HIGHER ORDER CURVATURE TERMS IN BORN-INFELD TYPE 
BRANE THEORIES

EFRAIN ROJAS
Departamento de Física, Facultad de Física e Inteligencia Artificial, Universidad Veracruzana
Xalapa, Veracruz, 91000, México
efrojas@uv.mx

The field equations associated to Born-Infeld type brane theories are studied by using
an auxiliary variables method. This approach hinges on the fact that the expressions
defining the physical and geometrical quantities describing the worldvolume are varied
independently. The general structure of the Born-Infeld type theories for branes contains
the square root of a determinant of a combined matrix between the induced metric
on the worldvolume swept out by the brane and a symmetric/antisymmetric tensor
depending of gauge, matter or extrinsic curvature terms taking place on the worldvolume.
The higher order curvature terms appearing in the determinant form come to play as
competition with other effective brane models. Additionally, we suggest a Born-Infeld-
Einstein type action for branes where the higher order curvature content is provided
by the worldvolume Ricci tensor. This action provide an alternative description of the
dynamics of braneworld scenarios.

1. Introduction

The Born-Infeld (BI) theory was originally designed to overcome the infinity prob-
lem of a point charge source in the standard Maxwell electromagnetism1,2. Over the
years this idea was almost forgotten until string theory revealed the existence of cer-
tain class of extended objects named $Dp$-branes which brought altogether a revival
of the original BI proposal3,4. These extended objects provide explicit realizations
of several interesting phenomena in a wide range of physical theories at the tini-
est scales, mainly in modern non-perturbative string theory. The simplest action
governing the dynamics of $Dp$-branes is the well known Dirac-Born-Infeld (DBI)
action which contains the square root of the determinant of a metric constructed
with the induced metric on the worldvolume swept out by the $Dp$-brane and the
electromagnetic field strenght. This action has given rise to an intense research
and we have witnessed a growing development of the subject which has triggered a
variety of suggestions akin to it. A related geometrical construction ignoring electro-
magnetism altogether but instead including higher order curvature terms in string
theory was proposed long time ago by Lindstrom et al.5,6. There, such structure
involves the determinant of a metric which is now the sum of the induced metric
and a symmetric tensor built from the extrinsic curvature of the worldsheets swept
out by the string. This effective BI type model suffers however severe drawbacks
mainly due to its second order derivative dependance and in consequence has not
received much attention. Needless to say, these are elegant gravitational theories defined on surfaces, with a determinant Lagrangian density form, which still deserve a careful analysis because of their attractive geometrical properties. This type of theories adapted to branes is the subject we will focus on this work.

As is by now well known in the brane context, despite its many successes, there are many contexts where the Dirac-Nambu-Goto (DNG) action is not adequate and it is natural to consider models that depend on higher derivatives of the embedding variables through the extrinsic curvature of the worldvolume swept out by the brane. The most general field theoretical effective action governing the dynamics of a brane results in an expansion constructed out of the geometrical scalars of its worldvolume. These extrinsic curvature correction terms have found extensive use in concrete applications. For instance, the addition of a rigidity term in an effective action for stringy QCD, the systematic approximation schemes that arise in expansions in the thickness of topological defects, or in actions appearing in the braneworld scenario, to mention some. Therefore, there is no field theoretical reason that does not preclude the possibility of consider an alternative expansion but, enclosed now in a geometrical BI type action solely, without introducing electromagnetism.

The mechanical content of an extended object is captured by its geometrical degrees of freedom where the conserved stress tensor plays a very important role. This determines the dynamics of the extended object thus providing a very powerful geometrical tool for the description of deformations of branes. On physical grounds such stress tensor is merely the momentum density of the brane. On the one hand, the dependence on the independent variables of the effective model governing the evolution of the brane is the first step to be recognized in order to perform a variational process. One then proceed in the usual way to obtain the dynamical laws that the system must obey. On the other hand, the standard variational process for a second order field theory is a non-trivial task which gives rise to annoying computations in order to derive equations of motion and it is sharply elaborated for determinant theories. Certainly this unpleasant fact appears to be a great difficulty but this is not the road we take here. To bypass this field theoretical problem we follow an original strategy for reaching geometrical and physical information, based in an auxiliary variables method (AVM) introduced by Guven in order to describe fluctuating surfaces. The cornerstone in this approach consists of promoting inherent geometrical quantities, defined on the worldvolume, to play the role of auxiliary variables, distributing the original deformation of the embedding variables among the auxiliary variables. The response of the action to a deformation of its worldvolume reflects in the conservation of the stress tensor.

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The criterion for rigidity that we will follow in this paper is that is as a correction to DNG objects in order to penalize singularities in their evolution and thus favouring a variety of configurations richer in geometrical and physical structure than DNG objects. On physical grounds, the inclusion of this type of correction terms simply means that rigid branes resist being bent.
In this paper, equipped with modern variational techniques, we provide the dynamical information supplied by the conserved stress tensor for BI type brane theories. We examine mainly second order brane theories that depend on the extrinsic curvature, which possess a determinant structure. We would like to point out that, with a suitable choice of constraints, the conserved stress tensor and the field equations are obtained effortless through the use of the AVM, elucidating also the geometrical nature of BI type brane models. The paper is structured as follows.

In Section 2, in order to gain insight into the AVM for BI objects, we adapt it to study the well known $D_p$-brane dynamics. We obtain with no effort the mechanical content of $D_p$-branes. We thus pave the way to the application of the AVM to more complex BI type theories. In Section 3 we follow closely the AVM introduced by Guven in order to study the geometrical properties of a rigid BI action for branes, originally proposed in the string theory context. By contrast, the resulting equations of motion are highly non-trivial. In Section 4 we introduce an action in the brane context that mimics the Born-Infeld-Einstein (BIE) suggestion by Deser and Gibbons to modify Einstein gravity. We suggest that an alternative approximation for the study of brane world scenarios lie in BIE type models. This action encodes a rich geometrical content which results promising to explore cosmological brane models which deserves a careful analysis. We conclude in Section 5 with some comments and we discuss briefly our findings. To make our work self-contained, we develop the general auxiliary variables method for a general relativistic extended object whose action include extrinsic curvature. This topic will be developed in Appendix A. Important mathematical expressions involving the determinant of a matrix have been collected in Appendix B.

2. DBI objects

Consider a $D_p$-brane, or in general a relativistic extended object denoted by $\Sigma$, of dimension $p$ evolving in a $N$-dimensional background spacetime endowed with an arbitrary metric $G_{\mu\nu}$ ($\mu, \nu = 0, 1, 2, \ldots, N - 1$). The trajectory, or worldvolume $m$ swept out by $\Sigma$ is an oriented timelike manifold of dimension $p + 1$, described by the embedding functions $x^\mu = X^\mu(\xi^a)$ where $x^\mu$ are local coordinates of the background spacetime, $\xi^a$ are local coordinates of $m$, and $X^\mu$ are the embedding functions $(a, b = 0, 1, 2, \ldots, p)$. The metric induced on the worldvolume from the background is given by

$$g_{ab} = G_{\mu\nu}e^\mu_a e^\nu_b := e_a \cdot e_b,$$

where $e^\mu_a = \partial_a X^\mu$ are the tangent vectors to $m$. Here and henceforth a central dot indicates contraction using the background metric. In this framework we introduce $N-p-1$ normal vectors to the worldvolume, denoted by $n^\mu_i$ $(i = 1, 2, \ldots, N-p-1)$. These are defined implicitly by $n \cdot e_a = 0$ and $n_i \cdot n_j = \delta_{ij}$ where $\delta_{ij}$ is the Kronecker delta symbol.
The well known DBI action governing the low energy dynamics of $Dp$-branes is
\[
S_{\text{DBI}}[X, A] = \beta_p \int_m d^{p+1} \xi \sqrt{-\det(g_{ab} + \mathcal{F}_{ab})},
\] (2)
where $\beta_p$ is the tension of the $Dp$-brane, $\mathcal{F}_{ab} = \alpha F_{ab} + B_{ab}$ is a gauge invariant quantity defined in terms of
\[
F_{ab} = 2\partial_{[a} A_b] \quad \text{and} \quad B_{ab} = B_{\mu\nu} e^\mu_a e^\nu_b,
\] (3)
where $F_{ab}$ is the electromagnetic field strength associated to a $U(1)$ gauge field $A_a$ living on $m$ and $B_{ab}$ is the pullback to the worldvolume of the Neveu-Schwarz (NS) 2-form $B_{\mu\nu}$ and $\alpha$ is the BI parameter related to the inverse tension of branes $3\beta_p$. Among the important features of the action (2) to be mentioned are that it is a first order derivative theory, its invariance under worldvolume reparametrizations and its invariance under a NS gauge transformation, $B_{ab} \rightarrow B_{ab} - 2\partial_{[a} \lambda_{b]}$ if we shift the $U(1)$ field $A_a \rightarrow A_a + \alpha^{-1}\lambda_a$ where $\lambda_a$ is a 1-form; in other words, $\mathcal{F}_{ab}$ is the gauge invariant quantity in the presence of NS background field.

The response of the action (2) to a deformation of the surface $X \rightarrow X + \delta X$, as well as to a deformation of the $U(1)$ gauge field $A \rightarrow A + \delta A$ turns out to be a rather involved computation. The source of this difficulty come from the definitions of the tensors $g_{ab}$ and $\mathcal{F}_{ab}$ which encode the derivatives of the field variables via the structural relationships (1) and (3) and also due to the fact that the background fields might depend of the embedding variables as occurs in more interesting $Dp$-brane scenarios.

To circumvent the usual variational procedure our strategy will be to adopt the action (2) as a functional of the independent variables $g_{ab}$ and $\mathcal{F}_{ab}$ instead of the usual ones, $X^\mu$ and $A_a$ through the Lagrangian density $\mathcal{L}_{\text{DBI}} = \mathcal{L}_{\text{DBI}}(g_{ab}, \mathcal{F}_{ab})$. We thus construct the new functional action
\[
S \begin{bmatrix} X, e_a, g_{ab}, \mathcal{F}_{ab}, \mathcal{G}^{ab}, \mathcal{T}^{ab} \end{bmatrix} = S_{\text{DBI}}[g_{ab}, \mathcal{F}_{ab}] + \int_m dV \mathcal{F}^a \cdot (e_a - \partial_a X) - \frac{1}{2} \int_m dV \mathcal{T}^{ab} (g_{ab} - e_a \cdot e_b) - \frac{1}{2} \int_m dV \mathcal{J}^{ab} (\mathcal{F}_{ab} - 2\alpha \partial_{[a} A_{b]} - B_{\mu\nu} e^\mu_a e^\nu_b),
\] (4)
where $dV = \sqrt{-g} d^{p+1} \xi$ is the worldvolume element. $\mathcal{F}^a$, $\mathcal{J}^{ab}$ and $\mathcal{T}^{ab}$ are Lagrange multipliers. Note that $\mathcal{T}^{ab}$ and $\mathcal{J}^{ab}$ are symmetric and antisymmetric, respectively. To deduce the dynamics we will take a shortcut treating $e_a, g_{ab}, \mathcal{F}_{ab}$ and $X$ as independent auxiliary variables. The implementation of an AVM to study another type of bosonic brane theories is described in Appendix A.

The variation of the total action (4) with respect to the embedding functions manifest as
\[
\nabla_a \mathcal{F}^a = -\frac{1}{2} \left( \mathcal{T}^{ab} \partial_\mu G_{a\beta} + \mathcal{J}^{ab} \partial_\mu B_{a\beta} \right) e^\alpha_a e^\beta_b,
\] (5)
where $\nabla_a$ is the covariant derivative compatible with $g_{ab}$. Now, the Euler-Lagrange (EL) derivative with respect to the tangent vectors $e_a$ is
\[
\mathcal{F}^a = -\left( \mathcal{T}^{ab} G_{\mu\nu} + \mathcal{J}^{ab} B_{\mu\nu} \right) e^\nu_b,
\] (6)
which results tangential to the worldvolume. To determine the form of the worldvolume stress tensor $\mathcal{T}^{ab}$ it is necessary to know completely $\mathcal{T}^{ab}$ and $\mathcal{J}^{ab}$. Varying over the induced metric we get

\[ \mathcal{T}^{ab} = \frac{2}{\sqrt{-g}} \left( \frac{\partial \mathcal{L}_{\text{DBI}}}{\partial g_{ab}} \right) = \beta_p \sqrt{-\mathcal{M}} \frac{1}{\sqrt{-g}} (\mathcal{M}^{-1})^{(ab)}, \]

which is nothing but the metric stress tensor associated to the DBI action. Here, $(\mathcal{M}^{-1})^{ab}$ denotes the inverse matrix of $\mathcal{M}_{ab} := g_{ab} + \mathcal{F}_{ab}$, such that $(\mathcal{M}^{-1})^{ac} \mathcal{M}_{cb} = \delta^a_b$ and $\mathcal{M} = \det(\mathcal{M}_{ab})$. Further, $(\mathcal{M}^{-1})^{(ab)}$ and $(\mathcal{M}^{-1})^{[ab]}$ denote the symmetric and the antisymmetric parts of the matrix $(\mathcal{M}^{-1})$, respectively. Now, with respect to the $U(1)$ field dependence of $\mathcal{G}$, we first compute the EL derivative with respect to $A$,

\[ \nabla_b \mathcal{J}^{ba} = 0. \] (8)

This reflects into a conservation law. The remaining EL derivative with respect to $\mathcal{F}_{ab}$ reads

\[ \mathcal{J}^{ab} = \frac{2}{\sqrt{-g}} \left( \frac{\partial \mathcal{L}_{\text{DBI}}}{\partial \mathcal{F}_{ab}} \right) = -\beta_p \sqrt{-\mathcal{M}} \frac{1}{\sqrt{-g}} (\mathcal{M}^{-1})^{[ab]}. \] (9)

This complements the geometrical information to determine the stress tensor $\mathcal{F}^{ab}$. $\mathcal{J}^{ab}$ is the excitation tensor on the worldvolume of $\mathcal{F}$. Thus, the stress tensor $\mathcal{T}^{ab}$ is written in terms of geometrical and physical tensors.

By means of a straightforward computation, the divergence expression (5) yields

\[ \nabla_a \mathcal{T}^{ab} e_b^\mu - \mathcal{T}^{ab} K^i_{ab} n^i_{\mu b} = \frac{1}{2} \mathcal{T}^{ab} \left( \partial_\mu B_{\beta\alpha} + \partial_\beta B_{\alpha\mu} + \partial_\alpha B_{\mu\beta} \right) e^\alpha_a e^\beta_b, \] (10)

where we have exploited the Gauss-Weingarten equations, $\nabla_a e^\mu_b = -K^j_{ab} n^i_{\mu b} - \Gamma^\mu_{\alpha\beta} e^\alpha_a e^\beta_b$, where $K^j_{ab}$ is the extrinsic curvature of the worldvolume and $\Gamma^\mu_{\alpha\beta}$ are the connection coefficients associated to $G_{\mu\nu}$. The worldvolume projections of the relation (10) are

\[ \mathcal{T}^{ab} K^i_{ab} = \mathcal{F}^i, \] (11)

\[ \nabla_a \mathcal{T}^{ab} = 0, \] (12)

where $\mathcal{F}^i = \frac{1}{2} \mathcal{J}^{ab} H_{\alpha\beta\mu} e^\alpha_a e^\beta_b n^i_{\mu b}$ with $H_{\alpha\beta\mu} = \partial_\mu B_{\beta\alpha} + \partial_\alpha B_{\beta\mu} + \partial_\beta B_{\alpha\mu}$ being the NS strength 3-form field which satisfies $dH = 0$. We can recognize immediately the equations of motion (11) as those appearing by using the original variables $X$. The equations (12) are consistency conditions which reduce to geometrical identities (17). These equations are accompanied with the conservation law for the bicurrent $\mathcal{J}^{ab}$.

Note therefore that we have $N - p - 1$ equations of motion for $X$, (11), and $p + 1$ equations of motion for $A$, (8). The consistency conditions (12) are consequence of the reparametrization invariance of the action $S_{\text{DBI}}$. Following Carter (22), it is worthy

\[ ^{b}\text{The symmetrization and the antisymmetrization notation over the indexes of tensors is the standard one, } S_{(ab)} = \frac{1}{2}(S_{ab} + S_{ba}) \text{ and } S_{[ab]} = \frac{1}{2}(S_{ab} - S_{ba}). \]
of mention that (11) resembles Newton’s second law where $T^{ab}$ plays the role of a mass, $K_{i}^{ab}$ the generalization of the acceleration and $F^{i}$ may roughly be viewed as a force density. In the same spirit, the equations (12) yield Maxwell equations with support on the worldvolume. For completeness in our geometrical description we would like to mention that the following identity holds

$$T^{a}_{\ c} = (-g)^{-1/2} \mathcal{L}_{\delta a} \delta^{a}_{\ c} - \alpha^{-1} f^{ab} F_{bc}. \quad (13)$$

3. Born-Infeld-Lindstrom-Roček-van Nieuwenhuizen branes

Instead of adding certain matter terms to the worldvolume metric, like scalar fields, into the combined BI matrix we can consider another tensors of different nature. In this section we study a more complex BI type theory for branes, originally proposed by Lindstrom et al in order to describe, in a first-order approach, a Weyl invariant string. The main feature of this theory is the inclusion of the extrinsic curvature of the worldsheet. We put forward for consideration such suggestion for extended objects of arbitrary dimension which evoke our interest in its geometrical properties.

Consider the action

$$S_{\text{BILRvN}}[X] = a \int_{m} d^{p+1} \xi \sqrt{-\det(g_{ab} + b f_{ab})}, \quad (14)$$

where

$$f_{ab} = K^{c}_{a} K^{b}_{c}, \quad (15)$$

is a worldvolume symmetric tensor, $a$ being the tension of the extended object and $b$ is the concomitant dimensional constant characterizing the relative weight of non-linear terms. Contrary to the DBI case, the extended metric $M_{ab} := g_{ab} + b f_{ab}$ becomes symmetric. We restrict ourselves in the rest of the work to consider a flat Minkowski as background spacetime for the sake of simplicity. Thus, $G_{\mu \nu} = \eta_{\mu \nu}$.

The action (14) will be referred hereafter to as Born-Infeld-Lindstrom-Roček-van Nieuwenhuizen action (BILRvN) for branes. The symmetry underlaying this action is the invariance under worldvolume reparametrizations. Because of the explicit extrinsic curvature dependence for $f_{ab}$, the BILRvN action when expanded, falls into the collection of second order actions for strings or membranes. The situation here is slightly different in comparison with the developed one for the case of $Dp$-branes. Now, both the tensors $g_{ab}$ and $K_{i}^{ab}$ encode derivatives of the original variables $X$. Bearing in mind the way followed for the $Dp$-brane case now we shall consider (14) as an action functional of the variables $g_{ab}$ and $K_{i}^{ab}$ through the Lagrangian density $\mathcal{L}_{\text{BILRvN}} = \mathcal{L}_{\text{BILRvN}}(g_{ab}, K_{i}^{ab})$. We concern now to compute the response of the action (14) to a deformation of the worldvolume. The convenient strategy will be to distribute the deformation $X \rightarrow X + \delta X$ among independent auxiliary variables. In Appendix A we have developed the general framework to describe actions involving extrinsic curvature by invoking an AVM. Here we shall use those general results.
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For our present case, by using the results (A.7) and (A.8), a straightforward computation leads to

\[
T^{ab} = \frac{2}{\sqrt{-g}} \left( \frac{\partial \mathcal{L}_{\text{BILRvN}}}{\partial g_{ab}} \right) = a \sqrt{-M} \left( M^{-1} \right)^{cd} \left[ \delta_c^a \delta_d^b - b K_c^a K^b_{d} \right],
\]

(16)

\[
\Lambda_i^{ab} = - \frac{1}{\sqrt{-g}} \left( \frac{\partial \mathcal{L}_{\text{BILRvN}}}{\partial K_i^{ab}} \right) = - ab \sqrt{-M} \left( M^{-1} \right)^{c(a} K^{b)_{c i}},
\]

(17)

where \((M^{-1})^{ab}\) stands for the inverse matrix of \(M_{ab}\) such that \((M^{-1})^{ac} M_{cb} = \delta^a_b\) and \(M = \det(M_{ab})\). We would like to remark that \(T^{ab}\) do correspond to the metric stress tensor.

From Eq. (A.11), the conserved stress tensor then acquires the form

\[
f^a = - \left( T^{ab} + ab \sqrt{-M} \left( M^{-1} \right)^{c(a} K^{d)_{c i} K_d^{b i} \right) \right) e_b - \frac{ab}{\sqrt{-g}} \nabla_b \left[ \sqrt{-M} \left( M^{-1} \right)^{c(a} K^{b)_{c i} \right] n^i,
\]

(18)

where \(\nabla_a\) is the \(O(N - p - 1)\) covariant derivative on \(m\) and also invariant under normal rotations. To determine the equations of motion as well the geometrical consistency conditions associated to this BI type action, we project the conservation law (A.5), taking into account (18), along the worldvolume basis. Whereas the part proportional to \(n^i\) implies the equations of motion

\[
T^{ab} K^i_{ab} = F^i,
\]

(19)

with

\[
\sqrt{-g} F^i = ab \nabla_a \nabla_b \left[ \sqrt{-M} \left( M^{-1} \right)^{c(a} K^{b)_{c i} \right] - ab \sqrt{-M} \left( M^{-1} \right)^{d(a} K^{c)_{d j} K_c^e K_c^{b j} K^{i}_{ab},
\]

the tangential part becomes geometrical consistency conditions (see (A.13)). The equations of motion (19) are the normal projections of the conservation law (A.5). Only the normal deformations are physical whereas the tangential deformations simply reduce to geometrical identities associated to reparametrizations of the worldvolume coordinates. Note that we have \(N - p - 1\) equations of motion, one for each normal. Despite that the relation (19) resembles Newton’s second law, as mentioned in previous section, this is not the case. The pitfall in this reasoning is that (19) comprises a set of \(N - p - 1\) partial differential equations of fourth order in derivatives of \(X\). Explicitly, the equations of motion (19) are given by

\[
\sqrt{-M} \left[ g^{ab} + \left( M^{-1} \right)^{ab} - b \left( M^{-1} \right)^{cd} K_c^a K^b_{d} \right] K^i_{ab}
\]

\[
= 2b \nabla_a \nabla_b \left[ \sqrt{-M} \left( M^{-1} \right)^{c(a} K^{b)_{c i} \right],
\]

(20)

where we have used the expressions (16) and (17). These equations of motion are in agreement with those equations of motion obtained by usual variation methods. Further, for this BILRvN theory, by defining \(\sqrt{-g} J^{ab} := a \sqrt{-M} \left( M^{-1} \right)^{cd} K_a^{c i} K_d^{b i} \), the following identity holds

\[
T^{ab} M_{bc} = (-g)^{-1/2} \mathcal{L}_{\text{BILRvN}} \delta^a_c - b J^{ab} M_{bc},
\]

(21)
3.1. Alternative geometry

We can adopt the viewpoint of an alternative volume element depending on the worldvolume metric and the term quadratic in the extrinsic curvature given by $f_{ab}$. By allowing for an extended metric, $M_{ab} = g_{ab} + b f_{ab}$, we go beyond the ordinary theory. Suppose now that $\mathcal{D}_a$ is the covariant derivative compatible with $M_{ab}$. Then, according to this new geometry the corresponding connection coefficients, $C_{bc}^a$, can be computed straightforwardly

$$C_{bc}^a = \Gamma_{bc}^a + b (M^{-1})^{ad} (\nabla_b f_{cd} + \nabla_c f_{bd} - \nabla_d f_{bc})$$

where we have introduced the shorthand notation $E_{a}^i := \tilde{\nabla}_a n^i$ and $\Gamma_{bc}^a$ are the connection coefficients compatible with the induced metric $g_{ab}$. Bearing in mind that $\Gamma_{bc}^a = g^{ad} e_d \cdot \partial_b e_c$ we note that $E_{a}^i$ plays the role of tangent vectors. The corresponding relationship between the Ricci tensors is the following

$$R_{ab} = R_{ab} + \mathcal{D}_a S_{bc}^d - \mathcal{D}_c S_{ab}^d - S_{ac}^d S_{bd}^e + S_{ab}^d S_{cd}^e,$$

where $R_{ab}$ is the Ricci tensor associated with the connection $C_{bc}^a$ and we have defined the tensor field

$$S_{bc}^a := b (M^{-1})^{ad} E_{a}^i \cdot \nabla_b E_{c}^i.$$

In closing this subsection we expand the integrand of the action (14) using the expression (24). It is straightforward to note that

$$\sqrt{-M} = \sqrt{-g} \left\{ 1 + \frac{b}{2} \left( K^i K_i - \mathcal{R} \right) + \frac{b^2}{8} \left[ (K_{ab} K_{ab})^2 - 2 K_{ab} K_{ac} K_{bd} K_{cd} \right] + \mathcal{O}(K^3) \right\}$$

where we have used the contracted Gauss-Codazzi condition, $\mathcal{R} = K^i K_i - K_{ab} K_{ab}$. We observe that this expansion produces a vast array of interactions. To first order in $b$ we have a mixture of the Dirac-Nambu-Goto model (DNG), the Polyakov action and the Regge-Teitelboim (RT) model. The case of a string ($p = 1$) is quite special since the integral of $\sqrt{-g} \mathcal{R}$ is a topological invariant, leading thus to empty equations of motion. To second order in $b$ we have quartic terms in the extrinsic curvature of the worldvolume. We can compare this expansion with the one developed for the cosmic string dynamics which involves finite width corrections by Anderson et al. Both expansions possess a slight discrepancy in a piece of the second order correction term.

3.2. The limit $|b f_{ab}| \ll |g_{ab}|$ for the string case

We survey the application of the formalism developed above by considering the string case ($p = 1$). For illustrative purposes, we only determine the expansions up
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to first order in $b$. We have

$$\sqrt{-M} = \sqrt{-g} \left( 1 + \frac{b}{2} f \right),$$

and

$$(M^{-1})^{ab} = g^{ab} - b f^{ab},$$

where we have used the notation $f = g^{ab} f_{ab}$. Putting these results together, it follows immediately that

$$\sqrt{-M} (M^{-1})^{ab} = \sqrt{-g} \left[ g^{ab} + \frac{b}{2} (f g^{ab} - 2 f^{ab}) \right]$$

(26)

We may therefore, by inserting the expansion (26) into (20), infer the equations of motion

$$\mathcal{E}^i_0 + b \mathcal{E}^i_1 = 0,$$

where

$$\mathcal{E}^i_0 = g^{ab} K^i_{ab},$$

$$\mathcal{E}^i_1 = -\tilde{\Delta} K^i - \left( f^{ab} - \frac{1}{2} f g^{ab} \right) K^i_{ab},$$

(27)

(28)

where $\tilde{\Delta} = g^{ab} \nabla_a \nabla_b$ is the worldvolume D’Alambertian operator. We identify immediately the DNG and Polyakov equations of motion in $\mathcal{E}^i_0$ and $\mathcal{E}^i_1$, respectively, when they vanish [12,13,17,22]. The expressions (28) result highly non-trivial. Up to second order in $b$, the associated expansions and the ensuing equations of motion are quite involved. We do not pursue this issue further here. It would be important to remark that another relevant expansions taking into account the string thickness built from the extrinsic curvature in order to describe cosmic strings can be found in [8,9].

4. Born-Infeld-Einstein type brane gravity

Some time ago, Deser and Gibbons suggested an elegant modification of the Einstein-Hilbert (EH) action of general relativity [20]. Their proposal has a determinant Lagrangian density form with the spacetime metric and the Ricci tensor adding to form a new metric. Many interesting properties like the ghost freedom, regularization of some singularities, supersymmetrizability and reduction to EH action at small curvatures, made their suggestion very especial indeed. Close in the spirit to the one developed by Deser and Gibbons, in the brane context we would like to explore such proposal. The action we then consider will be

$$S_{\text{BIE}}[X] = \alpha \int_m d^{p+1} \xi \sqrt{- \det(g_{ab} + \beta R_{ab})},$$

(29)

where $\alpha$ is the tension of the brane, $\beta$ is a concomitant dimensional parameter, and $R_{ab}$ denotes the worldvolume Ricci tensor. Hereafter, we will use the acronym BIE
to denote Born-Infeld-Einstein. \( R_{ab} \) depends explicitly of the extrinsic curvature of the worldvolume, courtesy of the Gauss-Codazzi integrability condition\[21,22\]. It is given by

\[
R_{ab} = K_i K_{iab} - K_a^{\;c} K_{cbi}. \tag{30}
\]

Note that the Ricci tensor is symmetric when it is expressed in the fashion (30).

In pure gravity this is not the case in general\[c\]. In order to introduce auxiliary variables in our description, as before, we first consider (29) as a functional with Lagrangian density \( L_{BIE} = L_{BIE}(g_{ab}, K_{iab}) \). To know the response of the action (29) to a deformation of the worldvolume, \( X \rightarrow X + \delta X \), among auxiliary variables, we will continue taking advantage of the results of Appendix A, as in the previous Section.

As before, we will use the notation \((M^{-1})^{ab}\) to denote the inverse matrix of \( M_{ab} := g_{ab} + \beta R_{ab} \), such that \((M^{-1})^{ac}(M^{-1})^{bd} = \delta^a_b\). Expressions (A.7) and (A.8) specialized to the BIE action result

\[
T_{ab} = \alpha \sqrt{-g} (M^{-1})^{cd} \left( \delta^a_c \delta^b_d - \beta R_{acbd} \right), \tag{31}
\]

\[
\Lambda_{ab} = -\frac{\alpha \beta}{\sqrt{-g}} G^{abcd} K_{cdi}, \tag{32}
\]

where we have used the worldvolume Riemann tensor expressed in terms of the extrinsic curvature via the Gauss-Codazzi equation,

\[
R_{abcd} = K_{i}^{\;ac} K_{bd}^{\;i} - K_{ad}^{\;i} K_{bc}^{\;i},
\]

and we have introduced the tensor field

\[
G^{abcd} = (M^{-1})^{ab} g^{cd} + (M^{-1})^{cd} g^{ab} - 2(M^{-1})^{(a(g)bd)}. \tag{33}
\]

The conserved stress tensor for our present case, from Eqs. (31), (32) and (A.11), is given by

\[
f^a = -\left\{ T^{ab} + \alpha \beta \sqrt{-g} \left( (M^{-1})^{ac} R_{c}^{\;b} + (M^{-1})^{cd} R_{cd}^{\;a} c \right) \right\} e_b
- \frac{\alpha \beta}{\sqrt{-g}} \nabla_b \left( \sqrt{-M} G^{abcd} K_{cdi} \right) n^i. \tag{34}
\]

As argued earlier, insisting on the conservation law for \( f^a \), its normal projection produces the equations of motion

\[
T^{ab} K_{ab}^{\;i} = F^i, \tag{35}
\]

where

\[
F^i = \frac{\alpha \beta}{\sqrt{-g}} \left[ \nabla_a \nabla_b \left( \sqrt{-M} G^{abcd} K_{cdi}^{\;j} \right) - \sqrt{-M} G^{abcd} K_{cdi}^{\;j} K_{bc}^{\;j} K_{ad}^{\;i} \right].
\]

Once more, apparently the \( N - p - 1 \) equations of motion (35) resembles Newton’s second law, but in general these equations are of fourth order in derivatives of \( X \).

\(^c\)Of course, the worldvolume Ricci tensor can be expressed also as \( R_{ab} = 2\partial_{[a} \Gamma_{b]}^{c} - 2\Gamma_{a}^{d} \Gamma_{b]d}^{c} - \Gamma_{a}^{c} \Gamma_{b]d}^{d} \).
There is comparison only up to first order in $\beta$. We will back on this point. Explicitly, the ensuing equations of motion (35) are

$$\sqrt{-M} \left[ g^{ab} + (M^{-1})^{ab} - \beta (M^{-1})^{cd} R_{c}^{\ a} \ b \right] K_{i}^{ab} = \beta \bar{\nabla}_{a} \bar{\nabla}_{b} \left( \sqrt{-M} G^{abcd} K_{i}^{cd} \right).$$

(36)

Once again, these equations of motion are in agreement with those equations of motion obtained by usual variation methods $^{17,22}$. To my knowledge, this form of the equations of motion has not been previously discussed. By the way, it is possible to exploit the Codazzi-Mainardi equation,

$$\bar{\nabla}_{a} K_{i}^{bc} = \bar{\nabla}_{b} K_{i}^{ac},$$

(37)

to separate multiplicatively the extrinsic curvature on the right-hand side of the previous relation. A straightforward computation yields

$$\bar{\nabla}_{a} \bar{\nabla}_{b} \left( \sqrt{-M} G^{abcd} K_{i}^{cd} \right) = \left\{ \nabla_{c} \nabla_{c} \left[ \sqrt{-M} (M^{-1})^{ab} \right] + \nabla_{c} \nabla_{d} \left[ \sqrt{-M} (M^{-1})^{cd} \right] g^{ab} - 2 \nabla_{c} \nabla_{d} \left[ \sqrt{-M} (M^{-1})^{ac(b} \right] g^{d)c} \right\} K_{i}^{ab}.$$ 

(38)

For completeness, as in previous cases, working out the expression defining the inverse matrix $(M^{-1})^{ab}$, we have the identity

$$T^{ab} M_{bc} = (-g)^{-1/2} F_{abc} - \beta P^{ab} M_{bc},$$

(39)

with $\sqrt{-g} P^{ab} := \alpha \sqrt{-M} (M^{-1})^{cd} R_{c}^{\ a} d^{b}$.

### 4.1. Alternative geometry

Analogous to the foregoing Section, for this model we can think also of an alternative volume element, now depending of the worldvolume metric and a term quadratic in the extrinsic curvature by the Ricci tensor. Suppose now that $D_{a}$ is the covariant derivative compatible with $M^{ab} = g^{ab} + \beta R_{ab}$. Then, according to this new geometry the corresponding connection coefficients $C^{a}_{bc}$ are given by

$$C^{a}_{bc} = \Gamma^{a}_{bc} + \frac{\beta}{2} (M^{-1})^{ad} \left( \nabla_{d} R_{cd} + \nabla_{c} R_{bd} - \nabla_{d} R_{bc} \right),$$

(40)

where $\Gamma^{a}_{bc}$ are the connection coefficients associated with the induced metric $g_{ab}$. The corresponding relationship between the Ricci tensors is now the following

$$R_{ab} = R_{ab} + D_{a} S^{c}_{b} - D_{b} S^{c}_{a} - S^{d}_{ac} S_{bd} + S^{d}_{ab} S_{cd},$$

(41)

where $R_{ab}$ is the Ricci tensor associated with the connection $C^{a}_{bc}$, and we have defined

$$S^{a}_{bc} := \frac{\beta}{2} (M^{-1})^{ad} \left( \nabla_{d} R_{cd} + \nabla_{c} R_{bd} - \nabla_{d} R_{bc} \right).$$

(42)

By considering this new structure it is straightforward to obtain the identity

$$\nabla_{a} \sqrt{-M} = \sqrt{-M} S_{ab}.$$

As before we turn now to expand the integrand of the action (14) using the expression $^{(B.4)}$. It is straightforward to note that

$$\sqrt{-M} = \sqrt{-g} \left\{ 1 + \frac{\beta}{2} R + \frac{\beta^{2}}{8} (R^{2} - 2 R_{ab} R^{ab}) + O(R^{3}) \right\}.$$
We observe that to first order in $\beta$ we have the sum of the DNG model and the RT model. Here it is worthy to note that the case of a string ($p = 1$) is quite special since the term $\sqrt{-g}R$ corresponds to a total derivative and in consequence we have a model classically equivalent to the DNG action. Furthermore, in general to first order in $\beta$ the equations of motion results of second order in the derivatives of the fields. Note that only up to second order in $\beta$, the strong second order derivative terms come into play where we have quartic terms in the extrinsic curvature of the worldvolume.

4.2. The limit $|\beta R_{ab}| \ll |g_{ab}|$ for the brane world scenarios

A special case worth pointing out is the $p = 3$ case. From the expansions up to second order in $\beta$ of the main geometrical components given by

$$\sqrt{-M} = \sqrt{-g} \left[ 1 + \frac{\beta}{2} R + \frac{\beta^2}{8} (R^2 - 2R_{ab}R^{ab}) \right],$$

and

$$(M^{-1})^{ab} = g^{ab} - \beta R^{ab} + \beta^2 R^a_c R^{cb},$$

we evidently get

$$\sqrt{-M}(M^{-1})^{ab} = \sqrt{-g} \left[ g^{ab} - \beta G^{ab} + \frac{\beta^2}{8} \left( (R^2 - 2R_{cd}R^{cd}) g^{ab} - 4RR^{ab} \right) ight. + \left. 8R^a_c R^{cb} \right].$$

where $G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}$ is the worldvolume Einstein tensor.

Therefore, taking into account the expression (47) and after some algebra, the equations of motion coming from (36) can be rearranged into

$$\mathcal{E}_0^i + \beta \mathcal{E}_1^i + \beta^2 \mathcal{E}_2^i = 0,$$

where

$$\mathcal{E}_0^i = g^{ab} K_{ab}^i, \quad \mathcal{E}_1^i = -G^{ab} K_{ab}^i, \quad \mathcal{E}_2^i = S^{ab} K_{ab}^i,$$

where we have introduced the tensor field

$$S_{ab} = \nabla^c \nabla_c R_{ab} - \frac{1}{2} g_{ab} \nabla^c \nabla_c R + 2R^{cd}R_{acbd} - R R_{ab} + \frac{1}{4} g_{ab} (R^2 - 2R_{cd}R^{cd}).$$

It must be noted that up to second order in $\beta$, the fourth-order equations of motion in the field variables appear. These equations are geometrical and physically correct. The $N-p-1$ equations (44), if vanish, correspond to the well known DNG equations of motion (22). Equally, when the expressions (45) vanish, these correspond to RT equations of motion (26) (27) (28). Notice that, the field equations for the string case ($p = 1$) are those of the DNG case. Moreover, the remaining relations (46) are quite...
involved. These are highly non-linear field equations. Such equations of motion are obtained from the model $L = \mathcal{R}^2 - 2\mathcal{R}_{ab}\mathcal{R}^{ab}$ which has recently proposed as a brane correction to the Dvali-Gabadadze-Porrati model proposed as a solution for the cosmological constant problem. Actually, the form as the equations of motion are presented allows us to identify the piece associated to the $\beta^2$ expression as a Weyl term, which is a geometric correction to the RT equations of motion. The dynamics of this type of extended objects is highly non-trivial but striking in the search of a proper solution of the cosmological constant problem. This model require a more careful and deep study under this approach which will be reported elsewhere.

5. Concluding remarks

A simple recipe has been presented for studying the dynamical structure of some BI type brane theories by exploiting the mechanical content that the conserved stress tensor possesses. This is achieved by using an auxiliary variables method. After considering appropriate quadratic geometrical constraints it is possible to obtain the response of every considered BI type brane action to a deformation $X \to X + \delta X$ once it is distributed among the auxiliary variables. We have seen that judicious choices for the constraints are intended to facilitate the calculation of the brane dynamics thus providing an economical and efficient way to explore the motion of interesting extended objects. In addition, we have presented a Born-Infeld-Einstein action adapted to the brane context, closed in spirit of the one developed by Deser and Gibbons. To my knowledge, this action has not been previously discussed in the brane content. This action contains a rich geometrical content and add new insights with likely benefit for the study of branes in a cosmological context. To lowest order in the rigidity parameter, it contains both DNG and RT models, which are first order theories. Nevertheless, the full second order derivative dependence is switched on up to second order terms in the rigidity term expansion. This modify substantially the RT gravity, by incorporating quadratic terms in the Ricci curvature. It would be interesting to investigate the relevance of this model in the brane world universes context where interesting physical background spacetimes must be considered. It is intended to discuss this in a separate work.

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Likewise to the Einstein tensor in the case that the field variable is the induced metric.
Appendix A. Auxiliary variables method

Consider a brane, denoted by $\Sigma$, of dimension $p$ evolving in a $N$-dimensional fixed Minkowski background spacetime with metric $\eta_{\mu\nu}$ ($\mu, \nu = 0, 1, \ldots, N-1$). The worldvolume $m$, swept out by $\Sigma$, is an oriented timelike manifold of dimension $p + 1$, described by the embedding functions $x^\mu = X^\mu(\xi^a)$ where $x^\mu$ are local coordinates of the background spacetime, $\xi^a$ are local coordinates of $m$, and $X^\mu$ represent the embedding functions ($a, b = 0, 1, \ldots, p$). The metric induced and the extrinsic curvature defined on the worldvolume are

$$g_{ab} = e_a \cdot e_b \quad K^i_{ab} = e_a \cdot \tilde{\nabla}_b n^i,$$

(A.1)

where $\tilde{\nabla}_a$ is the $O(N - p - 1)$ covariant derivative on $m$ and also invariant under normal rotations and we have exploited the Gauss-Weingarten equation, $\tilde{\nabla}_a n^i = K^i_{ab} g^{bc} e_c$.[22] Here, $e_a = \partial_a X$ and $n^i$ denote the tangent and unit normal vectors defined on $m$, respectively

$$n^i \cdot e_a = 0 \quad n^i \cdot n^j = \delta^{ij},$$

(A.2)

where $(i, j = 1, 2, \ldots, N - p - 1)$. Note that the tensors (A.1) encode derivatives of the field variables $X$. Suppose that the following generic action governs the dynamics of $\Sigma$

$$S_0[X] = \int_m d^{p+1}\xi \mathcal{L}(g_{ab}, K^i_{ab}),$$

(A.3)

where $\mathcal{L} = \mathcal{L}(g_{ab}, K^i_{ab})$ is the Lagrangian density of the field theory underlying the dynamics of $\Sigma$. At the technical level, this type of Lagrangians is complicated to handle.

In order to know the response of the action (A.3) to a deformation of the worldvolume, $X \rightarrow X + \delta X$, among auxiliary variables, we must first construct an extended action by considering constraints manifestly[18]

$$S[X, e_a, n^i, g_{ab}, f^a, \Lambda^i_{ab}, \Lambda^a_{ij}, \Lambda_e^a] = S_0[g_{ab}, K^i_{ab}] + \int_m dV f^a \cdot (e_a - \partial_a X) + \int_m dV \Lambda^a_i (e_a \cdot n^i) + \frac{1}{2} \int_m dV \Lambda_{ij} (n^i \cdot n^j - \delta^{ij}) - \frac{1}{2} \int_m dV T^{ab}(g_{ab} - e_a \cdot e_b) + \int_m dV \Lambda^a_i \left(K^i_{ab} - e_a \cdot \tilde{\nabla}_b n^i\right).$$

(A.4)

In this approach, $f^a, \Lambda^a_i, \Lambda_{ij}, T^{ab}$ and $\Lambda^a_{ib}$ are Lagrange multipliers enforcing the definitions of the auxiliary variables (A.1) and (A.2) whereas $X, e_a, n^i, g_{ab}$ and $K^i_{ab}$ are considered as independent fields. We perform the variation in steps. The variation of (A.3) with respect to $X$ reproduces the covariant conservation of $f^a$

$$\nabla_a f^a = 0.$$

(A.5)

Moreover, varying over tangent vectors we get that

$$f^a = - \left(T^{ab} - \Lambda^a_{ic} K^i_{bc}\right) e_b - \Lambda^a_i n^i.$$  

(A.6)
results an expression for $f^a$ as a linear expansion in terms of the worldvolume basis. This occurs since we are dealing with a second order derivative theory. The EL derivative of the action (A.4) with respect to the induced metric casts out

$$T^{ab} = \frac{2}{\sqrt{-g}} \left( \frac{\partial L}{\partial g_{ab}} \right).$$

This corresponds to the worldvolume stress tensor. Similarly, the EL derivative with respect to the extrinsic curvature results

$$\Lambda^{ab}_i = - \frac{1}{\sqrt{-g}} \left( \frac{\partial L}{\partial K^{ij}_{ab}} \right).$$

Moreover, the geometrical information contained in the Lagrange multipliers is associated with the EL derivative with respect to the normals $n^i$. They yield

$$\Lambda^a_i = - \tilde{\nabla}^b \Lambda^{ab}_i,$$

$$\Lambda_{ij} = \Lambda^{ab}_i K^j_{ab},$$

where we have considered, $\tilde{\nabla}_a e_b = -K^i_{ab} n_i$. Finally, the conserved stress tensor is given by

$$f^a = - (T^{ab} - \Lambda^{ac}_{i} K^{ij}_{cb}) e_b + \left( \tilde{\nabla}^b \Lambda^{ab}_i \right) n^i.$$

As argued earlier, in order to explore the contents of the conservation law for $f^a$, we observe that its normal projection produces the equations of motion, usually written as

$$T^{ab} K^i_{ab} = F^i,$$

with

$$F_i = - \tilde{\nabla}_a T^{ab} \Lambda^{ac}_{i} K^{bj}_{cb} + 2 \tilde{\nabla}_a \Lambda^{ac}_{i} K^{bj}_{cb}.$$

whereas the tangential part is given by

$$\nabla_a T^{ab} = \Lambda^{ac}_{i} \tilde{\nabla}_a K^{bi}_{cb} + 2 \tilde{\nabla}_a \Lambda^{ac}_{i} K^{bj}_{cb}.$$  

These latter are consistency conditions which are consequence of the reparametrization invariance of the action (A.3). Apparently the $N - p - 1$ equations of motion (A.4) resembles Newton’s second law, but in general these equations are of fourth order in derivatives of $X$.

### Appendix B. Determinant of a matrix

The Levi-Civita tensor in $n$-dimensions, $\epsilon^{a_1 a_2 \ldots a_n}$, is related to a totally antisymmetric pseudotensor by the relation

$$\epsilon^{a_1 a_2 \ldots a_n} = \sqrt{-g} \epsilon_{a_1 a_2 \ldots a_n},$$

where $\epsilon_{a_1 a_2 \ldots a_n}$ is the Levi-Civita pseudotensor which is a tensorial density of weight $\omega = -1$. 

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where $\epsilon_{a_1 a_2 \ldots a_n}$ is the Levi-Civita pseudotensor which is a tensorial density of weight $\omega = -1$. 

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The determinant of a matrix $M_{ab}$ can be defined in terms of the Levi-Civita pseudotensor by

$$ M := \det(M_{ab}) = \frac{1}{n!} \epsilon^{a_1, a_2, \ldots, a_n} \epsilon^{b_1, b_2, \ldots, b_n} M_{a_1 b_1} M_{a_2 b_2} \cdots M_{a_n b_n}, \quad (B.2) $$

where we assume that $\epsilon_{1,2,\ldots,n} = 1$. When $M_{ab}$ is non-singular, its inverse matrix has a representation in terms of the Levi-Civita pseudotensor

$$ (M^{-1})^{a_1 b_1} = \frac{1}{(n-1)!} \epsilon^{a_1, a_2, a_3, \ldots, a_n} \epsilon^{b_1, b_2, b_3, \ldots, b_n} M_{b_2 a_2} M_{b_3 a_3} \cdots M_{b_n a_n}. \quad (B.3) $$

Let $A$ be a $n \times n$ matrix. It results useful to expand the determinant of $M := I + a A$,

$$ [\det (I + a A)]^{1/2} = 1 + \frac{a}{2} \text{Tr} A + \frac{a^2}{8} \left[ (\text{Tr} A)^2 - 2\text{Tr} A^2 \right] + O(a^3), \quad (B.4) $$

where $a$ is a constant and $I$ being the $n \times n$ identity matrix.

References

1. M. Born, *Phys. Roy. Soc.* A143 (1934) 410.
2. M. Born and L. Infeld, *Phys. Roy. Soc.* A144 (1934) 425.
3. J. Polchinski, *Phys. Rev. Lett.* 75 (1995) 4724.
4. C. Johnson *D-branes* (Cambridge University Press, Cambridge, UK, 2003).
5. U. Lindstrom, M. Rocek and P. Van Nieuwenhuizen, *Phys. Lett. B* 199 (1987) 219; *Phys. Lett. B* 201 (1988) 63.
6. U. Lindstrom, *Int. J. Mod. Phys. A* 3 (1988) 2401.
7. B. Carter and R. Gregory, *Phys. Rev. D* 51 (1995) 5839.
8. M. Anderson, F. Bonjour, R. Gregory and J. Stewart, *Phys. Rev. D* 56 (1997) 8014.
9. M. Anderson, *Phys. Rev. D* 51 (1995) 2863.
10. A. M. Polyakov, *Nucl. Phys. B* 268 (1986) 406.
11. H. Kleinert, *Phys. Lett. B* 174 (1986).
12. D. Garfinkle and R. Gregory, *Phys. Rev. D* 41 (1990).
13. R. Gregory, *Phys. Rev. D* 43 (1991).
14. B. Carter and R. Gregory, *Phys. Rev. D* 51 (1995) 5839.
15. L. Randall and R. Sundrum, *Phys. Rev. Lett.* 83 3370 (1999); 83 4690 (1999).
16. M. Cadoni and P. Pani, *Phys. Lett. B* 674 (2009) 308.
17. G. Arreaga, R. Capovilla and J. Guven, *Ann. Phys. NY* 279 (2000) 126.
18. J. Guven, *J. Phys. A: Math. and Theor.* 37 (2004) L313.
19. R. Cordero, A. Molgado and E. Rojas, *Class. Quant. Grav.* 24 (2007) 1665.
20. S. Deser and G. W. Gibbons, *Class. Quant. Grav.* 15 (1998) L35-L39.
21. M. Spivak, *Comprehensive Introduction to Differential Geometry* Vol. 4 2nd edn (Publish or Perish, Boston, MA, 1970).
22. R. Capovilla and J. Guven, *Phys. Rev. D* D51 (1995) 6736.
23. B. Carter, *Int. J. Theor. Phys.* 40 (2001) 2099.
24. P. Bostock, R. Gregory, I. Navarro and J. Santiago, *Phys. Rev. Lett.* 92 (2004) 221601
25. R. Wald, *General Relativity* (The University of Chicago Press, 1986).
26. T. Regge and C. Teitelboim, Gravity à la string, in *Proceedings of the Marcel Grossman Meeting, Trieste, Italy* 1975, eds. R. Ruffini (North-Holland, Amsterdam, 1977), p. 77.
27. R. Cordero, A. Molgado and E. Rojas, *Phys. Rev. D* 79 (2009) 024024.
28. R. Capovilla, A. Escalante, J. Guven and E. Rojas, [hep-th/0605160](http://arxiv.org/abs/hep-th/0605160).