MINIMUM-TIME FRICCTIONLESS ATOM COOLING IN HARMONIC TRAPS

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Abstract. Frictionless atom cooling in harmonic traps is formulated as a time-optimal control problem and a synthesis of optimal controlled trajectories is obtained.

Key words. optimal control, optimal synthesis, atom cooling

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1. Introduction. During the last decades, a wealth of analytical and numerical tools from control theory and optimization have been successfully employed to analyze and control the performance of quantum mechanical systems, advancing quantum technology in areas as diverse as physical chemistry, metrology, and quantum information processing [1]. Although measurement-based feedback control [2] and the promising coherent feedback control [3] have gained considerable attention, open-loop control has been proven quite effective. Controllability results for finite- and infinite-dimensional quantum mechanical systems have been obtained, clarifying the control limits on these systems [4][11]. Analytical solutions for optimal control problems defined on low-dimensional quantum systems have been derived, leading to novel pulse sequences with unexpected gains compared with those traditionally used [12][20]. And numerical optimization methods, based on gradient algorithms or direct approaches, have been used to address more complex tasks and to minimize the effect of the ubiquitous experimental imperfections [21][29].

At the heart of modern quantum technology lies the efficient cooling of trapped atoms, since it has created the ultimate physical systems thus far for precision spectroscopy, frequency standards, and even tests of fundamental physics [30], as well as candidate systems for quantum information processing [31]. In the present article we study a time-optimal control problem related to the frictionless cooling of atoms trapped in a time-dependent harmonic potential. Frictionless atom cooling in a harmonic trapping potential is defined as the problem of changing the harmonic frequency of the trap to some lower final value, while keeping the populations of the initial and final levels invariant, thus without generating friction and heating. Conventionally, an adiabatic process is used where the frequency is changed slowly and the system follows the instantaneous eigenvalues and eigenstates of the time-dependent Hamiltonian. The drawback of this method is the long necessary times which may render it impractical. A way to bypass this problem is to use the theory of the time-dependent quantum harmonic oscillator [32] to prepare the same final states and energies as with the adiabatic process at a given final time, without necessarily following the instantaneous eigenstates at each moment. Achieving this goal in minimum time has many important potential applications. For example, it can be used to reach extremely low temperatures inaccessible by standard cooling techniques [33], to reduce the velocity dispersion and collisional shifts for spectroscopy and atomic clocks [34], and in adiabatic quantum computation [35]. It is also closely related to the problem of moving in
minimum time a system between two thermal states, as for example in the transition from graphite to diamond [36].

It was initially proved that minimum transfer time for the aforementioned problem can be achieved with “bang-bang” real frequency controls [36]. Later, it was shown that when the restriction for real frequencies is relaxed, allowing the trap to become an expulsive parabolic potential at some time intervals, shorter transfer times can be obtained, leading to a “shortcut to adiabaticity” [37]. In our recent work [38], we formulated frictionless atom cooling as a minimum-time optimal control problem, permitting the frequency to take real and imaginary values in specified ranges. We showed that the optimal solution has again a “bang-bang” form and used this fact to obtain estimates of the minimum transfer times for various numbers of switchings. In the present article we complete our previous work by fully solving the corresponding time-optimal control problem and obtaining the optimal synthesis. As the terminal point in the problem is varied, a rather unconventional and interesting switching structure involving cut-loci and discontinuous switching curves is revealed.

2. Formulation of the problem in terms of optimal control. The evolution of the wavefunction \( \psi(t, x) \) of a particle in a one-dimensional parabolic trapping potential with time-varying frequency \( \omega(t) \) is given by the Schrödinger equation

\[
\frac{i \hbar}{\partial t} \psi = \left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \frac{m\omega^2(t)}{2} x^2 \right] \psi, \tag{2.1}
\]

where \( m \) is the particle mass and \( \hbar \) is Planck’s constant; \( x \in \mathbb{R} \) and \( \psi \) is a square-integrable function on the real line. When \( \omega(t) \) is constant, the above equation can be solved by separation of variables and the solution is

\[
\psi(t, x) = \sum_{n=0}^{\infty} c_n e^{-i \frac{E_n^\omega}{\hbar} t} \Psi_n^\omega(x), \tag{2.2}
\]

where

\[
E_n^\omega = \left( n + \frac{1}{2} \right) \hbar \omega, \ n = 0, 1, \ldots \tag{2.3}
\]

are the eigenvalues and

\[
\Psi_n^\omega(x) = \frac{1}{\sqrt{2^n n!}} \left( \frac{m\omega}{\pi \hbar} \right)^{1/4} \exp \left( -\frac{m\omega}{2\hbar} x^2 \right) H_n \left( \frac{\sqrt{m\omega}}{\hbar} x \right) \tag{2.4}
\]

are the eigenfunctions of the corresponding time-independent equation

\[
\left( -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{m\omega^2}{2} x^2 \right) \Psi_n^\omega = E_n^\omega \Psi_n^\omega.
\]

Here \( H_n \) in (2.4) is the Hermite polynomial of degree \( n \). The coefficients \( c_n \) in (2.2) can be found from the initial condition

\[
c_n = \int_{-\infty}^{\infty} \psi(0, x) \Psi_n^\omega(x) dx.
\]

Consider now the case shown in Fig. 2.1 where \( \omega(t) = \omega_0 \) for \( t \leq 0 \) and \( \omega(t) = \omega_T < \omega_0 \) for \( t \geq T \). This corresponds to a temperature reduction by a factor \( \omega_T/\omega_0 \),
if the initial and final states are canonical [37]. For frictionless cooling, the path $\omega(t)$ between these two values should be chosen so that the populations of all the oscillator levels $n = 0, 1, 2, \ldots$ for $t \geq T$ are equal to the ones at $t = 0$. In other words, if

$$\psi(0, x) = \sum_{n=0}^{\infty} c_n(0) \Psi_{\omega_0}^{\omega_0}(x),$$

and

$$\psi(t, x) = \sum_{n=0}^{\infty} c_n(t) \Psi_{\omega_T}^{\omega_T}(x), \ t \geq T,$$

then frictionless cooling is achieved when

$$|c_n(t)|^2 = |c_n(0)|^2, \ t \geq T, \ n = 0, 1, 2, \ldots \quad (2.5)$$

Among all the paths $\omega(t)$ that result in (2.5), we would like to find one that achieves frictionless cooling in minimum time $T$. In the following we provide a sufficient condition on $\omega(t)$ for frictionless cooling and we use it to formulate the corresponding time-optimal control problem.

**Proposition 2.1.** If $\omega(t)$, with $\omega(0) = \omega_0$ and $\omega(t) = \omega(T) = \omega_T$ for $t \geq T$ is such that the Ermakov equation [39]

$$\ddot{b}(t) + \omega^2(t)b(t) = \frac{\omega_0^2}{b^3(t)} \quad (2.6)$$

has a solution $b(t)$ with $b(0) = 1, \dot{b}(0) = 0$ and $b(t) = b(T) = (\omega_0/\omega_T)^{1/2}, t \geq T$, then condition (2.5) for frictionless cooling is satisfied.

**Proof.** Without loss of generality we assume that the initial state is the eigenfunction corresponding to the $n$-th level $\psi(0, x) = \Psi_{\omega_0}^{\omega_0}(x)$. We will show that when the hypotheses of Proposition 2.1 hold then $\psi(t, x) = e^{i\alpha_n(t)}\Psi_{\omega_T}^{\omega_T}(x), t \geq T$, where $\alpha_n(t)$ is a global (independent of the spatial coordinate $x$) phase factor. This and the linearity of (2.1) imply that if $\psi(0, x) = \sum_{n=0}^{\infty} c_n(0) \Psi_{\omega_0}^{\omega_0}(x)$ then $\psi(t, x) = \sum_{n=0}^{\infty} c_n(0)e^{i\alpha_n(t)} \times \Psi_{\omega_T}^{\omega_T}(x), t \geq T$, thus condition (2.5) is satisfied.

*Fig. 2.1. Time evolution of the harmonic trap frequency.*


The frequency variations in the trapping potential change the time and distance scales and motivate the use of the following “ansatz”, introduced by Kagan et al. [40], in (2.1)

\[ \psi(t, x) = \frac{1}{\sqrt{b(t)}} \phi(\tau, \chi) \exp \left[ i \frac{m x^2 \dot{b}(t)}{2 \hbar} b(t) \right], \]

where \( \chi = x/b(t) \), \( \tau = \tau(t) \) is a time rescaling, and the distance scale \( b(t) \) satisfies (2.6) and the accompanying boundary conditions. We obtain

\[ i \hbar \frac{\partial \phi}{\partial \tau} \left( \frac{dt'}{dt} b^2 \right) = \left[ \frac{\hbar^2}{2m} \frac{\partial^2}{\partial \chi^2} + \frac{m (\ddot{b} + \omega^2 b^2)}{2} \phi \right] \]

(2.7)

If we choose the time scale \( \tau(t) \) such that

\[ \tau(t) = \int_0^t \frac{dt'}{b^2(t')}, \]

(2.8)

then (2.7) becomes

\[ i \hbar \frac{\partial \phi}{\partial \tau} = \left[ \frac{\hbar^2}{2m} \frac{\partial^2}{\partial \chi^2} + \frac{m \omega_0^2 \chi^2}{2} \right] \phi. \]

(2.7)

We will show that for \( t \geq T \), where \( b(t) = (\omega_0/\omega_T)^{1/2} \), \( \psi(t, x) \) has the desired form. We examine separately each of the three terms in (2.9). Since \( \dot{b}(t) = 0 \) in this time interval, the first exponential is equal to unity. About the second exponential, observe

\[ \tau(t) = \tau(T) + \frac{\omega_T}{\omega_0} (t - T), \]

(2.9)

We will show that for \( t \geq T \), where \( b(t) = (\omega_0/\omega_T)^{1/2} \), \( \psi(t, x) \) has the desired form. We examine separately each of the three terms in (2.9). Since \( \dot{b}(t) = 0 \) in this time interval, the first exponential is equal to unity. About the second exponential, observe

\[ \psi(t, x) = \exp \left[ i \frac{m x^2 \dot{b}(t)}{2 \hbar} \right] \times \exp \left[ -i \frac{E_n^{\omega_0}(\tau(t))}{\hbar} \right] \times \frac{1}{\sqrt{b(t)}} \Psi_n^{\omega_0}(x/b(t)) \]

(2.9)

The last term in (2.9) satisfies

\[ \left( \frac{\omega_T}{\omega_0} \right)^{1/4} \Psi_n^{\omega_0} \left( \frac{\omega_T}{\omega_0} x \right) = \Psi_n^{\omega_T}(x), \]

as it can be verified using (2.4). Putting all these together we see that \( \psi(t, x) \) has the desired form for \( t \geq T \).

In order to find the path \( \omega(t), 0 \leq t \leq T \), that accomplishes frictionless cooling in minimum time \( T \), we express the problem using the language of optimal control, incorporating possible restrictions on \( \omega(t) \) due, for example, to experimental limitations. If we set

\[ x_1 = b, \quad x_2 = \frac{\dot{b}}{\omega_0}, \quad u(t) = \frac{\omega^2(t)}{\omega_0^2}, \]

(2.10)
and rescale time according to \( t_{\text{new}} = \omega_0 t_{\text{old}} \), we obtain the following system of first order differential equations, equivalent to the Ermakov equation (2.6)

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -ux_1 + \frac{1}{x_1^4}.
\end{align*}
\] (2.11) (2.12)

If we set \( \gamma = (\omega_0 / \omega T)^{1/2} > 1 \), the time optimal problem takes the following form

**Problem 1.** Find \( -u_1 \leq u(t) \leq u_2 \) with \( u(0) = 1, u(T) = 1/\gamma^4 \) such that starting from \( (x_1(0), x_2(0)) = (1, 0) \), the above system reaches the final point \( (x_1(T), x_2(T)) = (\gamma, 0), \gamma > 1 \), in minimum time \( T \).

The boundary conditions on the state variables \( (x_1, x_2) \) are equivalent to those for \( b, \dot{b} \), while the boundary conditions on the control variable \( u \) are equivalent to those for \( \omega \), so the requirements of Proposition 2.1 are satisfied. Parameters \( u_1, u_2 > 0 \) define the allowable values of \( u(t) \) and it is \( u_2 \geq u(0) = 1 \). Note that the possibility \( \omega^2(t) < 0 \) (expulsive parabolic potential) for some time intervals is permitted, Chen et al. [37]. It is natural to consider that also \( u_1 \geq 1 \), i.e. we can at least achieve the negative potential \( V(x) = -m \omega_0^2 x^2 / 2 \). Finally observe that the above system describes the one-dimensional Newtonian motion of a unit-mass particle, with position coordinate \( x_1 \) and velocity \( x_2 \). The acceleration (force) acting on the particle is \( -ux_1 + 1/x_1^3 \). This point of view can provide useful intuition about the time-optimal solution, as we will see later.

In the next section we solve the following optimal control problem

**Problem 2.** Find \( -u_1 \leq u(t) \leq u_2 \), with \( u_1, u_2 \geq 1 \), such that starting from \( (x_1(0), x_2(0)) = (1, 0) \), the system above reaches the final point \( (x_1(T), x_2(T)) = (\gamma, 0), \gamma > 1 \), in minimum time \( T \).

In both problems the class of admissible controls formally are Lebesgue measurable functions that take values in the control set \( [-u_1, u_2] \) almost everywhere. However, as we shall see, optimal controls are piecewise continuous, in fact bang-bang. The optimal control found for problem 2 is also optimal for problem 1 with the addition of instantaneous jumps at the initial and final points, so that the boundary conditions \( u(0) = 1 \) and \( u(T) = 1/\gamma^4 \) are satisfied. Note that in connection with Fig. 2.1 a natural way to think about these conditions is that \( u(t) = 1 \) for \( t \leq 0 \) and \( u(t) = 1/\gamma^4 \) for \( t \geq T \); in the interval \( (0, T) \) we pick the control that achieves the desired transfer in minimum time.

3. **Optimal Solution.** The system described by (2.11), (2.12) can be expressed in compact form as

\[ \dot{x} = f(x) + ug(x), \] (3.1)

where the vector fields are given by

\[ f = \begin{pmatrix} x_2 \\ 1/x_1^3 \end{pmatrix}, \quad g = \begin{pmatrix} 0 \\ -x_1 \end{pmatrix} \] (3.2)

and \( x \in D = \{ (x_1, x_2) \in \mathbb{R}^2 : x_1 > 0 \} \) and \( u \in U = [-u_1, u_2] \). Admissible controls are Lebesgue measurable functions that take values in the control set \( U \). Given an admissible control \( u \) defined over an interval \( [0, T] \), the solution \( x \) of the system (3.1) corresponding to the control \( u \) is called the corresponding trajectory and we call the pair \( (x, u) \) a controlled trajectory. Note that the domain \( D \) is invariant in the
sense that trajectories cannot leave $D$. Starting with any positive initial condition $x_1(0) > 0$, and using any admissible control $u$, as $x_1 \to 0^+$, the “repulsive force” $1/x_1^3$ leads to an increase in $x_1$ that will keep $x_1$ positive (as long as the solutions exist).

For a constant $\lambda_0$ and a row vector $\lambda = (\lambda_1, \lambda_2) \in (\mathbb{R}^2)^*$ define the control Hamiltonian as

$$H = H(\lambda_0, \lambda, x, u) = \lambda_0 + \langle \lambda, f(x) + ug(x) \rangle.$$  

Then the conditions of the Pontryagin Maximum Principle [11] provide the following necessary conditions for optimality:

**Theorem 3.1 (Maximum principle for control affine time-optimal problems).** Let $(x_*(t), u_*(t))$ be a time-optimal controlled trajectory that transfers the initial condition $x(0) = x_0$ into the terminal state $x(T) = x_T$. Then it is a necessary condition for optimality that there exists a constant $\lambda_0 \leq 0$ and nonzero, absolutely continuous row vector function $\lambda(t)$ such that:

1. $\lambda$ satisfies the so-called adjoint equation:
   $$\dot{\lambda}(t) = -\frac{\partial H}{\partial x}(\lambda_0, \lambda(t), x_*(t), u_*(t)) = -\langle \lambda(t), Df(x_*(t)) + u_*(t)Dg(x_*(t)) \rangle$$

2. For $0 \leq t \leq T$ the function $u \mapsto H(\lambda_0, \lambda(t), x_*(t), u)$ attains its maximum over the control set $U$ at $u = u_*(t)$.

3. $H(\lambda_0, \lambda(t), x_*(t), u_*(t)) \equiv 0$.

We call a controlled trajectory $(x, u)$ for which there exist multipliers $\lambda_0$ and $\lambda(t)$ such that these conditions are satisfied an extremal. Extremals for which $\lambda_0 = 0$ are called abnormal. If $\lambda_0 < 0$, then without loss of generality we may rescale the $\lambda$’s and set $\lambda_0 = -1$. Such an extremal is called normal. Abnormal extremals typically correspond to some degeneracies in the structure of the optimal solution (often the value function is no longer differentiable along these paths), but they cannot be excluded a priori for time-optimal control problems. For example, the solution to the time-optimal control problem to the origin for the harmonic oscillator, a simple text book example, is largely characterised by two optimal abnormal controlled trajectories.

For the system [2.11], [2.12] we have

$$H(\lambda_0, \lambda, x, u) = \lambda_0 + \lambda_1 x_2 + \lambda_2 \left( \frac{1}{x_1^2} - x_1 u \right), \quad (3.3)$$

and thus

$$\dot{\lambda} = -\lambda \left[ \begin{array}{c} 0 \\ -\frac{3}{x_1^4} \\ 1 \\ 0 \end{array} \right] + u \left[ \begin{array}{c} 0 \\ 0 \\ -1 \\ 0 \end{array} \right] = -\lambda \left[ \begin{array}{c} 0 \\ -(u + 3/x_1^4) \\ 1 \\ 0 \end{array} \right] = -\lambda A \quad (3.4)$$

Observe that $H$ is a linear function of the bounded control variable $u$. The coefficient at $u$ in $H$ is $-\lambda_2 x_1$ and, since $x_1 > 0$, its sign is determined by $\Phi = -\lambda_2$, the so-called switching function. According to the maximum principle, point 2 above, the optimal control is given by $u = -u_1$ if $\Phi < 0$ and by $u = u_2$ if $\Phi > 0$. The maximum principle provides a priori no information about the control at times $t$ when the switching function $\Phi$ vanishes. However, if $\Phi(t) = 0$ and $\dot{\Phi}(t) \neq 0$, then at time $t$ the control switches between its boundary values and we call this a bang-bang switch. If $\Phi$ were to vanish identically over some open time interval $I$ the corresponding control is called singular.

**Proposition 3.2.** For Problem 3 optimal controls are bang-bang.
Proof. Whenever the switching function $\Phi(t) = -\lambda_2(t)$ vanishes at some time $t$, then it follows from the non-triviality of the multiplier $\lambda(t)$ that its derivative $\dot{\Phi}(t) = -\lambda_2(t) = \lambda_1(t)$ is non-zero. Hence the switching function changes sign and there is a bang-bang switch at time $t$. □

Thus optimal controls alternate between the boundary values $u = -u_1$ and $u = u_2$ of the control set and we shall see below that the number of switchings remains bounded on compact subsets of the domain $D$. Chattering controls that would have infinitely many switchings on a finite interval are not possible.

Definition 3.3. We denote the vector fields corresponding to the constant bang controls $-u_1$ and $u_2$ by $X = f - u_1 g$ and $Y = f + u_2 g$, respectively, and call the trajectories corresponding to the constant controls $u \equiv -u_1$ and $u \equiv u_2$ $X$- and $Y$-trajectories. A concatenation of an $X$-trajectory followed by a $Y$-trajectory is denoted by $XY$ while the concatenation in the inverse order is denoted by $YX$.

In this paper we establish the precise concatenation sequences for optimal controls and in particular calculate the times between switchings explicitly.

Proposition 3.4. All the extremals are normal.

Proof. If $(x,u)$ is an abnormal extremal trajectory that has a switching at time $t$, then, since $\lambda_2(t) = 0$, it follows from $H = 0$ that we must have $x_2(t) = 0$. The starting point is $(1,0)$ and suppose that $u = -u_1$ initially. From (2.12) it is $\dot{x}_2 > 0$ so $x_2 > 0$ and a switching at a point with $x_2(t) > 0$, not allowed for an abnormal extremal, is necessary in order to reach the target point $(\gamma,0)$. If $u = u_2$ initially then $\dot{x}_2(0) = 1 - u_2 < 0$ and $x_2 < 0$ for some time interval. During this time it is $\dot{x}_1 < 0$ and consequently $x_1 < 1 < \gamma$. A switching is necessary, which takes place on the $x_1$-axis for an abnormal extremal. The control changes to $u = -u_1$ and the situation is as before, where one more switching is necessary at a point with $x_2(t) > 0$, forbidden for abnormal extremals. Thus, there are no abnormal extremals in the optimal solutions. □

We henceforth only consider normal trajectories and set $\lambda_0 = -1$. For normal extremals, $H = 0$ then implies that for any switching time $t$ we must have $\lambda_1(t)x_2(t) = 1$. For an $XY$ junction we have $\dot{\Phi}(t) = \lambda_1(t) > 0$ and thus necessarily $x_2(t) > 0$ and analogously optimal $YX$ junctions need to lie in $\{x_2 < 0\}$. We now develop the precise structure of the switchings in a series of Lemmas. We start with computing the evolution of the state $x_1(t)$ along an $X$- or $Y$-trajectory.

Lemma 3.5 (Time evolution of $x_1$). The time evolution of $x_1$ along an $X$-trajectory in the upper quadrant starting from $(\alpha,0)$ is

$$x_1(t) = \sqrt{\frac{1}{2} \left( \alpha^2 - \frac{1}{u_1^2} \right) + \frac{1}{2} \left( \alpha^2 + \frac{1}{u_1^2} \right) \cosh(2\sqrt{u_1}t)},$$

while the corresponding evolution along a $Y$-trajectory in the lower quadrant starting from $(\beta,0)$ is

$$x_1(t) = \sqrt{\frac{1}{2} \left( \beta^2 + \frac{1}{u_2^2} \right) + \frac{1}{2} \left( \beta^2 - \frac{1}{u_2^2} \right) \cosh(2\sqrt{u_2}t)},$$

Proof. A first integral of the motion along the $X$-trajectory is

$$x_2^2 - u_1x_1^2 + \frac{1}{x_1^2} = c,$$
where \( c = -u_1 \alpha^2 + 1/\alpha^2 \). From (2.12) we observe that \( \dot{x}_2 \) is positive for \( u = -u_1 \) and since \( x_2(0) = 0 \) it follows that \( x_2(t) \) itself is positive. Hence

\[
x_2 = \frac{\sqrt{u_1 x_1^4 + cx_1^2 - 1}}{x_1}
\]

and (2.11) gives

\[
\dot{x}_1 = \frac{\sqrt{u_1 x_1^4 + cx_1^2 - 1}}{x_1}.
\]

Making a change of variables according to

\[
y = \frac{2u_1 x_1^2 + c}{\sqrt{c^2 + 4u_1}},
\]

the previous equation becomes

\[
\frac{dy}{\sqrt{y^2 - 1}} = 2\sqrt{u_1} dt.
\]

Integrating and using \( y(0) = 1 \) we obtain that

\[
\ln(y + \sqrt{y^2 - 1}) = 2\sqrt{u_1} t.
\]

From this and (3.8), equation (3.5) easily follows.

Similarly, a first integral of the motion along the \( Y \)-trajectory is given by

\[
x_2^2 + u_2x_1^2 + \frac{1}{x_1^2} = c,
\]

where now \( c = u_2 \beta^2 + 1/\beta^2 \). We are interested in the part of the trajectory in the lower quadrant, \( x_2 < 0 \), and thus

\[
x_2 = -\frac{\sqrt{-u_2 x_1^4 + cx_1^2 - 1}}{x_1}
\]

and

\[
\dot{x}_1 = -\frac{\sqrt{-u_2 x_1^4 + cx_1^2 - 1}}{x_1}.
\]

If we now make the change of variables

\[
y = \frac{2u_2 x_1^2 - c}{\sqrt{c^2 - 4u_2}}
\]

we obtain

\[
\frac{dy}{\sqrt{1 - y^2}} = -2\sqrt{u_1} dt.
\]

Integrating this and using \( y(0) = 1 \) we find that

\[
y = \cos(2\sqrt{u_2} t).
\]
From this and (3.10) we can easily derive (3.11). Note that in the calculation of \( y(0) \) we used that for evolution in the lower quadrant it necessarily holds that \( \dot{x}_2(0) < 0 \Rightarrow u_2\beta^2 > 1/\beta^2 \), so \( \sqrt{c^2 - 4u_2} = u_2\beta^2 - 1/\beta^2 \).

The times between consecutive switchings along optimal controls are determined by specific relations that we now derive.

**Lemma 3.6 (Inter-switching time).** Let \( p = (x_1, x_2) \) be a switching point and \( \tau \) denote the time to reach the next switching point \( q \). If \( \overrightarrow{pq} \) is a \( Y \)-trajectory, then

\[
\sin(2\sqrt{u_2}\tau) = -\frac{2\sqrt{u_2x_1x_2}}{x_2^2 + u_2x_1^2}, \quad \cos(2\sqrt{u_2}\tau) = \frac{x_2^2 - u_2x_1^2}{x_2^2 + u_2x_1^2} \tag{3.11}
\]

while, if \( \overrightarrow{pq} \) is an \( X \)-trajectory, then

\[
\sinh(2\sqrt{u_1}\tau) = -\frac{2\sqrt{u_1x_1x_2}}{x_2^2 - u_1x_1^2}, \quad \cosh(2\sqrt{u_1}\tau) = \frac{x_2^2 + u_1x_1^2}{x_2^2 - u_1x_1^2}. \tag{3.12}
\]

Note that the inter-switching times depend only on the ratio \( x_2/x_1 \).

**Proof.** These formulas are obtained as an application of the concept of a “conjugate point” for bang-bang controls as originally defined by Sussmann in [42] and [43]. For additional background on the synthesis of optimal controlled trajectories in the plane, we also refer the reader to the monograph [44] by Boscain and Piccoli that gives a comprehensive introduction to the theory of optimal control for 2-dimensional systems. In an effort to make the paper self-contained, we include Sussmann’s argument.

Without loss of generality assume that the trajectory passes through \( p \) at time 0 and is at \( q \) at time \( \tau \). Since \( p \) and \( q \) are switching points, the corresponding multipliers vanish against the control vector field \( g \) at those points, i.e., \( \langle \lambda(0), g(p) \rangle = \langle \lambda(\tau), g(q) \rangle = 0 \). We need to compute what the relation \( \langle \lambda(\tau), g(q) \rangle = 0 \) implies at time 0. In order to do so, we move the vector \( g(q) \) along the \( Y \)-trajectory backward from \( q \) to \( p \). This is done by means of the solution \( w(t) \) of the variational equation along the \( Y \)-trajectory with terminal condition \( w(\tau) = g(q) \) at time \( \tau \). Recall that the variational equation along \( Y \) is the linear system \( \dot{w} = Aw \) where \( A \) is given in (3.4). Symbolically, if we denote by \( e^{tY}(p) \) the value of the \( Y \)-trajectory at time \( t \) that starts at the point \( p \) at time 0 and by \( (e^{-\tau Y})_* \) the backward evolution under the linear differential equation \( \dot{w} = Aw \), then we can represent this solution in the form

\[
w(0) = (e^{-\tau Y})_* w(\tau) = (e^{-\tau Y})_* g(q) = (e^{-\tau Y})_* g(e^{\tau Y}(p)) = (e^{-\tau Y})_* \circ g \circ e^{\tau Y}(p).
\]

Since the “adjoint equation” of the Maximum Principle is precisely the adjoint equation to the variational equation, it follows that the function \( t \mapsto \langle \lambda(t), w(t) \rangle \) is constant along the \( Y \)-trajectory. Hence \( \langle \lambda(\tau), g(q) \rangle = 0 \) implies that

\[
\langle \lambda(0), w(0) \rangle = \langle \lambda(0), (e^{-\tau Y})_* g(e^{\tau Y}(p)) \rangle = 0
\]
as well. But the non-zero multiplier \( \lambda(0) \) can only be orthogonal to both \( g(p) \) and \( w(0) \) if these vectors are parallel, \( g(p) \parallel w(0) = (e^{-\tau Y})_* g(e^{\tau Y}(p)) \). It is this relation that defines the switching time.

It remains to compute \( w(0) \). For this we make use of the well-known relation [45]

\[
(e^{-\tau Y})_* \circ g \circ e^{\tau Y} = e^{\tau \text{ad}Y}(g) \tag{3.13}
\]

where the operator \( \text{ad}Y \) is defined as \( \text{ad}Y(g) = [Y, g] \), with \( [\cdot, \cdot] \) denoting the Lie bracket of the vector fields \( Y \) and \( g \). This representation is a consequence of the fact that
the derivative of the function \( \chi : t \mapsto (e^{-tY}, g(e^{tY}(p))) \) at \( t = 0 \) is given by \([Y, g](p)\) and iteratively the higher order derivatives of \( \chi \) at 0 are given by \( \chi^{(n)}(0) = \text{ad}^n Y(g) \) where, inductively, \( \text{ad}^n Y(g) = [Y, \text{ad}^{n-1} Y(g)] \). For our system, the Lie algebra \( \mathcal{L} \) generated by the fields \( f \) and \( g \) actually is finite dimensional: we have
\[
[f, g](x) = \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}
\]
and iteratively the higher order derivatives of \( \chi \) at 0 are given by
\[
\chi^{(n)}(0) = \text{ad}^n Y(g)
\]
where, inductively,
\[
\text{ad}^n Y(g) = \left[ Y, \text{ad}^{n-1} Y(g) \right].
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\]
and iteratively the higher order derivatives of \( \chi \) at 0 are given by
\[
\chi^{(n)}(0) = \text{ad}^n Y(g)
\]
where, inductively,
\[
\text{ad}^n Y(g) = \left[ Y, \text{ad}^{n-1} Y(g) \right].
\]
Fig. 3.1. Consecutive switching points lie on two opposite-slope lines through the origin. Blue curves correspond to X-segments, red curves to Y-segments.

for \( n = 0, 1, 2, \ldots \), and

\[
e^{t \sigma X}(g) = g + \frac{1}{2u_1} \sinh(2\sqrt{u_1}t)[f, g] + \frac{1}{2u_1} [\cosh(2\sqrt{u_1}t) - 1](f + u_1g).
\]

For \( t = \tau \) this field is parallel to \( g \) at \( p \) if and only if

\[
\sqrt{u_1}x_1 \sinh(2\sqrt{u_1}\tau) + x_2[\cosh(2\sqrt{u_1}\tau) - 1] = 0,
\]

from which we find

\[
\sinh(2\sqrt{u_1}\tau) = -\frac{x_2}{\sqrt{u_1}x_1} [\cosh(2\sqrt{u_1}\tau) - 1]. \tag{3.16}
\]

Using this relation we obtain (3.12). The solution \( \cosh(2\sqrt{u_1}\tau) = 1 \) corresponds to \( \tau = 0 \) and is rejected. ☐

**Lemma 3.7 (Main technical point).** The ratio of the coordinates of consecutive switching points has constant magnitude but alternating sign, while these points are not symmetric with respect to the \( x_1 \)-axis.

**Proof.** Consider the trajectory shown in Fig. 3.1 with switching points \((\kappa, \mu), (\zeta, \xi)\) and \((\lambda, \nu)\). Note that \((\kappa, \mu)\) is the intersection of an \( X \)-trajectory, passing from \((\alpha, 0)\) and a \( Y \)-trajectory passing from \((\beta, 0)\). From the first integrals \((3.7)\) and \((3.9)\) we have

\[
\mu^2 - u_1\kappa^2 + \frac{1}{\kappa^2} = -u_1\alpha^2 + \frac{1}{\alpha^2}, \tag{3.17}
\]

\[
\mu^2 + u_2\kappa^2 + \frac{1}{\kappa^2} = u_2\beta^2 + \frac{1}{\beta^2}. \tag{3.18}
\]

From these equations we obtain

\[
\frac{\mu}{\kappa} = \sqrt{(\kappa^2 - \alpha^2)(u_1\alpha^2\kappa^2 + 1)} \tag{3.19}
\]

and

\[
\beta^2 + \frac{1}{u_2\beta^2} = \left(\frac{u_1 + u_2}{u_2}\right)\alpha^2\kappa^2 - u_1\alpha^4 + 1. \tag{3.20}
\]
and after an elementary but a bit lengthy calculation

\[ \frac{1}{2} \left( \beta^2 - \frac{1}{u_2 \beta^2} \right) \cos(2\sqrt{u_2 \tau_0}) = \frac{(u_2 - u_1) \alpha^2 \kappa^2 + u_1 \alpha^4 - 1}{2u_2 \alpha^2}, \]

(3.22)

while from the corresponding relation for \( \kappa \)

\[ \kappa^2 = \frac{1}{2} \left( \beta^2 + \frac{1}{u_2 \beta^2} \right) + \frac{1}{2} \left( \beta^2 - \frac{1}{u_2 \beta^2} \right) \alpha \kappa^2 \cos(2\sqrt{u_2 \tau_0}). \]

Using (3.21), we obtain

\[ \frac{1}{2} \left( \beta^2 - \frac{1}{u_2 \beta^2} \right) \cos(2\sqrt{u_2 \tau_0}) = \frac{\alpha^2 \kappa^2}{2u_2 \alpha^2}, \]

(3.23)

Here we used that \( u_2 \beta^2 > 1/\beta^2 \), since \( \dot{x}_2 < 0 \) at \((\beta, 0)\), and \( \sin(2\sqrt{u_2 \tau_0}) > 0 \), since \( \tau_0 < T_0/2, T_0 = \pi/\sqrt{u_2} \) being the period of the closed orbit. For the terms involving the switching time we use (3.10), (3.13) to obtain

\[ \cos(2\sqrt{u_2 \tau_0}) = \frac{(u_1 - u_2) \alpha^2 \kappa^4 + (1 - u_1 \alpha^4) \kappa^2 - \alpha^2}{(u_1 + u_2) \alpha^2 \kappa^4 + (1 - u_1 \alpha^4) \kappa^2 - \alpha^2}, \]

(3.24)

and

\[ \sin(2\sqrt{u_2 \tau_0}) = \frac{-2 \kappa^2 \sqrt{u_2 \alpha^2(\kappa^2 - \alpha^2)(\alpha^2 \kappa^2 + 1)}}{(u_1 + u_2) \alpha^2 \kappa^4 + (1 - u_1 \alpha^4) \kappa^2 - \alpha^2}. \]

(3.25)

Observe from (3.10) that \( \sin(2\sqrt{u_2 \tau_0}) < 0 \) since \( \mu > 0 \). By using (3.22), (3.23), (3.24), (3.25) in (3.21), and the relations (3.20) and

\[ \xi^2 + u_2 \xi^2 + \frac{1}{\xi^2} = u_2 \beta^2 + \frac{1}{\beta^2}, \]

(3.26)

we obtain

\[ \zeta = \frac{\alpha \kappa}{\sqrt{(u_1 + u_2) \alpha^2 \kappa^4 + (1 - u_1 \alpha^4) \kappa^2 - \alpha^2}}, \]

(3.27)

\[ \xi = -\frac{\kappa \sqrt{(\kappa^2 - \alpha^2)(\alpha^2 \kappa^2 + 1)}}{(u_1 + u_2) \alpha^2 \kappa^4 + (1 - u_1 \alpha^4) \kappa^2 - \alpha^2}, \]

(3.28)

so

\[ \frac{\xi}{\zeta} = -\frac{\sqrt{(\kappa^2 - \alpha^2)(\alpha^2 \kappa^2 + 1)}}{\alpha \kappa^2} = -\frac{\mu}{\kappa}. \]

(3.29)
For the terms involving the switching time we use (3.12), (3.26) and find

\[ \xi^2 - u_1 \zeta^2 + \frac{1}{\zeta^2} = -u_1 \alpha_1^2 + \frac{1}{\alpha_1^2}. \] (3.30)

Using (3.26), (3.30) we obtain

\[ \alpha_1^2 - \frac{1}{u_1 \alpha_1^2} = \frac{(u_1 + u_2) \beta^2 \zeta^2 - u_2 \beta^4 - 1}{u_1 \beta^2}. \] (3.31)

and an alternative expression for \( \xi/\zeta \)

\[ \frac{\xi}{\zeta} = -\sqrt{\frac{(\beta^2 - \zeta^2)(u_2 \beta^2 \zeta^2 - 1)}{\beta \zeta^2}}. \] (3.32)

Starting from \((\zeta, \xi)\) let \(\tau_0, \tau_s\) now denote the time to reach the points \((\alpha_1, 0), (\lambda, \nu)\), respectively. Observe that \(\lambda\) satisfies (3.5) for \(t = \tau_s - \tau_0\) while \(\zeta\) satisfies this equation for \(t = -\tau_0\). From the first relation we get

\[ \lambda^2 = \frac{1}{2} \left( \alpha_1^2 - \frac{1}{u_1 \alpha_1^2} \right) + \frac{1}{2} \left( \alpha_1^2 + \frac{1}{u_1 \alpha_1^2} \right) \left[ \cosh(2\sqrt{u_1 \tau_0}) \cosh(2\sqrt{u_1 \tau_s}) - \sinh(2\sqrt{u_1 \tau_0}) \sinh(2\sqrt{u_1 \tau_s}) \right], \] (3.33)

while from the corresponding relation for \(\zeta\) and by using (3.34) we obtain

\[ \frac{1}{2} \left( \alpha_1^2 + \frac{1}{u_1 \alpha_1^2} \right) \cosh(2\sqrt{u_1 \tau_0}) = \frac{(u_1 - u_2) \beta^2 \zeta^4 + u_2 \beta^4 + 1}{u_1 \beta^2}, \] (3.34)

\[ \frac{1}{2} \left( \alpha_1^2 + \frac{1}{u_1 \alpha_1^2} \right) \sinh(2\sqrt{u_1 \tau_0}) = \sqrt{\frac{(\beta^2 - \zeta^2)(u_2 \beta^2 \zeta^2 - 1)}{u_1 \beta^2}}. \] (3.35)

For the terms involving the switching time we use (3.12), (3.26) and find

\[ \cosh(2\sqrt{u_1 \tau_s}) = \frac{(u_1 - u_2) \beta^2 \zeta^4 + (1 + u_2 \beta^4) \zeta^2 - \beta^2}{-(u_1 + u_2) \beta^2 \zeta^4 + (1 + u_2 \beta^4) \zeta^2 - \beta^2}; \] (3.36)

\[ \sinh(2\sqrt{u_1 \tau_s}) = \frac{2\zeta^2 \sqrt{u_1 \beta^2 (\beta^2 - \zeta^2)(u_2 \beta^2 \zeta^2 - 1)}}{-(u_1 + u_2) \beta^2 \zeta^4 + (1 + u_2 \beta^4) \zeta^2 - \beta^2}. \] (3.37)

By using (3.34), (3.35), (3.36), (3.37) in (3.33), and the relations (3.31) and

\[ \nu^2 - u_1 \lambda^2 + \frac{1}{\lambda^2} = -u_1 \alpha_1^2 + \frac{1}{\alpha_1^2}, \] (3.38)

we obtain

\[ \lambda = \frac{\beta \zeta}{\sqrt{-(u_1 + u_2) \beta^2 \zeta^4 + (1 + u_2 \beta^4) \zeta^2 - \beta^2}}, \] (3.39)

\[ \nu = \frac{\sqrt{(\beta^2 - \zeta^2)(u_2 \beta^2 \zeta^2 - 1)}}{\zeta \sqrt{-(u_1 + u_2) \beta^2 \zeta^4 + (1 + u_2 \beta^4) \zeta^2 - \beta^2}}. \] (3.40)
so

\[
\frac{\nu}{\lambda} = \frac{\sqrt{(\beta^2 - \zeta^2)(\beta^2 \zeta^2 - 1)}}{\beta \zeta^2} = -\frac{\xi}{\zeta}.
\]

(3.41)

Obviously, it is \( \lambda \neq \zeta \) in general so \( (\lambda, \nu) \neq (\zeta, -\xi) \), i.e. the subsequent switching point is different from the symmetric image of the previous switching point with respect to the \( x_1 \)-axis.

In the following proposition we use Lemma 3.7 to determine the form of the optimal trajectory.

**Proposition 3.8 (Form of the optimal trajectory).** The optimal trajectory can have the one-switching form \( XY \) or the spiral form \( YX \ldots YXY \) with an even number of switchings.

**Proof.** We first show that when the optimal trajectory has more than one switching, it cannot start with an \( X \)-segment. For just two switchings, consider the trajectory \( XYX \) depicted in Fig. 3.2, where \( \alpha = 1 \) (starting point), \((\gamma, 0), \gamma > 1 \) is the target point and \((\kappa, \mu), (\zeta, \xi) \) are the switching points. Since both of the switching points belong to the \( Y \)-segment passing through \((\beta, 0)\), their coordinates satisfy (3.18). If we denote by \( s \) the common ratio

\[
\frac{\mu^2}{\kappa^2} = \frac{\xi^2}{\zeta^2} = s,
\]

then both \( \kappa, \zeta \) satisfy the equation

\[
(s + u_2)x_1^4 - (u_2 \beta^2 + \frac{1}{\beta^2})x_1^2 + 1 = 0,
\]

so

\[
\kappa^2 \zeta^2 = \frac{1}{s + u_2} < 1,
\]

since \( u_2 \geq 1, s > 0 \). But also

\[
\kappa^2 \zeta^2 > 1,
\]

since \( \kappa^2 > 1 \) and \( \zeta^2 > \gamma^2 > 1 \). Thus this trajectory cannot be optimal.
For more switchings, consider the case shown in Fig. 3.1, where now \( \alpha = 1 \), and use \( s \) to denote the common ratio of the squares of the coordinates at the switching points. If \( \tau \) is the switching time between \( (\zeta, \xi) \) and \( (\lambda, \nu) \), then from (3.12) we obtain

\[
\frac{s}{u_1} = \frac{\cosh(2\sqrt{u_1}) + 1}{\cosh(2\sqrt{u_1}) - 1} > 1.
\]

But from (3.29) we find \( \alpha = 1 \)

\[
\frac{s}{u_1} = \frac{(u_1\kappa^2 + 1)(\kappa^2 - 1)}{u_1\kappa^4} < 1 \iff (u_1 - 1)\kappa^2 > -1,
\]

since \( u_1 \geq 1 \). Thus if the optimal trajectory has more than one switching, it needs to start with a \( Y \)-segment.

We next show that the optimal trajectory reaches the target point \((\gamma, 0)\), \( \gamma > 1 \) with a \( Y \)-segment. This is obviously the case for one switching, and also for two switchings since only the \( YXY \) trajectory is permitted (the \( XYX \) was excluded above). For more than two switchings consider the situation shown in Fig. 3.2. It is \( \mu^2/\kappa^2 = \xi^2/\zeta^2 = s \) and \( s > u_1 \) since at least one \( YXY \)-segment is included in the trajectory. Point \( (\zeta, \xi) \) belongs to the final \( X \)-segment ending to \((\gamma, 0)\), so

\[
(s - u_1)\xi^2 + \frac{1}{\xi^2} = -u_1\gamma^2 + \frac{1}{\gamma^2},
\]

The left hand side is positive, since \( s > u_1 \), while the right had side is negative, since \( \gamma > 1, u_1 \geq 1 \). Thus the optimal trajectory reaches the target point with a \( Y \)-segment.

Corollary 3.9. For \(|u| \leq 1\) the optimal solution has only one switching.

Proof. For \( u = u_2 = 1 \) the starting point \((1, 0)\) is an equilibrium point of system (2.11), (2.12). So the optimal trajectory cannot start with a \( Y \)-segment. The only trajectory thus permitted is \( XY \).

From Proposition 3.8 we see that the optimal trajectory can have aside from the expected one-switching form, shown in Fig. 3.3(a) the spiral form shown in Fig. 3.3(b) An intuitive understanding of this latter form can be obtained by viewing system equations (2.11), (2.12) as describing the motion of a unit mass particle with position \( x_1 \) and velocity \( x_2 \). In light of this interpretation we see that along a spiral trajectory the particle, instead of moving directly to the target, goes close to \( x_1 = 0 \), where there is a strong repulsive potential \( (1/x_1^3) \) to acquire speed and reach the target point faster. In the following theorem we calculate the transfer time for the candidate optimal trajectories.

Theorem 3.10. Starting from \((1, 0)\), the necessary time to reach the target point \((\gamma, 0), \gamma > 1 \) with one switching is

\[
T_0 = \frac{1}{\sqrt{u_1}} \sinh^{-1} \left( \frac{\sqrt{u_1(\gamma^2 - 1)(u_2\gamma^2 - 1)}}{\gamma^2(u_1 + u_2)(u_1 + 1)} \right) + \frac{1}{\sqrt{u_2}} \sin^{-1} \left( \frac{\sqrt{u_2(\gamma^2 - 1)(u_1\gamma^2 + 1)}}{(u_1 + u_2)(u_2\gamma^4 - 1)} \right).
\]

The necessary time to reach the target with \( n \) turns \((2n \text{ switchings})\) is

\[
T_n = T_I + nT_X + (n - 1)T_Y + T_F,
\]

(3.42)
where

\[
T_I = \frac{1}{2 \sqrt{u_2}} \cos^{-1} \left( \frac{-s c_1 + u_2 \sqrt{c_1^2 - 4(s + u_2)}}{(s + u_2) \sqrt{c_1^2 - 4u_2}} \right),
\]

\[
T_F = \frac{1}{2 \sqrt{u_2}} \cos^{-1} \left( \frac{-s c_{n+1} + u_2 \sqrt{c_{n+1}^2 - 4(s + u_2)}}{(s + u_2) \sqrt{c_{n+1}^2 - 4u_2}} \right),
\]

\[
T_X = \frac{1}{2 \sqrt{u_1}} \cosh^{-1} \left( \frac{s + u_1}{s - u_1} \right),
\]

\[
T_Y = \frac{1}{2 \sqrt{u_2}} \left( 2\pi - \cos^{-1} \left( \frac{s - u_2}{s + u_2} \right) \right),
\]

\[
c_1 = u_2 + 1,
\]

\[
c_{n+1} = u_2 \gamma^2 + \frac{1}{\gamma^2},
\]
and \( s \) is the solution of the transcendental equation
\[
\frac{c_1 + \sqrt{c_1^2 - 4(s + u_2)}}{c_{n+1} + \sqrt{c_{n+1}^2 - 4(s + u_2)}} = \left( \frac{s - u_1}{s + u_2} \right)^n
\]
(3.50)
in the interval \( u_1 < s \leq (u_2 - 1)^2/4 \). The constants \( c_1 \) and \( c_{n+1} \) characterize the first and the last Y-segments, respectively, of the trajectory. The number of turns satisfies the following inequality
\[
n \leq \left\lfloor \frac{T_0}{T_X(s_+)} \right\rfloor,
\]
(3.51)
where \( s_+ = (u_2 - 1)^2/4 \) and \( \lfloor \cdot \rfloor \) denotes the integer part.

Proof. In Fig. 3.3(a) we show a trajectory with one switching point \( B(\kappa, \mu) \). The coordinates of this point satisfy equations (3.17) and (3.18) with \( \alpha = 1 \) and \( \beta = \gamma \), from which we find
\[
\kappa^2 = \frac{u_2 \gamma^4 + 1 + \gamma^2(u_1 - 1)}{\gamma^2(u_1 + u_2)}
\]
Using (3.5) with \( \alpha = 1 \) and (3.6) with \( \beta = \gamma \), we find that the necessary transfer time is given by (3.42). Next consider the case with \( n \) turns and \( 2n \) switching points \( (\kappa_j, \mu_j) \), Fig. 3.3(b) with constant ratio \( \mu_j^2/\kappa_j^2 = s \). The first switching point satisfies the equations
\[
\begin{align*}
\mu_1^2 + u_2 \kappa_1^2 + \frac{1}{\kappa_1} &= c_1, \\
\mu_1^2 - u_1 \kappa_1^2 + \frac{1}{\kappa_1} &= c,
\end{align*}
\]
(3.52) (3.53)
where \( c_1 \) is given by (3.48) and \( c = -u_1 \alpha_1^2 + 1/\alpha_1^2 \), while the second switching point satisfies
\[
\begin{align*}
\mu_2^2 + u_2 \kappa_2^2 + \frac{1}{\kappa_2} &= c_2, \\
\mu_2^2 - u_1 \kappa_2^2 + \frac{1}{\kappa_2} &= c,
\end{align*}
\]
(3.54) (3.55)
where \( c_2 = u_2 \beta_1^2 + 1/\beta_1^2 \). The constants \( c_1 \) and \( c_2 \) characterize the first and second Y-segments of the trajectory, while the constant \( c \) characterizes the X-segment joining them. Subtracting (3.53) from (3.55) and using Lemma 3.7 which assures that \( \kappa_1 \neq \kappa_2 \) (consecutive switching points are not symmetric with respect to \( x_1 \)-axis) we find that
\[
s - u_1 - \frac{1}{\kappa_1 \kappa_2} = 0.
\]
(3.56)
But from (3.52), (3.54) and the constant ratio relation we find
\[
\begin{align*}
\kappa_1^2 &= \frac{2}{c_1 + \sqrt{c_1^2 - 4(s + u_2)}}, \\
\kappa_2^2 &= \frac{c_2 + \sqrt{c_2^2 - 4(s + u_2)}}{2(s + u_2)},
\end{align*}
\]
where, while solving the quadratic equations we used the $-\text{sign}$ for the first and the $+\text{sign}$ for the second switching point. The choice of sign for the first switching point will be justified below, while the choice of sign for consecutive switching points should be alternating to avoid picking the symmetric image of the previous point. Using these relations, (3.56) takes the form
\[
c_1 + \sqrt{c_1^2 - 4(s + u_2)} = \frac{s - u_1}{s + u_2}.
\]
By repeating the above procedure for all the consecutive pairs of switching points, we find
\[
\frac{c_i + \sqrt{c_i^2 - 4(s + u_2)}}{c_{i+1} + \sqrt{c_{i+1}^2 - 4(s + u_2)}} = \frac{s - u_1}{s + u_2}, \quad i = 1, 2, \ldots, n.
\]
Multiplying the above equations we obtain (3.50), one transcendental equation for the ratio $s$. If we choose the $+\text{sign}$ in the quadratic equation for $\kappa_1^2$, we obtain an equation similar to (3.50) but with inverted left hand side. It is $c_{n+1} > c_1 \Leftrightarrow (\gamma^2 - 1)(\gamma u_2^2 - 1) > 0$ and $c_1, c_{n+1} > 0$, so
\[
\frac{c_{n+1} + \sqrt{c_{n+1}^2 - 4(s + u_2)}}{c_1 + \sqrt{c_1^2 - 4(s + u_2)}} > 1 > \left(\frac{s - u_1}{s + u_2}\right)^n,
\]
and the corresponding transcendental equation has no solution. Note that the left hand side of (3.50) is a decreasing function of $s$ while the right hand side is an increasing one, so if a solution exists, it is unique. The ratio is bounded below by the requirement $s/u_1 > 1$ and above by $c_1^2 - 4(s + u_2) > 0 \Leftrightarrow s < s_+ = (u_2 - 1)^2/4$. This is also the maximum value of $s$ on the first $Y$-segment (3.52). Once we have calculated this ratio, we can find the time interval between consecutive switchings using (3.46) for an $X$-segment and (3.47) for a $Y$-segment, relations obtained from Lemma 3.6 on the inter-switching time. Observe that the times along all intermediate $X$- (respectively $Y$-) trajectories are equal. The initial time interval $T_I$ (from the starting point up to the first switching) and the final time interval $T_F$ (from the last switching up to the target point) can be easily calculated and are given in (3.44) and (3.45), respectively. The total duration $T_n$ of the trajectory with $n$ turns joining the points $(1,0)$ and $(\gamma,0)$ is given by (3.49). Observe that $T_n(s) > nT_X(s) \geq nT_X(s_+)$, where the last inequality follows from the fact that $T_X$ is a decreasing function of $s$, see (3.40). A solution with $n$ turns can be candidate for optimality only if the number of turns is bounded as in (3.51). Otherwise we have $T_n(s) > T_0$ and the one-switching strategy is faster.

We can find an approximate solution of (3.50) by setting $s = u_1$ (the lower limit) in the left hand side. We then obtain
\[
\hat{s}_n = \frac{u_1 + u_2 \sqrt{C}}{1 - \sqrt{C}}, \quad (3.57)
\]
where
\[
C = \frac{c_1 + \sqrt{c_1^2 - 4(u_1 + u_2)}}{c_{n+1} + \sqrt{c_{n+1}^2 - 4(u_1 + u_2)}}. \quad (3.58)
\]
Fig. 3.4. (a) Transfer times corresponding to zero, one and two turns for \( u_1 = 1, u_2 = 8, \gamma \in [1,10] \). (b) Switching curves (black curves) and characteristic optimal trajectories starting from \((1,0)\).

Due to the monotonicity of the right and left hand sides of (3.50), it is \( s_n \geq s_n \), where \( s_n \) is the exact solution. This approximation is good for \( s \) close to \( u_1 \) and thus for small \( n \). As the \( x_1 \) coordinate \( \gamma \) of the target point increases, the left hand side of (3.50) becomes less sensitive to variations in \( s \), making the approximation more accurate.

Using Theorem 3.10 we can find the times \( T_n \) for a specific target \((\gamma,0)\) and compare them to obtain the minimum time. This is done in the next section for specific values of the control bounds.

4. Examples. In Fig. 3.4(a) we plot the times \( T_0, T_1 \) and \( T_2 \) from Theorem 3.10 corresponding to zero, one and two turns, for \( u_1 = 1, u_2 = 8 \) and \( \gamma \in [1,10] \). For \( \gamma \leq \gamma_1 \) the strategy with zero turns (one switching) is optimal, while for \( \gamma \geq \gamma_1 \) it is the strategy with one turn (up to the range of \( \gamma \) plotted). The point \((\gamma_1,0)\) can be reached with both strategies in equal time, that is, it belongs to the cut-locus of these two control sequences from \((1,0)\). Note that the strategies with one and two turns are feasible after some \( \gamma > 1 \), where the transcendental equation (3.50) has a solution. In Fig. 3.4(b) we plot the switching curves (black curves) as well as some characteristic optimal trajectories starting from \((1,0)\). For \( \gamma \leq \gamma_1 \) the optimal
Fig. 3.5. (a) Transfer times corresponding to zero, one, two and three turns for $u_1 = 1$, $u_2 = 50$, $\gamma \in [1, 15]$. (b) Switching curves (black curves) and characteristic optimal trajectories starting from $(1, 0)$.

trajectory starts with an $X$-segment that coincides with the switching curve (black curve) passing from $(1, 0)$. It switches at some point and then travels along a $Y$-segment (red curve) to meet the $x_1$-axis. For $\gamma \geq \gamma_1$ the optimal trajectory starts with a $Y$-segment (red curve passing from $(1, 0)$) and switches at some point in the tiny black area of this curve to an $X$-segment (blue curve). Then it meets at some point the second switching curve on the upper quadrant and changes to a $Y$-segment (red curve) that hits the $x_1$-axis at the target point. Note that the optimal trajectories between the two switchings (blue curves) are very close to the second switching curve on the upper quadrant and they are not shown entirely.

In Fig. 3.5(a) we plot the times $T_0$, $T_1$, $T_2$ and $T_3$ from Theorem 3.10 corresponding to zero, one, two and three turns, for $u_1 = 1$, $u_2 = 50$ and $\gamma \in [1, 15]$. Again, for small $\gamma$ the one-switching strategy is optimal and after some $\gamma = \gamma_1$ the one-turn strategy becomes faster, but there is also a $\gamma = \gamma_2$ beyond which the two-turn strategy is optimal (up to the range of $\gamma$ plotted). The point $(\gamma_2, 0)$ thus belongs to the cut-locus of the one- and two-turn control sequences from $(1, 0)$ since it can be reached with one or two turns in equal time. In Fig. 3.5(b) we plot the switching curves (black curves) along with some characteristic optimal trajectories starting from $(1, 0)$. For
\[ \gamma \geq \gamma_2 \] the optimal trajectory makes an additional turn. This is demonstrated by the three adjacent Y-segments (red curves), which switch close to 0 to the corresponding X-segments (blue curves), on a tiny switching curve which is hardly seen. In turn, these trajectories switch on the third switching curve on the upper quadrant to Y-segments (red curves) that hit the \( x_1 \)-axis at the corresponding target points.

5. Conclusion. In this article we formulated frictionless atom cooling in harmonic traps in an optimal control language and solved the corresponding time-optimal problem for a fixed initial condition \((1, 0)\) and for varying terminal condition \((\gamma, 0), \gamma > 1\). The optimal synthesis was obtained and an interesting switching structure was revealed. The results presented here can be immediately extended to the frictionless cooling of a two-dimensional Bose-Einstein condensate confined in a parabolic trapping potential and even to the implementation of a quantum dynamical microscope, an engineered controlled expansion that allows to scale up an initial many-body state of an ultracold gas by a desired factor while preserving the quantum correlations of the initial state. The above techniques are not restricted to atom cooling but are applicable to areas as diverse as adiabatic quantum computing and finite time thermodynamic processes.

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