Problem of damping oscillations of a mechanical system with integral memory

Tatiana Bobyleva¹, and Alexey Shamaev²,³

¹ Moscow State University of Civil Engineering, Yaroslavskoe shosse, 26, Moscow, Russia
² Ishlinsky Institute for Problems in Mechanics of the Russian Academy of Sciences, 101-1, Pr. Vernadskogo, 119526, Moscow, Russia
³ Lomonosov Moscow State University, GSP-1, Leninskie Gory, Moscow, 119991, Russia

E-mail: tatyana2211@outlook.com

Abstract. The paper considers a mechanical system defined by a linear system of integro-differential equations with nonlocal convolution type terms. The problem of controllability is studied for various types of control (boundary and volume-distributed control) and various types of kernels. These kernels simulate the aftereffect of the system. It is proved that in some cases there is no possibility of damping the oscillations of the system for arbitrary initial conditions. In this work, those cases are distinguished when there is the possibility of damping oscillations over a finite period of time for any initial conditions.

1. Introduction
Calculation, design and construction of high-rise buildings pose such problems for engineers as ensuring the safety of buildings and structures under variable loads, especially wind and seismic, as well as ensuring their durability, ability to control strength, rigidity and basic dynamics, characteristics of building structures. The use of passive and active oscillation control systems can be considered as one of the approaches to solving the above problems. Control systems allow you to limit unwanted deformations, displacements and stresses, to control and change dynamic characteristics, using control forces that counteract external influences. Problems similar to those considered in this paper also arise in the case of damping oscillations of pipelines with a liquid. Currently, many high-rise buildings and other modern structures are not without control systems. The results of solving the problem of finding the optimal synthesizing control function of the damping process in the vibration isolation system are given in [1]. It is proved in the well-known monograph [2,3] and in the article [4] that by controlling the string at one of its ends with a small force or with the help of a small displacement it is possible to stop the oscillations of the string in a finite time. It is also easy to prove that a similar result is valid for the telegraph equation, which is a wave equation with an additional term that depends on the displacement rate [5]. Namely, by controlling a small force or a small displacement of one of the ends, it is possible to completely stop oscillations in a finite time. More precisely, in this case, the oscillatory system is modeled using the following equation $c^2\dddot{u} + \alpha \ddot{u} = u''$ on the segment $x \in [0, \pi], t \geq 0, u = u(t, x)$. Here, the dot means the derivative with respect to time, and the prime denotes the derivative with respect to the spatial variable.
The control action is \( u(0,t) = f(t) \) or \( u'(0,t) = g(t) \). For example, a next condition is specified \( u(\pi,t) = 0 \) at the right end of the segment. Instead of the term \( \hat{u} \) that simulates the resistance proportional to speed in a moving medium we consider the term \( \hat{u}' \) that simulates the Kelvin-Voigt friction. Kelvin-Voigt friction in this form is one of the models of internal friction in a material associated with the friction of different parts of the material against each other during deformation. Moreover, such friction also leads to damping of oscillations as well as friction against the external environment. However, studies show that the presence of Kelvin-Voigt friction in the system makes it impossible the damping of system oscillations in a finite time. In fact, in the work [4] a condition is established under which it is impossible to bring the system oscillations from an arbitrary initial state to complete rest. Namely, in this work, the characteristic of the spectrum density of the problem under consideration is introduced, which was previously introduced by the American mathematician N. Levinson to study other issues

\[
\hat{a} = \lim_{R \to \infty} \frac{2}{R} \int_{-R}^{R} \frac{n(t)}{t} dt,
\]

where \( n(t) \) is the number of spectrum points on the complex plane contained in a circle of radius \( t \) of the boundary value problem with homogeneous boundary conditions obtained from the original Laplace transform with respect to the variable \( t \). In this paper it was proved that if the spectrum is sufficiently “dense”, then the control problem we are considering does not have a solution for arbitrary initial conditions (for \( t = 0 \)). We note that in the presence of Kelvin – Voigt friction, the spectrum necessarily has an accumulation point on the real axis. Here, the spectrum is understood as the set of such complex values of the Laplace transform parameter that the corresponding boundary value problem with zero boundary conditions has a nonzero solution. You can verify this by noting that finding the spectrum in this case reduces to analyzing the solution of quadratic equations with respect to the spectral parameter. This simple analysis shows that there is a countable number of spectrum points on the real axis with a finite accumulation point, an accumulation point \(-\infty\), and also a finite number of complex conjugate pairs of spectrum points lying in the left half-plane. Thus, the characteristic of the density of the spectrum \( \hat{a} \) will be \( \infty \) in our case, and the problem of controllability of the system is unsolvable. It should be noted that the asymptotically solution of the problem, of course, exponentially quickly tends to zero. The problem of calming the oscillations is insoluble precisely in a finite time.

The situation changes if we consider external force as a control action \( f(x,t) \), distributed over the entire length of the segment. Similar problems were considered in [6] without taking into account the Kelvin-Voigt friction. All control problems considered in this paper have a solution. That is, the system can be brought to complete rest for a finite time by a control distributed and limited in size. In this paper, we show that the same result occurs in the presence of Kelvin-Voigt friction, i.e., Kelvin-Voigt friction for controlling distributed over the entire length of the segment is not an obstacle for bringing the system to rest for a finite time.

Integro-differential models are often used in the study of viscoelastic systems. So friction in the system is often described as a convolutional term

\[
\int_{0}^{t} K(t-\tau) u(x,\tau) d\tau.
\]

The relaxation core of the system is often chosen as the sum of decreasing exponential functions. Such a kernel does not have a singularity at zero. However in a number of studies the function \( K(t) \) is chosen as a function with an integrable singularity at the point zero [7]. We now consider similar problems of distributed boundary control in the case \( K(t) = \mathcal{E}_{\alpha}(-\lambda, t) \), where \( \mathcal{E} \) is the Rabotnov function [7]. This function does not have an explicit analytical expression, however, it can easily be
set using the Laplace transform: \( \hat{\mathcal{A}}_c(-\lambda, p) = (p^{1-a} + \lambda)^{-1} \). It turns out and this will be proved below for a given kernel neither the boundary control problem, nor the control problem distributed over the entire segment, the purpose of which is to bring peace to rest in a finite time, is unsolvable.

2. Problem specification and decision

We will seek a solution to the control problem, namely, the function \( f(x,t) \) in the form of an expansion according to the following system of functions: \( \{ \sin nx \}, \ n = 1,2,\ldots \). Moreover, we assume that the function \( u(t,x) \) has zero boundary values at \( x = 0, \pi \). We could consider other boundary values; in principle, this would not change the result of the existence of a control action \( f(x,t) \) that satisfies to the next restriction \( |f(t,x)| < \varepsilon \). Expanding the unknown solution \( u(t,x) \) in a series according to the same system of functions, we come to the need to solve a countable number of rest problems for second order ordinary linear differential equations with the help of limited force. Expanding the unknown solution \( u(t,x) \) in a series according to the same system of functions, we come to the need to solve a countable number of rest problems for the linear second order ordinary differential equations with the help of limited force. Moreover, it is easy to see that only a finite number of these ordinary differential equations will describe oscillatory motions, which corresponds to a finite number of complex points of the this problem spectrum. An infinite number of spectrum points corresponds to real negative eigenvalues. Below we show that such problems of calming the oscillations can be solved in a finite time, which can be chosen common for the entire countable number of these problems. We further consider the case of only purely real negative eigenvalues. The method of constructing a control action for the case of complex eigenvalues is absolutely similar.

Let's look at the problem of controlling system movement

\[
\ddot{u} + a\dot{u} + bu = f(t) \tag{3}
\]

\[
u(0) = A, \quad \dot{u}(0) = B, \quad |f(t)| < \varepsilon. \tag{4}
\]

In (3-4) \( u = u(t) \) is an unknown time dependent function, the dot denotes the time derivative, \( a,b,A,B,\varepsilon \) are constants and \( \varepsilon > 0 \), \( f(t) \) is a control function. Here \( u \) is a function of one variable \( t \), \( u \) describes the dynamics of the Fourier coefficient, that is the coefficient for one of the basis functions when the solution is expanded into a Fourier series.

The goal of control is to stop system oscillations, that is to find the function \( f(t) \), such that \( u(t) \equiv 0 \) for all \( t > T \), if \( T \) is a given constant.

Suppose that the characteristic equation for differential equation (3) has two different negative roots \( \lambda_1, \lambda_2 \). This assumption corresponds to the case of system motion without oscillations. For an oscillatory system, the solution of the problem of rest at a finite time is similar to the following. Then the solution of problem (3), (4) for a given function \( f(t) \) has the following form

\[
u(t) = \frac{B - A\lambda_2}{\lambda_1 - \lambda_2} e^{\lambda_1 t} + \frac{A\lambda_1 - B}{\lambda_1 - \lambda_2} e^{\lambda_2 t} + \int_0^t \frac{f(\tau)}{\lambda_1 - \lambda_2} (e^{\lambda_1 (t-\tau)} - e^{\lambda_2 (t-\tau)}) d\tau. \tag{5}
\]

We show there is a time moment \( T \) for which

\[
u(T) = \dot{u}(T) = 0 \tag{6}
\]

with the right choice of \( f(t) \) function. We introduce the following notation

\[
k_1 = \frac{\lambda_1 (B - A\lambda_2)}{\lambda_1 - \lambda_2}, k_2 = \frac{\lambda_2 (A\lambda_1 - B)}{\lambda_1 - \lambda_2}. \tag{7}
\]
Then the conditions for the system to be at rest (6) will take the following form

\[ k_1 e^{kT} + k_2 e^{kT} + R_1 e^{kT} \int_0^T e^{-k\tau} f(\tau) d\tau + R_2 e^{kT} \int_0^T e^{-k\tau} f(\tau) d\tau = 0 \]  

\[ \lambda_1 k_1 e^{kT} + \lambda_2 k_2 e^{kT} + \lambda_1 R_1 e^{kT} \int_0^T e^{-k\tau} f(\tau) d\tau + \lambda_2 R_2 e^{kT} \int_0^T e^{-k\tau} f(\tau) d\tau = 0. \]  

Here we have designated as \( R_1 = \frac{\lambda_1}{\lambda_1 - \lambda_2} \), \( R_2 = -\frac{\lambda_2}{\lambda_1 - \lambda_2} \).

To satisfy equalities (8), (9) we will seek a control function in the form

\[ f(t) = C_1 e^{-k_1 t} + C_2 e^{-k_2 t}, \]

where constants \( C_1, C_2 \) are to be determined.

We introduce the notation

\[ D_{ij} = \int_0^T e^{-(\lambda_i + \lambda_j)\tau} dt, \quad (i, j = 1, 2), \]

then conditions (8), (9) take the form

\[ k_1 e^{kT} + k_2 e^{kT} + (R_1 e^{kT} D_{11} + R_2 e^{kT} D_{21}) C_1 + (R_1 e^{kT} D_{12} + R_2 e^{kT} D_{22}) C_2 = 0 \]

\[ \lambda_1 k_1 e^{kT} + \lambda_2 k_2 e^{kT} + (\lambda_1 R_1 e^{kT} D_{11} + \lambda_2 R_2 e^{kT} D_{21}) C_1 + (\lambda_1 R_1 e^{kT} D_{12} + \lambda_2 R_2 e^{kT} D_{22}) C_2 = 0. \]

Thus, constants \( C_1, C_2 \) are a solution to a second order linear equation system \( M \cdot \vec{C} = \vec{r} \) with the matrix \( M \)

\[
M = \begin{pmatrix}
R_1 e^{kT} D_{11} + R_2 e^{kT} D_{21} & R_1 e^{kT} D_{12} + R_2 e^{kT} D_{22} \\
\lambda_1 R_1 e^{kT} D_{11} + \lambda_2 R_2 e^{kT} D_{21} & \lambda_1 R_1 e^{kT} D_{12} + \lambda_2 R_2 e^{kT} D_{22}
\end{pmatrix},
\]

right side \( \vec{r} \) and unknown vector \( \vec{C} \)

\[ \vec{r} = -\begin{pmatrix} k_1 e^{kT} + k_2 e^{kT} \\ \lambda_1 k_1 e^{kT} + \lambda_2 k_2 e^{kT} \end{pmatrix}, \quad \vec{C} = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}. \]

We study the behavior of the solution \( \vec{C} \) taking into account that the constants \( k_1, k_2, R_1, R_2 \) are independent of \( T \), and the constants \( D_{ij}(T) \) are exponentially increasing when \( T \to \infty \). Also the determinant of the matrix \( M \) will exponentially increase as \( e^{-(\lambda_1 + \lambda_2)T} \), when \( T \to \infty \), and the unknowns \( C_1, C_2 \) obtained, for example, according to the Cramer rule, will be limited. Therefore, the solution to the control problem \( f(t) = C_1(T) e^{k_1 T} + C_2(T) e^{k_2 T} \) will exponentially decrease on the \([0; T]\) interval. Hence it is easy to determine the control time needed to stop the system \( T \preceq -c \ln(\varepsilon), \) where \( c > 0 \) and \( c \) is independent of \( \varepsilon \).

Using the proved statement on the system that describes the motion of the individual Fourier components of the solution, it is easy to show that the motion of the entire system can be brought to rest in a finite time. It is only necessary to prove (and this is not difficult) that for all Fourier decomposition components, the rest time can be chosen as a common one.
Next, we consider a system with an integral term of convolution type, in the study of which the same approach can be applied, consisting in constructing a control for each term in the Fourier expansion of the desired solution. The system already contains a term corresponding to Kelvin–Voigt friction, and an integral member of the convolution type, which models the integral aftereffect in the system. It has the form of convolution, in which the role of the kernel is played by an exponential function. In this case, the task of bringing to rest with the help of a distributed force in a finite time is also solvable.

We now consider the following control problem

\[ \ddot{u} + \alpha\dot{u} + \beta^2 u - \beta^2 u + \beta^2 \lambda u = \lambda f(x,t), \]  

(16)

where \( f(x,t) \) is control, \( K(t) = C \exp(-\lambda t), \) \( u(t,x) = 0 \) at \( t = T \), \( u(0, x) = \varphi(x), \dot{u}(0, x) = \psi(x). \)

We differentiate the equation (16) with respect to \( t \). We get

\[ \ddot{u} + \alpha\dot{u} + \beta^2 u - \beta^2 u - \beta^2 \lambda u + \int_{0}^{t} \lambda e^{-\lambda(t - \tau)} u_{ss}(\tau) d\tau = \lambda f(x,t). \]  

(18)

We multiply the equation (16) by \( \lambda \) and add to equation (18). Then the integral terms mutually annihilate, and we get the following equation

\[ \ddot{u} + \alpha\dot{u} + (\alpha + C^2) u - C u + \lambda \ddot{u} - \beta^2 \lambda u = \lambda f(x,t) + \dot{f}(x,t). \]  

(19)

Then the solution to the problem (19) can be sought in the form \( u(t,x) = \sum_{k=1}^{\infty} u_k(t) \sin kx \). We obtain for the Fourier coefficients the equations

\[ u_k(t) = \int_{0}^{t} \lambda f(x,t) + \dot{f}(x,t) = \sum_{k=1}^{\infty} g_k(t) \sin kx. \]  

To solve the problem (20) i.e., construct a control \( |g_k(t)| < \varepsilon \) such that \( u_k(t) = 0 \) at \( t = T \), we represent a solution to a third-order equation with constant coefficients in the form

\[ u_k(t) = C_1 e^{-\mu_1 t} + C_2 e^{-\mu_2 t} + C_3 e^{-\mu_3 t} + \int_{0}^{t} P_k(t - \tau) g_k(\tau) d\tau, \]  

(21)

where \( P_k(t) \) is the fundamental solution to the equation (20), composed of exponential functions \( e^{-\mu_i t} (i = 1, 2, 3) \), so that \( P_k(0) = \dot{P}_k(0) = 0, \ddot{P}_k(0) = 1. \)

We will look for an unknown control \( g_k(\tau) \) in the form of a combination \( k_1 e^{-\mu_1 \tau} + k_2 e^{-\mu_2 \tau} + k_3 e^{-\mu_3 \tau} \) with unknown coefficients \( k_1, k_2, k_3 \) so that the following condition is satisfied: \( u_k(T) = \dot{u}_k(T) = \ddot{u}_k(T) = 0 \). This condition leads to a linear system for \( k_1, k_2, k_3 \), the coefficients of which can be easily expressed in terms of integrals of the form \( \int_{0}^{T} e^{\mu_i \tau} d\tau \cdot \int_{0}^{T} e^{\mu_i \tau} d\tau \). Using the explicit form of the system for \( k_1, k_2, k_3 \), we can prove (using the Cramer rule for solving linear systems) that the constant \( T > 0 \) can be chosen so that for all \( k = 1, 2, \ldots \), the solutions of these
linear systems will satisfy the inequality \(|g_\epsilon(t)|<\varepsilon^*\). With sufficient smoothness of the functions \(\varphi(x), \psi(x)\) (determining the initial conditions for the original system (16)), one can choose \(\varepsilon^*>0\) so that the following condition is satisfied: \(|f(x,t)|<\varepsilon^*\).

We show now that in the case when the internal friction is defined by an integral member of the convolution type, where the kernel is Rabotnov \(\mathcal{H}\)-function, neither the boundary control problem nor the distributed control problem will be solvable for arbitrary initial conditions. Suppose, on the contrary, that we have found the distributed control \(f(x,t)\) we need, we will make the Laplace transform in the variable \(t\). We obtain the next equality, using the properties of the Laplace transform of the derivative of the function and of the convolution of functions

\[
p^\alpha \hat{u}(p,x) + \alpha p \hat{u}_\alpha(p,x) - C(p^{1-\alpha} + \lambda)^{-1} \hat{u}_\alpha(p,x) = \hat{f}(p,x) - \varphi(x) - p\psi(x) . \tag{22}
\]

Here the functions \(\varphi(x), \psi(x)\) are the problem initial conditions. Note that the function \(\hat{f}(p,x)\) is an entire function of the complex variable \(p\) according to the Paley-Wiener theorem [11], since the original function \(f(x,t)\) is finite in \(t\) (since it only controls for a finite time). If now we go the curve around the point zero on the complex plane on the right side of the equality (22), then we will return to the original value. And if we make the same turn on the left side, then we will come to a different value. Indeed, functions \(\hat{u}(p,x), \hat{u}_\alpha(p,x)\) are entire functions for the same reason, and function \(p^{1-\alpha}\) with non-integer \(\alpha\) is a branching function for which the indicated traversal along the contour around the zero point leads to different value than the original. Consequently, the left-hand side will change its value with the indicated traversal along the contour around the point zero. We have obtained a contradiction, which proves the insolubility of the considered control problem.

3. Conclusion

In this paper, various control problems (namely, rest at a finite time) for systems with distributed parameters are considered. These are problems of boundary and distributed control; the system itself may contain terms that model the Kelvin – Voight friction and the integral aftereffect. The aftereffect kernel can be defined by both exponential functions and functions having a singularity at zero. Studies show that, depending on the convolution kernel, type of control, and the presence of Kelvin-Voight friction, different answers take place. Apparently the first works where it was pointed out that the problems of controllability by differential and integro-differential systems were qualitatively different were [12-14]. The behavior of such controlled systems is varied and qualitatively different from the previously studied cases of control of differential systems without integral aftereffect. In works [15-17] where systems of differential equations are considered, it is proved that even with toughening the requirements for the control action (small in absolute value boundary control and “smooth” boundary control), the controllability of the system is not lost and, on the whole, the qualitative picture corresponds to that established in [2]. These works are based on the methods developed in [18–20]. Problems of the dynamics of systems with integral delay were considered in [21-23].

References

[1] Chernyshev V and Fominova O 2019 Proc. 4th Int. Conf. on Ind. Eng. Control of damping process in system of vibration isolation (Cham: Springer) p 341-349
[2] Butkovskiy A G 1965 Theory of optimal control of distributed parameter systems (Moscow: Nauka)
[3] Lions J L 1972 Optimal control of systems governed by partial differential equations (Moscow: Mir)
[4] Lions J L 1988 Exact controllability, stabilization and perturbations for distributed systems SIAM Review 31 1-68
[5] Romanov I V and Shamaev A S 2018 Some problems of distributed and boundary control for
systems with integral aftereffect J. Math. Sci. 234:4 470-484
[6] Chernousko F L Bounded control in distributed-parameter systems J. Appl.Math. and Mech. 56:5 707-723
[7] Rabotnov Yu N 1968 Creep problems in structural members (Amsterdam: North-Holland Publ. Co)
[8] Ilyshin A A and Pobedrya B E 1970 Foundations of the mathematical theory of thermoviscoelasticity (Moscow: Nauka)
[9] Christensen R M 2010 Theory of viscoelasticity (New York: Dover)
[10] Rabotnov Yu N 1980 Elements of hereditary solid mechanics (Moscow: Mir)
[11] Yoshida K 1980 Functional Analysis (New York: Springer-Verlog)
[12] Ivanov S and Pandolfi L 2009 Heat equations with memory: lack of controllability to rest J. Math. Analysis and Appl. 355:1 1-11
[13] Pandolfi L 2005 The controllability of the Gurtin-Pipkin equations: a cosine operator approach Appl. Math. Optim. 52 143-165
[14] Ivanov S and Sheronova T 2010 Spectrum of the heat equation with memory arXiv: 0912.1818
[15] Romanov I V 2011 Control of system vibrations using boundary forces Bulletin of Moscow University, Series I, Mathematics and Mechanics 2 3-10
[16] Romanov I V and Shamaev A S 2016 Exact bounded boundary controllability of vibrations of a two-dimensional membrane Doklady Mathematics 94 607-610
[17] Romanov I V and Shamaev A S 2018 Exact bounded boundary controllability to rest for the two-dimensional wave equation arXiv:1603.01212
[18] Russel D L 1978 Controllability and stabilizability theory for linear partial differential equations: recent progress and open questions SIAM Review 20:4 639-739
[19] Lagnese J E 1989 Boundary stabilization of thin plates (Philadelphia: SIAM)
[20] Kapitonov B V 1993 Ates of the rate of stabilization of solutions of exterior mixed problems for a class of evolution systems Russian Acad. Sci. Sb. Math. 76:2 331-359
[21] Schmidt A and Gaul L 2006 On a critique of numerical scheme for the calculation of fractionally damped dynamical systems Mech. Research Communications 33 99-107
[22] Schmidt A and Gaul L 2002 Finite element formulation of viscoelastic constitutive equations using fractional time derivatives Nonlinear Dynamics 29 37-55
[23] Luo J, Schmidt A and Gaul L 2018 On the eigensolutions of circular plate with viscoelastic filling media using fractional derivatives 9th Vienna Int. Conf. Math. Modelling. IFAC-PapersOnLine 51:2 607-612