REGULAR GLOBAL SOLUTIONS FOR A GENERALIZED KDV EQUATION POSED ON A HALF-LINE

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Abstract. An initial-boundary value problem for a generalized KdV equation posed on a half-line is considered. Existence and uniqueness of global regular solutions for arbitrary smooth initial data are established.

1. Introduction

We are concerned with an initial-boundary value problem (IBVP) posed on the right half-line \( x > 0 \) for the generalized KdV equation

\[
Lu \equiv u_t + u^k u_x + u_{xxx} = 0. \tag{1.1}
\]

Equation (1.1) is a typical example of so-called dispersive equations attracting considerable attention of both pure and applied mathematicians. The KdV equation, \( k = 1 \), is more studied in this context. The theory of the initial-value problem (IVP henceforth) for (1.1) when \( k = 1 \) is considerably advanced today \([2, 8, 9, 17]\).

Although dispersive equations were deduced for the whole real line, necessity to calculate numerically the Cauchy problem approximating the real line by finite intervals implies to study initial-boundary value problems posed on bounded and unbounded intervals \([3, 12, 13]\). What concerns (1.1) with \( k > 1, l = 1 \), called generalized KdV equations, the Cauchy problem was studied in \([15, 16]\) and later in \([5, 6, 7, 9]\), where it has been established that for \( k = 4 \) (the critical case) the problem is well-posed for small initial data, whereas for arbitrary initial data solutions may blow-up in a finite time. The generalized Korteweg-de Vries equation was studied for understanding the interaction between...
the dispersive term and the nonlinearity in the context of the theory of nonlinear dispersive evolution equations [9]. In [14], the initial-boundary value problem for the generalized KdV equation with an internal damping posed on a bounded interval was studied in the critical case; exponential decay of weak solutions for small initial data has been established.

Recently, due to physics and numerics needs, publications on initial-boundary value problems in both bounded and unbounded domains for dispersive equations have been appeared [12, 13, 17]. In particular, it has been discovered that the KdV equation posed on a bounded interval possesses an implicit internal dissipation. This allowed to prove the exponential decay rate of small solutions for (1.1) with \( k = 1 \) posed on bounded intervals without adding any artificial damping term [13]. Similar results were proved for a wide class of dispersive equations of any odd order with one space variable [3, 12, 13].

Our work has been inspired by [14] where the critical KdV equation with internal damping posed on a bounded interval was considered and exponential decay of weak solutions has been established.

The main goal of our work is to prove for arbitrary smooth initial data the existence and uniqueness of global-in-time regular solutions for (1.1) posed on the right half-line. The paper is outlined as follows: Section I is the Introduction. Section 2 contains formulation of the problem and auxiliaries. In Section 3, regularization by an initial–boundary value problem for the Kawahara equation was used to prove the existence and exponential decay of regular global solutions without smallness conditions for initial data.

2. Problem and preliminaries

Let \( T > 0; \) and \( \mathbb{R}^+ = \{ x \in \mathbb{R}, x > 0 \}, \) \( Q_T = (0, T) \times \mathbb{R}^+ \). We use the usual notations of Sobolev spaces \( W^{k,p}, \) \( L^p \) and \( H^k \) and the following notations for the norms [1]:

\[
\| f \|_{L^p(\mathbb{R}^+)} = \int_{\mathbb{R}^+} |f|^p dx, \quad \| f \|_{W^{k,p}(\mathbb{R}^+)} = \sum_{0 \leq |\alpha| \leq k} \| D^\alpha f \|_{L^p(\mathbb{R}^+)}, \quad p \in (1, +\infty). 
\]

\[
H^k(\mathbb{R}^+) = W^{k,2}(\mathbb{R}^+); \quad \| f \|_{L^\infty(\mathbb{R}^+)} = \text{esssup}_{\mathbb{R}^+} |f(x,t)|.
\]
Consider the following IBVP:

\begin{align*}
Lu &\equiv u_t + u^k u_x + u_{xxx} = 0, \text{ in } Q_T; \\
u(0, t) &= 0, \ t > 0; \\
u(x, 0) &= u_0(x), \ x \in \mathbb{R}^+,
\end{align*}

where \( u_0 : \mathbb{R}^+ \to \mathbb{R} \) is a given function and \( k = 1, 2 \). When \( k \leq 3 \), (2.1) is called the regular KdV generalized equation while \( k = 4 \) corresponds to the critical KdV equation. The regular case \( k = 1 \) has been intensively studied, while published results on initial-boundary value problems in generalized case \( k \geq 2 \) in unbounded domains, such as half-line, are almost unknown.

Hereafter subscripts \( u_x, \) etc. denote the partial derivatives, as well as \( \partial_x \) when it is convenient. By \((\cdot, \cdot)\) and \( \| \cdot \| \) we denote the inner product and the norm in \( L^2(\mathbb{R}^+) \), and \( \| \cdot \|_{H^k(\mathbb{R}^+)} \) stands for the norm in \( L^2 \)-based Sobolev spaces.

We will need the following result [10].

**Lemma 2.1.** Let \( u \in H^1(\mathbb{R}^+) \) and \( u(0, t) = 0 \), then

\[
\|u\|_{L^4(\mathbb{R}^+)} \leq 2^{1/2}\|u_x\|^{1/2}\|u\|^{1/2}, \quad \|u\|_{L^8(\mathbb{R}^+)} \leq 4^{3/4}\|u_x\|^{3/4}\|u\|^{1/4}.
\] (2.4)

**Proposition 2.1.** Let for a.e. fixed \( t \) \( u(x, t) \in H^1(\mathbb{R}^+) \). Then

\[
\sup_{x \in \mathbb{R}^+} u^2(x, t) \leq 2\|u(t)\|\|u_x(t)\|.
\] (2.5)

### 3. Existence Theorem

**Theorem 3.1.** Given \( u_0 \in H^3(\mathbb{R}^+) \) such that \( u_0(0) = 0 \). Then for all positive \( T \) there exists a unique regular solution to (2.1)-(2.3) such that

\[
u \in L^\infty(\mathbb{R}^+; H^3(\mathbb{R}^+)) \cap L^2(\mathbb{R}^+; H^4(\mathbb{R}^+)); \\
u_t \in L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^+)) \cap L^2(\mathbb{R}^+; H^1(\mathbb{R}^+))
\]

To prove this theorem, we will use regularization of (2.1)-(2.3) by an initial-boundary value problem on the half-line for the following Kawahara equation

\begin{align*}
L_\epsilon u_\epsilon &\equiv u_\epsilon t + u_\epsilon^k u_\epsilon^x + u_\epsilon^x - \epsilon u_\epsilon^{xxxx} = 0 \text{ in } Q_T; \\
u_\epsilon(0, t) &= \epsilon u_\epsilon^x(0, t) = 0, \ t > 0; \\
u_\epsilon(x, 0) &= u_0^m(x), \ x \in \mathbb{R}^+,
\end{align*}

where \( \epsilon \in (0, 1] \) and \( u_0^m(x) \) is an approximation of \( u_0(x) \) by functions \( u_0^m \in H^5(\mathbb{R}^+) \) such that

\[
\lim_{m \to \infty} \|u_0 - u_0^m\|_{H^5(\mathbb{R}^+)} = 0.
\]
For $\epsilon = 1$, (3.1)-(3.3) has been studied in [4] where it has been established that for $k < 8$ this problem has a regular solution for all $T > 0$ provided a function $u^m_0(x)$ satisfies some compatibility conditions at $x = 0$. Problem (3.1)-(3.3) with $\epsilon \in (0, 1)$ can be treated with the same arguments.

Our goal is to prove necessary a priori estimates for $u_\epsilon(x, t)$ independent of $\epsilon$ that will allow us to pass to the limit as $\epsilon \to 0$ and $m \to \infty$ getting a unique global regular solution for $k = 1, 2$.

**Proof. Estimate I.** Multiply (3.1) by $u_\epsilon$ and integrate over $\Omega \times (0, t)$ to obtain

$$
\|u_\epsilon\|^2(t) + \epsilon \int_0^t |u_{x\epsilon}(0, \tau)|^2 d\tau = \|u_\epsilon^0\|^2 \leq \|u_0\|^2, \quad t > 0. \quad (3.4)
$$

**Estimate II.** Write the inner product

$$
2 \langle L_\epsilon u_\epsilon, (1 + x)u_\epsilon \rangle (t) = 0,
$$
dropping the index $\epsilon$, in the form:

$$
\frac{d}{dt} ((1 + x), u^2) (t) + \epsilon u_{xx}^2(0, t) + 3\|u_x\|^2(t) + 5\epsilon\|u_{xx}\|^2(t) = -2((1 + x)u^k u_x, u)(t). \quad (3.5)
$$

Taking into account (2.4) and (3.4), we obtain

$$
I = -2((1 + x)u^k u_x, u)(t) = \frac{4}{k + 2}(u^k u^2)(t)
\leq 2\sup_{(R^+)} u^2(x, t)\|u\|^2(t) \leq 4\|u\|^3(t)\|u_x\|(t)
\leq 2[\|u_x\|^2(t) + \|u_0\|^6].
$$

Substituting $I$ into (3.5), we get

$$
\frac{d}{dt} ((1 + x), u^2) (t) + \epsilon u_{xx}^2(0, t) + \|u_x\|^2(t) + 5\epsilon\|u_{xx}\|^2(t) \leq \|u_0\|^6.
$$

Hence

$$
((1 + x), u^2) (t) + \int_0^t [\epsilon u_{xx}^2(0, \tau) + \|u_x\|^2(\tau) + 5\epsilon\|u_{xx}\|^2(\tau)] d\tau \leq T\|u_0\|^6 + ((1 + x), u_0^2)
\leq (1 + T\|u_0\|^4)((1 + x), u_0^2) \equiv C_1((1 + x), u_0^2). \quad (3.6)
$$
Estimate III. Write the inner product
\[ 2 \langle L_\epsilon u^\epsilon, (1 + x)^2 u^\epsilon \rangle(t) = 0, \]
dropping the index \( \epsilon \), in the form:
\[ \frac{d}{dt} ((1 + x)^2, u^2)(t) + 2 \epsilon u^2_{xx}(0, t) + 6 (1 + x)^{1/2} u_x^2(t) 
+ 10 \epsilon (1 + x)^{1/2} u_{xx}^2(t) = -2((1 + x)^2 u^k u_x, u)(t) \] (3.7)

Taking into account (2.4) and (3.4), we obtain
\[ I = -2((1 + x)^2 u^k u_x, u)(t) = -\frac{4}{k+2}((1 + x) u^k u^2)(t) \]
\[ \leq \frac{4}{k+2} \sup_{\mathbb{R}^+} u^2(x,t)((1 + x), u^2(t)) \leq \frac{8}{3} \| u \|_1(t) \| u_x \|_1((1 + x), u^2)(t) \]
\[ \leq \frac{4}{3} \| (1 + x)^{1/2} u_x \|_2(t) + \frac{4}{3} \| u_0 \|_2^2((1 + x), u^2)(t). \]

Substituting \( I \) into (3.5), we obtain
\[ \frac{d}{dt} ((1 + x)^2, u^2)(t) + 2 \| (1 + x)^{1/2} u_x \|_2^2(t) 
+ 10 \epsilon \| (1 + x)^{1/2} u_{xx} \|_2^2(t) \leq 2 \| u_0 \|_2^2((1 + x)^2, u^2)(t) \]
\[ \leq 2 C_1^2 \| u_0 \|_2^2((1 + x), u^2_0)^2. \]

This implies
\[ ((1 + x)^2, |u^\epsilon|^2)(t) \]
\[ + \int_0^t \left[ 2 \| (1 + x)^{1/2} u_x^\epsilon \|_2^2(\tau) + 10 \epsilon \| (1 + x)^{1/2} u_{xx}^\epsilon \|_2^2(\tau) \right] d\tau \leq \left(1 + 2 C_1^2 T \| u_0 \|_2^2((1 + x)^2, u^2_0)^1 \right)((1 + x)^2, u^2_0) \]
\[ \equiv C_2((1 + x)^2, u^2_0). \] (3.8)

Estimate IV
Dropping the index \( \epsilon \), write the inner product
\[ 2 \langle (1 + x) u^k_t, \partial_t (L_\epsilon u^\epsilon) \rangle(t) = 0 \]
as
\[ \frac{d}{dt} ((1 + x), u_t^2)(t) + \epsilon u_{xx}^2(0, t) + 3 \| u_{xt} \|^2(t) + 5 \epsilon \| u_{xxt} \|^2(t) 
= 2 ((1 + x) u^k u_t, u_{xt}) (t) + 2 (u^k, u_t^2)(t). \] (3.9)
Taking \( k = 2 \), we estimate
\[
I_1 = 2 \left( (1 + x) u^k u_t, u_{xt} \right) (t)
\]
\[
\leq 2 \| u_{xt} \| (t) \sup_{(R^+)} |u(1 + x)^{1/2} u(x, t)|^2 \| (1 + x)^{1/2} u_t \| (t)
\]
\[
\leq 4 \| u_{xt} \| (t) \| (1 + x)^{1/2} u \| (t) \| ((1 + x)^{1/2} u)_x \| (t) \| (1 + x)^{1/2} u_t \| (t)
\]
\[
\leq 2 \| u_{xt} \|^2 (t) + 2 \| (1 + x)^{1/2} u \|^2 (t) \frac{u}{2(1 + x)^{1/2}}
\]
\[
+ (1 + x)^{1/2} u_x (t) (1 + x)^{1/2} u^2 (1 + x, u^2_t) (t)
\]
\[
\leq 2 \| u_{xt} \|^2 (t) + 4 \| (1 + x)^{1/2} u \|^2 (t) \| u \|^2 (t)
\]
\[
+ ((1 + x), u^2_x) (t) ((1 + x), u^2_t) (t).
\]

Similarly,
\[
I_2 = 2(u^k, u^2_t) (t) \leq 2 \sup_{R^+} u^2 (x, t) ((1 + x), u^2_t) (t)
\]
\[
\leq 4 \| u \| (t) \| u_x \| ((1 + x), u^2_t) (t)
\]
\[
\leq 2 \left( \| u_0 \|^2 + \| u_x \|^2 (t) \right) ((1 + x), u^2_t) (t).
\]

Substituting \( I_1, I_2 \) into (3.9), we come to the inequality
\[
\frac{d}{dt} ((1 + x), u^2_t) (t) + \epsilon u^2_{xt} (0, t) + \| u_{xt} \|^2 (t) + 5\epsilon \| u_{xx} \|^2 (t)
\]
\[
\leq C \left[ \| (1 + x)^{1/2} u_0 \|^2 + \| (1 + x)^{1/2} u_x \|^2 (t) \right] ((1 + x), u^2_t) (t). \tag{3.10}
\]

By (3.8), \( \| (1 + x)^{1/2} u_x \|^2 (t) \in L^1(0, T) \), hence
\[
\| u_t \|^2 (t) \leq ((1 + x), u^2_t) (t) \leq C((1 + x), \| u^2_t \| (0), \tag{3.11}
\]
where \( C \) depends on \( \| u_0 \|, T \) and
\[
((1 + x), \| u^2_t \| (0) = \int_{\mathbb{R}^+} (1 + x) \left[ - u_{0xxx} - u_0^k u_{0x} + \epsilon \partial_x^2 u_0 \right]^2 dx
\]
\[
\leq 2 \int_{\mathbb{R}^+} (1 + x) \left[ - u_{0xxx} - u_0^k u_{0x} \right]^2 dx + 2\epsilon^2 \int_{\mathbb{R}^+} (1 + x) \left[ \partial_x^2 u_0 \right]^2 dx. \tag{3.12}
\]

Returning to (3.10), we find
\[
((1 + x), \| u^2_t \| (t) + \int_0^T \{ \| u^2_{xt} \|^2 (t) + \epsilon \| u^2_{xxt} \|^2 (t) \} dt
\]
\[
\leq C((1 + x), \| u^2_t \| (0). \tag{3.13}
\]
Passage to the limit as $\epsilon \to 0$.

Rewrite (3.1) in the form

$$
(u^\epsilon_t, \phi)(t) + (u^\epsilon_x, \phi_{xx}(t)) - \frac{1}{k + 1}(u^{(k+1)}\epsilon, \phi_x)(t)
+ \epsilon(u^\epsilon_{xx}, \phi_{xxx})(t) = 0, \quad \text{for } (3.14)
$$

where $\phi(x)$ is an arbitrary function such that

$$
\phi \in H^3(\mathbb{R}^+); \quad \phi(0) = \phi_x(0) = \phi_{xx}(0) = 0.
$$

Making use of (3.8) and (3.13), we get

$$
\epsilon^{1/2}\|u^\epsilon_x\|(t) \leq C, \quad \|u^\epsilon_t\|(t) \leq C,
$$

hence

$$
\lim_{\epsilon \to 0} \epsilon\|u^\epsilon_x\|(t)\|\phi_{xxx}\| = 0,
$$

and we obtain from (3.14)

$$
(u_t, \phi)(t) + (u_x, \phi_{xx})(t) - \frac{1}{k + 1}(u^{(k+1)}\epsilon, \phi_x)(t) = 0. \quad \text{for } (3.15)
$$

Moreover,

$$
\lim_{\epsilon \to 0}((1 + x), |u^\epsilon_t|^2)(0) = ((1 + x), |u^m_t|^2)(0)
= \int_{\mathbb{R}^+} (1 + x)\{-u^m_{0xxx} - u^m ku^m_0x\}^2 \, dx, \quad \text{for } (3.16)
$$

consequently,

$$
\lim_{m \to \infty} \int_{\mathbb{R}^+} (1 + x)\{-u^m_{0xxx} - u^m ku^m_0x\}^2 \, dx = \int_{\mathbb{R}^+} (1 + x)\{-u_{0xxx} - u^k u_0x\}^2 \, dx.
$$

Multiplying (3.1) by $(1 + x)u$ and taking into account (3.7), (3.13) we come to the inequlaity

$$
\|u_x\|^2(t) \leq 2\|u_0\|^6 + 2|((1 + x)u_t, u)(t)| \leq C,
$$

then

$$
u_{xxx} = -u^k u_x - u_t \in L^\infty((0, T); L^2(\mathbb{R}^+)).
$$

In turn, multiplying this equality by $-2((1 + x)u_{xx}$, we find that

$$u_{xx} \in L^\infty((0, T); L^2(\mathbb{R}^+)), \quad u_t \in L^\infty((0, T); L^2(\mathbb{R}^+)) \cap L^2((0, T); H^1(\mathbb{R}^+))
$$

Since $u^k u_x \in L^2((0, T); H^1(\mathbb{R}^+))$, then $u_{xxx} \in L^2((0, T); H^1(\mathbb{R}^+))$ and

$$u \in L^\infty((0, T); H^3(\mathbb{R}^+)) \cap L^2((0, T); H^4(\mathbb{R}^+));
\quad u_t \in L^\infty((0, T); L^2(\mathbb{R}^+)) \cap L^\infty((0, T); H^1(\mathbb{R}^+)).$$
This proves the existence part of Theorem 3.1.

**Uniqueness** Let \( u_1, u_2 \) be two distinct solutions to (3.1)-(3.3) and \( z = u_1 - u_2 \). Considering more interesting case \( k = 2 \), we find

\[
Lz = z_t + z_{xxx} = -\frac{1}{3}(u_1^3 - u_2^3)_x = -\frac{1}{3}\{z(u_1^2 + u_1 u_2 + u_2^2)\}.
\]

Multiplying this equation by \( 2(1 + x)z \), we obtain

\[
\frac{d}{dt}((1 + x, z^2)(t) + 3\|z_x\|^2(t) + z_x^2(0, t)) = -\frac{2}{3}(z[u_1^2 + u_1 u_2 + u_2^2])_x, (1 + x)z(t).
\]

Estimate

\[
I = -\frac{2}{3}(z[u_1^2 + u_1 u_2 + u_2^2])_x, (1 + x)z(t) = \frac{1}{3}((u_1^2 + u_1 u_2 + u_2^2), z^2)(t)
\]

\[
+ \frac{1}{3}((2u_1 + u_2)u_1 x + (2u_2 + u_1)u_2 x, (1 + x)z^2)(t)
\]

\[
\leq \frac{2}{3}((u_1^2 + u_2^2), z^2)(t) + \frac{1}{6}((2u_1 + u_2)^2 + (2u_2 + u_1)^2
\]

\[
+ u_1^2_x + u_2^2_x], (1 + x)z^2)(t) \leq \frac{2}{3}[\sup_{R^+} u_1^2 + \sup_{R^+} u_2^2((1 + x), z^2)(t)
\]

\[
+ \frac{4}{3}\sup_{R^+}[u_1^2 + u_2^2 + u_1^2_x + u_2^2_x][(1 + x)z^2)(t)
\]

\[
\leq C\{\|u_1\|_{H^2(R^+)}(t) + \|u_2\|_{H^2(R^+)}(t)\}\{(1 + x), z^2)(t) \leq C((1 + x), z^2)(t).
\]

Substituting \( I \) into (3.17), we come to the inequality

\[
\frac{d}{dt}((1 + x), z^2)(t) \leq C((1 + x), z^2)(t).
\]

Since \( z(x, 0) \equiv 0, \) then

\[
\|z\|^2(t) \leq ((1 + x), z^2)(t) \equiv 0 \ t > 0.
\]

This completes the proof of Theorem 3.1

**Remark 3.1.** In the regular case \( k = 3 \), it is possible to prove local in \( t \) the existence and uniqueness of regular solutions. On the other hand, there are papers, [14], where a local dissipation term \( au, a > 0 \) is added to (1.1). It helps to prove the existence, uniqueness and exponential decay of small solutions.

**Conclusions.** An initial-boundary value problem for the generalized KdV equation posed on the right half-line has been considered. The existence and uniqueness of a regular global solution for all positive \( T \) and arbitrary smooth initial data have been established.
REFERENCES

[1] R. Adams, J. Fournier, Sobolev Spaces, Second Edition (2003), Elsevier Science Ltd.
[2] J. Colliander, M. Keel, G. Staffilani, H. Takaoka and T. Tao, Sharp global well-posedness results for periodic and non-periodic KdV and modified KdV on R and T., J. Amer. Math. Soc. 16 (2003), 705–749.
[3] Faminskii A. V., Larkin N. A.: Initial-boundary value problems for quasilinear dispersive equations posed on a bounded interval, Electron. J. Differ. Equations. 1–20 (2010).
[4] Faminskii A. V., Martynov E. V., On initial-boundary value problems on semi-axis for generalized Kawahara equation, Contemporary Mathematics. Fundamental directions, v. 65, No 4, 683-699 (2019). (Russian)
[5] Farah L. G., Linares F., Pastor A.: The supercritical generalized KDV equation: global well-posedness in the energy space and below, Math. Res. Lett. 18, no. 02, 357-377 (2011).
[6] G. Fonseca, F. Linares , G. Ponce, Global well-posedness for the modified Korteweg-de Vries equation, Comm. Part. Diff. Equats, 24 (3,4) (1999), 683-705.
[7] G. Fonseca, F. Linares , G. Ponce, Global existence for the critical generalized KDV equation, Proc. of the AMS, vol. 131, Number 6 (2002), 1847-1855.
[8] T. Kato, On the Cauchy problem for the (generalized) Korteweg-de Vries equations, Advances in Mathematics Supplementary Studies, Stud. Appl. Math. 8 (1983), 93–128.
[9] C. E. Kenig, G. Ponce and L. Vega, Well-posedness and scattering results for the generalized Korteweg-de Vries equation and the contraction principle, Commun. Pure Appl. Math. 46 (1993), 527–620.
[10] O. A. Ladyzhenskaya, V. A. Solonnikov and N. N. Uraltseva, Linear and Quasilinear Equations of Parabolic Type. American Mathematical Society, Providence, Rhode Island, 1968.
[11] N. A. Larkin; Correct initial boundary value problems for dispersive equations, J. Math. Anal. Appl. 344 (2008) 1079–1092.
[12] N. A. Larkin and J. Luchesi, Generalized dispersive equations of higher orders posed on bounded intervals: local theory, arXiv:1812.04146 v 1 [math. AP] 10 Dec 2018.
[13] N. A. Larkin and J. Luchesi, Initial-boundary value problems for generalized dispersive equations of higher orders posed on bounded intervals, Appl. Math and Optimiztion, doi.org/10.1007/s00245-019-09579-w.
[14] F. Linares and A. Pazoto; On the exponential decay of the critical generalized Korteweg-de Vries equation with localized damping. Proc. Amer. Math. Soc., 135, 1 (2007) 1515-1522.
[15] Y. Martel and F. Merle; Instability of solutions for the critical generalized Korteweg-de Vries equation, Geometrical and Funct. Analysis, 11 (2001) 74-123.
[16] F. Merle; Existence of blow up solutions in the energy space for the critical generalized KdV equation, J. Amer. Math. Soc., 14 (2001) 555-578.
[17] J. C. Saut, Sur quelques généralisations de l’équation de Korteweg-de Vries (French), J. Math. Pures Appl. 58 (1979), 21–61.