Quantum synchronization due to information backflow

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The phase synchronization of a single qubit in a dissipative bath in the absence of driving field is demonstrated. Using the Husimi $Q$-function we show that the phase preference is present in the long time limit only during non-Markovian evolution with a finite detuning. This happens due to the information backflow signifying that non-Markovianity is a resource for quantum synchronization. To quantify synchronization we use the shifted phase distribution as well as its maximal value. From the maximal value of the shifted phase distribution we observe the signatures of quantum synchronization viz the Arnold tongue. In our case the region of synchronization is outside the tongue region and the region inside the tongue is the desynchronized region. This is in contrast to the results in the literature, where the synchronization is within the tongue region.

PACS numbers: 03.67.-a, 03.30+p, 03.67.Hk
Keywords: Transient synchronization, dissipative environment, $Q$-function, shifted phase distribution, Arnold tongue

I. INTRODUCTION:

Synchronization is the phenomenon [1] in which a self-sustained oscillator adjusts its rhythm to an external weak perturbation. It is present in a wide variety of classical systems like electronic circuits [2], biological neural networks [3, 4] and circadian rhythms [5–7] in living systems. All these systems need to have a stable limit cycle and also be connected to an energy source to sustain the oscillators indefinitely. The van der Pol oscillator is a well known system in which classical synchronization has been examined in detail [8]. An emerging area of research is the synchronization of finite dimensional quantum systems like spins and nonlinear oscillators, which has applications in the field of quantum computation and quantum information. Initially synchronization in the quantum formulation of van der Pol oscillator [9, 10] was studied. Here in the regions far from the ground state the quantum synchronization was found to be the same as classical synchronization of the system under the influence of noise [9]. In the close to the ground state regime this correspondence does not exist and so we need to investigate quantum synchronization of systems with fewer energy levels.

Recently quantum synchronization in finite dimensional systems has been intensively studied [10–12]. An investigation on the smallest possible system that can be synchronized [10] and the conditions under which such synchronization can happen has been carried out. This is relevant because the quantum processor, fundamental to quantum computation is a collection of interconnected qubits in contact with an external environment. Hence in a quantum processor, synchronization of a qubit is an additional feature experienced due to the presence of other qubits, as also the driving due to an external field [9, 10, 14] and the environment to which the qubit is exposed [15, 16]. Initially in Ref. [17, 18], qubits were suggested to be the smallest possible system that can be synchronized. But later works [10] claimed that synchronization cannot occur in dissipative two level systems. Subsequently it was proved that synchronization of a qubit to an external signal is possible [19–21]. Generally synchronization can be classified into forced synchronization and mutual synchronization [22, 23]. In the forced synchronization or entrainment, there is a driving due to an external field. The work in Ref. [12] discusses this feature.

In the absence of a driving field, synchronization may emerge as a collective phenomenon due to coherent dynamics and this is known as mutual synchronization. Mutual synchronization occurs due to the open system dynamics between qubits and an external environment. An investigation of quantum synchronization induced by the environment was carried out in Ref. [15, 16]. In Ref. [15], the authors considered two spin-1/2 systems in the framework of a collision model. Here they investigated the synchronization between the spins induced by the environment. This they refer to as spontaneous mutual synchronization since the synchronization between the coupled spins is influenced by their interaction with their respective environments. A slightly different models of two interacting qubits in which only one qubit is interacting with a dissipative environment was studied in Ref. [16]. Here the authors explore the relation between non-Markovianity and two qubit synchronization by considering a phenomenological Lindblad-type master equation as well as a collision model. The results show that the information backflow and synchronization have an inverse relationship which can be captured by a trade-off relation. This raises a question as to whether a single quantum system exposed to

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a quantum environment can experience synchronization
due to quantum dynamics. In this letter we examine this
question by investigating the synchronization in the time
dynamics of a two level system in an external dissipative
environment. Our results show that there is no synchro-
nization in the Markov limit due to the dynamics. In the
non-Markov regime, when there is a finite detuning
the two level systems exhibits mutual synchronization due to
its interaction with the external environment.

In this letter to show phase localization we use the
Husimi Q-function [9]. The amount of synchronization is
measured using the shifted phase distribution $S(\phi,t)$. Finally
we observe the characteristic feature of quantum
synchronization via the Arnold tongue, we use the maxi-
mum value of the shifted phase distribution. A detailed
summary of our results are given at the end of the letter.

II. SYSTEM AND THE Q-FUNCTION:

For our investigations, we consider an open quantum system composed of a two level system interacting with
a dissipative environment. The Hamiltonian of the qubit
and the environment composite system reads:

$$H = \hbar \omega_0 \sigma_+ \sigma_- + \sum_k \hbar \omega_k b_k^\dagger b_k + \sum_k \hbar \left( g_k \sigma_+ b_k + g_k^* \sigma_- b_k^\dagger \right),$$

where $\omega_0$ is the transition frequency of the qubit and
$\sigma_\pm$ are its raising and lowering operators. The bath is
a collection of infinite bosonic modes with creation and
annihilation operators $b_k^\dagger$ and $b_k$. The coupling strength
between the system and the $k^{th}$ mode of the environment
with frequency $\omega_k$ is $g_k$. At zero temperature this model
can be solved exactly. The dynamics of the qubit is given
by the time evolved reduced density matrix:

$$\rho(t) = \begin{pmatrix} \rho_{11}(0) |h(t)|^2 & \rho_{10}(0) h(t) \\ \rho_{01}(0) h^*(t) & 1 - \rho_{11}(0) |h(t)|^2 \end{pmatrix}. \quad (1)$$

Here $h(t)$ is the time evolution function obtained from
$h(t) = -\int_0^t d\tau f(t-\tau) h(\tau)$ with $f(t-\tau)$ being the two
time bath correlation function. This correlation function is
related to the spectral density $J(\omega)$ of the reservoir as
$f(t-\tau) = \int d\omega J(\omega) \exp(-i(\omega_0 - \omega)(t-\tau))$. The
Lorentzian spectral density we consider in our work and
the corresponding time evolution function $h(t)$ are:

$$J(\omega) = \frac{1}{2\pi} \frac{\gamma_0 \lambda^2}{(\omega_0 - \omega - \Delta)^2 + \lambda^2},$$

$$h(t) = e^{-\frac{(\lambda+i\Delta)}{4}} \left[ \cosh \left( \frac{\Omega t}{2} \right) + \frac{\lambda - i\Delta}{\Omega} \sinh \left( \frac{\Omega t}{2} \right) \right],$$

where $\lambda$ represents the spectral width of the reservoir, $\gamma_0$
is the coupling strength between the system and the bath
and is related to the decay rate of the excited state of the
qubit in the Markovian limit of a flat spectrum. The factor
$\Delta = \omega_0 - \omega_c$, with $\omega_c$ being the central frequency of the
Lorentzian spectrum and $\Omega = \sqrt{(\lambda - i\Delta)^2 - 2\gamma_0 \lambda}$. There
are two distinct type of dynamics based on the value of
the system-environment parameters: (a) When $\lambda > 2\gamma_0$, the
reservoir correlation time is very small compared to the
relaxation time of the qubit and the dynamics is Markovian. (b) For $\lambda < 2\gamma_0$, the reservoir correlation time is larger than or of the same order as the relaxation
time and we observe non-Markovian effects.

Figure 1: The Husimi Q-function $Q(\theta,\phi,t)$ of a single qubit
system in the initial state $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ in contact with
a dissipative bath is given in the plot for (a) the initial time
$\gamma_0 t = 0$, (b) time $\gamma_0 t = 10$ in a Markov process with $\lambda = 5\gamma_0$
and $\Delta = 0$, (c) time $\gamma_0 t = 10$ for non-Markov dynamics with
$\lambda = 0.01\gamma_0$ and $\Delta = 0$, and (d) long time limit of $\gamma_0 t = 500$,
for the non-Markov process with $\Delta = \gamma_0$ and $\lambda = 0.01\gamma_0$.

Figure 2: A plot of the off-diagonal element $|\rho_{10}(t)|$ as a function of $\gamma_0 t$
is given for (a) Markovian and (b) non-Markovian
evolution.

To characterize the phase synchronization, we use the
Husimi Q-function [9], a quasi-probability distribution
which helps to visualize the phase space of the qubit. For
our system the Q-function is

$$Q(\theta,\phi,t) = \frac{1}{2\pi} \langle \theta,\phi | \rho(t) | \theta,\phi \rangle \quad (2)$$

where $|\theta,\phi\rangle = \cos(\theta/2)|1\rangle + \sin(\theta/2) \exp(i\phi)|0\rangle$ are the
spin-coherent states. For the qubit system, these are the eigenstates of the spin operator $\sigma_\theta = \vec{n} \cdot \vec{\sigma}$ along the axis
of the unit vector $\vec{n}$, with polar co-ordinates $\theta$ and $\phi$. Since
the spin coherent states are pure states on the Bloch sphere parametrized by the angles $\theta$ and $\phi$, the
Q-function gives the weights of the different pure states
contributing to the density matrices. The explicit form of
the time dependent Husimi $Q$-function is

$$Q(\theta,\phi, t) = \frac{1}{2\pi} \left[ \cos^2(\theta/2)\rho_{11}(t) + \sin^2(\theta/2)\rho_{00}(t) + \sin(\theta/2)\cos(\theta/2)(e^{i\phi}\rho_{10}(t) + e^{-i\phi}\rho_{01}(t)) \right]$$

The transient dynamics of the Husimi $Q$-function is shown through the plots in Figure 1, for the initial state $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. For this single qubit state, the $Q$-distribution at $t = 0$ is shown in Fig. 1(a). Here we observe the quasi-probability distribution is peaked at $\phi = 0$, indicating a non-uniform distribution of the phase of the system. This non-uniformity implies that the two level system has a phase preference at $\phi = 0$ at $t = 0$. The time evolved $Q$-function in the Markovian limit ($\lambda > 2\gamma_0$) is described in the plot Fig. 1(b) when there is no detuning ($\Delta = 0$). Here we find that at $\gamma_0 t = 10$, the $Q$-function is uniformly distributed in the $\phi$-axis and the phase preference is completely wiped out. This lack of phase preference is also seen in the Markov limit of the quantum system with a finite detuning ($\Delta \neq 0$) and the corresponding results are shown in the Appendix in Figures 7(a) and 7(b).

Next we investigate the phase preference in the non-Markovian limit ($\lambda < 2\gamma_0$), for the single qubit system. For $\gamma_0 t = 10$, the time evolved $Q$-distribution of the qubit system with zero detuning ($\Delta = 0$) is given in Fig. 1(c). At this time, while the phase preference is present in the system, it is not the same as the phase preference of the initial state. This phase preference decreases with increase in the evolution time $\gamma_0 t$ and vanishes in the long time limit and the plots illustrating this is given in the Appendix Fig. 8(a). The Husimi $Q$-function for the qubit system with finite detuning ($\Delta \neq 0$) in the non-Markovian limit is given in Fig. 1(d). Here we observe a dynamical phase lock in the system, i.e., only a specific region of $\phi$ contributes to the $Q$-distribution. This phase synchronization survives in the long time limit and in Fig. 1(d) we see the phase lock for $\gamma_0 t = 500$, where the localized peak of the Husimi $Q$-function has shifted towards $\phi = \pi$. In the Fig. 8(b) given in the appendix we show a series of plots displaying the evolution of the localized peak of the $Q$-distribution function and the system being phase locked at different $\phi$ for different times. Thus in a non-Markovian evolution the quantum system is phase synchronized only when $\Delta \neq 0$.

To understand this we plot $|\rho_{10}(t)|$ as a function of $\gamma_0 t$ in Fig. 2, for both Markovian and non-Markovian evolution. For the Markovian evolution we observe from Fig. 2(a) that irrespective of the value of detuning, $|\rho_{10}(t)| \to 0$ as $t$ increases. Similarly for a non-Markovian dynamics when $\Delta = 0$, $|\rho_{10}(t)| \to 0$ in the long-time limit. Thus for the Markovian evolution with any value of detuning and non-Markovian evolution with zero detuning, the system attains a diagonal steady state in the long-time limit. Consequently, there is no phase localization due to the lack of coherence in the system [11]. In the case of non-Markovian evolution with non-zero detuning, $|\rho_{10}(t)|$ is finite even in the long time limit as shown in Fig. 2(b). Due to the presence of the off-diagonal elements (coherences) in the steady state, phase localization and consequently quantum phase synchronization occurs.

The phase locking of the qubit system happens due to the information backflow from the environment. We consider an environment which is a collection of quantum harmonic oscillators obeying bosonic commutation rule. Since both the system and the bath are quantum mechanical by nature, their interaction leads to the synchronization of the qubit system under certain conditions. This synchronization is an emergent feature due to the collective dynamics between the bath and the system is caused by information backflow from the bath to the system. The aspect of synchronization discussed in Ref. [15, 16] are quite different in their origin and features. Here there are two quantum systems, out of which one system is influenced by an external bath which in turn synchronizes the two qubits. The authors refer to it as spontaneous mutual synchronization and find that the information backflow delays the synchronization. These two types of synchronizations are quite different in their origin and features [23].

III. SYNCHRONIZATION MEASURE AND ARNOLD’S TONGUE:

The Husimi $Q$-function explicitly shows the phase preference in the system. By integrating over the angular variable $\theta$ we can determine the phase distribution $P(\phi, \rho)$ for a given state $\rho$. For a limit cycle state $\rho_0$, $P(\phi, \rho_0) = 1/2\pi$ indicating a uniform phase distribution. Hence the synchronization of the qubit system can be measured using the shifted phase distribution

$$S(\phi, t) = \int_0^{\pi} d\theta \sin \theta Q(\theta, \phi, t) - \frac{1}{2\pi}.$$  

This function is zero if and only if there is no phase preference in the system implying the absence of phase

| $\gamma_0 t$ | $\phi$ |
|-------------|--------|
| 1           | -5π/2  |
| 2           | 0      |
| 3           | π/2    |
| 4           | -π/2   |

Figure 3: For the Markovian dynamics we show a contour plot describing the shifted phase distribution $S(\phi, t)$ as a function of $\Delta$ and $\phi$ for (a) $\gamma_0 t = 1$, (b) $\gamma_0 t = 2$, (c) $\gamma_0 t = 5$ and (d) $\gamma_0 t = 30$. The spectral width $\lambda = 5\gamma_0$ for all the plots.
synchronization. Evaluating the integral over the angular variable $\theta$, we find

$$S(\varphi, t) = \frac{1}{8} [\rho_{10}(t) \exp(i\varphi) + \rho_{01}(t) \exp(-i\varphi)]. \quad (5)$$

To estimate quantum synchronization, we plot $S(\varphi, t)$ as a function of $\Delta$ and $\varphi$ for different values of time. The plots for the Markovian dynamics ($\lambda > 2\gamma$) for the time values $\gamma_0 t = 1$, $\gamma_0 t = 2$, $\gamma_0 t = 5$ and $\gamma_0 t = 30$ are shown through the plots in Fig. 3 (a), 3 (b), 3 (c), and 3 (d) respectively. From the series of plots in Fig. 3 (b) - 3 (d) we observe the formation of small region around $\Delta = 0$ with no phase synchronization. The size of this region is proportional to $\gamma_0 t$ and in the long time limit, there is no phase synchronization for physically acceptable values of detuning as can be seen in Fig. 3(d).

**Figure 4:** The contour plot of the shifted phase distribution $S(\varphi, t)$ as a function of $\Delta$ and $\varphi$ for the non-Markovian dynamics for $\lambda = 0.01\gamma_0$ is given for (a) $\gamma_0 t = 1$, (b) $\gamma_0 t = 20$, (c) $\gamma_0 t = 100$ and (d) $\gamma_0 t = 500$.

The non-Markovian dynamics of the phase synchronization is shown in Fig. 4(a)-(d) using the shifted phase distribution. The different evolution times considered in Fig. 4(a), 4(b), 4(c), and 4(d) are $\gamma_0 t = 1, 20, 100$ and 500 respectively. At $\gamma_0 t = 1$ (4 (a)), the shifted phase distribution is peaked around $\phi = 0$, implying a synchronization in that region. In the neighborhood of $\phi = \pm\pi/2$, the function $S(\varphi, t) = 0$ signifying a lack of synchronization. Beyond this region and close to $\phi = \pm\pi$ the shifted phase distribution $S(\varphi, t) = -1/8$ due to antisynchronization. This behavior is uniformly seen for all values of detuning. As the system evolves, we observe the emergence of a region with no synchronization ($S(\varphi, t) = 0$) around $\Delta = 0$ and this region transitions from being a line of negligible width in Fig. 4 (b) to a narrow band of finite width in the detuning parameter as seen in Fig. 4 (d). So we conclude that in a non-Markovian evolution phase synchronization happens only for finite value of detuning and for zero or negligible detuning there is no phase synchronization.

The maximum of the shifted phase distribution $S_m(t) = \max S(\varphi, t)$ can also be used to characterize phase synchronization. We plot $S_m(t)$ in Fig. 5, as a function of the detuning parameter ($\Delta$) and the system-bath coupling strength ($\gamma$). To vary the coupling strength, we consider the spectral density $J(\omega)$ with coupling parameter $\gamma$ which varies in units of $\gamma_0$. In Fig. 5 (a) we consider a spectral width $\lambda = 0.01\gamma_0$ and in Fig. 5 (b) we consider $\lambda = 0.1\gamma_0$ and in both instances we consider $\gamma_0 t = 500$. We observe the formation of a Arnold tongue in both the plots. But in contrast to the results observed in the literature [10–12, 25] we find there is no synchronization in the tongue region, rather it is present only in the region outside of the tongue like formation. The central line of the tongue region is at $\Delta = 0$ and the width of the tongue increases with the spectral width ($\lambda$). This implies that the region of synchronization increases with the non-Markovian behavior. So in our model, the Arnold tongue marks out the unsynchronized region. The quantum synchronization in the system is dependent on the parameters viz detuning ($\Delta$), the system-environment coupling strength ($\gamma$) and the spectral width ($\lambda$). In Fig. 5, we investigated the variation of $S_m(t)$ as a function of $\Delta$ and $\gamma$. To have a complete picture about the synchronization we also plot $S_m(t)$ as a function of $\Delta$ and $\lambda$ in Fig. 6 for different values of $\gamma_0 t$. Here too we observe a triangular unsynchronized region similar to an Arnold tongue, and outside this region the system is synchronized. From the plots Fig. 6 (a) and 6 (b) we can see that the unsynchronized region increases with time. From Fig. 6 (b) we realize that to have synchronization in the long time limit, we need to
choose the spectral width $\lambda$ to be small.

IV. CONCLUSION:

In this letter we show that quantum mutual synchronization can be induced by information backflow. For this we consider a two level system exposed to an external environment modelled by an infinite bosonic modes. We consider a Lorentzian spectrum for the bath and examine the system for quantum synchronization under both Markovian and non-Markovian dynamics. We use the Husimi $Q$-function to plot the phase space of the qubit and we notice that when the system is under Markovian dynamics the initial phase preference is lost and consequently there is no transient phase synchronization. Under non-Markovian evolution for finite detuning, there is a phase preference even in the long-time limit, signalling the presence of a phase synchronization whose occurrence is due to the coherent dynamics and the information backflow into the system. This synchronization is an emergent phenomenon due to the coherent dynamics. In earlier works [15, 16] it was established that information backflow prevents or delays the onset of synchronization between quantum systems. Here, the main focal point of these investigations [15, 16] was on how the environment influences the synchronization between two quantum systems. In our work, we show that the environment is capable of directly inducing phase synchronization in a qubit. Thus we demonstrate that the information backflow has a positive effect on the collective synchronization and this contradicts the role of information backflow on spontaneous mutual synchronization in a two qubit system as described in Ref. [15, 16].

To quantify synchronization we use the shifted phase distribution and plot it as a function of the detuning and phase. In the Markov case there is no synchronization for physically acceptable values of detuning. During the non-Markovian evolution we observe synchronized regions in the long-time limit for finite values of detuning. Finally, we plot the maximum of the shifted phase distribution in two different ways: (a) By varying the detuning and the interaction strength and (b) through a variation of detuning and the spectral density width. In both the cases we observe the formation of Arnold tongue but in our study the qubit system is synchronized in the region outside the tongue formation and desynchronized within the tongue region. This is in contrast to the studies of synchronization so far [10–12, 25] where the synchronization happens to be small.

V. ACKNOWLEDGEMENTS:

Md. Manirul Ali was supported by the Centre for Quantum Science and Technology, Chennai Institute of Technology, India, vide funding number CIT/CQST/2021/RD-007. Po-Wen Chen was supported by the Division of Physics, Institute of Nuclear Energy Research, Taiwan. Chandrashekar Radhakrishnan was supported in part by a seed grant from IIT Madras to the Centre for Quantum Information, Communication and Computing.

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We go to the interaction picture with respect to $B$ where state system-plus-environment state is given by the system and the Bosonic bath is described by the $k$th mode with frequency $\omega_k$. The creation and annihilation operators of the transition frequency $\omega_0$ pressed in terms of the spin raising and lowering operators $\sigma_+ = |1\rangle\langle 0|$ and $\sigma_- = |0\rangle\langle 1|$. The qubit has a transition frequency $\omega_0$ between the states $|0\rangle$ and $|1\rangle$. Then the Schrödinger equation for the total density matrix reads:

$$H = H_S + H_E + H_I, \quad (A1)$$

where the system Hamiltonian $H_S = \hbar \omega_0 \sigma_+ \sigma_-$ is expressed in terms of the spin raising and lowering operators $\sigma_+ = |1\rangle\langle 0|$ and $\sigma_- = |0\rangle\langle 1|$. The qubit has a transition frequency $\omega_0$ between the states $|0\rangle$ and $|1\rangle$. The environment Hamiltonian $H_E = \sum_k \hbar \omega_k b_k^\dagger b_k$ describes a collection of Bosonic modes, where $b_k^\dagger$ and $b_k$ are the corresponding creation and annihilation operators of the $k$th mode with frequency $\omega_k$. The interaction between the system and the Bosonic bath is described by the Hamiltonian

$$H_I = \sum_k \left(g_k \sigma_+ \otimes b_k + g_k^* \sigma_- \otimes b_k^\dagger\right). \quad (A2)$$

We go to the interaction picture with respect to $H_0 = H_S + H_E$, then the Schrödinger equation for the total system-plus-environment state is given by

$$\frac{d}{dt} |\Psi(t)\rangle = -i \hbar H_I(t) |\Psi(t)\rangle, \quad (A3)$$

where

$$H_I(t) = e^{i H_0 t} H_I e^{-i H_0 t}, \quad (A4)$$

$$= \hbar \left[\sigma_+ \otimes B(t) + \sigma_- \otimes B^\dagger(t)\right].$$

with $B(t) = \sum_k g_k e^{i(\omega_0 - \omega_k) t} b_k$. We start with an initial state $|\Psi(0)\rangle = (d_0|0\rangle + d_1|1\rangle) \otimes |0\rangle_E$, the environment is initially in the vacuum state $|0\rangle_E = |000\ldots 0\ldots 0\rangle$. The exact time evolution of $|\Psi(0)\rangle$ is given by $|1\rangle$

$$|\Psi(t)\rangle = d_0|0\rangle \otimes |0\rangle_E + d_1(t)|1\rangle \otimes |0\rangle_E$$

$$+ \sum_k d_k(t) |0\rangle \otimes |k\rangle_E, \quad (A5)$$

where $|k\rangle_E = b_k^\dagger|0\rangle_E = |000\ldots 1_k\ldots 0\rangle$ is the state with one photon only in the mode $k$. The time evolved state is confined to the subspace spanned by the vectors $|0\rangle \otimes |0\rangle_E$, $|1\rangle \otimes |0\rangle_E$, and $|0\rangle \otimes |k\rangle_E$. This is because the Schrödinger equation is generated by the Hamiltonian $H_I(t)$ that conserves the total particle number. The amplitudes $d_k(t)$ and $d_k(t)$ depend on time, while $d_0$ is constant in time because $H_I(t)|0\rangle \otimes |0\rangle_E = 0$. Substituting $|\Psi(t)\rangle$ from Eq. (A5) into the Schrödinger equation (A3), one can obtain

$$\frac{d}{dt} d_1(t) = -i \sum_k g_k e^{i(\omega_0 - \omega_k) t} d_k(t), \quad (A6a)$$

$$\frac{d}{dt} d_k(t) = -i g_k e^{-i(\omega_0 - \omega_k) t} d_1(t). \quad (A6b)$$

Integrating Eq. (A6b) with the initial condition $d_k(t) = 0$ at $t = 0$, we have

$$d_k(t) = -i g_k \int_0^t e^{-i(\omega_0 - \omega_k) \tau} d_1(\tau) d\tau. \quad (A7)$$

We substitute this solution for $d_k(t)$ in Eq. (A6a) which is expressed as an integrodifferential equation for $d_1(t)$ and the equation reads:

$$\frac{d}{dt} d_1(t) = -\sum_k \int_0^t |g_k|^2 e^{i(\omega_0 - \omega_k)(t-\tau)} d_1(\tau) d\tau, \quad (A8)$$

where $f(t-\tau) = \sum_k |g_k|^2 e^{i(\omega_0 - \omega_k)(t-\tau)}$ is the two-time correlation function $\langle 0|B(t)B^\dagger(\tau)|0\rangle_E$ of the reservoir. If the reservoir spectrum is continuous, the two-time correlation function can be expressed through the spectral density $J(\omega)$ of the environment as $f(t-\tau) = \int d\omega J(\omega) e^{i(\omega_0 - \omega)(t-\tau)}$. The correlation function $f(t-\tau)$ characterizes all the non-Markovian back-action effects between the system and the reservoir. Here $J(\omega) = \sum_k |g_k|^2 \delta(\omega - \omega_k) = P(\omega)|g(\omega)|^2$ and $P(\omega)$ is the density of states of the reservoir and $g(\omega)$ is the frequency-dependent coupling between the system and the bath. The spectral density $J(\omega)$ contains the information about the frequency distribution of environmental modes and also the coupling between the system and the environment. Then the reduced density operator of the system

$$\rho(t) = \operatorname{Tr}_E \{ |\Psi(t)\rangle\langle \Psi(t)| \}$$

is determined by the function $d_1(t)$ and is

$$\rho(t) = |d_1(t)|^2 |1\rangle\langle 1| + d_1(t)d_1^\dagger(t) |0\rangle\langle 0|$$

$$+ d_1^\dagger(t) d_0 |0\rangle\langle 1| + (1 - |d_1(t)|^2) |0\rangle\langle 0|. \quad (A10)$$

One can define the time evolution function $h(t)$ as the solution of the equation

$$\frac{d}{dt} h(t) = -\int_0^t d\tau f(t-\tau) h(\tau), \quad (A11)$$

with the initial value $h(0) = 1$. Then the amplitude $d_1(t) = h(t)d_1(0)$ and the time dependent density matrix elements for the qubit are given by

$$\rho_{11}(t) = \langle 1|\rho(t)|1\rangle = |d_1(t)|^2 = |h(t)|^2 |d_1(0)|^2, \quad (A12)$$

$$\rho_{00}(t) = \langle 0|\rho(t)|0\rangle = d_1^\dagger(t) d_0 = h^*(t) d_1^\dagger(0) d_0,$$

$$\rho_{01}(t) = \langle 0|\rho(t)|1\rangle = d_1^\dagger(t) d_0 = h^*(t) d_1^\dagger(0) d_0,$$

$$\rho_{10}(t) = \langle 1|\rho(t)|0\rangle = (1 - |d_1(t)|^2) = 1 - |h(t)|^2 |d_1(0)|^2.$$
Hence the dissipative dynamics of the qubit is described by the time evolved reduced density matrix

$$\rho(t) = \begin{pmatrix} \rho_{11}(0) |h(t)|^2 & \rho_{10}(0) h(t) \\ \rho_{01}(0) h^*(t) & 1 - \rho_{11}(0) |h(t)|^2 \end{pmatrix},$$

(A13)

where $\rho_{11}(0) = |d_1(0)|^2$, $\rho_{10}(0) = d_1(0)d_0^*$, and $\rho_{01}(0) = d_1^*(0)d_0$. The above model of dissipation for a two-level quantum system is used in the context of non-Markovian dynamics [2–5] of open quantum systems. This exactly soluble model is also applied to investigate the measure of non-Markovianity [6–8] for two-level open quantum systems.

**Appendix B: Husimi distribution**

To visualize and characterize the phase synchronization behavior of a two-level quantum system, we use the Husimi $Q$ distribution [9, 25] defined by

$$Q(\theta, \phi, t) = \frac{1}{2\pi} \langle \rho(t)|\theta,\phi \rangle,$$

(B1)

where the spin-coherent states $|\theta, \phi\rangle$ are the eigenstates of the spin operator $\sigma_\alpha = \hat{n}_\alpha \hat{\sigma}$ along the unit vector $\hat{n}$ defined by the polar coordinates $\theta$ and $\phi$. These are the pure states $|\theta,\phi\rangle = \cos(\theta/2)|1\rangle + \sin(\theta/2) \exp(i\phi)|0\rangle$, representing a point on the Bloch sphere. Once the time evolved density matrix elements are determined, it is easy to obtain the time dynamics of $Q$ distribution as a function of $\theta$, $\phi$ and $t$ as follows:

$$Q(\theta, \phi, t) = \frac{1}{2\pi} \text{Tr} (|\theta, \phi\rangle\langle \theta, \phi| \rho(t)) = \frac{1}{2\pi} \text{Tr} \left[ \begin{pmatrix} \cos^2(\theta/2) & \sin(\theta/2) \cos(\theta/2) e^{i\phi} \\ \sin(\theta/2) \cos(\theta/2) e^{-i\phi} & \sin^2(\theta/2) \end{pmatrix} \begin{pmatrix} \rho_{11}(t) & \rho_{10}(t) \\ \rho_{01}(t) & \rho_{00}(t) \end{pmatrix} \right],$$

$$= \frac{1}{2\pi} \left[ \cos^2(\theta/2) \rho_{11}(t) + \sin(\theta/2) \cos(\theta/2) e^{-i\phi} \rho_{01}(t) + \sin(\theta/2) \cos(\theta/2) e^{i\phi} \rho_{10}(t) + \sin^2(\theta/2) \rho_{00}(t) \right].$$

(B2)

We start with an initial spin-coherent state corresponding to $\theta = \pi/2$ and $\phi = 0$. Figs. 7 and 8 show the time-dependent transient dynamics of $Q(\theta, \phi, t)$ for Markov and non-Markov evolution respectively. In Fig. 7, the Husimi $Q$ function is shown in the Markov regime ($\lambda > 2\gamma_0$) for two different values of the detuning (a) $\Delta = 0$ and (b) $\Delta = \gamma_0$ respectively. For zero detuning (Fig. 7) we see that initially at time $t = 0$, $Q$ function is nonuniform in phase and the distribution is peaked at $\phi = 0$. As time passes this phase preference of the Husimi distribution diminishes and is eventually wiped out at a time $\gamma_0 t = 10$, after which the $Q$ function becomes uniformly distributed along $\phi$-axis. Next in Fig. 7, we consider a nonzero finite value of the detuning $\Delta = \gamma_0$. Even for this nonzero value of detuning (Fig. 7), the transient dynamics of $Q$ distribution is quite similar to that of zero detuning case as long as the reservoir parameters are set in the Markov regime. The phase information of the state smears out quickly after a short interval of the system-environment interaction.

The non-Markovian counterpart of the dynamics is shown in Fig. 8 for the reservoir parameter with $\lambda < 2\gamma_0$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Figure7.png}
\caption{Temporal evolution of the Husimi distribution function $Q(\theta, \phi, t)$ is shown in the Markov regime ($\lambda > 2\gamma_0$) for (a) $\Delta = 0$ and (b) $\Delta = \gamma_0$. We have taken the spectral width $\lambda = 5\gamma_0$.}
\end{figure}
For zero detuning ($\Delta = 0$) we still observe a loss of phase information as time progresses, although this loss of phase preference in $Q$ function is delayed or slowed down (Fig. 8 (a)) here in the non-Markov regime. Finally, the nonuniform phase preference is ultimately lost and the $Q$ distribution gets delocalized with respect to $\phi$ at $\gamma_0 t = 200$. The situation is dramatically changed in the nonzero detuning ($\Delta \neq 0$). In Fig. 8 (b), we show the non-Markov dynamics of the $Q$ function for a nonzero finite value of the detuning. In the non-Markov regime ($\lambda < 2\gamma_0$) with nonzero detuning ($\Delta = \gamma_0$), we see a dynamical phase-locking for the two level system. The qubit is phase locked in the sense that the $Q$ distribution is mostly contributed from a specific $\phi$ region. The localized peak of the Husimi $Q$ distribution gradually shifts from $\phi = 0$ towards $\phi = \pi$ as time progress. Hence in the non-Markov regime for this finite detuning case ($\Delta = \gamma_0$), we observe here a transient phase-synchronization characterized by a single peak $Q$-distribution.

### Appendix C: synchronization measure and Arnold’s tongue

We have identified phase synchronization for a two-level quantum system through a phase-space quasiprobability distribution, called Husimi $Q$-function. Husimi $Q$-function precisely demonstrates the synchronization in the qubit system. Moreover, we adopt [10–12] a measure of synchronization to characterize the strength of the phase preference. Integrating the $Q$-function over the angular variable $\theta$ one can obtain a phase distribution $P(\phi, \rho)$ for a given state $\rho$. Corresponding to a limit cycle state, the phase distribution $P(\phi, \rho) = 1/2\pi$ that indicates a uniform phase distribution. Then the synchronization of the two-level quantum system can be measured using the shifted phase distribution

$$S(\phi, t) = \int_0^\pi d\theta \sin \theta Q(\theta, \phi, t) - \frac{1}{2\pi}.$$  \hfill (C1)

This function is zero in the absence of synchronization when there is no phase preference in the system. A positive maximum of this measure indicates a complete phase locking, while a negative maximum implies the anti-synchronization. Substituting $Q(\theta, \phi, t)$ from Eq. (B2) in Eq. (C1), we evaluate the integral over the angular variable $\theta$

$$S(\phi, t) = \frac{1}{2\pi} \left[ \rho_{11}(t) \int_0^\pi \cos^2(\theta/2) \sin \theta d\theta - 1 \right. \hfill (C2)$$
$$+ \rho_{00}(t) \int_0^\pi \sin^2(\theta/2) \sin \theta d\theta$$
$$+ \rho_{10}(t) e^{i\phi} \int_0^\pi \sin(\theta/2) \cos(\theta/2) \sin \theta$$
$$+ \rho_{01}(t) e^{-i\phi} \int_0^\pi \sin(\theta/2) \cos(\theta/2) \sin \theta \right].$$

By performing the above integral and using the trace preserving property $\rho_{11}(t) + \rho_{00}(t) = 1$, we have

$$S(\phi, t) = \frac{1}{8} \left[ \rho_{10}(t)e^{i\phi} + \rho_{01}(t)e^{-i\phi} \right].$$ \hfill (C3)

To estimate quantum synchronization, we plot $S(\phi, t)$ as a function of $\Delta$ and $\phi$ for different values of time (see Figure 3 and Figure 4 of the main manuscript). The plots for the Markovian ($\lambda > 2\gamma_0$) dynamics of $S(\phi, t)$ are shown in Figure 3 of the main manuscript, where we observe the formation of small region around $\Delta = 0$ with no phase synchronization. The size of this region is proportional to $\gamma_0 t$ and in the long time limit, there is no phase synchronization for physically acceptable values of detuning. The non-Markovian dynamics (for $\lambda < 2\gamma_0$) of the phase synchronization is also shown (see Figure 4 of the manuscript).
using this shifted phase distribution $S(\phi, t)$. We see that quantum phase synchronization region is enhanced in the non-Markovian regime, the function achieves a maximum value ($S(\phi, t) = 1/8$) if one avoids a narrow band region around $(\Delta = 0)$ just by introducing a small amount of detuning. The narrow band region in $S(\phi, t)$ around $\Delta = 0$ signifies a no-synchronization ($S(\phi, t) = 0$) zone. We also observed anti-synchronization ($S(\phi, t) = -1/8$) region for some specific range of values of $\phi$. So we conclude that for the Markov evolution there is no synchronization in the long-time limit for any value of detuning. For non-Markovian evolution phase synchronization happens only for finite value of detuning and for zero or negligible detuning there is no phase synchronization.

The maximum of the shifted phase distribution $S_m(t) = \max S(\phi, t)$ also provides an alternative measure of phase synchronization. The maximum value of $S(\phi, t)$ can be calculated as follows. From Eq. (C3), we have

$$S(\phi, t) = \frac{1}{4} \Re \left[ \rho_{10}(t) e^{i\phi} \right]. \quad (C4)$$

The density matrix element $\rho_{10}(t)$ is in general a complex number that can be expressed as $\rho_{10}(t) = r(t) e^{i\theta(t)}$, where $r(t) = \sqrt{x^2(t) + y^2(t)}$ and $\theta(t) = \tan^{-1}(y(t)/x(t))$. Here $x(t) = \Re \left[ \rho_{10}(t) \right]$ and $y(t) = \Im \left[ \rho_{10}(t) \right]$. Then from Eq. (C4), we have

$$S(\phi, t) = \frac{1}{4} \Re \left[ r(t) e^{i\theta(t)} e^{i\phi} \right] = \frac{1}{4} r(t) \cos \left( \theta(t) + \phi \right) \quad (C5)$$

To find the maximum value of $S(\phi, t)$ with respect to the variable $\phi$

$$\frac{\partial S}{\partial \phi} = -\frac{1}{4} r(t) \sin \left( \theta(t) + \phi \right) = 0. \quad (C6)$$

Hence the function $S(\phi, t)$ will have a maximum or minimum value at $\theta(t) + \phi = n\pi$, where $n$ is an integer. One can easily check that the function $S(\phi, t)$ has a maximum at $\phi = -\theta(t)$ for which the second derivative

$$\frac{\partial^2 S}{\partial \phi^2} \bigg|_{\phi=-\theta(t)} = -\frac{1}{4} r(t) \cos \left( \theta(t) + \phi \right) \bigg|_{\phi=-\theta(t)},$$

$$= -\frac{r(t)}{4}, \quad (C7)$$

is negative for any finite nonzero value of $r(t)$. The maximum value of $S(\phi, t)$ at $\phi = -\theta(t)$ is then given by

$$\max S(\phi, t) = \frac{1}{4} r(t) \cos \left( \theta(t) + \phi \right) \bigg|_{\phi=-\theta(t)},$$

$$= \frac{r(t)}{4} = \frac{1}{4} \sqrt{x^2(t) + y^2(t)}. \quad (C8)$$

We show here in Fig. 9 the maximum value of $S(\phi, t)$ as a function of the coupling strength $\gamma$ and the detuning parameter $\Delta$ with varying spectral width $\lambda$. To vary the coupling strength, we consider the spectral density $J(\omega)$ with coupling strength $\gamma$ which varies in units of $\gamma_0$. In Fig. 9, we consider four different values of the spectral width (a) $\lambda = 0.01\gamma_0$, (b) $\lambda = 0.05\gamma_0$, (c) $\lambda = 0.1\gamma_0$, and (d) $\lambda = 0.2\gamma_0$.

We consider $\gamma_0 t = 500$ for all the plots.

![Figure 9](image_url)

**Figure 9:** This plot describes the maximal value of the shifted phase distribution $S_m(t)$ as a function of the detuning parameter $\Delta$ and coupling strength $\gamma$ for the spectral widths (a) $\lambda = 0.01\gamma_0$, (b) $\lambda = 0.05\gamma_0$, (c) $\lambda = 0.1\gamma_0$, and (d) $\lambda = 0.2\gamma_0$. We have taken $\gamma_0 t = 500$ for all the plots.

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