ON THE FLAG $f$-VECTOR OF A GRADED LATTICE WITH NONTRIVIAL HOMOLOGY

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Abstract. It is proved that the Boolean algebra of rank $n$ minimizes the flag $f$-vector among all graded lattices of rank $n$, whose proper part has nontrivial top-dimensional homology. The analogous statement for the flag $h$-vector is conjectured in the Cohen-Macaulay case.

1. Introduction

Let $P$ be a finite graded poset of rank $n \geq 1$, having a minimum element $\hat{0}$, maximum element $\hat{1}$ and rank function $\rho : P \to \mathbb{N}$ (we refer to [12, Chapter 3] for any undefined terminology on partially ordered sets). Given $S \subseteq [n-1] := \{1, 2, \ldots, n-1\}$, the number of chains $C \subseteq P \setminus \{\hat{0}, \hat{1}\}$ such that $\{\rho(x) : x \in C\} = S$ will be denoted by $f_P(S)$. For instance, $f_P(S)$ is equal to the number of elements of $P$ of rank $k$, if $S = \{k\} \subseteq [n-1]$, and to the number of maximal chains of $P$, if $S = [n-1]$. The function which maps $S$ to $f_P(S)$ for every $S \subseteq [n-1]$ is an important enumerative invariant of $P$, known as the flag $f$-vector; see, for instance, [4].

The present note is partly motivated by the results of [2, 6]. There it is proven that the Boolean algebra of rank $n$ minimizes the cd-index, an invariant which refines the flag $f$-vector, among all face lattices of convex polytopes and, more generally, Gorenstein* lattices, of rank $n$. It is natural to consider lattices which are not necessarily Eulerian, in this context. To state our main result, we fix some more notation as follows. We denote by $\Delta(Q)$ the simplicial complex consisting of all chains in a finite poset $Q$, known as the order complex [5] of $Q$, and by $\tilde{H}_*(\Delta; k)$ the reduced simplicial homology over $k$ of an abstract simplicial complex $\Delta$, where $k$ is a fixed field or $\mathbb{Z}$. We denote by $B_n$ the Boolean algebra of rank $n$ (meaning, the lattice of subsets of the set $[n]$, partially ordered by inclusion) and recall that if $S = \{s_1 < s_2 < \cdots < s_l\} \subseteq [n-1]$, then $f_{B_n}(S)$ is equal to the multinomial coefficient $\alpha_n(S) = \binom{n}{s_1, s_2 - s_1, \ldots, n - s_l}$.

Theorem 1.1. Let $L$ be a finite graded lattice of rank $n$, with minimum element $\hat{0}$ and maximum element $\hat{1}$, and let $\bar{L} = L \setminus \{\hat{0}, \hat{1}\}$ be the proper part of $L$. If $\tilde{H}_{n-2}(\Delta(\bar{L}); k) \neq 0$, then

\begin{equation}
(1.1) \quad f_L(S) \geq \alpha_n(S)
\end{equation}

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for every $S \subseteq [n - 1]$. In other words, the Boolean algebra of rank $n$ minimizes the flag $f$-vector among all finite graded lattices of rank $n$ whose proper part has nontrivial top-dimensional reduced homology over $k$.

A similar statement, asserting that the Boolean algebra of rank $n$ has the smallest number of elements among all finite lattices $L$ satisfying $\tilde{H}_{n-2}(\Delta(L); \mathbb{Z}) \neq 0$, was proved by Meshulam [10]. The proof of Theorem 1.1 given in Section 2 is elementary and similar in spirit to (but somewhat more involved than) the proof of the result of [10]. A different (but less elementary) proof may be given using the methods of [6, Section 2]. In the remainder of this section we discuss some consequences of Theorem 1.1 and a related open problem.

The $f$-vector of a simplicial complex $\Delta$ is defined as the sequence $f(\Delta) = (f_0, f_1, \ldots)$, where $f_i$ is the number of $i$-dimensional faces of $\Delta$. We recall that the order complex $\Delta(\bar{\mathcal{L}}_n)$ is isomorphic to the barycentric subdivision of the $(n-1)$-dimensional simplex. The next statement follows from this observation, Theorem 1.1 and the fact (see, for instance, [13, p. 95]) that each entry of the $f$-vector of the order complex $\Delta(\bar{\mathcal{L}}_n)$ can be expressed as a sum of entries of the flag $f$-vector of $\mathcal{L}$.

Corollary 1.2. The barycentric subdivision of the $(n - 1)$-dimensional simplex has the smallest possible $f$-vector among all order complexes of the form $\Delta(\bar{\mathcal{L}})$, where $\mathcal{L}$ is a finite graded lattice of rank $n$ satisfying $\tilde{H}_{n-2}(\Delta(\bar{\mathcal{L}}); \mathbb{k}) \neq 0$.

Analogous results for the class of flag simplicial complexes have appeared in [1, 7, 9, 11].

Let $P$ be a graded poset of rank $n$, as in the beginning of this section. The flag $h$-vector of $P$ is the function assigning to each $S \subseteq [n - 1]$ the integer

$$h_P(S) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} f_P(T).$$

Equivalently, we have

$$f_P(S) = \sum_{T \subseteq S} h_P(T)$$

for every $S \subseteq [n - 1]$. We write $\beta_n(S)$ for the entry $h_{B_n}(S)$ of the flag $h$-vector of the Boolean algebra of rank $n$ and recall [12, Corollary 3.12.2] that $\beta_n(S)$ is equal to the number of permutations of $[n]$ with descent set $S$.

It is known that if $P$ is Cohen-Macaulay over $\mathbb{k}$ (see [5, Section 11] or [12, Section 3.8] for the definition), then $h_P(S) \geq 0$ for every $S \subseteq [n - 1]$. Moreover, in this case $\Delta(\bar{\mathcal{L}})$ has nontrivial top-dimensional reduced homology over $\mathbb{k}$ if and only if $\mu_P(\hat{0}, \hat{1}) \neq 0$, where $\mu_P$ is the Möbius function of $P$. Hence, Theorem 1.1 implies that the Boolean algebra of rank $n$ minimizes the flag $f$-vector among all Cohen-Macaulay lattices of rank $n$ with nonzero Möbius number. In view of (1.3), the following conjecture provides a natural strengthening of this statement.
Conjecture 1.3. Let $L$ be a finite lattice of rank $n$, with minimum element $\hat{0}$ and maximum element $\hat{1}$. If $L$ is Cohen-Macaulay over $k$ and $\mu_L(\hat{0}, \hat{1}) \neq 0$, then

\begin{equation}
    h_L(S) \geq \beta_n(S)
\end{equation}

for every $S \subseteq [n - 1]$. In other words, the Boolean algebra of rank $n$ minimizes the flag $h$-vector among all Cohen-Macaulay lattices of rank $n$ with nonzero Möbius number.

This conjecture was initially stated by the author under the assumption that $\mu_L(x, y) \neq 0$ holds for all $x, y \in L$ with $x \leq_L y$ and took its present form after a question raised by R. Stanley \cite{14}, asking whether this condition could be relaxed to $\mu_L(\hat{0}, \hat{1}) \neq 0$. It would imply that among all Cohen-Macaulay order complexes of the form $\Delta(L)$, where $L$ is a lattice of rank $n$ satisfying $\mu_L(\hat{0}, \hat{1}) \neq 0$, the barycentric subdivision of the $(n - 1)$-dimensional simplex has the smallest possible $h$-vector (the entries of the $h$-vector of this subdivision are the Eulerian numbers, counting permutations of the set $[n]$ by the number of descents). Conjecture 1.3 is known to hold for Gorenstein* lattices (in this case it follows from the stronger result \cite{6} Corollary 1.3, mentioned earlier, on the cd-index of such a lattice) and for geometric lattices \cite{3} Proposition 7.4.

2. Proof of Theorem 1.1

Throughout this section, $L$ is a lattice as in Theorem 1.1. For $a, b \in L$ with $a \leq_L b$, we denote by $\Delta(a, b)$ (respectively, by $\Delta(a, b]$) the order complex of the open interval $(a, b)$ (respectively, half-open interval $(a, b]$) in $L$. We say that an element $x \in L$ is good if $x = \hat{0}$ or $\tilde{H}_{k-2}(\Delta(\hat{0}, x); k) \neq 0$, where $k$ is the rank of $x$ in $L$, and otherwise that $x$ is bad.

The proof of Theorem 1.1 will follow from the next proposition.

Proposition 2.1. Under the assumptions of Theorem 1.1, the lattice $L$ has at least $\binom{n}{k}$ good elements of rank $k$ for every $k \in \{0, 1, \ldots, n\}$.

Proof. We proceed in several steps.

Step 1: We show that $L$ has at least one good coatom. Suppose, by the way of contradiction, that no such coatom exists. Suppose further that $L$ has the minimum possible number of coatoms among all lattices of rank $n$ which satisfy the assumptions of Theorem 1.1 and have no good coatom. Since $\Delta(L)$ is non-acyclic over $k$, the order complex $\Delta(L)$ cannot be a cone and hence $L$ must have at least two coatoms. Let $c$ be one of them and consider the complexes $\Delta(L \setminus \{c\})$ and $\Delta(\hat{0}, c]$. The union of these complexes is equal to $\Delta(L)$ and their intersection is equal to $\Delta(\hat{0}, c)$. Since $\Delta(\hat{0}, c]$ is a cone, hence contractible, and since $\tilde{H}_{n-3}(\Delta(\hat{0}, c); k) = 0$ by assumption, it follows from the Mayer-Vietoris long exact sequence in homology for $\Delta(L \setminus \{c\})$ and $\Delta(\hat{0}, c]$ that

\begin{equation}
    \tilde{H}_{n-2}(\Delta(L \setminus \{c\}); k) \cong \tilde{H}_{n-2}(\Delta(L); k) \neq 0.
\end{equation}

Since $L \setminus \{c\}$ may not be graded, we consider the subposet $M = J \cup \{\hat{1}\}$ of $L$, where $J$ stands for the order ideal of $L$ generated by all coatoms other than $c$. The poset $M$ is a finite meet-semilattice with a maximum element and hence it is a lattice by \cite{12} Proposition...
Since $L$ is graded of rank $n$, so is $M$ and the set of $(n-1)$-element chains of $\Delta(M)$ coincides with that of $\Delta(\bar{L} \setminus \{c\})$, where $\bar{M} = M \setminus \{\bar{0}, \bar{1}\}$ is the proper part of $M$. The last statement and (2.1) imply that
\[ \tilde{H}_{n-2}(\Delta(M); \mathbf{k}) \equiv \tilde{H}_{n-2}(\Delta(\bar{L} \setminus \{c\}); \mathbf{k}) \neq 0. \]

Clearly, all coatoms of $M$ are bad. Since $M$ has one coatom less than $L$, we have arrived at the desired contradiction.

**Step 2:** Assume that $n \geq 2$ and let $b$ be any coatom of $L$. We show that there exists an atom $a$ of $L$ which is not comparable to $b$ and satisfies $\tilde{H}_{n-3}(\Delta(a, \bar{1}); \mathbf{k}) \neq 0$. Arguing by contradiction, once again, suppose that no such atom exists. Suppose further that the last statement and (2.1) imply that an atom $a$ of $L$ which does not belong to the interval $[\bar{0}, b]$ is as small as possible for a graded lattice $L$ of rank $n$ and coatom $b$ which have this property and satisfy the assumptions of Theorem 1.1. Since $\Delta(L)$ is non-acyclic over $\mathbf{k}$, the Crosscut Theorem of Rota [3, Theorem 10.8] implies that there exists at least one atom of $L$ which does not belong to the interval $[\bar{0}, b]$. Let $a$ be any such atom and let $M$ be the subposet of $L$ consisting of $\bar{0}$ and the elements of the dual order ideal of $L$ generated by the atoms other than $a$. The arguments in Step 1, applied to the dual of $L$, show that $M$ is a graded lattice of rank $n$ which satisfies $\tilde{H}_{n-2}(\Delta(M); \mathbf{k}) \equiv \tilde{H}_{n-2}(\Delta(\tilde{L}); \mathbf{k}) \neq 0$. Since $M \setminus (\bar{0}, b]$ has one atom less than $L \setminus (\bar{0}, b]$, this contradicts our assumptions on $L$ and $b$.

**Step 3:** We now show that $L$ has at least $n$ good coatoms by induction on $n$. The statement is trivial for $n = 1$, so suppose that $n \geq 2$. By replacing $L \setminus \{\hat{1}\}$ with its order ideal generated by the good coatoms, as in Step 1, we may assume that all coatoms of $L$ are good. Let $b$ be any coatom of $L$. By Step 2, there exists an atom $a$ of $L$ which is not comparable to $b$ and satisfies $\tilde{H}_{n-3}(\Delta(a, \hat{1}); \mathbf{k}) \neq 0$. The interval $[a, \hat{1}]$ in $L$ is a graded lattice of rank $n - 1$ to which the induction hypothesis applies. Therefore, it has at least $n - 1$ coatoms and all of these are different from $b$. It follows that $L$ has at least $n$ coatoms, all of which are good.

**Step 4:** We prove the following: Given any integers $0 \leq r \leq k \leq n$ and any order ideal $I$ of $L \setminus \{\hat{1}\}$ generated by at most $r$ elements, there exist at least \( \binom{n-r}{k-r} \) good elements of $L$ of rank $k$ which do not belong to $I$. The special case $r = 0$ of this statement, in which $I$ is the empty ideal, is equivalent to the proposition. Thus, it suffices to prove the statement.

We proceed by induction on $n$ and $n - r$, in this order. The statement is trivial for $n = 1$ and for $r = n$, so we assume that $n \geq 2$ and $0 \leq r \leq n - 1$. Consider an order ideal $I$ of $L \setminus \{\hat{1}\}$ generated by at most $r$ elements and let $k$ be an integer in the range $r \leq k \leq n$. Since $I$ contains at most $r \leq n - 1$ coatoms of $L$, Step 3 implies that there exists a good coatom, say $b$, of $L$ which does not belong to $I$. The interval $[\bar{0}, b]$ of $L$ is a graded lattice of rank $n - 1$ whose proper part has nontrivial top-dimensional reduced homology over $\mathbf{k}$. Moreover, the intersection $I \cap [\bar{0}, b]$ is an order ideal of $[\bar{0}, b]$ which is generated by at most $r$ elements, namely the meets of $b$ with the maximal elements of $I$. By our induction on $n$, there exist at least \( \binom{n-r-1}{k-r-1} \) good elements of $[\bar{0}, b]$ of rank $k$ which do not belong to $I$. The union $J = I \cup [\bar{0}, b]$ is an order ideal of $L \setminus \{\hat{1}\}$ which is generated
by at most $r + 1$ elements. By our induction on $n - r$, there exist at least \( \binom{n-r-1}{k-r-1} \) good elements of $L$ of rank $k$ which do not belong to $J$. We conclude that there exist at least \( \left( \binom{n-r-1}{k-r-1} + \binom{n-r-1}{k-r-1} \right) = \binom{n-r}{k-r} \) good elements of $L$ of rank $k$ which do not belong to $I$. This completes the inductive step and the proof of the statement. \(\square\)

\textbf{Proof of Theorem 1.1.} We proceed by induction on $n$. The result is trivial for $n = 1$ and for $S = \emptyset$, so we assume that $n \geq 2$ and choose a nonempty subset $S$ of $[n-1]$. We denote by $k$ the largest element of $S$ and observe that $f_{\overline{L}}(S)$ is equal to the number of pairs $(x, C)$, where $x$ is an element of $L$ of rank $k$ and $C$ is a chain in the interval $[\hat{0}, x]$, such that the set of ranks of the elements of $C$ is equal to $S \setminus \{k\}$. By Proposition 2.1 there are at least \( \binom{n}{k} \) good elements $x$ of rank $k$ in $L$ and each of the intervals $[\hat{0}, x]$ is a graded lattice of rank $k$ whose proper part has nontrivial top-dimensional reduced homology over $k$. Thus, the induction hypothesis applies to these intervals and we may conclude that

$$f_{\overline{L}}(S) \geq \binom{n}{k} \alpha_k(S \setminus \{k\}) = \alpha_n(S).$$

This completes the induction and the proof of the theorem. \(\square\)

We end with a note on the case of equality in (1.1). It was shown in [10] that every lattice $L$ which satisfies $\tilde{H}_{n-2}(\Delta(\bar{L}); \mathbb{Z}) \neq 0$ and has cardinality $2^n$ must be isomorphic to the Boolean algebra $B_n$. As a result, if equality holds in (1.1) for every singleton $S \subseteq [n-1]$, then $L$ is isomorphic to $B_n$. Using the arguments in this section, as well as induction on $n$ and $k$, the following statement has been verified by Kolins and Klee [8]: if $L$ satisfies the assumptions of Theorem 1.1 and for some $k \in \{1, 2, \ldots, n-1\}$ equality holds in (1.1) for every subset $S$ of $[n-1]$ of cardinality $k$, then $L$ is isomorphic to the Boolean algebra of rank $n$.

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