On the unit distance problem

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Abstract. The Erdős unit distance conjecture in the plane says that the number of pairs of points from a point set of size $n$ separated by a fixed (Euclidean) distance is $\leq Cn^{1+\epsilon}$ for any $\epsilon > 0$. The best known bound is $Cn^{4/3}$. We show that if the set under consideration is well-distributed and the fixed distance is much smaller than the diameter of the set, then the exponent $4/3$ is significantly improved. Corresponding results are also established in higher dimensions. The results are obtained by solving the corresponding continuous problem and using a continuous-to-discrete conversion mechanism. The degree of sharpness of results is tested using the known results on the distribution of lattice points dilates of convex domains.

We also introduce the following variant of the Erdős unit distance problem: how many pairs of points from a set of size $n$ are separated by an integer distance? We obtain some results in this direction and formulate a conjecture.

1. Introduction

One of the hardest longstanding conjectures in extremal combinatorics is the Erdős unit distance conjecture ([2], see also [1]). It says that if $P$ is a planar point set with $n$ points, then the number of pairs of elements of $P$ a fixed Euclidean distance apart is bounded by $Cn^{1+\epsilon}$ for every $\epsilon > 0$. The best known bound, obtain by Spencer, Szemeredi and Trotter ([17]) is $Cn^{4/3}$. An interesting development occurred in 2005 when Pavel Valtr ([18]) proved that if the Euclidean distance is replaced by a distance induced by the norm defined by a bounded convex set with a smooth boundary and non-vanishing curvature, then the $Cn^{4/3}$ bound is, in general, best possible.

The purpose of this paper is to show in the realm of well-distributed sets that the $Cn^{4/3}$ can be significantly improved if we count the number of pairs of points separated by a distance that is much smaller than the diameter of the set. Our main result is the following.

Definition 1.1. We say that $P \subset \mathbb{R}^d$ of size $n$ is well-distributed if there exists $c > 0$ such that $|p-p'| \geq c$ and every unit lattice cube in $\mathbb{R}^d \cap [0, n^{1/d}]^d$ contains exactly one point of $P$.

Theorem 1.2. Let $B$ be a symmetric bounded convex set in $\mathbb{R}^d$, $d \geq 2$, with a smooth boundary and everywhere non-vanishing Gaussian curvature. Let $P$ be a well-distributed set of size $n$. Then for $k \in (1, n^{4/3})$,

\[
\# \left\{ (p, p') \in P \times P : k \leq ||p-p'||_B \leq k + n^{-\frac{d}{d+1}} \right\} \leq Cn^{2-d/2} \cdot \Lambda,
\]

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where

$$\Lambda = \left( \frac{k}{n^{2/3}} \right)^{\frac{d-1}{2}}.$$ 

In particular, if \( d = 2 \), the left hand side of (1.1) is bounded by \( Cn^\frac{2}{3} \cdot \left( \frac{k}{n^{2/3}} \right)^{\frac{1}{2}} \), which is an improvement over the known \( Cn^\frac{2}{3} \) bound when \( k = o(n^{\frac{2}{3}}) \).

**Remark 1.3.** When \( k \approx n^{\frac{2}{3}} \), Theorem 1.2 is implicit in the main result in [11], but the key feature here is the dependence on \( k \) with the resulting improvement when \( k = o(n^{\frac{2}{3}}) \). Also, we shall prove below that in the case \( k \approx n^{\frac{2}{3}} \), the estimate provided by Theorem 1.2 is sharp. See also [14] where the continuous-discrete correspondence is used in reverse in order to obtain sharpness examples for Falconer type estimates.

**Remark 1.4.** Note that the left hand side of (1.1) is trivially bounded by \( Cn \cdot k^{d-1} \). Therefore, the estimate in Theorem 1.2 is only interesting when \( k >> n^{\frac{2}{3}} \cdot \left( \frac{\log \log n}{\log n} \right)^{\frac{d-1}{d+1}} \). For example, in dimension two this threshold is \( n^{\frac{2}{3}} \).

We also study the following variant of the Erdős unit distance problem. How many pairs of points from a set of \( n \) points in \( \mathbb{R}^d \), \( d \geq 2 \), are separated by an integer distance? When \( P = \mathbb{Z}^2 \cap [0, n \frac{1}{d}]^d \), it is not difficult to see that the number of such pairs is \( \approx n^{\frac{5}{4}} \).

**Conjecture 1.5.** Let \( P \subset \mathbb{R}^d \), \( d = 2, 3 \), be a finite set of size \( n \). Then

$$\# \{(p, p') \in P \times P : |p - p'| \in \mathbb{Z}\} \leq Cn^{\frac{5}{4}}.$$

In higher dimensions this conjecture is not true, in general, due to the existence of the celebrated Lens example (see e.g. [1]) which shows that in dimensions 4 and higher there exists \( P \subset \mathbb{R}^d \) of size \( n \) such that \( \# \{(p, p') \in P \times P : |p - p'| = 1\} \geq cn^2 \). But in the setting of well-distributed sets, Conjecture 1.5 still makes sense when \( d \geq 4 \). The following result follows easily from Theorem 1.2.

**Theorem 1.6.** Let \( P \) be a well-distributed set of size \( n \). Then

$$\# \{(p, p') \in P \times P : \text{dist}(|p - p'|, \mathbb{Z}) < n^{-\frac{2+\varepsilon}{3-\varepsilon}} \} \leq Cn^{2-\frac{\varepsilon}{3-\varepsilon} + \frac{1}{2}} = n^{2-\frac{\varepsilon}{3-\varepsilon} \cdot n^{\frac{2}{3-\varepsilon}}}.$$

### 1.1. Sharpness of results.

The results associated with the lattice point counting problems provide a useful tool for testing sharpness of Theorem 1.2. Let \( n \approx q^d \) and \( P = \mathbb{Z}^d \cap B(0, 10q) \), the ball of radius 10\( q \) centered at the origin. Let \( N_d(R) \) denote the number of elements of \( \mathbb{Z}^d \) inside the ball of radius \( R \) centered at the origin. It is known (see e.g [8]) that

$$N_d(R) = \omega_d R^d + D_d(R),$$

where \( \omega_d \) is the volume of the unit ball, \( |D_2(R)| \leq C_\varepsilon R^{\frac{d}{d+1}+\varepsilon} \) ([9]), \( |D_3(R)| \leq C_\varepsilon R^{\frac{d}{d+1}+\varepsilon} \) ([6]), and \( |D_d(R)| \leq C_\varepsilon R^{d-2+\varepsilon} \) for \( d \geq 4 \) ([5]).

Then

$$\# \{(p, p') \in P \times P : q \leq |p - p'| \leq q + q^{-\frac{d+1}{d+2}} \} \geq Cq^d \cdot \left( N \left( q + q^{-\frac{d+1}{d+2}} \right) - N(q) \right).$$

We have

$$N \left( q + q^{-\frac{d+1}{d+2}} \right) - N(q) = \omega_d \left( \left( q + q^{-\frac{d+1}{d+2}} \right)^d - q^d \right) + D \left( q + q^{-\frac{d+1}{d+2}} \right) - D(q).$$
Using the bounds on $|D(R)|$ described above, we see that
\[ N \left( q + q^{-\frac{d+1}{d+s}} \right) - N(q) \geq cq^{d-1} - \frac{q^{d+1}}{d+s}, \]
which implies that
\[ \# \{(p, p') \in P \times P : q \leq |p - p'| \leq q + q^{-\frac{d+1}{d+s}} \} \geq Cq^{2d-1} + \frac{q^{d+1}}{d+s} \approx n^2 - \frac{n^2}{d+s}, \]
proving that Theorem 1.2 is sharp when $k \approx n^\frac{d}{s}$.

When $k \ll q$, we can conclude that
\[ N \left( k + q^{-\frac{d+1}{d+s}} \right) - N(k) \geq c k^{d-1} q^{d-1} \]
if the right hand side is larger than the error term measured in terms of the bounds on $|D_q(R)|$ described above. This happens for a range of $k$’s. In this range,
\[ \# \{(p, p') \in P \times P : q \leq |p - p'| \leq k + q^{-\frac{d+1}{d+s}} \} \geq C k^{d-1} q^d - \frac{q^{d+1}}{d+s}. \]

The right hand side is smaller than the bound obtained by Theorem 1.2 when $k = o(n^\frac{d}{s})$. This may indicate that Theorem 1.2 is not sharp in this range, but it is also possible that a more sophisticated sharpness example may be found.

The construction above applied to $\approx n^\frac{d}{s}$ annuli shows that when $k \approx n^\frac{d}{s}$, the conclusion of Theorem 1.6 is sharp.

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2. Proof of the main result

For $\frac{d}{2} < s < d$, define
\[ \mu_{q,s}(x) = q^{-d} q^{\frac{d^2}{q^2}} \sum_{p \in P} \phi \left( q^{\frac{d}{2}} \left( x - \frac{p}{q} \right) \right) \phi \left( \frac{p}{q} \right), \]
where $\phi$ is a smooth cut-off function supported in the ball of radius 2 and identically equal to 1 in the ball of radius 1.

This is a natural measure on the $q^{-\frac{d}{2}}$-neighborhood of $\frac{d}{q}P = \left\{ \frac{a}{q} : a \in P \right\}$. Our goal is to bound the expression
\[ \int \int \{ (x, y) : t \leq ||x - y||_B \leq t + q^{-\frac{d}{2}} \} d\mu_{q,s}(x)d\mu_{q,s}(y), \]
where $|| \cdot ||_B$ is the norm induced by a bounded symmetric convex set $B$ with a smooth boundary and everywhere non-vanishing curvature, and then relate it to the count for the number of pairs separated by a given distance.

Using a Fourier inversion type argument (see e.g. [16] or [19]), the expression (2.1) equals a constant multiple of
\[ \int |\hat{\mu}_{q,s}(\xi)|^2 \hat{\chi}_{A_1, q,s}(\xi) d\xi, \]
where
\[ A_{t,q,s} = \{ x \in \mathbb{R}^d : t \leq \| x \|_B \leq t + q^{-\frac{d}{2}} \} , \]
\( \chi \) denotes its indicator function, \( \| \cdot \|_B \) is the norm induced by a symmetric bounded convex set \( B \) with a smooth boundary and everywhere non-vanishing curvature and
\[ \hat{f}(\xi) = \int e^{-2\pi i x \cdot \xi} f(x) dx , \]
defined for functions in \( L^2(\mathbb{R}^d) \).

We first show that the \( s \)-energy integral of \( \mu_{q,s} \) is bounded independently of \( q \). See, for example, [4], [15] and [16] for the background on energy integrals in the setting of sets of a given Hausdorff dimension.

**Lemma 2.1.** For any \( s \in (\frac{d}{2}, d) \),
\[ \int |\hat{\mu}_{q,s}(\xi)|^2 |\xi|^{-d+s} d\xi = c_{d,s} \int \int |x-y|^{-s} d\mu_{q,s}(x)d\mu_{q,s}(y) \leq C < \infty \]
with a bound independent of \( q \).

We shall give the proof of Lemma 2.1 at the end of the paper. Next, we bound the Fourier transform of the indicator function of \( A_{t,q,s} \).

**Lemma 2.2.** [3] With the notation above,
\[ |\hat{\chi}_{A_{t,q,s}}(\xi)| \leq C t^{d-1} \| \xi \|^{-\frac{d+1}{2}} \min \left\{ q^{-\frac{d}{2}}, |\xi|^{-1} \right\} , \]
where \( C \) is a universal constant independent of \( t \) or \( q \).

Falconer proved this result in ([3]) in the case when \( B \) is the unit ball. The proof of the general case is similar.

With Lemma 2.1 and Lemma 2.2 in tow, we see that
\[ \int |\hat{\mu}_{q,s}(\xi)|^2 \hat{\chi}_{A_{t,q,s}}(\xi) d\xi \leq C t^{d-1} \| \xi \|^{-\frac{d+1}{2}} \int |\hat{\mu}_{q,s}(\xi)|^2 |\xi|^{-\frac{d+1}{2}} d\xi \]
\[ \leq C t^{d-1} q^{-\frac{d}{2}} \int |\hat{\mu}_{q,s}(\xi)|^2 |\xi|^{-d+s} d\xi \leq C' t^{d-1} q^{-\frac{d}{2}} \]
if \( s \geq \frac{d+1}{2} \).

We are now ready for the combinatorial conclusion. See [7], [12], [10] and [13] where various forms of the continuous to discrete conversion mechanisms are developed and applied. Observe that
\[ \# \{(p,p') \in P \times P : k \leq \| x-y \|_B \leq k + q^{-\frac{d}{2}+1} \} \]
\[ \leq C q^{2d} \int \int \left\{ (x,y) : \frac{k}{2} \leq \| x-y \|_B \leq \frac{k}{2} + q^{-\frac{d}{2}} \right\} d\mu_{q,s}(x)d\mu_{q,s}(y) . \]

By (2.4) this expression is bounded by
\[ C' q^{2d-\frac{d}{2} + \frac{d+1}{2}} k^{d-1} \frac{d+1}{2} = C' n^{2-\frac{d}{2}} \left( \frac{k}{n^{\frac{d}{2}}} \right)^{\frac{d+1}{2}} , \]
as desired. This completes the proof of Theorem 1.2 up the proof of Lemma 2.1.
2.1. Proof of Lemma 2.1. This result is proved in [13], but we include the proof for the sake of completeness. We have

\[
q^{-2d} q^{\frac{2d^2}{k}} \int \int |x - y|^{-s} d\mu_{q,s}(x) d\mu_{q,s}(y)
\]

\[
= \sum_{p, p' \in P} \phi \left( \frac{p}{q} \right) \phi \left( \frac{p'}{q} \right) \int \int |x - y|^{-s} \phi \left( q^{\frac{d}{q}} \left( x - \frac{p}{q} \right) \right) \phi \left( q^{\frac{d}{q}} \left( y - \frac{p'}{q} \right) \right) \, dx \, dy = I + II
\]

where

\[
I = q^{-2d} q^{\frac{2d^2}{k}} \sum_{p \in P} \phi^2 \left( \frac{p}{q} \right) \int \int |x - y|^{-s} \phi \left( q^{\frac{d}{q}} \left( x - \frac{p}{q} \right) \right) \phi \left( q^{\frac{d}{q}} \left( y - \frac{p}{q} \right) \right) \, dx \, dy
\]

and

\[
II = q^{-2d} q^{\frac{2d^2}{k}} \sum_{p \not\in P} \frac{2d^2}{k} \int \int |x - y|^{-s} \phi \left( q^{\frac{d}{q}} \left( x - \frac{p}{q} \right) \right) \phi \left( q^{\frac{d}{q}} \left( y - \frac{p'}{q} \right) \right) \, dx \, dy.
\]

By a direct calculation, \( I \) is bounded. Using the separation between \( p \) and \( p' \), we see that

\[
II \leq q^{-2d} q^{\frac{2d^2}{k}} \sum_{p \not\in P} \frac{2d^2}{k} \phi \left( \frac{p}{q} \right) \phi \left( \frac{p'}{q} \right) \left| \frac{p}{q} - \frac{p'}{q} \right|^{-s} q^{\frac{2d^2}{k}} = q^{-2d} q^{s} \sum_{p \not\in P} \phi \left( \frac{p}{q} \right) \phi \left( \frac{p'}{q} \right) |p - p'|^{-s}.
\]

Using the well-distributivity assumption on \( P \) we may replace the sum by the integral and thus the quantity is bounded. This completes the proof of Lemma 2.1.

3. Proof of Theorem 1.6

We have shown above that

\[
\# \{ (p, p') \in P \times P : k \leq |p - p'| \leq k + n^{-\frac{1}{2(d+1)}} \} \leq C n^{2-\frac{1}{d+1}} \cdot \left( \frac{k}{n^{\frac{1}{2}}} \right)^{\frac{d+1}{2}}.
\]

Summing both sides over \( k = 1, 2, \ldots, N \), where \( N \approx n^{\frac{1}{2}} \) yields the conclusion of Theorem 1.6.

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