PROBLEMS WITH POPPER

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Professor Karl R. Popper was one of the most influential philosophers of science in the twentieth century. He is most widely known for his doctrine that scientific theories are not provable, but to be accepted as scientific they must be falsifiable. The most-cited statement of this is from the Postscript to his *Logic of Scientific Discovery*:

... we adopt, as our criterion of demarcation, the criterion of falsifiability, i.e. of an (at least) unilateral or asymmetrical or one-sided decidability. According to this criterion, statements, or systems of statements, convey information about the empirical world only if they are capable of clashing with experience; or more precisely, only if they can be systematically tested, that is to say, if they can be subjected (in accordance with a 'methodological decision') to tests which might result in their refutation. (pp. 313-4, §4i(2); his italics, as they will be henceforth)

The book contains far more than this statement, however. I set out some problems with it below.

1 Experimental uncertainty

After asserting that every physical measurement is equivalent to noting a pointer’s position between two marks on a scale, which thus correspond to an interval within which the measurement lies, Popper continues:

It is the custom of physicists to estimate the interval for every measurement. (Thus following Millikan they give, for example, the elementary charge of the electron, measured in electrostatic units, as $e = 4.774 \times 10^{-10}$, adding that the range of imprecision is $\pm 0.005 \times 10^{-10}$.) But this raises a problem. What can be the purpose of replacing, as it were, one mark on a scale by two—to wit, the two bounds of the interval—when for each of these two bounds there must again arise the same question: what are the limits of accuracy for the bounds of the interval?

Giving the bounds of the interval is clearly useless unless these two bounds in turn can be fixed with a degree of precision greatly exceeding what we can hope to attain for the original measurement;
fixed, that is, within their own intervals of imprecision which should thus be smaller, by several orders of magnitude, than the interval they determine for the value of the original measurement. In other words, the bounds of the interval are not sharp bounds but are really very small intervals, the bounds of which are in their turn still much smaller intervals, and so on. In this way we arrive at the idea of what may be called the ‘unsharp bounds’ or ‘condensation bounds’ of the interval. (p. 125, §37)

There are two points I wish to make about this passage. First, while such a naive misconception of experimental uncertainty might be understandable in someone who had never had contact with science at all, it is bizarre in a professor who is writing a book purporting to set out very basic aspects of science. Second, this misunderstanding generates a whole conceptual structure (of ‘condensation bounds’), not only taking up space in itself but developing further ideas (notably §68).

A significant fraction of the book is given over to criticism of the quantum theory, in particular attempts to design thought-experiments to disprove the Heisenberg Uncertainty Principle. I will not go into them in any detail here, since it would require a great deal of time and attention; some criticism (too lenient, in my view) of his later writings on the subject has been published.

But a taste may be found on pp. 239-40, in which a monochromatic beam of particles has been prepared (thus they are all of a single momentum); we are then, by ‘focusing our attention’ on those within an arbitrarily small volume, to determine position and momentum to any desired accuracy. That Popper can set this (explicitly ‘unphysical’) procedure as a serious possibility shows something of his incomprehension of the basics of experimental physics.

2 Mathematics

Most of the book is taken up with an analysis of the mathematical theory of probability, including criticism of others’ formulations and a detailed presentation of Popper’s own construction. Before treating probability as such, I want to look at a couple of examples of Popper’s use of mathematics.

In the first, Popper is concerned with the definition of a probability apparently given by von Mises. In a sequence of events, the fraction is formed of ‘successes’ in which a particular thing occurs divided by the total number of events. If this fraction, called by Popper the ‘relative frequency,’ converges to a definite number in the limit of an infinitely long sequence, that number is the probability of success. The particular example used is the fraction of ones in a sequence of ones and zeroes.

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*Popper does not quote von Mises explicitly. From the description it appears that the latter dealt with ‘convergence in probability,’ a phrase that Popper does not mention, but which he would not have cared for.
Popper wishes to do without the requirement of convergence. That means, he argues, he needs a concept that can be used in place of a limiting frequency, applicable to all infinite sequences.

One frequency concept fulfilling these conditions is the concept of a *point of accumulation of the sequence of relative frequencies*. (A value \(a\) is said to be a point of accumulation of a sequence if after any given element there are elements deviating from \(a\) by less than a given amount, however small.) That this concept is applicable without restriction to all infinite reference sequences may be seen from the fact that for every finite alternative *at least one* such point of accumulation must exist for the sequence of relative frequencies which corresponds to it. Since relative frequencies can never be greater than 1 nor less than 0, a sequence of them must be bounded by 1 and 0. And as an infinite bounded sequence, it must (according to the famous theorem of Bolzano and Wierstrass) have *at least one* point of accumulation. (p. 185, §64)

The definition of accumulation point in the references I have at hand\(^3,4\) is not the same, nor equivalent. However, Popper’s definition is probably contained within it, so we will pass over that point. His statement of the Bolzano-Wierstrass theorem is accurate, except for the application to a ‘finite alternative.’

The theorem only holds for infinite sequences; in fact, a finite sequence has no accumulation points\(^3\). In the paragraph above, his mention of ‘finite sequences’ suggests (at least) that he has not grasped this requirement; this is proved on the next page (p. 186n4), where he applies the concept of accumulation points to finite and infinite sequences indiscriminantly. Let me be explicit: Popper is using a theorem in an area where it just isn’t true.

The real problem with using accumulation points is that, while the Bolzano-Wierstrass theorem assures us of at least one, there may be many. Popper realizes this (p. 185n2), and spends some time on the rather trivial point that, in this case, they are not useful in defining probability. He also recognises that to require a unique accumulation point is equivalent to requiring convergence (p. 186). Then he requires uniqueness anyway, asserting that it isn’t, and in any case that he is free to choose such sections of any sequence as have the behaviour he desires (p. 186n4).

I will pass over these last two problems with mathematical logic, because in fact accumulation points are irrelevant to the task Popper attempts in this section. The importance of this episode is not so much that Popper makes mistakes in the handling of accumulation points, which could be considered a rather esoteric bit of analysis; nor even in his failure to distinguish between finite and infinite, though that is certainly a serious drawback for anyone trying to do mathematics. It lies in the fact that he is not competent in this whole area of analysis, deploying irrelevant machinery and doing that improperly.

Let us look at another section of mathematics, set theory. Popper refers to
Kolmogorov’s development of a theory of probability that explicitly uses sets. But he does not like it:

And yet, he [Kolmogorov] assumes that, in ‘p(a, b)—I am using my own symbols, not his [that is, the probability of a given b]—a and b are sets; thereby excluding, among others, the logical interpretation according to which a and b are statements (or ‘propositions’, if you like). He says, rightly, ‘what the members of this set represent is of no importance’; but this remark is not sufficient to establish the formal character of the theory at which he aims; for in some interpretations, a and b have no members, nor anything that might correspond to members. (p. 327, §*iv)

Nowadays set theory is taught in elementary school; I am not sure what its status was when Popper wrote. But among the very first concepts one comes across is $\emptyset$, the empty set, the set with no members. Also among the first concepts is that a set may be made up of anything, including statements, propositions, truth-values, complex numbers, apples, oranges—or all at once. I cannot escape the conclusion that here Popper is criticising an approach knowing nothing about it. Indeed, in a later section (pp. 344-5, §*iv) he demonstrates the ‘superiority’ of his ‘Boolean’ approach over the ‘set-theoretic’ approach by performing set operations.

In presenting these two examples I am not asserting that Popper’s mistakes and misconceptions necessarily make all of his later work wrong. That would take a rather tedious effort of working through hundreds of pages of sometimes convoluted logic. I am asserting that he has attempted to produce results with mathematics that he does not understand or, worse, understands wrongly.

3 The probability of a hypothesis

Popper takes issue with the idea that a theory, a hypothesis, may be rendered more probable by a series of observations.

Let us now try to follow up the suggestion that the hypotheses themselves are sequences of statements. One way of interpreting it would be to take, as elements of the sequence, the various singular statements which can contradict, or agree with, the hypothesis. The probability of the hypothesis would then be determined by the truth-frequency of those among the statements which agree with it. But this would give the hypothesis a probability of 1/2 if, on the average, it is refuted by every second singular statement of the sequence! (p. 257, §80)

After considering a few modifications of this idea, he concludes,

This seems to me to exhaust the possibilities of basing the concept of the probability of a hypothesis on that of the frequency of true state-
ments (or the frequency of false ones), and thereby on the frequency
theory of the probability of events. (p. 260, §80)

There are two points to make about these statements immediately. One
is that Popper has set up an algorithm for the testing of a hypothesis that is
crude by any standard, and would not for a moment be entertained by anyone
actually attempting to test a hypothesis. The second is that, in presenting this
algorithm, he has set up a ‘strawman,’ any by refuting it has pretended to refute
all methods of testing a hypothesis (and giving it a greater or lesser probability)
by examining events.

Before returning to the question of the probability of a hypothesis I want to
bring out two other examples presented by Popper concerning basic calculations
in probability. First, he wants to disprove the ‘subjectivist theory of evidence,’
what we would now call the Bayesian approach (pp. 407-8, §ix). Given a coin,
what is the probability of heads? Well, one-half. Now given that the same
coin gave 500,000 heads (±1350) in one million trials, what is the probability?
One-half again. Hence, under the subjectivist theory, after a million coin flips
we have learned nothing.

Popper believes this exercise disproves the idea of associating a probability
with a subjective belief in an outcome (the initial guess of one-half), what we
would now call a Bayesian prior. What he has actually done, by setting up the
problem in a way that no one with a background in even classical probability
would do, is prove that if an initial guess proves correct, we do not change our
mind.

Before we go on to the question of the absolute probability of a theory, there
is one section that I think illuminates Popper’s preconceptions in an interesting
way. On p. 390, §ix, he deals with the throw of a die. We take \( p(x) \) as the
probability of throwing a six, \( p(\bar{x}) \) that of the negation (throwing anything else).
Initially, with no information, we set the probabilities as 1/6 and 5/6. Then, we
are given the information that that throw is an even number. The probabilities
are now 1/3 and 2/3. The information has supported the hypothesis \( x \) and
weakened its negation \( \bar{x} \); but the fact that the probability of \( x \) is still smaller
than \( \bar{x} \) is considered by Popper ‘clearly self-contradictory,’ and thus proof that
any calculation of the support or refutation of a hypothesis (a theory) cannot
be done within any conventional probability theory. Well, self-contradictory it
is not. He appears to require that any support of a theory (a hypothesis) make
it more probable than its negation, a rather absolute position. He neither states
this explicitly nor attempts to defend it (and defense it certainly requires), but
it underlies his work in this section.

We now come to Popper’s calculation of the probability of a theory, any
theory, in the universe. It is probably set out most clearly on pp. 363-8, §vii.

\[\text{He may have derived this formulation from an equation he attributes to Jeffries, p. 370,}
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\[\text{§vii, which he does not appear to understand.}\]

\[\text{The more useful question is, before the coin gets flipped, how sure are we of the initial guess}
\]

\[\text{of probability one-half? But there are many introductory texts that consider the problem of}
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\[\text{the possibly biased coin, so I will not go into more detail here.}\]
The probability of a theory is set equal to the product of the probabilities of all events predicted by the theory, \( p(a) = \prod p(a^n) \). Since each component \( p(a^n) \) can never be greater than unity, may in fact be less, and they will be infinite in number, the product will always go to zero. Hence the probability of any theory is exactly zero.

I have already noted that this is not the way to work out the probability of a hypothesis, given data. It also assumes that all events in the universe are independent. Popper formally recognises this, but asserts they must be, otherwise we could never learn anything new (p. 368). What it actually implies is total chaos—no event would have any relation to any other, and from one moment to another, anything could happen.

Popper concludes something almost as depressing. If \( a \) is any theory and \( b \) is any information, \( p(a) = 0 \) and \( p(a|b) = 0 \)

The consequences of this result will be traced in the next section. For the moment, let me emphasize that Popper’s understanding of elementary probability was inadequate and flawed, not attaining the point of being able to test a possibly biased coin.

### 4 Consequences

Popper’s conclusions to this point led him to construct a ‘calculus of relative probability’ in which all of the following formulae may be valid (p. 331, §iv):

\[
\begin{align*}
p(a, \bar{b}) &= 1 \\
\text{If } p(\bar{b}, b) &\neq 0 \quad \text{then } p(a, b) = 1 \\
\text{If } p(a, \bar{a}b) &\neq 0 \quad \text{then } p(a, b) = 1
\end{align*}
\]

These allow some very strange things to happen. In the first, some situation \( a \) is certain to happen, given both \( b \) and its negation. That is, the given situation requires a single flip of a coin to be both heads and tails. The second or third again allow a nonzero probability to a situation and its negation: given that a die has rolled a six, it is possible it rolled one through five.

In addition, one could also have simultaneously (for theories \( a_1 \) and \( a_2 \)),

\[
\begin{align*}
p(a_1, a_2) &= 0 \\
p(a_2, a_1) &= 1
\end{align*}
\]

\(^\dagger\)In passing, I note this logical fallacy: Popper is asserting that if we do not know everything, we are required to know nothing.
while at the same time $p(a_1) = p(a_2) = 0$ (p. 375, §vii). Eventually he introduces a notion to express the fact that, while the probability of every theory is zero, some zeroes are bigger than others (pp. 375-6, §vii).

The kindest thing to say about a system that purports to do anything with these statements is that it assigns meaning to something essentially meaningless. In fact one could say Popper has succeeded in his goal of creating something that is not as 'weak' as conventional probability (p. 330, §iv). By assigning a non-zero probability to a situation in which a statement and its negation are both true, his system can generate nonsense. He does not appear to notice the inconsistency of this with his claim to be implementing Boolean logic (p. 328, §iv). In any case, it is not something to be proud of.

5 Summary

In his most famous work, Karl Popper demonstrated an inadequate and occasionally incorrect understanding of science, mathematics and especially probability at an elementary level. I have not here looked at his philosophical ideas, but it would be surprising if anything of much use could be built on such a base.

References

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(4) Norman B. Haaser & Joseph A. Sullivan, Real Analysis, (New York: Dover), 1971, pp. 82-4