The finite-gap method
and the periodic NLS Cauchy problem
of anomalous waves for a finite number of unstable modes

P.G. Grinevich and P.M. Santini

Abstract. The focusing non-linear Schrödinger (NLS) equation is the simplest universal model describing the modulation instability (MI) of quasi-monochromatic waves in weakly non-linear media, and MI is considered to be the main physical mechanism for the appearance of anomalous (rogue) waves (AWs) in nature. In this paper the finite-gap method is used to study the NLS Cauchy problem for generic periodic initial perturbations of the unstable background solution of the NLS equation (here called the Cauchy problem of AWs) in the case of a finite number $N$ of unstable modes. It is shown how the finite-gap method adapts to this specific Cauchy problem through three basic simplifications enabling one to construct the solution, to leading and relevant order, in terms of elementary functions of the initial data. More precisely, it is shown that, to leading order, i) the initial data generate a partition of the time axis into a sequence of finite intervals, ii) in each interval $I$ of the partition only a subset of $\mathcal{N}(I) \leq N$ unstable modes are ‘visible’, and iii) for $t \in I$ the NLS solution is approximated by the $\mathcal{N}(I)$-soliton solution of Akhmediev type describing for these ‘visible’ unstable modes a non-linear interaction with parameters also expressed in terms of the initial data through elementary functions. This result explains the relevance of the $m$-soliton solutions of Akhmediev type with $m \leq N$ in the generic periodic Cauchy problem of AWs in the case of a finite number $N$ of unstable modes.

Bibliography: 118 titles.

Keywords: focusing non-linear Schrödinger equation, periodic Cauchy problem for anomalous waves, asymptotics in terms of elementary functions, finite-gap approximation, Riemann surfaces close to degenerate ones.

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1. Introduction

The two non-linear Schrödinger (NLS) equations

\[ iu_t + u_{xx} + 2\nu |u|^2 u = 0, \quad u = u(x, t) \in \mathbb{C}, \quad \nu = \pm 1, \] (1)

are the simplest universal models of the propagation of a quasi-monochromatic wave in a weakly non-linear medium. In particular, they are relevant in water waves ([111], [10]), in non-linear optics ([98], [28], [90]), in Langmuir waves in a plasma [101], and in the theory of Bose–Einstein condensates ([27], [92]). For instance, in a non-linear optics interpretation \( u(x, t) \) is the complex amplitude of the electric field, and the self-interacting term \( 2\nu |u(x, t)|^2 \) accounts for the non-linear response of the medium, which in a Kerr-like regime is proportional to the light intensity. In a quantum mechanical interpretation \( u(x, t) \) is the wave function and \( V(x, t) = -2\nu |u(x, t)|^2 \) is the self-induced potential proportional to the probability density. If \( \nu = 1 \), then the self-interacting term is self-focusing (thus the name ‘self-focusing NLS’) and corresponds to a potential well whose depth increases with the density. If \( \nu = -1 \), then the self-interacting term is defocusing (thus the name ‘defocusing NLS’) and corresponds to a potential barrier whose height increases with the density. It is clear from these considerations that the focusing and the defocusing regimes correspond to very different evolutions of the same initial data.

In particular, it is well known that the elementary solution

\[ u(x, t) = a \exp(2i\nu |a|^2 t), \quad a = \text{const} \in \mathbb{C}, \] (2)

of (1), which describes Stokes waves [100] in a water wave context, a state of constant light intensity in non-linear optics, and a state of constant boson density in a Bose–Einstein condensate, is stable in the defocusing case under perturbations of waves with arbitrary wave length, while it is unstable in the focusing case under perturbations of waves of sufficiently large wave length ([24], [21], [111], [116], [103], [94]). This modulation instability (MI), which is present only in the focusing
case, is considered to be the main cause for the formation of anomalous (rogue, extreme, freak) waves (AWs) in nature ([57], [41], [88], [68], [67], [87]). In this paper the term AW is deliberately used in a vague sense, and by it we just mean order-1 (or higher) coherent structures over the unstable background and generated by MI. This paper and our previous papers [52]–[55] are devoted to a complete analytic understanding of the deterministic aspects of the dynamics of these coherent structures when there are finitely many unstable modes.

We point out that in oceanography anomalous (or rogue) waves are defined as waves with a sufficiently high amplitude compared to average waves, and in this connection it should be noted that NLS soliton solutions over an unstable background can reach unusually large amplitudes if their parameters are correlated in a special way (see, for instance, [31]). An amplitude criterion for selecting such anomalously large amplitude waves among finite-gap solutions of the NLS equation over the background was recently used in [22]. Our long-term research plan, actually one of the main motivations of the studies made in this paper and in [52]–[55], is to understand the deterministic aspects of the theory when there are finitely many unstable modes, as the starting point for studying the statistical aspects of the theory when a large number of unstable modes are excited, with the particular goal of describing analytically the probability of the generation in space-time of waves with anomalously large amplitude, due to MI.

The integrable nature [117] of the focusing NLS equation enables one to construct a large zoo of exact solutions corresponding to perturbations of the background by using, for example, the following methods:

- the degeneration of finite-gap solutions ([63], [20], [73], [74]) when the spectral curve becomes rational;
- the use of classical Darboux transformations [81];
- dressing techniques ([118], [114], [115]);
- the Hirota method ([58], [59]).

Among these basic solutions, we mention the Peregrine soliton [89] rationally localized with respect to \(x\) and \(t\) over the background (2), the so-called Kuznetsov–Kawata–Inoue [65]–Ma [79] soliton exponentially localized in space over the background and periodic in time, and the solution found by Akhmediev, Eleonskii, and Kulagin in [14], which is periodic in \(x\) and exponentially localized in time over the background (4) (see below) and is known in the literature as the Akhmediev breather. Elliptic generalizations of these solutions were constructed in [15] and [16]. A more general one-soliton solution over the background (2) can be found, for instance, in [63] and [112] and corresponds to a spectral parameter in general position. These solutions have also been generalized to the case of multisoliton solutions describing their non-linear interaction (see, for instance, [63], [38], [66], and [113]). Finite-gap representations of AWs were constructed in [22], and elliptic solutions corresponding to special genus-2 curves with symmetries were studied in [97] using the method in [18]. We remark that the Peregrine solitons are homoclinic and describe AWs that apparently appear from nowhere and disappear in the future, while the multisoliton solutions of Akhmediev type are almost homoclinic, returning to the original background up to a multiplicative phase factor. Generalizations of these solutions to the case of integrable multicomponent NLS equations have also been found ([19], [35], [36]).
A special NLS Cauchy problem in which the initial condition is a small perturbation of the exact background (2) will be called the Cauchy problem of AWs. If such a perturbation is localized, then slowly modulated periodic oscillations described by elliptic solutions of (1) with $\nu = 1$ play an important role in the long-term regime ([25], [26]). The natural appearance of Kuznetsov–Kawata–Inoue–Ma solitons and superregular solitons (constructed by Zakharov and Gelash [112]; see also [113] and [50]) in this problem was demonstrated in [48]. In the case when the initial perturbation is $x$-periodic, numerical experiments and qualitative considerations indicate that the solutions of the NLS equation exhibit both time recurrence ([109], [77], [110], [13], [107], [76], [70], [84], [91]) and numerically induced chaos ([2], [9], [1]), in which almost homoclinic solutions of Akhmediev type seem to play an important role ([43], [46], [29]–[31]). There are reports of experiments in which the Peregrine and the Akhmediev solitons were observed ([32], [69], [110], [106], [70], [84], [91]). Their appearance for certain classes of localized initial data for the NLS problem in the small-dispersion regime was shown in [23] and [42] (see also [51] and [104] for investigations of their appearance in ocean waves and non-linear fibre optics).

The basic tool for investigating the periodic Cauchy problem for soliton equations is the finite-gap method. Its development started in the papers [86], [39], [62], [78], and [82], was first applied to the NLS equation in [61], and its generalization to 2+1 equations was first constructed in [72]. In [52] and the present paper we apply it to solve the periodic Cauchy problem of AWs for the focusing NLS equation

$$iu_t + u_{xx} + 2|u|^2u = 0, \quad u = u(x, t) \in \mathbb{C},$$

that is, we study the focusing NLS Cauchy problem on a segment $[0, L]$, with periodic boundary conditions for a small generic, smooth, periodic perturbation of the background solution

$$u_0(x, t) = e^{2it}$$

at the initial time:

$$u(x, 0) = 1 + \varepsilon v(x), \quad 0 < \varepsilon \ll 1, \quad v(x + L) = v(x),$$

where

$$v(x) = \sum_{j=1}^{\infty} (c_je^{ik_jx} + c_{-j}e^{-ik_jx}), \quad k_j = \frac{2\pi}{L}j,$$

and the average of the initial perturbation is assumed to be 0. We also assume the period $L$ to be generic, that is, $L/\pi$ is not an integer.

Remark 1. Due to the scaling symmetry of the NLS equation, one can without loss of generality use the simplified form (4) of the background (2) obtained by choosing $a = 1$. Again due to the scaling symmetry of the NLS equation, we have also assumed without loss of generality that the perturbation $v(x)$ in (6) has zero average,

$$\int_0^L v(x) \, dx = 0,$$
implying that the Fourier coefficient
\[
c_0 = \frac{1}{L} \int_0^L v(x) \, dx
\]
is zero.

It is well known that a monochromatic perturbation of (4) with wave number \(k\) is unstable if \(|k| < 2\), and therefore if \(N \in \mathbb{N}\) is given by
\[
N = \left\lfloor \frac{L}{\pi} \right\rfloor
\]
(9)
(where \([x], x \in \mathbb{R}\), denotes the largest integer not greater than \(x\)), then the first \(N\) modes \(\{\pm k_j\}, 1 \leq j \leq N\), are linearly unstable and give rise to exponentially growing and exponentially decaying waves of amplitudes \(O(\varepsilon e^{\pm \sigma_j t})\), where the growth rates \(\sigma_j\) are given by
\[
\sigma_j = k_j \sqrt{4 - k_j^2}, \quad 1 \leq j \leq N,
\]
while the remaining modes are linearly stable and give rise to small oscillations of amplitude \(O(\varepsilon e^{\pm i \omega_j t})\), where
\[
\omega_j = k_j \sqrt{k_j^2 - 4}, \quad j > N.
\]
(10)
For the unstable modes it is also convenient to introduce the angles \(\phi_j\) parametrizing them, defined by
\[
\phi_j = \arccos \frac{k_j}{2} = \arccos \left(\frac{\pi}{L} j\right), \quad 0 < \phi_j < \frac{\pi}{2}, \quad 1 \leq j \leq N,
\]
so that
\[
k_j = 2 \cos \phi_j \quad \text{and} \quad \sigma_j = 2 \sin(2\phi_j), \quad 1 \leq j \leq N.
\]
(12)
These well-known facts are summarized in the formula ([52], [53])
\[
u(x, t) = e^{2it} \left[ 1 + \sum_{j=1}^{N} \left( \frac{\varepsilon |\alpha_j|}{\sin(2\phi_j)} e^{\sigma_j t + i\phi_j} \cos[k_j(x - x_j)] + \frac{\varepsilon |\beta_j|}{\sin(2\phi_j)} e^{-\sigma_j t - i\phi_j} \cos[k_j(x - \tilde{x}_j)] + O(\varepsilon)\right) + O(\varepsilon^2), \right)
\]
valid for \(0 \leq t \leq O(1)\), where
\[
\alpha_j = e^{-i\phi_j} \tilde{c}_j - e^{i\phi_j} c_{-j}, \quad \beta_j = e^{i\phi_j} \tilde{c}_{-j} - e^{-i\phi_j} c_j,
\]
\[
x_j = \frac{\arg \alpha_j + \pi/2}{k_j}, \quad \tilde{x}_j = -\frac{\arg \beta_j + \pi/2}{k_j}, \quad j = 1, \ldots, N.
\]
(14)
This formula describes the first linear stage of MI, governed by the focusing NLS equation linearized about the solution (4):

$$\delta u_t + \delta u_{xx} + 2 \exp(4it)\overline{\delta u} + 4\delta u = 0.$$ 

Therefore, the initial data splits into exponentially growing and exponentially decaying waves (\(\alpha\)- and \(\beta\)-waves, respectively), each carrying half the information encoded in the unstable part of the initial data, plus small oscillations associated with the stable modes and remaining small during the evolution.

The \(j\)th unstable mode grows to order \(O(1)\) for times of order \(O(\sigma_j^{-1}|\log \varepsilon|)\), and therefore the most unstable modes, the ones appearing first, are the modes with larger growth rate \(\sigma_j\). It follows that for logarithmically large times one enters the (first) non-linear stage of MI, when the linearized NLS theory can no longer be used, and the full integrability machinery of the finite-gap method for the NLS equation must be used to describe the evolution.

In connection with the specific Cauchy problem of AWs (3)–(7), we found, remarkably, that the finite-gap method admits the following three basic simplifications enabling one to construct the solution, to leading and relevant order, in terms of elementary functions of the initial data (see [52] and this paper)

**Step 1. Finite-gap approximation.** A generic periodic \(O(\varepsilon)\) initial perturbation of the unstable background opens infinitely many \(O(\varepsilon)\) gaps in the spectral problem. They are organized in pairs, and each pair corresponds to the pair \(\pm k_j\) of excited modes of the linearized problem. A finite number \(N = \lfloor L/\pi \rfloor\) of these modes are unstable, and the remaining ones are stable. Since the stable modes, which give rise to oscillations of order \(O(\varepsilon)\), correspond to corrections of order \(O(\varepsilon)\) to the AWs, one can close the corresponding gaps, keeping open only the \(2N\) gaps corresponding to the unstable modes (no matter how small these gaps are, they will cause \(O(1)\) effects on the dynamics, due to instability). Therefore, using this recipe, we go from an infinite-gap theory to a \(2N\)-gap approximation of it. We point out that this finite-gap approximation, specific to the Cauchy problem of AWs, is non-standard. Indeed, in the usual finite-gap approximation, one closes gaps smaller than a certain constant, while in our case all gaps are small, and the criterion for closing a gap is the stability of the corresponding mode.

**Step 2. Explicit analytic approximation of the \(\theta\)-function parameters.** The above finite-gap approximation of the problem implies that the solutions are represented by ratios of \(\theta\)-functions of genus \(2N\). In general, the parameters in the \(\theta\)-function formulae are complicated transcendental expressions in terms of the Cauchy data. In our special setting good elementary formulae in terms of the initial data can be written, to leading and relevant order, for all the parameters used in the arguments of the \(\theta\)-functions.

**Step 3. Elementary approximation of the \(\theta\)-functions.** The Riemann \(\theta\)-functions are defined as infinite sums of exponentials over all integer points in \(\mathbb{R}^{2N}\). Due to the presence of the small parameter \(\varepsilon\), at each time it is sufficient to keep the summation only over a subset of the \(4^N\) vertices of the elementary hypercube of this multidimensional lattice containing the trajectory point. Thus, for a generic \(t \geq 0\), the infinite sum of exponentials reduces to a finite sum of \(4^{N(t)}\) exponentials,
with \( 0 \leq \mathcal{N}(t) \leq N \), whose arguments are given in terms of elementary functions of the Cauchy data. It turns out that this \( t \)-dependent representation of the solution in terms of elementary functions coincides, to leading order, with the \( \mathcal{N}(t) \)-soliton solution of Akhmediev type.

**Remark 2.** We remark that the first attempt to apply the finite-gap method to solve the NLS Cauchy problem on a segment for periodic perturbations of the background was made in \cite{105}; the fact that, in the \( \theta \)-function representation of the solution, different finite sets of lattice points are relevant in different time intervals was first observed there, but no connection was established between the initial data and the parameters of the \( \theta \)-function, and no description of the dynamics in terms of elementary functions was given.

Using the above three simplifying steps, in \cite{52} we studied the Cauchy problem of AWs (3)–(6) in the particular case where the initial perturbation excites just one of the unstable modes, say \( k_n, 1 \leq n \leq N \):

\[
 u(x, 0) = 1 + \varepsilon (c_n e^{i k_n x} + c_{-n} e^{-i k_n x}), \quad 1 \leq n \leq N. \tag{15}
\]

Here we must distinguish two cases:

1) the case where only the corresponding unstable gaps are opened by the initial condition (15), and

2) the case where more than one pair of unstable gaps is opened.

In case 1) the solution describes an exact deterministic alternate recurrence of linear and non-linear stages of MI, and the non-linear AW stages are described by the Akhmediev breather, whose parameters, different at each AW appearance, are always given in terms of the initial data through elementary functions \cite{52}. This result is summarized as follows.

**Theorem 1.** Consider the case in which the initial condition (15) opens only the pair of unstable gaps associated with the corresponding mode \( \pm k_n \) in the linearized problem, so that the associated finite-gap Riemann surface is a hyperelliptic curve of genus \( 2N = 2 \) with two handles of order \( O(\varepsilon) \). This happens, for instance, if \( N = n = 1 \ (\pi < L < 2\pi) \), the simplest case in which only the mode \( k_1 = 2\pi/L \) is unstable, or if \( L/2\pi < n < L/\pi \).

Define the parameters (for \( m \in \mathbb{N}_+ \))

\[
 X^{(m)}_n = X^{(1)}_n + (m - 1) \Delta X_n, \quad T^{(m)}_n = T^{(1)}_n + (m - 1) \Delta T_n,
\]

\[
 \Phi^{(m)}_n = 2\phi_n + (m - 1) \cdot 4\phi_n,
\]

\[
 X^{(1)}_n = x_n = \frac{\arg \alpha_n + \pi/2}{k_n} \quad \text{mod } L, \quad \Delta X_n = \frac{\arg(\alpha_n \beta_n)}{k_n} \quad \text{mod } L, \tag{16}
\]

\[
 T^{(1)}_n = \frac{1}{\sigma_n} \log \frac{\sigma^2_n}{2\varepsilon |\alpha_n|}, \quad \Delta T_n = \frac{2}{\sigma_n} \log \frac{\sigma^2_n}{2\varepsilon \sqrt{\varepsilon |\alpha_n \beta_n|}},
\]

where \( \alpha_n \) and \( \beta_n \) are defined in (14) in terms of the initial data.
Then for $0 \leq t \leq O(1)$ one is in the first linear stage of MI,

$$
u(x, t) = e^{2it} \left\{ 1 + \frac{\varepsilon}{\sin(2\phi_n)} \left[ |\alpha_n| \cos(k_n(x-x_n)) e^{\sigma_n t + i\phi_n} + |\beta_n| \cos(k_n(x-x_n)) e^{-\sigma_n t - i\phi_n} \right] \right\} + O(\varepsilon^2),$$

(17)

where $x_n$ and $\bar{x}_n$ are defined in (14), and for $|t - T_n^{(1)}| \leq O(1)$ one is in the first non-linear stage of MI, with the first appearance of an AW described by the formula

$$
u(x, t) = e^{2i\phi_n} F(x, t; \phi_n, X_n^{(1)}, T_n^{(1)}) + O(\varepsilon),$$

(18)

where the function $F$ is the Akhmediev breather

$$F(x, t; \theta, X, T) := e^{2it} \frac{\cosh[\sigma(\theta)(t-T) + 2i\theta] + \sin \theta \cos[k(\theta)(x-X)]}{\cosh[\sigma(\theta)(t-T)] - \sin \theta \cos[k(\theta)(x-X)]},$$

(19)

$$k(\theta) = 2 \cos \theta, \quad \sigma(\theta) = k(\theta) \sqrt{4 - k^2(\theta)} = 2 \sin(2\theta),$$

which is an exact solution of the focusing NLS equation for all real values of the parameters $\theta, X, \text{ and } T$. The subsequent evolution is completely fixed by the recurrence property of the solution

$$
u(x + \Delta X_n, t + \Delta T_n) = e^{2i\Delta T_n + 4i\phi_n} \nu(x, t) + O(\varepsilon), \quad x \in [0, L], \quad t \geq 0.$$  \hspace{1cm} (20)

Remark 3. We first observe that the first linear and non-linear stages of MI do match in the intermediate region $O(1) \ll t \ll T_n^{(1)} = O(\sigma_n^{-1} |\log \varepsilon|)$. We also remark that the periodicity property (20) implies that the solution describes an exact recurrence of AWs, and the $m$th AW of the sequence ($m \geq 1$) is described in the time interval $|t - T_n^{(m)}| \leq O(1)$ by the analytic deterministic formula

$$
u(x, t) = e^{i\phi_n^{(m)}} F(x, t; \phi_n, X_n^{(m)}, T_n^{(m)}) + O(\varepsilon), \quad m \geq 1.$$  \hspace{1cm} (21)

Therefore, we have the following simple picture.

The solution of the $x$-periodic Cauchy problem (3)–(7) describes, in the case where the initial condition (15) opens only the pair of gaps associated with $\pm k_n$, an exact recurrence of Akhmediev breathers whose parameters, which change at each appearance, are expressed in terms of the initial data via elementary functions. Furthermore, $T_n^{(1)}$ is the first appearance time of an AW (the time at which the AW achieves the maximum of its modulus), the $X_n^{(1)} + Lj/n, 1 \leq j \leq n - 1$, are the positions of such a maximum, $1 + 2 \sin \phi_n$ is the value of the maximum, $\Delta T_n$ is the recurrence time (the time interval between two consecutive AW appearances), and $\Delta X_n$ is the $x$-shift of the position of the maxima in the recurrence. In addition, after each appearance the AW changes the background by the multiplicative phase factor $\exp(4i\phi_n)$ (see Fig. 1).

Remark 4 (On the physical relevance of the above exact recurrence of AWs). It is very important to remark that if the number of unstable modes is greater than 1 ($N > 1$), then this $t$-uniform dynamics is sensibly affected by perturbations due
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Figure 1. The plot of $|u(x,t)|$, $x \in [-L/2, L/2]$, describing the alternate appearance of linear and non-linear stages of modulation instability (the AW sequence) and obtained by numerical integration of the NLS equation via the refined [64] Split Step Fourier Method ([12], [108], [102]). Here $L = 6$ (the case of one unstable mode $k_1$), with $c_1 = 1/2$, $c_{-1} = (0.3 - 0.4i)/2$, $\varepsilon = 10^{-4}$. The numerical output is in perfect qualitative and quantitative agreement with the theoretical formulae (16)–(21).

to numerical and/or real experiments. Indeed, these perturbations open generically other small unstable gaps, provoking corrections of order $O(1)$ to the result. Therefore, these analytic and $t$-uniform approximations are physically relevant (and numerically verifiable) only when $N = n = 1$ ($\pi < L < 2\pi$).

In case 2), when more than one unstable gap is opened by the initial condition (15), a detailed investigation of all these gaps is necessary to get a $t$-uniform dynamics, and such a study is postponed to a subsequent paper. It was, however, possible to obtain in [52] an elementary description of the first non-linear stage of MI, described again by the Akhmediev breather solution, and to understand how perturbations due to numerical and/or real experiments can affect this result.

Remark 5 (AW recurrence as a basic effect of non-linear MI in the periodic setting, and the finite-gap method). The recurrence of AWs can be predicted from simple qualitative considerations. The unstable mode grows exponentially and becomes of order $O(1)$ at logarithmically large times, when one enters the non-linear stage of MI, and one expects the generation of a transient coherent structure of order $O(1)$ over the unstable background (the AW). Since the Akhmediev breather describes the one-mode non-linear instability, it is a natural candidate to describe such a stage, to leading order. Due again to MI, this AW is expected to be destroyed on a finite time interval, and one enters the third asymptotic stage, characterized, like the first one, by the background plus an $O(\varepsilon)$ perturbation. This second linear stage is expected, due again to MI, to give rise to the formation of a second AW (the second non-linear stage of MI), described again by the Akhmediev breather, but in general with other parameters. And this procedure iterates forever in the integrable NLS model, giving rise to the generation of an infinite sequence of AWs.
described by different Akhmediev breathers. Thus, the AW recurrence is a characteristic effect of non-linear MI in the periodic setting, and the finite-gap method is a proper tool to give an analytic description of it.

**Remark 6** (Finite-gap approach vs matched asymptotic expansions). The above AW recurrence is described by an alternating sequence of linear and non-linear asymptotic stages of MI obviously matching in their overlapping time regions, and therefore this finite-gap result naturally motivates the introduction of a matched asymptotic expansions (MAEs) approach, presented in the paper [53] and involving more elementary mathematical tools. The advantages of the finite-gap approach are due to the fact that the \( \theta \)-function representation of the solution is uniform in time, and the analytic description of the non-linear stages of MI (of the sequence of AWs) does not require any guesswork. Such guesswork is instead needed in the MAE approach, when one has to select the proper non-linear coherent structure of the NLS equation describing a certain non-linear stage of MI and matching with the preceding linear stage. In all the situations in which such guesswork is no problem, the MAE approach becomes competitive, since it involves more elementary mathematics, as in the case of only one unstable mode [53]. But as we shall see in the following sections, if we have more than one unstable mode and the initial data are generic, then the dynamics is not described by a sequence of asymptotic stages of MI, and the MAE approach is not applicable, while the finite-gap method is able to provide a uniform representation of the solution. In the case of a finite number \( N > 1 \) of unstable modes the MAE approach works only for very special initial data [53].

**Remark 7** (AW recurrence as the stable output of the dynamics). Theorem 1 explains the relevance of the Akhmediev breather in a generic periodic Cauchy problem of AWs in the case of one unstable mode, and it leads to the following natural question: what happens if the initial condition is the highly non-generic Akhmediev breather? The answer is the following. The Akhmediev breather, a quasi-homoclinic solution of the NLS equation, is unstable, corresponding in the finite-gap spectral picture to the highly non-generic case where all the spectral gaps are closed. Any small perturbation (due, for instance, to the numerical scheme approximating the NLS equation, or to small corrections to the NLS equation coming from physics) opens small gaps, implying that, after the first appearance of the AW as predicted by the initial data, a recurrence of AWs described by (16)–(21) will be the stable output of the dynamics [55].

**Remark 8** (AW recurrence for other NLS-type models). It is natural to ask whether the above AW recurrence is typical of the NLS equation, or is shared by other integrable NLS-type models. In the case of the Ablowitz–Ladik model [4]

\[
i(u_n)_t + u_{n+1} + u_{n-1} - 2u_n + \nu|u_n|^2(u_{n+1} + u_{n-1}) = 0, \quad n \in \mathbb{Z}, \quad \nu = \pm 1,
\]

which is an integrable discretization of the NLS equation, the stability properties depend crucially on the amplitude \( a > 0 \) of the background solution (4), unlike the NLS case. If \( \nu = -1 \) (the case reducing to the defocusing NLS equation in the continuous limit, for which the background is stable with respect to any monochromatic perturbation), then the background is stable or unstable if
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0 < a < 1 or a > 1, respectively, for any monochromatic perturbation. If \( \nu = 1 \) (the case reducing to the focusing NLS equation in the continuous limit, for which the background is unstable with respect to monochromatic perturbations of sufficiently small wave number), then a similar criterion holds, but now it involves also the amplitude \( a \), and one obtains an AW recurrence of Narita-type solutions [34] (the Narita solution is the discrete analogue of the Akhmediev breather [85], [17]). In the case of the focusing PT-symmetric NLS equation (PT means parity-time) ([5]–[8])

\[
iw_t(x, t) + w_{xx}(x, t) + 2w^2(x, t)w(-x, t) = 0, \quad w = w(x, t) \in \mathbb{C},
\]

there are also important novelties in comparison to the NLS case. The analogue of the Akhmediev breather, which is always regular for all values of its parameters, is now a pair of exact solutions of the PT-NLS equation [96]. The first is singular in a certain domain of its parameter space, while the second is always singular, for all values of its arbitrary parameters; these singularities describe blow-ups at points of the \((x, t)\) plane. It follows that, depending on the initial data, the AWs are either regular, with arbitrarily large amplitude, or they blow up at a finite time [95]. This is a new phenomenon for integrable NLS-type systems, and a qualitative reason for it should be the gain-loss properties of the complex self-induced potential of the PT-NLS equation, causing extra-focusing effects in comparison to the usual NLS case. Thus, depending on the initial data, the AW recurrence for the PT-NLS equation may involve either both types of solution or only one type of solution.

Remark 9 (NLS exact recurrence vs Fermi–Pasta–Ulam recurrence in nature). From [52], [53] and the formulae (16)–(21) it follows that if the initial condition (5) excites only the unstable mode \( k_1 \),

\[u(x, 0) = 1 + \varepsilon(c_1 \exp(ik_1 x) + c_{-1} \exp(-ik_1 x)),\]

then the following assertions hold:

i) the energy is initially concentrated on the zero mode (the unstable background) and on the first mode (the monochromatic perturbation),

\[|u_0(0)|^2 = 1, \quad |u_m(0)|^2 = \delta_{m, \pm 1} \varepsilon^2 |c_{\pm 1}|^2, \]

(22)

where the \( u_m(t), m \in \mathbb{Z} \), are the Fourier coefficients of the NLS solution \( u(x, t) \);

ii) at the time \( T_1^{(1)} \) of first appearance of the AW (see (16)) the energy is distributed over all Fourier modes according to the simple law

\[|u_0(T_1^{(1)})|^2 = (2 \cos \phi_1 - 1)^2, \quad 0 < \phi_1 < \frac{\pi}{2}, \]

\[|u_m(T_1^{(1)})|^2 = 4(\cos \phi_1)^2 \left(\tan \frac{\phi_1}{2}\right)^{2|m|}, \quad m \neq 0; \]

(23)

iii) however, at the recurrence time \( \Delta T_1 \) the energy is re-absorbed by the zero and first modes,

\[|u_0(\Delta T_1)|^2 = 1, \quad |u_m(\Delta T_1)|^2 = \delta_{m, \pm 1} \varepsilon^2 |c_{\pm 1}|^2, \]

(24)

starting an exact recurrence.
AW recurrence in the periodic setting has already been considered in the literature (see, for instance, [109], [77], [110], [13], [107], [76]), and recent experiments in water waves [70], in fibre optics [84], and in the non-linear optics of a photorefractive crystal [91] accurately reproduce the recurrence phenomenon. In particular, the experimental findings obtained in [91] have been compared with the formulae (16)–(21) describing the NLS recurrence, and the qualitative and quantitative agreement is very good.

1) Since the NLS equation describes various physical phenomena only to leading order, one expects that instead of the exact NLS recurrence of AWs illustrated in (16)–(21) one will observe a ‘Fermi–Pasta–Ulam’ type of recurrence [47], until thermalization destroys the structure. Indeed, in [91] up to three recurrences were observed, and there was very good qualitative and quantitative agreement in comparison with the above NLS recurrence formulae. It was also shown there that the recurrent behavior disappears when the photorefractive crystal works instead in a non-linear regime different from the integrable (Kerr) regime.

2) A common feature of the experiments described in [70], [84], and [91] is the choice in most of the cases to work in a special symmetry of the experimental apparatus leading to particularly significant subcases in which the recurrence shift \( \Delta X_1 \) in (16) is either 0 or \( L/2 \), implying the respective time-periods \( \Delta T_1 \) or \( 2\Delta T_1 \), where \( \Delta T_1 \) is the recurrence time (16). This particular symmetry corresponds to the distinguished subcase in which \( |c_1| \sim |c_{-1}| \) in (6), and it leads to interesting phase resonances for the physical times \( T_1^{(1)} \) and \( \Delta T_1 \) as functions of \( \zeta = \frac{\arg c_1 + \arg c_{-1}}{2} \) for \( \zeta = -\phi_1, \pi - \phi_1 \) [54]; these resonances were experimentally observed in [91]. The above experimental findings, which are in good quantitative agreement with the theoretical formulae (16)–(21), are an important confirmation that the NLS equation is a good model in the description of non-linear modulation instabilities in non-linear optics and water waves.

Remark 10. The Cauchy problem (3)–(6) was numerically investigated in [11] for large \( N \) and random initial data. Numerical experiments studying the statistical properties of a gas of solitons over the unstable background were performed in [49].

In this paper we apply the finite-gap method to the generic periodic Cauchy problem of AWs (3)–(7) in the case of a finite number \( N \) of unstable modes. Qualitative considerations similar to those in Remark 5 suggest again that each unstable mode will appear recurrently in the dynamics, depending on its degree of instability; but this recurrence is now affected by the non-linear interactions with all the other unstable modes (we shall see that to leading order this interaction is pairwise). Again, the proper tool to describe all that is the finite-gap method, and again the finite-gap method adapts to this specific Cauchy problem through the three basic simplifications outlined above, enabling one to construct the solution to leading and relevant order in terms of elementary functions of the initial data also in this more complicated case.

More precisely, we will show that, to leading order:

i) the initial data generate a partition of the time axis into a sequence of finite intervals;

ii) in each interval \( I \) of the partition only a subset of \( \mathcal{N}(I) \leq N \) unstable modes are ‘visible’;
iii) for $t \in I$ the NLS solution is approximated by the $\mathcal{N}(I)$-soliton solution of Akhmediev type describing for these ‘visible’ unstable modes the non-linear interaction, and the parameters of this $\mathcal{N}(I)$-soliton solution are expressed in terms of the initial data through elementary functions.

This result explains why the $m$-soliton solutions of Akhmediev type with $m \leq N$ arise naturally in the generic periodic Cauchy problem of AWs in the case of a finite number $N$ of unstable modes. Therefore, in the case of finitely many unstable modes the theory of NLS anomalous waves is completely deterministic, and its analytic description is given, to leading and relevant order, in terms of elementary functions of the Cauchy data.

Our paper is organized as follows. In §2 we describe the main results. In §3 we summarize the classical features of the periodic Cauchy problem for the focusing and the defocusing NLS equations. In §4 we apply this theory to the Cauchy problem of anomalous waves, constructing the main and auxiliary spectra to leading and relevant order in terms of elementary functions of the initial data. In §5, after closing the infinitely many gaps corresponding to the stable modes and thereby obtaining a non-standard finite-gap approximation (the first basic simplification of the theory), we study the corresponding finite-gap curve. For it we construct the leading-order expression for the Riemann matrix, the vector of Riemann constants, and all the other quantities appearing in the parameters of the $\theta$-function formulae of the inverse problem in terms of elementary functions (this is the second basic simplification of the theory). In §5.5 we write the $\theta$-function formulae for the leading-order solution, and then §6 is devoted to the third basic simplification of this theory, in which the infinite sum of exponentials appearing in the definition of the $\theta$-function is reduced to a sum over a finite subset of exponentials, different in different time intervals.

This paper is appearing in Sergei Petrovich Novikov’s volume, on the occasion of his 80th birthday. Considering that the invention of the finite-gap approach is one of his most famous results, we note that:

1) he has always been interested in applying his results to real physics;

2) he has always stressed that the finite-gap formulae must in addition be made effective in order to be applicable.

We hope that here we have made serious progress in both directions, and we would like to dedicate this paper to Sergei Petrovich Novikov.

2. Results

The aim of this paper is to provide a solution, to leading order and in terms of elementary functions, of the generic periodic Cauchy problem of AWs (3)–(7) described in the Introduction, rewritten here for completeness:

$$iu_t + u_{xx} + 2|u|^2u = 0,$$

$u = u(x,t) \in \mathbb{C}, \ x \in [0,L], \ t \geq 0,$

$$u(x+L,t) = u(x,t),$$

$$u(x,0) = 1 + \varepsilon v(x), \ |\varepsilon| \ll 1,$$  

$$(25)$$

$$v(x) = \sum_{j \geq 1} (c_j e^{ik_j x} + c_{-j} e^{-ik_j x}), \quad k_j = \frac{2\pi}{L} j, \quad |c_j| = O(1),$$
where the period $L$ is generic ($L/\pi$ is not an integer).

We first list the ingredients we need to construct its solution.

1. The number $N$ of unstable modes

$$N = \left\lfloor \frac{L}{\pi} \right\rfloor. \quad (26)$$

2. Their wave numbers and growth rates

$$k_j = \frac{2\pi j}{L} \quad \text{and} \quad \sigma_j = k_j \sqrt{4 - k_j^2}, \quad 1 \leq j \leq N, \quad (27)$$

and the angles $\phi_j$ parametrizing them

$$\phi_j = \arccos \frac{k_j}{2} = \arccos \frac{\pi j}{L}, \quad 0 < \phi_j < \frac{\pi}{2}, \quad 1 \leq j \leq N, \quad (28)$$

so that

$$k_j = 2 \cos \phi_j \quad \text{and} \quad \sigma_j = 2 \sin(2\phi_j). \quad (29)$$

We also define the $2N$ angles $\widehat{\phi}_j$ by

$$\widehat{\phi}_j = \phi_j \quad \text{and} \quad \widehat{\phi}_{j+N} = -\phi_j, \quad j = 1, \ldots, N. \quad (30)$$

3. The following linear combinations of the Fourier coefficients of the unstable modes:

$$\alpha_j = e^{-i\phi_j}c_j - e^{i\phi_j}c_{-j}, \quad \beta_j = e^{i\phi_j}c_{-j} - e^{-i\phi_j}c_j, \quad j = 1, \ldots, N, \quad (31)$$

and the additional quantities

$$\widehat{\alpha}_j = \alpha_j, \quad \widehat{\alpha}_{j+N} = \overline{\beta}_j, \quad \widehat{\beta}_j = \beta_j, \quad \beta_{j+N} = \overline{\alpha}_j, \quad j = 1, \ldots, N. \quad (32)$$

4. The leading-order $2N \times 2N$ Riemann matrix $B = (b_{jk})$:

$$b_{jj} = 2 \log \frac{\varepsilon \sqrt{\alpha_j \beta_j}}{|4 \sin(2\phi_j) \cos \phi_j|}, \quad j = 1, \ldots, 2N, \quad (33)$$

$$b_{jk} = 2 \log \left| \frac{\sin((\widehat{\phi}_j - \widehat{\phi}_k)/2)}{\cos((\widehat{\phi}_j + \widehat{\phi}_k)/2)} \right|, \quad j \neq k, \quad j, k = 1, \ldots, 2N, \quad (34)$$

where we assume that

$$\text{Re} \sqrt{\alpha_j \beta_j} \geq 0, \quad j = 1, \ldots, 2N,$$

$$\sqrt{\alpha_{j+N} \beta_{j+N}} = \sqrt{\alpha_j \beta_j}, \quad j = 1, \ldots, N.$$

5. The following $2N$-dimensional complex vectors $\vec{z}_\pm(x,t)$:

$$(\vec{z}_-(x,t))_j = \frac{i\pi}{2} - \log \frac{\alpha_j}{\sqrt{\alpha_j \beta_j}} + 2i \cos(\widehat{\phi}_j)x - 2 \sin(2\widehat{\phi}_j)t, \quad (35)$$

$$(\vec{z}_+(x,t))_j = (\vec{z}_-(x,t))_j - \pi i - 2i \widehat{\phi}_j, \quad j = 1, \ldots, 2N,$$
which describe the leading-order linearized NLS evolution on the Jacobian variety. We see that
\[
\text{Re}(\vec{z}_+(x,t))_j = \text{Re}(\vec{z}_-(x,t))_j = -2\sin(2\hat{\phi}_j)t - \frac{1}{2}\log\left|\frac{\hat{\alpha}_j}{\beta_j}\right|,
\]
\[
\text{Re}(\vec{z}_\pm(x,t))_{j+N} = -\text{Re}(\vec{z}_\pm(x,t))_j, \quad j = 1, \ldots, N,
\]
and these expressions do not depend on \(x\).

6. The \(2N\)-dimensional real vector \(-\vec{w}(t)\), where
\[
\vec{w}(t) = (\text{Re } B)^{-1} \vec{z}_-(x,t) = (\text{Re } B)^{-1} \vec{z}_+(x,t),
\]
describes the corresponding straight-line time evolution in the space \(\mathbb{R}^{2N}\), equipped with the standard integer lattice \(\mathbb{Z}^{2N}\). Indeed, it is the position of the maximum of the real part of the argument of the exponentials in the \(\theta\)-function definition. From (31), (34), (36) it follows immediately that
\[
w_{j+N}(t) \equiv -w_j(t), \quad j = 1, \ldots, N.
\]
Equation (38) suggests that the construction of the relevant components of \(\vec{w}(t)\) can be simplified, involving the inversion of an \(N \times N\) matrix easier to handle. Indeed, if we introduce the \(N\)-vector \(w(t) \in \mathbb{R}^N\) whose components are the first \(N\) components of \(\vec{w}(t)\), then
\[
\vec{w}(t) = (w(t), -w(t))^T \in \mathbb{R}^{2N},
\]
\[
w(t) = w^{(1)}t + w^{(0)}, \quad w^{(1)} = \mathcal{B}^{-1} \sigma, \quad w^{(0)} = \mathcal{B}^{-1} \chi,
\]
where \(\mathcal{B}^{-1}\) is the inverse of the real symmetric \(N \times N\) matrix \(\mathcal{B}\) defined by
\[
\mathcal{B}_{jj} = \sigma_j \tau_j, \quad \mathcal{B}_{jk} = 2\log\left|\frac{\sin(\phi_j + \phi_k)}{\sin(\phi_j - \phi_k)}\right|, \quad 1 \leq j, k \leq N, \quad j \neq k,
\]
with
\[
\tau_j = \frac{2}{\sigma_j} \log\frac{\sigma_j^2}{2\varepsilon \sqrt{\alpha_j \beta_j}}, \quad 1 \leq j \leq N,
\]
and with the \(N\)-vectors \(\sigma, \chi \in \mathbb{R}^N\) defined by
\[
\sigma = \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_N \end{pmatrix}, \quad \chi = \begin{pmatrix} \log \sqrt{\alpha_1/\beta_1} \\ \vdots \\ \log \sqrt{\alpha_N/\beta_N} \end{pmatrix}.
\]

7. The real number \(p, 0 < p < 1\), which characterizes the accuracy of the approximation \(\hat{u}(x,t)\) we want to achieve in the construction of the solution \(u(x,t)\):
\[
u(x,t) = \hat{u}(x,t) + O(\varepsilon^p).
\]
It follows that if at a given time the contribution of a certain unstable mode to the solution is less than \(O(\varepsilon^p)\), then this mode is neglected as ‘invisible’ with respect to the above approximation.
Definition 1. An unstable mode is \( p\)-visible at a certain time \( t \) if its contribution to the solution is of order \( \varepsilon^p \) or greater. It is \( p\)-invisible otherwise.

8. The following \( 2N \) vector \( \vec{st}(t) \in \mathbb{R}^{2N} \) of components

\[
\begin{align*}
st_j(t) &= 1 - \Theta \left( -w_j(t) + \lfloor w_j(t) \rfloor + \frac{1-p}{2} \right) \\
&\quad + \Theta \left( w_j(t) - \lceil w_j(t) \rceil - \frac{1+p}{2} \right), \quad j = 1, \ldots, 2N, \tag{43}
\end{align*}
\]

where \( \Theta(x) \) denotes the standard step function

\[
\Theta(x) = \begin{cases} 
0, & x \leq 0, \\
1, & x > 0.
\end{cases}
\]

Equivalently,

\[
st_j(t) = \begin{cases} 
0 & \text{if } w_j(t) - \lfloor w_j(t) \rfloor < \frac{1-p}{2}, \\
1 & \text{if } \frac{1-p}{2} \leq w_j(t) - \lfloor w_j(t) \rfloor \leq \frac{1+p}{2}, \\
2 & \text{if } w_j(t) - \lfloor w_j(t) \rfloor > \frac{1+p}{2},
\end{cases}
\]

with

\[
st_{j+N}(t) + st_j(t) = 2 \quad \text{for all } t \text{ and } j = 1, \ldots, N.
\]

The vector (43) is used to detect which unstable modes are \( p\)-visible at a given time. The \( j \)th mode is \( p\)-visible at time \( t \) if and only if

\[
st_j(t) = 1,
\]

and it is ‘\( p\)-invisible’ if

\[
st_j(t) = 0 \quad \text{or} \quad st_j(t) = 2
\]

(see Fig. 2).

After the introduction of the above ingredients, all of them expressed in terms of the initial data via elementary functions, we are ready to formulate the main result, to be proved in the next sections.

Theorem 2. Consider the generic periodic Cauchy problem of anomalous waves (25), for \( x \in [0, L] \) and \( t \) in any finite time interval \([0, T_0]\). Let the positive quantities \( t_{2n-1}^{(j)} \), \( j = 1, \ldots, N \) and \( k \geq 1 \), be defined by

\[
t_{2n-1}^{(j)} = \frac{(1-p)/2 + n - 1 - w_j^{(0)}}{w_j^{(1)}} \quad \text{and} \quad t_{2n}^{(j)} = \frac{(1+p)/2 + n - 1 - w_j^{(0)}}{w_j^{(1)}}, \quad n \geq 1,
\]

where the \( w_j^{(0)} \) and \( w_j^{(1)} \), \( 1 \leq j \leq N \), are the \( N \) components of the vectors \( w^{(0)} \) and \( w^{(1)} \) defined in (39)–(42). They are the times at which the \( j \)th unstable mode
changes its status, from $p$-invisible to $p$-visible if $k$ is odd, and from $p$-visible to $p$-invisible if $k$ is even, exhibiting the following recurrence properties:

$$\begin{align*}
t^{(j)}_{2n} - t^{(j)}_{2n-1} &= \frac{p}{w^{(1)}_j}, & t^{(j)}_{2n} - t^{(j)}_{2n-1} &= \frac{1-p}{w^{(1)}_j}, \\
t^{(j)}_{2n+1} - t^{(j)}_{2n-1} &= t^{(j)}_{2n+2} - t^{(j)}_{2n} = \frac{1}{w^{(1)}_j}, & j &= 1, \ldots, N, \quad n \geq 1.
\end{align*}$$

(45)

They partition the interval $[0, T_0]$ naturally into a sequence of finite intervals. In a given interval $I$ of the partition, the $j$th unstable mode is $p$-visible if and only if

$$1 - \frac{p}{2} \leq w_j(t) - \lfloor w_j(t) \rfloor \leq \frac{1+p}{2}, \quad 1 \leq j \leq N, \quad t \in I,$n

and it is $p$-invisible otherwise. Let $\mathcal{N}(I)$ denote the number of $p$-visible modes in the interval $I$, and let $s_k(I), k = 1, \ldots, \mathcal{N}(I)$, denote the indices of these $p$-visible modes, assuming that $s_{k+\mathcal{N}(I)}(I) = s_k(I) + N$.

Then the solution of the Cauchy problem, to leading and relevant order, is given by the formula

$$u(x, t) = u_I(x, t) + O(\varepsilon^p), \quad t \in I,$$

(47)

where $u_I(x, t)$ is the exact $\mathcal{N}(I)$-soliton solution of Akhmediev type describing the non-linear interaction of the $\mathcal{N}(I)$ unstable modes that are visible in the interval $I$, $L$-periodic in $x$, and localized in time over the background:

$$u_I(x, t) = \exp(2i\Phi(I)) \cdot \mathcal{A}_I(x, t),$$

where $\mathcal{A}_I(x, t)$ is defined by

$$\mathcal{A}_I(x, t) = \frac{\hat{\theta}_{2\mathcal{N}(I)}(\hat{Z}_+(x, t) \mid B)}{\hat{\theta}_{2\mathcal{N}(I)}(\hat{Z}_-(x, t) \mid B)},$$

(48)
and where only the $p$-visible components in the sum are kept:

$$
\hat{\theta}_{2,N}(I)(\bar{Z} \mid B) = \sum_{j=1, \ldots, 2N(I)} \exp\left(\frac{1}{2} \sum_{l=1}^{2N(I)} \hat{n}_l \bar{Z}_l \right) + \sum_{1 \leq l, m \leq 2N(I), l \neq m} \log \left| \frac{\sin((\hat{\phi}_{sl}(I) - \hat{\phi}_{sm}(I))/2)}{\cos((\hat{\phi}_{sl}(I) + \hat{\phi}_{sm}(I))/2)} \right| \frac{\hat{n}_l \hat{n}_m}{4},
$$

(49)

$$
(\bar{Z}_-(x, t))_j = (\bar{z}_-(x, t))_{s_j(I)}, \quad j = 1, \ldots, 2N(I),
$$

$$
(\bar{Z}_+(x, t))_k = (\bar{z}_-(x, t))_k - \pi i - 2i \hat{\phi}_{sk(I)},
$$

$$
\bar{z}_-(x, t) = \bar{z}_-(x, t) - \sum_{k=1}^{2N} \left[ w_k(t) + \frac{\text{st}_k(I)}{2} \right] \bar{b}_k,
$$

(50)

$$
\Phi(I) = \sum_{k=1}^{N} \left[ 2[w_k(t)] + \text{st}_k(I) \right] \phi_k
$$

(51)

and $\bar{b}_k$ denotes the $k$th column of the matrix $B$.

Therefore, in each time interval $I$ of the partition we approximate $u(x, t)$ to within $O(\varepsilon^p)$, with $0 < p \leq 1$, by elementary functions explicitly defined in terms of the Cauchy data. For instance, in Fig. 3 we use the above analytic formulae to construct the leading-order solution in the case of three unstable modes ($L = 10$) with $p = 1/2$, and this is in very good agreement with the corresponding numerical experiment.

Remark 11. To the best of our knowledge, the exact $N$-soliton solution of Akhmediev type has the following history. The representation (48) was obtained by Its, Rybin, and Sall in [63], as a degeneration of the finite-gap formulae. The determinant form of this solution and a discussion of the connection between these two representations are also provided in [63]. We remark that the formulae (48) can be obtained from Hirota’s $N$-soliton solution [59] of the defocusing NLS equation by a complex rotation [30]. Determinant formulae for the $N$-soliton solution over the zero background for the self-focusing case were provided in the book [44] of Faddeev and Takhtadjan, and the determinant formula for the $N$-soliton solution over an arbitrary background, including the superregular solitons, was constructed by Zakharov and Gelash [112] (see also [113], [50]).

Remark 12. The formula (49) involves summation over $4^N$ terms, $N \leq N$, of order $O(1)$ growing exponentially with $N$ (for instance, for $N = 5$ the sum involves about $10^3$ terms). It follows that these sums could generate corrections comparable with $O(|\log \varepsilon|)$, for reasonable values of $\varepsilon$ coming from physics or from numerical simulations, and the results would become less reliable. Therefore, the number $N$ of unstable modes must be sufficiently small to avoid this problem.
The finite-gap method and the periodic NLS Cauchy problem

Figure 3. The graph of $|u(x,t)|$ to leading order, as given by the formulae (47)–(51), is in good agreement with the corresponding numerical experiment. Here $L = 10$ ($N = 3$), $0 \leq t \leq 60$, $x \in [-L/2, L/2]$, $\varepsilon = 10^{-6}$, $c_1 = 0.5$, $c_{-1} = 0.3 + 0.3i$, $c_2 = 0.5$, $c_{-2} = -0.03 + 0.03i$, $c_3 = 0.03$, $c_{-3} = 0.02 + 0.03i$, $p = 1/2$, and the short axis is the $x$-axis. The intervals of $p$-visibility for the three unstable modes are marked by bold lines below the graph. The boundary points of the partition intervals are, sequentially, $0$, $t^{(2)}_1$, $t^{(1)}_1$, $t^{(3)}_1$, $t^{(2)}_2$, $t^{(1)}_2$, $t^{(3)}_2$, $t^{(2)}_3$, $t^{(1)}_3$, $t^{(3)}_3$, $t^{(2)}_4$, $t^{(1)}_4$, $t^{(2)}_5$, $t^{(1)}_5$, $t^{(2)}_6$, $t^{(1)}_6$, $t^{(2)}_7$, and $t^{(1)}_7$. In the first interval $(0, t^{(2)}_1)$ we are in the first linear stage of MI, and all modes are invisible; in the interval $(t^{(2)}_1, t^{(1)}_1)$ the most unstable mode 2 is visible; in the interval $(t^{(1)}_1, t^{(3)}_1)$ the mode 1 is also visible; in the interval $(t^{(3)}_1, t^{(2)}_2)$ all three modes are visible; and so on. The unstable mode 2, characterized by two maxima in the period, is the most unstable mode ($\sigma_2 = \max\{\sigma_1, \sigma_2, \sigma_3\}$). It first appears in the interval $(t^{(2)}_1, t^{(2)}_2)$; it disappears in the interval $(t^{(2)}_2, t^{(2)}_3)$, reappearing again in $(t^{(2)}_3, t^{(2)}_4)$; and so on. Its recurrence properties are described by (45).

Remark 13. In this paper we use a more symmetric notation in comparison with the notation used in our previous papers on this subject ([52]–[55]) (see, for instance, equation (31)). We also use a different normalization for the $\theta$-functions: in the present text it coincides with the one used in [45], [20], and [40], while in our previous papers we used the normalization from [83].

3. Periodic problem for the non-linear Schrödinger equation

We recall the basic facts about the periodic theory of the non-linear Schrödinger (NLS) equation.

The NLS equation has two real forms:

$$
i u_t + u_{xx} - 2|u|^2u = 0, \quad u = u(x, t) \in \mathbb{C} \quad \text{(defocusing NLS)},$$

$$
i u_t + u_{xx} + 2|u|^2u = 0, \quad u = u(x, t) \in \mathbb{C} \quad \text{(self-focusing NLS).}$$

In non-linear optics both models describe media with refractive index depending on the electric field. If the refractive index decreases (increases) in the presence of
an electromagnetic wave, then we have the defocusing (respectively, self-focusing) NLS equation. Both forms can be treated as real reductions of the complex NLS equation
\[
\begin{align*}
  iq_t + q_{xx} + 2q^2r &= 0, \\
  -ir_t + r_{xx} + 2qr^2 &= 0,
\end{align*}
\] (53)
where
\[
q(x, t) = u(x, t), \quad r(x, t) = -\overline{u(x, t)} \quad \text{(the defocusing case)},
\] (54)
\[
q(x, t) = u(x, t), \quad r(x, t) = \overline{u(x, t)} \quad \text{(the self-focusing case)}.\] (55)

The zero-curvature representation for the NLS reductions (54) and (55) was found by Zakharov and Shabat in [117] and generalized for the complex NLS equation (53) by Ablowitz, Kaup, Newell, and Segur in [3].

A pair of functions \(q(x, t), r(x, t)\) satisfies the complex NLS equation (53) if and only if the following pair of linear problems is compatible:
\[
\begin{align*}
  \tilde{\Psi}_x(\lambda, x, t) &= U(\lambda, x, t)\tilde{\Psi}(\lambda, x, t), \\
  \tilde{\Psi}_t(\lambda, x, t) &= V(\lambda, x, t)\tilde{\Psi}(\lambda, x, t),
\end{align*}
\] (56)
where
\[
U = \begin{bmatrix} -\lambda & iq(x, t) \\ ir(x, t) & \lambda \end{bmatrix},
\]
\[
V(\lambda, x, t) = \begin{bmatrix}
-2i\lambda^2 + iq(x, t)r(x, t) & \lambda q(x, t) - q_x(x, t) \\
2i\lambda r(x, t) + r_x(x, t) & 2i\lambda^2 - iq(x, t)r(x, t)
\end{bmatrix}
\] (57)
and
\[
\tilde{\Psi}(\lambda, x, t) = \begin{bmatrix} \Psi_1(\lambda, x, t) \\ \Psi_2(\lambda, x, t) \end{bmatrix}.
\]
The first equation of the zero-curvature representation (56) can be rewritten as the following spectral problem:
\[
\mathfrak{L} \tilde{\Psi}(\lambda, x, t) = \lambda \tilde{\Psi}(\lambda, x, t),
\] (58)
where
\[
\mathfrak{L} = \begin{bmatrix} i\partial_x & q(x, t) \\ -r(x, t) & -i\partial_x \end{bmatrix}.
\]

The main tool for constructing periodic and quasi-periodic solutions of soliton systems is the finite-gap method, invented by Novikov [86] for the periodic Korteweg–de Vries (KdV) problem. Finite-gap solutions of the NLS equation were first constructed by Its and Kotljarov [61].

We recall the principal facts about the periodic direct and inverse spectral transform for the 1-dimensional Dirac operator (58).

Consider a fixed time \(t = t_0\). Let \(q(x)\) and \(r(x)\) denote the Cauchy data:
\[
q(x) = q(x, t_0), \quad r(x) = r(x, t_0), \quad q(x + L) = q(x), \quad r(x + L) = r(x).
\]
The direct spectral transform associates the following spectral data with these Cauchy data:
1) the spectral curve $\Gamma$, that is, the Riemann surface for the Bloch eigenfunctions;
2) the divisor, that is, the set of eigenvalues for an auxiliary Dirichlet-type spectral problem.

1) The spectral curve. The Bloch functions of the operator $\mathcal{L}$ are defined as the common eigenfunctions of $\mathcal{L}$ and the translation operator (the periodicity of $\mathcal{L}$ means that it commutes with translation by the basic period $L$, and therefore $\mathcal{L}$ and the translation operator have sufficiently many common eigenfunctions):

$$\mathcal{L}\Psi(x, t_0) = \lambda \Psi(x, t_0),$$

$$\Psi(x + L, t_0) = \kappa \Psi(x, t_0).$$

Equivalentally, the Bloch functions are eigenfunctions of the monodromy matrix $\hat{T}(\lambda, x_0, t_0)$ defined by

$$\hat{T}(\lambda, x_0, t_0) = \hat{\Psi}(\lambda, x_0 + L, t_0),$$

where the $2 \times 2$ invertible matrix $\hat{\Psi}(\lambda, x, t_0)$ is the solution of the matrix equation

$$\mathcal{L}\hat{\Psi}(\lambda, x, t_0) = \lambda \hat{\Psi}(\lambda, x, t_0)$$

with the initial condition

$$\hat{\Psi}(\lambda, x_0, t_0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The monodromy matrix $\hat{T}(\lambda, x_0, t_0)$ is holomorphic in $\lambda$ in the whole complex plane. It follows that the eigenvalues and eigenvectors of $\hat{T}(\lambda, x_0, t_0)$ (that is, the Bloch multipliers $\kappa$ and the Bloch eigenfunctions $\hat{\Psi}(x, t_0)$) are defined on a two-sheeted covering $\Gamma(x_0, t_0)$ of the $\lambda$-plane, and hence we shall write $\kappa(\gamma), \hat{\Psi}(\gamma, x, t_0), \gamma \in \Gamma$.

The Riemann surface $\Gamma(x_0, t_0)$ is called the spectral curve. If the potentials $q(x, t)$ and $r(x, t)$ satisfy the NLS equation (53), then the monodromy matrices corresponding to different $x_0$ and $t_0$ coincide up to conjugation. Therefore, $\Gamma(x_0, t_0)$ does not depend on $x_0$ and $t_0$ and will be denoted by $\Gamma$ in the rest of the text. In particular, this means that the curve $\Gamma$ provides an infinite set of conservation laws for the NLS hierarchy. This set is complete, and the standard local conservation laws can be easily obtained as expansion coefficients of $\Gamma$ near infinity.

The spectrum of the problem (58) in $L^2(\mathbb{R})$ is always continuous, and it is defined by the following property: $\lambda \in \mathbb{C}$ belongs to the spectrum of $\mathcal{L}$ if and only if equation (58) admits a solution growing no faster than some polynomial both as $x \to -\infty$ and as $x \to +\infty$. The endpoints of the spectral intervals coincide with the branch points of $\Gamma$. In [86] the spectral curve was introduced for the real Schrödinger operator, where the relation between the spectral curve and the classical spectrum is especially simple: the spectrum in $L^2(\mathbb{R})$ is the set of intervals on the real line which are bounded by branch points of $\Gamma$.

We use the following notation: if $\gamma \in \Gamma$ is a point of our spectral curve, then $\lambda(\gamma)$ denotes the projection of $\gamma$ to the $\lambda$-plane.

The multivalued function

$$p(\gamma) = \frac{1}{iL} \log \kappa(\gamma)$$

(60)
is called the quasi-momentum. Its differential $dp(\gamma)$ is well-defined and meromorphic on $\Gamma$, with two simple poles at the infinity points, and all periods of $dp$ are real. The trace of the matrix $U(\lambda, x, t)$ is zero; therefore,

$$\det \hat{T}(\lambda, x_0, t_0) \equiv 1,$$

and if $\lambda(\gamma_1) = \lambda(\gamma_2)$, then

$$\kappa(\gamma_1)\kappa(\gamma_2) = 1.$$

The ‘classical’ spectrum of $L$ in $L^2(\mathbb{R})$ coincides with the projection of the set $\{\gamma \in \Gamma, |\kappa(\gamma)| = 1\}$ to the $\lambda$-plane, or, equivalently, is defined by the condition

$$\text{Im} p(\gamma) = 0.$$

The branch points of $\Gamma$ coincide with the endpoints of the spectral gaps.

**Definition 2.** A pair of potentials $q(x), r(x)$ is said to be finite-gap if the spectral curve $\Gamma$ is algebraic, that is, it has only finitely many branch points and non-removable double points. In the real case $r(x) = \pm q(x)$ the second requirement is fulfilled automatically, and it is sufficient to assume that the number of branch points is finite.

**Remark 14.** The analytic properties of the Bloch eigenfunction for the 1-dimensional Schrödinger operator in the domain of complex energies were first studied in Kohn’s paper [71].

2) **Divisor.** The auxiliary spectrum is defined to be the set of points $\gamma \in \Gamma$ such that the first component of the Bloch eigenfunction is equal to 0 at the point $x_0$,

$$\mathcal{L}\bar{\Psi}(\gamma, x, t) = \lambda(\gamma)\bar{\Psi}(\gamma, x, t),$$

$$\bar{\Psi}(\gamma, x + L, t) = \kappa(\gamma)\bar{\Psi}(\gamma, x, t),$$

$$\Psi_1(\gamma, x_0, t_0) = 0. \tag{61}$$

Equivalently, the auxiliary spectrum coincides with the set of zeros of the first component of the Bloch eigenfunction:

$$\Psi_1(\gamma, x, t) = 0; \tag{62}$$

therefore, it is called the divisor of zeros. The zeros of $\Psi_1(\gamma, x, t)$ depend on $x$ and $t$. The $x$ and $t$ dynamics becomes linear after the Abel transform (for the KdV equation this was first established by Dubrovin [39] and Its and Matveev [62]).

**Remark 15.** We point out that a different auxiliary problem is used in some papers devoted to finite-gap NLS solutions. More precisely, one imposes the following symmetric boundary condition:

$$\Psi_1(\lambda, x_0, t_0) + \Psi_2(\lambda, x_0, t_0) = \Psi_1(\lambda, x_0 + L, t_0) + \Psi_2(\lambda, x_0 + L, t_0) = 0. \tag{63}$$

This approach has the advantage that all the divisor points are located in a compact area of the spectral curve, but it requires one extra divisor point and thus increases the complexity of the formulae.
The wave propagation in focusing and defocusing media is substantially different from the physical point of view. This difference means that the analytic properties of the solutions are also substantially different (this topic is discussed in detail in the paper [93] by Previato).

1. In the defocusing case:
   - The operator $\mathcal{L}$ is self-adjoint, and its spectrum is real.
   - All branch points of $\Gamma$ are real and simple.
   - If the normalization (63) is used, then each non-empty spectral gap contains exactly one divisor point.
   - The defocusing NLS equation possesses both regular and singular finite-gap solutions.

From the analytic point of view, the theory of the defocusing NLS equation is analogous to the theory of the real Korteweg–de Vries equation.

2. In the self-focusing case:
   - The operator $\mathcal{L}$ is not self-adjoint, and the matrix $U(\lambda, x, t)$ is skew-Hermitian for real $\lambda$. Thus, for each real $\lambda$ the monodromy matrix is unitary. This means that the whole real line lies in the $L^2(\mathbb{R})$ spectrum of $\mathcal{L}$. In addition, generically the $L^2(\mathbb{R})$ spectrum of $\mathcal{L}$ contains some arcs in the complex domain.
   - All branch points of $\Gamma$ are complex, and finitely many of them may be of high odd multiplicity. All real double points of $\Gamma$ are removable (the monodromy matrix has two different eigenvectors), but complex double points may not be removable, and such points are associated with homoclinic orbits. For sufficiently regular data, the number of complex double points is always finite. The curve $\Gamma$ is real; that is, it is invariant with respect to complex conjugation.
   - The characterization of the divisor is less explicit, and it can be provided in terms of Cherednik differentials [33].
   - All finite-gap solutions are automatically regular [33].

In contrast to the defocusing case, the self-focusing case is much richer and much more interesting from the analytic point of view.

For generic smooth NLS Cauchy data $q(x, t_0)$, $r(x, t_0)$, the surface $\Gamma$ has infinite genus, but for large values of the spectral parameter the branch points of $\Gamma$ form close pairs (see [37] for analytic estimates extending the results of [60] to the 1-dimensional Dirac operator), and one can construct an arbitrarily good finite-gap approximation for a given smooth potential by merging all but finitely many close pairs of branch points. ‘Naive’ merging does not respect spatial periodicity and provides an approximation only locally with respect to $x$. In the defocusing case, a period-preserving approximation can be obtained using a minor modification of the Marchenko–Ostrovskii approach [80]. In the focusing case, the analytic properties of the quasi-momentum are more complicated, and to construct a purely periodic finite-gap approximation one can use either the isoperiodic deformation technique (Grinevich–Schmid [56]) or a suitable adaptation of Krichever’s technique in [73].

Therefore, in the periodic problem for soliton equations the role of finite-gap potentials can be compared with the role of finite Fourier series in the theory of linear PDEs.
The solution of the inverse problem in the finite-gap case (the genus of $\Gamma$ is equal to $g < \infty$) is provided by the $\theta$-function formula ([61], and see also [93]):

$$u(x, t) = C \exp(\mathcal{U}x + \mathcal{V}t) \frac{\theta(\tilde{A}(\infty_+) - \tilde{U}_1x - \tilde{U}_2t - \tilde{A}(\mathcal{D}) - \tilde{K} | B)}{\theta(\tilde{A}(\infty_+) - \tilde{U}_1x - \tilde{U}_2t - \tilde{A}(\mathcal{D}) - \tilde{K} | B)}.$$  \hspace{1cm} (64)

Here $C$, $\mathcal{U}$, and $\mathcal{V}$ are constants defined in terms of the spectral curve, $\tilde{A}(\mathcal{D})$ and $\tilde{A}(\infty_+)$, $\tilde{A}(\infty_-)$ are the Abel transforms of the divisor and of the infinity points of $\Gamma$, respectively, $\tilde{K}$ is the so-called vector of Riemann constants, $B$ is the Riemann period matrix for $\Gamma$, $\tilde{U}_1$ and $\tilde{U}_2$ are the vectors of $b$-periods for the quasi-momentum and quasi-energy differentials, respectively (see the formulae (96) and (100)), and $\theta(z | B)$ denotes the Riemann $\theta$-function of genus $g$:

$$\theta(z | B) = \sum_{n_j \in \mathbb{Z}, \; j=1,\ldots,g} \exp\left[\frac{1}{2} \sum_{j,k=1}^{g} b_{jk} n_j n_k + \sum_{j=1}^{g} n_j z_j \right],$$  \hspace{1cm} (65)

where the $b_{jk}$ are the components of the matrix $B$. More information about $\theta$-function formulae can be found in [40] and [20]. Explicit approximate formulae for the parameters in (64) and (65) in the special Cauchy problem of anomalous waves are provided below in §4.

The normalization (65) implies the following periodicity properties:

$$\theta(\tilde{z} + \tilde{a}_l | B) = \theta(\tilde{z} | B),$$

$$\theta(\tilde{z} + \tilde{b}_l | B) = \theta(\tilde{z} | B) \exp\left(-\frac{1}{2} b_{ll} - z_l \right),$$  \hspace{1cm} (66)

where $l = 1, \ldots, g$.

4. Spectral transform for the Cauchy problem of anomalous waves

If one uses the focusing NLS equation as a mathematical model for anomalous waves, then special initial data (5)–(6) are considered. We show that the presence of a small parameter $\varepsilon$ in this problem enables one to construct a good approximation for the solutions in terms of elementary functions.

To construct the direct spectral transform for this problem, it is convenient to write

$$\mathcal{L} = \mathcal{L}_0 + \varepsilon \mathcal{L}_1,$$

where

$$\mathcal{L}_0 = \begin{bmatrix} i\partial_x & 1 \\ -1 & -i\partial_x \end{bmatrix}, \quad \mathcal{L}_1 = \begin{bmatrix} 0 & v(x) \\ -v(x) & 0 \end{bmatrix},$$

and to calculate the spectral data for $\mathcal{L}$ using the standard perturbation theory near the spectral data for $\mathcal{L}_0$. The leading-order formulae for the spectral curve and the divisor were provided in the recent paper [52] of the authors. Let us present a brief review of these results.
4.1. Spectral data for the unperturbed operator. The unperturbed spectral curve $\Gamma_0$ for $L_0$ is rational, and a point $\gamma \in \Gamma_0$ is a pair of complex numbers $\gamma = (\lambda, \mu)$ satisfying the quadratic equation

$$\mu^2 = \lambda^2 + 1.$$ 

The Bloch eigenfunctions for the operator $L_0$ can easily be calculated explicitly:

$$\psi^\pm(\gamma, x) = \left[ \frac{1}{\lambda(\gamma) \pm \mu(\gamma)} \right] e^{\pm i\mu(\gamma)x},$$

$$L_0\psi^\pm(\gamma, x) = \lambda(\gamma)\psi^\pm(\gamma, x).$$

These eigenfunctions are periodic (antiperiodic) if and only if $L = \frac{\pi n}{2\mu} \in \mathbb{Z}$ is an even (an odd) integer. Let us introduce the following notation:

$$\mu_n = \frac{\pi n}{L}, \quad \lambda_n = \sqrt{\mu_n^2 - 1} \quad (\text{Re} \lambda_n + \text{Im} \lambda_n > 0), \quad n = 0, 1, 2, \ldots, \infty.$$ 

Then the periodic and antiperiodic spectral points of $L_0$ are

$$\{ \pm \lambda_n \}, \quad n = 0, 1, 2, \ldots, \infty$$

(see Fig. 4). By analogy with [73] and [74] these points are called resonant points, because they split into pairs of branch points under small generic perturbations.

We also use the notation

$$\lambda_{-n} = \lambda_n, \quad \mu_{-n} = -\mu_n, \quad n = 1, 2, \ldots.$$ 

Recall that we assume the condition $L \neq \pi n$, where $n \in \mathbb{N}_+$, and therefore the point $\lambda = 0$ is not resonant. The case of a resonant point $\lambda = 0$ requires special treatment, and we plan to provide it in the near future.

We have the following basis of eigenfunctions for the periodic and antiperiodic problems:

$$\psi_n^\pm = \left[ \frac{1}{\mu_n \pm \lambda_n} \right] e^{i\mu_n x}, \quad n \in \mathbb{Z},$$

$$L_0\psi_n^\pm = \pm \lambda_n \psi_n^\pm.$$
The curve $\Gamma_0$ has two branch points,

$$E_0 = i \quad \text{and} \quad \overline{E_0} = -i,$$

which correspond to $n = 0$. If $n > 0$, then there is no branching at the resonant points $\pm \lambda_n$, but the monodromy matrix becomes diagonal with coinciding eigenvalues:

$$\hat{T}(\pm \lambda_n, 0, 0) = \begin{bmatrix} (-1)^n & 0 \\ 0 & (-1)^n \end{bmatrix}.$$ 

The resonant points are also the eigenvalues of the Dirichlet problem (61), that is, the divisor points for the unperturbed problem. The resonant points are also the eigenvalues of the Dirichlet problem (61), that is,

$$\mu$$

the linearized theory, we note that the Fourier modes correspond to the points

For a generic periodic perturbation, the matrix elements of

are unstable if the corresponding $\lambda_n$ are imaginary ($|\mu_n| < 1$) and stable if the $\lambda_n$ are real ($|\mu_n| \geq 1$); see §1. Therefore, the resonant points with $|n| < L/\pi$ are unstable, and the resonant points with $|n| \geq L/\pi$ are stable. For unstable modes we have

$$\lambda_j = i \sin \phi_j \quad \text{and} \quad \mu_j = \cos \phi_j, \quad j = 1, \ldots, N,$$

where the angles $\phi_j$ are the same as in (11).

4.2. Spectral data for the perturbed operator. To calculate the perturbed spectral curve we develop the perturbation theory for the periodic and antiperiodic problems using the basis (68).

It will be convenient to use the notation ($n \geq 1$)

$$\alpha_n = (\mu_n - \lambda_n)\overline{c_n} - (\mu_n + \lambda_n)c_{-n}, \quad \beta_n = (\mu_n + \lambda_n)\overline{c_n} - (\mu_n - \lambda_n)c_n, \quad (70)$$

$$\tilde{\alpha}_n = (\mu_n + \lambda_n)\overline{c_n} - (\mu_n - \lambda_n)c_{-n}, \quad \tilde{\beta}_n = (\mu_n - \lambda_n)\overline{c_n} - (\mu_n + \lambda_n)c_n, \quad (71)$$

which is consistent with (31), since $e^{\pm i\phi_n} = \mu_n \pm \lambda_n$ for unstable modes.

Also, let $|l_{\pm}\rangle$ denote the basis vector $\psi_l^\pm$ and $\langle l_{\pm}\rangle$ the adjoint basis vector:

$$\langle l_{+}\rangle m_+\rangle = \delta_{lm}, \quad (l_{-}\rangle m_-\rangle = \delta_{lm}, \quad (l_{+}\rangle m_-\rangle = (l_{-}\rangle m_+\rangle = 0.$$ 

For a generic periodic perturbation, the matrix elements of $\mathfrak{L}_1$ can be written in the following form:

$$\langle m_+\rangle \mathfrak{L}_1 \langle l_+\rangle = \frac{c_{(m-l)/2}(\lambda_m - \mu_m)(\lambda_l + \mu_l) - \overline{c_{-(m-l)/2}}}{2\lambda_m},$$

$$\langle m_-\rangle \mathfrak{L}_1 \langle l_+\rangle = \frac{c_{(m-l)/2}(\lambda_m + \mu_m)(\lambda_l + \mu_l) + \overline{c_{-(m-l)/2}}}{2\lambda_m},$$

$$\langle m_+\rangle \mathfrak{L}_1 \langle l_-\rangle = \frac{c_{(m-l)/2}(\lambda_m - \mu_m)(-\lambda_l + \mu_l) - \overline{c_{-(m-l)/2}}}{2\lambda_m},$$

$$\langle m_-\rangle \mathfrak{L}_1 \langle l_-\rangle = \frac{c_{(m-l)/2}(\lambda_m + \mu_m)(-\lambda_l + \mu_l) + \overline{c_{-(m-l)/2}}}{2\lambda_m}.$$
Here we use a slightly non-standard notation: $\langle f | L | g \rangle$ denotes the matrix element for both orthogonal and non-orthogonal bases. We also assume that $c_{-(m-l)/2} = 0$ if $m - l$ is odd.

![Diagram of spectrum](image)

Figure 5. The spectrum of the perturbed operator $L = L_0 + \varepsilon L_1$.

Using standard perturbation theory (see [52] for details), we easily show that the resonant points $\lambda_n$ and $-\lambda_n$ split generically into the pairs of branch points $\{E_{2n-1}, E_{2n}\}$ and $\{\bar{E}_{2n-1}, \bar{E}_{2n}\}$, where

$$
E_l = \lambda_n \pm \frac{\varepsilon}{2\lambda_n} \sqrt{\alpha_n\beta_n} + O(\varepsilon^2), \quad l = 2n - 1, 2n,
$$

and the branch points $E_0$ and $\bar{E}_0$ are shifted in the second order of smallness:

$$
E_0 = i + O(\varepsilon^2), \quad \bar{E}_0 = -i + O(\varepsilon^2)
$$

(see Fig. 5). Here we assume that

$$
\text{Re} \sqrt{\alpha_n\beta_n} \geq 0 \quad \text{and} \quad \text{Re} \sqrt{\bar{\alpha}_n\bar{\beta}_n} \geq 0
$$

for the unstable points, and

$$
\text{Im} \sqrt{\alpha_n\beta_n} < 0 \quad \text{and} \quad \text{Im} \sqrt{\bar{\alpha}_n\bar{\beta}_n} < 0
$$

for the stable points. To estimate the variation of the branch point $E_0$ we have also used the constraint $c_0 = 0$.

For the perturbations of unstable points we have

$$
\bar{E}_l = \bar{E}_l.
$$
We define the following enumeration for the unperturbed divisor points:

$$\lambda_{n}^{\text{div}} = \begin{cases} 
\lambda_{n}, & n > 0, \\
-\lambda_{n}, & n < 0 
\end{cases}, \quad \mu_{n}^{\text{div}} = \mu_{n}.$$ 

The calculation of the divisor position to within $O(\varepsilon^2)$ uses the same perturbation theory, with the following modification. In contrast to branch points, the Bloch multipliers for the Dirichlet spectrum are generically different from $\pm 1$; moreover, their absolute values do not have to be equal to 1. Thus, we use the additional constraint that the first component of the Dirichlet eigenfunction vanishes at the boundary of the interval, to determine simultaneously the variation of $\lambda$ and the variation of the Bloch multiplier $\kappa$, and we get that (see [52])

$$\lambda(\gamma_{n}) = \lambda_{n} + \frac{\varepsilon}{4\lambda_{n}}[(\mu_{n} + \lambda_{n})\alpha_{n} + (\mu_{n} - \lambda_{n})\beta_{n}] + O(\varepsilon^{2}),$$

$$p(\gamma_{n}) = \frac{\varepsilon}{4\mu_{n}}[(\mu_{n} + \lambda_{n})\alpha_{n} - (\mu_{n} - \lambda_{n})\beta_{n}] + O(\varepsilon^{2}), \tag{74}$$

and

$$\lambda(\gamma_{-n}) = -\lambda_{n} - \frac{\varepsilon}{4\lambda_{n}}[(\mu_{n} - \lambda_{n})\tilde{\alpha}_{n} + (\mu_{n} + \lambda_{n})\tilde{\beta}_{n}] + O(\varepsilon^{2}),$$

$$p(\gamma_{-n}) = \frac{\varepsilon}{4\mu_{n}}[(\mu_{n} - \lambda_{n})\tilde{\alpha}_{n} - (\mu_{n} + \lambda_{n})\tilde{\beta}_{n}] + O(\varepsilon^{2}). \tag{75}$$

5. The ‘unstable part’ of the spectral curve

A small generic perturbation of the constant solution generates a spectral curve of infinite genus. Of course, it is natural to approximate it by a curve of finite genus. The perturbations corresponding to stable resonant points remain of order $O(\varepsilon)$ for all $t$ and can be well described by the linear perturbation theory. Therefore, one can close all gaps associated with stable points, thereby obtaining a finite-gap approximation of the spectral curve.

Remark 16. This finite-gap approximation is rather non-standard. Usually, one keeps the gaps associated with sufficiently large harmonics and closes the gaps corresponding to the small ones; therefore, the genus of the approximating curve depends on the concrete perturbation. In our problem, all harmonics of the perturbation are small in amplitude, and the genus of the approximating curve is determined solely by the number of unstable modes and does not depend on the Cauchy data.

5.1. Finite-gap approximating curve. Starting from this point, we shall use the following $2N$-gap approximation of the spectral curve: we open only the resonant points associated with unstable modes, and do not perturb the stable double points.

Our next step is to calculate the leading-order approximation for the algebro-geometrical data.
We introduce the following notation:

\[
\begin{align*}
\tilde{\lambda}_j &= \begin{cases} 
\lambda_j & \text{if } 1 \leq j \leq N, \\
\lambda_{j-N} & \text{if } N + 1 \leq j \leq 2N;
\end{cases} \\
\tilde{\mu}_j &= \begin{cases} 
\mu_j & \text{if } 1 \leq j \leq N, \\
\mu_{j-N} & \text{if } N + 1 \leq j \leq 2N;
\end{cases} \\
\tilde{\gamma}_j &= \begin{cases} 
\gamma_j & \text{if } 1 \leq j \leq N, \\
\gamma_{j-N} & \text{if } N + 1 \leq j \leq 2N.
\end{cases}
\end{align*}
\]

There are \(4N+2\) branch points

\[E_j, \quad j = 0, \ldots, 2N, \quad \overline{E_j}, \quad j = 0, \ldots, 2N,\]

and the curve \(\Gamma\) is defined by the equation

\[
\mu^2 = \prod_{j=0}^{2N} (\lambda - E_j)(\lambda - \overline{E_j}).
\]

(76)
We also define
\[ \mu^{(0)}(\lambda) = \sqrt{\lambda^2 + 1} \quad \text{and} \quad z_n = \frac{E_{2n-1} + E_{2n}}{2} = \lambda_n + O(\varepsilon^2). \]

The natural compactification of \( \Gamma \) has two points at infinity,
\[ \infty_+: \mu \sim -\lambda^{2N+1} \quad \text{and} \quad \infty_-: \mu \sim \lambda^{2N+1}. \]

We have the following system of cuts (marked by the dashed lines in Fig. 6):
\[ [i\infty, E_0], \ [E_1, E_2], \ldots, \ [E_{2N-1}, E_{2N}], \ [E_{2N}, E_{2N-1}], \ [E_2, E_1], \ [E_0, -i\infty]. \]

Here we use a slightly non-standard convention: \( \infty_+ \) is located on Sheet 2 (dashed lines) and \( \infty_- \) is located on Sheet 1 (solid lines).

To obtain convenient formulae, it is essential to choose a proper basis of cycles. In our text we use the one given in Fig. 6.

To calculate finite-gap solutions we need the periods of the basic holomorphic differentials and some meromorphic differentials, the vector of Riemann constants, and the Abel transform of the Dirichlet spectrum. We calculate them to leading order. In our paper we use the following conventions.

1. The basic holomorphic differentials are normalized by the conditions
\[ (\bar{a}_k)_j = \oint_{a_j} \omega_k = 2\pi i. \quad (77) \]

In this normalization the real part of the Riemann matrix \( B = (b_{jk}) \) with
\[ b_{jk} = (\bar{b}_k)_j = \oint_{b_j} \omega_k \quad (78) \]
is negative definite.

2. In §§5.2 and 5.3 we assume that the initial point of the Abel transform is the branch point \( E_0 = i + O(\varepsilon^2) \):
\[ \bar{A}(\gamma) = \bar{A}_{E_0}(\gamma) = \begin{bmatrix} \int_{E_0}^{\gamma} \omega_1 \\ \vdots \\ \int_{E_0}^{\gamma} \omega_{2N} \end{bmatrix} = \int_{E_0}^{\gamma} \bar{\omega}, \quad (79) \]

where
\[ \bar{\omega} = \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_{2N} \end{bmatrix}, \quad \gamma \in \Gamma. \]
5.2. Riemann matrix. The calculation of the diagonal elements to leading order does not depend on $N$, and therefore we can use the results in [52]:

$$b_{jj} = 2 \log \frac{\varepsilon \sqrt{\hat{\alpha}_j \hat{\beta}_j}}{|4 \sin(2 \hat{\phi}_j) \cos \hat{\phi}_j|} + O(\varepsilon), \quad 1 \leq j \leq 2N,$$

$$\exp \left( \frac{b_{jj}}{2} \right) = \frac{\varepsilon \sqrt{\hat{\alpha}_j \hat{\beta}_j}}{|4 \sin(2 \hat{\phi}_j) \cos \hat{\phi}_j|} + O(\varepsilon^2), \quad 1 \leq j \leq 2N,$$

with $\hat{\alpha}_j$ and $\hat{\beta}_j$ defined in (32). Up to corrections of order $O(\varepsilon^2)$, the off-diagonal elements do not depend on the perturbation, and coincide with the limiting values from [63]:

$$b_{jk} = 2 \log \left| \frac{\sin((\hat{\phi}_j - \hat{\phi}_k)/2)}{\cos((\hat{\phi}_j + \hat{\phi}_k)/2)} \right| + O(\varepsilon^2), \quad j \neq k, \quad j, k = 1, \ldots, 2N. \quad (81)$$

5.3. Vector of Riemann constants. The calculation of the vector of Riemann constants is rather standard (see, for example, [40]), but we present it for completeness. It is based on the periodicity properties of the Riemann $\theta$-functions (66):

$$\theta(\vec{A}(\gamma) - \vec{C} + \vec{a}_k) = \theta(\vec{A}(\gamma) - \vec{C}), \quad (82)$$

$$\theta(\vec{A}(\gamma) - \vec{C} + \vec{b}_k) = \theta(\vec{A}(\gamma) - \vec{C}) \exp \left( -\frac{1}{2} b_{kk} - A_k(\gamma) + C_k \right), \quad (83)$$

where $\vec{C}$ is a generic vector and the $C_k$ are its components. Thus,

$$\log[\theta(\vec{A}(\gamma) - \vec{C} + \vec{a}_k)] = \log[\theta(\vec{A}(\gamma) - \vec{C})] + 2\pi i L_k, \quad L_k \in \mathbb{Z},$$

$$\log[\theta(\vec{A}(\gamma) - \vec{C} + \vec{b}_k)] = \log[\theta(\vec{A}(\gamma) - \vec{C})] - \frac{1}{2} b_{kk} - A_k(\gamma) + C_k + 2\pi i M_k, \quad M_k \in \mathbb{Z},$$

$$\int_{a_k} d\log[\theta(\vec{A}(\gamma) - \vec{C})] = 2\pi i L_k, \quad (84)$$

$$\int_{b_k} d\log[\theta(\vec{A}(\gamma) - \vec{C})] = -\frac{1}{2} b_{kk} - A_k(\text{starting point of the cycle } b_k) + C_k + 2\pi i M_k,$$

$$d\log[\theta(\vec{A}(\gamma) - \vec{C} + \vec{a}_k)] = d\log[\theta(\vec{A}(\gamma) - \vec{C})],$$

$$d\log[\theta(\vec{A}(\gamma) - \vec{C} + \vec{b}_k)] = d\log[\theta(\vec{A}(\gamma) - \vec{C})] - \omega_k.$$

Consider the $8N$-gon obtained from $\Gamma$ by cutting along the cycles. To do such cutting it is important to have a non-self-intersecting system of cycles starting from the point $E_0$ (see Fig. 7).

We remark that

$$\vec{A}(P_{k,0}) = \vec{0}, \quad \vec{A}(P_{k,1}) = \vec{a}_k,$$

$$\vec{A}(P_{k,2}) = \vec{a}_k + \vec{b}_k, \quad \vec{A}(P_{k,3}) = \vec{b}_k. \quad (85)$$
Figure 7. On the left: the system of cuts on \( \Gamma \) for \( N = 2 \). On the right above: the order of cycles near the point \( E_0 \). On the right below: the standard \( 8N \)-gon for \( N = 2 \). The starting point is also connected with the zeros \( \gamma_1, \ldots, \gamma_{2N} \) of the function \( \theta(\vec{A}(\gamma) - \vec{C}) \). The order of the basic cycles along the \( 8N \)-gon agrees with the order of the cycles in the picture above.

Denote by \( \mathcal{D} = \gamma_1 + \cdots + \gamma_{2N} \) the divisor of zeros of the function \( \theta(\vec{A}(\gamma) - \vec{C}) \) (since \( \vec{C} \) is generic, this function is not identically zero). Integrating the forms
\[ \omega_j \log[\theta(\tilde{A}(\gamma) - \tilde{C})] \text{ either along the paths connecting the point } E_0 \text{ with the divisor points or along the } 8N\text{-gon, we get that} \]

\[
- 2\pi i \tilde{A}_j(\mathcal{D}) = \int_{\partial \Gamma} \omega_j \log[\theta(\tilde{A}(\gamma) - \tilde{C})] \\
= \sum_{k=1}^{2N} \left[ \int_{a_k^+} \omega_j(\gamma) \log[\theta(\tilde{A}(\gamma) - \tilde{C})] + \int_{b_k^+} \omega_j \log[\theta(\tilde{A}(\gamma) - \tilde{C})] \\
- \int_{a_k^-} \omega_j \log[\theta(\tilde{A}(\gamma) - \tilde{C})] - \int_{b_k^-} \omega_j \log[\theta(\tilde{A}(\gamma) - \tilde{C})] \right] \\
= \sum_{k=1}^{2N} \int_{a_k^+} \omega_j \log[\theta(\tilde{A}(\gamma) - \tilde{C})] \\
- \int_{a_k^-} \omega_j \left( \log[\theta(\tilde{A}(\gamma) - \tilde{C})] - \frac{1}{2} b_{kk} - A_k(\gamma) + C_k + 2\pi i M_k \right) \\
- \int_{b_k^-} \omega_j \left( \log[\theta(\tilde{A}(\gamma) - \tilde{C})] + \int_{b_k^-} \omega_j \left[ \log[\theta(\tilde{A}(\gamma) - \tilde{C})] + 2\pi i L_k \right] \right) \\
= \sum_{k=1}^{2N} \left( \int_{a_k^+} \omega_j \left[ \frac{1}{2} b_{kk} + A_k(\gamma) - C_k - 2\pi i M_k \right] + \int_{b_k^-} \omega_j \cdot 2\pi i L_k \right) \\
= 2\pi i \left[ \frac{b_{jj}}{2} - C_j - 2\pi i M_j + \sum_{k=1}^{2N} b_{kj} L_k + \frac{1}{2\pi i} \sum_{k=1}^{2N} \int_{a_k^+} \omega_j A_k(\gamma) \right].
\]

Finally, modulo the periods we have

\[ \tilde{A}_j(\mathcal{D}) = -\frac{b_{jj}}{2} + C_j - \frac{1}{2\pi i} \sum_{k=1}^{2N} \int_{a_k^+} \omega_j A_k(\gamma) = C_j - K_j, \]

where

\[ K_j = -\frac{b_{jj}}{2} - \pi i + \frac{1}{2\pi i} \sum_{1 \leq k \leq 2N, k \neq j} \int_{a_k^+} A_k(\gamma) \omega_j. \tag{86} \]

In addition, if \( j \neq k \), then

\[ \int_{a_k^+} A_j(\gamma) \omega_k = 2\pi i A_j(\text{one of the branch points of the cycle } a_k), \]

and therefore

\[ K_j = -\frac{b_{jj}}{2} - \pi i - \sum_{1 \leq k \leq 2N, k \neq j} \begin{cases} A_j(E_{2k}), & k \leq N, \\ A_j(E_{2(k-N)-1}), & k > N. \end{cases} \tag{87} \]
From a simple direct calculation it follows that

\[
\vec{A}(E_0) = 0, \quad \vec{A}(E_1) = \frac{1}{2}[-\vec{b}_1], \quad \vec{A}(E_2) = \frac{1}{2}[-\vec{b}_1 + \vec{a}_1],
\]

\[
\vec{A}(E_3) = \frac{1}{2}[-\vec{b}_2 + \vec{a}_1], \quad \vec{A}(E_4) = \frac{1}{2}[-\vec{b}_2 + \vec{a}_1 + \vec{a}_2], \quad \ldots,
\]

\[
\vec{A}(E_{2k-1}) = \frac{1}{2} \left[ \sum_{j=1}^{k-1} \vec{a}_j - \vec{b}_k \right], \quad k \leq N,
\]

\[
\vec{A}(E_{2k}) = \frac{1}{2} \left[ \sum_{j=1}^{k} \vec{a}_j - \vec{a}_k \right], \quad k \leq N,
\]

\[
\vec{A}(E_{2k}) = \frac{1}{2} \left[ \sum_{j=1}^{2N-1} \vec{a}_j - \vec{b}_{N+1} \right], \quad \ldots,
\]

\[
\vec{A}(E_{2k-1}) = \frac{1}{2} \left[ \sum_{j=1}^{2N-k+1} \vec{a}_j - \vec{b}_{N+k} \right], \quad k \leq N,
\]

\[
\vec{A}(E_{2k}) = \frac{1}{2} \left[ \sum_{j=1}^{2N-k} \vec{a}_j - \vec{b}_{N+k} \right], \quad k \leq N.
\]

We see that the components \( a_k \) and \( b_k \) of \( \vec{K} \) can be calculated separately: \( \vec{K} = \vec{K}_a + \vec{K}_b \). For \( \vec{K}_b \) we immediately obtain

\[
\vec{K}_b = \frac{1}{2} \sum_{j=1}^{2N} \vec{b}_j.
\]

Let us calculate \( \vec{K}_a \). Modulo \( \vec{b}_k \) we have

\[
\frac{1}{2} = \begin{cases} 
A_j(E_{2j}), & j \leq N, \\
A_j(E_{2(j-N)-1}), & j > N,
\end{cases}
\]

and therefore, modulo \( \vec{b}_k \),

\[
(K_a)_j = -\sum_{k=1}^{2N} \begin{cases} 
A_j(E_{2k}), & k \leq N, \\
A_j(E_{2(k-N)-1}), & k > N,
\end{cases}
\]
and

\[
\vec{K}_a = \pi i \begin{bmatrix}
0 \\
1 \\
0 \\
\vdots \\
1 \\
0 \\
1 \\
\vdots \\
0 \\
1 \\
\end{bmatrix}
\]

if \(N\) is odd, \(\vec{K}_a = \pi i \begin{bmatrix}
0 \\
1 \\
0 \\
\vdots \\
1 \\
0 \\
1 \\
\vdots \\
0 \\
1 \\
\end{bmatrix}
\]

if \(N\) is even.

Finally, we get that

\[
\vec{K} = \sum_{k=1}^{N} [\vec{A}(E_{2k}) + \vec{A}(E_{2k-1})].
\]  

(88)

Here we have used the following argument. Each cycle \(a_k\) consists of three parts: a path \(\mathcal{P}_{k,1}\) connecting \(E_0\) to the point \(E_{2k-1}\) if \(k \leq N\) or to the point \(E_{2(k-N)}\) if \(k > N\), a closed contour \(\mathcal{P}_{k,2}\) going about either the interval \([E_{2k-1}, E_{2k}]\) if \(k \leq N\) or the interval \([E_{2(k-N)}, E_{2(k-N)-1}]\) if \(k > N\), and the path \(\mathcal{P}_{k,1}\) with the opposite orientation.

Assume that \(k \neq j\). Then the function \(A_j(\gamma)\) has the same values when we go forward and back along \(\mathcal{P}_{k,1}\), and hence the corresponding parts of the integral cancel:

\[
\int_{\mathcal{P}_{k,2}} A_j(\gamma) \omega_k(\gamma) = \int_{\mathcal{P}_{k,2}} A_j(BP) \omega_k(\gamma) + \int_{\mathcal{P}_{k,2}} [A_j(\gamma) - A_j(BP)] \omega_k(\gamma),
\]  

(89)

where BP denotes one of the branch points \(E_{2k-1}\) and \(E_{2k}\) (if \(k \leq N\)) or one of the branch points \(E_{2(k-N)}\) and \(E_{2(k-N)-1}\) (if \(k > N\)). The function \(A_j(\gamma) - A_j(BP)\) takes opposite values on the opposite sides of the cycle going around the cut. Thus, the second integral on the right-hand side of (89) is equal to zero and

\[
\int_{a_k} A_j(\gamma) \omega_k(\gamma) = 2\pi i A_j(BP).
\]

We also have

\[
\int_{a_k} A_k(\gamma) \omega_k(\gamma) = \frac{1}{2} \int_{a_k} d(A_k^2(\gamma))^2
\]

\[
= \frac{1}{2} [A_k^2(\text{final point of the path } a_k) - A_k^2(\text{starting point of the path } a_k)]
\]

\[
= \frac{(2\pi i)^2}{2}.
\]
The formula (88) can be interpreted as follows. Assume that a divisor $D = \gamma_1 + \cdots + \gamma_{2N}$ of degree $2N$ is such that each point $\gamma_j$ is located near the point $\lambda_j$. Then the formula (64) is essentially simplified:

$$\vec{A}(D) + \vec{K} = \sum_{j=1}^{N} \int_{E_{2j}}^{\gamma_j} \vec{\omega} + \sum_{j=1}^{N} \int_{E_{2j-1}}^{\gamma_j+N} \vec{\omega}. \quad (90)$$

**Convention.** In the remaining part of our paper we use the following convention: if a divisor point $\gamma_j$ is located close enough to the point $\lambda_j$, then we calculate its Abel transform starting from the corresponding branch point:

$$\vec{A}(\gamma_j) = \vec{A}_{E_{2j}}(\gamma_j), \quad \vec{A}(\gamma_{j+N}) = \vec{A}_{E_{2j-1}}(\gamma_{j+N}), \quad j = 1, \ldots, N; \quad (91)$$

and thus we can redefine the Abel transform of the divisor as

$$\vec{A}(D) = \sum_{j=1}^{N} [\vec{A}_{E_{2j}}(\gamma_j) + \vec{A}_{E_{2j-1}}(\gamma_{j+N})]. \quad (92)$$

The formula (90) means that if in the $\theta$-function formulae the Abel transform of the divisor is calculated using the convention (92), then the vector $\vec{K}$ of Riemann constants is identically zero:

$$\vec{K} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}. \quad (93)$$

This simplification was not used in [52].

**5.4. Periods of some meromorphic differentials.** The formulae for finite-gap solutions include the following meromorphic differentials.

1. A meromorphic differential $\Omega$ of the 3rd kind with zero $\alpha$-periods and first-order poles at the points $\infty_+$ and $\infty_-$:

$$\Omega = -\frac{d\lambda}{\lambda} + O(1) \quad \text{near } \infty_+, \quad \Omega = \frac{d\lambda}{\lambda} + O(1) \quad \text{near } \infty_-, \quad \oint_{a_{2j}} \Omega = 0, \quad j = 1, \ldots, 2N.$$  

From the Riemann bilinear relations (see [99]) it follows that

$$A_j(\infty_-) - A_j(\infty_+) = Z_j = \oint_{b_j} \Omega.$$  

A sufficiently standard calculation (for $N = 1$ the details can be found in [52]) gives us that

$$\vec{A}(\infty_-) = -\vec{A}(\infty_+), \quad (\vec{A}(\infty_+))_j = \frac{\pi i}{2} + i\hat{\phi}_j + O(\varepsilon^2). \quad (94)$$
2. The differential of the multivalued quasi-momentum function $p(\gamma)$ is the meromorphic differential $dp$ of the 2nd kind (quasi-momentum differential) such that

$$
dp = -d\lambda + O(1) \quad \text{near } \infty_+,
$$
$$
dp = d\lambda + O(1) \quad \text{near } \infty_-,
$$

$$
\oint_{a_j} dp = 0, \quad j = 1, \ldots, 2N.
$$

It is convenient to use the Riemann bilinear relations

$$
\oint_{b_j} dp = -\left[ \text{res}_{\infty_+} [p \omega_j] + \text{res}_{\infty_-} [p \omega_j] \right].
$$

Again, a sufficiently standard calculation (for $N = 1$ see [52]) gives us that

$$
\oint_{b_j} dp = 2 \cos b \phi_n + O(\varepsilon^2).
$$

Taking into account that

$$
(\vec{U}_1)_j = -i \oint_{b_j} dp,
$$
we immediately obtain

$$
(\vec{U}_1)_j = -2i \cos \phi_j + O(\varepsilon^2).
$$

Remark 17. If the finite-gap approximation is constructed using some technique preserving periodicity, for example, the isoperiodic deformation from [56], then instead of (95) we have an exact relation

$$
\oint_{b_j} dp = 2 \cos \phi_n.
$$

Otherwise, if we close the gaps corresponding to the stable modes ‘naively’, then the spatial periodicity of the leading-order solution is preserved only up to order $O(\varepsilon^2)$.

3. A meromorphic differential $dq$ of the 2nd kind such that

$$
dq = -d\lambda^2 + O(1) \quad \text{near } \infty_+,
$$
$$
dq = d\lambda^2 + O(1) \quad \text{near } \infty_-,
$$

$$
\oint_{a_j} dq = 0, \quad j = 1, \ldots, 2N.
$$

Again, using the Riemann bilinear relations

$$
\oint_{b_j} dq = -\left[ \text{res}_{\infty_+} [q \omega_j] + \text{res}_{\infty_-} [q \omega_j] \right],
$$
we obtain (for \( N = 1 \) see [52])

\[
\int_{b_j} dq = i \sin(2\tilde{\phi}_j) + O(\varepsilon^2). \tag{99}
\]

Taking into account that

\[
(\vec{U}_2)_j = -2i \int_{b_j} dq,
\]

we immediately obtain

\[
(\vec{U}_2)_j = 2 \sin(2\tilde{\phi}_j) + O(\varepsilon^2). \tag{101}
\]

5.5. Abel transform of the divisor. In this section we calculate the Abel transform of divisor points assuming the normalization (91) up to corrections of order \( O(\varepsilon) \).

First of all, under the given assumption we have

\[
A_k(\tilde{\gamma}_j) = O(\varepsilon), \quad k \neq j,
\]

so we can neglect this part.

Let us calculate \( A_j(\tilde{\gamma}_j) \). Near the point \( \tilde{\lambda}_j \) we have

\[
p(\gamma) = \begin{cases} 
  p(E_{2j}) + \frac{\tilde{\lambda}_j}{\mu_j} \nu(\gamma) + O(\varepsilon^2), & j \leq N, \\
  p(E_{2j-2N-1}) + \frac{\tilde{\lambda}_j}{\mu_j} \nu(\gamma) + O(\varepsilon^2), & j > N,
\end{cases}
\]

where

\[
\nu^2 = \begin{cases} 
  (\lambda - E_{2j-1})(\lambda - E_{2j}), & j \leq N, \\
  (\lambda - E_{2j-2N-1})(\lambda - E_{2j-2N}), & j > N.
\end{cases}
\]

We also have

\[
\omega_j = \frac{d\lambda}{\nu} + O(\varepsilon) = d \log[\lambda(\gamma) - \tilde{\lambda}_j + \nu(\gamma)] + O(\varepsilon)
\]

and

\[
A_j(\tilde{\gamma}_j) = \log \frac{\lambda(\tilde{\gamma}_j) - \tilde{\lambda}_j + \nu(\tilde{\gamma}_j)}{E_s - \tilde{\lambda}_j} + O(\varepsilon),
\]

where

\[
E_s = \begin{cases} 
  E_{2j}, & j \leq N, \\
  E_{2j-2N-1}, & j > N,
\end{cases}
\]

and up to corrections of order \( O(\varepsilon^2) \)

\[
E_s - \tilde{\lambda}_j = \begin{cases} 
  \frac{\varepsilon}{2\lambda_j} \sqrt{\alpha_j \beta_j}, & j \leq N, \\
  -\frac{\varepsilon}{2\lambda_j} \sqrt{\alpha_j \beta_j}, & j > N.
\end{cases}
\]
Here we assume that
\[ \text{Re} \sqrt{\alpha_j \beta_j} \geq 0, \quad \sqrt{\alpha_j + N \beta_j + N} = \sqrt{\alpha_j \beta_j}. \]
Using the relations (74) and (75), we obtain
\[
\lambda(\tilde\gamma_j) - \tilde\lambda_j + \nu(\tilde\gamma_j) = \begin{cases}
  e^{i\tilde{\phi}_j} \frac{\varepsilon}{2\lambda_j} \tilde{\alpha}_j + O(\varepsilon^2), & j \leq N, \\
  -e^{i\tilde{\phi}_j} \frac{\varepsilon}{2\lambda_j} \tilde{\alpha}_j + O(\varepsilon^2), & j > N,
\end{cases}
\]
and, finally,
\[
A_j(\tilde{\gamma}_j) = \log \frac{\tilde{\alpha}_j}{\tilde{\alpha}_j \tilde{\beta}_j} + i\tilde{\phi}_j + O(\varepsilon).
\]

The finite-gap leading-order solution of the Cauchy problem of anomalous waves is provided by the formula (64), with
\[
\mathcal{G} = \frac{\theta(\tilde{A}(\infty_+) - \tilde{A}(\mathcal{D}) | B)}{\theta(\tilde{A}(\infty_-) - \tilde{A}(\mathcal{D}) | B)} u(0, 0)(1 + O(\varepsilon^2)), \quad \mathcal{U} = O(\varepsilon^2), \quad \nu = 2i + O(\varepsilon^2).
\]
(102)

Considering the convention (92) and using (94), (97), and (101), we obtain the following approximation for the arguments of the \(\theta\)-function:
\[
\tilde{A}(\infty_+) - \tilde{U}_1 x - \tilde{U}_2 t - \tilde{A}(\mathcal{D}) - \tilde{K} = \tilde{z}_- (x, t) + O(\varepsilon),
\]
\[
\tilde{A}(\infty_-) - \tilde{U}_1 x - \tilde{U}_2 t - \tilde{A}(\mathcal{D}) - \tilde{K} = \tilde{z}_+(x, t) + O(\varepsilon),
\]
where
\[
(\tilde{z}_-(x, t))_j = \frac{i\pi}{2} - \log \frac{\tilde{\alpha}_j}{\sqrt{\tilde{\alpha}_j \tilde{\beta}_j}} + 2i \cos(\tilde{\phi}_j)x - 2\sin(2\tilde{\phi}_j)t,
\]
(103)
\[
(\tilde{z}_+(x, t))_j = (\tilde{z}_-(x, t))_j - \pi i - 2i\tilde{\phi}_j,
\]
(104)
and, finally, we obtain the leading-order solution
\[
u(x, t) = \exp(2it) \frac{\theta(\tilde{z}_+(x, t) | B)}{\theta(\tilde{z}_-(x, t) | B)} (1 + O(\varepsilon))
\]
in terms of \(\theta\)-functions of genus \(2N\).

6. The solution of the Cauchy problem in terms of elementary functions

The formula (105) provides the solution up to corrections of order \(O(\varepsilon)\). Therefore, it is enough to sum the exponentials in the \(\theta\)-function formula over the elementary hypercube in \(\mathbb{R}^{2N}\) containing the trajectory point \(-\tilde{w}(t)\) (this can easily be seen from (107)), where \(\tilde{w}(t)\) is defined in (37), and thus we obtain
\[
\theta(\tilde{z}_\pm(x, t) | B) = \tilde{\theta}(\tilde{z}_\pm(x, t) | B)(1 + O(\varepsilon)),
\]
where

$$\bar{\theta}(z_\pm(x, t) \mid B) = \sum_{n_j^{\text{min}}(t) \leq n_j \leq n_j^{\text{max}}(t)} \exp \left( \frac{1}{2} \sum_{l=1}^{2N} \sum_{s=1}^{2N} b_{ls} n_l n_s + \sum_{l=1}^{2N} n_l(z_\pm(x, t)) l \right).$$

Here

$$n_j^{\text{min}}(t) = -\left\lfloor w_j(t) + 1 \right\rfloor, \quad n_j^{\text{max}}(t) = -\left\lfloor w_j(t) \right\rfloor$$

and $[x]$ denotes the largest integer less than or equal to $x$.

If we are interested in constructing the solution up to corrections of order $|\varepsilon|^p$, $0 < p < 1$, then only a subset of vertices of the above hypercube contribute for generic $t$. Thus, the formula (106) admits a further simplification.

To estimate the summands in the $\theta$-function expansion, we use the identity

$$\text{Re} \left( \sum_{l=1}^{2N} \sum_{s=1}^{2N} b_{ls} n_l n_s + 2 \sum_{l=1}^{2N} n_l(z_\pm(x, t)) l \right)$$

$$= \sum_{l=1}^{2N} \sum_{s=1}^{2N} \text{Re}(b_{ls}) n_l n_s + 2 \sum_{l=1}^{2N} n_l \text{Re}(z_\pm(x, t)) l$$

$$= \sum_{l=1}^{2N} \sum_{s=1}^{2N} \text{Re}(b_{ls})(n_l + w_l)(n_s + w_s) - \sum_{l=1}^{2N} \sum_{s=1}^{2N} \text{Re}(b_{ls}) w_l w_s. \quad (107)$$

Consider the metric on $\mathbb{R}^{2N}$ with metric tensor

$$g_{kl} = -\frac{1}{2} \text{Re} b_{kl}.$$ 

We denote by $d_0$ the distance between the trajectory point $-\vec{w}(t)$ and the closest vertex of this hypercube. The relation (107) means that the real parts of the arguments of the exponentials in the $\theta$-function series are, up to a common multiple (the second term on the right-hand side of (107)), equal to minus the distance between $-\vec{w}$ and the corresponding lattice point. Hence, in (106) it is sufficient to keep only the vertices $\vec{n}$ such that the distance between $-\vec{w}$ and $\vec{n}$ is smaller than a critical distance

$$d_{cr} = d_0 + p|\log \varepsilon|,$$

and in (106) we sum only over the corresponding $2^N(t)$-dimensional subcube of the full hypercube, $N(t) \leq 2N(t)$ being the number of indices $j$ such that

$$\tilde{n}_j^{\text{min}}(t) < \tilde{n}_j^{\text{max}}(t),$$

where

$$\tilde{n}_j^{\text{min}}(t) = -\left\lfloor \frac{1}{2} w_j(t) + p \right\rfloor, \quad \tilde{n}_j^{\text{max}}(t) = -\left\lceil \frac{1}{2} w_j(t) - p \right\rceil,$$

and $[x]$ denotes the smallest integer greater than or equal to $x$. It is easy to verify that the number of such indices is always even. Restricting the summation to this
subset of vertices means that we are approximating the NLS solution by exact \( N(t) \)-soliton solutions of Akhmediev type (see the formula (48)) whose parameters are expressed through the Cauchy data in terms of elementary functions.

For numerical simulations, the representation (106) is not very convenient, since it involves ratios of exponentials with big arguments. To avoid this problem, one can use the periodicity properties of the \( \theta \)-functions (66) and shift the arguments to the basic elementary cell:

\[
(\tilde{z}_\pm)_j(x, t) = (\tilde{z}_\pm(x, t))_j - \sum_k b_{jk} [w_k(t)].
\]

In the notation

\[
\tilde{w}_j(t) = w_j(t) - [w_j(t)],
\]

the formula (105) becomes

\[
u(x, t) = \exp(2it + 2i\tilde{\Phi}) \theta(\tilde{z}_+(x, t) | B) \theta(\tilde{z}_-(x, t) | B) (1 + O(\varepsilon)),
\]

where

\[
\tilde{\Phi} = \sum_{j=1}^{2N} \left( \frac{\pi}{2} + \tilde{\phi}_j \right) [w_j(t)].
\]

If the argument \( z \) belongs to the basic elementary cell, then to obtain an \( O(\varepsilon) \) approximation it suffices to sum over the exponentials of the fundamental hypercube:

\[
\tilde{\theta}(z | B) = \sum_{n_j \in \{-1, 0\}} \sum_{j=1}^{2N} \exp \left( \frac{1}{2} \sum_{l=1}^{2N} \sum_{s=1}^{2N} b_{ls} n_l n_s + \sum_{l=1}^{2N} n_l z_l \right).
\]

Equivalently, shifting the hypercube by 1/2 in all directions (\( \tilde{n}_j = 2n_j + 1 \)), we get that

\[
u(x, t) = \exp(2it + 2i\tilde{\Phi}) \frac{\tilde{\theta}(\tilde{z}_+(x, t) | B)}{\tilde{\theta}(\tilde{z}_-(x, t) | B)} (1 + O(\varepsilon)),
\]

where

\[
\tilde{z} = \tilde{z} - \frac{1}{2} \sum_{k=1}^{2N} \tilde{b}_k,
\]

\[
\tilde{z}_\pm = \tilde{z}_\pm - \sum_{k=1}^{2N} \left( [w_k(t)] + \frac{1}{2} \right) \tilde{b}_k,
\]

\[
\tilde{\Phi} = \Phi + \frac{\pi N}{2}.
\]
From now on we assume that our time \( t \) is generic, that is, all the \( w_k(t) \) are non-integers, so that
\[
\lfloor w_{k+N}(t) \rfloor = -\lfloor w_k(t) \rfloor - 1
\]
and we have
\[
\hat{\Phi} = \sum_{j=1}^{N} \left( \lfloor w_j(t) \rfloor + \frac{1}{2} \right) \cdot 2\phi_j. \tag{113}
\]
For non-generic times, the final formulae follow from continuity arguments.

If the number of unstable modes is not too large (see Remark 12), then the off-diagonal elements in \( B \) can be omitted in the above rule for selecting the subset of hypercube vertices providing the main contribution (but of course the off-diagonal elements should be kept in the arguments of the exponentials in (111)). When we are interested in constructing the solutions up to corrections of order \( \varepsilon^p \), \( 0 < p < 1 \), we use the following rule:

1) if
\[
\tilde{w}_j < \frac{1-p}{2} \quad (st_j = 0),
\]
then we keep only the terms with \( \hat{n}_j = 1 \);
2) if
\[
\frac{1-p}{2} \leq \tilde{w}_j \leq \frac{1+p}{2} \quad (st_j = 1),
\]
then we keep the terms with \( \hat{n}_j = 1 \) or \( \hat{n}_j = -1 \);
3) if
\[
\tilde{w}_j > \frac{1+p}{2} \quad (st_j = 2),
\]
then we keep only the terms with \( \hat{n}_j = -1 \).

This rule follows from the estimates
\[
\text{Re} \hat{z}_k > -\frac{bp}{2} + O(1) \quad \text{if} \ st_k = 0
\]
and
\[
\text{Re} \hat{z}_k < \frac{bp}{2} + O(1) \quad \text{if} \ st_k = 2,
\]
where \( b = 2 \log \varepsilon \). This implies that
\[
\frac{\exp(\hat{n}_k\hat{z}_k/2)}{\exp(-\hat{n}_k\hat{z}_k/2)} = O(\varepsilon^p),
\]
and all the other terms in the exponentials in (111) are of higher order.
Then we have the approximation

$$
\tilde{\theta}(\hat{z}|B) = \sum_{\hat{n}_j = 1, \text{if \ } st_j = 0} \sum_{\hat{n}_j \in \{-1,1\}, \text{if \ } st_j = 1} \sum_{\hat{n}_j = -1, \text{if \ } st_j = 2} \exp\left( \sum_{1 \leq l, s \leq 2N} \frac{b_{ls} \hat{n}_l \hat{n}_s}{8} + \sum_{l=1}^{2N} \frac{\hat{n}_l \hat{z}_l}{2} \right) = D_1 \times D_2 \times D_3,
$$

(114)

where

$$
D_1 = \sum_{\hat{n}_j \in \{-1,1\}, \text{if \ } st_j = 1} \exp\left( \sum_{st_l = st_s = 1} \frac{b_{ls} \hat{n}_l \hat{n}_s}{8} + \sum_{st_l = 1, st_s = 1} (-1)^{st_s/2} \frac{b_{ls} \hat{n}_l}{4} + \sum_{st_l = 1} \frac{\hat{n}_l \hat{z}_l}{2} \right),
$$

$$
D_2 = \exp\left( \sum_{st_l \neq 1, st_s \neq 1} (-1)^{(st_l + st_s)/2} \frac{b_{ls}}{8} \right),
$$

$$
D_3 = \exp\left( \sum_{st_l \neq 1} (-1)^{st_l/2} \frac{\hat{z}_l}{2} \right).
$$

We see that

$$
D_1 = \sum_{\hat{n}_j \in \{-1,1\}, \text{if \ } st_j = 1} \exp\left( \sum_{st_l = st_s = 1} \frac{b_{ls} \hat{n}_l \hat{n}_s}{8} + \sum_{st_l = 1} \frac{\hat{n}_l \hat{z}_l}{2} \right),
$$
Figure 9. We compare the absolute values of the numerical solution, the full-hypercube finite-gap approximation (111), and the approximation involving only the relevant vertices. Here $L = 20$ (6 unstable modes), $0 \leq t \leq 30$, $c_1 = 0.5$, $c_{-1} = 0.3 + 0.3i$, $c_2 = 0.5$, $c_{-2} = -0.03 + 0.03i$, $c_3 = 0.3$, $c_{-3} = 0.2 + 0.3i$, $c_4 = 0.3$, $c_{-4} = -0.3 + 0.03i$, $c_5 = 0.3$, $c_{-5} = 0.2 + 0.3i$, $c_6 = 0.3$, $c_{-6} = -0.3 + 0.03i$, $\varepsilon = 10^{-6}$, and $p = 1/2$. The figure above shows the graph of $|u(x, t)|$ obtained using the analytic formula (108), coinciding pixel-to-pixel with numerical simulation. The middle picture shows the absolute value of the difference between the numerical solution and the full-hypercube finite-gap approximation, multiplied by $10^3$. The difference at the left-hand side of the picture, of order $10^{-3}$, is very likely a numerical artifact. The graph below shows the absolute value of the difference between the full-hypercube finite-gap approximation and the approximation involving only the relevant vertices, multiplied by $10^2$; it is of order $O(10^{-2})$, a little higher than $\varepsilon^{1/2}$, but considering that the full hypercube contains $4^6 = 4096$ points, the agreement is sufficiently good (see Remark 12).

where

$$\tilde{z}_l = \tilde{z}_l + \sum_{st, s \neq 1} (-1)^{st}/2 \frac{b_{ls}}{2}.$$  \hspace{1cm} (115)

If we assume that $\tilde{z} = \tilde{z}_-(x, t)$, then the vector $\tilde{z}_l$ defined by (115) coincides with the vector $\tilde{z}_-(x, t)$ in (49), and $\mathcal{Q}_1$ coincides with the function $\tilde{\theta}$ in (49). The
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The factor $Q_2$ does not depend on $\zeta$, and therefore it cancels in the ratio of $\theta$-functions in (110). Finally, the factor $Q_3$ yields the following phase correction through the ratio of $\theta$-functions:

$$
\Delta \Phi = -\frac{1}{2} \sum_{st_j \neq 1} (-1)^{st_j} / 2 \hat{\phi}_j = - \sum_{st_j \neq 1, j \leq N} (-1)^{st_j} / 2 \phi_j,
$$

which gives rise to the final phase

$$
\Phi = \hat{\Phi} + \Delta \Phi,
$$

coinciding with $\Phi(I)$ in the formula (51). Therefore, for each generic $t$ we have summation over the vertices of a hypercube of dimension $2N(t)$, with $N(t) \leq N$.

In our approximation the off-diagonal elements of the matrix $B$ do not depend on $\varepsilon$, and hence in each time interval $I$ the approximation function does not depend on $\varepsilon$, and in each time interval the solution is approximated by an exact NLS solution, the $N(I)$-soliton solution of Akhmediev type (47)–(51) (see Figs. 3, 8, and 9).

Suppose that, in some time interval $I$, only the $j$th unstable mode is visible (for example, in Fig. 8 only the first mode $k_1$ is visible and appears as an isolated peak in the center of the picture). Then in the interval $I$ the pair of variables $\tilde{w}_j$, $\tilde{w}_{j+N}$ are close to $1/2$ and all the other $\tilde{w}_k$ are either close to 0 or to 1. Thus, we have the following approximate formulae for the position $(X_{\text{max}}, T_{\text{max}})$ of the isolated $j$th mode:

$$
T_{\text{max}} = T_{j}^{(1)} + M_j \Delta T_j + \sum_{1 \leq k \leq N} M_k \Delta T(j, k),
$$

$$
X_{\text{max}} = X_{j}^{(1)} + M_j \Delta X_j,
$$

where $T_{j}^{(1)}$ and $X_{j}^{(1)}$ would be the coordinates of the first appearance of the $j$th mode, and $\Delta T_j$ and $\Delta X_j$ would be the recurrence time and the $x$ shift of the $j$th mode if it did not interact with the other modes, and therefore they are defined in (16). The number $M_k$ indicates how many times the mode $k$ was visible before the time $T_{\text{max}}$:

$$
M_k = \begin{cases} 
\text{the nearest integer to } w_k, & k \neq j, \\
[w_j], & k = j,
\end{cases}
$$

and

$$
\Delta T(j, k) = \frac{1}{\sigma_j} \log \left| \frac{\sin(\phi_j + \phi_k)}{\sin(\phi_j - \phi_k)} \right|
$$

is the time delay in the appearance of the $j$th mode due to its pairwise interaction with the $k$th mode, $j \neq k$. We see that the presence of the other modes delays the appearance of the $j$th mode (see (117)), but does not affect the $x$-shift (see (118)). We also see that the interaction of the unstable modes is pairwise.

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Petr G. Grinevich
Landau Institute for Theoretical Physics
of the Russian Academy of Sciences
E-mail: pgg@landau.ac.ru

Paolo Maria Santini
Università di Roma “La Sapienza”, Roma, Italy;
Istituto Nazionale di Fisica Nucleare (INFN),
Roma, Italy
E-mail: paolo.santini@roma1.infn.it