Discrete septic spline quasi-interpolants for solving generalized Fredholm integral equation of the second kind via three degenerate kernel methods

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Abstract Three main contributions are presented in this paper. First, the septic quasi-interpolants are calculated with all their coefficients. Second, we explore the results to solve a generalized and broad class of Fredholm integral equations of the second kind. Finally, we present three degenerate kernel methods; the latter is a combination of the two previously established methods in the literature. Moreover, we provide a convergence analysis and we give new error bounds. Finally, we exhibit some numerical examples and compare them with previous results in the literature.

Keywords Septic spline · Quasi-interpolation · Integral equations

Introduction

In the last years, many authors have presented different approaches to the solution of Fredholm integral equations using spline kernel approximations, to limit the number of terms in the approximate kernel. The results of this limitation provide high accuracy excluding unnecessary computational costs due to large linear systems.

The authors of [14] exploit Galerkin method to approximate the solution of the time-dependent Dirac equation in prolate spheroidal coordinates for an electron-molecular two-center system. They use balanced basis of kinetic-tomic to evaluate the initial state from a variational principle. Consequently, they obtain an exact and effective determination of the Dirac spectrum and eigenfunctions.

Using the polar method on parametric cubic spline technique, Prabhakar and Uma get wave resonating quadruplets calculated by the nonlinear source term of the wave model. They decide about the points of the locus for the two spacings, constant spacing and variable spacing. Refer to [27] for details.

The aim of [23] is to numerically solve the time fractional subdiffusion equation with Dirichlet boundary value conditions by use of the collocation method based on quadratic spline. In the paper, the authors explore, in details, the coefficient matrix of the discretized linear system.

In [10], the authors propose a collocation method based on the quadratic spline in the pricing problem under a finite activity jump diffusion model for the first time.

For the sake of analyzing the nonlinear elastoplastic behavior of prismatic thin-walled members, the authors of [12] found a method of beam finite element working on generalized beam theory.

In another work [21], the authors decide to use quadratic and cubic B-spline quasi-interpolants to achieve higher order numerical methods only for a limited equations of Sobolev type in one dimension. Their aim is to compare the accuracy and convergence rate of these methods’ performance.

Barton and Calo derive a new way for generating optimal quadrature rules for splines by making an association of source space and a known optimal quadrature and get the rule from the source space to the target one, but they maintain the point number and the optimality of quadrature. The aim of this process is to produce an optimal quadrature rule in a given spline space (see [7]).
The purpose of [25] is to develop the BS Hermite spline quasi-interpolation scheme, which is related to the continuous extension of the BS linear multistep methods for solving the ordinary differential equations.

The author of [15] explores the quadratic spline quasi-interpolants on bounded domains, and provides some applications to different areas of the approximation theory. In the same scope, a different work gives some significant applications to different areas of the approximation theory. Interpolants on bounded domains, and provide some contributions. The first one is to build the septic quasi-interpolant. Next, we use our new results to solve a class of Fredholm integral equations of the second kind. Finally, we present three degenerate kernel methods, where the latter method is original. We also provide a convergence analysis and we give new error bounds.

In Sect. 5, we illustrate our theory by different examples, comparing the results we achieved with already given results in the field to argue that ours are more efficient than the previous ones. Finally, some conclusions are presented.

Construction of discrete septic quasi-interpolant

B-splines and monomials of degree seven

Let \( X := C^0([a, b], \mathbb{R}) \) be the space of all continuous functions, equipped with the max-norm \( \| \cdot \|_{\infty} \). Let us consider the nodes \( x_0, x_1, x_2, \ldots, x_n \) in the interval \([a, b]\) with \( x_k = a + k h \) and \( h_k = x_k - x_{k-1} \) for all \( 1 \leq k \leq n \).

Let \( \mathcal{X}_n := \{ x_k, 0 \leq k \leq n \} \) denote the partition of the interval \([a, b]\) into \( n \) subintervals.

Define \( S_7 := S_7([a, b], X_n) \) to be the space of septic splines of class \( C^6 \) on this partition.

Consider the set \( \Gamma_n := \{ 1, 2, \ldots, n + 7 \} \).

Let a canonical basis of \( S_7 \) be \( \{ B_k, k \in \Gamma_n \} \), which is shaped by the \( n + 7 \) normalized B-splines. We add multiple knots at the endpoints to obtain the support of \( B_k \) which is the interval \([x_{k-8}, x_k]\) (see [28]). It is well-known that the representation of monomials using symmetric functions \( \text{symm}_r(N_k) \) of interior knots

\[
N_k := \{ x_{k-7}, x_{k-6}, x_{k-5}, x_{k-4}, x_{k-3}, x_{k-2}, x_{k-1} \} \text{ in } \text{Supp}(B_k).
\]

Denoting by \( D^j \) the derivation operator of the order \( j \), we consider the following function

\[
\phi_k(t) := (x_{k-7} - t)(x_{k-6} - t)(x_{k-5} - t)(x_{k-4} - t)(x_{k-3} - t)(x_{k-2} - t)(x_{k-1} - t).
\]

For \( 0 \leq r \leq 7 \), we have:

\[
m_r(x) = x^r = \sum_{k \in \Gamma_n} (-1)^{7-r} \frac{r!}{7!} D^{7-r} \phi_k(0) B_k(x)
\]

\[
= \sum_{k \in \Gamma_n} \gamma_r^k B_k(x),
\]

such that

\[
\gamma_r^k = \frac{\text{symm}_r(N_k)}{x_r}, \quad 0 \leq r \leq 7,
\]

where

\[
x_r := \binom{r}{7}.
\]

Specifically, if \( r = 0 \), we have \( \gamma_0^k = 1 \) for all \( k \in \Gamma_n \).

For \( r = 1 \), using the formula

\[
\sum_{k \in \Gamma_n} B_k(x) = 1,
\]

we obtain the Greville abscissae:

\[
\gamma_1^k = \frac{1}{x_7 - x_6} \frac{1}{x_6 - x_5} \frac{1}{x_5 - x_4} \frac{1}{x_4 - x_3} \frac{1}{x_3 - x_2} \frac{1}{x_2 - x_1} \frac{1}{x_1 - x_0}.
\]
We have calculated all the coefficients \( \theta'_j, j \in \Gamma_n, 0 \leq r \leq 7; \) the results are given in in Tables 1, 2, 3, 4, 5, 6 and 7 respectively.

### Discrete quasi-interpolant of degree 7

The discrete septic spline quasi-interpolant, abbreviated as dSSQI, is the operator

\[
Qf = \sum_{k \in \Gamma_n} \mu_k(f)B_k,
\]

whose coefficients are linear combinations of discrete values of \( f \) on the set of data points \( x'_n \).

The dSSQI is developed to be exact on \( P_7 \), i.e.

\[
Qp = p \quad \text{for all} \quad p \in P_7,
\]

in other words, \( Qm_r = m_r \), where

\[
m_r(x) = \sum_{k \in \Gamma_n} \mu_k(m_r)B_k(x) = \sum_{k \in \Gamma_n} \theta'_kB_k(x), \quad 0 \leq r \leq 7.
\]

Therefore, we obtain the following conditions

\[
\mu_k(m_r) = \theta'_k \quad \text{for} \quad k \in \Gamma_n, \quad 0 \leq r \leq 7.
\]

For \( 7 \leq k \leq n + 1 \), the functionals \( \mu_k(f) \) merely use values of \( f \) in a neighborhood of the support of \( B_k \), that is why it should be expressed as \( \mu_k(f) \) thus

\[
\mu_k(f) = \lambda_k f_{k-7} + \beta_k f_{k-6} + \gamma_k f_{k-5} + \delta_k f_{k-4} + \lambda_k f_{k-3} + \mu_k f_{k-2} + \nu_k f_{k-1},
\]

where \( f_k = f(x_k) \). The conditions above are the same as the systems of linear equations:

\[
\lambda_k x'_{k-7} + \beta_k x'_{k-6} + \gamma_k x'_{k-5} + \delta_k x'_{k-4} + \lambda_k x'_{k-3} + \mu_k x'_{k-2} + \nu_k x'_{k-1} = \theta'_k, \quad 0 \leq r \leq 7.
\]

For \( 1 \leq k \leq 6 \) and \( n + 2 \leq k \leq n + 7 \), we get the following equations, respectively

\[
\mu_k(f) = \lambda_k f_0 + \beta_k f_1 + \gamma_k f_2 + \delta_k f_3 + \lambda_k f_4 + \mu_k f_5 + \nu_k f_6 + \omega_k f_7.
\]

All these systems have Vandermonde determinants

\[
V_k(x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7) \neq 0.
\]

While the \( x_j, j \in J_n := \{0, 1, \ldots, n\} \), are different, they have unique solutions. Hence, the existence and uniqueness of the dSSQI. In the case of a uniform partition, the coefficient functionals are, respectively, defined by the following formulae:

\[
\mu_1(f) = f_0.
\]
Table 3 $\theta_j^n$

| $j$ | $\theta_j^n$ |
|-----|--------------|
| 1   | $a^1$        |
| 2   | $a^1 + \frac{2a^2}{f}$ |
| 3   | $a^1 + \frac{2a^2}{f} + \frac{2a}{f^2}$ |
| 4   | $a^1 + \frac{2a^2}{f} + \frac{14a^3}{f^3} + \frac{3a}{f^3}$ |
| 5   | $a^1 + \frac{2a^2}{f} + 5a^2 + \frac{10a}{f^3}$ |
| 6   | $a^1 + \frac{2a^2}{f} + \frac{15a^3}{f^3} + \frac{5a}{f^3}$ |

$7 \leq j \leq n + 1$

$\theta_j^n = (3ah^2 - 12h^3)j^2 + (3a^2 - 24ah^2 + 46h^3)j + a^3 - 12a^2h + 46ah^2 - 56h^3$

Table 4 $\theta_j^b$

| $j$ | $\theta_j^b$ |
|-----|--------------|
| 1   | $b^1$        |
| 2   | $b^1 + \frac{4b^2}{f}$ |
| 3   | $b^1 + \frac{12b^2}{f} + \frac{4b^3}{f^2}$ |
| 4   | $b^1 + \frac{24b^2}{f} + \frac{24b^3}{f^2} + \frac{24b^4}{f^3}$ |
| 5   | $b^1 + \frac{40b^2}{f} + 10a^2 + \frac{40b^4}{f^2} + \frac{24b^4}{f^3}$ |
| 6   | $b^1 + \frac{60b^2}{f} + 17b^3 + \frac{17b^4}{f^2} + \frac{27b^4}{f^3}$ |

$7 \leq j \leq n + 1$

$\theta_j^b = (4a^3 - 16h^3)j^2 + (6h^2a^2 - 48ah^3 + 92h^4)j^2 + (4a^3h - 48h^2a^2 + 184ah^3 - 224h^4)j + a^4 - 16a^3h + 92a^2h^2 - 224ah^3 + \frac{48a^4}{f^3}$

$\mu_2(f) = \frac{617}{980}f_0 + f_1 - \frac{3}{2}f_2 - \frac{5}{3}f_3 - \frac{5}{4}f_4 + \frac{3}{5}f_5 - \frac{1}{6}f_6 + \frac{1}{49}f_7.$

$\mu_3(f) = \frac{3623}{26.46f_0}f_0 + \frac{407}{210}f_1 - \frac{337}{140}f_2 + \frac{941}{378}f_3 - 151 \frac{f_4}{84}$

$+ \frac{59}{70}f_5 - \frac{871}{3780}f_6 + \frac{41}{1470}f_7.$

$\mu_4(f) = -\frac{23}{264.6}f_0 + \frac{2887}{210}f_1 - \frac{207}{700}f_2 - \frac{391}{3780}f_3$

$+ \frac{5}{24}f_4 - \frac{93}{700}f_5 + \frac{401}{9450}f_6 - \frac{83}{14,700}f_7.$

$\mu_5(f) = \frac{713}{52.92}f_0 - \frac{53}{1260}f_1 + \frac{241}{105}f_2 - \frac{8317}{3780}f_3$

$+ \frac{521}{360}f_4 - \frac{53}{84}f_5 + \frac{617}{3780}f_6 - \frac{167}{8820}f_7.$
Table 5 \( \theta_j^5 \)

| \( j \) | \( \theta_j^5 \) |
|---|---|
| 1 | \( a^5 \) |
| 2 | \( a^5 + \frac{5a^4h}{7} \) |
| 3 | \( a^5 + \frac{15a^4h}{7} + \frac{20a^3h^2}{7} \) |
| 4 | \( a^5 + \frac{30a^4h}{7} + \frac{10a^3h^2}{7} + \frac{12a^2h^3}{7} \) |
| 5 | \( a^5 + \frac{50a^4h}{7} + \frac{20a^3h^2}{7} + \frac{10a^2h^3}{7} + \frac{24ah^4}{7} \) |
| 6 | \( a^5 + \frac{75a^4h}{7} + \frac{80a^3h^2}{21} + \frac{45a^2h^3}{7} + \frac{27ah^4}{3} + \frac{40h^5}{7} \) |
| \( 7 \leq j \leq n + 1 \) | \( b^5 j^5 \) |

Table 6 \( \theta_j^6 \)

| \( j \) | \( \theta_j^6 \) |
|---|---|
| 1 | \( a^6 \) |
| 2 | \( a^6 + \frac{6a^5h}{7} \) |
| 3 | \( a^6 + \frac{18a^5h}{7} + \frac{10a^4h^2}{7} \) |
| 4 | \( a^6 + \frac{50a^5h}{7} + \frac{25a^4h^2}{7} + \frac{24ah^3}{7} \) |
| 5 | \( a^6 + \frac{90a^5h}{7} + \frac{45a^4h^2}{7} + \frac{27ah^3}{7} + \frac{40h^4}{7} \) |
| 6 | \( a^6 + \frac{135a^5h}{7} + \frac{225a^4h^2}{7} + \frac{90a^3h^3}{7} + \frac{336a^2h^4}{7} + \frac{27ah^5}{7} + \frac{40h^6}{7} \) |
| \( 7 \leq j \leq n + 1 \) | \( b^6 j^6 \) |

| \( n + 2 \) | \( b^6 j^6 - \frac{59b^5h}{7} - \frac{80b^4h^2}{21} - \frac{45b^3h^3}{7} - \frac{27bh^4}{3} - \frac{40h^5}{7} \) |
| \( n + 3 \) | \( b^6 j^6 - \frac{15b^5h}{7} - \frac{2b^4h^2}{7} \) |
| \( n + 4 \) | \( b^6 j^6 - \frac{15b^5h}{7} - \frac{2b^4h^2}{7} \) |
| \( n + 5 \) | \( b^6 j^6 - \frac{15b^5h}{7} - \frac{2b^4h^2}{7} \) |
| \( n + 6 \) | \( b^6 j^6 - \frac{15b^5h}{7} - \frac{2b^4h^2}{7} \) |
| \( n + 7 \) | \( b^6 j^6 \) |
For \( 7 \leq j \leq n + 1 \),

\[ \mu_j(f) = -\frac{311}{15,120} (j_{j-7} + j_{j-1}) + \frac{22}{105} (j_{j-6} + j_{j-2}) - \frac{1657}{1680} (j_{j-5} + j_{j-3}) + \frac{2452}{945} j_{j-4}. \]

\[ \mu_{n+2}(f) = \frac{3769}{105,840} f_0 - \frac{811}{2520} f_1 + \frac{263}{240} f_2 + \frac{5741}{7560} f_3 \]
\[ - \frac{683}{720} f_4 + \frac{85}{168} f_5 + \frac{311}{2160} f_6 + \frac{311}{7680} f_7. \]

\[ \mu_{n+4}(f) = -\frac{23,251}{264,600} f_0 + \frac{2887}{2100} f_{n-1} - \frac{207}{700} f_{n-2} \]
\[ - \frac{391}{3780} f_{n-3} + \frac{5}{24} f_{n-4} - \frac{93}{700} f_{n-5} \]
\[ + \frac{401}{9450} f_{n-6} - \frac{83}{14,700} f_{n-7}. \]

\[ \mu_{n+5}(f) = \frac{3623}{26,460} f_0 + \frac{407}{210} f_{n-1} - \frac{337}{140} f_{n-2} + \frac{941}{378} f_{n-3} \]
\[ - \frac{151}{84} f_{n-4} + \frac{59}{70} f_{n-5} \]
\[ - \frac{871}{3780} f_{n-6} + \frac{41}{1470} f_{n-7}. \]

\[ \mu_{n+6}(f) = \frac{617}{980} f_0 + f_{n-1} - \frac{3}{2} f_{n-2} + \frac{5}{3} f_{n-3} - \frac{5}{4} f_{n-4} + \frac{3}{5} f_{n-5} \]
\[ - \frac{1}{6} f_{n-6} + \frac{1}{49} f_{n-7}. \]

\[ \mu_{n+7}(f) = f_0. \]

It is noticeable that

\[ |\mu_5| = |\mu_{n+6}| \approx 6.8333, \]
\[ |\mu_5| = |\mu_{n+5}| \approx 9.8704, \]
\[ |\mu_4|_\infty = |\mu_{n+4}|_\infty \approx 2.2511, \]
\[ |\mu_5|_\infty = |\mu_{n+3}|_\infty \approx 6.8114, \]
\[ |\mu_6|_\infty = |\mu_{n+2}|_\infty \approx 3.8288. \]

For \( 7 \leq j \leq n + 1 \),
\[ |\mu_j|_\infty \approx 5.0275, \]
hence,
\[ \|Q\| \leq 9.8704. \]

Furthermore, for \( f \in C^8(I) \), we also have
\[ \|f - Qf\|_{\infty, h} \leq (1 + 9.87)d_{\infty, h}(f, \mathbb{P}_7) \leq 11d_{\infty, h}(f, \mathbb{P}_7), \]
where
\[ I_k = [x_{k-1}, x_k], \quad 1 \leq k \leq n, \]
and
\[ d_{\infty, h}(f, \mathbb{P}_7) = \max\{\|f - p\|_{\infty, h}; p \in \mathbb{P}_7\}. \]

Then
\[ \|f - Qf\|_{\infty} = O(h^8). \]

We can write the quasi-interpolant \( Q \) under the quasi-Lagrange form:
\[ Qf = \sum_{j \in J_n} f_j L_j, \]
where
\[ L_0 = B_1 + \frac{617}{980} B_2 + \frac{3623}{26,460} B_3 - \frac{23,251}{264,600} B_4 - \frac{713}{52,920} B_5 + \frac{3769}{105,840} B_6 - \frac{311}{15,120} B_7. \]
\[ L_1 = B_2 + \frac{407}{210} B_3 + \frac{2887}{2100} B_4 - \frac{53}{1260} B_5 - \frac{811}{2520} B_6 + \frac{22}{105} B_7 - \frac{311}{15,120} B_8. \]
\[ L_2 = -\frac{3}{2} B_2 - \frac{337}{140} B_3 - \frac{207}{700} B_4 + \frac{241}{105} B_5 + \frac{263}{240} B_6 - \frac{1657}{1680} B_7 + \frac{22}{105} B_8 - \frac{311}{15,120} B_9. \]
\[ L_3 = \frac{5}{3} B_2 + \frac{941}{378} B_3 - \frac{391}{3780} B_4 - \frac{8317}{3780} B_5 + \frac{5741}{7560} B_6 + \frac{2452}{945} B_7 - \frac{1657}{1680} B_8 + \frac{22}{105} B_9 - \frac{311}{15,120} B_{10}. \]
\[ L_4 = -\frac{5}{4} B_2 - \frac{151}{84} B_3 + \frac{5}{24} B_4 + \frac{521}{360} B_5 - \frac{683}{720} B_6 - \frac{1657}{1680} B_7 + \frac{2452}{945} B_8 - \frac{1657}{1680} B_9. \]
\[ + \frac{22}{105} B_{10} - \frac{311}{15,120} B_{11}. \]
\[ L_5 = \frac{3}{5} B_2 + \frac{59}{70} B_3 - \frac{93}{700} B_4 - \frac{53}{84} B_5 + \frac{85}{168} B_6 + \frac{22}{105} B_7 - \frac{1657}{1680} B_8 + \frac{2452}{945} B_9 - \frac{1657}{1680} B_{10}. \]
\[ + \frac{22}{105} B_{11} - \frac{311}{15,120} B_{12}. \]
\[ L_6 = -\frac{1}{6} B_2 + \frac{871}{3780} B_3 + \frac{401}{9450} B_4 + \frac{617}{3780} B_5 - \frac{311}{15,120} B_6 - \frac{311}{15,120} B_7 + \frac{22}{105} B_8 - \frac{1657}{1680} B_9 + \frac{2452}{945} B_{10} - \frac{1657}{1680} B_{11} + \frac{22}{105} B_{12}. \]
\[ L_7 = \frac{1}{49} B_2 + \frac{41}{1470} B_3 - \frac{83}{14} B_4 - \frac{167}{8820} B_5 + \frac{311}{17,640} B_6 - \frac{311}{15,120} B_7 + \frac{22}{105} B_8 - \frac{1657}{1680} B_9 + \frac{2452}{945} B_{10} - \frac{1657}{1680} B_{11} + \frac{22}{105} B_{12}. \]
\[ L_{n-7} = -\frac{1}{49} B_{n+6} + \frac{41}{1470} B_{n+5} - \frac{83}{14} B_{n+4} - \frac{167}{8820} B_{n+3} + \frac{311}{17,640} B_{n+2} - \frac{311}{15,120} B_{n+1} - \frac{22}{105} B_{n} - \frac{1657}{1680} B_{n-1} + \frac{2452}{945} B_{n-2} - \frac{1657}{1680} B_{n-3} + \frac{22}{105} B_{n-4} - \frac{311}{15,120} B_{n-5} - \frac{1657}{1680} B_{n-6}. \]
\[ L_{n-6} = -\frac{1}{6} B_{n+6} + \frac{871}{3780} B_{n+5} + \frac{401}{9450} B_{n+4} + \frac{617}{3780} B_{n+3} - \frac{311}{15,120} B_{n+2} - \frac{311}{15,120} B_{n+1} - \frac{22}{105} B_{n} - \frac{1657}{1680} B_{n-1} + \frac{2452}{945} B_{n-2} - \frac{1657}{1680} B_{n-3} + \frac{22}{105} B_{n-4} - \frac{311}{15,120} B_{n-5} - \frac{1657}{1680} B_{n-6}. \]
\[ L_{n-5} = \frac{3}{5} B_{n+6} + \frac{59}{70} B_{n+5} - \frac{93}{700} B_{n+4} - \frac{53}{84} B_{n+3} - \frac{85}{168} B_{n+2} + \frac{22}{105} B_{n+1} - \frac{1657}{1680} B_{n} + \frac{2452}{945} B_{n-1} - \frac{1657}{1680} B_{n-2} + \frac{22}{105} B_{n-3} - \frac{311}{15,120} B_{n-4}. \]
Solving Fredholm integral equations by degenerate kernel approximations

Consider the following generalized Fredholm integral equation of the second kind

\[ u(s) - \sum_{k=1}^{m} \int_{a}^{b} H_k(s,t)u(t)dt = f(s), \quad m \in \mathbb{N}^*, \quad a \leq s \leq b, \quad (1) \]

where \( f \) is a continuous function. We assume that \( H_k(\cdot, \cdot) \in C([a, b] \times [a, b], \mathbb{C}), \quad k = 1, \ldots, m \). Then the integral operator

\[ Tu(s) := \sum_{k=1}^{m} \int_{a}^{b} H_k(s,t)u(t)dt, \quad a \leq s \leq b, \]

is compact from \( \mathcal{X} \) into \( \mathcal{X} \), we assume that the Eq. (1) has a unique solution. Equation (1) can be rewritten in an operator form as:

\[
\begin{align*}
L_n &= -\frac{5}{4} B_{n+6} - \frac{151}{84} B_{n+5} + \frac{5}{24} B_{n+4} + \frac{521}{360} B_{n+3} - \frac{683}{720} B_{n+2} - \frac{1657}{1680} B_{n+1} + \frac{2452}{945} B_n - \frac{1657}{1680} B_{n-1} + \frac{22}{105} B_{n-2} - \frac{311}{15,120} B_{n-3}, \\
L_{n-3} &= \frac{5}{3} B_{n+6} - \frac{941}{378} B_{n+5} - \frac{391}{3780} B_{n+4} - \frac{8317}{3780} B_{n+3} + \frac{5741}{7560} B_{n+2} + \frac{2452}{945} B_{n+1} - \frac{1657}{1680} B_n + \frac{22}{105} B_{n-1} - \frac{311}{15,120} B_{n-2}, \\
L_{n-2} &= -\frac{3}{2} B_{n+6} - \frac{337}{140} B_{n+5} - \frac{207}{700} B_{n+4} + \frac{241}{105} B_{n+3} + \frac{263}{240} B_{n+2} - \frac{1657}{1680} B_{n+1} + \frac{22}{105} B_n - \frac{311}{15,120} B_{n-1}, \\
L_{n-1} &= B_{n+6} + \frac{407}{210} B_{n+5} + \frac{2887}{2100} B_{n+4} - \frac{53}{1260} B_{n+3} + \frac{81}{2520} B_{n+2} + \frac{22}{105} B_{n+1} - \frac{311}{15,120} B_n, \\
L_n &= B_{n+7} + \frac{617}{980} B_{n+6} + \frac{3623}{26,460} B_{n+5} - \frac{23,251}{264,600} B_{n+4} - \frac{713}{52,920} B_{n+3} + \frac{3769}{105,840} B_{n+2} - \frac{311}{15,120} B_{n+1}.
\end{align*}
\]

Degenerate kernel methods are crucial in approximation theory and in scientific computing. They have many interesting applications, particularly to solve integral and integro-differential equations. In [18], the authors explore the classical version of the degenerate kernel method to numerically solve the Hammerstein equations. Later, they extend the degenerate kernel method for single-variable Hammerstein equations to include multi-variable Hammerstein equations in [19]. The authors of [26] treat a degenerate approximation of the kernel using Taylor series and Lagrange interpolation for solving the general non-linear Fredholm integro-differential equations under mixed conditions. The degenerate kernel in the polar coordinates for two subdomains is adopted in [9] for the closed-form fundamental solution of null-field boundary integral equation method. Majidiana and Babolian [24] apply a degenerate kernel method with piecewise constant interpolation with respect to the second variable to approximate isolated eigenvalues of a class of noncompact linear operators. In [11], a new approach to the theory of kernel approximations is developed for the numerical solution of Fredholm integral equations of the second kind using a degenerate-kernel operator of fixed rank. Kalaba and Scott [17] use an initial-value method for integral equations with generalized degenerate kernels. Recently, the authors of [3] rely on two degenerate methods for solving the classical Fredholm integral equation of the second kind, based on (left and right) partial approximations of the kernel through a discrete quartic spline quasi-interpolant. From reviewing the literature, it is noticeable that the degenerate kernel method is commonly used in the development in the theory of approximation on a wide scale, mainly the resolution of integral equations. In a more modern sense, we intend to rely on the same degenerate kernel methods for solving (1) through our newly obtained results of discrete septic spline quasi-interpolants, unlike the previously mentioned work [3].

First septic spline degenerate kernel method

We first approximate the given continuous functions \( s \mapsto H_k(s, t) \) by the septic spline quasi-interpolant using quasi-Lagrange form:

\[
H_k^E(s, t) = \sum_{j \in J_n} H_k(s_j, t)L_j(s).
\]

The left degenerate kernel operator is defined by

\[
\tau_n^E u(s) := \sum_{j \in J_n} L_j(s) \sum_{k=1}^{m} \int_{a}^{b} H_k(s_j, t)u(t)dt.
\]
Approximating the Eq. (1) by

\[ u_n^F(s) - \sum_{j \in J_n} L_j(s) \sum_{k=1}^m \int_a^b H_k(s_j, t) u_n^F(t) \, dt = f(s), \quad (2) \]

the approximate solution \( u_n^F \) of Eq. (2) is given by

\[ u_n^F(s) = f(s) + \sum_{j \in J_n} c_j L_j(s), \]

for some scalars \( c_j \).

As a result, we obtain the linear system

\[ \sum_{i \in J_n} c_i \sum_{k=1}^m \int_a^b H_k(s_i, t) L_j(t) \, dt = \sum_{k=1}^m \int_a^b H_k(s_i, t) f(t) \, dt, \]

hence

\[ c_i = \sum_{j \in J_n} c_j \sum_{k=1}^m \int_a^b H_k(s_i, t) L_j(t) \, dt = \sum_{k=1}^m \int_a^b H_k(s_i, t) f(t) \, dt, \]

that is to say, the coefficients \( c_j \) are obtained by solving the following linear system

\[ (I - R_n)X_n^F = b_n^F, \]

where, for \( i \in J_n \) and \( j \in J_n \),

\[ F_n(i, j) := \sum_{k=1}^m \int_a^b H_k(s_i, t) L_j(t) \, dt, \]

\[ b_n^F(i) := \sum_{j=1}^m \int_a^b H_k(s_i, t) f(t) \, dt. \]

Second septic spline degenerate kernel method

Next, we consider the following right degenerate kernel operator

\[ T_n^R u(s) := \sum_{j \in J_n} \sum_{k=1}^m H_k(s, t_j) \int_a^b L_j(t) u(t) \, dt. \]

We approximate the given continuous functions \( t \rightarrow H_k(s, t) \) by a septic spline quasi-interpolant:

\[ H_k^R(s, t) = \sum_{j \in J_n} H_k(s, t_j) L_j(t), \]

and the approximate solution satisfies

\[ u_n^R(s) - \sum_{j \in J_n} \sum_{k=1}^m H_k(s, t_j) \int_a^b L_j(t) u_n^R(t) \, dt = f(s), \]

Therefore, \( u_n^R \) is of the form:

\[ u_n^R(s) = f(s) + \sum_{j \in J_n} r_j \sum_{k=1}^m H_k^R(s), \quad \text{with} \]

\[ H_k^R(s) := H_k(s, t_i), \]

for some scalars \( r_j \). Hence

\[ r_i - \sum_{j \in J_n} r_j \sum_{k=1}^m \int_a^b L_j(t) H_k^R(t) \, dt = \int_a^b L_i(t) f(t) \, dt, \quad i \in J_n. \]

The coefficients \( r_j \) are obtained by solving the following linear system

\[ (I - R_n)X_n^R = b_n^R, \]

where, for \( i \in J_n \) and \( j \in J_n \),

\[ R_n(i, j) := \sum_{k=1}^m \int_a^b L_j(t) H_k^R(t) \, dt, \]

\[ b_n^R(i) := \int_a^b L_i(t) f(t) \, dt. \]

Third septic spline degenerate kernel method

Finally, we approximate the given continuous functions \( s \rightarrow H_k(s, t) \) by the septic spline quasi-interpolant using quasi-Lagrange form:

\[ H_k^R(s, t) = \sum_{j \in J_n} \sum_{i \in J_n} H_k(s, t_i) L_j(s) L_i(t). \]

The third degenerate kernel operator is defined by

\[ T_n^R u(s) := H_k^R(s, t) = \sum_{j \in J_n} \sum_{k=1}^m H_k(s, t_i) \int_a^b L_i(t) u(t) \, dt. \]

Approximating the Eq. (1) by

\[ u_n^R(s) - \sum_{j \in J_n} \sum_{k=1}^m H_k(s, t_i) \int_a^b L_i(t) u_n^R(t) \, dt = f(s), \quad (3) \]

the approximate solution \( u_n^R \) of Eq. (3) is given by

\[ u_n^R(s) = f(s) + \sum_{j \in J_n} c_j L_j(s), \]

for some scalars \( c_j \).

As a result, we obtain the linear system

\[ \sum_{j \in J_n} c_j L_j(s) - \sum_{j \in J_n} \sum_{k=1}^m H_k(s, t_i) \int_a^b L_i(t) f(t) \]

\[ + \sum_{k=1}^m c_k L_i(t) \, dt = 0. \]

Hence
\[ c_j - \sum_{i \in J_n} \sum_{k=1}^{m} c_i H_k(s_i, t_i) \int_a^b L_i(t)L_k(t) \, dt \]
\[ = \sum_{i \in J_n} \sum_{k=1}^{m} H_k(s_i, t_i) \int_a^b L_i(t)f(t) \, dt, \]
that is to say, the coefficients \( c_j \) are obtained by solving the following linear system
\[ (I - A_n^{RF})x_n^{RF} = b_n^{RF}, \]
where, for \( j \in J_n \) and \( i \in J_n \),
\[ A_n^{RF}(j, i) := \sum_{k=1}^{m} H_k(s_i, t_i) \int_a^b L_i(t)L_k(t) \, dt, \]
\[ b_n^{RF}(j) := \sum_{k=1}^{m} H_k(s_i, t_i) \int_a^b L_i(t)f(t) \, dt. \]

\[ u - u_n^F = [(I - T)^{-1} - (I - T_n^{RF})^{-1}]f \]
\[ = (I - T_n^{RF})^{-1} [T - T_n^{RF}]^{-1}f \]
\[ = (I - T_n^{RF})^{-1} [Tu - T_n^{RF}u], \]
and
\[ H_k^n(s, t) = QH_k^i(s). \]

We obtain
\[ \|TFu - T_n^{RF}u\| \leq c_0 h^8 \sum_{k=1}^{m} \left| \frac{\partial^8 H_k}{\partial s^8} \right| \|u\|, \]
for some constant \( c_0 \) independent of \( n \), hence
\[ \|u - u_n^F\| \leq c_1 c_0 h^8 \sum_{k=1}^{m} \left| \frac{\partial^8 H_k}{\partial s^8} \right| \|u\|. \]
Letting \( c := c_1 c_0 \), the desired result is achieved.

**Theorem 2** Let \( H_k(.i, .) \in C^8([a, b] \times [a, b]), \quad k = 1...m. \)
The following estimate holds:
\[ \|u - u_n^F\| \leq \beta h^8 \sum_{k=1}^{m} \left| \frac{\partial^8 H_k}{\partial s^8} \right| \|u\|, \]
where \( \beta \) is a constant independent of \( n \).

**Proof** We have
\[ \|T - T_n^{RF}\| = \max_{a \leq s \leq b} \sum_{k=1}^{m} \int_a^b |H_k(s, t) - H_k^n(s, t)| \, dt, \]
and
\[ \|T - T_n^{RF}\| \leq \frac{\|T - (I - T_n^{RF})\|}{1 - \|T - (I - T_n^{RF})\|}. \]

Since \( T \) is compact, according to [6], the operator \( T - T_n^{RF} \) is invertible for \( n \) is large enough, and its inverse is uniformly bounded with respect to \( n \). Then there exists \( c_1 > 0 \), such as
\[ \sup_n \|T - T_n^{RF}\| \leq c_1. \]

Since
Letting $\beta := c_2 c_3$, the desired result is obtained.

**Theorem 3** Let $H_k(\cdot, \cdot) \in C^8([a, b] \times [a, b])$, $k = 1 \ldots m$. The following estimate holds:

$$\|u - u^\text{RF}_n\|_\infty \leq \gamma h^8 \sum_{k=1}^{m} \left| \frac{\partial^8 H_k}{\partial s^8} \right|_{\infty} + 10 \left| \frac{\partial^8 F_k}{\partial s^8} \right|_{\infty} \|u\|_\infty,$$

where $\gamma$ is a constant independent of $n$.

**Proof** Since

$$\|T - T^\text{RF}_n\|_\infty \leq \sum_{k=1}^{m} \left( \|(I - Q)H_k(\cdot)\|_\infty + \|Q\|(I - Q)H_k(\cdot)\|_\infty \right),$$

we obtain

$$\|T - T^\text{RF}_n\|_\infty \leq h^8 \sum_{k=1}^{m} \left| c_4 \frac{\partial^8 H_k}{\partial s^8} \right|_{\infty} + 10 c_5 \left| \frac{\partial^8 F_k}{\partial s^8} \right|_{\infty}.$$

On the other hand,

$$u - u^\text{RF}_n = (I - T^\text{RF}_n)^{-1} [Tu - T^\text{RF}_n u],$$

and

$$\|(I - T^\text{RF}_n)^{-1}\| \leq c_6$$

for $n$ sufficiently large.

Letting $\gamma := c_6 \max c_4 c_5$, the desired result is obtained.

**Numerical examples**

In Examples 1 and 2, we compare the results we obtained with previous results presented in [3]. For a high accuracy, we raised the degree to seven, while the mentioned paper worked on degree four. Examples 1 and 2 show the accuracy of our results vis-a-vis of [3]. Denote by $R^F_n$, $R^R_n$ and $R^\text{RF}_n$ error terms for the above three septic spline degenerate kernel method, respectively. We compare our methods with other methods such as discrete Galerkin methods and discrete collocation methods given in [8], Nyström methods given in [5], Iteration methods given in [4] and Petrov–Galerkin elements via Chebyshev polynomials described in [2].

**Example 1**

We consider the following Fredholm integral equation

$$u(s) - \frac{1}{2} \int_0^1 (s + 1) e^{-st} u(t) dt = e^{-s} - \frac{1}{2} + \frac{1}{2} e^{-s(t+1)}, \quad 0 \leq s \leq 1.$$

The exact solution is $u(s) = e^{-s}$. We present in Table 8 the corresponding absolute errors $R^F_n$, $R^R_n$ and $R^\text{RF}_n$ respectively for this example. We compare our results with the results given in [3].

**Example 2**

Consider the following Fredholm integral equation

$$u(s) - \int_0^1 \sin(s) \cos(t) u(t) dt = \sin s, \quad 0 \leq s \leq \frac{\pi}{2}.$$

The exact solution $u(s) = 2 \sin(s)$. We present in Table 9 the numerical results for this example.

**Example 3**

Consider the following Fredholm integral equation

$$u(s) + 2 \int_0^1 e^{-s-t} u(t) dt = 2se^t, \quad 0 \leq s \leq 1.$$

The exact solution is $u(s) = 2e^s(s - \frac{1}{2})$. We present in Table 10 the numerical results for this example.

**Table 8** Example 1

| $n$  | $R^F_n$ | $R^R_n$ in [3] | $R^R_n$ | $R^\text{RF}_n$ in [3] | $R^\text{RF}_n$ |
|------|---------|----------------|---------|----------------------|---------------|
| 8    | 1.2978e–10 | 4.5–8          | 4.2236e–11 | 1.2e–8               | 1.2533e–10    |
| 16   | 5.0865e–10 | 1.6–9          | 1.4638e–13 | 2.6e–10              | 5.2422e–13    |
| 32   | 9.8810e–15 | 4.8e–11        | 5.2736e–15 | 4.6e–12              | 2.0817e–15    |

**Table 9** Example 2

| $n$  | $R^F_n$ | $R^F_n$ in [3] | $R^R_n$ | $R^R_n$ in [3] | $R^\text{RF}_n$ |
|------|---------|----------------|---------|----------------|---------------|
| 8    | 3.0468e–008 | 2.7–6          | 4.2236e–11 | 9.5e–7           | 3.4793e–8     |
| 16   | 1.0291e–10  | 8.7e–8         | 5.4732e–11 | 9.5e–8           | 1.1094e–10    |
| 32   | 3.8120e–13  | 2.7e–9         | 2.9407e–13 | 2.8e–10          | 5.8328e–13    |

The exact solution is $u(s) = e^{-s}$. We present in Table 8 the corresponding absolute errors $R^F_n$, $R^R_n$ and $R^\text{RF}_n$ respectively for this example. We compare our results with the results given in [3].
Table 11 Example 4

| n  | $R_n^F$ | $R_n^R$ | $R_n^{RF}$ | Galerkin method in [8] | Collocation method in [8] |
|----|--------|--------|-----------|----------------------|------------------------|
| 8  | 2.8380e − 11 | 5.2729e − 11 | 7.1493e − 11 | 7.893595e − 05 | 6.855828e − 05 |
| 16 | 8.5876e − 14 | 2.4126e − 13 | 3.2456e − 13 | 9.778065e − 06 | 8.364251e − 06 |

Table 12 Example 5

| n  | $R_n^F$ | $R_n^R$ | $R_n^{RF}$ | $E_1$ in [4] | $E_2$ in [4] |
|----|--------|--------|-----------|-------------|-------------|
| 8  | 1.8865e − 8 | 7.0747e − 10 | 1.9520e − 08 | 3.05e − 02 | 7.02e − 05 |
| 16 | 5.5321e − 11 | 1.9194e − 12 | 5.3475e − 11 | 7.99e − 03 | 4.38e − 06 |
| 32 | 1.9158e − 13 | 1.4211e − 14 | 2.0393e − 13 | 2.05e − 03 | 2.74e − 07 |

Example 4 (cf.[8])

We consider the following Fredholm integral equation

$$u(s) = \frac{1}{2} \int_0^1 e^{t} u(t) dt = f(s), \quad 0 \leq s \leq 1.$$ 

The exact solution is $u(s) = e^{-s}\cos s$. We present in Table 11 the corresponding absolute errors $R_n^F$, $R_n^R$ and $R_n^{RF}$ respectively for this example. We compare our results with the results of discrete Galerkin methods and discrete collocation methods, respectively, given in [8].

Example 5 (cf.[4])

We consider the following Fredholm integral equation

$$u(s) - \int_0^1 s e^{s-t} u(t) dt = s e^s, \quad 0 \leq s \leq 1.$$ 

The exact solution is $u(s) = 2s e^s$. We present in Table 12 the corresponding absolute errors $R_n^F$, $R_n^R$ and $R_n^{RF}$ respectively for this example. We compare our results with the results given in [4] using iteration methods based on the classical continuous piecewise linear and quadratic Lagrange interpolants. We give the corresponding absolute errors $E_1$, $E_2$, respectively (Table 12).

Example 6 (cf.[2])

We consider the following Fredholm integral equation

$$u(s) - \int_0^1 \frac{e^{s} \sin s}{1 + r^2} u(t) dt = f(s), \quad 0 \leq s \leq 1.$$ 

The exact solution is $u(s) = s^3$. We present in Table 13 the corresponding absolute errors $R_n^F$, $R_n^R$ and $R_n^{RF}$ respectively for this example. We compare our results with the results of Petrov–Galerkin elements via Chebyshev polynomials described in [2], for $k = 1$ and $n = 10$.

Conclusions

In this paper, we present three degenerate kernel methods to numerically solve generalized Fredholm integral equation of the second kind, working on the septic spline quasi-interpolants. These methods are constructed to approach the kernel of the correspondent integral operator. While the first method is an approximation on the left, the second is on right. The last method, nevertheless, combines the former two methods. The strength of this combination lies in the reduction of integrals and calculations in the linear system.

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