A LIOUVILLE THEOREM ON COMPLETE NON-KÄHLER MANIFOLDS

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Abstract. In this paper, we prove a Liouville theorem for holomorphic functions on a class of complete Gauduchon manifolds. This generalizes a result of Yau for complete Kähler manifolds to the complete non-Kähler case.

1. Introduction

Let \((M, g)\) be a complete Riemannian manifold and \(\Delta\) be the Beltrami-Laplacian of the Riemannian metric. In [18], Yau studied the equation

\[(1.1) \quad \Delta \log u = f,\]

and proved that if \(0 < \int_M f dv_g \leq \infty\) or if \(f \equiv 0\), then the equation (1.1) has no non-constant \(L^p\)-solutions for \(0 < p < \infty\). As an application, Yau obtained the following \(L^p\) Liouville theorem.

**Theorem 1.1** ([18], Theorem 4). Let \((M, \omega)\) be a complete Kähler manifold. Then there is no non-constant \(L^p\) holomorphic functions for \(p > 0\).

It should be pointed out that, under the assumption that \((M, \omega)\) has sectional curvature of the same sign, Greene-Wu ([7]) got some lower bound for the \(L^p\) integral of holomorphic functions. In this paper, we want to study the \(L^p\) Liouville theorem of holomorphic functions for some complete non-Kähler manifolds.

From now on, let \((M, g)\) be a complete Hermitian manifold of complex dimension \(n\) and \(\omega\) the associated (1,1)-form. The metric \(g\) is called Gauduchon if \(\omega\) satisfies \(\partial\bar{\partial}\omega^{n-1} = 0\). In [6], Gauduchon proved that, when \((M, g)\) is compact, there must exist a Gauduchon metric \(g_0\) in the conformal class of \(g\). It is interesting to generalize some geometric results from Kähler manifolds to Gauduchon manifolds, for example, the Donaldson-Uhlenbeck-Yau theorem ([14], [4], [17]) is valid for Gauduchon manifolds (see [2, 10, 12, 11]).

Let \(\varphi\) be a holomorphic function on \(M\), it is easy to check that \(\partial\bar{\partial} \log |\varphi| \equiv 0\). Following Yau’s argument in [18], we consider the following equation

\[(1.2) \quad \tilde{\Delta} \log u = \sqrt{-1} \Lambda_w \partial\bar{\partial} \log u = f,\]

**2010 Mathematics Subject Classification:** 53C55; 53C07; 58E20.

**Key words and phrases:** holomorphic function; Gauduchon manifold; Liouville theorem

The authors were supported in part by NSF in China, No. 11625106, 11571332 and 11721101.
where $\Lambda_\omega$ denotes the contraction with $\omega$, $u \geq 0$ and the Hausdorff measure of $\{x \in M | u(x) = 0\}$ is zero. It is well known that the difference of two Laplacian is given by a first order differential operator as follows

$$\left(\tilde{\Delta} - \frac{1}{2}\Delta\right)\psi = V \cdot \nabla\psi,$$

where $V$ is a vector field on $M$. In the non-Kähler case, we should handle the first order term, the key is to control the vector field $V$ in order to use the Stokes’ theorem which was proved by Yau (Lemma in [18]). In fact, we can prove the following theorem.

**Theorem 1.2.** Let $(M, g)$ be a complete Gauduchon manifold of complex dimension $n$ with $|d(\omega^{n-1})| \in L^\infty(M)$. Suppose $f$ is bounded from below by a constant and is Lebesgue integrable with $0 < \int_M f dv_g \leq \infty$ or $f \equiv 0$. Then there is no non-constant $L^p$ smooth solution of the equation (1.2) for $0 < p < \infty$.

As an application, we obtain the following Liouville theorem.

**Theorem 1.3.** Let $(M, g)$ be a complete Gauduchon manifold of complex dimension $n$ with $|d(\omega^{n-1})| \in L^\infty(M)$. Then there is no non-constant $L^p$ holomorphic functions for $0 < p < \infty$.

**Remark:** Let $(M_1, g_1)$ be a compact Gauduchon manifold and $(M_2, g_2)$ be a complete Kähler manifold, it is easy to check that the product Riemannian manifold $(M_1 \times M_2, g_1 \times g_2)$ is a Gauduchon manifold satisfying the assumption in Theorem 1.2.

This paper is organized as follows. In Section 2, we give a proof of Theorems 1.2. In Section 3, we consider a vanishing theorem on Higgs bundles over complete non-Kähler manifolds.

2. Proof of Theorem 1.2

In [18], Yau established the following generalized Stokes theorem which is an extension of Gaffney’s result ([5]).

**Lemma 2.1 ([18, Lemma]).** Let $(N, h)$ be a complete Riemannian manifold and $\eta$ be a smooth integrable $(\dim N - 1)$-form defined on $M$. Then there exists a sequence of domains $B_i$ in $N$ such that $N = \cup_i B_i$, $B_i \subset B_{i+1}$ and $\lim_{i \to +\infty} \int_{B_i} d\eta = 0$.

Let $\epsilon > 0$, as that in [18], we set $v_\epsilon = (u + \epsilon)^\frac{1}{2}$. Form direct computations, it follows that

(2.1) \[\sqrt{-1}\Lambda_\omega \partial\bar{\partial} \log v_\epsilon = \sqrt{-1}\Lambda_\omega \partial\bar{\partial} \log u \cdot \frac{u}{2(u + \epsilon)} + \frac{\epsilon |\partial u|^2}{2u(u + \epsilon)}\]

and

(2.2) \[\sqrt{-1}\Lambda_\omega \partial\bar{\partial} \log v_\epsilon = \sqrt{-1}\Lambda_\omega \frac{\partial\bar{\partial} v_\epsilon}{v_\epsilon} - \sqrt{-1}\Lambda_\omega \frac{\partial v_\epsilon \land \bar{\partial} v_\epsilon}{v_\epsilon^2}.\]

Thus we have

(2.3) \[v_\epsilon \sqrt{-1}\Lambda_\omega \partial\bar{\partial} v_\epsilon = |\partial v_\epsilon|^2 + \frac{\sqrt{-1}\Lambda_\omega \partial\bar{\partial} \log u}{2} \cdot u + \frac{\epsilon |\partial u|^2}{2u}.\]
Let $0 < R_1 < R_2$ be any positive numbers. Fix a point $x_0 \in M$, and choose a nonnegative cut-off function $\varphi$ satisfying
\[
\varphi(x) = \begin{cases} 
1, & x \in B_{x_0}(R_1), \\
0, & x \in M \setminus B_{x_0}(R_2),
\end{cases}
\]
$0 \leq \varphi \leq 1$ and $|d\varphi|_\omega \leq \frac{C}{R_2 - R_1}$, where $C$ is a positive constant and $B_{x_0}(r)$ is the geodesic ball centered at $x_0$ with radius $r$. According to the Stokes formula and the condition $\partial\Omega(\omega^{n-1}) = 0$, one can check that
\[
\int_{B_{x_0}(R_2)} \varphi^2 u \sqrt{-1} \Lambda_\omega \partial \Omega u \omega^n \frac{n!}{n!} \\
= \int_M \varphi^2 v_\epsilon \sqrt{-1} \overline{\partial} \partial v_\epsilon \wedge \frac{\omega^{n-1}}{(n-1)!} \\
= \int_M (\partial(\varphi^2 \epsilon \sqrt{-1} \partial v_\epsilon \wedge \frac{\omega^{n-1}}{(n-1)!}) - \sqrt{-1} \partial \varphi^2 \wedge v_\epsilon \partial v_\epsilon \wedge \frac{\omega^{n-1}}{(n-1)!} \\
- \sqrt{-1} \varphi^2 \partial v_\epsilon \wedge \partial v_\epsilon \wedge \frac{\omega^{n-1}}{(n-1)!} + \varphi^2 v_\epsilon \sqrt{-1} \overline{\partial} \partial v_\epsilon \wedge \frac{\omega^{n-1}}{(n-1)!}) \\
= -2 \int_M \varphi v_\epsilon \langle \partial \varphi, \overline{\partial} v_\epsilon \rangle \omega^n \frac{n!}{n!} - \int_M \varphi^2 |\partial v_\epsilon|^2 \frac{\omega^n}{n!} + \frac{1}{2} \int_M \varphi^2 \sqrt{-1} \partial v_\epsilon \wedge \partial \varphi^2 \wedge \frac{\omega^{n-1}}{(n-1)!} \\
= -2 \int_M \varphi v_\epsilon \langle \partial \varphi, \overline{\partial} v_\epsilon \rangle \omega^n \frac{n!}{n!} - \int_M \varphi^2 |\partial v_\epsilon|^2 \frac{\omega^n}{n!} \\
+ \frac{1}{2} \int_M \sqrt{-1} \partial(\varphi^2 v_\epsilon \partial \varphi^2 \wedge \frac{\omega^{n-1}}{(n-1)!}) - \sqrt{-1} v_\epsilon^2 \overline{\partial} \varphi^2 \wedge \partial \varphi^2 \wedge \frac{\omega^{n-1}}{(n-1)!}) \\
\leq \int_M (v_\epsilon^2 |\partial \varphi|^2 + C^* v_\epsilon^2 |\partial \varphi|) \omega^n \frac{n!}{n!},
\]
where we have used the condition that $|d(\omega^{n-1})| \in L^\infty(M)$ in the last inequality.

Combining (1.2), (2.3) and (2.5), we obtain
\[
\frac{1}{4} \int_M \varphi^2 |\partial u|^2 \frac{\omega^n}{n!} \\
= \int_M \varphi^2 |\partial v_\epsilon|^2 \frac{\omega^n}{n!} \\
\leq \int_M v_\epsilon^2 (|\partial \varphi|^2 + C^* |\partial \varphi|) \frac{\omega^n}{n!} - \int_M \frac{f \varphi^2 u}{2} \\
\leq \left( \frac{C^2}{(R_1 - R_2)^2} + \frac{C^* C}{(R_2 - R_1)} \right) \int_{B_{x_0}(R_2)} v_\epsilon^2 \frac{\omega^n}{n!} - \frac{1}{2} \int_M \frac{f \varphi^2 u \omega^n}{n!},
\]
and
\[
\frac{1}{4} \int_{B_{x_0}(R_2)} \varphi^2 |\partial u|^2 \frac{\omega^n}{n!} \\
\leq \left( \frac{C^2}{(R_1 - R_2)^2} + \frac{C^* C}{(R_2 - R_1)} \right) \int_{B_{x_0}(R_2)} u^2 \frac{\omega^n}{n!} - \frac{1}{2} \int_{B_{x_0}(R_2)} \frac{f \varphi^2 u \omega^n}{n!}.
\]
Let \( R_1 = 2R_2 \to \infty \), we deduce
\[
\int_M |u|^2 \frac{\omega^n}{u} < \infty,
\]
and then
\[
\int_M |u|^n \frac{\omega^n}{n!} \leq \left( \int_M |u|^2 \frac{\omega^n}{u} \right)^{\frac{1}{2}} \left( \int_M |u|^n \frac{\omega^n}{n!} \right)^{\frac{1}{2}} < \infty.
\]

On the other hand, it is easy to see that
\[
\sqrt{-1} \Lambda \omega \partial \bar{\partial} \log u_{\epsilon} = \sqrt{-1} \partial \bar{\partial} (\omega^{n-1}) = 0
\]
and
\[
\int_M |\partial \log u_{\epsilon}| \frac{\omega^n}{n!} = \int_M \frac{|\partial u|}{u + \epsilon} \frac{\omega^n}{n!} \leq \int_M \frac{|\partial u|}{\epsilon} \frac{\omega^n}{n!} < \infty
\]
for any \( \epsilon > 0 \), where \( u_{\epsilon} = u + \epsilon \). Using the Stokes formula and the condition \( \partial \bar{\partial}(\omega^{n-1}) = 0 \) again, we derive
\[
\sqrt{-1} \Lambda \omega \partial \bar{\partial} \log u_{\epsilon} \frac{\omega^n}{n!} = \sqrt{-1} \partial \bar{\partial} (\omega^{n-1}) + \sqrt{-1} \partial \bar{\partial} (\omega_{\epsilon}^{n-1})
\]
\[
= \sqrt{-1} \partial \bar{\partial} (\log u_{\epsilon} \frac{\omega^{n-1}}{(n-1)!}) + \sqrt{-1} \partial \bar{\partial} (\log \frac{u}{\epsilon} + 1) \frac{\partial \omega^{n-1}}{(n-1)!}).
\]

Notice that
\[
\int_M \log (\frac{u}{\epsilon} + 1) \frac{\omega^n}{n!} \leq \int_M \frac{u \omega^n}{\epsilon n!} < \infty.
\]
This together with (2.11), the condition that \( |d(\omega^{n-1})| \in L^\infty(M) \) and lemma 2.1 gives us that
\[
0 = \lim_{i \to +\infty} \int_{B_i} \Delta \log u_{\epsilon} = \int_M \frac{fu}{u + \epsilon} \frac{\omega^n}{n!} + \int_M \frac{1}{u(u + \epsilon)^2} |du|^2 \frac{\omega^n}{n!}
\]
for all \( \epsilon > 0 \), where \( B_i \)'s are the ones in the lemma 2.1. By the same argument in [18], we know that \( u \) must be a constant. The proof of Theorem 1.2 is therefore completed.

3. A vanishing theorem on Higgs bundle

Let \((M, \omega)\) be an \( n \)-dimensional Hermitian manifold. A Higgs bundle \((E, \overline{\partial} E, \theta)\) over \( M \) is a holomorphic bundle \((E, \partial E)\) coupled with a Higgs field \( \theta \in \Omega^{1,0} \text{End}(E) \) such that \( \overline{\partial} E \theta = 0 \) and \( \theta \wedge \theta = 0 \). Higgs bundles first emerged thirty years ago in Hitchin’s ([8]) study of the self duality equations on a Riemann surface and in Simpson’s subsequent work ([15], [16]) on nonabelian Hodge theory. Such objects have rich structures and play an important role in many areas including gauge theory, Kähler and hyperkähler geometry, group representations and nonabelian Hodge theory. Let \( H \) be a Hermitian metric on the bundle \( E \), consider the Hitchin-Simpson connection
\[
\overline{\partial}_{\theta} := \overline{\partial} E + \theta, \quad D_{H,\theta}^{1,0} := D_{H}^{1,0} + \theta^* u, \quad D_{H,\theta} = \overline{\partial}_{\theta} + D_{H,\theta}^{1,0},
\]
where $D_H$ is the Chern connection of $(E, \overline{\partial}_E, H)$ and $\theta^{*H}$ is the adjoint of $\theta$ with respect to the metric $H$. The curvature of this connection is

\begin{equation}
F_{H,\theta} = F_H + [\theta, \theta^{*H}] + \partial_H \theta + \overline{\partial}_E \theta^{*H},
\end{equation}

where $F_H$ is the curvature of $D_H$ and $\partial_H$ is the $(1,0)$-part of $D_H$.

Let $s$ be a $\theta$-invariant holomorphic section of a Higgs bundle $(E, \overline{\partial}_E, \theta)$, i.e. there exists a holomorphic 1-form $\eta$ on $M$ such that $\theta(s) = \eta \otimes s$. When the base manifold $(M, \omega)$ is compact, following Kobayashi’s techniques [9], one can obtain vanishing theorems for $\theta$-invariant holomorphic sections on Higgs bundles or Higgs sheaves (see [1] [3] [13]). Now we consider the case that the base manifold is complete non-Kähler.

Since $s$ is $\theta$-invariant, by the formula (4.50) in [13], we know

\begin{equation}
\sqrt{-1} \Lambda_\omega \langle s, -[\theta, \theta^{*H}]s \rangle_H = |\theta^{*H} s - \langle \theta^{*H} s, s \rangle_H \frac{s}{|s|^2_H} |^2_{H,\omega} \geq 0.
\end{equation}

A straightforward calculation shows the following Weitzenböck formula

\begin{equation}
\sqrt{-1} \Lambda_\omega \partial \overline{\partial} \log |s|^2_H = |D_{H,\omega}^{1,0} s|^2_{H,\omega} + \sqrt{-1} \Lambda_\omega \langle s, F_H s \rangle_H
\end{equation}

where

\begin{align}
&= |D_{H,\omega}^{1,0} s|^2_{H,\omega} - \langle s, \sqrt{-1} \Lambda_\omega F_{H,\omega} s \rangle_H - \sqrt{-1} \Lambda_\omega \langle s, [\theta, \theta^{*H}]s \rangle_H \\
&\geq |D_{H,\omega}^{1,0} s|^2_{H,\omega} - \langle s, \sqrt{-1} \Lambda_\omega F_{H,\omega} s \rangle_H
\end{align}

on $M$. On the other hand, it holds that

\begin{equation}
\sqrt{-1} \Lambda_\omega \partial \overline{\partial} \log |s|^2_H \leq |s|^2_H |D_{H,\omega}^{1,0} s|^2_H.
\end{equation}

Then these yield that

\begin{equation}
\sqrt{-1} \Lambda_\omega \partial \overline{\partial} \log |s|^2_H
= \frac{1}{|s|^2_H} \sqrt{-1} \Lambda_\omega \partial \overline{\partial} |s|^2_H - \frac{1}{|s|^2_H} \sqrt{-1} \Lambda_\omega \partial |s|^2_H \wedge \overline{\partial} |s|^2_H
\geq - \frac{1}{|s|^2_H} \langle s, \sqrt{-1} \Lambda_\omega F_{H,\omega} s \rangle_H.
\end{equation}

Together with Theorem [1,2] we conclude

**Theorem 3.1.** Let $(M, \omega)$ be a complete Gauduchon manifold of complex dimension $n$ with $|d(\omega^{n-1})| \in L^\infty(M)$, and $(E, \overline{\partial}_E, \theta)$ be a Higgs bundle over $M$. If there exists a Hermitian metric $H$ on $E$ such that $\sqrt{-1} \Lambda_\omega F_{H,\omega} \leq -f I d_E$, where $f$ is a continuous function on $M$ which is bounded from below and $\int_M f \omega^n > 0$. Then $E$ admits no non-zero $\theta$-invariant holomorphic $L^p$-sections for $0 < p < \infty$.

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