Analysis of cellular automata governed by simple time-variant rules

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Abstract: This paper studies cellular automata governed by time-variant rules. The time-variant rule is constructed by switching two simple rules. Depending on the rule and initial conditions, the cellular automata can exhibit various spatiotemporal patterns. In order to investigate the dynamics, we present two simple feature quantities. The first quantity characterizes the expression ability of various spatiotemporal patterns. The second quantity characterizes the error correction ability of target spatiotemporal patterns. Performing numerical experiments, we have found three typical classes of the cellular automata. First, the expression ability is the highest and error correction ability is strong. Second, the error correction ability is very strong whereas the expression ability is very weak. Third, the expression ability is very high whereas the error correction ability is very weak.

Key Words: cellular automata, error correction ability, expression ability

1. Introduction

Cellular automata (CAs) are digital dynamical systems where time, space, and state variables are all discrete [1–3]. The dynamics is governed by simple rules. Depending on the rules and initial conditions, the CAs can generate various spatiotemporal patterns. Referring to the rules, the spatiotemporal patterns have been analyzed. Real/potential engineering applications are many, including sound data description, image processing, feature extraction, and block cipher [4–8]. Analysis of the CAs is important from both viewpoints of nonlinear dynamics and engineering applications.

This paper studies time-variant rules cellular automata (TCA). The time-variant rules are constructed by switching two rules. The two rules are selected from rules of the elementary cellular automata (ECA). In the ECA, the dynamics of each cell is governed by a time-invariant rule determined by the closest neighbors of the cell. The rules are described by Boolean functions from three inputs to one output and the number of rules is $2^{2^3} = 256$. The dynamics of the ECA has been analyzed in detail [2, 3]. Extracting two rules out of the 256 ECA rules, we obtain a rule pair. Applying either rule of the rule pair at each time step, we obtain a time-variant rule of the TCA. The TCA can exhibit various spatiotemporal patterns and can have excellent information expression ability.

In order to investigate the TCAs dynamics, we present two simple feature quantities. The first quantity characterizes expression ability. It is relevant to encoding capability of many data. The second quantity characterizes error correction ability of target spatiotemporal patterns. It is relevant
to a bit error correction in data sequences. Using the two feature quantities, we investigate TCAs of 16 cells. This size of TCA corresponds to a mapping on a set of 16-bit binary vectors and is applicable to sound data description [4, 5]. Based on the sound data, we have prepared target spatiotemporal patterns. Performing numerical experiments for many TCAs, we have calculated the two feature quantities. The results are summarized into a feature plane of the two feature quantities. The feature plane is useful to classify the TCAs. Especially, we have found three typical classes. First, we have found an optimal rule pair such that the expression ability is the highest and the error correction ability is strong. Such an optimal rule pair has not been found in previous works. Second, the error correction ability is very strong whereas the expression ability is very weak. Third, the expression ability is very high whereas the error correction ability is very weak.

These results give basic information to analyze expression ability and error correction ability in various TCAs, and to develop engineering applications of the TCAs. It should be noted that this is the first paper of the two feature quantities for analysis of TCAs. Preliminary results along these lines can be found in [9, 10].

2. Cellular automata governed by simple time-variant rules

First, we introduce the ECA defined on a ladder of \( N + 2 \) cells. Let \( x_i^t \in \{0, 1\} \equiv B \) be the binary state of the \( i \)-th cell at discrete time \( t \). The time evolution of \( x_i^t \) is governed by a Boolean function \( F \) of \( x_i^t \) and its closest neighbors:

\[
x_{i+1}^t = F(x_i^{t-1}, x_i^t, x_{i+1}^t), \quad i \in \{1, \cdots, N\}
\]

(1)

where the boundary condition on the ends of ladder is \( (x_0^t = 1, x_{N+1}^t = 0) \) for all \( t \) (see Fig. 1). The Boolean function \( F \) transforms three binary inputs to one binary output. We show two examples of \( F \):

\[
\begin{align*}
F(0, 0, 0) &= 1, \quad F(0, 0, 1) = 1, \quad F(0, 1, 0) = 1, \quad F(0, 1, 1) = 1, \\
F(1, 0, 0) &= 0, \quad F(1, 0, 1) = 0, \quad F(1, 1, 0) = 0, \quad F(1, 1, 1) = 0.
\end{align*}
\]

(2)

\[
\begin{align*}
F(0, 0, 0) &= 0, \quad F(0, 0, 1) = 0, \quad F(0, 1, 0) = 0, \quad F(0, 1, 1) = 0, \\
F(1, 0, 0) &= 1, \quad F(1, 0, 1) = 1, \quad F(1, 1, 0) = 1, \quad F(1, 1, 1) = 1.
\end{align*}
\]

(3)

The Boolean function \( F \) is referred to as a rule. The set of the 8 outputs are referred to as a rule table:

\[
\{f_0, f_1, f_2, f_3, f_4, f_5, f_6, f_7\}
\]

(4)

\[
\begin{align*}
F(0, 0, 0) &\equiv f_0, & F(0, 0, 1) &\equiv f_1, & F(0, 1, 0) &\equiv f_2, & F(0, 1, 1) &\equiv f_3, \\
F(1, 0, 0) &\equiv f_4, & F(1, 0, 1) &\equiv f_5, & F(1, 1, 0) &\equiv f_6, & F(1, 1, 1) &\equiv f_7
\end{align*}
\]

Fig. 1. Examples of spatiotemporal patterns for \( N = 8 \). Black and white cells mean output \( x_{i+1}^t = 1 \) and \( x_{i+1}^t = 0 \), respectively. (a) ECA(RN15). (b) ECA(RN240). (c) TCA(RN15, RN240), \( M = 3 \).
A rule table is equivalent to an 8 bits binary number whose decimal expression is referred to as the rule number (RN). There exist $2^8 = 256$ rules. The rule tables in Eqs. (2) and (3) are represented by RN15 and RN240, respectively. The rate of the positive outputs is referred to as the $\lambda$-parameter that is used to classify the rules. For example, the RN15 and RN240 are characterized by $\lambda = 0.5$. Figures 1(a) and (b) show examples of spatiotemporal patterns.

Next, we define the time-variant rules cellular automata (TCA). The TCA is governed by simple time-variant rules and is defined on a ladder of $N + 2$ cells. The dynamics is described by

$$x_i^{t+1} = F^t(x_{i-1}^t, x_i^t, x_{i+1}^t), \quad i \in \{1, \cdots, N\}$$

where the Boolean function $F^t$ depends on the discrete time $t$. Since it is hard to consider combination of 256 ECA rules, we extract two rules out of the 256 ECA rules. The two rules are denoted by $(F_A, F_B)$ and are referred to as a rule pair. The time-variant rule consists of a combination of the two rules for a time interval: $F^t \in \{ F_A, F_B \}$ for $t \in \{1, \cdots, M\}$. Figure 1 shows an example of spatiotemporal pattern from a TCA whose rule pair is given by $(F_A, F_B) = (\text{RN15}, \text{RN240})$.

Here, we consider generation of a spatiotemporal patterns of $M$ time steps. In order to generate the patterns, we define a TCA governed by a rule sequence with length $M$. As two rules $(F_A, F_B)$ are given, there exist $2^M$ kinds of rule sequences for the TCA. Let $k - 1$ denote a decimal expression of the binarized rule sequences:

$$k \in \{1, \cdots, 2^M\}, \quad k = 1: (F_A, \cdots, F_A), \quad k = 2: (F_A, \cdots, F_A, F_B), \cdots, k = 2^M: (F_B, \cdots, F_B)$$

The $2^M$ rule sequences are characterized by the index $k$. Using the one of the $2^M$ rule sequence, we define the TCA

$$x^{t+1} = F_l^k(x^t) \quad \text{for} \quad l \in \{1, \cdots, M\}, \quad l - 1 = t \mod M$$

Initial condition: $x^1 = x_0$

Boundary condition of ladder: $x_0 = 1, x_{N+1} = 0$ for all $t$

where $x^t \equiv (x_1^t, \cdots, x_N^t)$ and $F_l^k \in \{ F_A, F_B \}$ is given by a rule sequence represented by the index (suffix) $k$. The rule sequence is repeated with period $M$.

Since the dynamics of TCA depends on an initial condition $x_0$, we introduce the set of all initial conditions

$$x_0 \equiv \{ x_{01}, \cdots, x_{0N} \}, \quad x_0 \in \{ b_1, \cdots, b_{2^N} \} \quad (8)$$

where $b_i$ denote a binary expression of a nonnegative integer $i - 1 \in \{0, \cdots, 2^N - 1\}$. Figure 1(c) shows a simple example of spatiotemporal pattern for $(F_A, F_B) = (\text{RN15}, \text{RN240})$:

$$x^1 = b_{98} = (1, 0, 0, 0, 0, 1, 1, 0), \quad k = 3: (F_A, F_B, F_A)$$

$$x^2 = F_3^3(x^1), \quad F_3^1 = F_A$$

$$x^3 = F_3^3(x^2), \quad F_3^2 = F_B$$

$$x^4 = F_3^3(x^3), \quad F_3^3 = F_A. \quad (9)$$

It should be noted that the analysis of TCA becomes impossible as $N \to \infty$ and/or $M \to \infty$. For simplicity, we consider the TCAs in the range

$$4 \leq N \leq 16, \quad 1 \leq M \leq 16. \quad (10)$$

The TCA with $N = 16$ is applicable to sound data description [4].

3. Expression ability

In this section, we consider expression ability of the TCA. For $t \in \{1, \cdots, M\}$, $2^M \times 2^N$ spatiotemporal patterns (incl. overlaps) can be generated by the TCAs:

$$x^{t+1} = F_k^t(x^t), \quad k \in \{1, \cdots, 2^M\}, \quad x^1 \in \{ b_1, \cdots, b_{2^N} \}, \quad t \in \{1, \cdots, M\}$$

We define a set of reachable points $S_k$ within $M$ steps for initial condition $x^1$. 

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**Fig. 2.** Examples of return plots for $N = 4$, $M = 3$, $F_A = \text{RN15}$, and $F_B = \text{RN240}$. $C_k$ is the binary code of $k - 1$, e.g., $C_7 = (0, 1, 1, 0)$, “0” ≡ white, and “1” ≡ black. (a) $t = 1$. (b) $t = 2$. (c) $t = 3 = M$. (d) $1 \leq t \leq 3$.

$\gamma(3) = \frac{176}{256}$. The lower figure shows examples of transformation: $C_7$ is transformed to $C_5$ by $F_A$, $C_5$ is transformed to $C_{11}$ by $F_B$, and so on.

**S**$(M, k, i) = \{x^t | x^{t+1} = F_k^t(x^t), x^1 = b_i, t \in \{1, \cdots, M\}, k \in \{1, \cdots, 2^M\}, i \in \{1, \cdots, 2^N\}\}$ (12)

We define the first feature quantity for expression ability of the TCA

$$\gamma(M) = \frac{\text{The number of reachable points within M steps}}{2^N \times 2^N} = \frac{\bigcup_{k=1}^{2^M} \bigcup_{i=1}^{2^N} S(M, k, i)}{2^N \times 2^N}$$ (13)

where $\frac{1}{2^N} \leq \gamma(M) \leq 1$. Figure 2 illustrates return plots of reachable points for $M = 3$ and $N = 4$.

In order to investigate the expression ability of the TCA, we have performed numerical experiments for $N = 16$. We have found two rule pairs ($F_A, F_B$) = (N15, RN240) and (RN60, RN195) that realize the highest value $\gamma(M) = 1$ at $M = 16$. Such a pair has not been found in previous works. Figure 3 shows the feature quantity $\gamma(M)$ for 5 examples of rule pairs. We can see that the quantity $\gamma(M)$ increases as $M$ increases and the two rule pairs (RN15, RN240) and (RN60, RN195) reach the highest value at $M = 16$.

**Fig. 3.** Expressive ability $\gamma(M)$ in five rule pairs. Red plot: (RN15, RN240) and (RN60, RN195). Blue plot: (RN90, RN180), Green plot (RN118, RN152), Yellow plot: (RN31, RN224).
4. Error correction ability

In this section, we consider error correction ability of a target spatiotemporal pattern.

Let us consider a sequence of target data with length $Q$:

\[
(d^1, \ldots, d^Q), \quad d^s \in \left\{ \frac{0}{2^N}, \frac{1}{2^N}, \ldots, \frac{2^N - 1}{2^N} \right\}, \quad s \in \{1, \ldots, Q\}
\]

(14)

where each datum $d^s$ is normalized and quantized into $2^N$ points. For example, such a data is given by sampling and quantizing a sound data [4]. Applying a binary encoding to the data, we obtain a sequence of binary vectors with length $Q$:

\[
(D^1, \ldots, D^Q), \quad D^s = (D^s_1, \ldots, D^s_N), \quad D^s_i \in \{0, 1\}, \quad i \in \{1, \ldots, N\}.
\]

(15)

For example, $D^s = (0, 1, 0, 1)$ for $d^s = 10/16$ and $N = 4$. This sequence is referred to as a binarized data hereafter. We approximate the binarized data by a target spatiotemporal pattern (TSTP) of the TCA:

\[
X^t = (X^t_1, \ldots, X^t_N), \quad X^{t+1} = F^t_k(X^t), \quad t \in \{1, \ldots, T\}
\]

\[
X^1 = D^1, \quad X^{t+1} \text{ approximates } D^t, \quad t_s \in \{2, \ldots, T\}
\]

(16)

The binarized data is approximated at $Q$ steps of $t_s$ in $T$ time steps of $t$. Figure 5 (a) and (b) illustrate a binarized data and a TSTP. In order to show the approximation method, we introduce the following sequence with length $T$ from the TSTP:

\[
(\xi^1, \ldots, \xi^T), \quad \xi^t \in \left\{ \frac{0}{2^N}, \frac{1}{2^N}, \ldots, \frac{2^N - 1}{2^N} \right\}, \quad t \in \{1, \ldots, T\}, \quad \xi^1 = d^1.
\]

(17)

Applying the binary decoding to $X^t$, we obtain $\xi^t$.

We evaluate an error between the TSTP and the binarized data by the following error between the decoded elements $\xi^{t+s}$ from the TSTP and the real valued target data $d^s$:

\[
E(t_s) = |\xi^{t+s} - d^s|, \quad s \in \{1, \ldots, Q\}, \quad t_s \in \{2, \ldots, T\}.
\]

(18)

We define a method to determine a TCA with a rule sequence represented by $k$. If $d^s$ is approximated then $d^{s+1}$ is approximated as the following.

Step 1: Let $t = t_s$. Let $E(t_s) = |\xi^{t+s} - d^s| < \epsilon$ where $\epsilon$ is a small criterion parameter for the approximation.

Step 2: Apply $F_A$ and $F_B$ to $X^t$. Let $\xi_A$ and $\xi_B$ be binary decoding values of $F_A(X^t)$ and $F_B(X^t)$, respectively.

\[
F^t = F_A \text{ and } \xi^{t+1} = \xi_A \text{ if } |\xi_A - d^{s+1}| \leq |\xi_B - d^{s+1}|.
\]

\[
F^t = F_B \text{ and } \xi^{t+1} = \xi_B \text{ if } |\xi_A - d^{s+1}| > |\xi_B - d^{s+1}|.
\]

Step 3: If $E(t) = |\xi^{t+1} - d^{s+1}| < \epsilon$ then $t_{s+1} \leftarrow t + 1$ and $d^{s+1}$ is approximated. Otherwise go to Step 4.

Step 4: Let $t \leftarrow t + 1$, go to Step 2, and repeat until the time limit $t_s + t_{\max}$.

Applying this method to the binarized data in Fig. 4 (a), we obtain a TSTP in Fig. 4 (b).

We define the second feature quantity for error correction ability.

\[
\alpha(L) = \frac{\text{The number of initial errors corrected within } L \text{ steps}}{\text{The number of all initial errors}}, \quad 0 \leq \alpha \leq 1
\]

(19)

where an initial error at $t = 1$ is different from $D_1 = X_1$ in 1 bit (Hamming distance 1). In Fig. 5 (c), an initial error is corrected at $t = 5$.

In order to investigate the error correction ability, we prepare 100 samples of sound data. Binarizing and approximating them, we obtain 100 TSTPs where
Fig. 4. An example of TSTP for $N = 16$, $Q = 50$, and $T = 400$. (a) Binarized data. (b) TSTP. $t_{\text{max}} = 10$, $\epsilon = 100/2^N$.

Fig. 5. Error correction ability ($N = 8$, $Q = 3$, $T = 8$). (a) Binarized data. (b) TSTP. (c) Initial error correction at $t = 4$.

In the 100 TSTPs, we have confirmed approximation within $t_{\text{max}}$ steps for each binarized data. The parameter $T$ (the length of TSTP) varies depending on the original data.

For five rule pairs used in Fig. 3, we have calculated the feature quantity $\alpha(L)$ of the 100 TSTPs and have averaged them:

$$\alpha_{\text{avg}}(L) = \frac{1}{100} \sum_{i=1}^{100} \alpha^i(L), \quad 0 \leq \alpha_{\text{avg}}(L) \leq 1$$  \hspace{1cm} (21)

where $\alpha^i(L)$ denote $\alpha(L)$ of the $i$-th TSTP. Figure 6 shows the result of the calculation. We can see three types as $L$ increases: $\alpha_{\text{avg}}(L)$ remains to be low (black plots), $\alpha_{\text{avg}}(L)$ becomes high rapidly (yellow plot), and $\alpha_{\text{avg}}(L)$ increases (other three plots).

Table I shows expression ability $\gamma(16)$ and error correction ability $\alpha_{\text{avg}}(10)$ in the five rule pairs used in Figs. 3 and 6. The pairs (RN15, RN240) and (RN60, RN195) have the highest expression
Fig. 6. Error correction ability $\alpha_{\text{avg}}(L)$ in five rule pairs used in Fig. 3. Red plot: (RN15, RN240), Blue plot: (RN90, RN180), Green plot: (RN118, RN152), Yellow plot: (RN31, RN224), Black plot: (RN60, RN195).

Table I. Expression ability $\gamma(16)$ and error correction ability $\alpha_{\text{avg}}(10)$ in five rule pairs used in Figs. 3 and 6.

| $(F_A, F_B)$ | $\gamma(16)$ | $\alpha_{\text{avg}}(10)$ |
|-------------|--------------|---------------------------|
| (i) 15, 240 | 1.00         | 0.52                      |
| (ii) 90, 180| 0.55         | 0.43                      |
| (iii) 118, 152 | 0.34     | 0.53                      |
| (iv) 31, 224 | 0.02         | 0.91                      |
| (v) 60, 195  | 1.00         | 0.07                      |

ability $\gamma(16) = 1$, however, the error correction ability $\alpha_{\text{avg}}(10)$ is different: the rule pair (RN15, RN240) has much higher error correction ability than the pair (RN60, RN195).

In order to investigate the two feature quantities for various rule pairs, we have selected 200 rule pairs such that each rule has $\lambda$-parameter $\in \{3, 4, 5\}$ and a rule is different from its partner rule in Hamming distance 7 or 8. Using the feature quantities $\gamma(16)$ and $\alpha_{\text{avg}}(10)$, we have constructed a feature plane. Figure 7 shows the results on the feature plane with histograms of the two feature quantities. The feature plane is useful to classify the rule pairs and we can see three typical classes:

- In the rule pair (RN15, RN240), the expression ability ($\gamma(16)$) is the highest and the error correction ability ($\alpha_{\text{avg}}(10)$) is strong. Such a rule pair has not been found in previous works.
- As the rule pair (RN31, RN224), the error correction ability is very strong whereas the expression ability is very weak.
- As the rule pair (RN60, RN195), the expression ability is very high whereas the error correction ability is very weak.

5. Conclusions

Presenting the two feature quantities, the TCAs are analyzed in this paper. The first quantity characterizes expression ability. The second quantity characterizes error correction ability of TSTPs. Performing numerical experiments, we have found three typical classes of the TCAs. First, the expression ability is the highest and the error correction ability is strong. Second, the error correction ability is very strong whereas the expression ability is very weak. Third, the expression ability is very high whereas the error correction ability is very weak. These results will be basic information to analyze expression ability and error correction ability in various TCAs for engineering applications.
Future problems include analysis of the expression ability with information compression ability, analysis of the error correction ability in various TSTPs, application to CAs with space-dependent rules [11], and hardware implementation for engineering applications.

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