On a Multivalued Differential Equation with Nonlocality in Time

André Eikmeier · Etienne Emmrich

Received: 28 June 2019 / Accepted: 23 February 2020 / Published online: 5 May 2020 © The Author(s) 2020

Abstract
The initial value problem for a multivalued differential equation is studied, which is governed by the sum of a monotone, hemicontinuous, coercive operator fulfilling a certain growth condition and a Volterra integral operator in time of convolution type with exponential decay. The two operators act on different Banach spaces where one is not embedded in the other. The set-valued right-hand side is measurable and satisfies certain continuity and growth conditions. Existence of a solution is shown via a generalisation of the Kakutani fixed-point theorem.

Keywords Nonlinear evolution equation · Multivalued differential equation · Differential inclusion · Monotone operator · Volterra operator · Exponentially decaying memory · Existence · Kakutani fixed-point theorem

Mathematics Subject Classification (2010) 47J35 · 34G25 · 45K05 · 35R70

1 Introduction

1.1 Problem Statement and Main Result

We consider the multivalued differential equation
\begin{equation}
\begin{aligned}
v'(t) + Av(t) + (BKv)(t) &\in F(t, v(t)), \\
v(0) &= v_0,
\end{aligned}
\end{equation}

1 Also named differential inclusion by many authors. However, in order to distinguish this kind of problem from the ones containing subdifferentials or set-valued (maximal monotone) operators, we chose the name multivalued differential equation.

Dedicated to Volker Mehrmann on the occasion of his 65th birthday.

Etienne Emmrich
emmrich@math.tu-berlin.de

André Eikmeier
eikmeier@math.tu-berlin.de

Institut für Mathematik, Technische Universität Berlin, Straße des 17. Juni 136, 10623 Berlin, Germany
where 
\[(Kv)(t) = u_0 + \int_0^t k(t-s)v(s)ds, \quad k(z) = \lambda e^{-\lambda z}. \quad (2)\]

Here, $T > 0$ defines the considered time interval, $\lambda > 0$ is a given parameter and $v_0, u_0$ are the given initial data of the problem.

The operator $A : V_A \to V_A^*$ is a monotone, hemicontinuous, coercive operator satisfying a certain growth condition, where $V_A$ is a real, reflexive Banach space. The operator $B : V_B \to V_B^*$ is linear, bounded, strongly positive, and symmetric, where $V_B$ denotes a real Hilbert space. The space $V_A$ shall be compactly and densely embedded in a real Hilbert space $H$, whereas $V_B$ shall be only continuously and densely embedded in $H$. The dual of $H$ is identified with $H$ itself, such that both $V_A, H, V_A^*$ and $V_B, H, V_B^*$ form a so-called Gelfand triple. However, we do not assume any relation between $V_A$ and $V_B$ apart from $V = V_A \cap V_B$ being separable and densely embedded in both $V_A$ and $V_B$. We do not assume that $V_A$ is embedded into $V_B$ or the other way around. Overall, we have the scale
\[V_A \cap V_B = V \subset V_C \subset H = H^* \subset V_C^* \subset V^* = V_A^* + V_B^*, \quad C \in \{A, B\}, \quad (3)\]
of Banach and Hilbert spaces, where all embeddings are meant to be continuous and dense and the embedding $V_A \subset H$ is even meant to be compact.

The operator $F : [0, T] \times H \to P_{fc}(H)$ is measurable, fulfils a certain growth condition in the second argument and the graph of $v \mapsto F(t,v)$ is sequentially closed in $H \times H_w$ for almost all $t \in (0, T)$, where $H_w$ denotes the Hilbert space $H$ equipped with the weak topology. The set $P_{fc}(H)$ denotes the set of all nonempty, closed, and convex subsets of $H$.

Multivalued differential equations appear, e.g., in the formulation of optimal feedback control problems. If we consider the inclusion as an equation with a side condition on the right-hand side, i.e.,
\[
v'(t) + Av(t) + (BKv)(t) = f(t), \quad t \in (0, T),
\]
\[
f(t) \in F(t,v(t)), \quad t \in (0, T),
\]
\[v(0) = v_0,
\]
we can consider $f$ as the control of our system with the corresponding state $v$ and $F$ as the set of admissible controls, which, in the case of $F$ depending on $v$, leads to a feedback control system.

Physical applications of the system we are considering in this work are, e.g., heat flow in materials with memory (see, e.g., MacCamy [25], Miller [30]) or viscoelastic fluid flow (see, e.g., Desch, Grimmer, and Schappacher [11], MacCamy [26]). Another application related to that are non-Fickian diffusion models which describe diffusion processes of a penetrant through a viscoelastic material (see, e.g., Edwards [13], Edwards and Cohen [14], Shaw and Whiteman [39]). They also appear, e.g., in mathematical biology (see, e.g., Cushing [8], Fedotov and Iomin [18], Mehrabian and Abousleiman [27]).

Due to the specific form of the kernel $k$ given in (2), we can rewrite our system into the coupled system
\[
v'(t) + Av(t) + Cu(t) \in F(t,v(t)), \quad t \in (0, T),
\]
\[(u - Du_0)'(t) + \lambda (u - Du_0)(t) = \lambda Dv(t), \quad t \in (0, T),
\]
\[v(0) = v_0,
\]
\[u(0) = Du_0,
\]
where $C$ and $D$ are suitably chosen linear operators such that $B = CD$.

Instead of the kernel $k(z) = \lambda e^{-\lambda z}$, we might also consider $k(z) = ce^{-\lambda z}$ with $c, \lambda > 0$. However, for simplicity, we will stick to the first type of kernel. Actually, this type appears
naturally in many applications. In these applications, \( \lambda \) is often describing a relaxation or averaged delay time. If we consider the limit \( \lambda \to 0 \), the system (4) decouples such that \( u(t) = Du_0, \quad t \in [0, T] \), is the solution of the second equation. In the case \( \lambda \to \infty \), the system reduces to a single first-order equation for \( v \) without memory.

In the case of the kernel \( k(z) = ce^{-\lambda z} \), the behaviour for \( \lambda \to 0 \) is slightly different. The limit then yields a second-order in time equation for \( u \) (see, e.g., Emmrich and Thalhammer [16]).

1.2 Literature Overview

This work is a continuation of Eikmeier, Emmrich, and Kreusler [15]. There, the single-valued instead of the multivalued differential equation is considered in the same setting concerning the spaces \( V_A \) and \( V_B \). However, due to the structure of the proof in the present work, we additionally need the compact embedding \( V_A \subset H \) and we have to assume that the right-hand side is pointwisely \( H \)-valued.

Nonlinear integro-differential equations have been considered by many authors through the years. Results on well-posedness for more general classes of nonlinear evolution equations including Volterra operators, but only in the case of Hilbert spaces \( V_A = V_B \), can be found in, e.g., Gajewski, Gröger, and Zacharias [19]. In contrast to this, Crandall, Londen, and Nohel [7] study the case of a doubly nonlinear problem, where both nonlinear operators are assumed to be (possibly multivalued) maximal monotone subdifferential operators and the domain of definition of one of them has to be continuously and densely embedded in the domain of definition of the other one. For more references on nonlinear and also linear evolutionary integro-differential equations see Eikmeier, Emmrich, and Kreusler [15, Section 1.2].

Multivalued differential equations have also been studied by various authors. Basic results, also for set-valued analysis, can be found in, e.g., Aubin and Cellina [2], Aubin and Frankowska [3], or Deimling [9]. In O’Regan [32], some extensions of the results shown in Deimling [9] are presented. A semilinear multivalued differential equation with a linear, bounded, and strongly positive operator and a set-valued nonlinear operator is, e.g., considered in Beyn, Emmrich, and Rieger [4].

In particular, integro-differential equations in the multivalued case have been studied by, e.g., Papageorgiou [33–36]. The equations are considered under different assumptions with the set-valued operator appearing in the integral term. In most of the works mentioned, examples of applications in the theory of optimal control are given.

In the context of viscoelastic contact problems, evolution inclusions and hemivariational inequalities have been discussed in, e.g., Migórski [28] and Migórski, Ochal, and Sofonea [29].

The optimal feedback control of a motion of a viscoelastic fluid via a multivalued differential equation is, e.g., considered in Gori et al. [21] and Obukhovskiǐ, Zecca, and Zvyagin [31]. Existence of solutions for the equation are shown via topological degree theory.

1.3 Organisation of the Paper

The paper is organised as follows: In Section 2, we introduce the general notation and some basic results from set-valued analysis. In Section 3, we state our assumptions on the operators \( A, B, \) and \( F \) and some preliminary results concerning properties we need in the
following Section 4, where we prove existence of a solution to problem (1). This is done via a generalisation of the Kakutani fixed-point theorem.

2 Notation

Let $X$ be a Banach space with its dual $X^*$. The norm in $X$ and the standard norm in $X^*$ are denoted by $\| \cdot \|_X$ and $\| \cdot \|_{X^*}$, respectively. The duality pairing between $X$ and $X^*$ is denoted by $\langle \cdot , \cdot \rangle$. If $X$ is a Hilbert space, the inner product in $X$ is denoted by $(\cdot , \cdot )$. For the intersection $X \cap Y$ of two Banach spaces $X$ and $Y$, we consider the norm $\| \cdot \|_{X \cap Y} = \| \cdot \|_X + \| \cdot \|_Y$, and for the sum $X + Y$, we consider the norm

$$\| z \|_{X + Y} = \inf \{ \max (\| z_x \|_X, \| z_y \|_Y) \mid z = z_X + z_Y \text{ with } z_x \in X, z_y \in Y \}. $$

Note that $(X \cap Y)^* = X^* + Y^*$ if $X$ and $Y$ are embedded in a locally convex space and $X \cap Y$ is dense in $X$ and $Y$ with respect to the norm above, see, e.g., Gajewski et al. [19, pp. 12ff.].

Now, let $X$ be a real, reflexive, and separable Banach space and $1 \leq p \leq \infty$. By $L^p(0, T; X)$, we denote the usual space of Bochner measurable (sometimes also called strongly measurable), $p$-integrable functions equipped with the standard norm. For $1 \leq p < \infty$, the duality pairing between $L^p(0, T; X)$ and its dual space $L^q(0, T; X^*)$, where $\frac{1}{p} + \frac{1}{q} = 1$ for $p > 1$ and $q = \infty$ for $p = 1$, is also denoted by $\langle \cdot , \cdot \rangle$, and it is given by

$$\langle g, f \rangle = \int_0^T \langle g(t), f(t) \rangle dt,$$

see, e.g., Diestel and Uhl [12, Theorem 1 on p. 98, Corollary 13 on p. 76, Theorem 1 on p. 79].

By $W^{1,p}(0, T; X)$, $1 \leq p \leq \infty$, we denote the usual space of weakly differentiable functions $u \in L^p(0, T; X)$ with $u' \in L^p(0, T; X)$, equipped with the standard norm. By $\mathcal{C}([0, T]; X)$, we denote the space of functions that are continuous on $[0, T]$ with values in $X$, whereas $\mathcal{C}_w([0, T]; X)$ denotes the space of functions that are continuous on $[0, T]$ with respect to the weak topology in $X$. We have the continuous embedding $W^{1,1}(0, T; X) \subset \mathcal{C}([0, T]; X)$, see, e.g., Roubíček [38, Lemma 7.1]. Furthermore, a function $u \in W^{1,1}(0, T; X)$ is almost everywhere equal to a function that is absolutely continuous on $[0, T]$ with values in $X$, see, e.g., Brézis [5, Theorem 8.2]. We denote the set of all these functions by $\mathcal{A}_\mathcal{C}([0, T]; X)$. By $\mathcal{C}^1([0, T])$, we denote the space of real-valued functions that are continuously differentiable on $[0, T]$. By $c$, we denote a generic positive constant.

Now, let us recall some definitions from set-valued analysis. Let $(\Omega , \Sigma)$ be a measurable space and let $X$ be a complete separable metric space. By $\mathcal{L}([a, b])$ and $\mathcal{B}(X)$, we denote the Lebesgue $\sigma$-algebra on the interval $[a, b] \subset \mathbb{R}$ and the Borel $\sigma$-algebra on $X$, respectively. By $P_f(X)$, we denote the set of all nonempty and closed subsets $U \subset X$, and by $P_{fc}(X)$, we denote the set of all nonempty, closed, and convex subsets $U \subset X$.

For a set-valued function $F : \Omega \to 2^X \setminus \{ \emptyset \}$, let

$$|F(\omega)| := \sup \{ \| x \|_X \mid x \in F(\omega) \}, \quad \omega \in \Omega .$$

The graph of such a set-valued function is defined as

$$\text{graph}(F) = \{ (\omega, x) \in \Omega \times X \mid x \in F(\omega) \}.$$
A function \( F : \Omega \rightarrow P_f(X) \) is called measurable (sometimes also called weakly measurable) if the preimage of each open set is measurable, i.e.,

\[
F^{-1}(U) := \{ \omega \in \Omega \mid F(\omega) \cap U \neq \emptyset \} \in \Sigma
\]

for every open \( U \subset X \).\(^2\) A function \( f : \Omega \rightarrow X \) is called measurable selection of \( F \) if \( f(\omega) \in F(\omega) \) for all \( \omega \in \Omega \) and \( f \) is measurable. Each measurable set-valued function has a measurable selection, see, e.g., Aubin and Frankowska [3, Theorem 8.1.3].

Now, let \((\Omega, \Sigma, \mu)\) be a complete \( \sigma \)-finite measure space and let \( X \) be a separable Banach space. For a set-valued function \( F : \Omega \rightarrow P_f(X) \) and \( p \in [1, \infty) \), we denote by \( \mathcal{F}^p \) the set of all \( p \)-integrable selections of \( F \), i.e.,

\[
\mathcal{F}^p := \left\{ f \in L^p(\Omega; X) \mid f(\omega) \in F(\omega) \text{ a.e. in } \Omega \right\},
\]

where \( L^p(\Omega; X, \mu) \) denotes the space of Bochner measurable, \( p \)-integrable functions with respect to \( \mu \).\(^3\) If \( F \) is integrably bounded, i.e., there exists a nonnegative function \( m \in L^1(\Omega; \mathbb{R}) \) such that \( F(\omega) \subset m(\omega)B_X \) for \( \mu \)-almost all \( \omega \in \Omega \), where \( B_X \) denotes the unit ball in \( X \), each measurable selection of \( F \) is in \( \mathcal{F}^p \) due to Lebesgue’s theorem on dominated convergence. The integral of \( F \) is defined as

\[
\int_{\Omega} F \, d\mu := \left\{ \int_{\Omega} f \, d\mu \mid f \in \mathcal{F}^p \right\}.
\]

For properties of this integral, see, e.g., Aubin and Frankowska [3, Chapter 8.6].

For a set-valued function \( F : \Omega \times X \rightarrow P_f(X) \), a function \( v : \Omega \rightarrow X \) and \( p \in [1, \infty) \), we denote by \( \mathcal{F}^p(v) \) the set of all \( p \)-integrable selections of the mapping \( \omega \mapsto F(\omega, v(\omega)) \), i.e.,

\[
\mathcal{F}^p(v) := \left\{ f \in L^p(\Omega; X) \mid f(\omega) \in F(\omega, v(\omega)) \text{ a.e. in } \Omega \right\}.
\]

Finally, let \( X, Y \) be Banach spaces and \( \Omega \subset Y \). A set-valued function \( F : \Omega \rightarrow 2^X \setminus \{ \emptyset \} \) is called upper semicontinuous if \( F^{-1}(U) \) is closed in \( \Omega \) for all closed \( U \subset X \).

### 3 Main Assumptions and Preliminary Results

Throughout this paper, let \( V_A \) be a real, reflexive Banach space and let \( V_B \) and \( H \) be real Hilbert spaces, respectively. As mentioned in Section 1, we require that \( V = V_A \cap V_B \) is separable and the embeddings stated in (3) are fulfilled (with the embedding \( V_A \subset H \) meant to be compact). Let also \( 2 \leq p < \infty, 1 < q \leq 2 \) with \( \frac{1}{p} + \frac{1}{q} = 1 \).

For \( A : V_A \rightarrow V_A^* \), we say the assumptions (A) are fulfilled if

1. \( A \) is monotone,
2. \( A \) is hemicontinuous, i.e., \( \theta \mapsto \langle A(u + \theta v), w \rangle \) is continuous on \([0, 1]\) for all \( u, v, w \in V_A \),
3. \( A \) fulfils a growth condition of order \( p - 1 \), i.e., there exists \( \beta_A > 0 \) such that

\[
\| A v \|_{V_A^*} \leq \beta_A \left( 1 + \| v \|_{V_A}^{p-1} \right)
\]

for all \( v \in V_A \),

\(^2\)Depending on the assumptions on \((\Omega, \Sigma)\) and \( X \), there are many equivalent definitions of measurability for set-valued functions, see, e.g., Denkowski, Migórski, and Papageorgiou [10, Theorem 4.3.4].

\(^3\)Note that in the case of a separable Banach space \( X \), the Bochner measurability of \( f \) coincides with the \( \Sigma \)-\( B(X) \)-measurability, see, e.g., Amann and Escher [1, Chapter X, Theorem 1.4], Denkowski, Migórski, and Papageorgiou [10, Corollary 3.10.5], or Papageorgiou and Winkert [37, Theorem 4.2.4].
iv) $A$ is $p$-coercive, i.e., there exist $\mu_A > 0$, $c_A \geq 0$ such that

$$\langle Av, v \rangle \geq \mu_A \|v\|_{V_A}^p - c_A$$

for all $v \in V_A$.

One operator satisfying these assumptions is, e.g., the $p$-Laplacian $-\Delta_p = -\nabla \cdot (|\nabla|^{p-2} \nabla)$ acting between the standard Sobolev spaces $W^{1,p}_0(\Omega)$ and $W^{-1,p}(\Omega)$ for a bounded Lipschitz domain $\Omega$, see, e.g., Zeidler [40, p. 489]. For $B : V_B \to V_B^*$, we say the assumptions (B) are fulfilled if

i) $B$ is linear,

ii) $B$ is bounded, i.e., there exists $\beta_B > 0$ such that

$$\|Bv\|_{V_B^*} \leq \beta_B \|v\|_{V_B}$$

for all $v \in V_B$,

iii) $B$ is strongly positive, i.e., there exists $\mu_B > 0$ such that

$$\langle Bv, v \rangle \geq \mu_B \|v\|^2$$

for all $v \in V_B$,

iv) $B$ is symmetric.

Following these assumptions, $B$ induces a norm $\| \cdot \|_{B} := \langle B \cdot, \cdot \rangle^{1/2}$ in $V_B$ that is equivalent to $\| \cdot \|_{V_B}$. Therefore, we denote the space $L^2(0, T; (V_B, \| \cdot \|_B))$ by $L^2(0, T; B)$. An example for an operator satisfying these assumptions is the Laplacian $-\Delta$ acting between the standard Sobolev spaces $H^1_0(\Omega)$ and $H^{-1}(\Omega)$, again for a bounded Lipschitz domain $\Omega$, as well as the fractional Laplacian $(-\Delta)^s$ for $\frac{1}{2} < s < 1$, acting between the standard Sobolev–Slobodeckiǐ spaces $H^s_0(\Omega)$ and $H^{-s}(\Omega)$.

Finally, we say that $F : [0, T] \times H \to P_{f.c.}(H)$ fulfills the assumptions (F) if

i) $F$ is measurable,

ii) for almost all $t \in (0, T)$, the graph of the mapping $v \mapsto F(t, v)$ is sequentially closed in $H \times H_w$, where $H_w$ denotes the Hilbert space $H$ equipped with the weak topology,

iii) $|F(t, v)| \leq a(t) + b \|v\|_H^{2/q}$ a.e. with $a \in L^q(0, T)$, $a(t) \geq 0$ a.e. and $b > 0$.

Note that it is also possible to consider $A : [0, T] \times V_A \to V_A^*$ and $B : [0, T] \times V_B \to V_B^*$, where the mappings $t \mapsto A(t, v)$, $v \in V_A$, and $t \mapsto B(t, v)$, $v \in V_B$, are assumed to be measurable and all the assumptions above are assumed to hold uniformly in $t$. However, for simplicity, we will only consider the case of autonomous operators $A$ and $B$.

These operators can be extended to operators defined on $L^p(0, T; V_A)$ and $L^1(0, T; V_B)$, respectively. The monotonicity and hemicontinuity of $A : V_A \to V_A^*$ imply demicontinuity (see, e.g., Zeidler [40, Proposition 26.4 on p. 555]). Due to the separability of $V_A^*$, the theorem of Pettis (see, e.g., Diestel and Uhl [12, Theorem 2 on p. 42]) then implies that $A$ maps Bochner measurable functions $v : [0, T] \to V_A$ into Bochner measurable functions $Av : [0, T] \to V_A^*$, where $(Av)(t) = Av(t)$ for almost all $t \in (0, T)$. Due to the growth condition, we have the estimate

$$\|Av\|_{L^q(0,T; V_A^*)} \leq c \left(1 + \|v\|_{L^p(0,T; V_A)}^{p-1} \right)$$

for all $v \in L^p(0, T; V_A)$, i.e., $A$ maps $L^p(0, T; V_A)$ into $L^q(0, T; V_A^*)$.

Via the same definition $(Bv)(t) = Bv(t)$ for a function $v : [0, T] \to V_B$, we can extend the operator $B : V_B \to V_B^*$ to a linear, bounded, strongly positive, and symmetric operator.
mapping $L^2(0, T; V_B)$ into its dual or to a linear, bounded operator mapping $L^r(0, T; V_B^*)$, $1 \leq r \leq \infty$, respectively.

Due to the definition (2) of the operator $K$, we have the following lemma.

**Lemma 1** Let $X$ be an arbitrary Banach space, $k(z) = \lambda e^{-\lambda z}$, $\lambda > 0$, $u_0 \in X$. The operator $K : L^2(0, T; X) \to L^2(0, T; X)$ is well-defined, affine-linear, and bounded. The estimate

$$\|Kv - u_0\|_{L^2(0, T; X)} \leq \|k\|_{L^1(0, T)} \|v\|_{L^2(0, T; X)}$$

is satisfied for all $v \in L^2(0, T; X)$, where $\|k\|_{L^1(0, T)} = 1 - e^{-\lambda T}$. Further, the estimate

$$\|Kv - u_0\|_{\mathcal{C}([0, T]; X)} \leq \lambda \|v\|_{L^1(0, T; X)}$$

is satisfied for all $v \in L^1(0, T; X)$, i.e., $K$ is also an affine-linear, bounded operator of $L^1(0, T; X)$ into $\mathcal{C}([0, T]; X)$ (even $\mathcal{AC}([0, T]; X)$).

The proof is based on simple calculations, therefore we omit it here. Following this lemma, we obtain the following properties of the operator $BK$.

**Corollary 1** Let the assumptions of Lemma 1 (with $X = V_B$) and assumption (B) be fulfilled. Then the operator $BK : L^2(0, T; V_B) \to L^2(0, T; V_B^*)$ is well-defined, affine-linear, and bounded. The same holds for $BK : L^1(0, T; V_B) \to \mathcal{C}([0, T]; V_B^*)$.

One crucial relation in this setting, resulting from the exponential kernel, is the following one. Let $X$ be an arbitrary Banach space and $v \in L^1(0, T; X)$. Then we have

$$(Kv)'(t) = \lambda v(t) - ((Kv)(t) - u_0)$$

(6)

for almost all $t \in (0, T)$.

Concerning the operator $F$, we need a measurability result in order to be able to extract measurable selections of the multivalued mapping $t \mapsto F(t, u(t))$, where $u$ itself is a measurable function.

**Lemma 2** Let $X$ be a separable Banach space, let $F : [0, T] \times X \to P_f(X)$ be measurable and let $v : [0, T] \to X$ be Bochner measurable. Then the mapping $F_v : [0, T] \to P_f(X)$, $t \mapsto F(t, v(t))$, is measurable.

**Proof** Let $U \subset X$ be open. Consider

$$F_v^{-1}(U) = \{t \in [0, T] \mid F(t, v(t)) \cap U \neq \emptyset\}$$

$$= \pi_{[0, T]}(\{(t, x) \in [0, T] \times X \mid F(t, x) \cap U \neq \emptyset, x = v(t)\})$$

$$= \pi_{[0, T]}(\{(t, x) \in [0, T] \times X \mid F(t, x) \cap U \neq \emptyset\} \cap \text{graph}(v)),$$

where $\pi_{[0, T]}$ denotes the projection onto $[0, T]$. Since $v$ is Bochner measurable, its graph belongs to $\mathcal{L}([0, T]) \otimes \mathcal{B}(X)$, see, e.g., Castaing and Valadier [6, Theorem III.36]. Note again that for a separable Banach space $X$, Bochner measurability and $\mathcal{L}([0, T]) \otimes \mathcal{B}(X)$-measurability are equivalent, see, e.g., Denkowski, Migórski, and Papageorgiou [10, Corollary 3.10.5]. Due to the measurability of $F$, the intersection space in the equation above also belongs to $\mathcal{L}([0, T]) \otimes \mathcal{B}(X)$. Since the projection maps measurable sets into measurable sets (at least in this setting, see, e.g., Castaing and Valadier [6, Theorem III.23]), we have $F_v^{-1}(U) \in \mathcal{L}([0, T])$, which finishes the proof. □
Finally, we need an integration-by-parts formula similar to the one provided in Roubíček [38, Lemma 7.3] for functions in the spaces $W(0, T) := \{v \in L^p(0, T; V_A) | v' \in L^q(0, T; V^*)\}$ and $C^1([0, T]; V)$, respectively.

**Lemma 3** Let $v \in W(0, T)$, $w \in C^1([0, T]; V)$. Then the integration-by-parts formula

$$\int_{t_1}^{t_2} (\langle v'(t), w(t) \rangle + \langle w'(t), v(t) \rangle) \, dt = \langle v(t_2), w(t_2) \rangle - \langle v(t_1), w(t_1) \rangle$$

(7)

holds for all $t_1, t_2 \in [0, T]$.

**Proof** Due to the density of $C^1([0, T]; V_A)$ in $W(0, T)$ (see, e.g., Roubíček [38, Lemma 7.2]), there exists a sequence $\{v_n\} \subset C^1([0, T]; V_A)$ such that $v_n \to v \in W(0, T)$. The formula (7) obviously holds for $v_n$, $n \in \mathbb{N}$, and $w$ due to classical calculus. Therefore, we have

$$\int_{t_1}^{t_2} (\langle v_n'(t), w(t) \rangle + \langle w'(t), v_n(t) \rangle) \, dt = \langle v_n(t_2), w(t_2) \rangle - \langle v_n(t_1), w(t_1) \rangle$$

for all $n \in \mathbb{N}$. The convergence $v_n \to v$ in $W(0, T)$ yields

$$\int_{t_1}^{t_2} (\langle v_n'(t), w(t) \rangle + \langle w'(t), v_n(t) \rangle) \, dt \to \int_{t_1}^{t_2} (\langle v'(t), w(t) \rangle + \langle w'(t), v(t) \rangle) \, dt.$$  

On the other hand, the continuous embedding $W(0, T) \subset C([0, T]; V^*)$ (see, e.g., Roubíček [38, Lemma 7.1]) yields

$$\langle v_n(t), w(t) \rangle \to \langle v(t), w(t) \rangle$$

for all $t \in [0, T]$, in particular for $t_1, t_2$. This finishes the proof.

\[ \square \]

### 4 Existence of a Solution

**Theorem 1** Let the assumptions (A), (B), and (F) be fulfilled and let $u_0 \in V_B$, $v_0 \in H$ be given. Then there exists a solution $v \in L^p(0, T; V_A) \cap C^w([0, T]; H)$ to (1) with $Kv \in C^w([0, T]; V_B)$ and $v' \in L^q(0, T; V_A') + L^\infty(0, T; V_B^*)$, i.e., the initial condition is fulfilled in $H$ and there exists $f \in F^1(v)$ such that the equation

$$v' + Av + BKv = f$$

holds in the sense of $L^q(0, T; V_A^*)$, i.e.,

$$\langle v' + BKv, \varphi \rangle + \langle Av, \varphi \rangle = \langle f, \varphi \rangle$$

for all $\varphi \in L^p(0, T; V_A)$.

**Proof** Following the proof of [35, Theorem 3.1], we want to apply the Kakutani fixed-point theorem, generalised by Glicksberg [20] and Fan [17] to infinite-dimensional locally convex topological vector spaces. The same fixed-point theorem has also been applied in, e.g., Kalita, Migórski, and Sofonea [22] and Kalita, Szafraniec, and Shillor [23] to prove existence of solutions to partial differential inclusions.

First, we need a priori estimates for the solution. Assume $v \in L^p(0, T; V_A) \cap C^w([0, T]; H)$ solves problem (1) with the regularity stated in the theorem. Due to
Lemma 2, there exists a measurable selection \( f : [0, T] \to H \) of the mapping \( t \mapsto F(t, v(t)) \). The growth condition of \( F \) implies
\[
\| f(t) \|_H \leq a(t) + b \| v(t) \|_H^{2/q}
\]
for almost all \( t \in (0, T) \), and since \( a \in L^q(0, T) \) and \( v \in C_w([0, T]; H) \), we have \( f \in L^q(0, T; H) \). Now, test the equation
\[
v' + Av + BKv = f
\]
with \( v \) and integrate the resulting equation over \((0, t), t \in [0, T]\), which yields
\[
\int_0^t (v'(s) + (BKv)(s), v(s)) \, ds + \int_0^t (Av(s), v(s)) \, ds = \int_0^t (f(s), v(s)) \, ds.
\]
Since we neither know \( v' \in L^q(0, T; V_A^*) \) nor \( BKv \in L^q(0, T; V_A^*) \), it is not possible to do integration by parts for each term separately. However, [15, Lemma 4.3] yields
\[
\int_0^t (v'(s) + (BKv)(s), v(s)) \, ds = \frac{1}{2} \| v(t) \|_H^2 - \frac{1}{2} \| v_0 \|_H^2 + \frac{1}{2\lambda} \| (Kv)(t) \|_B^2 - \frac{1}{2\lambda} \| u_0 \|_B^2
\]
\[
- \int_0^t \langle (BKv)(s), u_0 \rangle \, ds + \int_0^t \| (Kv)(s) \|_B^2 \, ds.
\]
Due to the coercivity of \( A \) and Young's inequality, we have
\[
\frac{1}{2} \| v(t) \|_H^2 + \mu_A \int_0^t \| v(s) \|_{V_A^*}^p \, ds + \frac{1}{2\lambda} \| (Kv)(t) \|_B^2 + \int_0^t \| (Kv)(s) \|_B^2 \, ds
\]
\[
\leq c_A T + \frac{1}{2} \| v_0 \|_H^2 + \frac{1}{2\lambda} \| u_0 \|_B^2 + \int_0^t \| f(s) \|_{V_A^*}^q \| v(s) \|_{V_A^*} \, ds + \int_0^t \| (Kv)(s) \|_B \| u_0 \|_B \, ds
\]
\[
\leq c_A T + \frac{1}{2} \| v_0 \|_H^2 + \frac{1}{2\lambda} \| u_0 \|_B^2 + c \int_0^t \| f(s) \|_{V_A^*}^q \, ds + \frac{\mu_A}{2} \int_0^t \| (Kv)(s) \|_{V_A^*}^p \, ds
\]
\[
+ \frac{1}{2} \int_0^t \| (Kv)(s) \|_B^2 \, ds + \frac{T}{2} \| u_0 \|_B^2.
\]
After rearranging, the estimate on \( F \) yields
\[
\frac{1}{2} \| v(t) \|_H^2 + \frac{\mu_A}{2} \int_0^t \| v(s) \|_{V_A^*}^p \, ds + \frac{1}{2\lambda} \| (Kv)(t) \|_B^2 + \frac{1}{2} \int_0^t \| (Kv)(s) \|_B^2 \, ds
\]
\[
\leq c \left( 1 + \| v_0 \|_H^2 + \| u_0 \|_B^2 + \int_0^t \| f(s) \|_{V_A^*}^q \, ds \right)
\]
\[
\leq c \left( 1 + \| v_0 \|_H^2 + \| u_0 \|_B^2 + \int_0^t \left( a(s) + b \| v(s) \|_H^{2/q} \right)^q \, ds \right)
\]
\[
\leq c \left( 1 + \| v_0 \|_H^2 + \| u_0 \|_B^2 + \| a \|_{L^q(0, T)} + \int_0^t \| v(s) \|_H^2 \, ds \right).
\]
Applying Gronwall’s lemma, we obtain
\[
\| v(t) \|_H^2 \leq M_1
\]
for all \( t \in [0, T] \), where \( M_1 > 0 \) depends on the problem data. This also immediately yields
\[
\int_0^t \| v(s) \|_{V_A}^p \, ds \leq M_2
\]
as well as
\[
\| (Kv)(t) \|_B^2 \leq M_2
\]
We define the solution operator \( F \) of \( G(f) \) to prove the measurability, consider an arbitrary open subset \( U \) of Lebesgue-measure 0 such that the graph of an element of the same \( \hat{F} \) set-valued function in an element of the same \( \hat{F} \) set-valued function in an element of the same \( \hat{F} \) set-valued function. Since the radial retraction \( r_{M_i} \) in \( H \) is continuous, we have \( r_{M_i}(v_n) \to r_{M_i}(v) \) in \( H \). Then, \( w_n \in \hat{F}(t, v_n) = F(t, r_{M_i}(v_n)) \) and the continuity condition on \( F \) immediately yield \( w \in F(t, r_{M_i}(v)) = \hat{F}(t, v) \).

Due to the estimate on \( F \) in the assumptions (F), we have

\[
|\hat{F}(t, v)| \leq \hat{a}(t) := a(t) + bM_i^{2/q}
\]

for almost all \( t \in (0, T) \) and all \( v \in H \). Now, set

\[
E := \{ f \in L^q(0, T; H) \mid \| f(t) \|_H \leq \hat{a}(t) \text{ a.e.} \}.
\]

We define the solution operator

\[
G : E \to \overline{W}(0, T) := \{ v \in L^p(0, T; V_A) \mid v' \in L^q(0, T; V_A^*) + L^\infty(0, T; V_B^*) \}
\]

with \( G(f) = v \), where \( v \) is the unique solution to the problem

\[
v' + Av + BKv = f, \quad v(0) = v_0,
\]

\[13\]
in the sense of $L^q(0, T; V_A^*)$, which exists due to [15, Theorem 4.2, Corollary 5.2]. Note that

$$v' = f - Av - BKv \in L^q(0, T; H) + L^q(0, T; V_A^*) + L^\infty(0, T; V_B^*)$$

$$\subset L^q(0, T; V_A^*) + L^\infty(0, T; V_B^*)$$

in this case. Note also that $v \in \mathcal{C}_w((0, T); H)$ and $Kv \in \mathcal{C}_w((0, T); V_B)$ by [15, Theorem 4.2]. Now, the aim is to show that

$$v_n \rightharpoonup v \text{ in the sense of } V^*$$

Lemma 8.1. Note also that $v_n$ is a reflexive Banach space and $L^q(v)$ as $n \to \infty$. We obviously have $\hat{v} = v'$. As $v \in L^p(0, T; V_A)$ and $v' \in L^q(0, T; V^*)$, we have $v \in \mathcal{C}([0, T]; V^*)$, see, e.g., Roubíček [38, Lemma 7.1]. Together with $v \in L^\infty(0, T; H)$, this yields $v \in \mathcal{C}_w([0, T]; H)$, see, e.g., Lions and Magenes [24, Chapter 3, Lemma 8.1].

In order to show that $G$ is sequentially weakly continuous, we have to pass to the limit in the equation

$$v'_n + Av_n + BKv_n = f_n$$

(14)

and show that $v$ is a solution to problem (13). First, we want to show $u = Kv$. We know that the operator $\hat{K} : L^2(0, T; H) \to L^2(0, T; H)$ with $\hat{K}w := Kw - u_0$ is well-defined, linear, and bounded, see Lemma 1. Thus it is weakly-weakly continuous and $v_n \rightharpoonup v$ in $L^\infty(0, T; H)$ (and therefore $v_n \rightharpoonup v$ in $L^2(0, T; H)$) implies $Kv_n - Kv = \hat{K}v_n - \hat{K}v \rightharpoonup 0$ in $L^2(0, T; H)$. This yields $u = Kv$. Due to the linearity and boundedness of $B : V_B \to V_B^*$ and thus its weakly*-weakly* continuity, we also have $BKv_n \rightharpoonup Bu = BKv$ in $L^\infty(0, T; V_B^*)$.

Next, let us show $v(0) = v_0$ and $v_n(T) \rightharpoonup v(T)$ in $H$. Due to estimate (8), the sequence $\{v_n(T)\}$ is bounded in $H$, so there exists $v_T \in H$ such that, up to a subsequence, $v_n(T) \rightharpoonup v_T$ in $H$. Now, consider $\phi \in \mathcal{C}^1([0, T]), w \in V$ (recall that $V = V_A \cap V_B$). Since $v_n$ solves (14) in the sense of $L^q(0, T; V_A^*)$ and $v$ solves

$$v' + \tilde{a} + BKv = f$$

(15)
in the sense of $L^q(0, T; V^*_A)$, the integration-by-parts formula in Lemma 3 and the regularity $v \in C_w([0, T]; H)$ yield
\[
(v_n(T), w) \phi(T) - (v_n(0), w) \phi(0) = \int_0^T \langle v'_n(t), w \rangle \phi(t) dt + \int_0^T \langle v_n(t), w \rangle \phi'(t) dt
\]
\[
= \int_0^T \langle f_n - Av_n - BK v_n, w \rangle \phi(t) dt + \int_0^T \langle v_n(t), w \rangle \phi'(t) dt
\]
\[
\rightarrow \int_0^T \langle f - \tilde{a} - BK v, w \rangle \phi(t) dt + \int_0^T \langle v(t), w \rangle \phi'(t) dt
\]
\[
= \int_0^T \langle v', w \rangle \phi(t) dt + \int_0^T \langle v(t), w \rangle \phi'(t) dt
\]
\[
= (v(T), w) \phi(T) - (v(0), w) \phi(0)
\]
as $n \to \infty$. Choosing $\phi(t) = 1 - \frac{t}{T}$, this yields $(v_n(0), w) \to (v(0), w)$ for all $w \in V$. Due to $v_n(0) = v_0$ for all $n \in \mathbb{N}$, we have $v(0) = v_0$. Choosing $\phi(t) = \frac{t}{T}$, we get $(v_n(T), w) \to (v(T), w)$ for all $w \in V$ and therefore also $v_T = v(T)$ in $H$.

Next, let us show $(Kv_n(T)) \to (Kv)(T)$ in $V_B$. Estimate (10) implies the boundedness of the sequence $\{(Kv_n(T))\}$ in $V_B$, therefore there exists $u_T \in V_B$ such that, up to a subsequence, $(Kv_n(T)) \to u_T$ in $V_B$. Consider again $\phi(t) = \frac{t}{T}$, $w \in V$. Due to relation (6), we have
\[
((Kv_n)(T), w) = \int_0^T \langle (Kv_n)'(t), w \rangle \frac{t}{T} dt + \int_0^T \langle (Kv_n)(t), w \rangle \frac{1}{T} dt
\]
\[
= \lambda \int_0^T \langle v_n(t) - ((Kv_n)(t) - u_0), w \rangle \frac{t}{T} dt + \int_0^T \langle (Kv_n)(t), w \rangle \frac{1}{T} dt
\]
\[
\rightarrow \lambda \int_0^T \langle v(t) - ((Kv)(t) - u_0), w \rangle \frac{t}{T} dt + \int_0^T \langle (Kv)(t), w \rangle \frac{1}{T} dt
\]
\[
= \int_0^T \langle (Kv)'(t), w \rangle \frac{t}{T} dt + \int_0^T \langle (Kv)(t), w \rangle \frac{1}{T} dt
\]
\[
= ((Kv)(T), w)
\]
as $n \to \infty$. This immediately yields $u_T = (Kv)(T)$.

In order to show that $v$ is a solution to problem (13), it remains to show $\tilde{a} = Av$. Using the integration-by-parts formula [15, Lemma 4.3], we obtain
\[
\langle Av_n, v_n \rangle = \langle f_n, v_n \rangle - \langle v_n' + BK v_n, v_n \rangle
\]
\[
= \langle f_n, v_n \rangle - \frac{1}{2} \|v_n(T)\|^2_H + \frac{1}{2} \|v_0\|^2_H - \frac{1}{2\lambda} \|v_n(T)\|^2_B + \frac{1}{2\lambda} \|u_0\|^2_B
\]
\[
+ \int_0^T \langle BK v_n(s), u_0 \rangle ds - \|K v_n\|^2_{L^2(0,T;B)}.
\]

Since we have $v_n \rightharpoonup v$ in $\overline{W}(0, T)$ and since the embedding $\overline{W}(0, T) \subset L^p(0, T; H)$ is compact (see, e.g., Roubíček [38, Lemma 7.7]), there exists a subsequence, again denoted

\footnote{Note that $u_0$ cancels out in relation (6), so we consider $X = V_A$.}
by $n$, such that $v_n \to v$ in $L^p(0, T; H)$. This yields $\langle f_n, v_n \rangle \to \langle f, v \rangle$.\footnote{Here, we need the (in comparison to the single-valued case stronger) assumptions that the embedding $V_A \subset H$ is compact and that $f(t) \in H$ in order to identify the limit of the sequence $\{(f_n, v_n)\}$.} We also obviously have
\[
\int_0^T \langle (BKv_n)(s), u_0 \rangle ds \to \int_0^T \langle (BKv)(s), u_0 \rangle ds.
\]
Due to the convergences $v_n(T) \rightharpoonup v(T)$ in $H$, $(Kv_n)(T) \rightharpoonup (Kv)(T)$ in $V_B$ as well as $Kv_n \to Kv$ in $L^\infty(0, T; V_B)$ (and thus $Kv_n \to Kv$ in $L^2(0, T; V_B)$) and the lower semicontinuity of the norm, we obtain
\[
\limsup_{n \to \infty} \langle Av_n, v_n \rangle \leq \langle f, v \rangle - \frac{1}{2} \|v(T)\|^2_H + \frac{1}{2} \|v_0\|^2_H - \frac{1}{2\lambda} \|(Kv)(T)\|^2_B + \frac{1}{2\lambda} \|u_0\|^2_B
\]
and, together with (16),
\[
\langle Aw - \bar{a}, v - w \rangle \leq 0.
\]
Choosing $w = v \pm rz$ for an arbitrary $z \in L^p(0, T; V_A)$ and $r > 0$ and using the hemicontinuity as well as the growth condition of $A: V_A \to V_A^*$, Lebesgue’s theorem on dominated convergence yields for $r \to 0$
\[
\langle Av - \bar{a}, z \rangle = 0
\]
for all $z \in L^p(0, T; V_A)$, which implies $\bar{a} = Av$.

As the last step of this proof, consider the operator $R: E \to P_{fc}(E)$ with $R(f) = F^1(G(f))$, where the set $F^1(G(f))$ is meant with respect to the truncation $\hat{F}$ instead of $F$, i.e.,
\[
F^1(G(f)) = \left\{ f \in L^1(0, T; H) \mid f(t) \in \hat{F}(t, (G(f))(t)) \text{ a.e. in } (0, T) \right\}.
\]
Following the proof of [35, Theorem 3.1], this operator is upper semicontinuous on $E$ equipped with the weak topology. Thus, we can apply the generalisation of the Kakutani fixed-point theorem (see Glicksberg [20] and Fan [17]) to obtain the existence of $f \in E$ such that $f \in R(f) = F^1(G(f))$. This implies that $v = G(f)$ solves (1) with the right-hand side $\hat{F}$. However, due to the a priori estimate (8), we have $\hat{F}(t, v(t)) \leq F(t, v(t))$ for almost all $t \in [0, T]$ which proves the assertion.

\[\square\]
5 Example

We present an example for the problem discussed in this work. Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with boundary of class $C^{1,1}$ or a convex polyhedral and let $p > 6$. Consider

$$\frac{\partial}{\partial t} v(x,t) - \Delta_p v(x,t) + (\Delta^2 K v)(x,t) \in (F(t,v))(x,t), \quad (x,t) \in \Omega \times (0,T),$$

$$v(x,t) = \Delta v(x,t) = 0, \quad (x,t) \in \partial \Omega \times (0,T),$$

$$v(x,0) = v_0(x), \quad x \in \Omega,$$

where $K$ is given by (2) and

$$F(t,v) = \left\{ w \in L^2(\Omega) \mid \|g(v) - w\|_{L^2(\Omega)} \leq a(t) \right\} = B_{a(t)}^{L^2(\Omega)}(g(v))$$

with $g(v) = \|v\|^{1-2/p}_L v$ and a certain function $a \in L^q(0,T), a(t) \geq 0$ a.e.

Here, we have $A$ given by $-\Delta_p$ with $V_A = W^{1,p}_0(\Omega)$ and $B$ given by $\Delta^2$ with $V_B = H^2(\Omega) \cap H^1_0(\Omega)$. Obviously, $V_A$ is not embedded in $V_B$. As we have $d = 3$ and $p > 6$, $V_B$ is also not embedded in $V_A$. However, $V_A \cap V_B = H^2(\Omega) \cap W^{1,p}_0(\Omega)$ is separable and densely embedded in both $V_A$ and $V_B$. With $H = L^2(\Omega)$, we obtain the scale (3) of Banach and Hilbert spaces. As already mentioned, the $p$-Laplacian fulfills the assumptions (A). It is easy to see that $\Delta^2$ fulfills the assumptions (B). Thus, it remains to prove that $F$ fulfills the assumptions (F).

First, we prove the measurability of the mapping $(t,v) \mapsto d(x,F(t,v))$ for arbitrary fixed $x \in L^2(\Omega)$, which is equivalent to the measurability of $F$, see, e.g., Denkowski, Migórski, and Papageorgiou [10, Theorem 4.2.11]. Here, $d$ is the distance function in $L^2(\Omega)$, i.e., $d(x,A) = \inf_{y \in A} \|x - y\|_{L^2(\Omega)}$ for $x \in L^2(\Omega), A \subset L^2(\Omega)$. For $x \not\in F(t,v)$, we have

$$d(x,F(t,v)) = d\left(x,B_{a(t)}^{L^2(\Omega)}(g(v))\right) = d(x,g(v)) - a(t).$$

This is obviously measurable.

To prove the continuity condition, consider $\{v_n\},\{w_n\} \subset L^2(\Omega)$ and $v,w \in L^2(\Omega)$ with $v_n \rightarrow v$, $w_n \rightarrow w$, and $w_n \in F(t,v_n)$. We have to show $w \in F(t,v)$. As $w_n \in F(t,v_n)$, we have

$$\|g(v_n) - w_n\|_{L^2(\Omega)} \leq a(t).$$

The continuity of $g$ and the lower semicontinuity of the norm yield

$$\|g(v) - w\|_{L^2(\Omega)} \leq \liminf_{n \to \infty} \|g(v_n) - w_n\|_{L^2(\Omega)} \leq a(t),$$

i.e., $w \in F(t,v)$.

Finally, we have

$$|F(t,v)| = \sup_{w \in F(t,v)} \|w\|_{L^2(\Omega)}$$

$$\leq \sup_{w \in F(t,v)} \|g(v) - w\|_{L^2(\Omega)} + \|g(v)\|_{L^2(\Omega)}$$

$$\leq a(t) + \|v\|^{2-2/p}_{L^2(\Omega)}.$$
**Open Access** This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit [http://creativecommons.org/licenses/by/4.0/](http://creativecommons.org/licenses/by/4.0/).

**References**

1. Amann, H., Escher, J.: Analysis III. Birkhäuser, Basel (2001)
2. Aubin, J.-P., Cellina, A.: Differential Inclusions. Grundlehren der mathematischen Wissenschaften, vol. 264. Springer, Berlin (1984)
3. Aubin, J.-P., Frankowska, H.: Set-Valued Analysis. Birkhäuser, Boston (1990)
4. Beyn, W.-J., Emmrich, E., Rieger, J.: Semilinear parabolic differential inclusions with one-sided Lipschitz nonlinearities. J. Evol. Equ. **18**, 1319–1339 (2018)
5. Brézis, H.: Analyse Fonctionnelle: Théorie et Applications. Dunod, Paris (1999)
6. Castaing, C., Valadier, M.: Convex Analysis and Measurable Multifunctions. Lecture Notes in Mathematics, vol. 580. Springer, Berlin (1977)
7. Crandall, M.G., Londen, S.-O., Nohel, J.A.: An abstract nonlinear Volterra integrodifferential equation. J. Math. Anal. Appl. **64**, 701–735 (1978)
8. Cushing, J.M.: Integrodifferential Equations and Delay Models in Population Dynamics. Lecture Notes in Biomathematics, vol. 20. Springer, Berlin, Heidelberg (1977)
9. Deimling, K.: Multivalued Differential Equations. de Gruyter, Berlin (1992)
10. Denkowski, Z., Migórski, S., Papageorgiou, N.S.: An Introduction to Nonlinear Analysis: Theory. Kluwer Academic Publishers, Boston (2003)
11. Desch, W., Grimmer, R., Schappacher, W.: Wellposedness and wave propagation for a class of integrodifferential equations in Banach space. J. Differ. Equ. **74**, 391–411 (1988)
12. Diestel, J., Uhl, J.J.: Vector Measures. American Mathematical Society, Providence (1977)
13. Edwards, D.A.: Non-Fickian diffusion in thin polymer films. J. Polym. Sci. Part B: Polym. Phys. **34**, 981–997 (1996)
14. Edwards, D.A., Cohen, D.S.: A mathematical model for a dissolving polymer. AIChE J. **41**, 2345–2355 (1995)
15. Eikmeier, A., Emmrich, E., Kreusler, H.-C.: Nonlinear evolution equations with exponentially decaying memory: Existence via time discretisation, uniqueness, and stability. Comput. Methods Appl. Math. **20**, 89–108 (2020)
16. Emmrich, E., Thalhammer, M.: Doubly nonlinear evolution equations of second order: Existence and fully discrete approximation. J. Differ. Equ. **251**, 82–118 (2011)
17. Fan, K.: Fixed-point and minimax theorems in locally convex topological linear spaces. Proc. Nat. Acad. Sci. U. S. A. **38**, 121–126 (1952)
18. Fedotov, S., Iomin, A.: Probabilistic approach to a proliferation and migration dichotomy in tumor cell invasion. Phys. Rev. E **77**, 031911 (2008)
19. Gajewski, H., Gröger, K., Zacharias, K.: Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen. Akademie, Berlin (1974)
20. Glicksberg, I.L.: A further generalization of the Kakutani fixed theorem, with application to Nash equilibrium points. Proc. Amer. Math. Soc. **3**, 170–174 (1952)
21. Gori, C., Obukhovskii, V., Rubbioni, P., Zvyagin, V.: Optimization of the motion of a visco-elastic fluid via multivalued topological degree method. Dyn. Syst. Appl. **16**, 89–104 (2007)
22. Kalita, P., Migórski, S., Sofonea, M.: A class of subdifferential inclusions for elastic unilateral contact problems. Set-Valued Var. Anal. **24**, 355–379 (2016)
23. Kalita, P., Szafrieicz, P., Shillor, M.: A frictional contact problem with wear diffusion. Z. Angew. Math. Phys. **70**, 96 (2019)
24. Lions, J.-L., Magenes, E.: Non-Homogeneous Boundary Value Problems and Applications, vol. I. Springer, Berlin (1972)
25. MacCamy, R.C.: An integro-differential equation with application in heat flow. Q. Appl. Math. **35**, 1–19 (1977)
26. MacCamy, R.C.: A model for one-dimensional nonlinear viscoelasticity. Q. Appl. Math. 35, 21–33 (1977)
27. Mehrabian, A., Abousleiman, Y.: General solutions to poroviscoelastic model of hydrocephalic human brain tissue. J. Theor. Biol. 291, 105–118 (2011)
28. Migórski, S.: Dynamic hemivariational inequality modeling viscoelastic contact problem with normal damped response and friction. Appl. Anal. 84, 669–699 (2005)
29. Migórski, S., Ochal, A., Sofonea, M.: Nonlinear Inclusions and Hemivariational Inequalities: Models and Analysis of Contact Problems. Advances in Mechanics and Mathematics, vol. 26. Springer, New York (2013)
30. Miller, R.K.: An integrodifferential equation for rigid heat conductors with memory. J. Math. Anal. Appl. 66, 313–332 (1978)
31. Obukhovskii, V., Zecca, P., Zvyagin, V.G.: Optimal feedback control in the problem of the motion of a viscoelastic fluid. Topol. Methods Nonlinear Anal. 23, 323–337 (2004)
32. O’Regan, D.: Multivalued differential equations in Banach spaces. Comput. Math. Appl. 38, 109–116 (1999)
33. Papageorgiou, N.S.: Existence and convergence results for integral inclusions in Banach spaces. J. Integral Equ. Appl. 1, 265–285 (1988)
34. Papageorgiou, N.S.: Volterra integral inclusions in Banach spaces. J. Integral Equ. Appl. 1, 65–81 (1988)
35. Papageorgiou, N.S.: Nonlinear Volterra integrodifferential evolution inclusions and optimal control. Kodai Math. J. 14, 254–280 (1991)
36. Papageorgiou, N.S.: Volterra integrodifferential inclusions in reflexive Banach spaces. Funkcial. Ekvac. 34, 257–277 (1991)
37. Papageorgiou, N.S., Winkert, P.: Applied Nonlinear Functional Analysis. de Gruyter, Berlin (2018)
38. Roubíček, T.: Nonlinear Partial Differential Equations with Applications. Birkhäuser, Basel (2005)
39. Shaw, S., Whiteman, J.R.: Some partial differential Volterra equation problems arising in viscoelasticity. In: Agarwal, R.P., Neuman, F., Vosmanský, J. (eds.) Proceedings of Equadiff 9, pp. 183–s200. Masaryk University, Brno (1998)
40. Zeidler, E.: Nonlinear Functional Analysis and its Applications, II/B: Nonlinear Monotone Operators. Springer, New York (1990)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.