Multipliers over Fourier algebras of ultraspherical hypergroups

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Abstract

Let $H$ be an ultraspherical hypergroup associated to a locally compact group $G$ and let $A(H)$ be the Fourier algebra of $H$. For a left Banach $A(H)$-submodule $X$ of $VN(H)$, define $Q_X$ to be the norm closure of the linear span of the set $\{uf : u \in A(H), f \in X\}$ in $B_{A(H)}(A(H), X^*)$. We will show that $B_{A(H)}(A(H), X^*)$ is a dual Banach space with predual $Q_X$, we characterize $Q_X$ in terms of elements in $A(H)$ and $X$. Applications obtained on the multiplier algebra $M(A(H))$ of the Fourier algebra $A(H)$. In particular, we prove that $G$ is amenable if and only if $M(A(H)) = B_\lambda(H)$, where $B_\lambda(H)$ is the reduced Fourier-Stieltjes algebra of $H$. Finally, we investigate some characterizations for an ultraspherical hypergroup to be discrete.

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1 Introduction

Let $G$ be a locally compact group and let $A(G)$ and $B(G)$ be the Fourier and Fourier-Stieltjes algebras of $G$ introduced by Eymard [4]. Let $M(A(G))$ denote the multiplier algebra of $A(G)$. Then we have the following inclusions

$$A(G) \subseteq B(G) \subseteq M(A(G))$$

and $\|v\|_{A(G)} = \|v\|_{B(G)} \geq \|v\|_M$ for all $v \in A(G)$. It is known that if $G$ is amenable, then $B(G) = M(A(G))$ isometrically. Moreover, it is known from Losert [11] that $G$ is amenable, or equivalently $A(G)$ has a bounded approximate identity, whenever the norms $\|\cdot\|_{B(G)}$ and $\|\cdot\|_M$ are equivalent on $A(G)$. As in the group case, the Fourier space $A(H)$ of a locally compact hypergroup $H$, plays an important role in the harmonic analysis. A class of hypergroups, called tensor hypergroups, whose Fourier space forms a Banach algebra under pointwise multiplication first appeared in [2]. Another class, called ultraspherical hypergroups, was studied by Muruganandam [15]. In this work, we study ultraspherical hypergroups through multipliers of $A(H)$, denoted $M(A(H))$.

Let $A$ be a Banach algebra, and let $X$ and $Y$ be two right Banach $A$-modules. Suppose that $B_A(X, Y)$ is the Banach space of bounded right $A$-module maps with the operator norm denoted by $\|\cdot\|_M$. In recent years, people have become interested in studying the properties of $B_A(X, Y)$ for various classes of Banach algebras $A$ and right Banach $A$-modules $X$ and $Y$; see for example [3, 6, 7, 13].

In this paper, for a left Banach $A$-submodule $X$ of $A^*$ we study $B_A(A, X^*)$ as a dual Banach space, paying special attention to the Fourier algebra $A(H)$ of an ultraspherical hypergroup $H$ associated to a locally compact group $G$.

In Section 2, for a left Banach $A$-submodule $X$ of $A^*$, we show that $B_A(A, X^*)$ is a dual Banach space with predual $Q_X$, where $Q_X$ denote the norm closure of the linear span of the set $\{af : a \in A, f \in X\}$ in $B_A(A, X^*)^*$. We will obtain a characterization of $Q_X$.

In Section 3, we apply these results to Fourier algebra $A(H)$ of an ultraspherical hypergroup $H$. For the case of $X = C^*_\lambda(H)$, we show that the predual $Q_{C^*_\lambda(H)}$ of $M(A(H))$, the multiplier algebra of $A(H)$, is equal to the closure of $L^1(H)$ in $M(A(H))$ under the multiplier norm. We also prove that $G$ is amenable if and only if $M(A(H)) = B_\lambda(H)$, where $B_\lambda(H)$ is the reduced Fourier-Stieltjes algebra of $H$. In the case where $A(H)$ is $w^*$-dense in $M(A(H))$, we prove that $G$ is amenable if and only if the norms $\|\cdot\|_{A(H)}$ and $\|\cdot\|_M$ are equivalent on $A(H)$. For the case of $X = C_\delta(H)$, we study the
preidual of $B_{A(H)}(A(H), C_δ(H)^*)$. These results generalize some results of \cite{13} to ultraspherical hypergroups.

In Section 4, we shall define and study $UCB(A(\hat{H}))$, called uniformly continuous functionals on $A(H)$. We will focus in the relationship between $UCB(A(\hat{H}))$ and other subspaces of $VN(H)$. We extend various results of \cite{9} to the context of ultraspherical hypergroups. For example, we prove that $H$ is discrete if and only if $UCB(A(\hat{H})) = C_λ^*(H)$.

2 The dual Banach space $B_A(A, X^*)$

Let $A$ be a Banach algebra, and let $X$ and $Y$ be right and left Banach $A$-modules, respectively. The $A$-module tensor product of $X$ and $Y$ is the quotient space $X \hat{\otimes}_A Y = (X \hat{\otimes} Y)/N$, where

$$N = \langle x \cdot a \otimes y - x \otimes a \cdot y : x \in X, y \in Y, a \in A \rangle,$$

and $\langle \cdot \rangle$ denotes the closed linear span. It was shown in \cite{16} that

$$B_A(A, X^*) \cong N^⊥ \cong (X \hat{\otimes}_A Y)^*.$$

Let $X$ be a left Banach $A$-submodule of $A^*$. In this section we show that $B_A(A, X^*)$ is a dual Banach space and characterize its predual in terms of elements in $A$ and $X$. For every $a \in A$ and $f \in X$, define the bounded linear functional $af$ on $B_A(A, X^*)$ as follows:

$$\langle af, T \rangle = \langle f, T(a) \rangle \quad (T \in B_A(A, X^*)).$$

Moreover, it is easy to see that $\|af\|_M \leq \|a\| \|f\|_X$. Now, we denote the linear span of the set $\{af : a \in A, f \in X\}$ by $AX$ and define $Q_X$ to be the norm closure of $AX$ in $B_A(A, X^*)^*$.

**Theorem 2.1.** Let $A$ be a Banach algebra and let $X$ be a left Banach $A$-submodule of $A^*$. Then $B_A(A, X^*) = (Q_X)^*$.

**Proof.** Let $J : A \hat{\otimes} X \to Q_X$ be defined by $J(\sum_{i=1}^{\infty} a_i \otimes f_i) = \sum_{i=1}^{\infty} a_i f_i$. Then it is clear that $J$ is well defined and $\|J\| \leq 1$. As $B(A, X^*) = (A \hat{\otimes} X)^*$, we have the adjoint operator $J^* : (Q_X)^* \to B(A, X^*)$ with $\|J^*\| \leq 1$. Now, for each $T \in (Q_X)^*$, we show that $J^*(T) \in B_A(A, X^*)$. Let $a, b \in A$ and $f \in X$
Then

\[
\langle J^*(T)(ab), f \rangle = \langle J^*(T), (ab) \otimes f \rangle = \langle T, (ab)f \rangle \\
= \langle T, a(bf) \rangle = \langle T, J(a \otimes (bf)) \rangle \\
= \langle J^*(T), a \otimes (bf) \rangle = \langle J^*(T)(a), bf \rangle \\
= \langle J^*(T)(a) \cdot b, f \rangle.
\]

Therefore, \( J^*(T)(ab) = J^*(T)(a) \cdot b \) for all \( a, b \in \mathcal{A} \). Thus, \( J^*(T) \in B_{\mathcal{A}}(\mathcal{A}, X^*) \).

Let \( T \in B_{\mathcal{A}}(\mathcal{A}, X^*) \). Then the restriction of \( T \) to \( Q_X \) is in \( (Q_X)^* \) and we have

\[
\langle J^*(T), \sum_{i=1}^{\infty} a_i \otimes f_i \rangle = \langle T, \sum_{i=1}^{\infty} a_i f_i \rangle = \sum_{i=1}^{\infty} \langle T(a_i), f_i \rangle = \langle T, \sum_{i=1}^{\infty} a_i \otimes f_i \rangle,
\]

for all \( \sum_{i=1}^{\infty} a_i \otimes f_i \in \mathcal{A} \hat{\otimes} X \). It follows that \( J^*(T) = T \) and \( J^* \) is surjective.

Since \( J(\mathcal{A} \hat{\otimes} X) \) is dense in \( Q_X \), by \([12] \) Theorem 3.1.17 \( J^* \) is injective. Therefore, \( J^* \) is a surjective isometry. \[ \square \]

**Theorem 2.2.** Let \( \mathcal{A} \) be a Banach algebra and let \( X \) be a left Banach \( \mathcal{A} \)-submodule of \( \mathcal{A}^* \). Suppose that \( f \in B_{\mathcal{A}}(\mathcal{A}, X^*) \). Then \( f \in Q_X \) if and only if there are sequences \( (a_i) \subseteq \mathcal{A} \) and \( (f_i) \subseteq X \) with \( \sum_{i=1}^{\infty} \|a_i\| \|f_i\| < \infty \) such that \( f = \sum_{i=1}^{\infty} a_i f_i \) and

\[
\|f\|_M = \inf \left\{ \sum_{i=1}^{\infty} \|a_i\| \|f_i\| : f = \sum_{i=1}^{\infty} a_i f_i, \sum_{i=1}^{\infty} \|a_i\| \|f_i\| < \infty \right\}.
\]

**Proof.** By definition, each element of the form \( \sum_{i=1}^{\infty} a_i f_i \), as in the proof of Theorem 2.1 lies in \( Q_X \).

For the converse, let \( I : \mathcal{A} \hat{\otimes} X \to Q_X \) be defined by

\[
I(\sum_{i=1}^{\infty} a_i \otimes f_i + N) = \sum_{i=1}^{\infty} a_i f_i.
\]

Then it is routine to check that \( I \) is well defined and \( \|I\| \leq 1 \). In fact, if \( \sum_{i=1}^{\infty} a_i \otimes f_i \in N \), then for each \( T \in B_{\mathcal{A}}(\mathcal{A}, X^*) \), we have

\[
\langle \sum_{i=1}^{\infty} a_i f_i, T \rangle = \sum_{i=1}^{\infty} \langle T(a_i), f_i \rangle = \langle \sum_{i=1}^{\infty} a_i \otimes f_i, T \rangle = 0.
\]

Hence, \( I \) is well defined by duality.
We know from Theorem 2.11 that \((A \hat{\otimes}_A X)^* = B_A(A, X^*) = (Q_X)^*\). It follows that \(I^* : (Q_X)^* \to (A \hat{\otimes}_A X)^*\) is bijective. Hence, \(I\) is surjective by \cite{12} Theorem 3.1.22]. This proves first part of the theorem.

For the second part, let \(f \in Q_X\) and \(\epsilon > 0\) be given. Then by first part of theorem, there are sequences \((a_i) \subseteq A\) and \((f_i) \subseteq X\) such that \(f = \sum_{i=1}^{\infty} a_i f_i\) with \(\sum_{i=1}^{\infty} \|a_i\| \|f_i\| < \infty\). Let \(\xi = \sum_{i=1}^{\infty} a_i \otimes f_i + N\). Then \(\langle T, f \rangle = \langle T, \xi \rangle\) for all \(T \in B_A(A, X^*)\), which implies that \(\|f\|_M = \|\xi\|\). Now, as a consequence of the definition of quotient norm, there exist sequences \((b_i) \subseteq A\) and \((h_i) \subseteq X\) such that \(\sum_{i=1}^{\infty} \|b_i\| \|h_i\| < \|f\|_M + \epsilon\) and \(\xi = \sum_{i=1}^{\infty} b_i \otimes h_i + N\). Hence, \(f = \sum_{i=1}^{\infty} b_i h_i\) on \(B_A(A, X^*)\), as required. This completes the proof. \(\square\)

Suppose that \(X\) is a left Banach \(A\)-module. Then \(X^*\) is a right Banach \(A\)-module with the following module action

\[
\langle m \cdot a, f \rangle = \langle m, a \cdot f \rangle \quad (m \in X^*, f \in X, a \in A).
\]

By the above notions it is not hard to see that, if \(X\) is a left Banach \(A\)-submodule of \(A^*\), then the map

\[
i : X^* \to B_A(A, X^*), \quad m \mapsto m_L
\]

is a contractive linear map, where \(m_L\) is given by \(m_L(a) = m \cdot a\) for all \(a \in A\) and \(\|m_L\|_M \leq \|m\|_X\). Thus, we can assume that \(X^* \subseteq B_A(A, X^*)\). Moreover, the adjoint map \(i^* : B_A(A, X^*)^* \to X^{**}\) is simply the restriction map, say \(R\) and for every \(a \in A\), \(f \in X\) and \(m \in X^*\) we have

\[
\langle R(af), m \rangle = \langle af, m_L \rangle = \langle f, m_L(a) \rangle = \langle f, m \cdot a \rangle = \langle a \cdot f, m \rangle,
\]

which implies that \(R(Q_X) \subseteq X\).

**Proposition 2.3.** Let \(A\) be a Banach algebra and let \(X\) be a left Banach \(A\)-submodule of \(A^*\). Then \(R : Q_X \to X\) is surjective if and only if the norms \(\|\cdot\|_X\) and \(\|\cdot\|_M\) are equivalent on \(X^*\).

**Proof.** Let \(R\) be surjective. Then \(R^* : X^* \to (Q_X)^*\) is injective and \(R^*(X^*)\) is closed in \((Q_X)^*\) by \cite{12} Theorem 3.1.22. Since \(\|\cdot\|_M \leq \|\cdot\|_X\) on \(X^*\), the Open Mapping theorem shows that the norms \(\|\cdot\|_M\) and \(\|\cdot\|_X\) are equivalent on \(X^*\).

Conversely, let the norms \(\|\cdot\|_M\) and \(\|\cdot\|_X\) are equivalent on \(X^*\). Then \(R^*\) is injective and \(R^*(X^*)\) is closed in \((Q_X)^*\). It follows from \cite{12} Theorem 3.1.17 and \cite{12} Theorem 3.1.21 that \(R\) is surjective. \(\square\)
For every \( a \in A \) we can regard \( a \) as a functional on \( X \). It follows that the map
\[
\iota : A \to B_A(A, X^*), \quad a \mapsto a_L
\]
is a contractive linear map, where \( a_L \) is given by \( a_L(b) = ab \) for all \( b \in A \) and \( \|a_L\|_M \leq \|a\|_X \leq \|a\|_A \). This implies that \( A \subseteq B_A(A, X^*) \).

Define \( \tilde{Q}_X \) to be the range of the linear map \( \Gamma : \hat{A} \otimes X \to A^* \) defined by \( \Gamma(a \otimes f) = a \cdot f \). Then \( \tilde{Q}_X \) is a Banach space when equipped with the quotient norm from \( \hat{A} \otimes X \). Moreover, \( f \in \tilde{Q}_X \) if and only if there are sequences \( (a_i) \subseteq A \) and \( (f_i) \subseteq X \) with \( \sum_{i=1}^{\infty} \|a_i\| \|f_i\| < \infty \) such that \( f = \sum_{i=1}^{\infty} a_i \cdot f_i \).

**Theorem 2.4.** Let \( A \) be a Banach algebra and let \( X \) be a left Banach \( A \)-submodule of \( A^* \). Then \( A \) is \( w^* \)-dense in \( B_\mathcal{A}(A, X^*) \) if and only if \( \tilde{Q}_X \) is isometrically isomorphic to \( Q_X \).

**Proof.** Let \( A \) be \( w^* \)-dense in \( B_\mathcal{A}(A, X^*) \). Then it follows from \cite[Proposition 2.6.6]{12} that the annihilator \( \langle A \rangle \) of \( A \) in \( B_\mathcal{A}(A, X^*) \) can be identified with \( B_\mathcal{A}(A, X^*) = (Q_X)^* \), where
\[
\langle A \rangle = \{ f \in Q_X : \langle a_L, f \rangle = 0 \text{ for each } a \in A \}.
\]

Hence, \( A \) separates the points of \( Q_X \). Now, define \( \Lambda : Q_X \to \tilde{Q}_X \) by
\[
\Lambda\left( \sum_{i=1}^{\infty} a_i f_i \right) = \sum_{i=1}^{\infty} a_i \cdot f_i.
\]

If \( a \in A \) is arbitrary, then for each sequences \( (a_i) \subseteq A \) and \( (f_i) \subseteq X \) with \( \sum_{i=1}^{\infty} \|a_i\| \|f_i\| < \infty \), we have
\[
\langle a_L, \sum_{i=1}^{\infty} a_i f_i \rangle = \sum_{i=1}^{\infty} \langle a a_i, f_i \rangle = \sum_{i=1}^{\infty} \langle a, a_i \cdot f_i \rangle = \langle a, \sum_{i=1}^{\infty} a_i \cdot f_i \rangle.
\]

From this and the fact that \( A \) separates the points of \( Q_X \), we get that \( \Lambda \) is an isomorphism. Also, by Theorem 2.2 it is an isometry.

Conversely, let \( \tilde{Q}_X \) be isometrically isomorphic to \( Q_X \). Then \( A \) separates the points of \( Q_X \), which implies that \( \langle A \rangle = B_\mathcal{A}(A, X^*) \). Again by \cite[Proposition 2.6.6]{12}, \( A \) is \( w^* \)-dense in \( B_\mathcal{A}(A, X^*) \). \( \square \)

### 3 The multiplier algebra \( M(A(H)) \) and amenability

A bounded linear operator on commutative Banach algebra \( A \) is called a multiplier if it satisfies \( aT(b) = T(ab) \) for all \( a, b \in A \). We denote by
\( \mathcal{M}(\mathcal{A}) \) the space of all multipliers for \( \mathcal{A} \). Clearly \( \mathcal{M}(\mathcal{A}) \) is a Banach algebra as a subalgebra of \( B(\mathcal{A}) \) and \( \mathcal{M}(\mathcal{A}) = B_\mathcal{A}(\mathcal{A}) \). For the general theory of multipliers we refer to Larsen [8]. It is known that for a semisimple commutative Banach algebra \( \mathcal{A} \) every \( T \in \mathcal{M}(\mathcal{A}) \) can be identified uniquely with a bounded continuous function \( \hat{T} \) on \( \Delta(\mathcal{A}) \), the maximal ideal space of \( \mathcal{A} \). Moreover, if we denote by \( \mathcal{M}(\mathcal{A}) \) the normed algebra of all bounded continuous functions \( \varphi \) on \( \Delta(\mathcal{A}) \) such that \( \varphi \hat{A} \subseteq \hat{A} \), then \( \mathcal{M}(\mathcal{A}) = \hat{\mathcal{M}}(\mathcal{A}) \); see [8, Corollary 1.2.1].

Let \( H \) be an ultraspherical hypergroup associated to a locally compact group \( G \) and a spherical projector \( \pi : C_c(G) \to C_c(G) \) which was introduced and studied in [15]. Let \( A(H) \) denote the Fourier algebra corresponding to the hypergroup \( H \). A left Haar measure on \( H \) is given by \( \int_H f(x)dx = \int_G f(p(x))dx, \quad f \in C_c(H) \), where \( p : G \to H \) is the quotient map. The Fourier space \( A(H) \) is an algebra and is isometrically isomorphic to the subalgebra \( A_\pi(G) = \{ u \in A(G) : \pi(u) = u \} \) of \( A(G) \) [15, Theorem 3.10]. Recall that the character space \( \Delta(A(H)) \) of \( A(H) \) can be canonically identified with \( H \). The Fourier algebra \( A(H) \) is semisimple, regular and Tauberian [15, Theorem 3.13]. As in the group case, let \( \lambda \) also denote the left regular representation of \( H \) on \( L^2(H) \) given by

\[
\lambda(\hat{x})(f)(\hat{y}) = f(\hat{x} \ast \hat{y}) \quad (\hat{x}, \hat{y} \in H, f \in L^2(H))
\]

This can be extended to \( L^1(H) \) by \( \lambda(f)(g) = f \ast g \) for all \( f \in L^1(H) \) and \( g \in L^2(H) \). Let \( C^*_\lambda(H) \) denote the completion of \( \lambda(L^1(H)) \) in \( B(L^2(H)) \) which is called the reduced \( C^* \)-algebra of \( H \). The von Neumann algebra generated by \( \{ \lambda(\hat{x}) : \hat{x} \in H \} \) is called the von Neumann algebra of \( H \), and is denoted by \( VN(H) \). Note that \( VN(H) \) is isometrically isomorphic to the dual of \( A(H) \). Moreover, \( A(H) \) can be considered as an ideal of \( B_\lambda(H) \), where \( B_\lambda(H) \) is the dual of \( C^*_\lambda(H) \).

**Remark 3.1.** As \( A(H) \) is an ideal in \( B_\lambda(H) \), there is a canonical \( B_\lambda(H) \)-bimodule structure on \( VN(H) \). In particular, for \( f \in L^1(H) \) and \( \phi \in B_\lambda(H) \), we obtain

\[
\langle \phi \cdot \lambda(f), v \rangle = \langle \lambda(f), \phi v \rangle = \int f(\hat{x})\phi(\hat{x})v(\hat{x})d\hat{x} = \langle \lambda(\phi f), v \rangle
\]

for all \( v \in A(H) \). This shows that \( \phi \cdot \lambda(f) = \lambda(\phi f) \in \lambda(L^1(H)) \). Since \( \lambda(L^1(H)) \) is norm dense in \( C^*_\lambda(H) \), we conclude that \( C^*_\lambda(H) \) is a \( B_\lambda(H) \)-bimodule.

**Theorem 3.2.** Let \( H \) be an ultraspherical hypergroup. Then

\[
M(A(H)) = B_{A(H)}(A(H), C^*_\lambda(H))^*.
\]
Proof. Since $A(H)$ is commutative and semisimple, it suffices to show that $\mathcal{M}(A(H)) = B_{A(H)}(A(H), B_\lambda(H))$. To prove this, first note that $\mathcal{M}(A(H)) \subseteq B_{A(H)}(A(H), B_\lambda(H))$. Conversely, assume that $u \in A(H)$ has compact support. By regularity of $A(H)$, there exists $v \in A(H)$ such that $v(x) = 1$ for $x \in \text{supp}(u)$. Thus, for each $T \in B_{A(H)}(A(H), B_\lambda(H))$, we have

$$T(u) = T(vu) = vT(u).$$

Since $A(H)$ is an ideal in $B_\lambda(H)$, we conclude that $T(u) \in A(H)$. Moreover, since the set of compactly supported elements in $A(H)$ is dense in $A(H)$, a simple approximation argument shows that $T(u) \in A(H)$ for all $u \in A(H)$. Therefore, $T \in \mathcal{M}(A(H))$ as required.

Let $H$ ba an ultraspherical hypergroup and let $f \in L^1(H)$. Define a linear functional on $M(A(H))$ by

$$\langle f, \phi \rangle = \int f(\dot{x})\phi(\dot{x})d\dot{x} \quad (\phi \in M(A(H))).$$

Moreover, $|\langle f, \phi \rangle| \leq \|f\|_1\|\phi\|_\infty \leq \|f\|_1\|\phi\|_M$ for all $\phi \in M(A(H))$. Therefore, $f$ is in $M(A(H))^*$ and

$$\|f\|_M = \sup \{ |\langle f, \phi \rangle| : \phi \in M(A(H)), \|\phi\|_M \leq 1 \} \leq \|f\|_1.$$

Put

$$Q(H) := \overline{L^1(H)^||\cdot||_M} \subseteq M(A(H))^*.$$

Next we prove that $M(A(H))$ is a dual Banach space for any ultraspherical hypergroup $H$.

**Theorem 3.3.** Let $H$ ba an ultraspherical hypergroup. Then $Q_{C_\lambda(H)} = Q(H)$ and so $M(A(H)) = Q(H)^*$.

*Proof.* Suppose that $f \in C_c(H)$. Using the regularity of $A(H)$, there exists $u \in A(H)$ such that $u|_{\text{supp}(f)} \equiv 1$. Thus, $f = uf$ is in $Q_{C_\lambda(H)}$ and $\langle uf, \phi \rangle = \langle f, \phi \rangle = \int_H f(\dot{x})\phi(\dot{x})d\dot{x}$ for all $\phi \in M(A(H))$. Therefore, there is an isometry between the dense subspace of $Q_{C_\lambda(H)}$ and a dense subspace of $(L^1(H), \|\cdot\|_M)$. This shows that $Q_{C_\lambda(H)}$ is the completion of $L^1(H)$ with respect to the norm $\|\cdot\|_M$. \qed

**Theorem 3.4.** Let $H$ be an ultraspherical hypergroup on locally compact group $G$. Then $G$ is amenable if and only if $B_\lambda(H) = M(A(H))$. 
Proof. Suppose that \( G \) is amenable. Then \( B_\lambda(H) = M(A(H)) \) by [15, Theorem 4.2]. Conversely, assume that \( B_\lambda(H) = M(A(H)) \). Then the constant function 1 belongs to \( B_\lambda(H) \). Since \( A(H) \) is dense in \( B_\lambda(H) \) with respect to the \( \sigma(B_\lambda(H), C_\lambda(H)) \)-topology, there exists a net \( (u_\alpha) \) in \( A(H) \) such that \( u_\alpha \to 1 \) in the \( \sigma(B_\lambda(H), C_\lambda(H)) \)-topology and \( c = \sup_\alpha \|u_\alpha\|_{A(H)} < \infty \). Choose \( f \) in \( C_c(H) \) with \( f \geq 0 \) and \( \|f\|_1 = 1 \). For each \( \alpha \), define \( u'_\alpha = f * u_\alpha \).

Notice first that \( (u'_\alpha) \subseteq A(H) \) and

\[
\|u'_\alpha\|_{A(H)} \leq \|f\|_1 \|u_\alpha\|_{A(H)} \leq c
\]

for all \( \alpha \). In fact, for each \( g \in L^1(H) \) with \( \|\lambda(g)\|_{C_\lambda(H)} \leq 1 \), we have

\[
|\langle f * u_\alpha, \lambda(g) \rangle| = \left| \int_H \int_H f(y)u_\alpha(\tilde{y} \ast \tilde{x})g(\tilde{x})dyd\tilde{x} \right|
\]

\[
= \left| \int_H f(y)\langle \tilde{g}u_\alpha, \tilde{g} \rangle dy \right|
\]

\[
\leq \int_H |f(y)||\tilde{g}u_\alpha|_{A(H)}dy
\]

\[
\leq \|f\|_1 \|u_\alpha\|_{A(H)} \leq c.
\]

Let \( K \subseteq H \) be compact. Then the set \( \{\lambda(\tilde{z}f) : \tilde{z} \in K\} \) form a compact subset of \( C_\lambda(H) \), where the function \( \tilde{z}f \) on \( H \) is defined by \( \tilde{z}f(\tilde{y}) = f(\tilde{x} \ast \tilde{y}) \) for all \( \tilde{y} \in H \). Since \( u_\alpha \to 1 \) in the \( \sigma(B_\lambda(H), C_\lambda(H)) \)-topology and the net \( (u_\alpha) \) is bounded in \( B_\lambda(H) \), the convergence is uniform on compact subsets of \( C_\lambda(H) \). Hence,

\[
u'_\alpha(\tilde{x}) = \langle u_\alpha, \lambda(\tilde{z}f) \rangle \to \langle 1, \lambda(\tilde{z}f) \rangle = \int_H \tilde{z}f(\tilde{y})d\tilde{y} = 1
\]

uniformly on \( K \), where \( \tilde{u}_\alpha(\tilde{x}) = u_\alpha(\tilde{x}) \) for all \( \tilde{x} \in H \), and noticing that \( \tilde{u}_\alpha \in B_\lambda(H) \) by [14, Remark 2.9]. Again choose \( f \) in \( C_c(H) \) with \( f \geq 0 \) and \( \|f\|_1 = 1 \) and put \( w_\alpha = f * u'_\alpha \) for all \( \alpha \). Then \( \|w_\alpha\|_{A(H)} \leq c \). Assume that \( u \in A(H) \cap C_c(H) \). Next, we show that \( \|w_\alpha u - u\|_{A(H)} \to 0 \). In fact, if we put \( K = \text{supp}(f) \ast \text{supp}(u) \), then for each \( \tilde{x} \in \text{supp}(u) \) we have

\[
w_\alpha(\tilde{x}) = \int_H f(\tilde{y})u'_\alpha(\tilde{y} \ast \tilde{x})d\tilde{y}
\]

\[
= \int_H f(\tilde{y})(1_Ku'_\alpha)(\tilde{y} \ast \tilde{x})d\tilde{y}
\]

\[
= (f * (1_Ku'_\alpha))(\tilde{x}).
\]

Hence, \( uw_\alpha = u(f * (1_Ku'_\alpha)) \), where \( 1_K \) denote the characteristic function of \( K \). Similarly, \( u = u(f * 1_K) \). Since \( \|1_Ku'_\alpha - 1_K\|_2 \to 0 \), it follows that
\[ \|uw_\alpha - u\|_{A(H)} \to 0. \] Finally, since the net \((w_\alpha)\) is bounded and \(A(H) \cap C_0(H)\) is dense in \(A(H)\), a straightforward approximation argument shows that \(\|uw_\alpha - u\|_{A(H)} \to 0\) for all \(u\) in \(A(H)\). Thus, \(G\) is amenable by \([1, \text{Theorem 4.4}]\).

**Corollary 3.5.** Let \(H\) be an ultraspherical hypergroup on locally compact group \(G\). Then the following hold.

(i) Let \(f \in M(A(H))^*\). Then \(f \in Q(H)\) if and only if there exist sequences \((u_i) \subseteq A(H)\) and \((f_i) \subseteq C^*_\lambda(H)\) with \(\sum_{i=1}^{\infty} \|u_i\|_{A(H)} \|f_i\|_{C^*_\lambda(H)} < \infty\) such that \(f = \sum_{i=1}^{\infty} u_if_i\) and

\[
\|f\|_M = \inf \left\{ \sum_{i=1}^{\infty} \|u_i\|_{A(H)} \|f_i\|_{C^*_\lambda(H)} : f = \sum_{i=1}^{\infty} u_if_i, \sum_{i=1}^{\infty} \|u_i\|_{A(H)} \|f_i\|_{C^*_\lambda(H)} < \infty \right\}.
\]

(ii) \(G\) is amenable if and only if for any \(f \in C^*_\lambda(H)\) and \(\epsilon > 0\) there exist sequences \((u_i) \subseteq A(H)\) and \((f_i) \subseteq C^*_\lambda(H)\) such that \(f = \sum_{i=1}^{\infty} u_if_i\) on \(B_\lambda(H)\) with

\[
\sum_{i=1}^{\infty} \|u_i\|_{A(H)} \|f_i\|_{C^*_\lambda(H)} < \|f\|_{C^*_\lambda(H)} + \epsilon.
\]

**Proof.**

(i). It is an immediate consequence of Theorem \(2.2\).

(ii). It follows from (i) that the condition of (ii) is equivalent to \(C^*_\lambda(H) = Q(H)\) (equivalently, \(B_\lambda(H) = M(A(H))\)). However this is equivalent to \(G\) being amenable by Lemma \(3.1\).

**Proposition 3.6.** Let \(H\) be an ultraspherical hypergroup and let \(X\) be a Banach \(A(H)\)-submodule of \(VN(H)\) with \(C^*_\lambda(H) \subseteq X\). Then \(B_\lambda(H)\) is a subalgebra of \(B_{A(H)}(A(H), X^*)\) such that \(\|\phi\|_M \leq \|\phi\|_{B_\lambda(H)}\) for all \(\phi \in B_\lambda(H)\).

**Proof.** Let \(u \in A(H)\) and \(\phi \in B_\lambda(H)\). Then \(\phi u \in A(H) \subseteq VN(H)^*\). Thus \(\phi u \in X^*\). From this and the fact that \(C^*_\lambda(H) \subseteq X\), we get that

\[
\|\phi u\|_{A(H)} = \|\phi u\|_{C^*_\lambda(H)} \leq \|\phi u\|_X \leq \|\phi\|_{C^*_\lambda(H)} \|u\|_{A(H)}.
\]

Consequently, \(\|\phi\|_M \leq \|\phi\|_{B_\lambda(H)}\).

Let \(H\) be an ultraspherical hypergroup. We say that \(H\) has the approximation property if there is a net \((u_\alpha) \subseteq A(H)\) such that \(u_\alpha \overset{w^*}{\rightharpoonup} 1\) in \(M(A(H))\), i.e. in \(\sigma(M(A(H)), Q(H))\)-topology.

For an ultraspherical hypergroup \(H\), we put
\[ \overline{A(H)}^{w^*} := \text{the } w^*-\text{closure of } A(H) \text{ in } M(A(H)). \]

**Proposition 3.7.** Let \( H \) be an ultraspherical hypergroup. Then \( A(H) \) is \( w^* \)-dense in \( M(A(H)) \) if and only if \( H \) has the approximation property.

**Proof.** We know that \( 1 \in M(A(H)) \). Therefore, if \( A(H) \) is \( w^* \)-dense in \( M(A(H)) \), then \( 1 \in \overline{A(H)}^{w^*} \). Hence \( H \) has the approximation property. For the converse, assume that \( H \) has the approximation property. Since \( L^1(H) \) is dense in \( Q(H) \), a simple approximation argument shows that \( \phi f \in Q(H) \) for all \( \phi \in M(A(H)) \) and \( f \in Q(H) \). Now, if there exists a net \( (u_\alpha) \subseteq A(H) \) such that \( u_\alpha \wto 1 \) in \( M(A(H)) \), then for each \( \phi \in M(A(H)) \), we have

\[ \langle u_\alpha \phi, f \rangle = \langle u_\alpha, \phi f \rangle \to \langle 1, \phi f \rangle = \langle \phi f \rangle \quad (f \in Q(H)). \]

Consequently, \( \phi \) is \( w^* \)-limit of the net \( (u_\alpha) \subseteq A(H) \). Hence, \( \overline{A(H)}^{w^*} = M(A(H)) \), as required.

In what follows, for an ultraspherical hypergroup \( H \), we put

\[ Q^L(H) := \text{the Banach space of the restriction of elements in } Q(H) \text{ to } \overline{A(H)}^{w^*}. \]

**Proposition 3.8.** Let \( H \) be an ultraspherical hypergroup. Then the following hold.

(i) \( \overline{A(H)}^{w^*} \) is an ideal of \( M(A(H)) \).

(ii) \( \overline{A(H)}^{w^*} = Q^L(H)^\ast \).

Moreover, \( Q^L(H) \) is isometrically isomorphic to the completion of \( L^1(G) \) with respect to the norm

\[ \|f\|_L = \sup \left\{ \left| \int_H f(\hat{x})\phi(\hat{x})d\hat{x} \right| : \phi \in \overline{A(H)}^{w^*}, \|\phi\|_M \leq 1 \right\}. \]

**Proof.** (i). If \( \phi \in M(A(H)) \) and \( \psi \in \overline{A(H)}^{w^*} \), then there exists a net \( (u_\alpha) \subseteq A(H) \) such that \( u_\alpha \wto \psi \). By the same argument as used in the proof of Proposition 3.7 and using the fact that \( (u_\alpha \phi) \subseteq A(H) \), it is straightforward to conclude that \( \phi \psi \in \overline{A(H)}^{w^*} \). Hence, \( \overline{A(H)}^{w^*} \) is an ideal of \( M(A(H)) \).

(ii). As an immediate consequence of the Hahn-Banach theorem the identity map \( I : \overline{A(H)}^{w^*} \to Q^L(H)^\ast \) is an isometry. Let \( \psi \in Q^L(H)^\ast \) with \( \|\psi\| = 1 \). Since \( Q^L(H) \) is a subspace of \( \overline{(A(H))^{w^*}}^\ast \), we extend \( \psi \) to a linear functional \( \phi \) on \( \overline{(A(H))^{w^*}}^\ast \) with \( \|\phi\| = 1 \). By the Goldstine’s theorem, there is a net \( (u_\alpha) \) in unit ball of \( \overline{A(H)}^{w^*} \) such that \( u_\alpha \to \phi \) in the \( \sigma((A(H))^{w^*})^{**}, (A(H)^{w^*})^\ast \)-topology. In particular, \( \langle u_\alpha, f \rangle \to \langle \phi, f \rangle = \langle \psi, f \rangle \).
for all \( f \in Q(H) \). Consequently, \( \psi \in \overline{A(H)}^{w^*} \) and so \( I \) is onto. Hence, \( \overline{A(H)}^{w^*} \) is isometrically isomorphic to \( Q^2(H)^* \). Repeating the arguments in the proof of Theorem \ref{thm:3.3} it is straightforward to prove the last statement. \( \square \)

**Proposition 3.9.** Let \( H \) be an ultraspherical hypergroup on locally compact group \( G \). Then the following hold.

(i) \( A(H) \) is \( w^* \)-dense in \( M(A(H)) \) if and only if the restriction map \( R : Q(H) \to C^*_\lambda(H) \) is injective.

(ii) The norms \( \| \cdot \|_{A(H)} \) and \( \| \cdot \|_M \) are equivalent on \( A(H) \) if and only if the restriction map \( R : Q(H) \to C^*_\lambda(H) \) is surjective.

(iii) If \( A(H) \) is \( w^* \)-dense in \( M(A(H)) \), then \( G \) is amenable if and only if the norms \( \| \cdot \|_{A(H)} \) and \( \| \cdot \|_M \) are equivalent on \( A(H) \).

**Proof.** (i). Let \( A(H) \) be \( w^* \)-dense in \( M(A(H)) \). If \( f \in Q(H) \) with \( R(f) = 0 \), then we have \( \langle f, u \rangle = \langle R(f), u \rangle = 0 \) for all \( u \in A(H) \). Hence, a simple approximation argument gives that \( \langle R(f), \phi \rangle = 0 \) for all \( \phi \in M(A(H)) \). Therefore, \( R \) is injective. Conversely, if \( R \) is injective, then \( B_\lambda(H) \) is \( w^* \)-dense in \( M(A(H)) \) by \cite[Theorem 3.1.17]{12}. By an argument used in the proof of Proposition \ref{prop:3.8} we conclude that the identity map \( I : \overline{A(H)}^{w^*} \to C^*_\lambda(H)^* \) is surjective. It follows that \( \overline{A(H)}^{w^*} = B_\lambda(H)^{w^*} \). Hence, \( A(H) \) is \( w^* \)-dense in \( M(A(H)) \).

(ii). Let \( \| \cdot \|_{A(H)} \) and \( \| \cdot \|_M \) be equivalent on \( A(H) \). We first show that the norm on \( B_\lambda(H) \) is equivalent to the multiplier norm. Let \( i : A(H) \to M(A(H)) \) be the inclusion map. Then \( i \) is bounded and has \( \| \cdot \|_M \)-closed range. It follows from \cite[Theorem 3.1.21]{12} that \( i^*(M(A(H))^*) \) is \( w^* \)-closed in \( A(H)^* \). Again, by \cite[Theorem 3.1.21]{12}, \( i^{**}(A(H))^{**} \) is norm-closed in \( M(A(H))^{**} \). From this and the fact that \( B_\lambda(H) \) is norm-closed in \( A(H)^{**} \), we conclude that the \( \| \cdot \|_{B_\lambda(H)} \)-norm and the multiplier norm are equivalent on \( B_\lambda(H) \). Therefore, \( R \) is surjective by Proposition \ref{prop:2.3}.

Conversely, suppose that \( R \) is surjective. Then it follows from Proposition \ref{prop:2.3} that the norms \( \| \cdot \|_{B_\lambda(H)} \) and \( \| \cdot \|_M \) are equivalent on \( B_\lambda(H) \) and hence on \( A(H) \).

(iii). Suppose first that \( G \) is amenable. Then \( A(H) \) has a bounded approximate identity by \cite[Theorem 4.4]{13}. It follows easily that the norms \( \| \cdot \|_{A(H)} \) and \( \| \cdot \|_M \) are equivalent on \( A(H) \). Conversely, assume that the norms \( \| \cdot \|_{A(H)} \) and \( \| \cdot \|_M \) are equivalent on \( A(H) \). If \( A(H) \) is \( w^* \)-dense in \( M(A(H)) \), then by (i) and (ii) the restriction map is bijective. It follows that \( Q(H) \) is isometrically isomorphic to \( C^*_\lambda(H) \), which implies that \( 1 \in M(A(H)) = B_\lambda(H) \). Therefore, \( G \) is amenable by Theorem \ref{thm:3.4} \( \square \)
Remark 3.10. Identifying $\ell_1(H)$ with the subspace $\lambda(\ell_1(H))$ of $V_N(H)$, we denote the norm closure of $\ell_1(H)$ in $V_N(H)$ by $C_\delta(H)$. Let $f = \sum \alpha_i \lambda(x_i) \in \ell_1(H)$ and $u \in A(H)$. Then

$$u \cdot f = \sum \alpha_i u(x_i) \lambda(x_i) \in C_\delta(H),$$

and $\|u \cdot f\|_{C_\delta(H)} \leq \|u\|_\infty \|f\|_{C_\delta(H)} \leq \|u\|_{A(H)} \|f\|_{C_\delta(H)}$. Hence, $C_\delta(H)$ is a Banach $A(H)$-submodule of $V_N(H)$. Also, note that $C_\delta(H)^* \subseteq \ell_\infty(H)$.

**Proposition 3.11.** Let $H$ be an ultraspherical hypergroup on locally compact group $G$. Then the following hold.

(i) $B_{A(H)}(A(H), C_\delta(H)^*)$ consists of functions $\phi \in \ell_\infty(H)$ such that the pointwise multiplication map $T_\phi : A(H) \to C_\delta(H)^*, u \mapsto \phi u$ is a bounded operator.

(ii) $Q_{C_\delta(H)}$ is equal to the completion of $\ell_1(H)$ with respect to the norm

$$\|f\|_M = \sup \left\{ \left| \sum f(\hat{x}) \phi(\hat{x}) \right| : \phi \in B_{A(H)}(A(H), C_\delta(H)^*), \|\phi\| \leq 1 \right\}.$$ 

Furthermore, $M(A(H)) \subseteq B_{A(H)}(A(H), C_\delta(H)^*)$, and the corresponding inclusion map is contractive.

**Proof.** (i). Let $\phi \in \ell_\infty(H)$ be such that $T_\phi : A(H) \to C_\delta(H)^*$ is a bounded linear operator. Then since

$$T_\phi(uv) = \phi uv = u T_\phi(v) \quad (u, v \in A(H)),$$

it follows that $T_\phi \in B_{A(H)}(A(H), C_\delta(H)^*)$. For the reverse inclusion, let $\phi \in B_{A(H)}(A(H), C_\delta(H)^*)$. Define $\tilde{\phi} : H \to \mathbb{C}$ by $\tilde{\phi}(\hat{x}) = \langle \phi(u), \lambda(\hat{x}) \rangle$, where $u$ denotes a function in $A(H) \cap C_c(H)$ with $u(\hat{x}) = 1$. Then it is well defined. In fact, if $v$ is another function in $A(H) \cap C_c(H)$ such that $v(\hat{x}) = 1$, then we put $K = \text{supp}(u) \cup \text{supp}(v)$ and choose $w \in A(H) \cap C_c(H)$ such that $w|_K \equiv 1$. Then

$$\langle \phi(u), \lambda(\hat{x}) \rangle = \langle \phi(uw), \lambda(\hat{x}) \rangle = u(\hat{x}) \langle \phi(w), \lambda(\hat{x}) \rangle$$

$$= v(\hat{x}) \langle \phi(w), \lambda(\hat{x}) \rangle = \langle \phi(vw), \lambda(\hat{x}) \rangle$$

$$= \langle \phi(v), \lambda(\hat{x}) \rangle.$$ 

Observe next that if $u \in A(H)$, $\hat{x} \in H$ and $v \in A(H) \cap C_c(H)$ with $v(\hat{x}) = 1$, then

$$\langle \phi(u), \lambda(\hat{x}) \rangle = v(\hat{x}) \langle \phi(u), \lambda(\hat{x}) \rangle = \langle \phi(uv), \lambda(\hat{x}) \rangle$$

$$= u(\hat{x}) \langle \phi(v), \lambda(\hat{x}) \rangle = u(\hat{x}) \tilde{\phi}(\hat{x}).$$
This shows that \( \phi = T_\delta \).

(ii) Since \( C_\delta(H) \) is a Banach \( A(H) \)-submodule of \( VN(H) \), it follows from Theorem 2.1 that

\[
B_{A(H)}(A(H), C_\delta(H)^*) = Q_{C_\delta(H)}. \]

Let \( f \in \ell^1(H) \) be with finite support. Then \( f = uf \in Q_{C_\delta(H)} \), where \( u \in A(H) \) with \( u|_{\text{supp}(f)} \equiv 1 \). Consequently,

\[
\langle \phi, f \rangle = \langle \phi, uf \rangle = \langle \phi(u), f \rangle = \sum \phi(x)f(x),
\]

for all \( \phi \in B_{A(H)}(A(H), C_\delta(H)^*) \). Hence, there is an isometry between the dense subspace of \( \ell^1(H) \|M \) and a dense subspace of \( Q_{C_\delta(H)} \). Therefore, \( Q_{C_\delta(H)} = \ell^1(H) \|M \).

Since \( A(H) \subseteq C_\delta(H)^* \) and \( A(H) \) is an ideal in \( M(A(H)) \), it follows that \( \phi u \in C_\delta(H)^* \) for all \( \phi \in M(A(H)) \) and \( u \in A(H) \). This implies that \( M(A(H)) \subseteq B_{A(H)}(A(H), C_\delta(H)^*) \). Furthermore,

\[
\| \phi u \|_{C_\delta(H)} \leq \| \phi u \|_{A(H)} \leq \| \phi \|_M \| u \|_{A(H)}. \]

Hence, the inclusion map is contractive. \( \square \)

4 Introverted subspaces of \( VN(H) \) and discreteness

Let \( H \) be an ultraspherical hypergroup associated to a locally compact group \( G \). The Arens product on \( VN(H)^* \) is defined as following three steps. For \( u, v \in A(H), T \in VN(H) \) and \( m, n \in VN(H)^* \), we define \( u \cdot T, m \cdot T \in VN(H) \) and \( m \odot n \in VN(H)^* \) as follows:

\[
\langle u \cdot T, v \rangle = \langle T, uv \rangle, \quad \langle m \cdot T, u \rangle = \langle m, u \cdot T \rangle, \quad \langle m \odot n, T \rangle = \langle m, n \cdot T \rangle.
\]

A linear subspace \( X \) of \( VN(H) \) is called topologically invariant if \( u \cdot X \subseteq X \) for all \( u \in A(H) \). The topologically invariant subspace \( X \) of \( VN(H) \) is called topologically introverted if \( m \cdot T \in X \) for all \( m \in X^* \) and \( T \in X \). In this case, \( X^* \) is a Banach algebra with the multiplication induced by the Arens product \( \odot \) inherited from \( VN(H)^* \). Let \( W(\hat{H}) \) be the set of all \( T \) in \( VN(H) \) such that the map \( u \mapsto u \cdot T \) of \( A(H) \) into \( VN(H) \) is weakly compact. Let \( UCB(\hat{H}) \) denote the closed linear span of

\[
\{ u \cdot T : u \in A(H), T \in VN(H) \}.
\]
The elements in \( UCB(\hat{H}) \) are called uniformly continuous functionals on \( A(H) \). We also recall that, subspaces \( W(\hat{H}) \) and \( UCB(\hat{H}) \) of \( VN(H) \) are both topologically introverted.

**Proposition 4.1.** Let \( H \) be an ultraspherical hypergroup. Then \( C^*_\lambda(H) \subseteq W(\hat{H}) \).

*Proof.* It suffices to prove that if \( f \in L^1(H) \), then \( \lambda(f) \in W(\hat{H}) \). Let \( f \in L^1(H) \) be fixed. Then by Remark 3.1, for each \( \phi \in B_\lambda(H) \), we have \( \phi \cdot \lambda(f) = \lambda(\phi f) \). Consider the map \( \phi \mapsto \lambda(\phi f) \) from \( B_\lambda(H) \) into \( VN(H) \). This map is continuous when \( B_\lambda(H) \) has the \( \sigma(B_\lambda(H), C^*_\lambda(H)) \)-topology and \( VN(H) \) has the weak topology. Indeed, let \( \Psi \in VN(H)^* \) and \( (\phi_\alpha) \subseteq B_\lambda(H) \) be a net such that \( \langle \phi_\alpha, T \rangle \to \langle \phi, T \rangle \) for all \( T \in C^*_\lambda(H) \). Then the restriction of \( \Psi \) to \( C^*_\lambda(H) \) is in \( C^*_\lambda(H)^* = B_\lambda(H) \). Thus, there exists \( \psi \in B_\lambda(H) \) such that
\[
\langle \Psi, \lambda(h) \rangle = \langle \psi, \lambda(h) \rangle = \int h(\hat{x}) \psi(\hat{x}) d\hat{x} \quad (h \in L^1(H)).
\]

Hence,
\[
\langle \Psi, \lambda(\phi_\alpha f) \rangle = \langle \psi, \lambda(\phi_\alpha f) \rangle = \int \phi_\alpha(\hat{x}) f(\hat{x}) \psi(\hat{x}) d\hat{x}
= \langle \phi_\alpha, \lambda(\psi f) \rangle \to \langle \phi, \lambda(\psi f) \rangle
= \langle \psi, \lambda(\phi f) \rangle = \langle \Psi, \lambda(\phi f) \rangle.
\]

It follows that the set \( \{ \phi \cdot \lambda(f) : \phi \in B_\lambda(H), \|\phi\| \leq 1 \} \) is relatively compact in the weak topology of \( VN(H) \). The rest of the proof follows from the fact that \( A(H) \subseteq B_\lambda(H) \). \( \square \)

**Proposition 4.2.** Let \( H \) be an ultraspherical hypergroup. Then \( C^*_\lambda(H) \subseteq UCB(\hat{H}) \).

*Proof.* Let \( f \in C_\infty_c(H) \). By regularity of \( A(H) \), there exists \( u \in A(H) \) such that \( u|_{\text{supp}(f)} \equiv 1 \). Therefore,
\[
\langle u \cdot \lambda(f), v \rangle = \langle \lambda(f), uv \rangle = \int f(\hat{x}) u(\hat{x}) v(\hat{x}) d\hat{x}
= \int f(\hat{x}) v(\hat{x}) dt
= \langle \lambda(f), v \rangle
\]

for all \( v \in A(H) \). This implies that \( u \cdot \lambda(f) = \lambda(f) \). Hence, \( \lambda(f) \in UCB(\hat{H}) \). Consequently, \( C^*_\lambda(H) \subseteq UCB(\hat{H}) \) by the density of \( C_\infty_c(H) \) in \( C^*_\lambda(H) \). \( \square \)
Let $X$ be a closed topologically invariant subspace of $VN(H)$ containing $\lambda(\hat{e})$. Then $m \in X^*$ is called a topologically invariant mean on $X$ if:

(i) $\|m\| = \langle m, \lambda(\hat{e}) \rangle = 1$;

(ii) $\langle m, u \cdot T \rangle = u(\hat{e}) \langle m, T \rangle$ for all $T \in X$ and $u \in A(H)$.

We denote by $TIM(X)$ the set of all topologically invariant means on $X$. We also recall from Remark 3.1 that the space $C^*_\lambda(H)$ is an $A(H)$-submodule of $VN(H)$. The following proposition is a consequence of [3, Proposition 5.7] and [10, Proposition 6.3] and the fact that $A(H)$ is a commutative $F$-algebra.

**Proposition 4.3.** Let $H$ be an ultraspherical hypergroup. Then the following hold.

(i) The space $C^*_\lambda(H)$ is a topologically introverted subspace of $VN(H)$.

(ii) $W(\hat{H})$ admits a unique topologically invariant mean.

**Corollary 4.4.** Let $H$ be an ultraspherical hypergroup. Then $H$ is discrete if and only if $\lambda(\hat{e}) \in C^*_\lambda(H)$.

**Proof.** If $H$ is discrete, then $\ell^1(H) = L^1(H)$. Therefore, $\lambda(\hat{e}) \in C^*_\lambda(H)$. Conversely, assume that $\lambda(\hat{e}) \in C^*_\lambda(H)$, and $m$ denote the unique topologically invariant mean on $W(H)$. Then $\langle m, \lambda(\hat{e}) \rangle = 1$. It follows that $H$ must be discrete by [17, Theorem 4.4(iv)].

**Lemma 4.5.** Let $H$ be an ultraspherical hypergroup and let $R : VN(H)^* \to UCB(\hat{H})^*$ be the restriction map. Then $R : TIM(VN(H)) \to TIM(UCB(\hat{H}))$ is a bijection.

**Proof.** If $m_1, m_2 \in TIM(VN(H))$ with $m_1 \neq m_2$, then there exists $T \in VN(H)$ such that $\langle m_1, T \rangle \neq \langle m_2, T \rangle$. Given $u \in A(H)$ with $u(\hat{e}) = 1$, we have

$$\langle m_1, u \cdot T \rangle = \langle m_1, T \rangle \neq \langle m_2, T \rangle = \langle m_2, u \cdot T \rangle.$$ 

This implies that $R(m_1) \neq R(m_2)$, and hence $R$ is injective.

Suppose that $\tilde{m} \in TIM(UCB(\hat{H}))$. Choose $u \in A(H)$ with $\|u\|_{A(H)} = u(\hat{e}) = 1$; see [17, Proposition 3.4]. Define $m$ on $VN(H)^*$ by

$$\langle m, T \rangle = \langle \tilde{m}, u \cdot T \rangle \quad (T \in VN(H)).$$

Since $\|u\|_{A(H)} = 1$, it follows that $\|m\| \leq 1$. Moreover,

$$\langle m, \lambda(\hat{e}) \rangle = \langle \tilde{m}, u \cdot \lambda(\hat{e}) \rangle = u(\hat{e}) \langle \tilde{m}, \lambda(\hat{e}) \rangle = \langle \tilde{m}, \lambda(\hat{e}) \rangle = 1.$$
Therefore, \(|m| = 1\). Furthermore, for each \(v \in A(H)\) and \(T \in VN(H)\), we have

\[
\langle m, v \cdot T \rangle = \langle \tilde{m}, u \cdot (v \cdot T) \rangle = \langle \tilde{m}, v \cdot (u \cdot T) \rangle = v(\hat{e})\langle \tilde{m}, u \cdot T \rangle = v(\hat{e})\langle m, T \rangle.
\]

Consequently, \(m \in TIM(VN(H))\). Finally, if \(T \in UCB(\hat{H})\), then

\[
\langle R(m), T \rangle = \langle m, T \rangle = \langle \tilde{m}, u \cdot T \rangle = \langle \tilde{m}, T \rangle.
\]

Hence, \(R\) is surjective. 

**Proposition 4.6.** Let \(H\) be an ultraspherical hypergroup. Then the following are equivalent.

(i) \(H\) is discrete.

(ii) \(UCB(\hat{H}) = C^*_\lambda(H)\).

(iii) There is a unique topologically invariant mean on \(UCB(\hat{H})\).

**Proof.**

(i) \(\Rightarrow\) (ii). Assume that \(H\) is discrete. Then for each \(\hat{x} \in H\), the characteristic function \(\mathbf{1}_{\hat{x}} \in A(H)\); see [14, Proposition 2.22]. Let \(T \in VN(H)\) be fixed. Then for each \(v \in A(H)\), we get

\[
\langle \mathbf{1}_{\hat{x}} \cdot T, v \rangle = \langle T, \mathbf{1}_{\hat{x}} \rangle = \langle T, v(\hat{x}) \mathbf{1}_{\hat{x}} \rangle = v(\hat{x})\langle T, \mathbf{1}_{\hat{x}} \rangle.
\]

Hence, \(\mathbf{1}_{\hat{x}} \cdot T = \langle T, \mathbf{1}_{\hat{x}} \rangle \lambda(\hat{x}) \in C^*_\lambda(H)\). Let \(u \in A(H)\). Since \(A(H) \cap C_\epsilon(H)\) is dense in \(A(H)\), we can suppose that \(u\) has compact and hence finite support. Thus, \(u\) is a finite linear combination of characteristic functions on one point sets. Therefore, \(u \cdot T \in C^*_\lambda(H)\). It follows from Proposition 4.2 that \(UCB(\hat{H}) = C^*_\lambda(H)\).

(ii) \(\Rightarrow\) (iii). If \(UCB(\hat{H}) = C^*_\lambda(H)\), then \(UCB(\hat{H}) \subseteq W(\hat{H})\) by Proposition 4.1. Let \(m, n\) be topologically invariant means on \(VN(H)\). Then \(m = n\) when restricted to \(W(\hat{H})\) by Proposition 4.3(ii). Since \(UCB(\hat{H}) \subseteq W(\hat{H})\), we conclude that \(R(m) = R(n)\), and hence \(m = n\) by Lemma 4.5. Again Lemma 4.5 implies that there is a unique topological invariant mean on \(UCB(\hat{H})\).

(iii) \(\Rightarrow\) (i). This follows from Lemma 4.5 and [18, Theorem 1.7].

It is shown in [15, Theorem 3.15] that \(B_\lambda(H)\) is a Banach algebra under pointwise multiplication. As shown in Proposition 4.3, \(C^*_\lambda(H)\) is topologically introverted. In particular, \(C^*_\lambda(H)^* = B_\lambda(H)\) is a Banach algebra with the Arens Product. It is shown in [9, Proposition 5.3] that the Arens product on \(B_\lambda(G)\) is precisely the pointwise product on it. Following we show that the same is also true for an ultraspherical hypergroup \(H\).
Proposition 4.7. Let $H$ be an ultraspherical hypergroup. Then the Arens product and the pointwise multiplication on $B_\lambda(H)$ coincide.

Proof. Let $\phi, \psi \in B_\lambda(H)$. Then for each $f \in L^1(H)$, we have

$$\langle \phi \psi, \lambda(f) \rangle = \langle \phi, \lambda(\psi f) \rangle = \langle \psi, \lambda(\phi f) \rangle.$$ 

This shows that the pointwise multiplication on $B_\lambda(H)$ is separately continuous in the $w^*$-topology. Furthermore, for each $\psi \in B_\lambda(H)$, the map $\phi \mapsto \phi \odot \psi$ from $B_\lambda(H)$ into $B_\lambda(H)$ is $w^*-w^*$-continuous. Since $C_0^\lambda(H) \subseteq W(\widehat{H})$, it follows from [3] Proposition 3.11] that the map $\phi \mapsto \psi \odot \phi$ is continuous in the $w^*$-topology. Therefore, the Arens product also is separately continuous in the weak$^*$-topology. Since the Arens product and the pointwise multiplication on $A(H)$ coincide and $A(H)$ is $w^*$-dense in $B_\lambda(H)$, we conclude that $\phi \odot \psi = \phi \psi$ for all $\phi, \psi \in B_\lambda(H)$.

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Multipliers over Fourier algebras of ultraspherical hypergroups

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