Application of a weighted sensitivity approach for topology optimization analysis of time dependent problems based on the density method

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Abstract

In this study, we apply a weighted sensitivity approach to topology optimization analysis based on the density method in dynamic oscillation problems. In our proposed technique, strain energy is employed as the performance function. The self-adjoint relation is derived from the strain energy. However, when using this technique, the properties of the mesh prevent the acquisition of a non-oscillatory density distribution. Employing our proposed weighted sensitivity, however, reduces the numerical oscillation of the sensitivity in the search direction, resulting in a clear density distribution. We present the results of a comparison of the sensitivity and density distributions.

Keywords topology optimization, weighting sensitivity, strain energy minimization, dynamic oscillation problem, density method

Research Activity Group Mathematical Design

1. Introduction

Topology optimization \cite{1} gives the highest degree of freedom in design variables in structural optimization, and is anticipated to significantly improve performance. Topology optimization has solved various problems, such as minimizing the strain energy in steady problems and maximizing natural frequencies \cite{2}. In this study, topology optimization analysis based on the density method was applied to a dynamic oscillation problem with the aim of minimizing the strain energy. In other studies, when solving the problem, it is common to take as the performance function the square of the strain energy \cite{3}. However, many techniques require solving the adjoint equations. In this study, a self-adjoint relation approach that does not require to solve the adjoint equation was derived, and the optimal solution was found \cite{4}. However, this technique also has a drawback with respect to density distribution. To solve this problem, the sensitivity used in the density update is weighted to reduce the numerical oscillation of the sensitivity in the direction of iteration. In this report, topology optimization is formulated for dynamic oscillation problems and weighted sensitivity. Numerical experiments are described, and conclusions are drawn. We conclude that the result of the density distribution when weighted sensitivity is employed presents a clearer distribution than seen with results based on conventional density distribution.

2. Formulation

2.1 Topology optimization for dynamic oscillation problems

The goal of minimum strain energy design is to optimize the density distribution of a limited volume of material in the design domain $\Omega$. The topology optimization in this study employs the element-based density method. The density represents a non-dimensional density and is different from the mass density. Equation (1) shows the performance function for the optimization problem. The sum of the strain energies at an element can be zero. Thus, in this study, strain energy is considered as work. It is transformed into a problem that minimizes the work when the values are positive and a problem that maximizes the work when there are negative values, as shown in Equation (2). The governing equation, which are discretized in both spatial and time directions, uses the equation of motion shown in Equation (3). The constraint equations employ the volume constraint shown in Equation (4) and the density constraint shown in Equation (5). In this study, Galerkin method is employed for discretization in the spatial direction.

$$\min_{\rho_e} J = \int_0^t \sum_{e \in \Omega} J_e (t) \, dt \quad (1)$$
\[ J_e(t) = \begin{cases} \frac{1}{2} \{ f \}^T \{ u \} & (\{ f \}^T \{ u \} \geq 0) \\ -\frac{1}{2} \{ f \}^T \{ u \} & (\{ f \}^T \{ u \} < 0) \end{cases} \]  

subject to \[ [M] \{ \ddot{u} \} + [C] \{ \dot{u} \} + [K] \{ u \} = \{ f \} \]  

\[ V = \sum_{e \in \Omega} \rho_e V_{\text{total}} - \rho_0 \leq 0 \]  

where \([M], [C], [K], \{ \ddot{u} \}, \{ \dot{u} \}, \) and \{u\} represent the mass matrix, damping matrix, stiffness matrix, acceleration vector, velocity vector, and displacement vector, respectively. They are assumed to be functions of density \(\rho_e\). In addition, \{f\}, \(V_{\text{total}}\), \(\rho_0\), \(\nu_e\) and \(t_f\) are the load vector, total element volume in the design domain \(\Omega\), density value in an element \(e\), initial average density, volume in an element \(e\) and final time, respectively. Here, the subscript \(T\) represents transpose. Moreover, \(t_f\) sets the time when the displacement dissipates. Using a solid isotropic material with penalization method [5] which is a density method \([6]\), the coefficients for each matrix are defined as

\[ \gamma_e = (\gamma_0 - \gamma_{\text{min}}) \rho_e^p + \gamma_{\text{min}} \]  

\[ \zeta_e = (1 - \zeta_{\text{min}}) \rho_e^p + \zeta_{\text{min}} \]  

\[ E_e = (E_0 - E_{\text{min}}) \rho_e^p + E_{\text{min}} \]  

where \(\gamma_0, E_0\) and \(p\) are the mass density of the material, Young's modulus of the material, and penalization parameter, respectively. In addition, \(\gamma_{\text{min}}, \zeta_{\text{min}}\) and \(E_{\text{min}}\) are numerical stability parameters which should be as close to zero as possible. Penalization parameter \(p\) is set by Hashin-Shtrikman bounds. Rayleigh damping is used for the damping matrix and is determined as

\[ [C] = \zeta_e (\alpha_1 [M_0] + \alpha_2 [K_0]) \]  

where \(\alpha_1\) and \(\alpha_2\), \([M_0]\) and \([K_0]\) are Rayleigh damping, the mass matrix when \(\rho_e = 1\) and the stiffness matrix when \(\rho_e = 1\), respectively. The Lagrange multiplier method is employed to find an optimal solution. For the Lagrange function shown in Equation (10), the performance function shown in Equation (1) and the governing equation shown in Eq. (3) are used.

\[ L = \int_0^{t_f} \sum_{e \in \Omega} J_e \]  

\[ = \int_0^{t_f} \sum_{e \in \Omega} L_e dt \]  

where \([\lambda] \) and \(L_e\) are the Lagrange multiplier and the Lagrange function on an element, respectively. The gradient of the Lagrange function is calculated under the assumption that the smaller the value of each \(L_e\), the smaller the \(L\). In other words, \(L_e\) should be small to minimize \(L\). First, the gradient of the Lagrange function with respect to the displacement is derived. \(L_e\) is classified into \(L_{e+}\), which represents the Lagrange function when the work has a positive value, and \(L_{e-}\), which represents the Lagrange function when the work has a negative value. Equation (11) shows the gradient of the Lagrange function with respect to the displacement when the work has a positive value, and Equation (12) shows the gradient of the Lagrange function with respect to the displacement when the work has a negative value.

\[ \left\{ \frac{\partial L_{e+}}{\partial \rho_e} \right\}^T = \frac{1}{2} \int_0^{t_f} \left\{ \lambda \right\}^T \left\{ \frac{\partial [M]}{\partial \rho_e} \{ \ddot{u} \} + \frac{\partial [C]}{\partial \rho_e} \{ \dot{u} \} + \frac{\partial [K]}{\partial \rho_e} \{ u \} - \{ f \} \right\} dt \]  

\[ \left\{ \frac{\partial L_{e-}}{\partial \rho_e} \right\}^T = \frac{1}{2} \int_0^{t_f} \left\{ \lambda \right\}^T \left\{ \frac{\partial [M]}{\partial \rho_e} \{ \ddot{u} \} + \frac{\partial [C]}{\partial \rho_e} \{ \dot{u} \} + \frac{\partial [K]}{\partial \rho_e} \{ u \} - \{ f \} \right\} dt \]  

Finally, the density is updated using sensitivity. The optimality criteria method \([7]\), as shown in Equation (14) is employed for the update equation. The density update constraint shown in Equation (15) is set such that the solution does not deviate significantly when the density is updated.

\[ \rho_e^{(k+1)} = \rho_e^{(k)} \left( \frac{\frac{\partial L}{\partial \rho_e}}{-\Lambda(k) \frac{\partial V}{\partial \rho_e}} \right)^q \]  

\[ \max \left\{ \rho_e^{(k)} - \zeta, 0 \right\} \leq \rho_e^{(k+1)} \leq \min \left\{ \rho_e^{(k)} + \zeta, 1 \right\} \]  

2.2 Weighted sensitivity

Depending on the model and calculation conditions, the sensitivity of an element in the iteration direction may cause numerical oscillations. The numerical oscillation of the sensitivity prevents from converging to a smooth density distribution. To solve the numerical oscillation, the sensitivity of an element in the iteration direction is updated by

\[ \lambda^+(e) = \lambda^+(e) - \gamma \cdot \lambda^-(e) \]  

\[ \lambda^-(e) = \lambda^-(e) - \gamma \cdot \lambda^+(e) \]
oscillation, the sensitivity in the iteration direction is weighted [8]. The weighted sensitivity is obtained by

\[ \frac{\partial L_e^{(k)}}{\partial \rho_e} = \begin{cases} \frac{\partial L_e^{(k)}}{\partial \rho_e} - \frac{\partial L_e^{(k-1)}}{\partial \rho_e} & (k < 3) \\ \kappa^{(k)} \frac{\partial L_e^{(k)}}{\partial \rho_e} + (1 - \kappa^{(k)}) \frac{\partial L_e^{(k-1)}}{\partial \rho_e} & (k \geq 3) \end{cases} \]  

(16)

If the number of iterations \( k \) is less than three, the conventional sensitivity is used. In contrast, if the number of iterations \( k \) is three or more, the numerical oscillation relaxation parameter \( \kappa \) determines the weighted sensitivity. The numerical oscillation relaxation parameter \( \kappa \) is a variable function for each iteration and is calculated using the following steps. First, to determine whether if there is oscillation in the iteration from \( k \) to \( k-2 \), the parameter for the oscillation of sensitivity \( \xi_e \) is calculated by

\[ \xi_e^{(k)} = \text{sign} \left( \frac{\partial L_e^{(k)}}{\partial \rho_e} - \frac{\partial L_e^{(k-1)}}{\partial \rho_e} - \frac{\partial L_e^{(k-2)}}{\partial \rho_e} \right) \]  

(17)

Equation (17) is a sign function, and, if numerical oscillation occurs, \( \xi_e = -1 \). In contrast, if the numerical oscillation does not occur, \( \xi_e = 1 \). Next, the function \( \mu_e \) is determined.

\[ \mu_e^{(k)} = \begin{cases} c_a \kappa^{(k-1)} \xi_e^{(k)} = 1 \\ c_b \kappa^{(k-1)} \xi_e^{(k)} = -1 \end{cases} \]  

(18)

where \( c_a \) and \( c_b \) are the parameters for weighted sensitivity. Finally, the oscillation relaxation parameter \( \kappa \) is calculated to aggregate the function \( \mu_e \), which is calculated for each element. The equation of the oscillation relaxation parameter \( \kappa \) is written as

\[ \kappa^{(k)} = \min \left[ \left( \frac{1}{\Omega} \int_{\Omega} \mu_e^{(k)} \partial \Omega \right)^{-c_e}, 1 \right] \]  

(19)

where \( c_e \) is the parameter for weighted sensitivity.

### 3. Computational conditions

In this section, the computational conditions are described. The computational model employed is the cantilever beam model, which is a 3D linear elastic body, as shown in Fig. 1. The size of the model was \( 40[\text{mm}] \times 10[\text{mm}] \times 32[\text{mm}] \), and the size of the mesh was \( 1[\text{mm}] \times 1[\text{mm}] \times 1[\text{mm}] \). In addition, Fig. 2 shows the impact load applied to an element. As shown in Fig. 1, the \( Y-Z \) surface was completely fixed, and uniformly distributed load was applied to the central part, at the tip of the cantilever beam model. Other computational conditions for topology optimization are shown in Table 1. Because, in this study, the usefulness of weighted sensitivity was examined, four conditions were analyzed. Case 1 uses the conventional sensitivity, Cases 2–4 uses the weighted sensitivity as shown in Table 2. As parameter settings, Case 2 has a large weight of \( \partial L_e/\partial \rho_e \), Case 3 has an intermediate weight, and Case 4 has a large weight of \( \partial L_e/\partial \rho_e \).

### 4. Numerical experiments

In this section, the numerical experiments in the topology optimization analysis are described. Figs. 3 and 4 show the side views of the sensitivity distribution at convergence and the side views of the density distribution at convergence, respectively. When conventional sensitivity is used, the sensitivity distribution is presents some oscillations as shown in Fig. 3(a), hence, the density distribution is also oscillatory as shown in Fig. 4(a). However, when weighted sensitivity is used, the sensitivity distributions, as shown in Figs 3(b) to 3(d), and the density distributions, as shown in Figs 4(b) to 4(d), do not present any oscillation. By smoothing the numerical oscillation of the sensitivity in the iteration direction, it is considered that the sensitivity distribution of the
low value tends to be dense; as a result, a clear density distribution at convergence is obtained. The histories of the normalized performance functions and the displacement waveforms in the $Z$ direction at the load point, which is at $(40, 5, 16)$, are shown in Figs 5 and 6 for the validity of the distribution at convergence. Figs 5 and 6 confirm that the results do not deviate significantly from the results of Case 1. As a result, a solution was obtained with a small number of iterations by using the weighted sensitivity. In addition, in all cases, the value of the performance function at convergence was almost the same. However, the density distribution was slightly different, as shown in Figs 4(b) to 4(d), depending on the setting of the parameters $c_a$, $c_b$, and $c_c$. Moreover, if extreme parameters are set, the solution may deviate significantly.

5. Conclusions

In this study, the application of the weighted sensitivity approach was examined for topology optimization analysis in a 3D dynamic oscillation problem. The self-adjoint relation was derived, and the gradient of the Lagrange function for the density, which is the sensitivity, was calculated. The problem with this technique is that a non oscillatory density distribution cannot be obtained depending on the conditions. To solve this problem, the weighted sensitivity approach was employed. As a result, a clear density distribution was obtained by suppressing the numerical oscillation of the sensitivity in the search direction. Furthermore, the judgment of convergence was satisfied with a small number of iterations. Planned future work includes carrying out more parameter studies and investigating other models.

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