CURVATURE FLOW TO NIRENBERG PROBLEM

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Abstract. In this note, we study the curvature flow to Nirenberg problem on $S^2$ with non-negative nonlinearity. This flow was introduced by Brendle and Struwe. Our result is that the Nirenberg problems has a solution provided the prescribed non-negative Gaussian curvature $f$ has its positive part, which possesses non-degenerate critical points such that $\Delta_{S^2} f > 0$ at the saddle points.

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1. Introduction

In the interesting paper [7], M.Struwe studied a heat flow method to the Nirenberg problem on $S^2$. This kind of heat flow in conformal geometry was considered by S.Brendle in [2]. Given the Riemannian metric $g$ on $S^2$ with Gaussian curvature $K$. Using the well-known Gauss-Bonnet formula

$$\int_{S^2} K dv_g = 4\pi,$$

we know that $K$ has to be positive somewhere. This gives a necessary condition for the Nirenberg problem on $S^2$. Assuming the prescribed curvature function $f$ being positive on $S^2$, the heat flow for the Nirenberg problem $S^2$ is a family of metrics of the form $g = e^{2u(x,t)}c$ satisfying

(1) $$u_t = \alpha f - K, \quad x \in S^2, \quad t > 0,$$

where $c$ is the standard spherical metric on $S^2$, $u : S^2 \times (0, T) \rightarrow R$, and $\alpha = \alpha(t)$ is defined by

(2) $$\alpha \int_{S^2} f dv_g = 4\pi.$$

Here $dv_g$ is the area element with respect to the metric $g$. It is easy to see that

$$\alpha \int_{S^2} f dv_g = 2\alpha \int_{S^2} (K - \alpha f) f dv_g.$$

M.Struwe can show that the flow exists globally, furthermore, the flow converges at infinity provided $f$ is positive and possesses non-degenerate critical points.

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points such that $\Delta_{S^2} f > 0$ at the saddle points. Here $\Delta_{S^2} := \Delta$ is the Analyst’s Laplacian on the standard 2-sphere $(S^2, c)$. Recall that $\int_{S^2} dv_c = 4\pi$.

The purpose of this paper is to relax his assumption by allowing the function $f$ to have zeros.

Since we have

$$K = e^{2u}(-\Delta u + 1),$$

the equation \text{(1)} define a nonlinear parabolic equation for $u$, and the flow exists at least locally for any initial data $u|_{t=0} = u_0$. Clearly, we have

$$\partial_t \int_{S^2} dv_g = 2 \int S^2 u_t dv_g = 0.$$

We shall assume that the initial data $u_0$ satisfies the condition

\text{(3)}

$$\int f e^{2u} dv_c > 0.$$

We shall show that this property is preserved along the flow. It is easy to compute that

\text{(4)}

$$K_t = -2u_t K - \Delta_g u_t = 2K(K - \alpha f) + \Delta_g (K - \alpha f),$$

where $\Delta_g = e^{-2u} \Delta$. Using \text{(4)}, we can compute the growth rate of the Calabi energy $\int_{S^2} |K - \alpha f|^2 dv_g$.

Our main result is following

\textbf{Theorem 1.} Let $f$ be a positive somewhere, non-negative smooth function on $S^2$ with only non-degenerate critical points on the its positive part $f_+$. Suppose that there are at least two local positive maxima of $f$, and at all positive valued saddle points $q$ of $f$ there holds $\Delta_{S^2} f(q) > 0$. Then $f$ is the Gaussian curvature of the conformal metric $g = e^{2u}c$ on $S^2$.

Note that this result is an extension of the famous result of Chang-Yang \cite{3} where only positive $f$ has been considered. A similar result for Q-curvature flow has been obtained in \cite{9}.

For simplifying notations, we shall use the conventions that $dc = \frac{du}{4\pi}$ and $\bar{u} = \bar{u}(t)$ defined by

$$\int_{S^2} (u - \bar{u})dv_c = 0.$$

2. Basic properties of the flow

Recall the following result of Onofri-Hong \cite{5} that

\text{(5)}

$$\int_{S^2} (|\nabla u|^2 + 2u)dc \geq \log(\int_{S^2} e^{2u} dc) = 0,$$

where $|\nabla u|^2$ is the norm of the gradient of the function $u$ with respect to the standard metric $c$. Here we have used the fact that $\int_{S^2} e^{2u} dc = 1$ along the flow \text{(1)}. 


We show that this condition is preserved along the flow \((1)\). In fact, letting

\[ E(u) = \int_{S^2} (|\nabla u|^2 + 2u) \, dc \]

be the Liouville energy of \(u\) and letting

\[ E_f(u) = E(u) - \log(\int_{S^2} f e^{2u} \, dc) \]

be the energy function for the flow \((1)\), we then compute that

\[ \partial_t E_f(u) = -2 \int_{S^2} |\alpha f - K|^2 \, dv_g \leq 0. \]

One may see Lemma 2.1 in \([7]\) for a proof. Hence

\[ E_f(u(t)) \leq E_f(u_0), \quad t > 0. \]

After using the inequality \((5)\) we have

\[ \log(1/\int_{S^2} f e^{2u} \, dc) \leq E_f(u_0), \]

which implies that \(\int_{S^2} f e^{2u} \, dv_c > 0\) and

\[ e^{E_f(u_0)} \int_{S^2} e^{2u} \, dc \leq \int_{S^2} f e^{2u} \, dc. \]

Note also that \(\int_{S^2} f e^{2u} \, dc = 1/\alpha(t)\). Hence,

\[ \alpha(t) \leq \frac{1}{e^{E_f(u_0)}}. \]

Using the definition of \(\alpha(t)\) we have

\[ \alpha(t) \geq \frac{1}{\max_{S^2} f}. \]

We then conclude that \(\alpha(t)\) is uniformly bounded along the flow, i.e.,

\[ \frac{1}{\max_{S^2} f} \leq \alpha(t) \leq \frac{1}{e^{E_f(u_0)}}. \]

We shall use this inequality to replace (26) in \([7]\) in the study of the normalized flow, which will be defined soon following the work of M.Struwe \([7]\). If we have a global flow, then using \((6)\) we have

\[ 2 \int_0^\infty \int_{S^2} |\alpha f - K|^2 \, dv_g \leq 4\pi(E_f(u_0) + \log \max_{S^2} f). \]

Hence we have a suitable sequence \(t_l \to \infty\) with associated metrics \(g_l = g(t_l)\) and \(\alpha(t_l) \to \alpha > 0\), and letting \(K_l = K(g_l)\), such that

\[ \int_{S^2} |K_l - \alpha f|^2 \to 0, \quad (t_l \to \infty). \]

Therefore, once we have a limiting metric \(g_\infty\) of the sequence of the metrics \(g_l\), it follows that \(K(g_\infty) = \alpha f\). After a re-scaling, we see that \(f\) is the
Gaussian curvature of the metric $\beta g_{\infty}$ for some $\beta > 0$, which implies our Theorem II.

3. Normalized flow and the proof of Theorem II

We now introduce a normalized flow. For the given flow $g(t) = e^{2u(t)}c$ on $S^2$, there exists a family of conformal diffeomorphisms $\phi = \phi(t) : S^2 \to S^2$, which depends smoothly on the time variable $t$, such that for the metrics $h = \phi^* g$, we have

$$\int_{S^2} x dv_h = 0, \text{ for all } t \geq 0.$$ 

Here $x = (x^1, x^2, x^3) \in S^2 \subset \mathbb{R}^3$ is a position vector of the standard 2-sphere. Let

$$v = u \circ \phi + \frac{1}{2} \log(det(d\phi)).$$

Then we have $h = e^{2v}c$. Using the conformal invariance of the Liouville energy [3], we have

$$E(v) = E(u),$$

and furthermore,

$$Vol(S^2, h) = Vol(S^2, g) = 4\pi, \text{ for all } t \geq 0.$$ 

Assume $u(t)$ satisfies (1) and (2). Then we have the uniform energy bounds

$$0 \leq E(v) \leq E(u) = Ef(u) + \log(\int_{S^2} f e^{2u} dc) \leq Ef(u_0) + \log(\max_{S^2} f).$$

Using Jensen’s inequality we have

$$2\bar{v} := \int_{S^2} 2vd\sigma \leq \log(\int_{S^2} e^{2v} dc) = 0.$$ 

Using this we can obtain the uniform $H^1$ norm bounds of $v$ for all $t \geq 0$ that

$$\sup_t |v(t)|_{H^1(S^2)} \leq C.$$ 

See the proof of Lemma 3.2 in [7]. Using the Aubin-Moser-Trudinger inequality [1] we further have

$$\sup_t \int_{S^2} e^{2pv(t)} dc \leq C(p)$$

for any $p \geq 1$.

Note that

$$v_t = u_t \circ \phi + \frac{1}{2} e^{-2v} \text{div}_{S^2}(\xi e^{2v})$$

where $\xi = (d\phi)^{-1} \phi_t$ is the vector field on $S^2$ generating the flow $(\phi(t))$, $t \geq 0$, as in [7], formula (17), with the uniform bound

$$|\xi|_{L^\infty(S^2)} \leq C \int_{S^2} |\alpha f - K|^2 dv_g.$$
With the help of this bound, we can show (see Lemma 3.3 in [7]) that for any $T > 0$, it holds
\[ \sup_{0 \leq t < T} \int_{S^2} e^{4|u(t)|} dc < +\infty. \]

Following the method of M. Struwe [6] (see also Lemma 3.4 in [7]) and using the bound (7) and the growth rate of $\alpha$, we can show that
\[ \int_{S^2} |\alpha f - K|^2 dv_g \to 0 \]
as $t \to \infty$. Once getting this curvature decay estimate, we can come to consider the concentration behavior of the metrics $g(t)$. Following [6], we show that

**Lemma 2.** Let $(u_l)$ be a sequence of smooth functions on $S^2$ with associated metrics $g_l = e^{2u_l} c$ with $\text{vol}(S^2, g_l) = 4\pi$, $l = 1, 2, \ldots$. Suppose that there is a smooth function $K_\infty$, which is positive somewhere and non-negative in $S^2$ such that
\[ |K(g_l) - K_\infty|_{L^2(S^2, g_l)} \to 0 \]
as $l \to \infty$. Let $h_l = \phi_l^* g_l = e^{2u_l} c$ be defined as before. Then we have either

1) for a subsequence $l \to \infty$ we have $u_l \to u_\infty$ in $H^2(S^2, c)$, where $g_\infty = e^{2u_\infty} c$ has Gaussian curvature $K_\infty$, or

2) there exists a subsequence, still denoted by $(u_l)$ and a point $q \in S^2$ with $K_\infty(q) > 0$, such that the metrics $g_l$ has a measure concentration that
\[ dv_{g_l} \to 4\pi \delta_q \]
weakly in the sense of measures, while $h_l \to c$ in $H^2(S^2, c)$ and in particular, $K(h_l) \to 1$ in $L^2(S^2)$. Moreover, in the latter case the conformal diffeomorphisms $\phi_l$ weakly converges in $H^1(S^2)$ to the constant map $\phi_\infty = q$.

**Proof.** The case 1) can be proved as Lemma 3.5 in [7]. So we need only to prove the case 2). As in [7], we choose $q_l \in S^2$ and radii $r_l > 0$ such that
\[ \sup_{q \in S^2} \int_{B(q, r_l)} |K(g_l)| dv_{g_l} \leq \int_{B(q_l, r_l)} |K(g_l)| dv_{g_l} = \pi, \]
where $B(q, r_l)$ is the geodesic ball in $(S^2, g_l)$. Then we have $r_l \to 0$ and we may assume that $q_l \to q$ as $l \to \infty$. For each $l$, we introduce $\phi_l$ as in Lemma 3.5 in [7] so that the functions
\[ \hat{u}_l = u_l \circ \phi_l + \frac{1}{2} \log(det(d\phi_l)) \]
satisfy the conformal Gaussian curvature equation
\[ -\Delta_{R^2} \hat{u}_l = \hat{K}_l e^{2\hat{u}_l}, \text{ on } R^2, \]
where $\hat{K}_l = K(g_l) \circ \phi$ and $\Delta_{R^2}$ is the Laplacian operator of the standard Euclidean metric $g_{R^2}$. Note that for $\hat{g}_l = \phi_l^* g_l = e^{2\hat{u}_l} g_{R^2}$, we have
\[ \text{Vol}(R^2, \hat{g}_l) = \text{Vol}(S^2, g_l) = 4\pi. \]
Arguing as in [7], we can conclude a convergent subsequence \( \hat{u}_l \to \hat{u}_\infty \) in \( H^2_{loc}(\mathbb{R}^2) \) where \( \hat{u}_\infty \) satisfies the Liouville equation

\[-\Delta_{R^2} \hat{u}_\infty = \hat{K}_\infty(q) e^{2\hat{u}_\infty}, \text{ on } R^2,\]

with the finite volume \( \int_{R^2} e^{2\hat{u}_\infty} dz \leq 4\pi. \)

We remark that blow up point only occurs at point where \( K_\infty(q) > 0 \) provided we allow \( K_\infty(q) \) to change sign. We only need to exclude the case when \( K_\infty(q) \leq 0 \). It is clear that \( K_\infty(q) < 0 \) can not occur since there is no such a solution on the whole plane \( R^2 \) [8]. If \( K_\infty(q) = 0 \), then \( \tilde{u} := \hat{u}_\infty \) is a harmonic function in \( R^2 \). Let \( \bar{u}(r) \) be the average of \( u \) on the circle \( \partial B_r(0) \subset R^2 \). Then we have

\[ \Delta_{R^2} \bar{u} = 0. \]

Hence \( \bar{u} = A + B \log r \) for some constants \( A \) and \( B \), where \( r = |x| \). Since \( \bar{u} \) is a continuous function on \([0, \infty)\), we have \( \bar{u} = A \), which is impossible since we have by Jensen’s inequality that

\[ 2\pi \int_0^\infty e^{2\bar{u}(r)} r dr \leq \int_{R^2} e^{2\tilde{u}_\infty} dz \leq 4\pi. \]

We now have \( K_\infty(q) > 0 \). Recall that we have assumed \( K_\infty \geq 0 \). So we can follow the proof of Lemma 3.5 in [7]. In fact, using the classification result of Chen-Li [4] we know that \( \hat{u}_\infty \) can be obtained from stereographic projection \( S^2 \to R^2 \) with

\[ \int_{R^2} K_\infty(q) e^{2\tilde{u}_\infty} dz = 4\pi. \]

Thus for any large \( R > 0 \), we have error \( \circ(1) \to 0 \) as \( l \to \infty \) such that

\[ 4\pi = \int \int_{R^2} K_\infty(q) e^{2\tilde{u}_\infty} dz \leq \int_{B_R(q)} K_l dv_l + \circ(1) \leq \int_{B_R(q)} |K_l| dv_l + \circ(1). \]

Then the remaining part can be derived as in [7]. We confer to [7] for the full proof.

With this understanding, we can do the same finite-dimensional dynamics analysis as in section 4 in [7]. Then arguing as in section 5 in [7] we can prove Theorem 1. By now the argument is well-known, we omit the detail and refer to [7] for full discussion.

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