The minimum rank of universal adjacency matrices

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Abstract

In this paper we introduce a new parameter for a graph called the \textit{minimum universal rank}. This parameter is similar to the minimum rank of a graph. For a graph $G$ the minimum universal rank of $G$ is the minimum rank over all matrices of the form

$$U(\alpha, \beta, \gamma, \delta) = \alpha A + \beta I + \gamma J + \delta D$$

where $A$ is the adjacency matrix of $G$, $J$ is the all ones matrix and $D$ is the matrix with the degrees of the vertices in the main diagonal, and $\alpha \neq 0, \beta, \gamma, \delta$ are scalars. Bounds for general graphs based on known graph parameters are given, as is a formula for the minimum universal rank for regular graphs based on the multiplicity of the eigenvalues of $A$. The exact value of the minimum universal rank of some families of graphs are determined, including complete graphs, complete bipartite graph, paths and cycles. Bounds on the minimum universal rank of a graph obtained by deleting a single vertex are established. It is shown that the minimum universal rank is not monotone on induced subgraphs, but bounds based on certain induced subgraphs, including bounds on the union of two graphs, are given. Finally we characterize all graphs with minimum universal rank equal to 0 and to 1.

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1. Introduction

The minimum rank problem for a given graph is a well-studied problem in the spectral theory of graphs. The \textit{minimum rank of a graph} $G$ is the smallest rank among all real-valued, symmetric matrices that have the property: for $i \neq j$ the $(i,j)$-th entry is nonzero if and only if $\{i,j\}$ is an edge in the graph.
G. Such a quantity is denoted by \( \text{mr}(G) \). This number has been at the root of a number of studies over the past dozen years, and a complete resolution of determining \( \text{mr}(G) \) for all \( G \) seems essentially unattainable [4].

Haemers and Omidi in [6] defined a new family of matrices that is associated to a graph, these matrices are called the universal adjacency matrices of the graph. Consider a simple undirected graph \( G = (V, E) \) with \( V = \{v_1, v_2, \ldots, v_n\} \). Let \( A_G = [a_{ij}] \) be the \((0,1)\)-adjacency matrix of \( G \), that is \( a_{ij} \) equals one if \( \{i, j\} \in E \) and zero otherwise. Let \( D_G = \text{diag}[d_1, d_2, \ldots, d_n] \) with \( d_i = \text{deg}(v_i) \) be the degree matrix associated with \( G \), and denote by \( I \) and \( J \) the \( n \times n \) identity matrix and \( n \times n \) matrix of all ones. An \( n \times n \) matrix of the form

\[
U_G = U_G(\alpha, \beta, \gamma, \delta) = \alpha A_G + \beta I + \gamma J + \delta D_G,
\]

where \( \alpha, \beta, \gamma, \delta \) are scalars with \( \alpha \neq 0 \) is called a universal adjacency matrix of \( G \). We drop the subscript \( G \) when it is clear from the context. The entries of a universal adjacency matrix \( U = [u_{ij}] \) are then of the following form:

\[
u_{ij} = \begin{cases} 
\beta + \gamma + \delta d_i & \text{if } i = j \\
\alpha + \gamma & \text{if } \{i, j\} \in E \\
\gamma & \text{if } \{i, j\} \notin E.
\end{cases}
\]

Throughout this paper, graphs are considered to be simple and undirected. Thus a universal adjacency matrix is always a symmetric matrix.

The family of universal adjacency matrices is a generalization of several families of matrices associated to the graph. The following table shows that for specific values of the coefficients, the universal adjacency matrix is a well-known matrix associated with a graph:

| \((\alpha, \beta, \gamma, \delta)\) | resulting matrix |
|-------------------|-----------------|
| \((1, 0, 0, 0)\)   | adjacency matrix |
| \((-1, 0, 0, 1)\)  | Laplacian matrix |
| \((1, 0, 0, 1)\)   | signless Laplacian matrix |
| \((-2, -1, 1, 0)\) | Seidel matrix |
| \((\alpha, \beta, \gamma, 0)\) | generalized adjacency matrix |
| \((-1, -1, 1, 0)\) | adjacency matrix of the complement |

Figure 1: Specific cases of the universal adjacency matrix of a graph

Haemers and Omidi in [6] studied the number of distinct eigenvalues of a universal adjacency matrix associated to \( G \) and determined exactly which graphs have two distinct eigenvalues. In this paper we are concerned with the rank of universal adjacency matrices.

For a given graph \( G \), the minimum universal rank of \( G \), denoted by \( \text{mur}(G) \), is given by

\[
\text{mur}(G) = \min\{\text{rank}(U) \mid U \text{ is a universal adjacency matrix of } G\}.
\]
It is clear that the minimum rank of any of the matrices in Figure 1 is an upper bound on the minimum universal rank of a graph.

If \( U_G = \alpha A_G + \beta I + \gamma J + \delta D_G \) is a universal matrix for a graph \( G \), then, since \( \alpha \neq 0 \),
\[
A_G + \frac{\beta}{\alpha} I + \frac{\gamma}{\alpha} J + \frac{\delta}{\alpha} D_G
\]
is also a universal adjacency matrix for \( G \) with the same rank. Thus, in studying the minimum universal rank of a graph \( G \), we may assume without loss of generality that \( \alpha = 1 \) for a universal adjacency matrix of \( G \).

When considering a universal adjacency matrix, the off-diagonal entries also come in two types: if an entry corresponds to an edge it is a fixed number, otherwise it is required to be a different fixed value. This is in contrast to the off-diagonal entries of the matrices associated with the minimum rank of \( G \), in this case the entries that correspond to non-adjacent vertices must all be zero, while the entries corresponding to adjacent vertices are non-zero but otherwise independent (excluding their symmetric mate). Further, the main diagonal of a universal adjacency matrix is not completely free as it is in the matrices associated to minimum rank, but rather it depends on the degree of a vertex, and the parameters \( \beta, \gamma \) and \( \delta \). Consequently, the parameters \( \text{mr}(G) \) and \( \text{mur}(G) \) are not comparable in general (see examples throughout this work) and appear to not share any sort of strong relationship. For instance, a graph \( G \) in the assumption of Theorem 7.2 satisfies \( \text{mr}(G) \leq \text{mur}(G) \), while using Theorem 2.2 we have \( \text{mur}(K_n) \leq \text{mr}(K_n) \). However, note that for a given graph \( G \), the universal matrix \( U_G(1, \beta, \delta, 0) \) represents a zero-nonzero pattern for \( G \). So if \( \text{mr}(G) = \text{rank}(U_G(1, \beta, \delta, \gamma)) \), then
\[
\text{mr}(G) \leq \text{rank}(U_G(1, \beta, \delta, 0)) = \text{rank}(U_G(1, \beta, \delta, \gamma) - \gamma J) \leq \text{mur}(G) + 1.
\]

In our notation, \( J_{r,s} \) denotes the \( r \times s \) matrix of all entries equal to one and \( 0_{r,s} \) denotes the \( r \times s \) zero matrix. We use \( e \) to denote the all ones vector and add a subscript if it is necessary to specify the size of the vector.

2. Basic Results

In this section we give some basic results about minimum universal rank for general graphs. The first result shows that the minimum universal rank has an unusual property that neither the minimum rank nor the minimum rank of the generalized adjacency matrix has, namely that the minimum universal rank of a graph is equal to the minimum universal rank of its complement. We use \( \overline{G} \) to denote the complement of the graph \( G \).

**Lemma 2.1.** For any graph \( G \), \( \text{mur}(G) = \text{mur}(\overline{G}) \).

**Proof.** For any universal adjacency matrix
\[
U_G = \alpha A_G + \beta I + \gamma J + \delta D_G,
\]

...
using the facts that $A_{\overline{G}} = -A_G - I - J$ and $D_{\overline{G}} = (n - 1)I - D_G$, it follows that

$$U_G = (-\alpha)(-A_G - I - J) + (\beta - \alpha + (n - 1)\delta)I + (\gamma + \alpha)J + (-\delta)((n - 1)I - D_G)$$

$$= \alpha'(A_{\overline{G}}) + \beta'I + \gamma'J + \delta'D_{\overline{G}}$$

is a universal adjacency matrix for $\overline{G}$. This implies that for any set of scalars $\alpha, \beta, \gamma, \delta$ there is a set of scalars $\alpha', \beta', \gamma', \delta'$ such that the universal matrices $U_G = \alpha A_G + \beta I + \gamma J + \delta D_G$ and $U_{\overline{G}} = \alpha' A_{\overline{G}} + \beta' I + \gamma' J + \delta' D_{\overline{G}}$ are equal. Therefore, $G$ and $\overline{G}$ have the same minimum rank.

The proof of above lemma also shows that the set of universal adjacency matrices for a graph is equal to the set of universal adjacency matrices for its complement.

If a graph is disconnected, then its complement is connected, thus in discussing the minimum universal rank of a graph, we may assume that the graph is connected, although it may not always be convenient to do so.

The next result shows that it is possible for the minimum universal rank of a graph to be zero, but this can only happen in a specific case.

**Theorem 2.2.** For any graph $G$, $\operatorname{mur}(G) = 0$ if and only if $G$ or $\overline{G}$ is a complete graph.

**Proof.** Let $\alpha = 1$, $\beta = 1, \delta = 0$ and $\gamma = -1$, then the resulting universal adjacency matrix for $K_n$ is the zero matrix. For the converse, it suffices to note that if $G$ has edges and non-edges at the same time, then any universal adjacency matrix of $G$ will have a non-zero entry; therefore, if $\operatorname{mur}(G) = 0$, then $G$ is either complete graph or empty graph.

For a graph $G$ on $n$ vertices, the matrix $L_G = D_G - A_G$ is the Laplacian matrix of $G$. The Laplacian matrix of a graph is a universal adjacency matrix of the graph, so the rank of the Laplacian is an upper bound on the minimum universal rank of the graph. Much is known about the eigenvalues of a Laplacian matrix that can be used to bound the minimum universal adjacency matrix of a graph; see [8] for more details.

For example, it is known that $L_G$ is positive semi-definite. Moreover, the sum of the entries in each row of $L_G$ is zero which implies that zero is an eigenvalue for $L_G$ and $e$, a corresponding eigenvector. Furthermore, the multiplicity of zero as an eigenvalue of the Laplacian matrix is exactly the number of components of the graph. That is, if $c(G)$ denotes the number of components of $G$, and $m_A(\lambda)$ denotes the multiplicity of $\lambda$ as an eigenvalue of $A$, then we have $m_{L_G}(0) = c(G)$.

**Theorem 2.3.** For any graph $G$ on $n$ vertices

$$\operatorname{mur}(G) \leq n - c(G).$$

Note that, the upper bound above cannot be improved since equality holds for the empty graph. One interesting fact about Theorem 2.3 is that it relates
the minimum universal rank of a graph to a well-known graph parameter, but
sometimes it is possible to use the eigenvalues of the Laplacian matrix to get a
better bound on the minimum universal rank of a graph.

**Theorem 2.4.** Let $G$ be a connected graph on $n$ vertices, and let $m$ be the
maximum multiplicity of the nonzero eigenvalues of $L_G$. Then

$$\text{mur}(G) \leq n - m - 1.$$  

**Proof.** Let the eigenvalues of $L_G$ be $\lambda_1 = 0 \leq \lambda_2 \leq \ldots \leq \lambda_n$. The all ones
vector is an eigenvector for 0 and the eigenvectors for the nonzero eigenvalues
are orthogonal to the all ones vector. Suppose $\lambda_k$ is a nonzero eigenvalue of
$L_G$ with multiplicity $m$, then $\lambda_k$ is an eigenvalue of $L_G + \frac{\lambda_k}{n}J$ with multiplicity
$m + 1$ (to see this, consider the $m$ linearly independent eigenvectors for $L_G$
corresponding to $\lambda_k$ and the all ones vector). So, the matrix $L_G + \frac{\lambda_k}{n}J - \lambda_k I$
is a universal adjacency matrix for $G$ that has rank $n - m - 1$, and hence
$\text{mur}(G) \leq n - m - 1$. \hfill $\Box$

Note that, if $G$ is connected, then the multiplicity of the eigenvalue zero of
the Laplacian matrix is one (the number of components). So if zero has the
maximum multiplicity, then all eigenvalues are simple, which implies $m = 1$ in
above proof. We also note that Theorem 2.4 is valid even if $G$ is disconnected,
since then its complement $\overline{G}$ is connected and has the same minimum universal
rank as $G$. So if $G$ is disconnected we apply the above proof for $\overline{G}$.

The following is an immediate consequence of Theorem 2.4.

**Corollary 2.5.** For any graph $G$ on $n$ vertices

$$\text{mur}(G) \leq n - 2.$$  

It is known that the minimum rank of a graph on $n$ vertices is at most
$n - 1$, and $\text{mr}(G) = n - 1$ if and only if $G$ is a path on $n$ vertices; see [4]. In
the next section, it is shown that the upper bound in Corollary 2.5 is achieved
by paths. It is interesting to note that the graphs that achieve the maximum
possible minimum rank also achieve the maximum possible minimum universal
rank. But unlike minimum rank, where the paths are the only graphs that have
the maximum possible minimum rank, there are many graphs that achieve the
maximum possible minimum universal rank; see Example 3.5 for instance.

**3. Paths**

A *path* on $n$ vertices, denoted by $P_n$, is a graph with vertices $v_1, v_2, \ldots, v_n$
and edge set $\{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}\}$. The next result shows that if
a graph contains an induced path on $n$ vertices, then the minimum universal
rank of the graph is at least $n - 2$.

Let $A$ be an $m \times n$ matrix. For $\alpha \subseteq \{1, 2, \ldots, m\}$ and $\beta \subseteq \{1, 2, \ldots, n\}$,
the notation $A[\alpha, \beta]$ means the submatrix of $A$ lying in rows indexed by $\alpha$ and
columns indexed by $\beta$. 

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Lemma 3.1. If a graph $G$ contains the induced path $P_k$ ($k \geq 3$), then

$$\mu(G) \geq k - 2.$$ 

Proof. Suppose $v_1v_2 \ldots v_k$ is the induced path $P_k$. Order the vertices of $G$ such that the first $k$ vertices are $v_1, v_2, \ldots, v_k$. Then any universal adjacency matrix $U_G = A_G + \beta I + \gamma J + \delta D_G$ of the graph $G$ is of the form:

$$
\begin{bmatrix}
\beta + \gamma + \delta d_{v_1} & 1 + \gamma & \cdots & \gamma & \gamma \\
1 + \gamma & \beta + \gamma + \delta d_{v_2} & \cdots & \gamma & \gamma \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\gamma & \gamma & \cdots & \beta + \gamma + \delta d_{v_{k-1}} & 1 + \gamma \\
\gamma & \gamma & \cdots & 1 + \gamma & \beta + \gamma + \delta d_{v_k}
\end{bmatrix}
$$

Subtracting the $k$th column from each of the columns $2, \ldots, k-1$ results in the following matrix

$$U' = 
\begin{bmatrix}
\beta + \gamma + \delta d_{v_1} & 1 & 0 & \cdots & 0 & \gamma \\
1 + \gamma & * & 1 & \cdots & 0 & \gamma \\
\gamma & * & * & \cdots & 0 & \gamma \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\gamma & \cdots & 1 & \gamma \\
\gamma & \cdots & * & 1 + \gamma \\
\gamma & \cdots & * & \beta + \gamma + \delta d_{v_k}
\end{bmatrix}
$$

Since the submatrix $U'[(1, \ldots, k-2), \{2, \ldots, k-1\}]$ of $U$ has rank $k-2$, we have rank$(U_G) \geq \text{rank}(U') \geq k-2$, which implies $\mu(G) \geq k-2$. □

The *diameter* of a graph $G$, denoted by $\text{diam}(G)$, is the maximum distance between vertices of the graph. Since a path corresponding to the diameter is an induced path, we have the following consequence.

**Corollary 3.2.** For any graph $G$,

$$\mu(G) \geq \text{diam}(G) - 1.$$ □

The *union* of graphs $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2), \ldots, G_m = (V_m, E_m)$ is the graph

$$\bigcup_{i=1}^{m} G_i = \left( \bigcup_{i=1}^{m} V_i, \bigcup_{i=1}^{m} E_i \right).$$

If $G_1 = G_2 = \cdots = G_m$, then instead of $\bigcup_{i=1}^{m} G_i$, we use the notation $mG$. 6
Lemma 3.3. If a graph $G$ on $n$ vertices contains the induced subgraph $P_{k_1} \cup P_{k_2} \cup \cdots \cup P_{k_t}$ with $t \geq 2$, and $k_i \geq 2$, $i = 1, \ldots, t$, then

$$\text{mur}(G) \geq \left( \sum_{i=1}^{t} k_i \right) - (t + 1).$$

Proof. Order the paths as given and in each path order the vertices so that a pendant vertex comes first and every other vertex comes right after its previous neighbour. Then any universal adjacency matrix of $G$ is of the following form

$$U = \begin{bmatrix}
U_{11} & \gamma J_{k_1, k_2} & \cdots & \gamma J_{k_1, k_t} & U_{1(t+1)} \\
\gamma J_{k_2, k_1} & U_{22} & \cdots & \gamma J_{k_2, k_t} & U_{2(t+1)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\gamma J_{k_t, k_1} & \gamma J_{k_t, k_2} & \cdots & U_{tt} & U_{t(t+1)} \\
U_{11}^T & U_{21}^T & \cdots & U_{t1}^T & U_{(t+1)(t+1)}^T
\end{bmatrix}.$$

For each block $U_{ii}$, $i = 1, \ldots, t$, the super diagonal entries are $\gamma + 1$ and every other non-diagonal entry equals $\gamma$. Now subtracting the column $\sum_{i=1}^{t} k_i$ from each of the columns $1, 2, \ldots, \sum_{i=1}^{t} k_i - 1$, we produce a lower triangular submatrix of size $k_i - 1$ in the $(i, i)$ block for $i = 1, \ldots, k_i - 1$ and a lower triangular submatrix of size $k_i - 2$ in the $(t, t)$ block, with all ones on the main diagonal entries of each of the triangular matrices. Moreover, the entries of the blocks above these triangular matrices in the resulting matrix are all zero except possibly the last column. So $\text{rank}(U_G) \geq \sum_{i=1}^{t} k_i - (t + 1).$ \hfill \square

Lemma 3.4. If $G$ contains the induced subgraph $P_{k_1} \cup P_{k_2} \cup \cdots \cup P_{k_t} \cup mP_1$ with $t \geq 2$, $m \geq 1$, and $k_i \geq 2$, $i = 1, \ldots, t$, then

$$\text{mur}(G) \geq \left( \sum_{i=1}^{t} k_i \right) - t.$$

Proof. The proof is similar to the proof of Lemma 3.3. Using the same ordering for vertices, and subtracting the column corresponding to the column of one of the vertices in $mP_1$, results in a lower triangular of rank $k_i - 1$ matrix for the block corresponding to $P_{k_i}$ as well, which implies the inequality. \hfill \square

In some cases in Lemmas 3.3 and 3.4, the equality holds when the induced subgraph $P_{k_1} \cup P_{k_2} \cup \cdots \cup P_{k_t} \cup mP_1$, $(m \geq 0)$, is exactly the graph $G$. Some of these cases are listed below.
Example 3.5. For $n \geq 3$

(a) $\text{mur}(P_n) = n - 2$;

(b) $\text{mur}(P_{n-1} \cup P_1) = n - 2$;

(c) $\text{mur}(P_n \cup P_n) = 2n - 3$.

Proof. Equation (a) can be obtained from Lemma 3.1 and Corollary 2.5. Moreover, the universal adjacency matrix $-A - \lambda I + \frac{1}{n} J + D$ meets this bound, where $\lambda$ is an arbitrary nonzero eigenvalue of $L_{P_n}$. For instance, choosing $\lambda = 2 \left(1 - \cos\left(\frac{\pi}{n}\right)\right)$, we get the following universal adjacency matrix of rank $n - 2$

$$U_{P_n} = A_{P_n} + 2 \left(1 - \cos\left(\frac{\pi}{n}\right)\right) I - \frac{2}{n} \left(1 - \cos\left(\frac{\pi}{n}\right)\right) J - D_{P_n}.$$ 

Equation (b) is obtained from Lemma 3.3 and Corollary 2.5. The Laplacian matrix $L_{P_{n-1} \cup P_1}$ is an example of a universal adjacency matrix for the graph that has the minimum rank.

Finally, Equation (c) is obtained from Theorem 2.4 and Lemma 3.3. The universal adjacency matrix $-A - \lambda I + \frac{1}{n} J + D$ meets this bound, where $\lambda$ is an arbitrary nonzero eigenvalue of $L_{P_n}$.

4. Regular Graphs

A graph $G$ is called regular of degree $r$ if each vertex of $G$ is adjacent to exactly $r$ vertices. If $G$ is a regular graph of degree $r$, then it is evident that $A_{G}e = re$, $A_{G}J = J A_{G}$. Moreover, $D_{G} = rI$, so any universal adjacency matrix associated with a regular graph may be reduced to the form $U_{G} = A_{G} + \beta I + \gamma J$, which is the generalized adjacency matrix of $G$. Now we are able to derive the following result regarding the minimum universal rank of any regular graph.

Theorem 4.1. Suppose $G$ is a connected $r$-regular graph of degree $r$ on $n$ vertices. Let the spectrum of the adjacency matrix, $A_{G}$, be given by $r, \lambda_2, \lambda_3, \ldots, \lambda_n$ (these values may not be distinct), and assume that $m$ is the maximum multiplicity among the list \{\lambda_2, \lambda_3, \ldots, \lambda_n\}. Then

$$\text{mur}(G) = n - (m + 1).$$

Proof. Since $G$ is a regular graph of degree $r$, we know that $A_{G}e = re$. Thus for any other eigenvalue $\lambda_i$ if $x_i$ is a corresponding eigenvector, then $x_i$ is orthogonal to $e$ and hence $Jx_i = 0$. Furthermore, since $A_{G}$ and $J$ commute and are symmetric, it follows that the eigenvalues of $A_{G} + \gamma J$ are given by $r + \gamma n, \lambda_2, \lambda_3, \ldots, \lambda_n$. (Here we also used the facts that $J$ is rank one and $J e = ne$.) Thus the maximum number of zero eigenvalues admitted by any universal matrix $U_{G} = A_{G} + \beta I + \gamma J$ is equal to $m + 1$ by suitable choices of $\beta$ and $\gamma$. From which it follows that $\text{mur}(G) = n - (m + 1)$, which completes the proof.
The adjacency eigenvalues of \( K_n \) are \( \{n - 1, -1, -1, \ldots, -1\} \), where \(-1\) occurs with multiplicity \( n - 1 \), in this case \( m = n - 1 \), and \( \text{mur}(K_n) = 0 \). This gives an alternative proof of one of the directions in Theorem 2.2.

Since the adjacency eigenvalues of a cycle on \( n \) vertices are known, we have the following as an immediate consequence.

**Corollary 4.2.** For any \( n \geq 3 \), \( k \geq 1 \),

\[
\text{mur}(kC_n) = kn - 2k - 1.
\]

**Proof.** The adjacency eigenvalues of \( C_n \) are well known to be twice the real parts of the \( n \)-th roots of unity; see [3]. Thus, the maximum multiplicity among the eigenvalues different from the degree is \( m = 2 \). Moreover, \( kC_n \) has the same eigenvalues as \( C_n \) and each eigenvalue has multiplicity \( k \) times its multiplicity as an eigenvalue for \( C_n \). Applying Theorem 4.1 we have that \( \text{mur}(kC_n) = kn - (2k + 1) \).

In particular, this means that \( \text{mur}(C_n) = n - 3 \).

There are many large families of graphs for which all eigenvalues of their adjacency matrices are known and for these it is easy to determine the minimum universal rank. For example, the adjacency matrix of any strongly regular graph has exactly three distinct eigenvalues and the multiplicities of those that are not equal to the degree can be expressed in terms of the parameters for the graph; see [2]; thus the minimum universal rank can also be expressed in terms of the parameters of the strongly regular graph.

**5. Unions of Graphs**

We have seen several examples of the minimum universal rank for a graph that is the union of smaller graphs. This motivates us to consider bounds on the minimum universal rank of the union of two graphs; we start with what is a natural lower bound. Indeed we are considering unions of graphs as opposed to joins (see definition on page 11), as we feel this approach eases exposition.

**Lemma 5.1.** Let \( G \) and \( H \) be two graphs, then

\[
\text{mur}(G) + \text{mur}(H) \leq \text{mur}(G \cup H).
\]

**Proof.** Suppose \( G \) has \( n \) vertices and \( H \) has \( m \) vertices, and suppose that \( \text{mur}(G \cup H) \) is attained by the following universal adjacency matrix:

\[
U_{G\cup H}(1, \beta, \delta, \gamma) = \begin{bmatrix} U_G & \gamma J_{n,m} \\ \gamma J_{m,n} & U_H \end{bmatrix}.
\]

(1)

If \( \gamma = 0 \), then

\[
\text{mur}(G \cup H) = \text{rank}(U_G) + \text{rank}(U_H) \geq \text{mur}(G) + \text{mur}(H).
\]
Thus we may assume $\gamma \neq 0$. We consider two cases, first if the all ones vector is in the column space of either $U_G$ or $U_H$ and second when it is not. The column space of a matrix $A$ is denoted by $\text{col}(A)$.

**Case 1:** $e_n \in \text{col}(U_G)$ or $e_m \in \text{col}(U_H)$.

We only consider the case $e_n \in \text{col}(U_G)$, as the other case is similar. Note that $\gamma e_n$ must be a linear combination of the columns of $U_G$. Subtracting this combination from each column of $\gamma J_{n,m}$, we arrive at the following matrix, where $\zeta \gamma$ is the sum of the coefficients of the above linear combination.

$$
\begin{bmatrix}
U_G & 0_{n,m} \\
\gamma J_{m,n} & U_H - \zeta J_m
\end{bmatrix}.
$$

As $U_G$ is symmetric, subtracting the corresponding linear combination of the rows of $U_G$ from each row of $\gamma J_{m,n}$ we arrive at the matrix:

$$
\begin{bmatrix}
U_G & 0_{n,m} \\
0_{m,n} & U_H - \zeta J_m
\end{bmatrix}.
$$

So

$$\text{mur}(G \cup H) = \text{rank}(U_G) + \text{rank}(U_H - \zeta J_m) \geq \text{mur}(G) + \text{mur}(H).$$

Furthermore, if $\zeta \neq 0$ and $e_m \notin \text{col}(U_H)$, then $\text{rank}(U_H - \zeta J_m) = \text{rank}(U_H) + 1$, and hence

$$\text{mur}(G \cup H) \geq \text{mur}(G) + \text{mur}(H) + 1.$$
again, we get the following matrix:

\[
\begin{bmatrix}
\Lambda_{n'} & B & 0_{n', m'} \\
0_{n'-1, n'} & 0_{n'-1, n'-n'} & 0_{n'-1, m'} \\
0_{1, n'} & 0_{1, n'-n'} & e_m^T \\
\gamma J_{m, n'} & \gamma J_{m, n'-n'} & U_H
\end{bmatrix}.
\] (2)

Similarly, by applying some elementary row operations to the matrix in (2) on the rows corresponding to \(U_H\), and if necessary, permuting some columns of \(U_H\) (still preserving the zero-nonzero pattern of other blocks), we have the following matrix for some non-singular diagonal matrix \(\bar{\Lambda}\) of order \(m' < m\), and some matrix \(C\) of size \(m' \times (m-m')\):

\[
\begin{bmatrix}
\Lambda_{n'} & B & 0_{n', m'} & 0_{n', m-m'} \\
0_{n'-1, n'} & 0_{n'-1, n'-n'} & 0_{n'-1, m'} & 0_{n'-1, m-m'} \\
0_{1, n'} & 0_{1, n'-n'} & e_m^T & e_{m-m'}^T \\
0_{m', n'} & 0_{m', n'-n'} & \bar{\Lambda}_{m'} & C \\
e_m^T & e_{m-n'}^T & 0_{1, m'} & 0_{1, m-m'}
\end{bmatrix}.
\] (3)

By deleting the zero rows from the matrix in (3), we have

\[
\begin{bmatrix}
\Lambda_{n'} & B & 0_{n', m'} & 0_{n', m-m'} \\
0_{1, n'} & 0_{1, n'-n'} & e_m^T & e_{m-m'}^T \\
0_{m', n'} & 0_{m', n'-n'} & \bar{\Lambda}_{m'} & C \\
e_{n'}^T & e_{m-n'}^T & 0_{1, m'} & 0_{1, m-m'}
\end{bmatrix}.
\]

So

\[
\text{mur}(G \cup H) = \text{rank} \begin{bmatrix} \Lambda_{n'} & B \\ e_{n'}^T & e_{m-n'}^T \end{bmatrix} + \text{rank} \begin{bmatrix} e_{m'}^T & e_{m-m'}^T \\ \bar{\Lambda}_{m'} & C \end{bmatrix} = (\text{rank}\Lambda_{n'} + 1) + (\text{rank}\bar{\Lambda}_{m'} + 1) \\
\geq \text{mur}(G) + \text{mur}(H) + 2.
\]

Note that, from the proof we can conclude that if \(\gamma e \notin \text{col}(U_G)\) and \(\gamma e \notin \text{col}(U_H)\), then we actually have a stronger bound on \(\text{mur}(G \cup H)\).

The join of graphs \(G_1 = (V_1, E_1), G_2 = (V_2, E_2), \ldots, G_m = (V_m, E_m)\) is a graph on the vertices \(\cup_{i=1}^m V_i\) that includes the edges \(\cup_{i=1}^m E_i\) but also has all edges \(\{v_i, v_j\}\) where \(v_i \in V_i\) and \(v_j \in V_j\) with \(i \neq j\). The join is denoted by \(G_1 \vee G_2 \vee \cdots \vee G_m\). The join and the union are complementary operations in the sense that for any pair of graphs \(G_1\) and \(G_2\),

\[
\overline{G_1 \cup G_2} = \overline{G_1} \vee \overline{G_2}.
\]
This fact, together with Lemma 2.1, yields the following

\[ \text{mur}(G_1 \cup G_2) = \text{mur}(\overline{G_1} \cup \overline{G_2}) = \text{mur}(\overline{G_1} \lor \overline{G_2}). \]

This means that results about the union of graphs can be translated to results about joins of graphs, for example the next result is Lemma 5.1 stated for the join of two graphs

**Lemma 5.2.** For graphs \( G \) and \( H \)

\[ \text{mur}(G \lor H) \geq \text{mur}(G) + \text{mur}(H). \]

**Proof.** With the comments above, we simply note that

\[ \text{mur}(G \lor H) = \text{mur}(\overline{G} \cup \overline{H}) \geq \text{mur}(\overline{G}) + \text{mur}(\overline{H}) = \text{mur}(G) + \text{mur}(H). \]

Upper bounds on the minimum universal rank of the union of two graphs seem to be more difficult question. We have seen examples where it is possible to express the minimum universal rank of the union of graphs in terms of the minimum universal ranks of the graphs in the union. For example, \([\bar{1}]\) and \([\bar{3}]\) of Example 3.5 state that

\[ \text{mur}(P_{n-1} \cup P_1) = \text{mur}(P_{n-1}) + \text{mur}(P_1) + 1 \]

and

\[ \text{mur}(P_n \cup P_n) = 2 \text{mur}(P_n) + 1. \]

From Corollary 4.2

\[ \text{mur}(C_n \cup C_n) = 2 \text{mur}(C_n) + 1, \]

and more generally that \( \text{mur}(kC_n) = kmur(C_n) + k - 1. \)

This might lead one to conjecture that the minimum universal rank of the union of the two graphs is bounded above by the sum of the minimum universal ranks of the graphs in the union plus one, but the difference between \( \text{mur}(G \cup H) \) and \( \text{mur}(G) + \text{mur}(H) \) can be arbitrarily large. For example, take \( G = kC_3 \) and \( H = kC_4 \), so \( \text{mur}(kC_3) = k - 1 \) and \( \text{mur}(kC_4) = 2k - 1. \) But Theorem 4.1 implies that \( \text{mur}(kC_3 \cup kC_4) = 5k - 1 \) so

\[ \text{mur}(kC_3 \cup kC_4) - (\text{mur}(kC_3) + \text{mur}(kC_4)) = 2k + 1. \]

Even though the upper bound \( \text{mur}(G) + \text{mur}(H) + 1 \) on \( \text{mur}(G \cup H) \) may fail, there is an upper bound for the minimum universal rank of the union of graphs using the minimum universal rank of one and the number of vertices of the other.

**Proposition 5.3.** For an \( n \times n \) symmetric matrix \( A \), if \( e_n \notin \text{col}(A) \), then \( e \in \text{col}(A + \gamma J) \), for all \( \gamma \neq 0. \)
Proof. Since $A$ is symmetric, $\text{col}(A) = \text{nul}(A)^\perp$. So if $e \notin \text{col}A$, then there exists a vector $x \in \text{nul}(A)$ such that $e^T x \neq 0$. Thus, $(A + \gamma J)x = \gamma Jx = \gamma (e^T x)e$, which implies $e \in \text{col}(A + \gamma J)$.

**Theorem 5.4.** For graphs $G$ and $H$,$$
\text{mur}(G \cup H) \leq \text{mur}(G) + |V(H)| + 1.
$$

Proof. Assume that $G$ and $H$ have $m$ and $n$ vertices, respectively, and let $\text{mur}(G) = \text{rank}(U(1, \beta, \gamma, \delta))$. Order the vertices of $G \cup H$ such that the vertices of $G$ are the first $m$ vertices. Then

$$
U_{G \cup H}(1, \beta, \gamma, \delta) = 
\begin{bmatrix}
U_G & \gamma J_{m,n} \\
\gamma J_{n,m} & U_H
\end{bmatrix}.
$$

If $\gamma = 0$, then clearly $\text{mur}(G \cup H) \leq \text{mur}(G) + |V(H)|$. If $\gamma \neq 0$, we consider two cases:

If $e \in \text{col}(U_G)$, then by a similar method used in the proof of Lemma 5.1, the matrix $U_{G \cup H}$ can be reduced into the following form

$$
U_1 = 
\begin{bmatrix}
U_G & 0 \\
0 & U_H + pJ
\end{bmatrix}
$$

for some nonzero number $p$. Therefore,

$$
\text{mur}(G \cup H) \leq \text{rank}(U_1) \leq \text{mur}(G) + |V(H)| + 1.
$$

If $e \notin \text{col}(U_G)$, then using Proposition 5.3, $e \in \text{col}(U_G - \gamma J)$. Let $U' = U_G - \gamma J$, and subtract the $(n+1)$-st column of $U_{G \cup H}(1, \beta, \gamma, \delta)$ from each of the first $n$ columns. The result is the following matrix

$$
U_2 = 
\begin{bmatrix}
U' & \gamma J_{m,n} \\
R & U_H
\end{bmatrix}.
$$

Since $e \in \text{col}(U')$, the matrix $U_2$ can be reduced to

$$
U_3 = 
\begin{bmatrix}
U' & 0 \\
R & S
\end{bmatrix}.
$$

Using the fact that, $\text{rank}(U') \leq \text{mur}(G) + 1$, we have

$$
\text{mur}(G \cup H) \leq \text{rank}(U_3) \leq \text{rank}(U') + |V(H)| \leq \text{mur}(G) + |V(H)| + 1.
$$

6. Minimum Universal Rank Spread

The mur-spread of a graph $G$ at vertex $v$, denoted by $\text{mur}_v(G)$, is defined to be $\text{mur}(G) - \text{mur}(G \setminus \{v\})$. The following theorem establishes upper and lower bounds for the murm-spread of a vertex.
Theorem 6.1. If a vertex $v$ of $G$ has degree $d$, then

$$-d \leq \text{mur}_v(G) \leq d + 2.$$  

Proof. Let $U_G = U_G(1, \beta, \delta, \gamma)$ be a universal adjacency matrix associated with the graph $G$. Let $B$ be the submatrix of $U_G$ obtained by deleting the row and column of $U_G$ associated with the vertex $v$. Then $U_G$ has the following block form:

$$U_G(1, \beta, \delta, \gamma) = \begin{bmatrix} B & V \\ V^T & \beta + \gamma + d\delta \end{bmatrix}.$$  

Evidently,

$$\text{rank}(U_G) - 2 \leq \text{rank}(B) \leq \text{rank}(U_G).$$  

(4)

Let $N(v)$ denote the set of neighbours of $v$ and $D'$ be a diagonal matrix of the same size as $B$ whose diagonal entry $D'_{ii}$ is 1 if $v_i \in N(v)$ and 0 otherwise. Thus there is a universal adjacency matrix for the graph obtained by removing $v$ from $G$, namely $U_{G \setminus \{v\}}$, such that $B = U_{G \setminus \{v\}} + \delta D'$.

Using the subadditivity property of rank of matrices, we have the following inequalities

$$\text{rank}(B) - d \leq \text{rank}(U_{G \setminus \{v\}}) \leq \text{rank}(B) + d.$$  

(5)

Using equations (4) and (5), we have

$$\text{rank}(U_G) - (d + 2) \leq \text{rank}(U_{G \setminus \{v\}}) \leq \text{rank}(U_G) + d.$$  

(6)

Now in (6), if $U_G$ be a universal adjacency matrix with $\text{rank}(U_G) = \text{mur}(G)$, then

$$\text{mur}(G \setminus \{v\}) \leq \text{rank}(U_{G \setminus \{v\}}) \leq \text{mur}(G) + d.$$  

which implies $-d \leq \text{mur}_v(G)$. If $U_{G \setminus \{v\}}$ be a universal adjacency matrix with $\text{rank}(U_{G \setminus \{v\}}) = \text{mur}(G \setminus \{v\})$ in (4), then

$$\text{mur}(G) - (d + 2) \leq \text{rank}(U_G) - (d + 2) \leq \text{rank}(U_{G \setminus \{v\}}) = \text{mur}(G \setminus \{v\}),$$

which implies $\text{mur}_v(G) \leq d + 2$. \qed

Corollary 6.2. If a vertex $v$ of $G$ has degree $d$, then

$$\max\{-d, -(n - d - 1)\} \leq \text{mur}_v(G) \leq \min\{d + 2, (n - d - 1) + 2\}.$$  

Proof. Since

$$\text{mur}(G) - \text{mur}(G \setminus \{v\}) = \text{mur}(\overline{G}) - \text{mur}(\overline{G \setminus \{v\}}) = \text{mur}(\overline{G}) - \text{mur}(\overline{G \setminus \{v\}}),$$

simply apply Theorem 6.1 to $\overline{G}$, noting that the degree of $v$ in $\overline{G}$ is $n - d - 1$. \qed
In particular, the following holds:

**Corollary 6.3.** If \( v \) is a pendant vertex of the graph \( G \), then

\[ -1 \leq \text{mur}_v(G) \leq 3. \]

**Example 6.4.** The following examples show that there is a graph with mur-spread \( k \) for \(-1 \leq k \leq 2\). It is an open question to find a graph with mur-spread equal to 3 at a pendant vertex.

1. If \( r = s \geq 2 \), then \( \text{mur}_v(K_r \cup K_s) \cup \{v\} = -1 \), for every pendant vertex \( v \) (see Theorem 7.2 for a proof of this claim).
2. \( \text{mur}_v(K_{1,n}) = 0 \) for any pendant vertex \( v \) (see Theorem 8.1).
3. For \( n > 2 \), \( \text{mur}_v(P_n) = 1 \), for the end-point vertices \( v \).
4. A generalized star is a tree with at most one vertex of degree greater than or equal to three. If \( G \) is a generalized star on five vertices with the degree sequence \( 1, 1, 1, 2, 3 \), by calculation it can be shown that \( \text{mur}(G) = 2 \). So if \( v \) is a pendant vertex of \( G \) whose deletion leaves a star, using Theorem 8.1 we have \( \text{mur}_v(G) = 2 \).

The first example in the list is perhaps the most interesting, since it is an example of a graph that has a vertex that when removed leaves a graph that has a strictly larger minimum universal rank. We will consider this example in more detail in the next section.

### 7. Monotonicity

A parameter for a graph \( G \) is called *monotone on induced subgraphs* if the value of the parameter for the graph is never smaller than the value on an induced subgraph. In this section, we show that the minimum universal rank is not in general monotone on induced subgraphs. To see why this might be true, consider a graph \( G \) with a universal adjacency matrix \( U \). For any induced subgraph \( H \) of \( G \), there is a submatrix of \( U \) formed by taking all the rows and columns corresponding to vertices in \( H \). If this submatrix is a universal matrix for \( H \), then the minimum universal rank of \( H \) will be no larger than \( \text{mur}(G) \), but this submatrix may not be a universal matrix for \( H \). The problem is with the main diagonal entries, since these entries are based on the degree of the vertex, and a vertex may have different degrees in different subgraphs. To start, we will give an example of a graph that has an induced subgraph with a larger minimum universal rank.

**Theorem 7.1.** For all nonnegative integers \( r \) and \( s \), if \( s - r + 1 \neq 0 \), then

\[ \text{mur}((K_r \cup K_s) \cup \{v\}) \leq 2. \]

Further,
(a) \( \mu_r((K_r \cup \overline{K_s}) \vee \{v\}) = 0 \) if and only if \((s = 0)\), or \((s = 1 \text{ and } r = 0)\);

(b) \( \mu_r((K_r \cup \overline{K_s}) \vee \{v\}) = 1 \) if and only if \(s \neq 0 \text{ and } r = 0 \text{ or } 1\).

**Proof.** We first prove the statements (a) and (b). The graph \((K_r \cup \overline{K_s}) \vee \{v\}\) is isomorphic to a complete graph if and only if either \(s = 0\) or \(s = 1 \text{ and } r = 0\). Since these are the only two cases in which \((K_r \cup \overline{K_s}) \vee \{v\}\) is a complete graph, by Theorem 2.2, (a) holds.

If \(r = 0 \text{ or } r = 1\), then the graph \((K_r \cup \overline{K_s}) \vee \{v\}\) is a star with \(r + s\) edges. Provided that \(s > 1\), by Theorem 8.1 in the following section, the minimum universal rank of these graphs is one, thus (b) holds.

To show the general statement, assume that none of the above cases are satisfied. So either both of \(r\) and \(s\) are greater than or equal to 2 and \(s - r + 1 = 0\) or \(s = 1 \text{ and } r \geq 3\). Consider the following universal adjacency matrix:

\[
U = A + \left( \frac{-1}{r-1} \right) I + \frac{r-1}{s-r+1} J + \frac{1}{r-1} D,
\]

we claim that the rank of this matrix is 2.

Order the vertices of the graph so that the first \(r\) vertices are the vertices of \(K_r\), the next \(s\) vertices are the vertices of \(\overline{K_s}\) and the final vertex is \(v\). The first \(r-1\) diagonal entries are all \(\frac{s}{s-r+1}\), the next \(s\) diagonal entries are all equal to \(\frac{r-1}{s-r+1}\), and the final entry on the diagonal is

\[
\frac{s + r - 1}{r-1} + \frac{r - 1}{s-r+1} = \frac{s^2}{(r-1)(s-r+1)}.
\]

Adjacent vertices have the entry \(\frac{s}{s-r+1}\) and nonadjacent vertices have the entry \(\frac{r-1}{s-r+1}\). Thus, the matrix \(U\) can be written as

\[
U = \begin{bmatrix}
\frac{s}{s-r+1} T & \frac{r-1}{s-r+1} J_r & \frac{s}{s-r+1} C_r \\
\frac{r-1}{s-r+1} J_s & \frac{r-1}{s-r+1} J_s & \frac{s}{s-r+1} C_s \\
\frac{s}{s-r+1} C_T & \frac{s}{s-r+1} C_C & \frac{s^2}{(r-1)(s-r+1)}
\end{bmatrix}.
\]

Since \(s - r + 1 \neq 0\), the final row is a multiple of the rows in the middle block, which implies \(U\) has rank 2.

**Theorem 7.2.** For integers \(r\) and \(s\), if \(s \geq 3\) and \(s - r + 1 = 0\) then

\[
\mu_r((K_r \cup \overline{K_s}) \vee \{v\}) = 3.
\]

**Proof.** As in the proof of Theorem 7.1, order the vertices so that the first \(r\) vertices are from \(K_r\), the next \(s\) vertices are from \(\overline{K_s}\) and \(v\) is the last vertex.
Then, any universal adjacency matrix \( U = \alpha A + \beta I + \gamma J + \delta D \) for this graph has the form
\[
\begin{bmatrix}
(\gamma+1)J_{r,r} + (\beta + \gamma - 1)I_{r,r} & \gamma J_{s,s} & (\gamma+1)e_r \\
\gamma J_{s,r} & \gamma J_{s,s} + (\beta + \gamma - 1)I_{s,s} & (\gamma+1)e_s \\
(\gamma+1)e^T_r & (\gamma+1)e^T_s & \gamma + \beta + (r+s)\delta
\end{bmatrix}
\]
which can be row reduced to the following matrix
\[
\begin{bmatrix}
(\beta + \gamma - 1)I_{r,r} & 0 & (1 - (\beta + (r+s)\delta))e_r \\
0 & \gamma J_{s,s} + (\beta + \gamma - 1)I_{s,s} & (\gamma+1)e_s \\
(\gamma+1)e^T_r & (\gamma+1)e^T_s & \gamma + \beta + (r+s)\delta
\end{bmatrix}.
\]
If \( \beta + \gamma - 1 \neq 0 \), then the rank of \( U \) is at least \( r \) which is greater than four. So we assume that \( \beta + \gamma - 1 = 0 \), and further reduce the matrix to
\[
\begin{bmatrix}
0_{r,r} & (-1)J_{r,s} & [1 - (\beta + (r+s)\delta)]e_r \\
\gamma J_{s,r} & (\beta + \gamma - 1)I_{s,s} & (\gamma+1)e_s \\
(\gamma+1)e^T_r & 0_{1,s} & \gamma + \beta + (r+s)\delta
\end{bmatrix}.
\]
Since \( s \geq 3 \), if \( \beta + \gamma \neq 0 \) then the rank is at least 3, so we also assume that \( \beta + \gamma = 0 \). With this assumption and the assumption that \( \beta + \gamma - 1 = 0 \) we have \( \delta = \frac{1}{r-1} \) and \( \beta = -\delta \). The matrix then can be further reduced to
\[
\begin{bmatrix}
0_{r,r} & (-1)J_{r,s} & -e_r \\
\gamma J_{s,r} & 0_{s,s} & e_s \\
(\gamma+1)e^T_r & 0_{1,s} & 1
\end{bmatrix}.
\]
We also used the facts that \( 1 - (\beta + (r+s)\delta) = -1 \), and \( \frac{\gamma + 1}{r-1} = 2 \). Since there does not exist a value of \( \gamma \) such that \( \frac{\gamma + 1}{r-1} = 1 \) this matrix has rank 3.

This particular graph is of interest since it shows that the minimum universal rank of a graph is not monotone on induced subgraphs. For example, \( G_1 = (K_4 \cup T_3) \cup \{v\} \) is an induced subgraph of \( G_2 = (K_4 \cup T_4) \cup \{v\} \) but 3 = \( \mu(G_1) \) > \( \mu(G_2) = 2 \). (This example also shows that contraction of an edge of a graph can increase the minimum universal rank of a graph.) However, the minimum universal rank of a graph is monotone under certain conditions.

**Theorem 7.3.** If the minimum universal rank of a graph \( G \) is attained with a universal adjacency matrix of \( G \) with \( \delta = 0 \), then for any induced subgraph \( H \) of \( G \)
\[
\mu(H) \leq \mu(G).
\]

**Proof.** Let \( U = A + \beta I + \gamma J \) be a universal adjacency matrix for \( G \) that attains the minimum rank. Assume \( H \) is obtained from \( G \) by deleting the set of vertices
Then the principal submatrix, say $U_H$, of $U_G$ obtained by deleting the rows and columns corresponding to $R$, is a universal matrix for $H$. Since $\text{rank}(U_H) \leq \text{rank}(U_G)$,

$$\text{mur}(H) \leq \text{rank}(U_H) \leq \text{rank}(U_G) = \text{mur}(G).$$

By the discussion in the beginning of the Section 4, the universal adjacency matrix of a regular graph $G$ can always be written in the form $U = A + \beta I + \gamma J$. Therefore, Theorem 7.3 implies the following.

**Corollary 7.4.** If $G$ is a regular graph and $H$ is an induced subgraph of $G$, then

$$\text{mur}(H) \leq \text{mur}(G).$$

This corollary can be used to compute minimum universal rank of some graphs, for example it can be used to determine the minimum universal rank of the union of complete graphs with arbitrary sizes. The complement of such a graph is a complete multipartite graph, so this will also give the minimum universal rank of these graphs as well.

**Theorem 7.5.** For any integer $k$ and integers $n_1, \ldots, n_k > 1$,

$$\text{mur} \left( \bigcup_{i=1}^{k} K_{n_i} \right) = \text{mur} \left( K_{n_1, \ldots, n_k} \right) = k - 1.$$

**Proof.** Let $G = \bigcup_{i=1}^{k} K_{n_i}$ and $n = \max\{n_1, \ldots, n_k\}$. Define $G' = \bigcup_{i=1}^{k} K_n$, then $G$ is an induced subgraph of $G'$. By Theorem 7.3, $\text{mur}(G) \leq \text{mur}(G')$.

The eigenvalues of the adjacency matrix of $G'$ are $n - 1$ with multiplicity $k$, and $-1$ with multiplicity $k(n - 1)$. Therefore, using Theorem 4.1 we have

$$\text{mur}(G') = |V(G')| - (k(n - 1) + 1) = kn - (kn - k + 1) = k - 1.$$

Thus

$$\text{mur}(G) \leq k - 1. \quad (7)$$

If we order the vertices of $G$ so that the vertices in $K_{n_i}$ come before the vertices in $K_{n_{i+1}}$, then any universal adjacency matrix for $G$ has the form

$$U_G(1, \beta, \gamma, \delta) = \begin{bmatrix}
V_1 & \gamma J_{n_1, n_2} & \cdots & \gamma J_{n_1, n_k} \\
\gamma J_{n_2, n_1} & V_2 & \cdots & \gamma J_{n_2, n_k} \\
\vdots & \vdots & \ddots & \vdots \\
\gamma J_{n_k, n_1} & \gamma J_{n_k, n_2} & \cdots & V_k
\end{bmatrix},$$

where for any $i = 1, \ldots, k$, the $n_i \times n_i$ matrix $V_i$ is as follows:
Since $n_i > 1$, the $k \times k$ submatrix of $U_G$ that corresponds to the rows

$$\{1, n_1 + 1, n_1 + n_2 + 1, \ldots, n_1 + \cdots + n_{k-1} + 1\},$$

and columns

$$\{2, n_1 + 2, n_1 + n_2 + 2, \ldots, n_1 + \cdots + n_{k-1} + 2\}$$

is

$$\begin{bmatrix}
\gamma + 1 & \gamma & \cdots & \gamma \\
\gamma & \gamma + 1 & \cdots & \gamma \\
\vdots & \vdots & \ddots & \vdots \\
\gamma & \gamma & \cdots & \gamma + 1
\end{bmatrix},$$

subtracting the last column from the previous columns results in the following matrix

$$\begin{bmatrix}
1 & 0 & \cdots & 0 & \gamma \\
0 & 1 & \cdots & 0 & \gamma \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & \gamma \\
-1 & -1 & \cdots & -1 & \gamma + 1
\end{bmatrix},$$

whose rank is at least $k - 1$. This means that $\text{mur}(G) \geq k - 1$ and using (7) the result follows.

Note that Theorem 7.5 is not true if we drop the condition $n_i > 1$. For example, in the next section we show that $\text{mur}(K_n \cup K_1 \cup K_1) = 1$.

8. Graphs with Minimum Universal Rank Equal to One

In this section we characterize all graphs $G$ with $\text{mur}(G) = 1$.

**Theorem 8.1.** Let $G$ be a graph with $n = |V(G)| > 2$, then $\text{mur}(G) = 1$ if and only if $G$ or $\overline{G}$ is either $K_r \cup K_s$ for positive $r, s$, with $r + s > 2$, or $K_r \cup \overline{K_s}$ for $r, s$ with $1 \leq r < n$.

**Proof.** According to Theorem 7.5 if $G$ or $\overline{G}$ is $K_r \cup K_s$ for some $r, s$, with $r + s > 2$, then $\text{mur}(G) = 1$. Furthermore, if $G = K_r \cup \overline{K_s}$ for some $r, s$ with $1 < r < n$, then the universal adjacency matrix of $G$ with parameters $\alpha = 1$, $\beta = 0$, $\gamma = 0$ and $\delta = \frac{1}{r-1}$ is of the form

$$U = A + \frac{1}{r-1}D = \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix},$$

where $A$ is the adjacency matrix of $K_r$ and $D$ is the diagonal matrix with entries $\frac{1}{r-1}$.
whose rank is 1. Thus $\mu_r(G) \leq 1$, using Theorem 2.2 implies that $\mu_r(G) = \mu_r(G) = 1$.

To prove the converse, without loss of generality, assume that $G$ is connected and $\mu_r(G) = 1$. So there are real numbers $\beta, \gamma, \delta$ such that the rank of $U = A + \beta I + \gamma J + \delta D$ is one. Let $S$ be the largest independent set in $G$. Since $G$ is not a complete graph $s = |S| \geq 2$. Order the vertices of $G$ so that a universal matrix $U = A + \beta I + \gamma J + \delta D$ is of the following form,

$$U = \begin{bmatrix}
\beta + \gamma + \delta d_{v_1} & \gamma & \cdots & \gamma & * & * & \cdots & * \\
\gamma & \beta + \gamma + \delta d_{v_2} & \cdots & \gamma & * & * & \cdots & * \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\gamma & \gamma & \cdots & \beta + \gamma + \delta d_{v_s} & * & * & \cdots & * \\
* & * & \cdots & * & \beta + \gamma + \delta d_{v_{s+1}} & * & \cdots & * \\
* & * & \cdots & * & * & \beta + \gamma + \delta d_{v_{s+2}} & \cdots & * \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
* & * & \cdots & * & * & \cdots & \beta + \gamma + \delta d_{v_n} & \\
\end{bmatrix}.$$  

We claim that all the vertices in $S$ have the same set of neighbours. To show this, suppose that a vertex $x \in V(G) \setminus S$ is adjacent to $u \in S$ but not to $v \in S$. As the determinant of

$$U[\{u, v\}, \{v, x\}] = \begin{bmatrix}
\gamma & \gamma + 1 \\
\beta + \gamma + \delta d_v & \gamma \\
\end{bmatrix}$$

must be zero, we have

$$\beta(\gamma + 1) + \gamma + \delta d_v(\gamma + 1) = 0. \quad (8)$$

Since $G$ is connected, there is a vertex $y \in V(G) \setminus S$ adjacent to $v$. If $y$ is adjacent to $u$, then we have the following submatrix in $U$:

$$U[\{u, v\}, \{x, y\}] = \begin{bmatrix}
\gamma + 1 & \gamma + 1 \\
\gamma & \gamma + 1 \\
\end{bmatrix},$$

whose determinant being zero implies that $\gamma = -1$. Substituting this in (8) leads to a contradiction. Therefore, $y$ cannot be adjacent to $u$. Thus the above submatrix of $U$ is, as follows

$$U[\{u, v\}, \{x, y\}] = \begin{bmatrix}
\gamma + 1 & \gamma \\
\gamma & \gamma + 1 \\
\end{bmatrix}.$$
Since this matrix is singular we have $\gamma = -\frac{1}{2}$. Then

$$U[\{u,v\}, \{u,x\}] = \begin{pmatrix} \beta + \gamma + \delta_{d_u} & -1/2 \\ -1/2 & 1/2 \end{pmatrix}.$$  

Again using singularity we have $\beta + \gamma + \delta_{d_u} = 1/2$. Therefore, for any $z \neq u$, 

$$U[\{u,z\}, \{u,z\}] = \begin{pmatrix} \frac{1}{2} & \pm \frac{1}{2} \\ \pm \frac{1}{2} & \beta + \gamma + \delta_{d_z} \end{pmatrix},$$  

whose singularity results in that fact that $\beta + \gamma + \delta_{d_z} = \frac{1}{2}$. That is, all diagonal entries are equal to $\frac{1}{2}$.

Assume that $x_1$ and $x_2$ are adjacent vertices of $G$ such that $u$ is adjacent to $x_1$, but $u$ is not adjacent to $x_2$, then $U$ has the following submatrix:

$$U[\{u,x_1\}, \{x_1,x_2\}] = \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix},$$  

which is a contradiction with the rank of $U$ being 1. Thus, if $u$ is adjacent to a vertex $x_1$, then it must be adjacent to all the neighbors of $x_1$. But since $G$ is connected, there is a path $u, x_1, x_2, \ldots, x_t, v$ which implies that $u$ is adjacent to $v$. This is a contradiction since $u$ and $v$ are both in the independent set $S$, and so all the vertices in $S$ have the same set of neighbors.

As a result, if a vertex $z \in V(G) \setminus S$ is not adjacent to a vertex in $S$, then it is not adjacent to any of the vertices in $S$. So $S \cup \{z\}$ is an independent set, which contradicts the maximality of $S$. Therefore, all the vertices in $S$ are adjacent to all the vertices in $V(G) \setminus S$. This implies that $U$ is of the following form:

$$U = \begin{pmatrix} \beta + \gamma + \delta_{d_u} & \gamma & \cdots & \gamma & \gamma + 1 & \gamma + 1 & \cdots & \gamma + 1 \\ \gamma & \beta + \gamma + \delta_{d_u} & \cdots & \gamma & \gamma + 1 & \gamma + 1 & \cdots & \gamma + 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma & \gamma & \cdots & \beta + \gamma + \delta_{d_u} & \gamma + 1 & \gamma + 1 & \cdots & \gamma + 1 \\ \gamma + 1 & \gamma + 1 & \cdots & \gamma + 1 & \beta + \gamma + \delta_{d_1} & * & \cdots & * \\ \gamma + 1 & \gamma + 1 & \cdots & \gamma + 1 & * & \beta + \gamma + \delta_{d_2} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma + 1 & \gamma + 1 & \cdots & \gamma + 1 & * & * & \cdots & \beta + \gamma + \delta_{d_r} \end{pmatrix}.$$  

Finally, suppose that there is a vertex $x_1 \in V(G) \setminus S$ that is adjacent to a vertex $x_2 \in V(G) \setminus S$ but not adjacent to a vertex $x_3 \in V(G) \setminus S$. In this case, the singularity of 

$$U[\{u,x_1\}, \{x_2,x_3\}] = \begin{pmatrix} \gamma + 1 & \gamma + 1 \\ \gamma + 1 & \gamma \end{pmatrix}.$$  


results in $\gamma = -1$. But then we have

$$U[\{u, x_1\}, \{v, x_1\}] = \begin{bmatrix} -1 & 0 \\ 0 & \beta - 1 + d_{x_1} \end{bmatrix},$$

whose determinant being zero implies that

$$\beta - 1 + d_{x_1} = 0,$$

and so $U$ includes the following submatrix:

$$U[\{x_1, x_3\}, \{x_1, x_3\}] = \begin{bmatrix} 0 & -1 \\ 0 & \beta - 1 + d_{x_3} \end{bmatrix},$$

which has rank two, a contradiction. Hence, the subgraph induced by $V(G) \setminus S$ is either $K_s$ or $K_s$. In the first case $G = S \lor K_s$ (whose complement is $K_r \cup K_s$) and in the second case $G = K_{r,s}$ (whose complement is $K_r \cup K_s$).

Using Theorem 10 in [6], one can provide an alternative method to prove the “only if” part of Theorem 8. Indeed, $\mu r(G) = 1$ implies that there exists a universal adjacency matrix for $G$ which has exactly two distinct eigenvalues; namely 0 and a simple eigenvalue $\lambda \neq 0$.

9. Graphs with large minimum universal rank

It is known that the only graphs, whose minimum rank is one less than the number of vertices of the graph are the paths (this is the maximum possible minimum rank). For the case of minimum universal rank, the maximum possible value is two less than the number of vertices. It is an interesting question to ask which graphs on $n$ vertices have the maximum minimum universal rank $n - 2$? We have seen that the paths and paths with an isolated vertex achieve the maximum minimum universal rank; see Example 3. Are there any other graphs that also have the maximum possible minimum universal rank? We consider the paths with an additional edge. Define $P_n'$ to be the following graph:

For $n = 4, 5$ we know that $\mu r(P_n') = n - 1$ which is the number of vertices of the graph minus two in each case. So, there are graphs other than paths, with the maximum possible minimum universal rank. But $n = 4, 5$ are the only cases known for this family of graphs. Indeed, for infinitely many values of $n$ the minimum universal rank of $P_n'$ is three less than the number of vertices.
Proposition 9.1. (a) For \( n \geq 3 \), if \( n \equiv 0 \pmod{3} \), then \( \text{mur}(P'_n) = n - 2 \).

(b) For \( n \geq 6 \), if \( n + 1 = 4k \), then \( \text{mur}(P'_n) = n - 2 \).

Proof. Since \( P'_n \) has an induced path on \( n \) vertices, using Lemma 3.1 we have \( \text{mur}(P'_n) \geq n - 2 \). Under the assumption of part (a), the universal matrix \( U(1, 1, -\frac{1}{n+1}, -1) \) has rank \( n - 2 \), and under the assumption of part (b), the universal matrix \( U(1, 0, 0, -\frac{1}{2k}) \) has rank \( n - 2 \).

Moreover, for \( n = 8 \), the \( 9 \times 9 \) universal matrix with parameters \( \alpha = 1, -\beta = \delta = \frac{1 + \sqrt{5}}{2} \), and \( \gamma = -\frac{1}{3\delta - 5\delta + 1} \) has rank 6. And, for \( n = 10 \), any set of the parameters \( \alpha = 1, -\beta = \delta = -2 \cos \frac{2\pi}{7} \), or \( \alpha = 1, -\beta = \delta = -2 \cos \frac{6\pi}{7}, \gamma = -\frac{1}{3\delta - 5\delta + 1} \) gives the minimum universal rank equal to 8. This leads us to speculate that \( \text{mur}(P'_n) = n - 2 \), for \( n \geq 6 \).

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