RADIATIVE CORRECTIONS IN
NONRELATIVISTIC CHERN-SIMONS THEORY

DIDIER CAENEPEEL 1 and MARTIN LEBLANC 2,
1 Laboratoire de Physique Nucléaire,
2 Centre de Recherches Mathématiques,
Université de Montréal, Montréal, Qc, H3C-3J7

ABSTRACT

We present the one-loop scalar field effective potential for the $N = 2$ supersymmetric nonrelativistic self-interacting matter fields coupled to an Abelian Chern-Simons gauge field and for its generalization when bosonic matter fields are coupled to non-Abelian Chern-Simons field. In both models, Gauss’s law linearly relates the magnetic field to the matter field densities; hence, we also include radiative effects from the background gauge field. We compute the scalar field effective potentials in two gauge families, a gauge reminiscent of the $R\xi$-gauge in the limit $\xi \to 0$ and in the Coulomb family gauges. We regularize the theory with operator regularization and a cutoff to demonstrate that the results are independent of the regularization scheme.

1. The models.

1.1. $N = 2$ supersymmetric nonrelativistic Chern-Simons model

The first model we consider is composed of self-interacting scalar field and fermionic field coupled to an Abelian Chern-Simons gauge field $^1$ [diag $\eta = (+,-,-)$]

$$S = \int dt d^2x \left\{ \frac{\kappa}{2} (\partial_t A) \times A - \kappa A^0 \nabla \times A + i\phi^*(\partial_t + iA^0)\phi + i\psi^*(\partial_t + iA^0)\psi - \frac{1}{2} |D\phi|^2 - \frac{1}{2} |D\psi|^2 + \frac{1}{2} B |\psi|^2 - \frac{\lambda_1}{4} (|\phi|^2)^2 - \lambda_2 |\phi|^2 |\psi|^2 \right\}$$

(1.1)

where $D = \nabla - iA$ is the covariant derivative and $B = \nabla \times A$ is the magnetic field. Note that the fermionic field is non-minimally coupled to the gauge field through the Pauli term. We have omitted the mass parameter since in nonrelativistic systems, it is always possible to set it equal to unity. [We use a vector notation: in the plane the cross product is $V \times W = \epsilon^{ij} V^i W^j$, the curl of a vector is $\nabla \times V = \epsilon^{ij} \partial_j V^i$, the curl of a scalar is $(\nabla \times S)^i = \epsilon^{ij} \partial_j S$ and we shall introduce the notation $(A \times \hat{z})^i = \epsilon^{ij} A^j$. The notation $x = (t, x)$ will also be used unless stated otherwise.]
The system (1.1) enjoys several invariances at the classical level. The system is Galilean, conformal, gauge invariant \(^2,3\) and \(N = 2\) supersymmetric when \(\lambda_1 = -\frac{2}{\kappa}\) and \(\lambda_2 = \frac{4}{3}\lambda_1\) \(^1\). At those values of the coupling constants, the model admits static self-dual (soliton) solutions \(^3\).

1.2. Nonrelativistic non-Abelian Chern-Simons model.

The second model we consider is a self-interacting scalar field coupled to SU(2) non-Abelian Chern-Simons gauge fields, which generalizes the above system

\[
S = \int dt d^2 \mathbf{x} \left\{ \kappa \epsilon^{\alpha\beta\gamma} \text{Tr}(A_\alpha \partial_\beta A_\gamma + \frac{2}{3} A_\alpha A_\beta A_\gamma) + i \phi^\dagger D_t \phi - \frac{1}{2} |D \phi|^2 - \frac{\lambda_{pqrs}}{4} \phi_p^\dagger \phi_q^\dagger \phi_r \phi_s \right\} \tag{1.2}
\]

where the gauge fields belong to the su(2) Lie algebra \(A_\mu = i \frac{A^a_\mu}{2}\), and \(D_t = \partial_t + iA^a_0 \tau^a\) and \(D = \nabla - iA^a \tau^a\) are the time and space covariant derivatives respectively. \(\phi_p\) is the two component nonrelativistic scalar field, \(p = 1,2\). The self-interaction coupling constants satisfy \(\lambda_{pqrs} = \lambda_{qpsr}\) since the fields are bosonic and \(\lambda^*_{pqrs} = \lambda_{rspq}\) for the Lagrangian to be real. \(\tau^a\) are the Pauli matrices which satisfy the usual commutation relations \([\tau^a, \tau^b] = \epsilon^{abc} \tau^c\) and trace relation \(\text{Tr}(\tau^a \tau^b) = \frac{1}{2} \delta^{ab}\).

Again, the system (1.2) enjoys the Galilean and conformal symmetries but we speak of “gauge invariance” only when a special quantization condition holds for the Chern-Simons coupling constant \(4\pi\kappa = \text{const}\) \(^4\) and if

\[
\lambda_{1111} = 2\lambda_{1212} = \lambda_{2222} \equiv \lambda \tag{1.3}
\]

As above, self-dual (soliton) solutions exist in this model \(^5\).

Our goal is to compute the scalar field effective potentials for both nonrelativistic Chern-Simons matter systems using a functional method \(^6,7\).

2. Scalar field effective potential for the Abelian model.

One can ask whether any of the classical symmetries survive the quantization of the theory. Furthermore, one can wonder if the self-dual solutions are stable or even if there are any modifications to the form of the potential. Many methods have been constructed to assess those effects. For example, the well-know Feynman’s diagrammatic or the functional methods have been useful in such studies. Here, we will concentrate on the latter method to compute the effective potentials for both models.

The effective potential method starts with the definition of a new shifted action \(^8\)

\[
S_{\text{new}} = S \left\{ \phi(x) = \phi + \pi(x); \psi(x); A^\mu(x) = a^\mu(x) + Q^\mu(x) \right\} - S \left\{ \phi, a^\mu(x) \right\} - \text{terms linear in quantum fields}, \tag{2.1}
\]

where we shift the scalar field by a constant field and we shift the gauge field by a solution to the classical equations of motion for the electromagnetic fields. To maintain consistency with Gauss’s law, which relates linearly the magnetic field to
the matter field, we need to choose a background gauge field \( a^\mu(x) \) such that the magnetic field is constant throughout the plane. We set \( a(x) = -\frac{B}{2}\hat{z} \times \hat{z} = \frac{\kappa}{2\pi}\hat{z} \times \hat{z} \) where \( B \) is the constant magnetic field. Such a choice is also consistent with the electric field equation of motion if \( a^0(x) = \frac{\omega - \phi^2}{4\pi}x^2 \). The fermionic field is not shifted because we consider only quantum corrections to the scalar field effective potential. Note however, that this solution for the background vector potential and with a constant \( \phi \) does not provide a solution to the equation for the matter sector unless \( \phi = 0 \).

Next, one chooses a gauge-fixing condition. We performed the calculation with a Coulomb-gauge \( \mathcal{L}_{G.F.} = \frac{1}{4\pi}(∇ \cdot Q)^2 \) for arbitrary \( \xi \) and with an \( \text{R}\xi \)-gauge \( \mathcal{L}_{G.F.} = \frac{1}{4\pi}[∇ \cdot Q + i\xi\phi^*\pi^*] [∇ \cdot Q - i\xi\phi^*\pi] \) (in the \( \xi \to 0 \) limit). The second one turns out to be the most efficient gauge-fixing condition since it cancels unwanted cross terms. The shifted action then becomes (\( \text{R}\xi\)-gauge)

\[
S = \int dt\, d^2x \left\{ \frac{\kappa}{2}(\partial_t Q) \times Q - \kappa Q^0∇ \times Q + \frac{1}{2\xi}(∇ \cdot Q)^2 - \rho \frac{2}{2}Q \cdot Q
\right. \\
+ i\pi^*(\partial_t + i\rho^0)\pi - \frac{1}{2}|D\pi|^2 - \frac{\lambda_1}{4}((\phi^2(\pi^*))^2 + 4\rho|\pi|^2 + (\phi^*\pi^2)^2) + \frac{\xi}{2}|\rho|^2
\right.
\]
\[
+ i\phi^*(\partial_t + i\rho^0)\phi - \frac{1}{2}|D\phi|^2 - \lambda_2(\phi^2|\phi|^2 + \frac{B}{2}|\psi|^2
\]
\[
+ c^*(-∇^2 + \xi\rho)c + J^0Q^0 + J \cdot Q \right\} \tag{2.2}
\]

with \( \rho = \phi^*\phi \), \( J^0 = -[\phi^*\pi + \pi^*\phi] \) and \( J = a^0J^0 \). The c-field term is the ghost compensating term arising from the choice of gauge-fixing condition. After a change of variable, we can write the shifted action in a form readily integrable

\[
\int dt\, d^2x d^2x' \left\{ \frac{1}{2}\pi^a(x)D^{-1}_{ab}(x - x')\pi^b(x')
\right.
\]
\[
+ \frac{1}{2}\phi^a(x)S^{-1}_{ab}(x - x')\phi^b(x') - \frac{1}{2}Q^\mu(x)\Delta^{-1}_{\mu\nu}(x - x')Q^\nu(x')
\right.
\]
\[
+ c^*(x)P^{-1}(x - x')c(x') + \frac{1}{2}J^\mu(x)\Delta_{\mu\nu}(x - x')J^\nu(x') \right\} \tag{2.3}
\]

Upon performing the path integrals, we find the effective action to be

\[
\Gamma_{\text{eff}} = S(\phi, a^\mu(x)) + \frac{i}{2}\ln\det\{D^{-1}_{ab} + M_{ab}\} - i\ln\det S^{-1}_{ab} + \frac{i}{2}\ln\det\Delta^{-1}_{\mu\nu} - i\ln\det P^{-1} \tag{2.4}
\]

where the matrices are easily found from Eq.(2.2) and the presence of the \( M_{ab} \) matrix is due to the mixing between matter and gauge fields.

We now discuss the structure of the perturbative expansion. The effective potential is related to the effective action by \( V_{\text{eff}} \int d^3x = -\Gamma_{\text{eff}} \) when defined on constant background fields. In the present case, the background gauge field \( a^\mu(x) \) is space-dependent; hence, we cannot use directly the functional method of Jackiw. We adopt the following strategy. We will compute the effective action by factoring out a matrix that is background gauge field independent and perturbatively expand the gauge field dependent part in powers of small coupling constants \( \lambda_1 \ll 1, \kappa^{-1} \ll 1 \)
(recall that $a^0 \sim \frac{\xi^2}{\sqrt{a}}$ and $a \sim \frac{\xi}{\sqrt{a}}$). The computation is up to $O(\rho^3)$ because each term of $O(\rho^3)$ is either of $O(\lambda^3)$, $O(\sqrt{\lambda} \frac{a^3}{\sqrt{a}})$, or $O(\frac{1}{\sqrt{a}})$. Therefore, for the rest of the paper, we will use the terminology $O(\rho^3)$. We do not introduce the parameter $\lambda_2$ in these expressions for simplicity since $\lambda_2$ does not enter the scalar field effective potential, as we will see below. Thus, as again we will see, we need only to consider 1-point or 2-point functions in background gauge fields that contribute to the effective action. The gauge-independent part will be treated following Jackiw’s method. We will also take the special limit over the gauge parameter $\xi \to 0$. In this limit $\ln \det \Delta^{-1} = 0$ and $\ln \det \mathcal{P}^{-1} = 0$.

The contributions from the gauge field $a_\mu(x)$ can be obtained by factorization, for example, $\ln \det (D^{-1} + M) = \text{Tr} \ln \left( \Theta^{-1} (1 + \Theta X) \right)$ and then the $\ln (1 + \Theta X)$ can be expanded in power of $X$, which is gauge field dependent. Upon doing this, we find that all contribution from $a_\mu$ vanishes to the order considered. Thus, we are left only with matter contributions

$$V_{\text{eff}} = -\frac{\Gamma_{\text{eff}}}{\sqrt{a^3}} = V_0(\rho) - \frac{i}{2} \int \frac{d^4 p}{(2\pi)^2} \frac{d\omega}{2\pi} \ln \frac{1}{\mu^4} \left\{-\omega^2 + \left( \frac{p^2}{2} + \lambda_1 \rho \right)^2 + \left(-\frac{\lambda^2}{4} + \frac{1}{4\pi} \right) \rho^2 + O(\rho^3) \right\}$$

$$+ i \int \frac{d^4 p}{(2\pi)^2} \frac{d\omega}{2\pi} \ln \frac{1}{\mu^4} \left\{-\omega^2 + \left( \frac{p^2}{2} + \lambda_2 \rho - \frac{1}{2} B \right)^2 \right\}$$

(2.5)

where the first integral comes from the bosonic integration, while the second integral comes from the fermionic integration.

We pose for a moment to notice that so far, no regulator has been used. In fact, divergences occurs only in the matter sector. We have performed the regularization of the expression in Eq. (2.5) with a cut-off and operator regularization. The basic ingredient is the regularization of the logarithm

$$\text{Tr} \ln H = -\lim_{s \to 0} \frac{d}{ds} \text{Tr} \frac{1}{\Gamma(s)} \int_0^\infty dt \ t^{-1} \left\{ e^{-H_0 t} + e^{-H_0 t} (-t) H_I + e^{-H_0 t} \left( \frac{-t^2}{2} H_I^2 + \ldots \right) \right\}$$

(2.6)

Upon making the identification $H_0 = [-\omega^2 + (\frac{p^2}{2} + \lambda_1 \rho)]/\mu^4$ and $H_I = (-\frac{\lambda^2}{4} + \frac{1}{4\pi}) \rho^2/\mu^4$ for the boson, and $H_0 = [-\omega^2 + (\frac{p^2}{2} + \lambda_2 \rho - \frac{1}{2} B)]^2/\mu^4$ for the fermion, dropping an unimportant const.$\rho^2$-term coming from the first term in Eq. (2.6), computing the one-pts function, evaluating the energy/momentum integrals, and imposing the normalization condition $\frac{d^2}{d\rho^2} V_{\text{eff}} |_{\rho=\mu} = \frac{1}{2} \lambda_1(\mu)$, we find to $O(\rho^3)$ in the $\xi \to 0$ limit the result

$$V_{\text{eff}}(\rho) = \frac{1}{4} \lambda_1(\mu) \rho^2 + \frac{\hbar}{32\pi} \left( \lambda_1(\mu)^2 - \frac{4}{\kappa^2} \right) \rho^2 \left( \ln \frac{\rho}{\mu^2} - \frac{3}{2} \right).$$

(2.7)

3. Scalar field effective potential for the non-Abelian model.

We follow the same procedure used in the above section to compute the effective potential in the non-Abelian model. In the action (1.2), we shift the scalar field by
a constant but this time the gauge field also can be shifted by a constant since the
classical equation of motion for the electromagnetic field can be solved by a constant
gauge field. In fact, for $\lambda = -5/16\kappa$ even the equation of motion for the scalar field
is satisfied without having to resort to a vanishing scalar field. In such a choice
of background gauge field, only global gauge invariance is retained and again it is
sufficient to gauge-fix the action with $L_{G,F} = (1/2\xi)(\nabla \cdot Q^a + i\xi \pi^a \tau^a \phi)(\nabla \cdot Q^a - i\xi \phi^a \tau^a \pi)$. After writing the shifted action, adding the gauge-fixing term and compensating
ghosts, one can integrate over the three gauge fields one after the other and ghost
fields easily, since the form of the operators in the determinants are diagonal in
Fourier space. We find that the ghost contribution cancels against one contribution
coming from the gauge field integration, which is the pure background gauge field
dependent part. The other contribution coming from the gauge field integration
modifies the matter sector. Again, performing the integration over the matter
fields and regulating via operator regularization (or a cut-off), we find to $O(\rho^3)$ in
the $\xi \to 0$ limit and after (re)normalization

$$V_{\text{eff}}(\rho, a^a_\mu) = \frac{1}{4}\rho^2 \lambda(\mu) + \frac{h}{32\pi} \left( \lambda^2(\mu) - \frac{4}{\kappa^2} \frac{3}{16} \right) \rho^2 \left( \ln \frac{\rho}{\mu^2} - \frac{3}{2} \right). \quad (3.1)$$

4. Summary

We have computed the scalar field effective potential for matter fields coupled
to Abelian or non-Abelian Chern-Simons gauge field including radiative corrections
from a background gauge field consistent with Gauss’s law. In both models, we
choose to gauge-fix with an $R_\xi$-gauge in the limit $\xi \to 0$ or with a Coulomb gauge with
arbitrary $\xi$ and we regulate divergences of the matter sector either with operator
regularization or with a cut-off. We find that the answer is independent of the
gauge-fixing condition and independent of the regulator used. Our results agree
with the ones found in the literature.10–13

Note that no ultraviolet divergences occur in the course of the evaluation of the
effective potentials as it is expected from using operator regularization; however,
when a cut-off is used we need to renormalize the theory with adequate counter-
terms. Our results Eq. (2.7) and Eq. (3.1) are independent of the background gauge
fields and the fermions do not contribute to the effective potential in the Abelian
model. As a spin-off of our calculation, we find no infinite nor finite renormalization
of the Chern-Simons coupling constant in both models.

Finally, both models experiences radiative conformal symmetry-breaking for un-
related coupling constants. The $\beta$-function can be read from the renormalization
group equation and we find

$$\beta(\lambda(\mu)) = \frac{1}{4\pi} \left( \lambda^2(\mu) - \frac{4}{\kappa^2} \alpha^2 \right) \quad (4.1)$$

where the subscript 1 has been dropped for the Abelian case and the group factor
$\alpha^2$ is 1 for the Abelian model and 3/16 for the non-Abelian model. Notice that
in both models the conformal symmetry is restored upon choosing the self-dual
critical point.
Acknowledgements

We thank F. Gingras and D.G.C. McKeon for their collaboration and the Natural Sciences and Engineering Research Council of Canada and the Fonds pour la Formation de Chercheurs et l’aide à la Recherche for financial support.

References

1. G. Lozano, M. Leblanc and H. Min, Ann. of Phys. 219, 328 (1992).
2. C.R. Hagen, Phys. Rev. D31, 848 (1985).
3. R. Jackiw and S.Y. Pi, Phys. Rev. D49, 3500 (1990).
4. S. Deser, R. Jackiw and S. Templeton, Phys. Rev. Lett. 48, 975 (1982); Ann. Phys. (N.Y.) 140, 372 (1982).
5. G. Dunne, R. Jackiw, S. Pi and C. Trugenberger, Phys. Rev. D 43, 1332 (1991).
6. D. Caenepeel, F. Gingras, M. Leblanc and D.G.C. McKeon, Phys. Rev. D49, (1994) (in press).
7. D. Caenepeel and M. Leblanc, preprint UdeM-LPN-TH-94-200, (1994).
8. R. Jackiw, Phys. Rev. D9, 1686 (1974).
9. L.S. Brown and W.I. Weisberger, Nucl. Phys. B157, 285 (1979).
10. G. Lozano, Phys. Lett. B283, 70 (1992).
11. O. Bergman and G. Lozano, Ann. Phys. (N.Y.) 229, 416 (1994).
12. D. Freedman, G. Lozano and N. Rius, Phys. Rev. D49, 1054 (1994).
13. D. Bak and O. Bergman, preprint MIT-CTP-2283, (1994).