More gravity solutions of AdS/CMT

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Abstract
We have generalized the gravity solutions presented in arXiv:0808.1725 and arXiv:0808.3232 to arbitrary but even space time dimensions with the scaling symmetry $r \rightarrow \lambda r$, $x_i \rightarrow \lambda x_i$, $t \rightarrow \lambda^a t$. However, for $b = 0$, we have the solution in arbitrary space time dimension, not restricted to even dimensional. The physical meaning of this particular choice of $b$ is that we can have a solution with only temporal scale invariance. From the dual field theory point of view, the BF bound and the unitarity bound for operators dual to scalar field is discussed.
The usefulness of AdS/CFT correspondence [1],[2] and [3], lies in solving the strongly coupled problems. Moreover, it’s important to find the proper gravity solutions and extracting information from it by computing the correlation functions.

In this context it has been proposed in [4] that we can have a four dimensional gravitational description which exhibits the Lifshitz-like fixed points with an exponent. This particular gravity solution has been generalized in [5] to two exponents and more interestingly for a specific ratio of the exponents, there could be a possible dual field theory whose action is quadratic in fields. It could be that we may be able to understand the strongly coupled physics of some systems with quadratic in fields [6],[7] and [8]. In [5], the solution has been written in a better coordinate system, in the sense to have only temporal scale invariance along with the combination of spatial and temporal scale invariance.

In a related context, several gravitational solutions has been constructed in [9] and in [10] and is being embedded in string theory in [11] -[14] that shows the non-relativistic symmetry.

In all these solutions the system exhibits the scaling symmetry and we expect it to be preserved in both the bulk and boundary theories [1]. In this paper, we would like to generalize the solutions found in [4] and [5] to arbitrary but even $d$ dimensions with both temporal scale invariance and a combination of spatial and temporal scale invariance. Let us recall that these solutions have been generated in four space time dimensions using the electric type field for a two form field strength and a magnetic type field for a three form field. Here, we also generalize that and include the magnetic type field for the two form and electric type field for the higher form field strength.

The form of the scaling symmetry that we are interested in is

$$r \rightarrow \frac{r}{\lambda}, \quad x_i \rightarrow \lambda^b x_i, \quad t \rightarrow \lambda^a t$$

The action that we shall consider whose solution respects eq(1) is

$$S = \frac{1}{2\kappa^2} \int d^d x \sqrt{-g} (R - 2\Lambda) - \frac{1}{4\kappa^2} \int (F_2 \wedge \star F_2 + F_{d-1} \wedge \star F_{d-1}) - \frac{c}{2\kappa^2} \int B_{d-2} \wedge F_2,$$  \hspace{1cm} (2)

where $F_{d-1} = dB_{d-2}$ and $c$ is the topological coupling. The equations of motion that follows from it are

$$R_{MN} = g_{MN} \left[ \frac{2\Lambda}{d-2} + \frac{F_2^2}{4(2-d)} - \frac{F_{d-1}^2}{2(d-1)!} \right] + \frac{1}{2} F_{MM_1} F_N^{M_1},$$

$$+ \frac{1}{2(d-2)!} F_{MM_1\cdots M_{d-2}} F_N^{M_1\cdots M_{d-2}}.$$  \hspace{1cm} (3)
The ansatz to the solution consistent with the scaling symmetry are

\[ ds^2 = L^2[-r^{2a} dt^2 + r^{2b} \sum_{i=1}^{d-2} \delta_{ij} dx^i dx^j + \frac{dr^2}{r^2}] \]

\[ F_2 = \frac{AL^2}{r^{1-a}} dr \wedge dt, \quad F_{d-1} = \frac{BL^{d-1}}{r^{1-b(d-2)}} dr \wedge dx_1 \wedge \cdots \wedge dx_{d-2} \quad (4) \]

It is trivial to see that the fluxes obey Bianchi identity and we have considered electric field for the two form flux and magnetic field for the \((d-1)\)-form flux. The equations of motion associated to the fluxes give

\[ cBL = Ab(d - 2), \quad cAL = (-1)^{d-2} aB, \quad (5) \]

and follows from the \(F_2\) and \(F_{d-1}\) fluxes, respectively.

In what follows, we shall stick to even \(d\) dimensional space time. The topological coupling that follows from the fluxes is

\[ c^2 L^2 = ab(d - 2) \quad (6) \]

From the metric

\[ R_{00} = ar^{2a}[a + b(d - 2)], \quad R_{rr} = -\frac{1}{r^2}[a^2 + b^2(d - 2)] \]

\[ R_{ij} = -br^{2b}[a + b(d - 2)] \delta_{ij} \quad (7) \]

From the equation of motion to metric

\begin{align*}
R_{00} &= -L^2e^{2a}\left[ \frac{2\Lambda}{d-2} - \frac{3 - d A^2}{2 - d^2} - \frac{B^2}{2} \right] \\
R_{ij} &= L^2e^{2b}\left[ \frac{2\Lambda}{d-2} - \frac{1}{2 - d^2} \right] A^2 \\
R_{rr} &= L^2 \left[ \frac{2\Lambda}{d-2} - \frac{3 - d A^2}{2 - d^2} \right] \\
\end{align*}

Whose solution is

\[ A^2 = \frac{2a(a - b)}{L^2}, \quad B^2 = \frac{2b(a - b)}{L^2}(d - 2) \]

\[ \Lambda = -a^2 + ab(d - 2) + b^2(d - 2)^2 \quad (9) \]

It is interesting to note that for \(b = 0\) and \(c = 0\), imply

\[ A^2 = \frac{2a^2}{L^2}, \quad B^2 = 0, \quad \Lambda = -\frac{a^2}{L^2} \quad (10) \]
and this solution is valid in any arbitrary space time dimension, not necessarily restricted to even dimension. Moreover, in this case the spatial directions $x$, $y$ scales trivially, i.e. according to eq(26). This type of scale invariance suggests that we have a solution that possesses only the temporal scale invariance.

Let us construct a possible action of the dual field theory that live on the boundary of eq(4), consistent with the scaling symmetry written in eq(1)

$$S \sim \int dt d^{d-2}x \left[ (\partial_\mu^m \chi)^{\alpha} - K(\partial_\nu^m \chi)^{\beta} \right].$$

This form of the action gives the restriction

$$\alpha(a + h \beta) = \beta(b + \tilde{h} \beta) = a + b(d - 2),$$

where we have assumed that the field, $\chi$, transforms under scale transformation as $\chi \to \lambda^h \chi$. If the field $\chi$ transforms trivially then the conditions are $d - 2 = \frac{m_2 \beta}{m_1 \alpha} (m_1 \alpha - 1)$ and $\frac{a}{b} = \frac{m_2 \beta}{m_1 \alpha}$.

It is possible to have an action with first order in time and second order in space derivative and quadratic in fields, $\chi$, and which imply $a/b = 2$ and $d = 4 + 2\tilde{h}$. For either vanishing $h/b$ or any positive integer gives us the restriction that the dimension of the corresponding spacetime should be even. Whereas if we want to have an action with second order in time and second order in space derivative and quadratic in fields, $\chi$, gives $a = b$ and $d - 1 = 4 + 2\tilde{h}$. From this it follows that $d$ can be an odd integer, for either vanishing $h/b$ or any positive integer.

Now let us restrict our attention to a four dimensional theory with both electric and magnetic type of fluxes for both $F_2$ and $F_3$ form fluxes.

The ansatz to the fluxes are

$$F_2 = A_1 L^2 r^{a-1} dr \wedge dt + A_2 L^2 r^{b-1} dr \wedge dx + A_3 L^2 r^{\tilde{b}-1} dr \wedge dy,$$
$$F_3 = B_1 L^3 r^{a+b-1} dr \wedge dx \wedge dy + B_2 L^3 r^{a+b-1} dr \wedge dt \wedge dy + B_3 L^3 r^{a+b-1} dr \wedge dt \wedge dx$$

(13)

with the metric

$$ds^2 = L^2 \left[ -r^{2a} dt^2 + r^{2b} dx^2 + r^{2\tilde{b}} dy^2 + \frac{dr^2}{r^2} \right].$$

(14)

In this case the scaling symmetry is

$$r \to \frac{r}{\lambda}, \quad x \to \lambda^b x, \quad y \to \lambda^{\tilde{b}} y, \quad t \to \lambda^a t$$

(15)

It is trivial to see that the field strengths obey Bianchi identity and from the equation of motion that is

$$d \star F_2 = -c F_3, \quad d \star F_3 = c F_2,$$

(16)

we get

$$c^2 L^2 = a(b + \tilde{b}), \quad c^2 L^2 = b(a + \tilde{b}), \quad c^2 L^2 = \tilde{b}(a + b).$$

(17)
The first relation comes from the coefficient that appear in the fluxes $A_1$ and $B_1$, the second relation from $A_2$ and $B_2$ and the third relation from $A_3$ and $B_3$, respectively. Along with

$$\frac{A_1}{A_2} \left( \frac{b + \tilde{b}}{a + b} \right) = \frac{B_1}{B_2}, \quad \frac{A_1}{A_3} \left( \frac{b + \tilde{b}}{a + b} \right) = -\frac{B_1}{B_3}, \quad \frac{A_2}{A_3} \left( \frac{a + \tilde{b}}{a + b} \right) = -\frac{B_2}{B_3} \quad (18)$$

The various components of Ricci tensor that follows from the metric are

$$R_{00} = a(a + b + \tilde{b})r^{2a}, \quad R_{xx} = -b(a + b + \tilde{b})r^{2b}$$

$$R_{yy} = -\tilde{b}(a + b + \tilde{b})r^{2\tilde{b}}, \quad R_{rr} = -\frac{1}{r^2}(a^2 + b^2 + \tilde{b}^2) \quad (19)$$

From the equation of motion to metric

$$R_{00} = \frac{L^2}{2} r^{2a} \left[ \frac{1}{2} (A_1^2 + A_2^2 + A_3^2) + B_1^2 - 2\Lambda \right]$$

$$R_{xx} = \frac{L^2}{2} r^{2b} \left[ \frac{1}{2} (A_1^2 + A_2^2 - A_3^2) + B_2^2 + 2\Lambda \right]$$

$$R_{yy} = \frac{L^2}{2} r^{2\tilde{b}} \left[ \frac{1}{2} (A_1^2 - A_2^2 + A_3^2) + B_3^2 + 2\Lambda \right]$$

$$R_{rr} = \frac{L^2}{2r^2} \left[ \frac{1}{2} (-A_1^2 + A_2^2 + A_3^2) + 2\Lambda \right] \quad (20)$$

Finally comparing different components of Ricci tensor gives

$$L^2 \left[ \frac{1}{2} (A_1^2 + A_2^2 + A_3^2) + B_1^2 - 2\Lambda \right] = 2a(a + b + \tilde{b}),$$

$$L^2 \left[ \frac{1}{2} (A_1^2 + A_2^2 - A_3^2) + B_2^2 + 2\Lambda \right] = -2b(a + b + \tilde{b}),$$

$$L^2 \left[ \frac{1}{2} (A_1^2 - A_2^2 + A_3^2) + B_3^2 + 2\Lambda \right] = -2\tilde{b}(a + b + \tilde{b}),$$

$$L^2 \left[ \frac{1}{2} (-A_1^2 + A_2^2 + A_3^2) + 2\Lambda \right] = -2(a^2 + b^2 + \tilde{b}^2) \quad (21)$$

It is interesting to note that there are seven parameters, $A_i, B_i$ for $i=1, 2, 3$ and $\Lambda$ and as many equations. It is not that illuminating to find the most general solution. Instead, we shall solve these set of equations in different cases.

For vanishing of all the flux coefficients i.e. $A_i = 0$ and $B_i = 0$ give the cosmological constant

$$\Lambda = -\frac{3a^2}{L^2}, \quad \text{with} \quad a = b = \tilde{b} \quad (22)$$

Let us consider a situation when one pair of coefficients, $A_i, B_i$, of fluxes is non-zero. The consistent solution i.e. real flux is allowed when $A_1$ is non-zero and the rest coefficients are zero and similarly for $B_2$ and $B_3$ separately.
For $A_1$ non-zero, the solution is
\[ A_1 = \frac{2a^2}{L^2}, \quad \Lambda = -\frac{a^2}{2L^2}, \quad b = 0 = \bar{b} \quad (23) \]
and when $B_2$ is non-zero, the solution is
\[ B_2 = \frac{4a^2}{L^2}, \quad \Lambda = -\frac{2a^2}{L^2}, \quad b = 0 = \bar{b} \quad (24) \]
and when $B_3$ is non-zero, the solution is
\[ B_3 = \frac{4a^2}{L^2}, \quad \Lambda = -\frac{2a^2}{L^2}, \quad b = 0 = \bar{b} \quad (25) \]

It says that we can have a gravity solution by turning on either an electric 2-form flux or a magnetic 3-form flux with a scaling symmetry like
\[ r \rightarrow \frac{r}{\lambda}, \quad (x, y) \rightarrow (x, y), \quad t \rightarrow \lambda^a t \quad (26) \]
Instead of considering either a pure electric field or a magnetic field, if we consider only $A_1$ and $B_1$ then the solution is presented in [5]. There is not any consistent solution for considering $(A_2, B_2)$ and $(A_3, B_3)$ separately.

Let us take $(A_i, B_i)$ and $(A_j, B_j)$ for $i \neq j$. In this case the consistent solution is possible only for $(A_2, B_2)$ and $(A_3, B_3)$ and the solution is
\[ A_2^2 = A_3^2 = \frac{2b(a - b)}{L^2}, \quad B_2^2 = B_3^2 = \frac{2(a^2 - b^2)}{L^2}, \]
\[ \Lambda = -\frac{a^2 + ab + b^2}{L^2}, \quad b = \bar{b} \quad (27) \]
which respects the scaling symmetry as written in eq(1). From this study it follows that we can have a gravity solution with the scaling symmetry eq(1), with the help of a magnetic two form flux and an electric three form flux. Another thing to note that the “exponents” $b = \bar{b}$.

Now we would like to generalize it to arbitrary but even $d$ dimension. The ansatz to fluxes and geometry are
\[ F_2 = AL^2 r^{b-1} dr \wedge [dx_1 + \cdots + dx_{d-2}] \]
\[ F_{d-1} = BL^{d-1} r^{a-1+b(d-3)} dr \wedge dt \wedge [dx_2 \wedge \cdots \wedge dx_{d-2} + \cdots + dx_1 \wedge \cdots \wedge dx_{d-3}] \]
\[ ds^2 = L^2 [-r^{2a} dt^2 + r^{2b} \sum_{i=1}^{d-2} \delta_{ij} dx^i dx^j + \frac{dr^2}{r^2}] \quad (28) \]
The equation of motion to fluxes gives
\[ c^2 L^2 = b[a + b(d - 3)] \quad (29) \]
and the equation of motion to metric gives

\[ R_{00} = -L^2 r^{2a} \left[ \frac{2\Lambda}{d-2} - \frac{A^2}{2} \right], \quad R_{ij} = L^2 r^{2b} \left[ \frac{2\Lambda}{d-2} + (d-3) \frac{B^2}{2} \right] \delta_{ij} \]

\[ R_{rr} = \frac{L^2}{r^2} \left[ \frac{2\Lambda}{d-2} + (d-3) \frac{A^2}{2} \right] \] (30)

Solving for \( A^2, B^2 \) and \( \Lambda \), using the components of Ricci tensor as computed in eq(7), results in

\[ A^2 = \frac{2b(a-b)}{L^2}, \quad B^2 = \frac{2}{(d-3)L^2}[a^2 - b^2(d-3) + (d-4)ab], \]

\[ \Lambda = -\frac{(d-2)}{2L^2}[a^2 + b^2 + (d-3)ab] \] (31)

It is interesting to note that we can do a simple change of coordinates [5], to bring the metric to the following form.

\[ ds^2 = L^2[-\rho^2 \delta_{ij} dx^i dx^j + \frac{d\rho^2}{\rho^2}]. \] (32)

This form of the metric coincides with the one written in [4] by defining \( z = a/b \) in four spacetime dimensions and this change of coordinates makes sense only when \( b \neq 0 \). Which means in order to study only temporal scale invariance, the coordinate system written in eq(4) and eq(28) are better than eq(32).

## 1 Field theory observable

Generalizing the AdS/CFT prescription for the case that we are interested in makes us to identify the operators dual to bulk fields. In particular, we would like to identify the dimension, \( \Delta \), of an operator dual to scalar field of mass \( m \) obeying the minimal scalar field equation. In \( d \) space time dimension, the dimension of the operator is

\[ \Delta_\pm = \frac{a + b(d-2)}{2} \pm \sqrt{\frac{[a + b(d-2)]^2}{4} + m^2L^2} \] (33)

It follows that for the \( \Delta_+ \) branch there is a lower bound on the dimension of the scalar operators that is \( \frac{a+b(d-2)}{2} \).

The requirement of the finiteness of the Euclidean action of the scalar field imposes the restriction that if the mass of the scalar field obey

\[ m^2L^2 > 1 - \frac{[a + b(d-2)]^2}{4}, \] (34)
then only $\Delta_+$ branch is allowed, whereas if the mass stays
\[
-\frac{(a + b(d - 2))^2}{4} < m^2 L^2 < 1 - \frac{(a + b(d - 2))^2}{4},
\] (35)
then both $\Delta_+$ and $\Delta_-$ branches are allowed.

The analogue of Breitenlohner-Freedman bound [16] for this case is
\[
(mL)^2 \geq -\left(\frac{a + b(d - 2)}{2}\right)^2
\] (36)
and if the mass stays below this bound then there is an instability in the system.

The unitarity bound for the operators dual to scalar field depends on the ratio of $a$ and $b$. For $a = 2b$, the bound is
\[
\Delta > \frac{b(d - 2)}{2},
\] (37)
and for $a > 2b$, the bound is
\[
\Delta > -\frac{a + b(d - 2)}{2},
\] (38)
whereas for $a < 2b$, the bound is
\[
\Delta > \frac{a + b(d - 4)}{2}.
\] (39)
These bounds are computed following the analogous prescription for the asymptotically AdS space time in [15] i.e. demanding the positivity and finiteness of the action of the scalar field in Euclidean space time.

The normalized solution to the Green’s function of the scalar field in the mass less limit in the $a = 2b$ case
\[
G(u, k, \omega) = \frac{\Gamma\left[\frac{d}{2} + \frac{k^2}{4\omega} + \frac{d}{4}\right]}{\Gamma\left[\frac{d}{2}\right]} e^{-\frac{k^2}{4b\omega}} \left(U\left[\frac{k^2}{4b\omega}, \frac{d}{2} - \frac{d}{4}, 1 - \frac{d}{2}, \frac{\omega}{2b}\right] + 1\right),
\] (40)
where $U(\alpha, \beta, z)$ is the confluent hypergeometric function of the second kind.

Summarizing, we have presented solutions in arbitrary even dimensional spacetime dimensions with the scaling symmetry as written in eq(1) as well as solution in any arbitrary spacetime dimension with the scaling symmetry in eq(26). When the spacetime dimension is restricted to even dimensional, we can have both the spatial and temporal scale invariance and when $d$ is arbitrary, we have only the temporal scale invariance. From the field theory point of view, we have discussed the BF bound and the unitarity bound for operators dual to scalar field.
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