On the existence of finite-energy lumps in classic field theories

Roman V. Buniy and Thomas W. Kephart
Vanderbilt University, Nashville, TN 37235
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We show how the existence of non-trivial finite-energy time-dependent classical lumps is restricted by a generalized virial theorem. For simple model Lagrangians, bounds on energies follow.

INTRODUCTION

Obtaining exact solutions to most realistic field theories is a formidable task. However, limited information about such solutions is often available without solving the corresponding equations. One such general result is Derrick’s theorem [1], which precludes non-trivial static scalar field configurations in more than two space dimensions. Some theories have another general feature of their solutions being classified by their topological charges.

Several non-existence theorems have been proved too. For example, pure Yang-Mills fields do not hold themselves together to give finite-energy solutions that are either time-independent [2, 3] or periodic in time [4]. An even stronger result was proved: the only finite-energy non-singular non-radiating solutions with arbitrary time dependence are vacuum solutions [5]. This “no-go” theorem forbids the existence of classical glueballs in a pure Yang-Mills system. To support localized solutions, other fields have to be added.

Our particular interest [6] is in glueballs in QCD, where the $SU_C(3)$ gauge fields couple to quarks and confinement is involved. This work is a zeroth order analysis towards understanding this complete physical situation.

In this note we investigate conditions which are imposed on the energy-momentum tensor by the existence of such classical solutions (solitons or lumps). While these solitons are expected to be highly complicated objects, we do not address their existence nor attempt to find their explicit forms, but merely find the resulting restrictions on fields. This information can be used, for example, to find the lower bound for the energy of lumps as we demonstrate for two physically relevant systems: scalar fields and scalar fields coupled to gauge fields. Our theorem is a general result independent of a particular model, and subject to only mild requirements imposed on the fields.

THEOREM

Let us consider a classical field theory in $(n+1)$-dimensional space-time [9] characterized by the energy-momentum tensor $\theta^{\mu\nu}$ and define a quantity

$$G^{\mu i}(t, R) = \int_{r \leq R} d^nx x^i \theta^{0\mu},$$

where $\mu = 0, \ldots , n$; $i = 1, \ldots , n$ and $r = |x|$. Using conservation of the energy-momentum $\partial_\mu \theta^{\mu\nu} = 0$, we find

$$\partial_0 G^{\mu i}(t, R) = \int_{r \leq R} d^n x \theta^{i\mu} - \int_{r = R} d^{n-1} S_j x^i \theta^{j\mu}. \tag{2}$$

We specify the following asymptotic condition for the energy-momentum tensor,

$$\lim_{r \to \infty} r^{n+\delta} \theta^{\mu\nu}(t, x) = 0. \tag{3}$$

For the energy of the system to be finite we need $\delta \geq 0$; this also ensures that the surface term in Eq. (2) vanishes for large $R$. However, for the quantity $G^{\mu i}(t, \infty)$ to be finite we need $\delta \geq 1$. If the function $G^{\mu i}(t, R)$ is bounded, its derivative’s average value over a time interval $T$,

$$\langle \partial_0 G^{\mu i} \rangle_T = \frac{1}{T} \int_0^T dt \partial_0 G^{\mu i}(t, R) = \frac{1}{T} [G^{\mu i}(T, R) - G^{\mu i}(0, R)], \tag{4}$$

tends to zero as $T \to \infty$. Averaging Eq. (1) over infinite time interval and taking the limit $R \to \infty$, we find

$$\int d^n x \langle \theta^{\mu i} \rangle = 0. \tag{5}$$
Unfortunately, the above argument fails if $G^\mu(t, R)$ is unbounded. This is a case for many interesting physical situations, where the energy-momentum goes like $r^{−n−1}$ for large $r$. For example, the above argument cannot be used for time-dependent fields that asymptotically go like fields of monopoles or dyons in 4D, since under the assumption $\langle K^\mu \rangle$, $G^\mu(t, R)$ diverges as $R^{1−\delta}$ for large $R$. In what follows we will show how the result can still be proved in such situations.

For now we restrict our attention to lumps at rest; we can choose a frame such that this is the case for any lump moving slower than the speed of light. For localized and non-radiating lumps at rest, the energy-momentum must have an asymptotic behavior with $0 < \delta < 1$, which is uniform in time [10].

As a result of these assumptions, for a sphere of large radius $R$, the surface term vanishes and the right-hand side of Eq. (2) goes uniformly in time to

$$K^\mu(t) = \int d^n x \theta^{\mu}. \quad (6)$$

This means that for any positive $\epsilon$ there exists a time-independent $R_0$ such that for any $R > R_0$ we have $|\partial_0 G^\mu(t, R) − K^\mu(t)| < \epsilon$ for all $t \geq 0$. For such $R$,

$$|G^\mu(t, R) − G^\mu(0, R) − t\langle K^\mu \rangle_t| < \epsilon t. \quad (7)$$

On the other hand, using the relation $|P| \leq E$ between the momentum and energy of the system, we deduce that $G^\mu(t, R)$ is bounded,

$$|G^\mu(t, R)| \leq R \max_{t\geq 0} E(t, R) = \Delta(R); \quad (8)$$

$E(t, R)$ being the energy inside the sphere of radius $R$.

Without a detailed form for the function $K^\mu(t)$ we cannot conclude whether the bounds imposed on the function $G^\mu(t, R)$ are consistent or inconsistent. There is, however, a simple case where inconsistency is obvious. Let the average $\langle K^\mu \rangle$ be not zero. Since $\epsilon$ is arbitrary, we can choose $\epsilon < |\langle K^\mu \rangle|$. Then bounds (7) and (8) cannot possibly be consistent for all $t \geq 0$ (see Figure 1). It follows that the only resolution is to set $\langle K^\mu \rangle = 0$; this gives Eq. (5).

For pure Yang-Mills theory in 4D, $\langle K^i \rangle = 0$ leads to $E = 0$, which is Coleman’s conclusion in Ref. [5]. One can imagine several cases where $\langle K^\mu \rangle = 0$ is fulfilled: (1) $K^\mu(t)$ tends to zero as $t$ goes to infinity; (2) for large $t$, the function $K^\mu(t)$ approaches a periodic function that oscillates around zero.

We conclude this section with a remark on a system described by the Lagrangian density $L(\phi, \partial_\mu \phi)$. The dilatation transformation $\delta \phi = x^\mu \partial_\mu \phi$ of such density is

$$\delta L = \partial_\mu (x^\mu L) + \theta^{\mu \nu}. \quad (9)$$

From Eq. (9), it follows that the time average of the Lagrangian is the system’s energy, $\langle \delta L \rangle = E$.

**EXAMPLES**

We apply our result to two theories for which lumps exist (for a review see e.g. [8]).

![Diagram](image-url) FIG. 1: For $t > \tau$ the bound is valid and is not.
I.— For a scalar field theory with the Lagrangian density

$$L = \frac{1}{2} \partial_{\mu} \phi_{a} \partial^{\mu} \phi_{a} - U(\phi_{a}),$$

(10)

the requirement $\int d^{n}x \langle \theta_{i} \rangle = 0$ gives the following expression for the energy,

$$E = \int d^{n}x \langle 2U(\phi_{a}) + (1 - \frac{1}{n})(\partial_{i} \phi_{a})^{2} \rangle. $$

(11)

We thus have a bound

$$E \geq 2 \int d^{n}x \langle U(\phi_{a}) \rangle.$$  

(12)

For one space dimension the energy density becomes $\langle 2U(\phi_{a}) \rangle$. This agrees with a well-known result for a static scalar field with one component.

For a single scalar field in one space dimension, the potential $U(\phi_{a}) = 1 - \cos \phi$ leads to the sine-Gordon equation, which has a breather solution

$$\phi(t, x) = 4 \tan^{-1} \left[ \frac{\sqrt{1 - \omega^{2}} \sin \omega t}{\omega \cosh x \sqrt{1 - \omega^{2}}} \right].$$

(13)

It can be readily checked that this solution saturates the bound (12).

2.— For a theory of coupled scalar and gauge fields with the Lagrangian density

$$L = -\frac{1}{4} F_{\mu \nu}^{a} F_{a}^{\mu \nu} + \frac{1}{2} D_{\mu} \phi_{a} D^{\mu} \phi_{a} - U(\phi_{a})$$

(14)

we similarly find the total energy is

$$E = \int d^{n}x \langle 2U(\phi_{a}) + (1 - \frac{1}{n})(\partial_{i} \phi_{a})^{2} + (\frac{2}{n} - 1)F_{0i}^{a}F_{ai}^{0} + \frac{1}{2n} F_{ij}^{a} F_{ij}^{a} \rangle.$$  

(15)

For two and three spatial dimensions (the only interesting cases for the Lagrangian (14)) the sum of last three terms on the right-hand-side in Eq. (15) is positive. We thus have a bound

$$E \geq 2 \int d^{n}x \langle U(\phi_{a}) \rangle, \quad n = 2, 3.$$  

(16)

MASSLESS LUMPS

We now turn to the case of lumps moving with the speed of light. We carry calculations for a specific model, the 4D Yang-Mills system with sources, modifying the argument in Ref. [5].

By choosing the 3-axis in the direction of the momentum, we make the fields transverse with their components related by $E_{a} = \epsilon_{a\beta} H_{\beta}$. In terms of light-cone variables $x^{\pm} = x^{0} \pm x^{3}$, the only non-vanishing components of the field-strength are $F_{+a} = -2E_{a}$. Since $F_{12} = 0$, we can perform a gauge transformation depending on $x^{1}$ and $x^{2}$ to set $A_{1}$ and $A_{2}$ to zero. From $F_{-a} = 0$, it now follows that $A_{-}$ is independent of $x^{1}$ and $x^{2}$, so we make a gauge transformation depending on $x^{-}$ to set $A_{-} = 0$. Next, from the Yang-Mills equations of motion with sources $J_{\mu}$ we find $J_{-} = 0$ and

$$\partial^{\alpha} \partial_{\alpha} A_{+} = J_{+},$$

(17)

$$\partial^{+} \partial_{-} A_{+} + [A_{+}, \partial_{-} A_{+}] = -J_{\alpha}.$$  

(18)

Eq. (18) follows from Eq. (17) by differentiation and using covariant conservation of the current. We are left with only Eq. (17) to solve and its general solution is

$$A_{+}(x^{+}, x^{1}, x^{2}) = \tilde{A}_{+}(x^{+}, x^{1}, x^{2}) + \frac{1}{4\pi} \int d\xi^{1} d\xi^{2} J_{+}(x^{+}, \xi^{1}, \xi^{2}) \log \left[ (x^{1} - \xi^{1})^{2} + (x^{2} - \xi^{2})^{2} \right],$$  

(19)
where $\tilde{A}_+$ is a solution to the Laplace equation $(\partial^1 \partial_1 + \partial^2 \partial_2) \tilde{A}_+ = 0$. It is well known that the only non-singular solution to this equation is a function $\tilde{A}_+(x^+) = 0$. For a non-singular current, the second term in Eq. (14) can have a singularity only at infinity. For large $r$, the second term in Eq. (14) is asymptotically $\frac{1}{2\pi} I \log r$, where

$$I = \int d\xi^1 d\xi^2 J_+ = \oint_C (F_+ d\xi^1 - F^- d\xi^2), \quad (20)$$

and $C$ is an infinite contour in the $(\xi^1, \xi^2)$ plane, which encloses the sources. Observe that the contour integral form for $I$ is the gauge field flux in the transverse plane. Fields are non-singular only when the transverse flux vanishes, $I = 0$; the asymptotic $r^{-2}$ behavior of the field-strength ensures this. In contrast to the relativistic source-free case where only vacuum solutions exist, here we can only constrain the form of lumps moving with the speed of light.

CONCLUSIONS

To summarize, we have proved a generalized virial theorem which restricts possible forms of massive classical lumps. In particular, it establishes the lower energy bounds for such objects. Also, we have found only mild restrictions on massless solutions. These conclusions were reached for classical systems and they do not restrict forms of possible quantum lumps.

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[8] R. Rajaraman, Solitons and Instantons. An Introduction to Solitons and Instantons in Quantum Field Theory (Elsevier, Amsterdam, 1982).
[9] In QCD or the Standard Model $L_m$ depends on vectors, scalars, and spin-$\frac{1}{2}$ fermions in such a way that $L$ is renormalizable. However, the results given here are classical, so we need not require renormalizability.
[10] For a lump at rest, outside of a sphere with the radius independent of time, fields approach their asymptotic values with a given accuracy for all times. This guarantees absence of the outgoing radiation. For a lucid discussion of uniform convergence see R. Courant, Differential and Integral Calculus (Interscience, New York, 1937).