Infinite Charge Algebra of Gravitational Instantons

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ABSTRACT

Using a formalism of minitwistors, we derive infinitely many conserved charges for the $sl(\infty)$-Toda equation which accounts for gravitational instantons with a rotational Killing symmetry. These charges are shown to form an infinite dimensional algebra through the Poisson bracket which is isomorphic to two dimensional area preserving diffeomorphism with central extentions.

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It has been known for some time that certain large N limits of two dimensional field theories yield higher dimensional field theories (see e.g.\cite{1}). In particular, a large N limit of the two dimensional $sl(N)$-Toda equation becomes a three dimensional equation for a scalar field $u(w,\bar{w},t)^{[1,2]}$,
\[
\partial\bar{\partial}u = -\partial_t^2 e^u \ ; \ \partial = \frac{\partial}{\partial w}, \bar{\partial} = \frac{\partial}{\partial \bar{w}}
\]
which is also the self-dual Einstein equation with a rotational Killing symmetry and the metric:\cite{3}
\[
ds^2 = \frac{1}{u,t}(4e^udw d\bar{w} + dt^2) + u,t(d\theta + iu, w dw + iu, \bar{w} d\bar{w})^2.
\]
Eq.\eqref{eq:1} as a large N limit of the $sl(N)$-Toda equation is expected to possess infinite symmetries. Indeed, the infinitesimal action of such symmetries has been obtained previously and these were shown to form an algebra of area preserving diffeomorphisms.\cite{4} However, generators of such symmetries, i.e. conserved charges of Eq.\eqref{eq:1}, are not known explicitly except for few cases such as spin 2 charge.\cite{5} Moreover the associated charge algebra which is essential in understanding the quantum aspect of Eq.\eqref{eq:1} is presently unknown. Even though these charges are expected to arise from large N limits of conserved charges of the $sl(N)$-Toda equation, unlike the Toda equation case, the large N limit procedure for conserved charges is more involved. As explained in this letter, they are correctly described by the language of twistor theory.

In this letter, we first derive conserved charges explicitly through the first order differential equations which determine the infinitesimal symmetries of Eq.\eqref{eq:1}. Then, by defining a minitwistor space for Eq.\eqref{eq:1}, we provide a general closed form of spin-s conserved charges. These charges are also shown to form a symmetry algebra via a Poisson bracket which is isomorphic to the centrally extended area preserving diffeomorphisms.

We first recall that the infinitesimal spin-s symmetry of Eq.\eqref{eq:1},
\[
\partial\bar{\partial}\delta^{(s)}u = -\partial_t^2 (e^u \delta^{(s)}u) ,
\]
is given by the following recursive equations:\cite{4}
\[
\delta^{(s)}u = \partial_t A_0^{(s)} \\
\partial_t A_r^{(s)} = (\bar{\partial} + r\partial)A_r^{(s)} ; \ r = 1,2,\cdots s-1
\]
with $A_{s-1}^{(s)} = f(\bar{w})$ and $f$ an arbitrary anti-holomorphic function. Defining $\partial_t q \equiv u ; \ p \equiv \bar{\partial}q$ and
\[
K_r \equiv \frac{1}{\partial_t}(\bar{\partial} + rp_t) ; \ M_r \equiv (\bar{\partial} - rp_t)\frac{1}{\partial_t} ,
\]
\[
\text{ds}^2 = \frac{1}{u,t}(4e^udw d\bar{w} + dt^2) + u,t(d\theta + iu, w dw + iu, \bar{w} d\bar{w})^2 .
\]
the spin-s symmetry is formally given by

$$\delta^{(s)} q = K_1 \cdots K_{s-1} f(\bar{w}) .$$  

(6)

If we define the fundamental Poisson bracket by

$$\{ F(p(w, \bar{w}, t)) , G(p(w, \bar{w}', t')) \} = \int d\bar{v} d\tau \frac{\delta F}{\delta p(w, \bar{v}, \tau)} \frac{\partial}{\partial \bar{v}} \delta p(w, \bar{v}, \tau),$$  

(7)

the conserved spin-s charge $Q^{(s)}$ which acts as a generator of the infinitesimal symmetry of Eq.(4) may be introduced as follows:

$$\delta^{(s)} p = \bar{\partial} [K_1 \cdots K_s f(\bar{w})] = \{p , \int d\bar{v} f(\bar{v}) Q^{(s)}(\bar{v}) \}$$  

$$= \bar{\partial} [\int d\bar{v} f(\bar{v}) \frac{\delta Q^{(s)}(\bar{v})}{\delta p(w, \bar{w}, t)}],$$  

(8)

or

$$\delta^{(s)} q = \int d\bar{v} f(\bar{v}) \frac{\delta Q^{(s)}(\bar{v})}{\delta p(w, \bar{w}, t)} = K_1 \cdots K_{s-1} f(\bar{w})$$  

$$= \int d\bar{v} d\tau \frac{\delta p(w, \bar{v}, \tau)}{\delta p(w, \bar{w}, t)} K_1 \cdots K_{s-1} f(\bar{v})$$  

$$= \int d\bar{v} d\tau f(\bar{v}) M_{s-1}(w, \bar{v}, \tau) \cdots M_1 \frac{\delta p(w, \bar{v}, \tau)}{\delta p(w, \bar{w}, t)} .$$  

(9)

This implicit expression for the spin-s charge can be solved explicitly for lower spin cases. For example,

$$Q^{(2)} = \int d\tau \left[ \frac{1}{2} p^2 - \tau \bar{\partial} p \right]$$  

$$Q^{(3)} = \int d\tau \left[ \frac{1}{3} \bar{p}^3 + p \bar{\partial} \bar{\partial}^{-1} p - \tau \bar{\partial} pp + \frac{\tau^2}{2} \bar{\partial}^2 p \right]$$  

$$Q^{(4)} = \int d\tau \left[ \frac{1}{4} p^4 - \tau p^2 \bar{\partial} p + p^2 \bar{\partial} \bar{\partial}^{-1} p - p \bar{\partial} p \bar{\partial}^{-1} p \right.$$

$$+ \left. \frac{\tau^2}{2} (p \bar{\partial}^2 p + (\bar{\partial} p)^2) + \tau \bar{\partial}^2 p \bar{\partial}^{-1} p + (\bar{\partial} \bar{\partial}^{-1} p)^2 - \frac{\tau^3}{6} \bar{\partial}^3 p \right]$$  

(10)

Though one could essentially solve Eq.(9) for an arbitrary spin-s charge, in practice this is quite difficult. Moreover, the computation of the algebra among different charges through
the Poisson bracket is much more involved. Instead of trying to solve Eq.(9) for an arbitrary spin, in the following, we give yet another description of $Q(s)$; i.e. ‘the minitwistor method’ which provides a natural explanation for the recursive relations in Eq.(4) as well as other properties of Eq.(1). Such a description essentially arises from the identification of Eq.(1) as the Einstein equation with a rotational Killing symmetry.

In order to do so, we recall that most general self-dual Einstein equation can be given by\[6\]
\[
\Omega_{,yy\bar{z}} - \Omega_{,y\bar{z}y} = 1 
\]
which is the same as the integrability of the linear equations:
\[
\begin{align}
[\lambda \partial_{\bar{y}} + \Omega_{,y\bar{z}} \partial_y - \Omega_{,y\bar{y}} \partial_z] \Psi &= 0 ; \\
[\lambda \partial_z + \Omega_{,\bar{z}z} \partial_y - \Omega_{,\bar{y}z} \partial_z] \Psi &= 0
\end{align}
\]
where subscripts with a comma denote partial differentiation and $\lambda \in \mathbb{P}^1$ is a constant. We impose a rotational Killing symmetry on the metric by assuming $\Omega = \Omega(z, \bar{z}, r = y\bar{y})$ and change variables,
\[
e^u \equiv r \equiv y\bar{y}, \ w \equiv \bar{z}, \ \bar{w} \equiv z, \ \Omega = \Omega(r, z, \bar{z}), \ t \equiv r \Omega, \ s \equiv y ,
\]
as well as coordinates, $(y, z, \bar{y}, \bar{z}) \rightarrow (w, \bar{w}, t, s)$, so that
\[
\begin{align}
\partial_y &= \frac{1}{su,t} \partial_t + \partial_s ; \quad \partial_{\bar{y}} = \frac{s}{r,t} \partial_t \\
\partial_z &= \partial_{\bar{w}} - \frac{r,\bar{w}}{r,t} \partial_t ; \quad \partial_{\bar{y}} = \partial_w - \frac{r,w}{r,t} \partial_t \\
\partial_r &= \frac{1}{r,t} \partial_t ; \quad t,r = \frac{1}{r,t}
\end{align}
\]
Eq.(11) reduces to Eq.(1) after some calculation and Eq.(12) becomes
\[
\begin{align}
(\partial_{\bar{w}} - \frac{1}{\eta} \partial_t - \partial_{\bar{w}} u \eta \partial_\eta) \Psi &= 0 \\
(\partial_w + e^w \eta \partial_t - e^w u \eta^2 \partial_\eta) \Psi &= 0 ; \quad \eta = \frac{1}{\lambda s} .
\end{align}
\]
Note that $\Psi = \Psi(w, \bar{w}, t, \eta = 1/\lambda s)$ depends on $\eta$ while $u$ does not. In fact, Eq.(15) is a defining equation of the twistor space for the $sl(\infty)$-Toda equation which is known as the minitwistor space. To be more specific, let $F$ and $S$ be solutions of Eq.(15),i.e.
\[
(\partial_{\bar{w}} - \frac{1}{\eta} \partial_t - \partial_{\bar{w}} u \eta \partial_\eta)F = 0 = (\partial_w + e^w \eta \partial_t - e^w u \eta^2 \partial_\eta)F
\]
and the similar equations for $S$ with the asymptotic boundary conditions

$$F(\eta \to 0) \to \frac{1}{\eta} ; \quad S(\eta \to 0) \to \bar{w}.$$  \hspace{1cm} (17)

Then, $(F, S)$ provide local coordinates of the minitwistor space corresponding to rotationally symmetric gravitational instantons. In the case $u = 0$, which describes a three dimensional flat space $R^{2+1}$ degeneratedly embedded into $R^{2+2}$, $F$ and $S$ become

$$F = \frac{1}{\eta} \quad ; \quad S = \bar{w} + t\eta - w\eta^2.$$ \hspace{1cm} (18)

These are precisely the local coordinates of the minitwistor space $TP^1$, a tangent bundle over $P^1$, which corresponds to $R^{2+1}$.[7] Therefore, $(F, S)$ generalize the minitwistor space to the curved metric case. This also agrees with the work by Jones and Tod[8] where the minitwistor space for Einstein-Weyl spaces was identified with the factor space of a twistor space by a holomorphic vector field.

Having identified Eq.(15) as a defining equation of the minitwistor space for the $sl(\infty)$-Toda equation, we may write the most general solutions for $A_r^{(s)}$ in Eq.(4) in terms of contour integrals over the minitwistor space;

$$A_r^{(s)} = \frac{1}{2\pi i} \oint \frac{d\eta \eta^{r-1} F^{s-1} f(S)}{\eta} ; \quad r = 0, 1, \cdots, s - 1$$ \hspace{1cm} (19)

where the contour $\Gamma$ encloses $\eta = 0$ and $f(S)$ is an arbitrary function. A straightforward calculation shows that $A_{r-1}^{(s)} = f(\bar{w})$ and $A_r^{(s)}$ indeed satisfies Eq.(4). In particular,

$$\delta^{(s)} u = \partial_t A_0^{(s)} = \frac{1}{2\pi i} \frac{d}{\eta} \oint \frac{d\eta}{\eta} F^{s-1} f(S).$$ \hspace{1cm} (20)

This, when combined with eq.(8), gives

$$\int d\bar{v} f(\bar{v}) \frac{\delta Q^{(s)}}{\delta p(w, \bar{w}, t)} = \frac{1}{2\pi i} \oint \frac{d\eta}{\eta} F^{s-1} f(S)$$ \hspace{1cm} (21)

which is a contour integral expression of Eq.(9). In fact, Eq.(21) could be understood as a large N limit of conserved charges of $sl(N)$-Toda equation where the contour integral replaces the trace. On the other hand, Eq.(21) manifests a sheaf cohomological nature of conserved charges in such a way that the integrand of the contour integral represents an element of the sheaf cohomology group $H^1(T, \mathcal{O}(-2))$ of the minitwistor space $T$.[1]
Finally, we consider the charge algebra through the Poisson bracket eq.(7). It has been shown in [4] that the infinitesimal symmetries given in Eq.(4) close under commutation so as to form an algebra s.t.

\[ [\delta_f^{(k)}, \delta_g^{(l)}]q = \delta_f^{(k)}\delta_g^{(l)}q - \delta_g^{(l)}\delta_f^{(k)}q = \delta_h^{(k+l-2)}q \]

where \( h = (k-1)f'g - (l-1)f'g \). This can be proved directly by using the recursive relation Eq.(4) or by using the minitwistor formalism and the sheaf cohomology.[1] On the other hand, since

\[ \delta_f^{(k)}\delta_g^{(l)}q = \delta_f^{(k)}\{ \int gQ^{(l)} , \int fQ^{(k)} \} , \]

Eq.(22) gives rise to

\[ [\delta_f^{(k)}, \delta_g^{(l)}]q = \delta_f^{(k)}\{ \int gQ^{(l)} , \int fQ^{(k)} \} = \delta_\hat{h}^{(k+l-2)}q . \]

This determines the charge algebra up to a central term \( c \),

\[ \{ \int gQ^{(l)} , \int fQ^{(k)} \} = \int hQ^{(l+k-2)} + c \]

which is precisely the algebra of centrally extended area preserving diffeomorphisms. In order to determine the central charge in Eq.(25), we use the fact that the cocycle terms of area preserving diffeomorphisms over genus \( g \) surface have \( 2g \) independent cocycle terms.[9] For a cylinder, \( g = 1/2 \) and therefore we have only one central charge up to a multiplicative constant. We find that in equation (25),

\[ c = \int d\tau \frac{\tau^{k+l-2}}{(k-1)!(l-1)!} \int d\bar{v} \frac{\partial^{l-1}g}{\partial\bar{v}^{l-1}} \frac{\partial^k f}{\partial\bar{v}^k} \]

which agrees with the result from the lower spin calculation.

Let us conclude with a remark concerning the possibility[10] to transform Eq.(1) into a ‘Monge-Ampère form’ (and containing the exponential of one of the independent, rather than the dependent, variable),

\[ (V_{22}V_{33} - V_{23}^2) + (V_{11}V_{33} - V_{13}^2) + (V_{11}V_{22} - V_{12}^2)e^{p_1} = 0 \]

with \( V_{ij} := \frac{\partial^2 V}{\partial p_i \partial p_j} \), and \( V = V(p_1, p_2, p_3) \) being related to \( u(w = \frac{x_1+ix_2}{2}, \bar{w} = \frac{x_1-ix_2}{2}, t = x_3) = \frac{\partial}{\partial x_3}q(x_1, x_2, x_3) \) by a Legendre transformation, i.e. via inverting \( p_i(x) = \frac{\partial}{\partial x_i}q(x_1, x_2, x_3) \) to
obtain \( x_i(p) = \frac{\partial}{\partial p_i} V(p_1, p_2, p_3) \).

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