Cellular Automata: Wolfram’s Metaphors for Complex Systems

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In the late 1940’s while he was trying to construct a model for self-reproduction, a machine equivalent to biological systems, von Neumann invented a class of discrete mathematical systems called cellular automata [1]. Almost thirty years after the invention cellular automata underwent a radical reformation when, in the early 1980’s, Wolfram considered them as general mathematical representations of complex systems in nature [2, 3]. His investigations on cellular automata led him to the conviction that the laws for complex systems cannot be formulated as conventional mathematical equations; he proposed that the evolution of these systems can be correctly described only in the form of algorithms, the kind that are used in computer programs. It was the beginning of a new branch of science which Wolfram originally called the science of complexity. This new branch of science is based on the notion of computation [4]. According to Wolfram the evolution of any system, natural or artificial, can be viewed as a computation for which the initial state of the system is the input and the state that emerges after a given interval of time is the output. Cellular automata provided the ground for the discovery and the development of this new science.

Cellular automata are discrete dynamical systems defined on discrete space – an array of finite geometric cells, e.g, in the form of a lattice – and evolve in discrete time, i.e., time that changes in finite steps. The state of a cellular automaton is the set of the simultaneous states of its component cells. The function of each cell is defined in terms of a discrete dynamical variable that has a finite set of values and each value in this set denotes a distinct state of the cell. A cellular automaton evolves by updating the value of the dynamical variable simultaneously at all the cells forming the discrete space; this parallel updating is the primary feature of cellular automaton dynamics. The updating occurs at each step in discrete time. All updates follow a simple deterministic rule. The rule states, in terms of an algorithm, how the state of each cell (determined by the dynamical variable) changes after each time-step, influenced by itself and the states of the other cells in a well defined neighborhood. In the simplest cases, called elementary cellular automata, a two-valued variable $x \in \{0, 1\}$ evolves simultaneously at all the cells $i$ of a one-dimensional lattice by means of an updating rule $F$ that defines the interaction of each cell with its two nearest neighbors:

$$x_i^{(t+1)} = F [x_{i-1}^{(t)}, x_i^{(t)}, x_{i+1}^{(t)}].$$

(1)

Since each cell has two possible states there are $2^3 = 8$ different states of the three cell neighbourhood $\{x_{i-1}, x_i, x_{i+1}\}$ and each of them maps to a new state of the central cell $i$ for which there are the same two possibilities as there are for all other cells. Consequently there are $2^8 = 256$ different updating rules $F$ for elementary cellular automata. Evolution of two different kinds, reversible and irreversible, are shown in Figs. 1 and 2. Ref. [5] contains an overview of cellular automata.

The construction of cellular automata was motivated by the fact that natural systems, both physical and biological, are made of a large number of elementary units. Each unit has a simple structure and performs a simple function. However, when these are connected together by local interactions,
the resulting assembly (that forms a natural system) often produces extremely complex behavior. Besides, natural systems are inherently dissipative. Consequently they evolve in a manner that is irreversible and self-organizing, i.e., ordered structures are generated spontaneously from disordered initial forms. Cellular automata are mathematical systems that possess similar features. Like natural systems they are comprised of many identical units, each very simple, that evolve simultaneously by local interactions into complex ordered states. The unit of all cellular automata is a geometric cell in discrete space with a dynamical variable describing its state. Most of the updating rules are irreversible and generate self-organized states. Wolfram began his research on cellular automata with the aim of discovering the laws of self-organization. According to the second law of thermodynamics an isolated system spontaneously evolves to a state of maximum entropy and hence, of maximum disorder. The proof of this statement assumes that the microscopic evolution rule (i.e., the updating rule for each unit of the system) is reversible. Instead, if the microscopic evolution rule is irreversible, this particular restriction due to the second law of thermodynamics no longer exists and Wolfram showed that a system may evolve from a disordered state to a more ordered one. This is the origin of self-organization in most cellular automata. At each time-step of evolution the state of a cellular automaton has a unique successor, since the updating rule is deterministic. If the rule is reversible the predecessor of each state is also unique and the set of all allowed states of the cellular automaton remains constant under its evolution. However, if the updating rule is irreversible, several distinct states may evolve to one particular state which means that the predecessor of a state is not necessarily unique; unless the state in each time-step of evolution is memorized, the cellular automaton has no way of retracing its history when the direction of time is reversed. Therefore the set of allowed states of the cellular automaton contracts as it evolves and the limiting set of ordered states that ultimately remains is only a small subset of all possible initial states. This process of selecting a specific subset of all possible states forms the mechanism of self-organization. In cellular automata, the information on the specific subset to be selected is encoded in the updating rules. Though the updating rules are simple it appears that the outcomes of the evolution of most cellular automata are impossible to predict; this, according to Wolfram, is the mark of a complex system. Wolfram thus adopted cellular automata as the appropriate mathematical representations of the complex systems occurring in nature. However Wolfram’s definition of a complex system is only qualitative: a system that is not obviously simple; it still lacks a definition in quantitative terms.

The evolution of cellular automata are found to be equivalent to computations, i.e., cellular automata perform like digital computers. Besides self-organization, this is the other important feature of cellular automata. In general, each cellular automaton can perform a specific computation when provided with a specific form of the initial state. For example, the elementary cellular automaton that follows rule 132 (in Wolfram’s nomenclature scheme [2]) can effectively compute the remainder after dividing a natural number \( n \) by 2 if it is assigned the initial state that contains a block of \( n \) consecutive cells in state 1 and all other cells on both sides of the block are in state 0. The updating rule is expressed as \( x_i^{(t+1)} = \left[ x_{i-1}^{(t)} x_{i+1}^{(t)} + (1 + x_{i-1}^{(t)}) (1 + x_{i+1}^{(t)}) \right] x_i^{(t)} \mod 2 \). The outcome of the evolution of this cellular automaton tells whether a given natural number \( n \) is even or odd. If \( n \) is even, the cellular automaton evolves to a state where all the cells are in state 0; if \( n \) is odd, it evolves to a state that contains a single cell in state 1. Some cellular automata are known to be capable of universal computation, i.e., they can perform any possible computation with appropriate initial states. One of the earliest known examples is the two dimensional cellular automaton ‘Life’ invented by Conway [3, 7]. The simplest of all those that have been proved to be universal is the cellular automaton that follows elementary rule 110: \( x_i^{(t+1)} = \left[ (1 + x_{i-1}^{(t)}) x_i^{(t)} x_{i+1}^{(t)} + x_i^{(t)} + x_{i+1}^{(t)} \right] \mod 2 \). The proof is indirect: the elementary cellular automaton with rule 110 was shown to emulate any given cyclic tag system and it was possible to construct a cyclic tag system that emulates any given Turing machine; since some Turing machines are known to be universal computers, it establishes that rule 110 is capable of universal computation. The
computational capability of cellular automata led Wolfram to the view that all processes in nature are programs composed of simple algorithms in the form of cellular automata.

Wolfram’s research on cellular automata for almost two decades has been recorded in his book ‘A New Kind of Science’ [8]. The book is an outstanding collection of computer experiments and each set of experiments culminates in a remarkable discovery or proposition, two of which must be mentioned. While studying the evolution of reversible cellular automata from various initial states, Wolfram discovered that there are certain reversible cellular automata that, contrary to the existing belief, do not obey the second law of thermodynamics. Some of these reversible automata never evolve to disordered states from ordered ones; there are others that are found to self-organize from disordered initial states to configurations with ordered structures that are reminiscent of the outcomes of irreversible evolution. Though Wolfram has studied a vast number of cellular automata, the laws of self-organization have not been found. However, the results of these computer experiments led him to propose ‘the principle of computational equivalence’. In a general way, the principle states that ‘almost all processes that are not obviously simple can be viewed as computations of equivalent sophistication’ [8]. The principle makes a remarkable assertion that there is just one level of computational sophistication. Though it is still in the form of a hypothesis, Wolfram believes that this principle is a new law of nature.

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Figure 1: The evolution of two elementary cellular automata with reversible updating rules: (a) the left shift automaton: \( x_i^{(t+1)} = x_{i+1}^{(t)} \), and (b) the right shift automaton: \( x_i^{(t+1)} = x_{i-1}^{(t)} \). In Wolfram’s nomenclature scheme these are called rule 170 and rule 240 respectively. Each diagram shows an array of 65 cells evolving for 32 time-steps. Time increases in the downward direction. A white square denotes a cell in state 0 while a black square denotes a cell in state 1. In both cases the initial state of the cellular automaton contains a single cell in state 1 whereas the rest of the cells are in state 0. This simple structure of the initial state is maintained throughout the evolution. If the final state in each case is considered as the initial state by inverting the diagrams (equivalent to reversing the direction of time) the evolution of each automaton is retracted when the respective updating rules are applied; this happens because the shift automata are reversible.

Figure 2: The evolution of two elementary cellular automata with irreversible updating rules. In Wolfram’s nomenclature scheme these are called (a) rule 90: \( x_i^{(t+1)} = \left[ x_{i-1}^{(t)} + x_i^{(t)} \right] \mod 2 \), and (b) rule 150: \( x_i^{(t+1)} = \left[ x_{i-1}^{(t)} + x_i^{(t)} + x_{i+1}^{(t)} \right] \mod 2 \). As in Figure 1, white and black squares denote cells in the states 0 and 1 respectively and the evolution of an array of 65 cells is shown for 32 time-steps from an initial state which contains a single cell in state 1. Time increases downwards. It is clear that the simplicity of the initial state is destroyed as the automata evolve and an ordered structure emerges in each case. These cellular automata fail to retrace their evolution if the direction of time is reversed, which proves that they are irreversible.