ON BASE CHANGE OF LOCAL STABILITY IN POSITIVE CHARACTERISTICS

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ABSTRACT. We prove that a pointed one dimensional family of varieties $X \to b \in B$ in positive characteristics is locally stable iff the log pair $(X', X_b')$ arising from its base change to the perfectoid base $b' \in B_{perf}$ is log canonical.

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1. INTRODUCTION

Over an algebraically closed field with characteristic 0, the following notion of local stability is raised in the standard Minimal Model Program [3] [2] [1].

Definition 1.1. [Kollár] Let $k = \bar{k}$ be an algebraically closed field with char $k = 0$, and let $X \to b \in B$ be a pointed one-dimensional family of varieties over $k$, then the family is locally stable iff the log pair $(X, X_b)$ is log canonical.

Remark 1.2. Clearly, a well-defined notion of stability condition should behave well (esp. be invariant) under various base changes. In characteristic 0, the well-behavedness of the above notion of local stability under various base changes is guaranteed by Inversion of Adjunction [3] [2]. Especially, by Inversion of Adjunction, that the log pair $(X, X_b)$ is log canonical is equivalent to that the log special fiber $(X_b, D_b)$ with $D_b$ as the singular locus of $X_b$ is semi log canonical. Since the log special fiber $(X_b, D_b)$ is invariant under various base changes, we know that the notion of local stability is well-behaved (esp. invariant) under various base changes in characteristic 0.

Over an algebraically closed field with positive characteristic, the naive generalization of the above notion hardly works since currently we still don’t have Inversion of Adjunction in positive characteristics—neither do we know whether any kinds of Inversion of Adjunction exist in positive characteristics. Especially, in positive characteristics we even still don’t have resolution of singularities, which renders
much of the techniques of Minimal Model Program unavailable. So we make the following tentative notion of local stability in positive characteristics.

**Definition 1.3.** Let $k = \bar{k}$ be an algebraically closed field with $\text{char } k > 0$, and let $\mathcal{X} \to b \in B$ be a pointed one-dimensional family of varieties over $k$, then the family is locally stable iff for any base change $b' \in B' \to b \in B$, the log pair $(\mathcal{X}', \mathcal{X}'_{b'})$ where $\mathcal{X}' = \mathcal{X} \times_B B'$ is log canonical.

Except the part concerning log canonicity that parallels the zero characteristic case, our notion of local stability in positive characteristics is essentially a statement that the stability condition should be invariant under various base changes. The main technical result of this paper is the following Proposition 1.4.

**Proposition 1.4.** Let $k = \bar{k}$ be an algebraically closed field with $\text{char } k = p > 0$, and let $\mathcal{X} \to b \in B$ be a pointed one-dimensional family of varieties over $k$. Assume that $b' \in B' \to b \in B$ is a degree $p$ cover with Wild Ramification over $b$. Then $(\mathcal{X}', \mathcal{X}'_{b'})$ where $\mathcal{X}' = \mathcal{X} \times_B B'$ is log canonical, if $(\mathcal{X}'', \mathcal{X}'''_{b''})$ is log canonical for any Purely Inseparable base change $b'' \in B'' \to b \in B$ and $\mathcal{X}'' = \mathcal{X} \times_B B''$.

**Remark 1.5.** In our situation here, the only essentially new base changes in positive characteristics compared to the zero characteristic case are Purely Inseparable base changes and Wildly Ramified base changes. In particular, the case of Tamely Ramified base changes is trivial in our situation—the reader can check by a simple and direct computation that if $b' \in B' \to b \in B$ is a cover with Tame Ramification over $b$, then $(\mathcal{X}', \mathcal{X}'_{b'})$ where $\mathcal{X}' = \mathcal{X} \times_B B'$ is log canonical iff $(\mathcal{X}, \mathcal{X}_b)$ is log canonical.

Now consider the perfectoid[5] base $B_{\text{perf}}$ which comes from adding all the $p^n$-th roots of the local parameter $u$ of $b$ to $B$. In particular, for any $n \in \mathbb{N}$, let $B_n$ denote the new base which comes from adding the $p^n$-th root of the local parameter $u$ of $b$ to $B$, then the perfectoid base $B_{\text{perf}}$ is the inverse limit of $\{B_n\}_{n \in \mathbb{N}}$. And $\mathcal{X}' = \mathcal{X} \times_{B_{\text{perf}}} B_{\text{perf}}$ is the inverse limit of $\{\mathcal{X}_n\}_{n \in \mathbb{N}}$, with $\mathcal{X}_n = \mathcal{X} \times_{B_n} B_n$ for each $n \in \mathbb{N}$. Assume the point in $B_{\text{perf}}$ over $b \in B$ is $b'$, and the point in $B_n$ over $b \in B$ is $b_n$ for each $n \in \mathbb{N}$, and we define the log canonicity of the pair $(\mathcal{X}', \mathcal{X}'_{b'})$ through the obvious limiting procedure which depends on the log canonicity of the various pairs $(\mathcal{X}_n, \mathcal{X}_n_{b_n})$ for $n \in \mathbb{N}$, then by Remark 1.5 above, Proposition 1.4 implies the following Theorem 1.6.

**Theorem 1.6.** Let $k = \bar{k}$ be an algebraically closed field with $\text{char } k > 0$, and let $\mathcal{X} \to b \in B$ be a pointed one-dimensional family of varieties over $k$, let $b' \in B_{\text{perf}}$ be the perfectoid base of $b \in B$, then the family $\mathcal{X} \to b \in B$ is locally stable iff the log pair $(\mathcal{X}', \mathcal{X}'_{b'})$ where $\mathcal{X}' = \mathcal{X} \times_B B_{\text{perf}}$ is log canonical.

Theorem 1.6 above means that our notion of local stability in positive characteristics works well if we lift to the perfectoid base $b' \in B_{\text{perf}}$ and require log canonicity there.

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2. Preliminaries

Notation as in the statement of Proposition 1.4, firstly we have the following Lemma 2.1.
Lemma 2.1. Notation as in the statement of Proposition [4.4] for any local parameter \( r \) of \( b \in B \), there is a local parameter \( u \) of \( b' \in B' \) which is unique up to isomorphism such that the cover \( B' \to B \) is given by the following equation

\[
r = u^p + v \cdot u^{p+s} + \{\text{higher order terms in } u\}
\]

where \( v \in k - \{0\} \), \( s \in \mathbb{N} \), \( s > 0 \) and \( \gcd(s, p) = 1 \). \( s \) is unique and we call it the conductor of the \( p \)-th wildly ramified base change.

Proof. By Artin-Schreier theory[4], there are \( a,c \in K(B) \) such that the field extension \( K(B')/K(B) \) is defined by the equation

\[
T^p - a^{p-1}T + c = 0.
\]

Multiply both sides by \( r^lp \) for \( l \gg 0 \) if necessary, we may assume that \( a,c \in \mathcal{O}_{B,b} \). Since the base change is wildly ramified over the point \( b \), we should have \( a,c \in (r)\mathcal{O}_{B,b} \). Let \( a = u_a \cdot r^j, c = u_c \cdot r^j \), with \( u_a, u_c \) as units. Keep doing the transformations \( c \to c/r^p \), \( a \to a/r, T \to T/r \) and \( T \to T + r^j \) if necessary, we may assume that \( 1 \leq j < p \).

If \( j = 1 \), then we can finish the proof by picking \( u \) as a suitable element in \((T)k[[T]] - (T^2)k[[T]]\). Else we can suppose \( N_j = Mp + 1 \) with \( N > 0 \) and \( M > 0 \). Then one has

\[
r = (v \cdot T^N/r^M)^p(1 + \{\text{higher order terms in } T^N/r^M\})
\]

with \( v \in k - \{0\} \) by the Implicit Function Theorem. Then we can pick \( u \) as a suitable element in \((T^N/r^M)k[[T^N/r^M]] - (T^N/r^M)^2k[[T^N/r^M]]\). The non-existence of \( s \) will contradict the separability of the base change, and uniqueness of \( s \) follows immediately from the following formula

\[
K'_B = g^*KB + (s - 1) \cdot b'.
\]

\[\square\]

Notation as in the statement of Proposition [4.4] by Lemma 2.1 we may assume that the cover \( b' \in B' \to b \in B \) is given by the following equation

\[
r = u^p + v \cdot u^{p+s} + \{\text{higher order terms in } u\}
\]

where \( v \in k - \{0\} \), \( s \in \mathbb{N} \), \( s > 0 \) and \( \gcd(s, p) = 1 \). This implies the following Proposition 2.2.

Proposition 2.2. Notation as in the statement of Proposition [4.4] denote \( \mathcal{X}_b \) and \( \mathcal{X}'_b \) respectively by \( \Delta \) and \( \Delta' \), then we have

\[
K'_{\mathcal{X}'} = g^*_{\mathcal{X}}(K_{\mathcal{X}} + \Delta) + s \cdot \Delta'
\]

where \( g_{\mathcal{X}} \) denotes the morphism \( \mathcal{X}' = \mathcal{X} \times_B B' \to \mathcal{X} \).

Proof. From the assumptions we know that \( \mathcal{O}_{\mathcal{X}}/(r) \) is reduced. So at a general point \( x \in \mathcal{X} \) lying over \( b \), we can find \( n - 1 \) functions on \( \mathcal{X}_b \) which, together with \( r \), form a local coordinate system \((r, x_2, ..., x_n)\). Locally \( \mathcal{X}' \) is simply the normalization of the following equation

\[
r = u^p + v \cdot u^{p+s} + \{\text{higher order terms in } u\},
\]

which simply corresponds to a re-parametrization \((r, x_2, ..., x_n) \to (u, x_2, ..., x_n)\) by the Implicit Function Theorem. The conclusion then follows from

\[
(dr/r) \land dx_2 \land dx_3... \land dx_n = u^{s-1} \cdot du \land dx_2... \land dx_n.
\]
Notation as in the statement of Proposition 1.4 now let \( v_E \) be a divisorial valuation of \( K(\mathcal{X}) \) with \( E \subset Y \xrightarrow{f} \mathcal{X} \) a proper bi-rational morphism from a normal \( k \)-Variety \( Y \) to \( \mathcal{X} \), and \( E \) is an irreducible divisor in \( Y \). Then by the assumptions of Proposition 1.4, we have \( K_Y = f^*(K_X + \Delta) + a(E, \mathcal{X}, \Delta) \cdot E \) with \( a(E, \mathcal{X}, \Delta) \geq -1 \). Let \( Y' \) be the normalization of \( Y \times_Y B' \), and let \( E' \in Y' \) be the corresponding divisor in \( Y' \) lying over \( E \). We denote the morphism \( Y' \to Y \) by \( g_Y \), and denote the morphism \( Y' \to \mathcal{X}' \) by \( f' \). Around a general point \( e \in E \subset Y \) and a general point \( e' \in E' \subset Y' \) lying over \( e \), we have the following commutative diagram

\[
\begin{array}{ccc}
e' \in E' \subset Y' & \xrightarrow{f'} & \mathcal{X}'' \xrightarrow{\pi'} & b' \in B' \\
\downarrow g_Y & & & \downarrow g \\
e \in E \subset Y & \xrightarrow{f} & \mathcal{X} \xrightarrow{\pi} & b \in B.
\end{array}
\]

Denote \( \mathcal{X}_b \) and \( \mathcal{X}'_{b'} \) respectively by \( \Delta \) and \( \Delta' \), then we also have the following formula

\[
K_{Y'} = f'^*(K_{\mathcal{X}'} + \Delta') + a(E', \mathcal{X}', \Delta') \cdot E'
\]

\[
= f'^*(g_Y^*(K_X + \Delta) + s \cdot \Delta') + a(E', \mathcal{X}', \Delta') \cdot E'
\]

\[
= g_Y^* f^*(K_X + \Delta) + s \cdot f'^* \Delta' + a(E', \mathcal{X}', \Delta') \cdot E'
\]

together with

\[
K_{Y'} = g_Y^* K_Y + x \cdot E'
\]

\[
= g_Y^* f^*(K_X + \Delta) + a(E, \mathcal{X}, \Delta) \cdot g_Y^* E + x \cdot E'
\]

for some \( x \). We make the following Notation 2.3 about the \( x \) above.

**Notation 2.3.** We denote the ramification index \( x \) of \( E \) along \( E' \), which appears in the formula

\[
K_{Y'} = g_Y^* K_Y + x \cdot E',
\]

as \( x(E, E') \).

Now around \( e \in E \subset Y \) we have the local coordinate system \((x_E, x_2, ..., x_n)\), where \((x_E = 0)\) defines \( E \) locally. And we may assume that

\[
r = x_E^{t_E} \left(f_0(x_2, ..., x_n) + \{\text{higher order terms in } x_E\} \right).
\]

So locally around \( e' \in E' \subset Y' \) lying over \( e \), \( Y' \) will be the normalization of the following Equation 2.4.

**Equation 2.4.**

\[
u^p + v \cdot u^{p+s} + \{\text{higher order terms}\} = x_E^{t_E} \left(f_0(x_2, ..., x_n) \right.
\]

\[+ \{\text{higher order terms in } x_E\}\).

The proof of Proposition 1.4 is a detailed study of the normalization of Equation 2.4.
3. Proof of Proposition 1.4

Notation as in the statement of Proposition 1.4 before computing the normalization of Equation (2.4) we note that applying the divisorial valuation $v_{E'}$ to both sides of Equation (2.4) one gets the following relation

$$p \cdot v_{E'}(u) = t_E \cdot v_{E'}(x_E).$$

Since the base change $b' \in B' \to b \in B$ is of degree $p$ with $p$ as a prime number, $v_{E'}(x_E) = q_{E'}E|_{E'}$ is either 1 or $p$, so we have the following Proposition 3.1.

**Proposition 3.1. (Comparison of Log Discrepancies)** Notation as in the statement of Proposition 1.4 denote $X_0$ and $X'_0$ respectively by $\Delta$ and $\Delta'$, then $a(E', X', \Delta')$ is uniquely determined by $a(E, X, \Delta), v_{E'}(x_E)$ and $x(E, E')$. And one has 2 possible cases

- If $v_{E'}(x_E) = p$, then $v_{E'}(u) = t_E$ and:
  $$a(E', X', \Delta') + 1 = p \cdot (a(E, X, \Delta) + 1) + x(E, E') - st_E - p + 1;$$
- If $v_{E'}(x_E) = 1$, then $p|t_E, v_{E'}(u) = t_E/p$ and:
  $$a(E', X', \Delta') + 1 = (a(E, X, \Delta) + 1) + x(E, E') - st_E.$$

Now we study $Y'$ around $e'$ as follows.

3.1. The Case $\gcd(t_E, p) = 1$. Since $\gcd(t_E, p) = 1$, by Proposition 3.1 we have $v_{E'}(x_E) = p, v_{E'}(u) = t_E$, and we only need to compute $x(E, E')$ (Notation 2.3).

Consider the Following Algorithm

- Step 0: Let $t_E = l_0 \cdot p + r_1, 1 \leq r_1 < p, l_0 \geq 0$. Correspondingly, in the local coordinate system $(u, x_E, x_2, \ldots, x_n)$, we do a blow-up along the Weil divisor defined by the ideal sheaf $(x_E^0, u)$, then we get a finite morphism $Y_0 \to Y \times_B B'$, where $Y_0$ is defined in the local coordinate system $(u_0 = u/x_E^0, v_0 = x_E, x_2, \ldots, x_n)$ by the following equation
  $$a_0^1(1 + v \cdot v_0^{l_0} \cdot u_0^1 + \{\text{higher order terms in } u_0\}) = v_0^1(f_0(x_2, \ldots, x_n)$$
  $$+ \{\text{higher order terms in } x_E\}).$$

- Step 1: $p = l_1 \cdot r_1 + r_2, 1 \leq r_2 < r_1, l_1 > 0$. Correspondingly, in the local coordinate system $(u_0, v_0, \ldots, x_n)$, we do a blow-up along the Weil divisor defined by the ideal sheaf $(u_0^{l_1}, v_0)$, then we get a finite morphism $Y_1 \to Y_0$, where $Y_1$ is defined in the local coordinate system $(u_1 = u_0, v_1 = v_0/u_0^{l_1}, x_1, \ldots, x_n)$ by the following equation
  $$a_1^{l_1^2}(1 + v \cdot v_1^{l_0} \cdot u_1^{l_1 l_0} \cdot u_0^1 + \{\text{higher order terms in } u_1\}) = v_1^1(f_0(x_2, \ldots, x_n)$$
  $$+ \{\text{higher order terms in } x_E\}).$$

- ... 
- Step k: $r_k-1 = l_k \cdot r_k + r_{k+1}, 1 \leq r_{k+1} < r_k, r_{k+1} = 1$. We finally get $Y'$ in the local coordinate system $(u_k, v_k, \ldots, x_n)$, which is defined either by the following equation
  $$a_k^{l_k^2}(1 + v \cdot v_k^{N} \cdot u_k^{M} + \{\text{higher order terms in } u_k\}) = v_k^{r_k-1}(f_0(x_2, \ldots, x_n)$$
  $$+ \{\text{higher order terms in } x_E\})$$

or by the following equation
  $$a_k^{r_k-1}(1 + v \cdot v_k^{N} \cdot u_k^{M} + \{\text{higher order terms in } u_k\}) = v_k(f_0(x_2, \ldots, x_n)$$
In the first case $Y'$ has a local parameter system $(x_{E'} = v_k, x_2, ..., x_n)$, where $(x_{E'} = 0)$ defines $E'$. And in the second case $Y'$ has a local parameter system $(x_{E'} = u_k, x_2, ..., x_n)$ where $(x_{E'} = 0)$ also defines $E'$.

Tracing back the algorithm above, we can see that

\[ x_{E'} = x_E^N \cdot u^M \]

for some positive integer $N$ and $M$, and we have $pN + t_E M = 1$. So we have

\[ x_E = (x_E^N \cdot u^M)^p \left( f_0(x_2, ..., x_n) + \{ \text{higher order terms in } x_E \} \right). \]

**Remark 3.2.** This directly shows that $v_{E'}(x_E) = p$ if $gcd(t_E, p) = 1$.

Namely, we will have

\[ u = x_{E'}^{t_E} \cdot (f'_{0}(x_2, ..., x_n) + \sum_{i \geq 1} f'_i(x_2, ..., x_n) \cdot x_{E'}^i), \]

where $f'_i \in k[[x_2, ..., x_n]]$ for each $i \geq 0$. Inserting this into Equation 2.3 we have

\[ u^p \left( 1 + x_{E'}^{t_Es} \cdot f'_s \right) + \{ \text{higher order terms in } x_{E'} \} = x_{E'}^{t_E} \left( f_0(x_2, ..., x_n) \right. \]

\[ + \left. \{ \text{higher order terms in } x_E \} \right). \]

Now we apply the differential $d$ to both sides of the above equation and then wedge with $dx_2 \wedge dx_3 ... \wedge dx_n$, we get

\[ t_{E's} \cdot u^p \cdot x_{E'}^{t_Es-1} \cdot f'_s \cdot dx_{E'} \wedge dx_2 ... \wedge dx_n = t_E \cdot f_0 \cdot x_{E'}^{t_E-1} \cdot dx_E \wedge dx_2 ... \wedge dx_n. \]

This implies that

\[ K_{Y'} = g_Y^*K_Y + (t_E(p+s) - p(t_E - 1) - 1) \cdot E', \]

and hence we have (Notation 2.3)

\[ x(E, E') = t_E(p+s) + p(t_E - 1) - 1. \]

So by Proposition 3.1 we have

\[ a(E', \mathcal{X'}, \Delta') + 1 = p \cdot (a(E, \mathcal{X}, \Delta) + 1) \geq 0, \]

which means $(\mathcal{X}', \mathcal{X}'_b) = (\mathcal{X}', \Delta')$ is log canonical at the center of $E$.

3.2. The Case $p | t_E$: The Induction. In this case there is an integer $N > 0$ such that $t_E = pN$. Suppose we have the following expansion

\[ r = x_{E'}^{pN} \left( G(x_2, ..., x_n) + H(x_E, x_2, ..., x_n) \right), \]

where $G \in k[[x_2, ..., x_n]]$ and $H \in (x_E, x_2, ..., x_n)k[[x_E, x_2, ..., x_n]]$. Now we make the following Claim 3.3

**Claim 3.3.** In the case $p | t_E$, we can reduce to the following standard situation

\[ r = x_{E'}^{pN} \left( f_0(x_2, ..., x_n) + x_{E'}^{p} f_1(x_2, ..., x_n) + ... + x_{E'}^{pM} f_M(x_2, ..., x_n) \right. \]

\[ + x_{E'}^{pM+s'} f_{M+1}(x_2, ..., x_n) + \{ \text{higher order terms in } x_{E'} \}), \]

where $0 < s' < p$ and the right hand side (RHS) of the above expansion contains a monoid in $k[x_E, x_2, x_3, ..., x_n]$ which does not belong to $k[x_E, x_2', x_3', ..., x_n']$, i.e. there is an integer $i_0$ with $2 \leq i_0 \leq n$ such that $\partial_{i_0} \text{RHS} \neq 0$. 


Proof. We first observe that either $G$ or $H$ in the expansion
\[ r = x_E^{pN} \left( G(x_2, ..., x_n) + H(x_E, x_2, ..., x_n) \right) \]
above does not belong to $k[x_E^p, x_2^p, ..., x_n^p]$. Otherwise we would have
\[ r = \left( x_E^N G'(x_2, ..., x_n) + H'(x_E, x_2, ..., x_n) \right)^p \]
for some $H', G' \in k[x_2, ..., x_n]$. Then let $(r, y_2, ..., y_n)$ be a local coordinate system
around a smooth point, we would get
\[ f^*dr \wedge dy_2 \wedge ... \wedge dy_n = 0, \]
which is impossible since $f$ is a proper bi-rational morphism.

Now assume that $r = x_E^{pN} \left( G(x_2, ..., x_n) + H(x_E, x_2, ..., x_n) \right)$ as expanded above,
cannot be directly expressed in the form as claimed. Since either $G$ or $H$ does not belong to $k[x_E^p, x_2^p, ..., x_n^p]$,
we have the following two possible cases

Case 1—we have the following form of expansion
\[ r = x_E^{pN} \left( f_0(x_2, ..., x_n) + x_E^p f_1(x_2, ..., x_n) + x_E^{2p} f_2(x_2, ..., x_n) + x_E^{pM} f_M(x_2, ..., x_n) \right), \]
where at least one of $f_i$, with $1 \leq i \leq M$, does not belong to $k[x_2^p, ..., x_n^p]$.

Case 2—we have the following form of expansion
\[ r = x_E^{pN} \left( f_0(x_2, ..., x_n) + x_E^p f_1(x_2, ..., x_n) + ... + x_E^{pM} f_M(x_2, ..., x_n) + x_E^{pM+s'} f_{M+1}(x_2, ..., x_n) + \{ \text{higher order terms in } x_E \} \right), \]
where $0 < s' < p$ and the right hand side(RHS) of this expansion belongs to $k[x_E^p, x_2^p, ..., x_n^p]$, i.e. \( \partial_i \text{RHS} = 0 \) for any $i$ with $2 \leq i \leq n$.

In the first case, we can make the purely inseparable base change $r = w^{pN}$, then
do a blow-up along the Weil divisor defined by the ideal sheaf $(u, x_E)$. In the second case, suppose $f_0(x_2, ..., x_n) \neq 0$ (else $t_E$ will be strictly larger), and $f_0(x_2, ..., x_n) = (f_0^p(x_2, ..., x_n))^{pL}$ where $L$ is a positive integer and $f_0^p(x_2, ..., x_n) \in k[x_2, x_3, ..., x_n]$ does not belong to $k[x_2^p, x_3^p, ..., x_n^p]$, we can make the purely inseparable base change $r = w^{pL}$, then do a blow-up along the Weil divisor defined by the ideal sheaf $(u, f_0^p)$.

In both cases the corresponding purely inseparable base change and blow-up will result in an expansion of $r$ which is in the form as claimed.

Now by our assumptions in Proposition 1.4, $(X', X'_b) = (X', \Delta')$ is log canonical for any purely inseparably base change $b' \in B' \to b \in B$ where $X' = X \times_B B'$. So in order to prove Proposition 1.4, in the case $p|t_E$, it suffices to prove Proposition 1.4 for the purely inseparably base-changed families in the above two cases, thus
our claim is verified.

\[ \square \]

By Claim 3.3 above, we may assume
\[ r = x_E^{pN} \left( f_0(x_2, ..., x_n) + x_E^p f_1(x_2, ..., x_n) + ... + x_E^{pM} f_M(x_2, ..., x_n) + x_E^{pM+s'} f_{M+1}(x_2, ..., x_n) + \{ \text{higher order terms in } x_E \} \right), \]
where $f_i$’s on the right hand side(RHS) are possibly zero elements in $k[[x_2, ..., x_n]]$, $f_{M+1}$ is non-zero in $k[[x_2, ..., x_n]]$, $0 < s < p$, and there is an integer $i_0$ with $2 \leq i_0 \leq n$ such that $\partial_{i_0} \text{RHS} \neq 0$. 


Now we do a blow-up along the Weil divisor defined by the ideal sheaf \((u, x_E^N)\), then we can get a finite map around \(e'\): \(Y_0 \rightarrow Y \times_B B'\), where \(Y_0\) is defined in the local coordinate system \((u_1, x_E, ..., x_n)\) by the following Equation 3.4:

**Equation 3.4.**

\[ u_p^p (1 + u_1^s \cdot x_E^N + \{higher \ order \ terms \ in \ u_1 \cdot x_E^N\}) = f_0(x_2, ..., x_n) \]

\[ + x_E^{p} f_1(x_2, ..., x_n) + ... + x_E^{pM} f_M(x_2, ..., x_n) \]

\[ + x_E^{pM+s} f_{M+1}(x_2, ..., x_n) + \{higher \ order \ terms \ in \ x_E\} \]

Now there are two possible cases remained, as described and analyzed in the following.

3.2.1. *The case \( \partial_i f_0 \neq 0 \) for some \( i_0 \) with \( 2 \leq i_0 \leq m \).* In this case, we may assume that \( i_0 = 2 \). Then we can see that locally around \( e' \in E' \subset Y' \), \( Y' \) is defined by Equation 3.4 with \( E' \) defined by \((x_E = 0)\). Now we have a generator of \( K_Y \), given by

\[ dx_E \wedge du_1 \wedge dx_3 ... \wedge dx_n. \]

We apply the differential \( d \) to both sides of Equation 3.4 and then wedge with \( dx_E \wedge dx_3 ... \wedge dx_n \). We get the following

\[ K_{Y'} = g_Y^* K_Y + Ns \cdot E', \]

together with

\[ g_Y^* E = E', \]
\[ f'^* \Delta' = N \cdot E'. \]

So we have

\[ Ns + a(E, X, \Delta) = a'(E', X', \Delta') + Ns. \]

Namely, we have

\[ a'(E', X', \Delta') = a(E, X, \Delta) \geq -1, \]

which proves Proposition 1.4 in this case.

3.2.2. *The case \( f_0 = f_0^p \) for some \( f_0' \in k[[x_2, ..., x_n]] \).* In this case, we can do a re-parametrization of Equation 3.4 defined by \( u_2 = u_1 - f_0' \), then we get the following Equation 3.5:

**Equation 3.5.**

\[ u_2^p + (u_2 + f_0')^p \cdot \left( ((u_2 + f_0') \cdot x_E^N)^s + \{higher \ order \ terms \ in \ (u_2 + f_0') \cdot x_E^N\} \right) \]

\[ = x_E^{p} f_1(x_2, ..., x_n) + ... + x_E^{pM} f_M(x_2, ..., x_n) \]

\[ + x_E^{pM+s} f_{M+1}(x_2, ..., x_n) + \{higher \ order \ terms \ in \ x_E\}. \]

Now for the induction process to work we make the following Definition 3.6.

**Definition 3.6.** For an equation \( f \) with the form of Equation 3.5 we define \( \mathcal{I}(f) \) as follows

- If \( f_1 = f_2 = ... = f_M = 0 \),
  \[ \mathcal{I}(f) = pM + s' \; ; \]
- If \( f_i \neq 0 \) for some \( i \) with \( 1 \leq i \leq M \),
  \[ \mathcal{I}(f) = \min\{pi|f_i \neq 0, 1 \leq i \leq M\}. \]
Now the whole proof of Proposition 1.4 can be completed once the following detailed analysis of all the possible remaining cases is finished.

Case 1 — The case \( \gcd(p, N_s) = 1 \) and \( N_s < I \): in this case we have to deal with the normalization of an equation of the form

\[
u_2^p = x_{E'}^{N_s} \left( (f_0^{p+s} + \{ \text{higher order terms in } u_2 \}) + \{ \text{higher order terms in } x_E \} \right).
\]

Since \( \gcd(p, N_s) = 1 \), we can repeat our algorithm in “the case \( \gcd(p, t_E) = 1 \)” that we have discussed before, and conclude that \( Y' \) has a local parameter system \((x_{E'}, x_2, ..., x_n)\) around \( e' \), where \((x_{E'} = 0)\) defines \( E' \). And there is a positive integer \( L \) (which satisfies \( \gcd(p, L) = 1 \) by tracing back the algorithm) such that

\[
v_{E'}(x_E) = pL, \\
v_{E'}(u) = NL.
\]

Now we apply the differential \( d \) to both sides of Equation 3.5, and then wedge with \( dx_2 \wedge dx_3 \ldots \wedge dx_n \). Since \( u_1 = u_2 - f_0' \) is locally a unit and \( \gcd(p, s) = 1 \), we get the following relation

\[
x_{E'}^{N_s L p + N s L - 1} \cdot dx_{E'} \wedge dx_2 \ldots \wedge dx_n = c \cdot x_{E'}^{(N_s - 1)L_p} \cdot dx_E \wedge dx_2 \ldots \wedge dx_n,
\]

where \( c \) is a constant. So we have

\[
x = N s L + L p - 1.
\]

Then we have

\[
sNL + a(E', X', \Delta') = pL \cdot a(E, X) + N s L + L p - 1.
\]

This implies that

\[
a(E', X', \Delta') + 1 = pL \cdot (a(E, X, \Delta) + 1) \geq 0.
\]

So Proposition 1.4 is proved in this case.

Case 2 — The case when \( \gcd(p, N) = 1 \) and \( N_s = I \): in this case we can see that

\[
f_1 = f_2 = ... = f_M = 0,
\]

and

\[
N_s = pM + s'
\]

by Definition 8.6.

Now the equation that we have to study is locally of the form

\[
u_2^p = x_{E'}^{N_s} \left( (f_0^{p+s} - f_{M+1}) + \{ \text{higher order terms in } u_2 \} \right)
\]

\[
+ \{ \text{higher order terms in } x_E \} \right).
\]

Furthermore, there are two sub-cases in this case as follows.

Sub-Case 1 — If \( f_0^{p+s} - f_{M+1} \neq 0 \), then this situation can be reduced to the case which we just discussed before.

Sub-Case 2 — If \( f_0^{p+s} - f_{M+1} = 0 \), then we have to deal with the normalization of the following equation

\[
u_2^p = x_{E'}^{pM + s'} (s \cdot u_2 \cdot f_0^{p+s-1} + \{ \text{higher order terms in } u_2 \})
\]

\[
+ \{ \text{higher order terms in } x_E \} \text{ including } f_0^{p+s+1} \cdot x_{E'}^{pM + s' + N} \}.
\]
We do a blow-up along the Weil divisor defined by the ideal sheaf \((u_2, x_E^M)\), then we can reduce to the following equation
\[
u^p_3 = x_E^{s'} (s \cdot u_3 \cdot x_E^M \cdot f'_0 p^{p+s-1} + \{\text{higher order terms in } u_3 \cdot x_E^M\})
\]
\[+ \{\text{higher order terms in } x_E \text{ including } f'_0 p^{p+s+1} \cdot x_E^{s+N}\}.
\]

Now an algorithm similar to what we used in “the case \(gcd(p, t_E) = 1\)” can deal with the situation where \(N \geq M\). And when \(N < M\), the leading term of the right hand side of the equation above will not contain \(u_3\). Hence we can further reduce to the following equation
\[
u^p_3 = x_E^{s'+s''} (f''_0(x_2, ..., x_n) + ... + f'_0 p^{p+s+1} \cdot x_E^{N-s''} + \{\text{higher order terms in } x_E\})
\]
This new equation has a smaller degree of leading term in \(x_E\) and a strictly smaller \(I\), so by our induction this sub-case can be further reduced to a situation where the corresponding normalization gives a discrepancy formula in either one of the following form, i.e.
\[a(E', X', \Delta') + 1 \geq pL \cdot (a(E, X, \Delta) + 1) + (Ns - (N + s'))L
\]
\[> pL \cdot (a(E, X, \Delta) + 1) \geq 0,
\]
or
\[Ns + a(E, X, \Delta) = a'(E', X', \Delta') + Ns.
\]
If the discrepancy formula is of the second form above, we would have
\[a'(E', X', \Delta') = a(E, X, \Delta) \geq -1,
\]which would arise if \(p|(s' + s'')\) and \(\partial_i f''_0(x_2, ..., x_n) \neq 0\) for some \(i\) with \(2 \leq i \leq n\), or if similar cases arise in the full reduction process.

So Proposition 1.4 is proved in this case.

**Case 3**—The case \(gcd(p, N) = 1\) and \(Ns > I\): there are two further sub-cases in this case, as follows.

**Sub-Case 1**—If \(f_i = 0\) for all \(i\) with \(1 \leq i \leq M\), then we have \(s > s'\) by definition. By the same argument as we just discussed before, we get the following equation
\[a(E', X', \Delta') + 1 = pL \cdot (a(E, X, \Delta) + 1) + N(s - s')L
\]
\[> pL \cdot (a(E, X, \Delta) + 1) \geq 0.
\]
So Proposition 1.4 is proved in this sub-case.

**Sub-Case 2**—If \(f_i \neq 0\) for some \(i\) with \(1 \leq i \leq M\), then let \(i_0\) be the smallest among such \(i\)'s, we would have \(I = pi_0\). Now we do a blow-up along the Weil divisor defined by the ideal sheaf \((u_2, x_E^{i_0})\), then we get a new Equation 3.7

**Equation 3.7.**

\[u^p_3 + (u_3 \cdot x_E^{i_0} + f'_0)^p \cdot ((u_3 \cdot x_E^{i_0} + f'_0) \cdot x_E^N)^s + \{\text{higher order terms in}
\]
\[(u_3 \cdot x_E^{i_0} + f'_0) \cdot x_E^N = f_{i_0}(x_2, ..., x_n) + \sum_{f_i \neq 0, i_0 \leq i \leq M} x_E^{p(i-i_0)} f_i(x_2, ..., x_n)
\]
\[+ x_E^{p(M-i_0)+s'} f_{M+1}(x_2, ..., x_n) + \{\text{higher order terms in } x_E\}.
\]
Now if $f_{i_0} \neq f^{p_{i_0}}_{i_0}$ for any $f'_{i_0} \in k[[x_2, \ldots, x_n]]$ in the Equation 3.7 above, then this situation can be reduced to “the case $\partial_{i_0} f_0 \neq 0$ for some $i_0$ with $2 \leq i_0 \leq m$” which we have discussed before, so Proposition 1.4 is proved in this situation.

Else if $f_{i_0} = f^{p_{i_0}}_{i_0}$ for some $f'_{i_0} \in k[[x_2, \ldots, x_n]]$ in Equation 3.7 above, then we can do a re-parametrization defined by $u_4 = u_3 - f'_{i_0}$, and thus reduce to a new Equation 3.8.

**Equation 3.8.**

$$u_4^p + ((u_4 + f'_{i_0}) \cdot x_0^i + f_0')^p \cdot \left(\left((u_4 + f'_{i_0}) \cdot x_0^i + f_0'\right) \cdot x_E^N\right)^s + \{\text{higher order terms in } (u_4 + f'_{i_0}) \cdot x_0^i + f_0'\} = \sum_{f_i \neq 0, i_0 \leq M} x_E^{p(i-1)} f_i(x_2, \ldots, x_n)$$

$$+ x_E^{p(M-i_0)+s} f_{M+1}(x_2, \ldots, x_n) + \{\text{higher order terms in } x_E\}.$$  

Equation 3.8 above has a strictly smaller $\mathcal{I}$, and hence this situation can also be reduced to one of the cases or sub-cases we have discussed before in this induction process.

So Proposition 1.4 is also proved in this case.

**Case 4**—The case when $p|N$: there are three further sub-cases in this case, as follows.

**Sub-Case 1**—If $N < \mathcal{I}$: assume $N = p^i q$ for some positive integer $i_0$ such that $\gcd(p, q) = 1$, and let $N' = N/p$, then we have to deal with the following equation

$$u_2^p = x_E^{p^i q} \left((f_0^{p+s} + \{\text{higher order terms in } u_2\}) + \{\text{higher order terms in } x_E\}\right).$$

We do a blow-up along the Weil divisor defined by the ideal sheaf $(u_2, x_E^{N'}) = x_E^{p^i q - 1} q$, then we get a new Equation 3.9.

**Equation 3.9.**

$$u_3^p = (f_0^{p+s} + \{\text{higher order terms in } u_3 \cdot x_E^{N' s}\}) + (x_E^p f_1(x_2, \ldots, x_n))$$

$$+ x_E^{pM} f_M(x_2, \ldots, x_n) + x_E^{pM+s} f_{M+1}(x_2, \ldots, x_n) + \{\text{higher order terms in } x_E\}.$$  

Now if $f_{i_0}' \neq f_{i_0}^{p_{i_0}}$ for any $f''_{i_0} \in k[[x_2, \ldots, x_n]]$ in the Equation 3.9 above, then this situation can be reduced to “the case $\partial_{i_0} f_0 \neq 0$ for some $i_0$ with $2 \leq i_0 \leq m$” which we have discussed before, so Proposition 1.4 is proved in this situation.

Else suppose $f_{i_0}' = f_{i_0}^{p_{i_0}}$ for some $f''_{i_0} \in k[[x_2, \ldots, x_n]]$ in the Equation 3.9 above, then we can do a re-parametrization defined by $u_4 = u_3 - f''_{i_0}^{p+s}$, and thus reduce to a new Equation 3.10.

**Equation 3.10.**

$$u_4 = (0 + \{\text{higher order terms in } (u_4 + f''_{i_0}^{p+s}) \cdot x_E^{N' s}\}) + (x_E^p f_1(x_2, \ldots, x_n))$$

$$+ x_E^{pM} f_M(x_2, \ldots, x_n) + x_E^{pM+s} f_{M+1}(x_2, \ldots, x_n) + \{\text{higher order terms in } x_E\}.$$  

Equation 3.10 above has a strictly smaller $\mathcal{I}$, and hence this situation can also be reduced to one of the cases or sub-cases we have discussed before in this induction process.

**Sub-Case 2**—If $N = \mathcal{I}$, then since $p|N$, as we have discussed before—we can reduce to a new equation having the same form of Equation 3.8 with a strictly
smaller $I$, and hence this situation can also be reduced to one of the cases or sub-cases we have discussed before in this induction process.

**Sub-Case 3**—If $N_s > I$: this situation can be immediately reduced to one of the cases or sub-cases we have discussed before in this induction process.

So Proposition 1.4 is proved in this final case. Q.E.D.

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