Hydrogen atom in rotationally invariant noncommutative space

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Abstract

We consider the noncommutative algebra which is rotationally invariant. The hydrogen atom is studied in the rotationally invariant noncommutative space. We find the corrections to the energy levels of the hydrogen atom up to the second order in the parameter of noncommutativity. The upper bound of the parameter of noncommutativity is estimated on the basis of the experimental results for $1s - 2s$ transition frequency.

Key words: noncommutative space, rotational symmetry, hydrogen atom

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1 Introduction

The idea that space might have a noncommutative structure was proposed by Heisenberg and was formalized by Snyder [1]. In recent years noncommutativity has met considerable interest due to development of String Theory [2, 3] and Quantum Gravity [4].

In the canonical version of noncommutative space the coordinate and momentum operators satisfy the following commutation relations

$$[X_i, X_j] = i\hbar\theta_{ij},$$
$$[X_i, P_j] = i\hbar\delta_{ij} ,$$
$$[P_i, P_j] = 0,$$

where $\theta_{ij}$ is a constant antisymmetric matrix. Many physical problems have been considered in this space, among them the hydrogen atom [5, 6, 7, 8, 9, 10, 11, 12]. In [5] the corrections to the energy levels of this atom were found up to the first order in the parameter of noncommutativity. Also in this article the authors obtained the corrections to the Lamb shift within the noncommutative quantum electrodynamics theory. In [6] the hydrogen atom was studied as a two-particle system in the case when the particles of opposite charges feel opposite noncommutativity. The quadratic Stark effect was examined in [8]. New result for shifts in the spectrum of hydrogen atom in noncommutative space were presented in [9]. In [10] the hydrogen atom energy levels were calculated in the framework of the noncommutative Klein-Gordon equation. In [11, 12] the Dirac equation with a Coulomb field was considered in noncommutative space.

The hydrogen atom problem was also studied in the case of space-time noncommutativity [13, 14, 15, 16, 17], space-space and momentum-momentum noncommutativity [18, 19, 20, 21].

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It is worth noting that in a three-dimensional noncommutative space we face a problem of the rotational symmetry breaking \[5, 22\]. In order to preserve this symmetry new classes of noncommutative algebras were explored. For instance, in \[23\] the rotational invariance was preserved by foliating the space with concentric fuzzy spheres. In \[24\] the rotationally symmetric noncommutative space was constructed as a set of fuzzy spheres. In this case the exact solution of the hydrogen atom problem was found. In \[25\] the curved noncommutative space was introduced to maintain the rotational symmetry and the hydrogen atom spectrum was studied.

Also in order to preserve the rotation invariance the promotion of the parameter of noncommutativity to an operator in Hilbert space was suggested and the canonical conjugate momentum of this operator was introduced \[26\]. In this case the author considered an isotropic harmonic oscillator. The way to maintain the \(N\) dimensional rotation invariance was considered in \[27\]. In this article the coordinates are represented by operators. The measured values of these operators are expectations between generalized coherent states. In \[28\] the noncommutative coordinates invariant under rotations were presented. The author suggested identification of the coordinates with the boost operators in \(\text{SO}(1,3)\).

Note, however, that in a two-dimensional space the rotational symmetry survives even in the canonical version of noncommutativity \([X, Y] = i\hbar \theta\), where \(\theta\) is a constant.

Much attention has also been received to studying spherically symmetric noncommutative spaces \[29, 30, 31\], solving the problem of violation of the Lorentz invariance (see for example \[32, 33, 34\]). For instance, Lorentz invariant noncommutative gauge theory was proposed in \[32\]. The authors considered the promotion of \(\theta_{ij}\) to an antisymmetric tensor \(\hat{\theta}_{ij}\) that transforms as a Lorentz tensor. The generalizations of the operator trace and star product were found. Therefore, the Lagrangian was considered as a \(\theta\)-integrated quantity.

In this article we consider the way of constructing the rotationally invariant noncommutative space by the generalization of the constant matrix \(\theta_{ij}\) to a tensor. We propose to construct this tensor with the help of additional coordinates governed by the harmonic oscillators. The corresponding algebra is rotationally invariant. The hydrogen atom is considered in the rotationally invariant noncommutative space. We study the perturbation of the hydrogen atom energy levels caused by the noncommutativity of coordinates.

The article is organized as follows. In Section 2 the way to preserve the rotational symmetry is considered. In Section 3 the Hamiltonian of the hydrogen atom is studied in the rotationally invariant noncommutative space. The corrections to the energy levels of the hydrogen atom are found up to the second order in the parameter of noncommutativity in Section 4. In addition, the way to construct an effective Hamiltonian which is rotationally invariant is proposed. Section 5 is devoted to calculation of the corrections to the \(n_s\) levels. Conclusions are presented in Section 6.
2 Rotationally invariant noncommutative space

In order to preserve the rotational symmetry we propose to define the tensor of noncommutativity as follows

$$\theta_{ij} = \frac{\alpha}{\hbar}(a_i b_j - a_j b_i),$$

(4)

where $\alpha$ is a dimensionless constant. We suppose for simplicity that $a_i, b_i$ are governed by the harmonic oscillators

$$H_{\text{osc}} = \frac{(p_a^2)^2}{2m} + \frac{(p_b^2)^2}{2m} + \frac{m\omega^2 a^2}{2} + \frac{m\omega^2 b^2}{2}. \quad (5)$$

Note that harmonic oscillator has two independent units of measurement, namely the unit of length $\sqrt{\frac{\hbar}{m\omega}}$ and the unit of energy $\hbar \omega$. It is generally believed that the parameter of noncommutativity of coordinates is of the order of the Planck scale, therefore we put

$$\sqrt{\frac{\hbar}{m\omega}} = l_p, \quad (6)$$

where $l_p$ is the Planck length. Independently of (6) we also consider the limit $\omega \to \infty$. In this case the distance between the energy levels of harmonic oscillator tends to infinity. Therefore, harmonic oscillator put into the ground state remains in it.

So, we propose to consider the following commutation relations

$$[X_i, X_j] = i\alpha(a_i b_j - a_j b_i), \quad (7)$$

$$[X_i, P_j] = i\hbar\delta_{i,j}, \quad (8)$$

$$[P_i, P_j] = 0. \quad (9)$$

The coordinates $a_i, b_i$, and momenta $p_a^i, p_b^i$ satisfy the ordinary commutation relations $[a_i, a_j] = 0, [a_i, p_j^a] = i\hbar\delta_{i,j}, [b_i, b_j] = 0, [b_i, p_j^b] = i\hbar\delta_{i,j},$ also $[a_i, b_j] = [a_i, p_j^b] = [a_i, p_j^a] = [p_j^a, p_j^b] = 0$. It is worth noting that $a_i, b_i$ commute with $X_i$ and $P_i$ and therefore $\theta_{ij}$ given by (4) commutes with $X_i$ and $P_i$ too. So, algebra (7)-(9) is equivalent to (1)-(3).

It is convenient to use the following representation

$$X_i = x_i - \frac{\alpha}{2\hbar} \sum_j (a_i b_j - a_j b_i)p_j, \quad (10)$$

$$P_i = p_i, \quad (11)$$

where the coordinates $x_i$ and momenta $p_i$ satisfy the ordinary commutation relations $[x_i, x_j] = 0, [x_i, p_j] = i\hbar\delta_{i,j}$. Taking into account (11), it is clear that

$$[X_i, p_j^a] = \frac{i\alpha}{2} (b_j p_j - \delta_{i,j} (b \cdot p)), \quad (12)$$

$$[X_i, p_j^b] = \frac{-i\alpha}{2} (a_j p_j - \delta_{i,j} (a \cdot p)), \quad (13)$$

$$[P_i, p_j^a] = [P_i, p_j^b] = 0. \quad (14)$$

It is easy to show that noncommutative coordinates can be represented in a more convenient form

$$X_i = x_i + \frac{1}{2}[\theta \times p]_i, \quad (15)$$

$$p_i = p_i,$$
where $\theta = \frac{\alpha}{\hbar} [a \times b]$.

Algebra (7)-(9) is manifestly rotationally invariant. It is clear that commutation relation (7) remains the same after rotation $X_i' = U(\varphi)X_iU^+(\varphi)$, $a'_i = U(\varphi)a_iU^+(\varphi)$, $b'_i = U(\varphi)b_iU^+(\varphi)$

$$[X_i', X_j'] = i\alpha (a'_i b'_j - a'_j b'_i).$$

where the rotation operator reads $U(\varphi) = e^{i \bar{\theta} \hbar \cdot \tilde{L}}$. Here $\tilde{L}$ is the total angular momentum which we define as follows

$$\tilde{\mathbf{L}} = [\mathbf{r} \times \mathbf{p}] + [\mathbf{a} \times \mathbf{p}^a] + [\mathbf{b} \times \mathbf{p}^b],$$

or taking into account (11), (15)

$$\tilde{\mathbf{L}} = [\mathbf{R} \times \mathbf{P}] + \frac{1}{2} [\mathbf{P} \times [\mathbf{\theta} \times \mathbf{P}]] + [\mathbf{a} \times \mathbf{p}^a] + [\mathbf{b} \times \mathbf{p}^b],$$

where $\mathbf{r} = (x_1, x_2, x_3)$, and $\mathbf{R} = (X_1, X_2, X_3)$.

Note that $\tilde{\mathbf{L}}$ satisfies the following commutation relations

$$[X_i, \tilde{L}_j] = i\hbar \varepsilon_{ijk} X_k,$$
$$[P_i, \tilde{L}_j] = i\hbar \varepsilon_{ijk} P_k,$$
$$[a_i, \tilde{L}_j] = i\hbar \varepsilon_{ijk} a_k,$$
$$[p^a_i, \tilde{L}_j] = i\hbar \varepsilon_{ijk} p^a_k,$$
$$[b_i, \tilde{L}_j] = i\hbar \varepsilon_{ijk} b_k,$$
$$[p^b_i, \tilde{L}_j] = i\hbar \varepsilon_{ijk} p^b_k,$$

which are the same as in ordinary space. Also it is worth mentioning that the total angular momentum (17) is evidently commuting with $r^2$, $a^2$, and $b^2$. Besides, it is easy to check that $\tilde{\mathbf{L}}$ commutes with the scalar products

$$[\tilde{L}_i, (a \cdot p)] = [\tilde{L}_i, (a \cdot p)] = [\tilde{L}_i, (a \cdot b)] = [\tilde{L}_i, (r \cdot a)] = [\tilde{L}_i, (r \cdot b)] = 0.$$  

As a consequence, it is clear that $\tilde{\mathbf{L}}$ commutes with the operator of distance which can be written in the following form

$$R = \sqrt{\sum_i X_i^2} = \sqrt{\left(r + \frac{1}{2} [\theta \times \mathbf{p}]\right)^2} = \sqrt{\left(r - \frac{\alpha}{2\hbar} \mathbf{a}(\mathbf{b} \cdot \mathbf{p}) + \frac{\alpha}{2\hbar} \mathbf{b}(\mathbf{a} \cdot \mathbf{p})\right)^2}.$$  

So, the distance remains the same after rotation $R' = U(\varphi)RU^+(\varphi) = R$.

It is worth mentioning that there is also another way of generalization of the tensor of noncommutativity which gives the possibility to preserve the rotational symmetry

$$\theta_{ij} = \frac{\alpha}{\hbar} \varepsilon_{ijk} \tilde{a}_k,$$

here $\tilde{a}_k = \frac{a_k}{\langle a \rangle}$ are the dimensionless coordinates corresponding to the harmonic oscillator and $\alpha$ is a constant.

Note that in both cases (1), (27) in the limit $\alpha \to 0$ we obtain the ordinary commutation relations.
3 Hamiltonian of the hydrogen atom

In this section we study the Hamiltonian of the hydrogen atom in the rotationally invariant noncommutative space \( (7)-(9) \). We assume that in noncommutative space Hamiltonian has a similar form as in the ordinary space with commutative coordinates. So, the Hamiltonian of the hydrogen atom reads

\[
H_h = \frac{P^2}{2M} - \frac{e^2}{R}, \tag{28}
\]

where \( R = \sqrt{\sum_i X_i^2} \). Besides, in rotationally invariant noncommutative space we have to take into account additional terms that correspond to the harmonic oscillators \( (5) \). Therefore, we consider the total Hamiltonian as follows

\[
H = H_h + H_{osc}. \tag{29}
\]

Let us expand Hamiltonian \( (29) \) in the series over \( \theta \). First, let us find the expansion for the distance \( R \) up to the second order in \( \theta \). Using \( (15) \), we can write

\[
R = \sqrt{\left(r + \frac{1}{2}(\theta \times p)\right)^2} = \sqrt{r^2 - (\theta \cdot L) + \frac{1}{4}(\theta \times p)^2}, \tag{30}
\]

where \( r = \sqrt{\sum_i x_i^2}, \ L = [r \times p] \). Note, that the operators under the square root are noncommuting. Nevertheless it is possible to find the expansion of \( R \) over \( \theta \) in the similar form to the form which we have in the case of commuting operators but with the additional term \( \theta^2 f(r) \) with unknown function \( f(r) \).

\[
R = r - \frac{1}{2r}(\theta \cdot L) - \frac{1}{8r^3}(\theta \cdot L)^2 + \frac{1}{16} \left( \frac{1}{r} [\theta \times p]^2 + [\theta \times p]^2 \frac{1}{r} + \theta^2 f(r) \right). \tag{31}
\]

In order to find \( f(r) \) let us square left- and right-hand sides of equation \( (31) \). In the second order over \( \theta \) we obtain

\[
r^2 - (\theta \cdot L) + \frac{1}{4}[\theta \times p]^2 = r^2 - (\theta \cdot L) + \frac{1}{16} \left( 2[\theta \times p]^2 + r[\theta \times p]^2 \frac{1}{r} + \frac{1}{2} [\theta \times p]^2 r + 2r\theta^2 f(r) \right). \tag{32}
\]

Simplifying \( (32) \), we can write

\[
\frac{\hbar^2}{r^4} [\theta \times r]^2 - r \theta^2 f(r) = 0. \tag{33}
\]

Finally, from equation \( (33) \) we find

\[
\theta^2 f(r) = \frac{\hbar^2}{r^4} [\theta \times r]^2. \tag{34}
\]

Consequently, taking into account \( (34) \), we obtain the following expansion for the distance

\[
R = r - \frac{1}{2r}(\theta \cdot L) - \frac{1}{8r^3}(\theta \cdot L)^2 + \frac{1}{16} \left( \frac{1}{r} [\theta \times p]^2 + [\theta \times p]^2 \frac{1}{r} + \frac{\hbar^2}{r^4} [\theta \times r]^2 \right). \tag{35}
\]
Now it is straightforward to expand the inverse distance $R^{-1}$ in the series over $\theta$. Using (35), we find

$$
\frac{1}{R} = \frac{1}{r} + \frac{1}{2r^3}(\mathbf{\theta} \cdot \mathbf{L}) + \frac{3}{8r^5}(\mathbf{\theta} \cdot \mathbf{L})^2 - \frac{1}{16} \left( \frac{1}{r^2} [\mathbf{\theta} \times \mathbf{p}]^2 \frac{1}{r^2} + \frac{1}{r} [\mathbf{\theta} \times \mathbf{p}]^2 \frac{1}{r^2} + \frac{\hbar^2}{r^2} [\mathbf{\theta} \times \mathbf{r}]^2 \right) \tag{36}
$$

So, taking into account (36), we can rewrite Hamiltonian (29) in the following form

$$
H = H_0 + V, \tag{37}
$$

where

$$
H_0 = \frac{p^2}{2M} - \frac{e^2}{r} + H_{\text{osc}}, \tag{38}
$$

and $V$ is the perturbation caused by the noncommutativity of coordinates

$$
V = -\frac{e^2}{2r^3}(\mathbf{\theta} \cdot \mathbf{L}) - \frac{3e^2}{8r^5}(\mathbf{\theta} \cdot \mathbf{L})^2 + \frac{e^2}{16} \left( \frac{1}{r^2} [\mathbf{\theta} \times \mathbf{p}]^2 \frac{1}{r^2} + \frac{1}{r} [\mathbf{\theta} \times \mathbf{p}]^2 \frac{1}{r^2} + \frac{\hbar^2}{r^2} [\mathbf{\theta} \times \mathbf{r}]^2 \right). \tag{39}
$$

### 4 Perturbation of the energy levels

In order to find the corrections to the energy levels of the hydrogen atom let us use the perturbation theory.

The eigenvalues and eigenstates of the unperturbed Hamiltonian (38) read

$$
E_{n,n_a,n_b}^{(0)} = -\frac{e^2}{2a_B n^2} + \hbar \omega (n_1^a + n_2^a + n_3^a + n_1^b + n_2^b + n_3^b + 3), \tag{40}
$$

$$
\psi_{n,l,m,(n^a),(n^b)}^{(0)} = \psi_{n,l,m}^{(a)} \psi_{n_1^a,n_2^a,n_3^a,m^a}^{(b)} \psi_{n_1^b,n_2^b,n_3^b,m^b}, \tag{41}
$$

where $\psi_{n,l,m}$ are well known eigenfunctions of the hydrogen atom in ordinary space, $\psi_{n_1^a,n_2^a,n_3^a}$, $\psi_{n_1^b,n_2^b,n_3^b}$ are the eigenfunctions of the three-dimensional harmonic oscillators, and $a_B$ is the Bohr radius.

Let us find the corrections to the energy levels of the hydrogen atom in the case when the oscillators are in the ground state. According to the perturbation theory, in the first order in $V$ we have

$$
\Delta E_{n,l}^{(1)} = \left\langle \psi_{n,l,m,(0),(0)}^{(0)} \right| V \left| \psi_{n,l,m,(0),(0)}^{(0)} \right\rangle = \left\langle \psi_{n,l,m,(0),(0)}^{(0)} \right| -\frac{e^2}{2r^3}(\mathbf{\theta} \cdot \mathbf{L}) - \frac{3e^2}{8r^5}(\mathbf{\theta} \cdot \mathbf{L})^2 \left| \psi_{n,l,m,(0),(0)}^{(0)} \right\rangle + \frac{e^2}{16} \left\langle \psi_{n,l,m,(0),(0)}^{(0)} \right| \left[ \frac{1}{r^2} [\mathbf{\theta} \times \mathbf{p}]^2 \frac{1}{r^2} + \frac{1}{r} [\mathbf{\theta} \times \mathbf{p}]^2 \frac{1}{r^2} + \frac{\hbar^2}{r^2} [\mathbf{\theta} \times \mathbf{r}]^2 \right] \left| \psi_{n,l,m,(0),(0)}^{(0)} \right\rangle. \tag{42}
$$

It is clear that $\left\langle \psi_{0,0,0,0}^{(0)} \psi_{0,0,0}^{(b)} \right| \psi_{0,0,0}^{(a)} \psi_{0,0,0}^{(b)} \right\rangle = 0$. Therefore, in the first order in $\theta$ the corrections to the energy levels vanish

$$
\left\langle \psi_{n,l,m,(0),(0)}^{(0)} \right| \frac{e^2}{2r^3}(\mathbf{\theta} \cdot \mathbf{L}) \left| \psi_{n,l,m,(0),(0)}^{(0)} \right\rangle = 0. \tag{43}
$$
In order to calculate the correction caused by the second term \( \frac{3e^2}{8\pi^2} (\theta \cdot L)^2 \) we use the following result (see for instance \[35\])

\[
\left\langle \psi_{n,l,m} \left| \frac{1}{r^2} \right| \psi_{n,l,m} \right\rangle = \frac{4(5n^2 - 3l(l + 1) + 1)}{a_B^2 n^5 l(l + 1)(l + 2)(2l + 1)(2l + 3)(l - 1)(2l - 1)}.
\] \( (44) \)

It is easy to show that

\[
\left\langle \psi_{0,0,0}^a \psi_{0,0,0}^b \psi_{0,0,0}^a \psi_{0,0,0}^b \right\rangle = \frac{1}{2} \left( \frac{\alpha}{m \omega} \right)^2 \delta_{i,j} = \frac{1}{3} \langle \theta^2 \rangle \delta_{i,j},
\] \( (45) \)

where we use the notation

\[
\langle \theta^2 \rangle = \left\langle \psi_{0,0,0}^a \psi_{0,0,0}^b \right\rangle \left\langle \psi_{0,0,0}^a \psi_{0,0,0}^b \right\rangle = \frac{3}{2} \left( \frac{\alpha}{m \omega} \right)^2 = \frac{3\alpha^2 l^4}{2 \hbar^2}.
\] \( (46) \)

So, taking into account \((44), (45)\), we find

\[
\left\langle \psi_{n,l,m}^{(0)} \right| \left( \frac{3e^2}{8\pi^2} (\theta \cdot L)^2 \right) \left| \psi_{n,l,m}^{(0)} \right\rangle = \frac{\hbar^2 e^2 (5n^2 - 3l(l + 1) + 1) \langle \theta^2 \rangle}{2a_B^2 n^5 l(l + 2)(2l + 1)(2l + 3)(l - 1)(2l - 1)}.
\] \( (47) \)

In order to calculate the last term in \((42)\) it is convenient to rewrite it as follows

\[
\frac{1}{r^2} (\theta \times p)^2 \frac{1}{r^2} + \frac{1}{r} (\theta \times p)^2 \frac{1}{r^2} + \frac{\hbar^2}{r^2} (\theta \times r)^2 = \theta^2 p^2 \frac{1}{r^2} + \theta^2 p^2 \frac{1}{r^2} + \frac{\hbar^2}{r^2} - \frac{1}{r^2} (\theta \cdot p)^2 \frac{1}{r^2} - \frac{1}{r^2} (\theta \cdot p)^2 \frac{1}{r^2} - \frac{\hbar^2}{r^2} (\theta \cdot r)^2.
\] \( (48) \)

Then taking into account \((45)\), we obtain

\[
\left\langle \psi_{0,0,0}^a \psi_{0,0,0}^b \right| \frac{1}{r^2} (\theta \cdot p)^2 \frac{1}{r^2} + \frac{1}{r} (\theta \cdot p)^2 \frac{1}{r^2} + \frac{\hbar^2}{r^2} (\theta \cdot r)^2 \left| \psi_{0,0,0}^a \psi_{0,0,0}^b \right\rangle = \frac{1}{3} \left( \frac{1}{r^2} p^2 \frac{1}{r^2} + \frac{1}{r} p^2 \frac{1}{r^2} + \frac{\hbar^2}{r^2} \right) \langle \theta^2 \rangle.
\] \( (49) \)

As a result the last term in \((42)\) can be simplified to

\[
\left\langle \psi_{n,l,m}^{(0)} \right| \frac{1}{r^2} (\theta \times p)^2 \frac{1}{r^2} + \frac{1}{r} (\theta \times p)^2 \frac{1}{r^2} + \frac{\hbar^2}{r^2} (\theta \times r)^2 \left| \psi_{n,l,m}^{(0)} \right\rangle = \frac{2}{3} \left\langle \psi_{n,l,m} \right| \frac{1}{r^2} p^2 \frac{1}{r^2} + \frac{1}{r} p^2 \frac{1}{r^2} + \frac{\hbar^2}{r^2} \left| \psi_{n,l,m} \right\rangle \langle \theta^2 \rangle.
\] \( (50) \)

Let us rewrite \( \frac{1}{r^2} p^2 \frac{1}{r^2} + \frac{1}{r} p^2 \frac{1}{r^2} + \frac{\hbar^2}{r^2} \) in the following form

\[
\frac{1}{r^2} p^2 \frac{1}{r^2} + \frac{1}{r} p^2 \frac{1}{r^2} + \frac{\hbar^2}{r^2} = \frac{1}{r^3} p^2 + \frac{p^2}{r^3} + \frac{5\hbar^2}{r^3}.
\] \( (51) \)

Therefore, we can write

\[
\left\langle \psi_{n,l,m} \right| \frac{1}{r^2} p^2 \frac{1}{r^2} + \frac{1}{r} p^2 \frac{1}{r^2} + \frac{\hbar^2}{r^2} \left| \psi_{n,l,m} \right\rangle = -\frac{2\hbar^2}{a_B^2 n^2} \left\langle \psi_{n,l,m} \right| \frac{1}{r^3} \left| \psi_{n,l,m} \right\rangle + 4 \frac{\hbar^2}{a_B} \left\langle \psi_{n,l,m} \right| \frac{1}{r^4} \left| \psi_{n,l,m} \right\rangle + 5 \hbar^2 \left\langle \psi_{n,l,m} \right| \frac{1}{r^5} \left| \psi_{n,l,m} \right\rangle.
\] \( (52) \)
Using the following results (see for instance [35])

\[
\langle \psi_{n,l,m} | \frac{1}{r^3} | \psi_{n,l,m} \rangle = \frac{2}{2a_B^n n^3 l(l+1)(2l+1)}, \quad (53)
\]

\[
\langle \psi_{n,l,m} | \frac{1}{r^4} | \psi_{n,l,m} \rangle = \frac{4(3n^2 - l(l+1))}{a_B^n n^5 l(l+1)(2l+1)(2l+3)(2l-1)}, \quad (54)
\]

we can easily obtain an explicit expression for the last term in (42).

Finally in the first order of the perturbation theory we find the following corrections

\[
\Delta E_{n,l}^{(1)} = -\frac{\hbar^2 e^2}{a_B^n n^5} \left( \frac{1}{6l(l+1)(2l+1)} - \frac{6n^2 - 2l(l+1)}{3l(l+1)(2l+1)(2l+3)(2l-1)} \right) + \frac{5n^2 - 3l(l+1) + 1}{2(l+2)(2l+1)(2l+3)(l-1)(2l-1)} - \frac{5}{6} \frac{5n^2 - 3l(l+1) + 1}{l(l+1)(2l+1)(2l+3)(l-1)(2l-1)}, \quad (55)
\]

In the second order of the perturbation theory we have

\[
\Delta E_{n,l,\{0,\}}^{(2)} = \sum_{n',l',m',\{p\},\{p\}} \left| \frac{\langle \psi_{n',l',m',\{p\},\{p\}} | V | \psi_{n,l,m,\{0\},\{0\}} \rangle}{E_n^{(0)} - E_p^{(0)} - \hbar \omega (p_1^2 + p_2^2 + p_3^2 + p_1^2 + p_2^2 + p_3^2)} \right|^2 \quad (56)
\]

where \( n' \neq n, l' \neq l, m' \neq m, p_i^a \neq 0, p_i^b \neq 0, i = 1, 2, 3 \), and \( E_n^{(0)} = -\frac{e^2}{2a_B n^2} \) is the unperturbed energy of the hydrogen atom. It is worth noting that because of our assumption (6) matrix elements \( \left| \langle \psi_{n',l',m',\{p\},\{p\}} | V | \psi_{n,l,m,\{0\},\{0\}} \rangle \right| \) do not depend on \( \omega \).

Therefore, in the case of \( \omega \to \infty \) we obtain

\[
\lim_{\omega \to \infty} \Delta E_{n,l,\{0,\}}^{(2)} = 0. \quad (57)
\]

Consequently, taking into account (55), (57), the corrections to the hydrogen atom energy levels up to the second order in parameter of noncommutativity read

\[
\Delta E_{n,l} = \Delta E_{n,l}^{(1)}. \quad (58)
\]

At the end of this section we would like to note that one can obtain the same correction to the energy spectrum defining an effective Hamiltonian. As was mentioned above, in the case of \( \omega \to \infty \) the harmonic oscillator is always in the ground state. Therefore, we construct an effective Hamiltonian in the following form

\[
H_{\text{eff}} = \langle \psi_{0,0,0}^a \psi_{0,0,0}^b | H | \psi_{0,0,0}^a \psi_{0,0,0}^b \rangle, \quad (59)
\]

Taking into account (28), (36), (45), we obtain

\[
H_{\text{eff}} = \langle \psi_{0,0,0}^a \psi_{0,0,0}^b | H | \psi_{0,0,0}^a \psi_{0,0,0}^b \rangle = \frac{p^2}{2M} - \frac{e^2}{r} - \frac{e^2 L^2}{8r^5} \langle \theta^2 \rangle + \frac{\hbar^2}{24} \left( \frac{1}{r^2} + \frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^2} \right) \langle \theta^2 \rangle. \quad (60)
\]

It is important to note that effective Hamiltonian (60) is rotationally invariant. Using the first order perturbation theory, we find the same corrections to the energy levels as (58).
It is worth mentioning that in the case of \( l = 0 \) or \( l = 1 \) corrections \([55]\) are divergent. The problem of divergence of the corrections also arrives in the deformed space with minimal length. In order to overcome this problem the modified perturbation theory was proposed \([36,37]\).

In this article in order to estimate the upper bound of the parameter of noncommutativity we are interested in the corrections to the \( n \) \( s \) levels. In next section we propose the way to find the corrections to the \( n s \) levels of the hydrogen atom in the rotationally invariant noncommutative space. We plan to consider the problem of divergence of the corrections in the case of \( l = 1 \) in a forthcoming publication.

5 Corrections to the \( n s \) levels of the hydrogen atom

In order to find the corrections to the \( n s \) levels let us rewrite the perturbation caused by the noncommutativity of coordinates in the following form

\[
V = -\frac{e^2}{R} + \frac{e^2}{r} = -\frac{e^2}{\sqrt{r^2 - (\theta \cdot L) + \frac{1}{4}[\theta \times p]^2}} + \frac{e^2}{r}.
\]

(61)

Using the perturbation theory, and taking into account \([57]\), the corrections to the \( n s \) levels read

\[
\Delta E_{ns} = \langle \psi_{n,0,0,0}^{(0)},\{0\} | \frac{e^2}{r} - \frac{e^2}{\sqrt{r^2 - (\theta \cdot L) + \frac{1}{4}[\theta \times p]^2}} \rangle \vert \psi_{n,0,0,0}^{(0)},\{0\} \rangle.
\]

(62)

Note that \((\theta \cdot L)\) commutes with \([\theta \times p]^2\) and \(r^2\). Also it is clear that \((\theta \cdot L)\vert \psi_{n,0,0,0}^{(0)},\{0\} \rangle = 0\) because \(\psi_{n,0,0,0}^{(0)},\{0\} \) does not depend on angles. Therefore, we can rewrite (62) in the following form

\[
\Delta E_{ns} = \langle \psi_{n,0,0,0}^{(0)},\{0\} | \frac{e^2}{r} - \frac{e^2}{\sqrt{r^2 + \frac{1}{4}[\theta \times p]^2}} \rangle \vert \psi_{n,0,0,0}^{(0)},\{0\} \rangle.
\]

(63)

Using (45), up to the second order in the parameter of noncommutativity we can write

\[
\Delta E_{ns} = \langle \psi_{n,0} \vert \frac{e^2}{r} - \frac{e^2}{\sqrt{r^2 + \frac{1}{6}(\theta^2)p^2}} \rangle \vert \psi_{n,0} \rangle = \langle R_{n,0} \vert \frac{e^2}{r} - \frac{e^2}{\sqrt{r^2 + \frac{1}{6}(\theta^2)p^2}} \rangle \vert R_{n,0} \rangle,
\]

(64)

where \(p_r = -i\hbar \frac{\partial}{\partial r}\), and \(R_{n,0} = \sqrt{\frac{\alpha}{\beta n}} e^{-\frac{r}{\alpha}} L_{n-1}^1 \left( \frac{2r}{\alpha \beta n} \right)\) is the radial wavefunction, \(L_{n-1}^1 \left( \frac{2r}{\alpha \beta n} \right)\) are the generalized Laguerre polynomials.

First, let us find the correction to the 1s energy level of the hydrogen atom. It is convenient to introduce the dimensionless coordinate \(\rho = r \left( \frac{\alpha}{\hbar^2 \beta^2} \right)^{\frac{1}{4}}\). Therefore, we have

\[
\Delta E_{1s} = \langle R_{1,0} \vert \frac{e^2}{r} - \frac{e^2}{\sqrt{r^2 + \frac{1}{6}(\theta^2)p^2}} \rangle \vert R_{1,0} \rangle = \frac{4e^2 \beta^2}{\alpha \beta} \int_0^\infty d\rho \rho^2 e^{-\beta \rho} \left( \frac{1}{\rho} - \frac{1}{\sqrt{\rho^2 + p^2}} \right) e^{-\beta \rho}.
\]

(65)
where \( p_\rho = -i \frac{\partial}{\partial \rho} \rho \) and \( \beta = \left( \frac{\hbar^2 (\theta^2)}{6 a_B} \right)^{1/2} \).

In order to calculate \( (65) \) we propose to expand \( e^{-\beta \rho} \) over the eigenfunctions of \( \rho^2 + p_\rho^2 \) which we denote \( \phi_k \)

\[
e^{-\beta \rho} = \sum_{k=0}^{\infty} C_k \phi_k. \tag{66}
\]

The eigenfunctions and the eigenvalues of \( \rho^2 + p_\rho^2 \) are as follows (see for instance \[38\])

\[
\phi_k = \sqrt{\frac{2k!}{\Gamma(k + \frac{3}{2})}} e^{-\frac{\rho^2}{2}} L_k^\frac{1}{2} (\rho^2), \tag{67}
\]

\[
\lambda_k = 2 \left( 2k + \frac{3}{2} \right). \tag{68}
\]

So, using \( (67) \), from \( (66) \), we find

\[
C_k = \sqrt{\frac{2k!}{\Gamma(k + \frac{3}{2})}} \int_0^\infty d\rho \rho^2 e^{-\frac{\rho^2}{2} - \beta \rho} L_k^\frac{1}{2} (\rho^2). \tag{69}
\]

As a result, we can write

\[
\int_0^\infty d\rho \rho^2 e^{-\beta \rho} \frac{1}{\sqrt{\rho^2 + p_\rho^2}} e^{-\beta \rho} = \sum_{k=0}^{\infty} \frac{C_k^2}{\sqrt{\lambda_k}}. \tag{70}
\]

Similarly for the first term in \( (65) \) we have

\[
\int_0^\infty d\rho e^{-2\beta \rho} = \sum_{k=0}^{\infty} C_k \int_0^\infty d\rho e^{-\beta \rho} \phi_k = \sum_{k=0}^{\infty} C_k I_k, \tag{71}
\]

where

\[
I_k = \sqrt{\frac{2k!}{\Gamma(k + \frac{3}{2})}} \int_0^\infty d\rho e^{-\frac{\rho^2}{2} - \beta \rho} L_k^\frac{1}{2} (\rho^2). \tag{72}
\]

So, taking into account \( (70), (71) \), the correction to the 1s energy level reads

\[
\Delta E_{1s} = \frac{4 e^2 \beta^2}{a_B} \sum_{k=0}^{\infty} \left( C_k I_k - \frac{C_k^2}{\sqrt{\lambda_k}} \right) = \frac{e^2 \beta^2}{a_B} S_{1s}(\beta), \tag{73}
\]

where we use the notation

\[
S_{1s}(\beta) = 4 \int_0^\infty d\rho \rho^2 e^{-\beta \rho} \left( \frac{1}{\rho} - \frac{1}{\sqrt{\rho^2 + p_\rho^2}} \right) e^{-\beta \rho} = 4 \sum_{k=0}^{\infty} \left( C_k I_k - \frac{C_k^2}{\sqrt{\lambda_k}} \right). \tag{74}
\]

It is worth noting that \( S_{1s}(0) \) has a finite value

\[
S_{1s}(0) = 16 \sqrt{\frac{2}{\pi}} \sum_{k=0}^{\infty} \frac{\Gamma(k + \frac{3}{2})}{k!} \left( _2F_1 \left( -k, \frac{1}{2}, \frac{3}{2}, \rho^2 \right) - \sqrt{\frac{\pi}{8k + 6}} \right) = 1.72006 \ldots \tag{75}
\]
where \( _2F_1(-k, \frac{1}{2}, \frac{3}{2}; 2) \) is the hypergeometric function. Therefore it is clear that the asymptotic of \( \Delta E_{1s} \) for \( \beta \to 0 \) (\( \theta \to 0 \)) is as follows

\[
\Delta E_{1s} = \frac{e^2 \beta^2}{a_B} S_{1s}(0) = \frac{e^2}{a_B^3} \sqrt{\frac{\hbar^2 \langle \theta^2 \rangle}{6}} S_{1s}(0).
\]  

(76)

Likewise, we can find the corrections to the excited \( s \) levels

\[
\Delta E_{ns} = \frac{e^2 \beta^2}{a_B n^3} S_{ns}(\beta),
\]

where

\[
S_{ns}(\beta) = \frac{4}{\pi} \int_0^\infty d\rho \rho^2 e^{-\frac{\beta \rho}{n}} L_{n-1}^1 \left( \frac{2\beta \rho}{n} \right) \left( \frac{1}{\rho} - \frac{1}{\sqrt{\rho^2 + \rho^2}} \right) e^{-\frac{\beta \rho}{n}} L_{n-1}^1 \left( \frac{2\beta \rho}{n} \right).
\]

(78)

It is easy to show that

\[
S_{ns}(0) = S_{1s}(0)n^2 \simeq 1.72n^2.
\]

(79)

As a result we obtain the following corrections to the \( ns \) levels of the hydrogen atom

\[
\Delta E_{ns} = \frac{e^2}{a_B^3 n^3} \sqrt{\frac{\hbar^2 \langle \theta^2 \rangle}{6}} S_{1s}(0).
\]

(80)

6 Estimation of the upper bound of the parameter of noncommutativity

Let us estimate the upper bound of the parameter of noncommutativity. For this purpose we use the result of measurement of the hydrogen \( 1s - 2s \) transition frequency [39]. The authors obtained \( f_{1s-2s} = 2466061413187018(11) \text{Hz} \) with relative uncertainty of \( 4.5 \times 10^{-15} \).

Using (80), the correction to the energy of the \( 1s - 2s \) transition reads

\[
\Delta_{1,2} = \Delta E_{2s} - \Delta E_{1s} = -\frac{7e^2 S_{1s}(0)}{8\sqrt{6}a_B^3} \hbar \sqrt{\langle \theta^2 \rangle},
\]

(81)

\[
\frac{\Delta_{1,2}}{E_{2}^{(0)} - E_{1}^{(0)}} = -\frac{7}{3\sqrt{6}} \frac{S_{1s}(0)}{a_B^2} \hbar \sqrt{\langle \theta^2 \rangle},
\]

(82)

where \( E_n^{(0)} = -\frac{e^2}{2\alpha_B n^2} \) is the unperturbed energy of the hydrogen atom. Assuming that \( \frac{\Delta_{1,2}}{E_{2}^{(0)} - E_{1}^{(0)}} \) does not exceed \( 4.5 \times 10^{-15} \), we find

\[
\frac{7}{3\sqrt{6}} \frac{S_{1s}(0)}{a_B^2} \hbar \sqrt{\langle \theta^2 \rangle} \leq 4.5 \times 10^{-15},
\]

(83)

\[
\hbar \sqrt{\langle \theta^2 \rangle} \leq 7.7 \times 10^{-36} \text{m}^2.
\]

(84)

It is worth mentioning that our result (84) is stronger than the upper bound obtained on the basis of the data on the Lamb shift in [5].

Taking into account (46), we can also estimate the value of \( \alpha \)

\[
\alpha \leq 2.4 \times 10^{34}.
\]

(85)
7 Conclusion

In this article we have considered an important problem of the rotational symmetry breaking in noncommutative space. In order to preserve this symmetry we proposed the generalization of the tensor of noncommutativity (4) or (27) which gives the possibility to construct rotationally invariant algebras.

The hydrogen atom has been examined in the rotationally invariant noncommutative space. Using the perturbation theory, the corrections to the hydrogen atom energy levels have been found. It is worth noting that we have faced a problem of divergence of the corrections to the $ns$ and $np$ energy levels. For estimation of the upper bound of the parameter of noncommutativity we have used the experimental result for $1s - 2s$ transition frequency which is measured with a high accuracy [39]. Therefore, the way to find the corrections to the $ns$ levels has been proposed. Comparing our results with the experimental data we have found

$$\hbar \sqrt{\langle \theta^2 \rangle} \leq 7.7 \times 10^{-36} \text{ m}^2.$$  \hspace{1cm} (86)

This result is stronger then the upper bound estimated on the basis of the data on the Lamb shift in [5].

In addition, an effective rotationally invariant Hamiltonian has been constructed. We have defined the effective Hamiltonian of the hydrogen atom and obtained the corrections to the energy levels. It is worth noting that these results are in agreement with corrections [58] obtained from the total Hamiltonian [29].

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