On a characterisation theorem for $\alpha$-adic solenoids

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According to the Heyde theorem the Gaussian distribution on the real line is characterized by the symmetry of the conditional distribution of one linear form of independent random variables given another. We prove an analogue of this theorem for linear forms of two independent random variables taking values in an $\alpha$-adic solenoid $\Sigma_\alpha$ without elements of order 2, assuming that the characteristic functions of the random variables do not vanish, and coefficients of the linear forms are topological automorphisms of $\Sigma_\alpha$.

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1. Introduction

By the well-known Heyde theorem the Gaussian distribution on the real line is characterized by the symmetry of the conditional distribution of one linear form of independent random variables given another ([13], see also [14, §13.4.1]). For two independent random variables this theorem can be formulated as follows.

**Theorem A.** Let $\xi_1$ and $\xi_2$ be independent random variables with distributions $\mu_1$ and $\mu_2$. Assume that $\alpha \neq -1$. If the conditional distribution of the linear form $L_2 = \xi_1 + \alpha \xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric, then $\mu_1$ and $\mu_2$ are Gaussian distributions.

Group analogs of Heyde’s theorem in the case when independent random variables take values in a locally compact Abelian group $X$, and coefficients of the linear forms are topological automorphisms of $X$ were studied in the articles [3–5, 7–11, 15–17, see also [6, Chapter VI]). We remark that in all cited articles the corresponding characterization theorems were proved under certain restrictions on coefficients of the linear forms. In the article we prove an analogue of Heyde’s theorem for two independent random variables taking values in an $\alpha$-adic solenoid without elements of order 2, assuming that the characteristic function of considering random variables do not vanish. It is important that we do not impose any restrictions on coefficients of the linear forms.

We note that although our proofs of the main results use methods of abstract harmonic analysis, it was quite unexpected that some facts of complex analysis were also used, in particular, the well known Hadamard theorem on the representation of an entire function of finite order.

Before we formulate the main theorem recall some definitions and agree on notation. Let $X$ be a second countable locally compact Abelian group. We will consider only such groups, without mentioning it specifically. Denote by $\text{Aut}(X)$ the group of topological automorphisms of $X$, and by $I$ the identity automorphism of a group. Denote by $Y$ the character group of the group $X$, and by $(x,y)$ the value of a character $y \in Y$ at an element $x \in X$. If $G$ is a closed subgroup of $X$, denote by $A(Y,G) = \{y \in Y : (x,y) = 1 \text{ for all } x \in G\}$ its annihilator. Let $X_1$ and $X_2$ be
locally compact Abelian groups with character groups $Y_1$ and $Y_2$ respectively. Let $\alpha : X_1 \mapsto X_2$ be a continuous homomorphism. The adjoint homomorphism $\tilde{\alpha} : Y_2 \mapsto Y_1$ is defined by the formula $(\alpha x_1, y_2) = (x_1, \tilde{\alpha} y_2)$ for all $x_1 \in X_1$, $y_2 \in Y_2$. Denote by $\mathbb{C}$ the set of complex numbers, by $\mathbb{R}$ the group of real numbers, by $\mathbb{Z}$ the group of integers, by $\mathbb{Z}(m) = \{0, 1, \ldots, m - 1\}$ the group of residue classes modulo $m$, and by $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ the circle group. Let $f(y)$ be a function on the group $Y$, and let $h \in Y$. Denote by $\Delta_h$ the finite difference operator

$$\Delta_h f(y) = f(y + h) - f(y), \quad y \in Y.$$ 

Let $n$ be an integer. Denote by $f_n : X \mapsto X$ an endomorphism of the group $X$ defined by the formula $f_n x = nx$, $x \in X$. Put $X^{(n)} = f_n(X)$, $X^{(n)} = \text{Ker} f_n$.

Let $\mu$ be a measure or a signed measure on the group $X$. The characteristic function (the Fourier transform) of $\mu$ is defined by the formula

$$\hat{\mu}(y) = \int_X (x, y) d\mu(x), \quad y \in Y.$$ 

Denote by $M^1(X)$ the convolution semigroup of probability measures (distributions) on the group $X$. Let $\mu \in M^1(X)$. Denote by $\sigma(\mu)$ the support of $\mu$. Define the distribution $\tilde{\mu} \in M^1(X)$ by the formula $\tilde{\mu}(B) = \mu(-B)$ for any Borel subset $B$ of $X$. Note that $\hat{\tilde{\mu}}(y) = \tilde{\mu}(y)$. If $G$ is a Borel subgroup of $X$, we denote by $M^1(G)$ the subsemigroup of $M^1(X)$ of distributions concentrated on $G$. A distribution $\gamma \in M^1(X)$ is called Gaussian (see [18, Chapter IV, §6]) if its characteristic function is represented in the form

$$\hat{\gamma}(y) = (x, y) \exp \{-\varphi(y)\}, \quad y \in Y,$$

where $x \in X$, and $\varphi(y)$ is a continuous non-negative function on the group $Y$ satisfying the equation

$$\varphi(u + v) + \varphi(u - v) = 2[\varphi(u) + \varphi(v)], \quad u, v \in Y.$$  

Note that, in particular, the degenerate distributions are Gaussian. Denote by $\Gamma(X)$ the set of Gaussian distributions on $X$. Denote by $E_x$ the degenerate distribution concentrated at a point $x \in X$.

2. $a$-adic solenoids

Recall the definition of an $a$-adic solenoid. Put $a = (a_0, a_1, \ldots)$, where all $a_j \in \mathbb{Z}$, $a_j > 1$. Denote by $\Delta_a$ the group of $a$-adic integers. Consider the group $\mathbb{R} \times \Delta_a$. Denote by $B$ the subgroup of the group $\mathbb{R} \times \Delta_a$ of the form $B = \{(n, nu)\}_{n=-\infty}^{\infty}$, where $u = (1, 0, \ldots, 0, \ldots)$. The factor-group $\Sigma_a = (\mathbb{R} \times \Delta_a) / B$ is called an $a$-adic solenoid. The group $\Sigma_a$ is compact, connected and has dimension 1 ([12, (10.12), (10.13), (24.28)]). The character group of the group $\Sigma_a$ is topologically isomorphic to the discrete group of the form

$$H_a = \left\{ \frac{m}{a_0a_1 \cdots a_n} : n = 0, 1, \ldots; m \in \mathbb{Z} \right\}.$$ 

In order not to complicate the notation we will identify $H_a$ with the character group of the group $\Sigma_a$. We will also consider $H_a$ as a subset of $\mathbb{R}$. Any topological automorphism $\alpha$ of the group $\Sigma_a$ is of the form $\alpha = f_p f_q^{-1}$ for some mutually prime $p$ and $q$, where $f_p, f_q \in \text{Aut}(\Sigma_a)$. We will identify $\alpha = f_p f_q^{-1}$ with the real number $\frac{p}{q}$. If $\alpha = f_p f_q^{-1}$, then $\tilde{\alpha} = f_p f_q^{-1}$, and we will also identify $\tilde{\alpha}$ with the real number $\frac{p}{q}$. We note that if $\alpha \neq -I$, then either $\text{Ker}(I + \alpha) = \{0\}$ or $\text{Ker}(I + \alpha) \cong \mathbb{Z}(m)$ for some $m$. Put $G = (\Sigma_a)_{(2)}$. It is easy to verify that there are only two possibilities for $G$: either $G = \{0\}$ or $G \cong \mathbb{Z}(2)$. It is obvious that $G = \{0\}$ if and only if $f_2 \in \text{Aut}(\Sigma_a)$.
It follows from (1) and (2) that the characteristic function of a Gaussian distribution $\gamma$ on an $a$-adic solenoid $\Sigma_a$ is of the form
\[ \hat{\gamma}(y) = (x, y) \exp\{-\sigma y^2\}, \quad y \in H_a, \]
where $x \in \Sigma_a$, $\sigma \geq 0$.

Let $\xi_1$ and $\xi_2$ be independent random variables with values in a locally compact Abelian group $X$ and distributions $\mu_1$ and $\mu_2$. Let $\alpha_1, \beta_2 \in \text{Aut}(X)$. Consider the linear forms $L_1 = \alpha_1 \xi_1 + \alpha_2 \xi_2$ and $L_2 = \beta_1 \xi_1 + \beta_2 \xi_2$ and assume that the conditional distribution of the linear form $L_2$ given $L_1$ is symmetric. It is easy to see that the description of possible distributions $\mu_j$ is reduced to the case when $L_1 = \xi_1 + \xi_2$, $L_2 = \xi_1 + \alpha \xi_2$, where $\alpha \in \text{Aut}(X)$.

The main result of the article is the following theorem.

**Theorem 1.** Consider an $a$-adic solenoid $X = \Sigma_a$. Assume that $X$ contains no elements of order 2. Let $\alpha$ be a topological automorphism of the group $X$. Put $K = \text{Ker}(I + \alpha)$. Let $\xi_1$ and $\xi_2$ be independent random variables with values in the group $X$ and distributions $\mu_1$ and $\mu_2$ with nonvanishing characteristic functions. Assume that the conditional distribution of the linear form $L_2 = \xi_1 + \alpha \xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric. Then each of the distributions $\mu_j$ can be represented in the form $\mu_j = \gamma_j \ast \omega$, where $\gamma_j \in \Gamma(X)$, $\omega \in M^1(K)$. Moreover, if $\alpha > 0$, then $\mu_j = E_{x_j} \ast \omega$, where $x_j \in X$, $j = 1, 2$.

**Corollary 1.** Consider an $a$-adic solenoid $X = \Sigma_a$. Assume that $X$ contains no elements of order 2. Let $\alpha$ be a topological automorphism of the group $X$. Let $\xi_1$ and $\xi_2$ be independent random variables with values in the group $X$ and distributions $\mu_1$ and $\mu_2$ with nonvanishing characteristic functions. The symmetry of the conditional distribution of the linear form $L_2 = \xi_1 + \alpha \xi_2$ given $L_1 = \xi_1 + \xi_2$ implies that $\mu_j$ are Gaussian distributions if and only if $\text{Ker}(I + \alpha) = \{0\}$.

Theorem 1 will be proved in §3. In §4 we prove that generally speaking, Theorem 1 fails if an $a$-adic solenoid $X = \Sigma_a$ contains an element of order 2. Namely, we will prove for such groups that if $\alpha < 0$, then there exist independent random variables $\xi_1$ and $\xi_2$ with values in the group $X$ and distributions $\mu_j \notin \Gamma(X) \ast M^1(K)$ with nonvanishing characteristic functions such that the conditional distribution of the linear form $L_2 = \xi_1 + \alpha \xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric.

### 3. Proof of Theorem 1

To prove Theorem 1 we need some lemmas.

**Lemma 1.** (6, Lemma 16.1). Let $X$ be a locally compact Abelian group, $Y$ be its character group, $\alpha$ be a topological automorphism of $X$. Let $\xi_1$ and $\xi_2$ be independent random variables with values in the group $X$ and distributions $\mu_1$ and $\mu_2$. The conditional distribution of the linear form $L_2 = \xi_1 + \alpha \xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric if and only if the characteristic functions $\hat{\mu_j}(y)$ satisfy Heyde’s functional equation
\[ \hat{\mu_1}(u + v)\hat{\mu_2}(u + \alpha v) = \hat{\mu_1}(u - v)\hat{\mu_2}(u - \alpha v), \quad u, v \in Y. \quad (3) \]

It is convenient for us to formulate as lemmas the following well-known statements (see e.g. [6, §2]).

**Lemma 2.** Let $X$ be a locally compact Abelian group, $Y$ be its character group. Let $\mu \in M^1(X)$. Then the set $E = \{y \in Y : \hat{\mu}(y) = 1\}$ is a closed subgroup of $Y$, the characteristic function $\hat{\mu}(y)$ is $E$-invariant, i.e. $\hat{\mu}(y)$ takes a constant value on each coset of $E$ in the group $Y$, and $\sigma(\mu) \subset A(X, E)$.  

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Lemma 3. Let $X$ be a locally compact Abelian group, $G$ be a Borel subgroup of $X$, $μ ∈ M^1(G)$, $μ_1 = μ_1 * μ_2$, where $μ_j ∈ M^1(X)$. Then the distributions $μ_j$ can be replaced by their shifts $μ'_1 = μ_1 * E_x$, $μ'_2 = μ_2 * E_{-x}$, $x ∈ X$, in such a manner that $μ'_j ∈ M^1(G)$, $j = 1, 2$.

We will formulate as a lemma the following easily verifiable statement (see e.g. \cite{1} Lemma 6.9).

Lemma 4. Let $X = \mathbb{R} × G$, where $G$ is a locally compact Abelian group, $Y$ and $H$ be the character groups of the groups $X$ and $G$ respectively. Denote by $(t, g)$, $t ∈ \mathbb{R}$, $g ∈ G$, elements of the group $X$ and by $(s, h)$, $s ∈ \mathbb{R}$, $h ∈ H$, elements of the group $Y$. Let $μ ∈ M^1(X)$ and assume that $μ(s, 0)$, $s ∈ \mathbb{C}$, is an entire function in $s$. Then $μ(s, h)$ is an entire function in $s$ for every fixed $h ∈ H$, the representation

$$
\hat{μ}(s, h) = \int_X \exp(its)(g, h)dμ(t, g)
$$

holds for all $s ∈ \mathbb{C}$, $h ∈ H$, and the inequality

$$
\max_{|s| ≤ r} |\hat{μ}(s, h)| ≤ \max_{|s| ≤ r} |\hat{μ}(s, 0)|, \quad h ∈ H,
$$

(4)

is valid. Moreover, the function $\hat{μ}(−iy + x, h)/\hat{μ}(−iy, 0)$, where $x, y ∈ \mathbb{R}$, for any fixed $y$ is a characteristic function of variable $(x, h) ∈ \mathbb{R} × H$.

Proof of theorem 1. Denote by $Y$ the character group of the group $X$, and consider $Y$ as a subset of $\mathbb{R}$. Let $α = f_p f_q$ for some mutually prime $p$ and $q$, where $f_p, f_q ∈ \text{Aut}(X)$. By Lemma 1, the symmetry of the conditional distribution of the linear form $L_2$ given $L_1$ implies that the characteristic functions $μ_j(y)$ satisfy Heyde’s functional equation $\hat{μ}_j(y) = \hat{μ}_j(y)\exp(−iy) = \hat{μ}_j(y)$, $y ∈ Y$. Obviously, the characteristic functions $\hat{ν}_j(y)$ also satisfy Heyde’s functional equation $\hat{ν}_j(y)$, which takes the form

$$
\hat{ν}_1(u + v)\hat{ν}_2(u + \tilde{α}v) = \hat{ν}_1(u - v)\hat{ν}_2(u - \tilde{α}v), \quad u, v ∈ Y.
$$

(5)

Describe the scheme of the proof of Theorem 1. First we get some representation for the characteristic functions $\hat{ν}_j(y)$. By the topological automorphism $α$ we find natural numbers $m$ and $n$ and consider the group $\mathbb{R} × \mathbb{Z}(mn)$. Next we construct distributions $M_j ∈ M^1(\mathbb{R} × \mathbb{Z}(mn))$ and a continuous monomorphism $π : \mathbb{R} × \mathbb{Z}(mn) → X$ such that $ν_j = π(M_j)$. In so doing the characteristic functions of the distributions $M_j$ satisfy some Heyde’s functional equation. Next we solve the obtained Heyde functional equation and receive the representation for the characteristic functions of the distributions $M_j$. Finally, we find the desired representation for the distributions $μ_j$.

First prove the theorem assuming that $α ≠ ±I$. Put $φ_j(y) = \log \hat{ν}_j(y)$, $j = 1, 2$. It follows from (5) that the functions $φ_j(y)$ satisfy the equation

$$
φ_1(u + v) + φ_2(u + \tilde{α}v) - φ_1(u - v) - φ_2(u - \tilde{α}v) = 0, \quad u, v ∈ Y.
$$

(6)

We use the finite difference method to solve this equation. Let $k_1$ be an arbitrary element of $Y$. Put $h_1 = \tilde{α}k_1$. Hence, $h_1 - \tilde{α}k_1 = 0$. Substitute in (6) $u + h_1$ for $u$ and $v + k_1$ for $v$. Subtracting equation (6) from the resulting equation, we obtain

$$
Δ_l φ_1(u + v) + Δ_l φ_2(u + \tilde{α}v) - Δ_l φ_1(u - v) = 0, \quad u, v ∈ Y,
$$

(7)

where $l_{11} = (I + \tilde{α})k_1$, $l_{12} = 2\tilde{α}k_1$, $l_{13} = (\tilde{α} - I)k_1$.

Let $k_2$ be an arbitrary element of the group $Y$. Put $h_2 = k_2$. Hence, $h_2 - k_2 = 0$. Substitute in (7) $u + h_2$ for $u$ and $v + k_2$ for $v$. Subtracting equation (7) from the resulting equation, we find

$$
Δ_{l_1 2} Δ_l φ_1(u + v) + Δ_{l_1 2} Δ_l φ_2(u + \tilde{α}v) = 0, \quad u, v ∈ Y,
$$

(8)
where \( l_{21} = 2k_2, l_{22} = (I + \tilde{\alpha})k_2 \).

Let \( k_3 \) be an arbitrary element of \( Y \). Put \( h_3 = -\tilde{\alpha}k_3 \). Hence, \( h_3 + \tilde{\alpha}k_3 = 0 \). Substitute in (8) \( u + h_3 \) for \( u \) and \( v + k_3 \) for \( v \). Subtracting equation (8) from the resulting equation, we get

\[
\Delta_{l_{21}} \Delta_{l_{21}} \Delta_{l_{11}} \varphi_1 (u + v) = 0, \quad u, v \in Y,
\]

where \( l_{31} = (I - \tilde{\alpha})k_3 \). Putting in (9) \( v = 0 \), we find

\[
\Delta_{l_{31}} \Delta_{l_{21}} \Delta_{l_{11}} \varphi_1 (u) = 0, \quad u \in Y.
\]

Reasoning similarly we obtain from (8) that the function \( \varphi_2 (y) \) satisfies the equation

\[
\Delta_{l_{32}} \Delta_{l_{32}} \Delta_{l_{12}} \varphi_2 (u) = 0, \quad u \in Y,
\]

where \( l_{32} = -(I - \tilde{\alpha})k_3 \). Put

\[
H = (I + \tilde{\alpha})(Y) \cap Y^{(2)} \cap (I - \tilde{\alpha})(Y).
\]

It follows from (10) and (11) that the functions \( \varphi_j (y) \) satisfy the equation

\[
\Delta^3_h \varphi_j (y) = 0, \quad h \in H, \quad y \in Y, \quad j = 1, 2.
\]

It is useful to remark that we received equation (13) without using the fact that the group \( X \) contains no elements of order 2.

Since the group \( X \) contains no elements of order 2, we have \( Y^{(2)} = Y \). It is obvious that \((I + \tilde{\alpha})(Y) = Y^{(m)} \) for some natural \( m \). We can assume that \( m \) is the minimum possible. Similarly, \((I - \tilde{\alpha})(Y) = Y^{(n)} \) for some natural \( n \), and we can also assume that \( n \) is the minimum possible. It follows from \( Y^{(2)} = Y \) that \( m \) and \( n \) are odd. Since \( m \) and \( n \) are divisors of \( p + q \) and \( p - q \) respectively and \( p \) and \( q \) mutually prime, \( m \) and \( n \) are also mutually prime. This implies that \( H = Y^{(m)} \cap Y^{(n)} = Y^{(mn)} \).

Consider the factor-group \( Y/H \). It is obvious that \( Y/H \cong \mathbb{Z}(mn) \). Take an element \( y_0 \in Y \) in such a way that the coset \( y_0 + H \) be a generator of the factor-group \( Y/H \). Then

\[
Y = H \cup (y_0 + H) \cup (2y_0 + H) \cup \cdots \cup ((mn - 1)y_0 + H)
\]

is a decomposition of the group \( Y \) with respect to the subgroup \( H \). Put

\[
\psi_{lj}(y) = \varphi_j (ly_0 + y), \quad y \in H, \quad l = 0, 1, \ldots, mn - 1, \quad j = 1, 2.
\]

It follows from (13) that the functions \( \psi_{lj}(y) \) satisfy the equation

\[
\Delta^3_h \psi_{lj}(y) = 0, \quad h, y \in H, \quad l = 0, 1, \ldots, mn - 1, \quad j = 1, 2.
\]

This implies that for any coset \( ly_0 + H \) there exists the polynomial

\[
P_{lj}(y) = A_{lj} y^2 + B_{lj} y + C_{lj}, \quad j = 1, 2,
\]

on \( \mathbb{R} \) with real coefficients such that

\[
P_{lj}(y) = \varphi_j (y), \quad y \in ly_0 + H, \quad l = 0, 1, \ldots, mn - 1, \quad j = 1, 2.
\]

It follows from this that

\[
\hat{\nu}_j (y) = \exp \{ A_{lj} y^2 + B_{lj} y + C_{lj} \}, \quad y \in ly_0 + H, \quad l = 0, 1, \ldots, mn - 1, \quad j = 1, 2.
\]
Consider the group $\mathbb{R} \times \mathbb{Z}(mn)$ and denote by $(s,l)$, $s \in \mathbb{R}$, $l \in \mathbb{Z}(mn)$ its elements. Define the mapping $\tau : Y \mapsto \mathbb{R} \times \mathbb{Z}(mn)$ by the following way: $\tau y = (y,l)$ if $y \in l_{y_0} + H$, $l = 0,1,\ldots,mn - 1$. Obviously, $\tau$ is a monomorphism and the subgroup $\tau(Y)$ is dense in $\mathbb{R} \times \mathbb{Z}(mn)$. Since $m$ and $n$ are divisors of $p + q$ and $p - q$ respectively, and $p$ and $q$ are mutually prime, $p$ is mutually prime with $m$ and with $n$, and $q$ is mutually prime with $m$ and with $n$. Hence, $f_p, f_q \in \text{Aut}(\mathbb{Z}(mn))$.

Let $y \in l_{y_0} + H$. If we consider $l$ as an element of the group $\mathbb{Z}(mn)$, and $f_p$ and $f_q$ as automorphisms of the group $\mathbb{Z}(mn)$, then it is easy to see that $\tilde{\alpha}y \in (f_p f_q^{-1}) y_0 + H$. Therefore, on the one hand,

$$\tau \tilde{\alpha} y = (\tilde{\alpha} y, f_p f_q^{-1} l) = (f_p f_q^{-1} y, f_p f_q^{-1} l).$$

On the other hand, since $\tau y = (y,l)$ and $f_p f_q^{-1} (y,l) = (f_p f_q^{-1} y, f_p f_q^{-1} l)$, we have

$$\tau \tilde{\alpha} y = f_p f_q^{-1} \tau y, \quad y \in Y.$$  \hspace{1cm} (17)

Define on the set $\tau(Y)$ the functions $g_j(s,l)$ in the following way. Let $(s,l) = \tau y \in \tau(Y)$. Put $g_j(s,l) = \tilde{\nu}_j(y)$, $j = 1,2$. Since $\tau$ is a monomorphism, the functions $g_j(s,l)$ are correctly defined. It follows from (16) that the functions $g_j(s,l)$ are continuous on the subgroup $\tau(Y)$ in the topology induced on $\tau(Y)$ by the topology of the group $\mathbb{R} \times \mathbb{Z}(mn)$, and taking into account that the subgroup $\tau(Y)$ is dense in $\mathbb{R} \times \mathbb{Z}(mn)$, the functions $g_j(s,l)$ can be extended by continuity to some continuous positive definite functions $\tilde{g}_j(s,l)$ on the group $\mathbb{R} \times \mathbb{Z}(mn)$. By the Bochner theorem, there exist the distributions $\tilde{M}_j \in M^1(\mathbb{R} \times \mathbb{Z}(mn))$ such that $\tilde{M}_j(s,l) = \tilde{g}_j(s,l)$, $(s,l) \in \mathbb{R} \times \mathbb{Z}(mn)$. It is obvious that

$$\tilde{M}_j(\tau y) = \tilde{\nu}(y), \quad y \in Y, \quad j = 1,2.$$  \hspace{1cm} (18)

It follows from (16) and (18) that the characteristic functions $\tilde{M}_j(s,l)$ do not vanish, and hence $\tilde{M}_j(s,l) > 0$, $j = 1,2$.

Note that the character group of the group $\mathbb{R} \times \mathbb{Z}(mn)$ is topologically isomorphic to $\mathbb{R} \times \mathbb{Z}(mn)$. In order not to complicate the notation we will assume that it coincides with $\mathbb{R} \times \mathbb{Z}(mn)$. Put $\pi = \tilde{\tau}$, $\pi : \mathbb{R} \times \mathbb{Z}(mn) \mapsto X$. Since the subgroup $\tau(Y)$ is dense in $\mathbb{R} \times \mathbb{Z}(mn)$, $\pi$ is a continuous monomorphism generating an isomorphism of the semigroups of distributions $M^1(\mathbb{R} \times \mathbb{Z}(mn))$ and $M^1(\mathbb{R} \times \mathbb{Z}(mn))$. We shall also denote this isomorphism by $\pi$. It follows from (18) that

$$\nu_j = \pi(\tilde{M}_j), \quad j = 1,2.$$  \hspace{1cm} (19)

Find a representation for the characteristic functions $\tilde{M}_j(s,l)$. It follows from (5) and (18) that

$$\tilde{M}_1(\tau u + \tau v) \tilde{M}_2(\tau u + \tau \tilde{\alpha} v) = \tilde{M}_1(\tau u - \tau v) \tilde{M}_2(\tau u - \tau \tilde{\alpha} v), \quad u,v \in Y.$$  \hspace{1cm} (20)

Taking into account (17), we find from (20) that

$$\tilde{M}_1(\tau u + \tau v) \tilde{M}_2(\tau u + f_p f_q^{-1} \tau v) = \tilde{M}_1(\tau u - \tau v) \tilde{M}_2(\tau u - f_p f_q^{-1} \tau v), \quad u,v \in Y.$$  \hspace{1cm} (21)

Since the subgroup $\tau(Y)$ is dense in $\mathbb{R} \times \mathbb{Z}(mn)$, (21) implies that the characteristic functions $\tilde{M}_j(s,l)$ on the group $\mathbb{R} \times \mathbb{Z}(mn)$ satisfy the following Heyde’s functional equation

$$\tilde{M}_1(s_1 + s_2, l_1 + l_2) \tilde{M}_2(s_1 + p q^{-1} s_2, l_1 + f_p f_q^{-1} l_2) = \tilde{M}_1(s_1 - s_2, l_1 - l_2) \tilde{M}_2(s_1 - p q^{-1} s_2, l_1 - f_p f_q^{-1} l_2), \quad (s_1, l_1) \in X,$$  \hspace{1cm} (22)

Note that $\mathbb{Z}(mn) = \mathbb{Z}(m) \times \mathbb{Z}(n)$, and represent an element $l \in \mathbb{Z}(mn)$ in the form $l = (a,b)$, $a \in \mathbb{Z}(m)$, $b \in \mathbb{Z}(n)$. Since $m$ is a divisor of $p + q$, and $n$ is a divisor of $p - q$, it is easy to see that the automorphism $f_p f_q^{-1}$ acts on the group $\mathbb{Z}(m) \times \mathbb{Z}(n)$ in the following way $f_p f_q^{-1} (a,b) = (-a,b)$, and we can write equation (21) in the form

$$\tilde{M}_1(s_1 + s_2, a_1 + a_2, b_1 + b_2) \tilde{M}_2(s_1 + p q^{-1} s_2, a_1 - a_2, b_1 + b_2)$$
\[ \hat{M}_1(s_1 - s_2, a_1 - a_2, b_1 - b_2) \hat{M}_2(s_1 - pq^{-1}s_2, a_1 + a_2, b_1 - b_2), \quad (s_j, a_j, b_j) \in \mathbb{R} \times \mathbb{Z}(m) \times \mathbb{Z}(n). \] (23)

Putting in \( s_1 = s_2 = 0, a_1 = a_2 = 0 \), we get

\[ \hat{M}_1(0, 0, b_1 + b_2) \hat{M}_2(0, b_1 + b_2) = \hat{M}_1(0, 0, b_1 - b_2) \hat{M}_2(0, 0, b_1 - b_2), \quad b_j \in \mathbb{Z}(n). \] (24)

Putting in \( b_1 = b_2 = b \), we get \( \hat{M}_j(0, 0, 2b) = 1 \) for any \( b \in \mathbb{Z}(n) \), and taking into account that \( n \) is odd, we conclude that \( \hat{M}_j(0, 0, b) = 1, j = 1, 2, \) for \( b \in \mathbb{Z}(n) \). By Lemma 2, this implies that \( \hat{M}_j(s, a, b) = \hat{M}_j(s, a, 0) \) for all \( (s, a, b) \in \mathbb{R} \times \mathbb{Z}(m) \times \mathbb{Z}(n) \) and \( \sigma(M_j) \subset A(\mathbb{R} \times \mathbb{Z}(m) \times \mathbb{Z}(n), \mathbb{Z}(n)) = \mathbb{R} \times \mathbb{Z}(m), j = 1, 2 \). Therefore we can write the characteristic function \( \hat{M}_j(s, a, b) \) in the form \( \hat{M}_j(s, a) \).

Then equation (23) takes the form

\[ \hat{M}_1(s_1 + s_2, a_1 + a_2) \hat{M}_2(s_1 + pq^{-1}s_2, a_1 - a_2) \]

\[ \hat{M}_1(s_1 - s_2, a_1 - a_2) \hat{M}_2(s_1 - pq^{-1}s_2, a_1 + a_2), \quad (s_j, a_j) \in \mathbb{R} \times \mathbb{Z}(m). \] (25)

Put in \( a_1 = a_2 = 0 \). Taking into account Lemma 1, we get by Theorem A from the obtaining equation that

\[ \hat{M}_j(s, 0) = \exp\{-\sigma_j s^2\}, \quad s \in \mathbb{R}, \quad j = 1, 2. \] (26)

By Lemma 1, \( \hat{M}_j(s, a) \) are entire functions for any \( a \in \mathbb{Z}(m) \). As noted above the functions \( \hat{M}_j(s, a) \) do not vanish. It follows from (21) and (26) that the entire function \( \hat{M}_j(s, a) \) is of at most order 2. Applying the Hadamard theorem on the representation of an entire function of finite order, we obtain that

\[ \hat{M}_j(s, a) = \exp\{\alpha_{aj} s^2 + \beta_{aj} s + \gamma_{aj}\}, \quad s \in \mathbb{R}, \quad a \in \mathbb{Z}(m), \quad j = 1, 2. \] (27)

Put

\[ \phi_j(s, a) = \alpha_{aj} s^2 + \beta_{aj} s + \gamma_{aj}, \quad s \in \mathbb{R}, \quad a \in \mathbb{Z}(m), \quad j = 1, 2. \] (28)

It follows from (27) that the functions \( \phi_j(s, a) \) satisfy the equation

\[ \phi_1(s_1 + s_2, a_1 + a_2) + \phi_2(s_1 + pq^{-1}s_2, a_1 - a_2) \]

\[ -\phi_1(s_1 - s_2, a_1 + a_2) - \phi_2(s_1 - pq^{-1}s_2, a_1 + a_2) = 0, \quad (s_j, a_j) \in \mathbb{R} \times \mathbb{Z}(m). \] (29)

Substituting (28) into (28) and putting in the obtained equation \( a_1 = a_2 = a \), we get

\[ \alpha_{2a1}(s_1 + s_2)^2 + \beta_{2a1}(s_1 + s_2) + \gamma_{2a1} + \alpha_{2a2}(s_1 + pq^{-1}s_2)^2 + \beta_{2a2}(s_1 + pq^{-1}s_2) + \gamma_{2a2} \]

\[ -\alpha_{01}(s_1 - s_2)^2 - \beta_{01}(s_1 - s_2) - \gamma_{01} - \alpha_{2a2}(s_1 - pq^{-1}s_2)^2 - \beta_{2a2}(s_1 - pq^{-1}s_2) - \gamma_{2a2} = 0. \] (30)

Equating the coefficients of \( s_1^2 \) and \( s_2^2 \) on each side of (29), we get

\[ \alpha_{2a1} + \alpha_{2a2} = \alpha_{2a1} + \alpha_{2a2} = 0, \quad \alpha_{2a2} = \alpha_{2a2} = 0. \] (31)

Taking into account that \( (pq^{-1})^2 \neq 1 \) and the fact that \( m \) is odd, (31) and (26) imply that \( \alpha_{a1} = \alpha_{a1} = \sigma_1 \) and \( \alpha_{a2} = \alpha_{a2} = \sigma_2 \) for all \( a \in \mathbb{Z}(m) \). Equating the coefficients of \( s_1 \) and \( s_2 \) on each side of (29) we obtain

\[ \beta_{2a1} + \beta_{2a2} = \beta_{2a1} + \beta_{2a2} = 0, \quad \beta_{2a1} + pq^{-1}\beta_{01} + \beta_{2a2} + pq^{-1}\beta_{2a2} = 0. \] (32)

It follows from (28) that \( \beta_{01} = \beta_{2a2} = 0 \). Taking into account that \( pq^{-1} \neq -1 \) and the fact that \( m \) is odd, (32) implies that \( \beta_{2a1} = \beta_{a2} = 0 \) for all \( a \in \mathbb{Z}(m) \). Equating the constant term on each side of (29) we receive

\[ \gamma_{2a1} + \gamma_{2a2} = \gamma_{2a1} + \gamma_{2a2} = 0. \] (33)
Taking into account that in view of \(\text{(26)}\) \(\gamma'_{01} = \gamma'_{02} = 0\) and the fact that \(m\) is odd, \(\text{(33)}\) implies that 
\(\gamma_{a1} = \gamma_{a2} = 0\) for all \(a \in \mathbb{Z}(m)\). Put \(\gamma_{a1} = \gamma_{a2} = c_a,\ a \in \mathbb{Z}(m)\). Thus, we proved that the characteristic functions \(\hat{M}_j(s, a)\) have the following representation
\[
\hat{M}_j(s, a) = \exp\{-\sigma_j s^2 + c_a\}, \quad (s, a) \in \mathbb{R} \times \mathbb{Z}(m), \quad j = 1, 2.
\]  
(34)

It follows from \(\text{(19)}\) that the distributions \(\nu_j\) are concentrated on the subgroup \(\pi(\mathbb{R} \times \mathbb{Z}(mn))\). By Lemma \(\text{3}\) the distributions \(\mu_j\) can be substituted by their shifts \(\mu_j'\) in such a manner that \(\nu_j = \mu_j' \ast \hat{\mu}_j'\) and the distributions \(\mu_j'\) are also concentrated on the subgroup \(\pi(\mathbb{R} \times \mathbb{Z}(mn))\). Since \(\pi\) is an isomorphism of the semigroups of distributions \(\text{M}^1(\mathbb{R} \times \mathbb{Z}(mn))\) and \(\text{M}^1(\pi(\mathbb{R} \times \mathbb{Z}(mn)))\), we have
\[
M_j = \pi^{-1}((\nu_j) = \pi^{-1}((\mu_j') \ast \pi^{-1}((\mu_j')), \quad j = 1, 2.
\]  
(35)

Put \(N_j = \pi^{-1}(\mu_j')\), \(j = 1, 2\). We find from \(\text{(35)}\) that
\[
\hat{N}_j(s, a) = \hat{N}_j(s, a) \hat{N}_j(s, a), \quad (s, a) \in \mathbb{R} \times \mathbb{Z}(m), \quad j = 1, 2.
\]  
(36)

Putting in \(\text{(36)}\) \(a = 0\) and taking into account \(\text{(26)}\), by the Cramér theorem on decomposition of the Gaussian distribution, we get
\[
\hat{N}_j(s, 0) = \exp\left\{-\frac{\sigma_j}{2} s^2 + ib_0 js\right\}, \quad s \in \mathbb{R}, \quad j = 1, 2,
\]  
(37)

where \(b_{0j} \in \mathbb{R}\). We will assume without loss of generality that \(b_{0j} = 0\).

By Lemma \(\text{4}\) \(\hat{N}_j(s, a)\) are entire functions for any \(a \in \mathbb{Z}(m)\). It follows from \(\text{(34)}\) and \(\text{(36)}\) that they do not vanish, and \(\text{(4)}\) and \(\text{(37)}\) imply that the entire function \(\hat{N}_j(s, a)\) is of at most order 2 and type \(\frac{\sigma_j}{2}\). Applying the Hadamard theorem on the representation of an entire function of finite order, \(\text{(34)}\) and \(\text{(36)}\), we obtain that
\[
\hat{N}_j(s, a) = \exp\left\{-\frac{\sigma_j}{2} s^2 + ib_aj s + d_{aj}\right\}, \quad (s, a) \in \mathbb{R} \times \mathbb{Z}(m), \quad j = 1, 2,
\]  
(38)

where \(b_{aj} \in \mathbb{R}\). It follows from \(\text{(38)}\) that
\[
\hat{N}_j(-iy + x, a)/\hat{N}_j(-iy, 0) = \exp\left\{-\frac{\sigma_j}{2} x^2 + b_ayj + \sigma_ji xy + b_aj i x + d_{aj}\right\}.
\]

By Lemma \(\text{4}\) the function \(\hat{N}_j(-iy + x, a)/\hat{N}_j(-iy, 0)\) for any fixed \(y\) is a characteristic function of variable \((x, a) \in \mathbb{R} \times \mathbb{Z}(m)\). Thus, in particular, for any \(y\) the inequality \(|\hat{N}_j(-iy + x, a)/\hat{N}_j(-iy, 0)| \leq 1\) holds true. It follows from this that \(b_{aj} = 0\), \(j = 1, 2\), for any \(a \in \mathbb{Z}(m)\), and \(\text{(38)}\) implies that
\[
\hat{N}_j(s, a) = \exp\left\{-\frac{\sigma_j}{2} s^2 + d_{aj}\right\}, \quad (s, a) \in \mathbb{R} \times \mathbb{Z}(m), \quad j = 1, 2.
\]  
(39)

Let \(P_j\) be the Gaussian distributions on \(\mathbb{R}\) with the characteristic functions \(P_j(s) = \exp\{-\frac{\sigma_j}{2}s^2\},\ s \in \mathbb{R}\). Denote by \(Q_j\) the distributions on \(\mathbb{Z}(m)\) with the characteristic functions \(Q_j(a) = \exp\{d_{aj}\},\ a \in \mathbb{Z}(m)\). It follows from \(\text{(39)}\) that \(N_j = P_j \ast Q_j\), \(j = 1, 2\), and hence, \(\mu_j' = \pi(N_j) = \pi(P_j) \ast \pi(Q_j)\). It is obvious that \(\pi(P_j) \in \Gamma(X)\). Verify that \(\pi(\mathbb{Z}(m)) = K\). We have
\[
A(Y, K) = (I + \tilde{a})(Y) = Y^{(m)}.
\]  
(40)

Note that \(\text{(14)}\) implies the equality
\[
Y^{(m)} = H \cup (my_0 + H) \cup (2my_0 + H) \cup \cdots \cup ((n-1)my_0 + H),
\]
and hence, the equivalence
\[ y \in Y^{(m)} \iff \tau y \in \mathbb{R} \times \{0\} \times \mathbb{Z}(n). \]

We have, \( y \in A(Y, \pi(Z(m))) \iff (g, \tau y) = 1 \) for all \( g \in \mathbb{Z}(m) \iff \tau y \in \mathbb{R} \times \{0\} \times \mathbb{Z}(n). \) Taking into account (40) and (41), this implies the equality \( A(Y, \pi(Z(m))) = A(Y, K), \) and hence, \( \pi(Z(m)) = K. \) Thus, \( \pi(Q_j) \in M^1(K). \) So, we proved that \( \mu_j^i \in \Gamma(X) * M^1(K), \) and hence \( \mu_j \in \Gamma(X) * M^1(K), \) i.e. \( \mu_j = \lambda_j * \rho_j, \) where \( \lambda_j \in \Gamma(X), \rho_j \in M^1(K), \) \( j = 1, 2. \)

Let the characteristic functions \( \hat{\lambda}_j(y) \) be of the form
\[ \hat{\lambda}_j(y) = (x_j, y) \exp\{-\sigma_j y^2\}, \quad y \in Y, \quad j = 1, 2. \]  

Put \( L = A(Y, K) \) and note that \( \tilde{\alpha}(L) = L. \) Since \( \rho_j \in M^1(K), \) we have \( \hat{\rho}_j(y) = 1, \) \( y \in L, \) \( j = 1, 2. \)

Consider the restriction of Heyde’s functional equation (3) for the characteristic functions \( \hat{\mu}_j(y) \) to \( L. \) Taking into account (42), we find from the obtained equation that
\[ \sigma_1 + pq^{-1}\sigma_2 = 0, \]  
and hence,
\[ (x_1, u + v)(x_2, u + \tilde{\alpha}v) = (x_1, u - v)(x_2, u - \tilde{\alpha}v) \quad u, v \in L. \]  

It follows from (44) that \( 2(x_1 + \alpha x_2) \in K. \) Since \( f_2 \in \text{Aut}(X), \) we have \( x_1 + \alpha x_2 \in K. \) Put \( x_0 = x_1 + \alpha x_2, \gamma_1 = \lambda_1 * E_{-x_0}, \omega_1 = \rho_1 * E_{x_0}, \gamma_2 = \lambda_2, \omega_2 = \rho_2. \) It is obvious that \( \mu_j = \gamma_j * \omega_j \) and \( \omega_j \in M^1(K), \) \( j = 1, 2. \) It is easy to see that the characteristic functions \( \hat{\gamma}_j(y) \) satisfy Heyde’s functional equation (3). Hence, the characteristic functions \( \hat{\omega}_j(y) \) also satisfy equation (3). We have
\[ \hat{\omega}_1(u + v)\hat{\omega}_2(u + \tilde{\alpha}v) = \hat{\omega}_1(u - v)\hat{\omega}_2(u - \tilde{\alpha}v), \quad u, v \in Y. \]  

We consider here \( \omega_j \) as distributions on the group \( X. \) Denote by \( M \) the character group of the group \( K. \) Since \( \omega_j \in M^1(K), \) \( j = 1, 2, \) and \( \alpha x = -x \) for any \( x \in K, \) we can consider \( \omega_j \) as distributions on the group \( K. \) Then equation (45) takes the form
\[ \hat{\omega}_1(u + v)\hat{\omega}_2(u - v) = \hat{\omega}_1(u - v)\hat{\omega}_2(u + v), \quad u, v \in M. \]

Putting here \( u = v = y, \) we get \( \hat{\omega}_1(2y) = \hat{\omega}_2(2y), \) \( y \in M. \) Since \( K \cong Z(m), \) and \( m \) is odd, this implies that \( \hat{\omega}_1(y) = \hat{\omega}_2(y), \) \( y \in M, \) and hence \( \omega_1 = \omega_2 = \omega. \) Thus, \( \mu_j = \gamma_j * \omega, \) where \( \gamma_j \in \Gamma(X), \omega \in M^1(K), \) \( j = 1, 2. \)

Assume now that \( \alpha > 0. \) Obviously, (45) implies that \( \sigma_1 = \sigma_2 = 0, \) i.e. \( \gamma_j \) are degenerate distributions, and hence, the distributions \( \mu_j \) are of the required form. Thus, we proved Theorem 1 assuming that \( \alpha \neq \pm I. \)

It still remains the cases when \( \alpha = \pm I. \) Note that since the group \( X \) contains no elements of order 2, we have \( Y^{(2)} = Y. \) Let \( \alpha = I. \) Substituting in equation (5) \( \alpha = I, u = v = y, \) we receive \( \hat{\nu}_1(2y) = \hat{\nu}_2(2y) = 1, y \in Y. \) Hence, \( \hat{\nu}_1(y) = \hat{\nu}_2(y) = 1, y \in Y, \) and this means that \( \nu_1 = \nu_2 = E_0. \) Therefore, \( \mu_j \) are degenerate distributions. In this case Theorem 1 is also proved. Let \( \alpha = -I. \) Then \( K = X. \) Substituting in equation (6) \( \alpha = -I, u = v = y, \) we obtain \( \hat{\mu}_1(2y) = \hat{\mu}_2(2y), y \in Y. \) Hence, \( \mu_1(y) = \mu_2(y), y \in Y, \) and it means that \( \mu_1 = \mu_2. \) Theorem 1 is completely proved. \( \square \)

**Remark 1.** Consider an \( \alpha \)-adic solenoid \( X = \Sigma_\alpha, \) and assume that \( X \) contains no elements of order 2. Denote by \( Y \) the character group of the group \( X. \) Let \( \alpha = f_p f_q^{-1} \) be a topological automorphism of the group \( X. \) Put \( K = \text{Ker}(I + \alpha). \) Theorem 1 can not be strengthened. Indeed, let \( \gamma_j \) be Gaussian distributions on the group \( X \) with the characteristic functions of the form
\[ \hat{\gamma}_j(y) = (x_j, y) \exp\{-\sigma_j y^2\}, \quad y \in Y, \]
where $x_1 + \alpha x_2 = 0$ and $\sigma_1 + pq^{-1}\sigma_2 = 0$. Let $\omega \in M^1(K)$. Put $\mu_j = \gamma_j \ast \omega$, $j = 1, 2$. It easily follows from Lemma 1 that if $\xi_1$ and $\xi_2$ are independent random variables with values in the group $X$ and distributions $\mu_1$ and $\mu_2$, then the conditional distribution of the linear form $L_2 = \xi_1 + \alpha \xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric. Thus, we can not narrow the class of distributions in Theorem 1 which is characterized by the symmetry of the conditional distribution of the linear form $L_2$ given $L_1$.

4. Heyde’s characterisation theorem for $a$-adic solenoids containing an element of order 2

Let $X = \Sigma_a$ and $G = X(2)$. We discuss in this section the Heyde characterisation theorem for $a$-adic solenoids containing an element of order 2, i.e. the case when $G \cong \mathbb{Z}(2)$. Let $\alpha$ be a topological automorphism of the group $X$. Put $K = \text{Ker}(I + \alpha)$. It is easy to see that $K \supset G$ for any $\alpha \in \text{Aut}(X)$. Let $\xi_1$ and $\xi_2$ be independent random variables with values in the group $X$ and distributions $\mu_1$ and $\mu_2$ with nonvanishing characteristic functions. First we shall prove that generally speaking, Theorem 1 fails if the group $X$ contains an element of order 2. Secondly, in the case when $K = G$ we give a complete description of distributions which are characterized by the symmetry of the conditional distribution of the linear form $L_2 = \xi_1 + \alpha \xi_2$ given $L_1 = \xi_1 + \xi_2$. It turns out that the corresponding class of distributions is wider than the class $\Gamma(X) \ast M^1(K)$.

Let $Y$ be the character group of the group $X$. Note that a decomposition of the group $Y$ with respect to the subgroup $Y(2)$ consists of two cosets. Let $\mu \in M^1(X)$. It is easy to see that $\mu \in \Gamma(X) \ast M^1(G)$ if and only if the characteristic function $\hat{\mu}(y)$ is represented in the form

\[
\hat{\mu}(y) = \begin{cases} 
(x, y) \exp\{-\sigma y^2\}, & y \in Y^{(2)}, \\
(x, y)\kappa \exp\{-\sigma y^2\}, & y \notin Y^{(2)},
\end{cases}
\]

where $x \in X$, $\sigma \geq 0$, $\kappa \in \mathbb{R}$, $|\kappa| \leq 1$. Introduce into consideration a class of distributions on the group $X$ which is wider than the class $\Gamma(X) \ast M^1(G)$.

**Definition 1.** Let $X = \Sigma_a$, $G = X(2)$. Assume that $G \cong \mathbb{Z}(2)$. Let $Y$ be the character group of the group $X$. We say that a distribution $\mu$ on the group $X$ belongs to the class $Y$ if its characteristic function can be represented in the form

\[
\hat{\mu}(y) = \begin{cases} 
(x, y) \exp\{-\sigma y^2 + i\beta y\}, & y \in Y^{(2)}, \\
(x, y)\kappa \exp\{-\sigma y^2 + i\beta' y\}, & y \notin Y^{(2)},
\end{cases}
\]

for some $x \in X$, $\sigma \geq 0$, $\sigma' \geq 0$, $\beta, \beta', \kappa \in \mathbb{R}$, $|\kappa| \leq 1$.

Consider the group $\mathbb{R} \times \mathbb{Z}(2)$. Denote by $(t, k)$, $t \in \mathbb{R}$, $k \in \mathbb{Z}(2)$, its elements. The character group of the group $\mathbb{R} \times \mathbb{Z}(2)$ is topologically isomorphic to the group $\mathbb{R} \times \mathbb{Z}(2)$. Denote by $(s, l)$, $s \in \mathbb{R}$, $l \in \mathbb{Z}(2)$, elements of the character group of the group $\mathbb{R} \times \mathbb{Z}(2)$. Let $\mu \in \Gamma(\mathbb{R}) \ast M^1(\mathbb{Z}(2))$. It is easy to see that the characteristic function of the distribution $\mu$ is of the form

\[
\hat{\mu}(s, l) = \begin{cases} 
\exp\{-\sigma s^2 + i\beta s\}, & s \in \mathbb{R}, \ l = 0, \\
\kappa \exp\{-\sigma s^2 + i\beta s\}, & s \in \mathbb{R}, \ l = 1,
\end{cases}
\]

where $\sigma \geq 0$, $\beta, \kappa \in \mathbb{R}$, $|\kappa| \leq 1$. Introduce into consideration a class of distributions on the group $X$ which is wider than $\Gamma(\mathbb{R}) \ast M^1(\mathbb{Z}(2))$. We need this class of distributions to study distributions on $a$-adic solenoids containing an element of order 2 which are characterized by the symmetry of the conditional distribution of one linear form of independent random variables given another.
Lemma 5. Consider the group $\mathbb{R} \times \mathbb{Z}(2)$. Let $f(s, l)$ be a function on the character group of the group $\mathbb{R} \times \mathbb{Z}(2)$ of the form

$$f(s, l) = \begin{cases} \exp\{-\sigma s^2 + i\beta s\}, & s \in \mathbb{R}, \ l = 0, \\ \kappa \exp\{-\sigma' s^2 + i\beta' s\}, & s \in \mathbb{R}, \ l = 1, \end{cases}$$

(46)

where $\sigma \geq 0$, $\sigma' \geq 0$, $\beta, \beta', \kappa \in \mathbb{R}$. Then $f(s, l)$ is the characteristic function of a signed measure $\mu$ on the group $\mathbb{R} \times \mathbb{Z}(2)$. Moreover, $\mu$ is a measure if and only if 0 < $\sigma'$ < $\sigma$ and 0 < $|\kappa|$ $\leq \sqrt{\frac{\sigma'}{\sigma}} \exp\left\{-\frac{(\beta - \beta')^2}{4(\sigma - \sigma')}\right\}$ or $\sigma' = \sigma$, $\beta = \beta'$ and $|\kappa|$ $\leq 1$. In the latter case $\mu \in \Gamma(\mathbb{R}) \ast M^1(\mathbb{Z}(2))$.

Proof. Obviously, we can prove the lemma assuming that $\kappa \neq 0$. Multiplying if necessary the function $f(s, l)$ by a suitable character of the group $\mathbb{R} \times \mathbb{Z}(2)$, we can assume without loss of generality that $\kappa > 0$. Let $a \geq 0$. Denote by $\gamma_a$ the Gaussian distribution on the group $\mathbb{R}$ with the characteristic function

$$\hat{\gamma}_a(s) = \exp\{-as^2\}, \ s \in \mathbb{R}. \quad (47)$$

Consider on the group $\mathbb{R}$ the measure $\lambda_1 = \frac{1}{4}(\gamma_1 \ast E_\beta + \kappa \gamma_\sigma' \ast E_{\beta'})$ and the signed measure $\lambda_2 = \frac{1}{4}(\gamma_\sigma \ast E_{\beta} - \kappa \gamma_\sigma \ast E_{\beta'})$. Define a signed measure $\mu$ on the group $\mathbb{R} \times \mathbb{Z}(2)$ in the following way

$$\mu\{(E, 0)\} = \lambda_1\{E\}, \quad \mu\{(E, 1)\} = \lambda_2\{E\},$$

where $E$ is a Borel subset of $\mathbb{R}$. Taking into account that $\hat{\lambda}_1(s) + \hat{\lambda}_2(s) = \hat{\gamma}_a(s) \exp\{i\beta s\}$ and $\hat{\lambda}_1(s) - \hat{\lambda}_2(s) = \kappa \gamma_\sigma'(s) \exp\{i\beta' s\}$, we have

$$\hat{\mu}(s, l) = \int_{\mathbb{R} \times \mathbb{Z}(2)} e^{its}(k, l)d\mu(t, k) = \int_{\mathbb{R} \times \{0\}} e^{its}d\mu(t, 0) + \int_{\mathbb{R} \times \{1\}} e^{its}(1, l)d\mu(t, 1) = f(s, l).$$

Thus, $f(s, l)$ is the characteristic function of the signed measure $\mu$, and the signed measure $\mu$ is a measure if and only if the signed measure $\lambda_2$ is a measure. It is obvious that if the signed measure $\lambda_2$ is a measure, then either $\sigma > 0$ and $\sigma' > 0$ or $\sigma = \sigma' = 0$. It is clear that if $\sigma = \sigma' = 0$, then the signed measure $\mu$ is a measure if and only if $\beta = \beta'$ and $\kappa \leq 1$. The statement of the lemma is proved in this case.

Let $\sigma > 0$ and $\sigma' > 0$. Let $a > 0$. The density of the distribution $\gamma_a$ is of the form

$$\rho_a(t) = \frac{1}{2\sqrt{\pi a}} \exp\left\{-\frac{t^2}{4a}\right\}, \ t \in \mathbb{R}. \quad (48)$$

Taking into account (48), the signed measure $\lambda_2$ is a measure if and only if the inequality

$$\frac{1}{2\sqrt{\pi \sigma}} \exp\left\{-\frac{(t - \beta)^2}{4\sigma}\right\} - \frac{\kappa}{2\sqrt{\pi \sigma'}} \exp\left\{-\frac{(t - \beta')^2}{4\sigma'}\right\} \geq 0$$

holds for all $t \in \mathbb{R}$. This inequality is equivalent to the following

$$\kappa \leq \sqrt{\frac{\sigma}{\sigma'}} \exp\left\{-\frac{(t - \beta)^2}{4\sigma} + \frac{(t - \beta')^2}{4\sigma'}\right\}, \ t \in \mathbb{R}. \quad (49)$$

Let $\sigma \neq \sigma'$. Since $\kappa > 0$, we have $\sigma' < \sigma$. The function in the right-hand side of inequality (49) reaches its minimum at $t = \frac{\sigma \beta' - \sigma' \beta}{\sigma - \sigma'}$, and this minimum is equal to $\sqrt{\frac{\sigma}{\sigma'}} \exp\left\{-\frac{(\beta - \beta')^2}{4(\sigma - \sigma')}\right\}$. Assume that $\sigma = \sigma'$. Then (49) implies that $\beta = \beta'$ and $\kappa \leq 1$. Thus, the signed measure $\lambda_2$, and hence, the signed measure

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\(\mu\), is a measure if and only if either \(\sigma' < \sigma\) and \(d \leq \sqrt{\frac{2}{\pi} \exp \left\{ -\frac{(\beta - \beta')^2}{4(\sigma - \sigma')^2} \right\}}\) or \(\sigma' = \sigma\), \(\beta = \beta'\) and \(d \leq 1\). It is also obvious that in the latter case \(\mu \in \Gamma(\mathbb{R}) \ast M^1(\mathbb{Z}(2))\). Lemma 5 is proved.

We also note that if in (46) \(\sigma > 0\) and \(\kappa = 0\), then \(f(s, l)\) is the characteristic function of a convolution of a non-degenerate Gaussian distribution on the group \(\mathbb{R}\) and the Haar distribution on \(\mathbb{Z}(2)\). \(\square\)

We prove now that generally speaking, Theorem 1 fails if the group \(X = \Sigma_a\) contains an element of order 2.

**Proposition 1.** Let \(X = \Sigma_a\), \(G = X(2)\). Assume that \(G \cong \mathbb{Z}(2)\). Let \(\alpha\) be a topological automorphism of the group \(X\), \(\alpha < 0\), \(\alpha \neq -I\). Put \(K = \text{Ker}(I + \alpha)\). Then there exist independent random variables \(\xi_1\) and \(\xi_2\) with values in the group \(X\) and distributions \(\mu_j \in \Gamma\), \(\mu_j \notin \Gamma(X) \ast M^1(K)\), \(j = 1, 2\), with nonvanishing characteristic functions such that the conditional distribution of the linear form \(L_2 = \xi_1 + \alpha \xi_2\) given \(L_1 = \xi_1 + \xi_2\) is symmetric.

**Proof.** Consider the group \(R \times \mathbb{Z}(2)\). Let \(a \in \text{Aut}(R \times \mathbb{Z}(2))\). It is obvious that \(a\) is of the form \(a(t, k) = (c_a t, k)\), where \(c_a \in \mathbb{R}\), \(c_a \neq 0\). We will identify \(a\) and \(c_a\), i.e. we will write \(a(t, k) = (at, k)\) and assume that \(a \in \mathbb{R}\), \(a \neq 0\). The converse is also true. A nonzero real number corresponds to a topological automorphism of the group \(R \times \mathbb{Z}(2)\). We remark that if \(a \in \text{Aut}(R \times \mathbb{Z}(2))\), then \(\tilde{a}\) is of the form \(a(s, l) = (as, l)\), i.e. \(\tilde{a} = a\).

Choose numbers \(\sigma_j, \sigma_j'\) and \(\kappa\) in such a way that they satisfy the conditions: \(0 < \sigma_j' < \sigma_j, 0 < \kappa \leq \sqrt{\frac{\sigma_j'}{\sigma_j}}, j = 1, 2\). Consider on the group \(R \times \mathbb{Z}(2)\) the functions

\[
f_j(s, l) = \begin{cases} 
\exp\{-\sigma_j s^2\}, & s \in \mathbb{R}, \quad l = 0, \\
\kappa \exp\{-\sigma_j' s^2\}, & s \in \mathbb{R}, \quad l = 1, \quad j = 1, 2.
\end{cases}
\]

(50)

By Lemma 5 there exist distributions \(\lambda_j \in M^1(R \times \mathbb{Z}(2))\) such that \(\hat{\lambda_j}(s, l) = f_j(s, l)\). It is obvious that \(\lambda_j \notin \Gamma(R) \ast M^1(\mathbb{Z}(2))\), \(j = 1, 2\). By the condition \(\alpha = f_p f_q^{-1}\) for some mutually prime \(p\) and \(q\), where \(f_p, f_q \in \text{Aut}(X)\). It follows from \(G \cong \mathbb{Z}(2)\), that \(p\) and \(q\) are odd, and hence \(f_p\) and \(f_q\) are topological automorphisms of the group \(R \times \mathbb{Z}(2)\). Since \(\alpha < 0\), we can assume that the conditions

\[
\sigma_1 + pq^{-1} \sigma_2 = 0, \quad (51)
\]

\[
\sigma_1' + pq^{-1} \sigma_2' = 0 \quad (52)
\]

hold. Verify that the characteristic functions \(\hat{\lambda_j}(s, l)\) satisfy Heyde’s functional equation (3). Put \(u = (s_1, l_1), v = (s_2, l_2)\). Let either \(u, v \in \mathbb{R} \times \{0\}\) or \(u, v \in \mathbb{R} \times \{1\}\). Then \(u \pm v, u \pm \alpha v \in \mathbb{R} \times \{0\}\), and taking into account (50), equality (3) for these \(u, v\) is satisfied if the equality

\[
\sigma_1(s_1 + s_2)^2 + \sigma_2(s_1 + pq^{-1}s_2)^2 = \sigma_1(s_1 - s_2)^2 + \sigma_2(s_1 - pq^{-1}s_2)^2, \quad s_1, s_2 \in \mathbb{R},
\]

holds true. But this equality follows from (51). Let either \(u \in \mathbb{R} \times \{0\}, v \in \mathbb{R} \times \{1\}\) or \(u \in \mathbb{R} \times \{1\}, v \in \mathbb{R} \times \{0\}\). Then \(u \pm v, u \pm \alpha v \in \mathbb{R} \times \{1\}\), and taking into account (50), equality (3) for these \(u, v\) is satisfied if the equality

\[
\sigma_1'(s_1 + s_2)^2 + \sigma_2'(s_1 + pq^{-1}s_2)^2 = \sigma_1'(s_1 - s_2)^2 + \sigma_2'(s_1 - pq^{-1}s_2)^2, \quad s_1, s_2 \in \mathbb{R},
\]

is valid. But this equality follows from (52). It is obvious that we exhausted all possibilities for \(u\) and \(v\). Thus, we verified that the characteristic functions \(\hat{\lambda_j}(l, s)\) satisfy Heyde’s functional equation (3).
Let \( \zeta_1 \) and \( \zeta_2 \) be independent random variables with values in the group \( \mathbb{R} \times \mathbb{Z}(2) \) and distributions \( \lambda_1 \) and \( \lambda_2 \). Since the characteristic functions \( \hat{\lambda}_j(l, s) \) satisfy Heyde’s functional equation (3), by Lemma 1, the conditional distribution of the linear form \( T_2 = \zeta_1 + \alpha \zeta_2 \) given \( T_1 = \zeta_1 + \zeta_2 \) is symmetric.

Denote by \( Y \) the character group of the group \( X \). Let \( y_0 \notin Y^{(2)} \) and \( Y = Y^{(2)} \cup (y_0 + Y^{(2)}) \) be a decomposition of the group \( Y \) with respect to the subgroup \( Y^{(2)} \). Define the mapping \( \tau : Y \mapsto \mathbb{R} \times \mathbb{Z}(2) \) by the formula

\[
\tau(y) = \begin{cases} 
(y, 0), & y \in Y^{(2)}, \\
(y, 1), & y \in y_0 + Y^{(2)}. 
\end{cases}
\]  

(53)

It is obvious that \( \tau \) is a homomorphism. Put \( \pi = \bar{\tau}, \pi : \mathbb{R} \times \mathbb{Z}(2) \mapsto X \). Since the subgroup \( \tau(Y) \) is dense in \( \mathbb{R} \times \mathbb{Z}(2) \), \( \pi \) is a monomorphism. It is easy to see that

\[
\alpha \pi(t, k) = \pi f_p f_q^{-1}(t, k), \quad (t, k) \in \mathbb{R} \times \mathbb{Z}(2).
\]  

(54)

Put \( \xi_j = \pi(\zeta_j), j = 1, 2 \). Then \( \xi_j \) are independent random variables with values in the group \( X \) and distributions \( \mu_j = \pi(\lambda_j) \). We have \( \hat{\mu}_j(y) = \hat{\lambda}_j(\tau(y)) \), and (50) and (53) imply that

\[
\hat{\mu}_j(y) = \begin{cases} \exp(-\sigma_j y^2), & y \in Y^{(2)}, \\
\kappa \exp(-\sigma_j y^2), & y \in y_0 + Y^{(2)}, \quad j = 1, 2. 
\end{cases}
\]  

(55)

Obviously, the conditional distribution of the linear form \( L_2 = \pi(T_2) \) given \( L_1 = \pi(T_1) \) is symmetric. We have \( L_1 = \xi_1 + \xi_2 \), and it follows from (51) that \( L_2 = \xi_1 + \alpha \xi_2 \). Definition 1 and (55) imply that \( \mu_j \in \Upsilon \). Since \( \sigma_j \neq \sigma_j' \) and \( \alpha \neq -I \), we have \( \mu_j \notin \Gamma(X) \ast M^1(K), j = 1, 2 \). Proposition 1 is proved. \( \square \)

**Remark 2.** We consider here the case when \( \alpha = -I \) and complete Proposition 1 by the following statement.

Let \( X = \Sigma_{\alpha}, G = X^{(2)} \) and \( G \cong \mathbb{Z}(2) \). Let \( \xi_1 \) and \( \xi_2 \) be independent random variables with values in the group \( X \) and distributions \( \mu_1 \) and \( \mu_2 \) with nonvanishing characteristic functions. The conditional distribution of the linear form \( L_2 = \xi_1 - \xi_2 \) given \( L_1 = \xi_1 + \xi_2 \) is symmetric if and only if either \( \mu_1 = \mu_2 \ast \lambda \) or \( \mu_2 = \mu_1 \ast \lambda \), where \( \lambda \in M^1(G) \).

Let \( Y \) be the character group of the group \( X \), \( y_0 \notin Y^{(2)} \) and let \( Y = Y^{(2)} \cup (y_0 + Y^{(2)}) \) be a decomposition of the group \( Y \) with respect to the subgroup \( Y^{(2)} \). Assume that the conditional distribution of the linear form \( L_2 \) given \( L_1 \) is symmetric. By Lemma 1, the characteristic functions \( \hat{\mu}_j(y) \) satisfy Heyde’s functional equation (3) which takes the form

\[
\hat{\mu}_1(u + v)\hat{\mu}_2(u - v) = \hat{\mu}_1(u - v)\hat{\mu}_2(u + v), \quad u, v \in Y.
\]  

(56)

Substituting \( u = v = y \) in equation (56), we get \( \hat{\mu}_1(2y) = \hat{\mu}_2(2y), y \in Y \). Hence,

\[
\hat{\mu}_1(y) = \hat{\mu}_2(y), \quad y \in Y^{(2)}.
\]  

(57)

Substituting \( u = y, v = y + y_0 \) in equation (56), we obtain

\[
\hat{\mu}_1(2y + y_0)\hat{\mu}_2(y_0) = \hat{\mu}_1(y_0)\hat{\mu}_2(2y + y_0), \quad y \in Y.
\]  

(58)

Assume that \( |\hat{\mu}_1(y_0)| \leq |\hat{\mu}_2(y_0)| \). We find from (58) that

\[
\frac{\hat{\mu}_1(2y + y_0)}{\hat{\mu}_2(2y + y_0)} = \frac{\hat{\mu}_1(y_0)}{\hat{\mu}_2(y_0)}, \quad y \in Y.
\]  

(59)
Put $d = \hat{\mu}_1(y_0)/\hat{\mu}_2(y_0)$. It follows from $\hat{\mu}_j(-y) = \overline{\hat{\mu}_j(y)}$, $y \in Y$, that $d$ is a real number, and [59] implies that

$$\hat{\mu}_1(y) = d\hat{\mu}_2(y), \quad y \in y_0 + Y^{(2)}.$$  

(60)

Denote by $\lambda$ a distribution on $G$ with the characteristic function

$$\hat{\lambda}(y) = \begin{cases} 1, & y \in Y^{(2)}, \\ d, & y \in y_0 + Y^{(2)}. \end{cases}$$

(61)

It follows from (57), (60) and (61) that $\hat{\mu}_1(y) = d\hat{\mu}_2(y)\hat{\lambda}(y)$, $y \in Y$, and hence $\mu_1 = \mu_2 \ast \lambda$.

The converse statement follows directly from Lemma 11.

In the case when $K = G$ and $G \cong \mathbb{Z}(2)$ we give a complete description of distributions which are characterized by the symmetry of the conditional distribution of the linear form $L_2$ given $L_1$.

**Theorem 2.** Let $X = \Sigma_a$, $G = X^{(2)}$. Let $\alpha$ be a topological automorphism of the group $X$. Put $K = \text{Ker}(I + \alpha)$. Assume that $K = G$ and $G \cong \mathbb{Z}(2)$. Let $\xi_1$ and $\xi_2$ be independent random variables with values in the group $X$ and distributions $\mu_1$ and $\mu_2$ with nonvanishing characteristic functions. Assume that the conditional distribution of the linear form $L_2 = \xi_1 + \alpha\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric. Then $\mu_j \in \Upsilon$, $j = 1, 2$. Moreover, if $\alpha > 0$, then some sifts of the distributions $\mu_j$ are supported in $G$.

To prove Theorem 2 we need such lemmas.

**Lemma 6 ([15]).** Let $X$ be a locally compact Abelian group, $\alpha$ be a topological automorphism of $X$. Let $\xi_1$ and $\xi_2$ be independent random variables with values in the group $X$. If the conditional distribution of the linear form $L_2 = \xi_1 + \alpha\xi_2$ given $L_1 = \xi_1 + \xi_2$ is symmetric, then the linear forms $L'_1 = (I + \alpha)\xi_1 + 2\alpha\xi_2$ and $L'_2 = 2\xi_1 + (I + \alpha)\xi_2$ are independent.

It is well-known that the Gaussian distribution on the real line is characterized by the independence of two linear forms of $n$ independent random variables. We need a group analogue of this theorem for $n = 2$.

**Lemma 7 ([2], see also [6] Theorem 10.3]).** Let $X$ be a locally compact Abelian group containing no subgroup topologically isomorphic to the circle group $\mathbb{T}$. Let $\alpha_j$, $\beta_j$ be topological automorphisms of the group $X$. Assume that $\xi_1$ and $\xi_2$ are independent random variables with values in the group $X$ and distributions $\mu_1$ and $\mu_2$ with nonvanishing characteristic functions. Then the independence of the linear forms $L_1 = \alpha_1\xi_1 + \alpha_2\xi_2$ and $L_2 = \beta_1\xi_1 + \beta_2\xi_2$ implies that $\mu_j \in \Gamma(X)$, $j = 1, 2$.

We will prove now that the characteristic functions of divisors of a distribution on the group $\mathbb{R} \times \mathbb{Z}(2)$ with the nonvanishing characteristic function of the form (40) have the similar representation.

**Lemma 8.** Let $N \in M^1(\mathbb{R} \times \mathbb{Z}(2))$ and the characteristic function $\hat{N}(s,l) = f(s,l)$ is represented in the form (40), where either $0 < \sigma' < \sigma$ and $0 < |\kappa| \leq \sqrt{\frac{\sigma'}{\sigma}} \exp \left\{ -\frac{(\beta - \beta')^2}{4(\sigma - \sigma')} \right\}$ or $\sigma' = \sigma$, $\beta = \beta'$ and $0 < |\kappa| \leq 1$. Let $N = P_1 \ast P_2$, where $P_j \in M^1(\mathbb{R} \times \mathbb{Z}(2))$. Then each of the characteristic functions $\hat{P}_j(s,l)$ is of the form

$$\hat{P}_j(s,l) = \begin{cases} \exp \left\{ -\sigma_j s^2 + i\beta_j s \right\}, & s \in \mathbb{R}, \quad l = 0, \\ \kappa_j \exp \left\{ -\sigma'_j s^2 + i\beta'_j s \right\}, & s \in \mathbb{R}, \quad l = 1, \end{cases}$$

where either $0 < \sigma'_j < \sigma_j$ and $0 < |\kappa_j| \leq \sqrt{\frac{\sigma'_j}{\sigma_j}} \exp \left\{ -\frac{(\beta - \beta')^2}{4(\sigma - \sigma')} \right\}$, or $\sigma'_j = \sigma_j$, $\beta_j = \beta'_j$ and $0 < |\kappa_j| \leq 1$.  

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Proof. We have
\[ \hat{N}(s,l) = \hat{P}_1(s,l)\hat{P}_2(s,l), \quad s \in \mathbb{R}, \ l \in \mathbb{Z}(2). \]  
(62)
Putting in (62) \( l = 0 \), we obtain from (63) by the Cramér theorem on decomposition of the Gaussian distribution on the real line,
\[ \hat{P}_j(s,0) = \exp \{-\sigma_j s^2 + i\beta_j s\}, \quad s \in \mathbb{R}, \ j = 1, 2, \]  
(63)
where \( \sigma_j \geq 0, \ \beta_j \in \mathbb{R} \). By Lemma 4 \( \hat{P}_j(s,1) \) are entire functions, and (46) and (62) imply that they do not vanish. It follows from (4) and (63) that the entire function \( \hat{P}_j(s,1) \) is of at most order 2 and type \( \sigma_j \). Taking into account that \( \hat{P}_j(-s,1) = \hat{P}_j(s,1) \), by the Hadamard theorem on the representation of an entire function of finite order and Lemma 5 we obtain the desired statement. Lemma 8 is proved. □

Proof of Theorem 2. Denote by \( Y \) the character group of the group \( X \) and consider \( Y = H_\alpha \) as a subset of \( \mathbb{R} \). Let \( \alpha = f_p f_q^{-1} \) for some mutually prime \( p \) and \( q \), where \( f_p, f_q \in \text{Aut}(X) \). It follows from \( G \cong \mathbb{Z}(2) \) that \( p \) and \( q \) are odd. Consider the distributions \( \nu_j = \mu_j * \bar{\mu}_j \) and the functions \( \varphi_j(y) = \log \hat{\nu}_j(y) \), \( j = 1, 2 \). Reasoning as in the proof of Theorem 1, we see that the functions \( \varphi_j(y) \) satisfy equation (13), where \( H \) is of the form (12). Since \( G \cong \mathbb{Z}(2) \) and \( K = G \), it is easy to see that
\[ I + \alpha = f_2 \beta, \]  
(64)
where \( \beta \in \text{Aut}(X) \). Since \( p \) and \( q \) are odd, (12) and (51) imply that \( H = (I - \bar{\alpha})(Y) = Y^{(n)} \), where \( n \) is even, and we can assume that \( n \) is the minimum possible. Consider the factor-group \( Y/H \). It is obvious that \( Y/H \cong \mathbb{Z}(n) \). Take an element \( y_0 \in Y \) in such a way that the coset \( y_0 + H \) is a generator of the factor-group \( Y/H \). Let
\[ Y = H \cup (y_0 + H) \cup (2y_0 + H) \cup \cdots \cup ((n-1)y_0 + H) \]
be a decomposition of the group \( Y \) with respect to the subgroup \( H \). It follows from (13) that for any coset \( ly_0 + H \) there exist polynomials \( P_j(y) \) on \( \mathbb{R} \) with real coefficients of the form (15) such that
\[ P_j(y) = \varphi_j(y), \quad y \in ly_0 + H, \quad l = 0, 1, \ldots, n - 1, \ j = 1, 2. \]  
(65)

By Lemma 3 the linear forms \( L'_1 = (I + \alpha)\xi_1 + 2\alpha \xi_2 \) and \( L'_2 = 2\xi_1 + (I + \alpha)\xi_2 \) are independent. Consider the new independent random variables \( \eta_j = 2\xi_j, \ j = 1, 2 \). Taking into account (54), the linear forms \( L'_1 \) and \( L'_2 \) can be written as follows \( L'_1 = \beta \eta_1 + \alpha \eta_2 \) and \( L'_2 = \eta_1 + \beta \eta_2 \). Since \( \alpha, \beta \in \text{Aut}(X) \) and the group \( \tilde{X} \) contains no subgroup topologically isomorphic to the circle group \( \mathbb{T} \), by Lemma 7 the random variables \( \eta_j \) have Gaussian distributions. This easily implies that
\[ \varphi_j(y) = -\sigma_j y^2, \quad y \in Y^{(2)}, \quad j = 1, 2, \]  
(66)
where \( \sigma_j \geq 0 \).

It is obvious that \( Y = Y^{(2)} \cup (y_0 + Y^{(2)}) \) is a decomposition of the group \( Y \) with respect to the subgroup \( Y^{(2)} \). It follows from (66) and the inequality
\[ |\hat{\nu}_j(u) - \hat{\nu}_j(v)|^2 \leq 2(1 - \text{Re} \ \hat{\nu}_j(u - v)), \quad u, v \in Y, \ j = 1, 2, \]
which holds for any characteristic function on the group \( Y \), that each of the functions \( \hat{\nu}_j(y) \) is uniformly continuous on the coset \( y_0 + Y^{(2)} \) in the topology induced on \( y_0 + Y^{(2)} \) by the topology of the group \( \mathbb{R} \). Since
\[ y_0 + Y^{(2)} = (y_0 + H) \cup (3y_0 + H) \cup \cdots \cup ((y_0 - 1)c + H) \]

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and each coset \( l y_0 + H \) is a dense subset of \( \mathbb{R} \), taking into account (53) this implies that there exist polynomials \( A_j y^2 + B_j y + C_j, \ j = 1, 2 \), such that

\[
P_{ij}(y) = A_j y^2 + B_j y + C_j, \quad y \in l y_0 + H, \quad l = 1, 3, \ldots, n - 1, \quad j = 1, 2.
\]

and hence, the representations

\[
\varphi_j(y) = A_j y^2 + B_j y + C_j, \quad y \in y_0 + Y^{(2)}, \quad j = 1, 2, 
\]

(67) hold. Since \( \varphi_j(-y) = \varphi_j(y), \ y \in Y \), we have \( B_j = 0, \ j = 1, 2 \). Put \( A_j = -\sigma_j', \ \kappa_j = \epsilon C_j \). As a result, (66) and (67) imply the representations

\[
\nu_j(y) = \begin{cases} \exp\{-\sigma_j y^2\}, & y \in Y^{(2)}, \\ \kappa_j \exp\{-\sigma_j' y^2\}, & y \in y_0 + Y^{(2)}, \end{cases} 
\]

(68) where \( 0 < \kappa_j \leq 1, \ \sigma_j \geq 0, \ \sigma_j' \geq 0, \ j = 1, 2 \).

By Lemma 5, there exist signed measures \( M_j \) on the group \( \mathbb{R} \times \mathbb{Z}(2) \) with the characteristic functions of the form

\[
\tilde{M}_j(s,l) = \begin{cases} \exp\{-\sigma_j s^2\}, & s \in \mathbb{R}, \ l = 0, \\ \kappa_j \exp\{-\sigma_j' s^2\}, & s \in \mathbb{R}, \ l = 1, \ j = 1, 2. \end{cases} 
\]

(69)

Let a homomorphism \( \tau : Y \mapsto \mathbb{R} \times \mathbb{Z}(2) \) be defined by formula (53), and \( \pi = \tilde{\tau}, \ \pi : \mathbb{R} \times \mathbb{Z}(2) \mapsto X \). Taking into account (53), it follows from (68) and (69) that \( \nu_j = \pi(M_j), \ j = 1, 2 \). Since \( \pi \) is a continuous homomorphism, the signed measures \( M_j \) are measures. Obviously, the distributions \( \nu_j \) are concentrated on the Borel subgroup \( F = \pi(\mathbb{R} \times \mathbb{Z}(2)) \). By Lemma 3 the distributions \( \mu_j \) can be substituted by their shifts \( \mu_j' \) in such a way that the distributions \( \mu_j' \) are also concentrated on \( F \) and \( \nu_j = \mu_j' \ast \mu_j', \ j = 1, 2 \). Put \( N_j = \pi^{-1}(\mu_j') \in M^1(\mathbb{R} \times \mathbb{Z}(2)) \). It is obvious that \( M_j = N_j \ast \Pi_j \). By Lemma 8 the characteristic functions \( \hat{N}_j(s,l) \) are of the form

\[
\hat{N}_j(s,l) = \begin{cases} \exp\{-\frac{\sigma_j}{2} s^2 + i\beta_j s\}, & s \in \mathbb{R}, \ l = 0, \\ \sqrt{\kappa_j} \exp\{-\frac{\sigma_j'}{2} s^2 + i\beta_j' s\}, & s \in \mathbb{R}, \ l = 1, \end{cases} 
\]

(70)

where \( \beta_j, \beta_j' \in \mathbb{R}, \ j = 1, 2 \). Since \( \mu_j' = \pi(N_j') \), we have \( \hat{\mu}_j'(y) = \hat{N}_j(\tau(y)) \) and it follows from (53) and (70) that the characteristic functions \( \hat{\mu}_j'(y) \) are of the form

\[
\hat{\mu}_j'(y) = \begin{cases} \exp\{-\frac{\sigma_j}{2} y^2 + i\beta_j y\}, & y \in Y^{(2)}, \\ \sqrt{\kappa_j} \exp\{-\frac{\sigma_j'}{2} y^2 + i\beta_j' y\}, & y \in y_0 + Y^{(2)}, \ j = 1, 2. \end{cases} 
\]

Hence, \( \mu_j \in \Upsilon, \ j = 1, 2 \).

Let \( \alpha > 0 \). Assuming that \( u, v \in Y^{(2)} \), substitute (66) in (6). We obtain \( \sigma_1 + pq^{-1} \sigma_2 = 0 \), and this implies that \( \sigma_1 = \sigma_2 = 0 \). Hence, \( \nu_j(y) = 1, \ y \in Y^{(2)} \). It follows from Lemma 2 that \( \sigma(\nu_j) \subseteq A(X, Y^{(2)}) = X^{(2)} = G, \ j = 1, 2 \). It means by Lemma 3 that some shifts of the distributions \( \mu_j \) are supported in \( G \). Theorem 2 is completely proved. \( \square \)
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