The angular momentum of the gravitational field and the Poincaré group

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Abstract

We redefine the gravitational angular momentum in the framework of the teleparallel equivalent of general relativity. Similar to the gravitational energy–momentum, the new definition for the gravitational angular momentum is coordinate independent. By considering the Poisson brackets in the phase space of the theory, we find that the gravitational energy–momentum and angular momentum correspond to a representation of the Poincaré group. This result allows us to define Casimir type invariants for the gravitational field.

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1. Introduction

The teleparallel equivalent of general relativity (TEGR) is a viable alternative geometrical description of Einstein’s general relativity in terms of the tetrad field [1], and continues to be object of thorough investigations [2–5]. In the framework of the TEGR it has been possible to address the longstanding problem of defining the energy, momentum and angular momentum of the gravitational field [6–8]. The tetrad field seems to be a suitable field quantity to address this problem, because it yields the gravitational field and at the same time establishes a class of reference frames in spacetime [9]. Moreover, there are simple and clear indications that the gravitational energy–momentum defined in the context of the TEGR provides a unified picture of the concept of mass–energy in special and general relativity.

In special relativity the energy of an arbitrary body is frame dependent, and a similar property is expected to hold in general relativity. For instance, a black hole of mass $m$ that is distant from an observer behaves as a particle of mass $m$. If the observer is at rest with respect to the black hole, he or she will conclude that the energy of the black hole is $mc^2$. If, however, the observer is moving at a velocity $v$ with respect to the black hole, then for this observer the black hole energy is $\gamma mc^2$ (asymptotically), where $\gamma = (1 - v^2/c^2)^{-1/2}$. The
latter expression establishes the frame dependence of the black hole energy. This feature is naturally exhibited by the global SO(3, 1) covariance of the gravitational energy–momentum defined in the framework of the TEGR [10]. The frame dependence of the gravitational energy is not restricted to its total energy, neither to asymptotic regions. It holds in the consideration of the gravitational energy over finite regions of the three-dimensional spacelike hypersurface of an arbitrary spacetime (de Sitter or anti-de Sitter spacetimes, for instance). As an example of a finite region mentioned above, we may consider the gravitational energy contained within the (external) event horizon of a black hole, a quantity that defines its irreducible mass. We note that the dependence of the total Schwarzschild mass with respect to a boosted reference system has been addressed previously by York [12], who constructed the boosted Schwarzschild initial data by means of a suitable coordinate transformation in the Schwarzschild spacetime.

In this paper we redefine the angular momentum of the gravitational field in the framework of the TEGR. Similar to the definition of the gravitational energy–momentum, we interpret the appropriate constraint equations as equations that define the gravitational angular momentum. The latter turns out to be coordinate independent. We will show that the energy–momentum and angular momentum of the gravitational field satisfy the Poincaré algebra in the phase space of the theory. As a consequence of this result, we may define Casimir type invariants for an arbitrary configuration of the gravitational field.

The present definition is conceptually different from previous approaches to the gravitational angular momentum. The definition arises by considering the field equations of the theory. In contrast, other approaches are based on boundary terms to the Hamiltonian or to the action integral. Regge and Teitelboim [11] obtained a Hamiltonian formalism for general relativity that is manifestly invariant under Poincaré transformations at infinity by introducing ten new pairs of canonical variables, which yield ten surface integrals to the total Hamiltonian. The subsequent analysis by York [12] showed that a proper definition for the gravitational angular momentum requires a suitable asymptotic behaviour of the spatial components of the Ricci tensor (that ensures an ‘almost’ rotational symmetry of the spatial components \(g_{ij}\) of the metric tensor). A careful analysis of the exact form of the boundary conditions needed to define the energy, momentum and angular momentum of the gravitational field has been carried out by Beig and Ó Murchadha [13], and by Szabados [14], who found the necessary conditions that yield a finite value for the above mentioned quantities. In these analyses the Poincaré transformations and the Poincaré algebra are realized at the spacelike infinity. These are transformations of the Cartesian coordinates in the asymptotic region of the spacetime.

In the present analysis we show that the Poincaré algebra is realized in the full phase space of the TEGR. We evaluate the angular momentum of a simple configuration of the gravitational field, namely, of the spacetime of a rotating mass shell, in order to display the procedure regarding the present definition. We will pay special attention to the frame dependence of the latter.

**Notation:** spacetime indices \(\mu, \nu, \ldots\) and SO(3, 1) indices \(a, b, \ldots\) run from 0 to 3. Time and space indices are indicated according to \(\mu = 0, i, a = (0), (i)\). The tetrad field is denoted by \(e^a_{\mu}\), and the torsion tensor reads \(T_{a\mu\nu} = \partial_\mu e_{a\nu} - \partial_\nu e_{a\mu}\). The flat, Minkowski spacetime metric tensor raises and lowers tetrad indices and is fixed by \(\eta_{ab} = e_{a\mu} e_{b\nu} g^{\mu\nu} = (-+++).\) The determinant of the tetrad field is represented by \(\epsilon = \det(e^a_{\mu})\).

2. The Lagrangian and Hamiltonian formulations of the TEGR

We will briefly recall both the Lagrangian and Hamiltonian formulations of the TEGR. The Lagrangian density for the gravitational field in the TEGR is given by
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\[ L(e_{\mu \nu}) = -ke \left( \frac{1}{4} T^{abc} T_{abc} + \frac{1}{2} T^{abc} T_{bac} - T^a T_a \right) = L_M \]

\[ \equiv -ke \Sigma^{abc} T_{abc} - L_M, \]

where \( k = 1/(16\pi) \), and \( L_M \) stands for the Lagrangian density for the matter fields. As usual, tetrad fields convert spacetime into Lorentz indices and vice-versa. The tensor \( \Sigma^{abc} \) is defined by

\[ \Sigma^{abc} = \frac{1}{4}(T^{abc} + T^{bac} - T^{cab}) + \frac{1}{2}(\eta^{ac}T^{b} - \eta^{ab}T^{c}), \]

and \( T^a = T^b b^a \). The quadratic combination \( \Sigma^{abc} T_{abc} \) is proportional to the scalar curvature \( R(e) \), except for a total divergence. The field equations for the tetrad field read

\[ e_a(\Sigma^{abcd} T^{abcd} - \frac{1}{4} e_{aib} T^{bcd}) = \frac{1}{4} ke T^a, \]

where \( e_T a \equiv \delta L/\delta e_{aib} \). It is possible to prove by explicit calculations that the left-hand side of equation (3) is exactly given by \( \frac{1}{2} e \{ R(e) - \frac{1}{2} e_{ai} R(e) \} \). The field equations above may be rewritten in the form

\[ \partial_{\nu}(e_{a\lambda \nu}) = \frac{1}{4} ke e_{a\nu}(t^{a\nu} + T^{a\nu}), \]

where

\[ t^{a\nu} = k \left( 4 \Sigma_{b\lambda \nu} T_{b\lambda \nu} - g^{a\nu} \Sigma_{bcd} T_{bcd} \right), \]

is interpreted as the gravitational energy–momentum tensor [8]. (We remark that an energy–momentum tensor for cosmological perturbations has been considered in the framework of the Hilbert-Einstein formulation [15]. Such an energy–momentum tensor is defined with respect to certain backgrounds, and is related to Einstein’s equations via differential conservation laws.)

The Hamiltonian formulation of the TEGR is obtained by first establishing the phase space variables. The Lagrangian density does not contain the time derivative of the tetrad component \( e_{a0} \). Therefore this quantity will arise as a Lagrange multiplier. The momentum canonically conjugated to \( e_{ai} \) is given by \( \Pi_{ai} = \delta L/\delta e_{ai} \). The Hamiltonian formulation is obtained by rewriting the Lagrangian density in the form \( L = p \dot{q} - H \), in terms of \( e_{ai}, \Pi_{ai} \) and Lagrange multipliers. The Legendre transform can be successfully carried out, and the final form of the Hamiltonian density reads [16]

\[ H = \Pi_{ai} C^a + \alpha_{ik} \Gamma^{ik} + \beta_k \Gamma^k, \]

plus a surface term. \( \alpha_{ik} \) and \( \beta_k \) are Lagrange multipliers that (after solving the field equations) are identified as \( \alpha_{ik} = 1/2(T_{i0k} + T_{0ik}) \) and \( \beta_k = T_{00} \). \( C^a, \Gamma^{ik} \) and \( \Gamma^k \) are first class constraints. The Poisson brackets between any two field quantities \( F \) and \( G \) are given by

\[ \{ F, G \} = \int d^3x \left( \frac{\delta F}{\delta e_{ai}(x)} \frac{\delta G}{\delta \Pi_{ai}(x)} - \frac{\delta F}{\delta \Pi_{ai}(x)} \frac{\delta G}{\delta e_{ai}(x)} \right). \]

We recall that the Poisson brackets \( \{ \Gamma^{ij}(x), \Gamma^{kl}(y) \} \) reproduce the angular momentum algebra [16].

The constraint \( C^a \) is written as \( C^a = -\partial_a \Pi^a + h^a \), where \( h^a \) is an intricate expression of the field variables. The integral form of the constraint equation \( C^a = 0 \) motivates the definition of the gravitational energy–momentum tensor \( P^a \) [6],

\[ P^a = -\int_V d^3x \partial_a \Pi^a. \]

\( V \) is an arbitrary volume of the three-dimensional space. In the configuration space we have

\[ \Pi^a = -4ke \Sigma^{a0}. \]
The emergence of total divergences in the form of scalar or vector densities is possible in the framework of theories constructed out of the torsion tensor. Metric theories of gravity do not share this feature. We note that by making $\lambda = 0$ in equation (4) and identifying $\Pi^{a\mu}$ on the left-hand side of the latter, the integral form of equation (4) is written as

$$P^a = \int_V d^3x \, e e^a_{\mu}(T^{0\mu} + T^{00}).$$

(10)

It is important to rewrite the Hamiltonian density $H$ in the most simple form. We believe that the constraint $C^a$ admits a simplification, although we have not achieved it yet. However, we have been able to simplify the constraints $\Gamma_{ik}$ and $\Gamma_k$, which may be rewritten as a single constraint $\Gamma_{ab}$. It is not difficult to verify that the Hamiltonian density (6) may be written in the equivalent form

$$H = e_{a\mu}C^a + \frac{1}{2}\lambda_{ab}\Gamma_{ab},$$

(11)

where $\lambda_{ab} = -\lambda_{ba}$ are Lagrange multipliers that are identified as $\lambda_{ik} = \alpha_{ik}$ and $\lambda_{0k} = -\lambda_{k0} = \beta_k$. $\Gamma_{ab} = -\Gamma_{ba}$ embodies both constraints $\Gamma_{ik}$ and $\Gamma_k$ by means of the relations $\Gamma^{ik} = e^a_i e^b_k \Gamma_{ab}, \Gamma^k = \Gamma^{0k} = e_a^0 e_b^k \Gamma_{ab}$. It reads

$$\Gamma_{ab} = M_{ab} + 4k e(\Sigma_{a0} - \Sigma_{b0}),$$

(12)

with $M_{ab} = e^a_{\mu} e^b_{\nu} M^{\mu\nu} = -M_{ba}$. $M^{\mu\nu}$ is defined by

$$M^{ik} = 2\Pi^{i(k)} = e^i_0 \Pi^{ik} - e^i_a \Pi^{ai},$$

$$M^{0k} = \Pi^{0k} = e_a^0 \Pi^{ak}. (13)$$

Similar to the definition of $P^a$, the integral form of the constraint equation $\Gamma_{ab} = 0$ motivates the new definition of the spacetime angular momentum (in the expression previously defined [6], $2\Pi^{i(k)}$ was taken as the gravitational angular momentum density). The equation $\Gamma_{ab} = 0$ implies

$$M_{ab} = -4k e(\Sigma_{a0} - \Sigma_{b0}).$$

(15)

Therefore we define

$$L^{ab} = \int_V d^3x \, e e^a_{\mu} e^b_{\nu} M^{\mu\nu},$$

(16)

as the 4-angular momentum of the gravitational field. In contrast to the definition presented in [6], the expression above is invariant under coordinate transformations of the three-dimensional space. We note that on the right-hand side of equation (15), as well as on the right-hand side of equation (9), there arises the time index 0. We further note the presence of the determinant $e$ of the tetrad field in these quantities. This determinant can always be written as the product of the lapse function $N = (g^{00})^{-1/2}$ with the determinant of the triads restricted to the three-dimensional space $3e$. Thus, $e = N^3 e$. Because of the presence of the lapse function and of the time index 0, the right-hand sides of equations (9) and (15) are invariant under time reparametrizations.

Therefore $P^a$ and $L^{ab}$ are separately invariant under general coordinate transformations of the three-dimensional space and under time reparametrizations, which is an expected feature since these definitions arise in the Hamiltonian formulation of the theory. Moreover, these quantities transform covariantly under global SO(3, 1) transformations.

We emphasize that expressions (8) and (16) are defined in the phase space of the theory. In order to evaluate these expressions for a particular field configuration, we consider the right-hand side of equations (9) and (15) in the configuration space of the theory.
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3. The Poincaré structure in the phase space of the theory

An interesting result that follows from the definition above for the gravitational 4-angular momentum is that $L^{ab}$ and $P^a$ satisfy the algebra of the Poincaré group. By means of the Poisson bracket defined by equation (7) we find that the definitions (8), (13) (14) and (16) yield

$$\{ P^a, P^b \} = 0,$$

$$\{ P^a, L^{bc} \} = -\eta^{ac} P^b + \eta^{bc} P^a,$$

$$\{ L^{ab}, L^{cd} \} = -\eta^{ac} L^{bd} - \eta^{bd} L^{ac} + \eta^{ad} L^{bc} + \eta^{bc} L^{ad}. \tag{17}$$

Equations (17) may be easily verified by considering the following functional derivatives,

$$\frac{\delta L^{ab}}{\delta e_{ck}(z)} = \int d^3x \frac{\delta}{\delta e_{ck}(z)} \left[ e^a_{\mu} e^b_{\nu} \mathcal{M}^{\mu\nu} \right]$$

$$= \int d^3x \frac{\delta}{\delta e_{ck}(z)} \left[ e^a_{\mu} e^b_{\nu} \mathcal{M}^{\mu\nu} + e^a_{\mu} e^b_{\nu} \mathcal{M}^{00} + e^a_{\mu} e^b_{\nu} \mathcal{M}^{0j} \right]$$

$$= (\eta^{bc} e^a_{\mu} e^b_{\nu} 0(z) - \eta^{ac} e^a_{\mu} e^b_{\nu} 0(z)) \mathcal{M}_{0k}(z) + (\eta^{bc} e^a_{\mu} e^b_{\nu} j(z) - \eta^{ac} e^a_{\mu} e^b_{\nu} j(z)) \Pi^{kj}(z)$$

$$+ (\eta^{bc} e^a_{\mu} e^b_{\nu} j(z) - \eta^{ac} e^a_{\mu} e^b_{\nu} j(z)) \Pi^{kj}(z)$$

$$= -\eta^{ac} \Pi_{ck}(z) + \eta^{bc} \Pi_{ak}(z), \tag{18}$$

$$\frac{\delta L^{ab}}{\delta \Pi_{ck}(z)} = \delta^a_{\nu} \delta^b_{\mu} e^{\nu}(z) - \delta^b_{\nu} \delta^a_{\mu} e^{\nu}(z),$$

$$\frac{\delta P^a}{\delta e_{ck}(z)} = 0,$$

$$\frac{\delta P^a}{\delta \Pi_{ck}(z)} = - \int d^3x \frac{\partial}{\partial x^k} \delta^a_{\mu}(x - z).$$

We see that the gravitational energy–momentum and angular momentum constitute a representation of the Poincaré group. It is well known that the field quantities that satisfy the algebra above are intimately related to energy–momentum and angular momentum. Thus the Poincaré algebra of $P^a$ and $L^{ab}$ confirms the consistency of the definitions.

4. Tetrad fields as reference frames in spacetime and the gravitational angular momentum

The theory defined by equation (1) is invariant under general coordinate and global SO(3, 1) transformations. Because of the global SO(3, 1) invariance of the theory, two tetrad fields that (i) are solutions of the field equations, (ii) yield the same metric tensor and (iii) are not related by a global SO(3, 1) transformation, describe the same spacetime from the point of view of inequivalent reference frames. In view of this fact we must take into account the physical and geometrical meaning of tetrad fields as reference frames adapted to ideal observers in spacetime.

Before we proceed, we recall that the quadratic combination of the torsion tensor in equation (1) is equivalent to the scalar curvature except for a total divergence. The Lagrangian density (1) is invariant under infinitesimal Lorentz transformations only in the context of asymptotically flat spacetimes, in which case the total divergence plays no role, and provided both the tetrad field and the Lorentz transformation matrix satisfy appropriate boundary conditions [17]. In the general case, the Lagrangian density (1) is not invariant under local SO(3, 1) transformations.
Each set of tetrad fields defines a class of reference frames [9]. If we denote by \( x^\mu(s) \) the world line \( C \) of an observer in spacetime, and by \( u^\mu(s) = dx^\mu/ds \) its velocity along \( C \), we may identify the observer’s velocity with the \( a = (0) \) component of \( e_\mu \) [18]. Thus \( u^\mu(s) = e_{(0)} \mu \) along \( C \). The acceleration of the observer is given by \( a^\mu = Du^\mu/ds = De_{(0)} \mu /ds = u^\alpha \nabla_\alpha e_{(0)} \mu \), where the covariant derivative is constructed out of the Christoffel symbols. We see that \( e_\mu \mu \) determines the velocity and acceleration along the worldline of an observer adapted to the frame. From this perspective we conclude that a given set of tetrad fields, for which \( e_{(0)} \mu \) describes a congruence of timelike curves, is adapted to a particular class of observers, namely, to observers characterized by the velocity field \( u^\mu = e_{(0)} \mu \), endowed with acceleration \( a^\mu \). If \( e^\mu_\mu \rightarrow \delta_\mu^\mu \) in the limit \( r \rightarrow \infty \), then \( e^\mu_\mu \) is adapted to static observers at spacelike infinity.

Actual calculations of the energy–momentum and angular momentum of the gravitational field in the configuration space of the theory require the evaluation of the right-hand side of equations (9) and (15). The definitions of \( P^a \) and \( L^{ab} \) above are well defined if we consider tetrad fields \( e^a_\mu \) such that in the flat spacetime limit (i.e., in the absence of the gravitational field) we have \( T_{\mu\nu}(e) = 0 \). However there are flat spacetime tetrads \( E^a_\mu \), for which \( T_{\mu\nu}(E) \neq 0 \). Consequently, for such tetrads we obtain nonvanishing values of \( P^a \) and \( L^{ab} \) in the absence of the gravitational field. Therefore these expressions must be regularized. The regularization of the gravitational energy–momentum is discussed in detail in [9]. Conceptually it is the same regularization procedure that takes place in the Brown–York method [19], which may be understood as the subtraction of the flat spacetime energy. We denote \( T^a_\mu_\nu(E) = \partial_\mu E^a_\nu - \partial_\nu E^a_\mu \) and \( \Pi^{ab}(E) \) as the expression of \( \Pi^{ab} \) constructed out of flat tetrads \( E^a_\mu \). The regularized form of the gravitational energy–momentum \( P^a \) is defined by

\[
P^a = - \int_V d^3x \delta_k [\Pi^{ak}(e) - \Pi^{ak}(E)].
\]

(19)

This definition guarantees that the energy–momentum of the flat spacetime always vanishes. The reference spacetime is determined by the tetrad fields \( E^a_\mu \), obtained from \( e^a_\mu \) by requiring the vanishing of the physical parameters like mass, angular momentum, etc. The definition above for \( P^a \) has been investigated in [9] by considering a set of tetrad fields for the Kerr black hole for which \( T_{\mu\nu}(E) \neq 0 \). By subtracting the flat spacetime quantity \( \Pi^{ak}(E) \) we arrive at the expected value for the total gravitational energy of the Kerr spacetime. We remark that the regularization of the gravitational energy–momentum has been addressed recently in the analysis of [5].

We may likewise establish the regularized expression for the gravitational 4-angular momentum. It reads

\[
L^{ab} = \int_V d^3x [M^{ab}(e) - M^{ab}(E)].
\]

(20)

Expressions (19) and (20) allow the evaluation of the gravitational energy–momentum and 4-angular momentum out of an arbitrary set of tetrad fields.

5. The spacetime of the rotating mass shell

We will present in detail the application of definition (20) to the spacetime of a slowly rotating spherical mass shell, formulated by Cohen [20]. It describes a mathematically simple, nonsingular configuration of the gravitational field that exhibits rotational effects and is everywhere regular. In the limit of small angular momentum the metric for such spacetime corresponds to the asymptotic form of Kerr’s metric tensor. The main motivation [20] for considering this
metric is the construction of a realistic source for the exterior region of the Kerr spacetime, and therefore to match the latter region to a singularity-free spacetime. For a shell of radius 0 and total mass \( m = 2\alpha \) as seen by an observer at infinity, the metric reads
\[
d s^2 = -V^2 dr^2 + \psi^2 [dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta (d\phi - \Omega \, dr)^2],
\]
where
\[
V = \frac{r_0 - \alpha}{r_0 + \alpha}, \quad \psi = \psi_0 = 1 + \frac{\alpha}{r_0}, \quad \Omega = \Omega_0 = \text{constant},
\]
for \( r < r_0 \), and
\[
V = \frac{r - \alpha}{r + \alpha}, \quad \psi = 1 + \frac{\alpha}{r}, \quad \Omega = \left(\frac{r_0 \psi_0^2}{r \psi^2}\right)^3 \Omega_0,
\]
for \( r > r_0 \).

The metric tensor given by equation (21) is a solution of Einstein’s equations up to first order in \( \Omega \). We recall that \( \Omega_0 \) is the dragging angular velocity of locally inertial observers inside the shell. The contravariant components of the metric tensor will be useful in the following considerations. They read
\[
g^{\mu \nu} = \begin{pmatrix}
\frac{1}{V^2} & 0 & 0 & -\frac{\Omega}{V^2} \\
0 & \frac{1}{V^2} & 0 & 0 \\
0 & 0 & \frac{1}{r^2} & 0 \\
-\frac{\Omega}{V^2} & 0 & 0 & \frac{V^2 - r^2 \Omega^2 \psi^2 \sin^2 \theta}{V^2 r^2 \psi^2 \sin^2 \theta}
\end{pmatrix}.
\]

We will consider two simple configurations of tetrad fields and discuss their physical interpretation as reference frames. The first one is given by
\[
e_{a \mu} = \begin{pmatrix}
\Omega r^2 \psi^2 \sin \theta \sin \phi & \psi^2 \sin \theta \cos \phi & r \psi^2 \cos \theta \cos \phi & -r \psi^2 \sin \theta \sin \phi \\
-\Omega r^2 \psi^2 \sin \theta \cos \phi & \psi^2 \sin \theta \sin \phi & r \psi^2 \cos \theta \sin \phi & r \psi^2 \sin \theta \cos \phi \\
0 & \psi^2 \cos \phi & r \psi^2 \sin \theta \cos \phi & -r \psi^2 \sin \theta \sin \phi \\
0 & 0 & \psi^2 \sin \theta \cos \phi & r \psi^2 \sin \theta \sin \phi
\end{pmatrix}.
\]

The determinant of \( e_{a \mu} \) is \( e = V r^2 \psi^6 \sin \theta \). We find that the tetrad fields above are adapted to observers whose 4-velocity in spacetime is given by
\[
e_{(0)}^{\mu}(t, r, \theta, \phi) = \frac{1}{V} (1, 0, 0, \Omega).
\]
Therefore an observer at the radial position \( r \) rotates around the source along a circular trajectory, with angular velocity \( \Omega(r) \) (1/V is the normalization factor of the 4-velocity).

In order to obtain the spatial components of the gravitational angular momentum we need the evaluation of \( T_{a \mu} = e^a_b \, T_{b \mu \nu} \). We find
\[
T_{001} = V \, \partial_1 V - \frac{1}{2} \partial_1 (\Omega r \psi^2)^2 \sin^2 \theta, \quad T_{301} = r \psi^2 \partial_1 (\Omega r \psi^2)^2 \sin^2 \theta,
\]
\[
T_{002} = -\Omega r^2 \psi^2 \sin \theta \cos \theta, \quad T_{302} = \Omega r^2 \psi^2 \sin \theta \cos \theta,
\]
\[
T_{103} = -\Omega r \psi^4 \sin^2 \theta, \quad T_{203} = -\Omega r^2 \psi^4 \sin \theta \cos \theta,
\]
\[
T_{212} = r^2 \psi^2 (\partial_1 \psi^2), \quad T_{013} = -\Omega r^2 \psi^2 (\partial_1 \psi^2)^2 \sin^2 \theta, \quad T_{313} = r^2 \psi^2 (\partial_1 \psi^2)^2 \sin \theta.
\]

After long but simple calculations we obtain
\[
\Sigma_{001} = \frac{1}{2} (T_{001} - g_{00} T_1), \quad \Sigma_{301} = \frac{1}{2} (T_{301} - T_{013} + T_{103}) - \frac{1}{2} g_{03} T_1,
\]
\[
\Sigma_{002} = \frac{1}{2} T_{002}, \quad \Sigma_{103} = \frac{1}{2} (T_{103} + T_{013} + T_{301}), \quad \Sigma_{212} = \frac{1}{2} (T_{212} + g_{22} T_1),
\]
\[
\Sigma_{013} = \frac{1}{2} (T_{013} + T_{301} - T_{103}) + \frac{1}{2} g_{03} T_1, \quad \Sigma_{313} = \frac{1}{2} (T_{313} + g_{33} T_1), \quad \Sigma_{023} = \frac{1}{2} T_{203}.
\]
In view of the relations \( e^{(1)}_{\mu} g^{\mu 0} = e^{(2)}_{\mu} g^{\mu 0} = 0 \), the expression for \( M^{(2)(2)} \) is simplified to
\[
M^{(2)(2)} = -4k e^{(1)}_{\mu} \left[ (e^{(1)}_{\mu} g^{\mu 3}) e^{(2)}_{0} - (e^{(2)}_{\mu} g^{\mu 3}) e^{(1)}_{0} \right] \left[ g^{00} \Sigma_{01} - g^{00} \Sigma_{103} - g^{03} \Sigma_{313} \right].
\]
It turns out that \( g^{00} \Sigma_{01} - g^{00} \Sigma_{103} - g^{03} \Sigma_{313} = 0 \), and therefore
\[
M^{(2)(2)} = 0.
\]
The other components also vanish, \( M^{(1)(3)} = M^{(2)(3)} = 0 \), and consequently the angular momentum \( L^{(1)(j)} \) of the spacetime of a rotating mass shell vanishes if computed out of equation (25). However this result should come as no surprise, since the reference frame represented by equation (25) is rotating around the source with angular velocity \( \Omega \).

Now we address another configuration of tetrad fields that has a simple interpretation as reference frame. We consider
\[
e_{\mu} = \begin{pmatrix}
-X & 0 & 0 & Z \\
0 & \psi^2 \sin \theta \cos \phi & r \psi^2 \cos \theta \sin \phi & -Y \sin \theta \sin \phi \\
0 & \psi^2 \sin \theta \sin \phi & r \psi^2 \cos \theta \sin \phi & Y \sin \theta \cos \phi \\
0 & \psi^2 \cos \theta & -r \psi^2 \sin \theta & 0
\end{pmatrix},
\]
where
\[
X = (V^2 - r^2 \Omega^2 \psi^4 \sin^2 \theta)^{1/2}, \quad Y = -\frac{1}{X} \Omega r^2 \psi^4 \sin^2 \theta, \quad Z = \frac{V}{X} r \psi^2.
\]
This set of tetrad fields yields the velocity field given by
\[
e_{(0)}^\mu (t, r, \theta, \phi) = \left( \frac{1}{X}, 0, 0, 0 \right).
\]
According to the physical interpretation of equation (31), the latter is adapted to static observers in spacetime.

The nonvanishing components of \( T_{\mu \nu} \) are
\[
T_{001} = X \partial_1 X, \quad T_{301} = -Z \partial_1 X, \\
T_{202} = X \partial_2 X, \quad T_{302} = -Z \partial_2 X, \\
T_{212} = r^2 \psi^2 (\partial_1 \psi^2), \quad T_{013} = X \partial_1 Z, \\
T_{313} = -Z \partial_1 Z + (\partial_1 Y - \psi^2) Y \sin^2 \theta, \quad T_{023} = X \partial_2 Z, \\
T_{323} = -Z \partial_2 Z + Y (\partial_2 Y) \sin^2 \theta - (r \psi^2 - Y) Y \sin \theta \cos \theta.
\]
The quantities above yield the following nonvanishing components of \( \Sigma_{\mu \nu} \):
\[
\Sigma_{001} = \frac{1}{2} (T_{001} - g_{00} T_1), \quad \Sigma_{301} = \frac{1}{2} (T_{301} - T_{013}) - \frac{1}{2} g_{03} T_1, \\
\Sigma_{002} = \frac{1}{2} (T_{002} - g_{00} T_2), \quad \Sigma_{302} = \frac{1}{2} (T_{302} - T_{023}) - \frac{1}{2} g_{03} T_2, \\
\Sigma_{103} = \frac{1}{2} (T_{103} + T_{301}), \quad \Sigma_{112} = -\frac{1}{2} g_{11} T_2, \\
\Sigma_{212} = \frac{1}{2} (T_{212} + g_{22} T_1), \quad \Sigma_{013} = \frac{1}{2} (T_{013} - T_{301}) + \frac{1}{2} g_{03} T_1, \\
\Sigma_{313} = \frac{1}{2} (T_{313} + g_{33} T_1), \quad \Sigma_{023} = \frac{1}{2} (T_{023} - T_{302}) + \frac{1}{2} g_{03} T_2, \\
\Sigma_{323} = \frac{1}{2} (T_{323} + g_{33} T_2).
\]
The traces of the torsion tensor are
\[
T_1 = g^{00} T_{001} + g^{03} (T_{301} - T_{013}) - g^{22} T_{212} - g^{33} T_{313}, \\
T_2 = g^{00} T_{002} + g^{03} (T_{302} - T_{023}) - g^{33} T_{323}.
\]
The only physical parameter in equations (21) and (31) is $\alpha$, which is related to the mass of the shell by means of $m = 2\alpha$. By making $\alpha = 0$ we find that $T_{013} = \partial_1 Z$ and $T_{023} = \partial_2 Z$ are nonvanishing, and therefore $T_{013} \neq 0$ and $T_{023} \neq 0$ in this limit. The latter quantities behave as $O(r^{-2}\sin^2 \theta)$ and $O(r^{-1}\sin \theta \cos \theta)$, respectively. Consequently, we anticipate that it will be necessary to make use of the regularized definition of the gravitational angular momentum.

The exact expression of $M^{(1)}$ is given by

\[
M^{(1)} = -2ke \left( e^{(1)}_3 e^{(2)}_1 - e^{(1)}_1 e^{(2)}_3 \right) g^{00} g^{03} g^{11} (T_{001} - g_{00} T_1) - g^{00} g^{11} g^{33} (T_{013} + g_{03} T_1) + g^{03} g^{11} (T_{001} - g_{03} T_1) - g^{03} g^{11} g^{33} (T_{013} + g_{33} T_1) - 2ke \times \left( e^{(1)}_3 e^{(2)}_2 - e^{(1)}_1 e^{(2)}_3 \right) g^{00} g^{03} g^{22} (T_{002} - g_{03} T_2) + g^{00} g^{22} g^{33} \left( \frac{1}{2} (T_{023} - T_{032}) - g_{03} T_2 \right) - g^{03} g^{22} (T_{023} - T_{032}) + g^{03} g^{22} g^{33} (T_{323} + g_{33} T_2).
\]

In order to simplify the expression above we will make two assumptions. We will assume

\[
r^2 \Omega^2 \ll 1,
\]

\[
r_0 \gg \alpha.
\]

Equation (37) is necessary because the metric tensor given by equation (21) is a solution of Einstein’s equations in the limit of slow rotation. The assumption given by equation (38) simplifies several calculations, but we note that it holds in the consideration of ordinary laboratory objects, for which we know the value of the Newtonian angular momentum. Both conditions imply that $X = (V^2 - r^2 \Omega^2 \psi^2 \sin^2 \theta)^{1/2}$ is always real.

Condition (37) simplifies equation (36) to

\[
M^{(1)} = -2ke \left( e^{(1)}_3 e^{(2)}_1 - e^{(1)}_1 e^{(2)}_3 \right) g^{00} g^{03} g^{11} (T_{001} - g_{00} T_1) - g^{00} g^{11} g^{33} T_{013} + ke \left( e^{(1)}_3 e^{(2)}_2 - e^{(1)}_1 e^{(2)}_3 \right) g^{00} g^{03} g^{22} T_{002} - g^{00} g^{22} g^{33} T_{023}.
\]

Taking into account now condition (38) we obtain

\[
M^{(1)} = -4k \left[ 2\alpha \Omega \rho \sin \theta - \frac{1}{2} \Omega^2 \sin^3 \theta + \frac{1}{2} \Omega^2 r^2 \sin \theta \cos^2 \theta \right].
\]

Integration of the last two terms in the equation above diverges. These terms arise precisely because of the $T_{013}$ and $T_{023}$ components, and we see that they do not depend on the parameter $\alpha$. Thus these terms must be regularized, according to the discussion after equation (35).

The first term in equation (40) arises due to the spatial derivative of $\psi$ (specifically, there arises the term $\partial_1 \psi^2 = -(2\alpha)/r^2$ in the combination $T_{001} - g_{00} T_1$). Therefore the first term vanishes for $r < r_0$ because $\partial_1 \psi^2 = 0$ in this region, and we finally have

\[
M^{(1)}(\alpha) - M^{(1)}(\alpha = 0) \equiv \frac{1}{4\pi} 2\alpha (\Omega \rho) \sin^3 \theta,
\]

for $r > r_0$. In terms of the notation of equation (20) we have $M^{(1)}(\alpha) = M^{(1)}(\alpha = 0) = M^{(1)}(\epsilon) = M^{(1)}(E)$.

Integration of equation (41) yields

\[
L^{(1)}(\epsilon) \equiv -\frac{8\alpha}{3r_0} J = -\frac{4m}{3r_0} J,
\]

where

\[
J = \frac{1}{2} \left( r_0 \psi^2 \right)^2 \Omega_0.
\]

It is not difficult to verify that

\[
L^{(1)}(\epsilon) = L^{(2)}(\epsilon) = 0.
\]
We arrive at precisely the same value obtained in [6], except for the sign, in the analysis of the same gravitational field configuration. We recall that Cohen [20] identifies \( J \) given by equation (43) as the Newtonian value for the angular momentum of a rotating mass shell. Similar to the analysis of [6], it is possible to write \( L^{(1)} \) as the product of the moment of inertia of the source with \( \Omega_0 \), which is the induced angular velocity of inertial frames inside the shell. More precisely, we have [6]

\[
L^{(1)} = -\left( \frac{2mr_0^2}{\Omega_0} \right) \Omega_0. \tag{45}
\]

We can take into account the whole discussion presented in [6] and assert that \( L^{(1)} \) yields the value of the angular momentum of the field, not of the source (we have to reintroduce the constants \( c \) and \( G \) in all definitions). For weak gravitational fields we expect the gravitational angular momentum to be of small intensity in laboratory (cgs) units. The gravitational field of a mass shell of typical laboratory values is negligible, and consequently we would expect its gravitational angular momentum to be negligible as well. In contrast, the gravitational angular momentum of the spacetime of a rotating mass shell obtained by means of the Komar integral [21] yields a value that is of the same order of magnitude of the angular momentum of the source [6].

6. Final remarks

In this paper we have addressed the gravitational angular momentum in the framework of the TEGR. The definitions of energy–momentum and 4-angular momentum arise in the realm of the field equations of the TEGR and are separately invariant under general coordinate transformations of the three-dimensional space and under time reparametrizations (according to the discussion at the end of section 2). The gravitational angular momentum is frame dependent, and in the two cases studied above such dependence is clearly consistent with the physical interpretation of the frames. The general characterization of tetrad fields as reference frames in terms of the velocity and acceleration fields, \( u^\mu = e_{\{0\}}^\mu \) and \( a^\mu \), respectively, is an issue that must be better understood in general.

The gravitational 4-angular momentum in the context of the TEGR has already been investigated in [22, 23]. These analyses are essentially of the same nature as those developed in [11–14], namely, the gravitational angular momentum is identified with Hamiltonian boundary terms at spacelike infinity. In particular, the boundary terms considered in [22] and [23] agree with each other in the limit \( r \to \infty \).

It is very likely that the asymptotic boundary conditions and the parity conditions discussed by Beig and Ó Murchadha [13] and by Szabados [14] are necessary in order to arrive at a finite value for the gravitational 4-angular momentum, even in the context of the regularized definition given by equation (20). This issue must be investigated. It is not clear that the regularization procedure eliminates all types of infinities when the integrations are performed over the spacelike section of the spacetime.

The evaluation of the gravitational angular momentum of the spacetime of a rotating black hole, given by the Kerr solution, is not an easy procedure. The attempt towards this problem, briefly exposed in [6], led us to conclude that the essential singularity of the solution poses a problem in the integration of the angular momentum density. However, we believe that this problem may be overcome by adopting a more convenient set of coordinates [24] to the Kerr solution in the context of the regularized definition (20).
In view of equation (17) we define the gravitational Pauli–Lubanski vector $W_a$,

$$W_a = \frac{1}{2} \varepsilon_{abcd} P^b L^{cd}. \quad (46)$$

It is straightforward to verify that $W_a$ commutes with $P^b$,

$$\{W_a, P^b\} = 0. \quad (47)$$

Therefore we may also define the Casimir type quantities $P^2$ and $W^2$,

$$P^2 = \eta_{ab} P^a P^b, \quad W^2 = \eta^{ab} W_a W_b, \quad (48)$$

which commute with both $P^a$ and $W_a$. In this sense we may say that $P^2$ and $W^2$ are invariants. These quantities might play an important role in the characterization of the gravitational field. In [10] it was evaluated the value of $P^2$ for a Schwarzschild black hole of mass $m$. It was obtained $P^2 = -m^2 c^2$ in the framework of a static or moving observer. The generality of this result must be investigated. Another issue to be addressed is the polarization of gravitational waves. It should be analysed to what extent $W_a$ and $P^a$ yield information about the helicity of plane gravitational waves, for instance. We recall that investigations about the possible existence of spin-1 gravitational waves have been carried out in the literature [25]. This issue will considered in the context of the present analysis.

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