46L05 (Primary) 20M30, 46L30 (Secondary).

The author was supported by ERC through AdG 267079 and by RCN through FRIPRO 240362.
boundary quotient $\mathcal{T}_{\text{mult}}(N \times N^\times)$, obtained by imposing \((m)\), see [BaHLR12, Proposition 3.3]. Altogether, these form the boundary quotient diagram for $N \times N^\times$:

\[
\begin{array}{cccc}
\mathcal{T}(N \times N^\times) & \longrightarrow & \mathcal{T}_{\text{mult}}(N \times N^\times) & \\
\downarrow & & \downarrow & \\
\mathcal{T}_{\text{add}}(N \times N^\times) & \longrightarrow & Q(N \times N^\times) & \\
\end{array}
\]

This diagram was shown to exhibit interesting features with respect to KMS-states, see [BaHLR12, Section 4].

By now, several works on KMS-state structures on Toeplitz type algebras and their quotients have been influenced, if not very much inspired by the approach in [LR10], see for instance [LRR11, LRRW14, CaHR]. Somewhat intriguingly, the treatment for Baumslag-Solitar monoids $BS(c,d)^+$ features a boundary quotient diagram in disguise, see [CaHR, Corollary 5.3]. For both $N \times N^\times$ and $BS(c,d)^+$, the choices of the intermediate quotients are natural, yet based on the particular presentation of the semigroup. In addition, each of the aforementioned accounts on KMS-state structures remains an isolated case study for a specific family of right LCM semigroups, even though the similarities with regards to results and methods of proof are apparent.

One central aim of this work is to overcome this deficiency by introducing a boundary quotient diagram (2.2) for every right LCM semigroup with property (AR) that allows us to display the results on KMS-states from [LR10, BaHLR12, LRR11, LRRW14, CaHR] in a unified manner, see Theorem 4.1.

A convenient framework for this is provided through Li’s theory [Li12] of full semigroup $C^*$-algebras $C^*(S)$ and the notion of a boundary quotient $Q(S)$ for right LCM semigroups $S$ from [BRRW14], as these constructions generalize the corresponding ones for quasi-lattice ordered groups. Thus the task reduces to identifying two natural intermediate quotients that complement each other in a suitable sense. To this end, we recall that $Q(S)$ is obtained from $C^*(S)$ by imposing the boundary relation $\sum_{f \in F} e_{fS} = 1$ for all accurate foundation sets $F$, see Section 2 for details. The singleton foundation sets play a special role: They are given by the elements of the core subsemigroup $S_c = \{s \in S \mid sS \cap tS \neq 0 \text{ for all } t \in S\}$, and the boundary relation turns the corresponding generating isometries $v_s \in C^*(S)$ into unitaries. This will serve as our defining relation for the core boundary quotient $Q_c(S)$, which generalizes $\mathcal{T}_{\text{add}}(N \times N^\times)$.

The naive approach of defining the analogue of $\mathcal{T}_{\text{mult}}(N \times N^\times)$ as the quotient of $C^*(S)$ by the boundary relation for every accurate foundation set $F$ with $|F| \geq 2$, or equivalently $F \subset S \setminus S_c$, is bound to fail. Indeed, it is easy to see that we get nothing but $Q(S)$ in this case. Therefore, we propose a slightly more elaborate version: We call an element $s \in S \setminus S_c$ core irreducible if every factorization $s = vr$ with $r \in S_c$ satisfies $r \in S^\ast$, where $S^\ast$ denotes the subgroup of invertible elements in $S$. Since $S$ is assumed left cancellative, the core irreducible elements form a subsemigroup $S_d$ of $S$. We say that a foundation set $F$ is proper if $F$ consists of core irreducible elements, and then define the proper boundary quotient $Q_p(S)$ as the quotient of $C^*(S)$ by the boundary relation for all proper accurate foundation sets. Thus, the conditions
(c) For every \( s \in S_c \), the isometry \( v_s \) is a unitary.

(p) For every proper accurate foundation set \( F \), the Toeplitz-Cuntz family of isometries \( (v_f)_{f \in F} \) is a Cuntz family.

replace (a) and (m) from \( \mathbb{N} \times \mathbb{N}^\infty \) for a general right LCM semigroup with property (AR), and thus giving rise to the **boundary quotient diagram**

\[
\begin{array}{ccc}
C^*(S) & \rightarrow & \mathcal{Q}_p(S) \\
\downarrow & & \downarrow \\
\mathcal{Q}_c(S) & \rightarrow & \mathcal{Q}(S)
\end{array}
\]

We then show that, under mild assumptions, \( \mathcal{Q}_c(S) \) and \( \mathcal{Q}_p(S) \) are complementary quotients in between \( C^*(S) \) and \( \mathcal{Q}(S) \) in the sense that (c) and (p) together yield \( \mathcal{Q}(S) \), see Proposition 2.10. As a continuation of considerations from [BLS], we then describe sufficient conditions under which semigroup homomorphisms between right LCM semigroups give rise to \(*\)-homomorphisms between corresponding corners of the boundary quotient diagrams, see Remark 2.14.

In order to demonstrate the utility of our approach, various examples are discussed in Section 3. In particular, our boundary quotient diagram is shown to explain the appearance of the two intermediate quotients \( C^*_c(U \bowtie \bowtie A) \) and \( C^*_p(U \bowtie \bowtie A) \) in [BRRW14].

Remark 5.4] for rather special Zappa-Szép products \( U \bowtie \bowtie A \) of right LCM semigroups, see Example 3.4. More importantly, our perspective indicates that the focus on the two components \( U \) and \( A \) is somewhat misleading: The reason why the two quotients agree with \( \mathcal{Q}_c(U \bowtie \bowtie A) \) and \( \mathcal{Q}_p(U \bowtie \bowtie A) \) is that the prescribed conditions \( U \) and \( A \) need to satisfy force \( (U \bowtie \bowtie A)_c = U^* \bowtie \bowtie A_c \) and \( (U \bowtie \bowtie A)_p = U_c \bowtie \bowtie A^* \), so that proper accurate foundation sets of \( U \bowtie \bowtie A \) are essentially determined by proper accurate foundation sets of \( U \).

The structure of the subsemigroup of **core irreducible** elements \( S_{ci} \) is of independent interest. It appears to contain vital information on the semigroup \( S \) itself, see Remark 2.13. With regards to the associated \( C^*\)-algebras, we provide evidence that \( S_{ci} \) plays an important role in the quest for the \( K\)-theory of \( \mathcal{Q}(S) \) and \( \mathcal{Q}_p(S) \). This is discussed in Section 5 in the context of integral dynamics [BOS16] and Baumslag-Solitar monoids [Spi12].

We suspect the boundary quotient diagram to admit an elegant and equally useful description in the language of groupoids. In particular, this might be the key to studying a boundary quotient diagram for countable discrete left cancellative semigroups and to obtaining a vast generalization of Theorem 4.1. In this direction, the work of Laca and Neshveyev [LN11] may be crucial, and its appendix indicates the existence of a common theme for KMS-state structures of this kind. Having said that, we will not address this here for the sake of an elementary and brief exposition.

The paper is organized as follows: The boundary quotient diagram for right LCM semigroups with property (AR) is constructed in Section 2. The particular form of the boundary quotient diagram for a variety of examples is discussed in Section 3. As an application of the diagram, a unifying statement for the results on KMS-states from the
four different case studies [LR10, BaHLR12, LRR11, LRRW14, CaHR] is established in Section 4. In the final Section 5, these results are contrasted by $K$-theoretical consider-
atios, where we review the torsion subalgebra for integral dynamics in order to present two candidates for an analogue of this subalgebra for other right LCM semigroups.

Acknowledgements: The author would like to thank Nadia S. Larsen, Nathan Brown-
lowe, Selçuk Barlak, Dave Robertson, and Magnus Dahler Norling for stimulating dis-
cussions. Parts of this research were carried out during a visit to the University of
Wollongong, and the author would like to express his gratitude for the great hospitality
of its operator algebra group.

In [BRRW14], a boundary quotient $Q(S)$ was introduced for right LCM semigroups
as the quotient of the full semigroup $C^*$-algebra $C^*(S)$ by the relation $\prod_{f \in F}(1-e_{fS}) = 0$
for all foundation sets $F$ for $S$. Recall that a finite subset $F$ of $S$ is called a foundation
set, if, for every $t \in S$, there is $s \in F$ such that $sS \cap tS \neq \emptyset$. For convenience, let us
denote the set of all foundation sets for $S$ by $\mathcal{F}(S)$. This approach to defining $Q(S)$ was
inspired by the work of Crisp and Laca in the setting of quasi lattice-ordered groups
[CL07]. Shortly thereafter, it was observed in [BS16] that a broad class of right LCM
semigroups has the accurate refinement property, or property (AR) for short: For every
$F \in \mathcal{F}(S)$, there is $F' \in \mathcal{F}(S)$ such that

a) $F'$ is accurate ($fS$ and $f'S$ are disjoint for $f, f' \in F'$, $f \neq f'$), and
b) $F'$ refines $F$ (for every $f' \in F'$ there is $f \in F$ with $f' \in fS$).

If $S$ has property (AR), then the boundary relation for $Q(S)$ reduces to

\[ \sum_{f \in F} e_{fS} = 1 \]

for every accurate foundation set $F$, the collection of which we shall denote by $\mathcal{F}_a(S)$.

A simple, but important observation from [Star15] is the relevance of the core sub-
semigroup $S_c := \{ s \in S \mid sS \cap tS \neq \emptyset \text{ for all } t \in S \}$,
whose origin can again be traced back to [CL07]. The semigroup $S_c$ contains the group of
units $S^*$, and forms a right reversible semigroup, that is, finite intersections of nonempty
right ideals are nonempty. Hence $S_c$ is a right Ore semigroup provided that it has
right cancellation. With regards to the boundary quotient $Q(S)$, we note that every
generating isometry $v_s \in C^*(S)$ with $s \in S_c$ is turned into a unitary when passing to $Q(S)$. This motivates the definition of the first intermediate quotient between $C^*(S)$
and $Q(S)$.

**Definition 2.1.** The core boundary quotient $Q_c(S)$ is the quotient of $C^*(S)$ by the
relation $v_s v_s^* = 1$ for all $s \in S_c$.

One may be tempted to define the second intermediate quotient as the quotient of
$C^*(S)$ by (2.1) restricted to (accurate) foundation sets that do not contain any element
from the core $S_c$. However, this yields nothing but $Q(S)$ as we shall now see. The
starting point are the following two basic observations whose straightforward proofs are
left to the reader.
Lemma 2.2. For $F_1, F_2 \subseteq S$, the set $F_1 \cdot F_2 := \{st \mid s \in F_1, t \in F_2\} \in \mathcal{F}(S)$ is an accurate foundation set if and only if $F_1$ and $F_2$ are accurate foundation sets.

If $F_i = \{s\}$ for some $s \in S$, we shall simply write $s \cdot F_2$ or $F_1 \cdot s$, respectively.

Lemma 2.3. Let $F_1, F_2 \in \mathcal{F}_a(S)$. Then the boundary relation (2.1) for both $F_1$ and $F_2$ is equivalent to the boundary relation (2.1) for $F_1 \cdot F_2$.

As a direct consequence of Lemma 2.2 and Lemma 2.3, we get:

Corollary 2.4. Let $F \in \mathcal{F}_a(S)$ and $s \in S_c$. Then $s \cdot F, F \cdot s \in \mathcal{F}_a(S)$, and the boundary relation (2.1) for $s \cdot F$ or $F \cdot s$ is equivalent to the boundary relation (2.1) for $F$ and $v_s v_s^* = 1$.

By virtue of Corollary 2.4, we see that imposing the boundary relation (2.1) on $C^*(S)$ for all $F \in \mathcal{F}_a(S)$ with $F \cap S_c = \emptyset$ still yields $Q(S)$. Thus we need a more sophisticated approach, to this end, recall that a non-invertible element $s$ of a monoid $S$ is said to be irreducible if $s \notin S^*$ and any decomposition $s = tr$ in $T$ satisfies $t \in S^*$ or $r \in S^*$.

Definition 2.5. An element $s \in S \setminus S_c$ is called core irreducible if $s = tr$ for $t \in S$ and $r \in S_c$ implies $r \in S^*$. The set of core irreducible elements in $S$ is denoted by $S_{ci}$.

Note that the core irreducible elements are minimal representatives of the equivalence classes in $S/ \sim$, where $s \sim t$ if there are $r, r' \in S_c$ such that $sr = tr'$. In addition, let us remark that $S_{ci}$ is a semigroup (without identity) as $S$ is left cancellative, and we denote its unitization by $S_{ci}^1$.

Definition 2.6. A foundation set $F$ for $S$ is called proper if $F \subseteq S_{ci}$. The set of accurate proper foundation sets is denoted by $\mathcal{F}_a^{(p)}(S)$.

Definition 2.7. The proper boundary quotient $Q_p(S)$ is the quotient of $C^*(S)$ by the boundary relation (2.1) for all proper accurate foundation sets $F$.

We remark that Definition 2.7 does not cater for cases of type $s \cdot F$ with $s \in S_c \setminus S^*$ and $F \in \mathcal{F}_a^{(p)}(S)$ from Corollary 2.3 explicitly. The reason is that we always get $s \cdot F = F' \cdot s'$ for some $s' \in S_c \setminus S^*$ and $F' \in \mathcal{F}_a^{(p)}(S)$ in all the examples that we considered, see Section 3. This raises the question whether the definition of $Q_p(S)$ ought to be modified:

Question 2.8. Is there a right LCM semigroup $S$ (with property (AR)) for which there are $s \in S_c \setminus S^*$ and $F \in \mathcal{F}_a^{(p)}(S)$ with $s \cdot F \subseteq S_{ci}$, i.e. such that $s \cdot F \in \mathcal{F}_a^{(p)}(S)$?

With Definition 2.1 and Definition 2.7 at hands, we are ready for the main definition.

Definition 2.9. The boundary quotient diagram of a right LCM semigroup $S$ is given by:

\[
\begin{array}{ccc}
C^*(S) & \xrightarrow{\pi_p} & Q_p(S) \\
& \pi_c \downarrow & \downarrow \\
Q_c(S) & \rightarrow & Q(S)
\end{array}
\]

(2.2)
stationary. More precisely, this is true if the binary relation $s \rightarrow t :\iff s \in t(S_c \setminus S^*)$ is terminating as discussed in [Bri05, Subsection 2.5]: There is no infinite sequence $(s_n)_{n \geq 1}$ with $s_n \rightarrow s_{n+1}, s_n \neq s_{n+1}$ for all $n$. Clearly, if $\rightarrow$ is terminating, then $S = S^1_{ci}S_c$.

**Proposition 2.10.** Let $S$ be a right LCM semigroup with $S = S^1_{ci}S_c$. Then $Q(S)$ is the quotient of $Q_p(S)$ by the relation $\pi_p(v_s v^*_s) = 1$ for all $s \in S_c$. Equivalently, $Q(S)$ is the quotient of $Q_c(S)$ by the relation $\sum_{f \in F} \pi_c(e_f S) = 1$ for all accurate proper foundation sets $F$. In particular, this holds true if the relation $\rightarrow$ is terminating.

**Proof.** Let $Q'(S)$ be the quotient of $Q_p(S)$ obtained by imposing $\pi_p(v_s v^*_s) = 1$ for all $s \in S_c$. Then $Q(S)$ is a quotient of $Q'(S)$. Hence it suffices to show that (2.1) holds for every accurate foundation set in $Q'(S)$. Let $\pi'_c : Q_p(S) \rightarrow Q'(S)$ denote the quotient map and suppose $F$ is an accurate foundation set. If $F \cap S_c \neq \emptyset$, then necessarily $F = \{s\}$ for some $s \in S_c$ due to accuracy. But in this case, there is nothing to show. So let $F \subset S \setminus S_c$. By assumption, each $f \in F$ can be written as $f = f_i f_e$ with $f_i \in S_{ci}$ and $f_e \in S_e$. Noting that $F_i := \{f_i \mid f \in F\}$ is an accurate proper foundation set, and $\pi'_c(\pi_p(e_f S)) = \pi'_c(\pi_p(e_f S))$ as $e_f S - e_f S = v_{f_e}(1 - e_f S)v^*_f$, we get

$$\sum_{f \in F} \pi'_c(\pi_p(e_f S)) = \sum_{f_i \in F_i} \pi'_c(\pi_p(e_f S)) = 1.$$

Thus $Q'(S)$ coincides with $Q(S)$. \hfill $\square$

Given that $S = S^1_{ci}S_c$, Proposition 2.10 shows that (2.2) takes the form

$$C^*(S) \xrightarrow{\pi_p} Q_p(S) \xrightarrow{\pi_c} Q_c(S) \xrightarrow{\pi'_c} Q(S) \xrightarrow{\pi'_p} Q_p(S)$$

(2.3)

where $\pi'_p$ and $\pi'_c$ are induced by $\pi_p$ and $\pi_c$, respectively. As we shall see in Section 3 all our examples satisfy $S = S^1_{ci}S_c$. This motivates the following two questions:

**Question 2.11.** Is there a right LCM semigroup $S$ that does not satisfy $S = S^1_{ci}S_c$?

**Question 2.12.** Is there a right LCM semigroup $S$ for which (2.3) does not hold?

**Remark 2.13.** Let $S$ be right LCM with $S^* = \{1\}$. Then $S = S^1_{ci}S_c$ is precisely what is needed to display $S$ as the internal Zappa-Szép product $S^1_{ci} \bowtie S_c$, see [Bri05]. If $S^* \neq \{1\}$, but $S^1_{ci}$ admits a transversal $T \subset S^1_{ci}$ for the right action of $S^*$ which forms a semigroup, then $S$ is the internal Zappa-Szép product $T \bowtie S_c$. This is for instance the case for self-similar actions, see Example 3.5.

Let us now examine under which conditions a semigroup homomorphism $\phi : S \rightarrow T$ between two right LCM semigroups $S$ and $T$ induces maps of the quotients appearing...
in the boundary quotient diagram (2.2). This constitutes a natural continuation of [BLS, Section 3] leads to new applications, see Section 3. To begin with, observe that \( \phi \) is necessarily unital because \( T \) is assumed left cancellative and thus the only idempotent in \( T \) is \( 1_T \). Next, recall from [BLS, Theorem 3.3] that \( \phi \) induces a \( * \)-homomorphism \( \varphi: C^*(S) \to C^*(T) \) if and only if

\[
(2.4) \quad \phi(s_1)T \cap \phi(s_2)T = \phi(s_1S \cap s_2S)T \quad \text{for all } s_1, s_2 \in S.
\]

Remark 2.14. Let \( S \) and \( T \) be right LCM semigroups. A semigroup homomorphism \( \phi: S \to T \) satisfying (2.4) induces a \( * \)-homomorphism

\[ \begin{align*}
\varphi_c: & \quad Q_c(S) \to Q_c(T) \text{ if } \phi(S_c) \text{ is a subsemigroup of } T_c, \\
\varphi_p: & \quad Q_p(S) \to Q_p(T) \text{ if } \phi \text{ maps } \mathcal{F}^d(S) \text{ to } \mathcal{F}^d(T), \text{ and} \\
\varphi_q: & \quad Q(S) \to Q(T) \text{ if } \phi \text{ maps } \mathcal{F}(S) \text{ to } \mathcal{F}(T).
\end{align*} \]

If (2.2) is given by (2.3), e.g. if \( S = S^1_{ci}S_c \), then condition \( \phi(\mathcal{F}(S)) \subset \mathcal{F}(T) \) from c) is equivalent to

\[ c') \quad \phi(S_c) \subset T_c \text{ and } \phi \text{ maps } \mathcal{F}((S)_d) \text{ to } \mathcal{F}(T). \]

Question 2.15. Are the conditions presented in Remark 2.14 a)–c') necessary?

3. Examples

Within this section we discuss the boundary quotient diagram for a selection of right LCM semigroups encompassing integral dynamics [3.1], Baumslag-Solitar monoids [3.2], algebraic dynamical systems [3.3], Zappa-Szép products [3.4], and self-similar actions [3.5]. In addition, we mention right Ore semigroups in Example 3.6 and indicate obstructions to induced maps for the core boundary quotient for inclusions of right LCM subsemigroups in Example 3.7.

Example 3.1. Let \( P \subset \mathbb{N}^* \) be a monoid generated by a family \( \mathcal{P} \) of relatively prime numbers and consider \( S := \mathbb{N} \times P \subset \mathbb{N} \times \mathbb{N}^* \). Then \( S \) is right LCM, \( S^* \) is trivial, and \( S_c = \mathbb{N} \times \{1\} \). An element \((n,p) \in S\) is core irreducible if and only if \( 0 \leq n \leq p-1 \), and it is irreducible if, in addition, \( p \) is irreducible in \( P \). We note that \( S^* \) is trivial and \( S = S^1_{ci}S_c \) so that \( S = S^1_{ci} \preccurlyeq S_c \), a description that appeared already in [BS16, Subsection 3.2].

As \( P \) is directed, a finite set \( F \subset S \) is a foundation set if and only if it is refined by the elementary foundation set \( F' := \{(n,p_F) \mid 0 \leq n \leq p_F - 1\} \), where \( p_F \) is the least common multiple of \( \{p \mid (n,p) \in F \text{ for some } n \in \mathbb{N}\} \). A proof of this observation can be obtained along the lines of [BS16, Lemma 3.2 and Lemma 3.3]. Note that elementary foundation sets are accurate. As a consequence, it suffices to impose (2.1) for proper elementary foundation sets in order to form \( Q_p(S) \) (and \( Q(S) \)).

Let \( p = p_1 \cdots p_n \) be a factorization of \( p \in P \) with \( p_i \in \mathcal{P} \) for all \( i \). If \( F_i \) denotes the elementary foundation set for \( p_i \), then each \( F_i \) is a proper accurate foundation set and the elementary foundation set for \( p \) is given by \( F_1 \cdots F_n \). By Lemma 2.3 and Proposition 2.10, we get that:

\[ \begin{align*}
(a) \quad Q_c(S) & \text{ is the quotient by } v_{(1,1)}v^*_{(1,1)} = 1, \\
(b) \quad Q_p(S) & \text{ is the quotient by } \sum_{0 \leq k \leq p-1} v_{(k,p)}v^*_{(k,p)} = 1 \text{ for all } p \in \mathcal{P}, \\
(c) \quad Q(S) & \text{ is the quotient by } v_{(1,1)}v^*_{(1,1)} = 1 \text{ and } \sum_{0 \leq k \leq p-1} v_{(k,p)}v^*_{(k,p)} = 1 \text{ for all } p \in \mathcal{P}.
\end{align*} \]
In particular, Definition 3.4 recovers the boundary quotient of \cite{BaHLR12} in the case where $P$ is the set of all primes, see \cite{BaHLR12} Proposition 3.3.

**Example 3.2.** Consider the Baumslag-Solitar monoid $S = BS(c, d)^+ := \langle a, b \mid ab^l b = b^l a \rangle$ for $c, d \in \mathbb{N}^* \text{ with } cd > 1$. According to \cite{Spi12} Theorem 2.11, $S$ is quasi-lattice ordered, hence right LCM. By \cite{Spi12} Proposition 2.3, every $s \in S$ admits a unique normal form $s = w_1 w_2 \cdots w_n b^k$ with $w_k \in F_d := \{ b^l a \mid 0 \leq l \leq d-1 \}$ and $i \in \mathbb{N}$. In particular, $s \mapsto m$ gives a homomorphism $\ell : S \to \mathbb{N}$ and we call $\ell(s)$ the length of $s$.

Next we observe that for $s = v_1 v_2 \cdots v_m b^t$, $t = w_1 w_2 \cdots w_n b^t \in S$ with $m \leq n$, the intersection of $sS$ and $tS$ is non-empty if only if $v_k = w_k$ for all $1 \leq k \leq m$, in which case $sS \cap tS = tb^l S$ for a suitable $\ell \in \mathbb{N}$. Thus $S_c = \{ b \} \cong \mathbb{N}$ and $s = w_1 w_2 \cdots w_n b^t$ is core irreducible if and only if $i = 0$. The element $s$ is irreducible if and only if $m = 1$. Thus we see that $S^1_c = \langle F_d \rangle \cong \mathbb{F}_d^+$, the free monoid in $d$ generators. In particular, every accurate proper foundation set $F$ for $S$ is of the form $F = \langle F_d \rangle^k$ for some $k \geq 1$, i.e. all words in $F_d$ of length $k$. As for Example 3.1, $S^*$ is trivial, $S = S^1_c S_c$, and thus $S = S^1_c \bowtie S_c$.

Thus by Proposition 2.10 the boundary quotient diagram is characterized as follows:

(a) $Q_c(S)$ is the quotient by $v_i v_i^* = 1$.

(b) $Q_p(S)$ is the quotient by $\sum_{0 \leq k \leq d-1} v_i^{b^k a} v_i^{* b^k a} = 1$.

(c) $Q(S)$ is the quotient by $v_b v_b^* = 1$ and $\sum_{0 \leq k \leq d-1} v_b^{b^k a} v_b^{* b^k a} = 1$.

Implicitly, $Q_c(S)$ and $Q_p(S)$ have already appeared in \cite{CaHR} Corollary 5.3(a) and (c).

**Example 3.3.** For an algebraic dynamical system $(G, P, \theta)$, that is, a right LCM semigroup $P$ acting on a discrete group $G$ by injective group endomorphisms $\theta_p$ so that $pP \cap qP = rP$ forces $\theta_p(G) \cap \theta_q(G) = \theta_r(G)$, we consider the right LCM semigroup $S = G \rtimes_\theta P$, see \cite{BS16} for details. Let $N_p := [G : \theta_p(G)]$ for $p \in P$. We will assume that $N_p = 1$ implies $p \in P^*$, and that $P$ is directed with respect to $p \geq q :\iff q \in P q$. Then $S_c = S^* = G \rtimes_\theta P^*$ since for every $(g, p) \in S$ with $p \notin P^*$, we have $N_p \geq 2$, so there is $h \in G$ with $h^{-1} g \notin \theta_p(G)$ and hence $(g, p) S \cap (h, p) S = \emptyset$. Therefore, $S^*(S) = Q_c(S)$ and every element in $S \setminus S^*$ is core irreducible.

Let $P_{\text{fin}} := \{ p \in P \mid N_p < \infty \}$ denote the subsemigroup of finite elements in $P$. By \cite{BS16} Proposition 3.9, every foundation set for $S$ can be refined by an elementary foundation set in the sense of \cite{BS16} Definition 3.7 provided that every foundation set $F$ for $P$ with $F \subset P_{\text{fin}}$ admits an accurate refinement $F'$ that also satisfies $F' \subset P_{\text{fin}}$. In particular, $S$ has property (AR) in this case. Let us assume that this holds true. Then a proper elementary foundation set is of the form $F_p := \{(g_1, p), (g_2, p), \ldots, (g_{N_p}, p)\}$, where $2 \leq N_p < \infty$ and $\{g_1, \ldots, g_{N_p}\}$ forms a transversal for $G/\theta_p(G)$. If $p = p_1 \cdots p_n$ is a factorization into irreducible elements $p_i \in P$, then $F_{p_1} \cdots F_{p_n}$ forms an elementary foundation set for $p$, and each $F_{p_i}$ is a proper elementary foundation set. Thus, if every $p \in P$ admits a factorization into a product of finitely many irreducible elements, then it suffices to consider elementary foundation sets of irreducibles in $P$, thanks to Lemma 3.3 and therefore $Q_p(S) \cong Q(S)$ is the quotient of $S^*(S)$ by $\sum_{p \in G/\theta_p(G)} v_{(g, p)} v_{(g, p)}^* = 1$ for all irreducible $p \in P$.

**Example 3.4.** Let $S = U \bowtie A$ be a Zappa-Szép product with $U$ and $A$ right LCM semigroups, $J(A)$ totally ordered, and $U \to U, u \mapsto a \cdot u$ bijective for all $a \in A$. According to \cite{BRRW14} Lemma 3.3, $S$ is a right LCM semigroup. Then \cite{BRRW14}
Remark 3.4] gives $S^* = U^* \bowtie A^*$ and $(u, a)S \cap (v, b)S \neq \emptyset \iff uU \cap vU \neq \emptyset$, from which we deduce:

(a) The natural inclusions $\phi_U : U \to S$ and $\phi_A : A \to S$ satisfy (2.4),
(b) $S_c = U_c \times A$, and hence $\phi_U(U_c), \phi_A(A) \subset S_c$.
(c) A finite set $F \subset S$ is an accurate proper foundation set if and only if \{u \mid (u, a) \in F \text{ for some } a \in A\} is an accurate proper foundation set for $U$. In particular, $\phi$ maps $\mathcal{F}^{p}_a(U)$ to $\mathcal{F}^{p}_a(S)$. We have $A = A_c$ because $\mathcal{J}(A)$ is totally ordered, so $A$ does not have any proper foundation sets.

Hence we get a commutative diagram

\[
\begin{array}{c}
C^*(U) \\
Q_p(U) \\
Q_c(U) \\
Q(U)
\end{array} \quad \begin{array}{c}
C^*(S) \\
Q_p(S) \\
Q_c(S) \\
Q(S)
\end{array} \quad \begin{array}{c}
C^*(A) \\
C^*(A) \\
Q(A) \\
Q(A)
\end{array}
\]

Moreover, we also have $S_{ci} = U_{ci} \bowtie A^*$ and $S = S_{ci}^1S_c$ if and only if $U = U_{ci}^1U_c$ because $A^*U_c = U_cA^*$. In particular, we recover [BRRW14, Theorem 5.2] and get a conceptual approach to the intermediate quotients $C^*_A(U \bowtie A)$ and $C^*_U(U \bowtie A)$ from [BRRW14, Remark 5.4]. Note that the quotient $Q_c(U \bowtie A)$ is likely to be different from $C^*_A(U \bowtie A)$ as soon as $U_c \neq U^*$. More importantly, our approach provides candidates for intermediate quotients in the case where $A_c$ is a proper subsemigroup of $A$, i.e. outside the realm of [BRRW14, Lemma 3.3].

Example 3.5. Let $(G, X)$ be a self-similar action and denote by $X^*$ the free monoid in the alphabet $X$. Then $S = X^* \bowtie G$ is a right LCM semigroup which fits into the setup of Example 3.4 according to [BRRW14, Theorem 3.8], which is in fact a result due to Lawson, see [Law08, Proposition 3.5 and 3.6]. As $S_c = S^* = G, C^*(G) = \mathcal{Q}(G), Q_c(X^*) = C^*(X^*)$, and $Q_p(X^*) = \mathcal{Q}(X^*) \cong \mathcal{O}_{|X|}$, the diagram (3.1) simplifies to:

\[
\begin{array}{c}
C^*(X^*) \\
\mathcal{O}_{|X|} \\
C^*(S)
\end{array} \quad \begin{array}{c}
C^*(G) \\
\mathcal{Q}(S) \\
\mathcal{Q}(G)
\end{array}
\]

Example 3.6. Suppose $S$ is a right LCM semigroup that satisfies the right Ore condition, that is, there is an embedding of $S$ into a group $G$ such that $G = SS^{-1}$. Thanks to well-known results of Ore and Dubreil, the right Ore condition is equivalent to cancellation and left reversibility ($sS \cap tS \neq \emptyset$ for all $s, t \in S$). Under this assumption, we get $S_c = S$, and thus $C^*(S) = Q_p(S)$ as well as $Q_c(S) = \mathcal{Q}(S) \cong C^*(G)$.

Example 3.7. Choose a right LCM subsemigroup $S$ of a right LCM semigroup $T$ and let $\phi : S \to T$ be the natural inclusion. If the equation (2.4) is satisfied by $\phi$, then $S \cap T_c$ is a subsemigroup of $S_c$. If $S \cap T_c$ coincides with $S_c$, then $\phi$ induces a map $\varphi_c : Q_c(S) \to Q_c(T)$, see Remark 2.14. Note that $S \cap T_c$ can be a proper subsemigroup
of $S_c$, e.g. if $S$ is abelian but not contained in $T_c$. For instance, take $T$ to be the free monoid in two generators $a$ and $b$, and let $S$ the free abelian submonoid generated by $a$.

### 4. Toward a unified treatment for KMS states

In this section, we use the boundary quotient diagram $[2.2]$ to recast the essential results concerning KMS-states

- (a) for $\mathbb{N} \times \mathbb{N}^\times$ from [LR10, BaHLR12],
- (b) for $\mathbb{Z}^d \rtimes_A \mathbb{N}$ with $A \in M_d(\mathbb{Z}), |\det A| > 1$ from [LRR11],
- (c) for $X^* \Join G$, where $(G, X)$ is a self-similar action with $|X| > 1$, from [LRRW14], and
- (d) for Baumslag-Solitar monoids $BS(c, d)^+$ with $c, d \in \mathbb{N}^\times, d > 1$ from [CaHR].

Note that we appeal to the dual picture of $\mathbb{T}^d \rtimes A \mathbb{N}$ in (b) as opposed to the original treatment in [LRR11]. We remark that if $A$ is invertible in $M_d(\mathbb{Z})$ in case (b), $X$ is a singleton in case (c), or $d = 1$ in case (d), the study of KMS-states on the boundary quotient diagram essentially reduces to the study of traces on group $C^*$-algebras, compare [LRR11]. These cases will be excluded from our considerations as we intend to focus on proper semigroups.

In all the cases (a)-(d), the semigroup $S$ features a natural homomorphism $N: S \to \mathbb{N}^\times, s \mapsto N_s$ arising from a scaling factor $\kappa \in \mathbb{R}_{>0}$ and a length function $\ell: S \to \mathbb{N}$ as $N_s := \kappa^{\ell(s)}$. The map $N$ satisfies $N^{-1}(1) = S_c$ and yields a natural dynamics $\sigma$ of $\mathbb{R}$ on $C^*(S)$ via $\sigma_x(v_s) := N_s^{ix}v_s$ for $x \in \mathbb{R}$.

Define the $\zeta$-function for $S$ to be the formal series $\zeta_S(\beta) := \sum_{s \in S/S_c} N_s^{-\beta}$ for $\beta \in \mathbb{R}$. Note that $\zeta_S$ converges for all $\beta$ above a so-called critical inverse temperature $\beta_c$. The key to uniqueness of KMS$_\beta$-states on $C^*(S)$ for $\beta$ within a critical interval $[1, \beta_c]$ is a notion of minimality for the semigroups in (a)-(d). This data is displayed in the following table:

| type | $S_c$ | $\kappa$ | $\ell: S \to \mathbb{N}$ | $\beta_c$ | minimality |
|------|------|--------|-----------------|--------|-----------|
| (a) | $\mathbb{N}$ | 1 | $(m, p) \mapsto \log p$ | 2 | $\bigcap_{n \in \mathbb{N}^\times} \overline{p\mathbb{N}} = \{0\}$ |
| (b) | $\mathbb{Z}^d \rtimes A$ | $|\det A|$ | $(g, n) \mapsto n$ | 1 | $\bigcap_{n \in \mathbb{N}_0} A^n(\mathbb{Z}^d) = \{0\}$ |
| (c) | $G$ | $|X|$ | $(w, g) \mapsto \ell'(w)$ | 1 | $\forall g \in G: \{g\}_{w \in X^*} | w \in X^* \} < \infty$ |
| (d) | $\mathbb{N} \times d$ | length from $[3.2]$ | 1 | $\bigcap_{n \in \mathbb{N}} \frac{1}{n} \cdot \mathbb{N} = \{0\}$ |

Apparently, $\mathbb{N} \times \mathbb{N}$ is minimal. As opposed to [LRR11], we do not require that $A$ is a dilation matrix, that is, all its eigenvalues need to be larger than one in absolute value, because this feature is only used in [LRR11, Lemma 5.7], where the condition $\bigcap_{n \in \mathbb{N}_0} A^n(\mathbb{Z}^d) = \{0\}$ is then established. The point is that the latter condition is much more natural for the dynamical system $A : \mathbb{N} \to \mathbb{Z}^d$ as it expresses minimality of the dual system. The minimality condition for (d) is equivalent to $c \notin d\mathbb{N}$.

To distinguish between elements in $C^*(S)$ and $C^*(S_c)$, let the standard generating isometries for $C^*(S_c)$ be denoted by $w_s$.

**Theorem 4.1.** Suppose $S$ is a right LCM semigroup of type (a), (b), (c) or (d). Then the KMS-state structure on $C^*(S)$ with respect to the dynamics $\sigma$ given by $\sigma_x(v_s) := N_s^{ix}v_s$ is characterized by:

1. ...
(i) There are no KMS$_\beta$-states for $\beta < 1$.
(ii) For $\beta \in [1, \beta_c]$, there is a KMS$_\beta$-state $\psi_\beta$ given by $\psi_\beta(v_s^* v_t^*) = \delta_{st} N_s^{-\beta}$ for all $s, t \in S$. If $S$ is minimal, then $\psi_\beta$ is the only KMS$_\beta$-state.

(iii) For $\beta \in (\beta_c, \infty)$, there is an affine homeomorphism $\tau : \psi_\beta \mapsto \psi_{\beta,\tau}$ between the tracial states on $C^*(S_c)$ and the KMS$_\beta$-states given by

$$\psi_{\beta,\tau}(v_s^* v_t^*) = \begin{cases} N_s^{1-\beta} \tau(w_s w_t^*) & \text{if } sS \cap tS = sxS, \text{ } sx = ty \text{ with } x, y \in S_c, \\ 0 & \text{otherwise.} \end{cases}$$

(iv) There is a one-to-one correspondence $\phi : \psi_\beta \mapsto \psi_{\phi,\psi}$ between states on $C^*(S_c)$ and ground states on $C^*(S)$ given by $\psi_\phi(v_s^* v_t^*) = \chi_{S_c}(s) \chi_{S_c}(t) \phi(w_s w_t^*)$. A ground state is a KMS$_\infty$-state if and only if it comes from a tracial state on $C^*(S_c)$.

With regards to the boundary quotient diagram (2.2) the following statements hold:

(v) All the KMS$_\beta$-states for $\beta \in [1, \infty)$ factor through $\pi_c$.

(vi) A ground state factors through $\pi_c$ if and only if it is a KMS$_\infty$-state. In particular, every ground state on $Q_c(S)$ is a KMS$_\infty$-state.

(vii) If a KMS$_\beta$-state factors through $\pi_p$, then $\beta = 1$. In particular, $Q_p(S)$ and $Q(S)$ have a unique KMS$_\beta$-state corresponding to $\psi_1$ if $S$ is minimal.

(viii) There are no ground states on $Q_p(S)$, and hence none on $Q(S)$.

Proof. Using the descriptions obtained in Example 3.1, Example 3.3, Example 3.5, and Example 3.2 for (a)-(d), we embark on a reference chase, and leave it as an exercise to identify the semigroup $C^*$-algebra $C^*(S)$ with the $C^*$-algebra of Toeplitz type considered in the respective reference. We will prove part (viii) simultaneously for all cases at the end.

For $S = \mathbb{N} \times \mathbb{N}^\times$, the claims follow from [LR10, Theorem 7.1] and [BaHLR12, Section 4]. For $S = \mathbb{Z}^d \times_A \mathbb{N}$, we note that $\pi_c = \text{id}$ and hence $\pi = \pi_p$, so that [LRRW11, Theorem 1.1] yields (i)-(vii).

Next, let $S = X^* \cong G$. Part (i) is [LRRW14, Proposition 4.1(1)], and (iii) is [LRRW14, Theorem 6.1]. As $C^*(S) = Q_c(S)$, claim (v) is trivial. Also, (iv) and (vi) merge to a single statement that corresponds to [LRRW14, Proposition 5.3]: Ground states on $C^*(S)$ are KMS$_\infty$-states, and they are given by tracial states on $C^*(S) \cong C^*(G)$. The claims (ii) and (vii) are proven in [LRRW14, Proposition 7.1 and Theorem 7.3].

Now let $S = BS(c,d)^+$. Then [CaHR, Corollary 5.3(a) and (b)] gives (i) and (v). The combination of [CaHR, Corollary 5.3(c)] with [CaHR, Proposition 7.1] yields (ii) and (vii). Statement (iv) corresponds to [CaHR, Theorem 8.1], and (vi) is an immediate consequence of this. Claim (iii) is essentially provided by [CaHR, Theorem 6.1] except for the fact that the authors assume minimality of $S$ in order to show injectivity of the parametrization $\tau \mapsto \psi_{\beta,\tau}$. Thus we need to strengthen this result slightly. To this end,
we observe that the formula [CaHR, (6.1)] becomes
\[
\zeta_S(\beta)\psi_{\beta,\tau}(v^{n_1}) = \tau(v^{n_1}) + \sum_{k \geq 1: n \in d(d/c)^k \mathbb{N} \text{ for } 0 \leq j \leq k-1} d^{k(1-\beta)}\tau(v^{bn(c/d)^k})
\]
\[
= \tau(v^{n_1}) + \chi_{dn}(n) d^{1-\beta} \sum_{k \geq 0: (c/d)n \in d(d/c)^k \mathbb{N} \text{ for } 0 \leq j \leq k-1} d^{k(1-\beta)}\tau(v^{bn(c/d)^{k+1}})
\]
\[
= \tau(v^{n_1}) + \chi_{dn}(n) d^{1-\beta}\psi_{\beta,\tau}(v^{bn(c/d)})
\]
for every \( n \in \mathbb{N} \) within our notation. The analogous formula holds for \( v^* \). Hence, if \( \tau \) and \( \rho \) are tracial states on \( C^*(\mathbb{N}) \) with \( \psi_{\beta,\tau} = \psi_{\beta,\rho} \), then \( \tau = \rho \), without assuming that \( S \) is minimal, as in [LRR11, LRRW14]. This completes (iii).

In all cases, (viii) is a consequence of (iv) and the existence of accurate proper foundation sets, as indicated in [BaHLR12]. End of Section 4: Since \( \pi_p \) is a *-homomorphism, every ground state \( \phi \) on \( \mathcal{Q}_p(S) \) lifts to a ground state on \( C^*(S) \). We then conclude by (iv) that \( \phi \) vanishes on all projections \( \pi_p(e_{j_S}) \) with \( s \in S \setminus S_c \). So if there is an accurate proper foundation set \( F \), then \( 1 = \phi(1) = \phi(\sum_{f \in F} \pi_p(e_{f_S})) = 0 \). Thus there are no ground states on \( \mathcal{Q}_p(S) \) and \( \mathcal{Q}(S) \).

Assuming \( S_c \) to be right cancellative and hence right Ore, let \( G_c \) denote the group \( S_c^{-1} \). As the traces on \( C^*(S_c) \) correspond to traces on \( C^*(G_c) \), it may seem more natural to use the group \( C^* \)-algebra in Theorem 4.1. However, we emphasize the core subsemigroup \( S_c \) here because we like to think of the KMS-states as arising from the *-homomorphism \( \varphi: C^*(S_c) \to C^*(S) \) induced by the inclusion \( S_c \subset S \).

The parallels between the results in Theorem 4.1 for the different types (a)-(d) extend beyond their mere statements, for which the boundary quotient diagram provides a unifying framework. Indeed, there are striking analogies in the method of proof. It thus seems natural to ask whether there is a unified treatment for KMS-state structures on the proposed boundary quotient diagram 2.2 for general right LCM semigroups.

In an attempt to promote this perspective, the author is currently investigating how one may characterize the KMS-state structure for the boundary quotient diagram associated to certain algebraic dynamical systems \((G, P, \theta)\), without using the model \( S = G \rtimes \theta P \) explicitly [ABLS].

5. A FIRST LOOK AT THE K-THEORY

In Section 4 we learned that the left column of the boundary quotient diagram 2.2 typically has a significantly richer supply of KMS-states than the right column, see Theorem 4.1. With regards to K-theory however, the situation is almost the opposite:

**Theorem 5.1** ([BLS Theorem 5.3]). Suppose \( S \) is a left Ore right LCM semigroup. If the group \( S^{-1}S \) satisfies the Baum-Connes conjecture with coefficients in commutative \( C^* \)-algebras and the left regular representation \( \lambda: C^*(S) \to C_c^*(S) \) is an isomorphism, then \( K_*(C^*(S)) \cong K_*(C^*(S)) \).

In particular, Theorem 5.1 applies if \( S^{-1}S \) is amenable. For instance, \( S = \mathbb{N} \rtimes P \) as in Example 3.1 satisfies all these conditions. Moreover, \( \mathcal{Q}_c(\mathbb{N} \rtimes P) \cong C^*(\mathbb{Z} \rtimes P) \), and \( \mathbb{Z} \rtimes P \) also satisfies the prerequisites of Theorem 5.1. Thus we get \( K_*(C^*(S)) \cong K_*(C^*(S)) \) and \( K_*(\mathcal{Q}_c(S)) \cong K_*(C^*(\mathbb{Z})) \) in these cases.
On the other hand, the computation of the K-theory for $Q(S)$ can be a challenging task, already for the case where the semigroup arises from a singly generated dynamical system, see [CV13]. However, this is perhaps the most interesting case as $Q(S)$ happens to be a unital UCT Kirchberg algebra for many right LCM semigroups under mild assumptions, see [Star15, BS16].

For semigroups related to dynamical systems with higher complexity, very few is known. Therefore we will start with a very basic example, namely sub dynamics of $N \rtimes N = \mathbb{Z}_2$, i.e. $S = N \rtimes P$, where $P$ is the free abelian submonoid of $\mathbb{N}$ generated by a family $P$ of relatively prime numbers. For such semigroups, nontrivial partial results emerged in the recent past [LN16, BOS16]. These results lead to intriguing questions and relate to conjectures about $C^*$-algebras of $k$-graphs, see [BOS16, Conjecture 5.11].

**Remark 5.2.** In [LN16, 6.3], Li and Norling determined $K_*(Q_p(S))$ for $|P| = 2$ as well as $K_1(Q_p(S))$ for $|P| = 3$. In joint work with Barlak and Omland [BOS16], we obtained similar formulas for $K_*(Q(S))$ in the case of $|P| = 2$ or $g_P = 1$, where $g_P$ is the greatest common divisor of $P - 1 \subset \mathbb{N}$, i.e. $g_P := \gcd\{p - 1 \mid p \in P\}$. In addition, we proved a number of structural results about $K_*(Q(S))$, which we deem relevant here:

a) $K_i(Q(S)) \cong \mathbb{Z}^{|P| - 1} \oplus K_i(A(S))$ for $i = 0, 1$, where $A(S)$ is the subalgebra of $Q(S)$ generated by $\{\pi(v_{(n,p)}) \mid p \in P, 0 \leq n \leq p - 1\}$.

b) $A(S) \cong M_{d^\infty}(\mathbb{C} \rtimes P)$ with $d = \prod_{p \in P} p$, and hence $A(S)$ is a unital UCT Kirchberg algebra. In particular, $A(S)$ also embeds into $Q_p(S)$ due to simplicity. Thus there is a commutative diagram

$$
\begin{array}{ccc}
Q_p(S) & \xrightarrow{\cong} & \varinjlim M_{d^\infty}(C^*(\mathbb{N})) \rtimes P \\
& \swarrow & \searrow \\
\hat{A}(S) & \xrightarrow{\cong} & M_{d^\infty}(\mathbb{C} \rtimes P) \\
& \swarrow & \searrow \\
Q(S) & \xrightarrow{\cong} & \varinjlim M_{d^\infty}(C^*(\mathbb{Z})) \rtimes P
\end{array}
$$

given by the natural inclusions of the $P$-invariant subalgebra $M_{d^\infty}(\mathbb{C})$ into the Bunce-Deddens algebra of type $d^\infty$ and its extension $\varinjlim M_{d^\infty}(C^*(\mathbb{N}))$.

c) $S^1_S = \{(n,p) \in S \mid 0 \leq n \leq p - 1\}$ is a right LCM semigroup, $S = S^1_S \rtimes S_c$, and the inclusion $S^1_S \subset S$ induces an isomorphism $Q(S^1_S) \cong A(S)$.

d) $A(S)$ is isomorphic to $\bigotimes_{p \in P} O_p$ for $|P| \leq 2$. For $|P| \geq 3$, the order of every element in $K_*(A(S))$ is a divisor of $g_P$. This leads to the conjecture that $A(S)$ is isomorphic to $\bigotimes_{p \in P} O_p$ for all families $P$, see [BOS16, Conjecture 6.5].

e) By b), $A(S)$ embeds canonically into $Q_p(S)$ and this embedding yields an isomorphism in K-theory, at least for $|P| \leq 2$, see [LN16, BOS16].
The relevance of $A(S)$ for the K-theory of $Q(S)$ and $Q_p(S)$ is apparent. This shall serve as our motivation for investigating the potentials of two different options for defining $A(S)$ in a broader context:

Let $S = G \rtimes \theta P$ for an algebraic dynamical system $(G, P, \theta)$ and denote the diagonal subalgebra in $Q(S)$ by $D$. Using the approach from [Sta15], one can show that $Q(S) \cong D \rtimes S \cong (D \rtimes G) \rtimes P$. If, moreover, $P$ is directed with respect to $p \geq q :\iff p \in qP$ and $G/\theta_p(G)$ is finite for all $p \in P$, then we get a generalized Bunce-Deddens algebra $D \rtimes G = \lim_{p \in P} M_p(C^*(G))$, and its canonical AF-subalgebra $\lim_{p \in P} M_p(\mathbb{C})$ is invariant under the $P$-action. In this case, define $A(S)$ to be the resulting semigroup crossed, as this is a natural analogue for the torsion subalgebra $A(S)$ from Remark 5.2.

If $P$ is free abelian, a minor modification of [BOS16] Proposition 4.6 then shows that $K_* (A(S))$ is again a torsion group, which is finite if $P$ is finitely generated.

Next, suppose that $Q(S)$ to be a unital UCT Kirchberg algebra (or just simple). Then the corresponding property will pass to $A(S)$, basically by using the arguments from [BOS16] Proposition 5.1 and Corollary 5.2. In particular, $A(S)$ can be identified with a subalgebra of $Q(S)$ in this case.

An obvious alternative is to consider $Q(S^1_{c,d})$ for an arbitrary right LCM semigroup $S$. This is expected to be of greater interest if $S^* = \{1\}$ and $S = S^1_{c,d}$. Recall that this allows us to write $S = S^1_{c,d} \rtimes S_c$, see Remark 2.13. As an example, let us look at the outcome for Baumslag-Solitar monoids $S = BS(c, d)^+$ with $c, d \geq 1$: Thanks to the efforts of Spielberg, the $K$-theory for $Q(S)$ is already known:

**Theorem 5.3** ([Spi12] Theorem 4.8). For $c, d \geq 1$ and $S = BS(c, d)^+$, the K-theory of $Q(S)$ is given by

$$K_0(Q(S)) \cong \mathbb{Z}/(d - 1)\mathbb{Z} \oplus \delta_1, \mathbb{Z},$$

$$K_1(Q(S)) \cong \delta_1, \mathbb{Z} \oplus \mathbb{Z}/(c - 1)\mathbb{Z}.$$ 

Note that $c = d = 1$ gives $S = \mathbb{N}^2$. Let us focus on the case where $c, d \geq 2$. Recall from Example 3.2 that we have $S^1_{c,d} \cong F_d^+$. Thus $Q(S^1_{c,d}) \cong O_d$ is simple and embeds into $Q(S)$. More importantly, the formulas from Theorem 5.3 suggest that this map is injective in K-theory, which is true by [Spi12], where it is shown that $[1]_0$ is the generator of $\mathbb{Z}/(d - 1)\mathbb{Z} \subset K_0(Q(S))$. But this does not explain the parts of $K_*(Q(S))$ related to $c$. In that respect, we remark that the opposite semigroup $S^{opp}$ of $S$ coincides with $BS(d, c)^+$, and hence $(S^{opp})^1_{c,d} \cong F_d^+$. Another way of looking at $S^{opp}$ is to consider left ideals in $S$, and use the normal form $s = b^j w_n \cdots w_1$ with $w_i \in \{ab^j \mid 0 \leq j \leq c - 1\}$, which has analogous properties to the normal form used in Example 3.2. $S^{opp}$ plays the role of $S$ when considering right representations in place of left representations (bearing in mind that $C^*(S)$ is a universal model for the left regular representation of $S$ on $\ell^2(S)$).

As $Q((S^{opp})^1_{c,d}) \cong O_c$, the natural question arising from Theorem 5.3 is:

**Question 5.4.** Suppose $c, d > 1$. Is there a $\ast$-homomorphism from the suspension of $O_c$ to $Q(BS(c, d)^+)$ that is injective in K-theory?

Let us close by pointing out that $Q(S)$ is a Kirchberg algebra if and only if $d$ does not divide $c$, see [Spi12] Corollary 4.10. This minimality condition was identified as the prerequisite for uniqueness of the KMS$_1$-state on $C^*(S)$ in Theorem 1.1 (ii).
References

[ABLS] Zahra Afsar, Nathan Brownlowe, Nadia S. Larsen, and Nicolai Stammeier, *Equilibrium states on right LCM semigroup C*-algebras*, preprint, arxiv:1611.01052

[BOS16] Selçuk Barlak, Tron Omland, and Nicolai Stammeier, *On the K-theory of C*-algebras arising from integral dynamics*, Ergodic Theory Dynam. Systems, posted on 2016, DOI 10.1017/etds.2016.63 (published online).

[Bri05] Matthew G. Brin, *On the Zappa-Szép product*, Comm. Algebra 33 (2005), no. 2, 393–424, DOI 10.1081/AGB-200047404

[BaHLR12] Nathan Brownlowe, Astrid an Huef, Marcelo Laca, and Iain Raeburn, *Boundary quotients of the Toeplitz algebra of the affine semigroup over the natural numbers*, Ergodic Theory Dynam. Systems 32 (2012), no. 1, 35–62, DOI 10.1017/S0143385710000830.

[BLS17] Nathan Brownlowe, Nadia S. Larsen, and Nicolai Stammeier, *On C*-algebras associated to right LCM semigroups*, Trans. Amer. Math. Soc. 369 (2017), no. 1, 31–68, DOI 10.1090/tran/6638.

[BLS] _, *C*-algebras of algebraic dynamical systems and right LCM semigroups*, preprint, arxiv.org:1503.01599.

[BRRW14] Nathan Brownlowe, Jacqui Ramagge, David Robertson, and Michael F. Whittaker, *Zappa-Szép products of semigroups and their C*-algebras*, J. Funct. Anal. 266 (2014), no. 6, 3937–3967, DOI 10.1016/j.jfa.2013.12.025.

[BS16] Nathan Brownlowe and Nicolai Stammeier, *The boundary quotient for algebraic dynamical systems*, J. Math. Anal. Appl. 438 (2016), no. 2, 772–789, DOI 10.1016/j.jmaa.2016.02.015.

[CaHR] Lisa Orloff Clark, Astrid an Huef, and Iain Raeburn, *Phase transition on the Toeplitz algebra of Baumslag-Solitar semigroups*, to appear in Indiana Univ. Math. J. preprint arxiv:1503.04873.

[CL07] John Crisp and Marcelo Laca, *Boundary quotients and ideals of Toeplitz C*-algebras of Artin groups*, J. Funct. Anal. 242 (2007), no. 1, 127–156, DOI 10.1016/j.jfa.2006.08.001.

[Cun08] Joachim Cuntz, *C*-algebras associated with the ax + b-semigroup over N, K-theory and noncommutative geometry, 2008, pp. 201–215.

[CV13] Joachim Cuntz and Anatoly Vershik, *C*-algebras associated with endomorphisms and polymorphisms of compact abelian groups, Comm. Math. Phys. 321 (2013), no. 1, 157–179, DOI 10.1007/s00220-012-1647-0

[LN11] Marcelo Laca and Sergey Neshveyev, *Type III1 equilibrium states of the Toeplitz algebra of the affine semigroup over the natural numbers*, J. Funct. Anal. 261 (2011), no. 1, 169–187, DOI 10.1016/j.jfa.2011.03.009.

[LR10] Marcelo Laca and Iain Raeburn, *Phase transition on the Toeplitz algebra of the affine semigroup over the natural numbers*, Adv. Math. 225 (2010), no. 2, 643–688, DOI 10.1016/j.aim.2010.03.007.

[LRR11] Marcelo Laca, Iain Raeburn, and Jacqui Ramage, *Phase transition on Exel crossed products associated to dilation matrices*, J. Funct. Anal. 261 (2011), no. 12, 3633–3664, DOI 10.1016/j.jfa.2011.08.015.

[LRRW14] Marcelo Laca, Iain Raeburn, Jacqui Ramage, and Michael F. Whittaker, *Equilibrium states on the Cuntz-Pimsner algebras of self-similar actions*, J. Funct. Anal. 266 (2014), no. 11, 6619–6661, DOI 10.1016/j.jfa.2014.03.003.

[Law08] Mark V. Lawson, *A correspondence between a class of monoids and self-similar group actions*, I, Semigroup Forum 76 (2008), no. 3, 489–517, DOI 10.1007/s00233-008-9052-x

[Li12] Xin Li, *Semigroup C*-algebras and amenability of semigroups*, J. Funct. Anal. 262 (2012), no. 10, 4302–4340, DOI 10.1016/j.jfa.2012.02.020.

[LN16] Xin Li and Magnus Norling, *Independent resolutions for totally disconnected dynamical systems. II. C*-algebraic case*, J. Op. Th. 75 (2016), no. 1, 163–193, DOI 10.7900/jot.2014dec22.2061.

[Spi12] Jack Spielberg, *C*-algebras for categories of paths associated to the Baumslag-Solitar groups*, J. Lond. Math. Soc. (2) 86 (2012), no. 3, 728–754, DOI 10.1112/jlms/jds025
