REMARKS ON WELL-POSEDNESS OF THE GENERALIZED SURFACE QUASI-GEOSTROPHIC EQUATION

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Abstract. In this paper, we are concerned with the Cauchy problem of the generalized surface quasi-geostrophic (SQG) equation in which the velocity field is expressed as \( u = K \ast \omega \), where \( \omega = \omega(x,t) \) is an unknown function and \( K(x) = \frac{\delta^+}{|x|^{2+2\alpha}}, 0 \leq \alpha \leq \frac{1}{2} \). When \( \alpha = 0 \), it is the two-dimensional Euler equations. When \( \alpha = \frac{1}{2} \), it corresponds to the inviscid SQG. We will prove that if the existence interval of the smooth solution to the generalized SQG for some \( 0 < \alpha_0 \leq \frac{1}{2} \) is \([0, T]\), then under the same initial data, the existence interval of the generalized SQG with \( \alpha \) which is close to \( \alpha_0 \) will keep on \([0, T]\). As a byproduct, our result implies that the construction of the possible singularity of the smooth solution of the Cauchy problem to the generalized SQG with \( \alpha > 0 \) will be subtle, in comparison with the singularity presented in [13]. To prove our main results, the difference between the two solutions and meanwhile the approximation of the singular integrals will be dealt with. Some new uniform estimates with respect to \( \alpha \) on the singular integrals and commutator estimates will be shown in this paper.

1. Introduction and Main Results

We consider the Cauchy problem of the generalized SQG (Surface Quasi-geostrophic) equation in the plane as follows

\[
\begin{align*}
\omega_t + u \cdot \nabla \omega &= 0, \quad (x,t) \in \mathbb{R}^2 \times \mathbb{R}^+, \\
u &= \nabla^\perp(-\Delta)^{-1+\alpha} \omega, \\
\omega(x,0) &= \omega_0
\end{align*}
\]  
(SQG)

with \( 0 \leq \alpha \leq \frac{1}{2} \). Here, according to the second equation in (SQG), the unknown scalar function \( \omega = \omega(x,t) \) and vector field \( u = (u_1(x,t), u_2(x,t)) \) can be expressed as the singular integral

\[
u(x) = \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^{2+2\alpha}} \omega(y) \, dy.
\]  
(1.1)

Throughout our paper, we omit some constants before the singular integral (1.1) for conciseness. Meanwhile, the expression (1.1) implies that \( \nu \) is divergence-free, that is, \( \nabla \cdot \nu = \partial_{x_1} u_1 + \partial_{x_2} u_2 = 0 \).

When \( \alpha = 0 \), it is well-known that (SQG) corresponds to the two-dimensional incompressible Euler equations. In this case, the unknown functions \( \omega = \omega(x,t) \) and \( u = u(x,t) \) are the vorticity and the velocity field, respectively. When \( \alpha = \frac{1}{2} \), (SQG) corresponds to the surface quasi-geostrophic (SQG) equation which describes a famous approximation model of the nonhomogeneous fluid flow in a rapidly rotating 3D half-space (see [21]). In this case, the unknown functions \( \omega = \omega(x,t) \) and \( u = u(x,t) \) represent potential temperature and
velocity field, respectively. When \(0 < \alpha < \frac{1}{2}\), \((\text{SQG})\) is called the generalized (or modified) SQG equation.

The classical SQG and the generalized SQG equations have been widely studied in the past years and much more progress has been made. In [17, 18], it is proved that the generalized SQG in half space \(\mathbb{R}^2 = \{x = (x_1, x_2)|x_2 > 0\}\) has a unique local solution for vortex-patch initial data and will appear singularity in finite time for some such kind of initial data when \(0 < \alpha < \frac{1}{2}\). This strongly implies that the SQG equation will appear finite-time singularity (even for smooth initial data) since the velocity has less regularity when \(\alpha = \frac{1}{2}\). In fact, the singularity or formation of strong fronts has been suggested in [6] although the rigorous derivations have not been reached so far. We note that the global well-posedness or blow-up of the SQG equation is an important issue. As pointed out in [6], the singularity of the SQG equation will be similar to that of the three-dimensional Euler equations. Concerning the dissipative SQG equation, which enjoys a fractional dissipation term \(-(-\Delta)^\beta\) on the right hand side of the second equation of \((\text{SQG})\), the global well-posedness in the critical case \(\beta = \frac{1}{2}\) was proved independently by Caffarelli and Vasseur [2] and by Kiselev, Nazarov and Volberg [16] (see [7, 15] for different approaches). The proof of global regularity for the subcritical case \(\beta > \frac{1}{2}\) is standard (see e.g. [8]), while in the supercritical case \(\beta < \frac{1}{2}\) the global regularity of small solutions is obtained (see e.g. [3, 12, 13, 23]) and the slightly supercritical case is studied recently in [11].

In this paper, our target is to show that, for any \(T > 0\), if \(\varphi^{\alpha_0}, u^{\alpha_0}\) defined on \([0, T]\) is the unique smooth solution of \((\text{SQG})\) for some \(0 < \alpha_0 \leq \frac{1}{2}\), then there exists \(\delta > 0\) such that when \(0 < \alpha < \frac{1}{2}\) and \(0 < \alpha_0 - \alpha \leq \delta\), the problem \((\text{SQG})\) with same initial data has also a unique smooth solution \(\varphi^\alpha, u^\alpha\) defined on \([0, T]\), where we denote the solution of problem \((\text{SQG})\) corresponding to \(0 \leq \alpha \leq \frac{1}{2}\) by \(\varphi^\alpha, u^\alpha\) satisfying \(u^\alpha = \nabla (-\Delta)^{-1+\alpha}\varphi^\alpha\). This is motivated by [4] in which it is shown that if the Cauchy problem to the three-dimensional incompressible Euler equations have a unique smooth solution on \([0, T]\), then the corresponding three-dimensional incompressible Navier-Stokes equations with the same initial data will also have a unique smooth solution defined on \([0, T]\) when the viscosity is suitably small. Furthermore, our result implies that the construction of the possible singularity of the smooth solution of the Cauchy problem to the generalized SQG with \(\alpha > 0\) will be subtle (see Corollary 1.4), in comparison with the singularity result presented in [18]. To prove our main results, we consider the behavior of the difference between \(u^\alpha\) and \(u^{\alpha_0}\). Let us denote

\[ \varpi = \varphi^\alpha - \varphi^{\alpha_0} \quad \text{and} \quad \overline{u} = u^\alpha - u^{\alpha_0}, \]

we easily find that the couple \((\varpi, \overline{u})\) satisfies

\[ \varpi_t + (u^{\alpha_0} \cdot \nabla)\varpi + (\overline{u} \cdot \nabla)\varpi + (\overline{u} \cdot \nabla)\varphi^{\alpha_0} = 0. \]

With this equation, we can establish the following \(H^s\)-estimate of \(\varpi(t)\):

\[ \frac{1}{2} \frac{d}{dt} \|\varpi(t)\|_{H^s}^2 = - \int_{\mathbb{R}^2} J^s(u^{\alpha_0} \cdot \nabla \varpi) J^s \varpi dx - \int_{\mathbb{R}^2} J^s(\overline{u} \cdot \nabla \varpi) J^s \varpi dx \]

\[ - \int_{\mathbb{R}^2} J^s(\overline{u} \cdot \nabla \varphi^{\alpha_0}) J^s \varpi dx. \]

The difficulty for us is how to use the information of \(\|\varpi(t)\|_{H^s}\) to control \(\overline{u}\) in the nonlinear term. This requires us to consider the behavior in different scale in physical space or in
different frequency regime in frequency space. Thus, we decompose \( \tilde{u} \) in two parts

\[
\tilde{u} = u^\alpha - u^\alpha_0 = \nabla^\perp (-\Delta)^{-1+\alpha} \tilde{w} + (\nabla^\perp (-\Delta)^{-1+\alpha} - \nabla^\perp (-\Delta)^{-1+\alpha_0}) \omega^\alpha_0
\]

(1.2)

where \( \omega = \omega^\alpha - \omega^\alpha_0 \). In the above equalities (1.2), we see that the part \( u_I \) is related to the difference between the solutions, while the part \( u_{II} \) corresponds to the difference between the singular integrals. This observation together with the scale analysis enables us to establish some technique propositions, see Proposition 3.1, Proposition 4.1 and Proposition 4.2 which will play key roles in our proof of Theorem 1.2 and Theorem 1.3 respectively. More precisely, in view of Riesz potential (see (2.3)), the term \( \tilde{w} \) will play key roles in our proof of Theorem 1.2 and Theorem 1.3 respectively.

Let

\[
\omega^\alpha_0 = \nabla^\perp (-\Delta)^{-1+\alpha_0} \omega^\alpha_0 \text{ and } \omega_0 \in H^{s+1}, s > 2.
\]

Then, there exists \( \delta > 0 \) depending on \( T \) and \( J_0^T \| \omega^\alpha_0 \|_{H^{s+1}} dt \) such that if \( 0 < \alpha < \frac{1}{2} \) and \( |\alpha_0 - \alpha| \leq \delta \), the solution \( \omega^\alpha \) to (SQG) with

\[
u^a = \nabla^\perp (-\Delta)^{-1+\alpha} \omega^\alpha \text{ and the same initial data is smooth on } [0, T].
\]

Moreover, it holds that

\[
\| \omega^\alpha(t) - \omega^\alpha_0(t) \|_{H^s} \leq C \left( |\alpha_0 - \alpha|^{1-2\alpha_0} + |\alpha_0 - \alpha| \log |\alpha_0 - \alpha| \right),
\]

where \( C > 0 \) is a constant depending on \( T \) and \( J_0^T \| \omega^\alpha_0 \|_{H^{s+1}} dt \).

**Theorem 1.2.** Let \( 0 < \alpha < \alpha_0 = \frac{1}{2} \). Let \( \omega^\alpha_0 \) be a solution of (SQG) for \( 0 \leq t \leq T \) with \( u^\alpha_0 = \nabla^\perp (-\Delta)^{-1+\alpha_0} \omega^\alpha_0 \) and \( \omega_0 \in H^{s+1} \cap L^1 \), \( s > 2 \). Then, there exists \( \delta > 0 \) depending on \( T \) and \( J_0^T \| \omega^\alpha_0 \|_{H^{s+1}} dt \) such that if \( 0 < \alpha_0 - \alpha < \delta \), the solution \( \omega^\alpha \) to (SQG) with \( u^\alpha = \nabla^\perp (-\Delta)^{-1+\alpha} \omega^\alpha \) and the same initial data is smooth on \( [0, T] \). Moreover, it holds that

\[
\| \omega^\alpha(t) - \omega^\alpha_0(t) \|_{H^s} \leq C \left( \left( \frac{1}{2} - \alpha \right) + \left( \frac{1}{2} - \alpha \right) \log^2 \left( \frac{1}{2} - \alpha \right) \right),
\]

where \( C > 0 \) is a constant depending on \( T \) and \( J_0^T \| \omega^\alpha_0 \|_{H^{s+1}} dt \).

**Theorem 1.3.** Let \( 0 < \alpha < \alpha_0 = \frac{1}{2} \). Let \( \omega^\alpha_0 \) be a solution of (SQG) for \( 0 \leq t \leq T \) with \( u^\alpha_0 = \nabla^\perp (-\Delta)^{-1+\alpha_0} \omega^\alpha_0 \) and \( \omega_0 \in H^{s+2} \), \( s > 2 \). Then, there exists \( \delta > 0 \) depending on \( T \) and \( J_0^T \| \omega^\alpha_0 \|_{H^{s+2}} dt \) such that if \( 0 < \alpha_0 - \alpha < \delta \), the solution \( \omega^\alpha \) to (SQG) with \( u^\alpha = \nabla^\perp (-\Delta)^{-1+\alpha} \omega^\alpha \) and the same initial data is smooth on \( [0, T] \). Moreover, it holds that

\[
\| \omega^\alpha(t) - \omega^\alpha_0(t) \|_{H^s} \leq C \left( \left( \frac{1}{2} - \alpha \right) + \left( \frac{1}{2} - \alpha \right) \log^2 \left( \frac{1}{2} - \alpha \right) \right),
\]

where \( C > 0 \) is a constant depending on \( T \) and \( J_0^T \| \omega^\alpha_0 \|_{H^{s+2}} dt \).

**Remark 1.1.** In Theorem 1.1, we consider the case \( 0 < \alpha_0 < \frac{1}{2} \). In Theorem 1.2 and Theorem 1.3, we deal with the case \( \alpha_0 = \frac{1}{2} \). It is noted that in the proof of Theorem 1.1, the Hardy-Littlewood-Sobolev inequality (see Lemma 2.3) will be used. One point is that the expression \( \tilde{u}_I \) in (1.2) can be reduced to \( I_{1-2\alpha} \tilde{w} \), where \( I_{1-2\alpha} \) is a Riesz operator (see (2.3)) which is bounded from \( L^p \) to \( L^q \) with \( \frac{1}{q} = \frac{1}{p} - \frac{1-2\alpha}{n} \) satisfying \( 0 < 1 - 2\alpha < n \) and \( 1 < p < q < \infty \). However, it does not hold bounded uniformly with respect to \( \alpha \) when \( \alpha \)
tends to $\frac{1}{2}$. Hence, in the proof of Theorem 1.2 and Theorem 1.3 we can not use Hardy-Littlewood-Sobolev inequality directly. To prove Theorem 1.2 we will establish some new and uniform estimates with respect to $\alpha$ on the singular integral $\left(\ref{1.1}\right)$ (see Proposition 2.1). To prove Theorem 1.3, we will obtain some elegant estimates concerning $u_I$ with the help of Besov spaces of which definition is given in Appendix A (see Propositions 4.1 and 4.2).

Remark 1.2. In Theorem 1.2, we require that the initial data $\omega_0 \in H^{s+1} \cap L^1$ with $s > 2$. Thanks to the incompressible condition $\nabla \cdot u = 0$, the solution will stay in $L^1$. In Theorem 1.3, we drop the restriction on the initial data $\omega_0 \in L^1$, but more regularity of the initial data $\omega_0 \in H^{s+2}$ with $s > 2$ will be needed.

Remark 1.3. As mentioned above, (SQG) becomes the two-dimensional incompressible Euler equations when $\omega_0 = 0$, of which the global existence of smooth solutions has been known (see [19] and references therein). In comparison with the singularity for the patch solution with $0 < \alpha < \frac{1}{2}$ in half space obtained in [18], whether the patch or smooth solution of the Cauchy problem to (SQG) when $\alpha > 0$ appears singularity in finite time remains open. Theorem (1.11) implies that the possible blow-up time of the smooth solution to the Cauchy problem of (SQG) with $\alpha > 0$ can not be uniformly bounded when $\alpha \to 0^+$. More precisely, as a corollary of Theorem 1.1 we have

Corollary 1.4. Let $T^*_\alpha > 0$ the maximal existence time (may be $+\infty$) of the solution $\omega^* \in C([0, T]; H^{s+1}(s > 2))$ to (SQG) with $\alpha > 0$. Then $\lim_{\alpha \to 0^+} T^*_\alpha = +\infty$.

The rest of the paper is organized as follows. In Section 2 we will present some basic facts which will be needed later. In Section 3 we will investigate a singular integral which can be viewed as an approximation of the Riesz transform. In Section 4 we will obtain nonlinear terms and commutator estimates related to $u_I = \nabla_\perp (-\Delta)^{-\alpha/2} \omega$ in (1.2). The proof of the main results will be given in Section 5. In the end of the paper, Appendix A on the Littlewood-Paley decomposition, Besov spaces will be given.

2. Preliminaries

In this section, we present some basic analysis facts. First of all, we introduce

$$\Lambda^s = (-\Delta)^{\frac{s}{2}} \quad \text{and} \quad J^s = (I - \Delta)^{\frac{s}{2}},$$

where

$$\widehat{\Lambda^s f}(\xi) = |\xi|^s \hat{f}(\xi) \quad \text{and} \quad \widehat{J^s f}(\xi) = (1 + |\xi|^2)^{\frac{s}{2}} \hat{f}(\xi), \ s \in \mathbb{R}.$$

Definition 2.1 ([22]). Let $s \in \mathbb{R}$ and $1 \leq p \leq \infty$. We write

$$\|f\|_{W^{s,p}(\mathbb{R}^n)} := \|J^s f\|_{L^p(\mathbb{R}^n)}, \quad \|f\|_{\tilde{W}^{s,p}(\mathbb{R}^n)} := \|\Lambda^s f\|_{L^p(\mathbb{R}^n)}.$$

The nonhomogeneous Sobolev space $W^{s,p}(\mathbb{R}^n)$ is defined as

$$W^{s,p}(\mathbb{R}^n) = \{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{W^{s,p}(\mathbb{R}^n)} < \infty \}.$$

The homogeneous Sobolev space $\tilde{W}^{s,p}(\mathbb{R}^n)$ is defined as

$$\tilde{W}^{s,p}(\mathbb{R}^n) = \{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{\tilde{W}^{s,p}(\mathbb{R}^n)} < \infty \}.$$

Here $\mathcal{S}'$ is the Schwarz distributional function space.

With this definition in hand, we give a commutator estimate and product estimate (see, e.g., Kenig, Ponce and Vega [14]).
Lemma 2.1. Let \( s > 0 \) and \( 1 < p < \infty \). Then
\[
\|J^s(fg)\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{W^{s,p_1}(\mathbb{R}^n)} \|g\|_{L^{p_1}} + \|g\|_{W^{s,p_3}(\mathbb{R}^n)} \|f\|_{L^{p_4}(\mathbb{R}^n)},
\]
and
\[
\|J^s(fg) - f J^s g\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{W^{s,p_1}(\mathbb{R}^n)} \|g\|_{L^{p_2}} + \|g\|_{W^{s-1,p_3}(\mathbb{R}^n)} \|\nabla f\|_{L^{p_4}(\mathbb{R}^n)},
\]
where \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_1} + \frac{1}{p_4} \) with \( p_2, p_4 \in [1, \infty] \) and \( p_1, p_3 \in (1, \infty) \), and \( C' \) 's are constants depending on \( s, p_1, p_2, p_3 \) and \( p_4 \). In addition, these inequalities remain valid when \( J^s \) is replaced by \( \Lambda^s \).

We continue with the Sobolev embedding theorem [20].

Lemma 2.2. For \( p_2 \neq \infty \), \( W^{s_1,p_1}(\mathbb{R}^n) \hookrightarrow W^{s_2,p_2}(\mathbb{R}^n) \) if and only if
\[
s_1 - \frac{n}{p_1} \geq s_2 - \frac{n}{p_2}, \quad \frac{1}{p_1} \geq \frac{1}{p_2}.
\]

The following lemma is the so called Hardy-Littlewood-Sobolev inequality of fractional integration [22]. We begin with definition of the Riesz potential \( I_\alpha \).

Definition 2.2. Let \( 0 < \alpha < n \). The Riesz potential \( I_\alpha f \) is defined by
\[
I_\alpha f(x) = \frac{1}{\gamma(\alpha)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \, dy,
\]
with
\[
\frac{1}{\gamma(\alpha)} = \pi^{\frac{n}{2}} 2^{\alpha} \Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{n-\alpha}{2}\right),
\]
where \( \Gamma \) is Gamma function.

Lemma 2.3. Let \( 0 < \alpha < n \), \( 1 < p < q < \infty \), \( \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n} \). Then, there exists a constant \( A_{p,q} \) depending on \( p, q \) such that
\[
\|I_\alpha f\|_{L^q(\mathbb{R}^n)} \leq A_{p,q} \|f\|_{L^p(\mathbb{R}^n)}.
\]

Remark 2.1. It is noted that the constant \( A_{p,q} \) is unbounded as \( \alpha \to 0 \) here.

The following is an elementary result from [4] in which the case \( m = 1 \) is proved.

Proposition 2.4. Let \( T > 0 \), \( G > 0 \) and \( m > 0 \) be given constants and let \( F(t) \) be a nonnegative continuous function on \([0,T)\). Let \( \nu_0 \) be defined by
\[
\nu_0 = \frac{1}{4m(2mTG)^{\frac{1}{m}}} \int_0^T F(t) \, dt.
\]
Then, for all \( 0 < \nu \leq \nu_0 \), all nonnegative solution \( y(t) \) of the system
\[
\begin{cases}
\frac{dy(t)}{dt} \leq \nu F(t) + G y(t)^{1+m} \\
y(0) = 0
\end{cases}
\]
is uniformly bounded on \([0,T)\) and
\[
y(t) \leq \min\left\{ \frac{4^\frac{1}{m} - 1}{(2mTG)^{\frac{1}{m}}}, 4m(4^\frac{1}{m} - 1) \nu \int_0^T F(t) \, dt \right\}.
\]

Proof. Let us define
\[
\sigma = \min\left\{ \frac{1}{(2mT)^{1+m} G^{\frac{1}{m}}}, 4m \nu \int_0^T F(t) \, dt \right\}.
\]
Dividing the first equation of (2.5) by \((1 + (\frac{G}{\sigma})^{1+m} y)^{1+m}\) yields
\[
\frac{1}{m} \left( \frac{\sigma}{G} \right)^{\frac{1}{m+1}} \frac{d}{dt} \left( \frac{1}{1 + (\frac{G}{\sigma})^{1+m} y} \right)^{m} \geq -\nu F(t) - \sigma. \tag{2.7}
\]

By integrating from 0 to \(t\), we obtain
\[
\frac{1}{m} \left( \frac{\sigma}{G} \right)^{\frac{1}{m+1}} \left( \frac{1}{1 + (\frac{G}{\sigma})^{1+m} y} \right)^{m} \geq \frac{1}{m} \left( \frac{\sigma}{G} \right)^{\frac{1}{m+1}} - \nu \int_{0}^{t} F(t) \, dt - \sigma T. \tag{2.8}
\]
The choice \(\sigma \leq \frac{1}{(2mT)^{1+\frac{1}{m}G}}\) implies \(\sigma T \leq \frac{1}{2m} \left( \frac{\sigma}{G} \right)^{\frac{1}{m+1}}\).

For \(\nu \leq \nu_0\), we have
\[
\nu \int_{0}^{T} F(t) \, dt \leq \frac{1}{4m} \left( \frac{\sigma}{G} \right)^{\frac{1}{m+1}}.
\]
Indeed, if \(\sigma = \left(4m\nu \int_{0}^{T} F(t) \, dt\right)^{1+m} G\), the last inequality is indeed an equality and if \(\sigma = \frac{1}{(2mT)^{1+\frac{1}{m}G}}\), it follows from
\[
\nu \int_{0}^{T} F(t) \, dt \leq \nu_0 \int_{0}^{T} F(t) \, dt = \frac{1}{4m(2mT)^{\frac{1}{m}}} = \frac{1}{4m} \left( \frac{\sigma}{G} \right)^{\frac{1}{m+1}}.
\]
Thus, we get by (2.8) that
\[
\frac{1}{m} \left( \frac{\sigma}{G} \right)^{\frac{1}{m+1}} \left( \frac{1}{1 + (\frac{G}{\sigma})^{1+m} y} \right)^{m} \geq \frac{1}{4}
\]
which implies (2.6). \(\square\)

### 3. Estimates on a Singular Integral

In this section, we present some new results on a singular integral which will be needed in the proof of Theorem 1.2. We denote
\[
Tf(x) = K * f(x) = \int_{\mathbb{R}^n} K(x - y)f(y) \, dy \quad \text{with} \quad K(x) = \frac{x}{|x|^{n+1-\beta}}, \quad 0 < \beta < n, \tag{3.1}
\]
where \(x \in \mathbb{R}^n\).

Let
\[
\chi\chi(s) = \chi(\lambda s). \tag{3.2}
\]
where \(\chi(s) \in C_0^\infty(\mathbb{R})\) is the usual smooth cutting-off function which is defined as
\[
\chi(s) = \begin{cases} 
1, & |s| \leq 1, \\
0, & |s| \geq 2,
\end{cases}
\]
satisfying \(|\chi'(s)| \leq 2\).

Setting
\[
T_1f(x) := K_1 * f(x) \quad \text{with} \quad K_1(x) = K(x)\chi_\beta(|x|), \tag{3.3}
\]
\[
T_2f(x) := K_2 * f(x) \quad \text{with} \quad K_2(x) = K(x)(1 - \chi_\beta(|x|)). \tag{3.4}
\]
Then we have the following proposition which holds for general \(n\)-dimensional case.
Proposition 3.1. There exists a constant $C = C(n, s)$ independent of $\beta$ such that

$$\|T_1 f\|_{H^s(\mathbb{R}^n)} \leq C \|f\|_{H^s(\mathbb{R}^n)}, \quad s > 0, \; 0 < \beta < n; \quad (3.5)$$

$$\|T_2 f\|_{L^2(\mathbb{R}^n)} \leq C \frac{\beta^{\frac{2}{n - 2\beta}}}{\sqrt{n - 2\beta}} \|f\|_{L^1(\mathbb{R}^n)}, \quad s > 0, \; 0 < \beta < \frac{n}{2}; \quad (3.6)$$

$$\|T_2 f\|_{H^s(\mathbb{R}^n)} \leq C \left( \frac{\beta}{1 - \beta} \right) \|f\|_{H^{s-1}(\mathbb{R}^n)}, \quad s \geq 1, \; 0 < \beta < 1. \quad (3.7)$$

Remark 3.1. When $n = 2$, the result of Proposition 3.1 holds true if $K(x)$ in (3.1) is replaced by $K(x) = \frac{x^+}{|x|^{1-\beta}}$.

Remark 3.2. It is emphasized that the constants $C$ is independent of $\beta$, and what is more, $\frac{\beta^{\frac{2}{n - 2\beta}}}{\sqrt{n - 2\beta}}$ in (3.6) is sufficiently small, $\frac{\beta}{1 - \beta} + \frac{2^{\beta - 1}}{\beta} \beta^{-\beta}$ is uniformly bounded in (3.7) when $\beta$ tends to zero.

Remark 3.3. When $\beta = 0$, it follows from (3.1) that $T f = R f$ (in the sense that the integral takes principle values), where $R$ is a Riesz transformation which is a strong $(p, p)$ type operator with $1 < p < \infty$, that is,

$$\|R f\|_{L^p} \leq C \|f\|_{L^p}, \quad 1 < p < \infty \quad (3.8)$$

for some constant $C > 0$. By Proposition 3.1 it holds

$$\|T f\|_{L^2} \leq C \|f\|_{L^2} + \beta \|f\|_{L^1}, \quad (3.9)$$

where $C > 0$ is an absolute constant. This means that the estimate (3.9) recovers the corresponding one in (3.8) with $p = 2$.

Proof of Proposition 3.1. We firstly prove (3.6) and (3.7). Note that

$$T_2 f(x) = \int_{\mathbb{R}^n} \frac{x - y}{|x - y|^{n + 1 - \beta}} (1 - \chi_{\beta}(|x - y|)) f(y) \, dy.$$ 

For $0 < \beta < \frac{n}{2}$, we have

$$\|T_2 f\|_{L^2(\mathbb{R}^n)} \leq \left\| \int_{|x-y| \geq \frac{1}{2}} \frac{1}{|x-y|^{n-\beta}} |f(y)| \, dy \right\|_{L^2(\mathbb{R}^n)}$$

$$\leq \|f\|_{L^1(\mathbb{R}^n)} \left( \int_{|x-y| \geq \frac{1}{2}} \frac{1}{|x-y|^{2(n-\beta)}} \, dy \right)^{\frac{1}{2}}$$

$$\leq \frac{C}{\sqrt{n - 2\beta}} \frac{\beta^{\frac{2}{n - 2\beta}}}{\sqrt{n - 2\beta}} \|f\|_{L^1(\mathbb{R}^n)}.$$ 

Since

$$\lim_{\beta \to 0^+} \beta^{-\beta} = 1,$$ 

there exists an absolutely constant $C(n) > 0$ such that, for any $0 \leq \beta < \frac{n}{2},$

$$\|T_2 f\|_{L^2(\mathbb{R}^n)} \leq C \frac{\beta^{\frac{2}{n - 2\beta}}}{\sqrt{n - 2\beta}} \|f\|_{L^1(\mathbb{R}^n)}.$$ 

This means (3.6).
To prove (3.7), we note that for \( s \geq 1 \) and \( i = 1, 2, \ldots, n \),
\[
\partial_i \Lambda^{s-1} T_2 f(x) = \int_{\mathbb{R}^n} \partial_i \left( \frac{x - y}{|x - y|^{n+1-\beta}} (1 - \chi_\beta(|x - y|)) \Lambda_y^{s-1} f(y) \right) dy
\]
\[
= \int_{|x-y| \geq \frac{1}{\beta}} \partial_i \left( \frac{x - y}{|x - y|^{n+1-\beta}} (1 - \chi_\beta(|x - y|)) \Lambda_y^{s-1} f(y) \right) dy
\]
\[
+ \int_{\frac{1}{\beta} \leq |x-y| \leq \frac{2}{\beta}} \frac{x - y}{|x - y|^{n+1-\beta}} \partial_i \chi_\beta(|x - y|) \Lambda_y^{s-1} f(y) dy
\]
\[
:= J_1 + J_2,
\]
where \( \partial_i = \partial_{x_i}, i = 1, 2, \ldots, n \).

Then, for \( 0 < \beta < 1 \), we obtain
\[
\| J_1 \|_{L^2(\mathbb{R}^n)} \leq C \left\| \int_{|x-y| \geq \frac{1}{\beta}} \frac{1}{|x - y|^{n+1-\beta}} |\Lambda_y^{s-1} f(y)| |\Lambda_y^{s-1} f(y)| dx \right\|_{L^2(\mathbb{R}^n)}
\]
\[
\leq C \| \Lambda_y^{s-1} f \|_{L^2(\mathbb{R}^n)} \int_{|x-y| \geq \frac{1}{\beta}} \frac{1}{|x - y|^{n+1-\beta}} dy
\]
\[
\leq \frac{C}{1 - \beta} \| \Lambda_y^{s-1} f \|_{L^2(\mathbb{R}^n)}^2.
\]
The term \( J_2 \) can be bounded as
\[
\| J_2 \|_{L^2(\mathbb{R}^n)} \leq C \left\| \int_{\frac{1}{\beta} \leq |x-y| \leq \frac{2}{\beta}} \frac{1}{|x - y|^{n-\beta}} |\Lambda_y^{s-1} f(y)| dy \right\|_{L^2(\mathbb{R}^n)}
\]
\[
\leq C \| \Lambda_y^{s-1} f \|_{L^2(\mathbb{R}^n)} \int_{\frac{1}{\beta} \leq |x-y| \leq \frac{2}{\beta}} \frac{1}{|x - y|^{n-\beta}} dy
\]
\[
\leq \frac{C}{1 - \beta} \| \Lambda_y^{s-1} f \|_{L^2(\mathbb{R}^n)}^2.
\]
Substituting (3.11) and (3.12) into (3.10) and using the fact that
\[
\| \Lambda_y^{s-1} f \|_{L^2(\mathbb{R}^n)} \leq \| \nabla \Lambda_y^{s-1} T_2 f \|_{L^2(\mathbb{R}^n)},
\]
we finish the proof of (3.7).

Now we turn to prove (3.5). To do this, it suffice to show that there exists an absolute constant \( C > 0 \) independent \( \beta \) such that
\[
\| \hat{K}_1(y) \|_{L^\infty(\mathbb{R}^n)} \leq C, \ 0 < \beta < n.
\]
In fact, since \( \int_{\mathbb{S}^1} K_1(x) ds(x) = 0 \) (here \( \mathbb{S}^1 \) is the unit sphere surface in \( \mathbb{R}^n \) and \( K_1(x) \) is supported on \( |x| \leq \frac{2}{\beta} \), we have
\[
\hat{K}_1(y) = \int_{\mathbb{R}^n} e^{2\pi i x \cdot y} K_1(x) dx = \int_{|x| \leq \frac{2}{\beta}} (e^{2\pi i x \cdot y} - 1) K_1(x) dx
\]
\[
(3.14)
\]
If \( |y| < \frac{\beta}{2} \), it is direct to estimate
\[
|\hat{K}_1(y)| \leq C|y| \int_{|x| \leq \frac{2}{\beta}} |x| \frac{1}{|x|^{n-\beta}} dx \leq \frac{2^\beta}{\beta + 1} \beta^{-\beta}.
\]
Then there exists an absolute constant \( C > 0 \) such that
\[
|\hat{K}_1(y)| \leq C, \ 0 < \beta < n, \ |y| < \frac{\beta}{2}.
\]
\[
(3.15)
\]
For $\frac{\beta}{2} \leq |y| \leq \beta$, we rewrite $\hat{K}_1(y)$ as

$$
\hat{K}_1(y) = \int_{|x| < \frac{1}{|y|}} e^{2\pi i x \cdot y} K_1(x) \, dx + \int_{\frac{1}{|y|} \leq |x| \leq \frac{2}{\beta}} e^{2\pi i x \cdot y} K_1(x) \, dx
$$

$$
= \int_{|x| < \frac{1}{|y|}} (e^{2\pi i x \cdot y} - 1) K_1(x) \, dx + \int_{\frac{1}{|y|} \leq |x| \leq \frac{2}{\beta}} e^{2\pi i x \cdot y} K_1(x) \, dx
$$

Similar to (3.15), it deduces

$$
\left| \int_{|x| < \frac{1}{|y|}} (e^{2\pi i x \cdot y} - 1) K_1(x) \, dx \right| \leq C|y| \int_{|x| < \frac{1}{|y|}} \frac{1}{|x|^{n-\beta}} \, dx
$$

$$
\leq \frac{1}{\beta + 1} \frac{1}{|y|^\beta} \leq \frac{2^\beta}{\beta + 1} \beta^{-\beta}
$$

and

$$
\left| \int_{\frac{1}{|y|} \leq |x| \leq \frac{2}{\beta}} e^{2\pi i x \cdot y} K_1(x) \, dx \right| \leq \int_{\frac{1}{|y|} \leq |x| \leq \frac{2}{\beta}} \frac{1}{|x|^{n-\beta}} \, dx \leq C \frac{2^\beta - 1}{\beta} \beta^{-\beta}.
$$

Consequently, there exists an absolute constant $C > 0$ such that

$$
|\hat{K}_1(y)| \leq C \left( \frac{2^\beta}{\beta + 1} \beta^{-\beta} + \frac{2^\beta - 1}{\beta} \beta^{-\beta} \right) \leq C, \ 0 < \beta < n, \ \frac{\beta}{2} \leq |y| \leq \beta. \quad (3.17)
$$

As for $|y| > \beta$, $\hat{K}_1(y)$ can be divided into

$$
\hat{K}_1(y) = \int_{|x| < \frac{1}{|y|}} e^{2\pi i x \cdot y} K_1(x) \, dx + \int_{\frac{1}{|y|} \leq |x| \leq \frac{2}{\beta}} e^{2\pi i x \cdot y} K_1(x) \, dx
$$

$$
= \int_{|x| < \frac{1}{|y|}} (e^{2\pi i x \cdot y} - 1) K_1(x) \, dx + \int_{\frac{1}{|y|} \leq |x| \leq \frac{2}{\beta}} e^{2\pi i x \cdot y} K_1(x) \, dx. \quad (3.18)
$$

For the first term on the right hand of the above equality, we easily find that

$$
\left| \int_{|x| < \frac{1}{|y|}} (e^{2\pi i x \cdot y} - 1) K_1(x) \, dx \right| \leq C|y| \int_{|x| < \frac{1}{|y|}} \frac{1}{|x|^{n-\beta}} \, dx
$$

$$
\leq \frac{1}{\beta + 1} \frac{1}{|y|^\beta} \leq \frac{1}{\beta + 1} \beta^{-\beta}. \quad (3.19)
$$

For the second term, we choose $z = \frac{y}{2|y|^2}$ with $|z| = \frac{1}{2|y|} < \frac{1}{2\beta}$ such that $e^{2\pi i y \cdot z} = -1$ and

$$
\int_{\mathbb{R}^n} e^{2\pi i x \cdot y} K_1(x) \, dx = \frac{1}{2} \int_{\mathbb{R}^n} e^{2\pi i x \cdot y} (K_1(x) - K_1(x - z)) \, dx,
$$

moreover, we have

$$
\int_{\frac{1}{|y|} \leq |x| \leq \frac{2}{\beta}} e^{2\pi i x \cdot y} K_1(x) \, dx = \frac{1}{2} \int_{\frac{1}{|y|} \leq |x| \leq \frac{2}{\beta}} e^{2\pi i x \cdot y} (K_1(x) - K_1(x - z)) \, dx
$$

$$
- \frac{1}{2} \int_{|x| \leq \frac{1}{|y|}, \ |x| \leq \frac{1}{|y|}} e^{2\pi i x \cdot y} K_1(x) \, dx
$$

$$
+ \frac{1}{2} \int_{|x| \leq \frac{1}{|y|}, \ |x| \geq \frac{1}{|y|}} e^{2\pi i x \cdot y} K_1(x) \, dx
$$

$$
+ \frac{1}{2} \int_{|x| \geq \frac{2}{\beta}} e^{2\pi i x \cdot y} K_1(x) \, dx
$$

$$
:= I + J + K + L. \quad (3.20)
$$
To estimate the term $I$, we see that

$$I = \int_{\frac{1}{\beta} \leq |x| < \frac{1}{\beta}, \ |x-z| \leq \frac{1}{\beta}} \left( \frac{x}{|x|^{n+1-\beta}} - \frac{x-z}{|x-z|^{n+1-\beta}} \right) e^{2\pi i x \cdot y} \, dx$$

$$+ \int_{\frac{1}{\beta} \leq |x| \leq \frac{2}{\beta}, \ |x-z| \leq \frac{1}{\beta}} \left( \frac{x}{|x|^{n+1-\beta}} \chi_\beta(x) - \frac{x-z}{|x-z|^{n+1-\beta}} \chi_\beta(x-z) \right) e^{2\pi i x \cdot y} \, dx$$

$$+ \int_{\frac{1}{\beta} \leq |x| \leq \frac{2}{\beta}, \ |x-z| \geq \frac{1}{\beta}} \left( \frac{x}{|x|^{n+1-\beta}} - \frac{x-z}{|x-z|^{n+1-\beta}} \chi_\beta(x-z) \right) e^{2\pi i x \cdot y} \, dx$$

$$+ \int_{\frac{1}{\beta} \leq |x| \leq \frac{2}{\beta}, \ |x-z| \geq \frac{1}{\beta}} \left( \frac{x}{|x|^{n+1-\beta}} \chi_\beta(x) - \frac{x-z}{|x-z|^{n+1-\beta}} \chi_\beta(x-z) \right) e^{2\pi i x \cdot y} \, dx$$

$$:= I_1 + I_2 + I_3 + I_4.$$ 

We first estimate $I_2$. Thanks to $|x-z| \geq |x| - |z| \geq \frac{1}{\beta} - \frac{1}{2|y|} \geq \frac{1}{2\beta}$, one has

$$|I_2| \leq \int_{\frac{1}{\beta} \leq |x| \leq \frac{1}{\beta}} \frac{1}{|x|^{n-\beta}} \, dx + \int_{\frac{1}{\beta} \leq |x-z| \leq \frac{1}{\beta}} \frac{1}{|x-z|^{n-\beta}} \, dx$$

$$\leq C \frac{2^{\beta-1}}{\beta^{\beta-\beta}} + C \beta - \frac{2^{\beta-1}}{\beta^{\beta-\beta}}.$$

Then, thanks to $|x| = |x-z+z| \geq |x-z| - |z| \geq \frac{1}{\beta} - \frac{1}{2|y|} \geq \frac{1}{2\beta}$, $I_3$ can be estimated as follows:

$$|I_3| \leq \int_{\frac{1}{2\beta} \leq |x| \leq \frac{1}{\beta}} \frac{1}{|x|^{n-\beta}} \, dx + \int_{\frac{1}{\beta} \leq |x-z| \leq \frac{1}{2\beta}} \frac{1}{|x-z|^{n-\beta}} \, dx$$

$$\leq C \frac{2^{\beta-1}}{\beta^{\beta-\beta}} + C \beta - \frac{2^{\beta-1}}{\beta^{\beta-\beta}}.$$

The term $I_4$ is directly estimated as

$$|I_4| \leq \int_{\frac{1}{\beta} \leq |x| \leq \frac{1}{\beta}} \frac{1}{|x|^{n-\beta}} \, dx + \int_{\frac{1}{\beta} \leq |x-z| \leq \frac{1}{2\beta}} \frac{1}{|x-z|^{n-\beta}} \, dx$$

$$\leq C \frac{2^{\beta-1}}{\beta^{\beta-\beta}}.$$

Now we deal with $I_1$. Note that

$$\partial_i \left( \frac{x}{|x|^{n+1-\beta}} \right) = \frac{\mathbf{e}_i}{|x|^{n+1-\beta}} + (-n-1+\beta) \frac{x x_i}{|x|^{n+1-\beta}} \cdot i = 1, 2, \ldots, n.$$ 

In this case, since $|x-z| \geq |x| - |z| \geq 2|z| - |z| \geq |z|$, by Taylor’s expansion, one has

$$\left| \frac{x-z}{|x-z|^{n+1-\beta}} - \frac{x}{|x|^{n+1-\beta}} \right| \leq \sum_{i=1}^{n} \left( \frac{z x_i}{|x-z|^{n+1-\beta}} + (-n-1+\beta) \frac{(x-z)(x_i-z_i)z_i}{|x-z|^{n+1-\beta}} \right) + C \sum_{k=2}^{\infty} \frac{|z|^k}{k! |x-z|^{n+k-\beta}}.$$
Consequently,

$$|J| \leq (n + 2 - \beta)|z| \int_{|z| \leq |x - z| < \frac{1}{\beta}} \frac{1}{|x - z|^{n+1-\beta}} \, dx$$

$$+ C \sum_{k=2}^{\infty} \int_{|z| \leq |x - z| < \frac{1}{\beta}} \frac{|z|^k}{k!|x - z|^{n+k-\beta}} \, dx$$

$$\leq C \frac{|z|^\beta}{1 - \beta} \leq C \frac{\beta^{\beta}}{1 - \beta}.$$  \hspace{1cm} (3.25)

Substituting (3.22)-(3.25) into (3.21) yields

$$|I| = \left| \int_{|z| \leq |x - z| < \frac{1}{\beta}, |x - z| \leq \frac{1}{\beta}} \left( \frac{x}{|x|^{n+1-\beta}} - \frac{x - z}{|x - z|^{n+1-\beta}} \right) e^{2\pi i x \cdot y} \, dx \right|$$

$$\leq C \left( \frac{2^\beta - 1}{\beta} \beta^{-\beta} + \frac{1 - 2^\beta}{\beta} \beta^{-\beta} + \frac{\beta^{-\beta}}{1 - \beta} \right)$$

for some absolute constant $C > 0$.

Concerning the term $J$, thanks to $|x| \geq |x + z| - |z| \geq 2|z| - |z| \geq |z|$, one has

$$|J| \leq \int_{|z| \leq |x| \leq 2|z|} \frac{1}{|x|^{n-\beta}} \, dx \leq \frac{1 - 2^\beta}{\beta} \beta^{-\beta}. \hspace{1cm} (3.27)$$

Utilizing $|x| \leq |x + z| + |z| \leq 2|z| + |z| \leq 3|z|$, the term $K$ can be bounded by

$$|K| \leq \int_{|z| \leq |x - z| \leq 3|z|} \frac{1}{|x|^{n-\beta}} \, dx \leq \frac{3^\beta - 1}{\beta} \beta^{-\beta}. \hspace{1cm} (3.28)$$

As for the term $L$, the fact that $\frac{2}{\beta} \geq |x| \geq |x + z| - |z| \geq \frac{2}{\beta} - \frac{1}{2\beta} = \frac{3}{2\beta}$ enables us to conclude

$$|L| \leq \int_{\frac{1}{2\beta} \leq |x| \leq \frac{3}{2\beta}} \frac{1}{|x|^{n-\beta}} \, dx \leq \frac{1}{\beta} \left( \left( \frac{3}{2\beta} \right)^{\beta} - \left( \frac{2}{\beta} \right)^{\beta} \right). \hspace{1cm} (3.29)$$

Substituting (3.26)-(3.29) into (3.20), we readily obtain that there exists an absolute constant $C > 0$ such that

$$\left| \int_{|z| \leq |x| \leq \frac{3}{2\beta}} e^{2\pi i x \cdot y} K_1(x) \, dx \right| \leq C. \hspace{1cm} (3.30)$$

In view of (3.19), (3.30) and (3.18), there exists an absolute constant $C > 0$ such that

$$|\tilde{K}_1(y)| \leq C, \hspace{1cm} 0 < \beta < n, \hspace{1cm} |y| > \beta. \hspace{1cm} (3.31)$$

Combining (3.16), (3.17) with (3.31), we finish the proof of (3.13). Applying (3.13), one has

$$\|T_1 f\|_{L^2(\mathbb{R}^n)} = \|\tilde{K}_1 \tilde{f}\|_{L^2(\mathbb{R}^n)} \leq C \|\tilde{f}\|_{L^2(\mathbb{R}^n)} = C \|f\|_{L^2(\mathbb{R}^n)},$$

$$\|L^s T_1 f\|_{L^2(\mathbb{R}^n)} = \|\tilde{K}_1 L^s \tilde{f}\|_{L^2(\mathbb{R}^n)} \leq C \|L^s \tilde{f}\|_{L^2(\mathbb{R}^n)} = C \|L^s f\|_{L^2(\mathbb{R}^n)}$$

for any $0 < s < n$. Hence (3.5) is proved and the proof of the lemma is complete. \hspace{1cm} \Box
4. Estimates via Besov Spaces

In this section, we will establish two key estimates concerning

\[ \overline{\psi}_I = \nabla \Delta (-\Delta)^{-1+\alpha} \overline{\omega}, \quad 0 < \alpha < \frac{1}{2} \]

in nonhomogeneous Besov spaces (see the Appendix in the end of the paper) which will be needed in the proof of Theorem 1.3. The first proposition is to deal with the product of two functions. The second proposition is about a commutator estimate.

**Proposition 4.1.** For any \( s > 0 \), there exists a constant \( C \) depending only on \( s \) and \( \alpha \) such that

\[ \| \overline{\psi}_I \cdot \nabla \omega^\alpha \|_{H^s(\mathbb{R}^2)} \leq C \left( \| \overline{\omega} \|_{L^2(\mathbb{R}^2)} \| \omega^{\alpha} \|_{H^{s+2\alpha}(\mathbb{R}^2)} + \| \overline{\omega} \|_{H^{s}(\mathbb{R}^2)} \| \omega^{\alpha} \|_{B_{2,1}^{1+2\alpha}(\mathbb{R}^2)} \right). \]

**Remark 4.1.** Let us point out that the positive constant \( C \) is uniformly bounded as parameter \( \alpha \) goes to \( \frac{1}{2} \).

**Proof of Proposition 4.1.** In view of the Bony decomposition, one write

\[ \overline{\psi}_I \cdot \nabla \omega^\alpha = \sum_{i=1}^{2} \left( T_{\partial_i \omega^\alpha} \overline{\psi}_I + T_{\overline{\psi}_I} \partial_i \omega^\alpha + R(\overline{\psi}_I, \partial_i \omega^\alpha) \right), \]

where

\[ T_{\partial_i \omega^\alpha} \overline{\psi}_I = \sum_{q > 0} \Delta_q T_{q-1} \partial_i \omega^\alpha, \quad T_{\overline{\psi}_I} \partial_i \omega^\alpha = \sum_{q > 0} S_{q-1} \overline{\psi}_I \Delta_q \partial_i \omega^\alpha, \]

\[ R(\overline{\psi}_I, \partial_i \omega^\alpha) = \sum_{q \geq -1} \Delta_q \overline{\psi}_I \Delta_q \partial_i \omega^\alpha. \]

According to the Hölder inequality and Lemma 4.2, we obtain that for \( q > 0 \),

\[ 2^{qs} \| \Delta_q \overline{\psi}_I S_{q-1} \partial_i \omega^\alpha \|_{L^2} \leq 2^{qs} \| S_{q-1} \nabla \omega^\alpha \|_{L^\infty} \| \Delta_q \overline{\psi}_I \|_{L^2} \]

\[ \leq 2^{qs} \sum_{-1 \leq k \leq q-2} \| \Delta_k \nabla \omega^\alpha \|_{L^\infty} 2^{q(1+2\alpha)} \| \Delta_q \overline{\omega} \|_{L^2} \]

\[ \leq 2^{qs} \| \Delta_q \overline{\omega} \|_{L^2} \sum_{-1 \leq k \leq q-2} 2^{(q-k)(2\alpha-1)} 2^{k(1+2\alpha)} \| \Delta_k \omega^\alpha \|_{L^2}. \]

Therefore, Lemma 4.3 and the Young inequality for series yields

\[ \| T_{\nabla \omega^\alpha} \overline{\psi}_I \|_{H^s} \leq C_s \left\{ \left\{ 2^{qs} \| S_{q-1} \partial_i \omega^\alpha \Delta_q \overline{\psi}_I \|_{L^2} \right\}_{q > 0} \right\}_{q \geq 0} \|_{L^2} \]

\[ \leq C_s 2^{2(2\alpha-1)} \| \overline{\omega} \|_{H^s} \| \omega^\alpha \|_{B_{2,1}^{1+2\alpha}}. \quad (4.1) \]

Similarly, for \( 0 < \epsilon < 2\alpha \),

\[ 2^{qs} \| S_{q-1} \overline{\psi}_I \Delta_q \partial_i \omega^\alpha \|_{L^2} \leq 2^{qs} \| S_{q-1} \overline{\psi}_I \|_{L^\infty} \| \Delta_q \nabla \omega^\alpha \|_{L^2} \]

\[ \leq \sum_{-1 \leq k \leq q-2} \| \Delta_k \overline{\psi}_I \|_{L^\infty} \| \Delta_q \omega^\alpha \|_{L^2} 2^{q(s+1)} \]

\[ \leq C \| \Delta^{-(1-2\alpha+\epsilon)} \overline{\omega} \|_{L^{2\alpha/\epsilon}} \sum_{k \leq q-2} 2^{2k\alpha} 2^{q(s+1)} \| \Delta_q \omega^\alpha \|_{L^2} \]

\[ \leq C \| \overline{\omega} \|_{L^2} 2^{q(s+1+2\alpha)} \| \Delta_q \omega^\alpha \|_{L^2} \sum_{k \leq q-2} 2^{2\alpha(k-q)} \]
where Lemma 2.3 has been used in the last inequality. In addition, when \( \alpha \to \frac{1}{2} \), the constant \( C \) is uniformly bounded. Hence, by Lemma 2.3 we get

\[
\| T_{\partial_j} \partial_i \omega^{(0)} \|_{H^s} \leq C_s \| \{ 2^q \| S_q \partial_i \Delta_q \nabla \omega^{(0)} \|_{L^2} \}_{q \geq 0} \|_{L^2} \\
\leq C \| \bar{\omega} \|_{L^2} \| \omega^{(0)} \|_{H^{s+1+2\alpha}}.
\] (4.2)

For the reminder term, we see that

\[
2^q \| \Delta_q R_{\omega} (w_i, \partial_i \omega^{(0)}) \|_{L^2} \leq \sum_{q \leq q' + N_0} 2^q \| \Delta_q (\Delta_{q'} \bar{w}_i \Delta_{q'} \partial_i \omega^{(0)}) \|_{L^2} \\
\leq \sum_{q \leq q' + N_0} 2^q \| \Delta_q \bar{w}_i \|_{L^\infty} \| \bar{\Delta}_{q'} \nabla \omega^{(0)} \|_{L^2} \\
\leq C \| \Delta_{-(1-2\alpha)^+} \|_{L^2} \sum_{q \leq q' + N_0} 2^q 2^{(1+2\alpha)q'} \| \bar{\Delta}_{q'} \omega^{(0)} \|_{L^2} \\
\leq C \| \bar{\omega} \|_{L^2} \sum_{q \leq q' + N_0} 2^{(q-q')s} 2^{(s+1+2\alpha)q'} \| \bar{\Delta}_{q'} \omega^{(0)} \|_{L^2}.
\]

This ensures that by Lemma 2.2, for \( s > 0 \),

\[
\| R_{\omega} (w_i, \partial_i \omega^{(0)}) \|_{H^s} \leq \| \{ 2^q \| \Delta_q R_{\omega} (w_i, \nabla \omega^{(0)}) \|_{L^2} \}_{q \geq 1} \|_{L^2} \\
\leq C \| \bar{\omega} \|_{L^2} \| \omega^{(0)} \|_{H^{s+1+2\alpha}}.
\] (4.3)

Collecting (4.1), (4.2) and (4.3) above gives the proof of this proposition. \( \square \)

**Proposition 4.2.** For any \( s > 0 \), there exists a constant \( C \) depending only on \( s \) such that,

\[
\| J^s (w_i \cdot \nabla \bar{\omega}) - w_i \cdot J^s \nabla \bar{\omega} \|_{L^2(\mathbb{R}^2)} \\
\leq C \left( \| \bar{\omega} \|_{H^{2\alpha}(\mathbb{R}^2)}^2 + \| w_i \|_{H^s(\mathbb{R}^2)} \| \bar{\omega} \|_{H^s(\mathbb{R}^2)} + \| \bar{\omega} \|_{H^s(\mathbb{R}^2)} \| \nabla \bar{\omega} \|_{H^{2\alpha+1}(\mathbb{R}^2)} \right).
\] (4.4)

In particular, if \( s > 2 \), then we have

\[
\| J^s (w_i \cdot \nabla \bar{\omega}) - w_i \cdot J^s \nabla \bar{\omega} \|_{L^2(\mathbb{R}^2)} \leq C \| \bar{\omega} \|_{H^s(\mathbb{R}^2)}^2.
\] (4.5)

**Proof.** With the help of Bony’s decomposition, one writes

\[
J^s (w_i \cdot \nabla \bar{\omega}) - w_i \cdot J^s \nabla \bar{\omega} \\
= 2 \sum_{i=1}^2 \left( [J^s, T_{w_i} \partial_i] \bar{\omega} + J^s (T_{w_i} \bar{w}_i) - T_{w_i} \partial_i \bar{w}_i + J^s \left( R_{\omega} (w_i, \partial_i \bar{\omega}) \right) - R_{\omega} (w_i, J^s \partial_i \bar{\omega}) \right).
\]

The last two terms can be further decomposed into three parts

\[
J^s (R_{\omega} (w_i, \partial_i \bar{\omega})) - R_{\omega} (w_i, J^s \partial_i \bar{\omega}) \\
= \sum_{q' \geq 0} J^s (\Delta_{q'} \bar{w}_i \Delta_{q'} \partial_i \bar{\omega}) - \sum_{q' \geq 0} \Delta_{q'} \bar{w}_i \Delta_{q'} J^s \partial_i \bar{\omega} + [J^s, \Delta_{q'} \partial_i \bar{\omega}] \Delta_{-1} \bar{\omega}.
\]

Next, we are going to establish the standard inner \( L^2 \)-norm of the six terms above one by one.
Bounds for the term \([J^s, T_{\mathcal{M}} \partial_i] \overline{\omega}\). By virtue of Proposition \(\text{F.1}\) we can rewrite \([J^s, T_{\mathcal{M}} \cdot \partial_i]\) as a convolution operator. Indeed,

\[
[J^s, T_{\mathcal{M}} \partial_i] \overline{\omega} = \sum_{q > 0} [J^s \Delta_q, S_{q-1} u_I^j \partial_i] \Delta_q \overline{\omega}
\]

\[
= \sum_{q > 0} \int_{\mathbb{R}^2} 2^{2q} G_s(2^q y) (S_{q-1} u_I^j(x-y) - S_{q-1} u_I^j(x)) \Delta_q \partial_i \overline{\omega}(x-y) \, dy,
\]

where \(G_s\) is the inverse Fourier transform of \(\xi \mapsto (2^q \xi)^s \varphi(\xi)\).

From the first order Taylor formula, we deduce that

\[
||[J^s, T_{\mathcal{M}} \partial_i] \overline{\omega}||_{L^2} \leq \sum_{q > 0} \int_{\mathbb{R}^2} \int_0^1 2^{2q} |G_s(2^q y)||\nabla S_{q-1} u_I^j(x-\tau y)|| \Delta_q \partial_i \overline{\omega}(x-y) | d\tau dy.
\]

Now, taking the \(L^2\) norm of the above inequality, using the fact that \(L^2 \sim B^0_{2,2}\), and using Lemma \(\text{F.3}\) we get

\[
\left(\sum_{q > 0} \int_{\mathbb{R}^2} \int_0^1 2^{2q} |G_s(2^q y)||\nabla S_{q-1} u_I^j(x-\tau y)|| \Delta_q \partial_i \overline{\omega}(x-y) | d\tau dy\right)^{\frac{1}{2}}.
\]

Adopting to the fact that the norm of an integral is less than the integral of the norm and using Hölder’s inequality yield

\[
\left\| \int_{\mathbb{R}^2} \int_0^1 2^{2q} |G_s(2^q y)||\nabla S_{q-1} u_I^j(x-\tau y)|| \Delta_q \partial_i \overline{\omega}(x-y) | d\tau dy \right\|_{L^2}
\]

\[
\leq \int_{\mathbb{R}^2} \int_0^1 2^{2q} |G_s(2^q y)||\nabla S_{q-1} u_I^j(x-\tau y)||_{L^\infty} \Delta_q \partial_i \overline{\omega}(x-y) | d\tau dy
\]

\[
\leq 2^{2q_s} \||\nabla S_{q-1} u_I^j||_{L^\infty} \||\Delta_q \overline{\omega}||_{L^2},
\]

where the translation invariance of the Lebesgue measure is used in the last inequality.

Hence, the Hölder inequality and Bernstein’s inequality enable us to conclude that

\[
||[J^s, T_{\mathcal{M}} \partial_i] \overline{\omega}||_{L^2} \leq \left(\sum_{q > 0} 2^{2q_s} \||\nabla S_{q-1} u_I^j||_{L^\infty} \||\Delta_q \overline{\omega}||_{L^2}\right)^{\frac{1}{2}}
\]

\[
\leq \sup_{q > 0} \||\nabla S_{q-1} u_I^j||_{L^\infty} \||\overline{\omega}||_{H^s}
\]

\[
\leq C \||\overline{\omega}||_{B^{1+2\alpha}_{2,1}} \||\overline{\omega}||_{H^s}.
\]

Bounds for \(J^s(T_{\partial_i} \overline{\omega})\). By virtue of the Hölder inequality and Bernstein’s inequality, we get

\[
2^{q_s} \||\Delta_q \overline{\omega}||_{L^2} \leq 2^{q_s} \||S_{q-1} \partial_i \overline{\omega}||_{L^\infty} \||\Delta_q \overline{\omega}||_{L^2}
\]

\[
\leq 2^{q_s} \sum_{k \leq q-2} \|\Delta_k \overline{\omega}||_{L^\infty} 2^{(q-k)2(2\alpha+1)} \||\Delta_q \overline{\omega}||_{L^2}
\]

\[
\leq 2^{q_s} \||\Delta_q \overline{\omega}||_{L^2} \sum_{k \leq q-2} 2^{(q-k)(2\alpha+1)} 2^{k(2\alpha+2)} \||\Delta_k \overline{\omega}||_{L^2}.
\]
Hence, we have by Lemma F.3 that
\[
\| J^s(T_{\partial I^+_I}^T) \|_{L^2} \leq C_s \left\{ 2^{q_s} \| \Delta_q \overline{\partial_I^+ I} S_{q-1} \partial \overline{w} \|_{L^2} \right\}_{q>0} \leq C_s 2^{(2\alpha-1)} \| \overline{w} \|_{H^s} \| \overline{w} \|_{B^{1+2\alpha}_{2,1}}.
\]

A similar bound holds for both terms \( T_{J^s \partial I^+} \sum_{q' \geq 0} \Delta_q \overline{\partial_I^+ I} \cdot \Delta_{q'} J^s \partial \overline{w} \). By the Hörder inequality, one has
\[
\| \Delta_q \overline{\partial_I^+ I} S_{q-1} \partial \overline{w} \|_{L^2} \leq \| S_{q-1} \nabla J^s \overline{w} \|_{L^\infty} \| \Delta_q \overline{\partial_I^+ I} \|_{L^2} \\
\leq \sum_{k \leq q-2} \| \Delta_k \nabla J^s \overline{w} \|_{L^\infty} 2^{q(-1+2\alpha)} \| \Delta_q \overline{w} \|_{L^2} \\
\leq 2^{q(1+2\alpha)} \| \Delta_q \overline{w} \|_{L^2} \| J^s \overline{w} \|_{L^2} \sum_{k \leq q-2} 2^{(k-q)}.
\]
from which it follows that
\[
\| T_{J^s \partial I^+} \overline{w} \|_{L^2} \leq C \| \overline{w} \|_{H^s} \| \overline{w} \|_{B^{1+2\alpha}_{2,1}}.
\]
For the term \( \sum_{q' \geq 0} \Delta_q' \overline{\partial_I^+ I} \Delta_{q'} J^s \partial \overline{w} \), by the Hörder inequality, we immediately obtain
\[
\left\| \sum_{q' \geq 0} \Delta_q' \overline{\partial_I^+ I} \Delta_{q'} J^s \partial \overline{w} \right\|_{L^2} \leq \sum_{q' \geq 0} \| \Delta_q' \overline{\partial_I^+ I} \Delta_{q'} J^s \partial \overline{w} \|_{L^2} \\
\leq \sum_{q' \geq 0} \left( \sum_{q \leq q' + N_0} 2^{q_s} \| \Delta_q \overline{\partial_I^+ I} \Delta_{q'} \partial \overline{w} \|_{L^2} \right)^{\frac{1}{2}} \\
\leq \sum_{q' \geq 0} \left( \sum_{q \leq q' + N_0} 2^{q_s} \right)^{\frac{1}{2}} \| \Delta_q' \overline{\partial_I^+ I} \|_{L^2} \| \Delta_{q'} \partial \overline{w} \|_{L^\infty} \\
\leq \sum_{q' \geq 0} 2^{q_s} \| \Delta_q' \overline{w} \|_{L^2} 2^{q(1+2\alpha)} \| \Delta_{q'} \overline{w} \|_{L^2} \\
\leq \| \overline{w} \|_{H^s} \| \overline{w} \|_{H^{1+2\alpha}}.
\]

Bounds for the term \( \sum_{q' \geq 0} J^s \overline{\partial_I^+ I} \Delta_{q'} \partial \overline{w} \). Utilizing again the Hörder inequality and
Bernstein’s inequality gives
\[
\left\| \sum_{q' \geq 0} J^s \overline{\partial_I^+ I} \Delta_{q'} \partial \overline{w} \right\|_{L^2} \leq \sum_{q' \geq 0} \left( \sum_{q \leq q' + N_0} 2^{q_s} \| \Delta_q \overline{\partial_I^+ I} \Delta_{q'} \partial \overline{w} \|_{L^2} \right)^{\frac{1}{2}} \\
\leq \sum_{q' \geq 0} \left( \sum_{q \leq q' + N_0} 2^{q_s} \right)^{\frac{1}{2}} \| \Delta_q' \overline{\partial_I^+ I} \|_{L^2} \| \Delta_{q'} \partial \overline{w} \|_{L^\infty} \\
\leq \sum_{q' \geq 0} 2^{q_s} \| \Delta_q' \overline{w} \|_{L^2} 2^{q(1+2\alpha)} \| \Delta_{q'} \overline{w} \|_{L^2} \\
\leq \| \overline{w} \|_{H^s} \| \overline{w} \|_{H^{1+2\alpha}}.
\]

Bounds for the last term \( [J^s, \Delta_{-1} \overline{\partial_I^+ I} \Delta_{-1} \overline{w}] \). Adopting to the similar method to estimate \( [J^s, T_{\overline{\partial_I^+ I}} \partial \overline{w}] \), we get
\[
[J^s, \Delta_{-1} \overline{\partial_I^+ I} \Delta_{-1} \overline{w}] \\
= \sum_{|q+1| \leq 2} [J^s \Delta_q, \Delta_{-1} \overline{\partial_I^+ I} \Delta_{-1} \overline{w}]
\]

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Applying Lemma 2.2 and Lemma 2.3 gives the divergence-free condition and (2.2), we have

\[
L_\alpha \leq \sum_{|q+1| \leq 2} \int_{\mathbb{R}^2} 2^{2q} G_s(2^q y) (\nabla \Delta - \nabla \bar{\omega})(x - y) \, dy
\]

= \sum_{|q+1| \leq 2} \int_{\mathbb{R}^2} \int_0^1 2^{2q} G_s(2^q y) (\nabla \Delta - \nabla \bar{\omega})(x - \tau y) \, d\tau \, dy.

Based on this, the Minkowski inequality and the Hölder inequality allow us to infer that

\[
\left\| [J^s, \Delta \bar{\omega}] \mathcal{M} \right\|_{L^2} \leq \left( \sum_{|q+1| \leq 2} \left\| \int_{\mathbb{R}^2} \int_0^1 2^{2q} G_s(2^q y) (\nabla \Delta - \nabla \bar{\omega})(x - \tau y) \, d\tau \, dy \right\|_{L^2}^2 \right)^{\frac{1}{2}}
\]

\[
\leq C_s \left\| \Delta \nabla \bar{\omega} \right\|_{L^\infty} \left\| \Delta \nabla \bar{\omega} \right\|_{L^2} \leq C_s \| A^{2\alpha} \bar{\omega} \|_{L^2} \| \bar{\omega} \|_{L^2} \leq C_s \| \bar{\omega} \|_{H^{2\alpha}}^2.
\]

Combining these estimates above yields (4.4). This ends the proof. \( \square \)

5. PROOF OF MAIN THEOREMS

This section is devoted to showing the main theorems. Let us begin by proving Theorem 1.1.

5.1. Proof of Theorem 1.1

First of all, let us denote

\[
\bar{\omega} = \omega^\alpha - \omega^0 \quad \text{and} \quad \bar{\pi} = u^\alpha - u^0.
\]

Then, the couple \((\bar{\omega}, \bar{\pi})\) satisfies

\[
\bar{\omega} + u^0 \cdot \nabla \bar{\omega} + \bar{\pi} \cdot \nabla \bar{\omega} + \bar{\pi} \cdot \nabla \omega^0 = 0.
\]

Operating \(J^s\) on (5.1) and taking the scalar product of the resulting equation with \(J^s \bar{\omega}\) in \(L^2\), we get

\[
\frac{1}{2} \frac{d}{dt} \| \bar{\omega}(t) \|_{H^s}^2 = - \int_{\mathbb{R}^2} \nabla \cdot J^s(\bar{\omega}) \cdot \nabla \bar{\omega} \, dx - \int_{\mathbb{R}^2} \nabla \cdot J^s(\bar{\pi}) \cdot J^s \bar{\omega} \, dx
\]

\[
- \int_{\mathbb{R}^2} J^s(\bar{\pi} \cdot \nabla \omega^0) \cdot J^s \bar{\omega} \, dx
\]

\[
:= I_1 + I_2 + I_3.
\]

We are going to estimate the three terms on the right hand side of (5.2) one by one. By the divergence-free condition and (2.2), we have

\[
I_1 = - \int_{\mathbb{R}^2} (J^s(u^\alpha \cdot \nabla \bar{\omega}) - u^0 \cdot \nabla J^s \bar{\omega}) \cdot J^s \bar{\omega} \, dx
\]

\[
\leq \| J^s(u^\alpha \cdot \nabla \bar{\omega}) - u^0 \cdot \nabla J^s \bar{\omega} \|_{L^2} \| J^s \bar{\omega} \|_{L^2}
\]

\[
\leq \left( \| J^s u^\alpha \|_{L^\frac{2}{1+\alpha_0}} \| \nabla \bar{\omega} \|_{L^\frac{2}{1+\alpha_0}} + \| \nabla u^\alpha \|_{L^\infty} \| J^s \nabla \bar{\omega} \|_{L^2} \right) \| J^s \bar{\omega} \|_{L^2}.
\]

Note that \( 0 < \alpha_0 < \frac{1}{2} \) and

\[
u^\alpha_0 = \nabla^\bot (-\Delta)^{-1+\alpha_0} \omega^0.
\]

Using Lemma 2.3 yields

\[
\| J^s u^\alpha \|_{L^\frac{2}{1+\alpha_0}} \leq C \| \omega^0 \|_{H^s}.
\]

Applying Lemma 2.2 and Lemma 2.3 gives

\[
\| \nabla \bar{\omega} \|_{L^\frac{2}{1+\alpha_0}} \leq C \| \bar{\omega} \|_{H^s}.
\]
\[ \| \nabla u^\alpha \|_{L^\infty} \leq \| u^\alpha \|_{W^{1, \frac{1}{\alpha}}} \leq C \| \omega^\alpha \|_{H^s}. \]

Thus,
\[ |I_1| \leq C \| \omega \|_{H^s} \| \omega^\alpha \|_{H^s}. \tag{5.3} \]

By the decomposition (1.2), the second term can be written as
\[ I_2 = - \int_{\mathbb{R}^2} (J^s(\pi \cdot \nabla \omega) - \pi \cdot \nabla J^s \omega) J^s \omega \, dx \]
\[ = - \int_{\mathbb{R}^2} (J^s(\pi_I \cdot \nabla \omega) - \pi_I \cdot \nabla J^s \omega) J^s \omega \, dx - \int_{\mathbb{R}^2} (J^s(\pi_{II} \cdot \nabla \omega) - \pi_{II} \cdot \nabla J^s \omega) J^s \omega \, dx. \]

Using (2.2) and the Sobolev embedding inequalities, we obtain
\[ - \int_{\mathbb{R}^2} (J^s(\pi_I \cdot \nabla \omega) - \pi_I \cdot \nabla J^s \omega) J^s \omega \, dx \]
\[ \leq \left( \| J^s \pi_I \|_{L^\alpha} \| \nabla \omega \|_{L^1} + \| \nabla \pi_I \|_{L^\infty} \| J^{s-1} \nabla \omega \|_{L^2} \right) \| J^s \omega \|_{L^2} \]
\[ \leq \| \omega \|_{H^s}^2 \| J^s \pi_I \|_{L^\frac{1}{\alpha}}. \]

Similarly, for \( p > \frac{1}{\alpha} > 2, \)
\[ - \int_{\mathbb{R}^2} (J^s(\pi_{II} \cdot \nabla \omega) - \pi_{II} \cdot \nabla J^s \omega) J^s \omega \, dx \]
\[ \leq \left( \| J^s \pi_{II} \|_{L^p} \| \nabla \omega \|_{L^\frac{p}{p-2}} + \| \nabla \pi_{II} \|_{L^\infty} \| J^{s-1} \nabla \omega \|_{L^2} \right) \| J^s \omega \|_{L^2} \]
\[ \leq \| \omega \|_{H^s}^2 \| J^s \pi_{II} \|_{L^p}. \]

Therefore,
\[ |I_2| \leq \| \omega \|_{H^s}^2 (\| J^s \pi_I \|_{L^\frac{1}{\alpha}} + \| J^s \pi_{II} \|_{L^p}). \tag{5.4} \]

Concerning the third term, we use (2.1) and the Sobolev embedding inequalities to get
\[ |I_3| = \left| \int_{\mathbb{R}^2} J^s(\pi \cdot \nabla \omega^\alpha) J^s \omega \, dx \right| \]
\[ = \left| \int_{\mathbb{R}^2} J^s(\pi_I \cdot \nabla \omega^\alpha) J^s \omega \, dx + \int_{\mathbb{R}^2} J^s(\pi_{II} \cdot \nabla \omega^\alpha) J^s \omega \, dx \right| \tag{5.5} \]
\[ \leq \| \omega \|_{H^s} \| J^s \pi_I \|_{L^\frac{1}{\alpha}} + \| J^s \pi_{II} \|_{L^p} \| \omega^\alpha \|_{H^{s+1}}. \]

Plugging these estimates (5.3), (5.4), (5.5) into (5.2) yields
\[ \frac{d}{dt} \| \omega(t) \|_{H^s} \leq \| \omega \|_{H^s} \left( \| \omega^\alpha \|_{H^s} + \| J^s \pi_I \|_{L^\frac{1}{\alpha}} + \| J^s \pi_{II} \|_{L^p} \right) \]
\[ + \| \omega^\alpha \|_{H^{s+1}} \left( \| J^s \pi_I \|_{L^\frac{1}{\alpha}} + \| J^s \pi_{II} \|_{L^p} \right) \]
\[ := \tilde{I}_1 + \tilde{I}_2. \tag{5.6} \]

The integral form of \( \pi_I \) can be written as
\[ J^s \pi_I(x) = \int_{\mathbb{R}^2} \frac{(x - y)^1}{|x - y|^{2+2\alpha}} J^s \omega(y) \, dy. \]

Then, using Lemma 2.3 enables us to get
\[ \| J^s \pi_I \|_{L^\frac{1}{\alpha}} \leq \left\| \int_{\mathbb{R}^2} \frac{1}{|x - y|^{2-(1-2\alpha)}} J^s \omega(y) \, dy \right\|_{L^2} \leq C(\alpha) \| J^s \omega \|_{L^2}, \tag{5.7} \]
where \( C(\alpha) \) depends on \( \alpha \) and will be bounded if \( 0 \leq \alpha < \alpha_0 < \frac{1}{2} \) (but will be unbounded if \( \alpha \) tend to \( \frac{1}{2} \)). When \( p > \frac{1}{\alpha} > 2 \), adopting to the similar way to (5.7) gives

\[
\| J^s \overline{\Pi}_I \|_{L^p} \leq C(\alpha) \left( \| J^s \omega^{\alpha_0} \|_{L^{\frac{2p}{2p+(1-2\alpha)}}} + \| J^s \omega^{\alpha_0} \|_{L^{\frac{2p}{2p+(1-2\alpha)}}} \right) \\
\leq C(\alpha) \| \omega^{\alpha_0} \|_{H^{\alpha+1}}. 
\]

(5.8)

Hence,

\[
\tilde{I}_1 \leq C \| \omega \|_{H^2}^2 + C \| \omega \|_{H^1} \| \omega^{\alpha_0} \|_{H^{\alpha+1}}. 
\]

(5.9)

On the other hand, the estimate (5.8) is not adaptable to \( \tilde{I}_2 \) in (5.6). We will use a different way to estimate \( \| J^s \overline{\Pi}_I \|_{L^p} \) in (5.6). For \( 0 < \epsilon < 1 \) to be determined later, we write \( \overline{\Pi}_I \) as

\[
\overline{\Pi}_I = \int_{\mathbb{R}^2} \left( \frac{(x-y)_{+}}{|x-y|^{2+2\alpha}} - \frac{(x-y)_{+}}{|x-y|^{2+2\alpha}} \right) J^s \omega^{\alpha_0}(y) \, dy.
\]

Therefore,

\[
J^s \overline{\Pi}_I = \int_{\mathbb{R}^2} \left( \frac{(x-y)_{+}}{|x-y|^{2+2\alpha}} - \frac{(x-y)_{+}}{|x-y|^{2+2\alpha}} \right) J^s \omega^{\alpha_0}(y) \, dy \\
= \left( \int_{|x-y| \leq \epsilon} + \int_{1>|x-y| \geq \epsilon} + \int_{|x-y| \geq 1} \right) \left( \frac{(x-y)_{+}}{|x-y|^{2+2\alpha}} - \frac{(x-y)_{+}}{|x-y|^{2+2\alpha}} \right) J^s \omega^{\alpha_0}(y) \, dy \\
\leq H_1 + H_2 + H_3. 
\]

(5.10)

For the first term \( H_1 \), using the Young inequality and the Sobolev embedding, we get

\[
\| H_1 \|_{L^p} = \left\| \int_{|x-y| \leq \epsilon} \left( \frac{(x-y)_{+}}{|x-y|^{2+2\alpha}} - \frac{(x-y)_{+}}{|x-y|^{2+2\alpha}} \right) J^s \omega^{\alpha_0}(y) \, dy \right\|_{L^p} \\
\leq C \left( \frac{\epsilon^{1-2\alpha}}{1-2\alpha} + \frac{\epsilon^{1-2\alpha}}{1-2\alpha} \right) \| J^s \omega^{\alpha_0} \|_{L^p} \\
\leq C \left( \frac{\epsilon^{1-2\alpha}}{1-2\alpha} + \frac{\epsilon^{1-2\alpha}}{1-2\alpha} \right) \| \omega^{\alpha_0} \|_{H^{\alpha+1}}. 
\]

As for \( H_2 \) and \( H_3 \), it is divided into two cases.

**Case 1: \( \alpha_0 > \alpha \).** By the mean value theorem, we can obtain

\[
H_2 \leq |\alpha_0 - \alpha| \int_{1>|x-y| \geq \epsilon} \left| \frac{\log |x-y|}{|x-y|^{1+2\alpha}} \right| J^s \omega^{\alpha_0}(y) \, dy.
\]

Utilizing the Young inequality yields

\[
\| H_2 \|_{L^p} \leq |\alpha_0 - \alpha| \| J^s \omega^{\alpha_0} \|_{L^p} \int_{1>|x-y| \geq \epsilon} \left| \frac{\log |x-y|}{|x-y|^{1+2\alpha}} \right| \, dy, \\
\leq C \frac{1}{1-2\alpha} |\alpha_0 - \alpha| \| \omega^{\alpha_0} \|_{H^{\alpha+1}}.
\]

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To deal with $H_3$, we fix a small number $\sigma > 0$ such that $p > \frac{2}{2\alpha - \sigma} > 2$. Thanks to the fact that $\log |x - y| \leq C|x - y|^{\sigma}$ for any $\sigma > 0$ and $|x - y| \geq 1$, we have
\[
\|H_3\|_{L^p} \leq |\alpha_0 - \alpha| \left( \int_{|x-y| \geq 1} \frac{|\log |x-y||}{|x-y|^{1+2\alpha}} |J^s\omega^{\alpha_0}(y)| \, dy \right)_{L^p} \\
\leq |\alpha_0 - \alpha| \|J^s\omega^{\alpha_0}\|_{L^r} \left( \int_{|x-y| \geq 1} \left( \frac{|\log |x-y||}{|x-y|^{1+2\alpha}} \right)^q \, dy \right)^\frac{1}{q} \\
\leq |\alpha_0 - \alpha| \|J^s\omega^{\alpha_0}\|_{L^r} \left( \int_{|x-y| \geq 1} \frac{1}{|x-y|^{1+2\alpha-(\sigma/2)}} \, dy \right)^\frac{1}{q},
\]
where $\frac{1}{p} + 1 = \frac{1}{r} + \frac{1}{q}$, $q > \frac{2}{1+2\alpha - \sigma}$, $p > \frac{2}{2\alpha - \sigma}$, then we can choose some $r > 2$ such that
\[
H^{s+1}(\mathbb{R}^2) \hookrightarrow W^{s,r}(\mathbb{R}^2)
\]
holds (see Lemma 2.2).

**Case 2**: $\alpha_0 < \alpha < \frac{1}{2}$. It is similar to **Case 1** by exchanging the position of $\alpha$ and $\alpha_0$. For instance, by the mean value theorem, we can obtain
\[
H_2 \leq |\alpha_0 - \alpha| \int_{1>|x-y| \geq \epsilon} \frac{|\log |x-y||}{|x-y|^{1+2\alpha}} |J^s\omega^{\alpha_0}(y)| \, dy.
\]
Then
\[
\|H_2\|_{L^p} \leq |\alpha_0 - \alpha| \|J^s\omega^{\alpha_0}\|_{L^r} \int_{1>|x-y| \geq \epsilon} \frac{|\log |x-y||}{|x-y|^{1+2\alpha}} \, dy, \\
\leq \frac{C}{1-2\alpha} |\alpha_0 - \alpha| |\log \epsilon| \|\omega^{\alpha_0}\|_{H^{s+1}}.
\]
Now we fix a small number $\sigma > 0$ such that $p > \frac{2}{2\alpha_0 - \sigma} > 2$. Similarly, we have
\[
\|H_3\|_{L^p} \leq |\alpha_0 - \alpha| \left( \int_{|x-y| \geq 1} \frac{|\log |x-y||}{|x-y|^{1+2\alpha}} |J^s\omega^{\alpha_0}(y)| \, dy \right)_{L^p} \\
\leq |\alpha_0 - \alpha| \|J^s\omega^{\alpha_0}\|_{L^r} \left( \int_{|x-y| \geq 1} \frac{1}{|x-y|^{1+2\alpha-(\sigma/2)}} \, dy \right)^\frac{1}{q},
\]
where $\frac{1}{p} + 1 = \frac{1}{r} + \frac{1}{q}$, $q > \frac{2}{1+2\alpha - \sigma}$, $p > \frac{2}{2\alpha - \sigma}$, then we can choose some $r > 2$ such that
\[
H^{s+1}(\mathbb{R}^2) \hookrightarrow W^{s,r}(\mathbb{R}^2)
\]
holds (see Lemma 2.2).

As a consequence, we get
\[
\|J^s\overline{v}_{II}(t)\|_{L^p} \leq C \left( \epsilon^{1-2\alpha} + \epsilon^{1-2\alpha_0} + |\alpha_0 - \alpha| + |\alpha_0 - \alpha| |\log \epsilon| \right) \|\omega^{\alpha_0}\|_{H^{s+1}}.
\]
Hence,
\[
\tilde{I}_2 = \|\omega^{\alpha_0}\|_{H^{s+1}} (\|J^s\overline{v}_{II}\|_{L^\frac{1}{r}} + \|J^s\overline{v}_{II}\|_{L^p}) \\
\leq C \|\omega^{\alpha_0}\|_{H^{s+1}} \|\overline{v}\|_{H^s} + \|\omega^{\alpha_0}\|_{H^{s+1}}^2 \left( \epsilon^{1-2\alpha} + \epsilon^{1-2\alpha_0} + |\alpha_0 - \alpha| + |\alpha_0 - \alpha| |\log \epsilon| \right). \tag{5.11}
\]
Set $\epsilon = \alpha_0 - \alpha$. By plugging (5.9) and (5.11) into (5.6), we get
\[
\frac{d}{dt} \|\omega(t)\|_{H^s} \leq C \|\omega^{\alpha_0}\|_{H^{s+1}} \|\omega\|_{H^s} + C \|\omega\|_{H^s}^2 \\
+ \|\omega^{\alpha_0}\|_{H^{s+1}}^2 \left( |\alpha_0 - \alpha|^1 + |\alpha_0 - \alpha|^1 |\alpha_0 - \alpha| |\log |\alpha_0 - \alpha|\right).
\]

Multiply (5.12) by $\exp(-C \int_0^t \|\omega^{\alpha_0}\|_{H^{s+1}} \, ds)$ and consider the quantity
\[
y(t) = \|\omega\|_{H^s} \exp\left( - C \int_0^t \|\omega^{\alpha_0}\|_{H^{s+1}} \, ds \right).
\]
We then get the inequality
\[
\frac{dy(t)}{dt} \leq (|\alpha_0 - \alpha| |\alpha_0 - \alpha|^1 + |\alpha_0 - \alpha|^1 |\alpha_0 - \alpha|) F(t) + Gy^2(t),
\]
where
\[
F(t) = \|\omega^{\alpha_0}\|_{H^{s+1}}^2 \exp\left( - C \int_0^t \|\omega^{\alpha_0}(s)\|_{H^{s+1}} \, ds \right),
\]
and
\[
G = C \exp\left( C \int_0^T \|\omega^{\alpha_0}(t)\|_{H^{s+1}} \, dt \right).
\]
By Proposition 2.3, there exists a $\delta > 0$ depending on $T$ and $\int_0^T \|\omega^{\alpha_0}\|_{H^{s+1}} dt$ such that when $0 < \alpha < \frac{1}{2}$ and $|\alpha - \alpha_0| < \delta$,
\[
y(t) \leq C \left( |\alpha_0 - \alpha| |\alpha_0 - \alpha|^1 + |\alpha_0 - \alpha|^1 |\alpha_0 - \alpha| \right) \int_0^T F(t) \, dt,
\]
which implies that
\[
\|\omega\|_{H^s} \leq C \left( |\alpha_0 - \alpha| |\alpha_0 - \alpha|^1 + |\alpha_0 - \alpha|^1 |\alpha_0 - \alpha| \right).
\]
Here $C > 0$ is a constant depending on $T$ and $\int_0^T \|\omega^{\alpha_0}\|_{H^{s+1}} dt$ as well.

Assume that $\omega^{\alpha_0} \in C([0, T_0]; H^{s+1})$, $s > 2$. According to the local well-posedness theory, for $\alpha < \alpha_0$, $\omega^\alpha \in C([0, T_0]; H^{s+1})$ for some $T > 0$. If $T \geq T_0$, the proof is finished. If $T < T_0$, we are going to prove that $T$ can be extended to $T_0$. Denote $T_{\text{max}} > 0$ the maximal existence time satisfying $\omega^\alpha \in C([0, T_{\text{max}}]; H^{s+1})$. By performing the $(s + 1)$-order energy estimate, we get
\[
\frac{1}{2} \frac{d}{dt} \|\omega^\alpha(t)\|_{H^{s+1}}^2 = -\int_{\mathbb{R}^d} \Lambda^{s+1} (u^\alpha \cdot \nabla \omega^\alpha) \Lambda^{s+1} \omega^\alpha \, dx \\
= -\int_{\mathbb{R}^d} \Lambda^{s+1} (u^\alpha \cdot \nabla \omega^\alpha - u^\alpha \cdot \Lambda^{s+1} \nabla \omega^\alpha) \Lambda^{s+1} \omega^\alpha \, dx \\
\leq \|\omega^\alpha\|_{H^{s+1}} \left( \|\Lambda^{s+1} u^\alpha\|_{L^\infty} + \|\nabla \omega^\alpha\|_{L^{\frac{2m}{2m}}} + \|\nabla u^\alpha\|_{L^\infty} \|\omega^\alpha\|_{H^{s+1}} \right) \\
\leq \|\omega^\alpha\|_{H^{s+1}}^2 \left( \|\nabla \omega^\alpha\|_{L^{\frac{2m}{2m}}} + \|\nabla u^\alpha\|_{L^\infty} \right) \\
\leq \|\omega^\alpha\|_{H^{s+1}}^2 \|\omega^\alpha\|_{H^s}.
\]
By the Gronwall inequality, we have
\[
\|\omega^\alpha(t)\|_{H^{s+1}} \leq e^{T_{\text{max}}} \|\omega^\alpha_0(t)\|_{H^s} dt \|\omega^\alpha_0\|_{H^{s+1}} \\
\leq e^{T_{\text{max}}} (\|\omega(t)\|_{H^s} + \|\omega^{\alpha_0}(t)\|_{H^s}) dt \|\omega^\alpha_0\|_{H^{s+1}} \leq C
\]
for \( t \in [0, T_{\text{max}}] \) and hence \( \omega^\alpha(T_{\text{max}}) \) is finite. This deduces a contradiction with \( T_{\text{max}} \) is the maximal existence time by using the local well-posedness theory. In consequence, \( T = T_0 \) as required and the proof of the theorem is finished.

5.2. Proof of Theorem 1.2. Taking the scalar product of (5.1) with \( \overline{\omega} \) in \( H^s \) and using Lemma 2.1 and the Sobolev embedding inequalities enable us to get

\[
\frac{d}{dt} \| \overline{\omega}(t) \|_{H^s} \leq \| \overline{\omega} \|_{H^s} \| u^{\alpha_0} \|_{H^s} + (\| \overline{\omega} \|_{H^s} + \| \omega^\alpha \|_{H^{s+1}}) \| \nabla \|_{H^s}
\]

\[
\leq \| \overline{\omega} \|_{H^s} \| \omega^\alpha \|_{H^s} + (\| \overline{\omega} \|_{H^s} + \| \omega^\alpha \|_{H^{s+1}})(\| \nabla H^s + \| \nabla H^s \|)
\]

(5.13)

Here we have used the decomposition (1.2) with

\[
\overline{u}_I = \nabla^\perp (-\Delta)^{-1+\alpha} \overline{\omega}, \quad \overline{u}_{II} = \left( \nabla^\perp (-\Delta)^{-1+\alpha} - \nabla^\perp (-\Delta)^{-1+\alpha_0} \right) \omega^\alpha.
\]

By using Proposition 3.1 (Remark 3.1) with \( \beta = 1 - 2\alpha \), we obtain

\[
\| \overline{u}_{II}(t) \|_{H^s} \leq C(\| \overline{\omega} \|_{H^s} + (1 - 2\alpha) \| \overline{\omega} \|_{L^1}),
\]

(5.14)

where \( C = C(\alpha, s) \) is an absolutely constant when \( \alpha \to \frac{1}{2} \). By using Proposition 3.1 again (\( \beta = 1 - 2\alpha \) and \( \beta = 0 \) respectively), there also exists a uniformly bounded constant \( C = C(\alpha, s) \) when \( \alpha \to \frac{1}{2} \) such that

\[
\| \overline{u}_{II}(t) \|_{H^s} \leq C(\| \omega^\alpha \|_{H^s} + \| \omega^\alpha \|_{L^1}).
\]

It follows that

\[
\| \overline{\omega} \|_{H^s}(\| \overline{u}_I \|_{H^s} + \| \overline{u}_{II} \|_{H^s}) \leq C(\| \overline{\omega} \|_{H^s}^2 + C(\| \overline{\omega} \|_{H^s}(\| \omega^\alpha \|_{H^s} + \| \omega^\alpha \|_{L^1} + \| \omega^\alpha \|_{L^1})).
\]

(5.15)

Now we adopt to another way to estimate \( \| \overline{u}_{II} \|_{H^s} \) in order to deal with \( \| \omega^\alpha \|_{H^{s+1}} \| \overline{u}_{II} \|_{H^s} \) on the right-hand side of (5.13). The decomposition (5.10) will be applied, which is

\[
J^s \overline{u}_{II} = \int_{\mathbb{R}^2} \left( \frac{(x-y)^\perp}{|x-y|^{2+2\alpha}} - \frac{(x-y)^\perp}{|x-y|^{2+2\alpha}} \right) J^s \omega^\alpha(y) \, dy
\]

\[
= \left( \int_{|x-y| \leq \epsilon} + \int_{|x-y| \geq \epsilon} + \int_{|x-y| \geq 1} \right) \left( \frac{(x-y)^\perp}{|x-y|^{2+2\alpha}} - \frac{(x-y)^\perp}{|x-y|^{2+2\alpha}} \right) J^s \omega^\alpha(y) \, dy
\]

\[
= H_1 + H_2 + H_3,
\]

(5.16)

where \( 0 < \epsilon \leq 1 \) is to be determined later.

Performing the fact that

\[
\int_{|x| = 1} \frac{x^\perp}{|x|^{2+2\alpha}} \, ds = \int_{|x| = 1} \frac{x^\perp}{|x|^5} \, ds = 0,
\]

we get

\[
H_1 = \int_{|x-y| \leq \epsilon} \left( \frac{(x-y)^\perp}{|x-y|^{2+2\alpha}} - \frac{(x-y)^\perp}{|x-y|^{2+2\alpha}} \right) \left( J^s \omega^\alpha(y) - J^s \omega^\alpha(x) \right) \, dy
\]

\[
= \int_{|z| \leq \epsilon} \left( \frac{z^\perp}{|z|^{2+2\alpha}} - \frac{z^\perp}{|z|^3} \right) \left( J^s \omega^\alpha(x-z) - J^s \omega^\alpha(x) \right) \, dz.
\]

From the mean value theorem, we deduce that

\[
|H_1| \leq \int_0^1 \int_{|z| \leq \epsilon} \left( \frac{1}{|z|^{2\alpha}} + \frac{1}{|z|^3} \right) |\nabla J^s \omega^\alpha(x - \tau z)| \, dz \, d\tau.
\]

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Now, taking the $L^2$ norm of the above inequality, and using the fact that the norm of an integral is less that the integral of the norm, we get
\[
\|H_1\|_{L^2} \leq \int_0^1 \int_{|z| \leq \epsilon} \left( \frac{1}{|z|^{2\alpha}} + \frac{1}{|z|^2} \right) \|\nabla J^s \omega^{\alpha_0}(\cdot - \tau z)\|_{L^2} \, dz \, d\tau.
\]
The translation invariance of the Lebesgue measure then ensures that
\[
\|H_1\|_{L^2} \leq C \left( \frac{1}{2 - 2\alpha} \right)^{2-2\alpha} \|J^{s+1} \omega^{\alpha_0}\|_{L^2}. \tag{5.17}
\]
For $2 + 2\alpha \leq \xi \leq 3$, we estimate $H_2$ as follows,
\[
\|H_2\|_{L^2} = \left( \frac{1}{2} - \alpha \right) \left\| \int_{1>|x-y|\geq \epsilon} \frac{(x-y)^+}{|x-y|^{1+2\alpha}} J^s \omega^{\alpha_0}(y) \, dy \right\|_{L^2} \leq \left( \frac{1}{2} - \alpha \right) \left\| J^s \omega^{\alpha_0} \right\|_{L^p} \left( \int_{1>|x-y|\geq \epsilon} \frac{1}{|x-y|^{1+2\alpha}} \, dy \right)^{rac{1}{p}} \tag{5.18}
\]
where the Young inequality and the mean value theorem have been used. Adopting to the similar method to estimate $H_2$, we get
\[
\|H_3\|_{L^2} \leq \left( \frac{1}{2} - \alpha \right) \left\| \int_{|x-y|\geq 1} \frac{1}{|x-y|^{1+2\alpha}} J^s \omega^{\alpha_0}(y) \, dy \right\|_{L^2} \leq \left( \frac{1}{2} - \alpha \right) \left\| J^s \omega^{\alpha_0} \right\|_{L^p} \left( \int_{|x-y|\geq 1} \frac{1}{|x-y|^{1+2\alpha}} \, dy \right)^{rac{1}{p}} , \tag{5.19}
\]
where $\frac{3}{2} = \frac{1}{p} + \frac{1}{q}$, $q > \frac{2}{2+2\alpha + \sigma}$ whence $p < \frac{2}{2+2\alpha + \sigma} < 2$. By the Gagliardo-Nirenberg inequality, we have
\[
\|\Lambda^s \omega^{\alpha_0}\|_{L^p} \leq \|\omega^{\alpha_0}\|_{L^1}^{1-\theta} \|\Lambda^{s+1} \omega^{\alpha_0}\|_{L^2}^\theta ,
\]
where $\theta = 1 - \frac{2}{p_1(p+2)}$, $\frac{1}{p} \leq \frac{1}{p_1} + \frac{\theta}{2}$. Then, we conclude that $p \geq \frac{2(s+1)}{s+2}$. This enables us to choose some $\frac{2(s+1)}{s+2} \leq p < \frac{2}{2+2\alpha + \sigma}$. Combining the estimates (5.17)-(5.19) with (5.16) and choosing $\epsilon = (\frac{1}{2} - \alpha)$, we get
\[
\|\mathcal{W}_1(t)\|_{H^s} \leq CL(\alpha) (\|\omega^{\alpha_0}\|_{H^{s+1}} + \|\omega^{\alpha_0}\|_{L^1}) . \tag{5.20}
\]
Here and what in follow,
\[
L(\alpha) := \left( \frac{1}{2} - \alpha \right)^{2-2\alpha} + \left( \frac{1}{2} - \alpha \right) \left| \log \left( \frac{1}{2} - \alpha \right) \right|^2 + \left( \frac{1}{2} - \alpha \right).
\]
Hence, combining (5.14) and (5.20) yields
\[
\|\omega^{\alpha_0}\|_{H^{s+1}} \|\mathcal{W}_1\|_{H^s} + \|\omega^{\alpha_0}\|_{H^{s+1}} \|\mathcal{W}_1\|_{H^s} \leq \|\omega^{\alpha_0}\|_{H^{s+1}} \|\mathcal{W}\|_{H^s} + \left( \frac{1}{2} - \alpha \right) \left\| \mathcal{W} \right\|_{L^1} \left( \|\omega^{\alpha_0}\|_{H^{s+1}} + \|\omega^{\alpha_0}\|_{L^1} \right) + CL(\alpha) (\|\omega^{\alpha_0}\|_{H^{s+1}}^2 + \|\omega^{\alpha_0}\|_{L^1}^2) \tag{5.21}
\]
Plugging (5.15) and (5.21) into (5.13) gives
\[
\frac{d}{dt} \|\mathcal{W}(t)\|_{H^s} \leq \|\mathcal{W}\|_{H^s} (\|\omega^{\alpha_0}\|_{H^{s+1}} + \|\omega^{\alpha_0}\|_{L^1} + \|\omega^{\alpha_0}\|_{L^1}) + \|\mathcal{W}\|_{H^s}^2 + CL(\alpha) (\|\omega^{\alpha_0}\|_{H^{s+1}}^2 + \|\omega^{\alpha_0}\|_{L^1}^2 + \|\omega^{\alpha_0}\|_{L^1}^2) . \tag{5.22}
\]
Thanks to the incompressible condition that \( \nabla \cdot u = 0 \), it follows that \( \| \omega^\alpha \|_{L^1} + \| \omega^\alpha \|_{L^1} \) is bounded if the initial data \( \omega_0 \in L^1 \). Arguing similarly as in the last part of the proof of Theorem 1.1, we obtain
\[
\| \omega(t) \|_{H^s} \leq C \left( \left( \frac{1}{2} - \alpha \right) + \left( \frac{1}{2} - \alpha \right) \log^2 \left( \frac{1}{2} - \alpha \right) \right).
\]
Moreover, we can prove that \( \omega^\alpha \in C([0, T]; H^{s+1}) \). The proof of the theorem is finished.

5.3. Proof of Theorem 1.3. Similar to the proof Theorem 1.1 it follows from the difference equation (5.1) that (5.2) holds true. The three terms \( I_1, I_2, I_3 \) on the right side of (5.2) will be estimated as follows. Applying the commutator estimates in Lemma 2.1 and the Sobolev embedding inequalities, we immediately have
\[
|I_1| = \left| \int_{\mathbb{R}^2} (J^s(u^{\alpha_0} \cdot \nabla \omega) - u^{\alpha_0} \cdot \nabla J^s \omega) J^s \omega \, dx \right|
\leq \|J^s(u^{\alpha_0} \cdot \nabla \omega) - u^{\alpha_0} \cdot \nabla J^s \omega\|_{L^2} \|J^s \omega\|_{L^2}
\leq (\|J^s u^{\alpha_0}\|_{L^2} \|J^s \omega\|_{L^2} + \|J^s u^{\alpha_0}\|_{L^\infty} \|J^{s-1} \nabla \omega\|_{L^2}) \|J^s \omega\|_{L^2}
\leq \|\omega\|_{H^s}^2 \|J^s u^{\alpha_0}\|_{L^p},
\]
Substituting the decomposition (1.2) into \( I_2 \) and \( I_3 \), respectively, we have
\[
I_2 = - \int_{\mathbb{R}^2} J^s(\overline{\pi}_I \cdot \nabla \omega) J^s \omega \, dx - \int_{\mathbb{R}^2} J^s(\overline{\pi}_{II} \cdot \nabla \omega) J^s \omega \, dx := I_{21} + I_{22};
I_3 = - \int_{\mathbb{R}^2} J^s(\overline{\pi}_I \cdot \nabla \omega^\alpha) J^s \omega \, dx - \int_{\mathbb{R}^2} J^s(\overline{\pi}_{II} \cdot \nabla \omega^\alpha) J^s \omega \, dx := I_{31} + I_{32}.
\]
Choose some \( \sigma \in (0, 2\alpha) \). For any \( p > \frac{2}{2\alpha - \sigma} > 2 \), applying Lemma 2.1 and the Sobolev embedding inequalities again, we obtain
\[
|I_{22}| = \left| \int_{\mathbb{R}^2} (J^s(\overline{\pi}_{II} \cdot \nabla \omega) - \overline{\pi}_{II} \cdot \nabla J^s \omega) J^s \omega \, dx \right|
\leq \|J^s(\overline{\pi}_{II} \cdot \nabla \omega) - \overline{\pi}_{II} \cdot \nabla J^s \omega\|_{L^2} \|J^s \omega\|_{H^s}
\leq (\|J^s \overline{\pi}_{II}\|_{L^p} \|\nabla \omega\|_{L^{2\alpha/\sigma}} + \|J^s \overline{\pi}_{II}\|_{L^\infty} \|J^{s-1} \nabla \omega\|_{L^2}) \|J^s \omega\|_{L^2}
\leq \|\omega\|_{H^s}^2 \|J^s \overline{\pi}_{II}\|_{L^p},
\]
and
\[
|I_{32}| \leq \|J^s(\overline{\pi}_{II} \cdot \nabla \omega^\alpha)\|_{L^2} \|J^s \omega\|_{H^s}
\leq \|\omega\|_{H^s} \|\omega^\alpha\|_{H^{s+1}} \|J^s \overline{\pi}_{II}\|_{L^p}.
\]
Applying Proposition 4.2 gives
\[
|I_{21}| = \left| \int_{\mathbb{R}^2} (J^s(\overline{\pi}_I \cdot \nabla \omega) - \overline{\pi}_I \cdot \nabla J^s \omega) J^s \omega \, dx \right|
\leq \|J^s(\overline{\pi}_I \cdot \nabla \omega) - \overline{\pi}_I \cdot \nabla J^s \omega\|_{L^2} \|J^s \omega\|_{H^s} \leq \|\omega\|_{H^s}^3.
\]
By Proposition 4.1 one has
\[
|I_{31}| \leq \|J^s(\overline{\pi}_I \cdot \nabla \omega^\alpha)\|_{L^2} \|\omega^\alpha\|_{H^s}
\leq \|\omega\|_{H^s}^2 \|\omega^\alpha\|_{H^{s+2}}.
\]
Inserting the estimates of \( I_{21}, I_{22}, I_{31} \) and \( I_{32} \) into (5.24), we arrive at
\[
|I_2| + |I_3| \leq C (\|\omega\|_{H^s}^2 \|J^s \overline{\pi}_{II}\|_{L^p} + \|\omega\|_{H^s} \|\omega^\alpha\|_{H^{s+1}} \|J^s \overline{\pi}_{II}\|_{L^p} + \|\omega\|_{H^s}^3 + \|\omega\|_{H^s}^2 \|\omega^\alpha\|_{H^{s+2}}).
\]
In view of (5.2), (5.23) and (5.25), it deduces

\[
\frac{d}{dt} \| \varphi(t) \|_{H^s} \leq \| \varphi \|_{H^s}(\| u^{\alpha_0} \|_{H^s} + \| \omega^{\alpha_0} \|_{H^{s+2}}) + \| \varphi \|_{H^s}^2 \\
\quad + \| \varphi \|_{H^s} \| J^s \varphi \|_{L^p} + \| \omega^{\alpha_0} \|_{H^{s+1}} \| J^s \varphi \|_{L^p}. \tag{5.26}
\]

Now we estimate \( \| J^s \varphi \|_{L^p} \). Similar to the proof of Theorem 1.2, we use the integral form (5.16). Note that

\[
H_1 = \int_{|x-y| \leq \epsilon} \left( \frac{(x-y)^2}{|x-y|^2 + 2\alpha} - \frac{1}{|x-y|^3} \right) \left( J^s \omega^{\alpha_0}(y) - J^s \omega^{\alpha_0}(x) \right) \, dy.
\]

By the mean value formula and the Hölder inequality, we obtain

\[
\| H_1 \|_{L^p} \leq C \left( \frac{1}{2 - 2\alpha} \epsilon^{2-2\alpha} + \epsilon \right) \| J^{s+1} \omega^{\alpha_0} \|_{L^p} \\
\leq C \left( \frac{1}{2 - 2\alpha} \epsilon^{2-2\alpha} + \epsilon \right) \| \omega^{\alpha_0} \|_{H^{s+2}},
\]

where Lemma 2.2 is used in the last inequality. Similarly, for \( 2 + 2\alpha \leq \gamma \leq 3 \),

\[
\| H_2 \|_{L^p} = \left( \frac{1}{2} - \alpha \right) \left\| \int_{|x-y| \geq \epsilon} \frac{(x-y)^2 (|x-y|^\gamma \log |x-y|)}{|x-y|^{2+2\alpha} |x-y|^{2+2\alpha}} J^s \omega^{\alpha_0}(y) \, dy \right\|_{L^p} \\
\leq \left( \frac{1}{2} - \alpha \right) \left\| \int_{|x-y| \geq \epsilon} \log |x-y| \frac{|J^s \omega^{\alpha_0}(y)|}{|x-y|^{2}} \, dy \right\|_{L^p} \\
\leq \left( \frac{1}{2} - \alpha \right) \| \log \epsilon \|^2 \| J^s \omega^{\alpha_0} \|_{L^p} \\
\leq \left( \frac{1}{2} - \alpha \right) \| \log \epsilon \|^2 \| \omega^{\alpha_0} \|_{H^{s+1}}.
\]

Note that \( \log |x-y| \leq C |x-y|^\sigma \) for \( |x-y| \geq 1 \) and any \( \sigma > 0 \), where \( C \) may depend on \( \sigma \). We can apply the mean value formula and the Young inequality to obtain

\[
\| H_3 \|_{L^p} \leq \left( \frac{1}{2} - \alpha \right) \left\| \int_{|x-y| \geq 1} \frac{\log |x-y|}{|x-y|^{1+2\alpha}} |J^s \omega^{\alpha_0}(y)| \, dy \right\|_{L^p} \\
\leq \left( \frac{1}{2} - \alpha \right) \| J^s \omega^{\alpha_0} \|_{L^p} \left( \int_{|x-y| \geq 1} \left( \frac{\log |x-y|}{|x-y|^{1+2\alpha}} \right)^q \, dy \right)^{\frac{1}{q}} \\
\leq \left( \frac{1}{2} - \alpha \right) \| J^s \omega^{\alpha_0} \|_{L^p} \left( \frac{1}{(1 + 2\alpha - q)q - 2} \right)^{\frac{1}{q}},
\]

where \( \frac{1}{p} + 1 = \frac{1}{q} + \frac{1}{q}, q > \frac{2}{1 + 2\alpha - \sigma}, \ p > \frac{2}{2\alpha - \sigma} \), and we can choose some \( r > 2 \) such that the following embedding

\[
H^{s+1}(\mathbb{R}^2) \hookrightarrow W^{s,r}(\mathbb{R}^2)
\]

holds (see Lemma 2.2). Consequently,

\[
\| J^s \varphi \|_{L^p} \leq \| H_1 \|_{L^p} + \| H_2 \|_{L^p} + \| H_3 \|_{L^p} \\
\leq C \left( \frac{1}{2 - 2\alpha} \epsilon^{2-2\alpha} + \epsilon \right) \| \omega^{\alpha_0} \|_{H^{s+2}} \\
+ \left( \frac{1}{2} - \alpha \right) \| \log \epsilon \|^2 \| \omega^{\alpha_0} \|_{H^{s+1}} + \left( \frac{1}{2} - \alpha \right) \| \omega^{\alpha_0} \|_{H^{s+1}}. \tag{5.27}
\]
Let $\epsilon = \frac{1}{2} - \alpha$. The estimate (5.27) combined with (5.26) yields

$$\frac{d}{dt} \| \omega(t) \|_{H^s} \leq \| \omega(t) \|_{H^s} \left( \| u^\alpha \|_{H^s} + \| \omega^\alpha \|_{H^{s+2}} \right) + \| \omega \|_{H^s}^2 + \| \omega^\alpha \|_{H^{s+2}}^2 \left( \frac{1}{2} - \alpha \right) + \left( \frac{1}{2} - \alpha \right) \log \left( \frac{1}{2} - \alpha \right) \right).$$

Arguing similarly as the last part in the proof of Theorem 1.1, we obtain

$$\| \omega(t) \|_{H^s} \leq C \left( \frac{1}{2} - \alpha \right) + \left( \frac{1}{2} - \alpha \right) \log \left( \frac{1}{2} - \alpha \right).$$

Moreover, we can prove that $\omega^\alpha \in C([0, T]; H^{s+2})$ for any $t \in [0, T]$. The proof of the theorem is finished.

### 5.4.Proof of Corollary 1.4.

Suppose that the result is not true. Then there exists a $M > 0$ and a $\delta_0 > 0$ such that $T^*_{\alpha} \leq M$ for all $\alpha \in (0, \delta_0)$. But it is known that for any $T > 0$, the smooth solution of the Euler equations exists on $[0, T]$. Take $T = M + 1$. According to Theorem 1.1, there exists a $0 < \delta \leq \delta_0$ depending on $T$ such that the smooth solution of the generalized SQG exists on $[0, T]$ as well. This contradicts with the assumption that the maximal existence time $T^*_{\alpha} \leq M$. The proof of the corollary is complete.

### Appendix A. Littlewood-Paley theory and Besov spaces

In this appendix, we introduce Besov spaces which are a generalization of Sobolev spaces. We recall the dyadic decomposition of the unity in the whole space (see e.g. [1, 20]).

**Proposition F.1.** There exists a couple of smooth functions $(\chi, \varphi)$ with values in $[0, 1]$ such that $\text{supp} \chi \subset \{ \xi \in \mathbb{R}^n : |\xi| \leq \frac{4}{3} \}$, $\text{supp} \varphi \subset \{ \xi \in \mathbb{R}^n : \frac{3}{4} \leq |\xi| \leq \frac{3}{2} \}$ and

(i) $\chi(\xi) + \sum_{j \in \mathbb{N}} \varphi(2^{-j} \xi) = 1$, $\forall \xi \in \mathbb{R}^n$,

(ii) $\text{supp} \varphi(2^{-p} \cdot) \cap \text{supp} \varphi(2^{-q} \cdot) = \emptyset$, if $|p - q| \geq 2$,

(iii) $\text{supp} \chi(\cdot) \cap \text{supp} \varphi(2^{-q} \cdot) = \emptyset$, if $q \geq 1$.

For every $u \in S'(\mathbb{R}^n)$, we define the nonhomogeneous Littlewood-Paley operators by

$$\Delta_{-1} u := \chi(D) u \quad \text{and} \quad \Delta_j u := \varphi(2^{-j} D) u \quad \text{for each} \quad j \in \mathbb{N}.$$

We shall also use the following low-frequency cut-off:

$$S_j u := \chi(2^{-j} D) u.$$

It may be easily checked that

$$u = \sum_{j \geq -1} \Delta_j u$$

holds in $S'(\mathbb{R}^n)$.

The following lemma is the well-known Bernstein inequality which has been frequently used in the proof of Proposition 4.1 and Proposition 4.2.
We then have
\[
C^{-(k+1)\lambda} \|u\|_{L^\alpha(\mathbb{R}^n)} \leq \sup \|\partial^\alpha u\|_{L^\alpha(\mathbb{R}^n)} \leq C^{k+1} \lambda^{n+\frac{s}{2}s} \|u\|_{L^\alpha(\mathbb{R}^n)}, \quad \text{supp} \hat{u} \subset \lambda \mathcal{B},
\]

Lemma F.2 (Bernstein’s inequality). Let $\mathbf{B}$ be a ball of $\mathbb{R}^n$, and $\mathbf{C}$ be a ring of $\mathbb{R}^n$. There exists a positive constant $C$ such that for all integer $k \geq 0$, all $1 \leq a \leq b \leq \infty$ and $u \in L^a(\mathbb{R}^n)$, the following estimates are satisfied:

\[
\sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^b(\mathbb{R}^n)} \leq C^{k+1} \lambda^{n+\frac{s}{2}s} \|u\|_{L^a(\mathbb{R}^n)}, \quad \text{supp} \hat{u} \subset \lambda \mathbf{B},
\]

Let us now introduce the basic tool of the paradifferential calculus which is Bony’s decomposition. That is, for two tempered distributions $u$ and $v$,

\[
uv = T_u v + T_v u + R(u, v),
\]

where

\[
T_u v = \sum_j S_{j-1}u \Delta_j v, \quad R(u, v) = \sum_j \Delta_j u \tilde{\Delta}_j v,
\]

where $\Delta_j = \Delta_{j-1} + \Delta_j + \Delta_{j+1}$. In usual, $T_u v$ is called paraproduct of $v$ by $u$ and $R(u, v)$ the remainder term.

Definition F.1. For $s \in \mathbb{R}$, $(p, q) \in [1, +\infty]^2$ and $u \in \mathcal{S}'(\mathbb{R}^n)$, we set

\[
\|u\|_{B^s_{p,q}(\mathbb{R}^n)} := \left( \sum_{j \geq -1} 2^{jsq} \|\Delta_j u\|_{L^p(\mathbb{R}^n)}^q \right)^{\frac{1}{q}} \quad \text{if} \quad q < +\infty
\]

and

\[
\|u\|_{B^s_{p,\infty}(\mathbb{R}^n)} := \sup_{j \geq -1} 2^{js} \|\Delta_j u\|_{L^p(\mathbb{R}^n)}.
\]

Then we define inhomogeneous Besov spaces as

\[
B^s_{p,q}(\mathbb{R}^n) := \{ u \in \mathcal{S}'(\mathbb{R}^n) : \|u\|_{B^s_{p,q}(\mathbb{R}^n)} < +\infty \}.
\]

It should be remarked that the usual Sobolev spaces $H^s(\mathbb{R}^n)$ coincide with the inhomogeneous Besov spaces $B^s_{2,2}(\mathbb{R}^n)$. Also, by using Definition F.1, we get easily for any $s > 2$ and $\alpha < \frac{1}{2}$, the following embeds hold

\[
H^s(\mathbb{R}^n) \hookrightarrow B^{1+2\alpha}_{2,1}(\mathbb{R}^n) \hookrightarrow H^{1+2\alpha}(\mathbb{R}^n). \tag{D.1}
\]

Lastly, we turn to review two useful lemmas which have been used in foregoing sections.

Lemma F.3. Let $C'$ be an annulus of $\mathbb{R}^n$, $s$ be a real number, and $[p, r] \in [1, \infty]^2$. Assume $\{u_j\}_{j \geq -1}$ be a sequence of smooth functions such that

\[
\text{supp} \hat{u}_j \subset 2^j C' \quad \text{and} \quad \left\{2^{js} \|u_j\|_{L^p(\mathbb{R}^n)} \right\}_{j \geq -1} \|_{\ell^r} < \infty.
\]

We then have

\[
u := \sum_{j \geq -1} u_j \in B^s_{p, r}(\mathbb{R}^n) \quad \text{and} \quad \|u\|_{B^s_{p, r}(\mathbb{R}^n)} \leq C_s \left\{2^{js} \|u_j\|_{L^p(\mathbb{R}^n)} \right\}_{j \geq -1} \|_{\ell^r}.
\]

Lemma F.4. Let $\mathbf{B}'$ be a ball of $\mathbb{R}^n$, $s > 0$ be a real number, and $[p, r] \in [1, \infty]^2$. Let $\{u_j\}_{j \geq -1}$ be a sequence of smooth functions such that

\[
\text{supp} \hat{u}_j \subset 2^j \mathbf{B}' \quad \text{and} \quad \left\{2^{js} \|u_j\|_{L^p(\mathbb{R}^n)} \right\}_{j \geq -1} \|_{\ell^r} < \infty.
\]

We then have

\[
u := \sum_{j \geq -1} u_j \in B^s_{p, r}(\mathbb{R}^n) \quad \text{and} \quad \|u\|_{B^s_{p, r}(\mathbb{R}^n)} \leq C_s \left\{2^{js} \|u_j\|_{L^p(\mathbb{R}^n)} \right\}_{j \geq -1} \|_{\ell^r}.
\]
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References

[1] H. Bahouri, J.-Y. Chemin, R. Danchin, Fourier Analysis and Nonlinear Partial Differential Equations, Grundlehren Math. Wiss. vol. 343, Springer, 2011.

[2] L. Caffarelli, A. Vasseur, Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation, Ann. of Math. 171 (2010), no. 3, 1903-1930.

[3] Q. Chen, C. Miao, Z. Zhang, A new Bernstein's inequality and the 2D dissipative Quasi-Geostrophic equation, Comm. Math. Phys., 271(2014), 821-838.

[4] P. Constantin, Note on loss of regularity for solutions of the 3D incompressible Euler and related equations, Comm. Math. Phys. 104 (1986), 311-326.

[5] P. Constantin, G. Iyer, J. Wu, Global regularity for a modified critical dissipative quasi-geostrophic equation, Indiana Univ. Math. J. 57 (2008), 2681-2692.

[6] P. Constantin, A. Majda, E. Tabak, Formation of Strong fronts in the 2D quasi-geostrophic thermal active scalar, Nonlinearity, 7(1994), 1495-1533.

[7] P. Constantin, V. Vicol, Nonlinear maximum principles for dissipative linear nonlocal operators and applications, Geom. Funct. Anal. 22(2012), 1289-1321.

[8] P. Constantin, J. Wu, Behavior of solutions of 2D quasi-geostrophic equations, SIAM J. Math. Anal., 30(1999), 937-948.

[9] D. Cordoba, Noneexistence of simple hyperbolic blow-up for the quasi-geostrophic equation, Ann. of Math., 148(1998), 1135-1152.

[10] D. Cordoba, C. Fefferman, Growth of solutions for QG and 2D Euler equations, Journal of AMS, 15 (2002), 665-670.

[11] M. Dabkowski, A. Kiselev, L. Silvestre, V. Vicol, Global well-posedness of slightly supercritical active scalar equations, Anal. PDE, 7(2014), no. 1, 43-72.

[12] T. Hmidi, S. Keraani, Global solutions of the supercritical 2D dissipative quasi-geotrophic equation, Adv. Math., 214(2007), 618-638.

[13] N. Ju, Existence and uniqueness of the solution to the dissipative 2D quasi-geostrophic equations in the Sobolev space, Comm. Math. Phys., 251(2004), 365-376.

[14] C.E. Kenig, G. Ponce, L.Vega, Well-posedness of the initial value problem for the Kortewegde Vries equation, J. Amer. Math. Soc. 4(1991), 323-347.

[15] A. Kiselev, F. Nazarov, A variation on a theme of Caffarelli and Vasseur, Zap. Nauchn. Sem. POMI 370 (2010), 58-72.

[16] A. Kiselev, F. Nazarov, A. Volberg, Global well-posedness for the critical 2D dissipative quasi-geostrophic equation, Invent. Math. 167 (2007), 445-453.

[17] A. Kiselev, Y. Yao, A. Zlatoš, Local Regularity for the modified SQG patch equation, preprint, 2015.

[18] A. Kiselev, L. Ryzhik, Y. Yao, A. Zlatoš, Finite time singularity for the modified SQG patch equation, Ann. of Math., 184(2016), no.3, 909-948.

[19] A. Majda, A. Bertozzi, Vorticity and Incompressible Flow, Cambridge University Press, Cambridge, UK, 2002.

[20] C. Miao, J. Wu, Z. Zhang, Littlewood-Paley Theory and Applications to Fluid Dynamics Equations, Monographs on Modern Pure Mathematics, No. 142, Beijing: Science Press, 2012.

[21] J. Pedlosky, Geophysical Fluid Dynamics, Springer, New York, 1987.

[22] E. M. Stein, Singular integrals and differentiability properties of functions, Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.

[23] J. Wu, Global solutions of the 2D dissipative quasi-geostrophic equation in Besov spaces, SIAM J. Math. Anal., 36(2004), 1014-1030.
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