The Exact Solution of the $SU(3)$ Hubbard Model

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Abstract

The Bethe ansatz equations of the 1-D $SU(3)$ Hubbard model are systematically derived by diagonalizing the inhomogeneous transfer matrix of the $XXX$ model. We first derive the scattering matrix of the $SU(3)$ Hubbard model through the coordinate Bethe ansatz method. Then, with the help quantum inverse scattering method we solve the nested transfer matrix and give the eigenvalues, the eigenvectors and the Bethe ansatz equations. Finally, we obtain the exactly analytic solution for the ground state.

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1 Introduction

Strongly correlated electron systems have been an important research subject in the condensed matter physics and mathematical physics. One of the significant models is the 1-D Hubbard model for which the exact solution was first given by Lieb and Wu[1]. Based on the Bethe ansatz equations (Lieb-Wu’s equations), the excited spectrum (spin excitation and charge excitation) was discussed in Refs. [2, 3, 4, 5, 6].

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magnetic properties of the 1-D Hubbard model at zero temperature were investigated in Refs. [7, 8, 9], and the thermodynamics of the model was studied in Refs. [10, 11, 12] with the help of string hypothesis. The critical exponents of the system was found in Refs. [13, 14]. All physical properties can be discussed in the framework of Bethe ansatz equations. Although there are lots of works on Hubbard model, the integrability of the 1-D Hubbard model was finished until 1986 by Shastry [15]. Olmedilla and Wadati [16]. Moreover, the eigenvalue of the transfer matrix related to the Hubbard model was suggested in Ref. [15] and proved through different method in Ref. [17, 18]. Besides, the integrability and the exact solution of the 1-D Hubbard model with open boundary condition have been investigated by several authors. [19, 20, 21].

Recently, based on the Lie algebra knowledge, Maassarani and Mathieu constructed the Hamiltonian of the $SU(n)$ XX model and shown its integrability [22]. Considering two coupled $SU(n)$ XX models, Maassarani succeeded in generalizing Shastry’s method to construct the $SU(n)$ Hubbard model [23]. Furthermore, he found the related $R$-matrix which ensures the integrability of the one-dimensional $SU(n)$ Hubbard model [24]. (It was also proved by Martins for $n = 3, 4$ [25], and by Yue and Sasaki [26] for general $n$ in terms of Lax-pair formalism.) However, the eigenvalue and the eigenvectors of the $SU(n)$ Hubbard model have not been given yet.

In this paper, we first apply the coordinate Bethe ansatz method to construct the wave function and the scattering matrix of the $SU(3)$ Hubbard model in section 2, then with the help of the Yang-Baxter relation we list the algebra consisting the elements of the transfer matrix in section 3, which is the key relations for finding the solution of the nested transfer matrix. In section 4 we apply the algebraic Bethe ansatz method to discuss the eigenvalue and the Bethe ansatz equations of the model. Then the exact solution of the $SU(3)$ Hubbard model was given out. Under the thermodynamical limit, the explicitly analytic form of the ground state energy is given in section 5. In section 6, we make some conclusions and list some questions to be considered.

2 The Coordinate Bethe Ansatz

The Hamiltonian of the $SU(n)$ Hubbard model is

$$H = \sum_{k=1}^{L} \sum_{\alpha=1}^{n-1} \left( E_{a,k}^{\alpha} E_{a,k+1}^{\alpha} + E_{\sigma,k}^{\alpha} E_{\sigma,k+1}^{\alpha} + E_{\tau,k}^{\alpha} E_{\tau,k+1}^{\alpha} + E_{\tau,k}^{\alpha} E_{\tau,k+1}^{\alpha} + \frac{U n^2}{4} \right) + \sum_{k=1}^{L} C_{\sigma,k} C_{\tau,k}$$

(1)

where $U$ is the coulumb coupling constant, and $E_{a,k}^{\alpha}(a = \sigma, \tau)$ is a matrix with zeros everywhere except for an one at the intersection of row $\alpha$ and column $\beta$:

$$(E^{\alpha\beta})_{lm} = \delta_{l}^{\alpha} \delta_{m}^{\beta},$$

(2)
The subscripts \( a \) and \( k \) stand for two different \( E \) operators at site \( k \), \( k = 1, \cdots, L \). The \( n \times n \) diagonal matrix \( C \) is defined by \( C = \sum_{\alpha<n} E^{\alpha\alpha} - E^{nn} \). We have also assumed the periodic boundary condition \( E^{\alpha\beta}_{k+L} = E^{\alpha\beta}_k \).

It was shown that the Hamiltonian (1) has a \((su(n-1) \oplus u(1))_\sigma \oplus (su(n-1) \oplus u(1))_\tau\) symmetry. The generators are

\[
J_{\alpha\beta}^a = \sum_{k=1}^n E^{\alpha\beta}_{a,k},
\]

and

\[
K_a = \sum_{k=1}^L C_{a,k}, \quad \alpha, \beta = 1, \cdots, n-1, \quad a = \sigma, \tau.
\]

It is worthy to point out that the system enjoys the \( SO(4) \) symmetry when \( n = 2 \). But the generator are different from Eq. (3). In this paper, we just limit our attention on the \( SU(3) \) model.

Before proceeding the coordinate Bethe ansatz approach, we first begin by introducing some notations of our analysis. In \( SU(3) \) Hubbard model, there are two types of particle named \( \sigma \) and \( \tau \), and each type of particles can occupy one of two possible states. We denote \(|0\rangle\) the vacuum state, \(|1\rangle\) and \(|2\rangle\) the two particle states respectively. Under the appropriate basis

\[
|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |0\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\]

we can prove that \( E^{\alpha 3} \) and \( E^{3\alpha} \) act as a creating and a destroying operators of \(|\alpha\rangle\) state respectively.

In the coordinate Bethe ansatz method, the eigenstates of the Hamiltonian can be assumed as

\[
|\psi_{N_0}\rangle = \sum_{x_1 \leq x_2 \leq \cdots \leq x_{N_0} = 1} f^{\alpha_1 \alpha_2 \cdots \alpha_{N_0}}_{\sigma_1 \sigma_2 \cdots \sigma_{N_0}} E^{\alpha_1 3}_{\sigma_1 x_1} E^{\alpha_2 3}_{\sigma_2 x_2} \cdots E^{\alpha_{N_0} 3}_{\sigma_{N_0} x_{N_0}} |0\rangle,
\]

where

\[
f^{\alpha_1 \alpha_2 \cdots \alpha_{N_0}}_{\sigma_1 \sigma_2 \cdots \sigma_{N_0}} = \sum_{P,Q} [\epsilon_P \epsilon_Q A^{\alpha_{Q_1} \alpha_{Q_2} \cdots \alpha_{Q_{N_0}}} (k_{P_{Q_1}}, k_{P_{Q_2}}, \cdots, k_{P_{Q_{N_0}}})]
\times \theta(x_{Q_1} \leq x_{Q_2} \leq \cdots \leq x_{Q_{N_0}}) \exp(i \sum_{j=1}^{N_0} k_{P_{j}} x_{Q_j}),
\]

Here \( \alpha_i = 1, 2 (i = 1, 2, \cdots, N_0) \) stands for the different particle states, \( x_i \) the position of the particle, \( \sigma_i = \sigma, \tau \) the type of \( i \)th particle. The summation \( P \) and \( Q \) are taken over
all permutations of $N_0$ momenta $k_j$ and $N_0$ coordinates $x_j$ respectively. The symbols $\epsilon_P$ and $\epsilon_Q$ are the parities of two kinds of permutations. Substituting the wave function into the Schrodinger equation

$$H|\psi_{N_0}\rangle = E|\psi_{N_0}\rangle,$$  

we can get

$$A_{\ldots \alpha_i \sigma_i \ldots}(\ldots, k_i, k_j, \ldots) = S_{\alpha_i \sigma_i}^{\alpha_j \sigma_j}(\sin k_i, \sin k_j) A_{\ldots \alpha_i \sigma_i \ldots}(\ldots, k_j, k_i, \ldots). \quad (9)$$

with $S_{\alpha_i \sigma_i}^{\alpha_j \sigma_j}(\sin k_i, \sin k_j)$ being the two-particle scattering matrix:

$$S_{\alpha_i \sigma_i}^{\alpha_j \sigma_j}(\sin k_i, \sin k_j) = \sin k_i - \sin k_j + i\gamma P_{\alpha_i \alpha_j}^{\sigma_i \sigma_j} \sin k_i - \sin k_j + i\gamma \quad (10)$$

where $\gamma = \frac{9U}{2}$, $P_{\alpha_i \alpha_j}^{\sigma_i \sigma_j}$ is the direct product of two kinds of permutation operators and permutes the particle styles and particle states simultaneously. The energy of the Hamiltonian on this wave function is

$$E = 2 \sum_{i=1}^{N_0} \cos k_i + \frac{\gamma}{2}(L - 2N_0). \quad (11)$$

For the convenience, we denote by $\zeta_0$ the amplitude $A_{\ldots \alpha_j \sigma_j \ldots}(\ldots, k_j, k_i, \ldots)$. When the $j$-th particle moves across the else particle, it gets an $S$-matrix $S_{ji}(q_j - q_i)$. Here $q_j = \sin k_j$. The periodic boundary condition leads the following constrains on the amplitude $\zeta_0$

$$S_{jj+1}S_{jj+2} \cdots S_{jN_0}S_{j1}S_{j2} \cdots S_{jj-1} \zeta_0 = e^{ik_L} \zeta_0. \quad (12)$$

Eq. (12) is similar with the Yang’s eigenvalue problem[27]. Its solution will give out the Bethe ansatz equation.

3 The fundamental commutation rules

In the above section, we have obtained the scattering matrix. It can be proved that they satisfy the Yang-Baxter equation[27]

$$S_{jl}(q_j, q_l)S_{jk}(q_j, q_k)S_{lk}(q_l, q_k) = S_{lk}(q_l, q_k)S_{jk}(q_j, q_k)S_{jl}(q_j, q_l). \quad (13)$$

Define $\zeta = (S_{jj+1}S_{jj+2} \cdots S_{jN_0})^{-1}\zeta_0$, we can rewrite the eigenvalue problem (12) as

$$S_{j1}S_{j2} \cdots S_{jN_0} \zeta = X_j \zeta = \epsilon(q_j) \zeta. \quad (14)$$
We introduce an auxiliary space $\tau$ and $S_{\tau j} = S_{\tau j}(q - q_j)$, and define the monodromy matrix
\[ T_\tau(q) = S_{\tau 1}S_{\tau 2} \cdots S_{\tau N_0}. \] (15)
Obviously, $S_{\tau j}(q_j) = P_{\tau j}$ and
\[ tr_\tau T_\tau(q_j) = X_j. \] (16)
Since $S_{\tau i}$ satisfies the Yang-Baxter equation, we can easily prove that the monodromy matrix $T_\tau(q)$ also satisfies the Yang-Baxter relation
\[ S_{\tau \tau'}(q, q')T_{\tau j}(q, q_j)T_{\tau' j}(q', q_j) = T_{\tau' j}(q', q_j)T_{\tau j}(q, q_j)S_{\tau \tau'}(q, q'). \] (17)

From the point of view of a vertex model, we can interpret the matrix $S_{\tau j}$ as the vertex operator, the matrix $S_{\tau \tau'}$ as the $R$-matrix. So the $R$-matrix is an $16 \times 16$ matrix
\[
R_{j l}(q_j, q_l) = \begin{pmatrix}
\alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \alpha_2 & 0 & 0 & \alpha_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \alpha_2 & 0 & 0 & 0 & 0 & \alpha_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \alpha_2 & 0 & 0 & 0 & 0 & \alpha_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \alpha_3 & 0 & 0 & \alpha_2 & 0 & 0 & 0 & 0 & \alpha_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & 0 & \alpha_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & \alpha_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & \alpha_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_1 & 0 & \alpha_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_1 & 0 & \alpha_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_1 & 0 & \alpha_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_1 & 0 & \alpha_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_1 & 0 & \alpha_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_1 & 0 & \alpha_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_1 & 0
\end{pmatrix}. \] (18)

where
\[
\alpha_1(q_j, q_l) = 1, \quad \alpha_2(q_j, q_l) = \frac{q_j - q_l}{q_j - q_l + i\gamma}, \quad \alpha_3(q_j, q_l) = \frac{i\gamma}{q_j - q_l + i\gamma}. \] (19)

The vertex operator is
\[
L_{\tau j}(q - q_j) = S_{\tau j}(q - q_j)
\]
\[
= \alpha_2(q, q_j) + \alpha_3(q, q_j) \left\{ \frac{1}{2} (1 + \tau^+ \tau^z) + \sigma^+_j \sigma^-_j + \sigma^-_j \sigma^+_j \right\}
\]
\[
\otimes \left\{ \frac{1}{2} (1 + \tau^+ \tau^z) + \tau^+_j \tau^-_j + \tau^-_j \tau^+_j \right\}, \] (20)
where $\sigma_j, \tau_j$ are two different kinds of Pauli matrixes on $j$-th space.

In addition to the $L$-operator and the $R$-matrix the existence of a local reference state is another important object in the quantum inverse scattering program. In our case, we can define the local reference state by

$$|0\rangle_j = \left(\begin{array}{c} 1 \\ 0 \end{array}\right)_{\sigma_j} \otimes \left(\begin{array}{c} 1 \\ 0 \end{array}\right)_{\tau_j} = |\uparrow\rangle_{\sigma_j} \otimes |\uparrow\rangle_{\tau_j}$$

The action of $L$-operator on this state has the following property:

$$L_{\tau_j}(q)|0\rangle_j = \left(\begin{array}{cccc}
|0\rangle_j & |\uparrow\downarrow\rangle_j & |\downarrow\uparrow\rangle_j & |\downarrow\downarrow\rangle_j \\
0 & \frac{q}{q + i\gamma}|0\rangle_j & 0 & 0 \\
0 & 0 & \frac{q}{q + i\gamma}|0\rangle_j & 0 \\
0 & 0 & 0 & \frac{q}{q + i\gamma}|0\rangle_j
\end{array}\right)$$

(21)

The global reference state $|0\rangle$ is then defined by the tensor product of local reference states, i.e. $|0\rangle = \prod_{j=1}^{L} \otimes |0\rangle_j$. This state is an eigenstate of the transfer matrix which is the trace of monodromy matrix. In order to construct the other eigenstates, it is necessary to seek for an appropriate representation of the monodromy matrix. By this we mean a structure which is able to distinguish creation and annihilation operators as well as possible hidden symmetries. The property of $L$-operator suggests the monodromy matrix to take the following form [17]

$$T_{\tau}(q) = \begin{pmatrix}
B(q) & B(q) & F(q) \\
C(q) & A(q) & B^*(q) \\
C(q) & C^*(q) & D(q)
\end{pmatrix}_{4 \times 4}$$

(22)

where $B(q), C^*(q)$ and $B^*(q), C(q)$ are two component vectors with dimensions $1 \times 2$ and $2 \times 1$ respectively. The operator $A(q)$ is a $2 \times 2$ matrix and we shall denote its elements by $A_{ab}(q)$. The remaining operators $B(q), C(q), F(q), D(q)$ are scalars.

In the framework of the above partition the eigenvalue problem for the transfer matrix becomes

$$[B(q) + \sum_{a=1}^{2} A_{aa}(q) + D(q)]|\Phi\rangle = \Lambda(q)|\Phi\rangle.$$  

(23)

where $\Lambda(q)$ and $|\Phi\rangle$ correspond to the eigenvalue and the eigenvector respectively. From
Eq. (21), we know the action of the monodromy matrix on the reference state:

\[
\begin{align*}
B(q)|0\rangle &= |0\rangle, \\
D(q)|0\rangle &= \omega(q)|0\rangle, \\
A_{aa}(q)|0\rangle &= \omega(q)|0\rangle, \quad a = 1, 2. \\
C(q)|0\rangle &= 0, \\
C^*(q)|0\rangle &= 0, \\
C(q)|0\rangle &= 0, \\
A_{ab}(q)|0\rangle &= 0, \quad a \neq b.
\end{align*}
\]  

(24)

where the function \(\omega(q)\) is defined by

\[
\omega(q) = \prod_{j=1}^{N_q} \frac{q - q_j}{q - q_j + i\gamma}.
\]

(25)

The operator \(B(q)\) and \(F(q)\) play the role of creation operators over the reference state \(|0\rangle\).

To make further progress we have to recast the Yang-Baxter algebra in the form of commutation relations for the creation and annihilation operators. We shall start our discussion by the commutation rule between the operators \(B(q)\) and \(B(p)\):

\[
B(q) \otimes B(p) = [B(p) \otimes B(q)] \cdot \hat{r}(q,p).
\]

(26)

where \(\hat{r}(q,p)\) is an auxiliary 4 \(\times\) 4 matrix given by

\[
\hat{r}(q,p) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \hat{a}(q,p) & \hat{b}(q,p) & 0 \\
0 & \hat{b}(q,p) & \hat{a}(q,p) & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

(27)

and the functions \(\hat{a}(q,p)\) and \(\hat{b}(q,p)\) are defined by

\[
\hat{a}(q,p) = \frac{\alpha_3(q,p)}{\alpha_1(q,p)}, \quad \hat{b}(q,p) = \frac{\alpha_2(q,p)}{\alpha_1(q,p)}
\]

(28)

We can see that this auxiliary matrix \(r(q,p)\) is precisely the rational \(R\)-matrix of the isotropic 6-vertex model or the XXX spin chain. It is very easy to see out that

\[
\hat{a}(q,p) = 1 - \hat{b}(q,p),
\]

(29)
To solve the eigenvalue problem (24) we need several other commutation rules, especially the commutation rules between the diagonal and the creation operators. The commutation relations between the diagonal and creation operator $B(q)$ are

$$A(q) \otimes B(p) = \frac{\alpha_1(q,p)}{\alpha_2(q,p)}[B(p) \otimes A(q)] \cdot \hat{r}(q,p) - \frac{\alpha_3(q,p)}{\alpha_2(q,p)}B(q) \otimes A(p), \quad (30)$$

$$B(q)B(p) = \frac{\alpha_1(p,q)}{\alpha_2}B(p)B(q) - \frac{\alpha_3(p,q)}{\alpha_2(p,q)}B(q)B(p), \quad (31)$$

$$D(q)B(p) = B(p)D(q) + \frac{\alpha_3(q,p)}{\alpha_2(q,p)}F(p)C^*(q) - \frac{\alpha_3(q,p)}{\alpha_2(q,p)}F(q)C^*(p) \quad (32)$$

We also need the following commutation rule

$$C^*(q) \otimes B(p) = \frac{\alpha_1(q,p)}{\alpha_2(q,p)}[B(p) \otimes C^*(q)] \cdot \hat{r}(q,p) - \frac{\alpha_3(q,p)}{\alpha_2(q,p)}B(q) \otimes C^*(p) \quad (33)$$

We have set up the basic tools to construct the eigenvectors of the eigenvalue problem (23). In the next section we will show how this problem can be solved with the help of the commutations rules (26), (27), (30)-(33).

4 The eigenvectors and eigenvalue construction

The purpose of this section is to solve the eigenvalue problem for the transfer matrix. We shall begin by considering the construction of an ansatz for the corresponding eigenvectors.

4.1 The eigenvalue problem

The eigenvectors of the transfer matrix are in principle built up in terms of a linear combination of the products of many creation operators acting on the reference state, which are characterized by a set of rapidities parameterizing the creation operators. First, we define the eigenvector for an arbitrary $N_1$-particle state:

$$|\Phi_{N_1}(\{p_i\})\rangle = \Phi_{N_1}(p_1, p_2, \cdots, p_{N_1}) \cdot F|0\rangle \quad (34)$$

where the mathematical structure of the vector $\Phi_{N_1}(p_1, p_2, \cdots, p_{N_1})$ will be described in terms of the creation operators. We denote the components of vector $F$ by $F^{a_1\cdots a_{N_1}}$, which will be determined later on, where the index $a_i$ runs over two possible values $a_i = 1, 2$. 

8
To construct the eigenvectors of the transfer matrix, it is sufficient to look for combinations between the operators $B(q)$ and $F(q)$. The experience on constructing a few particle states suggests us the eigenvector of $N_1$-particle to be

$$\Phi_{N_1}(p_1, \cdots, p_{N_1}) = B(p_1) \otimes \Phi_{N_1-1}(p_2, \cdots, p_{N_1}). \quad (35)$$

Here we have formally identified $\Phi_0$ with the unity vector. The detailed construction for less than 3 particles was given in Appendix B. This vector has the following symmetry:

$$\Phi_{N_1}(p_1, \cdots, p_{j-1}, p_j, \cdots, p_{N_1}) = \Phi_{N_1}(p_1, \cdots, p_j, p_{j-1}, \cdots, p_{N_1}) \cdot \hat{r}_{j-1j}(p_{j-1}, p_j), \quad (36)$$

where the subscripts in $\hat{r}_{j-1j}(p_{j-1}, p_j)$ emphasize the positions in the $N_1$-particle space $V_1 \otimes \cdots \otimes V_{j-1} \otimes V_j \otimes \cdots \otimes V_{N_1}$ on which this matrix acts non-trivially. Here we have already assumed that the $(N_1 - 1)$-particle state was already symmetrized.

Applying the diagonal elements of monodromy matrix on this eigenstate, we can get

$$B(q)|\Phi_{N_1}(\{p_l\})) = \prod_{i=1}^{N_1} \frac{\alpha_1(p_i, q)}{\alpha_2(p_i, q)}|\Phi_{N_1}(\{p_l\}))$$

$$- \sum_{i=1}^{N_1} \frac{\alpha_3(p_i, q)}{\alpha_2(p_i, q)} \prod_{k=1, k \neq i}^{N_1} \frac{\alpha_1(p_k, p_i)}{\alpha_2(p_k, p_i)} \langle \Psi_{N_1-1}(q, p_i, \{p_l\}) \rangle. \quad (37)$$

$$D(q)|\Phi_{N_1}(\{p_l\})) = \omega(q)|\Phi_{N_1}(\{p_l\}))$$

$$\sum_{a=1}^{2} A_{aa}(q)|\Phi_{N_1}(\{p_l\})) = \omega(q) \prod_{i=1}^{N_1} \frac{\alpha_1(q, p_i)}{\alpha_2(q, p_i)} \Lambda^{(1)}(q, \{p_l\})|\Phi_{N_1}(\{p_l\}))$$

$$- \sum_{i=1}^{N_1} \omega(p_i) \frac{\alpha_3(q, p_i)}{\alpha_2(q, p_i)} \prod_{k=1, k \neq i}^{N_1} \frac{\alpha_1(p_k, p_i)}{\alpha_2(p_k, p_i)} \Lambda^{(1)}(q, \{p_l\})$$

$$\times |\Psi_{N_1-1}(q, p_i, \{p_l\}))|. \quad (39)$$

here

$$|\Psi_{N_1-1}^{(1)}(q, p_i, \{p_l\})) = B(q) \otimes \Phi_{N_1-1}(p_1, \cdots, \tilde{p}_i, \cdots, p_{N_1}) \hat{O}_i^{(1)}(p_i, \{p_l\}) \cdot \mathcal{F}[0]. \quad (40)$$

and

$$\hat{O}_i^{(1)}(p_i, \{p_k\}) = \prod_{k=1}^{i-1} \hat{r}_{k,k+1}(p_k, p_i) \quad (41)$$

where the symbol $\tilde{p}_i$ means that the rapidity $p_i$ is absent from the set $\{p_1, \cdots, p_{N_1}\}$.

The terms proportional to the eigenvector $|\Phi_{N_1}(\{p_l\}))$ are denominated the wanted terms because they contribute directly to the eigenvalue. From Eqs. (37)-(41), we can
directly get the eigenvalue of $N_1$-particle state

$$
\Lambda(q, \{p_i\}) = \prod_{i=1}^{N_1} \frac{\alpha_1(p_i, q)}{\alpha_2(p_i, q)} + \omega(q) + \omega(q) \prod_{i=1}^{N_1} \frac{\alpha_1(q, p_i)}{\alpha_2(q, p_i)} \Lambda^{(1)}(q, \{p_i\}).
$$

The remaining ones are called unwanted terms and can be eliminated by imposing further restrictions on the rapidities $p_i$. These restrictions, known as the Bethe Ansatz equations, are

$$
\frac{1}{\omega(p_i)} = \prod_{k=1,k\neq i}^{N_1} \frac{\alpha_2(p_k, p_i)}{\alpha_2(p_i, p_k)} \Lambda^{(1)}(p_i, \{p_i\}), \quad i = 1, \ldots, n.
$$

In fact, the undetermined eigenvalue $\Lambda^{(1)}(q, \{p_i\})$ must satisfy the following auxiliary problem

$$
T^{(1)}(q, \{p_i\})_{a_1 \ldots a_{N_1}}^{b_1 \ldots b_{N_1}} F^{b_1 \ldots b_{N_1}} = \Lambda^{(1)}(q, \{p_i\}) F_{a_1 \ldots a_{N_1}},
$$

where the inhomogeneous transfer matrix $T^{(1)}(q, \{p_i\})$ is

$$
T^{(1)}(q, \{p_i\})_{a_1 \ldots a_{N_1}}^{b_1 \ldots b_{N_1}} = \hat{r}_{a_1}^{c_1 b_1}(q, p_1) \hat{r}_{a_2}^{c_2 b_2}(q, p_2) \cdots \hat{r}_{a_{N_1}}^{c_{N_1} b_{N_1}}(q, p_{N_1}).
$$

This result is the direct extensions of that obtained from the two-particle state. The Bethe ansatz equations and the eigenvalue still depend on this additional auxiliary eigenvalue problem.

### 4.2 The eigenvalues and the nested Bethe ansatz

The purpose of this subsection will be the diagonalization of the auxiliary transfer matrix $T^{(1)}(q, \{p_i\})$. First, we write the transfer matrix $T^{(1)}(q, \{p_i\})$ as the trace of the following monodromy matrix:

$$
T^{(1)}(q, \{p_i\}) = \mathcal{L}^{(1)}_{\mathcal{A}^{(1)}}(q, p_{N_1}) \mathcal{L}^{(1)}_{\mathcal{A}^{(1)}}(q, p_{N_1-1}) \cdots \mathcal{L}^{(1)}_{\mathcal{A}^{(1)}}(q, p_1).
$$

where $\mathcal{A}^{(1)}$ is the auxiliary space. The $L$-operator $\mathcal{L}^{(1)}_{\mathcal{A}^{(1)}}(q, p_i)$ is related to the auxiliary matrix $\hat{r}(q, p_i)$ by a permutation operator

$$
\mathcal{L}^{(1)}_{\mathcal{A}^{(1)}}(q, p_i) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \hat{b}(q, p_i) & \hat{a}(q, p_i) & 0 \\
0 & \hat{a}(q, p_i) & \hat{b}(q, p_i) & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
$$

10
Comparing with 6-vertex model, one can find that this problem is equivalent to the 6-vertex model. So we can take the results of Ref. [17] directly. The eigenvalue of the auxiliary problem [44] is:

$$\Lambda^{(1)}(q, \{p_l\}, \{s_k\}) = \prod_{k=1}^{N_2} \frac{1}{b(s_k, q)} + \prod_{l=1}^{N_1} b(q, p_l) \prod_{k=1}^{N_2} \frac{1}{b(q, s_k)}.$$  \hspace{1cm} (48)

and the parameters \{s_k\} satisfy the following restrictions:

$$\prod_{l=1}^{N_1} b(s_k, p_l) = \prod_{i=1}^{N_2} \frac{b(s_k, s_i)}{b(s_i, s_k)}, \quad k = 1, \ldots, M.$$  \hspace{1cm} (49)

Now we use the auxiliary eigenvalue expression to rewrite our previous results for the eigenvalues and the Bethe ansatz equations. Substituting the expression [48] in Eqs. (42,43) and using the relation (28) we obtain that the eigenvalue is

$$\Lambda(q, \{p_l\}, \{s_k\}) = \prod_{l=1}^{N_1} \frac{(p_l - q + i\gamma)}{(p_l - q)} + \prod_{j=1}^{N_0} \frac{(q - q_j)}{(q - q_j + i\gamma)} + \prod_{j=1}^{N_0} \frac{(q - q_j)}{(q - p_j + i\gamma)}$$  

$$\times \left\{ \prod_{l=1}^{N_1} \frac{(q - p_l + i\gamma)}{(q - p_l)} \prod_{k=1}^{N_2} \frac{(s_k - q + i\gamma)}{(s_k - q)} + \prod_{k=1}^{N_2} \frac{(q - s_k + i\gamma)}{(q - s_k)} \right\}.$$  \hspace{1cm} (50)

In terms of the above eigenvalue, the periodic boundary condition (12) implies

$$e^{i k L} = \Lambda(q_j, \{p_l\}, \{s_k\}) = \prod_{l=1}^{N_1} \frac{\sin(k_j - \tilde{p}_l - i\gamma_2/2)}{\sin(k_j - \tilde{p}_l + i\gamma_2/2)}.$$  \hspace{1cm} (51)

The second set of Bethe ansatz equations Eq. (13) for the rapidities \(p_l\) changes into

$$\prod_{i=1, i\neq l}^{N_1} \frac{(\tilde{p}_l - \tilde{p}_i + i\gamma)}{(\tilde{p}_l - \tilde{p}_i - i\gamma)} = \prod_{j=1}^{N_0} \frac{(\tilde{p}_l - \sin k_j + i\gamma_2/2)}{(\tilde{p}_l - \sin k_j - i\gamma_2/2)} \prod_{k=1}^{N_2} \frac{(\tilde{p}_l - \tilde{s}_k + i\gamma_2/2)}{(\tilde{p}_l - \tilde{s}_k - i\gamma_2/2)}, \quad (l = 1, 2, \ldots, M).$$  \hspace{1cm} (52)

and the third set of Bethe ansatz equations Eq. (14) for the rapidities \(s_k\) is

$$\prod_{i=1, i\neq k}^{N_1} \frac{(\tilde{s}_k - \tilde{p}_l - i\gamma_2/2)}{(\tilde{s}_k - \tilde{p}_l + i\gamma_2/2)} = \prod_{i=1, i\neq k}^{N_2} \frac{(\tilde{s}_k - \tilde{s}_i - i\gamma)}{(\tilde{s}_k - \tilde{s}_i + i\gamma)}, \quad (k = 1, 2, \ldots, N_2).$$  \hspace{1cm} (53)

Here we have used the shifted parameters \(\tilde{p}_l = p_l - i\gamma/2\) and \(\tilde{s}_k = s_k - i\gamma\) to bring our equations to more symmetric forms.
5 The ground state

In the proceeding section, we have obtained the final Bethe ansatz equations of the $SU(3)$ Hubbard model. In this section, we will use them to analyse the ground state property of the model.

By taking the logarithm of the Bethe ansatz equations (51)-(53), we can get the following three sets of equations (In this section, we assume $U > 0$):

\[ k_j L = 2 \pi I_j + 2 \sum_{l=1}^{N_1} \tan^{-1} \left( \frac{\sin k_j - \tilde{p}_l}{\frac{\gamma}{2}} \right), \]  
\[ 2 \pi J_l + 2 \sum_{i=1,i \neq l}^{N_1} \tan^{-1} \left( \frac{\tilde{p}_l - \tilde{p}_i}{\gamma} \right) = 2 \sum_{j=1}^{N_0} \tan^{-1} \left( \frac{\tilde{p}_j - \sin k_j}{\frac{\gamma}{2}} \right) + 2 \sum_{k=1}^{N_2} \tan^{-1} \left( \frac{\tilde{p}_l - \tilde{s}_k}{\frac{\gamma}{2}} \right), \]  
\[ 2 \sum_{l=1}^{N_1} \tan^{-1} \left( \frac{\tilde{s}_k - \tilde{p}_l}{\frac{\gamma}{2}} \right) = 2 \sum_{i=1,i \neq k}^{N_2} \tan^{-1} \left( \frac{\tilde{s}_k - \tilde{s}_i}{\gamma} \right) + 2 \pi P_k, \]

where $I_j, J_l, P_k$ are integers or half-odd integers. Under the thermodynamic limits the Bethe ansatz equations for the ground state change into

\[ 2 \pi \rho(k) = 1 - \cos k \int_{-B}^{B} \frac{4 \gamma \sigma(\Lambda)}{\gamma^2 + 4(\sin k - \Lambda)^2} d\Lambda, \]  
\[ 2 \pi \sigma(\Lambda) + \int_{-B}^{B} \frac{2 \gamma \sigma(\Lambda')}{\gamma^2 + (\Lambda - \Lambda')^2} d\Lambda' = \int_{-Q}^{Q} \frac{4 \gamma \rho(k)}{\gamma^2 + 4(\Lambda - \sin k)^2} dk + \int_{-E}^{E} \frac{4 \gamma \tau(\Lambda')}{\gamma^2 + 4(\Lambda - \Lambda')^2} d\Lambda', \]  
\[ 2 \pi \tau(\Lambda) + \int_{-E}^{E} \frac{2 \gamma \tau(\Lambda')}{\gamma^2 + (\Lambda - \Lambda')^2} d\Lambda' = \int_{-B}^{B} \frac{4 \gamma \sigma(\Lambda')}{\gamma^2 + 4(\Lambda - \Lambda')^2} d\Lambda', \]

where $Q, B$ and $E$ are determined by the conditions

\[ \int_{-Q}^{Q} \rho(k) dk = \frac{N_0}{L}, \]  
\[ \int_{-B}^{B} \sigma(\Lambda) d\Lambda = \frac{N_1}{L}, \]  
\[ \int_{-E}^{E} \tau(\Lambda') d\Lambda' = \frac{N_2}{L}. \]
The function $\rho(k), \sigma(\Lambda), \tau(\Lambda')$ are the distribution functions of real parameters $k_j, \tilde{p}_l$ and $\tilde{s}_k$ respectively.

Eqs. (57)-(62) have a unique solution which is positive for all allowed $Q, B$ and $E$. $\frac{N_i}{N_j}, (i = 0, 1, 2)$ is a monotonically function of $Q, B$ and $E$ respectively. Thus the ground state is characterized by $Q = \pi, B = E = \infty$.

After taking Fourier transforms of the above equations we can obtain the result of the distribution functions

$$\rho(k) = \frac{1}{2\pi} - \frac{\cos k}{2\pi} \int_{-\infty}^{\infty} \frac{1 + e^{-\gamma|\omega|}}{1 + 2 \cosh(\gamma|\omega|)} e^{-i\omega \sin k} J_0(\omega) d\omega, \quad (63)$$

$$\sigma(\Lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2 \cosh(\frac{\gamma|\omega|}{2})}{4 \cosh^2(\frac{\gamma|\omega|}{2}) - 1} J_0(\omega) e^{-i\omega \Lambda} d\omega, \quad (64)$$

$$\tau(\Lambda') = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{4 \cosh^2(\frac{\gamma|\omega|}{2}) - 1} J_0(\omega) e^{-i\omega \Lambda'} d\omega. \quad (65)$$

From Eqs.(61)-(66) we have $N_0 = L$, this means that all lattice site is filled by one particles, $N_1 = \frac{2}{3}N_0$, and $N_2 = \frac{1}{3}N_0$.

The ground state energy then is given by

$$E = 2L \int_{-\infty}^{\infty} \rho(k) \cos kdk - \frac{\gamma}{2} L$$

$$= -2L \int_{-\infty}^{\infty} \frac{1 + e^{-\gamma|\omega|}}{\omega[1 + 2 \cosh(\gamma|\omega|)]} J_0(\omega) J_1(\omega) d\omega - \frac{\gamma}{2} L. \quad (66)$$

### 6 Conclusions

The main purpose of this paper was to investigate the eigenvalue of the $SU(3)$ Hubbard model. We have succeeded in finding the eigenvalue of the Hamiltonian [11] and obtained three sets of Bethe ansatz equations. Based on the Bethe ansatz equations, we have found the explicit expression of the energy and the distribution functions of the rapidities corresponding to the ground state for positive $U$.

One important question is to study the excitation spectrum and the low temperature thermodynamics of the model for both positive and negative $U$. The Bethe ansatz equations given in present paper will play an key role. As we know, the negative $U$ has a distinguished properties from positive $U$ case. The solution structure of Bethe ansatz equations corresponding to the ground state and excited spectrum for attractive case ($U < 0$) are different from those for repulsive case. It is worthy to be studied in future.

In Ref.[24], the $L$-operator and $R$-matrix were given, therefore, one can define the transfer matrix as done in usual Hubbard model ( $SU(2)$ case). But how to find
the eigenvalue of the transfer matrix is unknown. It may be solved by using the method proposed in Ref. [17]. We will consider this problem late. Another natural generalization the present paper is to find the eigenvalue of the Hamiltonian (1) for general \( n \), it will be considered elsewhere.

7 Appendix A:

First the one-particle eigenstate can be assumed to be

\[
|\psi_1\rangle = \sum_{x=1}^{L} f_{\sigma_1}^\alpha(x) E_{\sigma_1}^\beta |0\rangle.
\] (A.1)

Applying the Hamiltonian (1) into this ansatz, we can obtain

\[
E f_{\sigma_1}^\alpha(x) = f_{\sigma_1}^\alpha(x + 1) + f_{\sigma_1}^\alpha(x - 1) + \frac{9U}{4} (L - 2) f_{\sigma_1}^\alpha(x), \quad 1 < x < L.
\] (A.2)

The solution of above equation is

\[
f_{\sigma_1}^\alpha(x) = A_{\sigma_1}^\alpha e^{ikx} - A_{\sigma_1}^\alpha e^{-ikx},
\] (A.3)

with the eigenenergy

\[
E = 2 \cos k + \frac{9U}{4} (L - 2),
\] (A.4)

The periodic boundary condition governs the momentum to be

\[
k_j = \frac{2j\pi}{L}, \quad (j = 1, \ldots, L - 1).
\] (A.5)

For the two-particle state, the eigenstate is assumed as

\[
|\psi_2\rangle = \sum_{x_1,x_2=1}^{L} f_{\sigma_1\sigma_2}^{\alpha\beta}(x_1,x_2) E_{\sigma_1,x_1}^{\alpha\beta} E_{\sigma_2,x_2}^{\alpha\beta} |0\rangle.
\] (A.6)

From the Schrödinger equation (8) we have

\[
E f_{\sigma_1\sigma_2}^{\alpha\beta}(x_1,x_2) = f_{\sigma_1\sigma_2}^{\alpha\beta}(x_1 + 1,x_2) + f_{\sigma_1\sigma_2}^{\alpha\beta}(x_1 - 1,x_2) + \frac{9U}{4} (L - 4) f_{\sigma_1\sigma_2}^{\alpha\beta}(x_1,x_2)
+ f_{\sigma_1\sigma_2}^{\alpha\beta}(x_1,x_2 + 1) + f_{\sigma_1\sigma_2}^{\alpha\beta}(x_1,x_2 - 1), \quad x_1 \neq x_2,
\]

\[
E f_{\sigma_1\sigma_2}^{\alpha\beta}(x_1,x_1) = f_{\sigma_1\sigma_2}^{\alpha\beta}(x_1 + 1,x_1) + f_{\sigma_1\sigma_2}^{\alpha\beta}(x_1 - 1,x_1) + \frac{9U}{4} f_{\sigma_1\sigma_2}^{\alpha\beta}(x_1,x_1)
+ f_{\sigma_1\sigma_2}^{\alpha\beta}(x_1,x_1 + 1) + f_{\sigma_1\sigma_2}^{\alpha\beta}(x_1,x_1 - 1), \quad x_1 = x_2.
\] (A.7)
The fermionic property of particle requires the antisymmetry of the wave function. Therefore, we can assume the wave function $f_{\sigma_1 \sigma_2}^{\alpha_1 \alpha_2}(x_1, x_2)$ to be

$$f_{\sigma_1 \sigma_2}^{\alpha_1 \alpha_2} = \left\{ \begin{array}{ll}
A_{\alpha_1 \alpha_2}^{\sigma_1 \sigma_2}(k_1, k_2)e^{ik_1 x_1 + ik_2 x_2} - A_{\alpha_3 \alpha_2}^{\sigma_1 \sigma_2}(k_2, k_1)e^{ik_2 x_1 + ik_1 x_2}, & x_1 < x_2 \\
-A_{\alpha_2 \alpha_1}^{\sigma_2 \sigma_1}(k_1, k_2)e^{ik_1 x_2 + ik_2 x_1} + A_{\alpha_3 \alpha_1}^{\sigma_2 \sigma_1}(k_2, k_1)e^{ik_2 x_2 + ik_1 x_1}, & x_1 > x_2
\end{array} \right. \quad (A.8)$$

Substituting the above ansatz into the Eq. (A.7), we can obtain the eigenvalue of the Hamiltonian (1)

$$E = 2(\cos k_1 + \cos k_2) + \frac{9U}{4}(L - 4). \quad (A.9)$$

Imposing the continuous condition of the wave function

$$\lim_{x_2 \to x_1^+} f_{\sigma_1 \sigma_2}^{\alpha_1 \alpha_2}(x_1, x_2) = \lim_{x_1 \to x_2^-} f_{\sigma_1 \sigma_2}^{\alpha_1 \alpha_2}(x_1, x_2), \quad (A.10)$$

we have

$$A_{\alpha_1 \alpha_2}^{\sigma_1 \sigma_2}(k_1, k_2) = S_{\sigma_1 \sigma_2}^{\alpha_1 \alpha_2}(\sin k_1, \sin k_2)A_{\alpha_2 \alpha_1}^{\sigma_2 \sigma_1}(k_2, k_1), \quad (A.11)$$

where

$$S_{\sigma_1 \sigma_2}^{\alpha_1 \alpha_2}(\sin k_1, \sin k_2) = \frac{\sin k_1 - \sin k_2 + i\gamma P_{\sigma_1 \sigma_2}^{\alpha_1 \alpha_2}}{\sin k_1 - \sin k_2 + i\gamma} \quad (A.12)$$

is the scattering matrix of two particles with labels $(\sigma_1, \alpha_1)$ and $(\sigma_2, \alpha_2)$, and $\gamma = \frac{i\pi}{2}$, $P_{\sigma_1 \sigma_2}^{\alpha_1 \alpha_2}$ is the direct product of two kinds of permutation operators.

### 8 Appendix B:

Following the procedure of Ref. [17], the one-particle eigenstate is

$$|\Phi_1(p_1)\rangle = B(p_1) \cdot \mathcal{F}|0\rangle = B_a(p_1)\mathcal{F}^a|0\rangle. \quad (B.1)$$

where the repeated index means sum.

With the help of commutations (53)-(54) and the properties (20), (27), we find that the one-particle state satisfies the following relations

$$B(q)|\Phi_1(p_1)\rangle = \frac{\alpha_1(p_1, q)}{\alpha_2(p_1, q)}|\Phi_1(p_1)\rangle - \frac{\alpha_3(p_1, q)}{\alpha_2(p_1, q)}B(q) \cdot \mathcal{F}|0\rangle, \quad (B.2)$$

$$D(q)|\Phi_1(p_1)\rangle = \omega(q)|\Phi_1(p_1)\rangle, \quad (B.3)$$

$$\sum_{a=1}^{2} A_{aa}(q)|\Phi_1(p_1)\rangle = \omega(q)\frac{\alpha_1(q, p_1)}{\alpha_2(q, p_1)}e^{\alpha_1 b_1(q, p_1)b_1}|0\rangle - \omega(p_1)\frac{\alpha_3(q, p_1)}{\alpha_2(q, p_1)}B(q) \cdot \mathcal{F}|0\rangle. \quad (B.4)$$
The terms proportional to the eigenvector $|\Phi_1(p_1)\rangle$ denote the eigenvalue
\[
\Lambda(q, p_1) = \frac{\alpha_1(p_1, q)}{\alpha_2(p_1, q)} + \omega(q) + \sqrt{q} \frac{\alpha_1(q, p_1)}{\alpha_2(q, p_1)} \Lambda^{(1)}(q, p_1). \tag{B.5}
\]
where $\Lambda^{(1)}(q, p_1)$ is defined by the one-particle auxiliary eigenvalue problem
\[
T^{(1)}(q, p_1)\tilde{c}_{b_1} F^{b_1} = \tilde{r}^{b_1}(q, p_1) F^{b_1} = \Lambda^{(1)}(q, p_1) F^{c_1} \tag{B.6}
\]
The remaining ones will be eliminated by imposing the Bethe Ansatz equation
\[
\omega(p_1) = 1. \tag{B.7}
\]
The two-particle state should be a composition of two single hole excitations and a local hole pair. The former is made by tensoring two creating operators $B(q)$ while the latter represented by $F(q)$. Thus, we construct the two-particle vector as:
\[
\Phi_2(p_1, p_2) = B(p_1) \otimes B(p_2) + \xi F(p_1) B(p_2) \tilde{g}_0^{(0)}(p_1, p_2), \tag{B.8}
\]
where $\tilde{g}_0^{(0)}(p_1, p_2)$ is an arbitrary function to be determined. The vector $\xi$ plays the role of an “exclusion” principle, forbidding two same particles at the same site.

Putting the diagonal elements of transfer matrix on this state, we find that the diagonal operators acting on this state gives the following relations:

\[
B(q)|\Phi_2(p_1, p_2)\rangle = \prod_{i=1}^{2} \frac{\alpha_1(p_i, q)}{\alpha_2(p_i, q)} |\Phi_2(p_1, p_2)\rangle
\]
\[
- \sum_{i=1}^{2} \frac{\alpha_1(p_i, q)}{\alpha_2(p_i, q)} \prod_{k=1, k \neq i}^{2} \frac{\alpha_1(p_k, p_i)}{\alpha_2(p_k, p_i)} |\Psi_1^{(1)}(q, p_i, \{p_i\})\rangle \tag{B.10}
\]
\[
D(q)|\Phi_2(p_1, p_2)\rangle = \omega(q)|\Phi_2(p_1, p_2)\rangle, \tag{B.11}
\]
\[
\sum_{a=1}^{2} A_{aa}(q)|\Phi_2(p_1, p_2)\rangle = \omega(q) \prod_{i=1}^{2} \frac{\alpha_1(q, p_i)}{\alpha_2(q, p_i)} \Lambda^{(1)}(q, \{p_i\})|\Phi_2(p_1, p_2)\rangle
\]
\[
- \sum_{i=1}^{2} \omega(p_i) \frac{\alpha_1(q, p_i)}{\alpha_2(q, p_i)} \prod_{k=1, k \neq i}^{2} \frac{\alpha_1(p_k, p_i)}{\alpha_2(p_k, p_i)} \Lambda^{(1)}(p_i, \{p_i\}) |\Psi_1^{(1)}(q, p_i, \{p_i\})\rangle \tag{B.12}
\]
where we have used the two-particle auxiliary relation

$$T^{(1)}(q, \{p_i\})_{b_1b_2}F^{a_2a_1} = i^{c_{a_1}a_1}_{b_1d_1}(q, p_1)i^{d_{a_2}a_2}_{b_2c_1}(q, p_2)F^{a_2a_1} = \Lambda^{(1)}(q, \{p_i\})F^{b_2b_1},$$  \hspace{1cm} (B.13)

and

$$|\Psi^{(1)}(q, p_i, \{p_j\})\rangle = B(q) \otimes \Phi_1(p_k)\hat{O}^{(1)}_i(p_i, \{p_k\}) \cdot F|0\rangle, \quad k \neq i,$$  \hspace{1cm} (B.14)

The vanishing of unwanted terms gives us the Bethe ansatz equation

$$\frac{1}{\omega(p_i)} = \prod_{k=1, k \neq i}^{2} \frac{\alpha_2(p_k, p_i)}{\alpha_2(p_i, p_k)}\Lambda^{(1)}(p_i, \{p_j\}), \quad i = 1, 2$$  \hspace{1cm} (B.15)

Finally, From Eqs. (B.10)-(B.12) we can directly read the eigenvalue of two-particle state

$$\Lambda(q, \{p_i\}) = \prod_{i=1}^{2} \frac{\alpha_1(p_i, q)}{\alpha_2(p_i, q)} + \omega(q) + \omega(q) \prod_{i=1}^{2} \frac{\alpha_1(q, p_i)}{\alpha_2(q, p_i)}\Lambda^{(1)}(p_i, \{p_j\}).$$  \hspace{1cm} (B.16)

Before constructing the general multi-particle state, let us first analyse the symmetry of the two-particle state. From the above analysis, it is easy to prove the two-particle vector satisfying the following exchange property

$$\Phi_2(p_1, p_2) = \Phi_2(p_2, p_1) \cdot \hat{r}(p_1, p_2).$$  \hspace{1cm} (B.17)

In principle, such symmetrization mechanism can be implemented to any multi-particle state. This indeed help us to handle the problem of constructing a general multi-particle state ansatz. We construct the three-particle state as

$$\Phi_3(p_1, p_2, p_3) = B(p_1) \otimes B(p_2) \otimes B(p_3),$$  \hspace{1cm} (B.18)

satisfying

$$\Phi_3(p_1, p_2, p_3) = \Phi_3(p_1, p_3, p_2) \cdot \hat{r}_{2,3}(p_2, p_3).$$  \hspace{1cm} (B.19)

We also can rewrite the three-particle state in terms of the following recurrence relation:

$$\Phi_3(p_1, p_2, p_3) = B(p_1) \otimes \Phi_2(p_2, p_3).$$  \hspace{1cm} (B.20)

This expression is rather illuminating, because it suggests that a general \(N_1\)-particle state can be expressed in terms of the \(N_1 - 1\)-particle state via a recurrence relation. This recursive way not only help us to better simplify the wanted terms but also makes
it possible to gather the unwanted terms in rather closed forms. The action of diagonal operators on the three-particle state is

\[
B(q)\Phi_3(p_1, p_2, p_3) = \prod_{i=1}^{3} \frac{\alpha_1(p_i, q)}{\alpha_2(p_i, q)} \Phi_3(p_1, p_2, p_3) \\
- \sum_{i=1}^{3} \frac{\alpha_3(p_i, q)}{\alpha_2(p_i, q)} \prod_{k=1,k\neq i}^{3} \frac{\alpha_1(p_k, p_i)}{\alpha_2(p_k, p_i)} \\
\times |\Psi_2^{(1)}(q, p_i, \{p_l\})\rangle, \tag{B.21}
\]

\[
D(q)\Phi_3(p_1, p_2, p_3) = \omega(q)\Phi_3(p_1, p_2, p_3), \tag{B.22}
\]

\[
\sum_{a=1}^{2} A_{aa}(q)\Phi_3(p_1, p_2, p_3) = \omega(q)\prod_{i=1}^{3} \frac{\alpha_1(q, p_i)}{\alpha_2(q, p_i)} \Lambda^{(1)}(q, \{p_l\})\Phi_3(p_1, p_2, p_3) \\
- \sum_{i=1}^{3} \omega(p_i)\frac{\alpha_3(q, p_i)}{\alpha_2(q, p_i)} \prod_{k=1,k\neq i}^{3} \frac{\alpha_1(p_k, p_i)}{\alpha_2(p_k, p_i)} \Lambda^{(1)}(p_i, \{p_l\}) \\
\times |\Psi_2^{(1)}(q, p_i, \{p_l\})\rangle. \tag{B.23}
\]

In general, the knowledge of the 2-particle and the 3-particle results suggests the behaviour of the \(N_1\)-particle state. By using the mathematical induction we are able to determine the general structure for the multi-particle states to be Eq. (34).

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