Internal Josephson Effects in Spinor Dipolar Bose–Einstein Condensates

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We theoretically study the internal Josephson effect, which is driven by spin exchange interactions and magnetic dipole-dipole interactions, in a three-level system for spin-1 Bose–Einstein condensates, obtaining novel spin dynamics. We introduce single spatial mode approximations into the Gross–Pitaevskii equations and derive the Josephson type equations, which are analogous to tunneling currents through three junctions between three superconductors. From an analogy with two interacting nonrigid pendulums, we identify unique varied oscillations, namely the 0–π, 0–running, running–running, 2π & running–2π, single nonrigid pendulum, and two rigid pendulums phase modes. These Josephson modes in the three states are expected to be found in real atomic Bose gas systems.

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I. INTRODUCTION

The Josephson effect is a universal quantum phenomenon, defined as a current between two or more macroscopic quantum states with weak coupling driven by the relative phases of the macroscopic wave functions. The effect for a junction of two superconductors was originally predicted by Josephson [1] and discovered by Anderson and Rowell [2]. When the effect was first predicted and discovered it was considered a characteristic phenomenon of superconductors. However, the effect has been found in diverse fields since Feynman redefined the Josephson effect as the quantum tunneling between two levels [3]. In particular, Maki and Tsumeto have considered nonlinear ringing between equal spin pairing states |↑↑⟩ and |↓↓⟩ in superfluid 3He-A as a Josephson junction between the internal degrees of freedom [4]. Webb et al. subsequently observed the effect by NMR [5].

Such Josephson effects have also been studied in atomic Bose–Einstein condensates (BECs). Smerzi et al. derived Josephson type equations in a double well potential from Gross–Pitaevskii (GP) equations with two-mode approximations, revealing three oscillations, namely the 0, π and running phase modes [6, 7]. Macroscopic quantum self trapping from the running mode and AC Josephson effects from the 0 phase mode have been observed by Albiz et al. [8]. Furthermore, Zhang et al. considered transitions between three energy states for spin-1 BECs driven by spin exchange interactions as internal Josephson effects [9] and Chang et al. observed the internal Josephson oscillation [10].

In addition, studies have focused on the magnetic dipole–dipole interaction (MDDI) in BECs. The interaction between spins has a characteristic symmetry for spin and orbit. Therefore, the interaction is theoretically expected to give new quantum phases [11, 12, 13], the Einstein–de Haas effect [14] and so on. Griesmaier et al. realized spinor dipolar condensates using 52Cr atoms [15] which have larger magnetic moments than alkali atoms. The condensates clearly show shape-dependent anisotropy of the MDDI [16, 17]. Thus, the study of the MDDI has opened new avenues for investigating spinor condensates.

For the MDDI, we can demonstrate new internal Josephson effects in spin-1 BECs. The internal Josephson effect without the MDDI is analogous with two junctions between three superconductors because the weak coupling, namely the spin exchange interaction, permits transitions only from |0⟩ to |±1⟩ and from |±1⟩ to |0⟩. However, it is possible to transfer from |±1⟩ to |±1⟩ if the MDDI drives the transitions because the anisotropic interaction breaks the conservation law of magnetization, as will be discussed later. As the previous study, Cheng et. al. have discussed the Josephson effects driven by the MDDI in the conservation law [18]. Therefore, we consider out of the law that the transitions through the MDDI are analogous with Josephson effects for three cir-

FIG. 1: (color online) The transitions, driven by the Hamiltonians of a spin exchange interaction (red arrows) and an MDDI (red arrows), between the energy states (solid line) are analogous to the Josephson effect for cooper pairs tunneling through (a) and (b) three insulators (black boxes) between super conductors, named SC1, SC0, and SC-1 (red boxes). (a) and (b) compare differently with the MDDI, was the MDDI can also drives transition from |±1⟩ to |±1⟩. The blue circles represent the condensate.
In this paper, we introduce the spin dynamics in spinor dipolar BECs in Sec. III using the GP equations with single mode approximations, and we demonstrate new Josephson oscillations in Sec. IV. Section V is devoted to a conclusion.

II. FORMULATION

A. Single mode approximation

Before we consider the internal Josephson effect for spin-1 BECs, we analyze the spin-1 GP equations with an MDDI:

\[
\begin{align*}
  i\hbar \frac{\partial \psi_\alpha}{\partial t} &= \left( -\frac{\hbar^2}{2M} \nabla^2 + V - \mu \right) \psi_\alpha \\
  &- g_e \mu_B \mathbf{H} \cdot \mathbf{F}_{\alpha \beta} \psi_\beta \\
  &+ c_0 \psi_\beta^* \psi_\alpha + c_2 \mathbf{F} \cdot \mathbf{F}_{\alpha \beta} \psi_\beta \\
  &+ c_{dd} \int d\mathbf{r}' \frac{3e^{i\mathbf{r}'} \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \mathbf{F}_i(\mathbf{r}') \mathbf{F}_{j\beta} \psi_\beta, 
\end{align*}
\]

where the integers \( \alpha \) and \( \beta \) represent the spin \( 0 \) and \( \pm 1 \) states, \( V(\mathbf{r}) \) is the trapping potential, and \( \mu \) the chemical potential. The spin density vector \( \mathbf{F} = (F_x, F_y, F_z) \) and the vector of the spin components \( \mathbf{F}_{\alpha \beta} = (F_{x\beta}, F_{y\beta}, F_{z\beta}) \) are represented by \( F_{i=x,y,z} = \psi_\alpha^* F_{i\alpha} \psi_\beta \) given by the components \( F_{i\alpha} \) of the spin matrices \( \hat{F}_i \) for spin-1. The Zeeman term is presented by Lande's \( g \)-factor of an electron \( g_e \), Bohr magneton \( \mu_B \), and external magnetic field \( \mathbf{H} \).

The coefficients \( c_0 = (g_0 + 2g_2)/3 \) and \( c_2 = (g_2 - g_0)/3 \) are the interaction parameters for \( g_i = 4\hbar^2 a_i/M \) represented by the s-wave scattering lengths \( a_i \). The coefficient of the MDDI is given by \( c_{dd} = \mu_0 g_e^2 \mu_B^2 / 4\pi \) with \( \mathbf{e} = (\mathbf{r} - \mathbf{r}') / |\mathbf{r} - \mathbf{r}'| \) being a unit vector.

Since it is difficult to derive the internal Josephson effect from Eq. (1) directly, we introduce the single mode approximation [3]:

\[
\psi_i(\mathbf{r}, t) = \sqrt{N} \xi_i(t) \phi(\mathbf{r}) \exp \left( \frac{-i \mu t}{\hbar} \right),
\]

where \( \phi \) satisfies the eigenvalue equation \( -\hbar^2 \nabla^2 / 2M + V + a_0 \phi = \mu \phi \) with the relation \( \int d\mathbf{r} |\phi|^2 = 1 \). The approximations can be used when the shapes of the condensates are decided by the spin independent terms, namely \( |a_0| \gg |c_2| \). The condition is satisfied for \(^{87}\)Rb and \(^{23}\)Na. The validity of the approximation in spin-1 dipolar BECs has been discussed by Yi and Pu [10]. They found that the approximation is affected by the anisotropic parameter of the shape of the condensates \( \lambda \) and the ratio \( c_{dd}/|c_2| \). For small \( \lambda \), the approximation is well defined even if the ratio is large. Here, we assume that the ratio is smaller than 1, allowing the approximation to be used. Hence, introducing Eq. (2) into Eq. (1), we can derive the equation of the spinor \( |\xi\rangle = (\xi_1, \xi_0, \xi_{-1})^T \) in \( \mathbf{H} = \mathbf{H}_Z \):

\[
\begin{align*}
  i\hbar \frac{d}{dt} |\xi\rangle &= (\hat{H}_Z + \hat{H}_{SE} + \hat{H}_{dd}) |\xi\rangle, \\
  \hat{H}_Z &= -g_0 \mu_B \begin{pmatrix} H & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & -H \end{pmatrix}, \\
  \hat{H}_{SE} &= c \begin{pmatrix} 1 - 2\rho_{-1} & \rho_{-10} & 0 \\
  \rho_{0-1} & 1 - \rho_0 & \rho_{01} \\
  0 & \rho_{10} & 1 - 2\rho_1 \end{pmatrix}, \\
  \hat{H}_{dd} &= c_{dd}^+ \begin{pmatrix} \rho_0 & 0 & \rho_{-10} \\
  \rho_{01} & 0 & \rho_{01} \\
  \rho_{10} & 0 & \rho_0 \end{pmatrix} \\
  &+ c_{dd}^- \begin{pmatrix} 0 & \rho_{10} & 0 \\
  \rho_{01} & 0 & \rho_{01} \\
  \rho_{10} & 0 & \rho_0 \end{pmatrix}, \\
  &+ (c + 2c_{dd}^c) \begin{pmatrix} m & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & -m \end{pmatrix},
\end{align*}
\]

where \( \rho_{ij} = \xi_i^* \xi_j \) are components of the density matrix \( \rho = |\xi\rangle \langle \xi| \), \( c_{dd}^+ = c_{dd}^* + c_{dd}^c \), and \( c_{dd}^c \) given by

\[
c_{dd}^c = \frac{c_{dd}}{2N} \int d\mathbf{r} d\mathbf{r}' |\phi(\mathbf{r})|^2 |\phi(\mathbf{r}')|^2 (1 - 3 \sum_j e_j c_j),
\]

\[
c = c_2 N \int d\mathbf{r} |\phi|^4
\]

are the interaction parameters, and \( m = \rho_1 - \rho_{-1} \) is the magnetization. Comparing Eq. (3) with Eq. (6), we can conclude that the term with \( c_{dd} \) in \( \hat{H}_{dd} \) includes the operators \( |\pm 1\rangle \langle \mp 1| \), which project from \( |\mp 1\rangle \) to \( |\pm 1\rangle \), where \( |1\rangle = (1, 0, 0)^T \), \( |-1\rangle = (0, 0, 1)^T \) are unit vectors. These operators are not included in \( \hat{H}_{SE} \). We again emphasize that the transition from \( |\pm 1\rangle \) to \( |\mp 1\rangle \) is possible only by the MDDI.

Using the relation \( \sum_i \rho_i = 1 \) we can rewrite the spin exchange and the dipole term to give

\[
\begin{align*}
  \hat{H}_{SE} + \hat{H}_{dd} &= (c + c_{dd}^+) \begin{pmatrix} \rho_0 & \rho_{-10} & 0 \\
  \rho_{01} & 0 & \rho_0 \\
  0 & \rho_{10} & 0 \end{pmatrix} \\
  &+ c_{dd}^- \begin{pmatrix} 0 & \rho_{10} & 0 \\
  \rho_{01} & 0 & \rho_{01} \\
  \rho_{10} & 0 & \rho_0 \end{pmatrix}, \\
  &+ (c + 2c_{dd}^c) \begin{pmatrix} m & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & -m \end{pmatrix}.
\end{align*}
\]
exchange effect, as in Eq. (5), given by the $c_{dd}^+$ and $c_{dd}^-$ terms of $\hat{H}_{dd}$. The second is the transition from the $\pm 1$ to the $\mp 1$ states, given by the $c_{dd}^-$ term. The last is an interaction between the magnetization and the spin, given by the $c_{dd}^+$ term, which implies that the magnetization produces a molecular field.

**B. Josephson type equations**

In order to investigate the Josephson effect for transitions between the three states, we substitute $\xi_j = c\sqrt{\rho_j} e^{i\bar{\theta}_j}$ in Eq. (3), deriving the Josephson type equations for $\rho_0$, $m$ and relative phases $\bar{\theta} = \bar{\theta}_1 + \bar{\theta}_2 - 2\bar{\theta}_0$ and $\bar{\theta}_m = \bar{\theta}_1 - \bar{\theta}_2$;

\[
\dot{\rho}_0 = \frac{2}{\hbar} \left[ \rho_0 \sqrt{(1 - \rho_0)^2 - m^2} \sin \bar{\theta} + \frac{2}{\hbar} c_{dd} \rho_0 \{ (1 - \rho_0) \sin \varphi \cos \phi + m \cos \bar{\theta}_m \sin \rho_0 \} \right],
\]

\[
\dot{m} = \frac{2}{\hbar} c_{dd} \rho_0 \left\{ \sqrt{(1 - \rho_0)^2 - m^2} \sin \bar{\theta}_m + (1 - \rho_0) \cos \bar{\theta}_m \sin \rho_0 \right\},
\]

\[
\dot{\bar{\theta}} = \frac{2}{\hbar} \left[ c' (1 - \rho_0) + \frac{2}{\hbar} c' \sqrt{(1 - \rho_0)^2 - m^2} \cos \bar{\theta}_m \right] + \frac{2}{\hbar} c_{dd} \rho_0 \left\{ (1 - \rho_0) \cos \bar{\theta}_m - m \cos \bar{\theta}_m \sin \rho_0 \right\},
\]

\[
\dot{\bar{\theta}}_m = \frac{2}{\hbar} (E_Z + c'' m) + \frac{2}{\hbar} \left\{ \frac{m \rho_0}{\sqrt{(1 - \rho_0)^2 - m^2}} \cos \bar{\theta}_m + \frac{2}{\hbar} c_{dd} \rho_0 \sin \bar{\theta}_m \sin \rho_0 \right\},
\]

where $E_Z = -g_e \mu_B H$ is the Zeeman energy and $c' = c + c_{dd}^+$ and $c'' = c + 2c_{dd}^+$ are the interaction coefficients. The equations clearly demonstrate the definition of the Josephson effect, namely between a current state driven by a relative phase.

Firstly, from Eq. (12) we can easily see that conservation of the magnetization is broken by the $c_{dd}^-$ term. When $c_{dd} = 0$, the magnetization becomes a conserved value. Under the single mode approximation, $c_{dd}^+ = 0$ for uniform density and a spherical shape, which is obtained from the integration, Eq. (12). Secondly, we note that the variables $\rho_0$ and $m$ are canonical conjugates of $\bar{\theta}$ and $\bar{\theta}_m$, respectively. Therefore, these equations can be derived from the canonical equations of motions:

\[
\dot{\rho}_0 = \frac{2}{\hbar} \frac{\partial H}{\partial \bar{\theta}}, \quad \dot{\bar{\theta}} = \frac{2}{\hbar} \frac{\partial H}{\partial \rho_0}, \quad \dot{m} = \frac{2}{\hbar} \frac{\partial H}{\partial \bar{\theta}_m}, \quad \text{and} \quad \dot{\bar{\theta}}_m = -\frac{2}{\hbar} \frac{\partial H}{\partial \bar{\theta}_m}.
\]

**III. SPIN DYNAMICS**

In this section, we discuss solutions of the equations for $E_Z = 0$, namely small oscillations of two nonrigid pendulums in Sec. III A oscillations of a single nonrigid pendulum in Sec. III B and oscillations of two interacting nonrigid pendulums in Sec. III C.

**A. Small oscillation around stationary solutions**

Not being able to solve the Josephson type equation directly, we first obtain the solutions for the stationary
\[ \rho_0 = \rho_a, \quad m = 0, \quad \theta = (2n + 1)\pi, \quad \theta_m = \theta_a \]  
(17)

and

\[ \rho_0 = \frac{1}{2}, \quad m = 0, \quad \theta = 2n\pi, \quad \theta_m = n\pi. \]  
(18)

Here \( \rho_a \) and \( \theta_a \) are arbitrary constants. For \( E_Z = 0 \), we cannot obtain the stationary magnetization \( m \neq 0 \). In order to study the features of the oscillation of the pendulums, we perform linear analysis by introducing the approximations \( \rho_0 \approx \rho_a + \delta \rho, m \approx m_s + \delta m, \theta \approx \theta_s + \delta \theta \) and \( \theta_m \approx \theta_{ms} + \delta \theta_m \), where these deviations from the stationary solutions are very small, into the Josephson type equations. Linearizing the equations by ignoring squares of the deviations, we obtain the equations for the deviations; the solutions around Eq. (17) are

\[ \dot{\delta m} = \ddot{\delta \theta} = 0 \]  
(19)

and those around Eq. (18) are

\[ \dot{\delta \rho_0} = -\omega_{\rho \pm} \delta \rho \]  
(20)

\[ \dot{\delta m} = -\omega_{m \pm} \delta m, \]  
(21)

where \( \omega_{\rho \pm}^2 = 4(c' \pm c_\rho) / \hbar^2 \) and \( \omega_{m \pm}^2 = 4c_\rho(c_\rho \pm (c_d^\pm c_d^\pm) / 2) / \hbar^2 \) with \( \pm \) indicating the equations for \( \theta_{ms} = 2n\pi \) and \( (2n + 1)\pi \), respectively. From Eq. (19), we can conclude that there are no stable oscillational solutions around Eq. (17). On the other hand, Eqs. (20) and (21) are easily solved and have the solutions \( \delta \rho_0 = \Delta_\rho \cos \omega_{\rho \pm} t \) and \( \delta m = \Delta_m \cos \omega_{m \pm} t \), whose amplitudes are \( \Delta_\rho \) and \( \Delta_m \). Considering the deviations from Eqs. (19) to (21), we can conclude that there are no oscillations around \( \theta = (2n + 1)\pi \), namely the \( \pi \) phase mode in the canonically conjugate variable \( \{\rho_0, \theta\} \), whereas there are 0 and \( \pi \) phase modes in \( \{m, \theta_m\} \).

### B. Josephson effect for a nonrigid pendulum

Next, in order to review the 0, \( \pi \), and running phase modes in spin-1 BECs, we study the Josephson effect for \( c_{dd} = 0 \) obtained for spherically shaped condensates. Under this condition, the spin dynamics are represented by Eqs. (9) and (10) with \( c_{dd} = 0 \), where there is no need to consider \( \dot{m} \) and \( \theta_m \), because \( m \) is constant. The equations are the same as for the Josephson effect driven by the spin exchange interaction in Ref. [9] if the quadratic Zeeman effect, which is effective for the hyperfine interaction between the nuclear and electron spins in the Zeeman splits [10, 20], is introduced in the equation. Therefore, considering the quadratic Zeeman effect \( \delta_{dd} \propto H_{eff}^2 \) given by the effective dipole magnetic field \( H_{eff} \propto c_{dd} m \), we obtain

\[ \dot{\rho}_0 = \frac{2}{\hbar} c' \rho_0 \sqrt{(1 - \rho_0)^2 - m^2 \sin \theta} \]  
(22)

\[ \dot{\theta} = -\frac{2}{\hbar} \delta_{dd} + \frac{2}{\hbar} c' (1 - 2\rho_0) \]  
(23)

\[ + \frac{2}{\hbar c'} (1 - \rho_0)(1 - 2\rho_0) - m^2 \cos \theta, \]

where \( \rho_0 \) is the canonical conjugate of \( \theta \). Hence, these equations are derived from the canonical equations of motions for the Hamiltonian:

\[ \mathcal{H}' = \delta_{dd}(1 - \rho_0) + c' \rho_0 (1 - \rho_0) \]  
(24)

\[ + \rho_0 \sqrt{(1 - \rho_0)^2 - m^2} \cos \theta. \]

The Hamiltonian directly represents the system of a nonrigid pendulum. Using \( \mathcal{H}' = \mathcal{H}_0' = \mathcal{H}'(\rho_0(0), \theta(0)) \) for \( \mathcal{H}' = 0 \), we can solve Eq. (22) and obtain the solutions expressed by Jacobi’s elliptic function \( cn(a, k) \); for

![FIG. 3: (Color online) Contour lines of Eq. (24) for (a) \( m = 0 \) and (b) 0.3 with \( \delta_{dd} = 0.3c' > 0 \), represented as white. The blue, purple and magenta lines show the 0, running and \( \pi \) phase modes, respectively.](image-url)
\[ c_{dd}^+ > 0 \]
\[
\rho_0 = \rho_b + (\rho_c - \rho_b) \text{cn}^2 \left( \frac{1}{\hbar} \sqrt{2c'' \delta_{dd}(\rho_c - \rho_a) t}, k \right),
\]
and for \( c_{dd}^- < 0 \)
\[
\rho_0 = \rho_b - (\rho_c - \rho_a) \text{cn}^2 \left( \frac{1}{\hbar} \sqrt{-2c'' \delta_{dd}(\rho_c - \rho_a) t}, k \right),
\]
where \( \rho_{a,b,c} \) are the roots of \( \rho_0 = 0 \), and \( k = \{ (\rho_c - \rho_b)/(\rho_c - \rho_a) \}^{1/2} \). Pendulum-like oscillations occur on the contour lines of Eq. (24) for energy being conserved, shown in Fig. 3. The dynamics can be classified into three modes; the dynamics on the lines around \( \theta = 0 \) and \( \pi \) are called the 0 and \( \pi \) phase modes respectively and the line from \( \theta = -2\pi \) to \( 2\pi \) is the running phase mode. The 0 phase mode corresponds to the motion of a pendulum oscillating around \( \theta = 0 \) with a varying length. The \( \pi \) phase mode also shows an oscillation around \( \theta = \pi \) a characteristic dynamics for the nonrigid pendulum. Finally, the running phase mode represents the rotational dynamics.

### C. Josephson effect for two pendulums

Here we discuss the dynamics in Eqs. (9) to (12). Solving the equations numerically using the fourth-order Runge–Kutta method, we obtain several characteristic results for two interacting nonrigid pendulums. In the calculations, we estimate that the interaction parameters satisfy the relations \( c'/c_{dd}^- = 11 \) and \( c''/c_{dd}^- = 12 \), which are determined from the order estimations \( c_{dd}^+ \sim c_{dd}^- \sim c_{dd}^+ \) and \( c \sim 10c_{dd}^+ \). The orders between the spin exchange interactions and MDDIs have been discussed by Yi and Pu [21]. In this subsection, we denote the pendulums having angles \( \theta \) and \( \theta_m \) as \( \theta \) and \( \theta_m \) pendulums, respectively, considering the dynamics from the time developments of the canonically conjugate variables (in Sec. III C 1) and the Hamiltonian (in Sec. III C 2).

#### 1. Modes in the dynamics of four variables

First, we present the typical \( 0-\pi \) phase mode for which \( \theta \) and \( \theta_m \) oscillate around \( \theta = 0 \) and \( \theta_m = \pi \), respectively; shown in Fig. 4 (a) and (b). The phase mode is clearly obtained by applying small deviations from the stationary solution, Eq. (13). Considering the motion of the pendulums, the \( \theta \) and \( \theta_m \) pendulums make small oscillations around 0 and \( \pi \) as the lengths grow and shrink. The features of the oscillations are represented by trajectories projected in the phase spaces \( \{ \rho_0, \theta \} \) Fig. 4 (c) and \( \{ m, \theta_m \} \) (d). Since the trajectory is confined to a simple closed orbit, we can see that the motion of the pendulums is similar to that of two independent pendulums, namely a periodic motion. On the other hand, changing

\[ m(0) = 0.01 \text{ to } m(0) = 0.36, \theta(0) = \pi/18, \theta_m(0) = 19\pi/18. \]

![Fig. 4](color online) Typical 0–π phase mode. The dynamics of the four variables \( \rho_0, m, \theta \) and \( \theta_m \) in (a) and (b) and the phase projections of the motions in \( \{ \rho_0, \theta \} \) and \( \{ m, \theta_m \} \) in (c) and (d) are given for \( \rho_0(0) = 0.51, m(0) = 0.01, \theta(0) = \pi/18, \theta_m(0) = 19\pi/18. \]

![Fig. 5](color online) Quasi-0–π phase mode for \( \rho_0(0) = 0.51, m(0) = 0.36, \theta(0) = \pi/18, \theta_m(0) = 19\pi/18. \)
projections of the cross sections at \( t = nT \) (\( n = 1, 2, \ldots \)), where the period \( T \) is given by eigenfrequencies in the systems, providing direct evidence of chaotic motion for the \( 0-\pi \) phase modes, because there is only one point in the Poincaré mappings where the motions are not chaotic but periodic. However, it is unexpected that the typical \( 0-\pi \) phase mode should be chaotic. Therefore, we conclude that the motions of the two pendulums are essentially chaotic and thus the dynamics do not occur on a contour line of the \( 0 \) phase mode. The \( \theta_m \) pendulum is also not a dynamic on a contour line of the \( \pi \) phase mode.

Setting \( m(0) = 0.37 \), where the other initial conditions are same as the \( 0-\pi \) phase mode, we can produce the \( 0\text{-running} \) phase mode in Fig. 4. The phase mode shows that the \( \theta \) pendulum oscillates around \( \theta = 0 \) with a small amplitude chaotically, whereas the \( \theta_m \) pendulum rotates continuously, given by the dynamics of \( \theta_m \) in Fig. 4(b). The \( \theta_m \) pendulum has a long stay around \( \theta_m = 2n\pi \), which can be understood by Fig. 4(b).

Thus, we can obtain the transition from the \( 0-\pi \) phase mode to the \( 0\text{-running} \) phase mode by changing \( m(0) \).

Third, the \( \text{running-running} \) phase mode is shown in Fig. 8. The solutions show that the motions of the pendulums are rotations with different angular frequencies.

Fourth, we present the \( 2n\pi \) \& \( \text{running-2\pi} \) phase mode in Fig. 8 where \( n \) is an integer. In this mode, the dynamics of the two pendulums show strange and interesting motions. Especially, the \( \theta \) pendulum repeats an oscillation around \( \theta = 2n\pi \) and a rotation from \( \theta = 2n\pi \) to \( (2n+1)\pi \). The dynamics indicates a transition from the contour lines of the \( 2\pi \) phase mode to another line of the \( \text{running} \) phase mode, which is shown clearly in Fig. 8(c). Due to this transition, transition should occur for \( \mathcal{H} = 0 \) in \( \{m, \theta_m\} \). Also, the \( \theta_m \) pendulum has a time average of \( \langle \theta \rangle = 2\pi \). However, the pendulum repeats a rotation from \( \theta \approx 0 \) to \( 4\pi \) and returns to \( \theta \approx 0 \).

We obtain the motions of a single nonrigid pendulum in Fig. 10 and two rigid pendulum in Fig. 11 namely
the single nonrigid pendulum and two rigid pendulums phase modes. For the single pendulum, the $\theta$ pendulum oscillates around $\theta = 0$, whereas the length of the $\theta_m$ pendulum grows and shrinks without an oscillation because $\theta_m = 0$. The phase space in Fig. 11 (c) shows a single trajectory, indicating dynamics like that of a rigid pendulum. On the other hand, Fig. 11 (a) and (b) show the running-running mode for rigid pendulums given by the constants $\rho_0$ and $m$. Then, Fig. 11 (c) and (d) are very difficult solutions for understanding the dynamics.

The $\theta$ pendulum exponentially deviates from the initial angle to $\theta = \pi$. The $\theta_m$ pendulum, however, exhibits the rotation of a rigid pendulum.

2. Contour lines of the Hamiltonian

Here we discuss the various modes for two pendulums from the Hamiltonians.

It is impossible to plot the contour lines for the total Hamiltonian, namely Eq. (13), because of the four dimensions given by $\rho_0$, $m$, $\theta$ and $\theta_m$. By considering the Hamiltonian of the $\theta$ pendulum $\mathcal{H}_\theta$ and the $\theta_m$ pendulum $\mathcal{H}_{\theta_m}$, however, the features of the motions of the pendulums can be obtained.

As mentioned in Sec. II B, $\mathcal{H}$ is a conserved value. However, $\mathcal{H}_\theta$, $\mathcal{H}_{\theta_m}$ and $\mathcal{H}_{int}$ are not conserved. Figure 12 shows the time development of these Hamiltonians, indicating the non-conservation, noting though that the values in (a) exhibit only small oscillations that do suggest energy conservation. Therefore, the trajectory in the phase spaces does not have to follow a contour line. As examples, we show the trajectories and contour lines for the $0-\pi$ and $2\pi \& running-2\pi$ phase modes in Fig. 13. Naturally, the contour lines change with time because $\mathcal{H}_\theta$ and $\mathcal{H}_{\theta_m}$ include the parameters $m(t)$ and $\rho_0(t)$, which are time dependent. In Fig. 13, we plot the lines for the time averages $m = \langle m \rangle$ and $\rho_0 = \langle \rho_0 \rangle$ in $\mathcal{H}_\theta$ and $\mathcal{H}_{\theta_m}$. The trajectories of the $0-\pi$ phase mode almost follow the contour lines in Fig. 13 (a) and (b). On the other hand, for the $2\pi \& running-2\pi$ phase mode of (c) and (d), the

FIG. 9: (color online) $2\pi \& running-2\pi$ phase mode for $\rho_0(0) = 0.1$, $m(0) = 0$, $\theta(0) = \pi/2$, $\theta_m(0) = \pi/3$.

FIG. 10: (color online) single nonrigid pendulum phase mode for $\rho_0(0) = 0.2$, $m(0) = 0$, $\theta(0) = \pi/2$, $\theta_m(0) = \pi$.

FIG. 11: (color online) The motion of two rigid pendulums for the initial conditions $\rho_0(0) = 0$, $m(0) = 0.8$, $\theta(0) = 0$ and $\theta_m(0) = 0$ for (a) and (b), and $\rho_0 = 0$, $m = 0.01$, $\theta = \pi/4$, and $\theta_m = \pi$ for (c) and (d).
FIG. 12: (color online) Time development of the Hamiltonians. (a), (b), and (c) have the same initial conditions as for Fig. 4, Fig. 5, and Fig. 9, respectively.

FIG. 13: (color online) Trajectories and contour lines for the 0–π, (a) and (b), and 2nπ & running–2π phase modes, (c) and (d). The trajectories are shown as green dots in the 0–π mode and red dots in the 2nπ & running–2π mode. The lines are given by Eq. (14) for m = (m) ≃ 0 in (a) and (c) and Eq. (15) for ρ0 = ⟨ρ0⟩ ≃ 0.5 in (b) and 0.47 in (d). The magnitude of the Hamiltonians increases from black to white.

trajectories do not follow the lines. However, the transition from the 0 to running phase modes is clearly seen in {ρ0, θ}. An important result is that the dynamics of the pendulum cannot be discussed as transitions between the 0, π and running phase modes.

IV. CONCLUSION

By introducing the single mode approximation into the spin-1 GP equations with the magnetic dipole–dipole interactions, we show the transitions between the three states, which are analogous with Josephson junctions of a ring of three superconductors. Furthermore, deriving the canonical equations of motion from the Josephson equations, the Josephson junctions are found to also be analogous with two nonrigid pendulums having interactions with each other. First, in order to consider a simple nonrigid pendulum, we assume that the shapes of the condensates are spheres. In this way, we review well known three phase nodes, the 0, π and running phase modes. Second, we numerically solve the equations for two nonrigid pendulums, showing several motions: the 0–π, 0–running, running–running, 2π & running–2π, single pendulum, and two rigid pendulum phase modes. Finally, we discuss the transition between the modes from the non-conserved Hamiltonians H0, Hm, and Hnt.

We consider that the Josephson effect, which has been found in various phenomena from condensed matter physics to classical physics, is a universal and important physical phenomenon. In this study, the effect was discussed only in relation to BECs. However, we expect that the study of the Josephson effect of the three states will be useful in many other fields beyond atomic BECs.

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