Chern-Simons term in the 4-dimensional SU(2) Higgs Model

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Abstract

Using a variation of Luescher’s geometric charge definition for SU(2) lattice gauge theory, we have managed to give a geometric expression for it’s Chern-Simons term. From this definition we have checked the periodic structure. We determined the Chern-Simons density for lattices \(L^4\) and \(L^3 \times 2, 4\) with \(L = 4, 6, \) and \(8\) near the critical region in the SU(2) Higgs model. The data indicate that tunneling is increased at high temperature.

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1 Introduction

Quite some time ago ’t Hooft found that the baryon number and the lepton number are not conserved in the electroweak theory $B - L$ is of course conserved due to the anomaly cancellation. Baryon number violation is caused by the nontrivial topological winding of the SU(2) gauge fields. The anomaly of the fermionic current relates the winding of the gauge fields and the change in the baryon number by an amount

$$B(t_2) - B(t_1) = \frac{N_f}{16\pi^2} \int_{t_1}^{t_2} \int d^3 x \text{tr} [F_{\mu\nu} \tilde{F}_{\mu\nu}]$$  \hfill (1.1)

where $N_f$ is the number of families of quarks and leptons. Equivalent, in the temporal gauge $A_0 = 0$ we can relate the change in the baryon number to the change in the Chern-Simons number

$$B(t_2) - B(t_1) = N_f [N_{CS}(t_2) - N_{CS}(t_1)]$$  \hfill (1.2)

where the Chern-Simons number $N_{CS}$ is

$$N_{CS} = -\frac{1}{8\pi^2} \int d^3 x \epsilon_{ijk} \text{tr} [A_i (\partial_j A_k + \frac{2}{3} A_j A_k)].$$  \hfill (1.3)

At zero temperature such processes are exponentially suppressed as $\exp(-2\pi/\alpha_W)$, $\alpha \approx 1/30$. This is because any gauge field configuration which changes the winding number has an action at least that of the barrier height $2\pi/\alpha_W$.

At high temperatures which prevail in the early universe such an exponential suppression is absent since the system can pass over the barrier classically. The only suppression factor is the Boltzmann factor $\exp(-\beta E)$ where $E$ is the barrier height, and it was shown by the authors in ref. [2] that this factor goes to one. Thus, any baryon asymmetry generated at the GUT scale will get washed out as the universe approaches the electroweak phase transition from above. This happens typically around 10 Tev and is caused by static objects called sphalerons. These are unstable, but finite energy solutions of the classical Yang-Mills Higgs fields. While an instanton tunnel between two vacua, a sphaleron moves from one top of the barrier to the next. If we assume that the vacuum has $N_{CS} = 0$, then the sphaleron has a baryonic charge of 1/2.

The radical solution to the problem of the observed baryon asymmetry in the universe was put forward by the same authors using CP violating processes in the
electroweak theory and assuming thermal non-equilibrium provided by the expansion of the universe. In order for the whole scenario to work, it is necessary that the electroweak phase transition is of first order, which severely restricts the Higgs and the top quark masses. For more on the origin of the baryon asymmetry see ref. [3].

There has been some lattice studies of baryon number violating processes in the two dimensional Abelian Higgs model, as well as in the four dimensional SU(2) Higgs model ref. [4] and ref. [5]. The configurations are prepared at high temperature and the system is allowed either to change via the classical Hamiltonian equation of motion or by a Langevin equation. It is then possible to study how the Chern-Simons term $N_{CS}$ changes during the evolution as a function of the temperature. It was shown that when $\Delta N_{CS} = \pm 1$ the system passes through a sphaleron transition.

Since baryon number violating processes are of non-perturbative origin it is not clear if the sphaleron approximation is sufficient enough. There might be other relevant configurations contributing. This warrants a study in the full Higgs model without assuming high temperature. The quantity we have studied is the constraint effective potential $V(N_{CS})$

$$\exp(-V(N_{CS})) = \int dA d\phi \exp(-S(A, \phi)) \delta(N_{CS} + \frac{1}{8\pi^2} \int d^3 x \epsilon_{ijk} tr[A_i(\partial_j A_k + \frac{2}{3} A_j A_k)]). \quad (1.4)$$

As a measure for baryon number violating processes we can use the parameter $r = \exp(V(0) - V(\frac{1}{2}))$. To evaluate $N_{CS}$ we have used a geometric definition given in ref. [3]. After vectorizing the code first made by one of us (MLL), see ref. [7], we are able to calculate $N_{CS}$ efficiently.

## 2 Topological charge and the Chern-Simons term in the continuum

As a warm up, let us first consider the two dimensional Abelian gauge theory. We always work in Euclidean space time. The theory is determined from the gauge potentials $A_\mu(x)$, ($\mu = 0, 1$) which under a local gauge transformation $g(x)$ transform as:

$$A_\mu(x) \rightarrow A_\mu(x) + ig^{-1}(x)\partial_\mu g(x). \quad (2.1)$$
The gauge field tensor \( F_{\mu\nu}(x) \):
\[
F_{\mu\nu}(x) = \partial_{\mu}A_{\nu}(x) - \partial_{\nu}A_{\mu}(x)
\]  
(2.2)
is of course gauge invariant. From this we can define a gauge invariant and integer topological charge \( Q \):
\[
Q = \frac{1}{4\pi} \int_M d^2x \epsilon_{\mu\nu}F_{\mu\nu}(x) \in \mathbb{Z}
\]  
(2.3)
with \( M \) being the manifold. The topological charge density \( q(x) \) can be written as a perfect derivative
\[
q(x) = \frac{1}{4\pi} \epsilon_{\mu\nu}F_{\mu\nu}(x) = \partial_{\mu}K_{\mu}(x)
\]  
(2.4)
where we have introduced the Chern-Simons density \( K_{\mu}(x) \).
\[
K_{\mu} = \frac{1}{2\pi} \epsilon_{\mu\nu}A_{\nu}(x).
\]  
(2.5)
It transforms like:
\[
K_{\mu}(x) \rightarrow K_{\mu}(x) + \frac{i}{2\pi} \epsilon_{\mu\nu}g^{-1}(x)\partial_{\nu}g(x).
\]  
(2.6)
Now we can define the gauge dependent Chern-Simons number \( N_{CS} \)
\[
N_{CS} = \int_{\partial M = S^1} dx_1 K_0(x) \notin \mathbb{Z}
\]  
(2.7)
Here, the boundary of the manifold \( \partial M \) is assumed to be a one sphere \( S^1 \). In general \( N_{CS} \) is not an integer. This happens only if the the gauge field is pure gauge, that is \( A_{\mu}(x) = ig^{-1}(x)\partial_{\mu}g(x) \). However the gauge variation is always an integer
\[
\delta N_{CS} = \frac{i}{2\pi} \int_{\partial M = S^1} dx_1 g^{-1}(x)\partial_{1}g(x) \in \mathbb{Z}.
\]  
(2.8)
This follows from homotopy theory by considering the mapping
\[
g(x) : S^1 \rightarrow U(1) = S^1.
\]  
(2.9)
Such mappings are characterized with the homotopy class
\[
\Pi_1(S^1) \in \mathbb{Z}.
\]  
(2.10)
Now we are prepared for the four dimensional SU(2) gauge theory. The gauge fields are now non-abelian \( A_{\mu}(x) = A_{\mu}^{a}(x)T^{a} \) \((\mu = 0, 1, 2, 3 \text{ and } T^{a}, a = 1, 2, 3 \text{ are the Pauli-matrices})\), and they transform under a local gauge transformation \( g(x) \) as:
\[
A_{\mu}(x) \rightarrow g^{-1}(x)A_{\mu}(x)g(x) + g^{-1}(x)\partial_{\mu}g(x).
\]  
(2.11)
The gauge field tensor $F_{\mu\nu}(x)$:

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + [A_\mu(x), A_\nu(x)] \tag{2.12}$$

transforms now gauge covariantly

$$F_{\mu\nu}(x) \rightarrow g^{-1}(x)F_{\mu\nu}(x)g(x) \tag{2.13}$$

In analogy with two dimensions we have a gauge invariant and integer topological charge $Q$:

$$Q = -\frac{1}{32\pi} \int_M d^4x \epsilon_{\mu\nu\rho\sigma} tr[F_{\mu\nu}(x)F_{\rho\sigma}(x)] \in \mathbb{Z} \tag{2.14}$$

The topological charge density $q(x)$ is again a perfect derivative

$$q(x) = -\frac{1}{32\pi} \epsilon_{\mu\nu\rho\sigma} tr[F_{\mu\nu}(x)F_{\rho\sigma}(x)] = \partial_\mu K_\mu(x) \tag{2.15}$$

where the Chern-Simons density $K_\mu(x)$ is

$$K_\mu(x) = -\frac{1}{8\pi^2} \epsilon_{\mu\nu\rho\sigma} tr[A_\nu(x)(\partial_\rho A_\sigma(x) + \frac{2}{3} A_\rho(x)A_\sigma(x))] \tag{2.16}$$

It transforms like:

$$K_\mu(x) \rightarrow K_\mu(x) - \frac{1}{24\pi^2} \epsilon_{\mu\nu\rho\sigma} tr[g^{-1}(x)\partial_\nu g(x)g^{-1}(x)\partial_\rho g(x)g^{-1}(x)\partial_\sigma g(x)]$$

$$- \frac{1}{8\pi^2} \epsilon_{\mu\nu\rho\sigma} tr[\partial_\nu(\partial_\rho g(x)g^{-1}(x)A_\sigma(x))] \tag{2.17}$$

Finally the Chern-Simons number $N_{CS}$ is:

$$N_{CS} = \int_{\partial M = S^3} d^3x K_0(x) \notin \mathbb{Z} \tag{2.18}$$

This time the boundary of the manifold $\partial M$ is assumed to be a three sphere $S^3$. While $N_{CS}$ is only an integer for pure gauge configurations, the gauge variation is an integer (the boundary term vanishes)

$$\delta N_{CS} = -\frac{1}{24\pi^2} \int_{\partial M = S^3} d^3x \epsilon_{\mu\nu\rho\sigma} tr[g^{-1}(x)\partial_\nu g(x)g^{-1}(x)\partial_\rho g(x)g^{-1}(x)\partial_\sigma g(x)] \in \mathbb{Z} \tag{2.19}$$

This follows also from homotopy theory using the mapping

$$g(x) : S^3 \rightarrow SU(2) = S^3. \tag{2.20}$$

Such mappings are characterized with the homotopy class

$$\pi_3(S^3) \in \mathbb{Z}. \tag{2.21}$$
3 Topological charge and the Chern-Simons term on the lattice

We will now consider the lattice versions of the topological charge and the Chern-Simons number. We begin with the two dimensional U(1) theory and we assume that the manifold is a two torus \( M = T^2 \) and we shall cover \( M \) with cells (plaquettes) \( c(n) \). Let the gauge potential \( A_{n-\hat{\mu}} \) be defined on \( c(n-\hat{\mu}) \) and likewise \( A_{n} \) be defined on \( c(n) \). At the faces (links) \( f(n, \mu) = c(n-\hat{\mu}) \cap c(n) \), we relate the two potentials via a gauge transformation or transition function \( v_{n,\mu} \)

\[
A_{\nu-\hat{\mu}} = A_{\nu} + iv_{n,\mu}^{-1}\partial_{\nu}v_{n,\mu}. \tag{3.1}
\]

Starting from eqn. 2.3 we find for the topological charge

\[
Q = \frac{i}{2\pi} \sum_{n,\mu} \epsilon_{\mu\nu} \int_{f(n,\mu)} dx_{\nu}v_{n,\mu}^{-1}\partial_{\nu}v_{n,\mu} \in \mathbb{Z}. \tag{3.2}
\]

In the local axial gauge in \( c(n) \) one has at the corners

\[
v_{n,\mu}(x) = w_{n-\hat{\mu}}(x)w_{n}^{-1}(x) \tag{3.3}
\]

where \( w_{n}(x) \) is a parallel transporter. To get to the lattice result one must interpolate \( v_{n,\mu}(x) \) continuously between \( v_{n,\mu}(n) \) and \( v_{n,\mu}(n + \hat{\nu}) \) say

\[
v_{n,\mu}(x) = [v_{n,\mu}(n + \hat{\nu})v_{n,\mu}^{-1}(n)]^{x}v_{n,\mu}(n) \quad 0 \leq x \leq 1. \tag{3.4}
\]

After some simple algebra one arrives at:

\[
Q = \frac{1}{2\pi} \sum_{n} \arg[U_{n,0}U_{n+\hat{\nu},1}U_{n+1,0}^{-1}U_{n,1}^{-1}] \in \mathbb{Z}, \quad -\pi < \arg < \pi. \tag{3.5}
\]

Here \( U_{n,\mu} \) are the links and we notice that the charge only depends on the plaquette angle, hence it is gauge invariant. If we write \( U_{n,\mu} = \exp[ik_{n,\mu}] \) then the natural definition of Chern-Simons term is

\[
N_{CS} = \frac{1}{2\pi} \sum_{n} k_{n,0} \tag{3.6}
\]

where the sum is over the spatial lattice only. Since under a gauge transformation \( g(n) = \exp[i\theta(n)] \)

\[
k_{n,0} \to k_{n,0} + \theta(n) - \theta(n + \hat{\theta}) + 2\pi \times m, \quad m \in \mathbb{Z}, \tag{3.7}
\]
it is clear that $N_{CS}$ only can change by an integer.

This generalizes to SU(2) in four dimensions $M = T^4$. The cells are now hypercubes and the faces $f(n, \mu)$ are cubes. There are also plaquettes $p(n, \mu, \nu) = f(n) \cap f(n - \hat{\nu})$. For $Q$ one has, see ref. [6].

$$Q = -\frac{1}{24\pi^2} \sum_n \epsilon_{\mu\nu\rho\sigma} \int_{f(n,\mu)} d^3x [v_{n,\mu}^{-1}(x)\partial_{\nu}v_{n,\mu}(x)v_{n,\mu}^{-1}(x)\partial_{\rho}v_{n,\mu}(x)v_{n,\mu}^{-1}(x)\partial_{\sigma}v_{n,\mu}(x)] + 3 \int_{p(n,\mu,\nu)} d^2x [v_{n,\mu}(x)\partial_{\nu}v_{n,\mu}(x)v_{n,\mu}^{-1}(x)\partial_{\rho}v_{n-\hat{\mu},\nu}(x)v_{n-\hat{\mu},\nu}(x)]]. \quad (3.8)$$

At this point we shall deviate from Luescher's charge and use a slightly different procedure given by Seiberg. It has the advantage that one can define a Chern-Simons term also. The topological charge is:

$$Q = \sum_n q(n)$$
$$q(n) = \frac{1}{2\pi} \sum_\mu (-1)^\mu (k_{n,\mu} - k_{n+\mu,\mu}) - \pi < q(n) < \pi. \quad (3.9)$$

Here $k_{n,\mu}$ is the local Chern-Simons term which leads to

$$N_{CS} = \frac{1}{2\pi} \sum_n k_{n,0} \quad (3.10)$$

with the summation over the spatial lattice only. Explicitly:

$$k_{n,\mu} = -\frac{1}{12\pi} \sum_n \epsilon_{\mu\nu\rho\sigma} \int_{f(n,\mu)} d^3x [S(x)\partial_{\nu}S(x)^{-1}S(x)\partial_{\rho}S(x)^{-1}S(x)\partial_{\sigma}S(x)^{-1}] + 3 \int_{p(n,\mu,\nu)} d^2x [P^{-1}(x)\partial_{\rho}P(x)S^{-1}(x)\partial_{\sigma}S(x)]]]. \quad (3.11)$$

The function $P$ is defined on a plaquette with corners $a, b, c, d$:

$$P(x, y) = U_{ac}^y U_{ca}^y (U_{ab} U_{cd} U_{db} U_{ba})^y U_{ab} U_{bd}^y x, \quad 0 \leq (x, y) \leq 1. \quad (3.12)$$

In particular at the corners of the plaquette:

$$P(0, 0) = 1$$
$$P(1, 0) = U_{ab}$$
$$P(0, 1) = U_{ac}$$
$$P(1, 1) = U_{ac} U_{cd}. \quad (3.13)$$
Therefore, $P$ can be interpreted as the gauge transformation on the plaquette which brings the links into the complete axial gauge. Likewise $S$ is defined on the cube, such that it brings the links into the complete axial gauge. We have not given the expressions for $S$ since they are rather cumbersome. Letting $U_{n,\mu} = e^{aA_{n,\mu}}$, and taking the naive continuum limit $a \to 0$ one finds:

$$
\frac{1}{2\pi} k_{n,\mu}(x) = -\frac{a^3}{8\pi^2} \epsilon_{\mu\nu\rho\sigma} tr\left[ A_{n,\nu}(\partial_\rho A_{n,\sigma} + \frac{2}{3} A_{n,\rho} A_{n,\sigma}) \right]
$$

(3.14)

which agrees with the continuum expression eqn. 2.16. The only difference between Luescher’s and Seiberg’s definition is that Luescher gauge fixes to the complete axial gauge in the whole hypercube. All links in the hypercube are therefore typically of the form $U_{ac} U_{cd} U_{db} U_{ba}$. Thus the $k_{n,\mu}$ is by itself gauge invariant and it contributes $O(a^4)$ in the naive continuum limit. So it can not be interpreted as a Chern-Simons term. Also one can relax the restriction of the topological charge given in eqn. 3.8.

4 Tests and Monte-Carlo results for the Chern-Simons density

We have used an SU(2) Higgs model with action:

$$
S = -\beta \sum_{n,\mu<\nu} tr[U_{n,\mu\nu}] - \kappa \sum_\mu tr[\Phi_{n}^\dagger U_{n,\mu} \Phi_{n+\mu}] + \lambda (\Phi_{n}^\dagger \Phi_{n} - 1)^2.
$$

(4.1)

We always tried to work close to the Higgs phase transition, so the gauge Higgs couplings were chosen with: $(\beta, \lambda) = (2.25, 0.5)$ and $\kappa = 0.25, 0.30$.

To do the integrals in eqn. 3.10 we have used a vectorized version of the topology program used in ref. [7]. One integration is done in analytic form and we are left with a two dimensional integral. We have used the following strategy, which turned out to be the most efficient. Perform a Gaussian integration with $8 \times 8$ points and store the results for the eight $k_{n,\mu}$ in each cube. Redo the same thing with $16 \times 16$ points and compare the results for each $k_{n,\mu}$. If the difference is less than 0.001 we accept the contribution. Otherwise we collect the cubes who’s integrals have not yet converged and redo these with $32 \times 32$ points instead. Compare with the previous values. Usually, at this point only a few integrals have not converged. Those are normally quite tricky, so for these we use a library integration routine.
with interval adaption. The typically time for one topological charge on a $6^4$ lattice is 100 seconds on the CRAY-YMP. The charges are integers up to errors of the order $10^{-4}$. To make the Seiberg charge converge, it is necessary to perform a global Landau gauge fixing before the integration. This is allowed since the charge is gauge invariant. We have checked that Lueschers and Seibergs charge definitions agree in each hypercube up to an integer. Almost a few of hypercubes have a charge outside the interval $]-1/2, 1/2[$, so that often the two definitions agree. Now for the Chern-Simons term we only need to evaluate $k_{n,0}$ for one timeslice. That takes around 5 seconds. As we know the Chern-Simons term is gauge dependent, but for a check we looked at the periodic structure for a $4^3 \times 2$ lattice. We did 100 configurations and we first measured $N_{CS}$ without any gauge fixing. This required a much higher accuracy for our integrals before convergence. In Fig. 1 we have plotted the Chern-Simons density. Notice that there are many configurations which have $N_{CS}$ close to an integer. These configurations can be interpreted as being pure gauge. In Fig. 2 we have plotted the same quantity but with one Landau gauge fixing sweep. There are now more configurations with $N_{CS}$ centered around $0, \pm 1$. Most of the configurations have changed Chern-Simons number by an integer. In Fig. 3 we have performed 5 Landau gauge fixing sweeps and $N_{CS}$ is now centered at 0. Since we have demonstrated that the Chern-Simons term only changes by an integer under gauge transformations, we can safely restrict $N_{CS}$ to the interval $]-1/2, 1/2[$. We now compared the density for the two lattices $6^4$ and $6^3 \times 2$. For these and all the other lattices we used 50 Landau gauge fixing sweeps, this is sufficient for the integrals to converge. Typically we have around 1000 Chern-Simons numbers. See Fig. 4 and Fig. 5. There is a trend in the direction of a flatter distribution at finite temperature. We would like to interprete this as the system likes to tunnel more often. We have also done quite a number of other symmetric lattices $4^4, 6^4$ and $8^4$, as well as asymmetric $8^3 \times 2, 4$. We emphasize that these results are only preliminary and a lot has to be done before we can really say that tunneling is improved at high temperature. We also need to see how the density depends on the spatial volume. Finally we would like to mention that an alternative definition of the Chern-Simons term has been derived by the authors in ref. [8].
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Figure 1. The Chern-Simons density as a function of the Chern-Simons number $N_{CS}$ with zero gauge fixing sweeps. The volume is denoted $V$, and $\beta, \kappa, \lambda$ are the gauge Higgs couplings.

Figure 3. The Chern-Simons density as a function of the Chern-Simons number $N_{CS}$ with one Landau gauge fixing sweep. Same parameters as before.

Figure 3. The Chern-Simons density as a function of the Chern-Simons number $N_{CS}$ with five Landau gauge fixing sweeps. Same parameters as before.

Figure 3. The Chern-Simons density at zero temperature as a function of the restricted Chern-Simons number $N_{CS}$ with fifty Landau gauge fixing sweeps. Same parameters as before.

Figure 5. The Chern-Simons density at finite temperature as a function of the restricted Chern-Simons number $N_{CS}$ with fifty Landau gauge fixing sweeps. Same parameters as before.