Wave packet acceleration and inelastic scattering in non-Hermitian dynamics

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We discover the mechanism of the formation the dynamic skin effect in non-Hermitian dynamics, compare to the Hermitian counterpart, the Gaussian wave packet can be accelerated or decelerated during its time evolution. This extra motion is found to be related to the dynamic skin effect: the wave packet stays at the boundary during its time evolution and “inelastic scattering” seems to happen. We show in this letter that wave packet acceleration and inelastic scattering are result of the interplay of the non-Hermitian skin effect and the Hermitian wave packet spreading.

Introduction.—In solid physics, one of the fundamental principle is the Bloch theorem, it solves the problem of finding energy levels in translation invariant systems. However, in real world, translation symmetry is always broken at the boundary of the materials. Fortunately, it is found that the bulk energy levels of the Hermitian systems under the open boundary condition (OBC) is almost the same as the energy levels solved under the periodic boundary condition (PBC) \cite{1}. This fact broken down in non-Hermitian systems, in which it is found that the energy levels vary dramatically when the boundary condition is changed and a huge number of energy eigenstates under OBC are localized at the boundary. This phenomenon is defined as the non-Hermitian skin effect (NHSE) \cite{2, 12}. A complete theory called the non-Bloch band theory \cite{3, 4, 13, 22} has been developed to describe NHSE and the energy levels under OBC. NHSE is related to many physical properties such as chiral damping in open quantum systems \cite{10}, signal amplification \cite{23, 35} and spatial growing Green’s functions \cite{35, 37} and quantized responses \cite{38, 39}.

The NHSE is focused on the stationary states of the system, recently, numerical results show that non-Hermitian dynamics also have fascinating characters in one-dimensional systems \cite{10, 40, 41}, which are related to the NHSE. It is found that in non-Hermitian systems with the NHSE, unlike the Hermitian cases, the Gaussian wave packet may not be reflected back by the boundary and stay at the boundary. This phenomenon is dubbed as the non-Hermitian skin effect (NHSE) \cite{11}. However, although the numerical evidence is clear, this phenomenon still lack an analytical explanation and its mechanism is still unkown.

In this letter, we give a simple mechanism to the momentum-dependent dynamic sticky effect (MDDSE), it is found that MDDSE is due to the coexistence of the NHSE and wave packet spreading. In addition, we find a new phenomenon in non-Hermitian dynamics, compared to the Hermitian counterpart, the wave packets in non-Hermitian system can be accelerated or decelerated. It is a new finding both in numerical and theoretical sides.

We begin with the one-dimensional Hatano-Nelson (HN) model \cite{42}. In this system, we obtain exact results on the wave packet acceleration and inelastic scattering phenomenon. We show that they are both related to the NHSE and wave packet spreading. It is found that departure from the initial velocity of the wave packet, there is an extra motion caused by the NHSE and wave packet spreading. In the HN model, this extra motion is the uniformly acceleration or deceleration.

Model.—The continuous version of the HN model has the Hamiltonian

$$H = -\frac{\hbar^2 \nabla^2}{2m} + b \nabla + E_0,$$  \hspace{1cm} (1)

where $\nabla = \frac{d}{dx}$ in one-dimensional systems, the constant energy level $E_0 = -\frac{\hbar^2 m}{2}$ for our further convenience and we set $\hbar = 1$. The system is put in an one-dimensional infinitely deep square potential well with length $L$. Hence, the coordinate $x$ satisfies $x \in [0, L] = \Omega$, and the boundary condition should be the Dirichlet boundary condition

$$\psi|_{\partial \Omega} = 0.$$  \hspace{1cm} (2)

When $b = 0$, the Hamiltonian $H$ is Hermitian, otherwise, $H$ is non-Hermitian. While under PBC, the energy level can be obtained in $k$-space, i.e., $E(k) = \frac{k^2}{2} + ibk + E_0$.

The equation of motion is the effective Schrödinger equation

$$i\partial_t \psi = H \psi.$$ \hspace{1cm} (3)

Let the initial state be the Gaussian wave packet

$$\langle x | \psi(0) \rangle = \frac{1}{(2\pi\sigma^2)^{\frac{1}{4}}} e^{-\frac{x^2}{4\sigma^2} + i k_0 x}.$$ \hspace{1cm} (4)

It has a nonzero initial speed $v_0 = Re \left[ \left. \frac{\partial E(k)}{\partial k} \right|_{k_0} \right] = \frac{k_0}{m}$.

Using the finite difference method, the time evolution of the initial state $|\psi(0)\rangle$ can be numerically simulated as shown in Fig. 1. We take $m = b = 1$ and $L = 10$ in the simulation, and other detail of our numerical method can be found in \textsuperscript{43}.

First, let us consider the $k_0 = 0$ case, in this case, the initial velocity of the wave packet $v_0 = 0$. From our knowledge of the Hermitian dynamics, if the Hamiltonian is Hermitian, the wave packet will stay at the initial place. However, it can be seen in Fig. 1 that the wave packet is uniformly accelerated before it reaches the boundary. We
ask the following question: where does this extra motion of the wave packet come from? The answer will show not only the NHSE takes part in. It should also be noted that the initial Gaussian wave packet preserves the Gaussian type in the whole process. As time flows, it spreads and moves from left to right.

**Analytical solution.**—The Hamiltonian $H$ can be transformed to a Hermitian Hamiltonian $\tilde{H}$ using the following similarity transformation

$$\tilde{H} = S^{-1} HS,$$  \hspace{1cm} (5)

where $S(\psi)(x) = e^{bm\sigma_2} \psi(x)$ is a multiplication operator and $\tilde{H} = \sum m \sigma_2$ is the Hermitian Hamiltonian of the free particle. The time evolution can be calculated by

$$\langle x | e^{-i\tilde{H}t} | \psi \rangle = e^{bm\sigma_2} \langle x | e^{-iHt} | S^{-1}\psi \rangle.$$  \hspace{1cm} (6)

Note that $|S^{-1}\psi\rangle$ is also a Gaussian wave packet but with a constant center shift $-2bm\sigma^2$. Hence, $e^{-iHt} |S^{-1}\psi\rangle$ describes the time evolution of the Gaussian wave packet under Hermitian dynamics, it has the following form

$$| \langle x | e^{-iHt} | S^{-1}\psi \rangle |^2 \propto e^{-\frac{\sqrt{2}bm\sigma^2|x|^2}{2\sigma_2^2}},$$  \hspace{1cm} (7)

where $\sigma(t)^2 = \sigma_2^2 + \frac{t^2}{16m^2}$ describes the wave packet spreading under Hermitian dynamics \[43\], i.e., the standard deviation of the wave packet grows with time. After the final multiplication of the exponential factor $e^{2bm\sigma_2}$ by Eq. (6), the probability density is

$$| \langle x | e^{-iHt} | \psi \rangle |^2 = \frac{1}{2π\sigma(t)^2} e^{-\frac{(x-2bm\sigma(t)^2-x_0)^2}{2\sigma(t)^2}},$$  \hspace{1cm} (8)

and the center or peak of wave packet $x_p$ becomes time dependent, i.e., $x_p(t) = 2bm (\sigma(t)^2 - \sigma_2^2)$. The probability density as a function of $x$ at different time $t$ is shown in Fig. 2.

The analytical result of the wave packet trajectory is compared to the numerical result in Fig. 1. As it shows, the numerical result is quite close to the analytical result.

**Inelastic scattering.**—As shown in Fig. 1, in this non-Hermitian model, the velocity of the incident wave packet $v_{in}$ does not equal to the velocity of the reflected wave packet $v_{ref}$. This phenomenon is analogous to the inelastic scattering in classical mechanics.

Let us provide a simple mechanism of this inelastic scattering. As above, the time evolution can be written as Eq. (6). Apart from the exponential factor $e^{bm\sigma_2}$, the time evolution is described by the Hermitian dynamics $e^{-iHt} |S^{-1}\psi\rangle$. In the Hermitian system, the wave packet with the initial speed $v_0$ will be reflected by the boundary with the speed $-v_0$. Because of the exponential factor $e^{bm\sigma_2}$, the wave packet acquires an extra velocity

$$v_p(t) = \frac{dx_p(t)}{dt} = \frac{bt}{m\sigma_2^2}.$$  \hspace{1cm} (9)
both before it reaches the boundary and after it reaches the boundary. Hence, the incident velocity is

\[ v_{\text{in}}(t) = v_0 + v_p(t) = v_0 + \frac{bt}{m\sigma^2}, \tag{10} \]

and the reflected velocity is

\[ v_{\text{ref}}(t) = -v_0 + v_p(t) = -v_0 + \frac{bt}{m\sigma^2}. \tag{11} \]

We provide a rigorous derivation of Eq. (10) and Eq. (11) in [13]. Thus for a reflection process in this model, in general, \( v_{\text{in}}(t_1) \neq v_{\text{ref}}(t_2) \) \( (t_1 \text{ and } t_2 \text{ are two times that before and after the reflection), which explains the inelastic scattering in this non-Hermitian model.} \)

It is worth noting that consider the reflection process at the right boundary, if \( v_{\text{ref}}(t_2) \geq 0 \), the peak of the wave packet is stuck at the boundary. Fig. 1a is in such a case. For the reflection process at the left boundary, it can be seen from Eq. (10) and Eq. (11) that \( |v_{\text{in}}(t_1)| < |v_{\text{ref}}(t_2)| \), see Fig. 1b. For the reflection process at the right boundary, it can be seen from Eq. (10) and Eq. (11) that \( |v_{\text{in}}(t_1)| > |v_{\text{ref}}(t_2)| \), see Fig. 1c.

To further verify our analytical result, we take the numerical data in Fig. 1 to generate the \( x(t) \) and \( v(t) \) relations, which is shown in Fig. 3. Here, \( x \) is the peak of the wave packet (position where the probability density is maximal), and \( v(t) \) is the numerical differential of \( x(t) \). By taking \( t_1 = 0.1 \) and \( t_2 = 0.5 \), numerical results show that \( v(t_1) = 21.6 \) and \( v(t_2) = -12 \) which coincide exactly with \( v_{\text{in}}(t_1) \) and \( v_{\text{ref}}(t_2) \) respectively, and the numerical results fit well with the analytical results as shown in Fig. 3. Note that at \( t = 0.7 \), the width of the wave packet is large enough compared to the length \( L \) of the system, and extra non-wave packet reflection happens as shown in Fig. 3. Hence, the \( v(t) \) relation after \( t = 0.6 \) is not taken into count in Fig. 3.

**General result.**—The above findings can be generalized to other models. From the above exact result of the extra moving velocity of the HN model, we find that the wave packets spreading phenomenon is directly related to the extra wave packet motion. We want to mention that the wave packet spreading phenomenon is a result of Hermitian dynamics since it is caused by the time evolution operator \( e^{-i\hat{H}t} \) where \( \hat{H} \) is a Hermitian operator.

For general non-Hermitian models with uniform skin effect, assume that \( r = |\beta| \) as the exponential factor of the non-Hermitian skin modes, the time evolution can be written as

\[ \langle x | e^{-i\hat{H}t} | \psi \rangle = r^t \langle x | e^{-i\hat{H}t} | S^{-1}\psi \rangle. \tag{12} \]

The probability density of \( e^{-i\hat{H}t} | S^{-1}\psi \rangle \) is Gaussian,

\[ |\langle x | e^{-i\hat{H}t} | S^{-1}\psi \rangle|^2 \propto e^{-\frac{(x-x_p)^2}{2r^2\sigma^2}}. \tag{13} \]

Hence, the time evolution of the Gaussian wave packet with zero initial speed is approximately the following form,

\[ |\langle x | e^{-i\hat{H}t} | \psi \rangle|^2 \approx cr^2x e^{-\frac{(x-x_p)^2}{2r^2\sigma^2}}. \tag{14} \]

Before the wave packet reaches the boundary, the wave packet peak satisfying \( \frac{d}{dt} |\langle x | e^{-i\hat{H}t} | \psi \rangle|^2 = 0 \) is

\[ x_p(t) = 2\ln(r) \left[ \sigma(t)^2 - \sigma(0)^2 \right] \tag{15} \]

and the velocity of the wave packet peak is

\[ v_p(t) = 2\ln(r) \frac{d}{dt} \frac{\sigma(t)^2}{2}. \tag{16} \]

Hence, the coexistence of the NHSE and the wave packet spreading cause the extra wave packet motion. If the wave packet has the initial velocity \( v_0 \), then the incident velocity and reflected velocity of the wave packet are still

\[ v_{\text{in}}(t) = v_0 + v_p(t), \]

\[ v_{\text{ref}}(t) = -v_0 + v_p(t). \tag{17} \]

The above formulas are further verified in the non-Hermitian Su-Schrieffer-Heeger (SSH) model.

Consider the non-Hermitian SSH model [3] with the Bloch Hamiltonian

\[ H(k) = (t_1 + t_2 \cos k) \sigma_x + \left(t_2 \sin k + i \frac{\gamma}{2}\right) \sigma_y. \tag{18} \]

The exponential factor of the non-Hermitian skin modes is \( r = \sqrt{\frac{|t_1 - \gamma/2|}{|t_1 + \gamma/2|}} \). As discussed in Ref. [3], by taking
\[ S = \text{diag}(1, r, r^2, r^2, \ldots, r^{L-1}, r^{L-1}, r^L), \quad \tilde{H} = S^{-1}HS \]

is a Hermitian Hamiltonian with k-space form

\[ \tilde{H}(k) = (\tilde{t}_1 + t_2 \cos k) \sigma_x + t_2 \sin k \sigma_y, \quad (19) \]

where \( \tilde{t}_1 = \sqrt{(t_1 - \gamma/2)(t_1 + \gamma/2)} \). In the following, we numerically verify the extra motion formula \([15]\) in this model. Instead of deriving an analytical formula of \( \sigma(t)^2 \) for this model (which may be hard), we obtain \( \sigma(t) \) by using the numerical data. Our goal is to check the relation between the wave packet spreading and the extra motion, but not to obtain an exact formula for \( \sigma(t) \).

For a Gaussian wave packet, there is a relation between the half-wave width \( \delta \) and the standard deviation \( \sigma \), which is

\[ \frac{\delta}{2} = \sqrt{2 \ln(2)} \sigma. \quad (20) \]

The half-wave width is easy to obtain from the numerical data, hence, the standard deviation can also be calculated by using Eq. (20).

In our simulation, we take \( t_1 = 2, t_2 = 1 \), and the number of unit cell \( N \) is taken to be 500. The initial state is

\[ |A⟩\psi(0)⟩ = \frac{1}{(2\pi\sigma^2)^{1/4}} e^{-\frac{x^2}{4\sigma^2} + i k_0 x}, \]

\[ |B⟩\psi(0)⟩ = 0. \quad (21) \]

where \( A, B \) are two sites in the unit cell and \( \sigma = 20 \). The initial velocity \( v_{0,±} = \text{Re} \left( \frac{\partial E(k)}{\partial k} |_{k_0} \right) \), where \( E_±(k) = \pm \sqrt{(t_1 + t_2 \cos k)^2 + (t_2 \sin k + t_2^2/2)^2} \) are two energy levels of the Hamiltonian. For \( k_0 = 0, v_{0,±} = \pm \text{Re} \left( \frac{i \tilde{\gamma}/2 t_2}{\sqrt{(t_1 + t_2^2/2 - \gamma/2)^2}} \right) = 0. \) For \( k_0 \neq 0 \), in general, \( v_{0,+} \neq v_{0,-} \neq 0 \), there are two modes moving in different directions, and their velocities are

\[ v_+(t) = v_{0,+} + v_p(t), \]

\[ v_-(t) = v_{0,-} + v_p(t). \quad (22) \]

For \( k_0 = 0 \), because of \( v_{0,±} = 0 \), the trajectory of this two modes coincide and we only need to consider one mode. In the following, let us first consider the \( k_0 = 0 \) case.

Let \( \gamma = -0.2 \), the standard deviation \( \sigma(t) \) as a function of \( t \) is plotted in Fig. 4a, and the probability density \( |\psi(t, x)|^2 = |⟨x |A|ψ(t)⟩|^2 + |⟨x |B|ψ(t)⟩|^2 \) is plotted as a function of \( t \) and \( x \) in Fig. 4b. As shown in Fig. 4, Eq. (15) is quite accurate.

Now let us consider the \( k_0 \neq 0 \) case. Fig. 5a shows there are two modes moving in different directions. In the non-Hermitian case, due to the exponential factor \( e^{rt^2} \) in Eq. (14), one of the wave packet peaks is lower than another, and the wave packet with lower peak is always in the left. For a large enough \( r \), the lower wave packet peak mode can be unrecognizable, as shown in Fig. 5b; we can only see the trajectory of the higher wave packet peak mode. However, the wave packet with lower peak still exists, when the wave packet with lower peak and the wave packet with higher peak meet, they will pass through each other, and the wave packet with lower peak will become the one with higher peak. Specifically, the fold line pattern at \( (x, t) = (330, 500) \) in Fig. 5a is not due to the deceleration effect, actually, it is formed by the wave packet with lower peak comes from the left becoming the wave packet with higher peak. A similar example of wave meeting process is shown in Fig. 6.

Conclusions.—We find that the Gaussian wave packet can be accelerated or decelerated in non-Hermitian systems and give the mechanism of the formation the dynamic skin effect. The extra acceleration motion of the wave packet is found to be the reason of the dynamic skin effect. The wave packet acceleration and inelastic scattering are explained by the interplay of the NHSE and the Hermitian wave packet spreading. The complete analytical result is obtained in the HN model and a universal formula for systems with uniform skin effect is built. The physics can be generalized to other non-Hermitian systems, which will be left for future study.

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FIG. 5. (a) Probability density at time $t = 70$ of Hermitian case ($\gamma = 0$) and non-Hermitian case ($\gamma = -0.01$) are shown in yellow and blue respectively. It can be seen that two modes move in the different directions. In the non-Hermitian case, one of the wave packet peaks is lower than another. (b,c) Numerical simulation of the probability density $|\langle x | \psi(t) \rangle |^2$, it is plotted as a function of $x$ and $t$. $k_0 = 2$ in the simulation. (b) $\gamma = -0.01$, two independent modes moving in different directions are recognizable. Here, the non-Hermitian parameter $\gamma$ is small, hence the effect of wave packet acceleration and inelastic scattering are tiny. (c) $\gamma = -0.2$, we can only see the trajectory of the higher wave packet peak mode. In the first reflection process, the velocity of the incident wave packet $|v_{||}|$ is larger than the velocity of the reflected wave packet $|v_{ref}|$.

![Wave packet evolution](image1.png)

![Wave packet evolution](image2.png)

![Wave packet evolution](image3.png)

![Wave packet evolution](image4.png)

FIG. 6. Wave meeting process in the non-Hermitian SSH model. $k_0 = 2$ and $\gamma = -0.05$ in the simulation. (a) Numerical simulation of the probability density $|\langle x | \psi(t) \rangle |^2$, it is plotted as a function of $x$ and $t$. It can be seen that two wave packets meet each other at $t = 500$. (b) At $t = 460$, the probability density before two wave packets meet each other. (c) The probability density when two wave packets meet each other at $t = 500$. (d) At $t = 540$, the probability density after two wave packets meet. In (b), (c), and (d), the directions of the wave packet velocities are shown as arrows.

![Wave packet evolution](image5.png)

![Wave packet evolution](image6.png)

![Wave packet evolution](image7.png)

![Wave packet evolution](image8.png)

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RELATION WITH HERMITIAN DYNAMICS

Consider a non-Hermitian system which can be transformed to a Hermitian system by a similarity transformation. Assume the real-space single-particle Hamiltonian of the non-Hermitian system under the open boundary condition is $H$, the similarity transformation is $S$, and the Hamiltonian of the Hermitian system is $\tilde{H}$, by the previous assumption,

$$\tilde{H} = S^{-1}HS.$$  

(23)

Note that we require the Hermitian Hamiltonian $\tilde{H}$ having the translation symmetry except at the boundary.

Assume a set of observable $\psi_i$ evolving with time governed by the effective non-Hermitian Hamiltonian $H$, i.e., the equation of motion is

$$i\partial_t |\psi\rangle = H |\psi\rangle,$$

(24)

where $|\psi\rangle = (\psi_1, \psi_2, \ldots)^T$.

By Eq. (23) and Eq. (24), the initial state $|\psi\rangle$ evolves as

$$\langle x | e^{-iHt} |\psi\rangle = \langle x | Se^{-i\tilde{H}t}S^{-1} |\psi\rangle.$$  

(25)

The similarity transformation $S$ usually satisfies $S|x\rangle = r^x |x\rangle$, hence,

$$\langle x | Se^{-i\tilde{H}t}S^{-1} |\psi\rangle = \sum_{x'} \langle x | r^x e^{-i\tilde{H}t} S^{-1} |x'\rangle \langle x' |\psi\rangle = \sum_{x'} r^{x-x'} \langle x | e^{-i\tilde{H}t} |x'\rangle \langle x' |\psi\rangle.$$  

(26)

By Eq. (25) and Eq. (26), it follows that

$$\langle x | e^{-iHt} |\psi\rangle = \sum_{x'} r^{x-x'} \langle x | e^{-i\tilde{H}t} |x'\rangle \langle x' |\psi\rangle.$$  

(27)

It implies that the time evolution of any initial state $|\psi\rangle$ can be expressed by the Hermitian propagator $\langle x | e^{-i\tilde{H}t} |x'\rangle$, wave function amplitude at $x'$ and the exponential factor $r^{x-x'}$.

Examples of similarity transformable non-Hermitian systems

There are many similarity transformable non-Hermitian systems, in this section, we discuss two of them which are HN model and the non-Hermitian SSH model.

First, let us consider the HN model

$$H = \sum_{n=1}^{L-1} (t_n c_n^\dagger c_{n+1} + t_{n+1} c_n^\dagger c_{n+2}).$$  

(28)

Under the OBC, the real space Hamiltonian is

$$H = \begin{pmatrix} 0 & t_1 & 0 & \cdots & 0 \\ t_{L-1} & 0 & t_1 & \cdots & 0 \\ 0 & t_{L-1} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & t_{L-1} & 0 \end{pmatrix}_{L \times L}.$$  

(29)

Let $S = \text{diag}(r, r^2, \ldots, r^N)$, then

$$\tilde{H} = S^{-1}HS$$

$$= \begin{pmatrix} 0 & t_1 r & 0 & \cdots & 0 \\ t_{L-1} r^{-1} & 0 & t_1 r & \cdots & 0 \\ 0 & t_{L-1} r^{-1} & 0 & t_1 r & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & t_{L-1} r^{-1} & 0 \end{pmatrix}_{L \times L}.$$  

(30)

Taking $r = \sqrt{\frac{t_{L-1}}{t_1}}$ for positive $t_1$ and $t_{L-1}$, then $\tilde{H}$ becomes a Hermitian Hamiltonian.

Second, let us consider the non-Hermitian SSH model whose Bloch Hamiltonian is

$$H(k) = (t_1 + t_2 \cos k) \sigma_x + \left(t_2 \sin k + i \frac{\gamma}{2}\right) \sigma_y.$$  

(31)

As discussed in Ref. 33, by taking $S = \text{diag}(1, r, r^2, \ldots, r^{L-1}, r^{L-1}, r^L)$ and $r = \sqrt{\frac{t_{L-1} - \gamma/2}{t_{L-1} + \gamma/2}}$, $\tilde{H} = S^{-1}HS$ is a Hermitian Hamiltonian with k-space form

$$\tilde{H}(k) = (\tilde{t}_1 + t_2 \cos k) \sigma_x + t_2 \sin k \sigma_y,$$

(32)

where $\tilde{t}_1 = \sqrt{(t_1 - \gamma/2)(t_1 + \gamma/2)}$.

GENERALIZED BY SPECTRAL DECOMPOSITION

In general, a non-defective non-Hermitian Hamiltonian $H$ has the following spectral decomposition

$$H = \sum_n E_n |nR\rangle \langle nL|,$$

(33)

where $|nR\rangle$ is the right eigenstate of $H$ with eigenenergy $E_n$, and $|nL\rangle$ is the corresponding left eigenstate.
The time evolution operator of $H$ is
\[
e^{-iHt} = \sum_{m=0}^{\infty} \frac{(-iHt)^m}{m!}
\]
\[
= \sum_{m=0}^{\infty} \sum_{n_1, \ldots, n_m} \frac{(-iE_{n_1}t)(-iE_{n_2}t) \cdots (-iE_{n_m}t)}{m!} |n_1R\langle n_1L|n_2R\rangle \cdots |n_mR\rangle|n_mL\rangle
\]
\[
= \sum_{m=0}^{\infty} \sum_{n} \frac{(-iE_nt)^m}{m!} |nR\rangle\langle nL|
\]
\[
= \sum_{n} e^{-iE_nt}|nR\rangle\langle nL|, \quad (34)
\]
where in the third equality the biorthogonal relation $\langle n_1L|n_2R\rangle = \delta_{n_1,n_2}$ has been used.

We can use wave vector $\beta$ in the GBZ to label the bulk energy eigenstate $|nR\rangle$ and we simply denote it by $|\beta R\rangle$. In any system without topological edge states, any energy eigenstates $|nR\rangle$ corresponds to a $|\beta R\rangle$. At least in the bulk, $|\beta R\rangle$ and $|\beta L\rangle$ has the following form
\[
\langle x|\beta R\rangle = |\beta|^2 \phi_{\beta}(x),
\]
\[
\langle x|\beta L\rangle = |\beta|^{-2} \varphi_{\beta}(x), \quad (35)
\]
where $\phi_{\beta}(x)$ and $\varphi_{\beta}(x)$ are non-exponential functions (they are usually linear combinations of trigonometric functions).

Hence, the time evolution propagator $\langle x|e^{-iHt}|x'\rangle$ is given by
\[
\langle x|e^{-iHt}|x'\rangle = \sum_{\beta} e^{-iE(\beta)t}|\beta|^{-x-x'} \varphi_{\beta}^*(x') \phi_{\beta}(x). \quad (36)
\]

Eq. (36) is a generalization of Eq. (27), instead of the appearance of the single exponential factor in Eq. (27), Eq. (36) contains all contribution of the different exponential factors $|\beta|^x e^{-\beta x'}$. If there is an energy $E(\beta_0)$ with the maximal imaginary part, then the summation in Eq. (36) is dominated by the $\beta_0$ term after a long time period, i.e.,
\[
\langle x|e^{-iHt}|x'\rangle = e^{-iE(\beta_0)t}|\beta_0|^x \varphi_{\beta_0}(x') \phi_{\beta_0}(x). \quad (37)
\]

**EXTRA MOVING SPEED**

In this section, we show that wave packets have extra drifting speeds in non-Hermitian systems. Consider the Hamiltonian we discussed in the main text
\[
H = -\frac{\nabla^2}{2m} + b \nabla - \frac{b^2 m}{2}, \quad (38)
\]
where $\nabla = \frac{\partial}{\partial x}$ in one-dimensional systems. The boundary condition is taken to be the Dirichlet boundary condition
\[
\psi|_{\partial \Omega} = 0, \quad (39)
\]
which corresponds to systems in one-dimensional infinitely deep square potential well. The Hamiltonian (38) can be expressed in a simple form
\[
H = -\frac{1}{2m}(\frac{d}{dx} - bm)^2. \quad (40)
\]

Let $S$ be the following multiplication operator acting on the wave function
\[
S(\psi)(x) = e^{bmx} \psi(x). \quad (41)
\]

By applying $S$ an exponential factor $e^{bmx}$ is multiplied on $\psi(x)$ and the Dirichlet boundary condition is preserved by $S$. Let
\[
\tilde{H} = S^{-1}HS. \quad (42)
\]

Due to $(\frac{d}{dx} - bm)S = S(\frac{d}{dx})$, it follows that
\[
\tilde{H} = -\frac{1}{2m}(\frac{d}{dx})^2 = \frac{\nabla^2}{2m}, \quad (43)
\]
where $\tilde{H}$ is a Hermitian Hamiltonian.

The time evolution equation (27) away from the boundary (transforming to k-space is valid) can be written as
\[
\langle x|e^{-i\tilde{H}t}|\psi\rangle = \int dk dk' r^x \langle x|k'\rangle \langle k'\rangle e^{-i\tilde{H}t} |k\rangle \langle k|S^{-1}\psi\rangle = r^x \int dk \langle x|k\rangle e^{-i\tilde{H}(k)t} |k\rangle \langle k|S^{-1}\psi\rangle, \quad (44)
\]
where $r = e^{bm}$. Note that apart from the $r^x$ exponential factor, the above equation is about the Hermitian dynamics $e^{-i\tilde{H}t} |S^{-1}\psi\rangle$.

Consider the Gaussian wave packet
\[
\langle x|\psi\rangle = \frac{1}{(2\pi\sigma^2)^{\frac{3}{4}}} e^{-\frac{x^2}{4\sigma^2}}. \quad (45)
\]

It is a normalized wave function, i.e., $\langle \psi|\psi\rangle = 1$. Furthermore, $|S^{-1}\psi\rangle$ is also a Gaussian wave packet since
\[
\langle x|S^{-1}\psi\rangle = e^{-bmx} \frac{1}{(2\pi\sigma^2)^{\frac{3}{4}}} e^{-\frac{x^2}{4\sigma^2}} = \frac{1}{(2\pi\sigma^2)^{\frac{3}{4}}} e^{-\frac{(x+bmx)^2}{4\sigma^2}} e^{b^2m^2\sigma^2}. \quad (46)
\]

Compare to the initial wave packet, its center has a constant shift.

By Eq. (44),
\[
\langle x|e^{-i\tilde{H}t}|\psi\rangle = \sqrt{\frac{2}{\pi}} \exp\left(\frac{b^2 m^2 \sigma^2 - \frac{m(2bm+2x)^2}{4 \sigma^2 + 2it}}{2\sigma + \frac{id}{m}}\right) e^{bmx} \quad (47)
\]
and
\[
|\langle x| e^{-iHt} |\psi\rangle|^2 = c e^{2\sigma^2 m^2 \sigma^2 e^{2bm}\sigma^2} e^{-\frac{(x-2bm\sigma^2)^2}{2\sigma^2 t^2}}
\]
\[= c e^{-\frac{(x-2bm\sigma^2)^2}{2\sigma^2 t^2}} (48)
\]
where \(\sigma(t)^2 = \sigma^2 + \frac{t^2}{4\sigma^2 m^2}\) and \(c = \frac{1}{\sqrt{2\pi \sigma(t)^2}}\).

It can be seen from Eq. (48) that \(|\langle x| e^{-iHt} |\psi\rangle|^2\) is a stationary Gaussian wave packet \(e^{-\frac{(x-2bm\sigma^2)^2}{2\sigma^2 t^2}}\) multiplying an exponential factor \(e^{2bm\sigma^2}\). Due to this exponential factor \(e^{2bm\sigma^2}\), the peak \(x_p\) of the wave packet is not stationary,
\[
x_p(t) = 2bm \left[\sigma(t)^2 - \sigma^2\right] = \frac{bm t^2}{\sigma^2 m^2}, \quad (49)
\]
satisfying
\[
\frac{d}{dx}_{x=x_p} |\langle x| e^{-iHt} |\psi\rangle|^2 = 0. \quad (50)
\]
The velocity of the wave packet peak is
\[
v_p(t) = \frac{dx_p}{dt} = \frac{bm t}{\sigma^2 m^2}. \quad (51)
\]
Note that if \(b = 0\) (the Hermitian case), the wave packet peak is stationary and the velocity of the wave packet peak is zero. Otherwise, the wave packet peak is uniformly accelerated.

Nonzero \(b\) and the dependence of \(t\) in \(\sigma(t)^2\) lead to a nonzero \(v_p(t)\) in Eq. (51). Hence, it can be concluded that NHSE and wave packets spreading phenomenon make the wave packet acquire an extra moving speed.

**Incident and reflected velocity**

Now, consider the initial Gaussian wave packet with a nonzero central momentum \(k_0\)
\[
\langle x|\psi\rangle = \frac{1}{(2\pi \sigma^2)^{\frac{3}{2}}} e^{-\frac{x^2}{4\sigma^2}} e^{i k_0 x}. \quad (52)
\]
Under the time evolution governing by the Hermitian Hamiltonian \(\hat{H}\), it has a nonzero initial velocity
\[
v_0 = \frac{\partial \hat{H}(k)}{\partial k} |_{k_0} = \frac{k}{\sigma^2} m. \quad (53)
\]
The time evolution of \(\langle \psi \rangle\) can be calculated using
\[
\langle x| e^{-iHt} |\psi\rangle = e^{bm \sigma^2} \langle x| e^{-iH|\psi\rangle t} |\psi\rangle \quad (54)
\]
and its probability density is
\[
|\langle x| e^{-iHt} |\psi\rangle|^2 \approx c e^{-\frac{(x-2bm\sigma^2)^2}{2\sigma^2 t^2}}, \quad (55)
\]
where \(\sigma(t)^2 = \sigma^2 + \frac{t^2}{4\sigma^2 m^2}\), \(c = \frac{1}{\sqrt{2\pi \sigma(t)^2}}\) and \(x_p(t) = 2bm (\sigma(t)^2 - \sigma^2)\). This expression is valid before the wave packet touches the boundary and \(v_p(t)\) in the expression is the result of the Hermitian dynamics \(e^{-iHt} |\psi\rangle\) while \(x_p(t)\) is the result of the existence of the exponential factor \(e^{2bm\sigma^2}\) in Eq. (54). From above, the velocity before the wave packet reaches the boundary is
\[
v_{in}(t) = v_0 + 2bm \frac{ds(t)^2}{dt} = v_0 + \frac{b}{m\sigma^2} t. \quad (56)
\]
Now, let us consider the time evolution after the wave packet is bounced back by the boundary, and we assume the boundary is at \(x_b\)
\[
|\langle x| e^{-iHt} |\psi\rangle|^2 \propto e^{-\frac{(x-2bm\sigma^2)^2}{2\sigma^2 t^2}}. \quad (57)
\]
It can be obtained by the following argument. Before the wave packet reaches the boundary,
\[
e^{-iHt} |\psi\rangle = \int dk |k\rangle e^{-iHt} |\psi\rangle \quad (58)
\]
is a linear combination of a series of \(|k\rangle\) states. Assume the boundary is at \(x_b\), after the full reflection, these \(|k\rangle\) states are changed to \(|\tilde{k}\rangle\) states such that
\[
\langle x|\tilde{k}\rangle = e^{i(k-\langle k\rangle x_b+i(kx_b+\pi)}. \quad (59)
\]
In the reflection process, each \(|k\rangle\) state acquires a phase shift of \(\pi\). Hence, after the full reflection,
\[
e^{-i\tilde{H}t} |\tilde{\psi}\rangle = \int dk |\tilde{k}\rangle \langle k| e^{-i\tilde{H}t} |\tilde{\psi}\rangle \quad (60)
\]
and
\[
\langle x| e^{-i\tilde{H}t} |\tilde{\psi}\rangle = \int dk \langle x|\tilde{k}\rangle \langle k| e^{-i\tilde{H}t} |\tilde{\psi}\rangle
\]
\[
= - \int \frac{dk}{2\pi} e^{ik (2x_b-x)} \langle k| e^{-i\tilde{H}t} |\tilde{\psi}\rangle. \quad (61)
\]
It can be seen from Eq. (61) that compare to the time evolution before the reflection, Eq. (61) only substitutes \(x\) with \(2x_b-x\) and a minus sign is multiplied. Thus, after the full reflection,
\[
|\langle x| e^{-i\tilde{H}t} |\tilde{\psi}\rangle|^2 \propto e^{-\frac{(x-2bm\sigma^2)^2-\langle x\rangle^2}{2\sigma^2 t^2}}
\]
\[
= e^{-\frac{(x-2bm\sigma^2)^2-\langle k\rangle^2}{2\sigma^2 t^2}}. \quad (56)
\]
By Eq. (54), after the full reflection,
\[
|\langle x| e^{-i\tilde{H}t} |\tilde{\psi}\rangle|^2 \propto e^{2bm\sigma^2} e^{-\frac{(x-2bm\sigma^2)^2-\langle k\rangle^2}{2\sigma^2 t^2}}. \quad (62)
\]
The peak of the wave packet \(\tilde{x}_p(t)\) satisfying
\[
\frac{d}{dx}\tilde{x}_p \langle x| e^{-i\tilde{H}t} |\tilde{\psi}\rangle|^2 = 0 \quad (63)
\]
is
\[
\tilde{x}_p(t) = 2bm \left[\sigma(t)^2 + \sigma^2\right] + 2xb - v_0 t. \quad (63)
\]
Hence, the velocity of the wave packet after the full reflection is

\[ v_{\text{ref}}(t) = -v_0 + 2bm \frac{d\sigma(t)^2}{dt} = -v_0 + \frac{b}{m\sigma^2}t. \]  

(64)

From Eq. (56) and Eq. (64), it can be seen that \( v_{\text{in}} \neq -v_{\text{ref}} \), which is analogous to the inelastic scattering phenomenon in classical mechanics.

### SIMULATION METHOD

In our simulation, we use the finite difference method to express differential operators in terms of finite dimensional matrices. The Laplace operator can be approximately written as

\[ \nabla^2 \psi_n \approx \frac{\psi_{n+1} + \psi_{n-1} - 2\psi_n}{(\Delta x)^2}. \]  

(65)

Under the Dirichlet boundary condition (take \( \psi_{-1} = \psi_{N+1} = 0 \)), the Laplace operator can be written as

\[ \nabla^2 = \frac{1}{(\Delta x)^2} \begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots \end{pmatrix} \cdots \begin{pmatrix} 0 & \cdots & 0 & 1 & -2 \\ 0 & \cdots & 0 & 1 & -2 \end{pmatrix}_{N \times N}. \]  

(66)

\( \nabla \) can approximately written as

\[ \nabla \psi_n \approx \frac{\psi_{n+1} - \psi_n}{\Delta x}. \]  

(67)

Its matrix form is

\[ \nabla = \frac{1}{\Delta x} \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & 0 & \cdots \\ 0 & 0 & -1 & 1 & \cdots \end{pmatrix} \cdots \begin{pmatrix} 0 & \cdots & 0 & 0 & -1 \\ 0 & \cdots & 0 & 0 & -1 \end{pmatrix}_{N \times N}. \]  

(68)

The number of the lattice points \( N = \frac{L}{\Delta x} \). The accuracy of the finite difference method improves when reducing \( \Delta x \) and increasing the number of the mesh cells in the simulation. In our simulation, compared to the length \( L = 10 \), we take \( \Delta x = 0.01 \), in other words, there are 1000 lattice points in the simulation.