CLASSIFICATION OF $q$-PURE $q$-WEIGHT MAPS OVER FINITE DIMENSIONAL HILBERT SPACES

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ABSTRACT. An $E_0$-semigroup of $B(H)$ is a one parameter strongly continuous semigroup of *-endomorphisms of $B(H)$ that preserve the identity. Every $E_0$-semigroup that possesses a strongly continuous intertwining semigroup of isometries is cocycle conjugate to an $E_0$-semigroup induced by the Bhat induction of a $CP$-flow over a separable Hilbert space $K$. We say an $E_0$-semigroup $\alpha$ is $q$-pure if the $CP$-subordinates $\beta$ of norm one (i.e. $\|\beta_t(I)\| = 1$ and $\alpha_t - \beta_t$ is completely positive for all $t \geq 0$) are totally ordered in the sense that if $\beta$ and $\gamma$ are two $CP$-subordinates of $\alpha$ of norm one, then $\beta \geq \gamma$ or $\gamma \geq \beta$. This paper shows how to construct and classify all $q$-pure $E_0$-semigroups induced by $CP$-flows over a finite-dimensional Hilbert space $K$ up to cocycle conjugacy.

Introduction

An $E_0$-semigroup of $B(H)$ is a one parameter strongly continuous semigroup of *-endomorphisms of $B(H)$ (the set of all bounded operators on a Hilbert space $H$) that preserve the identity, (i.e. $\alpha_t(I) = I$ for all $t \geq 0$). Two $E_0$-semigroups $\alpha$ and $\beta$ of $B(H_1)$ and $B(H_2)$ are said to be conjugate if there is a unitary operator $U$ from $H_1$ onto $H_2$ so that $\beta_t(UA\alpha_t(U^{-1})) = U\alpha_t(A)U^{-1}$ for all $A \in B(H_1)$ and $t \geq 0$. If $\alpha$ is an $E_0$-semigroup of $B(H)$ we say $U = \{U(t) : t \geq 0\}$ is a cocycle for $\alpha$ if the operators $U(t)$ are strongly continuous in $t$ and satisfy the cocycle relation $U(t + s) = U(t)\alpha_t(U(s))$ for $t, s \geq 0$. The cocycle is said to be unitary if the $U(t)$ are unitary operators. Two $E_0$-semigroups $\alpha$ and $\beta$ are said to be cocycle conjugate if there is a unitary cocycle $U(t)$ for $\alpha$ so that the $E_0$-semigroup $\gamma$ given by $\gamma_t(A) = U(t)\alpha_t(A)U(t)^{-1}$ for $A \in B(H)$ and $t \geq 0$ is conjugate with $\beta$. The main problem in the theory of $E_0$-semigroups is to classify them up to cocycle conjugacy. For a discussion of $E_0$-semigroups we refer to the book of Arveson [Arv03].

An $E_0$-semigroup $\alpha$ of $B(H)$ is said to be spatial if there is a strongly continuous one parameter semigroup of isometries $U(t)$ which intertwine $\alpha$ so that $U(t)A = \alpha_t(A)U(t)$ for $A \in B(H)$ and $t \geq 0$. For spatial $E_0$-semigroups there is an integer valued index $n = 0, 1, \cdots$ and $\infty$ first suggested by Powers [PowS8] and later correctly defined by Arveson [Arv03]. Arveson showed that the index is additive under taking tensor products so if $\alpha$ and $\beta$ are spatial $E_0$-semigroups of index $n$ and $m$ then the tensor product $\alpha_t \otimes \beta_t$ is spatial and of index $n + m$. If an $E_0$-semigroup can be reconstructed from its intertwining semigroups it is said to be completely spatial. These are the $E_0$-semigroups of type I$_n$ where $n$ is the index. Arveson showed that the index $n$ is a complete cocycle conjugacy invariant for the $E_0$-semigroups of type I (i.e. if $\alpha$ and $\beta$ are of type I$_n$ then $\alpha$ and $\beta$ are cocycle conjugate.) Like the theory of factors the $E_0$-semigroups of type I are well understood.

An $E_0$-semigroup that is spatial but not completely spatial is said to be of type II and $E_0$-semigroups which are not spatial are said to be of type III. In this paper we focus on what we call the $q$-pure $E_0$-semigroups of type II. If $\alpha$ is an $E_0$-semigroup of $B(H)$ we say $C = \{C(t) : t \geq 0\}$ is a local cocycle for $\alpha$ if $C$ satisfies the cocycle condition $C(t + s) = C(t)\alpha_t(C(s))$ for $t, s \geq 0$ and $C(t)$ commutes with $\alpha_t(A)$ for $A \in B(H)$ (i.e., $C(t) \in \alpha_t(B(H))'$). Note if $C$ is a positive local cocycle for $\alpha$ and $s \leq 0$ then $D(t) = e^{st}C(t)$ for $t > 0$ is also a positive local cocycle for $\alpha$. We can exclude these trivial subordinates by only considering local cocycle of norm one (i.e. $\|C(t)\| = 1$ for $t \geq 0$). If $C_1$ and $C_2$ are local cocycles for $\alpha$ then we say $C_1 \geq C_2$ if $C_1(t) \geq C_2(t)$ for all $t \geq 0$. The positive local cocycles and their order structure is a cocycle conjugacy invariant for $\alpha$. We say
an $E_0$-semigroup is $q$-pure if the positive local cocycles of norm one are totally ordered. This means if $C_1$ and $C_2$ are positive local cocycles of norm one then either $C_1(t) \geq C_2(t)$ or $C_2(t) \geq C_1(t)$ for all $t \geq 0$. If $\alpha$ is a $q$-pure spatial $E_0$-semigroup then $\alpha$ is of index zero which means that either $\alpha$ is of type I$_0$ so $\alpha_t(A) = U(t)AU(t)^{-1}$ for $U$ a strongly continuous one parameter unitary group or $\alpha$ is an $E_0$-semigroup of type II$_\infty$.

In this paper we begin the classification of the $q$-pure spatial $E_0$-semigroups. We classify all the $q$-pure spatial $E_0$-semigroups that come from boundary weight maps over finite dimensional spaces. We find that all such $q$-pure $q$-weight maps are cocycle conjugate to $q$-weight maps of range rank one. These $q$-pure range rank one $q$-weight maps were first constructed by Powers in [Pow03a] and then by Jankowski [Jan10] in the case of weight maps given by a pair $(\varphi, \nu)$ where $\varphi$ is a completely positive map of the $(n \times n)$-matrices into themselves and $\nu$ is a pure weight on $B(L^2(0, \infty))$. The surprising result of Jankowski was that the $q$-weight map one constructs can be $q$-pure when $\varphi$ has a one dimensional range. This result runs counter to intuition in that a map $\varphi(A) = \text{tr}(A)I$ (which is in a sense the least pure completely positive map being the average of all pure maps) yields a $q$-pure weight map. In [JMP11] we classified all the range rank one $q$-weight maps over a finite dimensional Hilbert space $K$ up to cocycle conjugacy and in this paper we show that in the case of $q$-pure $q$-weight maps over a finite dimensional Hilbert space are all cocycle conjugate to $q$-pure range rank one $q$-weight maps.

What is surprising about this result is that a few years ago these problems seemed intractable. Even in the case of a two dimensional Hilbert space $\mathbb{C}^2$ it was more of a hope than a belief that $q$-pure weight maps were cocycle conjugate to range rank one $q$-weight maps. It took months of analysis to produce a single example of an non Shur $q$-pure $q$-weight map in the case of a Hilbert space $K$ of dimension two. It seemed then that the structure of these boundary weight maps was so complex any sort of classification was out of the question. This paper shows how to construct and classify all $q$-pure $E_0$-semigroups coming from boundary weight maps over finite dimensional Hilbert spaces up to cocycle conjugacy. Whether this analysis carries over to the infinite dimensional case is the next burning question.

1. Boundary weight maps

Each $q$-weight map over $K$ uniquely defines a $CP$-flow over $K$. A $CP$-flow $\alpha_t$ over $K$ is a strongly continuous one parameter semigroup of completely positive contractions of $B(H) = B(K \otimes L^2(0, \infty))$ into itself that is intertwined by translation. Specifically, if $U(t)$ is the translation isometry given by $(U(t)F)(x) = F(x - t)$ for $t \geq 0$ where $F \in H$ is represented by a $K$-valued function $F(x)$ then to say $\alpha_t$ is intertwined by $U(t)$ means that $U(t)A = \alpha_t(A)U(t)$ for $A \in B(H)$ and $t \geq 0$. A $CP$-flow $\alpha_t$ is unitif when $\alpha_t(I) = I$ for $t \geq 0$. By the Bhat induction theorem [Bha96] each unital $CP$-flow gives rise to a spatial $E_0$-semigroup, where spatial means the $E_0$-semigroup is of type I or II. It follows then that each unital $q$-weight map over $K$ gives rise to a $E_0$-semigroup which is determined up to cocycle conjugacy. Two such $E_0$-semigroups are cocycle conjugate if the $q$-weight maps are cocycle conjugate which is explained below. All this is explained in [Pow03b] in excruciating detail. Fortunately, for our purposes we will simply use the results and work directly with the $q$-weights. With this said we begin.

In this paper all Hilbert spaces $H$ are assumed to be separable. Finite dimensional Hilbert spaces of dimension $n$ are denoted by $\mathbb{C}^n$. The inner product $(f, g)$ with $f, g \in H$ is linear in $g$ and conjugate linear in $f$.

Of utmost importance the concept of a completely positive map. If $H$ and $K$ are Hilbert spaces a mapping $\phi$ of $B(H)$ into $B(K)$ is said to be completely positive if for all $n \in \mathbb{N}$ if $A_i \in B(H)$ and $f_i \in K$ for $i = 1, \cdots, n$ then

$$\sum_{i,j=1}^{n} (f_i, \phi(A_i^*A_j)f_j) \geq 0.$$
An alternative definition is as follows. If \( \phi \) is a linear mapping of \( B(H) \) into \( B(K) \) we say \( \phi \) is positive if \( \phi(A) \) is positive if \( A \) is positive. We denote by \( \phi_n = t_n \otimes \phi \) the mapping of \( (n \times n) \)-matrices with entries in \( B(H) \) into \( (n \times n) \)-matrices with entries in \( B(K) \) given by

\[
\phi_n \left( \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix} \right) = \begin{bmatrix} \phi(A_{11}) & \phi(A_{12}) & \cdots & \phi(A_{1n}) \\ \phi(A_{21}) & \phi(A_{22}) & \cdots & \phi(A_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ \phi(A_{n1}) & \phi(A_{n2}) & \cdots & \phi(A_{nn}) \end{bmatrix}.
\]

A mapping \( \phi \) is \( n \)-positive if \( \phi_n(A) \) is positive if \( A \) is positive. A mapping \( \phi \) is completely positive if \( \phi_n \) is positive for all \( n \).

An important estimate for completely positive maps is that if \( \phi \) is a completely positive mapping of \( B(H) \) into \( B(K) \) then

\[
\phi(A)^* \phi(A) \leq \phi(A^*A) \leq \|A\|^2 \|\phi(I)\| \phi(I)
\]

for \( A \in B(H) \). We will refer to the inequality on the left as the Schwarz inequality for completely positive maps. This estimate is derived by observing that

\[
\phi_2 \left( \begin{bmatrix} I & A \\ A^* & A^*A \end{bmatrix} \right) = \begin{bmatrix} \phi(I) & \phi(A) \\ \phi(A^*) & \phi(A^*A) \end{bmatrix} \leq \begin{bmatrix} \|\phi(I)\| I & \phi(A) \\ \phi(A^*) & \phi(A^*A) \end{bmatrix}
\]

so the matrix of operators on the right is positive. Taking the operator norm of the Schwarz inequality gives \( \|\phi(A)\| \leq \|A\| \|\phi(I)\| \) so the norm of a completely positive map \( \phi \) as a map is the norm \( \|\phi(I)\| \).

Every normal completely positive map \( \phi \) from \( B(H) \) to \( B(K) \) can be expressed in the form

\[
\phi(A) = \sum_{k \in J} S_k A S_k^*
\]

where \( J \) is a countable index set and the \( S_k \) are bounded linear operators from \( H \) to \( K \) which are linearly independent, by which we mean if \( c_k \in \mathbb{C} \) for \( k \in J \) so that

\[
\sum_{k \in J} |c_k|^2 < \infty
\]

and

\[
S = \sum_{k \in J} c_k S_k,
\]

then \( S = 0 \) if and only if \( c_k = 0 \) for all \( k \in J \). If one has a second decomposition of \( \phi \) in the above manner and \( J' \) is the index set for the second decomposition then there is a one to one mapping of \( J \) onto \( J' \). The number of elements of \( J \) is called the index of \( \phi \). If the index set has one element we say \( \phi \) is pure. If \( \phi \) is a completely positive map of \( B(H) \) into \( B(K) \) of the above form and \( \psi \) is a map of \( B(H) \) into \( B(K) \) of the form \( \psi(A) = SAS^* \) where \( S \) is a linear operator from \( H \) to \( K \) then \( \phi - \psi \) is completely positive if and only if there are complex numbers \( c_k \) for \( k \in J \) so that

\[
S = \sum_{k \in J} c_k S_k \quad \text{and} \quad \sum_{k \in J} |c_k|^2 \leq 1.
\]

We mention that given a map \( \phi \) of \( B(H) \) into \( B(K) \) we will assume it is normal unless stated otherwise.

We introduce the notation we will be using throughout this paper. First let \( K \) be a Hilbert space and let \( H = K \otimes L^2(0,\infty) \), so \( H \) can be thought of as Lebesgue measurable \( K \)-valued functions \( F(x) \) for \( x \in [0,\infty) \) with inner product

\[
(F,G) = \int_0^\infty (F(x),G(x)) \, dx.
\]

We define \( \Lambda \) as the mapping from \( B(K) \) to \( B(H) \) given by

\[
(\Lambda(A)F)(x) = e^{-x} AF(x)
\]

for \( F \in H, A \in B(K) \) and \( x \in [0,\infty) \). We often use the operator \( \Lambda(I) \) which will we often simply write as \( \Lambda \). We hope the reader is not confused as \( \Lambda \) can refer to the mapping \( \Lambda \) and the operator \( \Lambda(I) \) but the difference can be inferred from the context. In almost all cases where we use \( \Lambda \) to
denote \( \Lambda(I) \) it occurs in the form \( I - \Lambda \) in which case \( \Lambda \) is clearly \( \Lambda(I) \) since \( I \) is the unit operator so when we write the expression \( I - \Lambda \) it always means \( I - \Lambda(I) \). We will denote the identity mapping by \( \iota \) so when we write \( \iota + \omega \Lambda \) then \( \Lambda \) denotes the mapping \( \Lambda \).

Given a Hilbert space \( K \) we define \( \mathfrak{A}(K) \) as the set of operators in \( B(H) \) where \( H = K \otimes L^2(0, \infty) \) of the form

\[
A = (I - \Lambda)^{1/2} B(I - \Lambda)^{1/2}
\]

with \( B \in B(H) \). The norm on \( \mathfrak{A}(K) \) is given by

\[
\| A \|_+ = \|(I - \Lambda)^{-1/2} A(I - \Lambda)^{-1/2} \|
\]

where \( \| \cdot \| \) is the usual norm on \( B(H) \). We denote by \( \mathfrak{A}(K)_* \) the linear functionals \( \eta \) on \( \mathfrak{A}(K) \) so that the functional

\[
\rho(A) = \eta((I - \Lambda)^{1/2} A(I - \Lambda)^{1/2})
\]

for \( A \in B(H) \) is \( \sigma \)-weakly continuous on \( B(H) \). Such a linear functional \( \eta \) is called a boundary weight or \( b \)-weight. Now each element of \( \mathfrak{A}(K)_* \) is automatically bounded in the \( \| \cdot \|_+ \) norm given by

\[
\| \eta \|_1 = \sup \{ \| \eta(A) \| : A \in \mathfrak{A}(K), \| A \|_+ \leq 1 \}.
\]

When we say an element \( \eta \in \mathfrak{A}(K)_* \) is bounded we mean it is bounded with respect to the ordinary Hilbert space norm. So \( \eta \) is bounded if

\[
\| \eta \| = \sup \{ \| \eta(A) \| : A \in \mathfrak{A}(K), \| A \| \leq 1 \} < \infty.
\]

Recall \( H = K \otimes L^2(0, \infty) \) so we can think of \( H \) as \( K \)-valued functions of \( x \). We define \( L^2_+(0, \infty) \) as the set of Lebesgue measurable functions \( f \) so that

\[
\| f \|_+^2 = \int_0^\infty (1 - e^{-x}) |f(x)|^2 \, dx < \infty
\]

and \( \| \cdot \|_+ \) denotes the norm on \( L_+(0, \infty) \). We denote by \( H_+ = K \otimes L^2_+(0, \infty) \) where we can think of elements \( F, G \in H_+ \) as \( K \)-valued functions where the inner product is given by

\[
(F,G)_+ = \int_0^\infty (1 - e^{-x})(F(x), G(x)) \, dx.
\]

Note each \( F \in H_+ \) can be written as \( F = (I - \Lambda)^{-1/2} G \) with \( G \in H \).

Note each \( \eta \in \mathfrak{A}(K)_* \) can be expressed in the form

\[
\eta(A) = \sum_{k \in J} (F_k, AG_k)
\]

where

\[
\sum_{k \in J} \| F_k \|_+^2 = \sum_{k \in J} \| (I - \Lambda)^{-1/2} F_k \|^2 < \infty
\]

and

\[
\sum_{k \in J} \| G_k \|_+^2 = \sum_{k \in J} \| (I - \Lambda)^{-1/2} G_k \|^2 < \infty
\]

and \( J \) is a countable index set. Note if \( \eta \in \mathfrak{A}(K)_* \) and \( \eta \) is bounded then \( \eta \) can be expressed in the form

\[
\eta(A) = \sum_{k \in J} (F_k, AG_k)
\]

where

\[
\sum_{k \in J} \| F_k \|^2 < \infty \quad \text{and} \quad \sum_{k \in J} \| G_k \|^2 < \infty
\]

so \( \eta \) can be thought of as an element of the predual of \( B(K \otimes L^2(0, \infty)) \) (so \( \eta \in B(K \otimes L^2(0, \infty))_* \)).

We denote by \( B(K) \otimes \mathfrak{A}(K)_* \) the set of all linear mappings \( \phi \) of \( \mathfrak{A}(K) \) into \( B(K) \) so that for all \( f, g \in K \) the linear functional \( \langle f, \phi(A)g \rangle \) is in \( \mathfrak{A}(K)_* \). Again we say such a \( \phi \) is bounded if

\[
\| \phi \| = \sup \{ \| \phi(A) \| : A \in \mathfrak{A}(K), \| A \| \leq 1 \} < \infty.
\]
Next we introduce the cut off notation. Recall $H = K \otimes L^2(0, \infty)$. We define for $0 \leq a < b \leq \infty$ the projection $E(a, b)$ on $H$ as the hermitian projection onto functions $F \in H$ with support in the closed interval $[a, b]$ or $(a, \infty)$ in the case where $b = \infty$. If $A \in B(K) \otimes \mathfrak{A}(K)$ or $A \in B(K \otimes L^2(0, \infty))$ and $t > 0$ we denote by $A_t = E(t, \infty)AE(t, \infty)$. If $\eta$ is a $b$-weight we denote by $\eta_t$ the functional

$$\eta_t(A) = \eta(E(t, \infty)AE(t, \infty))$$

for all $A \in B(K \otimes L^2(0, \infty))$ and $t > 0$. Note that for all $t > 0$ if $A \in B(K \otimes L^2(0, \infty))$ then we have $A_t \in \mathfrak{A}(K)$ and if $\eta \in \mathfrak{A}(K)$, then $\eta_t \in B(K \otimes L^2(0, \infty))_s$.

Finally we will denote the identity mapping $A \to A$ of $B(K)$ into itself by $i$ so $i(A) = A$ for all $A \in B(K)$.

Armed with this notation we can now define a $q$-weight map over $K$.

**Definition 1.1.** Suppose $K$ is a separable Hilbert space. A $q$-weight map over $K$ is a completely positive element $\omega \in B(K) \otimes \mathfrak{A}(K)_s$ so that for each $t > 0$ the mapping $(i + \omega_t |A)$ of $B(K)$ into itself is invertible (i.e. this mapping has both a right and left inverse) and the mapping

$$\pi_t^\#(A) = (i + \omega_t |A)^{-1}\omega_t(A)$$

is a completely positive contractive normal linear mapping of $B(H) = B(K \otimes L^2(0, \infty))$ into $B(K)$. The $q$-weight map $\omega$ over $K$ is unital if $\omega(I - \Lambda) = I$. The mappings $\pi_t^\#$ are called the generalized boundary representation of $\omega$. If $\eta$ is an element of $B(K) \otimes \mathfrak{A}(K)_s$ we denote the fact that $\eta$ is completely positive by writing $\eta \geq 0$ and we denote the fact that the generalize boundary representation $\psi_t^\#$ constructed from $\eta$ as shown above is completely positive for all $t > 0$ by writing $\eta \geq_t 0$.

Given an $\omega \in B(K) \otimes \mathfrak{A}(K)_s$ to check that $\omega$ is a $q$-weight map it is only necessary to check that the generalized boundary representation $\pi_t^\#$ of $\omega$ is completely positive and $\pi_t^\#(I) \leq I$ for any sequence of $t_k > 0$ so that $t_k \to 0+$ as $k \to \infty$. This is because if $\pi_t^\#$ is a completely positive contraction then $\pi_t^\#$ is a completely positive contraction for $s \geq t$ so it is only necessary to check the condition for small $t$. We caution the reader that in checking that a weight map is a $q$-weight map that knowing that the limit of $\pi_t^\#(I)$ as $t \to 0+$ is $B$ and $B \leq I$ does not imply $\pi_t^\#(I) \leq I$ for $t > 0$. The important result of [Pow03b] is that every unital $q$-weight map over $K$ uniquely defines a spatial $E_0$-semigroup of $B(H)$ and up to cocycle conjugacy every spatial $E_0$-semigroup of $B(H)$ comes from a unital $q$-weight map over a Hilbert space $K$.

**Definition 1.2.** Suppose $K$ is a separable Hilbert space and $\omega$ and $\eta$ are $q$-weight maps over $K$ with generalized boundary representations $\pi_t^\#$ and $\psi_t^\#$, respectively. We say $\eta$ is a $q$-subordinate of $\omega$ (denoted $\omega \geq_q \eta$) if for all $t > 0$ we have $\pi_t^\# - \psi_t^\#$ is completely positive. We say a $q$-weight map $\omega$ is $q$-pure if the $q$-subordinates of $\omega$ are totally ordered so if $\omega \geq_q \eta_1 \geq_q 0$ and $\omega \geq_q \eta_2 \geq_q 0$ then either $\eta_1 \geq_q \eta_2$ or $\eta_2 \geq_q \eta_1$.

Again to check that $\eta$ is a $q$-subordinate of $\omega$ it is only necessary to check that $\pi_t^\# - \psi_t^\#$ is completely positive for any sequence $t_k > 0$ so that $t_k \to 0+$ as $k \to \infty$ in that if $\pi_t^\# \geq \psi_t^\#$ then $\pi_t^\# \geq \psi_s^\#$ for all $s \geq t$. Again the fact that $\pi_t^\# - \psi_t^\#$ is completely positive in the limit does not imply that $\eta$ is a $q$-subordinate of $\omega$.

If $\omega$ is a unital $q$-weight map and $\eta_1$ and $\eta_2$ are $q$-subordinates of $\omega$ then $\eta_1$ and $\eta_2$ are uniquely associated with a positive contractive local cocycles $C_1$ and $C_2$ of norm one of the $E_0$-semigroup $\alpha$ induced by $\omega$ so $0 \leq C_1(t) \leq I$, $0 \leq C_2(t) \leq I$ and $C_1(t) \geq C_2(t)$ for all $t > 0$ if and only if $\eta_1 \geq \eta_2$. Conversely, if $C$ is a positive contractive local cocycle of norm one for the $E_0$-semigroup $\alpha$ associated with $\omega$ then there is a $q$-subordinate $\eta$ of $\omega$ uniquely associated with $C$ so there is a one to one order preserving mapping from contractive local cocycles of norm one for $\alpha$ onto the $q$-subordinates of $\omega$.

The next theorem summarizes results in [Pow03b] and shows how to compute the index of the $E_0$-semigroup induced by a $q$-weight map.

**Theorem 1.3.** Suppose $\omega$ is a $q$-weight map over a Hilbert space $K$ and $\pi_t^\#$ is the generalized boundary representation of $\omega$. Then for each $s > 0$ the map $A \to \pi_t^# |_s(A)$ is non decreasing in $t$.
for $0 < t < s$. There is a completely positive $\sigma$-weakly continuous mapping $\pi_0^\#$ of $B(K \otimes L^2(0, \infty))$ into $B(K)$ so that for each $s > 0$ the map $A \mapsto \pi_0^\#_t(A)$ for $0 < t < s$ converges in the $\sigma$-strong operator topology to $\pi_0^\#(A)$ as $t \to 0^+$. The mapping $\pi_0^\#$ is called the normal spin of $\omega$. If $\omega$ is unital the index of the $E_0$-semigroup induced by $\omega$ is the rank of $\pi_0^\#$ as a completely positive map. We say the index of $\omega$ is the index of the normal spine of $\omega$. In particular we say $\omega$ is of index zero if normal spine of $\omega$ is zero.

It follows that to understand all $E_0$-semigroups of type II$_o$ one only has to understand all unital $q$-weight maps over a Hilbert space $K$ of index zero. The next theorem shows how to determine whether two $q$-weight maps of index zero induce $E_0$-semigroups that are cocycle conjugate. First we define $q$-corners between two $q$-weight maps.

**Definition 1.4.** Suppose $\omega_1$ and $\omega_2$ are $q$-weight maps over the Hilbert spaces $K_1$ and $K_2$, respectively. Let $K$ be the direct sum of $K_1$ and $K_2$ so $K = K_1 \oplus K_2$. Note every operator in $B(K)$ can be uniquely written in matrix form

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where $A_{ij}$ is a bounded linear operator from $K_i$ to $K_j$ for $i, j = 1, 2$. Similarly every operator in $B(K \otimes L^2(0, \infty))$ and every operator in $\mathfrak{A}(K)$ can be written in matrix form where $A_{ij}$ is a bounded linear operator from $K_i \otimes L^2(0, \infty)$ to $K_j \otimes L^2(0, \infty)$. Let $E_i$ be the hermitian projection of $K = K_1 \oplus K_2$ onto $K_i$ for $i = 1, 2$. Now consider the $b$-weight map on $\mathfrak{A}(K)$ given by

$$\omega(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}) = \begin{bmatrix} \omega_1(A_{11}) & \gamma(A_{12}) \\ \gamma^*(A_{21}) & \omega_2(A_{22}) \end{bmatrix}$$

where $\gamma \in B(K) \otimes \mathfrak{A}(K)_s$ and

$$\gamma(A) = E_1 \gamma((E_1 \otimes I)A(E_2 \otimes I))E_2$$

for $A \in \mathfrak{A}(K)$ so $\gamma$ only depends on the $A_{12}$ entry of $A$ and $\gamma(A)$ is a matrix with all zero entries except for the upper right entry. We mean by $\gamma^*$ the mapping given by $\gamma^*(A) = (\gamma(A^*))^*$. We say $\gamma$ is a corner from $\omega_1$ to $\omega_2$ if $\omega$ given above is completely positive. We say $\gamma$ is a $q$-corner from $\omega_1$ to $\omega_2$ if $\omega$ is $q$-positive, $\omega \geq 0$.

We say $\gamma$ is a maximal corner from $\omega_1$ to $\omega_2$ if $\eta$ is a subordinate $\eta$ of $\omega$ of the form

$$\eta(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}) = \begin{bmatrix} \omega_1'(A_{11}) & \gamma(A_{12}) \\ \gamma^*(A_{21}) & \omega_2(A_{22}) \end{bmatrix}$$

then $\omega_1' = \omega_1$. We say $\gamma$ is a maximal $q$-corner from $\omega_1$ to $\omega_2$ if $\eta$ is a $q$-subordinate of $\omega$ of the form

$$\eta(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}) = \begin{bmatrix} \omega_1'(A_{11}) & \gamma(A_{12}) \\ \gamma^*(A_{21}) & \omega_2(A_{22}) \end{bmatrix}$$

then $\omega_1' = \omega_1$.

We say $\gamma$ is a hyper maximal corner from $\omega_1$ to $\omega_2$ if $\eta$ is a subordinate $\eta$ of $\omega$ of the form

$$\eta(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}) = \begin{bmatrix} \omega_1'(A_{11}) & \gamma(A_{12}) \\ \gamma^*(A_{21}) & \omega_2(A_{22}) \end{bmatrix}$$

then $\omega_1' = \omega_1$ and $\omega_2' = \omega_2$. We say $\gamma$ is a hyper maximal $q$-corner from $\omega_1$ to $\omega_2$ if $\eta$ is a $q$-subordinate of $\omega$ of the form

$$\eta(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}) = \begin{bmatrix} \omega_1'(A_{11}) & \gamma(A_{12}) \\ \gamma^*(A_{21}) & \omega_2(A_{22}) \end{bmatrix}$$

then $\omega_1' = \omega_1$ and $\omega_2' = \omega_2$.

**Theorem 1.5.** Suppose $\omega_1$ and $\omega_2$ are unital $q$-weight maps over $K_1$ and $K_2$ of index zero, respectively. Then the $E_0$-semigroups induced by $\omega_1$ and $\omega_2$ are cocycle conjugate if and only if there is a hyper maximal $q$-corner from $\omega_1$ to $\omega_2$. 
Note that if $\omega$ is a $q$-weight map over $K$ then $\omega$ is a hyper maximal $q$-corner from $\omega$ to $\omega$ so every $q$-weight map over $K$ is cocycle conjugate to itself. Technically cocycle conjugacy refers to $E_0$-semigroups which correspond to unital $q$-weight maps but we extend the notion of cocycle conjugacy to arbitrary $q$-weight maps by saying the $q$-weight maps $\omega_1$ and $\omega_2$ over $K_1$ and $K_2$, respectively, are cocycle conjugate if there is a hyper maximal $q$-corner from $\omega_1$ to $\omega_2$. An important word of caution is that in the non unital case we do not know that cocycle conjugacy is an equivalence relation.

2. $Q$-pure $q$-weight maps

In this section we discuss the notion of $q$-pure $q$-weight maps over a Hilbert space $K$. We recall from the last section.

**Definition 2.1.** We say a $q$-weight map is $q$-pure if its $q$-subordinates are totally ordered.

We believe the next theorem is true for $q$-pure $q$-weight maps over $K$ where $K$ is infinite dimensional but so far we only have a proof for the case when $K$ is finite dimensional (i.e. $K = \mathbb{C}^p$ for $p$ a positive integer.)

**Theorem 2.2.** Suppose $\omega$ is $q$-pure $q$-weight map over $\mathbb{C}^p$ and $\rho$ is a faithful normal state on $B(\mathbb{C}^p)$. Then the mapping $\eta(\rho) \mapsto \eta(\rho)(I - \Lambda)$ is a one to one mapping of the $q$-subordinates $\eta$ of $\omega$ onto the interval $[0, \omega(\rho)(I - \Lambda)]$. Furthermore, if $\eta$ is a $q$-subordinate of $\omega$ then the range of $\eta$ is contained in the range of $\omega$.

**Proof.** Assume the hypothesis. Suppose $\eta$ and $\nu$ are $q$-weight maps so that $\omega \geq_{q} \eta \geq_{q} \nu \geq_{q} 0$. Since $\eta \geq \nu$ we have $\eta(\rho)(I - \Lambda) \geq \nu(\rho)(I - \Lambda)$ and if $(\eta - \nu)(\rho)(I - \Lambda) = 0$ it follows that $\eta = \nu$. Hence, we see the mapping $\eta \mapsto \eta(\rho)(I - \Lambda)$ is one to one. Next we show there are no gaps, which is to say that for every $s \in [0, \omega(\rho)(I - \Lambda)]$, there is a $q$-subordinate $\nu$ of $\omega$ so that $\nu(\rho)(I - \Lambda) = s$. If $s = \omega(\rho)(I - \Lambda)$ then $\nu = \omega$ provides the example and if $s = 0$ then $\nu = 0$ provides the example. We consider the case where $s \in (0, \omega(\rho)(I - \Lambda))$. Let $\pi_k^\#$ be the generalized boundary representation of $\omega$ and let $\eta_{(t,\lambda)} = \lambda(\iota - \lambda\pi_k^\#) - \pi_k^\#$ for $\lambda \in [0,1]$ and $t > 0$. Note that $\eta_{(t,\lambda)}$ is a $q$-subordinate of $\omega|_t$ and as $\lambda$ goes from 1 to 0, $\eta_{(t,\lambda)}(I - \Lambda)$ goes continuously from $\omega|_t(\rho)(I - \Lambda)$ to zero. Since $s < \omega(\rho)(I - \Lambda)$ and $\omega|_t(\rho)(I - \Lambda) \rightarrow \omega(\rho)(I - \Lambda)$ as $t \rightarrow 0$ we have that there is a $t_0$ so that $\omega|_t(\rho)(I - \Lambda) \geq s$ for $t < t_0$. Hence, for each $t \in (0, t_0]$ there is a $\lambda = \lambda_t$ so that $\eta_{(t,\lambda_t)}(I - \Lambda) = s$. To simplify notation we denote this $\eta_{(t,\lambda)}$ by $\eta_t$. Calculating $\eta_t$ in terms of $\omega$ we find

\[
\eta_t = \lambda_t(\iota + (1 - \lambda_t)\omega|_t\Lambda)^{-1}\omega|_t
\]

for $t > 0$.

Let $\{t_k : k = 1, 2, \ldots\}$ be decreasing sequence tending to zero (so $t_k \rightarrow 0$ as $k \rightarrow \infty$) and let $\sigma$ be an ultrafilter. To further simplify notation we let $\eta_k = \eta_{t_k}$. Since $\eta_k \in \mathfrak{A}(K)$, and we have the bound $||\eta_k(I - \Lambda)|| \leq 1$ the $\eta_k$ are in a compact set in the weak topology so $\lim_{k} \eta_k = \nu$ exists.

We claim $\nu$ is the desired $q$-subordinate of $\omega$. Since for $k$ sufficiently large we have $\eta_k(\rho)(I - \Lambda) = s$ it follows that $\nu(\rho)(I - \Lambda) = s$. To show $\nu$ is a $q$-subordinate of $\omega$ we must show that for each $t \in (0, t_0)$ that $(\iota + \nu|_t\Lambda)^{-1}$ exists and $\pi_k^\# \geq (\iota + \nu|_t\Lambda)^{-1}\nu|_t$. Let $\phi_k^\#$ be the generalized boundary representation of $\eta_k$. Note for each $t > 0$ and $k$ sufficiently large we have $\pi_k^\# \geq \phi_k^\# \geq 0$, $\pi_k^\# \Lambda \geq \phi_k^\# \Lambda$ and $I \geq \pi_k^\# \Lambda(I) \geq \phi_k^\# \Lambda(I)$. Now we have

\[
(\iota + \nu|_t\Lambda)^{-1} = \iota - \pi_k^\# \Lambda \quad \text{and} \quad (\iota + \eta_k|_t\Lambda)^{-1} = \iota - \phi_k^\# \Lambda.
\]

Since the mappings above are mappings of finite dimensional linear spaces into themselves the weak limit of such maps is also the strong limit and, therefore, the limit of the composition of two such maps is the composition of the limits so, for example,

\[
\lim_{\sigma} \phi_k \psi_k = \lim_{\sigma} \phi_k \lim_{\sigma} \psi_k
\]

and we will freely use this in our computations. Note that

\[
(\iota + \eta_k|_t\Lambda)^{-1} = \iota - \phi_k^\# \Lambda.
\]
Then if

\[ \psi_t = \lim_{\sigma}(t - \phi^*_{\mu}\Lambda) \]

we see that

\[ (t + \nu|\Lambda)\psi_t = \lim_{\sigma}(t + \eta_k|\Lambda)(t + \eta_k|\Lambda)^{-1} = \lim_{\sigma}t = t \]

and

\[ \psi_t(t + \nu|\Lambda) = \lim_{\sigma}(t + \eta_k|\Lambda)^{-1}(t + \eta_k|\Lambda) = \lim_{\sigma}t = t \]

and we conclude that \((t + \nu|\Lambda)^{-1}\) exists and

\[ (t + \nu|\Lambda)^{-1}\nu|_t = \lim_{\sigma}(t - \phi^*_{\mu}\Lambda)(t - \phi^*_{\mu}\Lambda)^{-1}\phi^*_{\mu} \]

and since \(\pi^*_t \geq \phi^*_{\mu} \geq 0\) for \(k\) sufficiently large we conclude that the generalized boundary representation \(\phi^*_{\mu}\) for \(\nu\) exist and satisfies

\[ \pi^*_t \geq \phi^*_{\mu} \geq 0 \]

and, hence, \(\nu\) is \(q\)-positive and \(\nu\) is a \(q\)-subordinate of \(\omega\).

To complete the proof we show that the range of any \(q\)-subordinate is contained in the range of \(\omega\). Recall that

\[ \eta_{(t,\lambda)} = \lambda(t - \lambda\pi^*_t\Lambda)^{-1}\pi^*_t = \lambda\pi^*_t + \lambda^2\pi^*_t\Lambda\pi^*_t + \cdots \]

so we see the range of \(\eta_{(t,\lambda)}\) is contained in the range of \(\pi^*_t\) which is contained in the range of \(\omega\). Since the range of \(\eta_{(t,\lambda)}\) is contained in the range of \(\omega\) for all \(t > 0\) and \(\lambda \in [0,1]\) and any \(q\)-subordinate \(\nu\) of \(\omega\) is the limit of \(\eta_{(t,\lambda)}\) it follows that the range of \(\nu\) is contained in the range of \(\omega\).

Consider \(\omega\) a \(q\)-weight map over \(C\) so \(\omega \in \mathfrak{A}(C)_*\) and \(\omega(I - \Lambda) \leq 1\). In Theorem 3.9 of [Pow03a] is was shown how to find all \(q\)-subordinates of \(\omega\).

**Theorem 2.3.** Suppose \(\omega\) is a \(q\)-weight map over \(C\) and \(\rho\) is a positive normal functional on \(B(L^2(0,\infty))\) so \(\rho \in B(L^2(0,\infty))_* \subset \mathfrak{A}(C)_*\) and \(\rho(I) < \infty\) and \(\omega \geq \rho\) so \(\omega(A) \geq \rho(A)\) for all positive \(A \in \mathfrak{A}(C)\). Then \(\eta = \lambda(1 + \rho(A))^{-1}(\omega - \rho)\) for \(0 \leq \lambda \leq 1\) is a \(q\)-subordinate of \(\omega\). Conversely, suppose \(\eta\) is a non zero \(q\)-subordinate of \(\omega\) then there is a positive normal functional \(\rho \in B(L^2(0,\infty))_*\) (so \(\rho(I) < \infty\)) and a real number \(\lambda \in (0,1]\) so that \(\eta = \lambda(1 + \rho(A))^{-1}(\omega - \rho)\). Furthermore, if \(\omega\) is unbounded then \(\rho\) and \(\lambda\) are unique.

One sees that an unbounded \(q\)-weight map \(\omega\) over \(C\) is \(q\)-pure if and only if \(\rho\) is a subordinate of \(\omega\) (so \(\omega(A) \geq \rho(A) \geq 0\) for all positive \(A \in \mathfrak{A}(C)\)) and \(\rho\) is bounded then \(\rho = 0\). This notion of purity comes up again and again as a condition for the \(q\)-purity of a \(q\)-weight map. For this reason we will give this notion a name.

**Definition 2.4.** Suppose \(K\) is a separable Hilbert space and \(\mu\) is a \(b\)-weight over \(K\) (i.e. \(\mu \in \mathfrak{A}(K)_*\)) then \(\mu\) is strictly infinite if \(\mu\) has no bounded subordinates. Similarly if \(\phi\) is a \(B(K)\) valued completely positive \(b\)-weight map on \(\mathfrak{A}(C^p)\) (so \(\phi \in B(K) \otimes \mathfrak{A}(C^p)_*\) and \(\phi \geq 0\)) we say \(\phi\) is strictly infinite if \(\phi\) has no bounded non zero subordinates so if \(\eta\) is a non zero subordinate of \(\phi\) (i.e. \(\eta\) is another \(B(K)\) valued completely positive \(b\)-weight map on \(\mathfrak{A}(C^p)\) and \(\phi \geq \eta\)) then \(\eta\) is unbounded.

In this terminology we see that an unbounded \(q\)-weight map \(\omega\) over \(C\) is \(q\)-pure if and only if it is strictly infinite. In [JMP11] we discussed range rank one \(q\)-weight maps over \(C^p\) and our results are summarized in the following theorems.

**Theorem 2.5.** Suppose \(\omega\) is a \(q\)-weight map of range rank one over \(K\) of index zero where \(K\) is a separable Hilbert space so \(\omega\) can be expressed in the form \(\omega(\rho)(A) = \rho(T)\mu(A)\) for \(\rho \in B(K)_*\) and \(A \in \mathfrak{A}(C^p)\) where \(T\) is a positive operator of norm one and \(\mu\) is a positive element of \(\mathfrak{A}(K \otimes L^2(0,\infty))_*\) and \(\mu(I - \Lambda(T)) \leq 1\) and \(\mu(I) = \infty\). Then \(\omega\) is \(q\)-pure if and only if the following three conditions are met.

(i) \(T\) is a projection.
(ii) \(\mu\) is strictly infinite.
Given such a mapping we define the super matrix associated with
\[ \text{Theorem 2.6.} \] Suppose \( A \) for \( \omega = \rho(T_1)\mu(A) \) for \( A \in \mathfrak{A}(K) \) and \( \rho \in B(K_1) \) and \( \eta(A) = \rho(T_2)\nu(A) \) for \( A \in \mathfrak{A}(K_2) \) and \( \rho \in B(K_2) \), and where \( T_1 \) and \( T_2 \) are hermitian projections and \( \mu \) and \( \nu \) are q-pure q-weight maps. Then \( \omega \) and \( \eta \) are cocycle conjugate (so there is a hyper maximal q-corner from \( \omega \) to \( \eta \)) if and only if there is a partial isometry \( U \) from \( K_1 \) to \( K_2 \) and a \( \lambda > 0 \) so that \( U^*U = T_1 \), \( UU^* = T_2 \), and \( \mu \) and \( \nu \) can be expressed in the form
\[
\mu(A) = \sum_{k \in J} (f_k, Af_k) \quad \text{and} \quad \nu(B) = \sum_{k \in J} (g_k, Bg_k)
\]
for \( A \in \mathfrak{A}(K_1) \) and \( B \in \mathfrak{A}(K_2) \) with \( g_k = \lambda(U \otimes I)f_k + h_k \) where \( h_k \in K_2 \otimes L^2(0, \infty) \) for \( k \in J \) and
\[
\sum_{k \in J} \|h_k\|^2 < \infty.
\]
In [JMP11] the above theorem was proved in the finite dimensional case. In this paper we will only make use of this theorem in the case where \( K_1 \) and \( K_2 \) are finite dimensional.

3. Completely positive and conditionally positive maps

This section we discuss completely positive maps with special emphasis on completely positive maps of the \((p \times p)\)-matrices \( B(C^p) \) into themselves. We denote by \( S(C^p) \) the space of all hermitian linear maps of \( B(C^p) \) into itself. If \( \phi \in S(C^p) \) we denote by \( \phi_{ij} \) the \((i,j)\)-entry of the matrix \( \phi(A) \). Given such a mapping we define the super matrix associated with \( \phi \) as
\[
S_{\phi_{ij}} = \phi_{ij}(e_{nm})
\]
where \( e_{nm} \) are the complete set of matrix units for \( B(C^p) \) i.e. \( (e_{nm}x)_k = \delta_{nk}x_m \) for \( n, m, k \in \{1, \ldots, p\} \) for \( x \in C^p \) and \( \delta_{nk} \) is the Kronecker delta which equals one for \( m = k \) and zero otherwise. Note the super matrix \( S \) is a \((p^2 \times p^2)\)-matrix. A very useful result of Choi is that \( \phi \) is completely positive if and only if the super matrix \( S \) is positive. Also the rank of a completely positive map of \( B(C^p) \) into itself is equal to the rank of its super matrix.

We consider ways of representing completely positive maps \( \phi \) of \( B(C^p) \) into itself. Since the combined action of \( B(C^p) \) on \( B(C^p) \) by both right and left multiplication is irreducible it follows that every linear mapping \( L \) of \( B(C^p) \) into itself can be written in the form
\[
L(A) = \sum_{i=1}^m B_i AC_i
\]
for \( A \in B(C^p) \) where the \( B_i, C_i \in B(C^p) \) for \( i = 1, \ldots, m \). If we require \( L \) to be hermitian so that \( L(A^*) = L(A)^* \) for \( A \in B(C^p) \) then we have
\[
L(A) = \frac{1}{2} \sum_{i=1}^m B_i AC_i + C_i^* AB_i^*
\]
and since
\[
BAC + C^* AB^* = \frac{1}{2}((B + C^*)A(B + C^*)^* - (B - C^*)A(B - C^*)^*)
\]
we see that every hermitian \( L \) can be written in the form
\[
L(A) = \sum_{i=1}^m r_i X_i AX_i^*
\]
for \( A \in B(C^p) \), where \( r_1, \ldots, r_m \in \mathbb{R} \). Now let \( G \) be the unitary group \( SU(p) \) and let \( \mu \) be Haar measure on \( G \). We note that for \( A \in B(C^p) \) we have
\[
\int_G U_g AU_g^* d\mu(g) = \text{tr}(A)I
\]
where \( tr \) is the trace normalized so that \( tr(I) = 1 \). Now given an hermitian linear map \( L \) of \( B(\mathbb{C}^p) \) into itself we define \( \Theta_L \) as

\[
\Theta_L(A) = \int_G L(U_g A) U_g^* d\mu(g)
\]

for \( A \in B(\mathbb{C}^p) \). If \( L \) is of the form given above then we have

\[
\Theta_L(A) = \sum_{i=1}^{m} r_i X_i U_g X_i^* U_g^* d\mu(g) = \sum_{i=1}^{m} r_i tr(X_i^* A) X_i
\]

for \( A \in B(\mathbb{C}^p) \). Considering \( B(\mathbb{C}^p) \) as a Hilbert space \( B(\mathbb{C}^p) \otimes B(\mathbb{C}^p) \) with inner product \( (A, B)_1 = tr(A^* B) \) we see that \( \Theta_L \) is an hermitian linear operator and as such it can be diagonalized so that we can write \( L \) in the form

\[
L(A) = \sum_{i=1}^{m} \lambda_i S_i A S_i^*
\]

for \( A \in B(\mathbb{C}^p) \) where \( S_i \in B(\mathbb{C}^p) \) and the \( \lambda_i \in \mathbb{R} \) and \( tr(S_i^* S_j) = \delta_{ij} \) for \( i, j = 1, \ldots, m \) and \( m \leq p^2 \). The numbers \( \lambda_i \) are the eigenvalues of \( \Theta_L \) and the \( S_i \) are the associated normalized eigenvectors.

We show that \( L \) is completely positive if and only if \( \lambda_i \geq 0 \) for \( i = 1, \ldots, m \). First we note that if \( \lambda_i \geq 0 \) in equation (3.1) then \( L \) is the sum of completely positive terms and, thus, \( L \) is completely positive. To prove the converse suppose \( \lambda_q < 0 \) in the sum (3.1). Let \( h_i \) be an orthonormal basis for \( \mathbb{C}^p \) and let \( A_i \) be the operator on \( \mathbb{C}^p \) given by \( A_i f = h_1(h_i, S_q f) \) for \( f \in \mathbb{C}^p \). We compute the sum

\[
\sum_{i,j=1}^{p} (h_i, L(A_i^* A_j) h_j) = \sum_{k=1}^{m} \sum_{i,j=1}^{p} \lambda_k (h_i, S_k A_i^* A_j S_k^* h_j).
\]

Now we have

\[
\sum_{j=1}^{p} A_j S_k^* h_j = \sum_{j=1}^{p} h_1(h_j, S_q S_k^* h_j) = p \cdot tr(S_q S_k^*) h_1 = p \delta_{qk} h_1.
\]

Combining this result with the above equation we find

\[
\sum_{i,j=1}^{p} (h_i, L(A_i^* A_j) h_j) = p^2 \lambda_q.
\]

We see that if \( \lambda_q < 0 \) then \( L \) is not completely positive. Hence, we have shown that \( L \) is completely positive if and only if \( \lambda_i \geq 0 \) for each term in equation (3.1).

The norm on \( \mathcal{S}(\mathbb{C}^p) \) is given by

\[
\| \phi(A) \| = \sup \{ \| \phi(A) \| : \| A \| \leq 1 \}.
\]

For a completely positive map \( \phi \) we have \( \| \phi \| = \| \phi(I) \| \). In general it is hard to compute the norm so we will often use a softer norm, the Hilbert Schmidt norm, which is the Hilbert Schmidt norm of the super matrix, so

\[
\| \phi \|_{H.S.}^2 = \frac{1}{p^2} \sum_{j,k,r,s=1}^{p} |\phi_{jkrst}|^2.
\]

One checks that if \( \phi \) is of the form given in (3.1) then

\[
\| \phi \|_{H.S.}^2 = \sum_{i=1}^{p} \lambda_i^2.
\]

The topology given by the Hilbert Schmidt norm and the norm is equivalent since

\[
p^{-3/2} \| \phi \| \leq \| \phi \|_{H.S.} \leq \| \phi \|
\]

and note that for the identity \( \iota(A) = A \) for \( A \in B(\mathbb{C}^p) \) we have \( \| \iota \|_{H.S.} = \| \iota \| \) and if \( \phi(A) = tr(A)e \) for \( A \in B(\mathbb{C}^p) \) where \( e \) is a rank one projection we have \( p^{-3/2} \| \phi \| = \| \phi \|_{H.S.} \). To prove the
right inequality we note that if \( \{ U_i : i = 1, \ldots, p^2 \} \) is an orthonormal basis for \( B(\mathbb{C}^p) \) of unitaries so \( \text{tr}(U_i^*U_j) = \delta_{ij} \) then \( \| \phi(U_i) \|_{\text{H.S.}}^2 \leq \| \phi \|^2 \) so we have

\[
\| \phi \|_{\text{H.S.}}^2 = \frac{1}{p^2} \sum_{i=1}^{p^2} \| \phi(U_i) \|_{\text{H.S.}}^2 \leq \frac{1}{p^2} \sum_{i=1}^{p^2} \| \phi \|^2 = \| \phi \|^2
\]

and to prove the left hand inequality we note that if \( A_o \in B(\mathbb{C}^p) \) and \( \| A_o \| = 1 \) and \( \| \phi(A_o) \| = \| \phi \| \) then there are unit vectors in \( \mathbb{C}^p \) so that \( (f, \phi(A_o)g) = \| \phi \| \). Now suppose \( e_1 \) and \( e_2 \) are the one dimensional hermitian projections so that \( e_1 f = f \) and \( e_2 g = g \) and we see that if \( \psi(A) = e_1 \phi(A)e_2 \) then \( \| \psi \| = \| \phi \| \) and \( \| \psi \|_{\text{H.S.}} \leq \| \phi \|_{\text{H.S.}} \) so we see that if we want to construct a \( \phi \) with \( \| \phi \| = 1 \) of smallest Hilbert Schmidt norm we should consider only \( \phi \) that have one dimensional range that is a multiple of a rank one operator and since the Hilbert Schmidt norm of \( \phi \) and \( \phi \cdot U \) where \( U \) is unitary are the same it is enough to consider a mapping \( \phi \) of the form \( \phi(A) = \text{tr}(XA)e \) for \( A \in B(\mathbb{C}^p) \) and we have \( \| \phi \| = \text{tr}((X^*X)^{\frac{1}{2}}) \) and

\[
\| \phi \|_{\text{H.S.}}^2 = \frac{1}{p^2} \sum_{i,j=1}^{p^2} p \cdot \text{tr}(\phi(e_{ij})^*\phi(e_{ij})) = \frac{1}{p^2} \sum_{i,j=1}^{p^2} |\text{tr}(Xe_{ij})|^2
\]

\[
= \frac{1}{p^2} \sum_{i,j=1}^{p^2} |x_{ij}|^2 = \frac{1}{p^3} \| X \|_{\text{H.S.}}^2.
\]

Now if \( (X^*X)^{\frac{1}{2}} \) has eigenvalues \( \lambda_i \) for \( i = 1, \cdots, p \) then

\[
\| \phi \| = \frac{1}{p} \sum_{i=1}^{p} \lambda_i \quad \text{and} \quad \| X \|_{\text{H.S.}}^2 = \frac{1}{p} \sum_{i=1}^{p} \lambda_i^2
\]

so to minimize \( \| \phi \|_{\text{H.S.}} \) subject to the fact that \( \| \phi \| \) is given one sets all the \( \lambda_i \) equal so \( \lambda_i = \| \phi \| \) and so \( \| X \|_{\text{H.S.}} = \| \phi \| \). Then we have

\[
\| \phi \|_{\text{H.S.}}^2 = \frac{1}{p^3} \| X \|_{\text{H.S.}}^2 = \frac{1}{p^3} \| \phi \|^2
\]

which establishes the left hand inequality of (3.2).

Next we consider the norm of the product of elements in \( S(\mathbb{C}^p) \). Clearly we have \( \| \phi \psi \| \leq \| \phi \| \cdot \| \psi \| \). To figure out the effect of the product on the Hilbert Schmidt norm consider \( \phi \in S(\mathbb{C}^p) \) and \( A \in B(\mathbb{C}^p) \) so we have

\[
\| \phi(A) \|_{\text{H.S.}}^2 = \text{tr}(\phi(A)^*\phi(A)) \leq \| \phi(A) \|^2 \leq \| \phi \|^2 \| A \|^2 = \| \phi \|^2 \| A^*A \|
\]

\[
\leq p \| \phi \|^2 \text{tr}(A^*A) = p \| \phi \|^2 \| A \|_{\text{H.S.}}^2.
\]

If \( \phi \in S(\mathbb{C}^p) \) we denote by

\[
B_{ij} = \phi(e_{ij}) \quad \text{and} \quad (f_i, \phi(A)f_j) = p \cdot \text{tr}(\Omega_{ij}A)
\]

where \( f_i \) and \( e_{ij} \) are the standard basis and matrix units for \( \mathbb{C}^p \) and \( B(\mathbb{C}^p) \). Then we note

\[
\| \phi \|_{\text{H.S.}}^2 = \frac{1}{p} \sum_{i,j=1}^{p} \| B_{ij} \|_{\text{H.S.}}^2 = \frac{1}{p} \sum_{i,j=1}^{p} \| \Omega_{ij} \|^2.
\]

Then we note that the Hilbert Schmidt norm of the product \( \psi \phi \) of two elements in \( S(\mathbb{C}^p) \) is given by

\[
\| \psi \phi \|_{\text{H.S.}}^2 = \frac{1}{p} \sum_{i,j=1}^{p} \| \psi(B_{ij}) \|_{\text{H.S.}}^2
\]

\[
\leq \| \psi \|^2 \frac{1}{p} \sum_{i,j=1}^{p} \| B_{ij} \|_{\text{H.S.}}^2 = p \| \psi \|^2 \| \phi \|_{\text{H.S.}}^2.
\]
Hence, we have $\|\psi\|_{H.S.} \leq \sqrt{p}\|\psi\| \cdot \|\phi\|_{H.S.}$ for $\psi, \phi \in S(C^p)$. An example where the inequality is an equality is when

$$\psi(A) = p \cdot tr(Ae_{11})e_{11} \quad \text{and} \quad \phi(A) = tr(A)I$$

for $A \in B(C^p)$.

Next we will prove the similar inequality, namely, that

$$\|\phi\psi\|_{H.S.} \leq \sqrt{p}\|\psi\| \cdot \|\phi\|_{H.S.}$$

for $\psi, \phi \in S(C^p)$. Now for $\psi \in S(C^p)$ we have

$$\psi(A) = \sum_{k=1}^{m} \lambda_k S_k A S_k^*$$

with $S_k \in B(C^p)$ and $tr(S_k^* S_n) = \delta_{kn}$ and the $\lambda_k$ real. We denote by $\tilde{\psi} \in S(C^p)$ the mapping

$$\tilde{\psi}(A) = \sum_{k=1}^{m} \lambda_k S_k A S_k$$

for $A \in B(C^p)$. Note that for $A, B \in B(C^p)$ we have

$$tr(A^* \psi(B)) = tr(\tilde{\psi}(A)^* B).$$

Then for $\psi, \phi \in S(C^p)$ we have

$$\|\phi\psi\|_{H.S.}^2 = \frac{1}{p} \sum_{i,j=1}^{p} \|\tilde{\psi}(\Omega_{ij})\|_{H.S.}^2.$$ 

So we need to estimate $\|\tilde{\psi}(A)\|_{H.S.}$ in terms of the Hilbert Schmidt norm of $A$. Now we have

$$\|\tilde{\psi}(A)\|_{H.S.}^2 \leq p\|\tilde{\psi}\|_{H.S.}^2 \|A\|_{H.S.}^2$$

but the norm of $\psi$ and $\tilde{\psi}$ need not be equal as when $\psi(A) = p \cdot tr(Ae_{11})I$ then $\|\psi\| = 1$ but $\|\tilde{\psi}\| = p$. Now we have

$$\|\tilde{\psi}(A)\|_{H.S.} = sup\{Re(tr(B^* \tilde{\psi}(A))) : B \in B(C^p), \|B\|_{H.S.} \leq 1\}$$

$$\leq sup\{\|\psi(B)\|_{H.S.} \|A\|_{H.S.} : B \in B(C^p), \|B\|_{H.S.} \leq 1\}$$

and, hence, we have

$$\|\phi\psi\|_{H.S.}^2 = \frac{1}{p} \sum_{i,j=1}^{p} \|\tilde{\psi}(\Omega_{ij})\|_{H.S.}^2.$$ 

Hence, we have shown that

(3.3) $\|\psi\|_{H.S.} \leq \sqrt{p}\|\psi\| \cdot \|\phi\|_{H.S.}$ and $\|\phi\psi\|_{H.S.} \leq \sqrt{p}\|\psi\| \cdot \|\phi\|_{H.S.}$

for $\psi, \phi \in S(C^p)$.

Next we turn to conditionally positive maps. A mapping of $B(K)$ into itself is said to be conditionally positive if $\phi$ is hermitian (so $\phi(A^*) = \phi(A)^*$ for $A \in B(K)$) and if $A_i \in B(K)$ and $f_i \in K$ for $i = 1, \cdots, n$ and

$$\sum_{i=1}^{n} A_i f_i = 0 \quad \text{then} \quad \sum_{i,j=1}^{n} (f_i, \phi(A_i^* A_j) f_j) \geq 0.$$ 

We really should call such maps conditionally completely positive maps but we will use the shorter term, conditionally positive, and hope the reader will remember we mean the longer expression.
We say a linear mapping $\phi$ of $B(K)$ into itself is conditionally negative if $-\phi$ is conditionally positive. We say such a map $\phi$ is conditionally zero if both $\phi$ and $-\phi$ are conditionally positive. If $K = \mathbb{C}^p$ then one discovers that a mapping $\phi$ is conditionally positive if and only if the super matrix $S$ associated with $\phi$ is hermitian and

$$(F, SF) \geq 0 \quad \text{for} \quad F \in \mathbb{C}^p \otimes \mathbb{C}^p \quad \text{with} \quad \sum_{i=1}^{p} f_{ii} = 0.$$  

Note this is not the same as saying that the super matrix is conditionally positive which would be

$$(F, SF) \geq 0 \quad \text{for} \quad F \in \mathbb{C}^p \otimes \mathbb{C}^p \quad \text{with} \quad \sum_{i,j=1}^{p} f_{ij} = 0.$$  

We consider the Shur product $A \circ B$ of two matrices in $B(\mathbb{C}^p)$ given by

$$(A \circ B)_{ij} = a_{ij}b_{ij}$$

for $i, j \in \{1, \cdots, p\}$. We note that the mapping $\psi_A$ given by

$$\psi_A(B) = A \circ B$$

is completely positive if and only if $A$ is positive and $\psi_A$ is conditionally positive if and only if $A$ is conditionally positive by which we mean $(f, Af) \geq 0$ for all $f \in \mathbb{C}_0^p$ where

$$\mathbb{C}_0^p = \{ f \in \mathbb{C}^p : \sum_{i=1}^{p} f_i = 0 \}.$$  

Next we give a brief discussion of ways to represent conditionally positive maps of $\mathbb{C}^p$ into itself. We will use a slightly different representation. Looking at equation (3.1) we note that each of the matrices $S_i$ can expressed as

$$S_i = s_i I + S'_i$$

where $tr(S'_i) = 0$. One can then write equation (3.1) with terms that have trace zero. The resulting $S'_i$ will not necessarily be orthonormal but by choosing a new basis one can write equation (3.1) in the form

$$L(A) = sA + YA + AY^* + K_L(A) \quad \text{where} \quad K_L(A) = \sum_{i=1}^{m} \lambda_i X_i AX_i^*$$

where $s$ is real the $\lambda_i$ are real and $Y$ and the $X_i$ are of trace zero. The $X_i$ can be chosen so that $tr(X_i^*X_j) = \delta_{ij}$ for $i, j = 1, \cdots, m$. We call $K_L$ the internal part of $L$ and $s$ the coefficient of the identical part of $L$. We note that $s$ and $Y$ are uniquely determined since

$$\Theta_L(I) = \int L(U_g)U_g^*d\mu(g) = sI + Y$$

so $s$ and $Y$ are uniquely determined. Note $s$ is determined since $Y$ is of trace zero.

Once $s$ and $Y$ are determined one can form the map

$$K_L(A) = L(A) - sA - YA - AY^* = \sum_{i=1}^{m} \lambda_i X_i AX_i^*$$

so $K$ the internal part of $L$ and $s$ the coefficient of the identical part of $L$ are uniquely determined.

Next we note that $L$ is conditionally positive if and only if $\lambda_i \geq 0$ for each $i = 1, \cdots, m$ (i.e. $L$ is conditionally positive if and only if $K_L$ is completely positive). Note if $\lambda_i \geq 0$ in (3.4) then $K_L$ is a completely positive map and one checks that the first three terms of (3.4) are conditionally zero. Hence, if the $\lambda_i \geq 0$ the mapping $L$ of (3.4) is the sum of a completely positive map and a conditionally zero map so it is conditionally positive.

Next we show that if one of the $\lambda_i$ is negative then $L$ is not conditionally positive. Suppose then that at least one of the $\lambda_i$, say $\lambda_q$, in (3.4) is negative. Let $h_i$ be an orthonormal basis for
$C^p$ and let $A_i$ be the operator given by $A_if = h_1(h_i,X_qf)$ for $f \in C^p$. We have

$$\sum_{i=1}^{p} A_i h_i = \sum_{i=1}^{p} h_1(h_i,X_qh_i) = p \cdot tr(X_q)h_1 = 0.$$  

(3.5)

Now if $L$ is conditionally positive we have

$$\sum_{i,j=1}^{p} (h_i, L(A_i^*A_j)h_j) = \sum_{k=1}^{m} \sum_{i,j=1}^{p} \lambda_k(h_i,X_kA_i^*A_jX_k^*h_j) \geq 0.$$  

where in the computation to the right of the equal sign we used the fact that condition (3.5) causes the first three terms in equation (3.4) to vanish. Now we have

$$\sum_{j=1}^{p} A_j X_k^* h_j = \sum_{j=1}^{p} h_1(h_j, X_kX_k^*h_j) = ptr(X_kX_k^*)h_1 = p\delta_{kq}h_1.$$  

Combining this result with the above equation we find

$$\sum_{i,j=1}^{p} (h_i, L(A_i^*A_j)h_j) = p^2 \lambda_q.$$  

We see that if $\lambda_q < 0$ then $L$ is not conditionally positive.

We note that if $L$ is conditionally zero then the $\lambda_i$ in the sum (3.4) are both positive and negative so we see that if $L$ is conditionally zero then $L$ is of the form

$$L(A) = sA + YA + AY^*$$  

for $A \in B(C^p)$ with $s$ real and $Y \in B(C^p)$ of trace zero. Note if $L$ is conditionally negative and $L$ is completely positive then $L$ is of the above form with $Y = 0$ and $s \geq 0$. To see this, note that if $L$ satisfies these conditions then $L$ is conditionally zero, so $L$ is of the above form, and one checks that if $L(e) \geq 0$ for all rank one projections then $s \geq 0$ and $Y$ is a multiple of the identity. Since $Y$ has trace zero we have $Y = 0$ so $L(A) = sA$ for $A \in B(C^p)$ with $s \geq 0$.

Given an element $\phi \in S(C^p)$ we will want to represent it the form $\phi = st + \psi$ where $\psi$ is small. Now in general it is difficult to compute the norm of an element of $S(C^p)$ so we will use the Hilbert Schmidt norm. We point out an advantage of equation (3.4) for representing elements of $S(C^p)$ is that if $\phi$ is of the form (3.4) and you wish to write $\phi = st + \psi$ where $\psi$ has the smallest Hilbert Schmidt norm then $s$ is precisely the $s$ in equation (3.4) so $\psi$ is simply the expression in equation (3.4) with the $sA$ term omitted.

Note if $\phi$ is given by equation (3.4) then one computes

$$\|\phi\|_{H.S.}^2 = s^2 + 2tr(Y^*Y) + \sum_{i=1}^{m} \lambda_i^2$$  

and if we write $\phi = st + \psi$ then

$$\|\psi\|_{H.S.}^2 = 2tr(Y^*Y) + \sum_{i=1}^{m} \lambda_i^2.$$  

Next we consider the question of when a map $\phi \in S(C^p)$ expressed in the form of equation (3.4) is completely positive. We claim the map

$$L(A) = sA + YA + AY^* + \sum_{i=1}^{m} \lambda_i X_i AX_i^*$$  

for $A \in B(C^p)$ is completely positive if and only if the $\lambda_i$ are positive and there are complex numbers $c_i$ so that

$$Y = \sum_{i=1}^{m} c_i X_i \quad \text{and} \quad \sum_{i=1}^{m} |c_i|^2/\lambda_i \leq s.$$  

(3.6)
To see this assume the conditions on $Y$ and $s$ are satisfied. Let $r$ be the square root of the second sum above so $0 \leq r^2 \leq s$. Then we have

$$L(A) = (s - r^2)A + (rI + Y/r)A(rI + Y^*/r) + \sum_{i=1}^{m} \lambda_i AX_i X_i^* - r^{-2} Y A Y^*.$$ 

Now

$$Y/r = \sum_{i=1}^{m} (\lambda_i^{1/2} c_i/r) \lambda_i^{1/2} X_i = \sum_{i=1}^{m} b_i \lambda_i^{1/2} X_i$$

where $b_i = \lambda_i^{-1/2} c_i/r$ for $i = 1, \cdots, m$. The last two terms in the expression for $L$ are completely positive if and only if

$$\sum_{i=1}^{m} |b_i|^2 \leq 1$$

and we have

$$\sum_{i=1}^{m} |b_i|^2 = r^{-2} \sum_{i=1}^{m} |c_i|^2 / \lambda_i = r^{-2} \cdot r^2 = 1$$

so $L$ is the sum of three completely positive maps so $L$ is completely positive.

In the other direction let $h_i$ be an orthonormal basis for $\mathbb{C}^p$ and let $A_i$ be the operator given by $A_i f = (h_i, (zI + Z)f)h_1$ for $f \in \mathbb{C}^p$ where $z \in \mathbb{C}$ and $Z \in B(\mathbb{C}^p)$ with $tr(Z) = 0$ of our choosing. We compute

$$\sum_{i,j=1}^{p} (h_i, L(A_i^* A_j) h_j) = \sum_{i,j=1}^{p} s(h_i, A_i^* A_j h_j) + \sum_{i,j=1}^{p} (h_i, Y A_i^* A_j h_j) + \sum_{i,j=1}^{p} (h_i, A_i^* A_j Y^* h_j) + \sum_{k=1}^{m} \sum_{i,j=1}^{p} \lambda_k (h_i, X_k A_i^* A_j X_k^* h_j).$$

Since $Z$ has trace zero we have

$$\sum_{j=1}^{p} A_j h_j = h_1 (h_j (zI + Z) h_j) = pzh_1$$

and using this we find

$$\sum_{i,j=1}^{p} (h_i, L(A_i^* A_j) h_j) = sp^2 |z|^2 + 2p^2 \text{Re} (\overline{z} tr(Y^* Z)) + \sum_{k=1}^{m} \sum_{i,j=1}^{p} \lambda_k (h_i, X_k A_i^* A_j X_k^* h_j).$$

We compute

$$\sum_{i=1}^{p} A_i X_i^* h_i = \sum_{i=1}^{p} h_1 (h_i, (zI + Z) X_i^* h_i) = p \cdot tr(X_i^* Z) h_1$$

so we find

$$\frac{1}{p^2} \sum_{i,j=1}^{p} (h_i, L(A_i^* A_j) h_j) = s |z|^2 + 2 \text{Re} (\overline{z} tr(Y^* Z)) + \sum_{k=1}^{m} \lambda_k tr(Z^* X_k) tr(X_k^* Z).$$
Since this expression must be positive for all complex $z$ we find
\[ |\text{tr}(Y^*Z)|^2 \leq s \sum_{k=1}^{m} \lambda_k \text{tr}(Z^*X_k)\text{tr}(X_k^*Z). \]

Now considering $B(\mathbb{C}^p)$ as a Hilbert space with inner product $(A, B) = \text{tr}(A^*B)$ we see that the matrices $X_k$ are orthogonal vectors so we can uniquely express $Y$ in the form
\[ Y = \sum_{k=1}^{m} c_k X_k + W \]
where $W$ is orthogonal to $X_k$ for $k = 1, \cdots, m$. Note that since $Y$ and the $X_k$ have trace zero then $W$ is of trace zero. Setting $Z = W$ in the above inequality we find $\text{tr}(W^*W) = 0$ so $W = 0$. Then setting
\[ Z = \sum_{k=1}^{m} \lambda_k^{-1} c_k X_k \]
we find
\[ \left( \sum_{k=1}^{m} |c_k|^2 / \lambda_k \right)^2 \leq s \sum_{k=1}^{m} |c_k|^2 / \lambda_k \]
and so we have the desired inequality
\[ \sum_{k=1}^{m} |c_k|^2 / \lambda_k \leq s. \]

A useful inequality concerning $Y$ for completely positive maps of the form (3.4) is the following. Suppose
\[ L(A) = sA + YA + AY^* + \rho(A) \quad \text{where} \quad \rho(A) = K_L(A) = \sum_{i=1}^{m} \lambda_i X_i A X_i^* \]
for $A \in B(\mathbb{C}^p)$ and $L$ is completely positive. Then the following $(2 \times 2)$-matrix with entries in $B(\mathbb{C}^p)$ is positive so
\[ \begin{bmatrix} sI & Y^* \\ Y & \rho(I) \end{bmatrix} \geq 0. \]
To see this we note since $L$ is completely positive there are complex numbers $c_k$ for $k = 1, \cdots, m$ so that
\[ Y = \sum_{i=1}^{m} c_i X_i \quad \text{and} \quad \sum_{i=1}^{m} |c_i|^2 / \lambda_i = r \leq s. \]
Then we have
\[ \sum_{i=1}^{m} \left[ \begin{bmatrix} c_i \lambda_i^{-1/2} I \\ \lambda_i X_i^* \end{bmatrix} \left[ \begin{bmatrix} \frac{r}{y} & Y^* \\ I & \rho(I) \end{bmatrix} \right] = \begin{bmatrix} \frac{r}{y} & Y^* \\ I & \rho(I) \end{bmatrix} \right] \geq 0 \]
and since $s \geq r > 0$ the result follows.

Another useful fact about completely positive map in $S(\mathbb{C}^p)$ is this. Suppose $\phi \in S(\mathbb{C}^p)$ is completely positive and $\|\phi(I)\| < 1$. Then $\tau - \phi$ is invertible and its inverse is completely positive. This is seen as follows. Since $\phi$ is completely positive $\|\phi\| = \|\phi(I)\| < 1$ and, hence, the series
\[ 1 + \|\phi\| + \|\phi\|^2 + \cdots \]
converges and since $\|\phi^n\| \leq \|\phi\|^n$ it follows that
\[ (\tau - \phi)^{-1} = \tau + \phi + \phi^2 + \cdots \]
where the series converges in norm. Since $\phi^n$ is completely positive for each $n$ it follows that $(\tau - \phi)^{-1}$ is completely positive.

We now prove a technical lemma which is of crucial importance in the next section.
Lemma 3.1. Suppose $\epsilon > 0$. Then there is a $\delta > 0$ so that if $\xi_1$ and $\xi_2$ are completely positive maps in $\mathcal{S}(\mathbb{C}^p)$ so that

$$\|\xi_1(I) + \xi_2(I)\| < 1 \quad \text{and} \quad \|\nu - (\nu - \xi_2)^{-1}\xi_1\| < \delta$$

then

$$\nu - \xi_2 = \kappa(\nu + \eta)$$

with $\kappa > 0$ and $\|\eta\| < \epsilon$.

Proof. Assume $\epsilon < 1$ and let

$$\delta = \min\left(\frac{\epsilon}{4p^2}, \frac{0.1}{\sqrt{p}}\right)$$

and suppose $\xi_1$ and $\xi_2$ satisfy the hypothesis of the lemma. Let

$$\nu - \xi = b^{-1}(\nu - \xi_2) \quad \text{with} \quad b = \|\nu - \xi_2\|_{H.S.}.$$  

Note that $b > 0$ since if $b = 0$ then $\xi_2 = \nu$ which would violate the assumption $\|\xi_1(I) + \xi_2(I)\| < 1$.

Note $\|\nu - \xi\|_{H.S.} = 1$. Let

$$\zeta = \nu - (\nu - \xi_2)^{-1}\xi_1$$

and by assumption $\|\zeta\| < \delta$. Note $\nu - \xi_2 = b(\nu - \xi)$ and we have

$$\xi_1 = (\nu - \xi_2)(\nu - \zeta) = b(\nu - \xi)(\nu - \zeta).$$

Since $\xi_1 \geq 0$ and $b > 0$ we have

$$\nu - \xi - \zeta + \zeta \geq 0.$$  

Let $\nu = (\nu - \xi)\zeta$. Since $\|\nu - \xi\|_{H.S.} = 1$ and from our previous estimate of the Hilbert Schmidt norm of the product of two elements of $\mathcal{S}(\mathbb{C}^p)$ (see inequality (3.3)) we have

$$\|\nu\|_{H.S.} = \|(\nu - \xi)\zeta\|_{H.S.} \leq \sqrt{p}\|\nu - \xi\|_{H.S.}\|\zeta\| = \sqrt{p}\|\zeta\| \leq \delta\sqrt{p}$$

and we have $\nu - \xi - \zeta \geq 0$.

Now $\nu - \xi = b^{-1}(\nu - \xi_2)$ and since $\xi_2 \geq 0$ and the identity map $\nu$ is conditionally zero we have $\nu - \xi$ is conditionally negative. Hence, we can write this map in the form of equation (3.4) as

$$(\nu - \xi)(A) = sA + YA + AY^* - \sum_{i=1}^{m} \lambda_i X_i AX_i^*$$

where $Y$ and the $X_i$ have trace zero and $s$ is real and $tr(X_i^* X_j) = \delta_{ij}$ and $\lambda_i > 0$ for $i, j = 1, \cdots, m$.

Since the Hilbert Schmidt norm of $\nu - \xi$ is one we have

$$\|\nu - \xi\|_{H.S.}^2 = s^2 + 2tr(Y^* Y) + \sum_{i=1}^{m} \lambda_i^2 = 1.$$  

Now we can express $\nu$ in the form

$$\nu(A) = rA + ZA + AZ^* - \sum_{i=1}^{q} \mu_i S_i AS_i^*$$

where $Z$ and the $S_i$ have trace zero and $tr(S_i^* S_j) = \delta_{ij}$. Note we have put a minus sign in front of the sum in anticipation of the fact that the $\mu_i$ will turn out to be positive. Since $\|\nu\|_{H.S.} < \delta p^{1/2}$

$$|\nu|^2 + 2\|Z\|_{H.S.}^2 + \sum_{i=1}^{m} \mu_i^2 < p\delta^2.$$  

Now we have

$$(\nu - \xi - \nu)(A) = (s - r)A + (Y - Z)A + A(Y - Z)^* - \sum_{i=1}^{m} \lambda_i X_i AX_i^* + \sum_{j=1}^{q} \mu_j S_j AS_j^*$$

for $A \in B(\mathbb{C}^p)$. Note the above map is completely positive. Now for the above mapping to be completely positive the two sums term above must add up to a positive quadratic form. Note
then if any of the $\mu_i$ are negative then the above expression is not completely positive. Let $A$ and $B$ correspond to the positive quadratic forms

$$A(Q) \iff \sum_{j=1}^q \mu_j S_j Q S_j^* \quad \text{and} \quad B(Q) \iff \sum_{i=1}^m \lambda_i X_i Q X_i^*.$$  

Since if $A \geq B \geq 0$ we have

$$\text{tr}(A^2) = \text{tr}(A^2 A A^2) \geq \text{tr}(A^2 B A A^2) = \text{tr}(AB)$$

$$\quad = \text{tr}(B^2 A B A^2) \geq \text{tr}(B^2 B B^2) = \text{tr}(B^2)$$

it follows that

$$\sum_{i=1}^m \lambda_i^2 \leq \sum_{j=1}^q \mu_j^2 < p \delta^2.$$  

Since the quadratic form $A - B$ is positive we can write

$$L(A) = -\sum_{i=1}^m \lambda_i X_i A X_i^* + \sum_{j=1}^q \mu_j S_j A S_j^* = \sum_{k=1}^r \sigma_k C_k A C_k$$

for $A \in B(\mathbb{C}^p)$ where the $C_k$ are of trace zero and $\text{tr}(C_k^* C_l) = \delta_{kl}$ and $\sigma_k > 0$ for $k, l = 1, \ldots, r$. We have

$$\sum_{j=1}^r \sigma_j^2 \leq \sum_{j=1}^q \mu_j^2 < p \delta^2$$

and we have

$$(\iota - \xi - \nu)(A) = (s - r) A + (Y - Z) A + A(Y - Z)^* + \sum_{j=1}^r \sigma_j C_j A C_j^*$$

for $A \in B(\mathbb{C}^p)$. Since the above map is completely positive we have from our previous discussion that $Y - Z$ must satisfy the conditions (3.6) so

$$Y - Z = \sum_{i=1}^r c_i C_i \quad \text{and} \quad \sum_{i=1}^r |c_i|^2 / \sigma_i \leq s - r.$$  

Note $s^2 \leq \|\iota - \xi\|_{H.S.}^2 = 1$ and $r^2 \leq \|\nu\|_{H.S.}^2 < p \delta^2$

$$s - r < 1 + \delta p^{\frac{1}{2}}$$

and

$$\sum_{i=1}^r |c_i|^2 / \sigma_i < 1 + \delta p^{\frac{1}{2}}.$$  

Now we have

$$\|Y - Z\|_{H.S.}^2 = \sum_{i=1}^r |c_i|^2.$$  

Maximizing the above sum subject the above inequality on the $|c_i|^2$ we see the maximum occurs when $c_i = 0$ except $c_q$ where $\sigma_q$ is the maximum of the $\sigma_i$. Hence, we have

$$\|Y - Z\|_{H.S.}^2 < (1 + \delta p^{\frac{1}{2}}) \sigma_{\text{max}}$$

where $\sigma_{\text{max}}$ is the largest of the $\sigma_i$. We have

$$\sigma_{\text{max}}^2 \leq \sum_{i=1}^r \sigma_i^2 < p \delta^2$$

we have

$$\|Y - Z\|_{H.S.} < \delta p^{\frac{1}{2}} \sqrt{1 + \delta p^{\frac{1}{2}}}$$
Lemma 3.2. Suppose $\phi$ is a completely positive map of $B(\mathbb{C}^p)$ into itself and $T$ is an hermitian operator. Then there is an hermitian linear map $\psi$ of $B(\mathbb{C}^p)$ into itself so that $\psi(I) = T$ and $\psi + \phi$ is conditionally zero. Furthermore if $\psi'$ is an hermitian linear map of $B(\mathbb{C}^p)$ into itself so that $\psi' + \phi$ is conditionally negative and $\psi' \geq \psi$ then $\psi' = \psi + si$ where $s \geq 0$ and $i$ is the identity map.

Proof. Assume the hypothesis and notation of the lemma. Let $\psi$ be given by

$$\psi(A) = YA + AY^* - \phi(A)$$

for $A \in B(K)$ where $Y = B + iC$ and $B = \frac{1}{2}(T + \phi(I))$ and $C = C^*$ can be freely chosen. Notice we can add a real multiple of the identity to $C$ and $\psi$ is unchanged. Now suppose $\psi'$ satisfies the hypothesis of the theorem. Then $\psi' = \psi + \eta$ where $\eta$ is completely positive and $\psi' + \phi$ is conditionally negative so we have the map

$$A \rightarrow YA + AY^* + \eta(A)$$

is conditionally negative. Since the map $A \rightarrow YA + AY^*$ is conditionally zero we have the map $\eta$ is conditionally negative. Hence, $\eta$ is conditionally zero and completely positive so

$$\eta(A) = sA$$

for $A \in B(\mathbb{C}^p)$ with $s \geq 0$.

An important theorem of Evans and Lewis [EL77] is that a mapping $\phi$ is the generator of a semigroup of completely positive maps if and only if $\phi$ is conditionally completely positive. We use this result in the follows lemma which we will need later.
Lemma 3.3. Suppose $\phi$ is a conditionally negative map of $B(\mathbb{C}^p)$ into itself and $\phi(I) \geq sI$ with $s > 0$. Then $\phi$ is invertible and its inverse $\phi^{-1}$ is completely positive. If $\phi'$ is also a conditionally negative map of $B(\mathbb{C}^p)$ into itself and $\phi \leq \phi'$ then $\phi'$ is also invertible and $\phi^{-1} \geq \phi'^{-1}$.

**Proof.** Assume the hypothesis and notation of the lemma. Choose $s'$ so that $0 < s' < s$. Since the identity map $\iota$ is conditionally zero $\phi - s'\iota$ is conditionally negative and the result of Evans and Lewis mentioned above states that the exponential of a conditionally positive linear mapping is completely positive. Hence, $\Psi_t = \exp(-t(\phi - s'\iota))$ is a completely positive map for $t > 0$. Since $\Psi_t$ is completely positive we have $\|\Psi_t\| = \|\Psi_t(I)\|$. We have

$$\Psi_t(I) = I - t(\phi - s'\iota)(I) + t^2/2!(\phi - s'\iota)^2(I) \cdots$$

$$\leq I - t(s - s')I + t^2/2!(\phi - s'\iota)^2(I) \cdots$$

$$\leq I - t(s - s')I + I((1/2!)t^2\|\phi - s'\iota\|^2 + (1/3!)t^3\|\phi - s'\iota\|^3 + \cdots)$$

$$= (1 - t(s - s') + r(t))I$$

where

$$r(t) = (e^t\|\phi - s'\iota\| - 1 - t\|\phi - s'\iota\|).$$

Since $r(t)/t^2 \to \frac{1}{2}\|\phi - s'\iota\|^2$ as $t \to 0$ there is a $\delta > 0$ so that $r(t) < t(s - s')$ for $0 < t < \delta$. Hence, $\Psi(t) \leq I$ for $0 < t < \delta$ and, hence, $\|\Psi(t)\| \leq 1$ for $0 < t < \delta$ and since $\Psi(t)$ is a semigroup we have $\|\Psi(t)\| \leq 1$ for all $t$. Since $e^{-t\phi} = e^{-s't(\phi - s'\iota)}$, we have $\|e^{-t\phi}\| \leq e^{-s't}$. Hence, we have

$$\phi^{-1} = \int_0^\infty e^{-t\phi} dt$$

where our estimate on the norm of $e^{-t\phi}$ insures that the integral exists and is equal to $\phi^{-1}$. Since $\phi^{-1}$ is the integral of completely positive maps $\phi^{-1}$ is completely positive.

Now suppose $\phi'$ has the properties stated in the lemma. Since $\phi' \geq \phi$ we have $\phi'(I) \geq \phi(I) \geq sI$ so $\phi'^{-1}$ exists and is completely positive. Consider the equation

$$\frac{d}{dt}e^{-t\phi'} = e^{-t\phi}(\phi' - \phi)e^{t\phi'}$$

and integrating we find

$$e^{-t\phi'}e^{t\phi'} - I = \int_0^t e^{-s\phi'}(\phi' - \phi)e^{s\phi'} ds$$

and multiplying by $e^{-t\phi'}$ on the right we have

$$e^{-t\phi} - e^{-t\phi'} = \int_0^t e^{-s\phi'}(\phi' - \phi)e^{(s-t)\phi'} ds.$$  

Since $e^{-s\phi}$, $\phi' - \phi$ and $e^{(s-t)\phi'}$ are completely positive for $0 \leq s \leq t$ the expression on the right hand side of the above equation is completely positive and, hence,

$$\phi^{-1} - \phi'^{-1} = \int_0^\infty (e^{-t\phi} - e^{-t\phi'}) dt$$

is completely positive so $\phi^{-1} \geq \phi'^{-1}$. \qed

4. **Index zero $q$-weight map of full rank**

We begin this section by introducing what will call the skeleton of a $q$-weight map where by the skeleton we mean the mapping $\phi_t = \omega|\Lambda$. This means we focus only on $\omega|\Lambda$ and ignore any further details contained in $\omega$. The notion of the skeleton is more psychological than mathematical. The idea is that $q$-weight map are fairly complicated objects and in focusing on the skeleton we concentrate on a less complicated object. We believe a complete understanding to skeletons would go a very long way toward understanding $q$-weight maps. We have discovered a great deal of information is contained in the skeleton. For example, in an earlier paper we showed that every range rank two $q$-weight map is never $q$-pure and some subordinants can be constructed with no
further knowledge than the skeleton. We will say that a q-weight map has a pure skeleton if the subordinates of the q-weight map than are constructed from only a knowledge of the skeleton are totally ordered. This is a somewhat vague statement but it will become precise in specific situations.

We begin with the general properties of skeletons. First we note that $\omega|_{t}(I)$ can be computed from the skeleton. This is because

$$\frac{d}{dt}\omega|_{t}(I) = e^{t} \frac{d}{dt}\phi_{t}(I)$$

and, therefore, $\omega|_{t}(I)$ can be calculated from $\phi_{t}(I)$ by integration

$$\omega|_{t}(I) = e^{t}\phi_{t}(I) + \int_{t}^{\infty} e^{s}\phi_{s}(I)ds.$$  \hspace{1cm} (4.1)

The important point is when we refer to $\omega|_{t}(I)$ we are referring to something that is computable from the skeleton so, for example, the fact that $\omega$ is unital is computable from its skeleton since $\omega$ is unital if and only if $\omega|_{t}(I - \Lambda) = \omega|_{t}(I) - \phi_{t}(I) \to I$ as $t \to 0+$.

**Theorem 4.1.** Suppose $\omega$ is a q-weight map over $\mathbb{C}^{p}$ and $\phi_{t} = \omega|_{t}\Lambda$ for $t > 0$. Then $\phi_{t}$ has the following properties.

(i) $\phi_{t}$ is completely positive.

(ii) $\phi_{t}$ is non increasing so $\phi_{t} \geq \phi_{s}$ for $0 < t \leq s$.

(iii) $(t + \phi_{t})^{-1}$ exists and $(t + \phi_{t})^{-1}\phi_{t}$ is a completely positive contractive map.

(iv) $(t + \phi_{t})^{-1}(\phi_{t} - \phi_{r})$ is completely positive for $0 < t \leq s \leq r$.

(v) $(t + \phi_{t})^{-1}\omega|_{t}(I) \leq I$.

(vi) $t - (t + \phi_{t})^{-1}(t + \phi_{s})$ is completely positive for $0 < t < s$.

(vii) $(t + \phi_{t})^{-1}(t + \phi_{s})$ is conditionally negative for $0 < t < s$.

**Proof.** Assume the hypothesis of the theorem. Conditions (i) and (iii) follow from the definition of a q-weight map. Condition (ii) follows from the fact that for $0 < t \leq s$ we have

$$(\phi_{t} - \phi_{s})(A) = \omega|_{t}\Lambda(A) - \omega|_{s}\Lambda(A) = \omega(E(t,s)\Lambda(A)E(t,s))$$

where $E(t,s)$ is the orthogonal projection in $B(\mathbb{C}^{p} \otimes L^{2}(0,\infty))$ onto functions with support in the interval $[t,s]$.

Condition (iv) follows from the fact that the generalized boundary representation $\pi^{\#}_{t}$ is completely positive and

$$(t + \phi_{t})^{-1}(\phi_{s} - \phi_{r})(A) = \pi^{\#}_{t}(E(s,r)\Lambda(A)E(s,r))$$

and condition (v) is just the statement that $\pi^{\#}_{t}$ is completely contractive.

Condition (vi) follows from the fact that $(t + \phi_{t})^{-1}\phi_{s}$ is non increasing in $s$ for $0 < t < s$ and $(t + \phi_{t})^{-1}(t + \phi_{t}) = t$ and since the identity map is conditionally zero condition (vii) follows from condition (vi). \hspace{1cm} $\square$

**Definition 4.2.** If $\phi = \{ \phi_{t} \}_{t \geq 0}$ is a one parameter family of maps satisfying the conditions of Theorem 4.1, where the formula for $\omega|_{t}(I)$ in condition (v) is replaced by the right-hand side of Equation (4.1), we say $\phi$ is an admissible skeleton over $\mathbb{C}^{p}$.

We believe that if $\phi$ is an admissible skeleton then there is a q-weight map $\omega$ over $\mathbb{C}^{p}$ so that $\phi_{t} = \omega|_{t}\Lambda$ for $t > 0$. We believe further that if $\phi$ is a pure admissible skeleton by which we mean you can not see that the q-weight map $\omega$ associated with $\phi$ is not q-pure then there is a q-pure q-weight map $\omega$ so that $\phi_{t} = \omega|_{t}\Lambda$ for $t > 0$. In the case of a q-weight map $\omega$ over $\mathbb{C}$, one sees that the q-weight map

$$\omega(A) = (f, Af) \quad \text{where} \quad f(x) = (-e^{x} \frac{d}{dx}\phi_{x}(I))^{\frac{1}{2}}$$

is a q-pure q-weight map so that $\phi_{t} = \omega|_{t}\Lambda$.

The next result except for the last sentence first appeared in [Jan10] and then in [JMP11] but because of its importance and the shortness of the proof we include it here.
Theorem 4.3. Suppose $\omega$ is a q-weight map over $\mathbb{C}^p$ of index zero and $\pi^t\#$ is the generalized boundary representation of $\omega$. Then $\pi^t\#(A) \rightarrow A$ as $t \rightarrow 0+$ for all $A$ in the range of $\omega$. There is a $\delta > 0$ so that the range of the skeleton $\phi_t = \omega|_t \Lambda$ is the range of $\omega$ for all $t \in (0, \delta)$. Furthermore, if $L$ is any limit point of the $\pi^t\#(A) \rightarrow t \rightarrow 0+$ then $L$ is a completely positive idempotent map with range $L$ the range of $\omega$ and $L$ is an $*$-algebra which we call the Choi-Effros algebra with multiplication given by $A \star B = L(AB)$. Furthermore, if $\omega'$ is a q-subordinate of $\omega$ then $\omega'$ is of index zero.

Proof. Assume the hypothesis and notation of the theorem. Note that the range of $\omega|_t$ is non increasing in $t$ so that for $0 < t < s$ we have $\text{Range}(\omega|_t) \supset \text{Range}(\omega|_s)$. Since $B(\mathbb{C}^p)$ is finite dimensional there is an $s > 0$ so that $\text{Range}(\omega|_t) = \text{Range}(\omega|_s)$ for $0 < t < s$. Now suppose $A \in \text{Range}(\omega)$ so there is an $B \in \mathfrak{A}(\mathbb{C}^p)$ so that $\omega|_s(B) = A$ and since $\omega|_s(B) = \omega(E(s, \infty))BE(s, \infty))$ (where $E(s, \infty)$ is the projection in $\mathfrak{A}(\mathbb{C}^p)$ onto functions with support in $[s, \infty)$) we may assume $B$ has support in $E(s, \infty)(\mathbb{C}^p \otimes L^2(0, \infty))$ so $B = E(s, \infty)BE(s, \infty)$. Then we have

$$A - \pi^t\#(A) = (t + \omega|_t(A) - \omega|_s(B))$$

$$= (t + \omega|_t(A) - \omega|_s(B)) = (t + \omega|_t(A) - \omega|_s(B))$$

as $t \rightarrow 0+$. Hence, $\pi^t\#(A) \rightarrow A$ as $t \rightarrow 0+$.

Clearly the range of $\pi^t\#(A)$ is contained in the range of $\omega$ for all $t > 0$ and since $B(\mathbb{C}^p)$ is finite dimensional and $\pi^t\#(A) \rightarrow A$ for all $A$ in the range of $\omega$ is follows there is a $\delta > 0$ so that the range of $\pi^t\#(A)$ is equal to the range of $\omega$ for $t \in (0, \delta)$. Recall in term of the skeleton $\phi_t = \omega|_t \Lambda$ we have $\pi^t\#(A) = (t + \phi_t)^{-1}(A) = \phi_t(t + \phi_t)^{-1}$ so the range of $\pi^t\#(A)$ is the range of $\phi_t$. Hence the range of $\phi_t$ is the range of $\omega$ for all $t \in (0, \delta)$.

Now if $L$ is any limit point of $\pi^t\#(A)$ as $t \rightarrow 0+$ then $L(A)$ for each $A \in B(\mathbb{C}^p)$ is the limit of operators in the range of $\omega$ and since $B(\mathbb{C}^p)$ is finite dimensional everything that is the limit of elements in the range of $\omega$ is, in fact, in the range of $\omega$. Hence, $L(A)$ is in the range of $\omega$ and, therefore, $\pi^t\#(L(A)) \rightarrow L(A)$ as $t \rightarrow 0+$. Hence, we have $L(L(A)) = L(A)$.

From the result of Choi and Effros [CE77] we have

$$L(AL(B)) = L(L(A)L(B)) = L(L(AB))$$

for $A, B \in B(\mathbb{C}^p)$. Note $L(A) = A$ for $A$ in the range of $\omega$. Choi and Effros define the multiplication as

$$A \star B = L(AB).$$

and equipped with this multiplication $\mathcal{L}(\omega)$ is a $C^*$-algebra. In the paper of Choi and Effros they assume that $I_o = L(I)$ is the unit $I$ of $B(\mathbb{C}^p)$ but in our finite dimensional case this assumption is not needed (see Theorem 5.1 in the next section).

Next suppose $\omega'$ is a q-subordinate of $\omega$. Suppose $A \in \mathfrak{A}(\mathbb{C}^p)$, $A \geq 0$ and $E(r, \infty)AE(r, \infty)$ for $r > 0$ where $E(r, \infty)$ is the orthogonal projection in $B(\mathbb{C}^p \otimes L^2(0, \infty))$ onto function with support in $[r, \infty)$. Suppose $\pi^t\#$ and $\pi^t\#$ are the generalized boundary representations of $\omega$ and $\omega'$, respectively. Since $\omega$ is of index zero we have $\pi^t\#(A) \rightarrow 0$ as $t \rightarrow 0+$ and since $\pi^t\# \geq \pi^t\#$ we have $\pi^t\#(A) \rightarrow 0$ as $t \rightarrow 0+$.

Now we further specialize our attention to q-weight maps of index zero and full range rank by which we mean for every $A \in B(\mathbb{C}^p)$ there is an $B \in \mathfrak{A}(\mathbb{C}^p)$ so that $\omega(B) = A$. It then follows for such q-weight maps that the skeleton $\phi_t$ has range $B(\mathbb{C}^p)$ for small $t$ and if $\pi^t\#$ is the generalized boundary representation of $\omega$ then $\pi^t\#(A) \rightarrow t$ as $t \rightarrow 0$. We prove an important theorem.

Theorem 4.4. Suppose $\omega$ is a q-weight map over $\mathbb{C}^p$ of index zero and the range of $\omega$ is $B(\mathbb{C}^p)$. For $t > 0$ let $\phi_t = \omega|_t \Lambda$ and let $\pi^t\#$ be the generalized boundary representation of $\omega$. Let

$$v_t = tr(I + \phi_t(I))$$

and

$$\Theta_t = v_t^{-1}(I + \phi_t)$$
where \( tr \) is the trace on \( B(\mathbb{C}^p) \) normalized so that \( tr(I) = 1 \). Then \( \Theta_t \) is completely positive and invertible and the inverse \( \Theta_t^{-1} \) is conditionally negative and \( \Theta_t \) converges to a limit \( \Theta \) as \( t \to 0^+ \) and \( \Theta \) is completely positive. The limit \( \Theta \) is invertible and the inverse \( \Theta^{-1} \) is conditionally negative and \( \Theta_t^{-1} \to \Theta^{-1} \) as \( t \to 0^+ \).

Furthermore, \( \nu = \Theta^{-1} \omega \) is a completely positive \( B(\mathbb{C}^p) \) valued \( \nu \)-weight map on \( \mathcal{A}(\mathbb{C}^p) \) (i.e. \( \nu \in B(\mathbb{C}^p) \otimes \mathcal{A}(\mathbb{C}^p)_s \)) with the property that
\[
\partial_t \lambda = \nu_t (t + \nu_t)
\]
and \( 1/\nu_t \to 0 \) and \( ||\nu_t|| \to 0 \) as \( t \to 0^+ \).

**Proof.** Assume the hypothesis and notation of the theorem. Note \( \Theta_t \) is completely positive and \( tr(\Theta_t(I)) = 1 \) for all \( t > 0 \). We will show that \( ||\Theta_t - \Theta_s|| \to 0 \) as \( t, s \to 0^+ \). Now suppose \( 0 < \epsilon < 0.1 \) and
\[
\epsilon_1 = \frac{\epsilon}{2p + \epsilon} \quad \text{and} \quad \delta_1 = \min\left(\frac{\epsilon_1}{4p^2}, \frac{0.1}{\sqrt{p}}\right).
\]
Since
\[
\pi_t^\# \Lambda = t - (t + \phi_t)^{-1} \to t
\]
as \( t \to 0^+ \), there is a \( \delta > 0 \) so that
\[
||(t + \phi_t)^{-1}|| < \delta_1
\]
for \( t \in (0, \delta) \). Now suppose that \( 0 < t < s < \delta \). Recalling the properties of the skeleton we have
\[
\xi_1 = (t + \phi_t)^{-1} \phi_s \geq 0 \quad \text{and} \quad \xi_2 = (t + \phi_t)^{-1}(\phi_t - \phi_s) \geq 0.
\]
Note that
\[
\nu_1 - \xi_2 = (t + \phi_t)^{-1}(t + \phi_s)
\]
and
\[
||\nu_1 - (t - \xi_2)^{-1}\xi_1|| = ||(t + \phi_s)^{-1}|| < \delta_1.
\]
Since \( ||\xi_1 + \xi_2|| = ||(t + \phi_t)^{-1}\phi_t|| = ||\pi_t^\# \Lambda|| < 1 \) we have by lemma 3.1
\[
(t + \phi_t)^{-1}(t + \phi_s) = \kappa(t + \eta) \quad \text{so} \quad t + \phi_s = \kappa(t + \phi_t)(t + \eta)
\]
where \( \kappa > 0 \) and \( ||\eta|| < \epsilon_1 \). Now for \( \Theta_t \) as defined above we have
\[
\Theta_s = \kappa v_t/v_s \Theta(t + \eta)
\]
and since \( tr(\Theta_s(I)) = tr(\Theta_t(I)) = 1 \) we have
\[
1 = \kappa v_t/v_s (1 + tr(\Theta_t(\eta(I))))
\]
and since
\[
-||\eta|| \leq \eta(I) \leq ||\eta|| I
\]
we have
\[
-\epsilon_1 < -||\eta|| \leq tr(\Theta_t(\eta(I))) \leq ||\eta|| < \epsilon_1
\]
so
\[
\left| \frac{1 - \nu_s}{\nu_s} \right| < \epsilon_1 \quad \text{and} \quad \left| \frac{\kappa v_t}{v_s} \right| < \epsilon_1
\]
and
\[
||\Theta_s - \Theta_t|| = ||\Theta_t((\kappa v_t/v_s - 1)\nu + \kappa v_t/v_s \eta)||
\leq ||\Theta_t||(\kappa v_t/v_s - 1) + \kappa v_t/v_s ||\eta||
\leq ||\Theta_t||(\frac{\epsilon_1}{1 - \epsilon_1} + \frac{||\eta||}{1 - \epsilon_1}) < ||\Theta_t||(\frac{2\epsilon_1}{1 - \epsilon_1}) \leq \frac{2p\epsilon_1}{1 - \epsilon_1} \leq \epsilon
\]
where we used the estimate \( ||\Theta_t|| = ||\Theta_t(I)|| \leq p \cdot tr(\Theta_t(I)) = p \). Hence, \( ||\Theta_s - \Theta_t|| < \epsilon \) for \( t, s \in (0, \delta) \) and, hence, \( \Theta_t \) converges to a limit \( \Theta \) as \( t \to 0^+ \). Since \( \Theta_t \) is the limit of completely positive maps we have \( \Theta \) is completely positive. For \( 0 < t < s < \delta \) we have
\[
(t + \phi_t)^{-1}(t + \phi_s) = \kappa(t + \eta)
\]
where \( ||\eta|| < \epsilon_1 \) which yields
\[
\Theta_t^{-1} \Theta_s = (\kappa v_t/v_s) (t + \eta) = t - (\kappa v_t/v_s - 1) \nu + \kappa v_t/v_s \eta.
\]
So
\[ \| \ell - \Theta_t^{-1} \Theta_s \| \leq |\kappa v_t/v_s - 1| + \kappa v_t/v_s \| \eta \| < \frac{2\epsilon_1}{1 - \epsilon_1} \leq \epsilon < 0.1. \]

Now if \( \| \ell - x \| < s \) with \( 0 < s < 1 \) then \( x \) is invertible with \( \| \ell - x^{-1} \| < s/(1 - s) \) so we have
\[ \| \ell - \Theta_s^{-1} \Theta_t \| < \frac{0.1}{1 - 0.1} = 1/9 \]
and letting \( t \to 0^+ \) we have
\[ \| \ell - \Theta_s^{-1} \Theta_t \| \leq 1/9 \]
and \( \Theta_s^{-1} \Theta \) is invertible so \( \Theta^{-1} = (\Theta_s^{-1} \Theta)^{-1} \Theta_s^{-1} \). Hence, \( \Theta \) is invertible and since \( \Theta_t \to \Theta \) as \( t \to 0^+ \) we have \( \Theta_t^{-1} \to \Theta^{-1} \) as \( t \to 0^+ \). Since \( \Theta \) is the limit of completely positive maps \( \Theta \) is completely positive.

Note that
\[ \Theta_t^{-1} \omega |t = v_t \pi_t^# \]
for \( t > 0 \) so \( \Theta_t^{-1} \omega |t \) is completely positive and since \( \Theta_t^{-1} \to \Theta^{-1} \) and \( \omega |t \to \omega \) in \( B(\mathbb{C}^p) \otimes \mathfrak{A}(\mathbb{C}^p)_* \) we have \( \vartheta = \Theta^{-1} \omega \) is a completely positive \( B(\mathbb{C}^p) \) valued \( b \)-weight map on \( \mathfrak{A}(\mathbb{C}^p) \).

Since \( \vartheta = \Theta^{-1} \omega \) we have
\[ \vartheta |t \Lambda = \Theta^{-1} \omega |t \Lambda = \Theta^{-1} \phi_t. \]

Now we have
\[ \pi_t^# \Lambda = (t + \phi_t)^{-1} \phi_t = v_t^{-1} \Theta_t^{-1} \phi_t \]
so
\[ \vartheta |t \Lambda = \Theta^{-1} \vartheta |t \pi_t^# \Lambda = v_t (\Theta^{-1} \Theta_t) \pi_t^# \Lambda. \]

Then we have
\[ \vartheta |t \Lambda = v_t (t + \nu_t) \quad \text{where} \quad \nu_t = \Theta^{-1} \Theta_t \pi_t^# \Lambda - t \]
and since \( \pi_t^# \Lambda \to t \) and \( \Theta^{-1} \Theta_t \to t \) as \( t \to 0^+ \) it follows that
\[ \| \nu_t \| \to 0 \]
as \( t \to 0^+ \). Note that \( (t + \nu_t)^{-1} = v_t^{-1} \Theta_t^{-1} \) and since \( \Theta_t^{-1} \to \Theta^{-1} \) and \( \| (t + \nu_t)^{-1} \| \to 0 \) as \( t \to 0^+ \) it follows that \( \nu_t^{-1} \to 0 \) as \( t \to 0^+ \).

To see that \( \Theta^{-1} \) is conditionally negative note that
\[ \pi_t^# \Lambda = t - (t + \phi_t)^{-1} = t - v_t^{-1} \Theta_t^{-1} \]
so if \( A_i \in B(\mathbb{C}^p) \) and \( f_i \in \mathbb{C}^p \) for \( i = 1, \cdots, n \) and
\[ \sum_{i=1}^n A_i f_i = 0 \]
then
\[ \sum_{i,j=1}^n v_t (f_i, \pi_t^# \Lambda (A_i^* A_j) f_j) = - \sum_{i,j=1}^n (f_i, \Theta_t^{-1} (A_i^* A_j) f_j) \geq 0 \]
so \( \Theta_t^{-1} \) is conditionally negative and its limit \( \Theta^{-1} \) is conditionally negative.

Now we make an important switch in notation. The previous theorem proved statements about the mapping \( \Theta \). Now we will focus on the mapping \( \psi \) which is the inverse of \( \Theta \). The reason for this change of focus is that it is much easier to characterize subordinates of a \( q \)-weight map in terms of the mapping \( \psi \). Express in terms of \( \psi \) (the inverse of \( \Theta \)) the conclusion of Theorem 4.4 is that every \( q \)-weight map \( \omega \) over \( \mathbb{C}^p \) of index zero and range of \( B(\mathbb{C}^p) \) can be written in the form \( \omega = \psi^{-1} \vartheta \) where \( \vartheta \in B(\mathbb{C}^p) \otimes \mathfrak{A}(\mathbb{C}^p)_* \) is completely positive and \( \psi \in \mathcal{S}(\mathbb{C}^p) \) is conditionally negative and \( \psi \) is invertible with \( \psi^{-1} \) completely positive and
\[ \vartheta |t \Lambda = v_t (t + \nu_t) \]
where \( v_t \to \infty \) and \( \| \nu_t \| \to 0 \) as \( t \to 0^+ \). As we will see it is far easier to discover the properties of \( \vartheta \) and \( \psi \) then to determine the properties of \( \omega \). The problem with the above expression for \( \vartheta |t \Lambda \) is that the definition of \( v_t \) is ambiguous since we can add or subtract a multiple of \( t \) to it. The method we will use is to choose \( v_t \) so as to minimize its Hilbert Schmidt norm. This then makes
where we first defined. This is because the conditions that the \( B \) weight map are most easily expressed in terms of \( \psi \) and \( \vartheta \). Note if we replace \( \psi \) by \( s\psi \) and \( \vartheta \) by \( s\vartheta \) then \( \omega = \psi^{-1}\vartheta \) remains unchanged. So from now on \( \psi \) and \( \vartheta \) are only unique up to multiplying both by a positive number. There are advantages and disadvantages to requiring a specific normalization of \( \psi \) and \( \vartheta \) and we decided in favor of more freedom particularly because given \( \omega = \psi^{-1}\vartheta \) we will want to consider other \( q \)-weight maps of the form \( \omega' = \psi^{-1}\vartheta' \).

Notice in the expression \( \vartheta|\Lambda \) if we change the normalization of \( \vartheta \) so \( \vartheta' = s\vartheta \) then we have

\[
v'_t|\Lambda = v'_t(t + \nu_t)
\]

where \( v'_t = sv_t \to \infty \) and \( \|\nu_t\| \to 0 \) as \( t \to 0+ \).

The reader may well wonder why we use \( \psi \) in the pair \( (\psi, \vartheta) \) when it was the inverse \( \psi^{-1} = \Theta \) which we first defined. This is because the conditions that the \( q \)-weight map \( \omega = \psi^{-1}\vartheta \) be a \( q \)-weight map are most easily expressed in terms of \( \psi \). In the next lemma there is very little mathematical content. It simply shows that a completely positive \( b \)-weight map can be expressed in a certain form. Yet putting \( \vartheta \) in this form proved to be of enormous importance. It had taken us almost three months of work using Maple and Matlab to construct a single example of a \( q \)-pure non Shur \( q \)-weight map of index zero for the simplest case of \( p = 2 \) but once we struck upon the idea of constructing examples of the form \( \psi^{-1}\vartheta \) with \( \vartheta \) of the form given in the next lemma we found we could construct barrel loads of examples for any \( p \). Calculation that had taken months could now be done in hours. To understand what is going on consider the case where a \( q \)-weight map \( \omega \) is bounded. Then one can take the limit of the generalized boundary representation \( \pi_t^\# \) as \( t \to 0+ \). Everything can be calculated from \( \pi_t^\# \). The index of \( \omega \) is just the index of \( \pi_t^\# \) and all \( q \)-subordinates of \( \omega \) can be found by taking subordinates of \( \pi_t^\# \). One can easily write down examples of \( \pi_t^\# \) and the associated \( \omega \) computed. In short \( \pi_t^\# \) a completely positive contractive map is easy to write down and analyze but the \( \omega \) is hard to find and hard to analyze. What the next lemma does is give us a form for \( \vartheta \) much like the form for \( \pi_t^\# \) which is relatively easy to construct. So even though the result of the next lemma is not at all deep the form we arrived at made problems that we had been struggling with for years suddenly tractable.

We should also mention that the next lemma contains a definition. We will use the decomposition in the lemma repeatedly in the rest of the paper.

**Lemma 4.5.** Suppose \( \vartheta \in B(C^p) \otimes A(C^p)_* \) is completely positive. Then \( \vartheta \) can be expressed in the form

\[
\vartheta_{ij}(A) = \sum_{k \in J} ((g_{ik} + h_{ik}), A(g_{jk} + h_{jk}))
\]

where the \( g_{ik}, h_{ik} \in C^p \otimes L^2_+ (0, \infty) \) and

\[
(g_{ik})_j(x) = \delta_{ij}g_k(x) \quad \text{and} \quad \sum_{i=1}^p (h_{ik})_i(x) = 0
\]

for \( A \in A(C^p), x \geq 0, i, j \in \{1, \cdots, p\} \) and \( k \in J \) a countable index set. Since \( \vartheta \in B(C^p) \otimes A(C^p)_* \) we have \( tr(\vartheta(I - \Lambda)) < \infty \) where

\[
tr(\vartheta(I - \Lambda)) = \frac{1}{p} \sum_{i=1}^p \vartheta_{ii}(I - \Lambda)
\]

\[
= \sum_{k \in J} \int_0^\infty (1 - e^{-x})|g_k(x)|^2 + \frac{1}{p} \sum_{i=1}^p \|h_{ik}(x)\|^2 \, dx.
\]

We define the completely positive \( B(C^p) \) valued \( b \)-weight map \( \rho \in B(C^p) \otimes A(C^p)_* \) given by

\[
\rho_{ij}(A) = \sum_{k \in J} (h_{ik}, A h_{jk})
\]

the definition of \( \nu_t \) unique. In Lemma 4.5 we will describe how to express \( \vartheta \) so that \( \nu_t \) is easily computable and afterwards we will adopt the conventions of that lemma.
for $A \in \mathfrak{A}(\mathbb{C}^p)$ and for $t > 0$ we define

$$w_t = \sum_{k \in J}(g_k, \Lambda|_tg_k), \quad (R_t)_{ij}(B) = \rho_{ij}|_t(\Lambda(B)), \quad (Y_t)_{ij} = \sum_{k \in J}(h_{ik})_j, \Lambda|_tg_k$$

and

$$\zeta_t(A) = Y_tA$$

for $A \in B(\mathbb{C}^p)$. Then we have

$$\vartheta|_t\Lambda(A) = w_tA + Y_tA + AY^*_t + R_t(A)$$

for $A \in B(\mathbb{C}^p)$ or

$$\vartheta|_t\Lambda = w_t + \zeta_t + \zeta^*_t + R_t$$

and $R_t$ is of the form

$$R_t(A) = \sum_{i=1}^m \lambda_i(t)X_i(t)AX_i(t)^*$$

and $Y_t$ and $X_i(i)$ are of trace zero and $\text{tr}(X_i(t)^*X_j(t)) = \delta_{ij}$.

From the expression for $\text{tr}(\vartheta(I - \Lambda)) < \infty$ above it follows that $\text{tr}(\rho(I - \Lambda)) \leq \text{tr}(\vartheta(I - \Lambda)) < \infty$.

**Proof.** Assume $\vartheta \in B(\mathbb{C}^p) \otimes \mathfrak{A}(\mathbb{C}^p)_*$ is a completely positive. From the general theory of completely positive maps we know that $\vartheta$ can be written in the form

$$\vartheta_{ij}(A) = \sum_{k \in J}(F_{ik}, AF_{jk})$$

with the $F_{ik} \in \mathbb{C}^p \otimes L^2_0(0, \infty)$ for $k \in J$ a countable index set. Furthermore, we know the $F_{ik}$ can be chosen so they are linearly independent over $l^2(J)$. Now we simply define

$$g_k(x) = \frac{1}{p} \sum_{i=1}^p (F_{ik})_i(x)$$

and then define

$$(g_{ik})_j(x) = \delta_{ij}g_k(x) \quad \text{and} \quad h_{ik}(x) = F_{ik}(x) - g_{ik}(x)$$

for $x \geq 0$, $i, j \in \{1, \ldots, p\}$ and $k \in J$.

The further results of the lemma follow from the discussion in section 3 and straightforward computation. \hfill \Box

In the above lemma we concluded that for $t > 0$

$$\vartheta|_t\Lambda = w_t + \zeta_t + \zeta^*_t + R_t = w_t + \gamma_t$$

where

$$\gamma_t = \omega_t^{-1}(\zeta_t + \zeta^*_t + R_t)$$

and in Theorem 4.4 we concluded that $\vartheta|_t\Lambda = v_t + \nu_t$ and $\|\nu_t\| \to 0$ as $t \to 0+$ where $v_t = \text{tr}(I + \omega|_t\Lambda(I))$ and $\nu_t = \Theta^{-1} \Theta_t r^\#_t \Lambda - t$. We remark on the connection between the two formulae. From equation 3.4 of section 3 we see that $R_t$ is the internal part of $\vartheta|_t\Lambda$ and $w_t$ is the coefficient of the identical part of $\vartheta|_t\Lambda$. Since both are equal to $\vartheta|_t\Lambda$ we have $w_t(t + \gamma_t) = v_t(t + \nu_t)$ for $t > 0$ and $v_t \to \infty$ and $\|\nu_t\| \to 0$ as $t \to 0+$ and $w_t$ is the coefficient of the identical part of $\vartheta|_t\Lambda$ it follows that $w_t/\nu_t \to 1$ and $\|\gamma_t\| \to 0$ so the two forms are roughly equivalent in the limit.

Once we wrote down $\vartheta$ in the form given in the lemma we considered what would happen if $\rho$ is bounded and we found we could construct $q$-weight maps fairly easily. Next we will need a technical lemma for making estimates.

**Lemma 4.6.** Suppose $\vartheta$ is a completely positive normal $B(\mathbb{C}^p)$ valued $b$-weight map of the form given in the previous lemma and suppose $\rho$ is a bounded $b$-weight map and suppose $w_t \to \infty$ as $t \to 0+$. Then

$$|(Y_t)_{ij}|^2/w_t \to 0$$

as $t \to 0+$ for each $i, j \in \{1, \ldots, p\}$. 
Proof. Assume the hypothesis and notation of the lemma. Then from inequality (3.7) of the last section we have the $(2 \times 2)$-matrix of matrices in $B(\mathbb{C}^p)$

$$M(t) = \begin{bmatrix} w_t I & Y_t^* \\ Y_t & R_t(I) \end{bmatrix}$$

is positive and non increasing so $M(t) \geq M(s)$ for $0 < t < s$. Now we have $w_t \to \infty$ and $R_t(I) \to R_o(I)$ as $t \to 0^+$. Choose a pair of indices $i, j \in \{1, \cdots, p\}$. and let $N(t)$ be the $(2 \times 2)$-matrix

$$N(t) = \begin{bmatrix} w_t & (Y_t^*)_{ij} \\ (Y_t)_{ji} & (R_t)_{jj} \end{bmatrix} = \begin{bmatrix} a(t) & b(t) \\ b(t) & c(t) \end{bmatrix}.$$ 

Where the second matrix on the right is the definition of $a(t)$, $b(t)$ and $c(t)$. Note $N(t)$ is positive and non increasing in $t$ so $N(t) \geq N(s)$ for $0 < t < s$. Now suppose $\epsilon > 0$. Consider the sequence $0 < t_2 < t_1 < t_o$ and let

$$a_k = a(t_k) - a(t_{k-1}), \quad b_k = b(t_k) - b(t_{k-1}), \quad c_k = c(t_k) - c(t_{k-1})$$

for $k = 1, 2$. Since $N(t_k) - N(t_{k-1}) \geq 0$ we have

$$|b_k|^2 \leq a_k c_k$$

and, hence

$$|b(t_2)| = |b_2 + b_1 + b(t_o)| \leq |b_2| + |b_1| + |b(t_o)| \leq \sqrt{a_2 c_2} + \sqrt{a_1 c_1} + |b(t_o)|.$$ 

Since $c(t) \to c(0)$ as $t \to 0^+$ we can choose $t_1$ so that $c(0) - c(t) < \epsilon^2/4$ for $0 < t < t_1$. Then

$$|b(t_2)| \leq \frac{\epsilon}{2} \sqrt{a(t_2) - a(t_1) + \sqrt{a_1 c_1} + |b(t_o)|}$$

so we have

$$\frac{|b(t_2)|}{\sqrt{a(t_2)}} \leq \frac{\epsilon}{2} \sqrt{1 - a(t_1)/a(t_2) + a(t_2)} - \frac{1}{2} \left( \sqrt{a_1 c_1} + |b(t_o)| \right)$$

and since $a(t_2) \to \infty$ as $t_2 \to 0^+$ there is a $\delta > 0$ so that the right hand side of the above inequality is less than $\epsilon$ for $0 < t_2 < \delta$. Hence, we have shown that $a(t)^{-\frac{1}{2}}|b(t)| \to 0$ as $t \to 0^+$ and, hence,

$$|Y_{ij}(t)|^2/w_t \to 0$$

as $t \to 0^+$. \qed

**Theorem 4.7.** Suppose $\omega$ is a $q$-weight map over $\mathbb{C}^p$ of index zero and the range of $\omega$ is $B(\mathbb{C}^p)$. Then $\omega$ is of the form $\omega = \psi^{-1}\vartheta$ where $\psi$ is an invertible conditionally negative map of $B(\mathbb{C}^p)$ into itself with a completely positive inverse and $\vartheta$ is of the form

$$\theta_{ij}(A) = \sum_{k \in J} ((g_{ik} + h_{ik}), A(g_{jk} + h_{jk}))$$

where the $g_{ik}, h_{ik} \in \mathbb{C}^p \otimes L^2(0, \infty)$ and

$$(g_{ik}(x) = \delta_{ij} g_k(x)) \quad \text{and} \quad \sum_{i=1}^p (h_{ik})_i(x) = 0$$

for $A \in \mathcal{A}(\mathbb{C}^p)$, $x \geq 0$, $i, j \in \{1, \cdots, p\}$ and $k \in J$ a countable index set and the $h_{ik} \in \mathbb{C}^p \otimes L^2(0, \infty)$ and if

$$w_t = \sum_{k \in J} (g_k, A|g_k) \quad \text{and} \quad \rho_{ij}(A) = \sum_{k \in J} (h_{ik}, A h_{jk})$$

then $\rho$ is bounded so

$$\sum_{k \in J} \|h_{ik}\|^2 < \infty \quad \text{and} \quad \sum_{k \in J} (g_k, (I - \Lambda) g_k) < \infty$$

and $1/w_t \to 0$ as $t \to 0^+$ and $\psi$ satisfies the conditions

$$\psi(I) \geq \vartheta(I - \Lambda) \quad \text{and} \quad \psi + \rho \Lambda$$

is conditionally negative. Furthermore, $\omega$ is unital if and only if $\psi(I) = \vartheta(I - \Lambda)$. 

Conversely, if $\vartheta$, $\rho$ and $\psi$ are as given above then $\omega$ is a $q$-weight map over $\mathbb{C}^p$ of index zero and the range of $\omega$ is all of $B(\mathbb{C}^p)$. Furthermore, if $\psi'$ is a second map satisfying the conditions above and $\omega' = \psi'^{-1}\vartheta$ then $\omega'$ is a $q$-subordinate of $\omega$ (i.e. $\omega \geq_q \omega'$) if and only if $\psi \leq \psi'$.

Proof. We begin with the proof in the paragraph beginning with the word, conversely. Suppose then that $\vartheta$, $\rho$ and $\psi$ have the properties given in the statement of the theorem. From the form of $\vartheta$ we see $\vartheta$ is the sum of completely positive maps and is, therefore, completely positive. Now we define $w_t$, $\rho$, $Y_t,R_t$ and $\zeta_t$ be defined as in Lemma 4.5 and we find

$$\vartheta|_t \Lambda = w_t + \zeta_t + \zeta_t^* + R_t$$

and we compute

$$t + \omega|_t \Lambda = \psi^{-1}(w_t + \zeta_t + \zeta_t^* + R_t + \psi).$$

Then

$$\pi_t^# = Z(t)^{-1}w_t^{-1}\vartheta|_t$$

where

$$Z_t = (t + w_t^{-1}(\zeta_t + \zeta_t^* + R_t + \psi)).$$

Recall in the discussion after Definition 1.1 to check that $\pi_t^#$ is a completely positive contraction for all $t > 0$ is only necessary to check this condition for $0 < t < \delta$ for some $\delta > 0$. Since $\vartheta|_t$ is completely positive and $w_t > 0$ it follows that $\pi_t^#$ is completely positive if $Z_t^{-1}$ is completely positive. Since $R_t$ is uniformly bounded and from Lemma 4.6 we have $\|\zeta_t\|^2/w_t \to 0$ and $1/w_t \to 0$ as $t \to 0+$ we have $Z_t \to t$ as $t \to 0+$ it follows that there are real numbers $s, \delta > 0$ so that $Z_t(I) \geq sI$ for $0 < t < \delta$. If $Z_t(I) \geq sI$ it follows from Lemma 3.3 that $Z_t^{-1}$ is completely positive provided $Z(t)$ is conditionally negative. We have $Z(t)$ is conditionally negative if and only if $R_t + \psi$ is conditionally negative. Since $R_t$ is non increasing as a completely positive map and $R_t \to R_o$ as $t \to 0+$ we have

$$R_t + \psi \leq R_o + \psi$$

and by assumption $R_o + \psi$ is conditionally negative. Hence, $Z_t^{-1}$ is completely positive and $\pi_t^#$ is completely positive for $0 < t < \delta$.

So all that remains to check is that $\pi_t^#(I) \leq I$ for $0 < t < \delta$. Now we have

$$\pi_t^#(\Lambda) = (t + w_t^{-1}(\zeta_t + \zeta_t^* + R_t + \psi))^{-1}(t + w_t^{-1}(\zeta_t + \zeta_t^* + R_t))^{-1}(I)$$

so it follows that

$$I - \pi_t^#(\Lambda) = (t + w_t^{-1}(\zeta_t + \zeta_t^* + R_t + \psi))^{-1}w_t^{-1}\psi(I) = Z_t^{-1}w_t^{-1}\psi(I)$$

and since

$$I - \pi_t^#(I) = I - \pi_t^#(\Lambda) - \pi_t^#(I - \Lambda)$$

we have

$$I - \pi_t^#(I) = Z_t^{-1}w_t^{-1}(\psi(I) - \vartheta|_t(I - \Lambda))$$

and since $Z_t^{-1}$ is completely positive and $\vartheta|_t(I - \Lambda)$ is non increasing in $t$ and $\psi(I) \geq \vartheta(I - \Lambda)$ it follows that $I \geq \pi_t^#(I - \Lambda)$ and $\vartheta|_t(I - \Lambda)$ is completely positive and completely contractive for $t > 0$. Hence, $\psi$ is a $q$-weight map over $\mathbb{C}^p$. The fact that the range of $\omega$ is $B(\mathbb{C}^p)$ is apparent since $\pi_t^# \Lambda \to \psi$ as $t \to 0+$. Since $w_t \to \infty$ as $t \to 0+$ it is apparent that $\pi_t^# (E(s,\infty)) \to 0$ as $t \to 0+$ where $E(s,\infty)$ is the projection in $B(\mathbb{C}^p \otimes L^2(0,\infty))$ onto functions with support to the right of $s$. Hence, the normal spine of $\pi_t^#$ is zero so $\omega$ is of index zero.

Finally since $\psi\omega = \vartheta$ we have $\psi\omega(I - \Lambda) = \vartheta(I - \Lambda)$ so if $\omega$ is unital we have $\psi(I) = \vartheta(I - \Lambda)$ and, conversely, if $\psi(I) = \vartheta(I - \Lambda)$ then $\omega(I - \Lambda) = \psi^{-1}\vartheta(I - \Lambda) = I$ so $\omega$ is unital.

Now suppose $\psi'$ is second invertible hermitian linear mapping of $B(\mathbb{C}^p)$ into itself, $\psi'(I) \geq \vartheta(I - \Lambda)$ and $\psi' + \rho \Lambda$ is conditionally negative. Let $\omega' = \psi'^{-1}\vartheta$. We show $\omega \geq_q \omega'$ if and only if $\psi \leq \psi'$.

First suppose $\psi \leq \psi'$. Then if $\pi_t^#$ and $\pi_t^*$ are the generalized boundary representations of $\omega$ and $\omega'$, respectively we have

$$\pi_t^# = (t + w_t^{-1}(\zeta_t + \zeta_t^* + R_t + \psi))^{-1}w_t^{-1}\vartheta|_t = Z_t^{-1}w_t^{-1}\vartheta|_t$$
and
\[ \pi_t^{\#} = (t + w_t^{-1} (\zeta_t + \zeta_t^* + R_t + \psi'))^{-1} w_t^{-1} \psi |_t = Z_t^{-1} w_t^{-1} \psi |_t \]
and
\[ \pi_t^{\#} - \pi_t^{\text{ref}} = (Z_t^{-1} - Z_t^{\text{ref}}) w_t^{-1} \psi |_t. \]

So \( \omega \geq \omega' \) if \( Z_t^{-1} - Z_t^{\text{ref}} \) is completely positive. We have
\[
Z_t^{-1} - Z_t^{\text{ref}} = Z_t^{-1}(Z_t' - Z_t)Z_t^{\text{ref}}^{-1}
\]
and since \( Z_t^{-1}, Z_t^{\text{ref}}^{-1} \) and \( \psi' - \psi \) are completely positive it follows that \( Z_t^{-1} - Z_t^{\text{ref}} \) is completely positive and \( \omega \geq \omega' \).

Now suppose \( \omega \geq \omega' \) and \( \pi_t^{\#} \) and \( \pi_t^{\text{ref}} \) are the generalized boundary representations of \( \omega \) and \( \omega' \), respectively. Then \( w_t \pi_t^{\#} \Lambda \geq w_t \pi_t^{\text{ref}} \Lambda \) for all \( t > 0 \). Then we have
\[
w_t(\pi_t^{\#} \Lambda - \pi_t^{\text{ref}} \Lambda) = w_t(Z_t^{-1} - Z_t^{\text{ref}})(t + w_t^{-1} (\zeta_t + \zeta_t^* + R_t))
\]
and taking the limit as \( t \to 0^+ \) we have
\[
Z_t \to \iota \quad Z_t' \to \iota \quad \text{and} \quad (t + w_t^{-1} (\zeta_t + \zeta_t^* + R_t)) \to \iota
\]
and, hence,
\[
w_t(\pi_t^{\#} \Lambda - \pi_t^{\text{ref}} \Lambda) \to \psi' - \psi
\]
so \( \psi' - \psi \) is the limit of completely positive maps so \( \psi' \geq \psi \).

Now we prove the first part of the theorem. Assume then that \( \omega \) is a \( q \)-weight map over \( \mathbb{C}^p \) of index zero and the range of \( \omega \) is \( B(\mathbb{C}^p) \). Then by Theorem 4.4 we can write \( \omega = \psi^{-1} \vartheta \) where \( \psi \) is conditionally negative and invertible and its inverse is completely positive and \( \vartheta \in B(\mathbb{C}^p) \otimes \mathfrak{A}(\mathbb{C}^p)_\ast \) is completely positive and
\[
\vartheta |_t \Lambda = v_t (t + \nu_t)
\]
where \( 1/v_t \to 0^+ \) and \( \|v_t\| \to 0 \) as \( t \to 0^+ \). (Note \( v_t \) and \( w_t \) can be slightly different.) From Lemma 4.5 we know \( \vartheta \) can be expressed in the form given in the statement of the theorem except that we can not conclude at this point that \( \rho \) is bounded. We do know that
\[
\vartheta |_t \Lambda = w_t + \zeta_t + \zeta_t^* + R_t
\]
and, therefore,
\[
\omega |_t \Lambda = \phi_t = \psi^{-1}(w_t + \zeta_t + \zeta_t^* + R_t)
\]
and, hence,
\[
(t + \phi_s)^{-1}(t + \phi_t) = (t + \phi_s^{-1})^{-1}(\psi + w_t + \zeta_t + \zeta_t^* + R_t)
\]
and for \( 0 < s < t \) by Theorem 4.1 the above expression is conditionally negative for all \( 0 < s < t \) so we have
\[
Q(\vartheta |_t \Lambda) = \text{tr}(I + \phi_s(I))(t + \phi_t)^{-1}(\psi + w_t + \zeta_t + \zeta_t^* + R_t)
\]
is conditionally negative for \( 0 < s < t \). Now as \( s \to 0^+ \) by Theorem 4.4 we have
\[
\text{tr}(I + \phi_s(I))(t + \phi_t)^{-1} \to \psi
\]
as \( s \to 0^+ \). Since \( Q(\vartheta |_t \Lambda) \) is conditionally negative we have taking the limit as \( s \to 0^+ \) that
\[
Q(\vartheta |_t \Lambda) = (\psi + w_t + \zeta_t + \zeta_t^* + R_t)
\]
is conditionally negative for all \( t > 0 \) and since \( w_t + \zeta_t + \zeta_t^* \) is conditionally zero we have \( \psi + R_t \) is conditionally negative for all \( t > 0 \). Now \( R_t \) is non increasing in \( t \) and we know that
\[
R_t(A) = \sum_{i=1}^m \lambda_i(t) X_i(t) AX_i^*(t)
\]
where $\lambda_i > 0$ and $tr(X_i(t)) = 0$ and $tr(X_i^*(t)X_j(t)) = \delta_{ij}$ and since $\psi$ is conditionally negative we know from the previous section that

$$\psi(A) = rA + WA + AW^* - \sum_{i=1}^{q} \sigma_i S_i AS_i^*$$

for $A \in B(\mathbb{C}^p)$ where $W$ and the $S_i$ are of trace zero and $tr(S_i^* S_j) = \delta_{ij}$ and $\sigma_i > 0$ for $i, j = 1, \cdots, q$. Since $\psi + R_\ell$ is conditionally negative we have $\lambda_i(t) \leq \sigma_{max}$ where $\sigma_{max}$ is the maximum of the $\sigma_i$s. Hence,

$$\|R_\ell\|_{H.S.}^2 = \sum_{i=1}^{m} \lambda_i^2(t) \leq m\sigma_{max}^2$$

and

$$\|R_\ell\| \leq p\|R_\ell\|_{H.S.} \leq p\sqrt{m\sigma_{max}} \leq p\sigma_{max}$$

so $R_\ell = \rho|_\ell(\Lambda)$ is uniformly bounded and since $\rho|_\ell$ is non increasing we have $\rho|_\ell\Lambda$ converges to a bounded b-weight map $\rho\Lambda$ as $t \to 0^+$ and $\psi + \rho\Lambda$ is conditionally negative. Finally, we show $\rho$ is bounded. Note from Lemma 4.5 we have $tr(\rho(I - \Lambda)) \leq tr(\vartheta(I - \Lambda))$ so $\|\rho(I - \Lambda)\| \leq \|\vartheta(I - \Lambda)\|$ so

$$\|\rho\| = \|\rho(I)\| = \|\rho(I - \Lambda) + \rho(\Lambda)\| \leq \|\rho(I - \Lambda)\| + \|\rho(\Lambda)\| \leq p\|\vartheta(I - \Lambda)\| + \|\rho(\Lambda)\|.$$ 

Then all that remains is to show that $\psi(I) \geq \vartheta(I - \Lambda)$. From our previous computations we have

$$I - \pi_t^\#(I) = Z_t^{-1} w_t^{-1}(\psi(I) - \vartheta_t(I - \Lambda))$$

and $Z_t^{-1} \to \iota$ and $\vartheta_t(I - \Lambda) \to \vartheta(I - \Lambda)$ as $t \to 0^+$ we have

$$w_t(I - \pi_t^\#(I)) \to \psi(I) - \vartheta(I - \Lambda)$$

and since the expression on the left is positive we have $\psi(I) \geq \vartheta(I - \Lambda)$. \hfill $\Box$

Given a $q$-weight map of the form $\omega = \psi^{-1} \vartheta$ as describe above we will call $\psi$ a coefficient map of $\omega$ and $\vartheta$ the corresponding limiting b-weight map. As mentioned before $\psi$ and $\vartheta$ are unique up to multiplying each by a positive number $s$ so if $\omega = \psi^{-1} \vartheta = \psi'^{-1} \vartheta'$ then $\psi' = s\psi$ and $\vartheta' = s\vartheta$ with $s > 0$.

We are particularly interested is the case when $\omega$ is $q$-pure. Note from the previous theorem we can easily construct $q$-subordinates of a $q$-weight map $\omega = \psi^{-1} \vartheta$ by simply finding $\psi'$ so that $\psi' \geq \psi$ and $\psi' + \rho\Lambda$ is conditionally negative. Note if the $\psi'$ satisfying these conditions are not totally ordered then $\omega$ will not be $q$-pure. From the discussion in the last section we see that the $\psi'$ satisfying these conditions are totally ordered if and only if $\psi + \rho\Lambda$ is conditionally zero in which case the $\psi'$ such that $\psi' + \rho\Lambda$ is conditionally negative and $\psi' \geq \psi$ consists of those $\psi'$ of the form $\psi' = \psi + s\iota$ with $s \geq 0$. If $\omega = \psi^{-1} \vartheta$ is of this form so $\psi' + \rho\Lambda$ is conditionally zero we say $\omega$ has a pure skeleton. Then $\omega = \psi^{-1} \vartheta$ has a pure skeleton if and only if its trivial subordinates are totally ordered.

We summarize these observations in a definition.

**Definition 4.8.** If $\omega$ is a $q$-weight map over $\mathbb{C}^p$ of index zero with range $B(\mathbb{C}^p)$ and $\omega = \psi^{-1} \vartheta$ where $\psi$ and $\vartheta$ were defined in Theorem 4.7. We call the pair $(\psi, \vartheta)$ a coefficient map of $\omega$ and $\vartheta$ the corresponding limiting b-weight map of $\omega$. The bounded map $\rho$ of Theorem 4.7 is constructed from the b-weight map $\vartheta$ and we denote this by writing $\rho_\vartheta$. If $\omega = \psi^{-1} \vartheta$ as just described and $\psi' \geq \rho\Lambda$ and $\psi' \geq \psi$ we call $\omega' = \psi'^{-1} \vartheta$ a trivial subordinate of $\omega$. The only trivial subordinates are of the form $\omega' = \psi'^{-1} \omega$ with $\psi' = \psi + s\iota$ with $s \geq 0$ if and only if $\psi + \rho\Lambda$ is conditionally zero. In this case we say $\omega$ has a pure skeleton.

Note that if $\omega$ is a $q$-weight map over $\mathbb{C}^p$ with index zero and range $B(\mathbb{C}^p)$ and $\omega = \psi^{-1} \vartheta$ with $\psi$ a coefficient map of $\omega$ and $\vartheta$ the corresponding limiting b-weight map then from Lemma 3.2 there is a map $\psi'$ so that $\psi' + \rho\Lambda$ is conditionally zero and $\psi'(I) = T$ where $T$ is any hermitian element of $B(\mathbb{C}^p)$ we choose. Then if we choose $T$ to be any positive operator with $T \geq \vartheta(I - \Lambda)$ then $\omega' = \psi'^{-1} \vartheta$ has a pure skeleton. If we choose $T = \vartheta(I - \Lambda)$ then $\omega'$ is unital. Simply put
given an index zero $q$-weight map $\omega$ over $\mathbb{C}^p$ with range $B(\mathbb{C}^p)$ then one can easily modify it to produce a $q$-weight map with a pure skeleton.

The next theorem shows how to find all subordinates in the case at hand.

**Theorem 4.9.** Suppose $\omega$ is a $q$-weight map over $\mathbb{C}^p$ of index zero and range $B(\mathbb{C}^p)$ and $\omega = \psi^{-1} \vartheta\eta$ with $\psi$ a coefficient map of $\omega$ and $\vartheta$ the corresponding limiting b-weight map. Suppose there is a bounded completely positive $B(\mathbb{C}^p)$ valued weight map $\eta \in B(\mathbb{C}^p) \otimes \mathcal{A}(\mathbb{C}^p)$, so that $\vartheta \geq \eta$. Then if

$$\psi' \geq \psi + \eta \Lambda \quad \text{and} \quad \psi' + \rho \vartheta \Lambda - \eta \Lambda$$

is conditionally negative then $\omega' = \psi'^{-1}(\vartheta - \eta)$ is a $q$-subordinate of $\omega$.

Conversely, if $\omega'$ is $q$-subordinate of $\omega$ there is a bounded completely positive $B(\mathbb{C}^p)$ valued b-weight map $\eta \in B(\mathbb{C}^p) \otimes \mathcal{A}(\mathbb{C}^p)$, with $\vartheta \geq \eta$ and $\psi'$ satisfying the conditions above so that $\omega' = \psi'^{-1}(\vartheta - \eta)$.

**Proof.** Assume the hypothesis of first paragraph of the theorem. We assume the notation of Theorem 4.7 so that

$$\vartheta'_{|\Lambda} = w_t + \zeta_t + \zeta_t^* + R_t$$

for $t > 0$. Let $\psi' = \vartheta - \eta$ and let $T_t = \eta_{|t} \Lambda$ so

$$\vartheta'_{|t} = w_t + \zeta_t + \zeta_t^* + R_t - T_t$$

and let $\psi'$ be such that $\psi' \geq \psi + \eta \Lambda = \psi + T_0$ and $\psi' + \rho \eta \Lambda - \eta \Lambda$ is conditionally negative. We show $\psi' + \rho \vartheta_{|t} \Lambda - \eta_{|t} \Lambda$ is conditionally negative for $t > 0$. To see this note that in constructing $\rho \vartheta_{|t} \Lambda$ from the definition of the $g$'s and $h$'s we find that $\rho \vartheta_{|t} \Lambda$ is the internal part of $\vartheta'_{|t} \Lambda$ where the internal part of a mapping of $\mathbb{C}^p$ into itself was defined in equation 3.4 of the last section. The reason we use the cut off at $t > 0$ is because $\vartheta$ is unbounded. To be more specific for the case at hand suppose $t > 0$ and let $L_t = \vartheta'_{|t} \Lambda$, $L'_t = \vartheta'_{|t} \Lambda = (\vartheta - \eta)_{|t} \Lambda$ and $L''_t = \eta_{|t} \Lambda$ and let $K_t$, $K'_t$ and $K''_t$ be the internal parts of $L_t$, $L'_t$ and $L''_t$, respectively, as defined in equation 3.4 of the last section. Note that $K_t = \rho \vartheta_{|t} \Lambda$, $K'_t = \rho \psi'_{|t} \Lambda$ and $K''_t = \eta_{|t} \Lambda$. Note that since $\rho_t$, $\rho_t \vartheta$ and $\eta$ are bounded $K_t$, $K'_t$ and $K''_t$ converge in norm as $t \to 0$. Note that $K''_t$ is non increasing in $t$ we have so $K''_t \geq K'_t$ for $t > 0$. Since the difference between a map and its internal part is conditionally zero we have $\psi' + \rho \vartheta_{|t} \Lambda - \eta_{|t} \Lambda$ is conditionally negative if and only if $\psi' + K'_t$ is conditionally negative and since we are given that $\psi' + K'_t$ is conditionally negative at $t = 0$ we have $\psi' + \rho \vartheta_{|t} \Lambda - \eta_{|t} \Lambda$ is conditionally negative for $t > 0$.

We have

$$\psi'(I) \geq \psi(I) + \eta \Lambda \geq \vartheta(I - \Lambda) + \eta \Lambda = \psi'(I - \Lambda) + \eta(I) \geq \psi'(I - \Lambda)$$

so by Theorem 4.7 we have $\omega' = \psi'^{-1} \psi'$ is a $q$-weight map over $\mathbb{C}^q$. Let $\pi_t^#$ and $\pi_t'^#$ be the generalized boundary representations of $\omega$ and $\omega'$, respectively. Then we have

$$\pi_t^# = (\nu + w_t^{-1}(\zeta_t + \zeta_t^* + R_t + \psi))^{-1} w_t^{-1} \vartheta'_{|t} \nu - w_t^{-1} Z_t^{-1} \vartheta'_{|t}$$

and

$$\pi_t'^# = (\nu + w_t^{-1}(\zeta_t + \zeta_t^* + R_t - T_t + \psi'))^{-1} w_t^{-1} \vartheta'_{|t} \nu - w_t^{-1} Z_t'^{-1} \vartheta'_{|t}$$

so

$$\pi_t^# - \pi_t'^# = w_t^{-1}(Z_t^{-1} \vartheta'_{|t} - Z_t'^{-1} \vartheta'_{|t} - Z_t'^{-1} \vartheta'_{|t} - Z_t^{-1} \vartheta'_{|t})$$

$$= w_t^{-1}(Z_t^{-1} \vartheta'_{|t} - Z_t'^{-1} \vartheta'_{|t} - Z_t'^{-1} \vartheta'_{|t} + Z_t'^{-1} \vartheta'_{|t})$$

$$= w_t^{-1}(Z_t^{-1} \vartheta'_{|t} - T_t - \psi) Z_t'^{-1} \vartheta'_{|t} + Z_t'^{-1} \vartheta'_{|t})$$

$$= w_t^{-1}(\psi' - T_t - \psi) Z_t'^{-1} \vartheta'_{|t} + Z_t'^{-1} \vartheta'_{|t})$$

Since $\psi' \geq \psi + \eta \Lambda \geq \psi + \eta_{|t} \Lambda = \psi + T_0$ we have $\psi' - T_t - \psi \geq 0$. Note $Z_t^{-1}$ and $Z_t'^{-1}$ are completely positive from Lemma 3.3 and since the other terms $\vartheta'_{|t}$ and $\eta_{|t}$ are completely positive we have $\pi_t^# - \pi_t'^#$ for $t > 0$ so $\omega'$ is a $q$-subordinate of $\omega$.

Now suppose $\omega$ is a $q$-weight map over $\mathbb{C}^p$ of index zero and the range of $\omega$ is $B(\mathbb{C}^p)$ and $\omega'$ is a non zero $q$-subordinate of $\omega$. Then from Theorem 4.3 we know that $\omega'$ is of index zero. Suppose $\pi_t^#$ and $\pi_t'^#$ are the generalized boundary representations of $\omega$ and $\omega'$, respectively. Since $\omega = \omega'$ we have $\pi_t^# \geq \pi_t'^#$ for $t > 0$ and we know $\pi_t^# \Lambda \to \nu$ and $\pi_t'^# \Lambda(A) \to A$ for all $A \in \text{Range}(\omega')$. 
Let $L'$ be a limit point of the $\pi_{t\#}A$ as $t \to 0^+$. Then we have $\iota \geq L' \geq 0$ and since $\iota$ is pure as a completely positive map we have $L' = sI$ for $s \in [0,1]$. Since $L'(A) = A$ for $A$ in the range of $\omega'$ and since $\omega'$ is not zero there are non zero $A$ in the range of $\omega'$. Hence $L' = \iota$ and since this is true for any limit point we have $\pi_{t\#}A \to \iota$ as $t \to 0^+$. Hence, $\omega'$ has range $B(C')$ so we know that $\omega' = \psi^{-1}d'$ where $\psi'$ is a coefficient map for $\omega'$ and $d'$ is the corresponding limiting b-weight map of $\omega'$.

In writing $\omega = \psi^{-1}d$ the maps $\psi$ and $d$ are not unique in that we can multiply both of them by the same positive number and $\omega$ is unchanged. Although the maps are not unique we can make them unique by requiring $tr(\psi^{-1}(I)) = 1$. We will assume that $\psi$ has been normalized so that $tr(\psi^{-1}(I)) = 1$ and prove the theorem under this additional hypothesis. Then at the end of the proof we will show that the truth of theorem under this hypothesis implies the theorem is true in general.

Then with this hypothesis on $\psi$ we see from the argument in Theorem 4.3 that

$$\psi_t = tr(I + \phi_t(I))(\iota + \phi_t)^{-1} \to \psi.$$

Now for the $q$-weight map $\omega'$ we express this in the form $\omega' = \psi'^{-1}d'$ where we require $tr(\psi'^{-1}(I)) = 1$. The reason we use the double prime is because $\omega'$ will turn out to be a multiple of the $\psi'$ in the statement of the theorem. Then in terms of $\omega'$ we have

$$\omega'_t = tr(I + \phi'_t(I))(\iota + \phi'_t)^{-1} \to \omega'.$$

Since $\omega \geq \omega'$ we have $\pi_{t\#} \geq \pi_{t\#}'$ so

$$tr(I + \phi_t(I))(\pi_{t\#}' - \pi_{t\#}) = \psi_t \omega_t - \frac{tr(I + \phi_t(I))}{tr(I + \phi'_t(I))} \psi'_t \omega'_t \geq 0$$

for $t > 0$. Since for some number $r$ it is not true that $\psi \omega \geq r \psi'' \omega''$ it follows that

$$\kappa = \lim sup_{t \to 0^+} \frac{tr(I + \phi_t(I))}{tr(I + \phi'_t(I))} \leq r.$$

Now let $t_k > 0$ be a decreasing sequence converging to zero and

$$\lim_{k \to \infty} \frac{tr(I + \phi_{t_k}(I))}{tr(I + \phi'_{t_k}(I))} = \kappa.$$

Then we find

$$\lim_{k \to \infty} \psi|_{t_k} = \frac{tr(I + \phi_{t_k}(I))}{tr(I + \phi'_{t_k}(I))} \psi'' |_{t_k} = \vartheta - \kappa \vartheta'' = \eta \geq 0.$$
the first paragraphs of this proof \( \rho_\theta \Lambda - \rho \varphi \Lambda - \eta \Lambda \) is conditionally zero which yields the desired result.

Now we have proved the theorem with this particular normalization of \( \psi \) and \( \vartheta \). What about other normalizations? If you examine the statement of the theorem and make the following replacement, \( \psi \to s \psi, \vartheta \to s \vartheta, \psi' \to s \psi', \eta \to s \eta \) and note with these replacements one has \( \rho_\theta \Lambda \to s \rho_\theta \Lambda \) you see all the statements in the theorem are replaced by equivalent statements. Therefore, it follows that since the theorem is true for a specific normalization for \( \psi \) and \( \vartheta \) it follows that the theorem is true in general.

\[ \square \]

Theorem 4.10. Suppose \( \omega \) is a q-weight map over \( \mathbb{C}^p \) of index zero and range \( B(\mathbb{C}^p) \) and \( \omega = \psi^{-1} \vartheta \) with \( \psi \) a coefficient map of \( \omega \) and \( \vartheta \) the corresponding limiting b-weight map. Then \( \omega \) is q-pure if and only if \( \psi + \rho \Lambda \) is conditionally zero and \( \vartheta \) is strictly infinite meaning that if \( \eta \) is a completely positive bounded \( B(\mathbb{C}^p) \) valued normal weight map on \( \mathfrak{A}(\mathbb{C}^p) \) and \( \eta \) is a subordinate of \( \vartheta \) so \( \vartheta \geq \eta \) then \( \eta = 0 \).

Proof. Assume the hypothesis of the theorem. From Theorem 4.7 we know that if \( \psi' \) is a hermitian map of \( B(\mathbb{C}^p) \) into itself and

\[ \psi' \geq \psi \quad \text{and} \quad \psi' + \rho \Lambda \]

is conditionally negative then \( \omega' = \psi'^{-1} \vartheta \) is a q-subordinate of \( \omega \) and if \( \psi'' \) is a second such map satisfying these conditions and \( \omega'' = \psi''^{-1} \vartheta \) then \( \omega' \geq_q \omega'' \) if and only if \( \psi' \leq \psi'' \). Let \( S \) be the set of \( \psi' \) satisfying the conditions above. It follows that if \( \omega \) is q-pure then \( S \) must be totally ordered. Note if \( \psi' = \psi + s \tau \) with \( s \geq 0 \) then \( \psi' \in S \) so if \( S \) is totally ordered then \( S \) simply consists of \( \psi' = \psi + s \tau \) with \( s \geq 0 \) and this is the case if and only if \( \psi + \rho \Lambda \) is conditionally zero. Hence, if \( \omega \) is q-pure we have \( \psi + \rho \Lambda \) is conditionally zero.

Next suppose there is a bounded subordinate \( \eta \) of \( \vartheta \) and \( \eta \neq 0 \). Let \( \psi' = \psi + \eta \Lambda \). Then

\[ \psi' + \rho \Lambda - \eta \Lambda = \psi + \rho \Lambda \]

is conditionally negative and by Theorem 4.9 it follows that \( \omega' = \psi'^{-1}(\vartheta - \eta) \) is a q-subordinate of \( \omega \). To show that \( \omega \) is not q-pure let \( \psi'' = \psi + s \tau \) where \( s \) is a positive number of our choosing and let \( \pi_{\Omega^\#} \) and \( \pi_{\Omega^\#} \) be the generalized boundary representation of \( \omega' \) and \( \omega'' \), respectively. Then recalling the computations of the previous theorem we have

\[ \pi_{\Omega^\#} = (t + w_t^{-1}(\zeta_t + \zeta_t + R_t + \psi + s \tau))^{-1}w_t^{-1}\vartheta |_t = w_t^{-1}Z_t^{-1}\vartheta |_t \]

and we have

\[ w_t(\pi_{\Omega^\#} - \pi_{\Omega^\#}) = Z_t^{-1}(T_o - T_t - s \tau)Z_t^{-1}\vartheta |_t + Z_t^{-1}\eta |_t \rightarrow \eta - s \vartheta. \]

We recall that \( E(r, \infty) \) is the projection in \( B(\mathbb{C}^p \otimes L^2(0, \infty)) \) on functions with support in \([r, \infty)\). Since \( \eta \neq 0 \) we there is a \( r > 0 \) so that \( \eta(E(r, \infty)) \neq 0 \). Now choose \( s > 0 \) so small that \( (s \vartheta - \eta)E(r, \infty) \) is not positive. For this choice of \( s \) we have it is not true that \( \omega' \geq_q \omega' \) since \( \eta - s \vartheta \) can not be positive since \( \eta \) is bounded and \( \vartheta \) is unbounded. It is also not true that \( \omega' \geq_q \omega'' \) since \( (s \vartheta - \eta)E(r, \infty) \) is not positive. Hence, there are two subordinates of \( \omega \) that are not ordered so \( \omega \) is not q-pure.

Hence, we have shown that if \( \omega \) is q-pure \( \psi + \rho \Lambda \) is conditionally zero and \( \vartheta \) is strictly infinite. Now suppose \( \omega \) satisfies the conditions of the theorem and \( \omega' \) is a q-subordinate of \( \omega \). By the previous theorem we have \( \omega' = \psi'^{-1}(\vartheta - \eta) \) where \( \eta \) is a bounded subordinate of \( \vartheta \) and

\[ \psi' \geq \psi + \eta \Lambda \quad \text{and} \quad \psi' + \rho \Lambda - \eta \Lambda \]

is conditionally negative. Since \( \vartheta \) is given to be strictly infinite we have \( \eta = 0 \) so \( \psi' \) satisfies the conditions above with \( \eta = 0 \). Since \( \psi + \rho \Lambda \) is conditionally zero we have \( \psi' - \psi \geq 0 \) and conditionally negative so \( \psi' = \psi + s \tau \) with \( s \geq 0 \). Hence, the q-subordinates of \( \omega \) are totally ordered.

\[ \square \]

In the next lemma we give conditions that \( \vartheta \) be strictly infinite.

Theorem 4.11. Suppose \( \vartheta \) is a \( B(\mathbb{C}^p) \) valued b-weight map on \( \mathfrak{A}(\mathbb{C}^p) \) of the form

\[ \vartheta_{ij}(A) = \sum_{k \in J} (g_{ik} + h_{ik}, A(g_{jk} + h_{jk})) \]
where the $g_{ik}, h_{ik} \in \mathbb{C}^p \otimes L^2_+(0, \infty)$ and
\[
(g_{ik})_j(x) = \delta_{ij} g_k(x) \quad \text{and} \quad \sum_{k \in J} \sum_{i=1}^p h_{ik}(x) = 0
\]
and
\[
\sum_{k \in J} \|h_{ik}\|^2 < \infty
\]
for $A \in \mathfrak{A}(\mathbb{C}^p)$, $x \geq 0$, $i, j \in \{1, \cdots, p\}$ and $k \in J$ a countable index. Let
\[
\mu(A) = \sum_{k \in J} (g_k, A g_k)
\]
for $A \in \mathfrak{A}(\mathbb{C})$. Then $\vartheta$ is strictly infinite if and only if $\mu$ is strictly infinite and the $h$'s are linearly independent over the $g$'s by which we mean that if $c \in \ell^2(J)$ and
\[
\sum_{k \in J} c_k g_k = 0
\]
then
\[
\sum_{k \in J} c_k h_{ik} = 0
\]
for each $i = 1, \cdots, p$.

**Proof.** Assume hypothesis and notation of the theorem. Now $\vartheta$ is strictly infinite if and only if for $c \in \ell^2(J)$ so that
\[
F_i = \sum_{k \in J} c_k (g_{ik} + h_{ik}) \in \mathbb{C}^p \otimes L^2(0, \infty)
\]
for each $i = 1, \cdots, p$ then $F_i = 0$ for each $i = 1, \cdots, p$. Suppose that $\mu$ is strictly infinite and the $h_{ik}$ are linearly independent over the $g_k$. Suppose $c \in \ell^2(J)$ and the $F_i$ above are in $\mathbb{C}^p \otimes L^2(0, \infty)$ for each $i = 1, \cdots, p$. Since the sum of the $c_k h_{ik}$ is in $\mathbb{C}^p \otimes L^2(0, \infty)$ it follows that the sum of the $c_k g_{ik}$ is in $\mathbb{C}^p \otimes L^2(0, \infty)$ and, hence,
\[
\sum_{k \in J} c_k g_k \in L^2(0, \infty)
\]
but since $\mu$ is strictly infinite the above sum is zero and since the $h_{ik}$ are linearly independent over the $g_k$ we have the sum of the $c_k h_{ik}$ is zero so $F_i = 0$ for $i = 1, \cdots, p$ and $\vartheta$ is strictly infinite.

Now suppose $\mu$ is not strictly infinite. Then there is a $c \in \ell^2(J)$ so that
\[
g = \sum_{k \in J} c_k g_k \in L^2(0, \infty) \quad \text{and} \quad \sum_{k \in J} |c_k|^2 = 1
\]
and $g \neq 0$. Let
\[
F_i = \sum_{k \in J} c_k (g_{ik} + h_{ik}).
\]
Since the sum of the $c_k h_{ik}$ is in $\mathbb{C}^p \otimes L^2(0, \infty)$ then $F_i \in \mathbb{C}^p \otimes L^2(0, \infty)$ for each $i = 1, \cdots, p$. Now from the condition on the $h_{ik}$ we have
\[
g(x) = \frac{1}{p} \sum_{i=1}^p (F_i)_i(x)
\]
so the $F_i$ can not all be zero so $\vartheta$ is not strictly infinite.

Now suppose the $h_{ik}$ are not linearly independent over the $g_k$. Then there is a $c \in \ell^2(J)$ so that
\[
\sum_{k \in J} c_k g_k = 0
\]
and
\[
\sum_{k \in J} c_k h_{ik} \neq 0
\]
for some \(i \in \{1, \cdots, p\}\). Then we have
\[
F_i = \sum_{k \in J} c_k (g_{ik} + h_{ik}) = \sum_{k \in J} c_k h_{ik} \in \mathbb{C}^p \otimes L^2(0, \infty)
\]
for each \(i \in \{1, \cdots, p\}\) and not all the \(F_i\) are zero. Hence, \(\vartheta\) is not strictly infinite. \(\square\)

5. 5. INDEX ZERO WITH GENERAL RANGE

As mentioned in the last section Choi and Effros showed that if \(L\) is a completely positive contractive linear mapping of \(B(\mathbb{C}^p)\) into itself that is idempotent, (i.e. \(L^2 = L\)) then \(L\) the range of \(L\) is a *-subalgebra of \(B(\mathbb{C}^p)\) where the product in \(L\) is given by \(A \ast B = L(AB)\). Because of the importance of this results and to establish notation we present a proof in the special simple case where \(B(H) = B(\mathbb{C}^p)\) is finite dimensional. Note the element \(I_o = L(I)\) is the unit of \(L\) since \(I_o \ast A = A \ast I_o = A\) for all \(A \in L\).

**Theorem 5.1.** Suppose \(L\) is a completely positive contractive linear mapping of \(B(\mathbb{C}^p)\) into itself that is idempotent (i.e. \(L^2 = L\)). Then there is a unique projection \(F\) so that \(L(A) = L(FAF)\) and \(F\) is the smallest projection so that \(L(F) = L(I) = I_o\). If \(\phi\) is the mapping of \(FB(\mathbb{C}^p)F\) into itself given by \(\phi(A) = FL(A)F\) then the range of \(\phi\) is a *-algebra so
\[
\phi(\phi(A)\phi(B)) = \phi(A)\phi(B) \quad \text{and} \quad \phi(A^*) = \phi(A)^*
\]
for \(A, B \in B(\mathbb{C}^p)\). The mapping \(\phi\) is faithful in that if \(A \in FB(\mathbb{C}^p)F\) and \(A \geq 0\) then if \(\phi(A) = 0\) then \(A = 0\).

Every operator in the range of \(L\) commutes with \(F\) so
\[
L(A) = \phi(A) + (I - F)L(\phi(A))(I - F)
\]
for all \(A \in B(\mathbb{C}^p)\). Both \(L\) and \(\phi\) have the Choi-Effros property that
\[
L(L(A)B) = L(L(A)L(B)) = L(AL(B))
\]
and
\[
\phi(\phi(A)B) = \phi(\phi(A)\phi(B)) = \phi(A\phi(B))
\]
for \(A, B \in B(\mathbb{C}^p)\).

Given the mapping \(\phi\) with the properties listed above one can specify any completely positive contractive map \(\psi\) from the range of \(\phi\) into \((I - F)B(\mathbb{C}^p)(I - F)\) and the mapping \(L'(A) = \phi(A) + \psi(\phi(A))\) is a completely positive contractive linear mapping of \(B(\mathbb{C}^p)\) into itself that is idempotent.

**Proof.** Suppose \(L\) is a linear completely positive contractive idempotent of \(B(\mathbb{C}^p)\). Let \(L(I) = I_o\). Since \(L\) is completely positive and contractive we have \(0 \leq I_o \leq I\). Since \(L\) has norm one we have \(\|I_o\| = 1\) so \(I_o = E_o + B\) where \(E_o\) is an hermitian projection and \(E_oB = BE_o = 0\) and \(0 \leq B \leq sI\) where \(s \in [0, 1]\) (i.e. \(s\) is strictly less than one).

Since \(L\) is completely positive we have
\[
L(A) = \sum_{i=1}^{m} S_iAS_i^*
\]
for \(A \in B(\mathbb{C}^p)\) where the \(S_i\) are linearly independent elements of \(B(\mathbb{C}^p)\). Since \(L(I_o) = L(L(I)) = L(I) = I_o\) we have \(L(I - I_o) = 0\) so
\[
L(I - I_o) = L((I - E_o)(I - B)(I - E_o)) = \sum_{i=1}^{m} S_i(I - E_o)(I - B)(I - E_o)S_i^* = 0
\]
and since \((I - E_o)(I - B)(I - E_o) \geq (1 - s)(I - E_o)\) it follows that \(S_i(I - E_o) = 0\) which yields
\[
S_i = S_iE_o \quad \text{and} \quad S_i^* = E_oS_i^*
\]
for \(i = 1, \cdots, m\). Recall that \(I_o = E_o + B\) so \(0 \leq E_o \leq I_o \leq I\) and we have
\[
I_o = L(I) = \sum_{i=1}^{m} S_iS_i^* = \sum_{i=1}^{m} S_iE_oS_i^* = L(E_o).
\]
Now let \( \mathfrak{M} \) be the linear span of the ranges of the \( S_i^* \) for \( i = 1, \cdots, m \) and let \( F \) be the hermitian projection of \( \mathbb{C}^p \) onto \( \mathfrak{M} \). Note \( F \) is the smallest projection so that \( FS_i^* = S_i^* \) for \( i = 1, \cdots, m \). Since for \( i = 1, \cdots, m \) we have \( E_oS_i^* = S_i^* \) it follows that \( E_o \geq F \). Note we have

\[
L(A) = \sum_{i=1}^{m} S_i A S_i^* = \sum_{i=1}^{m} S_i F A F S_i^* = L(F A F)
\]

for all \( A \in B(\mathbb{C}^p) \). Note that if \( P \) is an hermitian projection and \( F \geq P \geq 0 \) and \( L(P) = 0 \) then

\[
\sum_{i=1}^{m} S_i P P S_i^* = 0
\]

so for \( i = 1, \cdots, m \) we have \( P S_i^* = 0 \) so \( (F - P) S_i^* = F S_i^* = S_i^* \) so \( F - P \geq F \) which yields \( P = 0 \). Since any positive element of \( FB(\mathbb{C}^p)F \) is the sum projections in \( FB(\mathbb{C}^p)F \) with positive coefficients it follows that if \( A \in FB(\mathbb{C}^p)F \) and \( A \geq 0 \) and \( L(A) = 0 \) then \( A = 0 \). It follows that \( F \) is the smallest projection so that \( L(F) = L(I) = I_o \).

Now let \( \phi(A) = FL(A)F \) for \( A \in B(\mathbb{C}^p) \). Note \( \phi \) is completely positive and idempotent. And note that

\[
L(A) = L(L(A)) = L(FL(A)F) = L(\phi(A))
\]

for all \( A \in B(\mathbb{C}^p) \). This is an important equality that we will often use in the remainder of this proof.

We show the range of \( \phi \) is an algebra. Since \( \phi \) is completely positive \( \phi_2 = \nu_2 \otimes \phi \) is positive so we have for \( A \in B(\mathbb{C}^p) \) that

\[
\phi_2\left( \begin{bmatrix} F & \phi(A) \\ \phi(A)^* & \phi(A)^* \phi(\phi(A)) \end{bmatrix} \right) = \begin{bmatrix} F & \phi(A) \\ \phi(A)^* & \phi(\phi(A)^* \phi(\phi(A)) \end{bmatrix} \geq 0.
\]

So we have

\[
B = \phi(\phi(A)^* \phi(\phi(A)) = \phi(A)^* \phi(\phi(A)) \geq 0.
\]

But \( \phi(B) = 0 \) and since \( FBF = B \geq 0 \) we have \( B = 0 \). Thus, \( \phi(\phi(A)^* \phi(A)) = \phi(A)^* \phi(\phi(A)) \) for all \( A \in B(\mathbb{C}^p) \). It follows from complex linearity that

\[
\phi(\phi(A) \phi(B)) = \phi(A) \phi(B)
\]

for all \( A \in B(\mathbb{C}^p) \). Hence, the range of \( \phi \) is a *-subalgebra in \( FB(\mathbb{C}^p)F \) with unit \( F \). Suppose \( P \) is a hermitian projection in the range of \( \phi \). Then we have

\[
P = \phi(P) = \sum_{i=1}^{m} FS_i P S_i^* F
\]

and

\[
(I - P)(I - P) = \sum_{i=1}^{m} (I - P)FS_iP(I - P)FS_iP^* = 0
\]

and so we conclude \( (I - P)FS_iP = 0 \) for \( i = 1, \cdots, m \). Applying the same argument to \( F - P \) we find \( (I - F + P)FS_i(F - P) = PFS_i(I - P) = 0 \) for \( i = 1, \cdots, m \). Then from these two relations we find \( FS_i \) maps the range of \( P \) into itself and the range of \( I - P \) into itself so \( FS_i \) commutes with \( P \) for \( i = 1, \cdots, m \). Since the range of \( \phi \) is a finite dimensional \( C^* \)-algebra each \( A \) in the range of \( \phi \) can be written as a complex linear combinations of projections in the range of \( \phi \) and since each of those projections commute with the \( FS_i \) for \( i = 1, \cdots, m \) we see that \( \phi(A) \) commutes with both \( FS_i \) and \( S_i^* F \) for each \( i = 1, \cdots, m \). Hence, we see that

\[
\phi(\phi(A)B) = \phi(F \phi(A)BF) = \sum_{i=1}^{m} FS_i \phi(A) BS_i^* F
\]

\[
= \sum_{i=1}^{m} \phi(A) FS_i BS_i^* F = \phi(A) \phi(B) = \phi(\phi(A) \phi(B))
\]
and
\[ \phi(A\phi(B)) = \phi(FA\phi(B)F) = \sum_{i=1}^{m} FS_i A\phi(B)S_i^*F \]
\[ = \sum_{i=1}^{m} FS_i AS_i^*F\phi(B) = \phi(A)\phi(B) = \phi(\phi(A)\phi(B)) \]
and, hence, we have
\[ \phi(\phi(A)B) = \phi(\phi(A)\phi(B)) = \phi(A\phi(B)) \]
for all \( A, B \in B(C^p) \).

We prove that \( L(A) \) commutes with \( F \) for all \( A \in B(C^p) \). We have
\[ FL(A) = FL(\phi(A)) = \sum_{i=1}^{m} FS_i \phi(A)S_i^* = \sum_{i=1}^{m} \phi(A)FS_iS_i^* \]
\[ = \phi(A)FL(I) = \phi(A)FI_s = \phi(A)F = \phi(A) \]
and
\[ L(A)F = L(\phi(A))F = \sum_{i=1}^{m} S_i \phi(A)S_i^*F = \sum_{i=1}^{m} S_i S_i^* F\phi(A) = L(I)F\phi(A) \]
\[ = LF\phi(A) = F\phi(A) = \phi(A). \]
and, hence, \( FL(A) = \phi(A) = L(A)F \) for all \( A \in B(C^p) \). Hence, we have
\[ L(A) = FL(A)F + (I - F)L(A)F + FL(A)(I - F) + (I - F)L(A)(I - F) \]
\[ = \phi(A) + (I - F)L(A)(I - F) = \phi(A) + (I - F)L(\phi(A))(I - F). \]

Now we can prove the Choi-Effros identity that for \( A, B \in B(C^p) \) we have
\[ L(L(AB)) = L(FL(B)A) = L(\phi(AB)) = L(\phi(\phi(AB))) \]
\[ = L(\phi(\phi(A)\phi(B))) = L(\phi(A)\phi(B)) \]
and
\[ L(AL(B)) = L(AL(B))F = L(A\phi(B)) = L(\phi(A\phi(B))) \]
\[ = L(\phi(\phi(A))\phi(B))) = L(\phi(A)\phi(B)) \]
and
\[ L(L(AB)L(B)) = L(FL(AB)L(B)F) = L(\phi(AB)) \]
so we have
\[ L(L(AB)) = L(L(AB)) = L(AL(B)) \]
for all \( A, B \in B(C^p) \).

Now suppose \( \psi \) is a completely positive contractive linear mapping of the range of \( \phi \) into \( (I - F)B(C^p)(I - F) \). Let
\[ L'(A) = \phi(A) + \psi(\phi(A)) \]
for \( A \in B(C^p) \). We note \( L \) is completely positive and contractive and
\[ L'(L'(A)) = L'(\phi(A) + \psi(\phi(A))) = \phi(\phi(A)) + \psi(\phi(\phi(A))) \]
\[ = \phi(\phi(A)) = L'(A) \]
for \( A \in B(C^p) \). \[ \square \]

Our situation is further complicated by the fact that the limit \( L \) is not necessarily unique. In fact, by experimenting you can find examples where there are two limits \( L \) and \( L' \) with disjoint support projections \( F \) and \( F' \). What is unique is the range of \( L \) since this is equal to the range of \( \omega \). From the work of Jankowski we know that in the case of \( C^3 \) the Choi-Effros algebra can be isomorphic to the \( (2 \times 2) \)-matrices with elements \( \{ f_{ij} : i, j = 1, 2 \} \) given in terms of the \( (3 \times 3) \) matrix units \( \{ e_{ij} : i, j = 1, 2, 3 \} \) as follows
\[ f_{11} = e_{11} \quad f_{12} = e_{12} \quad f_{21} = e_{21} \quad f_{22} = e_{22} + \lambda e_{33} \]
with $\lambda \in [0, 1]$. In this case we can deduce that the support projection $F$ is $F = e_{11} + e_{22}$.

Since the limit $L$ is not unique we are faced with the following problem. Suppose there are two completely positive contractive idempotent maps $L_1$ and $L_2$ of $B(C^p)$ into itself with the same range $\mathcal{L}$. Then as we have seen $L_1$ and $L_2$ give us a multiplication on $\mathcal{L}$. The question is are they the same. In particular is

$$L_1(AB) = L_2(AB)$$

for $A, B \in \mathcal{L}$. Choi and Effros in [CE77] develop the theory of operator systems and we believe their results show the answer is yes. Unfortunately they do not specifically state the result we are after. Therefore, we will give a brief outline a proof in our finite dimensional case and apologize in advance for giving longer and more complicated argument when they would most probably give. In our discussion we will often be showing that $L_1$ and $L_2$ satisfy certain conditions such as $L_1(EFE) = \lambda E$ and $L_2(EFE) = \lambda E$ for certain elements $E$ and $F$. Rather than write out two derivation we will simply write out a derivation for $L$. So we will use the following notation. When we write $L$ we mean any completely positive contractive idempotent of $B(C^p)$ into $\mathcal{L}$ whose range is $\mathcal{L}$. Then any formula involving $L$ applies to both $L_1$ or $L_2$. There is one obvious word of caution, namely, if we have a collection of formulae involving $L$ the formulae are valid for both $L_1$ and $L_2$ but we can not mix the subscripts 1 and 2 (i.e. in all the formulae in a collection we must either use all 1's or all 2's).

Since $\mathcal{L}$ is a finite dimensional $C^*$-algebra every hermitian $A$ is the sum of minimal projections

$$A = \sum_{i=1}^{m} \lambda_i E_i$$

where the $\lambda_i$ are real and the $E_i$ are mutually orthogonal minimal projections. Note that such a decomposition does not require the idempotent $L_1$ or $L_2$ because a minimal projection $E$ can be characterized as follows. The element $E \in \mathcal{L}$ has the property that $E \geq 0$, $\|E\| = 1$ and if $A \in \mathcal{L}$ and $0 \leq A \leq E$ then $A = \lambda E$. One checks that if an element $E \in \mathcal{L}$ has these properties then $E$ is a minimal projection. Note the fact that $L(E^2) = E$ (where $L$ is any completely positive contractive idempotent with range $\mathcal{L}$) is automatically satisfied since $0 \leq E^2 \leq E$ and, therefore, $0 \leq L(E^2) \leq L(E)$ and, therefore, $L(E) = \lambda E$. And since $L$ is completely positive and the matrix of elements in $B(C^p)$

$$\begin{bmatrix} I & E \\ E & E^2 \end{bmatrix} \geq 0$$

the matrix obtained by replacing the above matrix elements of the above matrix with $L$ applied to that matrix element yields a positive matrix so

$$\begin{bmatrix} L(I) & L(E) \\ L(E) & L(E^2) \end{bmatrix} = \begin{bmatrix} L(I) & E \\ E & \lambda E \end{bmatrix} \geq 0$$

and since $I \geq L(I)$ this matrix can only be positive if $\lambda \geq 1$ and since $0 \leq \lambda \leq 1$ we have $\lambda = 1$ and $L(E^2) = E$.

Next if $E$ and $F$ are hermitian projections in $\mathcal{L}$ then $E$ and $F$ are orthogonal if and only if $\|E + F\| = 1$. This means that in the expression for $A$ above the fact that the $E_i$ are mutually orthogonal minimal projections can be determined without the use of $L_1$ or $L_2$. Now if $E$ and $F$ are orthogonal minimal hermitian projections in $\mathcal{L}$ then $L(EFE) = \lambda E$ with $\lambda \geq 0$ so $L(E(E + F)E) = (1 + \lambda)E$ and since $\|E + F\| = 1$ it follows that $\lambda = 0$ and since $L(EF(EF)^*) = 0$ it follows that $L(EF) = L(FE) = 0$. Then for the $E_i$ orthogonal minimal projections the product $L(E_iE_j) = \delta_{ij}E_i$ for $i, j = 1, \cdots, m$ (where $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$). Then we have

$$L(A^2) = L(\sum_{i,j=1}^{m} \lambda_i \lambda_j E_iE_j) = \sum_{i=1}^{m} \lambda_i^2 L(E_i).$$

Hence we see that $L(A^2)$ is the same element of $\mathcal{L}$ for any completely positive contractive idempotent map $L$ with range $\mathcal{L}$. Then, in particular, we have $L_1(A^2) = L_2(A^2)$. Applying this to the sum of two hermitian $A, B \in \mathcal{L}$ we see that the Jordan product is the same for $L_1$ and $L_2$, namely, $L_1(AB + BA) = L_2(AB + BA)$ for all hermitian $A, B \in \mathcal{L}$. 

38 CHRISTOPHER JANKOWSKI, DANIEL MARKIEWICZ, AND ROBERT T. POWERS
Next we use the fact that $L_1$ and $L_2$ are completely positive. Let $H_2 = H \oplus H$ and $L_2$ be the set of $(2 \times 2)$-matrices with entries in $L$ and let $L_{12} = \iota_2 \otimes L_1$ and $L_{22} = \iota_2 \otimes L_2$ (i.e. we apply $L_1$ and $L_2$ to $(2 \times 2)$-matrices with entries in $B(\mathbb{C}^p)$). Repeating the argument that showed $L_1(AB + BA) = L_2(AB + BA)$ for all hermitian $A, B \in L$ for $L_2$ we see that $L_{12}(AB + BA) = L_{22}(AB + BA)$ for hermitian $A, B \in L_2$. Now consider product of matrices in $L(H_2)$

$$X = \begin{bmatrix} A & B \\ B & -A \end{bmatrix} \begin{bmatrix} A & B \\ B & -A \end{bmatrix} = \begin{bmatrix} A^2 + B^2 & AB - BA \\ BA - AB & A^2 + B^2 \end{bmatrix}$$

for hermitian $A, B \in L$. Since $L_{12}(X) = L_{22}(X)$ we have that $L_1(AB - BA) = L_2(AB - BA)$ and since $L_1(AB + BA) = L_2(AB + BA)$ we have $L_1(AB) = L_2(AB)$ for all hermitian $A, B \in L$ and by complex linearity we have $L_1(AB) = L_2(AB)$ for all $A, B \in L$.

In summary we see that if $\omega$ is a non zero $q$-weight map of index zero and $\pi^\#_t$ is the generalized boundary representation of $\omega$ and $L$ is a limit point of $L_t = \pi^\#_t \Lambda$ as $t \to 0+$ then the range of $\omega$ which we denote by $L(\omega)$ is a $C^*$-algebra with multiplication defined by $A \star B = L(AB)$ for $A, B \in L(\omega)$. Note we have shown the multiplication defined by $L$ does not depend on which limit point of $L_t$ we take. Note the unit $I_o$ does not depend on which limit point of $L_t$ we take since the unit $I_o$ can be characterized without reference to $L$ in that $I_o$ is the unique positive norm one element of $L(\omega)$ with the property that $I_o \geq A$ for all positive norm one element of $L(\omega)$.

Armed with these results we find a projection $P$ that dominates all support projections.

**Theorem 5.2.** Suppose $L$ is a completely positive contractive linear idempotent mapping of $B(\mathbb{C}^p)$ into itself and $F$ is the support projection which is the smallest hermitian projection so that $L(F) = L(I) = I_o$ and $\phi(A) = FL(A)F$ for $A \in B(\mathbb{C}^p)$. Let $L$ denote the range of $L$. Let $P$ be the central support of $F$ in the center of the commutant $L'$ of $L$ (i.e. $P$ is the smallest projection in $L' \cap L''$ with $P \geq F$). Then $\psi(A) = PL(A)P$ is completely positive contractive linear mapping of $B(\mathbb{C}^p)$ into itself that is idempotent and the range of $\psi$ is a $*$-algebra so

$$\psi(\psi(A)\psi(B)) = \psi(A)\psi(B) \quad \text{and} \quad \psi(A^*) = \psi(A)^*$$

for $A, B \in B(\mathbb{C}^p)$. Furthermore, if $L'$ is another completely positive contractive idempotent map of $B(\mathbb{C}^p)$ into itself with range $L$ then

$$L'(A) = L'(PAP)$$

for $A \in B(\mathbb{C}^p)$ and if $F'$ is the support projection for $L'$ then $F' \leq P$ and $P$ is the smallest projection in $L' \cap L''$ with $P \geq F'$. If $I_o = L(I)$ is the unit of $L$ then $PI_o = I_oP = PI_oP = P$.

**Proof.** Assume the hypothesis and notation of the lemma. Let $\mathfrak{M}$ be the linear span of vectors of the form $Af$ with $A \in L'$ and $f$ in the range of $F$. Note a vector $f$ is in $\mathfrak{M}$ if and only if $f$ can be written in the form

$$f = \sum_{i=1}^m A_i F g_i$$

with $A_i \in L'$ and $g_i \in \mathbb{C}^p$ for $i = 1, \cdots, m$. Note from the previous theorem we have $F \in L'$. Then if $A \in L$ we have

$$Af = \sum_{i=1}^m A A_i F g_i = \sum_{i=1}^m A_i F g_i = \sum_{i=1}^m A_i F A g_i$$

so $Af \in \mathfrak{M}$ and if $B \in L'$ then since $BA_i \in L'$ we have $Bf \in \mathfrak{M}$. Hence, $A\mathfrak{M} \subset \mathfrak{M}$, $A^*\mathfrak{M} \subset \mathfrak{M}$, $B\mathfrak{M} \subset \mathfrak{M}$ and $B^*\mathfrak{M} \subset \mathfrak{M}$ for $A \in L$ and $B \in L'$. It follows that if $P$ is the orthogonal projection of $\mathbb{C}^p$ onto $\mathfrak{M}$ then $P \in L'$ and $P \in L''$ so $P$ is in the center of $L'$ which equals the center of $L''$. Note $P$ is the central support of $F$ since $P$ is the smallest projection $P \geq F$ which is contained in both $L'$ and $L''$.

To prove the last statement of the theorem we use another characterization of $P$ that $P$ is the smallest projection so that $P \geq U^*FU$ for all unitary $U \in L'$. Note from this definition of $P$ we have $P$ commutes with all unitary $U \in L'$ so $P \in L''$ and $P \in L'$ since $P$ is in the algebra generated by the $U^*FU \in L'$ and clearly $P$ is the smallest projection in $L' \cap L''$ with $P \geq F$. Note the $L(I) = L(I_o)$ so $L(I - I_o) = 0$. Since $F$ is the smallest projection so that $L(F) = I_o$ it
follows that \( F(I - I_o) = (I - I_o)F = 0 \). Hence, \( U^*FU (I - I_o) = U^*F (I - I_o)U = 0 \) for all \( U \in \mathcal{L}' \) so \( P (I - I_o) = (I - I_o)P = 0 \) and we have \( P = P_1 = P_{I_o} = P_{I_o}P = P_{I_o}P \).

Next we note that \( \mathcal{P} \mathcal{L} \mathcal{P} \) is a \( * \)-algebra. It is clear that \( \mathcal{P} \mathcal{L} \mathcal{P} \) is invariant under the \( * \)-operation. Note that if \( A, B \in \mathcal{L} \) then from Theorem 5.1 there is a \( C \in \mathcal{L} \) so that \( ABF = CF \). Then for \( f \in \mathcal{M} \) of the above form we have

\[
ABf = \sum_{i=1}^{m} ABA_i Fg_i = \sum_{i=1}^{m} A_i ABFg_i = \sum_{i=1}^{m} A_i C Fg_i = \sum_{i=1}^{m} CA_i Fg_i = Cf
\]

so \( \mathcal{P} \mathcal{L} \mathcal{P} \) is a \( * \)-algebra. Since the mapping \( A \to FAF \) is a completely positive one to one mapping from \( \mathcal{L} \) to \( F \mathcal{L} F \) it follows that the mapping \( A \to \mathcal{P} \mathcal{A} \) is a completely positive one to one mapping of \( \mathcal{L} \) onto \( \mathcal{P} \mathcal{L} \mathcal{P} \). From the previous lemma we have for \( A \in B(\mathbb{C}^p) \) that \( L(A) = L(FAF) \) and \( FP = F \) so if \( \psi(A) = \mathcal{P} \mathcal{L} \mathcal{P} \) we have

\[
\psi^2(A) = \mathcal{P}L(\mathcal{P}L(A)P)P = \mathcal{P}L(\mathcal{F}L(\mathcal{P}L(A)F)F)P = \mathcal{P}L(L(A))P = \mathcal{P}L(A)P = \psi(A)
\]

so we see that \( \psi \) is a completely positive contractive idempotent map of \( B(\mathbb{C}^p) \) into itself and the range of \( \psi \) is a \( * \)-algebra and \( \psi(A) = 0 \) if and only if \( L(A) = 0 \).

Now suppose \( L' \) is another completely positive contractive idempotent map of \( B(\mathbb{C}^p) \) into itself with range \( \mathcal{L} \) and \( F' \) is the support projection for \( L' \) and it is not true that \( F' \leq P \). Then repeating the previous argument we find there is a central projection \( P' \) so that \( P' \mathcal{L} P \) is a \( * \)-algebra and the mapping \( \psi'(A) = P' \mathcal{L} P \) is a completely positive contractive linear idempotent map of \( B(\mathbb{C}^p) \) so that the range of \( \psi' \) is a \( * \)-algebra and \( \psi(A) = 0 \) if and only if \( L(A) = 0 \).

Since it is not true that \( F' \leq P \) we have \( Q = P P' \) is central projection and \( Q \neq P' \). We will prove this in not the case because \( Q = P = P' \). Since every element in \( \mathcal{L}' \) can be written as a polynomial of elements of \( \mathcal{L} \) and the unit \( I \) of \( B(\mathbb{C}^p) \) it follows that

\[
Q = \sum_{k=1}^{m} A_k = B_{k1}B_{k2}\cdots B_{kn_k}
\]

with the \( B_{ki} \in \mathcal{L} \) or \( B_{ki} = I \). Now we claim that for each \( A_k \) above there is an element \( C_k \in \mathcal{L} \) so that \( P A_k = P C_k \) and \( P' A_k = P' C_k \) for each \( k = 1, \ldots, m \). This is seen as follows. Since \( \mathcal{P} \mathcal{L} \mathcal{P} \) is an algebra we have

\[
P A_k = P B_{k1}P B_{k2}P \cdots P B_{kn_k} = P C_k
\]

where \( C_k \in \mathcal{L} \) is unique. (Note \( P I = P I_o \) so any terms involving \( I \) can be replaced by \( I_o \).) Likewise we have \( P'A_k = P' C_k ' \) where \( C_k ' \in \mathcal{L} \) is unique. Since the multiplication defined by \( L \) and \( L' \) are the same it follows that \( C_k = C_k ' \) and, hence, if we define

\[
Q_o = \sum_{k=1}^{m} C_k
\]

we have \( P Q P = P Q_o P = P Q_o \) and \( P' Q P' = P' Q_o P' = P' Q_o \). Next suppose \( A \in \mathcal{L} \). Then \( QA = P P' A = P Q_o A = P L(Q_o A) \) and \( AQ = A P P' = A P Q_o = P A Q_o = P L(A Q_o) \) and since \( Q \in \mathcal{L}' \) we have \( L(Q_o A) = L(A Q_o) \) so \( Q_o \) is in the center of \( \mathcal{L} \) where we view \( \mathcal{L} \) as a \( * \)-algebra with the Choi-Effros multiplication.

Next we show \( Q_o = I_o \). If this is not the case then \( I_o - Q_o \) is a central projection in \( \mathcal{L} \). Since each projection in the center of \( \mathcal{L} \) is the sum of minimal central projections there is a non zero minimal central projection \( B_o \in \mathcal{L} \) with \( B_o \leq I_o - Q_o \). Note \( B_o P \) and \( B_o P' \) are central projections in \( \mathcal{L}' \). Since the mappings \( A \leftrightarrow P A \) and \( A \leftrightarrow P' A \) are \( * \)-isomorphisms of \( \mathcal{L} \) with \( \mathcal{P} \mathcal{L} \) and of \( \mathcal{L} \) with \( P' \mathcal{L} \) it follows that \( P B_o \) and \( P' B_o \) are both non zero. Note the product of these projections is zero since

\[
PB_o P' B_o = P P' B_o B_o = P Q_o B_o B_o = P L(Q_o B_o B_o) = P L(L(Q_o B_o) B_o)
\]

and since \( B_o \leq I_o - Q_o \) we have \( L(Q_o B_o) = 0 \) so \( P B_o P' B_o = 0 \). Now \( B_o \leftrightarrow \mathcal{L} \) (where \( B_o \leftrightarrow A = L(B_o A) \) for \( A \in \mathcal{L} \)) is a Choi-Effros factor of type I_o. Let \( E_{ij} \in B_o \leftrightarrow \mathcal{L} \) be a complete set of matrix units.
so \( E_{ij} \ast E_{nm} = \delta_{jn}E_{im} \) which span \( B_o \ast L \). Note there is a natural \(*\)-isomorphism \( \psi \) of \( B(\mathbb{C}^p) \) into \( B_o \ast L \) defined as follows. If \( A \in B(\mathbb{C}^p) \) corresponds to the matrix \( \{a_{ij}\} \) then
\[
\psi(A) = \sum_{i,j=1}^q a_{ij}E_{ij}.
\]
Since the mappings \( A \leftrightarrow PA \) and \( A \leftrightarrow P'A \) are \(*\)-isomorphisms of \( L \) it follows that \( \pi_1(A) = P\psi(A) \) and \( \pi_2(A) = P'\psi(A) \) are \(*\)-representations of \( B(\mathbb{C}^p) \) on \( PB_o \) and \( P'B_o \), respectively. As is well known any two \(*\)-representations of \( B(\mathbb{C}^p) \) are quasi-equivalent so there are intertwining operators. Specifically let \( f \) be a unit vector so that \( PE_{11}f = f \) and \( g \) be a unit vector so that \( P'E_{11}g = g \). Let
\[
f_i = PE_{11}f \quad \text{and} \quad g_i = P'E_{11}g
\]
for \( i = 1, \ldots, q \). We define the operator \( C \) as follows. We define \( Cf_i = g_i \) for \( i = 1, \ldots, q \) and \( CF = 0 \) if \( F \) is orthogonal to the \( f_i \). This defines an operator \( C \in B(\mathbb{C}^p) \) of rank \( q \). A little computation show that \( C\pi_1(A) = \pi_2(A)C \) for \( A \in B(\mathbb{C}^p) \). Now if \( A \in B_o \ast L \) then \( A \) is a linear combination of the \( E_{ij} \) so \( A = \psi(A) \) for a unique \( A_1 \in B(\mathbb{C}^p) \) so we can write \( A_1 = \psi^{-1}(A) \). Then \( \pi_1(A_1) = \pi_1(\psi^{-1}(A)) = PB_oA \) and \( \pi_2(A_1) = \pi_2(\psi^{-1}(A)) = P'B_oA \) and the fact that \( C\pi_1(A) = \pi_2(A)C \) means that \( CPB_oA = P'B_oAC \) for \( A \in L \). Notice that the range of \( C \) is contained the range of \( P'B_o \) and the range of \( C^* \) is contained in the range of \( PB_o \) so we have
\[
C = CPB_o = P'B_oC = P'B_oCPB_o.
\]
It follows that \( C \in L' \) since for \( A \in L \) we have
\[
CA = CPB_oA = P'B_oAC = AP'B_oC = AC.
\]
We have reached a contradiction. Recall \( PB_o \) is a central projection in \( L' \) and we see that \( C \in L' \) does not commute with it. Hence, the assumption that \( Q_o \neq I_o \) leads to a contradiction. Hence, \( PP' = PL_o = P'I_o \). But as we showed earlier in this proof we have \( PL_o = P \) and by the same argument \( P'I_o = P' \) so we have \( PP' = P = P' \). Since the support projection \( F' \) for \( L' \) satisfies \( F' \leq P' \) we have \( F' \leq P \).

We complete the proof by showing
\[
L'(A) = L'(PAP)
\]
for \( A \in B(\mathbb{C}^p) \). Now we have
\[
L'(PAP) = L'(F'PAPF') = L'(F'AF') = L'(A)
\]
for \( A \in B(\mathbb{C}^p) \).

**Definition 5.3.** Suppose \( L \) is a linear completely positive contractive idempotent map of \( \mathbb{C}^p \) into itself and \( L \) is the range of \( L \). Suppose \( F \) is the support projection for \( L \) which is the smallest hermitian projection \( F \in B(\mathbb{C}^p) \) so that \( L(F) = L(I) = I_o \). The projection \( P \) which is the smallest projection in \( L' \cap L'' \) with \( G \geq F \) is called the maximal support projection for \( L \).

In the above definition it appears as if the maximal support projection depends on \( L \) but from the previous theorem we see that any linear completely positive contractive idempotent map with the same range as \( L \) yields the same projection \( P \).

Next we consider the case where we have two completely positive contractive linear idempotent mappings \( L \) and \( L_1 \) of \( B(\mathbb{C}^p) \) into itself and \( L - L_1 \) is a completely positive map.

**Theorem 5.4.** Suppose \( L \) and \( L_1 \) are completely positive contractive linear idempotent mappings of \( B(\mathbb{C}^p) \) into itself and \( L - L_1 \) is a completely positive map. Suppose further that \( L_1(I) = E \) and \( E \) is a projection. Let \( L \) be the range of \( L \). Then \( E \) commutes with every element of \( L \) (i.e., \( E \in L' \)) and \( E \leq L(I) \) and \( L_1(A) = EL(A) = EL(A)E \) for \( A \in B(\mathbb{C}^p) \).

**Proof.** Assume the hypothesis and notation of the theorem. Suppose \( A \in B(\mathbb{C}^p) \) and \( 0 \leq A \leq I \) then since \( L_1 \) is completely positive we have \( 0 \leq L_1(A) \leq E \) from which it follows that \( L_1(A) = \)
\( EL_1(A)E \) and since \( B(\mathbb{C}^p) \) is the complex linear span of its positive elements we have \( L_1(A) = EL_1(A)E \) for all \( A \in B(\mathbb{C}^p) \). Now consider the mapping
\[
\phi(A) = E(L(A) - L_1(A))E = EL(A)E - L_1(A)
\]
for \( A \in B(\mathbb{C}^p) \). Since \( L \geq L_1 \) this mapping is completely positive. Note \( \phi(I) = EL(I)E - E \) and since \( L \) is contractive we have \( \phi(I) \leq 0 \) but since \( \phi \) is completely positive we have \( \phi(I) = 0 \) so \( \phi = 0 \) and
\[
L_1(A) = EL(A)E
\]
for all \( A \in B(\mathbb{C}^p) \). Now suppose \( A \in \mathcal{L} \) and \( A \geq 0 \). Then
\[
L(A) - L_1(A) = A - EAE \geq 0
\]
from which we conclude that \( E \) commutes with \( A \). Since \( \mathcal{L} \) is the complex linear span of its positive elements we have \( E \in \mathcal{L'} \).

Notice that if \( \omega \) is a \( q \)-weight map over \( \mathbb{C}^p \) with range \( \mathcal{L}(\omega) \) when we compute the generalized boundary representation of \( \omega \) given by
\[
\pi_t^\# = (\iota + \omega|t\Lambda)^{-1}\omega|t
\]
in computing the inverse of \( \iota + \omega|t\Lambda \) we only need compute \( (\iota + \omega|t\Lambda) \) on \( \mathcal{L}(\omega) \). We know from the general theory that the inverse exists but for calculational purposes we only care about the inverse on \( \mathcal{L}(\omega) \). In the case where \( \omega \) is of index zero then we know that the map \( A \leftrightarrow \text{PAP} \) where \( P \) is the maximal support projection for \( \mathcal{L}(\omega) \) is completely positive and one to one in both directions. So to parameterize \( \mathcal{L}(\omega) \) we can take a complete set of matrix units \( e_{ij}^r \) for \( \mathcal{L}(\omega) \) chosen so that \( Pe_{ij}^rP \) are a complete set of matrix units for \( P\mathcal{L}(\omega)P \). Now when we analyze a \( q \)-weight map over \( \mathbb{C}^p \) of index zero when we speak of the map \( \omega|A \) we will often consider this to be a map of \( \mathcal{L}(\omega) \) into itself rather than a map of \( B(\mathbb{C}^p) \) into itself. This may seem an obvious observation but it took us some time to realize this. To give an example. Note that limit points \( L \) of \( \pi_t^\# \Lambda \) as \( t \to 0^+ \) are not unique but if we restrict our attention to \( \mathcal{L}(\omega) \) the limit is unique and the limit is the identity map.

Next we show that if \( \omega \) is \( q \)-pure \( q \)-weight map over \( \mathbb{C}^p \) of index zero then range of \( \omega \) is a factor with the Choi-Effros product. First we prove a routine lemma.

**Lemma 5.5.** Suppose \( B(X) \) is the Banach space of all linear maps of a finite dimensional Banach space \( X \) into itself and \( A_n \in B(X) \) for \( n = 1, 2, \cdots \) is a sequence of invertible elements and \( A_n \to A \) as \( n \to \infty \) then \( A \) is invertible and \( A_n^{-1} \to A^{-1} \) as \( n \to \infty \) if and only if there is a constant \( K \) so that \( \|A_n^{-1}\| \leq K \) for all \( n \).

**Proof.** Assume \( A_n \) is a sequence as stated above. If \( A \) is invertible and \( A_n^{-1} \to A^{-1} \) as \( n \to \infty \) then \( \|A_n^{-1}\| \) is uniformly bounded. Now suppose there is a constant \( K \) so that \( \|A_n^{-1}\| \leq K \) for all \( n \). Then we have
\[
\|A_n^{-1} - A_m^{-1}\| = \|A_n^{-1}(A_n - A_m)A_m^{-1}\| \leq \|A_n^{-1}\| \|A_n - A_m\| \|A_m^{-1}\| \leq K^2 \|A_n - A_m\| \to 0
\]
as \( n, m \to \infty \). Since \( B(X) \) is complete there is a \( B \in B(X) \) so that \( A_n^{-1} \to B \) as \( n \to \infty \). We show \( A \) is invertible and \( B = A^{-1} \). Since \( A_n \to A \) there is a constant \( K' \) so that \( \|A_n\| \leq K' \) for all \( n \). Now we have
\[
\|AB - I\| = \|(A - A_n)B + A_n(B - A_n^{-1})\| \leq \|A - A_n\| \|B\| + K' \|B - A_n^{-1}\| \to 0
\]
as \( n \to \infty \). Hence \( AB = I \) and \( B = A^{-1} \).

**Theorem 5.6.** Suppose \( \omega \) is a \( q \)-weight map over \( \mathbb{C}^p \) and \( \pi_t^\# \) is the generalized boundary representation of \( \omega \). Suppose that \( \psi_t^\# \) is a completely positive map of \( \mathfrak{A}(\mathbb{C}^p) \) into \( B(\mathbb{C}^p) \) which is subordinate to \( \pi_t^\# \) so \( \pi_t^\# \geq \psi_t^\# \) for each \( t > 0 \). Let
\[
\eta_t = (\iota - \psi_t^\# \Lambda)^{-1}\psi_t^#
\]
for $t > 0$. Then $\omega|t \geq q \eta$ for $t > 0$ and if $\eta$ is a weak limit point in $B(\mathbb{C}^p) \otimes \mathfrak{A}(\mathbb{C}^p)_s$ of $\eta_k$ as $t \to 0$ then $\eta$ is a $q$-subordinate of $\omega$ so $\omega \geq q \eta$.

**Proof.** Assume the hypothesis and notation of the theorem. Since the generalized boundary representation $\phi_k^\#$ of $\eta_k$ is $\psi^\#_k$ for $t \leq s$ and the boundary representation $\pi^\#_t$ of $\omega|s$ is $\pi^\#_t$ for $t \leq s$ it follows that $\omega|s \geq q \eta$ for all $s > 0$. Now suppose that $\eta$ is a weak limit point of $\eta_k$ as $t \to 0^+$. Since $B(\mathbb{C}^p)$ is finite dimensional there is a decreasing sequence of $t_k > 0$ so that $\eta_{t_k}(A) \to \eta(A)$ as $k \to \infty$ for each $A \in \mathfrak{A}(\mathbb{C}^p)$. To simplify notation we let $\eta_k = \eta_{t_k}$ and $\psi^\#_k = \psi^\#_{t_k}$, $\pi^\#_k = \pi^\#_{t_k}$ and $\omega_k = \omega|_{t_k}$ for $k = 1, 2, \ldots$. Suppose $s > 0$. Since $\pi^\#_k \geq \psi^\#_k$ we have for $k$ large enough so that $t_k < s$ that

$$(\iota + \eta_k|s\Lambda)^{-1}\eta_k|s \leq (\iota + \omega|s\Lambda)^{-1}\omega|s.$$

Now we have $\eta_k|s \Lambda \to \eta|s \Lambda$ as $k \to \infty$ and from lemma 5.5 we have $(\iota + \eta|s\Lambda)$ is invertible and

$$(\iota + \eta_k|s\Lambda)^{-1} \to (\iota + \eta|s\Lambda)^{-1}$$

as $k \to \infty$ if and only if there is a constant $K$ so that

$$\|(\iota + \eta_k|s\Lambda)^{-1}\| \leq K$$

for all $k = 1, 2, \ldots$. Now we have

$$\eta_k|s = (\iota - \psi^\#_k \Lambda)^{-1}\psi^\#_k|s$$

so

$$\iota + \eta_k|s\Lambda = (\iota - \psi^\#_k \Lambda)^{-1}(\iota - (\psi^\# - \psi^\#_k|s)\Lambda)$$

which yields

$$(\iota + \eta_k|s\Lambda)^{-1} = (\iota - (\psi^\# - \psi^\#_k|s)\Lambda)^{-1}(\iota - \psi^\#_k \Lambda) = \iota - (\iota - (\psi^\# - \psi^\#_k|s)\Lambda)^{-1}\psi^\#_k|s\Lambda$$

so if we let $T_k = \iota - (\iota + \eta_k|s\Lambda)^{-1}$ we have

$$T_k = (\iota - (\psi^\# - \psi^\#_k|s)\Lambda)^{-1}\psi^\#_k|s\Lambda$$

and using the geometric series expansion $(1 - x)^{-1} = 1 + x + x^2 + \cdots$ which converges since $\|(\psi^\# - \psi^\#_k|s)\Lambda\| \leq \|\pi^\#_k \Lambda\| < 1$ we have

$$T_k = (\iota + (\psi^\# - \psi^\#_k|s)\Lambda + ((\psi^\# - \psi^\#_k|s)\Lambda)^2 + \cdots)\psi^\#_k|s\Lambda$$

so $T_k$ is the sum of completely positive terms so $T_k$ is completely positive and, hence, $\|T_k\| = \|T_k(I)\|$. Note that if $\phi_i$ and $\phi'_i$ are completely positive maps and $\phi_i \geq \phi'_i$ for $i = 1, \ldots, n$ then

$$\phi_1\phi_2\cdots\phi_n \geq \phi'_1\phi'_2\cdots\phi'_n$$

so we have

$$T_k \leq (\iota + \psi^\#_k \Lambda + (\psi^\#_k \Lambda)^2 + \cdots)\psi^\#_k|s\Lambda = (\iota - \psi^\#_k \Lambda)^{-1}\psi^\#_k|s\Lambda = \eta_k|s\Lambda \leq \omega|s\Lambda$$

and, hence,

$$\|T_k\| \leq \|\omega|s\Lambda\|$$

for all $k$ with $t_k \leq s$. Hence, we have

$$\|(\iota + \eta_k|s\Lambda)^{-1}\| = \|\iota - T_k\| \leq \|\iota\| + \|T_k\| \leq 1 + \|\omega|s\Lambda\|$$

for $t_k \leq s$. Hence, from Lemma 5.5 we have $\iota + \eta|s\Lambda$ is invertible and

$$(\iota + \eta|s\Lambda)^{-1} \to (\iota + \eta|s\Lambda)^{-1}$$

as $k \to \infty$ and, hence,

$$(\iota + \eta_k|s\Lambda)^{-1}\eta_k|s \to (\iota + \eta|s\Lambda)^{-1}\eta|s$$

as $k \to \infty$. Since

$$(\iota + \eta_k|s\Lambda)^{-1}\eta_k|s \leq (\iota + \omega|s\Lambda)^{-1}\omega|s$$

for all $k$ we have

$$(\iota + \eta|s\Lambda)^{-1}\eta|s \leq (\iota + \omega|s\Lambda)^{-1}\omega|s$$
and, hence, \( \eta \) is a \( q \)-subordinate of \( \omega \).

**Theorem 5.7.** Suppose \( \omega \) is a \( q \)-weight map over \( \mathbb{C}^p \) of index zero and \( L \) is the range of \( \omega \) which is a Choi-Effros algebra with the multiplication \( A \cdot B = L(AB) \) where \( L \) is a limit point of \( \pi_t^\# \Lambda \) as \( t \to 0^+ \). Let \( Q \) be a non-zero central projection in \( L \). Then there is a \( q \)-subordinate \( \eta \) of \( \omega \) so that the range of \( \eta \) is \( Q \ast L \). In particular, if \( \omega \) is a \( q \)-pure \( q \)-weight map over \( \mathbb{C}^p \) of index zero then the range \( L \) of \( \omega \) is a Choi-Effros factor.

**Proof.** Assume the hypothesis and notation of the theorem. If \( L \) is a factor with the Choi-Effros product then \( \eta = \omega \) and the proof is complete. Suppose that \( L \) is not a factor with the Choi-Effros product and \( Q \in L \) is a central projection in \( L \). Let \( I_o \) be the unit of \( L \) and note \( I_o - Q \) is not zero. Let \( \pi_t^\# \) be the generalized boundary representation of \( \omega \) and let \( \psi_t^\# = Q \ast \pi_t^\# \) and note that \( \pi_t^\# \ast \pi_t^\# = (I_o - Q) \ast \pi_t^\# \geq 0 \) for \( t > 0 \) and let

\[
\eta_t = (t - \psi_t^\#)^{-1} \psi_t^\# = \psi_t^\# + \psi_t^\# \Lambda \psi_t^\# + \cdots.
\]

Note the range of \( \eta_t \) is contained in \( Q \ast L \). Then from Theorem 5.6 we know that any weak limit \( \eta \) of \( \eta_t \) is a \( q \)-subordinate of \( \omega \).

Next we show that any limit point \( \eta \) is not zero and the range of \( \eta \) is \( Q \ast L \). Now we have

\[
\eta_t = (t - \psi_t^\#)^{-1} \psi_t^\# \quad \text{and} \quad \psi_t^\# = Q \ast \pi_t^\# = Q \ast (t + \omega_t \Lambda)^{-1} \omega_t
\]

so

\[
\eta_t = (t - Q \ast \pi_t^\#)^{-1} Q \ast (t + \omega_t \Lambda)^{-1} \omega_t
\]

and since

\[
(t + \omega_t \Lambda)^{-1} = t - \pi_t^\#
\]

we have

\[
\eta_t = (t - Q \ast \pi_t^\#)^{-1} Q \ast (t - \pi_t^\#) \omega_t = R_t \omega_t
\]

and

\[
R_t = (t + Q \ast \pi_t^\# \Lambda + (Q \ast \pi_t^\# \Lambda)^2 + \cdots) Q \ast (t - \pi_t^\# \Lambda)
\]

\[
= (t + Q \ast \pi_t^\# \Lambda + (Q \ast \pi_t^\# \Lambda)^2 + \cdots) Q \ast -(Q \ast \pi_t^\# \Lambda + (Q \ast \pi_t^\# \Lambda)^2 + \cdots)
\]

\[
= Q \ast -Q \ast \pi_t^\# \Lambda (I_o - Q) \ast -(Q \ast \pi_t^\# \Lambda)^2 (I_o - Q) \ast -(Q \ast \pi_t^\# \Lambda)^3 (I_o - Q) \ast -\cdots
\]

\[
= Q \ast -Q \ast (t - Q \ast \pi_t^\# \Lambda)^{-1} (I_o - Q)
\]

so we have

\[
\eta_t = Q \ast \omega_t - Q \ast (t - Q \ast \pi_t^\# \Lambda)^{-1} (\omega_t - Q \ast \omega_t)
\]

\[
= Q \ast (t - (t - Q \ast \pi_t^\# \Lambda)^{-1} (I_o - Q)) \ast \omega_t
\]

for \( t > 0 \). Let \( \eta \) be a limit point of \( \eta_t \) as \( t \to 0^+ \). Since the range of \( \omega \) is \( L \) for any \( A \in L \) there is a \( t_0 > 0 \) and a \( B \in A(\mathbb{C}^p) \) so that \( E(t_0, \infty)BE(t_0, \infty) = B \) (where \( E(s, \infty) \) is the projection in \( A(\mathbb{C}^p) \) onto function with support in \( [s, \infty) \)) and \( \omega(B) = A \). Now let \( A \) be any element of \( L \) so that \( Q \ast A = A \). Then \( (I_o - Q) \ast A = 0 \) and we see that for \( t \in (0, t_0) \) we have \( \eta_t(B) = A \). This then shows that every limit point \( \eta \) is not zero and the range of \( \eta \) is \( Q \ast L \). Then \( \eta \) is a \( q \)-subordinate of \( \omega \) with range \( Q \ast L \).

Now we prove the last sentences of the theorem. Suppose that \( \omega \) is a \( q \)-pure \( q \)-weight map over \( \mathbb{C}^p \) with range \( L \) which is an algebra with the Choi-Effros multiplication. Suppose \( L \) is not a Choi-Effros factor. Then there are at least two minimal central projections \( Q_1 \) and \( Q_2 \) so that \( Q_1 \ast Q_2 = 0 \). By what we have just proved there are \( q \)-subordinates \( \eta_1 \) and \( \eta_2 \) of \( \omega \) with ranges \( Q_1 L \) and \( Q_2 L \), respectively. Since \( Q_1 \) and \( Q_2 \) are disjoint central projections it is immediately clear that it is not true that \( \eta_1 \geq \eta_2 \) or \( \eta_2 \geq \eta_1 \). Hence, \( \omega \) is not \( q \)-pure. \( \square \)

Our ultimate goal is to understand \( q \)-pure \( q \)-weight maps \( \omega \) of index zero and from the last theorem we know that for such maps the range of \( \omega \) is a Choi-Effros factor (i.e. \( L \) the range of \( \omega \) is a factor with the Choi-Effros product). The remainder of this section we will focus on this situation. The main results of the next few theorems and lemmas is to carry over the results of the last section to this new setting. What is surprising is that the arguments of the last section
carry over to our new setting with virtually no change except for a change in notation. What we will do is to replace $B(C^p)$ of the last section by $\mathcal{L}$ the range of $\omega$ so instead of considering a mapping for $\mathfrak{A}(C^p)$ into $B(C^p)$ we will consider the same mapping from $\tilde{\mathfrak{A}}(C^p)$ to $\mathcal{L}$ where $\tilde{\mathfrak{A}}(C^p)$ is $\mathcal{L} \otimes \mathfrak{A}(C)$. Note $\mathfrak{A}(C^p) = B(C^p) \otimes \mathfrak{A}(C)$ so $\tilde{\mathfrak{A}}(C^p)$ is obtained by restriction. We will spell this out in detail but for the moment we what to emphasize the main point that if you see a tilde on a mapping it means the same mapping only restricted $\mathcal{L}$. The mapping $\phi_t = \omega|_t \Lambda$ is a mapping of $B(C^p)$ into itself. The mapping $\phi_t$ is the same mapping restricted to $\mathcal{L}$. The mapping $L_t = \pi_t \Lambda$ is a mapping of $B(C^p)$ into itself and the mapping $L_t$ is the same mapping restricted to $\mathcal{L}$.

The basic rule is that the tilde means restrict to $\mathcal{L}$ so $\tilde{B}(C^p) = \mathcal{L}$. The mapping $\Lambda$ is a mapping of $B(C^p)$ into it $B(C^p \otimes L^2(0, \infty))$. The mapping $\tilde{\Lambda}$ is the same mapping restricted to $\mathcal{L}$. The range of $\tilde{\Lambda}$ is in $B(C^p \otimes L^2(0, \infty)) = B(C^p) \otimes B(L^2(0, \infty))$ but it is also in $B(C^p \otimes L^2(0, \infty)) = \tilde{B}(C^p) \otimes B(L^2(0, \infty)) = \mathcal{L} \otimes B(L^2(0, \infty))$. Here is the tricky part. In order to specify an element of $A \in B(C^p)$ we specify a $(p \times p)$-matrix $a_{ij} \in \mathbb{C}$ for $i, j = 1, \ldots, p$. Now $\mathcal{L}$ is a Choi-Effros factor of type $I_q$ which means there are matrix units $E_{ij} \in \mathcal{L}$ which span $\mathcal{L}$ so to specify an element $A \in \mathcal{L}$ we specify a $(q \times q)$-matrix $a_{ij} \in \mathbb{C}$ for $i, j = 1, \ldots, q$. Similarly each element of $B(C^p \otimes L^2(0, \infty))$ can be specified by a $(p \times p)$-matrix of elements of $B(L^2(0, \infty))$ and each $A \in \tilde{B}(C^p \otimes L^2(0, \infty))$ can be specified by a $(q \times q)$-matrix of elements of $A_{ij} \in B(L^2(0, \infty))$ by the formula

$$A = \sum_{i,j=1}^q E_{ij} A_{ij}.$$  

Notice there is a bijection from $\tilde{B}(C^p \otimes L^2(0, \infty))$ to $B(C^q \otimes L^2(0, \infty))$ and this bijection is a completely positive contraction in both directions. In the language of Choi and Effros it is a relations for example it involves the choice of matrix unit $E$. The possibility that the term corresponding to $(g_{ik})_{j}(x)$ would involve multiple terms $(g_{ik})_j$ with different $k$’s. This is both the beauty and draw back of the tilde notation. The draw back is that understanding formulas in terms of $\tilde{\Lambda}$ in terms of $\Lambda$ can be quite complicated. The beauty is that calculations with $\tilde{\Lambda}$ can be extremely simple where as the corresponding calculation with $\Lambda$ would be so complicated that one would not have the courage to undertake them.

We formalize this with the following extended definition.

**Definition 5.8.** A subset $\mathcal{L}$ of $B(C^p)$ is called a Choi-Effros factor if there is a completely positive contractive idempotent mapping $L$ of $B(C^p)$ into itself with range $\mathcal{L}$ and $\mathcal{L}$ equipped with the Choi-Effros multiplication $A \star B = L(AB)$ is a $(q \times q)$-matrix algebra with unit $I_q$. A complete set of matrix units for $\mathcal{L}$ consist of elements $E_{ij}$ for $i, j = 1, \ldots, q$ which span $\mathcal{L}$ and satisfying the relations

$$E_{ij}^* = E_{ji}, \quad E_{ij} \star E_{nm} = \delta_{in} E_{jm}$$

and

$$E_{11} + E_{22} + \cdots + E_{qq} = I_q.$$

If $A \in \mathcal{L}$ the matrix entries of $A$ are complex numbers $a_{ij}$ so that

$$A = \sum_{i,j=1}^q a_{ij} E_{ij}.$$  

If $\psi$ is a mapping of $B(C^p)$ into another space we denote by $\tilde{\psi}$ the same mapping restricted to $\mathcal{L}$. We denote,

$$\tilde{B}(C^p \otimes L^2(0, \infty)) = \mathcal{L} \otimes B(L^2(0, \infty))$$

$$\tilde{\mathfrak{A}}(C^p) = \mathcal{L} \otimes \mathfrak{A}(C) \subset \mathcal{L} \otimes B(L^2(0, \infty)) \subset \tilde{B}(C^p \otimes L^2(0, \infty)).$$
Each element \( A \in \tilde{B}(\mathbb{C}^p \otimes L^2(0, \infty)) \) or \( A \in \tilde{\mathcal{A}}(\mathbb{C}^p) \) can be written as a \((q \times q)\)-matrix of elements in \( B(L^2(0, \infty)) \)

\[
A = \sum_{i,j=1}^{q} E_{ij} \otimes A_{ij}
\]

and the operators \( A_{ij} \in B(L^2(0, \infty)) \) are called the coefficients of \( A \) in \( \mathcal{L} \otimes B(L^2(0, \infty)) \). The mapping from \( A \) to the matrix of coefficients of \( A \) gives us a complete order isomorphism of \( B(\mathbb{C}^p \otimes L^2(0, \infty)) \) and \( \tilde{\mathcal{A}}(\mathbb{C}^p) \) with \( B(\mathbb{C}^q \otimes L^2(0, \infty)) \) and \( \mathcal{A}(\mathbb{C}^q) \), respectively.

Given an element \( A \in \mathcal{L} \) we denote by \( \tilde{tr}(A) \) the trace of \( A \) in \( \mathcal{L} \) defined as follows. If

\[
A = \sum_{i,j=1}^{q} a_{ij} E_{ij} \quad \text{then} \quad \tilde{tr}(A) = \frac{1}{q} \sum_{i=1}^{q} a_{ii}.
\]

Technically when we refer to the matrix elements \( a_{ij} \) of \( A \) we should say with respect to the matrix units \( E_{ij} \) but we will forgo repeating this. Note the trace \( \tilde{tr}(A) \) does not depend on the particular choice of matrix unit \( E_{ij} \) for \( \mathcal{L} \) and \( \tilde{tr}(I_o) = 1 \).

In summary all the calculations involving \( \phi_t, \Lambda, \tilde{\mathcal{A}}(\mathbb{C}^p) \) and \( \mathcal{L} \) are the same as calculations involving \( \phi_t, \Lambda, \mathcal{A}(\mathbb{C}^p) \) and \( B(\mathbb{C}^p) \) except \( \mathbb{C}^p \) is replaced by \( \mathbb{C}^q \) and we think of \( \mathcal{L} \) as \( B(\mathbb{C}^q) \). For us the fact that this works is a triumph of the work of Choi and Effros without which we would never have come this far.

The first example which shows the advantage of the tilde notation comes from the very basic property of the \( q \)-weight map \( \omega \). Recall this is a completely positive mapping of \( \mathcal{A}(\mathbb{C}^p) \) into \( B(\mathbb{C}^p) \) which satisfies inequality \( \omega(I - \Lambda) \leq I \) where \( \Lambda = \Lambda(I) \) by which we mean

\[
\omega|_t(I - \Lambda(I_o)) \leq I_t
\]

for all \( t > 0 \). The corresponding inequality the tilde notation is

\[
\omega|_t(I - \Lambda(I_o)) \leq I_o
\]

for all \( t > 0 \) where \( I_o \) is the identity \( \mathcal{L} \) the range of \( \omega \). The next theorem shows this is true and not only in the factor case but for the general case as well.

**Theorem 5.9.** Suppose \( \omega \) is a \( q \)-weight map over \( \mathbb{C}^p \) of index zero and \( \mathcal{L} \) is the range of \( \omega \) and \( I_o \in \mathcal{L} \) is the unit of \( \mathcal{L} \) (i.e. \( I_o \ast A = A \ast I_o = A \) for all \( A \in \mathcal{L} \)). Then \( \omega(I - \Lambda(I_o)) \leq I_o \) by which we mean \( \omega|_t(I - \Lambda(I_o)) \leq I_o \) for all \( t > 0 \).

**Proof.** Assume the hypothesis and notation of the theorem and let \( \pi_t^\# \) be the generalized boundary representation of \( \omega \). First we claim that \( \pi_t(I) \leq I_o \) for all \( t \). Let \( P \) be the maximal support projection for \( \mathcal{L} \). Since \( \pi_t^\#(I) \leq I \), we clearly have \( P \pi_t(I) = P \pi_t(I)P \leq P \) for \( t > 0 \) and since the mapping \( A \to PA = PAP \) is an order isomorphism for \( A \in \mathcal{L} \) and \( PI_o = P \) and the range of \( \pi_t^\# \) is \( \mathcal{L} \) it follows that \( \pi_t^\#(I) \leq I_o \) for all \( t > 0 \). Now we have

\[
\omega|_t(I - \Lambda(I_o)) = (\iota - \pi_t^\# \Lambda)^{-1} \pi_t^\#(I - \Lambda(I_o))
\]

\[
= (\iota - \pi_t^\# \Lambda)^{-1}(\iota - \pi_t^\# \Lambda)\pi_t^\#(I)
\]

\[
- (\iota - \pi_t^\# \Lambda)^{-1}(\pi_t^\# \Lambda(I_o) - \pi_t^\# \Lambda \pi_t^\#(I))
\]

\[
= \pi_t^\#(I) - (\iota - \pi_t^\# \Lambda)^{-1} \pi_t^\# \Lambda(I_o - \pi_t^\#(I))
\]

and since \( \pi_t^\#(I) \leq I_o \) and the mapping \( (\iota - \pi_t^\# \Lambda)^{-1} \pi_t^\# \Lambda \) is completely positive we have

\[
\omega|_t(I - \Lambda(I_o)) \leq I_o
\]

for all \( t > 0 \). \( \square \)

The next theorem translates the results of the last section where \( \mathcal{L} \) was \( B(\mathbb{C}^p) \) to our new setting where \( \mathcal{L} \) is a Choi-Effros factor. In an earlier version of this paper we laboriously translated each of the theorems and lemmas. What the next theorem shows is we get it all for free. The simple trick is defining \( \tilde{\omega} \) as explained in the proof of the theorem.
Theorem 5.10. Suppose \( \omega \) is a \( q \)-weight map over \( \mathbb{C}^p \) of index zero over \( B(\mathbb{C}^p) \) and the range of \( \omega \) denoted by \( L \) is a Choi-Effros factor of type \( I_q \) (where \( q < p \)). For \( t > 0 \) let \( \phi_t = \omega|_t \Lambda \) and let \( \pi^\#_t \) be the generalized boundary representation of \( \omega \). We denote by \( \Lambda_t \) and \( \tilde{\phi}_t \) these mappings restricted to \( L \) as described above. Let

\[
v_t = tr(I_o + \tilde{\phi}_t(I_o)) \quad \text{and} \quad \Theta_t = v_t^{-1}(t + \tilde{\phi}_t).
\]

Then \( \Theta_t \) is completely positive invertible mapping of \( L \) onto \( L \) and the inverse \( \Theta_t^{-1} \) is conditionally negative and \( \Theta_t \) converges to a limit \( \Theta \) as \( t \to 0^+ \) and \( \Theta \) is completely positive. The limit \( \Theta \) is invertible and the inverse \( \Theta^{-1} \) is conditionally negative and \( \Theta_t^{-1} \to \Theta^{-1} = \psi \) as \( t \to 0^+ \).

We define \( \tilde{\psi} = \Theta^{-1} \omega = \psi \omega \) where \( \tilde{\psi} \in B(\mathbb{C}^p) \otimes \mathfrak{A}(\mathbb{C}^p)_* \) is completely positive and

\[
\tilde{\psi}(A) = \lim_{t \to 0^+} v_t \pi^\#_t(A)
\]

for \( A \in \mathfrak{A}(\mathbb{C}^p) \). We define \( \vartheta_{ij} \) by the formula

\[
\vartheta(A) = \sum_{i,j=1}^q \vartheta_{ij}(A) E_{ij}
\]

and we have \( \omega = \Theta \vartheta = \psi^{-1} \vartheta \). We have \( \tilde{\psi} \) the restriction of \( \vartheta \) to \( \tilde{\mathfrak{A}}(\mathbb{C}^p) \) can be expressed in the form

\[
\tilde{\psi}_{ij}(A) = \sum_{k \in J} ((g_{ik} + h_{ik}), A(g_{jk} + h_{jk}))
\]

where the \( g_{ik}, h_{ik} \in \mathbb{C}^q \otimes L^2_+(0, \infty) \) and

\[
(g_{ik})_j(x) = \delta_{ij} g_k(x) \quad \text{and} \quad \sum_{i=1}^q (h_{ik})_i(x) = 0
\]

for \( A \in \tilde{\mathfrak{A}}(\mathbb{C}^p), x \geq 0, i, j \in \{1, \ldots, q\} \) and \( k \) a countable index set and the \( h_{ik} \in \mathbb{C}^q \otimes L^2(0, \infty) \) and if

\[
w_t = \sum_{k \in J} (g_k, \Lambda|_t g_k), \quad \tilde{\rho}_{ij}(A) = \sum_{k \in J} (h_{ik}, A h_{jk})
\]

then \( \tilde{\rho} \) is bounded so

\[
\sum_{i,j} \|h_{ik}\|^2 < \infty \quad \text{and} \quad \sum_{k \in J} (g_k, (I - \Lambda) g_k) < \infty
\]

and \( 1/w_t \to 0 \) as \( t \to 0^+ \) and \( \psi \) satisfies the conditions

\[
\psi(I_o) \geq \vartheta(I - \Lambda(I_o)) \quad \text{and} \quad \psi + \rho \tilde{\Lambda}
\]

is conditionally negative.

Proof. Assume the hypothesis and notation of the theorem. Now we define a \( \tilde{\omega} \) which will turn out to be a \( q \)-weight map over \( \mathbb{C}^q \). Given \( \omega \) as stated we define \( \tilde{\omega} \) by simply restricting \( \omega \) to \( \tilde{\mathfrak{A}}(\mathbb{C}^p) \) which we identify with \( \mathfrak{A}(\mathbb{C}^q) \). Note each element in \( \tilde{\mathfrak{A}}(\mathbb{C}^p) \) can be uniquely expressed as a \( (q \times q) \)-matrix with entries in \( \mathfrak{A}(\mathbb{C}) \). Specifically if \( A \in \tilde{\mathfrak{A}}(\mathbb{C}^p) \) it can be written in the form

\[
A = \sum_{i,j=1}^q E_{ij} A_{ij}
\]

with the \( A_{ij} \in \mathfrak{A}(\mathbb{C}) \). In this way we can think of \( \tilde{\omega} \) as an element of \( B(\mathbb{C}^q) \otimes \mathfrak{A}(\mathbb{C}) \). Then one simply checks that \( \tilde{\omega} \) is a \( q \)-weight map of index zero over \( \mathbb{C}^q \). The reason this works is because in the definition of the generalized boundary representation

\[
\pi^\#_t = (t + \omega|_t \Lambda)^{-1} \omega|_t = (t + \tilde{\phi}_t)^{-1} \omega|_t = (t + \tilde{\phi}_t)^{-1} \omega|_t = v_t^{-1} \Theta_t^{-1} \omega|_t
\]

one only has to compute the inverse above on the range of \( \omega \) which allows us to replace \( \phi_t \) by \( \tilde{\phi}_t \) in the above equation. Notice that in the last section \( L \) was \( B(\mathbb{C}^p) \) so the unit \( I_o \) of \( L \) was the unit \( I \) of \( B(\mathbb{C}^p) \). Now in applying the arguments of the lemmas and theorems 4.5 to 4.7 to our present situation we make the following changes. We replace the unit \( I \) of \( B(\mathbb{C}^p) \) by the unit \( I_o \) of \( L \). Note the unit \( I \) of \( B(\mathbb{C}^p \otimes L^2(0, \infty)) \) is not replaced so, for example, the unit \( I \) in \( \pi^\#_t(I) \) is
not replaced. However, the expression $\pi^\#_t(\Lambda)$ which is a shorten form of $\pi^\#_t(\Lambda(I))$ is replaced by $\pi^\#_t(\Lambda(I_o))$. Notice the inequality $\omega(I - \Lambda) \leq I$ translates to the inequality $\omega(I - \Lambda(I_o)) \leq I_o$ in our new setting and this inequality was proved in Theorem 5.9. Notice that in an expression like $\psi(I)$ the $I$ should be replaced by $I_o$ since $\psi$ is a map of $L$ into itself. We see that the statement involving $\psi$ in the last sentence of this theorem is simply the translation of the corresponding statement in the statement of Theorem 4.7. Note that in the statement

$$\psi(I_o) \geq \vartheta(I - \Lambda(I_o))$$

since $I_o \in L$ we can replace $\vartheta$ by $\tilde{\vartheta}$ since $I_o \in L$ so the inequality follows from the tilde calculations discussed above.

Now we address the results involving $\vartheta$ (without the tilde). First $\vartheta$ is defined as $\vartheta = \Theta^{-1}\omega$ so $\vartheta \in B(\mathbb{C}^p) \otimes \mathfrak{A}(\mathbb{C}^p)$. We show $\vartheta$ is completely positive. Since $\pi^\#_t = v_t^{-1}\Theta_t^{-1}\omega|_t$ we have $\vartheta|_t = v_t\Theta^{-1}\Theta_t\pi^\#_t$ so for $0 < t \leq s$ we have

$$\vartheta|_s = v_t\Theta^{-1}\Theta_t s|_s.$$ 

Now taking the limit as $t \to 0^+$ and using the fact that $\Theta^{-1}\Theta_t \to \iota$ as $t \to 0^+$ we have

$$\vartheta|_s = \lim_{t \to 0^+} v_t\pi^\#_t|_s$$

and since $\vartheta \in B(\mathbb{C}^p) \otimes \mathfrak{A}(\mathbb{C}^p)$ so $\vartheta|_s(A) \to \vartheta(A)$ as $s \to 0^+$ for $A \in \mathfrak{A}(\mathbb{C}^p)$ we have

$$\vartheta(A) = \lim_{t \to 0^+} v_t\pi^\#_t(A)$$

and since $\pi^\#_t$ is completely positive it follows that $\vartheta$ is completely positive. \hfill $\Box$

Our goal is to show that if $\omega$ is a $q$-weight map of index zero then $\omega$ has a $q$-subordinate where $L$ the range of $\omega$ is not only a Choi-Effros factor but a factor which means that if $P$ is the maximal support projection then $P \mathcal{L} = \mathcal{L}$. In order to prove this we will need to show that $\omega(\Lambda(I - P))$ is finite. Now if $E$ is a minimal $L$-projection (i.e. $E \star E = E$ and $E$ is minimal) then $P(E - E^2) = 0$ so $E - E^2 \leq I - P$ so if $\omega(\Lambda(I - P))$ is finite then $\omega(\Lambda(E - E^2))$ is finite for all minimal $L$-projections $E$. We will prove this but first we need a routine lemma for making norm estimates.

**Lemma 5.11.** Suppose $H$ is a finite dimensional Hilbert space and $L$ is a Choi-Effros algebra which is the range of a completely positive contractive idempotent map $\mathcal{L}$ of $B(H)$ into itself and for $A, B \in L$ the Choi-Effros product $A \star B = L(AB)$ and $I_o$ is the unit of $L$. Suppose $E \in \mathcal{L}$ is an hermitian projection by which we mean $E = L(E^*E)$. Suppose $T \in \mathcal{L}$ is positive. Then

$$\|T\| \leq \|E \star T \star E\| + \|(I_o - E) \star T \star (I_o - E)\|.$$ 

**Proof.** Assume the hypothesis and notation of the lemma. From the Choi-Effros theory we know that $L$ equipped with the $\star$-multiplication is a finite dimensional $C^*$-algebra and as such it has a faithful $\star$-representation $\pi$ on a finite dimensional Hilbert space $K$. Since $\pi$ is faithful $\pi$ preserves norms so $\|T\| = \|\pi(T)\|$. Then to prove the lemma we need only estimate the norm of $\pi(T)$. To simplify notation rather that writing $\pi(T)$ or $\pi(E)$ in our calculations we will simply write $T$ or $E$ (without the $\pi$). This means that in estimating the norm of $T$ we will simply consider $T$ to be a positive operator and $E$ a hermitian projection acting on a finite dimensional Hilbert space. Then to prove the lemma we need to prove the estimate

$$\|T\| \leq \|ETE\| + \|(I_o - E)T(I_o - T)\|.$$ 

Let

$$A = ETE, \quad B = ET(I - E), \quad B^* = (I - E)TE, \quad C = (I - E)T(I - E).$$

Note that $T = A + B + B^* + C$. Now let $H_1 = \text{Range}(E)$ and $H_2 = \text{Range}(I - E)$. We can now write $T$ as a $(2 \times 2)$-matrix

$$T = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$$

where the entries $T_{ij}$ of $T$ map $H_i$ into $H_j$. To prove the lemma we show $\|T\| \leq \|A\| + \|C\|$. 

Since $T \geq 0$ we have $(F,TF) \geq 0$ where $F = \{f,zg\}$ with $f \in H_1,g \in H_2$, and $z \in \mathbb{C}$ which yields 

$$(f,Af) + |z|^2(g,Cg) + 2Re(z(f,Bg)) \geq 0.$$ 

Note that if either $(g,Cg) = 0$ or $(f,Af) = 0$ otherwise you can find a value of $z$ so that the expression is less than zero. Now assuming $(g,Cg) \neq 0$ we find by setting $z = -(f,Bg)/(g,Cg)$ that 

$$(f,Af)(g,Cg) \geq 0.$$ 

There is a sequence of unit vectors $f_n,g_n \in H_1 \oplus H_2$ so that $(f_n,Bg_n) \to ||B||$. Then we have 

$$||A|| \cdot ||C|| \geq (f_n,Af_n)(g_n,Cg_n) \geq ||(f_n,Bg_n)||^2 \to ||B||^2$$ 

as $n \to \infty$ so $||A|| \cdot ||C|| \geq ||B||^2$. Now if $F = \{f,g\}$ we have 

$$(F,TF) = (f,Af) + (g,Cg) + 2Re((f,Bg))$$

$$\leq ||A|| \cdot ||f||^2 + ||C|| \cdot ||g||^2 + 2||B|| \cdot ||f|| \cdot ||g||$$

$$\leq ||A|| \cdot ||f||^2 + ||C|| \cdot ||g||^2 + 2||A|| \frac{1}{2} ||C||^\frac{1}{2} ||f|| \cdot ||g||$$

$$= (||A|| \frac{1}{2} ||f|| + ||C|| \frac{1}{2} ||g||)^2.$$ 

Maximizing this expression subject to the constraint $||f||^2 + ||g||^2 = 1$ we find the maximum occurs when 

$$||f|| = ||A|| \frac{1}{2} (||A|| + ||C||)^{-\frac{1}{2}} \quad \text{and} \quad ||g|| = ||C|| \frac{1}{2} (||A|| + ||C||)^{-\frac{1}{2}}$$ 

which gives $(F,TF) \leq ||A|| + ||C||$ and since $||T||$ is the supremum of $(F,TF)$ for all unit vectors $F \in H_1 \oplus H_2$ we have $||T|| \leq ||A|| + ||C||$. 



\textbf{Theorem 5.12.} Suppose $\omega$ is a $q$-weight map over $\mathbb{C}^p$ of index zero and $\mathcal{L}$ the range $\omega$ is a Choi-Effros factor of type $I_q$. Suppose $E$ is a minimal $\mathcal{L}$-projection so $E \in \mathcal{L}$ and $E \ast E = E$. Let $F$ be the support projection for $E$ so $F$ is smallest projection in $B(\mathbb{C}^p)$ with $F \geq E$. Then $\omega(\Lambda(F - E))$ is finite meaning there is a constant $K$ so that $||\omega||_t(\Lambda(F - E)) || \leq K$ for all $t > 0$. Note that $F - E \geq E - E^2$ so we have shown that $\omega(\Lambda(F - E^2))$ is finite.

\textbf{Proof.} Assume the hypothesis and notation of the theorem. Suppose $E$ is a minimal $\mathcal{L}$-projection. Assume $\psi$, $\vartheta$ and the $g_{ik}$ and $h_{ik}$ are defined as in Theorem 5.10. We will show $||\vartheta||_t(\Lambda(F - E)) ||$ is bounded for $t > 0$ and since $\omega = \psi^{-1} \vartheta$ we have $||\omega||_t(\Lambda(F - E)) || \leq ||\psi^{-1}|| \cdot ||\vartheta||_t(\Lambda(F - E)) ||$ so we will have proved the theorem. In the remainder of this proof we will assume $t > 0$.

Recall $F$ is the support projection for $E$ the smallest projection in $B(\mathbb{C}^p)$ with $F \geq E$ so $F$ is not necessarily in $\mathcal{L}$. We begin by obtaining estimates of $\Lambda(F - E)$ in terms of elements of $\mathcal{L}$. Since $I \geq F$ we have 

$$\Lambda(F - E) \leq \Lambda(I - E) = \Lambda(I - I_o) + \Lambda(I_o - E) \leq I - \Lambda(I_o) + \Lambda(I_o - E)$$ 

so we have 

$$\vartheta||_t(\Lambda(F - E)) \leq \vartheta||_t(I - \Lambda(I_o)) + \vartheta||_t\Lambda(I_o - E)$$

and since $\vartheta(I - \Lambda(I_o)) \leq \psi(I_o)$ we have 

$$\vartheta||_t(\Lambda(F - E)) \leq \vartheta||_t\Lambda(I_o - E) + \psi(I_o).$$ 

Next we get an estimate in terms of $\Lambda(E)$. Let $\lambda_i \in [0,1]$ for $i = 0,1,\cdots,m$ be the spectrum of $E$ in increasing order so $\lambda_0 = 0$ and $\lambda_m = 1$ so $\lambda_i$ is the smallest positive eigenvalue of $E$. Note the spectrum of $F - E$ consists of the numbers $\lambda_o = 0$ and $1 - \lambda_i$ for $i = 1,2,\cdots,m-1$. Note we do not include $\lambda_m$ since $\lambda_m = 1$ so $1 - \lambda_m = 0$ and this has already been listed in the spectrum of $F - E$. Let $\kappa$ be a positive real number which we will define shortly. Then the spectrum of $\kappa(E - F) = (\kappa + 1)E - F$ consists of the numbers $0$ and $(\kappa + 1)\lambda_i - 1$ for $i = 1,\cdots,m$. Now let 

$$\kappa = \lambda_1^{-1} - 1$$
and we see the spectrum of $\kappa E - (F - E)$ consist of the numbers 0 and $(\lambda_i/\lambda_1 - 1)$ for $i = 2, \ldots, m$. Note we do not include $i = 1$ since $(\lambda_1/\lambda_1 - 1) = 0$ and this has already been listed. Hence, the spectrum of $\kappa E - (F - E)$ is non negative so we have

$$\kappa E \geq F - E$$

and, hence, we have

$$\vartheta [I \Lambda (F - E) \leq \kappa \vartheta [I \Lambda (E).$$

Now that we have above two estimates for $\vartheta [I \Lambda (F - E)$ in terms of elements of $\mathcal{L}$ we can use the formula for $\vartheta$ in terms of the $g'$s and $h'$s in Theorem 5.10. First note that the tilde mappings are mapping of $B(\mathbb{C}^q)$ into itself and so in our calculation we will think of $B(\mathbb{C}^q)$ as $(q \times q)$-matrices. Recalling the formula for $\vartheta$ in Theorem 5.10 we find that for $A \in B(\mathbb{C}^q)$ representing an element of $\mathcal{L}$

$$\tilde{\vartheta} [I \Lambda (A) = w_t A + Y_t A + \lambda Y_t^* + \rho [I \Lambda (A)$$

where $\rho$ is completely positive and uniformly bounded and $1/w_t \to 0$ as $t \to 0+$ and $Y_t \in B(\mathbb{C}^q)$ is given by

$$(Y_t)_{ij} = \int_{t}^{\infty} e^{-x} (h_{ik}) \overline{g_k (x)} dx$$

and from the condition on the $h'$s it follows that $Y_t$ is of trace zero. There are further properties of $\rho [I \Lambda$ but we only need that it is bounded meaning there is a constant $K$ so that $\rho [I \Lambda (I_\omega) \leq K I_\omega$ for all $t > 0$. Now since $\vartheta [I \Lambda (F - E) \leq \kappa \vartheta [I \Lambda (E)$ we have

$$\vartheta [I \Lambda (F - E) \leq \kappa w_t E + \kappa Y_t E + \kappa E Y_t^* + \kappa \rho [I \Lambda (E)$$

and since $\vartheta [I \Lambda (F - E) \leq \vartheta [I \Lambda (I_\omega - E) + \psi (I_\omega)$ we have

$$\vartheta [I \Lambda (F - E) \leq w_t (I_\omega - E) + Y_t (I_\omega - E) + (I_\omega - E) Y_t^* + \rho [I \Lambda (I_\omega - E) + \psi (I_\omega).$$

Note $\vartheta [I \Lambda (F - E) \in \mathcal{L}$ where $\mathcal{L}$ is the Choi-Effros algebra. We will estimate its norm using the previous lemma. Then sandwiching the bottom inequality between $E$ on the right and left using the Choi-Effros product $A \ast B = L(AB)$ we have

$$E \ast \vartheta [I \Lambda (F - E) \ast E \leq E \ast \tilde{\vartheta} [I \Lambda (I_\omega - E) \ast E + E \ast \psi (I_\omega) \ast E \leq (K + \| \psi (I_\omega) \|) E$$

and sandwiching the top inequality between $(I_\omega - E)$ on the right and left we have

$$(I_\omega - E) \ast \vartheta [I \Lambda (F - E) \ast (I_\omega - E) \leq (I_\omega - E) \ast \kappa \tilde{\vartheta} [I \Lambda (E) \ast (I_\omega - E) \leq \kappa K (I_\omega - E).$$

Now if $0 \leq A \leq B$ we have $0 \leq \| A \| \leq \| B \|$ which gives us the estimates

$$\| E \ast \vartheta [I \Lambda (F - E) \ast E \| \leq (K + \| \psi (I_\omega) \|)$$

and

$$\| (I_\omega - E) \ast \vartheta [I \Lambda (F - E) \ast (I_\omega - E) \| \leq \kappa K$$

so by the previous lemma we have

$$\| \vartheta [I \Lambda (F - E) \| \leq (1 + \kappa) K + \| \psi (I_\omega) \|$$

and this bound in independent of $t$. 

In the previous theorem we showed that $\omega(\Lambda (E - E^2))$ is finite for $E$ a minimal $\mathcal{L}$-projection. Next we will show that if $\omega(\Lambda (E - E^2))$ is finite for all minimal $\mathcal{L}$-projections then $\omega(\Lambda (I - P))$ is finite where $P$ is the maximal support projection for $\mathcal{L}$. Our strategy is as follows. Note that $P(I - I_\omega) = 0$ and $P(E - E^2) = 0$ for all minimal projections $E \in \mathcal{L}$. We define a projection $Q$ as the largest projection in $B(\mathbb{C}^q)$ with these properties and then show that $Q = P$. We will restrict our attention to the case where $\mathcal{L}$ is a Choi-Effros factor but we believe the results are valid in the more general case.

Here we introduce some notation that we will be using for calculations.

**Definition 5.13.** If $A \in B(\mathbb{C}^q)$ is positive we denote by

$$A^+ = \lim_{n \to \infty} A^{1/n}$$

the support projection of $A$. We denote by $A^-$ the projection on the subspace spanned by the eigenvectors of $A$ of eigenvalue $\lambda \geq 1$. 


Here we present various properties of $A^+$ and $A^-$ that we will use in our calculations. We assume $A \in B(\mathbb{C}^p)$ and $A \geq 0$. First note that since $\mathbb{C}^p$ is finite dimensional $A^+$ and $A^-$ can be expressed as polynomials in $A$. Note that if $C$ commutes with $A$ then $C$ commutes with $A^+$ and $A^-$. Note that if $Q$ is a projection then $QA = 0$ if and only if $QA^+ = 0$. Note that if $A \geq 0$ and $\|A\| \leq 1$ then

$$A^- = \lim_{n \to \infty} A^n.$$ 

Note if $A_1, A_2, \ldots, A_m$ are positive operators in $B(\mathbb{C}^p)$ then the projection onto the subspace spanned by the ranges of the $A_i$ is $(A_1 + A_2 + \cdots + A_m)^+$. If $A_i \in B(\mathbb{C}^p)$ and $0 \leq A_i \leq I$ for $i = 1, \ldots, m$ then the projection onto the intersection of the ranges of the $A_i^\perp$ is $((1/m)(A_1 + A_2 + \cdots + A_m))^\perp$.

**Lemma 5.14.** Suppose $Q \in B(\mathbb{C}^p)$ is an hermitian projection and $A \in B(\mathbb{C}^p)$ is positive then $QA = 0$ if and only if $QAQ = 0$. Suppose further that $A \in B(\mathbb{C}^p)$ is positive and of norm one, $(A \geq 0$ and $\|A\| = 1)$. Then the following statements are equivalent.

1. $Q(A - A^2) = 0$
2. $Q(A - A^2)^+ = 0$
3. $Q(A^+ - A^-) = 0$
4. $Q(A - A^-) = 0$
5. $Q(A^+ - A) = 0$

**Proof.** Assume the hypothesis and notation of the theorem. We prove $QA = 0$ if and only if $QAQ = 0$. Now if $QA = 0$ then multiplying by $Q$ on the right we have $QAQ = 0$. Now if $QAQ = 0$ we have $XX^* = 0$ where $X = QA^{\frac{1}{2}}$ from which we conclude that $X = 0$ from which we conclude $XA^\frac{1}{2} = QA = 0$.

Now to prove the equivalence of the five conditions of the lemma we further assume that $A$ is of norm one. From the spectral decomposition of $A$ we know

$$A = \sum_{i=1}^{m} \lambda_i F_i$$

with $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_{m-1} < \lambda_m = 1$ and the $F_i$ are mutually orthogonal projections. Then

$$A - A^2 = \sum_{i=1}^{m-1} (\lambda_i - \lambda_i^2) F_i \quad (A^+ - A^-) = (A - A^2)^+ = \sum_{i=1}^{m-1} F_i$$

and

$$A - A^- = \sum_{i=1}^{m-1} \lambda_i F_i \quad A^+ - A = \sum_{i=1}^{m-1} (1 - \lambda_i) F_i.$$ 

Since in all cases the coefficients of the $F_i$ are positive we have $Q$ times any of these expressions is zero if and only if $QF_i = 0$ for $i = 1, \ldots, m-1$. Hence, the five conditions above are equivalent. $\square$

**Lemma 5.15.** Suppose $E \in B(\mathbb{C}^p)$ is positive and of norm one, $(E \geq 0$ and $\|E\| = 1)$ and $T \in B(\mathbb{C}^p)$ satisfies $0 \leq T \leq E$. Suppose $Q \in B(\mathbb{C}^p)$ is an hermitian projection. If $Q(E - E^-) = 0$ and $Q(E - T) = 0$ then $Q(E^+ - T^-) = Q(E - T^-) = 0$. Note if $\|T\| < 1$ then $T^- = 0$ so $QE = QE^+ = 0$.

**Proof.** Assume the hypothesis an notation of the lemma. We show there is a constant $b \geq 1$ so that $b(E^+ - T) \geq E^+ - T^-$. Note that the range of $E^-$ is the set of vectors $f \in \mathbb{C}^p$ so that $(f, Ef) = (f, f)$ and the range of $T^-$ is the set of vectors $f \in \mathbb{C}^p$ so that $(f, Tf) = (f, f)$. Since $E \geq T$ the range of $T^-$ is contained in the range of $E^-$ so $T^- \leq E^-$. Then $E^+ - T^- = T^-$. Since

$$I \geq E^+ \geq E \geq E^- \geq T^- \quad \text{and} \quad I \geq E^+ \geq E \geq T \geq T^-$$

and $T^-$ is a projection so its eigenvalues are zero and one it follows that $T^-$ commutes with $E^+, E, E^-, T$ and $T^-$ and the product of any of these operators with $T^-$ is $T^-$. Note that $E^+$ is a unit for all of these operators and $E^+ - T^-$ is a unit for the difference of any two of these operators.
Lemma 5.16. Suppose \( \mathcal{L} \subset B(\mathbb{C}^p) \) is a Choi-Effros factor of type \( I_q \) and \( E_{ij} \) are a complete set of matrix units for \( \mathcal{L} \). Then if \( i \neq j \)

\[
E_{ii}E_{jk} = 0 \quad \text{and} \quad E_{kj}E_{ii} = 0.
\]

Proof. Assume the hypothesis and notation of the lemma. First we prove \( E_{11}E_{22} = 0 \). Since \( \|E_{11} + E_{22}\| = 1 \) if \( f \in \mathbb{C}^p \) is a unit vector so that \( E_{11}f = f \) we have \( (f, (E_{11} + E_{22})f) = 1 + (f, E_{22}f) \leq 1 \) so \( (f, E_{22}f) = 0 \) and \( f \) is orthogonal to the range of \( E_{22} \). Hence \( E_{22}E_{11} = 0 \) so \( E_{11}E_{22} = 0 \). Let \( L \) be a completely positive contractive idempotent with range \( \mathcal{L} \) then \( E_{22} = L(E_{22}) = L(E_{21}E_{12}) \) and from the Schwarz inequality for completely positive maps we have (note \( \|L\| = 1 \))

\[
E_{2k}E_{k2} = L(E_{k2})L(E_{k2}) \leq L(E_{2k}E_{k2}) = E_{22}
\]

so \( E_{11}E_{2k}E_{k2} = 0 \). By the polar decomposition in a finite dimensional Hilbert space we have \( E_{2k} = (E_{2k}E_{k2})^{\frac{1}{2}}U \) where \( U \) is a unitary so the range of \( E_{2k} \) is the range of \( (E_{2k}E_{k2})^{\frac{1}{2}} \) and since \( \mathbb{C}^p \) is finite dimensional the range of \( (E_{2k}E_{k2})^{\frac{1}{2}} \) is the range of \( E_{2k}E_{k2} \). Since \( E_{11}f = 0 \) in the range of \( E_{2k}E_{k2} \) it follows that \( E_{11}f = 0 \) for \( f \) in the range of \( E_{2k}E_{k2} \). Hence, \( E_{11}E_{2k} = 0 \). The proof for general \( i, j, k \) with \( i \neq j \) follows from replacing 1 and 2 by \( i \) and \( j \). Taking adjoints we obtain the second equalities. \( \Box \)

Lemma 5.17. Suppose \( \mathcal{L} \subset B(\mathbb{C}^p) \) is a Choi-Effros factor of type \( I_q \). Let \( Q \) be the largest projection so that \( Q(I - I_o) = 0 \) and \( Q(E - E^2) = 0 \) for all minimal \( \mathcal{L} \)-projections \( E \in \mathcal{L} \). Then if \( E_{ij} \) are a complete set of matrix units for \( \mathcal{L} \) then \( Q(E_{ii} - E_{ij}E_{ji}) = 0 \) and \( Q(E_{ii}^+ - (E_{ij}E_{ji})^-) = 0 \) for \( i, j = 1, \ldots, q \)

Proof. Assume the hypothesis and notation of the theorem. We prove the lemma for \( i, j = 1, 2 \). Suppose \( x \) and \( \theta \) are real numbers with \( x^2 \leq 1 \) (where we think of \( x \) as a real variable and \( \theta \) as a constant) and

\[
E_x = s_x^2E_{11} + x e^{i\theta} s_x E_{12} + x e^{-i\theta} s_x E_{21} + x^2 E_{22} \quad \text{with} \quad s_x = \sqrt{1 - x^2}.
\]

Since \( E_x \) is a minimal \( \mathcal{L} \)-projection we have \( Q(E_x - E_x^2) = 0 \). Making a power series expansion in \( x \) we have

\[
Q(E_x - E_x^2) = Q(A_0 + x A_1 + x^2 A_2 + \cdots) = 0.
\]

Since this expression is real analytic in \( x \) (meaning it has a convergent power series expansion for \( -1 < x < 1 \)) it follows that each of the coefficients \( QA_n \) must vanish so \( QA_2 = 0 \) and calculating \( A_2 \) we find

\[
QA_2 = Q(2E_{11}^2 - E_{11} - E_{12}E_{21} + E_{22} - E_{21}E_{12} - E_{11}E_{22} - E_{22}E_{11} - e^{2i\theta} E_{12}E_{12} - e^{-2i\theta} E_{21}E_{21}) = 0.
\]

Averaging over \( \theta \) from \( \theta = -\pi \) to \( \theta = +\pi \) the last two terms average to zero. From the pervious lemma we have \( Q(E_{11}E_{22}) = Q(E_{22}E_{11}) = 0 \) so we have

\[
Q(2E_{11}^2 - E_{11} - E_{12}E_{21} + E_{22} - E_{21}E_{12}) = 0.
\]

Since \( E_{11} \) is a minimal projection in \( \mathcal{L} \) we also have \( Q(E_{11} - E_{11}^2) = 0 \) and adding twice this equation to the above equation and multiplying by \( Q \) on the right we find

\[
Q(E_{11} - E_{12}E_{21})Q + Q(E_{22} - E_{21}E_{12})Q = 0.
\]

As we saw in the previous lemma we have from the Schwarz inequality \( E_{22} \geq E_{21}E_{12} \) and \( E_{11} \geq E_{12}E_{21} \) so the expression above is the sum of two positive terms so both terms must be zero. From
Lemma 5.14 we know that if \( A \in B(\mathbb{C}^p) \) is positive and \( QAQ = 0 \) then \( QA = 0 \) from which we conclude

\[
Q(E_{11} - E_{12}E_{21}) = 0 \quad \text{and} \quad Q(E_{22} - E_{21}E_{12}) = 0.
\]

Since \( Q(E_{11} - E_{12}^2) = Q(E_{22} - E_{21}^2) = 0 \) we have proved the lemma for \( i, j = 1, 2 \). The argument we have just given will apply to any two values for \( i \) and \( j \) from 1 to \( q \). Now if \( E = E_{ik} \) and \( T = E_{ij}E_{ji} \) then we have shown \( Q(E - E^2) = 0 \) and \( Q(E - T) = 0 \). Lemma 5.14 shows that \( Q(E - E^2) = 0 \) implies \( Q(E - E^-) = 0 \) so we now have shown that \( Q(E - E^-) = 0 \) and \( Q(E - T) = 0 \) and Lemma 5.15 shows these two conditions imply \( Q(E^+ - T^-) = 0 \). Hence, we have shown \( Q(E_{ik}^+ - (E_{ij}E_{ji})^-) = 0 \). \( \square \)

In the next lemma we define a projection \( Q_1 \) which will turn out to be the maximal support projection. The lemma is also a definition.

**Lemma 5.18.** Suppose \( \mathcal{L} \) is a Choi-Effros factor of type \( I_q \) and \( E_{ij} \) are a complete set of matrix units for \( \mathcal{L} \) and let \( I_o \) be the unit of \( \mathcal{L} \). Let \( Q \) be the largest projection so that \( Q(I - I_o) = 0 \) and \( Q(E - E^2) = 0 \) for every minimal projection \( E \in \mathcal{L} \). For each \( i = 1, \cdots, q \) let \( T_i \) be the projection onto the intersection of the ranges of \( (E_{ij}E_{ji})^- \) for \( j = 1, \cdots, q \) and let \( Q_1 = T_1 + T_2 + \cdots + T_q \). Then \( Q_1 \geq Q \).

**Proof.** Assume the hypothesis and notation of the lemma. From the previous lemma we have \( Q(E_{11} - (E_{11}E_{11})^-) = 0 \) for \( i = 1, \cdots, q \) and, hence,

\[
Q(E_{11} - R_1) = 0 \quad \text{where} \quad R_1 = \frac{1}{q} \sum_{i=1}^{q} (E_{11}E_{i1})^-
\]

and then we have from Lemma 5.15 that \( Q(E_{11} - R_1) = 0 \) and \( R_1^- \) is \( T_1 \) so \( Q(E_{11} - T_1) = 0 \). Repeating this argument with 1 replaced by \( i \) yields the result that \( Q(E_{ii} - T_i) = 0 \) for \( i = 1, \cdots, q \). Since the sum of the \( E_{ii} \) is \( I_o \) we have \( Q(I_o - Q_1) = 0 \) and since \( Q(I - I_o) = 0 \) it follows that \( Q(I - Q_1) = 0 \). Hence \( Q_1 \geq Q \). \( \square \)

**Theorem 5.19.** Suppose \( \mathcal{L} \subset B(\mathbb{C}^p) \) is a Choi-Effros factor of type \( I_q \) with unit \( I_o \) and \( Q \) is the largest projection so that \( Q(I - I_o) = 0 \) and \( Q(E - E^2) = 0 \) for all minimal projections \( E \in \mathcal{L} \). Then \( Q = P \) where \( P \) is the maximal support projection for \( \mathcal{L} \).

**Proof.** Assume the hypothesis and notation of the theorem. Let \( E_{ij} \) be a complete set of matrix units for \( \mathcal{L} \) and let \( T_i \) be the projection onto the intersection of the ranges of \( (E_{ij}E_{ji})^- \) for \( j = 1, \cdots, q \). Let \( g_1, g_2, \cdots, g_r \) be an orthonormal basis for the range of \( T_i \). Let \( f_{ik} = E_{i1}g_k \) for \( i = 1, \cdots, q \) and \( k = 1, \cdots, r \). We show that if \( i \neq j \) then \( E_{mi}f_{jk} = 0 \). First note that since \( T_1g_k = g_k \) we have \( (g_k, E_{1j}E_{j1}g_k) = 1 \) so \( E_{1j}E_{j1}g_k = g_k \) so

\[
1 = (E_{j1}g_k, E_{j1}E_{j1}E_{j1}g_k) \leq (E_{j1}g_k, E_{jj}E_{j1}g_k) \leq 1
\]

so \( E_{jj}g_k, E_{j1}E_{j1}g_k = 1 \) so \( E_{j1}^*E_{j1}g_k = E_{j1}g_k \). Then we have

\[
E_{mi}f_{jk} = E_{mi}E_{i1}g_k = E_{mi}E_{jj}E_{j1}g_k
\]

and since \( i \neq j \) we have from Lemma 5.16 that \( E_{mi}E_{jj}^- = 0 \) and, hence, \( E_{mi}f_{jk} = 0 \) for \( i \neq j \).

Now since \( f_{ik} = E_{i1}g_k \) we have for \( i \neq j \) that

\[
(f_{ik}, f_{jm}) = (E_{i1}g_k, f_{jm}) = (g_k, E_{i1}f_{jm}) = 0
\]

for \( k, m = 1, \cdots, r \) and for \( i = j \) we have

\[
(f_{ik}, f_{im}) = (E_{i1}g_k, E_{i1}g_m) = (g_k, E_{i1}E_{i1}g_m) = (g_k, g_m) = \delta_{km}
\]

so the \( f_{ik} \) form an orthonormal set of vectors.

Now let

\[
C = \sum_{i,j=1}^{q} E_{ij}.
\]
Note $C$ is positive so if $f = z_1 f_{1k} + z_2 f_{2k} + \cdots + z_q f_{qk}$ for $z_i$ complex numbers then $(f, Cf) \geq 0$ and calculating this using the above relations we find

$$(f, Cf) = \sum_{i,j=1}^{q} \overline{z}_i z_j (f_{ik}, E_{ij} f_{jk})$$

so $(f, Cf) \geq 0$ for all choices of the $z_i$ if and only if the matrix $A$ with matrix elements

$$a_{ij} = (f_{ik}, E_{ij} f_{jk})$$

is positive. Now $a_{ii} = a_{i1} = 1$ and $a_{ii} \leq 1$ for $i = 1, \cdots, q$. Looking at the $(2 \times 2)$-matrix

$$\begin{bmatrix} a_{11} & a_{1i} \\ a_{i1} & a_{ii} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & a_{ii} \end{bmatrix} \geq 0$$

and given $a_{ii} \leq 1$ we see the determinant is non-negative if and only if $a_{ii} = 1$ and looking at the matrix

$$\begin{bmatrix} a_{11} & a_{1i} & a_{1j} \\ a_{i1} & a_{ii} & a_{ij} \\ a_{j1} & a_{ji} & a_{jj} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & a_{ij} \\ 1 & a_{ji} & 1 \end{bmatrix} \geq 0$$

we see this matrix has a non-negative determinant if and only if $a_{ij} = a_{ji} = 1$. It then follows that since $A$ is positive we have $a_{ij} = 1$ for all $i$ and $j$. Note this is true for all $k = 1, \cdots, r$. Hence, we have

$$E_{ij} f_{mk} = \delta_{jm} f_{ik}$$

so if $Q_1$ is the projection onto the span of the $f_{ik}$ then $Q_1$ commutes with the $E_{ij}$ and $L$ restricted to the range of $Q_1$ is an actual type $I_1$ factor. 

Note we have $E_{jk} E_{i1} = Q_1 E_{21}$ and $T_2$ which is the largest projection so that $T_2 \leq (E_{21} E_{i2})^{-1}$ for $i = 1, \cdots, q$ satisfies the relation $T_2 \geq Q_1 E_{21}$. Note $\|E_{21} T_1 f\| = \|T_1 f\|$ so it follows that $\dim(E_{21} T_1 E_{i2}) = \dim(T_1)$. Since $T_2 \geq E_{21} T_1 E_{i2}$ it follows that $\dim(T_2) \geq \dim(T_1)$. But we can repeat all of the above arguments with the indices 1 and 2 interchanged and reach the conclusion that $\dim(T_1) \geq \dim(T_2)$ so we conclude that $\dim(T_1) = \dim(T_2)$. Since $T_2 \geq E_{21} T_1 E_{i2}$ and $\dim(E_{21} T_1 E_{i2}) = \dim(T_1)$ we have $T_2 = E_{21} T_1 E_{i2}$. And replacing 1 and 2 by indices $i$ and $j$ we have $T_i = E_{ij} T_j E_{ji}$.

Now recall that $Q_1$ is the projection onto the span of the $f_{ik}$ for $i = 1, \cdots, q$ and $k = 1, \cdots, r$. Now $T_i$ is the projection onto the span of the $f_{ik}$ for $k = 1, \cdots, r$ so

$$T_1 + T_2 + \cdots + T_q = Q_1$$

and $Q_1$ is the projection $Q_1$ of Lemma 5.18 so $Q_1 \geq Q$ and since $P$ satisfies all the conditions defining $Q$ we have $Q \geq P$. Now we will produce a completely positive contractive idempotent map of $B(CP)$ into itself with range $L$ and support projection $Q_1$. Let

$$L(A) = \frac{1}{r} \sum_{i,j=1}^{q} \sum_{k=1}^{r} (f_{ik}, A f_{jk}) E_{ij}$$

for $A \in B(CP)$. Note $L(E_{ij}) = E_{ij}$ so it is clear that the range of $L$ is $L$ and $L(L(A)) = L(A)$, $L(I) = I_o$ and $Q_1$ is the smallest projection so that $L(Q_1) = I_o$. Now let $F_k$ be the projection onto the vectors $f_{ik}$ for $i = 1, \cdots, q$. Now $F_k A F_k$ is a $(q \times q)$ matrix with matrix elements $(f_{ik}, A f_{jk})$. Clearly the mapping $A \mapsto F_k A F_k$ is completely positive and, hence, the sum of the maps

$$\phi(A) = \frac{1}{r} \sum_{k=1}^{r} F_k A F_k$$

is a completely positive map from $B(CP)$ to the $(q \times q)$-matrices and since the map from $(q \times q)$ matrices $A = \{a_{ij}\}$ given by

$$\psi(A) = \sum_{i,j=1}^{q} a_{ij} E_{ij}$$

is completely positive it follows that the map $L = \psi \phi$ is completely positive. Hence, $Q_1$ is the support projection for a completely positive contractive idempotent mapping of $B(CP)$ into itself.
with range $\mathcal{L}$. Hence, by Theorem 5.2 we have $P \geq Q_1$. Hence, we have $Q_1 \geq Q \geq P \geq Q_1$ so $P = Q$.

**Theorem 5.20.** Suppose $\omega$ is a $q$-weight map over $\mathbb{C}^p$ of index zero and $\mathcal{L}$ the range $\omega$ is a Choi-Effros factor of type $I_q$ and suppose $P$ is the maximal support projection for $\mathcal{L}$. Then $\omega(\Lambda(I-P))$ is finite meaning there is a constant $K$ so that $\|\omega|_{\Lambda(I-P)}\| \leq K$ for all $t > 0$.

**Proof.** Assume the hypothesis and notation of the theorem. Then from the previous theorem we know that $P$ is the largest projection so that $P(I-I_o) = 0$ and $P(E - E^2) = 0$ for all minimal projections. This means that $I - P$ is the projection onto the space spanned by the ranges of $I - I_o$ and $E - E^2$ for all minimal projections $E \in \mathcal{L}$. If $I - I_o \neq 0$ let $\lambda$ be the smallest positive eigenvalue of $I - I_o$ otherwise let $\lambda = 1$. Then $\lambda^{-1}(I - I_o) \geq (I - I_o)^+ = R_o$. If $E - E^2 \leq R_o$ for all minimal projections then $R_o = I - P$. If this is not the case then there is a minimal projection $E_1$ so that this is not true. Then let $R_1 = (R_o + (E_1 - E_1^2)^+)^+$ and note $R_1 \leq \lambda^{-1}(I - I_o) + (E_1 - E_1^2)^+$. If $E - E^2 \leq R_1$ for all minimal projections $E$ then $R_1 = I - P$ and if not we can find a minimal projection $E_2$ and form the bigger projection $R_2 = (R_1 + (E_2 - E_2^2)^+)^+$ and note $R_2 \leq \lambda^{-1}(I - I_o) + (E_1 - E_1^2)^+ + (E_2 - E_2^2)^+$. We can continue this process each time increasing the dimension of $R_i$ by at least one and since $\mathbb{C}^p$ is finite dimensional this process must terminate after a finite number of steps. Then we have

$$I - P \leq \lambda^{-1}(I - I_o) + (E_1 - E_1^2)^+ + (E_2 - E_2^2)^+ + \cdots + (E_m - E_m^2)^+. $$

For the rest of this proof when write $\|\omega(\Lambda(A))\| \leq K$ we mean $\|\omega|\Lambda(\Lambda(A))\| \leq K$ for all $t > 0$. Now from Theorem 5.9 we know $\omega(I - \Lambda(I_o)) \leq I_o$ so we know $\omega(I - \Lambda(I) + \lambda(I - I_o)) \leq I_o$ and since $I \geq \Lambda(I)$ we have $\|\omega(\Lambda(I_o))\| \leq 1$. From Theorem 5.12 we know there are constants $\lambda_i$ so that $\|\omega(\Lambda(E_i^+ - E_i))\| \leq K_i$ for $i = 1, \ldots, m$. Let $\lambda_i$ be the largest eigenvalue of $E_i$ that is less than one and let $C_i = K_i/(1 - \lambda_i)$. Since $(E_i - E_i^2)^+ \geq (1 - \lambda_i)^{-1}(E_i^+ - E_i)$ it follows that $\|\omega(\Lambda((E_i - E_i^2)^+))\| \leq C_i$ and, hence, we have

$$\|\omega(\Lambda(I-P))\| \leq \lambda^{-1} + C_1 + C_2 + \cdots + C_m. $$

We remark that if we were being graded on how good a bound we have obtained we would get a pretty low grade. Fortunately we only need a bound as the next lemma shows.

**Lemma 5.21.** Suppose $\omega$ is a $q$-weight map over $\mathbb{C}^p$ of index zero and $\mathcal{L}$ the range $\omega$ is a Choi-Effros factor of type $I_q$ and suppose $P$ is the maximal support projection for $\mathcal{L}$. Then there is a $q$-subordinate $\eta$ (i.e. $\omega \geq q \eta$) with range $\mathcal{L}$ so that $\|\eta(\Lambda(I-P))\| \leq \frac{1}{2}$.

**Proof.** Assume the hypothesis and notation of the lemma. From the previous theorem we there is a constant $K$ so that $\|\omega|\Lambda(I-P)\| \leq K$ for all $t > 0$. Assume $K$ is the least such constant. If $K \leq \frac{1}{2}$ then $\eta = \omega$ satisfies the conclusion of the theorem so we assume $K > \frac{1}{2}$. Note $\omega|\Lambda(I-P)$ is non increasing in $t$. Hence, there is a $t_o > 0$ so that $t > t_o$ then $\omega|\Lambda-I(\Lambda(I-P)) \leq (1/4)I$ for $0 < t < t_o$. Let $\pi_t^\#$ be the generalized boundary representation of $\omega$. For $t > 0$ let

$$\eta_t = (t - \lambda t \pi_t^\#)^{-1} \lambda t \pi_t^\#$$

where $\lambda_t$ for $0 < t < t_o$ is the largest number in $(0, 1]$ so that $\|\eta_t|\Lambda(I-P)\| \leq 1/4$. Let $\eta_t$ be a weak limit point of $\eta_t$ as $t \to 0+$. From Theorem 5.6 we know that $\eta$ is a $q$-subordinate of $\omega$. From the construction of $\eta_t$ we have $\|\eta_t|\Lambda(I-P)\| \leq 1/4$. Since $\eta \leq \omega$ we have $\|\eta|\Lambda-I(\Lambda(I-P)) \leq (1/4)I$ for $0 < t < t_o$ so we have $\|\eta|\Lambda(I-P)\| \leq \frac{1}{2}$ for all $t > 0$.

Note if $\eta = 0$ then $\|\eta|\Lambda(I-P)\| = 0$ for all $t > 0$ and recalling how $\lambda_t$ were chosen this implies $K \leq \frac{1}{2}$ and for this case the proof is trivial. From the construction of $\eta$ it is clear that the range of $\eta_t$ is contained in $\mathcal{L}$. Now let $\psi_t^\#$ be the generalized boundary representation of $\eta_t$. Since $L_t = \pi_t^\# \Lambda \geq \psi_t^\# \Lambda = L_t^\#$ for $t > 0$ so taking a sequence $t_k \to 0+$ so that both $L_{t_k}$ and $L_{t_k'}$ converge to limits $L$ and $L'$ we have $L \geq L'$. Now $L$ restricted to $\mathcal{L}$ is the identity map and since the identity map is pure as a completely positive map we have $L'(A) = \Lambda(L(A)$ for $A \in \mathcal{L}$ with $0 \leq \lambda \leq 1$ and since $L' \neq 0$ we have $\lambda > 0$. (Actually $\lambda = 1$.) Hence, the range of $L'$ is $\mathcal{L}$ so the range of $\eta$ contains $\mathcal{L}$ so the range $\eta$ is $\mathcal{L}$. □
We remark in the previous lemma we did not specifically use the fact that $P$ was the maximal support projection. The properties of $P$ that we did use were the fact that $P \in \mathcal{L}'$ and $\|\omega(\Lambda(I - P))\| < \infty$. This means the conclusions of Lemma 5.21 remain valid if we only assume these weaker conditions on $P$.

**Theorem 5.22.** Suppose $\omega$ is a $q$-weight map of index zero over $\mathbb{C}^p$ with range $\mathcal{L}$ which is a Choi-Effros factor of type $I_q$ with $P$ the maximal support projection for $\mathcal{L}$. Then there is a $q$-subordinate $\eta$ of $\omega$ so that the range of $\eta$ is $\mathcal{L}_\eta = P\mathcal{L}$. It follows that Choi-Effros product for $\mathcal{L}_\eta$ is the ordinary operator product. Furthermore, it follows that if $\omega$ is a $q$-pure $q$-weight map of index zero over $\mathbb{C}^p$ then $\mathcal{L}$ the range of $\omega$ is a factor and the unit $I_\omega$ of $\mathcal{L}$ equals the maximal support projection of $\mathcal{L}$ (i.e. the range of $\omega$ is a Choi-Effros factor and $A \ast B = AB$ for $A, B \in \mathcal{L}$ and $I_\omega = P$).

**Proof.** Assume the hypothesis and notation of the theorem. We make a change in notation. We replace $\omega$ in the statement of the theorem by $\omega'$. Then by the previous lemma we know that $\omega'$ has a $q$-subordinate $\omega$ with range $\mathcal{L}$ so that $\|\omega\Lambda(I - P)\| \leq \frac{1}{2}$. Since any $q$-subordinate of $\omega$ is a $q$-subordinate of $\omega'$ we only need to prove the theorem for $\omega$ where $\omega$ now satisfies the additional the requirement that $\|\omega\Lambda(I - P)\| \leq \frac{1}{2}$.

We show how to construct $\eta$. Suppose $\pi_t^{\#}$ is the generalized boundary representation of $\omega$ and let $\psi_t^{\#} = P\pi_t^{\#}$ for $t > 0$ and let

$$\eta_t = (t - \psi_t^{\#}\Lambda)^{-1}\psi_t^{\#}.$$  

Then we have

\[
\eta_t = (t - P\pi_t^{\#}\Lambda)^{-1}P\pi_t^{\#} = (t - P\pi_t^{\#}\Lambda)^{-1}P(t + \omega_t\Lambda)^{-1}\omega\big|_t \\
= (t - P\pi_t^{\#}\Lambda)^{-1}P(t - \pi_t^{\#}\Lambda)\omega\big|_t \\
= P(t - P\pi_t^{\#}\Lambda)^{-1}((t - P\pi_t^{\#}\Lambda) - (I - P))\omega\big|_t \\
= P\omega\big|_t - P(t - P\pi_t^{\#}\Lambda)^{-1}(I - P)\omega\big|_t \\
= (P - P(t - P\pi_t^{\#}\Lambda)^{-1}(I - P))\omega\big|_t = (\xi - \zeta_t)\omega\big|_t
\]

where for $A \in B(\mathbb{C}^p)$ the maps $\xi$ and $\zeta_t$ are given by $\xi(A) = PAP$ and

\[
\zeta_t(A) = P(t - P\pi_t^{\#}\Lambda)^{-1}((I - P)A(I - P)) \\
= P\pi_t^{\#}\Lambda(t - P\pi_t^{\#}\Lambda)^{-1}((I - P)A(I - P)) \\
= (P\pi_t^{\#}\Lambda + (P\pi_t^{\#}\Lambda)^2 + \cdots)((I - P)A(I - P)).
\]

Note $\zeta_t$ is completely positive. We estimate the norm $\|\zeta_t\| = \|\zeta_t(I)\|$. Since $\pi_t^{\#}\Lambda - P\pi_t^{\#}\Lambda$ is a completely positive map if we replace $P\pi_t^{\#}$ by $\pi_t^{\#}$ in the above formula for $\zeta_t(A)$ with $A = I$ we obtain the estimate

\[
\zeta_t(I) = (P\pi_t^{\#}\Lambda + (P\pi_t^{\#}\Lambda)^2 + \cdots)((I - P)I(I - P)) \\
\leq P(\pi_t^{\#}\Lambda + (\pi_t^{\#}\Lambda)^2 + \cdots)(I - P) \\
= P(t - \pi_t^{\#}\Lambda)^{-1}\pi_t^{\#}\Lambda(I - P) = P\omega\big|_t\Lambda(I - P)
\]

so we have $\|\zeta_t(I)\| \leq \|P\omega|\Lambda(I - P)\| \leq \|\omega|\Lambda(I - P)\| \leq \frac{1}{2}$ for all $t > 0$. Note that $\eta_t = (\xi - \zeta_t)\omega\big|_t$ so the limit points of $\eta_t$ as $t \to 0^+$ are in one to one correspondence with limit points of $\zeta_t$. Then let $\zeta$ be a limit point of $\zeta_t$ as $t \to 0^+$. Then from Theorem 5.6 we have $\eta = (\xi - \zeta)\omega$ is a $q$-subordinate of $\omega$. Since the $\zeta_t$ are completely positive maps with $\|\zeta_t\| \leq \frac{1}{2}$ it follows that $\zeta$ is completely positive with $\|\zeta\| \leq \frac{1}{2}$. We will show that the range of $\eta$ is $P\mathcal{L}$.

Here is the tricky part. We are given that the range of $\omega$ is $\mathcal{L}$ so for each $A \in \mathcal{L}$ there is a $B \in \mathfrak{A}(\mathbb{C}^p)$ so that $A = \omega(B)$. Now $\eta(B) = (\xi - \zeta)\omega(B) = (\xi - \zeta)(A)$ so $\eta(B)$ does not depend on the actual operator $B$ but only on the fact that $\omega(B) = A$. This means that $\zeta$ can be viewed as a map from $\mathcal{L}$ to $P\mathcal{L}$ and since $A \in \mathcal{L}$ is uniquely determined by $PA$ this means that $\zeta$ can be viewed as a mapping of $P\mathcal{L}$ into itself. Finally, since $\zeta$ is completely positive we can view $\zeta$ as a completely positive mapping of $P\mathcal{L}$ into itself. The strange thing is that $\zeta(A)$ is computed from $(I - P)A$ and $(I - P)A$ may be zero while $A$ is not zero. This is not an error since if $(I - P)A = 0$
then $\zeta(A) = 0$. We have worked very hard to arrange it so that $\|\zeta\| \leq \frac{1}{2}$ and without such an estimate we could not conclude this proof. Now as a mapping of $PL$ into itself the mapping $\xi$ is just the unit mapping and the mapping under consideration is $\xi - \zeta$. Since even for $\zeta$ considered as a mapping of $PL$ into itself we still have $\|\zeta\| \leq \frac{1}{2}$ the mapping $\xi - \zeta$ is invertible with
\[(\xi - \zeta)^{-1} = \xi + \zeta + \zeta^2 + \zeta^3 + \cdots\]
where the series converges in norm. Hence, for each $A \in L$ there is a $B \in L$ so that $(\xi - \zeta)(B) = PA$. Since the range of $\omega$ is $L$ there is a $C \in \mathcal{A}(C^p)$ so that $\omega(C) = B$ and then
\[\eta(C) = (\xi - \zeta)(\omega(C)) = (\xi - \zeta)(B) = PA\]
so the range of $\eta$ is $PL$.

Now we prove the last statement of the theorem. Suppose $\omega$ is a $q$-pure $q$-weight map over $C^p$ of index zero. From Theorem 5.5 we know that the range $L$ of $\omega$ is a factor with the Choi-Effros multiplication. Now suppose $\eta$ is the $q$-subordinate of $\omega$ we just constructed. From Theorem 2.2 we know that the range of $\eta$ is contained in $L$ and, therefore, $PL \subset L$ and since the mapping $A \leftrightarrow PA$ is a $*$-isomorphism in each direction it follows that $PL = L$. Hence, the range of $\omega$ is a factor and the unit $I_o$ of $L$ is $PI_o = P$.

We remark that if in the previous theorem all we wish to prove is that there is a $q$-subordinate $\eta$ with range $L_{\eta} = PL$ then all we need to assume about the projection $P$ is that $P \in L'$ and $\|\omega(\Lambda(I - P))\| < \infty$.

6. The factor case

As we saw in the last section if $\omega$ is a $q$-pure $q$-weight map over $C^p$ of index zero then the range of $\omega$ is a factor of type $I_q$ contained in $B(C^p)$ with $q \leq p$. For these range algebras $L$ the Choi-Effros product is just the ordinary product so $A \star B = AB$ for $A, B \in L$. Also the maximal support projection $P$ for $L$ is the unit $I_o$ of $L$. Since we are primarily interested in finding the $q$-pure $q$-weight map over $C^p$ we will restrict our attention to these $q$-weight maps.

As mentioned in the last section when computing the generalized boundary representation $\pi_i^\# = (i + \omega|i\Lambda)^{-1}\omega|i$ we need only consider how $(i + \omega|i\Lambda)^{-1}$ acts on $L$ the range of $\omega$. Therefore, in our computation we only need know the action of $\omega|i\Lambda$ on $L$ which can be parameterized by matrix units $E_{ij}$ for $i, j = 1, \ldots, q$. Given a mapping $\phi$ of $B(C^p)$ into itself we denote by $\phi$ the restriction of $\phi$ to $L$. If $\phi$ maps $L$ into itself then we can think of $\tilde{\phi}$ as a mapping of the $(q \times q)$-matrices into themselves. So in what follows when we work with maps $\phi$ we will parameterize them as mappings of the $(q \times q)$-matrices into themselves. Note the order relations on such maps is the same as the order relations on the corresponding matrix maps. Notice that if $\phi_t = \omega|i\Lambda$ then $\phi_t = \omega|i\Lambda$.

First we formalize these ideas with the following definition.

Definition 6.1. Suppose $L$ is a type $I_q$ factor with $q \leq p$ contained in $B(C^p)$ where the identity $I_o$ of $L$ is a projection (not necessarily the unit of $B(C^p)$). We say $\omega$ is a $q$-weight map over $L$ if $\omega$ is a $q$-weight map over $C^p$ with values in $L$. We say $\omega$ is $q$-pure over $L$ if the $L$ valued $q$-subordinates of $\omega$ are totally ordered.

The next theorem is Theorem 5.10 only instead of getting an expression for $\tilde{\theta}$ we get an expression for $\vartheta$. Note in our decomposition of $\vartheta$ in terms of $g$’s and $h$’s we need the $E_{ij}$ to be actual partial isometries which they are due to the fact that $L$ is an algebra with the ordinary operator product.

Theorem 6.2. Suppose $\omega$ is a $q$-weight map over $C^p$ of index zero and the range of $\omega$ is $L$ which is a factor of type $I_q$. Then $\omega$ is of the form $\omega = \psi^{-1}\vartheta$ where $\psi$ is an invertible conditionally negative map of $L$ into itself with a completely positive inverse and $\vartheta$ is of the form
\[\vartheta(A) = \sum_{i,j=1}^q E_{ij}\vartheta_{ij}(A)\]
where the $E_{ij}$ are a complete set of matrix units for $L$ and

$$\vartheta_{ij}(A) = \sum_{k \in J}((g_{ik} + h_{ik}), A(g_{jk} + h_{jk}))$$

for $A \in \mathfrak{A}(\mathbb{C}^p)$ where the $g_{ik}, h_{ik} \in \mathbb{C}^p \otimes L_+^2(0, \infty)$ and

$$g_{ik}(x) = E_{11}g_k(x) \quad \text{and} \quad E_{11}g_k(x) = g_k(x)$$

and

$$\sum_{i=1}^q E_{11}h_{ik}(x) = 0$$

for $A \in \mathfrak{A}(\mathbb{C}^p)$, $x \geq 0$, $i, j \in \{1, \cdots, q\}$ and $k \in J$ a countable index set and the $h_{ik} \in \mathbb{C}^p \otimes L_+^2(0, \infty)$ and if

$$w_t = \sum_{k \in J}(g_k, \Lambda|tg_k) \quad \rho_{ij}(A) = \sum_{k \in J}(h_{ik}, Ah_{jk})$$

then $\rho$ is bounded so

$$\sum_{k \in J}\|h_{ik}\|^2 < \infty \quad \text{and} \quad \sum_{k \in J}(g_k, (I - \Lambda)g_k) < \infty$$

and $1/w_t \rightarrow 0$ as $t \rightarrow 0+$ and $\psi$ satisfies the conditions

$$\psi(I_o) \geq \vartheta(I - \Lambda(I_o)) \quad \text{and} \quad \psi + \rho \tilde{\Lambda}$$

is conditionally negative. (Recall $\tilde{\Lambda}$ is $\Lambda$ restricted to $L$.)

Conversely, if $\vartheta$, $\rho$ and $\psi$ are as given above then $\omega$ is a $q$-weight map over $\mathbb{C}^p$ of index zero and the range of $\omega$ is $L$. Furthermore, if $\psi'$ is a second map satisfying the conditions above and $\omega' = \psi'^{-1}\vartheta$ then $\omega'$ is a $q$-subordinate of $\omega$ (i.e. $\omega \geq_q \omega'$) if and only if $\psi \leq \psi'$. \[\text{Proof.} \] Assume the hypothesis and notation of the theorem. Since $\omega$ satisfies the hypothesis of Theorem 5.10 we have $\omega = \Theta \vartheta = \psi^{-1}\vartheta$ where $\Theta, \psi$ and $\vartheta$ are defined in Theorem 5.10. Note $\vartheta$ is a completely positive $L$ valued $b$-weight map on $\mathfrak{A}(\mathbb{C}^p)$. In Theorem 5.10 there is a decomposition of $\vartheta$ in terms of $g$’s and $h$’s. Since the $E_{ij}$ are actual partial isometries we can make a finer decomposition of $\vartheta$ in terms of different $g$’s and $h$’s as follows. From the general theory of completely positive maps we know that $\vartheta$ can be written in the form

$$\vartheta_{ij}(A) = \sum_{k \in J}(F_{ik}, AF_{jk})$$

with the $F_{ik} \in \mathbb{C}^p \otimes L_+^2(0, \infty)$ for $k \in J$ a countable index set. Furthermore, we know the $F_{ik}$ can be chosen so they are linearly independent over $l^2(J)$. Now we define

$$g_k = \frac{1}{q} \sum_{i=1}^q E_{11}F_{ik}$$

and

$$g_{ik}(x) = E_{11}g_k(x) \quad \text{and} \quad h_{ik} = F_{ik} - g_{ik}.$$ 

One checks from the definition of the $h$’s that

$$\sum_{i=1}^q E_{11}h_{ik}(x) = 0.$$

We define the completely positive $B(\mathbb{C}^p)$ valued $b$-weight map $\rho \in B(\mathbb{C}^p) \otimes \mathfrak{A}(\mathbb{C}^p)^*$ given by

$$\rho_{ij}(A) = \sum_{k \in J}(h_{ik}, Ah_{jk})$$

for $A \in \mathfrak{A}(\mathbb{C}^p)$ and for $t > 0$ we define

$$w_t = \sum_{k \in J}(g_k, \Lambda|tg_k), \quad (R_t)_{ij}(A) = \rho_{ij}|t(\Lambda(A)), \quad (Y_t)_{ij} = \sum_{k \in J}((h_{ik})_j, \Lambda|tg_k)$$

and

$$\zeta_t(A) = Y_tA.$$
for $A \in B(\mathbb{C}^p)$. Then we have

$$\vartheta|_t \Lambda(A) = w_t A + Y_t A + AY_t^* + R_t(A)$$

for $A \in B(\mathbb{C}^p)$ or

$$\vartheta|_t \Lambda = w_t t + \zeta_t + \zeta_t^* + R_t$$

There is no simple relation between the $g$’s and $h$’s above and in Theorem 5.10. In fact in Theorem 5.10 they are $\mathbb{C}^q$ valued functions of $x$ and here they are $\mathbb{C}^p$ valued functions of $x$. What is unique is the functionals they generate when restricted to $\mathcal{L}$ (i.e. the tilde maps). We see this by noting that in the decomposition

$$\tilde{\vartheta}|_t \Lambda = w_t \tilde{\zeta} + \tilde{\zeta}^* + \tilde{R}_t$$

that the completely positive mapping $\tilde{\rho}|_t \Lambda = \tilde{R}_t$ is the internal part of $\tilde{\vartheta}|_t \Lambda$ where the internal part of a mapping was defined equation 3.4 of section 3 of this paper. We recall that in defining the internal part of a mapping we decomposed operators in $B(\mathbb{C}^q)$ to a multiple of the unit $I$ plus an operator of trace zero. The trace zero conditions for the $h$’s corresponds to the condition

$$\sum_{i=1}^{q} E_{ik} h_{ik}(x) = 0.$$ 

In Theorem 5.10 we saw that $\tilde{\rho}|_t \Lambda$ is the internal part of $\tilde{\vartheta}|_t \Lambda$. Recall in Theorem 5.10 the $h$’s satisfied the condition

$$\sum_{i=1}^{q} (h_{ik})_i(x) = 0$$

which corresponds to the trace zero condition in the setting of Theorem 5.10. We see then in both cases we have $\tilde{\rho}|_t \Lambda$ is the internal part of the mapping $\tilde{\vartheta}|_t \Lambda$ so the results of Theorem 5.10 apply to our $\tilde{\rho}|_t \Lambda$ and, hence, there is a constant $K$ such that

$$\tilde{\rho}|_t \Lambda(I_0) = \rho|_t \Lambda(I_0) \leq K I_0$$

for $t > 0$. Now we have $\rho|_t(I) = \rho|_t(\Lambda(I_0)) + \rho|_t(I - \Lambda(I_0)) \leq K I_o + \rho|_t(I - \Lambda(I_0))$ and

$$\sum_{i=1}^{q} \rho|_t(I - \Lambda(I_0)) \leq \sum_{i=1}^{q} \vartheta|_t(I - \Lambda(I_0)) = \sum_{i=1}^{q} (\psi \omega)_{ii}(I - \Lambda(I_0))$$

$$\leq ||\psi|| \sum_{i=1}^{q} ||\omega||_i(I - \Lambda(I_0)) \leq ||\psi|| \sum_{i=1}^{q} ||I_o|| = q ||\psi||$$

so

$$||\rho|_t|| = ||\rho|_t(I)|| \leq K + q ||\psi||$$

for $t > 0$ so $\rho$ is bounded.

The fact that

$$\psi(I_0) \geq \vartheta(I - \Lambda(I_0)) \quad \text{and} \quad \psi + \rho \tilde{\Lambda}$$

is conditionally negative was established in Theorem 5.10.

Now for the proof of the last paragraph. Here we return to the first part of the proof of Theorem 4.7. The proof in our case follows line by line the proof of Theorem 4.7 making the following changes. We replace $Z(t)$ by the corresponding $\tilde{Z}(t)$. This replacement is automatic because $Z(t)^{-1}$ acts on $\vartheta$ which has range $\mathcal{L}$. The unit $I \in B(\mathbb{C}^p)$ is replaced by $I_o$. Note the unit $I$ of $B(\mathbb{C}^p \otimes L^2(0, \infty))$ is not replaced so, for example, the unit $I$ in $\pi^h(I)$ is not replaced. Expressions like $\pi^h(\Lambda)$ are replaced by $\pi^h(\Lambda(I_0))$. In proving $\pi^h(I) \leq I$ one proves the stronger inequality $\pi^h(\Lambda) \leq I_o \leq I$ using the inequality $\psi(I_0) \geq \vartheta(I - \Lambda(I_0))$ given in the statement of the present theorem. Note this inequality is the translation of the inequality $\psi(I) \geq \vartheta(I - \Lambda)$ that was used in the proof of Theorem 4.7. The proof of the last sentence of this theorem follows from the Theorem 4.7 with the replacements described.

The next theorem shows how to find all $q$-subordinates in the case at hand that have the same range $\mathcal{L}$. 


Theorem 6.3. Suppose $\omega$ is a $q$-weight map over $\mathbb{C}^p$ of index zero and range $\mathcal{L}$ which is a factor of type $I_q$ and $\omega = \psi \bar{\vartheta}$ with $\psi$ a coefficient map of $\omega$ and $\vartheta$ the corresponding limiting $b$-weight map. Suppose there is a bounded completely positive $\mathcal{L}$ valued weight map $\eta \in B(\mathbb{C}^p) \otimes \mathfrak{A}(\mathbb{C}^p)_*$ so that $\vartheta \geq \eta \geq 0$. Then if
\[
\psi' \geq \psi + \eta \Lambda \quad \text{and} \quad \psi' + \rho_0\Lambda - \eta \Lambda
\]
is conditionally negative (where $\Lambda$ is $\Lambda$ restricted to $\mathcal{L}$) then $\omega' = \psi'^{-1}(\vartheta - \eta)$ is a $q$-subordinate of $\omega$.

Conversely, if $\omega'$ is $q$-subordinate of $\omega$ whose range is contained in $\mathcal{L}$ there is a bounded completely positive $\mathcal{L}$ valued $b$-weight map $\eta \in B(\mathbb{C}^p) \otimes \mathfrak{A}(\mathbb{C}^p)_*$ with $\vartheta \geq \eta \geq 0$ and $\psi'$ satisfying the conditions above so that $\omega' = \psi'^{-1}(\vartheta - \eta)$.

Proof. The proof of this theorem follows from the proof of Theorem 4.9 where each of the equations and inequalities in 4.9 are reinterpreted in our new setting. Rather than write out the new proof we will explain how to make the translation. The first lines of the proof of Theorem 4.9 produce formulae for $\vartheta|_1 \Lambda$ and $\psi'|_1 \Lambda$. Instead those formulae should be replaced by the corresponding formulae for $\vartheta|_1 \Lambda$ and $\psi'|_1 \Lambda$ and the reference to Theorem 4.7 should be replaced by a reference to Theorem 6.2. The mapping $T_t = \eta|_1 \Lambda$ should be defined as $T_t = \eta|_1 \Lambda$. This brings us to the first rule in translating the proof of Theorem 4.9 to our new setting, namely replace the mapping $\Lambda$ by $\Lambda$. This replacement is justified because in all the calculation involving $\Lambda$ the calculated mappings act on the range of $\vartheta$, $\psi'$ or $\eta$. Note that since $\phi_t = \omega|_1 \Lambda$ and $\phi'_t = \omega'|_1 \Lambda$ the replacement $\Lambda$ by $\Lambda$ means these mappings should be replace by $\tilde{\phi}_t$ and $\tilde{\phi}'_t$.

The next replacement is the unit $I$ of $B(\mathbb{C}^p)$ which should be replaced by $I_o$ the unit of $\mathcal{L}$. Note in our case $I_o = P$ the maximal support projection. To give an example this means the expression $\psi(I)$ in the proof of Theorem 4.9 should be replaced by $\psi(I_o)$. Another example is an expression like $\eta \Lambda$ in Theorem 4.9 where $\Lambda$ is not the mapping $\Lambda$ but short for $\Lambda(I)$ which should be replaced by $\eta(\Lambda(I_o))$ or $\Lambda(\Lambda(I_o))$. Note $\Lambda(I_o)$ and $\Lambda(I_o)$ are interchangeable since $I_o \in \mathcal{L}$.

Next an important rule we noted earlier. The unit $I$ of $B(\mathbb{C}^p \otimes L^2(0, \infty))$ is not changed so, for example, an expression like $\eta I$ in Theorem 4.9 remains $\eta I$ in translation. Note the inequality $\pi_t^#(I) \leq I$ in Theorem 4.9 becomes $\pi_t^#(I) \leq I_o$ in our new setting.

To see how these rules apply consider the inequality $\omega(I - \Lambda) \leq I$ in Theorem 4.9. Applying the translation rules we have described this expression becomes $\omega(I - \Lambda(I_o)) \leq I_o$ which was proved in Theorem 5.9. Notice then that an inequality like $\psi(I) \geq \vartheta(I - \Lambda)$ in Theorem 4.9 becomes $\psi(I_o) \geq \vartheta(I - \Lambda(I_o))$ in our new setting. Also the references to previous theorems in Theorem 4.9 should be updated to the corresponding theorems in sections five and six in our new setting. Then following these rules the proof of Theorem 4.9 gives us a proof of the present theorem. □

Theorem 6.4. Suppose $\omega$ is a $q$-weight map over $\mathbb{C}^p$ of index zero and the range $\mathcal{L}$ the range of $\omega$ is a factor of type $I_q$ with $q \leq p$. We say $\omega$ is $q$-pure over $\mathcal{L}$ if the $\mathcal{L}$-valued $q$-subordinates are totally ordered. Then $\omega$ is $q$-pure over $\mathcal{L}$ if and only if $\omega$ can be written in the form given in Theorem 6.2 so $\omega = \psi \bar{\vartheta}$ where $\psi + \rho_0\Lambda$ is conditionally zero and $\psi(I_o) \geq \vartheta(I - \Lambda(I_o))$ and $\vartheta$ is strictly infinite $\mathcal{L}$-valued $b$-weight map meaning that if $\eta$ is a finite completely positive $\mathcal{L}$ valued $b$-weight map and $\vartheta \geq \eta \geq 0$ then $\eta = 0$.

Proof. The proof of the theorem follows from the proof of Theorem 4.10 where we translate the proof of Theorem 4.10 following the procedure just described. □

Theorem 6.5. Suppose $\omega$ is a $q$-weight map over $\mathbb{C}^p$ of index zero and $\mathcal{L}$ is the range of $\omega$. Then $\omega$ is $q$-pure if and only if the following conditions are satisfied.

(i) $\mathcal{L}$ is a factor of type $I_q$ with $1 \leq q \leq p$ and $\omega$ is of the form $\omega = \psi \bar{\vartheta}$ with $\psi$, $\vartheta$ and $\rho$ are as given in Theorem 6.2 and the map $\psi + \rho\Lambda$ is conditionally zero.

(ii) The $b$-weight map $\vartheta$ is strictly infinite as an $\mathcal{L}$ valued $b$-weight map.

(iii) If $e \in \mathcal{L}'$ is a non zero hermitian projection and $e \leq I_o$ the unit of $\mathcal{L}$ then $\|\omega(\Lambda(e))\| = \infty$.

Proof. Suppose $\omega$ is a $q$-pure $q$-weight map over $\mathbb{C}^p$ of index zero. Then from Theorem 6.4 we see that $\omega$ satisfies conditions (i) and (ii).

Now suppose $\omega$ fails to satisfy condition (iii) so there is a projection $e \in \mathcal{L}'$ with $e \leq I_o$ and $\|\omega(\Lambda(e))\| < \infty$. Note since $\omega$ is of index zero $\omega(\Lambda(I_o))$ is infinite so $e \neq I_o$. Repeating the
argument of Lemma 5.21 (see the remark after Lemma 5.21) we find there is a \( q \)-subordinate \( \omega' \) of \( \omega \) so that \( \| \omega' (\Lambda (e)) \| \leq \frac{1}{2} \). Let \( P = I - e \), let \( \pi_i^# \) be the boundary representation of \( \omega' \) and let \( \psi_i^# = P \pi_i^# \) for \( t > 0 \). Since \( P \in \mathcal{L}' \) we have \( \psi_i^# \leq \pi_i^# \) for \( t > 0 \). Let

\[
\eta_t = (t - \psi_i^# \Lambda)^{-1} \psi_i^#
\]

for \( t > 0 \) and by Theorem 5.6 we know that any weak limit point of \( \eta_t \) as \( t \to 0^+ \) is a \( q \)-subordinate of \( \omega' \) and since \( \omega \geq \eta \omega' \) it is a \( q \)-subordinate of \( \omega \). Calculating \( \eta_t \) as we did in Theorem 5.22 (see the remark after Theorem 5.22) we find

\[
\eta_t = (P - P (I - P \pi_i^# \Lambda)^{-1} (I - P)) \omega \|_{\tau} = (P - \zeta_t) \omega \|_{\tau} = P (P - \zeta_t) \omega \|_{\tau}
\]

where

\[
\zeta_t (A) = P (I - P \pi_i \Lambda)^{-1} ((I - P) A (I - P))
\]

and repeating the argument in the proof of Theorem 5.22 we find that \( \zeta_t \) is a completely positive contraction and we see that the limit points of \( \eta_t \) as \( t \to 0^+ \) are of the form \( \eta = (P - \zeta) \omega \) where \( \zeta \) is limit point of \( \zeta_t \) as \( t \to 0^+ \) and what is more important is \( \| \zeta_t (I) \| \leq \frac{1}{2} \). As in the proof of Theorem 5.22 this bound insures that inverse

\[(t - \zeta)^{-1} = t + \zeta + \zeta^2 + \cdots \]

exists since the series converges. Since the range of \( \omega \) is \( \mathcal{L} \) the range of \( \eta \) is \( P \mathcal{L} \). But this is a contradiction since from Theorem 2.2 we know that the range of any \( q \)-subordinate of \( \omega \) is contained in \( \mathcal{L} \). Hence, the assumption that \( \| \omega (\Lambda (e)) \| < \infty \) is false.

Now suppose \( \omega \) satisfies the three conditions of the theorem. Since \( \omega \) satisfies conditions (i) and (ii) then by Theorem 6.4 we have \( \omega \) is \( q \)-pure over \( \mathcal{L} \). Then if \( \omega \) is not \( q \)-pure it has a \( q \)-subordinate \( \tau \) whose range is not contained in \( \mathcal{L} \) and by Theorem 5.22 there is a \( q \)-subordinate of \( \nu \) of \( \tau \) so that the range of \( \nu \) is a factor. Since \( \omega \) is of index zero it follows that \( \nu \) is of index zero. Let \( \pi_i^# \) and \( \phi_i^# \) be the generalized boundary representations of \( \omega \) and \( \nu \), respectively. By routine compactness arguments we can find a decreasing sequence \( t_k \to 0^+ \) so that \( \pi_k^# \Lambda \to L \) and \( \phi_k^# \Lambda \to L_2 \) as \( k \to \infty \). Since \( \pi_k^# \Lambda \geq \phi_k^# \Lambda \) for \( t > 0 \) we have \( L \geq L_2 \) and \( L_2 \) are completely positive contractive idempotent maps. Let \( \mathcal{L}_1 \) be the range of \( \nu \). Since \( \mathcal{L}_1 \) is a factor the unit \( P \) of \( \mathcal{L}_1 \) is a projection so \( L_2 (I) = P \) and by Theorem 5.4 we have \( P \in \mathcal{L}' \), \( P \leq I_0 = L(I) \) and \( L_2 (A) = P L(A) P \) for \( A \in B (\mathcal{C}^p) \) and the range of \( \nu \) is \( P \mathcal{L} \).

Notice that the mapping \( M \to PA = PAP \) is a \( * \)-isomorphism of \( \mathcal{L} \) onto \( \mathcal{L}_1 \). Since \( \pi_i^# \geq \phi_i^# \) and the range of \( \phi_i^# \) is \( P \mathcal{L} = \mathcal{L}_1 \) it follows that \( P \pi_i^# \geq \phi_i^# \) for all \( t > 0 \). Now let \( \psi_i^# = P \pi_i^# \) and as before let

\[
\eta_t = (t - \psi_i^# \Lambda)^{-1} \psi_i^#.
\]

Since \( \pi_i^# \geq \psi_i^# \geq \phi_i^# \) it follows that \( \eta_t \geq \nu \|_{\tau} \) for all \( t > 0 \) and by Theorem 5.6 we see that if \( \eta \) is a limit point of \( \eta_t \) as \( t \to 0^+ \) then \( \omega \geq \eta \geq \nu \) and since \( \eta \geq \nu \) we see any limit point of \( \eta_t \) is not zero. Let \( \eta \) be a limit point of \( \eta_t \) as \( t \to 0^+ \). Repeating our earlier argument we see \( \eta \) is of the form \( \eta = (P - \zeta) \omega \) where \( \zeta \) is a completely positive contractive map of \( \mathcal{L} \) into \( \mathcal{L}_1 \). (This time we only have the bound \( \| \zeta (I) \| \leq 1 \) but we do not need smaller bound than one since we know that \( \eta \geq \nu \) so \( \eta \) is not zero).

Now we need to look more closely at the map \( (P - \zeta) \omega \) which is a map of \( \mathcal{L} \) into \( \mathcal{L}_1 \). Now \( \zeta \) is a completely positive contractive map of \( \mathcal{L} \) into \( \mathcal{L}_1 \) and since \( P \in \mathcal{L}' \) the mapping \( A \to PA \) is a \( * \)-isomorphism of \( \mathcal{L} \) into \( \mathcal{L}_1 \) so we can consider \( \zeta \) to be a completely positive contractive map of \( \mathcal{L}_1 \) into itself. Then \( (P - \zeta) \) restricted to \( \mathcal{L}_1 \) is the mapping \( (t - \zeta) \). Since \( \| \zeta \| \leq 1 \) we can not immediately conclude that this mapping has an inverse. But we are in luck since \( \eta = (P - \zeta) \omega \geq \nu \) and \( \nu \) has range \( P \mathcal{L}_1 \). Now let \( \Theta_i^# \) be the generalized boundary representation of \( \eta \). Note that \( \pi_i^# \geq \Theta_i^# \geq \phi_i^# \) for \( t > 0 \). Recall that for the sequence \( t_k \to 0 \) as \( k \to \infty \) we have \( \pi_k^# \Lambda \to L \) and \( \phi_k^# \Lambda \to L_2 \) as \( k \to \infty \). By a routine compactness argument we can pass to a subsequence of \( t_k \) (which we also denote by \( t_k \)) so that \( \Theta_t^# \Lambda \) converges to a limit \( L_1 \) as \( k \to \infty \) and since we have passed to a subsequence \( \pi_k^# \Lambda \to L \) and \( \phi_k^# \Lambda \to L_2 \) as before. Since \( \pi_i^# \geq \Theta_i^# \geq \phi_i^# \) for \( t > 0 \) we have \( L \geq L_1 \geq L_2 \) and by Theorem 5.4 we have \( L_1 (A) = P_1 L (A) P_1 = P_1 L (A) \) for \( A \in B (\mathcal{C}^p) \) with \( P_1 \in \mathcal{L}' \). Since \( L \geq L_1 \geq L_2 \) we have \( I_0 \geq P_1 \geq P \). Recalling the construction of
we see the range of \( \eta \) is contained in \( PL \). Hence, we have \( P_1 \leq P \) so \( P_1 = P \). Hence, the range of \( \eta \) is \( PL \) and thus the range of \( (I - \zeta) \) is \( PL \) and since \( PL \) is finite dimensional we conclude the inverse \( (I - \zeta)^{-1} \) exists. Now for \( 0 < \lambda < 1 \) we have

\[
(i - \lambda \zeta)^{-1} = i + \lambda \zeta + \lambda^2 \zeta^2 + \cdots
\]

where the series converges in norm and since \( (I - \zeta)^{-1} \) exists and the resolvent set is open and the resolvent is continuous it follows that \( (i - \lambda \zeta)^{-1} \) converges to \( (I - \zeta)^{-1} \) as \( \lambda \to 1 \) and since \( (i - \lambda \zeta)^{-1} \) is completely positive being the sum of completely positive terms it follows that \( (I - \zeta)^{-1} \) is completely positive. Since \( \eta = (P - \zeta)\omega \) we have \( P\omega = (i - \zeta)^{-1} \eta \). Now since \( \eta \) is a \( q \)-weight map we have from Theorem 5.9 that \( \eta(I - \Lambda(P)) \leq P \). Hence

\[
P\omega(I - \Lambda(P)) = (i - \zeta)^{-1} \eta(I - \Lambda(P)) \leq (I - \zeta)^{-1}(P).
\]

Since the mapping \( A \to PA \) is a *-isomorphism of \( L \) with \( PL \) for \( A \in L \) we have \( ||A|| = ||PA|| \) so we have

\[
||\omega(I - \Lambda(P))|| = ||P\omega(I - \Lambda(P))|| \leq ||(i - \zeta)^{-1}(P)|| = ||(I - \zeta)^{-1}||.
\]

Now let \( e_o = I_o - P \) and we have \( e_o \) is projection in \( L' \) and \( e_o \leq I_o \) and

\[
\omega(I - \Lambda(P)) = \omega(I - \Lambda(I_o)) + \omega(\Lambda(e_o)) \leq ||(I - \zeta)^{-1}|| I
\]

and since \( \omega(I - \Lambda(I_o)) \geq 0 \) we have \( 0 \leq \omega(\Lambda(e_o)) \leq ||(I - \zeta)^{-1}|| I \) so \( ||\omega(\Lambda(e_o))|| \) is finite which violates condition (iii) of the theorem. Hence, if \( \omega \) is not \( q \)-pure then conditions (i), (ii) and (iii) can not be satisfied.

**Theorem 6.6.** Suppose \( \vartheta \) is a \( L \) valued \( b \)-weight map on \( \mathfrak{A}(\mathbb{C}^p) \) where \( L \) is a factor of type \( I_q \) of the form given in Theorem 6.2 and

\[
\mu(A) = \sum_{k \in J}(g_k, A g_k)
\]

for \( A \in \mathfrak{A}(\mathbb{C}^p) \) where the \( g_k \) are as given in Theorem 6.2. Notice that \( \mu \) is supported on \( E_{11} \otimes I \). Then \( \vartheta \) is strictly infinite if and only if \( \mu \) is strictly infinite and the \( h's \) (as given in Theorem 6.2) are linearly independent over the \( g's \) by which we mean that if \( c \in l^2(I) \) and

\[
\sum_{k \in J} c_k g_k = 0
\]

then

\[
\sum_{k \in J} c_k h_{ik} = 0
\]

for each \( i = 1, \cdots, p \). Furthermore, the \( q \)-weight map \( \omega \) constructed from \( \vartheta \) by Theorem 6.2 will satisfy condition (iii) of Theorem 6.5 if and only if \( \mu(\Lambda(f)) = \infty \) for every non zero projection \( f \leq E_{11} \).

**Proof.** The proof of the lemma excluding the last sentence is simply a repeat of the proof of Theorem 4.11 in our new setting so all we need do is prove the last sentence, the sentence beginning with the word furthermore. Suppose then that \( \mu(\Lambda(f)) < \infty \) with \( f \) non zero and \( f \leq E_{11} \). Let

\[
e = \sum_{i=1}^q E_{11} f E_{11}.
\]

We have \( e \in L' \) and one calculates that

\[
\vartheta_{ii}(\Lambda(e)) = \sum_{k \in J}((g_{ik} + h_{ik}), \Lambda(e)(g_{ik} + h_{ik}))
\]

\[
= \sum_{k \in J}((e \otimes I)(g_{ik} + h_{ik}), \Lambda(e)(e \otimes I)(g_{ik} + h_{ik}))
\]

\[
\leq \sum_{k \in J}||((e \otimes I)(g_{ik} + h_{ik}))||^2.
\]
Now since \( \mu(\Lambda(f)) < \infty \) and \( \mu(I - \Lambda) < \infty \) we have \( \mu(f \otimes I) < \infty \) which means
\[
\sum_{k \in J} \|f g_k\|^2 < \infty
\]
and since \( g_{ik} = (E_{i1} \otimes I)g_k \) we have
\[
\sum_{k \in J} \|(e \otimes I)g_{ik}\|^2 = \sum_{k \in J} \|f g_k\|^2 < \infty
\]
and since
\[
\sum_{k \in J} \|h_{ik}\|^2 < \infty
\]
we have
\[
\sum_{k \in J} \|(e \otimes I)h_{ik}\|^2 < \infty
\]
from which we conclude that \( \vartheta_i(\Lambda(e)) < \infty \) for each \( i = 1, \cdots, q \). Since \( \omega = \psi \vartheta \) we conclude that \( \omega(\Lambda(e)) \) is bounded. So we have shown that if \( \mu(\Lambda(f)) < \infty \) for a non zero projection \( f \) with \( f \leq E_{11} \) then there is a projection \( e \in \mathcal{L}' \) with \( \|\omega(\Lambda(e))\| < \infty \).

Conversely, if there is a non zero projection \( e \in \mathcal{L}' \) with \( \|\omega(\Lambda(e))\| < \infty \) then if \( f = E_{11}e \) then one sees that \( \mu(\Lambda(f)) < \infty \).

We are now ready to prove the main result of this paper that every unital \( q \)-pure \( q \)-weight map of index zero is cocycle conjugate to a unital \( q \)-pure \( q \)-weight map of index zero of range rank one. We remark that the proof is mostly notation. What is really going on is that the real work in proving the theorem has already been done in the characterization of \( q \)-pure \( q \)-weight maps.

**Theorem 6.7.** Suppose \( \omega \) is a unital \( q \)-pure \( q \)-weight map over \( \mathbb{C}^m \) of index zero and the range \( \mathcal{L} \) of \( \omega \) is a factor of type \( I_q \). Then \( \omega \) is cocycle conjugate to a unital \( q \)-pure \( q \)-weight map \( \eta \) of range rank one over \( \mathbb{C}^m \).

**Proof.** Suppose \( \omega \) is as stated in the theorem. Then from Theorem 6.2 \( \omega \) is of the form \( \omega = \psi^{-1}\vartheta \) where \( \psi \) is an invertible conditionally negative map of \( \mathcal{L} \) into itself with a completely positive inverse and \( \vartheta \) is of the form given in Theorem 6.2. Since \( \omega \) is unital \( I_q = I \) and \( \psi \) satisfies the additional requirement that \( \psi(I) = \vartheta(I - \Lambda) \). Furthermore, we know from Theorem 6.5 that \( \psi + \rho(\Lambda) \) is conditionally zero and \( \vartheta \) is strictly infinite as an \( \mathcal{L} \) valued \( b \)-weight map. Let the \( g_k \) be as given in Theorem 6.2 and let \( \mu \) be the \( b \)-weight on \( \mathfrak{A}(\mathbb{C}^p) \) defined by
\[
\mu(A) = \sum_{k \in J}(g_k, Ag_k)
\]
for \( A \in \mathfrak{A}(\mathbb{C}^p) \). Note \( \mu \) is supported on \( E_{11} \otimes I \) so we can identify \( \mu \) as a \( b \)-weight on \( \mathfrak{A}(\mathbb{C}^m) \) where \( m \) is the rank of \( E_{11} \). Let \( \eta \) be the range rank one \( q \)-weight map on \( \mathfrak{A}(\mathbb{C}^m) \) given by \( \eta(A) = s_o^{-1}I_m \mu(A) \) for \( A \in \mathfrak{A}(\mathbb{C}^m) \) where \( s_o = \mu(I - \Lambda) \) and \( I_m \) is the unit of \( B(\mathbb{C}^m) \). Note we have \( \eta(I - \Lambda) = I_m \) so \( \eta \) is unital \( q \)-weight map over \( \mathbb{C}^m \). From Theorem 6.6 we know that \( \mu \) is strictly infinite and if \( f \) is a non zero projection in \( B(\mathbb{C}^m) \) then \( \mu(\Lambda(f)) \) is infinite so \( \eta \) satisfies the conditions of Theorem 2.5 so \( \eta \) is a unital \( q \)-pure range rank one \( q \)-weight map of index zero.

We introduce the notation we will be using in this proof. The variable \( q \) is fixed as \( \mathcal{L} \) is a factor of type \( I_q \). The variable \( r = q + 1 \) is fixed. We think of \( \mathcal{L} \) the factor of type \( I_q \) with matrix units \( E_{ij} \) as sitting in \( \mathcal{L}_1 \) the factor of type \( I \) with matrix units \( E_{ij}' \) and the two sets of matrix units match up so that \( E_{ij}' = E_{ij} \) for \( i, j = 1, \cdots, q \) (i.e. we think of \( \mathcal{L} \) as the top left corner of \( \mathcal{L}_1 \)).

We introduce two important projections if \( B(\mathbb{C}^m) \)
\[
E = E_{11}' + E_{22}' + \cdots + E_{qq}' \quad \text{and} \quad F = E_{rr}'.
\]
Note \( E \) is the unit of \( \mathcal{L} \) and \( E + F \) is the unit of \( \mathcal{L}_1 \). We will often consider an operator \( A \in B(\mathbb{C}^m) \) as a \((2 \times 2)\)-matrix with entries
\[
A = \begin{bmatrix}
EAE & EAF \\
FAE & FAF
\end{bmatrix}
\]
and we will call \( EAE \) the top diagonal of \( A \), we call \( EAF \) the upper right corner of \( A \), we call \( FAE \) the bottom left corner of \( A \) and we call \( FAF \) the bottom diagonal of \( A \).
We say a mapping \( \phi \) of \( B(\mathbb{C}^m) \) into itself is a Shur mapping if \( \phi \) preserves this structure meaning

\[
E \phi(A)E = \phi(EAE) \quad E \phi(A)F = \phi(EAF)
\]

for \( A \in B(\mathbb{C}^m) \). We say a mapping \( \phi \) of \( B(\mathbb{C}^m) \otimes \mathfrak{A}(\mathbb{C}) \) into \( B(\mathbb{C}^m) \) is a Shur mapping if \( \phi \) preserves this structure meaning

\[
E \phi(A)E = \phi((E \otimes I)A(E \otimes I)) \quad E \phi(A)F = \phi((E \otimes I)A(F \otimes I))
\]

\[
F \phi(A)E = \phi((F \otimes I)A(E \otimes I)) \quad F \phi(A)F = \phi((F \otimes I)A(F \otimes I))
\]

for \( A \in B(\mathbb{C}^m) \otimes \mathfrak{A}(\mathbb{C}) \).

Now we will prove the theorem by constructing a unital \( q \)-pure \( q \)-weight map \( \theta' \) where \( \theta' \) is \( b \)-weight map over \( \mathbb{C}^m \) given by

\[
\theta'(A) = \sum_{i,j=1}^r E_{ij} \theta'_{ij}(A)
\]

for \( A \in \mathfrak{A}(\mathbb{C}^m) \) and

\[
\theta'_{ij}(A) = \sum_{k \in J} ((g_{ik} + h'_{ik}), A(g'_{jk} + h'_{jk}))
\]

where \( g'_{ik} = g_{ik}, h'_{ik} = h_{ik} \) for \( k \in J \) and \( i = 1, \ldots, q \) where the \( g_{ik} \) and \( h_{ik} \) are from the expression for \( \theta \) in Theorem 6.2 and

\[
(g'_{rk})_j(x) = E_{r1} g_k(x) \quad \text{and} \quad h'_{rk} = 0
\]

for \( x \geq 0 \). Note we have constructed \( \theta' \) so that \( \theta' \) is a Shur mapping so that the top left diagonal of \( \theta' \) is \( \theta \) and the bottom right diagonal of \( \theta' \) is \( \eta \).

Now we define \( \psi' \). We have seen that \( \psi(I) = \theta(I - \Lambda) \) and \( \psi + \rho_\theta(\Lambda) \) is conditionally zero. This means that there is a \( Q \in B(\mathbb{C}^m) \) so that

\[
\psi(A) = QA + A Q^* - \rho_\theta(\Lambda(A))
\]

for \( A \in \mathcal{L} \). Calculating \( \theta(I - \Lambda) \) we find

\[
\theta(I - \Lambda) = I \mu(I - \Lambda) + Y + Y^* + \rho_\theta(I - \Lambda) = s_o I + Y + Y^* + \rho_\theta(I - \Lambda)
\]

where

\[
Y_{ij} = \sum_{q \in J} \int_0^\infty (1 - e^{-x})(h_{ik}(x), E_{j1} g_k(x)) \, dx
\]

from which we find

\[
Q + Q^* - \rho_\theta(\Lambda) = s_o I + Y + Y^* + \rho_\theta(I - \Lambda)
\]

so

\[
Q = \frac{1}{2}s_o I + B + iC \quad \text{where} \quad B = \frac{1}{2}(Y + Y^* + \rho_\theta(I))
\]

and \( C = C^* \). Note the if you replace \( C \) by \( C + \lambda I \) for real \( \lambda \) the mapping \( \psi \) unchanged so \( C \) is only determined up to adding a multiple real multiple of \( I \).

Since the \( h_{rk} \) are zero it follows that \( \rho_{\theta'}(\Lambda(\Lambda(A))) \) is a Shur mapping namely

\[
\rho_{\theta'}(\Lambda(A)) = E \rho_{\theta'}(\Lambda(EAE))E.
\]

Now it is clear how to define \( \psi' \). We define

\[
\psi'(A) = Q'A + AQ'^* - \rho_{\theta'}(\Lambda(A))
\]

and we define \( Q' \) so that

\[
EQ'E = Q, \quad EQ'F = 0, \quad FQ'E = 0, \quad FQ'F = \frac{1}{2}s_o F
\]
so
\[ \psi'(I) = \vartheta'(I - \Lambda(I)) \]
\[ Q' = Q + \frac{1}{2} s_o F = \frac{1}{2} s_o I + B + iC \]
Note in the above equation \( I = E + F \) the unit of \( \mathcal{L}_1 \) and \( B = EBE \) and \( C = ECE \). We note that \( \psi' \) is a Shur mapping meaning
\[ E\psi'(A)E = \psi'(EAE)E \]
\[ F\psi'(A)F = \psi'(FAF)F = s_o FAF \]
\[ E\psi'(A)F = EQ'AF + \frac{1}{2} s_o EAF = (s_o E + B + iC)AF = \psi'(EAF)F \]
\[ F\psi'(A)E = \frac{1}{2} s_o FAF + FAQ'\ast E = FA(s_o E + B - iC) = \psi'(FAE) = F\psi'(FAE)E \]
for \( A \in \mathcal{L}_1 \) so \( \psi' \) is a Shur mapping. Now that we have specified \( \omega' \) we check from Theorems 6.2 to 6.5 that \( \omega' \) is a unital \( q \)-pure \( q \)-weight map over \( \mathbb{C}^m \).

To give it a name let \( \gamma = E\omega'F \) be the top right corner of \( \omega' \). We claim \( \gamma \) is a hyper maximal \( q \)-corner from \( \omega \) to \( \eta \). Since \( \omega' \) is Shur mapping and a \( q \)-weight map \( \gamma \) is a \( q \)-corner from \( \omega \) to \( \eta \).

To see if \( \gamma \) is hyper maximal we examine the \( q \)-subordinates of \( \omega' \) whose top right corner equals \( \gamma \). But since \( \omega' \) is \( q \)-pure the only \( q \)-subordinates of \( \omega' \) are of the form \( \omega'' = \psi''_\lambda \psi' \) where \( \psi''_\lambda = \psi' + \lambda \) with \( \lambda \geq 0 \). Calculating \( E\psi''_\lambda F \) we find
\[ E\psi''_\lambda(A)F = ((s_o + \lambda)E + B + iC)AF = Z_\lambda AF \]
for \( A \in \mathcal{L}_1 \) so \( E\psi''_\lambda F \) is given by
\[ E\psi''_\lambda(A)F = ((s_o + \lambda)E + B + iC)^{-1}AF = Z_\lambda^{-1} AF \]
where by \( Z_\lambda^{-1} \) we mean the inverse of this operator \( Z_\lambda \) considered as an operator from the range of \( E \) to the range of \( E \). (The inverse on the larger space make no sense as \( Z_\lambda F = 0 \).) Let \( z_i \) be the eigenvalues of \( Z_\lambda \). Note that \( \text{Re}(z_i) > 0 \) and the eigenvalues of \( Z_\lambda = z_i + \lambda \) and the eigenvalues of \( Z_\lambda^{-1} = (z_i + \lambda)^{-1} \). In any event it is clear that \( E\psi''_\lambda F \neq E\psi''_\lambda F \) for \( \lambda > 0 \) so the only \( q \)-subordinate of \( \omega' \) with corner \( \gamma \) is \( \omega' \) and, hence, \( \gamma \) is a hyper maximal \( q \)-corner from \( \omega \) to \( \eta \). Hence, \( \omega' \) and \( \eta \) are cocycle conjugate.

In conclusion we have shown that every \( q \)-pure \( q \)-weight map over a finite dimensional Hilbert space \( K \) is cocycle conjugate to a \( q \)-pure rank one \( q \)-weight map and these are described in Theorem 2.5. Theorem 2.6 gives necessary and sufficient conditions for two such \( q \)-weight maps to be cocycle conjugate. It follows that \( q \)-pure spatial \( E_0 \)-semigroups coming from \( q \)-weight maps over finite dimensional Hilbert spaces are fairly well understood. The next burning question is whether this remains true in the infinite dimensional case.

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