Quantum spin liquid near Mott transition with fermionized $\pi$-vortices

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In this paper, we study the non-magnetic insulator state near Mott transition of 2D $\pi$-flux Hubbard model on square lattice and find that such non-magnetic insulator state is quantum spin liquid state with nodal fermionic excitations - nodal spin liquid (NSL). When there exists small easy-plane anisotropic term, the ground state becomes $Z_2$ topological spin liquid (TSL) with full gapped excitations. The $U(1) \times U(1)$ mutual–Chern-Simons (MCS) theory is obtained to describe the low energy physics of NSL and TSL.

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I. INTRODUCTION

People have been looking for quantum spin liquid states in frustrated spin models for more than 20 years. For example, various approaches show that quantum spin liquids may exist in two-dimensional (2D) $S = 1/2$ J$_1$–J$_2$ model or the Heisenberg model on Kagomé lattice. In these models, the quantum spin liquids are accessed (in principle) by appropriate frustrating interactions. However, the nature of the quantum disordered ground state is still much debated.

On the other hand, the experiments in the organic material $\kappa$-(BEDT–TTF)$_2$Cu$_2$(CN)$_3$ indicate the realization of the quantum spin liquid states. Then, $U(1)$ and $SU(2)$ slave-roton theories of the Hubbard model were formulated on the triangular or honeycomb lattices. Recently, the quantum spin liquid state near Mott transition of the Hubbard model on honeycomb lattice has been conformed by different approaches. In particular, in Ref. quantum spin liquid state has been predicted by quantum Monte Carlo (QMC) simulation.

In Ref., the possible non-magnetic state in the nodal antiferromagnetic (AF) insulator (an insulator with an AF spin density wave ordering and massive Dirac fermionic excitations) is predicted to near the Mott transition of the $\pi$-flux Hubbard model (See Fig.1). Such type of quantum disordered states in bipartite lattices is also not driven by frustrations, as people have done in varied spin models. Instead, they come from quantum fluctuations of a relatively small effective spin-moments near Mott transition. What’s the nature of the possible non-magnetic state here? In this paper we will answer the question. We find that for the isotropic case, the non-magnetic state with $SU(2)$ spin rotation symmetry is a new type of spin liquid - nodal spin liquid (NSL) with gapless fermionic excitations and roton-like excitations; on the other hand, for the anisotropic case by adding small easy-plane anisotropic term, the non-magnetic state with weakly breaking $SU(2)$ spin rotation symmetry becomes $Z_2$ topological spin liquid (TSL) with topological degenerate ground states.

The paper is organized as follows. In Sec.II, we study the spin fluctuations of the nodal AF insulator (NAI) in $\pi$-flux Hubbard model on square lattice based on a formulation by keeping spin rotation symmetry. In Sec.III, we study the properties of half skyrmions - topological solitons of the nodal AF insulator and show their induced quantum numbers and statistics. In Sec.IV, nodal spin liquid is proposed to be the ground state of the quantum non-magnetic insulator state in the nodal AF insulator. In this section, we use $U(1) \times U(1)$ mutual–Chern-Simons theory to learn the properties of NSL and TSL. Finally, the conclusions are given in Sec.V.

II. NON-MAGNETIC STATE IN NODAL AF INSULATOR OF THE HUBBARD MODEL ON $\pi$-FLUX LATTICE

In this paper we will focus on the $\pi$-flux Hubbard model on square lattice. The Hamiltonian of it is

$$
\mathcal{H} = - \sum_{\langle i,j \rangle} \left( t_{ij} \hat{c}^\dagger_{i\uparrow} \hat{c}_{j\uparrow} + h.c. \right) + U \sum_i \hat{n}_{i\uparrow} \hat{n}_{i\downarrow}.
$$

Here $\hat{c}_i = (\hat{c}_{i\uparrow}, \hat{c}_{i\downarrow})^T$ are defined as electronic annihilation operators. $U$ is the on-site Coulomb repulsion. $\langle i, j \rangle$ denotes two sites on a nearest-neighbor link. $\hat{n}_{i\uparrow}$ and $\hat{n}_{i\downarrow}$ are the number operators of electrons at site $i$ with up-spin and down-spin, respectively. There is a $\pi$-flux phase when a particle hops around a plaquette in a $\pi$-flux lattice. So the nearest-neighbor hopping $t_{i,j}$ in a $\pi$-flux lattice could be chosen as $t_{i,i+\hat{x}} = t, t_{i,i+\hat{y}} = t$.

![FIG. 1: The scheme of the spin liquid state near Mott transition of the $\pi$-flux Hubbard model](image-url)
te^{±i\frac{3\pi}{4}}. Although π-flux phase does not break translational symmetry, we may still divide the square lattice into two sublattices, $A$ and $B$. After transforming the hopping term into momentum space, we obtain \( \xi_k = \pm \sqrt{4t^2(\cos^2 k_x + \cos^2 k_y)} \). So there exist two nodal fermi-points at \( k_1 = (\frac{\pi}{3}, \frac{\pi}{3}) \), \( k_2 = (\frac{\pi}{3}, -\frac{\pi}{3}) \) and the spectrum of fermions becomes linear in the vicinity of the two nodal points. In the non-interacting limit, the π-flux Hubbard model is reduced into a free fermion model with nodal fermions at half filling. Here the momentum cutoff is introduced as \( \Lambda = \min(\xi_k) \). will set the spin velocity to be unit.

Because the Hubbard model on bipartite lattices is unstable against antiferromagnetic instability, at half-filling, the ground state may be an insulator with AF SDW order with increasing interacting strength. Such AF SDW order is described by the following mean field Hamiltonian of the 2D π-flux Hubbard model is obtained as

\[
\mathcal{H} = -\sum_{\langle ij \rangle} (t_{ij} c_i^\dagger c_j + h.c.) - \sum_i (-1)^i \Delta c_i^\dagger \sigma_z c_i \tag{2}
\]

where \( \Delta = \frac{UM}{2} \) leads to the energy gap of electrons and \( \sigma_z \) is the Pauli matrix. After diagonalization, the spectrum of the electrons is obtained as \( E_k = \pm \sqrt{\xi_k^2 + \Delta^2} \). By minimizing the free energy at zero temperature, the self-consistent equation of (2) is reduced into

\[
\sum_{k} \frac{1}{2\xi_k} = 1 \quad \text{where} \quad N = \text{number of particles}.
\]

From mean field approach, one may see that the MI transition of the π-flux Hubbard model occurs at a critical value about \( U/t \approx 3.11 \). In the weakly coupling limit \( (U/t < 3.11) \), the ground state is a semi-metal (SM) with nodal fermi-points. In the strong coupling region \( (U/t > 3.11) \), due to \( M \neq 0 \), the ground state becomes an insulator with relativistic massive fermionic excitations. However, the non-zero value of \( M \) only means the existence of effective spin moments. It does not necessarily imply that the ground state of NAI is a long range AF order in the mean field theory. Thus one needs to examine stability of magnetic order against quantum fluctuations of effective spin moments based on a formulation by keeping spin rotation symmetry, \( \sigma \rightarrow \Omega \cdot \sigma \).

Within the haldane’s mapping, the spins are parametrized as \( \Omega_i = (-1)^i n_i \sqrt{1 - L_{i}^{\text{eff}}} + L_{i}^{\text{eff}} \). Here \( n_i \) is the AF order parameter and \( |n_i| = 1 \), \( L_i \) is the transverse canting field, which is chosen to \( L_i \cdot n_i = 0 \). By replacing the electronic operators \( c_i^\dagger \) and \( c_j \) by Grassmann variables \( c_i^\dagger \) and \( c_j \), we get the effective Lagrangian with spin rotation symmetry as

\[
L_{\text{eff}} = \sum_i c_i^\dagger \partial_\tau c_i - \sum_{\langle ij \rangle} (t_{ij} c_i^\dagger c_j + h.c.) - \sum_i (-1)^i \Delta c_i^\dagger \Omega_i \cdot \sigma c_i. \tag{3}
\]

After integrating the massive fermions and transverse canting field, an effective O(3) nonlinear σ model appears that describes the long-wavelength spin fluctuations with spin rotation symmetry:

\[
L_s = \frac{1}{2g} (\partial_\mu n^\mu)^2. \tag{4}
\]

The coupling constant is obtained as \( g = \sqrt{\frac{1}{\chi^s_{\perp}}} \) where \( 2\Delta_{\text{38-39}}^L (\sigma \text{ is lattice constant}) \). The existence of a non-magnetic insulator state provides an alternative candidate for finding spin liquid state. An interesting issue is the nature of the non-magnetic insulator. Is it a valence-band crystal, or algebra spin liquid state, or a new type of quantum state? In the following parts we will answer the question.

III. HALF-SKYRMION AS FERMIONIC EXCITATION

To learn the nature of the non-magnetic insulator, one may study its excitations. In the non-magnetic insulator, we get the effective model of massive spin-1 excitations

\[
\mathcal{L}_s = \frac{1}{2g} [(\partial_\mu n^\mu)^2 + m_s^2 n^2]. \tag{7}
\]
Or using the CP(1) representation, we have

\[ \mathcal{L}_z = \frac{2}{g} \left[ (\partial_\mu - ia_\mu)z |^2 + m_z^2 z^2 \right] \tag{8} \]

where \( z \) is a bosonic spinon, \( z = (z_1, z_2), n_i = \bar{z}_i \sigma z_i, \bar{z} z = 1, a_\mu = -\frac{i}{2}(\bar{z} \partial_\mu z - \partial_\mu \bar{z} z). \) Here \( a_\mu \) is introduced as an assistant gauge field. Because the bosonic spinons have mass gap, we may integrate them and get \( \mathcal{L}_{\text{eff}} = \frac{1}{4g^2} (\partial_\mu a_\nu)^2 \) with \( e_i^2 \sim m_z = \frac{1}{2} m_s. \) Due to the instanton effect, the gauge field \( a_\mu \) obtains a mass gap and bosonic spinons that couple the gauge field \( a_\mu \) are confined. Now the lowest energy excitations are gapped spin wave. However, the answer is not quite right. Applying the Oshikawa’s commensurability condition and Hastings’ theorem to the present case, the quantum state without spontaneously symmetry breaking can be either a topological order with degenerate ground state or a uniform ground state with triplet excitation gap must be accompanied with other gapless excitations.\(^{47,48}\) In this paper we indeed find that non-magnetic insulator is either topological spin liquid with degenerate ground state or a nodal spin liquid with gapless fermionic excitations.

### A. half-skyrmion

In the following parts, we focus on the half-skyrmions (topological vortices with half topological charge),

\[ Q = \int d^2r \frac{1}{4\pi} \epsilon_{0\nu\lambda} n \cdot \partial^\nu n \times \partial^\lambda n = \pm \frac{1}{2} \tag{9} \]

In the vector \( n \) representation, the solutions of the half-skyrmion of the continuum limit are \(^{49-56}\)

\[ n_{hs} = \left( \frac{\lambda(x-x_0)}{\sqrt{|r-r_0|^2 + \lambda^2}}, \pm \frac{\lambda(y-y_0)}{\sqrt{|r-r_0|^2 + \lambda^2}}, \pm \frac{\lambda}{\sqrt{|r-r_0|^2 + \lambda^2}} \right). \tag{10} \]

Here \( \lambda \) is the radius of the half-skyrmion at \( r_0 = (x_0, y_0). \) Inside the core \( |r-r_0|^2 < \lambda, \) the spin is polarized; outside it \( |r-r_0|^2 > \lambda, \) one gets a vortex-like spin configuration on \( XY \) plane. To stabilize topological vortices (half-skyrmions), a small easy-plane anisotropic term should be added to the original model phenomenally, \( H' = \kappa \sum n_i^2 \) \((\kappa > 0, \frac{\lambda}{\kappa} \ll 1).\)

From the exact solutions, there exist two types of merons: one has an up-spin polarized core \( Q = \frac{1}{2}, \)

\[ (n_{hs})_1 = \left( \frac{\lambda(x-x_0)}{\sqrt{|r-r_0|^2 + \lambda^2}}, \frac{\lambda(y-y_0)}{\sqrt{|r-r_0|^2 + \lambda^2}}, \frac{\lambda}{\sqrt{|r-r_0|^2 + \lambda^2}} \right), \tag{11} \]

the other a down-spin polarized core \( Q = -\frac{1}{2}, \)

\[ (n_{hs})_2 = \left( \frac{\lambda(x-x_0)}{\sqrt{|r-r_0|^2 + \lambda^2}}, \frac{\lambda(y-y_0)}{\sqrt{|r-r_0|^2 + \lambda^2}}, \frac{\lambda}{\sqrt{|r-r_0|^2 + \lambda^2}} \right). \tag{12} \]

Fig.2 and Fig.3 show the two types of merons. From them, one can see that the topological charge of a half-skyrmion is determined by both the spin configuration and the polarized direction of AF order in the core.

![FIG. 2: The scheme of \((n_{hs})_1\), the meron with topological charge \(-1/2\). up-spins locate at the center.](image)

![FIG. 3: The scheme of \((n_{hs})_2\), the meron with topological charge \(1/2\). Down-spins locate at the center.](image)
In AF ordered state, the mass of half-skyrmion $m_{hs}$ is associated with the ordered staggered moment:

$$m_{hs} = 2\pi \tilde{\rho}_s + E_{\text{core}} \ (g < g_c) \quad (15)$$

where $\tilde{\rho}_s = \rho_s(1 - \frac{\lambda}{g_c})$ is the renormalized spin stiffness and $E_{\text{core}}$ is the core energy. For the isotropic case $\kappa = 0$, the energy of the half-skyrmion does not depend on its radius $\lambda$ and the core energy is zero, $E_{\text{core}} = 0$. For the anisotropic case $\kappa > 0$, the scale invariance is broken at $H' = \kappa \sum_i n_i^2 \sim H_{\text{kinetic}}$ where $H_{\text{kinetic}}$ is the kinetic energy, $H_{\text{kinetic}} = \frac{\lambda}{2g}(\nabla \mathbf{n})^2$. A new scale $\lambda^2 = \frac{\sigma^2 A}{2g\kappa}$ appears $(a$ is lattice constant). Now, the core energy is estimated as $E_{\text{core}} = \text{const} \cdot \left(\frac{\lambda}{\Lambda}\right)^2$ which indicates the instability of the static soliton against collapse, $\lambda \to 0$. The half-skyrmion without a core $\lambda \to 0$ turns into a spin-vortex.

$$n_{hs} = \left(\frac{x - x_0}{\sqrt{|\mathbf{r} - \mathbf{r}_0|^2 + \Lambda^2}} \pm \frac{y - y_0}{\sqrt{|\mathbf{r} - \mathbf{r}_0|^2 + \Lambda^2}} 0\right). \quad (16)$$

See Fig.4, all spins of the spin-vortex are suppressed onto the $XY$ plane. On the other hand, for the easy-axis anisotropic case, $\kappa < 0$, the energy of spin-vortex will diverge, $m_{hs} \sim N\kappa \to \infty$.

![FIG. 4: The scheme of the spin-vortex. All spins are suppressed onto the XY plane.](image)

### B. Fermionic zero modes

In this part we calculate the fermionic zero modes around the spin-vortex. By replacing the operator $\hat{c}_i$ and site number $i$ by Grassmann number $\psi(x)$ and continuum coordinates $x, y$, we calculate the fermion bound state on the spin-vortex by the continuum formula of the following effective Hamiltonian

$$\mathcal{H}_{\text{eff}} = -\sum_{(ij)}(t_{ij} \hat{c}_j^\dagger \psi_i + h.c.) + \sum_i (-1)^i \Delta \hat{c}_i^\dagger n_{hs} \cdot \sigma \hat{c}_i \quad (17)$$

In the continuum limit, the two-flavor Dirac-like effective Lagrangian\cite{28,29} describes the low energy fermionic excitations at two nodes $k_1 = (\frac{\pi}{2}, \frac{\pi}{2})$, $k_2 = (\frac{\pi}{2}, -\frac{\pi}{2})$.

$$\mathcal{L}_{\text{eff}} = i\bar{\psi}_1 \gamma_\mu \partial_\mu \psi_1 + i\bar{\psi}_2 \gamma_\mu \partial_\mu \psi_2 + m_c(\bar{\psi}_1 n_{hs} \cdot \sigma \psi_1 - \bar{\psi}_2 n_{hs} \cdot \sigma \psi_2) \quad (18)$$

where $\bar{\psi}_1 = \psi_1^\dagger \gamma_0 = (\bar{\psi}_{1A}, \bar{\psi}_{1B}, \bar{\psi}_{1A}, \bar{\psi}_{1B})$ and $\bar{\psi}_2 = \psi_2^\dagger \gamma_0 = (\bar{\psi}_{2A}, \bar{\psi}_{2A}, \bar{\psi}_{2B}, \bar{\psi}_{2A})$. $\gamma_\mu$ is defined as $\gamma_0 = \sigma_0 \otimes \tau_z$, $\gamma_1 = \sigma_0 \otimes \tau_y$, $\gamma_2 = \sigma_0 \otimes \tau_x$, $\sigma_0 = (1 \ 0 \ 0 \ 1)$.

On the other hand, the double zero modes of a spin-vortex on a $21 \times 21$ lattice was shown in Fig.5. The exact charge is localized around the defect center within a length-scale $\sim \Delta^{-1}$. For the two spin-vortices, the fermion zero modes are slightly split due to tunneling between them. From numerical results, we find that there exist two zero modes on each spin-vortex. The fermion zero modes around a $\pi$ spin-vortex match the results in\cite{56,61}.

![FIG. 5: The fermionic zero mode of spin-vortex on a 21 x 21 lattice.](image)

### C. Induced quantum numbers and statistics

Next, we calculate the induced quantum number on the spin-vortices and show the relationship between the induced quantum numbers and the topological charge of them\cite{62}.

For the solutions of zero modes, there are four zero-energy soliton states around a spin-vortex which are denoted by

$$|a_1 \otimes b_1\rangle, \quad |a_0 \otimes b_0\rangle, \quad |a_0 \otimes b_1\rangle, \quad |a_1 \otimes b_0\rangle. \quad (19)$$
Here $|a_0⟩$ and $|b_0⟩$ are empty states of the zero modes $ψ_1^0(\mathbf{r})$ and $ψ_2^0(\mathbf{r})$; $|a_1⟩$ and $|b_1⟩$ are occupied states of them. Without doping, the soliton states of a spin-vortex $|\text{sol}⟩$ are denoted by $|a_0⟩⟨a_0|⊗|b_1⟩$ and $|a_1⟩⟨a_1|⊗|b_0⟩$.

In Ref.56, the induced quantum numbers on the solitons states including total induced fermion number $N_F = \sum_{\alpha,i} c_{\alpha,i}^\dagger c_{\alpha,i}$ and the induced staggered spin number $\hat{S}_z^{(\pi,x)} = 1 2 \sum_{i \in A} c_{z,i}^\dagger \sigma_z c_{z,i} - 1 2 \sum_{i \in B} c_{z,i}^\dagger \sigma_z c_{z,i}$ has been calculated. The total induced fermion number on the solitons is zero from the cancelation effect between two nodals $N_F |\text{sol}⟩ = 0$. However, there exists an induced staggered spin moment on the soliton states56, $\hat{S}_z^{(\pi,x)} |\text{sol}⟩ = ± 1 2 |\text{sol}⟩$.

In particular, we will show the relationship between the induced staggered spin moment $\hat{S}_z^{(\pi,x)}$ and the topological charge of the spin-vortex $Q$. Remember we have study the zero modes and the induced quantum number on a spin-vortex. Now the spin polarization inside the core of the spin-vortex will be determined by the induced spin moment : for the case of $\hat{S}_z^{(\pi,x)} = - 1 2$, one has a up-spin polarized core, $Q = - 1 2$; for the case of $\hat{S}_z^{(\pi,x)} = - 1 2$, one has a down-spin polarized core, $Q = 1 2$. As a result, the spin-vortex with induced staggered spin moment becomes a half-skyrmion with a narrow core and the topological charge of such half-skyrmion dependents on the induced spin moment (See Fig.6).

In the long range AF ordered state ($g < g_c$), the spin-vortex will be confined and the total energy of it diverges. On the other hand, in the region of $g > g_c$, the spin-vortex has finite energy that is $E_{\text{core}} = \kappa$. Thus in the quantum non-magnetic insulator state ($g > g_c$), the mass of half-skyrmion vanishes in the isotropic limit57, $m_{hs} = 0$ and survives when there is a small easy-plane anisotropic term $m_{hs} = E_{\text{core}} = \kappa$. For the easy-axis anisotropic case, $\kappa < 0$, the energy of spin-vortex always diverge, $m_{hs} \to \infty$.

\section{IV. QUANTUM SPIN LIQUID STATE}

Whether the non-magnetic insulator is a VBC state? Because the VBC state is characterized by the condensation of the half-skyrmions, if half-skyrmions are bosons, they will condense. Then one gets a VBC state (or quantum dimer state)53,54. However, in the non-magnetic insulator half-skyrmion obeys fermionic statistics by trapping a bosonic spinon $z$ onto its core. The massless fermionic vortices lead to a new story of the quantum spin liquid states. Consequently, a new type of quantum spin liquid state - nodal spin liquid (NSL) is explored. In addition, after adding a small easy-plane anisotropic term, the fermionic vortices get energy gap, then the ground state becomes a topologically ordered spin liquid.

\subsection{A. Effective model of half-skyrmions}

Because the spin-vortices may have zero energy, to learn the low energy physics of non-magnetic insulator, we need to know quantum dynamics of them. In the following parts we will consider the spin-vortex as a quantum object and obtain its effective model.

As shown in Fig.6, the center of a spin-vortex is the plaquette rather than the original sites. So we may define dual lattice by $I$. The spin-vortex will hop from one dual lattice to another. Thus the spin-vortices show similar behavior of vortices in XY model : it can move on dual lattice with the same lattice constant and feel an effective $\pi$-flux phase51,52 (See detail in Appendix C).

We may use the operator $f_{I\sigma}$ to describe such neutral fermionic particle with half spin at dual lattice $I$. Here $\sigma$ is the spin index. The relation between the zero energy states and the fermionic states is given as

$$|a_1⟩⟨a_0|⊗|b_1⟩ = f_{I\uparrow}^\dagger |0⟩_f$$

and

$$|a_0⟩⟨a_1|⊗|b_0⟩ = f_{I\downarrow}^\dagger |0⟩_f$$

(The state $|0⟩_f$ is defined through $f_{I\uparrow}|0⟩_f = f_{I\downarrow}|0⟩_f = 0$). We call such neutral object at dual lattice $I$ (fermion with $± 1 2$ spin degree freedom) a "fermionic spinon". Then the leading order of the hopping term of the half-skyrmions...
is
\[ \mathcal{H}_{hs} = - \sum_{\langle i,j \rangle} \left( \bar{f}_{i} \gamma_0 f_{j} + h.c. \right) + \sum_{i} m_{hs} (-1)^{i} f_{i} \]

where \( f_{i} = (f_{i \uparrow}, f_{i \downarrow})^T \) are defined as fermionic spinon's annihilation operators. \( I \) and \( J \) denote two nearest-neighbor dual sites. Here \( t_{i,j} \) is the effective hopping of the fermionic spinons. For the \( \pi \)-flux phase for the half-skymrons, one also needs to divide the dual-lattice into two sublattices, \( A \) and \( B \). After transforming the hopping term into momentum space, we obtain a d-wave like dispersion of the fermionic spinons \( \epsilon_{k} = \pm \sqrt{4t^{2}(\cos^{2} p_{x} + \cos^{2} p_{y})} \). So there also exist two nodal fermi-points at \( p_{1} = (\frac{\pi}{2}, \frac{\pi}{2}) \), \( p_{2} = (\frac{\pi}{2}, -\frac{\pi}{2}) \) and the spectrum of fermions spinons becomes linear in the vicinity of the two nodal points.

In the continuum limit, we get a Dirac-like effective Lagrangian that describes the low energy fermionic spinons
\[ \mathcal{L}_{hs} = \sum_{a} i \bar{\Psi}_{a} \gamma_{\mu} \partial_{\mu} \Psi_{a} + \sum_{a} m_{hs} \bar{\Psi}_{a} \Psi_{a} \]  

where \( \Psi_{1} = \bar{\Psi}_{1}^{\dagger} \gamma_{0} = (\bar{f}_{1 \uparrow A}, \bar{f}_{1 \uparrow B}, \bar{f}_{1 \downarrow A}, \bar{f}_{1 \downarrow B}) \) and \( \Psi_{2} = \Psi_{2}^{\dagger} = (\bar{f}_{2 \uparrow A}, \bar{f}_{2 \uparrow B}, \bar{f}_{2 \downarrow A}, \bar{f}_{2 \downarrow B}) \). \( \gamma_{\mu} \) is defined as \( \gamma_{0} = \sigma_{0} \otimes \tau_{z}, \gamma_{1} = \sigma_{0} \otimes \tau_{y}, \gamma_{2} = \sigma_{0} \otimes \tau_{x}, \sigma_{0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \). \( \tau^{x}, \tau^{y}, \tau^{z} \) are Pauli matrices. For simplicity, we set the Fermi velocity to be unit. Thus in the isotropic limit \( \kappa \to 0 \), we may ignore the mass gap of \( \Psi_{a} \) and consider the fermionic spinons as gapless particles.

\[ \mathcal{L}_{eff} = \mathcal{L}_{hs} + \mathcal{L}_{s} + \mathcal{L}_{constraint} \]
\[ = i \sum_{a} \bar{\Psi}_{a} \gamma_{\mu} \partial_{\mu} + i \frac{\sigma^{z}}{2} \gamma_{\mu} A_{\mu} + m_{hs} \bar{\Psi}_{a} \Psi_{a} + \frac{1}{2g} (\partial_{\mu} n_{\nu})^{2} + \mathcal{L}_{MCS} \]
\[ = i \sum_{a} \bar{\Psi}_{a} \left( \gamma_{\mu} \partial_{\mu} + i \frac{\sigma^{z}}{2} \gamma_{\mu} A_{\mu} + m_{hs} \right) \Psi_{a} + \frac{2}{g} \langle \partial_{\mu} - ia_{\mu} \rangle |n_{\nu}|^{2} + \mathcal{L}_{MCS}. \]  

In particular, there exists a mutual-Chern-Simons term,
\[ \mathcal{L}_{MCS} = \frac{i}{\pi} \epsilon^{\mu \nu \lambda} A_{\mu} \partial_{\nu} a_{\lambda} = \frac{i}{\pi} \epsilon^{\mu \nu \lambda} a_{\mu} \partial_{\nu} A_{\lambda}. \]

To obtain above low energy effective theory, we have use the equation \( \partial_{\nu} a_{\lambda} = \partial_{\lambda} a_{\nu} = \frac{1}{2} n \cdot \partial_{\nu} n \times \partial_{\lambda} n \).

Then the effective model becomes an \( U(1) \times U(1) \) mutual-Chern-Simons (MCS) gauge theory. Fermions \( \Psi_{a\sigma} \) couple to an \( U(1) \) gauge field \( A_{\mu} \); boson spinon \( z \) couples to an \( U(1) \) gauge field \( a_{\mu} \). The results say that \( \Psi_{a\sigma} \) act as half topological vortices for \( a_{\mu} \) and \( z \) act as half topological vortices of \( A_{\mu} \). This effective Lagrangian proposed in here and earlier papers\(^{25, 26} \) retains the full symmetries of translation, parity, time-reversal, and global spin rotation, in contrast to the conventional Chern-Simons theories where the second and third symmetries are usually broken. In the following parts of the paper, we use \( U(1) \times U(1) \) MCS theory to learn the quantum non-magnetic state near Mott transition.

\[ \mathcal{L}_{constraint} = i A_{\mu} \left( \frac{1}{8\pi} \epsilon_{\mu \nu \lambda} n_{\nu} \times \partial_{\lambda} n + i \sum_{a} \bar{\Psi}_{a} \frac{\sigma^{z}}{2} \gamma_{\mu} \Psi_{a} \right). \]  

B. Mutual Chern-Simons theory

From above results, there exist two types of fields, the bosonic spinon \( z \) and the fermionic spinon \( f_{i} \). Now the low energy effective model becomes
\[ \mathcal{L}_{eff} = \mathcal{L}_{hs} + \mathcal{L}_{s} = \frac{1}{2g} (\partial_{\mu} n_{\nu})^{2} + i \sum_{a \sigma} \bar{\Psi}_{a} \gamma_{\mu} \partial_{\mu} + m_{hs} \bar{\Psi}_{a} \Psi_{a}. \]

However, there exists non-trivial topological relationship between \( z \) and \( \Psi_{a\sigma} \) - the fields \( \Psi_{a\sigma} \) that carry \( \pm \frac{1}{4} \) winding number of AF vector \( n \),
\[ \frac{1}{8\pi} \epsilon_{\mu \nu \lambda} n_{\nu} \times \partial_{\lambda} n = -i \sum_{a} \bar{\Psi}_{a} \frac{\sigma^{z}}{2} \gamma_{\mu} \Psi_{a}. \]

Here the operator \( \frac{\sigma^{z}}{2} \) means that the topological charge of the fermionic spinons is spin-dependence : For the meron with \( \frac{1}{4} \) (\( -\frac{1}{4} \)) spin number, the topological charge \( Q = \frac{1}{4} \) (\( Q = -\frac{1}{4} \)) with up-spin (down-spin) polarized in the core.

To ensure such constraint in Eq.(25), we add a new term in the effective Lagrangian,
\[ \mathcal{L}_{constraint} = i A_{\mu} \left( \frac{1}{8\pi} \epsilon_{\mu \nu \lambda} n_{\nu} \times \partial_{\lambda} n + i \sum_{a} \bar{\Psi}_{a} \frac{\sigma^{z}}{2} \gamma_{\mu} \Psi_{a} \right). \]

Finally the low energy effective theory is obtained as
\[ \mathcal{L}_{constraint} = i A_{\mu} \left( \frac{1}{8\pi} \epsilon_{\mu \nu \lambda} n_{\nu} \times \partial_{\lambda} n + i \sum_{a} \bar{\Psi}_{a} \frac{\sigma^{z}}{2} \gamma_{\mu} \Psi_{a} \right). \]  

C. AF order

If \( g < g_{c} \), the ground state has a long range Néel order, \( \langle z \rangle = z_{0} \neq 0 \) or \( \langle n \rangle \neq 0 \). Hence the effective model turns
into

\[ \mathcal{L}_{\text{eff}} \approx \frac{g_{\text{eff}}}{g_{\mu}} |a_\mu|^2 + \frac{i}{\pi} \mu^\lambda \lambda \mathcal{A}_\mu \mathcal{A}_\lambda \]

where \( g_{\text{eff}} \) is renormalized coupling constant as \( g_{\text{eff}} = \frac{g_{\text{eff}}}{g_\mu} \). Here the mass term of the gauge field \( a_\mu \) is caused by the condensation. After integrating \( a_\mu \) we get a renormalized kinetic term for gauge field \( A_\mu \),

\[ \mathcal{L}_{\text{eff}} = \frac{1}{4e_A^2} (\partial_\mu A_\nu)^2 + \mathcal{L}_{hs} \]

with \( \frac{1}{e_A^2} = \frac{g_{\text{eff}}}{2\pi^2} \). Due to the instanton effect, the gauge field \( A_\mu \) also obtains a mass gap and a linear confinement appears for fermions. The low energy physics is dominated only by spin waves and the effective Lagrangian becomes \( \mathcal{L}_{\text{eff}} \rightarrow \frac{1}{2} \rho_\mu (\nabla \mathbf{n})^2 \). Here \( \rho_\mu \) is the renormalized spin stiffness as \( \rho_\mu = \rho_\mu (1 - \frac{\Delta}{\mu}) \). At low temperatures, fermionic vortices are paired by the logarithmic-attractive interaction \( V(\mathbf{r}) \rightarrow 2\pi \rho_\mu \ln \frac{\mu}{\Delta} \). With the increase of temperature, neutral vortex-antivortex pairs like those in the XY model are thermally excited, leading to a conventional contribution to the screening effect. By the conventional Kosterlitz-Thouless (KT) theory, there exists a “deconfining” temperature, \( T_{\text{de}} \approx \frac{\Delta}{\mu} \). Above \( T_{\text{de}} \), free excited fermionic vortices exist.

D. Nodal spin liquid - isotropic case \( \kappa = 0 \)

If \( g > g_c \), the ground state has no AF long range order. Now \( \mathbf{z} \) has a mass gap \( m_z \), or one has the massive spin-1 quanta,

\[ \mathcal{L}_s = \frac{1}{2g} \left[ \left( \partial_\mu - ia_\mu \right) \mathbf{z} \right]^2 + m_z^2 \mathbf{z}^2 \].

For the isotropic case \( \kappa = 0 \), the fermionic vortex has no energy gap. Thus the effective model becomes

\[ \mathcal{L}_{\text{eff}} = i \sum_a \bar{\Psi}_a \left( \gamma_\mu \partial_\mu + i \frac{\sigma^\lambda}{2} \gamma_\lambda A_\mu \right) \Psi_a + \frac{1}{2g} \left[ \left( \partial_\mu - ia_\mu \right) \mathbf{z} \right]^2 + m_z^2 \mathbf{z}^2 + \frac{i}{\pi} \epsilon_{\mu\lambda\lambda} A_\mu \partial_\nu a_\lambda \].

After integrating the massless fermions and the massive bosonic spinons, the effective model turns into

\[ \mathcal{L}_{\text{eff}} = \frac{i}{4e_A^2} (\partial_\mu a_\mu)^2 - \frac{1}{4e_A^2} (\partial_\mu A_\mu)^2 + \frac{i}{\pi} \epsilon_{\mu\lambda\lambda} a_\mu \partial_\nu A_\lambda \]

where \( e_A^2 \approx 3\pi m_z \), \( \frac{1}{e_A^2} = \frac{1}{4\sqrt{\pi}} \) and \( p^2 = p_\mu^2 \) is the momentum. Are bosonic spinons and fermionic vortices real quasi-particle? To answer the question, we need to calculate the energy gap of the gauge fields furthermore.

Firstly, after integrating \( a_\mu \) we obtain a mass term for gauge field \( A_\mu \),

\[ \mathcal{L}_{\text{eff}} = i \sum_a \bar{\Psi}_a \left( \gamma_\mu \partial_\mu + i \frac{\sigma^\lambda}{2} \gamma_\lambda A_\mu \right) \Psi_a - \frac{1}{4e_A^2} (\partial_\nu A_\mu)^2 + \frac{e_A^2}{\pi^2} A_\mu^2 \].

Due to exchanging the gauge field \( A_\mu \), a short range interaction is induced between fermions. It is obvious that the short range interaction is irrelevant. As a result, the massless Dirac particles \( \Psi_a \) that couple to \( A_\mu \) become real low energy degrees of freedom.

Secondly, after integrating \( A_\mu \) we get the effective model of \( a_\mu \) as the

\[ \mathcal{L}_{\text{eff}} = \frac{1}{4e_a^2} (\partial_\mu a_\mu)^2 + \frac{e_a^2}{\pi^2} a_\mu^2 \].

This Lagrangian has the schematic form \(-\frac{1}{2}\pi^2 p^2 + \frac{1}{2}p\). That means the momentum of the gauge field \( a_\mu \) is not zero, \( p = \frac{2\pi^2}{\pi^2} \neq 0 \). Then the gauge field \( a_\mu \) has a finite energy gap \( \mathcal{L}_{\text{eff}} \approx \frac{e_a^2}{\pi^2} a_\mu^2 \) and shows roton-like behavior. As a result, the induced interaction by exchanging the gauge field \( a_\mu \) is irrelevant and the bosonic spinons are real quasi-particles.

Finally, the low energy effective theory of nodal spin liquid becomes

\[ \mathcal{L}_{\text{eff}} = i \sum_a \bar{\Psi}_a \gamma_\mu \partial_\mu \Psi_a + \frac{1}{2g} \left[ (\partial_\mu \mathbf{z})^2 + m_z^2 \mathbf{z}^2 \right] + \frac{1}{2g} \left[ (\partial_\mu - ia_\mu \mathbf{z})^2 + m_z^2 \mathbf{z}^2 \right] + \frac{i}{\pi} \epsilon_{\mu\lambda\lambda} A_\mu \partial_\nu a_\lambda \].

There are three types of quasi-particles: two flavor gapless fermionic spinons \( \Psi_\alpha \), gapped bosonic spinons \( \mathbf{z} \) and the roton-like gauge fields \( a_\mu \). Therefore, from the effective theory we conclude that NSL state is stable and can be considered as a new type of quantum spin liquid.

E. Topological spin liquid - anisotropic case \( \kappa > 0 \)

For the anisotropic case \( \kappa \neq 0 \), the fermionic vortex has finite energy gap. Thus the effective model becomes

\[ \mathcal{L}_{\text{eff}} = i \sum_a \bar{\Psi}_a \left( \gamma_\mu \partial_\mu + i \frac{\sigma^\lambda}{2} \gamma_\lambda A_\mu + m_h \mathbf{z} \right) \Psi_a + \left[ (\partial_\mu - ia_\mu \mathbf{z})^2 + m_z^2 \mathbf{z}^2 \right] + \frac{i}{\pi} \epsilon_{\mu\lambda\lambda} A_\mu \partial_\nu a_\lambda \].

After integrating the bosonic spinons \( \mathbf{z} \) and fermionic spinons \( \Psi_a \), there appear the kinetic terms for gauge field \( a_\mu \) and \( A_\mu \),

\[ \mathcal{L}_{\text{eff}} = -\frac{1}{4e_A^2} (\partial_\mu A_\mu)^2 + \frac{1}{4e_A^2} (\partial_\mu a_\mu)^2 + \frac{i}{\pi} \epsilon_{\mu\lambda\lambda} A_\mu \partial_\nu a_\lambda \]
where $c_n^2 \simeq 3\pi m_n$ and $c_A^2 \simeq \frac{3}{2}\pi m_{hs}$.

In particular, one may find that due to the mutual CS term, the mass term of one gauge field can be obtained by integrating the other: by integrating $a_\mu$, the effective model of $A_\mu$ becomes

$$L_{\text{eff}} = -\frac{1}{4e_A^2}(\partial_\mu A_\mu)^2 + \frac{e^2}{\pi^2} A^2.$$

On the other hand, by integrating $A_\mu$ we get the effective model of $a_\mu$ as

$$L_{\text{eff}} = -\frac{1}{4e_A^2}(\partial_\mu a_\mu)^2 + \frac{e^2}{\pi^2} a^2.$$

Thus the gauge fields have mass gap

$$m_a = m_A = \frac{eAe}{2\pi}.$$

Thus we find that the mutual U(1) model of $A_\mu$ and $a_\mu$, a short range interaction is induced between fermionic spinons and bosonic spinons. It is obvious that the short range interaction is irrelevant. As a result, the spinons that couple to $A_\mu$ or $a_\mu$ become real low energy degrees of freedom.

To show the topological properties of the topological spin liquid, we calculate ground state degeneracy on a torus. For periodic boundary condition, we can expand the gauge fields as

$$(A_x, A_y) = \left(\frac{1}{L_x} \Theta_x + \sum_k A_k^x e^{i\vec{k} \cdot \vec{r}}, \frac{1}{L_y} \Theta_y + \sum_k A_k^y e^{i\vec{k} \cdot \vec{r}}\right),$$

$$(a_x, a_y) = \left(\frac{1}{L_x} \phi_x + \sum_k a_k^x e^{i\vec{k} \cdot \vec{r}}, \frac{1}{L_y} \phi_y + \sum_k a_k^y e^{i\vec{k} \cdot \vec{r}}\right)$$

(39) (40)

where $k = (k_x, k_y) = (\frac{2\pi}{L_x} n_x, \frac{2\pi}{L_y} n_y)$ where $n_x, n_y$ are integers. $(A_k^x, A_k^y)$ and $(a_k^x, a_k^y)$ are the gauge fields with non-zero momentum and $(\Theta_x, \Theta_y)$ and $(\phi_x, \phi_y)$ are the zero modes with zero momentum for the gauge fields $A_i$ and $a_i$. Because the existence of the mass gap, the degree freedoms for gauge fields with non-zero momentum $(A_k^x, A_k^y)$ and $(a_k^x, a_k^y)$ have nothing to do with the low energy physics. The low energy physics is determined by $(\Theta_x, \Theta_y)$ and $(\phi_x, \phi_y)$.

For the temporal gauge $A_0 = 0$, after the mode expansion, we write down the following effective Hamiltonian to describe the low energy physics of the mutual U(1) × U(1) CS theory

$$H_{\text{eff}} = \frac{(P_{\Theta_x} - \frac{\Theta_x}{2\pi})^2}{2M_x} + \frac{P_{\Theta_y}^2}{2m_y} + \frac{(P_{\Theta_x} - \frac{\Theta_x}{2\pi})^2}{2m_x} + \frac{P_{\Theta_y}^2}{2M_y},$$

where the conjugate momentum for $(\Theta_x, \Theta_y)$ and $(\phi_x, \phi_y)$ are defined as

$$P_{\Theta_x} = M_x \dot{\Theta}_x - \frac{\Theta_x}{2\pi} P_{\Theta_y} = M_y \dot{\Theta}_y - \frac{\Theta_y}{2\pi}.$$

Thus we map the original mutual U(1) × U(1) CS theory to a quantum mechanics model of two particles in two dimensions. The effective Hamiltonian of $(\Theta_x, \Theta_y)$ and $(\phi_x, \phi_y)$ corresponds to two particles on a torus through two-unit flux: $(\Theta_x, \Theta_y)$ are the coordinates of the first particle, and $(\phi_x, \phi_y)$ are the coordinates of the second particle. The degeneracy for $(\Theta_x, \Theta_y)$ degrees of freedom and the degeneracy for $(\phi_x, \phi_y)$ degrees of freedom are given as $D(\Theta_x, \Theta_y) = 2$ and $D(\phi_x, \phi_y) = 2$. As a result, for the mutual U(1) × U(1) CS theory, the ground states have four-fold degeneracy. That means the ground state of the anisotropic case $\kappa > 0$ is a $Z_2$ topological spin liquid.

For the easy-axis anisotropic case, $\kappa < 0$, the situation changes. For this case, the effective nonlinear $\sigma$ model is not available for the temperature below the energy scale of the anisotropic $\kappa$. Thus the ground state is always long range (Ising) AF order in the insulator phase $M \neq 0$.

## F. Experimental predictions

Firstly, we discuss the experimental predictions in the NSL.

In NSL, the spin-correlation decays exponentially

$$\langle S^+(x,y)S^-(0) \rangle = e^{iQ \cdot R} \langle n^+(x,y)n^-(0) \rangle \sim e^{iQ \cdot R \cdot r / \xi}$$

(41)

with $Q = (\pi, \pi)$, $r = \sqrt{x^2 + y^2}$ and $n^\pm = n^x \pm i n^y$. Here $\xi$ is spin correlated length, $\xi = \frac{\pi}{\xi_{\text{spin}}}$. In contrast, in algebraic spin liquid or algebraic vortex liquid, the spin-correlation shows critical behavior.

Secondly, we calculate the special heat. Because the fermionic spinons have no energy gap, the special heat is dominated by them. So at low temperature, the special heat is

$$C_V = \frac{12\zeta}{\pi} k_B T^2$$

(42)

where $\zeta = \int_0^\infty \frac{x^2 dx}{e^{x+1}} = \frac{3}{4}\Gamma(3)\zeta(3) \simeq 1.803$.

Thirdly, we calculate the spin susceptibility. The definition of the spin susceptibility is

$$F = F(B = 0) - \frac{1}{2}\chi B^2...$$

(43)

where $F$ is free energy and $B$ is the external magnetic field. There are two contributions to the total spin susceptibility $\chi$, one from bosonic spinons, the other is from the fermionic spinons as

$$\chi = \chi_b + \chi_f.$$

(44)
Here $\chi_b$ is given by\(^{46,48}\)
\[
\chi_b = \frac{2}{3} \chi + \mu^2_B M^2 + \frac{2(2\mu_B M)^2}{\pi\beta} \left[ \frac{m_s \beta}{1 - e^{-m_s \beta}} - \ln(e^{m_s \beta} - 1) \right]
\]  
with $\beta \equiv 1/k_B T$. The contribution from the fermionic spinons is almost linear temperature dependence as
\[
\chi_f = \frac{4 \ln 2}{\pi \beta}
\]  
in unit of $(g\mu_B)^2$.

On the other hand, for TSL, the spin-correlation also decays exponentially \((S^+ (x, y) S^- (0)) \sim e^{Q R_c} e^{-r/\xi}\) with $\xi = \frac{\pi}{2\beta_{\text{eff}}}$. However, due to mass gap, $m_{hs} \neq 0$, at low temperature, the special heat and the spin susceptibility from the fermionic spinons are all proportion to $e^{-\beta m_{hs}}$ and disappear at zero temperature.

\section{V. CONCLUSION}

Let us draw a conclusion. In this paper, we study the non-magnetic insulator state near Mott transition of 2D $\pi$-flux Hubbard model on square lattice and find that for the isotropic case such non-magnetic insulator state is quantum spin liquid state with nodal fermionic excitations - NSL; for the anisotropy case it is TSL with full gapped excitations. The low energy physics is basically determined by its $U(1) \times U(1)$ mutual Chern-Simons gauge theory. There exist both fermionic spinons and bosonic spinons. And it is just the mutual semion statistics between fermionic spinons and bosonic spinons that guarantee the stability of quantum spin liquid states.

Because NSL state represents a new class of quantum state which may be applied to learn the nature of the spin liquid state in other systems, for example, the Hubbard model on honeycomb lattice. People have proposed that quantum non-magnetic state of the Hubbard model on honeycomb lattice is really a $Z_2$ topological spin liquid ordered state, which is robust against arbitrary perturbation including the anisotropic term\(^{28,32}\). However, our results of the quantum spin liquid state near the Mott transition of the $\pi$-flux Hubbard model are different - for the isotropic case, the non-magnetic state with SU(2) spin rotation symmetry is nodal spin liquid (NSL) with gapless fermionic excitations and roton-like excitations; for the anisotropic case by adding arbitrary small easy-plane anisotropic term, the non-magnetic state becomes a $Z_2$ topological spin liquid with topological degenerate ground states. People may check the different theories by QMC approach in the future.

Finally, we give a comparison on different quantum orders with $\pi$-vortex (half-skyrmion or vison) and bosonic spinon. In general, for a system with $\pi$-vortex and bosonic spinon, there exist five types of quantum orders as\(^{24}\):

1. **VBC state**: If one has gapless bosonic $\pi$-vortices and massive bosonic spinons, then the ground state is always VBC state with spontaneous translation symmetry breaking;
2. **AF order**: If one has massive bosonic $\pi$-vortices and gapless bosonic spinons, then the ground state is an SDW order with spontaneous spin rotation symmetry breaking;
3. **Algebraic vortex liquid**: if one has gapless fermionic $\pi$-vortices and massive bosonic spinons, then ground state may be an algebraic vortex liquid (AVL)\(^{20}\). In algebraic vortex liquid, fermionic excitations themselves couple to massless $U(1)$ gauge fields, as gives an example to algebraic spin liquid.
4. **Topological spin liquid**: if one has massive fermionic $\pi$-vortex and massive bosonic spinon, the ground state is a topological order with topologically degenerate ground state;
5. **Nodal spin liquid**: if one has gapless fermionic $\pi$-vortices (the fermionic spinons in this paper) and massive bosonic spinons, the ground state is a nodal spin liquid without any spontaneous symmetry breaking. In particular, there exist gapped roton-like gauge modes.

In order to giving a clear comparison, we give Table.2:

|        | $\pi$-vortex | bosonic spinon |
|--------|--------------|----------------|
| VBC state | massless Boson | massive Boson |
| AF order | massive Boson | massless Boson |
| AVL state | massless fermion | massive Boson |
| TSL order | massive Boson(fermion) | massive Boson |
| NSL state | massless fermion | massive Boson |

We give a short remark on the relation between NSL state and the AVL in Ref.\(^{16,17,21}\). In AVL, fermionic excitations coupling to a massless $U(1)$ gauge field and cannot be real quasi-particles. In NSL state, due to the protection from the mutual semion statistics between fermionic spinons and bosonic spinons, fermionic excitations are real excitations. There is neither bosonic spinon nor roton-like gauge mode in AVL state and the low energy effective theory of AVL state is also different from that of NSL. In AVL, the spin-correlation shows critical behavior, while in NSL state, although there exist gapless fermionic spinons, the spin-correlation decays exponentially.

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\section{VI. APPENDIX A: EFFECTIVE NLRM}

In this appendix we use the path-integral formulation of electrons with spin rotation symmetry to obtain the
effective NLσM of the spin fluctuations\textsuperscript{23,38–41,56}. The interaction term in Eq.[1] can be handled by using the SU(2) invariant Hubbard-Stratonovich decomposition in the arbitrary on-site unit vector $\Omega$,

$$\hat{n}_{i\uparrow}\hat{n}_{i\downarrow} = \frac{(c^+_{i\uparrow}c^+_{i\downarrow})^2}{4} - \frac{1}{4}[\Omega_i, c^+_i \sigma c^+_i]^2. \quad (47)$$

Here $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices. By replacing the electronic operators $c^+_i$ and $c_i$ by Grassmann variables $c^+_i$ and $c^\dagger_i$, the effective Lagrangian of the 2D generalized Hubbard model at half filling is obtained:

$$\mathcal{L}_{\text{eff}} = \sum_i c^+_i \partial_x c_i - \sum_{\langle ij \rangle} (t_{ij} \sigma^e c^+_i c_j + h.c.) - \Delta \sum_i c^+_i \Omega_i \sigma c_i. \quad (48)$$

To describe the spin fluctuations, we use the Haldane’s mapping\textsuperscript{36,37,39}:

$$\Omega_i = (-1)^i n_i \sqrt{1 - L_i^2} + L_i \quad (49)$$

where $n_i = (n^x_i, n^y_i, n^z_i)$ is the Neel vector that corresponds to the long-wavelength part of $\Omega_i$, with a restriction $n^2_i = 1$. $L_i$ is the transverse canting field that corresponds to the short-wavelength parts of $\Omega_i$, with a restriction $L_i \cdot n_i = 0$. We then rotate $\Omega_i$ to $\hat{z}$-axis for the spin indices of the electrons at $i$-site\textsuperscript{23,38–41,56}:

$$\begin{align*}
\psi_i &= U^\dagger_i c_i \\
U_i \cdot \sigma U_i &= \sigma_z \\
U_i L_i \cdot \sigma U_i &= 1 \cdot \sigma
\end{align*} \quad (50)$$

where $U_i \in \text{SU}(2)/\text{U}(1)$.

One then can derive the following effective Lagrangian after such spin transformation:

$$\begin{align*}
\mathcal{L}_{\text{eff}} &= \sum_i \psi^+_i \partial_x \psi_i + \sum_i \psi^+_i a_0 (i) \psi_i \\
&\quad - \sum_{\langle ij \rangle} (t_{ij} \psi^e_i \psi_j + h.c.) \\
&\quad - \Delta \sum_i \psi^+_i \left[ (-1)^j \sigma_z \sqrt{1 - L_i^2} + L_i \cdot \sigma \right] \psi_i
\end{align*} \quad (51)$$

where the auxiliary gauge fields $a_{ij} = a_{ij,1} \sigma_x + a_{ij,2} \sigma_y$ and $a_0 (i) = a_{0,1} (i) \sigma_x + a_{0,2} (i) \sigma_y$ are defined as

$$e^{i a_{ij}} = U^\dagger_i U_j, \ a_0 (i) = U^\dagger_i \partial_x U_i. \quad (52)$$

In terms of the mean field result $M = (-1)^i \langle \psi^+_i \sigma_z \psi_i \rangle$ as well as the approximations,

$$\sqrt{1 - L_i^2} \simeq 1 - \frac{L_i^2}{2}, \ e^{ia_{ij}} \simeq 1 + ia_{ij},$$

we obtain the effective Hamiltonian as:

$$\begin{align*}
\mathcal{L}_{\text{eff}} &\simeq \sum_i \psi^+_i \partial_x \psi_i + \sum_i \psi^+_i [a_0 (i) - \Delta \sigma \cdot L_i] \psi_i \\
&\quad - \sum_{\langle ij \rangle} [t_{ij} \psi^+_i (1 + ia_{ij}) \psi_j + h.c.] \\
&\quad - \Delta \sum_i (-1)^i \psi^+_i \sigma_z \psi_i + \Delta M \sum_i \frac{L_i^2}{2}
\end{align*} \quad (53)$$

By integrating out the fermion fields $\psi^+_i$ and $\psi_i$, the effective action with the quadric terms of $[a_0 (i) - \Delta \sigma \cdot L_i]$ and $a_{ij}$ becomes

$$S_{\text{eff}} = \frac{1}{2} \int_0^\beta d\tau \sum_i [-4\kappa (a_0 (i) - \Delta \sigma \cdot L_i)^2 + 4\rho_s a_{ij} + \frac{2\Delta^2}{U} L_i^2]. \quad (54)$$

To give $\rho_s$ and $\zeta$ for calculation, we choose $U_i$ to be

$$U_i = \left( \begin{array}{cc} z^\dagger_{i\uparrow} & z^\dagger_{i\downarrow} \\
-\bar{z}_{i\downarrow} & \bar{z}_{i\uparrow} \end{array} \right), \quad (55)$$

where $n_i = z_i \sigma z_i$, $z_i = (z^\dagger_{i\uparrow}, z^\dagger_{i\downarrow})^T$, $\bar{z}_i z_i = 1$. And the spin fluctuations around $n_i = \bar{z}_i$ are

$$n_i = \bar{z}_i + \text{Re} (\phi_i) \hat{x} + \text{Im} (\phi_i) \hat{y} \quad (56)$$

$$z_i = \left( 1 - |\phi_i|^2/8 \right) + O (\phi_i^3). \quad (57)$$

Then the quantities $U^\dagger_i U_i$ and $U^\dagger_i \partial_x U_i$ can be expanded in the power of $\phi_i - \phi_j$ and $\partial_x \phi_i$,

$$U^\dagger_i U_j = e^{-\frac{\phi^\dagger_i - \phi^\dagger_j}{2} \sigma_y} \quad (58)$$

$$U^\dagger_i \partial_x U_i = \left( \begin{array}{c} 0 \\
-\frac{\phi^\dagger}{2} \partial_x \phi_i \end{array} \right). \quad (59)$$

According to Eq.[52], the gauge field $a_{ij}$ and $a_0 (i)$ are given as

$$a_{ij} = -\frac{1}{2} (\phi_i - \phi_j) \sigma_y \quad (60)$$

$$a_0 (i) = \frac{i}{2} \partial_x \phi_i \sigma_y. \quad (61)$$

Supposing $a_{ij}$ and $a_0 (i)$ to be a constant in space and denoting $\partial_i \phi_i = a$ and $\partial_x \phi_i = iB_y$, we have

$$a_{ij} = \frac{1}{2} a \cdot (i - j) \sigma_y \quad (62)$$

$$a_0 (i) = -\frac{1}{2} B_y \sigma_y. \quad (63)$$

The energy of Hamiltonian of Eq.[54] becomes

$$E (B_y, a) = \frac{1}{2} \zeta B_y^2 + \frac{1}{2} \rho_s a^2. \quad (64)$$

Then one could get $\zeta$ and $\rho_s$, from the following equations by calculating the partial derivative of the energy

$$\zeta = \frac{1}{N} \left. \frac{\partial^2 E (B_y, a)}{\partial B_y^2} \right|_{B_y=0} \quad (65)$$

$$\rho_s = \frac{1}{N} \left. \frac{\partial^2 E (a)}{\partial a^2} \right|_{a=0}. \quad (66)$$
Here $E_0(B_y)$ and $E_0(a)$ are the energy of the lower Hubbard band

$$E_0(B_y) = \sum_k \left( E^\xi_{\pm,k} + E^\zeta_{\pm,k} \right)$$  \hspace{1cm} (67)

$$E_0(a) = \sum_k \left( E^\rho_{\pm,k} + E^\varphi_{\pm,k} \right)$$  \hspace{1cm} (68)

where $E^\xi_{\pm,k}$, $E^\zeta_{\pm,k}$ and $E^\rho_{\pm,k}$, $E^\varphi_{\pm,k}$ are the energies of the following Hamiltonian $\mathcal{H}_\xi$ and $\mathcal{H}_\rho$

$$\mathcal{H}_\xi = \sum_{<ij>} (t_{ij} \psi^*_i \psi_j + h.c.) - \Delta \sum_i (-1)^i \psi^*_i \sigma_z \psi_i$$

$$+ \sum_i \psi^*_i a_0(i) \psi_i$$  \hspace{1cm} (69)

$$\mathcal{H}_\rho = \sum_{<ij>} (t_{ij} \psi^*_i e^{\alpha_{ij}} \psi_j + h.c.) - \Delta \sum_i (-1)^i \psi^*_i \sigma_z \psi_i$$

Using the Fourier transformations for $\mathcal{H}_\xi$, we have the spectrum of the lower band of $\mathcal{H}_\xi$

$$E^\xi_{\pm,k} = -\sqrt{\left( |\xi_k| + \frac{B_y}{2} \right)^2 + \Delta^2}$$  \hspace{1cm} (71)

where $\xi_k = \pm \sqrt{4t^2 \cos^2 k_x - \cos^2 k_y}$. And $\xi$ is obtained as

$$\xi = \frac{1}{N} \sum_k \frac{\Delta^2}{4 (|\xi_k|^2 + \Delta^2)^{\frac{1}{2}}}$$  \hspace{1cm} (72)

Similarly, using the Fourier transformations for $\mathcal{H}_\rho$, we obtain the spectrum of the lower band of $\mathcal{H}_\rho$

$$E^\rho_{\pm,k} = -\sqrt{\Delta^2 + |\vartheta|^2 + |\varphi|^2} \pm \left[ 4\Delta^2 |\vartheta|^2 - (\varphi \vartheta - \varphi \vartheta^*)^2 \right]^{\frac{1}{2}}$$  \hspace{1cm} (73)

where $\varphi$ and $\vartheta$ are defined as

$$\varphi = -t \sum_\delta e^{ik \cdot \delta} \cos \left( \frac{1}{2} a \cdot \delta \right)$$  \hspace{1cm} (74)

$$\vartheta = -t \sum_\delta e^{ik \cdot \delta} \sin \left( \frac{1}{2} a \cdot \delta \right)$$  \hspace{1cm} (75)

where $\delta = a(e_x, e_y)$ and $e^2_x = e^2_y = 1$. Using Eq. (68), $\rho_s$ is given as

$$\rho_s = \frac{1}{N} \sum_k \frac{\varphi^2}{4(|\xi_k|^2 + \Delta^2)^{\frac{1}{2}}}.$$  \hspace{1cm} (76)

For the $\pi$-flux Hubbard model, we get the corresponding coefficient $\varphi^2$ as

$$\varphi^2 = t^2 \cos (2k_x) \left( \Delta^2 + 8t^2 + 4t^2 \cos (2k_y) \right)$$

$$+ \Delta^2 + 3t^2 + t^2 \cos (4k_x) \right].$$  \hspace{1cm} (77)

Next, to learn the properties of the low energy physics, we study the continuum theory of the effective action in Eq. (54). In the continuum limit, we denote $n_i$, $l_i$, $a_{ij} \approx U_i^0 U_j - 1$ and $\alpha_i(i) = U_i^0 \partial_x U_i$, respectively. From the relations between $U_i^0 \partial_x U_i$ and $\partial_x n_i$, $\alpha_i \cdot l_i = \frac{1}{2} (n \times \partial_x n) \cdot l_i$, (80)

the continuum formulation of the action in Eq. (51) turns into

$$S_{\text{eff}} = \frac{1}{2} \int_0^1 d\tau \int d^2r [\xi (\partial_x n)^2 + \rho_s (\nabla n)^2$$

$$+ 4i \Delta \nabla (n \times \partial_x n)] + \left( \frac{2\Delta^2}{U} - 4\Delta^2 \right) \right] \right]$$  \hspace{1cm} (81)

where the vector $a_0$ is defined as $a_0 = (a_{0,1}, a_{0,2}, 0)$. Finally we integrate the transverse canting field $l$ and obtain the effective NL$\sigma$M as

$$S_{\text{eff}} = \frac{1}{2} g \int_0^1 d\tau \int d^2r [\xi (\partial_x n)^2 + c (\nabla n)^2]$$  \hspace{1cm} (82)

with a constraint $n^2 = 1$. The coupling constant $g$ and spin wave velocity $c$ are defined as $g_b, g_c, g_a, g_b, g_c, g_a$:

$$g = \sqrt{\frac{1}{\lambda^2 - \rho_s}}$$

$$c^2 = \frac{\rho_s}{\lambda^2}$$  \hspace{1cm} (83)

where $\lambda^2$ is the transverse spin susceptibility

$$\lambda^2 = \left( \frac{1}{N} \sum_k \frac{\Delta^2}{4(|\xi_k|^2 + \Delta^2)^{\frac{1}{2}}} \right)^{-1} - 2U^{-1}.$$  \hspace{1cm} (84)

Then we use the effective NL$\sigma$M to study the magnetic properties of the insulator state. The Lagrangian of NL$\sigma$M with a constraint $(\mathbf{n}^2 = 1)$ by a Lagrange multiplier $\lambda$ becomes

$$L_{\text{eff}} = \frac{1}{2c} \left[ (\partial_x n)^2 + c (\nabla n)^2 + i\lambda(1 - n^2) \right]$$  \hspace{1cm} (85)

where $i\lambda = m_s^2$ and $m_s$ is the mass gap of the spin fluctuations. Using the large-$N$ approximation we rescale the field $\mathbf{n} \rightarrow \sqrt{N} \mathbf{n}$ and obtain the saddle-point equation of motion as $n_0^0 = 0$.

$$n_0^0 + k_B T \sum_{\omega_n \neq 0} \frac{g \omega_n}{\omega_n^2 + c^2 \omega_n^2 + m_s^2} = 1.$$  \hspace{1cm} (86)

In Eq. (86), $n_0$ is the mean field value of $\mathbf{n}$ and $\omega_n = 2\pi nk_BT$, $n$ are integers.

From Eq. (86), we may get the solution of $m_s$ as

$$m_s = 2k_B T \sinh^{-1} \left[ e^{\frac{2\pi a}{k_B T}} \sinh \left( \frac{c \Delta}{2k_B T} \right) \right].$$  \hspace{1cm} (87)
At zero temperature the solutions of \( n_0 \) and \( m_s \) of Eq.\(^{[66]}\) are determined by the coupling constant \( g \). There exists a critical point \( g_c = \frac{4\pi}{3} \): For the case of \( g < \frac{4\pi}{3} \), we get a non-magnetic insulator with solutions of \( n_0 \) and \( m_s \):

\[
n_0 = (1 - \frac{g}{g_c})^{1/2}, \quad m_s = 0 \tag{88}
\]

For the case of \( g > \frac{4\pi}{3} \), we get a long range AF order with solutions of \( n_0 \) and \( m_s \):

\[
n_0 = 0, \quad m_s = 4\pi c(\frac{1}{g_c} - \frac{1}{g}) \tag{89}
\]

### VII. APPENDIX B: THE INDUCED KINETIC TERM OF GAUGE FIELD

Starting from the CP(1) model,

\[
\mathcal{L}_s = \frac{1}{2g} \left[ |(\partial_\mu - ia_\mu)z|^2 + m_0^2 z^2 \right],
\]

we calculate the induced kinetic term of the gauge field \( a_\mu \).

We obtain the expression of the expansion of the renormalization 2-point function at one-loop order:

\[
\frac{\pi}{(2\pi)^2} \int d^3k \left[ \frac{(k_\mu + 2p_\mu)(k_\nu + 2p_\nu)}{(k^2 + m_0^2)((k + p)^2 + m_0^2)} - 2\delta_\mu_\nu \frac{1}{k^2 + m_0^2} \right]
\]

At low energy limit as \( p^2 \to 0 \), we simplify the above integral to the following one

\[
\frac{\pi}{(2\pi)^2} \left[ \frac{1}{3m_0^2} (p_\mu p_\nu - \delta_\mu_\nu p^2) \right] \tag{90}
\]

At low energy limit ( \( p^2 \to 0 \)), thus from \(-\frac{1}{2m_0^2} (\partial_\mu a_\mu)^2\), the induced coupling constants of three dimensional gauge field is obtained as

\[
e_a^2 = 3\pi m_0^2. \tag{92}
\]

Using similar approach, one may get the induced kinetic term of gauge field \( A_\mu \).

### VIII. APPENDIX C: PROJECTIVE SYMMETRY GROUP OF HALF SKYRMIONS

In the insulator state (non-magnetic or magnetic order), there exists a bosonic spinon on each site. Because, the half skyrmion is really the vortex, we may consider the dynamics of half skyrmions as that of the vortices of the insulator state of bosons on lattice with unit filling (See Fig.7). As a result, the square lattice of bosonic spinons plays a role of \( \pi \)-flux phase of half skyrmion on dual lattice \( \mathbf{I} = (I_x, I_y) \). So we may use the approach of the projective representation of the space group (PSG) for vortex to learn the dynamics of half skyrmion here. The PSG denotes the phase transformations associated with the operations in the space group.

For the \( \pi \)-flux phase of half skyrmions, we have \( x \) and \( y \) translations, and a \( \pi/2 \) rotation of the PSG on the dual square lattice as

\[
T_y : f(I_x, I_y) \rightarrow f(I_x, I_y - 1)
\]

\[
T_x : f(I_x, I_y) \rightarrow f(I_x + 1, I_y)(-1)^{I_y}
\]

\[
R_{\pi/2} : f(I_x, I_y) \rightarrow f(I_x, -I_y)(-1)^{I_x + I_y} \tag{93}
\]

where \( f(I_x, I_y) \) is the wave-function of half skyrmions and \( \mathbf{I} = (I_x, I_y) \) are the dual lattices. One may check that the operations associated with translations in the PSG do not commute as \( T_x T_y = -T_y T_x \) that means the half skyrmion obtains a \( \pi \) phase encircling a site of the direct lattice \((i)\) with one bosonic spinon.

We are looking at the half skyrmions hopping around dual square lattice in the presence of \( \pi \)-flux per plaquette. Then the effective action should be invariant under PSG and the dual \( U(1) \) gauge symmetry. For the simplest gauge and nearest neighbor hopping, the effective Hamiltonian of half skyrmions is

\[
\mathcal{H}_{hs} = - \sum_{(i,j)} \left( \tilde{t}_{i,j} f_{i}^\dagger f_{j} + h.c. \right) \tag{94}
\]

where the effective nearest-neighbor hopping \( \tilde{t}_{i,j} \) could be chosen as \( \tilde{t}_{i,i+\hat{x}} = t, \tilde{t}_{i,i+\hat{y}} = t e^{\pm i\frac{\pi}{2}} \).
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