THE GENERIC FLAT PREGEOMETRY

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Abstract. We examine the first order structure of pregeometries of structures built via Hrushovski constructions. In particular, we show that the class of flat pregeometries is an amalgamation class such that the pregeometry of the unbounded arity Hrushovski construction is precisely its generic.

We show that the generic is saturated, provide an axiomatization for its theory, show that the theory is \(\omega\)-stable, and has quantifier-elimination down to boolean combinations of \(\exists\forall\)-formulas. We show that the pregeometries of the bounded-arity Hrushovski constructions satisfy the same theory, and that they in fact form an elementary chain.

1. Introduction

We investigate the pregeometries associated to the hypergraph Hrushovski constructions, the strongly minimal structures used to refute Zilber’s conjecture. In particular, we show that these pregeometries are generic structures for classes of flat pregeometries, with the pregeometry associated to the unbounded arity Hrushovski construction being the generic for the class of all finite flat pregeometries. Using this characterization, we show that this pregeometry is saturated and \(\omega\)-stable, and provide its theory and quantifier elimination down to boolean combinations of \(\exists\forall\)-formulas. Finally, we present the pregeometries of Hrushovski constructions of bounded arity as an elementary chain limiting to the unbounded arity pregeometry, showing that all are models of the same \(\omega\)-stable theory, differing in degree of saturation.

There are few known prototypical examples of \(\omega\)-stable theories, and even fewer which are not uncountably categorical. The two canonical such examples are everywhere infinite forests, the generic structure for the amalgamation class of finite trees, and \(\text{DCF}_0\), the theory of differentially closed fields of characteristic 0. It is surprising to find any truly new \(\omega\)-stable theory and it is strong evidence that despite their reputation, the pregeometries of Hrushovski constructions are nice and natural objects.

A combinatorial pregeometry, or matroid, is an abstract dependence relation on a set (see subsection 2.1). Classical examples are linear dependence in vector spaces, and algebraicity in field extensions. A pregeometry associated to a structure illustrates the degree of interaction between given elements — e.g. linear independence indicates zero interaction, whereas a large set of a small linear dimension indicates many linear dependencies between the elements. In model theory, structures...
(types) that are sufficiently well behaved model theoretically have a naturally associated pregeometry called the **forking** pregeometry. In the examples above, as well as many others, the forking pregeometry coincides with the classical dependence relation intrinsic to the structure.

A major program in model theory is to classify structures via their associated pregeometries. There are three types of pregeometries that arise ubiquitously in the model theoretic analysis of mathematical structures: disintegrated (set-like), locally modular (linear space-like), and field-like.

Zilber conjectured [Zil84] that among the strongly minimal theories, those theories that are most model theoretically tame, these are the only types of pregeometries that arise. Hrushovski [Hru93] showed that the conjecture is false by producing a strongly minimal theory which was non-disintegrated yet interpreted no algebraic structures. Instead, his new strongly minimal theory is inherently combinatorial in nature. This was seen as the end for the hope of an orderly classification of all strongly minimal theories in terms of simple, understandable pregeometries. In this paper, we counter this position, giving an analysis of the pregeometry Hrushovski constructed, showing that it is in fact tame.

Where we can understand strongly minimal sets in terms of their pregeometries, many fruitful applications of model theory have been found. In many restricted cases – see for example [Rab93, KR16, HZ96] – the suggested trichotomy does in fact hold: Any strongly minimal subset of a tame enough structure must have one of these three sorts of pregeometries. This is fertile ground for interaction of model theory with other fields such as algebraic geometry, differentially closed fields, valuation theory, and many more – see for example [HZ96, Hru96, Zil14]. In an attempt to salvage the trichotomy in a large class of cases, Hrushovski and Zilber [HZ96] showed that the trichotomy holds for all Zariski structures. We hope that with our analysis of the pregeometries of structures arising from Hrushovski constructions, we can again pursue a general theory.

While the stated motivation is model theoretic, the paper reads as a study of flat pregeometries (known as strict gammoids or cotransversal matroids to matroid theorists) with the intended purpose of constructing generic objects. Pregeometries and hypergraphs are the stars of the show, with model theory confined almost exclusively to subsection 3.2.

To discuss the results of the paper, we must briefly survey Hrushovski’s construction. For a finite hypergraph $\mathcal{A} = (M, R)$ ($R$ can be any set of finite subsets of $M$), we define the **predimension** $\delta(\mathcal{A}) = |M| - |R|$ to be the difference between the number of vertices of $\mathcal{A}$ and the number of edges of $\mathcal{A}$. We write $\mathcal{A} \leq \mathcal{B}$ if there is no finite intermediate $\mathcal{A} \subseteq \mathcal{D} \subseteq \mathcal{B}$ such that $\delta(\mathcal{D}) < \delta(\mathcal{A})$. In the context of a hypergraph $\mathcal{A} = (M, R)$, a finite subset $X \subseteq \mathcal{A}$ is deemed independent if $X \leq \mathcal{A}$ and there are no edges contained in $X$. In this way, we associate to each hypergraph a pregeometry $G_{\mathcal{A}}$. These associated pregeometries were proven to have the property of flatness by Hrushovski [Hru93], which is key in showing Hrushovski’s strongly minimal construction interprets no algebraic structures. The class $C$ of all finite hypergraphs such that $\emptyset \leq \mathcal{A}$ is an amalgamation class under the notion of embedding $\leq$, and so has a unique countable generic structure $\mathcal{M}$. 
This is Hrushovski’s (non-collapsed\textsuperscript{1}) construction for hypergraphs of unbounded arity.

The pregeometry \( G_M \), denoted \( G \), is our main object of study in this paper. To allow detailed analysis of \( G \), we build it as the generic structure for \( \mathcal{C} \), the class of all finite flat pregeometries. We achieve this by following the Hrushovski construction blueprint: an amalgamation class with respect to a distinguished notion of embedding, paralleling the procedure for hypergraphs.

In the first part of the paper, we introduce flatness and its accompanying \( \sqsubseteq^* \), \( \sqsubseteq \), \( \rho \) — the geometric analogues of “induced subgraph”, \( \leq \), \( \delta \). The contents of Section 2 are the proofs that these are indeed adequate analogues, showing in Corollary 2.6.7 that \( H \sqsubseteq G \) is equivalent to \( \rho(H) \) being minimal among \( H' \sqsubseteq^* G \) containing \( H \), and to the existence of hypergraphs \( A \leq B \) such that \( G_A = H \), \( G_B = G \). In subsection 3.1, we show that \( (\mathcal{C}, \sqsubseteq) \) is an amalgamation class and that its generic structure is isomorphic to \( G \).

The construction echoes previous works by Evans [Eva11, Section 5], in which the author explores geometric characterizations of \( \leq \) in the finite case, and Evans and Ferreira [EF11, Section 6], in which the authors prove \( G \) has a weak form of genericity by applying a forgetful functor to the construction of \( \mathcal{M} \). However, lack of ability to depart from the hypergraph scaffolding leaves \( G \) impervious to further analysis. Using the technology developed in this paper to overhaul Evans and Ferreira’s attempt yields what they were after [EF11, Section 6, Problem], and opens \( G \) up to the model theoretic analysis they did not have the means to conduct.

For the second part of the paper, in subsection 3.2, we use our detailed construction and characterization of \( G \) to analyze it model theoretically: we show it is saturated, we axiomatize its theory, and we show that its theory is \( \omega \)-stable and has quantifier elimination down to boolean combinations of a specified set of \( \exists \forall \) formulas.

Finally, we develop a geometrical definition of arity in the flat setting. Having a thorough analysis of \( G \), we consider the pregeometries of the structures built by the bounded-arity Hrushovski constructions, as studied by Evans and Ferreira [EF11]. Let \( \mathcal{M}_n \) be Hrushovski’s construction for hypergraphs of arity up to \( n \). Let \( G_n \) be its pregeometry. Evans and Ferreira showed that \( G_n \not\equiv G_m \) whenever \( n \neq m \). Moreover, they showed that even after localizing at finite sets, \( G_n \) and \( G_m \) remain non-isomorphic, seemingly demonstrating that there are \( \omega \) many fundamentally different pregeometries arising from Hrushovski constructions. In contrast, we show that there are natural elementary embeddings: \( G_3 \prec G_4 \prec \cdots \prec G \). We conclude that not only is the theory of \( G \) \( \omega \)-stable, but it is the theory of all of the pregeometries of the canonical Hrushovski constructions, differing only by their level of saturation.

2. Flatness

2.1. Pregeometry. Write \( A \subseteq_{\text{fin}} B \) to say that \( A \) is a finite subset of \( B \).

Definition 2.1.1. A combinatorial pregeometry \( G \) is a set \( X \) with a dimension function \( d : \text{Fin}(X) \to \mathbb{N} \) such that

\textsuperscript{1}Evans and Ferreira [EF12] showed that, under minor assumptions on the multiplicity function \( \mu \), the pregeometry associated to Hrushovski’s strongly minimal construction is identical to the one associated to the non-collapsed construction, which is easier to work with.
(1) \( d(\emptyset) = 0 \)
(2) \( d(A) \leq d(Ax) \leq d(A) + 1 \)
(3) \( d(A \cup B) + d(A \cap B) \leq d(A) + d(B) \)

Say that \( A \subseteq X \) is independent if \( d(A) = |A| \). Say that \( Y \subseteq X \) is independent if every finite subset of \( Y \) is independent. For \( A \subseteq X \) define \( \text{cl}(A) = \{ x \in X \mid d(Ax) = d(A) \} \). For \( Y \subseteq X \) define \( \text{cl}(Y) = \bigcup_{Y_0 \subseteq Y} \text{cl}(Y_0) \). For \( Z \subseteq Y \subseteq X \), say that \( Z \) is a basis for \( Y \) if \( Z \) is independent and \( Y \subseteq \text{cl}(Z) \). Say that \( Y \subseteq X \) is closed if \( \text{cl}(Y) = Y \). Observe that an arbitrary intersection of closed sets is closed. We may extend \( d \) to infinite subsets by taking \( d(Y) = \sup\{|Y_0| : Y_0 \subseteq Y \text{ is independent}\} \), this definition coincides with \( d \) on finite sets.

We interchangeably think of a pregeometry on a set \( X \) as:

(1) A closure operator \( \text{cl} : P(X) \to P(X) \);
(2) A dimension function \( d : P(X) \to \text{Card} \);
(3) A first order structure with relations \( \{ I_n \mid n \in \mathbb{N} \} \) where \( I_n \subseteq X^n \) is the set of independent \( n \)-tuples.
(4) A first order structure with relations \( \{ D_n \mid n \in \mathbb{N} \} \) where \( D_n \subseteq X^n \) is the set of dependent \( n \)-tuples.

A subpregeometry \( H \subseteq G \) is a substructure of a first order representation of \( G \), and it is itself the pregeometry gotten by restricting the dimension function of \( G \) to subsets of \( H \). When no confusion arises, we may omit distinction between subsets and subpregeometries.

By convention, if several pregeometries \( H, G, \ldots \) are discussed simultaneously, we differentiate their dimension functions and closure operators with a subscript, i.e., \( d_H, d_G, \text{cl}_H, \text{cl}_G \), etc. If all dimension functions in discussion are restrictions of some ambient dimension \( d \), we omit the subscripts.

2.2. Definition of flatness.

Notation 2.2.1. Let \( G \) be a pregeometry and let \( \Sigma = \{ E_1, \ldots, E_k \} \) be some ambient collection of closed subsets of \( G \). For each non-empty set of indices \( s \subseteq [k] \) we denote \( E_s = \bigcap_{i \in s} E_i \), and for \( s = \emptyset \) we denote \( E_\emptyset = \bigcup_{i=1}^k E_i \). Denote also

\[
\Delta_G(\Sigma) = \sum_{\emptyset \neq s \subseteq [k]} (-1)^{|s|+1} d_G(E_s)
\]

The alternating sum \( \Delta_G(\Sigma) \) is the inclusion-exclusion principle, where the dimension function \( d_G \) replaces cardinality. Like in inclusion-exclusion, the alternating sum should be thought of as reconstructing the dimension of \( E_\emptyset \), the union of the sets \( E_i \), based on the sum of the information found within each individual \( E_i \). With this intuition in mind, it should make no difference whether we add to \( \Sigma \) closed subsets of \( E_i \), as we are adding no new information.

Observation 2.2.2. In the context of a pregeometry \( G \) and \( \Sigma = \{ E_1, \ldots, E_k \} \), if there are some \( i, j \leq k \) distinct with \( E_i \subseteq E_j \), then \( \Delta_G(\Sigma \setminus \{ E_i \}) = \Delta_G(\Sigma) \).

Proof. Observe that

\[
\Delta_G(\Sigma) - \Delta_G(\Sigma \setminus \{ E_i \}) = \sum_{\emptyset \neq s \subseteq [k]\setminus\{i\}} (-1)^{|s|+1} d(E_{s \cup \{i\}})
\]

Note that \( E_{s \cup \{i,j\}} = E_{s \cup \{i\}} \) for every \( s \subseteq [k]\setminus\{i,j\} \). Since the dimensions of these sets appear with opposite signs in the sum, they cancel each other out. \( \square \)
Cardinality is the simplest of dimension functions, but for an arbitrary dimension function there is no reason why $\Delta_G(\Sigma)$ should evaluate to the precise dimension of $E_0^2$. Flatness is the statement that whenever we use an alternating sum to “guess” the dimension of a union, we may be overestimating, but never underestimating.

**Definition 2.2.3.** Say that a pregeometry $G$ is *flat* if whenever $\Sigma$ is a finite collection of finite dimensional closed sets in $G$, then $\Delta_G(\Sigma) \geq d(\bigcup \Sigma)$.

In the next observation and its corollary we see how adding information to $\Sigma$ changes the estimate $\Delta_G(\Sigma)$, given that $G$ is flat.

**Observation 2.2.4.** Let $G$ be a flat pregeometry and let $\Sigma = \{E_1, \ldots, E_k\}$ be a collection of finite dimensional closed sets in $G$. Let $X$ be a closed finite-dimensional subset of $G$ and denote $\Sigma_X = \{E_i \cap X \mid E_i \in \Sigma\}$. Observe $\bigcup \Sigma_X = X \cap \bigcup \Sigma$. Then

$$\Delta_G(\Sigma \cup \{X\}) - \Delta_G(\Sigma) = d_G(X) - \sum_{\emptyset \neq s \subseteq [k]} (-1)^{|s|+1} d_G(E_s \cap X)$$

$$= d_G(X) - \Delta_X(\Sigma_X)$$

$$\leq d_G(X) - d_G(X \cap \bigcup \Sigma)$$

**Corollary 2.2.5.** If $G$ is flat, $\Sigma = \{E_1, \ldots, E_k\}$ a collection of finite dimensional closed sets in $G$, and $X = \text{cl}_G(Y)$ for some $Y \subseteq \bigcup \Sigma$, then $\Delta_G(\Sigma \cup \{X\}) \leq \Delta_G(\Sigma)$. Moreover, the inequality is strict if and only if $d_G(X) \neq \Delta_X(\Sigma_X)$, where $\Sigma_X = \{E_i \cap X \mid E_i \in \Sigma\}$.

**Notation 2.2.6.** Let $G$ be a pregeometry and let $\Sigma$ be a finite collection of finite dimensional closed sets in $G$. As above, we will often be interested in the dimensional information the closed sets of $\Sigma$ capture of some set other than $G$. We denote the relativization of $\Sigma$ with a subscript. If a subscript is already present, i.e., $\Sigma_G$ instead of $\Sigma$, replace it.

For $X \subseteq G$ a subpregeometry of $G$, denote

$$\Sigma_X = \{E \cap X \mid E \in \Sigma\}.$$  

For $H \supset G$ a pregeometry containing $G$, denote

$$\Sigma_H = \{\text{cl}_H(E) \mid E \in \Sigma\}.$$  

2.3. **Distinguished embeddings.** In observation 2.2.4, $\Sigma_X$ is the restriction of the elements of $\Sigma$ to $X$. For an arbitrary $X$, the value of $\Delta_X(\Sigma_X)$ may differ from $\Delta_G(\Sigma)$. One obvious reason is that intersecting $\bigcup \Sigma$ with a smaller set may result in a drop in dimension. Bar that, a subtler possibility is that the dimension of intersections between the elements of $\Sigma$ is not witnessed in full in $X$.

Given our intuition regarding inclusion-exclusion, if $d(\bigcup \Sigma_X) = d(\bigcup \Sigma)$, we should have $\Delta_X(\Sigma_X) \geq \Delta_G(\Sigma)$, as the restrictions to $X$ hold less information than the unrestricted sets in $G$. However, this non-witnessing of intersections may result in $\Delta_X(\Sigma_X)$ being strictly smaller than $\Delta_G(\Sigma)$. Indeed, this would imply that the pregeometry on $X$ is displaying non-flat behaviour. This motivates the next definition.

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In fact, inclusion-exclusion always being correct is equivalent to a disintegrated dimension function
Definition 2.3.1. For pregeometries $H \subseteq G$ write $H \subseteq^* G$ if whenever $X_1, X_2 \subseteq H$ are closed in $H$ then

\[ d_H(X_1 \cap X_2) = d_G(\text{cl}_G(X_1) \cap \text{cl}_G(X_2)). \]

Observation 2.3.2. The relation $\subseteq^*$ is transitive.

Observation 2.3.3. Assume $F \subseteq^* G$ and $F \subseteq H \subseteq G$. Then $F \subseteq^* H$.

Lemma 2.3.4. If $H \subseteq^* G$ and $E_1, \ldots, E_k$ are closed subsets in $H$, then

\[ \text{cl}_G \left( \bigcap_{i=1}^k E_i \right) = \bigcap_{i=1}^k \text{cl}_G(E_i). \]

Proof. We prove by induction on $k$. If $k = 1$, then there is nothing to show. Otherwise, by $H \subseteq^* G$ and induction hypothesis,

\[ d_H \left( E_1 \cap \bigcap_{i=2}^k E_i \right) = d_G \left( \text{cl}_G(E_1) \cap \text{cl}_G \left( \bigcap_{i=2}^k E_i \right) \right) = d_G \left( \bigcap_{i=1}^k \text{cl}_G(E_i) \right). \]

As $\bigcap_{i=1}^k E_i \subseteq \bigcap_{i=1}^k \text{cl}_G(E_i)$, the desired equality is evident. \qed

Corollary 2.3.5. Let $H \subseteq^* G$ and let $\Sigma$ be a finite set of finite dimensional closed subsets of $H$. Then

\[ \Delta_G(\Sigma_G) = \Delta_H(\Sigma) \]

Proof. Denoting $\Sigma = \{E_1, \ldots, E_k\}$ and $\Sigma_G = \{F_1, \ldots, F_k\}$, where $F_i = \text{cl}_G(E_i)$, Lemma 2.3.4 gives $d_H(E_s) = d_G(F_s)$ for every non-empty $s \subseteq [k]$. Therefore, the sum $\Delta_G(\Sigma_G)$ is precisely the sum $\Delta_H(\Sigma)$. \qed

Corollary 2.3.6. If $H \subseteq^* G$ and $G$ is flat, then also $H$ is flat.

Proof. Let $\Sigma$ be a finite collection of finite dimensional closed sets in $H$. By monotonicity of dimension, flatness of $G$, and $H \subseteq^* G$,

\[ d(\bigcup \Sigma) \leq d(\bigcup \Sigma_G) \leq \Delta_G(\Sigma_G) = \Delta_H(\Sigma). \]

The next definition is a significant strengthening of $\subseteq^*$ that will later allow amalgamation of flat pregeometries.

Definition 2.3.7. For pregeometries $H \subseteq G$, write $H \subseteq^* G$ if whenever $\Sigma$ is a finite collection of finite dimensional closed sets in $G$, then

\[ \Delta_H(\Sigma_H) \leq \Delta_G(\Sigma). \]

Say that $H$ is strongly embedded in $G$.

Observation 2.3.8. $H \subseteq G \implies H \subseteq^* G$.

Proof. Let $X_1, X_2$ be closed subsets of $H$ and denote $Y_i = \text{cl}_G(X_i)$. Note that $Y_i \cap H = X_i$. Then by $H \subseteq G$

\[ d(Y_1) + d(Y_2) - d(Y_1 \cap Y_2) \geq d(X_1) + d(X_2) - d(X_1 \cap X_2). \]

By $d(Y_i) = d(X_i)$ we get $d(X_1 \cap X_2) \geq d(Y_1 \cap Y_2)$. The equality is then immediate by $X_1 \cap X_2 \subseteq Y_1 \cap Y_2$. \qed
Corollary 2.3.9. If \( H \subseteq G \) and \( G \) is flat, then also \( H \) is flat.

Observation 2.3.10. \( \subseteq \) is transitive.

Lemma 2.3.11. If \( F \subseteq G \) and \( F \subseteq H \subseteq G \), then \( F \subseteq H \).

Proof. Let \( \Sigma \) be a finite set of finite dimensional closed subsets of \( H \). Observe that for every \( E \in \Sigma \), it holds that \( \text{cl}_G(E) \cap F = E \cap F \). Hence, relativizing \( \Sigma_G \) to \( F \) results in \( \Sigma_F \).

By Corollary 2.3.5, \( \Delta_G(\Sigma_G) = \Delta_H(\Sigma) \). By \( F \subseteq G \), \( \Delta_F(\Sigma_F) \leq \Delta_G(\Sigma_G) \). Therefore, \( \Delta_F(\Sigma_F) \leq \Delta_H(\Sigma) \). We conclude \( F \subseteq H \).

2.4. Hypergraphs. A hypergraph \( \mathcal{A} = (M, R) \) is a set of vertices \( M \) and a set \( R \subseteq [M]^{<\omega} \) of non-empty (hyper)edges. For \( P \subseteq M \), we write \( R[P] \) for \( \{ e \in R | e \subseteq P \} \). We write \( B \subseteq \mathcal{A} \) if \( B = (P, R[P]) \) for some \( P \subseteq M \). We write \( \mathcal{A}[P] \) for the hypergraph \( (P, R[P]) \) induced on \( P \) by \( \mathcal{A} \).

For a finite hypergraph \( \mathcal{A} = (M, R) \) define its predimension

\[
\delta(\mathcal{A}) = |M| - |R|
\]

For a (possibly infinite) hypergraph \( \mathcal{A} = (M, R) \) define its associated dimension function \( d_\mathcal{A} : \text{Fin}(M) \to \mathbb{Z} \cup \{-\infty\} \) by

\[
d_\mathcal{A}(X) = \inf \{ \delta(B) | \mathcal{A}[X] \subseteq B \subseteq_{\text{fin}} \mathcal{A} \}.
\]

If \( d_\mathcal{A} \) is non-negative, then it is the dimension function of a pregeometry \( G_\mathcal{A} \) on \( M \). Call this \( G_\mathcal{A} \) the pregeometry associated to the hypergraph \( \mathcal{A} \). We say that a set \( X \) is closed (independent) in \( \mathcal{A} \) if it is closed (independent) in \( G_\mathcal{A} \). Say that \( \mathcal{A} \) is a representation of a pregeometry \( G \) if \( G_\mathcal{A} = G \). We say that \( \mathcal{A} \) is a good representation of \( G_\mathcal{A} \) if whenever \( e \in R \), then \( d_\mathcal{A}(e) = |e| - 1 \).

For \( \mathcal{A} = (M, R) \) and \( P \subseteq M \), denote \( \delta_\mathcal{A}(P) = \delta(\mathcal{A}[P]) \). For \( L \subseteq M \) such that \( |L \setminus P| < \infty \) denote \( \delta_\mathcal{A}(L/P) = |L \setminus P| - |R[L \cup P] \setminus R[P]| \).

Write \( P \preceq \mathcal{A} \) and say that \( P \) is self-sufficient or strongly embedded in \( \mathcal{A} \), if for every finite \( X \subseteq M \), \( \delta_\mathcal{A}(X/P) \geq 0 \). For \( \mathcal{B} = (N, R[N]) \) write \( \mathcal{B} \preceq \mathcal{A} \) for \( N \preceq \mathcal{A} \). The function \( d_\mathcal{A} \) being non-negative is equivalent to \( \emptyset \preceq \mathcal{A} \). For \( P \) finite, \( P \preceq \mathcal{A} \) is equivalent to \( d_\mathcal{A}(P) = \delta(\mathcal{A}[P]) \).

Fact 2.4.1. Let \( \mathcal{A} = (M, R) \). The following are well known and readily follow from the definitions:

1. The function \( \delta_\mathcal{A} \) is submodular, i.e., \( \delta_\mathcal{A}(X/Y) \leq \delta_\mathcal{A}(X/X \cap Y) \) for all \( X, Y \subseteq M \).
2. \( B \preceq \mathcal{A} \) if and only if \( B \subseteq \mathcal{A} \) and \( d_B \) is the restriction of \( d_\mathcal{A} \) to subsets of \( B \).
3. The relation \( \preceq \) is transitive.
4. If \( X \) is closed in \( \mathcal{A} \), then \( X \subseteq \mathcal{A} \).
5. If \( X, Y \subseteq \mathcal{A} \), then \( X \cap Y \subseteq \mathcal{A} \).
6. If \( X \subseteq \mathcal{A} \) and \( \delta_\mathcal{A}(Y_i/Y_i \cap X) \leq 0 \) for every \( i \leq n \), then \( X \cup \bigcup_{i=1}^n Y_i \subseteq \mathcal{A} \) and the \( Y_i \) are freely joined over \( X \), i.e., \( R[\bigcup_{i=1}^n Y_i \cup X] = R[X] \cup R[X] \).

From (5) of Fact 2.4.1, we get that every subset \( X \) has a self-sufficient closure in \( \mathcal{A} \) given by \( \bigcap \{ Y \supseteq X : X \subseteq \mathcal{A} \} \), which is a non-empty intersection by \( \mathcal{A} \subseteq \mathcal{A} \). If \( X \) is finite and \( d_\mathcal{A} \) is bounded from below, then the self-sufficient closure of \( X \) is also finite. Denote the self-sufficient closure of the set \( X \) in \( \mathcal{A} \) by \( \Lambda_\mathcal{A}(X) \).

Hrushovski showed that the pregeometry associated to a hypergraph is flat. As we need a slightly stronger statement (the additional part of the proposition below),
and as the original proof contains an imprecision\(^3\) despite being morally correct, we bring the proof in full.

**Proposition 2.4.2** ([Hru93, Lemma 15]). Let \( \mathcal{A} = (M, R) \) be a hypergraph with \( \emptyset \subseteq \mathcal{A} \) and associated pregeometry \( G \). Let \( \Sigma = \{ E_1, \ldots, E_k \} \) be a set of finite dimensional closed sets in \( G \). Then

\[
\Delta_G(\Sigma) \geq d_G(E_{\emptyset}).
\]

Additionally, equality holds iff \( E_{\emptyset} \subseteq \mathcal{A} \) and \( R[\bigcup_{i=1}^k E_i] = \bigcup_{i=1}^k R[E_i] \).

**Proof.** Recall that \( E_{\emptyset} = \bigcup_{i=1}^k E_i \) need not be closed. For each \( s \subseteq [k] \), let \( F_s \subseteq E_s \) be a finite set such that \( \text{cl}_G(F_s) \supseteq E_s \) and let \( \mathcal{F} = \bigcup_{s \subseteq [k]} F_s \). Let \( P_{\emptyset} = \Lambda_\mathcal{A}(\mathcal{F}) \), for each \( \emptyset \neq s \subseteq [k] \) let \( P_s = P_{\emptyset} \cap E_s \), and let \( \mathcal{P} = \bigcup_{i=1}^k P_{\{i\}} = P_{\emptyset} \cap E_{\emptyset} \). Then for every \( s \subseteq [k] \)

1. \( P_s \subseteq \mathcal{A} \), as an intersection of self-sufficient sets;
2. \( F_s \subseteq P_s \subseteq E_s \), hence \( \text{cl}_G(P_s) \supseteq E_s \), implying \( \delta_\mathcal{A}(P_s) = d_G(E_s) \) by (1).

We compute, using inclusion-exclusion between the third and fourth lines,

\[
d_G(E_{\emptyset}) - \Delta_G(\Sigma) = \sum_{s \subseteq [k]} (-1)^{|s|} d_G(E_s)
= \sum_{s \subseteq [k]} (-1)^{|s|} \delta_\mathcal{A}(P_s)
= \sum_{s \subseteq [k]} (-1)^{|s|}|P_s| - \sum_{s \subseteq [k]} (-1)^{|s|}|R[P_s]|
= (|P_{\emptyset}| - \bigcup_{i=1}^k P_i) - (|R[P_{\emptyset}]| - \bigcup_{i=1}^k R[P_i])
= |P_{\emptyset} \setminus \mathcal{P}| - |R[P_{\emptyset}] \setminus R[\mathcal{P}]| - |R[\mathcal{P}] \setminus \bigcup_{i=1}^k R[P_i]|
= \delta(P_{\emptyset}/\mathcal{P}) - |R[\mathcal{P}] \setminus \bigcup_{i=1}^k R[P_i]|
\]

Noting that \( d_\mathcal{A}(\mathcal{P}) = d_\mathcal{A}(E_{\emptyset}) = \delta(P_{\emptyset}) \), the first summand is non-positive, proving the main statement.

We prove the additional part by examining each of the two summands, beginning with the second.

We wish to apply (6) of Fact 2.4.1 to see that the sets \( E_i \) are freely joined over \( P_{\emptyset} \subseteq \mathcal{A} \). Although each \( E_i \) may be infinite, it may be presented as the union of a properly increasing chain of finite sets \( P_i \subseteq Y_1^j \subseteq Y_2^j \subseteq \ldots \). Since fact 2.4.1 implies that the sets \( Y_j^j \) are freely joined over \( P_{\emptyset} \) for every \( j \), this is true also for the sets \( E_i \) in their entirety. Thus, \( R[E_{\emptyset} \cup P_0] = R[P_{\emptyset}] \cup \bigcup_{i=1}^k R[E_i] \). Intersecting both sides with \( R[E_{\emptyset}] \), we get \( R[E_{\emptyset}] = R[P] \cup \bigcup_{i=1}^k R[E_i] \). Subtracting \( \bigcup_{i=1}^k R[E_i] \) from both resulting sides gives \( R[E_{\emptyset}] \setminus \bigcup_{i=1}^k R[E_i] = R[P] \setminus \bigcup_{i=1}^k R[P_i] \). Thus, the second summand equals zero if and only if \( R[E_{\emptyset}] = \bigcup_{i=1}^k R[E_i] \).

\(^3\)The last line of the original proof [Hru93] implicitly assumes \( P_{\emptyset} = \mathcal{P} \) (in the original notation, \( \bigcup_i G_i = G_{\emptyset} \)), which need not be true.
The first summand equals zero if and only if $\delta_A(P) = \delta_A(P_0) = d_A(P)$, i.e., if $P \subseteq A$. We claim that this is equivalent to $E_0 \subseteq A$. If $E_0 \subseteq A$, then $P \subseteq A$ as an intersection of self-sufficient sets. If $P \subseteq A$, then by using (6) of Fact 2.4.1 again, this time with $P$ and the sets $E_i$, we get that $E_0 = P \cup \bigcup^k_{i=1} E_i \subseteq A$.

As $d_G(E_0) = \Delta_G(\Sigma)$ if and only if both summands are zero, we are done. □

It is known that every finite flat pregeometry has a good representation [Eva11]. The proof uses Hall’s Marriage Theorem, which is not applicable to infinite pregeometries, and does not allow the control we will later need. Instead, we give a different proof by inductive construction, that applies to any arbitrary flat pregeometry. We will thus receive the characterization

**Theorem 2.4.3.** A pregeometry $G$ has a good representation if and only if $G$ is flat.

We only need to show right to left. We execute the construction by laying down one edge at a time. The following is the key lemma allowing us to see the construction through.

**Lemma 2.4.4.** Let $G$ be a flat pregeometry on the set $M$. Let $A = (M, R)$ be a hypergraph such that $\emptyset \subseteq A$. Assume for some fixed $n$ that

1. $R \subseteq [M]^\leq n+1$  
2. $d_A(e) = |e| - 1$, whenever $e \in R$.  
3. For any $X \subseteq M$, $cl_A(X) \subseteq cl_G(X)$. Moreover, if $d_G(X) < n$, equality holds.

Assume $S \in [M]^n$ is independent in $G$ satisfying

4. If $r \in R \cap [M]^{n+1}$ with $r \not\subseteq cl_G(S)$, then $cl_G(r) = cl_A(r)$.

such that there exists $t \in cl_G(S) \setminus cl_A(S)$. Let $B = (M, R \cup \{St\})$ be the hypergraph obtained by adding the edge $St := S \cup \{t\}$ to $A$. Then assertions (1)-(3) above hold with respect to $B$.

**Proof.** We begin with assertion (3). It will suffice to show that for any finite $X \subseteq M$, $cl_B(X) \subseteq cl_G(X)$. Assume to the contrary there is some $X \subseteq M$ such that there exists $y \in cl_B(X) \setminus cl_G(X)$. Choose $X$ to be of minimal size, hence independent in $B$, so also independent in $A$.

Let $Y_0 = A_B(Xy)$. It must be that $St \subseteq Y_0$, since otherwise $\delta_A(Y_0) = \delta_B(Y_0)$ and $y \in cl_A(X) \subseteq cl_G(X)$. In particular, $St \subseteq Y_0$ implies $Y_0 \subseteq A$. Define

$$\Sigma = \{cl_G(r) \mid r \in R[Y_0] \cup \{St\}\}$$

and, on account of Observation 2.2.2, thin $\Sigma$ out by removing any $Z \in \Sigma$ that is not maximal under inclusion in $\Sigma$. Observe that $cl_G(St)$ remains in $\Sigma$, because it is of dimension $n$, which is maximal in $\Sigma$ by assertion (1). Enumerate $\Sigma = \{E_1, \ldots, E_{k+1}\}$ with $E_{k+1} = cl_G(St)$. For each $1 \leq i \leq k$ fix some $r_i \in R[Y_0]$ such that $E_i = cl_G(r_i)$, and observe that by assertion (4), also $E_i = cl_A(r_i)$.

We would like to have for each non-empty $s \subseteq [k+1]$ the equality

$$cl_G(E_s \cap Y_0) = E_s$$

To achieve this, we will replace $Y_0$ with a bigger set $Y$ such that $\delta_B(Y) = d_B(Xy)$, $R[Y] \subseteq \bigcup^{k+1}_{i=1} R[E_i]$, and the equality above holds for every $s \subseteq [k+1]$.

The equality already holds whenever $|s| = 1$, and increasing $Y_0$ will not change that. For each $s \subseteq [k+1]$ with $|s| \geq 2$ fix some $m_s \in s$ such that $m_s \neq$
We conclude that $d_{\mathcal{A}}(Y/Y_0) = 0$. Moreover, $Y_0 \subseteq \mathcal{A}$ implies that equality holds, and for each $r \in R(Y) \setminus R(Y_0)$ there is some $s$ such that $r \in R[A]$ (using (6) of 2.4.1). In particular, for each such an $r$ there is some $i \leq k$ such that $r \subseteq R[A]$. Denote $F_i = E_i \cap Y$ and let $\Sigma_Y = \{F_1, \ldots, F_{k+1}\}$. We observe that every element of $Y$ appears in some $r \in R[Y] \cup \{St\}$. For elements of $Xy$, this is by assumption on $y$ and minimality of $X$. For $a \in Y \setminus Xy$, since $\delta_{\mathcal{A}}(Y) = d_{G}(Xy) \leq d_{\mathcal{A}}(Y)$ it must be that $\delta_{\mathcal{A}}(Y \setminus \{a\}) \geq \delta_{\mathcal{A}}(Y)$ so $a \in r$ for some $r \in R[Y] \cup \{St\}$. Thus, $F_0 = Y$.

By flatness of $G$, $d_{G}(Y) = d_{G}(E_0) \leq \Delta_{G}(\Sigma)$. By $d_{G}(E_s) = d_{G}(F_s)$ for every $\emptyset \neq s \subseteq [k+1]$, $\Delta_{G}(\Sigma) = \sum_{\emptyset \neq s \subseteq [k+1]} (-1)^{|s|+1} d_{G}(F_s)$.

Denote $\mathcal{Y} = \mathcal{A}[Y]$. Since $Y \subseteq \mathcal{A}$, we have $d_{\mathcal{Y}} = d_{\mathcal{A}} \upharpoonright P(Y)$, as well as $cl_{\mathcal{Y}}(Z) = cl_{\mathcal{A}}(Z) \cap Y$ for any $Z \subseteq Y$. For each $i \leq k$, recall that $E_i = cl_{\mathcal{A}}(r_i)$. For each $s \subseteq [k+1]$ such that $|s| \geq 2$, because $d_{G}(E_s) < n$, by assertion (3) the set $E_s$ is closed in $\mathcal{A}$ and $d_{\mathcal{A}}(E_s) = d_{G}(E_s)$. Therefore, in either case, $F_s$ is closed in $Y$ with $d_{\mathcal{Y}}(F_s) = d_{\mathcal{A}}(F_s) = d_{G}(F_s)$. Then

$$\Delta_{G}(\Sigma) = \left( \sum_{\emptyset \neq s \subseteq [k+1]} (-1)^{|s|+1} d_{\mathcal{Y}}(F_s) \right) + d_{G}(F_{k+1})$$

$$\leq \left( \sum_{\emptyset \neq s \subseteq [k+1]} (-1)^{|s|+1} d_{\mathcal{Y}}(F_s) \right) + \left( d_{\mathcal{A}}(F_{k+1}) - 1 \right)$$

$$= \left( \sum_{\emptyset \neq s \subseteq [k+1]} (-1)^{|s|+1} d_{\mathcal{Y}}(F_s) \right) - 1$$

$$= \Delta_{\mathcal{Y}}(\Sigma_{Y}) - 1$$

For any edge $r \in R$ there is some $i \leq k+1$ such that $r \subseteq F_i$, and $F_0 \subseteq \mathcal{Y}$ because $F_0 = Y$. These two facts, by the additional part of Proposition 2.4.2, imply that $\Delta_{\mathcal{Y}}(\Sigma_{Y}) = d_{\mathcal{Y}}(F_0)$. Hence,

$$\Delta_{G}(\Sigma) \leq d_{\mathcal{Y}}(F_0) - 1 = \Delta_{\mathcal{A}}(Y) - 1 = \left( d_{G}(Y) + 1 \right) - d_{G}(Y)$$

We conclude that $d_{G}(Y) \leq d_{G}(Y) = |X|$, so $y \in cl_{G}(X)$ in contradiction to our assumption.
This proves assertion (3) holds in \( \mathcal{B} \). Assertion (1) is immediate by construction. Lastly, assertion (3) gives us that \( d_G(e) = d_G(e) = d_A(e) = |e| - 1 \) for every \( e \in R \). As \( d_G(St) = n \) by construction, assertion (2) holds in \( \mathcal{B} \) as well.

Now, to prove Theorem 2.4.3 all that we need is an enumeration of all finite tuples in \( G \) that is favourable to applications of Lemma 2.4.4.

**Definition 2.4.5.** Let \( G \) be a flat pregeometry on a set \( M \). Let \((I,<)\) be well ordered. Consider a bijection \( f : I \rightarrow [M]^{<\omega} \). For every \( n \in \mathbb{N} \), denote \( I_n^f := \{ i \in I \mid f(i) \in [M]^n \} \) and for every \( F \), an \( n \)-dimensional closed set in \( G \), denote \( I^F_n := \{ i \in I \mid f(i) \in [F]^{n+1} \} \).

Say that \( f \) is a valid enumeration for \( G \) if the sets of the form \( I_n^f, I^F_n \) are intervals in \( I \) such that \( I_n^f < I_m^f \) whenever \( n < m \).

**Definition 2.4.6** (The Enumerative Construction). Let \( G \) be a flat pregeometry on a set \( M \) and let \( f : I \rightarrow [M]^{<\omega} \) be a valid enumeration for \( G \). We define an inductive construction of a hypergraph, one edge at a time. Let \( A_0 = (M, \emptyset) \). For each \( i \in I \), we construct \( A_i = (M, R_i) \) such that, denoting \( n = |f(i)| - 1 \),

i. \( R_i \subseteq R_j \)

ii. Assertions (1) - (3) of lemma 2.4.4 hold with respect to \( n, A_i \) and \( G \).

iii. \( d_{A_{i+1}}(f(i)) = d_G(f(i)) \).

At stage \( i \), check whether \( \exists_{A_i}(f(i)) = d_G(f(i)) \). If equality holds, we define \( R_{i+1} = R_i \). Otherwise, by assertion (3) it must be that \( f(i) \) is independent in \( A_i \), and so we define \( R_{i+1} = R_i \cup \{ f(i) \} \). For \( i \in I \) a limit in \((I,<)\), define \( R_i = \bigcup \{ R_j : j < i \} \).

It is easy to verify that the conditions above hold, with the second condition given by Lemma 2.4.4. Letting \( R = \bigcup_{i \in I} R_i \), we define \( A_f = (M, R) \), the \( f \)-construction of \( G \).

The next proposition concludes the proof of Theorem 2.4.3:

**Proposition 2.4.7.** For \( G \) a flat pregeometry on a set \( M \) and \( f \) a valid enumeration for \( G \), the hypergraph \( A_f \) is a good representation of \( G \).

**Proof.** Let \( X \subseteq_{\text{fin}} M \). Since \( f \) is surjective, let \( i \in I \) be such that \( f(i) = X \). Then for every \( j > i \), Lemma 2.4.4 guarantees \( d_{A_j}(X) = d_G(X) \), so in particular \( d_{A_f}(X) = d_G(X) \). Conclude that \( G_{A_f} = G \). \( \square \)

**Observation 2.4.8.**

i. Every \( \mathcal{B} = (M, S) \), a good representation of \( G \), can be attained as an enumerative construction — choose \( f \) such that for every closed set \( F \), the set of edges \( S \cap [F]^{|d(F)|+1} \) is an initial segment of \( I^F_n \).

ii. Let \( A_1 = (M, R_1) \) and \( A_2 = (M, R_2) \) be two good representations of \( G \) and let \( F \) be a closed \( n \)-dimensional subset of \( G \). Denoting \( R^F_i = R_i \cap [F]^{n+1} \) for \( i \in \{1, 2\} \), the hypergraph \((M, (R_1 \setminus R_1^F) \cup R_2^F)\) is also a good representation of \( G \). See this by replacing the enumeration of \([F]^{n+1}\) used for the construction of \( A_1 \) with that used for the construction of \( A_2 \).

Our next goal is to attach hypergraph characterizations to the geometric embeddings \( \subseteq^* \) and \( \subseteq \). We do this by carefully choosing the enumeration used in the construction.

**Definition 2.4.9.** Let \( G \) be a flat pregeometry on a set \( M \) and let \( f \) be a valid enumeration for \( G \).
(1) Say that $f$ is a \textit{hydra} if for each $F$ closed in $G$ there exists an independent set $Z_f^I \subseteq [F]^{d(F)}$ such that the set $\{ i \in I \mid f(i) = Z_f^I \cup \{a\}, a \in F \}$ is an initial segment of $I_f^I$.

(2) For a set $P \subseteq M$, say that $f$ is \textit{centered} at $P$ if for every $F$ closed in $G$, the sets $\{ i \in I_f^I \mid f(i) \subseteq P \}$, $\{ i \in I_f^I \mid d_G(f(i) \cap P) = d_G(F \cap P) \}$ are initial segments of $I_f^I$. If $f$ is a hydra, require also that $|Z_f^I \cap P| = d_G(F \cap P)$.

(3) For $P \subseteq M$, let $J = \{ i \in I \mid f(i) \subseteq P \}$, call the function $f|_J$ the restriction of $f$ to $P$.

\textbf{Observation 2.4.10.} Let $G$ be a flat pregeometry defined on a set $M$ and let $H \subseteq G$ be the restriction of $G$ to $P \subseteq M$. Assume $H$ is flat. Then whenever $f : I \to [M]^{<\omega}$ is a valid enumeration for $G$, the restriction of $f$ to $P$ is a valid enumeration for $H$. Moreover, if $f$ is a hydra for $G$ centered at $P$, then the restriction of $f$ to $P$ is a hydra for $H$.

Using a hydra $f$ for the enumeration makes the resulting hypergraph easier to understand. For a closed set $F$, looking only at the edges $e \subseteq F$ with $|e| = d(F) + 1$, we get a (partial) “sun” shape, with the edges connecting elements of $F$ to the basis $Z_f^I$. Those elements who do not have an edge going to them, are already in the closure of the “sun” due to existing edges of lower dimension.

\textbf{Lemma 2.4.11.} Let $G$ be a flat pregeometry on $M$, let $f : I \to [M]^{<\omega}$ be a hydra for $G$. Then for $F$ closed in $G$, if $e$ is an edge in $A_f$ such that $\text{cl}_G(e) = F$, then $Z_f^I \subseteq e$.

In particular, if $f$ is centered at $P \subseteq M$, $d_G(e \cap P) = d_G(F \cap P)$.

\textbf{Proof.} Let $F$ be an $n$-dimensional closed set in $G$ and let $i \in I_f^I$. It will be enough to show that, if $Z_f^I \not\subseteq f(i)$, then when $f(i)$ is examined during the construction of $A_f$, already $d_A(f(i)) \leq n$.

Let $a \in f(i) \setminus Z_f^I$. Since we have examined the set $Z_f^I \cup \{a\}$ in a previous stage, we have $a \in \text{cl}_A(Z_f^I)$. Thus, $f(i) \subseteq \text{cl}_A(Z_f^I)$ and $d_A(f(i)) \leq d_A(Z_f^I) = n$. ~\(\square\)

This subsection culminates in the proof of Corollary 2.4.15, summarizing the analogy between $\subseteq$ and $\leq$. Propositions 2.4.12 and 2.4.14 are the two directions of the proof.

\textbf{Proposition 2.4.12.} Let $A = (M, R)$, $B = (P, R[P])$ be such that $B \leq A$. Then $G_B \subseteq G_A$. In particular, $G_B \subseteq^* G_A$.

\textbf{Proof.} Let $\Sigma$ be a finite collection of finite dimensional closed subsets of $G_A$. Define $R' = \bigsqcup_{E \in \Sigma} R[E]$ and consider the hypergraphs $A' = (M, R')$, $B' = (P, R'[P])$ obtained by removing from $A$ and $B$ all edges not contained in some $E \in \Sigma$. Observe the following easy facts:

(1) Since $R' \subseteq R$, if $Y \leq A$, then also $Y \leq A'$.

(2) For any $E \in \Sigma$ and $Y \leq A[E]$, because $R'[Y] = R[Y]$, we have that $d_A(Y) = d_{A'}(Y)$.

(3) Because $R' = R[\bigcup \Sigma]$, clearly $\bigcup \Sigma \leq A'$.

From applying (1) to $P$, we get $B' \leq A'$, hence $G_{B'} \subseteq G_{A'}$ by (2) of Fact 2.4.1. Applying (2) to $E_s := \bigcap S$, for any non-empty $S \subseteq \Sigma$, we get $d_A(E_s) = d_{A'}(E_s)$ and $d_B(E_s \cap P) = d_{A'}(E_s \cap P) = d_{B'}(E_s \cap P)$, where (2) is again used for the middle equality. Therefore, we see that $\Delta_A(\Sigma) = \Delta_{A'}(\Sigma)$.
and $\Delta_B(\Sigma_P) = \Delta_B(\Sigma_P)$. By the additional part of Proposition 2.4.2, we have $\Delta_A(\Sigma) = d_A(\Sigma)$ and $\Delta_B(\Sigma_P) = d_B(\Sigma_P) = d_A(\Sigma_P)$, where the last equality is by (3). By monotonicity, $d_A(\Sigma_P) \leq d_A(\Sigma)$, so $\Delta_B(\Sigma_P) \leq \Delta_A(\Sigma)$. We conclude that $G_B \subseteq G_A$. □

Lemma 2.4.13. Let $A = (M, R)$. Let $X \subseteq M$, $Y \supseteq X$ be such that $\delta_A(Y) = d_A(X)$, and let $\Gamma = \{ cl_A(r) \mid r \in R[Y] \}$. If $\Sigma \supseteq \Gamma$ is a finite collection of finite dimensional closed subsets of $A$ with $X \subseteq \bigcup \Sigma$ and $d_A(E \cap Y) = d_A(E)$ for every $E \in \Sigma$, then $\Delta_A(\Sigma) = d_A(X)$.

In particular, if $Y = M$ and $\Sigma \supseteq \Gamma$, then $\Delta_A(\Sigma) = d_A(M) = \delta(A)$.

Proof. Denote $B = A[Y]$. By Proposition 2.4.12, $G_B \subseteq^* G_A$. Since by assumption $\Sigma = \{ cl_A(F) \mid F \in \Sigma_Y \}$, Corollary 2.3.5 gives $\Delta_A(\Sigma) = \Delta_B(\Sigma_Y)$.

Every $y \in Y \setminus X$ is involved in some relation $r \in R[Y]$, or otherwise we would have $\delta(B) > d_A(X)$. Thus, $\bigcup \Sigma_Y = Y \subseteq B$, and by definition of $\Gamma$, $R[Y] \subseteq \bigcup_{E \in \Sigma} R[E]$. Then Proposition 2.4.2 guarantees that $\Delta_B(\Sigma_Y) = d_B(Y) = d_A(X)$. □

Proposition 2.4.14. Let $H \subseteq G$ be flat pregeometries on $P \subseteq M$, respectively. Let $f : I \to [M]^{<\omega}$ be a hydra for $G$ centered at $P$. Let $g : J \to [P]^{<\omega}$ be the restriction of $f$ to $P$. Let $A = (M, R)$ be the $f$-construction of $G$ and let $B = (P, S)$ be the $g$-construction of $H$. Then

1. If $H \subseteq G$, then $R[P] \subseteq S$.
2. $H \subseteq G$ if and only if $R[P] = S$ if and only if $B \subseteq A$.

Proof. Assume first that $H \subseteq^* G$. For each $j \in J \subseteq I$, denote by $A_j = (M, R_j)$ and $B_j = (P, S_j)$ the $j$-th stages of the construction of $A$ and $B$, respectively.

Claim. For each stage $j \in J$ of the construction, whenever $Y \subseteq P$ is finite, then $d_{A_j}(Y) \leq d_{B_j}(Y)$.

Proof. We prove by induction. This is clear for $j := \min J$. If $j \in J$ is a limit stage and $d_{A_j}(Y) > d_{B_j}(Y)$ for some $Y \subseteq P$, then there is already some successor stage $j' < j$ in which $R_{j'}[A_{j'}(Y)] = R_j[A_{j'}(Y)]$ and $S_{j'}[B_{j'}(Y)] = S_j[B_{j'}(Y)]$, implying $d_{A_{j'}}(Y) > d_{B_{j'}}(Y)$. Thus, we only need to take care of successor stages.

Assume the claim is true for stage $j \in J$, but not $j + 1$. Let $Y \subseteq P$ be finite such that $d_{A_{j+1}}(Y) > d_{B_{j+1}}(Y)$. By replacing it with its self-sufficient closure in $B_{j+1}$, we may assume that $Y \subseteq B_{j+1}$, and so $Y \subseteq B_j$. Let $e := f(j) = g(j)$, $n := |e| - 1$, and $F = cl_G(e)$. It must be that $e$ is an edge in $B_{j+1}$, but not in $A_{j+1}$, and $e \subseteq Y$.

Let $\Gamma_j$ be the collection of closed sets in $H$ whose tuples we finished enumerating prior to stage $j$ of the construction. Let $\Sigma_H = \{ E \in \Gamma_j \mid d_H(E \cap Y) = d_H(E) \}$. For each $E \in \Sigma_H$, denote $E_G = cl_G(E)$. Then for each $E \in \Sigma_H$, $cl_{B_j}(E) = E$, $cl_{A_j}(E) = E_G$, and $d_{B_j}(E) = d_{A_j}(E_G) = d_G(E_G)$. By $H \subseteq^* G$, we have $d_H(E \cap F) = d_G(E_G \cap F)$. Again by $H \subseteq^* G$ and choice of $\Gamma_j$, using Corollary 2.3.5, we have $\Delta_{B_j}(\Sigma_H) = \Delta_H(\Sigma_H) = \Delta_G(\Sigma_G) = \Delta_{A_j}(\Sigma_G)$.

We’d like to have $d_{B_j}(E \cap F) = d_{A_j}(E_G \cap F)$, for every $E \in \Sigma_H$. We achieve this by increasing $Y$ to a superset $\bar{Y}$. For each $E \in \Sigma_H$, let $K_E \subseteq E$ be finite such that $cl_G(K_E) \supseteq E \cap F$. In particular, $cl_G(K_E) = E_G \cap F$. Let $L_E \subseteq E$ be finite containing $K_E$ such that $\delta_{B_j}(L_E \cap E) \leq 0$. Let $\bar{Y} := Y \cup \bigcup_{E \in \Sigma_H} L_E$. Since $Y \subseteq B_j$ and $\delta_{B_j}(\bar{Y} \setminus Y) \leq 0$, we have $\bar{Y} \subseteq B_j$, and every edge in $S_j[\bar{Y}] \setminus S_j[Y]$ is in $S_j[L_E \cup Y]$ for some $E \in \Sigma_H$. Observe that still $d_{A_{j+1}}(\bar{Y}) \geq d_{A_{j+1}}(Y) > d_{B_{j+1}}(Y) = d_{B_{j+1}}(Y)$.
Let $K$, containing $e$, be a basis for $\bigcup_{E \in \Sigma_H} K_E$ in $B_j$. Observe that since $e$ is not an edge in $A_{j+1}$, the set $K$ is not independent in $A_j$, hence $d_{A_j}(K) < d_{B_j}(K)$.

Let $C = \text{cl}_{A_j}(K)$, $D = \text{cl}_{B_j}(K)$, and note that $C = \text{cl}_{A_j}(D)$. For each $E \in \Sigma_H$, as $K_E \subseteq C, D$, we have $d_{B_j}(D \cap E) = d_{A_j}(C \cap E_G) = d_G(F \cap E_G)$. Since $e \subseteq D$ and $g$ is a hydra, $Z_f \subseteq e \subseteq D$, and so Lemma 2.4.11 gives us that any $e' \in S_j[F]$ with $|e'| = n + 1$, is contained in $D$. Now all conditions of Lemma 2.4.13 hold with respect to $B_j$, $Y$, $\bar{Y}$, and $\Sigma \cup \{D\}$. Thus, $\Delta_{B_j}(\Sigma_H \cup \{D\}) = d_{B_j}(Y) = d_{B_j}(\bar{Y})$. Note that since $Y \subseteq B_j$, the set $\Sigma_H$ is closed under intersections. Thus,

\[ \Delta_{A_j}(\Sigma_G \cup \{C\}) - \Delta_{B_j}(\Sigma_H \cup \{D\}) = d_{A_j}(C) - d_{B_j}(D) < 0. \]

On the other hand, flatness of the pregeometry associated to $A_j$ gives $d_{A_j}(\bar{Y}) \leq \Delta_{A_j}(\Sigma_G \cup \{C\})$. We conclude that $d_{A_j}(\bar{Y}) < d_{B_j}(\bar{Y})$. Since $d_{A_{j+1}}(\bar{Y}) = d_{A_j}(\bar{Y})$ and $d_{B_{j+1}}(\bar{Y}) = d_{B_j}(\bar{Y}) - 1$ this is in contradiction to $d_{A_{j+1}}(\bar{Y}) > d_{B_{j+1}}(\bar{Y})$. □

Part (1) of the main statement is now clear. For every $j \in J$, if $e := f(j) = g(j)$ is an edge in $R[P]$, then $d_{B_j}(e) \geq d_{A_j}(e) = |e|$, so by construction $e \in S_{j+1} \subseteq S$.

We prove part (2). We assume $S \setminus R[P] \neq \emptyset$ and show $H \not\subseteq G$. We may assume $H \supseteq G$, for otherwise this is clear. Let $e \in S \setminus R[P]$ be such that $j := f^{-1}(e)$ is minimal, i.e., $S_j = R_j[P]$. Denote $n := |e| - 1$, let $X = \Lambda_{A_j}(e)$, and observe that $X \cap P \subseteq B_j$. If $X \subseteq P$, then $\delta_{B_j}(X \cap P) = \delta_{A_j}(X) = d_{A_j}(X) = n$, which is not the case, because $e \in S_{j+1}$.

We strip away from $A_j$ and $B_j$ the edges of full dimension in cl$_G(e)$. Denote $R' = \{r \in R_j \mid \text{cl}_{A_j}(r) \neq \text{cl}_G(e)\}$, $S' = R'[P]$ and let $A' = (M, R')$, $B' = (P, S')$. Note that by Lemma 2.4.11 and $f$ being centered at $P$, it must be that $R_j[X] \setminus R' = S_j \setminus S' \subseteq [P]^{n+1}$. Denoting $m = |S_j[X] \setminus S'|$, the number of edges we have removed, we have $d_{A'}(X) = \delta_{A'}(X) = n + m$ and $d_{B'}(X \cap P) = \delta_{B'}(X \cap P) = n + 1 + m$.

By construction, for every $k < n$, $G$ and $A'$ have exactly the same $k$-dimensional subsets. The same goes for $H$ and $B'$. Let $\Sigma_G = \{\text{cl}_G(X_0) \mid X_0 \in [X]^{<n}\}$. Then $\Delta_{B'}(\Sigma_G) = \Delta_{A'}(\Sigma_G)$ as well as $\Delta_{H}(\Sigma_H) = \Delta_{B'}(\Sigma_H)$. Since $X \subseteq A'$ and $R'[X] \subseteq [X]^{\leq n}$, by Lemma 2.4.13 (for the special case $X = Y$, in the notation of the lemma), we have $d_{A'}(X) = \Delta_{A'}(\Sigma_G)$). Similarly, $d_{B'}(X \cap P) = \Delta_{B'}(\Sigma_H)$. Conclude that $\Delta_{G}(\Sigma_G) < \Delta_{H}(\Sigma_H)$ and $H \not\subseteq G$.

If $R[P] = S$, then $B \subseteq A$. The restriction of $d_A = d_G$ to subsets of $P$ is precisely $d_H = d_B$. By (2) of Fact 2.4.1, this means $B \subseteq A$.

Finally, $B \subseteq A$ implies $H \not\subseteq G$ by Proposition 2.4.12. □

**Corollary 2.4.15.** Let $G$ be a flat pregeometry and let $H \subseteq G$. Then $H \subseteq G$ if and only if there exist hypergraphs $B \subseteq A$ such that $G_B = H$ and $G_A = G$. □

### 2.5. The $\alpha$-function.

The $\alpha$-function was defined by Mason [Mas72] in order to characterize the class of flat pregeometries — strict gammoids, in matroid theoretic terminology. However, Mason’s definition of a strict gammoid was distinct from ours, going through linkages in directed graphs. From the point of view of our presentation, the $\alpha$ function is a measure of how much the dimension of a set deviates from the sum of “dimensional data” contained in its subsets of smaller dimension, under the assumption “flat” interaction between these lower dimensional subsets.
Definition 2.5.1. In the context of an ambient pregeometry \( G \), for \( X \subseteq G \), write \( Y \leq X \) to indicate that \( Y \) is a closed set in \( G \) such that \( Y \subseteq X \). Write \( Y < X \) to mean \( Y \preceq X \) and \( Y \neq X \).

Remark 2.5.2. When there is ambiguity with respect to the ambient pregeometry with which \( Y \leq X \) is used, we dispel it like so: \( Y \leq X \subseteq H \).

Definition 2.5.3. Let \( G \) be a pregeometry. For every \( X \subseteq G \) finite, define recursively
\[
\alpha_G(X) = |X| - d(X) - \sum_{Y \prec X} \alpha_G(Y)
\]
In this definition an empty sum is taken to equal zero.

In the flat context, \( \alpha(X) \) is the number of “edges” that must be put on \( X \), on top of edges contained in its closed proper subsets, in order to achieve its dimension.

Proposition 2.5.4. Let \( G \) be a flat pregeometry on \( M \). Let \( \mathcal{A} = (M, R) \) be a representation of \( G \) and let \( F \) be a finite closed subset of \( G \). Then
\[
\alpha_G(F) = |\{ r \in R \mid \text{cl}(r) = F \}|
\]
Proof. By induction on \( d(F) \).
\[
d(F) = 0: \quad F = \text{cl}(\emptyset) \text{ so } F \leq \mathcal{A} \text{ and } 0 = d(F) = \delta(F) = |F| - |R[F]|, \text{ hence } |\text{cl}(F)| = |F|. \quad \text{On the other hand, } \alpha_G(F) = |F| - d(F) - \sum_{X \prec F} \alpha_G(X) = |F|.
\]
\[
d(F) > 0: \quad \text{For any } r \in R[F] \text{ such that cl}(r) \neq F, \text{ the set cl}(r) \text{ is a closed proper subset of } F. \text{ Therefore, by induction hypothesis, the number of edges on } F \text{ whose closure is not } F \text{ is precisely } \sum_{X \prec F} \alpha_G(X). \quad \text{Also, since } F \leq \mathcal{A}, \text{ we have } d(F) = \delta(F) = |F| - |R[F]|, \text{ so } |\text{cl}(F)| = |F| - d(F). \text{ We get that the number of edges in } R[F] \text{ whose closure is } F \text{ is}
\]
\[
|R[F]| - \sum_{X \prec F} \alpha(X) = |F| - d(F) - \sum_{X \prec F} \alpha_G(X) = \alpha_G(F)
\]
\[\square\]

The weakness of the \( \alpha \)-function is that it only sees finite closed sets. In his preprint, Evans [Eva11] explores the connection between flatness, hypergraphs and the \( \alpha \)-function for finite pregeometries. The following characterization of flatness can be found in section 4.

Proposition 2.5.5. A finite pregeometry \( G \) is flat if and only if whenever \( X \) is a union of closed sets, then \( \alpha_G(X) \geq 0 \).

To better understand the \( \Delta \) operation, and for the sake of completeness, we strengthen the key lemma [Eva11, Lemma 4.2] and bring the proof of Proposition 2.5.5 in full.

In light of Proposition 2.5.4, Lemma 2.5.6 is best understood in the setting of \( G \) flat, and holding in mind some good representation of \( G \). In that case, the alternating sum \( \Delta_G(\Sigma) \) is truly an inclusion-exclusion on sets of edges. With that said, the lemma holds also when \( G \) is not flat.

Lemma 2.5.6. Let \( G \) be a finite pregeometry. Let \( \Sigma \) be a collection of closed subsets of \( G \) such that if \( Y \leq X \in \Sigma \), then \( Y \in \Sigma \). Then
\[
\Delta_G(\Sigma) = d(G) - \sum_{X \subseteq G \setminus \bigcup \Sigma} \alpha_G(X) - |G \setminus \bigcup \Sigma|
\]
Proof. We construct \( \Sigma \) inductively and show that the equation holds with respect to each intermediate stage. Denote \( F = \text{cl}(\emptyset) \) and let \( \Sigma_0 = \{ F \} \). Observe \( \Delta_G(\Sigma_0) = 0 \) and recall \( \alpha_G(F) = |F| \). Then

\[
\sum_{X \subseteq G} \sum_{X \neq F} \alpha_G(X) - |G \setminus F| = \sum_{X \subseteq G} \alpha_G(X) - |G| = -d(G) = \Delta_G(\Sigma_0) - d(G)
\]

where the second equality is by definition of \( \alpha_G(G) \).

Assume now that we have constructed \( \Sigma_i \subseteq \Sigma \), downwards-closed with respect to \( \subseteq \), such that the statement holds for every \( \Gamma \subseteq \Sigma_i \). Choose some \( X \in \Sigma \setminus \Sigma_i \) such that if \( X' \subseteq X \), then already \( X' \in \Sigma_i \). Let \( \Sigma_{i+1} = \Sigma_i \cup \{ X \} \).

Assume first that \( d(X) = 1 \). Then \( X \) intersects \( \bigcup \Sigma_i \), and every element of \( \Sigma_i \), in \( F \). In particular, \( d(\bigcap S \cap X) = 0 \) for every \( \emptyset \neq S \subseteq \Sigma \). By definition, \( d(X) = |X| - \alpha_G(X) - \alpha_G(F) = |X \setminus F| - \alpha_G(X) \). We compute

\[
\Delta_G(\Sigma_{i+1}) - d(G) = \Delta_G(\Sigma_i) + d(X) - d(G)
\]

\[
= \left( \sum_{Y \subseteq G} \alpha_G(Y) - |G \setminus \bigcup \Sigma_i| \right) - (|X \setminus F| - \alpha_G(X))
\]

\[
= \sum_{Y \subseteq G} \sum_{Y \not\subseteq \Sigma_i} \sum_{Y \subseteq G} \alpha_G(Y) - |G \setminus \bigcup \Sigma_{i+1}|
\]

Now assume \( d(X) > 1 \). Let \( \Sigma_X = \{ E \cap X \mid E \in \Sigma_i \} = \{ E \mid E \preceq X \} \). Note that \( \bigcup \Sigma_X = X \subseteq \bigcup \Sigma_i \). Denote \( A = \bigcup \Sigma_i \cup \Sigma_{i+1} \). By assumption,

\[
\Delta_G(\Sigma_i) - d(G) = \sum_{Y \subseteq G} \sum_{Y \not\subseteq \Sigma_i} \alpha_G(Y) - |G \setminus A| = \sum_{Y \subseteq G} \alpha_G(Y) + \alpha_G(X) - |G \setminus A|
\]

and

\[
(*) \quad \Delta_G(\Sigma_X) = \left( \sum_{Y \subseteq G} \sum_{Y \not\subseteq \Sigma_X} \alpha_G(Y) + d(G) \right) - |G \setminus X|
\]

\[
= \left( |G| - \sum_{Y \subseteq G} \alpha_G(Y) \right) - |G \setminus X|
\]

\[
= |X| - \sum_{Y \subseteq X} \alpha_G(Y) = \alpha_G(X) + d(X)
\]

where the first equality is by induction hypothesis, the second by definition of \( \alpha_G(G) \), and the fourth by definition of \( \alpha_G(X) \). Thus,

\[
\Delta_G(\Sigma_{i+1}) - d(G) = (\Delta_G(\Sigma_i) + d(X) - \Delta_G(\Sigma_X)) - d(G)
\]

\[
= (\Delta_G(\Sigma_i) - d(G)) - (\Delta_G(\Sigma_X) - d(X))
\]

\[
= \sum_{Y \subseteq G} \alpha_G(Y) - |G \setminus A|
\]
Now that Lemma 2.5.6 is proved, we may apply the equality (*) whenever \( \Sigma = \{ E \mid E \triangleleft X \} \), for some arbitrary \( X \).

**Corollary 2.5.7** ([Eva11], Lemma 4.2). Let \( X \) be a union of closed sets in a finite pregeometry \( G \) and let \( \Sigma = \{ E \mid E \triangleleft X \} \). Then \( \alpha_G(X) = \Delta_G(\Sigma) - d(X) \).

**Proof of Proposition 2.5.5.** Let \( \Sigma \) be some collection of closed sets, denote \( X = \bigcup \Sigma \). Assume \( X \notin \Sigma \), for otherwise clearly \( \Delta_G(\Sigma) = d(X) \). By Corollary 2.2.5, increasing \( \Sigma \) to \( \{ E \mid E \triangleleft X \} \) does not restrict generality, as it only decreases \( \Delta_G(\Sigma) \). Then \( \alpha_G(X) \geq 0 \) if and only if \( \Delta_G(\Sigma) \geq d(\bigcup \Sigma) \).

\[ \square \]

2.6. **Geometric prerank.** We now define a notion of **prerank**, which will be to flat pregeometries what \( \delta \) is to hypergraphs. Much in the same way that the **predimension** \( \delta \) approximates dimension (Morley rank, in Hrushovski’s non-collapsed construction), our prerank \( \rho \) will be closely related to Morley rank\(^4\) and quantifier elimination in the soon-to-come generic construction.

**Definition 2.6.1.** Define \( \mathfrak{O} \) to be the free \( \mathbb{Z} \)-module generated by \( \{ \omega^i : i < \omega + 1 \} \) and endowed with the order where \( \sum_{i < \omega + 1} a_i \omega^i < \sum_{i < \omega + 1} b_i \omega^i \) if and only if \( a_i < b_i \), where \( j = \max\{i : a_i \neq b_i \} \). That is, the reverse-lexicographical order, i.e.,

\[
3\omega^5 + 7\omega^5 < 4\omega^5 + 6\omega^5 < 4\omega^5 + \omega^5.
\]

When all coefficients of \( \alpha, \beta \in \mathfrak{O} \) are non-negative, addition in \( \mathfrak{O} \) is precisely the **natural sum** (or **Hessenberg sum**).

**Definition 2.6.2.** For every finite pregeometry \( G \) assign \( \rho(G) \in \mathfrak{O} \) by

\[
\rho(G) = d(G)\omega^\omega + \sum_{X \subseteq G} \alpha_G(X)\omega^{d(X)}
\]

For \( H \sqsubseteq^* G \) pregeometries with \( H \) finite, write \( H \leq_r G \) if \( \rho(H') \geq \rho(H) \) for every finite intermediate \( H \subseteq H' \subseteq^* G \).

Our goal now is to show that, in the flat context, \( \sqsubseteq \) and \( \leq_r \) are equivalent. The following two lemmas lead up to Corollary 2.6.5, the left-to-right implication, and Proposition 2.6.6 is the right-to-left implication.

**Lemma 2.6.3.** If \( H \sqsubseteq G \) is the induced pregeometry on a closed subset of \( G \), then \( \alpha_H \) is the restriction of \( \alpha_G \) to finite subsets of \( H \).

**Proof.** Observe that for any \( F \subseteq H \), \( F \) is closed in \( H \) if and only if it is closed in \( G \). We prove inductively. Let \( X \subseteq H \) be such that for every \( F \triangleleft X \) we have \( \alpha_H(F) = \alpha_G(F) \).

Then

\[
\alpha_G(X) = |X| - d(X) - \sum_{Y \triangleleft X} \alpha_G(Y) = |X| - d(X) - \sum_{Y \triangleleft X} \alpha_H(Y) = \alpha_H(X)
\]

\[ \square \]

**Lemma 2.6.4.** Let \( H \sqsubseteq G \) be finite pregeometries and let \( X \subseteq H \). Then \( \alpha_H(X) \leq \alpha_G(\text{cl}_G(X)) \).

\(^4\)A full analysis of Morley rank is not included in this text. Morley rank is “shifted” with respect to \( \rho \), namely \( \omega^4 \cdot \rho \), but we find our definition of \( \rho \) more convenient to work with in the context of this paper. See Digression immediately after the proof of Lemma 3.3.11 for an explanation.
Proposition 2.6.6. If \( \Sigma = \{ E \mid E \in Y \subseteq G \} \). Observe that whenever \( E \in \Sigma \), then \( d(E) < d(Y) = d(X) \), so \( E \cap X \subseteq X \subseteq H \). Hence, \( \Sigma_H = \{ F \mid F \subseteq X \subseteq H \} \). By \( H \subseteq G \), we have \( \Delta_H(\Sigma_H) \leq \Delta_G(\Sigma) \). By Lemma 2.6.3 and Corollary 2.5.7,

\[
\begin{align*}
\alpha_G(Y) &= \alpha_Y(Y) = \Delta_Y(\Sigma) - d(Y) = \Delta_G(\Sigma) - d(Y) \\
\alpha_H(X) &= \alpha_X(X) = \Delta_X(\Sigma_H) - d(X) = \Delta_H(\Sigma_H) - d(X)
\end{align*}
\]

Thus, as \( d(X) = d(Y) \), we get \( \alpha_H(X) \leq \alpha_G(Y) \).

\[\Box\]

**Corollary 2.6.5.** If \( G \) is a flat pregeometry and \( H \subseteq G \) is finite, then \( H \leq G \).

**Proof.** Let \( H \subseteq H' \subseteq G \) be finite. Then \( H' \) is flat and, by Lemma 2.3.11, \( H \subseteq H' \).

By Proposition 2.5.5 Flatness gives that whenever \( Y \subseteq H' \), then \( \alpha_H(Y) \geq 0 \). Then, using Lemma 2.6.4,

\[
\rho(H) = d(H)\omega^\omega + \sum_{X \subseteq H} \alpha_H(X)\omega^{d(X)}
\]

\[
\leq d(H')\omega^\omega + \sum_{X \subseteq H} \alpha_H'(\text{cl}_H'(X))\omega^{d(X)}
\]

\[
\leq d(H')\omega^\omega + \sum_{Y \subseteq H'} \alpha_H'(Y)\omega^{d(Y)} = \rho(H')
\]

\[\Box\]

**Proposition 2.6.6.** If \( G \) is a flat pregeometry and \( H \subseteq G \) is finite such that \( H \leq_G G \), then \( H \subseteq G \).

**Proof.** It will be enough to show that \( H \subseteq G_0 \) for some \( H \subseteq G_0 \subseteq G \), so we may assume there is no such \( G_0 \) distinct from \( G \). In particular, \( G \) is finite and \( d(H) = d(G) \).

Let \( P, M, R, S, f, g, A, B \) be as in the statement of Proposition 2.4.14, and recall \( R[P] \subseteq S \). Let \( R' = R \setminus R[P] \), \( S' = S \setminus R[P] \), \( A' = (M, R') \), \( B' = (P, S') \), \( G' = G_A' \), \( H' = G_{B'} \). Note that \( H' \subseteq G' \).

Let \( \Sigma = \{ \text{cl}_H(r) \mid r \in S' \} \cup \{ \text{cl}_H(a) \mid a \in P \} \) and observe that every set that is closed in \( H \) is also closed in \( H' \). Note that \( \delta_A(X/X \cap P) = \delta_{A'}(X/X \cap P) \) for every \( X \subseteq M \), hence \( \Sigma_G = \Sigma_{G'} \). For every \( X \) closed in \( G \), it holds that

\[
d_{G'}(X) - d_{H'}(X \cap P) = \delta_{A'}(X) - \delta_{B'}(X \cap P)
\]

so \( \Delta_{G'}(\Sigma_G) - \Delta_{H'}(\Sigma) = \Delta_G(\Sigma_G) - \Delta_H(\Sigma) \). By \( H \leq_G G \), we have \( \Delta_H(\Sigma) = \Delta_G(\Sigma_G) \), hence \( \Delta_{H'}(\Sigma) = \Delta_{G'}(\Sigma_G) \).

By applying the additional part of Proposition 2.4.2 in \( B' \), we get \( \Delta_{H'}(\Sigma) = d_{H'}(P) = d_{G'}(P) \). Denote \( E = \bigcup \Sigma_G \). By flatness, \( d_G'(E) \leq \Delta_{G'}(\Sigma_G) = d_{G'}(P) \).

Since \( P \subseteq E \), in fact \( d_G'(E) = d_{G'}(P) \), so \( d_G'(\bigcup \Sigma_G) = \Delta_{G'}(\Sigma_G) \). Using the additional part of Proposition 2.4.2, this time in the other direction, we get \( E \leq A' \) and \( R'[E] = \bigcup_{F \in \Sigma_G} R'[F] \). By construction, since \( P \subseteq E \), we have \( E \leq A \). By the minimality assumption on \( G \), this means \( E = M \).

Assuming \( S' \neq \emptyset \), denote \( n := \max\{|e| : e \in S'\} \).

**Claim.** \( R' \subseteq \{ M \}^{<n} \).
Corollary 2.6.7. For a pregeometry $G$ and a finite $H \subseteq G$, the following are equivalent:

1. $H \subseteq G$
2. $H \subseteq^\star G$ and $H \leq^r G$
(3) There exist good representations $A$ and $B$ of $G$ and $H$, respectively, such that $B \leq A$.

3. Generic flat pregeometries

3.1. Construction of $G$. In the context of a class of relational structures $D$ and $\leq_D$ — a transitive, invariant-under-isomorphism notion of distinguished substructure between elements of $D$ — the following special case of Fraïssé’s Theorem gives a method of constructing a generic structure for a subclass $C$.

**Theorem 3.1.1.** Let $C \subseteq D$ be a countable (up to isomorphism) class of finite structures, closed under isomorphisms and taking $\leq_D$-substructures. Assume

1. $\emptyset \leq_D A$, for every $A \in C$
2. For every $A, B_1, B_2 \in C$ with embeddings $f_i : A \to B_i$ such that $f_i[A] \leq_D B_i$, there exists $D \in C$ and $g_i : B_i \to D$ such that $g_i[B_i] \leq_D D$ and $g_1 f_1 = g_2 f_2$.

Then there exists a unique (up to isomorphism) countable structure $M$ such that $C = \{ A \leq_D M : |A| < \infty \}$ and

(*) Whenever $A \leq_D M, A \leq_D B \in C$, then there is an embedding $f : B \to M$ fixing $A$ such that $f[B] \leq_D M$.

Call $M$ the generic structure for $C$.

**Remark 3.1.2.** If $M$ is generic for $C$, a standard back and forth argument shows that any isomorphism between finite $\leq_D$-embedded substructures extends to an automorphism of $M$.

The procedure with which Hrushovski’s non-collapsed construction is attained is an application of the theorem to a class of hypergraphs. From the properties in Fact 2.4.1, it is not hard to show that the conditions of Theorem 3.1.1 hold.

**Definition 3.1.3.** For every $n \in \mathbb{N}$, define $C_n$ to be the class of finite hypergraphs $A = (M, R)$ such that $\emptyset \leq A$ and $R \subseteq [M]^{\leq n}$. For $n = \omega$, define $C_\omega = \bigcup_{n \in \mathbb{N}} C_n$. Denote by $M_n$ the generic structure for $C_n$.

When this causes no confusion, we omit the subscript and write by convention $C, M$ for $C_\omega, M_\omega$, respectively.

Model theoretically, the structures $M_n$ are saturated, $\omega$-stable, and almost model complete (have quantifier elimination up to boolean combinations of existential formulas). We will similarly construct generic flat pregeometries $G_n$, sharing similar traits, and demonstrate that in fact $G_n = G_{M_n}$. Since the procedure goes through regardless of arity, we do the work with unbounded arity. We geometrically define and address bounded arities in a later subsection.

**Definition 3.1.4.** Define $\mathcal{C}$ to be the class of all finite flat pregeometries.

By Proposition 2.4.3, in fact $\mathcal{C} = \{ G_A | A \in C \}$. By Corollary 2.6.7, it is clear that $G$ is flat if and only if $\emptyset \sqsubseteq G$. In order to apply Theorem 3.1.1, we only need to show amalgamation. We will go through hypergraphs to do this.

**Definition 3.1.5.** Let $B_1 = (M_1, R_1), B_2 = (M_2, R_2)$ be hypergraphs such that $\emptyset \leq B_1, B_2$. Denote $M_0 = M_1 \cap M_2$ and assume $M_0 \leq B_1$. Define the hypergraph

$$B_1 \sqcup B_2 = (M_1 \cup M_2, R_2 \cup (R_1 \setminus R_1[M_0]))$$
While this is not necessarily an amalgam of hypergraphs, if both $B_1$ and $B_2$ induce the same pregeometry on their intersection, $G_{B_1 \cup B_2}$ will be an amalgam of pregeometries. Before stating the definition of the amalgam for pregeometries in Definition 3.1.9, we first show it is well-defined in Corollary 3.1.7, and capitalize on that to get a short useful result in Corollary 3.1.8. The following Lemma demonstrates that the amalgam is a “free” amalgam.

**Lemma 3.1.6.** In the notation of Definition 3.1.5 above, assume that $G_{B_1[M_0]} \subseteq G_{B_2}$. That is, $G_{B_1}$ and $G_{B_2}$ restrict to the same pregeometry on $M_0$. Then for any $X \subseteq M_1 \cup M_2$ closed in $\mathcal{D} := B_1 \cup B_2$ of finite dimension,

$$d_\mathcal{D}(X) = d_{B_1}(X \cap M_2) + d_{B_2}(X \cap M_1) - d_{B_1}(X \cap M_0)$$

**Proof.** Assume for a moment that $\mathcal{D}$ is finite. Since $X$ is closed in $\mathcal{D}$, $d_\mathcal{D}(X) = \delta_\mathcal{D}(X)$. By construction, $X \cap M_2$ is clearly closed in $B_2$, so

$$d_{B_2}(X \cap M_2) = \delta_{B_2}(X \cap M_2) = \delta_\mathcal{D}(X \cap M_2).$$

Also, $X \cap M_1$ is closed in $B_1$ and since $M_0 \subseteq B_1 \cup B_2$, the set $X \cap M_0$ is closed in $B_1[M_0]$. So

$$\delta_\mathcal{D}(X \cap M_1/X \cap M_0) = \delta_{B_1}(X \cap M_1/X \cap M_0)$$

$$= \delta_{B_1}(X \cap M_1) - \delta_{B_1}(X \cap M_0)$$

$$= d_{B_1}(X \cap M_1) - d_{B_1}(X \cap M_0)$$

By construction, $\delta_\mathcal{D}(X/X \cap M_2) = \delta_\mathcal{D}(X \cap M_1/X \cap M_0)$. Since, by definition, $\delta_\mathcal{D}(X) = \delta_\mathcal{D}(X \cap M_2) + \delta_\mathcal{D}(X/X \cap M_2)$, we are done.

Now, if $\mathcal{D}$ is infinite, reduce to a self-sufficient subgraph of $\mathcal{D}$ containing bases for $X$, $X \cap M_1$, $X \cap M_2$, and $X \cap M_0$ for the argument to go through. \( \square \)

**Corollary 3.1.7.** In the notation of Definition 3.1.5, assuming $G_{B_1[M_0]} \subseteq G_{B_2}$, the pregeometry $G := G_{B_1 \cup B_2}$ does not depend on the structure of $B_1$ and $B_2$, but only on $G_{B_1}$ and $G_{B_2}$.

**Proof.** Denote $G := G_{\mathcal{D}}$. Let $B_1'$, $B_2'$ be such that $M_0 \subseteq B_i'$ and $G_{B_i'} = G_{B_i}$ for $i = 1, 2$. Denote $G' := G_{B_1' \cup B_2'}$. Let $X \subseteq M_1 \cup M_2$ be closed in $G'$. Then as before $X \cap M_i$ is closed in $B_i'$ for $i = 1, 2$, $X \cap M_0$ is closed in $B_1'[M_0]$, and so

$$d_{G'}(X) = d_{B_1'}(X \cap M_2) + d_{B_2'}(X \cap M_1) - d_{B_1'}(X \cap M_0)$$

$$= d_{B_2}(X \cap M_2) + d_{B_1}(X \cap M_1) - d_{B_1}(X \cap M_0)$$

$$= \delta_{B_1}(X \cap M_1) + \delta_{B_2}(X \cap M_2) + \delta_{B_1}(X \cap M_1/X \cap M_0)$$

$$= \delta_\mathcal{D}(X) = d_\mathcal{D}(X) = d_G(X)$$

By symmetry of the argument, taking $Y = c_G(X)$, we have $d_{G'}(Y) \leq d_{G'}(Y) = d_G(Y)$. By definition, $d_G(X) \leq d_{G'}(Y)$ so we get that $d_{G'}(Y) = d_G(X)$, i.e., $Y = X$. Since $G$ and $G'$ have the exact same closed sets, $G = G'$. \( \square \)

**Corollary 3.1.8.** If $H \subseteq G$ are flat and $B$ is a representation of $H$, then there exists $A$ a representation of $G$ such that $B \subseteq A$.

**Proof.** Choose some representations $B' \subseteq A'$ of $H \subseteq G$. Note $A' = A' \cup B'$. Let $A = A' \cup B$, clearly $B \subseteq A$. Because $G_B = G_{B'}$, we also have $G_{A' \cup B'} = G_{A' \cup B}$, so $G_A = G_{A'B'} = G$. \( \square \)
We can now rigorously define the geometric amalgam and show that it indeed (strongly) extends the component pregeometries.

**Definition 3.1.9.** For flat pregeometries \( H, G_1, G_2 \) such that \( H \sqsubseteq G_1, H \sqsubseteq G_2 \) and \( H = G_1 \cap G_2 \). We define \( G_1 \Pi_H G_2 \), the amalgam of \( G_1 \) and \( G_2 \) over \( H \), to be the pregeometry associated to \( B_1 \Pi B_2 \), where \( B_i \) is a good representation of \( G_i \) and \( B_1[H] \sqsubseteq B_1 \).

**Lemma 3.1.10.** In the notation of Definition 3.1.9, letting \( G := G_1 \Pi_H G_2 \),

i. \( G_2 \not\sqsubseteq G \)

ii. \( G_1 \not\subseteq G \)

iii. Whenever \( X \) is a closed set in \( G_i \), then \( \text{cl}_G(X) = X \cup \text{cl}_{G_{3-i}}(X \cap H) \)

iv. If \( H \not\sqsubseteq G_2 \), then \( G_1 \not\subseteq G \)

v. If \( H \not\subseteq G_2 \), then \( G_1 \not\sqsubseteq G \)

**Proof.** Let \( B_1 = (M_1, R_1) \), \( B_2 = (M_2, R_2) \) be good representations of \( G_1, G_2 \) such that \( M_0 := M_1 \cap M_2 \not\subseteq B_1 \), and let \( D := B_1 \Pi B_2 \).

i. As \( B_2 \not\subseteq D \), it is clear that \( G_2 \not\subseteq G \).

ii. Let \( X \subseteq M_1 \). Letting \( Y = \text{cl}_G(X) \), from Lemma 3.1.6 we have that \( d_G(Y) \geq d_{B_1}(Y \cap M_1) \geq d_{B_1}(X) \). So \( d_G(X) \geq d_{B_1}(X) \). Now let \( Z_1 = \text{cl}_{B_1}(X) \), \( Z_2 = \text{cl}_{B_2}(Z_1 \cap M_0) \), and \( Z = Z_1 \cup Z_2 \). Observe that \( Z_2 \cap M_0 = Z_1 \cap M_0 \). Then

\[
\delta_D(Z) = \delta_{B_1}(Z_2) + \delta_{B_1}(Z_1/Z_1 \cap M_0) = d_{B_2}(Z_1 \cap M_0) + d_{B_1}(Z_1) - d_{B_1}(Z_1 \cap M_0) = d_{B_1}(Z_1).
\]

So \( d_G(X) \leq \delta_D(Z) = d_{B_1}(Z_1) = d_{B_1}(X) \). Conclude \( d_G(X) = d_{B_1}(X) \).

iii. Fix \( i \in \{1, 2\} \). Let \( X \) be closed in \( G_i \) and let \( Y = \text{cl}_G(X) \). By Lemma 3.1.6,

\[
d(Y) = d(Y \cap M_i) + d(Y \cap M_{3-i}) - d(Y \cap M_0).
\]

Since \( d(Y) = d(X) = d(Y \cap M_i) \), we get that \( d(Y \cap M_{3-i}) = d(Y \cap M_0) \), i.e., \( Y \cap M_{3-i} = \text{cl}_{G_{3-i}}(Y \cap M_0) \). As \( Y \cap M_0 = X \cap M_0 \), we have

\[
Y = (Y \cap M_i) \cup (Y \cap M_{3-i}) = X \cup \text{cl}_{G_{3-i}}(X \cap M_0).
\]

iv. Assume \( H \not\subseteq G_2 \). Let \( X_1, X_2 \) be closed in \( G_1 \), and denote \( Y_i = \text{cl}_G(X_i) \). Then by the previous item,

\[
Y_1 \cap Y_2 = (M_1 \cap Y_1 \cap Y_2) \cup (M_2 \cap Y_1 \cap Y_2)
= (X_1 \cap X_2) \cup (\text{cl}_{G_2}(X_1 \cap M_0) \cap \text{cl}_{G_2}(X_2 \cap M_0))
= (X_1 \cap X_2) \cup (\text{cl}_{G_2}(X_1 \cap X_2 \cap M_0)) = \text{cl}_G(X_1 \cap X_2)
\]

where going from the second line to the third is by \( H \not\subseteq G_2 \). Then \( d(X_1 \cap X_2) = d(Y_1 \cap Y_2) \) and \( G_1 \not\subseteq G \).

v. Assume \( H \not\subseteq G_2 \). By Corollary 3.1.8 we may assume that \( M_0 \not\subseteq B_2 \) and \( B_1[M_0] = B_2[M_0] \). So by construction \( B_1 \not\subseteq D \) and \( G_1 \not\subseteq G \).

**Corollary 3.1.11.** Let \( H \not\subseteq G_1, G_2 \) be flat pregeometries with \( H = G_1 \cap G_2 \). Then there exists a flat pregeometry \( G \) such that \( G_1, G_2 \not\subseteq G \). Moreover, \( G \) is defined on the union of the sets on which \( G_1, G_2 \) are defined.

We have proven that \( \mathcal{C} \) is an amalgamation class. We denote by \( \mathcal{G} \) the countable generic structure guaranteed by Theorem 3.1.1. We dub \( \mathcal{G} \) the *generic flat pregeometry of unbounded arity.*
Now that we have constructed $\mathcal{G}$ independently, we show that it is in fact the pregeometry of Hrushovski’s non-collapsed construction for hypergraphs of unbounded arity.

**Proposition 3.1.12.** $G_M \cong \mathcal{G}$.

*Proof.* Since $\mathcal{M}$ is a hypergraph, $G_M$ is flat, and so whenever $H \subseteq G_M$, also $H$ is flat. Thus, $\{H \varsubsetneq G_M : |H| < \infty\} \subseteq \mathcal{C}$. Conversely, if $H \in \mathcal{C}$, then for $A$ a good representation of $H$, $A \in \mathcal{C}$. Without loss of generality, we may assume $A \subseteq M$ and so $G_A \subseteq G_M$.

We show that (⋆) of Theorem 3.1.1 holds with respect to $G_M$, $\mathcal{C}$ and $\subseteq$. Assume $F \subseteq G_M$ and $F \varsubsetneq H \in \mathcal{C}$. Let $B = \Lambda_M(X_F)$ where $X_F$ is the underlying set of $F$. Denote $K = G_{M[B]}$. Then $K \subseteq G_M$ and by Lemma 2.3.11, also $F \varsubsetneq K$. By renaming elements of $H$, we may assume $H \cap K = F$. Let $L = K \sqcup_F H$. Then $K \subseteq L$ and so by Corollary 3.1.8 choose some $D$ a good representation of $L$ such that $M[B] \subseteq D$. By genericity of $\mathcal{M}$, we may strongly embed $D$ into $\mathcal{M}$ over $\mathcal{M}[B]$, so without loss of generality $D \subseteq \mathcal{M}$. Now $L \subseteq G_M$ and $F \varsubsetneq H \subseteq L$. Hence, we have strongly embedded $H$ into $G_M$ over $F$. \hfill \qed

### 3.2. Model theory of $\mathcal{G}$

We now examine pregeometries as first order objects. Fix the language $\mathcal{L} = \{I_n : n \in \omega\}$ where $I_n$ is an $n$-ary relation symbol. We consider a pregeometry $G$ as an $\mathcal{L}$-structure by interpreting $I_n$ as the set of independent $n$-tuples in $G$.

**Observation 3.2.1.** Let $G$ be a pregeometry. Then

1. If $X$ is definable in $G$ and $n \in \mathbb{N}$, then “$d(X) \geq n$” is an $\mathcal{L}$-formula. Hence “$d(X) = n$” is an $\mathcal{L}$-formula.
2. For points $a_1, \ldots, a_n$, the set $\text{cl}(a_1 \ldots a_n)$ is definable.
3. Using the first two items, for fixed $m, d \in \mathbb{N}$, we can quantify over $m$ closed sets of dimension at most $d$, and speak of the dimension of their intersections and unions.
4. For each $n$, there is an $\mathcal{L}$-formula $\varphi_n(x_1, \ldots, x_n)$ stating $\{x_1, \ldots, x_n\} \sqsubseteq^* G$.

The class of pregeometries/matroids is an elementary class of $\mathcal{L}$-structures. From the definition of flatness (Definition 2.2.3), we see that the class of flat pregeometries is also an elementary class, given by an infinite scheme of axioms.

We set out to axiomatize the theory of $\mathcal{G}$. The axiomatization is similar to Hrushovski’s first order axiomatization of his construction [Hru93]. The genericity is expressed by a scheme of axioms that, paraphrased to invoke the definition of continuity, state that to achieve an embedding of an extension $H$ over the base $F$ that is at most $\epsilon$ away from being strong, the base $F$ needs to be at most $\delta$ away from being strong in the ambient structure, where $\delta$ depends only on $\epsilon$ and $|H|$. This is (T3) of Definition 3.2.4.

**Definition 3.2.2.** Write $X \sqsubseteq^n G$ to mean that $X \sqsubseteq^* G$ and whenever $Y \subseteq G$ contains $X$ such that $Y \sqsubseteq^* G$ and $|Y \setminus X| \leq n$, then $X \subseteq Y$.

**Observation 3.2.3.** In an ambient structure $G$, “$X \sqsubseteq^n G$” is a first order formula in $|X|$ many variables, for $X$ finite of a fixed size.

Observe further that $X \sqsubseteq^n G$ holds for arbitrarily large $n$ if and only if $X \subseteq G$. From left to right, see this equivalence by choosing some finite $Y \varsubsetneq G$ containing $X$ and using that $X \sqsubseteq^n G$, where $n \geq |Y \setminus X|$, to get $X \subseteq Y \subseteq G$. From right to left, the implication is immediate by Lemma 2.3.11.
Definition 3.2.4. For a fixed $\tau: \mathbb{N} \to \mathbb{N}$ with $\tau(n) \geq n$, let $T_\tau$ be the $L$-theory stating (in an ambient structure $G$):

(T1) $G$ is a flat pregeometry.

(T2) $G$ is infinite-dimensional.

(T3) Suppose $F \subseteq^* G$, $F \subseteq H \in \mathcal{C}$. Then for every natural $n$, if $F \subseteq^\tau(|H|+n) G$, then there exists an embedding $f: H \to G$ fixing $F$ such that $f[H] \subseteq^n G$.

The definition of $T_\tau$ a priori depends on the choice of $\tau$. We will see that if $T_\tau$ is at all satisfiable, then it implies a complete theory independent of the choice of $\tau$, namely the theory of $G$. We prove a series of lemmas, geometric analogues of hypergraph trivialities, to show that a good $\tau$ exists.

We observe that the operation of amalgamating from the left with a fixed pregeometry preserves the $\subseteq^*$ and $\subseteq$ relations.

Observation 3.2.5. Let $F \subseteq G_1$, $F \subseteq H \subseteq G_2$ be pregeometries such that $G_1, G_2$ are flat and $F = G_1 \cap G_2$. Then $G_1 \Pi_F G_2 \subseteq^* G_1 \Pi_F G_2$. Furthermore, if $H \subseteq G_2$, then $G_1 \Pi_F H \subseteq G_1 \Pi_F G_2$.

Proof. Since $H \subseteq^* G_2$ and $F \subseteq G_1$, both $F$ and $H$ are flat. Let $\mathcal{B}_1 \subseteq \mathcal{A}_1$ be hypergraphs representing $F \subseteq G_1$ and let $\mathcal{D}, \mathcal{A}_2$ be hypergraphs representing $H, G_2$. Note that $\mathcal{D} \subseteq \mathcal{A}_1 \Pi \mathcal{D}$ and observe that, by definition, $\mathcal{A}_1 \Pi \mathcal{A}_2 = (\mathcal{A}_1 \Pi \mathcal{D}) \Pi \mathcal{A}_2$. Thus, $G_1 \Pi_F G_2 = (G_1 \Pi_F H) \Pi H G_2$. Lemma 3.1.10 finishes the proof. \qed

Corollary 3.2.7 and its preceding lemma describe how $\subseteq^*/\subseteq$-embededness of a set $K$ in an amalgam reflects on its intersection with each component, given that $K$ contains the base of the amalgam.

Lemma 3.2.6. Let $G = G_1 \Pi_F G_2$ be an amalgam of flat pregeometries. Let $K \subseteq G$ contain $F$ and, for $i \in \{1, 2\}$, denote $K_i = K \cap G_i$. Then whenever $\Sigma$ is a finite collection of finite dimensional closed sets in $G_2$, denoting $\Sigma_K = \{\text{cl}_G(E) \cap K | E \in \Sigma\}$, the equality $\Delta_K(\Sigma_K) = \Delta_{K_2}(\Sigma_{K_2})$ holds.

Furthermore, if $F \subseteq^* G_2$, whenever $\Sigma$ is a finite collection of finite dimensional closed sets in $G_1$, then $\Delta_K(\Sigma_K) = \Delta_{K_1}(\Sigma_{K_1})$.

Proof. Enumerate $\Sigma = \{E_1, \ldots, E_k\}$ and recall the notation $E_s = \bigcap_{i \in s} E_i$ for $\emptyset \neq s \subseteq [k]$. By $G_2 \subseteq^* G$, we have $\bigcap_{i \in s} \text{cl}_G(E_i) = \text{cl}_G(E_s)$. By the third item of Lemma 3.1.10, we have

$$\text{cl}_G(E_s) \cap K = (\text{cl}_{G_1}(E_s \cap F) \cup E_s) \cap (K_1 \cup K_2)$$

$$= \text{cl}_{K_1}(E_s \cap F) \cup (E_s \cap K_2)$$

$$\subseteq \text{cl}_K(E_s \cap K_2).$$

Therefore, $d(\bigcap_{i \in s} \text{cl}_G(E_i) \cap K) = d(E_s \cap K_2)$ for every $\emptyset \neq s \subseteq [k]$, proving $\Delta_K(\Sigma_K) = \Delta_{K_2}(\Sigma_{K_2})$.

For the additional part, by the fourth item of Lemma 3.1.10, $F \subseteq^* G_2$ implies $G_1 \subseteq^* G$, which enables a symmetric argument. \qed

Corollary 3.2.7. Let $G := G_1 \Pi_F G_2$ be an amalgam of flat pregeometries and let $F \subseteq K \subseteq G$. Then

(1) if $K \subseteq^* G$, then $K_2 \subseteq^* G_2$. Moreover, if $K \subseteq G$ then $K_2 \subseteq G_2$.

(2) Assuming $F \subseteq^* G_2$, if $K \subseteq^* G$, then $K_1 \subseteq^* G_1$. Moreover, under the same assumption, if $K \subseteq G$ then $K_1 \subseteq G_1$. 

3.2.6, both parts of (1) are immediate. Item (2) is the same, by the additional part of Lemma 3.2.6.

The next two lemmas are an analogue to the fact that in an ambient hypergraph \( A \), if \( D \subseteq B \subseteq A \) and \( D \not\subseteq C \subseteq A \), then \( D \not\subseteq B \cap C \).

**Lemma 3.2.8.** There exists a fixed function \( g : \mathbb{N} \to \mathbb{N} \) such that if \( G \) is flat and \( F \subseteq G \), then there is some \( H \subseteq^* G \) containing \( F \) with \( |H| \leq g(|F|) \).

**Proof.** Assume the contrary. Let \( n \) be such that for every \( k \in \mathbb{N} \) there exists \( G_k \) flat with some \( F \in |G_k|^n \) such that whenever \( H \subseteq^* G_k \) contains \( F \), then \( |H| > k \). Adding to \( L \) constant symbols \( c_1, \ldots, c_n \), consider the theory stating (see Observation 3.2.1) that the ambient structure \( G \) is a flat pregeometry and that for every natural \( k \)

\[
\forall x_1, \ldots, x_k \{ c_1, \ldots, c_n, x_1, \ldots, x_k \} \not\subseteq^* G.
\]

Then this theory is finitely satisfiable and has a model \( G \). Since \( G \) is flat, there is a hypergraph \( A \) representing it. Let \( X = \{ c^G_1, \ldots, c^G_n \} \). Then \( \Lambda_A(X) \subseteq G \), so in particular \( \Lambda_A(X) \subseteq^* G \). But \( \Lambda_A(X) \) is finite, a contradiction. \( \square \)

**Lemma 3.2.9.** There exists a fixed function \( h \) such that if \( G \) is a flat pregeometry and

\[
(1) \ X \subseteq Y \subseteq G \\
(2) \ X \not\subseteq Z \subseteq^* G
\]

then there is some \( V \subseteq^* Y \) containing \( X \) such that \( X \not\subseteq V \) and \( |V| \leq h(|Z|) \).

**Proof.** Fix \( g \) as in the statement of Lemma 3.2.8. For each natural number \( n \), set \( h(n) = \max \{ g(k) \mid k \leq 2^n n \} \).

Let \( \Sigma_Z \) witness that \( X \not\subseteq Z \) and denote \( \Sigma = \{ \text{cl}_G(E) \mid E \in \Sigma_Z \} \). Since \( Z \not\subseteq^* G \), we have \( \Delta_G(\Sigma_Z) = \Delta_G(\Sigma) \). By \( Y \subseteq G \), we know \( \Delta_Y(\Sigma_Y) \leq \Delta_G(\Sigma) \). By choice of \( \Sigma, \Delta_G(\Sigma_Z) = \Delta_X(\Sigma) \). Conclude \( \Delta_Y(\Sigma_Y) < \Delta_X(\Sigma_X) \), so \( \Sigma_Y \) witnesses that \( X \not\subseteq Y \).

There are at most \( 2^{|Z|} \) elements in \( \Sigma \) and each is of dimension at most \( d(Z) \), so there is some \( V' \subseteq Y \) of size at most \( 2^{|Z|} d(Z) \) such that for every \( E \in \Sigma \),

\[
d(V' \cap E) = d(Y \cap E).
\]

Now, take a minimal \( V \subseteq^* Y \) containing \( V' \). Then \( |V| \leq g(|V'|) \leq h(|Z|) \) and \( \Sigma_Y \) witnesses \( X \not\subseteq V \). \( \square \)

**Remark 3.2.10.** In Lemma 3.2.8 and Lemma 3.2.9, there is no harm in assuming the functions \( g \) and \( h \) are non-decreasing.

We now have the components required for the proof.

**Proposition 3.2.11.** There exists \( \tau : \mathbb{N} \to \mathbb{N} \) such that \( G \models T_\tau \). In particular, \( T_\tau \) is satisfiable.

**Proof.** We only need to address T3.

Fix some \( n \geq |H| \). Let \( F \subseteq^* G, F \subseteq H \in \mathcal{C} \) and let \( E \subseteq G \) contain \( F \). By genericity of \( G \), we may assume \( H \Pi_F E \) is strongly embedded into \( G \) over \( E \).

Assume now that \( H \not\subseteq^* G \). Let \( Z \subseteq^* G \), \( |Z \setminus H| \leq n \) contain \( H \) such that \( H \not\subseteq Z \). By Lemma 3.2.9, there is some \( V \subseteq^* H \Pi_F E \) containing \( H \) of size at most \( h(|Z|) \leq h(|H| + n) \) such that \( H \not\subseteq V \). By (1) of Corollary 3.2.7, we have \( V_0 := V \cap E \subseteq^* E \). Then by Observation 3.2.5, \( V = H \Pi_F V_0 \). If \( F \subseteq V_0 \), the last
item of Lemma 3.1.10 implies $H \subseteq V$, which is not the case. So $F \nsubseteq V_0$, where $V_0 \subseteq G$ with $|V_0| \leq h(|H| + n)$.

Conclude that for choosing $\tau$ greater or equal to the $h$ of Lemma 3.2.9, $G \models T_\tau$. □

**Notation 3.2.12.** From now on, fix $T := T_\tau$ for some $\tau$ such that $T_\tau$ is satisfiable (not necessarily the $\tau$ of the above lemma).

We will show that $G$ is a saturated model for $T$. In the case of Hrushovski’s construction, it is easy to show that $\mathcal{M}$ is saturated, since $\mathcal{M}$ is isomorphic to each of its elementary extensions. This is not the case for $G$. We instead use the weaker property stated in Proposition 3.2.14. The proposition is proved by constructing an increasing chain, with each step constructed using the next lemma.

**Lemma 3.2.13.** Let $L \models T$, $F \subseteq L$, $F \subseteq H \in \mathcal{C}$. Then there exists some elementary extension $L \preceq L'$ and $f : H \rightarrow L'$ fixing $F$ such that $f[H] \subseteq L'$.

**Proof.** Observe that although the language $\mathcal{L}$ is infinite, the atomic type of a finite pregeometry is given by a single finite formula. Denote by $F(\bar{x})$ the atomic type of $F$. Denote by $H(\bar{x}\bar{y})$ the atomic type of $H$, where the induced structure on $\bar{x}$ is that of $F$, and the elements of $\bar{y}$ realize over $\bar{x}$ the atomic type of $H$ over $F$. Denote by "$\bar{x} \sqsubseteq_n G$" a first order formula in variables $\bar{x}$ stating that $\bar{x}$ is $\subseteq^n$-embedded in the ambient structure. Denote by "$\bar{x} \sqsubseteq G$" the partial type $\{\bar{x} \sqsubseteq_n G \mid n \in \omega\}$.

Observe that by the axiom scheme T3, for any fixed natural $n$

$$T \cup \{F(\bar{x})\} \cup \"\bar{x} \sqsubseteq G\" \models \exists \bar{y}(H(\bar{x}\bar{y}) \land \"\bar{x}\bar{y} \sqsubseteq G\")$$

Whenever $i > j$, $\exists \bar{y}(H(\bar{x}\bar{y}) \land \"\bar{x}\bar{y} \sqsubseteq^i G\") \implies \exists \bar{y}(H(\bar{x}\bar{y}) \land \"\bar{x}\bar{y} \sqsubseteq^j G\")$. Consequently, if $G \models T \cup \{F(\bar{a})\} \cup \"\bar{a} \sqsubseteq G\",$ then the type $\{H(\bar{a}\bar{y})\} \cup \"\bar{a}\bar{y} \sqsubseteq G\"$ over $\bar{a}$ is finitely satisfiable. Since $L \models T$ and $F \subseteq L$, we may realize $H(F\bar{y}) \land \"F\bar{y} \sqsubseteq G\"$ in some elementary extension $L \preceq L'$. This finishes the proof. □

**Proposition 3.2.14.** Whenever $L \models T$ is countable, there exists an elementary extension $L \preceq G$ that is generic for $\mathcal{C}$. In particular $G \equiv \mathcal{G}$.

**Proof.** We construct an elementary chain, similarly to a Fraïssé construction, but starting from a model and realizing types instead of amalgamating.

Let $M_0 = L$. Assume $M_i$ countable, $F_i \subseteq M_i$ and $F \subseteq H_i \in \mathcal{C}$ are given. Use Lemma 3.2.13 to get a countable elementary extension $M_i \preceq M_{i+1}$ into which $H$ can be strongly embedded over $F$. Choose an enumeration so that for every $i < \omega$, every $A \subseteq M_i$ and $A \subseteq B \in \mathcal{C}$, the pair $(A,B)$ is chosen as $(F_i, H_i)$ infinitely often.

Let $G = \bigcup_{i<\omega} M_i$. Then $L \preceq G$. Observe that being $\sqsubseteq$-embedded in a model is a first order property (type) preserved by elementary extension, so by construction $G$ is clearly generic for $\mathcal{C}$. □

Before proceeding with the proof of saturation, we note an immediate corollary of Proposition 3.2.14.

**Corollary 3.2.15.** $T \equiv Th(\mathcal{G})$. In particular, $T = T_\tau$ is a complete theory independent of the specific choice of $\tau$. □

**Theorem 3.2.16.** $\mathcal{G}$ is saturated.
Proof. Let \( p(x) \) be a complete type over a finite set \( F \subseteq G \). By increasing \( F \), we may assume \( F \subseteq G \). Let \( L \) be an elementary extension of \( G \) in which \( p(x) \) is realized, say by \( a \in L \). Let \( B \subseteq L \) contain \( F \cup \{a\} \). Since \( F \subseteq L \), in particular \( F \subseteq B \). By genericity of \( G \), we may strongly embed \( B \) into \( G \) over \( F \), call the image of this embedding \( H \subseteq G \).

By Proposition 3.2.14, let \( G \) be an elementary extension of \( L \) such that \( G \cong G \). Then \( H, B \subseteq G \) and \( H, B \) are isomorphic over \( F \). As in Remark 3.1.2, there exists an automorphism \( f \) of \( G \) extending the isomorphism between \( B \) and \( H \). In particular, \( tp^G(f(a)/F) = tp^G(f(f(a))/F) = tp^G(a/F) = tp^L(a/F) = p(x) \). So the arbitrary type \( p(x) \) is realized in \( G \), hence \( G \) is saturated. \( \square \)

Now that we have saturation of \( G \), we can show that \( T \) is \( \omega \)-stable and has quantifier elimination up to a set of formulas, reminiscent of the case of \( Th(M) \). We lead with \( \omega \)-stability. The next lemma shows that the type of a strongly embedded set in a model of \( T \) is determined by its atomic diagram.

**Lemma 3.2.17.** Let \( H, F \subseteq G \) (possibly infinite) be such that \( f : H \to F \) is an isomorphism of pregeometries. Then, seen as \( \omega \)-tuples, \( tp^G(H) = tp^G(F) \).

**Proof.** Let \( X \subseteq H \) be arbitrary. It is enough to show that \( tp^G(X) = tp^G(f(X)) \). Let \( Y \subseteq H \subseteq G \) be finite containing \( X \). Then \( f[Y] \subseteq f[H] = F \), and since \( F \subseteq G \), also \( f[Y] \subseteq G \). Then the restriction of \( f \) to elements of \( Y \) is an isomorphism between \( \subseteq \)-embedded finite substructures of \( G \). As in Remark 3.1.2, there is an automorphism of \( G \) taking \( Y \) to \( f[Y] \), hence they have the same type. Since \( X \subseteq Y \), also \( tp^G(X) = tp^G(f(X)) \). \( \square \)

**Theorem 3.2.18.** \( T \) is \( \omega \)-stable.

**Proof.** Identify the underlying set of \( G \) and the underlying set of \( M \), so that \( M \) is a representation of \( G \), and call that set \( M \). Since \( G \) is saturated, it will be enough to show that \( S^G_1(M) \) is countable. We do this by injectively mapping \( S^G_1(M) \) to pairs \((H, a)\) where \( H \) is an isomorphism type of an element of \( C \) and \( a \in H \).

Let \( p(x) \in S^G_1(M) \). By Proposition 3.2.14 let \( G \cong G \) be generic for \( C \) such that \( G \models p(a) \) for some \( a \in G \). Observe that since \( G \) is elementarily embedded in \( G \), any finite set strongly embedded in \( G \) is also strongly embedded in \( G \), i.e., \( G \subseteq G \). By Corollary 3.1.8, let \( \mathcal{N} = (N, S) \) represent \( G \) such that \( M \subseteq \mathcal{N} \).

Assume first that \( a \in cl_G(M) \). Then there is some finite independent \( X \subseteq M \) such that \( a \in cl_X(X) \). Let \( Y = \Lambda_X(X \cup \{a\}) \) and let \( H \) be the pregeometry induced on \( Y \) by \( G \). Then because \( X \subseteq Y \cap M \), it must be that \( \delta_X(Y/M) \leq 0 \), and since \( M \subseteq \mathcal{N} \), we have \( \delta_X(Y/M) = 0 \), \( M \cup Y \subseteq \mathcal{N} \), and \( \mathcal{N}[M \cup Y] = \mathcal{M} \upharpoonright \mathcal{N}[Y] \). So \( G_{\mathcal{N}[M \cup Y]} = G \upharpoonright \mathcal{N}[H] \). Define \( \Theta(p) = (H, a) \).

Observe that by Lemma 3.2.17, the type \( p \) is recoverable from \( \Theta(p) \) — it is the type of the image of \( a \) over the image of \( M \) in \( a \)-embedded copy of \( G \upharpoonright \mathcal{N}[H] \) in \( G \), which by Lemma 3.2.17 is unique. So \( \Theta \) is injective on types of \( S^G_1(M) \) whose realizations depend on \( M \). But, if \( a \notin cl_G(M) \), then \( M \cup \{a\} \subseteq \mathcal{N} \) and \( G_{\mathcal{N}[M \cup \{a\}]} = G \upharpoonright \mathcal{N}[a] \). Thus, again by Lemma 3.2.17, there is a unique type in \( S^G_1(M) \) whose realizations are independent of \( M \). All in all, \( S^G_1(M) \) is countable. \( \square \)

In order to address quantifier elimination for \( T \), we only need to be able to speak of finite \( \subseteq \)-extensions.
For each finite pregeometry $H$, letting $\bar{h}$ be an enumeration of the elements of $H$ as an ordered tuple, let $\Phi_{H}(\bar{x})$ be the full\textsuperscript{5} atomic type of $\bar{h}$. Let $\Phi_{H}^{*}(\bar{x})$ be the formula stating additionally that $\bar{x} \subseteq^* G$ in the ambient structure $G$, namely: “$\Phi_{H}$ holds, and whenever $X_{1}, X_{2} \subseteq \bar{x}$, denoting $n := d(\text{cl}_{H}(X_{1}) \cap \text{cl}_{H}(X_{2}))+1$, every $y_{1}, \ldots, y_{n} \in \text{cl}_{G}(X_{1}) \cap \text{cl}_{G}(X_{2})$ are dependent”. Since $X_{1}$ is a subtuple of $\bar{x}$, the set $\text{cl}_{G}(X_{1})$ is definable by a quantifier free formula, and since $\bar{x}$ is isomorphic to $H$, the set $\text{cl}_{H}(X_{1})$ is known. Thus, the formula $\Phi_{H}^{*}$ is a conjunction of universal formulas, i.e., universal.

**Definition 3.2.19.** Define $\mathcal{L}_{EX}$ to be the language $\mathcal{L}$ enriched by a predicate symbol for each formula of the form $\exists \bar{y} \Phi_{H}^{*}(\bar{x}\bar{y})$, where $H \in \mathcal{C}$.

We interpret a pregeometry as an $\mathcal{L}_{EX}$ structure in the obvious way, implicitly assuming that every $\mathcal{L}_{EX}$-theory forces the “correct” interpretation.

**Proposition 3.2.20.** $T$ has quantifier elimination in the language $\mathcal{L}_{EX}$.

**Proof.** Since $\mathcal{G}$ is saturated, also its definable expansion to the language $\mathcal{L}_{EX}$ is saturated. Therefore, it is enough to show that the quantifier free $\mathcal{L}_{EX}$-type of a finite tuple $\bar{a} \in \mathcal{G}$ implies the full type of $\bar{a}$.

Let $\bar{a}, \bar{b} \subseteq \mathcal{G}$ be finite such that they have the same quantifier free $\mathcal{L}_{EX}$-type. Let $A \subseteq^* \mathcal{G}$ be an extension of $\bar{a}$ such that $\rho(A)$ is minimal. In particular, $A \preceq_{r} \mathcal{G}$, so by Corollary 2.6.7, $A \subseteq \mathcal{G}$. Since $\mathcal{G} \models \exists \bar{y} \Phi_{A}^{*}(\bar{a}, \bar{y})$, also $\mathcal{G} \models \exists \bar{y} \Phi_{A}^{*}(\bar{b}, \bar{y})$. Let $B \subseteq^* \mathcal{G}$ witness this. Clearly, $\rho(B)$ has to also be minimal among $\subseteq^*$-embedded extension of $\bar{b}$, so also $B \subseteq \mathcal{G}$. As $A$ and $B$ are finite, isomorphic, and strongly embedded in $\mathcal{G}$, by genericity there is an automorphism of $\mathcal{G}$ extending the isomorphism between $A$ and $B$. In particular, $\text{tp}^{\mathcal{G}}(\bar{a}) = \text{tp}^{\mathcal{G}}(\bar{b})$.

So in the language $\mathcal{L}$, the theory $T$ has quantifier elimination up to boolean combinations of $\mathcal{L}_{EX}$ quantifier free formulas, which in particular are $\exists \forall \mathcal{L}$-formulas. The $\mathcal{L}$-theory $T$ is not model complete in general, but it is with respect to $\subseteq$-embeddings.

**Lemma 3.2.21.** For each $H \in \mathcal{C}$ there exists a unique type $p_{H} \in S(T)$ such that $G \models p_{H}(\bar{a})$ if and only if $\bar{a} \subseteq G$ and $\bar{a} \cong H$.

**Proof.** The existence of such a type is clear. Take the type of some strongly embedded copy of $H$ in $\mathcal{G}$. To see that this type is unique, use saturation of $\mathcal{G}$ and apply Lemma 3.2.17.

For the explicit definition of the type, recall T3 of the definition of $T$ and observe that for each $A \in \mathcal{C}$, we have $G \models \exists \bar{y} \Phi_{A}^{*}(\bar{a}, \bar{y})$ precisely when $\bar{a} \subseteq A$. By quantifier elimination, this gives the full type of $\bar{a}$.

**Theorem 3.2.22.** If $G_{1} \subseteq G_{2}$ with $G_{1}, G_{2} \models T$, then $G_{1} \preceq G_{2}$.

**Proof.** Let $F \subseteq G_{1}$. Since we need to show $\text{tp}^{G_{1}}(F) = \text{tp}^{G_{2}}(F)$, there is no harm in increasing $F$, so assume $F \subseteq G_{1}$. By transitivity, also $F \subseteq G_{2}$. By Lemma 3.2.21 above, $\text{tp}^{G_{1}}(F) = p_{F} \neq \text{tp}^{G_{2}}(F)$. 

\footnote{While technically the atomic type is not finite, it is isolated by its restriction to the finite sublanguage $\mathcal{L}^{\leq |X|} = \{ t_{n} \mid n \leq |X| \}$.
3.3. **Geometric arity.** Evans and Ferreira [EF11,EF12] showed that when bounding the arity of the hypergraphs in the amalgamation class $\mathcal{C}$, different associated pregeometries arise. To be precise, whenever $k > n$ are non-negative, the pregeometries $G_{M_k}$ and $G_{M_n}$ are not isomorphic, even up to localization in a finite set. The argument hinges on the existence of a self-sufficient edge of maximal arity. We will show that this is the only difference, in the sense that $G_{M_k}$ is a (geometrically) homogeneous elementary extension of $G_{M_n}$ realizing the non-isolated type of such an edge.

We define “arity” of a flat pregeometry as a purely geometric notion, and show in Proposition 3.3.5 that the definition indeed coincides with a definition by the arity of hypergraph representations.

**Definition 3.3.1.** Let $n \in \mathbb{N}$ be non-negative. Say that a flat pregeometry $G$ is of **arity at most** $n$ and write $a(G) \leq n$ if whenever $H \subseteq G$ is finite, then $\Delta_H(\Sigma) = d(H)$, where $\Sigma$ is the collection of closed sets in $H$ of dimension less than $n$. Define $a(G)$, the **arity** of $G$, to be the least $n \in \mathbb{N} \cup \{\omega\}$ such that $a(G) \leq n$.

The meaning of $G$ being of arity at most $n$ is that the pregeometry is completely determined by independence of $n$-tuples, or in other words, by its reduct to the language $\mathcal{L}_n := \{I_k \mid k \leq n\} \subseteq \mathcal{L}$. This reflects in the automorphism group of the pregeometry.

**Proposition 3.3.2.** If $G$ is flat with $a(G) \leq n$, then a bijection $f : G \to G$ is an automorphism of $G$ if and only if $d(X) = d(f[X])$ whenever $X \subseteq [G]^{\leq n}$.

**Proof.** Assume the right hand side. So $f$ preserves the dimension and closedness of every closed set in $G$ of dimension less than $n$. Then for every $\Sigma$, finite collection of closed subsets of dimension less than $n$, we have $\Delta_G(\Sigma) = \Delta_G(f(\Sigma))$, where $f(\Sigma) = \{f[E] \mid E \in \Sigma\}$. Since $a(G) \leq n$, the dimension of any finite set is given by $\Delta_G(\Sigma)$ for such a $\Sigma$, hence $f$ preserves dimension of all tuples. The left to right implication holds by definition. \(\square\)

**Observation 3.3.3.** Let $G$ be a flat pregeometry with $a(G) \leq n$. If $H \subseteq G$, then $a(H) \leq n$. However, $H \subseteq^* G$ does not imply any finite bound on $a(H)$.

**Definition 3.3.4.** We say that a pregeometry $H$ is a **circuit** if $H$ is not independent, but every proper subset of $H$ is independent.

**Proposition 3.3.5.** For $G$ a flat pregeometry. The following are equivalent:

1. $a(G) \leq n$
2. If $H \subseteq G$ is such that every $X \in [H]^{\leq n}$ is independent, then $H$ is independent
3. If $H \subseteq G$ is a circuit, then $[H]^{\leq n}$ is independent
4. Whenever $A = (M,R)$ is a good representation of $G$, then $R \subseteq [M]^{\leq n}$
5. There exists $A = (M,R)$ a representation of $G$ with $R \subseteq [M]^{\leq n}$

**Proof.** (1) $\Rightarrow$ (2): First assume $H$ is finite. Then, letting $\Sigma$ be the collection of all closed sets in $H$ of dimension less than $n$, in fact $\Sigma = [H]^{<n}$. Thus, the alternating sum $\Delta_H(\Sigma)$ is a true inclusion-exclusion, which by $a(G) \leq n$ results in $d(H) = \Delta_H(\Sigma) = |\bigcup \Sigma| = |H|$. Since an infinite set is independent if and only if each of its finite subsets is independent, (2) also holds for $H$ of an arbitrary size.
Fact 2.4.1, \( \bigcup \) of Proposition 2.4.2, \( \Delta \) the additional part of Proposition 2.4.2 to \( A \) Corollary 3.3.6. Use (4) of Proposition 3.3.5 in constructing the hypergraph proof.

By (5) of Proposition 3.3.5, a(\( A \)) is a circuit, and \( \delta_A(e) = |e| - 1 = d_A(e) \), so \( e \subseteq A \) and \( e \subseteq G \). Thus, \( |e| \leq n \).

(4) \( \Rightarrow \) (1): Let \( H \subseteq G \). By Corollary 2.6.7, let \( A = (M, R) \) be a good representation of \( G \), such that \( X \subseteq A \), where \( X \) is the underlying set of \( H \). Let \( \Sigma \) be as in Definition 3.3.1. Note \( \Delta_H(\Sigma) = \Delta_G(\Sigma_G) \), because \( H \subseteq^* G \). Now apply Lemma 2.4.13 to \( \Sigma_G \), to get \( \Delta_A(\Sigma_G) = d_A(X) = d(H) \).

(4) \( \Rightarrow \) (5): Immediate.

(5) \( \Rightarrow \) (4): Let \( A = (M, R) \) be a representation of \( G \) with \( R \subseteq [M]\leq^n \) and let \( A' = (M, R') \) be some good representation of \( G \). Let \( e \in R' \) be arbitrary and let \( X = \Lambda_A(e) \) and \( \Sigma = \{ cl_G(X_0) \mid X_0 \in [X]^<n \} \). Since \( X \subseteq A \), we have \( \delta_A(cl_G(X_0)) = d_G(X_0) = \delta_A(cl_G(X_0) \cap X) \) for every \( X_0 \subseteq X \). Then by (6) of Fact 2.4.1, \( \bigcup \Sigma \subseteq A \) and \( R[\bigcup \Sigma] = \bigcup_{E \in \Sigma} R[E] \). Therefore, by the additional part of Proposition 2.4.2, \( \Delta_G(\Sigma) = d_G(\bigcup \Sigma) \). Now with the equality known, applying the additional part of Proposition 2.4.2 to \( A' \) yields that \( R'[\bigcup \Sigma] = \bigcup_{E \in \Sigma} R'[E] \subseteq [M]\leq^n \). In particular, \( e \in E \) for some \( E \) with \( d_G(E) < n \). Since \( A' \) is a good representation, this means \( |e| \leq n \).

Towards defining the \( n \)-ary generic pregeometry, we observe that taking an amalgam does not increase arity.

**Corollary 3.3.6.** For flat pregeometries \( H, G_1, G_2 \) such that \( G_1 \Pi_H G_2 \) is well-defined, \( a(G_1 \Pi_H G_2) \leq \max\{a(G_1), a(G_2)\} \).

**Proof.** Use (4) of Proposition 3.3.5 in constructing the hypergraph \( B_1 \Pi B_2 \) of which \( G_1 \Pi_H G_2 \) is the associated pregeometry.

**Definition 3.3.7.** Define \( \mathcal{C}_n \) to be the class of finite flat pregeometries of arity at most \( n \). Equivalently, by (5) of Proposition 3.3.5, \( \mathcal{C}_n = \{ G_A \mid A \in \mathcal{C}_n \} \).

Corollary 3.3.6 gives that \( \mathcal{C}_n \) is an amalgamation class. Theorem 3.1.1 thus guarantees a unique countable generic structure for \( \mathcal{C}_n \), which we denote \( G_n \) and call the generic flat \( n \)-ary pregeometry.

**Proposition 3.3.8.** \( G_{M_n} \cong G_n \)

**Proof.** By (5) of Proposition 3.3.5, \( a(G_{M_n}) \leq n \), so whenever \( H \subseteq G_{M_n} \) is finite, \( a(H) \leq n \), i.e., \( H \in \mathcal{C}_n \). Now, letting \( H \in \mathcal{C}_n \) be arbitrary, there is some \( A \in \mathcal{C}_n \) with \( H = G_A \), so by genericity of \( M_n \) we may assume \( A \subseteq M_n \), hence \( H = G_A \subseteq G_{M_n} \). From both inclusions conclude that \( \{ H \subseteq G_{M_n} \mid |H| < \infty \} = \mathcal{C}_n \).

The rest is exactly the same as in the proof of \( \mathcal{G} \cong G_M \) (Proposition 3.1.12), keeping Corollary 3.3.6 in mind.

In order to show \( G_n \models T \), we will go through some technical lemmas showing that for every \( k \) and \( n \geq 3 \), every flat pregeometry \( G \) of arbitrary arity is \( \subseteq^k \)-embedded in some pregeometry \( H \) of arity at most \( n \).

**Definition 3.3.9.** Let \( F \) be a flat pregeometry. We say that \( H \in \mathcal{C}_n \) is an \( n \)-resolution of \( F \) if \( F \subseteq^* H \) and there is no \( F \subseteq F' \subseteq H \) distinct from \( H \).

**Remark 3.3.10.** If \( a(F) \leq n \), then \( F \) is an \( n \)-resolution of itself.
In order to find resolutions of arbitrary pregeometries, we will use resolutions of circuits. The next lemma provides an explicit construction of such resolutions.

**Lemma 3.3.11.** For $m > n \geq 3$, let $F$ be the unique circuit of size $m$. Then there exists $G_F$, an $n$-resolution of $F$. Moreover, for every natural $p$, $G_F$ can be chosen such that $F \subseteq^p G_F$.

**Proof.** Denote the underlying set of $F$ by $M$. Enumerate $M = \{f_1, \ldots, f_m\}$ and for every natural non-negative number of the form $l = r \cdot m + s$, identify $f_l := f_s$. Fix some $k > 3(m+1)$ divisible by $3$ such that $f_k \neq f_1$. Let $a_1, \ldots, a_k$ be new elements. Denote $b_1 = a_1$, $b_2 = a_{\frac{k}{3}}$, $b_3 = a_{2\frac{k}{3}}$. Define

\[
N = M \cup \{a_i : 1 \leq i \leq k\}
\]

\[
R = \{\{a_i, f_{i+1}, f_{i+(n-3)}, a_{i+1} \} : 1 \leq i \leq k\}
\]

\[
\cup \{\{a_k, f_k, f_{k+(n-3)}, a_1\}, \{b_1, b_2, b_3, f_1, \ldots, f_{n-3}\}\}
\]

and $\mathcal{A} = (M, R)$. We show that $G_F := G_A \in \mathcal{C}_n$ is as desired.

**Claim 1.** For every $X \subseteq N$,

(i) If $M \nsubseteq X$ and $X \nsubseteq M$, then $\delta_A(X) > |X \cap M|$.

(ii) If $M \subseteq X \subseteq N$, then $\delta_A(X) \geq m$.

(iii) If $|X| > n$, then $\delta_A(X) \geq n$.

**Proof of claim.** Note that (i), (ii) are true when $|X \setminus M| = 1$, because $R[X]$ is empty. Let $X$ be minimal contradicting either (i) or (ii), in particular $|X \setminus M| > 1$. Then for each $a_i \in X$, we may assume $a_i$ appears in at least two edges in $R[X]$, or else $X \setminus \{a_i\}$ also contradicts either (i) or (ii). Thus, by construction:

(1) If $1 < i < \frac{k}{3}$ and $a_i \in X$, then $a_j \in X$ for each $1 \leq j < \frac{k}{3}$.

(2) If $\frac{k}{3} < i < \frac{2k}{3}$ and $a_i \in X$, then $a_j \in X$ for each $\frac{k}{3} \leq j \leq \frac{2k}{3}$.

(3) If $\frac{2k}{3} < i \leq k$ and $a_i \in X$, then $a_1 \in X$ and $a_j \in X$ for each $\frac{2k}{3} \leq j \leq k$.

Assume for a moment $X$ fails (i). As $\frac{k}{3} > m$, the conclusion of (1) would imply $M \subseteq X$, and similarly for the conclusions of (2) and (3). Then $X \setminus M \subseteq \{b_1, b_2, b_3\}$. However, if $b_1 \in X$, then either $a_2$ or $a_3$ is an element of $x$, which cannot be, hence $b_1 \notin X$ and similarly $b_2, b_3 \notin X$. So $X \subseteq M$, in contradiction. This proves (i).

Now assume $X$ fails (ii). If $\{b_1, b_2, b_3, f_1, \ldots, f_{n-3}\} \notin R[X]$, then mapping each $a_i \in X \setminus M$ to $\{a_i, f_i, \ldots, f_{i+(k-2)}, a_{i+1}\}$ is a surjection onto a superset of $R[X]$, hence $|X \setminus M| \geq |R[X]|$ and $\delta_A(X) \geq \delta_A(M) = m$. So it must be that $b_1, b_2, b_3 \in X$. Denote $X_1 = \{a_1, \ldots, a_{\frac{k}{3}}\}$, $X_2 = \{a_{\frac{k}{3}}, \ldots, a_{2\frac{k}{3}}\}$, $X_3 = \{a_{2\frac{k}{3}}, \ldots, a_k, a_1\}$. Since each $a_i$ appears in at least two edges in $X$, by (1)-(3) above, at least two of $Y_1, Y_2, Y_3$ are subsets of $X$. Since $X \neq N$, at most two of $Y_1, Y_2, Y_3$ are subsets of $X$, hence $X = Y_i \cup Y_j$ for distinct $i,j \in \{1,2,3\}$. But $\delta_A(Y_i \cup Y_j) = m$, in contradiction to our choice of $X$. This finishes (ii).

Assume (iii) does not hold for some $X$. If $M \subseteq X$, then $\delta_A(X) \geq d_A(M)$, and by part (ii) we observe $d_A(M) = d_A(N) = m - 1 \geq n$. If $X \subseteq M$, then clearly $\delta_A(X) = |X|$. So by part (i) it must be that $|X \cap M| < \delta_A(X) \leq n - 1$. By assumption, it must be that $R[X]$ is not empty, so $|X \cap M| = n - 2$. If $a_i \in X$ is such that $i \neq 1, \frac{k}{3}, 2\frac{k}{3}$, then $a_i$ appears in at most one edge in $R[X]$, i.e., $\delta_A(X \setminus \{a_i\}) \leq \delta_A(X)$. Then for any $X' \subseteq X$ such that $X' \cap (M \cup \{b_1, b_2, b_3\}) = X \cap (M \cup \{b_1, b_2, b_3\})$, we have $\delta_A(X') \leq \delta_A(X) < n$. In particular, we may assume $|X| = n + 1$. Since there
are no two edges $e_1, e_2 \in R[X]$ such that $|e_1 \cap e_2| \geq n - 1$, clearly $|R[X]| \leq 1$ and $\delta_\mathcal{A}(X) \geq n$, in contradiction. \hfill \Box

Part (i) of Claim 1 immediately yields that for every $M_0 \subseteq M$, either $M_0$ is closed in $\mathcal{A}$, in which case $d_\mathcal{A}(M_0) = |M_0|$, or $M \subseteq \text{cl}_\mathcal{A}(M_0)$. Consequently, $F \subsetneq^* G_F$.

Part (iii) of Claim 1 shows that the non-trivial $(n - 1)$-dimensional closed sets in $\mathcal{A}$ are precisely the edges. Thus, $\mathcal{A}$ is the unique good representation of $G_\mathcal{A}$. By Corollary 2.6.7, this means there exists $F \subseteq F' \subseteq G_\mathcal{A}$ distinct from $G_\mathcal{A}$ if and only if there exists some $M \subseteq M' \not\subseteq \mathcal{A}$ distinct from $N$. But $\delta(\mathcal{A}) = m - 1$, so (ii) guarantees no such $M'$ exists. Then $G_F$ is an $n$-resolution of $F$.

We address the additional part of the statement.

**Claim 2.** If $X \subseteq N$ with $d_\mathcal{A}(X) < m - 1$, then $|X| \leq (m + 2)m$.

**Proof of claim.** When writing $a_i$ for some $i > k$, we mean $a_{i-k}$. Without loss of generality, by increasing $X$ assume $d_\mathcal{A}(X) = d_\mathcal{A}(X)$.

Since $|M| = m$, we may assume $X \not\subseteq M$. Let $C = \{a_i, \ldots, a_{i+l}\}$ be maximal such that for each $j < l$, the elements $a_{i+j}$ and $a_{i+j+1}$ appear together in an edge in $R[X]$. If $l \geq m$, then each element of $M$ appears in some edge in $R[X]$, so $M \subseteq X$ in contradiction to $d(X) < m - 1$. Therefore $|C| < m$.

Assume $\{b_1, b_2, b_3, f_1, \ldots, f_{n-3}\} \not\subseteq R[X]$. Then $d_\mathcal{A}(C/(X \setminus C)) = 1$. Repeating this process for $X \setminus C$ in the role of $X$, stripping away sets of the form of $C$ until we are only left with elements of $M$, we have removed at most $d_\mathcal{A}(X) < m - 1$ many sets of size at most $m$ and are left with at most $m$ elements, so $|X| \leq (m - 2)m + m$.

Now, if $\{b_1, b_2, b_3, f_1, \ldots, f_{n-3}\} \subseteq R[X]$, for $i \in \{1, 2, 3\}$ let $C_i$ containing $b_i$ be of the form of $C$, and denote $D = C_1 \cup C_2 \cup C_3$. Then $d_\mathcal{A}(D/(X \setminus D)) = 2$, and as we’ve seen $|X \setminus D| \leq (m - 2)m + m$. Then $|X| \leq 3m + (m - 2)m + m = (m + 2)m$. \hfill \Box

Fix some natural $p$. We show that if $k$ is large enough, then $F \subsetneq^p G_F$. Choose $k$ to be large enough so that, by Claim 2, $G_F$ is not a union of $2^{(|F|+p)}$ closed sets of dimension less than $m - 1$.

Assume for a contradiction that $F \not\subseteq Z \subsetneq^* G_F$, $|Z \setminus F| \leq p$ and let $\Sigma$ be a collection of closed sets in $Z$ witnessing this. Note that $|\Sigma| \leq 2^{2^{|Z|}}$ and every $E \in \Sigma$ has $d_\mathcal{A}(E) < m - 1$, or otherwise $E = Z$. Recall that $M$ is the underlying set of $F$ and that every proper subset of $F$ is independent. Then

$$|M \cap \bigcup \Sigma| = \Delta_F(\Sigma_F) > \Delta_Z(\Sigma) \geq d_\mathcal{A}(\bigcup \Sigma) \geq d_\mathcal{A}(M \cap \bigcup \Sigma).$$

Since $M \cap \bigcup \Sigma$ is not independent, it must be that $M \subseteq \bigcup \Sigma$ and $\Delta_Z(\Sigma) = m - 1$. By $Z \subsetneq^* G_F$, we have that $\Delta_Z(G_F) = \Delta_Z(\Sigma) = m - 1 = d(\bigcup \Sigma_{G_F})$. By the additional part of Proposition 2.4.2, we get that $\bigcup \Sigma_{G_F} \subseteq \mathcal{A}$. Since $G_F$ is an $n$-resolution of $F$, this means $\bigcup \Sigma_{G_F} = N$. By choice of $k$, this cannot be. Conclude $F \subsetneq^p G_F$. \hfill \Box

**Digression.** Resolutions motivate Morley rank in $\mathcal{G}$ and its similarity to $\rho$. We can see inductively that an $n$-sized circuit has Morley rank $\geq \omega^{n-4}$.

An $n + 1$-sized circuit can resolve in infinitely many mutually exclusive ways into a finite configuration of $n$-sized circuits. For a configuration with $k$ $n$-sized circuits, each one can resolve independently of the others, which by induction hypothesis gives the configuration Morley rank at least $k \cdot \omega^{n-4}$. Since $k$ is unbounded, the Morley rank of the $n + 1$-sized circuit limits to at least $\omega^{n-3}$. As a base for the
induction, it is enough to note that a 4-sized circuit has infinitely many distinct
3-resolutions, i.e., Morley rank at least 1 = \omega^0.

This means that the Morley rank of \( T \) is at least \( \omega^\omega \), since the type of a point
independent from (a basis of) \( G \) is, for example, the limit of increasingly large
circuits involving \( b \) and elements (of a basis of) \( G \).

In an \( n \)-resolution of \( F \) with some \( F_0 \sqsubseteq F \), if we want to preserve strong embed-
dedness of \( F_0 \), we cannot require \( H \in \mathcal{C}_n \), because \( F_0 \) may have strongly embedded
circuits of size greater than \( n \). This issue arises when wishing to replace \( F \) in the
amalgam \( F \sqcup F_0 H \) with \( F' \), an \( n \)-resolution of \( F \), for the sake of lower arity — if \( F_0 \)
is no longer strongly embedded in \( F' \), the amalgam \( F' \sqcup F_0 H \) does not necessarily
exist. There will be no harm in leaving \( F_0 \) “unresolved” in \( F' \), because what seems
like a circuit in \( F \), may in fact be a part of a low-arity configuration in \( H \).

To proceed, we generalize the notion of arity and resolution to make sense over
some strongly embedded subpregeometry.

**Definition 3.3.12.** Say that a flat pregeometry \( G \) is of arity at most \( n \) over \( F \sqsubseteq G \n\)
if there exist good representations \( (M_0, R_0) \leq (M, R) \) of \( F \sqsubseteq G \) with \( R \setminus R_0 \subseteq [M]\leq n \). Write this statement as \( a(G/F) \leq n \) and let \( a(G/F) \) be the least \( n \) such
that \( a(G/F) \leq n \).

**Observation 3.3.13.** In the vein of Corollary 3.3.6, for flat pregeometries \( H, G_1, G_2 \n\)
such that \( G_1 \sqcup H G_2 \) is well-defined, \( a(G_1 \sqcup H G_2) \leq \max\{a(G_1/H), a(G_2)\} \).

**Definition 3.3.14.** For \( F_0 \sqsubseteq F \), say that \( H \in \mathcal{C} \) is an \( n \)-resolution of \( F \) over \( F_0 \) if
\( a(H/F_0) \leq n \), \( F \sqsubseteq^* H \), there is no \( F \subseteq F' \sqsubseteq H\) distinct from \( H \), and \( F_0 \sqsubseteq H \).

**Observation 3.3.15.** For a flat pregeometry \( F \), \( a(F) = a(F/\emptyset) \) and \( H \) is an
resolution of \( F \) if and only if \( H \) is an \( n \)-resolution of \( F \) over \( \emptyset \sqsubseteq F \).

We now show that an \( n \)-resolution over a strongly embedded subpregeometry
always exists. In particular, the observation immediately above implies that every
flat pregeometry has an \( n \)-resolution.

**Proposition 3.3.16.** For every \( F \in \mathcal{C} \), \( F_0 \subseteq F \) and \( n \in \mathbb{N} \), there exists \( H \in \mathcal{C} \), an
\( n \)-resolution of \( F \) over \( F_0 \). Moreover, for every natural \( p \), \( H \) can be chosen so that
\( F \sqsubseteq^p H \).

**Proof.** Let \( \mathcal{A} = (M, R) \), \( \mathcal{A}_0 = (M_0, R_0) \) be a good representations of \( F, F_0 \) respec-
tively, such that \( \mathcal{A}_0 \leq \mathcal{A} \). For each \( e \in R \setminus R_0 \), seen as a subpregeometry of \( F \),
using 3.3.11, let \( G_e \) be an \( n \)-resolution of \( e \). If \( |e| \leq n \), take \( G_e = e \). Enumerate
\( R \setminus R_0 = \{e_1, \ldots, e_k\} \), define \( H_1 = F \) and inductively define \( H_{i+1} = H_i \sqcup_{e_i} G_{e_i} \).
Denote \( H = H_{k+1} \).

Recalling the explicit construction of an amalgam of pregeometries (definitions
3.1.5, 3.1.9), at every stage the pregeometry \( H_{i+1} \) is represented by a hypergraph
\( \mathcal{D}_{i+1} = (N, S_{i+1}) \) which is \( \mathcal{D}_i \sqcup \mathcal{B}_i \), where \( \mathcal{B}_i \) is a (good) representation of \( G_{e_i} \). Then
for every \( j \geq i \) we have that still \( e_j \in S_j[M] \), and in particular \( \delta_{\mathcal{D}_j}(e_j) = |e_j| - 1 = \delta_F(e_j) = \delta_{\mathcal{D}_i}(e_j) \). Therefore, \( e_j \subseteq H_i \), implying \( e_j \subseteq H_i \). This means that \( H_{i+1} \) is
well defined at every step.

Unraveling the construction, we see that the order in which we enumerate the
edges makes no difference to the resulting hypergraph \( \mathcal{D}_{k+1} \) and associated prege-
ometry \( H \). Thus, given any \( e \in R \setminus R_0 \), we may re-enumerate so that \( e = e_k \) and get
\( H = H_K \sqcup G_e \).

Now we check that \( H \) is an \( n \)-resolution of \( F \) over \( F_0 \). Clearly, \( H \in \mathcal{C} \).
• By Lemma 3.1.10, since \( e_i \sqsubseteq^* G_{e_i} \), we have \( H_i \sqsubseteq^* H_{i+1} \), so inductively \( F \sqsubseteq^* H \).

• By construction, inductively, for each \( i \) the restriction of \( H_i \) to \( M_0 \) is \( F_0 \). Since \( D_{k+1} [M_0] = \mathcal{A}_0 \), we have \( G_{D_{k+1} [M_0]} = G_{\mathcal{A}_0} = F_0 \). By (2) of Fact 2.4.1, this implies \( \mathcal{A}_0 \leq D_{k+1} \) and consequently \( F_0 \subseteq H \).

• For each edge in \( D_{k+1} \), its dimension remains unchanged from the stage when it was introduced into the construction. By choosing only good representations of the pregeometries \( G_e \) during construction, the resulting \( D_{k+1} \) is a good representation of \( H \). In the previous item we saw \( \mathcal{A}_0 \leq D_{k+1} \), so \( a(H/F_0) \leq n \).

• Lastly, let \( F \subseteq F' \sqsubseteq H \), then by (1) of Corollary 3.2.7, \( F' \cap G_e \subseteq G_e \) for every edge \( e \). But \( G_e \) is an \( n \)-resolution of \( e \) so it must be that \( G_e \subseteq F' \). Then \( F' = H \).

We prove the additional part. Fix \( p \) and assume we had chosen all the \( G_e \) such that \( e \sqsubseteq^p G_e \). Let \( Z \sqsubseteq^* H \) contain \( F \) such that \( |Z \setminus F| \leq p \). For each \( e \), by (2) of Corollary 3.2.7, since \( e \sqsubseteq^* G_e \), we have \( Z \cap G_e \sqsubseteq^* G_e \). By \( e \sqsubseteq^p G_e \), in fact \( e \subseteq Z \cap G_e \). For each \( e \) denote \( Z_e = Z \cap G_e \). Applying Observation 3.2.5 iteratively, we see that the restriction of \( H \) to \( Z \) is the amalgam of the pregeometries \( F \sqcup Z_e \) over \( F \). By the last item of Lemma 3.1.10, for each \( e \) it holds that \( F \sqsubseteq F \sqcup Z_e \), so inductively we get \( F \subseteq Z \).

We can finally determine the theory of \( \mathcal{G}_n \). The proof below is similar to that of \( \mathcal{G} \models T \) (Proposition 3.2.11), but does require more consideration.

**Proposition 3.3.17.** There exists some \( \tau \) such that \( \mathcal{G}_n \models T_\tau \) (Recall Definition 3.2.4). In particular \( \mathcal{G}_n \models T \).

**Proof.** Both T1 and T2 are clear. We show T3 holds. Suppose \( F \sqsubseteq^* \mathcal{G}_n \), \( F \sqsubseteq H \subseteq \mathcal{E} \) and fix some natural \( k \). We want to find a sufficient condition on \( \tau(k) \) so that if \( F \sqsubseteq^{\tau([H]+k)} \mathcal{G}_n \), then there is an embedding \( f : H \to \mathcal{G}_n \) such that \( f[H] \sqsubseteq^k \mathcal{G}_n \).

By Proposition 3.3.16, let \( G_0 \) be an \( n \)-resolution of \( H \) over \( F \) such that \( H \sqsubseteq^p G_0 \) for some \( p > h(h([H]+k)) \), where \( h \) is a function as in 3.2.9. Choose some finite \( E \subseteq \mathcal{G}_n \) containing \( F \), and let \( G = G_0 \sqcup F \sqcup E \). Observation 3.3.13 gives \( a(G) \leq n \), so by genericity we may assume \( G \) is strongly embedded into \( \mathcal{G}_n \) over \( E \).

Assume that \( H \not\subseteq^k \mathcal{G}_n \) and let \( Z \sqsubseteq^* \mathcal{G}_n \) contain \( H \) such that \( |Z \setminus H| \leq k \) and \( H \not\subseteq Z \). Then by Lemma 3.2.9 there is some \( V \sqsubseteq^* G \) containing \( H \) such that \( |V| \leq h([H]+k) \) and \( H \not\subseteq V \). By (1) of Corollary 3.2.7, \( V_0 := V \cap E \sqsubseteq^* E \). Observation 3.2.5 now implies \( G' := G_0 \sqcup F \sqcup V_0 \sqsubseteq^* G \). Note that \( p > h(|V|) \).

Assume for a moment \( F \subseteq V_0 \). Then \( G_0 \subseteq G' \) and, considering \( H \not\subseteq V \sqsubseteq^* G' \), an application of Lemma 3.2.9 yields some set \( V' \sqsubseteq^* G_0 \) containing \( H \) such that \( H \not\subseteq V' \). But \( |V'| \leq h(|V|) \) so \( p \), in contradiction to \( H \not\subseteq^p G_0 \). So it must be that \( F \not\subseteq V_0 \), hence \( F \not\subseteq^{h([H]+k)} \mathcal{G}_n \). As before, setting \( \tau \) greater or equal to the \( h \) function of Lemma 3.2.9 gives us what we want. \( \square \)

While tempting to call \( T \) the theory of generic flat pregeometries, there are still more generic flat pregeometries – in the sense that they are generic structures for amalgamation classes – not sharing this theory. Like in the case of the strongly minimal Hrushovski construction, one can enforce finite multiplicities on certain configurations – see [Hru93, Lemmas 17,18] for good representations of such. Also, there are models of \( T \) besides those that we’ve seen — an elementary extension of
Lemma 3.3.20. \(G_3\) realizing a unique strongly-embedded “4-ary” circuit, say. The study of these variants is left for a different paper.

To conclude our investigation, we show that the generic flat pregeometries we have studied form an elementary chain. Fix until the end of this section copies of \(\mathcal{M}\) and \(\mathcal{G}\) such that \(\mathcal{G} = G_\mathcal{M}\) and \(\mathcal{M} = (M, R)\). Also fix \(I \subseteq M\), a basis for \(\mathcal{G}\).

**Observation 3.3.18.** Let \(A_1 = (M, R_1), A_2 = (M, R_2)\) be good representations of the same pregeometry and let \(B_i = (M, R_i \cap [M]^{\leq n})\). Then \(G_{B_1} = G_{B_2}\).

Being good representations, clearly \(G_{B_1}\) and \(G_{B_2}\) have the same closed sets of dimension less than \(n\). By definition of \(a(B_i) \leq n\), this uniquely determines the entire pregeometry associated to \(B_i\).

**Definition 3.3.19.** Let \(G\) be a flat pregeometry, let \(A = (M, R)\) be a good representation of \(G\) and let \(A^{\leq n} = (M, R \cap [M]^{\leq n})\). For every \(X \subseteq M\) define \(\text{cl}_G^{|X|}(X) = \text{cl}_{A^{\leq n}}(X)\), the \(n\)-ary closure of \(X\) in \(G\).

The structure \(\mathcal{M}_n\) can be identified in \(\mathcal{M}\) as the \(n\)-ary closure of an infinite independent set.

**Lemma 3.3.20.** Let \(N = \text{cl}_G^{|I|}(I)\), for some \(n \geq 3\). Then \(N \subseteq \mathcal{M}\) and \(\mathcal{M}[N] \cong \mathcal{M}_n\).

**Proof.** For every finite \(X \subseteq \text{fin} N\), there is \(Y \subseteq \text{fin} N\) containing \(X\) such that \(\delta_{\mathcal{M}}(Y/Y \cap I) \leq 0\). To be precise, \(|Y \setminus I| \leq |R[Y] \cap [M]^{\leq n}|\). As \(Y \cap I \subseteq \mathcal{M}\), we have \(Y \subseteq \mathcal{M}\) and \(R[Y] = R[Y] \cap [M]^{\leq n}\). By \(Y \subseteq \mathcal{M}\), we get that \(d_{\mathcal{M}[N]}(X) = d_{\mathcal{M}[Y]}(X) = d_{\mathcal{M}}(X)\). Therefore, since \(X\) was arbitrary, \(N \subseteq \mathcal{M}\) and \(R[N] = R[N] \cap [M]^{\leq n}\).

Now, we only need to show extension – (\(\ast\)) of Theorem 3.1.1 – to get that \(\mathcal{M}[N]\) is generic for \(C_\mathcal{n}\). Let \(A \leq \mathcal{M}[N] \leq \mathcal{M}\) and let \(A \leq B \in C_\mathcal{n}\). Increasing \(A\) by mapping points in \(B\) independent from \(A\) in \(B\) to elements of \(I\) that are independent from \(A\) in \(\mathcal{M}\), we may assume \(\delta(A) = \delta(B)\). By genericity of \(\mathcal{M}\), we may assume \(B\) is strongly embedded into \(\mathcal{M}\) over \(A\). But then the embedding is clearly into \(\mathcal{M}[N]\).

**Theorem 3.3.21.** \(\mathcal{G}_3 \prec \mathcal{G}_4 \prec \mathcal{G}_5 \prec \cdots \prec \mathcal{G}\)

**Proof.** By the above lemma, identifying \(\mathcal{M}_n\) with \(\text{cl}_{\mathcal{G}}^{|I|}(I)\), we get

\[\mathcal{M}_3 \leq \mathcal{M}_4 \leq \mathcal{M}_5 \leq \cdots \leq \mathcal{M}\]

which immediately gives

\[\mathcal{G}_3 \subseteq \mathcal{G}_4 \subseteq \mathcal{G}_5 \subseteq \cdots \subseteq \mathcal{G}\]

Since \(\mathcal{G}_n \models T\) for every \(n \geq 3\), by Theorem 3.2.22 we are done.

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