Some results on the generalized inverse of tensors and idempotent tensors

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Abstract

Let \( A \) be an order \( t \) dimension \( m \times n \times \cdots \times n \) tensor over complex field. In this paper, we study some generalized inverses of \( A \), the \( k \)-T-idempotent tensors and the idempotent tensors based on the general tensor product. Using the tensor generalized inverse, some solutions of the equation \( A \cdot x^{t-1} = b \) are given, where \( x \) and \( b \) are dimension \( n \) and \( m \) vectors, respectively. The generalized inverses of some block tensors, the eigenvalues of \( k \)-T-idempotent tensors and idempotent tensors are given. And the relation between the generalized inverses of tensors and the \( k \)-T-idempotent tensors is also showed.

Keywords: Tensor, Generalized inverse, Idempotent tensor, Tensor eigenvalue

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1. Introduction

In recent years, there has been extensive attention and interest in the work of spectral theory of tensors and hypergraphs \([4]-[10]\), since the research of Qi \([1]\), Lim \([2]\) and Shao \([3]\).

For a positive integer \( n \), let \( [n] = \{1, \ldots, n\} \). An order \( t \) tensor \( \mathcal{A} = (a_{i_1 \cdots i_t})_{1 \leq i_j \leq n_j} \ (j = 1, \ldots, t) \) is a multidimensional array with \( n_1 n_2 \cdots n_t \) entries. Let \( \mathbb{C}^{n_1 \times \cdots \times n_t} \) be the set of all the order \( t \) dimension \( n_1 \times \cdots \times n_t \) tensors.
over complex field $\mathbb{C}$. Clearly, an order 2 tensor is a matrix. Let $\mathbb{C}^{m,n}$ denote the set of all the $m \times n \times \cdots \times n$ tensors of order $t$ over complex field. Let $\mathcal{D} = \text{diag}(d_{1\cdots 1}, \ldots, d_{n\cdots n}) \in \mathbb{C}^{n,n}_t$ be a diagonal tensor whose entries are all zero except for $d_{ii\cdots i}, i = 1, \ldots, n$. If $d_{ii\cdots i} = 1, i = 1, \ldots, n$, then $\mathcal{D}$ is the unit tensor with order $t$, denoted by $I$. For the vector $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{C}^n$ and the tensor $\mathcal{A} \in \mathbb{C}^{m,n}_t$, let $\mathcal{A} \cdot x^{t-1}$ be a dimension $m$ vector whose $i$-th component is

$$
(\mathcal{A} \cdot x^{t-1})_i = \sum_{i_2, \ldots, i_t \in [n]} a_{i_2 \cdots i_t} x_{i_2} x_{i_3} \cdots x_{i_t},
$$

where $i \in [m]$. In [1], Qi defined the eigenvalue of tensors. For $\mathcal{A} \in \mathbb{C}^{n,n}_t$, $\lambda \in \mathbb{C}$ is called the eigenvalue of $\mathcal{A}$ if there exists nonzero vector $x \in \mathbb{C}^n$ such that $\mathcal{A} \cdot x^{t-1} = \lambda x^{[t-1]}$, where $x^{[r]} = (x_1^r, x_2^r, \ldots, x_n^r)^T$.

Shao defined the general tensor product in [3]. For tensors $\mathcal{A} \in \mathbb{C}^{n,n}_t$ and $\mathcal{B} \in \mathbb{C}^{n,n}_k$, the general tensor product of them is $\mathcal{A} \mathcal{B} \in \mathbb{C}^{n,n}_{(t-1)(k-1)+1}$ with entries as

$$
(\mathcal{A} \mathcal{B})_{i_1 \alpha_1 \cdots \alpha_{t-1}} = \sum_{i_2, \ldots, \alpha_t \in [n]} a_{i_2 \cdots i_t} b_{i_2 \alpha_1} \cdots b_{i_t \alpha_{t-1}},
$$

where $i \in [n], \alpha_1, \ldots, \alpha_{t-1} \in [n]^{k-1}$. Bu et al. showed that the general product of $\mathcal{A} \in \mathbb{C}^{n,n}_t$, $\mathcal{B} \in \mathbb{C}^{n,m}_k$ also can be written as Eq.(2), where $i \in [m], \alpha_1, \ldots, \alpha_{t-1} \in [m]^{k-1}$.

The definition of the inverse of tensors was given in [11]. For $\mathcal{A} \in \mathbb{C}^{n,n}_t$, $\mathcal{B} \in \mathbb{C}^{n,n}_k$, if $\mathcal{A} \mathcal{B} = I$, then $\mathcal{A}$ is called an order $t$ left inverse of $\mathcal{B}$, denoted by $\mathcal{B}^L$; $\mathcal{B}$ is called an order $k$ right inverse of $\mathcal{A}$, denoted by $\mathcal{A}^R$.

It is well known that there are many types of generalized inverses of matrices (operators) [12]. Let $\mathbb{H}$ be a Hilbert space and $\mathcal{L}(\mathbb{H})$ be the set of the linear operators on $\mathbb{H}$. Let $A \in \mathbb{C}^{n \times n} (\in \mathcal{L}(\mathbb{H})), X \in \mathbb{C}^{n \times m} (\in \mathcal{L}(\mathbb{H}))$,

$$
(1) AXA = A; \quad (2) XAX = X; \quad (5) XA = AX.
$$

The matrix (operator) $X$ is said to be the $\{i\}$ inverse of the matrix (operator) $A$ if the above equation (i) holds. $A\{i\}$ is the set of all the $\{i\}$ inverses of $A$. The matrix (operator) $X$ is called the group inverse of the matrix (operator) $A$ if the equations (1), (2) and (5) hold, denoted by $A^\#$. And the group inverse is a kind of spectral generalized inverse [12].

The generalized inverses of block matrices (operators) have important applications in algebraic connectivity and algebraic bipartiteness of graphs.
Markov chains ([14,15]), resistance distance [16] and so on. Scholars gave many results on the representations of the generalized inverses of block matrices [17]-[19].

In this paper, we study the generalized inverses of tensors, the $k$-T-idempotent tensors and the idempotent tensors. This paper is organized as follows. In section 2, we show the definitions of some generalized inverses of tensors. And using the generalized inverse, some solutions of the equation \( A \cdot x^{-1} = b \) is gotten, where \( A \in \mathbb{C}^{m,n} \), the vectors \( x \in \mathbb{C}^n \), \( b \in \mathbb{C}^m \). The tensor generalized inverses of some block tensors are given. In the section 3, we give the definitions of the $k$-T-idempotent tensors and idempotent. There also some results on the eigenvalues of the $k$-T-idempotent tensors and idempotent tensors; the relation between the $k$-T-idempotent tensors (idempotent tensors) and the tensor generalized inverses. In the section 4, some examples of tensor generalized inverses and idempotent tensors are presented.

2. Generalized inverse of tensors

In this section, we show the \( \{1\} \) inverse of a tensor \( A \in \mathbb{C}^{m,n}_t \) first.

For a tensor \( A \in \mathbb{C}^{m,n}_t \), let \( R(A) = \{ A \cdot y^{-1} | y \in \mathbb{C}^n \} \), \( N(A) = \{ x | A \cdot x^{-1} = 0, \ x \in \mathbb{C}^n \} \). Obviously, the equation \( A \cdot x^{-1} = b \) is solvable iff \( b \in R(A) \), where \( x \in \mathbb{C}^n \). Next, we consider the problem that for the tensor \( A \in \mathbb{C}^{m,n}_t \), whether \( G \cdot (b^{[\frac{1}{k}]})^{k-1} \) is a solution of \( A \cdot x^{-1} = b \) for all the vectors \( b \in R(A) \), where \( G \in \mathbb{C}_{k,m}^n \) and \( s = (t-1)(k-1) \). If it is a solution of \( A \cdot x^{-1} = b \) for all the \( b \in R(A) \). Then, \( AG \cdot (b^{[\frac{1}{k}]})^s = b \), that is

\[
AG \cdot [(A \cdot y^{-1})^{[\frac{1}{k}]}]^s = A \cdot y^{-1} \quad \text{for all } y \in \mathbb{C}^n.
\]

On the other hand, if \( G \) satisfies \( AG \cdot [(A \cdot y^{-1})^{[\frac{1}{k}]}]^s = A \cdot y^{-1} \) for all \( y \in \mathbb{C}^n \). Since \( b \in R(A) \), there exists a vector \( y \in \mathbb{C}^n \) such that \( AG \cdot (b^{[\frac{1}{k}]})^s = AG \cdot [(A \cdot y^{-1})^{[\frac{1}{k}]}]^s = A \cdot y^{-1} = b \). It implies that \( G \cdot (b^{[\frac{1}{k}]})^{k-1} \) is a solution of \( A \cdot x^{-1} = b \).

From the above discussion, we give the concept of the \( \{1\} \) inverse of tensors as follows.

**Definition 2.1.** Let \( A \in \mathbb{C}^{m,n}_t \) and \( X \in \mathbb{C}^{k,n}_m \). If the equation

\[
AX \cdot [(A \cdot y^{-1})^{[\frac{1}{k}]}]^s = A \cdot y^{-1}
\]

holds for all \( y \in \mathbb{C}^n \), where \( s = (t-1)(k-1) \), then \( X \) is called the order \( k \) \( \{1\} \) inverse of \( A \), denoted by \( A^{(1)}k \). Denote the set of all the order \( k \) \( \{1\} \) inverses of \( A \) by \( A\{1\}_k \).
If an order $k$ right inverse of the tensor $\mathcal{A} \in \mathbb{C}^{m,n}_t$ exists, then $\mathcal{A} \mathcal{A}^R_k = \mathcal{I}$, $\mathcal{A} \mathcal{A}^R_k \cdot [(\mathcal{A} \cdot y^t)^{1/2}]^s = \mathcal{I} \cdot [(\mathcal{A} \cdot y^t)^{1/2}]^s = \mathcal{A} \cdot y^t$, where $y \in \mathbb{C}^n$ is an arbitrary vector and $s = (t-1)(k-1)$. Hence, $\mathcal{A}^R_k$ is a $\{1\}$ inverse of $\mathcal{A}$. The $\{1\}$ inverse (with fixed order) of a tensor is not unique in general. If an order $k$ tensor $\mathcal{X}$ is a $\{1\}$ inverse of $\mathcal{A}$, we write $\mathcal{A}^{(1)_k} = \mathcal{X}$. When $t = k = 2$, it is easy to see that Definition 2.1 is the definition of the $\{1\}$ inverse of matrices (see [12]).

**Proposition 2.2.** Let $\mathcal{A} \in \mathbb{C}^{m,n}_t$ and $x \in \mathbb{C}^n$. Let $\mathcal{A}^{(1)_k}$ denote a $\{1\}$ inverse of $\mathcal{A}$. If the equation $\mathcal{A} \cdot x^{t-1} = b$ is solvable, then $x = \mathcal{A}^{(1)_k} \cdot (b^{1/2})^{k-1}$ is a solution of it, where $s = (t-1)(k-1)$.

For $a \in \mathbb{C}$, let $a^+ = \{a^{-1}$, $a \neq 0$, $0$, $a = 0$. 

**Proposition 2.3.** Let $\mathcal{A} = \text{diag}(a_1, a_2, \ldots, a_n) \in \mathbb{C}^{n,n}_t$ be a diagonal tensor. Then 

$$\text{diag}((a_1^+)^{-1/2}, (a_2^+)^{-1/2}, \ldots, (a_n^+)^{-1/2}) \in \mathbb{C}^{n,n}_k$$

is an order $k$ $\{1\}$ inverse of $\mathcal{A}$.

**Proof.** Let $\mathcal{X} = \text{diag}((a_1^+)^{-1/2}, (a_2^+)^{-1/2}, \ldots, (a_n^+)^{-1/2})$. It follows from Eq. (1) that $(\mathcal{A} \cdot y^t)^{1/2} = a_i y_i^{t-1}$, $i \in [n]$, $y \in \mathbb{C}^n$. By Eq. (2), it yields that $\mathcal{A} \mathcal{X}$ is a diagonal tensor with the diagonal entries 1 or 0. By directly computation, we have 

$$\mathcal{A} \mathcal{X} \cdot [(\mathcal{A} \cdot y^t)^{1/2}]^s = \mathcal{A} \cdot y^t,$$

for all the $y \in \mathbb{C}^n$, where $s = (k-1)(t-1)$. Hence, $\mathcal{X}$ is an order $k$ $\{1\}$ inverse of $\mathcal{A}$.

The order $k$ right inverse of unit tensor exists and is not unique in general (see [11]), the order $k$ right inverse of a unit tensor is the order $k$ $\{1\}$ inverse of itself.

Next, we show some results on the $\{1\}$ inverse of block tensors. Let $\mathcal{A} = (a_{i \ldots i}) \in \mathbb{C}^{m,n}_t$. Let $\tilde{\mathcal{A}} = (a_{i \ldots i}) \in \mathbb{C}^{r,l}_t$ be a subtensor of $\mathcal{A}$, where $r \leq m$, $l \leq n$ (see [21]).

**Theorem 2.4.** Let $\mathcal{A} \in \mathbb{C}^{m,n}_t$ and let $\tilde{\mathcal{A}} \in \mathbb{C}^{r,l}_t$ be the subtensor of $\mathcal{A}$. If the entries of $\mathcal{A}$ are all zero except for $\tilde{\mathcal{A}}$. Then each tensor $\mathcal{G} \in \mathcal{A} \{1\}_k$ is a tensor with the subtensor $\tilde{\mathcal{G}} \in \mathcal{A} \{1\}_k$ and all the other entries are arbitrary.
Proof. Let $G = (g_{i_1 \cdots i_k}) \in \mathbb{C}_k^{n,m}$ be an order $k\{1\}$ inverse of $A$. $\tilde{G} \in \mathbb{C}_k^{l,r}$ denotes the subtensor of $G$. Let $y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \in \mathbb{C}^a$ be an arbitrary vector, where $Y_1 = (y_1, \ldots, y_l)^T$ and $Y_2 = (y_{l+1}, \ldots, y_n)^T$.

By the general tensor product, it yields that the $i$-th component of vector $A \cdot y^{t-1}$ is

$$(A \cdot y^{t-1})_i = \begin{cases} \sum_{i_2, \ldots, i_t \in [l]} a_{i_2 \cdots i_t} y_{i_2} \cdots y_{i_t} = (\tilde{A} \cdot Y_1^{t-1})_i, & i \leq r; \\
0, & i > r. \end{cases}$$

That is $A \cdot y^{t-1} = \begin{pmatrix} \tilde{A} \cdot Y_1^{t-1} \\ 0 \end{pmatrix}$. By computing, we get

$$(A G)_{i \alpha_1 \cdots \alpha_{t-1}} = \sum_{i_2, \ldots, i_t \in [l]} a_{i_2 \cdots i_t} g_{i_2 \alpha_1} \cdots g_{i_t \alpha_{t-1}} = (\tilde{A} \tilde{G})_{i \alpha_1 \cdots \alpha_{t-1}},$$

if all the indices in $i, \alpha_1, \ldots, \alpha_{t-1}$ are less than or equal to $r$; $(A G)_{i \alpha_1 \cdots \alpha_{t-1}} = 0$ if $i > r$.

Let $z = (A \cdot y^{t-1})^{[\frac{1}{2}]} = \begin{pmatrix} (\tilde{A} \cdot Y_1^{t-1})^{[\frac{1}{2}]} \\ 0 \end{pmatrix}$, where $s = (t-1)(k-1)$. By Eq.(1), it yields that

$$(A G \cdot z^s)_i = \begin{cases} (\tilde{A} \tilde{G} \cdot [(\tilde{A} \cdot Y_1^{t-1})^{[\frac{1}{2}]}]^s)_i, & i \leq r; \\
0, & i > r. \end{cases}$$

That is $A G \cdot z^s = \begin{pmatrix} \tilde{A} \tilde{G} \cdot [(\tilde{A} \cdot Y_1^{t-1})^{[\frac{1}{2}]}]^s \end{pmatrix}$. Since $G$ is an order $k\{1\}$ inverse of $A$, it yields that $A G \cdot z^s = A \cdot y^{t-1}$, so $\tilde{A} \tilde{G} \cdot [(\tilde{A} \cdot Y_1^{t-1})^{[\frac{1}{2}]}]^s = \tilde{A} \cdot Y_1^{t-1}$. Thus, we get $\tilde{G}$ is an order $k\{1\}$ inverse of $\tilde{A}$ and all the other entries of $G$ are arbitrary.

When the tensor $A$ in Theorem 2.4 is an order 2 tensor, the following result can be gotten.

**Corollary 2.5.** Let the block matrix $A = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{C}^{m \times n}$ and $A_1 \in \mathbb{C}^{r \times l}$. Then

$$A\{1\} = \left\{ \begin{pmatrix} W & X \\ Y & Z \end{pmatrix} \in \mathbb{C}^{n \times m} \middle| \begin{array}{c} W \in A_1\{1\}, \ X, Y \text{ and } Z \text{ are proper matrices with arbitrary entries} \end{array} \right\}. $$

5
The tensor $\mathcal{A} \in \mathbb{C}_t^{n,n}$ is a diagonal block tensor as

$$\mathcal{A} = \text{diag}(\mathcal{A}_1, \mathcal{A}_2),$$

where $\mathcal{A}_1 = (a_{i_1 \cdots i_r})$ ($i_1, \ldots, i_r \leq r \leq n$, $r$ is a positive integer); $\mathcal{A}_2 = (a_{i_1 \cdots i_t})$ ($i_1, \ldots, i_t > r$). And the other entries of $\mathcal{A}$ are all zero (see [22]).

**Theorem 2.6.** Let $\mathcal{A}$ be the form as in [3]. Then $\text{diag}(\mathcal{A}_1^{(1)}, \mathcal{A}_2^{(1)}) \in \mathbb{C}_k^{n,n}$ is an order $k$ $\{1\}$ inverse of $\mathcal{A}$.

**Proof.** Let $\mathcal{G}_1 = \mathcal{A}_1^{(1)}, \mathcal{G}_2 = \mathcal{A}_2^{(1)}$ and $\mathcal{G} = \text{diag}(\mathcal{G}_1, \mathcal{G}_2)$. And $y = \left( \begin{array}{c} Y_1 \\ Y_2 \end{array} \right) \in \mathbb{C}^n$ is an arbitrary vector, where $Y_1 = (y_1, \ldots, y_r)^T$ and $Y_2 = (y_{r+1}, \ldots, y_n)^T$.

By calculating, it yields that

$$(\mathcal{A} \cdot y^{t-1}) = \left( \begin{array}{c} (A_1 \cdot Y_1^{t-1})_i, \text{ if } i \leq r; \\ (A_2 \cdot Y_2^{t-1})_i, \text{ if } i > r. \end{array} \right)$$

That is $\mathcal{A} \cdot y^{t-1} = \left( \begin{array}{c} A_1 \cdot Y_1^{t-1} \\ A_2 \cdot Y_2^{t-1} \end{array} \right)$. It follows from Eq.(2) that: if all the indices in $i, \alpha_1, \ldots, \alpha_{t-1}$ are less than or equal to $r$, then

$$(\mathcal{A}\mathcal{G})_{i\alpha_1 \cdots \alpha_{t-1}} = (\mathcal{A}_1\mathcal{G}_1)_{i\alpha_1 \cdots \alpha_{t-1}};$$

if all the indices in $i, \alpha_1, \ldots, \alpha_{t-1}$ are greater than $r$, then

$$(\mathcal{A}\mathcal{G})_{i\alpha_1 \cdots \alpha_{t-1}} = (\mathcal{A}_2\mathcal{G}_2)_{i\alpha_1 \cdots \alpha_{t-1}};$$

the other entries of $\mathcal{A}\mathcal{G}$ are all zero. It implies that $\mathcal{A}\mathcal{G} = \text{diag}(\mathcal{A}_1\mathcal{G}_1, \mathcal{A}_2\mathcal{G}_2)$.

Let $z_1 = A_1 \cdot (Y_1^{t-1})_1, z_2 = A_2 \cdot (Y_2^{t-1})_1$ and $z = \left( \begin{array}{c} z_1 \\ z_2 \end{array} \right) = (\mathcal{A} \cdot y^{t-1})_{11}$. By Eq.(1), we have

$$(\mathcal{A}\mathcal{G} \cdot z^s)_i = \left\{ \begin{array}{ll} (A_1\mathcal{G}_1 \cdot z_1^s)_i = (A_1\mathcal{G}_1 \cdot [(A_1 \cdot Y_1^{t-1})_{11}])^s)_i = (A_1 \cdot Y_1^{t-1})_i, \text{ if } i \leq r; \\ (A_2\mathcal{G}_2 \cdot z_2^s)_i = (A_2\mathcal{G}_2 \cdot [(A_2 \cdot Y_2^{t-1})_{11}])^s)_i = (A_2 \cdot Y_2^{t-1})_i, \text{ if } i > r. \end{array} \right.$$
We partition a tensor $\mathbf{A} \in \mathbb{C}^{m,n}$ into the row blocks as

$$
\mathbf{A} = \left( \begin{array}{c}
\mathbf{A}_1 \\
\mathbf{A}_2
\end{array} \right),
$$

(4)

where $\mathbf{A}_1 = (a_{i_1 \ldots i_t})$ ($i_1 \leq r$); $\mathbf{A}_2 = (a_{i_1 \ldots i_t})$ ($i_1 > r$, $r \leq m$, $r$ is a positive integer). And we also can partition $\mathbf{A}$ into the column blocks as

$$
\mathbf{A} = \left( \begin{array}{c}
\mathbf{A}_1 \\
\mathbf{A}_2
\end{array} \right),
$$

(5)

where $\mathbf{A}_1 = (a_{i_1 \ldots i_t})$ ($i_1, \ldots, i_t \leq r \leq n$); $\mathbf{A}_2 = (a_{i_1 \ldots i_t})$ otherwise (see [11]).

**Theorem 2.7.** (1) Let $\mathbf{A}$ be the form as in (4) and $\mathbf{A}_1 \{1\}_k$ be the set of all the order $k \{1\}$ inverses of $\mathbf{A}_1$. If $\mathbf{A}_2 = 0$, then

$$
\mathbf{A} \{1\}_k = \left\{ \left( \begin{array}{c}
\mathbf{W} \\
\mathbf{X}
\end{array} \right) \in \mathbb{C}^{n,m} \right| \mathbf{W} \in \mathbf{A}_1 \{1\}_k, \text{ } \mathbf{X} \text{ is a proper tensor with arbitrary entries} \right\};
$$

(2) Let $\mathbf{A}$ be the form as in (5) and $\mathbf{A}_1 \{1\}_k$ be the set of all the order $k \{1\}$ inverses of $\mathbf{A}_1$. If $\mathbf{A}_2 = 0$, then

$$
\mathbf{A} \{1\}_k = \left\{ \left( \begin{array}{c}
\mathbf{W} \\
\mathbf{Y}
\end{array} \right) \in \mathbb{C}^{n,m} \right| \mathbf{W} \in \mathbf{A}_1 \{1\}_k, \text{ } \mathbf{Y} \text{ is a proper tensor whose entries are arbitrary} \right\}.
$$

**Proof.** (1) Let the column blocks tensor $\mathbf{G} = (\mathbf{G}_1, \mathbf{G}_2) \in \mathbb{C}^{n,m}$ be an order $k \{1\}$ inverse of $\mathbf{A}$, where $\mathbf{G}_1 \in \mathbb{C}^{n,r}$. And $\mathbf{y} = \left( \begin{array}{c}
\mathbf{Y}_1 \\
\mathbf{Y}_2
\end{array} \right) \in \mathbb{C}^{n}$ is an arbitrary vector, where $\mathbf{Y}_1 = (y_1, \ldots, y_r)^T$ and $\mathbf{Y}_2 = (y_{r+1}, \ldots, y_n)^T$.

By calculating, it yields that the $i$-th component of vector $\mathbf{A} \cdot \mathbf{y}^{t-1}$ is

$$
(\mathbf{A} \cdot \mathbf{y}^{t-1})_i = \left\{ \begin{array}{ll}
\sum_{i_2, \ldots, i_t \in [n]} a_{ii_2 \ldots i_t} y_{i_2} y_{i_3} \cdots y_{i_t} = (\mathbf{A}_1 \cdot \mathbf{y}^{t-1})_i, & \text{if } i \leq r; \\
(\mathbf{A} \cdot \mathbf{y}^{t-1})_i = 0, & \text{if } i > r.
\end{array} \right.
$$

That is $\mathbf{A} \cdot \mathbf{y}^{t-1} = \left( \begin{array}{c}
\mathbf{A}_1 \cdot \mathbf{y}^{t-1} \\
0
\end{array} \right)$. It follows from the general tensor product that

$$
(\mathbf{A} \mathbf{G})_{i_1 \ldots i_t t-1} = \left\{ \begin{array}{ll}
\sum_{i_2, \ldots, i_t \in [n]} a_{ii_2 \ldots i_t} g_{i_2 a_1} \cdots g_{i_t a_t-1} = (\mathbf{A}_1 \mathbf{G})_{i_1 \ldots i_t-1}, & \text{if } i \leq r; \\
0, & \text{if } i > r.
\end{array} \right.
$$
It implies that \( \mathcal{A} \mathcal{G} = \begin{pmatrix} \mathcal{A}_1 \mathcal{G} \\ 0 \end{pmatrix} \). Let \( z = (\mathcal{A} \cdot y^{t-1})^{[\frac{1}{2}]} = \begin{pmatrix} (\mathcal{A}_1 \cdot y^{t-1})^{[\frac{1}{2}]} \\ 0 \end{pmatrix} \), where \( s = (t - 1)(k - 1) \). By calculating, we have

\[
(\mathcal{A} \mathcal{G} \cdot z^s)_i = \begin{cases} (\mathcal{A}_1 \mathcal{G}_1 \cdot [(\mathcal{A}_1 \cdot y^{t-1})^{[\frac{1}{2}]}]^s)_{i} & \text{if } i \leq r; \\ 0 & \text{if } i > r. \end{cases}
\]

Since \( \mathcal{G} \) is a \( \{1\} \) inverse of \( \mathcal{A} \), that is \( \mathcal{A} \mathcal{G} \cdot z^s = \mathcal{A} \cdot y^{t-1} \), so \( \mathcal{A}_1 \mathcal{G}_1 \cdot [(\mathcal{A}_1 \cdot y^{t-1})^{[\frac{1}{2}]}]^s = \mathcal{A}_1 \cdot y^{t-1} \). Thus, we get \( \mathcal{G}_1 \) is a \( \{1\} \) inverse of \( \mathcal{A}_1 \) and \( \mathcal{G}_2 \) is arbitrary.

(2) Let the row blocks tensor \( \mathcal{G} = \begin{pmatrix} \mathcal{G}_1 \\ \mathcal{G}_2 \end{pmatrix} \in \mathbb{C}^{m \times m} \) be an order \( k \) \{1\} inverse of \( \mathcal{A} \), where \( \mathcal{G}_1 \in \mathbb{C}_k^{m \times m} \). And \( y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \in \mathbb{C}^m \) is an arbitrary vector, where \( Y_1 = (y_1, \ldots, y_r)^T \) and \( Y_2 = (y_{r+1}, \ldots, y_n)^T \).

By calculation, it yields that

\[
(\mathcal{A} \cdot y^{t-1})_i = \sum_{i_2 \cdots i_r \in [n]^{t-1}} a_{i_2 \cdots i_r} y_{i_2} y_{i_3} \cdots y_{i_t} \\
+ \sum_{i_2 \cdots i_r \in \mathbb{Z}^r} a_{i_2 \cdots i_r} y_{i_2} y_{i_3} \cdots y_{i_t} \\
= (A_1 \cdot Y_1^{t-1} + 0)_i.
\]

It follows from the general tensor product that

\[
(\mathcal{A} \mathcal{G})_{i_1 \alpha_1 \cdots \alpha_{t-1}} = \sum_{i_2 \cdots i_r \in [n]^{t-1}} a_{i_2 \cdots i_r} g_{i_2 \alpha_1} \cdots g_{i_r \alpha_{t-1}} \\
+ \sum_{i_2 \cdots i_r \in \mathbb{Z}^r} a_{i_2 \cdots i_r} g_{i_2 \alpha_1} \cdots g_{i_r \alpha_{t-1}} \\
= (A_1 \mathcal{G}_1 + 0)_{i_1 \alpha_1 \cdots \alpha_{t-1}}.
\]

Let \( z = (\mathcal{A} \cdot y^{t-1})^{[\frac{1}{2}]} = (A_1 \cdot Y_1^{t-1})^{[\frac{1}{2}]} \), where \( s = (t - 1)(k - 1) \). Then \( (\mathcal{A} \mathcal{G} \cdot z^s)_i = (A_1 \mathcal{G}_1 \cdot [(A_1 \cdot Y_1^{t-1})^{[\frac{1}{2}]}]^s)_{i} \). Since \( \mathcal{G} \) is a \( \{1\} \) inverse of \( \mathcal{A} \), it yields that \( \mathcal{A} \mathcal{G} \cdot z^s = \mathcal{A} \cdot y^{t-1} \), so \( A_1 \mathcal{G}_1 \cdot [(A_1 \cdot Y_1^{t-1})^{[\frac{1}{2}]}]^s = A_1 \cdot Y_1^{t-1} \). Thus, we get \( \mathcal{G}_1 \) is a \( \{1\} \) inverse of \( \mathcal{A}_1 \) and \( \mathcal{G}_2 \) is arbitrary.

When the tensor \( \mathcal{A} \) in Theorem 2.7 is an order 2 tensor, the following result can be gotten.
Corollary 2.8. (1) Let the block matrix $A = \begin{pmatrix} A_1 & 0 \end{pmatrix} \in \mathbb{C}^{m \times n}$ and $A_1 \in \mathbb{C}^{m \times r}$. Then

$A\{1\} = \left\{ \begin{pmatrix} W \\ Y \end{pmatrix} \in \mathbb{C}^{n \times m} \mid W \in A_1\{1\}, Y \text{ is a proper matrix with arbitrary entries} \right\} ;$

(2) Let the block matrix $A = \begin{pmatrix} A_1 & 0 \end{pmatrix} \in \mathbb{C}^{m \times n}$ and $A_1 \in \mathbb{C}^{r \times n}$. Then

$A\{1\} = \left\{ \begin{pmatrix} W & X \end{pmatrix} \in \mathbb{C}^{n \times m} \mid W \in A_1\{1\}, X \text{ is a proper matrix with arbitrary entries} \right\} .

Theorem 2.9. Let $\mathcal{A} = \text{diag}(a_1, a_2, \ldots, a_n) \in \mathbb{C}_t^{n,n}$ be a diagonal tensor, where $a_i \neq 0$ ($i = 1, \ldots, n$). Then the order $2 \{1\}$ inverse of $\mathcal{A}$ is the following diagonal matrix

$$\text{diag}(a_1^{\frac{-1}{t-1}}, a_2^{\frac{-1}{t-1}}, \ldots, a_n^{\frac{-1}{t-1}}) \in \mathbb{C}^{n \times n}.$$ 

Proof. Let $X = (x_{ij}) \in \mathbb{C}^{n \times n}$ be the order $2 \{1\}$ inverse of $\mathcal{A}$ and $y \in \mathbb{C}^n$ be an arbitrary vector. By computation, we have the components of $\mathcal{A} \cdot y^{t-1}$ and $X(\mathcal{A} \cdot y^{t-1})[\frac{1}{t-1}]$ are

$$(\mathcal{A} \cdot y^{t-1})_i = a_i y_i^{t-1} \text{ and } (X(\mathcal{A} \cdot y^{t-1})[\frac{1}{t-1}])_i = \sum_{j=1}^n x_{ij} a_j^{\frac{1}{t-1}} y_j$$

respectively. Then the component of $\mathcal{A}X \cdot [(\mathcal{A} \cdot y^{t-1})[\frac{1}{t-1}]]^{t-1}$ is

$$(\mathcal{A}X \cdot [(\mathcal{A} \cdot y^{t-1})[\frac{1}{t-1}]]^{t-1})_i = a_i \left( \sum_{j=1}^n x_{ij} a_j^{\frac{1}{t-1}} y_j \right)^{t-1} .$$

It follows from the Definition of tensor $\{1\}$ inverse that $\mathcal{A}X \cdot [(\mathcal{A} \cdot y^{t-1})[\frac{1}{t-1}]]^{t-1} = \mathcal{A} \cdot y^{t-1}$, that is

$$a_i \left( \sum_{j=1}^n x_{ij} a_j^{\frac{1}{t-1}} y_j \right)^{t-1} = a_i y_i^{t-1}, \quad i = 1, \ldots, n.$$ 

Note that the above equation holds for all $y \in \mathbb{C}^n$, it is easy to see that $x_{ij} = 0$ if $i \neq j$ and $x_{ii} = a_i^{\frac{1}{t-1}}$ ($i, j = 1, \ldots, n$). \hfill \Box
By the above theorem and Theorem \[2.4\], the following result can be gotten.

**Theorem 2.10.** Let \( A \in \mathbb{C}^{m,n} \) and let \( \tilde{A} = \text{diag}(a_1, a_2, \ldots, a_r) \in \mathbb{C}^{r,r} \) be the subtensor of \( A \), where \( a_i \neq 0, \ i = 1, \ldots, r \). If the entries of \( A \) are all zero except for \( \tilde{A} \), then the set of the order 2 \( \{1\} \) inverse of \( A \) is

\[
A\{1\}_2 = \left\{ \begin{pmatrix} X & Y \\ Z & W \end{pmatrix} \in \mathbb{C}^{n \times m} \mid X = \text{diag}(a_1^{-1}, \ldots, a_r^{-1}, \ldots), \ Y, Z \text{ and } W \text{ are proper matrices with arbitrary entries} \right\}.
\]

**Theorem 2.11.** Let \( A \in \mathbb{C}^{m,n} \), \( B \in \mathbb{C}^{m,n} \) and let \( B^{(1)}_k \in \mathbb{C}^{n_1,m} \) be an order \( k \) \( \{1\} \) inverse of \( B \). If \( A = PBQ \), then \( A^{(1)}_k = Q^{(1)}B^{(1)}_k P^T \), where \( P \in \mathbb{C}^{m \times m} \) is a permutation matrix, \( Q \in \mathbb{C}^{n_1 \times n} \) is a matrix with full row rank.

**Proof.** Since \( Q \) is a full row rank matrix, then \( QQ^{(1)} = I \), where \( I \) is a unit matrix (see [12]). Let \( G = Q^{(1)}B^{(1)}_k P^T \) and \( y \in \mathbb{C}^n \) is an arbitrary vector, we have

\[
(PBQ \cdot y^{t-1})^\frac{1}{k^s} = P(BQ \cdot y^{t-1})^\frac{1}{k^s},
\]

where \( s = (t-1)(k-1) \). By computation, it yields that

\[
AG \cdot [(A \cdot y^{t-1})^\frac{1}{k^s}] = PBQQ^{(1)}B^{(1)}_k P^T \cdot [(PBQ \cdot y^{t-1})^\frac{1}{k^s}] = PBB^{(1)}_k \cdot [(BQ \cdot y^{t-1})^\frac{1}{k^s}]^s.
\]

It follows from Definition \[2.1\] that

\[
PBB^{(1)}_k \cdot [(BQ \cdot y^{t-1})^\frac{1}{k^s}]^s = PBQ \cdot y^{t-1} = A \cdot y^{t-1}.
\]

Thus, we get \( AG \cdot [(A \cdot y^{t-1})^\frac{1}{k^s}]^s = A \cdot y^{t-1} \), so \( G \) is an order \( k \) \( \{1\} \) inverse of \( A \).

**Corollary 2.12.** Let \( A \in \mathbb{C}^{m,n} \), \( B \in \mathbb{C}^{m,n} \) and let \( B^{(1)}_k \in \mathbb{C}^{n,m} \) is an order \( k \) \( \{1\} \) inverse of \( B \). If \( A = PBQ \), then \( A^{(1)}_k = Q^{-1}B^{(1)}_k P^T \), where \( P \in \mathbb{C}^{m \times m} \) is a permutation matrix, \( Q \in \mathbb{C}^{n \times n} \) is an invertible matrix.

Obviously, when \( P, Q \) are both permutation matrices, Theorem \[2.11\] also holds.

In the following, we show the definitions of the \( \{i\} \) inverse and group inverse of tensors, \( k\)-T-idempotent tensors and idempotent tensors.
Definition 2.13. Let $\mathcal{A} \in \mathbb{C}^{m,n}_t$ and $\mathcal{X} \in \mathbb{C}^{n,m}_k$.

1. $\mathcal{A}\mathcal{X} \cdot ([A \cdot y^{t-1}]^{(2)})^{s} = \mathcal{A} \cdot y^{t-1}$, for all $y \in \mathbb{C}^n$;
2. $\mathcal{X}\mathcal{A}\mathcal{X} \cdot (y^{1/2})^{s(k-1)} \mathcal{X} \cdot y^{k-1}$, for all $y \in \mathbb{C}^m$;
3. $\mathcal{A}\mathcal{X} \cdot (y^{1/2})^{s} = \mathcal{X} \cdot ([A \cdot y^{t-1}]^{(2)})^{k-1}$, for all $y \in \mathbb{C}^n$;

where $s = (t - 1)(k - 1)$. If the equation (i) holds, then $\mathcal{X}$ is called the order $k$ \{i\} inverse of $\mathcal{A}$, denoted by $\mathcal{X} = \mathcal{A}^{(i)k}$. And the set of all the order $k$ \{i\} inverses of $\mathcal{A}$ is denoted by $\mathcal{A}\{i\}_k$. For a tensor $\mathcal{A} \in \mathbb{C}^{n,m}_t$, if the equations (1), (2) and (3) hold for all $y \in \mathbb{C}^n$, then the tensor $\mathcal{X} \in \mathbb{C}^{n,m}_k$ is called an order $k$ group inverse of $\mathcal{A}$, denoted by $\mathcal{A}\#^k$. And the set of all the order $k$ group inverse of $\mathcal{A}$ is denoted by $\mathcal{A}\{\#\}_k$.

If an order $k$ right inverse of $\mathcal{A} \in \mathbb{C}^{n,m}_t$ exists, then $\mathcal{A}\mathcal{A}^{Rk} = \mathcal{I}$, $\mathcal{A}^{Rk}\mathcal{A}\mathcal{A}^{Rk} \cdot (y^{1/2})^{s(k-1)} = \mathcal{A}^{Rk} \cdot y^{k-1}$, where $s = (t - 1)(k - 1)$ and $y \in \mathbb{C}^n$ is an arbitrary vector. Hence, $\mathcal{A}^{Rk}$ is a \{2\} inverse of $\mathcal{A}$. It is clear that the \{i\} inverse of a tensor $\mathcal{A} \in \mathbb{C}^{n,m}_t$ is not unique in general and the group inverse of a tensor $\mathcal{A} \in \mathbb{C}^{n,m}_t$ (t $\geq$ 3) is not unique in general. When $t = k = 2$, the Definition 2.13 is the definitions of the \{i\} inverse and the group inverse of matrices (see [12]).

Similar to Proposition 2.3, we can obtain the following result.

Proposition 2.14. Let $\mathcal{A} = \text{diag}(a_1, a_2, \ldots, a_n) \in \mathbb{C}^{n,m}_t$ is a diagonal tensor. Then

$$\text{diag}([a_1^{(1)}]^{(t)}, [a_2^{(1)}]^{(t)}, \ldots, [a_n^{(1)}]^{(t)})$$

is an order $k$ group inverse of $\mathcal{A}$.

Similar to the \{1\} inverse of a unit tensor, the group inverse (with fixed order) of a unit tensor is not unique in general.

Theorem 2.15. Let $\mathcal{A}$ be the diagonal block tensor as in [3]. Then $\text{diag}(\mathcal{A}_1\#, \mathcal{A}_2\#) \in \mathbb{C}^{n,m}_k$ is an order $k$ group inverse of $\mathcal{A}$.

Proof. Let $\mathcal{G}_1 = \mathcal{A}_1\#$, $\mathcal{G}_2 = \mathcal{A}_2\#$ and $\mathcal{G} = \text{diag}(\mathcal{G}_1, \mathcal{G}_2)$. And $y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \in \mathbb{C}^n$ is an arbitrary vector, where $Y_1 = (y_1, \ldots, y_t)^T$ and $Y_2 = (y_{r+1}, \ldots, y_n)^T$.

It follows from Theorem 2.6 that $\mathcal{G}$ is an order $k$ \{1\} inverse of $\mathcal{A}$. Similar to Theorem 2.6, we have the vector $\mathcal{G} \cdot y^{k-1} = \begin{pmatrix} \mathcal{G}_1 \cdot Y_1^{k-1} \\ \mathcal{G}_2 \cdot Y_2^{k-1} \end{pmatrix}$ and $\mathcal{G}\mathcal{A} = \text{diag}(\mathcal{G}_1\mathcal{A}_1, \mathcal{G}_2\mathcal{A}_2)$. Let $z_1 = \mathcal{G}_1 \cdot Y_1^{[1]}(k-1)$, $z_2 = \mathcal{G}_2 \cdot Y_2^{[1]}(k-1)$ and
\[ z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \mathcal{G} \cdot (y^{[\frac{1}{s}]})^{k-1}, \text{ where } s = (t-1)(k-1). \] By calculating, it yields that
\[
(\mathcal{G} \mathcal{A} \cdot z^s)_i = \begin{cases} 
(\mathcal{G}_1 \mathcal{A}_1 \mathcal{G}_1 \cdot (y^{[\frac{1}{s}]})^{s(k-1)})_i = (\mathcal{G}_1 \cdot Y_1^{k-1})_i, & \text{if } i \leq r; \\
(\mathcal{G}_2 \mathcal{A}_2 \mathcal{G}_2 \cdot (y^{[\frac{1}{s}]})^{s(k-1)})_i = (\mathcal{G}_2 \cdot Y_2^{k-1})_i, & \text{if } i > r.
\end{cases}
\]

From the above discussion, we have \( \mathcal{G} \mathcal{A} \cdot z^s = \begin{pmatrix} \mathcal{G}_1 \cdot Y_1^{k-1} \\ \mathcal{G}_2 \cdot Y_2^{k-1} \end{pmatrix} = \mathcal{G} \cdot y^{k-1}, \) so \( \mathcal{G} \mathcal{A} \mathcal{G} \cdot (y^{[\frac{1}{s}]})^{s(k-1)} = \mathcal{G} \cdot y^{k-1}. \) Hence, we have \( \mathcal{G} \) is an order \( k \) \{2\} inverse of \( \mathcal{A}. \)

By the general tensor product, it yields that
\[
\mathcal{G} \cdot [(\mathcal{A} \cdot y^{t-1})^{[\frac{1}{s}]})^{k-1}] = \begin{pmatrix} 
\mathcal{G}_1 \cdot [(\mathcal{A}_1 \cdot Y_1^{t-1})^{[\frac{1}{s}]})^{k-1}] \\
\mathcal{G}_2 \cdot [(\mathcal{A}_2 \cdot Y_2^{t-1})^{[\frac{1}{s}]})^{k-1}] 
\end{pmatrix}
\]
and
\[
\mathcal{A} \mathcal{G} \cdot (y^{[\frac{1}{s}]})^s = \begin{pmatrix} 
\mathcal{A}_1 \mathcal{G}_1 \cdot (y^{[\frac{1}{s}]})^s \\
\mathcal{A}_2 \mathcal{G}_2 \cdot (y^{[\frac{1}{s}]})^s 
\end{pmatrix}.
\]
According to the definition of the tensor \{5\} inverse, we get \( \mathcal{G} \cdot [(\mathcal{A} \cdot y^{t-1})^{[\frac{1}{s}]})^{k-1} = \mathcal{A} \mathcal{G} \cdot (y^{[\frac{1}{s}]})^s. \) Hence, we get \( \mathcal{G} \) is an order \( k \) \{5\} inverse of \( \mathcal{A}. \)

Thus, we get \( \mathcal{G} \) is an order \( k \) group inverse of \( \mathcal{A}. \) \( \square \)

**Theorem 2.16.** Let \( \mathcal{A} \in \mathbb{C}^{m,n}, \mathcal{B} \in \mathbb{C}^{n,m} \) and let \( \mathcal{B}^{(2)_k} \in \mathbb{C}^{n_1,m} \) be an order \( k \) \{2\} inverse of \( \mathcal{B}. \) If \( \mathcal{A} = PBQ, \) then \( \mathcal{A}^{(2)_k} = Q^{(2)} \mathcal{B}^{(2)_k} P^T, \) where \( P \in \mathbb{C}^{m \times m} \) be a permutation matrix, \( Q \in \mathbb{C}^{n_1 \times n} \) be a matrix with full row rank.

**Proof.** Since \( Q \) is a full row rank matrix, then \( QQ^{(2)} = I, \) where \( I \) is a unit matrix (see [12]). Let \( \mathcal{G} = Q^{(2)} \mathcal{B}^{(2)_k} P^T. \) By computation, it yields that
\[
\mathcal{G} \mathcal{A} \mathcal{G} \cdot (y^{[\frac{1}{s}]})^{s(k-1)} = Q^{(2)} \mathcal{B}^{(2)_k} P^T P B Q Q^{(2)} \mathcal{B}^{(2)_k} P^T \cdot (y^{[\frac{1}{s}]})^{s(k-1)}
\]
\[
= Q^{(2)} \mathcal{B}^{(2)_k} B B^{(2)_k} : [(P^T y)^{[\frac{1}{s}]})]^{s(k-1)},
\]
where \( s = (t-1)(k-1) \) and \( y \in \mathbb{C}^m \) is an arbitrary vector. It follows from the definition of tensor \{2\} inverse that
\[
Q^{(2)} \mathcal{B}^{(2)_k} B B^{(2)_k} : [(P^T y)^{[\frac{1}{s}]})]^{s(k-1)} = Q^{(2)} \mathcal{B}^{(2)_k} P^T \cdot y^{k-1} = \mathcal{G} \cdot y^{k-1}.
\]
Thus, we get $GAG \cdot (y^{[1]}^{s(k-1)}) = G \cdot y^{k-1}$, so $G$ is an order $k \{2\}$ inverse of $A$.

\[\square\]

**Theorem 2.17.** Let $A \in \mathbb{C}^{n \times n}$ and $G \in \mathbb{C}^{n,n}$ be an order $k$ group inverse of $A$. If $\lambda$ is an eigenvalue of $A$, then $\lambda^+$ is an eigenvalue of $G$.

**Proof.** Let $\lambda$ be an eigenvalue of $A$, then $Ax = \lambda x$, $0 \neq x \in \mathbb{C}^n$. Clearly, $(Ax)^{[1]}^{s(k-1)} = \lambda^{x^{[1]}(x-1)}^{s(k-1)}$. By the (1) in Definition 2.13, it yields that $AG \cdot [(Ax)^{[1]}^{s(k-1)}]^{k-1} = Ax = \lambda x$. Since $AG \cdot (\lambda^{x^{[1]}(x-1)}^{s(k-1)}) = \lambda AG \cdot (x^{[1]}^{s(k-1)})^{k-1}$, it is obtained that

$$\lambda AG \cdot (x^{[1]}^{s(k-1)})^{k-1} = \lambda x. \quad (6)$$

It follows from the (5) in Definition 2.13 that

$$AG \cdot (x^{[1]}^{s(k-1)})^{k-1} = G \cdot [(Ax)^{[1]}^{s(k-1)}]^{k-1} = G \cdot (\lambda^{x^{[1]}(x-1)}^{s(k-1)})^{k-1} = \lambda G \cdot (x^{[1]}^{s(k-1)})^{k-1}. \quad (7)$$

That is

$$AG \cdot (x^{[1]}^{s(k-1)})^{k-1} = \lambda G \cdot (x^{[1]}^{s(k-1)})^{k-1}. \quad (7)$$

Applying Eq. (6) and (7), it yields that

$$\lambda^2 G \cdot (x^{[1]}^{s(k-1)})^{k-1} = \lambda x.$$

If $\lambda \neq 0$, then $G \cdot (x^{[1]}^{s(k-1)})^{k-1} = \lambda^{-1}(x^{[1]}^{s(k-1)})^{k-1}$. Hence, $\lambda^{-1}$ is an eigenvalue of $G$.

If $\lambda = 0$, then there exists a vector $0 \neq x \in \mathbb{C}^n$ such that $Ax = 0$. Substituting it into the (5) of Definition 2.13 it yields that $AG \cdot (x^{[1]}^{s(k-1)})^{k-1} = G \cdot [(Ax)^{[1]}^{s(k-1)}]^{k-1} = 0$. And it follows from the (2) of Definition 2.13 that $G \cdot x^{k-1} = GAG \cdot (x^{[1]}^{s(k-1)})^{k-1} = 0$. Hence, 0 is an eigenvalue of $G$. \[\square\]

From the above theorem, it is easy to see that if $\lambda$ is an eigenvalue of a matrix $A$, then $\lambda^+$ is an eigenvalue of the matrix $A^\#$.

Let $\mathbb{H} \subset \mathbb{C}^n$ and $\mathbb{H}[s] = \{x[s] | x \in \mathbb{H}\}$, where $s \geq 0$. 

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Proposition 2.18. Let \( A \in \mathbb{C}^{m,n} \), let \( A^{(i)k} \in \mathbb{C}^{n,m} \) and \( A^{\#k} \in \mathbb{C}^{n,n} \) be the order \( k \) \{i\} inverse and group inverse of \( A \), respectively. Then the following results hold:

1. \( R(\mathcal{A}A^{(1)k}) = R(A) \);
2. \( R(\mathcal{A}A^{(2)k}A) = R(\mathcal{A}A^{(2)k}) \);
3. \( R(\mathcal{A}\mathcal{A}^{\#k}) = R(\mathcal{A}) \), \( R(\mathcal{A}^\#kA) = R(\mathcal{A}^\#k) \);
4. \( R(A) \subset R(\mathcal{A}A^{\#k}) \);
5. \( N(A) \subset \left( N(AA^{(5)k}) \right)_{[s]} \);
6. \( \left( N(AA^{(2)k}) \right)_{[s]} \subset N(A^{(2)k}) \);
7. \( N(A) \subset N(A^{(2,5)k}) \), (\( A^{(2,5)k} \in \mathbb{C}^{n,n} \) is an order \( k \) \{2\} inverse and \{5\} inverse of \( A \));

where \( s = (t-1)(k-1) \).

Proof. (1) It is easy to see that \( R(\mathcal{A}A^{(1)k}) \subset R(A) \). For each \( y \in R(A) \), there exists a vector \( x \in \mathbb{C}^n \) such that \( y = A \cdot x^t - 1 \). Since \( y = A \cdot x^t - 1 = A\mathcal{A}^{(1)k} \cdot [(\mathcal{A} \cdot x^t - 1)^{[\frac{1}{2}]}]^s \in R(\mathcal{A}A^{(1)k}) \), we have \( R(A) \subset R(\mathcal{A}A^{(1)k}) \).

(2) It is clear that \( R(\mathcal{A}^{(2)k}A) \subset R(A^{(2)k}) \). For each \( y \in R(A^{(2)k}) \), there exists \( x \in \mathbb{C}^m \) such that \( y = A^{(2)k} \cdot x^k - 1 \). Since \( y = A^{(2)k} \mathcal{A}A^{(2)k} \cdot (x^k)^{s(k-1)} \in R(A^{(2)k}) \), we get \( R(A^{(2)k}) \subset R(A^{(2)k}) \).

(3) From the above results (1) and (2), it is easy to see that (3) holds.

(4) For each \( y \in R(A) \), it follows from (3) that there exist a vector \( x \in \mathbb{C}^n \) such that \( y = \mathcal{A}A^{\#k} \cdot [(x^s)^{[\frac{1}{2}]]}^s = \mathcal{A}^{\#k} \cdot [(\mathcal{A} \cdot (x^s)^{t-1})^{[\frac{1}{2}]}]^k - 1 \), it yields that \( y \in R(\mathcal{A}^{\#k}) \). That is \( R(A) \subset R(\mathcal{A}^{\#k}) \).

(5) For each \( x \in N(A) \), we have \( A \cdot x^t - 1 = 0 \), so

\[
\mathcal{A}A^{(5)k} \cdot (x^{[\frac{1}{2}]})^s = \mathcal{A}^{(5)k} \cdot [(A \cdot x^t - 1)^{[\frac{1}{2}]}]^k - 1 = 0.
\]

Hence, \( x^{[\frac{1}{2}]} \in N(\mathcal{A}A^{(5)k}) \). That is \( x \in \left( N(\mathcal{A}A^{(5)k}) \right)_{[s]} \).

(6) For each \( x \in \left( N(\mathcal{A}A^{(2)k}) \right)_{[s]} \), we have \( x^{[\frac{1}{2}]} \in N(\mathcal{A}A^{(2)k}) \), so \( \mathcal{A}A^{(2)k} \cdot (x^{[\frac{1}{2}])}^s = 0 \). Multiplying by \( A^{(2)k} \) on the left hand side of the above equation, it yields that

\[
\mathcal{A}^{(2)k} \mathcal{A}A^{(2)k} \cdot (x^{[\frac{1}{2}])}^s = A^{(2)k} \cdot x^{k-1} = 0.
\]

Hence, \( x \in N(\mathcal{A}^{(2)k}) \).
For each \( x \in \mathbb{N}(A) \), there exists a vector \( x \in \mathbb{C}^{n} \) such that \( A \cdot x^{t-1} = 0 \). So \( (A \cdot x^{t-1})^{\frac{1}{2}} = 0 \). It follows from Definition 2.13 that

\[
A^{(2,5)k} \cdot x^{k-1} = A^{(2,5)k} A A^{(2,5)k} \cdot ((A \cdot x^{t-1})^{\frac{1}{2}})^{(k-1)^2} = 0.
\]

Hence, \( x \in \mathbb{N}(A^{(2,5)k}) \).

3. \( k \)-T-idempotent tensors and idempotent tensors

In this section, we give the definitions of the \( k \)-T-idempotent tensors and idempotent tensors first.

**Definition 3.1.** Let \( A \in \mathbb{C}_t^{n,n} \) and the positive integer \( k \geq 2 \). If the equation

\[
A^{k} \cdot (y^{\frac{1}{(t-1)^{k-1}}})^{(t-1)^k} = A \cdot y^{t-1}
\]

holds for all \( y \in \mathbb{C}^{n} \), where \( s = (t - 1)^{(k-1)} \), then \( A \) is called the \( k \)-T-idempotent tensor. When \( k = 2 \), \( A \) is called the T-idempotent tensor.

If \( A \in \mathbb{C}_t^{n,n} \) is a T-idempotent tensor, we have \( A^{2} \cdot (y^{\frac{1}{(t-1)^{k-1}}})^{(t-1)^2} = A \cdot y^{t-1} \) for all \( y \in \mathbb{C}^{n} \). By computing, it yields that

\[
A^{k} \cdot (y^{\frac{1}{(t-1)^{k-1}}})^{(t-1)^k} = A^{k-2} A^{2} \cdot [(y^{\frac{1}{(t-1)^{k-2}}})^{(t-1)^2}]^{(t-1)^k}
\]

\[
= A^{k-2} A \cdot (y^{\frac{1}{(t-1)^{k-2}}})^{(t-1)^2}
\]

\[
= A^{k-1} \cdot (y^{\frac{1}{(t-1)^{k-2}}})^{(t-1)^2}
\]

\[
= \cdots = A \cdot y^{t-1},
\]

where \( s = (t - 1)^{(k-1)} \). Hence, a T-idempotent tensor is a \( k \)-T-idempotent tensor for all the positive integer \( k \geq 2 \).

**Definition 3.2.** Let \( A \in \mathbb{C}_t^{n,n} \), if the equation \( A \cdot [(A \cdot y^{t-1})^{\frac{1}{t-1}}]^{t-1} = A \cdot y^{t-1} \) holds for all \( y \in \mathbb{C}^{n} \). Then \( A \) is called the idempotent tensor.

**Proposition 3.3.** Let \( A \in \mathbb{C}_t^{n,n} \).

(1) If \( A \) is a T-idempotent tensor, then \( A \) is an order \( t \) \( \{2\} \) inverse of itself;

(2) If \( A \) is both idempotent tensor and T-idempotent tensor, then \( A \) is an order \( t \) \( \{1\} \) inverse of itself and \( A \cdot (b^{\frac{1}{(t-1)^{t-1}}})^{t-1} \) is a solution of the solvable equation \( A \cdot x^{t-1} = b \), where \( x \in \mathbb{C}^{n} \).
Proof. (1) Since $\mathcal{A}$ is a T-idempotent tensor, it yields that $\mathcal{A}^2 \cdot (y^{[t^{-1}]})^{(t-1)^2} = \mathcal{A} \cdot y^{t-1}$, for all $y \in \mathbb{C}^n$. Multiplying by $\mathcal{A}$ on the left hand side of the above equation, we get $\mathcal{A}^3 \cdot (y^{[t^{-1}]})^{(t-1)^3} = \mathcal{A}^2 \cdot y^{(t-1)^2}$, then $\mathcal{A}^3 \cdot (y^{[t^{-1}]})^{(t-1)^3} = \mathcal{A} \cdot (y^{[t^{-1}]})^{(t-1)^3}$. Let $z = y^{[t-1]}$, then $\mathcal{A}^3 \cdot (z^{[t^{-1}]})^{(t-1)^3} = \mathcal{A} \cdot z^{t-1}$. Thus, we have $\mathcal{A} = \mathcal{A}^{(2t)}$.

(2) Since $\mathcal{A}$ is a T-idempotent tensor, then $\mathcal{A}^2 \cdot (z^{[t^{-1}]})^{(t-1)^2} = \mathcal{A} \cdot z^{t-1}$, for all $z \in \mathbb{C}^n$. Let $z = (\mathcal{A} \cdot y^{[t^{-1}]})^{[t^{-1}]}, y \in \mathbb{C}^n$ is an arbitrary vector, then

$$\mathcal{A}^2 \cdot [(\mathcal{A} \cdot y^{[t^{-1}]})^{[t^{-1}]})^{(t-1)^2} = \mathcal{A} \cdot [(\mathcal{A} \cdot y^{[t^{-1}]})^{[t^{-1}]})^{(t-1)^2}.$$

Since $\mathcal{A}$ is an idempotent tensor, it yields that $\mathcal{A} \cdot [(\mathcal{A} \cdot y^{[t^{-1}]})^{[t^{-1}]})^{(t-1)^2} = \mathcal{A} \cdot y^{t-1}$. So $\mathcal{A}^2 \cdot [(\mathcal{A} \cdot y^{[t^{-1}]})^{[t^{-1}]})^{(t-1)^2} = \mathcal{A} \cdot y^{t-1}$. Hence, $\mathcal{A}$ is an order $t \{1\}$ inverse of itself. From Proposition 2.2, it is easy to see that $\mathcal{A} \cdot (b^{[t^{-1}]})^{t-1}$ is a solution of the solvable equation $\mathcal{A} \cdot x^{t-1} = b$. \hfill \square

Theorem 3.4. If $\mathcal{A} \in \mathbb{C}_{t}^{n,n}$ is a $k$-T-idempotent tensor, then the eigenvalues of $\mathcal{A}$ are the roots of $\lambda^{(t-1)^k} = 1$ or 0.

Proof. Let $\lambda$ be an eigenvalue of $\mathcal{A}$, then $\mathcal{A} \cdot x^{t-1} = \lambda x^{[t-1]}, 0 \neq x \in \mathbb{C}^n$. Multiplying by $\mathcal{A}^k$ on the left hand side of it, we get

$$\mathcal{A}^{k+1} \cdot x^{(t-1)^{k+1}} = \mathcal{A}^k \cdot (\lambda x^{[t-1]})^{(t-1)^k} = \lambda^{(t-1)^k} \mathcal{A}^k \cdot (x^{[t-1]})^{(t-1)^k}$$

$$= \lambda^{(t-1)^k} \mathcal{A}^k \cdot \left(\left(x^{[t-1]}\right)^{[t^k]}\right)^{(t-1)^k},$$

where $s = (t-1)^{(k-1)}$. Since $\mathcal{A}$ is a $k$-T-idempotent tensor, it yields that

$$\mathcal{A}^k \cdot (y^{[t^{-1}]})^{(t-1)^k} = \mathcal{A} \cdot y^{t-1},$$

for all $y \in \mathbb{C}^n$. Then

$$\mathcal{A}^{k+1} \cdot x^{(t-1)^{k+1}} = \lambda^{(t-1)^k} \mathcal{A}^k \cdot \left(\left(x^{[t-1]}\right)^{[t^k]}\right)^{(t-1)^k}$$

$$= \lambda^{(t-1)^k} \mathcal{A} \cdot \left(x^{[t-1]}\right)^{t-1}$$

$$= \lambda^{(t-1)^k} \mathcal{A} \cdot (x^{[t-1]^{(k-1)}})^{t-1}.$$
So, $\mathcal{A}^{k+1} \cdot x^{(t-1)^{k+1}} = \lambda^{(t-1)^k} \mathcal{A} \cdot (x^{(t-1)^k})^{t-1}$. By the $k$-T-idempotency, we have

$$
\mathcal{A}^{k+1} \cdot x^{(t-1)^{k+1}} = \mathcal{A}^{k+1} \cdot \left[ \left( x^{(t-1)^k} \right)^{(t-1)^k} \right]^{(t-1)^k} = \mathcal{A} \cdot (x^{(t-1)^k})^{t-1}.
$$

Hence,

$$
\lambda^{(t-1)^k} \mathcal{A} \cdot (x^{(t-1)^k})^{t-1} = \mathcal{A} \cdot (x^{(t-1)^k})^{t-1},
$$

that is

$$
(\lambda^{(t-1)^k} - 1) \mathcal{A} \cdot (x^{(t-1)^k})^{t-1} = 0.
$$

If $\mathcal{A} \cdot (x^{(t-1)^k})^{t-1} \neq 0$, then $\lambda^{(t-1)^k} = 1$. If $\mathcal{A} \cdot (x^{(t-1)^k})^{t-1} = 0$, then 0 is an eigenvalue of $\mathcal{A}$. \hfill \Box

**Corollary 3.5.** (1) If $\mathcal{A} \in \mathbb{C}^{n,n}$ is a $T$-idempotent tensor, then the eigenvalues of $\mathcal{A}$ are the roots of $\lambda^{(t-1)^2} = 1$ or 0;

(2) If $\mathcal{A} \in \mathbb{C}^{n \times n}$ is a $k$-idempotent matrix, then the eigenvalues of $\mathcal{A}$ are the roots of $\lambda^k = 1$ or 0.

**Theorem 3.6.** If $\mathcal{A} \in \mathbb{C}^{n,n}$ is an idempotent tensor, then the eigenvalues of $\mathcal{A}$ are 1 or 0.

**Proof.** Let $\lambda$ be an eigenvalue of $\mathcal{A}$, then $\mathcal{A} \cdot x^{t-1} = \lambda x^{t-1}$, $0 \neq x \in \mathbb{C}^n$. So $\mathcal{A} \cdot x^{t-1} = \lambda x^{(t-1)}$. Multiplying by $\mathcal{A}$ on the left hand side of it, we obtain $\mathcal{A} \cdot [(\mathcal{A} \cdot x^{t-1})^{(t-1)}]^{(t-1)} = \lambda \mathcal{A} \cdot x^{t-1}$. Since $\mathcal{A}$ is an idempotent tensor, $\mathcal{A} \cdot x^{t-1} = \lambda \mathcal{A} \cdot x^{t-1}$, that is $(\lambda - 1) \mathcal{A} \cdot x^{t-1} = 0$. If $\mathcal{A} \cdot x^{t-1} \neq 0$, then $\lambda = 1$. If $\mathcal{A} \cdot x^{t-1} = 0$, then 0 is an eigenvalue of $\mathcal{A}$. \hfill \Box

**Proposition 3.7.** Let $\mathcal{A} \in \mathbb{C}^{m,n}_t$ and let $\mathcal{A}^{(1)}_k$, $\mathcal{A}^{(2)}_k \in \mathbb{C}^{n,m}_k$ be the order $k$ {1} inverse and {2} inverse of $\mathcal{A}$, respectively. Then

(1) $\mathcal{A} \mathcal{A}^{(2)}_k$ is a $T$-idempotent tensor;

(2) $\mathcal{A} \mathcal{A}^{(1)}_k$ is an idempotent tensor and the eigenvalues of it are 1 or 0.

**Proof.** (1) Form the definition of the tensor {2} inverse, it yields that

$$
(\mathcal{A} \mathcal{A}^{(2)}_k)^2 \cdot (y_j^{[k]})^{s^2} = \mathcal{A} (\mathcal{A}^{(2)}_k \mathcal{A} \mathcal{A}^{(2)}_k) \cdot (y_j^{[k]})^{s^2} = \mathcal{A} \mathcal{A}^{(2)}_k \cdot y^s,
$$

for all $y \in \mathbb{C}^n$, where $s = (t-1)(k-1)$. Hence, $\mathcal{A} \mathcal{A}^{(2)}_k$ is a T-idempotent tensor.

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(2) By the definition of the tensor \( \{1\} \) inverse, we have \( \mathcal{A} \mathcal{A}^{(1)_k} \cdot \left[ (\mathcal{A} \cdot \mathcal{z}^{-1})^{(1)_k} \right] = \mathcal{A} \cdot \mathcal{z}^{t-1} \), for all \( z \in \mathbb{C}^n \), where \( s = (t-1)(k-1) \). Let \( z = \mathcal{A}^{(1)_k} \cdot y^{k-1} \), \( y \in \mathbb{C}^n \) is an arbitrary vector. Then \( \mathcal{A} \mathcal{A}^{(1)_k} \cdot \left[ (\mathcal{A} \mathcal{A}^{(1)_k} \cdot y^*)^{(1)_k} \right] = \mathcal{A} \mathcal{A}^{(1)_k} \cdot y^* \). It yields that \( \mathcal{A} \mathcal{A}^{(1)_k} \) is an idempotent tensor. And it follows from Theorem 3.3 that the eigenvalues of it are 1 or 0.

4. Some examples

Let the vector \( \alpha_i \in \mathbb{C}^{m_i} \) \( (i = 1, \ldots, t) \), the outer product of \( \alpha_1, \ldots, \alpha_t \), denoted by \( \alpha_1 \otimes \cdots \otimes \alpha_t \), is a tensor \( \mathcal{A} = (a_{i_1 \cdots i_t}) \in \mathbb{C}^{n_1 \times \cdots \times n_t} \) with entries \( a_{i_1 \cdots i_t} = (\alpha_1)_{i_1} \cdots (\alpha_t)_{i_t} \), where \( (\alpha_i)_{j} \) is the \( j \)-th component of \( \alpha_i \). The tensor \( \mathcal{A} = (a_{i_1 \cdots i_t}) \in \mathbb{C}^{n_1 \times \cdots \times n_t} \) can be decomposed into the form as

\[
\mathcal{A} = \sum_{j \in [r]} \alpha_j^1 \otimes \cdots \otimes \alpha_j^t,
\]

where \( \alpha_j^i \in \mathbb{C}^{m_i} \) \( (i \in [t], j \in [r]) \) (see 20). Let the matrix \( B_k \in \mathbb{C}^{m_k \times n_k} \) \( (k \in [t]) \) and let \( (B_k)_{j,i} \) be the \( (j, i) \)-entry of \( B_k \). By the Tucker’s product, we get a tensor \( \mathcal{A}' = (a'_{j_1 \cdots j_t}) \in \mathbb{C}^{m_1 \times \cdots \times m_t} \) as follows (see 20)

\[
\mathcal{A}' = (B_1, \ldots, B_t) \cdot \mathcal{A} = \sum_{j \in [r]} B_1 \alpha_j^1 \otimes \cdots \otimes B_t \alpha_j^t,
\]

where

\[
a'_{j_1 \cdots j_t} = \sum_{i_1, \ldots, i_t = 1}^{n_1, \ldots, n_t} (B_1)_{j_1, i_1} \cdots (B_t)_{j_t, i_t} \alpha_{i_1 \cdots i_t}.
\]

If a tensor \( \mathcal{A} \in \mathbb{C}^{n_1 \times \cdots \times n_t} \) can be decomposed into the form as

\[
\mathcal{A} = \sum_{i \in [r]} \lambda_i e_i \otimes \alpha_i \otimes \alpha_i \otimes \cdots \otimes \alpha_i,
\]

where the vectors \( \alpha_1, \ldots, \alpha_r \in \mathbb{C}^n \) are linearly independent, \( \lambda_i \in \mathbb{C} \). Let the matrix \( A = (\alpha_1, \alpha_2, \ldots, \alpha_r) \in \mathbb{C}^{n \times r} \) and \( B = (e_1 e_2 \cdots e_r) \in \mathbb{C}^{m \times r} \), \( e_i \) is the unit vector with \( i \)-th component being 1, then

\[
\mathcal{A} = (B, A, \ldots, A) \cdot \mathcal{D} = B \mathcal{D} A^T,
\]

where \( \mathcal{D} \in \mathbb{C}^{m \times r} \) is a diagonal tensor with the diagonal entries \( \lambda_1, \ldots, \lambda_r \) (see 23). Let \( \mathcal{D}^{(1)_k}, \mathcal{D}^{(2)_k} \in \mathbb{C}^{r \times r} \) be the order \( k \) \{1\} inverse and \{2\} inverse of \( \mathcal{A} \), respectively. Similar to Theorem 2.11 and Theorem 2.16 we get

\[
\mathcal{A}^{(1)_k} = (A^T)^{(1)_k} \mathcal{D}^{(1)_k} B^T \in \mathbb{C}^{n \times m}.
\]
\[ \mathcal{A}^{(2)k} = (A^T)^{(2)k} D^{(2)k} B^T \in \mathbb{C}^{n,m} \]

Next, two examples are showed by the above discussion. Let \( A = (A_1 | A_2 | \cdots | A_t) \in \mathbb{C}^{n_1 \times \cdots \times n_t} \), where \( A_i = (a_{i_1 \cdots i_t}), i \in [n_1] \).

**Example 4.1.** Let \( A \) be a 3 \( \times \) 3 \( \times \) 3 tensor as follows

\[
A = \begin{pmatrix}
1 & 2 & 3 & 16 & 8 & 4 & 4 & 6 & 8 \\
2 & 4 & 6 & 8 & 4 & 2 & 6 & 9 & 12 \\
3 & 6 & 9 & 4 & 2 & 1 & 8 & 12 & 16
\end{pmatrix},
\]

then the below tensor \( B \) is both a \( \{1\} \) inverse and a \( \{2\} \) inverse of \( A \),

\[
B = \begin{pmatrix}
5 & -5 & 0 & -14 & 14 & 0 & 8 & -8 & 0 \\
5 & 1 & 0 & 14 & -2 & 0 & 8 & 1 & 0 \\
0 & 0 & -4 & 0 & 0 & 11 & 0 & 0 & -6
\end{pmatrix}.
\]

In the following, the examples of T-idempotent tensor and idempotent tensor are given.

**Example 4.2.**

The tensor \( A = (A_1 | A_2) = \begin{pmatrix}
25 & -15 & 100 & -60 \\
-15 & 9 & -60 & 36
\end{pmatrix} \) is an idempotent tensor;

The tensor \( B = (B_1 | B_2) = \begin{pmatrix}
9 & -6 & 36 & -24 \\
-6 & 4 & -24 & 16
\end{pmatrix} \) is a T-idempotent tensor.

**References**

[1] L. Qi, Eigenvalues of a real supersymmetric tensor, J. Symbolic Comput. 40(2005) 1302-1324.

[2] L. Lim, Singular values and eigenvalues of tensors, a variational approach, in: Proceedings 1st IEEE international Workshop on computational Advances of Multitensor Adaptive Processing, (2005) 129-132.

[3] J. Shao, A general product of tensors with applications, Linear Algebra Appl. 439(2013) 2350-2366.
[4] W. Ding, L. Qi, Y. Wei, M-tensors and nonsingular M-tensors, Linear Algebra Appl. 439(2013) 3264C3278.

[5] J. Shao, H. Shan, L. Zhang, On some properties of the determinants of tensors, Linear Algebra Appl. 439(2013) 3057C3069.

[6] Y. Yang, Q. Yang, Further results for Perron-CFrobenius theorem for nonnegative tensors, SIAM J. Matrix Anal. Appl. 31(5)(2010) 2517C2530.

[7] L. Qi, Symmetric nonnegative tensors and copositive, Linear Algebra Appl. 439(2013) 228C238.

[8] K.C. Chang, K. Pearson, T. Zhang, Perron Frobenius Theorem for nonnegative tensors, Commun. Math. Sci. 6(2008) 507C520.

[9] C. Bu, J. Zhou, Y. Wei, E-cospectral hypergraphs and some hypergraphs determined by their spectra, Linear Algebra Appl. 459(2014) 397C403.

[10] L. Qi, J. Shao, Q. Wang, Regular uniform hypergraphs, s-cycles, s-paths and their largest Laplacian H-eigenvalues, Linear Algebra Appl. 443(15)(2014) 215 - 227.

[11] C. Bu, X. Zhang, J. Zhou, W. Wang, Y. Wei, The inverse, rank and product of tensors, Linear Algebra Appl. 446(2014) 269C280.

[12] A. Ben-I’sral, T.N.E. Greville, Generalized Inverse: Theory and Applications, Wiley New York 1974.

[13] S.J. Kirkland, M. Neumann, B.L. Shader, On a bound on algebraic connectivity: the case of equality, Czech. Math. J. 48 (1998) 65C76.

[14] J.J. Hunter, Generalized inverses of Markovian kernels in terms of properties of the Markov chain, Linear Algebra Appl. 447 (2014) 38C55.

[15] S.L. Campbell, C.D. Meyer, Generalized Inverses of Linear Transformations, Dover, New York, 1991 (Originally published: Pitman, London, 1979).

[16] L. Sun, W. Wang, J. Zhou, C. Bu, Some results on resistance distances and resistance matrices, Linear and Multilinear Algebra, doi.org/10.1080/03081087.2013.877011.
[17] C. Bu, L. Sun, J. Zhou, Y. Wei, A note on the block representations of the group inverse of Laplacian Matrices, Electron. J. Linear Algebra 23 (2012) 866C876.

[18] Y. Wei, A characterization and representation of the generalized inverse $A_{T,S}^{(2)}$ and its applications, Linear Algebra Appl. 280(1998) 87C96.

[19] C. Bu, Y. Wei, Sign Pattern of the Generalized Inverse, Science Press, Beijing, 2014 (in Chinese).

[20] L.H. Lim, Tensors and hypermatrices, in: L. Hogben (Ed.), Handbook of Linear Algebra, 2nd Ed., CRC Press, Boca Raton, FL, 2013.

[21] S. Ragnarsson, C.F. Van Loan, Block tensors and symmetric embeddings, Linear Algebra Appl. 438 (2013) 853C874.

[22] S. Hu, Z. Huang, C. Ling, L. Qi, On determinants and eigenvalue theory of tensors, J. Symbolic Comput. 50(2013) 508C531.

[23] J. Chen, Y. Saad, On the tensor SVD and the optimal low rank orthogonal approximation of tensors, SIAM J. Matrix Anal. Appl. 30(4)(2009) 1709C1734.