On PBZ*–lattices

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Abstract

We continue our investigation of paraorthomodular BZ*-lattices (PBZ*–lattices), started in [18, 19, 20, 21, 33]. We shed further light on the structure of the subvariety lattice of the variety PBZL* of PBZ*–lattices; in particular, we provide axiomatic bases for some of its members. Further, we show that some distributive subvarieties of PBZL* are term-equivalent to well-known varieties of expanded Kleene lattices or of non-classical modal algebras. By so doing, we somehow help the reader to locate PBZ*–lattices on the atlas of algebraic structures for nonclassical logics.

1 Introduction

One of the core topics within the impressive corpus of Mohammad Ardeshir’s contributions to mathematical logic is the algebraic semantics of non-classical logics. In particular, Ardeshir and his collaborators intensively investigated the relationships between Visser’s basic propositional calculus [39] and its algebraic counterpart, basic algebras, generalisations of Heyting algebras where only the left-to-right direction of the residuation equivalence $x \land y \leq z \iff x \leq y \to z$ is retained [2, 3, 4]. Also, in a basic algebra $A$ there may be $a \in A$ such that $1 \to a \neq a$. Crucially, the introduction of these structures is not motivated by abstraction per se: Ardeshir argues that basic algebras can contribute to a deeper understanding of constructive mathematics, whence they can have a paramount foundational interest.

The approach that led to the introduction of paraorthomodular BZ*-lattices (PBZ*–lattices) [18, 19, 20, 21, 33] is similar. The key motivation for this particular generalisation of orthomodular lattices, in fact, comes from the foundations of quantum mechanics. Consider the structure

$E(H) = (\mathcal{E}(H), \land, \lor, ', \sim, \emptyset, I)$,

where:

- $\mathcal{E}(H)$ is the set of all effects of a given complex separable Hilbert space $H$, i.e., positive linear operators of $H$ that are bounded by the identity operator $I$;
• $\land_s$ and $\lor_s$ are the meet and the join, respectively, of the spectral ordering $\leq_s$ so defined for all $E, F \in \mathcal{E}(H)$:

$$E \leq_s F \iff \forall \lambda \in \mathbb{R} : M^F(\lambda) \leq M^E(\lambda),$$

where for any effect $E$, $M^E$ is the unique spectral family \[28, \text{Ch. 7}\] such that $E = \int_{-\infty}^{\infty} \lambda dM^E(\lambda)$ (the integral is here meant in the sense of norm-converging Riemann-Stieltjes sums \[28, \text{Ch. 1}\]);

• $\mathbb{0}$ and $\mathbb{1}$ are the null and identity operators, respectively;

• $E' = \mathbb{1} - E$ and $E'' = P_{\ker(E)}$ (the projection onto the kernel of $E$).

The operations in $\mathcal{E}(H)$ are well-defined. The spectral ordering is indeed a lattice ordering \[34, 15\] that coincides with the usual ordering of effects induced via the trace functional when both orderings are restricted to the set of projection operators of the same Hilbert space.

A PBZ$^*$–lattice can be viewed as an abstraction from this concrete physical model, much in the same way as an orthomodular lattice can be viewed as an abstraction from a certain structure of projection operators in a complex separable Hilbert space. The faithfulness of PBZ$^*$–lattices to the physical model whence they stem is further underscored by the fact that they reproduce at an abstract level the "collapse" of several notions of sharp physical property that can be observed in $\mathcal{E}(H)$.

Referring the reader to \[18\] for a more detailed discussion of the previous issues, we now summarise the discourse of the present paper. In Section 2 we collect some preliminaries, with the twofold aim of fixing the notation to be used throughout the article and of making the article itself sufficiently self-contained — although we will occasionally need to refer the reader to results included in the previous papers on the subject. In Section 3 we zoom in on some subvarieties of the variety $\text{PBZL}^*$ of PBZ$^*$–lattices. First, we axiomatise the subvariety of $\text{PBZL}^*$ generated by a particular algebra whose role in the context of $\text{PBZL}^*$ is analogous to the role of the benzene ring in the context of ortholattices. Next, we prove that the subvariety of $\text{PBZL}^*$ generated by the (unique PBZ$^*$–lattice over the) 4-element Kleene chain is the unique antiorthomodular cover of the variety generated by the (unique PBZ$^*$–lattice over the) 3-element Kleene chain. Finally, we put to good use the construction of subdirect products of varieties of PBZ$^*$–lattices, employing them to characterise some joins of subvarieties of PBZ$^*$–lattices. Section 4 is devoted to term-equivalence results that establish connections between distributive varieties of PBZ$^*$–lattices and some known expansions of Kleene lattices, on the one hand, and nonclassical modal algebras — i.e., modal algebras whose nonmodal reducts are generic De Morgan algebras rather than Boolean algebras — on the other. We hope that these equivalences can help readers to make out the whereabouts of PBZ$^*$–lattices in the vast landscape of algebraic structures for nonclassical logic, a territory whose exploration has been decisively aided by the research work of Mohammad Ardeshir.

2 Preliminaries

For further information on the notions recalled in this section, we refer the reader to \[18, 19, 20, 21, 33\].
We denote by \( \mathbb{N} \) the set of the natural numbers and by \( \mathbb{N}^* = \mathbb{N} \setminus \{0\} \). If \( A \) is an algebra, then \( A \) will denote its universe. We call trivial algebras the singleton algebras. For any \( n \in \mathbb{N}^* \), \( D_n \) will denote the \( n \)-element chain, as well as any bounded lattice-ordered structure having this chain as a bounded lattice reduct. For any lattice \( L \), we denote by \( L^d \) the dual of \( L \). For any bounded lattices \( L \) and \( M \), we denote by \( L \oplus M \) the ordinal sum of \( L \) with \( M \), obtained by glueing together the top element of \( L \) and the bottom element of \( M \), thus stacking \( M \) on top of \( L \), and by \( L \oplus M \) the universe of the bounded lattice \( L \oplus M \); clearly, the ordinal sum of bounded lattices is associative.

Let \( V \) be a variety of algebras of similarity type \( \tau \) and \( C \) a class of algebras with \( \tau \)-reducts. We denote by \( I_V(C) \), \( H_V(C) \), \( S_V(C) \) and \( P_V(C) \) the classes of the isomorphic images, homomorphic images, subalgebras and direct products of \( \tau \)-reducts of members of \( C \), respectively, and by \( V_L(C) = H_V S_V P_V(C) \) the subvariety of \( V \) generated by the \( \tau \)-reducts of the members of \( C \). For any class operator \( O \) and any \( A \in C \), the notation \( O_V(\{A\}) \) will be streamlined to \( O_V(A) \). If \( A \) is an algebra having a \( \tau \)-reduct, \( n \in \mathbb{N} \) and \( \kappa_1, \ldots, \kappa_n \) are constants over \( \tau \), then we denote by \( \text{Con}_V(A) \) the complete lattice of the congruences of the \( \tau \)-reduct of \( A \), as well as the set reduct of this congruence lattice, and by \( \text{Con}_{\kappa_1, \ldots, \kappa_n}(A) \) the complete sublattice of \( \text{Con}_V(A) \) consisting of the congruences with singleton classes of \( \kappa_1^A, \ldots, \kappa_n^A \), as well as its set reduct. If \( V \) is the variety of lattices or that of bounded lattices, then the subscript \( V \) will be eliminated from the previous notations. If \( C \subseteq V \), then we denote by \( S_i(C) \) the class of the members of \( C \) which are subdirectly irreducible in \( V \). The lattice of subvarieties of \( V \) and its set reduct will be denoted by \( \text{Subvar}(V) \).

An involution lattice (in brief, \( I \)-lattice) is an algebra \( L = (L, \wedge, \vee, ^\prime) \) of type \( (2, 2, 1) \) such that \( (L, \wedge, \vee) \) is a lattice and \( ^\prime : L \to L \) is an order-reversing operation that satisfies \( a^{\prime \prime} = a \) for all \( a \in L \). This makes \( ^\prime \) a dual lattice automorphism of \( L \), called involution.

A bounded involution lattice (in brief, \( BI-L \)-lattice) is an algebra \( L = (L, \wedge, \vee, ^\prime, 0, 1) \) of type \( (2, 2, 1, 0, 0) \) such that \( (L, \wedge, \vee, 0, 1) \) is a bounded lattice and \( (L, \wedge, \vee, ^\prime) \) is an involution lattice. A distributive bounded involution lattice is called a De Morgan algebra.

For any \( BI-L \)-lattice \( L \), we denote by \( S(L) \) the set of the sharp elements of \( L \), that is: \( S(L) = \{x \in L : x \vee x^\prime = 1\} \). A \( BI-L \)-lattice \( L \) is called an ortholattice if all its elements are sharp, and it is called an orthomodular lattice iff, for all \( a, b \in L \), \( a \leq b \) implies \( b = (b \wedge a^\prime) \vee a \).

A pseudo–Kleene algebra is a \( BI-L \)-lattice \( L \) that satisfies \( a \wedge a^\prime \leq b \vee b^\prime \) for all \( a, b \in L \). The involution of a pseudo–Kleene algebra is called Kleene complement. Distributive pseudo–Kleene algebras are called Kleene algebras or Kleene lattices.

Clearly, for any bounded lattice \( L \) and any \( BI-L \)-lattice \( K \), if \( K_L \) is the bounded lattice reduct of \( K \), then the bounded lattice \( L \oplus K_L \oplus L^d \) becomes a \( BI-L \)-lattice with the involution that restricts to the involution of \( K \) on \( K \), to a dual lattice isomorphism from \( L \) to \( L^d \) on \( L \) and to the inverse of this lattice isomorphism on \( L^d \). This \( BI-L \)-lattice, which we denote by \( L \oplus K \oplus L^d \), is a pseudo–Kleene algebra iff \( K \) is a pseudo–Kleene algebra.

We denote by \( \mathbb{B}A \), \( \mathbb{O}ML \), \( OL \), \( \mathbb{K}A \), \( \mathbb{P}KA \), \( \mathbb{B}I \) and \( \mathbb{I} \) the varieties of Boolean algebras, orthomodular lattices, ortholattices, Kleene algebras, pseudo–Kleene algebras, \( BI-L \)-lattices and \( I-L \)-lattices, respectively. Note that \( \mathbb{B}A \subseteq \mathbb{O}ML \subseteq OL \subseteq \)
PKA $\subseteq$ BI and BA $\subseteq$ KA $\subseteq$ PKA.

An algebra $A$ having a BI–lattice reduct is said to be paraorthomodular iff, for all $a, b \in A$, if $a \leq b$ and $a' \land b = 0$, then $a = b$. Note that orthomodular lattices are paraorthomodular and that paraorthomodular ortholattices are orthomodular lattices.

A Brouwer–Zadeh lattice (in brief, BZ–lattice) is an algebra $L = (L, \land, \lor, ', \sim, 0, 1)$ of type $(2, 2, 1, 1, 0, 0)$ such that $(L, \land, \lor, ', 0, 1)$ is a pseudo–Kleene algebra and $\sim : L \to L$ is an order–reversing operation, called Brouwer complement, that satisfies: $a \land a\sim = 0$ and $a \leq a\sim\sim = a'\sim$ for all $a \in L$. In any BZ–lattice $L$, we denote by $\Box a = a\sim$ and by $\Diamond a = a\sim\sim$ for all $a \in L$. Note that, in any BZ–lattice $L$, we have, for all $a, b \in L$: $a\sim\sim\sim = a\sim \leq a'$, $(a \lor b)\sim = a\sim \lor b\sim$ and $(a \land b)\sim \geq a\sim \lor b\sim$. The class of BZ-lattices is a variety, hereafter denoted by $\mathbb{BZL}$.

We consider the following equations over $\mathbb{BZL}$, out of which SDM (the Strong De Morgan identity) clearly implies $(\ast)$, as well as SK, while J0 implies J2:

\[
\begin{align*}
(\ast) & \quad (x \land x')\sim \approx x\sim \lor x'\sim \\
SDM & \quad (x \land y)\sim \approx x\sim \lor y\sim \\
SK & \quad x \land y \leq \Box x \lor y \\
DIST & \quad x \land (y \lor z) \approx (x \land y) \lor (x \land z) \\
J0 & \quad (x \land y\sim) \lor (x \land \Diamond y) \approx x \\
J2 & \quad (x \land (y \lor y')\sim) \lor (x \land \Diamond (y \land y')) \approx x
\end{align*}
\]

A $\text{PBZ}^\ast$–lattice is a paraorthomodular BZ–lattice that satisfies equation $(\ast)$. In any $\text{PBZ}^\ast$–lattice $L$,

\[S(L) = \left\{a\sim : a \in L\right\} = \left\{a \in L : a\sim\sim = a\right\} = \left\{a \in L : a' = a\sim\sim\right\}\]

and $S(L)$ is the universe of the largest orthomodular subalgebra of $L$, that we denote by $S(L)$.

We denote by $\text{PBZL}^\ast$ the variety of $\text{PBZ}^\ast$–lattices; note that paraorthomodularity becomes an equational condition under the $\mathbb{BZL}$ axioms and condition $(\ast)$. We also denote by $\text{DIST} = \left\{L \in \text{PBZL}^\ast : L \models \text{DIST}\right\}$. By the above, OML can be identified with the subvariety $\left\{L \in \text{PBZL}^\ast : L \models x' \approx x\sim\sim\right\}$ of $\text{PBZL}^\ast$, by endowing each orthomodular lattice, in particular every Boolean algebra, with a Brouwer complement equaling its Kleene complement. With the same extended signature, OL becomes the subvariety $\left\{L \in \mathbb{BZL} : L \models x' \approx x\sim\sim\right\}$ of $\mathbb{BZL}$.

A $\text{PBZ}^\ast$–lattice $A$ with no nontrivial sharp elements, that is with $S(A) = \{0, 1\}$, is called an antiortholattice. A $\text{PBZ}^\ast$–lattice $A$ is an antiortholattice iff it is endowed with the following Brouwer complement, called the trivial Brouwer complement: $0\sim = 1$ and $a\sim = 0$ for all $a \in A \setminus \{0\}$. Every paraorthomodular pseudo–Kleene algebra with no nontrivial sharp elements becomes an antiortholattice when endowed with the trivial Brouwer complement. In particular, any BZ–lattice with the 0 meet–irreducible, and thus any BZ–chain, is an antiortholattice. Moreover, BZ–lattices with the 0 meet–irreducible are exactly the antiortholattices that satisfy SDM. Also, if $L$ is a nontrivial bounded lattice and $K$ is a pseudo–Kleene algebra, then the pseudo–Kleene algebra $L \oplus K \oplus L'^\ast$, endowed with the trivial Brouwer complement, becomes an antiortholattice, that we will also denote by $L \oplus K \oplus L'^\ast$.
Antiortholattices form a proper universal class, denoted by $\text{AOL}$. Clearly, $\text{AOL} \cup \text{OML} \subseteq \text{PBZL}^*$ $\subseteq \text{BZL} \supseteq \text{OL}$. Note, also, that $\text{OML} \cap \text{DIST} = \text{BA}$, hence $\text{DIST} \subseteq \text{PBZL}(\text{AOL})$. We denote by $\text{SDM} = \{L \in \text{PBZL}^* : L \models \text{SDM}\}$ and by $\text{SAOL} = \text{SDM} \cap \text{PBZL}(\text{AOL})$.

If $L$ is a nontrivial bounded lattice and $\mathcal{C}$ is a class of bounded lattices, BI–lattices or pseudo–Kleene algebras, then we denote by $L \oplus \mathcal{C} \oplus L^d$ the following class of bounded lattices, BI–lattices or antiortholattices:

$$L \oplus \mathcal{C} \oplus L^d = \{L \oplus A \oplus L^d : A \in \mathcal{C}\}.$$

3 A Study of Some Subvarieties

Throughout this section, the results cited from [33] will be numbered as in the third arXived version of this paper.

3.1 The $F_8$ Problem

There is a long and time-honoured tradition that aims at characterising subvarieties of varieties of ordered algebras in terms of “forbidden configurations”, harking back to Dedekind’s celebrated result to the effect that the distributive subvariety of the variety of lattices is the one whose members do not contain as subalgebras $M_3$ or $N_5$, while the modular subvariety is the one whose members do not contain $N_5$. Other important results in the same vein appear in the theory of ortholattices. For example, the benzene ring $B_6$:

$$B_6 : \begin{array}{c}
1 \\
b' \\
a' \\
a \\
b \\
0
\end{array}$$

is a forbidden configuration for the orthomodular subvariety of the variety of ortholattices; more precisely,

$$\text{OML} = \{L \in \text{OL} : B_6 \notin S_1(L)\}.$$

Consequently:

**Lemma 1** ($\text{OML}, V_{\text{BI}}(B_6)$) is a splitting pair in $\text{Subvar}(\text{OL})$.

In this subsection, we intend to give a first, limited application of this method, by means of a forbidden configuration consisting of a “paraorthomodular analogue” of $B_6$: the antiortholattice $D_2 \oplus B_6 \oplus D_2$, hereafter denoted by $F_8$, along with any of its reducts, for the sake of brevity:
Since it has the 0 meet–irreducible, the antiortholattice $F_8$ satisfies SDM, thus $F_8 \in \text{SAOL}$. The question arises naturally as to which subvarieties $V$ of PBZL* are maximal with respect to the property that $F_8 \notin S_1(V)$, i.e. $F_8 \notin S_1(A)$ for any $A \in V$. This problem will be referred to as the “$F_8$ problem”. Although we will not give an answer to this question, we provide a quasiequational characterisation of paraorthomodular bounded involution lattices that do not contain $F_8$ as a bounded involution sublattice and we study the varieties of PBZ* –lattices that contain the antiortholattice $F_8$.

Clearly, for any $L, M \in \mathcal{B}I$, we have: $D_2 \oplus M \oplus D_2 \in S_1(L)$ iff $D_2 \oplus M \oplus D_2 \in S_{B\mathcal{I}}(L)$. The right-to-left direction is trivial, while, if $D_2 \oplus M \oplus D_2 \in S_{\mathcal{I}}(L)$ and $A = M \cup \{0, 1\}$, then $D_2 \oplus M \oplus D_2 \cong_{\mathcal{B}\mathcal{I}} A \in S_{\mathcal{B}\mathcal{I}}(L)$. In particular, for any $A \in \mathcal{BZL}$, we have that $F_8 \in S_1(A)$ iff $F_8 \in S_{\mathcal{B}\mathcal{I}}(A)$; also, if $F_8 \in S_{\mathcal{BZL}}(A)$, then $F_8 \in S_{\mathcal{B}\mathcal{I}}(A)$, while, if $A$ is an antiortholattice, then $F_8 \in S_{\mathcal{BZL}}(A)$ iff $F_8 \in S_{\mathcal{B}\mathcal{I}}(A)$.

Observe what follows:

- no distributive PBZ* –lattice can contain $B_6$ or $F_8$ as sublattices, in particular as sub-involution lattices;

- since $B_6$ is a sub-involution lattice of $F_8$ and $B_6$ is not a sub-involution lattice of any orthomodular lattice, no orthomodular lattice can contain $F_8$ as a sub-involution lattice;

- by the above, any subvariety $V$ of PBZL* such that $V \subseteq \text{DIST} \cup \text{OML}$ satisfies $F_8 \notin S_1(V)$;

- $F_8 \in \text{SAOL}$, whence any subvariety $V$ of PBZL* such that SAOL $\subseteq V$ satisfies $F_8 \in S_1(V)$.

Let us now consider the following quasiequations in the language of $I$–lattices:

1. $x \leq y' \text{ and } x' \wedge y' \leq x \wedge y \Rightarrow x = y'$

2. $x' \wedge (x' \wedge u)' \leq x \wedge (x' \wedge u) \Rightarrow u \leq x'$

Note that 1 is equivalent to 2.

**Lemma 2** If $A \in \mathcal{I}$ and $a, b \in A$ are such that $a \leq b'$ and $a' \wedge b' \leq a \wedge b$, then $a \wedge a' = b \wedge b' = a' \wedge b' = a \wedge b$.

**Proof.** Let $c = a' \wedge b'$. Then $c \leq a \wedge b$ by the choice of $a$ and $b$, therefore, since we also have $a \leq b'$ and thus $b \leq a'$: $a \wedge a' = a \wedge b' \wedge a' = a \wedge c = c$; $b \wedge b' = b \wedge a' \wedge b' = b \wedge c = c$; $a \wedge b = a \wedge b' \wedge b = a \wedge c = c$. $\blacksquare$
Lemma 3 For any $A \in \text{PBI}$, we have:

$$B_6 \in S_3(A) \iff F_8 \in S_{61}(A).$$

**Proof.** The right-to-left direction is trivial. Now assume that $B_6 \in S_3(A)$, with $B_6 = \{c, a, b, a', b', c'\} \subseteq A$, where $c = a \land b$ and $a < b'$. Assume ex absurdo that $c = 0$, so that $a' \land b' = 0$. Since $A$ is paraorthomodular, it follows that $a = b'$, and we have a contradiction. Therefore $c \neq 0$, so, if we denote by $L = \{0, c, a, b, a', b', c', 1\}$, then $F_8 \cong_{S_{61}} L \in S_{61}(A)$. □

Proposition 4 For any $A \in I$, we have:

$$A \vdash \emptyset \iff B_6 \not\in S_1(A).$$

**Proof.** For the direct implication, assume that $B_6 \in S(A)$, with $B_6 = \{c, a, b, a', b', c'\} \subseteq A$, where $c = a \land b$ and $a < b'$. Then $a \leq b'$ and $a' \land b' = a \land b \leq a \land b$, but $a \neq b'$, hence $A \not\models \emptyset$.

For the converse, assume that $A \not\models \emptyset$, so that there exist $a, b \in A$ with $a' \land b' \leq a \land b$ and $a < b'$, so $b < a'$. Then, by Lemma 3 if we denote by $c = a' \land b'$, then $c = a \land b = a \land a' = b \land b'$. Since $a < b'$, $a \land b \leq a' \lor b'$; were it the case that $a \land b = a' \lor b'$, we would have that $a' \leq a' \lor b' = a \land b \leq b$, a contradiction. Hence $c' = (a \land b)' = a' \lor b' > a \land b = c$. Also, $a \lor b = (a' \lor b)' = c'$, $a \lor a' = (a \land a')' = c'$ and $b \lor b' = (b \lor b')' = c'$. If we had $a \leq b$, then $a \leq b \land b' = c = a \land a' \leq a$, hence $c = a \land a' = a < b' \leq a' \land b' = c$, and we have a contradiction again. Similarly, $b \not\leq a$. Hence $a$ and $b$ are incomparable. Were it $a \leq a'$, then $c = a \land a' = a$, which would lead to the same contradiction as above. On the other hand, if $a' \leq a$, then $c = a \land a' = a' > b \geq b \land b' = c$, which gives us another contradiction. Hence $a$ and $a'$ are incomparable and so are, analogously, $b$ and $b'$. Therefore, if we denote by $L = \{c, a, b, a', b', c', 1\}$, then $B_6 \cong_{S_{61}} L \in S_{61}(A)$. □

Theorem 5 For any $A \in \text{PBI}$, we have:

$$A \models \emptyset \iff F_8 \not\in S_{61}(A).$$

**Proof.** By Lemma 8 and Proposition 3. □

Example 6 Here is an antiortholattice (in particular, a paraorthomodular BI-lattice) $A$ such that $F_8 \not\in S_{61}(A)$, but $F_8 \in H_{61}(A)$, in particular $F_8 \in S_{61}(H_{61}(A)) \subseteq S_{61}(H_{61}(A))$:

![Diagram of a lattice](image-url)
The equivalence relation $\theta$ with cosets

$$\{0\}, \{a\}, \{c, e\}, \{b, d\}, \{b', d'\}, \{c', e'\}, \{a'\}, \{1\}$$

belongs to $\Con_{\BOL}(A) \subset \Con_{\BZL}(A)$ and $A/\theta \cong F_8$, but, as announced above, $F_8 \notin S_{\BLO}(A)$.

**Corollary 7** $\circ$ is not an equational condition in $\BLO$ or $\BZL^*$.

Now let us investigate the subvarieties of $\BZL^*$ that contain $F_8$. We consider the following equation in the language of BZ-lattices:

$$D2OL \lor (x \land x') \sim \lor (y \land y') \sim \lor x \lor x' \approx (x \land x') \sim \lor (y \land y') \sim \lor y \lor y'$$

By [20], $V_{\BZL}(AOL)$ is axiomatised by $J_0$ relative to $\BZL^*$. By [33], $V_{\BZL}(D_2 \oplus \OL \oplus D_2)$ is axiomatised by $D2OL \lor$ relative to $S_{\OL}$.

We use the following notation from [33]: for any $k, n, p \in \N$ and any equation $t \approx u$, where $t(x_1, \ldots, x_k, z_1, \ldots, z_p)$ and $u(y_1, \ldots, y_n, z_1, \ldots, z_p)$ are terms in the language of $\BLO$ having the arities $k + p$, respectively $n + p$, and $p$ common variables $z_1, \ldots, z_p$, we denote by $m(t, u)$ the following $(k + n)$-ary term in the language of $\BLO$:

$$m(t, u)(x_1, \ldots, x_k, y_1, \ldots, y_n, z_1, \ldots, z_p) =
\bigvee_{i=1}^{k}(x_i \land x'_i) \sim \lor \bigvee_{j=1}^{n}(y_j \land y'_j) \sim \lor \bigvee_{h=1}^{p}(z_h \land z'_h) \sim \lor t(x_1, \ldots, x_k, z_1, \ldots, z_p).$$

Note that:

$$m(u, t)(x_1, \ldots, x_k, y_1, \ldots, y_n, z_1, \ldots, z_p) =
\bigvee_{i=1}^{k}(x_i \land x'_i) \sim \lor \bigvee_{j=1}^{n}(y_j \land y'_j) \sim \lor \bigvee_{h=1}^{p}(z_h \land z'_h) \sim \lor u(y_1, \ldots, y_n, z_1, \ldots, z_p).$$

**Lemma 8** [33] Corollary 6.14/ For any $C \subseteq \BLO$ and any $D \subseteq \PKA$, $V_{\BLO}(D_2 \oplus C \oplus D_2) = V_{\BLO}(D_2 \oplus V_{\BLO}(C) \oplus D_2)$ and $V_{\BZL}(D_2 \oplus \D \oplus D_2) = V_{\BZL}(D_2 \oplus V_{\BLO}(D) \oplus D_2)$.

**Proposition 9** $V_{\BLO}(F_8) = V_{\BLO}(D_2 \oplus V_{\BLO}(B_6) \oplus D_2)$ and $V_{\BZL}(F_8) = V_{\BZL}(D_2 \oplus V_{\BLO}(B_6) \oplus D_2)$.

**Proof.** By Lemma 8 and the fact that $F_8 = D_2 \oplus B_6 \oplus D_2$. ■

The following consequence of results from [33] shows that we can obtain an axiomatisation for $V_{\BZL}(F_8)$ relative to $\BZL^*$ from an axiomatisation of $V_{\BLO}(B_6)$ relative to $\OL$; note that any such axiomatisation can be written with nonnullary terms over $\BLO$, since $\OL$ satisfies the equations $x \lor x' \approx 1$ and $x \land x' \approx 0$.

**Corollary 10** $\{t_i \approx u_i : i \in I\}$ is an axiomatisation of $V_{\BLO}(B_6)$ relative to $\OL$ such that, for each $i \in I$, the terms $t_i$ and $u_i$ have nonzero arities iff $\{m(t_i, u_i) \approx m(u_i, t_i) : i \in I\}$, and $\{J_0, D2OL \lor\}$ is an axiomatisation of $V_{\BZL}(F_8)$ relative to $\BZL^*$. 

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Proof. By Proposition 9 the fact that $V_{\mathfrak{B}L}(B_6) \subseteq \mathcal{O}L$ and 33 Theorem 6.38.(ii). ■

Theorem 11 [33, Theorem 6.25] The operator $\mathcal{V} \mapsto V_{\mathfrak{B}ZL}(D_2 \oplus \mathcal{V} \oplus D_2)$ is a bounded lattice embedding from the lattice of subvarieties of $\mathfrak{P}K\mathfrak{A}$ to the principal filter generated by $V_{\mathfrak{B}ZL}(D_3)$ in the lattice of subvarieties of $\mathcal{S}\mathcal{A}\mathcal{O}L$.

Corollary 12 ($V_{\mathfrak{B}ZL}(D_2 \oplus \mathcal{O}M\mathcal{L} \oplus D_2), V_{\mathfrak{B}ZL}(F_8)$) is a splitting pair in the lattice of subvarieties of $\mathcal{O}L$.

Proof. By Lemma 1 Proposition 9 and Theorem 11. ■

Proposition 13 • $V_{\mathfrak{B}L}(B_6) \subseteq V_{\mathfrak{B}L}(F_8) = V_{\mathfrak{B}L}(D_n \oplus F_8 \oplus D_n)$ for any $n \in \mathbb{N}^*$;

• $V_{\mathfrak{B}L}(F_8) \subseteq V_{\mathfrak{B}L}(D_2 \oplus F_8 \oplus D_2) = V_{\mathfrak{B}L}(D_n \oplus F_8 \oplus D_n)$ for any $n \in \mathbb{N} \setminus \{0, 1, 2\}$.

Proof. By Proposition 9 the fact that $V_{\mathfrak{B}L}(B_6) \subseteq \mathcal{O}L$, while $D_3 \in V_{\mathfrak{B}L}(F_8)$, and 33 Corollary 6.23, we get that $V_{\mathfrak{B}L}(B_6) \subseteq V_{\mathfrak{B}L}(F_8) = V_{\mathfrak{B}L}(D_2 \oplus F_8 \oplus D_2)$ and hence $V_{\mathfrak{B}L}(F_8) = V_{\mathfrak{B}L}(D_n \oplus F_8 \oplus D_n)$ for any $n \in \mathbb{N}^*$. This, Theorem 11 and again Proposition 9 show that $V_{\mathfrak{B}L}(F_8) \subseteq V_{\mathfrak{B}L}(D_2 \oplus F_8 \oplus D_2) = V_{\mathfrak{B}L}(D_n \oplus F_8 \oplus D_n)$ for any $n \in \mathbb{N} \setminus \{0, 1, 2\}$. ■

3.2 Covers in the Lattice of Subvarieties of $\mathfrak{P}B\mathfrak{Z}L^*$

In this subsection, we continue the study of the lattice $\text{Subvar}(\mathfrak{P}B\mathfrak{Z}L^*)$ of subvarieties of $\mathfrak{P}B\mathfrak{Z}L^*$–lattices, started in 18 19 20 21 33. We begin by recapitulating a few known results.

Lemma 14 (i) [18, Subsection 5.3] $\mathcal{B}\mathcal{A}$ is the unique atom of $\text{Subvar}(\mathfrak{P}B\mathfrak{Z}L^*)$.

(ii) [18, Theorem 5.4.(2)] $\mathcal{B}\mathcal{A} = \mathcal{O}M\mathcal{L} \cap V_{\mathfrak{B}ZL}(\mathcal{A}\mathcal{O}L)$.

(iii) [7, Corollary 3.6] The unique cover of $\mathcal{B}\mathcal{A}$ in the ideal $\mathcal{O}M\mathcal{L}$ of $\text{Subvar}(\mathfrak{P}B\mathfrak{Z}L^*)$ is $V_{\mathfrak{B}ZL}(\mathcal{M}\mathcal{O}2)$.

(iv) [18, Theorem 5.5] For any $L \in \mathfrak{P}B\mathfrak{Z}L^* \setminus \mathcal{O}M\mathcal{L}$, we have $D_3 \in \text{HS}(L) \subseteq V_{\mathfrak{B}ZL}((L))$, so the unique non–orthomodular cover of $\mathcal{B}\mathcal{A}$ in $\text{Subvar}(\mathfrak{P}B\mathfrak{Z}L^*)$ is $V_{\mathfrak{B}ZL}(D_3)$.

By the above, in Subvar($\mathfrak{P}B\mathfrak{Z}L^*$) $V_{\mathfrak{B}ZL}(\mathcal{M}\mathcal{O}2)$ and $V_{\mathfrak{B}ZL}(D_3)$ are the only covers of $\mathcal{B}\mathcal{A}$, and $\mathcal{O}M\mathcal{L} \lor V_{\mathfrak{B}ZL}(D_3)$ is the unique cover of $\mathcal{O}M\mathcal{L}$.

Lemma 15 [18, Lemma 3.3.(1)] All subdirectly irreducible members of $V_{\mathfrak{B}ZL}(\mathcal{A}\mathcal{O}L)$ belong to $\mathcal{A}\mathcal{O}L$.

Lemma 16 [33]

(i) $\mathcal{B}\mathcal{A} = \mathcal{O}M\mathcal{L} \cap V_{\mathfrak{B}ZL}(\mathcal{A}\mathcal{O}L) = V_{\mathfrak{B}ZL}(D_2) \subseteq V_{\mathfrak{B}ZL}(D_3) \subseteq V_{\mathfrak{B}ZL}(D_4) \subseteq V_{\mathfrak{B}ZL}(D_3)$.

(ii) $\text{Si}(V_{\mathfrak{B}ZL}(D_3)) = V_{\mathfrak{B}ZL}(D_3) \cap \mathcal{A}\mathcal{O}L = I_{\mathfrak{B}ZL}([D_1, D_2, D_3])$. 

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We now prove the main result of this subsection.

**Theorem 17** The only cover of $V_{\text{BZL}}(D_3)$ in Subvar($\text{PBZL}^*$) included in $V_{\text{BZL}}(AOL)$ is $V_{\text{BZL}}(D_4)$.

**Proof.** For any subvariety $W$ of $V_{\text{BZL}}(AOL)$ such that $V_{\text{BZL}}(D_3) \subseteq W$, there exists an $A \in S(V_{\text{BZL}}(D_3))$ by Lemma 15 and Lemma 16.(ii), thus $A$ is an antiortholattice with $|A| > 3$. Hence, there exists an $a \in A \setminus \{0, 1\} = A \setminus S_{\text{BZL}}(A)$ with $a \neq a'$, so that $0 < a \land a' < a \lor a' < 1$. Therefore $\{0, a \land a', a \lor a', 1\}$ is the universe of a subalgebra of $A$ isomorphic to $D_4$, i.e., $D_4 \in S_{\text{BZL}}(A)$, thus $V_{\text{BZL}}(D_4) \subseteq V_{\text{BZL}}(A) \subseteq W$. Since $V_{\text{BZL}}(D_3) \subseteq V_{\text{BZL}}(D_4)$ by Lemma 15(iii), it follows that $V_{\text{BZL}}(D_1)$ is the only cover of $V_{\text{BZL}}(D_3)$ in Subvar($V_{\text{BZL}}(AOL)$), which is, of course, a convex sublattice of Subvar($\text{PBZL}^*$), thus $V_{\text{BZL}}(D_4)$ is a cover of $V_{\text{BZL}}(D_3)$ in Subvar($\text{PBZL}^*$). $\blacksquare$

It remains open to determine whether $V_{\text{BZL}}(D_4)$ is the only cover of $V_{\text{BZL}}(D_3)$ in Subvar($\text{PBZL}^*$). Recall, also, that $V_{\text{BZL}}(D_5) = \text{SDM} \cap \text{DIST}$ contains all antiortholattice chains, i.e., all $\text{PBZ}^*$–chains.

**Example 18** Let us consider the following example of a PBZ$^*$–lattice from $\mathbb{Z}_7$.

![Diagram of a lattice](image)

Note that:

- $H \models \{\text{SDM}, \text{SK}\}$, thus $\text{OML} \lor V_{\text{BZL}}(H) \models \{\text{SDM}, \text{SK}\}$ since $\text{OML} \models \{\text{SDM}, \text{SK}\}$;

- $H \not\models J_2$, thus $H \not\models \text{OML} \lor V_{\text{BZL}}(AOL) \models J_2$, in particular $H \not\models \text{OML} \lor V_{\text{BZL}}(D_3)$;

- $D_3 \in S(H)$, hence $\text{OML} \lor V_{\text{BZL}}(D_3) \subseteq \text{OML} \lor V_{\text{BZL}}(H)$, therefore $\text{OML} \lor V_{\text{BZL}}(D_3) \subseteq \text{OML} \lor V_{\text{BZL}}(H)$ by the above;

- since $\text{OML} \lor V_{\text{BZL}}(H) \models \text{SK}$ and $D_4 \not\models \text{SK}$, we have $D_4 \not\models \text{OML} \lor V_{\text{BZL}}(H)$, hence $D_4$ does not belong to every proper supervariety of $\text{OML} \lor V_{\text{BZL}}(D_3)$.

$H \models \{\text{SDM}, \text{SK}\}$, $H \not\models J_2$ and $\text{OML} \lor V_{\text{BZL}}(AOL) \models J_2$, hence $H \in (\text{SDM} \cap \text{SK}) \setminus (\text{OML} \lor V_{\text{BZL}}(AOL))$, thus $\text{SDM} \cap \text{SK} \not\models \text{OML} \lor V_{\text{BZL}}(AOL)$, $\text{AOL} \not\models \text{SDM}$ and $\text{AOL} \not\models \text{SK}$, thus $\text{AOL} \not\models \text{SDM} \cap \text{AOL} \not\models \text{SK}$, in particular $\text{OML} \lor V_{\text{BZL}}(AOL) \not\models \text{SDM} \cap \text{SK}$. Therefore $\text{SDM} \cap \text{SK} \not\models \text{OML} \lor V_{\text{BZL}}(AOL)$. Now let $V = V_{\text{BZL}}(M_2) \lor V_{\text{BZL}}(D_3) \subseteq \text{SDM} \cap \text{SK}$. $D_3 \not\models \text{OML}$, thus $V \not\models \text{OML}$. $M_2 \not\models V_{\text{BZL}}(AOL)$, thus $V \not\subseteq V_{\text{BZL}}(AOL)$. Finally, $V$ satisfies the modular law, while both $\text{OML}$ and $V_{\text{BZL}}(AOL)$ fail it, hence $\text{OML} \not\models V$ and $V_{\text{BZL}}(AOL) \not\models V$. Therefore $\text{OML} \lor V \models V_{\text{BZL}}(AOL)$.
We list hereafter a few problems that remain open at the time of writing:

- Is OML ∨ V_{BZL}(D_1) a successor of OML ∨ V_{BZL}(D_3) in Subvar(PBZL^*)? Is it its only successor?
- Is Subvar(PBZL^*) strongly atomic? If so, then OML ∨ V_{BZL}(H) includes a successor of OML ∨ V_{BZL}(D_3) which differs from OML ∨ V_{BZL}(D_4).

### 3.3 Subdirect Products and Varieties of PBZ^*–lattices

Let V and W be varieties of the same type. Obviously, if V and W are incomparable, then there exist A ∈ (V ∨ W) \ V and B ∈ (V ∨ W) \ W, so that A × B ∈ (V ∨ W) \ (V ∪ W) and thus V ∪ W ⊈ V ∨ W. Recall that the subdirect product of V and W is the class, denoted by V × s W, whose members are isomorphic images of subdirect products of a member of V and a member of W. Clearly, V ∪ W ⊆ V × s W ⊆ V ∨ W, so that

\[ Si(V) ∪ Si(W) = Si(V ∪ W) ⊆ Si(V × s W) ⊆ Si(V ∨ W). \]

For any M ∈ Si(V × s W), M is a subdirect product of an A ∈ V and a B ∈ W, so that A is trivial, case in which M ∈ Si(W), or B is trivial, case in which M ∈ Si(V). Thus Si(V × s W) ⊆ Si(V) ∪ Si(W), hence Si(V × s W) = Si(V) ∪ Si(W). Since V × s W ⊆ V ∨ W, we get that the following equivalence holds: V ∨ W = V × s W iff Si(V ∨ W) = Si(V) ∪ Si(W).

Sufficient Mal’tsev-type conditions for the equivalence V ∨ W = V × s W to hold are available in the literature; see [35, 26, 27]. These contributions are all inspired by the celebrated result by Grätzer, Lakser and Plonka according to which two independent similar varieties V and W are such that every member of V ∨ W is isomorphic to the direct product of a member of V and a member of W [24]. Of course, the notion of independence is of limited use in the context of PBZ^*–lattices, since EA is the unique atom in Subvar(PBZL^*) and thus there are no two nontrivial disjoint (hence, no two independent) varieties of PBZ^*–lattices. The investigation of subdirect products of varieties of PBZ^*–lattices, however, can be carried out with more ad hoc methods, yielding useful information on joins of specific subvarieties.

If V ∨ W = V × s W and U is a variety of the same type as V and W, then (U ∩ V) × s (U ∩ W) ⊆ (U ∩ V) ∨ (U ∩ W) ⊆ U ∩ (V ∨ W) and

\[ Si(U ∩ (V ∨ W)) = Si(U) \cap Si(V ∨ W) \]
\[ = Si(U) \cap (Si(V) ∪ Si(W)) \]
\[ = (Si(U) \cap Si(V)) ∪ (Si(U) ∩ Si(W)) \]
\[ = Si(U ∩ V) ∪ Si(U ∩ W) \]
\[ = Si((U ∩ V) × s (U ∩ W)), \]

hence U ∩ (V ∨ W) = (U ∩ V) ∨ (U ∩ W) = (U ∩ V) × s (U ∩ W). For instance, since OML ∨ V_{BZL}(AOL) = OML × s V_{BZL}(AOL) (see Lemma [21] below), it follows that

\[ SDM \cap (OML ∨ V_{BZL}(AOL)) = (SDM \cap OML) \cap (SDM \cap V_{BZL}(AOL)) \]
\[ = OML \cap SAOL = OML × s SAOL. \]

As a consequence of the above, if V ∨ W = V × s W and Subvar(V) and Subvar(W) are distributive, then Subvar(V ∨ W) is distributive.
Problem 19  If $V \lor W = V \times_s W$, $C$ is a subvariety of $V$ and $D$ is a subvariety of $W$, under what conditions does it follow that $C \lor D = C \times_s D$? Does the condition that $C \cap D = V \cap W$ suffice? A partial answer to this question is given by Lemma 20 below.

If $V \lor W = V \times_s W$ and $U$ is a subvariety of $V$, then $U \lor W$ is a subvariety of $V \lor W$, so that

$$Si(U \lor W) = (U \lor W) \cap Si(V \lor W) = (U \lor W) \cap (Si(V) \lor Si(W)) = ((U \lor W) \cap Si(V)) \cup ((U \lor W) \cap Si(W)) = ((U \lor W) \cap Si(V)) \cup Si(W).$$

Lemma 20  Let $V$ and $W$ be varieties of a similarity type $\tau$, $U$ a subvariety of $V$ and $\Gamma$ a set of equations over $\tau$ such that $V \lor W = V \times_s W$, $W \Vdash \Gamma$ and $U = \{ A \in V : A \Vdash \Gamma \}$. Then:

- $U \lor W = U \times_s W = \{ A \in V \lor W : A \Vdash \Gamma \};$
- $U = V$ iff $U \lor W = V \lor W$.

**Proof.** Of course, $Si(U) \cup Si(W) \subseteq Si(U \lor W)$. For all $A \in Si(U \lor W)$, we have: $A \in Si(V \lor W) = Si(V) \lor Si(W)$ and $A \Vdash \Gamma$, so that either $A \in Si(W)$ or $A \in Si(V) \subset V$ and $A \Vdash \Gamma$, the latter of which implies that $A \in Si(V) \cap U = Si(U)$. Therefore $Si(U \lor W) = Si(U) \cup Si(W)$, thus $U \lor W = U \times_s W$. We have that:

$$Si(\{ A \in V \lor W : A \Vdash \Gamma \}) = \{ A \in Si(V \lor W) : A \Vdash \Gamma \} = \{ A \in Si(V) \cup Si(W) : A \Vdash \Gamma \} = \{ A \in Si(V) : A \Vdash \Gamma \} \cup Si(W) = Si(\{ A \in V : A \Vdash \Gamma \}) \cup Si(W) = Si(U) \cup Si(W) = Si(U \lor W),$$

hence $U \lor W = \{ A \in V \lor W : A \Vdash \Gamma \}$.

Trivially, $U = V$ implies $U \lor W = V \lor W$. Conversely, if $V \lor W = U \lor W = \{ A \in V \lor W : A \Vdash \Gamma \}$, then $V \lor W \Vdash \Gamma$, thus $V \Vdash \Gamma$, hence $U = \{ A \in V : A \Vdash \Gamma \} = V$.

Lemma 21  All subdirectly irreducible members of $OML \lor V_{BZL}(AOL)$ belong to $OML \lor AOL$, in particular $OML \lor V_{BZL}(AOL) = OML \times_s V_{BZL}(AOL)$.

We can derive from the above the following result from \[20\]:

Proposition 22  $OML \lor V_{BZL}(AOL) = OML \times_s V_{BZL}(AOL)$.

- $OML \lor V_{BZL}(D_3) \subseteq OML \lor SAOL \subseteq OML \lor V_{BZL}(AOL)$.

**Proof.** Recall from \[19\] Corollary 3.3 that $V_{BZL}(D_3) = \{ A \in V_{BZL}(AOL) : A \Vdash \{ SDM, SK \} \}$. Now apply the fact that $OML \Vdash \{ SDM, SK \}$ and Lemmas 21 and 20 to obtain first that $OML \lor SAOL = OML \times_s SAOL$, then that $OML \lor V_{BZL}(D_3) = OML \times_s V_{BZL}(D_3)$. Recall that $D_3 \in SAOL \setminus V_{BZL}(D_3)$,
which is easily noticed from the fact that $D_5 \not\subseteq SK$. The antiortholattice $D_3 \oplus D_2 \in V_{SZL}(AOL) \setminus SAOL$. Hence $V_{SZL}(D_3) \subseteq SAOL \subseteq V_{SZL}(AOL)$, thus $OML \lor V_{SZL}(D_3) \subseteq OML \lor SAOL \subseteq OML \lor V_{SZL}(AOL)$ by Lemma 20 and the above.

Let us consider the identities:

- **WDSDM**
  $$(x \land (y \lor z))^\approx \approx (x \land y)^\approx \land (x \land z)^\approx$$

- **DIST**
  $$(x \lor x^\approx) \land (y \land y^\approx \lor z \lor z^\approx) \approx$$
  $$((x \lor x^\approx) \land (y \lor y^\approx)) \lor ((x \land z^\approx) \land (z \lor z^\approx))$$

Note that **WDSDM** implies **DIST** and that, in **DIST**, **DIST** implies **DIST**.

Also, recall from [19, 33] that $V_{SZL}(D_3) = SAOL \cap DIST$.

**Proposition 23**
$V_{SZL}(D_3) = SAOL \cap DIST \subseteq SAOL, DIST \subseteq SAOL \lor DIST \subseteq V_{SZL}(AOL)$.

**Proof.** Observe that the identity **WDSDM** is satisfied both in **SAOL** and in **DIST**. The antiortholattice on $M_3 \oplus M_3$ fails **WDSDM**, because, if $a, b, c$ are its three atoms, then $(a \land (b \lor c))^\approx = a^\approx = 0$, yet $(a \land b)^\approx \land (a \land c)^\approx = 0^\approx \land 0^\approx = 1$. Hence $M_3 \oplus M_3 \in AOL \setminus (SAOL \lor DIST) \subseteq V_{SZL}(AOL) \setminus (SAOL \lor DIST)$.

The antiortholattice $D_2 \oplus M_3 \oplus D_2 \in SAOL \setminus DIST$, while the antiortholattice $D_2 \oplus D_2 \in DIST \setminus SAOL$, hence **SAOL** and **DIST** are incomparable, thus $SAOL \cap DIST \subseteq SAOL, DIST \subseteq SAOL \lor DIST$.

**Proposition 24**
- **OML** $\land$ **DIST** $= OML \times_s DIST$ and **OML** $\lor V_{SZL}(D_3) = OML \times_s V_{SZL}(D_3)$;
- **OML** $\lor V_{SZL}(D_3) \subseteq OML \lor V_{SZL}(D_3) = OML \lor (SAOL \cap DIST) = (OML \lor SAOL) \cap (OML \lor DIST) \subseteq OML \lor SAOL, OML \lor DIST \subseteq OML \lor SAOL \lor DIST \subseteq OML \lor V_{SZL}(AOL)$, in particular the varieties **OML** $\lor$ **SAOL** and **OML** $\lor$ **DIST** are incomparable.

**Proof.** Note that **OML** $\vdash$ **DIST** and that, in **AOL**, **DIST** is equivalent to **DIST**, that is **DIST** $\cap$ **AOL** $= \{A \in AOL : A \vdash$ **DIST**$\}$. The latter, along with the fact that **DIST** is a subvariety of $V_{SZL}(AOL)$ and Lemma 13 give us:

$$Si(DIST) = DIST \cap Si(V_{SZL}(AOL))$$
$$= DIST \cap Si(AOL)$$
$$= Si(DIST \cap AOL)$$
$$= Si(\{A \in AOL : A \vdash$ **DIST**$\})$$
$$= Si(\{A \in V_{SZL}(AOL) : A \vdash$ **DIST**$\}),$$

therefore $DIST = \{A \in V_{SZL}(AOL) : A \vdash$ **DIST**$\}$. By Lemmas 21 and 20 it follows that **OML** $\lor$ **DIST** $= OML \times_s DIST$. By the above, **OML** $\vdash$ \{**SDM**, **DIST**\} and $V_{SZL}(D_3) = SAOL \cap DIST = \{A \in V_{SZL}(AOL) : A \vdash$ \{**SDM**, **DIST**\}$\}$, hence $OML \lor V_{SZL}(D_3) = OML \times_s V_{SZL}(D_3)$ by Lemmas 21 and 20. By the above, Propositions 22 and 24 and again Lemma 20 it follows that:

$$OML \lor V_{SZL}(D_3) \subseteq OML \lor V_{SZL}(D_3)$$
$$\subseteq OML \lor SAOL, OML \lor DIST$$
$$\subseteq OML \lor V_{SZL}(AOL).$$
antiortholattice, hence there exist a \( D_k \) such that there exists a

\[ \exists D_k \text{ such that } \exists a \]

\[ a \in D_k \]

Lemma 26

Proof. The second statement obviously follows from the first. Now assume \( a \) be a nonempty family of antiortholattices. Then:

\[ (\exists D_k) \text{ such that } \exists a \in D_k \]

\[ a \in D_k \]

hence \((\exists D_k) \in (\exists a)\) or \( D_k = a \). Hence, by the proof of Proposition 23, the antiortholattice

\[ M_3 \oplus M_3 \]

fails \( W D I S T \). It follows that \( M_3 \oplus M_3 \in A O L \setminus \{ O M L \cap S A O L \cap D I S T \} \subset \{ O M L \cap V_B Z L (A O L) \} \setminus \{ O M L \cap S A O L \cap D I S T \} \), therefore \( O M L \cap S A O L \cap D I S T \subset O M L \cap V_B Z L (A O L) \). ■

Lemma 25 For any subvariety \( V \) of \( O M L \vee V_B Z L (A O L) \), \( S_i(V) = V \cap S_i(O M L \cup A O L) \).

Proof. By Lemma 21 ■

Note that, if a PBZ*-lattice \( L \) satisfies the SDM, then 0 is meet–irreducible in the join–subsemilattice \( T(L) \) of \( L \), but the converse does not hold.

Lemma 26 Let \( A \) be an antiortholattice without SDM and \( (A_i)_{i \in I} \) be a nonempty family of antiortholattices. Then:

- if \( A \in S_B Z L (\prod_{i \in I} A_i) \), then the family \( (A_i)_{i \in I} \) contains no nontrivial antiortholattice with SDM;

- \( A \in S_B Z L (\prod_{i \in I} A_i) \) iff \( A \in S_B Z L (\prod_{i \in I, A_i \# S D M} A_i) \).

Proof. The second statement obviously follows from the first. Now assume that \( A \in S_B Z L (\prod_{i \in I} A_i) \), let \( J = \{ j \in I : A_j \models S D M \} \) and assume ex absurdo that there exists a \( k \in J \) such that \( A_k \) is nontrivial. We may consider \( A \subseteq \prod_{i \in I, A_i \# S D M} A_i \) is an antiortholattice that fails SDM, in particular a nontrivial antiortholattice, hence there exist \( a = (a_i)_{i \in I}, b = (b_i)_{i \in I} \in A \setminus \{ 0 \} = D(A) \)

\[ D(\prod_{i \in I} A_i) = \prod_{i \in I} D(A_i) = \prod_{i \in I} ((A_i \setminus \{ 0 \}) \cup \{ 1 \}) \]

such that \( a \land b = 0 \), so that \( a_k \land b_k = 0 \) and \( a_k \in D(A_k) = A_k \setminus \{ 0 \} \), which contradicts the fact that \( A_k \) satisfies the SDM. ■

Proposition 27 If \( V \) is a subvariety of \( V_B Z L (A O L) \), then: \( V \vee S A O L = V \times_s S A O L \) if \( (V \vee S A O L) \cap A O L = (V \cup S A O L) \cap A O L \).

Proof. By the above, \( V \vee S A O L = V \times_s S A O L \) if \( S_i(V \vee S A O L) = S_i(V \cup S A O L) \). Since \( S_i(V_B Z L (A O L)) \subset A O L \), the right-to-left implication holds. Now assume that \( S_i(V \vee S A O L) = S_i(V \cup S A O L) \), and assume ex absurdo that there exists
an $L \in ((\forall \vee \text{SAOL}) \cap \text{AOL}) \setminus (\forall \cup \text{SAOL})$. Then $L \in \forall \times \text{SAOL}$, hence $L \in \mathcal{S}_{\mathcal{BL}}(A \times \prod_{j \in J} B_j)$ for some family $(B_j)_{j \in J} \subseteq \text{SAOL} \cap \text{AOL}$. Thus $L \in \mathcal{S}_{\mathcal{BL}}(A)$ by Lemma 26 so that $L \in \mathcal{V}$, a contradiction. Hence $(\forall \vee \text{SAOL}) \cap \text{AOL} \subseteq (\forall \cup \text{SAOL}) \cap \text{AOL}$. ■

4 Comparison with Other Structures

4.1 Distributive Lattices with Two Unary Operations

Bounded distributive lattices expanded both by a De Morgan complementation and a unary operation with Stone-like properties have been the object of rather intensive investigations over the past decades. In particular, Blyth, Fang and Wang [6] have studied, under the label of quasi-Stone De Morgan algebras, bounded distributive lattices with two unary operations that make their appropriate reducts, at the same time, De Morgan algebras and quasi-Stone algebras [37, 17, 13]. Quasi-Stone De Morgan algebras that are simultaneously Stone algebras and Kleene algebras are known under the name of Kleene-Stone algebras; they have been studied in [25] and, more recently, in the already quoted [6]. We begin this section by showing that the variety of antiortholattices generated by the algebra $D_5$ coincides with the variety of Kleene-Stone algebras. This fact explains the similarity of some results independently obtained in [6, 19, 33].

Definition 28 A quasi-Stone algebra is an algebra $A = (A, \wedge, \vee, \sim, 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice and the unary operation $\sim$ satisfies the following conditions for all $a, b \in A$:

\begin{align*}
QS1 & \quad 0^\sim = 1 \text{ and } 1^\sim = 0; \\
QS2 & \quad (a \vee b)^\sim = a^\sim \wedge b^\sim; \\
QS3 & \quad (a \wedge b^\sim)^\sim = a^\sim \vee b^\sim^\sim; \\
QS4 & \quad a \leq a^\sim^\sim; \\
QS5 & \quad a^\sim \wedge a^\sim^\sim = 1.
\end{align*}

A quasi-Stone algebra $A$ is a Stone algebra if it additionally satisfies SDM.

The following useful lemma contains results to be found in [37] and [6]:

Lemma 29 Let $A = (A, \wedge, \vee, \sim, 0, 1)$ be a quasi-Stone algebra. Then:

(i) $A$ satisfies the following conditions for all $a, b \in A$:

\begin{align*}
QS6 & \quad \text{if } a \leq b, \text{ then } b^\sim \leq a^\sim; \\
QS7 & \quad a \wedge a^\sim = 0; \\
QS8 & \quad a^\sim^\sim = a^\sim; \\
QS9 & \quad a \wedge b^\sim = 0 \iff a \leq b^\sim.
\end{align*}

(ii) The set $B(A) = \{a^\sim : a \in A\} = \{a \in A : a = a^\sim^\sim\}$ is a Boolean subuniverse of $A$.

Clearly, in case $A$ is a Stone algebra, the condition QS9 can be strengthened to the pseudocomplementation equivalence:

\begin{align*}
S1 & \quad a \wedge b = 0 \iff a \leq b^\sim \text{ for all } a, b \in A.
\end{align*}
Definition 30 A quasi-Stone De Morgan algebra is an algebra \( A = (A, \wedge, \vee, \sim, 0, 1) \) of type \( (2, 2, 1, 1, 0, 0) \) such that \((A, \wedge, \vee, 0, 1)\) is a De Morgan algebra, 
\((A, \wedge, \vee, \sim, 0, 1)\) is a quasi-Stone algebra, and \( a' \in B(A) \) whenever \( a \in B(A) \), 
If \((A, \wedge, \vee, \sim, 0, 1)\) is a Kleene algebra and \((A, \wedge, \vee, \sim, 0, 1)\) is a (quasi-)Stone algebra, then \( A \) is said to be a Kleene-(quasi-)Stone algebra.

Lemma 31 If \( A \) is a quasi-Stone De Morgan algebra, then for all \( a \in A \) we have that \( a \sim \sim = a \sim' \sim' \).

Recall from Proposition 23 that the variety generated by the 5-element antiortholattice chain \( D_5 \) is axiomatised relative to \( \mathbb{PBZL}^* \) by the lattice distribution axiom \( \text{DIST} \) and the Strong De Morgan law \( \text{SDM} \) (J0 easily follows from these assumptions in the context of \( \mathbb{PBZL}^* \)). We now show that:

Theorem 32 \( V_{\mathbb{BZL}}(D_5) \) coincides with the variety of Kleene-Stone algebras.

Proof. It is readily seen that \( D_5 \) satisfies all the defining conditions of Kleene-Stone algebras. Conversely, by the above remark, it will be sufficient to show that Kleene-Stone algebras satisfy all the axioms of \( \mathbb{PBZL}^* \)-lattices, since they are clearly distributive as lattices and satisfy SDM by definition. We confine ourselves to the sole nontrivial items. (i) The condition \((\ast), (x \land x') \sim = x' \lor x' \sim \) directly follows from SDM. (ii) We show that \( a \sim \sim = a \sim' \). By QS5, \( a \sim \lor a \sim \sim = 1 \), whence \( a \sim' \land a \sim \sim' = 0 \). By S1, \( a \sim \sim \leq a \sim' \), whence, given the fact that \( a \sim \sim \in B(A) \),
\[
a \sim' \leq (QS4) a \sim \sim \leq (QS6) a \sim \sim = a \sim.
\]
From this inequality, QS6 and QS8 we obtain that \( a \sim = a \sim \sim \leq a \sim' \) and thus, by Lemma 31 \( a \sim \sim = a \sim \sim \leq a \sim' \). The converse inequality follows from S1 and the fact that \( a \sim \in B(A) \). (iii) To round up our proof, it will suffice to show that any Kleene algebra is paraorthomodular. Thus, let \( a \leq b \) and \( a' \land b = 0 \). Then \( a' \land a \leq a' \land b = 0 \), whence \( a \) is sharp and thus \( a \lor a' = 1 \). As \( a \land b = a \) and \( a' \land b = 0 \), distributivity implies that
\[
a = (a \land b) \lor (a' \land b) = (a \lor a') \land b = 1 \land b = b.
\]

The question as to whether the distributive subvariety \( \text{DIST} \) of \( V_{\mathbb{BZL}}(\mathbb{AO}) \) coincides with the variety of Kleene-quasi-Stone algebras is of a certain interest. The next Example answers this problem in the negative.

Example 33 The BZ-lattice \( \mathbb{BZ}_4 \) (see [18, Figure 5]) is a Kleene-quasi-Stone algebra, yet it is not even a member of \( \mathbb{PBZL}^* \). In fact, call \( a \) and \( a' \) its two atoms. We have that:
\[
(a \land a')' \sim = 0' \sim = 1 \neq 0 = a' \sim \lor a' \sim.
\]

Finally, we prove that the variety generated by the 3-element antiortholattice chain \( D_3 \) is a discriminator variety [40].

Proposition 34 \( V_{\mathbb{BZL}}(D_3) \) is a discriminator variety.
Proof. Clearly, it suffices to find a ternary term that realizes the discriminator function on $D_3$. Let first

$$e(x, y) = (x \sim \Diamond y) \vee (y \sim \Diamond x) \vee (\Box x \wedge (\Box y) \sim) \vee (\Box y \wedge (\Box x) \sim).$$

It is a routine matter to check that for all $a, b \in D_3$, $e^{D_3}(a, a) = 0$ and $e^{D_3}(a, b) = 1$ if $a \neq b$. It follows that

$$t(x, y, z) = (e(x, y) \vee z) \wedge (e(x, y)' \vee x)$$

realizes the discriminator function on $D_3$. \(\blacksquare\)

Observe that the algebra $D_3$ fails to be primal, because it has the nontrivial proper subuniverse $\{0, 1\}$. Nonetheless, upon identifying $D_3$ with the set of rational numbers $\{0, \frac{1}{2}, 1\}$, the truncated sum operation is definable as follows:

$$x \oplus y = \min (1, x + y) = (x \vee \Diamond y) \wedge (y \vee \Diamond x).$$

It is easy to check that, upon expanding its signature by this binary operation, $D_3$ becomes an instance of a De Morgan Brouwer-Zadeh MV-algebra \cite{10, 11} and, therefore, generates a subvariety of such. The interest of this remark lies in the fact that the variety of De Morgan Brouwer-Zadeh MV-algebras is known to be term-equivalent to other well-known varieties of algebras of logic, including Heyting-Wajsberg algebras, Stonean MV-algebras and MV algebras with Baaz Delta \cite{9}. In the next section, we will see that $V_{BZL}(D_3)$ is term-equivalent to another well-known variety of algebras of logic.

4.2 Modal Algebras

The standard examples of modal algebras (monadic algebras or interior algebras, to name a few examples) were devised as the algebraic counterparts of normal modal logics, which are extensions of classical propositional logic — therefore, they all have a Boolean algebra reduct. There is a thriving literature, however, on “nonstandard” modal algebras based on generic De Morgan algebras: see below for the appropriate references. The aim of this section is to chart this area of research and locate term-equivalent counterparts of some distributive subvarieties of PBZ$^*$-lattices on this map. We consider algebras $M = (M, \wedge, \vee, \Diamond, 0, 1)$ of type $(2, 2, 1, 1, 0, 0)$, where $(M, \wedge, \vee, \Diamond, 0, 1)$ is a De Morgan algebra. We assume that $\Diamond$ binds stronger than $\Diamond$, to reduce the number of parentheses. The following list of identities will be crucial for defining the varieties that follow; henceforth, $\Box x$ is short for $(\Diamond x)'$.

\begin{align*}
M1 \quad \Diamond 0 & \approx 0 \\
M2 \quad \Diamond (x \vee y) & \approx \Diamond x \vee \Diamond y \\
M3 \quad x & \leq \Diamond x \\
M4 \quad \Diamond x & \approx \Diamond \Diamond x \\
M5 \quad \Diamond x \wedge (\Diamond x)' & \approx 0 \\
M6 \quad \Diamond x & \approx \Box \Diamond x
\end{align*}
\begin{align*}
M7 \quad & \Diamond (x \land x') \approx \Diamond x \land \Diamond x' \\
M8 \quad & x' \lor \Diamond x \approx 1 \\
M9 \quad & \Diamond (x \land y) \approx \Diamond x \land \Diamond y \\
M10 \quad & x \land x' \approx \Diamond x \land x'
\end{align*}

**Definition 35**  
(i) A \Diamond-De Morgan algebra is an algebra \( M = (M, \land, \lor, ', \Diamond, 0, 1) \) of type \( (2, 2, 1, 1, 0, 0) \), where \( (M, \land, \lor, ', 0, 1) \) is a De Morgan algebra and the identities \( M1 \) and \( M2 \) are satisfied.

(ii) A topological quasi-Boolean algebra is a \Diamond-De Morgan algebra satisfying the identities \( M3 \) and \( M4 \).

(iii) A classical \Diamond-De Morgan algebra is a topological quasi-Boolean algebra satisfying the identity \( M5 \).

(iv) A monadic De Morgan algebra is a classical \Diamond-De Morgan algebra satisfying the identity \( M6 \).

\Diamond-De Morgan algebras and classical \Diamond-De Morgan algebras were introduced in dual form by Sergio Celani \cite{13}, pp. 253-254]. Topological quasi-Boolean algebras were first investigated by Banerjee and Chakraborty in the context of the theory of rough sets \cite{5}. The authors of \cite{36} also introduce, under the label of topological quasi-Boolean algebras \cite{5}, a subvariety of topological quasi-Boolean algebras that satisfy \( M6 \) but not \( M5 \). Clearly, topological quasi-Boolean algebras are meant to be a nonclassical counterpart of interior algebras, while monadic De Morgan algebras can be viewed as a nonclassical counterpart of monadic algebras. Condition \( M5 \), which is of course trivial once our algebras have a Boolean nonmodal reduct, is there to restore the Boolean behaviour of the nonmodal operators, when applied to arguments of the form \( \Diamond x \). Observe that all classical \Diamond-De Morgan algebras satisfy the identity \( M8 \) \cite{13}, Lemma 2.3].

There are several ways to strengthen the defining conditions of classical \Diamond-De Morgan algebras with an eye to obtaining varieties with more interesting properties.

(i) A possible avenue is to impose on the possibility operator properties that would determine a collapse of modality when the underlying structures are Boolean algebras. For example, tetravalent modal algebras \cite{32, 29} are classical \Diamond-De Morgan algebras that satisfy \( M10 \), although they are usually presented in a streamlined axiomatisation containing only the axioms for De Morgan algebras plus \( M8 \) and \( M10 \). They form a discriminator variety, generated by a quasiprimal four-element algebra (see item (iv) of the proof of Theorem \ref{thm:axiomatisation} below).

(ii) On the other hand, one can enforce what Cattaneo et al. \cite{8} call a "deviant" behaviour of the possibility operator, requesting that it distribute not only over joins, but over meets as well. Involutive Stone algebras \cite{14}; cp. also \cite{8}, where these structures are called MDS5-algebras, thus, are classical \Diamond-De Morgan algebras satisfying \( M9 \). It is known that both involutive Stone algebras and tetravalent modal algebras are monadic De Morgan algebras: see \cite{14} and \cite{10}, Proposition 1.2, respectively.
We now introduce the modal analogue of distributive PBZ\(^*\)-lattices.

**Definition 36** A weak Lukasiewicz algebra is a classical \(\Diamond\)-De Morgan algebra \(M = (M, \land, \lor, \Diamond, 0, 1)\) such that its \(\Diamond\)-free reduct is a Kleene algebra and the identity M7 is satisfied.

**Theorem 37** (i) Every weak Lukasiewicz algebra \(M\) is a monadic De Morgan algebra.

(ii) The variety of weak Lukasiewicz algebras is term-equivalent to \(\mathbb{D}\text{DIST}\).

**Proof.** (i) Let \(a \in M\). Using M1, M5, M7 and M4, we have that

\[
0 = \Diamond 0 = \Diamond (\Diamond a \land (\Diamond a)') = \Diamond \Diamond a \land \Diamond ((\Diamond a)') = \Diamond a \land \Diamond ((\Diamond a)') .
\]

Thus \((\Diamond a)' \lor \Box \Diamond a = 1\), whence, by M5,

\[
\Diamond a = \Diamond a \land ((\Diamond a)' \lor \Box \Diamond a) = (\Diamond a \land (\Diamond a)') \lor (\Diamond a \land \Box \Diamond a) = \Diamond a \land \Box \Diamond a.
\]

Consequently, \(\Diamond a \leq \Box \Diamond a\). The converse inequality follows from M3.

(ii) Let \(M = (M, \land, \lor, \Diamond, 0, 1)\) be a weak Lukasiewicz algebra. We define \(f(M)\) as the algebra \((M, \land, \lor, \Diamond f(M), 0, 1)\), where for all \(a \in M\), \(a^{\sim f(M)\Diamond M} = (\Diamond a)^{f(M)}\). Conversely, given a distributive PBZ\(^*\)-lattice \(L = (L, \land, \lor, \Diamond L, 0, 1)\), we define \(g(L)\) as the algebra \((L, \land, \lor, \Diamond g(L), 0, 1)\), where for all \(a \in L\), \(\Diamond g(L) a = a^{\sim L \Diamond L} = a\). Clearly, \(f(M)\) has a Kleene lattice reduct. If \(a \in M\), then \(a \land a^{\sim f(M)\Diamond M} = a \land (\Diamond M a)' \leq \Diamond M a \land (\Diamond M a)' = 0\), by M3 and M5. Moreover,

\[
a^{\sim f(M)\Diamond M} = (\Diamond (\Diamond M a)')' = \Diamond M a \geq a,
\]

by M3 and item (1). For the same reason, \(a^{\sim f(M)\Diamond M} = (\Diamond M a)^{f(M)} = \Diamond M a = a^{\sim f(M)\Diamond M}\). Finally, by M2, whenever \(a \leq b\),

\[
\Diamond M b = \Diamond M (a \lor b) = \Diamond M a \lor \Diamond M b,
\]
i.e. \(\Diamond M a \leq \Diamond M b\), whence \(b^{\sim f(M)\Diamond M} = (\Diamond M b)' \leq (\Diamond M a)' \leq a^{\sim f(M)\Diamond M}\). In sum, \(f(M)\) is a distributive BZ-lattice. Condition (\(\ast\)) holds because of M7. Similarly, by reverse-engineering \(g(L)\), it is not hard to show that it is a weak Lukasiewicz algebra. To round off the proof, observe that for \(a \in L\),

\[
a^{\sim f(g(L))} = (\Diamond g(L) a)' = a^{\sim L \Diamond L} = a^{\sim L \Diamond L} = a^{\sim L},
\]

\[
\Diamond g(f(M)) a = a^{\sim f(M)\Diamond M} = (\Diamond M (\Diamond M a)')' = \Diamond M a .
\]

Thus, \(f\) and \(g\) are mutually inverse functions. \(\blacksquare\)

Similar term-equivalence results with subvarieties of PBZL\(^*\) are obtained in [10] and [12] for two special subvarieties of weak Lukasiewicz algebras.

**Definition 38** (i) \[\text{Definition 4.2}]\ A Lukasiewicz algebra is a weak Lukasiewicz algebra that satisfies the identity M9.
(ii) A three-valued Lukasiewicz algebra is a Lukasiewicz algebra that satisfies the identity M10.

Clearly, Lukasiewicz algebras are exactly the involutive Stone algebras whose \(\Diamond\)-free reduct is a Kleene lattice. There is a burgeoning literature on three-valued Lukasiewicz algebras, see e.g. \(\llbracket 1, 31, 30 \rrbracket\). Three-valued Lukasiewicz algebras can be equivalently characterised as tetravalent modal algebras satisfying M9, in which case, the Kleene identity follows from the axioms. They are also called pre-rough algebras in the literature \(\llbracket 36 \rrbracket\).

**Theorem 39** \(\llbracket 10 \rrbracket\) Theorems 4.3 and 5.7

(i) The variety of Lukasiewicz algebras is term-equivalent to \(V_{\text{LBZL}}(D_5)\).

(ii) The variety of three-valued Lukasiewicz algebras is term-equivalent to \(V_{\text{LBZL}}(D_3)\).

Taking into account the remarks at the end of last section, it is evident that \(V_{\text{LBZL}}(D_5)\) and \(V_{\text{LBZL}}(D_3)\) have repeatedly resurfaced in many different incarnations, with different choices of primitives or with different axiomatisations. We collect many of the observations made thus far in the following result.

**Theorem 40** The strict inclusions and incomparabilities depicted in the following diagram all hold:

\[
\begin{array}{c}
\Diamond - \text{De Morgan algebras} \\
\text{topological quasi-Boolean algebras} \\
\text{classical } \Diamond - \text{De Morgan algebras} \\
\text{monadic } \Diamond - \text{De Morgan algebras} \\
\text{involutive Stone algebras} \\
\text{Lukasiewicz algebras} \\
\text{three-valued Lukasiewicz algebras} \\
\end{array}
\]

**Proof.** All that remains to be proved is that the inclusions are strict and that the varieties not connected by upward chains are incomparable.

(i) Consider the algebra \(D_2\) as a De Morgan algebra, and let \(\Diamond 0 = \Diamond 1 = 0\). This algebra is a \(\Diamond\)-De Morgan algebra which is not a topological quasi-Boolean algebra.

(ii) Consider the algebra \(D_3\) as a De Morgan algebra, and let \(\Diamond x = x\) for all \(x \in D_3 = \{0, a, 1\}\). This algebra is a topological quasi-Boolean algebra which is not a classical \(\Diamond\)-De Morgan algebra. In fact, \(\Diamond a \land (\Diamond a)' = a \neq 0\).

(iii) Consider the algebra \(D_2^*\) as a De Morgan algebra with universe \(\{0, a, a', 1\}\), and let \(\Diamond x = x\) for all \(x \in \{0, a, 1\}\), and \(\Diamond a' = 1\). This algebra is a topological quasi-Boolean algebra which is not a monadic De Morgan algebra. In fact, \(\Box \Diamond a = 0 \neq a = \Diamond a\).
(iv) Let $B_4$ be the four-element algebra on \{0, a, b, 1\} that generates De Morgan algebras, with $a = a'$ and $b = b'$. Let $\Diamond 0 = 0$ and $\Diamond x = 1$ for all $x \neq 0$. This is a tetravalent modal algebra (actually, it generates this variety), hence a monadic De Morgan algebra, but not an involutive Stone algebra. In fact, $\Diamond (a \land b) = 0 \neq 1 = \Diamond a \land \Diamond b$. Having two fixpoints for the involution, it also fails to be a weak Lukasiewicz algebra, hence a Lukasiewicz algebra or a three-valued Lukasiewicz algebra.

(v) Consider the algebra $D_2^2$ as a De Morgan algebra with universe \{0, a, a', 1\}, and let $\Diamond 0 = 0$, and $\Diamond x = 1$ for all $x \neq 0$. This algebra is a monadic De Morgan algebra which is not a tetravalent modal algebra. In fact, $\Diamond a \land \Diamond a' = a' \neq 0 = \Diamond a \land \Diamond a'$.

(vi) Consider the ordinal sum $D_2^2 \oplus B_4 \oplus D_2$ as a De Morgan algebra with universe \{0, a, b, c, a', 1\}, with $b = b'$ and $c = c'$, and let $\Diamond 0 = 0$, and $\Diamond x = 1$ for all $x \neq 0$. This algebra is an involutive Stone algebra which is not a weak Lukasiewicz algebra (or a Lukasiewicz algebra) since it has two fixpoints for the involution.

(vii) Consider the ordinal sum $D_2^2 \oplus D_2^2$ as a De Morgan algebra on \{0, a, b, c, b', a', 1\}, with $c = c'$, and let $\Diamond 0 = 0$, and $\Diamond x = 1$ for all $x \neq 0$. This is a weak Lukasiewicz algebra which is not an involutive Stone algebra, for $\Diamond (a \land b) = 0 \neq 1 = \Diamond a \land \Diamond b$. A fortiori, it fails to be a Lukasiewicz algebra.

(viii) Finally, consider the algebra $D_4$ as a De Morgan algebra on \{0, a, a', 1\}, and let $\Diamond 0 = 0$, and $\Diamond x = 1$ for all $x \neq 0$. This is a Lukasiewicz algebra, hence both an involutive Stone algebra and a weak Lukasiewicz algebra. However, it fails to be a tetravalent modal algebra (hence a three-valued Lukasiewicz algebra), for $\Diamond a \land \Diamond a' = a' \neq 0 = a \land a'$.

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