Universal relations of charged-rotating-AdS black holes with small correction in presence of quintessence and cloud of string with verification of WGC

J. Sadeghi, S. Noori Gashti, E. Naghd Mezerji

*Department of Physics, Faculty of Basic Sciences, University of Mazandaran P. O. Box 47416-95447, Babolsar, Iran

Abstract

Researchers have recently applied a series corrections to general relativity to find the relationship between entropy and extremality bound of black hole. This relationship has been investigated for many black holes, such as charged AdS, rotating and massive gravity black holes. Now we give small constant correction to the action and confirm these universal relations for a charged-rotating-AdS black hole. Then we examine these calculations for the black hole which is surrounded by the quintessence and also cloud of string. In this paper, we evaluate a new universal relation. It means that we find the relation between the extremity mass of the black hole and factor of cloud string $-\zeta \frac{\partial b}{\partial \epsilon} = \frac{\partial M_{ext}}{\partial \epsilon}$ and observe that the corresponding universal relation is well established. In fact, we note here that the added constant correction is inversely related to the entropy of the black hole. It leads us to see that the mass-to-charge ratio decreases and fully confirms the weak gravity conjecture (WGC) issue in the black hole.

Keywords: Universal relations, Charged-rotating-AdS black holes, Quintessence, Cloud of string, WGC.

1Email: pouriya@ipm.ir
2Email: saeed.noorigashti@stu.umz.ac.ir
3Email: e.n.mezerji@stu.umz.ac.ir
1 Introduction

As we know, the black hole is best object for the developing of gravity as form of quantum gravity. Recently, quantum gravity has been studied from different perspectives, such as the framework of low-energy effective field theories. In that case, string theory provides a complete description of quantum gravity. Therefore, for studies of quantum gravity in string theory, various conjectures and methods are always used, including the swampland program and weak gravity conjecture [1–19]. If we want to have quantum gravity as a form of low-energy theories, we need some object as form of black hole background with special constraint as the charge-to-mass ratio must always be greater than one $Q \geq M$. This will be best constraint for the extremal black holes to have evaporating. If this conjecture is not correct, the concepts related to black hole evaporation and many other concepts that have been approved by researchers will face major problems. For example, one of these issues is the violation of cosmic censorship. The development of this conjecture can suggested by the string theory for introducing new way to find some compatible theory [20–34]. As mentioned in the text, many attempts have been done to calculate the universal relations in recent years. Recently Goon and Penco [35] have presented a universal thermodynamic relation due to the perturbative corrections to the thermodynamic relations. This universal relation has been proven for charged AdS black holes and investigated in other works related to rotating as well as massive gravity black holes [36–39]. Now with all the above mentioned concepts, we want to prove this universal relation for a Kerr-Newman-AdS black hole which is surrounded by quintessence as well as quintessence with cloud of string. After than we will try to compare the universal relation of different form of black holes. Before going further, we first give some review to these physical subject. As we know, all current observations represent an expanding universe that has acceleration due to negative pressure. This negative pressure can have interpretations such that one of these interpretations is quintessence [40–44]. Quintessence is actually described by a typical scalar field which is minimally coupled with gravity, due to the anti-gravity nature of dark energy, which always has certain potentials that lead to late time inflation. In the case of extremal black holes surrounded by quintessence, very specific changes can be considered, such as black holes that do not have a singularity [44–46]. Other case we consider here is the black hole with the cloud of string and the analysis of these string clouds was first reviewed by Letelier [47]. He studied Schwarzschild’s expansion kind of black hole is investigated by [47]. Also the solutions of black hole surrounded by spherical symmetry cloud of string is studied by [48, 49]. In this article, our main goal is to explore a new implication for this string cloud. In fact, we are looking for a new universal thermodynamic relation for the corresponding system. All of the above concepts give us motivation to confirm universal relations for the charged-rotating-AdS black hole. Hence we first examine the universal relations for this black hole, so we add a small constant correction to the action and we obtain the modified thermodynamic relations. By performing a series of straightforward calculations, we obtain the universal relation. Then,
taking into account the quintessence, we calculate all the mentioned steps for this black hole. Then we consider the cloud of string as a new feature for Kerr-Newman-AdS black hole. We investigate the all of the universal relation, especially we evaluate a new universal relation. It is relation between the mass of the black hole and the factor related to cloud of string, ie $-\zeta \frac{\partial b}{\partial \epsilon} = \frac{\partial M_{ext}}{\partial \epsilon}$. By solving some complicated equations we obtain the modified thermodynamic parameters as mass, entropy and etc. Here we note that when the added correction constant has a negative value, the entropy of the black hole increases. Also the mass of the black hole and the mass-charge ratio decrease. Hence these black holes show WGC-like behavior. Finally when we compare the obtained results from different black hole from universal relation point of view we will see that the additional correction constant plays a very influential role in the thermodynamic parameters of the black hole.

All above information give us motivation to organize this article in this way. In Sections 2, we confirm the universal relations for the Kerr-Newman-AdS black hole due to a small constant correction. This calculation help us to obtain the new universal relation of above mentioned black hole surrounded by quintessence and quintessence with cloud of string which are discussed by section 3 and 4 respectively. Finally in section 5, we describe the results of universal relation and compare the results of different black hole.

2 Kerr-Newman-AdS black hole

In this section, we are going to investigate the universal relation and also show the weak gravity conjecture how come to game in the corresponding black holes. In order to study the universal relation, we have to consider generally thermodynamic relations such as temperature and mass and angular velocity according to the solution of action which is given by $dM = TdS + \Phi dQ + VdP + \Omega dJ + \eta d\alpha + \zeta db$. On the other hand, we need to some black hole like solution to prove the universal relations. So, for these reasons, we introduce the Einstein-Maxwell-AdS action in four dimensions, which is given by [50, 51],

$$S = \frac{1}{16\pi G} \int_M d^4x \sqrt{-g} (R - F^2 + 2\Lambda)$$

where $F = dA$, $A$ is the potential 1-form and $\Lambda = -\frac{3}{l^2}$. The solution of action (1) will be Kerr-Newman AdS black hole which is given by [50, 51],

$$ds^2 = -\frac{f(r)}{\rho^2} (dt - \frac{a \sin^2 \theta}{\Xi} d\phi)^2 + \frac{\rho^2}{f(r)} dr^2 + \frac{\rho^2}{f(\theta)} d\theta^2 + \frac{f(\theta) \sin^2 \theta}{\rho^2} (adt - \frac{r^2 + a^2}{\Xi} d\phi)^2,$$
and $f(r)$ is,

$$f(r) = r^2 - 2Mr + a^2 + Q^2 + \frac{r^2}{l^2}(r^2 + a^2), \quad \Xi = 1 - \frac{a^2}{l^2}$$

$$\rho^2 = r^2 + a^2 \sin^2 \theta, \quad f(\theta) = 1 - \frac{a^2}{l^2} \cos^2 \theta,$$

where the $M$, $Q$ and $a$ are the mass, charge and rotational parameter respectively. The outer and inner event horizons associated with a black hole are calculated from $f(r) = 0$. So, in that case we can easily calculate the thermodynamic relation of Kerr-Newman-AdS black hole such as mass, temperature and angular velocity which are given by,

$$M = \frac{a^2 \sqrt{\pi}}{2 \sqrt{S}} + \frac{\sqrt{\pi}Q^2}{2 \sqrt{S}} + \frac{\sqrt{S}}{2 \sqrt{\pi}} + \frac{a^2 \sqrt{\pi}}{2 \sqrt{l^2 \sqrt{\pi}}} + \frac{S^3}{2 l^2 \pi^2},$$

$$T = -\frac{a^2 \sqrt{\pi}}{4S^{\frac{3}{4}}} - \frac{\sqrt{\pi}Q^2}{4S^{\frac{3}{4}}} + \frac{S^3}{4 \sqrt{\pi} \sqrt{l}} + \frac{a^2 \sqrt{S}}{4l^2 \sqrt{\pi} \sqrt{l}} + \frac{3\sqrt{S}(1 + \epsilon)}{4l^2 \pi^2},$$

and

$$\Omega = +\frac{a\sqrt{\pi}}{\sqrt{S}} + \frac{a\sqrt{S}}{l^2 \sqrt{\pi}}.$$

Now we are going to give some small correction as $\epsilon$ to the above action, so the modified form of action will be as,

$$S = -\frac{1}{16\pi G} \int_M d^4x \sqrt{-g} (R - F^2 + (1 + \epsilon) \times 2\Lambda).$$

Due to the modification of the action, the black hole solution is also modified. Therefore, all thermodynamic quantities of black holes will be also modified. Hence, the modified mass and temperature are obtained by the following equations,

$$M = \frac{a^2 \sqrt{\pi}}{2 \sqrt{S}} + \frac{\sqrt{\pi}Q^2}{2 \sqrt{S}} + \frac{\sqrt{S}}{2 \sqrt{\pi}} + \frac{a^2 \sqrt{\pi}}{2 \sqrt{l^2 \sqrt{\pi}}} + \frac{(1 + \epsilon)S^3}{2 l^2 \pi^2},$$

$$T = -\frac{a^2 \sqrt{\pi}}{4S^{\frac{3}{4}}} - \frac{\sqrt{\pi}Q^2}{4S^{\frac{3}{4}}} + \frac{S^3}{4 \sqrt{\pi} \sqrt{l}} + \frac{a^2 \sqrt{S}}{4l^2 \sqrt{\pi} \sqrt{l}} + \frac{3\sqrt{S}(1 + \epsilon)}{4l^2 \pi^2},$$

and

$$\Omega = +\frac{a\sqrt{\pi}}{\sqrt{S}} + \frac{a\sqrt{S}}{l^2 \sqrt{\pi}}.$$

According to the above modified expressions, the mass and the entropy of a black hole are modified with a small constant correction. It can be stated that when the added correction is
continuously negative, the mass of the black hole decreases and the mass-charge ratio of
the black hole decreases and approaches one. In fact, these change of mass and the mass-charge
ratio of the black hole are essentially a confirmation of the weak gravity conjecture (WGC). In
order to obtain new universal relation, we use equation (8) and one can write the correction
parameter as,
\[ \epsilon = -\frac{a^2l^2\pi^2 - l^2Q^2\pi^2 + 2l^2M\pi\frac{3}{2}\sqrt{S} - a^2\pi S - l^2\pi S - S^2}{S^2}. \] (11)
Here we take derivative with respect to \( S \) according to equation (11), we will have,
\[ \frac{\partial \epsilon}{\partial S} = \frac{\pi(a^2(2l^2\pi + S) + l^2(2\pi Q^2 - 3M\sqrt{\pi}\sqrt{S} + S))}{S^3}. \] (12)
Now we use the equations (9) and (12) and obtain \( T\frac{\partial S}{\partial \epsilon} \). In that case we arrive long terms,
this lead us to have manipulation with some limitation for the corresponding terms. All above
information help us to simplify the obtained results. So, finally one can obtain the following
relation,
\[ -T\frac{\partial S}{\partial \epsilon} = S^3 \frac{2}{2l^2\pi^2} \] (13)
In order to obtain the second part of the universal relation, we use equations (8) and (9). In
that case, the extremal mass help us the obtain following equation,
\[ \frac{\partial M_{\text{ext}}}{\partial \epsilon} = S^3 \frac{2}{2l^2\pi^2}. \] (14)
Here, we see that two equations (13) and (14) are exactly same. In fact, we first proved the
Goon-Penco universal extremality relation for this black hole. Now we are going to examine
another universal relation. In that case just like the previous results, we use relation (11). So,
we will have,
\[ \frac{\partial \epsilon}{\partial Q} = \frac{2l^2\pi^2 Q}{S^2}. \] (15)
By considering the electric potential as \( \Phi = \frac{\sqrt{\pi}Q}{\sqrt{S}} \) from the Kerr-Newman-AdS black hole
and assuming externality bound, one can rewrite equation (15) as,
\[ -\Phi \frac{\partial Q}{\partial \epsilon} = S^3 \frac{2}{2l^2\pi^2}. \] (16)
As we can see, the equation (16) and (14) are same. In this way another universal relation is
also proved.
In the following we will try to seek another universal relation. This universal relation coming
from relation between mass and pressure $P = \frac{3}{8\pi l^2} = -\frac{\Lambda}{8\pi}$. Also here we use equation (11) and write the following derivative,

$$\frac{\partial \epsilon}{\partial P} = \frac{3(\pi(a^2 + Q^2) - 2M\sqrt{\pi}S + S}{8P^2S^2}.$$  \hspace{1cm} (17)

Therefore, according to the thermodynamic relation related to the black hole such as $V = \frac{4}{3}a^2\sqrt{\pi}S + \frac{4(1+\epsilon)S^2}{3\sqrt{\pi}}$ as well as extremal bound, we will arrive following equation,

$$-V\frac{\partial P}{\partial \epsilon} = \frac{S^3}{2l^2\pi^{\frac{3}{2}}}.$$  \hspace{1cm} (18)

Here also we see that the two equations (18) and (14) are exactly same. So, we see here that another universal relation is also confirmed. Now we are going to consider rotation of black hole and prove last universal relation. In that case also we use equation (11) and give opportunity to parameter of rotation to play important role in universal relation, so we have following relation,

$$\frac{\partial \epsilon}{\partial a} = \frac{-2a l^2\pi^2 - 2a\pi S}{S^2}.$$  \hspace{1cm} (19)

In order to achieve the last universal relation we take equation (10) and (19) for the corresponding black hole, so one can obtain,

$$-\Omega\frac{\partial a}{\partial \epsilon} = \frac{S^3}{2l^2\pi^{\frac{3}{2}}}.$$  \hspace{1cm} (20)

Also here we see that two equations (20) and (14) are exactly same. We note here in the next section we obtain the universal relations for this black hole which is surrounded by quintessence and compare it with the results obtained in this section. The corresponding figures lead us to compare two cases as unmodified and modified mass. Here, therefor we fix some parameters and we plot the mass in terms of charge $Q$ of black holes. The initial critical state is when the mass-to-charge ratio is one, shown as dash form. It can be seen that the mass ratio of the unmodified black hole to the amount of charge is more than one. We consider different modes of AdS space radius $l$ as 0.1, 0.2, 0.3. As we can see in the figure (a), the curvature of the mass lines is different from the charge of the black hole for different modes. As we can see in the figure (b), by using a small constant correction for the mass, we compare that with its unmodified state. As shown in Figure (b), when the constant correction is positive, the mass of the black hole increases, and when this constant correction is negative, the mass of the black hole decreases. Of course, in the numerical calculation of entropy, this happens in reverse. It means that the entropy decreases with a positive correction. In fact, according to the concepts mentioned above, when we consider the small correction as negative, the mass of a black hole decreases to one and the charge-to-mass ratio increases or the mass-charge ratio decreases, which is completely satisfied by the weak gravity conjecture (WGC).
In the previous section, we examined the universal relationship for the Kerr-Newman-AdS black hole without any extra terms. Now in this section we will confirm these universal relations for this black hole while surrounded by quintessence. So the action for Kerr-Newman-AdS black hole surrounded by quintessence dark energy likewise the previous section is expressed in the following form [51, 52],

$$S = -\frac{1}{16\pi G} \int_M d^4x \{ \sqrt{-g} (R - F^2 + 2\Lambda) + L_q \},$$  \hspace{1cm} (21)

where the $L_q$ is Lagrangian of quintessence as a barotropic perfect fluid, which is given by [53],

$$L_q = -\rho(c^2 + \int \frac{P(\rho)}{\rho^2} d\rho) = -\rho_q(1 + \omega_q \ln(\rho_q/\rho_0)),$$  \hspace{1cm} (22)

where $\rho_0$ and $\rho_q$ are the constant parameter of integral and energy density. $\omega_q$ is the barotropic index. The equation of state is given by $\frac{P}{\rho_q} = \omega_q$, which is bounded by $-1 < \omega_q < -\frac{1}{3}$ for the quintessence dark energy. Of course, the equation of state also is bounded by $-1 > \omega_q$ for the phantom dark energy. Therefore, the metric of Kerr-Newman-AdS black hole surrounded by quintessence dark energy is given by [51, 52]

$$ds^2 = \frac{\Sigma^2}{f(r)} dr^2 + \frac{\Sigma^2}{f(\theta)} d\theta^2 + \frac{f(\theta) \sin^2 \theta}{\Sigma^2} (a \frac{dt}{\Xi} - (r^2 + a^2) \frac{d\phi}{\Xi})^2 - \frac{f(r)}{\Sigma^2} \left( \frac{dt}{\Xi} - a \sin^2 \frac{d\phi}{\Xi} \right)^2,$$  \hspace{1cm} (23)
and $f(r)$ is,

$$f(r) = r^2 - 2Mr + a^2 + Q^2 - \frac{\Lambda}{3} r^2 (r^2 + 2) - \alpha r^{1-3\omega}$$

$$f(\theta) = 1 + \frac{\Lambda}{3} a^2 \cos^2 \theta, \quad \Xi = 1 + \frac{\Lambda}{3} a^2$$

Here also one can write the thermodynamic quantities of Kerr-Newman-AdS black hole surrounded by quintessence such as mass, temperature and angular velocity which are given by,

$$M = \frac{a^2 \sqrt{\pi}}{2 \sqrt{S}} + \frac{\sqrt{\pi} Q^2}{2 \sqrt{S}} + \frac{\sqrt{S}}{2 \sqrt{\pi}} + \frac{a^2 \sqrt{\pi}}{2l^2 \sqrt{\pi}} + \frac{S^\frac{3}{2}}{2l^2 \pi^\frac{3}{2}} - \frac{1}{2} \pi^{\frac{3}{2}} S^{-\frac{3}{2}} \alpha,$$

$$T = -\frac{a^2 \sqrt{\pi}}{4S^\frac{3}{2}} - \frac{\sqrt{\pi} Q^2}{4S^\frac{3}{2}} + \frac{1}{4 \sqrt{\pi} \sqrt{S}} + \frac{a^2}{4l^2 \sqrt{S} \sqrt{\pi}} + \frac{3\sqrt{S}}{4l^2 \pi^\frac{3}{2}} + \frac{3}{4} \pi^{\frac{3}{2}} S^{-1-\frac{3}{2}} \alpha \omega,$$

and

$$\Omega = \frac{a \sqrt{\pi}}{\sqrt{S}} + \frac{a \sqrt{S}}{l^2 \sqrt{\pi}}$$

Now we are going to give small constant correction as $\varepsilon$ to the corresponding action. So the corrected action will be as,

$$S = -\frac{1}{16\pi G} \int_M d^4x \{-\sqrt{-g} (R - F^2 + 2(1+\varepsilon) \times \Lambda) + L_q\}$$

So, we have modified thermodynamic quantities for the corresponding black hole in the following,

$$M = \frac{a^2 \sqrt{\pi}}{2 \sqrt{S}} + \frac{\sqrt{\pi} Q^2}{2 \sqrt{S}} + \frac{\sqrt{S}}{2 \sqrt{\pi}} + \frac{a^2 \sqrt{\pi}}{2l^2 \sqrt{\pi}} + \frac{S^\frac{3}{2}(1+\varepsilon)}{2l^2 \pi^\frac{3}{2}} - \frac{1}{2} \pi^{\frac{3}{2}} S^{-\frac{3}{2}} \alpha,$$

$$T = -\frac{a^2 \sqrt{\pi}}{4S^\frac{3}{2}} - \frac{\sqrt{\pi} Q^2}{4S^\frac{3}{2}} + \frac{1}{4 \sqrt{\pi} \sqrt{S}} + \frac{a^2}{4l^2 \sqrt{S} \sqrt{\pi}} + \frac{3(1+\varepsilon) \sqrt{S}}{4l^2 \pi^\frac{3}{2}} + \frac{3}{4} \pi^{\frac{3}{2}} S^{-1-\frac{3}{2}} \alpha \omega,$$

and

$$\Omega = \frac{a \sqrt{\pi}}{\sqrt{S}} + \frac{a \sqrt{S}}{l^2 \sqrt{\pi}}$$

With respect to the above thermodynamic expression, we will introduce some quantities such as the conjugate to $\alpha$ that means the $\eta = -\frac{1}{2} \pi^{\frac{3}{2}} S^{-\frac{3}{2}}$, electric potential $\Phi = \frac{\sqrt{\pi} Q}{\sqrt{S}}$ and volume $V = \frac{4}{3} a^2 \sqrt{\pi} S + \frac{4(1+\varepsilon) S^\frac{3}{2}}{3 \pi^\frac{3}{2}}$ as well as $P = -\frac{\Lambda}{8\pi}$. The mass and the entropy of black hole are modified with a small constant correction. It can be stated that when the added correction is continuously negative, the mass of the black hole decreases and the mass-charge ratio of the
black hole decreases and approaches one. In fact, these changes give us a clue to the weak
gravity conjecture, exactly like the pervious section. Also, by solving the equation (29), the
constant correction parameter $\epsilon$ is calculated by,

$$
\epsilon = -1 - \frac{a^2 l^2 \pi^2}{S^2} - \frac{Q^2 l^2 \pi^2}{S^2} + \frac{2 l^2 M \pi^\frac{3}{2}}{S^\frac{5}{2}} - \frac{a^2 \pi}{S} - \frac{l^2 \pi^\frac{1}{2} + \frac{3\omega}{2} S^{-\frac{1}{2}}}{S^{-\frac{1}{2}}} \alpha.
$$

(32)

Then we take the derivative from $S$, so we will have,

$$
\frac{\partial \epsilon}{\partial S} = \frac{1}{2} \frac{a^2 l^2 \pi^2}{S^3} + \frac{2 Q^2 l^2 \pi^2}{S^3} - \frac{3 l^2 M \pi^\frac{3}{2}}{S^\frac{5}{2}} + \frac{a^2 \pi}{S^2} + \frac{l^2 \pi^\frac{1}{2} + \frac{3\omega}{2} S^{-\frac{1}{2}}}{S^{-\frac{1}{2}}} (-\frac{3}{2} - \frac{3\omega}{2})
$$

(33)

Now by combining the two equation (30) and (33), the first equation for universal relation is,

$$
-T \frac{\partial S}{\partial \epsilon} = \frac{S^\frac{3}{2}}{2 l^2 \pi^\frac{7}{2}}
$$

(34)

In order to obtain the second relation for the completing this universal relation, we have to
use the $T = 0$ and solve the temperature. To obtain modify entropy, the calculation is very
difficult, so we need to do some simplification. Hence we obtain the extremal mass with respect
to $T = 0$, so the equation (29) lead us to arrive following equation,

$$
\frac{\partial M_{\text{ext}}}{\partial \epsilon} = \frac{S^\frac{3}{2}}{2 l^2 \pi^\frac{7}{2}}
$$

(35)

Here we note that two equations (34) and (35) are exactly same. So, we first confirmed the
Goon-Penco universal extremality relation for this black hole. To investigate another universal
relation, just like the previous part with respect to relation (32), we will have,

$$
\frac{\partial \epsilon}{\partial Q} = -\frac{2 l^2 \pi^2 Q^2}{S^2}
$$

(36)

By considering electric potential $\Phi$, assuming externality bound and using equation (36), we
will obtain following equation,

$$
-\Phi \frac{\partial Q}{\partial \epsilon} = \frac{S^\frac{3}{2}}{2 l^2 \pi^\frac{7}{2}}
$$

(37)

Here we see that the equations (37) and (35) are same. So the second universal relation is
also proved. In the following, we seek to confirm other universal relations. So, with respect to
pressure $P = \frac{3}{8 \pi l^2} = -\frac{\lambda}{8 \pi}$ as well as equation (32) we will have,

$$
\frac{\partial \epsilon}{\partial P} = \frac{3 (a^2 \pi + \pi Q^2 - 2 M \sqrt{\pi} \sqrt{S} + S - \pi^\frac{1}{2} + \frac{3\omega}{2} \pi^\frac{1}{2} - \frac{3\omega}{2} \alpha)}{8 P^2 S^2}
$$

(38)
according to the thermodynamic relation of black hole as well as extremal bound, one can obtain,

\[-V \frac{\partial P}{\partial \epsilon} = \frac{S^{\frac{3}{2}}}{2l^2 \pi^{\frac{3}{2}}}\]  

(39)

Also here we see that two equations (39) and (35) are exactly same. After confirming the universal relations stated in this section, given the rotation feature of mentioned black hole, we also study the universal relationship associated with this feature of the black hole. So, according to relation (32) we have,

\[\frac{\partial \epsilon}{\partial a} = -\frac{2a \pi (l^2 \pi + S)}{S^2}\]  

(40)

So by using the equation (31) and (40), one can obtain,

\[-\Omega \frac{\partial a}{\partial \epsilon} = \frac{S^{\frac{3}{2}}}{2l^2 \pi^{\frac{3}{2}}}\]  

(41)

As we can see, two equations (41) and (35) are exactly same. After reviewing and confirming the universal relations for the charged-rotating-AdS black hole surrounded by quintessence, now we want to examine the new universal relation that is related to the dark energy parameter. Hence according to equation (32), we have following equation,

\[\frac{\partial \epsilon}{\partial \alpha} = l^2 \pi^{\frac{3}{2}} + \frac{3 \omega}{2} S - \frac{3}{2} - \frac{3 \omega}{2}\]  

(42)

So by using the \(\eta\) and the equation (42), we confirmed the last universal relation for this black hole which is calculated as follow,

\[-\eta \frac{\partial \alpha}{\partial \epsilon} = \frac{S^{\frac{3}{2}}}{2l^2 \pi^{\frac{3}{2}}} = \frac{\partial M_{ext}}{\partial \epsilon}\]  

(43)

Here also we use figures and compare the two unmodified and modified mass. Therefor, we fix some parameters and we plot the mass diagram in terms of Q. It can be seen that the mass ratio of the unmodified black hole to the amount of charge is more than one. We consider different modes of AdS space radius l as 0.1, 0.2, 0.3, fixed density of quintessence \(\rho_q = 1.0\) and \(\omega_q = -\frac{2}{3}\). As we can see in the figure (a), the curvature of the mass lines is different from the charge of the black hole for different modes. Now, we will perform a numerical analysis for the mass of the black hole and we give some description to the entropy. As we can see in the figure (b), by using a small constant correction for the mass, we compare that with its unmodified state. As shown in Figure (b), when we consider the small correction as negative, the mass of a black hole decreases to one, and in the same way, the charge-to-mass ratio increases, which is satisfied by the weak gravity conjecture. As we can see, the obtained results show us the quintessence terms play a very effective role in the calculation of the mass-charge ratio and concept of weak gravity conjecture of black holes.
4 Kerr–Newman–AdS black hole with quintessence and cloud of strings

In this section, we will do exactly same as previous two sections, except that in this section we will add another term, i.e., cloud of strings to Kerr–Newman–AdS black hole surrounded by quintessence. Therefore, the metric for Kerr–Newman–AdS black hole with quintessence and cloud of strings is expressed in the following form [54],

\[
\begin{align*}
    ds^2 &= \frac{\Sigma}{f(r)}(dt^2 + \frac{\Sigma}{f(\theta)}d\theta^2 + \frac{f(\theta)\sin^2 \theta}{\Sigma}(a \frac{dt}{\Xi} - (r^2 + a^2) \frac{d\phi}{\Xi}))^2 \\
    &= \frac{f(r)}{\Sigma}(\frac{dt}{\Xi} - a \sin^2 \theta \frac{d\phi}{\Xi})^2
\end{align*}
\]

(44)

where

\[
    f(r) = (1 - b)r^2 + a^2 + Q^2 - 2Mr - \frac{\Lambda}{3}r^2(r^2 + a^2) - \alpha r^{1-3\omega_q}
\]

\[
    f(\theta) = 1 + \frac{\Lambda}{3}a^2 \cos^2 \theta,
\]

\[
    \Xi = 1 + \frac{\Lambda}{3}a^2
\]

(45)

where the \( M, Q \) and \( a \) are the mass, charge and rotational parameter respectively. By using the \( f(r) = 0 \) we obtain the temperature, entropy and etc. Here also we apply a small constant correction to the action and we calculate the modified thermodynamic relation for the
corresponding black hole. So, same as before we have following equations,

\[ M = \frac{a^2\sqrt{\pi}}{2\sqrt{S}} + \frac{\sqrt{\pi}Q^2}{2\sqrt{S}} + \frac{\sqrt{S}}{2\sqrt{\pi}} - \frac{b\sqrt{S}}{2\sqrt{\pi}} + \frac{a^2\sqrt{S}}{2l^2\sqrt{\pi}} + \frac{(1 + \epsilon)S^\frac{3}{2}}{2l^2\pi^\frac{1}{2}} - \frac{1}{2} \pi^{\frac{3}{2}} S^{-\frac{3}{2}} \alpha, \quad (46) \]

\[ T = -\frac{a^2\sqrt{\pi}}{4S^{\frac{3}{2}}} - \frac{\sqrt{\pi}Q^2}{4S^{\frac{3}{2}}} + \frac{1}{4\sqrt{S}\pi} - \frac{b}{4\sqrt{S}\pi} + \frac{a^2}{4l^2\sqrt{S}\pi} + \frac{3(1 + \epsilon)S}{4l^2\pi^{\frac{1}{2}}} + \frac{4}{3} \pi^{\frac{3}{2}} S^{-1 - \frac{3}{2}} \alpha \omega, \quad (47) \]

and

\[ \Omega = \frac{a(l^2\pi + S)}{l^2\pi S} \quad (48) \]

Then we will investigate some quantities with respect to thermodynamic relation such as, \( \eta = -\frac{1}{2} \pi \frac{d\ln S}{d\pi} \frac{3\omega}{2} \) and electric potential \( \Phi = \frac{\sqrt{\pi}Q}{\sqrt{S}} \) and volume \( V = \frac{4\sqrt{S}(a^2\pi + S + \delta)}{3\sqrt{\pi}} \) as well as \( P = -\frac{A}{8\pi} \) and the important relation for this section \( \zeta = -\frac{\sqrt{S}}{2\sqrt{\pi}} \) that conjugate to \( b \). The mass and entropy of black hole are modified with a small constant correction. It can be stated that when the added correction is continuously negative, the mass of the black hole decreases and the mass-charge ratio of the black hole decreases and approaches one. In fact, these changes give us a clue to the weak gravity conjecture. By solving the equation (46), the constant correction parameter \( \epsilon \) is calculated by following equation,

\[ \epsilon = -1 + \frac{a^2\pi(2\pi + S) + l^2\pi(-\pi Q^2 + 2M\sqrt{\pi}S + (-1 + b)S + \pi^{\frac{3}{2}} + \frac{3\omega}{2} S^{\frac{3}{2}} - \frac{3\omega}{2} \alpha (1 + \omega))}{S^2} \quad (49) \]

So, we take derivative with respect to \( S \) and we have,

\[ \frac{\partial \epsilon}{\partial S} = \frac{2a^2l^2\pi^2 + 2l^2\pi^2Q^2 - 3l^2M\pi^\frac{3}{2}\sqrt{S} + a^2\pi S + l^2\pi S - bl^2\pi S - \frac{3}{2}l^2\pi^\frac{3}{2} S^{\frac{3}{2}} - \frac{3\omega}{2} \alpha (1 + \omega)}{S^3} \quad (50) \]

Now by combining the two equation (47) and (50), on can rewrite following equation,

\[ -T \frac{\partial S}{\partial \epsilon} = \frac{S^\frac{3}{2}}{2l^2\pi^\frac{1}{2}} \quad (51) \]

In order to obtain the second relation for completing the corresponding universal relation, we assume \( T = 0 \) we solve the temperature equation. But this calculation is very difficult and needs to be simplified for the obtain modify entropy. Hence, we employ extremal mass and equation (47), we obtain,

\[ \frac{\partial M_{\text{ext}}}{\partial \epsilon} = \frac{S^\frac{3}{2}}{2l^2\pi^\frac{1}{2}} \quad (52) \]
The two equations (51) and (52) are exactly the same. We first confirm the Goon-Penco
universal extremality relation for this black hole. Now we want confirm the other universal
relation, likewise the pervious sections. So with respect to relation (49), we will have.

\[
\frac{\partial \epsilon}{\partial Q} = -\frac{2l^2 \pi^2 Q}{S^2}
\]  

(53)

Also by using the equation (53), the electric potential \( \Phi \) and extremality bound, we have,

\[
-\Phi \frac{\partial Q}{\partial \epsilon} = \frac{S^2}{2l^2 \pi^2}
\]  

(54)

As we can see the equation (54) and (52) are same. In the following, we use the pressure \( P = \frac{3}{8 \pi l^2} = -\frac{\Lambda}{8 \pi} \) and equation (49), one can obtain,

\[
\frac{\partial P}{\partial \epsilon} = -\frac{3S}{8l^2 \pi (a^2 \pi + S + S \epsilon)}
\]  

(55)

Here, according to the thermodynamic relation of black hole such as \( V \) as well as extremal
bound one can obtain following equation,

\[
-\frac{\partial P}{\partial \epsilon} = \frac{S^2}{2l^2 \pi^2}
\]  

(56)

Here two equations (56) and (52) are exactly same. Also according to relation (49), we will have

\[
\frac{\partial \epsilon}{\partial a} = -\frac{2a \pi (l^2 \pi + S)}{S^2}
\]  

(57)

So by using the equation (48) and (57), one can obtain,

\[
-\frac{\partial a}{\partial \epsilon} = \frac{S^2}{2l^2 \pi^2}
\]  

(58)

As we see here, the two equations (58) and (52) are exactly same. Now we are going to examine
the another universal relation that is related to the dark energy parameter. Hence according
to equation (49), one can we have,

\[
\frac{\partial \epsilon}{\partial \alpha} = l^2 \pi \frac{3}{2} (1+\omega) S^{-\frac{3}{2}(1+\omega)}
\]  

(59)

So by using the \( \eta \) and the equation (59), we confirmed the other relation for this black hole
which is calculated by,

\[
-\eta \frac{\partial \alpha}{\partial \epsilon} = \frac{S^2}{2l^2 \pi^2}
\]  

(60)
As we can see, two equation (60) and (52) are extremely same. After proving all the universal relations, we now obtain a new universal relation that arises from the new property of the black hole as cloud of string. In that case, we prove the corresponding universal relation and also show how such black hole is satisfied by weak gravity conjecture. So, we use relation (49) and obtain,

\[
\frac{\partial \epsilon}{\partial b} = \frac{l^2 \pi}{S} \tag{61}
\]

We note here the above equation and relation of \( \zeta \) in text lead us to confirm the last universal relation. So, it is given by,

\[
- \zeta \frac{\partial b}{\partial \epsilon} = \frac{S \frac{4}{3}}{2l^2 \pi \frac{2}{3}} = \frac{\partial M_{ext}}{\partial \epsilon} \tag{62}
\]

Here the same as before we have some figures, these show us how the cloud of string terms will effect to the corresponding system. Therefore, as we can see from all the above relations, we examined the universal relationships for Kerr-Newman-AdS black holes in three form. We obtained the universal relations in the different cases and we compared them with each other. By using thermodynamic relations, we observe that when a new property is added to the black hole, a new universal relation can be considered for it. In this paper, we also calculated the new universal relations by considering the rotating and quintessence dark energy as well as the cloud of string of the black hole. We observed that this universal relation is well established. As it turned out, when the constant correction is a negative, the mass decreases and the mass-charge ratio also decreases to equal one. In fact, there is a sign of weak gravity conjecture behavior, which was examined for all cases with respect to the extremal bound. Therefore, by
considering new modes such as high dimensions and black holes with different structures new universal relations can be evaluated, we will study this problem in future.

5 Conclusion

Recently researchers have confirmed that general relativity corrections lead to a glance relationship between entropy and extremality bound. This relationship has been investigated for many black holes, such as charged AdS, rotating and massive gravity black holes. In this article, we confirmed these universal relations for a charged-rotating-AdS black hole, that is actually done by adding a small constant correction to the action. Then we examined these calculations for this black hole while it was surrounded by the quintessence and quintessence with cloud of strings. Also we evaluated a new universal relation which relation between the mass of the black hole and the factor related to quintessence density. Also we got a new universal relation for cloud of string which is given by,

\[-\zeta \frac{\partial b}{\partial \epsilon} = \frac{M_{\text{ext}}}{\partial \epsilon}\]

We observed that this universal relation is well established. In fact, we know that the added constant correction is inversely related to the entropy of the black hole. In that case the mass-to-charge ratio decreases, this lead us to conclude that the black hole has a WGC-like behavior. Also here we note that the calculating of universal relations by considering some features such as higher dimensions, strings fluid mimics, van der Waals fluid behavior and black holes with other structures such as Einstein-Gauss-Bonnet may be interesting research for future.

References

[1] A. Strominger and C. Vafa, Phys. Lett.B 379, 99-104 (1996).

[2] C. Vafa, arXiv:hep-th/0509212 (2005).

[3] H. Ooguri and C. Vafa, Nucl. Phys. B 766, 21-33 (2007).

[4] N. Arkani-Hamed, L. Motl, A. Nicolis and C. Vafa, JHEP 06, 060 (2007).

[5] M. Orellana, F. Garcia, F. Teppa Pannia and G. Romero, Gen. Rel. Grav 45, 771-783 (2013).

[6] S. Nojiri and S. D. Odintsov, Phys. Rept. 505, 59 (2011).
[7] S. Capozziello, M. De Laurentis, S. D. Odintsov and A. Stabile, Phys. Rev. D 83, 064004 (2011).

[8] Salvatore Capozziello, Mir. Faizal, Mir. Hameeda, Behnam Pourhassan, Vincenzo Salzano and Sudhaker Upadhyay, Mon. Not. Roy. Astron. Soc. 474, 2430-2443 (2018).

[9] A. Arapoglu, C. Deliduman and K. Y. Eksi, JCAP 1107, 020 (2011).

[10] S. Capozziello, R. D’Agostino, O. Luongo, Int. J. Mod. Phys. D, 28, 1930016 (2019).

[11] S. Capozziello, R. D’Agostino, O. Luongo, JCAP 1805, 008 (2018).

[12] M. Khurshudyan, A. Pasqua, and B. Pourhassan, Can. J. Phys. 1107, 449-455 (2015).

[13] Ph. Channuie, Eur. Phys. J. C, 79, 508 (2019).

[14] S. Capozziello, R. D’Agostino, O. Luongo, Gen. Rel. Grav. 51, 2 (2019).

[15] J. Sadeghi, E. Naghd Mezerji and S. Noori Gashti, arXiv:1910.11676 (2019).

[16] R. Myrzakulov, L. Sebastian and S. Vagnozzi, Eur. Phys. J.C 75, 444 (2015).

[17] J. Sadeghi, B. Pourhassan, A. S. Kubeka and M. Rostami, Int. J. Mod. PhysD, 25, 1650077 (2015).

[18] S. Nojiri and S. D. Odintsov, Gen. Rel. Grav. 36, 1765 (2004).

[19] S. D. Odintsov, V. K. Oikonomou and L. Sebastiani, Nucl. Phys. B, 923, 608 (2017).

[20] W. M. Chen, Y. T. Huang, T. Noumi and C. Wen, Phys. Rev. D 100 no.2, 025016 (2019).

[21] L. Aalsma, A. Cole and G. Shiu, JHEP 1908 022 (2019).

[22] S. J. Lee, W. Lerche and T. Weigand, Nucl. Phys. B 938 321 (2019).

[23] S. J. Lee, W. Lerche and T. Weigand, JHEP 1908 104 (2019).

[24] Y. Kats, L. Motl and M. Padi, JHEP 0712 068 (2007).

[25] G. J. Loges, T. Noumi and G. Shiu, arXiv:1909.01352 (2019).

[26] M. Montero, T. Van Riet and G. Venken, JHEP 2001 039 (2020).

[27] P. A. Cano, S. Chimento, R. Linares, T. Ortn and P. F. Ramirez, JHEP 2002 031 (2020).

[28] P. A. Cano, T. Ortn and P. F. Ramirez, JHEP 2002 175 (2020).
[29] H. S. Reall and J. E. Santos, JHEP 1904 021(2019).
[30] B. Heidenreich, M. Reece and T. Rudelius, JHEP 1910 055 (2019).
[31] S. Brahma and M.W. Hossain, Phys. Rev. D 100 no.8, 086017 (2019).
[32] E. Gonzalo and L. E. Ibez, JHEP 1908 118 (2019).
[33] Y. Hamada, T. Noumi and G. Shiu, Phys. Rev. Lett. 123 no.5, 051601(2019).
[34] C. Cheung, J. Liu and G. N. Remmen, JHEP 1810 004 (2018).
[35] G. Goon and R. Penco, Phys. Rev. Lett. 124 no.10, 101103 (2020).
[36] S. W. Wei, K. Yang and Y. X. Liu, arXiv:2003.06785 (2020).
[37] D. Chen, J. Tao and P. Wang, arXiv:2004.10459 (2020).
[38] J. Sadeghi, S. Noori Gashti and E. Naghd Mezerji. Phys. Dark Univ 30, 100626, (2020).
[39] S. M. Carroll, W. H. Press and E. L. Turner, Ann. Rev. Astron. Astrophys. 30 499 (1992).
[40] B. Ratra and P. J. E. Peebles, Phys. Rev. D 37, 3406 (1988).
[41] S. M. Carroll, Phys. Rev. Lett. 81 3067 (1998).
[42] I. Zlatev, L. M. Wang and P. J. Steinhardt, Phys. Rev. Lett. 82 896(1999).
[43] P. J. E. Peebles and A. Vilenkin, Rev. D 59 063505 (1999).
[44] E. Ayon-Beato and A. Garcia, Phys. Rev. Lett. 80 , 5056-5059 (1998).
[45] E. Ayon-Beato and A. Garcia, Phys. Lett. B 464 25 (1999).
[46] J. Bardeen, in Proceedings of GR5, Tbilisi, U.S.S.R. (1968).
[47] P.S. Letelier, Phys. Rev. D 20, 1294 (1979).
[48] P.S. Letelier, Il Nuovo Cim. B (1971–1996) 63, 519 (1981).
[49] P.S. Letelier, Phys. Rev. D 28, 2414 (1983).
[50] Caldarelli Marco M, Cognola Guido and Klemm Dietmar, Class. Quant. Grav.17, 399–420 (2000).
[51] A. Belhaj, M. Chabab, H. El. Moumni, L. Medari and M. B. Sedra, Chin. Phys. Lett.30 090402 (2013)
[52] Xu Zhaoyi and Wang Jiancheng, Phys. Rev. D95, 6 064015 (217).

[53] Minazzoli Olivier and Harko Tiberiu, Phys. Rev. D86, 087502 (2012).

[54] J.M. Toledo and V.B. Bezerra, Gen. Rel. Grav. 52, 4 34 (2020).