Characterization of the boundedness of generalized fractional integral and maximal operators on Orlicz–Morrey and weak Orlicz–Morrey spaces

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Funding information
Grant-in-Aid for Scientific Research (B), Grant/Award Number: 15H03621; Japan Society for the Promotion of Science

Abstract
We give necessary and sufficient conditions for the boundedness of generalized fractional integral and maximal operators on Orlicz–Morrey and weak Orlicz–Morrey spaces. To do this, we prove the weak–weak type modular inequality of the Hardy–Littlewood maximal operator with respect to the Young function. Orlicz–Morrey spaces contain $L^p$ spaces ($1 \leq p \leq \infty$), Orlicz spaces, and generalized Morrey spaces as special cases. Hence, we get necessary and sufficient conditions on these function spaces as corollaries.

KEYWORDS
fractional integral, fractional maximal operator, modular inequality, Orlicz–Morrey space, weak Orlicz–Morrey space

MSC (2020)
42B35, 46E30, 42B20, 42B25

1 INTRODUCTION

In this paper, we consider the generalized fractional integral operator $I_\rho$ and the generalized fractional maximal operator $M_\rho$. We give necessary and sufficient conditions for the boundedness of $I_\rho$ and $M_\rho$ on Orlicz–Morrey spaces $L^{(\Phi,\varphi)}(\mathbb{R}^n)$ and weak Orlicz–Morrey spaces $wL^{(\Phi,\varphi)}(\mathbb{R}^n)$. More precisely, we investigate the following boundedness of the operator $T = I_\rho$ or $M_\rho$:

$$
\|Tf\|_{wL^{(\Phi,\varphi)}} \leq C\|f\|_{L^{(\Phi,\varphi)}},
$$

$$
\|Tf\|_{L^{(\Phi,\varphi)}} \leq C\|f\|_{L^{(\Phi,\varphi)}},
$$

$$
\|Tf\|_{wL^{(\Phi,\varphi)}} \leq C\|f\|_{wL^{(\Phi,\varphi)}}.
$$

We treat a wide class of Young functions as $\Phi : [0, \infty] \to [0, \infty]$. Orlicz–Morrey spaces contain $L^p$ spaces ($1 \leq p \leq \infty$), Orlicz spaces and generalized Morrey spaces as special cases. Hence we get necessary and sufficient conditions for the boundedness of $I_\rho$ and $M_\rho$ on these function spaces as corollaries.

Now, we recall the definitions of the Orlicz–Morrey and weak Orlicz–Morrey spaces. For a measurable set $G \subset \mathbb{R}^n$, we denote by $|G|$ its Lebesgue measure. We denote by $B(\alpha, r)$ the open ball centered at $\alpha \in \mathbb{R}^n$ and of radius $r$. For a function
For a measurable set $G \subset \mathbb{R}^n$, a measurable function $f$ and $t \geq 0$, let

$$m(G, f, t) = |\{ x \in G : |f(x)| > t \}|.$$

In the case $G = \mathbb{R}^n$, we briefly denote it by $m(f, t)$.

**Definition 1.1.** For a Young function $\Phi : [0, \infty) \to [0, \infty)$, a function $\varphi : (0, \infty) \to (0, \infty)$ and a ball $B = B(a, r)$, let

$$\| f \|_{\Phi, \varphi, B} = \inf \left\{ \lambda > 0 : \frac{1}{\varphi(r)} \int_B \Phi \left( \frac{|f(x)|}{\lambda} \right) dx \leq 1 \right\},$$

$$\| f \|_{\Phi, \varphi, B, \text{weak}} = \inf \left\{ \lambda > 0 : \sup_{t \in (0, \infty)} \frac{\Phi(t) m(B, f/\lambda, t)}{|B| \varphi(r)} \leq 1 \right\}.$$

Let $L^{(\Phi, \varphi)}(\mathbb{R}^n)$ and $wL^{(\Phi, \varphi)}(\mathbb{R}^n)$ be the set of all measurable functions $f$ on $\mathbb{R}^n$ such that the following functionals are finite, respectively,

$$\| f \|_{L^{(\Phi, \varphi)}} = \sup_B \| f \|_{\Phi, \varphi, B},$$

$$\| f \|_{wL^{(\Phi, \varphi)}} = \sup_B \| f \|_{\Phi, \varphi, B, \text{weak}},$$

where the suprema are taken over all balls $B$ in $\mathbb{R}^n$. (For the definition of the Young function, see Definition 2.2 below.)

Then, $\| \cdot \|_{L^{(\Phi, \varphi)}}$ is a norm and thereby $L^{(\Phi, \varphi)}(\mathbb{R}^n)$ is a Banach space. On the other hand $\| \cdot \|_{wL^{(\Phi, \varphi)}}$ is a quasi norm with the inequality

$$\| f + g \|_{wL^{(\Phi, \varphi)}} \leq 2 \left( \| f \|_{wL^{(\Phi, \varphi)}} + \| g \|_{wL^{(\Phi, \varphi)}} \right),$$

and thereby $wL^{(\Phi, \varphi)}(\mathbb{R}^n)$ is a quasi Banach space.

If $\varphi(r) = 1/r^n$, then $L^{(\Phi, \varphi)}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ and $wL^{(\Phi, \varphi)}(\mathbb{R}^n) = wL^p(\mathbb{R}^n)$, which are the usual Orlicz and weak Orlicz spaces, respectively. In this paper, we treat a wide class of Young functions such as

$$\Phi_1(t) = \begin{cases} \max(0, t^2 - 4), & \text{for } t \geq 0, \\ t, & \text{for } 0 \leq t \leq 1, \\ \infty, & \text{for } t > 1, \end{cases}$$

and

$$\Phi_2(t) = \begin{cases} t, & \text{for } 0 \leq t \leq 1, \\ \infty, & \text{for } t > 1. \end{cases}$$

We also treat generalized Young functions such as

$$\Phi_3(t) = \begin{cases} e^{1-t/p}, & \text{for } 0 \leq t \leq 1, \\ e^{1/(t-1)}, & \text{for } 1 < t < p, \\ e^{p-1}, & \text{for } t > p. \end{cases} \quad (1.1)$$

which is not convex near $t = 1$.

If $\Phi(t) = t^p (1 \leq p < \infty)$, then we denote $L^{(\Phi, \varphi)}(\mathbb{R}^n)$ and $wL^{(\Phi, \varphi)}(\mathbb{R}^n)$ by $L^{(p, \varphi)}(\mathbb{R}^n)$ and $wL^{(p, \varphi)}(\mathbb{R}^n)$, which are the generalized Morrey and weak Morrey spaces, respectively. If $\varphi(r) = r^\lambda$, $-n \leq \lambda < 0$, then $L^{(p, \varphi, \lambda)}(\mathbb{R}^n)$ is the classical Morrey space. In particular, if $\lambda = -n$, then it is the Lebesgue space $L^p(\mathbb{R}^n)$.

Orlicz spaces were introduced by [46, 47]. For the theory of Orlicz spaces, see [25, 26, 28, 32, 51] for example. Weak Orlicz spaces were studied in, for example, [21, 31, 40]. Morrey spaces were introduced by [35]. For their generalization, see, for
example, [33, 36, 48, 49]. Weak Morrey spaces were studied in, for example, [13, 14, 53, 58, 63]. The Orlicz–Morrey space $L^{(p,\varphi)}(\mathbb{R}^n)$ was first studied in [41]. The spaces $L^{(p,\varphi)}(\mathbb{R}^n)$ and $wL^{(p,\varphi)}(\mathbb{R}^n)$ were investigated in, for example, [17, 42, 43, 52]. For other kinds of Orlicz–Morrey spaces, see, for example, [4, 5, 8, 11, 55]. For the study related to weak Orlicz and weak Morrey spaces, see, for example, [7, 10, 15, 16, 19, 20, 29, 30, 64].

Next, we recall the generalized fractional integral operator $I_\rho$. For a function $\rho : (0, \infty) \to (0, \infty)$, the operator $I_\rho$ is defined by

$$I_\rho f(x) = \int_{\mathbb{R}^n} \frac{\rho(|x-y|)}{|x-y|^n} f(y) \, dy, \quad x \in \mathbb{R}^n,$$

where we always assume that

$$\int_0^1 \frac{\rho(t)}{t} \, dt < \infty. \quad (1.4)$$

If $\rho(r) = r^\alpha$, $0 < \alpha < n$, then $I_\rho$ is the usual fractional integral operator $I_\alpha$. The condition (1.4) is needed for the integral in (1.3) to converge for bounded functions $f$ with compact support. In this paper, we also assume that there exist positive constants $C, K_1$ and $K_2$ with $K_1 < K_2$ such that, for all $r > 0$,

$$\sup_{r \leq t \leq 2r} \rho(t) \leq C \int_{K_1 r}^{K_2 r} \frac{\rho(t)}{t} \, dt. \quad (1.5)$$

The condition (1.5) was considered in [50]. If $\rho$ satisfies the doubling condition (2.12) below, then $\rho$ satisfies (1.5). Let $\rho(r) = \min(r^\alpha, e^{-r/2})$ with $0 < \alpha < n$, which controls the Bessel potential (see [59]). Then, $\rho$ also satisfies (1.5). The operator $I_\rho$ was introduced in [38] to extend the Hardy–Littlewood–Sobolev theorem to Orlicz spaces whose partial results were announced in [37]. For example, the generalized fractional integral $I_\rho$ is bounded from $\exp L^p(\mathbb{R}^n)$ to $\exp L^q(\mathbb{R}^n)$, where

$$\rho(r) = \begin{cases} 1/(\log(1/r))^{\alpha+1} & \text{for small } r, \\ (\log r)^{\alpha-1} & \text{for large } r, \end{cases} \alpha > 0, \quad (1.6)$$

$p, q \in (0, \infty), -1/p + \alpha = -1/q$ and $\exp L^p(\mathbb{R}^n)$ is the Orlicz space $L^\Phi(\mathbb{R}^n)$ with

$$\Phi(t) = \begin{cases} 1/\exp(1/t^p) & \text{for small } t, \\ \exp(t^p) & \text{for large } t. \end{cases} \quad (1.7)$$

See also [39–42, 44].

We also consider the generalized fractional maximal operator $M_\rho$ and compare its boundedness with $I_\rho$. For a function $\rho : (0, \infty) \to (0, \infty)$, the operator $M_\rho$ is defined by

$$M_\rho f(x) = \sup_{B(a,r) \ni x} \rho(r) \int_{B(a,r)} |f(y)| \, dy, \quad x \in \mathbb{R}^n,$$

where the supremum is taken over all balls $B(a,r)$ containing $x$. We need neither the condition (1.4) nor (1.5) on the definition of $M_\rho$. The operator $M_\rho$ was studied in [54] on generalized Morrey spaces. See also [18]. If $\rho(r) = |B(0,r)|^{\alpha/n}$, then $M_\rho$ is the usual fractional maximal operator $M_\alpha$. If $\rho \equiv 1$, then $M_\rho$ is the Hardy–Littlewood maximal operator $M$. It is known that the usual fractional maximal operator $M_\alpha$ is dominated pointwise by the fractional integral operator $I_\alpha$, that is, $M_\alpha f(x) \leq C I_\alpha |f|(x)$ for all $x \in \mathbb{R}^n$. Then, the boundedness of $M_\alpha$ follows from one of $I_\alpha$. However, we have a better estimate of $M_\rho$ than $I_\rho$.

To prove the boundedness of $I_\rho$ and $M_\rho$ on $L^{(p,\Phi)}(\mathbb{R}^n)$ and $wL^{(p,\Phi)}(\mathbb{R}^n)$ we show the pointwise estimate by the Hardy–Littlewood maximal operator $M$ and use the modular inequality of $M$ with respect to Young functions $\Phi$. The strong and weak-type modular inequalities are known. In this paper, we show the weak–weak-type modular inequality. In general, the modular inequality is stronger than the norm inequality. For the boundedness on generalized Morrey spaces $L^{(p,\Phi)}(\mathbb{R}^n)$, we only need the $L^p$-norm inequality of $M$. However, for $L^{(p,\Phi)}(\mathbb{R}^n)$ and $wL^{(p,\Phi)}(\mathbb{R}^n)$, we need the modular inequality.
The organization of this paper is as follows: In the next section, we state precise definitions of the functions $\Phi$ and $\varphi$ by which we define $L^{(\Phi,\varphi)}(\mathbb{R}^n)$ and $wL^{(\Phi,\varphi)}(\mathbb{R}^n)$. Then, we state the main results in Section 3. To prove them in the final section, we give properties of Young functions, Orlicz–Morrey and weak Orlicz–Morrey spaces in Section 4.

At the end of this section, we make some conventions. Throughout this paper, we always use $C$ to denote a positive constant that is independent of the main parameters involved but whose value may differ from line to line. Constants with subscripts, such as $C_p$, are dependent on the subscripts. If $f \leq Cg$, we then write $f \lesssim g$ or $g \gtrsim f$; and if $f \lesssim g \lesssim f$, we then write $f \sim g$.

### 2 ON THE FUNCTIONS $\Phi$ AND $\varphi$

In this section, we state on the functions $\Phi$ and $\varphi$ by which we define $L^{(\Phi,\varphi)}(\mathbb{R}^n)$ and $wL^{(\Phi,\varphi)}(\mathbb{R}^n)$. We first recall the Young function and its generalization.

For an increasing (i.e., nondecreasing) function $\Phi : [0, \infty) \to [0, \infty)$, let

$$a(\Phi) = \sup\{t \geq 0 : \Phi(t) = 0\}, \quad b(\Phi) = \inf\{t \geq 0 : \Phi(t) = \infty\},$$

with convention $\sup\emptyset = 0$ and $\inf\emptyset = \infty$. Then, $0 \leq a(\Phi) \leq b(\Phi) \leq \infty$.

Let $\Phi$ be the set of all increasing functions $\Phi : [0, \infty) \to [0, \infty]$ such that

$$0 \leq a(\Phi) < \infty, \quad 0 < b(\Phi) \leq \infty,$$

$$\lim_{t \to 0^+} \Phi(t) = \Phi(0) = 0,$$

$$\Phi$$

is left continuous on $[0, b(\Phi))$,

$$\text{if } b(\Phi) = \infty, \text{ then } \lim_{t \to \infty} \Phi(t) = \Phi(\infty) = \infty,$$

$$\text{if } b(\Phi) < \infty, \text{ then } \lim_{t \to b(\Phi)^-0} \Phi(t) = \Phi(b(\Phi)) \leq \infty.$$

In what follows, if an increasing and left continuous function $\Phi : [0, \infty) \to [0, \infty)$ satisfies (2.3) and $\lim_{t \to \infty} \Phi(t) = \infty$, then we always regard that $\Phi(\infty) = \infty$ and that $\Phi \in \Phi$.

For $\Phi \in \Phi$, we recall the generalized inverse of $\Phi$ in the sense of O’Neil [45, Definition 1.2].

**Definition 2.1.** For $\Phi \in \Phi$ and $u \in [0, \infty]$, let

$$\Phi^{-1}(u) = \begin{cases} \inf\{t \geq 0 : \Phi(t) > u\}, & u \in [0, \infty), \\ \infty, & u = \infty. \end{cases}$$

Let $\Phi \in \Phi$. Then, $\Phi^{-1}$ is finite, increasing, and right continuous on $[0, \infty)$ and positive on $(0, \infty)$. If $\Phi$ is bijective from $[0, \infty]$ to itself, then $\Phi^{-1}$ is the usual inverse function of $\Phi$. Moreover, if $\Phi \in \Phi$, then

$$\Phi(\Phi^{-1}(u)) \leq u \leq \Phi^{-1}(\Phi(u)) \quad \text{for all } u \in [0, \infty],$$

which is a generalization of Property 1.3 in [45]. For its proof, see [56, Proposition 2.2].

For $\Phi, \Psi \in \Phi$, we write $\Phi \approx \Psi$ if there exists a positive constant $C$ such that

$$\Phi(C^{-1}t) \leq \Psi(t) \leq \Phi(Ct) \quad \text{for all } t \in [0, \infty].$$

For functions $P, Q : [0, \infty) \to [0, \infty]$, we write $P \sim Q$ if there exists a positive constant $C$ such that

$$C^{-1}P(t) \leq Q(t) \leq CP(t) \quad \text{for all } t \in [0, \infty].$$
Then, for $\Phi, \Psi \in \Phi$, 

$$\Phi \approx \Psi \iff \Phi^{-1} \sim \Psi^{-1},$$

(2.9)

see [56, Lemma 2.3].

Now we recall the definition of the Young function and give its generalization.

**Definition 2.2.** A function $\Phi \in \Phi$ is called a Young function (or sometimes also called an Orlicz function) if $\Phi$ is convex on $[0, b(\Phi))$. Let $\Phi_Y$ be the set of all Young functions. Let $\Phi_Y$ be the set of all $\Phi \in \Phi$ such that $\Phi \approx \Psi$ for some $\Psi \in \Phi_Y$.

For example, $\Phi_1$ and $\Phi_2$ defined by (1.1) are in $\Phi_Y$, and $\Phi_3$ defined by (1.2) is in $\Phi_Y \setminus \Phi_Y$. 

Similar to Definition 1.1 we also define $L(\Phi, \varphi)(\mathbb{R}^n)$ and $wL(\Phi, \varphi)(\mathbb{R}^n)$ by using generalized Young functions $\Phi \in \Phi_Y$ together with $\|\cdot\|_{\Phi, \varphi, B}$ and $\|\cdot\|_{\Phi, \varphi, B, \text{weak}}$, respectively. Then, $\|\cdot\|_{\Phi, \varphi, B}$ and $\|\cdot\|_{\Phi, \varphi, B, \text{weak}}$ are quasi norms and thereby $L(\Phi, \varphi)(\mathbb{R}^n)$ and $wL(\Phi, \varphi)(\mathbb{R}^n)$ are quasi Banach spaces. Note that, for $\Phi, \Psi \in \Phi_Y$, if $\Phi \approx \Psi$ and $\varphi \sim \psi$, then $L(\Phi, \varphi)(\mathbb{R}^n) = L(\Psi, \psi)(\mathbb{R}^n)$ and $wL(\Phi, \varphi)(\mathbb{R}^n) = wL(\Psi, \psi)(\mathbb{R}^n)$ with equivalent quasi norms.

**Definition 2.3.**

(i) A function $\Phi \in \Phi$ is said to satisfy the $\Delta_2$-condition, denoted by $\Phi \in \Delta_2$, if there exists a constant $C > 0$ such that

$$\Phi(2t) \leq C\Phi(t) \quad \text{for all } t > 0.$$  

(2.10)

(ii) A function $\Phi \in \Phi$ is said to satisfy the $\nabla_2$-condition, denoted by $\Phi \in \nabla_2$, if there exists a constant $k > 1$ such that

$$\Phi(t) \leq \frac{1}{2k}\Phi(kt) \quad \text{for all } t > 0.$$  

(2.11)

(iii) Let $\Delta_2 = \Phi_Y \cap \Delta_2$ and $\nabla_2 = \Phi_Y \cap \nabla_2$.

Next, we say that a function $\vartheta : (0, \infty) \to (0, \infty)$ satisfies the doubling condition if there exists a positive constant $C$ such that, for all $r, s \in (0, \infty)$,

$$\frac{1}{C} \leq \frac{\vartheta(r)}{\vartheta(s)} \leq C, \quad \text{if } \frac{1}{2} \leq \frac{r}{s} \leq 2.$$  

(2.12)

We say that $\vartheta$ is almost increasing (resp. almost decreasing) if there exists a positive constant $C$ such that, for all $r, s \in (0, \infty)$,

$$\vartheta(r) \leq C\vartheta(s) \quad \text{(resp. } \vartheta(s) \leq C\vartheta(r),) \quad \text{if } r < s.$$  

(2.13)

In this paper, we consider the following class of $\varphi : (0, \infty) \to (0, \infty)$.

**Definition 2.4.** Let $\varphi^{\text{dec}}$ be the set of all functions $\varphi : (0, \infty) \to (0, \infty)$ such that $\varphi$ is almost decreasing and that $r \mapsto \varphi(r)r^n$ is almost increasing. That is, there exists a positive constant $C$ such that, for all $r, s \in (0, \infty)$,

$$C\varphi(r) \geq \varphi(s), \quad \varphi(r)r^n \leq C\varphi(s)s^n, \quad \text{if } r < s.$$  

If $\varphi \in \varphi^{\text{dec}}$, then $\varphi$ satisfies doubling condition. Let $\varphi : (0, \infty) \to (0, \infty)$. If $\varphi$ for some $\varphi \in \varphi^{\text{dec}}$, then $\varphi \in \varphi^{\text{dec}}$.

**Remark 2.5.** Let $\varphi \in \varphi^{\text{dec}}$. Then, there exists $\hat{\varphi} \in \varphi^{\text{dec}}$ such that $\varphi \sim \hat{\varphi}$ and that $\hat{\varphi}$ is continuous and strictly decreasing, see [42, Proposition 3.4]. Moreover, if

$$\lim_{r \to 0} \varphi(r) = \infty, \quad \lim_{r \to \infty} \varphi(r) = 0,$$

(2.14)

then $\hat{\varphi}$ is bijective from $(0, \infty)$ to itself.
At the end of this section, we note that, for $\Phi \in \Phi_Y$ and a ball $B$, the following relation holds:

$$\sup_{t \in (0, \infty)} \Phi(t) m(B, f, t) = \sup_{t \in (0, \infty)} t m(B, f, \Phi^{-1}(t)) = \sup_{t \in (0, \infty)} t m(B, \Phi(\lfloor f \rfloor), t).$$

(2.15)

See [22] for the proof of (2.15). Hence, the norm inequality $\|Tf\|_{wL_\Phi(\Psi, \varphi)} \leq C \|f\|_{L_\Phi(\Psi, \varphi)}$ holds if and only if

$$tm\left( B, \Psi\left( \frac{|Tf|}{C \|f\|_{L_\Phi(\Psi, \varphi)}} \right), t \right) \leq |B| \varphi(r)$$

holds for all balls $B = B(a, r)$ and $t \in (0, \infty)$.

### 3 MAIN RESULTS

For a measurable function $f$ and $t \geq 0$, recall that $m(f, t) = |\{x \in \mathbb{R}^n : |f(x)| > t\}|$. First we state known modular inequalities for the Hardy–Littlewood maximal operator $M$. For their proofs, see [26, Theorem 1.2.1 and Lemma 1.2.4] for example.

**Theorem 3.1** ([26, 62]). Let $\Phi \in \Phi_Y$. Then, there exists a positive constant $C_{\Phi}$ such that

$$\sup_{t \in (0, \infty)} \Phi(t) m(Mf, t) \leq \int_{\mathbb{R}^n} \Phi(C_{\Phi} |f(x)|) \, dx. \quad (3.1)$$

Moreover, if $\Phi \in \nabla_2$, then

$$\int_{\mathbb{R}^n} \Phi(Mf(x)) \, dx \leq \int_{\mathbb{R}^n} \Phi(C_{\Phi} |f(x)|) \, dx. \quad (3.2)$$

If $\Phi \in \nabla_2$ and $\varphi \in \mathcal{G}_{\text{dec}}$, then $wL_{(\Phi, \varphi)}(\mathbb{R}^n) \subset L^1_{\text{loc}}(\mathbb{R}^n)$, see Lemma 4.7 below. Hence, $Mf$ is well defined for $f \in wL_{(\Phi, \varphi)}(\mathbb{R}^n)$, in particular, for $f \in wL_{\Phi}(\mathbb{R}^n)$.

Our first result is the following modular inequality.

**Theorem 3.2.** If $\Phi \in \nabla_2$, then there exists a positive constant $C_{\Phi}$ such that

$$\sup_{t \in (0, \infty)} \Phi(t) m(Mf, t) \leq \sup_{t \in (0, \infty)} \Phi(t) m(C_{\Phi} f, t). \quad (3.3)$$

Liu and Wang [31] proved the norm inequality for $wL_{\Phi}(\mathbb{R}^n)$ with $\Phi \in \Delta_2 \cap \nabla_2$. Theorem 3.2 is its extension. By Theorems 3.1 and 3.2, we will prove the following boundedness, which is an extension of [42, Theorem 6.1].

**Theorem 3.3.** Let $\Phi \in \Phi_Y$ and $\varphi \in \mathcal{G}_{\text{dec}}$. Then, the Hardy–Littlewood maximal operator $M$ is bounded from $L_{(\Phi, \varphi)}(\mathbb{R}^n)$ to $wL_{(\Phi, \varphi)}(\mathbb{R}^n)$. Moreover, if $\Phi \in \nabla_2$, then the operator $M$ is bounded from $L_{(\Phi, \varphi)}(\mathbb{R}^n)$ to itself and from $wL_{(\Phi, \varphi)}(\mathbb{R}^n)$ to itself.

Next, we state the boundedness of generalized fractional integral operators $I_{\varphi}$.

**Theorem 3.4.** Let $\Phi, \Psi \in \Phi_Y$, $\varphi \in \mathcal{G}_{\text{dec}}$, and $\rho : (0, \infty) \rightarrow (0, \infty)$. Assume that $\rho$ satisfies (1.4) and (1.5).

(i) Assume that $\varphi$ satisfies (2.14) and that there exists a positive constant $A$ such that, for all $r \in (0, \infty)$,

$$\int_0^r \frac{\rho(t)}{t} \, dt \Phi^{-1}(\varphi(t)) + \int_r^\infty \frac{\rho(t)}{t} \Phi^{-1}(\varphi(t)) \, dt \leq A \Psi^{-1}(\varphi(r)). \quad (3.4)$$
Then, for any positive constant $C_0$, there exists a positive constant $C_1$ such that, for all $f \in L^{(\Phi, \varphi)}(\mathbb{R}^n)$ with $f \neq 0$,

$$\Psi\left(\frac{|I_{\rho}f(x)|}{C_1\|f\|_{L^{(\Phi, \varphi)}}}\right) \leq \Phi\left(\frac{Mf(x)}{C_0\|f\|_{L^{(\Phi, \varphi)}}}\right), \quad x \in \mathbb{R}^n. \quad (3.5)$$

Consequently, $I_{\rho}$ is bounded from $L^{(\Phi, \varphi)}(\mathbb{R}^n)$ to $wL^{(\Psi, \varphi)}(\mathbb{R}^n)$. Moreover, if $\Phi \in \nabla^2$, then, for all $f \in wL^{(\Phi, \varphi)}(\mathbb{R}^n)$ with $f \neq 0$,

$$\Psi\left(\frac{|I_{\rho}f(x)|}{C_1\|f\|_{wL^{(\Phi, \varphi)}}}\right) \leq \Phi\left(\frac{Mf(x)}{C_0\|f\|_{wL^{(\Phi, \varphi)}}}\right), \quad x \in \mathbb{R}^n. \quad (3.6)$$

Consequently, if $\Phi \in \nabla^2$, then $I_{\rho}$ is bounded from $L^{(\Phi, \varphi)}(\mathbb{R}^n)$ to $L^{(\Psi, \varphi)}(\mathbb{R}^n)$ by (3.5) and from $wL^{(\Phi, \varphi)}(\mathbb{R}^n)$ to $wL^{(\Psi, \varphi)}(\mathbb{R}^n)$ by (3.6).

(ii) If $I_{\rho}$ is bounded from $L^{(\Phi, \varphi)}(\mathbb{R}^n)$ to $wL^{(\Psi, \varphi)}(\mathbb{R}^n)$, then there exists a positive constant $A'$ such that, for all $r \in (0, \infty)$,

$$\int_0^r \frac{\rho(t)}{t} \, dt \Phi^{-1}(\varphi(r)) \leq A'\Psi^{-1}((\varphi(r)). \quad (3.7)$$

Moreover, under the assumption that there exists a positive constant $C$ such that, for all $r \in (0, \infty)$,

$$\int_0^r \varphi(t)r^{n-1} \, dt \leq C\varphi(r)r^n, \quad (3.8)$$

if $I_{\rho}$ is bounded from $L^{(\Phi, \varphi)}(\mathbb{R}^n)$ to $wL^{(\Psi, \varphi)}(\mathbb{R}^n)$, then (3.4) holds for some $A \in (0, \infty)$ and for all $r \in (0, \infty)$.

(iii) In part (ii), the boundedness from $L^{(\Phi, \varphi)}(\mathbb{R}^n)$ to $wL^{(\Psi, \varphi)}(\mathbb{R}^n)$ can be replaced by the boundedness from $L^{(\Phi, \varphi)}(\mathbb{R}^n)$ to $L^{1, (\Psi^{-1}(\varphi))}(\mathbb{R}^n)$.

Third, we state the boundedness of the generalized fractional maximal operators $M_{\rho}$. In this case, we need neither the assumption (1.4) nor (1.5) on the function $\rho : (0, \infty) \to (0, \infty)$.

**Theorem 3.5.** Let $\Phi, \Psi \in \tilde{\Phi}_Y, \varphi \in \mathcal{C}^{\text{dec}}$, and $\rho : (0, \infty) \to (0, \infty)$.

(i) Assume that $\lim_{r \to \infty} \varphi(r) = 0$ or that $\Psi^{-1}(t)/\Phi^{-1}(t)$ is almost decreasing on $(0, \infty)$. If there exists a positive constant $A$ such that, for all $r \in (0, \infty)$,

$$\left(\sup_{0 < t \leq r} \rho(t)\right)\Phi^{-1}(\varphi(r)) \leq A\Psi^{-1}(\varphi(r)), \quad (3.9)$$

then, for any positive constant $C_0$, there exists a positive constant $C_1$ such that, for all $f \in L^{(\Phi, \varphi)}(\mathbb{R}^n)$ with $f \neq 0$,

$$\Psi\left(\frac{M_{\rho}f(x)}{C_1\|f\|_{L^{(\Phi, \varphi)}}}\right) \leq \Phi\left(\frac{Mf(x)}{C_0\|f\|_{L^{(\Phi, \varphi)}}}\right), \quad x \in \mathbb{R}^n. \quad (3.10)$$

Consequently, $M_{\rho}$ is bounded from $L^{(\Phi, \varphi)}(\mathbb{R}^n)$ to $wL^{(\Psi, \varphi)}(\mathbb{R}^n)$. Moreover, if $\Phi \in \nabla^2$, then, for all $f \in wL^{(\Phi, \varphi)}(\mathbb{R}^n)$ with $f \neq 0$,

$$\Psi\left(\frac{M_{\rho}f(x)}{C_1\|f\|_{wL^{(\Phi, \varphi)}}}\right) \leq \Phi\left(\frac{Mf(x)}{C_0\|f\|_{wL^{(\Phi, \varphi)}}}\right), \quad x \in \mathbb{R}^n. \quad (3.11)$$

Consequently, if $\Phi \in \nabla^2$, then $M_{\rho}$ is bounded from $L^{(\Phi, \varphi)}(\mathbb{R}^n)$ to $L^{(\Psi, \varphi)}(\mathbb{R}^n)$ by (3.10) and from $wL^{(\Phi, \varphi)}(\mathbb{R}^n)$ to $wL^{(\Psi, \varphi)}(\mathbb{R}^n)$ by (3.11).

(ii) If $M_{\rho}$ is bounded from $L^{(\Phi, \varphi)}(\mathbb{R}^n)$ to $wL^{(\Psi, \varphi)}(\mathbb{R}^n)$, then (3.9) holds for some $A \in (0, \infty)$ and for all $r \in (0, \infty)$.

(iii) In part (ii), the boundedness from $L^{(\Phi, \varphi)}(\mathbb{R}^n)$ to $wL^{(\Psi, \varphi)}(\mathbb{R}^n)$ can be replaced by the boundedness from $L^{(\Phi, \varphi)}(\mathbb{R}^n)$ to $L^{1, (\Psi^{-1}(\varphi))}(\mathbb{R}^n)$. 
Part (i) in the above theorem is an extension of [57, Theorem 5.1]. If $\Psi \in \nabla_2$, then part (iii) implies part (ii) as same as Theorem 3.4.

Remark 3.6. From (1.5) and (3.4), it follows that

$$
\left( \sup_{0 < t < r} \frac{\rho(t)}{t} \right) \Phi^{-1}(\varphi(r)) \lesssim \int_0^{K_2r} \frac{\rho(t)}{t} \, dt \Phi^{-1}(\varphi(r)) \lesssim \Psi^{-1}(\varphi(r)),
$$

which is the condition (3.9). If $\rho(r) = (\log(1/r))^{-\alpha}$ for small $r > 0$ or $\rho(r) = (\log r)^{\alpha}$ for large $r > 0$ with $\alpha \geq 0$, then the condition (3.9) is strictly weaker than (3.4).

For the case $\varphi(r) = 1/r^n$, we have the following corollaries.

**Corollary 3.7** ([9, 26, 27, 31]). Let $\Phi \in \nabla_Y$. Then the Hardy–Littlewood maximal operator $M$ is bounded from $L^\Phi(\mathbb{R}^n)$ to $wL^\Phi(\mathbb{R}^n)$. Moreover, if $\Phi \in \nabla_2$, then the operator $M$ is bounded from $L^\Phi(\mathbb{R}^n)$ to itself and from $wL^\Phi(\mathbb{R}^n)$ to itself.

For the boundedness of the operator $M$ on Orlicz spaces, see also Cianchi [3] and Kita [23, 24]. Cianchi [3] also investigated the operators $M_{\alpha}$ and $I_{\alpha}$ on Orlicz spaces.

**Corollary 3.8.** Let $\Phi, \Psi \in \Phi_Y$ and $\rho : (0, \infty) \to (0, \infty)$. Assume that $\rho$ satisfies (1.4) and (1.5).

(i) Assume that there exists a positive constant $A$ such that, for all $r \in (0, \infty)$,

$$
\int_0^r \frac{\rho(t)}{t} \, dt \Phi^{-1}(1/r^n) + \int_r^\infty \frac{\rho(t)}{t} \Phi^{-1}(1/t^n) \, dt \leq A \Psi^{-1}(1/r^n). \tag{3.12}
$$

Then, for any positive constant $C_0$, there exists a positive constant $C_1$ such that, for all $f \in L^\Phi(\mathbb{R}^n)$ with $f \not\equiv 0$,

$$
\Psi\left( \frac{||I_\rho f(x)||}{C_1 ||f||_{L^\phi}} \right) \leq \Phi\left( \frac{Mf(x)}{C_0 ||f||_{L^\phi}} \right), \quad x \in \mathbb{R}^n. \tag{3.13}
$$

Consequently, $I_\rho$ is bounded from $L^\Phi(\mathbb{R}^n)$ to $wL^\Psi(\mathbb{R}^n)$. Moreover, if $\Phi \in \nabla_2$, then, for all $f \in wL^\Phi(\mathbb{R}^n)$ with $f \not\equiv 0$,

$$
\Psi\left( \frac{||I_\rho f(x)||}{C_1 ||f||_{wL^\phi}} \right) \leq \Phi\left( \frac{Mf(x)}{C_0 ||f||_{wL^\phi}} \right), \quad x \in \mathbb{R}^n. \tag{3.14}
$$

Consequently, if $\Phi \in \nabla_2$, then $I_\rho$ is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$ and from $wL^\Phi(\mathbb{R}^n)$ to $wL^\Psi(\mathbb{R}^n)$.

(ii) If $I_\rho$ is bounded from $L^\Phi(\mathbb{R}^n)$ to $wL^\Psi(\mathbb{R}^n)$, then there exists a positive constant $A'$ such that, for all $r \in (0, \infty)$,

$$
\int_0^r \frac{\rho(t)}{t} \, dt \Phi^{-1}(1/r^n) \leq A' \Psi^{-1}(1/r^n). \tag{3.15}
$$

Moreover, under the assumption that there exists a positive constant $A''$ such that, for all $r \in (0, \infty)$,

$$
\int_r^\infty \frac{\rho(t)}{t} \Phi^{-1}(1/t^n) \, dt \leq A'' \Psi^{-1}(1/r^n), \tag{3.16}
$$

if $I_\rho$ is bounded from $L^\Phi(\mathbb{R}^n)$ to $wL^\Psi(\mathbb{R}^n)$, then (3.12) holds for some $A \in (0, \infty)$ and for all $r \in (0, \infty)$.

(iii) In part (ii), the boundedness from $L^\Phi(\mathbb{R}^n)$ to $wL^\Psi(\mathbb{R}^n)$ can be replaced by the boundedness from $L^\Phi(\mathbb{R}^n)$ to $L^{(1, \Psi)}(\mathbb{R}^n)$ with $\Psi(r) = \Psi^{-1}(1/r^n)$.

The above corollary is an extension of [4, Theorem 3]. See [39], for examples of $\Phi, \Psi \in \Phi_Y$ which satisfy the assumption in Corollary 3.8. The boundedness of the usual fractional integral operator on Orlicz spaces was given by O’Neil [45]. See also [34] for the boundedness of $I_\rho$ on Orlicz space $L^\Phi(\Omega)$ with bounded domain $\Omega \subset \mathbb{R}^n$. 
Corollary 3.9. Let $\Phi, \Psi \in \Phi_Y$ and $\rho : (0, \infty) \to (0, \infty)$.

(i) If there exists a positive constant $A$ such that, for all $r \in (0, \infty)$,
\[
\left( \sup_{t \in \mathbb{R}^n} \rho(t) \right) \Phi^{-1}(1/r^n) \leq A \Psi^{-1}(1/r^n),
\]  
then, for any positive constant $C_0$, there exists a positive constant $C_1$ such that, for all $f \in L^\Phi(\mathbb{R}^n)$ with $f \neq 0$,
\[
\Psi \left( \frac{M_\rho f(x)}{C_1 \|f\|_{wL^\Phi}} \right) \leq \Phi \left( \frac{M f(x)}{C_0 \|f\|_{L^\Phi}} \right), \quad x \in \mathbb{R}^n.
\]  
Consequently, $M_\rho$ is bounded from $L^\Phi(\mathbb{R}^n)$ to $wL^\Psi(\mathbb{R}^n)$. Moreover, if $\Phi \in \nabla^2$, then, for all $f \in wL^\Phi(\mathbb{R}^n)$ with $f \neq 0$,
\[
\Psi \left( \frac{M_\rho f(x)}{C_1 \|f\|_{wL^\Phi}} \right) \leq \Phi \left( \frac{M f(x)}{C_0 \|f\|_{wL^\Phi}} \right), \quad x \in \mathbb{R}^n.
\]

(ii) If $M_\rho$ is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$ and from $wL^\Phi(\mathbb{R}^n)$ to $wL^\Psi(\mathbb{R}^n)$.

(iii) In part (ii) the boundedness from $L^\Phi(\mathbb{R}^n)$ to $wL^\Psi(\mathbb{R}^n)$ can be replaced by the boundedness from $L^\Phi(\mathbb{R}^n)$ to $L^{(1,\Psi)}(\mathbb{R}^n)$ with $\psi(r) = \Psi^{-1}(1/r^n)$.

The above corollary is an extension of [56, Theorem 3.8]. For the case $\Phi(t) = t^p$, we have the following corollaries.

Corollary 3.10. Let $1 \leq p < \infty$ and $\varphi \in G^{\infty}_e$. Then, the Hardy–Littlewood maximal operator $M$ is bounded from $L^{(p,\varphi)}(\mathbb{R}^n)$ to $L^\varphi(\mathbb{R}^n)$. Moreover, if $1 < p < \infty$, then the operator $M$ is bounded from $L^{(p,\varphi)}(\mathbb{R}^n)$ to itself and from $wL^{(p,\varphi)}(\mathbb{R}^n)$ to itself.

The above corollary is an extension of [36, Theorem 1] and [42, Corollary 6.2]. The boundedness of the Hardy–Littlewood maximal operator on the classical Morrey space was proven by Chiarenza and Frasca [2].

Corollary 3.11. Let $1 \leq p < q < \infty$, $\varphi \in G^{\infty}_e$ and $\rho : (0, \infty) \to (0, \infty)$. Assume that $\rho$ satisfies (1.4) and (1.5).

(i) Assume that $\varphi$ satisfies (2.14) and that there exists a positive constant $A$ such that, for all $r \in (0, \infty)$,
\[
\int_0^r \frac{\rho(t)}{t} \, dt \varphi(r)^{1/p} + \int_r^\infty \frac{\rho(t) \varphi(t)^{1/p}}{t} \, dt \leq A \varphi(r)^{1/q}.
\]  

Then, there exists a positive constant $C$ such that, for all $f \in L^{(p,\varphi)}(\mathbb{R}^n)$,
\[
|I_\rho f(x)| \leq C M f(x)^{p/q} \left( \|f\|_{L^{(p,\varphi)}} \right)^{1-p/q}, \quad x \in \mathbb{R}^n.
\]
Consequently, $I_\rho$ is bounded from $L^{(p,\varphi)}(\mathbb{R}^n)$ to $wL^{(q,\varphi)}(\mathbb{R}^n)$. Moreover, if $p > 1$, then, for all $f \in wL^{(p,\varphi)}(\mathbb{R}^n)$,
\[
|I_\rho f(x)| \leq C M f(x)^{p/q} \left( \|f\|_{wL^{(p,\varphi)}} \right)^{1-p/q}, \quad x \in \mathbb{R}^n.
\]

Consequently, if $p > 1$, then $I_\rho$ is bounded from $L^{(p,\varphi)}(\mathbb{R}^n)$ to $L^{(q,\varphi)}(\mathbb{R}^n)$ by (3.21) and from $wL^{(p,\varphi)}(\mathbb{R}^n)$ to $wL^{(q,\varphi)}(\mathbb{R}^n)$ by (3.22).

(ii) If $I_\rho$ is bounded from $L^{(p,\varphi)}(\mathbb{R}^n)$ to $wL^{(q,\varphi)}(\mathbb{R}^n)$, then there exists a positive constant $A'$ such that, for all $r \in (0, \infty)$,
\[
\int_0^r \frac{\rho(t)}{t} \, dt \varphi(r)^{1/p} \leq A' \varphi(r)^{1/q}.
\]

Moreover, under the assumption (3.8), if $I_\rho$ is bounded from $L^{(p,\varphi)}(\mathbb{R}^n)$ to $wL^{(q,\varphi)}(\mathbb{R}^n)$, then (3.20) holds for some $A \in (0, \infty)$ and for all $r \in (0, \infty)$. 

(iii) In part (ii) the boundedness from \( L^{(p,\varphi)}(\mathbb{R}^n) \) to \( wL^{(q,\varphi)}(\mathbb{R}^n) \) can be replaced by the boundedness from \( L^{(p,\varphi)}(\mathbb{R}^n) \) to \( L^{(1,\varphi^{-1/q})}(\mathbb{R}^n) \).

The above corollary is an extension of [6, Theorems 1.1, 1.2, 1.8, and 1.9]. Condition (3.20) was first given by Gunawan [12]. The boundedness of the usual fractional integral operator on the classical Morrey space was proven by Adams [1].

**Corollary 3.12.** Let \( 1 \leq p \leq q < \infty, \varphi \in \mathcal{G}^{\text{dec}} \) and \( \rho : (0,\infty) \to (0,\infty) \).

(i) If there exists a positive constant \( A \) such that, for all \( r \in (0,\infty) \),

\[
\left( \sup_{0 < t < r} \rho(t) \right) \varphi(r)^{1/p} \leq A \varphi(r)^{1/q},
\]

(3.24)

then there exists a positive constant \( C \) such that, for all \( f \in L^{(p,\varphi)}(\mathbb{R}^n) \),

\[
M_\rho f(x) \leq CMf(x)^{p/q} \left( \|f\|_{L^{(p,\varphi)}} \right)^{1-p/q}, \quad x \in \mathbb{R}^n.
\]

(3.25)

Consequently, \( M_\rho \) is bounded from \( L^{(p,\varphi)}(\mathbb{R}^n) \) to \( wL^{(q,\varphi)}(\mathbb{R}^n) \). Moreover, if \( p > 1 \), then, for all \( f \in wL^{(p,\varphi)}(\mathbb{R}^n) \),

\[
M_\rho f(x) \leq CMf(x)^{p/q} \left( \|f\|_{wL^{(p,\varphi)}} \right)^{1-p/q}, \quad x \in \mathbb{R}^n.
\]

(3.26)

Consequently, if \( p > 1 \), then \( M_\rho \) is bounded from \( L^{(p,\varphi)}(\mathbb{R}^n) \) to \( L^{(q,\varphi)}(\mathbb{R}^n) \) by (3.25) and from \( wL^{(p,\varphi)}(\mathbb{R}^n) \) to \( wL^{(q,\varphi)}(\mathbb{R}^n) \) by (3.26).

(ii) If \( M_\rho \) is bounded from \( L^{(p,\varphi)}(\mathbb{R}^n) \) to \( wL^{(q,\varphi)}(\mathbb{R}^n) \), then (3.24) holds for some \( A \in (0,\infty) \) and for all \( r \in (0,\infty) \).

(iii) In part (ii), the boundedness from \( L^{(p,\varphi)}(\mathbb{R}^n) \) to \( wL^{(q,\varphi)}(\mathbb{R}^n) \) can be replaced by the boundedness from \( L^{(p,\varphi)}(\mathbb{R}^n) \) to \( L^{(1,\varphi^{-1/q})}(\mathbb{R}^n) \).

### 4 PROPERTIES OF YOUNG FUNCTIONS AND ORLICZ–MORREY SPACES

In this section, we state the properties of Young functions, Orlicz–Morrey and weak Orlicz Morrey spaces. By the convexity, any Young function \( \Phi \) is continuous on \([0,b(\Phi))\) and strictly increasing on \([a(\Phi),b(\Phi))\). Hence, \( \Phi \) is bijective from \([a(\Phi),b(\Phi))\) to \([0,\Phi(b(\Phi))]\).  

**Definition 4.1.** Let

\[
\mathcal{Y}^{(1)} = \{ \Phi \in \Phi_Y : b(\Phi) = \infty \},
\]

\[
\mathcal{Y}^{(2)} = \{ \Phi \in \Phi_Y : b(\Phi) < \infty, \Phi(b(\Phi)) = \infty \},
\]

\[
\mathcal{Y}^{(3)} = \{ \Phi \in \Phi_Y : b(\Phi) < \infty, \Phi(b(\Phi)) < \infty \}.
\]

**Remark 4.2.**

(i) If \( \Phi \in \mathcal{Y}^{(1)} \cup \mathcal{Y}^{(2)} \), then \( \Phi(\Phi^{-1}(u)) = u \) for all \( u \in [0,\infty) \).

(ii) If \( \Phi \in \mathcal{Y}^{(3)} \) and \( 0 < \delta < 1 \), then there exists a Young function \( \Psi \in \mathcal{Y}^{(2)} \) such that \( b(\Phi) = b(\Psi) \) and

\[
\Psi(\delta t) \leq \Phi(t) \leq \Psi(t) \quad \text{for all } t \in [0,\infty).
\]

To see this, we only set \( \Psi = \Phi + \Theta \), where we choose \( \Theta \in \mathcal{Y}^{(2)} \) such that \( a(\Theta) = \delta b(\Phi) \) and \( b(\Theta) = b(\Phi) \).

(iii) \( \overline{\Delta}_2 \subset \Phi_Y \) (26, Lemma 1.2.3).

(iv) Let \( \Phi \in \Phi_Y \). Then, \( \Phi \in \overline{\Delta}_2 \) if and only if \( \Phi \approx \Psi \) for some \( \Psi \in \Delta_2 \), and, \( \Phi \in \overline{\Delta}_2 \) if and only if \( \Phi \approx \Psi \) for some \( \Psi \in \Delta_2 \).

(v) If \( \Phi \in \Phi_Y \), then \( \Phi^{-1} \) satisfies the doubling condition by its concavity.

(vi) Let \( \Phi \in \Phi_Y \). Then, \( \Phi \in \mathcal{V}_2 \) if and only if \( t \mapsto \frac{\Phi(t)}{t^p} \) is almost increasing for some \( p \in (1,\infty) \).
For a Young function $\Phi$, its complementary function is defined by

$$
\tilde{\Phi}(t) = \begin{cases} 
\sup\{tu - \Phi(u) : u \in [0, \infty)\}, & t \in [0, \infty), \\
\infty, & t = \infty.
\end{cases}
$$

Then, $\tilde{\Phi}$ is also a Young function, and $(\Phi, \tilde{\Phi})$ is called a complementary pair. For example, if $\Phi(t) = t^p/p$, then $\tilde{\Phi}(t) = t^{p'}/p'$ for $p, p' \in (1, \infty)$ and $1/p + 1/p' = 1$. If $\Phi(t) = t$, then $\tilde{\Phi}(t) = \begin{cases} 
0, & t \in [0,1], \\
\infty, & t \in (1,\infty].
\end{cases}$

Let $(\Phi, \tilde{\Phi})$ be a complementary pair of functions in $\Phi_Y$. Then, the following inequality holds:

$$
t \leq \Phi^{-1}(t) \tilde{\Phi}^{-1}(t) \leq 2t \quad \text{for} \quad t \in [0, \infty],
$$

which is (1.3) in [60].

Orlicz and weak Orlicz spaces on a measure space $(\Omega, \mu)$ are defined by the following: For $\Phi \in \Phi_Y$, let $L^\Phi(\Omega, \mu)$ and $wL^\Phi(\Omega, \mu)$ be the set of all measurable functions $f$ such that the following functionals are finite, respectively:

$$
\|f\|_{L^\Phi(\Omega, \mu)} = \inf \left\{ \lambda > 0 : \int_\Omega \Phi\left(\frac{|f(x)|}{\lambda}\right) d\mu(x) \leq 1 \right\},
$$

$$
\|f\|_{wL^\Phi(\Omega, \mu)} = \inf \left\{ \lambda > 0 : \sup_{t \in (0, \infty)} \Phi(t) \mu(\{x \in \Omega : |f(x)| > t\}) \leq 1 \right\},
$$

where

$$
\mu(f, t) = \mu(\{x \in \Omega : |f(x)| > t\}).
$$

Then, $\| \cdot \|_{L^\Phi(\Omega, \mu)}$ and $\| \cdot \|_{wL^\Phi(\Omega, \mu)}$ are quasi norms and thereby $L^\Phi(\Omega, \mu)$ and $wL^\Phi(\Omega, \mu)$ are quasi Banach spaces. If $\Phi \in \Phi_Y$, then $\| \cdot \|_{L^\Phi(\Omega, \mu)}$ is a norm and thereby $L^\Phi(\Omega, \mu)$ is a Banach space. For $\Phi, \Psi \in \Phi_Y$, if $\Phi \approx \Psi$, then $L^\Phi(\Omega, \mu) = L^\Psi(\Omega, \mu)$ and $wL^\Phi(\Omega, \mu) = wL^\Psi(\Omega, \mu)$ with equivalent quasi norms, respectively.

For a measurable set $G \subset \Omega$, let $\chi_G$ be its characteristic function. For $\Phi \in \Phi_Y$, it is known that

$$
\|\chi_G\|_{wL^\Phi(\Omega, \mu)} = \|\chi_G\|_{L^\Phi(\Omega, \mu)} = \frac{1}{\Phi^{-1}(1/\mu(G))}, \quad \text{if} \quad \mu(G) > 0. \tag{4.2}
$$

Let $(\Phi, \tilde{\Phi})$ be a complementary pair of functions in $\Phi_Y$. Then, the following generalized Hölder’s inequality holds:

$$
\int_\Omega |f(x)g(x)| d\mu(x) \leq 2\|f\|_{L^\Phi(\Omega, \mu)}\|g\|_{\tilde{\Phi}(\Omega, \mu)}. \tag{4.3}
$$

For a ball $B = B(a, r) \subset \mathbb{R}^n$, let $\mu_B = dx/(|B|\varphi(r))$. Then, we have the following relations.

$$
\|f\|_{\Phi, \mu_B} = \|f\|_{L^\Phi(B, \mu_B)}, \quad \|f\|_{\Phi, \mu_B, \text{weak}} = \|f\|_{wL^\Phi(B, \mu_B)}. \tag{4.4}
$$

By relations (4.2) and (4.4), we have

$$
\|\chi_B\|_{\Phi, \mu_B, \text{weak}} = \|\chi_B\|_{\Phi, \mu_B} = \frac{1}{\Phi^{-1}(1/\mu_B(B))} = \frac{1}{\Phi^{-1}(\varphi(r))}. \tag{4.5}
$$

For functions $f, g$ on $\mathbb{R}^n$, by (4.3) and (4.4), we have

$$
\frac{1}{|B|\varphi(r)} \int_B |f(x)g(x)| \, dx \leq 2\|f\|_{\Phi, \mu_B}\|g\|_{\tilde{\Phi}, \mu_B}. \tag{4.6}
$$
Lemma 4.3. Let \( \Phi \in \Phi_Y \) and \( \varphi \in \mathcal{O}^{\text{dec}} \). Then, there exists a constant \( C \geq 1 \) such that, for any ball \( B = B(a, r) \),
\[
\frac{1}{\Phi^{-1}(\varphi(r))} \leq \| \chi_B \|_{wL(\Phi, \varphi)} \leq \| \chi_B \|_{L(\Phi, \varphi)} \leq C \frac{1}{\Phi^{-1}(\varphi(r))}.
\] (4.7)

Proof. By (4.5), we have
\[
\frac{1}{\Phi^{-1}(\varphi(r))} = \| \chi_B \|_{\Phi, \varphi, B, \text{weak}} \leq \| \chi_B \|_{L(\Phi, \varphi)} \leq \| \chi_B \|_{L(\Phi, \varphi)}.
\]
The last inequality in (4.7) has been proven in [57, Lemma 4.1].

Corollary 4.4. Let \( \varphi \in \mathcal{O}^{\text{dec}} \). Then, there exists a constant \( C \geq 1 \) such that, for any ball \( B = B(a, r) \),
\[
\frac{1}{\varphi(r)} \leq \| \chi_B \|_{wL(1, \varphi)} \leq \| \chi_B \|_{L(1, \varphi)} \leq C \varphi(r).
\]
The following relation between Orlicz–Morrey spaces and generalized Morrey spaces is known:

Lemma 4.5 ([42, Corollary 4.7]). Let \( \Phi \in \Phi_Y \) and \( \varphi : (0, \infty) \rightarrow (0, \infty) \). Then, there exists a positive constant \( C \) such that, for all \( f \in L(\Phi, \varphi)(\mathbb{R}^n) \) and for all balls \( B = B(a, r) \),
\[
\frac{1}{\Phi^{-1}(\varphi(r))} \leq \| \chi_B \|_{\Phi, \varphi, B, \text{weak}} \leq \| \chi_B \|_{L(\Phi, \varphi)} \leq C \frac{1}{\Phi^{-1}(\varphi(r))}.
\] (4.8)

Consequently, \( L(\Phi, \varphi)(\mathbb{R}^n) \) continuously embeds in \( L^{(1, \Phi^{-1}(\varphi))}(\mathbb{R}^n) \).

In the above lemma, we do not need to assume \( \varphi \in \mathcal{O}^{\text{dec}} \), while it was assumed in [42, Corollary 4.7]. Actually, we can prove (4.8) by (4.1), (4.5), and (4.6).

Remark 4.6. If \( \Phi \in \Phi_Y \) and \( \varphi \in \mathcal{O}^{\text{dec}} \), then \( \Phi^{-1}(\varphi) \in \mathcal{O}^{\text{dec}} \), see [42, Corollary 4.7].

The following lemma shows that any weak Orlicz–Morrey space continuously embeds in a generalized Morrey space, whenever \( \Phi \in \mathcal{V}_2 \), and it is an extension of [40, Theorem 3.4], which treats weak Orlicz spaces. Note that if \( \Phi \in \mathcal{V}_2 \), then \( \Phi \approx \Phi_1 \) for some \( \Phi_1 \in \mathcal{V}_2 \), see Remark 4.2, (iii) and (iv). In this case \( \| f \|_{\Phi, \varphi, B, \text{weak}} \sim \| f \|_{\Phi_1, \varphi, B, \text{weak}} \) and \( \Phi^{-1} \sim \Phi^{-1}_1 \).

Lemma 4.7. Let \( \Phi \in \mathcal{V}_2 \) and \( \varphi : (0, \infty) \rightarrow (0, \infty) \). Then, there exists a positive constant \( C \) such that, for all \( f \in wL(\Phi, \varphi)(\mathbb{R}^n) \) and for all balls \( B = B(a, r) \),
\[
\int_B |f(x)| \, dx \leq C \Phi^{-1}(\varphi(r)) \| f \|_{\Phi, \varphi, B, \text{weak}}.
\]

Consequently, \( wL(\Phi, \varphi)(\mathbb{R}^n) \) continuously embeds in \( L^{(1, \Phi^{-1}(\varphi))}(\mathbb{R}^n) \).

Proof. Case 1: \( \Phi \in \mathcal{Y}(1) \cup \mathcal{Y}(2) \). We may assume that \( \| f \|_{\Phi, \varphi, B, \text{weak}} = 1 \). If \( b(\Phi) < \infty \) and \( t \in [b(\Phi), \infty) \), then \( m(f, t) = 0 \). For any ball \( B \) of radius \( r \), let \( t_0 = \Phi^{-1}(\varphi(r)) \). Then, \( \Phi(t_0) = \varphi(r) \in (0, \infty) \) by Remark 4.2 (i). That is, \( t_0 \in (a(\Phi), b(\Phi)) \). Take \( p \in (1, \infty) \) such that \( \Phi(t)/t^p \) is almost increasing, see Remark 4.2 (vi). Then,
\[
\int_B |f(x)| \, dx = \int_0^{t_0} m(B, f, t) \, dt + \int_{t_0}^{b(\Phi)} m(B, f, t) \, dt \leq t_0 |B| + \int_{t_0}^{b(\Phi)} \frac{\varphi(r)|B|}{\Phi(t)} \, dt = t_0 |B| + \varphi(r)|B| \int_{t_0}^{b(\Phi)} \frac{t^p}{\Phi(t)} \, t^{-p} \, dt.
\]
\[ \leq t_0 |B| + \varphi(r)|B| \frac{t_0^p}{\Phi(t_0)} \int_{t_0}^{b(\Phi)} t^{-p} \, dt \]
\[ = t_0 |B| + \frac{t_0 |B| \varphi(r)}{(p-1)\Phi(t_0)} \]
\[ = t_0 |B| + \frac{t_0 |B|}{(p-1)}. \]

This shows the conclusion.

Case 2: \( \Phi \in \mathcal{Y}^{(3)} \). In this case, for any \( \delta \in (0, 1) \), there exists \( \Phi_1 \in \mathcal{Y}^{(2)} \) such that
\[ \Phi_1(\delta t) \leq \Phi(t) \leq \Phi_1(t), \quad t \in [0, \infty], \]
see Remark 4.2 (ii). It follows that
\[ \delta \Phi^{-1}(u) \leq \Phi_1^{-1}(u) \leq \Phi^{-1}(u) \quad \text{and} \quad \delta \|f\|_{wL(\Phi_1, \varphi)} \leq \|f\|_{wL(\Phi, \varphi)} \leq \|f\|_{wL(\Phi_1, \varphi)}. \]

By Case 1, we have
\[ \int_B |f(x)| \, dx \leq C\Phi_1^{-1}(\varphi(r)) \|f\|_{wL(\Phi_1, \varphi)} \leq C\Phi^{-1}(\varphi(r)) \|f\|_{wL(\Phi, \varphi)} / \delta. \]

Letting \( \delta \to 1 \), we have the conclusion. \( \square \)

**Lemma 4.8** ([42, Lemma 9.4]). Let \( \Phi \in \Phi_{Y} \) and \( \varphi \in \mathcal{O}_{\text{dec}} \), and let \( f \in L^{(\Phi, \varphi)}(\mathbb{R}^n) \). For a ball \( B = B(a, r) \), if \( \text{supp} f \cap 2B = \emptyset \), then
\[ Mf(x) \leq C\Phi_1^{-1}(\varphi(r)) \|f\|_{L^{(\Phi_1, \varphi)}} \quad \text{for} \quad x \in B, \]
where the constant \( C \) depends only on \( \Phi \) and \( \varphi \).

**Lemma 4.9.** Let \( \Phi \in \nabla_2 \) and \( \varphi \in \mathcal{O}_{\text{dec}} \), and let \( f \in L^{(\Phi, \varphi)}(\mathbb{R}^n) \). For a ball \( B = B(a, r) \), if \( \text{supp} f \cap 2B = \emptyset \), then
\[ Mf(x) \leq C\Phi_1^{-1}(\varphi(r)) \|f\|_{wL(\Phi_1, \varphi)} \quad \text{for} \quad x \in B, \]
where the constant \( C \) depends only on \( \Phi \) and \( \varphi \).

**Proof.** For any ball \( B' \ni x \) whose radius is \( s \), if \( s \leq r/2 \), then \( \int_{B'} |f(x)| \, dx = 0 \), and, if \( s > r/2 \), then, using Lemma 4.7, we have
\[ \int_{B'} |f(x)| \, dx \leq \Phi^{-1}(\varphi(s)) \|f\|_{wL(\Phi, \varphi)} \leq \Phi^{-1}(\varphi(r)) \|f\|_{wL(\Phi, \varphi)}, \]
since \( r \mapsto \Phi^{-1}(\varphi(r)) \) is almost decreasing and satisfies the doubling condition. \( \square \)

## 5 PROOFS

We first note that, to prove the theorems, we may assume that \( \Phi, \Psi \in \Phi_{Y} \) instead of \( \Phi, \Psi \in \Phi_{Y} \). For example, if \( \Phi \) and \( \Psi \) satisfy (3.4) and \( \Phi \approx \Phi_1, \Psi \approx \Psi_1 \), then \( \Phi_1 \) and \( \Psi_1 \) also satisfy (3.4) by the relation (2.9). Moreover, \( L^{(\Phi, \varphi)}(\mathbb{R}^n) = L^{(\Phi_1, \varphi)}(\mathbb{R}^n) \), \( L^{(\Psi, \varphi)}(\mathbb{R}^n) = L^{(\Psi_1, \varphi)}(\mathbb{R}^n) \), \( wL^{(\Phi, \varphi)}(\mathbb{R}^n) = wL^{(\Phi_1, \varphi)}(\mathbb{R}^n) \), and \( wL^{(\Psi, \varphi)}(\mathbb{R}^n) = wL^{(\Psi_1, \varphi)}(\mathbb{R}^n) \) with equivalent quasi norms. Similarly, by Remark 4.2 (iv), we may assume that \( \Phi \in \nabla_2 \) instead of \( \Phi \in \nabla_2 \). By Remark 2.5, we may also assume that \( \varphi \) is continuous and strictly decreasing.
5.1 Proof of Theorem 3.2

To prove Theorem 3.2, we use the following lemma.

Lemma 5.1 ([61, p. 92]). If \( f \in L^1(\mathbb{R}^n) \), then

\[
m(\mathcal{M}f, t) \leq \frac{C}{t} \int_{|f| > t/2} |f(x)| \, dx, \quad t \in (0, \infty).
\]

Proof of Theorem 3.2. We may assume that \( \Phi \in \nabla_z \) and \( f \geq 0 \). By Remark 4.2 (vi) we can take \( p > 1 \) such that \( \Phi(t)/t^p \) is almost increasing. Then,

\[
\frac{\Phi(t)}{t^p} \leq \frac{\Phi(s)}{s^p}, \quad t < s.
\]

Let

\[
N(f) = \sup_{t \in (0, \infty)} \Phi(t)m(2f, t) < \infty \quad \text{and} \quad G_t = \{2f > t\},
\]

and let \( g_t(x) = (f(x) - t/2)\chi_{G_t}(x) \) and \( h_t(x) = f(x) - g_t(x) \). Then, \( Mh_t(x) \leq t/2 \), which implies

\[
m(\mathcal{M}f, t) \leq m(Mg_t, t/2) + m(Mh_t, t/2) = m(Mg_t, t/2).
\]

We consider three cases \( t \in (0, \alpha(\Phi)] \), \( t \in (\alpha(\Phi), \beta(\Phi)) \), and \( t \in [\beta(\Phi), \infty) \).

Case 1: If \( \alpha(\Phi) > 0 \) and \( t \in (0, \alpha(\Phi)] \), then \( \Phi(t)m(Mf, t) = 0 \).

Case 2: If \( 0 \leq \alpha(\Phi) < \beta(\Phi) \leq \infty \) and \( t \in (\alpha(\Phi), \beta(\Phi)) \), then \( |G_t| = m(2f, t) < \infty \). Hence,

\[
2 \int_{\mathbb{R}^n} g_t(x) \, dx = \int_{G_t} (2f(x) - t) \, dx = \int_0^t m(G_t, 2f, s) \, ds + \int_t^{\beta(\Phi)} m(G_t, 2f, s) \, ds - t|G_t|
\]

\[
\leq \int_t^{\beta(\Phi)} m(2f, s) \, ds = \frac{1}{\Phi(t)} \int_t^{\beta(\Phi)} \frac{\Phi(t)}{\Phi(s)} \Phi(s)m(2f, s) \, ds
\]

\[
\leq \frac{C_p N(f)}{\Phi(t)} \int_t^{\beta(\Phi)} \left( \frac{1}{s} \right)^p \, ds \leq \frac{C_p N(f) t}{(p - 1)\Phi(t)} < \infty.
\]

That is, \( g_t \in L^1(\mathbb{R}^n) \) for \( t \in (\alpha(\Phi), \beta(\Phi)) \). By Lemma 5.1, we have

\[
\Phi(t)m(Mf, t) \leq \Phi(t)m(Mg_t, t/2) \leq \frac{2C\Phi(t)}{t} \int_{\mathbb{R}^n} g_t(x) \, dx \leq \frac{CC_p}{p - 1} N(f).
\]

Case 3: If \( \beta(\Phi) < \infty \) and \( t \in [\beta(\Phi), \infty) \), then \( |G_t| = m(2f, t) = 0 \) and \( g_t = 0 \) a.e. Hence, \( Mg_t = 0 \) and

\[
\Phi(t)m(Mf, t) \leq \Phi(t)m(Mg_t, t/2) = 0.
\]

In conclusion, for some \( C' \geq 1 \) and all \( t \in (0, \infty) \),

\[
\Phi(t)m(Mf, t) \leq C'N(f)
\]

\[
\leq \sup_{s \in (0, \infty)} \Phi(C's)m(2f, s)
\]

\[
= \sup_{s \in (0, \infty)} \Phi(s)m(2C'f, s).
\]

This shows the conclusion. \( \square \)
5.2 Proof of Theorem 3.3

In this subsection, we prove Theorem 3.3 by using Theorems 3.1 and 3.2.

Proof of Theorem 3.3. We may assume that $\Phi \in \Phi_Y$. Let $f \in L^{(\Phi, \varphi)}(\mathbb{R}^n)$ and $||f||_{L^{(\Phi, \varphi)}} = 1$. To prove the norm inequality $\|Mf\|_{wL^{(\Phi, \varphi)}} \leq C_0$, it is enough to prove that, for any ball $B = B(a, r)$,

$$\|Mf\|_{\Phi, \varphi, B, \text{weak}} \leq C_0 \tag{5.1}$$

Let $f = f_1 + f_2$, $f_1 = f \chi_{2B}$. Then, we have

$$\int_{\mathbb{R}^n} \Phi(|f_1(x)|) \, dx \leq \int_{2B} \Phi \left( \frac{|f(x)|}{\|f\|_{\Phi, \varphi, 2B}} \right) \, dx \leq |2B| \varphi(2r) \leq C_{n, \varphi} |B| \varphi(r), \tag{5.2}$$

for some constant $C_{n, \varphi} \geq 1$ by the doubling condition of $\varphi$. Hence, (3.1) yields

$$\sup_{t \in (0, \infty)} \Phi(t) m(B, \frac{Mf_1}{C_{n, \varphi} C_{\Phi}}, t) \leq \frac{1}{C_{n, \varphi}} \int_{\mathbb{R}^n} \Phi(|f_1(x)|) \, dx \leq |B| \varphi(r).$$

Next, since $\text{supp} \ f_2 \cap 2B = \emptyset$, by Lemma 4.8, we have $Mf_2(x) \leq C \Phi^{-1}(\varphi(r))$ for $x \in B$. Then,

$$\sup_{t \in (0, \infty)} \Phi(t) m(B, \frac{Mf_2}{C}, t) \leq \int_B \Phi \left( \frac{Mf_2(x)}{C} \right) \, dx \leq \int_B \varphi(r) \, dx = |B| \varphi(r).$$

In the above, we use (2.8) for the second inequality. Therefore,

$$\|Mf\|_{\Phi, \varphi, B, \text{weak}} \leq 2 \left( \|Mf_1\|_{\Phi, \varphi, B, \text{weak}} + \|Mf_2\|_{\Phi, \varphi, B, \text{weak}} \right) \leq 2(C_{n, \varphi} C_{\Phi} + C),$$

which shows (5.1). If $\Phi \in \nabla_2$, then by the same way we have the norm inequality $\|Mf\|_{L^{(\Phi, \varphi)}} \leq C_0$, using (3.2) instead of (3.1).

Next, let $f \in wL^{(\Phi, \varphi)}(\mathbb{R}^n)$ and $||f||_{wL^{(\Phi, \varphi)}} = 1$. To prove the norm inequality $\|Mf\|_{wL^{(\Phi, \varphi)}} \leq C_0$, it is enough to prove (5.1) for any ball $B = B(a, r)$. Let $f = f_1 + f_2$, $f_1 = f \chi_{2B}$. Then, we have

$$\sup_{t \in (0, \infty)} \Phi(t) m(f_1, t) \leq \sup_{t \in (0, \infty)} \Phi(t) m \left( \frac{f}{\|f\|_{\Phi, \varphi, 2B, \text{weak}}}, t \right) \leq |2B| \varphi(2r) \leq C_{n, \varphi} |B| \varphi(r),$$

instead of (5.2). Hence, (3.3) yields

$$\sup_{t \in (0, \infty)} \Phi(t) m \left( B, \frac{Mf_1}{C_{n, \varphi} C_{\Phi}}, t \right) \leq \frac{1}{C_{n, \varphi}} \sup_{t \in (0, \infty)} \Phi(t) m \left( \frac{Mf_1}{C_{\Phi}}, t \right) \leq \frac{1}{C_{n, \varphi}} \sup_{t \in (0, \infty)} \Phi(t) m(f_1, t) \leq |B| \varphi(r).$$

We also have $Mf_2(x) \leq C \Phi^{-1}(\varphi(r))$ for $x \in B$ by Lemma 4.9. Then,

$$\sup_{t \in (0, \infty)} \Phi(t) m \left( B, \frac{Mf_2}{C}, t \right) \leq \sup_{t \in (0, \infty)} \Phi(t) m \left( B, \Phi^{-1}(\varphi(r)), t \right) \leq |B| \varphi(r).$$

Therefore, we have the conclusion. \qed
5.3 | Proof of Theorem 3.4

We may assume that \( \Phi, \Psi \in \Phi_Y \) and that \( \varphi \) is continuous and strictly decreasing. First, we state a lemma.

Lemma 5.2 ([4, Proposition 1]). Let \( \rho, \tau : (0, \infty) \to (0, \infty) \). Assume that \( \rho \) satisfies (1.5) and that \( \tau \) satisfies the doubling condition (2.12). Define

\[
\tilde{\rho}(r) = \int_{K_1 r}^{K_2 r} \frac{\rho(t)}{t} \, dt, \quad r \in (0, \infty).
\]

(5.3)

Then, there exists a positive constant \( C \) such that, for all \( r \in (0, \infty) \),

\[
\sum_{j=-\infty}^{-1} \tilde{\rho}(2^j r) \lesssim \int_0^{K_2 r} \frac{\rho(t)}{t} \, dt,
\]

(5.4)

\[
\sum_{j=0}^\infty \tilde{\rho}(2^j r) \tau(2^j r) \lesssim \int_{K_1 r}^{\infty} \frac{\rho(t) \tau(t)}{t} \, dt.
\]

(5.5)

Proof of Theorem 3.4 (i). By the assumption (2.14) we may assume that \( \varphi \) is bijective from \((0, \infty)\) to itself. If \( \Phi^{-1}(0) > 0 \), then, from (3.4) it follows that

\[
0 < \int_0^\infty \frac{\rho(t)}{t} \, dt \Phi^{-1}(0) \lesssim \Psi^{-1}(0),
\]

since \( \lim_{r \to \infty} \varphi(r) = 0 \). Let \( f \in L^{(\Phi, \varphi)}(\mathbb{R}^n) \) and \( \|f\|_{L^{(\Phi, \varphi)}} = 1 \), and let \( x \in \mathbb{R}^n \). We may assume that

\[
0 < \frac{Mf(x)}{C_0} < \infty \quad \text{and} \quad 0 \leq \Phi \left( \frac{Mf(x)}{C_0} \right) < \infty,
\]

otherwise there is nothing to prove.

If \( \Phi(Mf(x)/C_0) = 0 \), then, by (2.8) we have

\[
\frac{Mf(x)}{C_0} \leq \Phi^{-1}(0) = \sup\{u \geq 0 : \Phi(u) = 0\}.
\]

Hence, using (5.4), we have

\[
|I_\rho f(x)| \leq \sum_{j=-\infty}^\infty \int_{2^j|x-y| < 2^{j+1}} \frac{\rho(|x-y|)}{|x-y|^n} |f(x)| \, dx
\]

\[
\lesssim \sum_{j=-\infty}^\infty \tilde{\rho}(2^j) \int_{|x-y| < 2^{j+1}} |f(y)| \, dy \lesssim \int_{0}^{\infty} \frac{\rho(t)}{t} \, dt Mf(x)
\]

\[
\leq C_0 \int_{0}^{\infty} \frac{\rho(t)}{t} \, dt \Phi^{-1}(0) \lesssim \Psi^{-1}(0) \lesssim \Psi^{-1} \left( \Phi \left( \frac{Mf(x)}{C_0} \right) \right),
\]

which shows (3.5).

If \( \Phi(Mf(x)/C_0) > 0 \), choose \( r \in (0, \infty) \) such that

\[
\varphi(r) = \Phi \left( \frac{Mf(x)}{C_0} \right),
\]

(5.6)

and let
\[ J_1 = \sum_{j=-\infty}^{1} \frac{\hat{\rho}(2^j r)}{(2^j r)^n} \int_{|x-y|<2^{j+1} r} |f(y)| \, dy, \]
\[ J_2 = \sum_{j=0}^{\infty} \frac{\hat{\rho}(2^j r)}{(2^j r)^n} \int_{|x-y|<2^{j+1} r} |f(y)| \, dy. \]

Then,
\[ |I_{\rho} f(x)| \leq J_1 + J_2. \]

By (5.6) and (2.8), we have \( Mf(x) \leq C_0 \Phi^{-1}(\varphi(r)) \). Then, using (5.4), we have
\[ J_1 \lesssim \int_0^{K_2 r} \frac{\rho(t)}{t} \, dt \, Mf(x) \lesssim \int_0^{K_2 r} \frac{\rho(t)}{t} \, dt \, \Phi^{-1}(\varphi(r)). \]

Next, by Lemma 4.5, \( \|f\|_{L(\Phi, \varphi)} \leq 1 \), and (5.5), we have
\[ J_2 \lesssim \sum_{j=0}^{\infty} \frac{\rho(2^j r) \Phi^{-1}(\varphi(2^{j+1} r))}{\Phi^{-1}(\varphi(2^j r))} \lesssim \int_{K_1 r}^{\infty} \frac{\rho(t) \Phi^{-1}(\varphi(t))}{t} \, dt. \]

Then by (3.4), the doubling condition of \( \Phi^{-1}(\varphi(r)) \) and \( \Psi^{-1}(\varphi(r)) \), and (5.6), we have
\[ J_1 + J_2 \lesssim \int_0^{K_2 r} \frac{\rho(t)}{t} \, dt \, \Phi^{-1}(\varphi(r)) + \int_{K_1 r}^{\infty} \frac{\rho(t) \Phi^{-1}(\varphi(t))}{t} \, dt \]
\[ \lesssim \frac{\Psi^{-1}(\Phi(K_2 r))}{\Phi^{-1}(\varphi(K_2 r))} \Phi^{-1}(\varphi(r)) + \Psi^{-1}(\varphi(K_1 r)) \]
\[ \sim \Psi^{-1}(\varphi(r)) = \Psi^{-1}\left( \Phi\left( \frac{Mf(x)}{C_0} \right) \right). \]

Combining this inequality with (2.8), we have (3.5).

The proof of (3.6) is almost the same as one of (3.5). The only difference is that we use Lemma 4.7 instead of Lemma 4.5 to estimate \( J_2 \).

□

To prove Theorem 3.4 (ii) we state three lemmas.

**Lemma 5.3** ([6, Lemma 2.1]). There exists a positive constant \( C \) such that, for all \( R > 0 \),
\[ \int_0^{R/2} \frac{\rho(t)}{t} \, dt \, \chi_{B(0,R/2)}(x) \leq CI_{\rho} \chi_{B(0,R)}(x), \quad x \in \mathbb{R}^n. \]

The following lemma is an extension of [6, Lemma 2.4] and gives a typical element in \( L^{(\Phi, \varphi)}(\mathbb{R}^n) \).

**Lemma 5.4.** For \( \Phi \in \Phi_Y \) and \( \varphi \in G^{dec} \), let \( g(x) = \Phi^{-1}(\varphi(|x|)) \). If \( \varphi \) satisfies (3.8), then \( g \in L^{(\Phi, \varphi)}(\mathbb{R}^n) \).

**Proof.** By Remark 2.5, we may assume that \( \varphi \) is decreasing. In this case \( x \mapsto \varphi(|x|) \) is radial decreasing, so that, for any ball \( B(a, r) \),
\[ \int_{B(a,r)} \varphi(|x|) \, dx \leq \int_{B(0,r)} \varphi(|x|) \, dx \sim \frac{1}{r^n} \int_0^r \varphi(t) t^{n-1} \, dt \leq \varphi(r). \]

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\[ \int_0^{R/2} \frac{\rho(t)}{t} \, dt \, \chi_{B(0,R/2)}(x) \leq CI_{\rho} \chi_{B(0,R)}(x), \quad x \in \mathbb{R}^n. \]

The following lemma is an extension of [6, Lemma 2.4] and gives a typical element in \( L^{(\Phi, \varphi)}(\mathbb{R}^n) \).

**Lemma 5.4.** For \( \Phi \in \Phi_Y \) and \( \varphi \in G^{dec} \), let \( g(x) = \Phi^{-1}(\varphi(|x|)) \). If \( \varphi \) satisfies (3.8), then \( g \in L^{(\Phi, \varphi)}(\mathbb{R}^n) \).

**Proof.** By Remark 2.5, we may assume that \( \varphi \) is decreasing. In this case \( x \mapsto \varphi(|x|) \) is radial decreasing, so that, for any ball \( B(a, r) \),
\[ \int_{B(a,r)} \varphi(|x|) \, dx \leq \int_{B(0,r)} \varphi(|x|) \, dx \sim \frac{1}{r^n} \int_0^r \varphi(t) t^{n-1} \, dt \leq \varphi(r). \]
In the above, we used (3.8) for the last inequality. Then, taking a suitable constant $C_g \geq 1$, and using the convexity of $\Phi$ and (2.8), we have

$$
\frac{1}{\varphi(r)} \int_{B(a,r)} \Phi\left(\frac{\Phi^{-1}(\varphi(|x|))}{C_g}\right) dx \leq \frac{1}{C_g \varphi(r)} \int_{B(a,r)} \Phi(\Phi^{-1}(\varphi(|x|))) dx
$$

$$
\leq \frac{1}{C_g \varphi(r)} \int_{B(a,r)} \varphi(|x|) dx \leq 1.
$$

This shows $g \in L^{(\Phi, \varphi)}(\mathbb{R}^n)$ with $\|g\|_{L^{(\Phi, \varphi)}} \leq C_g$. \hfill \Box

**Lemma 5.5.** Let $\Phi \in \Phi_Y$ and $\varphi \in \mathcal{G}^{\text{dec}}$. For $R > 0$, let $g_R(x) = \Phi^{-1}(\varphi(|x|))\chi_{\mathbb{R}^n \setminus B(0,R)}(x)$. Then, there exists a positive constant $C$ such that, for all $R > 0$,

$$
\int_{2R}^{\infty} \frac{\rho(t)\Phi^{-1}(\varphi(t))}{t} dt \chi_{B(0,R)}(x) \leq C I_{\rho} g_R(x).
$$

**Proof.** Let $x \in B(0,R)$. Then, $B(0,R) \subset B(x,2R)$ and $|x-y| \sim |y|$ for all $y \notin B(0,2R)$. Hence,

$$
I_{\rho} g_R(x) = \int_{\mathbb{R}^n \setminus B(0,R)} \frac{\rho(|x-y|)\Phi^{-1}(\varphi(|y|))}{|x-y|^n} dy
$$

$$
\geq \int_{\mathbb{R}^n \setminus B(x,2R)} \frac{\rho(|x-y|)\Phi^{-1}(\varphi(|y|))}{|x-y|^n} dy
$$

$$
= \int_{\mathbb{R}^n \setminus B(0,2R)} \frac{\rho(|y|)\Phi^{-1}(\varphi(|x-y|))}{|y|^n} dy
$$

$$
\sim \int_{2R}^{\infty} \frac{\rho(t)\Phi^{-1}(\varphi(t))}{t} dt.
$$

This shows the conclusion. \hfill \Box

**Proof of Theorem 3.4 (ii).** First, by Lemma 5.3 and the boundedness of $I_{\rho}$, we have

$$
\int_{0}^{r} \frac{\rho(t)}{t} dt \|\chi_{B(0,r)}\|_{wL(\Psi, \varphi)} \lesssim \|I_{\rho} \chi_{B(0,2r)}\|_{wL(\Psi, \varphi)} \lesssim \|\chi_{B(0,2r)}\|_{L^{(\Psi, \varphi)}}.
$$

By Lemma 4.3 and the doubling condition of $\Phi^{-1}(\varphi(r))$ we have

$$
\int_{0}^{r} \frac{\rho(t)}{t} dt \Phi^{-1}(\varphi(r)) \lesssim \Psi^{-1}(\varphi(r)).
$$

Second, under the assumption (3.8), let $g$ and $g_R$ be functions as in Lemmas 5.4 and 5.5, respectively. Then, by the boundedness of $I_{\rho}$ we obtain

$$
\int_{r}^{\infty} \frac{\rho(t)\Phi^{-1}(\varphi(t))}{t} dt \|\chi_{B(0,r/2)}\|_{wL(\Psi, \varphi)} \lesssim \|I_{\rho} g_{r/2}\|_{wL(\Psi, \varphi)} \lesssim \|g_{r/2}\|_{L^{(\Psi, \varphi)}} \lesssim \|g\|_{L^{(\Phi, \varphi)}}.
$$

By Lemma 4.3 and the doubling condition of $\Psi^{-1}(\varphi(r))$, we have

$$
\int_{r}^{\infty} \frac{\rho(t)\Phi^{-1}(\varphi(t))}{t} dt \lesssim \Psi^{-1}(\varphi(r)).
$$

Thus, we obtain the conclusion. \hfill \Box
Proof of Theorem 3.4 (iii). By Lemma 4.3, Corollary 4.4, and Remark 4.6 we have
\[ \|X_B(0,r)\|_{wL^p(\varphi,\psi)} \sim \|X_B(0,r)\|_{L^{1,\psi^{-1}(\varphi)}(\Omega)} \sim \frac{1}{\psi^{-1}(\varphi(r))}. \]
Then, we can replace \( \|X_B(0,r)\|_{wL^p(\varphi,\psi)} \) by \( \|X_B(0,r)\|_{L^{1,\psi^{-1}(\varphi)}(\Omega)} \) in the proof of (ii).

5.4 Proof of Theorem 3.5

We may assume that \( \Phi, \Psi \in \Phi_Y \) and that \( \varphi \) is continuous and strictly decreasing.

Proof of Theorem 3.5 (i). The pointwise estimate (3.10) was already proven in [57, Theorem 5.1]. The pointwise estimate (3.11) can be proven by almost the same way as (3.10). To prove (3.11), we use
\[ \int_B |f(x)| \, dx \leq C \Phi^{-1}(\varphi(r)) \|f\|_{L^p(\Phi,\varphi)} \]
by Lemma 4.7 instead of
\[ \int_B |f(x)| \, dx \leq C \Phi^{-1}(\varphi(r)) \|f\|_{L^p(\Phi,\varphi)}. \]
Other parts are the same as the proof of (3.10).

For the proof of Theorem 3.5 (ii), we use the following lemma.

Lemma 5.6 ([56, Lemma 5.1]). Let \( \rho : (0,\infty) \to (0,\infty) \). Then, for all \( r \in (0,\infty) \),
\[ \left( \sup_{0 < t \leq r} \rho(t) \right) \chi_B(0,r)(x) \leq (M_\rho \chi_B(0,r))(x), \quad x \in \mathbb{R}^n. \] (5.7)

Proof of Theorem 3.5 (ii). By Lemma 5.6 and the boundedness of \( M_\rho \) from \( L^p(\Phi,\varphi)(\mathbb{R}^n) \) to \( wL^p(\Psi,\varphi)(\mathbb{R}^n) \), we have
\[ \left( \sup_{0 < t \leq r} \rho(t) \right) \|X_B(0,r)\|_{wL^p(\varphi,\psi)} \leq \|M_\rho X_B(0,r)\|_{wL^p(\varphi,\psi)} \lesssim \|X_B(0,r)\|_{L^{1,\psi^{-1}(\varphi)}(\Omega)}. \]
Then, by Lemma 4.3 we have the conclusion.

Proof of Theorem 3.5 (iii). The same as proof of Theorem 3.4 (iii).

ACKNOWLEDGMENT

The authors would like to thank the referees for their careful reading and many useful comments, by which we could add the third parts in Theorems 3.4 and 3.5, respectively. The second author was supported by Grant-in-Aid for Scientific Research (B), No. 15H03621, Japan Society for the Promotion of Science.

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How to cite this article: R. Kawasumi, E. Nakai, and M. Shi, Characterization of the boundedness of generalized fractional integral and maximal operators on Orlicz–Morrey and weak Orlicz–Morrey spaces, Math. Nachr. 296 (2023), 1483–1503. https://doi.org/10.1002/mana.202000332