Since the early days of General Relativity (GR) there has been attempts to modify gravity at long distances. The problem is not just a theoretical challenge, but it could bring to important phenomenological consequences. The only ghost free massive deformation of linearized GR in the Minkowski background was given in [1] by Pauli and Fierz (PF). The peculiarity of gravity became apparent when it was realized that in the zero mass limit the PF theory was discontinuous [2] due the presence of an extra polarization state. The mystery of massive gravity deepened when it was reexamined in the framework of effective field theory realizing that the propagation of five degrees of freedom is spoiled by interactions and a sixth ghost-like state turns on [3]. The problem was reexamined in the framework of effective field theory realizing that the reason behind the misbehavior of PF massive gravity was strong coupling [4]. It remains an open question whether a well behaved non-linear extension of the PF theory exists (see for instance [6]) but Lorentz invariance clearly plays a crucial restrictive role. If one gives up Lorentz invariance a trouble-free massive deformation of gravity where the strong coupling and discontinuity issues are disentangled can be found [7, 8].

In this letter we describe a mechanism of spontaneous lorentz breaking that provides mass to the graviton in a consistent way.

Let us consider a theory with two dynamical metrics $g_{1\mu\nu}$, $g_{2\mu\nu}$ [15] each interacting with its own matter, a so called bigravity theory. The action contains two Einstein-Hilbert (EH) terms, and a mixed term [5]:

$$S = \int d^4x \left[ \sqrt{g_1} \left( M_1^2 R_1 + L_1 \right) + \sqrt{g_2} \left( M_2^2 R_2 + L_2 \right) - 4(g_1 g_2)^{1/4} V(g_1, g_2) \right],$$

(1)

where $L_{1,2}$ are the corresponding matter lagrangians.

The mixed term $V$ contains only non-derivative couplings between two metrics, therefore it can only be function of the tensor $X_{\mu}^{\alpha} = g_{1\nu}^{\alpha} g_{2\nu\mu} [9]$. The cosmological terms can be included in $V$, e.g. $V_{\Lambda_1} = \Lambda_1 g^{-1/4}$, with $g = \det X = (g_1 g_2)^{1/4}$. The equations of motion (EoM) are

$$M_1^2 E_{\mu}^{\nu} + q^{1/4} (V \delta_{\mu}^{\nu} - 4V_{\alpha}^{\nu} X_{\mu}^{\alpha}) = \frac{1}{2} T_{\mu}^{\nu},$$

(2)

$$M_2^2 E_{2\mu}^{\nu} + q^{1/4} (V \delta_{\mu}^{\nu} + 4V_{\alpha}^{\nu} X_{\mu}^{\alpha}) = \frac{1}{2} T_{2\mu}^{\nu},$$

where $V_{\alpha}^{\mu}$ is the derivative of $V$ with respect to $X_{\mu}^{\alpha}$.

The indices of the two equations are raised/lowered with the corresponding metrics.

The action is invariant under a generic infinitesimal diffeomorphism (diff) generated by $\xi^\mu$:

$$\delta g_{\mu\nu} = \partial_\mu \xi^\alpha g_{\alpha\nu} + \partial_\nu \xi^\alpha g_{\alpha\mu} + \xi^\alpha \partial_\mu g_{\alpha\nu},$$

(3)

with $\alpha = 1, 2$. Notice that in the absence of $V$ the system has a larger gauge symmetry: one can use different $\xi_1^\mu$ and $\xi_2^\mu$ for $g_1$ and $g_2$. The “diagonal” diff invariance is encoded in a set of generalized Bianchi identities.

The two metrics can be diagonalized simultaneously, but in general their eigenvalues will not be proportional, and thus local Lorentz invariance will be broken. For vacuum solutions, we will assume that rotational invariance is preserved and that the two metrics have the same signature.

**The vacuum.** The EoM (2) always admit constant curvature solutions for both $g_1$ and $g_2$, and in addition curvatures are proportional [10]. For simplicity here we focus only on the flat limit of those solutions, for which one fine tuning on $V$ is necessary, as in standard GR for setting the cosmological term to zero. In this biflat case, the EoM are simply

$$\bar{V} = 0, \quad \bar{V}_{\mu}^{\nu} = 0,$$

(4)

where the bar stands for the background values.

For rotationally invariant backgrounds, these are three independent equations. One of the equations corresponds to the mentioned fine-tuning for biflat backgrounds. Then assuming that we live in sector 1 we can set

$$g_{1\mu\nu} = \text{diag}(-1, 1, 1),$$

$$\bar{g}_{2\mu\nu} = \omega^2 \text{diag}(-c^2, 1, 1, 1).$$

(5)

Thus, the two remaining equations determine the constants $c$ and $\omega$ for any given $V$. Physically $c$ is the speed of light in sector 2, while $\omega$ parametrizes the relative conformal factor. However, a solution with $c = 1$ is always present, since in this case two equations coincide (and determine $\omega$).

Summarizing, we have two branches of solutions: Lorentz Invariant (LI) for $c = 1$, and Lorentz Breaking (LB) for $c \neq 1$ [16].

The LB branch is of particular interest since it naturally allows for consistent massive deformations of gravity.
Linearized analysis. Let us consider perturbations around the background (5), defined as $g_{1\mu\nu} = \bar{g}_{1\mu\nu} + h_{1\mu\nu}$, and $g_{2\mu\nu} = g_{2\mu\nu} + \omega^2 h_{2\mu\nu}$. The diffeomorphisms (3) act at lowest order as $\delta h_{1\mu\nu} = 2\bar{g}_{1\alpha\beta} \partial_{\mu} \xi^\alpha \partial_{\nu} \xi^\beta$ and $\delta h_{2\mu\nu} = 2\omega^2 \bar{g}_{2\alpha\beta} \partial_{\mu} \partial_{\nu} \xi^\alpha$. Since the background preserves rotations, we decompose both perturbations according to their spin content:

$$h_{a0} = \psi_a, \quad h_{a0i} = u_{ai} + \partial_i \psi_a,$$

$$h_{a\beta} = \chi_{a\beta} + \partial_i S_{a\beta} + \partial_j S_{ai} + \partial_j \sigma_{ai} + \delta_{ij} \tau_a,$$  
(6)

with $\partial_i u_{ai} = \partial_j S_{a\beta} = \partial_j \chi_{a\beta} = \delta_{ij} \chi_{a\beta} = 0$. We have thus 1+1 gauge invariant tensors, 2+2 vectors, 4+4 scalars. Also the 4 diagonal diffs can be split into one vector $\xi^i$ and 2 scalars $\xi^0, \partial_i \xi^0$. As a result, we expect that 2 tensors, 3 vectors and 6 scalars are physically states determined by the EoM, while the remaining vector and 2 scalars can be gauged away.

At quadratic level, in general the Lagrangian has the form $L = L_{\text{kin}} + L_{\text{mass}} + L_{\text{source}},$ where $L_{\text{kin}}$ contains derivative terms (in space and time) emerging from the EH actions, $L_{\text{mass}}$ comes the the quadratic expansion of the mixed term, while $L_{\text{source}}$ describes the gravitational coupling with matter. In field space, each field is a 2-component column vector, and it is convenient to define the following 2×2 matrices: $C = \text{diag}(1, c), M^2 = M_0^2 \text{diag}(1, \kappa)$ where $\kappa = M_0^2 / M_0^2 \omega^2 c$. Then the kinetic term has the compact form

$$L_{\text{kin}} = \frac{1}{4} \bar{X} \chi_{ij} M^2 \left( C^2 \Delta - \partial^2 \right) \chi_{ij} - \frac{1}{2} W_i^t M^2 \Delta W_i +$$

$$+ \frac{1}{2} \left[ 2 \Phi M^2 \Delta \tau - \tau^t M^2 \left( C^2 \Delta - 3 \partial^2 \right) \tau \right],$$  
(7)

where $W_i = u_i - \partial_i S_i$, $C = \psi - 2 \partial_i v + \partial^2 \sigma$. Notice that $\chi_{ij}, W_i, \Phi, \tau$ are all gauge invariant fields making the invariance of $L_{\text{kin}}$ manifest. When the matter energy-momentum tensors are conserved also the source lagrangian is expressed in terms of gauge invariant fields:

$$L_{\text{source}} = \frac{1}{2} \left( - T_{0i} C \chi_{ij} + 2 T_{00} \Omega C^{-1} W_i + - T_{ii} C \tau - T_{00} C^{-3} \Phi \right).$$  
(8)

The crucial term, $L_{\text{mass}}$, in the flat limit stems from the expansion of $V$. For the rotational invariant biflat background (5), for which $V = \bar{V} = 0$, it reads

$$L_{\text{mass}} = -2 \left( \bar{g}_{1\mu\nu} \bar{g}_{2\alpha\beta} \right)^{1/4} \text{Tr} \left( X_1 \bar{V}^\prime X_1 \right)$$

$$= \frac{1}{4} \left( M^{ab} h_{00} h_{00} + 2 M^{ab} h_{00} h_{00} - M^{ab} h_{ij} h_{ij} + M^{ab} h_{ij} h_{ij} - 2 M^{ab} h_{ij} h_{ij} \right),$$  
(9)

where $X_1 = \bar{X} \bar{g}_{2\mu\nu}^{-1/2} h^{-1/2} b_1 \bar{X}$ is the fluctuation of $X$. The second line defines the Lorentz breaking masses $M$ [7] that can be computed explicitly once a $V$ and a consistent background are given. Their pattern is drastically different in the two branches.

Gauge invariance (3) gives crucial constraints:

$$M_{0,1} \left( \frac{1}{c^2} \right) = 0, \quad M_{1,2,3,4} \left( \frac{1}{1} \right) = 0,$$

$$M_{4} \left( \frac{1}{c^2} \right),$$  
(10)

In the LI phase, $c \neq 1$, this implies that $M_1 = 0$ and $M_0 = \lambda_0 \left( \frac{1}{1} - \frac{1}{c^2} \right)$. $M_{4} = \lambda_4 \left( \frac{1}{1} - \frac{1}{c^2} \right)$.

In the LI phase instead $c = 1$ and two conditions in (10) coincide, thus allowing a non-vanishing $M_1$. Also, all the $M$’s are proportional to the same projector (11). Moreover, in this branch the mass term reduces to the generalized PF form: $L_{\text{mass}} = \gamma^{\mu \nu} \eta^{\alpha \beta} h_{\mu \nu} A_{\alpha \beta} + h^{\Omega} B h$ ($A$ and $B$ being 2×2 matrices). This is equivalent to $M_0 = A + B, M_{1,2} = -A, M_{3,4} = B$. We remark that the limit $c \rightarrow 1$ is discontinuous.

Let us analyze in detail the linearized theory, separately in the two branches.

Lorentz invariant phase. Since here all the mass matrices are proportional to the same projector with a null eigenvalue, the linearized theory is diagonalized by the mass eigenstates

$$h_{\mu \nu} = \cos \theta h_{\mu \nu}^{c1} + \sin \theta h_{\mu \nu}^{c2},$$

$$h_{\mu \nu}^{c-} = \sin \theta h_{\mu \nu}^{c1} - \cos \theta h_{\mu \nu}^{c2},$$  
(12)

where $h^{c1,2}$ are the canonically normalized fields and the mixing angle is $\sin \theta = (1 + c^2)^{-1/2}$. The state $h^{c+}$ is massless, while $h^{c-}$ has a generalized PF mass term.

The massive sector $h^{c-}$ will be plagued by the known problems related to the Pauli-Fierz case. In the general case, with $A \neq -B$, it contains ghosts [17]. If we remove the ghosts setting $A = -B$, we face the vDVZ discontinuity problem [2]. In this theory however, since the original gravitons have different Planck scales, the discontinuity in our sector can be controlled by the mixing angle. The present bounds [13] would require $M_2 \sim 30 M_1$. In addition the Newton constant becomes distance dependent: it changes by a factor of $1 + 4 \tan^2 \theta / 3$ from short to large distance, the critical distance being the inverse graviton mass. The matter of type 2, if it exists, will interact with ours in distance dependent way [14].

Finally, the sixth mode [3] of $h^{c-}$ will manifest once interactions are considered, and in general will propagate in both gravities, leading to strong coupling [4].

Lorentz breaking phase. We give an overall analysis of the static and propagating modes in the LB branch, and refer to table I for the full summary of all phases; details will be written in [10].

a. Tensors

Each tensor has two independent components. They propagate according to the EoM:

$$\left[ \begin{array}{cc} (k_0^2 - \vec{k}^2) & 0 \\ 0 & \kappa (k_0^2 - c^2 \vec{k}^2) \end{array} \right] \chi_{ij} = 0, \quad (13)$$

describing two gravitons with different "speeds of light". The gravitons are mixed by the non-diagonal mass matrix $M_2$, and thus oscillate and have a non-linear dispersion relation. Since $M_2$ has a zero eigenvalue, expanding the dispersion relation in powers of $k^2$ one finds at low energy a massless graviton that...
travels with speed $v^2 = (1 + c^2\kappa)/(1 + \kappa)$, and a massive one, of mass $m^2 = (1 + \kappa^{-1})\lambda_3/M_2^2$. In the high energy limit two states propagate with different speeds and also oscillate.

Interestingly, when $c > 1$, the second graviton propagates faster than "our" light, though this will not lead to causality violations. Indeed, in the coordinates where the metrics are (5), the Cauchy problem is globally well posed in terms of the preferred time $t$. In principle this scenario could be tested by observing the time of flight difference between gravitational waves and optical signals, or frame dependence of the gravitational waves propagation that would provide the evidence for a preferred frame.

b. Vectors Thanks to $M_1 = 0$, the vector states do not propagate. The EoM however determine three static potentials: the two $W_{i1}$, $W_{2i}$ that couple to the respective $T_{0i}$’s in standard GR, without mixing, and the combination $S_{1i} = (S_{1i} - S_{2i})$, that is zero.

c. Scalars Also scalar modes do not propagate, again because of $M_1 = 0$; they however mediate the static potentials. By defining $\lambda_3 = \lambda_2 + \lambda_0(\lambda_2 - \lambda_3)$, $\lambda_\mu = 3\lambda_3^2 + \lambda_0(\lambda_2 - 3\lambda_3)$ and $T_{\mu\nu} = (T_{100} - T_{200}/c^4\omega^2\kappa)$, the solutions of the EoM are

$$\Phi_1 = \frac{1}{2M_2^2} \left( T_{000} + T_{111} - 3T_{100} \right) + \frac{1}{M_2^2} \mu^2 T_{000},$$
$$\Phi_2 = \frac{c^{-1}}{2M_2^2} \left( T_{200} + c^2 T_{222} - 3T_{200} \right) - \frac{1}{M_2^2} \mu^2 T_{000},$$
$$\tau_1 = \frac{1}{2M_2^2} T_{000}, \quad \tau_2 = \frac{c^{-3}}{2M_2^2} T_{200},$$

where

$$\mu^2 = \frac{\lambda_2 - \lambda_0}{2M_2^2} \equiv \frac{\lambda_2 - \lambda_0(3\lambda_3 - \lambda_2)}{2M_2^2}. \quad (15)$$

The two remaining gauge invariant fields, $\psi_\pm \equiv (c^2\psi_1 - \psi_2)$ and $\sigma_\pm \equiv (\sigma_1 - \sigma_2)$ are determined in terms of $\tau_\pm \equiv (\tau_1 - \tau_2)$. Since they have no source, the explicit expression is omitted. From $\Phi_1$ in (14), $M_1$ is identified as the Planck mass, $M_2^2 = \pi G^{-1}$.

Since in this phase only gravitons propagate, strong coupling problems [4] are absent.

**Linear term.** In (14) we recognize the standard GR potentials plus, in $\Phi_{1,2}$, a term proportional to $1/\Delta^2$, that represents a linearly growing potential at large distances [11]. For generic $\lambda_0, 2, 3, 4$, the scale at which this force sets out is proportional to $\mu$, (15). Since this linear growth signals the breakdown of perturbation theory at large distances from sources, it is interesting to study the conditions under which $\mu$ vanishes. One, cheap, possibility is to have massless gravitons, $\lambda_2 = 0$: the other possibility is that the numerator vanishes, $\mu_\mu = 3\lambda_3^2 - \lambda_0(3\lambda_3 - \lambda_2) = 0$. The latter condition can be understood as a symmetry requirement on the potential $V$ by expressing it as a function of the fluctuation $X_1$: the most general potential preserving rotations, with vanishing $V$ and $V'$, can be written directly in terms of the lorentz breaking masses $\lambda_1$:

$$V = \lambda_1 tr[X_1 P_1 t^2] - 2\lambda_2 tr[X_1 P_1 X_1 P_1] + \lambda_3 tr[X_1 P_2 t^2] + + 2\lambda_4 tr[X_1 P_1 t] tr[X_1 P_1] + 2\lambda_5 tr[X_1 P_2 X_1 P_1] + \cdots (16)$$

with $P_i = \text{diag}(1/c^2, 0, 0, 0), P_s = \text{diag}(0, 1, 1, 1)$. In the LB phase the last term is absent. Requiring $\mu_\mu = 0$, we find that the potential is invariant under

$$\delta X_1 = \epsilon \delta X \leftrightarrow X \rightarrow (1 + \epsilon)X,$$  \quad (17)

that shifts only the scalar fields $\psi_+$ and $\sigma_-$. This transformation matches the one encountered for goldstone fields in the effective description of [11].

However, if we further restrict $3\lambda_4 = -\lambda_0$ ($\gamma = 1$ in [11]) we find the “anti-diagonal” Weyl transformation

$$\text{Weyl}_- : \delta X_1 = \epsilon X \leftrightarrow X \rightarrow (1 + \epsilon)X,$$

generated by two opposite rescalings of $g_1$ and $g_2$. We have thus an explicit non-perturbative symmetry protecting $\mu = 0$, that gives in addition $\lambda_0 = -3\lambda_4$. Still the graviton mass $\lambda_2$ can be nonzero and arbitrary. One example of a full potential invariant under Weyl_-

$$V = a_0 + a_1 tr[X] tr[X^{-1}] + a_2 tr[X]^2 tr[X^{-2}].$$
In the LI phase also there are interesting consequences: WeyL makes all masses proportional, so that one cannot get rid of ghosts (A + B = 0) without setting all masses to zero. This suggests that the LB phase is the only physical one with massive gravitons.

It is interesting to note that μ = 0 even when the potential is homogeneous of a generic degree α ≠ 0 under WeyL: V(ΛX) = λαV(X). Interestingly enough, for ansätze of constant curvatures this symmetry leads to a constraint on the curvatures as a function of α, with possible phenomenological consequences. For instance, potentials of degree ±1 lead to vanishing curvature for g1,2. This opens up even the possibility to investigate the stability of the LB phase under quantum corrections effects from matter [10].

**Phenomenology.** First of all it is clear that since the lagrangian of our matter Λ1 is defined only with metric g1, no lorentz breaking effects can be observed in its propagation and interactions, and the Weak Equivalence Principle is satisfied [18]. The same happens also for matter 2, though the speeds of light in two sectors are different. Due to the interaction between metrics, the Strong Equivalence Principle is violated in both sectors.

In the weak-field regime, besides the newtonian interaction α/ r we have an additional term that grows linearly with r. Moreover, through this term the two kinds of matter see each other. For instance, an ordinary source M will produce potentials in both sectors, $\Phi_1 = -GM(1/r + \mu^2r)$ and $\Phi_2 = +k_0c^2GM\mu^2r$. Notice that the effect of the linear term is opposite in the two sectors.

Of course if μ = 0 as discussed above this effect is absent: both potentials are newtonian, without mixing between the two sectors. On the other hand if μ ≠ 0 the linear term generates a constant acceleration in direction of the source, attractive or repulsive depending on the sign of $\mu^2$ [11]. (Amusingly, this constant acceleration can explain the Pioneer anomaly, $a \sim 10^{-9}m/s^2$, for μ ~ $10^{-21}eV$).

The linearized solution can be trusted only from the Schwarzschild radius $r_{uv} = GM$ up to distances where the linear term drives Φ to be ~ 1; for instance, with matter of type 1, $r_{1R} = (GM\mu^2)^{-1}$. In particular for the sun $r_{1R} \sim 10^{12}pc \times (\mu/10^{-21}eV)^{-2}$, and clearly the linearized approximation works fine for μ ~ $10^{-21}eV$; on the other hand, taking the galaxy as the source, $r_{1R}$ drops well below the galactic radius. In such a situation a non-perturbative solution is needed [10]. We just observe here that since the curvature of the linear term vanishes as 1/r, one can hope to match with a well-behaved solution at infinity.

Pulsar binary systems (BPS) set stringent limits on the rate of gravitational waves (GWs) emission [13]. In our theory the main effect is due to the fact that matter in sector 1 emits a combination of graviton mass eigenstates, and the massive one is forbidden if the energy is too low. The emission rate will thus be modified only if $m_g$ is higher than the GW energy [19]. For BPS a rough limit is $m_g < 10^{-20}eV$. However the BPS analysis involves the periastron advance rate [13] that is expected to be modified as well in our case, therefore a complete study is needed. Even if $m_g$ is below such limit, there could be observable effects in GW detection: as mentioned, the two graviton states have different speeds, therefore it would be possible to observe a delay or anticipation of the GWs with respect to optical signals coming from the same source. Finally, supposing that also type-2 matter can be a source of GWs, due to oscillations one could see GW signals in our detectors not associated with any otherwise visible sources like neutron stars etc.

We conclude that in this theory the spontaneous breaking of lorentz, required to obtain an healthy massive gravity theory, turns into an rich phenomenology, namely massive, oscillating and lorentz breaking gravitational waves.

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[16] The special case when $V = V(det X)$ is exactly solvable: Bianchi identities force det X to be constant, and there are additional gauge symmetries that allow to set c = 1. As a result both branches are equivalent [10].
[17] Ghosts are absent in the LI phase also with $A = 0$, but the running theory is rather boring: there are additional gauge symmetries and the spin 2 states stay massless and decoupled.
[18] This would not be the case if one admits terms that couple our matter directly to $g_2$.
[19] $m_g$ is linked to μ by order one factors, $m_g \sim \mu$, when the X’s are all of the same order.