Automodel solutions for Lévy flight-based transport on a uniform background

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Abstract

A wide class of non-stationary superdiffusive transport on a uniform background with a power-law decay at large distances of the step-length probability distribution function (PDF) is shown to possess an approximate automodel solution. The solution for the Green’s function is constructed using the scaling laws for the propagation front (relevant-to-superdiffusion average displacement) and asymptotic solutions far beyond and far in advance of the propagation front. These scaling laws are determined essentially by the long-free-path carriers (Lévy flights). The validity of the suggested automodel solution is proved by its comparison with numerical solutions in the one-dimensional (1D) case of the transport equation with a simple long-tailed PDF with various power-law exponents and in the 3D case of the Biberman–Holstein equation of the resonance radiation transfer for various (Doppler, Lorentz, Voigt and Holtsmark) spectral line shapes.

Keywords: Lévy flights, automodel solution, superdiffusive transport, Green’s function

(Some figures may appear in colour only in the online journal)

1. Introduction

The self-similarity of random walks is expressed in the fact that the space-time evolution of the transport on a uniform background from an instant point source (i.e. Green’s function) is a function of a single variable. In the case of normal (or ordinary) diffusion, defined as the Brownian motion described by the differential equation of the Fokker–Planck type, the Green’s function is a Gaussian whose argument determines the scaling law for the propagation front, $r_{fr} \sim (Dt)^{1/2}$, where $D$ is the diffusion coefficient.
In a broader frame, beyond the Brownian motion, the diversity of the free-path lengths (step lengths) provided by the long-tailed, power-law probability distribution function (PDF) leads to a superdiffusive transport (see, e.g. [1–4]). In this case the dominant contribution to the transport comes from the long-free-path carriers (named, by Mandelbrot [5], Lévy flights, see p IX in [1]). In various physics problems, superdiffusion was qualified/named as a non-local transport which is described by an integral equation, in space variables, which is non-reducible to a differential one.

This is the case, e.g. for the Biberman–Holstein equation [6, 7] which is derived from a couple of differential kinetic equations for photons and atoms, or ions, and describes the space-time evolution of the density of excited atoms/ions produced by the radiative transfer in the spectral lines in gases and plasmas. The Biberman–Holstein equation assumes complete redistribution over frequency (within spectral line width) in the elementary act of the resonance scattering (i.e. absorption and subsequent emission) of the photon by an atom/ion. It is the loss of memory by the photon during its trapping by the atom/ion that makes the process a Markovian one. The respective value of $D$ in the differential equation, derived by a proper expansion/simplification of the integral equation, explicitly depends on the size, $L$, of a finite medium and tends to infinity with $L \to \infty$ (see, e.g. [8, 9]). The latter makes the very concept of diffusivity irrelevant to such a mechanism of transport (however, the term diffusion is sometimes applied to such phenomena). The respective mean squared displacement in an infinite medium also diverges so that the propagation front should be defined in a way relevant to superdiffusion [10] (see also [8, 9]). The role of long-free-path photons, i.e. those emitted in the far wings of the spectral line shape, has been recognized [11, 12] and was the basis of the escape probability (EP) approaches [13, 14]. We can also mention an extension of the Biberman–Holstein approach from the line to continuous spectra: the EP method for non-stationary heat transport by the longitudinal (electron Bernstein) waves in plasmas [15], which extended the integral equation approach to the similar steady-state transport [16], and the EP approach to the transport by plasma waves [17], which extended/modified the approach to the transport by electron cyclotron waves in nuclear fusion plasmas [18]. Despite the fact that the term ‘Lévy flights’ was not used in [6] to [18], the dominant mechanism of the transport considered in these papers actually gets to this category.

As an alternative to the Biberman–Holstein equation widely used in laboratory plasmas, in astrophysics the above-mentioned couple of differential kinetic equations for photons and atoms/ions is often reduced to an integral equation, in space variables, for radiation intensity (see, e.g. [19, 20]). Here the role of Lévy flights is identified, as a rule without naming, as well.

Now the situation in the literature appears to be ambiguous. On the one hand, in [21] it was claimed that ‘... to date, it has not seemed possible to observe and study Lévy transport in actual materials. For example, experimental work on heat, sound, and light diffusion is generally limited to normal, brownian, diffusion’. On the other hand, the role of Lévy flight for light in the conventional frames of radiative transfer in spectral lines is identified, e.g. in [22]. Here the multiple scattering of near-resonant light in a Doppler-broadened hot atomic vapor experimentally proved theoretical analysis [23] where the photon trajectories in the Biberman–Holstein radiative transfer for Doppler, Lorentz and Voigt line shapes were shown to be Lévy flights.

In recent decades, the original version of superdiffusion in the formalism of fractional order derivatives went far beyond pure mathematics and now is widely used in many applications (see, e.g. [1, 2, 24]). In this formalism the scaling laws (i.e. self-similarity of solutions) are derived more easily than in the case of integral equations; however, derivation of an exact solution is often as difficult as in the integral equation formalisms.
In the present paper we formulate a method for derivation of an approximate automodel solution for a wide class of non-stationary superdiffusive transport on a uniform background in the frame of integral equation formalism. The method allows the derivation of the solution in the form which relies on the scaling law for the propagation front, defined as a relevant-to-superdiffusion average displacement. The algorithm of constructing such a solution uses also the asymptotic behavior of the Green’s function far beyond and far in advance of the propagation front. All the building blocks of the model are shown to be determined by the long-free-path carriers in the power-law tail of the PDF (i.e. by Lévy flights). The validity of the suggested automodel solution is proved by its comparison with the numerical solutions of transport equations. First, the main principles of the method are presented for the one-dimensional (1D) transport with a simple long-tailed PDF (section 2). A test of the proposed automodel solution is carried out via comparison with a direct numerical solution for various power-law exponents (section 3). A 3D transport is addressed in the case of the Biberman–Holstein equation with various spectral line shapes (section 4). A test of the general automodel solution is made via comparison with analytic solutions by Veklenko [10] for the Green’s function for the Doppler and Lorenz spectral line shapes, and with similar solutions for the Voigt and Holtsmark spectral line shapes.

2. A method of deriving an automodel Green’s function (simple PDF, 1D case)

We consider the 1D transport on a uniform background, described by the equation for spatial density $f(x, t)$ of an excitation of the background medium, which may evolve due to the exchange of excitation between various points of the medium via emission and absorption of the carriers (here the retardation caused by the finite velocity of carriers is neglected):

$$\frac{\partial f(x, t)}{\partial t} = \frac{1}{\tau} \int_{-\infty}^{\infty} W(|x - x_1|) f(x_1, t) dx_1 - \left(\frac{1}{\tau} + \sigma\right) f(x, t) + q(x, t),$$

where $W(x)$ is a step-length PDF (i.e. the probability that the carrier, emitted at some point, is absorbed at a distance $x$ from that point), $1/\tau$ is the absolute value of the emission rate (i.e. $\tau$ is the average waiting time between the absorption and reemission of the carrier), $q$ is the source function, which is the rate of production of excitation by an external source (i.e. a source which differs from the excitation of the medium due to absorption described by the $W$ function), and $\sigma$ is the rate of the quenching of excitation. The possible physics model for such a transport is described in the appendix. The uniformity of the background assumes that, first, $W$ is a function of only one variable—the distance between the points of emission and absorption—and, second, $\tau$ and $\sigma$ are the constants. The latter makes the role of quenching simply described by the time exponent $\exp(-\sigma t)$, therefore in what follows we omit this process. Hereafter we use the dimensionless time and space coordinate, assuming the normalization of time by $\tau$ and using a dimensionless PDF. We will seek for the Green’s function, taking, respectively, the source function as a point instant source,

$$q(x, t) = \delta(x) \delta(t).$$

We take the PDF in the following simple form which possesses a long-tail and the infinite value of the mean square displacement:
\[ W(\rho) = \frac{\gamma}{2(1 + \rho)^{1+\Gamma}}, \quad 0 < \gamma < 2, \quad \rho = |x - x'|, \]

\[ \int_{-\infty}^{\infty} W(|x - x'|)dx' = 1. \] (3)

It is worth introducing the probability, \( T(\rho) \), for the carrier to pass, without any absorption, the distance not exceeding a certain value, \( \rho \). This function may be expressed in terms of the PDF of equation (3):

\[ W(\rho) = -\frac{dT(\rho)}{2d\rho}, \quad T(0) = 1. \] (4)

Further, we suggest the following equation for the propagation front, \( \rho_{fr}(t) \), which, as shown below, appears to be close to the time dependence of the mean displacement:

\[ (t + 1)T(\rho_{fr}(t)) = 1, \quad \rho = |x|. \] (5)

Equation (5) may be obviously considered as a definition of the time evolution of the front, \( t_{fr}(\rho) \). Note that equation (5) may be substantiated for large values of dimensionless time, whereas for \( t \sim 1 \) it is interpolated to an obvious condition \( \rho_{fr}(0) = 0 \).

Equation (5) is suggested by the success of the escape probability methods in the theory of radiative transfer in spectral lines. These methods have been anticipated by the approximate solution \([11]\), obtained for the steady-state problem by taking \( f(x) \) out of the integral term, and by the analysis \([12]\) of the validity of the solution \([11]\).

Equation (1) allows the derivation of the asymptotic behavior of the Green’s function far beyond and far in advance of the propagation front defined by equation (5). Far in advance of the propagation front arrival at a distance \( \rho, \rho \gg \rho_{fr}(t) \gg 1 \) (or, equivalently, for a short time, \( 1 \ll t \ll t_{fr}(\rho) \)), the density is determined by the direct population by the carriers emitted by the source. At this stage the distant points see almost a point source of carriers because the diffusion of the excitation around the origin \((x = 0)\) may be neglected. The exchange of the neighboring distant points may also be neglected. This corresponds to the excitation of the distant points by the rare long-free-path carriers (i.e. Lévy flights) whereas the frequent exchange with short-free-path carriers, both around the source and the every distant point, may be neglected. This gives a simple relation

\[ f(x, t) \approx t W(\rho), \quad \rho = |x|, \quad \rho \gg \rho_{fr}(t) \gg 1. \] (6)

The asymptotics of the Green’s function far behind the propagation front, \( \rho \ll \rho_{fr}(t) \), or equivalently \( t \gg t_{fr}(\rho) \gg 1 \), may be found taking into account the above-mentioned frequent exchange with short-free-path carriers. The latter produces local uniformity of the density. Assuming a plateau-like spatial distribution around the origin, one has:

\[ f(x, t) \sim \frac{1}{2\rho_{fr}(t)}\eta(\rho_{fr}(t) - |x|) = \rho \ll \rho_{fr}(t), \] (7)

where \( \eta \) is the Heaviside step function.

It appears that the asymptotic laws of equations (6) and (7) may be unified in a single interpolation formula, e.g. of the following type:

\[ f(x, t) \sim t W(\sqrt{x^2 + C\rho_{fr}^2(t)}), \] (8)

where the constant \( C \) should give equality to equation (7). It is worth, however, leaving a functional freedom of the interpolation between equations (6) and (7). This gives
where \( g \) is a function of a single variable, and its asymptotic behavior is known:
\[
g(s) = 1, \quad s \ll 1, \tag{10}
\]
\[
g(s) \propto s, \quad s \gg 1. \tag{11}
\]
A test of solutions (9)–(11), (5) via comparison with the direct numerical solution of equation (1) may be considered as an inverse problem of reconstructing the function \( g \) or, equivalently, as a proof of the scaling law (self-similarity of solutions) formulated with equations (9)–(11), (5).

3. Validation and reconstruction of an automodel Green’s function from a numerical solution (simple PDF, 1D case)

We try the validity of the method of section 2 in the case of a simple PDF with a power-law tail. We consider the following PDF:
\[
T(\rho) = \frac{1}{(1 + \rho)^\gamma}, \quad W(\rho) = \frac{\gamma}{2(1 + \rho)^{\gamma+1}},
\]
\[
\gamma > 0, \quad \rho = |x - x'|. \tag{12}
\]
Equation (1) may be solved via Fourier transformation. For the Fourier transform
\[
\hat{y}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y(x) e^{-ipx} \, dx \tag{13}
\]
with allowance for the relation
\[
(y * z)(x) = \sqrt{2\pi} \hat{y} \hat{z}, \tag{14}
\]
where
\[
(y * z)(x) = \int_{-\infty}^{\infty} y(x - x') z(x') \, dx',
\]
equation (1) turns into
\[
\frac{\partial \hat{f}(p, t)}{\partial t} = \sqrt{2\pi} \hat{W}(p) \hat{f}(p, t) - \hat{f}(p, t), \tag{15}
\]
with the boundary condition \( \hat{f}(p, 0) = 1/\sqrt{2\pi} \). Solving equation (15) and taking the inverse Fourier transform, one has:
\[
f(x, t) = \frac{1}{2\pi} e^{-t} \int_{-\infty}^{\infty} e^{i\sqrt{2\pi} \hat{W}(p) \psi} \, dp, \tag{16}
\]
where
\[
\hat{W}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} W(x) e^{-ipx} \, dx
\]
\[
= \frac{1}{\sqrt{2\pi}} \left\{ 1 - p \int_{0}^{\infty} \frac{\sin px}{(1 + x)^\gamma} \, dx \right\}. \tag{17}
\]
This finally leads to an analytic solution of equation (1) with the PDF of equation (12):

\[
f(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \cos px \exp \left( -tp \int_{0}^{\infty} \frac{\sin px_1}{(1 + x_1)^\gamma} dx_1 \right) dp.
\]

(18)

In the case of \( t \to \infty, x \to 0, 0 < \gamma < 2 \) equation (18) takes the form

\[
f(x, t) = \frac{\Gamma(1 + 1/\gamma)}{\pi I(1/\gamma)^{1/\gamma}}.
\]

(19)

where \( \Gamma \) is the gamma function, and

\[
I(\gamma) = \int_{0}^{\infty} \frac{\sin x}{x^\gamma} dx.
\]

(20)

For the alternative asymptotics, \( t \to 0, x \to \infty \), equation (18) gives the scaling law of equation (6):

\[
f(x, t) = \frac{\gamma t}{2(1 + |x|)^{\gamma+1}}.
\]

(21)

The automodel solution of equation (1) with the kernel (12) may be obtained from equation (9):

\[
f(x, t) = \frac{\gamma}{2} \left[ 1 + \rho g \left( \frac{\rho}{\rho_0}(t) \right) \right]^{-1/\gamma}, \quad \rho = |x|,
\]

(22)

where asymptotics of the function \( g \) obeys equations (10), (11), and

\[
\rho_0(t) = (t + 1)^{1/\gamma} - 1.
\]

(23)

In the case of PDF (12), equation (11) may be specified:

\[
g(s) = \alpha s, \quad s \gg 1,
\]

\[
\alpha(\gamma) = \frac{\pi \gamma I(1/\gamma)^{1/\gamma}}{2 \Gamma(1 + 1/\gamma)}^{1/\gamma+1}.
\]

(24)

It appears that the asymptotic solution far behind the front deviates from the qualitative scaling law of equation (7) by the constant, a factor \( 2\Gamma(1 + 1/\gamma)/[\pi I(1/\gamma)^{1/\gamma}] \) which is of the order of unity, \( I(1/2) \approx 1.25, I(1) \approx 1.56, I(3/2) \approx 2.31 \).

Comparison of equations (22) and (23) with the numerical calculation of equation (18) enables us, first, to prove the validity of the automodel solution (22), (23) and, second, reconstruct the function \( g \). The relation between \( g \) and the exact solution of equation (18), \( f_{\text{exact}} \), is described by the following equations:

\[
Q_W(x, t) \equiv \frac{1}{\rho} \hat{W}^{-1} \left( \frac{f_{\text{exact}}(x, t)}{t} \right)
\]

\[
= \frac{1}{\rho} \left[ \frac{\gamma t}{2f_{\text{exact}}(x, t)} \right]^{1/\gamma} - 1, \quad \rho \equiv |x|.
\]

(25)

where \( \hat{W}^{-1} \) is the function reciprocal to the \( W \) function,

\[
Q_W(\rho, f(\rho, s)) \equiv Q_W(\rho, s) = g(s).
\]

(26)
where the functions \( \rho(t, s) \) and \( \rho(s, t) \) are determined by the relation

\[
\rho = \frac{(t + 1)^{\frac{1}{\gamma}} - 1}{\rho},
\]

(the propagation front \( \rho(t) = \rho(t, s = 1) \)). To prove the automodel solution one has to show weak dependence (independence) of \( Q_{W1} \) and \( Q_{W2} \) functions on, respectively, space coordinate and time. The \( Q_{W1} \) function for various values of space coordinates is shown in figure 1 for three values of the exponent \( \gamma \).

It is seen that for given values of \( \gamma \) the function (22) is indeed an automodel solution of equation (1) with the PDF of equation (12) with an accuracy defined by the coincidence of the curves \( Q_{W1}(s, \rho) \) for various values of the distance \( \rho \). According to exact asymptotic solutions (6) and (7), used for constructing solution (9), one may expect high accuracy of the respective solution (22) for large enough values of dimensionless time and distance from the source. Indeed, we have the following numbers for the maximum deviation of \( Q_{W1} \) and \( Q_{W2} \) functions from the function of \( s \) only. For instance, for \( \gamma = 1 \) this deviation amounts to 3\% for \( \{ \rho > 30, t > 30 \} \) and 1\% for \( \{ \rho > 30, t > 270 \} \) and \( \{ \rho > 100, t > 30 \} \). The accuracy for a weaker tail appears to be worse: for \( \gamma = 1.5 \) and \( \rho > 30 \) the deviation amounts to 12\%, 7\% and 1\% for, respectively, \( t > 30 \), \( t > 100 \), and \( t > 2400 \). For a stronger tail, \( \gamma = 0.5 \), the propagation front moves substantially faster, and high accuracy is achieved at larger distances: 5\% for \( \rho > 100, t > 30 \) and 1\% for \( \rho > 100, t > 100 \). The largest deviations take place at \( s \sim 1 \).

4. Automodel solution of the Biberman–Holstein equation

The Biberman–Holstein equation for radiative transfer in a uniform medium of two-level atoms/ions is obtained from a system of equations for spatial density of excited atoms, \( F(r, t) \), and spectral intensity of resonance radiation. This system is reduced to a single equation for \( F(r, t) \), which appears to be an integral equation, non-reducible to a differential diffusion-type equation:

\[
\frac{\partial F(r, t)}{\partial t} = \frac{1}{\tau} \int V G(|r - r|)F(r, t) dV - \left( \frac{1}{\tau} + \sigma \right)F(r, t) + q(r, t).
\]

where \( \tau \) is the lifetime of the excited atomic state with respect to spontaneous radiative decay; \( \sigma \) is the rate of the collisional quenching of excitation; \( q \) is the source of excited atoms different from population by the absorption of the resonant photon (e.g. collisional excitation). The kernel \( G \) is determined by the (normalized) emission spectral line shape \( \varepsilon_{o} \) and the absorption coefficient \( k_{\omega} \). In homogeneous media, \( G \) depends on the distance between the points of emission and the absorption of the photon:

\[
G(r) = -\frac{1}{4\pi r^2} \frac{dT(r)}{dr}, \quad T(r) = \int_{0}^{\infty} \varepsilon_{\omega} \exp(-k_{\omega}r) d\omega.
\]

The non-locality of the Biberman–Holstein radiative transfer demands special definition of the mean time \( \tilde{t}(r) \) needed for a photon to pass the distance \( r \) from a point instant source \( q(r, t) = \delta(r - r_{0})\delta(t - t_{0}) \). The respective scalings for various line broadening mechanisms strongly deviate from the diffusion law (see [8–12]). For Doppler and Lorentz line
shapes, the results [10] may be written in the unified form: \( \tilde{F}(\rho) \approx 1/[AT_{\text{ms}}(\rho)] \) [12], where \( \rho \equiv k(\omega_0)r \), and \( T_{\text{ms}}(\rho) \) are the asymptotics of the Holstein functional \( T \) at \( \rho \gg 1 \).

Our numerical analysis of the Veklenko’s Green’s function [10] for various line shapes shows that the scaling defined by equation (5) gives good approximation for the time moment when \( F(\rho, t) \) attains its maximum value at the distance \( \rho \) from the source. It is the analysis
that enabled us to suggest using equation (5) in automodel solutions in this and other radiative transfer problems.

Let us introduce dimensionless time and use the former notation, \( t \equiv t/\tau \). For a short time, \( 1 \ll t \ll t_2(\rho) \) (or, equivalently, far in advance of front propagation coming at the distance \( r, \rho \gg \rho_0(t) \gg 1 \)), the asymptotics of the Green’s function [10] for Doppler and Lorentz line shapes may be written in the form

\[
F \approx t \, G(\rho),
\]

which corresponds to the direct excitation of distant atoms by the photons in the far wings of the spectral line shape (i.e. by Lévy flights). Asymptotic solutions in these particular cases suggested an extension of this law to a much broader class of non-local transport.

The Green’s function far behind the propagation front, \( \rho \ll \rho_0(t) \), or equivalently \( t \gg t_2(\rho) \gg 1 \), may be estimated assuming the local uniformity of the excitation due to the fast exchange of atoms in the core of the spectral line shape. The respective quasi-plateau solution in the 3D case takes the form:

\[
F(\rho, t) \approx \frac{1}{4\pi (\rho_0(t))^3} \eta(\rho_0(t) - \rho).
\]

(32)

where \( \rho_0(t) \) is defined by equation (5). Comparison of equation (32) with numerical calculations of the exact Green’s function [10] proves that this asymptotic gives a good scaling for the time dependence for various line shapes. However, the absolute values of the plateau in equation (32) and the asymptotics of the exact Green’s function may differ by a constant which amounts to a factor of unity for a Doppler line shape and \( \sim 200 \) for a Lorentz line shape. The large value of the constant may be explained by the longer precursor to the excitation front of equation (32) in the case of the longer PDF tail that, in turn, stems from the wider wings of the Lorentz line shape.

The automodel solution is suggested in the form of equation (9) which in the 3D case and for arbitrary space-time coordinates of the instant point source takes the form:

\[
F_{\text{auto}}(r, t; r_0, t_0) = (t - t_0) G\left( \frac{r - r_0}{|r - r_0|} \right). G_1(\rho_0(t - t_0)).
\]

(33)

The results of the validation of the automodel solution and the reconstruction of function \( g \) from comparison of function (33) with computations of the Green’s function [10] for the Doppler, Lorentz, Voigt, and Holtsmark line shapes are shown in figure 2. The procedure of the reconstruction of function \( g \) is quite similar to that of equations (25)–(28):

\[
Q_G(\rho, t) \equiv R^{-1} \left( Q_{\text{exact}}(\rho, t) \right),
\]

(34)

where \( R^{-1} \) is the function reciprocal to the \( R \) function,

\[
Q_G(\rho, t(\rho, s)) \equiv Q_{G1}(s, \rho) = g(s),
\]

\[
Q_G(\rho(t, s), t) \equiv Q_{G2}(s, t) = g(s),
\]

(35)

(36)

where the functions \( t(\rho, s) \) and \( \rho(t, s) \) are determined by the relation

\[
s = \frac{\rho_0(t)}{\rho}.
\]

(37)
Figure 2. The automodel function $g$ in equation (33), reconstructed with the help of equations (34) and (35) for various spectral line shapes: (a) Lorentz; (b) Doppler; (c) Voigt for $\alpha = \sqrt{\ln 2} \Delta \omega_{\text{Lorentz}}/\Delta \omega_{\text{Doppler}} = 1$; (d) Holtsmark.
It is seen that function (33) is indeed an automodel Green’s function of equation (29) with an accuracy defined by the coincidence of the curves $Q_{G1}(s, \rho)$ for various values of the distance $\rho$. Maximum deviation of $Q_{G1}$ and $Q_{G2}$ functions from the function of $s$ only is as follows. For $\{\rho > 30, \ t > 40\}$ we have 0.2% for the Lorentz line shape, and 4%, 3%, and 8% for, respectively, Doppler, Voigt ($a = 1$), and Holtsmark line shapes. The maximum deviation amounts to 1% for the last-named three line shapes if we, for $\rho > 30$, restrict the time to $t > 950, 280, 700$, respectively. The largest deviations take place at $s \sim 1$.

5. Conclusions

The approximate automodel solution of a wide class of non-stationary superdiffusive transport on a uniform background with a power-law decay at large distances of the step-length PDF is found. The success of identifying such solutions is based on the identification of the dominant role of the long-free-path carriers (i.e. Lévy flights) in all three scaling laws used to construct the automodel solution; namely, the scalings for the propagation front (i.e. relevant-to-superdiffusion average displacement) and asymptotic solutions far beyond and far in advance of the propagation front.

The simplicity of the algorithm suggests the possibility of substantially broadening the applicability of the method. In particular, the method may be extended to Lévy walks which, in contrast to Lévy flights, take into account a finite velocity of carriers (see, e.g. [25]).

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Appendix

Equation (1) for medium’s excitation transport may be applied to a wide range of transport by such fast carriers that one may neglect the respective retardation effects. Here we give an example of a possible physics model. This is a 1D model of the processes similar to the medium’s excitation transport by the photons emitted and absorbed in a resonance transition which has a spectral distribution around the mean value. The space-time evolution of the excitation is described by a couple of equations for, respectively, the density of the holders of the excitation in the medium, $f(x, t)$, and the spectral intensity (i.e. spectral density of the flux) of the carriers moving in positive, $I_+(\omega, x, t)$, and negative, $I_-(\omega, x, t)$, directions with respect to the space coordinate $x$:

$$\frac{\partial f(x, t)}{\partial t} = -\left(\frac{1}{\tau} + \sigma\right)f(x, t) + q(x, t) + \int_{-\infty}^{\infty} a(\omega)[I_+(\omega, x, t) + I_-(\omega, x, t)]d\omega;$$  \hspace{1cm} (A.1)

$$\frac{\partial I_s(\omega, x, t)}{\partial x} = \frac{1}{2\tau}f(x, t)e(\omega) - a(\omega)I_s(\omega, x, t), s = +, -;$$  \hspace{1cm} (A.2)
Equations (A.1)–(A.3) describe the stochastic evolution of the system where the successive displacements of the medium’s excitation are (i) random, (ii) independent (due to the loss of memory in every act of absorption-reemission of the carrier), (iii) statistically identical over different time intervals of the same length, and (iv) Lévy-distributed (i.e. ‘heavy-tailed’). Equation (A.1) describes the evolution of the medium’s excitation. Here, the sink is due to spontaneous decays, with the waiting time $\tau$, of the excitation, which produces the carriers, and due to such a quenching, with inverse lifetime $\sigma$, which does not produce the carriers. The source of the medium’s excitation is due to the absorption of carriers by the medium and due to an external source of the excitation, which is not caused by the absorption of the carriers. Equation (A.2) describes the evolution of the carriers. Here, the source is due to the above-mentioned spontaneous decay, with the waiting time $\tau$ of the medium’s excitation, and the sink is due to the absorption of carriers by the medium. The probability distribution functions $e(\omega)$ and $a(\omega)$ in equations (A.1) and (A.2) describe the spectral distributions of, respectively, the emission and absorption of carriers (these functions are the PDFs of these processes; e.g. in the radiative transfer in atomic spectral lines, these are the normalized spectral line shapes of, respectively, the emission and absorption of photons).

The substitution of the solution of equation (A.2) into equation (A.1) gives equation (1) with the function $T(\rho)$ of the following form:

$$T(\rho) = \int_{-\infty}^{\infty} e(\omega) \exp(-a(\omega)\rho) d\omega, \quad \rho = |x - x'|,$$

(A.4)

which is quite similar to the Holstein function $T(\rho)$ in equation (30), where $\omega$ is the detuning of the dimensionless photon’s energy (or frequency) from the center of the transition between the ground and excited states of atoms/ions, and the distance $\rho$ is expressed in the units of the free path of carriers with the frequency detuning $\omega = 0$.

The power-law dependence assumed in equation (3),

$$T(\rho) = \frac{1}{(1 + \rho)^\gamma},$$

(A.5)

may be generated by various sets of functions $e(\omega)$ and $a(\omega)$. To illustrate the possibility of a heavy-tailed $T(\rho)$ function, we recall two well-known cases of the transport by the resonant photons, namely, Lorentz and Doppler line shapes, respectively (see, e.g. [9]):

$$e(\omega) = a(\omega) = \frac{1}{\pi(1 + \omega^2)},$$

$$T(\rho) \to \frac{1}{|\sqrt{\rho}|}, \quad \rho \gg 1,$$

(A.6)

$$e(\omega) = a(\omega) = \frac{1}{\sqrt{\pi}} \exp(-\omega^2),$$

$$T(\rho) \to \frac{1}{\rho \sqrt{\pi \ln \rho}}, \quad \rho \gg 1.$$

(A.7)

It is seen that the exponent $\gamma$ essentially depends on the tail of the line shape while the behavior of $T(\rho)$ in the core, i.e. at small and moderate values of $\rho$, is determined by the core of the line shape. The simplified form of equation (3) may be easily extended to an arbitrary monotonic function with a heavy tail $W(\rho) \propto 1/\rho^{\gamma+1}$, $0 < \gamma < 2$ at large distances. Note
also that there is an example of a physics model which gives the power-law PDF of equation (3); see equation (1) in [26] for photon-assisted transport of minority carriers in semiconductors (photo-excited holes in n-type InP [27]).

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