Nonperturbative Flipped $SU(5)$ Vacua in Heterotic M–Theory

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Abstract

The evidence for neutrino masses in atmospheric and solar neutrino experiments provides further support for the embedding of the Standard Model fermions in the chiral $16$ $SO(10)$ representation. Such an embedding is afforded by the realistic free fermionic heterotic-string models. In this paper we advance the study of these string models toward a non–perturbative analysis by generalizing the work of Donagi, Pantev, Ovrut and Waldram from the case of $G = SU(2n + 1)$ to $G = SU(2n)$ stable holomorphic vector bundles on elliptically fibered Calabi–Yau manifolds with fundamental group $\mathbb{Z}_2$. We demonstrate existence of $G = SU(4)$ solutions with three generations and $SO(10)$ observable gauge group over Hirzebruch base surface, whereas we show that certain classes of del Pezzo base surface do not admit such solutions. The $SO(10)$ symmetry is broken to $SU(5) \times U(1)$ by a Wilson line. The overlap with the realistic free fermionic heterotic–string models is discussed.

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1 Introduction

Over the past few years a profound new understanding of string theory has emerged. In this picture the different perturbative string theories, together with eleven dimensional supergravity, are limits of a single underlying quantum theory \[1\]. While the rigorous formulation of this theory is still elusive, this development means that we can utilize any of the perturbative string limits to probe the features of the more fundamental structure. In particular, we can probe those properties that pertain to the phenomenological and cosmological features, as we observe them in our experimental apparatus, and by using the low energy effective field theory parameterization. One of these properties, indicated by the observed experimental data, is the embedding of the Standard Model matter states in the chiral 16 representation of \(SO(10)\). This embedding received in recent years additional strong support from the evidence for neutrino masses in atmospheric and solar neutrino experiments \[2\]. This embedding also yields the canonical GUT normalization of the weak hypercharge and consequently qualitative agreement with the measured values of \(\sin^2 \theta_W(M_Z)\) and \(\alpha_s(M_Z)\) \[3\].

The perturbative string limit which may preserve this \(SO(10)\) embedding is the heterotic string. It should be emphasized, however, that the heterotic string in itself does not guarantee the preservation of the \(SO(10)\) embedding and indeed many quasi-realistic models have been constructed that do not maintain the \(SO(10)\) embedding \[4\]. A class of realistic string models that do preserve the \(SO(10)\) embedding are
the free fermionic heterotic string models [5]. In these three generation models the $SO(10)$ symmetry is broken to one of its subgroups by utilizing Wilson line symmetry breaking. A generic feature, in fact, of heterotic string unification, with profound phenomenological and cosmological implications [3], is precisely the utilization of Wilson–line GUT symmetry breaking, rather than GUT symmetry breaking by the Higgs mechanism. In summary, there are two pivotal ingredients that we would like the realistic string vacuum to possess. First, it should admit the $SO(10)$ embedding of the Standard Model spectrum, which is motivated by the observed experimental data. Second, it should allow for the Wilson breaking of the $SO(10)$ symmetry.

These two ingredients are in general not accommodated in generic string vacua, but are afforded by the realistic free fermionic models. The free fermionic models are, however, constructed in the perturbative heterotic string limit and it is therefore natural to examine which of their structures is preserved in the nonperturbative limit. The nonperturbative limit of the heterotic string is conjectured to be given by the heterotic M–theory limit, or by compactifications of the Hořava–Witten model [7] on Calabi–Yau threefolds.

We further remark that one should not expect heterotic M–theory compactifications to compete with the perturbative heterotic string in trying to calculate properties of the vacuum that are more readily obtained in the perturbative limit. Indeed, we may not even know how or whether some of these properties are defined in the nonperturbative limit. Thus, details of the particle spectrum and the superpotential interactions are more readily obtained in the perturbative heterotic string limit. The merit of the nonperturbative limit will be in trying to gain insight into phenomena which are intrinsically nonperturbative in nature. Specifically, in trying to elucidate the dynamical mechanism which is responsible for selecting a specific string vacuum and the related topology changing transitions. As they have brought to the fore the relevance of string compactifications to the details of the Standard Model data, it is plausible that the free fermionic models will also be instrumental to shed light on these nontrivial issues. We will expand on this aspect in future publications.

In this paper we make the first steps towards studying compactifications of Hořava–Witten theory on manifolds that are associated with the realistic free fermionic models [5]. Heterotic M–theory compactifications to four dimensions have been studied by Donagi et al. [8, 9], on manifolds that do not admit Wilson line breaking and yield $SU(5)$, $SO(10)$ or $E_6$ grand unified gauge groups [8], as well as construction of $SU(5)$ grand unified models that can be broken to the Standard Model gauge group by Wilson line breaking [9]. In this paper we extend the work of Donagi et al., to the case of $SO(10)$ models that allow Wilson line breaking. This entails the generalization of the gauge bundle analysis of ref. [8] from $G = SU(2n + 1)$ to $G = SU(2n)$ in the decomposition of $E_8 \supset G \times H$, where $H = SO(10)$ in our case. As a result we advance the analysis toward relating heterotic M–theory compactification to the realistic free fermionic models. It should also be remarked, as will be further discussed below and in future publications, that the Calabi–Yau manifolds that are
associated with the free fermionic compactifications realize the structure of the mani-

folds constructed by Donagi et al. More precisely, they correspond to manifolds with

fundamental group $\mathbb{Z}_2$, which is necessary for Wilson line breaking. As our concrete

element in this paper we discuss the breaking of the $SO(10)$ symmetry by Wilson

lines to $SU(5) \times U(1)$, which is the flipped $SU(5)$ breaking pattern [10].

Our paper is organized as follows: in section 2 we review the construction of the

standard model of particle physics from M–theory. Section 3 presents a classification

of nonperturbative, heterotic M–theory vacua of toroidally fibre d Calabi–Yau 3–folds

over Hirzebruch surfaces $F_r$. Del Pezzo surfaces $dP_r$ have recently been shown to

exhibit a mysterious duality with toroidalcompactifications of M–theory [11]; in

section 4 we extend our analysis to the case of the del Pezzo surface $dP_3$. The

overlap between these constructions and the free fermionic models is highlighted in

section 5. Finally, section 6 contains a discussion and conclusions.

2 Review of standard models from heterotic M–theory

This section summarizes the relevant information to construct the standard model

elementary particles from heterotic M–theory. We follow ref. [9] closely.

2.1 The anomaly–cancellation condition

The 11–dimensional spacetime $M_{11}$ of M–theory is taken to be

$$M_{11} = M_4 \times \frac{S^1}{\mathbb{Z}_2} \times Z,$$

where $M_4$ is 4–dimensional Minkowski spacetime, the compact eleventh dimension $S^1$

is modded out by the action of $\mathbb{Z}_2$, and $Z$ is a Calabi–Yau (complex) 3–fold. There is

a semistable holomorphic vector bundle $V_i$, $i = 1, 2$ over the 3–fold $Z$ on the orbifold

fixed plane at each of the two fixed points of the $\mathbb{Z}_2$–action on $S^1$. The structure

group $G_i$ of $V_i$ is a subgroup of $E_8$.

Fivebranes exist in the vacuum, which wrap holomorphic 2–cycles within $Z$ and

are parallel to the orbifold fixed planes. The fivebranes are represented by a 4-form

cohomology class $[W]$.

The Calabi–Yau 3–fold $Z$, the gauge bundles $V_i$ and the fivebranes are subject to

the cohomological constraint on $Z$

$$c_2(V_1) + c_2(V_2) + [W] = c_2(TZ),$$

where $c_2(V_i)$ is the second Chern class of the $i$–th gauge bundle and $c_2(TZ)$ is the

second Chern class of the holomorphic tangent bundle to $Z$. Equation (2.2) above is

referred to as the anomaly–cancellation condition.
2.2 Elliptically–fibred Calabi–Yau 3–folds

An elliptically–fibred Calabi–Yau complex 3–fold $X$ consists of a base $B$, which is a complex 2–fold, together with an analytic map

$$\pi : X \to B \quad (2.3)$$

such that the fibre $\pi^{-1}(b)$ at a generic point $b \in B$ is an elliptic curve. We also require the existence of a section, i.e., an analytic map

$$\sigma : B \to X \quad (2.4)$$

such that $\pi \circ \sigma = 1_B$.

Given this elliptic fibration, a line bundle $L$ is defined over $B$ as the conormal bundle to the section in $X$. The Calabi–Yau condition $c_1(TX) = 0$ then implies that

$$c_1(L) = c_1(B). \quad (2.5)$$

The second Chern class $c_2(TX)$ has been found in ref. [12] to be

$$c_2(TX) = c_2(B) + 11c_1(B)^2 + 12\sigma c_1(B). \quad (2.6)$$

The Calabi–Yau condition further requires that the base be a del Pezzo, an Enriques, or a Hirzebruch surface, or a blowup of a Hirzebruch surface [15]. For all these the Chern classes $c_1(B)$ and $c_2(B)$ are known.

2.3 The spectral cover

The method of spectral covers provides an effective construction of holomorphic, semistable gauge bundles $V_X$ on elliptically–fibred manifolds $X$ [12, 13, 14, 9, 16]. The gauge bundle breaks the observable $E_8$ to $G \times H$. For the gauge group $G = SU(n)$, the required data are a divisor $C \subset X$, called spectral cover, plus a line bundle $N$ on $C$. The divisor $C$ is an $n$–fold covering of the base $B$, i.e., the restriction $\pi_C : C \to B$ of the elliptic fibration $\pi$ is an $n$–sheeted branched covering. We will not distinguish notationally between $\pi$ and its restriction $\pi_C$. We will denote the pullback and the pushforward of (co)homology classes between $C$ and $B$ by $\pi^*$ and $\pi_*$, respectively.

The requirement that

$$c_1(V_X) = 0, \quad (2.7)$$

imposed by $G = SU(n)$, implies that the line bundle $N$ has a first Chern class given by

$$c_1(N) = -\frac{1}{2} (c_1(C) - \pi^* c_1(B)) + \gamma. \quad (2.8)$$

In the above equation,

$$c_1(C) = -n\sigma - \pi^* \eta \quad (2.9)$$
is the first Chern class of the surface $C$. The latter reads in homology

$$C = n\sigma + \pi^*\eta$$

(2.10)

and, since $C$ is an actual surface within $X$, we must impose the condition that $\eta$ be an effective class in $B$. In cohomology, $\eta$ is the first Chern class of a certain line bundle $\mathcal{M}$ on $B$,

$$\eta = c_1(\mathcal{M}),$$

(2.11)

and $\gamma \in H^2(C, \mathbb{Q})$ is a class whose pushforward to $H^2(B)$ vanishes,

$$\pi_*(\gamma) = 0.$$  

(2.12)

The general solution to eqn. (2.12) is

$$\gamma = \lambda (n\sigma - \pi^*\eta + n\pi^*c_1(B)) \cdot C,$$

(2.13)

where $\lambda$ is a rational parameter. Substituting eqns. (2.9) and (2.13) into (2.8) we obtain

$$c_1(\mathcal{N}) = n\left(\frac{1}{2} + \lambda\right)\sigma + \left(\frac{1}{2} - \lambda\right)\pi^*\eta + \left(\frac{1}{2} + n\lambda\right)\pi^*(c_1(B)).$$

(2.14)

Now $c_1(\mathcal{N})$ must be an integer class. This leads to various sets of sufficient conditions on $\lambda$ and $\eta$ that ensure the integrality of $c_1(\mathcal{N})$. Following ref. [9] we find that, when $n = 2k$, one such set is

$$\lambda \in \mathbb{Z}, \quad \eta = c_1(B) \text{ mod } 2,$$

(2.15)

where the modding is by an even element of $H^2(B, \mathbb{Z})$. There are, however, alternative sets of sufficient conditions that ensure the integrality of $c_1(\mathcal{N})$. When $n = 2k$, one such alternative set is

$$\lambda = \frac{2m + 1}{2}, \quad m \in \mathbb{Z}, \quad c_1(B) \text{ even.}$$

(2.16)

This alternative set will be analyzed extensively in section 3.

### 2.4 Torus–fibred Calabi–Yau 3–folds

The breaking of $SO(10)$ to $SU(5) \times U(1)$ is done by means of Wilson lines on the Calabi–Yau manifold $X$. Nontriviality of the Wilson lines requires a nontrivial fundamental group $\pi_1(X)$. For the base manifolds $B$ enumerated above, only the Enriques surface gives rise to a nontrivial $\pi_1(X)$. However, one proves that the Enriques surface is ruled out for the reasons explained in ref. [9].

Over a base manifold given by a Hirzebruch surface $F_r$ or a del Pezzo surface $dP_r$, a Calabi–Yau 3–fold $Z$ with $\pi_1(Z) = \mathbb{Z}_2$ can be constructed as the quotient
of an elliptically–fibred Calabi–Yau $X$ by a freely–acting involution $\tau_X$, i.e., a map $\tau_X : X \to X$ satisfying $\tau_X^2 = 1$. This construction necessitates a second section $\zeta$,

$$\zeta : B \to X, \quad \pi \circ \zeta = 1_B,$$

such that $\zeta + \zeta = \sigma$ under fibrewise addition. It turns out that the involution preserves the fibration structure but exchanges the two sections,

$$\tau_X(\zeta) = \sigma, \quad \tau_X(\sigma) = \zeta,$$

so the quotient space

$$Z = X/\tau_X$$

is torus fibred instead of elliptically fibred. Let $\tau_B$ denote the involution of the base $B$ induced by the involution $\tau_X$ of $X$. Since $\tau_X$ preserves the fibration $\pi$ of $X$, it follows that $Z$ is a torus fibration over the base $B/\tau_B$.

Additional effects of the second section $\zeta$ are the following.

A curve of singularities appears in the section $\zeta$, which has to be blown up for $X$ to be smooth. The general elliptic fiber $F$ splits into two spheres: the new fiber $N$, plus the proper transform of the singular fiber, which is in the class $F - N$. The union of these new fibers $N$ over the curve of singularities forms the exceptional divisor $E$. The latter intersects the spectral cover $C$ in such a way that, as a class in $X$,

$$E|_C = E \cdot C = 4(\eta \cdot c_1(B))N = \sum_i N_i$$

for some new curves $N_i \in H_2(C, \mathbb{Z})$. Their number is $4\eta \cdot c_1(B)$, and the class

$$\frac{1}{2} \sum_i N_i$$

has the important property of being integral.

The fivebranes physically wrap a holomorphic curve within the 3–fold. This means that the cohomology class $[W]$ of the wrapped fivebranes must be Poincaré–dual to the homology class of a set of (complex) curves in the Calabi–Yau space, i.e., $[W]$ must be effective as a homology class. In general, the class of a curve in $H_2(X, \mathbb{Z})$ can be written as

$$[W] = \sigma^* (\omega) + c(F - N) + dN,$$

where $c, d$ are integers, $\omega$ is a class in the base $B$, and $\sigma^* (\omega)$ is its pushforward to $X$ under the section (2.4). One can show [8] that a sufficient condition for $[W]$ to be effective is that $\omega$ be effective in $B$, plus

$$c \geq 0, \quad d \geq 0.$$

Finally, the second Chern class of $X$ also gets modified by the presence of the new section $\zeta$,

$$c_2(TX) = 12\sigma^* c_1(B) + \left(c_2(B) + 11c_1^2(B)\right)(F - N) + \left(c_2(B) - c_1^2(B)\right) N.$$
2.5 Bundles on torus–fibred 3–folds

We will consider semistable, holomorphic vector bundles \( V_Z \) over the 3–fold \( Z = X/\tau_X \). This corresponds to constructing supersymmetric vacua of the gauge theory. Call \( q \) the quotient map

\[
q : X \to Z.
\]

Then Chern classes \( c_i(TZ) \), \( c_i(V_Z) \) of the tangent bundle and of the gauge bundle on \( Z \) can be determined from those corresponding to \( X \) by pushforward:

\[
c_i(TZ) = \frac{1}{2} q_* (c_i(TX)), \quad c_i(V_Z) = \frac{1}{2} q_* (c_i(V_X)).
\]

The spectral–cover construction summarized in section 2.3 for elliptically–fibred 3–folds \( X \) may be adapted to torus–fibred 3–folds \( Z \) [9]. One continues to have a line bundle \( \mathcal{N} \) on \( C \) whose first Chern class is given by eqns. (2.8) – (2.12). One can prove that the general solution to eqn. (2.12) picks up terms proportional to the new classes \( N_i \), so the new \( \gamma \) reads

\[
\gamma = \lambda \left( n\sigma - \pi^* \eta + n\pi^* c_1(B) \right) \cdot C + \sum_i \kappa_i N_i,
\]

for arbitrary rational coefficients \( \kappa_i \). As in section 2.3 we have to impose the condition that \( c_1(\mathcal{N}) \) (given by eqns. (2.8), (2.27))

\[
c_1(\mathcal{N}) = n \left( \frac{1}{2} + \lambda \right) \sigma + \left( \frac{1}{2} - \lambda \right) \pi^* \eta + \left( \frac{1}{2} + n\lambda \right) \pi^* (c_1(B)) + \sum_i \kappa_i N_i
\]

be an integer class. As in eqns. (2.15) and (2.16), we find various sets of sufficient conditions that ensure the integrality of \( c_1(\mathcal{N}) \). In the case when \( n = 2k \) one such set is

\[
\lambda \in \mathbb{Z}, \quad \eta = c_1(B) \mod 2,
\]

and the \( \kappa_i \) can be either all integer or all one–half an odd integer. Another set of sufficient conditions when \( n = 2k \) is

\[
\lambda = \frac{2m + 1}{2}, \quad m \in \mathbb{Z}, \quad c_1(B) \text{ even},
\]

with the same requirements on the \( \kappa_i \) as before.

Finally every semistable, holomorphic vector bundle \( V_Z \) over the 3–fold \( Z = X/\tau_X \) can be pulled back to a bundle \( V_X \) over \( X \) that is invariant under \( \tau_X \), and conversely. This invariance condition is expressed as

\[
\tau_X^*(V_X) = V_X.
\]

An analysis of eqn. (2.31) has been performed in ref. [9]. Here we will just quote the result that a set of necessary conditions for \( V_X \) to be invariant is

\[
\tau_B(\eta) = \eta, \quad \sum_i \kappa_i = \eta \cdot c_1(B).
\]
2.6 Observable and hidden sectors

In the Hořava–Witten picture [7], the gauge bundles on the Calabi–Yau 3–folds appear as subbundles of the $E_8$ bundles on the orbifold fixed planes. For simplicity, following [9], we will place a trivial, unbroken $E_8$–bundle on one of the two fixed planes (the hidden sector). On the remaining orbifold plane (the observable sector) the structure group will be an $SU(4)$ subgroup of $E_8$. Within $E_8$, the group $H = SO(10)$ is the commutant of the structure group $G = SU(4)$. Throughout this paper, when referring to an $SU(n)$–bundle on the 3–fold, we refer to this particular $SU(4)$ whose commutant leaves behind an $SO(10)$ grand unified gauge theory in the observable sector.

Picking the trivial $E_8$–bundle on the hidden sector simplifies the anomaly–cancellation condition (2.2) on the 3–fold $Z$ to

$$ [W_Z] = c_2(TZ) - c_2(V_Z). $$

(2.33)

Using eqn. (2.26), this can be expressed as an anomaly–cancellation condition on the covering 3–fold $X$,

$$ [W_X] = c_2(TX) - c_2(V_X). $$

(2.34)

Second Chern classes for $SU(n)$ gauge bundles over $X$ were computed in refs. [12, 9, 17, 18]. For the second Chern class in the case when $X$ admits only one section $\sigma$ we have

$$ c_2(V_X) = \eta\sigma - \frac{1}{24}(n^3 - n)c_1^2 - \frac{1}{2}\left(\lambda^2 - \frac{1}{4}\right)n\eta(\eta - nc_1), $$

(2.35)

where $c_1$ denotes $c_1(B)$. When $X$ admits two sections $\sigma, \zeta$ the result is [9]

$$ c_2(V_X) = \sigma \cdot \pi^* \eta - \left[\frac{1}{24}(n^3 - n)c_1^2 - \frac{1}{2}\left(\lambda^2 - \frac{1}{4}\right)n\eta(\eta - nc_1) - \sum_i \kappa_i^2\right](F - N) $$

$$ - \left[\frac{1}{24}(n^3 - n)c_1^2 - \frac{1}{2}\left(\lambda^2 - \frac{1}{4}\right)n\eta(\eta - nc_1) - \sum_i \kappa_i^2 + \sum_i \kappa_i\right]N, $$

(2.36)

which reduces to eqn. (2.35) on setting $\kappa_i = 0$. Substituting eqn. (2.36) into (2.34) and denoting $c_2(B)$ simply by $c_2$, we can identify the coefficients $c, d$ and the curve $\omega$ in eqn. (2.22):

$$ c = c_2 + \left(\frac{1}{24}(n^3 - n) + 11\right)c_1^2 - \frac{1}{2}\left(\lambda^2 - \frac{1}{4}\right)n\eta(\eta - nc_1) - \sum_i \kappa_i^2, $$

(2.37)

$$ d = c_2 + \left(\frac{1}{24}(n^3 - n) - 1\right)c_1^2 - \frac{1}{2}\left(\lambda^2 - \frac{1}{4}\right)n\eta(\eta - nc_1) $$

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A physical requirement is the existence of three families of quarks and leptons in the observable sector. The net number $N_{\text{gen}}$ of generations is related to the third Chern class of the gauge bundle through

$$N_{\text{gen}} = \frac{1}{2} \int_Z c_3(V_Z).$$  
(2.40)

Now the third Chern class reads \[17, \ 19\]

$$c_3(V_Z) = 2 \lambda \sigma \eta (\eta - nc_1(B)).$$  
(2.41)

Using eqns. 2.26, 2.40, 2.41 and integrating over the fibre $F$ yields the requirement

$$\lambda \eta (\eta - nc_1(B)) = 6.$$  
(2.42)

In ref. 20 a necessary condition has been worked out to ensure that the commutant $H$ is actually the largest preserved subgroup of $E_8$, for a given choice of $G$. In our case this condition reads

$$\eta \geq 4c_1(B).$$  
(2.43)

2.7 Summary of rules

The rules presented above allow one to construct realistic particle physics vacua with $N = 1$ supersymmetry, three families of quarks and leptons and a grand unified gauge group $SO(10)$. We refer to 4 for the geometric conditions needed on the elliptically–fibred 3–fold $X$ to produce a torus–fibred 3–fold $Z$. Let us summarize the remaining rules before working out some explicit examples.

a) Semistability condition: the spectral data $(C, N)$ specifying a semistable, holomorphic vector bundle can be written, via eqns. 2.8, 2.10, 2.27 in terms of an effective class $\eta \in H^2(B, \mathbb{Z})$ and coefficients $\lambda, \kappa_i$ satisfying eqn. 2.29,

$$\lambda \in \mathbb{Z}, \quad \eta = c_1(B) \mod 2,$$

(2.44)

(and the $\kappa_i$ either all integer or all half an odd integer) or eqn. 2.30,

$$\lambda = \frac{2m + 1}{2}, \ m \in \mathbb{Z}, \quad c_1(B) \text{ even},$$

(2.45)

(with the same requirements on the $\kappa_i$ as above).

b) Involution conditions: for a vector bundle $V_X$ on $X$ to descend to a vector bundle $V_Z$ on $Z$ it is necessary that

$$\tau_B(\eta) = \eta, \quad \sum_i \kappa_i = \eta \cdot c_1(B).$$

(2.46)
c) Effectiveness condition: a sufficient condition for $[W]$ in eqn. (2.22) to be an effective class is
\[ 12c_1(B) \geq \eta, \quad c \geq 0, \quad d \geq 0. \] (2.47)
d) Commutant condition: for the gauge group $SO(10)$ this condition reads
\[ \eta \geq 4c_1(B). \] (2.48)
e) Three–family condition:
\[ \lambda \eta (\eta - nc_1(B)) = 6. \] (2.49)

3 Vacua over Hirzebruch surfaces $F_r$

We take the base manifold $B$ to be the Hirzebruch surface $F_r$, $r \geq 0$ [21]. The latter is a $\mathbb{CP}^1$–fibration over $\mathbb{CP}^1$. A basis for $H_2(F_r, \mathbb{Z})$ composed of effective classes is given by the class of the base $\mathbb{CP}^1$, denoted $S$, plus the class of the fibre $\mathbb{CP}^1$, denoted $E$. Their intersections are
\[ S \cdot S = -r, \quad S \cdot E = 1, \quad E \cdot E = 0. \] (3.1)
All effective curves in $F_r$ are linear combinations of $S$ and $E$ with nonnegative coefficients. The Chern classes of $F_r$ are
\[ c_1(F_r) = 2S + (r + 2)E, \quad c_2(F_r) = 4. \] (3.2)
It is proved in ref. [9] that, over the base $F_r$, one can construct torus–fibred Calabi–Yau 3–folds $Z$ whose fundamental group is $\mathbb{Z}_2$ when $r = 0, 2$. For those allowed values of $r$, any class $\eta \in H_2(F_r, \mathbb{Z})$ is $\tau_B$–invariant [10]. In what follows we will work with an arbitrary allowed value of $r$. Let us write $\eta \in H_2(F_r, \mathbb{Z})$ as
\[ \eta = sS + eE \] (3.3)
for some integers $s, e$ to be determined imposing the conditions summarized in section 2.7.

For the semistability condition we have a choice. Either we impose eqn. (2.44), which implies that
\[ \lambda \in \mathbb{Z}, \quad s \text{ even}, \quad e - r \text{ even}, \] (3.4)
or we impose condition (2.45),
\[ \lambda = \frac{2m + 1}{2}, \quad m \in \mathbb{Z}, \quad r \text{ even}. \] (3.5)
The involution conditions (2.46) are
\[ \sum_i \kappa_i = 2s + 2e - sr, \] (3.6)
while the effectiveness conditions (2.47) read
\[ 24 \geq s, \quad 12r + 24 \geq e, \quad (3.7) \]
\[ \sum_{i} \kappa_i^2 \leq 112 + \frac{3}{\lambda} - 12\lambda \quad (3.8) \]
and
\[ \sum_{i} \kappa_i^2 \leq 16 + \sum_{i} \kappa_i + \frac{3}{\lambda} - 12\lambda. \quad (3.9) \]
The commutant condition (2.48) requires
\[ s \geq 8, \quad e \geq 4r + 8. \quad (3.10) \]
Finally we analyze the three–family condition (2.49):
\[ -rs^2 + 4rs + 2es - 8e - 8s = \frac{6}{\lambda}. \quad (3.11) \]
Now the left–hand side of eqn. (3.11) is always an even integer, whatever our choice for the semistability condition (eqn. (3.4) or (3.5)). If we make the choice (3.4), \( \lambda \) can only take the values \( \pm 1, \pm 2, \pm 3, \pm 6 \). Parity rules out the values \( |\lambda| = 2, 6 \) and allows \( |\lambda| = 1, 3 \). We denote this family of solutions as class A:
\[
\text{class A} : \quad s \text{ even}, \quad e - r \text{ even}, \quad \lambda = \pm 1, \pm 3. \quad (3.12)
\]
Choosing (3.5) instead, then integrality of \( \frac{6}{\lambda} = \frac{12}{2m + 1} \) restricts \( m \) to
\( -2, -1, 0, 1, \quad i.e., \quad |\lambda| = 1/2, 3/2 \). This we call class B:
\[
\text{class B} : \quad r \text{ even}, \quad \lambda = \pm 1/2, \pm 3/2. \quad (3.13)
\]
Next we solve the three–family condition (3.11) for \( e \), assuming a given value for \( s \). This gives \( e \) as a function of \( r \) and \( \lambda \):
\[ e(r; \lambda) = \frac{1}{2s - 8} \left( rs^2 - 4rs + 8s + \frac{6}{\lambda} \right). \quad (3.14) \]
For each value of \( 8 \leq s \leq 24 \) and the corresponding appropriate choice for \( \lambda \) we present below the solutions for \( e(r; \lambda) \). We indicate it whenever the solution is not an integer for any allowed value of \( \lambda \).

Class A)
\[
\begin{align*}
\text{s} &= 8, \quad e(r; \lambda) = 4r + 8 + 3/4\lambda \notin \mathbb{Z} \\
\text{s} &= 10, \quad e(r; \lambda) = 5r + 20/3 + 1/2\lambda \notin \mathbb{Z} \\
\text{s} &= 12, \quad e(r; \lambda) = 6r + 6 + 3/8\lambda \notin \mathbb{Z} \\
\text{s} &= 14, \quad e(r; \lambda) = 7r + 28/5 + 3/10\lambda \notin \mathbb{Z} \\
\text{s} &= 16, \quad e(r; \lambda) = 8r + 16/3 + 1/4\lambda \notin \mathbb{Z}
\end{align*}
\]
\[ s = 18, \quad e(r; \lambda) = 9r + 36/7 + 3/14\lambda \notin \mathbb{Z} \]
\[ s = 20, \quad e(r; \lambda) = 10r + 5 + 3/16\lambda \notin \mathbb{Z} \]
\[ s = 22, \quad e(r; \lambda) = 11r + 44/9 + 1/6\lambda \notin \mathbb{Z} \]
\[ s = 24, \quad e(r; \lambda) = 12r + 24/5 + 3/20\lambda \notin \mathbb{Z}. \] \quad (3.15)

There are no integer solutions in this class. We conclude that class A contains no vacua over the Hirzebruch surface \( F_r \), for any allowed value of \( r \).

Class B)
\[
\begin{align*}
  s &= 8, \quad e(r; \lambda) = 4r + 8 + 3/4\lambda \notin \mathbb{Z} \\
  s &= 9, \quad e(r; \lambda = -1/2) = 9r/2 + 6 \\
  s &= 10, \quad e(r; \lambda = 3/2) = 5r + 7 \\
  s &= 11, \quad e(r; \lambda = -3/2) = 11r/2 + 6 \\
  s &= 12, \quad e(r; \lambda) = 6r + 6 + 3/8\lambda \notin \mathbb{Z} \\
  s &= 13, \quad e(r; \lambda = 3/2) = 13r/2 + 6 \\
  s &= 14, \quad e(r; \lambda = -1/2) = 7r + 5 \\
  s &= 15, \quad e(r; \lambda = 1/2) = 15r/2 + 6 \\
  s &= 16, \quad e(r; \lambda) = 8r + 16/3 + 1/4\lambda \notin \mathbb{Z} \\
  s &= 17, \quad e(r; \lambda) = 17r/2 + 68/13 + 3/13\lambda \notin \mathbb{Z} \\
  s &= 18, \quad e(r; \lambda = -3/2) = 9r + 5 \\
  s &= 19, \quad e(r; \lambda) = 19r/2 + 76/15 + 1/5\lambda \notin \mathbb{Z} \\
  s &= 20, \quad e(r; \lambda) = 10r + 5 + 3/16\lambda \notin \mathbb{Z} \\
  s &= 21, \quad e(r; \lambda) = 21r/2 + 84/17 + 3/17\lambda \notin \mathbb{Z} \\
  s &= 22, \quad e(r; \lambda = 3/2) = 11r + 5 \\
  s &= 23, \quad e(r; \lambda) = 23r/2 + 92/19 + 3/19\lambda \notin \mathbb{Z} \\
  s &= 24, \quad e(r; \lambda) = 12r + 24/5 + 3/20\lambda \notin \mathbb{Z}. \] \quad (3.16)

This class does lead to integer solutions to the three–family equation (3.14). In all cases the commutant condition (3.10) is satisfied, as well as eqn. (3.7) about the effectiveness of \( \omega \) in \( B \).

We can now return to eqn. (2.22) and write explicit expressions for the homology class \([W]\) that is being wrapped by the fivebranes on the torus–fibre Calabi–Yau 3–fold \( Z \). We have for the class \( \omega \)

\[
\omega = (24 - s)S + (12r + 24 - e)E, \quad (3.17)
\]

and for the coefficients \( c, d, \)

\[
c = 112 + \frac{3}{\lambda} - 12\lambda - \sum_i \kappa_i^2, \quad (3.18)
\]

\[
d = 16 + \frac{3}{\lambda} - 12\lambda + \sum_i \kappa_i - \sum_i \kappa_i^2. \quad (3.19)
\]
Evaluating eqns. (3.17), (3.18) and (3.19) at the integer solutions for \( e(r; \lambda) \) tabulated in eqn. (3.16), we arrive at the following vacua \([W]\). Every allowed choice of \( r \), plus every choice of the rational coefficients \( \kappa_i \), subject to the conditions indicated in each case, gives rise to a different class \([W]\):

- \( s = 9 \): \( \sum_i \kappa_i = 30 \) and \( \sum_i \kappa_i^2 \leq 46 \),

\[
[W] = \sigma_*(15S + \left(\frac{15}{2} r + 18\right)E) + (112 - \sum_i \kappa_i^2)(F - N) + (46 - \sum_i \kappa_i^2)N. \quad (3.20)
\]

- \( s = 10 \): \( \sum_i \kappa_i = 34 \) and \( \sum_i \kappa_i^2 \leq 34 \),

\[
[W] = \sigma_* (14S + (7r + 17)E) + (96 - \sum_i \kappa_i^2)(F - N) + (34 - \sum_i \kappa_i^2)N. \quad (3.21)
\]

- \( s = 11 \): \( \sum_i \kappa_i = 34 \) and \( \sum_i \kappa_i^2 \leq 66 \),

\[
[W] = \sigma_* (13S + \left(\frac{13}{2} r + 18\right)E) + (128 - \sum_i \kappa_i^2)(F - N) + (66 - \sum_i \kappa_i^2)N. \quad (3.22)
\]

- \( s = 13 \): \( \sum_i \kappa_i = 38 \) and \( \sum_i \kappa_i^2 \leq 38 \),

\[
[W] = \sigma_* (11S + \left(\frac{11}{2} r + 18\right)E) + (96 - \sum_i \kappa_i^2)(F - N) + (38 - \sum_i \kappa_i^2)N. \quad (3.23)
\]

- \( s = 14 \): \( \sum_i \kappa_i = 38 \) and \( \sum_i \kappa_i^2 \leq 54 \),

\[
[W] = \sigma_* (10S + (5r + 19)E) + (112 - \sum_i \kappa_i^2)(F - N) + (54 - \sum_i \kappa_i^2)N. \quad (3.24)
\]

- \( s = 15 \): \( \sum_i \kappa_i = 42 \) and \( \sum_i \kappa_i^2 \leq 58 \),

\[
[W] = \sigma_* (9S + \left(\frac{9}{2} r + 18\right)E) + (112 - \sum_i \kappa_i^2)(F - N) + (58 - \sum_i \kappa_i^2)N. \quad (3.25)
\]

- \( s = 18 \): \( \sum_i \kappa_i = 46 \) and \( \sum_i \kappa_i^2 \leq 78 \),

\[
[W] = \sigma_* (6S + (3r + 19)E) + (128 - \sum_i \kappa_i^2)(F - N) + (78 - \sum_i \kappa_i^2)N. \quad (3.26)
\]

- \( s = 22 \): \( \sum_i \kappa_i = 54 \) and \( \sum_i \kappa_i^2 \leq 54 \),

\[
[W] = \sigma_* (2S + (r + 19)E) + (96 - \sum_i \kappa_i^2)(F - N) + (54 - \sum_i \kappa_i^2)N. \quad (3.27)
\]

Each one of the above classes represents a nonperturbative vacuum of an \( SO(10) \) grand unified theory of particle physics.
4 Vacua over the del Pezzo surface $dP_3$

As our next example we choose the base $B$ to be the del Pezzo surface $dP_3$ (for nice reviews on del Pezzo surfaces see [11, 9]). The latter can be thought of as complex projective space $\mathbb{CP}^2$, blown up at three points. A basis of $H_2(dP_3, \mathbb{Z})$ composed entirely of effective classes is given by the hyperplane class $l$, plus three exceptional divisors $E_i$, $i = 1, 2, 3$. Their intersections are

$$l \cdot l = 1, \quad E_i \cdot E_j = -\delta_{ij}, \quad E_i \cdot l = 0.$$  \hspace{1cm} (4.1)

The first and second Chern classes are given by

$$c_1(dP_3) = 3l - E_1 - E_2 - E_3, \quad c_2(dP_3) = 6.$$  \hspace{1cm} (4.2)

It is proved in ref. [9] that, over the base $dP_3$, one can construct torus–fibred Calabi–Yau 3–folds $Z$ whose fundamental group $\pi_1(Z)$ is $\mathbb{Z}_2$. It is convenient to consider the independent curves

$$M_1 = l + E_1 - E_2 - E_3$$
$$M_2 = l - E_1 + E_2 - E_3$$
$$M_3 = l - E_1 - E_2 + E_3,$$  \hspace{1cm} (4.3)

whose intersection matrix is

$$M_1 \cdot M_1 = M_2 \cdot M_2 = M_3 \cdot M_3 = -2, \quad M_1 \cdot M_2 = M_1 \cdot M_3 = M_2 \cdot M_3 = 2.$$  \hspace{1cm} (4.4)

It turns out that the $M_j$ are effective classes and that they satisfy $\tau_B(M_j) = M_j$. They also generate all other $\tau_B$–invariant classes, so the most general $\tau_B$–invariant class $\eta \in H_2(dP_3, \mathbb{Z})$ is a linear combination

$$\eta = m_1M_1 + m_2M_2 + m_3M_3,$$  \hspace{1cm} (4.5)

for some arbitrary integer coefficients $m_j$ to be determined imposing the requirements summarized in section 2.7. The first Chern class $c_1(dP_3)$ reads, in terms of the $M_j$,

$$c_1(dP_3) = M_1 + M_2 + M_3.$$  \hspace{1cm} (4.6)

Next we impose the rules summarized in section 2.7. For the semistability condition we cannot impose eqn. (2.45) in view of $c_1(dP_3)$, so we have (2.44) instead:

$$\lambda \in \mathbb{Z}, \quad m_j \text{ odd}, \quad j = 1, 2, 3,$$  \hspace{1cm} (4.7)

and either all the $\kappa_i$ integer or all half an odd integer. With our choice (4.5) for $\eta$, the involution condition (2.46) reduces to

$$\sum_i \kappa_i = 2(m_1 + m_2 + m_3).$$  \hspace{1cm} (4.8)
The effectiveness conditions (2.47) read

\[ m_j \leq 12, \quad j = 1, 2, 3 \]  \hspace{1cm} (4.9)

and

\[ \sum_i \kappa_i^2 \leq 87 + \frac{3}{\lambda} - 12\lambda, \]  \hspace{1cm} (4.10)

\[ \sum_i \kappa_i^2 \leq 15 + \frac{3}{\lambda} - 12\lambda + \sum_i \kappa_i. \]  \hspace{1cm} (4.11)

The commutant condition (2.48) requires

\[ m_j \geq 4, \quad j = 1, 2, 3. \]  \hspace{1cm} (4.12)

Finally, the three–family condition (2.49) is expressed as

\[ -m_1^2 - m_2^2 - m_3^2 + 2(m_1m_2 + m_1m_3 + m_2m_3) - 4(m_1 + m_2 + m_3) = 3/\lambda. \]  \hspace{1cm} (4.13)

Given that the \( m_j \) are odd, the left–hand side of the above is always odd, so the allowed values for \( \lambda \) are \( \pm 1, \pm 3 \).

Odd integer solutions to eqn. (4.13) in the range \( 4 \leq m_j \leq 12 \), for (at least) one of the allowed values for \( \lambda \), correspond to nonperturbative vacua of heterotic M–theory, compactified on a torus–fibred Calabi–Yau 3–fold over a \( dP_3 \) surface, with \( SO(10) \) as GUT group and 3 families of chiral matter. One can see that for \( \lambda = \pm 1, \pm 3 \) there are no solutions in the required range, and hence no vacua. A simple proof of this fact is as follows. Assume that all solutions \( m_j \) are equal to a certain value \( m \):

\[ m_1 = m_2 = m_3 = m, \]  \hspace{1cm} then solve eqn. (4.13) for \( m \). This gives \( m = 2 \pm \sqrt{4 + 1/\lambda} \), which for \( \lambda = \pm 1, \pm 3 \) is not an integer. Next assume that two solutions are equal, say \( m_1 = m_2 = m \), and that \( m_3 \neq m_1 \). Then eqn. (4.13) becomes linear in \( m \), but its solutions when \( m_3 = 5, 7, 9, 11 \) are never an odd integer for \( \lambda = \pm 1, \pm 3 \). Finally, when all the \( m_j \) are pairwise different, one can easily check numerically for \( m_j = 5, 7, 9, 11 \) that there is no solution for \( \lambda = \pm 1, \pm 3 \).

5 Overlap with the free fermionic models

In this section we elaborate briefly on the overlap with the free fermionic models. Amazingly enough, the structure of the manifolds constructed by Donagi et al , up to the imposition of the three generation condition, precisely coincides with the structure of the manifold that underlies the free fermionic models.

In the free fermionic formalism [22] a model is specified in terms of a set of boundary condition basis vectors and one–loop GSO projection coefficients. These fully determine the partition function of the models, the spectrum and the vacuum structure. The three generation models of interest here are constructed in two stages.
The first corresponds to the NAHE set of boundary basis vectors \{1, S, b_1, b_2, b_3\}\)\cite{23}. The second consists of adding to the NAHE set three additional boundary condition basis vectors, typically denoted \{\alpha, \beta, \gamma\}. The gauge group at the level of the NAHE set is \(SO(6)^3 \times SO(10) \times E_8\), which is broken to \(SO(4)^3 \times U(1)^3 \times SO(10) \times SO(16)\) by the vector \(2\gamma\). Alternatively, we can start with an extended NAHE set \{1, S, \xi_1, \xi_2, b_1, b_2\}, with \(\xi_1 = 1 + b_1 + b_2 + b_3\). The set \{1, S, \xi_1, \xi_2\} produces a toroidal Narain model with \(SO(12) \times E_8 \times E_8\) or \(SO(12) \times SO(16) \times SO(16)\) gauge group depending on the GSO phase \(c(\xi_1, \xi_2)\). The basis vectors \(b_1\) and \(b_2\) then break \(SO(12) \rightarrow SO(4)^3\), and either \(E_8 \times E_8 \rightarrow E_6 \times U(1)^2 \times E_8\) or \(SO(16) \times SO(16) \rightarrow SO(10) \times U(1)^3 \times SO(16)\). The vectors \(b_1\) and \(b_2\) correspond to \(Z_2 \times Z_2\) orbifold modding. The three sectors \(b_1, b_2\) and \(b_3\) correspond to the three twisted sector of the \(Z_2 \times Z_2\) orbifold, with each producing eight generations in the \(27\), or \(16\), representations of \(E_8\), or \(SO(10)\), respectively. In the case of \(E_6\) the untwisted sector produces an additional \(3 \times (27 + 27)\), whereas in the \(SO(10)\) model it produces \(3 \times (10 + 10)\). Therefore, the Calabi–Yau manifold that corresponds to the \(Z_2 \times Z_2\) orbifold at the free fermionic point in the Narain moduli space has \((h_{11}, h_{21}) = (27, 3)\).

To note the overlap with the construction of Donagi et al we construct the \(Z_2 \times Z_2\) at a generic point in the moduli space. For this purpose, let us first start with the compactified torus \(T_1^2 \times T_2^2 \times T_3^2\) parameterized by three complex coordinates \(z_1, z_2\) and \(z_3\), with the identification

\[
z_i = z_i + 1 \quad ; \quad z_i = z_i + \tau_i, \tag{5.1}
\]

where \(\tau\) is the complex parameter of each torus \(T^2\). We consider \(Z_2\) twists and possible shifts of order two:

\[
z_i \rightarrow (-1)^{\xi_i} z_i + \frac{1}{2} \delta_i, \tag{5.2}
\]

subject to the condition that \(\Pi_i (-1)^{\xi_i} = 1\). This condition insures that the holomorphic three–form \(\omega = dz_1 \wedge dz_2 \wedge dz_3\) is invariant under the \(Z_2\) twist. Under the identification \(z_i \rightarrow -z_i\), a single torus has four fixed points at

\[
z_i = \{0, 1/2, \tau/2, (1 + \tau)/2\}. \tag{5.3}
\]

The first model that we consider is produced by the two \(Z_2\) twists

\[
\alpha : (z_1, z_2, z_3) \rightarrow (-z_1, -z_2, z_3) \\
\beta : (z_1, z_2, z_3) \rightarrow (z_1, -z_2, -z_3). \tag{5.4}
\]

There are three twisted sectors in this model, \(\alpha, \beta\) and \(\alpha \beta = \alpha \cdot \beta\), each producing 16 fixed tori, for a total of 48. The untwisted sector adds three additional Kähler and complex deformation parameters producing in total a manifold with \((h_{11}, h_{21}) = (51, 3)\). We refer to this model as \(X_1\). This manifold admits an elliptic fibration over
a base $F_0 = \mathbb{CP}^1 \times \mathbb{CP}^1$. This can be seen from the Borcea–Voisin classification of elliptically fibered Calabi–Yau manifolds [24] and from ref. [25]. We emphasize that in the discussion here our intention is not to reproduce the generic M–theory elliptic fibration model, but rather to indentify the overlap with the specific geometrical structures that underly the realistic free fermionic models. We further remark that here we only discuss the correspondence with the compactified geometry and it would be of further interest to elucidate how the gauge bundles are related in the fermionic and the M-theory models.

Next we consider the model generated by the $\mathbb{Z}_2 \times \mathbb{Z}_2$ twists in (5.4), with the additional shift

$$\gamma : (z_1, z_2, z_3) \rightarrow (z_1 + \frac{1}{2}, z_2 + \frac{1}{2}, z_3 + \frac{1}{2}).$$

(5.5)

This model again has fixed tori from the three twisted sectors $\alpha$, $\beta$ and $\alpha \beta$. The product of the $\gamma$ shift in (5.5) with any of the twisted sectors does not produce any additional fixed tori. Therefore, this shift acts freely. Under the action of the $\gamma$ shift, half the fixed tori from each twisted sector are paired. Therefore, the action of this shift is to reduce the total number of fixed tori from the twisted sectors by a factor of $1/2$. Consequently, in this model $(h_{11}, h_{21}) = (27, 3)$. This model therefore reproduces the data of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold at the free-fermion point in the Narain moduli space. We refer to this model as $X_2$.

The manifold $X_1$ therefore coincides with the manifold $X$ of Donagi et al, the manifold $X_2$ coincides with the manifold $Z$, and the $\gamma$–shift in eq. (5.5) coincides with the freely acting involution $\tau_X$ in eqns. (2.18,2.19). Thus, the free fermionic models admit precisely the structure of the Calabi-Yau manifolds considered in ref. [9].

6 Discussion and conclusions

The role of the freely acting shift discussed in the previous section and employed in sections 3, 4 is to produce a manifold which is not simply connected, with $\pi_1(Z) = \mathbb{Z}_2$. This enables the use of Wilson lines to break the $SO(10)$ symmetry to one of its subgroups. For example, up to $SO(10)$ automorphisms, the only generator of $SO(10)$ that leaves $SU(5)$ unbroken is given by [26]

$$-iH = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$ (6.1)
Other breaking patterns, that break the $SO(10)$ symmetry to one of its other subgroups, as in ref. \[3\], are possible, and in some cases by utilizing two independent Wilson lines \[4\]. Here the third Chern class counts the number of $16$ minus $\overline{16}$ representations of $SO(10)$. The net number $N_{\text{gen}} = 3$ then contains the three chiral Standard Model generations. The additional $10 \oplus \overline{10}$ representations of $SU(5)$ needed to break the $SU(5) \times U(1)$ symmetry arise from additional $16 \oplus \overline{16}$ representations that are obscured from $N_{\text{gen}}$ but, in general, appear in the physical spectrum. The adjoint $248$ representation of $E_8$ decomposes as $248 = (45, 1) + (1, 15) + (10, 6) + (16, 4) + (\overline{16}, \overline{4})$ under $SO(10) \times SU(4)$. As is the case in the example of the free fermionic string models the $(10, 6)$ component in this decomposition can produce the $SO(10)$ vectorial representation, which contains the electroweak Higgs multiplets of the Minimal Supersymmetric Standard Model, and may produce the fermion masses from the $16 \cdot 16 \cdot 10$ superpotential term.

The question, however, is the utility of the sophisticated mathematical tools employed in this paper. It is rather plausible that details of the massless spectrum and fermion masses can be more readily obtained by using the conformal field theory based formulations of the perturbative string limits. Indeed, this lesson we can already infer from heterotic string studies that yielded detailed fermion mass textures, addressing issues like Cabibbo mixing and fermion mass hierarchy \[27\]. It is doubtful that the geometry based formalism can explore a similar level of detail.

As we saw in sections \[3\] and \[4\], one utility of the geometrical approach is in classification of the available geometries by the phenomenological criteria that they allow or disallow. While Hirzebruch surfaces $F_r$, $r = 0, 2$, provide three generation solutions with $SO(10)$ symmetry that may be broken by Wilson lines, the del Pezzo surface $dP_3$ does not admit such solutions, within the classes that we have analyzed here. Whether this observation can be generalized to any del Pezzo surface $dP_r$ is an interesting question that will be addressed in a forthcoming publication \[28\]. Thus, the geometrical insight provides a tool to classify the manifolds according to very basic phenomenological criteria.

More importantly, however, it is apparent that the power of the complex manifold analysis, in the particle phenomenology context, will be revealed in trying to elucidate basic issues like the topology changing transitions and vacuum selection. In this respect we can promote the following view of the utility of the M–theory picture that emerged in recent years. It is now conjectured that the different string theories are limits of one single, still elusive, more basic theory. Each limit can then be utilized to probe the properties of the more fundamental theory, or its properties that may pertain to the observed particle physics phenomenology. Thus, one limit may be utilized to extract classes of manifolds that possess appealing phenomenological characteristics, whereas another limit may be useful to investigate dynamical transitions between nearby manifolds.

In this context, it has long been argued that the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold compactification naturally gives rise to three chiral generations \[29\]. Modding the 6–dimensional
compactified space by the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold projection produces three twisted sectors. In free fermionic string models that are connected to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold each twisted sector gives rise to one chiral generation. Thus, in these models the existence of three generations is correlated with the underlying geometrical structure.

The existence of three twisted sectors is a generic property of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold of a 6–dimensional compactified space. At a generic point of the compactified space one can take the moduli space to be that of $T^2 \times T^2 \times T^2$ yielding a model with $(h_{21}, h_{11}) = (51, 3)$. At the free fermionic point the symmetry is enhanced, producing an $SO(12)$ lattice. Taking the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold of the $SO(12)$ lattice then produces a model with $(h_{21}, h_{11}) = (27, 3)$. Each of these models has three twisted sectors with 16 and 8 fixed points in their respective twisted sectors. Depending on a discrete torsion phase which commutes with the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold projection we can set the 4–dimensional nonabelian observable symmetry to be $SO(10)$ or $E_6$. As we discussed in this paper, in order to allow Wilson line breaking of the nonabelian $SO(10)$ or $E_6$ GUT symmetry, a Calabi–Yau manifold has to be nonsimply connected. From refs. [8, 17, 30] we learned that a simple way to achieve this is by modding a simply connected Calabi–Yau manifold by a freely–acting involution, the latter being $\mathbb{Z}_2$ in the models studied here and in [9, 17, 30]. Now, the $(51, 3)$ and $(27, 3)$ $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold compactifications are connected by precisely such a freely–acting involution. Thus, what is remarkable is that precisely at the free fermionic point in the moduli space, we find that the model naturally accommodates three generations due to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold structure, with the desirable $SO(10)$ Grand Unified symmetry, while at the same time it allows for the inclusion of the Wilson line to break the GUT symmetry, due to the freely–acting involution.

Thus, we see that precisely at the free fermionic point in the string moduli space, some of the needed ingredients coalesce to produce the phenomenologically required features. This remarkable coincidence is, however, valid at weak coupling. The issue to understand is whether it remains valid in the strong coupling regime. The compactification of Hořava–Witten theory, as discussed here, provides the means to investigate such questions.

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