ON THE SPECTRAL ESTIMATES FOR THE
SCHRÖDINGER OPERATOR ON \( \mathbb{Z}^d \), \( d \geq 3 \)

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ABSTRACT. For the discrete Schrödinger operator we obtain sharp estimates for the number of negative eigenvalues.

1. Introduction

We study the estimates of the number of negative eigenvalues of the discrete Schrödinger operator

\[
H_{\alpha V} = -\Delta - \alpha V
\]

in the Hilbert space \( \ell_2(\mathbb{Z}^d) \), \( d \geq 3 \). We use the standard notation for the lattice points: \( x = (x_1, \ldots, x_d) \), with \( x_j \in \mathbb{Z} \) for each \( j \). The discrete Laplacian is

\[
(\Delta u)(x) = \sum_{j=1}^d (u(x + 1_j) + u(x - 1_j) - 2u(x)), \quad x \in \mathbb{Z}^d,
\]

where \( 1_j \) is the multi-index with all zero entries except 1 in the position \( j \). The Laplacian \( \Delta \) is a bounded operator, and the spectrum of \( -\Delta \) is absolutely continuous and coincides with \([0, 2d] \). The corresponding quadratic form is

\[
Q_0[u] = \sum_{x \in \mathbb{Z}^d} \sum_{j=1}^d |u(x + 1_j) - u(x)|^2.
\]

Being considered on the set \( \mathcal{F} \) of sequences \( u \) with finite support, the quadratic form \( Q_0 \) is non-degenerate. For \( u \in \mathcal{F} \) the Hardy inequality is fulfilled,

\[
Q_0[u] \geq c_d \sum_{x \in \mathbb{Z}^d} \frac{|u(x)|^2}{|x|^2 + 1}, \quad d \geq 3;
\]

see Section 4 where we discuss this inequality and its generalizations. It follows from (1.3) that the completion of \( \mathcal{F} \) in the metric generated by the quadratic form \( Q_0 \) is some Hilbert space of number sequences. We denote it by \( \mathcal{H}^1 = \mathcal{H}^1(\mathbb{Z}^d) \). The symbol \( V \) in (1.1) stands for the discrete potential \( V = V(x) \geq 0 \). We usually assume that \( V(x) \to 0 \) as
|x| \to \infty; \text{ then the operator of multiplication by } V \text{ is compact in } \ell_2(\mathbb{Z}^d). 

Finally, } \alpha \geq 0 \text{ in } (1.1) \text{ is a large parameter (the coupling constant).}

If } V \to 0 \text{ at infinity then for any } \alpha > 0 \text{ the essential spectrum } \sigma_{\text{ess}}(H_{\alpha V}) \text{ is the same as for } \alpha = 0, \text{ i.e., it is } [0, 2d]. \text{ The negative spectrum consists of a finite or countable set of eigenvalues, each of a finite multiplicity, with the only possible accumulation point at } l = 0. \text{ We denote by } N_-(H_{\alpha V}) \text{ the number of negative eigenvalues, counted with their multiplicities. If } V \text{ has infinite support, then } N_-(H_{\alpha V}) \to \infty \text{ as } \alpha \text{ grows. For finitely supported } V \text{ one obviously has}

\begin{equation}
N_-(H_{\alpha V}) \leq \# \{ x \in \mathbb{Z}^d : V(x) \neq 0 \}, \quad \forall \alpha > 0.
\end{equation}

Our main goal is to find order-sharp estimates for } N_-(H_{\alpha V}), \text{ depending on the properties of the potential } V \text{ and on the value of } \alpha.

A similar problem for the ‘continuous’ Hamiltonian is well studied, see e.g. the survey paper [18] and references therein. Here we recall the basic results for the latter problem, since the comparison of these two problems will be one of our main concerns.

The most important result for the continuous case is given by the Rozenblum – Lieb – Cwikel inequality. We present its most complete formulation, see Theorem 2.1 in [18]. This formulation goes back to the lectures [1], see Theorems 4.14, 4.15, and 4.17 there. For the Schrödinger operator on } \mathbb{R}^d \text{ we use the same notation (1.1) as for its discrete counterpart; the meaning of all terms in the formulas (1.5) and (1.6) below should be clear from the context.

**Theorem 1.1.** Let } d \geq 3. \text{ Then there exists a constant } C_{1.5} = C_{1.5}(d) \text{ such that for any } V \in L^2_\alpha(\mathbb{R}^d), \ V \geq 0,

\begin{equation}
N_-(H_{\alpha V}) \leq C_{1.5} \alpha \frac{d}{2} \int_{\mathbb{R}^d} V^{\frac{d}{2}} dx,
\end{equation}

and moreover, the Weyl asymptotic formula holds:

\begin{equation}
N_-(H_{\alpha V}) \sim \omega_d \alpha \frac{d}{2} \int_{\mathbb{R}^d} V^{\frac{d}{2}} dx, \quad \alpha \to \infty.
\end{equation}

Conversely, suppose that } d \geq 3, \text{ for a certain } V \geq 0 \text{ the operator } H_{\alpha V} \text{ is well defined (via its quadratic form) and bounded below for all } \alpha > 0, \text{ and } N_-(H_{\alpha V}) = O(\alpha^{\frac{d}{2}}) \text{ as } \alpha \to \infty. \text{ Then } V \in L^2_\alpha(\mathbb{R}^d), \text{ and, therefore, estimate (1.5) and asymptotic formula (1.6) are fulfilled.}

We immediately conclude from Theorem [1.1] that in the continuous case the behavior } N_-(H_{\alpha V}) = o(\alpha^{\frac{d}{2}}) \text{ is impossible, unless } V \equiv 0. \text{ The
growth faster than $N_{-}(H_{\alpha V}) = O(\alpha^{\frac{d}{2}})$ is possible, see the results in [3, 4]; some of them are also presented in [18].

In the discrete case an analogue of (1.5) remains valid, but other statements of Theorem 1.1 fail to be true. Let us formulate the result.

**Theorem 1.2.** Let $d \geq 3$. Then there exists a constant $C_{1.7} = C_{1.7}(d)$ such that for any $V \in \ell_{\frac{d}{2}}(\mathbb{Z}^d)$, $V \geq 0$,

$$(1.7) \quad N_{-}(H_{\alpha V}) \leq C_{1.7}\alpha^{\frac{d}{2}} \sum_{x \in \mathbb{Z}^d} V(x)^{\frac{d}{2}}.$$  

At the same time,

$$(1.8) \quad N_{-}(H_{\alpha V}) = o(\alpha^{\frac{d}{2}}).$$

The estimate (1.7) is known, see e.g. [11, 18]. The property (1.8) is quite elementary, but, probably, was observed for the first time in the survey paper [18]. See Section 2.2 below for a more detailed discussion of Theorem 1.2.

Comparing these theorems, we readily see not only analogies but also distinctions between the continuous and the discrete cases. There are also others facts of a similar nature which we are going to discuss.

Let us describe the structure of the paper. In the next Section 2 we present the necessary auxiliary material, and then we explain the proof of Theorem 1.2. In Section 3 we obtain the estimates of the type $N_{-}(H_{\alpha V}) = O(\alpha^{q})$ with $2q < d$. They have no analogues in the continuous case.

Technically, the case $2q > d$ is more involved than the previous one. The general multidimensional discrete Hardy type inequalities are an important ingredient of our approach. They are known in ‘mathematical folklore’, but we could not find any exposition of this material in the literature. To fill this gap, we give such an exposition in Section 4. We describe a wide class of discrete Hardy inequalities that can be derived directly from their classical analogues for $\mathbb{R}^d$.

Section 5 is devoted to the estimates $N_{-}(H_{\alpha V}) = O(\alpha^{q})$ with $2q > d$. Based upon the Hardy inequalities established in Section 4 we obtain the upper estimates similar to those known for the continuous Schrödinger operator. We also analyze an example showing that these estimates are order-sharp.

In Section 6 we analyze a special class of so-called sparse potentials. The estimates we derive for such potentials, depend not only on the values of $V(x)$ (as, say, in (1.7)), but also on the geometry of the support $\{x: V(x) \neq 0\}$. In particular, the results of this section allow
us to construct potentials $V$ such that

$$N_{-}(\mathbf{H}_{\alpha V}) = O(\alpha^{\frac{d}{2}}) \quad \text{but} \quad N_{-}(\mathbf{H}_{\alpha V}) \neq o(\alpha^{\frac{d}{2}}).$$

By (1.8), such potentials cannot lie in $\ell_{d}^{2}$.

The results of Section 6 show also that the class of Hardy inequalities, described in Section 4, does not cover all the possibilities. More precisely, we prove that any bounded sparse potential is a discrete Hardy weight, which may be considered as a rather unexpected result. We consider the material of Section 6 as the most important in the whole paper.

The concluding Section 7 is devoted to a discussion of some related results.

Our notation is rather standard, or it is explained in the course of the presentation. We denote by $C, c$, etc., without an index, various constants whose value is indifferent for us. Our notation for the more important constants is clear from the following example: $C_{1.5}$ is the constant appearing for the first time in the inequality (1.5). Sometimes, we explicitly indicate the parameters affecting the value of $C$.

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2. Preliminaries. Proof of Theorem 1.2

2.1. Birman – Schwinger operator. When studying the quantity $N_{-}(\mathbf{H}_{\alpha V})$, it is usually convenient to move on to another, but equivalent, setting of the problem.

Let $d \geq 3$, and let $V$ be a bounded, real-valued function on $\mathbb{Z}^{d}$. In the Hilbert space $\mathcal{H}^{1}(\mathbb{Z}^{d})$ we consider the quadratic form

$$b_{V}[u] = \sum_{x \in \mathbb{Z}^{d}} V(x)|u(x)|^{2}.$$  

If $b_{V}[u]$ is bounded in $\mathcal{H}^{1}(\mathbb{Z}^{d})$, it generates in this space a bounded linear operator, say $B_{V}$. We call it the Birman – Schwinger operator for our original problem. Since $V$ is supposed to be real-valued, the operator $B_{V}$ is self-adjoint. If $V \geq 0$, it is non-negative.

For a non-negative compact operator $T$, we denote by $l_{j}(T)$ its positive eigenvalues, counted according to their multiplicities and numbered in order of decrease. The symbol $n_{+}(s, T), s > 0$, stands for the
distribution function of the eigenvalues:
\[ n_+(s, \mathbf{T}) = \#\{ j : l_j(\mathbf{T}) > s \}. \]

The following statement is a particular case of the general Birman–Schwinger principle; see, e.g., [4] for its exposition.

**Proposition 2.1.** Let \( V \geq 0 \). If the operator \( B_V \) is compact, then for the Schrödinger operator (1.1) the number of negative eigenvalues is finite for any \( \alpha > 0 \). Moreover, the equality holds:

\[ N_-(H_\alpha V) = n_+(\alpha^{-1}, B_V), \quad \forall \alpha > 0. \] (2.2)

Conversely, if \( N_-(H_\alpha V) < \infty \) for all \( \alpha > 0 \), then the operator \( B_V \) is compact and, therefore, (2.2) is valid.

Recall that the modulus of an operator \( \mathbf{T} \) is defined as
\[ |\mathbf{T}| = (\mathbf{T}^*\mathbf{T})^{\frac{1}{2}}. \]
Evidently, \( |\mathbf{T}| = \mathbf{T} \) if the latter operator is non-negative. The compact operators, such that for some \( q > 0 \)
\[ n_+(s, |\mathbf{T}|) = O(s^{-q}), \]
or, equivalently, \( l_j(|\mathbf{T}|) = O(j^{-\frac{1}{q}}) \), form a linear space (an ideal in the algebra of all bounded operators). This space is usually denoted by \( \mathcal{S}_{q, \infty} \), or by \( \Sigma_q \). The latter notation is used in the book [2], and we keep it here. The spaces \( \Sigma_q \) are the ‘weak analogues’ of the classical Schatten ideals \( \mathcal{S}_q \). The functional

\[ \|\mathbf{T}\|_{\Sigma_q} = \sup_{s > 0} s n_+(s, |\mathbf{T}|)^{\frac{1}{q}} = \sup_j j^{\frac{1}{q}} l_j(|\mathbf{T}|) \] (2.3)

defines a quasi-norm on \( \Sigma_q \). If \( q > 1 \), and only in this case, an equivalent norm exists on \( \Sigma_q \). However, the quasi-norm (2.3) is more convenient for estimates.

The spaces \( \Sigma_q \) are complete and non-separable. The condition
\[ n_+(s, |\mathbf{T}|) = o(s^{-q}), \quad s \to 0, \]
or, equivalently, \( l_j(|\mathbf{T}|) = o(j^{-\frac{1}{q}}) \), singles out a closed separable subspace of \( \Sigma_q \), which we denote by \( \Sigma_q^\circ \). See Section 11.6 in the book [2] for more detail on the classes \( \Sigma_q \) and \( \Sigma_q^\circ \).

Along with the quasi-norm (2.3), let us consider the functionals
\[ \Delta_q(\mathbf{T}) = \limsup_{s \to 0} (s^q n_+(s, \mathbf{T})), \quad \delta_q(\mathbf{T}) = \liminf_{s \to 0} (s^q n_+(s, \mathbf{T})). \]

They are well-defined for any \( \mathbf{T} \in \Sigma_q \). The following elementary fact, see Theorem 11.6.7 in [2], will be useful later.
Proposition 2.2. The functionals $\Delta_q, \delta_q$ are continuous in the topology of the space $\Sigma_q$.

Proposition 2.1 allows one to replace the study of the function $N_-(H_\alpha V)$ with the investigation of the compact operator $B_V$. This is convenient, since it makes it possible to use the powerful machinery of the theory of compact operators.

For instance, for any $q > 0$ the powerlike estimate

$$N_-(H_\alpha V) \leq C\alpha^q, \quad \forall \alpha > 0$$

is equivalent to $B_V \in \Sigma_q$, with $\|B_V\|_{\Sigma_q} \leq C^1 q$. In a similar way,

$$N_-(H_\alpha V) = o(\alpha^q) \iff B_V \in \Sigma^\circ_q.$$

In particular, the estimate (1.7), being reformulated in terms of the operator $B_V$, reads in either of two equivalent forms:

$$(2.4) \quad \|B_V\|_{\Sigma^\circ_q} \leq C^\frac{1}{1,q} \|V\|_{\ell^q_d}; \quad n_+(s, B_V) \leq C_1 s^{\frac{-1}{d}} \sum_{x \in \mathbb{Z}^d} V(x)^{\frac{-1}{q}}.$$

The spaces $\Sigma_q$ and $\Sigma^\circ_q$ have their counterparts $\ell_{q,w}$ and $\ell^\circ_{q,w}$ in the theory of function spaces on countable sets. Let a function $V$ on $\mathbb{Z}^d$ be such that $V(x) \to 0$ as $|x| \to \infty$. By re-arranging the numbers $|V(x)|$, $x \in \mathbb{Z}^d$, in the non-increasing order, we obtain a sequence of non-negative numbers (notation $V^*_j$, $j \in \mathbb{N})$. Then $V^*_j \to 0$ as $j \to \infty$.

Besides, for any $s > 0$ we denote

$$(2.5) \quad E(s, V) = \{x \in \mathbb{Z}^d : |V(x)| > s\}; \quad \nu(s, V) = \#E(s, V).$$

By definition,

$$\{V \in \ell_{q,w}\} \iff \{\nu(s, V) = O(s^{-q})\} \iff \{V^*_j = O(j^{-\frac{1}{q}})\},$$

and

$$\{V \in \ell^\circ_{q,w}\} \iff \{\nu(s, V) = o(s^{-q})\} \iff \{V^*_j = o(j^{-\frac{1}{q}})\}.$$

The functional

$$\|V\|_{\ell_{q,w}} = \sup_{s > 0} s \nu(s, V)^{\frac{1}{q}} = \sup_j \frac{1}{q} V^*_j$$

defines the standard quasi-norm in $\ell_{q,w}$.

The next statement elucidates the role of classes $\ell_{q,w}$ and $\ell^\circ_{q,w}$ in the problems we are studying. For its proof, see Section 4.2 below.

Proposition 2.3. Let $d \geq 3$ and $V \in \ell_{\frac{d}{q},w}(\mathbb{Z}^d)$, $V \geq 0$. Then the operator $B_V$ is bounded. If $V \in \ell^\circ_{\frac{d}{q},w}(\mathbb{Z}^d)$, this operator is compact.
Note that the conditions given by this proposition are only sufficient, but not necessary. Say, they are violated for the sparse potentials considered in Section 6.

2.2. On Theorem 1.2. As it was already mentioned in Section 1, Theorem 1.2 is basically known. Still, in order to make our exposition self-contained, we outline its proof.

Proof of Theorem 1.2. It is well known that the discrete Laplacian generates a positivity preserving semigroup in the space $\ell^2(\mathbb{Z}^d)$. If $d \geq 3$, the following lower estimate, with some $c > 0$, is satisfied for the quadratic form $Q_0$ in (1.2):

$$Q_0[u] \geq c \left( \sum_{x \in \mathbb{Z}^d} |u(x)|^\sigma \right)^{\frac{2}{\sigma}}, \quad \sigma = \frac{2d}{d-2}, \quad \forall u \in H^1(\mathbb{Z}^d).$$

This is a particular case of Theorem IV.5.2 in [19]. Thus, the assumptions of Theorem 1.2 in [11] are fulfilled, whence the estimate (1.7).

Another way to obtain the latter estimate is to derive it from the results of the paper [17]; see also Section 9 in [18]. It is also possible to deduce (1.7) directly from (1.5). This will be explained in Section 5.1.

For justifying (1.8), let us consider the operator $B_V$. The sequences $V$ with finite support form a dense subset of $\ell^2(\mathbb{Z}^d)$. By (1.4), for such $V$ the non-zero spectrum of $B_V$ is finite and therefore, (1.8) is satisfied, so that $\Delta_B(V) = 0$. Using the inequality (2.4) and Proposition 2.2, we conclude that (1.8) extends to all non-negative potentials $V \in \ell^2(\mathbb{Z}^d)$.

The proof of Theorem 1.2 is complete.

Note that the estimate (1.8) does not replace the, formally weaker, estimate (1.7). An important property of (1.7) is the explicit dependence on the function $V$. It is this dependence that enabled us, by approximating $V$ by potentials with finite support, to pass to the limit and obtain (1.8).

3. Estimates in $\Sigma_q$, $q < \frac{d}{2}$

As it is clear from the comparison of Theorems 1.1 and 1.2, the estimates obtained in this section have no analogues for the operators on $\mathbb{R}^d$. We are going to show that, unlike in the continuous case, the function $N_{\pm}(H_{\alpha V})$ can grow as $O(\alpha^q)$ with arbitrarily small $q > 0$, which corresponds to the arbitrarily fast powerlike decay of the eigenvalues $l_j(B_V)$. 
**Theorem 3.1.** Let $d \geq 3$, $q < \frac{d}{2}$, and $V \in \ell_{q,w}(\mathbb{Z}^d)$, $V \geq 0$. Then $B_V \in \Sigma_q$, and

$$\|B_V\|_{\Sigma_q} \leq C_{3.1} \|V\|_{\ell_{q,w}},$$

with a constant depending only on $d$ and $q$. Equivalently,

$$N_-(H_{AV}) \leq C_{3.1}^q \alpha^q \|V\|_{\ell_{q,w}}^q.$$

For $q = \frac{d}{2}$ the statement is no more true: the inclusion $V \in \ell_{\frac{d}{2},w}$ guarantees only the boundedness of $B_V$.

**Proof of Theorem 3.1.** For a fixed $s$, let us split $V$ into two terms,

$$V = V_1 + V_2,$$

where $V_1(x) = V(x)$ at the vertices $x \in E(s,V)$, cf. (2.5), and $V_1(x) = 0$ otherwise. By the Weyl inequality (see, e.g., Theorem 9.2.9 in [2]),

$$n_+(s, B_V) \leq n_+(s/2, B_{V_1}) + n_+(s/2, B_{V_2}).$$

The first term in (3.2) is estimated by (1.4),

$$n_+(s/2, B_{V_1}) \leq \nu(s/2, V) \leq \left(2P/s\right)^q, \quad P = \|V\|_{\ell_{q,w}}.$$

For the second term we apply the estimate (2.4), with $V$ replaced by $V_2$. This gives

$$n_+(s/2, B_{V_2}) \leq 2^{\frac{d}{2}}C_{1.7}s^{-\frac{d}{2}} \sum_{V(x) \leq s} V(x)^{\frac{d}{2}}.$$

Now we transform the latter sum:

$$\sum_{V(x) \leq s} V(x)^{\frac{d}{2}} = -\int_0^s \tau^{\frac{d}{2}} d\nu(\tau, V) = \frac{d}{2} \int_0^s \tau^{\frac{d}{2} - 1} \nu(\tau, V) d\tau$$

$$\leq \frac{d}{2} P^q \int_0^s \tau^{\frac{d}{2} - 1 - q} d\tau = \frac{d}{d - 2q} s^{\frac{d}{2} - q} P^q.$$
Proof. For a fixed $\tau$, we consider the set $E(\tau, V)$. Let the sublattice $(2Z)^d$ consist of the points in $Z^d$ with all components divisible by 2. For any $k \in \{0, 1\}^d$ consider the set $(2Z)^d + k$. These sets are mutually disjoint. Therefore, the sets $\Omega_k(V) = E(\tau, V) \cap ((2Z)^d + k)$ are disjoint as well, and at least for one value of $k \in \{0, 1\}^d$, we have
$$\#\Omega_k(V) \geq 2^{-d} \nu(\tau, V).$$

Now, consider the subspace $L \subset H^1(Z^d)$ formed by the functions
$$u(x) = \sum_{y \in \Omega_k(V)} c_y \delta(x - y).$$
The functions $\delta(x - y)$, $y \in \Omega_k(V)$, are mutually orthogonal both in the metric of $H^1(Z^d)$ and with respect to the quadratic form $b_V$ in (2.1). So, for any $u \in L$ we have $Q_0[u] = 2d \sum_{y \in \Omega_k(V)} |c_y|^2$, while $b_V[u] = \sum_{y \in \Omega_k(V)} |c_y|^2 V(y)$. Since $V(y) \geq \tau$ for $y \in \Omega_k(V)$, we have constructed a subspace of dimension greater than $2^{-d} \nu(\tau, V)$ such that $b_V[u] \geq (2d)^{-1} \tau Q_0[u]$. This immediately implies (3.5) by the variational principle. □

Corollary 3.3. Let $d \geq 3$, $0 < q < \frac{d}{2}$, and $V \geq 0$. Then $B_V \in \Sigma_q$ if and only if $V \in \ell_{q,w}(Z^d)$, and, moreover, there are constants $c_0, c_1 > 0$ such that
$$c_0 \|V\|_{\ell_{q,w}(Z^d)} \leq \|B_V\|_{\Sigma_q} \leq c_1 \|V\|_{\ell_{q,w}(Z^d)},$$
and also
$$c_0 \limsup_{s \to 0} s^q \nu(s, V) \leq \Delta_q(B_V) \leq c_1 \limsup_{s \to 0} s^q \nu(s, V).$$

In particular,
$$N_-(H_{\alpha V}) = o(\alpha^q) \iff V \in \ell_{q,w}^c(Z^d).$$

Here only the relation (3.6) needs a justification. To this end, let us denote
$$R(V) = \limsup_{s \to 0} s^q \nu(s, V)^{\frac{1}{q}},$$
then $R(V) \leq \|V\|_{\ell_{q,w}}$. For any $\varepsilon > 0$, one can change the values of $V(x)$ at a finite number of points $x \in Z^d$ in such a way that for the new potential, say $V_\varepsilon$, we have
$$\|V_\varepsilon\|_{\ell_{q,w}} \leq R(V) + \varepsilon.$$
Then $\Delta_q(B_{V_\varepsilon}) = \Delta_q(B_V)$ and, by (3.1),
$$\Delta_q(B_V) \leq \|B_{V_\varepsilon}\|_{\Sigma_q}^q \leq C_{3.1}^q (R(V) + \varepsilon)^q.$$
Since $\varepsilon$ is arbitrary, we come to the right inequality in (3.6). The left inequality is a direct consequence of (3.5).
4. Discrete Hardy inequalities

Here we collect the material on the discrete multidimensional Hardy type inequalities. We need them for studying the estimates of the operator $B_V$ in the classes $\Sigma_q$ with $q > \frac{d}{2}$. As it was mentioned in the Introduction, this material should be considered as known on the ‘folklore level’. However, we could not find its exposition in the literature, and decided to present it here.

In this section we describe a rather simple class of discrete Hardy inequalities. Namely, with any function $W(x) \geq 0$ on $\mathbb{Z}^d$ we associate a function $W(\xi)$ on $\mathbb{R}^d$, which assumes the value $W(x)$ on the unit cell determined by the vertex $x$; see Subsection 4.1 for details. Then we show that if the “continuous” Hardy inequality is satisfied with the weight $W(\xi)$, then the discrete Hardy inequality with the weight $W(x)$ holds on $\mathbb{Z}^d$. Naturally, this class of discrete Hardy inequalities is rather restricted. Later, in Section 6.3, we will show that there exist discrete Hardy inequalities of a different origin.

4.1. Operator of poly-linear interpolation. Let $\mathcal{H}^1(\mathbb{R}^d), d \geq 3$, stand for the homogeneous Sobolev space. The metric in $\mathcal{H}^1(\mathbb{R}^d)$ is defined by the standard Dirichlet integral,

$$D[U] = \int_{\mathbb{R}^d} |\nabla U(\xi)|^2 d\xi.$$  

We introduce an operator $I : \mathcal{H}^1(\mathbb{Z}^d) \rightarrow \mathcal{H}^1(\mathbb{R}^d)$ in the following way. In every elementary cubic cell in $\mathbb{R}^d$ we interpolate $2^d$ values of $u$ at the vertices of the cell by a poly-linear function. For instance, the function $u(x), x \in \{0,1\}^d$, such that $u(0,\ldots,0) = 1$ and $u(x) = 0$ at the remaining vertices, interpolates as

$$U(\xi) = (Ju)(\xi) = \prod_j (1 - \xi_j), \quad \xi = (\xi_1, \ldots, \xi_d) \in [0, 1]^d.$$  

Such interpolation defines a mapping $I : u \mapsto U$ of the space $\mathcal{H}^1(\mathbb{Z}^d)$ to a certain subspace of piecewise poly-linear, continuous functions on $\mathbb{R}^d$.

**Lemma 4.1.** The image of the mapping $I$ is the space $\mathcal{H}^p_\infty(\mathbb{R}^d)$ of all piecewise poly-linear, continuous functions with the finite Dirichlet integral. The quadratic forms $Q_0[u]$, see (1.2), and $D[Ju] = \int_{\mathbb{R}^d} |\nabla (Ju)|^2 dx$ are equivalent: there are constants $c, c' > 0$ such that

$$cQ_0[u] \leq D[Ju] \leq c'Q_0[u], \quad \forall u \in \mathcal{H}^1(\mathbb{Z}^d).$$  


Proof. Consider the space $L(\mathcal{C})$ of poly-linear functions on the unit cell $\mathcal{C} = [0, 1]^d$. Clearly, $\dim L(\mathcal{C}) = 2^d$. On $L(\mathcal{C})$ we consider the quadratic forms

$$
\tilde{Q}[U; \mathcal{C}] = \sum_{x,y \in \{0, 1\}^d} \sum_{x \sim y} |U(x) - U(y)|^2; \quad \tilde{D}[U; \mathcal{C}] = \int_{\mathcal{C}} |\nabla U(\xi)|^2 d\xi.
$$

These two quadratic forms vanish on the same subspace in $L(\mathcal{C})$, consisting of constant functions. Therefore, they are equivalent, i.e., with some $c, c' > 0$ we have

$$
c\tilde{Q}[U; \mathcal{C}] \leq \tilde{D}[U; \mathcal{C}] \leq c'\tilde{Q}[U; \mathcal{C}].
$$

By adding up similar inequalities for all the cells $\mathcal{C} + x$, $x \in \mathbb{Z}^d$, we arrive at (4.1).

It remains to check that $Ju \in \mathcal{H}^1(\mathbb{R}^d)$ for any $u \in \mathcal{H}^1(\mathbb{Z}^d)$. It is sufficient to show this for the dense in $\mathcal{H}^1(\mathbb{Z}^d)$ subset of functions with finite support. But for any such $u$ the function $Ju$ has compact support and, therefore, it can be approximated in $\mathcal{H}^1$ by functions in $C_0^\infty$, in the metric of the Dirichlet integral. Hence, it lies in $\mathcal{H}^1(\mathbb{R}^d)$. $\square$

Now, with any non-negative function $W(x)$, $x \in \mathbb{Z}^d$, we associate a function $W = JW$ on $\mathbb{R}^d$, setting $W(\xi) = W(x)$ for $\xi \in \mathcal{C} + x$ where $\mathcal{C}$ is the same as above.

**Lemma 4.2.** For any weight $W$ on $\mathbb{Z}^d$ and the corresponding weight $W = JW$ on $\mathbb{R}^d$, one has

$$
\sum_{x \in \mathbb{Z}^d} W(x)|u(x)|^2 \leq C_{4.2} \int_{\mathbb{R}^d} W(\xi)|U(\xi)|^2 d\xi, \quad U = Ju,
$$

for any function $u$ on $\mathbb{Z}^d$. The constant in (4.2) depends only on $d$.

**Proof.** By linearity, it is sufficient to prove (4.2) for $W(x)$ having support at one point, say, at $0 \in \mathbb{Z}^d$. Then the quadratic forms in (4.2) define Hilbert seminorms on a $2^d$-dimensional space, with the null subspace for the quadratic form on the right-hand side contained in the null subspace for the one on the left-hand side. This gives the required inequality. $\square$

Note that the estimate inverse to (4.2) is impossible. Indeed, the null subspace for the quadratic form on the left-hand side is strictly wider than for the other one.
4.2. Hardy type inequalities. We will call a non-negative function \( W(x), \ x \in \mathbb{Z}^d \) a Hardy weight on \( \mathbb{Z}^d \), if for some constant \( H(W) \) the inequality is satisfied:

\[
\sum_{x \in \mathbb{Z}^d} W(x) |u(x)|^2 \leq H(W) Q_0[u], \quad \forall u \in \mathcal{H}^1(\mathbb{Z}^d).
\]

The best possible constant \( H(W) \) in (4.3) will be called the Hardy constant for \( W \). We will say that a Hardy weight \( W \) is normalized, if \( H(W) = 1 \). This definition carries over to the lattice case the classical definition of Hardy weights in \( \mathbb{R}^d \):

\[
\int_{\mathbb{R}^d} W(\xi)|U(\xi)|^2 d\xi \leq H(W) \int_{\mathbb{R}^d} |\nabla U|^2 d\xi, \quad \forall U \in \mathcal{H}^1(\mathbb{R}^d).
\]

In the continuous case the complete description of Hardy weights was found by Maz’ya, see [12], Ch. 8. There, the necessary and sufficient condition on a function \( W(\xi) \) to be a Hardy weight in \( \mathbb{R}^d \) is given in the terms of the capacity. By means of the well-known relation between capacity and measure, a sufficient condition for \( W \) to be a Hardy weight can be expressed in more elementary terms, see, e.g., Proposition 5.1 in [4].

**Proposition 4.3.** Any function \( W \geq 0 \) in the weak class \( L^d_{d,w}(\mathbb{R}^d) \), \( d \geq 3 \), is a Hardy weight on \( \mathbb{R}^d \) with Hardy constant \( H(W) \) satisfying \( H(W) \leq C(d) \| W \|_{L^d_{d,w}(\mathbb{R}^d)} \).

The discrete analogue of Proposition 4.3 also holds.

**Proposition 4.4.** Let \( W \geq 0 \) be a function defined on \( \mathbb{Z}^d \), \( d \geq 3 \). Suppose that \( W = JW \) is a Hardy weight on \( \mathbb{R}^d \). Then \( W \) is a Hardy weight on \( \mathbb{Z}^d \), and \( H(W) \leq C H(W) \). In particular, any \( W \in L^d_{d,w}(\mathbb{Z}^d) \) is a Hardy weight on \( \mathbb{Z}^d \) with \( H(W) \leq C(d) \| W \|_{L^d_{d,w}(\mathbb{Z}^d)} \).

**Proof.** The first statement follows immediately from the inequalities (4.1) and (4.2). As for the second statement, note that the functions \( W \) on \( \mathbb{Z}^d \) and \( W = JW \) on \( \mathbb{R}^d \) are equimeasurable, therefore \( W \in L^d_{d,w}(\mathbb{R}^d) \), with the same quasi-norm, so we can use Proposition 4.3. \( \square \)

The simplest example of a function in \( L^d_{d,w}(\mathbb{Z}^d) \) is \( W(x) = (|x|^2 + 1)^{-1} \). By Proposition 4.4 it is a Hardy weight on \( \mathbb{Z}^d \). This justifies the inequality (1.3).

Now we are in a position to prove Proposition 2.3. Indeed, the boundedness of the operator \( \mathbf{B}_V \) is just a re-formulation of the property
of $V$ to be a Hardy weight. Evidently, $B_V$ is compact for any $V$ with finite support. This property extends to the whole of $L^q_{2, w}$ by continuity.

5. **Estimates in $\Sigma_q$, $q > \frac{d}{2}$**

Having the Hardy type inequalities at our disposal, we now move on to the estimation of the operator $B_V$ in the classes $\Sigma_q$ with $q > \frac{d}{2}$. Results in this section are the direct analogues (actually, immediate consequences) of the corresponding results for the operators on $\mathbb{R}^d$, obtained in [3, 4]. In order to distinguish between the discrete and the continuous Laplacians, we (in this section only) will denote them by $\Delta_d$ and by $\Delta_c$ respectively.

5.1. **Eigenvalue estimates.** The material of Section 4 allows one to prove that any statement on the eigenvalue behavior of the operator $B_V$, expressed in terms of the standard, or weak $L^q$-classes, automatically implies its discrete counterpart. In particular, this shows that the estimate (1.7) follows directly from (1.5), thus giving one more proof of Theorem 1.2. Below we demonstrate, how this idea implements for obtaining the discrete analogues of the results in [3, 4].

Let a function $W$ be a normalized Hardy weight on $\mathbb{R}^d$, $d \geq 3$. The following is the formulation of Theorem 4.1 in [4]. For a function $F \geq 0$ on $\mathbb{R}^d$ we say that $F \in L^q,w(\mathbb{R}^d)$ if

$$\|F\|_{L^q,w(\mathbb{R}^d)}^q = \sup_{t > 0} \left( t^q \int_{F(\xi) > tW(\xi)} W^{d/2} d\xi \right) < \infty.$$ 

**Proposition 5.1.** Let $d \geq 3$, and let $W > 0$ be a normalized Hardy weight on $\mathbb{R}^d$. Suppose $\frac{d}{2} < q < \infty$ and $V \geq 0$. Then

$$V/W \in L^q,w(\mathbb{R}^d) \Rightarrow N_\alpha(-\Delta_c - \alpha V) \leq C_{5,1} \alpha^q \|V/W\|^q_{L^q,w(\mathbb{R}^d)}$$

where the constant depends on $d$ and $q$.

If, besides, $\lim_{t \to 0, \infty} \left( t^q \int_{V(\xi) > tW(\xi)} W^{d/2} d\xi \right) = 0$, then $N_\alpha(-\Delta_c - \alpha V) = o(\alpha^q)$.

We now can prove the discrete version of this Proposition. Let $W > 0$ be a Hardy weight on $\mathbb{Z}^d$, $d \geq 3$. For a function $F \geq 0$ on $\mathbb{Z}^d$, we say that $F \in \ell^q,w(\mathbb{Z}^d)$ if
Theorem 5.2. Let $d \geq 3$, and let $W > 0$ be a function on $\mathbb{Z}^d$, such that $W = \beta W$ is a normalized Hardy weight on $\mathbb{R}^d$. Suppose $\frac{d}{2} < q < \infty$ and $V \geq 0$. Then

\begin{equation}
\frac{V}{W} \in \ell_{q,w}(W^{d/2}) \Rightarrow N_-(-\Delta_d - \alpha V) \leq C_{5.1} \alpha^q \|V/W\|_{\ell_{q,w}(W^{d/2})}^q.
\end{equation}

If, besides, $\lim_{t \to 0, \infty} \left( t^q \sum_{V(x) > tW(x)} W(x)^{d/2} \right) = 0$, then $N_-(-\Delta_d - \alpha V) = o(\alpha^q)$.

Proof. Consider the operator $B_V$ in the space $\mathcal{H}^1(\mathbb{R}^d)$, defined by the quadratic form

$$b_V[U] = \int V(\xi)|U(\xi)|^2d\xi, \quad V = \beta V.$$

By the Birman-Schwinger principle, see Proposition 2.1 (or, more exactly, its continuous version), we have

$$N_-\left(\Delta - \alpha V \right) = n_+\left(\alpha^{-1}, B_V \right).$$

The function $V$ is equimeasurable with $V$, $W$ is equimeasurable with $W$, and $V/W$ is equimeasurable with $V/W$. Therefore the condition $V/W \in \ell_{q,w}(W^{d/2})$ implies $V/W \in L_{q,w}(W^{d/2})$, so we can apply Theorem 5.1 and obtain the estimate

\begin{equation}
n_+\left(\alpha^{-1}, B_V \right) \leq C_{5.1} \alpha^q \|V/W\|_{\ell_{q,w}(W^{d/2})}^q.
\end{equation}

Now we consider the operator $B^0_V$ defined by the same quadratic form $b_V[U]$, but restricted to the space $\mathcal{H}\mathcal{P}^1(\mathbb{R}^d)$ of piecewise poly-linear functions in $\mathcal{H}^1(\mathbb{R}^d)$. Since we narrowed the domain of the quadratic form, the eigenvalues of the operator cannot grow, so (5.2) leads to

\begin{equation}
n_+\left(\alpha^{-1}, B^0_V \right) \leq C_{5.1} \alpha^q \|V/W\|_{\ell_{q,w}(W^{d/2})}^q.
\end{equation}

By (4.1) and (4.2), the quadratic form $b_V[u]$ in (2.1) is estimated from above by $b_V[Iu]$, and the quadratic form $Q_0[u]$ in (1.2) is estimated from below by $\int |\nabla (Ju)|^2 d\xi$. Therefore the eigenvalues of the operator $B_V$ are majorized by the eigenvalues of the operator $B_V$, and the estimate (5.1) follows from (5.3). The second part of Theorem 5.2 is proved by a similar reasoning. \qed
Theorem 5.2 gives much freedom in choosing the Hardy weight $W$. The most standard choice is $W(x) = c(|x|^2 + 1)^{-1}$ where $c > 0$ is the normalizing constant. For this $W$, the corresponding particular case of Theorem 5.2 was presented in [18] as Theorem 9.2. Below we repeat its formulation, in order to correct a misprint in [18].

**Corollary 5.3.** Let $d \geq 3$ and $2q > d$, and let $V \geq 0$. Then

\[
N_-(H_\alpha V) \leq C(d, q) \alpha^q \sup_{t>0} \left( t^q \sum_{(|x|^2+1)V(x)>t} (|x|^2 + 1)^{-\frac{d}{2}} \right).
\]

5.2. **An example.** Here we analyze an example which shows that the estimate (5.4) is order-sharp for certain potentials $V$. Note that in the setting of Section 4, such examples are unnecessary, since its results give two-sided estimates of the quasi-norm $\|B_V\|_{\Sigma_q}$ for $2q < d$.

What we give below, is an analogue of Example 4.1 in [3]. However, in the discrete case we were not able to establish the asymptotic behavior of the eigenvalues $l_j(B_V)$, and we only give for them some lower estimate.

In our example the potential, for $|x| > 1$, has the form $V(x) = |x|^{-2}(\log(|x|))^{-\frac{1}{q}}, \quad 2q > d$.

By Corollary 5.3, for the corresponding Birman-Schwinger operator $B_V$, the upper eigenvalue estimate holds,

\[
n_+(s, B_V) \leq Cs^{-q}.
\]

We will show that this estimate cannot be improved. More specifically, we will prove that $B_V \not\in \Sigma_q^0$. To this end, we will construct an orthonormal in $H^1$ sequence of functions $\varphi_n$ such that

\[
b_V[\varphi_n] \geq cn^{-\frac{1}{q}}, \quad c > 0.
\]

Then the desired lower estimate will follow from the well known results on compact operators in a Hilbert space. Namely, it follows from Lemma II.4.1 in [8], or Theorem 11.5.7 in [2], that

\[
\sum_{n=1}^{m} l_n(B_V) \geq \sum_{n=1}^{m} b_V[\varphi_n], \quad \forall m \in \mathbb{N}.
\]

Since $\frac{1}{q} < \frac{2}{d} < 1$, this inequality and (5.5) imply

\[
\sum_{n=1}^{m} l_n(B_V) \geq cm^{1-\frac{1}{q}},
\]

which is inconsistent with $B_V$ being in the class $\Sigma_q^0$. 

So, it remains to construct the functions $\varphi_n$. We obtain them by normalizing a certain sequence $\{u_n\}$. For constructing the latter, we use the Fourier representation of functions on the lattice. With a function $u \in \mathcal{H}^1$ we associate its Fourier transform,

$$\hat{u}(z) = (2\pi)^{-d/2} \sum_{x \in \mathbb{Z}^d} u(x) e^{-ixz}, \quad z \in \mathbb{T}^d,$$

where $\mathbb{T}^d$ stands for the $d$-dimensional torus. Conversely,

$$u(x) = (2\pi)^{-d/2} \int_{\mathbb{T}^d} \hat{u}(z) e^{ixz} dz.$$

Then

$$Q_0[u] = \int_{\mathbb{T}^d} |\hat{u}(z)|^2 \omega(z) dz, \quad \omega(z) = 4 \sum_{j=1}^d \sin^2(z_j/2).$$

We take $h_n = 4^{-n}$ and set

$$v_n(z) = \hat{u}_n(z) = \chi \left( \frac{z_1}{h_n} - 3 \right) \prod_{j=2}^d \chi \left( \frac{z_j}{h_n} \right),$$

where $\chi$ is the characteristic function of the interval $(-1, 1)$. So, the function $v_n(z)$ has support in the square with the side length $2h_n$ and with the center moved by $3h_n$ in the direction of $z_1$ axis. Obviously, the supports are disjoint, so the functions $v_n$ are orthogonal in $L^2\omega$. The latter notation stands for the $L^2$-space with the weight $\omega$. Thus, the functions $u_n$ are mutually orthogonal in $\mathcal{H}^1$.

The functions $u_n$ can be calculated explicitly:

$$u_n(x) = (2\pi)^{-d/2} e^{3ih_n x_1} \prod_{j=1}^d 2 \sin(h_n x_j) \frac{\sin(h_n x_j)}{x_j}.$$

We have $\|u_n\|_{\mathcal{H}^1}^2 = C \|v_n\|_{L^2\omega}^2$. Since $\omega(z) \asymp h_n^2$ on the support of $u_n$, this gives $\|v_n\|_{L^2\omega} \asymp h_n^2 \|v_n\|_{L^2} \asymp h_n^{2+d}$. Now, from (5.7) we derive that

$$b_V[u_n] \geq c \sum_{x \in \mathbb{Z}^d} V(x) \prod_{j=1}^d \frac{4 \sin^2(h_n x_j)}{|x_j|^2}.$$

We need to estimate the last expression from below. To this end, we restrict summation in the last sum to the region $\Omega_n \subset \mathbb{Z}^d$ where all
\[ |h_nx_j| < \pi/2 \text{ and } |x| > 4. \]

In \( \Omega_n \), we have \( \sin^2(h_nx_j) \geq \left( \frac{2}{\pi} h_nx_j \right)^2 \).

Therefore,
\[
b_V[u_n] \geq c h_n^{2d} \sum_{x \in \Omega_n} V(x) = \sum_{4 < |x| \leq ch_n^{-1}} |x|^{-2} (\log |x|)^{-\frac{1}{q}}.
\]

A lower bound is given by the integral
\[ h_n^{2d} \int_4^{ch_n^{-1}} r^{-3+d} (\log r)^{-\frac{1}{q}} dr \asymp h_n^{-2+d} |\log h_n|^{-\frac{1}{q}}. \]

Therefore,
\[ b_V[u_n]/\|u_n\|_{H^1}^2 \geq c |\log h_n|^{-\frac{1}{q}} \geq cn^{-\frac{1}{q}}. \]

So, by normalizing the functions \( u_n \) we obtain the sequence \( \{\varphi_n\} \) satisfying (5.5).

It is interesting to notice that the test functions guaranteeing the lower estimate for \( 2q < d \) were constructed as having disjoint supports, while for \( 2q > d \) such functions have disjoint supports of their Fourier transform.

6. Sparse potentials

The results of Sections 3, 5 allow one to construct, for any prescribed value of \( q \neq \frac{d}{2} \), the potentials \( V \) such that \( B_V \in \Sigma_q \), but \( B_V \notin \Sigma_0^q \). For the borderline value \( q = \frac{d}{2} \), Theorem 1.2 leaves open the question of existence of such potentials.

In this section we consider a special class of potentials for which the theory can be advanced much further. In particular, we answer the above question by showing that for any number sequence \( p_j \not\to 0 \) such that \( p_{j+1}/p_j \to 1 \), a potential \( V \) does exist, such that the sequence of eigenvalues \( l_j(B_V) \) asymptotically behaves as \( \{p_j\} \).

6.1. Green function of the discrete Laplacian. The operator \((-\Delta)^{-1}\) acts as a discrete convolution, its kernel can be represented by the explicit formula:
\[ h_y(x) = h_0(x - y), \quad \forall y \in \mathbb{Z}^d, \]

where
\[ h_0(x) = (2\pi)^{-d} \int_{T^d} \frac{e^{ixz}}{4 \sum_{j=1}^{d} \sin^2(z_j/2)} dz. \]

Note that here the denominator coincides with the weight function \( \omega \) in (5.6).

We will call \( h_0 \) the Green function. One should be careful when studying its properties, since the point \( l = 0 \) lies in the spectrum of
However, all the difficulties can be easily overcome by systematic use of the representation (6.1). See, in particular, [5], where the case $d = 3$ is analyzed. For any $d > 3$ the reasoning is similar.

The function $h_0(x)$ lies in $\mathcal{H}^1(\mathbb{Z}^d)$, is harmonic outside the point $x = 0$, and its value at this point is

$$h_0(0) = \mu^2 = (2\pi)^{-d} \int_{\mathbb{T}^d} \frac{dz}{4 \sum_{j=1}^{d} \sin^2(z_j/2)}.$$

Then also $h_y(y) = h_0(0) = \mu^2$ for any $y \in \mathbb{Z}^d$. Besides,

$$(\Delta h_y)(y) = -1, \quad \forall y \in \mathbb{Z}^d.$$ 

Let $u$ be a function with finite support. Then summation by parts leads to the equality

$$\begin{align*}
(6.2) \quad (u, h_y) &= - \sum_{x \in \mathbb{Z}^d} u(x)(\Delta h_y)(x) = u(y).
\end{align*}$$

Starting from (6.2), the scalar products, and also the norms, are taken in $\mathcal{H}^1$. Since finitely supported functions are dense in $\mathcal{H}^1$, the equality (6.2) extends by continuity to all $u \in \mathcal{H}^1$. In particular,

$$\begin{align*}
(6.3) \quad (h_y, h_{y_1}) &= h_y(y_1) = h_0(y - y_1).
\end{align*}$$

Taking in (6.3) $y_1 = y$, we find that

$$\|h_y\| = \mu, \quad \forall y \in \mathbb{Z}^d.$$ 

We have $0 < h_0(x) \leq \mu^2$ for all $x \in \mathbb{Z}^d$. It also follows from the representation (6.1) that

$$\begin{align*}
(6.4) \quad h_0(x) &\leq C_{6.4}|x|^{-(d-2)}, \quad x \neq 0,
\end{align*}$$

with a constant depending only on $d$. The relations (6.3) and (6.4) show that for the points $y, y_1 \in \mathbb{Z}^d$ lying far enough from each other, the functions $h_y, h_{y_1}$, are ‘almost orthogonal’. It is convenient to normalize these functions, so that further on we work with

$$\begin{align*}
\tilde{h}_y = \mu^{-1} h_y.
\end{align*}$$

6.2. Sparse subsets in $\mathbb{Z}^d$. Let $Y$ be a subset in $\mathbb{Z}^d$, $d \geq 3$, and let $\mathcal{H}^1_Y$ stand for the subspace in $\mathcal{H}^1(\mathbb{Z}^d)$ spanned by the functions $h_y$, $y \in Y$. We say that the set $Y$ is sparse (or, in more detail, strongly sparse), if in $\mathcal{H}^1(\mathbb{Z}^d)$ there exists a compact operator $T$, such that the operator $I - T$ has bounded inverse and the functions

$$\begin{align*}
(6.5) \quad e_y = (I - T)^{-1} \tilde{h}_y, \quad y \in Y,
\end{align*}$$

form an orthonormal system in $\mathcal{H}^1$. 
We say that \( Y \) is weakly sparse, if in the above definition we replace the requirement of \( T \) being compact by its boundedness.

We are going to describe (in Lemma 6.2 below) a rather general way to construct sparse subsets. To this end, we need some preliminary material.

Suppose that a sequence \( \{ \psi_j \}, j \in \mathbb{N} \), of elements of a separable Hilbert space \( \mathcal{H} \) can be represented as

\[
\psi_j = (I - T)\varphi_j, \quad j \in \mathbb{N},
\]

where \( \{ \varphi_j \} \) is an orthonormal basis in \( \mathcal{H} \) and \( T \) is a linear operator of the class \( \mathcal{S}_q \) with some \( q, 0 < q \leq \infty \). Then we say that \( \{ \psi_j \} \) is a \( q \)-basis. We recall that \( \mathcal{S}_\infty \) stands for the space of all compact linear operators in \( \mathcal{H} \), and \( \mathcal{S}_q, \ 0 < q < \infty \), stands for the Schatten ideal; see, e.g., [2, 8].

The following result is due to Prigorski, see Theorem 5 in [16].

**Proposition 6.1.** Let \( \{ \psi_j \}, j \in \mathbb{N} \), be a complete and \( \omega \)-linearly independent sequence in \( \mathcal{H} \). It is a \( q \)-basis if and only if the matrix

\[
((\psi_j, \psi_k)_{\mathcal{H}} - \delta_{jk})
\]

belongs to the class \( \mathcal{S}_q \).

Here the property of \( \omega \)-linear independence of the system \( \{ \psi_j \} \) means that the assumptions

\[
\eta = \{ \eta_j \} \in \ell_2, \quad \sum_j \eta_j \psi_j = 0
\]

yield \( \eta = 0 \), see definition in [8], Section VI.2.4.

Note that for \( q = 2 \) the statement of Proposition 6.1 turns into that of Theorem VI.3.3 in [8]. The proof for the general case follows the same scheme as in [8].

Now we introduce the quantities that appear in the formulation of Lemma 6.2 below. Let \( Y \) be a subset of \( \mathbb{Z}^d \). Given a \( y \in Y \), we denote

\[
r_y = \text{dist}(y, Y \setminus \{y\})
\]

and

\[
[y] = \#\{x \in Y : |x| \leq |y|\}.
\]

**Lemma 6.2.** Suppose \( d \geq 3 \), and let \( Y \subset \mathbb{Z}^d \) be a set, such that \( [y]r_y^{-(d-2)} \to 0 \) as \( y \in Y, \ |y| \to \infty \). Suppose also that

\[
\sum_{y \in Y} r_y^{-(d-2)} \leq A, \quad \sup_{y \in Y} ([y]r_y^{-(d-2)}) \leq A, \quad \forall y \in Y,
\]
with a constant $A$ satisfying
\begin{equation}
2AC_{6.4} < \mu^2.
\end{equation}
Then the set $Y$ is strongly sparse.

\textbf{Proof.} Consider the Gram matrix
\begin{equation*}
G = \left( \langle \tilde{h}_x, \tilde{h}_y \rangle \right)_{x,y \in Y}.
\end{equation*}
Its diagonal elements are equal to one, and by (6.3), for every $y \in Y$ we have
\begin{equation}
\sum_{x \in Y, x \neq y} |\langle \tilde{h}_x, \tilde{h}_y \rangle| \leq C_{6.4} \mu^{-2} \left( \sum_{x \in Y : |x| \leq |y|} + \sum_{x \in Y : |x| > |y|} \right) |x-y|^{-(d-2)}
\end{equation}
(6.7)
\begin{equation*}
\leq C_{6.4} \mu^{-2} \left( [y] r_y^{-(d-2)} + \sum_{x \in Y : |x| > |y|} r_x^{-(d-2)} \right).
\end{equation*}

Since the matrix $G$ is Hermitian, the same inequality holds if the roles of $x$ and $y$ are interchanged. By the Shur test (see, e.g., Theorem 2.5.6 in [2]), this implies that the matrix $G-I$ defines a bounded operator in $\ell_2(Y)$ and, moreover, (6.6) yields $\|G-I\| < 1$.

In order to prove that $G-I \in \mathcal{S}_\infty$, we show that the ‘truncated’ matrices
\begin{equation*}
G_N = \left( \langle \tilde{h}_x, \tilde{h}_y \rangle \right)_{x,y \in Y : |x|,|y| \leq N}
\end{equation*}
converge to $G$ in the operator norm. For estimating $\|G-G_N\|$ we again use the Shur test. For the rows with $|y| > N$ the estimate (6.7) survives, and under the assumptions of Lemma the sums on the right are small if $N$ is large enough. If $|y| \leq N$, then this sum should be replaced by $\sum_{x \in Y : |x| > |y|} r_x^{-(d-2)}$ which is also small. So, $G-I \in \mathcal{S}_\infty$.

Next, we check that the system $\{\tilde{h}_y\}_{y \in Y}$ is $\omega$-linearly independent. Indeed, by taking the scalar products with $\tilde{h}_x$, $x \in Y$, we come to the infinite system of linear homogeneous equations, $G\eta = 0$. Since $\|G-I\| < 1$, the operator $G$ is invertible in $\ell_2$, so that the only $\ell_2$-solution of this system is trivial: $\eta = 0$, and we are done.

Now Proposition [6.1] applies, with $q = \infty$. So, we get an operator $T_0 \in \mathcal{S}_\infty$ in the space $\mathcal{H}_1^\perp$, such that the system $\{(I-T_0)\tilde{h}_y\}_{y \in Y}$ is orthonormal. Extending $T_0$ by zero to the orthogonal complement of $\mathcal{H}_1^\perp$, we obtain the operator $T$ that satisfies all the properties we need.

We do not have a method that would allow us to construct subsets that are weakly sparse but not sparse. Still, we consider the notion
of weak sparseness useful, since it gives us a way to see the difference between implications of both types of sparseness.

6.3. **Sparse potentials.** We say that $V \geq 0$ is a sparse (a weakly sparse) potential on $\mathbb{Z}^d$, if its support

$$Y_V := \{ x \in \mathbb{Z}^d : V(x) > 0 \}$$

is a sparse (respectively, weakly sparse) subset.

Let $V$ be a weakly sparse potential. Then, using the equalities (6.2) and (6.5), we see that the quadratic form $b_V[u]$ can be written as

$$b_V[u] = \sum_{y \in Y_V} V(y)(u, (I - T)e_y)^2 = \mu^2 \sum_{y \in Y_V} V(y)(u, (I - T)e_y)^2.$$

Along with the operator $B_V$ generated by this quadratic form, consider also the self-adjoint in $H^1(\mathbb{Z}^d)$ operator

$$N_V = \mu \sum_{y \in Y_V} \sqrt{V(y)}(\cdot, e_y)e_y.$$

We have

$$N_V(I - T^*) = \mu \sum_{y \in Y_V} \sqrt{V(y)}(\cdot, \tilde{h}_y)e_y,$$

whence

$$\|N_V(I - T^*)u\|^2 = \mu^2 \sum_{y \in Y_V} V(y)(u, \tilde{h}_y)^2 = b_V[u].$$

This means that

(6.9) $$B_V = (I - T)N_V^2(I - T^*).$$

This representation allows one to obtain the following simple result that is not covered by Proposition 4.4.

**Theorem 6.3.** Let $V \geq 0$ be a function on $\mathbb{Z}^d$, $d \geq 3$, such that its support $Y_V$ is weakly sparse. Then $V$ is a discrete Hardy weight if and only if $V$ is bounded. Moreover, the following two-sided inequality is satisfied for the Hardy constant $H(V)$:

(6.10) $$C\|V\|_{\ell_\infty} \leq H(V) \leq C'\|V\|_{\ell_\infty},$$

where $C = \|(I - T)^{-1}\|^{-2}\mu^2$ and $C' = \|I - T\|^2\mu^2$.

**Proof.** In the equivalent terms, we have to find the boundedness conditions of the operator $B_V$. They immediately follow from the representation (6.9) and the equality $\|N_V\| = \mu\|V\|_{\ell_\infty}^{1/2}$. \qed
The next result is also a consequence of (6.9). Recall that for a function $V$, such that $V(x) \to 0$ as $|x| \to \infty$, we write $V_j, j \in \mathbb{N}$, for the numbers $|V(x)|$ rearranged in the non-increasing order, see Section 2.1.

**Theorem 6.4.** Let $V \geq 0$ be a function on $\mathbb{Z}^d$, $d \geq 3$, such that its support $Y_V$ is weakly sparse. Then the operator $B_V$ is compact if and only if $V(x) \to 0$ as $|x| \to \infty$. Moreover, the following two-sided inequality is satisfied for the eigenvalues $l_j(B_V)$:

$$CV_j^* \leq l_j(B_V) \leq C'V_j^*, \quad \forall j \in \mathbb{N},$$

with the same constants as in (6.10).

The proof is the same as for Theorem 6.3.

The next result is more advanced, and it requires the potential to be sparse but not weakly sparse. We derive this result for potentials $V$ which meet an additional condition: we assume that the corresponding sequence $\{V_j^*\}$ is *moderately varying*. We say that an (infinite) sequence $\{p_j\}$ of positive numbers is moderately varying, if $p_j \downarrow 0$ and $p_{j+1}/p_j \to 1$. We shall use a result of M.G. Krein, see Theorem 5.11.3 in [8]. Below we reproduce its formulation, restricting ourselves to the situation we need.

**Proposition 6.5.** Let $H \geq 0$ and $S$ be self-adjoint, compact operators. Suppose $\text{rank } H = \infty$ and the sequence of non-zero eigenvalues $l_j(H)$ is moderately varying. Then for the operator $M = H(I + S)H$ one has

$$\lim_{j \to \infty} \frac{l_j(M)}{l_j^2(H)} = 1.$$ 

The next theorem is the main result of this section.

**Theorem 6.6.** Let $V \geq 0$ be a sparse potential, such that the numbers $V_j^*$ form a moderately varying sequence. Then

$$\lim_{j \to \infty} \frac{l_j(M)}{V_j^*} = 1.$$ 

**Proof.** The non-zero spectrum of the operator (6.9) coincides with that of the operator

$$(6.11) \quad M_V := N_V(I - T^*)(I - T)N_V = N_V(I + S)N_V,$$ 

where

$$S = -T - T^* + T^*T.$$
Now we apply Proposition 6.5 to the operators $H = N_V$ and $M_V$, given by (6.8) and (6.11) respectively. All the assumptions of proposition are evidently satisfied, and we get the desired result. □

**Corollary 6.7.** For any moderately varying sequence $\{p_j\}$ there exists a sparse potential $V \geq 0$, such that $l_j(B_V) \sim p_j$.

Taking here $p_j = j^{-q}$ with an arbitrary $q > 0$, we obtain a potential $V$ such that the eigenvalues $l_j(B_V)$ behave as $p_j$ and, therefore,

$$N_-(H_{\alpha V}) = O(\alpha^q); \quad N_-(H_{\alpha V}) \neq o(\alpha^q).$$

For $q = \frac{d}{2}$, this solves the problem stated in the end of Introduction. This corollary allows one also to construct potentials $V$, such that the eigenvalues $l_j(B_V)$ decay arbitrarily slowly, say, $l_j(B_V) \sim (\log j)^{-1}$. This corresponds to the exponential growth of the function $N_-(H_{\alpha V})$. Applying Theorem 6.4 we easily obtain also potentials with the logarithmic growth of this function.

## 7. Discussion and concluding remarks

### 7.1. Let us briefly summarize our main results.

1. Each estimate of the type

$$N_-(H_{\alpha V}) = O(\alpha^q)$$

with $q \geq \frac{d}{2}$, known for the continuous Hamiltonian (1.1), has an analogue for its discrete counterpart. What is more, such analogue can be derived directly from its prototype.

2. For $q \geq \frac{d}{2}$ there are also estimates (7.1) of a different origin, that hold for the discrete Hamiltonian (1.1) but do not have continuous prototypes. These are estimates for the sparse potentials. Unlike the results mentioned above, these ones give also lower estimates of the same order.

3. Estimates of the type (7.1) with $q < \frac{d}{2}$, that hold for the discrete Hamiltonian, have no continuous analogues either. They give necessary and sufficient condition on the potential, in order that $B_V \in \Sigma_q$ with a given $q$ to be satisfied.

### 7.2. Here we discuss the case of non-sign-definite potentials $V \to 0$. As usual, we denote

$$V_\pm = \frac{1}{2}(|V| \pm V).$$

For the operator $H_{\alpha V}$ with such potential, the eigenvalues may appear also to the right of the point $2d = \max \sigma_{\text{ess}}(H_{\alpha V})$, and along with $N_-(H_{\alpha V})$ one should consider the function $N_+(2d; H_{\alpha V})$ defined as the number of such eigenvalues, counted with multiplicities.
Consider the mapping
\[ \Gamma : u(x) \mapsto (-1)^{x_1+\ldots+x_d}u(x). \]
It defines a unitary operator in \( \ell_2(\mathbb{Z}^d) \), and it is easy to see that
\[ \Gamma(-\Delta - \alpha V) = (\Delta + 2dI - \alpha V)\Gamma, \]
therefore,
\[ N_+(2d; H_{\alpha V}) = N_-(H_{-\alpha V}). \]

The inequalities
\[ N_-(H_{\alpha V}) \leq N_-(H_{\alpha V_+}), \quad N_+(2d; H_{\alpha V}) \leq N_-(H_{\alpha V_-}) \]
reduce the problem of estimating the functions on the left to the similar problem for the non-negative \( V \). The first inequality in (7.3) follows directly from the variational principle. To derive the second, one uses also the equality (7.2).

Sometimes, lower estimates of the same type are also possible. In particular, this is the case for the results in Section 3.

7.3. Up to now, the problem of counting the number of negative eigenvalues of the multidimensional discrete Schrödinger operator did not attract much attention of the specialists. The only paper we are aware of, is [10], where the operator (1.1) with \( \alpha = 1 \) was considered, under the assumptions that can be formulated as follows:
\[ V = V_1 - V_2, \quad V_1, V_2 \geq 0, \]
and, moreover,
\[ V_2 \geq \nu_0 > 0; \quad V_1 \in \ell_r, \quad r > 1. \]

The main result consists in the estimate
\[ N_-(H_V) \leq (e/\nu_0)^r \sum_{x \in \mathbb{Z}^d} V_1(x)^r \]
that holds for all \( d \geq 1 \), rather than only for \( d \geq 3 \). In the latter case, the estimate (7.4) is much weaker than any of our estimates.

Below we analyze the estimate (7.4). The conditions on \( V \) imply, for each \( n \in \mathbb{Z}^d \), that \( V(n) \leq V_1(n) - \nu_0 \). Hence, \( V_+(n) > 0 \) yields \( V_1(n) > \nu_0 \). The number \( N \) of such indexes \( n \) is always finite, and moreover,
\[ N \leq \nu_0^{-r} \sum_{x \in \mathbb{Z}^d} V_1(x)^r. \]

Using now (7.3) and the trivial inequality (1.4), we come to an estimate similar to (7.4), but without the redundant factor \( e \).
In other words, the estimate (7.4) needs nothing for its validity, except for the non-negativity of the operator $-\Delta$. In particular, this explains why it holds in any dimension.

7.4. A problem, closely related to the one of counting eigenvalues, concerns the Lieb – Thirring type inequalities. This problem was considered in [9]. Let $E_j^-(H_V), E_j^+(H_V)$ stand for the eigenvalues of the operator $H_V$, lying on $(-\infty, 0)$ and on $(2d, \infty)$ respectively. The following estimates were established in [9] for any $d \geq 1$, along with other interesting results: if $V \in \ell_q(Z^d)$ for some $q \geq 1$, then

$$\sum_j |E_j^-(H_V)|^q + \sum_j (E_j^+(H_V) - 2d)^q \leq \sum_{x \in Z^d} |V(x)|^q,$$

if $V \in \ell_{q+d}(Z^d)$, again for some $q \geq 1$, then

$$\sum_j |E_j^-(H_V)|^q + \sum_j (E_j^+(H_V) - 2d)^q \leq C(d, q) \sum_{x \in Z^d} |V(x)|^{q+\frac{d}{2}},$$

with some, explicitly given constant factor.

In this respect we note that from our results it easily follows that in dimensions $d \geq 3$ a similar inequality holds for any $q > 0$, but with an additional and unspecified factor $C'(d, q)$ on the right. Moreover, in the second estimate the assumption $V \in \ell_{q+d}(Z^d)$ can be replaced by $V \in \ell_{q+r}(Z^d)$ with an arbitrary $r \in (0, \frac{d}{2}]$, and with the corresponding change in the right-hand side.

In this paper we do not look for the optimal constants in such estimates. Note that traditionally just seeking the optimal constants is the main issue when dealing with the Lieb – Thirring inequalities. So, in this respect our results do not give anything new.

7.5. Our initial goal when starting the work on this paper was to construct potentials on $Z^d$, $d \geq 3$, such that $N_-(H_{\alpha V}) = O(\alpha^\frac{d}{2})$ but $N_-(H_{\alpha V}) \neq o(\alpha^\frac{d}{2})$. In [18] this problem was mentioned as unsolved. We solved it here (in Section 6) by using the sparse potentials.

Sparse potentials were already used in the spectral theory of the multidimensional Schrödinger operator in the papers [13, 14]. The authors studied the continuous spectrum of such operators, including the scattering for the pair $(H_0, H_V)$. Their definition of sparseness always requires $\# \{y \in Y_V : |y| < r\} = o(r^d)$. Further requirements depend on the problem considered. The property (6.3) of almost orthogonality, which is the basis for our analysis, does not appear in their approach.
Note that the use of sparse potentials in the theory of one-dimensional operators has a long history, starting from the seminal work \cite{15} by Pearson.

7.6. The spectrum (including its discrete component) of the Hamiltonian of the form \cite{11} in dimension one is well studied, mostly due to the possibility to use the theory of Jacobi matrices. The first results on counting the number of eigenvalues outside the interval \([0, 2]\) are due to Geronimo \cite{6}, \cite{7}, see also \cite{9}, where a minor error in \cite{7} was corrected. Other results in \cite{9} concern the Lieb–Thirring type inequalities, similar to those for \(d \geq 3\) described in Section 7.4 but much more elaborated.

7.7. We do not touch upon the case \(d = 2\) in this paper. Some facts can be derived from the corresponding results for the continuous case, by means of the technique developed in Section 4. However, the results for \(d = 2\) are incomplete, as well as in the continuous version of the theory, see the survey \cite{18}, and they leave many natural questions unanswered. In particular, the approach based upon sparse potentials does not work for \(d = 2\), so that in this case we still have no examples giving \(N(H_{\alpha V}) = O(\alpha)\) but \(N(H_{\alpha V}) \neq o(\alpha)\).

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SCHRÖDINGER OPERATOR ON $\mathbb{Z}^d$

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