NONEQUILIBRIUM FLUCTUATIONS FOR A TAGGED PARTICLE IN MEAN-ZERO ONE-DIMENSIONAL ZERO-RANGE PROCESSES

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ABSTRACT. We prove a non-equilibrium functional central limit theorem for the position of a tagged particle in mean-zero one-dimensional zero-range process. The asymptotic behavior of the tagged particle is described by a stochastic differential equation governed by the solution of the hydrodynamic equation.

1. INTRODUCTION

Informally, the zero-range particle system follows a collection of dependent random walks on the lattice $\mathbb{Z}^d$ where, from a vertex with $k$ particles, one of the particles displaces by $j$ with rate $(g(k)/k)p(j)$. The function on the non-negative integers $g: \mathbb{N}_0 \rightarrow \mathbb{R}_+$ is called the process “rate”, and $p(\cdot)$ denotes the translation-invariant single particle transition probability. The name “zero-range” derives from the observation that, infinitesimally, the interaction is only with respect those particles at the particular vertex. The case when $g(k)$ is proportional to $k$ describes the situation of completely independent particles.

The problem of the asymptotics of a distinguished, or tagged particle interacting with others has a long history and was even mentioned in Spitzer’s paper [14] (see also chapters 8.I, 6.II [15]). The main analytical difficulty is that the tagged particle motion is not in general Markovian due to the interaction with other particles. However, the intuition is that in a scale the tagged particle behaves as a random walk with certain “homogenized” parameters reflecting the system dynamics.

We prove in this article a nonequilibrium invariance principle, with respect to a diffusion process whose coefficients depend on the hydrodynamic density, for the diffusively rescaled position of the tagged particle in one-dimensional zero-range processes when the transition probability $p$ is finite-range and mean-zero. This invariance principle is the first result which captures the nonequilibrium fluctuations of a single, given particle in a general finite-range interacting particle system. We remark, however, in [4], a nonequilibrium central limit theorem was proved for a tagged particle in the nearest-neighbor symmetric one-dimensional simple exclusion model by completely different methods which rely on the special structure of the nearest-neighbor one-dimensional dynamics. Also, we note, in [9], a “propagation of chaos” type nonequilibrium result was shown for finite-range symmetric $d \geq 1$
dimensional simple exclusion processes which gives the fluctuations for a tagged particle selected at random, or in other words the average tagged particle position; however, this result, which makes key use of the “averaging,” does not convey the fluctuations of any fixed, given particle and so is weaker than the one we state in this paper.

We mention also, with respect to zero-range tagged particles, previous results on laws of large numbers, in equilibrium \[11\], \[12\] and non-equilibrium \[10\], and equilibrium central limit theorems when the jump probability \(p\) is mean-zero, \(\sum jp(j) = 0\) \[11\], \[12\], and also when \(p\) is totally asymmetric and nearest-neighbor in \(d = 1\) \[13\], and also some diffusive variance results when \(p\) has a drift \(\sum jp(j) \neq 0\) in \(d = 1\) and \(d \geq 3\) \[13\].

Denote by \(\xi \in \mathbb{N}_0^2\), \(\mathbb{N}_0 = \{0, 1, \ldots\}\), the states of the zero-range process, so that \(\xi(x), x \in \mathbb{Z}\), stands for the total number of particles at site \(x\) for the configuration \(\xi\).

Fix an integer \(N \geq 1\), scale space by \(N^{-1}\) and assume that the zero-range process rescaled diffusively, \(\{\xi^N_t : t \geq 0\}\), starts from a local equilibrium state with density profile \(\rho_0 : \mathbb{R} \rightarrow \mathbb{R}_+\). Denote by \(\{\pi^N_{t,0} : t \geq 0\}\) its empirical measure. It is well known that \(\pi^N_{t,0}\) converges in probability to the absolutely continuous measure \(\rho(t, u) du\), where \(\rho(t, u)\) is the solution of a non-linear parabolic equation with initial condition \(\rho_0\).

Tag a particle initially at the origin and denote by \(X^N_t\) its position at time \(t\). It is relatively simple to show that the rescaled trajectory \(\{X^N_t/N : 0 \leq t \leq T\}\) is tight for the uniform topology. In particular, to prove convergence, one needs only to characterize the limit points.

In contrast with other models, in zero-range processes \(X^N_t\) is a square integrable martingale with a bounded quadratic variation \(\langle X^N \rangle_t\) given by the time integral of a local function of the process as seen from the tagged particle:

\[
\langle X^N \rangle_t = \sigma^2 N^2 \int_0^t \frac{g(\eta^N_s(0))}{\eta^N_s(0)} ds,
\]

where \(\sigma^2\) is the variance of the transition probability \(p(\cdot)\), \(g(\cdot)\) is the jump rate mentioned before, and \(\eta^N_s = \pi_{X^N_s}^{N,0}\) is the state of the process as seen from the tagged particle. Here \(\{\tau_x : x \in \mathbb{Z}\}\) stands for the group of translations. In particular, if the rescaled position of the tagged particle \(x^N_t = X^N_t/N\) converges to some path \(x_t\), this process \(x_t\) inherits the martingale property from \(X^N_t\). If in addition \(x_t\) is continuous, to complete the characterization, one needs to examine the asymptotic behavior of its quadratic variation.

Denote by \(\{\nu_\rho : \rho \geq 0\}\) the one-parameter family, indexed by the density, of invariant states for the process as seen from the tagged particle. Let \(\pi^N_t\) be the empirical measure associated to this process: \(\pi^N_t = \tau_{X^N_t} \pi^{N,0}_t\) and suppose that one can replace the local function \(g(\eta^N_s(0))/\eta^N_s(0)\) by a function of the empirical measure. If we assume conservation of local equilibrium for the process as seen from the tagged particle, this function should be \(\bar{h}(\lambda(s,0))\), where \(\bar{h}(\rho)\) is the expected value of \(g(\eta(0))/\eta(0)\) under the invariant state \(\nu_\rho\) and \(\lambda(s,0)\) is the density of particles around the tagged particle, i.e., the density of particles around the origin for the system as seen from the tagged particle.
As we are assuming that \( X^N_t / N \) converges to \( x_t \), since \( \pi^N_t = \tau_X \pi^{N,0}_t \) and \( \pi^{N,0}_t \) converges to \( \rho(t,u)du \), we must have \( \lambda(s,0) = \rho(s,x_s) \). Therefore, if the quadratic variation of \( X^N_t / N \) converges to the quadratic variation of \( x_t \), \( \langle x \rangle_t = \sigma^2 \int_0^t h(\rho(s,x_s))ds \). In particular, by the characterization of continuous martingales, \( x_t \) satisfies the stochastic differential equation

\[
\mathrm{d}x_t = \sigma \sqrt{h(\rho(s,x_s))} \, dB_t,
\]

where \( \rho \) is the solution of the hydrodynamic equation, \( h \) is defined above and \( B \) is a Brownian motion.

We see from this sketch that the main difficulty consists in proving the conservation of local equilibrium around the tagged particle, without assuming any type of attractiveness, which is relied upon in \cite{7}. The absence of a space average creates a major obstacle in this step. In contrast with the proof of the hydrodynamic limit, we need to replace a local function instead of a space average of translations of a local function. We may, therefore, only use the bonds close to the origin of the Dirichlet form to perform the replacement and we may not exclude large densities of particles close to the origin. In particular, all estimates (equivalence of ensembles and local central limit theorems) need to be uniform over the density. This lack of translation invariance confines us to one-dimension.

The method presented here may apply to other one-dimensional mean-zero interacting particle systems. However, instead of replacing a local function by a function of the empirical measure, one will need to replace a current multiplied by \( \rho \) by a function of the empirical measure, as what it is done for non-gradient systems, but without any space average.

2. Notation and Results

We consider one-dimensional zero-range processes with periodic boundary conditions to avoid unnecessary technicalities. This process is a system of random walks on the discrete torus \( T_N = \mathbb{Z}/NZ \) where particles interact infinitesimally only when they are at the same site. Fix a rate function \( g : \mathbb{N}_0 = \{0, 1, \ldots \} \to \mathbb{R}_+ \) with \( g(0) = 0 \), \( g(k) > 0 \), \( k \geq 1 \), and a finite range probability measure \( p(\cdot) \) on \( \mathbb{Z} \). The particle dynamics is described as follows. If there are \( k \) particles at a site \( x \), one of these particles jumps to site \( y \) with an exponential rate \( (g(k)/k)p(y-x) \).

For simplicity, we assume that \( p(\cdot) \) is symmetric, but our results remain true, with straightforward modifications, for any irreducible, finite-range, mean-zero transition probability \( p(\cdot) \). For the rate function \( g \), we assume the next conditions:

\[
(LG) \quad \exists a_1 > 0 \text{ such that } |g(n+1) - g(n)| \leq a_1 \text{ for } n \geq 0, \\
(M) \quad \exists a_0 > 0, b \geq 1, \text{ such that } g(n+b) - g(n) > a_0 \text{ for } n \geq 0.
\]

A consequence of (LG), (M) is that \( g \) is bounded between two linear slopes: There is a constant \( 0 < a < \infty \) such that \( a^{-1}n \leq g(n) \leq an \) for all \( n \geq 0 \).

Denote by \( \Omega_N = \mathbb{N}_0^{T_N} \) the state space and by \( \xi \) the configurations of \( \Omega_N \) so that \( \xi(x), x \in T_N \), stands for the number of particles in site \( x \) for the configuration \( \xi \). The zero-range process is a continuous-time Markov chain \( \xi_t \) generated by

\[
(L_N f)(\xi) = \sum_{x \in T_N} \sum_{z \in \mathbb{Z}} p(z) g(\xi(x)) \left[ f(\xi^x, x+z) - f(\xi) \right], \tag{2.1}
\]
where \( \xi^{x:y} \) represents the configuration obtained from \( \xi \) by displacing a particle from \( x \) to \( y \):

\[
\xi^{x:y}(z) = \begin{cases} 
\xi(x) - 1 & \text{for } z = x \\
\xi(y) + 1 & \text{for } z = y \\
\xi(z) & \text{for } z \neq x, y.
\end{cases}
\]

Now consider an initial configuration \( \xi \) such that \( \xi(0) \geq 1 \). Tag one of the particles initially at the origin, and follow its trajectory \( X_t \) jointly with the evolution of the process \( \xi_t \). Specially convenient for our purposes is to consider the process as seen by the tagged particle defined by \( \eta_t(x) = \xi_t(x + X_t) \). This process is again Markovian, now on the set \( \Omega_N^\ast = \{ \eta \in \Omega_N; \eta(0) \geq 1 \} \) and generated by the operator \( L_N = L_N^{nv} + L_N^{ip} \), where \( L_N^{nv} \) are defined by

\[
(L_N^{nv} f)(\eta) = \sum_{x \neq 0} \sum_{z \in \mathbb{Z}} p(z) \left( \eta^{x} \cdot \eta^{x+z} - f(\eta) \right)
+ \sum_{z \in \mathbb{Z}} \eta(0) \left( \eta^{0} \cdot \eta_{z} - f(\eta) \right),
\]

\[
(L_N^{ip} f)(\eta) = \sum_{z \in \mathbb{Z}} p(z) \left( \eta(0) \left( \theta_z \eta \right) - f(\eta) \right).
\]

In this formula, the translation \( \theta_z \) is defined by

\[
(\theta_z \eta)(x) = \begin{cases} 
\eta(x+z) & \text{for } x \neq 0, -z \\
\eta(z) + 1 & \text{for } x = 0 \\
\eta(0) - 1 & \text{for } x = -z.
\end{cases}
\]

The operator \( L_N^{ip} \) corresponds to jumps of the tagged particle, while \( L_N^{nv} \) corresponds to jumps of the other particles, called environment.

In order to recover the position of the tagged particle from the evolution of the process \( \eta_t \), let \( N_t^z \) be the number of translations of length \( z \) up to time \( t \): \( N_t^z = N_t^{z} + 1 \Leftrightarrow \eta_t = \theta_z \eta_{t-} \). In this case, \( X_t = \sum z N_t^z \). As jumps are not simultaneous, the processes

\[
N_t^z - \int_0^t p(z) \frac{g(\eta_s(0))}{\eta_s(0)} \, ds
\]

are orthogonal martingales and, as \( \sum z p(z) = 0 \), we see that \( X_t \) is a martingale with quadratic variation

\[
\langle X \rangle_t = \sigma^2 \int_0^t \frac{g(\eta_s(0))}{\eta_s(0)} \, ds,
\]

where \( \sigma^2 = \sum z |z|^2 p(z) \).

We now discuss the invariant measures. For each \( \varphi \geq 0 \), consider the product probability measures \( \bar{\mu}_\varphi = \bar{\mu}_N \cdot \varphi \) in \( \Omega_N \) defined by

\[
\bar{\mu}_\varphi(\xi(x) = k) = \frac{1}{Z(\varphi)} \varphi^k g(k)!,
\]

where \( g(k)! = g(1) \cdots g(k) \) for \( k \geq 1 \), \( g(0)! = 1 \) and \( Z(\varphi) \) is the normalization constant. \( Z(\varphi) \) and \( \bar{\mu}_\varphi \) are well defined for all \( \varphi \geq 0 \) due to conditions (LG), (M). Let \( \rho = \rho(\varphi) = \int \eta(0) d\bar{\mu}_\varphi \). By conditions (LG), (M), \( \varphi \mapsto \rho \) is a diffeomorphism from \([0, \infty)\) into itself. Define then \( \mu_\rho = \bar{\mu}_{\rho(\varphi)} \), since \( \rho \) corresponds to the density
of particles at each site. The measure \( \{ \mu_\rho : \rho \geq 0 \} \) are invariant for the process \( \xi_t \) (cf. [1]).

Due to the inhomogeneity introduced at the origin by the tagged particle, \( \mu_\rho \) is no longer invariant for the process \( \eta_t \). However, a computation shows that the size biased measures \( \nu_\rho \) defined by \( d

\nu_\rho / d\mu_\rho = \eta(0)/\rho \) are invariant for the process as seen by the tagged particle, reversible when \( \rho(\cdot) \) is symmetric. Here, we take \( \nu_0 = \delta_{0_0} \), the Dirac measure concentrated on the configuration \( 0_0 \) with exactly one particle at the origin, and note \( \nu_\rho \Rightarrow \delta_{0_0} \) as \( \rho \downarrow 0 \).

From now on, to avoid uninteresting compactness issues, we define every process in a finite time interval \([0, T]\), where \( T < \infty \) is fixed. Let \( T \) be the unit torus and let \( \mathcal{M}_+(T) \) be the set of positive Radon measures in \( T \).

Consider the process \( \xi_t^N =: \xi_{tN^2} \), generated by \( \nu^N \). Define the process \( \pi_t^N,0 \) in \( \mathcal{D}([0, T], \mathcal{M}_+(T)) \) as

\[
\pi_t^N,0(du) = \frac{1}{N} \sum_{x \in \mathbb{T}_N} \xi^N_t(x) \delta_{x/N}(du),
\]

where \( \delta_u \) is the Dirac distribution at point \( u \).

For a continuous function \( \rho_0 : T \to \mathbb{R}_+ \), define \( \mu^N_{\rho_0(\cdot)} \) as the product measure in \( \Omega_N \) given by \( \mu^N_{\rho_0(\cdot)}(\eta(x) = k) = \mu^N_{\rho_0(x/N)}(\eta(x) = k) \). The next result is well known (cf. Chapter V [6]; see also [2], [3]).

**Theorem 2.1.** For each \( 0 \leq t \leq T \), \( \pi_t^N,0 \) converges in probability to the deterministic measure \( \rho(t, u)du \), where \( \rho(t, u) \) is the solution of the hydrodynamic equation

\[
\begin{cases}
\partial_t \rho = \sigma^2 \partial_x^2 \varphi(\rho) \\
\rho(0, u) = \rho_0(u),
\end{cases}
\]

and \( \varphi(\rho) = \int g(\xi(0))d\mu_\rho \).

Define now the product measure \( \nu^N = \nu^N_{\rho_0(\cdot)} \) in \( \Omega^*_N \) given by \( \nu^N_{\rho_0(\cdot)}(\eta(x) = k) = \nu^N_{\rho_0(x/N)}(\eta(x) = k) \), and let \( \eta^N =: \eta^N_{N^2} \) be the process generated by \( \nu^N \), starting from the initial measure \( \nu^N \). Define the empirical measure \( \pi_t^N \) in \( \mathcal{D}([0, T], \mathcal{M}_+(T)) \) by

\[
\pi_t^N(du) = \frac{1}{N} \sum_{x \in \mathbb{T}_N} \eta^N_t(x) \delta_{x/N}(du).
\]

Define also the continuous function \( \psi : \mathbb{R}_+ \to \mathbb{R}_+ \) by

\[
\psi(\rho) = \int (g(\eta(0)))/\eta(0))d\nu_\rho = \begin{cases}
\varphi(\rho)/\rho & \text{for } \rho > 0 \\
g(1) & \text{for } \rho = 0.
\end{cases}
\]

The next theorems are the main results of this article. We first identify the scaling limit of the tagged particle as a diffusion process:

**Theorem 2.2.** Let \( x^N_t = X^N_t/N \) be the rescaled position of the tagged particle for the process \( \xi^N_t \). Then, \( \{x^N_t : t \in [0, T]\} \) converges in distribution in the uniform topology to the diffusion \( \{x_t : t \in [0, T]\} \) defined by the stochastic differential equation

\[
dx_t = \sigma \sqrt{\psi(\rho(t, x_t))} dB_t,
\]

where \( B_t \) is a standard Brownian motion on \( T \).
Through this characterization we can describe the evolution of the empirical measure as seen from the tagged particle:

**Theorem 2.3.** \( \{ \pi_t^N : t \in [0, T] \} \) converges in distribution on \( D([0, T], \mathcal{M}_+(\mathbb{T})) \) to the measure-valued process \( \{ \rho(t, u + x_t) du : t \in [0, T] \} \), where \( \rho(t, u) \) is the solution of the hydrodynamic equation (2.2) and \( x_t \) is given by (2.8).

Recall that \( \eta_0^N \) is distributed according to \( \nu_{\rho_0(\cdot)}^N \). Denote by \( \mathbb{P}^N \) the probability measure in \( D([0, T], \Omega_N^\rho) \) induced by the process \( \eta_t^N \), and by \( \mathbb{E}^N \) the expectation with respect to this process. Denote also by \( \mathbb{E}_\mu[h] \) and \( \langle h \rangle_\mu \) the expectation of a function \( h : \Omega_N \to \mathbb{R} \) with respect to the measure \( \mu \); when \( \mu = \nu_\rho \), let \( \mathbb{E}_\rho[h], \langle h \rangle_\rho \) stand for \( \mathbb{E}_{\nu_\rho}[h], \langle h \rangle_{\nu_\rho} \). Finally, since in the next sections we consider only the speeded-up process \( \eta_t^N \) we omit hereafter the superscript \( N \).

The plan of the paper is now the following. After some tightness estimates in Section 3 certain limits are established in Theorem 4.1 in Section 4—with the aid of “global” and “local” hydrodynamics results in Sections 5 and 6—which give the main Theorems 2.2 and 2.3.

3. Tightness

To keep notation simple, in this section we assume the transition probability \( p(\cdot) \) to be nearest neighbor:

\[
p(1) = p(-1) = 1/2
\]

so that \( \sigma^2 = 1 \). Denote by \( C(\mathbb{T}) \) the space of real continuous functions on \( \mathbb{T} \) and by \( C^2(\mathbb{T}) \) the space of twice continuously differentiable functions on \( \mathbb{T} \). For a function \( G \) in \( C(\mathbb{T}) \), denote by \( \pi_t^N(G) \) the integral of \( G \) with respect to \( \pi_t^N \):

\[
\pi_t^N(G) = \int G(u)\pi_t^N(du) = \frac{1}{N} \sum_{x \in \mathbb{T}_N} G(x/N)\eta_t^N(x).
\]

For \( T > 0 \), denote by \( D_T = D([0, T], \mathcal{M}_+(\mathbb{T}) \times \mathcal{M}_+(\mathbb{T}) \times \mathbb{T} \times \mathbb{R}_+) \) the path space of càdlàg trajectories endowed with the Skorohod topology. For \( N \geq 1 \), let \( Q_N \) be the probability measure on \( D_T \) induced by the process \( \{ \pi_0^{N,0}, \pi_t^N, x^N_t, (x^N)_t \} \), where \( \langle x^N \rangle_t \) stands for the quadratic variation of the martingale \( x^N_t \). We prove in this section that the sequence \( \{ Q_N : N \geq 1 \} \) is tight, which follows from the tightness of each component of \( \{ \pi_0^{N,0}, \pi_t^N, x^N_t, (x^N)_t \} \).

Let \( Q_1^N \) be the probability measure in \( D([0, T], \mathcal{M}_+(\mathbb{T})) \) corresponding to the process \( \pi_1^{N,0} \). As mentioned in Theorem 2.1, \( Q_1^N \) converges to the Dirac-\( \delta \) measure concentrated on the path \( \rho(t, u)du \), where \( \rho \) is the solution of (2.2). Hence, the sequence \( \{ Q_1^N : N \geq 1 \} \) is tight.

On the other hand, as \( \mathcal{M}_+(\mathbb{T}) \) is a metrizable space under the dual topology of \( C(\mathbb{T}) \), to show that \( \{ \pi_1^N : N \geq 1 \} \) is tight, it is enough to prove tightness of the projections \( \{ \pi_t^N(G) : N \geq 1 \} \) for a suitable set of functions \( G \), dense in \( C(\mathbb{T}) \). For \( G \) in \( C(\mathbb{T}) \), let \( Q_G^N \) be the measure in \( D([0, T], \mathbb{R}) \) corresponding to the process \( \{ \pi_t^N(G) : 0 \leq t \leq T \} \). Tightness of the sequence \( \{ Q_G^N : N \geq 1 \} \) follows from Aldous’s criteria in the next lemma.

**Lemma 3.1.** The sequence \( \{ Q_G^N : N \geq 1 \} \) is tight if

(i) For every \( t \in [0, T] \) and every \( \epsilon > 0 \), there exists \( M > 0 \) such that

\[
\sup_N \mathbb{P}^N \left[ |\pi_t^N(G)| > M \right] < \epsilon.
\]
Let $\mathcal{T}_T$ be the set of stopping times bounded by $T$. Then, for every $\epsilon > 0$,
\[
\lim_{\gamma \to 0} \limsup_{N \to \infty} \sup_{\tau \in \mathcal{T}_T, \theta \leq \gamma} \mathbb{P}^N \left[ |\pi^{N}_{\tau + \theta}(G) - \pi^{N}_\tau(G)| > \epsilon \right] = 0.
\]

Lemma 3.2. The sequence $\{Q^G_N : N \geq 1\}$, $G \in C^2(T)$, is tight.

Proof. An elementary computation shows that for each $G$ in $C(T)$,
\[
M^t_{N,G} = \pi^{N}_t(G) - \pi^0_N(G) - \int_0^t \frac{1}{N} \sum_{x \in \mathbb{T}_N} (\Delta_N G)(x/N) g(\eta_s(x)) \, ds
\]
\[
- \int_0^t \frac{g(\eta_s(0))}{\eta_s(0)} \pi^{N}_s(\Delta_N G) \, ds + \int_0^t \frac{2}{N} \frac{g(\eta_s(0))}{\eta_s(0)} (\Delta_N G)(0) \, ds
\]
is a martingale of quadratic variation $\langle M^{N,G} \rangle_t$ given by
\[
\langle M^{N,G} \rangle_t = \frac{1}{N^2} \int_0^t \sum_{z \in \mathbb{Z}} p(z) g(\eta_s(x)) [\nabla_{N,z} G(x/N)]^2 \, ds
\]
\[
+ \frac{1}{N^2} \int_0^t \sum_{z \in \mathbb{Z}} p(z) g(\eta_s(0)) \eta_s(0) - \frac{1}{\eta_s(0)} [\nabla_{N,z} G(0)]^2 \, ds
\]
\[
+ \int_0^t \sum_{z \in \mathbb{Z}} p(z) \frac{g(\eta_s(0))}{\eta_s(0)} \left( \frac{1}{N} \sum_{x \in \mathbb{T}_N} (\nabla_{N,z} G)(x/N) \eta_s(x) - \frac{1}{N} (\nabla_{N,z} G)(0) \right)^2 \, ds.
\]
In these formulas, $\nabla_{N,z} G$, $\Delta_N G$ correspond to the discrete first and second derivatives of $G$:
\[
(\nabla_{N,z} G)(u) = N[G(u + z/N) - G(u)],
\]
\[
(\Delta_N G)(u) = N^2 \sum_{z \in \mathbb{Z}} p(z) [G(u + z/N) - G(u)].
\]

Since the rate function $g$ grows at most linearly and since the total number of particles is preserved by the dynamics,
\[
\langle M^{N,G} \rangle_t = \int_0^t \sum_{z \in \mathbb{Z}} p(z) \frac{g(\eta_s(0))}{\eta_s(0)} \left( \frac{1}{N} \sum_{x \in \mathbb{T}_N} \nabla_{N,z} G(x/N) \eta_s(x) \right)^2 \, ds + R^N_t(G),
\]
where $|R^N_t(G)| \leq C_0 N^{-2} \sum_{x \in \mathbb{T}_N} \eta_0(x)$ and $C_0$ is a finite constant which depends only on $G$, $g$, $p$ and $T$. In particular, $\mathbb{E}^N \|R^N_t(G)\| \leq C_1 N^{-1}$.

Note that in contrast with the martingale associated to empirical measure $\pi_{t,0}^N$, due to the jumps of the tagged particle, the martingale $M^{N,G}$ does not vanish in $L^2(\mathbb{P}^N)$. In particular, we may not expect the convergence of the empirical measure $\pi_{t}^N$ to a deterministic trajectory.

We are now in a position to prove the lemma. Condition (i) of Lemma 3.1 is a direct consequence of the conservation of the total number of particles. In order to prove condition (ii), recall the decomposition (3.1) of $\pi_t^N(G)$ as an integral term plus a martingale. The martingale term can be estimated by Chebychev's inequality.
and the explicit form of its quadratic variation:
\[
\mathbb{E}^N \left[ |M^N_{t+\theta} - M^N_t|^2 \right] \leq \frac{1}{\epsilon^2} \mathbb{E}^N \left[ (M^N_{t+\theta} - M^N_t)^2 \right] \\
\leq \frac{C_1}{\epsilon^2} \left| G' \right|^2 \mathbb{E}^N \left[ \int_t^{t+\theta} \left( \frac{1}{N} \sum_{x \in \mathbb{T}_N} \eta(x) \right) ds \right] + \frac{C_1}{\epsilon^2 N} \left| G' \right|^2 \theta \mathbb{E}^N_{\nu_\theta(\cdot)} \left[ \left( \frac{1}{N} \sum_{x \in \mathbb{T}_N} \eta(x) \right)^2 \right] + \frac{C_1}{\epsilon^2 N}
\]
which converges to 0 as $N \uparrow \infty$ and $\gamma \downarrow 0$. The integral term can be estimated in the same way, using again the conservation of the total number of particles. This proves the lemma.

It remains to consider the scaled position of the tagged particle $x_t^N$ and its quadratic variation. We recall that $x_t^N$ is a martingale with quadratic variation
\[
\langle x^N \rangle_t = \int_0^t \frac{g(\eta^N_t(0))}{\eta^N_t(0)} - ds.
\]

**Lemma 3.3.** The process $\{(x^N, \langle x^N \rangle) : N \geq 1\}$ is tight for the uniform topology.

**Proof.** We need to show that
\[
\lim_{\epsilon \to 0} \limsup_{N \to \infty} \mathbb{E}^N \left[ \sup_{|t-s| \leq \epsilon} |x^N_t - x^N_s| > \delta \right] = 0
\]
for all $\delta > 0$ and a similar statement for the quadratic variation $\langle x^N \rangle_t$. Recall that $\sup_{k \geq 1} g(k)/k \leq a < \infty$ and consider a symmetric random walk $Z^N_t$ on the discrete torus $\mathbb{T}_N$ with jump rate $a$ and transition probability $p(\cdot)$. We may couple $Z^N_t$ and $X^N_t$ in such a way that the skeleton chains are equal, i.e., that the sequence of sites visited by both processes are the same, and the holding times of $Z^N_t$ are always less than or equal to the holding times of $X^N_t$. In particular,
\[
\sup_{|t-s| \leq \epsilon} |x^N_t - x^N_s| \leq \sup_{|t-s| \leq \epsilon} |z^N_t - z^N_s|
\]
if $z^N_t = Z^N_t/N$. Therefore, (3.3) follows from the tightness in the uniform topology of a rescaled symmetric random walk.

Tightness of the quadratic variation $\langle x^N \rangle_t$ in the uniform topology is an elementary consequence of its explicit expression (3.2) and the boundedness of $g(k)/k$. \hfill \Box

4. Limit Points and Proof of Theorems 2.2, 2.3

The following, which characterizes certain limit points, is the main result of this section, which yields Theorems 2.2 and 2.3

**Theorem 4.1.** The sequence $Q_N$ converges in the Skorohod topology to the law $Q$ concentrated on trajectories $\{\{\pi^N_t, \pi_t, x_t, A_t\} : 0 \leq t \leq T\}$ such that $\pi^N_t(du) = \rho(t, u)du$, where $\rho$ is the unique weak solution of \[2.2\]: $\pi_t(du) = \rho(t, x_t+u)du$ and $A_t = \sigma^2 \int_0^t \psi(\rho(s, x_s)) ds$.

**Proof of Theorems 2.2 and 2.3.** As the limit $x_t$ is concentrated on continuous paths, Theorem 4.1 straightforwardly implies Theorems 2.2 and 2.3. \hfill \Box
The proof of Theorem 4.1 is now divided in a sequence of lemmata. Denote by \( \{ \tau_u : u \in \mathbb{T} \} \) the group of translations in \( \mathbb{T} \) acting on points, functions and measures.

**Lemma 4.2.** All limit points \( Q \) of the sequence \( \{ Q_N : N \geq 1 \} \) are concentrated on trajectories \( \{ (\pi^0_t, \pi_t, x_t, A_t) : 0 \leq t \leq T \} \) in which \( x_t \) is a continuous square integrable martingale.

**Proof.** Assume, without loss of generality, that \( Q_N \) converges to \( Q \). Since, by Lemma 3.3 \( \{ x^N : N \geq 1 \} \) is tight for the uniform topology, \( Q \) is concentrated on continuous paths \( x_t \). In particular, \( x^N_t \) converges in law to \( x_t \) for all \( 0 \leq t \leq T \).

The martingale property is inherited by \( x_t \) because \( x^N_t \) converges in law to \( x_t \) and

\[
\mathbb{E}^N \left[ (x^N_t)^2 \right] = \mathbb{E}^N \left[ \sigma^2 \int_0^t \frac{g(\eta_u(0))}{\eta_t(0)} \, ds \right] \leq a\sigma^2 t
\]

uniformly in \( N \). Therefore, \( x_t \) is a square integrable martingale relative to its natural filtration. \( \Box \)

**Lemma 4.3.** All limit points \( Q \) of the sequence \( \{ Q_N : N \geq 1 \} \) are concentrated on trajectories \( \{ (\pi^0_t, \pi_t, x_t, A_t) : 0 \leq t \leq T \} \) in which \( \pi_t^0(du) = \rho(t, u)du \), where \( \rho \) is the unique weak solution of (2.2) and \( \pi_t^0(du) = \tau_x, \pi_t^0(du) = \rho(t, x_t)du \).

**Proof.** Assume, without loss of generality, that \( Q_N \) converges to \( Q \). The first statement follows from Theorem 2.1. On the other hand, by Lemma 3.2 and since \( \rho \) is continuous, \( Q \) is concentrated on continuous trajectories \( \{ (\pi^0_t, x_t) : 0 \leq t \leq T \} \). Hence, all finite dimensional distributions (f.d.d.) of \( (\pi_t^{0,N}, x^N_t) \) (and therefore of \( \tau_{x_t} \pi_t^{0,N} \)) converge to the f.d.d. of \( (\pi_t^0, x_t) \) (and since the f.d.d. characterize a measure on \( \mathcal{D}([0, T]) \), the lemma is proved. \( \Box \)

For \( \varepsilon > 0 \), denote \( \varepsilon = \varepsilon^{-1}1\{(0, \varepsilon]\}(u) \) and \( \alpha = (2\varepsilon)^{-1}1\{(-\varepsilon, \varepsilon]\}(u) \). For \( l \geq 1 \) and \( x \in \mathbb{Z} \), denote by \( \eta^l(x) \) the mean number of particles in a cube of length \( 2l + 1 \) centered at \( x \in \Omega_N \) at time \( s \geq 0 \):

\[
\eta^l(x) = \frac{1}{2l+1} \sum_{|y-x| \leq l} \eta_u(y).
\]

When \( s = 0 \), we drop the suffix “\( s \)” for simplicity.

A function \( h : \Omega_N \rightarrow \mathbb{R} \) is said to be local if it depends only on a finite number of sites. For a local, bounded function \( h : \Omega_N \rightarrow \mathbb{R} \), denote by \( H(\rho) \) and \( \tilde{h}(\rho) \) its expectations with respect to \( \nu_\rho \) and \( \mu_\rho \), respectively. Thus, \( H, \tilde{h} : \mathbb{R}_+ \rightarrow \mathbb{R} \) are the functions defined by

\[
H(\rho) = E_{\nu_\rho}[h(\eta)], \quad \text{and} \quad \tilde{h}(\rho) = E_{\mu_\rho}[h(\xi)].
\]

Also, define for \( l \geq 1 \) the local function \( H_l : \Omega_N \rightarrow \mathbb{R} \) given by

\[
H_l(\eta) = H(\eta^l(0)).
\]

Then, \( \tilde{H}_l : \mathbb{R}_+ \rightarrow \mathbb{R} \) is the function \( \tilde{H}_l(\rho) = E_{\mu_\rho}[H_l] \).

A local function \( h : \Omega_N \rightarrow \mathbb{R} \) is said to be Lipschitz if there exists a finite subset \( A \) of \( \mathbb{Z} \) and a finite constant \( C_0 \) such that

\[
|h(\xi) - h(\xi')| \leq C_0 \sum x \in A |\xi(x) - \xi'(x)|
\]
for all configurations $\xi, \xi' \in \Omega_N$.

Consider in particular the local function $h_0(\eta(0)) = g(\eta(0))/\eta(0)$. It follows from assumptions (LG), (M) that $h_0(\cdot)$ is a Lipschitz function, bounded above by a finite constant and below by a strictly positive constant.

We now characterize the quadratic variation of $x_t$.

**Lemma 4.4.** All limit points $Q$ of the sequence $\{Q_N : N \geq 1\}$ are concentrated on trajectories $\{(\pi_0^t, \pi_t, x_t, A_t) : 0 \leq t \leq T\}$ such that

$$A_t = \sigma^2 \int_0^t \psi(s, \sigma s) \, ds$$

for all $0 \leq t \leq T$. Moreover, $A_t$ is the quadratic variation of the martingale $x_t$.

*Proof.* Assume, without loss of generality, that $Q_N$ converges to $Q$. Since $\langle x^N \rangle_t$ is tight for the uniform topology by Lemma 4.3, $\langle x^N \rangle_t$ converges to a limit $A_t$ for all $0 \leq t \leq T$. By Proposition 6.1, with respect to $h(\eta(0)) = g(\eta(0))/\eta(0)$ and $H(\rho) = \psi(\rho)$, and since for each $0 \leq t \leq T$ the map $\pi \to \int_0^t ds \int \psi(x) \psi(t, x) \, dx$ is continuous for the Skorohod topology,

$$\lim_{l \to \infty} \lim_{\delta \to 0} \left| A_t - \sigma^2 \int_0^t ds \int \psi(x) \psi(t, x) \, dx \right| T \leq \delta = 0$$

for all $0 \leq t \leq T$ and $\delta > 0$. By Lemma 4.3, $\pi_l(du) = \rho(t, x_l + u) \, du$. Also, $\rho(s, \cdot)$ is continuous for $0 \leq s \leq T$, and $\psi(t, x) \to \psi(t)$ as $l \to \infty$ by bounded convergence. Then, as $\epsilon \downarrow 0$, $\epsilon \downarrow 0$, and $l \to \infty$, we have a.s.

$$\int \psi(t, x) \int_0^t \psi(t, x) \, dx = \int_0^t \psi(t, x) \, dx \to \int_0^t \psi(t, x) \, dx.$$

It remains to show that $A_t$ corresponds to the quadratic variation of the square integrable martingale $x_t$. By [5, Corollary VI.6.6], $\{(x_t^N, \langle x^N \rangle_t) : 0 \leq t \leq T\}$ converges in law to $\{(x_t, \langle x \rangle_t) : 0 \leq t \leq T\}$. Since by the first part of the lemma, $\{(x_t^N, \langle x^N \rangle_t) : 0 \leq t \leq T\}$ converges to $\{(x_t, \langle x \rangle_t) : 0 \leq t \leq T\}$, $\langle x \rangle_t = A_t$. This concludes the proof of the lemma.

Recall that the quadratic variation $\langle x \rangle_t$ of a martingale $x_t$ is equal to $x_t^2 - x_0^2 - 2 \int_0^t x_s \, ds$ and that $\langle x \rangle_t$ can be approximated in $L^2$ by the sequence of Riemannian sums $\sum_{j=1}^{M} (x_{t_{j+1}} - x_{t_j})^2$, as the mesh of a partition $\{t_j : 1 \leq j \leq M\}$ of the interval $[0, t]$ vanishes. In particular, one can prove directly in our context the identity between $A_t$ and the quadratic variation $\langle x \rangle_t$.

It follows from the characterization of continuous martingales that $x_t$ is a time-changed Brownian motion.

**Corollary 4.5.** The rescaled position of the tagged particle $\{x_t^N : 0 \leq t \leq T\}$ converges in law to the solution of the stochastic differential equation

$$dx_t = \sigma \sqrt{\psi(t, x_t)} \, dB_t,$$

where $B_t$ is a Brownian motion and $\psi$ is the solution of the differential equation (2.2).

**Proof of Theorem 4.1** By Section 4.3, the sequence $Q^N$ is tight. On the other hand, by Lemma 4.3 and Corollary 4.5, the law of the first and the third components of the vector $(\pi_0^t, \pi_t, x_t, A_t)$ are uniquely determined. Since, by Lemmatta...
the distribution of the second and fourth components are characterized by
the distribution of \( x_t \), and \( \rho(t, x) \), the theorem is proved.

5. Global replacement lemma

In this section, we replace the full empirical average of a local, bounded and
Lipschitz function in terms of its density field. The proof involves only a few
modifications of the standard hydrodynamics proof of [6, Lemma V.1.10, Lemma
V.5.5].

Proposition 5.1 (Global replacement). Let \( r : \Omega_N \to \mathbb{R} \) be a local, bounded and
Lipschitz function. Then, for every \( \delta > 0 \),

\[
\lim_{\varepsilon \to 0} \limsup_{N \to \infty} \mathbb{P}_N \left[ \int_0^T \frac{1}{N} \sum_{x \in T_N} \tau_x \mathcal{V}_N(\eta) \, ds \geq \delta \right] = 0,
\]

where

\[
\mathcal{V}_l(\eta) = \left| \frac{1}{2l + 1} \sum_{|y| \leq l} \tau_y r(\eta) - \bar{r}(\eta(0)) \right|,
\]

and \( \bar{r}(a) = \mathbb{E}_\mu a[r] \).

For two measures \( \mu, \nu \) defined on \( \Omega_N \) (or \( \Omega_N^* \)), denote by \( \mathcal{H}(\mu|\nu) \) the entropy of
\( \mu \) with respect to \( \nu \):

\[
\mathcal{H}(\mu|\nu) = \sup_f \left\{ \int f \, d\mu - \log \int e^f \, d\nu \right\},
\]

where the supremum is carried over all bounded continuous functions \( f \).

A simple computation shows that the initial entropy \( \mathcal{H}(\nu_{\rho(\cdot)}|\nu_\rho) \) is bounded by
\( C_0 N \) for some finite constant \( C_0 \) depending only on \( \rho_0(\cdot) \) and \( g \). Let \( f_t^N(\eta) \) be the
density of \( \eta_t \) under \( \mathbb{P}_N \) with respect to a reference measure \( \nu_\rho \) for \( \rho > 0 \), and let
\( \hat{f}_t^N(\eta) = t^{-1} \int_0^t f_s^N(\eta) \, ds \). By standard arguments (cf. Section V.2 [6]),

\[
\mathcal{H}_N(\hat{f}_t^N) := \mathcal{H}(\hat{f}_t^N \, d\nu_\rho|\nu_\rho) \leq C_0 N \quad \text{and} \quad \mathcal{D}_N(\hat{f}_t^N) := \left\langle \int \hat{f}_t^N(\eta) \, d\nu_\rho \right\rangle \leq \frac{C_0}{N}.
\]

Consequently, by Chebyshev inequality, to prove Proposition 5.1 it is enough to show, for all finite constants \( C \), that

\[
\lim_{\varepsilon \to 0} \limsup_{N \to \infty} \sup_{\mathcal{H}_N(f) \leq C, \mathcal{D}_N(f) \leq C/N} \int \frac{1}{N} \sum_{x \in T_N} \tau_x \mathcal{V}_N(\eta) f(\eta) \, d\nu_\rho = 0
\]

where the supremum is with respect to \( \nu_\rho \)-densities \( f \). Notice that we may remove
from the sum the integers \( x \) close to the origin, say \( |x| \leq 2\varepsilon N \), because \( \mathcal{V}_N \) is bounded. After removing these sites, we are essentially in the space homogeneous
case. Proposition 5.1 follows from the two standard lemmata below as in the proof
of [6, Lemma V.1.10].

Lemma 5.2 (Global 1-block estimate).

\[
\lim_{k \to \infty} \limsup_{N \to \infty} \sup_{\mathcal{H}_N(f) \leq C, \mathcal{D}_N(f) \leq C/N} \int \frac{1}{N} \sum_{|x| > 2\varepsilon N} \tau_x \mathcal{V}_k(\eta) f(\eta) \, d\nu_\rho = 0.
\]
Lemma 5.3 (Global 2-block estimate).

\[
\lim sup \lim sup \lim sup \sup_{k \to \infty} \varepsilon \to 0 \quad N \to \infty \quad \mathcal{H}_N(f) \leq C N \\
\mathcal{D}_N(f) \leq C/N \quad \frac{1}{2N\varepsilon + 1} \sum_{|y| \leq N\varepsilon} \int \frac{1}{N} \sum_{|x| > 2N\varepsilon} |\eta^k(x + y) - \eta^k(x)| f(\eta) d\nu_\rho = 0.
\]

We now indicate the proofs of Lemmata 5.2 and 5.3 in relation to [6, Sections V.4, V.5].

Proofs of Lemmata 5.2 and 5.3. To be brief, we discuss only the proof of Lemma 5.2 through some modifications of the argument in [6, Section V.4], as the proof of Lemma 5.3, using the modifications for Lemma 5.2 given below, is on similar lines to that in [6, Section V.5].

In the first step of the 1-block estimate we cut-off high densities. We claim that

\[
\lim sup \lim sup \lim sup \sup_{A \to \infty} k \to \infty \quad N \to \infty \quad \mathcal{H}_N(f) \leq C N \\
\mathcal{D}_N(f) \leq C/N \quad \int \frac{1}{N} \sum_{|x| > 2N\varepsilon} \tau_x \mathcal{V}_k(\eta) 1\{|\eta^k(x) > A\} f(\eta) d\nu_\rho = 0.
\]

Since \( \mathcal{V}_k \) is bounded, we may replace it by a constant and estimate the indicator by \( A^{-1} \eta^k(x) \). After a summation by parts, the expression is easily shown to be less than or equal to \( C_0 (AN)^{-1} \sum_{x \neq 0} \eta(x) \) for some finite constant \( C_0 \). To conclude it remains to follow the proof of [6, Lemma V.4.1], applying the entropy inequality with respect to \( \nu_\rho \) and keeping in mind that the marginals of \( \mu_\rho \) and \( \nu_\rho \) coincide on sites \( x \neq 0 \).

Define now \( \mathcal{V}_{k,A}(\eta) = \mathcal{V}_k(\eta) 1\{|\eta^k(0) \leq A\} \). By the previous argument, it is enough to show that for every \( A > 0 \),

\[
\lim sup \lim sup \lim sup \sup_{k \to \infty} \quad N \to \infty \quad \mathcal{D}_N(f) \leq C/N \\
\mathcal{V}_{k,A}(\eta) = \mathcal{V}_k(\eta) f(\eta) d\nu_\rho = 0.
\]

(5.1)

The proof is analogous to the homogeneous case. Since the origin does not appear, both the Dirichlet form \( \mathcal{D}_N \) and the measure \( \nu_\rho \) coincide with the Dirichlet form of the space homogeneous zero-range process and the stationary state \( \mu_\rho \). In particular, all estimates needed involve only the functionals of the space-homogeneous process already considered in [6]. □

6. Local replacement lemma

In this section, we replace a bounded, Lipshitz function supported at the origin by a function of the empirical density.

Proposition 6.1 (Local replacement). For any bounded, Lipschitz function \( h : \mathbb{N}_0 \to \mathbb{R} \), and any \( t > 0 \),

\[
\lim sup \lim sup \lim sup \lim sup_{l \to \infty} \varepsilon \to 0 \quad \varepsilon \to 0 \quad N \to \infty \mathbb{P}^N \left[ \left| \int_0^t h(\eta_0(s)) - \frac{1}{\varepsilon N} \sum_{x=1}^{\varepsilon N} \bar{H}_l(\eta_0^N(x)) \, ds \right| \right] = 0,
\]

where \( H(\rho) = E_{\nu_\rho}[h] \), \( H_l(\eta) = H(\eta^l(0)) \), and \( \bar{H}_l(\rho) = E_{\mu_\rho}[H_l] \).

In the proof of this lemma, there are two difficulties. The first and the most important one is the absence of a spatial average, a crucial point in the standard one and two blocks estimates since it allows a cut-off of large densities and a reduction to translation-invariant densities in the estimation of the largest eigenvalue of a
local perturbation of the generator of the process. Without the density cut-off, the equivalence of ensembles, and therefore the local central limit theorem, has to be proved uniformly over all densities. Moreover, this absence of space average confines us to one-dimension.

A second obstacle is the lack of translation invariance of the stationary state, turning the origin into a special site. Functions \( h(\eta(0)) \) and \( h(\eta(x)) \), for instance, have different distributions. In particular, in contrast with the original zero-range process, the integral \( \int \{ g(\eta(0)) - g(\eta(x)) \} f d\rho \) cannot be estimated by the Dirichlet form of \( f \).

The proof of Proposition 6.1 is divided in several steps. We start with a spectral gap for the evolution of the environment restricted to a finite cube. For \( l \geq 1 \), denote by \( \Lambda_l \) a cube of length \( 2l + 1 \) around the origin: \( \Lambda_l = \{-l, \ldots, l\} \) and by \( L_{\Lambda_l}^{env} \) the restriction of the environment part of the generator to the cube \( \Lambda_l \):

\[
(L_{\Lambda_l}^{env} f)(\eta) = \sum_{x \in \Lambda_l} \sum_{y \in \Lambda_l} p(y - x) g(\eta(x)) \left[ f(\eta^x, y) - f(\eta) \right] + \sum_{z \in \Lambda_l} p(z) g(\eta(0)) \frac{\eta(0) - 1}{\eta(0)} \left[ f(\eta^{0,z}) - f(\eta) \right].
\]

We assume above, without loss of generality, that \( l \) is larger than the range of \( p(\cdot) \).

Let \( \nu_{\rho, l}^{\Lambda_l} \) be the measure \( \nu_{\rho} \) restricted to the set \( \Lambda_l \). For \( j \geq 1 \), denote by \( \Sigma_{\Lambda_l,j} \) the set of all configurations in \( \Lambda_l \) with at least one particle at the origin and \( j \) particles in \( \Lambda_l \), and by \( \nu_{\Lambda_l, j} \) the measure \( \nu_{\rho, l}^{\Lambda_l} \) conditioned to \( \Sigma_{\Lambda_l,j} \):

\[
\Sigma_{\Lambda_l,j} = \left\{ \eta \in \mathbb{N}_0^{\Lambda_l} : \eta(0) \geq 1, \sum_{x \in \Lambda_l} \eta(x) = j \right\}, \quad \nu_{\Lambda_l, j}(\cdot) = \nu_{\rho, l}^{\Lambda_l}(\cdot | \Sigma_{\Lambda_l,j}). \tag{6.1}
\]

Note that \( \nu_{\Lambda_l, j} \) does not depend on the parameter \( \rho \).

**Lemma 6.2.** There exists a finite constant \( C_0 \) such that

\[
\langle f; f \rangle_{\nu_{\Lambda_l, j}} \leq C_0 l^2 \langle f(-L_{\Lambda_l}^{env} f) \rangle_{\nu_{\Lambda_l, j}}
\]

for all \( j \geq 1 \), all \( l \geq 1 \) and all functions \( f \) in \( L^2(\nu_{\Lambda_l, j}) \). In this formula, \( \langle f; f \rangle_{\nu_{\Lambda_l, j}} \) stands for the variance of \( f \) with respect to \( \nu_{\Lambda_l, j} \).

**Proof.** This result follows from the spectral gap of the zero-range process proved in [8]. Since \( g(k)/k \) is bounded above and below by finite strictly positive constants, an elementary computation shows that

\[
\langle f; f \rangle_{\nu_{\Lambda_l, j}} = \inf_c \langle (f - c)^2 \rangle_{\nu_{\Lambda_l, j}} \leq a^2 \inf_c \langle (f' - c)^2 \rangle_{\mu_{\Lambda_l,j-1}} = a^2 \langle f'; f' \rangle_{\mu_{\Lambda_l,j-1}}
\]

and

\[
\langle f'(-L_{\Lambda_l} f') \rangle_{\mu_{\Lambda_l,j-1}} \leq a^2 \langle f(-L_{\Lambda_l}^{env} f) \rangle_{\nu_{\Lambda_l, j}}
\]

provided \( 0 < a^{-1} \leq g(k)/k \leq a \) for all \( k \geq 1 \). In this formula, \( L_{\Lambda_l} \) is the generator of the zero-range process \( (2.1) \) restricted to the set \( \Lambda_l \), \( \mu_{\Lambda_l,j-1} \) is the canonical measure associated to the zero-range process restricted to the set \( \Lambda_l \) with \( j - 1 \) particles, and \( f'(\eta) = f(\eta + \delta_0) \), where \( \delta_0 \) is the configuration with exactly one particle at the origin and summation of configurations is performed componentwise. \( \square \)
6.1. **Local one-block estimate.** For \( l \geq 1 \), define the function \( V_l(\eta) \) by
\[
V_l(\eta) = h(\eta(0)) - H(\eta(0))
\]
where we recall \( h \) is a bounded, Lipschitz function, and \( H(\eta) = E_{\nu_\eta}[h(\eta(0))] \). In this subsection we give the second step for the proof of Proposition 6.1.

**Lemma 6.3** (One-block estimate). For every \( 0 \leq t \leq T \),
\[
\limsup_{l \to \infty} \limsup_{N \to \infty} \mathbb{E}^N \left[ \left| \int_0^t V_l(\eta_s) \, ds \right| \right] = 0.
\]

**Proof.** Since the initial entropy \( \mathcal{H}(\nu_{\gammaN_0(\cdot)}|\nu_\rho) \) is bounded by \( C_0 N \), by the entropy inequality,
\[
\mathbb{E}^N \left[ \left| \int_0^t V_l(\eta_s) \, ds \right| \right] \leq \frac{C_0}{\gamma} + \frac{1}{\gamma N} \log \mathbb{E}_\rho \left[ \exp \left\{ \gamma N \left| \int_0^t V_l(\eta_s) \, ds \right| \right\} \right],
\]
where \( \mathbb{E}_\rho \) denotes expectation with respect to the process starting from the invariant measure \( \nu_\rho \). Using the elementary inequality \( e^{|x|} \leq e^x + e^{-x} \), we can get rid of the absolute value in the previous integral, considering \( h \) and \( -h \). In this case, by Feynman-Kac formula, the second term on the right hand side is bounded by \( (\gamma N)^{-1} T \lambda_{N,l} \), where \( \lambda_{N,l} \) is the largest eigenvalue of \( N^2 L_N + \gamma N V_l \). Therefore, to prove the lemma, it is enough to show that \( (\gamma N)^{-1} \lambda_{N,l} \) vanishes, as \( N \uparrow \infty, l \uparrow \infty \), for every \( \gamma > 0 \).

By the variational formula for \( \lambda_{N,l} \),
\[
(\gamma N)^{-1} \lambda_{N,l} = \sup_f \left\{ \gamma N \left( V_l f \right)_\rho - \gamma^{-1} N \sqrt{\mathbb{E}^N \left[ \left( \sqrt{-L_N^\text{env}} \right) \right]} \rho \right\},
\]
where the supremum is carried over all densities \( f \) with respect to \( \nu_\rho \). Recall that we denote by \( L_{N,i}^\text{env} \) the restriction of the environment part of the generator to the cube \( A_i \). As the Dirichlet forms satisfy \( \mathbb{E}^N \left[ \left( \sqrt{\mathbb{E}^N \left[ \left( \sqrt{-L_N^\text{env}} \right) \right]} \right) \right] \rho \leq \mathbb{E}^N \left[ \left( \sqrt{-L_N^\text{env}} \right) \right] \rho \), we may bound the previous expression by a similar one where \( L_N \) is replaced by \( L_{N,i}^\text{env} \).

Denote by \( \hat{f}_l \) the conditional expectation of \( f \) given \( \{ \eta(z) : z \in A_l \} \). Since \( V_l \) depends on the configuration \( \eta \) only through \( \{ \eta(z) : z \in A_l \} \) and since the Dirichlet form is convex, the expression inside braces in (6.2) is less than or equal to
\[
\int V_l \hat{f}_l \, d\nu^\Lambda_{\rho} - \gamma^{-1} N \int \sqrt{\hat{f}_l (-L_{N,i}^\text{env} \sqrt{\hat{f}_l})} \, d\nu^\Lambda_{\rho},
\]
where, as in (6.1), \( \nu^\Lambda_{\rho} \) stands for the restriction of the product measure \( \nu_\rho \) to \( \hat{A}_l^\Lambda \).

The linear term in this formula is equal to
\[
\sum_{j \geq 1} c_{l,j}(f) \int V_l \hat{f}_{l,j} \, d\nu^\Lambda_{\rho},
\]
where \( \nu^\Lambda_{\rho} \) is the canonical measure defined in (6.1) and
\[
c_{l,j}(f) = \int_{\Sigma_{\Lambda_{l,j}}} \hat{f}_{l,j} \, d\nu^\Lambda_{\rho}, \quad \hat{f}_{l,j}(\eta) = c_{l,j}^{-1}(f) \nu^\Lambda_{\rho}(\Sigma_{\Lambda_{l,j}}) \hat{f}_{l}(\eta) .
\]
The sum starts at \( j = 1 \) because there is always a particle at the origin. Note also that \( \sum_{j \geq 1} c_{l,j}(f) = 1 \) and that \( \hat{f}_{l,j}(\cdot) \) is a density with respect to \( \nu^\Lambda_{\rho} \).
By the same reasons, the quadratic term of (6.3) can be written as
\[ \gamma^{-1} N \sum_{j \geq 1} c_{l,j}(f) \int \sqrt{\hat{f}}_{l,j} (-L_{\Lambda_i}^{env} \sqrt{\hat{f}}_{l,j}) \, d\nu_{\Lambda_i,j} \, . \]
In view of this decomposition, (6.2) is bounded above by
\[
\sup_{j \geq 1} \sup_{f} \left\{ \int V_{l,j} \, d\nu_{\Lambda_i,j} - \gamma^{-1} N \int \sqrt{\hat{f}} (-L_{\Lambda_i}^{env} \sqrt{\hat{f}}) \, d\nu_{\Lambda_i,j} \right\},
\]
where the second supremum is carried over all densities with respect to \( \nu_{\Lambda_i,j} \).

Recall that \( V_l(\eta) = h(\eta(0)) - H(\eta'(0)) \). Let \( \hat{H}_l(j/2l + 1) = \int h(\eta(0)) \, d\nu_{\Lambda_i,j} \). By Lemma 6.4 below, we can replace \( H(\eta'(0)) \) by \( \hat{H}_l(\eta'(0)) \) in the previous expression. Let \( V_{l,j}(\eta) = h(\eta(0)) - \hat{H}_l(j/2l + 1) \) and notice that \( V_{l,j} \) has mean zero with respect to \( \nu_{\Lambda_i,j} \) for all \( j \geq 1 \). By Lemma 6.2, \( L_{\Lambda_i}^{env} \) has a spectral gap on \( \Sigma_{\Lambda_i,j} \) of order \( C_0 l^{-2} \), uniformly in \( j \). In particular, since \( h \) is bounded, and the inverse spectral gap of \( L_{\Lambda_i}^{env} \) is order \( l^2 \) uniformly in \( j \), by Rayleigh expansion [6, Theorem A3.1.1], for sufficiently large \( N \),
\[
\int V_{l,j} \, d\nu_{\Lambda_i,j} - \gamma^{-1} N \int \sqrt{\hat{f}} (-L_{\Lambda_i}^{env} \sqrt{\hat{f}}) \, d\nu_{\Lambda_i,j} 
\leq \frac{\gamma N^{-1}}{1 - 2 \| V_l \|^2 \Lambda_i} C_0 l^2 \gamma N^{-1} \int V_{l,j} (-L_{\Lambda_i}^{env})^{-1} V_{l,j} \, d\nu_{\Lambda_i,j} 
\leq 2 \gamma N^{-1} \int V_{l,j} (-L_{\Lambda_i}^{env})^{-1} V_{l,j} \, d\nu_{\Lambda_i,j}
\]
uniformly in \( j \geq 1 \). By the spectral gap of \( L_{\Lambda_i}^{env} \) again, this expression is less than or equal to
\[ C_0 l^2 \gamma N^{-1} \int V_{l,j}^2 \, d\nu_{\Lambda_i,j} \leq C_0 \| h \|^2 \gamma N^{-1} \]
because \( h \) is bounded. This proves that (6.2) vanishes as \( N \uparrow \infty \), \( l \uparrow \infty \), and therefore the lemma.

**Lemma 6.4.** For bounded, Lipschitz function \( h : \mathbb{N}_0 \to \mathbb{R}, \)
\[ \lim_{l \to \infty} \sup_{k \geq 0} \left| E_{\nu_{\Lambda_i,k}}[h(\eta(0))] - E_{\nu_{k/|\Lambda_i|}}[h(\eta(0))] \right| = 0 . \]

**Proof.** Fix \( \epsilon > 0 \) and consider \( (l, k) \) such that \( k/|\Lambda_i| \leq \epsilon \). We may subtract \( h(1) \) to both expectations. Since \( h \) is Lipschitz, the absolute value appearing in the statement of the lemma is bounded by
\[ C(h) \left\{ \int \{ \eta(0) - 1 \} \, d\nu_{\Lambda_i,k} + \int \{ \eta(0) - 1 \} \, d\nu_{k/|\Lambda_i|} \right\} . \] (6.4)

Note that both terms are positive because both measures are concentrated on configurations with at least one particle at the origin. We claim that each term is bounded by \( a^2 \epsilon \).

On the one hand, since \( \nu_{\rho}(d\eta) = \{ \eta(0)/\rho \} \, \mu_{\rho}(d\eta) \), the second term inside braces is equal to \( \rho^{-1} \int \eta(0) \, d\mu_{k/|\Lambda_i|} \), where \( \rho = k/|\Lambda_i| \). Since \( k \leq a \varphi(k) \), we may replace \( \eta(0) \) by \( a \varphi(\eta(0)) \) and perform a change of variables \( \eta' = \eta - \eta_0 \) to bound the second term in (6.4) by \( a^2 \epsilon \).

\[ \int \{ \eta(0) - 1 \} \, d\nu_{\Lambda_i,k} \]
On the other hand, by the explicit formula for \( \nu_{\Lambda_{l,k}} \), the first term in (6.3) is equal to
\[
\sum_{x \in \Lambda_l} \eta(0) \left( \eta(0) - 1 \right) \prod_{x \in \Lambda_l} \frac{1}{g(\eta(x))!} / \sum_{x \in \Lambda_l} \eta(0) \prod_{x \in \Lambda_l} \frac{1}{g(\eta(x))!},
\]
where both sums are performed over \( \Sigma_{\Lambda_{l,k}} \). Replacing \( \eta(0) \) by \( a^{\pm 1} \eta(0) \) in the numerator and in the denominator, we obtain that the previous expression is less than or equal to \( a^2 E_{\mu_{\Lambda_{l,k}}}[\eta(0)] \leq a^2 k/|\Lambda_l| \). In last formula, \( \mu_{\Lambda_{l,k}} \) is the product measure \( \mu_{l,k} \) conditioned on the hyperplane \( \Sigma_{\Lambda_{l,k}} = \{ \xi \in \mathbb{N}_{\Lambda_l} : \sum_{x \in \Lambda_l} \xi(x) = k \} \).

For \( (l,k) \) such that \( k/|\Lambda_l| \geq \epsilon \), write
\[
E_{\nu_{\Lambda_{l,k}}}[h(\eta(0))] - E_{\nu_{\Lambda_{l,k}}}[h(\eta(0))] = \frac{1}{\rho} \left( E_{\mu_{\Lambda_{l,k}}}[h'(\eta(0))] - E_{\mu_{\Lambda_{l,k}}}[h'(\eta(0))] \right),
\]
where \( \rho = k/|\Lambda_l| \) and \( h'(j) = h(j)j \). By Corollary 6.1 (parts a,b) \( \mathbb{E}[h(\eta(0))] \) is bounded by \( C(h)\epsilon^{-1/2} \).

**6.2. Local two-blocks estimate.** In this subsection we show how to go from a box of size \( l \) to a box of size \( \epsilon N \):

**Lemma 6.5** (Two-blocks estimate). Let \( H : \mathbb{R}_+ \to \mathbb{R} \) be a bounded, Lipschitz function. For every \( t > 0 \),
\[
\lim_{l \to \infty} \sup_{N \to \infty} \sum_{l \to \infty} \sum_{\epsilon \to 0} \mathbb{E}^N \left[ \left| \int_0^t \left\{ H(\eta^l(0)) - \frac{1}{\epsilon N} \sum_{x=1}^{\epsilon N} H(\eta^l_x(x)) \right\} ds \right| \right] = 0. \quad (6.5)
\]
The proof of this lemma is very similar to the proof of Lemma 6.3. The expectation in (6.5) is bounded by
\[
\frac{1}{\epsilon N} \sum_{x=2l+1}^{\epsilon N} \mathbb{E}^N \left[ \left| \int_0^t \left\{ H(\eta^l(0)) - H(\eta^l_x(x)) \right\} ds \right| \right] + \frac{C(H)(2l+1)}{\epsilon N}.
\]
Following the proof of the one-block estimate, we see that it is enough to estimate, uniformly in \( 2l+1 \leq x \leq \epsilon N \), the quantity
\[
\sup \left\{ \langle V_{l,x} f \rangle_\rho - N\gamma^{-1} \langle \sqrt{f}(-L_N \sqrt{f}) \rangle_\rho \right\},
\]
where the supremum, as before, is over all density functions \( f \) with \( \int f d\nu_\rho = 1 \) and \( V_{l,x} \) is defined by
\[
V_{l,x}(\eta) = H(\eta^l(0)) - H(\eta^l(x)).
\]
Notice that the blocks \( \Lambda_l \) and \( \Lambda_l(x) = \{ -l+x, \ldots, l+x \} \) are disjoint. Let \( L_{\Lambda_{l,x}} \) be the restriction of \( L_{\Lambda_{l,x}}^{\nu_\rho} \) to the set \( \Lambda_{l,x} = \Lambda_l \cup \Lambda_l(x) \) and define the operator \( L_{l,x} \) by
\[
L_{l,x} f(\eta) = L_{\Lambda_{l,x}}^{\nu_\rho} f(\eta) + g(\eta(0)) \left| f(\eta^{l,x-1}) - f(\eta) \right| + g(\eta(x-l)) \left| f(\eta^{x-l,l}) - f(\eta) \right|.
\]
The operator \( L_{l,x} \) corresponds to the environment generator of a zero-range dynamics on which particles can jump between adjacent sites on each box, and between endpoints \( l \) and \( x-l \). Since \( x \leq \epsilon N \), we see, by adding and subtracting at most \( \epsilon N \) terms, that
\[
\langle f(-L_{l,x} f) \rangle_\rho \leq (1 + \epsilon N) \langle f(-L_N f) \rangle_\rho.
\]
Then, it is enough to prove that
\[
\sup_f \left\{ \left\langle \{ H(\eta'(0)) - H(\zeta'(0)) \} f \right\rangle_{\nu_\Lambda^*_{\gamma}} - \frac{1}{2\epsilon} \langle \sqrt{f} (-L_{l,l} \sqrt{f}) \rangle_{\nu_{\Lambda^*_{\gamma}}} \right\}
\]
vanesces as \( \epsilon \downarrow 0 \) and \( l \uparrow \infty \). In this formula, the state space is \( \mathbb{N}_0^{\Lambda^*_{\gamma}} \), where \( \Lambda^*_{\gamma} = \{-l, \ldots, 3l + 1\} \), the configurations of this space are denoted by the pair \( \beta = (\eta, \zeta) \), where \( \eta \) belongs to \( \mathbb{N}_0^{\Lambda^*_{\gamma}} \) and \( \zeta \) belongs to \( \mathbb{N}_0^{l+1\ldots3l+1} \), expectation is taken with respect to the measure \( \nu_{\Lambda^*_{\gamma}} \), the projection of \( \nu_{\rho} \), the generator of the environment restricted to the set \( \Lambda^*_{\gamma} \):
\[
(L_{l,l} f)(\beta) = \sum_{x \neq 0, x \in \Lambda^*_{\gamma}} p(y - x) g(\beta(x)) \left[ f(\beta^{x,y}) - f(\beta) \right] + \sum_{x \in \mathbb{Z}} p(z) g(\beta(0)) \frac{\beta(0) - 1}{\beta(0)} \left[ f(\beta^{0,z}) - f(\beta) \right];
\]
and the supremum is carried over all densities \( f \) with respect to \( \nu_{\Lambda^*_{\gamma}} \).

Following the proof of the one-block Lemma [6,3], we need only to prove that
\[
\sup_{k \geq 1} \sup_f \left\{ \int \{ H(\eta'(0)) - H(\zeta'(0)) \} f \, d\nu_{\Lambda^*_{\gamma},k} - \frac{1}{2\epsilon} \int \sqrt{f} (-L_{l,l} \sqrt{f}) \, d\nu_{\Lambda^*_{\gamma},k} \right\}
\]
vanesces with limits on \( \epsilon \) and \( l \) where the supremum is on densities \( f \) with respect to the canonical measure \( \nu_{\Lambda^*_{\gamma},k} \) defined similarly as in \( 6.1 \), where \( \Sigma_{\Lambda^*_{\gamma},k} = \{ \beta \in \mathbb{N}_0^{\Lambda^*_{\gamma}} : \beta(0) \geq 1, \sum_{y \in \Lambda^*_{\gamma}} \beta(y) = k \} \).

Let \( W_l(\beta) = H(\eta'(0)) - H(\zeta'(0)) \). By the Rayleigh expansion \([6, \text{ Theorem A3.1.1}]\), spectral gap estimate Lemma \([6,3] \) applied to \( L_{l,l} \) (which can be thought of as the environment generator on a block of length \( 2|\Lambda_l| \)) and boundedness of \( H \), for large \( N \) and small \( \epsilon \),
\[
\int W_l f \, d\nu_{\Lambda^*_{\gamma},k} - \frac{1}{2\epsilon} \int \sqrt{f} (-L_{l,l} \sqrt{f}) \, d\nu_{\Lambda^*_{\gamma},k}
\leq \int W_l \, d\nu_{\Lambda^*_{\gamma},k} + \frac{2\epsilon}{1 - C(H)^2} \int W_l \{(-L_{l,l})^{-1} W_l\} \, d\nu_{\Lambda^*_{\gamma},k}
\leq \int W_l \, d\nu_{\Lambda^*_{\gamma},k} + C(H)^2 \epsilon
\]
for some finite constant \( C(H) \) depending on \( H \). The last term vanishes as \( \epsilon \downarrow 0 \), while the first term vanishes uniformly in \( k \) as \( l \uparrow \infty \) by Lemma \([6,3] \) below. \( \square \)

**Lemma 6.6.** For a bounded, Lipschitz function \( H : \mathbb{R}_+ \to \mathbb{R} \), we have that
\[
\lim \sup_{l \to \infty} \sup_{k \geq 0} \left| E_{\nu_{\Lambda^*_{\gamma},k}} \left[ H(\eta'(0)) - H(\zeta'(0)) \right] \right| = 0.
\]

**Proof.** Fix \( \epsilon > 0 \). Using that \( H \) is Lipschitz, we have that \( |H(\eta'(0)) - H(\zeta'(0))| \leq C(H) \{ \eta'(0) + |\zeta'(0)| \} \), and so the expectation appearing in the statement of the lemma is less than or equal to \( C(H) E_{\nu_{\Lambda^*_{\gamma},k}} \{ \eta'(0) + |\zeta'(0)| \} \). A computation, similar to the one presented in the proof of Lemma \([6,4] \) shows that
\[
E_{\nu_{\Lambda^*_{\gamma},k}} \{ \beta(0) \} \leq a^2 \left\{ 1 + E_{\nu_{\Lambda^*_{\gamma},k-1}} \{ \xi(0) \} \right\}; \quad E_{\nu_{\Lambda^*_{\gamma},k}} \{ \beta(y) \} \leq a^2 E_{\nu_{\Lambda^*_{\gamma},k-1}} \{ \xi(0) \}
\]
for all \( y \neq 0 \). In this formula, \( \mu_{n,k} \) stands for the canonical measure defined by \( \mu_{n,k} = \mu_{n} = \sum_{x \in n} \beta(x) = k \), where \( \mu_{n} \) is the product measure \( \mu_{n} \) restricted to the set \( n \). In particular, the expectation appearing in the statement of the lemma is less than or equal to \( C(H) a^2 \{ 1 + k \} / \{ 2 |n| \} \). This concludes the proof of the lemma in the case \( k/2 |n| \leq \epsilon \).

Assume now that \( k/2 |n| \geq \epsilon \). By definition of the canonical measure \( \nu_{n,k} \) and the grand-canonical measure \( \nu_{n}^{\infty} \), the expectation appearing in the statement of the lemma is equal to

\[
\frac{1}{E_{\nu_{n,k}^{\infty}}[\beta(0)]} E_{\nu_{n,k}^{\infty}} \left[ \beta(0) \left\{ H(\eta_{l}(0)) - H(\zeta_{l}(0)) \right\} \right].
\] (6.6)

Since the measure is space homogeneous, the denominator is equal to \( \rho_{l,k} = k/2 |n| \), while in the numerator we may replace \( \beta(0) \) by \( \eta_{l}(0) \). The numerator can therefore be rewritten as

\[
E_{\nu_{n,k}^{\infty}} \left[ \left\{ \eta_{l}(0) - \rho_{l,k} \right\} \left\{ H(\eta_{l}(0)) - H(\zeta_{l}(0)) \right\} \right] + \rho_{l,k} E_{\nu_{n,k}^{\infty}} \left[ H(\eta_{l}(0)) - H(\zeta_{l}(0)) \right].
\]

The second term vanishes because the measure \( \nu_{n,k} \) is space homogeneous, while the first one is absolutely bounded by \( C(H) \rho_{l,k} \). By [8, Corollary 6.1 (C)], this expression is less than or equal to

\[
C'(H) \sigma(\rho_{l,k}) l^{-1/2},
\]

where \( \sigma(\rho) \) stands for the variance of \( \xi(0) \) under \( \mu_{n} \). By [8, (5.2)] and since \( \varphi(\rho)/\rho \) is bounded below and above, \( \sigma(\rho_{l,k}) l^{2} \leq C \rho_{l,k} \). Therefore, if we recall the denominator in (6.6), we obtain that

\[
E_{\nu_{n,k}^{\infty}} \left[ H(\eta_{l}(0)) - H(\zeta_{l}(0)) \right] \leq \frac{C(H)}{\sqrt{\rho_{l,k}}},
\]

which concludes the proof of the lemma since we assumed the density to be bounded below by \( \epsilon \). \( \square \)

6.3. Proof of Proposition 6.1. Recall \( H(\rho) = E_{\nu_{l}}[h], \ H_{l}(\eta) = H(\eta_{l}(0)), \) and \( H_{l}(\rho) = E_{\mu_{l}}[H_{l}] \). Then, we have that

\[
\begin{align*}
\mathbb{E}^{N} \left[ \left| \int_{0}^{t} \{ h(\eta_{l}) - \frac{1}{\epsilon N} \sum_{x=1}^{\epsilon N} \tilde{H}_{l}(\eta_{l}^{N}(x)) \} ds \right| \right] \\
\leq \mathbb{E}^{N} \left[ \left| \int_{0}^{t} \{ h(\eta_{l}) - H(\eta_{l}^{N}(0)) \} ds \right| \right] \\
+ \mathbb{E}^{N} \left[ \left| \int_{0}^{t} \left\{ H(\eta_{l}^{N}(0)) - \frac{1}{\epsilon N} \sum_{x=1}^{\epsilon N} H(\eta_{l}^{N}(x)) \right\} ds \right| \right] \\
+ \mathbb{E}^{N} \left[ \left| \int_{0}^{t} \left\{ \frac{1}{\epsilon N} \sum_{x=1}^{\epsilon N} \left( H(\eta_{l}^{N}(x)) - \tilde{H}_{l}(\eta_{l}^{N}(x)) \right) \right\} ds \right| \right].
\end{align*}
\]

As \( h \) is bounded, Lipschitz, we have \( H \) is bounded, Lipschitz by Lemma 6.7 below, and so the first and second terms vanish by Lemmata 6.3 and 6.5. For the
third term, we can rewrite it as
\[ \mathbb{E}^N \left[ \left| \int_0^t \left\{ \frac{1}{N} \sum_{x \in \mathbb{Z}} \iota_x(x/N) \left( H(\eta_t(x)) - \bar{H}(\eta_t(x)) \right) \right\} ds \right| \right] \]
where \( \iota_x(\cdot) = \epsilon^{-1}\{0, x\} \). In fact, as \( h \) is bounded, we can replace \( \iota_x \) in the last expression by a smooth approximation. Then, for fixed \( \epsilon > 0 \) and \( l \geq 1 \), treating \( H_i(\eta) = H(\eta(0)) \) as a local function, which is also bounded, Lipschitz, \( h \) is bounded, Lipschitz, the third term vanishes using Proposition 5.1 by taking \( N \uparrow \infty \), and \( \epsilon \downarrow 0 \).

\[ \square \]

Lemma 6.7. Let \( h : \Omega_N \to \mathbb{R} \) be a local, Lipschitz function. Then, \( H : \mathbb{R}_+ \to \mathbb{R} \) given by \( H(\rho) = E_{\nu_\rho}[h] \) is also Lipschitz.

Proof. The proof is similar to that of Corollary II.3.7 which shows \( \bar{h}(\rho) = E_{\nu_\rho}[h] \) is Lipschitz. Following the proof of Corollary II.3.7, it is not difficult to show \( \{ \nu_\rho : \rho \geq 0 \} \) is a stochastically increasing family, and for \( \rho_1 < \rho_2 \) that
\[ |H(\rho_1) - H(\rho_2)| \leq C \sum_{x \in A} |E_{\nu_{\rho_1}}[\eta(x)] - E_{\nu_{\rho_2}}[\eta(x)]| \]
where \( C \) is the Lipschitz constant of \( h \), and \( A \subset \mathbb{Z} \) corresponds to the support of \( h \). If \( A \) does not contain the origin, the proof is the same as for Corollary II.3.7.

Otherwise, it is enough to estimate the difference \( |E_{\nu_{\rho_1}}[\eta(0)] - E_{\nu_{\rho_2}}[\eta(0)]| \). When \( 0 = \rho_2 < \rho_1 \), the difference equals \( E_{\mu_{\rho_1}}[\eta(0)(\eta(0) - 1)]/\rho_1 \leq a\varphi(\rho_1) \leq a^2\rho_1 \) as \( a^{-1} \leq g(k)/k, \varphi(\rho)/\rho \leq a \) through (LG), (M), and \( E_{\mu_\rho}[g(\eta(0))f(\eta)] = \varphi(\rho)E_{\mu_\rho}[f(\eta + \delta_0)] \) where \( \delta_0 \) is the configuration with exactly one particle at the origin. When \( 0 < \rho_2 < \rho_1 \), the difference equals
\[ |\rho_1^{-1}E_{\mu_{\rho_1}}[\eta(0)^2] - \rho_2^{-1}E_{\mu_{\rho_2}}[\eta(0)^2]| \leq |\sigma^2(\rho_1)/\rho_1 - \sigma^2(\rho_2)/\rho_2| + |\rho_1 - \rho_2| \]
where \( \sigma^2(\rho) = E_{\mu_\rho}[(\eta(0) - \rho)^2] \). The Lipschitz estimate now follows by calculating a uniform bound on the derivative
\[ \frac{\partial_\rho \sigma^2(\rho)}{\rho} = \frac{m_3(\rho)}{\rho \sigma^2(\rho)} - \frac{\sigma^2(\rho)}{\rho^2} \]
where \( m_3(\rho) = E_{\mu_\rho}[(\eta(0) - \rho)^3] \). For \( \rho \) large, under assumptions (LG), (M), this is in order \( O(\rho^{-1/2}) \) from Lemma 5.2 and bound \( a^{-1} \leq \varphi(\rho)/\rho \leq a \); on the other hand, as \( \rho \downarrow 0 \), the derivative is also bounded.

\[ \square \]

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