Testing Broken $U(1)$ Symmetry in a Two-Component Atomic Bose-Einstein Condensate

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Abstract

We present a scheme for determining if the quantum state of a small trapped atomic Bose-Einstein condensate is state with a well defined number of atoms, a Fock state, or a state with a broken $U(1)$ gauge symmetry, a coherent state. The proposal is based on the observation of Ramsey fringes. The population difference measured by a Ramsey fringe experiment will exhibit collapse and revivals due to the mean-field interactions. The collapse and revival times depend on the relative strengths of the mean-field interactions for the two components and the initial quantum state of the condensate.

I. INTRODUCTION

Since the observation of Bose-Einstein condensation in trapped atomic gases in 1995 [1], an unresolved issue in the theory of atomic Bose-Einstein condensates (BEC) is the quantum state of the condensate [2] [3] [4] [5] [6] [7] [8] [9]. In the standard theory of BEC, which applies in the thermodynamic limit, the quantum state is one of well defined phase, $\phi$. This state corresponds to a broken $U(1)$ gauge symmetry [10] [11]. Because particle number, $N$, and phase obey the uncertainty relation $\Delta N \Delta \phi \geq 1$ [12], a state of well defined phase implies uncertainty in the particle number.

Again, this result is based on assumptions which are valid only in the thermodynamic limit in which $N \to \infty$. The second quantized Hamiltonian for a system of bosons is expressed in terms of the bosonic field operator, $\hat{\Psi}(\mathbf{r}, t)$, and its adjoint which annihilate and create a particle at position $\mathbf{r}$, respectively. The Hamiltonian is invariant under the global $U(1)$ transformation, $\hat{\Psi}(\mathbf{r}, t) \to e^{i\chi}\hat{\Psi}(\mathbf{r}, t)$. The $U(1)$ symmetry implies a conserved Noether charge which corresponds to the total number of particles. When a Bose-Einstein condensate is present, a single quantum state becomes macroscopically occupied and it is assumed that the field operator acquires a nonvanishing expectation value,
\[ \langle \hat{\Psi}(r, t) \rangle = \psi(r, t) \], \hspace{1cm} (1)

with respect to the condensate state. Here, \( \psi(r, t) \) is the order parameter for the condensate. However, \( \psi(r, t) \) is no longer invariant under the transformation, \( \psi(r, t) \rightarrow e^{i\chi}\psi(r, t) \) which implies that the \( U(1) \) symmetry is spontaneously broken in the condensate. Equation (1) may be rewritten as

\[ \langle \hat{\Psi}(r, t) \rangle = \langle \varphi(N-1)|\psi(r, t)|\varphi(N)\rangle \]

where \( |\varphi(N)\rangle \) and \( |\varphi(N-1)\rangle \) are "like" condensate states which differ by one particle [13] [14]. In the thermodynamic limit, the difference between \( |\varphi(N)\rangle \) and \( |\varphi(N-1)\rangle \) disappears, in which case the condensate is in a coherent state, \( \hat{\Psi}(r, t)|\varphi(N)\rangle = \psi(r, t)|\varphi(N)\rangle \) and \( \psi(r, t) \) may be identified with the wave function for the quantum state in which Bose condensation has occurred (with the wave function normalization \( \int d^3r |\psi(r, t)|^2 = N \)). However, it is not clear that Eq. (1) is still applicable when \( N \) is finite. Examples of BEC in condensed matter physics such as superfluid He may have \( N \sim 10^{20} \) whereas trapped atomic gases typically have \( N \sim 10^3 - 10^6 \).

In fact, there are two immediate objections to the use of a coherent state for finite particle number. First, at zero temperature, one expects that the true ground state of the condensate will be a number state (i.e. a Fock state) with no quantum fluctuations in the particle number even if we are ignorant of what that number is. Also, since the Hamiltonian for the system is \( U(1) \) symmetric, one must introduce a symmetry breaking field into the Hamiltonian in order to define the phase of the condensate [15]. The symmetry breaking field vanishes in the thermodynamic limit. However, there is no physical interaction which corresponds to this symmetry breaking term and as such, it simply amounts to a mathematical trick [16].

The general definition of a Bose-Einstein condensate due to Penrose and Onsager [17] is that the single particle density matrix, \( \rho_1(r, r', t) = \langle \hat{\Psi}^\dagger(r, t)\hat{\Psi}(r', t) \rangle \), does not vanish as for large separations,

\[ \lim_{|r-r'| \to \infty} \rho_1(r, r', t) = \Phi^*(r, t)\Phi(r', t). \] \hspace{1cm} (2)

Although Eq. (2) is consistent with Eq. (1), Eq. (2) simply requires the macroscopic occupation of a single quantum state and will therefore be true for a condensate that is in
a number state. Wright et al. have shown that, for small condensates \(N \sim 10^3\) described initially by a coherent state, the order parameter undergoes collapses and revivals such that \(\psi(r, t) \rightarrow 0\) during the collapse but Eq. (2) remains valid at all times since \(\rho_1(r, r', t)\) is unaffected by the phase diffusion which causes the collapse and revivals \[3\]. This implies that a coherent state description is inappropriate for small \(N\) since it is not an energy eigenstate of the system. Similar results were obtained in \[8\] where the dependence of the collapse and revival times on the dimensionality of the condensate and the trapping potential was studied. In contrast, Barnett et al. have argued that the best pure state description of a condensate is the coherent state since it is the most robust state with respect to interactions with the environment \[2\]. In short, there appear to be no conclusive arguments for or against a coherent state description of atomic BEC’s.

Since a coherent state has a well defined phase, the appearance of interference fringes in the atomic density for two overlapping condensates would be an indication that the condensates were in coherent states (or some other superposition of number states such that one could ascribe a phase to the condensate). However, Javanainen and Yoo have shown that even if the two condensate states are initially in number states, there will be an observable interference pattern \[4\]. This is because the destructive detection of atoms creates an uncertainty in the relative number of atoms in the two condensates since it not known from which condensate the detected atom came from. Consequently, with each atom detection, the relative phase between the two condensates becomes more precisely defined. Thus, any interference experiment based on destructive detection of atoms will not be able to distinguish between two condensates initially in number states or coherent states. Similar work has shown that the detection of spontaneously scattered photons between two condensates can establish a relative phase between the condensates even when the condensates are initially in number states \[18\].

In this paper, we propose a method for distinguishing between a condensate that is in a number state and a coherent state. The method is based Ramsey’s separated oscillatory field technique \[19\] in a two-component condensate such as \(^{87}\text{Rb}\) \[21\]. Such an experiment
has recently been performed by Hall et al. at JILA [20]. Ramsey’s technique, as applied to two-level atoms initially in their ground states, consists of applying two ”$\pi$-pulses” generated by an external field of frequency $\omega_e$ which couple the ground state and excited state. These pulses are separated by a time $T$. The first pulse puts each of the atoms into a superposition of the ground and excited states with equal population. The relative phase between the two states then evolves as $\omega_o T$ where $\hbar \omega_o$ is the energy difference between the two states. The second pulse creates a population difference between the two states which measures the relative phase accumulated by the atoms as compared to the phase accumulated by the external field during the period $T$. The population difference is then $\cos(\delta T)$ where $\delta = \omega_o - \omega_e$. In a condensate, the population difference after the second pulse will be affected by two-body interactions which cause a phase diffusion of the relative phase of the two components in the interval $T$ between the pulses. As such, the population difference will experience collapse and revivals as function of $T$. The collapse and revival times depend on the strength of the two-body interactions and the initial state of the condensate such that the collapse and revival times for a coherent state are different from a number state.

Wright et al. predicted a similar effect for the interference fringe visibility of two spatially overlapping condensates [5]. They showed that the revival time for the fringe visibility for condensates initially in coherent states was twice that of condensates in number states. However, their result was based on the assumption that the intra-condensate interactions were the same and that inter-condensate interactions could be ignored. They also assumed that the coherence between the two number state condensates was established by measurement of the interference pattern in the same manner as described in [4].

The key advantage of the Ramsey fringe technique is that the collapse and revivals manifest themselves in the population difference between the two condensate components which is readily measured used absorptive or dispersive imaging of the condensate. Proposals to directly measure the order parameter, in order to detect collapse and revivals, such as [22] usually rely on the detection of scattered light from the condensate. Reference [22] involves two independent condensates that are in spatially separated potentials. Such an experiment
would be technically difficult.

The remainder of this paper is organized as follows: In section II we present the second quantized Hamiltonian for a two-component condensate and derive a two-mode model for the ground states of the two components. The two-mode Hamiltonian is then represented in terms of angular momentum operators by exploiting the equivalence between the algebra of two harmonic oscillators and the angular momentum algebra. In section III, we consider a condensate prepared in one of the modes with a state vector given by either a number state or a coherent state and study the time evolution of these states subject to two $\frac{\pi}{2}$ pulses. In section IV and V, we discuss the collapse and revivals as well as relevant time scales for observing them.

II. PHYSICAL MODEL

A. Derivation of two-mode Hamiltonian

We consider a collection of bosonic atoms that have internal states $|1\rangle$ and $|2\rangle$ with energies $\hbar \omega_o/2$ and $-\hbar \omega_o/2$, respectively. There is a spatially uniform time dependent radiation field with frequency $\omega_e$ which couples the two internal states with a Rabi frequency $\Omega(t)$. The atom field detuning is denoted by $\delta = \omega_o - \omega_e$. The atoms in states $|1\rangle$ and $|2\rangle$ are subject to isotropic harmonic trapping potentials $V_i(r) = \frac{1}{2} m \omega_i r^2$ for $i = \{1, 2\}$, respectively. Furthermore, the atoms interact via elastic two-body collisions through the interaction potentials $V_{ij}(r - r') = U_{ij} \delta(r - r')$ where $U_{ij} = \frac{4 \pi \hbar^2 a_{ij}}{m}$ and $a_{ij}$ is the s-wave scattering length between atoms in states $i$ and $j$. It is assumed that $a_{ij} > 0$ corresponding to repulsive interactions. The Hamiltonian operator describing the system is given by,

$$
\hat{H} = \hat{H}_{\text{atom}} + \hat{H}_{\text{coll}}
$$

$$
\hat{H}_{\text{atom}} = \int d^3 r \left\{ \hat{\Psi}_1^\dagger(\mathbf{r}) \left[ -\frac{\hbar^2}{2m} \nabla^2 + V_1(\mathbf{r}) + \frac{\hbar \delta}{2} \right] \hat{\Psi}_1(\mathbf{r}) + \hat{\Psi}_2^\dagger(\mathbf{r}) \left[ -\frac{\hbar^2}{2m} \nabla^2 + V_2(\mathbf{r}) - \frac{\hbar \delta}{2} \right] \hat{\Psi}_2(\mathbf{r}) + \frac{\hbar}{2} \Omega^*(t) \hat{\Psi}_1^\dagger(\mathbf{r}) \hat{\Psi}_2(\mathbf{r}) + \frac{\hbar}{2} \Omega(t) \hat{\Psi}_2^\dagger(\mathbf{r}) \hat{\Psi}_1(\mathbf{r}) \right\}
$$
\[
\hat{H}_{\text{coll}} = \frac{1}{2} \int d^3r \left\{ U_{11} \hat{\Psi}_1^\dagger(\mathbf{r}) \hat{\Psi}_1^\dagger(\mathbf{r}) \hat{\Psi}_1(\mathbf{r}) \hat{\Psi}_1(\mathbf{r}) + U_{22} \hat{\Psi}_2^\dagger(\mathbf{r}) \hat{\Psi}_2^\dagger(\mathbf{r}) \hat{\Psi}_2(\mathbf{r}) \hat{\Psi}_2(\mathbf{r}) \\
+ 2U_{12} \hat{\Psi}_1^\dagger(\mathbf{r}) \hat{\Psi}_2^\dagger(\mathbf{r}) \hat{\Psi}_1(\mathbf{r}) \hat{\Psi}_2(\mathbf{r}) \right\}. 
\]

(3c)

Here, \( \hat{H}_{\text{atom}} \) is the single particle Hamiltonian and \( \hat{H}_{\text{coll}} \) represents two-body interactions.

The operators \( \hat{\Psi}_i(\mathbf{r}) \) and \( \hat{\Psi}_i^\dagger(\mathbf{r}) \) are bosonic annihilation and creation operators for an atom in state \( i = \{1, 2\} \) at position \( \mathbf{r} \) which satisfy the commutation relations \([\hat{\Psi}_i(\mathbf{r}), \hat{\Psi}_j^\dagger(\mathbf{r}')] = \delta_{ij} \delta(\mathbf{r} - \mathbf{r}') \) and \([\hat{\Psi}_i(\mathbf{r}), \hat{\Psi}_j(\mathbf{r}')] = 0 \). The operators, \( \hat{\Psi}_i(\mathbf{r}) \), have been written in a field interaction representation which is rotating at the frequency of the external field, \( \omega_e \), so that \( \hat{\Psi}_1(\mathbf{r}) = \hat{\Psi}_1^{(N)}(\mathbf{r}) e^{i\omega_e t/2} \) and \( \hat{\Psi}_2(\mathbf{r}) = \hat{\Psi}_2^{(N)}(\mathbf{r}) e^{-i\omega_e t/2} \). Here, \( \hat{\Psi}_i^{(N)}(\mathbf{r}) \) are the field operators in the normal representation. This explains the appearance of the detuning in Eq. (3b).

In the presence of the condensate, we assume that the field operators may be approximated using a two-mode model such that \( \hat{\Psi}_1(\mathbf{r}) = a_1 \phi_1(\mathbf{r}) + \delta \hat{\Psi}_1(\mathbf{r}) \) and \( \hat{\Psi}_2(\mathbf{r}) = a_2 \phi_2(\mathbf{r}) + \delta \hat{\Psi}_2(\mathbf{r}) \) where the \( a_i \) are the mode annihilation operators for the condensate modes which obey bosonic commutation relations \([a_i, a_j^\dagger] = \delta_{ij} \) and \([a_i, a_j] = 0 \). The \( \delta \hat{\Psi}_i(\mathbf{r}) \) represent the field operator for the non-condensate modes and will be neglected since the number of atoms in these modes is assumed to be negligible compared to the condensate modes. For small condensates, such that \( N a_{ij}/a_{ho,i} \lesssim 1 \) where \( a_{ho,i} = \sqrt{\hbar/ \mu m_{ho,i}} \) is the harmonic oscillator length, \( \phi_i(\mathbf{r}) \) are given by the harmonic oscillator ground states of the trap, \([ -\frac{\hbar^2}{2m} \nabla^2 + V_i(\mathbf{r}) ] \phi_i(\mathbf{r}) = \frac{3\hbar \omega_i}{2} \phi_i(\mathbf{r}) \]

Assuming a weak trap, \( a_{ho,i} \approx 10 \mu \text{m} \) and \( a_{ij} \approx 5 \mu \text{m} \) for \(^{87}\text{Rb}\), one has \( N \lesssim 2000 \). In the two-mode approximation the Hamiltonian becomes (with \( \hbar = 1 \)),

\[
\hat{H} = \frac{1}{2} (\delta + 3\omega_1) a_1^\dagger a_1 + \frac{1}{2} (\delta + 3\omega_2) a_2^\dagger a_2 + \frac{1}{2} \Omega(t) a_1^\dagger a_2 + \frac{1}{2} \bar{\Omega}(t) a_2^\dagger a_1 \\
+ \frac{1}{2} \left[ \chi_{11} a_1^\dagger a_1^\dagger a_1 a_1 + \chi_{22} a_2^\dagger a_2^\dagger a_2 a_2 + 2 \chi_{12} a_1^\dagger a_1^\dagger a_2 a_2 \right]; 
\]

(4)

where

\[
\bar{\Omega}(t) = \Omega(t) \int d^3r \phi_2^\dagger(\mathbf{r}) \phi_1(\mathbf{r}); 
\]

(5a)
\[
\chi_1 = U_{11} \int d^3r \, |\phi_1(r)|^4; 
\]
\[
\chi_2 = U_{22} \int d^3r \, |\phi_2(r)|^4; 
\]
\[
\chi_{12} = U_{12} \int d^3r \, |\phi_1(r)|^2 |\phi_2(r)|^2. 
\]

For \(\Omega(t) = 0\), the eigenstates of Eq. (4) are simply the number states \(|n_1, n_2\rangle_F\) such that 
\[
\hat{N}_i |n_1, n_2\rangle_F = n_i |n_1, n_2\rangle_F \text{ where } \hat{N}_i = a_i^{\dagger} a_i. 
\]
One may note that Eq. (4) with \(\chi_{12} = 0\) also describes tunnelling between two condensates in a double well potential [24].

**B. Angular momentum representation**

Equation (4) may be expressed in a more convenient form by taking advantage of the mapping between the algebra for two independent harmonic oscillators and the algebra for angular momentum [25]. The mapping between the two algebras is achieved by making the following definitions

\[
J_+ = a_1^{\dagger} a_2; 
\]
\[
J_- = a_2^{\dagger} a_1; 
\]
\[
J_z = \frac{1}{2} \left( a_1^{\dagger} a_1 - a_2^{\dagger} a_2 \right); 
\]

By noting that the x and y components of the angular momentum are given by the operators

\[
J_x = \frac{1}{2}(J_+ + J_-) \text{ and } J_y = \frac{1}{2i}(J_+ - J_-),
\]

it follows that

\[
J^2 = \frac{\hat{N}}{2} \left( \frac{\hat{N}}{2} + 1 \right); 
\]

where \(\hat{N} = \hat{N}_1 + \hat{N}_2\) is the total number operator which commutes with the two-mode Hamiltonian. Thus \(J^2\) is a constant of motion with eigenvalues \(j(j+1)\) and \(j = N/2\). Consequently, for a state with definite \(N\), Eq. (4) has the form

\[
\hat{H} = \Delta \omega J_z + \chi_+ J^2 + \frac{1}{2} \tilde{\Omega}(t) J_+ + \frac{1}{2} \tilde{\Omega}(t) J_- + \frac{1}{2} (\chi_1 + \chi_2 + 2\chi_{12}) j^2 + \frac{1}{2} (3\omega_1 + 3\omega_2 - \chi_1 - \chi_2) j;
\]
\[
\Delta \omega = \delta + \frac{3}{2} (\omega_1 - \omega_2) + \chi_- (2j - 1); \\
\chi_+ = \frac{1}{2} (\chi_1 + \chi_2 - 2\chi_{12}); \\
\chi_- = \frac{1}{2} (\chi_1 - \chi_2);
\]

(9a) \hspace{1cm} (9b) \hspace{1cm} (9c)

In writing Eq. (8) we have made the replacement $\hat{N} \to 2j$. This limits Eq. (8) to the subspace of states with the same total number of atoms. However, in the following section, we are only interested in calculating the expectation values of the operators $J_z$ and $J_\pm$ after some time $t$ so that the replacement $\hat{N} \to 2j$ does not affect any of our results even when we choose an initial state that is a superposition of number states. Therefore, we can drop the last two terms Eq. (8) so that our effective Hamiltonian is given by

\[
\hat{H} = \Delta \omega J_z + \chi_+ J_z^2 + \frac{1}{2} \Omega^r(t) J_+ + \frac{1}{2} \Omega(t) J_-.
\]

(10)

For $\Omega(t) = 0$, the eigenstates of Eq. (8) are simply the eigenstates of $J_z$, $|j, m\rangle$ with $m = -j, ..., j$. By noting that $n_1 = j + m$ and $n_2 = j - m$, it follows that $|j, m\rangle = |j + m, j - m\rangle_F$. In order to avoid confusion, we use the subscript $F$ (F as in Fock) on the kets for the number state basis in order to distinguish them from the angular momentum kets. When $\bar{\Omega}(t) = 0$, the condensate ground state is simply the lowest energy eigenstate of Eq. (8) for a fixed total number of atoms. This corresponds to the $|j, m\rangle$ state with $m = \text{integer} \left( -\frac{\Delta \omega}{2\chi_+} \right)$ for $\text{integer} \left( -\frac{\Delta \omega}{2\chi_+} \right) < j$ and $m = \pm j$ otherwise. Here, integer() denotes the integer part of the number in parentheses. However, we assume that $-\frac{\Delta \omega}{2\chi_+} < -j$ so that the ground state, $|j, -j\rangle = |0, N\rangle_F$, is fully polarized. When the resonance condition $\delta + \frac{3}{2} (\omega_1 - \omega_2) = 0$ is satisfied, $|j, -j\rangle$ will be the ground state when (i) $\chi_2 < \chi_{12}$ and $\chi_+ > 0$ or (ii) $\chi_2 > \chi_{12}$ and $\chi_+ < 0$.

One might also consider a coherent state in the number state basis given by

\[
|0, \alpha_2\rangle = e^{-|\alpha_2|^2/2} \sum_{n_2=0}^\infty \frac{\alpha_2^{n_2}}{\sqrt{n_2!}} |0, n_2\rangle_F = e^{-|\alpha_2|^2/2} \sum_{n_2=0}^\infty \frac{\alpha_2^{n_2}}{\sqrt{n_2!}} \delta_{n_2,2j'} |j', -j'\rangle,
\]

(11)
as a variational wave-function that minimizes $\hat{H}$ with $\alpha_2 = \sqrt{N} e^{i\varphi}$. One may show using Eq. (8) that
\[ \delta E = \langle j, -j | \hat{H} | j, -j \rangle - \langle 0, \alpha_2 | \hat{H} | 0, \alpha_2 \rangle = -\chi_2 N / 2. \]

Notice that $\chi_2 = U_{22}^2 / [(2\pi)^{3/2} a_{ho,2}^3] \sim U_{22} / V$ where $V$ is the volume of the trap. Therefore $\delta E \sim U_{22} (N/V)$ which is an intensive quantity. Since $\langle j, -j | \hat{H} | j, -j \rangle$ is extensive, $\delta E$ is negligible in the thermodynamic limit. Therefore, the coherent state represents a good variational wavefunction for the the ground state energy in the thermodynamic limit.

The primary advantage of the angular momentum representation is that the dynamics of the condensate can be understood in terms of a spin vector on a Bloch sphere. For strong external pulses of duration $t_p$ such that $\int_0^{t_p} dt |\bar{\Omega}(t)| \gg |\Delta \omega| t_p$, $|\chi| t_p$, the time evolution operator, $U = \hat{T} e^{-i \int \hat{H} dt}$, is simply a rotation operator in spin-space

\[ R(\theta, \phi) = \exp[-i \theta (J_x \sin \phi - J_y \cos \phi)], \tag{12a} \]

where $\theta \sin \phi = \int_0^{t_p} dt \text{Re} (\bar{\Omega}(t))$ and $\theta \cos \phi = \int_0^{t_p} dt \text{Im} (\bar{\Omega}(t))$. We have neglected the time ordering operator, $\hat{T}$ in Eq. (12a). This is justified if $\bar{\Omega}(t)$ is a square pulse so that the Hamiltonian commutes with itself at different times in the interval $0 \leq t \leq t_p$. Equation (12a) is a rotation in spin-space through an angle $\theta$ about the rotation axis $\mathbf{n} = (\sin \phi, -\cos \phi, 0)$.

When there is no external field present, the time evolution operator is simply

\[ U_o(t) = e^{-i(\Delta \omega J_z + \chi J_x^2)t} \tag{13} \]

which is diagonal in the $|j, m\rangle$ basis.

### III. CONDENSATE DYNAMICS

In this section, we consider the dynamics of the condensate subject to two external pulses separated by a time interval $T$. For $t \leq 0$ we assume that there have been no pulses applied and that the condensate is in the ground state. We denote the two ground states, considered in the last section, at $t = 0$ by

\[ |\Psi_N\rangle = |j, -j\rangle; \tag{14a} \]

\[ |\Psi_C\rangle = |0, \alpha_2\rangle; \tag{14b} \]
These two states correspond to a spin vector that is pointing towards the south pole of the Bloch sphere. For the number state, \(|\Psi_N\rangle\), the length of this vector in the \(-z\) direction is \(N/2\). The coherent state, \(|\Psi_C\rangle\), also points in the \(-z\) direction but with an *average* length of \(N/2\) and an uncertainty in the \(z\)-component of the length of \(\Delta J_z = \frac{1}{2}\sqrt{N}\).

At time \(t = 0^+\), a pulse is applied that rotates the system about the \(y\)-axis through an angle of \(\pi/2\). This pulse is described by the rotation operator \(R(\pi/2, \pi)\). It has the effect of transferring half of the condensate population from state 2 into state 1 so that after the pulse \(\langle J_z \rangle = 0\) with the spin vector now pointing in the \(+x\) direction. The condensate is then allowed to evolve freely for a time \(T\) according to Eq. (13). After this period of free evolution, a second \(\pi/2\)-pulse is applied, again given by the rotation \(R(\pi/2, \pi)\). After the second pulse, the population difference, \(\langle \hat{N}_1 - \hat{N}_2 \rangle = 2 \langle J_z(T) \rangle\) as a function of \(T\) is measured.

For \(T = 0\), the effect of the two pulses would just be a spin-flip so that the spin vector would now be pointing in the \(+z\) direction. Because the state of the system following the first pulse (for both \(|\Psi_N\rangle\) and \(|\Psi_C\rangle\)) is a superposition of \(|j, m\rangle\) states for all \(m\) values, the free evolution due to Eq. (13) causes the spin vector to diffuse in the equatorial plane as the different \(|j, m\rangle\) states get out of phase with each other. Because the \(m\) are discrete integers, the \(|j, m\rangle\) states can re-phase leading to a revival of the spin vector. This dephasing and rephasing of the spin vector manifests itself as collapse and revivals of the population difference following the second pulse. However, the two initial states, \(|\Psi_N\rangle\) and \(|\Psi_C\rangle\), have very different collapse and revival times owing to the fact that \(|\Psi_C\rangle\) is a superposition of states with different \(j\).

The calculation of \(\langle J_z(T) \rangle\) is straightforward and we outline the calculation for the two initial states in the following two subsections.

### A. Number state, \(|\Psi_N\rangle\).

The quantity we wish to calculate is

\[
\langle J_z(T) \rangle_N = \langle \Psi_N | R(\pi/2, \pi)^\dagger U_o(T)^\dagger R(\pi/2, \pi)^\dagger J_z R(\pi/2, \pi) U_o(T) R(\pi/2, \pi) | \Psi_N \rangle.
\]

(15)
First we note that
\[ R(\beta, \pi) J_z R(\beta, \pi) = \cos \beta J_z - \sin \beta J_x \]  
so that \[ R(\frac{\pi}{2}, \pi) J_z R(\frac{\pi}{2}, \pi) = - \text{Re} \{ J_z \} . \] Consequently, Eq. (15) reduces to
\[ \langle J_z(T) \rangle_N = - \text{Re} \left\{ \langle \Psi_N | R(\frac{\pi}{2}, \pi)^\dagger J_z U_0(T) R(\frac{\pi}{2}, \pi) | \Psi_N \rangle \right\} = - \text{Re} \left\{ \langle J_+ (T) \rangle_N \right\} \]  
The matrix elements of \( R(\frac{\pi}{2}, \pi) \) are easily calculated for arbitrary \( j \), so that the state of the system following the first pulse and the free evolution period is simply,
\[ U_0(T) R(\frac{\pi}{2}, \pi) | \Psi_N \rangle = \sum_{m=-j}^{j} \frac{(-1)^{m+j}}{2^j} \sqrt{\frac{(2j)!}{(j-m)!(j+m)!}} \exp \left[ -i \left( \Delta \omega m + \chi_m^2 \right) \right] | j, m \rangle . \]  
Finally, one obtains
\[ \langle J_+ (T) \rangle_N = - j e^{i \Delta \omega T} \cos^{2j-1} (\chi_T) ; \]  
so that the population difference, \( \langle \hat{N}_1 - \hat{N}_2 \rangle_N = 2 \langle J_z(T) \rangle_N \) is then
\[ \langle \hat{N}_1 - \hat{N}_2 \rangle_N = N \cos \left( \left( \delta + \frac{3}{2} (\omega_1 - \omega_2) + \chi_- (N - 1) \right) T \right) \cos^{N-1} (\chi_T) . \]  

**B. Coherent State, \( |\Psi_C\rangle \).**

The calculation of \( \langle J_z(T) \rangle_C \) is similar to that of \( \langle J_z(T) \rangle_N \), the main difference being an average over a Poissonian distribution of number states. As before, we only need to calculate the expectation value of \( J_+ \) following the free evolution period,
\[ \langle J_+ (T) \rangle_C = - \text{Re} \left\{ \langle \Psi_C | R(\frac{\pi}{2}, \pi)^\dagger J_+ U_0(T) R(\frac{\pi}{2}, \pi) | \Psi_C \rangle \right\} = - \text{Re} \left\{ \langle J_+ (T) \rangle_C \right\} \]  
Following the first pulse and the free evolution period, the state of the system is
\[ U_0(T) R(\frac{\pi}{2}, \pi) | \Psi_C \rangle = e^{-|\alpha|^2/2} \sum_{n_2=0}^{\infty} \frac{\alpha_2^{n_2}}{\sqrt{n_2!}} \delta_{n_2, 2j} \]  
\[ \times \left( \sum_{m=-j}^{j} \frac{(-1)^{m+j}}{2^j} \sqrt{\frac{(2j)!}{(j-m)!(j+m)!}} \exp \left[ -i \left( \Delta \omega m + \chi_m^2 \right) \right] | j, m \rangle \right) ; \]
Again, it should be emphasized that Eq. (22) is only valid for calculating matrix elements for operators that are diagonal in $j$, which include all angular momentum operators. The evaluation of $\langle J_+ (T) \rangle_C$ can be done in two steps,

$$\langle J_+ (T) \rangle_C = e^{-|\alpha_2|^2} \sum_{n_2=0}^{\infty} \frac{|\alpha_2|^{2n_2}}{n_2!} \delta_{n_2,2j} (-je^{i\Delta \omega T} \cos^{2j-1} (\chi_T)), \quad (23)$$

where the term in parantheses is the same as Eq. (19a). The final step is just an averaging over a Poissonian distribution of the number of atoms ($|\alpha_2|^2 = N$),

$$\langle J_+ (T) \rangle_C = -\frac{N}{2} e^{i(\delta + \frac{3}{2}(\omega_1 - \omega_2))T} \exp [N (-1 + \cos (\chi_T) e^{i\chi_T})]. \quad (24)$$

The population difference following the second pulse is then,

$$\langle \hat{N}_1 - \hat{N}_2 \rangle_C = N \exp [N (-1 + \cos (\chi_T) \cos (\chi_T))] \times \cos \left( \left( \delta + \frac{3}{2}(\omega_1 - \omega_2) \right) T + N \cos (\chi_T) \sin (\chi_T) \right). \quad (25)$$

Equations (20) and (25) represent the central results of this paper. Note that although the number of atoms, $N$, appearing in the two equations has the same value, the meaning of $N$ is different. In Eq. (20), $N$ is the exact number of particles while for Eq. (25), $N$ is the average number of particles for a superposition of number states.

**IV. COLLAPSE AND REVIVALS**

From Eqs. (20) and (25) one can see that the population difference involves a rapidly oscillating part and an envelope function that is responsible for the collapse and revival of the population difference. For simplicity we assume that the external field is on resonance so that $\delta + \frac{3}{2}(\omega_1 - \omega_2) = 0$ and the population difference for the two cases simplify to

$$\langle \hat{N}_1 - \hat{N}_2 \rangle_N = N \cos (\chi_T (N-1)T) \cos^{N-1} (\chi_T); \quad (26)$$

$$\langle \hat{N}_1 - \hat{N}_2 \rangle_C = N \exp [N (-1 + \cos (\chi_T) \cos (\chi_T))] \cos (N \cos (\chi_T) \sin (\chi_T)); \quad (27)$$

In the following sub-sections we consider several limiting cases.
A. $\chi_-=0$

The simplest nontrivial case to consider is $\chi_1 = \chi_2 \neq \chi_{12}$ (this corresponds to Ref. \[5\] where $\chi_{12} = 0$) so that $\langle \hat{N}_1 - \hat{N}_2 \rangle_n = N \cos^{N-1}(\chi_+ T)$ and $\langle \hat{N}_1 - \hat{N}_2 \rangle_c = N \exp[N(-1 + \cos(\chi_+ T))]$. This case is shown in Figure 1. The population difference quickly decays to zero for both cases as soon as $\cos(\chi_+ T)$ deviates significantly from 1. The collapse time may be estimated by making a Gaussian approximation for small times. One finds then for $\chi_+ T \ll 1$

\[
\langle \hat{N}_1 - \hat{N}_2 \rangle_n \approx N e^{-(T/\tau_N)^2};
\]
\[
\langle \hat{N}_1 - \hat{N}_2 \rangle_c \approx N e^{-(T/\tau_C)^2};
\]

and the collapse times are $\tau_N = \chi_+^{-1}\sqrt{2/(N-1)}$ and $\tau_C = \chi_+^{-1}\sqrt{2/N}$ which are indistinguishable for $N \gg 1$. The variance in $J_z$ following the first pulse is given by

\[
\Delta J_z = \sqrt{\langle \Psi_i | R(\frac{\pi}{2}, \pi) J_z^2 R(\frac{\pi}{2}, \pi) | \Psi_i \rangle} = \sqrt{j/2} = \sqrt{N/2}
\]

for $i = N, C$. The collapse times can be expressed as $\tau_N = \tau_C = 1/ (\sqrt{2} \chi_+ \Delta J_z)$ which shows that the collapses are attributable to the dephasing of the different $J_z$ states due to the $\chi_+ J_z^2$ term in the Hamiltonian.

The revival times are quite different for the two states. For the number state, the revivals occur whenever $T_N = n\pi/\chi_+$ where $n$ is an integer. However, when the number of atoms is even, $N - 1$ is odd and the condensate will undergo anti-revivals when $T = (2n + 1)\pi/\chi_+$ so that $\langle \hat{N}_1 - \hat{N}_2 \rangle_n = -N$ at these times. (Note that a similar affect was described in Ref. \[5\] in which the fringe visibility of the interference pattern could undergo a revival with a $\pi$ phase shift when the number of atoms that had been detected in order to establish an interference pattern was even). In contrast, the coherent state undergoes revivals at the times $T_C = 2n\pi/\chi_+$ which is twice the revival time of the number state. In addition, $\langle \hat{N}_1 - \hat{N}_2 \rangle_c > 0$ regardless of $N$. Therefore, there appear to be two key differences that distinguish a number state from a coherent state for $\chi_- = 0$: (i) The occurrence of a negative
population difference when \( N \) is even and (ii) revival times that are half the revival times of a coherent state.

Since the revivals are determined by the \( \chi_+ J_z^2 \) in Eq. (10), there is a simple explanation for the factor of two difference in the revival times. The state \( U_o(T) R(\frac{\pi}{2}, \pi) |\Psi_N\rangle \) involves a superposition of \(|j, m\rangle\) in which the \( m \) values all differ by an integer. Consequently, the revivals occur when the relative phase between all of the \(|j, m\rangle\) states is an integer multiple of \( 2\pi \). This corresponds to the condition 

\[
[(m+1)^2 - m^2] \chi_+ T = 2n'\pi + \phi \quad \text{for all } m
\]

where \( \phi \) is a global phase factor that is independent of \( m \) and \( n' \) is an integer. By taking \( n' = mn \) and \( \phi = \chi_+ T \), one sees that the revivals occur at integer multiples of the time \( \pi/\chi_+ \).

However, for the coherent state one has instead \( U_o(T) R(\frac{\pi}{2}, \pi) |\Psi_C\rangle \) which is a superposition of states with different \( j \) and \( m \) values so that in this case the values of \( m \) in the superposition need only differ by a half-integer. Therefore, for the coherent state the condition for the occurrence of a revival is 

\[
[(m+1/2)^2 - m^2] \chi_+ T = 2n'\pi + \phi.
\]

Again taking \( n' = mn \) but with \( \phi = \chi_+ T/4 \), the revivals occur at integer multiples of \( 2\pi/\chi_+ \).

**B. \( \chi_- \neq 0 \) and \( \chi_- N \gg 1 \)**

In this case Eqs. (26-27) consist of a rapid oscillations modulated by a slowly varying envelope function which gives rise to the collapse and revivals. As such, we only consider the behavior of the envelope functions in this sub-section which is given by \( f_N(T) \) and \( f_C(T) \) for the number state and coherent state, respectively. The envelope functions are given by

\[
f_N(T) = \cos^{N-1} (\chi_+ T), \quad \text{(31a)}
\]

\[
f_C(T) = \exp \left[ N (-1 + \cos (\chi_+ T) \cos(\chi_- T)) \right]. \quad \text{(31b)}
\]

The collapse and revival times for the number state are the same as what was found in the previous subsection. The only difference is that there are no antirevivals since \( f_N(T) = -1 \) simply corresponds to a \( \pi \) phase shift in the rapidly oscillating part of Eq. (26).

However, the behavior of the coherent state is quite different. One can see that for
short times, \( f_C(T) \approx e^{-(T/\tau_C)^2} \) where the collapse time is \( \tau_C = \sqrt{\frac{2}{N(\chi_+^2 + \chi_-^2)}} \). Therefore, increasing \( \chi_- \) decreases the collapse time. This is attributable to the \( \chi_-(2j-1)J_z \) term in the Hamiltonian which causes the states with different \( j \) but the same \( m \) to get out of phase with each other in a time \( \sim 1/(\chi_-\Delta N) = 1/(\chi_-\sqrt{N}) \). The reduction in the collapse time is illustrated in Figure 2 for the case \( \chi_-/\chi_+ = 2 \).

Revivals occur when \( \cos(\chi_+T)\cos(\chi_-T) = 1 \) which can only be satisfied if \( \chi_-/\chi_+ = p/q \) where \( p \) and \( q \) are integers. When this condition is satisfied, the revivals occur at times \( T_C = n\pi/\chi_+ \) where \( n \) is a positive integer and \( (p/q) n \) is an integer such that if \( n \) is odd (even) then \( (p/q) n \) is also odd (even). When \( \chi_-/\chi_+ \) is irrational, there are no revivals and even for rational values of \( \chi_-/\chi_+ \), the period between revivals can differ significantly from the revival period for the number state, \( \pi/\chi_+ \). For example, if \( \chi_-/\chi_+ = 1/4 \) then the first revival will occur at \( 8\pi/\chi_+ \) for the coherent state. In Figure 3, the revivals are shown for the ratio \( \chi_-/\chi_+ = 1/3 \). One can see that the number state has revivals at all integer multiples of \( \pi/\chi_+ \) while the first revival for the coherent state occurs at \( 3\pi/\chi_+ \).

V. DISCUSSION

In the previous section it was shown that the population difference in a Ramsey fringe experiment can exhibit collapses and revivals with times that can be very different for a number state and a coherent state. For a number state, we will, in general, be ignorant of the exact number of atoms so that we can only say that the state \( |0, N\rangle \) occurs with some probability \( p(N) \). Consequently, an ensemble average over many different experimental runs will lead to statistical fluctuations in the number of atoms even though for the number state, the number of atoms in any given experiment would be precisely defined. Suppose \( p(N) \) obeys a Poisson distribution, will there be any difference now between the number states and the coherent state? There is in fact a difference since the quantum fluctuations of the coherent state are present in each experimental run whereas the statistical fluctuations of the number states only become manifest after averaging over many such runs. As an
example, consider the case of the anti-revivals for the number state, \( \langle \hat{N}_1 - \hat{N}_2 \rangle_N = -N \), which can occur when \( \chi_1 = \chi_2 \neq \chi_{12} \). In this case, one might observe such anti-revivals in any single experiment with a number state but such anti-revivals would never be observed in any experiment for a coherent state since it is the averaging over the Poisson number distribution which prevents the anti-revivals. Even though the anti-revivals may be observed in each experimental run for the number states, the average over many experimental runs, \( \sum_N p(N) \langle \hat{N}_1 - \hat{N}_2 \rangle_N = \langle \hat{N}_1 - \hat{N}_2 \rangle_C \), will not exhibit the anti-revivals since they get averaged out.

One may also consider initial states of the condensate given by \( |\Psi\rangle = \sum_n c_n |0, n\rangle_F \) where \( c_n \) is sharply peaked around \( n = N \) (such as \( c_n \propto e^{-(n-N)/4\sigma^2} \)). This state satisfies Eq. (1) for finite \( N \). However, \( |\Psi\rangle \) will exhibit qualitatively similar behavior to the coherent state \( |C\rangle \) since the critical difference in the results of the Ramsey fringe experiment lie in the quantum fluctuations in the particle number of the coherent state.

Finally, we estimate the order of magnitude of the collapse and revival times. The fundamental time scale which determines the collapses and revivals is \( \hbar/\chi_+ \) (where we now explicitly include that factors of \( \hbar \)). If the trapping potentials for the two components are the same, \( \omega_1 = \omega_2 = \omega \), then, \( \phi_1(r) = \phi_2(r) = \phi(r) \) and \( \int |\phi(r)|^4 d^3 r = 1/ \left[ (2\pi)^{3/2} a_{ho}^3 \right] \) where \( a_{ho} = \sqrt{\hbar/m\omega} \). If we assume a relatively weak trap, \( a_{ho} = 5\mu m \) and if we take \( U_{ij} = \frac{4\pi\hbar^2 a_{ij}}{m} \sim \frac{4\pi\hbar^2 a}{m} \) with \( a = 5nm \) as an estimate for \(^{87}Rb\), one finds \( \chi_i/\hbar \sim \left( \frac{4\pi a_{ho}}{m} \right)/ \left[ (2\pi)^{3/2} a_{ho}^3 \right] = 0.023s^{-1} \). Therefore the collapse times will be on the order of \( (\chi_i/\hbar)^{-1} \sqrt{1/N} \sim 1s \) for \( N = 1000 \) and the revival times will be on the order of \( \pi (\chi_i/\hbar)^{-1} \sim 100s \). Note that since the scattering lengths are nearly equal for the two hyperfine states of the \(^{87}Rb\) condensate \([21]\), \( \chi_\pm \approx 0 \) for \( \omega_1 = \omega_2 \). However, this may be overcome by either manipulating one of the scattering lengths using a Feshbach resonance \([29]\) or by changing the trapping potentials so that \( \phi_1(r) \neq \phi_2(r) \).

As mentioned, a Ramsey fringe experiment has been performed at JILA \([20]\). However, this experiment was performed with a relatively large condensate \((N = 5 \times 10^5) \) such that
\[ Na_{ij}/a_{ho,i} \gg 1 \] 23. The theoretical analysis of the experiment \[ 20 \] \[ 27 \] was based on the Gross-Pitaevskii equation which is a mean-field equation for the order parameter and, as such, already assumes a state of broken symmetry in the condensate. In addition, the duration of this experiment (i.e. the period between pulses, \( T \)) was too short to observe the phase diffusion due to the quantum dynamics of the condensate (see footnote [23] in Ref. [20]).

VI. CONCLUSION

In this paper we have shown that Ramsey’s separated oscillatory field technique applied to a small atomic Bose-Einstein condensate exhibits collapse and revivals in the population difference between the two internal states of the condensate. The collapse and revival times depend on the strength of the two-body interactions and the initial state of the condensate so that one may potentially distinguish between a condensate state that is a number state or a coherent state. Since absorptive and dispersive imaging of atomic BEC measure the density of atoms in the condensate, the Ramsey fringe experiment proposed here should be easier to perform than an experiment which tries to directly observe the collapse and revivals in the order parameter.
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Figure 1. Plot of the normalized population difference, $\langle \hat{N}_1 - \hat{N}_2 \rangle / N$ as a function of $T$ for the case $\chi_\pm = 0$ and $N = 1000$ for the number state, (a), and the coherent state, (b).

Figure 2. Plot of the normalized population difference, $\langle \hat{N}_1 - \hat{N}_2 \rangle / N$ for $\chi_+ T \ll 1$, $\chi_- / \chi_+ = 2$, and $N = 1000$ illustrating the rapid oscillations and the envelope function which leads to the collapse. The collapse time for the coherent state is significantly smaller than the number state.

Figure 3. Plot of the normalized population difference, $\langle \hat{N}_1 - \hat{N}_2 \rangle / N$ as a function of $T$ for $\chi_- / \chi_+ = 1/3$ and $N = 1000$ which shows that revivals at multiples of $\pi / \chi_+$. (a) is the number state and (b) is the coherent state.