Yang–Mills at strong vs. weak coupling
Renormalons, OPE and all that

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Abstract I discuss various situations in which perturbative expansions are used in Yang–Mills theories with asymptotic freedom and establish the limits of its applicability.

1 Introduction

The notion of resurgence and trans-series associated with it was a breakthrough discovery1 in constructive mathematics in the 1980s mostly associated with the name of Jean Ecalle. Resurgence is based on the idea that divergent perturbative series can be made well-defined by invoking additional data on the structure of singularities in the coupling constant complex plane plus certain quasiclassical information. In this way, the standard perturbative series becomes a trans-series,

\[ E(g^2) = E_{PT, regularized}(g^2) + \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{1}{g^2 N+1} \exp\left[ \frac{-c}{g^2} \right]^{k} \times \]

\[ \left( \log \frac{c}{g^2} \right)^{l} c_{k,l,p} g^{2p} \text{ regularized PT} \]  

(1.1)

The role of previously neglected exponential terms of the type \( \exp\left(-\frac{1}{g^2}\right) \) in the divergent coupling constant expansion is fully revealed in trans-series which allow one to achieve any preset accuracy of expansion.

The method was scarcely known in modern quantum theory and related areas of physics until 2010s when this construction—being appropriately adjusted and extended—was introduced in this context in \cite{2-4}. Now it is being further developed by many followers in multiple applications.

The most essential advances have been achieved in such areas as quantum mechanics (QM), partial differential equations, integrable nonlinear systems and orthogonal polynomials. Much more modest is the progress along these lines in asymptotically free quantum field theories, such as Yang–Mills theory, with running constants and complete dynamical restructuring at large distances.

There is a profound difference between the perturbative expansion, say, for the energy eigenvalues of an anharmonic oscillator and in problems arising in asymptotically free field theories. In the former case, a coupling constant is well-defined. It is a small number \( g \) which, in principle can be complexified, if necessary, and continued in the complex plane. Perturbative series in \( g \) which are usually plagued by factorial divergences in high orders can be made well-defined based on the quasiclassical data obtained in the complex plane. In this way one arrives at trans-serries, including both regularized perturbation theory and exponential terms \( e^{-c/g^2} \) in a systematic manner.

In Yang–Mills theory there is no dimensionless coupling constant to form a perturbative expansion in the strict sense of this word. If we ignore quarks for the time being then the only parameters of the theory are the dynamical scale \( \Lambda \) and the vacuum angle \( \theta \). The only expansion parameter appearing in the 't Hooft limit \cite{5,6} is \( 1/N \) where \( N \) is the number of colors. Quantitative methods for construction of perturbative series in \( 1/N \) have not yet been developed although a number of qualitative observations exist. If such a series could be obtained, say, for the mass of the lightest glueball, we would arrive at

\[ M_{\text{glueball}} = A \sum_{j=0}^{\infty} \frac{c_{j}}{Nk_{j}} \]  

(1.2)

1 For a pedestrian review understandable to physicists (at least, in part) and an exhaustive list of references see \cite{1}.

2 From experiment we know that \( \theta \leq 10^{-10} \) and is irrelevant in phenomenology. However, it plays a role in theoretical studies of Yang–Mills dynamics at strong coupling.
where \( c_j \) are purely numerical dimensionless coefficients depending only on the quantum numbers of the glueball under consideration. Needless to say, the question of convergence in (1.2) at large \( j \) will arise. I think exponential terms of the type \( \exp(-cN) \) do appear, but at the moment it would be prudent to refrain from more detailed speculations.

Passing from Yang–Mills to QCD with massless quarks aggravates the situation. As is well-known, spontaneous breaking of the continuous chiral symmetry (\( \chi \)SB) is not seen in perturbation theory in the gauge coupling, whatever this coupling might mean. Nobody in one’s right mind would write that, say, the \( \rho \) meson mass can be extracted from a perturbative series even amended by additional quasiclassical information. However, there is a range of questions in which QCD perturbation theory is widely used.

In QCD and similar theories, it is quite common to add external sources to use them as tools. For instance, a virtual photon produces a quark-antiquark pair eventually evolving in a cascade of hadrons. If the momentum \( Q \) injected in this way can be translated in Euclidean we acquire a large external parameter \( Q/\Lambda \) and can develop a perturbation theory in \( \alpha_s(Q) \sim (\log Q/\Lambda)^{-1} \). In a large number of problems (e.g. moments of the structure functions in deep inelastic scattering) people learned how to build expansions in \( \alpha_s(Q) \) up to three or even four loops. These results can be compared with experiment. With time such comparisons becomes more and more accurate.

I hasten to add that even in the problems with a large parameter \( Q/\Lambda \) the \( \alpha_s(Q) \) expansion cannot be made closed—i.e. cannot be continued to any desirable accuracy, as is the case in QM trans-series. The problem lies deeper than just the existence of \( \sim k! \) graphs with \( k \) loops. Were this the only obstacle, it could be dealt with either through resurgence or by passing to the ’t Hooft limit [5,6]. In this limit, only planar graphs survive, and the number of the planar graphs does not grow factorially [7,5].

More serious issues exist. Any physical observable is measured in Minkowski space, not in Euclidean. Conversion from a Euclidean calculation to Minkowski predictions is controllable in a limited sense at best. Second, the \( \alpha_s(Q) \) expansion is intrinsically ill-defined [8] because even if \( \alpha_s(Q) \ll 1 \) any Feynman diagram saturated at \( p \sim Q \) (I stress, \emph{any}) still contains contributions from virtual momenta \( p \sim \Lambda \). In this domain the coupling \( \alpha_s \) is not defined, simply because the Lagrangian formulated in terms of quarks and gluons ceases to exist.

Therefore, Feynman graphs are, in fact, represented by a complicated combination: the \( \alpha_s(Q) \) series, (i.e. a series of terms of the type \( \sum_k \{\log(Q/\Lambda)^{-1}\} k \) which may also contain \( \log\log \)'s, etc.), exponential terms in \( \alpha_s \) of the type \( \sum_k (A/\Lambda)^k \), coming from IR contributions, and conversion factors from Euclidean to Minkowski of the type \( \exp(-Q/\Lambda)^{\gamma} \). Both power and exponential terms also are generated by the UV physics, namely tails of the UV renormalons and small-size instantons. All the expansions above depend on \( N \). For conversion factors from Euclidean to Minkowski this dependence is most dramatic and so are deviations from perturbative expansions. I am aware of only one example when this conversion is smooth (see Sect. 5).

Can we amend the \( \alpha_s(Q) \) series transforming it in a trans-series, such as in Eq. (1.1) by invoking quasiclassical information?

The resurgence and trans-series program (such as in quantum mechanics) can \emph{not} be fully successful in QCD-like theories at strong coupling. Underlying dynamics in confining theories at large distances in no way reduces to expansion in \( \alpha_s \) even being supplemented by additional quasiclassical analyses. Confinement of the Nambu–Mandelstam–’t Hooft type in the SUSY setting was demonstrated [9,10] to emerge from the dual Meißner effect—a very special non-perturbative feature of the Yang–Mills vacuum—and so is \( \chi \)SB. Both are crucial at distances \( \gg \Lambda^{-1} \) and leave no trace in perturbation theory.

Summarizing, if one has an external large parameter such as the momentum transfer \( Q^2 \) or \( r(S_1)^{-1} \), (here \( r(S_1) \) is the radius of the compactified dimension) one can classify various expansions: (i) logarithmic, i.e. in powers of \( 1/(\log(Q^2/A^2)) \) and \( \log\log's/\log(Q^2/A^2) \); (ii) in powers of \( A^2/Q^2 \) or \( (A^2/Q^2) (\log(Q^2/A^2))^{\gamma} \); and, finally, possible exponential effects of the type \( \exp(-Q/\Lambda)^{\gamma} \). The series in power terms (1.3) is very likely to be divergent [11,12].

In the absence of the solution of strong coupling Yang–Mills theories in 4D, the best one can do is the OPE in the form described in detail in the reviews [13,14] (see also references therein) which requires a new delimiting parameter \( \mu \),

\[
\Lambda \ll \mu \ll Q.
\]

The latter drops out from all measurable quantities. Introduction of the above parameter eliminates IR renormalons. The conspiracy between perturbative series and purely gluon operators in OPE [15] remains valid.

However, if we force the coupling constant to stop running in the infrared (IR) by e.g. Higgsing the gauge theory or formulating it on \( R_3 \times S_1 \) with \( r(S_1) \ll \)}
Λ⁻¹ situation changes to mostly quasiclassical and the study of perturbative expansions and isolating non-perturbative terms may turn out productive and give us certain insights.

Putting Yang–Mills on \( R \times S_1 \) with \( r(S_1) \ll Λ⁻¹ \) is one of the most popular techniques exploited today. It was pioneered by Ünsal et al. \([16–18]\). In this set-up, the coupling constant ceases running at the scale \( \sim r⁻¹ \gg Λ \), i.e. at weak coupling. I will return to this set-up later, starting from the most obvious option in which we stay in \( R \) but Higgs the gauge theory at hand.

2 Yang–Mills at weak coupling

More exactly, I should say “gauge theories mostly at weak coupling.” Even if all gauge bosons are Higgsed, nontrivial large-distance effects may show up. The most well known example is two-dimensional Higgsed QED in which linear confinement is still present \([19]\) (see also \([20]\)). A less known example is the phenomenon of the baryon number nonconservation at high energies in the Standard Model (see reviews \([21,22]\) and references therein, also \([20]\), Section 22). At low energies \( E/m_{W} \sim 1 \) this cross section is of the order of \( \exp (−2S_{\text{inst}}) \) where \( S_{\text{inst}} = \frac{8π^2}{9r} \) is the instanton action. As the energy grows the above cross section grows exponentially until it reaches its maximum \( \sim \exp (−S_{\text{inst}}) \) at the energy around the sphaleron mass,

\[
M_{\text{sph}} = \text{const} \times m_{W} \frac{2π}{α(m_{W})} \quad (2.1)
\]

where the numerical constant is of the order of \( 1 \).\(^6\) The difference between \( \exp (−2S_{\text{inst}}) \) and \( \exp (−S_{\text{inst}}) \) is enormous, 90 orders of magnitude or so. What is sphaleron? It is a configuration in the space of fields which corresponds to the top of the barrier separating distinct instanton pre-vacua. The latter have the Chern–Simons numbers \( K = 0, ±1, ±2, \ldots \) while the sphaleron Chern-Simons numbers are half-integer. Further details are discussed in Sect. 2.1.

The typical number of \( W \) bosons and Higgs bosons produced in this process scales as \( 1/α(m_{W}) \). The exact maximal value of the \textit{baryon number nonconservation cross section}\(^7\) of the type (for one generation)

\[
1 \text{ leptons } + 1 \text{ quark } \rightarrow 2 \text{ antiquarks } + 1/α_{W} \text{ bosons} \quad (2.2)
\]
cannot be obtained in perturbation theory, nor by its quasiclassical extension. Even the logarithm of this cross section at \( E \sim M_{\text{sph}} \) in fact contains an unknown constant of the order of \( 1 \) (although less than \( 2 \)).

However, such examples are rather exotic. Barring such exotic situations, the vast majority of phenomena in Higgsed Yang–Mills or Yang–Mills theories on a small-radius cylinder can be treated semiclassically.

2.1 Higgsing

An example known to everybody is the SU(2) sector of the Standard Model. If we put the Weinberg angle \( θ_W = 0 \), all three gauge bosons have the same masses. We will refer to them as \( W \) bosons and assume that \( m_{W} \gg Λ \). The maximal value of the running coupling constant is \( α_{\text{max}} = α(m_{W}) \ll 1 \). Its running towards the IR is frozen at \( m_{W} \).

Under the circumstances there are no IR renormalons in this theory since

\[
(β_1α(Q^2)/4π) \log(Q^2/k^2)
\]

in Eq. (3.11) never approaches unity due to the fact that effectively \( k^2 \gtrsim m_{W}^2 \) (see Fig. 2 and Sect. 3 below). The instanton configuration becomes well defined. Its contribution to low-energy quantities is of the order of

\[
∫ d^4x Λ^{43/6} e^{−19/6} \quad (2.3)
\]

The overall nonperturbative effect in the vacuum is an extremely rarified instanton gas. Only if we approach energies of the order of (2.1) will the strong instanton-antinstanton interaction manifest itself as was discussed above. Considering the rarified instanton gas we could say that it represents the unit operator or any other purely bosonic operator. The difference is only in the pre-exponent. The \( μ \) evolution below \( m_{W} \) is frozen and we can descend all the way down to \( μ = m_{W} \). The situation changes, however, if we add massless (or very light compared to \( m_{W} \)) fermions.

The instanton–induced interaction contains all doublets of left-handed SM fermions, say, for the first generation it has the form

\[
(qq)(ℓ) e^{−S_{\text{inst}}} \quad (2.4)
\]

The fermion pre-factor conserves the electric charge and \( B−L \). However, obviously it violates the baryon charge conservation by one unit. The operator appearing in (2.4) is four-fermion and, generally speaking, does not reduce to the unit operator.

The instanton effects considered above are obtained in Euclidean calculations. Instantons are Euclidean objects. Nevertheless, each non-perturbative effect seen in Euclidean must have a clear-cut reflection in Minkowski physics. And, indeed, they do have a reflection in the spectrum of the theory. In addition to the \( W \)-boson triplet and the Higgs scalar this theory has an unstable soliton called a \textit{sphaleron} of a huge mass (2.1) and a typical size \( \sim m_{W}^{-1} \) (see Fig. 1). The ratio

\(^6\) This phenomenon is called \textit{premature unitarization} \([23–26]\).

\(^7\) Baryon number violation at high temperatures, of the order of \( M_{\text{sph}} \), can be treated quasiclassically, though \([27–30]\).
$M_{\text{sph}}/m_W \gg 1$. Such a relation between characteristic parameters is typical for quasiclassical objects. The physical meaning of the sphaleron is the height of the top of a barrier in the functional space [31–34] separating pre-vacua with the Chern-Simons charges $K = K' \pm 1$. The sphaleron is a coherent state of $\sim 1/\alpha(m_W)$ W bosons. The sphaleron and the associated phenomenon of exponential growth of the W-boson production at energies below the sphaleron mass is indirectly related to the Euclidean instanton (see [20], Sections 21.14 and 22, and references therein).

In the ‘t Hooft limit, the sphaleron mass (which scales as $N$ if $m_W$ is fixed) becomes infinite and it disappears from the spectrum; simultaneously, the instanton contribution vanishes as $e^{-N}$.

Another non-perturbative effect surviving in the Higgsed Yang–Mills theory is that of the UV renormalon. There is no new physics in it, see Sect. 3.1. The $g^2$ series coming from the UV renormalon is summable (although not expandable in a regular $g^2$ series).

Long ago Fradkin and Shenker argued [35] that the following continuity takes place.\footnote{Fradkin and Shenker argument referred to Yang–Mills on the lattice; later it was extended.} Suppose that, in addition to gauge fields, say in the SU($N$) group, a given non-Abelian theory contains a set of Higgs fields in the fundamental representation, which, by developing vacuum expectation values (VEVs) completely Higgs the gauge group. The theory is at weak coupling provided all gauge bosons are heavy.

Then, upon decreasing all the above VEVs in proportion to each other from large to small values triggering a strong coupling regime we do not pass through a Higgs-confinement phase transition. Rather, a crossover from weak to strong coupling takes place.

One can argue that this is the case in many different ways. Perhaps the most straightforward line of reasoning is as follows. Using the Higgs field in the fundamental representation one can build gauge-invariant interpolating operators for all possible physical states. They span the space of all possible global quantum numbers (such as spin and discrete symmetries of the Lagrangian).

The Källen–Lehmann spectral functions corresponding to these operators, which carry complete information on the spectrum, depend smoothly on $\nu$. When the latter parameter is large the Higgs description is more convenient; when it is small it is more convenient to think in terms of bound states. There is no sharp boundary and no phase transition.

This seems quite surprising—the Higgs regime does not look even remotely similar to that occurring at strong coupling. And, nevertheless, the passage from weak to strong coupling proceeds without phase transition.\footnote{However, if we add massless quarks the $\chi_{SB}$ phase transition can occur.} The same phenomenon takes place on a cylinder under certain conditions.

2.2 Yang–Mills on a cylinder

As was suggested in [16–18], if one considers Yang–Mills theory on $R_3 \times S_1$ rather than on $R_4$ under certain conditions the theory becomes weakly coupled, with no phase transition on the way from weak to strong coupling. This is called adiabatic continuity. The most important condition is conservation of the center symmetry which is equivalent to a vanishing Polyakov line. In Yang–Mills theory on $R_3 \times S_1$ one can force the Polyakov line to vanish either by adding massless adjoint quarks (one is enough) or through a double trace deformation. As we continuously change $r(S_1)$ from very small (weak coupling) to very large (strong coupling) there is no phase transition, in much the same way as in the example of Sect. 2.1. The change of the regimes proceeds through a crossover. Qualitative regularities established at weak coupling are also applicable to strong coupling. In pure Yang–Mills on $R_3 \times S_1$ the center symmetry is spontaneously broken at small $r(S_1)$ and therefore the confining limit $r(S_1) \to \infty$ with restored center symmetry is separated by a phase transition.

The compactified dimension is assumed to be spatial, $r(S_1) \ll (N\Lambda)^{-1}$, with judiciously chosen boundary conditions on fermion fields (in most cases periodic, sometimes twisted). Thus, the boundary conditions are non-thermal. Then, the SU($N$) gauge group is Higgsed, SU($N$) $\to$ U($1$)$^{N-1}$, all gauge bosons not belonging to the Cartan subalgebra acquire masses $m_W \gg \Lambda$.

The advantages of this method of the IR regularization are as follows:

(i) One can consider $N$ as a free parameter, in particular, explore the limit $N \to \infty$. The ‘t Hooft limit (large-$N$) is much easier to implement on $R_3 \times S_1$ than through Higgsing on $R_4$. In this limit only planar diagrams survive; their number does not grow factorially\footnote{The number $\nu$ of graphs of genus $h$ with $k$ vertices grows at large $k$ as $\nu_{n}(k) \sim (k)^{-b_0} \exp(ck)$. A sphere (relevant to} with $\ell$ where $\ell$ is the number of loops...
In what follows I will not dwell on this interesting aspect, referring the reader to the original papers.

(ii) The gauge coupling constant is frozen at \( g^2(r(S_1)^{-2}) \). To guarantee the weak coupling one must choose \( r(S_1) \ll \Lambda^{-1} \) (remember, \( N \) is not a large parameter in my considerations. Otherwise, one should choose \( r(S_1) \ll (AN)^{-1} \). This allows one to carry out a quasiclassical analysis. The class of relevant quasiclassical solutions on \( R_{3 \times S_1} \) is richer than on \( R_4 \). In addition to instanton-monopoles there are bions \(^{36} \) which carry both topological and magnetic charges.

(iii) The action of these objects is \( N \) times smaller than that of the instanton. More exactly,

\[
S_{\text{frac}} = \frac{k}{N} S_{\text{inst}} = k \frac{8\pi^2}{g_0^2 N} \tag{2.5}
\]

where \( k = 1 \) for instanton-monopole, \( k = 2 \) for bions and so on. That’s why sometimes they are referred to as fractons. Say, in pure Yang–Mills theory the instanton-monopole contribution to the vacuum energy density is proportional to

\[
\Delta \mathcal{E}_{\text{vac}} \sim M_0^4 \exp \left[ -\frac{8\pi^2}{g_0^2 N} \left( 1 - \frac{1}{12} \right)^{-1} \right] \tag{2.6}
\]

where \( -\frac{1}{12} \) in the parentheses presents the contribution of non-zero modes. The above contribution does not vanish at large \( N \).

(iv) If we add a massless fermion \( \psi \), instanton-monopoles produce a bifermion condensate \( \bar{\psi} \psi \). If we add a number of massless fermions \( N_f > 1 \), the global flavor \( SU(N_f) \) symmetry can be explicitly broken by twisted boundary conditions along the compact dimension. This can extend adiabatic continuity to include Yang–Mills with massless quarks.

Renormalons are absent provided \( r(S_1) \) is sufficiently small. Nonperturbative effects come from fractons, vacuum expectation values of various local operators, UV renormalons, and (at finite \( N \)) from a factorially large number of the multiloop diagrams. The above effects are conceptually similar to those discussed in Sect. 2.1.

\section{3 Renormalons}

The issue of renormalons was raised by ’t Hooft \(^{[8]} \) (see also \(^{11} \) \([37–39]\), detailed reviews can be found in \([40, 41]\) and \([13,14]\)). My point of view is that in Yang–Mills theory this problem is misformulated by many.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig2.png}
\caption{The bubble-chain diagrams for the Adler function \( D \) representing renormalons. Wavy lines stand for an external “photon”. Solid lines denote quark propagators, while dashed lines are for gluons. The quark bubbles are also to be added to the gluon bubbles.}
\end{figure}

To explain it I will first outline the original idea and then try to present it from the modern standpoint on confining dynamics.

The authors of \([8,37]\) observed that a special class of the so-called bubble diagrams lead to a factorial divergence of the coefficients of \( (g^2)^k \) at large \( k \). Then the Borel summation of the above formal series leads to ambiguities.

After one integrates over the loop momentum of the “large” fermion loop and the angles of the gluon momentum \( k \) in Fig. 2 one arrives at

\[
D \propto \int \frac{dk^2}{k^2} F(k^2) \alpha_s(k^2) \tag{3.1}
\]

where the function \( F(k^2) \) calculated in \([42]\) has the limits

\[
F(k^2) \to \begin{cases} c_1 \left( k^4/Q^4 \right), & k^2 \ll Q^2, \\ c_2 Q^2 \left( k^2 \right)^{-1} \log k^2, & k^2 \gg Q^2, \end{cases} \tag{3.2}
\]

and to the leading order

\[
\alpha(k^2) = \alpha(Q^2) \left[ 1 - \frac{\beta_1 \alpha(Q^2)}{4\pi} \log(Q^2/k^2) \right]^{-1}. \tag{3.3}
\]

Here \( \beta_1 \) is the first coefficient of the \( \beta \) function, \( \beta_1 = \frac{11}{3} N - \frac{2}{3} N_f \). The upper line in (3.2) is relevant to the IR renormalon, while the lower line to the UV renormalon which will be briefly discussed later.

The confining regime at large distances in Yang–Mills theory at strong coupling implies that Eq. (3.11) below is totally inapplicable at \( k^2 \lesssim \Lambda^2 \) since \( \alpha(k^2) \) is not defined.

The ’t Hooft renormalon analysis is a formal procedure completely neglecting the above circumstance. Indeed, people formally expand Eq. (3.11) in \( \alpha(Q^2) \) and then integrate over \( k^2 \) from zero (I emphasize, zero) to \( Q^2 \).

\[
D(Q^2) \propto \frac{1}{Q^2} \sum_{n=0}^{\infty} \left( \frac{\beta_1 \alpha}{4\pi} \right)^n \int_0^{Q^2} dk^2 k^2 \left( \ln \frac{Q^2}{k^2} \right)^n, \\
\alpha \equiv \alpha(Q^2), \tag{3.4}
\]

\footnote{In fact, Lautrup discovered what would now be called UV renormalons appering in QED due to the Landau pole, in his analysis of \( g - 2 \).}
which can be identically rewritten as

\[ D(Q^2) \propto \frac{1}{2} \sum_{n=0}^{\infty} \left( \frac{\beta_0 \alpha}{8\pi} \right)^n \int_0^{\infty} dy \, y^n \, e^{-y}, \]

\[ y = 2 \ln \frac{Q^2}{k^2}, \]  

(3.5)

Note that \( k^2 \sim 0 \) corresponds to \( y \sim \infty \). Formally, the integral in (3.5) is \( n! \) which makes the sum in (3.4) divergent. It is quite obvious, however, that the integral in (3.4) cannot run from \( k^2 = 0 \) just because Eq. (3.11) is defined only provided

\[ \frac{\beta_1 \alpha(Q^2)}{4\pi} \log(Q^2/k^2) < 1 \]  

(3.6)

which in turn requires

\[ k^2 > \mu^2, \quad \mu^2 = cA^2, \quad c \gg 1. \]  

(3.7)

The IR cut off of this type is implemented in \textit{bona fide} QCD OPE, see e.g. the review papers [13,14]. Then the factorial growth of the coefficients is cut off at a critical value

\[ n_* = 2 \ln \frac{Q^2}{\mu^2}, \]  

(3.8)

and the series in \( \alpha(Q^2) \) in the OPE coefficients is well-defined. (See Fig. 9 in the second paper in Refs. [13,14].) Alternatively, one can say that the integral in (3.1) is well defined provided we set the lower limit of integration at \( \mu^2 \gg A^2 \). Moreover, if we use the exact expression for the function \( F(k^2) \) we can integrate over \( k^2 \in [\mu^2, \infty] \), the integral will be convergent and well defined. In addition, it will include the UV renormalon in its entirety. However, if we try to expand it in the series in the coupling constant two problems arise: (i) it is not clear what is exactly the coupling constant we expand in; and (ii) the series will not be convergent, with factorially growing coefficients due to UV renormalons, see below.

In weakly coupled Yang-Mills theories an auxiliary IR cut off \( \mu^2 \) is not needed. A natural cut off is provided by either \( \nu^2 \) or \( \lfloor r(S_1) \rfloor^{-2} \).

### 3.1 A few words about UV renormalons

In the previous section, I emphasized that the IR renormalon is a formal issue provided the OPE is properly built, including the delimitating parameter \( \mu^2 \gg A^2 \). At weak coupling no IR renormalons exist to begin with.

UV renormalons do exist at weak coupling since they emerge due to the fact of the slow approach of the gauge coupling \( \alpha(k^2) \) to zero as \( k^2 \to \infty \). In asymptotically free theories we do not expect any unknown phenomena in the extreme UV. The coupling constant (3.11) is well defined at \( k^2 \to \infty \).

Usually (see e.g. [41]) when discussing UV renormalons people start from (3.1) and the second line in (3.2) and arrive at

\[ D(Q^2) \propto Q^2 \sum_{n=0}^{\infty} \left( \frac{\beta_0 \alpha}{4\pi} \right)^n \int_0^{\infty} dk^2 \frac{1}{k^4} (-1)^n (\ln k^2/Q^2)^{n+1}. \]  

(3.9)

The integral in (3.9) is indeed factorial, but sign alternating. It is Borel-summable. However, the whole procedure is totally unreasonable. It is like going from A to B for no reason and then returning back to A. Indeed, instead of expansion (3.9) we should have stared directly from

\[ D(Q^2) \propto Q^2 \int_{Q^2}^{\infty} dk^2 \frac{1}{k^4} \alpha(k^2) \left( \ln k^2/Q^2 \right) \]  

(3.10)

where

\[ \alpha(k^2) = \alpha(Q^2) \left[ 1 + \frac{\beta_1 \alpha(Q^2)}{4\pi} \log(k^2/Q^2) \right]^{-1}. \]  

(3.11)

The integrand in (3.10) is nonsingular, the integral is convergent but not expandable in the series in the gauge coupling.

### 4 Yang–Mills in Banks–Zaks limit at large \( N \)

It is instructive to discuss the Banks–Zaks limit [43] at large \( N \).\footnote{This section resulted from a discussion with M. Ünsal.}

At large \( N \) only the planar graphs survive and their number does \textit{not} grow factorially in high orders [7]. Moreover, instantons disappear in the 't Hooft limit. There are no renormalons either.

Indeed, as was noted by Banks and Zaks, at \( N_f \ll \frac{11}{2} N \) we are close to the right edge of the conformal window, see Fig. 3. Let us have a closer look at the beta function \( \beta(a) \),

\[ \beta(a) = - \beta_1 a^2 - \beta_2 a^3 + \cdots, \quad a = \frac{N\alpha}{2\pi}, \]

\[ \beta_1 = \frac{11}{3} \left( 1 - 2 \frac{N_f}{11N} \right), \]

\[ \beta_2 = \frac{17}{3} - \frac{N_f}{6N} \left( 13 - \frac{3}{2\pi} \right). \]  

(4.1)

If we choose

\[ N_f = \frac{11}{2} N - \nu, \quad 0 < \nu \ll \frac{11}{2} N \]  

(4.2)
then the first coefficient, $\beta_1$, is anomalously small,

$$\beta_1 = \frac{2}{3} \frac{\nu}{N},$$

(4.3)

while the second coefficient is of a normal order of magnitude,

$$\beta_2 \approx -\frac{25}{4}$$

(4.4)

and is negative. This results in an infrared fixed point at a small value of $a$,

$$a = a_* = \frac{8}{75} \frac{\nu}{N}.$$ \hspace{1cm} (4.5)

Since $a_*$ is small, both its value and the very existence are reliable. Therefore, if in the UV we start from $a_0 > a_*$ then in the IR the maximal value of the coupling constant is $a_*$, and we flow to the conformal theory with small anomalous dimensions. The corresponding RG flow is depicted in Fig. 4.

While the approach to the UV fixed point is logarithmic and is determined by the dynamical scale $\Lambda$, the approach to the IR conformal point is power-like,

$$a(\mu) \to a_* - \left( \frac{\mu}{\Lambda} \right)^\gamma, \quad \mu \to 0.$$ \hspace{1cm} (4.6)

The dimensional parameter $\tilde{\Lambda}$ is determined also by the value of $a_0 = a(M_{uv})$. Typically, $\tilde{\Lambda} \gg \Lambda$. The power $\gamma$ in Eq. (4.6) is positive, its numerical value depends on parameters. In the regime discussed above the approach to $a_*$ is from below, see Fig. 3.

The fact that $a(\mu) < a_* \ll 1$ explains the absence of renormalons. Seemingly, there is no source for the factorial divergence of the perturbation theory coefficients at large order. Nevertheless, two distinct dimensional parameters appear in this theory through dimensional transmutation. If we started from the initial condition $a_0 = a(M_{uv}) > a_*$ the corresponding regime would dramatically change, from conformal to Landau, $a(\mu)$ would approach $a_*$ from above in the IR, asymptotic freedom would be lost and we acquire the Landau growth in the UV.

A remark is in order regarding possible compactification from $R^4$ to $R^3 \times S^1$ with $r(S_1) \ll \Lambda^{-1}$. This does not do any good in the discussion of the Banks–Zaks limit. Indeed, the RG flow is interrupted and frozen at the scale $r(S_1)$ and we never approach the conformal limit unless we tend $r(S_1) \to \infty$. In this aspect one can compare the situation with that in fully Higgsed Yang–Mills theory.

Another question to ask is about the emergence of $\tilde{\Lambda}$ as we approach the conformal regime. Perturbation theory seems well-defined in IR and yet a dimensional parameter shows up. In the conformal regime with anomalous dimensions it should appear in the coefficients in the laws of power fall-off of the correlation functions. A related question is what happens to OPE in this regime. In strongly coupled Yang–Mills it is defined through separation of short and large distances [13,14] which is not applicable near the conformal point. On the other hand, the OPE is well formulated in the conformal regime per se, and even simpler than at strong coupling.

Needless to say, UV renormalons are present in the theory at hand. They generate factorials in the perturbation series, which is Borel summable, i.e. well defined. The remnant of the Borel summation can be represented by the dynamical scale $\Lambda$.

5 Smooth passage from Euclidean to Minkowski in the conformal window

In the Banks–Zaks regime with $a_* \ll 1$ (see (4.1), (4.5)) the theory is conformal, with small anomalous dimensions. There is no confinement and no mass gap. Under these circumstances, the quark-gluon description of the theory should be applicable not only in the Euclidean space but also in Minkowski, with a smooth transition between them. This should be true for any $N$. There
are no new thresholds opening as we move in energy in "experiment."

This issue was discussed in detail in Refs. [44, 45] in the framework of $\mathcal{N} = 1$ supersymmetric QCD (SQCD). An exact formula was derived for the photon-induced Adler $D$ function,

$$ D(Q^2) = \frac{3}{2} N \sum_f q_f^2 \left[ 1 - \gamma \left( \alpha_s(Q^2) \right) \right] , \quad (5.1) $$

where $f$ is the flavor index, and $q_f$ is the corresponding electric charge (in the units of $e$). Equation (5.1) assumes that all matter fields are in the fundamental representation of SU($N$), although their electric charges can be different. Above, $\gamma (\alpha_s(Q^2))$ is the anomalous dimension of the matter superfield. It is known only in the form of expansion in $\alpha_s(Q^2)$.

In this section, we will study some consequences ensuing [44, 45] from the exact formula (5.1) in the so-called conformal window (more exactly, close to the right edge of the window, where the theory is weakly coupled).

$$ \frac{3}{2} N < N_f < 3N . \quad (5.2) $$

Inside this window SQCD flows to the conformal points: $\gamma = 0$ in the ultraviolet (asymptotic freedom) and to

$$ \gamma_s = -\frac{3N - N_f}{N_f} \quad (5.3) $$

in the infrared. Given $N$ and $N_f$ the maximal value of $\gamma$ is $\gamma_s$. The equation (5.1) then implies that the Adler function

$$ D(Q^2) \to \frac{3}{2} N \sum_f q_f^2 \times \left\{ \frac{1}{3N} , \quad Q^2 \to \infty , \quad Q^2 \to 0 . \right. \quad (5.4) $$

The $Q^2$ evolution of the Adler function in the conformal window is sketched in Fig 5.

Of course, we do not know the analytic value of $\gamma (\alpha_s(Q^2))$ except in two limiting points (5.4). However, we can expect (at least in the 't Hooft limit of large $N$) that the only source of non-perturbative corrections is the UV renormalon which smoothly continues from Euclidean to Minkowski so that the predicted curve for

$$ R(s) = \frac{\sigma(e^+ e^- \to (s)\text{quarks, gluons}(\text{inos}) \to \text{hadrons})}{\sigma(e^+ e^- \to \mu^+ \mu^-)} \quad (5.5) $$

obtained from the imaginary part of $D$ is smooth.

### 6 Exactly solvable models

In special cases of exactly solvable asymptotically free models, the analysis of OPE can be further extended.\(^{13}\) Among this class are two-dimensional sigma models, for instance, the $O(N)$ and $CP(N-1)$ models in the limit of large $N$. Typically we can solve them in the leading and next-to leading order in $1/N$. With the exact solutions in hand we can first check the $\mu$ independence of OPE [48] and after this is done answer the question what happens in the limit $\mu \to 0$.

The question was raised long ago [46, 47] and recently discussed more than once by Ünsal et al. in various models. Here we will briefly review the $O(N)$ model in which we will discuss the limit of $\mu \to 0$. The most challenging case of the supersymmetric $O(N)$ sigma model is presented in [50].

Since one knows the exact answer for all correlation functions, one can define the “coupling constant” for perturbation theory,

$$ g(p^2) \overset{\text{def}}{=} \frac{4\pi}{N \log \frac{p^2}{\Lambda^2}} , $$

$$ \Lambda^2 \equiv m^2 = M_0^2 \exp \left( -\frac{4\pi}{N \gamma_0} \right) . \quad (6.1) $$

At $p^2 \gg \Lambda^2$ the definition in (6.1) coincides with the standard perturbative one.

Dependence on $p/m$ in the exact solution appears in a two-fold way. The exact formula contains logarithms of the type $\log p^2/m^2$ and powers of $m^2/p^2$. Of course, in mathematical sense $m^2/p^2$ is just the exponent of $\log p^2/m^2$. However, it is instructive to keep a double expansion

$$ \langle n^a(-p) n^a(p) \rangle = \sum_{j,\ell} C_{j\ell} \left( \frac{1}{\log p^2/m^2} \right)^j \left( \frac{m^2}{p^2} \right)^{d_{h^2}} \quad (6.2) $$

\(^{13}\) This section is based on discussions with Daniel Schubring and Chao-Hsiang Sheu. Detailed results based on these discussions are published in a separate paper [50].
where the first factor represents coefficients while the second matrix element in the limit $\mu \to 0$.

Now we can make use of the knowledge of the exact Green’s functions in this model [48–50]. For instance, for the two-point function of the $n$ fields we have

$$\langle n^a(-p), n^a(p) \rangle = \frac{N}{p^2 + m^2} \left[ \frac{8m^2\pi}{(p^2 + m^2)^2} \int \frac{d^2k}{4\pi^2} \frac{1}{k^2 + 4m^2} \right.$$

$$\left. - \frac{2}{p^2 + m^2} \int \frac{d^2k}{4\pi^2} \frac{1}{(k^2 + 4m^2)} J(k^2) \left[ (k^2 + p^2 + m^2) \right] \right.$$  

$$= \frac{1}{p^2 + m^2} \left[ N + \frac{2m^2}{p^2 + m^2} \log \frac{M^2 + 4m^2}{4m^2} \right.$$

$$\left. - \frac{1}{2} \gamma + \log \frac{Ng(p)}{4\pi} \right]$$

$$- 2 \sum_{j=1}^{\infty} (2j)! \zeta(2j + 1) \left( \frac{Ng(p)}{4\pi} \right)^{2j+1} \right] \quad (6.3)$$

where

$$J(k^2) = \frac{1}{2\pi \sqrt{k^2 + 4m^2}} \log \left( \frac{\sqrt{k^2 + 4m^2} + \sqrt{k^2}}{\sqrt{k^2 + 4m^2} - \sqrt{k^2}} \right) \quad (6.4)$$

and after integration in the second and third lines we performed a (formal) expansion in $1/\log \frac{p^2}{4\pi}$, see Eq. (6.1). Moreover, $\zeta(s)$ in (6.3) is the $\zeta$ Riemann function and $\gamma$ is the Euler’s constant. The expansion over $1/\log \frac{p^2}{4\pi}$ can be viewed as an exact contribution to the unit operator.

Now, the result (6.3) can be also expanded in $m^2/p^2$. The latter expansion can be viewed as coming from higher dimension operators. Let us limit ourselves by the identity and dimension-2 operators in OPE. The latter is $(\partial_\alpha n^a)^2$. In the leading order in $1/N$

$$\langle (\partial_\alpha n^a)(\partial_\alpha n^a) \rangle = -\frac{m^2}{g} + O(1/N) \quad (6.5)$$

We will consider the subleading in $1/N$ order in this matrix element below. The OPE takes the form

$$\langle n^a(-p), n^a(p) \rangle = \frac{N}{p^2} C_1 I + C_{(\partial_\alpha n^a)} \langle (\partial_\alpha n^a)^2 \rangle \quad (6.6)$$

where

$$C_1(p^2) = 1 - \frac{1}{N} \left( ... + 2 \sum_{j=1}^{\infty} (2j)! \zeta(2j + 1) \left( \frac{Ng(p)}{4\pi} \right)^{2j+1} \right) \quad (6.7)$$

$$C_{(\partial_\alpha n^a)} = \frac{1}{p^4} + O(1/N) \quad (6.8)$$

Both $C_1$ and $\langle (\partial_\alpha n^a)^2 \rangle$ have formal (purely imaginary) ambiguities associated with the Borel summation of the series at the order $O(1/N)$. The coefficient $C_1$ formally has an ambiguity

$$\delta C_1 = \mp i \pi \frac{m^2}{p^2} \cdot \frac{1}{N} \quad (6.9)$$

The matrix element of the operator $(\partial_\alpha n^a)(\partial_\alpha n^a)$ also has a purely imaginary ambiguity,

$$\langle (\partial_\alpha n^a)^2 \rangle = \pm i \pi m^2 \quad (6.10)$$

If we assemble them together we observe perfect cancellation (to the given order in $1/N$). The emergence of purely imaginary ambiguities is the price we have to pay for our unwillingness to explicitly introduce the sliding (delimitating) parameter $\mu^2 \gg \Lambda^2$, where $\Lambda^2$ is defined in (6.1). Were $\mu^2$ introduced we would have no ambiguities, but will have to trace cancellation of $\mu$ in OPE. Both cancellations are manifestation of the conspiracy inherent to OPE.

I started Sect. 6 with the statement: “With the exact solutions in hand we can first check the $\mu$ independence of OPE [48] and after this is done answer the question what happens in the limit $\mu \to 0$.” Unfortunately, today we still cannot answer the last part of this question in QCD and similar theories which operate in the strong coupling regime.

### 7 Conclusions

Various sources of nonperturbative effects in Yang–Mills theories at weak coupling are revisited. Although perturbative expansions are widely used in asymptotically free Yang–Mills theories it should be realized that their accuracy is limited even at weak coupling.

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