Characterizing Some Rings of Finite Order

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Abstract. In this paper, we compute the number of distinct centralizers of some classes of finite rings. We then characterize all finite rings with \( n \) distinct centralizers for any positive integer \( n \leq 5 \). Further we give some connections between the number of distinct centralizers of a finite ring and its commutativity degree.

1 Introduction

Finite abelian groups have been completely characterized up to isomorphism for a long time but finite rings have yet to be characterized. The problem of characterizing finite rings up to isomorphism has received considerable attention in recent years (see [2, 8, 10, 12, 13]) starting from the works of Eldridge [11] and Raghavendran [15]. In this paper we characterize finite rings in terms of their number of distinct centralizers. Given a ring \( R \) and an element \( r \in R \), the subrings \( C(r) = \{ s \in R : rs = sr \} \) and \( Z(R) = \{ s \in R : rs = sr \text{ for all } r \in R \} \) are known as centralizer of \( r \) in \( R \) and center of \( R \) respectively. We write \( \text{Cent}(R) \) to denote the set of all centralizers in \( R \). Firstly we compute the order of \( \text{Cent}(R) \) for some classes of finite rings \( R \). Motivated by the works of Belcastro and Sherman [3] and Ashrafi [1], we define \( n \)-centralizer ring for any positive integer \( n \). A ring \( R \) is said to be \( n \)-centralizer ring if \( |\text{Cent}(R)| = n \), for any positive integer \( n \). We then characterize \( n \)-centralizer finite rings for all \( n \leq 5 \), adapting similar techniques that are used by Belcastro and Sherman [3] in order to characterize \( n \)-centralizer finite groups for \( n \leq 5 \). It is worth mentioning that 6, 7-centralizer finite rings have been characterized in [9].

Further, we conclude the paper by noting some interesting connections between \( d(R) \) and \(|\text{Cent}(R)|\), where \( d(R) \) is the probability that a randomly chosen pair of elements of \( R \) commute. For any finite ring \( R \) we have \( d(R) = \frac{1}{|R|^2} \sum_{r \in R} |C(r)| \). This \( d(R) \) is also known as commutativity degree.

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degree or commuting probability of $R$ and it was introduced by MacHale [14] in the year 1976. Some characterizations of finite rings in terms of commutativity degree can be found in [14, 5, 6].

Throughout the paper $R$ denotes a finite ring possibly non-associative and non-unital. For any subring $S$ of $R$, $R/S$ and $\frac{R}{S}$ denote the additive quotient group and $|R : S|$ denotes the index of the additive subgroup $S$ in the additive group $R$. Note that the isomorphisms considered are the additive group isomorphisms. Also for any two non-empty subsets $A$ and $B$ of a ring $R$, we write $A + B = \{a + b : a \in A, b \in B\}$. We shall use the fact that for any non-commutative ring $R$, the additive group $\frac{R}{Z(R)}$ is not a cyclic group (see [14, Lemma 1]).

2 Some computations of $|Cent(R)|$

In this section, we compute $|Cent(R)|$ for some classes of finite rings. However, first we prove some results which are useful for subsequent results as well as for the next sections.

Proposition 2.1. $R$ is a commutative ring if and only if $R$ is a 1-centralizer ring.

Proof. The proposition follows from the fact that a ring $R$ is commutative if and only if $C(r) = R$ for each $r \in R$. □

Proposition 2.2. Let $R, S$ be two rings. Then

$$Cent(R \times S) = Cent(R) \times Cent(S).$$

Proof. It can be easily seen that $C((r, s)) = C(r) \times C(s)$ for any $r \in R$ and $s \in S$. This proves the proposition. □

The following lemmas play an important role in finding lower bound of $|Cent(R)|$ for any non-commutative ring $R$.

Lemma 2.1. Let $R$ be a ring. Then $Z(R)$ is the intersection of all centralizers in $R$.

Proof. It is clear that $Z(R) \subseteq \bigcap_{r \in R} C(r)$. Now, for any $s \in \bigcap_{r \in R} C(r)$ we have $rs = sr$ for all $r \in R$. Therefore $s \in Z(R)$. Hence the lemma follows. □

Lemma 2.2. If $R$ is a ring, then $R$ is the union of centralizers of all non-central elements of $R$.

Proof. It is clear that $\bigcup_{r \in R - Z(R)} C(r) \subseteq R$. Again, for any $s \in Z(R)$, we have by Lemma 2.1, $s \in C(r)$ for all $r \in R$. So $s \in \bigcup_{r \in R - Z(R)} C(r)$. Also for any $s \in R - Z(R)$, we have $s \in C(s)$ and so $s \in \bigcup_{r \in R - Z(R)} C(r)$. Hence the lemma follows. □
Lemma 2.3. A ring $R$ cannot be written as a union of two of its proper subrings.

Proof. The lemma follows from the well-known fact that a group cannot be written as a union of two of its proper subgroups. 

Theorem 2.1. For any non-commutative ring $R$, $|\text{Cent}(R)| \geq 4$.

Proof. Since $R$ is non-commutative, so $|\text{Cent}(R)| \geq 2$. If $|\text{Cent}(R)| = 2$, then, by Lemma 2.2, $R$ is equal to a proper subset of itself, which is not possible. Also by Lemma 2.3, $|\text{Cent}(R)| \neq 3$. Hence the theorem follows. 

Note that the ring $R = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right\}$, where $0, 1 \in \mathbb{Z}_2$, has 4 distinct centralizers. So the above result is the best one possible.

At this point, the following question, similar to the question posed by Belcastro and Sherman [3, p. 371], arises naturally.

Question 2.2. Does there exist an $n$-centralizer ring for any positive integer $n \neq 2, 3$? Can we characterize an $n$-centralizer ring?

The following results show the existence of $n$-centralizer rings for some values of $n$.

Proposition 2.3. There exists a $(p + 2)$-centralizer ring for any prime $p$.

Proof. We consider the ring $R = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in \mathbb{Z}_p \right\}$.

For any element $\begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix}$ of $C \left( \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \right)$ we have $xb - ay = 0$.

Clearly, $C \left( \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right) = R$. Using simple calculations, we have for any $a \neq 0$ and $l \in \mathbb{Z}_p$,

$C \left( \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \right) = \left\{ \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} : x \in \mathbb{Z}_p \right\}$ and $C \left( \begin{bmatrix} la & a \\ 0 & 0 \end{bmatrix} \right) = \left\{ \begin{bmatrix} lx & x \\ 0 & 0 \end{bmatrix} : x \in \mathbb{Z}_p \right\}$.

Hence $|\text{Cent}(R)| = p + 2$. 

The above proposition is a particular case of the following theorem.

Theorem 2.3. Let $R$ be a non-commutative ring of order $p^2$, where $p$ is a prime. Then $|\text{Cent}(R)| = p + 2$. 
Proof. For any \( x \in R - Z(R) \), we consider \( C(x) \). As \( C(x) \) is an additive subgroup of \( R \) we have \( |C(x)| = 1, p \) or \( p^2 \). Clearly, \( |C(x)| \neq 1, p^2 \), as \( x, 0_R \in C(x) \) and \( R \) is non-commutative, where \( 0_R \) is the additive identity in \( R \). Hence \( C(x) \) is additive cyclic group of order \( p \) and so \( Z(R) = \{ 0_R \} \).

Let \( x, y \in R - Z(R) \). If there exists an element \( t \neq 0_R \in C(x) \cap C(y) \) then \( C(x) = C(y) \), as \( C(x), C(y) \) are additive cyclic groups of order \( p \). Thus for any \( x, y \in R - Z(R) \) we have either \( C(x) \cap C(y) = \{ 0_R \} \) or \( C(x) = C(y) \). Therefore the number of centralizers of non-central elements is

\[
\frac{|R| - |Z(R)|}{p - 1} = \frac{p^2 - 1}{p - 1} = p + 1.
\]

Hence \( |\text{Cent}(R)| = p + 2 \).

Theorem 2.4. Let \( p \) be a prime number and \( R \) be a non-commutative ring of order \( p^3 \) with unity \( 1_R \). Then \( |\text{Cent}(R)| = p + 2 \).

Proof. Let \( x \) be an arbitrary element of \( R - Z(R) \). Then \( C(x) \) is an additive subgroup of \( R \) and so \( |C(x)| = 1, p, p^2 \) or \( p^3 \). Here \( |C(x)| \neq 1, p^3 \) as \( x, 0_R \in C(x) \), where \( 0_R \) is the additive identity in \( R \) and \( R \) is non-commutative. If \( |C(x)| = p \) then \( |Z(R)| = 1 \), which is not possible as \( 0_R, 1_R \in Z(R) \). So \( |C(x)| = p^2 \) and this gives \( |Z(R)| = p \).

Now, we suppose that \( y \in R - Z(R) \) and \( y \in C(x) \). Let \( z \in C(x) \) be an arbitrary element. We know that \( Z(R) \subset Z(C(x)) \) and so \( |Z(C(x))| > 1 \). Therefore \( |C(x) : Z(C(x))| = 1 \) or \( p \) and so \( C(x) \) is commutative. Thus \( z \in C(y) \), as \( y \in C(x) \). So \( C(x) \subseteq C(y) \). Also \( |C(x)| = |C(y)| \). Hence, \( C(x) = C(y) \); and if \( y \notin C(x) \) then \( C(x) \cap C(y) = Z(R) \). Therefore the number of centralizers of non-central elements of \( R \) is

\[
\frac{|R| - |Z(R)|}{|C(x)| - |Z(R)|} = \frac{p^3 - p}{p^2 - p} = p + 1.
\]

Thus \( |\text{Cent}(R)| = p + 2 \).

As an application of the above theorem, it follows that the ring

\[
R = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in \mathbb{Z}_p \right\}
\]

having order \( p^3 \) is a \((p + 2)\)-centralizer ring. The following theorem, which is a generalization of Theorem 2.3, gives another class of \((p + 2)\)-centralizer rings.

Theorem 2.5. Let \( R \) be a ring and \( \frac{R}{Z(R)} \cong \mathbb{Z}_p \times \mathbb{Z}_p \), where \( p \) is a prime. Then \( |\text{Cent}(R)| = p + 2 \).
Proof. We write $Z := Z(R)$. Since $R/Z \cong \mathbb{Z}_p \times \mathbb{Z}_p$ we have

$$\frac{R}{Z} = \langle Z + a, Z + b : p(Z + a) = p(Z + b) = Z; a, b \in R \rangle.$$ 

If $S/Z$ is additive non-trivial subgroup of $R/Z$ then $|S/Z| = p$. Therefore any additive proper subgroup of $R$ properly containing $Z$ has $p$ disjoint right cosets. Hence the proper additive subgroups of $R$ properly containing $Z$ are

$$S_m = Z \cup (Z + (a + mb)) \cup (Z + 2(a + mb)) \cup \cdots \cup (Z + (p - 1)(a + mb)),$$

where $1 \leq m \leq (p - 1)$,

$$S_p = Z \cup (Z + a) \cup (Z + 2a) \cup \cdots \cup (Z + (p - 1)a)$$

and

$$S_{p+1} = Z \cup (Z + b) \cup (Z + 2b) \cup \cdots \cup (Z + (p - 1)b).$$

Now for any $x \in R - Z$, we have $Z + x$ is equal to $Z + k$ for some $k \in \{ma, mb, a + mb, 2(a + mb), \ldots, (p - 1)(a + mb) : 1 \leq m \leq (p - 1)\}$. Therefore $C(x) = C(k)$. Again, let $y \in S_j - Z$ for some $j \in \{1, 2, \ldots, (p + 1)\}$, then $C(y) \neq S_q$, where $1 \leq q \neq j \leq (p + 1)$. Thus $C(y) = S_j$. Hence $|\text{Cent}(R)| = p + 2$. 

Further, we have the following theorem analogous to Lemma 2.7 of [1, p. 142].

**Theorem 2.6.** Let $R$ be a non-commutative ring whose order is a power of a prime $p$. Then $|\text{Cent}(R)| \geq p + 2$, and equality holds if and only if $\frac{R}{Z(R)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

Proof. Let $R$ be a non-commutative ring whose order is a power of a prime $p$. Suppose $k = |\text{Cent}(R)|$. Let $A_1, \ldots, A_k$ be the distinct centralizers of $R$ such that $|A_1| \geq \cdots \geq |A_k|$ and $A_1 = R$. So $R = \bigcup_{i=2}^{k} A_i$ and by Cohn’s Theorem in [7, p. 44], we have $|R| \leq \sum_{i=3}^{k} |A_i|$ (as $A_i$’s are additive groups). Also $|A_i| \leq \frac{|R|}{p}$, where $i \neq 1$. Hence

$$|R| \leq \frac{|R|}{p} + \cdots + \frac{|R|}{p} \underbrace{\text{ times}}_{(k - 2)\text{-times}},$$

which implies $|R| \leq (k - 2)\frac{|R|}{p}$ and so $k \geq p + 2$. That is $|\text{Cent}(R)| \geq p + 2$.

For the equality, if $\frac{R}{Z(R)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$ then by Theorem 2.5, we have $|\text{Cent}(R)| = p + 2$. Conversely, we assume that $l = |\text{Cent}(R)| = p + 2$. Suppose $A_1, A_2, \ldots, A_l$ are distinct centralizers of $R$ such that $|A_1| \geq \cdots \geq |A_l|$ and $A_1 = R$. So $R = \bigcup_{i=2}^{l} A_i$ and by Cohn’s Theorem in [7, p.
44], we have $|R| \leq \sum_{i=3}^{l} |A_i|$. Also $|A_i| \leq \frac{|R|}{p}$, where $i \neq 1$. Suppose, there exists an $A_i$ such that $|A_i| < \frac{|R|}{p}$ for $3 \leq i \leq l$ then

$$|R| < \frac{|R|}{p} + \cdots + \frac{|R|}{p} = (l-2)\frac{|R|}{p} = |R|,$$

a contradiction. Hence $|A_3| = \frac{|R|}{p}, \ldots, |A_l| = \frac{|R|}{p}$. Also $|A_2| \geq \cdots \geq |A_l|$, so $|A_i| = \frac{|R|}{p}$, where $2 \leq i \leq l$. Hence $\sum_{i=3}^{l} |A_i| = (l-2)\frac{|R|}{p} = |R|$. Therefore $\sum_{i=3}^{l} |A_i| = |R|$ if and only if $A_2 + A_m = R$, for all $m \neq 2$ and $A_k \cap A_l \subseteq A_2$ for all $k \neq l$ (By Cohn’s Theorem in [7, p. 44]). Interchanging $A_i$’s we have $A_2 \cap A_3 = Z(R)$. Thus

$$|R| = |A_2 + A_3| = \frac{|A_2||A_3|}{|A_2 \cap A_3|} = \frac{|R|^2}{p^2|Z(R)|}$$

which gives $|R : Z(R)| = p^2$. Hence $\frac{R}{Z(R)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$, since $R$ is non-commutative. This completes the proof. \hfill $\square$

3 \hspace{1cm} 4-centralizer rings

In this section, we give a characterization of finite 4-centralizer rings analogous to Theorem 2 of [3, p. 367]. The following lemma is useful in characterization of 4-centralizer rings.

**Lemma 3.1.** Let $R$ be a 4-centralizer finite ring. Then at least one of the centralizers of non-central elements has index 2 in $R$.

**Proof.** Let $A, B, C$ be the three proper centralizers of $R$. Suppose none of $A, B, C$ has index 2, that is $|R : A| \geq 3, |R : B| \geq 3, |R : C| \geq 3$. Then as $R = A \cup B \cup C$, we have

$$|R| \leq |A| + |B| + |C| - 2|Z(R)| \leq \frac{|R|}{3} + \frac{|R|}{3} + \frac{|R|}{3} - 2|Z(R)| < |R|,$$

which is a contradiction. Hence the lemma follows. \hfill $\square$

We have the following characterization of finite 4-centralizer rings.

**Theorem 3.1.** Let $R$ be a non-commutative finite ring. Then $|\text{Cent}(R)| = 4$ if and only if $\frac{R}{Z(R)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.\hfill $\square$
Proof. If \( \frac{R}{Z(R)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \) then by Theorem 2.5, we have \( |\text{Cent}(R)| = 4 \).

Conversely, let \( |\text{Cent}(R)| = 4 \) then \( R \) has exactly four distinct centralizers, say \( R, A, B, C \) where \( A, B, C \) are three distinct centralizers of non-central elements of \( R \).

By Lemma 2.3, \( R \) can not be written as the union of two of its proper subrings of \( R \). Therefore we may choose \( a \in A - (B \cup C) \), \( b \in B - (C \cup A) \) and \( c \in C - (A \cup B) \) respectively. It can be easily seen that \( C(a) = A \), \( C(b) = B \) and \( C(c) = C \). By Lemma 3.1, at least one of the centralizers \( A, B, C \), say \( A \) has index 2 in \( R \), that is \( |R : A| = 2 \).

Now, let \( x \in (A \cap B) - Z(R) \) then \( C(x) \neq R \). If \( C(x) = A \) then \( a, b \in C(x) \). So, \( C(x) \neq A \). Similarly it can be seen that \( C(x) \neq B \). If \( C(x) = C \) then \( x \in A \cap B \cap C = Z(R) \) (using Lemma 2.1), which is a contradiction. Therefore \( |\text{Cent}(R)| \) must be at least 5, which is again a contradiction. So \( A \cap B = A \cap B \cap C = Z(R) \). Similarly it can be seen that \( B \cap C = Z(R) \), \( A \cap C = Z(R) \). Again \( A, B, C \) are additive subgroups of \( R \), therefore

\[
|R| \geq |A + B| = \frac{|A||B|}{|A \cap B|} = \frac{|A||B|}{|Z(R)|}
\]

which gives \( |B| \leq 2|Z(R)| \). Since \( Z(R) \subset B \), so \( \frac{|B|}{2} \leq |Z(R)| < |B| \). Hence \( |B| = 2|Z(R)| \). Similarly \( |C| = 2|Z(R)| \). Therefore

\[
|R| = |A| + |B| + |C| - 2|Z(R)| = \frac{|R|}{2} + 2|Z(R)|
\]

which gives \( |R : Z(R)| = 4 \) and hence \( \frac{R}{Z(R)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \). \( \square \)

4 5-centralizer rings

In this section, we give a characterization of finite 5-centralizer rings analogous to Theorem 4 of [3, p. 369]. The following lemmas are useful in this regard.

Lemma 4.1. Let \( R \) be a ring and \( R = A \cup B \cup C \), where \( A, B, C \) are the proper distinct subrings. We put \( K = A \cap B \cap C, L = A \cap B - K, M = A \cap C - K, N = B \cap C - K \) and \( A' = A - (B \cup C), B' = B - (A \cup C), C' = C - (A \cup B) \). Then

(a) \( L = M = N = \emptyset \),
(b) \( A' + B' \subseteq C', B' + C' \subseteq A' \) and \( C' + A' \subseteq B' \),
(c) \( A' + A' \subseteq K, B' + B' \subseteq K \) and \( C' + C' \subseteq K \),
(d) \( |R : K| = 4 \).
Proof. (a) We consider \( l \in L \) and \( c' \in C' \). Then \( c' + l \in A \) or \( B \) or \( C \). If \( c' + l \in A \) then \( c' + l + (-l) = c' \in A \), a contradiction. If \( c' + l \in B \) then \( c' + l + (-l) = c' \in B \), a contradiction. If \( c' + l \in C \) then \( (-c') + c' + l = l \in C \), a contradiction. Since \( C' \neq \emptyset \), we must have \( L = \emptyset \). Similarly \( M = N = \emptyset \).

(b) Let \( a' \in A' \), then \( a' \in A \Rightarrow -a' \in A \Rightarrow -a' \in K \) or \( A' \). If \( -a' \in K \) then \( a' \in K \), a contradiction. Hence \( a' \in A' \). Similarly if \( b' \in B' \) then \( -b' \in B' \) and if \( c' \in C' \) then \( -c' \in C' \). Suppose \( a' \in A' \), \( b' \in B' \) then \( a' + b' \in K \) or \( A' \) or \( B' \) or \( C' \). If \( a' + b' \in A' \subseteq A \) then \( b' = -a' + a' + b' \in A \), a contradiction. If \( a' + b' \in B' \subseteq B \), then \( a' = a' + b' + (-b') \in B \), a contradiction. If \( a' + b' \in K \), then \( a' + b' \in A \), a contradiction. Hence \( a' + b' \in C' \). Thus \( A' + B' \subseteq C' \). Similarly it can be seen that \( B' + C' \subseteq A' \) and \( C' + A' \subseteq B' \).

(c) Let \( a', a_1' \in A' \subseteq A \). So \( a' + a_1' \in A \Rightarrow a' + a_1' \in A' \) or \( K \). Let \( a' + a_1' \in A' \). We consider \( b' + a' + a_1' \), for some \( b' \in B' \). Then by second part we have \( b' + (a' + a_1') \in C' \) and \( (b' + a') + a_1' \in B' \). So \( b' + a' + a_1' \in B' \cap C' \), a contradiction. Similarly we can show the other two.

(d) From part (a), we have \( R = K \cup A' \cup B' \cup C' \). Let \( k + a' \in K \cup A' \) where \( k \in K \), \( a' \in A' \) then \( k + a' \in A = K \cup A' \). If \( k + a' \in K \) then \( a' \in K \), a contradiction. So \( K + a' \subseteq A' \). Again \( x' \in A' \) gives \( x' + (-a') \in K \) (by part (c)). So, \( x' \in K + a' \). Hence \( K + a' = A' \). Similarly it can be seen that \( K + b' = B' \), \( K + c' = C' \), where \( b' \in B' \), \( c' \in C' \). Therefore \( |R : K| = 4 \).

Lemma 4.2. Let \( R \) be a 5-centralizer finite ring and \( A, B, C, D \) be the four proper centralizers of \( R \). Then

(a) \( |R| = |A| + |B| + |C| + |D| - 3|Z(R)| \).

(b) If \( S \) and \( T \) are distinct proper centralizers of \( R \), then

\[
\frac{|S| |T|}{|R|} \leq |Z(R)| \leq \frac{|R|}{6}.
\]

Proof. Let \( a \in A - (B \cup C) \), \( b \in B - (A \cup C) \) and \( c \in C - (A \cup B) \). Suppose there does not exist any \( a \in A - (B \cup C) \) such that \( C(a) = A \). Then \( C(a) = D \) for all \( a \in A - (B \cup C) \). Therefore \( A - (B \cup C) \subseteq D - (B \cup C) \). Interchanging the roles of \( A \) and \( D \) we get \( A - (B \cup C) = D - (B \cup C) \), which gives \( A \cup B \cup C = D \cup B \cup C = R \). Again, by Lemma 4.1(a), we have \( B \cap C = C \cap D \) and so \( Z(R) = A \cap B \cap C \). Therefore, by Lemma 4.1(d), we have \( R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \). This gives \( |\text{Cent}(R)| = 4 \), contradiction. Hence \( C(a) = A \). Similarly \( C(b) = B \) and \( C(c) = C \).

(a) Let us assume without loss of generality that \( D \) is a subset of \( A \cup B \cup C \). Then \( R = A \cup B \cup C \cup D = A \cup B \cup C \). Now, by Lemma 4.1, we have \( |R : K| = 4 \) where \( K = A \cap B \cap C = Z(R) \). Thus by Theorem 3.1, \( |\text{Cent}(R)| = 4 \), which is a contradiction. Therefore no one of \( A, B, C \) or \( D \) is contained in the union of the other three.
Let \( r \in (A \cap B) - (C \cup D) \) then \( r \in C(a) \cap C(b) \) which gives \( a, b \in C(r) \). But \( a \notin C(b) \), so \( C(r) \neq A, B \). Again \( r \notin C(b) \); so \( C(r) \neq B, C \). Also \( C(r) \neq R \), since \( r \in R - Z(R) \). Therefore \( |\text{Cent}(R)| \) must be at least 6, a contradiction. Hence \((A \cap B) - (C \cup D) = \emptyset \). This shows that no element of \( R \) is in exactly two proper centralizers.

Let \( r \in (A \cap B \cap C) - D \) then \( r \in C(a) \cap C(b) \cap C(c) \). Therefore \( a, b, c \in C(r) \). But \( b \notin C(a), c \notin C(b) \). So \( C(r) \neq A, B, C \). Also \( C(r) \neq D, R \); as \( r \notin D \) and \( r \notin Z(R) \). Therefore \( |\text{Cent}(R)| \) must be at least 6, a contradiction. Hence \( A \cap B \cap C - D = \emptyset \). Thus no element of \( R \) is in exactly three proper centralizers.

From above, it can be seen clearly that

\[
|R| = |A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - 3|Z(R)|.
\]

(b) Note that for any two proper centralizers \( S \) and \( T \) of \( R \) we have \( S \cap T = Z(R) \), since no element of \( R \) is in exactly two as well as three proper centralizers. Also any proper centralizers of \( R \) are additive subgroups of \( R \), so \( \frac{|S|\cdot|T|}{|S \cup T|} = |S \cap T| = |Z(R)| \). Since \( S + T \subseteq R \) we have \( |Z(R)| \geq \frac{|S|\cdot|T|}{|R|} \).

Again by part (a),

\[
|R| = |A| + |B| + |C| + |D| - 3|Z(R)|
\geq 2|Z(R)| + 2|Z(R)| + 2|Z(R)| + 2|Z(R)| - 3|Z(R)|.
\]

Thus \( |R : Z(R)| \geq 5 \). If \( |R : Z(R)| = 5 \) then \( \frac{R}{Z(R)} \cong \mathbb{Z}_5 \), a contradiction. Therefore \( |Z(R)| \leq \frac{|R|}{6} \). So, \( \frac{|S|\cdot|T|}{|R|} \leq |Z(R)| \leq \frac{|R|}{6} \). \( \square \)

We would like to mention here that the group theoretic analogues of Lemma 4.1 and Lemma 4.2 can be found in [4, p. 52-53] and [3, p. 370] respectively. Now we prove the main theorem of this section which characterizes finite 5-centralizer rings.

**Theorem 4.1.** Let \( R \) be a finite ring. Then \( |\text{Cent}(R)| = 5 \) if and only if \( \frac{R}{Z(R)} \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \).

**Proof.** Let \( \frac{R}{Z(R)} \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \), then by Theorem 2.5, we get \( |\text{Cent}(R)| = 5 \).

Conversely, let \( |\text{Cent}(R)| = 5 \). Let \( A, B, C, D \) be the four proper centralizers of \( R \). Then by Lemma 4.2(b), \( \frac{|A||B|}{|R|} \leq |Z(R)| \leq \frac{|R|}{6} \). Our aim is to get more near lower bound for \( |Z(R)| \). We may assume without loss of generality that \( |A| \geq |B| \geq |C| \geq |D| \). Suppose \( |A| < \frac{|R|}{3} \), as \( 1 < |A| \leq \frac{|R|}{2} \). That is \( |A| \leq \frac{|R|}{4} \). Now by Lemma 4.2(a), \( |R| \leq |R| - 3Z(R) \) \( < |R| \), a contradiction.

Hence \( |A| = \frac{|R|}{2} \) or \( |A| = \frac{|R|}{3} \). If \( |A| = \frac{|R|}{2} \), then \( |R| = |A| + |B| + |C| + |D| - 3|Z(R)| \) gives \( \frac{|R|}{2} < |B| + |C| + |D| \) and so \( \frac{|R|}{6} < |B| \). Also, applying Lemma 4.2(b) on \( A \) and \( B \) we have \( \frac{|R|}{6} < |B| \leq \frac{|R|}{3} \). So \( |B| \) is one of \( \frac{|R|}{3}, \frac{|R|}{4}, \) or \( \frac{|R|}{5} \). Reapplying Lemma 4.2(b) on \( A \) and \( B \) we have,

\[
\frac{|A||B|}{|R|} \leq |Z(R)| \leq \frac{|R|}{6}
\]
which gives \( \frac{|R|}{10} \leq |Z(R)| \leq \frac{|R|}{6} \). Thus \( |Z(R)| \) is one of \( \frac{|R|}{6}, \frac{|R|}{7}, \frac{|R|}{8}, \frac{|R|}{9}, \text{ or } \frac{|R|}{10} \). Let \( |Z(R)| = \frac{|R|}{7} \), \( \frac{|R|}{8} \) then 2 divides 7 and 9, which is not possible. If \( |Z(R)| = \frac{|R|}{6} \) then \( \frac{|R|}{Z(R)} \equiv Z_6 \), a contradiction. Let \( |Z(R)| = \frac{|R|}{8} \) then \( \frac{|R|}{3} \) divides \( |B| \). If \( |B| = \frac{|R|}{7}, \frac{|R|}{5} \) then 3, 5 divides 8, a contradiction. Therefore \( |B| = \frac{|R|}{8} \). By Lemma 4.2(a), we have \( \frac{5|R|}{8} = |C| + |D| \). Also \( |B| \geq |C| \geq |D| \). So \( |C| + |D| \leq \frac{|R|}{2} < \frac{5|R|}{8} = |C| + |D| \), a contradiction. If \( |Z(R)| = \frac{|R|}{10} \), then \( \frac{|R|}{Z(R)} \equiv Z_8 \). If \( |B| = \frac{|R|}{5} \) then 3, 4 divides 10, a contradiction. Therefore \( |B| = \frac{|R|}{8} \). Now Lemma 4.2(a) gives, \( |C| + |D| = \frac{6|R|}{10} \). Also \( |B| \geq |C| \geq |D| \), therefore \( |C| + |D| \leq \frac{2|R|}{5} < \frac{6|R|}{10} = |C| + |D| \), a contradiction.

If \( |A| = \frac{|R|}{3} \) then Lemma 4.2(a) gives, \( \frac{2|R|}{3} < |B| + |C| + |D| \) which gives \( \frac{2|R|}{3} < 3|B| \) and so \( |B| \geq \frac{|R|}{4} \). Also \( |A| \geq |B| \), so \( |B| = \frac{|R|}{3} \) or \( \frac{|R|}{4} \). Again, applying Lemma 4.2(b) on \( A \) and \( B \) we get,

\[
\frac{|A||B|}{|R|} \leq |Z(R)| \leq \frac{|R|}{6}
\]

which gives \( \frac{|R|}{12} \leq |Z(R)| \leq \frac{|R|}{6} \). Therefore \( |Z(R)| \) is one of \( \frac{|R|}{6}, \frac{|R|}{7}, \frac{|R|}{8}, \frac{|R|}{9}, \frac{|R|}{10}, \frac{|R|}{11} \) or \( \frac{|R|}{12} \). Now if \( |Z(R)| = \frac{|R|}{7}, \frac{|R|}{8}, \frac{|R|}{10}, \frac{|R|}{11} \) then 3 divides 7, 8, 10, 11, a contradiction. Let \( |Z(R)| = \frac{|R|}{9} \) then as above we get a contradiction. Let \( |Z(R)| = \frac{|R|}{9} \) then \( \frac{|R|}{Z(R)} \equiv Z_3 \times Z_3 \). Let \( |Z(R)| = \frac{|R|}{12} \) and \( |B| = \frac{|R|}{3} \) then applying Lemma 4.2(b) on \( A \) and \( B \) we have, \( \frac{|R|}{9} \leq \frac{|R|}{12} \), a contradiction. If \( |B| = \frac{|R|}{4} \) then Lemma 4.2(a) gives, \( |C| + |D| = \frac{4|R|}{6} \). Also \( |C|, |D| \leq \frac{|R|}{4} \), so \( |C| + |D| \leq \frac{3|R|}{6} < \frac{4|R|}{6} = |C| + |D| \), which is not possible. Hence \( \frac{|R|}{Z(R)} \equiv Z_3 \times Z_3 \).

5 Relation between \( |\text{Cent}(R)| \) and \( d(R) \)

Note that \( d(R) = 1 \) if and only if \( R \) is commutative. Therefore, by Proposition 2.1, we have \( |\text{Cent}(R)| = 1 \) if and only if \( d(R) = 1 \). By Theorem 3.1 and Theorem 1 of [14, p. 31], we have the following result.

**Proposition 5.1.** Let \( R \) be a non-commutative finite ring. Then \( |\text{Cent}(R)| = 4 \) if and only if \( d(R) = \frac{5}{8} \).

In [14, p. 31], MacHale also proved the following theorem:

**Theorem 5.1.** Let \( R \) be a non-commutative finite ring and \( p \) the smallest prime dividing the order of \( R \). Then \( d(R) \leq \frac{1}{p^2}(p^2 + p - 1) \), with equality if and only if \( |R : Z(R)| = p^2 \).

Now by Theorem 2.5 and Theorem 5.1, we have the following interesting connection between \( d(R) \) and \( |\text{Cent}(R)| \).

**Proposition 5.2.** Let \( R \) be a non-commutative finite ring and \( p \) the smallest prime dividing the order of \( R \). If \( d(R) = \frac{1}{p^2}(p^2 + p - 1) \) then \( |\text{Cent}(R)| = p + 2 \).
We conclude the paper by noting that the converse of Proposition 5.2 holds for some finite non-commutative rings. In particular, by Theorem 2.6 and Theorem 5.1, we have the following result.

**Proposition 5.3.** Let $R$ be a non-commutative ring whose order is a power of a prime $p$. If $|\text{Cent}(R)| = p + 2$ then $d(R) = \frac{1}{p^3}(p^2 + p - 1)$.

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