Topological edge states in acoustic Kagome lattices

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Abstract

We demonstrate that an acoustic Kagome lattice formed by an array of interconnected resonant cavities exhibits a new class of topological states protected by C₃ symmetry, and it is characterised by a topological invariant in the form of a winding number in Pauli vector space. This acoustic topological metamaterial can be considered as the two-dimensional analogue of the Su–Schrieffer–Heeger model, exhibiting a topological transition when a detuning is introduced between the inter-cell and intra-cell hopping amplitudes. The topological transition caused by such detuning is accompanied by the opening of a complete topological band gap, which may host edge states. The edge states emerge on either truncated ends of the lattice terminated by a cladding layer or at the domain walls between topologically nontrivial and trivial domains. First-principles simulations based on full-wave finite element method are used to design the lattice and confirm our analytical predictions.

1. Introduction

Recently there has been a significant surge of interest in topological states supported by classical systems, which was inspired by their unique properties such as robustness against various forms of perturbations, as well as the ability to emulate exotic quantum states of matter. A variety of topological systems, including those described by the Su–Schrieffer–Heeger (SSH) model [1], quantum Hall effect [2–4] and quantum spin-Hall effect [5–12], as well as topological crystalline insulators [13–15], have been emulated with great success in mechanical, acoustic and electromagnetic structures [16–29, 35–45], encouraging the use of their exotic characteristics in practical applications.

While the origin of the topological properties varies dramatically in the proposed structures, from breaking time-reversal symmetry and engineering synthetic gauge fields [2–4, 16–23], to topological transitions self-induced by nonlinearities [46–48], the approach relying on spatial symmetries has been proven to be one of the most accessible to achieve topological order, since it relies on simply manipulating the symmetry of the underlying lattice. For example, chiral symmetries have been used to emulate SSH states in electromagnetics and acoustics [30, 31], and space group symmetry [13–15] to emulate spin-Hall effects in photonic two-dimensional lattices [32].

In this work, we study the emergence of a topological state in an acoustic Kagome lattice [33] induced by altering inter-cell and intra-cell hopping amplitudes; the system represents the two-dimensional analogue of the well-established 1D SSH model [1]. While in the SSH chain the zero energy edge states are protected by the chiral symmetry, in the system proposed here, as we show below, C₃ spatial symmetry of the lattice is at the origin of the topological order and topological edge modes. We develop an analytical approach, which allows characterizing the topological state of the system by calculating its topological invariant. The presence of topological edge states on the truncated end interfaced with the cladding layer and at the domain wall, respectively, validates the...
prediction of existence of the described topological transition. Although any cut of the finite system breaks the C3 symmetry, in the nontrivial situation topological edge states can still survive, as long as the C3 symmetry with the fixed point at the edge is respected locally. Numerical simulations on the existence of topological edge states and the robustness of edge states to the structure defects with and without C3 symmetry at the boundary are performed and also agree with our analytical prediction.

2. Analytical and numerical approaches

The geometry of an acoustic realization of the proposed structure is shown in figure 1(a), and the schematic of its tight-binding description is plotted as an inset to figure 1(b). The lattice is formed by an array of tall hollow cylinders, whose dimensions are chosen such that their lowest frequency modes are sustained by vertical standing pressure waves with resonance frequencies lying near the resonances defined by \( \nu_0 = (n + \frac{1}{2}) c_0 / H \), where \( c_0 = 343.2 \text{ m s}^{-1} \) is the speed of sound, height \( H = 10 \text{ cm} \). Higher-order modes with horizontal modulation of pressure resonate at higher frequencies defined by the radius of the cylinder \( R = 2.5 \text{ cm} \), where \( R < H/2 \). When the cylinders are interconnected by cylindrical hard walled tubes, the exchange of pressure and

Figure 1. Schematics of the structure. (a) 3D unit cell (b) bulk energy bands of the unit cell for \( \nu_0 \) modes with simple 2d Kagome lattice inset. (c) Field profiles of monopolar singlet and dipolar doublets at \( \Gamma' \) from bottom to top. (d) Bulk band structures corresponding to three cases, where black dash lines are gapless bands, blue dot lines represent the topologically trivial bands and red lines the topologically nontrivial bands. (e) Pressure profiles of the nontrivial acoustic modes 1, 2 and 3 from the lowest frequency to highest one at \( K' \) and \( K \) points, respectively. The height of cylinder in figure (a) is \( H = 10 \text{ cm} \), the radius \( R = 2.5 \text{ cm} \), connectors have the same vertical displacements \( h_1 = h_2 = 2.89 \text{ cm} \) and the diameter of the inner and outer connectors are (i) for gapless band structure \( d_1 = d_2 = 1.21 \text{ cm} \), (ii) for nontrivial gapped band structure \( d_1 = 1.4 \text{ cm} \), \( d_2 = 1 \text{ cm} \), and (iii) trivial gapped band structure \( d_1 = 1.4 \text{ cm} \), \( d_2 = 1 \text{ cm} \), respectively.
resulting coupling results in the emergence of another set of modes with modulation within the cylinders neither in the vertical nor horizontal directions, and which are sustained by the pressure waves propagating through the connecting tubes and the cylinders. These modes are defined by the dimensions of the unit cell as a whole, and appear at lower frequencies, below the resonant modes of the individual cylinders. In what follows we will focus on the two cases: (i) the low-frequency modes of the lattice stemming from the hybridization of the individual cylinder resonances; (ii) the low-frequency collective modes emerging in the presence of energy exchange in the lattice. This opens the possibility of examining the physics of the same topological state in two distinct regimes, namely (i) the regime of highly localized acoustic cavity modes weakly coupled through the connectors, which is expected to agree well with the tight-binding model, and (ii) the regime of nearly freely propagating acoustic waves for the case of collective acoustic modes.

We begin our analysis by developing an effective Hamiltonian approach based on the tight-binding model, and consider each cylinder as a resonator with resonant frequency \( \omega_0 \)

\[
\mathcal{H} = \sum_{m,n} \hat{\mathcal{G}}_{mn} \langle m, n \rangle \langle m, n + 1 \rangle + \hat{\mathcal{F}}_{mn} \langle m, n \rangle \langle m - 1, n + 1 \rangle + \text{h.c.},
\]

where \( m = 1, \ldots, M \) is the index of a unit cell along \((1, 0, 0)\), \( n = 1, \ldots, N \) is the index of a unit cell along \((\frac{1}{2}, \frac{\sqrt{3}}{2}, 0)\) as specified in figure 1(b). \( \hat{\mathcal{G}} \) is given by

\[
\hat{\mathcal{G}} = \begin{pmatrix} \omega_0 & \kappa & \kappa^* \\ \kappa^* & \omega_0 & \kappa \\ \kappa & \kappa^* & \omega_0 \end{pmatrix} \quad \text{and} \quad \hat{\mathcal{F}}_1 = \begin{pmatrix} 0 & 0 & \gamma \\ 0 & 0 & 0 \\ \gamma^* & 0 & 0 \end{pmatrix}, \quad \hat{\mathcal{F}}_2 = \begin{pmatrix} 0 & \gamma & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

are on-site and off-site contributions, with \( \kappa \) and \( \gamma \) playing the role of intra-cell and inter-cell hopping amplitudes.

To demonstrate the effect of lattice symmetry reduction (\( C_6 \) symmetry is broken while \( C_2 \) is preserved) due to shrunken or expanded geometry on the band structure, we first consider the case of bands originating from resonant modes, which appear near the frequency \( \nu_0 = \omega_0 / 2H \), the case most closely described by the tight-binding Hamiltonian \((1)\). These modes are characterized by an antisymmetric pressure profile in the vertical (\( z \)-axis) direction, as shown in figure 1(c), and thus possess nodal points in the centre of the cylinders. In order to enable the coupling of the resonators, we displace the tubes connecting the resonators vertically from this nodal point. The magnitude of this shift, together with the change of diameter of the on-site and off-site connectors, allows tuning the strength of inter-cell and intra-cell coupling and it is sufficient to realize the desirable symmetry reduction. Figure 1(b) shows the results of full-wave numerical calculations obtained in COMSOL Multiphysics for the cases of equal inter- and intra-cell couplings. We use the COMSOL Acoustic Module to solve the pressure acoustic equation. The lattice primitive vectors for Kagome lattice are chosen as

\[
\vec{a}_1 = a_0 (0, 1) \quad \text{and} \quad \vec{a}_2 = a_0 (\frac{\sqrt{3}}{2}, \frac{1}{2}),
\]

where \( a_0 = 11.26 \text{ cm} \) is the lattice constant. The Floquet periodic boundary conditions are applied for calculating the acoustic band structure of the system. In this case, the connectors have the same vertical displacements above the ground \( h_1 = h_2 = 2.89 \text{ cm} \) and diameters \( d_1 = d_2 = 1.21 \text{ cm} \), and we find three bands located near the frequency of the individual resonators \( \nu_0 = 1800 \text{ Hz} \). These bands originate from coupling of individual modes of the cylinders comprising the trimer constituting the unit cell (figure 1(a)), with the dipolar and monopolar profiles formed as a result of such coupling, as shown in figure 1(c). The bands appear to cross pairwise, forming degeneracies between dipolar modes at the \( \Gamma \) point, and between monopolar and each of the dipolar modes at \( K \) and \( K' \) points, respectively. It is worth mentioning that one of the dipolar modes interferes destructively in real space due to the Kagome lattice symmetry and is trapped inside the hexagon formed by the cylinders of nearby trimers, as represented by the flat band in figure 1(b). Changing the relative strength of the inter- and the intra-hopping allows reducing the \( C_6 \) space group symmetry of the lattice. When the \( C_6 \) symmetry is reduced, either by detuning vertical placements of the connectors \( h_1 \neq h_2 \) or their diameters \( d_1 \neq d_2 \) (in our case, the diameter of the inner and outer connectors are \( d_1 = 1 \text{ cm} \) and \( d_2 = 1.4 \text{ cm} \)), this reduction induces a repulsion of the bands at \( K \) and \( K' \) and opens a complete acoustic bandgap. The modes at \( K \) and \( K' \) form time reversal partners, which indicates the conservation of time reversal symmetry, and are numbered by 1, 2 and 3 from lowest frequency to highest one in figure 1(e). It is shown by figures 1(c) and (e) that dipolar and monopolar modes are mixed somewhere between \( \Gamma \) and \( K(K') \), leading to the flip of the modes 1 and 2 at \( K(K') \). On the other hand, the degeneracy between dipolar modes at \( \Gamma \) points, protected by time-reversal symmetry and \( C_2 \) symmetry, is unaffected by the reduction of \( C_6 \) symmetry.

It is instructive to look into the most extreme case \( \kappa = 0, \gamma = 0 \), for which we obtain a totally ‘trimerized’ lattice with unit cells disconnected from each other, which is reminiscent of the case of a fully dimerized 1D SSH model. In this case the modes of the lattice form a dipolar doublet and a monopolar singlet (see acoustic pressure
field profiles in figure 1(c)). The opposite case of $\kappa = 0, \gamma \neq 0$ may be expected to yield an identical band structure, but in fact gives rise to a topologically distinct state. As it will be shown below, the two states can be easily distinguished by a topological invariant, which represents a 2D generalization of the winding number, and changes its value at $\kappa = \gamma$, when the crossing of the acoustic bands occur at $K$- and $K'$-points. The magnitude of $\kappa = \gamma$ is obtained by numerically fitting the parameters and subsequent matching the band diagram from tight binding method with the simulation data. The tunnelling amplitude was estimated as $\kappa = \gamma \approx 30$ Hz.

The kernel of the bulk Hamiltonian in $k$-space obtained from (1) is expressed as

$$
\mathcal{H}(k) = \begin{pmatrix}
\omega_0 & \kappa + \gamma e^{i(k_x + \sqrt{3}k_y)\alpha_0} \\
\kappa + \gamma e^{-i(k_x + \sqrt{3}k_y)\alpha_0} & \omega_0 \\
\kappa + \gamma e^{i(-k_x + \sqrt{3}k_y)\alpha_0} & \kappa + \gamma e^{-i(-k_x + \sqrt{3}k_y)\alpha_0}
\end{pmatrix}.
$$

The energy dispersion is obtained by solving the secular equation for the Hamiltonian (2), following which an analytic expression for the bulk band can be derived from (2). As noted above, the cases of stronger intra-cell coupling $\kappa > \gamma (d_1 > d_2)$ and stronger inter-cell coupling $\kappa < \gamma (d_1 < d_2)$ produce identical gapped band structures, which are shown in figure 1(d). Since the Kagome lattice always preserves $C_3$ symmetry during an adiabatic transformation (i.e., changing the relative strength of inter-cell and intra-cell hopping amplitudes), we can apply the unitary transformation $H' = UHU^{-1}$ to diagonalize the Hamiltonian at $K$, where

$$
U = \frac{1}{\sqrt{3}} \begin{pmatrix}
1 & 1 & 1 \\
e^{\frac{2\pi i}{3}} & e^{\frac{2\pi i}{3}} & e^{-\frac{2\pi i}{3}} \\
e^{\frac{4\pi i}{3}} & e^{-\frac{2\pi i}{3}} & e^{-\frac{4\pi i}{3}}
\end{pmatrix}.
$$

Dirac points at $K$ and $K'$ consist of different sets of bands: we choose to work with the band set at $K$-point, which consists of the monopolar mode and the dipolar mode. By using the fact that the $C_6$-symmetry reduction does not result in mixing of dipolar modes (as indicated by the degeneracy at $\Gamma$-point in figure 1(d)), we truncate the basis to two bands, and treat pairs of bands crossings at $K$ and $K'$ points separately. This treatment remains valid as long as the time reversal symmetry and $C_3$-symmetry is preserved. For simplicity, we also shift the Dirac point to zero energy (frequency) by setting effective onsite frequency $\omega'_0 = 0$. This allows mapping the Hamiltonian from $k$-space to the Pauli vector space, and the Hamiltonian assumes the simple form

$$
H'_k(k) = D(k) \cdot \sigma,
$$

where $\sigma$ is the Pauli matrix, and the vector field $D(k)$ parametrized by the wavenumber $k$ can be expressed as

$$
D_x = \frac{1}{3} \gamma \left\{ - \cos(k_x \alpha_0) + \sqrt{3} \sin(k_x \alpha_0) + \cos\left(\frac{\sqrt{3}k_y \alpha_0}{2}\right) \right\},
$$

$$
D_y = \frac{\sqrt{3}}{2} \gamma \left[ \sqrt{3} \cos\left(\frac{k_y \alpha_0}{2}\right) - \sin\left(\frac{k_y \alpha_0}{2}\right) \right] \sin\left(\frac{\sqrt{3}k_y \alpha_0}{2}\right),
$$

$$
D_z = \gamma \left\{ \cos\left(\frac{k_y \alpha_0}{2}\right) \left[ \cos\left(\frac{k_x \alpha_0}{2}\right) + \cos\left(\frac{\sqrt{3}k_y \alpha_0}{2}\right) \right] + \frac{\sqrt{3}}{2} \sin\left(\frac{k_y \alpha_0}{2}\right) \right\}.
$$

Although the reduced Hamiltonian does not exactly match the original one in the whole Brillouin zone, it preserves the degeneracy of the bands at $\Gamma$-point, which is crucial to reflect the symmetries of the lattice. Interestingly, the non-zero value of $D_z$ except at $K$ and $K'$ points leads to an unusual topological edge state distinct from those of the SSH model, which will be discussed below. The Hamiltonian (4) has two eigenstates $| \pm \rangle$ of the form

$$
| + \rangle = e^{i\lambda(\varphi, \theta)} \begin{pmatrix}
e^{-i\theta/2} \cos\left(\frac{\varphi}{2}\right) \\
e^{i\theta/2} \sin\left(\frac{\varphi}{2}\right)
\end{pmatrix},
$$

$$
| - \rangle = e^{i\lambda(\varphi, \theta)} \begin{pmatrix}
-e^{-i\theta/2} \sin\left(\frac{\varphi}{2}\right) \\
e^{i\theta/2} \cos\left(\frac{\varphi}{2}\right)
\end{pmatrix}.
$$
where \( e^{i\theta} = \frac{D_3 + iD_4}{\sqrt{D_3^2 + D_4^2}} \cos(\varphi) = \frac{D_3}{|D|} \), and \( \lambda(\varphi, \theta) \) is the gauge phase, which depends on the gauge transformation. The eigenvalues of the two states are \( \Omega = \pm |D| \), respectively.

To reveal the difference in the topological states induced by the lattice symmetry reduction, we calculated the topological invariant \( \Omega \), defined as a generalized winding number. To this end, we first calculated the Berry curvature [34]

\[
\Omega \equiv \frac{1}{2\pi} \text{Im} \left( \nabla_{\mathbf{k}} \cdot \mathbf{A}(\mathbf{k}) \right),
\]

(7)

The topological invariant phase \( \Omega \) was calculated numerically by integrating the Berry curvature over the orientated area \( S \) in the Brillouin zone. After lengthy but straightforward calculation, the topological invariant for lower (indicated by \(+\) ) and upper (indicated by \(-\) ) bands can be written as

\[
w_{\pm} = \frac{\Omega_{\pm}}{4\pi} = \pm[1 - \text{sign}(\kappa - \gamma)]/2,
\]

(8)

which reveals its dependence on the relative magnitude of \( \kappa \) and \( \gamma \). Thus, if \( \kappa > \gamma \), \( w_{\pm} = 0 \), which corresponds to the topological trivial case. If the system undergoes adiabatic changes in its parameters towards \( \kappa < \gamma \), it will first pass through the case of gapless Kagome lattice (\( \kappa = \gamma \)) with degeneracies at \( K \) point, and then the gap will reopen for \( \kappa < \gamma \). In this new gapped state one can find that \( w_{\pm} = \pm 1 \), which indicates that the system enters a nontrivial topological phase. For another set of bands at \( K' \), we have obtained the same expression for topological invariant \( w \). It is forbidden to add or subtract the two invariant numbers \( w \) since their corresponding subspaces are different. The above argument will become void if the degeneracy is broken at \( \Gamma \)-point. As shown later, the breaking of \( C_3 \) symmetry for \( \kappa = \gamma \), will lift the degeneracy at \( \Gamma \)-point, and break the topological nature of the system.

3. Topological edge states and their robustness

As in the case of 1D SSH arrays, we expect that in this 2D system the topological transition will be accompanied by the emergence of edge states when a finite system is considered. To verify their existence from first principles, we performed finite element method simulations of a supercell with different boundary conditions. It is expected that because the topological state is symmetry protected, the existence of the topological edge modes also sensibly rely on the presence of the sublattice symmetry with the fixed point at the boundary. However, both the hard wall and soft wall cuts of the lattice reduce \( C_3 \) symmetry at the boundary. The domain wall, on the other hand, has less detrimental effect on the \( C_3 \) symmetry. Yet another way to create interface is using a cladding layer as in the bulk region, except for the shorter height of the connectors, which is also less destructive for the symmetry. The dimensions in the cladding region are the same as the transition region, with hard wall boundary conditions imposed at the openings of the outermost connectors, which is also less destructive for the symmetry. The dimensions in the cladding region are the same as in the bulk region, except for the shorter height \( H_0 = 9 \) cm of the cylinders leading to a shift of the eigenmodes of the cladding layer to a higher frequency range. A periodic boundary condition was imposed on connectors in the horizontal direction to form an acoustic ribbon that is infinite in the \( \left( \frac{\sqrt{3}}{2}, \frac{1}{2}, 0 \right) \) direction and has a finite width in the vertical (0, 1, 0) direction. The number of unit cells for bulk region and cladding region are 15 and 3, respectively.

We examined both situations of topologically trivial (with \( d_1 > d_2 \) ) and topologically nontrivial lattices (with \( d_1 < d_2 \) ), and found that only in the topologically nontrivial case (characterized by \( w = 1 \) ) the bulk band gap hosts additional acoustic modes localized at the opposite truncated ends of the ribbon. Figures 2(a) and (b) show the dispersion of the acoustic ribbon for both respective cases of \( d_1 > d_2 \) and \( d_1 < d_2 \). As shown in figure 2(a), edge states are absent inside the bandgap, while figure 2(b) reveals the presence of the pair of the edge states (shown by red and blue lines, localized at zigzag and armchair cut, respectively) inside the complete acoustic band gap of the bulk acoustic bands (shown by black lines). The field profiles corresponding to the two modes localized at the interfaces of the ribbon are shown in figure 2(c). Note that while there are also other non-topological edge modes that appear at the interface of the ribbon in both situations, and are shown by pink bands in figures 2(a) and (b), these modes are always present, and coexist with the bulk modes regardless of topological phase transition, which are similar to Tamm surface states in condensed matter. They therefore represent topologically trivial edge modes not of interest in this paper.

In addition to the example of the ribbon interfaced with a cladding layer, the case of domain wall between topologically trivial and topologically nontrivial acoustic domains is also investigated. In this case the numerical studies are performed for a linear array of acoustic resonators consisting of \( 1 \times 22 \) unit cells with the domain wall in the middle. The domain wall is created by interchanging the inter-cell and intra-cell coupling strengths in the two domains, which is achieved by shrinking and expanding the diameters of the respective connectors in the lower half of the supercell. The diameter of the connectors for the domain wall composed of 2 unit cells is 5.9 cm, which is larger than the shrunk value, but less than the expanded one. Periodic boundary conditions are imposed.
Figure 2. Two situations of topologically trivial and topologically nontrivial lattice composed of 15 unit cells terminated by cladding layer consisting of 3 unit cells with the cylinder height $H_z = 8$ cm along $y$ direction, and periodic boundary condition is applied along $(\frac{\pi}{2}, \frac{1}{2}, 0)$. (a) Trivial band diagram—no edge states go through the complete band gap; (b) nontrivial—edge states present inside the complete band gap; and (c) two field profiles corresponding to the blue and red bands in figure 2(b), respectively. Pink bands represent the topologically trivial edge modes mixed with bulk modes, and light green region indicates the band gap region.

Figure 3. Interface between topologically trivial and topologically nontrivial acoustic domains consisting of $1 \times 22$ unit cells, with periodic boundary condition applied along $(0, 1, 0)$ and $(\frac{\pi}{2}, \frac{1}{2}, 0)$. (a) Band diagram for supercell (b) field profile corresponding to the edge modes inside the complete band gap in figure 3(a), depicted by blue and red lines, respectively. Pink bands represent the topologically trivial edge modes mixed with bulk modes, and light green region indicates the band gap.

In both $(\frac{\pi}{2}, \frac{1}{2}, 0)$ and $(0, 1, 0)$ directions, effectively introducing another domain wall at the outer boundaries. In this geometry, we expect the emergence of two edge states localized to inner and outer domain walls, respectively. Figure 3 confirms that the edge states are present in this configuration and are confined to the topological domain walls, as illustrated by the corresponding field profiles plotted in figure 3(b) for both modes. We therefore can conclude that the analytically calculated topological invariant obtained from the truncated basis provides an accurate description of the topological nature of the system and allows predicting the emergence of topologically protected edge states. As expected from the parametric dependence of the effective Hamiltonian on wavenumber $k$, the topological edge modes exhibit dispersion, and their eigenfrequencies vary over the Brillouin zone. However, it is important to notice that, in contrast to other classes of topological states such as QHE and QSHE, which are characterized by Chern and spin-Chern numbers, respectively, the edge states of our system do not cross the bulk band gap. Indeed, the non-vanishing topological invariant solely predicts the existence of the edge states, while their dispersion can be understood based on the analogy with the generalized SSH model\textsuperscript{[31]}. Thus, a deviation of the edge state from zero energy occurs due to a non-vanishing $z$-component of the vector field $D'$ mapped from the $2 \times 2$ Hamiltonian (4) projected in the direction along the interface. Note that the modes in figure 3(b) on the two opposite interfaces are not equivalent and are not degenerate at $K$ point, but they become degenerate at some point in the Brillouin zone. Figures 3(a) and 5(d) illustrate the case of the domain wall, the edge states are degenerate at the $K$ point.

We also examined the cases of a topologically non-trivial truncated system ($\kappa < \gamma$) with soft wall and hard wall boundary conditions, respectively. The strip consists of 15 unit cells along $(0, 1, 0)$, and periodic boundary condition was applied along $(\frac{\pi}{2}, \frac{1}{2}, 0)$. The numerical study shows that the edge mode localized at a zigzag cut is dispersive and goes into the bandgap for both cases of soft and hard wall boundary conditions. Modes cannot penetrate through hard wall, they can still vibrate at the soft wall, leading to a different dispersion of the edge...
modes in figures 4(a) and (b). As predicted by previous analysis, although the bulk region has topological nontrivial properties, the edge states with soft and hard wall boundary conditions are topological trivial since both boundary conditions breaks $C_3$ symmetry with fixed point at the boundary. We prove this conclusion by the numerical simulation with monopole excitation source below.

Although our analytical approach is based on a tight-binding approximation, its validity extends far beyond the limits of this model, as it is expected for topological systems [28]. Here we demonstrate that the above conclusions remain valid even in the opposite limit of nearly free acoustic waves propagating in the lattice at lower frequencies. Since $C_3$ symmetry of Kagome lattice is responsible for the topological properties, no matter what type of modes (nearly free propagation mode or standing wave modes inside the cavity) are considered, as long as the topological transition takes place during the adiabatic process without breaking $C_3$ symmetry, the topological edge modes will exist and be protected by this symmetry.

The lowest frequency modes supported by the lattice correspond to waves with pressure modulation taking place in the $x$–$y$ plane on the scale of the unit cell, as shown in figure 5(a). These modes are enabled by the presence of pressure oscillations that propagate through the lattice via the connectors. Unlike the localized modes considered above, these modes do not stem from localized resonances of the standalone cylinders, and therefore do not exhibit significant modulation in either vertical or horizontal direction within them. Because of their non-resonant nature, these modes are not confined to the cylinders, which rather become scattering objects for these waves. Also, because the energy of such delocalized modes is evenly distributed between cylinders and connectors, their spectrum shows significant dependence on the geometrical dimensions of the connectors. To confirm that our conclusions regarding the topological effects extend to the delocalized modes, we performed numerical simulations around low frequencies, ranging from 0 to 800 Hz. The corresponding band structure is shown in figure 5(b) and it reveals clear differences from the one of the confined modes, as it does not have a low frequency cut-off. The field configurations for these modes, shown in figure 5(a), again have dipolar and monopolar profiles, assuming that the same classification as for localized modes can be used. Combined with the same symmetry arguments for the lattice, we therefore assume that reduction of $C_6$ symmetry by detuning intra- and inter-cell coupling (changing the couplers’ diameter in our cases) leads to a topological transition of the same nature as for the case of confined modes. Indeed, the calculations of eigenmodes for the cases of an acoustic ribbon interfaced with cladding layer and of a domain wall, shown in figures 5(c) and (d), reveal the presence of edge states only in the case when the intra-cell coupling is smaller than the inter-cell one, $d_1 < d_2$. Note that the strip interfaced with the cladding layer consists of 15 unit cells along (0, 1, 0), while the cladding layer has 3 unit cells with the height of the cylinder $H_2 = 18$ cm leading to the shift of the eigenmodes to lower frequency region, as shown by orange bands in figure 6(c).

Since the time-reversal symmetry is preserved, edge modes propagate reciprocally along the interface or the cut. The conservation of $C_1$ symmetry of interface with fixed point at the boundary is crucial to distinguish whether edge modes are topologically nontrivial and protected against the $C_3$ symmetry preserving perturbation. The large scale numerical simulations for the case of monopolar source excitation were performed to test the existence of topological edge states and their robustness against the sharp bend and impurity. Figure 6 shows the numerical experiment carried out on the finite system, which includes topologically trivial and nontrivial regions in the centre consisting of 14 × 18 unit cells, terminated by the cladding layer with height $H_1$ around the tetragon shaped domain (light blue region). The domain wall shaded by light green region is formed.

Figure 4. Band diagram for the strip composed of 1 × 15 unit cells, which is terminated by (a) hard wall boundary, (b) soft wall at the cut along $y$ direction, and periodic boundary condition is applied along $(\frac{1}{2}, \frac{1}{2}, 0)$. Blue and red bands represents the edge modes localized at the armchair end and zigzag end, respectively. Pink bands are edge modes mixed with bulk modes, and light green region indicates the band gap region.
in the middle, and monopole source with frequency chosen as 1795 Hz (midgap) is placed on the right side of the domain wall, depicted by the star in figures 6(a) and (b). Edge modes excited by the source propagate along the interface and encircle around the upper topologically nontrivial region. By making the speed of sound 20% higher than \( c_0 \), impurity defects are created in a way that they either respect or break \( C_3 \) symmetry at the interfaces, and placed at the upper interface and in the middle of the domain wall, as shown by blue circles in figures 6(a) and (b). In the first case, edge modes continue flowing along the interface and circumvent the defects though the amplitude of edge modes is diminished after passing the defects due to the backscattering. On the other hand, the edge modes appear to be blocked from propagating when they encounter the defects which break the \( C_3 \) symmetry, as shown in figures 6(c) and (d).

Meanwhile, edge modes existing inside the bandgap for the cases of hard wall and soft wall boundary are trivial and not robust even against perturbations with preserved \( C_3 \) symmetry. We consider the finite topological nontrivial system \(( \kappa < \gamma )\) which consists of 10 \times 15 \) unit cells with soft(hard) wall boundary applied at the cut to demonstrate this point (figure 7). The monopole source with the frequency of 1800 Hz (1805 Hz) is placed at the left upper corner of the system where the zigzag and armchair cuts meet. Modes propagate along the zigzag cut but not along the armchair cut, validating that edge modes inside the bandgap only localize near the zigzag cut for soft (hard) wall (figure 7(a)). The \( C_3 \) symmetry preserving defect is introduced at the zigzag cut, and is depicted by light blue circle in figure 7(b). Edge modes for the case of soft (hard) wall terminations fail to bypass the defect and are scattered around, vividly revealing the difference between the topologically nontrivial and trivial edge modes, the latter of which is alike to Tamm surface states in condensed matters, which are not protected by any symmetry.

In the discussion above we limited our consideration to the case of lossless propagation. However, thermal and viscous losses are found in all realistic small footprint acoustic structures, such as rigid slabs whose size is
comparable to the wavelength of the acoustic wave \[49\]. The length scale which characterizes the viscothermal losses for our model is \(50 \mu m\) (corresponding to the frequency 1800 Hz), which is much smaller than the finest size of our structure. Therefore, the effects of viscothermal losses at 1800 Hz can be ignored. Although the presence of the viscothermal losses is unavoidable for the nearly free propagation modes at lower frequency, we will not discuss this effect here since the effects of dissipation go beyond the scope of this work.

4. Conclusions

We have demonstrated the emergence of a topologically nontrivial acoustic phase in distorted Kagome lattices of acoustic resonators. The topological classification based on a tight-binding description and the consequent definition of a meaningful topological invariant were applied to predict the topological transition of this acoustic metamaterial and the existence of topologically protected edge states. To support our analytical results, we performed numerical simulations confirming the presence of topological edge states on both interfaces of the lattice terminated by cladding layer and at domain walls separating topological and trivial crystals. Edge states inside the bandgap are not topologically protected for the cases of hard wall and soft wall boundaries since the wall conditions break the \(C_3\) symmetry at the boundary. Further numerical simulations for the case of monopole source excitation were carried out and verified the protection of topological edge modes against the \(C_3\) symmetry preserving perturbations. Reduction of \(C_3\) symmetry at the boundary turns the edge modes into the trivial ones, which cannot be protected even if the perturbation keeps \(C_3\) symmetry. The proposed structure is amenable to
physical implementation and the topological edge modes of the Kagome lattice discovered here are worth further investigation based on a real experimental system.

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