ON THE AXIOM OF SPHERES IN KÄHLER GEOMETRY

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1. Introduction. Let $M$ be a $2m$-dimensional Kähler manifold with a Riemannian metric $g$ and a complex structure $J$. Let $R$, $S$ and $\tau(R)$ denote the curvature tensor, the Ricci tensor and the scalar curvature of $M$, respectively. The Bochner curvature tensor $B$ is defined by

$$B(X,Y,Z,U) = R(X,Y,Z,U) - \frac{1}{2(m+2)} \{ g(X,U)S(Y,Z) - g(X,Z)S(Y,U) + g(Y,U)S(X,Z) + g(X,JU)S(Y,JZ) - g(X,JZ)S(Y,JU) + g(Y,JZ)S(X,JU) - g(Y,JU)S(X,JZ) - 2g(X,JY)S(Z,JU) - 2g(Z,JU)S(X,JY) \}$$

Let $N$ be an $n$-dimensional submanifold of $M$. The second fundamental form $\alpha$ of the immersion is defined by $\alpha(X,Y) = \tilde{\nabla}_X Y - \nabla_X Y$ for $X, Y \in \mathfrak{X}(N)$, where $\tilde{\nabla}$ (resp. $\nabla$) is the covariant differentiations on $M$ (resp. $N$). The submanifold $N$ is said to be totally umbilical, if $\alpha(X,Y) = g(X,Y)H$, $H$ being the mean curvature vector of $N$ in $M$, i.e. $H = (1/n)\text{trace} \alpha$. Let $\xi$ be a vector field normal to $N$. Then the Weingarten formula is

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi,$$

where $-A_\xi X$ (respectively, $D_X \xi$) denotes the tangential (resp. the normal) component of $\tilde{\nabla}_X \xi$. The vector field $\xi$ is said to be parallel, if $D_X \xi = 0$ for each $X \in \mathfrak{X}(N)$.

The equation of Codazzi is given by

$$\{ R(X,Y)Z \}^\perp = (\tilde{\nabla}_X \alpha)(Y,Z) - (\tilde{\nabla}_Y \alpha)(X,Z),$$

where $\{ R(X,Y)Z \}^\perp$ denotes the normal component of $R(X,Y)Z$ and

$$(\tilde{\nabla}_X \alpha)(Y,Z) = D_X \alpha(Y,Z) - \alpha(\nabla_X Y, Z) - \alpha(Y, \nabla_X Z).$$

By an $n$-plane in $T_p(M)$ we mean an $n$-dimensional linear subspace of $T_p(M)$. An $n$-plane $\pi$ is said to be holomorphic (resp. antiholomorphic) if $\pi = J\pi$ (resp. $\pi \perp J\pi$).
A $2m$-dimensional Kähler manifold $M$ is said to satisfy the axiom of holomorphic (resp. antiholomorphic) $2n$-spheres (resp. $n$-spheres), where $n$ is a fixed integer, $1 \leq n \leq m$, if for each point $p \in M$ and for any $2n$-dimensional holomorphic (resp. $n$-dimensional antiholomorphic) plane $\pi$ in $T_p(M)$ there exists a totally umbilical submanifold $N$ of $M$ with a parallel mean curvature vector, such that $p \in N$ and $T_pN = \pi$.

As was proved in [2], if a $2m$-dimensional Kähler manifold $M$ satisfies the axiom of holomorphic $2n$-spheres for some $n$, $1 \leq n < m$ or the axiom of antiholomorphic $n$-spheres for some $n$, $1 < n \leq m$, then $M$ is of constant holomorphic sectional curvature.

We shall prove the following theorems:

**Theorem 1.** Let $M$ be a $2m$-dimensional Kähler manifold, $m > 2$, and let $n$ be a fixed integer, $2 \leq n < m$. If for each point $p \in M$ and for any holomorphic $2n$-plane $\pi$ in $T_p(M)$ there exists a totally umbilical submanifold $N$ of $M$, such that $p \in N$ and $T_pN = \pi$, then $M$ is of constant holomorphic sectional curvature.

**Theorem 2.** Let $M$ be a $2m$-dimensional Kähler manifold, $m > 2$, and let $n$ be a fixed integer, $2 < n \leq m$. If for each point $p \in M$ and for any antiholomorphic $n$-plane $\pi$ in $T_p(M)$ there exists a totally umbilical submanifold $N$ of $M$, such that $p \in N$ and $T_pN = \pi$, then $M$ is of constant holomorphic sectional curvature.

2. A Lemma and Proofs of the Theorems. As is known, a $2m$-dimensional Kähler manifold $M$ has vanishing Bochner curvature tensor, iff for each point $p$ of $M$ the sum $\sum_{i=1}^{m} R(e_i, Je_i, Je_i, e_i)$ is independent of the orthonormal basis $\{e_i, Je_i; i = 1, ..., m\}$ of $T_p(M)$ [5]. Hence it is not difficult to prove the following

**Lemma.** A Kähler manifold $M$ of dimension $2m \geq 6$ has a vanishing Bochner curvature tensor, iff for each point $p \in M$ and for all unit vectors $x, y, z \in T_p(M)$ which span an antiholomorphic 3-plane

$$R(x, Jx, y, z) = 2R(x, y, Jx, z)$$

holds good.

Let $N$ be a totally umbilical submanifold of $M$. Then Codazzi’s equation reduces to

$$\{R(X, Y)Z\}^\perp = g(Y, Z)D_XH - g(X, Z)D_YH .$$

Now we can proceed to prove Theorem 1. For a point $p \in M$ we take arbitrary unit vectors $x, y, z \in T_p(M)$ which span an antiholomorphic 3-plane. Let $N$ be a totally umbilical submanifold of $M$ such that $p \in N$, $x, y, Jx, Jy \in T_p(N)$ and $z \perp T_p(N)$. Then, from (2.1) we obtain

$$R(x, Jx, y, z) = 0 ,$$

$$R(x, y, Jx, z) = 0$$
and, according to the Lemma, $M$ has vanishing Bochner curvature tensor. Hence

$$R(X, Y, Z, U) = \frac{1}{4(m+2)} \{g(X, U)S(Y, Z)$$
$$- g(X, Z)S(Y, U) + g(Y, Z)S(X, U) - g(Y, U)S(X, Z)$$
$$+ g(X, JU)S(Y, JZ) - g(X, JZ)S(Y, JU) + g(Y, JZ)S(X, JU)$$
$$- g(Y, JU)S(X, JZ) - 2g(X, JY)S(Z, JU) - 2g(Z, JU)S(X, JY)\}$$

From (2.2) and (2.3)

$$S(y, z) = 0$$

for all $y, z \in T_p(M)$ with $g(y, z) = g(y, Jz) = 0$ and for each point $p \in M$. Consequently, $M$ is an Einsteinian manifold and because of $B = 0$ $M$ is of constant holomorphic sectional curvature.

The proof of Theorem 2 is similar.

**Remark 1.** If $M$ is a Kähler manifold of constant holomorphic sectional curvature, then every totally umbilical submanifold of $M$ has parallel mean curvature vector, see [3].

**Remark 2.** It is known that any $2m$-dimensional Kähler manifold of constant holomorphic sectional curvature satisfies the axiom of holomorphic $2n$-spheres and the axiom of antiholomorphic $n$-spheres for each $n$, $1 \leq n \leq m$, see e.g. [3].

**Remark 3.** For a Riemannian manifold the condition analogous to that in Theorem 1 or Theorem 2 is equivalent to the requirement that the manifold has vanishing Weil curvature tensor, see [1].

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References

[1] J. Schouten. Ricci-Calculus. Springer-Verlag. Berlin, 1954.
[2] S. Goldberg, E. Moskal. Kodai Math. Sem. Rep. **27**, 1976, 188.
[3] B.-Y. Chen, K. Ogiue. Michigan Math. J. **21**, 1974, 225.
[4] K. Nomizu. J. Differ. Geom. **8**, 1973, 335.
[5] G. Stanoilov. Pure and Appl. Math. Sci. **5**, 1977, 7.