Global properties of asymptotically de Sitter and
Anti de Sitter spacetimes

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Ad Majorem Dei Gloriam.

To all those who seek to fulfill God’s will in their lives.
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## Contents

1 Introduction  
1.1 Lorentz vector spaces  
1.2 Spacetimes  
1.2.1 The Levi-Civita connection  
1.2.2 Covariant derivative  
1.2.3 Geodesics  
1.3 Causal theory  
1.3.1 Definitions  
1.3.2 Global causality conditions  
1.3.3 Achronal sets  
1.3.4 Domains of dependence and Cauchy surfaces  
1.4 Smooth null hypersurfaces  
1.5 The Einstein equations  

2 Asymptotically de Sitter Spacetimes  
2.1 Conformal boundaries  
2.2 Rigidity of de Sitter space
2.3 Globally hyperbolic and asymptotically de Sitter spacetimes  . . . . . . 40
2.4 The initial value problem  . . . . . . . . . . . . . . . . . . . . . . . . . 53
2.5 Matter fields in asymptotically simple spacetimes  . . . . . . . . . . . 56

3 Asymptotically Anti de Sitter Spacetimes  . . . . . . . . . . . . . . . . . 72

3.1 Spacetimes with timelike boundary  . . . . . . . . . . . . . . . . . . . 73
   3.1.1 Causal theory  . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 75
   3.1.2 Global Hyperbolicity  . . . . . . . . . . . . . . . . . . . . . . . . 83
   3.1.3 Limit of curves  . . . . . . . . . . . . . . . . . . . . . . . . . . . 86
   3.1.4 Covering Spacetimes  . . . . . . . . . . . . . . . . . . . . . . . . 91
   3.1.5 Smooth null hypersurfaces  . . . . . . . . . . . . . . . . . . . . . 96
3.2 The Principle of Topological Censorship  . . . . . . . . . . . . . . . . . 100
3.3 PTC on spacetimes with timelike boundary  . . . . . . . . . . . . . . . 101
   3.3.1 Fastest causal curves  . . . . . . . . . . . . . . . . . . . . . . . . 102
   3.3.2 Strong form of the PTC  . . . . . . . . . . . . . . . . . . . . . . 108
3.4 Future research  . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 112

Bibliography  . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 113

A Extensions of spacetimes with boundary  . . . . . . . . . . . . . . . . . 118
B Focal points along null geodesics  . . . . . . . . . . . . . . . . . . . . 121
C Conformally related metrics  . . . . . . . . . . . . . . . . . . . . . . . . 128
Chapter 1

Introduction

As a means to acquaint the reader to the subject, we begin this work by providing a quick survey on definitions and standard results in Lorentzian geometry. The proofs of most results listed in this chapter can be found in any of the standard references \cite{2,32,46,48}. For consistency, we will follow as much as we can the conventions given in \cite{46}.

1.1 Lorentz vector spaces

In the special relativity, spacetime is modelled after Minkowski space $\mathbb{M}^n$, that is, $\mathbb{R}^n$ endowed with the Lorentz product $L : \mathbb{M}^n \times \mathbb{M}^n \to \mathbb{R}$,

$$L(v, w) = -v_0w_0 + v_1w_1 + \ldots + v_{n-1}w_{n-1}. \quad (1.1.1)$$

In this context, the first coordinate $v_0$ of $v \in \mathbb{M}^n$ represents time while the remaining coordinates represent space.
From the algebraic point of view, $L$ is just a symmetric bilinear form on the vector space $\mathbb{R}^n$. Moreover $L$ is non-degenerate, in the sense that the linear functional $L_v: \mathbb{M}^n \to \mathbb{R}$, $L_v(w) = L(v, w)$ is identically zero if and only if $v = 0$.

In greater generality, a non-degenerate symmetric bilinear form $g: V \times V \to \mathbb{R}$ on a real vector space $V$ is called an inner product in $V$. By a well known result in linear algebra, every inner product $g$ admits an orthonormal basis, i.e. a basis $\{e_i\}$, $i = 1, 2, \ldots, n$ satisfying

$$g(e_i, e_j) = \begin{cases} 
-1 & \text{if } i = j \text{ and } 1 \leq i \leq r \\
1 & \text{if } i = j \text{ and } r + 1 \leq i \leq n \\
0 & \text{if } i \neq j 
\end{cases}$$  \hspace{1cm} (1.1.2)

Further, the number $r$ is an invariant of $g$, called the signature of $g$.

**Definition 1.1** A Lorentz vector space $(V, g)$ is a vector space $V$ of dimension $n \geq 2$ together with an inner product of signature $r = 1$.

Hence, it follows that any Lorentz vector space of dimension $n$ is isometric to $\mathbb{M}^n$. Therefore $V$ splits in three disjoint sets, just as Minkowski space.

**Definition 1.2** Let $(V, g)$ be a Lorentz vector space, we say that $v \neq 0$ is timelike, spacelike or null if $g(v, v) < 0$, $g(v, v) > 0$ or $g(v, v) = 0$, respectively. The zero vector will be considered spacelike. A non-spacelike vector will be also called causal.

Notice that the set of null vectors form a cone. This is true also for timelike and causal vectors.
Definition 1.3 Let \((V, g)\) be a Lorentz vector space, the sets

\[
\Lambda(V) = \{v \in V \setminus \{0\} \mid g(v, v) = 0\}, \quad \mathcal{T}(V) = \{v \in V \mid g(v, v) < 0\}, \quad (1.1.3)
\]

\[
C(V) = \{v \in V \mid g(v, v) \leq 0\},
\]

are called the null, timelike and causal cone of \(V\), respectively.

The existence of these cones is a distinctive feature of Lorentzian vector spaces. At this point it is also important to observe that \(\mathcal{T}(V)\) has exactly two connected components. This simple observation enables us to develop the notion of past and future directions.

Definition 1.4 A time orientation on a Lorentz vector space \(V\) of dimension \(n \geq 3\) is a choice of one connected component \(\mathcal{T}^+(V)\) of \(\mathcal{T}(V)\). This component will be called the future timelike cone of \(V\), while \(\mathcal{T}^-(V) = \mathcal{T}(V) - \mathcal{T}^+(V)\) is called the past timelike cone of \(V\).
Since any null vector can be approximated by a sequence of timelike vectors, a
time orientation on $V$ induces in a natural way a choice of a component $C^+(V)$ of
the causal cone $C(V)$.

The following proposition enumerates some of the most important properties re-
lating timelike cones (see lemmas 5.29-5.31 in [46])

**Proposition 1.5** Let $(V, g)$ be a Lorentz vector space and for $v \in C(V)$ define $|v| := \sqrt{-g(v,v)}$. Thus the following results hold:

1. $C^+(V)$ and $C^-(V)$ are convex sets.
2. $v \in C^+(V)$ if and only if $-v \in C^-(V)$.
3. Let $v, w \in C^+(V)$, then $v$ and $w$ are in the same causal cone if and only if $g(v,w) < 0$ or $v$ and $w$ are two collinear null vectors.
4. $|g(v,w)| \geq |v| \cdot |w|$ for all $v, w \in C(V)$, with equality if and only if $v$ and $w$ are
collinear.
5. $|v + w| \geq |v| + |w|$ for all $v, w \in C^+(V)$.

To finish our discussion on Lorentz vector spaces, let us notice that a Lorentz
metric $g$ on $V$ does not always induce a Lorentz metric on a vector subspace $W \subset V$.
There are actually three different possibilities: $(g|_W, W)$ can be either a Lorentz
vector space, a Riemann vector space (that is, $g|_W$ is a positive definite bilinear form
on $W$) or $g|_W$ may be a degenerate form on $W$. In the latter case, we say that $W$ is
a null subspace of $V$. 
The next proposition gives a criterion to determine in which category $W$ falls, depending on the causal vectors of $V$.

**Proposition 1.6** Let $(V, g)$ be a Lorentz vector space and $W \subset V$ a vector subspace of $V$ then

1. $(W, g|_W)$ is a Riemann vector space if and only if $W$ contains no causal vector.

2. $(W, g|_W)$ is a Lorentz vector space if and only if $W$ contains a timelike vector.

3. $(W, g|_W)$ is null if $W$ contains a null vector but no timelike vector.

**Remark 1.7** If $(W, g|_W)$ is a degenerate subspace of $V$ then $W$ intersects the null cone $\Lambda(V)$ in a distinctive null direction. More precisely, $W \cap (\Lambda(V) \cup \{0\})$ is a one dimensional (null) vector subspace of $V$. As a consequence, if $v \in W \cap \Lambda(V)$ then the map $w \mapsto v + w$ is a linear monomorphism from $W$ to $T_v\Lambda(V)$. (See figure 1.2).
1.2 Spacetimes

We move now into studying spaces that locally look like Minkowski space.

A metric \( g \) on a connected smooth manifold \( M \) is a non-degenerate, symmetric bilinear smooth tensor field of type \((0,2)\) on \( M \). In other words, \( g \) smoothly assigns to each \( p \in M \) an inner product \( g_p \) on the vector space \( T_pM \). Notice that by the connectedness of \( M \) the signatures of \( g_p \) are all equal, and hence this common signature is referred to as the signature of \( g \). A smooth manifold endowed with a metric is called a semi-Riemannian manifold.

**Definition 1.8** A Lorentzian manifold is a semi-Riemannian manifold of signature \( r = 1 \).

Then the metric \( g \) on a Lorentzian manifold \( M \) can be thought as a device that transplants smoothly the geometry of Minkowski space to the tangent spaces of \( M \).

For a Lorentz manifold to adequately describe our universe, it must have a globally defined “future”. As discussed above, we can give a time orientation to each tangent space on a Lorentzian manifold \((M, g)\). Moreover, in light of proposition 1.5, a time orientation on \( V \) is equivalent to the selection of a timelike vector \( v \in T(V) \). Thus the existence of a global time orientation on \((M, g)\) is equivalent to the existence of a smooth timelike vector field on \( M \).

**Definition 1.9** A Lorentzian manifold is said to be time orientable if there exists a smooth timelike vector field \( X \) on \( M \).
Notice that the existence of a time orientation is independent of the existence of an orientation of \( M \) as a smooth manifold. Nevertheless, in analogy with non-orientable smooth manifolds, if \((M, g)\) does not admit a time orientation, then it has a time orientable double cover.

Because of its physical significance, on chapters 2 and 3 we will be dealing exclusively with spacetimes.

**Definition 1.10** A spacetime is an oriented and time oriented Lorentzian manifold.

We finish this section by making the following observation:

**Remark 1.11** Any embedded submanifold \( \iota: N \hookrightarrow M \) of a Riemannian manifold \((M, g)\) is a Riemannian manifold itself since the pullback \( \iota^* g \) is a positive definite symmetric form on \( N \). This does not hold in the Lorentzian setting, since the pullback \( \iota^* g \) may be degenerate. Further, it may not have a constant signature; and even if it does, \( \iota^* g \) could be either Lorentzian or Riemannian. In the former case we say \( N \hookrightarrow M \) is timelike while in the latter we say \( N \) is spacelike. (See figure 1.3).

### 1.2.1 The Levi-Civita connection

Just as in the more familiar Riemannian setting, a Lorentz metric \((M, g)\) not only gives us a way to measure lengths of tangent vectors, but it also can be used to build the machinery that enables us to differentiate vector fields.

**Definition 1.12** A connection on a smooth manifold \( M \) is a function \( \nabla: \mathcal{X}(M) \times \mathcal{X}(M) \rightarrow \mathcal{X}(M), (X, Y) \mapsto \nabla_X Y \) such that
Figure 1.3:

Three submanifolds of $M = \mathbb{M}^3$ with different causal character. $A$ is the upper half of the hyperboloid $L(v, v) = -1$, $B$ is the hyperboloid $L(v, v) = 1$, and $W$ is the plane $y + t = 0$. $A$ is spacelike, $B$ is timelike and $W$ is null (degenerate).
1. \( \nabla_{fX + gY} Z = f \nabla_X Z + g \nabla_Y Z \) for all \( f, g \in C^\infty(M) \).

2. \( \nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z \).

3. \( \nabla_X fZ = f \nabla_X Z + X(f)Z \) for all \( f \in C^\infty(M) \).

Thus a connection is a device that takes “directional derivatives” of smooth vector fields. Notice that besides the notion of differentiability introduced by a connection, we also have a simpler notion of derivation given by the Lie bracket \([X,Y]\). The difference between these two ways of taking derivatives is given by a tensor, called the torsion

\[
\text{Tor}(X, Y) = \nabla_X Y - \nabla_Y X - [X,Y] \quad X, Y \in \mathfrak{X}(M). \tag{1.2.1}
\]

A connection is torsion free if \( \text{Tor} \equiv 0 \). In the case of a connection on a semi-Riemannian manifold \((M, g)\), it is desirable that \( \nabla \) relates to the metric \( g \) via a “product rule”. We will say that \( \nabla \) is adapted to \( g \) if

\[
Z(g(X,Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y) \quad \text{for all } X, Y, Z \in \mathfrak{X}(M) \tag{1.2.2}
\]

A fundamental result due to T. Levi-Civita establishes the existence and uniqueness of a torsion free connection adapted to the metric in any semi-Riemannian manifold.

**Theorem 1.13** Let \((M, g)\) be a semi-Riemannian manifold \((M, g)\). Then there exists a unique torsion free connection \( \nabla \) on \( M \) adapted to \( g \). Such connection is called the Levi-Civita connection of \( M \).

Just as in Riemannian geometry, curvature is measured as the difference between \([\nabla_X, \nabla_Y]\) and \( \nabla_{[X,Y]} \).
**Definition 1.14** Let $(M, g)$ be a semi-Riemannian manifold. The Riemann curvature morphism is the map that assigns to each pair of smooth vector fields $(X, Y)$ the endomorphism $R(X, Y): \mathcal{X}(M) \to \mathcal{X}(M)$ given by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \quad (1.2.3)$$

Thus if $\mathcal{U}$ is a coordinate chart we have $R(\partial_\alpha, \partial_\beta)\partial_\gamma = R^\varsigma_{\alpha\beta\gamma} \partial_\varsigma$ for some $R^\varsigma_{\alpha\beta\gamma} \in C^\infty(\mathcal{U})$.

A number of different tensors can be constructed using the Riemann morphism.

**Definition 1.15**

1. The Riemann curvature tensor is the $(0, 4)$ tensor metrically related to $R(X, Y)Z$, that is $R(X, Y, Z, W) := g(R(X, Y)Z, W)$.

2. The Ricci tensor $\text{Ric}$ is found by contracting the Riemann endomorphism. In coordinates $\text{Ric}_{\alpha\beta} = R^\varsigma_{\alpha\beta\varsigma}$.

3. The Weyl tensor $W$ is the trace free part of the Riemann curvature tensor, hence it is characterized by the requirement that any of its contractions vanishes. It is given by the formula

$$W := R - \frac{2}{n-2} g \otimes \text{Ric} + \frac{R}{(n-1)(n-2)} g \otimes g, \quad (1.2.4)$$

where $\otimes$ denotes the Kulkarni-Nomizu product (see paragraph 1.110 in [4]).

4. The scalar curvature $R$ is the trace of the Ricci tensor. Thus $R = g^{\alpha\beta} \text{Ric}_{\alpha\beta}$.

---

1Recall Einstein summation convention: summation is performed over an abstract index when it appears twice in the same expression, once up and once down.
5. Let \( \Pi \subset T_pM \) be a two dimensional plane of non-zero area. The sectional curvature \( K \) of \( \Pi \) is defined by

\[
K(\Pi) := \frac{g(R(v, w)v, w)}{g(v, v)g(w, w) - g(v, w)^2},
\]

where \( v, w \in T_pM \) are any two vectors spanning \( \Pi \).

**Definition 1.16** An Einstein manifold is a semi-Riemannian manifold \((M, g)\) for which \( \text{Ric} = kg \), for some constant \( k \).

It follows that Einstein manifolds have constant scalar curvature. As the name suggests, Einstein manifolds arise as solution of the Einstein equations of general relativity.

**Remark 1.17** An Einstein manifold whose Weyl tensor vanishes everywhere has constant (sectional) curvature (see paragraphs 1.114 and 1.118 in [4]).

### 1.2.2 Covariant derivative

The Levi-Civita connection can also be used to take derivatives on tangent directions along curves.

**Theorem 1.18** Let \( \alpha: I \to M \) be a smooth curve and let us denote by \( \mathcal{X}(M) \) the set of smooth vector fields along \( \alpha \). There is a unique operator \( D/dt: \mathcal{X}(\alpha) \to \mathcal{X}(\alpha) \) satisfying

1. \( \frac{D}{dt}(X + Y) = \frac{DX}{dt} + \frac{DY}{dt} \) for all \( X, Y \in \mathcal{X}(\alpha) \).

2. \( \frac{D}{dt}(fX) = \frac{df}{dt}X + f \frac{DX}{dt} \) for all \( f \in C^\infty(\alpha), X \in \mathcal{X}(\alpha) \).
3. \( \frac{D}{dt}X|_{\alpha(t)} = \nabla_{\alpha'(t)}X \) for all \( X \in \mathcal{X}(M) \).

\( D/dt \) is called the covariant derivative along \( \alpha \).

A vector field \( X \in \mathcal{X}(\alpha) \) is said to be parallel if \( X' := DX/dt \) vanishes along \( \alpha \). Given \( v \in T_pM \) with \( p = \alpha(0) \), basic ODE theory guarantees the existence of a unique parallel vector field \( V \in \mathcal{X}(\alpha) \) with \( V(0) = v \). If \( q = \alpha(a) \) then \( V(a) \in T_qM \) is called the parallel translate of \( v \) to \( q \) along \( \alpha \). Hence by considering the parallel translates of all vectors at \( p \) along \( \alpha \), we construct a map \( P: T_pM \to T_qM \) called parallel translation from \( p \) to \( q \) along \( \alpha \). It is easy to check that \( P \) is a linear isometry.

### 1.2.3 Geodesics

By taking the covariant derivative of the tangent vector field to a curve \( \alpha \) we get an analog to the acceleration of \( \alpha \). From our experience in Euclidean space, a straight line can be characterized as a curve with zero acceleration. We can extend this idea to semi-Riemannian geometry as follows:

**Definition 1.19** A geodesic is a smooth curve \( \alpha: I \to M \) such that \( \alpha'' \equiv 0 \)

Geodesics play a fundamental role in geometry. From existence and uniqueness results in ODE theory, it follows that given \( p \in M \) and \( v \in T_pM \) there exists a unique geodesic \( \gamma_v: I \to M \) with \( \gamma_v(0) = p \) and \( \gamma'_v(0) = v \) whose domain of definition is maximal. In other words, if \( \alpha: J \to M \) is another geodesic with \( \alpha(0) = p \) and \( \alpha'(0) = v \), then \( J \subset I \) and \( \alpha = \gamma_v|_J \). Such a geodesic is said to be inextendible. Furthermore, it turns out that every continuous extension of a geodesic \( \gamma \) to one of its endpoints is a geodesic itself (refer to lemma 5.8 in [46]).
**Proposition 1.20** Let $\gamma: [0, a) \to M$ be a geodesic and let $\tilde{\gamma}: [0, a] \to M$ be a continuous curve with $\tilde{\gamma}|_{[0,a)} = \gamma$. Then $\tilde{\gamma}$ is a geodesic.

We can group all the geodesics emanating from $p \in M$ in a single map $\exp_p: T_pM \to M$ called the exponential map at $p$.

**Definition 1.21** The exponential map at $p$, $\exp_p: T_pM \to M$, is defined by $\exp_p(v) = \gamma_v(1)$.

The differential of the exponential map $(\exp_p)_*: T_0(T_pM) \to T_pM$ is the canonical isomorphism $T_0(T_pM) \simeq T_pM$. Thus, by the inverse function theorem, there is a neighborhood $\tilde{U}$ of $0_p \in T_pM$ such that $\exp_p|_{\tilde{U}}$ is a diffeomorphism onto $U := \exp_p(\tilde{U})$.

Such a neighborhood $U$ is called a normal neighborhood of $p$. A more detailed analysis shows every point $p \in M$ has a neighborhood $U$ that is normal with respect to any of its points. These latter normal neighborhoods are called convex. The terminology comes from the fact that there exists a unique geodesic segment contained in $U$ joining any pair of points $x, y \in U$.

At this point we notice that the exponential map based at $p \in M$ is a particular case of a more general construction. Given a semi-Riemannian submanifold $P \subset M$ of dimension $k$ and $p \in P$, we have the splitting $T_pM = T_pP \oplus T_pP^\perp$, where $T_pP^\perp = \{v \in T_pM \mid g(v, w) = 0, \forall w \in T_pP\}$. Hence $NP := \bigcup_{p \in P} T_pP^\perp$ is a rank $n-k$ vector bundle over $P$. We then define the normal exponential map to $P$, $\exp^\perp: NP \to M$ by $\exp^\perp(v_p) := \gamma_{v_p}(1)$. Thus, if we consider $\{p\}$ as a 0-dimensional submanifold of $M$, the normal exponential map to $\{p\}$ is just $\exp_p$. 
**Definition 1.22** A Jacobi field on a geodesic $\alpha$ is a vector field $Y \in \mathcal{X}(\alpha)$ that satisfies the Jacobi equation $Y'' + R(Y, \alpha')\alpha' = 0$.

Jacobi fields arise in a natural way in various branches of semi-Riemannian geometry, such as comparison theory. They also play a critical role in the calculus of variations, since the variation vector field of a geodesic $\alpha$ by geodesics is a Jacobi field on $\alpha$. The next result shows that the existence of certain Jacobi fields are related to the degeneracy of the exponential map.

**Definition 1.23** Two points $p$ and $q$ along a geodesic $\alpha$ are conjugate if there exists a nontrivial Jacobi field $Y$ on $\alpha$ such that $Y_p = 0 = Y_q$.

**Proposition 1.24** Let $\alpha$ be a geodesic with $\sigma(0) = p$ and $\sigma(a) = q$. Then the following are equivalent:

1. $q$ is conjugate to $p$ along $\alpha$.

2. There exists a nontrivial variation through geodesics from $p$ whose variation vector field vanishes at $q$.

3. The exponential map $\exp_p$ is singular at $a\alpha'(0) \in T_pM$.

In analogy to the concept of conjugate points, we encounter the notion of focal points when considering the points where the normal exponential map degenerates. Appendix B is devoted to the treatment of null focal points to spacelike submanifolds.

Note that geodesics in a Lorentzian manifold posses a causal character. In other words, $\alpha$ is either timelike, spacelike or null, according to the character of its tangent vector field.
Moreover, geodesics in a Lorentzian manifold are radically different from its Riemannian counterparts. More notably, timelike geodesics on a Lorentzian manifold locally \textit{maximize} arc length. That is, if \( \alpha \) is a timelike geodesic in a Lorentzian manifold \((M, g)\), \( U \) is a normal neighborhood around \( p = \alpha(0) \) and \( I \subset \mathbb{R} \) is an interval such that \( 0 \in I \) and \( \alpha(I) \subset U \) then the geodesic segment \( \alpha|_{[0,a]} \) is the unique longest timelike curve in \( U \) joining \( p \) to \( \alpha(a) \) for all \( a \in I \).

1.3 Causal theory

1.3.1 Definitions

A time orientation can be used to causally relate pair of points on a spacetime \((M, g)\). We say that \( p \) \textit{chronologically precedes} \( q \), denoted by \( p \ll q \), if there is a future timelike path\[^1\] from \( p \) to \( q \). Likewise, we say \( p \) \textit{causally precedes} \( q \) if there is a future causal path from \( p \) to \( q \) and denote this by \( p \leq q \). These relations define the \textit{causal structure} of \( M \).

Notice that we can also define the relations \( \ll \) and \( \leq \) relative to a set \( A \subset M \). In other words, \( p \ll_A q \) \((p \leq_A q)\) will mean that there is a future timelike (causal) path contained in \( A \) from \( p \) to \( q \).

Intuitively, if \( p \ll q \) then \( q \) is at the future of \( p \). We formalize this observation in the next definition.

\[^1\]A future timelike path is a piecewise smooth curve with finitely many segments, all of which are future timelike.
**Definition 1.25** Let \((M, g)\) be a spacetime and \(p \in M\). The timelike future of \(p\), \(I^+(p)\), and causal future of \(p\), \(J^+(p)\), are given by:

\[
I^+(p) = \{ q \in M \mid p \ll q \}, \quad J^+(p) = \{ q \in M \mid p \leq q \}. \tag{1.3.1}
\]

The timelike and causal pasts of \(p\) are defined time dually:

\[
I^-(p) = \{ q \in M \mid q \ll p \}, \quad J^-(p) = \{ q \in M \mid q \leq p \}. \tag{1.3.2}
\]

Notice the above definitions extend to arbitrary sets \(A \subset M\) in a natural way. For instance \(I^+(A) = \bigcup_{p \in A} I^+(p)\).

The causal structure of \(M\) is closely related to its manifold topology.

**Proposition 1.26** Let \((M, g)\) be a spacetime and \(p \in M\). Then

1. \(I^+(p)\) is an open set in \(M\).

2. \(\text{int}(J^+(p)) = I^+(p)\).

3. \(J^+(p) \subset I^+(p)\).

Notice that in general \(J^+(p)\) is not a closed subset of \(M\). On the other hand, \(J^+(p, U)\) is closed in \(U\), where \(U\) is a convex normal neighborhood of \(p\).

Note also that any timelike curve can be approximated by a sequence of broken null geodesics. Conversely, using variational arguments we can deform a causal path \(\alpha\) into a timelike curve while keeping its endpoints fixed, unless \(\alpha\) is a smooth null geodesic (see proposition 10.46 in [46]).
Figure 1.4:

Let \( M = M^3 - \{q\} \) and \( p \) be the origin, then \( J^+(p) \) is not a closed subset of \( M \).

**Proposition 1.27** Let \((M, g)\) be a spacetime. If \( \alpha \) is a future causal path from \( p \) to \( q \) that is not a null geodesic, then there is a timelike curve from \( p \) to \( q \) arbitrarily close to \( \alpha \).

The following results follow immediately:

**Proposition 1.28** Let \( q \in J^+(p) - I^+(p) \), then every future causal curve from \( p \) to \( q \) can be parameterized as a null geodesic.

**Proposition 1.29** Let \( p, q, r \in M \) with \( p \leq q \) and \( q \ll r \), then \( p \ll r \).

**Proposition 1.30** Let \( \alpha \) and \( \gamma \) be two distinct future null geodesics meeting at \( q \in M \). If \( p \) comes before \( q \) along \( \alpha \) and \( r \) comes after \( q \) along \( \gamma \), then \( p \ll r \).

Thus proposition 1.30 implies that if two future directed null geodesics emanating from \( p \) meet at \( q \), then every point to the future of \( q \) along any of them is on the timelike future of \( p \).
Moreover, since in light of proposition 1.24 conjugate points can be viewed as almost meeting points of geodesics, we also have the following:

**Proposition 1.31** Let $\alpha$ be a null geodesic emanating from $p \in M$. If $q \in \alpha$ comes after the first conjugate point to $p$ along $\alpha$, then $q \in I^+(p)$.

Notice also that by considering the normal exponential map to a spacelike submanifold we get a similar result in terms of focal points:

**Proposition 1.32** Let $P$ be a spacelike submanifold of $M$ and $\alpha : [0,1] \rightarrow M$ a future causal curve from $p$ to $q \in S$. Then either $I^+(p) \cap S \neq \emptyset$ or $\gamma$ is a null geodesic normal to $S$ with no focal points to $S$ on $(0,1]$.

We end up this section by defining the Lorentz distance function and listing some of its properties.

**Definition 1.33** Let $(M,g)$ be a spacetime, then $\tau : M \times M \rightarrow [0,\infty]$ defined by

$$\tau(p,q) = \begin{cases} 
\sup \{ L(\gamma) \mid \gamma \text{ is a causal curve from } p \text{ to } q \} & \text{if } p \leq q \\
0 & \text{otherwise}
\end{cases}$$

is called the Lorentz distance function of $M$.

**Proposition 1.34** Let $(M,g)$ be a spacetime and $\tau$ its Lorentz distance function, then

1. $\tau(p,q) > 0$ if and only if $p \ll q$.

2. If $p \leq q$ and $q \leq r$ then $\tau(p,r) \geq \tau(p,q) + \tau(q,r)$.

3. $\tau$ is lower semicontinuous, although it is not continuous in general.
1.3.2 Global causality conditions

For a spacetime \((M, g)\) to adequately depict the universe we live in, it must satisfy certain causality conditions. For instance, it is clear that \(M\) should not contain any closed causal path. Further, we would like \(M\) not to have “almost closed” causal curves.

**Definition 1.35** A set \(A \subset M\) is causally convex if every causal curve intersects it in a connected set. Strong causality is said to hold at \(p \in M\) if \(p\) has arbitrarily small causally convex neighborhoods. A spacetime is strongly causal if strong causality holds in all of its points.

Thus in strongly causal spacetimes, any future causal curve beginning near \(p\) must stay within a neighborhood of \(p\) or else leave it never to return. The following result characterizes strong causality failure (see lemma 4.16 in [48]).

**Proposition 1.36** Strong causality fails at \(p \in M\) if and only if there exists \(q \leq p\), with \(p \neq q\) such that \(x \ll y\) for all \(x \in I^-(p), y \in I^+(q)\).

Notice also that no future inextendible causal curve\(^†\) can be “imprisoned” in a compact subset of a strongly causal spacetime (refer to lemma 14.13 in [46]).

**Proposition 1.37** Suppose strong causality holds on a compact set \(K \subset M\). If \(\alpha\) is a future inextendible causal curve with \(\alpha(0) \in K\) then \(\alpha\) leaves \(K\) never to return.

Throughout this thesis a yet stronger form of causality is used. For the most part, we will be studying globally hyperbolic spacetimes.

\(^†\)A causal curve is said to be inextendible if it doesn’t have any continuous extension.
Definition 1.38 A spacetime \((M, g)\) is said to be globally hyperbolic if it strongly causal and the sets \(J^+(p) \cap J^-(q)\) are compact for all \(p, q \in M\).

Although global hyperbolicity seems to be a very stringent condition, as it turns out it is a reasonable assumption to be imposed on a spacetime modelling our universe. From the results presented in section 1.3.4 below, it follows that a non globally hyperbolic spacetime presents a breakdown in predictability: complete knowledge of a snapshot of spacetime will never be enough to fully determine the history of the universe.

From the mathematical point of view, globally hyperbolic spacetimes have very nice properties.

Proposition 1.39 Let \((M, g)\) be a globally hyperbolic spacetime. Then

1. \((M, g)\) is causally simple. In other words, \(J^+(p)\) and \(J^-(p)\) are closed for all \(p \in M\).

2. The sets \(J^+(A) \cap J^-(B)\) are compact for all compact \(A, B \subset M\).

3. The Lorentzian distance function \(\tau\) is continuous and finite on \((M, g)\).

4. Given \(p, q \in M\) with \(p \leq q\), there is a future causal geodesic \(\gamma\) from \(p\) to \(q\) that realizes Lorentzian distance, i.e. \(L(\gamma) = \tau(p, q)\).

1.3.3 Achronal sets

Definition 1.40 A set \(A \subset M\) is achronal if no two of its points are chronologically related. In other words, there is no timelike curve joining any two of its points. A is
said to be acausal if no two of its points are causally related.

We could be led to think that spacelike surfaces are achronal, but this is not the case. Notice though that spacelike surfaces are locally achronal. The following result will prove useful in later chapters.

**Proposition 1.41**

1. An achronal spacelike hypersurface is acausal.

2. If $M$ is simply connected then every closed spacelike hypersurface is achronal (hence acausal).

Achronal sets arise in many different settings, for instance, sets of the form $\partial I^+(A)$ are achronal. Notice also that achronal subsets usually have “edges”, but if they don’t then they are topological hypersurfaces (see figure 1.5).

**Definition 1.42** The edge $\text{edge}(A)$ of an achronal set $A$ consist on all points $p \in \overline{A}$ such that every neighborhood $U$ of $p$ contains a future timelike curve from $I^-(p,U)$ to $I^+(p,U)$ that does not meet $A$.

**Proposition 1.43** An achronal set $A \subset M$ is a topological hypersurface if and only if $A \cap \text{edge}(A) = \emptyset$. Further, $A$ is a closed topological hypersurface if and only if $\text{edge}(A) = \emptyset$.

The sets of the form $\partial I^+(A)$ are called achronal boundaries and play an important role in causal theory. We state now two of the most important results pertaining achronal boundaries (refer to results 3.15, 3.17 and 3.20 in [48]).
Let $A$ be the union of two closed disks in $\mathbb{M}^3$. Notice $\partial I^+(A)$ is a topological hypersurface but not a smooth manifold. Note also that all the null generators of $\partial I^+(A)$ have past endpoints in $A$. Finally observe that $\text{edge}(\partial I^+(A)) = \emptyset$. 

Figure 1.5:
**Proposition 1.44** Let $M$ be a spacetime and let $B \neq \emptyset$ be an achronal boundary. Then there exists a unique future set $F$ and a unique past set $P$ such that $F$, $P$ and $B$ are pairwise disjoint and $M = P \cup B \cup F$. Moreover, $B = \partial F = \partial P$.

**Proposition 1.45** Let $A \subset M$, then $\partial I^+(A)$ is a topological hypersurface. Further, $\partial I^+(A) - \overline{A}$ is ruled by null geodesics. More precisely, if $x \in \partial I^+(A) - \overline{A}$ then there exists a null geodesic $\eta \subset \partial I^+(A)$ with future endpoint $x$ that is either past inextendible or else has a past endpoint in $\overline{A}$.

### 1.3.4 Domains of dependence and Cauchy surfaces

Closely related to global hyperbolicity are the concepts of Cauchy surface and domain of dependence. We begin by defining the domain of dependence of an achronal set $A \subset M$.

**Definition 1.46** Let $A \subset M$ be an achronal set. We define the future domain of dependence $D^+(A)$ of $A$ as follows: $p \in D^+(A)$ if every past inextendible causal curve starting at $p$ intersects $A$. The past domain of dependence $D^-(A)$ is defined in a time dual fashion. Finally, we define the domain of dependence $D(A)$ by $D(A) := D^+(A) \cup D^-(A)$.

On physical grounds, the future domain of dependence $D^+(A)$ is the set of points of $M$ which are causally determined by $A$. In other words, any signal sent to $p \in D^+(A)$ must have passed through $A$ before reaching $p$, hence information stored on $A$ should suffice to predict what happens at $p$.

As it turns the topological interior of $D(A)$ is globally hyperbolic.
**Theorem 1.47** Let $A \subset M$ be a closed achronal set. Then $\text{int}(D(A))$ is globally hyperbolic, if non-empty. Moreover, if $A$ is acausal then $D(A)$ is open, hence globally hyperbolic.

Similarly, we can also show (see proposition 6.6.6 in [32]):

**Proposition 1.48** Let $S$ be a closed achronal subset of $M$. If $p \in \text{int}(D(S))$, then $J^+(p) \cap J^-(S)$ is compact.

Now we turn our attention to Cauchy surfaces.

**Definition 1.49** An achronal set $A \subset M$ is called a Cauchy surface if every inextendible causal curve intersects it.

As the name suggests, any Cauchy surface is necessarily a closed topological hypersurface.

It is clear that not all spacetimes admit a Cauchy surface. Actually, from the discussion above we have that $M$ admits a Cauchy surface $S$ if and only if $M = D(S)$. Thus if a spacetime $M$ admits a Cauchy surface, it is globally hyperbolic.

The converse is also true: every globally hyperbolic spacetime $M$ admits a Cauchy surface $S$. In this case, a result due to R. Geroch [28] establishes that $M$ is homeomorphic to $\mathbb{R} \times S$. Recently, M. Sanchez and A. Bernal have settled the long standing question as to if a globally hyperbolic spacetime $M$ is diffeomorphic to $\mathbb{R} \times S$ [5]. We summarize our discussion in the next theorem:

**Theorem 1.50** A spacetime $M$ is globally hyperbolic if and only if it admits a Cauchy surface $S$. Moreover, if $M$ is globally hyperbolic then $M$ is diffeomorphic
to a product $\mathbb{R} \times S$, where $\{t\} \times S$ is a spacelike Cauchy surface and $\nabla t$ is a past timelike vector field on $M$.

Notice that a function $f \in C^\infty(M)$ for which $\nabla f$ is past timelike is a time function, i.e. $f$ is strictly increasing along any future causal curve. Thus, the above result establishes the existence of a smooth time function on every globally hyperbolic spacetime.

1.4 Smooth null hypersurfaces

As discussed in section 1.2, a null hypersurface $S$ in $(M, g)$ is a codimension one embedded submanifold such that the pullback of the metric is degenerate. In this section we briefly describe some aspects on the geometry of null surfaces. For a detailed discussion consult [37].

First notice that at any point $p$, there is a distinguished future null direction in which $g$ degenerates. Thus any null hypersurface admits a unique, up to a positive scaling factor $f$, future directed null vector field $K$ such that $K_p^\perp = T_pS$ for all $p \in S$. It can be shown that the integral curves of $K$ are null geodesics, henceforth called the null generators of $S$.

Note also that by proposition 1.6 above all vectors in $T_pS$ not collinear to $K_p$ are spacelike. We define a positive definite metric on a quotient of $T_pS$ as follows: given $X, Y \in T_pS$ we say $X \sim Y$ if $X - Y = aK$ for some $a \in \mathbb{R}$. $\sim$ is readily seen to be an equivalence relation on $T_pS$, thus let us denote by $\overline{X}$ the equivalence class of $X$ and let us define $T_pS/K := T_pS/ \sim$ and $TS/K := \bigcup_p T_pS/K$. Observe $TS/K$ is
actually independent of the choice of the null vector field $K$. Finally, let $h$ be defined by $h(X, Y) = g(X, Y)$. A straightforward computation shows $h$ is a well defined symmetric and positive definite bilinear form.

On the other hand, we can check the map $b_K: T_pS/K \to T_pS/K, b_K(X) = \nabla_X K$ is well defined and depends only on the value of $K$ at $p$. Moreover, in similarity to the Riemannian setting, $b$ is self adjoint with respect to $h$. Thus we call $b$ the Weingarten map of $S$ relative to $K$; its associated bilinear form $B_K: T_pS/K \times T_pS/K \to \mathbb{R}$, $B_K(X, Y) = h(b_K(X), Y)$ is called the null second fundamental form. Since $b_K = fb_K$, then the condition $B_K \equiv 0$ does not depend on the choice of $K$, and thus it is an intrinsic property of $S$.

**Definition 1.51** We say that $S$ is totally geodesic if $B_K \equiv 0$ for some null future vector field $K$.

As expected, if $S$ is totally geodesic then any geodesic in $M$ starting tangent to $S$ remains in $S$.

**Definition 1.52** The trace of the Weingarten map $\theta_K = \text{Tr}(b_K)$ with respect to $h$ is called the null mean curvature.

Note $\theta_{fK} = f\theta_K$, hence null mean curvature inequalities like $\theta_K \geq 0$ actually do not depend on the choice of $K$.

Let $\Sigma$ be a codimension one submanifold of $S$ that is transverse to $K$ near $p \in S$, thus $\Sigma$ is a spacelike submanifold of dimension $n - 2$. Let then $\{e_1, \ldots, e_{n-2}\}$ be an orthonormal basis of $T_p\Sigma$. It follows that $\{\overline{e}_1, \ldots, \overline{e}_{n-2}\}$ is an orthonormal basis of
\( T_pS/K \) and

\[
\theta = \sum_{i=1}^{n-2} h(b(e_i, e_i)) = \sum_{i=1}^{n-2} g(\nabla_{e_i} K, e_i) = \text{div}_K K. \tag{1.4.1}
\]

Therefore, we can interpret \( \theta \) as the divergence towards the future of the null generators of \( S \).

Now let \( \eta \) be an affinely parameterized null generator of \( S \) and let \( b(s) := b_{\eta'}(s) \). Then the one parameter family of Weingarten maps \( s \mapsto b(s) \) satisfy a Ricatti type equation

\[
b' + b^2 + R = 0 \tag{1.4.2}
\]

where \( \overline{X}' := \overline{X}' \) denotes covariant differentiation along \( \eta \) and \( b', R \) are defined by

\[
b'(\overline{X}) := b(\overline{X}') - b(\overline{X}'), \quad R(\overline{X}) := R(\overline{X}, \eta')\eta'. \tag{1.4.3}
\]

Finally, by taking the trace of equation \( 1.4.2 \) we obtain the Raychaudhuri equation

\[
\theta' = -\text{Ric}(\eta', \eta') - \sigma^2 - \frac{1}{n-2} \theta^2 \tag{1.4.4}
\]

where \( \theta(s) = \theta_{\eta'(s)} \), and the shear

\[
\sigma := B - \frac{\theta}{(n-1)h} \tag{1.4.5}
\]

is the trace free part of \( B \).

On the other hand, the trace free part of equation \( 1.4.2 \) gives rise to a propagation equation, where a null tetrad is chosen so that \( e_0 \) agrees with \( \eta' \). (Compare to equation 4.36 in [32]).

\[
\sigma'_{\alpha \beta} = -W_{\alpha \beta 0} - \theta \sigma_{\alpha \beta} - \sigma_{\alpha \gamma} \sigma_{\gamma \beta} + \delta_{\alpha \beta} \sigma^2, \tag{1.4.6}
\]
1.5 The Einstein equations

A cornerstone in the theory of general relativity is the idea that gravity and matter shape the universe we live in. According to general relativity, our universe is modelled by a spacetime \((M, g)\). The effects of gravity are accounted by the Lorentz metric \(g\), while the matter content of spacetime is described by the energy-momentum tensor \(T\). The interaction between gravity and matter is given by the Einstein equations:

\[
\text{Ric} - \frac{1}{2} Rg + \Lambda g = T. \tag{1.5.1}
\]

where \(\Lambda\) is a constant called the cosmological constant.

Notice that the left hand side is purely geometrical, whereas the right hand side is physical in nature. Thus the Einstein equations establish a dictionary between the physics and geometry of spacetime.

To illustrate this correspondence, let us notice that the Einstein tensor

\[
G := \text{Ric} - \frac{1}{2} Rg \tag{1.5.2}
\]

is divergence free; hence by the Einstein equations we have the relation \(\text{div}T \equiv 0\), which in physical terms is interpreted as the Law of Conservation of Energy.

Conversely, note that some restrictions have to be imposed to \(T\) in order for it to describe physically reasonable matter. Among such restrictions we find the so-called energy conditions. For instance, the null energy condition asserts that the local contribution of energy must be non-negative, i.e. \(T(X, X)_p \geq 0\) for all null vectors \(X \in T_p M\). Thus, via the Einstein equations, the null energy condition translates to the curvature inequality \(\text{Ric}(X, X) \geq 0\) for all null \(X \in T_p M\).
In section 2.5 we will be dealing primarily with a particular form of matter model, called *perfect fluid*. The idea behind this model is that galaxies move like particles of a homogeneous fluid with no viscosity. In this case the energy-momentum tensor takes the form

\[ T = (\rho + p)U^* \otimes U^* + pg \]  

(1.5.3)

where \( U \) is a timelike vector field representing the fluid’s flow, and \( \rho, p \in C^\infty(M) \) represent the density and pressure of the fluid, respectively.

As pointed out in [50], a large class of models (called quasi-gases) that include perfect fluids have non positive trace. In the latter case we have \( \text{Tr}(T) = (n-1)p - \rho \), hence the pressure is bounded by the density: \( p \in [0, \rho/(n-1)] \). The case \( p = 0 \) is called *dust* and it represents incoherent matter. On the other hand, the case \( p = \rho/(n-1) \) models the effects of *pure radiation* and is particularly useful in the study of the early universe arising from a big-bang.

A universe devoid of matter is called *vacuum*. Such a spacetime satisfies the Einstein equations with \( T \equiv 0 \). Notice that in this case by contracting both sides of the Einstein equations we find \( R = 2n\Lambda/(n-2) \). Thus \( \text{Ric} = \lambda g \), with \( \lambda = 2\Lambda/(n-2) \). In other words, if \( (M, g) \) satisfies the Einstein vacuum equations then it is an Einstein manifold.
Chapter 2

Asymptotically de Sitter Spacetimes

A Lorentzian space form is a geodesically complete and connected Lorentzian manifold of constant sectional curvature $C$. A classical result in semi-Riemannian geometry states that two simply connected Lorentzian space forms of the same dimension having the same constant curvature $C$ must be isometric \cite{46, 55}.

De Sitter space $dS^n$ is the simply connected $n$-dimensional space form of constant curvature $C \equiv 1$. In this chapter we will analyze some global properties of spacetimes that look like de Sitter space “close to infinity”. To make this notion precise, we first introduce the concept of conformal infinity \cite{47}. 
2.1 Conformal boundaries

As a means to motivate the definition of conformal infinity, we start by looking at a Riemannian example: the hyperbolic half space $H_n$. Recall $H_n$ is just the manifold

$$H_n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n > 0\} \quad (2.1.1)$$
endowed with the metric

$$ds^2 = \frac{dx_1^2 + dx_2^2 + \ldots + dx_n^2}{x_n^2}, \quad (2.1.2)$$

hence $ds^2$ is conformal to the standard Euclidean metric

$$\tilde{ds}^2 = dx_1^2 + dx_2^2 + \ldots + dx_n^2. \quad (2.1.3)$$

Moreover, notice that $H_n$ has a topological boundary $J := \partial H_n$ and that it is defined in terms of the conformal factor $x_n$, namely

$$J = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n = 0\}. \quad (2.1.4)$$

As it can be checked, the inextendible geodesics in $H_n$ are either circles orthogonal to $J$ or straight vertical lines. Further, all these geodesics are complete. Hence $J$ is actually “at infinity” since it is made of the endpoints of complete rays.

Now we introduce the concept of conformal infinity as developed by Penrose [17]:

**Definition 2.1** We say that a spacetime $(\tilde{M}, \tilde{g})$ admits a conformal boundary $J$ if there exists a spacetime with non-empty boundary $(M, g)$ such that

1. $\tilde{M}$ is the interior of $M$ and $J = \partial M$, thus $M = \tilde{M} \cup J$.

2. There exists $\Omega \in C^\infty(M)$ such that
(a) $g = \Omega^2 \tilde{g}$ on $\tilde{M}$,

(b) $\Omega > 0$ on $\tilde{M}$,

(c) $\Omega = 0$ and $d\Omega \neq 0$ on $\mathcal{J}$.

In this setting $g$ is referred to as the unphysical metric, $\mathcal{J}$ is called the conformal boundary of $\tilde{M}$ in $M$ and $\Omega$ its defining function.

Assume that the future inextendible null geodesic $\gamma_1 : [0, a_1) \to \tilde{M}$ has a future endpoint $p \in \mathcal{J}$. Let $\gamma_2 : [0, a_2) \to M$ the future inextendible null geodesic in $(M, g)$ with the same initial conditions as $\gamma_1$. By the conformal invariance of null geodesics $\gamma_1$ and $\gamma_2$ coincide on $\tilde{M}$, hence there exists $b \in (0, a_2)$ such that $\gamma_2(b) = p$. Since the affine parameters $\lambda_i$ of $\gamma_i$ are related by

\[
\frac{d\lambda_1}{d\lambda_2} = \frac{C}{\Omega^2}
\]

for some constant $C$ [54], we have $\lambda_1 \to \infty$ as $\lambda_2 \to b$. Thus $a_1 = \infty$, so then $p$ is “at infinity” and hence the use of the term conformal infinity when referring to $\mathcal{J}$ is justified.

Now we turn back our attention to de Sitter space and show it admits a spacelike conformal boundary. First recall that $dS^n$ can be realized as the hyperboloid

\[- x_0^2 + x_1^2 + \ldots + x_n^2 = 1\]  

(2.1.6)

embedded in $\mathbb{M}^{n+1}$. By letting $\sinh t = x_0$ the metric of $dS^n$ takes the form

\[ds^2 = -dt^2 + \cosh^2 t d\omega^2\]  

(2.1.7)
where $d\omega^2$ is the standard metric in $S^{n-1}$. By further setting $\tan(u/2 + \pi/4) = e'$ we get

$$ds^2 = \frac{1}{\cos^2 u} (-du^2 + d\omega^2). \quad (2.1.8)$$

From here it is clear that $dS^n$ admits a *spacelike* conformal boundary

$$\mathcal{J} = \{ u = \pm \pi/2 \} \quad (2.1.9)$$

in the Einstein static universe $(E^n, g) = (\mathbb{R} \times S^{n-1}, -du^2 + d\omega^2)$.

Hence the following definition formalizes the intuitive idea of a spacetime “having a structure at infinity similar to de Sitter space”.

**Definition 2.2** A spacetime $(\bar{M}, \bar{g})$ with conformal boundary $\mathcal{J}$ is said to be asymptotically de Sitter if $\mathcal{J}$ is spacelike.

We present now the Schwarzchild de Sitter spacetime $SdS^n$ as our first nontrivial example of an asymptotically de Sitter spacetime.

Physically, this spacetime represents a black hole sitting in a de Sitter background.
Mathematically, it is a particular case of the so-called Kottler metrics [36]. The line element is given by

$$ds^2 = -\left(1 - \frac{2m}{r^{n-3}} - \frac{\Lambda}{3} r^2\right) dt^2 + \left(1 - \frac{2m}{r^{n-3}} - \frac{\Lambda}{3} r^2\right)^{-1} dr^2 + r^2 d\omega^2 \quad (2.1.10)$$

where $m > 0$ is a constant and $d\omega^2$ is the standard metric on $S^{n-1}$.

Another source of examples are the Robertson Walker spacetimes. They are used in physics to model an isotropic and spatially homogeneous universe, hence are of key importance in the study of cosmology.

**Definition 2.3** The Robertson Walker spacetime $RW(a, k)$ is the warped product $\mathbb{R} \times_a N_k$ where $a(t)$ is a smooth non-vanishing function on $\mathbb{R}$ and $N_k$ is a Riemannian space form of constant curvature $k$.

Assume $(\tilde{M}, \tilde{g}) = RW(a, 0)$ satisfies the Einstein equations with $\Lambda > 0$. By imposing conditions on the energy-momentum tensor (for example, when $T$ represents vacuum, dust or radiation) we get an asymptotic behavior of $a(t)$ that forces $\tilde{M}$ to
be asymptotically de Sitter. This can be done explicitly in particular examples by making the coordinate change

$$u = \int \frac{dt}{a(t)}.$$  \hfill (2.1.11)

To illustrate this process consider \((\tilde{M}, \tilde{g}) = RW(e^t, 0)\) and notice that under the coordinate change \(u = -e^{-t}\) the Robertson Walker metric \(ds^2 = -dt^2 + e^{2t}d\sigma^2\) is transformed to

$$ds^2 = \frac{1}{u^2}(-du^2 + d\sigma^2)$$ \hfill (2.1.12)

which is clearly asymptotically de Sitter.

Because of the spacelike character of \(J\) in an asymptotically de Sitter spacetime, every timelike vector \(v \in T_pM\) with \(p \in J\) is transversal to \(J\). Thus, since \(\nabla\Omega|_J\) is outward pointing and timelike, \(J\) can be decomposed as the union of the disjoint sets

$$J^+ = \{p \in J \mid \nabla\Omega_p \text{ is future pointing}\}$$ \hfill (2.1.13)

$$J^- = \{p \in J \mid \nabla\Omega_p \text{ is past pointing}\}$$

and as a consequence, \(J^+ \subset I^+(\tilde{M}, M), J^- \subset I^-(\tilde{M}, M)\). It follows as well that both sets \(J^+, J^-\) are acausal in \(M\).

Notice though that one of the sets \(J^\pm\) might be empty. We say that \((\tilde{M}, \tilde{g})\) is a past asymptotically de Sitter spacetime if \(J^- \neq \emptyset\). Future asymptotically de Sitter spaces are defined similarly. For instance \(RW(e^t, 0)\) is future but not past asymptotically de Sitter.

A very special feature of de Sitter space is that all inextendible null geodesics in \(dS^n\) have endpoints in \(J\) and hence are complete. This observation in turn motivates
the following definition:

**Definition 2.4** A spacetime $(\tilde{M}, \tilde{g})$ admitting a conformal boundary $\mathcal{J}$ is asymptotically simple if every inextendible null geodesic has endpoints on $\mathcal{J}$.

As a counterexample, we can easily see from the Penrose diagram that $SdS^n$ is not asymptotically simple, since there are future inextendible null geodesics entering the black hole region and hence never intersecting $\mathcal{J}^+$.

Asymptotic simplicity is a strong global condition. As it was pointed out before, asymptotically simple spacetimes are null geodesically complete. Moreover, it also follows that asymptotically simple and de Sitter spacetimes satisfy $\mathcal{J}^- \neq \emptyset \neq \mathcal{J}^+$.

It was shown in [1] that every asymptotically simple spacetime $(\tilde{M}, \tilde{g})$ is globally hyperbolic as well. We include the statement of this result for further reference.

**Theorem 2.5** Let $(\tilde{M}, \tilde{g})$ be an asymptotically simple and de Sitter spacetime. Then $(\tilde{M}, \tilde{g})$ is globally hyperbolic.

### 2.2 Rigidity of de Sitter space

Some of the most important results in semi-Riemannian geometry require a priori bounds on geometrical quantities. However, it may happen that one of these results involving a strict bound fails to be true only under very special circumstances when one relaxes the strict bound condition to a weak inequality. Such results are known as *rigidity theorems*. As an example of the rigidity philosophy we have the Cheeger-Gromoll Splitting Theorem in Riemannian geometry:
**Theorem 2.6** Let \((N, h)\) be a complete Riemannian manifold with \(\text{Ric}(v, v) \geq 0\) for all \(v \in TM\) and which contains a line (that is, a complete geodesic that minimizes distance between any of its points). Then \((M, g)\) can be written uniquely as a metric product \(N' \times \mathbb{R}^k\) where \(N'\) contains no lines and \(\mathbb{R}^k\) is given the standard flat metric.

Here, rigidity comes in the following way: it has been established [29] that a complete Riemannian manifold \(M\) satisfying \(\text{Ric}(v, v) > 0, \forall v \in TM\) is connected at infinity; but if one relaxes this condition to \(\text{Ric}(v, v) \geq 0\) and assumes that \(M\) is disconnected at infinity, then one is able to prove the existence of a line in \(M\) and hence the Splitting Theorem asserts that \(M\) is isometric to a product. In a nutshell, connectedness at infinity will fail when the curvature bound is relaxed only in the very special case of a product \(N' \times \mathbb{R}^k\) having all the properties stated at the conclusion of the Splitting Theorem.

In the Lorentzian setting, a number of rigidity results have been proven in recent times. For instance, the Lorentzian Splitting Theorem proven in the 1990’s [15, 20, 45] is the Lorentzian analog of the Cheeger-Gromoll theorem. More recently, G. Galloway proved the following Null Splitting Theorem [24].

**Definition 2.7** A null line is an achronal inextendible null geodesic; or equivalently, an inextendible null geodesic that maximizes Lorentzian distance between any of its endpoints.

**Theorem 2.8** Let \((M, g)\) be a null geodesically complete spacetime which obeys the null energy condition. If \(M\) admits a null line \(\eta\), then \(\eta\) is contained in a smooth properly embedded, achronal and totally geodesic null hypersurface \(S\).
Remark 2.9 Let $\partial_0 I^\pm(\eta)$ be the connected components of $\partial I^\pm(\eta)$ containing $\eta$. The proof of the Null Splitting Theorem shows $\partial_0 I^+(\eta) = S = \partial_0 I^-(\eta)$. Moreover, the proof also shows that future null completeness of $\partial_0 I^-(\eta)$ and past null completeness of $\partial_0 I^+(\eta)$ are sufficient for the result to hold.

This result is a consequence of the Maximum Principle for $C^0$ Null Hypersurfaces [24]. We will be needing later on a weaker version of this result, namely the Maximum Principle for Smooth Null Hypersurfaces.

Theorem 2.10 Let $S_1$ and $S_2$ be smooth null hypersurfaces in a spacetime $(M, g)$.

Suppose that $S_1$ and $S_2$ meet in $p$ and

1. $S_2$ lies to the future of $S_1$ near $p$.

2. The null mean curvature scalars $\theta_i$ satisfy $\theta_2 \leq 0 \leq \theta_1$.

Then $S_1$ and $S_2$ coincide near $p$ and this common hypersurface has null mean curvature $\theta = 0$.

As an application of the Null Splitting Theorem, the following rigidity result for asymptotically de Sitter spacetimes has been established [25].

Theorem 2.11 Let $(\tilde{M}, \tilde{g})$ be an asymptotically simple and de Sitter spacetime of dimension $n = 4$ that satisfies the vacuum Einstein equations with positive cosmological constant. If $\tilde{M}$ contains a null line, then $\tilde{M}$ is isometric to de Sitter space $dS^4$.

This theorem can be interpreted in terms of the initial value problem in the following way: H. Friedrich has shown that the set of asymptotically simple solutions
to the Einstein equations with positive cosmological constant is open in the set of all maximal globally hyperbolic solutions with compact spatial sections [17]. As a consequence, by slightly perturbing the initial data on a fixed Cauchy surface of $dS^4$ we should get an asymptotically simple solution of the Einstein equations different from $dS^4$. Thus in virtue of theorem 2.11 such a spacetime has no null lines. In other words, a small perturbation of the initial data destroys all null lines.

Alternatively, we could say that no other asymptotically simple solution of the Einstein equations besides $dS^4$ develops eternal observer horizons. By definition, an observer horizon $\mathcal{A}$ is the past achronal boundary $\partial I^- (\gamma)$ of a future inextendible timelike curve, thus $\mathcal{A}$ is ruled by future inextendible null geodesics. In the case of de Sitter space, observer horizons are eternal, that is, all null generators of $\mathcal{A}$ extend from $\mathcal{J}^+$ all the way back to $\mathcal{J}^-$ and hence are null lines.

Since the observer horizon is the boundary of the region of spacetime that can be observed by $\gamma$, the question arises as to whether at one point $\gamma$ would be able to observe the full space. More precisely, we want to know if there exists $q \in \mathcal{M}$ such that $I^- (q)$ would contain a Cauchy surface of spacetime. S. Gao and R. Wald were able to answer this question affirmatively for globally hyperbolic spacetimes with compact Cauchy surfaces, assuming null geodesic completeness, the null energy condition and the null generic condition‡[27]. Refer also to [6] for a discussion on the relationship between the existence of eternal observer horizons and entropy bounds on asymptotically de Sitter spacetimes.

‡The null generic condition is the statement that each null geodesic contains a point at which $k_{[\alpha R,\beta] \gamma} \delta_{[\eta k_\xi]} k_\gamma k_\delta \neq 0$ where $k_\alpha$ denotes the tangent to the geodesic.)
Though no set of the form \( I^{-}(q) \subset dS^n \) contains a Cauchy surface, \( I^{-}(q) \) gets arbitrarily close to do so as \( q \to J^{+} \). However, notice that de Sitter space is not a counterexample to [27], since \( dS^n \) does not satisfy the null generic condition. Actually, the latter remark enables us to interpret theorem 2.11 as a rigidity result in the asymptotically simple context: by dropping the null generic hypothesis in [27] the conclusion will only fail if \((\tilde{M}, \tilde{g})\) is isometric to \( dS^4 \).

Notice also that the stability of the null generic condition under small perturbations of the initial data has not yet been established. In this sense, the null generic condition is not “generic”.

Since asymptotically simple spacetimes are globally hyperbolic [1], the above discussion prompts us to study the structure of globally hyperbolic and asymptotic de Sitter spacetimes to later investigate the consequences of relaxing the hypotheses of theorem 2.11.

### 2.3 Globally hyperbolic and asymptotically de Sitter spacetimes

Notice that surjectivity doesn’t hold anymore when the hypothesis of asymptotic simplicity in theorem 2.11 is weakened to global hyperbolicity. Actually, there are globally hyperbolic and asymptotically de Sitter proper subsets of \( dS^n \) admitting null lines with endpoints in \( J \). (See fig. 2.3 below).

The main goal of this section is to prove that surjectivity is the only conclusion of theorem 2.11 that does not carry over to the more general globally hyperbolic
setting. In turn, by relaxing the hypothesis of asymptotical simplicity in theorem 2.11 we broaden its scope of application, since we will not be ruling out a priori the appearance of black holes.

To make proofs easier, it is convenient to embed the spacetime with boundary $(M, g)$ in an open spacetime. This can always be done in light of theorem A.1. In the case of globally hyperbolic and asymptotically de Sitter spacetimes, this latter result can be improved, as the next proposition shows.

**Proposition 2.12** Let $(\tilde{M}, \tilde{g})$ be a globally hyperbolic and asymptotically de Sitter spacetime, then $(\tilde{M}, \tilde{g})$ can be embedded in a globally hyperbolic spacetime $(N, h)$ such that $J$ topologically separates $\tilde{M}$ and $N - \tilde{M}$. 

Figure 2.3:

\[ D = I^+ (B) \cap I^- (A) \] is an asymptotically de Sitter and globally hyperbolic open subset of $dS^n$ with Cauchy surface $S$. 
Proof: It suffices to show $(\tilde{M}, \tilde{g})$ can be extended past $J^-$, since a similar procedure can be used to extend $(\tilde{M}, \tilde{g})$ beyond $J^+$. Thus without loss of generality we can assume $J = J^-$.

By proposition [A.1] there is an open spacetime $(\overline{M}, h)$ extending $(M, g)$. Since $\overline{M}$ is obtained from $M$ by attaching collars, the separation part of the proposition holds.

As a consequence, $J$ is acausal in $\tilde{M}$, then the Cauchy development $D(J, \overline{M})$ is an open subset of $\overline{M}$. Hence $N = M \cup D(J, \overline{M})$ is an open spacetime containing $M$.

We proceed to show $(N, h)$ is globally hyperbolic. To this end, let us note that $D := D(J, N)$ is globally hyperbolic by construction.

First we show $N$ is strongly causal. Thus let $p \in N$. If $p \in \tilde{M}$, we claim that any causally convex neighborhood $V$ of $p$ with respect to $\tilde{M}$ is causally convex with respect to $N$ as well. In order to prove this claim, let us consider a future directed causal curve $\gamma$ with endpoints in $V$. If $\gamma \subset \tilde{M}$ then $\gamma \subset V$ by the causal convexity of $\gamma$. On the other hand, if $\gamma$ leaves $\tilde{M}$ then it has to intersect $J$ at least twice to be able to come back to $V$, thus violating the acausality of $J$. A similar argument can be used in the case $p \in N - M$ to show that any causally convex neighborhood of $p$ with respect to $D^-(J, N)$ is causally convex with respect to $N$.

Thus $p \in J$ is the only case that remains to be checked. We proceed by contradiction. Assume strong causality fails at $p$, then by proposition [1.36] there exists $q \in J^-(p, N)$, $q \neq p$ with the property that $y \in I^+(x, N)$ for all $x \in I^-(p, N)$, $y \in I^+(q, N)$. Thus, $q \in D^-(J, N) - J$ by the acausality and separating properties of $J$. Now, since strong causality holds on $D$, it follows that we can choose $q$ in such a way that $p \not\in J^+(q, D)$. Now let $\{p_n\}$ be a sequence in $I^-(p, N)$ converging to $p$
and \(\{q_n\} \subset I^+(q,N)\) with \(q_n \to q\). Notice \(q_n \in D^-(\mathcal{J},N) - \mathcal{J}\) for large \(n\) by the separating properties of \(\mathcal{J}\). Since \(p_n \in D^-(\mathcal{J},N) - \mathcal{J}\) for all \(n \in \mathbb{N}\), then we can assume without loss of generality that both sequences are in \(D^-(\mathcal{J},N) - \mathcal{J}\). Now let \(\gamma_n\) be an inextendible future timelike curve emanating from \(p_n\) and passing through \(q_n\). By definition, \(\gamma_n\) must intersect \(\mathcal{J}\) at a point \(z_n\). Moreover, notice that \(z_n\) has to lie at the future of \(q_n\) along \(\gamma_n\), since acausality of \(\mathcal{J}\) would be violated otherwise.

Further, note \(J^+(q,D) \cap \mathcal{J}\) is compact in virtue of proposition 1.48 being a closed subset of the compact set \(J^+(q,D) \cap J^-(\mathcal{J},D)\). Since \(\{z_n\} \subset J^+(q,D) \cap \mathcal{J}\) then there is a subsequence \(\{z_m\}\) converging to \(z \in J^+(q,D) \cap \mathcal{J}\). Finally notice that \(\gamma_m|_{p_m} \subset D^-(\mathcal{J},N) \subset D\). Then since \(D\) is globally hyperbolic a strong version of the Limit Curve Lemma applies (see corollary 3.32 in [2]) to guarantee the existence of a future causal curve from \(p\) to \(z\), a clear contradiction to the acausality of \(\mathcal{J}\). Thus strong causality holds in all of \(N\).

Let \(p, q \in N\), then we want to show \(A := J^+(p,N) \cap J^-(q,N)\) is compact. If \(p \in \tilde{M}\) then by observations above \(J^+(p,N) = J^+(p) \subset \tilde{M}\)\(^1\) hence as an easy consequence we have \(A = J^+(p) \cap J^-(q)\), thus \(A\) is compact due to the global hyperbolicity of \(\tilde{M}\). Likewise, if \(q \in D\) then \(J^-(q,N) \subset D\) and hence \(A = J^+(p,D) \cap J^-(q,D)\). So compactness of \(A\) follows as well since \(D\) is globally hyperbolic.

Therefore, the only case left to consider is when \(p \in N - \tilde{M}\) and \(q \in N - D\). Take now a Cauchy surface \(S\) of \(D\) slightly to the future of \(\mathcal{J}\) so that \(S \subset \tilde{M}\). Let

\(^1\)In the remaining sections of this chapter all causal relations are taken with respect to the physical spacetime \((\tilde{M}, \tilde{g})\) unless otherwise specified. For instance, \(J^+(A) = J^+(A, \tilde{M})\), \(\forall A \subset \tilde{M}\).
\( S' := J^+(p, N) \cap J^-(q, N) \cap S \) and let us prove that

\[
A = [J^+(p, D) \cap J^-(S', D)] \cup [J^-(q) \cap J^+(S')]\)  \tag{2.3.1}
\]

First notice that one inclusion is trivial. On the other hand, let \( x \in A \) and consider a future causal curve \( \gamma \) from \( p \) to \( q \) passing through \( x \). Since \( p \in N - \tilde{M} \), \( q \in N - D \) and \( S \) is a Cauchy surface of \( D \), \( \gamma \) must intersect \( S \) in a point \( y \in S' \). If \( y \) comes before \( x \) along \( \gamma \) then \( \gamma|_y \subset \tilde{M} \), thus \( x \in J^-(q) \cap J^+(S') \). Alternatively, if \( y \) comes after \( x \) along \( \gamma \) then \( x \in D \), since \( J^-(S, N) = J^-(S, D) \). But then \( J^+(p, N) \cap D = J^+(p, D) \) implies \( x \in J^+(p, D) \cap J^-(S', D) \), thus proving equation (2.3.1).

To finish the proof, notice that \( J^+(p, N) \cap S = J^+(p, D) \cap S \), hence \( J^+(p, N) \cap S \) is compact. Also note that \( J^-(q, N) \cap S = J^-(q) \cap S \), so then \( J^-(q, N) \cap S \) is a closed set as well. Hence \( S' \) is compact. Thus by propositions 1.39 and 1.41 we have that the sets \( J^+(p, D) \cap J^-(S', D) \) and \( J^-(q) \cap J^+(S') \) are compact, and the result follows.

\[\square\]

We analyze now the structure of a globally hyperbolic and asymptotically de Sitter spacetime near the past endpoint of a causal curve.

**Lemma 2.13** Let \((\tilde{M}, \tilde{g})\) be a globally hyperbolic and asymptotically de Sitter spacetime and \( \eta \) a future directed causal curve in \( M \). Further assume \( p \in J^- \) is the past endpoint of \( \eta \). Then

1. \( \partial I^+(\eta) = J^+(p, N) - (I^+(p, N) \cup \{p\}) \),

2. \( J^+(N_p, N) \cap \tilde{M} \subset D^+(N_p, N) \cap \tilde{M} \),

where \( N_p := \partial_N I^+(p, N) \) and \( N \) is a globally hyperbolic extension of \((M, g)\).
Proof: First, let us consider a globally hyperbolic extension \((N,h)\) of \((M,g)\) as described in proposition \ref{2.12}. Then by global hyperbolicity the set \(J^+(p,N)\) is closed in \(N\), and as a consequence

\[
\partial_N I^+(p,N) = J^+(p,N) - I^+(p,N). \tag{2.3.2}
\]

Thus by the acausality of \(\mathcal{J}^-\) we have

\[
\tilde{M} \cap \partial_N I^+(p,N) = \partial_N I^+(p,N) - \{p\}. \tag{2.3.3}
\]

On the other hand, let us show \(I^+(\eta) = I^+(p,N)\). It is clear that \(I^+(\eta) \subset I^+(p,N)\). Conversely, let \(x \in I^+(p,N)\). Since \(I^-(x,N)\) is open and \(p \in I^-(x,N)\), there is \(y \in \eta \cap I^-(x,N)\). But any future timelike curve from \(y\) to \(x\) has to be contained in \(\tilde{M}\) due to the separating properties of \(\mathcal{J}^-\), hence \(x \in I^+(\eta)\) and the inclusion \(I^+(p,N) \subset I^+(\eta)\) is proven. As a consequence \(\partial I^+(\eta) = \partial_{\tilde{M}} I^+(p,N)\).

Finally, since \(I^+(p,N)\) is an open set in \(N\), by a standard topological result \cite{44} we get

\[
\partial_{\tilde{M}} I^+(p,N) = \tilde{M} \cap \partial_N I^+(p,N). \tag{2.3.4}
\]

Then the first assertion follows by combining equations (2.3.2), (2.3.3) and (2.3.4).

To prove the second part of the lemma we proceed by contradiction. Thus let us assume \(x \in J^+(N_p,N) \cap \tilde{M} - D^+(N_p,N) \cap \tilde{M}\), hence \(x \in J^+(N_p,N) - N_p\). It follows \(x \in I^+(p,N)\). On the other hand, since \(x \not\in D^+(N_p,N) \cap \tilde{M}\) there is a past inextendible causal curve \(\gamma: [0,a) \to N\) starting at \(x\) that does not intersect \(N_p\). Notice \(\gamma\) never leaves \(I^+(p,N)\), since otherwise it had to intersect \(N_p = \partial_N I^+(p,N)\). Thus \(\gamma\) is contained in the compact set \(J^+(p,N) \cap J^-(x,N)\), which by proposition \ref{1.37} contradicts strong causality. \(\square\)
In a time dual manner, we get $\partial I^-(\eta) = J^-(q, N) - (I^-(q, N) \cup \{q\})$ where $q \in J^+$ is a future endpoint of $\eta$ in a future asymptotically de Sitter spacetime.

Now notice that in $dS^n$ the set $N_p$ is just the local causal cone at $p$. This generalizes to globally hyperbolic and asymptotically de Sitter spacetimes satisfying the null energy condition.

**Lemma 2.14** Let $(\tilde{M}, \tilde{g})$ be a globally hyperbolic and asymptotically de Sitter spacetime and let $\eta$ be a future directed null line in $\tilde{M}$ having endpoints $p \in J^-$ and $q \in J^+$. Further assume $(\tilde{M}, \tilde{g})$ satisfies the null energy condition. Then $\partial I^+(\eta)$ is the diffeomorphic image under the exponential map $\exp_p$ of the set $\Lambda^+_p \cap O$ where $\Lambda^+_p \subset T_p M$ is the future null cone based at $0_p$ and $O$ is the biggest open set on which $\exp_p$ is defined.

**Proof:** Let $(N, h)$ be as in the previous lemma. Hence by proposition 1.28 and lemma 2.13, any point in $\tilde{M} \cap \partial_N I^+(p, N)$ is the future endpoint of a future null geodesic segment emanating from $p$. Thus

$$\partial I^+(\eta) \subset \exp_p(\Lambda^+_p \cap O) \cap \tilde{M}. \quad (2.3.5)$$

Now let $\gamma$ be a null generator of $\partial I^+(\eta)$ passing through $x \in \partial I^+(\eta)$. Let $y \in \gamma$ a point slightly to the past of $x$ and notice $y \in \partial_N I^+(p, N)$ by equation (2.3.3). On the other hand, let $\tau(t)$ be a null geodesic emanating from $p$ and passing through $y$. Then $\gamma$ coincides with $\tau \subset \tilde{M}$ since otherwise we would have two null geodesics meeting at an angle in $y$ and hence $x \in I^+(p, N)$ by proposition 1.30, a clear contradiction. Thus, $\gamma$ can be extended to $p \in J^-$ and thus it is past complete. In a time dual fashion, the generators of $\partial I^-(\eta)$ are future complete.
Let $S$ be the component of $\partial I^+(\eta)$ containing $\eta$. By the proof of the Null Splitting Theorem, $S$ is a closed smooth totally geodesic null hypersurface in $\tilde{M}$. As a consequence, the null generators of $S$ do not have future endpoints in $\tilde{M}$ and hence are future inextendible in $S$. Furthermore, by the argument in the previous paragraph, each of these generators is the image under $\exp_p$ of the set $V \cap O$, where $V$ is an inextendible null ray in $\Lambda_p^+$.

Let $\gamma$ be a generator of $S$, then $\gamma \cap I^+(p, N) = \emptyset$. Thus $\gamma$ is conjugate point free and does not intersect with any other generator of $S$, since otherwise $\gamma$ would enter $I^+(p, N)$ (see propositions [1.30] and [1.31]). As a result we have that $S$ is the diffeomorphic image under $\exp_p$ of an open subset of $\Lambda_p^+$.

To check that $S$ encompasses the whole local null cone at $p$ consider a causally convex normal neighborhood $\mathcal{V}$ of $p$ and a spacelike hypersurface $\Sigma$ slightly to the future of $J^-$. Thus $\Sigma_0 := \Sigma \cap \exp_p(\Lambda_p^+ \cap \mathcal{V})$ is connected. Moreover, by the way $\mathcal{V}$ and $\Sigma$ were chosen we have $\Sigma_0 \subset J^+(p) - (I^+(p) \cup \{p\}) = \partial I^+$. Thus

$$\exp_p(\Lambda_p^+ \cap \mathcal{O}) \cap \tilde{M} \subset S \quad (2.3.6)$$

since every future null geodesic emanating from $p$, including $\eta$, must intersect $\Sigma_0$. It follows $S = \partial I^+ (\eta)$ and the proof is complete. $\square$

**Corollary 2.15** Let $(\tilde{M}, \tilde{g})$ be a globally hyperbolic and asymptotically de Sitter spacetime satisfying the null energy condition. Let $\eta$ be a future directed null line in $\tilde{M}$ having endpoints $p \in J^-$ and $q \in J^+$ and let $N_p = \partial N I^+(p, N),$ $N_q = \partial N I^-(q, N)$. Then $N_p - \{p\}$ and $N_q - \{q\}$ agree and form a smooth totally geodesic null hypersurface $S$. Furthermore, $S \cup \{p, q\}$ is homeomorphic to $S^{n-1}$. 
Proof: Let $\partial_0 I^+(\eta), \partial_0 I^-(\eta)$ be the components of $\partial I^+(\eta), \partial I^-(\eta)$ containing $\eta$ respectively. By the proof of the previous lemma and the Null Splitting Theorem, we have $\partial_0 I^+(\eta) = \partial_0 I^-(\eta)$, and this common null hypersurface is closed, smooth and totally geodesic. Moreover, by the previous lemma we also conclude $S := \partial I^+(\eta)$ is connected, thus $S = \partial I^+(\eta) = \partial_0 I^+(\eta)$. Lastly, by lemma 2.13 we have $\partial I^+(\eta) = N_p - \{p\}$ and $\partial I^-(\eta) = N_q - \{q\}$. Thus $N_p - \{p\} = S = N_q - \{q\}$ as desired.

On the other hand, notice that $N_p - \{p\} = N_q - \{q\}$ in conjunction with lemma 2.14 imply that every point in $S$ is at the same time the future endpoint of a null geodesic emanating from $p$ and the past endpoint of a null geodesic from $q$. These geodesic segments must form a single geodesic, otherwise achronality of $\eta$ would be violated. Hence, all future null geodesics emanating from $p$ meet again at $q$. Then $S \cup \{p, q\}$ is homeomorphic to a sphere. □

**Corollary 2.16** Let $(\tilde{M}, \tilde{g})$ be a globally hyperbolic and asymptotically de Sitter spacetime satisfying the null energy condition. If $\tilde{M}$ has a future directed null line $\eta$ with endpoints $p \in J^- \cap J^+$, then $\tilde{M}$ has Cauchy surfaces homeomorphic to $S^{n-1}$. In particular, $\tilde{M}$ is simply connected.

Proof: Let $N_p$ be a normal neighborhood of $p$, let $w \in T_p M$ be a unit future timelike vector and consider $r > 0$ small enough so that $S_r = \exp_p(C_r) \subset N_p$, where $C_r$ is the slice given by $C_r := \{v \in C^+_p \mid g(v, w) = -r\}$. Further let $\Lambda_r = \{v \in \Lambda_p^+ \cup \{0\} \mid -r \leq g(v, w) \leq 0\}$, then since $N_p$ is the future local null cone at $p$, by projecting $\exp_p(\Lambda_r)$ onto $S_r$ we obtain a homeomorphism between $N_p$ and an achronal set $N'_p \subset \tilde{M}$. Let then $N'_q$ be an achronal set homeomorphic to $N_q$ obtained in a similar fashion.
Thus $S' := N'_p \cup N'_q$ is homeomorphic to $N_p \cup N_q = S \cup \{p, q\}$. It follows from corollary 2.15 that $S' \approx S^{n-1}$.

Finally, notice $S'$ is an embedded compact hypersurface in $\tilde{M}$. Since it is achronal in $\tilde{M}$, it has to be a Cauchy surface for $\tilde{M}$ \footnote{15}. Thus $\tilde{M}$ is homeomorphic to $S \times \mathbb{R}$, hence simply connected. $\square$

We now proceed to prove the main result of this section.

**Theorem 2.17** Let $(\tilde{M}, \tilde{g})$ be a globally hyperbolic and asymptotically de Sitter spacetime of dimension $n = 4$ satisfying the vacuum Einstein equations with positive cosmological constant. If $\tilde{M}$ has a null line with endpoints $p \in J^-$, $q \in J^+$ then $(\tilde{M}, \tilde{g})$ is isometric to an open subset of de Sitter space containing a Cauchy surface.

**Proof:** Let $(N, h)$ be a globally hyperbolic extension of $(M, g)$ and let $S = \partial I^+(\eta)$. Further let $N_p = S \cup \{p\}$. Then by lemma 2.13 we have $I^+(S) \subset D^+(N_p, N) \cap \tilde{M}$. In a time dual fashion $I^-(S') \subset D^-(N_q, N) \cap \tilde{M}$ where $S' = \partial I^-(\eta)$ and $N_q = S' \cup \{q\}$.

Since $(\tilde{M}, \tilde{g})$ satisfies the null energy condition, by corollary 2.15 we have $S = S'$, and as a consequence of proposition 1.44 we get $\tilde{M} = I^+(S) \cup S \cup I^-(S)$. Hence

$$\tilde{M} \subset D^+(N_p, N) \cup D^-(N_q, N) \quad \text{(2.3.7)}$$

We show now $(\tilde{M}, \tilde{g})$ has constant sectional curvature. First notice that corollary 2.15 establishes that $S$ is a totally geodesic null hypersurface. As a consequence the shear tensor $\tilde{\sigma}_{\alpha \beta}$ of $S$ in the physical metric $\tilde{g}$ vanishes. Since the shear scalar $\tilde{\sigma} = \tilde{\sigma}_{\alpha \beta} \tilde{\sigma}^{\alpha \beta}$ is a conformal invariant we have $\sigma_{\alpha \beta} \equiv 0$ as well. Recall the shear satisfies the propagation equation \footnote{14.6} 

$$\sigma'_{\alpha \beta} = -W_{\alpha \beta 0} - \theta \sigma_{\alpha \beta} - \sigma_{\gamma \gamma} \sigma_{\gamma \beta} + \delta_{\alpha \beta} \sigma^2, \quad \text{(2.3.8)}$$
Then it follows $W_{\alpha_0\beta_0} = 0$ on $S$.

In [16] H. Friedrich used the conformal field equations

$$\nabla_\alpha d^{\alpha}_{\beta\gamma\zeta} = 0, \quad d^{\alpha}_{\beta\gamma\zeta} = \Omega^{-1}W^{\alpha}_{\beta\gamma\zeta}$$

(2.3.9)

along with a recursive ODE argument to guarantee the vanishing of the rescaled conformal tensor $d$ on $D^+(S \cup \{p\}, N)$ given that $W_{0000}$ vanishes on $S$.

Hence, we have shown $d \equiv 0$ on $D^+(N_p, N)$. Since $\widetilde{W} = W$ on $\tilde{M}$ we have $\tilde{W} \equiv 0$ on $D^+(N_p, N) \cap \tilde{M}$. By a time dual argument we conclude $\tilde{W} \equiv 0$ on $D^-(N_q, N) \cap \tilde{M}$, thus $\tilde{W} \equiv 0$ on $\tilde{M}$ by (2.3.7) above. Since $(\tilde{M}, \tilde{g})$ is an Einstein manifold, the vanishing of the Weyl tensor implies that $(\tilde{M}, \tilde{g})$ has constant curvature $C > 0$ (see remark 1.17). Thus, without loss of generality we can assume $C = 1$.

By corollary 2.16 $(\tilde{M}, \tilde{g})$ is simply connected, hence there exists a local isometry $\Phi: \tilde{M} \to dS^4$ by the Cartan-Ambrose-Hicks Theorem [9, 55]. Notice that in particular, $\Phi$ is an open map.

Let us denote by $S$ a fixed Cauchy surface of $\tilde{M}$. In virtue of [5] we can assume $S$ is spacelike and $\tilde{M} = S \times \mathbb{R}$. We proceed to show that $\Phi_S := \Phi|_S$ is a topological embedding.

To fix some notation, let us consider the conformal embedding of $dS^4$ into the Einstein static universe, as described in section 2.1. Let $\mathcal{S}$ be the Cauchy surface of $dS^4$ given by $\mathcal{S} = u^{-1}(0)$, hence $dS^4 \approx \mathcal{S} \times \mathbb{R}$. Let $\pi: dS^4 \to \mathcal{S} \times \mathbb{R} = \mathcal{S}$ be the projection on the first factor. Note that the fibers of $\pi$ are all timelike curves. Further, let $\hat{S} := \Phi(S)$.

We first show $\pi|_{\hat{S}}$ is a local homeomorphism. Since manifolds are locally compact Hausdorff spaces, it suffices to show $\pi$ is locally one to one.
Thus let \( y \in \hat{S} \). Take then \( x \in S \) with \( \Phi(x) = y \) and consider a neighborhood \( \mathcal{V} \) of \( x \) such that \( \Phi: \mathcal{V} \to U_0 \) is an isometry. Further, since \( dS^4 \) is globally hyperbolic there is a causally convex neighborhood \( \mathcal{U} \) of \( y \) contained in \( U_0 \). Let then \( a, b \in \mathcal{U} \) such that \( \pi(a) = z = \pi(b) \). If \( a \neq b \) let us denote by \( \gamma \) the portion of \( \pi^{-1}(z) \) from \( a \) to \( b \), then \( \gamma \) is a timelike curve connecting \( a \) and \( b \). Thus by causal convexity, \( \gamma \) must be contained in \( \mathcal{U} \subset U_0 \). Hence \( \Phi^{-1}(\gamma) \cap \mathcal{V} \) is a timelike curve joining two points of \( S \). But \( S \) is achronal, being a Cauchy surface for \( \tilde{M} \). Thus \( a = b \) so \( \pi|_{\hat{S} \cap \mathcal{U}} \) is injective.

Hence \( F: S \to \mathcal{G} \) defined by \( F = \pi \circ \Phi|_S \) is a local homeomorphism, therefore \( F \) is an open map, i.e. \( F(S) \) is an open subset of \( \mathcal{G} \). On the other hand, since \( S \) is compact then \( F(S) \) is compact, hence closed in \( \mathcal{G} \). It then follows from connectedness of \( \mathcal{G} \) that \( F \) is surjective.

Thus by a standard topological result (refer for instance to proposition 2.19 in [40] and notice that the proof works as well in the continuous setting) we have that \( F \) is a topological covering map. Moreover, by corollary 2.15 we have \( \mathcal{G} \approx S^3 \). Thus \( \mathcal{G} \) is simply connected and as a consequence, \( F \) is a homeomorphism. Hence \( \Phi|_S \) is injective as well, therefore a topological embedding since \( S \) is compact.

Then \( \hat{S} \) is a closed embedded spacelike hypersurface. Thus since \( dS^4 \) is simply connected, we have that \( \hat{S} \) is achronal by proposition 1.41.

Let \( S_a, a \in \mathbb{R} \), be the foliation of \( \tilde{M} \) induced by \( S \). Since \( \hat{S}_a := \Phi(S_a) \) is achronal for all \( a \in \mathbb{R} \) it follows that no two of these surfaces can intersect. Thus \( \Phi \) is injective.

The result now follows since every injective local isometry is an isometry into an open subset of the codomain. \( \square \)
Remark 2.18 G. Mess points out in [43] the existence of simply connected and locally de Sitter spacetimes of constant curvature $\equiv 1$ that can not be embedded in 3-dimensional de Sitter space. In [3], I. Bengtsson and S. Holst were able to construct a similar example in dimension four. Moreover, this latter spacetime occurs as a Cauchy development (see section 2.4 below) of a Cauchy surface $S$ with topology $\mathbb{H}^2 \times \mathbb{R}$, which is clearly non compact. On the other hand, theorem 2.17 above shows that no such example can be found in our setting.

Theorem 2.17 can be interpreted in terms of the Cosmic Censor Conjecture. This notion is due to R. Penrose [49] who conjectured the existence of a “cosmic censor” who would forbid the appearance of naked singularities. In its strong form, the cosmic censor conjecture just says that all reasonable spacetimes must be globally hyperbolic.

The weak form of the Cosmic Censor Conjecture states that apart from a possible initial singularity (a “big bang” singularity), the region of spacetime away from black or white holes must be globally hyperbolic. Thus, if a singularity is present it should be contained either inside a black hole or inside a white whole. In that sense “naked” singularities are not allowed, since all of them are hidden by an event horizon.

Equivalently, we have the following mathematical formulation:

**The Weak Cosmic Censor Conjecture** Let $(\bar{M}, \bar{g})$ be a spacetime admitting a conformal boundary $\mathcal{J}$. Then the Domain of Outer Communication $\mathcal{D}(\bar{M}) := I^+(\mathcal{J}^-, M) \cap I^-(\mathcal{J}^+, M)$ is globally hyperbolic.

Note that both de Sitter space and Schwarzchild-de Sitter satisfy the weak cosmic censor conjecture. Thus we have this immediate corollary to theorem 2.17.
Corollary 2.19 Let $(\tilde{M}, \tilde{g})$ be an asymptotically de Sitter spacetime satisfying the Einstein equations. Further assume $(\tilde{M}, \tilde{g})$ admits a null line with endpoints $p \in J^-, q \in J^+$. If the weak cosmic censor conjecture holds in $(\tilde{M}, \tilde{g})$ then the domain of outer communication $D(\tilde{M})$ is isometric to an open subset of de Sitter space that contains a Cauchy surface.

2.4 The initial value problem

Notice that when put in coordinate form, the Einstein equation $G = T$ gives rise to a second order PDE system $G_{\alpha\beta} = T_{\alpha\beta}$. Thus, from the viewpoint of PDE theory we can ask ourselves when this system admits an initial value formulation; that is, we want to find out under what circumstances initial data will determine a unique solution to the Einstein equations.

In the above setting, we are aiming at solving for the functions $g_{\alpha\beta}$, i.e. for spacetime itself. Thus from a physical perspective, it is natural to assign our initial conditions on a Riemannian manifold $S$, which represents a frozen picture of spacetime. We then use the Einstein equations to determine how $S$ evolves in time, and thus we generate a spacetime $(M, g)$ in which $S$ sits as a spacelike hypersurface.

At this point we need to determine the nature of the initial conditions. Notice first that a spacetime $(M, g)$ induces in a natural way two geometrically meaningful objects on a spacelike hypersurface: the first and second fundamental forms. Thus, a good candidate for an initial data set would be a Riemannian manifold $S$ endowed with two tensor fields $h$ and $K$ that will be the induced metric and second fundamental form of $S$ in $(M, g)$.
However, these initial conditions can not be freely chosen. Indeed, since the Gauss-Codazzi equations must hold, $h$ and $K$ must satisfy the constraint equations:

$$r + (\text{Tr}_h K)^2 - |K|^2 = 2T_{00}$$

$$D^j K_{ij} - D_i \text{Tr}_h K = T_{0i}$$

where $i, j = 1, 2, \ldots, n - 1$ and $r, D$ are the scalar curvature and Levi-Civita connection of $(S, h)$ respectively.

The fundamental result of Y. Choquet-Bruhat [10] shows that the initial value problem in $(3 + 1)$ general relativity is well posed for initial conditions $(S, h, K)$ satisfying the constraint equations and the vacuum Einstein equation $G = 0$.

Moreover, R. Geroch and Y. Choquet-Bruhat [11] have proven the existence of a maximal Cauchy development $\mathcal{M}^*$ relative to a initial data set $(S, h, K)$ satisfying the vacuum Einstein equation. As pointed out in [11], the argument used in [11] is valid when considering the Einstein equations with cosmological constant $G + \Lambda g = 0$.

Following [54] we summarize our discussion in the following theorem:

**Theorem 2.20** Let $(S, h)$ be a 3-dimensional Riemannian manifold and $K$ a smooth symmetric tensor field on $S$. Suppose $(S, h, K)$ satisfy the constraint equations, then there exists a spacetime $(\mathcal{M}^*, g^*)$, called the maximal domain of dependence of $(S, h, K)$, satisfying all of the following:

1. $(\mathcal{M}^*, g)$ is a solution of the vacuum Einstein equation with cosmological constant.

2. $(\mathcal{M}^*, g^*)$ is globally hyperbolic with Cauchy surface $S$. 
3. The induced metric and second fundamental form of $S$ are $h$ and $K$.

4. Any spacetime satisfying the above three conditions is isometric to a subset of $(\mathcal{M}^*, g^*)$.

Notice that the above definition doesn’t give us any information on how big $\mathcal{M}^*$ is, since in principle $\mathcal{M}^*$ could be isometrically embedded in a strictly larger spacetime $M_0$. However, if this happens then $S \subset M_0$ can not be a Cauchy surface of $M_0$. In other words, $\mathcal{M}^*$ satisfies a domain of dependence condition (refer to theorem 10.2.2 in [54]).

**Theorem 2.21** Let $(S_i, h_i, K_i)$, $i = 1, 2$, be two initial data sets with maximal Cauchy developments $(\mathcal{M}_i^*, g_i^*)$. Let $A_i \subset S_i$ and assume there is a diffeomorphism sending $(A_1, h_1, K_1)$ to $(A_2, h_2, K_2)$. Then $D(A_1, \mathcal{M}_1^*)$ is isometric to $D(A_2, \mathcal{M}_2^*)$.

Now, we can state theorem 2.17 in terms of the initial value problem.

**Theorem 2.22** Let $(S, h, K)$ be an initial data set and $(\mathcal{M}^*, g^*)$ its maximal Cauchy development. Suppose $(\mathcal{M}^*, g^*)$ is asymptotically de Sitter and satisfies the null energy condition. If $(\mathcal{M}^*, g^*)$ contains a null line from $\mathcal{J}^-$ to $\mathcal{J}^+$, then it is isometric to $dS^4$.

**Proof:** By theorem 2.17 there is an isometry $\Phi: (\mathcal{M}^*, g^*) \rightarrow \mathcal{A}$, where $\mathcal{A}$ is an open subset of $dS^4$. Furthermore, by the proof of theorem 2.17 we also know $\Phi(S)$ is a Cauchy surface of $dS^4$, hence $D(\Phi(S), dS^4) = dS^4$. Then the result follows from theorem 2.21. $\square$
2.5 Matter fields in asymptotically simple spacetimes

In this section we will be considering matter fields on an asymptotically de Sitter spacetime \((\tilde{M}, \tilde{g})\) satisfying all four of the following hypotheses:

A. The Dominant Energy Condition.

**Definition 2.23** The energy-momentum tensor \(T\) satisfies the Dominant Energy Condition if for all timelike \(X \in \mathcal{X}(\tilde{M})\), \(T(X, X) \geq 0\) and the vector field metrically related to \(T(X, -)\) is causal.

The dominant energy condition is believed to hold for all known forms of matter. It may be interpreted as saying that no observer detects a negative local energy density. Actually the dominant energy condition is equivalent to the statement that the speed of the energy flow is less or equal than the speed of light \([54, 52]\).

Notice that the dominant energy condition is stated solely in terms of the causal structure of \((\tilde{M}, \tilde{g})\). It is also easy to check that the following equivalence holds as well \([32, 14]\):

**Proposition 2.24** A symmetric tensor \(T\) satisfies the dominant energy condition if and only if its components \(T_{\alpha\beta}\) with respect to an orthonormal frame \(\{e_\alpha\}\) in which \(e_0\) is timelike satisfy \(T_{00} \geq |T_{\alpha\beta}|\).

From this proposition it is easy to see that a perfect fluid satisfies the dominant energy condition if and only if \(\rho \geq |p|\).
B. $\tilde{\text{Tr}}T \leq 0$ on a neighborhood of $\mathcal{J}$.

This hypothesis is satisfied for a wide variety of fields. It holds for photon gases, electromagnetic fields $^{50}$ $^{35}$ $^{54}$ as well as for quasi-gases $^{50}$. In particular it holds for dust, pure radiation and all perfect fluids satisfying $0 \leq p \leq \rho/(n-1)$.

C. If $K$ is a null vector at $p \in \tilde{M}$ with $T(K, K) = 0$, then $T \equiv 0$ at $p$.

Recall that a Type I energy-momentum tensor is by definition diagonalizable $^{32}$. With the exception of a null fluid, all energy-momentum tensors representing reasonable matter are diagonalizable $^{54}$. Let $\{\rho, p_1, \ldots, p_{n-1}\}$ be the eigenvalues of such a tensor with respect to an orthonormal basis $\{e_0, e_1, \ldots, e_{n-1}\}$, where $e_0$ is timelike. Then for a Type I tensor the existence of $\lambda \in (0, 1)$ satisfying $\lambda \rho \geq |p_i|$, $i = 1, \ldots, n - 1$ prevents the vanishing of $T_x$ in null directions, unless $T_x \equiv 0$. In particular, perfect fluids with $0 \leq p \leq \rho/(n-1)$ satisfy this condition.

Assumption C carries a very important consequence: it guarantees the vanishing of the energy momentum tensor along totally geodesic null hypersurfaces.

**Proposition 2.25** Let $(\tilde{M}, \tilde{g})$ be a spacetime satisfying assumption C in which the Einstein equations hold. Let $S$ be a totally geodesic null hypersurface in $(\tilde{M}, \tilde{g})$, then $T \equiv 0$ on $S$.

**Proof:** Let $p \in S$. In virtue of assumption C, it suffices to show that $T(K, K) = 0$ for some null vector $K \in T_p \tilde{M}$. Hence let us consider a future null generator $\gamma$ of $S$ trough $p$ and recall from section 1.4 that $\gamma'$ satisfies the Raychaudhuri equation

$$\frac{d\theta}{ds} = -\text{Ric}(\gamma', \gamma') - \sigma^2 - \frac{1}{n-2}\theta^2. \quad (2.5.1)$$

Since $S$ is totally geodesic we must have $\theta \equiv 0$ and $\sigma \equiv 0$, thus $\text{Ric}(\gamma', \gamma') = 0$. 
Finally, since $\gamma'$ is null the Einstein equations imply $\text{Ric}(\gamma',\gamma') = T(\gamma',\gamma')$, and thus $T(\gamma',\gamma') = 0$, as desired. $\square$

D. The following falloff condition holds:

$$\lim_{p \to J} \Omega T(\nabla \Omega, \nabla \Omega)_p = 0. \quad (2.5.2)$$

As a way of motivation, let us check that perfect fluid models on Robertson-Walker spacetimes (see definition 2.3) satisfy this condition.

Thus let $(\tilde{M}, \tilde{g}) = \text{RW}(a,k)$ and assume $\tilde{M}$ satisfies the Einstein equations with $\Lambda > 0$ with matter content given by a perfect fluid. In this setting, we have the equations [14, 35]

$$\rho + \Lambda = \frac{(n-1)(n-2)}{2} \left( \frac{\dot{a}^2 + k}{a^2} \right) \quad (2.5.3)$$

$$(n-1) \frac{\dot{a}}{a} = -\frac{\dot{\rho}}{\rho + p}$$

so, if we specialize further and consider dust or radiation, equations 2.5.3 integrates to give rise to conservation laws:

$$\rho a^{n-1} = C \quad \text{(dust)} \quad \rho a^n = C \quad \text{(radiation)} \quad (2.5.4)$$

and hence equations (2.5.3) can be written as

$$\dot{a}^2 = \frac{2a^{3-n}}{(n-1)(n-2)} + \frac{2\Lambda a^2}{(n-1)(n-2)} - k \quad \text{(dust)} \quad (2.5.5)$$

$$\dot{a}^2 = \frac{2a^{2-n}}{(n-1)(n-2)} + \frac{2\Lambda a^2}{(n-1)(n-2)} - k \quad \text{(radiation)}$$

As $t \to \infty$ the solution to either of these two ODE’s grows without bound. Hence the term involving $a^2$ in equations (2.5.5) dominates the remaining ones, thus in both
cases we have $\dot{a}^2 \sim Pa^2$, with $P = \sqrt{2\Lambda/(n-1)(n-2)}$. Thus $a \sim e^{Pt}$ and then $\tilde{g}$ approaches the metric $-dt^2 + e^{2Pt}d\sigma^2$. Hence

$$\tilde{g} \sim \frac{1}{(Pu)^2}(-du^2 + d\sigma^2), \quad u = -e^{-Pt}. \quad (2.5.6)$$

Let $g = (Pu)^2\tilde{g}$, so $g \sim -du^2 + d\sigma^2$. Notice $e_0 = u\nabla u$ is a unit timelike vector field with respect to $\tilde{g}$. Thus

$$T(\nabla u, \nabla u) = \frac{1}{u^2}T(e_0, e_0) \sim \frac{\rho}{u^2}; \quad (2.5.7)$$

hence the conservation laws (2.5.4) imply $T(\nabla u, \nabla u) \to 0$ as $u \to 0$ and thus the falloff condition $D$ holds as well.

Assumption $D$ enables us to give a nice description of the unphysical metric near $\mathcal{J}$ as the following result shows.

**Lemma 2.26** Let $(\tilde{M}, \tilde{g})$ be a past asymptotically de Sitter spacetime satisfying the Einstein equations with $\Lambda > 0$. Assume that for a completion $(M, g)$ with defining function $\Omega$ the past conformal boundary $\mathcal{J}^-$ is compact. Further assume the decay condition

$$\lim_{p \to \mathcal{J}^-} \Omega T(\nabla \Omega, \nabla \Omega)_p = 0 \quad (2.5.8)$$

holds, then there exist a defining function $\overline{\Omega} \in C^\infty(M)$ satisfying 2.5.2 and a neighborhood $\mathcal{U}$ of $\mathcal{J}^-$ such that

$$\tilde{g} = \frac{1}{\overline{\Omega}^2}[-d\overline{\Omega}^2 + h(u)] \quad \text{on } \mathcal{U} \quad (2.5.9)$$

where $h(u)$ is a Riemannian metric on the slice $S_u = \overline{\Omega}^{-1}(u)$. 
Proof: First notice that by a constant rescaling of the physical metric we can assume
\[ \Lambda = \frac{(n - 1)(n - 2)}{2}. \] (2.5.10)

Since \( J = \Omega^{-1}(0) \) and \( d\Omega \neq 0 \) on \( J \) then \( \nabla \Omega \) is normal to \( J \). Let \((x_0, x_1, \ldots, x_{n-1})\) be the slice coordinates in a neighborhood \( W \) of \( J^- \) adapted in such a way that \( \partial_0|_{J^-} = \nabla \Omega|_{J^-} \). Let \( X = \Omega \partial_0 \) and notice that the component functions \( X^\alpha \) satisfy
\[ X^\alpha = \Omega \nabla^\alpha \Omega + O(\Omega^2) \quad \text{on } W, \] (2.5.11)
thus the Einstein equations in conjunction with proposition C.1 yield the following estimate
\[ T(X, X) = \left[ \frac{(n - 1)(n - 2)}{2} g(\nabla \Omega, \nabla \Omega) + \Lambda \right] g(X, X) + O(\Omega) \quad \text{on } W. \] (2.5.12)

As we approach \( J^- \) the falloff condition (2.5.2) implies
\[ \lim_{p \to J^-} T(X, X)_p = \lim_{p \to J^-} \Omega^2 T(\nabla \Omega, \nabla \Omega)_p = 0. \] (2.5.13)

On the other hand, by (2.5.12) we have
\[ T(X, X) \to \left[ \frac{(n - 1)(n - 2)}{2} g(\nabla \Omega, \nabla \Omega) + \Lambda \right] g(\nabla \Omega, \nabla \Omega) \quad \text{as } \Omega \to 0. \] (2.5.14)

hence we must have
\[ g(\nabla \Omega, \nabla \Omega)|_{J^-} = -1. \] (2.5.15)

Consider now the conformally rescaled quantities
\[ \bar{\Omega} = \frac{\Omega}{\theta}, \quad \bar{g} = \frac{g}{\theta^2} \] (2.5.16)
then we want to find \( \theta \) smooth in a neighborhood \( U \) of \( J \) such that \( \bar{\Omega} \) agrees with \( \Omega \) on \( J^- \) and
\[ \bar{g}(\nabla \Omega, \nabla \Omega) = -1 \quad \text{on } U. \] (2.5.17)
To do so, first notice $\nabla \Omega = \theta \nabla \Omega - \Omega \nabla \theta$ so then equation (2.5.17) gives rise to the first order PDE

$$2\theta g(\nabla \Omega, \nabla \theta) - \Omega g(\nabla \theta, \nabla \theta) - \frac{\theta^2}{\Omega} (1 + g(\nabla \Omega, \nabla \Omega)) = 0$$

(2.5.18)

and since

$$\frac{1 + g(\nabla \Omega, \nabla \Omega)}{\Omega} \in C^\infty(\mathcal{W})$$

(2.5.19)

then a standard PDE result (refer to the generalization of theorem 10.3 on page 36 in [52]) guarantees that equation (2.5.18) subject to the initial condition $\theta|_{\mathcal{J}^-} = 1$ has a unique solution in a neighborhood $\mathcal{U}$ of $\mathcal{J}^-$. Notice that, by shrinking $\mathcal{U}$ if necessary, we can extend $\theta$ smoothly to a positive function in all of $M$.

Observe that the integral curves of the gradient $\nabla \Omega$ are unit speed timelike curves in $\mathcal{U}$, hence geodesic segments with respect to the unphysical metric $\tilde{g}$. Moreover, all this geodesics emanate from and are normal to $\mathcal{J}^-$. By further restricting $\mathcal{U}$ to a normal neighborhood of $\mathcal{J}^-$, we can take the slices $S_u$ to be the normal gaussian foliation of $\mathcal{U}$ with respect to $\mathcal{J}^-$. Hence (2.5.9) follows.

Finally, notice that

$$T(\nabla \Omega, \nabla \Omega) = \theta^2 T(\nabla \Omega, \nabla \Omega) + O(\Omega) \quad \text{on } \mathcal{U}$$

(2.5.20)

hence the fall-off condition (2.5.2) holds for $\nabla \Omega$ as well. $\Box$

To finish our discussion, notice that a strengthened version of assumption $D$ causes $\mathcal{J}$ to be totally geodesic.

**Proposition 2.27** Let $(\tilde{M}, \tilde{g})$ be a past asymptotically de Sitter spacetime satisfying the Einstein equations with $\Lambda > 0$. Assume $T$ satisfies the dominant energy condition and that $T(\nabla \Omega, \nabla \Omega)$ is bounded in a neighborhood of $\mathcal{J}^-$. Then $\mathcal{J}^-$ is totally geodesic.
Proof: As before, we can assume \( \Lambda = (n - 1)(n - 2)/2 \). We proceed to show the Hessian of \( \Omega \) vanishes at every point of \( \mathcal{J} \). Thus, without loss of generality, let \( p \in \mathcal{J}^- \). Let \( V \) be a paracompact neighborhood of \( p \) in \( \mathcal{J}^- \) and let \( \mathcal{V} = D(V, N) \). Hence \( (\mathcal{V} \cap \tilde{M}, \tilde{g}|_{\mathcal{V}\cap\tilde{M}}) \) is a past asymptotically de Sitter spacetime in its own right with defining function \( \Omega \) and past conformal boundary \( V \).

In virtue of lemma 2.26 we can choose the defining function \( \Omega \) in a way that \( g(\nabla \Omega, \nabla \Omega) = -1 \) in a neighborhood of \( V \). Further, by contracting the Einstein equations we get

\[
\tilde{\text{Tr}} T = \frac{n - 2}{2} (n(n - 1) - \tilde{R}). \tag{2.5.21}
\]

Since \( e_0 = \Omega \nabla \Omega \) is a unit timelike vector with respect to \( \tilde{g} \), the dominant energy condition implies

\[
|\tilde{\text{Tr}} T| \leq n\Omega^2 T(\nabla \Omega, \nabla \Omega), \tag{2.5.22}
\]

but \( T(\nabla \Omega, \nabla \Omega) \) is bounded as \( \Omega \to 0 \), hence \( \tilde{\text{Tr}} = O(\Omega^2) \), which in turn leads us to

\[
\tilde{R} - n(n - 1) = O(\Omega^2) \tag{2.5.23}
\]

On the other hand, since \( g(\nabla \Omega, \nabla \Omega) = -1 \) near \( V \), proposition C.1 implies

\[
\tilde{R} + 2(n - 1) \frac{\Delta \Omega}{\Omega} = \frac{1}{\Omega^2} (\tilde{R} - n(n - 1)), \tag{2.5.24}
\]

thus by (2.5.23) we have \( \Delta \Omega = O(\Omega) \).

Finally by combining the Einstein equations and the formulas from propositions C.1 and C.2 we get

\[
T = G + \frac{n - 2}{\Omega} \text{Hess}_\Omega - \frac{n - 1}{\Omega} \Delta \Omega, \tag{2.5.25}
\]
then by considerations above Hess$_Q/\Omega$ is bounded near $V$, hence Hess $\Omega \equiv 0$ on $p$. Thus the second fundamental form of $\mathcal{J}$ in $M$ vanishes as well. □

Now let us consider a spacetime $(\tilde{M}, \tilde{g})$ satisfying all the assumptions of lemmas 2.14 and 2.26 above. That is,

- $(\tilde{M}, \tilde{g})$ is globally hyperbolic and asymptotically de Sitter.
- $(\tilde{M}, \tilde{g})$ contains a null line $\eta$ with past endpoint $p \in \mathcal{J}^-$ and future endpoint $q \in \mathcal{J}^+$.
- $(\tilde{M}, \tilde{g})$ satisfies the Einstein equations with $\Lambda > 0$.
- The energy momentum tensor $T$ has the decay rate (2.5.2).

In this set up $\mathcal{S} := \partial I^+(\eta)$ is just the future null cone at $p$, i.e. $\mathcal{S} = \exp_p(\Lambda_p^+ \cap \mathcal{O}) \cap \tilde{M}$ where $\mathcal{O}$ is the maximal set in which $\exp_p$ is defined. Let us denote now the local causal cone at $p$ by $\mathcal{U} := \exp_p(C_p^+ \cap \mathcal{O}) \cap \tilde{M}$, hence $\mathcal{U} - \{p\}$ is a manifold-with-boundary and $\partial(\mathcal{U} - \{p\}) = \mathcal{S}$. Let us choose now a neighborhood $\mathcal{U}$ of $p$ and $\Omega$ so that $\mathcal{U}$ is foliated by $\Omega$ and let $t_0 > 0$ such that $\mathcal{U}' := \mathcal{U} \cap \Omega^{-1}([0, t_0]) \subset \mathcal{U}$.

**Definition 2.28** For $s, t \in (0, t_0)$ with $s < t$ we define $\mathcal{U}(s, t) := \mathcal{U}' \cap \Omega^{-1}([s, t])$, $\mathcal{S}(s, t) := \mathcal{S} \cap \Omega^{-1}([s, t])$ and $\Sigma(t) = \mathcal{U}' \cap \Omega^{-1}(t)$. (See figure 2.4.)
Thus $\mathcal{U}(s, t)$ is a compact manifold with corners and

$$
\partial \mathcal{U}(s, t) = \mathcal{S}(s, t) \cup \Sigma(s) \cup \Sigma(t) 
$$

(2.5.26)

**Lemma 2.29** Let $(\tilde{M}, \tilde{g})$ be a spacetime satisfying the three conditions listed above. Let $N_p = \partial I^+(p, N)$ and for $0 < t_1 < t_0$ define $S' := [N_p - \Omega^{-1}([0, t_1))] \cup \Sigma(t_1)$, $\mathfrak{U}'' := \mathfrak{U} \cap \Omega^{-1}([0, t_1])$. Then $J^+(p, N) \cap \tilde{M} - \mathfrak{U}'' \subset D^+(S', N) \cap \tilde{M}$.

**Proof:** Let $x \in J^+(p, N) \cap \tilde{M} - \mathfrak{U}''$ and let $\gamma$ be a past inextendible timelike curve with future endpoint $x$. Since $J^+(p, N) \cap \tilde{M} \subset D^+(N_p, N)\tilde{M}$ by lemma 2.13 we have that $\gamma$ must intersect $N_p$, say at $y$. If $\Omega(y) \geq t_1$ then $y \in S'$. If $\Omega(y) < t_1$ then notice that $\Omega(x) > t_1$ since $x \notin \mathfrak{U}''$. Now, since the function $t \mapsto \Omega(\gamma(t))$ is continuous there exist a point $z \in \gamma$ between $x$ and $y$ such that $\Omega(z) = t_1$. Hence $z \in \Sigma(t_1) \subset S'$. \(\square\)

Now we can prove the main result of this section.

**Theorem 2.30** Let $(\tilde{M}, \tilde{g})$ be an asymptotically simple spacetime of dimension $n = 4$ which is a solution of the Einstein equations with positive cosmological constant

$$
R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} + \Lambda g_{\alpha\beta} = T_{\alpha\beta},
$$

(2.5.27)
where the energy-momentum tensor $T$ satisfies the following:

**A.** The Dominant Energy Condition.

**B.** $\tilde{\text{Tr}} T \leq 0$ on a neighborhood of $\mathcal{J}$.

**C.** If $K$ is a null vector at $p \in \tilde{M}$ with $T(K,K) = 0$, then $T \equiv 0$ at $p$.

**D.** The falloff condition $\lim_{p \to \mathcal{J}} \Omega T(\nabla\Omega,\nabla\Omega)_p = 0$ holds.

If $(\tilde{M}, \tilde{g})$ contains a null line $\eta$ then $(\tilde{M}, \tilde{g})$ is isometric to de Sitter space.

**Proof:** We first show that the hypotheses of the theorem imply $T \equiv 0$ on asymptotically simple and de Sitter spacetimes of any dimension. Then we can invoke theorem 2.11 to conclude $\tilde{M}$ is de Sitter space when $n = 4$.

By asymptotic simplicity $\eta$ has a past endpoint $p \in \mathcal{J}^-$ and a future endpoint $q \in \mathcal{J}^+$. We first show that the assumptions on $T$ causes the energy momentum tensor to vanish in a neighborhood of $p$.

Consider the notation of definition 2.28. For the time being, let $s \in (0,t_0)$ be fixed and let $\mathcal{U}(t) := \mathcal{U}(s,t)$, $S(t) := S(s,t)$ for all $t \in (s,t_0)$. Let $A$ be the vector field related to $T(\nabla\Omega, -)$ via the unphysical metric, in other words,

$$g(A,X) = T(\nabla\Omega,X) \quad \text{for all } X \in \mathcal{X}(\tilde{M}).$$  \hfill (2.5.28)

Let $\text{dv}$ denote the volume element of $(M,g)$. Then $d(i_A\text{dv}) = \text{div} A\text{dv}$, hence by Stokes theorem

$$\int_{\mathcal{U}(t)} \text{div} A\text{dv} = \int_{\mathcal{U}(t)} d(i_A\text{dv}) = \int_{\partial \mathcal{U}(t)} i_A\text{dv}. \hfill (2.5.29)$$

Furthermore, the integral over $\partial \mathcal{U}(t)$ can be computed as

$$\int_{\partial \mathcal{U}(t)} i_A\text{dv} = \int_{\Sigma(s)} i_A\text{dv} + \int_{\Sigma(t)} i_A\text{dv} + \int_{S(t)} i_A\text{dv}. \hfill (2.5.30)$$
Notice $\nabla \Omega$ is a unit vector field normal to the surfaces $\Sigma(a)$. Since $\nabla \Omega$ is outward pointing at $\Sigma(s)$ and inward pointing at $\Sigma(t)$ we have

$$i_A dv = -g(A, -\nabla \Omega)d \sigma = T(\nabla \Omega, \nabla \Omega)d \sigma \quad \text{at } \Sigma(t) \quad (2.5.31)$$

$$i_A dv = -g(A, \nabla \Omega)d \sigma = -T(\nabla \Omega, \nabla \Omega)d \sigma \quad \text{at } \Sigma(s)$$

where $d \sigma$ is the volume form on $\Sigma(a)$.

On the other hand, let $B = \{e_0, e_1, \ldots, e_{n-1}\}$ be a positively oriented frame at $p \in M$. A direct computation shows

$$i_A dv = \sum_{i=0}^{n-1} (-1)^i T(\nabla \Omega, e_i)e_0^* \wedge \cdots \wedge e_i^* \wedge e_{i-1}^* \wedge e_{i+1}^* \wedge \cdots \wedge e_{n-1}^*, \quad (2.5.32)$$

but in virtue of proposition 2.25 we have $T \equiv 0$ on $S$, hence the above equation implies $i_A dv|_S \equiv 0$. Thus

$$\int_{\partial(t)} \text{div}A \ dv = \int_{\Sigma(t)} T(\nabla \Omega, \nabla \Omega)d \sigma - \int_{\Sigma(s)} T(\nabla \Omega, \nabla \Omega)d \sigma. \quad (2.5.33)$$

Now let $\hat{T}$ be the $(1,1)$ tensor $g$-equivalent to $T$ and let $C$ denote tensor contraction with respect to $g$. Since $A = C(\hat{T} \otimes \nabla \Omega)$ we have

$$\text{div}A = C \nabla A = C \nabla C(\hat{T} \otimes \nabla \Omega) = C^2 \nabla (\hat{T} \otimes \nabla \Omega)$$

$$= C^2(\nabla \hat{T} \otimes \nabla \Omega + \hat{T} \otimes \nabla(\nabla \Omega))$$

$$= C^2(\nabla \hat{T} \otimes \nabla \Omega) + C^2(\hat{T} \otimes \nabla(\nabla \Omega)) \quad (2.5.34)$$

$$= \text{div}T (\nabla \Omega) + C^2(\hat{T} \otimes \nabla(\nabla \Omega)).$$
Hence

\[
\int_{\mathcal{U}(t)} \text{div} T(\nabla \Omega) dv + \int_{\mathcal{U}(t)} C^2(\hat{T} \otimes \nabla(\nabla \Omega)) dv = \int_{\Sigma(t)} T(\nabla \Omega, \nabla \Omega) d\sigma dv - \int_{\Sigma(s)} T(\nabla \Omega, \nabla \Omega) d\sigma
\]

(2.5.35)

Since \( \mathcal{U} \) is compact, the components \( \Omega_{\alpha}^{\beta} \) of \( \nabla(\nabla \Omega) \) in any \( g \)-orthonormal frame field are bounded from above, say by \( Q \). Similarly, \( T(\nabla \Omega, \nabla \Omega) \geq |T_{\alpha}^{\beta}| \) on \( \tilde{M} \) by the dominant energy condition, hence by continuity \( \lim_{z \to p} T(\nabla \Omega, \nabla \Omega)_z \geq \lim_{z \to p} |T_{\alpha}^{\beta}(z)| \) as well. Then

\[
C^2(\hat{T} \otimes \nabla(\nabla \Omega)) = T_{\beta}^{\alpha} \Omega_{\alpha}^{\beta} \leq P T(\nabla \Omega, \nabla \Omega)
\]

(2.5.36)
on \( \mathcal{U} \), where \( P := n^2Q \). Thus

\[
\int_{\mathcal{U}(t)} C^2(\hat{T} \otimes \nabla(\nabla \Omega)) dv \leq \int_{\mathcal{U}(t)} P T(\nabla \Omega, \nabla \Omega) dv,
\]

(2.5.37)

On the other hand, recall the conformal relation \[C.2\]

\[
\text{div} T(\nabla \Omega) = \frac{1}{\Omega^2} \tilde{\text{div}} T(\nabla \Omega) + \frac{n-2}{\Omega} T(\nabla \Omega, \nabla \Omega) + \frac{1}{\Omega^2} \tilde{\text{Tr}} \hat{T}.
\]

(2.5.38)

Since the physical metric satisfies the Einstein equations, the energy-momentum tensor is divergence free. Thus

\[
\tilde{\text{div}} T(\nabla \Omega) \equiv 0 \quad \text{in} \quad \tilde{M}
\]

(2.5.39)

Moreover, by assumption \( B \) \( \tilde{\text{Tr}} \hat{T} \leq 0 \), thus \( \text{(2.5.38)} \) and \( \text{(2.5.42)} \) give rise to the inequality

\[
\int_{\mathcal{U}(t)} \text{div} T(\nabla \Omega) dv \leq \int_{\mathcal{U}(t)} \frac{n-2}{\Omega} T(\nabla \Omega, \nabla \Omega) dv
\]

(2.5.40)
Hence equation (2.5.35) along with (2.5.37) and (2.5.40) yield

\[ \int_{\Sigma(t)} T(\nabla \Omega, \nabla \Omega) d\sigma dv - \int_{\Sigma(s)} T(\nabla \Omega, \nabla \Omega) d\sigma \leq \int_{\mathcal{U}(t)} \left( \frac{n-2}{2} + P \right) T(\nabla \Omega, \nabla \Omega) dv \]

(2.5.41)

and then Fubini’s theorem implies

\[ \int_{\Sigma(t)} T(\nabla \Omega, \nabla \Omega) d\sigma - \int_{\Sigma(s)} T(\nabla \Omega, \nabla \Omega) d\sigma \leq \int_{\Sigma(\tau)} \left( \frac{n-2}{2} + P \right) T(\nabla \Omega, \nabla \Omega) d\sigma d\tau. \]

(2.5.42)

Now, we would like to analyze the limit of both sides of relation (2.5.42) as \( s \to 0 \).

Let then \( p(s) \in \Sigma(s) \) be such that \( T(\nabla \Omega_z, \nabla \Omega_z) \leq T(\nabla \Omega_{p(s)}, \nabla \Omega_{p(s)}) \) for all \( z \in \Sigma(s) \). Such \( p(s) \) always exists since \( \Sigma(s) \) is compact. Thus

\[ \int_{\Sigma(s)} \frac{1}{\Omega} T(\nabla \Omega, \nabla \Omega) d\sigma \leq \frac{1}{s} \int_{\Sigma(\tau)} T(\nabla \Omega_{p(s)}, \nabla \Omega_{p(s)}) d\sigma \]

\[ = \frac{1}{s} T(\nabla \Omega_{p(s)}, \nabla \Omega_{p(s)}) \text{Vol}(\Sigma(s)) \]

(2.5.43)

Let us consider now a small normal neighborhood \( \mathcal{N} \) around \( p \). It is known [51] that the metric volume of the local causal cone truncated by a timelike vector \( X_p \) is of the same order as the volume of the corresponding truncated cone in \( T_pM \). As a consequence, if \( g(X_p, X_p) = -r^2 \) then the volumes of the slices \( S_r := \exp_p(C_r) \) and \( C_r := \{ v \in C^p_+ \mid g(v, X_p) = -r^2 \} \) are also of the same order in \( r \), and hence \( \text{Vol}(S_r) = O(r^{n-1}) \). Hence by considering \( s \) very small and setting \( X_p = s \nabla \Omega_p \), we get the estimate

\[ \text{Vol}(\Sigma(s)) = O(s^{n-1}), \]

(2.5.44)

since \( \text{Vol}(S_s) \sim \text{Vol}(\Sigma(s)) \) for very small \( s > 0 \). Thus without loss of generality we can take \( t_0 > 0 \) such that \( \mathcal{U}' \) is contained in such a normal neighborhood \( \mathcal{N} \).
Therefore by letting \( s \to 0 \), equations (2.5.43) and (2.5.44) above imply

\[
\frac{1}{s}T(\nabla \Omega_p(s), \nabla \Omega_p(s)) \text{Vol}(\Sigma(s)) \leq C \left[ T(\nabla \Omega_p(s), \nabla \Omega_p(s)) s^{n-2} \right] (2.5.45)
\]

for some positive constant \( C \). Hence

\[
\lim_{s \to 0^+} \int_{\Sigma(s)} \frac{1}{\Omega} T(\nabla \Omega, \nabla \Omega) d\sigma = 0 \quad (2.5.46)
\]

in virtue of assumption B.

Thus

\[
x(t) := \lim_{s \to 0} \int_t^s \int_{\Sigma(\tau)} \left( \frac{n-2}{\Omega} + P \right) T(\nabla \Omega, \nabla \Omega) d\sigma \ d\tau \quad (2.5.47)
\]

is a well defined function of \( t \). Further, since inequality (2.5.42) holds for all \( s > 0 \) it will hold on the limit as well. Hence by taking limits to both sides of inequality (2.5.42) we get

\[
\int_{\Sigma(t)} T(\nabla \Omega, \nabla \Omega) d\sigma \leq x(t) \quad (2.5.48)
\]

and further notice 2.5.48 can be expressed as differential inequality:

\[
\frac{dx}{dt} \leq \left( \frac{n-2}{t} + P \right) x, \quad t \in (0, t_0) \quad (2.5.49)
\]

which in turn implies

\[
\frac{d}{dt} \left( \frac{e^{-Pt}}{t^{n-2}} x \right) = \frac{e^{-Pt}}{t^{n-2}} \left[ \frac{dx}{dt} - \left( \frac{n-2}{t} + P \right) x \right] \leq 0 \quad (2.5.50)
\]

i.e. the function

\[
I(t) = \frac{x(t)e^{-Pt}}{t^{n-2}} \quad (2.5.51)
\]

is decreasing near \( J^- \).
Thus, we analyze \( \lim_{t \to 0^+} I(t) \). Notice first that estimate (2.5.45) yields

\[
\int_{\Sigma(t)} \left( \frac{n-2}{\Omega} + P \right) T(\nabla \Omega, \nabla \Omega) d\sigma \leq C' T(\nabla \Omega_{p(t)}, \nabla \Omega_{p(t)}) t^{n-2} \tag{2.5.52}
\]

for some constant \( C' \geq 0 \). Thus by L'Hospital rule we get

\[
\lim_{t \to 0^+} \frac{x(t)}{t^{n-2}} = \lim_{t \to 0^+} \frac{1}{(n-2)t^{n-3}} \int_{\Sigma(t)} \left( \frac{n-2}{\Omega} + P \right) T(\nabla \Omega, \nabla \Omega) d\sigma \tag{2.5.53}
\]

\[
\leq \frac{C'}{n-2} t T(\nabla \Omega_{p(t)}, \nabla \Omega_{p(t)})
\]

then \( \lim_{t \to 0^+} \frac{x(t)}{t^{n-2}} = 0 \) and hence \( \lim_{t \to 0^+} I(t) = 0 \).

It follows that \( I(t) \equiv 0 \) on \( \mathcal{U}' \), and consequently \( T(\nabla \Omega, \nabla \Omega) \equiv 0 \) on \( \mathcal{U}' \). Therefore \( T \equiv 0 \) on \( \mathcal{U}' \) by the dominant energy condition.

Now let \( 0 < t_1 < t_0 \) and consider \( \mathcal{S}' \) and \( \mathcal{U}'' \) as in lemma 2.29. Hence it is clear that \( T \equiv 0 \) on \( \mathcal{U}'' \). Further, let \( x \) be in the topological interior of \( D^+(\mathcal{S}', N) \), hence \( W = J^-(x, N) \cap J^+(\mathcal{S}', N) \) is compact. Then \( T \equiv 0 \) on \( W \) by the conservation theorem of Hawking and Ellis (cfr. page 93 in [32]), thus \( T \equiv 0 \) on \( \text{int}D^+(\mathcal{S}', N) \).

Hence by continuity we have \( T \equiv 0 \) on \( D^+(\mathcal{S}', N) \cap \tilde{M} \).

On the other hand, by the lemmas 2.13 and 2.29 we have the inclusions

\[
I^+(S) \subset J^+(p, N) \cap \tilde{M} \subset \mathcal{U}'' \cup (D^+(\mathcal{S}', N) \cap \tilde{M}) \tag{2.5.54}
\]

where \( S = \partial I^+(\eta) \) as in lemma 2.14. Then we just showed \( T \equiv 0 \) on \( I^+(S) \).

On a time dual fashion, we can show \( T \) vanishes in a neighborhood of \( q \) and consequently on the whole set \( I^- (S) \). To finish the proof, recall that since \( \partial I^+(\eta) = S = \partial I^- (\eta) \) then \( \tilde{M} = S \cup I^+(S) \cup I^-(S) \) in virtue of proposition 3.15 in [48]. Therefore \( T \equiv 0 \) on \( \tilde{M} \) and the result follows. \( \square \)
Finally, notice that by theorem 2.17 and the proof of theorem 2.30, we have the following corollary:

**Corollary 2.31** Let $(\tilde{M}, \tilde{g})$ be a globally hyperbolic and asymptotically de Sitter spacetime of dimension $n = 4$, which is a solution of the Einstein equations with positive cosmological constant

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} + \Lambda g_{\alpha\beta} = T_{\alpha\beta}, \quad (2.5.55)$$

where the energy-momentum tensor $T$ satisfies conditions A to D above. If $(\tilde{M}, \tilde{g})$ contains a null line $\eta$ with endpoints $p \in J^- \text{ and } q \in J^+$ then $(\tilde{M}, \tilde{g})$ is isometric to an open subset of de Sitter space.
Chapter 3

Asymptotically Anti de Sitter

Spacetimes

Anti de Sitter space $AdS^n$ is the simply connected space form of constant curvature $C \equiv -1$. It can be realized as the hyperboloid

\[ x_0^2 - x_1^2 - \ldots - x_{n-1}^2 + x_n^2 = 1 \quad (3.0.1) \]

embedded in semi-Euclidean space $\mathbb{R}^{2,n-1}$. Then by considering the parametrization

\[ x_0 = \cosh \rho \cos t \quad x_n = \cosh \rho \sin t \quad (3.0.2) \]

the $AdS^n$ metric takes the form

\[ ds^2 = -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\omega^2. \quad (3.0.3) \]

By further considering the change of variables $\sinh \rho = \tan \sigma$ we get

\[ ds^2 = \frac{1}{\cos^2 \sigma}(-dt^2 + d\sigma^2 + \sin^2 \sigma d\omega^2), \quad (3.0.4) \]
Figure 3.1:

$AdS^n$ embedded in the Einstein static universe. Notice $\mathcal{J} \approx S^{n-2} \times R$ is timelike.

thus $AdS^n$ is readily seen to admit a conformal boundary in the Einstein static universe (see figure 3.1 above). Notice that unlike de Sitter space, $AdS^n$ has a timelike conformal boundary $\mathcal{J}$.

### 3.1 Spacetimes with timelike boundary

In order to be able to study in depth those spacetimes whose structure at infinity is similar to $AdS^n$, we have to gain first some understanding on the topology and causal structure of spacetimes having a timelike boundary. Thus the following definitions arise naturally:

**Definition 3.1** A manifold-with-timelike-boundary $(M, g)$ is a Lorentzian manifold-
Extended Schwarzschild spacetime showing the region $M = \{X^2 - T^2 \leq C^2\}$, $C = (R/2m - 1) \exp(R/2m)$, $R > 2m$. Notice that $M$ is a spacetime-with-timelike-boundary and that $\partial M$ has two components.

**Definition 3.2** A manifold with timelike boundary is time-orientable if there exists a smooth timelike vector field $X$ defined in all $M$. A spacetime-with-timelike-boundary is an orientable and time-orientable manifold with timelike boundary.

As mentioned earlier, the conformal completion of $AdS^n$ in the Einstein static universe is a spacetime-with-timelike-boundary. Another example is provided by the region $X^2 - T^2 \leq C^2$ in Kruskal coordinates on the extended Schwarzschild spacetime. See figure 3.2

Notice that since $\partial M$ is not necessarily given by a defining function, there is no notion of “infinity” in a spacetime-with-timelike-boundary.
It is clear from definition \[3.2\] that \( \partial M \) is time-orientable as a Lorentzian manifold with the time orientation given by \( \tan X \), where \( X \) is a time orientation for \( M \) and \( \tan: T_pM \to T_p\partial M \) is the standard projection. Recall also that an orientation on a smooth manifold-with-timelike-boundary \( M \) gives rise in a natural way to an orientation on \( \partial M \), the so called Stokes orientation. Hence we conclude that the boundary \( \partial M \) of a spacetime-with-timelike-boundary \( M \) is a spacetime in its own right.

In the remainder of this section we generalize important results in Lorentzian geometry to the spacetime-with-timelike-boundary scenario.

### 3.1.1 Causal theory

**Remark 3.3** In order to make proofs simpler, from now on we will always assume the spacetime-with-timelike-boundary \((M, g)\) is embedded in an open spacetime \((\overline{M}, \overline{g})\) (refer to theorem \[A.1\]). Also, unless otherwise explicitly stated, all causal relations and sets are taken with respect to \( M \).

We begin by proving that the relation \( I^+ \) is open in \( M \), but before we do so, let us prove a technical lemma:

**Proposition 3.4** Let \( p \in M \), then \( I^+(p) \cap \partial M \) is open in \( \partial M \).

**Proof:** Let \( x \in I^+(p) \cap \partial M \) and consider a future pointing timelike curve \( \gamma: [0, 1] \to M \) from \( p \) to \( x \). We first assume that \( \gamma|_{1-\varepsilon}^{1} \subset \partial M \) for some \( \varepsilon > 0 \). Let \( y = \gamma(1 - \varepsilon) \), then \( I^+(y, \partial M) \) is an open neighborhood of \( x \) in \( \partial M \) and \( I^+(y, \partial M) \subset I^+(p) \cap \partial M \), which proves the claim in this case.
To deal with the general case, we first proceed to deform \( \gamma \) into a timelike curve in \( M \) intersecting \( \partial M \) only once near \( x \). This can be accomplished in the following way: Let \( w \in T_xM \) be an inward pointing vector and let \( W \) be its parallel translate along \( \gamma \). Then there is an \( \varepsilon > 0 \) such that \( x_0 := \gamma(1-\varepsilon) \in \text{Int}(M) \) and \( W \) is inward pointing whenever \( \gamma_{|1-\varepsilon} \) intersects \( \partial M \). Consider now the variation \((u,v) \mapsto x(u,v)\) of \( \gamma_{|1-\varepsilon} \) with variational vector field \( fW \) where \( f \) is a smooth bump function vanishing at endpoints. By construction, for small values of \( v \), the longitudinal curves \( u \mapsto x(u,v) \) remain in \( M \), are timelike and meet \( \partial M \) only at the endpoint \( x \).

Now, let us consider a \( \overline{M} \)-neighborhood \( U \) of \( x \) not containing \( x_0 \) and a past pointing timelike segment \( \alpha : [0,T] \to U \cap \partial M \) starting at \( x \). First we extend \( \alpha' \) to a vector field \( X \in \mathcal{X}(\partial M) \) with \( \text{supp}(X) \subset U \cap \partial M \), then we further extend \( X \) to \( \overline{X} \in \mathcal{X}(\overline{M}) \) with \( \text{supp}(\overline{X}) \subset U \).

By the way \( \overline{X} \) was constructed, all of its integral curves starting at \( \partial M \) remain on \( \partial M \). Thus by uniqueness of ODE’s, none of the integral curves with initial points in \( \text{Int}(M) \) can intersect \( \partial M \) in a positive time, hence all these curves remain in \( \text{Int}(M) \).

For \( v \in [1-\varepsilon,1] \) let \( \varphi_v(t) \) be the image under the flow \( \varphi_{\overline{X}} \) at time \( t \) of the point \( \gamma(v) \in M \), and notice that \( \varphi_{1-\varepsilon}(t) = x_0 \) for all \( t \). By continuity of the metric tensor, there is \( t_0 \in (0,T) \) such that the curves \( v \mapsto \varphi_v(t) \) are timelike for all \( t \in [0,t_0] \). Let \( q = \varphi_1(t_0) \), then \( q \in I^+(x_0) \) hence \( q \in I^+(p) \). Notice also \( \alpha \upharpoonright q \) is a past pointing curve in \( \partial M \). Hence we are back to the first scenario and the result follows. \( \square \)

**Proposition 3.5** The relation \( I^+ \) is open in \( M \).

**Proof:** Let \( x \in I^+(p) \). If \( x \in \text{Int}(M) \) the result follows immediately. Thus let
$x \in \partial M$, so by lemma \[3.4\] there is a $\partial M$-neighborhood $\mathcal{W}$ of $x$ such that $\mathcal{W} \subset I^+(p)$.

Let $v \in T_x M$ be a future timelike and inward pointing vector, and extend it to a vector field $V \in \mathcal{X}(M)$. By continuity we can find a $\partial M$-neighborhood $\mathcal{V}$ of $x$ and $T > 0$ so that the integral curves $\varphi_y(t)$ of $V$ are future timelike for all $t \in [0, T]$ and for all $y \in \mathcal{V}$. Further, by considering smaller $\mathcal{V}$ and $T$ if necessary, we assure $V$ is inward pointing along $\mathcal{V}$ and hence $\varphi_y(t) \in \text{Int}(M)$ for all $t \in (0, T)$.

Thus, let $\mathcal{U} = \mathcal{W} \cap \mathcal{V}$ and $\mathcal{U}_0 = \varphi_V(\mathcal{U} \times [0, T])$. Clearly, $\mathcal{U}_0$ is open in $M$ and $x \in \mathcal{U}_0$. Finally, the discussion above shows $\mathcal{U}_0 \subset I^+(\mathcal{U})$. On the other hand, $\mathcal{U} \subset \mathcal{W} \subset I^+(\mathcal{U})$, hence $\mathcal{U}_0 \subset I^+(p)$. □

Notice that since $\partial M$ is a spacetime in its own right the sets $I^-(q)$ are never empty, even when $q \in \partial M$. Thus, as a consequence of proposition \[3.5\] we have that $\{I^+(p)\}_{p \in M}$ is an open cover of $M$.

**Proposition 3.6** Let $p \ll q$ and $q \leq r$. Then $p \ll r$.

**Proof:** First notice that the general case reduces to the case in which $r \in \partial M$ and $q \in \text{Int}(M)$. Thus, let $\alpha : [0, 1] \to M$ be a future causal curve from $q$ to $r$ and let us analyze first the particular case in which $\alpha$ meets $\partial M$ only at $r$. Thus let $w \in T_r M$ be a past timelike and interior pointing vector. Let $W$ be the parallel translate of $w$ along the reverse of $\alpha$, thus by continuity $g(W(1 - t), \alpha') > 0$. Consider now $V(t) = (2 - t)W(1 - t)$ and observe that $V(1) = W(0) = w$ and $V' = -W(1 - t)$, hence $g(V', \alpha') = -g(W(1 - t), \alpha') < 0$.

Let $(t, s) \mapsto x(t, s)$ be a variation of $\alpha$ with variational vector field $V$. Then we can apply theorem 10.45 in [46] to find $\varepsilon > 0$, such that the longitudinal curves
\( \alpha_s(t) = x(t,s) \) are timelike for all \( s \in (0, \varepsilon] \), hence \( x(1,s) \in I^+(x(0,s)) \) for all \( s \in (0, \varepsilon] \). Notice as well that since the final transversal curve \( \beta(s) = x(1,s) \) is inward pointing, we can assume \( x(t,s) \in M \) for all \( s \in [0, \varepsilon] \) and \( t \in [0, 1] \). Further, since \( w \) is past pointing, we can also assume that \( \beta \) is past timelike on the interval \( [0, \varepsilon] \), hence \( r \in I^+(\beta(s)) \subset I^+(x(0,s)) \) for all \( s \in (0, \varepsilon] \).

Since \( q \in I^+(p) \) and \( I^+(p) \) is open we know there exists a \( \varepsilon_0 \in (0, \varepsilon) \) such that \( x(0,\varepsilon_0) \in I^+(p) \). Hence by considerations above we have \( r \in I^+(p) \). This finishes the proof of the special case.

Now let us treat the more general case, thus let assume \( A := \alpha \cap \partial M \) has at least two points. Further, let \( B = A \cap I^+(p) \) and by a slight abuse of notation, we identify \( \alpha(t) \in A \) with \( t \in [0, 1] \). Since \( A \subset [0, 1] \) is compact, it has a minimum \( t_0 < 1 \), thus by the special case we just proved above we have \( t_0 \in B \). Then \( B \subset [0, 1] \) is a non empty bounded set, so let \( T = \sup B \) and notice \( T \in A \) since \( A \) is closed. We want to show \( T = 1 \), but first we proceed to show \( T \in B \). To this end, let \( v \in T_{\alpha(T)} \overline{M} \) be a past timelike and inward pointing vector and let \( V(s), s \in [-T, 1 - T] \) be its parallel translate along \( \alpha \). By continuity, there is a small interval \( I \) around \( T \) such that \( (1 + T - s)V(s) \) is past timelike and inward pointing for \( s \in I \cap A \). On the other hand, since \( T = \sup B \) it follows \( I \cap B \neq \emptyset \). If \( T \in B \) there is nothing to prove, so let \( T_0 \in I \cap B, T_0 < T \). By the same variation argument used in the previous case we can construct a future timelike curve with final point \( T \) and initial point arbitrary close to \( T_0 \). Therefore \( T_0 \in I^+(p) \) implies \( T \in I^+(p) \), hence \( T \in B \).

To finish the proof, assume \( T < 1 \). Then since \( I^+(p) \) is open and \( T \in I^+(p) \) there exists \( \varepsilon > 0 \) such that \( T + \varepsilon < 1 \) and \( [T, T + \varepsilon) \subset I^+(p) \). Therefore \( t \notin A \) for
that is, \( \alpha(t) \not\in \partial M \) for all \( t \in (T, T + \varepsilon) \). Thus if \( t_1 \leq 1 \) denotes the first point in \( A \) after \( T \), then \( \alpha|_{T+\varepsilon/2} \) intersects \( \partial M \) only at \( t_1 \). Hence we are once again back to our special case with \( q = \alpha(T + \varepsilon/2) \) and \( r = \alpha(t_1) \), thus \( t_1 \in I^+(T) \), hence \( t_1 \in I^+(p) \), and therefore \( t_1 \in B \), a contradiction. \( \square \)

Notice that a similar argument can be used to prove \( p \ll r \) whenever \( p \leq q \) and \( q \ll r \). As the first application of proposition 3.6 we have the following result:

**Proposition 3.7** Let \( S \subset M \), then \( J^+(S) \subset I^+(S) \)

**Proof:** Let \( x \in J^+(S) \). We proceed to construct a sequence \( \{x_n\} \) in \( I^+(S) \) converging to \( x \). Thus, let us consider \( q \in I^+(x) \) and let \( \gamma: [0, a] \to M \) be a timelike segment from \( x \) to \( q \). Now let \( t_n \downarrow 0 \) be a sequence in \( (0, a) \), hence \( x_n \to x \) and \( x_n \in I^+(x) \), where \( x_n := \gamma(t_n) \). Take now \( p \in S \) with \( x \in I^+(p) \), therefore by proposition 3.6 we have \( x_n \in I^+(p) \). \( \square \)

At this point it is worthwhile noticing that the analog to proposition [1.28] does not hold on the spacetime-with-timelike-boundary case, as the example depicted in figure 3.3 shows. However, we do get a statement in the spirit of proposition 1.28 when our causal curve touches \( \partial M \) only at its endpoints.

**Proposition 3.8** Let \( \gamma: [0, 1] \to M \) be a future causal curve joining \( p \) to \( q \) such that \( \gamma(0, 1) \subset \text{Int}(M) \). Then either \( q \in I^+(p) \) or \( \gamma \) is a smooth null geodesic.

**Proof:** Let us assume \( q \not\in I^+(p) \). Let \( t_n \downarrow 0 \) and \( s_n \not\uparrow 1 \) be two sequences on \([0, 1]\) and let us define \( p_n = \gamma(t_n) \), \( q_n = \gamma(s_n) \). By proposition 3.6 and our assumption we have that \( q_n \in J^+(p_n) \) \( - I^+(p_n) \). Thus by proposition 1.28 we have \( \gamma|_{p_n} \subset \text{Int}(M) \) is a null geodesic segment. It follows that \( \gamma|_{(0,1)} \) is a null geodesic. Finally, consider
Let \( M \) be Minkowski space \( \mathbb{M}^3 \) with an open cylinder removed. Observe that \( q \in J^+(p) - I^+(p) \), but no future null geodesic in \( M \) connects \( p \) to \( q \).

a spacetime extension \( \overline{M} \) and notice \( \gamma \) continuously extends to its endpoints in \( \overline{M} \), then by proposition 1.20 \( \gamma \) is a geodesic in \( \overline{M} \), hence it is a geodesic in \( M \) as well. \( \square \)

We can also generalize lemma 10.50 in [46] to the spacetime-with-timelike-boundary context.

**Proposition 3.9** Let \( S \) be a spacelike submanifold of \( \partial M \) and \( \gamma: I \to M \) a causal curve joining \( p \) to \( q \in S \) that intersects \( \partial M \) only at \( q \). Then either there is a timelike curve from \( p \) to \( S \) or \( \gamma \) is a null geodesic meeting \( S \) orthogonally.

**Proof:** By the previous proposition, \( \gamma \) must be a null geodesic or otherwise \( q \in I^+(p) \), hence we only need to prove the statement about the orthogonality of \( \gamma \). Assume
then that $\gamma$ is a null geodesic not normal to $S$. We will show $S \cap I^+(p) \neq \emptyset$.

Since $S \subset \partial M$ is spacelike, there is a neighborhood $U'$ of $q$ in $\partial M$ such that $S \cap U'$ is acausal and closed in $U'$. Then $U := D(S \cap U', U')$ is a globally hyperbolic neighborhood of $q$ in $\partial M$. Now, by means of the normal exponential map we can extend $S \cap U$ to a spacelike hypersurface $S_0$ in $\overline{M}$. Consider now a $\overline{M}$-neighborhood $\mathcal{V}_0$ of $q$ such that $\mathcal{V}_0 \cap \partial M \subset U$. Since $S_0 \cap \mathcal{V}_0 \subset \mathcal{V}_0$ is spacelike, there is a $\overline{M}$-neighborhood $\mathcal{U}_0 \subset \mathcal{V}_0$ of $q$ such that $S_0 \cap \mathcal{U}_0$ is acausal in $\mathcal{U}_0$. As we did before, by replacing $\mathcal{U}_0$ by $D(S_0 \cap \mathcal{U}_0, \mathcal{U}_0)$ we can assume $\mathcal{U}_0$ is a globally hyperbolic $\overline{M}$-neighborhood of $q$.

Let us prove now $J^+(S_0 \cap \mathcal{U}_0, \mathcal{U}_0) \cap \partial M \subset I^+(S \cap U, \partial M) \cup S$. Let $x \in J^+(S_0 \cap \mathcal{U}_0, \mathcal{U}_0) \cap \partial M$, then if $x \in S$ the claim follows immediately, so let us assume $x \notin S$ and let $\alpha : I \to U$ be a future directed and inextendible timelike curve passing through $x$. Since $x \in U - S$ then by definition of domain of dependence, $\alpha$ must intersect $S \cap U$ exactly once, say at $y$. Assume $y$ comes strictly after $x$ along $\alpha$, then $\alpha$ must have left $\mathcal{U}_0$ at a point before $y$, otherwise the acausality of $S_0 \cap \mathcal{U}_0$ in $\mathcal{U}_0$ would be violated. Hence $\alpha$ -considered as an inextendible curve in $\mathcal{U}_0$- must intersect $S_0 \cap \mathcal{U}_0$ to the past of $x$, say at $z \in S \cap U$, thus $z \neq y$. But then $\alpha|_z^y$ would be a future timelike curve in $U$ joining two points of $S \cap U$, contradicting the acausality of $S \cap U$ in $U$. Therefore $y$ has to come before $x$ along $\alpha$, i.e $x \in J^+(y, U)$ and the claim follows. Similarly we can show the time dual $J^-(S_0 \cap \mathcal{U}_0, \mathcal{U}_0) \cap \partial M \subset I^-(S \cap U, \partial M) \cup S$. 
Thus let $p' \in \gamma \cap U_0$, $p' \neq q$ be such that $\gamma|_{p'}^{q} \subset U_0$. Hence, proposition 10.48 of [46] applied to $\gamma|_{p'}^{q}$ and $S \cap U_0$ assures the existence of a future timelike curve $\sigma$ in $U_0$ joining $p'$ to $q' \in S \cap U_0$. For simplicity, assume $\sigma$ intersects $S$ only at $q'$.

If $\sigma$ stays in $M$ we are done. Otherwise $\sigma$ has to intersect $\partial M$ for the first time at a point $r$ strictly before $q'$. Notice $r \not\in J^+(S_0 \cap U_0, U_0)$ since $S_0 \cap U_0$ is acausal in $U_0$, hence $r \in J^-(S_0 \cap U_0, U_0)$ and hence $r \in I^-(S \cap U, U) \cup S$ by the claim above, hence the result follows in this scenario as well. $\Box$

With similar techniques we can prove the following analogue of proposition 1.32.

**Proposition 3.10** Let $S$ be a spacelike submanifold of $\partial M$ and $\gamma: I \rightarrow M$ a future causal curve joining $p$ to $q \in S$ that intersects $\partial M$ only at $q$. If $I^+(p) \cap S = \emptyset$ then $\gamma$ is a null geodesic orthogonal to $S$ at $q$ with no focal points of $S$ strictly after $p$.

Now we turn our attention to the study of the Lorentzian distance function $\tau$. Recall that $\tau$ is known to be lower semicontinuous for open spacetimes. The same result holds in the spacetime-with-timelike-boundary case.

**Proposition 3.11** The Lorentzian distance $\tau$ is lower semicontinuous on $M$.

**Proof:** Let $p, q \in M$. If $\tau(p, q) = 0$ the result holds trivially, so let us assume $\tau(p, q) > 0$. By proposition 3.6, this in turn implies $q \in I^+(p)$.

Let $\varepsilon > 0$ and suppose $\tau(p, q) < \infty$, then we want to find neighborhoods $U$ and $V$ around $p$ and $q$ such that $\tau(x, y) \geq \tau(p, q) - \varepsilon$ for all $x \in U$ and $y \in U$. Thus consider a future pointing timelike curve $\gamma$ from $p$ to $q$ with $L(\gamma) > \tau(p, q) - \varepsilon/3$.

Since the the function $t \rightarrow L(\gamma|_{p}^{t})$ is continuous there is $a \in \gamma$, $a \neq p$ such that $L(\gamma|_{p}^{a}) < \varepsilon/3$. Similarly, we can find $b \in \gamma$, $b \neq q$ with $L(\gamma|_{b}^{q}) < \varepsilon/3$. 


Let $\mathcal{U} = I^-(a)$ and $\mathcal{V} = I^+(b)$, then given $x \in \mathcal{U}$ there is a future timelike curve $\alpha$ from $x$ to $a$. Furthermore, such $\alpha$ trivially satisfies $L(\alpha) > L(\gamma|_p^0) - \varepsilon/3$. Likewise, given $y \in \mathcal{V}$ there is a future timelike curve from $b$ to $y$ with $L(\beta) > L(\gamma|_b^q) - \varepsilon/3$.

By concatenating $\alpha$, $\gamma|_a^b$ and $\beta$ we obtain a future timelike curve $\gamma_0$ from $x$ to $y$ with

$$L(\gamma_0) > L(\gamma) - 2\varepsilon/3 > \tau(p, q) - \varepsilon$$

(3.1.1)

hence $\tau(x, y) > \tau(p, q) - \varepsilon$ for all $x \in \mathcal{U}$ and all $y \in \mathcal{V}$.

If $\tau(p, q) = \infty$ then there are timelike curves of arbitrary large length joining $p$ to $q$. Thus given $A > 0$ there is a causal curve from $p$ to $q$ with $L(\gamma) > A + 2$. Following the same procedure as above we find neighborhoods $\mathcal{U}$ and $\mathcal{V}$ around $p$ and $q$ such that $\tau(x, y) > A$, and the result follows. $\square$

### 3.1.2 Global Hyperbolicity

Continuing our program, now we extend to the spacetime-with-boundary context the concept of global hyperbolicity.

**Definition 3.12** A set $B \subset M$ is causally convex if for any pair $x, y \in B$ we have that the condition $p \in J^+(x) \cap J^-(y)$ implies $p \in B$. Thus, any causal curve with endpoints in $B$ stays in $B$.

**Definition 3.13** We say $M$ is strongly causal at $p \in M$ if $p$ has arbitrarily small causally convex neighborhoods; that is, given a neighborhood $\mathcal{U}$ of $p$, there exists a causally convex neighborhood $\mathcal{V}$ of $p$ contained in $\mathcal{U}$. $M$ is said to be strongly causal if it is strongly causal at any of its points.
**Definition 3.14** A spacetime-with-timelike-boundary \((M, g)\) is said to be globally hyperbolic if strong causality holds on \(M\) and the sets \(J^+(p) \cap J^-(q)\) are compact for all \(p, q \in M\).

We first prove that the boundary of a globally hyperbolic spacetime-with-timelike-boundary inherits this property.

**Proposition 3.15** Let \((M, g)\) be a globally hyperbolic spacetime-with-timelike-boundary. Then \(\partial M\) is a globally hyperbolic spacetime.

**Proof:** It is clear that \(\partial M\) is strongly causal, so only the compactness property needs to be shown. Thus, let \(p, q \in \partial M\), then by lemma 4.29 in \([2]\) it suffices to show \(A = J^+(p, \partial M) \cap J^-(q, \partial M)\) has compact closure in \(\partial M\).

Notice that \(J^+(p, \partial M) \subset J^+(p) \cap \partial M\). By global hyperbolicity, \(J^+(p) \cap J^-(q)\) is compact, hence closed in \(M\). Thus \(J^+(p) \cap J^-(q) \cap \partial M\) is closed as well. Then

\[
\overline{J^+(p, \partial M) \cap J^-(q, \partial M)} \subset J^+(p) \cap J^-(q) \cap \partial M \quad (3.1.2)
\]

where the upper bar indicates the closure with respect to \(M\). Therefore

\[
B = \overline{J^+(p, \partial M) \cap J^-(q, \partial M)} \quad (3.1.3)
\]

is compact, being a closed subset of the compact set \(J^+(p) \cap J^-(q)\).

Further, the closure of \(A\) in \(\partial M\) is just the intersection of the compact set \(B\) and the closed set \(\partial M\), hence it is compact as well. The proof is complete. \(\square\)
**Proposition 3.16** Let $M$ be globally hyperbolic and $p \in M$, then $J^+(p)$ is closed.

**Proof:** Let $x \in \overline{J^+(p)}$ and consider $q \in I^+(x)$. Let $\gamma$ be a future directed timelike curve from $x$ to $q$, and let $\{x_n\}$ be a sequence in $\gamma$ such that $x_{n+1} \ll x_n$ and $x_n \to x$. Thus $x_n \in J^+(p) \cap I^-(q) \subset J^+(p) \cap J^-(q)$ and hence $x \in \overline{J^+(p) \cap J^-(q)}$. Since $M$ is globally hyperbolic, the set $J^+(p) \cap J^-(q)$ is compact, hence closed, thus $\overline{J^+(p) \cap J^-(q)} = J^+(p) \cap J^-(q)$. Therefore $x \in J^+(p) \cap J^-(q) \subset J^+(p)$. □

**Proposition 3.17** Let $M$ be globally hyperbolic, then $M$ is causally simple, i.e. $J^+(A)$ is closed for all compact $A$.

**Proof:** Let $A \subset M$ be compact and let $x \in J^+(A)$. Just as in the previous result, let $\{x_n\}$ be a sequence in $I^+(x)$ converging to $x$ with $x_{n+1} \ll x_n$, hence $J^-(x_{n+1}) \subset J^-(x_n)$. Further, since $J^-(x_n)$ is closed and $A$ is compact, then $J^-(x_n) \cap A$ is compact. Notice also that $x \in J^-(x_n) \cap A$ for all $n$, therefore $\{J^-(x_n) \cap A\}$ is a nested sequence of nonempty compact sets. Thus by a standard topological result $\bigcap_n (J^-(x_n) \cap A) \neq \emptyset$, so let $p \in \bigcap_n (J^-(x_n) \cap A)$. It follows that $x_n \in J^+(p)$ for all $n$. By the previous result $J^+(p)$ is closed, so we have $x \in J^+(p)$ as desired. □

**Proposition 3.18** Let $M$ be globally hyperbolic, then $J^+(A) \cap J^-(B)$ is compact for any pair of compact sets $A, B \subset M$.

**Proof:** First notice that by the previous result $J^+(A) \cap J^-(B)$ is closed. Now, for each $a \in A$ let $a' \in M$ be a point in the chronological past of $a$. Then $A \subset \bigcup I^+(a')$. Since $A$ is compact and each set $I^+(a')$ is open, there are finitely many $a'_i$ such that $A \subset \bigcup_i I^+(a'_i)$. Likewise, there exist finitely many $b'_j$ with $B \subset \bigcup_j I^+(b'_j)$. Thus
\[ J^+(A) \cap J^-(B) \subset \bigcup_{i,j} J^+(a'_i) \cap J^-(b'_j). \] Each set \( J^+(a'_i) \cap J^-(b'_j) \) is compact, thus \( J^+(A) \cap J^-(B) \) is a closed subset of a compact set, hence it is compact. \( \square \)

### 3.1.3 Limit of curves

In this section we analyze how causal curves arise as limits of causal curves in globally hyperbolic spacetimes. Our discussion will follow the lines of chapter 3 in [2].

Since a limit of smooth curves need not be smooth, we need to extend our definition of causality to continuous curves.

**Definition 3.19** A continuous curve \( \gamma: I \to M \) on a strongly causal spacetime-with-timelike-boundary \( M \) is said to be future timelike if for any \( t_0 \in I \) there exists a \( \overline{M} \)-convex neighborhood \( U_0 \) around \( \gamma(t_0) \) and an interval \([a, b] \subset I \) around \( t_0 \) such that for all \( t, s \in [a, b] \) with \( t \leq t_0 \leq s \) we have \( \gamma(s) \in I^+(\gamma(t), U_0 \cap M) \).

Obviously, replacing \( I^+ \) by \( J^+ \) in the previous definition gives us the notion of future causal continuous curve. The time dual concepts are defined in a similar way.

**Remark 3.20** Because of strong causality this concept is well defined, i.e. this definition actually does not depend on the extension \( \overline{M} \).

**Remark 3.21** Causal relations are preserved. In other words, \( q \in I^+(p) \) if and only if there exists a continuous future timelike curve from \( p \) to \( q \). A similar statement holds for \( J^+(p) \).

**Definition 3.22** Let \( \{\gamma_n\} \) be a sequence of curves on \( M \). A curve \( \gamma \) is a limit of \( \{\gamma_n\} \) if there is a subsequence \( \{\gamma_m\} \) such that for all \( p \in \gamma \), each neighborhood of \( p \) intersects all but finitely many of the \( \gamma_m \)'s. Such subsequence is said to distinguish \( \gamma \).
As can be shown, a given sequence \( \{\gamma_n\} \) may have many different limit curves, or no limit curve at all. However, in the open spacetime context the existence of such a limit curve is guaranteed when \( \{\gamma_n\} \) has an accumulation point (for a proof, consult proposition 3.31 in [2]). A similar result holds in the spacetime-with-timelike-boundary case.

**Lemma 3.23** Let \( \{\gamma_n\} \) be a sequence of future inextendible causal curves in \( M \) having an accumulation point \( p \), then there exists a limit curve \( \gamma \) of \( \{\gamma_n\} \). Further, \( \gamma \) is future inextendible causal and \( p \in \gamma \).

**Proof:** Consider an extension \( \overline{M} \) and apply the aforementioned Limit Curve Lemma to the sequence \( \{\gamma_n\} \) to obtain a future inextendible limit curve \( \gamma : \mathbb{R} \to \overline{M} \) passing through \( p \). Finally, since \( M \) is a closed subset of \( \overline{M} \) and each point on \( \gamma \) is a limit point of a sequence in \( M \) we have \( \gamma \subset M \). \( \square \)

The following is a straightforward modification of the analogous results in [2] to fit in the spacetime-with-timelike-boundary case.

**Proposition 3.24** Let \( M \) a be globally hyperbolic spacetime-with-timelike-boundary. Let \( \{p_n\} \) and \( \{q_n\} \) be two sequences converging to \( p \) and \( q \), respectively with \( p \neq q \). Further assume that \( q_n \in J^+(p_n) \) for all \( n \). If \( \{\gamma_n\} \) is a sequence of future directed causal curves from \( p_n \) to \( q_n \), then \( \{\gamma_n\} \) has a future directed causal limit curve \( \gamma \) from \( p \) to \( q \).

**Proof:** First notice \( q \in J^+(p) \). Suppose otherwise, then since \( J^+(p) \) is closed there is a neighborhood \( U \) of \( q \) contained in \( M - J^+(p) \). Let \( a, b \in U \) such that \( a \in I^+(q) \) and \( b \in I^+(a) \). Since \( I^-(a) \) is open, \( q \in I^-(a) \) and \( q_n \to q \) we have \( q_n \in I^-(a) \) for all
\( n \geq N \), hence \( p_n \in J^-(a) \), \( n \geq N \) as well. But, since \( J^-(a) \) is closed and \( p_n \to p \) we must have \( p \in J^-(a) \) therefore \( b \in J^+(p) \), a contradiction.

Consider now an extension \((\overline{M}, \overline{g})\) of \((M, g)\). For each \( x \in M \) let \( \mathcal{U}'_x \) be a precompact \( \overline{M} \)-convex neighborhood around it. Due to strong causality there is a causally convex neighborhood \( \mathcal{U}_x \) around \( x \) satisfying \( \mathcal{U}_x \subset \mathcal{U}'_x \cap M \). Cover \( J^+(p) \cap J^-(q) \) with a collection \( \{ \mathcal{U}_i \} \) of such neighborhoods. Then by compactness \( J^+(p) \cap J^-(q) \) can be covered by finitely many of them, say \( \{ \mathcal{U}_i \} 1 \leq i \leq n \).

Let \( h \) be a complete Riemannian metric on \( \overline{M} \). Every smooth curve \( \sigma \) on a normal neighborhood \( \mathcal{U}' \) satisfies a Lipschitz condition, thus \( h \)-arc length is bounded in any precompact convex neighborhood \( \mathcal{U}' \). Let \( B_i > 0 \) be such that \( L_h(\sigma) \leq B_i \) for all causal \( \sigma: I \to \overline{U}_i \) and set \( B = \sum B_i, U = \bigcup_i \mathcal{U}_i \). Since each \( \mathcal{U}_i \) is causally convex we have \( L_h(\sigma) \leq B \) for all causal \( \sigma: I \to U \).

Let \( \gamma_n: [0, a_n] \to M \) be a future causal curve from \( p_n \) to \( q_n \) parameterized with respect to \( h \)-arc length and let \( \overline{\gamma}_n: [0, \infty) \to \overline{M} \) be a future inextendible extension of it. By applying the limit curve lemma to \( \overline{\gamma}_n \) with accumulation point \( p \) we get a (continuous) causal curve \( \gamma: [0, \infty) \to \overline{M} \) with \( \gamma(0) = p \) and a subsequence \( \{ \overline{\gamma}_m \} \) of \( \{ \overline{\gamma}_n \} \) converging uniformly to \( \gamma \) in compact subsets of \([0, \infty)\).

First notice that \( \gamma \) passes through \( q \): Since \( \gamma_m \subset U \) we have \( 0 < a_m = L(\gamma_m) \leq B \). Thus \( q_m \to q \) together with uniform convergence on \([0, B]\) yield \( q = \lim_m q_m = \lim_m \gamma_m(a_m) = \gamma(a) \) for some \( a \in (0, B] \). \( \square \)

In order to relate causality and the Lorentzian distance function, we need to consider a kind of convergence of curves that guarantees favorable properties of the arc length functional.
We topologize the space $C(M)$ of all continuous causal curves in $M$ in the following way: let $U, V, M$ be open sets in $M$, and let us denote by $C_M(U, V)$ the set of all continuous future causal curves contained in $M$ whose initial points lie $U$ and its final points lie in $V$. The collection of all such sets form a basis for a topology in $C(M)$.

Definition 3.25 The topology on $C(M)$ defined above is called the $C^0$ topology of curves in $M$.

The two different types of convergence we have been discussing are closely related as the following result shows. For a proof, refer to proposition in [2] and apply the same type of modifications we used in propositions 3.23 and 3.24.

Proposition 3.26 Let $M$ be a strongly causal and assume $\gamma: [a, b] \to M$ is a limit curve of the sequence of causal curves $\{\gamma_n\}$. If $\gamma_n(a) \to \gamma(a)$ and $\gamma_n(b) \to \gamma(b)$ then there is a subsequence $\{\gamma_m\}$ that converges to $\gamma$ on the $C^0$ topology of curves.

To extend the notion of length to continuous causal curves we first observe that any smooth causal curve can be approximated in the $C^0$ topology of curves by a broken causal geodesic, thus its length can be viewed as the limit of the lengths of such broken geodesics.

To be more precise, let $\gamma: I \to M$ be a future continuous causal curve from $p$ to $q$ and consider a collection of points $\Upsilon := \{x_i\}$, $0 \leq i \leq k$ on $\gamma$ such that $x_0 = p, x_k = q$ and there exist convex neighborhoods $\{\overline{U}_i\}$ in an extension $\overline{M}$ with the property that there is a geodesic segment $\gamma_i \subset \overline{U}_i$ joining $x_i$ to $x_{i+1}$. If we denote by $\gamma_\Upsilon$ the
concatenation of the $\gamma_i$’s then we define the length of $\gamma$ by

$$L(\gamma) = \inf_{\mathcal{Y}} L(\gamma_{\mathcal{Y}}). \quad (3.1.4)$$

Notice as well that the infimum on the right hand side is independent of the choice of the extending spacetime $\overline{M}$.

It is known that the arc length functional is upper semicontinuous in the $C^0$ topology of curves in strongly causal spacetimes ([2, 54]). Only slight changes on the proof of theorem 7.5 in [45] are needed to prove the analogous result for spacetimes-with-timelike-boundary.

**Proposition 3.27** The arc length functional $L$ is continuous on strongly causal spacetimes-with-timelike-boundary. That is, for any sequence of causal curves $\{\gamma_n\}$ converging to $\gamma$ in the $C^0$ topology of curves we have $L(\gamma) \geq \limsup L(\gamma_n)$.

We finish this section with a couple of useful results regarding the Lorentzian distance function and maximal curves in globally hyperbolic spacetimes-with-timelike-boundary.

**Proposition 3.28** Let $M$ be a globally hyperbolic spacetime-with-timelike-boundary, $S \subset \partial M$ be a compact submanifold and $p \in M$ such that $p \in J^+(S)$. Then there exists a causal curve $\gamma$ from $S$ to $p$ such that $\tau(S, p) = L(\gamma)$.

*Proof:* For each $n \in \mathbb{N}$ there is $\gamma_n$ from $S$ to $p$ such that $\tau(S, p) - 1/n < L(\gamma_n)$. Since $S$ is compact, the sequence $\{\gamma_n(0)\}$ has a subsequence $\{\gamma_m(0)\}$ converging to $x \in S$. By proposition 3.24 there is a causal limit curve $\gamma$ of the sequence $\{\gamma_m\}$ joining $x$ and $p$. Further, in light of proposition 3.26 we may assume $\{\gamma_m\}$ converges to $\gamma$ in the $C^0$
topology of curves. Thus by the upper semicontinuity of $L$ we have

$$\tau(S, p) \leq \limsup L(\gamma_m) \leq L(\gamma).$$  \hspace{1cm} (3.1.5)

It follows $\tau(S, p) = L(\gamma)$. \square

**Proposition 3.29** Let $M$ be a globally hyperbolic spacetime-with-timelike-boundary. Then $\tau$ is continuous on $M$.

**Proof**: We know already $\tau$ is lower semicontinuous (proposition 3.11), thus we only need to check $\tau$ is upper semicontinuous.

We proceed by contradiction. Let $p, q \in M$ and $\{p_n\}, \{q_n\}$ be sequences in $M$ converging to $p$ and $q$ respectively, and assume there is $\varepsilon > 0$ such that $\tau(p_n, q_n) \geq \tau(p, q) + \varepsilon$ for all $n$. Thus $\tau(p_n, q_n) > 0$ and hence $q_n \in J^+(p_n)$. Therefore, by proposition 3.28 above, there exist a future causal curve $\gamma_n$ from $p_n$ to $q_n$ such that $L(\gamma_n) = \tau(p_n, q_n)$. By propositions 3.24 and 3.27 there is a future causal curve $\gamma$ from $p$ to $q$ satisfying $L(\gamma) \geq \limsup L(\gamma_n)$. Then we find

$$L(\gamma) \geq \limsup L(\gamma_n) = \limsup \tau(p_n, q_n) \geq \tau(p, q) + \varepsilon$$  \hspace{1cm} (3.1.6)

which is a clear contradiction. \square

### 3.1.4 Covering Spacetimes

Our main goal in this section will be to construct a globally hyperbolic spacetime-with-timelike-boundary $\pi: \tilde{M} \to M$ having an isometric copy of a fixed component of $\partial M$. In order to attain this goal, we need to prove a monodromy type lemma first:
Lemma 3.30 Let $\pi: \hat{M} \to M$ a topological covering map of a globally hyperbolic spacetime-with-timelike-boundary $M$. Let $\gamma_n: [0, t_n] \to \hat{M}$ be a sequence of curves satisfying the following properties:

1. The sequence $\{\gamma_n(0)\}$ converges to $x \in \hat{M}$.
2. The curves $\alpha_n := \pi \circ \gamma_n$ are future causal.
3. The sequence $\{\alpha_n(t_n)\}$ converges to $q \in M$, $q \neq \pi(x)$.

Then a subsequence $\{\gamma_m(t_m)\}$ converges to $y \in \pi^{-1}(q)$.

Proof: Let us fix some notation: let $x_n = \gamma_n(0)$, $y_n = \gamma_n(t_n)$, $p_n = \alpha_n(0)$, $q_n = \alpha_n(t_n)$ and $p = \pi(x)$.

Let $U$ be an evenly covered neighborhood around $p$ and let $\hat{U}$ be the unique component of $\pi^{-1}(U)$ containing $x$. Since $\pi|_{\hat{U}}: \hat{U} \to U$ is a homeomorphism and $x_n \to x$ we have $p_n \to p$, thus by proposition 3.24 there is a causal limit curve $\alpha: [0, b] \to M$ of the sequence $\{\alpha_n\}$ joining $p$ to $q$. Let $\gamma$ be the unique lift of $\alpha$ to $\hat{M}$ with base point $x$. We shall show below that a subsequence of $\{y_n\}$ converges to $y = \gamma(b)$.

Next let us consider an evenly covered neighborhood $U_s$ around each $\alpha(s) = \alpha(s) \in V_s$. By compactness, there will be finitely many such $V_i$, $i = 1, \ldots, k$ covering $\alpha$. Further, we can choose $V_i$ in such a way that $p \in V_1$, $q \in V_k$ and $V_i$ only intersects $V_{i-1}$ and $V_{i+1}$. Now, let $a_0 = p$, $a_k = q$ and $a_i \in \alpha \cap V_i \cap V_{i+1}$ for each $1 \leq i \leq k-1$. Notice $\pi^{-1}(a_i) \cap \gamma$ consist of a single point, since $\alpha$ would contain closed loops otherwise. Hence let us define $b_i := \pi^{-1}(a_i) \cap \gamma$ and let $\hat{U}_i$ be the unique component of $\pi^{-1}(U_i)$ containing $b_{i-1}$.
Now we proceed by induction on $i$. As our induction hypothesis, we’ll assume there is a subsequence $\{\alpha_{i-1,m}\}$ of $\{\alpha_n\}$ converging to $\alpha$ and points $b_{i-1,m} \in \gamma_{i-1,m}$ such that $b_{i-1,m} \to b_{i-1}$.

Since $\mathcal{V}_i \cap \mathcal{V}_{i+1}$ is open and $\alpha$ is a limit curve of $\alpha_{i,m}$, there is a subsequence $\{a_{i,m}\} \subset \mathcal{V}_i \cap \mathcal{V}_{i+1}$ converging to $a_i$ such that $a_{i,m} \in \alpha_{i,m}$. By further taking a subsequence we may assume $a_{i,m} \in J^+(a_{i-1,m})$ since otherwise strong causality at $a_{i-1}$ would not hold. Thus since $\mathcal{V}_i$ is causally convex we have that the entire segment of $\alpha_{i,m}$ from $a_{i-1,m}$ to $a_{i,m}$ is contained in $\mathcal{V}_i$, hence in $\mathcal{U}_i$.

Since $b_{i-1,m} \to b_{i-1}$ then $\hat{\mathcal{U}}_i$ will contain $b_{i-1,m}$ for all $m$ large. Since $\pi|_{\hat{\mathcal{U}}_i}$ is a homeomorphism and $\gamma_{i,m}$ is the unique lift of $\alpha_{i,m}$ with base point $x_{i,m}$ we conclude $b_{i,m} := \pi^{-1}(a_{i,m}) \cap \gamma_{i,m} \in \hat{\mathcal{U}}_i$ and $b_{i,m} \to b_i$.

We can now repeat this process on each $a_i$ and at the end, causality and the unique lifting property will guarantee $b_{k,m} = y_{k,m}$ and $b_k = b$ as claimed. □

**Remark 3.31** As can be seen from the proof, this result holds for open spacetimes as well.

**Remark 3.32** Notice that if $\hat{M}$ happens to be a spacetime in which $\pi$ is a local isometry, we have that the curve $\gamma$ joining $x$ to $y$ is causal, thus $y \in J^+(x)$.

**Proposition 3.33** Let $M$ be a globally hyperbolic spacetime-with-timelike-boundary and let $\pi: \hat{M} \to M$ be a (topological) covering map. Then there is a smooth structure on $\hat{M}$ in which $\hat{M}$ is a globally hyperbolic spacetime-with-timelike-boundary and $\pi$ is a local isometry.
Proof: Let \( p \in \mathcal{M} \) and consider \( \hat{p} \in \pi^{-1}(p) \). Let \( \hat{U} \) be a neighborhood of \( \hat{p} \) such that \( \pi|_{\hat{U}} \) is a homeomorphism and consider a coordinate chart \((\mathcal{V}, \phi)\) around \( p \). Then it is clear that the collection \( \{(\hat{U} \cap \pi^{-1}(\mathcal{V}), \phi \circ \pi)\} \) is a smooth atlas for \( \hat{M} \). Notice as well that \( \pi^{-1}(\partial \mathcal{M}) = \partial \hat{M} \).

By taking the pullback metric \( \pi^*g \) we turn \( \hat{M} \) into a lorentzian manifold in which \( \pi \) is a local isometry, hence \( \partial \hat{M} \) is timelike. The orientation and time orientation of \( M \) lift as well to make \( \hat{M} \) a spacetime-with-timelike-boundary.

We now prove \( \hat{M} \) is strongly causal. Thus, let \( x \in \hat{M} \) and \( \hat{U} \) a neighborhood containing \( x \). Let \( p = \pi(x) \) and choose a neighborhood \( \mathcal{V} \) of \( p \) such that \( \mathcal{V} \subset \pi(\hat{U}) \) and it is evenly covered by \( \pi \). By further reducing \( \mathcal{V} \) if necessary, we can assume the component \( \hat{V} \) of \( \pi^{-1}(\mathcal{V}) \) containing \( x \) is a subset of \( \hat{U} \). Since \( M \) is strongly causal, there is a causally convex neighborhood \( \mathcal{W} \) of \( p \) contained in \( \mathcal{V} \), thus \( \pi^{-1}(\mathcal{W}) \cap \mathcal{V} \) is a causally convex neighborhood of \( x \).

Finally, let us show the sets \( J^+(\hat{p}, \hat{M}) \cap J^-(\hat{q}, \hat{M}) \) are compact. Thus let us consider a sequence \( \{\hat{a}_n\} \) in \( J^+(\hat{p}, \hat{M}) \cap J^-(\hat{q}, \hat{M}) \) and let \( \hat{\alpha}_n, \hat{\beta}_n \) be causal curves joining \( \hat{p} \) to \( \hat{a}_n \) and \( \hat{a}_n \) to \( \hat{q} \), respectively. Now denote by non hatted symbols the corresponding objects on the base spacetime \( M \). Thus the sequence \( \{a_n\} \) is contained in the compact set \( J^+(p) \cap J^-(q) \), as a consequence, there is a subsequence \( \{a_m\} \) converging to \( x \in J^+(p) \cap J^-(q) \).

Now we are in a situation in which we can apply lemma 3.30. We first apply it to the sequence of curves \( \{\hat{\alpha}_m\} \) to obtain a subsequence, also denoted by \( \{\hat{a}_m\} \) for brevity, converging to some \( \hat{x} \in \pi^{-1}(x) \) such that \( \hat{x} \in J^+(\hat{p}, \hat{M}) \). Then an application of the time dual version to the sequence of reverse curves \( \{\hat{\beta}_m^{-}\} \) will yield a subsequence
\{\hat{b}_k\} of \{\hat{a}_m\} converging to \hat{y} \in J^-(\hat{q}, \hat{M}). But since \{a_m\} is already a convergent sequence we must have \hat{x} = \hat{y}. The proof is complete. \(\square\)

**Proposition 3.34** Let \(M\) be a globally hyperbolic spacetime-with-timelike-boundary with \(\partial M\) connected. Then there exists a covering map \(\pi: \hat{M} \to M\) with \(\hat{M}\) globally hyperbolic such that

1. \(\hat{M}\) contains a copy of \(\partial M\), i.e. there is a component \(S\) of \(\pi^{-1}(\partial M)\) such that \(\pi|_S\) is an isometry onto \(\partial M\).

2. The induced map \(\iota_*: \Pi_1(S, x) \to \Pi_1(\hat{M}, x)\) is surjective.

**Proof:** For any \(p \in M\) and a curve \(\gamma\) from \(\partial M\) to \(p\), let us denote by \([\gamma]\) the homotopy class of \(\gamma\) modulo \(\partial M\) and \(p\). Define \(\hat{M}\) as the set of all pairs \((p, [\gamma])\) and given a simply connected neighborhood \(V\) of \(p\), let \((p, [\gamma], V) \subset \hat{M}\) denote the set of all pairs of the form \((q, [\gamma \ast \sigma])\) where \(q \in V\) and \(\sigma\) is a curve in \(V\) joining \(p\) to \(q\). Endow \(\hat{M}\) with the topology generated by the collection \((q, [\gamma], V)\).

In this context [42] shows that \(\hat{M}\) is a topological covering space of \(M\) such that \(\iota_*: \Pi_1(S, x) \to \Pi_1(\hat{M}, x)\) is surjective. Hence by proposition [3.33] above we conclude \(\hat{M}\) is a globally hyperbolic spacetime-with-timelike boundary as well.

Moreover, it can be shown (see [30]) that given \(p \in \partial M\), the component \(S\) of \(\pi^{-1}(\partial M)\) containing \((p, [1_p])\) is homeomorphic to \(\partial M\) via \(\pi\). Further, since \(\pi|_S\) is both a homeomorphism and a local isometry, it is an isometry from \(S\) onto \(\partial M\). \(\square\)

**Remark 3.35** The spacetime-with-timelike-boundary \(\hat{M}\) can be characterized as the largest spacetime-with-timelike-boundary covering \(M\) containing an isometric copy of \(\partial M\) (see proposition 2.4 in [30]).
Remark 3.36 Notice also that any curve in $\hat{M}$ with endpoints in $S$ is fixed endpoint homotopic to a curve in $S$. This property is closely related to the concept of Topological Censorship that will be explored in the next section.

3.1.5 Smooth null hypersurfaces

We finish our discussion on spacetimes-with-timelike-boundary by proving a couple of technical lemmas on smooth null surfaces.

Lemma 3.37 Let $M$ be a strongly causal spacetime-with-timelike-boundary and let $\gamma: [0, a) \to M$ be a future null geodesic orthogonal to a spacelike surface $S \subset \partial M$ such that $\gamma \cap \partial M = \{\gamma(0)\}$. Let $0 < T < a$, then there exists a neighborhood $U$ of $p := \gamma(0)$ in $S$ and a future and normal null vector field $X$ along $U$ such that $X_p = \gamma'(0)$ and all the null geodesics with initial velocity $X_q$ remain within a neighborhood of $\gamma|_0^T$ and intersect $\partial M$ only at the base point $q \in U$.

Proof: Let us consider an extension $\overline{M}$ of $M$ and let us denote by $\exp^\perp$ the normal exponential map to $S$ in $\overline{M}$. Then PDE theory guarantees the existence of a neighborhood $V$ of $p$ in $S$ and a null future vector field $Y$ along $V$ extending $\gamma'(0)$ such that $\gamma_q(t) := \exp^\perp(tX_q)$ is defined up to $t = T$ for all $q \in V$. Moreover, since geodesics depend smoothly on initial conditions, the geodesic segments $\gamma_q|_0^T$ remain within an open neighborhood of $\gamma|_0^T$. It remains to show that we can choose $V$ in such a way that all the geodesics $\gamma_q$ stay in $M$ and touch $\partial M$ only at $q$.

Assume the contrary, then there would be a sequence $\{p_n\}$ converging to $p$ such that the geodesics $\gamma_n := \gamma_{p_n}$ meet $\partial M$ at a time $t_n \in (0, T]$. By compactness of $[0, T]$
there is a subsequence \( \{t_m\} \) converging to \( T_0 \in [0,T] \). Since \( \{p_m\} \) converges to \( p \) we have \( x_m := \gamma_m(t_m) \to \gamma(T_0) \). Notice that \( \gamma(T_0) \in \partial M \) since \( \partial M \) is closed, thus \( T_0 = 0 \) since by hypothesis \( \gamma \) intersects \( \partial M \) only at \( p \).

Let \( \mathcal{N}_0 \) be a normal neighborhood of \( S \) in \( \overline{M} \). We proceed to show that \( p \) has no causally convex neighborhood contained in \( \mathcal{N}_0 \cap M \), and hence \( M \) can not be strongly causal at \( p \). Thus, let \( U \subset \mathcal{N}_0 \cap M \) be a \( M \)-neighborhood of \( p \) and let us consider \( U \), a \( \partial M \)-neighborhood of \( p \) such that \( D(U, \partial M) \) is globally hyperbolic with Cauchy surface \( S \cap U \), and a \( \overline{M} \)-neighborhood \( \mathcal{U}_0 \), \( \mathcal{U}_0 \cap M \subset U \) with \( \mathcal{U}_0 \cap \partial M \subset U \) and such that \( S \cap \mathcal{U}_0 \) is acausal in \( U \) (for a construction of such neighborhoods \( U \) and \( \mathcal{U}_0 \), refer to the proof of proposition 3.9).

Since both sequences \( \{x_m\} \) and \( \{p_m\} \) converge to \( p \), we have that \( \mathcal{U}_0 \cap M \) contains \( x_m \) and \( p_m \) for all \( m \geq N \). Notice that \( \gamma_N \) can not intersect \( \partial M \) to the past of \( S \cap U \) because of the acausality of \( S \cap \mathcal{U}_0 \) in \( \mathcal{U}_0 \), therefore we must have \( x_N \in I^+(S \cap U, U) \).
It follows \( \tau_{\overline{N}}(S,x_N) > 0 \), thus \( \gamma_N \) fails to realize Lorentzian distance in \( \mathcal{N}_0 \) between \( x_N \) and \( S \). However, by properties of normal neighborhoods with respect to \( \exp^\perp \), the initial segment of \( \gamma_N \cap \mathcal{N}_0 \) is distance realizing in \( \overline{N} \). Therefore \( \gamma_N|_{x_N^0} \not\subset \mathcal{N}_0 \), hence \( \gamma_N|_{x_N^0} \) intersects \( \mathcal{U} \) in a disconnected set. The proof is complete. \( \square \)

**Lemma 3.38** Let \( M \) be a globally hyperbolic spacetime-with-timelike-boundary and \( \gamma : [0,a) \to M \) be a null geodesic that meets \( \partial M \) only at \( p := \gamma(0) \). Further, let \( \partial_0 \) be the component of \( \partial M \) containing \( p \) and let \( S \) be a Cauchy surface of \( \partial_0 \) passing through \( p \); and assume \( \gamma(t) \in J^+(S) - I^+(S) \forall t \in [0,a) \). Then for every \( T \in (0,a) \) there exists a smooth null hypersurface \( N \subset M \) for which \( S \cap N \) is a spacelike cut
and $\gamma|_0^T$ is one of its null generators.

Proof: First notice that by propositions 3.8 and 3.9, $\gamma$ is a null geodesic that meets $S$ orthogonally. Furthermore, by proposition 3.10 we know the segment $\gamma|_0^T$ is focal point free.

Next, let us consider a neighborhood $U \subset S$ of $p$ and a null normal vector field $X$ along $U$ as established by the previous lemma and consider a neighborhood $V$ of $p$ with compact closure such that $\overline{V} \subset U$. Let us define $\Phi: U \times [0,a) \to M$ by $\Phi(q,t) = \exp^\perp(tX_q)$, then $t \mapsto \Phi(q,t)$ is just the null geodesic normal to $S$ with initial velocity $X_q$.

Let $t \in [0,T]$. Since $\gamma(t)$ is not a focal point of $S$ along $\gamma$, $\exp^\perp$ is not singular at $t\gamma'(0)$ thus in virtue of theorem B.7 and the inverse function theorem, there exists a neighborhood $U_t$ of $t\gamma'(0)$ in the normal null cone $\mathcal{N}(S)$ such that $\exp^\perp|_{U_t}$ is a diffeomorphism, hence injective.

Now we show there is a neighborhood $U_t \subset \Phi(\overline{V} \times [0,T])$ of $\gamma(t)$, such that $\Phi(q_1,t_1) = \Phi(q_2,t_2)$ implies $(q_1,t_1) = (q_2,t_2)$. The existence of such a neighborhood for $t = 0$ is assured by the existence of normal neighborhoods (see lemma 7.26 on [16]), so let us consider now the case $t > 0$. Assume no such $U_t$ exists, then there would be a sequence of points $\{x_n\}$ converging to $\gamma(t)$ and two sequences $\{(p_n,t_n)\}$, $\{(q_n,s_n)\}$ such that $(p_n,t_n) \neq (q_n,s_n)$ and $\Phi(p_n,t_n) = x_n = \Phi(q_n,s_n)$. By compactness of $\overline{V}$ there are subsequences $\{p_m\}$, $\{q_m\}$ converging to $p'$ and $q'$ respectively. Then, since $x_n \in J^+(p_n)$ and $p' \neq \gamma(t)$, by proposition 3.24 there is a limit causal curve $\alpha$ joining $p'$ to $\gamma(t)$. Moreover, since $\alpha \subset \Phi(U \times [0,T])$ we know $\alpha$ only meets $\partial M$ at $\alpha(0)$.
Hence, since $\gamma(t) \in J^+(S) - I^+(S)$ propositions 3.8 and 3.9 guarantee that $\alpha$ is a null geodesic normal to $S$. If $\alpha \neq \gamma$ then $\gamma(t + \varepsilon) \in I^+(S)$ by 1.30. Hence $\alpha = \gamma$ and therefore $p_m \to p$. Thus by further taking a subsequence we can assume $X_{p_m} \to \gamma'(0)$. A similar argument applied to the sequence $\{q_m\}$ shows $q_m \to q$ and $X_{q_m} \to \gamma'(0)$. This in turn implies that $\exp^\perp$ is not injective in any neighborhood of $t\gamma'(0)$ in $\mathcal{H}(S)$. A contradiction.

Choose now $U_t$ so that $U_t \subset \exp^\perp(U_t)$ and consider neighborhoods $V_t$ of $\gamma(t)$ such that $\overline{V_t} \subset U_t$. Since $V \times [0, T]$ is compact and $\Phi$ is continuous, there are finitely many $U_i$ such that $\Phi(V \times [0, T]) \subset \cup_i V_i$. Let $W' = \bigcup_i \Phi^{-1}(U_i)$ and $W = \bigcup_i \overline{V_i}$, then $\Phi|_{W' \times [0, T]}$ is a one to one local diffeomorphism, hence a diffeomorphism onto its image. In fact, by the compactness of $\overline{W}$ we conclude $\Phi|_{W \times [0, T]}$ is an embedding. Therefore $N = \Phi(W \times [0, T])$ is the desired null hypersurface. □

**Remark 3.39** Notice that the affine parameter on the null generators of $N$ can be rescaled so that the slices $\Phi(W \times \{s\})$ are all diffeomorphic to $W \subset S$. Notice as well that such $N$ always exists locally. Indeed, because of properties of the null normal exponential map, given $v \in T_pM \cap \mathcal{H}(S)$ we can always find a $S$-neighborhood $U$ of $p$, a null normal vector field $V$ along $U$ extending $v$ and $T_0 > 0$ small enough such that $\exp^\perp(tV)$, $t \in [0, T_0]$ defines a smooth null surface.

Let $S \subset \partial M$ be spacelike hypersurface of $\partial M$. Then there are, up to positive scaling, two future directed null normal vector fields along $S$, one of which is inward pointing while the the other is outward pointing. Let $K_1$ and $K_2$ be these vector fields, with $K_1$ inward pointing. For $i = 1, 2$ define the null Weingarten map of $S$ relative to
\( K_i \) by \( b_i: T_pS \to T_pS, b_i(X) := \tan \nabla_X K_i, \) where \( \tan: T_pM \to T_pS \) is the standard projection. The corresponding \textit{null second fundamental forms} \( B_i: T_pS \times T_pS \to \mathbb{R} \) are defined by \( B_i(X,Y) = g(b_i(X),Y). \) Finally the \textit{null expansions of} \( S, \) are given by \( \theta_i = \text{Tr}(b_i). \)

**Definition 3.40** We say \( S \) is null convex if \( B_1 \) is negative definite and \( B_2 \) is positive definite. We say \( S \) is weakly null convex if \( \theta_1 < 0 \) and \( \theta_2 \geq 0. \)

**Remark 3.41** Let \( N \subset M \) is a null surface that meets \( \partial M \) transversally at \( p \in \partial M. \) Then \( S := N \cap \partial M \) is spacelike near \( p. \) Let \( K \) be the null vector field \( K \) associated to \( N. \) If \( K|_S \) agrees with \( K_i \) then by equation \([1.4.1]\) we readily see that the null expansion \( \theta_K \) of \( K \) agrees with \( \theta_i \) at \( p. \)

### 3.2 The Principle of Topological Censorship

The singularity theorems of Gannon \[26\] and Lee \[38\] establish that non trivial spatial topology on a spacetime leads to the formation of singularities. More precisely, if a spacetime \( M \) satisfying the null energy condition admits an asymptotically flat Cauchy surface \( S \) with \( \pi(S) \neq 0 \) then \( M \) is null geodesically incomplete.

On the other hand, one expects that our universe does not present “naked” singularities. In other words, the physical processes responsible of the formation of singularities (e.g. gravitational collapse) would also induce the formation of an event horizon hiding the singularity from view.
Thus, in view of the aforementioned singularity theorems we should expect that the region $\mathcal{D}$ of spacetime outside all black holes and white holes to be topologically trivial. This is the essence of the Principle of Topological Censorship. Following [18], we state the Principle of Topological Censorship in the following way:

**Principle of Topological Censorship (PTC):** Let $(\tilde{M}, \tilde{g})$ be a spacetime admitting a conformal completion $(M, g)$. Then every causal curve on $M$ whose initial and final endpoints lie on $\mathcal{J}$ is fixed endpoint homotopic to a curve on $\mathcal{J}$.

Topological censorship has proved to be an important tool to relate the topology at infinity with the topology of black hole horizon. Actually, under the assumption of topological censorship the famous result by S. Hawking on the spherical topology of stationary black holes [31] has been generalized in different directions [12, 34, 21].

### 3.3 PTC on spacetimes with timelike boundary

In the seminal work of J. Friedman, K. Schleich and D. Witt [18], the PTC is showed to hold for asymptotically flat spacetimes satisfying an energy condition. Later developments extended the field of application of the PTC to globally hyperbolic spacetimes with a *timelike* conformal structure [23, 7].

It is of special interest the quasilocal version of the PTC presented in [22]. In this context, the conformal boundary $\mathcal{J}$ is replaced by a timelike boundary $\partial M \approx S^{n-2} \times \mathbb{R}$ with null convex Cauchy surfaces. PTC then follows as a consequence of the following theorem:
Theorem 3.42 Let $(M, g)$ be a globally hyperbolic spacetime-with-timelike-boundary which obeys the null energy condition, such that each component $T_\alpha$ of $\partial M$ has a spacelike Cauchy surface $S_\alpha \approx S^{n-1}$. Further, assume each of these Cauchy surfaces is null convex and acausal in $M$. Then for all $\alpha \neq \beta J^+(T_\alpha) \cap J^-(T_\beta) = \emptyset$.

In physical terms, this result implies the non traversability of wormholes whose “mouths” are weakly null convex spheres in globally hyperbolic spacetimes satisfying the null energy condition. In fact, if $S$ is a Cauchy surface of spacetime and $N \subset S$ is such a wormhole, then $M = N \times \mathbb{R}$ is a spacetime-with-timelike-boundary satisfying all the hypotheses of theorem 3.42.

In this section we will improve the main result on [22] in two different ways. More precisely, we will show PTC holds in the finite infinity setting when the spatial sections of $\partial M$ are compact and weakly null convex. In particular, no assumption on its homotopy type is made. Moreover, we fill in a technical gap in the proof of theorem 3.42.

3.3.1 Fastest causal curves

A fundamental problem related to the PTC consists in proving the existence of “fastest” curves communicating two different boundary components of $M$. In other words, if we assume $\partial_\alpha$ and $\partial_\beta$ are two components of $\partial M$ with $J^+(S) \cap \partial_\beta \neq \emptyset$, where $S$ is a Cauchy surface of $\partial_\alpha$; then we want to find a null geodesic $\eta \subset \partial I^+(S)$ connecting $S$ to $\partial_\beta$ at the earliest time possible.

Notice that by proposition 3.15 each component of $(\partial M, \iota^*g)$ is a globally hyperbolic spacetime in its own right. Hence each component of $\partial M$ has a Cauchy surface and a smooth time function.
To be more precise, let us denote by $\Sigma_t$ the level sets of the time function $t$ of $\partial \beta$ and let $B = \{t \in \mathbb{R} \mid J^+(S) \cap \Sigma_t \neq \emptyset\}$, hence $B \neq \emptyset$. Then if $t_0 := \inf B$ exists, we would like to be able to construct a null geodesic $\eta \subset \partial I^+(S)$ joining $S$ to $\Sigma_{t_0}$. Such a geodesic will be called a fastest geodesic from $S$ to $\partial \beta$.

Before proceeding any further, let us prove some useful lemmas.

**Lemma 3.43** Suppose $t_0 \neq -\infty$ and $\gamma$ is a causal curve joining $S$ to $\Sigma_{t_0}$ that intersects $\partial M$ only at its endpoints. Then $\gamma$ is a null geodesic contained in $\partial I^+(S)$ that meets both $\Sigma_{t_0}$ and $S$ orthogonally.

**Proof:** Let $q \in \Sigma_{t_0}$ be the future endpoint of $\gamma$. First notice that $q \notin I^+(S)$. In fact, if this was not the case by proposition 3.5 we would have $I^+(S) \cap \Sigma_t \neq \emptyset$ for $t$ slightly less than $t_0$. Moreover, this fact and proposition 3.6 guarantee that no pair of points of $\gamma$ can be chronologically related. Therefore $\gamma \subset J^+(S) - I^+(S)$.

Further, since $J^+(S)$ is closed in virtue of proposition 3.17 we have $J^+(S) \supset \overline{I^+(S)}$. On the other hand $J^+(S) \subset \overline{I^+(S)}$ by proposition 3.7 then $J^+(S) = \overline{I^+(S)}$ so

$$\partial I^+(S) = \overline{I^+(S)} - I^+(S) = J^+(S) - I^+(S),$$

(3.3.1)

thus $\gamma \subset \partial I^+(S)$ as claimed.

Finally, since $\gamma$ only meets $\partial M$ at endpoints we can apply propositions 3.8 and 3.9 above to conclude $\gamma$ is a null geodesic meeting $\Sigma_{t_0}$ and $S$ orthogonally. $\Box$

**Lemma 3.44** Let $M$ be a globally hyperbolic spacetime-with-timelike-boundary satisfying the null energy condition and $\partial_\alpha$, $\partial_\beta$ two different connected components of $\partial M$. Further, let $S_\alpha \subset \partial_\alpha$, $S_\beta \subset \partial_\beta$ be weakly null convex Cauchy surfaces with
\( J^+(S_\alpha) \cap S_\beta \neq \emptyset \). If \( I^+(S_\alpha) \cap S_\beta = \emptyset \) then every future causal curve joining \( S_\alpha \) and \( S_\beta \) must meet \( \partial M \) at a point other than its endpoints.

**Proof:** We proceed by contradiction. Assume there is a future causal curve \( \gamma: [0, 1] \to M \) from \( S_\alpha \) to \( S_\beta \) that meets \( \partial M \) only at endpoints. By propositions 3.8 and 3.10, \( \gamma \) must be a null geodesic, orthogonal to both \( S_\alpha \) and \( S_\beta \) and with no focal points to \( S_\alpha \) on \([0, 1)\).

First notice that by remark 3.39 above, given \(-\gamma'(1) \perp S_\beta\) there is a \( S_\beta\)-neighborhood \( U_\beta \) of \( q \), \( \varepsilon > 0 \) and a smooth surface \( N_\beta \) of the form \( N_\beta = \Psi(U_\beta \times (1 - 2\varepsilon, 1)) \), \( \Psi(y, s) = \exp^\perp(sV_y) \) such that \( \gamma|_{1-2\varepsilon} \) is one of its generators. We can further assume that the slices \( U_{\beta,s} := \Psi(U_\beta \times \{s\}) \) are all diffeomorphic to \( U_\beta \).

On the other hand, by proposition 3.38 there is a surface \( N_\alpha \) emanating from a neighborhood \( U_\alpha \subset S_\alpha \) of the form \( N_\alpha = \Phi(U_\alpha \times [0, 1 - \varepsilon/2]) \), \( \Phi(x, t) = \exp^\perp(tZ_x) \) having \( \gamma|_{1-\alpha/2} \) as one of its generators. Moreover, we can assume \( U_\alpha \approx U_{\alpha,t} \), where \( U_{\alpha,t} := \Phi(U_\alpha \times \{t\}) \). Observe \( N_\alpha \) and \( N_\beta \) meet on \( r = \gamma(1 - \varepsilon) \).

Furthermore, since \( S_\beta \) is null convex and \( \gamma'(1) \) is future outward pointing we have that \( \theta_\beta(1) \geq 0 \), where \( \theta_\beta \) is the null expansion of \( U_{\beta,s} \) along \( \gamma \). Hence, by the Raychaudhuri equation and the null energy condition we have

\[
\frac{d\theta_\beta}{dt} \leq -\text{Ric}(\alpha', \alpha') \leq 0 \tag{3.3.2}
\]

thus \( \theta_\beta(s) \) is non increasing, therefore \( \theta_\beta(r) = \theta_\beta(1 - \varepsilon) \geq 0 \). In a similar fashion, since \( S_\alpha \) is null convex and \( \gamma'(0) \) is future inward pointing we have \( \theta_\alpha(0) < 0 \), thus \( \theta_\alpha(r) = \theta_\alpha(1 - \varepsilon) < 0 \).

Finally, let us notice \( N_\alpha \) is to the future of \( N_\beta \) near \( r \). To this end, let us a consider
a neighborhood $\mathcal{V}$ of $r$ such that $N_\beta$ is closed and achronal in $\mathcal{V}$, hence $r \not\in \text{edge}(S)$, thus there exists a neighborhood $\mathcal{W} \subset \mathcal{V}$ of $r$ such that every timelike curve from $I^-(r, \mathcal{W})$ to $I^+(r, \mathcal{W})$ intersects $N_\beta$. Let $a \in I^+(r, \mathcal{W})$ and $b \in I^-(r, \mathcal{W})$ and let $\mathcal{U} := I^-(a, \mathcal{W}) \cap I^+(b, \mathcal{W})$. We will show $N_\alpha \cap \mathcal{U} \subset J^+(N_\beta \cap \mathcal{U}, \mathcal{U})$.

Let $x \in N_\alpha \cap \mathcal{U}$, then there is a future timelike curve $\sigma$ from $b \in I^-(r, \mathcal{W})$ to $a \in I^+(r, \mathcal{W})$ passing through $x$. Thus, by the way $\mathcal{W}$ was chosen, $\sigma$ has to intersect $N_\beta$ at some point, say $y$. If $y \in I^+(x, \mathcal{W})$ then by concatenating $\sigma|_y^x$ to the generator of $N_\alpha$ through $y$ and the corresponding generator through $x$ we get a causal curve from $S_\alpha$ to $S_\beta$ that is not a null geodesic and meets $\partial M$ only at endpoints. As a consequence, $I^+(S_\alpha) \cap S_\beta \neq \emptyset$, contradicting one of our hypothesis. Thus $y$ comes before $x$ along $\sigma$, hence $x \in J^+(N_\beta \cap \mathcal{U}, \mathcal{U})$ as desired.

Therefore, by the maximum principle for smooth null hypersurfaces \[2.10\] we have that $\theta_\alpha(r) = 0 = \theta_\beta(r)$. This contradicts $\theta_\alpha(r) < 0$. □

Returning to our original problem; we analyze the simplest case of all, that is, when $\partial M$ consists of two components $\partial_1$ and $\partial_2$, both acausal in $M$.

**Theorem 3.45** Let $M$ be a globally hyperbolic spacetime-with-timelike-boundary and assume $\partial M$ has only two components $\partial_1$ and $\partial_2$. Further assume the Cauchy surfaces of both $\partial_1$ and $\partial_2$ are compact and acausal in $M$. Let $S$ be a Cauchy surface of $\partial_1$ with $J^+(S) \cap \partial_2 \neq \emptyset$. Then there exists a fastest null geodesic $\eta \subset \partial I^+(S)$ from $S$ to $\partial_2$. Moreover, $\eta$ is normal to both $S$ and $\Sigma_{t_0}$. 
In order to be able to use lemma 3.43, we must first prove that \( \inf B \) exists under the hypotheses of the theorem.

**Lemma 3.46** Let \( B = \{ t \mid J^+(S) \cap \Sigma_t \neq \emptyset \} \). Then \( B \) is bounded below and \( J^+(S) \cap \Sigma_{t_0} \neq \emptyset \), where \( t_0 = \inf B \).

**Proof:** We proceed by contradiction. If \( B \) is unbounded from below, then there exist sequences \( t_n \downarrow -\infty \) and \( q_n \in \Sigma_{t_n} \) with \( q_n \in J^+(S) \). Fix \( T \geq t_1 \) and let \( p_n \in \Sigma_T \) be the projection of \( q_n \) under the time function \( t \) of \( \partial_2 \). Since \( \Sigma_T \) is compact, \( \{p_n\} \) has a convergent subsequence \( p_m \to p \). Let \( \gamma_m : [0, T - t_m] \to \partial_2 \) be the integral line segments of \( \nabla t \) from \( p_m \) to \( q_m \). Then by the limit curve lemma there is a limit curve \( \gamma : [0, a] \to \partial_2 \) of \( \{\gamma_m\} \). Since \( t_m \to -\infty \) we have that \( a = \infty \), thus \( \gamma \) is inextendible.

Let \( s \in [0, \infty) \) and consider \( N \) such that \( \gamma_m \) is defined at \( s \) for all \( m \geq N \). Since \( \gamma_m(s) \in J^+(S) \) for all \( m \geq N \) and \( J^+(S) \) is closed we have \( \gamma(s) = \lim_m \gamma_m(s) \in J^+(S) \). Thus \( \gamma \subset J^+(S) \). On the other hand \( \gamma(0) = p \) so \( \gamma \subset J^-(p) \). Therefore \( \gamma \subset J^-(p) \cap J^+(S) \cap \partial_2 \).

Now note \( J^-(p) \cap J^+(S) \) is a compact set by proposition 3.18 and \( \gamma \) is a future inextendible causal curve contained in it. This contradicts strong causality on \( \partial_2 \).

Finally, let us consider a sequence \( \{t_n\} \subset B \) converging to \( t_0 = \sup B \) and \( y_n \in J^+(S) \cap \Sigma_{t_n} \). Let \( x_n \) be the projection of \( y_n \) to \( \Sigma_{t_0} \) by the time function \( t \). Since \( \Sigma_{t_0} \) is compact there is a subsequence \( \{x_m\} \) converging to \( x \in \Sigma_{t_0} \), hence \( y_m \to x \) as well. Thus causal simplicity implies that \( x \in J^+(S) \). \( \square \)

**Proof:** (of theorem 3.45) Since \( J^+(S) \cap \Sigma_{t_0} \neq \emptyset \), let \( \gamma \) be a causal curve from \( p \in S \) to \( q \in \Sigma_{t_0} \). By lemmas 3.43 and 3.46 it suffices to show \( \gamma \) meets \( \partial M \) only at \( p \) and \( q \).
Let $p' \in \partial M$ be the last point in which $\gamma$ intersects $\partial_1$. If $p' \neq p$ then since $S$ is acausal in $M$ we must have $p' \in J^+(p) - S$. Even further, since $\partial_1$ is globally hyperbolic, we can use the time function of $\partial_1$ to show $p' \in I^+(S, \partial_1)$. Hence by proposition 3.6 we have $q \in I^+(S)$, which will lead to a contradiction to the minimality of $t_0$. Therefore $p = p'$, i.e. $\gamma$ only touches $\partial_1$ at its initial point.

On the other hand, let $q' \in \Sigma_T$ the first point in which $\gamma$ meets $\partial_2$. Clearly $T \geq t_0$ is by the way $t_0$ is defined. Finally, if $q' \neq q$ then the acausality of $\Sigma_T$ is violated. This completes the proof. □

As an application of theorem 3.45 we prove the non traversability of wormholes in the two boundary component case.

**Corollary 3.47** Let $M$ be a globally hyperbolic spacetime-with-timelike-boundary and assume $\partial M$ has only two components $\partial_1$ and $\partial_2$. Further assume the Cauchy surfaces of both $\partial_1$ and $\partial_2$ are compact, weakly null convex and acausal in $M$. Then $J^+(\partial_1) \cap J^-(\partial_2) = \emptyset$.

Proof: Assume $J^+(\partial_1) \cap J^-(\partial_2) \neq \emptyset$, then there is a causal curve from $p \in \partial_1$ to $q \in \partial_2$. Let $S$ be a Cauchy surface of $\partial_1$ containing $p$. Then by theorem 3.45 there is a future null geodesic $\eta$ joining $S$ to $\Sigma_{t_0}$. Moreover, the proof of theorem 3.45 also shows that $\eta$ meets $\partial M$ only at its endpoints. Hence by the contrapositive of lemma 3.44 we have $I^+(S) \cap \Sigma_{t_0} \neq \emptyset$. Since $I^+(S)$ is open, there would be a point $x \in \partial_2 \cap I^+(S)$ slightly to the past of $\Sigma_{t_0}$, thus contradicting the choice of $t_0$ as the infimum of the set $\{t \mid J^+(S) \cap \Sigma_t \neq \emptyset\}$. □
3.3.2 Strong form of the PTC

We prove now the main result of this chapter, namely that a stronger version of the PTC holds on the domain of outer communication relative to a boundary component.

**Definition 3.48** Let \((M, g)\) be a spacetime-with-timelike-boundary and \(A \subset \partial M\).

The domain of outer communication relative to \(A\) is given by \(D(A) := I^+(A) \cap I^-(A)\).

Thus, let \((M, g)\) be a spacetime with a connected timelike boundary. \(\mathcal{I} := \partial M\).

First notice that since \(\mathcal{I}\) is timelike we have \(\mathcal{I} \subset D(\mathcal{I})\), hence \(D(\mathcal{I})\) is a spacetime-with-timelike-boundary and \(\partial D(\mathcal{I}) = \mathcal{I}\).

**Strong form of PTC** Any curve in \(D(\mathcal{I})\) with endpoints in \(\mathcal{I}\) is fixed endpoints homotopic to a curve in \(\mathcal{I}\).

As pointed out in [23], the strong form of the PTC is a consequence of the PTC in its standard form and the following topological result (see lemma 3.2 in [23]):

**Proposition 3.49** Let \(M\) and \(S\) be topological manifolds, \(\imath: S \hookrightarrow M\) an embedding and \(\pi: M^* \to M\) the universal cover of \(M\). If \(\pi^{-1}(S)\) is connected then the induced group homomorphism \(\imath_*: \pi_1(S) \to \pi_1(M)\) is surjective.

We thus begin by showing that the standard version of PTC holds on \(D(\mathcal{I})\).

**Theorem 3.50** Let \(M\) be a spacetime-with-timelike-boundary with \(\mathcal{I} = \partial M\) connected and assume \(\mathcal{D} := D(\mathcal{I})\) is globally hyperbolic. Further assume the Cauchy surfaces of \(\mathcal{I}\) are compact, weakly null convex and acausal in \(\mathcal{D}\). Then the Principle of Topological Censorship holds on \(\mathcal{D}\).
Proof: Let $\gamma: [0, 1] \rightarrow \mathcal{D}$ be a causal curve from $p \in \mathcal{I}$ to $q \in \mathcal{I}$, we want to show that $\gamma$ is fixed endpoint homotopic to a curve $\gamma_0: [0, 1] \rightarrow \mathcal{I}$.

First of all, by proposition 3.34 there is a globally hyperbolic covering spacetime-with-timelike-boundary $\pi: \hat{\mathcal{D}} \rightarrow \mathcal{D}$ having a copy $\hat{\mathcal{I}}$ of $\mathcal{I}$. Now let $\hat{p} \in \hat{\mathcal{I}}$ such that $\pi(\hat{p}) = p$ and lift $\gamma$ to a curve $\hat{\gamma}: [0, 1] \rightarrow \hat{\mathcal{D}}$ starting at $\hat{p}$, then $\hat{q} := \hat{\gamma}(1) \in \partial \hat{\mathcal{M}}$.

If $\hat{\gamma}(1) \in \hat{\mathcal{I}}$, by remark 3.34 we know $\hat{\gamma}$ is fixed endpoints homotopic to a curve $\hat{\gamma}_0: [0, 1] \rightarrow \hat{\mathcal{I}}$, in which case we are done since $\gamma_0 = \pi \circ \hat{\gamma}_0$ would be the desired curve.

Now, let us consider the case $\hat{\gamma}(1) \in \partial \hat{\mathcal{M}} - \hat{\mathcal{I}}$.

Let $T$ be a time function on $\mathcal{I}$ such that the Cauchy surface $S_0 := T^{-1}(0)$ contains $p$, and let us define $S_t := T^{-1}(t)$. Consider now $\hat{S}_0 := \pi^{-1}(S_0) \cap \hat{\mathcal{I}}$ and $B := J^+(\hat{S}_0, \hat{\mathcal{D}}) \cap (\partial \hat{\mathcal{M}} - \hat{\mathcal{I}})$, hence $\hat{q} \in B$.

Since $B \neq \emptyset$ then $(T \circ \pi)(B) \neq \emptyset$. Further, $(T \circ \pi)(B)$ is bounded from below by 0, since the acausality of $S_0$ in $\mathcal{M}$ would be violated otherwise. Hence $t_0 = \inf\{(T \circ \pi)(B)\}$ exists.

By definition, consider a decreasing sequence $\{t_n\} \subset (T \circ \pi)(B)$ converging to $t_0$. Thus, let $\hat{\gamma}_n$ be a future causal curve in $\hat{\mathcal{D}}$ joining $\hat{p}_n \in \hat{S}_0$ to $\hat{q}_n \in \partial \hat{\mathcal{D}} - \hat{\mathcal{I}}$, $t_n = (T \circ \pi)(\hat{q}_n)$ and let us denote by unhatted symbols the corresponding objects in $\mathcal{D}$.

Since $S_0$ is compact and $\pi|_{\hat{\mathcal{I}}}$ is a homeomorphism we know $\hat{S}_0$ is compact as well, hence there is a subsequence $\{\hat{p}_m\}$ converging to $\hat{p}_0 \in \hat{S}_0$.

On the other hand, let $q'_m$ be the projection of $q_m$ into $S_{t_0}$ by the time function $T$. By compactness of $S_{t_0}$, after further taking a subsequence, we may assume $q'_m$ converges to $q_0 \in S_{t_0}$. Hence $q_m \rightarrow q_0$ as well.
Further notice that \( q_0 \neq p_0 \). Assume the contrary, and then consider a neighborhood \( U \) of \( p_0 \) evenly covered by \( \pi \) and let \( \mathcal{V} \subset U \) be a causally convex neighborhood of \( p_0 \). Since both sequences \( \{p_m\} \) and \( \{q_m\} \) converge to \( p_0 \) we have that \( p_N, q_N \in \mathcal{V} \) for some \( N \) sufficiently large. Thus \( \gamma_N \subset \mathcal{V} \); hence \( \hat{\gamma}_N \subset \hat{\mathcal{U}} \), where \( \hat{\mathcal{U}} \) is the unique component of \( \pi^{-1}(U) \) containing \( \hat{p}_0 \). It follows \( \hat{q}_N \in \hat{\mathcal{I}} \) which is a contradiction.

Therefore, all the hypothesis of the monodromy lemma (proposition 3.30) are met, hence we can assert the existence of a future causal curve \( \hat{\gamma}_0 \) from \( \hat{p}_0 \) to \( \hat{q}_0 \) and thus \( T \circ \pi(\hat{q}_0) = t_0 \). Hence \( t_0 \in (T \circ \pi)(B) \).

Next, let us denote by \( \partial_0 \) the component of \( \partial \hat{M} \) containing \( \hat{q}_0 \) and let \( \Sigma_0 \) be a Cauchy surface of \( \partial_0 \) through \( \hat{q}_0 \). Notice that \( \hat{\gamma}_0 \) touches \( \partial \hat{D} - \hat{\mathcal{I}} \) only at \( \hat{q}_0 \in \partial_0 \). In fact, if \( \hat{x} \in \hat{\gamma}_0 \cap \partial \hat{D} - \hat{\mathcal{I}} \), \( \hat{x} \neq \hat{q}_0 \), then \( t_1 := T \circ \pi(\hat{x}) \geq t_0 \) and thus the segment of \( \gamma_0 \) from \( x \) to \( q_0 \) would be a future causal curve connecting \( S_{t_1} \) to \( S_{t_0} \), violating the acausality of \( S_{t_0} \).

Finally, note that since \( \pi \) is a local isometry, then both \( \hat{S}_0 \) and \( \Sigma_0 \) are null convex. Hence by lemma 3.44 above we conclude \( I^+(\hat{S}_0) \cap \Sigma_0 \neq \emptyset \), but then since \( I^+(\hat{S}_0) \) is an open set we would have \( I^+(\hat{S}_0) \cap I^-(\Sigma_0, \partial_0) \neq \emptyset \); which in turns yield a contradiction to the fact that \( t_0 = \inf T \circ \pi(B) \).

Hence we just ruled out the case \( \hat{\gamma}(1) \in \partial \hat{M} - \hat{\mathcal{I}} \), thus \( \hat{\gamma}(1) \in \hat{\mathcal{I}} \) and the proof is complete. \( \square \)

**Theorem 3.51** Let \( D \) be as in theorem 3.50, then the strong form of the principle of topological censorship holds on \( D \).

**Proof:** Let \( \pi: D^* \to D \) be the universal cover of \( D \). Then \( D^* \) is a spacetime with
timelike boundary $\mathcal{I}^* := \pi^{-1}(\mathcal{I})$. By proposition 3.49 it suffices to show $\mathcal{I}^*$ is connected.

Thus, let $\{\mathcal{I}_\alpha^*\}$, $\alpha \in A$, be the collection of connected components of $\mathcal{I}^*$ and let us define $\mathcal{D}_\alpha^* := I^+(\mathcal{I}_\alpha^*, \mathcal{D}^*) \cap I^-(\mathcal{I}_\alpha^*, \mathcal{D}^*)$.

Let $x \in \mathcal{D}^*$. Since $\pi(x) \in \mathcal{D}$, there is a future causal curve $\gamma^* : I \to \mathcal{D}^*$ through $x$ with endpoints on $\mathcal{I}^*$. Hence $\gamma := \pi \circ \gamma^*$ is a future causal curve in $\mathcal{D}$ with endpoints in $\mathcal{I}$, then by theorem 3.50 $\gamma$ is fixed endpoint homotopic to a causal curve $\beta : I \to \mathcal{I}$. By lifting this homotopy to $\mathcal{D}^*$, we find that $\gamma^*$ is homotopic to a curve in $\mathcal{I}^*$. Thus the endpoints of $\gamma^*$ lie on the same component $\mathcal{I}_\alpha^*$, hence $x \in \mathcal{D}_\alpha^*$. It follows $\{\mathcal{D}_\alpha^*\}$ is an open cover of $\mathcal{D}^*$.

Moreover, we have also shown that $I^+(\mathcal{I}_\alpha^*, \mathcal{D}^*) \cap I^-(\mathcal{I}_\beta^*, \mathcal{D}^*) = \emptyset$ if $\alpha \neq \beta$. As a consequence the sets $\mathcal{D}_\alpha^*$ are disjoint. However $\mathcal{D}^*$ is connected, thus we must have $|A| = 1$, i.e. $\mathcal{I}^*$ is connected.

Hence by proposition 3.49 we have that any loop in $\mathcal{D}$ based at $\mathcal{I}$ is homotopic to a loop in $\mathcal{I}$ based at the same point. Let now $\gamma$ be any curve in $\mathcal{D}$ from $p \in \mathcal{I}$ to $q \in \mathcal{I}$. Since $\mathcal{I}$ is connected, there is a curve in $\sigma$ in $\mathcal{I}$ from $q$ to $p$, hence the concatenation $\gamma \ast \sigma$ is a loop in $\mathcal{D}$ based at $p \in \mathcal{I}$. Therefore there is a loop $c$ in $\mathcal{I}$ so that $\gamma \ast \sigma$ is fixed endpoint homotopic to $c$. Thus $\gamma$ is homotopic to $c \ast \sigma^-$, completing the proof. □
3.4 Future research

We end this dissertation by making some comments on possible directions for future research related to the new results established here.

I. Rigidity in asymptotically flat spacetimes

A spacetime \((M, g)\) admitting a conformal completion is \textit{asymptotically flat} if the conformal boundary \(J\) is null. Some of the classical models in general relativity, such as Schwarzschild, are asymptotically flat. The study of such spacetimes is an important branch in mathematical relativity, that dates back to the origins of the field. Moreover, some notions like topological censorship were first developed in the asymptotically flat setting.

Recall that in section 2.3 we extended the rigidity result presented in the form of theorem 2.11 from the asymptotically simple and de Sitter to the more relaxed globally hyperbolic and asymptotically de Sitter case. However, theorem 2.11 is also valid for asymptotically flat and simple spacetimes. Thus is natural to expect to get a rigidity result analogous to theorem 2.17 in the asymptotically flat case.

Notice however, that the proof of theorem 2.17 relies heavily on the construction of a globally hyperbolic extension and on the rather simple causal structure of this new spacetime near the endpoints of a null line \(\eta\) connecting \(J^+\) and \(J^-\). A more careful analysis is needed in the asymptotically flat setting, since \(\partial I^+ (\eta)\) intersects \(J\) in a null geodesic in this case.
II. Other models

In section 2.5 we explored asymptotically de Sitter spacetimes which were solution to the Einstein equations with an energy-momentum tensor satisfying some properties. We believed similar results may be obtained in the asymptotically flat case as well, but furthermore, by suitable changing the restrictions on $T$ we can obtain rigidity results for other choices of $T$, such as electromagnetic fields.

III. Asymptotically Anti de Sitter black strings

Intuitively, a black string is formed when the result of gravitational collapse is not a point (i.e. a black hole) but rather a one dimensional object. Lately, the conjectured AdS/CFT correspondence has sparked much interest in the study of black strings that approach Anti de Sitter space outside a compact set.

More formally, an asymptotically de Sitter black string is a spacetime $(M, g)$ asymptotic to $AdS^n \times N$, where $(AdS^n, g_0)$ is the standard Anti de Sitter spacetime and $(N, h)$ is a compact Riemannian manifold, whose metric $g$ approaches $g_0 + h$ near $\mathcal{J}$.

In this setting, we feel a quasi-local form of topological censorship can be established as a consequence of theorem 3.50, once appropriate decay conditions on the metric are imposed. Such result would settle a question on the traversability of worm holes posed by J. Maldacena [11].
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Appendix A

Extensions of spacetimes with boundary

A good part of the constructions and results in Lorentzian geometry rely on “open conditions” such as variational principles or the existence of open normal neighborhoods.

In the process of extending the main results of Lorentzian geometry to the spacetime-with-boundary setting, we are faced with the technical difficulty that such open conditions in principle may not hold in the boundary. Hence a much more delicate analysis has to be performed.

Another way to deal with this issue consists in embedding our spacetime-with-boundary in an open spacetime of the same dimension. This latter approach enables us to apply the standard results and methods in the extended spacetime and then interpret the results so obtained back in to our original spacetime-with-boundary setting.
As the next result shows, any spacetime-with-boundary can be extended to an open spacetime.

**Theorem A.1** Every spacetime-with-boundary \((M, g)\) admits an extension to a spacetime \((\overline{M}, \overline{g})\).

**Proof:** First extend \(M\) to a smooth manifold \(M'\) by means of attaching collars to all the components of \(\partial M\). Since \(M\) is time orientable, there exists a timelike vector field \(V \in \mathcal{X}(M)\). Let us extend \(V\) to all of \(M'\) and let \(W = \{p \in M' \mid V_p \neq 0\}\). Clearly \(W\) is an open subset of \(M'\) containing all of \(M\), so without loss of generality we can assume \(M' = W\).

Let \(p \in \partial M\) and choose a \(M'\)-chart \(U_p\) around it. Now let \(g = g_{ij}dx^idx^j\) be the coordinate expression of \(g\) in the \(M\)-chart \(M \cap U_p\). Since the \(g_{ij}\)'s are smooth functions on \(M \cap U_p\), they can be smoothly extended to an \(M'\)-neighborhood \(U'_p \subset U_p\) with \(M \cap U'_p = M \cap U_p\). Let us denote by \(g'_{ij}\) such extensions. It is important to notice that \(U'_p\) can be chosen in such a way that \(g' = g'_{ij}dy^idy^j\) is a Lorentz metric on \(U'_p\) with \(g'(V, V) < 0\).

Now, let \(e_0\) be the unit vector field (with respect to \(g'\)) in the direction of \(V\) and let \(B = \{e_0, e_1, e_2, e_3\}\) be an orthonormal basis of \(T_pM'\). Choose an \(M'\)-neighborhood \(V_p \subset U'_p\) in which a local frame field relative to \(g'\) and adapted to \(B\) is defined. Thus

\[
g' = -e_0^* \otimes e_0^* + \sum_i e_i^* \otimes e_i^* \text{ on } V_p, \tag{A.0.1}\]

where \(e_i^*\) denotes the covector \(g'\)-related to \(e_i\). Choose a cover \(\{V_\alpha\}\) of \(\partial M\) by such open sets and let \(h_\alpha = 2e_0^* \otimes e_0^* + g'_\alpha\). Further consider a smooth partition of unity \(f_\alpha\) subordinated to \(\{V_\alpha\}\), thus \(h = \sum_\alpha f_\alpha h_\alpha\) is a Riemannian metric on \(V = \cup_\alpha V_\alpha\).
Finally, let $X$ be the unit vector field (with respect to $h$) in the direction of $V$, let $\omega$ be the covector $h$-related to $X$ and let $g'' = h - 2\omega \otimes \omega$. It is a straightforward computation to check that $g''$ is a Lorentz metric on $\mathcal{V}$ that agrees with $g$ on the overlap $\mathcal{V} \cap M$. Thus by gluing $g''$ and $g$ together we obtain a Lorentz metric $\overline{g}$ on $\overline{M} = \mathcal{V} \cup M$. Notice $\overline{g}$ is smooth since $\mathcal{V}$ is open. □
Appendix B

Focal points along null geodesics

Here we specialize the classical result relating focal points, normal variations and degeneracy of the normal exponential map to the case in which the base geodesic is null.

In what follows, $\sigma$ is a geodesic normal to $P$ at $p = \sigma(0)$ and $P \subset M$ is a spacelike submanifold of codimension at least 2.

Let us first recall some definitions and standard results concerning Jacobi fields. For a reference, consult chapters 8 and 10 on [46] or chapter 10 on [2].

**Lemma B.1** Let $p \in M$ and $x \in T_p M$. For $v_x \in T_x(T_p M)$ we have

$$(\exp_p)_*(v_x) = V(1) \quad (B.0.1)$$

where $V$ is the unique Jacobi field on the geodesic $\gamma_x$ such that $V(0) = 0$ and $V'(0) = v \in T_p M$.

**Lemma B.2** Let $Y$ be a Jacobi field on the geodesic $\gamma$. Then $Y \perp \gamma$ if and only if there exist $a \neq b$ such that $Y(a) \perp \gamma$ and $Y(b) \perp \gamma$. 

121
Definition B.3 The tensor $\tilde{II}: \mathcal{X}(P) \times \mathcal{X}(P) \perp \to \mathcal{X}(P)$ is defined by

$$\tilde{II}(X,Y) = \tan \nabla_X Y \quad (B.0.2)$$

where $\nabla$ is the Levi-Civita connection on $M$.

Definition B.4 A $P$-Jacobi field on a geodesic $\sigma$ is a Jacobi field that satisfies the following initial conditions:

1. $V(0)$ is tangent to $P$.

2. $\tan V'(0) = \tilde{II}(V(0), \sigma'(0))$

Definition B.5 The point $q = \sigma(r), r \neq 0$ is a focal point of $P$ along $\sigma$ provided there exists a nonzero $P$-Jacobi field $J$ on $\sigma$ with $J(r) = 0$.

Definition B.6 The normal null cone to $P$ is the manifold

$$\mathfrak{N}(P) = \{v \in NP \mid g(v,v) = 0, \ v \neq 0\} \quad (B.0.3)$$

Theorem B.7 Let $\sigma$ be a null geodesic normal to $P$. The following are equivalent:

1. $\sigma(1)$ is a focal point of $P$ along $\sigma$.

2. There exists a nontrivial variation $x$ of $\sigma$ through $P$-normal null geodesics such that $V(1) = 0$.

3. The normal exponential map $\exp^\perp$ restricted to $\mathfrak{N}(P)$ is singular. (i.e. there is $x \neq 0$ tangent to $\mathfrak{N}(P)$ such that $\exp^\perp(x) = 0$).
Proof: We divide the proof in four parts.

I. (1) implies (2):

Let \( Y \) be a \( P \)-Jacobi vector field on \( \sigma \) such that \( Y(1) = 0 \). Then \( Y(0) \perp \sigma \) and \( Y(1) \perp \sigma \) so by lemma [B.2] above \( Y \perp \sigma \). Thus

\[
0 = \frac{d}{dt} g(Y(t), \sigma'(t)) = g(Y', \sigma') + g(Y, \sigma'') = g(Y', \sigma'). \tag{B.0.4}
\]

In particular, since \( \sigma'(0) \perp P \) we have

\[
0 = g(Y'(0), \sigma'(0)) = g(\text{nor} Y'(0), \sigma'(0)). \tag{B.0.5}
\]

Now let \( A = T_p(P) \perp \) and \( W = \text{span}_A \{\sigma'(0)\} \). It is clear that \( W^\perp := \{x \in A \mid g(x, w) = 0 \forall w \in W\} \) is a degenerate subspace of \( A \) hence by remark [1.7] translation by \( \sigma'(0) \) gives rise to an isomorphism

\[
W^\perp \approx T_{\sigma'(0)} (\Lambda_p(A)). \tag{B.0.6}
\]

As we have noted before \( \text{nor} Y'(0) \in W^\perp \), thus by the isomorphism just mentioned there exists a curve \( \lambda \) in \( \Lambda_p(A) \) with \( \lambda(0) = \sigma'(0) \) and \( \frac{d\lambda}{ds}(0) = \text{nor} Y'(0) \).

Let \( \{e_i\} \) be a basis of \( T_p(P) \perp \), then since \( \lambda(s) \in T_p(P) \perp \) we have \( \lambda(s) = \lambda^i(s)e_i \).

Consider now a curve \( \alpha \) in \( P \) such that \( \alpha(0) = p \) and \( \alpha'(0) = Y(0) \) and let \( \{E_i\} \) be a normal parallel frame along \( \alpha \) adapted to \( \{e_i\} \) and define \( Z \in \mathcal{X}(\alpha) \) by \( Z(t) = \lambda^i(t)E_i \).

Hence \( \text{nor } Z'(t) = \frac{d\lambda^i}{dt} E_i(t) \), then

\[
\text{nor } Z'(0) = \lambda'(0) = \text{nor } Y'(0). \tag{B.0.7}
\]

Notice also

\[
\tan Z'(0) = \tilde{\Pi} (\alpha'(0), Z(0)) = \tilde{\Pi} (Y(0), \sigma'(0)) = \tan Y'(0) \tag{B.0.8}
\]
Thus $Z$ is a null vector field on $\alpha$ normal to $P$ such that

$$Z(0) = \sigma'(0) \quad \text{and} \quad Z'(0) = Y'(0). \tag{B.0.9}$$

Let $x(u, v) = \exp^{-1}(uZ(v))$. Note $x_u(0, v) = Z(v)$ therefore $\sigma_v(u) := x(u, v)$ is a null geodesic normal to $P$ with initial velocity $Z(v)$. In particular $\sigma_0(u) = \exp^+(u\sigma'(0)) = \sigma(u)$ so $x$ is a variation of $\sigma$.

Finally, let $V$ be the variation vector field of $x$. By standard results, $V$ is a Jacobi vector field on $\sigma$. Notice $x(0, v) = \exp^{-1}(0_{\pi\circ Z(v)}) = \alpha(v)$ thus

$$V(0) = x_v(0, v) = \alpha'(0) = Y(0). \tag{B.0.10}$$

Moreover,

$$V'(0) = x_{vu}(0, 0) = x_{uv}(0, 0) = Z'(0) = Y'(0) \tag{B.0.11}$$

Thus, $V(u) = Y(u)$ by uniqueness of Jacobi fields. Hence $V(1) = Y(1) = 0$ so $x$ is a variation of $\sigma$ with the desired properties.

**II.** (2) implies (1):

Let $x(u, v)$ be such a variation. Let $\alpha$ be the curve in $P$ defined by $\alpha(v) = x(0, v)$ and define $Z \in \mathcal{X}(\alpha)$ by $Z(v) = x_u(0, v)$.

Consider now the variation vector field $V$ of $\sigma$. Then

$$V(0) = x_v(0, 0) = \alpha'(0) \in T_p(P). \tag{B.0.12}$$

Moreover,

$$V'(0) = x_{vu}(0, 0) = x_{uv}(0, 0) = Z'(0) \tag{B.0.13}$$

thus, since $Z(0) = x_u(0, 0) = \sigma'(0)$ we have

$$\tan V'(0) = \tan Z'(0) = \tilde{\Pi} (\alpha'(0), Z(0)) = \tilde{\Pi} (V(0), \sigma'(0)) \tag{B.0.14}$$
so $V(u)$ is a $P$-Jacobi field that vanishes at $u = 1$.

III. (3) implies (2):

Let $x \in T_{\sigma'(0)}\mathfrak{N}(P)$, $x \neq 0$, with $\exp^\perp_\pi(x) = 0$. Let $\pi : NP \to P$ be the standard projection. Since $\pi$ is a submersion, we have that $\pi_*(x) = 0$ if and only if $x$ is tangent to the fiber $T_p(P)^\perp$.

Let us first consider the case $\pi_*(x) = 0$. Then $x \in T_{\sigma'(0)}(T_p(P)^\perp)$, so let $x_0 \in T_p(P)^\perp$ be the vector that corresponds canonically to $x$. By lemma [B.1]

$$0 = \exp^\perp_\pi(x) = (\exp_p)_*(x) = J(1) \quad (B.0.15)$$

where $J$ is the unique Jacobi field on $\sigma$ with the initial conditions $J(0) = 0$ and $J'(0) = x_0$.

Since $x_0 \in T_p(P)^\perp$, then

$$\tan J'(0) = 0 = \tilde{\Pi}(J(0), \sigma'(0)). \quad (B.0.16)$$

Thus $J$ is a $P$-Jacobi field on $\sigma$ vanishing at $J(1)$. It is clear that $J \neq 0$ since $J'(0) = x_0 \neq 0$ and the result follows from (I) above.

We now consider the case $\pi_*(x) \neq 0$. Let $Z : I \to \mathfrak{N}(P)$ be a curve such that $Z(0) = \sigma'(0)$ and $Z'(0) = x$. Let $x(u,v) = \exp^\perp(uZ(v))$ and note $x(u,0) = \exp^\perp(u\sigma'(0)) = \sigma(u)$, so $x$ is a variation of $\sigma$ through null geodesics normal to $P$.

Further, $x(0,v) = \exp^\perp(0_{\pi \circ Z(v)}) = (\pi \circ Z)(v)$, then

$$V(0) = x_v(0,0) = (\pi \circ Z)'(0) = \pi_*(x) \neq 0 \quad (B.0.17)$$

and, as a consequence, $V \neq 0$.
Finally, observe that $x(1,v) = \exp^+(Z(v))$ hence

$$V(1) = x_v(1,0) = \exp^+(Z'(0)) = \exp^+(x) = 0 \quad \text{(B.0.18)}$$

**IV.** (2) implies (3):

Let $x$ be a variation with the given properties and denote by $\alpha$ its initial curve (i.e. $\alpha(v) = x(0,v)$). Let us define a variation on $\mathcal{H}(P)$ by $\tilde{x}(u,v) = u x_u(0,v)$ and notice $x = \exp^+ \circ \tilde{x}$. Thus

$$x_v = x_u(\partial_v) = (\exp^+ \circ \tilde{x})_u(\partial_v) = \exp^+(\mathcal{H}_u(\partial_v)) = \exp^+(\tilde{x}_v) \quad \text{(B.0.19)}$$

hence

$$\exp^+(\tilde{x}_v(0,1) = x_v(0,1) = 0. \quad \text{(B.0.20)}$$

Now let $\beta: I \to \mathcal{H}(P)$ be the final transversal curve of $\tilde{x}$, that is, $\beta(v) = \tilde{x}(1,v)$. Then we have $\beta'(0) = \tilde{x}_v(1,0)$. Moreover, by definition $\tilde{x}(1,v) = x_u(0,v)$ hence $\alpha = \pi \circ \beta$.

Therefore

$$x_v(0,0) = \alpha'(0) = \pi_u(\beta'(0)) = \pi_u(\tilde{x}_v(1,0)). \quad \text{(B.0.21)}$$

In virtue of the last two equations, we have that $x_v(0,0) \neq 0$ implies $\tilde{x}_v(1,0) \neq 0$ and we are done.

On the other hand assume $x_v(0,0) = 0$ and consider the variation vector field $V = x_v(u,0)$, hence $V(0) = 0$. By (II) above $V$ is a $P$-Jacobi field, so in particular

$$\tan V'(0) = \tilde{I}(V(0), \sigma'(0)) = 0 \quad \text{(B.0.22)}$$

thus $V'(0)$ is normal to $P$ at $p$. Since $V$ is perpendicular to $\sigma$ at both $u = 0$ and $u = 1$ the relation $g(V'(0), \sigma'(0)) = 0$ follows easily from lemma B.2 (see equation
Further notice $V'(0) \neq 0$, since otherwise $V \equiv 0$. Hence $V'(0)$ is the desired vector due to the isomorphism (B.0.6). \( \square \)
Appendix C

Conformally related metrics

Here we carry out some computations that will enable us see how the mathematical objects present in the Einstein equations change under conformal transformations of the metric. Thus, let us consider

\[ g = \Omega^2 \tilde{g} \]  

(C.0.1)

and let us denote by a tilde \( \tilde{\cdot} \) the quantities related to the physical metric \( \tilde{g} \).

First, we write down the formula for conformally related curvature tensors. For a proof, refer to [13].

**Proposition C.1** Let \( g \) and \( g_0 \) be two conformally related Lorentz metrics in \( M^n \), i.e. \( g = \Omega^2 \tilde{g} \) for some positive \( \Omega \in C^\infty(M) \), then

\[ \tilde{\text{Ric}} = \text{Ric} + \frac{n-2}{\Omega} \text{Hess} \Omega \left[ \frac{\Delta \Omega}{\Omega} - \frac{n-1}{\Omega^2} g(\nabla \Omega, \nabla \Omega) \right] g \]  

(C.0.2)

Now, we turn our attention to the energy-momentum tensor.
Proposition C.2 Let $T$ be any symmetric $(0,2)$ tensor on $M$. Then

$$\operatorname{div} T(X) = \frac{1}{\Omega^2} \tilde{\operatorname{div}} T(X) + \frac{(n-2)}{\Omega} T(\nabla \Omega, X) - \frac{X(\Omega)}{\Omega^3} \tilde{\operatorname{Tr}} T \tag{C.0.3}$$

Proof: Let $\{E_i\}$ be a $g$-orthonormal frame. Then $\{\tilde{E}_i\}$ is a $\tilde{g}$-orthonormal frame, where $\tilde{E}_i = \Omega E_i$ and notice also that $\tilde{\nabla} \Omega = \Omega^2 \nabla \Omega$. Consider now the formula for the divergence of $T$ in an orthonormal frame \[46\]:

$$\operatorname{div} T(X) = \sum_{i=1}^{n} \epsilon_i \nabla_{E_i} T(E_i, X) \tag{C.0.4}$$

$$= \sum_{i=1}^{n} \epsilon_i[ E_i(T(E_i, X)) - T(\nabla_{E_i} E_i, X) - T(E_i, \nabla_{E_i} X)]$$

We proceed to compute each one of the summands in the right hand side of \[C.0.4\].

The first term is easy to deal with:

$$E_i(T(E_i, X)) = \frac{1}{\Omega} \tilde{E}_i \left( \frac{1}{\Omega} T(\tilde{E}_i, X) \right)$$

$$= \frac{1}{\Omega^2} \tilde{E}_i(T(\tilde{E}_i, X)) + \frac{1}{\Omega} \tilde{E}_i \left( \frac{1}{\Omega} T(\tilde{E}_i, X) \right) \tag{C.0.5}$$

$$= \frac{1}{\Omega^2} \tilde{E}_i(T(\tilde{E}_i, X)) - \frac{1}{\Omega^2} \tilde{E}_i(\Omega) T(\tilde{E}_i, X)$$

To find a formula for the third term, let us recall first that the connections $\nabla$ and $\tilde{\nabla}$ are related by (see 1.159 in \[41\]):

$$\nabla_X Y = \tilde{\nabla}_X Y + \frac{X(\Omega)}{\Omega} Y + \frac{Y(\Omega)}{\Omega} X - \frac{\tilde{g}(X, Y)}{\Omega} \tilde{\nabla}(\Omega), \tag{C.0.6}$$

hence

$$\nabla_{E_i} X = \tilde{\nabla}_{E_i} X + \frac{\tilde{E}_i(\Omega)}{\Omega} X + \frac{X(\Omega)}{\Omega} \tilde{E}_i - \tilde{g}(\tilde{E}_i, X) \frac{\tilde{\nabla}(\Omega)}{\Omega}, \tag{C.0.7}$$

Thus

$$T(E_i, \nabla_{E_i} X) = T \left( \frac{\tilde{E}_i}{\Omega} \nabla_{\tilde{E}_i/\Omega} X \right) = \frac{1}{\Omega^2} T(\tilde{E}_i, \nabla_{\tilde{E}_i} X) \tag{C.0.8}$$
and then
\[
T(E_i, \nabla E_i, X) = \frac{1}{\Omega^2} T(\tilde{E}_i, \tilde{\nabla} E_i, X) + \frac{\tilde{E}_i(\Omega)}{\Omega^3} T(\tilde{E}_i, X) \tag{C.0.9}
\]
\[
+ \frac{X(\Omega)}{\Omega^3} T(\tilde{E}_i, \tilde{E}_i) - \frac{\tilde{g}(\tilde{E}_i, X)}{\Omega^3} T(\tilde{E}_i, \tilde{\nabla} \Omega).
\]

Finally, let us tackle the second term. First notice that
\[
T(\nabla E_i, E_i, X) = T(\tilde{\nabla}_{\tilde{E}_i/\Omega} \tilde{E}_i/\Omega + \frac{2\tilde{E}_i(\Omega)}{\Omega^3} \tilde{E}_i - \tilde{g}(\tilde{E}_i/\Omega, \tilde{E}_i/\Omega) \tilde{\nabla} \Omega, X), \tag{C.0.10}
\]
and also
\[
T(\tilde{\nabla}_{\tilde{E}_i/\Omega} \tilde{E}_i/\Omega, X) = \frac{1}{\Omega} T \left( \frac{1}{\Omega} \tilde{\nabla}_{\tilde{E}_i} \tilde{E}_i + \tilde{E}_i \left( \frac{1}{\Omega} \right) \tilde{E}_i, X \right)
\]
\[
= \frac{1}{\Omega^2} T(\tilde{\nabla}_{\tilde{E}_i} \tilde{E}_i, X) + \frac{1}{\Omega} T(\tilde{E}_i \left( \frac{1}{\Omega} \right) \tilde{E}_i, X) \tag{C.0.11}
\]
\[
= \frac{1}{\Omega^2} T(\tilde{\nabla}_{\tilde{E}_i} \tilde{E}_i, X) - \frac{1}{\Omega^3} \tilde{E}_i(\Omega) T(\tilde{E}_i, X),
\]
\[
T \left( \frac{2}{\Omega E_i(\Omega)} \Omega^3 \tilde{E}_i, X \right) = \frac{2}{\Omega^3} \tilde{E}_i(\Omega) T(\tilde{E}_i, X), \tag{C.0.12}
\]
\[
T(\tilde{g}(\tilde{E}_i/\Omega, \tilde{E}_i/\Omega) \tilde{\nabla} \Omega \left( \frac{\tilde{\nabla} \Omega}{\Omega^2} \right), X) = \frac{\tilde{g}(\tilde{E}_i, \tilde{E}_i)}{\Omega} T \left( \frac{\tilde{\nabla} \Omega}{\Omega^2}, X \right) = \frac{\epsilon_i}{\Omega} T(\nabla \Omega, X). \tag{C.0.13}
\]

Hence equations (C.0.11), (C.0.12) and (C.0.13) imply
\[
T(\nabla E_i, E_i, X) = \frac{1}{\Omega^2} T(\tilde{\nabla}_{\tilde{E}_i} \tilde{E}_i, X) + \frac{1}{\Omega^3} \tilde{E}_i(\Omega) T(\tilde{E}_i, X) - \frac{\epsilon_i}{\Omega} T(\nabla \Omega, X) \tag{C.0.14}
\]
Now we group all the above terms. First notice
\[
\sum_{i=1}^{n} \epsilon_i \left( \frac{1}{\Omega^3} \tilde{E}_i(\Omega) T(\tilde{E}_i, X) \right) = \frac{1}{\Omega^3} T \left( \sum_{i=1}^{n} \epsilon_i \tilde{E}_i(\Omega) \tilde{E}_i, X \right) \tag{C.0.15}
\]
\[
= \frac{1}{\Omega^3} T(\tilde{\nabla} \Omega, X) = \frac{1}{\Omega} T(\nabla \Omega, X)
\]
and similarly
\[
\sum_{i=1}^{n} \epsilon_i \left( \frac{\bar{g}(\bar{E}_i, X)}{\Omega^3} T(\bar{E}_i, \bar{\nabla} \Omega) \right) = \frac{1}{\Omega^3} T \left( \sum_{i=1}^{n} \epsilon_i \bar{g}(\bar{E}_i, X) \bar{E}_i, \bar{\nabla} \Omega \right)
\]  
(C.0.16)
\[
= \frac{1}{\Omega^3} T(X, \bar{\nabla} \Omega) = \frac{1}{\Omega} T(X, \nabla \Omega).
\]

On the other hand
\[
\sum_{i=1}^{n} \epsilon_i \left( \frac{X(\Omega)}{\Omega^3} T(\bar{E}_i, \bar{E}_i) \right) = \frac{X(\Omega)}{\Omega^3} \sum_{i=1}^{n} \epsilon_i T(\bar{E}_i, \bar{E}_i) = \frac{X(\Omega)}{\Omega^3} \bar{\text{Tr}} T,
\]  
(C.0.17)

hence by putting (C.0.9), (C.0.15), (C.0.16) and (C.0.19) together we get
\[
\sum_{i=1}^{n} \epsilon_i T(E_i, \nabla E_i) X = \frac{X(\Omega)}{\Omega^3} \bar{\text{Tr}} + T \frac{1}{\Omega^2} \sum_{i=1}^{n} T(\bar{E}_i, \bar{\nabla} \bar{E}_i) X.
\]  
(C.0.18)

Moreover, since \(\epsilon_i^2 = 1\) we have
\[
\sum_{i=1}^{n} \epsilon_i \left[ \epsilon_i T(\nabla \Omega, X) \right] = \frac{n}{\Omega} T(\nabla \Omega, X),
\]  
(C.0.19)

thus by (C.0.14), (C.0.15) and (C.0.19) we have
\[
\sum_{i=1}^{n} \epsilon_i T(\nabla E_i, E_i, X) = \frac{1-n}{\Omega} T(\nabla \Omega, X) + \frac{1}{\Omega^2} \sum_{i=1}^{n} \epsilon_i T(\bar{\nabla} \bar{E}_i, X).
\]  
(C.0.20)

Further combining (C.0.5) and (C.0.15) yields
\[
\sum_{i=1}^{n} \epsilon_i E_i(T(E_i, X)) = -\frac{1}{\Omega} T(\nabla \Omega, X) + \frac{1}{\Omega^2} \sum_{i=1}^{n} \epsilon_i \tilde{E}_i(T(\bar{E}_i, X))
\]  
(C.0.21)

Finally,
\[
\frac{1}{\Omega^2} \sum_{i=1}^{n} \epsilon_i \left( \tilde{E}_i(T(\bar{E}_i, X)) - T(\tilde{\nabla} \tilde{E}_i, X) - T(\bar{E}_i, \tilde{\nabla} \tilde{E}_i, X) \right) = \frac{1}{\Omega^2} \text{div} T(X).
\]  
(C.0.22)

Hence, by substituting (C.0.18), (C.0.20), (C.0.21) and (C.0.22) into (C.0.4) we get
\[
\text{div} T(X) = \frac{1}{\Omega^2} \text{div} T(X) + \frac{(n-2)}{\Omega} T(\nabla \Omega, X) - \frac{X(\Omega)}{\Omega^3} \bar{\text{Tr}} T
\]  
(C.0.23)

as desired. □