GEOMETRIC INEQUALITIES AND RIGIDITY OF GRADIENT SHRINKING RICCI SOLITONS

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ABSTRACT. In this paper we prove that the Sobolev inequality, the logarithmic Sobolev inequality, the Schrödinger heat kernel upper bound, the Faber-Krahn inequality, the Nash inequality and the Rozenblum-Cwikel-Lieb inequality all equivalently exist on complete gradient shrinking Ricci solitons. We also obtain some integral gap theorems for compact shrinking Ricci solitons.

1. Introduction

In this paper we will investigate the equivalence of various geometric inequalities on gradient shrinking Ricci solitons. As applications, We apply the Sobolev inequality to give some integral gap theorems for compact shrinking Ricci solitons. Recall that an $n$-dimensional Riemannian manifold $(M, g)$ is called a gradient shrinking Ricci soliton or shrinkers (see [28]) if there exists a smooth function $f$ on $M$ such that the Ricci curvature $\text{Ric}$ and the Hessian of $f$ satisfy

$$\text{Ric} + \text{Hess} f = \lambda g$$

for some positive number $\lambda$. Function $f$ is often called a potential of the shrinker. For simplicity, we often normalize $\lambda = \frac{1}{2}$ by scaling the metric $g$ so that

$$\text{Ric} + \text{Hess} f = \frac{1}{2}g.$$ 

According to the work of [28, 9], without loss of generality, adding $f$ by a constant if necessary, we can assume that equation (1.2) simultaneously satisfies

$$R + |\nabla f|^2 = f \quad \text{and} \quad (4\pi)^{-\frac{n}{2}} \int_M e^{-f} dv = e^\mu,$$

where $R$ is the scalar curvature of $(M, g)$ and $\mu = \mu(g, 1)$ is the entropy functional of Perelman [16]; see also the detailed explanation in [37] or [53, 56]. For a compact shrinker, $\mu$ has a lower bound; but for the non-compact case, we generally need to assume $\mu > -\infty$ so that our discussion makes sense. In particular, for the Euclidean space, we have $\mu = 0$. In [37], Li, Li and Wang proved that $e^\mu$ is nearly equivalent to $V(p_0, 1)$, i.e., the volume of geodesic ball $B(p_0, 1)$ centered at point $p_0 \in M$ and radius 1. Here $p_0 \in M$ is a point where $f$ attains its infimum, which always exists on shrinkers but possibly is not unique; see [29].

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In this paper we always let the triple \((M, g, f)\) denote the shrinking gradient Ricci soliton (or shrinker) with (1.2) and (1.3) satisfying \(\mu > -\infty\).

Shrinkers are natural extension of Einstein manifolds and can be regarded as the critical point of Perelman’s \(W\)-functional [46]. In addition, shrinkers are self-similar solutions to the Ricci flow and naturally rise as singularity analysis of the Ricci flow [28]. For example, Enders, Müller and Topping [20] proved that the proper rescaling limits of a type-I singularity point always converge to non-trivial shrinkers. At present, one of main issue in the Ricci flow theory is the understanding on the geometry and classification of shrinkers. For dimensions 2 and 3, the classification is complete by the works of [27], [35], [46], [44] and [5]. For dimension 4 and higher, the classification remains open, though much progress has been made. The interested reader can refer to [4] for an excellent survey.

In recent years, many geometric and analytic results about shrinkers have been investigated. Wylie [61] proved that any complete shrinker has finite fundamental group (the compact case due to Derdziński [18]). Chen [14] showed that the scalar curvature \(R \geq 0\); Pigola, Rimoldi and Setti [48] proved that \(R > 0\) unless \((M, g, f)\) is the Gaussian shrinker; Chow, Lu and Yang [3] showed that the scalar curvature of non-trivial shrinkers has at least quadratic decay of distance function. Chen and Zhou [7] showed that the potential function \(f\) is uniformly equivalent to the distance squared; they [7] also showed that all shrinkers have at most Euclidean volume growth by combining an observation of Munteanu [41]. Later, Munteanu and Wang [42] proved that shrinkers have at least linear volume growth. These volume growth properties are similar to manifolds with nonnegative Ricci curvature.

On the other hand, Haslhofer and Müller [29, 30] proved a Cheeger-Gromov compactness theorem of shrinker. Huang [33] proved an \(\epsilon\)-regularity theorem for 4-dimensional shrinkers, which was later improved by Ge and Jiang [22]. Their result gives an answer to Cheeger-Tian’s question [13]. In [53], the author applied gradient estimate technique to prove a Liouville type theorem for ancient solutions to the weighted heat equation on shrinkers. In [57, 58], P. Wu and the author applied weighted heat kernel upper estimates to give a sharp weighted \(L^1\)-Liouville theorem for weighted subharmonic functions on shrinkers.

By analyzing the Perelman’s functional under the Ricci flow, Li and Wang [39] obtained a sharp logarithmic Sobolev inequality on complete (possible non-compact) shrinkers, which says that for any \(\tau > 0\),

\[
\int_M \varphi^2 \ln \varphi^2 dv \leq \tau \int_M \left( 4|\nabla \varphi|^2 + R \varphi^2 \right) dv - \left( \mu + n + \frac{n}{2} \ln(4\pi\tau) \right).
\]

for any compactly supported locally Lipschitz function \(\varphi\) with \(\int_M \varphi^2 dv = 1\). The equality case could be attained when \(\tau = 1\) (see Carrillo and Ni [9]). This sharp inequality is useful for understanding shrinkers. Indeed the author [56] was able to apply (1.4) to give sharp upper diameter bounds of compact shrinkers in terms of the integral of scalar curvature. The author [55] also used (1.4) to study the Schrödinger heat kernel on shrinkers. Here the definition of the Schrödinger heat kernel is similar to the classical heat kernel. That is, for each \(y \in M\), we say that \(H^R(x, y, t)\) is called the Schrödinger heat kernel if \(H^R(x, y, t) = u(x, t)\) is a minimal positive smooth solution of the Schrödinger heat equation

\[-\Delta u + \frac{R}{4} u + \partial_t u = 0\]

satisfying \( \lim_{t \to 0} u(x, t) = \delta_y(x) \), where \( \delta_y(x) \) is the delta function defined as
\[
\int_M \phi(x) \delta_y(x) dv = \phi(y)
\]
for any \( \phi \in C^\infty_0(M) \). In general, the Schrödinger heat kernel always exists on compact shrinkers. For non-compact shrinkers, we do not know if it still exists without any assumption but it indeed exists when the scalar curvature of \( (M, g, f) \) is bounded. The Schrödinger heat kernel of shrinkers shares many kernel properties of the classical Laplacian heat kernel on manifolds; see [55]. In this paper, we always assume that the Schrödinger heat kernel exists on \( n \)-dimensional complete shrinker \( (M, g, f) \).

In [55] the author applied (1.4) to prove that
\[
H^R(x, y, t) \leq \frac{e^{-\mu}}{(4\pi t)^{\frac{n}{2}}}
\]
for all \( x, y \in M \) and \( t > 0 \). We remark that this type upper bound of the conjugate heat kernel under the Ricci flow was obtained by Li and Wang [39]. The author also obtained its Gaussian type upper bounds by the iteration argument. That is, for any \( \alpha > 4 \), the author showed that there exists a constant \( A = A(n, \alpha) \) depending on \( n \) and \( \alpha \) such that
\[
H^R(x, y, t) \leq Ae^{-\mu} \exp \left( -\frac{d^2(x, y)}{\alpha t} \right)
\]
for all \( x, y \in M \) and \( t > 0 \), where \( d(x, y) \) denotes the geodesic distance between \( x \) and \( y \). Considering the classical Laplace heat kernel of Euclidean space, estimate (1.6) is obvious sharp. Moreover the heat kernel estimate is useful for analyzing eigenvalues of the Schrödinger operator. Indeed the author [55] used the upper bounds to get lower bounds of their eigenvalues. Namely, for any open relatively compact set \( \Omega \subset M \), let \( 0 < \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \ldots \) be the Dirichlet eigenvalues of the Schrödinger operator in \( \Omega \). Then we have
\[
\lambda_k(\Omega) \geq \frac{2n\pi}{e} \left( \frac{k e^{\mu}}{V(\Omega)} \right)^{2/n}, \quad k \geq 1,
\]
where \( V(\Omega) \) is the volume of \( \Omega \). Recall that the classical Weyl’s asymptotic formula of the \( k \)-th Dirichlet eigenvalue of Laplacian in open relatively compact set \( \Omega \subset \mathbb{R}^n \) states that
\[
\lambda_k(\Omega) \sim c(n) \left( \frac{k}{V(\Omega)} \right)^{2/n}, \quad k \to \infty,
\]
which indicates that (1.7) is sharp for the exponent \( 2/n \). We remark that by the Rozenblum-Cwikel-Lieb inequality [49, 16, 40] (see also estimate (3.1) in Section 3), we easily get that eigenvalues of the Schrödinger operator \( -\Delta + \frac{R}{4} \) on non-trivial shrinkers are all positive.

Besides of the above results, combining (1.4) and the Markov semigroup technique of Davies [17], Li and Wang [39] proved a local Sobolev inequality of shrinkers. Namely, for each compactly supported locally Lipschitz function \( \varphi \) in \( M \),
\[
\left( \int_M \varphi^{2n/n} dv \right)^{\frac{n-2}{n}} \leq C(n)e^{-\frac{2}{\mu}} \int_M (4|\nabla \varphi|^2 + R\varphi^2) dv
\]
for some constant $C(n)$ depending only on $n$. Here $e^\mu$ can be viewed as the volume of unit geodesic ball and hence this Sobolev inequality is very similar to the classical Sobolev inequality on manifolds, which plays an important role in some PDE ways. For example, P. Wu and the author [59] applied (1.8) to study the dimensional estimates for the spaces of harmonic functions and Schrödinger functions with polynomial growth. In [54], the author used (1.8) to derive a mean value type inequality and further study the analyticity in time for solutions of the heat equation on shrinkers.

In this paper we continue to study geometric inequalities and their relations on shrinkers. We first show that the above geometric inequalities, and the Nash inequality, the Rozenblum-Cwikel-Lieb inequality all equivalently exist on shrinkers, which may be regarded as natural generalizations of the case of manifolds with nonnegative Ricci curvature [23, 50, 62].

**Theorem 1.1.** Let $(M, g, f)$ be an $n$-dimensional complete (compact or noncompact) shrinker. The following six properties are equivalent up to constants.

(I) The Sobolev inequality (1.8) holds.

(II) The logarithmic Sobolev inequality (1.4) holds.

(III) The Schrödinger heat kernel upper bound (1.6) holds.

(IV) The Faber-Krahn inequality holds. That is, for all open relatively compact set $\Omega \subset M$ with smooth boundary,

$$
\lambda_1(\Omega) \geq \frac{2n\pi}{e} \left( \frac{e^\mu}{V(\Omega)} \right)^\frac{1}{n},
$$

where $\lambda_1(\Omega)$ is the lowest Dirichlet eigenvalue of the Schrödinger operator in $\Omega$.

(V) The Nash inequality holds. That is, there exists a constant $c(n)$ depending on $n$ such that

$$
\| \varphi \|_2^{2 + \frac{4}{n}} \leq c(n)e^{-\frac{2\mu}{n}} \| \varphi \|_1^{\frac{2}{n}} \int_M \left( 4|\nabla \varphi|^2 + R\varphi^2 \right) dv
$$

for any compactly supported locally Lipschitz function $\varphi$ in $M$.

(VI) The Rozenblum-Cwikel-Lieb inequality holds. That is, there exists a constant $c(n)$ depending on $n$ such that

$$
\mathcal{N}\left(-\Delta + \frac{n}{4} + V\right) \leq c(n)e^{-\mu} \int_M V_{-}^\frac{n}{2} dv,
$$

for any function $V \in L^1_{loc}(M)$, where $V_- := \max\{0, -V\} \in L^{n/2}(M)$ is the non-positive part of $V$, and $\mathcal{N}(A)$ is the number of non-positive $L^2$-eigenvalues of the operator $A$, counting multiplicity.

**Remark 1.2.** For (III), (IV) and (VI), we need to assume the existence of Schrödinger heat kernel on complete shrinkers because our proof will be involved with the Schrödinger heat kernel. For compact shrinkers, the Schrödinger heat always exists; but for the non-compact case, we only know that it exists when scalar curvature is bounded (see [55]).

**Remark 1.3.** We point out that (III) is equivalent to (1.5). Indeed, (III) $\Rightarrow$ (1.5) is obvious, while (1.5) $\Rightarrow$ (III) due to the work [55]. We also point out that (IV) is equivalent to (1.7). Indeed, from Theorem 1.1, we first have (IV) $\Rightarrow$ (III), and from the work [55], we then know (III) $\Rightarrow$ (1.7). Combining two parts finally yields (IV) $\Rightarrow$ (1.7). The converse is trivial.
Remark 1.4. Estimates in (III) and (IV) are both sharp; see [55]. In addition (1.10) is also sharp in some sense. Indeed on Gaussian shrinker \((\mathbb{R}^n, \delta_{ij}, |x|^2)\), where \(\delta_{ij}\) is the standard flat Euclidean metric, we know \(R = 0\) and \(\mu = 0\). For a large parameter \(\alpha\), replacing \(V\) by \(\alpha V\) in (1.10),
\[
\alpha^{-\frac{n}{2}} \mathcal{N}(-\Delta + \alpha V) \leq c(n) \int_{\mathbb{R}^n} V^{-\frac{n}{2}} dv.
\]
In turns out that for any potential \(V\) with \(V^- \in L^{n/2}(\mathbb{R}^n)\) and \(V \in L^1_{loc}(\mathbb{R}^n)\), we have the Weyl asymptotics
\[
\lim_{\alpha \to \infty} \alpha^{-\frac{n}{2}} \mathcal{N}(-\Delta + \alpha V) = \left(\frac{2\sqrt{\pi}}{\Gamma(1 + \frac{n}{2})}\right) \int_{\mathbb{R}^n} V^{-\frac{n}{2}} dv.
\]
This indicates that (1.10) is sharp in order in \(\alpha\) for a class function of \(V\).

The proof strategy of Theorem 1.1 is as follows. Li and Wang [39] proved that (II) holds on shrinkers and they confirmed that (II) \(\Rightarrow\) (I). The author [55] proved that (II) \(\Rightarrow\) (III) \(\Rightarrow\) (IV). The rest part is the following.

(1) We will apply the Jensen inequality to confirm (I) \(\Rightarrow\) (II).
(2) We shall apply the Davies’ argument [17] and the Markov semigroup to reprove Li-Wang’s Sobolev inequality (1.8), i.e., (III) \(\Rightarrow\) (I). In particular we will talk about the scope of Sobolev constant, which will be useful to study gap theorems.
(3) We will apply the Schrödinger heat kernel and the level set method to prove (IV) \(\Rightarrow\) (III) by the approximation argument.
(4) We will use the Hölder inequality to prove (I) \(\Rightarrow\) (V).
(5) By the approximation argument, we only need to apply the Schrödinger heat kernel and analytical technique to prove (V) \(\Rightarrow\) (III) with Dirichlet condition.
(6) We will apply Schrödinger heat kernel properties and some functional theory to prove (I) \(\iff\) (VI).

We remark that the proof of (III) \(\Rightarrow\) (I) will be separately provided in Section 2; the proof of (I) \(\iff\) (VI) will be given in Section 3; the rest cases will be discussed in Section 4.

As applications, we will apply the Sobolev inequality of shrinkers to give integral gap results for the Weyl tensor on compact shrinkers by adapting the proof strategy of [10]. To state the result, we fix some notations. We denote by \(W\), \(\text{Ric}\) and \(V(M)\) the Weyl tensor, traceless Ricci tensor and the volume of manifold \((M, g)\) respectively. The norm of a \((k, s)\)-tensor \(T\) of \((M, g)\) is defined by \(|T|_g^2 := g^{im_1} \cdots g^{k m_k} g_{j_1 m_1} \cdots g_{j_s m_s} T_{i_1 \cdots i_k}^{j_1 \cdots j_s} T_{m_1 \cdots m_k}^{m_1 \cdots m_k}\).

Theorem 1.5. Let \((M, g, f)\) be an \(n\)-dimensional, \(4 \leq n \leq 8\), compact shrinker. If
\[
\left( \int_M \left| W + \frac{\sqrt{2}}{\sqrt{n(n-2)}} \text{Ric} \circ g \frac{|T|^2}{2} dv \right|^\frac{2}{n} \right)^\frac{n}{2} + \left( \sqrt{n} - \sqrt{\frac{n-1}{2(n-2)}} \right) V(M)^{\frac{n}{2}} < \left( \frac{1}{\sqrt{n}} - \frac{1}{n} \sqrt{\frac{2(n-2)}{n-1}} \right) \frac{C(n)}{C(n)}
\]
where \(\circ\) denotes the Kulkarni-Nomizu product and \(C(n)\) is the constant in the Sobolev inequality (1.8), then \((M, g, f)\) is isometric to a quotient of the round sphere.
Remark 1.6. Chang, Gursky and Yang [12] obtained integral gap results for compact manifolds in terms of the Yamabe constant. Catino [10] proved some integral gap results for compact shrinkers, which was later improved in [21]. Our result involves the the Sobolev constant of shrinkers rather than the Yamabe constant.

The main ingredients of proving Theorem 1.5 are Bochner-Weitzenböck type formulas for the norm of curvature tensors, curvature algebraic inequalities and Kato type inequalities. The above pinching assumption is not true when \( n \geq 9 \). Indeed we will see that constant \( C(n) \) in Sobolev inequality (1.8) cannot be sufficiently small (see Remark 2.5), i.e.,
\[
C(n) \geq \frac{n - 1}{2n(n - 2)\pi e}.
\]
This range obviously affects the dimensional valid of the pinching assumption. On the other hand, algebraic curvature inequalities and elliptic equation of the norm of traceless Ricci tensor also restrict the choice of dimension \( n \), such as inequality (5.5) in Section 5.

In particular, inspired by Cao-Tran’s result [8], we apply the Sobolev inequality of shrinker to get an integral gap result for the half Weyl tensor on compact four-dimensional shrinkers.

**Theorem 1.7.** Let \((M, g, f)\) be a four-dimensional oriented compact shrinker. Let \(C(4)\) denote the constant in the Sobolev inequality (1.8) in \((M, g, f)\). If
\[
\left( \int_M |W^{\pm}|^2 dv \right)^{\frac{1}{2}} < \frac{e^{\frac{\mu}{2}}}{4\sqrt{6}C(4)}
\]
and
\[
\int_M |\delta W^{\pm}|^2 dv \leq \frac{1}{8} \int_M R|W^{\pm}|^2 dv,
\]
then \(W^{\pm} \equiv 0\) and hence \((M^4, g, f)\) is isometric to a finite quotient of the round sphere or the complex projective space.

**Remark 1.8.** For relevant notations, see Section 6. Gursky [25] proved an integral pinching result for four-dimensional Einstein manifolds involving the norm of half Weyl tensor in terms of the Euler characteristic and the signature. Cao and Tran [8] generalized Gursky’s result to shrinkers. Our gap result depends on the Sobolev constant of shrinkers rather than topological invariants.

Inspired by Catino’s result [8], if we use the Yamabe constant instead of the Sobolev inequality, then we have another integral gap result.

**Theorem 1.9.** Let \((M, g, f)\) be a four-dimensional oriented compact shrinker satisfying (1.1). If
\[
216 \int_M |W^{\pm}|^2 dv + 12 \int_M |\overset{\circ}{\text{Ric}}|^2 dv \leq \int_M R^2 dv
\]
and
\[
\int_M |\delta W^{\pm}|^2 dv \leq \frac{1}{6} \int_M R|W^{\pm}|^2 dv,
\]
then \((M, g, f)\) is isometric to a finite quotient of the round sphere or the complex projective space.
Remark 1.10. In Theorems 1.7 and 1.9, the first assumption is an pinching condition of half Weyl tensor; while the second assumption is an pinching condition of harmonic half Weyl tensor. It is an interesting question if the second assumption is unnecessary.

There are many gap results for Einstein manifolds, Ricci solitons and closed manifolds; such as Catino and Mastroia [11], Hebey and Vaugon [32], Li and Wang [38, 39], Munteanu and Wang [43], Petersen and Wylie [47], Singer [51], Tran [52], Zhang [63] and their references. In this paper we provide a different gap criterion, which depends on the constant $C(n)$ of Sobolev inequality. It is an interesting question to estimate a best upper bound of $C(n)$ on shrinkers.

The structure of this paper is the following. In Section 2, we will recall some basic results about algebraic inequalities of curvature tensors and some formulas of shrinkers. In particular, we will reprove the Sobolev inequality by the Schrödinger heat kernel upper bound. Meanwhile, we will discuss the best Sobolev constant of shrinkers. In Section 3, we will apply the Schrödinger heat kernel to study the equivalence between (I) and (VI) of Theorem 1.1. In Section 4, we will prove the rest cases of Theorem 1.1. In Section 5, we will apply the Sobolev inequality of shrinkers and Weitzenböck formulas for curvature tensors to prove Theorem 1.5. In Section 6, we will study the gap results for half Weyl tensor. We shall prove Theorems 1.7 and 1.9.

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2. Decomposition and Sobolev inequality

In this section we first give a brief introduction of curvature notations of the Riemannian manifold $(M^n, g)$ and some algebraic inequalities of curvature tensors. Then we review some geometric equations and formulas about shrinkers, especially for the Sobolev inequality and its explicit coefficient. These results will be used in the following sections. For more related results, see [39, 55].

We use $g_{ij}$ to be the local components of metric $g$ and its inverse by $g^{ij}$. Let $Rm$ be the $(4,0)$ Riemannian curvature tensor, whose local components denoted by $R_{ijkl}$. Let $Ric$ denote the Ricci curvature with local components $R_{ik} = g^{jl}R_{ijkl}$, and let $R = g^{ij}R_{ik}$ be the scalar curvature. The traceless Ricci tensor is denoted by

$$^\circ \text{Ric} = \text{Ric} - \frac{1}{n}Rg,$$

whose local components

$$^\circ R_{ik} = R_{ik} - \frac{1}{n}Rg_{ik}.$$

When $n \geq 4$, the Weyl tensor $W$ is defined by the orthogonal decomposition

$$W = Rm - \frac{R}{2n(n-1)}g \circ g - \frac{1}{n-2}^\circ \text{Ric} \circ g,$$

where $\circ$ denotes the Kulkarni-Nomizu product for two symmetric tensors $A$ and $B$, which is defined as:

$$(A \circ B)_{ijkl} = A_{ik}B_{jl} - A_{il}B_{jk} - A_{jk}B_{il} + A_{jl}B_{ik}.$$
In local coordinates, we can write $W$ as

$$W_{ijkl} = R_{ijkl} - \frac{1}{n-2} (g_{ik} R_{jl} + g_{jl} R_{ik} - g_{il} R_{jk} - g_{jk} R_{il})$$

$$+ \frac{1}{(n-1)(n-2)} R (g_{ik} g_{jl} - g_{il} g_{jk}).$$

The Weyl tensor has the same algebraic symmetries as the Riemannian curvature tensor. It is well-known that the Weyl tensor is totally trace-free and it is conformal invariant:

$$W(e^{2\varphi} g) = e^{2\varphi} W(g)$$

for any smooth function $\varphi$ on $M$.

In [10], Catino proved two algebraic curvature inequalities for any $n$-dimensional Riemannian manifold, which will be used in the gap theorems.

**Lemma 2.1.** Each $n$-dimensional Riemannian manifold $(M^n, g)$ satisfies the estimate

$$\left| -W_{ijkl} R_{ik} R_{jl} + \frac{2}{n-2} W_{ij} R_{jk} R_{ik} \right| \leq \sqrt{\frac{n-2}{2(n-1)}} \left( |W|^2 + \frac{8|Ric|^2}{n(n-2)} \right)^{\frac{1}{2}} |Ric|^2.$$

**Lemma 2.2.** On an $n$-dimensional Riemannian manifold $(M^n, g)$, there exists a positive constant $c(n)$ such that

$$2W_{ijkl} W_{ipkj} W_{pjql} + \frac{1}{2} W_{ijkl} W_{klpq} W_{pqlj} \leq c(n)|W|^3.$$

We can take $c(4) = \frac{\sqrt{6}}{4}$, $c(5) = 1$, $c(6) = \frac{\sqrt{70}}{2\sqrt{3}}$ and $c(n) = \frac{5}{2}$ for $n \geq 7$.

On shrinker $(M, g, f)$, by Proposition 2.1 in [19], we have the following basic formulas, which will also be used in the proof of gap theorems.

**Lemma 2.3.** Let $(M, g, f)$ be an $n$-dimensional complete shrinker. Then,

$$\Delta f = \frac{n}{2} - R,$$

$$\Delta f R = R - 2|Ric|^2,$$

$$\Delta f R_{ik} = R_{ik} - 2W_{ijkl} R_{jl} + \frac{2}{(n-1)(n-2)}$$

$$\times \left( R^2 g_{ik} - nRR_{ik} + 2(n-1) g^{mn} R_{im} R_{nk} - 2(n-1)|Ric|^2 g_{ik} \right),$$

where $\Delta f := \Delta - \nabla f \cdot \nabla$.

In the end of this section, we will give the following Sobolev inequality of shrinkers, which was proved by Li and Wang [39] by using the logarithmic Sobolev inequality. Here we shall reprove it by using the upper bound of Schrödinger heat kernel. Meanwhile we will provide an explicit Sobolev constant and discuss its range. This constant will play a key role in proving gap results of shrinkers.
Lemma 2.4. Let \((M, g, f)\) be an \(n\)-dimensional complete shrinker. Then for each compactly supported locally Lipschitz function \(\varphi\) in \(M\),

\[
\left( \int_M \varphi^{\frac{2n}{n-2}} \, dv \right)^{\frac{n-2}{n}} \leq C(n) e^{-\frac{2n}{n-2}} \int_M \left( 4|\nabla \varphi|^2 + R \varphi^2 \right) \, dv
\]

for some dimensional constant \(C(n)\). In particular, we can take

\[
C(n) = \frac{1}{\pi^2} \left( \frac{2}{n-2} \right)^{\frac{4}{n}}.
\]

Remark 2.5. For an \(n\)-sphere \(S^n\) of radius \(\sqrt{2(n-1)}\) with its standard metric, we have \(\text{Ric} = \frac{1}{2} g\). Recall that on \(S^n\), Aubin [1] (see also Proposition 4.21 in [31]) proved that for any \(\varphi \in W^{1,2}(S^n)\),

\[
\left( \int_{S^n} \varphi^{\frac{2n}{n-2}} \, dv \right)^{\frac{n-2}{n}} \leq \frac{8(n-1)}{n(n-2)} V(S^n)^{-\frac{2}{n-2}} \int_{S^n} |\nabla \varphi|^2 \, dv + V(S^n)^{-\frac{2}{n-2}} \int_{S^n} \varphi^2 \, dv.
\]

This inequality is optimal in the sense that the two constants \(\frac{8(n-1)}{n(n-2)} V(S^n)^{-\frac{2}{n-2}}\) and \(V(S^n)^{-\frac{2}{n-2}}\) can not be lowered. On the other hand, from [13], we see that

\[
R = f = \frac{n}{2} \quad \text{and} \quad (4\pi e)^{-\frac{n}{2}} V(S^n) = e^{\mu}.
\]

Substituting them into (2.1) and comparing with (2.2), we easily conclude that

\[
C(n) \geq \frac{n-1}{2n(n-2)\pi e}.
\]

Proof of Lemma 2.4. We essentially follow the argument of Davies [17] (see also [39, 62]). Since \(H^R = H^R(x, y, t)\) is the Schrödinger heat kernel of the operator \(-\Delta + \frac{R}{4}\), then

\[
\int_M H^R(x, y, t) \, dv(y) \leq 1.
\]

In [55] we proved an upper of the Schrödinger heat kernel

\[
H^R(x, y, t) \leq \frac{e^{-\mu}}{(4\pi t)^{\frac{n}{2}}}
\]

for all \(x, y \in M\) and \(t > 0\). In the following we will use this estimate to prove (2.1).

Now using the Hölder inequality, for any \(u(x) \in L^2(M)\), we have

\[
\| H^R * u \|_\infty = \sup_{x \in M} \left| \int_M H^R(x, y, t) u(y) \, dv(y) \right| \leq \sup_{x \in M} \left( \int_M (H^R)^2(x, y, t) \, dv(y) \right)^{\frac{1}{2}} \cdot \| u \|_2.
\]
By (1.5), we further have
\[
\| H^R * u \|_\infty \leq \frac{e^{-\mu}}{(4\pi t)^{\frac{2}{n}}} \sup_{x \in M} \left( \int_M H^R(x, y, t) dv(y) \right)^{\frac{1}{2}} \cdot \| u \|_2
\]
\[
\leq \frac{c_1^{1/2}}{t^{n/4}} \| u \|_2,
\]
where \( c_1 = \frac{e^{-\mu}}{(4\pi)^{n/4}} \). Similarly, by the Hölder inequality, for any \( u(x) \in L^q(M) \), \( q \in [1, n] \) and \( \tilde{q} = q/(q-1) \), we may obtain another estimate
\[
\| H^R * u \|_\infty \leq \sup_{x \in M} \left( \int_M (H^R)^{\tilde{q}}(x, y, t) dv(y) \right)^{1/\tilde{q}} \cdot \| u \|_q
\]
\[
\leq \sup_{x \in M} \left( \sup_{y \in M} (H^R)^{\tilde{q}}(x, y, t) \int_M H^R(x, y, t) dv(y) \right)^{1/\tilde{q}} \cdot \| u \|_q
\]
\[
\leq \sup_{x \in M} \left( \sup_{y \in M} (H^R)^{\tilde{q}}(x, y, t) \right)^{1/\tilde{q}} \cdot \sup_{x \in M} \left( \int_M H^R(x, y, t) dv(y) \right)^{1/\tilde{q}} \cdot \| u \|_q.
\]
Using (1.5) again, for any \( u(x) \in L^q(M) \) and \( q \in [1, n] \), we finally get
\[
(2.3) \quad \| H^R * u \|_\infty \leq \frac{c_1^{1/q}}{t^{\frac{n}{2q}}} \| u \|_q.
\]
Now we consider the integral operator
\[
L := \left( \sqrt{-\Delta + \frac{R}{4}} \right)^{-1}.
\]
Since \(-\Delta + \frac{R}{4}\) is a self-adjoint operator, by the eigenfunction expansion, for any \( u(x) \in C^\infty_0(M) \), we have
\[
(Lu)(x) = \Gamma(1/2)^{-1} \int_0^\infty t^{-\frac{1}{2}} (e^{(\Delta-R/4)t} u)(x, t) dt
\]
\[
= \Gamma(1/2)^{-1} \int_0^\infty t^{-\frac{1}{2}} (H^R * u)(x, t) dt,
\]
where \( e^{(\Delta-R/4)t} u \) denotes the semigroup of \( H^R * u \). Fix \( T > 0 \), which will be determined later and let
\[
(Lu)(x) = \Gamma(1/2)^{-1} \int_0^T t^{-\frac{1}{2}} (H^R * u)(x, t) dt + \Gamma(1/2)^{-1} \int_T^\infty t^{-\frac{1}{2}} (H^R * u)(x, t) dt
\]
\[
:= (L_1 u)(x) + (L_2 u)(x).
\]
For any \( \lambda > 0 \), we see that
\[
(2.4) \quad \{ x \left| (Lu)(x) \geq \lambda \right. \} \leq \{ x \left| (L_1 u)(x) \geq \lambda/2 \right. \} + \{ x \left| (L_2 u)(x) \geq \lambda/2 \right. \}.
\]
By (2.3) and the definition of $L_2 u$, since $\Gamma(1/2) = \sqrt{\pi}$, we have
\[
\| L_2 u \|_\infty \leq \Gamma(1/2)^{-1} \int_0^T t^{-1/2} \left( \frac{c_1^{1/q}}{t^{3/2}} \| u \|_q \right) dt
\]
\[
= \frac{2qc_1^{1/q}}{(n-q)\sqrt{\pi}} \cdot T^{1-n/2} \| u \|_q .
\]
We now choose $T$ such that
\[
\frac{\lambda}{2} = \frac{2qc_1^{1/q}}{(n-q)\sqrt{\pi}} \cdot T^{1-n/2} \| u \|_q .
\]
Then the set $\{ \{ x \mid (L_2 u)(x) > \lambda/2 \} = \emptyset$ and (2.4) becomes
\[
\left| \{ x \mid |(Lu)(x)| \geq \lambda \} \right| \leq \left| \{ x \mid |(L_1 u)(x)| \geq \lambda/2 \} \right|
\]
\[
\leq (\lambda/2)^{-q} \int_M |(L_1 u)(x)|^q dv(x).
\]
We will estimate the right hand side of the above inequality. By the Minkowski inequality for two measured spaces and the H"older inequality, we get that
\[
\| L_1 u \|_q \leq \Gamma(1/2)^{-1} \int_0^T t^{-1/2} \| H^R * u(\cdot, t) \|_q dt
\]
\[
\leq \Gamma(1/2)^{-1} \int_0^T t^{-1/2} \sup_{x \in M} \| H^R(x, \cdot, t) \|_1 \cdot \| u \|_q dt
\]
\[
\leq \frac{2}{\sqrt{\pi}} T^{1/2} \| u \|_q .
\]
Hence,
\[
\left| \{ x \mid |(Lu)(x)| \geq \lambda \} \right| \leq \left( \frac{4}{\sqrt{\pi}} \right)^q \lambda^{-q} T^{q/2} \| u \|_q^q .
\]
According to the above choice of $T$, we have
\[
\left| \{ x \mid (Lu)(x) \geq \lambda \} \right| \leq c(n,q)c_{\frac{q}{n-q}} \lambda^{-r} \| u \|_q^r ,
\]
where $r = qn/(n-q)$ and
\[
c(n,q) = \left( \frac{4}{\sqrt{\pi}} \right)^{\frac{q}{n-q}} \left( \frac{q}{n-q} \right)^{\frac{q^2}{n-q}} .
\]
We see that for all $q \in [1, n)$, the linear operator $L$ could map the space $L^q(M)$ into the weak $L^r(M)$ space. That is,
\[
\| Lu \|_{r,w} \leq c(n,q)^{\frac{1}{r}} c_{\frac{1}{q}} \| u \|_q
\]
\[
= \frac{4}{\sqrt{\pi}} \left( \frac{q}{n-q} \right)^{\frac{n}{\pi}} c_1^{\frac{1}{n}} \| u \|_q .
\]
For any $0 < \epsilon << 1$, letting $q_1 = q - \epsilon$, $q_2 = q + \epsilon$, $r_i = q_i n / (n - q_i)$ ($i = 1, 2$), we indeed have

$$\| Lu \|_{r_i, w} \leq \frac{4}{\sqrt{\pi}} \left( \frac{q_i}{n - q_i} \right)^\frac{\frac{4}{3}}{c_1^n} \| u \|_{q_i}.$$  

Applying the Marcinkiewicz interpolation theorem to the above case, for any $0 < t < 1$, we get that

$$\| Lu \|_{b} \leq \frac{4}{\sqrt{\pi}} \left[ \left( \frac{q_1}{n - q_1} \right)^\frac{\frac{2}{3}}{c_1^n} \right]^t \left[ \left( \frac{q_2}{n - q_2} \right)^\frac{\frac{2}{3}}{c_1^n} \right]^{1-t} \| u \|_{a},$$

where

$$\frac{1}{a} = \frac{t}{q_1} + \frac{1 - t}{q_2}, \quad \frac{1}{b} = \frac{t}{r_1} + \frac{1 - t}{r_2}.$$ 

Since the coefficient is continuous with respect to $\epsilon$ at $\epsilon = 0$, letting $\epsilon \to 0+$ and choosing $q = 2$ and $p = 2n / (n - 2)$, then $a \to 2$ and $b \to 2n / (n - 2)$ and we finally get

$$\| Lu \|_{p} \leq \frac{4}{\sqrt{\pi}} \left( \frac{2}{n - 2} \right)^\frac{\frac{2}{3}}{c_1^n} \| u \|_2,$$

where $c_1 = \frac{e^{-n}}{(4\pi)^\frac{n}{2}}$. Let $\varphi = Lu$ and then $u = L^{-1} \varphi$ and

$$\| u \|_2^2 = \langle L^{-1} \varphi, L^{-1} \varphi \rangle = \langle L^{-2} \varphi, \varphi \rangle = \langle -\Delta \varphi + \frac{R}{4} \varphi, \varphi \rangle = \int_M (|\nabla \varphi|^2 + \frac{R}{4} \varphi^2) dv.$$

Substituting this into the above inequality proves the theorem. \(\square\)

In the above proof course, if we let $q_1 = 3$, $q_2 = 1$, $t = 3/4$, $r_1 = \frac{3n}{n-3}$ and $r_2 = \frac{n}{n-1}$, we can also take

$$C(n) = \frac{1}{\pi^2} \left( \frac{3}{n - 3} \right)^\frac{\frac{9}{2n}}{c_1^n} \left( \frac{1}{n - 1} \right)^\frac{\frac{1}{2n}}{c_1^n}.$$ 

Obviously, when $4 \leq n \leq 14$, this constant is bigger than the one in Lemma 2.4 but when $n \geq 15$, it is reverse. Naturally it is to ask

**Question.** For a complete gradient shrinking Ricci soliton $(M, g, f)$, specially for the compact case, What is the best constant $C(n)$?

### 3. Rozenblum-Cwikel-Lieb inequality

It is well-known that the spectrum of the Laplacian $-\Delta$ on Riemannian manifold $(M, g)$ is contained in the internal $[0, \infty]$. This can be proved by taking the Fourier transform and using the Plancherel theorem. If one considers the Schrödinger operator $-\Delta + V$ for some function $V$ on $M$, then $-\Delta + V$ may have some negative spectrum. However, if we have some restriction on $V$, like decay conditions at infinity, we may hope that the essential spectra of $-\Delta + V$ and $-\Delta$ coincide. In this case, the negative spectrum of $-\Delta + V$ is a discrete set with possibly an accumulation point at 0 (if 0 is indeed the bottom of the spectrum of $-\Delta$). It is an important question in mathematical physics to estimate the
number of these negative eigenvalues. One of beautiful results about this question is the Rozenblum-Cwikel-Lieb (RCL) inequality

$$\mathcal{N}(-\Delta + V) \leq c(n) \int_M V_+^2 dv,$$

where $V_-$ is the negative part of function $V \in L^1_{\text{loc}}(M)$, and $\mathcal{N}(A)$ is the number of non-positive $L^2$-eigenvalues of the operator $A$. The RCL inequality was first established by Rozenblum [49], and it was independently found by Lieb [40] and Cwikel [16] for $n \geq 3$. Afterwards, another remarkable proof with sharper constant were provided by Li and Yau [36], where their proof relies only on the positive of heat kernel and the Sobolev inequality.

On a shrinker $(M, g, f)$, one may would like to consider the special Schrödinger operator

$$-\Delta^R := -\Delta + \frac{R}{4}$$

instead of the usual Laplacian. Since scalar curvature $R \geq 0$ on $(M, g, f)$, by the RCL inequality, it is easy to see that its eigenvalues are all nonnegative. If one considers a perturbation of $-\Delta^R$ by a real-valued potential $V$ and defines another Schrödinger operator $-\Delta^R + V$, then the nonnegative property of eigenvalues is not necessarily satisfied. Naturally one may ask:

**What assumption on function $V$ will imply a bound on the number of negative eigenvalues of $-\Delta^R + V$ on shrinkers?**

In the following we will give an answer to this question. i.e., Theorem 1.1: (I) $\Rightarrow$ (VI). To prove this result, we start with an important proposition, which is a key step of proving the RCL type inequality on shrinkers.

**Proposition 3.1.** Let $D$ be a bounded domain in a shrinker $(M^n, g, f)$, where $n \geq 3$. Assume that $q(x)$ is a positive function defined on $D$. Let $\lambda_k$ be the $k$-th eigenvalue of the equation

$$-\Delta^R \phi(x) = \lambda q(x) \phi(x)$$

on $D$ with the Dirichlet boundary condition $\phi|_{\partial D} \equiv 0$. Then,

$$\lambda_k^{n/2} \int_D q^{n/2}(x) dv(x) \geq c(n) e^{\mu_k}$$

for some dimensional constant $c(n)$.

**Proof of Proposition 3.1.** Inspired by the Li-Yau work [36], we consider the “heat” kernel of the parabolic operator

$$-\Delta^R/q + \partial_t$$

on shrinker $(M, g, f)$. Let $\{\phi_i(x)\}_{i=1}^\infty$ be a set of orthonormal eigenfunctions such that

$$-\Delta^R \phi_i = \lambda_i q \phi_i,$$

where $\lambda_i$ denote the eigenvalues of the corresponding eigenfunctions $\{\phi_i(x)\}_{i=1}^\infty$. Then the kernel $\tilde{H}(x, y, t)$ of $-\Delta^R/q + \partial_t$ must have the following expression

$$\tilde{H}(x, y, t) = \sum_{i=1}^\infty e^{-\lambda_i t} \phi_i(x) \phi_i(y).$$
By the property of Schrödinger heat kernel \(H^R(x, y, t)\) (see [55]), we have \(\tilde{H}(x, y, t) > 0\) in the interior of \(D \times D\) and \(\tilde{H}(x, y, t) \equiv 0\) on \(\partial D \times \partial D\) for any \(t\). At this time, the \(L^2\)-norm is given by the weighted volume \(q(x)dv\) and
\[
\int_D \phi_i(x)\phi_j(x)q(x)dv = \delta_{ij}
\]
Since
\[
h(t) := \sum_{i=1}^{\infty} e^{-2\lambda_i t} = \int_D \int_D \tilde{H}^2(x, y, t)q(x)q(y)dv(x)dv(y),
\]
then we have
\[
\frac{dh}{dt} = 2 \int_D \int_D \tilde{H}(x, y, t)\tilde{H}_t(x, y, t)q(x)q(y)dv(x)dv(y)
= 2 \int_D \int_D \tilde{H}(x, y, t)\Delta_y^R \tilde{H}(x, y, t)q(x)dv(y)dv(x)
= -2 \int_D \int_D \left(\nabla \tilde{H}(x, y, t)\right)^2 + \frac{R}{4} \tilde{H}^2(x, y, t) q(x)dv(y)dv(x),
\]
where we used
\[
\left(\frac{\Delta_y^R}{q(y)} - \partial_t\right) \tilde{H}(x, y, t) = 0.
\]
On the other hand, using the Cauchy-Schwarz inequality, we have
\[
h(t) = \left[ \int_D q(x) \left( \int_D \tilde{H}^{\frac{2n}{n+2}}(x, y, t)dv(y) \right)^{\frac{n+2}{n+2}} dv(x) \right]^{\frac{n}{n+2}}
\times \left[ \int_D q(x) \left( \int_D \tilde{H}(x, y, t)q^{\frac{n+2}{2}}(y)dv(y) \right)^2 dv(x) \right]^{\frac{2}{n+2}}.
\]
Let us now analyze the above inequality. Consider the quantity
\[
Q(x, t) := \int_D \tilde{H}(x, y, t)q^{\frac{n+2}{2}}(y)dv(y)
\]
and it satisfies
\[
\left(\frac{\Delta_x^R}{q(x)} - \partial_t\right) Q(x, t) = 0
\]
with \(Q(x, t) \equiv 0\) on \(\partial D\) for \(t > 0\) and \(Q(x, 0) = q^{\frac{n+2}{2}}(x)\). We observe that
\[
\partial_t \int_D q(x)Q^2(x, t)dv(x) = 2 \int_D q(x)Q(x, t)\partial_t Q(x, t)dv(x)
= 2 \int_D Q(x, t)\Delta_x^R Q(x, t)dv(x)
= -2 \left( \int_D |\nabla_x Q(x, t)|^2 + \frac{R}{4} Q^2(x, t) \right) dv(x)
\leq 0,
\]
where we used the scalar curvature $R \geq 0$ on shrinkers. This implies that
\[
\int_D q(x)Q^2(x,t)dv(x) \leq \int_D q(x)Q^2(x,0)dv(x)
= \int_D q^{n/2}(x)dv(x).
\]
Using this, from (3.4) we have
\[
(3.5) \quad h^{\frac{n+2}{n}}(t) \left( \int_D q^{n/2}(x)dv(x) \right)^{-\frac{2}{n}} \leq \int_D q(x) \left( \int_D \tilde{H}^{\frac{2n}{n-2}}(x,y,t)dv(y) \right)^{\frac{n-2}{n}} dv(x).
\]
Recall that the Sobolev inequality (1.8) of shrinkers by letting $\varphi = \tilde{H}(x,y,t)$ says that
\[
\left( \int_D |\tilde{H}|^{\frac{2n}{n-2}} dv(y) \right)^{\frac{n-2}{n}} \leq C(n) e^{-\frac{2\mu}{n}} \int_D \left( 4|\nabla \tilde{H}|^2 + R \tilde{H}^2 \right) dv(y).
\]
Combining this with (3.5) and (3.3) yields
\[
\frac{dh}{dt} \leq -\frac{e^{\frac{2\mu}{n}}}{2C(n)} \left( \int_D q^{n/2}(x)dv(x) \right)^{-\frac{2}{n}} \cdot h^{\frac{n+2}{n}}(t).
\]
Dividing this by $h^{\frac{n+2}{n}}(t)$ and integrating with respect to $t$,
\[
h(t) \leq (nC(n))^{\frac{2}{n}} e^{-\mu} \left( \int_D q^{n/2}(x)dv(x) \right) t^{-\frac{n}{2}}.
\]
Combining this with (3.2), we get
\[
\sum_{i=1}^{\infty} e^{-2\lambda_i t} \leq (nC(n))^{\frac{n}{2}} e^{-\mu} \left( \int_D q^{n/2}(x)dv(x) \right) t^{-\frac{n}{2}}.
\]
Setting $t = \frac{n}{4\lambda_k}$, we conclude that
\[
\sum_{i=1}^{\infty} e^{-\frac{n\lambda_i}{4\lambda_k}} \leq (nC(n))^{\frac{n}{2}} e^{-\mu} \left( \int_D q^{n/2}(x)dv(x) \right) \left( \frac{n}{4\lambda_k} \right)^{-\frac{n}{2}}.
\]
Noticing that
\[
\sum_{i=1}^{\infty} e^{-\frac{n\lambda_i}{4\lambda_k}} \geq ke^{-n/2},
\]
so we have
\[
\lambda_k^{n/2} \int_D q^{n/2}(x)dv(x) \geq c(n) e^{\mu} k
\]
for some dimensional constant $c(n)$. \qed

Now we will apply Proposition 3.1 to prove Theorem 1.1: (I) $\Rightarrow$ (VI).

Proof of Theorem 1.1: (I) $\Rightarrow$ (VI). By the monotonicity of $\mathcal{N}(-\Delta^R + V)$ with respect to the function $V(x)$ on shrinker $(M, f, g)$, we may assume $V(x) \leq 0$ by replacing $V(x)$ by $-V_-(x)$. Moreover $-V_-(x)$ can be approximated by a sequence of strictly negative function.
So we can assume $V(x) < 0$ for all $x \in M$. By the exhausting argument (see Lemma 3.2 in [45]), we only need to prove

$$\mathcal{N}(-\Delta^R + V) \leq c(n) \int_D V_+^2 \, dv$$

for the equation

$$(3.6) \quad (-\Delta^R + V) \phi = \lambda \phi$$

with $\phi|_{\partial D} \equiv 0$ for any fixed domain $D \in M$.

It is easy to see that the number of non-positive eigenvalues $\mathcal{N}(-\Delta^R + V)$ for (3.6), counting multiplicity, is equal to the number of eigenvalues less than 1 for the case in Proposition 3.1 by considering $q(x) = -V(x)$.

Indeed, since $V(x) < 0$, from the relation

$$\int_D (|\nabla \phi|^2 + \frac{R}{4} \phi^2) \, dv + \int_D V \phi^2 \, dv = \frac{\int_D |\nabla \phi|^2 \, dv}{\int_D \phi^2 \, dv} \left( \frac{\int_D (|\nabla \phi|^2 + \frac{R}{4} \phi^2) \, dv}{\int_D |\nabla \phi|^2 \, dv} - 1 \right),$$

we conclude that the dimension of the subspace on which the left hand side is non-positive is equal to the dimension of the subspace on which the quadratic form

$$\frac{\int_D (|\nabla \phi|^2 + \frac{R}{4} \phi^2) \, dv}{\int_D |\nabla \phi|^2 \, dv}$$

associated to Proposition 3.1 is not more than 1. Now we let $\lambda_k$ be the greatest eigenvalue which is not more than 1. By Proposition 3.1, we have

$$\int_D |V|^{n/2} \, dv(x) \geq \lambda_k^{n/2} \int_D |V|^{n/2} \, dv(x) \geq c(n) e^{\mu k} \geq c(n) e^{\mu} \cdot \mathcal{N}(-\Delta^R + V),$$

which completes the proof of (I) $\Rightarrow$ (VI). \hfill \square

Next we will prove the easy implication (VI) $\Rightarrow$ (I).

**Proof of Theorem 1.1:** (VI) $\Rightarrow$ (I). We assume (1.10) holds for all potentials $V \in C^\infty_0(M)$. Then for all $V \in C^\infty_0(M)$ satisfying

$$\|V\|_{n/2} < c(n) e^{-\frac{2a}{n}},$$

we know that $-\Delta^R + V$ is a non-negative operator. That is,

$$\int_M |\nabla \varphi|^2 + \frac{R}{4} \varphi^2 \, dv + \int_M V \varphi^2 \, dv \geq 0$$

for all such $V$ and all $\varphi \in C^\infty_0(M)$. In other words,

$$\int_M |\nabla \varphi|^2 + \frac{R}{4} \varphi^2 \, dv \geq \sup_{V \in C^\infty_0(M)} \left\{ \int_M -V \varphi^2 \, dv \right\}$$

satisfying

$$\|V\|_{n/2} < c(n) e^{-\frac{2a}{n}}.$$
Since the dual of $L^{n/2}(M)$ is $L^{n/(n-2)}(M)$, the above functional inequality implies
\[ c(n) e^{-2\mu} \int_M \left( |\nabla \varphi|^2 + \frac{R}{4} \varphi^2 \right) dv \geq \left( \int_M \varphi^{\frac{2n}{n-2}} dv \right)^{\frac{n-2}{n}} \]
and Theorem 1.1 (I) follows. \hfill \Box

4. Equivalence of geometric inequalities

In this section, we will give rest proofs of Theorem 1.1. This part of theorem mainly says that the (logarithmic) Sobolev inequality, the Schrödinger heat kernel upper bound, the Faber-Krahn inequality and the Nash inequality are all equivalent up to possible different constants. Notice that (II) $\Rightarrow$ (I) was proved in [39]; (II) $\Rightarrow$ (III) $\Rightarrow$ (IV) was proved in [55]. So we only need to consider the following cases: (I) $\Rightarrow$ (II), (III) $\Rightarrow$ (I), (IV) $\Rightarrow$ (III), (I) $\Rightarrow$ (V), (V) $\Rightarrow$ (III).

Proof of Theorem 1.1. (I) $\Rightarrow$ (II): We may assume (1.8) holds on shrinkers. That is, for each compactly supported locally Lipschitz function $\varphi$ in $(M, g, f)$,
\[ \left( \int_M \varphi^{\frac{2n}{n-2}} dv \right)^{\frac{n-2}{n}} \leq C(n) e^{-2\mu} \int_M (4|\nabla \varphi|^2 + R \varphi^2) dv \]
for some dimensional constant $C(n)$. Given function $\varphi$ with $\| \varphi \|_2 = 1$, we introduce the weighted measure $d\mu = \varphi^2 dv$ on shrinker $(M, g, f)$, then $\int_M d\mu = 1$. Since function $\ln G$ is concave with respect to parameter $G$, letting $G = \varphi^{q-2}$, where $q = \frac{2n}{n-2}$, and applying the Jensen inequality
\[ \int_M \ln G d\mu \leq \ln \left( \int_M G d\mu \right), \]
we have that
\[ \int_M (\ln \varphi^{q-2}) \varphi^2 dv \leq \ln \left( \int_M \varphi^{q-2} \varphi^2 dv \right) = \ln \| \varphi \|^q, \]
That is,
\[ \int_M \varphi^2 \ln \varphi dv \leq \frac{q}{q-2} \ln \| \varphi \|^q = \frac{n}{2} \ln \| \varphi \|^q. \]
Combining this with the Sobolev inequality (1.8), we get
\[ \int_M \varphi^2 \ln \varphi^2 dv \leq \frac{n}{2} \ln \| \varphi \|^q \]
\[ \leq \frac{n}{2} \ln \left[ C(n) e^{-2\mu} \int_M (4|\nabla \varphi|^2 + R \varphi^2) dv \right] \]
\[ = \frac{n}{2} \ln C(n) - \mu + \frac{n}{2} \ln \left[ \int_M (4|\nabla \varphi|^2 + R \varphi^2) dv \right]. \]
Using an elementary inequality:
\[ \ln x \leq -1 + \ln \sigma \]
for any \( \sigma > 0 \), the above estimate can be further reduced to
\[ \int_M \varphi^2 \ln \varphi^2 \, dv(x) \leq \frac{n}{2} \ln C(n) - \mu + \frac{n\sigma}{2} \int_M \left( 4|\nabla \varphi|^2 + R \varphi^2 \right) \, dv - \frac{n}{2}(1 + \ln \sigma). \]
Setting \( \tau = \frac{n\sigma}{2} \), we obtain
\[ \int_M \varphi^2 \ln \varphi^2 \, dv \leq \tau \int_M \left( 4|\nabla \varphi|^2 + R \varphi^2 \right) \, dv - \mu - n - \frac{n}{2} \ln(4\pi \tau) + \frac{n}{2} \ln(2n\pi \cdot C(n)) \]
and hence (b) follows with possible different constants.

(III) \( \Rightarrow \) (I): Since (III) implies (1.5) with different constants, then (1.5) further implies (I) by following the proof of Lemma 2.4 in Section 2.

(IV) \( \Rightarrow \) (III): Since (1.5) is equivalent to (III), we only need to apply (IV) to prove (1.5). By the approximation argument, we only need to prove (1.5) for the Dirichlet Schrödinger heat kernel \( H_{\Omega}^R(x, y, t) \) of any relatively compact set \( \Omega \) in \( (M, g, f) \). In fact, let \( \Omega_i, i = 1, 2, ..., \) be a sequence of compact exhaustion of \( M \) such that \( \Omega_i \subset \Omega_{i+1} \) and \( \cup \Omega_i = M \). If we are able to prove (1.5) for the Dirichlet Schrödinger heat kernel \( H_{\Omega_i}^R(x, y, t) \) for any \( i \), then the result follows by letting \( i \to \infty \).

For a fixed point \( y \in \Omega \), let \( u = u(x, t) = H_{\Omega}^R(x, y, t) \) and consider the integral
\[ I(t) := \int_{\Omega} u^2(x, t) \, dv. \]
Then,
\[ I'(t) = 2 \int_{\Omega} uu_t \, dv = -2 \int_{\Omega} \left( |\nabla u|^2 + \frac{1}{4} Ru^2 \right) \, dv. \]
For any positive number \( s \), we have
\[ u^2 \leq (u - s)_+^2 + 2su \]
and therefore,
\[ I(t) \leq \int_{\Omega} (u - s)_+^2 \, dv + 2s \int_{\Omega} u \, dv. \]
Now for fixed \( s, t > 0 \), consider the set
\[ D(s, t) := \{ x \in M, u(x, t) > s \} \]
and its the first eigenvalue
\[ \lambda(D(s, t)) = \inf_{0 \neq \varphi \in C_0^\infty(D(s, t))} \frac{\int_{D(s, t)} (|\nabla \varphi|^2 + \frac{R}{4} \varphi^2) \, dv}{\int_{D(s, t)} \varphi^2 \, dv}. \]
Letting $\varphi = (u - s)_+$, then
\[
\lambda(D(s, t)) (I(t) - 2s) \leq \int_{D(s, t)} \left( |\nabla (u - s)_+|^2 + \frac{R}{4} ((u - s)_+)^2 \right) dv \\
\leq \int_{\Omega} \left( |\nabla u|^2 + \frac{R}{4} u^2 \right) dv.
\]
Note that in the above first inequality, we threw away a positive term and used the Schrödinger heat kernel property $\int_{\Omega} u(x, t) dv \leq 1$. This property also indicates that $V(D(s, t)) \leq s^{-1}$.

On the other hand, by the Faber-Krahn inequality, we have
\[
\lambda(D(s, t)) \geq \frac{2n\pi}{e} \left( \frac{e^{\mu}}{V(D(s, t))} \right)^\frac{2}{n} \\
\geq \frac{2n\pi}{e} (e^{\mu} \cdot s)^\frac{2}{n}.
\]
We remark that if the set $D(s, t)$ is not relatively compact, we can choose a sequence of relatively compact sets which converges to it such that the Faber-Krahn inequality remains valid for $D(s, t)$. Combining the above two inequalities, we obtain
\[
I(t) \leq \frac{e^{1-2\mu/n}}{2n\pi} \int_{\Omega} \left( |\nabla u|^2 + \frac{R}{4} u^2 \right) dv \cdot s^{-2/n} + 2s.
\]
Minimizing the right hand side of the above inequality,
\[
I(t) \leq c(n) e^{-\frac{2\mu}{n}} \left[ \int_{\Omega} \left( |\nabla u|^2 + \frac{R}{4} u^2 \right) dv \right]^{\frac{n}{n+2}}.
\]
Combining this with (4.1), we have
\[
I'(t) \leq c(n) e^{\frac{2\mu}{n}} I^{\frac{n+2}{n}}.
\]
Integrating this from $t/2$ to $t$,
\[
I(t) \leq c(n) e^{-\mu \frac{t}{2n}}
\]
for $t > 0$. In other words, we in fact get
\[
\int_{\Omega} H^R_{\Omega}(x, y, t)H^R_{\Omega}(x, y, t)dv(y) \leq c(n) \frac{e^{-\mu}}{t^{n/2}}.
\]
By the Schrödinger heat kernel property, we indeed show that
\[
H^R_{\Omega}(x, x, 2t) = \int_{\Omega} H^R_{\Omega}(x, y, t)H^R_{\Omega}(y, x, t)dv(y) \\
\leq c(n) \frac{e^{-\mu}}{t^{n/2}}.
\]
This further implies

\[ H^R_\Omega(x, y, t) = \int_\Omega H^R_\Omega(x, z, t/2) H^R_\Omega(z, y, t/2) \, dv(z) \]

\[ \leq \left( \int_\Omega (H^R_\Omega)^2(x, z, t/2) \, dv(z) \right)^{1/2} \left( \int_\Omega (H^R_\Omega)^2(z, y, t/2) \, dv(z) \right)^{1/2} \]

\[ = (H^R_\Omega)^{1/2}(x, x, t)(H^R_\Omega)^{1/2}(y, y, t) \]

\[ \leq c(n) \frac{e^{-\mu t}}{t^{n/2}}. \]

Next we apply the same argument of proving Theorem 1.1 in [55] to get an upper bound with a Gaussian exponential factor and finally (III) follows.

(I) \( \Rightarrow \) (V): We remark that the Nash inequality can be viewed as an interpolation between the Hölder inequality and the Sobolev inequality. Assume that \((M, g, f)\) admits (1.8). By the Hölder inequality, for \(p_1 = \frac{n+2}{n-2}\) and \(p_2 = \frac{n+2}{4}\), we have

\[ \int_M \varphi^2 \, dv = \int_M \varphi^{\frac{2n}{n+2}} \varphi^{\frac{4}{n+2}} \, dv \]

\[ \leq \left( \int_M \varphi^{\frac{2n}{n+2}p_1} \, dv \right)^{1/p_1} \left( \int_M \varphi^{\frac{4}{n+2}p_2} \, dv \right)^{1/p_2} \]

\[ = \left( \int_M \varphi^{\frac{2n}{n+2}} \, dv \right)^{\frac{n-2}{n+2}} \left( \int_M |\varphi| \, dv \right)^{\frac{4}{n+2}} \]

and hence,

\[ \| \varphi \|_2^{2+\frac{4}{n}} \leq \left( \int_M \varphi^{\frac{2n}{n+2}} \, dv \right)^{\frac{n-2}{n}} \left( \int_M |\varphi| \, dv \right)^{\frac{4}{n}}. \]

Combining this with the Sobolev inequality (1.8) gives the Nash inequality (1.9).

(V) \( \Rightarrow \) (III): Using the approximation argument, it suffices to prove (1.6) for the Dirichlet Schrödinger heat kernel \(H^R_\Omega(x, y, t)\) of any relatively compact set \(\Omega\) in \((M, g, f)\).

For any \(y \in M\), let \(\varphi = \varphi(x, t) = H^R_\Omega(x, y, t)\). Then

\[ \frac{\partial}{\partial t} \left( \int_\Omega \varphi^2 \, dv \right) = \int_\Omega 2\varphi \varphi_t \, dv \]

\[ = \int_\Omega 2\varphi (\Delta \varphi - \frac{1}{4} R \varphi) \, dv \]

\[ = -\frac{1}{2} \int_\Omega \left( 4|\nabla \varphi|^2 + R \varphi^2 \right) \, dv. \]

Scaling function \(\varphi\) such that \(\| \varphi \|_1 = 1\), by our assumption, we may assume the Nash inequality

\[ \| \varphi \|_2^{2+\frac{4}{n}} \leq c(n) e^{-\frac{2\mu}{n}} \int_\Omega \left( 4|\nabla \varphi|^2 + R \varphi^2 \right) \, dv. \]
Combining the above estimates, we have
\[
\frac{\partial}{\partial t} \left( \int_{\Omega} \varphi^2 dv \right) \leq -\frac{e^{2\mu}}{2c(n)} \| \varphi \|_2^{2+\frac{4}{n}}.
\]
Let
\[
F(s) := \int_{\Omega} \varphi^2(x,s) dv,
\]
where \( s \in (0, t] \). Then
\[
\frac{\partial}{\partial s} F(s) \leq -\frac{e^{2\mu}}{2c(n)} F(s)^{1+\frac{2}{n}}.
\]
Integrating it from \( t/2 \) to \( t \) yields
\[
-\frac{n}{2} \left( F(t) - \frac{2}{n} F(t/2) - \frac{2}{n} \right) \leq -\frac{e^{2\mu}}{2c(n)} \cdot \frac{t}{2},
\]
which implies that
\[
F(t) \leq \left[ 2nc(n) \right]^{n/2} e^{-\mu \cdot \frac{t}{n^2}}.
\]
This estimate is the same as \( I(t) \) in the proof of the case (IV) \( \Rightarrow \) (III). Therefore we only use the same strategy of proving Theorem 1.1 in [55] to get an upper bound with a Gaussian exponential factor and finally prove (III).

5. Gap result for Weyl tensor

In this section we will prove Theorem 1.5 by using the arguments of [10, 21]. We first recall the elliptic equation of the norm of traceless Ricci tensor on shrinkers, which can be directly computed by Lemma 2.3 (see also Lemma 3.2 in [10]).

Lemma 5.1. If \((M^n, g, f)\) be an \(n\)-dimensional shrinker satisfying (1.2), then
\[
\frac{1}{2} \Delta f |\tilde{\text{Ric}}|^2 = |\nabla \tilde{\text{Ric}}|^2 + |\tilde{\text{Ric}}|^2 - 2W_{ijkl} R_{ik} R_{jl} + \frac{4}{n-2} R_{ij} R_{jk} R_{ki} - \frac{2(n-2)}{n(n-1)} R |\tilde{\text{Ric}}|^2.
\]

In the following we will apply the similar arguments of [10, 21] to prove gap theorems on shrinkers. In our case, we need to carefully deal with the constant \( C(n) \) of Sobolev inequality (1.8).

Proof of Theorem 1.5. By Lemma 5.1 using
\[
\frac{1}{2} \Delta |\tilde{\text{Ric}}|^2 = |\nabla |\tilde{\text{Ric}}||^2 + |\tilde{\text{Ric}}| \Delta |\tilde{\text{Ric}}|
\]
and the Kato inequality
\[
|\nabla |\tilde{\text{Ric}}||^2 \geq |\nabla |\tilde{\text{Ric}}||^2
\]
at each point where \(|\tilde{\text{Ric}}| \neq 0\), we obtain
(5.1)
\[
|\tilde{\text{Ric}}| \Delta |\tilde{\text{Ric}}| \geq |\tilde{\text{Ric}}|^2 - 2W_{ijkl} R_{ik} R_{jl} + \frac{4}{n-2} R_{ij} R_{jk} R_{ki} - \frac{2(n-2)}{n(n-1)} R |\tilde{\text{Ric}}|^2 + \frac{1}{2} \langle \nabla f, \nabla |\tilde{\text{Ric}}|^2 \rangle.
\]
To simplify the notation, we let $u := |\text{Ric}|$. Then for any positive number $s$, which will be determined later, by (5.1), we compute that

$$u^s \Delta u^s = s(s - 1)u^{2s - 2} |\nabla u|^2 + su^{2s - 1} \Delta u$$

$$= \left(1 - \frac{1}{s}\right) |\nabla u|^2 + su^{2s - 2} u \Delta u$$

$$\geq \left(1 - \frac{1}{s}\right) |\nabla u|^2 + su^{2s} + s\left(-2W_{ijkl} \bar{\omega}_{ik} \bar{\omega}_{jl} + \frac{4}{n - 2} \bar{\omega}_{ij} \bar{\omega}_{jk} \bar{\omega}_{ki}\right)u^{2s - 2}$$

$$- \frac{2(n - 2)}{n(n - 1)} s Ru^{2s} + \frac{s}{2} u^{2s - 2} \langle \nabla f, \nabla u^2 \rangle.$$

Using Lemma 2.1, we further have

$$u^s \Delta u^s \geq \left(1 - \frac{1}{s}\right) |\nabla u|^2 + su^{2s} - \sqrt{\frac{2(n - 2)}{n - 1}} s \left(|W|^2 + \frac{8u^2}{n(n - 2)}\right)^{\frac{1}{2}} u^{2s}$$

$$- \frac{2(n - 2)}{n(n - 1)} s Ru^{2s} + \frac{1}{2} \langle \nabla f, \nabla u^2 \rangle.$$

Since $M^n$ is closed, integrating by parts over $M^n$ and using the equality

$$\Delta f = \frac{n}{2} - R$$

from Lemma 2.3, we have that

$$0 \geq \left(2 - \frac{1}{s}\right) \int_M |\nabla u|^2 dv + s \int_M u^{2s} dv - \sqrt{\frac{2(n - 2)}{n - 1}} s \int_M \left(|W|^2 + \frac{8u^2}{n(n - 2)}\right)^{\frac{1}{2}} u^{2s} dv$$

$$- \frac{2(n - 2)}{n(n - 1)} s \int_M Ru^{2s} dv - \frac{1}{2} \int_M u^{2s} \Delta f dv$$

$$= \left(2 - \frac{1}{s}\right) \int_M |\nabla u|^2 dv - \sqrt{\frac{2(n - 2)}{n - 1}} s \int_M \left(|W|^2 + \frac{8u^2}{n(n - 2)}\right)^{\frac{1}{2}} u^{2s} dv$$

$$- \left(\frac{n}{4} - s\right) \int_M u^{2s} dv + \frac{n(n - 1) - 4(n - 2)s}{2n(n - 1)} \int_M Ru^{2s} dv.$$

For $2 - 1/s > 0$, by the Sobolev inequality of shrinker using $\varphi = u^s$

$$\int_M |\nabla u|^2 dv \geq \frac{e^{2s}}{4C(n)} \left(\int_M u^{2s} dv\right)^{\frac{n-2}{n}} - \frac{1}{4} \int_M Ru^{2s} dv,$$

the above inequality becomes

$$0 \geq \left(2 - \frac{1}{s}\right) \frac{e^{2s}}{4C(n)} \left(\int_M u^{2s} dv\right)^{\frac{n-2}{n}} - \sqrt{\frac{2(n - 2)}{n - 1}} s \int_M \left(|W|^2 + \frac{8u^2}{n(n - 2)}\right)^{\frac{1}{2}} u^{2s} dv$$

$$- \left(\frac{n}{4} - s\right) \int_M u^{2s} dv + \frac{n(n - 1) - 8(n - 2)s^2}{4n(n - 1)s} \int_M Ru^{2s} dv.$$
By the Hölder inequality, for \( n - 4s \geq 0 \), we get that
\[
0 \geq \left\{ \left( 2 - \frac{1}{s} \right) \frac{e^{\frac{2n}{4C(n)}}}{4C(n)} - \sqrt{\frac{2(n-2)}{n-1}} s \left[ \int_M \left( |W|^2 + \frac{8u^2}{n(n-2)} \right) \frac{n}{s} \, dv \right]^{\frac{n}{2}} - \left( \frac{n}{4} - s \right) V(M) \right\} \\
	imes \left( \int_M u^{\frac{2ns}{n-2}} \, dv \right) \frac{n-2}{n} + \frac{n(n-1) - 8(n-2)s^2}{4n(n-1)s} \int_M R u^{2s} \, dv.
\]
Now we choose
\[
s = \sqrt{\frac{n(n-1)}{8(n-2)}} \in \left( \frac{1}{2}, \frac{n}{4} \right),
\]
and the last term of the above inequality vanishes. Moreover, notice that the curvature integral assumption of theorem is equivalent to
(5.3)
\[
\left( 2 - \frac{1}{s} \right) \frac{e^{\frac{2n}{4C(n)}}}{4C(n)} - \sqrt{\frac{2(n-2)}{n-1}} s \left[ \int_M \left( |W|^2 + \frac{8u^2}{n(n-2)} \right) \frac{n}{s} \, dv \right]^{\frac{n}{2}} - \left( \frac{n}{4} - s \right) V(M) > 0,
\]
where we used the equality
\[
\left| W + \sqrt{\frac{2}{n(n-2)}} \circ \text{Ric} \circ g \right| \leq \left| |W|^2 + \frac{8}{n(n-2)} \circ |\text{Ric}|^2 \right|
\]
due to the totally trace-free tensor \( W \). Therefore, we conclude that \( |\circ \text{Ric}| \equiv 0 \) and \((M, g, f)\) is Einstein.

Now we have \( \text{Ric} = \frac{1}{2} g \). By (1.3), we know
\[
f = \frac{n}{2} \quad \text{and} \quad (4\pi e)^{-\frac{2}{n}} V(M) = e^n
\]
and the pinching condition (5.3) reduces to
(5.4)
\[
\left( \int_M |W|^\frac{2}{n} \, dv \right)^{\frac{n}{2}} \leq \epsilon_1(n) := \sqrt{\frac{n-1}{32(n-2)} \left[ \left( \frac{2}{s} - \frac{1}{s^2} \right) \frac{C(n)^{-1}}{4\pi e} + 4 - \frac{n}{s} \right]},
\]
where \( \mu_M \) denotes the average of the integration, i.e.,
\[
\mu_M |W|^\frac{2}{n} \, dv = \frac{1}{V(M)} \int_M |W|^\frac{2}{n} \, dv.
\]
By Remark 2.5 we see
\[
C(n) \geq \frac{n-1}{2n(n-2)\pi e}
\]
and hence
\[
\epsilon_1(n) \leq \sqrt{\frac{n-1}{32(n-2)} \left[ \left( \frac{2}{s} - \frac{1}{s^2} \right) \frac{n(n-2)}{2(n-1)} + 4 - \frac{n}{s} \right]}.
\]
Notice that the right hand side of the above inequality is nonnegative if
(5.5)
\[
s \geq \frac{n + \sqrt{n^2 + 8n(n-1)(n-2)}}{8(n-1)}.
\]
Proposition 5.2. Let \((M^n, g)\) be an \(n\)-dimensional Einstein manifold with \(\text{Ric} = kg\), where \(k > 0\) is a constant. There exists a positive constant \(\epsilon_2(n)\) depending only on \(n\) such that if

\[
\left( \int_M |W|^{\frac{2}{n}} \, dv \right)^{\frac{n}{2}} < \epsilon_2(n),
\]

where \(\int_M |W|^{\frac{2}{n}} \, dv = \frac{1}{V(M)} \int_M |W|^{\frac{2}{n}} \, dv\), then \((M^n, g)\) is isometric to a quotient of the round sphere with radius \(\sqrt{\frac{n-1}{k}}\). We can take \(\epsilon_2(4) = \frac{14}{5\sqrt{6}} k\), \(\epsilon_2(5) = \frac{4\sqrt{6}}{5} k\), \(\epsilon_2(6) = \frac{16\sqrt{5}}{9\sqrt{6}} k\), \(\epsilon_2(7) = \frac{49}{125} k\), \(\epsilon_2(8) = \frac{267}{625} k\), \(\epsilon_2(9) = \frac{22}{50} k\) and \(\epsilon_2(n) = \frac{2n}{5(n-1)} k\) if \(n \geq 10\).

Proof of Proposition 5.2. Following the argument in [32, 10], we have the Bochner type formula for \(|W|^2\),

\[
\frac{1}{2} \Delta |W|^2 = |\nabla W|^2 + 2k|W|^2 - 2 \left( 2W_{ijkl}W_{ipkq}W_{pjql} + \frac{1}{2}W_{ijkl}W_{klpq}W_{pqij} \right).
\]

Since

\[
\frac{1}{2} \Delta |W|^2 = |\nabla |W||^2 + |W| \Delta |W|,
\]

then we have

\[
|W| \Delta |W| = |\nabla W|^2 - |\nabla |W||^2 + 2k|W|^2 - 2 \left( 2W_{ijkl}W_{ipkq}W_{pjql} + \frac{1}{2}W_{ijkl}W_{klpq}W_{pqij} \right).
\]

Using Lemma 2.2 and the refined Kato inequality

\[
|\nabla W|^2 \geq \frac{n+1}{n-1} |\nabla |W||^2
\]

at every point where \(|W| \neq 0\), we get

\[
(5.6) \quad |W| \Delta |W| \geq \frac{2}{n-1} |\nabla W|^2 + 2k|W|^2 - 2c(n)|W|^3,
\]

where \(c(n)\) is a dimensional constant, which is defined by \(c(4) = \frac{\sqrt{6}}{4}\), \(c(5) = 1\), \(c(6) = \frac{\sqrt{70}}{2\sqrt{3}}\) and \(c(n) = \frac{\alpha}{2}\) for \(n \geq 7\).
GEOMETRIC INEQUALITIES AND RIGIDITY OF SHRINKERS

Since $M^n$ is closed, integrating the above inequality over $M^n$ yields

\[
0 \geq \left( 2 - \frac{n-3}{(n-1)s} \right) \int_M |\nabla u|^2 dv + 2ks \int_M u^{2s} dv - 2c(n)s \int_M u^{2s+1} dv.
\]

Recall that Ilias [34] proved the Sobolev inequality of manifolds satisfying $\text{Ric} \geq kg$, where $k > 0$ (by letting $f = |W|^s = u^s$ in [34]), which can be stated that in our setting

\[
\left( \int_M u^{2\alpha s} dv \right)^{\frac{n-2}{n}} \leq 4(n-1)(n(n-2)k) V(M)^{-2/n} \int_M |\nabla u|^2 dv + V(M)^{-2/n} \int_M u^{2s} dv.
\]

By the proposition assumption, we have the following equivalent form

\[
\left( \int_M u^{2s} dv \right)^{\frac{n}{n-2}} < \epsilon_2(n) V(M)^{2/n}.
\]

Therefore, for $s > 0$, if $\epsilon_2(n)$ satisfies

\[
\begin{align*}
2 - \frac{n-3}{(n-1)s} - 8c(n)s\epsilon_2(n) \frac{(n-1)}{n(n-2)k} & \geq 0, \\
2ks - 2c(n)\epsilon_2(n) & \geq 0,
\end{align*}
\]

we immediately have $W \equiv 0$ and $g$ has constant positive sectional curvature. Here we give explicit constants such that the above two inequalities holds.

- When $n = 4$ and $s = \frac{7}{10}$, since $c(4) = \frac{\sqrt{6}}{4}$, we can take $\epsilon_2(4) = \frac{14}{5\sqrt{6}} k$.
- When $n = 5$ and $s = \frac{4}{5}$, since $c(5) = 1$, we can take $\epsilon_2(5) = \frac{4}{5} k$.
- When $n = 6$ and $s = \frac{9}{10}$, since $c(6) = \frac{\sqrt{70}}{2\sqrt{3}}$, we can take $\epsilon_2(6) = \frac{16\sqrt{3}}{9\sqrt{70}} k$.
- When $n = 7$ and $s = \frac{19}{20}$, since $c(7) = \frac{5}{2}$, we can take $\epsilon_2(7) = \frac{49}{120} k$.
- When $n = 8$ and $s = \frac{26}{29}$, since $c(8) = \frac{5}{2}$, we can take $\epsilon_2(8) = \frac{267}{325} k$.
- When $n = 9$ and $s = \frac{29}{30}$, since $c(9) = \frac{5}{2}$, we can take $\epsilon_2(9) = \frac{23}{50} k$.
- When $n \geq 10$ and $s = \frac{n}{n-1}$, since $c(n) = \frac{n}{2}$, we can take $\epsilon_2(n) = \frac{2n}{5(n-1)} k$. 

\[\square\]
6. Gap result for half Weyl tensor

In this section we will apply the similar argument of Section 5 to talk about an gap phenomenon for shrinkers under the integral condition of half Weyl tensor.

Recall that, on an oriented 4-dimensional shrinker \((M, g, f)\), the bundle of 2-forms \(\wedge^2 M\) can be decomposed as a direct sum

\[ \wedge^2 M = \wedge^+ M + \wedge^- M, \]

where \(\wedge^\pm M\) is the \((\pm 1)\)-eigenspace of the Hodge star operator

\[ * : \wedge^2 M \to \wedge^2 M. \]

Let \(\{e_i\}_{i=1}^4\) be an oriented orthonormal basis of tangent bundle \(TM\). For any pair \((ij)\), \(1 \leq i \neq j \leq 4\), let \((i'j')\) denote the dual of \((ij)\), i.e., the pair such that

\[ e_i \wedge e_j \pm e_{i'} \wedge e_{j'} \in \wedge^\pm M. \]

In other words, \((iji'j') = \sigma(1234)\) for some even permutation \(\sigma \in S_4\). For the Weyl tensor \(W\), its (anti-)self-dual part is

\[ W^\pm_{ijkl} = \frac{1}{4} (W_{ijkl} \pm W_{ij'k'l'} \pm W_{i'j'k'l} + W_{i'j'k'l'}), \]

It is easy to check that

\[ W^\pm_{ijkl} = \pm W^\pm_{i'j'k'l'} = \pm W^\pm_{i'j'k'l'} = \frac{1}{2} (W_{ijkl} \pm W_{ij'k'l}). \]

On shrinkers, we have the following Weitzenböck formula for \(W^\pm\) (see \([8]\) or its generalization \([60]\)), and it is useful for analyzing the structure of shrinkers.

Lemma 6.1. Let \((M, g, f)\) be a four-dimensional shrinker satisfying (1.1). Then

\[ \frac{1}{2} \Delta_f |W^\pm|^2 = |\nabla W^\pm|^2 + 2\lambda |W^\pm|^2 - 18 \det W^\pm - \frac{1}{2} \langle (\circ \Ric \circ \Ric)^\pm, W^\pm \rangle. \]

Using Lemma 6.1, we can prove Theorem 1.7 in the introduction.

Proof of Theorem 1.7. By Lemma 6.1, using the following algebraic inequality observed by Cao and Tran \([8]\)

\[ \det W^\pm \leq \frac{\sqrt{6}}{18} |W^\pm|^3 \]

and the Kato inequality

\[ |\nabla W^\pm|^2 \geq |\nabla|W^\pm||^2 \]

at every point where \(|W^\pm| \neq 0\), we get

\[ \frac{1}{2} \Delta_f |W^\pm|^2 \geq |\nabla|W^\pm||^2 + 2\lambda |W^\pm|^2 - \sqrt{6} |W^\pm|^3 - \frac{1}{2} \langle (\circ \Ric \circ \Ric)^\pm, W^\pm \rangle. \]

Since

\[ \frac{1}{2} \Delta |W^\pm|^2 = |\nabla|W^\pm||^2 + |W^\pm| \Delta |W^\pm|, \]

then we have

\[ |W^\pm| \Delta |W^\pm| \geq 2\lambda |W^\pm|^2 - \sqrt{6} |W^\pm|^3 - \frac{1}{2} \langle (\circ \Ric \circ \Ric)^\pm, W^\pm \rangle + \frac{1}{2} \langle f, \nabla|W^\pm||^2 \rangle. \]
Since $M^n$ is closed, integrating the above inequality and integrating by parts over $M^n$, we have

$$0 \geq \int_M |\nabla |W|^2|^2 dv + 2\lambda \int_M |W|^2 dv - \sqrt{6} \int_M |W|^3 dv$$

(6.1)

$$- \frac{1}{2} \int_M \langle \circ (\circ \circ R \circ \circ R), W \rangle dv - \frac{1}{2} \int_M |W|^2 \Delta f dv.$$ 

Note that Cao and Tran (Corollary 5.8 in [8]) observed

$$\int_M \langle \circ (\circ \circ R \circ \circ R), W \rangle dv = 4 \int_M |\delta W|^2 dv,$$

and hence the second assumption of theorem in fact is

$$\int_M \langle \circ (\circ \circ R \circ \circ R), W \rangle dv \leq \frac{1}{2} \int_M R |W|^2 dv.$$ 

Using this, (6.1) becomes

$$0 \geq \int_M |\nabla |W|^2|^2 dv + 2\lambda \int_M |W|^2 dv - \sqrt{6} \int_M |W|^3 dv$$

$$- \frac{1}{4} \int_M R |W|^2 dv - \frac{1}{2} \int_M |W|^2 \Delta f dv.$$ 

Using the equality of shrinkers

$$\Delta f = 4\lambda - R,$$

we further have

(6.2) $$0 \geq \int_M |\nabla |W|^2|^2 dv - \sqrt{6} \int_M |W|^3 dv + \frac{1}{4} \int_M R |W|^2 dv.$$ 

By the Sobolev inequality of shrinker (1.8) by letting $\varphi = |W|^2$,

$$\int_M |\nabla |W|^2|^2 dv \geq \frac{e^2}{4C(4)} \left( \int_M |W|^4 dv \right)^{\frac{1}{2}} - \frac{1}{4} \int_M R |W|^2 dv,$$

then (6.2) can be simplified as

$$0 \geq \frac{e^2}{4C(4)} \left( \int_M |W|^4 dv \right)^{\frac{1}{2}} - \sqrt{6} \int_M |W|^3 dv.$$ 

Using the H"{o}lder inequality,

$$0 \geq \left[ \frac{e^2}{4C(4)} - \sqrt{6} \left( \int_M |W|^2 dv \right)^{\frac{1}{2}} \right] \left( \int_M |W|^4 dv \right)^{\frac{1}{2}}.$$ 

By the first assumption of theorem, we immediately get $W^\pm \equiv 0$. Finally we apply the classification of Chen and Wang [15] to conclude that the shrinker is isometric to a finite quotient of the round sphere or the complex projective space. \(\square\)

In the end of this section, following the argument of Catino [10], we can apply the Yamabe constant to give another gap result, i.e., Theorem 1.9 in introduction.
Proof of Theorem 1.9. Similar to the argument of Theorem 1.7, using the second assumption of theorem, (6.1) can also be written as

\[ 0 \geq \int_M |\nabla W^\pm|^2 dv + 2\lambda \int_M |W^\pm|^2 dv - \sqrt{6} \int_M |W^\pm|^3 dv - \frac{1}{3} \int_M R|W^\pm|^2 dv - \frac{1}{2} \int_M |W^\pm|^2 \Delta f dv. \]

Using the shrinker’s equality \( \Delta f = 4\lambda - R \), we obtain

(6.3) \[ 0 \geq \int_M |\nabla W^\pm|^2 dv - \sqrt{6} \int_M |W^\pm|^3 dv + \frac{1}{6} \int_M R|W^\pm|^2 dv. \]

We will apply the Yamabe constant to estimate the first gradient term in the above inequality. Recall that the Yamabe constant \( Y(M, [g]) \) is defined by

\[ Y(M, [g]) := \inf_{\varphi \in W^{1,2}(M)} \frac{4^{(n-1)/n-2} \int_M |\nabla \varphi|^2 dv_g + \int_M R \varphi^2 dv_g}{(\int_M |\varphi|^{2n/(n-2)} dv_g)^{(n-2)/n}}, \]

where \([g]\) denotes the conformal class of \( g \). As we all known, \( Y(M, [g]) \) is positive on a compact manifold if and only if there exists a conformal metric in \([g]\) whose scalar curvature is positive everywhere. Hence the compact shrinker has positive Yamabe constant \( Y(M, [g]) \).

If we let \( \varphi = |W^\pm| \) on a four-dimensional compact shrinker, then the Yamabe constant \( Y(M, [g]) \) implies the following inequality

\[ \int_M |\nabla W^\pm|^2 dv \geq \frac{Y(M, [g])}{6} \left( \int_M |W^\pm|^4 dv \right)^{\frac{1}{2}} - \frac{1}{6} \int_M R|W^\pm|^2 dv. \]

Using this, (6.3) can be reduced to

\[ 0 \geq \frac{Y(M, [g])}{6} \left( \int_M |W^\pm|^4 dv \right)^{\frac{1}{2}} - \sqrt{6} \int_M |W^\pm|^3 dv. \]

By the Hölder inequality, we have

(6.4) \[ 0 \geq \left[ \frac{Y(M, [g])}{6} - \sqrt{6} \left( \int_M |W^\pm|^2 dv \right)^{\frac{1}{2}} \right] \left( \int_M |W^\pm|^4 dv \right)^{\frac{1}{2}}. \]

Recall that Gursky [24] proved the following estimate on a compact four-dimensional manifold

\[ \int_M R^2 dv - 12 \int_M |\text{Ric}|^2 dv \leq Y^2(M, [g]). \]

Here this inequality is strict unless the manifold is conformally Einstein. Combining this with the first assumption of theorem, we have

\[ 6\sqrt{6} \left( \int_M |W^\pm|^2 dv \right)^{\frac{4}{3}} \leq Y(M, [g]). \]

Combining this with (6.4) we conclude that \( W^\pm \equiv 0 \) or \((M, g)\) is conformally Einstein.

When \( W^\pm \equiv 0 \), by Theorem 1.7, \((M^4, g, f)\) is isometric to a finite quotient of the round sphere or the complex projective space.
When \((M, g)\) is conformally Einstein, it is naturally Bach flat (see Proposition 4.78 in [2]) and hence is Einstein (see Theorem 1.1 in [6]). Now since \((M^4, g)\) is Einstein, combining the first pinching condition of theorem and a gap result of Gursky and Lebrun (see Theorem 1 in [26]), we also get \(W^\pm \equiv 0\) and hence \((M^4, g, f)\) is also isometric to a finite quotient of the round sphere or the complex projective space. 

\[\Box\]

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