Abstract. Plasticity is a classic hysteresis phenomenon. We represent elasto-plasticity with kinematic strain-hardening for multiaxial systems via the differential inclusion
\[ a \frac{d\varepsilon}{dt} - b \frac{d\sigma}{dt} \in \partial I_K(c\sigma - d\varepsilon) \ (\text{with } a, b, c, d \text{ constants } \geq 0); \]
here \( I_K \) is the indicator function of a (nonempty) closed convex set \( K \) of the space of symmetric \( 3 \times 3 \)-deviators. We formulate a hyperbolic initial- and boundary-value problem, and provide a well-posedness result. We then assume that the material coefficients rapidly oscillate w.r.t. space, formulate a corresponding two-scale problem, and discuss the homogenization.

Introduction

Plasticity was one of the very first hysteresis phenomena to be studied. Early models due to Tresca, Saint Vénant and Lévy date back to mid XIX century. In 1913 von Mises proposed his well-known yield criterion for isotropic solids; in 1924 Prandtl introduced a scalar model of elasto-plasticity, which was then extended to tensors by Reuss in 1930. In 1928 Prandtl proposed a more general model accounting for work-hardening, which was then also studied by Timoshenko and Ishlinskii. Another model of work-hardening was proposed by Melan in 1938, and in 1949 Prager introduced the model of kinematic strain-hardening (which we deal with in this paper, too). In 1948 Hill formulated the maximal dissipation principle of plasticity; in 1949 Hodge, Prager and Greenberg proposed two minimum principles for elasto-plasticity. In the 1960’s plasticity was formulated in terms of variational inequalities. References to these works may be found e.g. in [28]; [15] provides a picture of the development of research in plasticity up to 1960; more recent developments are illustrated e.g. in [13]. Other hysteresis behaviours also occur in continuum mechanics; these include that of the recently developed shape memory alloys, that are of theoretical and applied interest.

Let us denote the stress tensor by \( \sigma \), the linearized strain tensor by \( \varepsilon \). In this note we represent the classic Prager model of elasto-plasticity with linear kinematic strain-hardening for multiaxial systems in the form
\[ a \frac{d\varepsilon}{dt} - b \frac{d\sigma}{dt} \in \partial I_K(c\sigma - d\varepsilon) \ (\text{with } a, b, c, d \text{ constants } \geq 0). \] (0.1)

Here \( I_K \) is the indicator function of a (nonempty) closed convex set \( K \) of the space of symmetric \( 3 \times 3 \)-deviators (i.e., \( I_K = 0 \) in \( K \), \( I_K = +\infty \) outside \( K \)). By \( \partial I_K \) we denote its subdifferential; for
Here we deal with the evolution of a space-distributed system. We denote by \( \Omega \) the domain occupied by the body, by \( \vec{u} \) the displacement vector, by \( \rho \) the density, and by \( \vec{f} \) a distributed load. We couple (0.1) with the dynamical equation
\[
\rho \frac{\partial^2 \vec{u}}{\partial t^2} - \nabla \cdot \sigma = \vec{f},
\]
and with boundary conditions, provide a weak formulation and illustrate its well-posedness.

This also applies to composite materials, and we deal with a nonhomogeneous material that exhibits fast oscillations w.r.t. space. More specifically, we assume that the yield criterion \( K \) and the coefficients \( a, b, c \) and \( d \) depend not only on the coarse-scale \( x \) but also on a finer length-scale, that we represent by a variable \( y \) that ranges through a reference cell \( Y := [0,1]^3 \). We also assume that the latter dependence is periodic. We then relate the two scales \( x, y \) by setting \( y = x/\eta \), let the parameter \( \eta \) vanish, and derive a two-length-scale model via Nguetseng’s notion of two-scale convergence [1, 23].

Aside from a large literature dealing with elasto-plasticity, cf. e.g. [7, 12, 13, 18, 19, 22], the monographs [5, 16, 17, 28] and the abovementioned works [10, 11] applied the point of view of hysteresis. In [4] Blanchard, Le Tallec and Ravachol considered a more general constitutive law than (0.1), proved existence of a solution for the dynamical problem and studied its numerical approximation; see also [3, 26]. The mechanical behaviour of composite materials was studied via homogenization in a number of works, see e.g. [2, 6, 21, 25, 27]. In the univariate setting the homogenization of elasto-plasticity was already studied by Franci and Krejčí in [10, 11], who derived a Prandtl-Ishlinskii model.

This note develops a formulation that was introduced in [29] and illustrates results of [31], where the more general setting of nonlinear elasto-viscosity is studied and a single-scale homogenized problem is also derived under suitable restrictions.

1. A model of elasto-plasticity
We assume that deformations are so small that the Euler and Lagrange coordinates may be assumed to be identical. In most of the models of continuum mechanics a linear elastic relation is assumed between the spherical components \( \sigma_s \) and \( \varepsilon_s \) of the stress and strain tensors:
\[
\sigma_s = a \varepsilon_s \quad \text{for some constant } a > 0.
\]
On the other hand several constitutive laws have been proposed to relate the deviatoric components \( \sigma_d \) and \( \varepsilon_d \). We just consider linear elasticity and rigid perfect plasticity that we represent via
\[
\sigma_d = 2 \mu \varepsilon_d \quad \text{for some constant } \mu > 0,
\]
Here the yield criterion \( K \) is a (nonempty) closed convex subset of \( D_s^3 \), the linear space of symmetric 3×3 deviatoric tensors (henceforth we denote the subspace of symmetric tensors by the index \( s \)). This inclusion is equivalent to the variational inequality
\[
\sigma_d \in K, \quad \frac{d\varepsilon_d}{dt} : (\sigma_d - v) \geq 0 \quad \forall v \in K,
\]
or
\[
\sigma_d \in K, \quad \frac{d\varepsilon_d}{dt} : \sigma_d = I_K^* \left( \frac{d\varepsilon_d}{dt} \right).
\]
The convex conjugate function $I^*_K$ is a positively homogenous function of degree one, which is called the support function of the convex set $K$.

More complex constitutive models can be obtained by composing these basic behaviours. This may conveniently be represented by means of so-called rheological models, which typically relate an applied force to the corresponding deformation and/or to the time derivative of the latter. Strain and stress relations are then derived on the basis of the identifications force $\leftrightarrow$ stress and deformation $\leftrightarrow$ strain. These models are uniaxial, but the corresponding constitutive relations are extrapolated to the deviatoric component of stress and strain tensors. However in a multiaxial setting series and parallel arrangements lose their original configurational meaning.

The inclusion (0.1) can be represented via the parallel combination between an elastic element $E_1$ and a model consisting in the series arrangement of an elastic element $E_2$ with a rigid perfectly-plastic element $P$; this corresponds to the rheological formula $E_1|_{(E_2 - V)}$. Let

$$\alpha, \beta \in \mathbb{R}, \quad 0 \leq \alpha < \beta, \quad K (\neq \emptyset)$$

be a closed convex subset of $D^0_s$, and consider the differential inclusion

$$\beta \frac{d\varepsilon}{dt} - \frac{d\sigma}{dt} \in \partial I_K(\sigma - \alpha \varepsilon). \quad (1.2)$$

This inclusion is obviously of the form (0.1), which is only apparently more general: it is easy to see that two of the parameters \{a, b, c, d\} may be normalized, by rescaling the convex set $K$ and by noticing that $\lambda \partial I_K = \partial I_K$ for any $\lambda > 0$.

By the Fenchel equality (a classic result of convex analysis [8, 14, 24]) this is tantamount to the inclusion $\sigma - \alpha \varepsilon \in K$ coupled with the equality

$$\left(\beta \frac{d\varepsilon}{dt} - \frac{d\sigma}{dt}\right): (\sigma - \alpha \varepsilon) = I_K(\sigma - \alpha \varepsilon) + I^*_K(\beta \frac{d\varepsilon}{dt} - \frac{d\sigma}{dt}). \quad (1.3)$$

By the next statement under this constitutive relation the spheric and deviatoric components decouple.

**Proposition 1.1.** The inclusion (1.2) is equivalent to the system

$$\begin{align*}
\sigma(s) &= \alpha \varepsilon(s) \\
\beta \frac{d\varepsilon(d)}{dt} - \frac{d\sigma(d)}{dt} &\in \partial I_K(\sigma(d) - \alpha \varepsilon(d)).
\end{align*} \quad (1.4)$$

**Proof.** As $K \subset D^0_s$, (1.2) entails (1.4)1. By the orthogonality between deviatoric and spheric tensors, this equality yields

$$\left(\beta \frac{d\varepsilon(d)}{dt} - \frac{d\sigma(d)}{dt}\right): (\sigma(d) - \alpha \varepsilon(d)) = \left(\beta \frac{d\varepsilon}{dt} - \frac{d\sigma}{dt}\right): (\sigma - \alpha \varepsilon).$$

By the definition of a convex conjugate function we have

$$I^*_K(v) = \sup_{u \in \mathbb{R}^d_+} \{v: u - I_K(u)\} = \sup_{u \in K} v: u$$

$$= \sup_{u \in K} v(d): u = \sup_{u \in \mathbb{R}^d_+} \{v(d): u - I_K(u)\} = I^*_K(v(d)) \quad \forall v \in \mathbb{R}^d_+;$$
hence

\[ I_K \left( \beta \frac{d\varepsilon}{dt} - \frac{d\sigma}{dt} \right) = I_K^* \left( \beta \frac{d\varepsilon}{dt} - \frac{d\sigma}{dt} \right). \]

Then (1.3) reads

\[ \left( \beta \frac{d\varepsilon}{dt} - \frac{d\sigma}{dt} \right) : (\sigma_{(d)} - \alpha \varepsilon_{(d)}) = I_K (\sigma_{(d)} - \alpha \varepsilon_{(d)}) + I_K^* \left( \beta \frac{d\varepsilon}{dt} - \frac{d\sigma}{dt} \right), \]

and by the Fenchel equality this is equivalent to (1.4)2.

**Proposition 1.2.** If two elastic elements \( E_1 \) and \( E_2 \) and a plastic element \( P \) are respectively characterized by the relations

\[ \sigma_{(d)} = \alpha \varepsilon_{(d)}, \quad \sigma_{(d)} = (\beta - \alpha) \varepsilon_{(d)}, \quad \partial I_K (\sigma_{(d)}) \ni (\beta - \alpha) \frac{d\varepsilon}{dt}, \]

then (1.4)2 is the rheological equation of the model \( E_1 \mid (E_2 - P) \) (i.e., the parallel combination of \( E_1 \) and the serial arrangement of \( E_2 \) and \( P \)), and represents a nonlinear elasto-plastic behaviour. For \( \alpha = 0 \) this model is reduced to \( E_2 - P \).

**Proof.** In order to simplify formulas, in this argument we omit the index \((d)\). We denote the deformation of the elements \( E_1, E_2, P \) and \( E_1 \mid (E_2 - P) \) by \( \varepsilon_1, \varepsilon_2, \varepsilon_3 \) and \( \varepsilon \), resp., and the forces acting on these elements by \( \sigma_1, \sigma_2, \sigma_3 \) and \( \sigma \), resp. It is then easy to see that

\[ \sigma_1 = \alpha \varepsilon_1, \quad \sigma_2 = (\beta - \alpha) \varepsilon_2, \quad \partial I_K (\sigma_3) \ni (\beta - \alpha) \frac{d\varepsilon_3}{dt}, \]

\[ \sigma = \sigma_1 + \sigma_2, \quad \sigma_2 = \sigma_3, \quad \varepsilon = \varepsilon_1 = \varepsilon_2 + \varepsilon_3. \]

Setting \( \gamma := (\beta - \alpha)^{-1} \), we get

\[ \sigma_3 = \sigma_2 = \sigma - \sigma_1 = \sigma - \alpha \varepsilon_1 = \sigma - \alpha \varepsilon, \]

\[ \varepsilon_3 = \varepsilon - \varepsilon_2 = \varepsilon - \gamma \sigma_2 = \varepsilon + \gamma (\sigma_1 - \sigma) \]

\[ = \varepsilon + \gamma (\alpha \varepsilon_1 - \sigma) = \varepsilon + \gamma (\alpha \varepsilon - \sigma) = \gamma (\beta \varepsilon - \sigma). \]

The inclusion (1.2) then follows.

It should be noticed that no internal variable occurs in this model. We also note that the above formulas yield

\[ \varepsilon_{(d)} = \varepsilon, \quad \varepsilon_{(d)2} = (\beta - \alpha)^{-1} (\sigma_{(d)} - \alpha \varepsilon_{(d)}), \quad \varepsilon_{(d)3} = (\beta - \alpha)^{-1} (\beta \varepsilon_{(d)} - \sigma_{(d)}), \]

\[ \sigma_{(d)} = \alpha \varepsilon_{(d)}, \quad \sigma_{(d)2} = \sigma_{(d)3} := \sigma_{(d)} - \alpha \varepsilon_{(d)}. \]

The constitutive relation (1.2) can be represented via hysteresis operators.

**Proposition 1.3.** Let (1.1) be satisfied, and \( \sigma^0, \varepsilon^0 \in \mathbb{R}^d \) be such that \( \sigma^0 - \alpha \varepsilon^0 \in K \). Then:

(i) For any \( \varepsilon \in H^1(0,T)^d \) such that \( \varepsilon(0) = \varepsilon^0 \), there exists one and only one \( \sigma \in H^1(0,T)^d \) that satisfies (1.2) and such that \( \sigma(0) = \sigma^0 \). This defines a causal operator named stop (see Figure 1(a) below):

\[ \mathcal{G}^\sigma_{\varepsilon^0} : H^1(0,T)^d \rightarrow H^1(0,T)^d : \varepsilon \mapsto \sigma. \]

(ii) If \( \alpha > 0 \) then for any \( \sigma \in H^1(0,T)^d \) such that \( \sigma(0) = \sigma^0 \), there exists one and only one \( \varepsilon \in H^1(0,T)^d \) that satisfies (1.2) and such that \( \varepsilon(0) = \varepsilon^0 \). This also defines a causal operator named play (see Figure 1(b) below):

\[ \mathcal{E}^\varepsilon_{\sigma^0} : H^1(0,T)^d \rightarrow H^1(0,T)^d : \sigma \mapsto \varepsilon. \]
Proof. (i) Setting \( \tilde{\sigma} := \sigma - \alpha \varepsilon \), (1.2) reads

\[
(\beta - \alpha) \frac{d\tilde{\sigma}}{dt} \in \partial I_K(\tilde{\sigma}).
\]

Adding the initial condition \( \tilde{\sigma}(0) = \sigma^0 - \alpha \varepsilon^0 \), this inclusion determines a Lipschitz-continuous causal operator \( H^1(0, T)_s^9 \rightarrow L^\infty(0, T)_s^9 \): formally this claim is easily checked by multiplying (1.5) by \( d\tilde{\sigma}/dt \); this might be made rigorous along the lines of [28]. Therefore \( \sigma = \tilde{\sigma} + \alpha \varepsilon \in H^1(0, T)_s^9 \).

(ii) Setting \( \tilde{\varepsilon} := \beta \varepsilon - \sigma \), (1.4) reads

\[
\frac{d\tilde{\varepsilon}}{dt} \in \partial I_K((1 - \frac{\alpha}{\beta})\sigma - \frac{\alpha}{\beta} \varepsilon), \quad \text{i.e.} \quad \partial I_K^*(\frac{d\tilde{\varepsilon}}{dt}) + \frac{\alpha}{\beta} \varepsilon = (1 - \frac{\alpha}{\beta})\sigma.
\]

We now prescribe the initial condition \( \tilde{\varepsilon}(0) = \beta \varepsilon^0 - \sigma^0 \). Taking the incremental ratio of the latter inclusion and multiplying by the incremental ratio of \( \tilde{\varepsilon} \), formally it is easy to see that \( d\tilde{\varepsilon}/dt \in L^2(0, T)_s^9 \). Thus \( \varepsilon = (\tilde{\varepsilon} + \sigma)/\beta \in H^1(0, T)_s^9 \). This argument also may be made rigorous via the techniques of [28].

\[\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1}
\caption{A stop and a play operator}
\end{figure}\]

The latter statement might be improved, but it suffices for our present purposes. By combining several stops (plays, resp.), the more general class of Prandtl-Ishlinskii models of stop-type (play-type, resp.) are obtained; cf. e.g. [5, 16, 17, 28].

2. Weak formulation

Henceforth we deal with the evolution of a (possibly heterogeneous) elasto-plastic material having density \( \rho \), that occupies a bounded domain \( \Omega \subset \mathbb{R}^3 \). We assume that

\[
\rho, \alpha, \beta \in L^\infty(\Omega), \quad \rho \geq \text{constant} > 0, \quad 0 < \alpha < \beta \quad \text{a.e. in } \Omega,
\]

and require the constitutive law (1.2) to hold pointwise:

\[
\beta \frac{\partial \varepsilon}{\partial t} - \frac{\partial \sigma}{\partial t} \in \partial I_K(\sigma - \alpha \varepsilon) \quad \text{a.e. in } \Omega_T := \Omega \times ]0, T[.
\]

We denote an applied load by \( \tilde{f} : \Omega_T \rightarrow \mathbb{R}^3 \), assume that the fields \( \tilde{u}^0, \tilde{v}^0, \tilde{g} \) are prescribed, denote by \( \tilde{\nu} \) the outward-oriented unit normal vector, and couple (2.2) with the following dynamical equation and initial- and boundary-conditions:

\[
\rho \frac{\partial^2 \tilde{u}}{\partial t^2} - \nabla \cdot \sigma = \tilde{f} \quad \text{in } \Omega_T,
\]
\[ \ddot{u}(\cdot,0) = \dot{u}^0, \quad \frac{\partial \ddot{u}}{\partial t}(\cdot,0) = \dot{v}^0 \quad \text{in } \Omega, \]  
\[ \sigma \cdot \dot{v} = \tilde{g} \quad \text{on } \partial \Omega \times ]0,T[. \]  
(2.4)

In view of the weak formulation of this problem, we define the density of elastic energy, \( U \), and that of total mechanical energy, \( E \) (kinetic plus elastic energy):

\[ U(\varepsilon, \sigma) := \frac{\alpha}{2} |\varepsilon|^2 + \frac{1}{2(\beta - \alpha)} |\sigma - \alpha \varepsilon|^2 \quad \forall (\varepsilon, \sigma) \in (\mathbb{R}^9_s)^2, \]  
(2.6)

\[ E(\frac{\partial \ddot{u}}{\partial t}, \varepsilon, \sigma) := \rho \left( \frac{\partial \ddot{u}}{\partial t} \right)^2 + U(\varepsilon, \sigma) \quad \text{a.e. in } \Omega. \]

(The derivation of (2.6) may be found in [31].)

**Proposition 2.1.** The constitutive relation (2.2) is equivalent to the condition

\[ \sigma - \alpha \varepsilon \in K \quad \text{a.e. in } \Omega_T, \]

coupled with the power balance

\[ \frac{d}{dt} \int_{\Omega} E\left( \frac{\partial \ddot{u}}{\partial t}, \varepsilon, \sigma \right) dx + \int_{\Omega} \Psi^* \left( \frac{1}{(\beta - \alpha)} \frac{\partial}{\partial t}(\beta \varepsilon - \sigma) \right) dx \]

\[ = \int_{\Omega} \tilde{f} \cdot \frac{\partial \ddot{u}}{\partial t} dx + \int_{\partial \Omega} \tilde{g} \cdot \frac{\partial \ddot{u}}{\partial t} dx \quad \text{a.e. in } ]0,T[. \]  
(2.7)

This equality is also equivalent to the inequality

\[ \frac{d}{dt} \int_{\Omega} E\left( \frac{\partial \ddot{u}}{\partial t}, \varepsilon, \sigma \right) dx + \int_{\Omega} \Psi^* \left( \frac{1}{(\beta - \alpha)} \frac{\partial}{\partial t}(\beta \varepsilon - \sigma) \right) dx \]

\[ \leq \int_{\Omega} \tilde{f} \cdot \frac{\partial \ddot{u}}{\partial t} dx + \int_{\partial \Omega} \tilde{g} \cdot \frac{\partial \ddot{u}}{\partial t} dx \quad \text{a.e. in } ]0,T[. \]

At first sight the equivalence between an equality and an inequality may look unexpected. This follows from a classic result of convex analysis, namely the equivalence between Fenchel’s equality and inequality, cf. [31] and the above-quoted monographs on convex analysis.

**A Boundary- and Initial-Value Problem.** We assume that \( \Omega \) is of Lipschitz class, and set \( L^2(\Omega)^9_s := L^2(\Omega; \mathbb{R}^9_s) \). We also identify \( H := L^2(\Omega)^3 \) with its dual \( H' \), which in turn can be identified with a subspace of the dual space \( V' \) of \( V := H^1(\Omega)^3; \) this yields the Hilbert triplet \( V \subset H = H' \subset V' \), with continuous injections. Denoting by \( \langle \cdot, \cdot \rangle \) the duality pairing between \( V' \) and \( V \), we define the linear and continuous operator

\[ \nabla^s : L^2(\Omega)^9_s \to V' : \quad \langle \nabla^s \cdot w, \tilde{v} \rangle := - \int_{\Omega} w : \nabla \tilde{v} \ dx \quad \forall w \in L^2(\Omega)^9_s, \forall \tilde{v} \in V. \]

We assume that (2.1) is satisfied and that

\[ \ddot{u}^0, \dot{v}^0 \in L^2(\Omega)^3, \quad \varepsilon^0 := \nabla^s \dot{u}^0 \in L^2(\Omega)^9_s, \quad \sigma^0 \in L^2(\Omega)^9_s, \]  
(2.8)

\[ \tilde{f} := f + \tilde{h}, \quad \tilde{f} \in L^1(0,T; L^2(\Omega)^3), \quad \tilde{h} \in W^{1,1}(0,T; V'). \]  
(2.9)

Now we introduce a weak formulation of the initial- and boundary-value problem (2.3)—(2.5).
Problem 2.2. Find \((\bar{u}, \sigma)\) such that, setting \(\varepsilon := \nabla^s \bar{u}\) and \(\bar{v} := \partial \bar{u}/\partial t\),

\[
\bar{u} \in W^{1,\infty}(0, T; L^2(\Omega)^3), \quad \sigma, \varepsilon \in L^\infty(0, T; L^2(\Omega))_x,
\]

\[
\sigma - \alpha \varepsilon \in K \quad \text{a.e. in } \Omega_T,
\]

\[
\int \Omega E(\bar{v}, \varepsilon, \sigma)(x, t) \, dx + \int_0^t \int \Omega \left( \frac{\partial}{\partial \tau} (\beta \varepsilon - \sigma) \right) \, dx \, dt 
\leq \int \Omega E(\bar{v}^0, \varepsilon^0, \sigma^0)(x) \, dx + \int_0^t \int \Omega (\varphi, \bar{v}) \, dx \, dt \quad \forall t \in [0, T],
\]

\[
\int \Omega \left( \rho \frac{\partial \bar{u}}{\partial t} \cdot \bar{w} + \sigma : \nabla \bar{w} \right) \, dx = \int \Omega (\bar{v}, \bar{w}) \quad \forall \bar{w} \in V, \ a.e. \ in \ [0, T],
\]

\[
\bar{u}(\cdot, 0) = \bar{u}^0, \quad \bar{v}(\cdot, 0) = \bar{v}^0 \quad a.e. \ in \ \Omega.
\]

**Interpretation.** The equation (2.11) yields the dynamical equation

\[
\rho \frac{\partial^2 \bar{u}}{\partial t^2} - \nabla^s \cdot \sigma = \varphi \quad \text{in } H^{-1}(0, T; L^2(\Omega)^3) + L^2(0, T; V').
\]
Interpretation. The equation (2.16) implies (2.12). If \( \vec{\varphi} \in L^1(0,T;L^2(\Omega)^3) \), by the time regularity of \( \vec{u} \) we infer that \( \nabla \cdot \sigma \in L^1(0,T;L^2(\Omega)^3) \), and that (2.3) holds pointwise in \( \Omega_T \). We can then multiply this equation by \( \partial \vec{u}/\partial t \), and thus get (2.7). By Proposition 2.1 the equation (2.15) is then equivalent to

\[
\int_I \int_{\Omega_T} I_n^k \left( \frac{\partial}{\partial t} (\beta \varepsilon - \sigma) \right) \, dx \, dt = \int_I \int_{\Omega_T} (\sigma - \alpha \varepsilon) \frac{\partial}{\partial t} (\beta \varepsilon - \sigma) \, dx \, dt;
\]

coupled with (2.14), this equality is tantamount to the inclusion (2.2).

Theorem 2.5. (Regularity) Let (2.1), (2.8), (2.9) hold, and Theorem 2.5.

Then Problem 2.4 has a solution such that

\[
\vec{u} \in W^{2,\infty}(0,T;L^2(\Omega)^3), \quad \sigma, \varepsilon \in W^{1,\infty}(0,T;L^2(\Omega)^9).
\]

For this problem regularity yields well-posedness.

Theorem 2.6. (Well-Posedness) Let (2.1) hold and two sets of data \((\vec{u}_i, \varepsilon_i, \sigma_i, \vec{\varphi}_i) (i = 1, 2)\) fulfill the assumptions (2.8), (2.9), (2.17) and (2.18). Let \((\vec{u}_1, \varepsilon_1, \sigma_1)\) be corresponding solutions of Problem 2.2 with the regularity (2.19) (which hold by Theorem 2.3), and set

\[
\tilde{\vec{u}} := \vec{u}_1 - \vec{u}_2, \quad \tilde{\varepsilon} := \varepsilon_1 - \varepsilon_2, \quad \tilde{\sigma} := \sigma_1 - \alpha \varepsilon_1 - (\sigma_2 - \alpha \varepsilon_2), \quad \tilde{\vec{\varphi}} := \vec{\varphi}_1 - \vec{\varphi}_2, \quad \text{and so on.}
\]

Then, for a suitable constant \( C \) independent of the data,

\[
\int_{\Omega} \left( \left( \frac{\partial \tilde{u}}{\partial t} \right)^2 + |\tilde{\varepsilon}|^2 + |\tilde{\sigma}|^2 \right) \, dx
\]

\[
\leq \int_{\Omega} \left( (|\vec{u}_1|^2 + |\varepsilon_1|^2 + |\sigma_1|^2)^2 \right) \, dx + C \|\tilde{\vec{\varphi}}\|^2_{L^1(0,T;L^2(\Omega)^3) + W^{1,1}(0,T;V')}.
\]

Problem 2.4 has thus a unique solution with the regularity (2.19).

Proposition 2.7. (Local Regularity in Space) Let the hypotheses of Theorem 2.5 be fulfilled, and

\[
\alpha, \beta \text{ be constant } (> 0), \quad \rho \in W^{1,\infty}(\Omega), \quad \sigma^0 \in H^1(\Omega)^9,
\]

and

\[
\vec{\varphi} = \vec{f} + \vec{h}, \quad \vec{f} \in L^1(0,T;H^1(\Omega)^3), \quad \vec{h} \in W^{1,1}(0,T;L^2(\Omega)^3).
\]

Then Problem 2.4 has a solution such that

\[
\varepsilon \in W^{1,\infty}(0,T;L^2_{\text{loc}}(\Omega)^9), \quad \sigma, \varepsilon \in L^\infty(0,T;H^1_{\text{loc}}(\Omega)^9).
\]
3. Two-scale homogenization

Now we deal with two length-scales: a (macroscopic) coarse and a (mesoscopic) fine length-scale; we denote the respective variables by \(x\) and \(y\). We let \(y\) range through a reference cell \(\mathcal{Y} := [0,1]^3\), that we equip with the structure of the torus. In analogy with (2.1), we assume that

\[
\rho, \alpha, \beta \in L^\infty(\mathcal{Y}), \quad \rho \geq \text{constant} > 0, \quad 0 < \alpha < \beta \quad \text{a.e. in } \mathcal{Y},
\]

and that \(K : \mathcal{Y} \rightarrow \mathcal{P}(\mathbb{R}^3)\) is a measurable multi-valued mapping such that \(K(y)\) is closed for a.a. \(y\). We then restate the constitutive law (2.2) in two-scale form as follows:

\[
\beta(y) \frac{\partial \varepsilon}{\partial t}(x,y,t) - \frac{\partial \sigma}{\partial t}(x,y,t) \in \partial I_{K(y)}(\sigma(x,y,t) - \alpha(y)\varepsilon(x,y,t))
\]

for a.a. \((x,y,t) \in \Omega \times \mathcal{Y} \times [0,T]\) \((=: \Omega_T \times \mathcal{Y})\).

As \(\alpha, \beta\) and \(K\) explicitly depend on \(y\) but not on \(x\), this may be interpreted as the constitutive behaviour of a macroscopically homogeneous and mesoscopically heterogeneous elasto-plastic material. This setting might easily be extended to account for macroscopic heterogeneity, too.

Now we provide a two-scale formulation of our problem.

**Problem 3.1.** Find \(\tilde{\mathbf{u}}, \tilde{\mathbf{u}}, \tilde{\mathbf{v}}, \varepsilon, \sigma, \Sigma_1\) such that \(\tilde{\mathbf{u}}_1 = 0\), \(\tilde{\Sigma}_1 = 0\) a.e. in \(\Omega_T\) and, setting \(\Sigma := \int_0^t \sigma \, dt\),

\[
\tilde{\mathbf{u}} \in W^{1,\infty}(0,T; L^2(\Omega; H^1(\mathcal{Y})^3) \cap L^\infty(0,T; H^1(\Omega)^3)),
\]

\[
\nabla \cdot \tilde{\Sigma} \in L^\infty(0,T; L^2(\Omega)^3),
\]

\[
\tilde{\mathbf{u}}_1, \tilde{\mathbf{v}}, \nabla_y \Sigma_1 \in L^\infty(0,T; L^2(\Omega \times \mathcal{Y})^3),
\]

\[
\varepsilon, \sigma, \Sigma_1 \in L^\infty(0,T; L^2(\Omega \times \mathcal{Y})^9),
\]

\[
\varepsilon = \nabla^s \tilde{\mathbf{u}} + \nabla_y^s \tilde{\mathbf{u}}_1, \quad \sigma - \alpha \varepsilon \in K(y) \quad \text{a.e. in } \Omega_T \times \mathcal{Y},
\]

where \(\nabla^s \) and \(\nabla_y^s \) denote the symmetrized gradients.
\[ \frac{\partial \tilde{u}}{\partial t} = \tilde{\nu} \quad \text{a.e. in } \Omega_T, \quad (3.11) \]

\[ \int_{\Omega \times Y} E(\tilde{v}, \varepsilon, \sigma)(x, y, t) \, dx \, dy + \int_{\Omega \times Y} I_{K(y)}^{\ast} \left( \frac{\partial}{\partial t} (\beta \varepsilon - \sigma) \right) \, dx \, dy \, d\tau \]
\[ \leq \int_{\Omega \times Y} E(\tilde{v}^{0}, \varepsilon^{0}, \sigma^{0})(x, y) \, dx \, dy + \int_{\Omega \times Y} \frac{\partial \bar{F}}{\partial t} \cdot \bar{v} \, dx \, dy \, d\tau \quad \forall t \in ]0, T], \quad (3.12) \]

\[ \int_{\Omega \times Y} \left[ \rho(y)(\bar{u}^{0} - \bar{u}) \frac{\partial \bar{w}}{\partial t} - (\nabla \cdot \Sigma) \cdot \bar{w} - (\nabla_y \cdot \Sigma_1) \cdot \bar{w} \right] \, dx \, dy \, dt \]
\[ = \int_{\Omega \times Y} \bar{F} \cdot \bar{w} \, dx \, dy \, dt \quad \forall \bar{w} \in H^1(0, T; L^2(\Omega \times Y)^3) \text{ such that } \bar{w}|_{t=T} = \bar{0}, \quad (3.13) \]

**Interpretation.** The equation (3.13) yields

\[ \rho(y) \frac{\partial \tilde{u}}{\partial t} - \nabla \cdot \dot{\Sigma} - \nabla_y \cdot \Sigma_1 = \bar{F} \quad \text{a.e. in } \Omega_T \times Y, \]

and the initial condition \( \bar{w}|_{t=0} = \bar{u}^{0} \) a.e. in \( \Omega \). The initial condition on \( \tilde{v} \) is also implicitly contained in this equation, cf. (3.4). Differentiating in time we then get the two-scale dynamical equation

\[ \rho(y) \frac{\partial^2 \tilde{u}}{\partial t^2} - \nabla \cdot \dot{\sigma} - \nabla_y \cdot \frac{\partial \Sigma_1}{\partial t} = \tilde{f} \quad \text{in } H^{-1}(0, T)^3, \text{ a.e. in } \Omega \times Y. \]

If \( \rho \) is independent of \( y \), the extra-term \( \nabla_y \cdot \frac{\partial \Sigma_1}{\partial t} \) drops out, since the other terms are independent of \( y \); we thus get the coarse-scale dynamical equation

\[ \frac{\rho \partial^2 \tilde{u}}{\partial t^2} - \nabla \cdot \dot{\sigma} = \tilde{f} \quad \text{in } H^{-1}(0, T)^3, \text{ a.e. in } \Omega. \]

Consistently with Proposition 2.1, the inequality (3.12) is a two-scale formulation of the energy balance, and thus accounts for the constitutive law (3.2) a.e. in \( \Omega_T \times Y \).

### 4. Derivation of Problem 3.1

We prove existence of a solution of this problem via an asymptotic procedure. First, we fix a small \( \eta > 0 \) and relate the two length-scales by setting \( y = x/\eta \) modulo \( \eta \); that is, \( y \in Y \), and for any \( i \) there exists \( k_i \in \mathbb{Z} \) such that \( x_i = k_i \eta + y_i \). This is tantamount to setting \( y = x/\eta \) and then extending any function \( Y \to \mathbb{R} \) by \( Y \)-periodicity. This allows us to provide a single-length-scale representation of elasto-plastic evolution. Displaying the dependence on the scale-parameter \( \eta \), the constitutive equation (3.2) reads

\[ \beta(x/\eta) \frac{\partial \varepsilon_\eta}{\partial t}(x, t) - \frac{\partial \sigma_\eta}{\partial t}(x, t) \in \partial I_{K(x/\eta)}(\sigma_\eta(x, t) - \alpha(x/\eta)\varepsilon_\eta(x, t)) \]

for a.a. \( (x, t) \in \Omega_T, \forall \eta > 0 \).
We couple this inclusion with the dynamical equation
\[ \rho(x/\eta) \frac{\partial^2 \tilde{u}_\eta}{\partial t^2}(x, t) - \nabla \cdot \sigma_\eta(x, t) = \tilde{f}(x, t) \quad \text{for a.a. } (x, t) \in \Omega_T, \forall \eta > 0, \]
and formulate a single-length-scale initial- and boundary-value problem analogous to Problem 2.2, that we label as Problem 2.2. By Theorem 2.3 for any \( \eta > 0 \) this problem has a solution \((\tilde{u}_\eta, \sigma_\eta)\) that fulfils the estimate (2.13) uniformly w.r.t. \( \eta \). Setting \( \varepsilon_\eta := \nabla^s u_\eta \) this reads
\[ \int_{\Omega} \left( \frac{\partial^2 \tilde{u}_\eta}{\partial t^2} \right)^2 + |\varepsilon_\eta|^2 + |\sigma_\eta|^2 \)(x, t) \, dx \leq C \quad \text{(independent of } \eta, t) \quad \forall t > 0. \tag{4.1} \]

**Theorem 4.1.** (Convergence and Existence) Let (3.5) and (3.6) be satisfied, and for any \( \eta \) let \((\tilde{u}_\eta, \sigma_\eta)\) be a solution of Problem 2.2. Then there exist \( \bar{u}, \bar{u}_1, \bar{v}, \varepsilon, \sigma, \Sigma_1 \) such that, setting \( \Sigma_\eta := \int_0^t \sigma_\eta \, d\tau \) for any \( \eta \) and \( \Sigma := \int_0^t \sigma \, d\tau \), (3.7)–(3.11) are satisfied and, as \( \eta \to 0 \) along a suitable sequence,
\[ \tilde{u}_\eta \rightharpoonup \bar{u} \quad \text{in } W^{1,\infty}(0, T; L^2(\Omega)^3) \cap L^\infty(0, T; H^1(\Omega)^3), \]
\[ \bar{v}_\eta := \frac{\partial \tilde{u}_\eta}{\partial t} \rightharpoonup 2\bar{v} \quad \text{in } L^\infty(0, T; L^2(\Omega \times \mathcal{Y})^3), \]
\[ \sigma_\eta \rightharpoonup 2\sigma \quad \text{in } L^\infty(0, T; L^2(\Omega \times \mathcal{Y})_s^9), \]
\[ \varepsilon_\eta := \nabla^s u_\eta \rightharpoonup 2\nabla^s \bar{u} + \nabla^s \bar{u}_1 =: \varepsilon \quad \text{in } L^\infty(0, T; L^2(\Omega \times \mathcal{Y})_s^9), \]
\[ \nabla \cdot \Sigma_\eta \rightharpoonup 2\nabla \cdot \dot{\Sigma} + \nabla_y \cdot \Sigma_1 \quad \text{in } L^\infty(0, T; L^2(\Omega \times \mathcal{Y})^3). \]

This implies that \((\bar{u}, \bar{u}_1, \bar{v}, \varepsilon, \sigma, \Sigma_1)\) is a solution of Problem 3.1.

The final statement is based on the technique of two-scale convergence of [1, 23], and of some properties of [30].

**Theorem 4.2.** (Regularity) Assume that the hypotheses of Theorem 4.1 are fulfilled, and that
\[ \tilde{v}^0 \in L^2(\Omega)^3, \quad \nabla^s \tilde{v}^0 \in L^2(\Omega)_s^9, \quad \sigma^0 \in L^2(\Omega)_s^9, \quad \nabla \cdot \sigma^0 \in L^2(\Omega)^3, \]
\[ \exists \xi \in L^2(\Omega)_s^9 \quad \text{such that } \xi \in \partial I_{K(y)}(\sigma^0 - \alpha e^0) \quad \text{a.e. in } \Omega \times \mathcal{Y}, \]
\[ F \in W^{2,1}(0, T; L^2(\Omega)^3). \tag{4.3} \]

Then Problem 3.1 has a solution such that
\[ \bar{u} \in W^{2,\infty}(0, T; L^2(\Omega)^3), \quad \sigma, \varepsilon \in W^{1,\infty}(0, T; L^2(\Omega \times \mathcal{Y})_s^9), \quad \nabla \cdot \dot{\sigma}, \nabla_y \frac{\partial \Sigma_1}{\partial t} \in L^\infty(0, T; L^2(\Omega)^3), \]
\[ \nabla_y \cdot \sigma = 0 \quad \text{in } L^2(\Omega_T \times \mathcal{Y})^3. \tag{4.5} \]
Theorem 4.3. (Well-Posedness) Let (3.1) hold and two sets of data $(\vec{u}_i^0, \vec{v}_i^0, \sigma_i^0, f_i)$ $(i = 1, 2)$ fulfil the assumptions of Theorem 3.1 and (4.2) and (4.3). Let $(\vec{u}_i, \vec{v}_i, \epsilon_i, \sigma_i, \Sigma_i)$ be corresponding solutions of Problem 3.1 with the regularity (4.4) and (4.5) (which exist by Theorem 3.2), and set

$$\tilde{u} := u_1 - u_2, \quad \tilde{\epsilon} := \epsilon_1 - \epsilon_2, \quad \tilde{\sigma} := \sigma_1 - \alpha \epsilon_1 - (\sigma_2 - \alpha \epsilon_2), \quad \tilde{f} := f_1 - f_2,$$

and so on.

Then, for a suitable constant $L$ independent of the data,

$$\int_{\Omega \times Y} \left( \frac{\partial \tilde{u}}{\partial t} \right)^2 + |\tilde{\epsilon}|^2 + |\tilde{\sigma}|^2 \, dx \, dy \leq \int_{\Omega \times Y} \left( |\tilde{v}_0|^2 + |\tilde{\epsilon}_0|^2 + |\tilde{\sigma}_0|^2 \right) \, dx \, dy + L\|\tilde{f}\|_{L^1(0,T;L^2(\Omega)^3)} + W_{1,1}(0,T;V),$$

Therefore Problem 3.1 has a unique solution with the regularity (4.4) and (4.5).

The above results can be extended to the quasi-static problem, that we do not address in this note.

Scalar Setting. For $N = 1$, $\nabla^s$ and $\nabla \cdot$ are both replaced by the ordinary derivative $D_x = d/dx$; similarly, $\nabla^y_s$ and $\nabla_y \cdot$ are replaced by $D_y = d/dy$. The equation (3.14) then entails

$$D_y \sigma = 0 \quad \text{in } D'(\Omega_T \times Y),$$

so that $\sigma$ is independent of $y$ (i.e. $\tilde{\sigma} = \sigma$), at variance with the multiaxial problem. An analogous conclusion was attained by Francu and Krejčí in [10, 11] via a different approach.

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