SPECTRAL DETERMINATION OF ANALYTIC BI-AXISYMMETRIC PLANE DOMAINS

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Abstract. Let $\mathcal{D}$ denote the class of bounded real analytic plane domains with the symmetry of an ellipse. Under generic conditions, we prove that if $\Omega_1, \Omega_2 \in \mathcal{D}$ and if the Dirichlet spectra coincide, $\text{Spec}(\Omega_1) = \text{Spec}(\Omega_2)$, then $\Omega_1 = \Omega_2$ up to rigid motion.

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1. Introduction

In this paper we give a positive solution to the inverse spectral problem for the class $\mathcal{D}$ of analytic axi-symmetric plane domains $\Omega$ satisfying:

\begin{itemize}
  \item $\Omega$ is real analytic
  \item $\Omega$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$ -- symmetric
  \item at least one axis of $\Omega$ is a non-degenerate bouncing ball orbit $\gamma$
  \item $L_\gamma$ has multiplicity one in $\text{Lsp}(\Omega)$.
\end{itemize}

Let $\text{Spec}(\Omega)$ denote the spectrum of the Laplacian $\Delta_\Omega$ of the domain $\Omega$ with Dirichlet boundary conditions. Our main result is the following theorem (announced in [Z.5]):

**Theorem 1.1.** $\text{Spec}: \mathcal{D} \mapsto \mathbb{R}^N_+$ is 1-1.

Let us clarify the assumptions. The symmetry assumption is that there is an up/down reflection symmetry across a horizontal axis and a left/right reflection symmetry across a vertical axis. Both axes intersect the boundary at right angles, hence are projections to $\Omega$ of ‘bouncing ball orbits’ of the billiard flow $G^t$ on $T^*\Omega$, with the usual law of reflection at the boundary. Associated to any periodic reflecting ray $\beta$ of $G^t$ is its Poincare map $P_\beta$, defined as the first return map on a local transversal to $\beta$ in $S^*\Omega$, and its linear Poincare map $P_\beta = d_{\beta(0)}P_\beta$. A periodic orbit $\beta$ is said to be non-degenerate if no eigenvalue of $P_\beta$ is a root of unity. Thus, it is non-degenerate elliptic if the eigenvalues of $P_\beta$ have the form $\{e^{\pm i\alpha}\}$ with $\alpha/\pi \notin \mathbb{Q}$, or non-degenerate hyperbolic if they have the form $\{e^{\pm \lambda}\}$ with $\lambda \neq 0$. Our non-degeneracy assumption is that at least one of the axes, which we will denote by $\overline{AB}$, is the projection to $\Omega$ of a non-degenerate bouncing ball orbit $\gamma$. Also, $\text{Lsp}(\Omega)$ denotes the length spectrum of $\Omega$, i.e. the set of lengths of periodic billiard trajectories (including the boundary). Our length spectrum assumption is that $\gamma$ is of multiplicity one in $\text{Lsp}(\Omega)$, i.e. that it is the unique trajectory of its length $L_\gamma$. With no loss of generality we will assume $\overline{AB}$ is the vertical axis, and will denote its length by $L$; thus $L_\gamma = 2L$.

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The proof of Theorem (1.1) is based on the method of normal forms, which was introduced into inverse spectral theory by Colin de Verdiere [CV] and by Guillemin [G]. The basic idea of Guillemin [G] is to construct a quantum analogue of the Birkhoff normal form of $\Delta$ $\Omega$ around each closed geodesic and to prove that the coefficients $B_{\gamma k}$ of the normal form are spectral invariants. The latter is accomplished by relating these coefficients $B_{\gamma k}$ to the wave invariants $a_{\gamma k}$ of $\Delta$ $\Omega$ at $\gamma$, i.e. the coefficients of singularity expansion of the trace of the wave group $U(t)$ at the time $t = L_\gamma$. In [C], Guillemin constructed the normal form and proved that it is a spectral invariant in the case of non-degenerate elliptic closed geodesics on boundaryless manifolds. A somewhat different construction of the normal form and a new proof that the coefficients are spectral invariants was given in [Z.1] [Z.2] for general non-degenerate closed geodesics. The inverse spectral problem is then reduced to determining the metric from the normal form.

The latter inverse problem remains difficult since the different closed geodesics do not easily ‘communicate’ with each other and since the $B_{\gamma k}$’s for a fixed $\gamma$ (and its iterates) do not appear to give enough information to determine the metric, even locally. Therefore it is natural to consider the problem first for classes of real analytic metrics with one functional degree of freedom, where it is plausible that the normal form coefficients at just one closed geodesic should determine the metric. This motivated our study in [Z.3] of real analytic ‘simple’ surfaces of revolution. We proved there that a simple analytic surface of revolution is determined by the normal form coefficients at just one closed geodesic should determine the metric. This should imply that the formal function $\hat{\Delta}$ $\Omega$ around each closed geodesic and to prove that the coefficients $B_{\gamma k}$ of the normal form are spectral invariants. The latter is accomplished by relating these coefficients $B_{\gamma k}$ to the wave invariants $a_{\gamma k}$ of $\Delta$ $\Omega$ at $\gamma$, i.e. the coefficients of singularity expansion of the trace of the wave group $U(t)$ at the time $t = L_\gamma$. In [C], Guillemin constructed the normal form and proved that it is a spectral invariant in the case of non-degenerate elliptic closed geodesics on boundaryless manifolds. A somewhat different construction of the normal form and a new proof that the coefficients are spectral invariants was given in [Z.1] [Z.2] for general non-degenerate closed geodesics. The inverse spectral problem is then reduced to determining the metric from the normal form.

In this paper, we extend the method of normal forms to the case of bounded analytic plane domains with a non-degenerate bouncing ball orbit $\gamma$. We do not use the symmetry assumption on the domain in the construction of the normal form at $\gamma$, but only in the last step of deducing the domain from the normal form. Since the construction of the normal form and the various technical pitfalls may make the proof difficult to follow, let us give a brief summary here of the main ideas. See also [Z.3] for a somewhat less technical outline.

As mentioned above, the main idea is to introduce a notion of Birkhoff normal form $F(|D|, \hat{I})^2$ for $\Delta$ at a bouncing ball orbit $\gamma$. The normal form lives on the model space $\Omega_s := |0, L| \times \mathbb{R}_y$, which carries a natural abelian algebra of pseudodifferential operators $A = |D_s|, \hat{I}$, where: $|D| = |D_s| = \sqrt{-D_s^2}$ with Dirichlet boundary conditions on $[0, L]$ and where $\hat{I}$ is a quantum action operator. Here, $D_s = \frac{1}{2} \frac{\partial}{\partial y}$. In the elliptic case, $\hat{I} = \frac{1}{2}(|D|^{-1} D_s^2 + |D| y^2)$ is the ‘transverse’ homogeneous harmonic oscillator, while in the hyperbolic case $\hat{I} = \frac{1}{2}(|D|^{-1} D_s^2 - |D| y^2)$. By definition, the normal form of $\sqrt{\Delta}$ is a first order polyhomogeneous symbol

$$F(|D|, \hat{I}) \sim |D| + \frac{\alpha}{L} \hat{I} + \frac{p_1(\hat{I})}{|D|} + \frac{p_2(\hat{I})}{|D|^2} + \ldots$$

in $|D|, \hat{I}$ with $p_j$ a polynomial of degree $j + 1$. The coefficients $B_{\gamma k}$ mentioned above are the coefficients of these polynomials (cf. [G] [Z.3] [Z.4]).

Heuristically, $\sqrt{\Delta}$ should be microlocally conjugate to $F(|D|, \hat{I})$ near corresponding bouncing ball orbits. This should imply that $\sqrt{\Delta}$ and $F(|D|, \hat{I})$ have the same wave invariants at iterates of their corresponding bouncing ball orbits, and consequently that the formal function $F$ is a spectral invariant. As may be anticipated the boundary gives rise to many complications and we cannot quite implement this
outline. In fact, we work as much as possible in the open space containing the bounded domain. The normal form is actually used to define a parametrix for the Dirichlet wave kernel near $\gamma$.

1.1. **Outline of the proof.** The main contribution of this paper is the construction of a normal form $F(|D|, \hat{I})$ of $\Delta$ around a bouncing ball orbit $\gamma$, and the proof that the normal form is a spectral invariant. Here are the main steps.

1.1.1. **Straightening the domain.** To conjugate to the normal form, we first use a special map $\Phi$ introduced by Lazutkin in [Lazutkin] to ‘straighten the domain’ near $\overline{AB}$, i.e. to carry an open neighborhood of $\gamma$ in $\Omega$ to an open neighborhood of $[0,L] \times \{0\}$ in $\Omega_0$. In fact, $\Phi$ will be defined in a neighborhood in $\mathbb{R}^2$ of $\overline{AB}$. In the elliptic case, Lazutkin’s map additionally puts the metric into the normal form $ds^2 + b(s,y)[y^2 ds^2 + dy^2]$, or equivalently the Laplacian has the normal form $D_s^2 + B(s,y)(y^2D_s^2 + D_y^2) + \text{LOT}$ (lower order terms). We will modify Lazutkin’s construction so that in the hyperbolic case the Laplacian has the normal form $D_s^2 + B(s,y)(-y^2D_s^2 + D_y^2) + \text{LOT}$. We do not use the metric normal form in this paper beyond the quadratic term and we do not really need to define the straightening map by Lazutkin’s method. We do so anyway because there is no advantage to constructing another map and because we believe the details of Lazutkin’s construction could be useful in the general inverse problem.

The straightening map carries $\Delta$ to a variable coefficient Laplacian $\Delta$ in a neighborhood of $[0,L] \times \{0\}$ in $\mathbb{R}^2$. A given wave invariant depends only on a certain germ of the Laplacian at the orbit, and consequently only on a certain germ of the straightening. So with no loss of generality we may assume that $\Delta$ is a polynomial differential operator in the $\dot{y}$ variable with analytic coefficients in $s$. We define the Dirichlet Laplacian $\Delta_{\Omega_0}$ to be this operator with Dirichlet boundary conditions on $\partial\Omega_0$.

1.1.2. **Conjugation to normal form.** The next step is roughly to conjugate $\Delta_{\Omega_0}$ to a microlocal normal form near $\gamma_0 := [0,L] \times \{0\}$ by an FIO (Fourier integral operator) on the model space. Intuitively, we would like to construct a microlocally invertible FIO $W$ which ‘preserves Dirichlet boundary conditions’ and such that $W^{-1}\Delta W \sim F(|D|, \hat{I})^2$ modulo an ‘acceptable remainder’ near $\gamma_0$.

Let us be more precise. First, by ‘preserving Dirichlet boundary conditions’ we mean that $W$ should carry the domain of the Dirichlet Laplacian $\Delta$ as an unbounded operator on $\Omega_0$ to the domain of $F(|D|, \hat{I})^2$ as an unbounded operator on $\Omega_0$. Most significantly, $Wu = 0$ on $\partial\Omega_0$ if $u \in H^1_0(\Omega_0)$.

Second, let us be more precise about the sense in which we are conjugating $\Delta$ to $F(|D|, \hat{I})$. It is technically complicated to conjugate a boundary value problem by Fourier integral operator methods, so we do something simpler which is sufficient for the proof of our theorem. Namely we observe that both the Dirichlet wave operator $\cos t\sqrt{\Delta_{\Omega_0}}$ and the normal form wave operator $\cos tF(|D|, \hat{I})$ are restrictions to their domains of well-defined Fourier integral operators in microlocal neighborhoods in the open space of the bouncing ball orbits. We can use the intertwining operator $W$ on the open space to ‘pull back’ $\cos tF(|D|, \hat{I})$ (or more precisely its ‘odd part’) to a parametrix for $\cos t\sqrt{\Delta_{\Omega_0}}$ in the interior of $\Omega_0$. It is only in this weak sense that we conjugate the wave group to normal form. Since the wave trace $Tr \cos t\sqrt{\Delta}$ at $\gamma$ involves the wave kernel only in the interior, we can compute it in terms of this parametrix and hence in terms of the normal form.
Our goal then is the construction of an intertwining operator to normal form in this weak sense and the characterization of the error term in the conjugation. There are two main issues we would like to emphasize in this introduction. The first has to do with solvability of the conjugation equations. Those familiar with the conjugation to normal form in the boundaryless case will recall that (just as in the classical conjugation to Birkhoff normal form) it is based on solving a sequence of homological equations \[ Q, |D|^2 + \frac{\partial}{\partial t} = \text{KNOWN} \] for the infinitesimal intertwining operator \( Q \). This is a first order equation for the symbol of \( Q \) and it may seem mysterious that one can solve these equations with two boundary conditions (one at each boundary component). The second point to explain is the relevant notion of ‘acceptable remainder’.

1.1.3. Acceptable remainders. To clarify the second of these points, we recall that the link between the spectrum and normal form is through the coefficients in the singularity expansion

\[
\text{Tr} U(t) = c_\gamma (t - L_\gamma + i0)^{-1} + a_{i0} \log(t - L_\gamma + i0) + \sum_{k=1}^{\infty} a_{\gamma k} (t - L_\gamma + i0)^k \log(t - L_\gamma + i0)
\]

of the trace of the wave group \( U(t) = \exp(it\sqrt{\Delta}) \) at \( t = L_\gamma \). When \( L_\gamma \) is the length of a bouncing ball orbit (or in general a periodic reflecting ray), the wave trace expansion is very similar to the boundaryless case in that the singularity is Lagrangean and the coefficients may be calculated by the stationary phase method (see Corollary (2.5)). In another language (cf. Corollary (2.6)), the wave invariants \( a_{\gamma k} \) are non-commutative residues \( \text{res}(\frac{d}{dt}U(t))|_{t=L_\gamma} \) of the wave group and its time derivatives. As already proved in [GZ], it follows that only a certain amount of data from the Taylor expansion of the symbol of \( \Delta \) along \( \gamma \) goes into a given wave invariant \( \gamma \). The precise statement is that \( a_{\gamma k} \) depends only on the class of \( \sigma_\Delta \mod{S^{2,2(k+2)}(V, \mathbb{R}\gamma)} \) of Boutet de Monvel [BM]. Here, \( V \) is a conic neighborhood of the symplectic cone \( \mathbb{R}\gamma \). The bigrading of symbols is in terms of symbol order and order of vanishing along \( \gamma \) (cf. §2). Terms of low symbolic order or of high vanishing order along \( \gamma \) do not contribute to \( a_{\gamma k} \) (cf. Proposition (2.7)). Thus we need to construct an FIO \( W \) which preserves Dirichlet boundary conditions and which conjugates \( \Delta_{\Omega_0} \rightarrow F(|D|, \hat{I})^2 \) modulo a remainder in \( S^{2,2(k+2)} \).

1.1.4. Reduction to a semiclassical problem. To do this, we convert the problem to a semiclassical conjugation problem as in [Z1]. With the proper semiclassical scaling of symbols, elements of \( S^{2,K} \) are detected by their coefficient in the semiclassical parameter \( N^{-1} \). The goal then is to construct a semiclassical intertwining operator \( W_N \) preserving Dirichlet boundary conditions and conjugating the scaled version of \( \Delta \) to normal form modulo a sufficiently high power of \( N^{-1} \).

As in [GZ] and elsewhere, we construct \( W_N \) as a product \( W_N = \prod_{j=1}^{\infty} e^{N^{-j}(P+iQ)_{j/2}} \) as a product of elliptic semiclassical pseudodifferential operators. The exponents will just be Weyl pseudodifferential operators \( (P+iQ)_{j/2}(s,y,D_y) \) on the transverse space \( \mathbb{R} \); here, \( P, Q \) are assumed to have real-valued symbols. The real part \( P \) is of two lower orders in \( (y,D_y) \) than is \( Q \). As mentioned above, the condition that \( W_N \) intertwines to normal form translates into homological equations for the exponents. The reason why we can solve these equations while preserving Dirichlet
boundary conditions is that the boundary condition only affects the ‘odd terms’ in $P_{j/2}$ and the ‘even terms’ in $Q_{j/2}$ with respect to the involution $(y, \eta) \rightarrow (y, -\eta)$. The equations for the even/odd parts of $P_{j/2}, Q_{j/2}$ are coupled except for ‘diagonal’ terms which are powers of $\hat{I}$. Hence one can eliminate the ‘non-diagonal’ even parts to get second order ordinary differential equations for the odd parts. The boundary problem for these can be solved in the hyperbolic case, and in the elliptic case they can be solved as long as $\alpha$ is independent of $\pi$ over $Q$. The remaining powers of $\hat{I}$ constitute the terms in the normal form.

1.1.5. Spectral invariance of the normal form. The intertwining operator $W$ conjugating $\Delta_{\Omega}$ to $F(|D|, \hat{I})$ modulo $S^{2,2(k+2)}$ can be used to construct a microlocal parametrix for the Dirichlet wave group modulo similar acceptable errors. The definition of the parametrix is $E(t) = WF_o(t)\psi_v 1_{\Omega_o}W^{-1}$, where $W^{-1}$ is a microlocal inverse to $W$, where $F_o(t)$ is the odd part of the normal form wave group cost$F(|D|, \hat{I})$ (the odd part satisfying Dirichlet boundary conditions at $s = 0, L$), where $1_{\Omega_o}$ is the characteristic function of $\Omega_o$ and where $\psi_v$ is a microlocal cutoff to a suitably small conic neighborhood of $\gamma_o$. The wave invariants of the Dirichlet Laplacian $\Delta_{\Omega}$ can then be calculated in terms of the normal form coefficients, and conversely (as in [G] (see also [Z.1] [Z.2]) the normal form coefficients can be determined from the wave invariants. Hence, the coefficients of the normal form $F(|D|, \hat{I})$ are spectral invariants of $\Delta_{\Omega}$; a fortiori, the classical normal form of the Poincare map $\mathcal{P}_\gamma$ is a spectral invariant.

1.1.6. Conclusion of the proof. Theorem (1.1) then follows from a theorem of Colin de Verdiere [CV] that a bi-axisymmetric analytic plane domain is determined by the Birkhoff normal form of $\mathcal{P}_\gamma$ of a non-degenerate elliptic axial orbit $\gamma$. His proof works as well when $\gamma$ is hyperbolic and therefore we can conclude the proof in either case.

1.2. Future problems. It should be remarked that $F(|D|, \hat{I})$ is explicitly constructed by the algorithm of this paper. In conjunction with Lazutkin’s construction of a metric normal form, one can get explicit albeit complicated formulae for the normal form coefficients as polynomials in the Taylor coefficients of the boundary defining functions at the points $A, B$. In the future we plan to take up the obvious question of whether one can determine the Taylor coefficients from the normal form coefficients. Colin de Verdiere’s theorem shows that the principal symbol of the normal form alone is enough to determine these coefficients when the Taylor coefficients at $A$ and $B$ are the same and when the odd coefficients at $A$ vanish. In less symmetric cases one will have to go into lower order terms in the normal form. In the simpler but somewhat analogous case of surfaces of revolution, it was necessary to go two steps below the principal symbol level to determine all of Taylor coefficients from the normal form. Rotational symmetry is analogous to left-right symmetry in a plane domain, so we suspect one can solve the inverse spectral problem at least for left-right symmetric analytic domains using just the wave invariants at one orbit.

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2. Preliminaries

2.1. Billiards on plane domains. First we recall the definition of a non-degenerate bouncing ball orbit for billiards on a smooth domain in $\mathbb{R}^n$

Let $\Omega \subset \mathbb{R}^n$ be a smooth domain, and let $\partial \Omega$ denote its boundary. Let $g$ be a smooth metric defined in a neighborhood of $\Omega$. The billiard flow $\Phi^t$ of $(\Omega, g)$ is the geodesic flow of $g$ in the interior $T^* \mathrm{int} \Omega$, with the usual law of reflection at the boundary (i.e., the tangential component of the velocity remains the same but the normal component changes its sign). Here, int $\Omega$ denotes the interior of $\Omega$.

From the symplectic point of view, there are two important hypersurfaces in $T^*(\Omega)$: the manifold $T^* \partial \Omega$ of (co)vectors with footpoints on $\partial \Omega$ and the unit cotangent bundle $S^* \Omega$. We understand by $T^* \Omega$ the restriction $T^* \mathbb{R}^2 |\Omega$. The intersection $S^* \partial \Omega = T^* \partial \Omega \cap S^* \Omega$ is transversal. The characteristic foliation of $S^* \Omega$ is spanned by the generator of the geodesic flow. The projection $\pi : T^* \partial \Omega \rightarrow T^* \Omega$ which projects a (co)vector at $x \in \partial \Omega$ to the (co)tangent hyperplane defines a real line whose fiber at $x$ is the normal bundle $N_x(\partial \Omega)$. We denote by $\nu_x$ the inward unit normal so that $N_x(\partial \Omega) = \mathbb{R}\nu_x$. This line bundle coincides with the characteristic foliation of $T^* \partial \Omega$. The image of $S^* \partial \Omega$ under $\pi$ is the unit disc bundle $D^*(\partial \Omega)$.

We will identify it with the space $S^* \partial \Omega$ of inward unit tangent (co)vectors with footpoints on $\partial \Omega$.

One then defines the billiard ball map $\beta : D^*(\partial \Omega) \rightarrow D^*(\partial \Omega)$ as follows: lift a tangent (co)vector $v$ to $\partial \Omega$ of length $< 1$ to $S^* \partial \Omega$ and move along the corresponding geodesic until the ball hits the boundary. Then project its tangent vector to $D^* \partial \Omega$.

By a reflecting ray one means a broken geodesic or billiard trajectory of the billiard flow whose intersections with the boundary are all transversal. We denote the successive points of contact with the boundary by $q_0, q_1, q_2, \ldots$. Of special importance here are the periodic reflecting rays where $q_n = q_0$ for some $n > 1$. We will denote such a periodic trajectory by $\gamma : [O, T] \rightarrow S^* \Omega$. By a bouncing ball orbit $\gamma$ one means a periodic reflecting ray where $q_0 = q_2$. The projection to $\Omega$ consists of a segment $q_0q_1$ which is orthogonal to the boundary at both endpoints, i.e., $q_0q_1$ is an extremal diameter. The period is of course twice the length of the segment, which we denote by $L$. We write $q_0 = A, q_1 = B$.

The Poincare map $\mathcal{P}_\gamma$ of a periodic reflecting ray $\gamma$ is the first return map to a symplectic transversal. A natural symplectic transversal is given by a neighborhood of $\gamma(0)$ in $S^* \partial \Omega$. Thus for $v \in S^* \partial \Omega$, $\mathcal{P}_\gamma(v)$ is obtained by following the broken geodesic thru $v$ until it reflects from $\partial \Omega$ the $n$th time in an inward vector. The linear Poincare map $P_\gamma$ is defined to be $d\mathcal{P}_\gamma(\gamma(0))$.

Equivalently, one can identify $S^* \partial \Omega$ with $D^*(\Omega)$ as above and define $\mathcal{P}_\gamma$ as a map on a neighborhood $S_0$ of the zero vector at an endpoint of $\gamma$ in $D^*(\Omega)$. Then for a periodic reflecting ray $\gamma$ with $n$ reflections, $\mathcal{P}_\gamma$ may be identified with $\beta^n|_{S_0}$.

**Definition 2.1.** A bouncing ball orbit $\gamma$ is said to be non-degenerate if the eigenvalues of $P_\gamma$ are not roots of unity. There are two cases: $\gamma$ is
(i) non-degenerate elliptic if the eigenvalues of $P_\gamma$ are of the form \( \{e^{\pm i\alpha}\} \) with $\alpha/\pi \notin \mathbb{Q}$;
(ii) non-degenerate hyperbolic if the eigenvalues of $P_\gamma$ are of the form \( \{e^{\pm \lambda}\} \) for some $\lambda \in \mathbb{R}^+$. 

For background on bouncing-ball orbits we refer to [B.B] (see §5.2). If we denote by $R_A$, resp. $R_B$ the radii of curvature of the boundary at the endpoints $A$, resp. $B$ of an extremal diameter, then one finds that ellipticity is equivalent to the condition that $L < R_A + R_B$, $L > R_A$, $L > R_B$ or else that $L < R_A + R_B$, $L < R_A$, $L < R_B$. If $\bar{AB}$ is a local minimum diameter then $L < R_A + R_B$ while if it is a local maximum diameter then $L < R_A + R_B$. So a (non-degenerate) local maximum diameter must be hyperbolic; a local minimum diameter is elliptic if it satisfies the additional inequalities above. Under our symmetry assumption $R_A = R_B$, so ellipticity is equivalent to the statement that the center of curvature at $A$ lies below the horizontal axis.

2.2. Wave trace on a manifold with boundary. Let $\Delta$ denote the Laplacian of a metric $g$ defined in an open neighborhood of a bounded smooth domain $\Omega \subset \mathbb{R}^2$. The Dirichlet Laplacian $\Delta_\Omega$ of $\Omega$ is then defined to be the self-adjoint operator on $L^2(\Omega)$ with domain
\[
\text{Dom}(\Delta_\Omega) = \{ u \in H^1_0(\Omega) : \Delta u \in L^2(\Omega) \}
\]
where $H^1_0(\Omega)$ denotes the closure of $C^\infty_0(\Omega)$ under the norm
\[
||u||_{1}^2 = \sum_{j \leq 2} ||\frac{\partial u}{\partial x_j}||_{L^2}^2.
\]
and where $\Delta u$ is taken in the sense of distributions. We recall that $H^1_0(\Omega) = \{ u \in H^1(\mathbb{R}^2) : \text{supp } u \subset \Omega \}$.

Now let $E(t,x,y)$ be the fundamental solution of the mixed wave equation with Dirichlet boundary conditions:
\[
\frac{\partial^2 E}{\partial t^2} = \Delta E \quad \text{on } \mathbb{R} \times \Omega \times \Omega
\]
\[
E(0,x,y) = \delta(x-y) \quad \frac{\partial E}{\partial t}(0,x,y) = 0
\]
\[
E(t,x,y) = 0 \quad (t,x,y) \in \mathbb{R} \times \partial \Omega \times \Omega.
\]

As discussed in ([GM], §5) ([PS]), the fundamental solution is not globally a Lagrangean distribution. However, for any $T > 0$ there exists a conic neighborhood $\Gamma_T$ of the bouncing ball orbit $\gamma$ so that the microlocalization of $E(t,x,y)$ to $\Gamma_T$ is Lagrangean for $|t| \leq T$. More precisely,

**Theorem 2.2.** ([GM], Theorem 4.1 and Proposition; or [PS], §6) Let $(x,\xi) \in T^*(\text{int} \Omega)$ or let $x \in \partial \Omega$ and suppose that $\xi$ is a non-glancing (co-) direction at $x$. Then there exists a conic neighborhood $\mathcal{O}$ of $(x,\xi)$ in $T^*\mathbb{R}^n - 0$ and a Fourier integral distribution $\hat{V}(T,x,y)$ essentially supported in $\mathcal{O}$ such that for any pseudodifferential operator $\chi_\mathcal{O}(x,D)$ essentially supported in $\mathcal{O}$, $\chi_\mathcal{O}(x,D)\hat{V}(T,x,y) - \chi_\mathcal{O}(x,D)E(T,x,y) \in C^\infty(\mathbb{R} \times \Omega \times \Omega)$.

By a partition of unity, this leads to a Fourier integral formula for the wave trace near a periodic reflecting ray:
Theorem 2.3. Suppose that $T$ is an isolated point of the length spectrum of $\Omega$ with the following properties:
(i) $T \neq |\partial\Omega|$;
(ii) All of the closed billiard trajectories of length $T$ are non-degenerate reflecting rays.

Then modulo smooth functions in $t$ near $t = T$ we have:

$$\sum_{\lambda_j \in \mathrm{Sp}(\sqrt{\mathcal{E}})} \cos \lambda_j t = \int_{\Omega} E(t, x, x) dx = \sum_j \int_{\Omega} \tilde{V}_\pm^{(j)}(t, x, x) dx$$

where $\tilde{V}_\pm^{(j)}(t, x, y)$ are Fourier integral distributions associated to the broken geodesic flow near the closed trajectories of length $T$.

The wave invariants can therefore be obtained by applying the method of stationary phase to an oscillatory integral. An apparent obstruction is that the domain of integration is a manifold with boundary. However, Guillemin-Melrose prove:

**Lemma 2.4.** Let $\Omega \subset \mathbb{R}^n$ be a smooth domain with boundary and let $\phi \in C^\infty(\mathbb{R}^n)$ have clean critical point sets. For each critical value $\lambda$ let $C_\lambda$ denote the critical set in $\mathbb{R}^n$. Suppose that for each $\lambda$, $C_\lambda$ intersects $\partial\Omega$ transversally. Let $U_\lambda$ be a neighborhood of $C_\lambda$ with the properties:
(i) $U_\lambda$ contains no critical points of $\phi$ except $C_\lambda$;
(ii) $U_\lambda \cap \partial\Omega$ contains no critical points of $\phi|_{\partial\Omega}$ except $C_\lambda \cap \partial\Omega$. Then the usual stationary phase expansion is valid:

$$\int_{\Omega} a(x)e^{i\tau \phi(x)} dx \sim e^{i\tau \lambda \tau - k/2} \sum_{i=0}^{\infty} \alpha_i \tau^{-i}, \quad (\tau \to \infty)$$

where $k$ is the codimension of $C_\lambda$ in $\mathbb{R}^n$ and where $\alpha_i$ are integrals of $a$ and its derivatives over $C_\lambda \cap \Omega$.

2.3. Wave trace invariants as non-commutative residues. We may summarize the relevant result on the Poisson formula as follows:

**Corollary 2.5.** Let $\gamma$ be a non-degenerate billiard trajectory whose length $L_\gamma$ is isolated and of multiplicity one in $\mathrm{Lsp}(\Omega)$. Let $\Gamma_L$ be a sufficiently small conic neighborhood of $\mathbb{R}^+ \gamma$ and let $\psi$ be a microlocal cutoff to $\Gamma_L$. Then for $t$ near $L_\gamma$, the trace of the wave group has the singularity expansion

$$\mathrm{Tr}\psi E(t) = c_\gamma (t-L_\gamma+i0)^{-1} + a_{\gamma 0} \log(t-L_\gamma+i0) + \sum_{k=1}^{\infty} a_{\gamma k} (t-L_\gamma+i0)^k \log(t-L_\gamma+i0)$$

where the coefficients $a_{\gamma k}$ are calculated by the stationary phase method from a Lagrangean parametrix.

This corollary allows us to identify the wave invariants as non-commutative residues as in [GZ1]. Recall that if $A$ is a Fourier integral operator, and if $P$ is any positive elliptic first order pseudodifferential operator, then the zeta function $\zeta(z, A, P) := \mathrm{Tr} AP^{-2}$ has meromorphic extension to $\mathbb{C}$ with at most simple poles. The residue at $z = 0$ is referred to as the non-commutative residue $\mathrm{res}(A)$ of $A$. It is independent of $A$ and is a tracial invariant, i.e. $\mathrm{res}(WAW^{-1}) = \mathrm{res}A$. In the boundaryless case, one has $a_{\gamma k} = \mathrm{res}(\frac{1}{(2\pi)^k} E(t)|_{t=L_\gamma})$. The only ingredients in the proof are the Lagrangean property of $E$ and a canonical transform between
TrAe^{itP} and TrAP^{-s}. Hence the same result remains valid in the case of periodic reflecting rays of the boundary case:

**Corollary 2.6.** If γ is a periodic reflecting ray, then \( \alpha_{\gamma_k} = \text{res} \left( \frac{d}{dt} \right)^k E(t)|_{t=L} \).

Now we recall some elementary results from [3] and [Z.1] on the data of the domain or metric which go into a given wave invariant \( \alpha_{\gamma_k} \) in the boundaryless case. Analogous results hold in the boundary case, but we postpone stating them.

In the following, let \( P \) be any first order pseudodifferential operator of real principal type on a boundaryless manifold \( U \) and assume for simplicity that all closed orbits of its bicharacteristic flow are non-degenerate. Let \( \alpha_{\gamma_k}(P) \) denote the kth wave invariant at a closed orbit \( \gamma \) for \( e^{itP} \). Since it is calculated by a stationary phase expansion at \( \gamma \), it is obvious that \( \alpha_{\gamma_k}(P) \) depends only on certain part of the jet of the complete symbol of \( P \) around \( \gamma \).

To state the precise result, let us Taylor expand each term in the complete (Weyl) symbol \( p(s, \sigma, y, \eta) \sim p_1 + p_o + \ldots \) of \( P \) at \( \mathbb{R}^+\gamma \):

\[
p_j(s, \sigma, y, \eta) = \sigma^j p_j(s, 1, y, \frac{\eta}{\sigma}) = \sigma^j (p_j^{[a]} + p_j^{[1]} + \ldots)
\]

with \( p_j^{[m]}(s, 1, y, \frac{\eta}{\sigma}) \) the part which is homogeneous of degree \( m \) in \( (y, \frac{\eta}{\sigma}) \). Set \( P_j := Op(p_j), P_j^{[m]} := Op(p_j^{[m]}) \) and \( P_j^{\leq N} = \sum_{m \leq N} P_j^{[m]} \). Then we have:

**Proposition 2.7.** ([Z.1], Proposition 4.2) \( \alpha_{\gamma_k}(P) = \alpha_{\gamma_k}(P_1^{\leq 2(k+2)} + P_0^{\leq 2(k+1)} + \ldots P_0^{\leq 2k-1}) \).

Thus, it is sufficient to define the normal form and the intertwining operator in a microlocal (conic) neighborhood \( V \) of \( \mathbb{R}^+\gamma \). In \( (s, \sigma, y, \eta) \) coordinates, we may define the cone by:

\[
V = \{(s, \sigma, y, \eta) \in T^*U \cup : |y| \leq \epsilon, |\eta| \leq \epsilon|\sigma|\}.
\]

Let \( \psi_r \) be a microlocal cutoff to \( V \), with symbol identically equal to one in a slightly smaller open cone around \( \mathbb{R}\gamma \) and put \( \Delta_r := \psi_r \Delta \psi_r \). \( \Delta_r \) and \( \Delta \) of course have the same microlocal normal form around \( \gamma \) so for notational simplicity we often drop the subscript and leave it the reader to recall that the operator is cutoff. We will re-instate the cutoff at the crucial point of calculating the residue.

Since terms which vanish to too high order at \( \gamma \) or which have too low a pseudodifferential order do not contribute to \( \alpha_{\gamma_k} \) we introduce a bi-grading on symbols in terms of order as a symbol and order of vanishing along \( \gamma \). Let \( \mathbb{R}^+\gamma \subset T^*U \) be the cone thru an embedded curve \( \gamma \in T^*U \) and \( O_j S^m(U) \) denote the class of symbols of order \( m \) over \( U \) which vanish to order \( j \) along \( \gamma \). Following ([BM], see also [BMGH]) we denote by \( S^{m,k}(V, \mathbb{R}^+\gamma) \) the the class of symbols microsupported in \( V \) which admit asymptotic expansions

\[
a \sim \sum_{j=0}^{\infty} a_{m-j}, \quad a_{m-j}(x, r\xi) = r^{m-j} a(x, \xi), \quad a_{m-j} \in O_{k-2j} S^{m-j}.
\]

Here, \( k \in \mathbb{N} \) and is no condition if \( 2j \geq k \). We denote by \( Op^w S^{m,k} \) the Weyl pseudodifferential operators with complete symbols in \( S^{m,k} \).
In local coordinates \((s, \eta)\) with \(\gamma = \{ y = 0 \}\) and with dual symplectic coordinates \((\sigma, \eta)\), \(a \in S^m(V, \mathbb{R}^\gamma)\) if
\[
a(s, \sigma, y, \eta) \sim \sum_{j=0}^{\infty} a_{m-j,k}(s, \sigma, y, \eta) \sim \sum_{j=0}^{\infty} \sigma^m \sigma^{-j} a_{m-j,k}(s, 1, y, \eta/\sigma)
\]
(7) \[a_{m-j,k}(s, 1, y, \eta/\sigma) \in O_{k-2j} S^0.
\]
The coefficient \(a_{m-j}(s, 1, y, \eta/\sigma)\) is homogeneous of degree 0 and the assumption that \(a\) is microsupported in \(V\) becomes that \(a_{m-j}\) is supported in the transverse ball \(B_\epsilon = \{(y', \eta')| < \epsilon \}\).

Any symbol \(a \in S^m(V)\) may be expanded as a sum of symbols in \(S^m\). Indeed, let \(a_{m-j,k}\) be the term of degree \(k-2j\) in its Taylor expansion for \(k \geq 2j\)
\[
a_{m-j,k} = \sum_{|\alpha|+|\beta|=k-2j} a_{m-j,k-2j,\alpha,\beta}(s) \gamma^\alpha (\eta/\sigma)^\beta.
\]
Then the asymptotic sum \(a_{(m,k)} \sim \sum_{j=0}^{[k/2]} a_{m-j,k}\) belongs to \(S^m\) (sharp) and \(a = \sum_{k=0}^{\infty} a_{m,k}\). In particular, we may expand \(\Delta\) in this form, and \(a_{\gamma,k}\) depends only on the class of \(\Delta\) modulo \(S^{2,k}\).

We summarize the discussion by restating Proposition (2.7) in terms of these symbol classes:

**Corollary 2.8.** Suppose that \(A, B \in Op^m S^1(V)\) and that \(A \equiv B \mod OpS^{1,k}(V)\). Then \(a_{\gamma,k}(A) = a_{\gamma,k}(B)\).

### 3. Straightening the Domain

As discussed in the introduction, the normal form lives on the model domain \(\Omega_o = [0, L] \times \mathbb{R}\), or more precisely in a microlocal neighborhood of \(T^*[0, L]\) in its cotangent bundle. In this section we introduce the analytic objects on the model domain and explain how to transfer \(\Delta\), in a neighborhood in \(\mathbb{R}^2\) of a non-degenerate (elliptic or hyperbolic) bouncing ball orbit \(\gamma\), to a variable coefficient Laplacian \(\Delta\) in a neighborhood of \([0, L] \times \{0\}\) in the model. We emphasize that all maps and operators that we discuss in this section extend to open domains containing the various manifolds with boundary.

#### 3.1. The model domain \(\Omega_o\).

The configuration space of the model is the infinite strip \(\Omega_o\). We denote the coordinate on \([0, L]\) by \(s\) and that on \(\mathbb{R}\) by \(\eta\), with dual cotangent coordinates \(\sigma, \eta\) on \(T^*[0, L] \times T^* \mathbb{R}\). We also denote by \(I_e = \frac{1}{2}(\eta^2 + y^2)\) the elliptic action variable and by \(I_h = \frac{1}{2}(\eta^2 - y^2)\) the hyperbolic action variable on \(T^* \mathbb{R}\). To simplify notation we often just write \(I\) for the relevant action variable. In the Poisson algebra of \(T^*(\mathbb{R}^2)\) we consider the maximal abelian subalgebra \(A_{cl} = \langle |\sigma|, I \rangle\). The model (classical) Hamiltonians are those of the form \(H_\alpha = |\sigma| + \frac{\alpha}{2} I\) which generate linear Hamiltonian flows. As in the case of straightline motion, \(H_\alpha\) generates a broken Hamiltonian flow on \(T^*[0, L] \times T^* \mathbb{R}\) when equipped with the boundary condition that the trajectory is reflected by \(\tau\) when it hits the boundary.

We will view \(\Omega_o\) as a submanifold with boundary of the ‘open space’ \(S^1_{2L} \times \mathbb{R}\) with \(S^1_{2L} = \mathbb{R}/2\mathbb{Z} \cong [-L, L]\). Many of our operators will be supported in a neighborhood \(U_\epsilon = (-\epsilon, L + \epsilon) \times \mathbb{R}\) of \(\Omega_o\) in \(S^1_{2L} \times \mathbb{R}\). We denote by \(\tau_\epsilon(s)\) a smooth cutoff to \(U_\epsilon\) with \(\tau_\epsilon \equiv 1\) in a smaller neighborhood of \(\Omega_o\).
Since $\Omega_o$, $U_e$, and $S^1_{2L} \times \mathbb{R}$ are products, their algebras of pseudodifferentials operators are easily described in terms of pseudodifferential operators along $S^1_{2L}$ and transverse pseudodifferential operators on $\mathbb{R}$.

In the direction of $S^1_{2L}$, we introduce the algebra $\Psi^*(S^1_{2L}) = \langle s, D_s \rangle$ of standard pseudodifferential operators on $S^1_{2L}$. We distinguish the element $|D_s| = \sqrt{-D_s^2D_s}$ with eigenfunctions $e^{\frac{\pi k}{L}}$ associated eigenvalues $|k|$. We also define the subspaces and projections:

\[
\begin{align*}
H^2_\pm(S^1_{2L}) &= \bigoplus_{k=0}^\infty \mathbb{C} e^{\pm i \frac{\pi k}{L}} \\
\Pi_\pm : L^2 \to H^2_\pm \\
\Pi_{\pm k} : L^2(S^1_{2L}) \to \mathbb{C} e^{\pm i \frac{\pi k}{L}} \\
\Pi_{\mp k} : L^2(S^1_{2L}) \to \mathbb{C} e^{-i \frac{\pi k}{L}} \\
L^2_{\text{odd}}(S^1_{2L}) &= \bigoplus_{k=0}^\infty \mathbb{C} \sin(\frac{\pi k}{L}) \\
\Pi_o : L^2 \to L^2_{\text{odd}}, \quad \Pi_o f(s) = \frac{1}{2}(f(s) - rf(s)) \\
\Pi_{\pm k} : L^2(S^1_{2L}) \to \mathbb{C} \sin(\frac{\pi k}{L})
\end{align*}
\]

Here, $rf(s) = f(-s)$ where $-s$ is taken modulo $2L$, i.e. $r$ is reflection through the boundary. We may tensor the subspaces with $L^2(\mathbb{R})$ to get corresponding subspaces of and operators on $L^2(S^1_{2L} \times \mathbb{R})$ and we use the same notation for these.

We will sometimes identify functions on $\Omega_o$ with odd functions on $S^1_{2L} \times \mathbb{R}$. More precisely, let us put:

\[
\begin{align*}
\Pi_k : L^2([0, L] \times \mathbb{R}) &\to \mathbb{C} \sin(\frac{\pi k}{L}) \otimes L^2(\mathbb{R}) \\
1_{[0, L]}(s) : L^2(S^1_{2L}) &\to L^2([0, L]) \\
A : L^2([0, L]) &\to L^2\text{odd}(S^1_{2L}) \\
A^* : L^2\text{odd}(S^1_{2L}) &\to L^2([0, L]) \\
|D_s| : H^0_1([0, L]) &\to L^2([0, L]) \\
|D_s| \sin(\frac{\pi k}{L}) &= \frac{\pi k}{L} \sin(\frac{\pi k}{L})
\end{align*}
\]

We use the notation $|D_s|$ (or simply $|D|$) simulteneously for the ‘Laplacian’ on $S^1_{2L}$ and the Dirichlet Laplacian on $[0, L]$. No confusion should result since they are defined on different domains, and moreover the definitions are compatible under the above identification, as the following proposition shows.

**Proposition 3.1.** We have:

(i) $A : L^2([0, L]) \to L^2\text{odd}(S^1_{2L})$ and $AA^* = \Pi_o, A^*A = Id$.

(ii) $A^*|D_s|A = |D_s|$

The proof is obvious so we omit it.

In the transverse direction, we first introduce the isotropic Weyl algebra $\mathcal{W}^*$.

This is a completion of the algebra $\mathcal{E} := \langle y, D_y \rangle$ of polynomial differential operators on $\mathbb{R}$. We denote by $\mathcal{E}^n$ denote the subspace of polynomial differential operators of degree $n$ in the variables $y, D_y$. We also denote by $\mathcal{E}^n_e$ the polynomials all of whose terms have the same parity as $n$. In the isotropic Weyl algebra $\mathcal{W}^*$, the operators $y, D_y$ are given the order $\frac{1}{2}$, so that

\[
\mathcal{E}^n \subset \mathcal{W}^{n/2}, \quad [\mathcal{E}^m, \mathcal{E}^n] \subset \mathcal{E}^{m+n-2}.
\]
We note that such operators are not standard homogeneous pseudodifferential operators, but can be rescaled to this form. The rescaled algebra is generated by the first order homogeneous pseudodifferential operators $y|D_s|^\frac{1}{2}$ and $|D_s|^{-\frac{1}{2}}D_y$. In the open space the relevant algebra is the algebra of homogeneous pseudodifferential operators over $U_e$ generated by $D_s, y|D_s|^\frac{1}{2}, |D_s|^{-\frac{1}{2}}D_y$. To be more precise, this construction is only well-defined in a microlocal neighborhood of $\gamma$ where $|D_s|$ is elliptic.

It should also be recalled that $a^w(y, D_y) \in \mathcal{E}^n$ with real-valued symbols are essentially self-adjoint operators on $\mathcal{S}(\mathbb{R})$ (the Schwartz space), i.e. have a unique self-adjoint extension to $L^2(\mathbb{R})$. Hence their exponentials $e^{ia^w(y, D_y)}$ are unambiguously defined.

### 3.2. Half-density Laplacian.

In order to deal with self-adjoint operators with respect to the Lebesgue density $dsdy$, we pass from the scalar Laplacian to the (unitarily equivalent) $1/2$-density Laplacian

$$\Delta_{\frac{1}{2}} := J^{1/2} \Delta J^{-1/2}$$

$$(12) \quad = \sum_{i,j=1}^2 J^{-1/2} D_x^i g^{ij} JD_x^j J^{-1/2}$$

$$= g^{11} D_x^2 + g^{22} D_y^2 + 2g^{12} D_x D_y + \frac{1}{4} \Gamma^1 D_x + \frac{1}{4} \Gamma^2 D_y + \sigma_0.$$ 

Here we write $(s, y) = (x_1, x_2), D_{x_j} = \frac{\partial}{\partial x_j}$. The functions $\Gamma_j$ are real valued. Since it is self-adjoint relative to the Lebesgue density, its complete Weyl symbol is real valued. Henceforth we denote the $1/2$-density Laplacian simply by $\Delta$.

### 3.3. Straightening the domain.

Let us now explain how to transfer the Laplacian to the model domain. As in the boundaryless case, the model space is in some sense the normal bundle of the orbit. This is literally correct in the boundaryless case and the exponential map along the normal bundle $N_\gamma$ of $\gamma$ can be used to transfer $\Delta$ to the normal bundle. In the case of a bouncing ball orbit in the boundary case, the normal bundle and exponential map are ill-defined at the reflection points but Lazutkin has constructed a nice replacement for them. Namely, he constructs a map $\Phi$ which straightens the domain to a strip near a stable elliptic bouncing ball orbit and which simultaneously puts the Laplacian into a preliminary normal form. We will modify his method to encompass hyperbolic bouncing ball orbits as well. We will modify his method to encompass hyperbolic bouncing ball orbits as well. As mentioned above, we do not need the full details of the map or metric normal form here. Hence we only sketch the construction of $\Phi$, referring the reader to [L] for the details. In the following, $\Omega_\epsilon$ denotes an $\epsilon$-neighborhood of $AB$ in $\Omega$.

**Definition 3.2.** By a transversal power series map from $\Omega_\epsilon$ to $\Omega_0$ we mean a formal power series

$$\Phi : \Omega_\epsilon \to U_e, \quad \Phi(s, y) = (\bar{s}, \bar{y})$$

of the form

$$\bar{s} = s + \sum_{m=2}^\infty \kappa_m(s)y^m$$

$$\bar{y} = \sum_{p=1}^\infty \psi_p(s)y^p$$

with real valued analytic coefficients in $s$ extending analytically to a neighborhood of $[0, L]$ in $\mathbb{C}$ and satisfying the boundary conditions:

$$\{\bar{s} = 0\} \cup \{\bar{s} = L\} = \Phi(\partial \Omega_\epsilon).$$
We use the language of formal power series since we only need to use a polynomial part of the map to construct the normal form up to a desired accuracy. Indeed, a given wave invariant \( a_{\gamma_k} \) only involves the \( 2k + 4 \)-jet of \( \Phi \). The convergence of the series is irrelevant to our purposes and we will not discuss it. The following Lemma was in effect proved by Lazutkin in \([L]\) in the elliptic case.

**Lemma 3.3.** Suppose that \( AB \) is a bouncing ball orbit. Then there exists a transversal power series map \( \Phi : (\Omega, \partial \Omega) \to (\Omega, \partial \Omega) \) which straightens the domain and puts \( \Delta \) in the form:

\[
\bar{\Delta} \sim D_s^2 + B(\bar{s}, \bar{y})(\bar{y}^2 D_s^2 + D_{\bar{y}}^2) + \Gamma_s D_s + \Gamma_y D_y \quad \text{elliptic case}
\]

\[
\bar{\Delta} \sim D_s^2 + B(s, y)(D_y^2 - \bar{y}^2 D_s^2) + \Gamma_s D_s + \Gamma_y D_y \quad \text{hyperbolic case}
\]

in the sense that the left and right sides agree to infinite order at \( y = 0 \). Here, \( B(s, y) \) is a transversal power series.

Of course, we only need this form of \( \bar{\Delta} \) to order \( K \) construct the normal form modulo \( O_p S^{2, K} \), and only in the principal terms do we need to know the exact form. Therefore we only briefly recall the proof of the lemma and only discuss the principal terms in detail.

Under any map \( \Phi \), the usual (scalar) \( \Delta \) conjugates to

\[
\tilde{\Delta}_0 := \Phi^{-1}_* \Delta \Phi^* = |\nabla \bar{s}|^2 D_s^2 + |\nabla \bar{y}|^2 D_{\bar{y}}^2 + 2 \langle \nabla \bar{s}, \nabla \bar{y} \rangle D_s D_y + \frac{1}{i} \Delta \bar{s} D_s + \frac{1}{i} \Delta \bar{y} D_y.
\]

In the case of a transversal power series map, the coefficients are also transversal power series of the form:

\[
|\nabla \bar{s}|^2 = 1 + \sum_{m=2}^\infty a_{mm}(\bar{s}) \bar{y}^m
\]

\[
|\nabla \bar{y}|^2 = \sum_{m=0}^\infty b_{mm}(\bar{s}) \bar{y}^m
\]

\[
\Delta \bar{s} = \sum_{m=0}^{\infty} c_{mm}(\bar{s}) \bar{y}^m
\]

\[
\Delta \bar{y} = \sum_{m=0}^\infty d_{mm}(\bar{s}) \bar{y}^m
\]

\[
\langle \nabla \bar{s}, \nabla \bar{y} \rangle = \sum_{m=0}^\infty e_{mm}(\bar{s}) \bar{y}^m
\]

The volume density in the new coordinates, \( J(\bar{s}, \bar{y}) \) has a similar form. The \( 1/2 \)-density Laplacian \( \Phi^{-1}_* \Delta \Phi^* \) in the transformed coordinates is given by

\[
\bar{\Delta} := J^{-\frac{i}{2}}(\Phi^{-1}_* \Delta \Phi^*)(J^\frac{i}{2}) = |\nabla \bar{s}|^2 D_s^2 + |\nabla \bar{y}|^2 D_{\bar{y}}^2 + 2 \langle \nabla \bar{s}, \nabla \bar{y} \rangle D_s D_y + \frac{1}{i} \Gamma_s D_s + \frac{1}{i} \Gamma_y D_y + K
\]

where

\[
\Gamma_s = -|\nabla \bar{s}|^2 \frac{\partial}{\partial \bar{s}} \log J - \langle \nabla \bar{s}, \nabla \bar{y} \rangle \frac{\partial}{\partial \bar{y}} \log J + \Delta \bar{s}
\]

\[
\Gamma_y = -|\nabla \bar{y}|^2 \frac{\partial}{\partial \bar{y}} \log J - \langle \nabla \bar{s}, \nabla \bar{y} \rangle \frac{\partial}{\partial \bar{s}} \log J + \Delta \bar{y}
\]

\[
K = J^{-\frac{i}{2}}(\Phi^{-1}_* \Delta \Phi^*)(J^\frac{i}{2}).
\]

Combining with the power series expressions in \([L]\) one has an expression for \( \bar{\Delta} \) modulo terms vanishing to order \( K \) at \( y = 0 \).
The principal term (modulo $S^2, S^3$) is given by

$$D_s^2 + \frac{1}{2} b_{00}(s)[\bar{y}^2 D_s^2 + D_y^2] \quad \text{elliptic case}$$

$$D_s^2 + \frac{1}{2} b_{00}(s) [D_y^2 - \bar{y}^2 D_s^2] \quad \text{hyperbolic case}$$  

(17)

This requires the coefficients to solve the equations:

- $e_{11} = 0$

(18)

- $a_{22} = b_{00} \quad \text{elliptic case}$

- $a_{22} = -b_{00} \quad \text{hyperbolic case}$

To solve these equations we use the following expressions from ([L], (4.5) – (4.9)):

$$a_{22}(s) = (2\kappa'_{22} + 4\kappa_{22}^2)\psi_{11}^{-2}$$

$$b_{00}(s) = \psi_{11}^2(s)$$

$$c_{00}(s) = 2\kappa'_{22}(s)$$

(19)

Since they are obtained purely algebraically, the same equations hold in both the elliptic and hyperbolic cases. They are easiest to solve if we make the substitution $\theta_{11} = \psi_{11}^{-1}$. One then has:

$$e_{11} = 0 \Rightarrow \kappa_{22}(s) = -\frac{1}{2} \frac{\psi'(s)}{\psi(s)}$$

(20)

$$a_{22} = b_{00} \Rightarrow \theta''_{11} = \frac{1}{\theta_{11}} \quad \text{elliptic case}$$

$$a_{22} = -b_{00} \Rightarrow \theta''_{11} = -\frac{1}{\theta_{11}} \quad \text{hyperbolic case}$$

The solutions have the form:

$$\theta_{11} = \sqrt{\ell + \frac{(\bar{s} - s_0)^2}{\ell}} \quad \text{elliptic case}$$

$$\theta_{11} = \sqrt{-\ell + \frac{(\bar{s} - s_0)^2}{\ell}} \quad \text{hyperbolic case}$$

(21)

for some constants $\ell, s_0$. To verify this it is easiest to substitute $\xi = \theta_{11}^2$. The equation for $\xi$ is then $\frac{1}{2}\xi'' = \frac{1}{4}(\xi')^2 \pm 1 \ (+= \text{elliptic}, - = \text{hyperbolic})$. Solving for $b_{00}$ gives

$$b_{00}(\bar{s}) = \frac{1}{\ell + \frac{(\bar{s} - s_0)^2}{\ell}} \quad \text{(elliptic)}, \quad b_{00}(\bar{s}) = \frac{1}{-\ell + \frac{(\bar{s} - s_0)^2}{\ell}} \quad \text{(hyperbolic)}$$

(22)

hence

$$\int_{\bar{s}_0}^{\bar{s}} b_{00}(\bar{s}) d\bar{s} = \frac{1}{\ell} \tan^{-1}\left(\frac{\bar{\bar{s}} - s_0}{\ell}\right) \quad \text{(elliptic)}, \quad \int_{\bar{s}_0}^{\bar{s}} b_{00}(\bar{s}) d\bar{s} = -\frac{1}{\ell} \tanh^{-1}\left(\frac{\bar{s} - s_0}{\ell}\right) \quad \text{(hyperbolic)}.$$  

In the elliptic case one finds that

$$\ell = \sqrt{x_0(R_A - s_0)} = \sqrt{(L - s_0)(R_B - L + s_0)}.$$  

(23)
These conditions uniquely determine \( x_0, \ell \) (see the pictures on p.135 of [L.1]). It also follows that \( c_{00}(s) = -\frac{s_{11}}{s_{11}^2} = -\frac{\epsilon_{20}}{\epsilon_{11} + \epsilon_{22}} \).

In §4.2 we will put the leading term into a canonical normal form with coefficients independent of \( \bar{s} \).

### 3.4. Wave invariants revisited.

Having straightened the Laplacian, and hence having transferred the information about the boundary into the metric, we can now state precisely just how much data of the boundary goes into a given wave invariant \( a_{\gamma k} \). Since the wave invariants at \( \{ \bar{y} = 0 \} \) of \( \bar{\Delta} \) are calculated by the stationary phase method, we have, as in the boundaryless case:

**Proposition 3.4.** \( a_{\gamma k}(\bar{\Delta}) = a_{\gamma k}(\bar{\Delta}_{1}^{\leq 2(k+2)} + \bar{\Delta}_{0}^{\leq 2(k+1)} + \cdots + \bar{\Delta}_{0}^{o_{k-1}}) \).

As \( \bar{\Delta} \) is a well-defined partial differential operator in a neighborhood of \( \Omega_o \) we can again reformulate the conclusion in terms of the symbol classes \( S^{2,2}_{2,2(k+2)}(V, \mathbb{R}^+ \gamma_o) \) where \( \gamma_o \) denotes the bouncing ball orbit \([0, L] \times \{0\}\). The discussion in the boundaryless case remains valid, although the microlocal neighborhood \( V \) now acquires two components: Since \( |\sigma| \geq \frac{1}{2} > 0 \) in \( V, V = V_+ \cup V_- \) where \( V_+ = V \cap \{ \sigma > 0 \}, V_- = V \cap \{ \sigma < 0 \} \). The two components are obviously interchanged by the canonical involution \( \tau(s, \sigma, y, \eta) = (s, -\sigma, y, -\eta) \) of \( T^*((0, L] \times \mathbb{R}) \).

**Corollary 3.5.** \( a_{\gamma k}(\bar{\Delta}) \) depends only on the class of \( \bar{\Delta} \) modulo \( S^{2,2}_{2,2(k+2)} \).

In view of the metric normal form, the data which goes in to the k-jet of \( \bar{\Delta} \) along \( \{ y = 0 \} \) is precisely the \( k + 2 \)-jet of the function \( B(\bar{s}, \bar{y}) \) and hence the \( k + 2 \)-jet of the boundary defining functions. Hence \( a_{\gamma k} \) depends only on the \( 2k + 4 \)-jet of the boundary at \( \{ y = 0 \} \).

We will actually need a slight generalization of (3.5) which is proved in precisely the same way.

**Proposition 3.6.** Suppose that \( A \in Op(S^{2,2}_{2,2(k+2)}) \). Then \( resFAG = 0 \) for any bounded Fourier integral operators \( F, G \).

### 4. Semiclassical normal form

As mentioned in the introduction, our approach is to convert the conjugation to normal form to a semiclassical problem.

#### 4.1. Semiclassical scaling.

To introduce the semiclassical parameter, we make a semiclassical scaling of operators on the model space. Roughly, the scaling weights the tangential derivative \( D_{\bar{s}} \) by \( N^2 \), the normal derivative \( D_{\bar{y}} \) by \( N \) and \( \bar{y} \) by \( N^{-1} \).

We define operators \( T_N, M_N \) on the model space \( L^2(\Omega_o) \) by

\[
T_N f(\bar{s}, \bar{y}) := N f(\bar{s}, N \bar{y})
\]

\[
M_N f(\bar{s}, \bar{y}) := e^{iN^2 \bar{s}} f(\bar{s}, \bar{y})
\]

We then have:

\[
T_N^* D_{\bar{y}} T_N = N D_{\bar{y}}
\]

\[
T_N^* \bar{y} T_N = N^{-1} \bar{y}
\]

\[
M_N^* D_{\bar{s}} M_N = (N^2 + D_{\bar{s}})
\]
Definition 4.1. The rescaling of an operator \( a^w(s, D_s, \bar{y}, D_{\bar{y}}) \) is given by
\[
a^w_N(s, D_s, \bar{y}, D_{\bar{y}}) := M_N^* T_N^* a^w(s, D_s, \bar{y}, D_{\bar{y}}) T_N M_N.
\]

We have:

Proposition 4.2. The complete symbol \( a^w_N(s, \bar{\sigma}, \bar{y}, \bar{\eta}) \) of \( M_N^* T_N^* a^w(s, D_s, \bar{y}, D_{\bar{y}}) T_N M_N \) is given by
\[
a^w_N(s, \bar{\sigma}, \bar{y}, \bar{\eta}) = a^w(s, \bar{\sigma} + N^2, \frac{1}{N} \bar{y}, N \bar{\eta}).
\]

Proof The operator kernel of \( a^w(s, D_s, \bar{y}, D_{\bar{y}}) \) is equal to
\[
\int_{\mathbb{R}} \int_{\mathbb{R}} a(\frac{1}{2}(\bar{s} - \bar{s}'), \bar{\sigma}, \frac{1}{2}(\bar{y} - \bar{y}'), \bar{\eta}) e^{i(\bar{s} - \bar{s}') \bar{\sigma} + i(\bar{y} - \bar{y}') \bar{\eta}} d\bar{\sigma} d\bar{\eta}.
\]

Conjugating with \( M_N \) amounts to adding \( N^2(\bar{s} - \bar{s}') \) to the phase and hence to a translation \( \bar{\sigma} \to \bar{\sigma} + N^2 \). Conjugating with \( T_N \) amounts to changing the phase \( \langle \bar{y} - \bar{y}', \bar{\eta} \rangle \) to \( \frac{1}{N} \langle \bar{y} - \bar{y}', \bar{\eta} \rangle \) and the amplitude to \( a(\frac{1}{2}(\bar{s} - \bar{s}'), \bar{\sigma}, \frac{1}{2}(\bar{y} - \bar{y}'), \bar{\eta}) \). Change variables \( \bar{\eta} \to N \bar{\eta} \) to get a Weyl pseudodifferential operator with amplitude \( a(\frac{1}{2}(\bar{s} - \bar{s}'), \bar{\sigma}, \frac{1}{2}(\frac{1}{N} \bar{y} - \frac{1}{N} \bar{y}'), N \bar{\eta}) \).

\[\square\]

Definition 4.3. The semiclassically scaled (1/2-density) Laplacian is the Weyl pseudodifferential operator on \( \Omega_\alpha \) defined by
\[
\tilde{\Delta}_N = M_N^* T_N^* \Delta T_N M_N.
\]

In the straightened form, we have
\[
\tilde{\Delta}_N \sim \langle D_s + N^2 \rangle^2 + B(s, N^{-1} y)[\pm \bar{y}^2 N^{-2} (D_s + N^2)^2 + N^2 D_{\bar{y}}^2]
\]
(26)
\[
+ \{ \Gamma_s \} N [D_s + N] + N \{ \Gamma_{\bar{y}} \} N D_{\bar{y}} + \{ K \} N
\]
where \( \{ f(s, \bar{y}) \} \) is related to the elliptic/hyperbolic dichotomy.

The Weyl symbol of \( \tilde{\Delta} \) has the simple form:
\[
\sigma^w_N(s, \bar{\sigma}, \bar{y}, \bar{\eta}) := \sigma^2 + B(s, y) I + K, \quad K = \tilde{\Delta} \cdot 1
\]
(27)
The linear terms vanish because they give the subprincipal symbol in the Weyl calculus and that of \( \tilde{\Delta} \) equals zero. By the above proposition, we get upon rescaling
\[
\sigma^w_N \sim (\sigma + N)^2 + B(s, \frac{1}{N} y)[\pm y^2 N^{-2}(\sigma + N^2)^2 + N^2 \eta^2] + K(s, \frac{1}{N} y).
\]
(28)

4.2. Linearized problem. To get a sense of what is involved in putting \( \Delta \) into normal form, let us first consider the ‘linearized’ problem which involves only the highest powers of \( N \):
\[
\sigma^w_N \sim N^4 + 2 N^2 [D_s + \hat{b}_{00}(s)] \hat{\tilde{I}} \bmod{N}
\]
(29)
where \( \hat{I} \) denotes the quantum action operator: \( \hat{I}^e = \frac{1}{2}(D_{\bar{y}}^2 + y^2) \) in the elliptic case and \( \hat{I}^h = \frac{1}{2}(D_{\bar{y}}^2 - y^2) \) in the hyperbolic case.

As in the boundaryless case \([Z.1, \S1]\) we can complete the conjugation of the linear/quadratic term to normal form by a moving metaplectic conjugation. When results apply mutatis-mutandis to both elliptic and hyperbolic cases, we denote the action simply by \( \hat{I} \). In the following proposition, \( \mu \) denotes the metaplectic representation of \( SL(2, \mathbb{R}) \) (strictly speaking, of its double cover but signs are irrelevant here).
Proposition 4.4. There exists an $SL(2, \mathbb{R})$-valued function $a_\alpha(s)$ so that

$$\mu(a_\alpha)^*[D_s + b_{00}(s)\hat{I}]\mu(a_\alpha) = \mathcal{R}$$

$$\mu(a_\alpha)(0) = \mu(a_\alpha)(L) = Id$$

where $\mathcal{R} = D_s + \frac{\alpha}{T} \hat{I}$ and where $\alpha = \int_0^L b_{00}(s) ds$.

Proof: We first construct a function $a(s)$ so that $\mu(a)^*[D_s + b_{00}(s)\hat{I}]\mu(a) = D_s$. The desired metaplectic operator is obviously given by

$$\mu(a(s)) = \exp(-i\int_0^s b_{00}(\bar{s})d\bar{s}\hat{I}).$$

Hence $a(s) = \exp(-J/2 \int_0^s b_{00}(\bar{s})d\bar{s})$ where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ elliptic case}$$

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ hyperbolic case.}$$

The boundary values of $\alpha$ are given by $\alpha(0) = I, \alpha(L) = \exp(-J \int_0^L b_{00}(\bar{s})d\bar{s}) = \exp(-\alpha I)$. Now let $r_\alpha(s) = \exp(\frac{J}{2}Js)$ and put $a_\alpha(s) = a(s)r_\alpha(s)^{-1}$. Then $\mu(a_\alpha(0)) = Id$ and $\mu(a)L = \exp(a\hat{I}) \circ \exp(-a\hat{I}) = Id$. Further, $\mu(r_\alpha(s))$ conjugates $D_s$ to $D_s + \frac{\alpha}{T} \hat{I}$. \(\Box\)

4.3. Semiclassical pseudodifferential operators. Semiclassical scaling produces a partial differential operator $\hat{\Delta}_N$ depending on a small parameter $1/N$ and in the conjugation to normal form we will introduce other such operators. Let us pause to clarify the kinds of semiclassical pseudodifferential operators which will be of concern to us. Our discussion is based on ideas and notation from [BMGH].

First, let us recall that the usual (admissible) semiclassical $h = 1/N$-pseudodifferential operators of order $m$ on $\mathbb{R}^n$ are the Weyl quantizations

$$a^w(x, \frac{1}{N}D; N)u(x) = (2\pi h)^{-n} \int \int e^{i(x-y, \xi)} a\left(\frac{1}{2}y, \frac{1}{N}\xi; N\right)u(y)dyd\xi$$

of amplitudes $a$ belonging to the space $S^m_n(T^*\mathbb{R}^n)$ of asymptotic sums

$$a(x, \xi, N) \sim N^m \sum_{j=0}^\infty a_j(x, \xi)N^{-j}$$

with $a_j \in C^\infty(R^{2n})$.

Semiclassical scaling gives rise to symbols of the form $a(\bar{s}, \bar{\sigma} + N^2, \frac{1}{N}\bar{y}, N\bar{\eta})$ where $a$ is a polyhomogeneous symbol $a \sim \sum_{m-j}^\infty a_{m-j}$ of order $m$. As in [BMGH] we may write:

$$a(\bar{s}, \bar{\sigma} + N^2, \frac{1}{N}\bar{y}, N\bar{\eta}) \sim \sum_{j=0}^\infty N^{2m-2j}a_{m-j}(\bar{s}, 1 + \frac{\sigma}{N^2}, \frac{1}{N}\bar{y}, \frac{1}{N}\bar{\eta}).$$

(31)

It is evident that semiclassical scaling produces symbols which behave in both transverse variables $(\bar{y}, \bar{\eta})$ like standard semiclassical symbols in the fiber variable $\eta$. Thus, scaled symbols are isotropic analogues of semiclassical admissible symbols in the transverse $\mathbb{R}^n$. They also have a tangential dependence in $1 + \frac{\sigma}{N^2}$ which seems to have no precise analogue for standard semiclassical symbols.
Let $\psi(\bar{s}, \sigma, \bar{y}, \eta)$ be the homogeneous cutoff to the cone $V$. Under rescaling it goes over to the symbol $N^2 \psi(\bar{s}, 1 + \frac{\bar{s}}{N}, \eta, \frac{\eta}{N})$. For symbols independent of $\sigma$ or for $\sigma < cN^2$ the scaled symbol is supported in the transverse ball
\begin{equation}
V_N := \{(\bar{s}, \sigma, \bar{y}, \eta) : |(\frac{1}{N}\bar{y}, \frac{1}{N}\eta)| \leq \epsilon \}.
\end{equation}

It is useful to reformulate the condition that $A \in OpS^{m,k}$ in terms of the scaled symbol. Given $A \in OpS^{m}$ we define (with some modifications to \[BMGH\]) the formal differential operator
\begin{equation}
\sigma^\infty_{(\bar{s}, \sigma, x, \xi)}(A) = \sum_{j, \alpha, \beta} \frac{1}{\alpha! \beta!} (\frac{\partial}{\partial x})^\alpha (\frac{\partial}{\partial \xi})^\beta a_j(\bar{s}, \sigma, x, \xi)\bar{y}^\alpha D_y^\beta N^{-(|\alpha| + |\beta| + 2j)}
\end{equation}
and put
\begin{equation}
\sigma^k_{(\bar{s}, \sigma, x, \xi)}(A) = \sum_{|\alpha| + |\beta| + 2j = k} \frac{1}{\alpha! \beta!} (\frac{\partial}{\partial x})^\alpha (\frac{\partial}{\partial \xi})^\beta a_j(\bar{s}, \sigma, x, \xi)\bar{y}^\alpha D_y^\beta.
\end{equation}

Then $a \in S^{m,k}(V, \mathbb{R}^+) \text{ if and only if } \sigma^\infty(\bar{s}, \sigma, x, \xi)$ is divisible by $N^{-k}$ for $(x, \xi) = (0, 0)$; hence $\sigma^\infty(\bar{s}, \sigma, \bar{y}, \eta)(A) = \sigma^k(\bar{s}, \sigma, \bar{y}, \eta)(A)N^{-k}$ mod $N^{-(k+1)}$ for $(s, \sigma, x, \xi) \in \mathbb{R}^{+} \gamma_o$. We thus have

**Proposition 4.5.** If $a \in S^{m}(V)$ then $a \in S^{m,k}(V, \mathbb{R}^+) \text{ if and only if the formal Taylor expansion of } a(\bar{s}, N^2, \frac{\bar{s}}{N}, \bar{y}, N\eta) \text{ along } \mathbb{R}^{+} \gamma_o \text{ is divisible by } N^{-(k+1)}$.

4.4. **Conjugation to a semiclassical normal form.** We now come to the principal step in the conjugation to normal form: the conjugation of $\Delta_N$ to a semiclassical normal form.

We first conjugate by $\mu(a_\alpha)$ as in the linearization step to get the somewhat simpler form:
\begin{equation}
\mathcal{R}_N := \mu(a_\alpha)^* \Delta_N \mu(a_\alpha).
\end{equation}
We then wish to conjugate $\mathcal{R}_N$ to the semiclassical normal form
\begin{equation}
F_N(\hat{I})^2 \sim N^4 + N^2\frac{\alpha\hat{I}}{L} + p_1(\hat{I}) + N^{-2}p_2(\hat{I}) + \cdots
\end{equation}
by means of a semiclassical pseudodifferential intertwining operator $W_N(\bar{s}, \bar{y}, D_y)$ which preserves Dirichlet boundary conditions. Thus the full intertwining operator is $\mu(a_\alpha)W_N$. To avoid encumbering the notation we will also denote this full intertwining operator in (5) by $W_N$.

Let us now explain what we mean by semiclassical conjugation to normal form. We assume for simplicity that the bouncing ball orbit is elliptic, but the same argument and result hold in the hyperbolic case. Roughly speaking, our object is to produce bounded semiclassical pseudodifferential operators $W^+_k(\bar{s}, \bar{y}, D_y), W^-_k(\bar{s}, \bar{y}, D_y)$ defined in the open neighborhood (even $U_\epsilon$ of $\Omega_o$ and satisfying the following asymptotic relations on this domain:
\begin{align}
(i) \quad & \hat{\Delta}W^+_k e^{i\pi k \hat{z}} D_q(N_k \bar{y}) \sim F(k, q + \frac{1}{2}) + e^{i\pi k \hat{z}} W^+_k D_q(N_k \bar{y}) \\
(ii) \quad & \hat{\Delta}W^-_k e^{-i\pi k \hat{z}} D_q(N_k \bar{y}) \sim F(k, q + \frac{1}{2}) + e^{-i\pi k \hat{z}} W^-_k D_q(N_k \bar{y}) \\
(iii) \quad & W^+_k e^{i\pi k \hat{z}} D_q(N_k \bar{y}) - W^-_k e^{-i\pi k \hat{z}} D_q(N_k \bar{y}) = 0 \text{ at } \bar{s} = 0, \bar{s} = L.
\end{align}
Here, $F(k, q + \frac{1}{2}) = F_{N_k}(q + \frac{1}{2})$ with $N_k = \sqrt{k}$ and $D_q$ is the qth normalized Hermite function. The precise meaning of $\sim$ will be clarified below. The Laplacian $\Delta$ is the Laplacian acting on the open space $U_\epsilon$. The condition (iii) implies that $W_k^+ e^{i\pi k} \frac{\partial}{\partial x} D_q(N_k \bar{y}) - W_k^- e^{-i\pi k} \frac{\partial}{\partial x} D_q(N_k \bar{y}) \in H^2_{\Omega}(\Omega)$ so that it lies in the domain of $\Delta_\Omega$.

Since $C \Delta C = \Delta$ (with $C$ the operator of complex conjugation), it suffices to construct $\tilde{W}_k^+$ and to put $\tilde{W}_k^- = C \tilde{W}_k^+$. Then,

$$\tilde{W}_k^+ e^{i\pi k} \frac{\partial}{\partial x} D_q(N_k \bar{y}) - \tilde{W}_k^- e^{-i\pi k} \frac{\partial}{\partial x} D_q(N_k \bar{y}) = 2i \tilde{W}_k^+ e^{i\pi k} \frac{\partial}{\partial x} D_q(N_k \bar{y})$$

In order that $3 \tilde{W}_k^+ e^{i\pi k} \frac{\partial}{\partial x} D_q(N_k \bar{y}) \in \text{Dom}(\Delta_\Omega)$ it is thus sufficient that

$$C \tilde{W}_k^+(0)C = \tilde{W}_k^+(0), \quad C \tilde{W}_k^+(L)C = \tilde{W}_k^+(L).$$

This boundary condition (38) is correct in both the elliptic and hyperbolic cases, although the quasimode construction which motivates it only works in the elliptic case.

As a further preliminary, let us rewrite the equations (37) and the boundary conditions (39) in terms of the semiclassically scaled Laplacian. We observe that $D_q(N_k \bar{y}) = T_{N_k} D_q(\bar{y})$. Let us define:

$$W_k^+ = T_{N_k}^{-1} \tilde{W}_k^+ T_{N_k}.$$

Then (37)-(39) is equivalent to

$$\begin{align*}
(i) & \quad \Delta_{N_k} W_k^+ D_q(\bar{y}) \approx F(k, q + \frac{1}{2})^2 W_k^+ D_q(\bar{y}) \\
(ii) & \quad \Delta_{N_k} W_k^- D_q(\bar{y}) \approx F(k, q + \frac{1}{2})^2 W_k^- D_q(\bar{y}) \\
(iii) & \quad CW_k^+(0)C = W_k^+(0), \quad CW_k^+(L)C = W_k^+(L).
\end{align*}$$

The operators $W_k^+ = W_k^+(\bar{s}, \bar{y}, D_{\bar{y}})$ will be essentially a family of pseudodifferential operators on the transverse space, parametrized by $\bar{s}$. Hence there are no subtleties involving the definition of pseudodifferential operators on manifolds with boundary. Moreover, $W_k^+$ is essentially applied only to a function of $\bar{y}$. To be more precise, we first conjugate by $\mu(a_{\alpha})$ to put the linear term in normal form, and then $W_k^+$ is applied to a function in the kernel of $R$. Hence we only require that the conjugation identity hold as operators applied to functions in the kernel of $R$. Therefore we introduce the following notation: Given an operator $A(s, y, D_y)$, we denote by $A|_{\alpha}$ the restriction of $A$ to functions of $y$ only, i.e. $|\alpha$ denotes the restriction to functions in the kernel of $R$.

The following lemma proves the existence of such a conjugating operator. We emphasize that the conjugation of $\Delta$ takes place over $U_\epsilon$.

**Lemma 4.6.** Let $R_N = \mu(a_{\alpha})^* \Delta_N \mu(a_{\alpha})$ with $\frac{\alpha}{\gamma} \notin \mathbb{Q}$ in the elliptic case. Then there exist polynomial differential operators $P^{w}_{j/2}(\bar{s}, \bar{y}, D_{\bar{y}})$ and $Q^{w}_{j/2}(\bar{s}, \bar{y}, D_{\bar{y}})$ of degree $2j + 2$ on $L^2(\mathbb{R}_y)$ with smooth coefficients in $\bar{s} \in (-\epsilon, L + \epsilon)$ and polynomials $f_j(I)$ of degree $j + 2$ such that:

(a) $P^{w}_{j/2}(\bar{s}, \bar{y}, \eta)$ and $Q^{w}_{j/2}(\bar{s}, \bar{y}, \eta)$ are real-valued;

(b) For each $\bar{s}$, there is a formal $N$-expansion:

$$\begin{align*}
(W_N^+)^{-1} R_N W_N^+ & \sim -N^4 + 2N^2 R + \sum_{j=0}^{\infty} N^{-j} R_{2-\frac{j}{4}}(\bar{s}, D_{\bar{s}}, \bar{y}, D_{\bar{y}})
\end{align*}$$

where

(i) $\mathcal{R}^k_{2-j/2}(\tilde{s}, D\tilde{s}, \tilde{y}, D\tilde{y}) = \mathcal{R}^{\infty, 2}_{2-j/2} + \mathcal{R}^{\infty, 1}_{2-j/2} + \mathcal{R}^{\infty, o}_{2-j/2}$, with $\mathcal{R}^{\infty, k}_{2-j/2} \in C^\infty([0, L], \mathcal{E}_t^{-2k})$;

(ii) $\mathcal{R}^k_{2-j/2}(\tilde{s}, D\tilde{s}, \tilde{y}, D\tilde{y})|_{o} = \mathcal{R}^{\infty, o}_{2-j/2}(\tilde{s}, \tilde{y}, D\tilde{y})|_{o} = f_j(\hat{I})$ for certain polynomials $f_j$ of degree $j+2$ on $\mathbb{R}$;

(iii) $\mathcal{R}^k_{2-j/2}(\tilde{s}, D\tilde{s}, \tilde{y}, D\tilde{y})|_{o} = \mathcal{R}^{\infty, o}_{2-j/2}(\tilde{s}, \tilde{y}, D\tilde{y})|_{o} = 0$;

(c) $W^+_N$ satisfies the boundary condition: $CW^+_N(0)C = W^+_N(0), \quad CW^+_N(L)C = W^+_N(L)$;

(d) We have

$$
[(W^+_N)^{-1}\mathcal{R}_N W^+_N]|_{o} \sim -N^4 + 2N^2R + F^+_N(\hat{I}) + E^+_N
$$

where $F^+_N(\hat{I}) := \sum_{j=0}^K N^{-2j} f_j(\hat{I})$ and where

(e) The error term $E^+_N(\tilde{s}, \tilde{y}, \tilde{\eta}) = N^{-(K+1)} \tilde{E}_N$ where $\hat{I}^{-(K+1)} \tilde{E}_N$ is a bounded operator on $L^2(\mathbb{R})$ for each $\tilde{s}$ with a uniform bound in $N$.

**Proof**

After the linearization step, the scaled Laplacian has the perturbative form

$$
(42) \quad \mathcal{R}^+_N = N^4 + N^2R + E^+_N, \quad E^+_N := \sum_{m=2}^\infty N^{(4-m)} \mathcal{R}^{m}_{2-m/2}
$$

where

$$
(43) \quad \mathcal{R}^{m}_{2-m/2} \in C^\infty(\mathbb{R}, \mathcal{E}_t^{m-4})\mathcal{R}^2 + C^\infty(\mathbb{R}, \mathcal{E}_t^{m-2})\mathcal{R} + C^\infty(\mathbb{R}, \mathcal{E}_t^{m})
$$

Actually, as discussed in (1.7), $a_{\gamma k}$ depends only on the class of $\tilde{\Delta}$ in $S^{2,2(k+2)}(T^*U, \mathbb{R}^{+}\gamma)$ and by (3.1), this is the same as the class of the complete symbol $\sigma_{\bar{\Delta}_N}$ of $\bar{\Delta}_N$ at $\sigma = 0$ modulo $N^{-2(k+2)}$. We may therefore drop higher order terms to get a polynomial partial differential operator $\tilde{\Delta}^{2k+2}$ with the same wave invariants $a_{\gamma j}$ for $j \leq k$. For simplicity we do not indicate this truncation in our notation, but the reader is invited to think of $\bar{\Delta}$ as a polynomial differential operator in the $(\tilde{y}, D\tilde{y})$ variables.

We now construct Weyl symbols $P_{j/2}, Q_{j/2}$ so that iterated composition with $e^{N^{-1}(P+iQ)j/2}$ will successively remove the lower order terms in $\mathcal{R}_N$ after restriction by $|o|$ and so that the boundary condition is satisfied. As mentioned above, all operators will be standard Weyl pseudodifferential operators defined in a neighborhood of $\Omega_0$ and acting only on the transverse space $\mathbb{R}_y$ with coefficients in $\tilde{s}$. Hence we may construct them using the symbolic calculus.

Let us first rewrite the boundary condition in terms of Weyl symbols: On the symbol level, we have

$$
(44) \quad Ca^u(\tilde{s}, \tilde{y}, D\tilde{y})C = \bar{a}^u(\tilde{s}, \tilde{y}, -D\tilde{y})
$$

where $\bar{a}$ is the complex conjugate of $a$. Hence it is natural to split up a Weyl symbol into its real/imaginary parts and into its even/odd parts with respect to the canonical involution $(\tilde{y}, \tilde{\eta}) \rightarrow (\tilde{y}, -\tilde{\eta})$. We note that this splitting is invariantly defined since $a$ is real if and only if $a^u(\tilde{y}, D\tilde{y})$ is self-adjoint and since the even/odd parts of a real symbol are its even/odd parts under conjugation by $C$. Since our
symbols are always Weyl symbols in this section, we omit the superscript $w$ in the future.

By assumption $P_{j/2}$ and $Q_{j/2}$ are real symbols so it remains to split them into their even/odd parts:

$$P_{j/2} = P_{j/2}^e + P_{j/2}^o, \quad Q_{j/2} = Q_{j/2}^e + Q_{j/2}^o. \quad \text{(45)}$$

Now we observe that for general Weyl pseudodifferential operators $P, Q$

$$C(P + iQ)C = iQ \iff P^o = Q^e = 0.$$ 

Therefore, the boundary condition on $W^+_N$ is equivalent to:

$$P_{j/2}^o(0, \bar{y}, \bar{\eta}) = P_{j/2}^o(L, \bar{y}, \bar{\eta}) = 0, \quad Q_{j/2}^e(0, \bar{y}, \bar{\eta}) = Q_{j/2}^e(L, \bar{y}, \bar{\eta}) = 0. \quad \text{(46)}$$

We emphasize that there is no condition on $Q_{j/2}^o$ or $P_{j/2}^e$.

To begin the induction, let us construct $P_{\frac{j}{2}}(\bar{s}, \bar{y}, D_{\bar{y}})$, $Q_{\frac{j}{2}}(\bar{s}, \bar{y}, D_{\bar{y}}) \in C^\infty([0, L]) \otimes \mathcal{E}^3$ so that the boundary conditions are satisfied and so that

$$e^{-N^{-1}(P+iQ)\frac{j}{2}} R_N e^{N^{-1}(P+iQ)\frac{j}{2}}|_o = \{N^4 + N^2 R + E_N^1\}|_o. \quad \text{(47)}$$

Expanding the exponential, we get to leading order the homological equation:

$$\{[R, (P + iQ)\frac{j}{2}] + R\frac{j}{2}\}|_o = 0. \quad \text{(48)}$$

Taking the complete symbol of both sides we get the symbolic homological equation:

$$i\{\sigma + \frac{\alpha}{L} I, P\frac{j}{2} + iQ\frac{j}{2}\} + R\frac{j}{2}|_o = 0. \quad \text{(49)}$$

The equation may be rewritten in the form:

$$\partial_s (P + iQ)\frac{j}{2}(\bar{s}, r_{\alpha}(\bar{s})(\bar{y}, \bar{\eta})) = -i R\frac{j}{2}|_o. \quad \text{(50)}$$

The solution has the form:

$$P + iQ\frac{j}{2}(\bar{s}, r_{\alpha}(\bar{s})(\bar{y}, \bar{\eta})) = (P + iQ)\frac{j}{2}(0) - i \int_0^s R\frac{j}{2}(u, \bar{y}, \bar{\eta})du. \quad \text{(51)}$$

We need to determine $(P + iQ)\frac{j}{2}(0)$ so that the boundary conditions at $\bar{s} = 0$ and $\bar{s} = L$ are satisfied. To clarify the boundary conditions and their solvability, we separate out real/imaginary and even/odd parts in the equations to get:

$$\partial_s P\frac{j}{2} + \frac{\alpha}{L} I, P\frac{j}{2} = \Im R\frac{j}{2}|_o \quad \partial_s Q\frac{j}{2} + \frac{\alpha}{L} I, Q\frac{j}{2} = \Im R\frac{j}{2}|_o \quad \text{(52)}$$

$$\partial_s Q\frac{j}{2} + \frac{\alpha}{L} I, Q\frac{j}{2} = -\Re R\frac{j}{2}|_o \quad \partial_s Q\frac{j}{2} + \frac{\alpha}{L} I, Q\frac{j}{2} = -\Re R\frac{j}{2}|_o$$

together with the boundary conditions

$$P\frac{j}{2}(0) = P\frac{j}{2}(L) = 0, \quad Q\frac{j}{2}(0) = Q\frac{j}{2}(L) = 0. \quad \text{(53)}$$

We now observe as in the boundaryless case (cf. [Z.1], Lemma (2.22)) that $R\frac{j}{2}(u, \bar{y}, \bar{\eta}) = \bar{y} \circ (a r_{\alpha}^{-1}(u)) I$ is a polynomial of degree 3 in $(\bar{y}, \bar{\eta})$ in which every term is of odd degree in $(\bar{y}, \bar{\eta})$. By (53) it follows that $P\frac{j}{2}, Q\frac{j}{2}$ are also odd polynomials of degree 3.
To analyse the equations further we change coordinates. In the elliptic case, we use the complex cotangent variables $z = y + i\eta, \bar{z} = y - i\eta$ which satisfy $\{l, z^m \bar{z}^n\} = i(z \frac{\partial}{\partial z} - \bar{z} \frac{\partial}{\partial \bar{z}})z^m \bar{z}^n = i(m - n)z^m \bar{z}^n$. We then write:

$$P^e_\frac{1}{2}(\tilde{s}, z, \bar{z}) = \sum_{m,n,m+n \leq 3} p^e_{\frac{1}{2}mn}(\tilde{s})(z^m \bar{z}^n + \bar{z}^m z^n),$$  
$$P^o_\frac{1}{2}(\tilde{s}, z, \bar{z}) = \sum_{m,n,m+n \leq 3} p^o_{\frac{1}{2}mn}(\tilde{s})(z^m \bar{z}^n - \bar{z}^m z^n).$$

In the hyperbolic case, we first use real coordinates $(y, \eta)$ in which $I^h = \frac{1}{2}(\eta^2 - y^2)$. We then define $w = y + \eta, \bar{w} = y - \eta$ so that $I^h = \frac{1}{2}|w|^2 := \frac{1}{2}w\bar{w}$ and so that $\{l, w^m \bar{w}^n\} = (m - n)y^m \eta^n$. Since the elliptic and hyperbolic cases are quite similar, we only carry out the details in the elliptic case and refer to [Z.2] (Lemma 3.1) for complete details on the hyperbolic case. We will refer to $m = n$ terms as the ‘diagonal terms’. They only occur for even $j$ and are terms which are polynomials in the action variables.

Then (52) may be rewritten in terms of these coordinates. In the elliptic case, we have:

$$\frac{d}{ds}p^e_{\frac{1}{2}mn}(s) + i\frac{1}{2}(m-n)p^o_{\frac{1}{2}mn}(s) = \Im R^e_{\frac{1}{2}}|_o$$
$$\frac{d}{ds}p^o_{\frac{1}{2}mn}(s) + i\frac{1}{2}(m-n)p^e_{\frac{1}{2}mn}(s) = \Im R^o_{\frac{1}{2}}|_o$$

Similarly in the hyperbolic case, although there is no factor of $i$ in the second terms on the left sides. Since $m \neq n$, we can immediately solve for $p^e_{\frac{1}{2}mn}, q^e_{\frac{1}{2}mn}$:

$$p^e_{\frac{1}{2}mn}(s) = \frac{-L}{2\alpha(m-n)}\frac{d}{ds}p^o_{\frac{1}{2}mn}(s) + \Im R^e_{\frac{1}{2}}|_o$$
$$q^e_{\frac{1}{2}mn}(s) = \frac{-L}{2\alpha(m-n)}\frac{d}{ds}q^o_{\frac{1}{2}mn}(s) + \Im R^e_{\frac{1}{2}}|_o$$

We thus eliminate the $p^e_{\frac{1}{2}mn}, q^e_{\frac{1}{2}mn}$ variables and reduce to (uncoupled) second order equations for the independent variable $p^o_{\frac{1}{2}mn}, q^o_{\frac{1}{2}mn}(s)$. In the elliptic case, they read:

$$-\frac{d^2}{ds^2}p^o_{\frac{1}{2}mn}(s) - \frac{1}{2}(m-n)^2p^o_{\frac{1}{2}mn}(s) = \frac{-L}{\alpha(m-n)}\frac{d}{ds}\Im R^e_{\frac{1}{2}}|_o + \Im R^e_{\frac{1}{2}}|_o$$
$$-\frac{d^2}{ds^2}q^o_{\frac{1}{2}mn}(s) - \frac{1}{2}(m-n)^2q^o_{\frac{1}{2}mn}(s) = -\frac{-L}{\alpha(m-n)}\frac{d}{ds}\Re R^e_{\frac{1}{2}}|_o + \Re R^e_{\frac{1}{2}}|_o$$
In the hyperbolic case, the operator on the left side is \(-\frac{d^2}{dz^2} + [\frac{a}{\pi}(m-n)]^2\). The boundary conditions on \(p_{2mn}^q, q_{2mn}^q\) are (in both elliptic and hyperbolic cases):

\[
p_{2mn}^q(0) = 0, \quad p_{2mn}^q(L) = 0
\]

\[
q_{2mn}^q(0) = 0, \quad q_{2mn}^q(L) = 0.
\]

The boundary value problem (57) - (58) is always solvable unless 0 is an eigenvalue of the operator \(D_2^2 - [\frac{a}{\pi}(m-n)]^2\) (elliptic case), resp. \(D_2^2 + [\frac{a}{\pi}(m-n)]^2\) (hyperbolic case) with boundary conditions \(q(0) = 0 = q(L)\). The eigenfunction would have to have the form \(\sinh(\frac{a}{\pi}(m-n)s)\) in the elliptic case or \(\sinh(\frac{a}{\pi}(m-n)s)\) in the hyperbolic case. Hence in the elliptic case, a sufficient condition for solvability is that \(\alpha/\pi \notin \mathbb{Q}\) while in the hyperbolic case there is no obstruction if \(\alpha \neq 0\).

Thus we have solved the conjugation problem to third order. The exponents \(P_{1/2}, Q_{1/2}\) are odd polynomial differential operators of degree 3 with smooth coefficients defined in a neighborhood of \([0, L]\). By construction, the \(Q_{1/2}\)'s will always have the same order, same order of vanishing, and same parity as the restriction \(R_{1/2}^q\). It follows that

\[
N^{-1}ad((P + iQ)_{1/2}) : N^{-k}\Psi^l(R) \otimes \mathcal{E}^m \rightarrow N^{-(k+1)}[\Psi^{l-1}(R) \otimes \mathcal{E}^{m+3} + \Psi^l(R) \otimes \mathcal{E}^{m+1}].
\]

We now carry the process forward one more step because, as in the boundaryless case, the even steps require something new. In the second step, \(R_N\) is replaced by

\[
R_N^\frac{1}{2} = e^{-N^{-1}(P + iQ)_{1/2}}R_N e^{N^{-1}(P + iQ)_{1/2}} \in \Psi_N^2(R^1 \times R).
\]

We expand in powers of \(N\) to get:

\[
R_N^\frac{1}{2} \sim \sum_{n=0}^{\infty} N^{-4+n} \sum_{j+m=n} \frac{i^j}{j!}(ad(P + iQ)_{1/2})^j R_{2-n^2} = N^{-4} + N^{-2} R + \sum_{n=3}^{\infty} N^{-4+n} R_N^\frac{1}{2} - \frac{1}{2}.
\]

An obvious induction as in \([Z, 1]\) gives that

\[
ad(P + iQ)_{1/2})^j R_{2-n^2} \in C^\infty(R, \mathcal{E}^{m+j-4})R^2 + C^\infty(R, \mathcal{E}^{m+j-2})R + C^\infty(R, \mathcal{E}^{m+j}).
\]

It follows that \(R_N^\frac{1}{2} - \frac{1}{2}\) has the same filtered structure as \(R_2 - \frac{1}{2}\).

We now conjugate \(R_N^\frac{1}{2}\) with an exponential of the form \(e^{-N^{-2}(P + iQ)_{1/2}}\). As above, it leads to a boundary problem analogous to the previous case with \(j = 1\) except that now diagonal terms with \(m = n\) do occur. In the elliptic (resp. hyperbolic) case, a diagonal term is a function of \(|z|^2\) (resp. \(|w|^2\)), hence is even under the involution \(z \rightarrow \bar{z}\) (resp. \(w \rightarrow \bar{w}\)). Since the equations for the even/odd coefficients decouple completely, the boundary condition reduces to:

\[
P_{1d}^q(0, |z|^2) = P_{1d}^q(L, |z|^2) = 0
\]

\[
Q_{1d}^q(0, |z|^2) = Q_{1d}^q(L, |z|^2) = 0.
\]

Similarly in the hyperbolic case with \(w\) in place of \(z\).
The same is true for any even \(j\) so let us consider the general elliptic case. We write the diagonal terms in the form:

\[
P^d_j(s, \abs{z}^2) = \sum_{m:m \leq j} p^d_{jm}(s) \abs{z}^{2m}
\]

We note that \(O^p\).

In the case which we must address here is that the real parts \(P\), \(Q\) and we therefore omit the details. An important point is that the real parts \(P\), \(Q\) and \(R\) are real, all three operators being self-adjoint with zero boundary conditions in general. To satisfy the boundary condition we need to add terms \(f_j(\abs{z}^2)\) to the right side. Then we get:

\[
\frac{d}{ds} p^d_{jm}(s) = \Im R^d_{2-j}\big|_o
\]

with zero boundary conditions in general. To satisfy the boundary condition we need to add terms \(f_j(\abs{z}^2)\) to the right side. Then we get:

\[
\frac{d}{ds} q^d_{jm}(s) = -\Re R^d_{2-j}\big|_o
\]

with the boundary condition above on \(p^d_{jm}, q^d_{jm}\). We can solve the equations with

\[
p^d_{jm}(s) = \int_0^s \{-\Im R^d_{2-j}\big|_o(u, \abs{z}^2) + \Re f_j(\abs{z}^2)\} du,
\]

\[
q^d_{jm}(s) = \int_0^s (\Re R^d_{2-j}\big|_o(u, \abs{z}^2) + \Im f_j(\abs{z}^2)) du,
\]

where

\[
f_j(\abs{z}^2) = -\frac{1}{L} \int_0^L R^d_{2-j}\big|_o(u, \abs{z}^2) du.
\]

In the case \(j = 1\), we have then conjugated \(R_N\) to \(N^4 + N^2 R + Op^\infty(f_1(I)) + O(N^{-1})\).

We note that \(Op^\infty(f_1(I))\) is a function of \(I\), so we have conjugated to normal form to fourth order.

We then proceed inductively to define polynomial symbols \((P + iQ)_2(s, \bar{y}, \bar{\eta})\), polynomials \(f_j(I)\) and unitary \(N\)-pseudodifferential operators \(W_{N, j} := \exp(N^{-j}(P + iQ)_2)\) such that:

1. \(\mathcal{R}^\infty_{N, j} := W_{N, j}^{-1} W_{N, j-1} \cdots W_{N, j} W_{N, j+1} \cdots W_{N, j} W_{N, j-1} W_{N, j}^{-1}\);
2. \(\mathcal{R}^\infty_N \sim N^4 + N^2 R + \sum_{n=3}^{2K} N^{4-n} \mathcal{R}^\infty_{2-j} + E^{+}_{N, j}\);
3. For \(K \geq n - 2, \mathcal{R}^\infty_{2-j} = \mathcal{R}^\infty_{2-j} = [\mathcal{R}, (P + iQ)_2] + \mathcal{R}^\infty_{2-j}\);
4. \(\mathcal{R}^\infty_{2-j}(\bar{s}, \bar{D}_s, \bar{y}, \bar{D}_y)\big|_o = \mathcal{R}^\infty_{2-j}(\bar{s}, \bar{y}, \bar{D}_y)\big|_o = f_j(I)\big|_o\);
5. \(\mathcal{R}^\infty_{2-j+1}(\bar{s}, \bar{D}_s, \bar{y}, \bar{D}_y)\big|_o = \mathcal{R}^\infty_{2-j+1}(\bar{s}, \bar{y}, \bar{D}_y)\big|_o = 0\);
6. \(\mathcal{R}^\infty_{N} \mathcal{R}^\infty_{N} = C^\infty(\mathbb{R}, \mathcal{E}^{m-4}) \mathcal{R} + C^\infty(\mathbb{R}, \mathcal{E}^{m-2}) \mathcal{R} + C^\infty(\mathbb{R}, \mathcal{E}^{m})\);
7. \((P + iQ)_2\big|_o = C^\infty(\mathbb{R}, \mathcal{E}^{m+1})\);
8. \((P + iQ)_2\big|_o = (P + iQ)_2\big|_o = 0\);
9. \(E^+_{N, j}\) is divisible by \(N^{4-(2K+1)}\).

The details about the degrees and parities of the polynomials are similar to the boundaryless case of \([\mathbb{R}]\) and we therefore omit the details. An important point which we must address here is that the real parts \(P_{j/2}\) do not contribute to the principal part of the exponent \((P + iQ)_{j/2}\). More precisely, the homogeneous part of \(P_{j/2}\) of leading order \(2j + 2\) equals zero. This happens essentially because the Weyl symbols of \(\Delta, \Delta_N\) and \(\mathcal{R}^\infty_N\) are real, all three operators being self-adjoint with
respect to Lebesgue measure. Conjugation by unitary operators \( e^{iN^{-1}Q_{j/2}} \) preserves this reality. At first sight it appears that the \( R_{N/2}^k \) operators continue to have real symbols and therefore that \( P_{j/2} \) is zero. However, the \(|o|\) operation kills part of the operator and in particular it kills the self-adjointness and reality of symbols. The \( R_{N/2}^k \) generally have complex symbols and one does needs to conjugate with \( e^{-iN^{-1}(P_{j/2}+iQ_{j/2})} \).

What is true is that the leading order terms of order \( m \) the Weyl symbol of \( R_{N/2}^k \) in \((y, \eta)\) is real. As is visible from (vi) above, such terms have no factor of \( R \) in front while any term with a factor of \( R \) has a lower degree in \((y, \eta)\). Hence taking \(|o|\) commutes with setting \( \sigma_R = 0 \) in terms of top degree. But setting \( \sigma_R = 0 \) in a real symbol produces another real symbol, so the terms of top degree in \((y, \eta)\) must be real.

Let us consider the statement (ix) about the error terms. In each conjugation we expand the exponential up to order \( N^{-K} \) and absorb the remainder in the higher powers of \( N^{-1} \).

To analyse the error, we write the partial Taylor expansion with remainder of the exponential function as:

\[
e^{ix} = e_M(ix) + r_M(ix), \quad e_M(ix) = 1 + ix + \cdots + \frac{(ix)^M}{M!},
\]

\[
r_M(ix) = (ix)^{M+1} b_M(ix), \quad b_M(ix) = \int_0^1 \cdots \int_0^1 t_{M+1} \cdots t_0 e^{i\sum t_j} dt_0 \cdots dt_M.
\]

Clearly \( b_M \) is a bounded function on the axis of real \( x \). Now plug in \( ix = N^{-j} \text{ad}(P + iQ)_{j/2} \); although it is complex, its principal part is real. We get that

\[
e^{N^{-j} \text{ad}(P + iQ)_{j/2}} = I + N^{-j} \text{ad}(P + iQ)_{j/2} + \cdots + \left[ \frac{(N^{-j} \text{ad}(P + iQ)_{j/2})^M}{M!} \right]
\]

\[
+ [N^{-j} \text{ad}(P + iQ)_{j/2}]^{M+1} b_M(N^{-j} \text{ad}(P + iQ)_{j/2}).
\]

At the \( 2K \)th stage we are conjugating with \( 2K \) factors \( e^{iN^{-j}Q_{j/2}} \), \( j = 1, \ldots, K \). To get a normal form up to order \( N^{4-(2K+1)} \) at the \( K \)th stage it suffices to choose each \( M_j \) such that \( j(M_j + 1) > K \). The \( e_M \) terms give a polynomial differential operator satisfying (vi) and the \( b_M \) terms give the error \( E^+_{N\frac{k}{2}} \). By construction, it is \( N^{4-(2K+1)} \) times a conjugate of a polynomial differential operator of degree \( K + 2 \) in \((\bar{y}, \eta)\).

4.4.1. Remarks. (a) The operators \( \tilde{W}^+_{k} \) with which we began are now determined by (44), i.e. by \( \tilde{W}^+_k = T_{N_k} W^+_k T_{N_k}^{-1} \). We have:

\[
\tilde{W}^+_k = \prod_{j=1}^\infty e^{-N^{-j}T_{N_k}(P + iQ)_{j/2}T_{N_k}^{-1}}.
\]

As will be clear below the \( \tilde{W}^+_k \)'s belong to the class of homogeneous Fourier integral operators, unlike the \( W^+_k \)'s, which are exponentials of isotropic pseudodifferential operators.

(b) In the next section we will glue together the \( \tilde{W}^+_k \)'s into a Fourier integral operator. The vanishing of the principal part of \( P_{j/2} \) then implies that it does not contribute to the principal symbol of \((P + iQ)_{j/2}\), but only to the amplitude. It will also imply that \( b_M(N^{-j} \text{ad}(P + iQ)_{j/2}) \) is a bounded FIO.
Finally, let us note that $CW_N^e i N^2 L C = W_N^e i N^2 L$ if and only if $e^{-i N^2 L} = e^{i N^2 L}$, i.e. if and only if $N^2 = \frac{\pi}{2} k$ for some $k$. Henceforth we put $N_k = \sqrt{\frac{\pi}{2k}}$ and only consider these special values of $N$. For notational simplicity we write $W_k^+$ for $W_{N_k}^+$. We also define $F(k, \hat{I})$ as the formal asymptotic series whose $K$th partial sum is $F_{N_k K}(\hat{I})$.

5. Conjugation of the Dirichlet Wave group

So far, we have conjugated the semiclassical Laplacian on the open space in a microlocal neighborhood of $\gamma$ to a semiclassical normal form modulo a small semiclassical remainder. We now glue together the component intertwining operators $\tilde{W}_k^+ = T_{N_k} W_k^+ T_{N_k}^{-1}$ into a homogeneous Fourier integral intertwining operator of the Dirichlet wave group to its normal form. We emphasize that the intertwining operators are defined in the open space.

Conjugation by the scaling operator $T$ and its local form $T_{N_k}$ plays an important role since it converts isotropic pseudodifferential operators into standard homogeneous pseudodifferential operators. This will be discussed further below. Since it becomes heavy notationally to distinguish the action operators $\hat{I}$ from their conjugates we abuse notation somewhat by denoting both $\hat{I}$ and $T\hat{I}T^{-1}$ by the same symbol $\hat{I}$.

By the Dirichlet wave kernel we mean the fundamental solution $E(t, x, y)$ of the mixed wave equation (4) with Dirichlet boundary conditions, i.e. the kernel of $\cos t \sqrt{-\Delta \Omega}$. The ‘free’ Laplacian $\Delta$ on the open space is only well-defined in a neighborhood of $\gamma$, so to be precise we need to cut it off to $U_{\epsilon}$. We will not indicate this cutoff in the notation because we will explicitly microlocalize it later on.

5.1. Local components on $S^{1}_{2L} \times \mathbb{R}$. By ‘local component’ we mean an operator which only acts on a specific Fourier coefficient.

Definition 5.1. Define the operators: $\tilde{W}_k : L^2(S^{1}_{2L} \times \mathbb{R}) \to L^2(S^{1}_{2L} \times \mathbb{R})$

$$\tilde{W}_k = \tau_c (\tilde{W}_k^+ \Pi_{+k} - \tilde{W}_k^- \Pi_{-k})$$

Here, $\tilde{W}_k^- = CW_k^+ \Pi_{+k} C$.

Proposition 5.2. We have:

(i) $\tau_c \Delta \tilde{W}_k^\pm \Pi_{\pm k} = \tau_c \{\tilde{W}_k^\pm F(|D|, \hat{I})^2 \Pi_{\pm k} + \tilde{W}_k^\pm E_{kk}^\pm \Pi_{\pm k}\}$.

Here, $|D| = \sqrt{-D^* D}$ on $S^{1}_{2L}$.

(ii) $\tau_c \Delta \tilde{W}_k \Pi_o = \tau_c \{\tilde{W}_k F(|D|, \hat{I})^2 \Pi_o + \tilde{W}_k E_{ko}^+ \Pi_o\}$.

(iii) The Schwarz kernel of $\tilde{W}_k \Pi_o$ vanishes on $\partial \Omega_o$.

Proof

(i) This is essentially just a restatement of Lemma (4.6). We can prove it by testing both sides against functions of the form $e^{ik s} f(N_k y)$. We do the case $k > 0$. On the
set \( \tau_\epsilon \equiv 1 \) we have:
\[
\Delta \tilde{W}_k^+ e^{i \frac{2\pi k}{L}} f(N_ky) = e^{i \frac{2\pi k}{L}} e^{-i \frac{2\pi k}{L}} \Delta e^{i \frac{2\pi k}{L}} T_{N_k} W_k^{+1} T_{N_k}^{-1} f(N_ky)
\]
\[
= e^{i \frac{2\pi k}{L}} T_{N_k} \Delta N_k W_k^{+} f(y) = e^{i \frac{2\pi k}{L}} T_{N_k} W_k^{+} (F(k, \tilde{I})^2 + E_k^L) f(y)
\]
\[
= \tilde{W}_k^+ (F(k, \tilde{I}))^2 e^{i \frac{2\pi k}{L}} f(N_ky) + \tilde{W}_k^+ E_k^L e^{i \frac{2\pi k}{L}} f(N_ky)
\]
\[
= \{ \tilde{W}_k^+ (F(|D|, \tilde{I}))^2 + \tilde{W}_k^+ E_k^L \} \Pi_{\pm k} e^{i \frac{2\pi k}{L}} f(N_ky).
\]

Between lines two and three, the action operator \( \hat{T} \) changed from its isotropic form to its homogeneous form. The case of \( k < 0 \) follows by taking complex conjugates.

(ii) This follows immediately from (i) by taking the difference of the + and – cases.

(iii) The Schwarz kernel of \( \tilde{W} \Pi_\epsilon \) is given by \( \sum_k \tilde{W}_k \Pi_\epsilon \), so it suffices to show that \( \tilde{W} \sin(\frac{2\pi k}{L}) f(N_ky) = 0 \) on \( \tilde{s} = 0, L \). This follows from Lemma (4.6), since \( (\tilde{W}_k \sin(\frac{2\pi k}{L}) f(N_ky) = 0 \) on \( \tilde{s} = 0, L \). Also, we obviously have \( \sin(\frac{2\pi k}{L}) f(N_ky') = 0 \) on \( \tilde{s} = 0, L \).

5.2. The homogeneous intertwiner. We glue the \( \tilde{W}_k \) together as follows:

**Definition 5.3.** Define the operators \( \tilde{W}^\pm, \tilde{W} : L^2(S^1_{2L} \times \mathbb{R}_y) \to L^2(S^1_{2L} \times \mathbb{R}_y) \) by

(i) \( \tilde{W}^+ = \sum_{k=1}^{\infty} \tilde{W}_k^+ \Pi_k^+ \)

(ii) \( \tilde{W}^- = \sum_{k=1}^{\infty} \tilde{W}_k^- \Pi_k^- \)

(iii) \( \tilde{W} = \tilde{W}^+ - \tilde{W}^- \).

We now analyse the Fourier integral nature of \( \tilde{W}, \tilde{W}^\pm \). A key point is that the operators \( T_{N_k} \) glue together to form a ‘global’ transverse scaling operator
\[
T : L^2([0, L]_x \times \mathbb{R}_y) \to L^2([0, L]_x \times \mathbb{R}_y)
\]
\[
(68)
T \sin(\frac{2\pi k}{L}) f(y) := N_k \sin(\frac{2\pi k}{L}) f(N_ky).
\]

Conjugation with \( T \) transforms the isotropic Weyl calculus into the usual homogeneous pseudodifferential calculus. For instance
\[
(69)
T(D_y^2 + \tilde{y}^2)^* = |D_s|^{-1} D_y^2 + \tilde{y}^2 |D_s|, \quad T(D_y^2 - \tilde{y}^2)^* = |D_s|^{-1} D_y^2 - \tilde{y}^2 |D_s|.
\]

As is verified in (3) and elsewhere, \( T \) is an oscillatory integral operator associated to the canonical transformation
\[
(70)
\psi(s, \sigma, y, \eta) = (s + \frac{|\eta|}{2\sigma}, \sigma, \sqrt{\sigma y}, \sqrt{\sigma^{-1} \eta}).
\]

The following proposition is analogous way to (3.1). Below, the notation \( A \sim B \) in \( V \) means that \( A - B \) is smoothing in \( V \). In the following proposition we assume the order \( K \) of the Birkhoff normal form is infinity.

**Proposition 5.4.** There exist conic neighborhoods \( V^\pm, V^{\pm'} \) of \( \gamma^\pm := \gamma_o \cap \{ \pm \sigma > 0 \} \) and canonical transformations \( \chi^\pm : V^\pm \to V^{\pm'} \) such that:

(i) \( \chi^\pm = Id \) on \( R_{\gamma^\pm} := \{ s, \sigma, \tilde{y}, \tilde{\eta} \} \in V^\pm : \tilde{y} = \tilde{\eta} = 0 \};

(ii) \( \tilde{W}^\pm \in L^0(S^1_{2L} \times \mathbb{R} \times S^1_{2L} \times \mathbb{R}; gr\chi^\pm) \) where \( gr\chi^\pm \subset V^\pm \times V^\pm' \) is the graph of \( \chi^\pm \).
(iii) \( \tilde{W}^\pm \) is elliptic in \( V^\pm \times V^\pm \).

**Proof**

To analyse the sum over \( k \), we enlarge the ‘open model space’ \( S^1 \times \mathbb{R} \) to the ‘space-time’ \( S^1 \times \mathbb{R} \times S^1 \) and define the operator

\[
e^{i[t|D_s]} : L^2(S^1 \times \mathbb{R}) \rightarrow L^2(S^1 \times \mathbb{R} \times S^1), \quad e^{i[t|D_s]} e^{i\frac{\pi i}{k} f(y)} = e^{i\frac{\pi i}{k} f(y)}.
\]

The range of \( e^{i[t|D_s]} \) is contained in the kernel \( \mathcal{H} \) of the ‘wave operator’ \(|D_s|^2 - |D_t|^2\) on \( S^1 \times \mathbb{R} \times S^1 \). We also denote by \( P \) the orthogonal projection to \( \mathcal{H} \). We then introduce the further operators:

**Definition 5.5.** Define \( \tilde{W}^\pm_{|D_s|}((\bar{s}, \bar{y}, D_y)) \) and \( \tilde{W}_{|D_s|} \) on \( L^2(S^1 \times \mathbb{R} \times S^1) \)

(a) For \( k \geq 0 \) put

\[
\tilde{W}^+_{|D_s|}((\bar{s}, \bar{y}, D_y)) = \Pi^+ f((\bar{s}, N_k \bar{y})) e^{i\frac{\pi i}{k} \hat{W}^+_k f((\bar{s}, N_k \bar{y}))};
\]

For \( k < 0 \) put it equal to zero.

(b) \( \tilde{W}^-_{|D_s|}((\bar{s}, \bar{y}, D_y)) = \tilde{W}^+_{|D_s|}((\bar{s}, \bar{y}, D_y)) C; \)

(c) \( \tilde{W}_{|D_s|}((\bar{s}, \bar{y}, D_y)) = \tilde{W}^+_{|D_s|}((\bar{s}, \bar{y}, D_y)) \Pi^+ - \tilde{W}^-_{|D_s|}((\bar{s}, \bar{y}, D_y)) \Pi^- \).

These definitions suggest the introduction of a scaling operator \( \tilde{T} \) adapted to the \( t \)-variable. We therefore define \( T e^{it \text{m}} f(s, y) = N_m e^{it \text{m}} f(s, N_m y) \). We are only interested in its action on the invariant subspace \( \mathcal{H} \) where the frequency in \( s \) and \( t \) are the same. As with \( T \), \( \tilde{T} \) is an oscillatory integral operator with underlying canonical transformation

\[
\psi(s, \sigma, t, \tau, y, \eta) = (s, \sigma, t + \frac{y \eta}{2 \tau}, \tau, \sqrt{\tau y}, \sqrt{\tau^{-1} y}).
\]

We then have:

\[
\tilde{W}^+_{|D_s|}((s, y, D_y)) = \Pi^+_{j \geq 0} e^{i|D_s|^{-1/2} (\tilde{P} + i \tilde{Q})}_{j/2}((s, \bar{y}, D_y)), \quad \text{with} \quad \tilde{P} = \tilde{T} P \tilde{T}^{-1}, \quad \tilde{Q} = \tilde{T} Q \tilde{T}^{-1}.
\]

The operators \( \tilde{P}, \tilde{Q} \) are (usual) homogeneous pseudodifferential operators, as we will argue below.

The following identity is the key to the Fourier integral properties of \( W^\pm, W \):

Let \( j : S^1 \times \mathbb{R} \rightarrow S^1 \times \mathbb{R} \) be the inclusion \( j(s, y) = (s, 0, y) \). Then:

\[
\tilde{W}^\pm((\bar{s}, \bar{y}, D_y)) = j^* \tilde{W}^\pm_{|D_s|}((\bar{s}, \bar{y}, D_y)) e^{i[t|D_s]}
\]

(72)

\[
\tilde{W} = j^* \tilde{W}_{|D_s|} e^{i[t|D_s]}
\]

To prove it, we apply both sides to functions of the form \( e^{i\frac{\pi i}{k} \eta} f(N_k y) \). In the plus case, we have (for \( k \geq 0 \)),

\[
j^* \tilde{W}^+_{|D_s|}((\bar{s}, \bar{y}, D_y)) e^{i[t|D_s]} e^{i\frac{\pi i}{k} \eta} f(N_k y)
\]

\[
= j^* e^{i\frac{\pi i}{k} \eta} \tilde{W}^+_k((\bar{s}, \bar{y}, D_y)) e^{i\frac{\pi i}{k} \eta} f(N_k y)
\]

\[
= \tilde{W}^+ e^{i\frac{\pi i}{k} \eta} f(N_k y)
\]

and similarly in the minus case.
We now complete the proof of (i)-(iv). For simplicity we only consider the + case, the other being essentially the same. By definition, $\tilde{W}^+_|\tilde{D}_4|$ is a product of factors of the form $e^{i|D_t|^{-j/2}}(\tilde{P}_{j/2} + i\tilde{Q}_{j/2})$ with $\tilde{P}_{j/2}, \tilde{Q}_{j/2}$ scaled polynomial differential operators of degree $j + 2$ in the variables $|D_t|^{\frac{j}{2}} y$ and $|D_t|^{-\frac{j}{2}} y$. As noted in the proof of Lemma (4.6), the terms in $\tilde{P}_{j/2}$ have the parity of $j$ and the leading order term in $(y, D_y)$ vanishes. Hence, $|D_t|^{-j/2} \tilde{P}_{j/2}$ is actually a pseudodifferential operator of at most zero order and its does not affect the status of $\tilde{W}^+_|\tilde{D}_4|$ as a Fourier integral operator. Since $\tilde{Q}_{j/2}$ is a polynomial order $j + 2$ in $(|D_t|^{\frac{j}{2}} y, |D_t|^{-\frac{j}{2}} y)$, it follows that $|D_t|^{-j/2} (\tilde{P} + i\tilde{Q})_{j/2}$ is a first order (homogeneous) pseudodifferential operator of real principal type for each $j$. Its exponential $e^{i|D_t|^{-j/2}(\tilde{P} + i\tilde{Q})_{j/2}}$ is therefore a Fourier integral operator. Hence the product of any finite number of factors is a Fourier integral operator. As discussed above, we only need to deal with finitely many factors, but we note that the full infinite product can be regularized as a Fourier integral operator. Indeed, one may apply the Baker-Campbell-Hausdorff formula to any finite number of factors to rewrite $\tilde{W}^+_|\tilde{D}_4|$ in the form $\tilde{W}^+_|\tilde{D}_4| = e^{\sum_{j=1}^{\infty} |D_t|^{-j/2}(P_{j/2} + iQ_{j/2})}$. If one interprets the sum as given by the Borel summation method then the exponent is well-defined as a pseudodifferential operator of order one and of real principal type. The principal symbol of the series $\sum_j |D_t|^{-j/2}(P_{j/2} + iQ_{j/2})$ is then well-defined as an element of $S^{1,2}$ of the form $H(s, y, \eta, \tau) := \sum_j \tau^{-j/2} q_j(s, \sqrt{T} y, \sqrt{T}^{-1} \eta)$ in $(y, \eta)$, with $q_j$ the leading order homogeneous part of the Weyl symbol of $Q_{j/2}$. The symbol of $P_{j/2}$ is of order at most zero as a symbol and hence does not contribute to $H$.

The operators $e^{it\tilde{D}_4}$ and $j^*$ are clearly Fourier integral operators, with associated canonical relations

$$C = \{(s, \sigma, y, \eta) : (s + t, \sigma, t, \sigma, y, \eta)\} \subset T^*(S^1_{2L} \times \mathbb{R} \times S^1_{2L} \times \mathbb{R})$$

$$\Gamma_{j^*} = \{(s, \sigma, y, \eta) : (s, \sigma, 0, \sigma, y, \eta)\} \subset T^*(S^1_{2L} \times \mathbb{R} \times S^1_{2L} \times \mathbb{R} \times S^1_{2L}).$$

The canonical relation underlying $j^* \tilde{W}^+_|\tilde{D}_4| e^{it\tilde{D}_4}$ is therefore given by the composite relation

$$\Lambda = \Gamma_{j^*} \circ \text{gr } \phi^1 \circ C$$

where $\phi^u = \text{exp } u\Xi_H$ with $\Xi_H$ the Hamilton vector field of $H$ and $\text{exp } u\Xi_H$ its flow. The first two factors compose to the relation $\{(s, \sigma, y, \eta) : \phi^1(s + t, \sigma, t, \sigma, y, \eta)\}$. It is easy to see that $\phi^1(s + t, \sigma, t, \sigma, y, \eta)$ has the form $(s + t, \sigma(1), t(1), \sigma(y(1), \eta(1))$ where $x(1)$ stands for the value at $u = 1$ of the of $x$-coordinate of the orbit of $\phi^u$ through the initial point $(s + t, \sigma, t, \sigma, y, \eta)$. To contribute to $\Lambda$ one must have $0 = t(1) = t + \int_0^1 \frac{DH}{\sigma}(\phi^u(s, \sigma, t, \sigma, y, \eta)) du$. We observe that $H$ vanishes to order at least two along $\mathbb{R}^+ \gamma_0 = \{y = \eta = 0\}$ and hence that $\phi^u$ acts as the identity on this set. In particular, $t(1) = t = 0$ there. Futhermore, for $(y, \eta)$ sufficiently small, the derivative of $t(1)$ with respect to $t$ is non-zero and hence there exists a unique solution $t(s, \sigma, y, \eta)$ of the equation relating $t(1)$ and $t$. It follows that (at least) in a sufficiently small cone around $(y = \eta = 0)$, $\Lambda$ is the graph of the canonical transformation $\psi^+(s, \sigma, y, \eta) = \phi^1(s + t(s, \sigma, y, \eta), \sigma(t(s, \sigma, y, \eta), \sigma, y, \eta)$ (where the $(t, \sigma)$-coordinates are omitted on the left side).

The $-$ component has a similar description. Therefore, $W$ is a microlocal Fourier integral operator associated to the graph of a canonical transformation defined in
a small cone around $\mathbb{R}^+ \gamma_o$. Each of the three operators in its composition has a canonical principal symbol and it follows in a standard way that $\sigma(W)$ is a graph 1/2-density. Hence $W$ is microlocally elliptic, concluding the proof. □

Below we will need a somewhat smaller open cone with the following property:

**Proposition 5.6.** Let $\text{int}(\gamma_o) = \gamma_o \cap T^* \text{int} \Omega_o$ where $\text{int} \Omega_o$ denotes the interior of $\Omega_o$. Then there exists an open conic neighborhood $V_o$ of $\text{int}(\gamma_o)$ with the property that $\chi(V_o)$ lies in the interior of $T^*(\Omega_o)$.

**Proof** The point we must address is that $\chi$ could carry a covector at $(s, y) \in \text{int}(\Omega_o)$ to a covector on $\partial \Omega_o$ or in the exterior. To determine whether this happens we must study $s' = \frac{ds}{dy} \bar{H}$. We observe that this component is homogeneous of degree 0 on $T^* \Omega_o$ and that it vanishes to order three along $y = \eta = 0$. It follows that there is a neighborhood $V_o$ of $\gamma_o$ of the form $\text{max}\{|s|, |L - s|\} \leq C \text{min}\{|y^3|, |y|^2 \frac{|y|}{\sigma^2}, |y|^2 \frac{|y|^3}{\sigma^3}\}$ with the property that $\phi(s, y, \sigma, \eta) \in T^* \text{int} \Omega_o$ if $(s, y, \sigma, \eta) \in V_o$. □

### 5.3. The error term.

We now make a similar analysis of the error term. We first define operators $\hat{E}^{\pm K}$ on $H^2(S^1_{2L} \times \mathbb{R})$ and $\hat{E}_K$ on $L^2(S^1_{2L} \times \mathbb{R})$ by

$$
\hat{E}^{\pm K} e^{ik \frac{d}{dy}} f(N_ky) = \hat{W}_k^{\mp} \hat{E}^{\pm K} e^{ik \frac{d}{dy}} f(N_ky), \quad (k \geq 0)
$$

$$
\hat{E}^{-K} e^{ik \frac{d}{dy}} f(N_ky) = C\hat{W}_k^{+} \hat{E}^{-K} C e^{ik \frac{d}{dy}} f(N_ky), \quad (k < 0)
$$

$$
\hat{E}_K = \hat{E}^{+ K} \Pi_+ - \hat{E}^{- K} \Pi_-
$$

To analyse the kernels of $\hat{E}^{\pm K}$ and $\hat{E}_K$ we introduce the operator on $S^1_{2L} \times \mathbb{R} \times S^1_{2L}$:

$$(74) \quad \hat{E}^{\pm K}_{|D_i|} = \hat{W}^{\mp}_{|D_i|} \Delta \hat{W}^{\pm}_{|D_i|} - F_k^2 (|D_i|, \hat{1}).$$

Recall here that $F_k^2 (|D_i|, \hat{1}) = \sum_{j=1}^{2K} f_j (|D_i|, \hat{1})$.

**Proposition 5.7.** As an operator on $L^2(S^1_{2L} \times \mathbb{R})$, we have: $\hat{E}^{+ K} = j^* \hat{W}^{+}_{|D_i|} \hat{E}^{\pm K}_{|D_i|} e^{2ik \frac{d}{dy}}$.

**Proof**

As above, we test both sides on functions of the form $e^{ik \frac{d}{dy}} f(N_ky)$ for $k > 0$:

$$
= j^* \hat{W}^{+}_{|D_i|} \hat{E}^{K}_{|D_i|} e^{2ik \frac{d}{dy}} e^{ik \frac{d}{dy}} f(N_ky) = j^* \hat{W}^{+}_{|D_i|} \hat{E}^{K}_{|D_i|} e^{2ik \frac{d}{dy}} e^{ik \frac{d}{dy}} f(N_ky)
$$

$$
= \hat{W}_k^{+} \hat{E}_K e^{ik \frac{d}{dy}} f(N_ky) = \hat{E}_K e^{ik \frac{d}{dy}} f(N_ky).
$$

Similarly for $k < 0$.

□

In the following we denote by $\hat{V}$ the cone in $T^*(S^1_{2L} \times \mathbb{R} \times S^1_{2L})$ defined by $(s, \sigma, y, \eta, t, \tau) \in \hat{V}$ if $(s, \sigma, y, \eta) \in V, C < |\tau|/\sigma \leq 1/C$ for some $C < 1$. We also denote by $\mathbb{R}^+ \gamma \times T^* S^1_{2L}$ the symplectic symplectic subcone of $\hat{V}$ which is defined by $y = \eta = 0$.

**Proposition 5.8.** $\hat{E}^{+}_{|D_i|} \in \text{Op} S^2.K(\hat{V}, \mathbb{R}^+ \gamma \times T^* S^1_{2L})$. 
A We use the notation homogeneous analogue of Proposition (5.2) and the proof is essentially the same.

5.4. Conclusion. The following proposition sums up the discussion. It gives the homogeneous analogue of Proposition (5.2) and the proof is essentially the same.

Proof By (74), by Proposition (5.6) and by Egorov’s theorem, we see that $\tilde{E}^+_{|D_k|K}$ is a homogeneous pseudodifferential operator of order 2. To show that its symbol lies in $S^{2,K}$ it suffices by proposition (5.1) to show that the Taylor expansion of its scaled symbol is divisible by $N^{-(K+1)}$. The symbol of $\tilde{E}^+_{|D_k|K}$ equals $E^+_{NK}$ with $|\tau|$ substituted for $N^2$ and we will denote it by $E^+_{NK}(s,\bar{y},\eta)$. Here we use that $E^+_{NK}$ is independent of $\sigma$. Rescaling in $\tilde{V}$ replaces $|\tau|$ by $N^2$. Moreover, the transversal scaling $(\frac{1}{N}y, N\eta)$ is canceled by the operator $T$. Hence the scaled symbol is precisely the original semiclassical symbol $E_{NK}(s,\bar{y},\eta)$. By lemma (4.6) it is divisible by $N^{-(K+1)}$.

From the previous two propositions we have:

Corollary 5.9. $E_K$ is a sum of terms of the form $ARB$ where $A,B$ are bounded Fourier integral operators and where $R \in OpS^{2,K}(V,\mathbb{R}^+\gamma)$.

5.4. Conclusion. The following proposition sums up the discussion. It gives the homogeneous analogue of Proposition (5.2) and the proof is essentially the same.

We use the notation $A \sim B$ in $V$ to mean that $A - B$ is a Fourier integral operator of order $-\infty$ in $V$. Also we emphasize that the action variables are the scaled ones, i.e. functions of $|D|^{\frac{1}{2}}y$ and $|D|^{-\frac{1}{2}}y$.

Proposition 5.10. We have:

(i) $\bar{\Delta}W \sim \tilde{W}[|D|,\hat{I}^2 + E_K]$ in $V$.
(ii) If $u \in H^1_0(\Omega_0)$ then $\tilde{W}u = 0$ on $\partial\Omega_0$.
(iii) The canonical transformation $\chi$ conjugates the Hamiltonian $|\xi|_\phi$ on $V^\perp$ to the Hamiltonian $F(\sigma,I)$ on $V^\perp_0$ modulo an error which vanishes to infinite order on $\mathbb{R}^+\gamma_0$.

Proof

(i) Since $\tilde{W} = \sum_k (\tilde{W}_k^{+}\Pi_k + \tilde{W}_k^{-}\Pi_{-k})$ it follows as in Proposition (5.2) that $\tau_{e}\Delta\tilde{W}e^{ik\frac{\phi}{\|\cdot\|}}f(N_ky) = \tau_{e}\tilde{W}[|D|,\hat{I}^2 + E_K]e^{ik\frac{\phi}{\|\cdot\|}}f(N_ky)$. (ii) This follows by taking the principal symbol of the equation in (i). In the principal symbol, the remainder is homogeneous of order 1 and vanishes to order $K$ at $\gamma_0$. (iii) It suffices to show that $W \sin k\frac{\phi}{\|\cdot\|}f(N_ky) = 0$ on $\partial\Omega_0$. But this follows from the fact that $W_k \sin k\frac{\phi}{\|\cdot\|}f(N_ky) = 0$ on $\partial\Omega_0$ for all $k$. □

5.5. Normal form and microlocal parametrix. We now use $\tilde{W}$ to conjugate the (odd part of the) wave group of the normal form to a kind of parametrix for the mixed problem on $\Omega_0$. First, let us define the odd Dirichlet normal form wave group:

Definition 5.11. By $F_0(t,x,y)$ we denote the odd part of the fundamental solution of the wave equation:

$$\frac{\partial^2 F_0(t)}{\partial t^2} = F(|D|,\hat{I})^2 F_0(t) \quad \text{on } \mathbb{R} \times (S^1_{2L} \times \mathbb{R}) \times (S^1_{2L} \times \mathbb{R})$$

$$F_0(0) = \Pi_0 \quad \frac{\partial F_0}{\partial t}(0) = 0 \quad F_0(t,\cdot,z) = 0 = F_0(t,z,\cdot) \quad \text{on } \partial\Omega_0.$$

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To be correct, $F$ is only defined in a microlocal neighborhood of $\gamma_o$. We do not indicate this in the notation since we will microlocalize it later on. In the elliptic case, for instance, $F_o$ has the eigenfunction expansion

$$F_o(t, \bar{s}, \bar{y}; \bar{s}', \bar{y}') = \sum_{k,q} \cos t(F(k, q + \frac{1}{2}))\phi_{kq}(\bar{s}, \bar{y})\phi_{kq}(\bar{s}', \bar{y}')$$

where $\phi_{kq}(\bar{s}, \bar{y}) = \sin k\frac{\bar{s} - \bar{y}}{2}D_q(Nk\bar{y})$. By odd part we refer to the projection under $\Pi_o$.

We now wish to conjugate the odd normal form wave group to a microlocal parametrix for the Dirichlet wave group $E(t)$.

**Proposition 5.12.** There exists a conic neighborhood $V' \subset V$ of $\gamma_o$ and a zeroth order pseudodifferential operator $G$ such that:

$$\left\{ \begin{array}{l}
\check{W}\check{W}'G \sim \check{W}'G\check{W} = I + R \text{ with } WF(R) \cap V' = \emptyset \\
\check{W}F^2\check{W}'G = \Delta + A_1E^KA_2 + R_1, \quad WF(R_1) \cap V' = \emptyset
\end{array} \right.$$

where $A_1, A_2$ are zeroth order Fourier integral operators.

**Proof:** By proposition 5.4, $\check{W}$ is a zeroth order Fourier integral operator microsupported in $V \times \chi(V)$ and associated to the graph of the canonical transformation $\chi$. Hence $W\check{W}'$ is a zeroth order pseudodifferential operator microsupported in $V$ with symbol identically equal to one in a smaller cone $V' \subset V$. Hence there exists a positive zeroth order self-adjoint pseudodifferential operator $G$ microsupported in $V$ such that $W\check{W}'G + R$ with $WF(R) \cap V' = \emptyset$.

Let us put $\check{W}_{-1} = \check{W}G$. Then by proposition 5.10,

$$\check{W}F^2\check{W}_{-1} = \check{W}F^2\check{W}'G$$

(76)

$$\sim \check{\Delta}\check{W}'G + \check{W}E^K\check{W}'G \text{ in } V$$

$$\sim \check{\Delta} + R \text{ in } V'$$

where $R = \check{W}E^K\check{W}'G$.

We now introduce a kind of microlocal parametrix for the wave kernel. First we define a microlocal cutoff $\psi$ to the conic neighborhood $V$ of $\mathbb{R}\gamma_o$, with $\psi \equiv 1$ in a conic neighborhood of $\text{int}\mathbb{R}^+ \gamma_o$.

**Definition 5.13.** The parametrix is defined by: $E(t) = \check{W}F_{\psi T}(t)\psi V 1_{\Omega_o}\check{W}_{-1}$.

The extent to which $E(t)$ is indeed a microlocal parametrix near $\gamma_o$ is given in the following lemma:

**Lemma 5.14.** For any $A \in \Psi^*(\Omega)$ with microsupport $WF(A) \subset V_o$ (defined in proposition 5.4), $E(t)A$ and $E(t)A$ are Fourier integral operators in the class $I^0((S^1_{2L} \times \mathbb{R}) \times (S^1_{2L} \times \mathbb{R}), gr\chi) + I^0((S^1_{2L} \times \mathbb{R}) \times (S^1_{2L} \times \mathbb{R}), gr\chi \circ r)$. We have

$$\frac{1}{2}E(t)A \sim E(t)A + R(t)A$$

where $R(t)$ is a finite sum (or smooth compactly supported integrals) of terms $B_1R_1(t)B_2$ where $B_1, B_2$ are bounded Fourier integral operators and $R_1 \in \text{Op}S^{2,K}$. 
Proof: $E(t)$ is uniquely characterized as the microlocal solution of the Cauchy problem \( (\hat{\mathcal{P}}) \) on $\Omega_o$ with initial condition equal to the identity operator on $L^2(\Omega_o)$. Hence it suffices to show that $\mathcal{E}$ is a also a microlocal solution modulo errors in the stated class.

We first verify that $\mathcal{E}(t)$ is a microlocal solution of the wave equation in $V'$. But

\[
\frac{d^2}{dt^2} \mathcal{E}(t) = \hat{W} F(|D|, \hat{I})^2 F_o \psi_\nu \Pi_\nu \hat{W}_{-1} = \hat{W} F(|D|, \hat{I})^2 \hat{W}_{-1} W F_o \psi_\nu \Pi_\nu \hat{W}_{-1} + \hat{W} F(|D|, \hat{I})^2 (I - \hat{W}_{-1} \hat{W}) F_o \psi_\nu \Pi_\nu \hat{W}_{-1} = \Delta \mathcal{E}(t) + E^K \mathcal{E}(t) + R_1(t),
\]

where $R_1(t) = \hat{W} F(|D|, \hat{I})^2 (I - \hat{W}_{-1} \hat{W}) F_o \psi_\nu \Pi_\nu \hat{W}_{-1} + R_2 \mathcal{E}(t)$ with $WF(R_2) \cap V' = \emptyset$. By shrinking $V$ if necessary, the first term also has order $-\infty$ in $V$. Hence $\mathcal{E}(t)$ is a microlocal solution of the forced wave equation with forcing term of the form $E^K \mathcal{E}(t) + R_1(t)$.

Regarding the boundary condition, we observe that $W \Pi_o(x, y) = 0$ if $x \in \partial \Omega$ and so $\mathcal{E}(x, y) = 0$ if $x \in \partial \Omega$.

The initial condition is more difficult. We must show that $W \Pi_o \psi_\nu \Pi_\nu \hat{W}_{-1} A \sim \frac{1}{2} A$ for a pseudodifferential operator $A$ microsupported in $V_o$. To this end we first recall that $\Pi_o = \frac{1}{2} (I - \hat{r})$ where $\hat{r}$ is the reflection $(s, \hat{y}) \rightarrow (-s, \hat{y})$ (with $s \mod 2L$). Hence $W \Pi_o \psi_\nu \Pi_\nu \hat{W}_{-1} A = \frac{1}{2} \hat{W} \psi_\nu \Pi_\nu \hat{W}_{-1} A - \frac{1}{2} \hat{W} r \psi_\nu \Pi_\nu \hat{W}_{-1} A$. It suffices to show that $\hat{W} \psi_\nu \Pi_\nu \hat{W}_{-1} A \sim A$ and that $\hat{W} r \psi_\nu \Pi_\nu \hat{W}_{-1} A \sim 0$.

The operator $\hat{W} \psi_\nu \Pi_\nu \hat{W}_{-1} A$ is a pseudodifferential operator with a singular symbol. Temporarily ignoring the singularity along $\partial \Omega_o$ we may apply Egorov’s theorem formally to find that the complete symbol is identically equal to one in $V_o$. To see this, it is convenient (although not necessary) to use that $\hat{W} = e^{iP}$ for some first order pseudodifferential $P$ of real principal type (see proposition (5.4)). We then put $\hat{W}(u) = e^{iuP}$ and consider the conjugation $\hat{W}(u) \psi_\nu \Pi_\nu \hat{W}_{-1}(u)$. In a well-known way (cf. [1], §7.8), the complete symbol expansion of this operator is obtained recursively by solving transport equations along the orbits of the Hamilton flow $\varphi^\nu$ of $\sigma_P = H$. The principal symbol equals $\varphi^\nu(\psi_\nu \Pi_\nu \hat{W}_{-1})$. Similarly, the complete symbol at $\varphi^\nu(x, \xi)$ of $\hat{W} \psi_\nu \Pi_\nu \hat{W}_{-1} A$ depends only on the germ of the complete symbol of $\psi_\nu \Pi_\nu \hat{W}_{-1}$ at $\varphi^\nu(x, \xi)$ and on the germ of $\varphi^\nu$ at $(x, \xi)$. Since $\varphi^\nu(V_o) \subset V \cap T^*(\text{int} \Omega_o)$, the germ of the complete symbol of $\psi_\nu \Pi_\nu \hat{W}_{-1}$ is identically one at $(x, \xi) \in V_o$ and hence the germ of the complete symbol of $\hat{W} \psi_\nu \Pi_\nu \hat{W}_{-1}$ equals that of $\hat{W} \psi_\nu \Pi_\nu \hat{W}_{-1}$ at $\varphi(x, \xi)$, hence equals one identically. Since we have composed with $A$ satisfying $WF(A) \subset V_o$ this calculation is valid on microsupport of $A$ and thus $\hat{W} \psi_\nu \Pi_\nu \hat{W}_{-1} A \sim A$.

The statement $\hat{W} r \psi_\nu \Pi_\nu \hat{W}_{-1} A \sim 0$ follows from the fact that the underlying canonical transformation of this operator is $(\varphi^\nu)^{-1} r^* \varphi^\nu$, and the graph of this canonical transformation is disjoint from the $T^*(\text{int} \Omega_o) \cap V_o \times T^*(\text{int} \Omega_o) \cap V_o$. This follows since the graph of $r^*$ does not intersect $T^*(\text{int} \Omega_o) \times T^*(\text{int} \Omega_o)$.

Thus $\mathcal{E}(t)$ is a microlocal solution of the mixed Cauchy problem modulo errors in $\text{OpS}^{2,K}$. Since the Cauchy problem is well-posed, we have by Duhamel’s principle
that
\[(78) \quad (\mathcal{E}(t) - E(t)) \sim \int_0^t G_o(t-u)R(u)\mathcal{E}(u)du\]

where \(R(u)\) is an error in the stated class and where \(G_o\) is the kernel of \(\sin t(\sqrt{\Delta_o})\) i.e. of the mixed problem
\[(\partial^2_t - \Delta)G_o = 0 \quad \text{on } \Omega_o \times \Omega_o\]

\[(79) \quad G_o(t, \bar{s}, \bar{y}, \bar{s}', \bar{y}')|_{t=0} = 0 \quad \partial_t G_o(t, \bar{s}, \bar{y}, \bar{s}', \bar{y}')|_{t=0} = Id\]

\[G_o(t, \bar{s}, \bar{y}, \bar{s}', \bar{y}') = 0 \quad \text{for } \bar{s} = 0, L.\]

Since we are working microlocally near \(\gamma\), we may replace \(G_o\) modulo smoothing operators by a Fourier integral parametrix (cf \(\text{GM} \quad \text{PS}\)). Thus \(\int_0^t G_o(t-u)R(u)\mathcal{E},Q)(u, \bar{s}, \bar{y}, \bar{s}', \bar{y}')du\) is a sum of terms of the form \(AR_1B\) with \(R_1\) as above.

\[\square\]

6. Wave invariants and normal form

We now prove the crucial lemma that the wave invariants of the normal form at \(\gamma_o^m\), i.e. the singularities of \(Tr\cos t(F(|D|, \hat{I})\) at \(\bar{s} = 2mL\), agree with the wave invariants of \(\sqrt{\Delta}\) at \(\gamma^m\), i.e. the singularity of \(Tr e^{it\sqrt{\Delta}}\) at \(t = 2mL\). Here, the \(|D|\) in \(F(|D|, \hat{I})\) refers to the Dirichlet \(D\) on \([0, L]\) and the \(\hat{I}\) as above refers to the homogeneous action operator.

**Lemma 6.1.** Let \(\gamma\) denote a non-degenerate bouncing ball orbit of \(\Omega\) and let \(\gamma_o\) denote the corresponding bouncing ball orbit \(\gamma_o\) of the normal form glow of \(\Omega_o\). Then for all \(m\), we have \(a_{\gamma^m,k}(\sqrt{\Delta}) = a_{\gamma^m,k}(F(|D|, \hat{I})).\)

**Proof**

To prevent confusion between traces on \([0, L] \times \mathbb{R}\) and traces on \(S^1 \times \mathbb{R}\) we will exclusively use the notation 'res' for the latter space and will explicitly put in the cutoff \(1_{\Omega_o}\) (the characteristic function of \(\Omega_o\)) to indicate traces on the former. We have:

\[a_{\gamma^m,k}(\sqrt{\Delta}) = a_{\gamma^m,k}(\sqrt{\Delta_{\Omega_o}})\]

\[(80) \quad = res \ (D^k_t E(t)|_{t=2mL})1_{\Omega_o}.\]

The residue is given by the integral of a density over the fixed point set of the underlying canonical relation. The density is calculated using the method of stationary phase on a manifold with boundary (see Theorem (2.2) or \(\text{GM}\)), essentially as in the boundaryless case. Thus the residue depends only on the kernel of \(E(t)\) microlocally in the cone \(V_o\) of proposition (5.6). Hence by lemma (6.1) we have

\[(81) \quad res \ (D^k_t E(t)|_{t=2mL})1_{\Omega_o} = 2res \ (D^k_t \mathcal{E}(t)|_{t=2mL})1_{\Omega_o} + 2res \ (D^k_t R(t)|_{t=2mL})1_{\Omega_o}\]

with \(R(t) \in A_1 OpS^{2,K}A_2\). By Proposition (3.3) we can drop the second term. Then by the definition of \(\mathcal{E}(t)\) we get that \((E)\) equals

\[(82) \quad 2res 1_{\Omega_o} \hat{W}(D^k_t F_o(t)|_{t=2mL})\psi V 1_{\Omega_o} \hat{W}^-1.\]
To evaluate this residue let us approximate $1_{\Omega_0}$ by a smooth cutoff $\tau_\epsilon(s)$, equal to one on $\Omega_0$ and supported for $s \in (-\epsilon, \pi + \epsilon)$. We have:

$$a_{\gamma,m,k}(\sqrt{\Delta}) = 2 \text{res} \ 1_{\Omega_0} \tilde{W}(D_k^b F_o(t)|_{t=2mL})\psi/V 1_{\Omega_0} \tilde{W}^{-1}_1$$

$$= 2 \lim_{\epsilon \to 0} \text{res} \tau_\epsilon(x) \tilde{W}(D_k^b F_o(t)|_{t=2mL})\psi/V 1_{\Omega_0} \tilde{W}^{-1}_1$$

$$= 2 \lim_{\epsilon \to 0} \text{res} \ 1_{\Omega_0} \tilde{W}^{-1}_1 \tau_\epsilon \tilde{W}(D_k^b F_o(t)|_{t=2mL})\psi/V$$

where in the last line we used the tracial property of $\text{res}$. We note that

$$\tilde{W}^{-1}_1 \tau_\epsilon \tilde{W}(D_k^b F_o(t)|_{t=2mL})\psi/V$$

is a (standard) Fourier integral operator on the boundaryless manifold $S_{2L}^1 \times \mathbb{R}$. As mentioned above, the residue of $1_{\Omega_0}$ times this operator equals the integral of a density over the fixed point set of the underlying canonical relation. Since $\tilde{W}^{-1}_1 \tau_\epsilon \tilde{W}$ is a pseudodifferential operator, the canonical relation is simply that of $(D_k^b F_o(t)|_{t=2mL})$. Due to the factor of $1_{\Omega_0}$ the residue density may be calculated as if $\tau_\epsilon \equiv 1$. But then the factors of $\tilde{W}$ and $\tilde{W}^{-1}_1$ cancel and we are left with

$$a_{\gamma,m,k}(\sqrt{\Delta}) = 2 \text{res} \ 1_{\Omega_0} (D_k^b F_o(t)|_{t=2mL})\psi/V.$$  

To complete the proof we must show that this residue equals

$$\text{res} \ 1_{\Omega_0} (D_k^b F_o(t)|_{t=2mL})\psi/V.$$  

The difference is just that the odd kernel equals $\Pi_0 = \frac{1}{2}(I - r)$ composed with the full kernel. The two terms of $\Pi_0$ give rise to two components to the canonical relation of $F_o$. They consist of the graph of the Hamilton flow of the normal form, which we denote by $G_{\text{nor}}^{2mL}$, and the graph of $G_{\text{nor}}^{2mL} \circ r$. The latter map has no fixed points in $T*(\text{int}\Omega_0)$. Hence the residue of the second term equals zero. In the first term the factors of 2 cancel we give the result claimed in (85). \hfill \Box

**Corollary 6.2.** The quantum Birkhoff normal coefficients, i.e. the coefficients of the polynomials $p_k(\hat{I})$ are spectral invariants of $\Delta$.

The proof of the corollary from the lemma is identical to that in (3) (see also 8.2), so we omit the proof.

### 6.1. Conclusion of Proof of Theorem

We now complete the proof of the main result. Since the quantum Birkhoff normal coefficients of $\sqrt{\Delta}$ at $\gamma$ are spectral invariants, it follows a fortiori (by taking the interior symbol) that the Birkhoff normal form for the metric Hamiltonian $H = |\xi|^2 g$ of the metric $g_{\Phi_0}$ at the bouncing ball orbit is a spectral invariant. Thus we may write:

$$H = |\sigma| + \frac{\alpha}{L} I + \frac{p_1(I)}{|\sigma|} + \cdots + \frac{p_k(I)}{|\sigma|^k} + \cdots$$

where $p_k$ is the homogeneous polynomial part of degree $k + 1$ in the quantum normal form. We now observe that this normal form of the Hamiltonian induces the Birkhoff normal form of the Poincare map $P_\gamma$.

Indeed, we first note that the coordinate $s$ is dual to $\sigma$ and hence is an angle variable. Hence the Hamilton flow in action-angle variables takes the form:

$$\phi^\epsilon(s, \phi, |\sigma|, I) = (s + t\omega|\sigma|, \phi + t\omega I, |\sigma|, I), \quad \omega|\sigma| = \partial_{\sigma|} H, \omega I = \partial_I H.$$
The billiard map from the bottom component of the boundary to the top component is given by:

\[ \beta_1(\phi, I) = (\phi + t_1(\phi, |\sigma|, I) \omega_1, I) \]

where \( t_1(\phi, |\sigma|, I) \) is the time until the trajectory through the initial vector defined by \((0, \phi, |\sigma|, I)\) hits the upper part of the boundary. We obviously have:

\[ t_1(\phi, |\sigma|, I) = L/|\omega_1| \]

Similarly, the billiard map for the return trip is given by

\[ \beta_2(\phi, I) = (\phi + t(\phi, |\sigma|, I) \omega_I, I) \]

It follows that the Poincare map has the Birkhoff normal form

\[ P_\gamma(\phi, I) = (\phi + t_1 \omega_I + t_2 \omega_I \circ \beta_1, I). \]

Thus, the Birkhoff normal form of \( P_\gamma \) is a spectral invariant. We now want to apply the argument of Colin de Verdière (\[CV\], §4, THEOREME) to conclude that the domain \( \Omega \) is determined by its spectrum. Although Colin de Verdière makes the assumption that \( \gamma \) is a non-degenerate elliptic orbit, his argument is equally valid for hyperbolic bouncing ball orbits. Let us briefly explain the necessary modifications.

Assuming that the non-degenerate bouncing ball orbit is the vertical axis, we write the graph of \( \Omega \) over the horizontal axis as \( y = f(x) = 1 + a_0 x^2 + \cdots + a_n x^{2n-2} \). Only even terms appear due to the left/right symmetry assumption. By definition, the Birkhoff normal form of the (non-linear) Poincare map \( P_\gamma \) is an expression for this map in local action-angle variables \((I, \theta)\) on the transversal. In the elliptic case we need to assume (as above) that \( \alpha/\pi \notin \mathbb{Q} \) to ensure that the normal form exists; no assumption is needed in the hyperbolic case. Following \[CV\] we write the normal form as

\[ T(I, \theta) = (I + O(I^\infty), \theta + b_0 + b_1 I + \cdots + O(I^\infty)). \]

The key assertion is that there is an ‘upper-triangular’ bijection between the Taylor coefficients \( a_j \) and the Birkhoff normal form coefficients \( b_k \). This means (i) that \( b_n = B + C a_n \) where \( B, C \) depend only on \( a_0, \ldots, a_{n-1} \) and (ii) that \( C \neq 0 \). Granted (i) - (ii), \( \{a_0, \ldots, a_n\} \rightarrow \{b_0, \ldots, b_n\} \) can be inverted and the Taylor coefficients are determined by the normal form coefficients.

The proof of (i)–(ii) is almost exactly the same in the non-degenerate elliptic and hyperbolic cases. To convince the reader of this, we briefly recall the argument in \[CV\] and extend it to the hyperbolic case. The starting point is that one may write down a generating function \( \phi \) for \( T \) in terms of \( f \):

\[ \text{graph } T = \{(x, \frac{\partial \phi}{\partial x}, x_1, -\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial s} = 0) \}
\]

with \( \phi(x, x_1, s) = [(x - s)^2 + (f(s)^2)^2]^{\frac{1}{2}} + [(x_1 - s)^2 + (f(s)^2)^2]^{\frac{1}{2}} \). Here, \( x, x_1, s \) denote points on the horizontal axis. In either the elliptic or hyperbolic case, one may expand \( \phi = 2 + \phi_0 + \cdots + \phi_n + \cdots \) where \( \phi_0 \) (resp. \( \phi_n \)) is homogeneous of degree 2 (resp. degree \( n + 2 \)) in \((x, x_1, s)\). One easily finds that \( \phi_0(x, x_1, s) = \frac{1}{2}(x^2 + x_1^2) - s(x + x_1) + (2a_0 + 1)s^2 \) and that \( \phi_n = \psi_n + 2a_n s^{2n+2} \) where \( \psi_n \) depends only on \( a_0, \ldots, a_{n-1} \). The linear Poincare map \( P_\gamma \) has generating function \( \phi_0 \) and one finds that

\[ P_\gamma = \begin{pmatrix} A - 1 & -A \\ 2 - A & A - 1 \end{pmatrix} \]
where $A = 2(2a_0 + 1)$. In the elliptic case $a_0 \in (-\frac{1}{2}, 0)$ while in the hyperbolic case $a_0 > 0$ or $a_0 < -\frac{1}{2}$. We assume $A \neq 0$, i.e. $a_0 \neq -\frac{1}{2}$.

Using the generating function one shows (\cite{CV}, Lemma 1) that

$$
T \begin{bmatrix} x \\ \xi \end{bmatrix} = T^0 \begin{bmatrix} x \\ \xi \end{bmatrix} + C(x - \xi)^{2n+1} a_n \begin{bmatrix} 1 \\ -1 \end{bmatrix} + O((x, \xi)^{2n+2})
$$

where $T^0$ is the $(2n + 1)$ jet of $T$ computed with the assumption that $a_n = 0$ and where $C \neq 0$. The proof is a formal manipulation with Taylor series and only in the evaluation of $C$ does it matter whether the linear part of $T$ is elliptic or hyperbolic. The constant $C$ has the form $C_1 A^{2n+2}$ where $C_1$ is universal. Hence it is non-vanishing as long as $A \neq 0$.

From the construction of the Birkhoff invariants (cf. \cite{SM}) and from (92) it follows that $b_n = B + C a_n$. It remains to show that $C \neq 0$. To prove this, we recall that two germs of area-preserving transformations of $(\mathbb{R}^2, 0)$ for which the eigenvalues of the linear parts are not roots of unity are symplectically equivalent if and only if they have identical normal forms (\cite{SM}, p. 162). To bring this down to a finite dimensional statement, we define $G_{2n+1}$ as the group of $(2n+1)$-jets of area-preserving transformations of $(\mathbb{R}^2, 0)$, and $\Gamma_{2n+1} \subset G_{2n+1}$ as the subgroup of elements of the form $Id + O(((x, \xi)^{2n+1})$. Also let $\mathcal{O}_T = \{T' \in G_{2n+1}: T' - T = O(((x, \xi)^{2n+1})$. Assuming the eigenvalues of the linear part of $T' \in \mathcal{O}_T$ are not roots of unity, the orbit $\Gamma_{2n+1} \cdot T' \subset \mathcal{O}_T$ consists of elements $T'' \in \mathcal{O}_T$ with the same Birkhoff normal form invariants up to and including $b_n$. Put $T' = T(a_0, \ldots, 0)$. If $C = 0$, then $T(a_0, a_1, \ldots, a_n)$ would on the orbit $\Gamma_{2n+1} \cdot T(a_0, \ldots, 0)$ for all $a_n$, i.e. the curve $a_n \to T(a_0, a_1, \ldots, a_n)$ would be tangent to this orbit. However, it is shown in \cite{CV} that it is transversal to the orbit. In both elliptic and hyperbolic cases, we can choose coordinates $(u, v)$ in which the linear part of $T$ is diagonal and observe that the tangent vector to $a_n \to T(a_0, a_1, \ldots, a_n)$ contains the monomials $u^n v^{n+1}$ and $u^{n+1} v^n$ with non-zero coefficients. However, these monomials cannot occur in tangent vectors to $\Gamma_{2n+1} \cdot T(a_0, \ldots, a_n)$. This statement only involves the conjugation of the linear part of $T$ by $\Gamma_{2n+1}$ and is valid (with the same proof) in both elliptic and hyperbolic cases. Therefore (i) - (ii) are valid in both cases and \{a_0, \ldots, a_n\} \to \{b_0, \ldots, b_n\} can be inverted.

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