A more general Pandora’s rule?

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Abstract

In a classic model analysed by Weitzman an agent is presented with boxes containing prizes. She may open boxes in any order, discover prizes within, and optimally stop. She wishes to maximize the expected value of the greatest prize found, minus costs of opening boxes. The problem is solved by a so-called Pandora’s rule, and has applications to searching for a house or job. However, this does not model the problem of a student who searches for the subject to choose as her major and benefits from all courses she takes while searching.

So motivated, we ask whether there exist any problems for which a generalized Pandora’s rule is optimal when the objective is a more general function of all the discovered prizes. We show that if a generalized Pandora’s rule is optimal for all specifications of costs and prize distributions, then the objective function must take a special form. We also explain how the Gittins index theorem can be applied to an equivalent multi-armed bandit problem to prove optimality of Pandora’s rule for the student’s problem. However, we also show that there do exist some problems which are not of multi-armed bandit type for which Pandora’s rule is optimal.

1 Weitzman’s problem and its generalization

1.1 Hunting the best prize

In a classic problem that was first analyzed by Weitzman [7] an agent called Pandora is presented with \( n \) boxes, each of which contains a prize. Pandora can, by paying a known cost \( c_i \), open box \( i \) to reveal its prize. The nonnegative value of the prize, denoted \( x_i^o \), is not known until the box is opened, but ex ante it has known distribution \( F_i \). The superscript ‘o’ is provided as a mnemonic for ‘opened’ or ‘observed’. Pandora wishes to choose the order of opening the boxes, and when to stop opening, so as

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to maximize the expected value of the greatest discovered prize, net of the sum of the costs paid to open boxes. Weitzman’s problem is attractive for two reasons. Firstly, it has an enormous number of applications, such as to searching for a house, job, or research project to conduct. A key feature is that it combines problems of scheduling (in what order should the boxes be opened?) and stopping (when should one be content to take the greatest prize found thus far?)

The second reason that the problem is attractive is that it has a remarkably simple solution, which we now describe. Suppose there is just one unopened box, say box $i$. However, there is also a reservation prize already on the table, of value $y$ that may be taken at any time. It is optimal not to open the box $i$ if and only if

$$y \geq -c_i + E \max[y, x_i^o].$$

The expectation is taken over $x_i^o$ according to the distribution $F_i$. The inequality is equivalent to $c_i \geq E \max[0, x_i^o - y]$, whose right-hand is decreasing in $y$. So there is a least nonnegative $y$ for which is true:

$$x_i^* = \min\{y : c_i \geq E \max[0, x_i^o - y], \ y \geq 0\}, \quad (1)$$

and $x_i^*$ is called the reservation value (or reservation prize) of $x_i$.

The so-called Pandora’s rule, which is optimal for Weitzman’s problem, is to first compute the reservation value of each box, as if each were the only box, and then open boxes in descending order of these values until a prize is found whose value exceeds the reservation value of any unopened box.

Attractive as it is, the Weitzman model does not cover an important and large class of applications in which the agent’s utility is not merely a function of the one prize the agent takes at the end of search, but of all prizes uncovered. Such problems contain features of both Weitzman’s problem and the celebrated multi-armed bandit problem as solved by Gittins and Jones [6]. For example, a student benefits from the courses she takes while searching for the subject to choose as major; or people obtain a flow utility of dating with different partners in the process of looking for a spouse; or an institution which experiments with different forms of organization, before adopting a more permanent form, is affected by those temporary forms.

Motivated by this, and other applications to be described in the following sections, we now consider a possible generalization of Weitzman’s model. Suppose that $S$ is the set of opened boxes at the point

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1 Weitzman anticipated that Pandora’s rule would not generalize to such problems. He wrote: “If some fraction of its reward can be collected from a research project before the sequential search procedure as a whole is terminated, that could negate Pandora’s rule in extreme cases.” However, he gave no supporting detailed analysis, and it turns out to be difficult to say whether or not some interesting generalization might be possible.
we stop, and the vector of the prize values found is \( x^o_S = (x^o_i, \ i \in S) \). In Weitzman’s problem the aim is to maximize the expected value of

\[
R(x^o_S) = \max_{i \in S} x^o_i - \sum_{i \in S} c_i. \tag{2}
\]

(Strictly speaking, \( R \) is a function of \( S \) and \( x^o_S \). However, in (2), and in similar definitions that depend on \( (S, x^o_S) \), it is convenient to suppress the \( S \).) Now consider a more general reward, expressed as a utility that depends on all the prizes discovered.

\[
R(x^o_S) = u(x^o_S) - \sum_{i \in S} c_i. \tag{3}
\]

We shall use the notation \( \emptyset \) to denote both the empty set and the empty vector; so may write \( R(\emptyset) = 0 \).

The paper is primarily concerned with the following question.

**Question 1**  
For what utility functions \( u \) is a simple (generalized) Pandora’s rule optimal?

### 1.2 A generalized Pandora’s rule

To answer Question 1 we must start by saying what a ‘generalized Pandora’s rule’ might be. Suppose a set of boxes \( S \subset N = \{1, \ldots, n\} \) has been opened, and \( i \not\in S \). We might ask, what is the smallest prize whose addition to the set of prizes already discovered would cause it to be optimal to stop rather than open box \( i \) and then stop? This defines a reservation value (or prize) for \( x_i \), of

\[
x^*_i = \min \{ y : u(x^o_S, y) \geq -c_i + Eu(x^o_S, y, x^o_i), \ y \geq 0 \} \tag{4}
\]

with the expectation being taken over \( x^o_i \). Notice that for this to make sense we must assume, as we now do, that \( u \) is a function that maps a vector of any length to a real value. Note that if \( x^o_S \) is a vector of length \( n - 1 \) then the right-hand side of (4) requires \( u \) to be defined over vectors of length \( n + 1 \). It is straightforward to see that for Weitzman’s problem (1) and (4) coincide, as do the following definitions of Weitzman’s Pandora’s rule and a generalized Pandora’s rule.

**Definition 1** (Weitzman’s Pandora’s rule). Open the unopened box with greatest reservation value, as defined by (4), until there is no unopen box whose reservation value exceeds the greatest prize that has been found.
**Definition 2** (Generalized Pandora’s rule). Open the unopened box with greatest reservation value, as defined by (4), until there is no unopened box whose reservation value exceeds 0.

Notice that $x^*_i$ is not an index in the most usual sense. Unlike the reservation prize $l$ in Weitzman’s problem, or a Gittins index, $x^*_i$ is a function not only of $c_i$ and $F_i$, but also $x^*_S$, the vector of values of prizes that have already been uncovered. Consequently, after opening a box, the reservation values of all the unopened boxes must be recomputed from $l$.

### 1.3 A generalized utility function

We cannot hope that the generalized Pandora’s rule should be optimal unless we place some restrictions on the utility function $u$. What minimal constraints should we impose on $u$ to obtain a nice answer? Let us take as a guide the fact that the utility $u(x_1,\ldots,x_k) = \max_i x_i$ of Weitzman’s problem has several special properties, which we now group under the heading of Assumption 1 and will also wish to require subsequently.

**Assumption 1.**

1. $u(\emptyset) = 0$ and $u(0,x_2,\ldots,x_k) = u(x_2,\ldots,x_k)$.
2. $u$ is continuous, nonnegative, symmetric, nondecreasing and submodular in its arguments;

By ‘symmetric’ we mean that the value of $u$ for any $k$-tuple of arguments is the same as its value for any permutation of that $k$-tuple. So utility depends only the set of prizes found, not on the order in which they are found. By ‘submodular’, we mean that for any vectors $x$ and $y$ of the same length

$$u(x) + u(y) \geq u(x \wedge y) + u(x \vee y)$$

where $x \wedge y$ and $x \vee y$ denote the minimum and maximum of $x$ and $y$ taken component-wise. An equivalent statement is that the increase in $u(x)$ obtained by increasing one component of $x$ becomes no greater as any other component becomes greater. That is, for any $x^*_S$, $\underline{x}_1 < \underline{x}_1$ and $\underline{x}_2 < \underline{x}_2$,

$$u(x^*_S,\underline{x}_1,\underline{x}_2) - u(x^*_S,\underline{x}_1,\underline{x}_2) \leq u(x^*_S,\underline{x}_1,\underline{x}_2) - u(x^*_S,\underline{x}_1,\underline{x}_2).$$

We shall assume throughout the rest of the paper that Assumption 1 holds. It is helpful to remember that $u$ is symmetric when reading some of the expressions below.

There is a further property of (2), which we now state as distinct from Assumption 1.
Assumption 2. The benefit of increasing a component of $x$ from 0 to some positive value $x_i^0$ is independent of the values of other components of $x$ which are greater than $x_i^0$. That is, for $x_S^0$ and any $x_i \leq x_j < \overline{x}_j$, with $i, j \not\in S$,

$$u(x_S^0, \overline{x}_j, x_i) - u(x_S^0, \overline{x}_j, 0) = u(x_S^0, \underline{x}_j, x_i) - u(x_S^0, \underline{x}_j, 0).$$  \hspace{1cm} (5)

At first sight this assumption appears very strong, but we shall see that if a generalized Pandora’s rule is to be optimal for all choices of $c_i$ and $F_i$, then it is necessary.

Consider now the following special case of utility. Since it plays an important role in our paper we define a name for it. In the proof of Theorem 2 below, we see that this form of $u$ poses a problem that is equivalent to a multi-armed bandit problem.

Definition 3. A utility function $u$ is said to be ‘Strongly Pandora’s Rule compatible’ (SPR) if

$$u(x_S^0) = u(\max_{i \in S} x_i^0) - f(\max_{i \in S} x_i^0) + \sum_{i \in S} f(x_i^0)$$ \hspace{1cm} (6)

where $f$ denotes some function, and then $u, f, u - f$ are all nonnegative and nondecreasing functions, and $u(0) = f(0) = 0$.

It is straightforward to check that if $u$ is SPR compatible then it satisfies Assumptions 1 and 2. Moreover, if $u$ is SPR compatible then the generalized reservation value is equal to Weitzman’s reservation value. To see this, we note that the reservation value of $x_i$ is the least $y$ such that

$$u(x_S^0, y, 0) \geq -c_i + Eu(x_S^0, y, x_i^0),$$ \hspace{1cm} (7)

or equivalently, with $x_1 = \max_{j \in S} x_j^0$, the least $y$ such that

$$u(x_1) \geq -c_i + E \max\{u(x_1) + f(x_i^0), u(x_i^0) + f(x_1)\}, \hspace{1cm} y \leq x_1,$$

$$u(y) \geq -c_i + E \max\{u(y) + f(x_i^0), u(x_i^0) + f(y)\}, \hspace{1cm} y > x_1.$$ \hspace{1cm} (8)

or, in a form that we wish to compare later to (13),

$$u(y) - f(y) \geq -c_i + Ef(x_i^0) + E \max\{u(y) - f(y), u(x_i^0) - f(x_i^0)\}, \hspace{1cm} y > x_1.$$ \hspace{1cm} (9)

In the special case of Weitzman’s problem $f = 0$ and $u(x) = x$, and then the second line in (8) (or (6)).
reduces to $y \geq -c_i + E \max\{x_i^0, y\}$, which agrees with the calculation of Weitzman’s reservation prize value in (1).

Throughout the paper we assume Assumption 1 holds. In Section 2, we show if the generalized Pandora’s rule is to be optimal for all possible choices of $((c_i, F_i), \ i \in N)$ then the utility $u$ must be SPR compatible. That is, if $u$ satisfies Assumption 1, but is not SPR compatible, then there exist some costs $(c_i, \ i \in N)$ and prize distributions $(F_i, \ i \in N)$ such that Pandora’s rule is not optimal.

In Theorem 2 of Section 3 we present a converse to the above: namely, that if $u$ is SPR compatible then Pandora’s rule is optimal for all $((c_i, F_i), \ i \in N)$. We prove Theorem 2 by recasting the problem as an equivalent multi-armed bandit problem and applying the Gittins index theorem. The connection between the multi-armed bandit problem and Weitzman’s problem has been previously noticed by Chad and Smith [2] who remarked that “Weitzman’s method is a nice application of Gittins’ solution of the bandit problem”. However they focused upon a problem of simultaneous search, rather than Weitzman’s sequential search, and did not actually explain how the solution to Weitzman’s problem can be obtained from the Gittins index theorem.

Of course, the results of Section 2 leave open the possibility that there might exist interesting problems in which $u$ is not SPR compatible, and yet Pandora’s rule is optimal for some, but not all, $(c_i, F_i), \ i \in N)$. In Theorem 3 of Section 4 we establish that this happens if Assumptions 1, 2 and a further Assumption 3 (which we also denote as ‘ORD’) are all satisfied. Theorem 3 is stronger than Theorem 2 because if $u$ is SPR compatible then ORD is satisfied. Theorem 3 cannot be proved by applying the Gittins index theorem or by some adaptation of Weitzman’s proof. We now conclude this introduction by describing a problem of this type to which Theorem 3 provides the optimal strategy.

**Example 1.** Suppose that each $F_i$ is the two-point distribution such that $x_i^0 = 0$ or $x_i^0 = 1$ with given probabilities $q_i$ and $p_i = 1 - q_i$, respectively. One has in mind that each box may or may not contain a prize, but all prizes are the same. Let $\psi$ be a concave increasing function of the total value of prizes found and consider the objective function

$$R(x^0_S) = \psi\left(\sum_{i \in S} x_i^0\right) - \sum_{i \in S} c_i.$$  \hspace{1cm} (10)

This $u = \psi$ does not obey Assumptions 1 and 2, so by Theorem 1 Pandora’s rule is not optimal for all $c_i$ and $F_i$. However, for the form of $F_i$ given, we will see that it is easy to check that the sufficient conditions of Theorem 3 are met, and so we may conclude that Pandora’s rule is optimal. The rule takes the following form. Suppose $c_1/p_1 \leq \cdots \leq c_n/p_n$. Then one should open boxes in the order
1, 2, ..., but stop when we are about to open some box $j$, have thus far found $k \leq j - 1$ prizes, and

$$
\psi(k) \geq -c_i + p_i \psi(k + 1) + q_i \psi(k),
$$

or equivalently,

$$
c_i / p_i \geq \psi(k + 1) - \psi(k).$$

While it would be possible to guess and establish this fact by a fairly short tailored proof, using induction on $n$, the sufficient conditions provided by Theorem 4 are quick to check.

## 2 Necessary conditions for strong Pandora’s rule optimality

We state some preliminary lemmas, whose proofs are in Appendices A and B.

**Lemma 1.** Suppose the utility $u$ satisfies Assumption 1 and Pandora’s rule maximizes expected utility for all costs $c_i$ and distributions $F_i$. Then $u$ also satisfies Assumption 2.

**Lemma 2.** Suppose the utility $u$ satisfies Assumptions 1 and 2. For any $(x^0_S : i \in S)$, let $x_\ell$ denote the $\ell$th greatest element. Then,

(a) there exist functions $f_\ell : \mathbb{R} \rightarrow \mathbb{R}$, $\ell = 1, 2, \ldots$, such that for any $x^0_S$ we have

$$
u(x^0_S) = \sum_{\ell=1}^{\vert S \vert} f_\ell(x_\ell), \tag{11}
$$

(b) $f_\ell(x)$ is (weakly) increasing in $x$ and (weakly) decreasing in $\ell$,

(c) $f_\ell(x) - f_{\ell+1}(x)$ is weakly increasing in $x$.

The main theorem of this section now follows. The fact that it concerns conditions which are necessary if Pandora’s rule is to be optimal for all costs $c_i$ and distributions $F_i$ is signaled by the word ‘strong’ that appears in the title to this section. Similarly, ‘strong’ appears in the title to Section 3, but not in the title to Section 4, which concerns sufficient conditions, depending upon specific $c_i, F_i$, for which Pandora rule is optimal.

**Theorem 1.** Suppose the utility $u$ satisfies Assumption 1, and Pandora’s rule maximizes expected utility for all costs $c_i$ and distributions $F_i$. Then necessarily $u$ must be SPR compatible (Definition 3).
Remark. Note that $u(x_1, x_1) = u(x_1) + f(x_1) \implies f(x) = u(x, x) - u(x)$. As in Lemma 2, we let $x_1 \geq x_2 \geq \cdots \geq x_{|S|}$ denote the ordered $(x_i : i \in S)$. Then we may write (6), as

$$u(x^0_S) = u(x_1) + \sum_{\ell=2}^{|S|} f(x_\ell),$$

where $f(x) = u(x, x) - u(x)$.

Part proof of Theorem 1. We give in Appendix C the proof that $u$ must satisfy (6). For now, we prove that if (6) holds then the subsequent statements within Definition 3 are true. Clearly, $u(0) = f(0) = 0$, and $u, f$ are nonnegative. The other facts are proved as follows.

- $f(x)$ is nondecreasing in $x$, since for $x < x'$

$$f(x) = u(x, x) - u(x) = u(x, x') - u(x') \leq u(x', x') - u(x'),$$

where the equality is by Assumption 2, and the inequality is by Assumption 1.

- $u - f$ is nonnegative since by an application of Assumption 1 (submodularity)

$$u(x) - f(x) = u(x) - [u(x, x) - u(x, 0)] \geq u(x) - [u(0, x) - u(0, 0)] = 0.$$

- $u(x) - f(x)$ is nondecreasing in $x$, since for $x < x'$

$$u(x) - f(x) = u(x) - [u(x, x) - u(x)]$$

$$\leq u(x) - [u(x', x) - u(x')]$$

$$\leq u(x') - [u(x', x') - u(x')]$$

$$= u(x') - f(x'),$$

where the first and second inequalities follow from Assumption 1, by submodularity and monotonicity, respectively.

3 Sufficient conditions for strong Pandora’s rule optimality

Theorem 2 is a converse to Theorem 1.
Theorem 2. If utility \( u \) is SPR compatible then, for all costs \( c_i \) and distributions \( F_i \), Pandora’s rule is optimal.

Proof. We prove this by mapping the problem to an instance of a multi-armed bandit problem and applying the Gittins index theorem. In particular, we use the fact that the Gittins index theorem is true for undiscounted target processes (Gittins, Glazebrook and Weber [5], Chapter 7). In a problem about target processes, reward accrues only until the first time that one of the processes reaches a target (or certain state). This is also the problem that Dumetriu, Tetali and Winkler [3] cutely call ‘playing golf with more than one ball’, in which the aim is to minimize costs (or maximize reward) until one of several golf balls is first sunk in a hole.

Consider a family of \( n \) alternative bandits processes, each evolving on its own state space. We will think of the covered variables as bandits in state 0, and the target or hole as a certain state 1. Bandit \( i \) starts in its initial state 0. When it is continued for the first time a reward \(-c_i\) accrues and the state makes a random transition to a new state which we denote as \((x_0^i, 1)\), where \(x_0^i\) is chosen according to distribution \(F_i\). When the bandit is continued from this state a reward \(f(x_0^i)\) accrues and the state makes a deterministic transition to \((x_0^i, 2)\). When the bandit is continued from this state a reward \(u(x_0^i) - f(x_0^i)\) accrues and the state makes a deterministic transition to state 1. As this state is the target the problem comes to an end. So only one of the bandits can make a third step. (Note that the Gittins index theorem is true for an uncountable state space, as we may have here.)

It is clear that in maximizing reward it will be optimal to continue some set of bandits twice each, and then one of these a third time (the one we choose to be the one to enter state 1 and bring the problem to its end). If we were to continue bandits 1, \ldots, \(k\) twice, and then pick \(i\) for continuation a third time, we would achieve reward

\[-(c_1 + \cdots + c_k) + f(x_1^i) + f(x_2^i) + \cdots + f(x_k^i) + u(x_0^i) - f(x_0^i),\]

which is the same objective function as we seek to minimize when \( u \) is SPR compatible. Since \( u - f \) is nondecreasing the process we should choose for continuation a third time is clearly the one having greatest uncovered value \(x_1^0\).

Having mapped our problem to an instance of a multi-armed bandit problem we can appeal to the Gittins index theorem, which says that expected total reward is maximized by continuing at each stage a bandit with the greatest Gittins index (with ties broken arbitrarily). The Gittins index of bandit \( i \) in this problem can be found by the ‘calibration method’ which computes the Gittins index as the least
\[ \lambda \geq -c_i + E \max \{ f(x_i^o) + \lambda, f(x_i^o) + [u(x_i^o) - f(x_i^o)] \}. \tag{13} \]

The right hand side of (13) is the maximum expected reward which can be obtained by continuing bandit \( i \) once, and thereafter either continuing it once more to take reward \( f(x_i^o) \) and then retiring with reward \( \lambda \), or continuing it twice more until reaching state 1. Comparing (13) to (9), and identifying \( \lambda \) with \( u(y) - f(y) \) (which is nondecreasing in \( y \)) we see that the Gittins indices do indeed prescribe exactly the same policy as does Pandora’s rule for generalized reservation values and a \( u \) that is SPR compatible.

\[ \square \]

### 4 Sufficient conditions for Pandora’s rule optimality

We require a new assumption.

**Assumption 3 (ORD).** *(History-Independence of the Ordering of Reservation Values):*

The ordering of reservation values \( x^*_k \) of the covered variables is independent of both the number of variables that have already been uncovered and their realizations. That is, for any \( S \), \( x^*_S \), and \( k, j \notin S \),

\[ x^*_k(x^*_S) \geq x^*_j(x^*_S) \iff x^*_k(\emptyset) \geq x^*_j(\emptyset). \]

We denote this property by the abbreviation ORD. Admittedly, it is very strong assumption. Unlike Assumptions 1 and 2, this assumption is not simply a property of the utility function alone, but a joint property of the utility function \( u \), and the \((c_i, F_i)\). In most problems it is easy to check whether or not ORD is satisfied. In particular, it is satisfied if \( u \) is SPR compatible.

We now state the main theorem of this section. Its proof is in Appendix B.

**Theorem 3.** *If Assumptions 1, 2, and 3 are satisfied, then the generalized Pandora’s rule maximizes expected utility.*
4.1 Application

We conclude this section by applying Theorem 3 to Example 1 introduced in Section 1, in which we presented the problem of maximizing the expected value of

\[ R(x^o_S) = \psi \left( \sum_{i \in S} x^o_i \right) - \sum_{i \in S} c_i \]  

(14)

where \( \psi \) is a concave increasing function of the total value of prizes found. However, since we assume \( x^o_i \) is either 0 or 1, we might also pose it as the problem of maximizing the expected value of

\[ R(x^o_S) = \sum_{i=1}^{\left| S \right|} w_ix_t - \sum_{i \in S} c_i \]

where \( w_i = \psi(i) - \psi(i - 1) \). Given that \( k \) prizes have already been found the reservation value \( x^*_i \) is the least nonnegative \( y \) such that

\[ \sum_{j=1}^{k} w_i + w_{k+1}y \geq -c_i + \sum_{j=1}^{k} w_i + w_{k+1}y + p_i \left( (1 - y)w_{k+1} + w_{k+2}y \right) \]

and hence

\[ x^*_i = \max \left\{ 0, \frac{w_{k+1} - c_i/p_i}{w_{k+1} - w_{k+2}} \right\}. \]

Thus the reservation values of the covered variables can be ranked greatest to smallest in the order that their \( c_i/p_i \) are ranked least to greatest, independently of which other variables have been uncovered or not, and the values taken by the uncovered variables. This means that ORD holds and so by Theorem 3 Pandora’s rule is optimal.

While one could establish this fact by a hands-on and fairly short proof, using an induction on \( n \), the sufficient conditions provided by Theorem 3 are quicker to check.

5 Conclusions

Other researcher have also proved ‘negative results’, similar to our Theorem 1. For example, Banks and Sundaram [1] have shown that the Gittins index cannot be extended to bandits with switching costs. Also, Gittins, et al. [5, Chapter 3] show that the Gittins index theorem holds only when one makes assumptions of infinite time horizon, constant exponential discounting and that only one bandit is continued at a time. Example 1 could be solved by a Pandora’s rule because of the simplified
form of $F_i$. In other research index policies are sometimes found to be optimal by imposing so-called ‘compatibility constraints’ on the parameters of the problem.

There are other interesting Pandora box problems that remain unsolved. For example, it would be very interesting to address a version of Weitzman’s problem in which the prizes (offers) do not remain permanently available. When a box is opened its prize must be taken immediately or permanently lost. This problem is unlikely to have a simple answer, except in very special cases.

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Appendix

In Sections A–D of this appendix we prove results described in Section 2, showing that if Pandora’s rule is to be optimal for all choices of costs and distributions, $\{(c_i, F_i), i \in N\}$, then this places very severe restrictions on the admissible form of $u$. Section F contains the proof of Theorem 3.
A Proof of Lemma 1.

We wish to show that if Assumption 1 holds, and the Pandora’s rule maximizes expected utility for all costs $c_i$ and distributions $F_i$ then $u$ must satisfy Assumption 2.

In the following proof, as in that of Theorem 1 below, we will use examples in which $F_i$ is a degenerate distribution. We might rewrite the proof using continuous random variables instead, by making perturbations in which absolutely continuous distributions approximate our degenerate ones. But to give a proof using degenerate distributions is both simpler and stronger. It is stronger since any random variable with a degenerate distribution can be approximated (arbitrarily closely) by a continuous random variable (but not vice versa). So if we show that the assumption that Pandora’s rule is optimal for all $c_i, F_i$, with the $F_i$ assumed ‘degenerate’, then the same is true if ‘degenerate’ is replaced by ‘absolutely continuous’. The implication would not be true the other way around.

Proof. Consider an arbitrary $S$, $x_0^j$, and $j, k \notin S$, with $j \neq k$, and numbers $x_j^0 \leq x_k^0 < \overline{x}_k^0$. Suppose there were a violation of Assumption 2 of the form

$$u(x_S^0, x_j^0, x_k^0) - u(x_S^0, x_j^0) > u(x_S^0, x_k^0) - u(x_S^0, \overline{x}_k^0).$$

Notice that we could not have the opposite strict inequality, by Assumption 1 (submodularity). By Assumption 1, we can increase $x_j^0$ to $\overline{x}_k^0$ and the same inequality will hold. This implies that there exists $\epsilon > 0$ such that

$$u(x_S^0, x_j^0, x_k^0) - u(x_S^0, x_j^0) > u(x_S^0, x_k^0, \overline{x}_k^0) - u(x_S^0, \overline{x}_k^0) + \epsilon. \quad (15)$$

We now show that if (15) is true then Pandora’s rule cannot be optimal for all $(c_i, F_i, i \in N)$. To this end, suppose $F_j$ and $F_k$ are degenerate, with $x_j^0 = x_k^0$ and $x_k^0 = \overline{x}_k^0$ with probability 1. Let

$$c_j = u(x_S^0, x_k^0, x_k^0) - u(x_S^0, x_k^0) - \epsilon, \quad c_k = u(x_S^0, x_k^0, \overline{x}_k^0) - u(x_S^0, \overline{x}_k^0).$$

The reservation price of $x_j$ is the least nonnegative $y$ such that

$$c_j = u(x_S^0, x_k^0, x_k^0) - u(x_S^0, x_k^0) - \epsilon \geq u(x_S^0, x_k^0, y) - u(x_S^0, y). \quad (16)$$

Since (16) is false for $y = x_k^0$ we must have $x_j^0 > x_k^0$. 

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The reservation price of $x_k$ is the least nonnegative $y$ such that

$$c_k = u(x^0_S, x^0_k, x_k^0) - u(x^0_S, x_k^0) \geq u(x^0_S, x_k^0, y) - u(x^0_S, y). \quad (17)$$

Suppose $y$ is such that (16) holds, and therefore $y \geq x^*_j > x_k^0$. Then (17) also holds, since

$$u(x^0_S, x_k^0, y) - u(x^0_S, y)$$

$$= u(x^0_S, x_k^0, y) - u(x^0_S, x_k^0, y) + u(x^0_S, x_k^0, y) - u(x^0_S, y)$$

$$\leq u(x^0_S, x_k^0, y) - u(x^0_S, x_k^0, y) + u(x^0_S, x_k^0, y) - u(x^0_S, y) - \epsilon$$

$$= c_k + [u(x^0_S, x_k^0, y) - u(x^0_S, x_k^0, y)] - [u(x^0_S, x_k^0, x_k^0) - u(x^0_S, x_k^0, y)] - \epsilon$$

$$\leq c_k,$$

where the first inequality is by (16) and the second inequality is by using Assumption 1 (submodularity) to see that since $y \geq x_k^0$ the first square-bracketed term is no greater than the second.

From this it follows that $x_k^* \leq x_j^*$. Thus, according to Pandora’s rule, it would be optimal to next uncover $x_j$.

However, the payoff obtained by uncovering $x_k$ first and then stopping search is strictly greater than the payoff obtained by uncovering $x_j$ first and then stopping search if

$$u(x^0_S, x_k^0) - c_k > u(x^0_S, x_k^0) - c_j. \quad (18)$$

On substituting for $c_j$ and $c_k$ we find that (18) is the same as (15).

Note also that the right-hand side of (18) is nonnegative since

$$u(x^0_S, x_k^0) - c_k = u(x^0_S, x_k^0) - [u(x^0_S, x_k^0, x_k^0) - u(x^0_S, x_k^0)] + \epsilon$$

$$\geq u(x^0_S, x_k^0) - [u(x^0_S, x_k^0, 0) - u(x^0_S, 0)] + \epsilon$$

$$= u(x^0_S) + \epsilon,$$

where the inequality is by Assumption 1 (submodularity). The payoff obtained by uncovering $x_k$ first and then stopping search is also strictly greater than the payoff of uncovering both if

$$u(x^0_S, x_k^0) - c_k > u(x^0_S, x_k^0, x_k^0) - c_j - c_k. \quad (19)$$
On substituting for \( c_j \) we find that (19) is also the same as (15).

Assume the parameters of all uncovered variables, apart from \( x_j \) and \( x_k \), are such that they should certainly stay covered. For example, each such uncovered \( x_i \) might have \( x_i^0 = 0 \) and \( c_i > 0 \). We have argued that uncovering \( x_k \) first and then stopping search is strictly better than uncovering \( x_j \) first and then either stopping search or uncovering \( x_k \). As \( x_k^* \leq x_j^* \), Pandora’s rule dictates that it is optimal to uncover variable \( x_j \) first. As this is false, we must conclude that if Pandora’s rule is optimal then (15) must be false, and thus there can be no violation to Assumption 2.

\[ \square \]

**B Proof of Lemma 2.**

*Proof.* Define

\[
g_\ell(x) = u(x, \ldots, x)\]

\[
f_\ell(x) = g_\ell(x) - g_{\ell-1}(x)
\]

Then for \( S = \{1, \ldots, k\} \) and \( x_S^0 = (x_i, i \in S) \),

\[
u(x_S^0) = u(x_1, \ldots, x_k) = u(x_1, \ldots, x_k) - u(x_1, \ldots, x_{k-1}, 0) + u(x_1, \ldots, x_k)\]

\[
= u(x_k, \ldots, x_k) - u(x_{k-1}, 0) + u(x_1, \ldots, x_k) - u(x_k, \ldots, x_k)\]

\[
= g_k(x_k) - g_{k-1}(x_k) + u(x_1, \ldots, x_k) - u(x_k, \ldots, x_k)\]

\[
= f_k(x_k) + u(x_1, \ldots, x_k) - u(x_k, \ldots, x_k)\]

\[
= \sum_{\ell=1}^{k} f_\ell(x_\ell).
\]

where (21) follows from (20) by repeated application of Assumption 2. This proves (a).

For (b), the fact that \( f_\ell(x) \) is a decreasing function of \( \ell \) follows from Assumption 1 (submodularity). The fact that \( f_\ell(x) \) is increasing in \( x \) can be seen by taking \( x < x' \), and observing that

\[
g_\ell(x) - g_{\ell-1}(x) = u(x, x', \ldots, x') - u(x', \ldots, x') \leq g_\ell(x') - g_{\ell-1}(x'),
\]

where (21) follows from (20) by repeated application of Assumption 2. This proves (a).
where the equality is by Assumption 2, and the inequality is by Assumption 1.

Finally, for (c), we note that if $x < x'$,

$$f_\ell(x) - f_{\ell+1}(x) = [u(x, \ldots, x) - u(x, \ldots, x)] - [u(x, \ldots, x) - u(x, \ldots, x)]$$

The second line is by Assumption 2, and the third line by Assumption 1 (submodularity).

## C Proof of Theorem 1

We complete the proof of Theorem 1. As in the proof of Lemma 1 it is convenient to use degenerate distributions in constructing counterexamples.

**Proof of Theorem 1** To complete the proof begun in Section 2 it remains to show that $u$ has form of (6), i.e. that in Lemma 2 we can put $f_2 = \cdots = f_n$, where we have established in Lemma 2 that $f_1 \geq f_2 \geq \cdots \geq f_n$ and $f_2(0) = \cdots = f_n(0) = 0$. We do this for $f_2 = f_3$. The proof of $f_\ell(x_0) = f_{\ell+1}(x_0)$, $\ell > 2$ follows by examining an instance in which the first $\ell - 2$ variables to be uncovered are ones with $c_i = 0$ (and reservation values $\infty$) and their uncovered values are greater than any values that can be found amongst the variables which remain uncovered at that point.

(i) Assume $f_3 \neq 0$ and that there exists $x_0$ such that $f_1(x_0) \geq f_2(x_0) > f_3(x_0) > 0$. Consider three variables, $x_1$, $x_2$ and $x_3$ with the same degenerate distribution, having $x_0^i = x_0$, with probability 1, $i = 1, 2, 3$. Let costs be chosen so

$$c_3 = 0 \leq c_1 < f_3(x_0) < c_2 < f_2(x_0). \quad (22)$$

We proceed to show the generalized Pandora’s rule cannot be optimal.

Firstly, it follows from (22) that for all $y$ we have $u(y) < -c_i + Eu(y, x_0^i)$. Hence initially, when $S = \emptyset$, all three variables have reservation value $\infty$. So if Pandora’s rule is optimal then it must be optimal to uncover any of them first. Suppose $x_2$ is uncovered first, and then $x_3$ (which still has
reservation value $\infty$). It is now strictly best to uncover $x_1$ iff

$$f_1(x_0) + f_2(x_0) < -c_1 + f_1(x_0) + f_2(x_0) + f_3(x_0)$$

which is true because $c_1 < f_3(x_0)$. The payoff is that of uncovering all three variables.

Alternatively, if we uncover $x_1$ first, followed by $x_3$, it is now strictly best not to uncover $x_2$, since $c_2 > f_3(x_0)$. The difference between the expected payoffs of the strategy which uncovers $x_1, x_3$ and of that which uncovers $x_2, x_3, x_1$ is

$$\left[-c_1 - c_3 + f_1(x_0) + f_2(x_0)\right] - \left[-c_1 - c_3 - c_2 + f_1(x_0) + f_2(x_0) + f_3(x_0)\right] = c_2 - f_3(x_0),$$

which is positive, whereas if Pandora’s rule were optimal this difference should be no greater than 0.

(ii) Now consider the special case in which $f_3 = 0$ and $x_0$ is such that $f_1(x_0) \geq f_2(x_0) > f_2(x_0) = 0$. Suppose $x_i$ is a variable such that $x_i^0$ is equal to 0 or $x_0$ with probabilities $q_i = 1 - p_i$ and $p_i$. Consider the class of variables like this, for varying $p_i$ and $c_i$. All have initial reservation value $\infty$. Suppose a variable in this class is uncovered and reveals value $x_0$. Subsequent to this, the reservation value of another variable in the class is now the least $y$, with $y \leq x_0$, such that

$$f_1(x_0) + f_2(y) \geq -c_i + f_1(x_0) + p_i f_2(x_0) + (1 - p_i)f_2(y)$$

i.e. the least $y$ such that $f_2(y) \geq -c_i/p_i + f_2(x_0)$. Suppose $c_i/p_i$ is chosen just a bit less than $f_2(x_0)$, in such a way that the reservation value is positive. Recall that $f_2(y)$ is nondecreasing and continuous. Since $c_i/p_i$ may differ and $f_2 \neq 0$, variables in this class may now have different reservation values.

So suppose we start with three variables in this class. We uncover one and it takes value $x_0$. The other two now have positive reservation values, the greatest of which is for the variable with least value of $c_i/p_i$. In following Pandora’s rule we may start by uncovering any variable initially, and then continue by uncovering variables in increasing order of $c_\ell/p_\ell$, until either two values of $x_0$ have been revealed or all three variables have been uncovered.
If we uncover the variables in the order \( x_i, x_j, x_k \) then the expected payoff is

\[
\begin{align*}
- c_i + p_i \left[ f_1(x_0) - c_j + p_j f_2(x_0) + q_j \left[ -c_k + p_k f_2(x_0) \right] \right] \\
+ q_i \left[ -c_j + p_j \left[ f_1(x_0) - c_k + p_k f_2(x_0) \right] + q_j \left[ -c_k + p_k f_1(x_0) \right] \right] \\
= \left( \frac{c_k}{p_k} \right) p_i p_j p_k + \sigma,
\end{align*}
\]

where \( \sigma \) is an expression that is symmetric in \( i, j, k \). So if \( c_i / p_i < c_j / p_j < c_k / p_k < f_2(x_0) \) then it is strictly better to begin by uncovering \( x_i \) or \( x_j \), than to begin by uncovering \( x_k \). Thus optimality of Pandora’s rule is incompatible with \( f_2(x_0) > f_3(x_0) = 0 \).

D A special case of Theorem 1

Some readers might consider the proof of Theorem 1 to be slightly unsatisfactory because it refers to variables whose reservation value is \( \infty \). We conjecture, but have not been able to prove that Theorem 1 is true even if we restrict attention to variables with finite reservation values, at least under some mild restrictions on utility \( u \).

We can, however, prove Theorem 1 for the special utility described in Theorem 4. The proof has similarities to the proof of Theorem 1, and is omitted.

**Theorem 4.** Suppose \( w_1 \geq w_2 \geq \cdots \) are given, and for any \( x^0_S \) the utility is

\[
u(x^0_S) = \sum_{i=1}^{\left| S \right|} x_i w_i,\]

where \( x_1 \geq x_2 \geq \cdots \geq x_{\left| S \right|} \) are the ordered values of the components of \( x^0_S = (x^0_i : i \in S) \). Then

(a) \( u(\cdot) \) satisfies Assumptions 1 and 2;

(b) if Pandora’s rule is optimal for all \( \{(c_i, F_i), i \in N\} \) then necessarily \( w_2 = w_3 = \cdots \).

E Proof of Theorem 3

The proof of Theorem 3 is by induction with respect to the number of remaining, uncovered variables. Fix a set of already uncovered variables \( x^0 \), and suppose that Pandora’s rule applies to cases in which there are fewer remaining variables. We compare the payoff of uncovering first the variable \( k \) with the greatest reservation value \( x^*_k \), followed by some (possibly suboptimal) strategy, which will be specified
later, to the payoff of uncovering first a variable $\ell$, followed by an optimal continuation strategy, and we will show that the former payoff is no less than the latter payoff.

We need the following property of reservation values.

**Lemma 3.** The reservation value $x^*_k(x^o)$ is a (weakly) decreasing function of the number of uncovered variables. That is, for any $x^o = x^o_S$, $x^o_j$ and $x_k$ such that $j,k \notin S$ and $j \neq k$,

$$x^*_k(x^o) \geq x^*_k(x^o, x^o_j).$$

**Proof.** By Assumption 1 (submodularity),

$$\int u(x^o, x^*_k, x^o_k) dF_k(x^o_k) - u(x^o, x^*_k) \geq \int u(x^o, x^o_j, x^*_k, x^o_k) dF_k(x^o_k) - u(x^o, x^o_j, x^*_k)$$

for every value of $x^*_k$. Therefore, Lemma 3 follows from (4), defining $x^*_k$. Indeed, $x^*_k$ is determined by the intersection (more precisely, the point most to the left) of the graph of a decreasing function

$$\int u(x^o, x^*_k, x^o_k) dF_k(x^o_k) - u(x^o, x^*_k)$$

of variable $x^*_k$ with the horizontal line with intercept $c_k$ (assuming the usual convention regarding 0 and $+\infty$).

Equipped with Lemma 3, we can now prove Theorem 3.

**Proof of Theorem 3** If $x^*_k = 0$, and so all reservation values are equal to 0, it is an optimal strategy to stop search without uncovering any additional variable. Indeed, since

$$c_k \geq \int [u(x^o, x^*_k, x^o_k) - u(x^o, x^*_k)] dF_k(x^o_k)$$

for every $x^*_k$,

$$c_k \geq \int [u(x^o, 0, x^o_k) - u(x^o, 0)] dF_k(x^o_k) = \int [u(x^o, x^o_k) - u(x^o)] dF_k(x^o_k).$$

Thus, it is no worse to stop search without uncovering any variable than to stop search after uncovering variable $x_k$; or analogously, than to stop search after uncovering any other variable. However, by Lemma 3 and the inductive hypothesis, after uncovering any other variable, for all values of that variable, it is optimal to stop search.
In what follows, we assume that $x_{k}^{*} > 0$. Consider the strategy of uncovering variable $x_{\ell}$ first. We now characterize an optimal continuation strategy. It follows from the inductive assumption and ORD that an optimal continuation strategy is to uncover variable $x_{k}$ if $x_{k}^{*}(x_{o}, x_{\ell}^{0}) > 0$ and to stop search if $x_{k}^{*}(x_{o}, x_{\ell}^{0}) = 0$.

For any $x_{\ell}^{0} < x_{k}^{*}(x_{o})$,

$$u(x_{o}, x_{\ell}^{0}) < -c_{k} + \int u(x_{o}, x_{\ell}^{0}, x_{\ell}) dF_{k}(x_{\ell}^{0}).$$

Since

$$u(x_{o}, 0, x_{\ell}^{0}) = u(x_{o}, x_{\ell}^{0})$$

and $u(x_{o}, 0, x_{\ell}^{0}, x_{k}) = u(x_{o}, x_{\ell}^{0}, x_{k}^{0})$,

by Assumption 1 (its first part) we have that

$$u(x_{o}, y, x_{\ell}^{0}) < -c_{k} + \int u(x_{o}, x_{\ell}^{0}, y, x_{k}) dF_{k}(x_{\ell}^{0})$$

for sufficiently small $y$, which means that $x_{k}^{*}(x_{o}, x_{\ell}^{0}) > 0$.

For any $x_{\ell}^{0} \geq x_{k}^{*}(x_{o})$,

$$u(x_{o}, x_{\ell}^{0}) \geq -c_{k} + \int u(x_{o}, x_{\ell}^{0}, x_{k}) dF_{k}(x_{\ell}^{0}).$$

By Assumption 1 (submodularity),

$$u(x_{o}, y, x_{\ell}^{0}, x_{k}) - u(x_{o}, y, x_{\ell}^{0}) \leq u(x_{o}, x_{\ell}^{0}, x_{k}^{0}) - u(x_{o}, x_{\ell}^{0})$$

for every $y$. Thus,

$$u(x_{o}, y, x_{\ell}^{0}) \geq -c_{k} + \int u(x_{o}, x_{\ell}^{0}, y, x_{k}) dF_{k}(x_{\ell}^{0})$$

for every $y$, which means that $x_{k}^{*}(x_{o}, x_{\ell}^{0}) = 0$.

Thus, it is an optimal continuation strategy (after uncovering variable $x_{\ell}$) to uncover variable $x_{k}$ if $x_{\ell}^{0} < x_{k}^{*}(x_{o})$ and to stop search if $x_{\ell}^{0} \geq x_{k}^{*}(x_{o})$. We compare the strategy of uncovering variable $x_{\ell}$ first (followed by this optimal continuation strategy) to the strategy of uncovering variable $x_{k}$ first, followed by stopping search if the realization of this variable exceeds $x_{k}^{*}(x_{o})$, and uncovering variable $x_{\ell}$ next if the realization of variable $x_{k}$ falls below $x_{k}^{*}(x_{o})$. In what follows, we let $x_{k}^{*} = x_{k}^{*}(x_{o})$ and assume that $x_{k}^{*} < +\infty$. If $x_{k}^{*} = +\infty$, the two strategies obviously yield the same payoff.

The comparison of the two strategies yields Figure 1(a), where the vertical and horizontal lines are at the level of $x_{\ell} = x_{k}^{*}$ and $x_{k} = x_{k}^{*}$, respectively. Within each cell the upper line is the payoff under
the policy of uncovering $x_\ell$ first and the lower line is the payoff under the policy of uncovering $x_k$ first. In the south west cell of the diagram, both strategies yield the same continuation payoff, which is denoted by $U(x^o, x^o_\ell, x^o_k)$. This is the greatest continuation payoff, contingent on the realizations $x^o_\ell, x^o_k < x^*_k = x^*_k(x^o)$.

\[ u(x^o, x^o_\ell, x^o_k) - c_\ell \]

\[ u(x^o, x^o_\ell, x^o_k) - c_k \]

\[ U(x^o_\ell, x^o_k) - c_\ell \]

\[ U(x^o_\ell, x^o_k) - c_k \]

\[ U(x^o_\ell, x^o_k) - c_\ell \]

\[ U(x^o_\ell, x^o_k) - c_k \]

\[ U(x^o_\ell, x^o_k) - c_\ell \]

\[ U(x^o_\ell, x^o_k) - c_k \]

\[ u(x^o_\ell, x^o_k) - c_\ell \]

\[ u(x^o_\ell, x^o_k) - c_k \]

\[ u(x^o_\ell, x^o_k) - c_\ell \]

\[ u(x^o_\ell, x^o_k) - c_k \]

Figure 1(a)

Figure 1(b) is obtained from Figure 1(a) by deleting common terms within upper and lower rows of each cell.

\[ x^o_\ell < x^*_k \]

\[ x^o_\ell \geq x^*_k \]

\[ x^o_k \geq x^*_k \]

\[ x^o_k < x^*_k \]

\[ u(x^o, x^o_\ell, x^o_k) - c_\ell \]

\[ u(x^o, x^o_\ell, x^o_k) - c_k \]

\[ U(x^o_\ell, x^o_k) - c_\ell \]

\[ U(x^o_\ell, x^o_k) - c_k \]

\[ u(x^o_\ell, x^o_k) - c_\ell \]

\[ u(x^o_\ell, x^o_k) - c_k \]

\[ 0 \]

\[ 0 \]

\[ u(x^o_\ell, x^o_k) - c_\ell \]

\[ u(x^o_\ell, x^o_k) - c_k \]

Figure 1(b)

Figure 1(c) is obtained from 2(b) by recalling that

\[ u(x^o, x^*_k) = -c_k + \int u(x^o, x^*_k, x^o_k) dF_k(x^o_k) \]

\[ u(x^o, x^*_k) \geq -c_\ell + \int u(x^o, x^*_k, x^o_\ell) dF_\ell(x^o_\ell). \]

Since $0 < x^*_k < \infty$, (23) holds. Inequality (24) holds by the definition of $x^*_k$, Assumption 1 (submodularity), and the assumption that $x^*_\ell < x^*_k$. Figure 1(c) is obtained from 2(b) by using (23) to replace $-c_k$ in the two right-column cells and using (24) to replace $-c_\ell$ in the two top-row cells with something...
that is no less.

\[
\begin{array}{c|c|c}
\text{ } & x^o_k < x^*_k & x^o_k \geq x^*_k \\
--- & --- & --- \\
x^o_k \geq x^*_k & u(x^o, x^o_k, x^o_k) & u(x^o, x^o_k) \\
& -u(x^o, x^*_k, x^o_k) + u(x^o, x^*_k) & -u(x^o, x^*_k, x^o_k) + u(x^o, x^*_k) \\
& u(x^o, x^o_k) & u(x^o, x^o_k) \\
x^o_k < x^*_k & 0 & 0 \\
& 0 & 0 \\
\end{array}
\]

Figure 1(c)

Notice finally that the entries in the top and bottom row of each cell of Figure 1(c) are equal due to Assumption 2. Indeed, in the north west cell

\[
u(x^o, x^o_k, x^o_k) - u(x^o, x^o_k) = u(x^o, x^o_k, x^*_k) - u(x^o, x^*_k)
\]
because \(x^o_k, x^*_k \geq x^o_\ell\) in that cell. In the north east cell

\[
u(x^o, x^o_k, x^*_{\ell}) - u(x^o, x^*_{\ell}) = u(x^o, x^o_k, x^*_{\ell}) - u(x^o, x^*_{\ell})
\]
because \(x^o_\ell, x^o_k \geq x^*_k\) in that cell. Finally, in the south east cell,

\[
u(x^o, x^*_{\ell}, x^o_k) - u(x^o, x^o_k) = u(x^o, x^*_{\ell}, x^o_k) - u(x^o, x^o_k)
\]
because \(x^*_{\ell}, x^o_k \geq x^o_\ell\) in that cell.

Thus, for any given \(x^o\), the strategy of uncovering the variable with the greatest \(x^*_k\) is no worse than the strategy of uncovering any other variable. To complete the inductive proof, we need to show that the strategy of uncovering the variable with the greatest \(x^*_k\) is no worse than stopping search.

First note that by Assumption 1 (continuity), we have in the case that \(0 < x^o_k < \infty\)

\[
u(x^o, x^o_k) = -c_k + \int u(x^o, x^o_k, x^o_k) dF_k(x^o_k).
\]

(25)
So by \(25\),

\[
c_k = \int \left[ u(x^o, x_k^*, x_k^o) - u(x^o, x_k^0) \right] dF_k(x_k^0),
\]

which by Assumption 1 (submodularity) yields

\[
c_k \leq \int \left[ u(x^o, x_k^o) - u(x^o) \right] dF_k(x_k^0).
\]

This implies that uncovering variable \(x_k\) is no worse than stopping search. \(\square\)