Quantum theory with arrow of time: symmetry breaking and non-local spinor realization with non-commuting operators of energy and decay

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Abstract.

We present a modified quantum dynamics with a non-Hermitian Hamiltonian $\hat{H} \neq \hat{H}^+$, and a complex parameter of evolution. The theory is developed with $[\hat{H}^+, \hat{H}] = \kappa^2$, where $\kappa$ is the new fundamental constant. It is shown that this theory loses its symmetry in the case of $\kappa \neq 0$. We demonstrate that this theory allows a thermodynamical interpretation and possesses an ‘arrow of time’ for isothermal and adiabatic regimes of evolution. A realization of this theory is a modified four-dimensional Dirac spinor system, which appears non-local in the physical three-dimensional space.

1. Introduction

General principles of quantum theory in Dirac’s understanding provide a universal framework for the construction of any fundamental theory [1–3]. Thermodynamics is another ‘supertheory’, which also imposes restrictions on a realistical theoretical scheme [4, 5]. Both of these theories deal with statistical ensembles in their description of reality, and both have other remarkable similarities. It seems possible to construct a theory, which will unify quantum and thermodynamic principles and which will provide a dynamical realization of the thermodynamic laws in a consequent and precise form.

We think that this unified theory can be constructed by ‘minimal’ extensions of the standard quantum mechanics formalism. This extensions do not destroy the general and fundamental principles of quantum theory. In fact, we try to realize the idea of a total ‘analytic continuation’ of the conventional quantum theoretical scheme. We consider theories with a complex parameter of evolution and a non-Hermitian Hamiltonian and study the dynamics of the resulting system with respect to its reversibility and other properties that are interesting in the thermodynamic framework.

In our previous report, given at ‘Symmetries in Science XIV’, we have presented results obtained for theories with

$$[\hat{H}^+, \hat{H}] = 0,$$

where $\hat{H}$ and $\hat{H}^+$ are the Hamiltonian and its Hermitian conjugate, respectively [6]. The presence of an arrow of time in the dynamics of this quantum system was established. We have
also constructed explicitly several realizations of above mentioned quantum theory and have presented their results. This activity was highly motivated by the study of hidden symmetries of modified quantum systems (see [7–9] for the general symmetry context). Particularly, $\hat{H}$ and $\hat{H}^+$ are the symmetry operators in the dynamics under consideration [10–14]. The problem of time reversal invariance violation was also studied in [15–19].

In this article we discuss results obtained by a modified quantum theory with

$$\left[ \hat{H}^+, \hat{H} \right] = \kappa^2,$$

where $\kappa \neq 0$ is the new fundamental constant of the theory. We prove the presence of an arrow of time in the modified dynamics and study general properties of this new fundamental quantum scheme, including the possibility to explicitly solve the Cauchy problem. Our interest to this version of the theory is related to the closure of the set of fundamental constants in relativistic quantum theories of this type: taking $\kappa$ into account as a third fundamental parameter besides the reduced Planck constant $\hbar$ and the speed of light in vacuum $c$, one fixes the ‘physical scales’ in natural way (we use the notation where $\hbar = 1$ and $c = 1$). It is important to note that in this theory only the Hamiltonian $\hat{H}$ is the symmetry operator – the symmetry that corresponds to the operator $\hat{H}^+$ becomes broken if $\kappa \neq 0$. We discuss this in detail below.

2. General formalism

2.1. Quantum theory in bra and ket vector terms

Conventional quantum theory is based on principles that must be preserved in all possible realistic modifications (superposition principle, relativistic invariance, etc.). We modify the general quantum theory scheme using an analytic continuation of all its ‘building blocks’. The resulting ‘minimally extended’ quantum theory possesses an arrow of time and gives the dynamical base for non-equilibrium statistical mechanics.

2.1.1. Standard elements

A state vector $|\Psi\rangle$ depends on the evolutionary parameter $\tau$ according to the Hamiltonian $\hat{H}$ in quantum theory. The corresponding dynamics is covered by the linear and homogenous Schrödinger equation

$$i |\Psi\rangle, _{\tau} = \hat{H} |\Psi\rangle,$$

which defines the state vector $|\Psi\rangle = |\Psi(\tau)\rangle$ completely with initial data $|\Psi(\tau_0)\rangle = |\Psi_0\rangle$. In fact, quantum theory deals with a Cauchy problem of first order in $\tau$. The most interesting and actually fundamental quantum systems have a Hamiltonian independent of $\tau$, i.e. they yield the additional stationarity condition

$$\hat{H}, _{\tau} = 0.$$

For such dynamical systems the solution of the Cauchy problem reads:

$$|\Psi(\tau)\rangle = \exp \left[ -i \hat{H} (\tau - \tau_0) \right] |\Psi_0\rangle.$$

The expectation value $\bar{A}$ of the observable $\hat{A} = \hat{A}^+$ corresponding to the (non-normalized) state vector $|\Psi\rangle$ is defined by the relation

$$\bar{A} = \frac{\hat{A}}{Z},$$

with $Z = \langle \Psi | \hat{1} | \Psi \rangle$. The corresponding stationarity condition

$$\hat{A}, _{\tau} = 0,$$
where
\[ A = \langle \Psi | \hat{A} | \Psi \rangle \] (7)
and
\[ Z = \langle \Psi | \Psi \rangle . \] (8)

2.1.2. Analytic extension

Conventional quantum theory deals with some complex linear space of state-vectors, i.e. \(| \Psi \rangle \neq | \Psi \rangle^* \). We propose to extend it to the theory with a complex parameter of evolution \( \tau \neq \tau^* \) and a non-Hermitian Hamiltonian \( \hat{H} \neq \hat{H}^+ \). It can be performed in the ‘minimal’ analytic form, which is defined by the additional relations
\[ | \Psi \rangle_{\tau^*} = 0 \text{ and } \hat{H}_{\tau^*} = 0 . \] (9)

Note that the solution of the initial problem is given by Eq. (5). Using the parametrization
\[ \tau = t - \frac{i}{2} \beta \text{ and } \hat{H} = \hat{E} - \frac{i}{2} \hat{\Gamma} \] (10)
with the real variables \( t \) and \( \beta \) and the Hermitian operators \( \hat{E} \) and \( \hat{\Gamma} \), one obtains the following results for the expectation values of the corresponding observables:
\[ \bar{E} = - \langle \log Z \rangle_\beta \text{ and } \bar{\Gamma} = - \langle \log Z \rangle_t . \] (11)

This leads to the interpretation of \( \bar{E} \) and \( \beta \) as the energy operator and the inverse temperature of a system, whereas \( t \) and \( \hat{\Gamma} \) corresponds to the ‘usual’ time and the operator of decay, respectively. The quantity \( Z \) is not only the norm of the state vector \( | \Psi \rangle \), but also the statistical sum of the quantum ensemble under consideration in the framework of this interpretation. We work with systems with \( Z > 0 \) in this article.

2.1.3. Definition of energy and decay operators

We consider quantum systems with the operators \( \hat{E} \) and \( \hat{\Gamma} \) that are canonically conjugated up to the constant
\[ \left[ \hat{E}, \hat{\Gamma} \right] = ik, \] (12)
where \( k = \text{const} \) (note that \( \hbar = 1 \) in the our notation). Specifically, we extend the standard quantum system with time parameter \( t \) and Hermitian Hamiltonian \( \hat{E} \) using the modified one by the substitutions \( \hat{E} \rightarrow \hat{H} \) and \( t \rightarrow \tau \) according to Eq. (10). Here, Eq. (12) plays the role of the defining relation for the construction of the decay operator \( \hat{\Gamma} \) in the problem with the given operator \( \hat{E} \). This equation is linear and non-homogenous. We construct its solution in a form that guarantees a ‘minimal generalization’ of the starting conventional quantum system.

It is important to note that \( \hat{H} \) is the symmetry operator of the system for arbitrary values of the parameter \( k \), whereas \( \hat{H}^+ \) has this property only for \( k = 0 \). This means that the last symmetry breaks if the constant \( k \) becomes nontrivial. Actually, let us suppose that the state vector \( | \Psi \rangle \) satisfies the Schrödinger equation. Then the map
\[ | \Psi \rangle \rightarrow | \Psi' \rangle = \hat{H}^+ | \Psi \rangle \] (13)
defines the transformed state vector. This state vector satisfies the equation
\[ i | \Psi' \rangle_{\tau} = \hat{H} | \Psi' \rangle + k | \Psi \rangle , \] (14)
which coincides with Eq. (3) in the case of \( k = 0 \) only.
2.1.4. Arrow of time It is clear that the quantities $Z$, $\bar{E}$, and $\bar{\Gamma}$ are the functions of evolutionary parameters $t$ and $\beta$ on the dynamics of the quantum system. Performing the calculations one obtains:

$$\bar{\Gamma}_{,t} = -k - \bar{E} \circ \bar{\Gamma}, \quad \bar{E}_{,\beta} = -\frac{k}{4} - \bar{E} \circ \bar{E}, \quad (15)$$

$$\bar{\Gamma}_{,\beta} = \bar{E}_{,t} = -\bar{E} \circ \bar{\Gamma}.$$ 

Here $E \circ \Gamma$ means a correlation between the observables $\bar{E}$ and $\bar{\Gamma}$; it is defined as

$$E \circ \Gamma = \frac{1}{2} \left[ (\hat{E} - \bar{E})(\hat{\Gamma} - \bar{\Gamma}) + (\hat{\Gamma} - \bar{\Gamma})(\hat{E} - \bar{E}) \right], \quad (16)$$

whereas

$$E \circ E = D_E^2, \quad \Gamma \circ \Gamma = D_{\Gamma}^2 \quad (17)$$

are the corresponding dispersions.

Let us consider two temperature regimes, i.e. a dynamics with one additional relation $\beta = \beta(t)$ imposed. We study the isothermal and the adiabatic evolution of the system.

(i) In the isothermal case one has $\beta = \text{const}$, and for the total time derivative of the function $\bar{\Gamma}[t] = \bar{\Gamma}[t, \beta(t)]$ one has:

$$\frac{d\bar{\Gamma}}{dt} = -k - \bar{E} \circ \bar{\Gamma}. \quad (18)$$

(ii) For the adiabatic evolution we put $\bar{E} = \bar{E}[t, \beta(t)] = \text{const}$, and for the total time derivative of the same quantity $\bar{\Gamma}$ one obtains the following result:

$$\frac{d\bar{\Gamma}}{dt} = -\frac{k^2}{4} + k(E \circ E + \frac{1}{2} \Gamma \circ \Gamma) + \frac{E \circ E \cdot \Gamma \circ \Gamma - (E \circ \Gamma)^2}{k + E \circ E}. \quad (19)$$

From Eqs. (18) and (19) it follows that

$$\frac{d\bar{\Gamma}}{dt} \leq 0 \quad (20)$$

if $k \geq 0$ for an arbitrary initial state $|\Psi_0\rangle$ that specifies the solution of a Cauchy problem completely.

Thus, the quantity $\bar{\Gamma}[t]$ detects the presence of an arrow of time in dynamics of the quantum system in these two temperature regimes. We fix the non-negative sign of the constant $k$ and put

$$k = \kappa^2, \quad (21)$$

where $\kappa$ is the real non-negative constant.

2.2. Ladder operators analysis

Let us put

$$\hat{H} = \kappa \hat{a}^+, \quad (22)$$

then

$$[\hat{a}, \hat{a}^+] = 1. \quad (23)$$

From the formal point of view $\hat{a}$ and $\hat{a}^+$ are lowering and raising operators, respectively.
2.2.1. ‘Oscillatory’ basis  

Let us define a linear subspace $|0\rangle$ by the relation

$$\hat{a} |0\rangle = 0. \quad (24)$$

Let the number $K$ denote the dimensionality of this subspace and $\{ |0, k\rangle \}$ its orthonormal basis ($k = 1, \ldots, K$). Thus, we have the corresponding scalar products

$$\langle 0, k_1 | 0, k_2 \rangle = \delta_{k_1 k_2}, \quad (25)$$

and the decomposition

$$|0\rangle = \sum_k C_{0k} |0, k\rangle \quad (26)$$

for an arbitrary vector $|0\rangle$ with the coefficients $C_{0k}$. Let us introduce the infinite set of vectors

$$|n, k\rangle = \frac{a^n}{\sqrt{n!}} |0, k\rangle, \quad (27)$$

where $n = 1, \ldots$ is an arbitrary integer number. It is possible to show that these vectors are orthonormal too for any $k = 1, \ldots, K$ and $n \in \mathbb{N}$:

$$\langle n_1, k_1 | n_2, k_2 \rangle = \delta_{n_1 n_2} \delta_{k_1 k_2}. \quad (28)$$

The statement is that these vectors with the starting subset $|0, k\rangle$ form the general solution of the eigenproblem

$$\hat{n} |n, k\rangle = n |n, k\rangle \quad (29)$$

for a Hermitian operator

$$\hat{n} = \hat{a}^+ \hat{a} \quad (30)$$

with the eigenvalues $n = 0, 1, \ldots$. Finally, this set of orthonormal vectors is a basis in the linear space of the state vectors of quantum theory in accordance to Steklov’s theorem with the decomposition formula:

$$|\Psi(\tau)\rangle = \sum_{n=0}^{+\infty} \sum_{k=1}^{K} C_{n,k}(\tau) |n, k\rangle. \quad (31)$$

The expectation values of the energy and decay operators calculated over the state vector $\{ |n, k\rangle \}$ are obtained as

$$\bar{E} = \Gamma = 0 \quad (32)$$

and the dispersions as

$$\bar{E}^2 = \frac{1}{4} \bar{\Gamma}^2 = \frac{k^2}{2} \left( n + \frac{1}{2} \right). \quad (33)$$

It is also interesting to note that

$$\bar{E} \circ \bar{\Gamma} = 0. \quad (34)$$

To prove these statements one must calculate averaged values of the operators $\hat{H}$, $\hat{H}\hat{H}$ and $\hat{H}\hat{H}^+$ and take into account the underlying commutation relation (12) and Eq. (21).
2.2.2. Exact solution: ‘Big Bang of vacuum state’ In terms of the coefficient functions \( C_{n,k} = C_{n,k}(\tau) \) from the decomposition formula (31), Schrödinger’s equation (3) reads:

\[
\frac{dC_{0,k}}{d\tau} = 0, \quad \frac{dC_{n+1,k}}{d\tau} = -i\kappa \sqrt{n+1} C_{n,k}.
\]

Let us construct a special exact solution of the problem with initial data specified by the relations

\[
C_{n,k}(0) = \delta_{n,0} c_k \quad \text{and} \quad \sum_{k=1}^{\infty} |c_k|^2 = 1,
\]

where \( c_k = \text{const} \). This solution describes the dynamics of the quantum system with only ‘vacua modes’ \(|0, k\rangle\) excited at the initial moment of evolution \( \tau = 0 \). This moment corresponds physically to an initial time \( t_0 = 0 \) and an absolute temperature \( T_0 = 1/\beta_0 = \infty \). The solution of the Cauchy problem (35)–(36) reads:

\[
C_{n,k}(\tau) = c_k \frac{(-i\kappa \tau)^n}{\sqrt{n!}}.
\]

For the statistical sum, which is also the norm of the constructed solution, one has:

\[
Z(t, \beta) = \exp \left( \kappa^2 |\tau|^2 \right) = \exp \left[ \kappa^2 \left( t^2 + \frac{\beta^2}{4} \right) \right],
\]

so

\[
\bar{\Gamma}_t = -2\kappa^2 < 0
\]

and

\[
\bar{E}_T = \frac{\kappa^2}{2T^2} > 0.
\]

It is seen that this solution describes the dynamics of the consecutive excitation of the originally non-excited modes \(|n, k\rangle\) of an explosion type. The system demonstrates an irreversibility (see Eq. (39)) and has a positive heat capacity – as it follows from Eq. (40).

2.3. Analysis in energy representation

Dirac’s theory of quantum representations is based on the use of the complete set of commuting Hermitian variables \( \hat{Q}_\alpha = \hat{Q}^\dagger_\alpha \):

\[
[\hat{Q}_\alpha, \hat{Q}_\beta] = 0,
\]

and the related basis of eigenvectors \(|Q\rangle\):

\[
\hat{Q}_\alpha |Q\rangle = Q_\alpha |Q\rangle.
\]

The wave function in the \( Q \)-representation is defined by the relation

\[
\Psi(\tau, Q) = \langle Q | \Psi(\tau) \rangle.
\]
2.3.1. Energy representation  The energy operator $\hat{E}$ in the energy representation is included in the set $\{\hat{Q}_\alpha\}$. Let us put $\hat{Q}_1 = \hat{E}$; then $Q_1 = E$ and for the wave function we have the notation $\Psi(\tau, Q) = \Psi(\tau, E, \ldots)$. In this representation

$$\hat{E} = E \quad \text{and} \quad \hat{\Gamma} = -i\kappa^2 \frac{\partial}{\partial E}. \quad (44)$$

Actually, the second relation in Eq. (44) defines the Hermitian decay operator that yields the underlying condition (12).

2.3.2. Solution of initial problem  Schrödinger’s equation in the energy representation reads:

$$\Psi_{,\tau} - i\kappa^2 \frac{E}{2} \Psi_{,E} = -iE\Psi. \quad (45)$$

It has the simplest form

$$\Phi_{,v} = \frac{u + v}{2\kappa^2} \Phi \quad (46)$$

in terms of the new variables $u = E + i\kappa^2 \frac{\tau}{2}$ and $v = E - i\kappa^2 \frac{\tau}{2}$.

and the new function $\Phi(u, v, \ldots) = \Psi(\tau, E, \ldots)$. Solving Eq. (46) yields

$$\Phi(u, v, \ldots) = \phi(u, \ldots) \exp \left[\frac{(u + v)^2}{4\kappa^2}\right], \quad (48)$$

where $\phi$ is an arbitrary function of all variables excepting $v$. Thus, the general solution of Eq. (45) is

$$\Psi(\tau, E, \ldots) = \phi \left( E + i\kappa^2 \frac{\tau}{2}, \ldots \right) \exp \left( E^2 / \kappa^2 \right). \quad (49)$$

Taking into account the initial data $\Psi_0(E, \ldots) = \Psi(0, E, \ldots)$, where we put $\tau_0 = 0$ again, one can fix the introduced function $\phi$:

$$\phi(E, \ldots) = \Psi_0(E, \ldots) \exp \left( -E^2 / \kappa^2 \right). \quad (50)$$

Finally, for the solution of the quantum Cauchy problem we obtain:

$$\Psi(\tau, E, \ldots) = \Psi_0 \left( E + i\kappa^2 \frac{\tau}{2}, \ldots \right) \exp \left( -iE\tau + \kappa^2 \tau^2 / 4 \right). \quad (51)$$

3. Two specific realizations

In this section we give two examples of the modified quantum theory considered above. These systems have ‘truncations’ (defined by the map $\tau \to t, \quad \hat{H} \to \hat{E}$) to the conventional relativistic quantum mechanics.
3.1. Modified transport equation

This model is intended for $(1 + 1)$–space-time and its energy operator is equal to the momentum defined by a single component:

\[ \hat{E} = \hat{p}. \]  

(52)

We introduce the decay operator as

\[ \hat{\Gamma} = -\kappa^2 \hat{x}, \]  

(53)

where \( \hat{x} \) is the operator of the coordinate that is canonically conjugated to \( \hat{p} \). In fact, the momentum representation coincides with the energy one for this theory, so the solution of Cauchy problem using the wave function \( \Psi (\tau, p) \) can be found in the previous section.

3.2. Modified Dirac equation

Our second example is related to conventional Dirac’s spinor theory in four dimensions. We show that its modified version with an arrow of time is non-local in physical three-dimensional space.

3.2.1. 4D spinor theory

The energy operator for Dirac’s spinor theory is defined as

\[ \hat{E} = \gamma^0 \left( m + \gamma^k \hat{p}^k \right), \]  

(54)

where \( m \) is the mass parameter, \( k = 1, 2, 3 \), and \( \gamma^\mu \) are the Dirac matrices \( (\mu = 0, \ldots, 3) \). The wave function \( \Psi \) is the 4-column in the representation of the lowest matrix dimensionality.

3.2.2. Construction of decay operator

An analysis shows that the introduction of the decay operator to this quantum system, i.e. the map

\[ \hat{E} \rightarrow \hat{H}, \]  

(55)

is realized by the substitution

\[ m \rightarrow \hat{M} = m \left( 1 + \frac{i\kappa^2}{2} \hat{\alpha} \right) \text{ and } \hat{p}^k \rightarrow \hat{P}^k = \hat{p}^k + \frac{i\kappa^2}{4} \hat{\pi}^k, \]  

(56)

with

\[ \hat{\alpha} = \frac{1}{6} \left[ \left( \hat{p}^l \right)^{-1} \hat{x}^l + \hat{x}^l \left( \hat{p}^l \right)^{-1} \right] \text{ and } \hat{\pi}^k = \left( \hat{\alpha} \hat{p}^k + \hat{p}^k \hat{\alpha} \right). \]  

(57)

For the multiplicity number (see Eq. (31)) one obtains \( \mathcal{K} = 4 \) in this approach.

3.2.3. Non-locality in coordinate representation

The momentum representation (where \( \hat{\pi}^k = \partial \hat{p}^k / \partial p^k \), \( \hat{x}^k = i \partial / \partial p^k \), and \( \hat{\Psi} = \Psi \left( t, p^k \right) \)) for the mass- and momentum-like operators (56) gives:

\[ \hat{M} = m \left( 1 + \frac{i\kappa^2}{12} \right) \left( \frac{1}{p^l} \right)^2 - 2 \frac{\partial}{p^l \partial p^l} \right), \quad \hat{P}^k = \hat{p}^k - \frac{i\kappa^2}{12} \left[ \frac{1}{p^k} - \frac{\partial}{p^l \partial p^l} \left( \frac{1}{p^l} \right)^2 + \frac{p^k}{p^l} \frac{\partial}{p^l} \right]. \]  

(58)

Subjecting the standard modified Dirac equation to the map (56), (58), we obtain

\[ i\Psi_{\tau} = \gamma^0 \left( \gamma^k \hat{p}^k + m \right) \left[ 1 + \kappa^2 \left( \frac{1}{(p^l)^2} \right) - 2 \frac{\partial}{(p^l)^2} \right] - \kappa^2 \left( \frac{\gamma^k}{12} \frac{\gamma^k}{p^k} \right) \Psi. \]  

(59)
This equation must be supplemented by the Cauchy-Riemann condition
\[ \Psi_{,\beta} = -\frac{i}{2} \Psi_{,\bar{\beta}} \] (60)
instead of the first relation in Eq. (9). Eq. (59) is of the same type as the conventional Dirac equation in the coordinate representation. Actually, it is of first order with respect to the derivatives, but its coefficients are no longer constant. It is important to note that Eq. (59) corresponds to the action
\[ S = \int dt d^3 p \mathcal{L} \] (61)
with the Lagrangian
\[ \mathcal{L} = \frac{i}{2} \left( \bar{\Psi} \gamma^0 \Psi_{,t} - \bar{\Psi} i \gamma^0 \Psi - \frac{1}{2} \left[ \bar{\Psi} \gamma^k \left( \hat{P}^k \Psi + \hat{M}^k \Psi \right) + \bar{\Psi} \gamma^k \Psi \left( \hat{M}^k \Psi + \hat{M}^k \bar{\Psi} \right) \right] \right). \] (62)
Thus, the modified Dirac theory in the \((t, p^k)\)-representation possesses a Lagrangian formulation despite it has an arrow of time.

The modified Dirac equation in the coordinate representation, where \(\hat{x}^k = x^k, \hat{p}^k = -i \partial / \partial x^k\), and \(\bar{\Psi} = \Psi \left( t, x^k \right) = \Psi \left( x^\mu \right)\), has the following form:
\[ (i \gamma^\mu \partial_{x^\mu} - m) \Psi = \frac{\kappa^2}{24} \left( m - i \gamma^k \partial_{x^k} \right) \int_0^{+\infty} d\xi \sum_{l=1}^{3} \left[ \xi \Psi_{t+} - \left( 2x^l + i \gamma^l \right) \Psi_{t-} \right]. \] (63)
Here we put \(\Psi_{\pm k} = \Psi_{\pm k} \left( t, x^k, \xi \right)\), where
\[
\Psi_{\pm 1} = \Psi \left( t, x^1 - \xi, x^2, x^3 \right) \pm \Psi \left( t, x^1 + \xi, x^2, x^3 \right), \\
\Psi_{\pm 2} = \Psi \left( t, x^1, x^2 - \xi, x^3 \right) \pm \Psi \left( t, x^1, x^2 + \xi, x^3 \right), \\
\Psi_{\pm 3} = \Psi \left( t, x^1, x^2, x^3 - \xi \right) \pm \Psi \left( t, x^1, x^2, x^3 + \xi \right). \] (64)
To obtain Eq. (63) from Eq. (59) one must Fourier transform and take into account the convolution theorem. It is seen that Eq. (63) is essentially an integro-differential equation that is non-local with respect to the physical three-dimensional space.

4. Conclusions
The main results of this article are:

(i) A quantum theory with a complex parameter of evolution and a non-Hermitian Hamiltonian does provide a dynamical framework for a non-equilibrium statistical mechanics and thermodynamics.

(ii) This dynamical framework possesses an arrow of time for the quantum theory with operators of energy and decay conjugated canonically up to some positive constant.

(iii) This nontrivial constant is an additional fundamental constant in the theory, which leads to some hidden dynamical symmetry breaking.

(iv) A realization of this theory can be constructed on the base of the four-dimensional Dirac theory. This realization is a Lagrangian one in momentum space.

(v) The constructed modified Dirac theory is essentially non-local in the physical three-dimensional space. This fact seems really interesting for a potential resolution of the Einstein-Podolsky-Rosen paradox [20, 21].
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