BLOW-UP OF SOLUTIONS TO THE ONE-DIMENSIONAL SEMILINEAR WAVE EQUATION WITH DAMPING DEPENDING ON TIME AND SPACE VARIABLES

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Abstract. In this paper, we give a small data blow-up result for the one-dimensional semilinear wave equation with damping depending on time and space variables. We show that if the damping term can be regarded as perturbation, that is, non-effective damping in a certain sense, then the solution blows up in finite time for any power of nonlinearity. This gives an affirmative answer for the conjecture that the critical exponent agrees with that of the wave equation when the damping is non-effective in one space dimension.

1. Introduction

We consider the initial value problem of the one-dimensional semilinear damped wave equation

\begin{equation}
\begin{cases}
  u_{tt} - u_{xx} + a(t, x)u_t = |u|^p, & (t, x) \in (0, \infty) \times \mathbb{R}, \\
  (u, u_t)(0, x) = (u_0, u_1)(x), & x \in \mathbb{R},
\end{cases}
\end{equation}

where \( u = u(t, x) \) is real-valued unknown and \( p > 1 \). We assume that \( a(t, x) \in C^2([0, \infty) \times \mathbb{R}) \) satisfies

\begin{equation}
|\frac{\partial^\alpha t \partial^\beta x}{t^\alpha} a(t, x)| \leq \frac{\delta}{(1+t)^{k+\alpha}} \quad (\alpha, \beta = 0, 1)
\end{equation}

with some \( k > 1 \) and small \( \delta > 0 \).

In this paper we will prove a small data blow-up result, that is, if the initial data satisfy a certain condition, which is independent of their amplitude, then the corresponding solution blows up in finite time.

For the \( n \)-dimensional semilinear damped wave equation

\begin{equation}
\begin{cases}
  u_{tt} - \Delta u + a(t, x)u_t = |u|^p, \\
  \alpha + \beta < 1.
\end{cases}
\end{equation}

the critical exponent \( p_c \) is well studied. Here "critical" means that if \( p_c < p \), all small data solutions of (1.1) are global; if \( 1 < p \leq p_c \), the local solution cannot be extended globally even for small data.

When \( a \equiv 1 \), Todorova and Yordanov \[22\] and Zhang \[29\] determined \( p_c = 1 + 2/n \). Ikehata, Todorova and Yordanov \[27\] treated the case \( a \sim (x)^{-\alpha} \) with \( 0 \leq \alpha < 1 \) and proved that \( p_c = 1 + 2/(n - \alpha) \). For the time-dependent case \( a = (1 + t)^{-\beta} \), Lin, Nishihara and Zhai \[15\] determined \( p_c = 1 + 2/n \) (see also \[3, 2\] for more general types of damping depending on the time variable).

However, there are only few results for the damping depending on both time and space variables. The author \[23\] considered \( a = (x)^{-\alpha}(1 + t)^{-\beta} \) with \( 0 \leq \alpha, \beta \) and \( 0 \leq \alpha + \beta < 1 \). In this case it is conjectured that \( p_c \) is given by \( 1 + 2/(n - \alpha) \).

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He proved that if $p > 1 + 2/(n - \alpha)$, then there is a unique global solution for any small data. Recently, a similar result is obtained by Khader [13] independently. However, we do not know whether the solution blows-up in finite time when $1 < p \leq 1 + 2/(n - \alpha)$.

In the previous results [29] [3] [15], the blow-up parts were obtained by a test-function method developed by [29]. As we will see in Section 3, in order to apply the test function method, we have to transform the equation (1.1) into divergence form and the nonlinear term must be positive. When the damping term depends only on the time variable, Lin, Nishihara and Zhai [15] used a positive solution $g(t)$ of an appropriate ordinary differential equation and transformed the equation into divergence form.

Turning back to our problem, we follow [15] and transform the equation (1.1) into divergence form. Multiplying (1.1) by a positive function $g = g(t, x)$, we have

$$
(\varphi)_{tt} - (\varphi)_{xx} + 2(\varphi)_{tx} + ((-2\varphi_t + ga)\varphi) + (\varphi - \varphi_{xx} - (ga)\varphi) = g|\varphi|^p.
$$

Thus, if $g$ satisfies

$$
g_{tt} - g_{xx} - (ga)_t = 0,
$$

then (1.3) becomes divergence form and we can apply the test function method. We will find a solution $g$ of (1.5) having the form

$$
g(t, x) = 1 + h(t, x),
$$

where $h$ has small amplitude, more precisely, $|h(t, x)| \leq \theta$ with some $\theta \in (0, 1)$. This ensures the positivity of $g$ and so the nonlinearity $g|\varphi|^p$. Then $h$ must satisfy

$$
h_{tt} - h_{xx} - a(t, x)h_t - a_t(t, x)(1 + h) = 0.
$$

We can find a classical solution $h$ of (1.7) having desired property by the method of characteristics.

**Lemma 1.1.** Let $\theta \in (0, 1)$ and $k > 1$. Then there exists $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$ the following holds: if $a$ satisfies (1.2), then there exists a solution $h \in C^2([0, \infty) \times \mathbb{R})$ of (1.7) satisfying

$$
|h(t, x)| \leq \frac{\theta}{(1 + t)^{k-1}}, \quad |\partial^\alpha_t \partial_x^\beta h(t, x)| \leq \frac{C}{(1 + t)^k} \quad (\alpha + \beta = 1)
$$

for all $(t, x) \in [0, \infty) \times \mathbb{R}$ with some constant $C > 0$.

Using this $h$, we can obtain a blow-up result for (1.1). To state our result precisely, we define the solution of (1.1). Let $T \in (0, \infty]$. We say that $u \in X(T) := C([0, T]; H^1(\mathbb{R})) \cap C^1([0, T]; L^2(\mathbb{R}))$ is a solution of the initial value problem (1.1) on the interval $[0, T]$ if it holds that

$$
\begin{align}
\int_{[0,T] \times \mathbb{R}} u(t, x)(\partial_t^2 \psi(t, x) - \partial_x^2 \psi(t, x) - \partial_t(a(t, x)\psi(t, x)))dxdt &= \int_{\mathbb{R}} \{(a(0, x)u_0(x) + u_1(x))\psi(0, x) - u_0(x)\partial_t \psi(0, x)\} dx \\
&\quad + \int_{[0,T] \times \mathbb{R}} |u(t, x)|^p \psi(t, x)dxdt
\end{align}
$$

for any $\psi \in C^2_0([0, T] \times \mathbb{R})$. In particular, when $T = \infty$, we call $u$ a global solution.

We first recall a local existence result:
Proposition 1.2. Let $1 < p < \infty$ and $(u_0, u_1) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$. Then there exists $T^* \in (0, +\infty)$ and a unique solution $u \in X(T^*)$. Moreover, if $T^* < +\infty$, then it follows that

$$\lim_{t \to T^* - 0} \|(u, u_t)(t)\|_{H^1 \times L^2} = +\infty.$$  

For the proof, see for example [8]. We put an assumption on the data

$$\liminf_{R \to \infty} \int_{-R}^{R} \left( (\pm g_t(0, x) + g(0, x)a(0, x))u_0(x) + g(0, x)u_1(x) \right) dx > 0,$$

where $g$ is defined by (1.6) with $h$ in Lemma 1.1. Our main result is the following.

Theorem 1.3. Let $1 < p < \infty$. Under the same situation as Lemma 1.1, let $(u_0, u_1) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ satisfy (1.10). Then the local solution $u$ of (1.1) blows up in finite time, that is, $\lim_{t \to T^* - 0} \|(u, u_t)(t)\|_{H^1 \times L^2} = +\infty$ holds for some $T^* \in (0, +\infty)$.

Remark 1.1. (i) For Lemma 1.1, our method does not work in higher dimensional cases $n \geq 2$ and we have no idea to find an appropriate solution $g$ of (1.5). (ii) We expect that the assumption on the smallness of $\delta$ is removable.

Theorem 1.3 is closely related to so-called diffusion phenomenon, which means that the solution of the damped wave equation

$$u_{tt} - \Delta u + a(t, x)u_t = f(u),$$

behaves like a solution of the corresponding heat equation $-\Delta v + a(t, x)v_t = f(v)$. Here $f(u)$ denotes a nonlinear term.

For the linear and constant coefficient case, that is $f(u) = 0$ and $a \equiv 1$, the asymptotic behavior of the solution was initiated by Matsumura [17]. He showed that some decay rates of solution are same as that of corresponding heat equation and applied these estimates to semilinear problems. After that more specific asymptotics were given by [28, 16, 6, 20, 19]. They showed that the asymptotic profile of solution is actually given by that of the corresponding heat equation.

For the linear and variable coefficient case, Mochizuki [18] proved that if $a(t, x)$ has bounded derivatives and satisfies $a(t, x) \lesssim (1 + |x|)^{-k}$ with some $k > 1$, then the solution of (1.1) is asymptotically equivalent to a solution of the free wave equation $w_{tt} - \Delta w = 0$. This means that if the damping term decays sufficiently fast, then the friction becomes non-effective and the equation recovers its hyperbolic structure. Wirth [26, 27] considered time-dependent dampings, for example, $a = (1 + t)^{-k}$. He showed that if $k > 1$ (resp. $k < 1$), then the asymptotic profile of solution is given by that of the free wave equation (resp. the corresponding heat equation). When the space-dependent damping case $a = a(x) \sim \langle x \rangle^{-\alpha}$ with $0 \leq \alpha < 1$, Todorova and Yordanov [22] obtained an energy decay estimate of the solution by a weighted energy method. The decay rate they obtained is almost same as that of the corresponding heat equation. This shows that in this case the equation has a diffusive structure (see also [10, 12]).

For the semilinear case $f(u) = |u|^p$, the results [21, 29, 9, 15] we mentioned above show the critical exponents coincide those of the corresponding semilinear heat equations (see [4] for the heat equation).

On the contrary, by the results of [18, 26], it is expected that when the damping term decays sufficiently fast, the critical exponent agrees with that of the wave equation. However, we do not know any results for this conjecture (see [24] for
a partial result). In particular, in one-dimensional case, Kato [11] proved that
the critical exponent of the wave equation is given by $+\infty$, that is, the blow-up
result holds for any $1 < p < \infty$. Therefore, Theorem 1.3 can be interpreted as an
affirmative answer for this conjecture in one-dimensional case.

We can also treat other types of damping. We give two examples. These examples
have the shape $a(t, x) = \mu/(1 + t) + b(t, x)$, here $b$ denotes a perturbation term.
The wave equation with the damping term $\frac{\mu}{1 + t}u_t$ was investigated by [25, 1, 2, 24].
Wirth [25] obtained several $L^p-L^q$ estimates of solutions to the linear problem. Using
these estimates, recently, D’Abbicco [1] proved several global existence results
(see also [2] for blow-up results). The author [24] also obtained a certain global
existence result by a weighted energy method.

The first example is the case that $a(t, x)$ is a perturbation of $2/(1 + t)$, that is,
$a$ is given by

$$a(t, x) = \frac{2}{1 + t} + b(t, x)$$

and $b(t, x) \in C^2([0, \infty) \times \mathbb{R})$ satisfies (1.2).

In this case by putting

$$g(t, x) = (1 + t)(1 + h(t, x)),$$

the equation (1.7) becomes

$$h_{tt} - h_{xx} - b(t, x)h_t - \left(\frac{b(t, x)}{1 + t} + b_t(t, x)\right)(1 + h) = 0.$$  

In the same way as in the proof of Lemma 1.1, we can obtain a solution $h$ of (1.14):  

**Lemma 1.4.** Let $\theta \in (0, 1)$ and $k > 1$. Then there exists $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$ the following holds: if $b$ satisfies (1.2) and $a$ is given by (1.12), then there exists a solution $h \in C^2([0, \infty) \times \mathbb{R})$ of (1.14) satisfying

$$|h(t, x)| \leq \frac{\theta}{(1 + t)^{k-1}}, \quad |\partial_t^\alpha \partial_x^\beta h(t, x)| \leq \frac{C}{(1 + t)^k} \quad (\alpha + \beta = 1)$$

for all $(t, x) \in [0, \infty) \times \mathbb{R}$ with some constant $C > 0$.

Using this $h$, we can apply a test function method and obtain a blow-up result.

**Theorem 1.5.** Let $1 < p \leq 3$. Under the same situation as Lemma 1.4, let $(u_0, u_1) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ satisfy (1.10). Then the local solution $u$ of (1.1) blows up in finite time.

The second example is

$$a(t, x) = \frac{\mu}{1 + t} + b(t, x)$$

with $\mu > 0$ and $b(t, x) \in C^2([0, \infty) \times \mathbb{R})$ satisfying (1.2). By putting

$$g(t, x) = (1 + t)^\mu(1 + h(t, x)),$$

we have

$$h_{tt} - h_{xx} + \left(\frac{\mu}{1 + t} - b\right)h_t - \left(\frac{\mu}{1 + t}b + b_t\right)(1 + h) = 0.$$ 

In a similar way to Lemma 1.4 with some technical argument, we can find an appropriate solution $h$ of (1.18).
Lemma 1.6. Let \( \theta \in (0, 1) \), \( \mu > 0 \) and \( k > \max\{1, \mu\} \). Then there exists \( \delta_0 > 0 \) such that for all \( \delta \in (0, \delta_0) \) the following holds: if \( b \) satisfies (1.2) and \( a \) is given by (1.16), then there exists a solution \( h \in C^2([0, \infty) \times \mathbb{R}) \) of (1.18) satisfying

\[
|h(t,x)| \leq \frac{\theta}{(1+t)^{k-1}}, \quad |\partial_t^\alpha \partial_x^\beta h(t,x)| \leq C \frac{1}{(1+t)^k} \quad (\alpha + \beta = 1)
\]

for all \( (t,x) \in [0, \infty) \times \mathbb{R} \) with some constant \( C > 0 \).

This lemma and the test function method imply

Theorem 1.7. Let \( 1 < p \leq 1 + 2/\mu \) Under the same situation as Lemma 1.6, let \( (u_0, u_1) \in H^1(\mathbb{R}) \times L^2(\mathbb{R}) \) satisfy (1.10). Then the local solution \( u \) of (1.1) blows up in finite time.

Remark 1.2. When \( \mu = 2 \), Theorem 1.5 is better than Theorem 1.7.

At the end of this section, we explain some notation and terminology used throughout this paper. We put

\[
\|f\|_{L^p(\mathbb{R}^n)} := \left( \int_{\mathbb{R}^n} |f(x)|^p \, dx \right)^{1/p}
\]

for \( 1 < p < \infty \) and \( \|f\|_\infty := \sup_{x \in \mathbb{R}} |f(x)| \). We denote the usual Sobolev space by \( H^1(\mathbb{R}) \). For an interval \( I \) and a Banach space \( X \), we define \( C^r(I; X) \) as the Banach space whose element is an \( r \)-times continuously differentiable mapping from \( I \) to \( X \) with respect to the topology in \( X \) (if \( I \) is semi-open or closed interval, the differential at the endpoint is interpreted as one-sided derivative). The letter \( C \) indicates the generic constant, which may change from line to line. We also use the symbols \( \lesssim \) and \( \sim \). The relation \( f \lesssim g \) means \( f \leq Cg \) with some constant \( C > 0 \) and \( f \sim g \) means \( f \lesssim g \) and \( g \lesssim f \).

2. Proof of Lemma 1.1

In this section, we construct a solution of (1.7) by the method of characteristics. First, we diagonalize (1.7). Put

\[
\begin{pmatrix}
  v_1 \\
  v_2
\end{pmatrix}
= \begin{pmatrix}
  h_t + h_x \\
  h_t - h_x
\end{pmatrix}.
\]

Then \( v_1, v_2 \) satisfies

\[
\partial_t v_1 = \partial_x v_1 + \frac{a(t,x)}{2}(v_1 + v_2) + a(t,x)(1 + h)
\]

and

\[
\partial_t v_2 = -\partial_x v_2 + \frac{a(t,x)}{2}(v_1 + v_2) + a(t,x)(1 + h),
\]

respectively. We can rewrite (2.1)-(2.2) as

\[
\partial_t(v_1(t,x-t)) = \frac{a(t,x-t)}{2}(v_1(t,x-t) + v_2(t,x-t)) + a(t,x-t)(1 + h(t,x-t)),
\]

\[
\partial_t(v_2(t,x+t)) = \frac{a(t,x+t)}{2}(v_1(t,x+t) + v_2(t,x+t)) + a(t,x+t)(1 + h(t,x+t)).
\]
We seek solutions satisfying $\lim_{t \to +\infty} (v_1, v_2) = 0$ uniformly in $x$. Integrating the above identities over $[t, \infty)$ and changing variables, one has a system of integral equation

\begin{align}
(2.3) \quad v_1(t, x) &= -\int_t^\infty \left\{ \frac{a}{2} (v_1 + v_2) + a_t (1 + h) \right\} (s, x + t - s) ds,
\end{align}

\begin{align}
(2.4) \quad v_2(t, x) &= -\int_t^\infty \left\{ \frac{a}{2} (v_1 + v_2) + a_t (1 + h) \right\} (s, x - (t - s)) ds.
\end{align}

Next, we construct solutions to (2.3), (2.4) by an iteration argument in an appropriate Banach space. We define a function space $Y$. We say $V = (v_1, v_2, h) \in Y$ if $V \in (C([0, \infty) \times \mathbf{R}))^3$, $V$ is differentiable with respect to $x$ for all $(t, x) \in [0, \infty) \times \mathbf{R}$, $\partial_t V \in (C([0, \infty) \times \mathbf{R}))^3$, and $\|V\|_Y = \|(v_1, v_2, h)\|_Y < +\infty$, where

$$
\|v_1, v_2, h\|_Y = \sup_{t \in [0, \infty)} \left\{ (1 + t)^k \|v_1(t)\|_{\mathcal{G}_1} + (1 + t)^k \|v_2(t)\|_{\mathcal{G}_1} + (1 + t)^{k-1} \|h(t)\|_{\mathcal{G}_1} \right\},
$$

$$
\|h(t)\|_{\mathcal{G}_1} = \|h(t, \cdot)\|_Y + \|\partial_x h(t, \cdot)\|_Y.
$$

Then $Y$ is a Banach space with norm $\|V\|_Y$. Let $\theta \in (0, 1)$ and let

$$
K_\theta = \{ (v_1, v_2, h) \in Y \mid \sup_{t \in [0, \infty)} (1 + t)^k \|v_1(t)\|_\infty \leq \theta, \sup_{t \in [0, \infty)} (1 + t)^k \|v_2(t)\|_\infty \leq \theta, \sup_{t \in [0, \infty)} (1 + t)^k \|h(t)\|_\infty \leq \theta \}.
$$

Take $(v_1^{(0)}, v_2^{(0)}, h^{(0)}) \in K_\theta$ arbitrarily and define $V^{(n)} = (v_1^{(n)}, v_2^{(n)}, h^{(n)})$ inductively by

\begin{align}
(2.5) \quad v_1^{(n)}(t, x) &= -\int_t^\infty \left\{ \frac{a}{2} (v_1^{(n-1)} + v_2^{(n-1)}) + a_t (1 + h^{(n-1)}) \right\} (s, x + t - s) ds,
\end{align}

\begin{align}
\quad v_2^{(n)}(t, x) &= -\int_t^\infty \left\{ \frac{a}{2} (v_1^{(n-1)} + v_2^{(n-1)}) + a_t (1 + h^{(n-1)}) \right\} (s, x - (t - s)) ds,
\end{align}

\begin{align}
\quad h^{(n)}(t, x) &= -\frac{1}{2} \int_t^\infty (v_1^{(n)} + v_2^{(n)})(s, x) ds.
\end{align}

The following proposition shows that if the coefficient of damping term $\delta$ is sufficiently small, then $\{V^{(n)}\}_{n=0}^\infty$ is a Cauchy sequence.

**Proposition 2.1.** Let $k > 1$ and $\theta \in (0, 1)$. Then there exists $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0]$ the following holds: if $a$ satisfies (1.2), then $\{V^{(n)}\}_{n=0}^\infty \in K_\theta$ for all $n$ and $\{V^{(n)}\}_{n=0}^\infty$ is a Cauchy sequence with respect to the norm $\| \cdot \|_Y$.

**Proof.** Frist, we prove that if $V^{(n-1)} \in K_\theta$, then $V^{(n)} \in K_\theta$. Assume $V^{(n-1)} \in K_\theta$. It is obvious that $V^{(n)} \in (C([0, \infty) \times \mathbf{R}))^3$. We have

$$
|v_1^{(n)}(t, x)| \leq \int_t^\infty \left\{ \frac{a}{2} (|v_1^{(n-1)}| + |v_2^{(n-1)}|) + |a_t| (1 + |h^{(n)|} \right\} (s, x + t - s) ds
\leq \int_t^\infty \frac{\theta \delta}{(1 + s)^k} + \frac{\delta}{(1 + s)^{k+1}} (1 + \theta(1 + s)^{-(k-1)}) ds
= \delta C_1 (1 + t)^{-k}
$$
with some $C_1 > 0$. Hereafter, $C_j$ ($j = 1, 2, \ldots$) denotes a constant depending only on $k, \theta$. Moreover, differentiating under the integral sign, we obtain
\[
\partial_x v_1^{(n)}(t, x) = -\int_t^\infty \frac{a_x}{2}(v_1^{(n-1)} + v_2^{(n-1)}) + \frac{a}{2}(\partial_x v_1^{(n-1)} + \partial_x v_2^{(n-1)})
+ a_{tx}(1 + h^{(n-1)}) + a_1 h_x^{(n-1)}) (s, x + t - s) ds.
\]
This implies $\partial_x v_1^{(n)} \in C([0, \infty) \times \mathbb{R})$ and
\[
|\partial_x v_1^{(n)}(t, x)| \leq \delta C_2(1 + t)^{-k}.
\]
We can also obtain the same estimates for $v_2^{(n)}$. By differentiating under the integral sign again, we have
\[
\partial_x h^{(n)}(t, x) = -\frac{1}{2} \int_t^\infty (\partial_x v_1^{(n)} + \partial_x v_2^{(n)})(s, x) ds.
\]
Thus, we have
\[
|h^{(n)}(t, x)| \leq \delta C_1 \int_t^\infty \frac{ds}{(1 + s)^k} = \delta C_3(1 + t)^{-(k-1)},
\]
\[
|\partial_x h^{(n)}(t, x)| \leq \delta C_2 \int_t^\infty \frac{ds}{(1 + t)^k} = \delta C_4(1 + t)^{-(k-1)}.
\]
The above estimates show $V^{(n)} \in Y$. Moreover, taking $\delta_0$ so small that $\delta_0 \max\{C_1, C_3\} \leq \theta$, we have $V^{(n)} \in K_\delta$ for all $\delta \in (0, \delta_0]$.

Next, we prove that $\{V^{(n)}\}_{n=0}^{\infty}$ is a Cauchy sequence with respect to the norm $\| \cdot \|_Y$. It follows that
\[
|v_1^{(n)}(t, x) - v_1^{(n-1)}(t, x)| \leq \int_t^\infty \left\{ \frac{\delta}{2(1 + s)^k} (|v_1^{(n-1)} - v_2^{(n-1)}| + |v_2^{(n-1)} - v_2^{(n-2)}|) + \frac{\delta}{(1 + s)^{k+1}} |h^{(n-1)} - h^{(n-2)}| \right\} (s, x + t - s) ds
\]
\[
\leq \delta C_5(1 + t)^{-(2k-1)} \|V^{(n-1)} - V^{(n-2)}\|_Y.
\]
In the same way, we have
\[
|\partial_x v_1^{(n)}(t, x) - \partial_x v_1^{(n-1)}(t, x)| \leq \delta C_6(1 + t)^{-(2k-1)} \|V^{(n-1)} - V^{(n-2)}\|_Y.
\]
and the same estimates holds for $v_2^{(n)} - v_2^{(n-1)}$. We also obtain
\[
|h^{(n)}(t, x) - h^{(n-1)}(t, x)| \leq \delta C_7(1 + t)^{-(2k-1)} \|V^{(n-1)} - V^{(n-2)}\|_Y,
\]
\[
|\partial_x h^{(n)}(t, x) - \partial_x h^{(n-1)}(t, x)| \leq \delta C_8(1 + t)^{-(2k-1)} \|V^{(n-1)} - V^{(n-2)}\|_Y.
\]
Consequently, taking $\delta_0$ smaller so that $r = \delta_0(2C_5 + 2C_6 + C_7 + C_8) < 1$, we have
\[
\|V^{(n)} - V^{(n-1)}\|_Y \leq r \|V^{(n-1)} - V^{(n-2)}\|_Y,
\]
which shows that $\{V^{(n)}\}_{n=0}^{\infty}$ is a Cauchy sequence.

Proof of Lemma 1.1. By the above proposition, $\{V^{(n)}\}_{n=0}^{\infty}$ is a Cauchy sequence and converges to some element $(v_1, v_2, h) \in K_\theta$. Therefore, $(v_1, v_2, h)$ satisfies the integral equation (2.3)–(2.4). By noting the differentiability with respect to $t$ of the right-hand-side of (2.3)–(2.4), we have $v_1, v_2 \in C^1([0, \infty) \times \mathbb{R})$. Differentiating
and (2.6), the estimate (1.8) is obvious. By the equation of \( h \), we also have

\[
\partial_th(t, x) = \frac{1}{2}(v_1(t, x) + v_2(t, x)), \quad \partial_xh(t, x) = \frac{1}{2}(v_1(t, x) - v_2(t, x)).
\]

Thus, \( h \in C^2([0, \infty) \times \mathbb{R}) \) and \( h \) is a classical solution of (1.7). By (v_1, v_2, h) \in K_\theta and (2.6), the estimate (1.8) is obvious. \( \square \)

3. Proof of Theorem 1.3

In this section, we give a proof of Theorem 1.3. By Lemma 1.1 there exists \( h \) satisfying (1.7). Thus, (1.5) holds for \( g \) given by (1.6) and we can transform the equation (1.1) into divergence form

\[
(gu)_t - (gu)_{xx} + 2(g_xu)_x + ((-2g_t + ga)u)_t = g|u|^p.
\]

We apply a text function method to (3.1). Since \( g \) is defined by (1.6) and \( h \) satisfies (1.8), we have

\[
C^{-1} \leq g(t, x) \leq C, \quad |g_t(t, x)| \leq \frac{C}{(1 + t)^\ell_X}, \quad |g_x(t, x)| \leq \frac{C}{(1 + t)^\ell_X}
\]

with some constant \( C > 0 \). We define test functions

\[
\phi(x) = \begin{cases}
1, & (|x| \leq 1/2) \\
\exp(-1/(1 - x^2)) \exp(-1/(x^2 - 1/4)) + \exp(-1/(1 - x^2)) & (1/2 < |x| < 1), \\
0, & (|x| \geq 1),
\end{cases}
\]

\[
\eta(t) = \begin{cases}
1, & (0 \leq t \leq 1/2), \\
\exp(-1/(1 - t^2)) \exp(-1/(t^2 - 1/4)) + \exp(-1/(1 - t^2)) & (1/2 < t < 1), \\
0, & (t \geq 1).
\end{cases}
\]

It is obvious that \( \phi \in C_0^\infty(\mathbb{R}), \eta \in C_0^\infty([0, \infty)) \). We also see that

\[
|\phi'(x)| \lesssim \phi(x)^{1/p}, \quad |\phi''(x)| \lesssim \phi(x)^{1/p},
\]

\[
|\eta'(t)| \lesssim \eta(t)^{1/p}, \quad |\eta''(t)| \lesssim \eta(t)^{1/p}.
\]

Indeed, let \( q, r \) satisfy \( 1/p + 1/q = 1, 1/p + 2/r = 1 \) and let \( \mu = \phi^{1/q}, \nu = \phi^{1/r} \). Then we have

\[
|\phi'| = |(\mu^q)'| = |q\mu^{q-1}\mu'| \lesssim \mu^{q-1} = \phi^{1/p}
\]

and

\[
|\phi''| = |(\nu^r)'| \lesssim |\nu^r|/\nu^r - 1 + |\nu^r|^2\nu^{r-2} \lesssim \nu^{r-2} = \phi^{1/p}.
\]

To prove Theorem 1.3 we use a contradiction argument. Suppose \( u \in X(\infty) \) is a global solution to (1.1) with initial data \((u_0, u_1)\) satisfying (1.10). Let \( \tau, R \) be parameters such that \( \tau \in (\tau_0, \infty), R \in (R_0, \infty) \), where \( \tau_0 \geq 1, R_0 > 0 \) are defined later. We put

\[
\eta_\tau(t) = \eta(t/\tau), \quad \phi_R(x) = \phi(x/R), \quad \psi_{\tau, R}(t, x) = \eta_\tau(t)\phi_R(x)
\]
and

\[ I_{\tau,R} := \int_0^{\tau} \int_{-R}^R g|u|^p \psi_{\tau,R} dx dt, \]

\[ J_R := \int_{-R}^R (-g_t(0,x) + g(0,x)\alpha(0,x))u_0(x) + g(0,x)u_1(x)) \phi_R(x) dx, \]

Substituting the test function \( g(t,x)\psi_{\tau,R}(t,x) \) into the definition of solution (1.9), we see that

\[ I_{\tau,R} + J_R = \int_0^{\tau} \int_{-R}^R (gu\partial_t^2 \psi_{\tau,R} - gu\partial_x^2 \psi_{\tau,R} - 2(g_x u)\partial_x \psi_{\tau,R} - (-2g_t + ga)u\partial_t \psi_{\tau,R}) \]

\[ dx dt =: K_1 + K_2 + K_3 + K_4. \]

Next, we estimate the terms \( K_1, \ldots, K_4 \). Let \( q \) be the dual of \( p \), that is \( q = p/(p - 1) \). By using the Hölder inequality and (3.2), (3.3), it follows that

\[ K_1 \leq \tau^{-2} \int_0^{\tau} \int_{-R}^R |g||u||\eta''(t/\tau)||\phi_R(x) dx dt \]

\[ \lesssim \tau^{-2} \left( \frac{1}{p} \right) \left( \int_{-R}^R \left( \int_0^t g(t,x) dt \right) \phi_R(x) dx \right)^{1/q} \]

\[ \lesssim \tau^{-2+1/q} R^{1/q} \int_{\tau,R}^{1/p}, \]

\[ K_2 \leq R^{-2} \int_0^{\tau} \int_{-R}^R |gu|\phi''(x/R)||\eta(t) dx dt \]

\[ \lesssim R^{-2} \left( \frac{1}{p} \right) \left( \int_{-R}^R \left( \int_0^t g(t,x) \eta(t) dt \right) dx \right)^{1/q} \]

\[ \lesssim \tau^{1/q} R^{-2+1/q} \int_{\tau,R}^{1/p}, \]

\[ K_3 \leq R^{-1} \int_0^{\tau} \int_{-R}^R |g_x u|\phi'(x/R)||\eta dx dt \]

\[ \lesssim R^{-1} \left( \frac{1}{p} \right) \left( \int_{-R}^R \left( \int_0^t (1 + t)^{-q_k} dt \right) dx \right)^{1/q} \]

\[ \lesssim R^{-1+1/q} \int_{\tau,R}^{1/p}. \]

Finally, we estimate \( K_4 \). Noting that \( \text{supp} \ \eta'(t) \subset [1/2, 1] \), we have

\[ K_4 \leq \tau^{-1} \int_0^{\tau} \int_{-R}^R (2|g_t| + |ga|)|u||\eta'(t/\tau)||\phi_R dx dt \]

\[ \lesssim \tau^{-1} \left( \frac{1}{p} \right) \left( \int_{-R}^R \left( \int_{\tau/2}^\tau (1 + t)^{-q_k} dt \right) \phi_R(x) dx \right)^{1/q} \]

\[ \lesssim \tau^{-1-k+1/q} R^{1/q} \int_{\tau,R}^{1/p}. \]

Therefore, putting

\[ D(\tau,R) := \tau^{-2+1/q} R^{1/q} + \tau^{1/q} R^{-2+1/q} + R^{-1+1/q}, \]
we obtain
\begin{equation}
I_{\tau,R} + J_R \leq CD(\tau,R)I_{\tau,R}^{1/p}.
\end{equation}
By the assumption on the data \((1.10)\), there exists \(R_0 > 0\) such that \(J_R > 0\) holds for \(R \geq R_0\). This implies
\[I_{\tau,R} \leq CD(\tau,R)^q.\]
Putting \(\tau_0 = R_0\) and \(R = \tau\), we have
\begin{equation}
I_{\tau,\tau} \leq C\tau^{-1+1/q}
\end{equation}
for \(\tau \geq \tau_0\). In particular, \(I_{\tau,\tau} \leq C\) with some \(C > 0\) and hence, \(g|u|^p \in L^1([0,\infty) \times \mathbf{R})\) and \(\lim_{\tau \to +\infty} I_{\tau,\tau} = \|g|u|^p\|_{L^1([0,\infty) \times \mathbf{R})}\). Moreover, since \(-1 + 1/q < 0\), by letting \(\tau \to +\infty\), the right-hand-side of \((4.5)\) tends to 0. This gives \(\|g|u|^p\|_{L^1([0,\infty) \times \mathbf{R})} = 0\), that is \(u \equiv 0\). However, in view of \((1.9)\), it contradicts \((u_0, u_1) \neq 0\). This completes the proof.

4. Proof of Theorem 1.5

In this section, we give a proof of Theorem 1.5. Note that we can prove Lemma 1.3 by the same argument as the proof of Lemma 1.1. The only difference between the proofs of Lemmas 1.1 and 1.3 is that of the coefficients of \((1.7)\) and \((1.14)\). However, by the assumption on \(b(t, x)\), it follows that
\[|b(t, x)| \leq \frac{\delta}{(1 + t)^k}, \quad \left|\frac{b(t, x)}{1 + t} + b_1(t, x)\right| \leq \frac{2\delta}{(1 + t)^{k+1}}.
\]
Using this estimate, we can prove Lemma 1.4 in the same way as Section 2 and hence, we omit the detailed proof.

Now we prove Theorem 1.5. By \((1.15)\) and \((1.13)\), we have
\begin{equation}
g \sim (1 + t), \quad |g| \lesssim 1, \quad |g| \lesssim (1 + t)^{-k+1}.
\end{equation}
We use the same notation as in Section 3 and suppose that \(u\) is a global solution. The main difference with the previous section lies in the estimate of the terms \(K_1, \ldots, K_4\). In this case, we have
\[K_1 \lesssim \tau^{-2+2/q}R^{1/q}I_{\tau,R}^{1/p},
\]
\[K_2 \lesssim \tau^{2/q}R^{-2+1/q}I_{\tau,R}^{1/p},
\]
\[K_3 \lesssim F(\tau)R^{-1+1/q}I_{\tau,R}^{1/p},
\]
\[K_4 \lesssim \tau^{-1+1/p+1/q}R^{1/q}I_{\tau,R}^{1/p},
\]
where
\begin{equation}
F(\tau) = \begin{cases} 
1 & (-q(1/p + k - 1) < -1), \\
(\log \tau)^{1/q} & (-q(1/p + k - 1) = -1), \\
\tau^{-(1/p+k-1)+1/q} & (-q(1/p + k - 1) > -1) 
\end{cases}
\end{equation}
and we have used that \(\text{supp} \eta'(t) \subset [1/2, 1]\) and \((4.1)\) for the estimate of \(K_4\). Let
\begin{equation}
D(\tau,R) := \tau^{-2+2/q}R^{1/q} + \tau^{2/q}R^{-2+1/q} + F(\tau)R^{-1+1/q}.
\end{equation}
We note that the powers of each terms of \(D(\tau, \tau)\) are negative if \(1 < p < 3\). Thus, by putting \(R = \tau\) and the same argument as the previous section, we can lead a contradiction.
When \( p = 3 \), we need a certain modification of the above argument. We put

\[
I'_{\tau,R} = \int_{\tau/2}^{\tau} \int_{-R}^{R} g|u|^p \psi_{\tau,R} dx dt, \quad I''_{\tau,R} = \int_{0}^{\tau} \int_{R/2 < |x| < R} g|u|^p \psi_{\tau,R} dx dt.
\]

Then we can improve the estimates of \( K_1, \ldots, K_4 \) as

\[
\begin{align*}
K_1 & \leq \tau^{-2+2/q} R^{1/q} (I'_{\tau,R})^{1/p}, \\
K_2 & \leq \tau^{2/q} R^{-2+1/q} (I'_{\tau,R})^{1/p}, \\
K_3 & \leq F(\tau) R^{-1+1/q} (I''_{\tau,R})^{1/p}, \\
K_4 & \leq \tau^{-1-1/p+1/q} R^{1/q} (I''_{\tau,R})^{1/p}.
\end{align*}
\]

Thus, we have

\[
I_{\tau,R} \leq C \left( \tau^{-2+2/q} R^{1/q} (I'_{\tau,R})^{1/p} + (\tau^{2/q} R^{-2+1/q} + F(\tau) R^{-1+1/q}) (I''_{\tau,R})^{1/p} \right).
\]

Substituting \( p = 3 \) and \( R = \tau \), we obtain

\[(4.4) \quad I'_{\tau,R} \leq C ((I'_{\tau,R})^{1/3} + (I''_{\tau,R})^{1/3}).\]

In particular, we see that \( I'_{\tau,R} \leq C \) with some constant \( C > 0 \), since \( I''_{\tau,R} \leq I'_{\tau,R} \). Hence \( g|u|^3 \in L^1([0, \infty) \times \mathbb{R}) \) and \( \lim_{\tau \to \infty} I'_{\tau,R} = \|g|u|^3\|_{L^1([0, \infty) \times \mathbb{R})} \). However, by noting the integral region of \( I'_{\tau,R}, I''_{\tau,R} \), we can see that the integrability of \( g|u|^3 \) shows

\[
\lim_{\tau \to \infty} I'_{\tau,R} = 0, \quad \lim_{\tau \to \infty} I''_{\tau,R} = 0.
\]

Therefore, turning back to (3.34), we obtain \( \lim_{\tau \to \infty} I_{\tau,R} = 0 \). This implies \( u \equiv 0 \).

In view of (3.21), this contradicts \( (u_0, u_1) \neq 0 \). This completes the proof.

5. PROOF OF THEOREM 1.7

In this section, we give a proof of Theorem 1.7. In order to prove Lemma 1.6, we modify the argument in Section 2. Following the argument in Section 2, we look for an appropriate solution by the following iteration:

\[
(5.1) \quad v^{(n)}(t, x) = \int_{t}^{\infty} \left\{ \frac{1}{2} \left( \frac{\mu}{1 + b} - b \right) (v^{(n-1)}_1 + v^{(n-1)}_2) \\
- \left( \frac{\mu}{1 + b} - b \right) (1 + h^{(n-1)}) \right\} (s, x + t - s) ds,
\]

\[
v^{(n)}_2(t, x) = \int_{t}^{\infty} \left\{ \frac{1}{2} \left( \frac{\mu}{1 + b} - b \right) (v^{(n-1)}_1 + v^{(n-1)}_2) \\
- \left( \frac{\mu}{1 + b} - b \right) (1 + h^{(n-1)}) \right\} (s, x - (t - s)) ds,
\]

\[
h^{(n)}(t, x) = -\frac{1}{2} \int_{t}^{\infty} (v^{(n)}_1 + v^{(n)}_2)(s, x) ds.
\]

We modify the definition of function space \( Y \) in Section 2 as follows. We say \( V = (v_1, v_2, h) \in Y \) if \( V \in (C([0, \infty) \times \mathbb{R}))^3 \), \( V \) is differentiable with respect to \( x \) for all \( (t, x) \in [0, \infty) \times \mathbb{R}, \partial_x V \in (C([0, \infty) \times \mathbb{R}))^3 \), and \( \|V\|_Y = \|(v_1, v_2, h)\|_Y < +\infty \), where

\[
\|(v_1, v_2, h)\|_Y = \sup_{t \in [0, \infty)} \{ \lambda(1 + t)^k \|v_1(t)\|_{B^1} + \lambda(1 + t)^k \|v_2(t)\|_{B^1} + (1 + t)^{k-1} \|h(t)\|_{B^1} \},
\]
where $\lambda$ is a large parameter fixed later. We put
\begin{equation}
K_\theta := \{ (v_1, v_2, h) \in Y \mid \sup_{t \in [0, \infty)} (1 + t)^k \| v_1(t) \|_{\infty} \leq \theta' \},
\end{equation}
\begin{equation}
\sup_{t \in [0, \infty)} (1 + t)^k \| v_2(t) \|_{\infty} \leq \theta', \quad \sup_{t \in [0, \infty)} (1 + t)^{k-1} \| h(t) \|_{\infty} \leq \theta,
\end{equation}
where $\theta' := \min\{k - 1, 1\}$. We take $(v_1^{(0)}, v_2^{(0)}, h^{(0)}) \in K_\theta$ arbitrarily and define $V^{(n)} = (v_1^{(n)}, v_2^{(n)}, h^{(n)})$ inductively by (5.1). Now we prove that $\{V^{(n)}\}_{n=0}^{\infty}$ is a Cauchy sequence in $K_\theta$ for sufficiently large $\lambda$ and small $\delta$.

**Proposition 5.1.** If $k \geq \max\{1, \mu\}$, then there exist $\lambda$ and $\delta_0$ having the following property: if $\delta \in (0, \delta_0]$, then $\{V^{(n)}\}_{n=0}^{\infty}$ is a Cauchy sequence in $K_\theta$ with respect to the norm $\| \cdot \|_Y$.

**Proof.** We first show that if $V^{(n-1)} \in K_\theta$, then $V^{(n)} \in K_\theta$. We calculate
\[ |v_1^{(n)}(t, x)| \leq \left( \frac{\mu}{k} \theta' + \delta C \right) (1 + t)^{-k}. \]
In view of $k > \mu$, by taking $\delta$ sufficiently small, we obtain
\[ (1 + t)^{k} |v_1^{(n)}(t, x)| \leq \theta'. \]
By the same way, we also have $(1 + t)^{k} |v_2^{(n)}(t, x)| \leq \theta'$. Noting that $\theta'/(k - 1) \leq \theta$, we obtain
\[ |\partial_x h^{(n)}(t, x)| \leq \int_t^\infty \frac{\theta'}{(1 + s)^k} ds \leq \theta (1 + t)^{-(k-1)}. \]
By differentiating under the integral sign and noting that $V^{(n-1)} \in Y$, we have
\[ (1 + t)^{k} |\partial_x v_1^{(n)}(t, x)| \leq C, \quad (1 + t)^{k} |\partial_x v_2^{(n)}(t, x)| \leq C, \quad (1 + t)^{k-1} |\partial_x h^{(n)}(t, x)| \leq C \]
with some constant $C > 0$. Therefore we have $V^{(n)} \in K_\theta$.

Next, we prove that $\{V^{(n)}\}_{n=0}^{\infty}$ is a Cauchy sequence. By a straightforward calculation, we can estimate
\[ \sum_{\alpha=0,1} \sum_{j=1,2} \| \partial_x^\alpha v_j^{(n)}(t, x) - \partial_x^\alpha v_j^{(n)}(t, x) \|
\leq \int_t^\infty \frac{\mu}{1 + s} \sum_{\alpha=0,1} \sum_{j=1,2} \| \partial_x^\alpha v_j^{(n-1)} - \partial_x^\alpha v_j^{(n)} \|_{\infty} ds \]
\[ + \delta C \int_t^\infty \frac{1}{(1 + s)^k} \sum_{\alpha=0,1} \sum_{j=1,2} \| \partial_x^\alpha v_j^{(n-1)} - \partial_x^\alpha v_j^{(n-2)} \|_{\infty} ds \]
\[ + \delta C \int_t^\infty \frac{1}{(1 + s)^{k+1}} \sum_{\alpha=0,1} \| \partial_x^\alpha h^{(n-1)} - \partial_x^\alpha h^{(n-2)} \|_{\infty} ds. \]

Since $k > 1$, this implies
\[ \lambda (1 + t)^k \sum_{\alpha=0,1} \sum_{j=1,2} \| \partial_x^\alpha v_j^{(n)}(t, x) - \partial_x^\alpha v_j^{(n-1)}(t, x) \| \leq \left( \frac{\mu}{k} + \delta \lambda C \right) \| V^{(n-1)} - V^{(n-2)} \|_Y. \]

Using this, we can estimate the difference of $h^{(n)}$ and $h^{(n-1)}$ as
\[ (1 + t)^{k-1} \sum_{\alpha=0,1} \| \partial_x^\alpha h^{(n)}(t, x) - \partial_x^\alpha h^{(n-1)}(t, x) \| \leq \frac{1}{2\lambda(k-1)} \left( \frac{\mu}{k} + \delta \lambda C \right) \| V^{(n)} - V^{(n-1)} \|_Y. \]
Proof of Theorem 1.7. First, we note that we can prove Lemma 1.6 by the same argument as the proof of Lemma 1.1. Thus, we find a solution \( g \) of \( (5.3) \) global solution. Using the estimates \( (5.3) \), we can obtain
\[
\| V^{(n)} - V^{(n-1)} \|_Y \leq \left( 1 + \frac{1}{2\lambda(k-1)} \right) \left( \frac{\mu}{k} + \delta \lambda C \right) \| V^{(n-1)} - V^{(n-2)} \|_Y.
\]
Thus, by taking \( \lambda \) sufficiently large and then \( \delta \) sufficiently small, we obtain
\[
\| V^{(n)} - V^{(n-1)} \|_Y \leq \tau \| V^{(n-1)} - V^{(n-2)} \|_Y
\]
with some \( 0 < \tau < 1 \). This completes the proof. \( \Box \)

Proof of Theorem 1.7. First, we note that we can prove Lemma 1.6 by the same argument as the proof of Lemma 1.1. Thus, we find a solution \( g \) of \( (1.6) \) satisfying
\[
(5.3) \quad g \sim (1 + t)^\mu, \quad \| g(t) \|_\infty \lesssim (1 + t)^{\mu-1}, \quad \| g_x(t) \|_\infty \lesssim (1 + t)^{\mu-k}.
\]
In what follows, we use the same notation as in Section 3. Suppose that \( u \) is a global solution. Using the estimates \( (5.3) \), we can obtain
\[
\begin{align*}
K_1 & \lesssim \tau^{-2+\mu+q}/q R^{1/q} I_{\tau,R}^{1/p}, \\
K_2 & \lesssim \tau^{\mu+q}/q R^{-2+q}/q I_{\tau,R}^{1/p}, \\
K_3 & \lesssim G(\tau) R^{-1+q}/q I_{\tau,R}^{1/p}, \\
K_4 & \lesssim \tau^{-2+\mu+q}/q R^{1/q} I_{\tau,R}^{1/p},
\end{align*}
\]
where
\[
G(\tau) = \begin{cases} 
1 & (\mu - kq < -1), \\
(\log \tau)^1/q & (\mu - kq = -1), \\
\tau^{-k+\mu+q}/q & (\mu - kq > -1).
\end{cases}
\]
In this case we put
\[
(5.4) \quad D(\tau,R) := \tau^{-2+\mu+q}/q R^{1/q} + \tau^{\mu+q}/q R^{-2+q}/q + G(\tau) R^{-1+q}/q.
\]
We note that the powers of each terms of \( D(\tau,R) \) do not exceed 0 if \( 1 < p \leq 1 + 2/\mu \). Thus, by putting \( R = \tau \) and the same argument as Sections 3 and 4, we can lead a contradiction and complete the proof. \( \Box \)

6. Estimates of Lifespan

We can also give an upper estimate of the lifespan of the solution. In this section, we follow the argument in \[7\] (see also \[14\]). We consider the initial value problem \( (1.1) \) with the initial data \((u,u_1)(0,x) = \varepsilon(u_0,u_1)(x)\) instead of \((u_0,u_1)(x)\), where \( \varepsilon \) denotes a positive small parameter. For the sake of simplicity, we treat only the case that \( a \) satisfies \( (1.2) \). We define the lifespan of the solution by
\[
T_\varepsilon := \sup\{ T \in (0,\infty) \mid \text{there is a unique solution } u \in X(T) \}.
\]
By Proposition 1.2 if \((u_0,u_1) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})\), then \( T_\varepsilon > 0 \) for any \( \varepsilon > 0 \). We have an upper bound of \( T_\varepsilon \) as follows.

**Proposition 6.1.** Let \( 1 < p < \infty \) and let \((u_0,u_1) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})\) satisfy \((1.10)\) with \( g \) defined by \((1.6)\) and \( h \) in Lemma 1.7. Then \( T_\varepsilon \) is estimated as
\[
(6.1) \quad T_\varepsilon \leq C \varepsilon^{-1/\kappa}
\]
with some constant \( C > 0 \) and \( \kappa = \frac{1}{p-1}(1 + 1/p) \).
Proof. Again we use the same notation as in Section 3. Here we note that if $T \epsilon \leq \tau_0$, then the estimate (6.1) is obvious. Thus, we may assume that $T \epsilon > \tau_0$.

Now we use a fact that the inequality
\[ dc^b - c \leq (1 - b) b^{(1-b)} d^{1/(1-b)} \]
holds for all $d > 0$, $0 < b < 1$, $c \geq 0$. We can immediately prove it by considering the maximal value of the function $f(c) = dc^b - c$. From this and (3.4), we obtain
\[ J_R \lesssim D(\tau, R)^q. \]

On the other hand, by the assumption on the data, there exist $C > 0$ and $R_0$ such that $J_R \geq C \epsilon$ holds for all $R > R_0$. Consequently, we have
\[ \epsilon \lesssim D(\tau, R)^q \]
for all $\tau \in (\tau_0, T \epsilon), R \in (R_0, \infty)$. We put $R = \tau^\alpha$ with $\alpha > 0$ and $\tau_0 := \max\{1, R_0^{1/\alpha}\}$. Then we obtain
\[ D(\tau, \tau^\alpha) \leq \tau^{\max\{-1-1/p+(1-1/p)\alpha, 1-1/p+(1-1/p)\alpha, -\alpha/p\}}. \]

Now we take $\alpha = 1 + 1/p$, which minimizes the power of $\tau$ in (6.4). From (6.3) we see that
\[ \epsilon \lesssim D(\tau, \tau^{1+1/p})^q \lesssim \tau^{(-1/p)(1+1/p)} = \tau^{-1}. \]

Therefore, we have
\[ \tau \leq C \epsilon^{-1}. \]

Since $\tau$ is arbitrarily in $(\tau_0, T \epsilon)$, it follows that
\[ T \epsilon \leq C \epsilon^{-1}, \]
which completes the proof. \qed

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