Convergence properties of data augmentation algorithms for high-dimensional robit regression

Sourav Mukherjee ¹, Kshitij Khare¹ and Saptarshi Chakraborty²

¹Department of Statistics, University of Florida
²Department of Biostatistics, State University of New York at Buffalo

e-mail: souravmukherjee@ufl.edu; kdkhare@stat.ufl.edu; chakrab@buffalo.edu

Abstract: The logistic and probit link functions are the most common choices for regression models with a binary response. However, these choices are not robust to the presence of outliers/unexpected observations. The robit link function, which is equal to the inverse CDF of the Student’s t-distribution, provides a robust alternative to the probit and logistic link functions. A multivariate normal prior for the regression coefficients is the standard choice for Bayesian inference in robit regression models. The resulting posterior density is intractable and a Data Augmentation (DA) Markov chain is used to generate approximate samples from the desired posterior distribution. Establishing geometric ergodicity for this DA Markov chain is important as it provides theoretical guarantees for asymptotic validity of MCMC standard errors for desired posterior expectations/quantiles. Previous work [1] established geometric ergodicity of this robit DA Markov chain assuming (i) the sample size \( n \) dominates the number of predictors \( p \), and (ii) an additional constraint which requires the sample size to be bounded above by a fixed constant which depends on the design matrix \( X \). In particular, modern high-dimensional settings where \( n < p \) are not considered. In this work, we show that the robit DA Markov chain is trace-class (i.e., the eigenvalues of the corresponding Markov operator are summable) for arbitrary choices of the sample size \( n \), the number of predictors \( p \), the design matrix \( X \), and the prior mean and variance parameters. The trace-class property implies geometric ergodicity. Moreover, this property allows us to conclude that the sandwich robit chain (obtained by inserting an inexpensive extra step in between the two steps of the DA chain) is strictly better than the robit DA chain in an appropriate sense, and enables the use of recent methods to estimate the spectral gap of trace class DA Markov chains.

MSC2020 subject classifications: Primary 60J05, 60J20.

Keywords and phrases: Markov chain Monte Carlo, geometric ergodicity, High-dimensional binary regression, robust model, trace class.

Contents

1 Introduction .................................................. 2
2 Trace-class property for the DA chain .................................. 5
3 Numerical Illustrations ............................................. 21
   3.1 Low Dimensional (\( n > p \)) Setting: Lupus Data Set ............................. 22
   3.2 High Dimensional (\( n < p \)) Setting: Prostate Data Set ....................... 25
A A Mill’s ratio type result for Student’s t distribution .................. 26
References ......................................................... 29
1. Introduction

Consider a regression setting with \( n \) independent binary responses \( Y_1, Y_2, \ldots, Y_n \) and corresponding predictor vectors \( \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \in \mathbb{R}^p \), such that

\[
P(Y_i = 1 \mid \beta) = F(\mathbf{x}_i^T \beta)
\]

for \( 1 \leq i \leq n \). Here \( F \) is a strictly increasing cumulative distribution function, and \( G = F^{-1} \) is referred to as the link function. Two popular choices of \( F \) are given by \( F(x) = \frac{e^x}{1 + e^x} \) (logistic link) and \( F(x) = \Phi(x) \) (probit link) where \( \Phi(x) \) denotes the standard normal CDF. However, it is well known that estimates of \( \beta \) produced for both these choices are not robust to outliers [4, 9]. To address such settings, \( F \) is set to be the CDF of the Student’s \( t \)-distribution, and the corresponding model is referred to as the robit regression model [6]. For binary responses, an outlier is an unexpected observation with large value(s) of the predictor(s) and a misclassified response, and the robit model effectively down-weights such outliers to produce a better fit [10].

Following [1, 5] we consider a Bayesian robit regression model specified as follows.

\[
P(Y_i = 1 \mid \beta) = F_{\nu}(\mathbf{x}_i^T \beta) \quad \text{for} \quad 1 \leq i \leq n
\]

\[
\beta \sim \mathcal{N}_p(\beta_0, \Sigma^{-1})
\]

where \( F_\nu \) denotes the CDF of the Student’s \( t \)-distribution with \( \nu \) degrees of freedom (with location 0, scale 1) and \( \mathcal{N}_p \) denotes the \( p \) variate normal distribution. Let \( \mathbf{Y} \) denote the response vector, and let \( \pi(\beta \mid \mathbf{y}) \) denote the posterior density of \( \beta \) given \( \mathbf{Y} = \mathbf{y} \). As demonstrated in [1], the posterior density \( \pi(\beta \mid \mathbf{y}) \) is intractable in the sense that relevant posterior expectations are ratios of two intractable integrals and are not available in closed form. Also, generating IID samples from this density is computationally feasible even for moderately large values of \( p \). To resolve this, [1] develops a clever and effective Data Augmentation (DA) approach which can be used to construct a computationally tractable Markov chain which has \( \pi(\beta \mid \mathbf{y}) \) as its stationary density. We describe this Markov chain below.

Let \( t_{\nu}(\mu, \sigma) \) denote the Student’s \( t \)-distribution with \( \nu \) degrees of freedom, location \( \mu \) and scale \( \sigma \). Consider unobserved latent variables \( (Z_1, \lambda_1), (Z_2, \lambda_2), \ldots, (Z_n, \lambda_n) \) which are mutually independent and satisfy \( Z_i \mid \lambda_i \sim \mathcal{N}(\mathbf{x}_i^T \beta, 1/\lambda_i) \) and \( \lambda_i \sim \text{Gamma}(\nu/2, \nu/2) \). Then, it can be shown that the marginal distribution of \( Z_i \) is given by \( Z_i \sim t_{\nu}(\mathbf{x}_i^T \beta, 1) \). If \( Y_i \) is defined as the indicator of \( Z_i \) taking positive values, i.e., \( Y_i = 1 \{Z_i > 0\} \), then \( P(Y_i = 1 \mid \beta) = P(z_i > 0) = F_{\nu}(\mathbf{x}_i^T \beta) \), which is consistent with the robit regression model specified in (1.1). Straightforward calculations (see [1]) now show the following.

- \( (Z_1, \lambda_1), (Z_2, \lambda_2), \ldots, (Z_n, \lambda_n) \) are conditionally independent given \( \beta, \mathbf{Y} = \mathbf{y} \). Also,

\[
Z_i \mid \beta, \mathbf{y} \sim \text{t}_{\nu}(\mathbf{x}_i^T \beta, y_i)
\]

\[
\lambda_i \mid Z_i = z_i, \beta, \mathbf{y} \sim \text{Gamma}
\left(\frac{\nu + 1}{2}, \frac{\nu + (z_i - \mathbf{x}_i^T \beta)^2}{2}\right).
\]

Here \( \text{t}_{\nu}(\mathbf{x}_i^T \beta, y_i) \) denotes the \( t_{\nu} \) distribution with location \( \mathbf{x}_i^T \beta \) and scale 1, truncated to \( \mathbb{R}_+ \) if \( y_i = 1 \) and to \( \mathbb{R}_- \) if \( y_i = 0 \).
Let \( Z = (Z_1, \ldots, Z_n)^T \), \( \lambda = (\lambda_1, \ldots, \lambda_n)^T \) and \( \Lambda \) denote a diagonal matrix whose diagonal is given by entries of \( \lambda \). The conditional distribution of \( \beta \) given \( Z = z, \lambda, Y = y \) is 
\[
N_p \left( (X^T \Lambda X + \Sigma_a)^{-1} (X^T \Lambda z + \Sigma_a \beta_a), (X^T \Lambda X + \Sigma_a)^{-1} \right).
\]

These observations are used in [1] to construct a DA Markov chain \( \{\beta^{(m)}\}_{m \geq 0} \) on \( \mathbb{R}^p \) (with stationary density \( \pi(\beta \mid y) \)) whose one step transition from \( \beta^{(m)} \) to \( \beta^{(m+1)} \) is described in the following Algorithm 1. Harris ergodicity of the robit DA chain is critical for obtaining asymptotically valid standard errors for Markov chain based estimates of posterior quantities.

### Algorithm 1: \((m+1)\)-st Iteration of the Robit Data Augmentation Algorithm

1. Make independent draws from the \( Tt_{\nu}(x_i^T \beta^{(m)}, y_i) \) distributions for \( 1 \leq i \leq n \). Denote the respective draws by \( z_1, z_2, \ldots, z_n \). Draw \( \lambda_i \) from the Gamma \( \left( \frac{\nu+1}{2}, \frac{\nu+(x_i - x_i^T \beta^{(m)})^2}{2} \right) \) distribution.

2. Draw \( \beta^{(m+1)} \) from the \( N_p \left( (X^T \Lambda X + \Sigma_a)^{-1} (X^T \Lambda z + \Sigma_a \beta_a), (X^T \Lambda X + \Sigma_a)^{-1} \right) \) distribution.

Algorithm 1 is established in [1]. Suppose a posterior expectation \( E_{\pi(y \mid y)}[h(\beta)] \) (assumed to exist) is of interest. Then by Harris ergodicity, the cumulative averages \( \frac{1}{m+1} \sum_{r=0}^{m} h(\beta^{(r)}) \) converge to \( E_{\pi(y \mid y)}[h(\beta)] \) as \( m \to \infty \), and can be used to approximate the desired posterior expectation. However, any such approximation is useful only with an estimate of the associated error. The standard approach for obtaining such error estimates is to establish a Markov chain central limit theorem (CLT) which guarantees that
\[
\sqrt{m} \left( \frac{1}{m+1} \sum_{r=0}^{m} h(\beta^{(r)}) - E_{\pi(y \mid y)}[h(\beta)] \right) \xrightarrow{D} \mathcal{N}(0, \sigma_h^2)
\]
as \( m \to \infty \), and then construct a consistent estimate \( \hat{\sigma}_h \) of the asymptotic standard deviation \( \sigma_h \).

A key sufficient condition for establishing a Markov chain CLT is geometric ergodicity. A Markov chain is geometrically ergodic if the total variation distance between its distribution after \( m \) steps and the stationary distribution converges to 0 as \( m \to \infty \). To summarize, establishing geometric ergodicity of the robit DA chain is critical for obtaining asymptotically valid standard errors for Markov chain based estimates of posterior quantities.

With this motivation [1] investigated and established geometric ergodicity of the robit DA chain. However, it is assumed that the design matrix is full rank (which implies \( n \geq p \) and rules out high-dimensional settings), \( \Sigma_a = g^{-1}X^TX \) and that
\[
n \leq \frac{g^{-1} \nu}{(\nu+1)(1 + 2 \sqrt{\beta_a^p X^T X \beta_a})}.
\]
The last upper bound on \( n \) involving the design matrix, the prior mean and covariance and the degrees of freedom \( \nu \) is in particular very restrictive. While we found that these conditions can be relaxed to some extent by a tighter drift and minorization analysis, the resulting constraints still remain quite restrictive. Geometric ergodicity results for the related probit DA chain (see [8]) do not require such assumptions, and quoting from [1, Page 2469] “Ideally, we would like to be able to say that the DA algorithm is geometrically ergodic for any \( n, \nu, y, X, \beta_a, \Sigma_a \)”.
In the probit DA setting (see [5]) the latent variables $Z_i$ have a normal distribution, and there is no need to introduce the additional latent variables $\lambda_i$. This additional layer of latent variables creates additional complexity in the structure of the robit DA chain which makes the convergence analysis significantly more challenging compared to the probit DA chain analyses undertaken in [2, 8].

It is not clear if the restrictive conditions listed above for geometric ergodicity of the robit DA chain are really necessary or if they are an artifact of the standard drift and minorization technique used in [1] for establishing geometric ergodicity. We take a completely different approach, and focus on investigating the trace class property for the robit DA chain. A Markov chain with stationary density $\pi$ is trace class if the corresponding Markov operator on $L^2(\pi)$ has a countable spectrum and the corresponding eigenvalues are summable. The trace class property implies geometric ergodicity, and can be established by showing that an appropriate integral involving the transition density of the Markov chain is finite (see Section 2). As the main technical contribution of this paper, we establish that the robit DA chain is trace class for any $n, \nu > 2, y, X, \beta, \Sigma_a$. This in particular establishes geometric ergodicity for any $n, \nu > 2, y, X, \beta, \Sigma_a$, and significantly generalizes existing results in [1].

The trace class property is much stronger than geometric ergodicity, and the bounding of the relevant integral can get quite involved and challenging (see for example the analysis is [8] and in Section 2). It is therefore not surprising that the conditions needed for establishing geometric ergodicity through the trace class approach have typically been stronger than those needed to establish geometric ergodicity using the drift and minorization approach (for chains where both such analyses have been successful). For example, [8] considers convergence analysis of the probit DA chain with a proper prior for $\beta$ as in (1.1). Using drift and minorization geometric ergodicity is established for all $n, y, X, \beta, \Sigma_a$, but the trace class property was only established under some constraints on $X$ and $\Sigma_a$ (see [8, Theorem 2]). Hence, it is quite interesting that for the robit DA chain the reverse phenomenon holds: the drift and minorization approach needs stronger conditions to succeed than the trace class approach. Essentially, the additional layer of latent variables $\lambda_i$ introduced in the robit setting severely hampers the drift and minorization analysis, but (with careful additional analysis) eases the path for showing the finiteness of the relevant trace class integral.

Establishing the trace class property of the DA chain gives additional benefits on top of geometric ergodicity. Using results from [11], it can now be concluded that the sandwich robit DA chain constructed in [1] is also trace class and is strictly better than the robit DA chain (in the sense that the spectrum of the latter strictly dominates the spectrum of the former). Also, the trace class property is a key sufficient condition for using recent approaches in [12, 13] to estimate the spectral gap of Markov chains. The remainder of the paper is organized as follows. Section 2 contains the proof of the trace class property for the robit DA chain. Section 3 provides numerical illustrations of various chains on two real datasets, one with $n \geq p$, and one with $n < p$. Additional mathematical results needed for the proof of the trace class property are provided in an appendix.
2. Trace-class property for the DA chain

Recall from [1] that the DA Markov chain has associated transition density given by

\[
k(\beta, \beta') = \int_{\mathbb{R}^p} \int_{\mathbb{R}^n} \pi(\beta' | \lambda, z, y) \pi(\lambda | \beta, y) \, dz \, d\lambda
= \int_{\mathbb{R}^p} \int_{\mathbb{R}^n} \pi(\beta' | \lambda, z, y) \pi(\lambda | z, \beta, y) \pi(z | \beta, y) \, dz \, d\lambda
\tag{2.1}
\]

Let \(L^2_0(\pi(.|y))\) denote the space of square-integrable functions with mean zero (with respect to the posterior density \(\pi(.|y)\)). Let \(K\) denote the Markov operator on \(L^2_0(\pi(.|y))\) associated with the transition density \(k\). Note that the Markov transition density \(k\) is reversible with respect to its invariant distribution, and \(K\) is a positive, self-adjoint operator. The operator \(K\) is trace class (see Jörgens [3]) if

\[
I := \int_{\mathbb{R}^p} k(\beta, \beta) \, d\beta < \infty
\tag{2.2}
\]

If the trace-class property holds, then the spectrum of \(K\) is countable and the corresponding eigenvalues are summable. This in particular implies that \(K\) is compact, and the associated Markov chain is geometrically ergodic.

The following theorem shows that the Markov operator \(K\) corresponding to the robit DA chain is trace class under very general conditions.

**Theorem 1.** For \(\nu > 2\), the Markov operator \(K\) corresponding to the DA Markov chain is trace-class for an arbitrary choice of the design matrix \(X\), sample size \(n\), number of predictors \(p\), prior mean vector \(\beta\), and (positive definite) prior precision matrix \(\Sigma\).

**Proof.** We shall show that (2.2) holds for the DA Markov chain. The proof is quite lengthy and involved, and we have tried to make it accessible to the reader by highlighting the major steps/milestones.

We begin by fixing our notations. Let \(I_A(.)\) be the indicator function of the set \(A\) and \(\phi(x; a, b)\) be the univariate normal density evaluated at point \(x\) with mean \(a\) and variance \(b\). Further, let \(\phi_p(x; \mu, \Sigma)\) denote the \(p\)-variate normal density with mean vector \(\mu\), covariance matrix \(\Sigma\), evaluated at a vector \(x \in \mathbb{R}^p\). Finally, let \(q(\omega; a, b) = b^a \omega^{a-1} e^{-b\omega}/\Gamma(a)\) be the gamma density evaluated at \(\omega\) with shape parameter \(a\) and rate parameter \(b\).

Note from Section 2.1 of [1] that the joint posterior density of \((\beta, \lambda, z)\) is given by

\[
\pi(\beta, \lambda, z | y) = \frac{1}{m(y)} \prod_{i=1}^n \left\{ I_{\mathbb{R}^+}(z_i) I_{(1)}(y_i) + I_{\mathbb{R}^-}(z_i) I_{(0)}(y_i) \right\} \phi(z_i; \lambda_i^T \beta, 1/\lambda_i) q\left(\lambda_i; \frac{\nu}{2}, \frac{\nu}{2}\right)
\times \phi_p(\beta; \beta_a, \Sigma_a^{-1})
= \frac{1}{m(y)} \prod_{i=1}^n \left\{ I_{\mathbb{R}^+}(z_i) I_{(1)}(y_i) + I_{\mathbb{R}^-}(z_i) I_{(0)}(y_i) \right\} \times \frac{\sqrt{\lambda_i}}{\sqrt{2\pi}} \exp\left\{ -\frac{\lambda_i}{2} (z_i - x_i^T \beta)^2 \right\}
\]
moreover,

Step I: A useful linear reparametrization to adjust for the prior mean $\beta_a$ and derivation of associated conditionals. Consider the following reparametrization

$$\lambda$$

$$\lambda$$

$$\lambda$$

$$\lambda$$

The absolute value of the Jacobian of this transformation is one, and the joint posterior density of $\left(\tilde{z}, \lambda, \beta\right)$ is given by

$$\pi \left(\tilde{\beta}, \lambda, \tilde{z}|y\right) = \frac{1}{m(y)} \prod_{i=1}^{n} \left\{ I_{R^+} \left( z_i + x_i^T \beta_a \right) I_{\{1\}} \left( y_i \right) + I_{R^-} \left( z_i + x_i^T \beta_a \right) I_{\{0\}} \left( y_i \right) \right\}$$

$$\times \frac{\sqrt{\lambda_i}}{\sqrt{2\pi}} \exp \left\{ -\frac{\lambda_i}{2} \left( z_i - x_i^T \tilde{\beta} \right)^2 \right\} \times \frac{\left( \frac{\nu}{2} \right)^{\frac{\nu}{2}}}{\Gamma \left( \frac{\nu}{2} \right)} \lambda_i^{\nu - 1} \exp \left\{ -\frac{\nu}{2} \lambda_i \right\}$$

$$\times (2\pi)^{-\frac{\nu}{2}} \sqrt{\det \left( \Sigma_a \right)} \exp \left[ -\frac{1}{2} \tilde{\beta}^T \Sigma_a \tilde{\beta} \right]$$

Straightforward calculations using (2.3) show that

$$\tilde{\beta} | \lambda, \tilde{z}, y \sim N_p \left( X^T \Lambda X + \Sigma_a \right)^{-1} X^T \Lambda \tilde{z}, \left( X^T \Lambda X + \Sigma_a \right)^{-1}$$

and

$$\pi \left( \tilde{\beta} | \lambda, \tilde{z}, y \right) = (2\pi)^{-\frac{\nu}{2}} \sqrt{\det \left( X^T \Lambda X + \Sigma_a \right)} \exp \left[ -\frac{1}{2} \left\{ \tilde{\beta}^T \left( X^T \Lambda X + \Sigma_a \right) \tilde{\beta} - 2 \tilde{\beta}^T X^T \Lambda \tilde{z} + \tilde{z}^T \Lambda X \left( X^T \Lambda X + \Sigma_a \right)^{-1} X^T \Lambda \tilde{z} \right\} \right]$$

It is easy to notice from (2.3) that $(\lambda_i, \tilde{z}_i)$’s are conditionally independent given $(\tilde{\beta}, y)$, and moreover,

$$\pi \left( \lambda_i, \tilde{z}_i | \tilde{\beta}, y_i \right) \propto \left\{ I_{R^+} \left( z_i + x_i^T \beta_a \right) I_{\{1\}} \left( y_i \right) + I_{R^-} \left( z_i + x_i^T \beta_a \right) I_{\{0\}} \left( y_i \right) \right\}$$

$$\times \frac{\nu_i + 1}{\nu_i} \lambda_i^{\nu_i - 1} \exp \left[ -\frac{\lambda_i}{2} \left\{ \nu + \left( z_i - x_i^T \tilde{\beta} \right)^2 \right\} \right].$$
Hence, \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are conditionally independent given \((\tilde{z}, \tilde{\beta}, y)\), and

\[
\lambda_i \mid \tilde{z}_i, \tilde{\beta}, y_i \sim \text{Gamma} \left( \frac{\nu + 1}{2}, \frac{\nu + (\tilde{z}_i - x_i^T \tilde{\beta})^2}{2} \right)
\]

for each \( i \in \{1, 2, \ldots, n\} \), which implies that

\[
\pi \left( \lambda \mid \tilde{z}, \tilde{\beta}, y \right) = K_1 \left[ \prod_{i=1}^n \left( \frac{\nu + (\tilde{z}_i - x_i^T \tilde{\beta})^2}{2} \right)^{\frac{\nu}{2}} \right] \times \prod_{i=1}^n \lambda_i^{\frac{\nu - 1}{2}} 
\]

\[
\times \exp \left\{ -\frac{1}{2} \sum_{i=1}^n \lambda_i \left( \nu + (\tilde{z}_i - x_i^T \tilde{\beta})^2 \right) \right\}
\]

\[
= K_1 \left[ \prod_{i=1}^n \left( \nu + (\tilde{z}_i - x_i^T \tilde{\beta})^2 \right)^{\frac{\nu}{2}} \right] \times \prod_{i=1}^n \lambda_i^{\frac{\nu - 1}{2}} \times \exp \left\{ -\frac{1}{2} \left( \tilde{z}^T \Lambda \tilde{z} - 2 \beta^T X^T \Lambda \tilde{z} + \tilde{\beta}^T X^T \Lambda \tilde{\beta} \right) \right\}
\]

(2.6)

where \( K_1 \) and \( K'_1 \) are appropriate constants.

To find \( \pi \left( \tilde{z} \mid \tilde{\beta}, y \right) \), we use (2.5) to get for each \( i \in \{1, 2, \ldots, n\} \),

\[
\pi \left( \tilde{z}_i \mid \tilde{\beta}, y_i \right) = \int_{R^+} \pi \left( \lambda_i, \tilde{z}_i \mid \tilde{\beta}, y_i \right) d\lambda_i
\]

\[
= C_{\tilde{\beta}, y_i} \times \left[ I_{R^+} \left( \tilde{z}_i + x_i^T \beta \right) I_{1}(y_i) + I_{R^-} \left( \tilde{z}_i + x_i^T \beta \right) I_{0}(y_i) \right]
\]

\[
\times \int_{R^+} \lambda_i^{\frac{\nu + 1}{2} - 1} \exp \left\{ -\frac{\lambda_i}{2} \left( \nu + (\tilde{z}_i - x_i^T \beta)^2 \right) \right\} d\lambda_i
\]

[where, \( C_{\tilde{\beta}, y_i} \) is a constant which depends on \( \tilde{\beta}, y_i \)]

\[
= C_{\tilde{\beta}, y_i}' \times \left[ I_{R^+} \left( \tilde{z}_i + x_i^T \beta \right) I_{1}(y_i) + I_{R^-} \left( \tilde{z}_i + x_i^T \beta \right) I_{0}(y_i) \right] \times \frac{\Gamma \left( \frac{\nu + 1}{2} \right)}{\left( \nu + (\tilde{z}_i - x_i^T \beta)^2 \right)^{\frac{\nu + 1}{2}}}
\]

\[
= C_{\tilde{\beta}, y_i}' \times \left[ I_{R^+} \left( \tilde{z}_i + x_i^T \beta \right) I_{1}(y_i) + I_{R^-} \left( \tilde{z}_i + x_i^T \beta \right) I_{0}(y_i) \right] \times \left( 1 + \frac{(\tilde{z}_i - x_i^T \tilde{\beta})^2}{\nu} \right)^{-\frac{\nu + 1}{2}}
\]

(2.7)

where \( C_{\tilde{\beta}, y_i}' \) is the product of all constant terms that are free of \( \tilde{z}_i \). We conclude from (2.3) and (2.7) that conditional on \((\tilde{\beta}, y), \tilde{z}_1, \tilde{z}_2, \ldots, \tilde{z}_n \) are independent with \( \tilde{z}_i \mid \tilde{\beta}, y \) following a truncated \( t \) distribution with location \( x_i^T \tilde{\beta} \), scale 1 and degrees of freedom \( \nu \) that is truncated left at \( -x_i^T \beta \) if \( y_i = 1 \) and truncated right at \( -x_i^T \beta \) if \( y_i = 0 \).
Now, if we denote $t_\nu(\mu,1)$ to be the univariate Student’s $t$ distribution with location $\mu$, scale 1 and degrees of freedom $\nu$, and $F_\nu$ to be the cdf of the $t_\nu(0,1)$ distribution, then for $y_i = 0$,

\[
\pi\left(\tilde{z}_i | \tilde{\beta}, y_i \right) = \frac{K_2 \left( 1 + \frac{(\tilde{z}_i - x_i^T \tilde{\beta})^2}{\nu} \right)^{-\frac{\nu+1}{2}}}{P\left( t_\nu \left( x_i^T \tilde{\beta}, 1 \right) \leq -x_i^T \beta_a \right)} \\
= \frac{K_2 \left( 1 + \frac{(\tilde{z}_i - x_i^T \tilde{\beta})^2}{\nu} \right)^{-\frac{\nu+1}{2}}}{1 - F_\nu \left( x_i^T \tilde{\beta} + \beta_a \right)}
\]

(2.8)

and, for $y_i = 1$,

\[
\pi\left(\tilde{z}_i | \tilde{\beta}, y_i \right) = \frac{K_2 \left( 1 + \frac{(\tilde{z}_i - x_i^T \tilde{\beta})^2}{\nu} \right)^{-\frac{\nu+1}{2}}}{P\left( t_\nu \left( x_i^T \tilde{\beta}, 1 \right) \geq -x_i^T \beta_a \right)} \\
= \frac{K_2 \left( 1 + \frac{(\tilde{z}_i - x_i^T \tilde{\beta})^2}{\nu} \right)^{-\frac{\nu+1}{2}}}{F_\nu \left( x_i^T \tilde{\beta} + \beta_a \right)}
\]

(2.9)

Combining (2.8) and (2.9), we have for any $y_i$,

\[
\pi\left(\tilde{z}_i | \tilde{\beta}, y_i \right) = \frac{K_2 \left( 1 + \frac{(\tilde{z}_i - x_i^T \tilde{\beta})^2}{\nu} \right)^{-\frac{\nu+1}{2}}}{P\left( t_\nu \left( x_i^T \tilde{\beta}, 1 \right) \leq -x_i^T \beta_a \right)} \\
= \frac{K_2 \left( 1 + \frac{(\tilde{z}_i - x_i^T \tilde{\beta})^2}{\nu} \right)^{-\frac{\nu+1}{2}}}{1 - F_\nu \left( x_i^T \tilde{\beta} + \beta_a \right)}
\]

(2.10)

which implies,

\[
\pi\left(\tilde{z} | \tilde{\beta}, y \right) = K_2^n \nu^{\frac{n(\nu+1)}{2}} \prod_{i=1}^{n} \left[ \left( \nu + \left( \tilde{z}_i - x_i^T \tilde{\beta} \right)^2 \right)^{-\frac{\nu+1}{2}} \frac{1}{F_\nu \left( x_i^T \tilde{\beta} + \beta_a \right)} \right]^{y_i} \\
\times \left\{ \frac{1}{1 - F_\nu \left( x_i^T \tilde{\beta} + \beta_a \right)} \right\}^{1-y_i}
\]

(2.10)
Let \( S := \{ i : y_i = 0 \} \). Then, \( S^c = \{ i : y_i = 1 \} \). Using (2.1), (2.4), (2.6) and (2.10), we get the following form for the integral \( I \) in (2.2) under the new parametrization.

\[
I := \int_{\mathbb{R}^p} k(\tilde{\beta}, \beta) \, d\tilde{\beta}
\]

\[
= \int_{\mathbb{R}^p} \int_{\mathbb{R}_+^n} \int_{\mathbb{R}^n} \pi(\tilde{\beta} | \lambda, \bar{z}, y) \, \pi(\lambda | \tilde{z}, \beta, y) \, \pi(\bar{z} | \beta, y) \, d\bar{z} \, d\lambda \, d\tilde{\beta}
\]

\[
= C_0 \int_{\mathbb{R}^p} \int_{\mathbb{R}_+^n} \int_{\mathbb{R}^n} \sqrt{\det (X^T \Lambda X + \Sigma_a)} \times \left[ \prod_{i=1}^n \lambda_i^{\nu_i-1} \right] \times \exp\left[ -\frac{1}{2} \sum_{i=1}^n \lambda_i \right] \times \prod_{i \in S} \left[ \frac{1}{1 - F_{\nu} (x_i^T (\tilde{\beta} + \beta_a))} \right] \times \prod_{i \in S^c} \left[ \frac{1}{F_{\nu} (x_i^T (\beta + \beta_a))} \right] \times \exp\left[ -\frac{1}{2} \left\{ \tilde{\beta}^T (X^T \Lambda X + \Sigma_a) \tilde{\beta} - 2 \tilde{\beta}^T X^T \bar{z} \bar{z}^T \Lambda X^{-1} X^T \Lambda^{-1} \bar{z} \right\} \right] \times \exp\left[ -\frac{1}{2} \left\{ \bar{z}^T \Lambda \bar{z} - 2 \bar{z}^T X^T \Lambda \bar{z} + \Lambda^{-1} \bar{z}^T \Lambda \bar{z} \right\} \right] \, d\bar{z} \, d\lambda \, d\tilde{\beta}
\]

(2.11)

Here, \( C_0 \) denotes the product of all constant terms (independent of \( \tilde{\beta}, \lambda, \) and \( \bar{z} \)) appearing in the conditional densities \( \pi(\tilde{\beta} | \lambda, \bar{z}, y), \pi(\lambda | \tilde{z}, \beta, y), \) and \( \pi(\bar{z} | \beta, y) \).

**Step II: Another reparametrization to adjust for the prior precision matrix \( \Sigma_a \).** Now, let us define \( \theta = \Sigma_a^{1/2} \beta, W = X \Sigma_a^{-1/2}, \) and \( \bar{c} = \Sigma_a^{1/2} \beta_a \). Absolute value of the Jacobian of the transformation \( \tilde{\beta} \rightarrow \theta \) is \( \{ \det (\Sigma_a) \}^{-1/2} > 0 \). Therefore, the right hand side of (2.11), after this further reparametrization becomes

\[
I = C_0 \int_{\mathbb{R}^p} \int_{\mathbb{R}_+^n} \int_{\mathbb{R}^n} \sqrt{\det (W^T \Lambda W + I_p)} \times \left[ \prod_{i=1}^n \lambda_i^{\nu_i-1} \right] \times \exp\left[ -\frac{1}{2} \sum_{i=1}^n \lambda_i \right] \times \prod_{i \in S} \left[ \frac{1}{1 - F_{\nu} (w_i^T (\theta + \bar{c}))} \right] \times \prod_{i \in S^c} \left[ \frac{1}{F_{\nu} (w_i^T (\theta + \bar{c}))} \right] \times \exp\left[ -\frac{1}{2} \left\{ \theta^T (W^T \Lambda W + I_p) \theta - 2 \theta^T W^T \Lambda \bar{z} \right\} \right] \times \exp\left[ -\frac{1}{2} \left\{ \bar{z}^T \Lambda W (W^T \Lambda W + I_p)^{-1} W^T \Lambda \bar{z} \right\} \right] \times \exp\left[ -\frac{1}{2} \left\{ \bar{z}^T \Lambda \bar{z} - 2 \bar{z}^T W^T \Lambda \bar{z} + \theta^T W^T \Lambda \bar{z} \right\} \right] \, d\bar{z} \, d\lambda \, d\theta
\]

\[
= C_0 \int_{\mathbb{R}^p} \int_{\mathbb{R}_+^n} \sqrt{\det (W^T \Lambda W + I_p)} \times \left[ \prod_{i=1}^n \lambda_i^{\nu_i-1} \right] \times \exp\left[ -\frac{1}{2} \sum_{i=1}^n \lambda_i \right]
\]
Combining (2.12) and (2.15), we get

\[ \times \prod_{i \in S} \left[ \frac{1}{1 - F_{\nu} \left( \mathbf{w}_i^T (\mathbf{\theta} + \mathbf{\tilde{c}}) \right)} \right] \times \prod_{i \in S'} \left[ F_{\nu} \left( \mathbf{w}_i^T (\mathbf{\theta} + \mathbf{\tilde{c}}) \right) \right] \]

\[ \times \exp \left[ - \frac{1}{2} \left\{ \mathbf{\theta}^T \left( 2W^T \Lambda W + I_p \right) \mathbf{\theta} \right\} \right] \]

\[ \times \left( \int_{\mathbb{R}^n} \exp \left[ 2\mathbf{\theta}^T W^T \mathbf{\Lambda} \mathbf{\tilde{z}} - \frac{1}{2} \mathbf{\tilde{z}}^T \left( \Lambda + \Lambda W (W^T \Lambda W + I_p)^{-1} W^T \Lambda \right) \mathbf{\tilde{z}} \right] d\mathbf{\tilde{z}} \right) \]

\[ d\mathbf{\lambda} \ d\mathbf{\theta} \]  

(2.12)

**Step III: An upper bound for the innermost \( \mathbf{\tilde{z}} \) integral in (2.12).** We now derive an upper bound for the innermost integral in (2.12). Note that

\[ \int_{\mathbb{R}^n} \exp \left[ 2\mathbf{\theta}^T W^T \mathbf{\Lambda} \mathbf{\tilde{z}} - \frac{1}{2} \mathbf{\tilde{z}}^T \left( \Lambda + \Lambda W (W^T \Lambda W + I_p)^{-1} W^T \Lambda \right) \mathbf{\tilde{z}} \right] d\mathbf{\tilde{z}} \]

\[ = \exp \left[ \frac{1}{2} 4\mathbf{\theta}^T W^T \Lambda \left( \Lambda + \Lambda W (W^T \Lambda W + I_p)^{-1} W^T \Lambda \right)^{-1} \Lambda \mathbf{\theta} \right] \times (C_1)^{-1} \]

\[ \times \int_{\mathbb{R}^n} C_1 \exp \left[ - \frac{1}{2} (\mathbf{\tilde{z}} - \mathbf{a}_i^*)^T \left( \Lambda + \Lambda W (W^T \Lambda W + I_p)^{-1} W^T \Lambda \right) \left( \mathbf{\tilde{z}} - \mathbf{a}_i^* \right) \right] d\mathbf{\tilde{z}} \]

\[ \text{where, } \mathbf{a}_i^* = 2 \left( \Lambda + \Lambda W (W^T \Lambda W + I_p)^{-1} W^T \Lambda \right)^{-1} \Lambda W \mathbf{\theta} \]

\[ = (C_1)^{-1} \exp \left[ \frac{1}{2} 4\mathbf{\theta}^T W^T \Lambda \left( \Lambda + \Lambda W (W^T \Lambda W + I_p)^{-1} W^T \Lambda \right)^{-1} \Lambda \mathbf{\theta} \right] \]

(2.13)

where, \( C_1 = (2\pi)^{-n/2} \left\{ \det \left( \Lambda + \Lambda W (W^T \Lambda W + I_p)^{-1} W^T \Lambda \right) \right\}^{1/2} \). The last equality follows from the fact that the integrand is a normal density. However,

\[ C_1 = (2\pi)^{-n/2} \left\{ \det \left( \Lambda + \Lambda W (W^T \Lambda W + I_p)^{-1} W^T \Lambda \right) \right\}^{1/2} \]

\[ \geq (2\pi)^{-n/2} \left\{ \det (\Lambda) \right\}^{1/2} \]

\[ = (2\pi)^{-n/2} \left( \prod_{i=1}^{n} \lambda_i \right) \]

(2.14)

From (2.13) and (2.14), we have an upper bound for the inner integral in (2.12) as follows

\[ \int_{\mathbb{R}^n} \exp \left[ 2\mathbf{\theta}^T W^T \mathbf{\Lambda} \mathbf{\tilde{z}} - \frac{1}{2} \mathbf{\tilde{z}}^T \left( \Lambda + \Lambda W (W^T \Lambda W + I_p)^{-1} W^T \Lambda \right) \mathbf{\tilde{z}} \right] d\mathbf{\tilde{z}} \]

\[ \leq (2\pi)^{n/2} \prod_{i=1}^{n} \lambda_i^{-1/2} \exp \left[ \frac{1}{2} 4\mathbf{\theta}^T W^T \Lambda \left( \Lambda + \Lambda W (W^T \Lambda W + I_p)^{-1} W^T \Lambda \right)^{-1} \Lambda \mathbf{\theta} \right] \]

(2.15)

Combining (2.12) and (2.15), we get

\[ I \leq C_0 (2\pi)^{n/2} \int_{\mathbb{R}^p} \int_{\mathbb{R}^n} \sqrt{\det (W^T \Lambda W + I_p)} \times \left[ \prod_{i=1}^{n} \lambda_i^{\nu/2 - 1} \right] \times \exp \left[ -\frac{\nu}{2} \sum_{i=1}^{n} \lambda_i \right] \]
Similarly for $i$.

From (2.18) and (2.19), we have for any $i$ where $C_i$ and, if $w_i > 0$, then by Corollary A.1 in Appendix A we have

$$
\int \sqrt{\det (W^T \Lambda W + I_p)} \times \prod_{i=1}^{n} \frac{1}{\lambda_i^{\nu/2-1}} \times \exp \left[ -\frac{\nu}{2} \sum_{i=1}^{n} \lambda_i \right]
$$

$$
= C_2 \int_{\mathbb{R}^p} \int_{\mathbb{R}^q} \sqrt{\det (W^T \Lambda W + I_p)} \times \prod_{i=1}^{n} \frac{1}{\lambda_i^{\nu/2-1}} \times \exp \left[ -\frac{\nu}{2} \sum_{i=1}^{n} \lambda_i \right]
$$

$$
\times \prod_{i \in S} \left[ \frac{1}{1 - F_{\nu}(w_i^T (\theta + \check{c}))} \right] \times \prod_{i \in S^c} \left[ \frac{1}{F_{\nu}(w_i^T (\theta + \check{c}))} \right]
$$

$$
\times \exp \left[ -\frac{1}{2} \left\{ G(\theta, \lambda) \right\} \right] \, d\lambda \, d\theta
$$

(2.16)

where $C_2 = C_0 (2\pi)^{n/2}$ and

$$
G(\theta, \lambda) = \theta^T (2W^T \Lambda W + I_p) \theta - 4\theta^T W^T \Lambda \left( \Lambda + \Lambda W (W^T \Lambda W + I_p)^{-1} W^T \Lambda \right)^{-1} \Lambda W \theta
$$

$$
= \theta^T \left[ (2W^T \Lambda W + I_p) - 4W^T \Lambda \left( \Lambda + \Lambda W (W^T \Lambda W + I_p)^{-1} W^T \Lambda \right)^{-1} \Lambda W \right] \theta \quad (2.17)
$$

Step IV: An upper bound for the products involving the cdf $F_{\nu}$. We now target the product terms in the integrand involving the t-cdf $F_{\nu}$. Note that for $i \in S$, if $w_i^T (\theta + \check{c}) \leq 0$, then

$$
F_{\nu}(w_i^T (\theta + \check{c})) \leq F_{\nu}(0) = \frac{1}{2} \implies \frac{1}{1 - F_{\nu}(w_i^T (\theta + \check{c}))} \leq 2. \quad (2.18)
$$

and, if $w_i^T (\theta + \check{c}) > 0$, then by Corollary A.1 in Appendix A we have

$$
\frac{1}{1 - F_{\nu}(w_i^T (\theta + \check{c}))} \leq \frac{\left( (w_i^T \theta + w_i^T \check{c})^2 + \nu \right)^{\nu/2}}{\kappa}. \quad (2.19)
$$

From (2.18) and (2.19), we have for any $i \in S$,

$$
\frac{1}{1 - F_{\nu}(w_i^T (\theta + \check{c}))} \leq \max \left\{ 2, \frac{\left( (w_i^T \theta + w_i^T \check{c})^2 + \nu \right)^{\nu/2}}{\kappa} \right\}
$$

$$
\leq \left( 2 + \frac{\left( (w_i^T \theta + w_i^T \check{c})^2 + \nu \right)^{\nu/2}}{\kappa} \right). \quad (2.20)
$$

Similarly for $i \in S^c$, if $w_i^T (\theta + \check{c}) \geq 0$, then

$$
F_{\nu}(w_i^T (\theta + \check{c})) \geq F_{\nu}(0) = \frac{1}{2}
$$
\[ \Rightarrow \frac{1}{F_\nu \left( w_i^T (\theta + \tilde{c}) \right)} \leq 2. \]  \hspace{1cm} (2.21)

and, if \( w_i^T (\theta + \tilde{c}) < 0 \), i.e., \(-w_i^T (\theta + \tilde{c}) > 0\), then by Corollary A.1 in Appendix A we have

\[ \frac{1}{F_\nu \left( w_i^T (\theta + \tilde{c}) \right)} = \frac{1}{1 - F_\nu \left(-w_i^T (\theta + \tilde{c})\right)} \leq \left( \frac{\left(w_i^T \theta + w_i^T \tilde{c}\right)^2 + \nu}{\kappa} \right)^{\frac{\nu}{2}} \]  \hspace{1cm} (2.22)

From (2.21) and (2.22), we have for any \( i \in S^c \),

\[ \frac{1}{F_\nu \left( w_i^T (\theta + \tilde{c}) \right)} \leq \max \left\{ 2, \left( \frac{\left(w_i^T \theta + w_i^T \tilde{c}\right)^2 + \nu}{\kappa} \right)^{\frac{\nu}{2}} \right\} \]

\[ \leq \left( 2 + \left( \frac{\left(w_i^T \theta + w_i^T \tilde{c}\right)^2 + \nu}{\kappa} \right)^{\frac{\nu}{2}} \right) \]  \hspace{1cm} (2.23)

Finally from (2.20) and (2.23), we have

\[ \prod_{i \in S} \left[ \frac{1}{1 - F_\nu \left( w_i^T (\theta + \tilde{c}) \right)} \right] \times \prod_{i \in S^c} \left[ \frac{1}{F_\nu \left( w_i^T (\theta + \tilde{c}) \right)} \right] \leq \prod_{i = 1}^{n} \left( 2 + \left( \frac{\left(w_i^T \theta + w_i^T \tilde{c}\right)^2 + \nu}{\kappa} \right)^{\frac{\nu}{2}} \right) \]  \hspace{1cm} (2.24)

Note that

\[ \left( w_i^T \theta + w_i^T \tilde{c} \right)^2 = (\theta + \tilde{c})^T w_i w_i^T (\theta + \tilde{c}) \]

\[ \leq (\theta + \tilde{c})^T \left( \sum_{i = 1}^{n} w_i w_i^T \right) (\theta + \tilde{c}) \]

\[ = (\theta + \tilde{c})^T W^T W (\theta + \tilde{c}) \]  \hspace{1cm} (2.25)

for every \( 1 \leq i \leq n \). It follows from (2.24), (2.25), and the \( c_\nu \)-inequality that

\[ \prod_{i \in S} \left[ \frac{1}{1 - F_\nu \left( w_i^T (\theta + \tilde{c}) \right)} \right] \times \prod_{i \in S^c} \left[ \frac{1}{F_\nu \left( w_i^T (\theta + \tilde{c}) \right)} \right] \]

\[ \leq \prod_{i = 1}^{n} \left( 2 + \left( \frac{\left((\theta + \tilde{c})^T W^T W (\theta + \tilde{c}) + \nu\right)^{\frac{\nu}{2}}}{\kappa} \right) \right) \]

\[ = \left( 2 + \left( \frac{\left((\theta + \tilde{c})^T W^T W (\theta + \tilde{c}) + \nu\right)^{\frac{\nu}{2}}}{\kappa} \right) \right)^{n} \]

\[ \leq 2^n \left( 2^n + \left( \frac{\left((\theta + \tilde{c})^T W^T W (\theta + \tilde{c}) + \nu\right)^{\frac{\nu n}{2}}}{\kappa^n} \right) \right) \]
\[
\leq 2^n \left[ 2^n + \frac{2^{n+2} \left\{ \left( \begin{bmatrix} \theta + \tilde{c} \end{bmatrix}^T W^T W (\theta + \tilde{c}) \right)^{\frac{\nu}{2}} + \nu^{n+2} \right\}}{\kappa^n} \right]
\]

\[
= C_3 + C_4 \left( \left( \begin{bmatrix} \theta + \tilde{c} \end{bmatrix}^T W^T W (\theta + \tilde{c}) \right)^{\frac{\nu}{2}} \right)
\]

\[
= C_3 + C_6 \left( \left( \begin{bmatrix} \theta + \tilde{c} \end{bmatrix}^T (\theta + \tilde{c}) \right)^{\frac{\nu}{2}} \right),
\]

where \( C_3 = 2^{2n} + \frac{2^{2n} (2\nu)^{n+2}}{\kappa^n} \), \( C_4 = \frac{2^{n+2} (2\nu)^{n+2}}{\kappa^n} \), \( C_5 \) denotes the largest eigenvalue of \( W^T W \), and \( C_6 = C_4 C_5^{\frac{\nu}{2}} \).

Now, from (2.16), (2.26) and Fubini’s theorem, we get

\[
I \leq C_2 \int_{\mathbb{R}^p} \int_{\mathbb{R}_+^n} \sqrt{\det (W^T A W + I_p)} \times \left[ \prod_{i=1}^n \lambda_i^{\frac{\nu}{2}} \right] \times \exp \left[ -\frac{\nu}{2} \sum_{i=1}^n \lambda_i \right]
\times \left[ C_3 + C_6 \left( \left( \begin{bmatrix} \theta + \tilde{c} \end{bmatrix}^T (\theta + \tilde{c}) \right)^{\frac{\nu}{2}} \right) \right]
\times \exp \left[ -\frac{1}{2} \left\{ G(\theta, \lambda) \right\} \right] \ d\lambda \ d\theta
\]

\[
= C_7 \int_{\mathbb{R}_+^n} \int_{\mathbb{R}_+^p} \max \left\{ 1, \lambda_{\max}^{\nu/2} \right\} \sqrt{\det (W^T A W + I_p)} \times \left[ \prod_{i=1}^n \lambda_i^{\frac{\nu}{2}} \right] \times \exp \left[ -\frac{\nu}{2} \sum_{i=1}^n \lambda_i \right]
\times \left[ C_3 + C_6 \left( \left( \begin{bmatrix} \theta + \tilde{c} \end{bmatrix}^T (\theta + \tilde{c}) \right)^{\frac{\nu}{2}} \right) \right]
\times \exp \left[ -\frac{1}{2} \left\{ G(\theta, \lambda) \right\} \right] \ d\theta \ d\lambda
\]

Let \( \lambda_{\max} \) denote the largest diagonal entry of the diagonal matrix \( \Lambda \). Using \( (W^T A W + I_p) \leq \max(1, \lambda_{\max}) (W^T W + I_p) \), it follows that

\[
I \leq C_7 \int_{\mathbb{R}_+^n} \int_{\mathbb{R}_+^p} \max \left\{ 1, \lambda_{\max}^{\nu/2} \right\} \sqrt{\det (W^T W + I_p)} \times \left[ \prod_{i=1}^n \lambda_i^{\frac{\nu}{2}} \right] \times \exp \left[ -\frac{\nu}{2} \sum_{i=1}^n \lambda_i \right]
\times \left[ C_3 + C_6 \left( \left( \begin{bmatrix} \theta + \tilde{c} \end{bmatrix}^T (\theta + \tilde{c}) \right)^{\frac{\nu}{2}} \right) \right]
\times \exp \left[ -\frac{1}{2} \left\{ G(\theta, \lambda) \right\} \right] \ d\theta \ d\lambda
\]

where \( C_7 = C_2 \sqrt{\det (W^T W + I_p)} \) is a constant term free of \( \theta \) and \( \lambda \).
Step V: Showing $G(\theta, \lambda)$ is positive definite quadratic form in $\theta$. In order to show the finiteness of the upper bound for $I$ in (2.27), we will first prove that $G(\theta, \lambda)$ is a positive definite quadratic form in $\theta$ for all $\theta \in \mathbb{R}^p$ and $\lambda \in \mathbb{R}^n_+$. For that, it is enough to show by (2.17) that the matrix

$$\begin{bmatrix} (2W^T \Lambda W + I_p) - 4W^T A \left( \Lambda + \Lambda W (W^T \Lambda W + I_p)^{-1} W^T \Lambda \right)^{-1} A W \end{bmatrix}$$

is a positive definite matrix for all $\lambda \in \mathbb{R}^n_+$. We show this by working out the spectral decomposition of this matrix separately in the low and high-dimensional settings.

**Low-dimensional setting:** When $n \geq p$

$$(2W^T \Lambda W + I_p) - 4W^T A \left( \Lambda + \Lambda W (W^T \Lambda W + I_p)^{-1} W^T \Lambda \right)^{-1} A W$$

$$= (2W^T \Lambda W + I_p) - 4W^T \Lambda^{1/2} \left( I_n + \Lambda^{1/2} W (W^T \Lambda W + I_p)^{-1} W^T \Lambda^{1/2} \right)^{-1} \Lambda^{1/2} W$$

$$= (2A^T A + I_p) - 4A^T \left( I_n + A (A^T A + I_p)^{-1} A^T \right)^{-1} A$$

(2.28)

where $A = \Lambda^{1/2} W$. Now, by the Singular Value Decomposition, $A$ can be written as

$$A_{n \times p} = U_{n \times p} D_{p \times p} V_{p \times p}^T$$

(2.29)

where, $U$ is a semi-orthogonal matrix i.e. $U^T U = I_p$, $V$ is an orthogonal matrix i.e. $V^T V = V V^T = I_p$, and $D$ is a diagonal matrix with singular values of $A$ being the diagonal entries. Since, $U_{n \times p}$ is a semi-orthogonal matrix with full column rank $p$, there exists a matrix $U_0_{n \times (n-p)}$ such that the matrix $\begin{bmatrix} U_{n \times p} & U_0_{n \times (n-p)} \end{bmatrix}$ becomes an orthogonal matrix, i.e.,

$$\begin{bmatrix} U & U_0 \end{bmatrix} \begin{bmatrix} U^T \\ U_0^T \end{bmatrix} = \begin{bmatrix} U^T \\ U_0^T \end{bmatrix} \begin{bmatrix} U & U_0 \end{bmatrix} = I_n$$

which implies that

$$U_0^T U = 0_{(n-p) \times p}, \quad U^T U_0 = 0_{p \times (n-p)}.$$ Using $A = UDV^T$ in (2.28), and standard matrix algebra leveraging the various orthogonality properties discussed above, we get

$$(2W^T \Lambda W + I_p) - 4W^T A \left( \Lambda + \Lambda W (W^T \Lambda W + I_p)^{-1} W^T \Lambda \right)^{-1} A W$$

$$= (2A^T A + I_p) - 4A^T \left( I_n + A (A^T A + I_p)^{-1} A^T \right)^{-1} A$$

$$= 2VD^2 V^T + I_p$$

$$- 4VDU^T \begin{bmatrix} U & U_0 \end{bmatrix} \begin{bmatrix} U^T \\ U_0^T \end{bmatrix}$$

$$+ \begin{bmatrix} U & U_0 \end{bmatrix} UDV^T (VD^2 V^T + I_p)^{-1} VDU^T \begin{bmatrix} U & U_0 \end{bmatrix} \begin{bmatrix} U^T \\ U_0^T \end{bmatrix}^{-1} UDV^T$$
Again, using properties discussed above, we get which implies that
\[ V = \begin{bmatrix} V_1 & V_2 \end{bmatrix}, \]
where, \( V_1 \) is a semi-orthogonal matrix with full column rank \( n \), there exists a matrix \( U_0 \) such that the matrix \( \begin{bmatrix} U_0 & U_0 \end{bmatrix} \) becomes an orthogonal matrix, i.e.,
\[
\begin{bmatrix} U & U_0 \end{bmatrix} \begin{bmatrix} U^T \\ U_0^T \end{bmatrix} = \begin{bmatrix} U^T \\ U_0^T \end{bmatrix} \begin{bmatrix} U & U_0 \end{bmatrix} = I_p
\]
which implies that
\[
U_0^T U = 0_{(p-n) \times n}, \quad U^T U_0 = 0_{n \times (p-n)}.
\]
Again, using \( A = VDU^T \) in (2.28), and standard matrix algebra leveraging the various orthogonality properties discussed above, we get
\[
(2W^T \Lambda W + I_p) - 4W^T \Lambda \left( \Lambda + \Lambda W (W^T \Lambda W + I_p)^{-1} W^T \Lambda \right)^{-1} \Lambda W
\]
\[
= (2A^T A + I_p) - 4A^T \left( I_n + A (A^T A + I_p)^{-1} A^T \right)^{-1} A
\]
\[
= (2UD^2U^T + I_p) - 4UD \left( I_n + DU^T \left( UD^2U^T + I_p \right)^{-1} UD \right)^{-1} DU^T
\]
\[
\begin{align*}
\lambda & \in \mathbb{R}^p + n + \nu + n = U \begin{bmatrix} 0 & 0 \\ I_n^T & U_0^T \end{bmatrix}^{-1} U D \begin{bmatrix} U^T \\ U_0^T \end{bmatrix}^{-1} D U T \\
\Omega (\Lambda) & = (2D^2 + I_n) U^T + U_0 U_0^T - U \left[ 4D^2 (D^2 + I_n) (2D^2 + I_n)^{-1} \right] U T \\
& = U \begin{bmatrix} 2D^2 + I_n \end{bmatrix} U^T + U_0 U_0^T \\
& = U \left[ \frac{I_n}{2D^2 + I_n} \right] U^T + U_0 U_0^T
\end{align*}
\]

Now, if we denote
\[
\Omega (\Lambda) := (2W^T \Lambda W + I_p) - 4W^T \Lambda \left( \Lambda + \Lambda W (W^T \Lambda W + I_p)^{-1} W^T \Lambda \right)^{-1} \Lambda W
\]
then from (2.30) and (2.31), it follows that
\[
\Omega (\Lambda) = \begin{cases} 
V \left[ \frac{I_n}{2D^2 + I_n} \right] V^T & \text{if } n \geq p \\
U \left[ \frac{I_n}{2D^2 + I_n} \right] U^T + U_0 U_0^T & \text{if } n < p
\end{cases}
\]

Then, clearly \( \Omega (\Lambda) \) is a positive definite matrix for both the cases \( n \geq p \) and \( n < p \), and for all \( \Lambda \in \mathbb{R}^n_+ \). This implies that \( G (\theta, \lambda) = \theta^T \Omega (\Lambda) \theta \) is a positive definite quadratic form in \( \theta \) for all \( \theta \in \mathbb{R}^p \) and \( \lambda \in \mathbb{R}^n_+ \). Moreover,
\[
\Sigma (\Lambda) := (2D^2 + I_n) U^T + U_0 U_0^T
\]

Now, from (2.27) we get
\[
I \leq C_7 \int_{\mathbb{R}^n_+} \int_{\mathbb{R}^p} \max \left\{ 1, \frac{\nu}{\nu_\max} \right\} \times \left[ \prod_{i=1}^{\nu} \lambda_i^{\nu} \right] \times \exp \left[ -\frac{\nu}{2} \sum_{i=1}^{\nu} \lambda_i \right] \\
\times \left[ C_3 + C_6 \left( (\theta + \tilde{c})^T (\theta + \tilde{c}) \right)^{\frac{\nu}{2}} \right] \\
\times \exp \left[ -\frac{1}{2} G (\theta, \lambda) \right] \, d\theta \, d\lambda
\]
\[
= C_7 \int_{\mathbb{R}^n_+} \int_{\mathbb{R}^p} \max \left\{ 1, \frac{\nu}{\nu_\max} \right\} \times \left[ \prod_{i=1}^{\nu} \lambda_i^{\nu} \right] \times \exp \left[ -\frac{\nu}{2} \sum_{i=1}^{\nu} \lambda_i \right] \\
\times \left[ C_3 + C_6 \left( (\theta + \tilde{c})^T (\theta + \tilde{c}) \right)^{\frac{\nu}{2}} \right] \\
\times \exp \left[ -\frac{1}{2} \theta^T \Omega (\Lambda) \theta \right] \, d\theta \, d\lambda
\]
Step VI: An upper bound for the inner integral in (2.34). We now derive an upper bound for the inner integral in (2.34) using properties of the multivariate normal distribution. Note that

\[
\int_{\mathbb{R}^p} \left[ C_3 + C_6 \left( (\theta + \bar{c})^T (\theta + \bar{c}) \right)^{\frac{p}{2}} \right] \exp \left[ -\frac{1}{2} \theta^T \Omega (\Lambda) \theta \right] \, d\theta
\]

\[
= (2\pi)^{p/2} \sqrt{\det(\Sigma(\Lambda))} \int_{\mathbb{R}^p} (2\pi)^{-p/2} \det(\Sigma(\Lambda))^{-1/2} \left[ C_3 + C_6 \left( (\theta + \bar{c})^T (\theta + \bar{c}) \right)^{\frac{p}{2}} \right] \exp \left[ -\frac{1}{2} \theta^T \Omega (\Lambda) \theta \right] \, d\theta
\]

\[
= (2\pi)^{p/2} \sqrt{\det(\Sigma(\Lambda))} \times E_{N_p(0, \Sigma(\Lambda))} \left[ C_3 + C_6 \left( (\theta + \bar{c})^T (\theta + \bar{c}) \right)^{\frac{p}{2}} \right]
\]

where, \( N_p(0, \Sigma(\Lambda)) \) stands for multivariate normal distribution with mean vector \( 0 \) and covariance matrix \( \Sigma(\Lambda) \). Observe that

\[
E_{N_p(0, \Sigma(\Lambda))} \left[ C_3 + C_6 \left( (\theta + \bar{c})^T (\theta + \bar{c}) \right)^{\frac{p}{2}} \right]
\]

\[
= C_3 + C_6 \left( \frac{\lambda_{\text{max}}}{\lambda_{\text{max}}} \right) E_{N_p(0, \Sigma(\Lambda))} \left[ (\theta + \bar{c})^T \Omega (\Lambda)^{1/2} \Sigma(\Lambda)^{1/2} (\theta + \bar{c}) \right]^{\frac{p}{2}}
\]

\[
\leq C_3 + C_6 \left( \frac{\lambda_{\text{max}}}{\lambda_{\text{max}}} \right) E_{N_p(0, \Sigma(\Lambda))} \left[ (\theta + \bar{c})^T \Omega (\Lambda)^{1/2} \Sigma(\Lambda)^{1/2} (\theta + \bar{c}) \right]^{\frac{p}{2}}
\]

\[
= C_3 + C_6 \left( \frac{\lambda_{\text{max}}}{\lambda_{\text{max}}} \right) E_{N_p(0, \Sigma(\Lambda))} \left[ (\theta^T \Omega (\Lambda) \theta + 2\bar{c}^T \Omega (\Lambda) \theta + \bar{c}^T \Omega (\Lambda) \bar{c} \right]^{\frac{p}{2}}
\]

(2.36)

since \( \Sigma(\Lambda) \leq \left( \frac{\lambda_{\text{max}}}{\lambda_{\text{max}}} \right) I_p \), and

\[
E_{N_p(0, \Sigma(\Lambda))} \left[ (\theta^T \Omega (\Lambda) \theta + 2\bar{c}^T \Omega (\Lambda) \theta + \bar{c}^T \Omega (\Lambda) \bar{c} \right]^{\frac{p}{2}}
\]

\[
\leq E_{N_p(0, \Sigma(\Lambda))} \left[ (\theta^T \Omega (\Lambda) \theta + 2\bar{c}^T \Omega (\Lambda) \theta + \bar{c}^T \Omega (\Lambda) \bar{c} \right]^{\frac{p}{2}}
\]

\[
\leq 3^{p/2} E_{N_p(0, \Sigma(\Lambda))} \left[ (\theta^T \Omega (\Lambda) \theta \right]^{\frac{p}{2}} + E_{N_p(0, \Sigma(\Lambda))} \left[ (2 |\bar{c}^T \Omega (\Lambda) \theta | \right]^{\frac{p}{2}} + E_{N_p(0, \Sigma(\Lambda))} \left[ (\bar{c}^T \Omega (\Lambda) \bar{c} \right]^{\frac{p}{2}}
\]

\[
\text{[since, for any } a, b, c, n \in \mathbb{R}_+ \cup \{0\}, \ (a + b + c)^n \leq 3^n (a^n + b^n + c^n) \]

\[
= 3^{p/2} E_{N_p(0, \Sigma(\Lambda))} \left[ (\theta^T \Omega (\Lambda) \theta \right]^{\frac{p}{2}} + E_{N_p(0, \Sigma(\Lambda))} \left[ (2 |\bar{c}^T \Omega (\Lambda) \theta | \right]^{\frac{p}{2}} + E_{N_p(0, \Sigma(\Lambda))} \left[ (\bar{c}^T \Omega (\Lambda) \bar{c} \right]^{\frac{p}{2}}
\]

(2.37)
We will show that each term in (2.37) is uniformly bounded in $\lambda$. Note that if $\theta \sim \mathcal{N}_p(0, \Sigma(\Lambda))$, then $\theta^T \Omega(\Lambda) \theta \sim \chi^2_p$. Hence $E_{\mathcal{N}_p(0, \Sigma(\Lambda))} \left[ \theta^T \Omega(\Lambda) \theta \right]^{\frac{nu}{2}}$ is a finite quantity free of $\lambda$.

Again, using the fact that $2 |a^T b| \leq a^T a + b^T b$, and taking $a = \Omega(\Lambda)^{\frac{1}{2}} \tilde{c}$ and $b = \Omega(\Lambda)^{\frac{1}{2}} \theta$, we get

$$2 |\tilde{c}^T \Omega(\Lambda) \theta| \leq \tilde{c}^T \Omega(\Lambda) \tilde{c} + \theta^T \Omega(\Lambda) \theta$$

$$\leq \text{eig}_{\max}(\Omega(\Lambda)) \tilde{c}^T \tilde{c} + \theta^T \Omega(\Lambda) \theta$$

[since, $\Omega(\Lambda) \preceq \text{eig}_{\max}(\Omega(\Lambda)) I_p$]

However, from the expression of $\Omega(\Lambda)$ in (2.32), it is easy to see that $\Omega(\Lambda) \preceq I_p$ and hence $\text{eig}_{\max}(\Omega(\Lambda)) \leq 1$. It follows that

$$E_{\mathcal{N}_p(0, \Sigma(\Lambda))} \left[ (2 |\tilde{c}^T \Omega(\Lambda) \theta|) \right]^{\frac{nu}{2}} \leq 2^{\nu/2} \text{eig}_{\max}(\Omega(\Lambda)) \left( \tilde{c}^T \tilde{c} + \left( \theta^T \Omega(\Lambda) \theta \right) \right) \frac{nu}{2}$$

$$= 2^{\nu/2} \left[ (\tilde{c}^T \tilde{c})^{\frac{nu}{2}} + E_{\mathcal{N}_p(0, \Sigma(\Lambda))} \left[ \left( \theta^T \Omega(\Lambda) \theta \right) \right]^{\frac{nu}{2}} \right]$$

$$= C_9 \quad \text{(say)}$$

where $C_9$ is finite and independent of $\lambda$ based on the observations above. Finally, since all eigenvalues of $\Omega(\Lambda)$ are non-negative and bounded above by 1, it follows that

$$(\tilde{c}^T \Omega(\Lambda) \tilde{c})^{\frac{nu}{2}} \leq (\tilde{c}^T \tilde{c})^{\frac{nu}{2}}$$

Using (2.36) and (2.37), we get

$$E_{\mathcal{N}_p(0, \Sigma(\Lambda))} \left[ C_3 + C_6 \left( (\theta + \tilde{c})^T (\theta + \tilde{c}) \right)^{\frac{nu}{2}} \right]$$

$$\leq C_3 + C_{12} \text{eig}_{\max}^{\nu/2}(\Sigma(\Lambda))$$

$$= C_3 + C_{12} \left( 2d_{\max}^2 + 1 \right)^{\frac{nu}{2}} \quad \text{(by the expression of } \Sigma(\Lambda) \text{ in (2.33))}$$

where $C_{12}$ is an appropriate constant independent of $\lambda$ and $d_{\max}$ is the largest element of the diagonal matrix $D$ in the expression of (2.33). Plugging this in (2.35), we get the following upper bound for the inner integral in (2.34).

$$\int_{\mathbb{R}^p} \left[ C_3 + C_6 \left( (\theta + \tilde{c})^T (\theta + \tilde{c}) \right)^{\frac{nu}{2}} \right] \exp \left[ -\frac{1}{2} \theta^T \Omega(\Lambda) \theta \right] d\theta$$

$$\leq \left( 2\pi \right)^{p/2} \sqrt{\det(\Sigma(\Lambda))} \left[ C_3 + C_{12} \left( 2d_{\max}^2 + 1 \right)^{\frac{nu}{2}} \right].$$

This in turn leads to the following upper bound for the integral $I$ of interest.

$$I \leq C_7 \left( 2\pi \right)^{p/2} \max_{\mathbb{R}_+^n} \left\{ 1, \lambda_{\max}^{p/2} \right\} \times \left[ \prod_{i=1}^{n} \lambda_i^{\nu-1} \right] \times \exp \left[ -\frac{\nu}{2} \sum_{i=1}^{n} \lambda_i \right]$$

$$\times \sqrt{\det(\Sigma(\Lambda))} \left[ C_3 + C_{12} \left( 2d_{\max}^2 + 1 \right)^{\frac{nu}{2}} \right] d\lambda. \quad (2.38)$$
Our final steps will be to bound the (single) integral in (2.38) separately in the low and high-dimensional settings.

**Step VII: An upper bound for the integral in (2.38) in the low-dimensional setting.**

Suppose \( n \geq p \). By the expression of \( \Sigma (\Lambda) \) in (2.33) for \( n \geq p \), we have

\[
\det (\Sigma (\Lambda)) = \det \left( V \left( 2D^2 + I_p \right) V^T \right) = \prod_{i=1}^{p} (2d_i^2 + 1) \leq (2d_{\max}^2 + 1)^p,
\]

(2.39)

where \( \{d_i\}_{i=1}^{p} \) are the singular values of \( A \), and \( d_{\max} \) is the largest singular value of \( A = \Lambda^{1/2}W \) in (2.29). Note that

\[
d_{\max}^2 = \text{eig}_\max \left( AA^T \right) = \text{eig}_\max \left( \Lambda^{1/2}WW^T\Lambda^{1/2} \right).
\]

(2.40)

Since \( WW^T \) is a fixed positive semi-definite matrix, there exists a large positive real number \( C_{13} \)

such that

\[
\Lambda^{1/2}WW^T\Lambda^{1/2} \leq C_{13} \Lambda \leq C_{13} \lambda_{\max} I_n,
\]

where \( \lambda_{\max} \) is the largest element in the matrix \( \Lambda \). It follows from (2.40) that \( d_{\max}^2 \leq C_{13} \lambda_{\max} \).

Using (2.38), (2.39) and the \( c_\nu \)-inequality, we get

\[
I \leq C_7 (2\pi)^{p/2} \int_{\mathbb{R}^n_{\nu}} \max \left\{ 1, \lambda_{\max}^{p/2} \right\} \times \left[ \prod_{i=1}^{n} \lambda_i^{\nu-1} \right] \times \exp \left[ -\frac{\nu}{2} \sum_{i=1}^{n} \lambda_i \right] \times (2d_{\max}^2 + 1)^{p/2} \left[ C_3 + C_{12} \left( 2d_{\max}^2 + 1 \right)^{\nu/2} \right] d\lambda \]
\leq C_7 (2\pi)^{p/2} \int_{\mathbb{R}^n_{\nu}} \max \left\{ 1, \lambda_{\max}^{p/2} \right\} \times \left[ \prod_{i=1}^{n} \lambda_i^{\nu-1} \right] \times \exp \left[ -\frac{\nu}{2} \sum_{i=1}^{n} \lambda_i \right] \times (2C_{13} \lambda_{\max} + 1)^{p/2} \left[ C_3 + C_{12} \left( 2C_{13} \lambda_{\max} + 1 \right)^{\nu/2} \right] d\lambda \]
\leq C_7 (2\pi)^{p/2} \int_{\mathbb{R}^n_{\nu}} \left( 1 + \lambda_{\max}^{p/2} \right) \times \left[ \prod_{i=1}^{n} \lambda_i^{\nu-1} \right] \times \exp \left[ -\frac{\nu}{2} \sum_{i=1}^{n} \lambda_i \right] \times 2^{p/2} \left( C_{14} \lambda_{\max}^{p/2} + 1 \right) \left[ C_3 + C_{12} 2^{\nu/2} \left( C_{14}' \lambda_{\max}^{\nu/2} + 1 \right) \right] d\lambda,
\]

where \( C_{14} = (2C_{13})^{p/2} \) and \( C_{14}' = (2C_{13})^{\nu/2} \). Expanding the product of the polynomial terms in \( \lambda_{\max} \) in the integrand gives

\[
I \leq C_{15} \int_{\mathbb{R}^n_{\nu}} \left( \lambda_{\max}^{p/2} + \lambda_{\max}^{(p+\nu)/2} + \lambda_{\max}^{p/2} + \lambda_{\max}^{\nu/2} + 1 \right) \times \left[ \prod_{i=1}^{n} \lambda_i^{\nu-1} \right] \times \exp \left[ -\frac{\nu}{2} \sum_{i=1}^{n} \lambda_i \right] d\lambda
\]
for an appropriate finite constant $C_{15}$ which does not depend on $\lambda$. Using the fact that $\lambda_{\text{max}} \leq \sum_{j=1}^{n} \lambda_j^r$ for any positive $r$, we get

$$I \leq C_{15} \sum_{j=1}^{n} \int_{\mathbb{R}_{+}^n} \left( \lambda_j^{p/n^*} + \lambda_j^{p/(p+n)/2} + \lambda_j^{p/2} + \lambda_j^{n^*/2} + 1 \right) \times \left[ \prod_{i=1}^{n} \lambda_i^{\frac{r}{2} - 1} \right] \times \exp \left[ -\frac{\nu}{2} \sum_{i=1}^{n} \lambda_i \right] d\lambda$$

$$\leq C_{15} \sum_{j=1}^{n} \left( \int_{\mathbb{R}_{+}} \left( \lambda_j^{p/n^*} + \lambda_j^{p/(p+n)/2} + \lambda_j^{p/2} + \lambda_j^{n^*/2} + 1 \right) \lambda_j^{\frac{r}{2} - 1} \exp \left[ -\frac{\nu}{2} \lambda_j \right] d\lambda_j \right)$$

$$\times \left( \prod_{i=1, i \neq j}^{n} \int_{\mathbb{R}_{+}} \lambda_i^{\frac{r}{2} - 1} \exp \left[ -\frac{\nu}{2} \lambda_i \right] d\lambda_i \right). \quad (2.41)$$

Since all of the terms in the above bound are integrals over unnormalized gamma densities (with strictly positive and finite shape and rate parameters), it follows that $I < \infty$.

**Step VIII: An upper bound for the integral in (2.38) in the high-dimensional setting.**

Suppose $n < p$. By the expression of $\Sigma (\Lambda)$ in (2.33) for $n < p$, we have

$$\det (\Sigma (\Lambda)) = \det (U (2D^{2} + I_{n}) U^{T} + U_{0} U_{0}^{T})$$

$$= \det \left( \begin{array}{cc} (2D^{2} + I_{n}) & 0 \\ 0 & I_{p-n} \end{array} \right)$$

$$= \prod_{i=1}^{n} (2d_i^2 + 1)$$

$$\leq (2d_{\text{max}}^2 + 1)^n. \quad (2.42)$$

By a similar argument as for the $n \geq p$ setting, we get $d_{\text{max}}^2 \leq C'_{13} \lambda_{\text{max}}$ for an appropriate constant $C'_{13}$ not depending on $\lambda$. Using (2.38), (2.42) and the $c_{r}$-inequality, we get

$$I \leq C_{7} (2\pi)^{p/2} \int_{\mathbb{R}_{+}^n} \max \left\{ 1, \lambda_{\text{max}}^{p^*/2} \right\} \times \left[ \prod_{i=1}^{n} \lambda_i^{\frac{r}{2} - 1} \right] \times \exp \left[ -\frac{\nu}{2} \sum_{i=1}^{n} \lambda_i \right]$$

$$\times (2d_{\text{max}}^2 + 1)^{\frac{n}{2}} \left[ C_{3} + C_{12} \left( 2d_{\text{max}}^2 + 1 \right)^{\frac{n}{2}} \right] d\lambda$$

$$\leq C_{7} (2\pi)^{p/2} \int_{\mathbb{R}_{+}^n} \max \left\{ 1, \lambda_{\text{max}}^{p^*/2} \right\} \times \left[ \prod_{i=1}^{n} \lambda_i^{\frac{r}{2} - 1} \right] \times \exp \left[ -\frac{\nu}{2} \sum_{i=1}^{n} \lambda_i \right]$$

$$\times \left( 2C'_{13} \lambda_{\text{max}} + 1 \right)^{\frac{n}{2}} \left[ C_{3} + C_{12} \left( 2C'_{13} \lambda_{\text{max}} + 1 \right)^{\frac{n}{2}} \right] d\lambda$$

$$\leq C_{7} (2\pi)^{p/2} \int_{\mathbb{R}_{+}^n} \left( 1 + \lambda_{\text{max}}^{p^*/2} \right) \times \left[ \prod_{i=1}^{n} \lambda_i^{\frac{r}{2} - 1} \right] \times \exp \left[ -\frac{\nu}{2} \sum_{i=1}^{n} \lambda_i \right]$$

$$\times 2^{n/2} \left( C'_{13} \lambda_{\text{max}}^{p^*/2} + 1 \right) \left[ C_{3} + C_{12} \lambda_{\text{max}}^{p^*/2} \left( 2C'_{13} \lambda_{\text{max}}^{p^*/2} + 1 \right) \right] d\lambda,$$
where \( C_{13}^* = (2C_{13}^t)^{n/2} \) and \( C_{14}^* = (2C_{13}^t)^{nu/2} \) similar to the low-dimensional setting. Expanding the product of the polynomial terms in \( \lambda_{\max} \) in the integrand gives

\[
I \leq C_{15}' \int_{\mathbb{R}^d_+} \left( \lambda_{\max}^{\frac{p+n+nu}{2}} + \lambda_{\max}^{\frac{n+nu}{2}} + \lambda_{\max}^{\frac{n}{2}} + \lambda_{\max}^{(p+nu)/2} + \lambda_{\max}^{nu/2} + \lambda_{\max}^{p/2} + 1 \right) \\
\times \left[ \prod_{i=1}^{n} \lambda_i^{\frac{\nu}{2} - 1} \right] \exp \left[ -\frac{\nu}{2} \sum_{i=1}^{n} \lambda_i \right] d\lambda,
\]

for an appropriate finite constant \( C_{15}' \) which does not depend on \( \lambda \). Using the fact that \( \lambda_{\max}^r \leq \sum_{j=1}^{n} \lambda_j^r \) for any positive \( r \), and similar arguments regarding gamma integrals as in (2.41) for the low-dimensional \( n \geq p \) setting, it follows that \( I < \infty \) in the high-dimensional \( n < p \) setting as well. This establishes the trace-class property of the DA Markov chain.

As discussed in the introduction, the trace-class property established above implies compactness of the Markov operator \( K \), which implies that the corresponding DA Markov chain is geometrically ergodic.

**Corollary 1.** For \( \nu > 2 \), the DA Markov chain with transition density \( k \) (in (2.1)) is geometrically ergodic for an arbitrary choice of the design matrix \( X \), sample size \( n \), number of predictors \( p \), prior mean vector \( \beta_a \), and (positive definite) prior precision matrix \( \Sigma_a \).

### 3. Numerical Illustrations

This section presents numerical illustrations to compare/contrast the convergence properties of the robit DA and some other relevant Markov chains. To examine both the low and high-dimensional settings, we consider two real data sets, viz., the Lupus data \( (n > p) \) from [14] and the prostate cancer data \( (n < p) \) from [15]. We note at the outset that with a view to the main goal/contribution of this paper, viz., theoretical convergence analysis for (robit) Markov chains, our focus in this section centers entirely around exemplification of the said convergences. Prior elicitation for statistical inference are beyond the scope of this section and the paper; we consider the data-driven Zellner’s \( g \)-prior following [8] in the first example, and independent standard normal priors in the second example to run the respective Markov chains.

As noted in the Introduction, a trace-class property ensures guaranteed improvements in the convergence of a DA algorithm by *sandwiching* (see Corollary 2 below). To exemplify/visualize this improvement in the current context we alongside consider a sandwich algorithm obtained by inserting an inexpensive random generation step in between the two steps of Algorithm 1. More specifically, we consider the sandwich algorithm from [1] which inserts a univariate random gamma generation as presented in Algorithm 2. Note that both the original DA algorithm and its sandwiched version share the same target stationary distribution for \( \beta \).
Algorithm 2: (m + 1)-st Iteration of the Robit Sandwich Algorithm

1. Make independent draws from the \( T_t(x^T \beta^{(m)}, y_i) \) distributions for \( 1 \leq i \leq n \). Denote the respective draws by \( z_1, z_2, \cdots, z_n \). Draw \( \lambda_i \) from the Gamma \( \left( \frac{\nu+1}{2}, \frac{\nu+(x_i^T \beta^{(m)})^2}{2} \right) \) distribution.

2. Generate \( h^2 \sim \text{Gamma} \left( \frac{1}{2}, \frac{1}{2} \right) \), where \( Q = \Lambda^{1/2} X (X^T \Lambda X + \Sigma_a)^{-1} X^T \Lambda^{1/2} \), and subsequently define \( z_i = h z_i; 1 \leq i \leq n \).

3. Draw \( \beta^{(m+1)} \) from the \( N_p \left( (X^T \Lambda X + \Sigma_a)^{-1} (X^T \Lambda z^{1/2} + \Sigma_a \beta_a), (X^T \Lambda X + \Sigma_a)^{-1} \right) \) distribution.

The trace class property of the robit DA chain (Theorem 1) along with results in [11] imply that the following properties hold for the sandwich chain.

**Corollary 2.** For \( \nu > 2 \), the sandwich Markov chain described in Algorithm 2 is trace class (and hence geometrically ergodic) for an arbitrary choice of the design matrix \( X \), sample size \( n \), number of predictors \( p \), prior mean vector \( \beta_a \), and (positive definite) prior precision matrix \( \Sigma_a \). Furthermore, if \( (\lambda_i)_{i=0}^{\infty} \) and \( (\lambda_i^*)_{i=0}^{\infty} \) denote the non-increasing sequences of eigenvalues corresponding to the robit DA and sandwich operators respectively, then \( \lambda_i^* \leq \lambda_i \) for every \( i \geq 0 \), with at least one strict inequality.

To facilitate model comparison in each example below we consider two robit models: one with a small degrees of freedom (\( \nu = 3 \)) and one with a large degrees of freedom (\( \nu = 1000 \)). The latter is expected to perform similarly to the probit model, owing to the relationship between a \( t \) distribution with large degrees of freedom and the standard normal distribution, which ensures the corresponding likelihoods to be close. To document this similarity herein we also consider the DA and sandwich Markov chains for the probit model from [8], and note their comparative convergence behaviors vis-a-vis the robit model chains. Collectively we thus consider three models: (a) probit, (b) robit with small \( \nu \) and (c) robit with large \( \nu \), and for each model we consider two Markov chains – the original DA chain and a corresponding sandwich chain.

### 3.1. Low Dimensional (n > p) Setting: Lupus Data Set

The Lupus data set of [14] comprises observations on two antibody molecule predictor variables and a binary outcome variable cataloging occurrences of latent membranous lupus nephritis among \( n = 55 \) patients. Interest lies in regressing the binary outcome on the predictors; in this regression we include an intercept term which effectively makes the number of predictors to be \( p = 3 \). The data set has been previously considered in the context of convergence analyses of the probit model DA and sandwich Markov chains [14, 2, 8]. Here we use it to illustrate convergences of the robit model Markov chains. Following [8] we assign on the regression coefficient vector \( \beta \) the Zellner’s \( g \)-prior \( \beta \sim N_p(0, g(X^T X)^{-1}) \) with two choices of \( g \): (a) \( g = 1000 \) which induces a diffuse prior on \( \beta \), and (b) \( g = 3.49 \) which ensures the trace-class property of the probit DA algorithm [8]. For this data set we thus collectively consider 12 Markov chains from 3 models, 2 priors, and 2 Markov chain types (DA or sandwich). All 12 chains are initiated at the the maximum likelihood
estimates $\beta_0 = -1.778$ (intercept), $\beta_1 = 4.374$, and $\beta_2 = 2.428$ obtained from the probit model. As the DA Markov chains for the Lupus data are known to be slow-mixing [2], we run each chain for $10^6$ iterations, after discarding the first $2 \times 10^6$ iterations as burn-in. We subsequently use the retained realizations from all the 12 Markov chains to compute (a) Markov chain autocorrelations up to lag 50, and (b) running means for each of the two non-intercept regression coefficients $\beta_1$ and $\beta_2$.

![Autocorrelation plots for Markov chains run on the Lupus data set.](image)

The autocorrelations and running means are displayed in Figures 1 and 2 as plot-matrices with the rows corresponding to priors ($g$ priors with $g = 1000$ and $g = 3.49$) and columns corresponding to $\beta$ components. Individual plots in the plot-matrices display as line diagrams autocorrelations ($y$-axis) plotted against lags ($x$-axis) in Figure 1, or running means ($y$-axis) plotted against Markov chain iterations ($x$-axis) in Figure 2. A separate line is drawn for each model/Markov chain type combination (6 lines in total in each plot). The lines are color coded by models with the probit model, the robit model with small $\nu$, and the robit model with large $\nu$ being displayed as red, gray, and blue lines respectively. On the other hand, Markov chain types are displayed via line types: solid and dashed lines are used for DA and sandwich chains respectively.

The following three observations are made from these two plots. First, for all three models
sandwiching appears to aid substantial improvements in convergence and mixing over the original DA algorithm when the underlying prior is vague ($g = 1000$). This is demonstrated by both lowered autocorrelations in Figure 1 (top rows/panels) and stabler running means in Figure 2 (top rows/panels) for the $\beta$-components in the sandwich chains. The improvements are not noticeable when the more informative prior with $g = 3.49$ is used. There, the DA and the sandwich chains display similar convergence properties, and the lines from the different model/Markov chain type combinations all effectively get superimposed at the displayed scale (bottom rows/panels in Figures 1 and 2). Interestingly, whereas the gain in sandwiching over the original DA algorithm is theoretically guaranteed for all choices of $g$ in the robit model (Theorem 1), it has so far been theoretically guaranteed only for $g < 3.5$ in the probit model [8]. Second, the running means from the DA chain are less stable than the sandwich chain; however, they do converge to the same limit within the same model. This is clearly expected since both the DA and sandwich chain have the same stationary distribution for $\beta$. This is particularly well-documented in the upper panels of Figure 2. Third, when the vague prior ($g = 1000$) is used, the running means from the analogous chains for the probit and the robit model with large $\nu$ become nearly identical with increasing it-

Fig 2: Running mean plots for the Markov chains run on the Lupus data set.
eration sizes, but they appear systematically different from the running means for the robit model with small $\nu$. As noted before, this of course points to the relationship between a $t$-distribution with a large degrees of freedom and a standard normal distribution; as expected a robit model with a large $\nu$ gets well approximated by a probit model. When a more informative ($g = 3.45$) prior is used the posteriors become less impacted by the systematic differences in the likelihoods and are more driven by the prior information; a fact well visualized in the lower panels of Figure 2. There, all chains from all models appear to share the same limiting running means at the scale displayed.

3.2. High Dimensional ($n < p$) Setting: Prostate Data Set

In the second example we consider the prostate cancer dataset from [15]. The dataset records gene expressions of 50 normal and 52 prostate tumor samples at 6033 arrays, of which we select the first 150 arrays for our analysis. We are interested in regressing the binary cancer status (normal = 0, tumor = 1) on these selected 150 expression arrays (predictors). Similar to the analysis done in the previous section, we include an intercept term to the regression model to obtain the number of predictors $p = 151$ which is bigger than the total sample size of $n = 102$. We consider three models as before: (a) the probit model, (b) the robit model with a small $\nu$ ( = 3), and (c) the robit model with a large $\nu$ ( = 1000), and in each model assign independent standard normal priors on the components of the regression coefficient vector $\beta$. For each model we then run two Markov chains – the original DA chain, and the corresponding sandwich chain. All 6 chains are initiated at $\beta = 0$ and are run for $10^5$ iterations, after discarding the first $2 \times 10^5$ iterations as burn-in. Subsequently, the (un-normalized) log-likelihood $\text{lik}(\beta)$ and (un-normalized) log-posterior density $\text{lpd}(\beta)$ values are calculated as univariate functions of $\beta$ on the retained realizations of each Markov chain. Here

$$\text{lik}(\beta) = \sum_{i=1}^{n} \left\{ y_i \log F(x_i^T \beta) + (1 - y_i) \log[1 - F(x_i^T \beta)] \right\},$$

and $F$ is the normal/t CDF associated with the probit/robit model. Finally for these computed log-likelihoods and log-posterior densities we calculate the Markov chain autocorrelations up to lag 50 and running means as done in the previous Lupus example. The resulting values are displayed in Figures 3 and 4 respectively. These figures follow the same color and line type conventions as considered in Figures 1 and 2.

The following observations are made from these figures. First, as expected, the sandwich chains (broken lines) are observed to have better convergences, i.e., smaller auto-correlations (Figure 3) and stabler running means (Figure 4) than the original DA chains (solid lines). This is particularly well reflected among the log-posterior density values. Second, when focusing on the original DA chains, autocorrelations observed in both log-likelihood and log-posterior density values for the two robit chains are similar and are on average higher than the probit chain. By contrast all three sandwich chains appear to have similar auto-correlations. Third, for both the DA and sandwich chains, the “limiting” running means for the log-likelihood and log-posterior density values from
the two robit models differ, but the probit model and the robit model with large $\nu$ appear similar. As noted before, this similarity in the likelihoods (and hence the posteriors) between the latter two models are attributable to the similarity between a $t$-distribution with large degrees of freedom and a standard normal distribution.

**Appendix A: A Mill’s ratio type result for Student’s $t$ distribution**

**Lemma A.1.** For $t > 0$ we have

$$
\frac{1}{(1 - F_{\nu}(t))(t^2 + \nu)^{\nu/2}} \leq \frac{\sqrt{t^2 + \nu - m}}{\kappa_m},
$$

where $\kappa_m = \Gamma((\nu + 1)/2)(\nu - m)m^{\nu/2 - 1}/(2\sqrt{\pi}\Gamma(\nu/2))$, $m$ is any arbitrary real number $\in (0, \nu)$ and as before $F_{\nu}(\cdot)$ is the cdf of $t_{\nu}(0,1)$.

**Proof.** Firstly, let us introduce the incomplete beta function ratio, defined by

$$
I_p(a, b) = \frac{1}{B(a, b)} \int_0^p u^{a-1}(1 - u)^{b-1} du, \quad 0 < p < 1 \tag{A.1}
$$

where, $a, b \in \mathbb{R}_+$, and $B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$.

Let, $f_{\nu}(\cdot)$ be the pdf of $t_{\nu}(0,1)$. The cumulative distribution function $F_{\nu}(\cdot)$ can be written in terms of $I$, the incomplete beta function ratio. From Chapter 28 of [7], we have for $t > 0$,

$$
F_{\nu}(t) = \int_{-\infty}^t f_{\nu}(u) du = 1 - \frac{1}{2} I_{x(t)}\left(\frac{\nu}{2}, 1/2\right),
$$
Fig 4: Running mean plots for the Markov chains run on the Prostate data set.

where

\[ x(t) = \frac{\nu}{t^2 + \nu} \]

Now, from the definition (A.1) of the incomplete beta function ratio, it follows that

\[ I_{x(t)} \left( \frac{\nu}{2}, \frac{1}{2} \right) = \frac{1}{B \left( \frac{\nu}{2}, \frac{1}{2} \right)} \int_0^{x(t)} u^{\frac{\nu}{2} - 1} \left( 1 - u \right)^{\frac{1}{2} - 1} du \]

\[ = \frac{\Gamma \left( \frac{\nu+1}{2} \right)}{\Gamma \left( \frac{\nu}{2} \right) \Gamma \left( \frac{1}{2} \right)} \int_0^{x(t)} u^{\frac{\nu}{2} - 1} \left( 1 - u \right)^{\frac{1}{2} - 1} du \]

\[ = \frac{\Gamma \left( \frac{\nu+1}{2} \right)}{\Gamma \left( \frac{\nu}{2} \right) \Gamma \left( \frac{1}{2} \right)} \int_0^{\nu t^2 + \nu} u^{\frac{\nu}{2} - 1} \left( \frac{1}{\sqrt{1-u}} \right) du \]  

\[ \text{(A.2)} \]

Again, if we consider the function \( g(u) = \frac{u^{\nu/2} - 1}{\sqrt{1-u}} \), then since

\[ g'(u) = \frac{\left( \sqrt{1-u} \right) \left( \frac{\nu}{2} - 1 \right) u^{\frac{\nu}{2} - 2} + u^{\frac{\nu}{2} - 1} \frac{1}{2\sqrt{1-u}}}{1-u} > 0 \text{ for all } u \in (0, 1), \text{ as } \nu > 2 \]

which implies \( g(u) \) is a strictly increasing function in \( u \) for \( u \in (0, 1) \). Therefore, for the integral in the right hand side of (A.2), we have

\[ \int_0^{\nu t^2 + \nu} u^{\frac{\nu}{2} - 1} \frac{1}{\sqrt{1-u}} du = \int_0^{m \nu t^2 + \nu} u^{\frac{\nu}{2} - 1} \frac{1}{\sqrt{1-u}} du + \int_{m \nu t^2 + \nu}^{\nu t^2 + \nu} u^{\frac{\nu}{2} - 1} \frac{1}{\sqrt{1-u}} du \]

\[ \text{[where } m \text{ is an arbitrary real number in } (0, \nu)] \]

\[ \geq \int_{m \nu t^2 + \nu}^{\nu t^2 + \nu} u^{\frac{\nu}{2} - 1} \frac{1}{\sqrt{1-u}} du \]
\[
I_x(t) \left( \frac{\nu}{2}, 1 \right) \geq \frac{\Gamma \left( \frac{\nu + 1}{2} \right)}{\sqrt{\pi} \Gamma \left( \frac{\nu}{2} \right)} \frac{m^{\nu/2} - 1}{\sqrt{t^2 + \nu - m}} \times \frac{1}{\sqrt{t^2 + \nu - m}} \times \frac{1}{(t^2 + \nu)^{\frac{\nu - 1}{2}}}
\]

Hence, from (A.2) and (A.3), we have

\[
1 - F_{\nu}(t) = \frac{1}{2} I_x(t) \left( \frac{\nu}{2}, 1 \right)
\]

\[
\Rightarrow \frac{1}{(1 - F_{\nu}(t)) (t^2 + \nu)^{\frac{\nu - 1}{2}}} \leq \frac{\sqrt{t^2 + \nu - m}}{\kappa_m}
\]

where, \( \kappa_m = \Gamma \left( \frac{\nu + 1}{2} \right) \frac{(\nu - m) m^{\nu/2 - 1}}{2\sqrt{\pi} \Gamma \left( \frac{\nu}{2} \right)} \), and \( m \) is any arbitrary real number \( \in (0, \nu) \).

**Corollary A.1.** For \( t > 0 \) we have

\[
\frac{1}{1 - F_{\nu}(t)} \leq \frac{(t^2 + \nu)^{\frac{\nu}{2}}}{\kappa}
\]

where \( \kappa = \Gamma \left( \frac{(\nu + 1)/2}{2} \right) (\nu - 1) / (2\sqrt{\pi} \Gamma (\nu/2)) \) and as before \( F_{\nu}(.) \) is the cdf of \( t_{\nu}(0, 1) \).

**Proof.** From Lemma A.1, we know that for \( t > 0 \) and for any arbitrary real number \( m \in (0, \nu) \)

\[
\frac{1}{1 - F_{\nu}(t)} (t^2 + \nu)^{\frac{\nu - 1}{2}} \leq \frac{\sqrt{t^2 + \nu - m}}{\kappa_m}
\]

Since \( \nu > 2 \), we can take \( m = 1 \) to get the following

\[
\frac{1}{1 - F_{\nu}(t)} (t^2 + \nu)^{\frac{\nu - 1}{2}} \leq \frac{\sqrt{t^2 + \nu - 1}}{\kappa}
\]

\[
\leq \frac{\sqrt{t^2 + \nu}}{\kappa}
\]

\[
\Rightarrow \frac{1}{1 - F_{\nu}(t)} \leq \frac{(t^2 + \nu)^{\frac{\nu}{2}}}{\kappa}
\]

where, \( \kappa = \Gamma \left( \frac{(\nu + 1)/2}{2} \right) (\nu - 1) / (2\sqrt{\pi} \Gamma (\nu/2)) \).
References

[1] Vivekananda Roy. Convergence rates for MCMC algorithms for a robust Bayesian binary regression model. *Electronic Journal of Statistics*, Vol. 6, 2463 – 2485, 2012.

[2] Vivekananda Roy and James P. Hobert. Convergence Rates and Asymptotic Standard Errors for Markov Chain Monte Carlo Algorithms for Bayesian Probit Regression. *Journal of the Royal Statistical Society. Series B (Statistical Methodology)*, Vol. 69(4), 607 – 623, 2007.

[3] K Jörgens. Linear integral operators. Surveys and reference works in mathematics. *Pitman Advanced Pub. Program*, 1982.

[4] Daryl Pregibon. Resistant Fits for Some Commonly Used Logistic Models with Medical Applications. *Biometrics*, Vol. 38(2), 485 – 498, 1982.

[5] James H. Albert and Siddhartha Chib. Bayesian Analysis of Binary and Polychotomous Response Data. *Journal of the American Statistical Association*, Vol. 88(422), 669–679, 1993.

[6] Chuanhai Liu. Robit Regression: A Simple Robust Alternative to Logistic and Probit Regression. *John Wiley & Sons, Ltd*, chapter 21, pages 227-238, 2004.

[7] N.L. Johnson, S. Kotz, N. Balakrishnan. Continuous Univariate Distributions, Volume 2, 2nd Edition. *Wiley*, 1995.

[8] Saptarshi Chakraborty and Kshitij Khare. Convergence properties of Gibbs samplers for Bayesian probit regression with proper priors. *Electronic Journal of Statistics*, Vol. 11, 177-210, 2017.

[9] Andrew Gelman and Jennifer Hill. Data Analysis Using Regression and Multilevel/Hierarchical Models. *Cambridge University Press, Cambridge*, 2007.

[10] Andrew Gelman, Jennifer Hill and Aki Vehtari. Regression and Other Stories. *Cambridge University Press, Cambridge*, 2020.

[11] Khare, K. and Hobert, J. P. A spectral analytic comparison of trace-class data augmentation algorithms and their sandwich variants. *The Annals of Statistics*, Vol. 39, 2585-2606, 2011.

[12] Chakraborty, Saptarshi and Khare, Kshitij. Consistent estimation of the spectrum of trace class Data Augmentation algorithms. *Bernoulli*, Vol. 25, 3832-3863, 2019.

[13] Qian Qin and James P. Hobert and Kshitij Khare. Estimating the spectral gap of a trace-class Markov operator. *Electronic Journal of Statistics*, Vol. 13, 1790 – 1822, 2019.

[14] David A. van Dyk and Xiao-Li Meng. The Art of Data Augmentation. *Journal of Computational and Graphical Statistics*, Vol. 10(1), 1–50, 2001.

[15] Chun, Hyonho and Keleç, Sündüz. Sparse Partial Least Squares Regression for Simultaneous Dimension Reduction and Variable Selection. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, Vol. 72(1), 3–25, 2010.