Minimal Representations of Tropical Rational Signomials

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Abstract

This paper studies the following question: given a piecewise-linear function, find its minimal algebraic representation as a tropical rational signomial. We put forward two different notions of minimality, one based on monomial length, the other based on factorization length. We show that in dimension one, both notions coincide, but this is not true in dimensions two or more. We prove uniqueness of the minimal representation for dimension one and certain subclasses of piecewise-linear functions in dimension two. As a proof step, we obtain counting formulas and lower bounds for the number of regions in an arrangement of tropical hypersurfaces, giving a small extension for a result by Montúfar, Ren and Zhang. As an equivalent formulation, it gives a lower bound on the number of vertices in a regular mixed subdivision of a Minkowski sum, giving a small extension for Adiprasito’s Lower Bound Theorem for Minkowski sums.

Keywords: Arrangements of tropical hypersurfaces, Minkowski sums, tropical rational signomials, mixed subdivisions, tropical methods in machine learning

1 Introduction

Piecewise-linear functions are an important class of functions featured in machine learning [ZJS15],[HLM17], statistics [BR69] and optimization [MB09]. Training a deep ReLU network with a fixed architecture, for instance, can be understood as an optimization problem over a subset of such functions. One promising way to parametrize this space, put forward independently by Zhang et al.[ZNL18] and Maragos et al.[MCT21], is through the tropical algebra \( \mathbb{T} = (\mathbb{R}, \oplus, \odot, \oslash) \), where \( a \oplus b := \max(a, b) \), \( a \odot b := a + b \) and \( a \oslash b := a - b \). Over this algebra, a piecewise-linear function \( \varphi \) can be written as \( g \oslash h \), where \( g \) and \( h \) are tropical signomials [Ovc02; Mel86; KS87; GZ94]. Tropical methods have shown many promises in machine learning [Mar19; MT20b; SMR20; MT20a; SM20], one of which is the simplification of network structure. Motivated by this ongoing effort, the goal of this paper is to explore the notion of a minimal pair \((g, h)\) for a given \( \varphi \).

Concretely, let \( \varphi : \mathbb{R}^d \to \mathbb{R} \) be a piecewise linear function. Let

\[
\mathcal{F}(\varphi) = \{(g, h) : f(x) = g(x) \odot h(x) \text{ for all } x \in \mathbb{R}^d\}
\]

be the set of all representations of \( \varphi \) as a tropical rational signomial. Let \( \text{len} : g \mapsto \mathbb{N} \) be some length function that measures the complexity required to algebraically represent \( g \). Then \( \text{len} \) naturally induces the following preorder on \( \mathcal{F}(\varphi) \)

\[
g_1 \odot h_1 \preceq_{\text{len}} g_2 \odot h_2 \text{ if and only if } \text{len}(g_1) \leq \text{len}(g_2), \text{len}(h_1) \leq \text{len}(h_2).
\]

We consider two natural notions of length for a tropical signomial, the monomial length \( \text{mlen} \) and the factorization length \( \text{flen} \).

**Definition 1.** Let \( g \) be a reduced tropical signomial (c.f. Section 2) given by a tropical sum of monomials. Its monomial length \( \text{mlen}(g) \) is the number of monomials in \( g \). For a factorization \( g = h_1 \odot \cdots \odot h_m \), its factorization length \( \text{flen}(g) \) is \( \sum_{i=1}^m \text{mlen}(h_i) - (m - 1) \).

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The $m - 1$ term in the above definition of factorization length is a normalizing factor. Factorization of tropical signomials is simple in dimension 1 but is NP-Hard in dimension 2 or more [GL01], [Tiw08]. Therefore, one may suspect that theory of minimal representation is already quite rich in dimension $d = 2$. Our investigation supports this finding. We show in Proposition 15 that when $d = 1$, the two notions of length coincide and there is a unique minimal representation. However, when $d \geq 2$, factorization and monomial lengths can differ (c.f. Proposition 5). Our first result gives a sufficient condition for there to be a unique minimal representation with respect to either notion of length when $d = 2$. Let $\mathcal{V}(\varphi) \subset \mathbb{R}^d$ be the locus of non-linearity of a piecewise linear function $\varphi$.

**Theorem 1.** Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be a piecewise linear function. If $\mathcal{V}(\varphi)$ is an irreducible polyhedral fan, then $\varphi$ has a unique minimal representation $g \circ h$ w.r.t. the monomial length $\text{mlen}$, which is also minimal w.r.t. the factorization length $\text{flen}$. Moreover, if $(g_1 \circ \cdots \circ g_k) \circ (h_1 \circ \cdots \circ h_l)$ is minimal w.r.t. $\text{flen}$, then $g_1 \circ \cdots \circ g_k$ is a factorization of $g$ and $h_1 \circ \cdots \circ h_l$ is a factorization of $h$, up to a translation.

We define irreducibility in Section 4. To prove Theorem 1, we work with weighted polyhedral complexes and formulate the following minimal balancing problem. For any tropical signomial $g : \mathbb{R}^d \to \mathbb{R}$, $\mathcal{V}(g)$, also called a tropical hypersurface, is a pure weighed balanced polyhedral complex of codimension 1 in $\mathbb{R}^d$, which satisfies the relation $\mathcal{V}(g \circ h) = \mathcal{V}(g) \cup \mathcal{V}(h)$. Therefore, we migrate the concept of monomial (resp. factorization) length to balanced polyhedral complexes (c.f. Section 2).

**Definition 2.** Let $X$ be a weighted polyhedral complex pure of codimension 1 and $w_X$ be its weight function. A balancing of $X$ is a balanced polyhedral complex $Y$ such that $w_X \leq w_Y$. A minimal balancing of $X$ is a balancing that has minimal monomial (resp. factorization) length.

$\mathcal{V}(\varphi)$ has the structure of a sign-balanced polyhedral complex (c.f. Section 2). It can be written as $\mathcal{V}_+ - \mathcal{V}_-$ where $\mathcal{V}_+, \mathcal{V}_- \subset \mathcal{V}(\varphi)$ are the subcomplexes with positive and negative weights, respectively. To prove Theorem 1, we construct the unique minimal balancing for $\mathcal{V}_+$ and $\mathcal{V}_-$ simultaneously, which constitute the minimal representation for $\varphi$. For an extension to $d \geq 3$, we construct a conjectural minimal representation (c.f. Conjecture 1). The main idea is to construct the 2-faces of two polytopes $P$ and $Q$ minimally using Theorem 1, and represent $\varphi$ as $g \circ h$ such that $P$ and $Q$ are the Newton polytopes of $g$ and $h$, respectively.

In general, minimal balancings and minimal representations overlap, but are not identical. There exists a minimal balancing of $\mathcal{V}_+$ or $\mathcal{V}_-$ that induces a minimal representation of $\varphi$. This relation is explained by Figure 5. Besides its relation to minimal representations, the minimal balancing problem models many problems in various fields, thus worth studying separately. One question in machine learning is if one can recover the architecture and parameters of a neural network by simply querying the network [RK20; OSF19]. The minimal balancing problem, as a comparison, asks if one can construct a network that is compatible with the queries with the simplest structure. Another example is the Inverse Voronoi Problem [Alo+13], which asks how to fit a Voronoi diagram to a subdivision of the plane, where the subdivision is not necessarily convex. In that context, the minimal balancing problem asks for the Voronoi diagram with the fewest sites.

Our next theorem says The minimal balancing problem w.r.t. the monomial length and w.r.t. the factorization length can be different.

**Theorem 2.** There exist weighted polyhedral complexes $X$ with minimal balancing $Y_1$ w.r.t. the monomial length $\text{mlen}$, and minimal balancing $Y_2$ w.r.t. the factorization length $\text{flen}$, such that $\text{mlen}(Y_1) < \text{mlen}(Y_2)$ and $\text{flen}(Y_1) > \text{flen}(Y_2)$.

Theorem 3 gives comparison bounds between the monomial length and the factorization length. Under mild conditions, the factorization length of a signomial is much smaller than its monomial length.

**Theorem 3.** Let $g_1, \ldots, g_m : \mathbb{R}^d \to \mathbb{R}$ be tropical signomials with generic parameters. Then

$$\text{mlen}(g_1 \circ \cdots \circ g_m) \geq \text{flen}(g_1 \circ \cdots \circ g_m) + \sum_{k=2}^{d} \binom{m}{k}$$

The equality holds if and only if all the intersections of $\mathcal{V}(g_i)$’s are affine subspaces.
When all $q_i$'s are binomials, Theorem 3 recovers the classic region counting formula for arrangements of hyperplanes. As part of the proof, we derive a counting formula for the number of regions in an arrangement of tropical hypersurfaces, or equivalently, the number of vertices in the regular mixed subdivision of a Minkowski sum, an important topic in polyhedral geometry [GS93; Wei12; Fuk04; FW07]. Using a series of theories developed in [AS16] and [AY21] and the Lefschetz theorem for simplicial polytopes, Adiprasito gave a lower bound for the number of vertices of Minkowski sums [Adi17]. In comparison, Theorem 3 also applies to regular mixed subdivisions of Minkowski sums. We recover part of his result as a special case (c.f. Theorem 13), as well as give a lower bound that is asymptotically slightly better. Our proof is based on another result by Adiprasito on the topology of intersections of tropical hypersurfaces [Adi20]. Using the lower bound in [Adi17] and a result in [MRZ21], Tseran and Montúfar derived a lower bound for the number of linear regions of maxout networks ([TM21] Theorem 8). Our proof thus provides another path to their result. Theorem 3 is also complementary to Corollary 4.11 in [MRZ21], an upper bound for the number of regions in arrangements of tropical hypersurfaces derived from Weibel’s result [Wei12] for counting the number of upper vertices of a Minkowski sum.

We also find that the factorization length has some nice properties not shared by the monomial length (c.f. Proposition 16), which suggests that the factorization length is a more natural notion for developing an algorithmic solution to the minimal balancing problem. Moreover, every weighted polyhedral complex $X$ has a natural balancing consisting of the affine spans of all its maximal cells, which we call the canonical arrangement of $X$, denoted $A_X$. With respect to the factorization length, the following result shows that in dimension 2, the canonical arrangement is often a good solution, if not a minimal one.

**Proposition 4.** Let $X \subset \mathbb{R}^2$ be a weighted polyhedral complex pure of dimension 1. Let $V$ be its minimal balancing. Then

$$\text{flen}(A_X) \leq 3\text{flen}(V)$$

**Organization.** In Section 2, we explain notations and basic facts that will be used in this paper. In Section 3, we prove Theorem 3. In Section 4, we prove Theorem 1, Theorem 2, Proposition 4 and other properties of the factorization length. We conclude with open questions in Section 5.

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**2 Notations and preliminaries**

We assume that the readers have some familiarity with constructions in polyhedral geometry such as polyhedral complexes, recession fans, Newton polytopes, and regular subdivisions etc. For a comprehensive treatment, see [MS15; Jos21]. We point out that the balancing condition for balanced polyhedral complexes is slightly different from the one in classic tropical algebraic geometry, where complexes are naturally rational. Regardless, the core results about polyhedral complexes and tropical hypersurfaces hold verbatim in our setting. The balancing condition suitable in our setting can be found, for example, in [BIS22] Definition 3.3.

We explain some notations besides those in the introduction. Let $X$ be a weighted pure polyhedral complex and $w_X$ be its weight function defined on the set of all top dimensional cells. Unless otherwise stated, by a weighted polyhedral complex, we mean it is pure and $w_X$ is positive. The sum (resp. difference) $X + Y$ (resp. $X - Y$) of two weighted complexes $X$ and $Y$ is the weighted complex $X \cup Y$ with weight function $w_X + w_Y$ (resp. $w_X - w_Y$), after proper refinement, if necessary. Note that $w_X - w_Y$ may not be positively weighted. The support of a weighted complex is the closure of the union of all maximal cells with nonzero weight. The underlying polyhedral complex of a weighted polyhedral complex is its support. Therefore, as a set, $X - Y$ might be a proper subset of $X + Y$ due to cancellation.

If $X \subset \mathbb{R}^d$ is balanced and has codimension 1, then we define its monomial length $\text{mlen}(X)$ to be the number of regions it divides $\mathbb{R}^d$ into. If $X = X_1 + \cdots + X_m$ where $X_1, \ldots, X_m$ are balanced codimension 1 polyhedral complexes, then the factorization length of $X$ is $\sum_{i=1}^m \text{mlen}(X_i) - (m - 1)$. Let $\chi(X)$ be the Euler characteristic of $X$, the alternating sum of the numbers of cells in different dimensions. Let $\text{rec}(X)$ be the recession fan of $X$. 

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The reason why we call $\text{mlen}(X)$ its monomial length is due to the Structure Theorem for Tropical Hypersurfaces ([MS15] Section 3.3). The tropical hypersurface of a tropical signomial is a balanced polyhedral complex of codimension 1, whose weight of each cell is given by the magnitude of the gradient change when crossing that cell. Conversely, every balanced polyhedral complex of codimension 1 is a tropical hypersurface of a tropical signomial. For any tropical signomials $g, h$, their tropical hypersurfaces $V(g), V(h)$, their Newton polytopes $\Delta(g), \Delta(h)$, and their lifted Newton polytopes $\Delta^\uparrow(g), \Delta^\uparrow(h)$ satisfy the following relations.

\[
\begin{align*}
V(g \circ h) &= V(g) + V(h) \\
\Delta(g \circ h) &= \Delta(g) + \Delta(h) \\
\Delta^\uparrow(g \circ h) &= \Delta^\uparrow(g) + \Delta^\uparrow(h)
\end{align*}
\]

The monomial (resp. factorization) length of $g$ is the same as the monomial (resp. factorization) length of $V(g)$. The monomial length of $g$ is the same as the number of upper vertices of $\Delta^\uparrow(g)$, which is also the number of vertices in the regular subdivision of $\Delta(g)$ induced by $\Delta^\uparrow(g)$. For any polytope $P$, we will use $|P|$ to denote the number of its vertices.

We will also use an analogous fact for tropical rational signomials. The corner locus of a tropical rational signomial $\varphi$ is also a weighted polyhedral complex, but not necessarily positively weighted. If we allow negative weights in the balancing condition, then the weight function of $V(\varphi)$ satisfies the balancing condition verbatim. For that reason, we call $V(\varphi)$ \textit{sign-balanced}. Conversely, every sign-balanced polyhedral complex pure of codimension 1 is the corner locus of some tropical rational signomial. Any sign-balanced polyhedral complex is the difference of two balanced polyhedral complexes. In particular, whenever $\varphi = g \circ h$, $V(\varphi) = V(g) - V(h)$.

On the other hand, $V(\varphi)$ can be written as $V_+ - V_-$. $V_+$ is where $V(\varphi)$ has positive weight, or equivalently where $\varphi$ is strictly convex; $V_-$ is where $V(\varphi)$ has negative weight, or where $\varphi$ is strictly concave. As weighted polyhedral complexes, $V_+ \subset V(g)$ and $V_- \subset V(h)$, whereas $V_+$ and $V_-$ are not usually balanced. In other words, $V_+$ and $V_-$ are what remain visible after the cancellation between $V(g)$ and $V(h)$. This cancellation makes the geometry of tropical rational signomials much richer than that of tropical signomials.

\textbf{Example 1.} Consider

\[
\varphi(x) = \begin{cases} 
x, & 0 \leq x \leq y \\
y, & 0 \leq y \leq x \\
0, & \text{otherwise}
\end{cases}
\]

The following are two ways to represent $\varphi$ as a tropical rational signomial. The corresponding decomposition of $V(\varphi)$ is shown in Figure 1. The black subcomplex of $V(\varphi)$ is $V_+$ and the red subcomplex is $V_-$. \[
\varphi(x) = (x \oplus 0) \circ (y \oplus 0) \circ (x \oplus y \oplus 0) = (x \circ y \oplus x \oplus y) \circ (x \oplus y)
\]

![Figure 1: Two ways of representing $\varphi$ from the perspective of $V(\varphi)$.](image)

We call a tropical signomial \textit{reduced} if none of its factor contains redundant monomials, meaning removing any of the monomials will change the underlying real-valued function. We call a tropical rational signomial...
φ = g ⊙ h reduced if both g and h are reduced, and they don’t have common factors. In the above example, both of the two representations are reduced. For convenience, we occasionally call the tuple \((\text{mlen}(g), \text{mlen}(h))\) the monomial length of \(φ\), likewise for factorization length.

3 The length of tropical signomials

This section is devoted to proving Theorem 3, along with the following proposition, showing the distinct behavior of monomial length and factorization length.

Proposition 5. For tropical signomials in one variable, the monomial length and the factorization are the same for any factorization. There exist tropical signomials \(g, h\) in two variables such that \(\text{mlen}(g) < \text{mlen}(h)\) whereas \(\text{flen}(g) > \text{flen}(h)\).

Proof. It is a well known fact that tropical signomials in one variable factor uniquely into linear factors. The number of tropical roots is the number of linear pieces minus 1, and the factorization length is the number of tropical roots plus 1. Hence, for any tropical signomial \(g\) in one variable, \(\text{flen}(g) = \text{mlen}(g)\) for the irreducible factorization and the result follows for an arbitrary factorization. To see how the above fails for higher dimensions, consider the following example. Let

\[
g(x, y) = (1 ⊙ x^1 ⊙ y^{-1}) ⊕ (1 ⊙ y^{0-2}) ⊕ (1 ⊙ x^{-1} ⊙ y^{-1}) ⊕ 1 ⊙ y
\]

\[
h(x, y) = (x^{-1} ⊙ y^{0-1} ⊙ 0) ⊕ ((-2) ⊙ x^{01} ⊙ y^{0-1} ⊙ 0) ⊕ (y^{01} ⊙ 0)
\]

One can check that \(g\) is irreducible, and \(\text{mlen}(g) = 5 < \text{mlen}(h) = 6\), whereas \(\text{flen}(g) = 5 > \text{flen}(h) = 4\). Figure 2 shows \(V(g)\) and \(V(h)\) and the dual subdivision of their Newton polytopes. The subdivision defined by \(g\) has fewer vertices, but it takes more room to store in a computer since it exhibits less regularity, compared to the zonotope on the right.

![Figure 2: V(g), V(h) and the dual subdivision of the corresponding Newton polytopes](image)

The idea for proving Theorem 3 is the following. First, we establish the relation between the monomial length and the factorization length by appealing to the Euler-Poincaré relation of balanced polyhedral complexes and the inclusion-exclusion property of the Euler characteristic. Then, we control the topology of intersections of tropical hypersurfaces and bound their Euler characteristic.

Lemma 6 (Euler-Poincaré relation). If \(X ⊂ \mathbb{R}^d\) is a balanced polyhedral complex of codimension 1, then

\[
\chi(X) + (-1)^d \text{mlen}(X) = (-1)^d
\]

Lemma 7 (Inclusion-exclusion). Let \(X, Y ⊂ \mathbb{R}^d\) be two polyhedral complexes. Then

\[
\chi(X ∪ Y) = \chi(X) + \chi(Y) - \chi(X ∩ Y)
\]

Theorem 8 (Region counting formula for arrangements of tropical hypersurfaces). Let \(g_1, \ldots, g_m : \mathbb{R}^d → \mathbb{R}\) be tropical signomials. Then

\[
\text{mlen}(g_1 ⊙ \cdots ⊙ g_m) = \text{flen}(g_1 ⊙ \cdots ⊙ g_m) + \sum_{k=2}^{m} (-1)^{k+d} \sum_{S ∈ \binom{[m]}{k}} \chi(\bigcap_{i ∈ S} V(g_i))
\]
Proof. Combine the Euler-Poincaré relation and Lemma 7.

Recall that $\text{mlen}(g_1 \odot \cdots \odot g_m)$ is the number of regions the tropical hypersurfaces $V(g_1) \cup \cdots \cup V(g_m)$ cuts $\mathbb{R}^d$ into. One should compare Theorem 8 to Theorem 5.5 in [MRZ21] (c.f. Example 2). There, they express the number of regions in an arrangement of tropical hypersurfaces as the sum of the Möbius function of each cell weighted by its Euler characteristic, an idea based on Zaslavsky’s topological dissection theory. The advantage of that formula is that it is more general and can apply to arbitrary complexes, while Theorem 8 only applies to tropical hypersurfaces. The advantages of Theorem 8 are twofold. First, it is easier to compute. Second, it allows one to read off more information by analyzing the topology of intersections of tropical hypersurfaces, which has been studied previously [Adi20; AB14; BB07].

Example 2. The following is an example in [MRZ21]. Their picture shows an arrangement of tropical curves in $\mathbb{R}^2$ and its intersection poset.

Each tropical curve divides $\mathbb{R}^2$ into three regions. Their intersection is three points, having Euler characteristic 3. Theorem 1 then tells us the arrangement cuts $\mathbb{R}^2$ into $3 + 3 - 2 + 1 + 3 = 8$ regions. Using the formula given in [MRZ21], we need to compute the Möbius function and the Euler characteristic of the intersections of the cells. $\mu(\bullet) = 1$, $\mu(\ast) = 1$, $\mu(H_{ab}) = -1$, $\mu(\mathbb{R}^2) = 1$. This yields

$$r = 1 + 0(-1 - 1 - 1 - 1 - 1 - 1) + (-1)^0(2 + 1 + 1 + 1) = 8$$

regions.

In machine learning, the number of linear regions a neural network has is a measure of its expressivity. Given a tropical rational signomial $\varphi = g \odot h$, one can associate a surrogate signomial $g \odot h$. The number of linear regions is bounded from above by the number of linear regions of $g \odot h$, which can be computed using Theorem 8. This implies the following.

Corollary 9. Let $\varphi = g \odot h$ be a tropical rational signomial on $\mathbb{R}^d$ and $r(V(\varphi))$ be the number of linear regions it has. Then

$$r(V(\varphi)) \leq \text{mlen}(g) + \text{mlen}(h) + (-1)^d 2^d \chi(V(g) \cap V(h))$$

Now we prepare another ingredient for the proof to Theorem 8. We call a polyhedral complex $X$ a generic intersection if it is the intersection of tropical hypersurfaces $V(g_1), \ldots, V(g_m)$ where $g_1, \ldots, g_m$ are tropical signomials with generic parameters. By generic parameters, we mean that the coefficients and the exponents are generic. Using Stratified Morse Theory, Adiprasito shows that if all the tropical hypersurfaces in a complete intersection are pointed, meaning that they divide the ambient space $\mathbb{R}^d$ into pointed polyhedra, then their intersection is homotopy Cohen-Macaulay ([Adi20], Theorem 1.1), which implies that the one-point compactification of the intersection is a wedge of spheres of the same dimension. Moreover, the bounded part of the intersection, meaning the subcomplex consisting of bounded polyhedra, is also homotopy equivalent to a wedge of spheres. By examining the approach he used there, we get the following result.

Lemma 10. The one-point compactification of a generic intersection $X$ of dimension $n$ is homotopy equivalent to a wedge of $n$-spheres, so is its bounded part (the subcomplex of $X$ consisting of polytopes).

Proof. The key components of the proof for Theorem 1.1 in [Adi20] are the construction of the Morse function and an induction on the dimension and the number of hypersurfaces. When the tropical hypersurfaces are generic, there are obvious ways to construct Morse functions. For example, if $P$ is a $d$-dimensional polyhedron in $\mathbb{R}^d$, one can define a smooth function $f$ on $P$ to be the product of the distance to the defining hyperplanes of $P$, as is defined in [AB14] Lemma 7.8. Then $f$ restricted to $P \cap X$ will be a Morse function as long as $X$ is generic. In this way, the argument in [Adi20] applies and the conclusion follows. □
**Lemma 11.** Let $X$ be a generic intersection of dimension $n$. Then $\chi(\text{rec}(X)) \leq \chi(X)$ if $n$ is even, and $\chi(\text{rec}(X)) \geq \chi(X)$ if $n$ is odd.

**Proof.** Let $Y$ be the subcomplex of $X$ consisting all the bounded polyhedra. Then $\chi(X \setminus Y) = \chi(\text{rec}(X)) - 1 = \chi(X) - \chi(Y)$. If $Y$ is a point, then we are done. If $Y$ is non-empty, then $Y$ is homotopy equivalent to a wedge of $n$-spheres, so $\chi(Y) \geq 2$ if $n$ is even, and $\chi(Y) \leq 0$ if $n$ is odd. This finishes the proof. \hfill $\Box$

**Lemma 12.** Let $X$ be a generic intersection of dimension $n$. Then $\chi(X) \geq 1$ if $n$ is even, and $\chi(X) \leq -1$ if $n$ is even. The equality holds if and only if $X$ is an affine subspace.

**Proof.** By Lemma 10, the weak inequalities hold. We only need to show that strict inequalities hold when $X$ is not an affine subspace. For $n = 0$, $X$ consists of finitely many points. $X$ is not an affine subspace if and only if $X$ contains more than one point, so the claim is obvious. Suppose the claim holds for $n \leq k$ and that $X$ has dimension $k + 1$. Consider $\text{rec}(X)$. If $X$ is not an affine subspace, neither is $\text{rec}(X)$. By Lemma 11, when $k + 1$ is even, $\chi(\text{rec}(X)) \leq \chi(X)$, whereas when $k + 1$ is odd, $\chi(\text{rec}(X)) \geq \chi(X)$. Therefore, it suffices to prove the claim for fans. Now assume $X$ is a fan. Consider the following construction. Intersect $X$ with a generic hyperplane $H$ that goes through the base point of $X$. We decompose $X$ into three disjoint sub-complexes $X \cap H$, $X \cap H^{>0}$ and $X \cap H^{<0}$, where $H^{>0}$ and $H^{<0}$ are the half-spaces defined by $H$. Let $H_+$ be any hyperplane parallel to $H$ that lives in $H^{>0}$ and $H_-$ be any hyperplane parallel to $H$ that lives in $H^{<0}$. Let $X_1 = X \cap H_+$ and $X_2 = X \cap H_-$. Then $X \cap H^{>0}$ is homeomorphic to $X_1 \times \mathbb{R}$ and $X \cap H^{<0}$ is homeomorphic to $X_2 \times \mathbb{R}$. Therefore, $\chi(X \cap H^{>0}) = -\chi(X_1), \chi(X \cap H^{<0}) = -\chi(X_2)$. Moreover, $X \cap H = \text{rec}(X_1) = \text{rec}(X_2)$. Putting everything together, we get

$$\chi(X) = \chi(X \cap H^{>0}) + \chi(X \cap H^{<0}) + \chi(X \cap H) = -\chi(X_1) - \chi(X_2) + \chi(\text{rec}(X_1))$$

Note that $X_1, X_2$ and $\text{rec}(X_1)$ are generic intersections of dimension $k$. Moreover, at least one of $X_1$ and $X_2$ is not an affine subspace. If $k$ is even, then

$$-\chi(X_1) - \chi(X_2) + \chi(\text{rec}(X_1)) \leq \min\{-\chi(X_1), -\chi(X_2)\} < -1$$

while if $k$ is odd,

$$-\chi(X_1) - \chi(X_2) + \chi(\text{rec}(X_1)) \geq \max\{-\chi(X_1), -\chi(X_2)\} > 1$$

This completes the proof. \hfill $\Box$

**Proof of Theorem 3.** Since all the tropical hypersurfaces are in general position, any $k$ of them have empty intersection if $k \geq d + 1$. By Lemma 12,

$$\chi(\bigcap_{i \in S} V(g_i)) \leq -1$$

if $d - k$ is odd, and

$$\chi(\bigcap_{i \in S} V(g_i)) \geq 1$$

if $d - k$ is even. Therefore, we have

$$(-1)^{k+d} \chi(\bigcap_{i \in S} V(g_i)) \geq 1$$

which yields the lower bound by Theorem 8. By Lemma 12, the equality holds if and only if all the intersections $\bigcap_{i \in S} V(g_i)$ are affine subspaces. \hfill $\Box$

It is not true in general if $g_i$’s don’t have generic parameters. Consider two triangles with only one pair of unparallel edges. Their Minkowski sum is a 4-gon, having monomial length 4, while the factorization length is 5.

Theorem 3 shows that factorization length is usually much smaller than monomial length. When the number of factors grows, the factorization length grows linearly while the monomial length grows in $O(m^d)$. This also means in the preorder induced by factorization length, a minimal representation of $g$ must be an irreducible factorization of $g$, whenever all factors have generic parameters. However, we note that the converse is not true, as different irreducible factorizations may have different lengths, as is shown in the
following example. Let \( g = (x \oplus y) \odot (x \oplus 0) \odot (y \oplus 0) = (x \oplus y \oplus 0) \odot (x \odot y \oplus x \odot y) \). The first factorization has length 4, while the second has length 5.

Theorem 3 has several interpretations. First, it says the number of regions in an arrangement of tropical hypersurfaces in general orientations is no less than the number of regions in an arrangement of hyperplanes, a very intuitive fact that doesn’t seem easy to prove directly. Second, recall that the number of regions in an arrangement of tropical hypersurfaces corresponds to the number of vertices in the regular mixed subdivision of the Minkowski sum of the corresponding Newton polytopes. Therefore, Theorem 3 is a lower bound in that setting. The most interesting case is when the mixed subdivision is trivial. This corresponds to arrangements of balanced polyhedral fans which are based at the same point. As a special case, we have

\[ \text{Theorem 13. Let } P_1, \ldots, P_m \subset \mathbb{R}^d \text{ be polytopes in general position. Then} \]

\[ |P_1 + \cdots + P_m| \geq \sum_{i=1}^{m} |P_i| + 2 \sum_{k=0}^{d-1} \binom{m-1}{k} - 2m \]

\[ \text{Proof. Consider the dual tropical hypersurfaces of those polytopes. Consider the intersection of } k \text{ of them. If } k \leq d, \text{ the argument is the same as in Theorem 3. If } k > d, \text{ then their intersection is a single point. Applying Lemma 12 and Theorem 8, one gets} \]

\[ |P_1 + \cdots + P_m| \geq \sum_{i=1}^{m} |P_i| + \sum_{k=2}^{d} \binom{m}{k} + \sum_{k=d+1}^{m} (-1)^k \binom{d}{k} = \sum_{i=1}^{m} |P_i| + 2 \sum_{k=0}^{d-1} \binom{m-1}{k} - 2m \]

\[ \square \]

If all \( P_i \)'s are polytopes of positive dimension, then \( |P_i| \geq 2 \) for all \( i \), which implies

\[ |P_1 + \cdots + P_m| \geq 2 \sum_{k=0}^{d-1} \binom{m-1}{k} \]

This gives another proof for the Lower Bound Theorem by Adiprasito (Corollary 8.2 [Adi17]).

\[ \text{Corollary 14. A Minkowski sum of polytopes of positive dimension in general position has at least the same number of vertices as the Minkowski sum of the same number of line segments in general position.} \]

\[ \text{Remark 1. When all the polytopes } P_1, \ldots, P_m \text{ are full-dimensional, the precise lower bound obtained in [Adi17] is} \]

\[ |P_1 + \cdots + P_m| \geq \sum_{i=1}^{m} |P_i| + \binom{d+m-1}{d-1} - d - m + 1 \]

For a fixed \( d \), this bound is better than Theorem 13 when \( m \) is small. When \( m \) is large, note that

\[ \binom{d+m-1}{d-1} - d - m + 1 \sim \frac{m^{d-1}}{(d-1)!}, \quad 2 \sum_{k=1}^{d-1} \binom{m-1}{k} - 2m \sim \frac{2m^{d-1}}{(d-1)!} \]

Therefore, Theorem 13 is a slightly better bound asymptotically.

\[ \text{Remark 2. In [TM21] Theorem 8, Tseran and Montúfar obtain a lower bound for the number of activation regions for a maxout network. Their proof is based on the result in [Adi17] and a lower bound on the number of strictly upper vertices of a Minkowski sum ([MRZ21] Theorem 6.9). Their result can be re-obtained by applying Theorem 3.} \]
4 Minimal representations

This section is devoted to proving Theorem 1 and Proposition 15, along with which we develop languages regarding the minimal representation problem and the minimal balancing problem. To start with, we note that a minimal representation is necessarily reduced, whereas a reduced representation is not necessarily minimal. Recall the function in Example 1

$$x \oplus y \oplus 0 - x \oplus y = x \odot y \odot x \odot y \oplus 0 - x \odot y \odot x \odot y$$

Both sides are reduced and represent the same piecewise-linear function, but the right hand side has monomial length $(4, 3)$ while the left hand side has monomial length $(3, 2)$. If we factor $x \odot y \odot x \odot y \odot 0$ into $(x \odot 0) \odot (y \odot 0)$, then the right hand side has factorization length $(3, 3)$. In whichever case, the left hand side is a shorter representation. In fact, it is the unique minimal representation w.r.t. either of the monomial length or the factorization length. Minimal representations are not unique, in general, but they are when $d = 1$.

**Proposition 15.** Every tropical rational signomial in one variable has a unique minimal representation up to a translation. Namely, if $\varphi = g_1 \odot h_1$ and $\varphi = g_2 \odot h_2$ are two minimal representations, then $g_1 \odot g_2$ is a linear function. Moreover, a representation for $\varphi$ is minimal if and only if its reduced.

**Proof.** By Proposition 5, monomial length and factorization length agree when $d = 1$. Suppose $\varphi = g \odot h$ is reduced where $g$ and $h$ are in the standard form. This means that $V(\varphi) = V(g) \cup V(h)$, $V(g)$ is the positive part of $V(\varphi)$ while $V(h)$ is the negative part of $V(\varphi)$. Therefore, $\text{mlen}(g) = |V(g)| + 1$ and $\text{mlen}(h) = |V(h)| + 1$. Note that this is the lower bound for the length of any representation. Hence, $g \odot h$ must already be minimal. For uniqueness, suppose $g_1 \odot h_1$ and $g_2 \odot h_2$ are two minimal representations. Then $V(g_1) = V(g_2)$ as weighted polyhedral complex. This means $g_1 \odot g_2$ must be a linear function. \hfill $\Box$

Proposition 15 also gives a quick way for representing a piecewise linear function in one variable as a tropical rational signomial. Suppose we are given such a function $\varphi$, knowing $V(\varphi) = \{x_1, \ldots, x_n\} \cup \{y_1, \ldots, y_m\}$, where $\varphi$ is strictly convex at $x_i$’s and strictly concave at $y_i$’s. Let

$$\psi(x) = \frac{\prod_{i=1}^{a_1} (x \oplus x_1)^{a_1} \odot \cdots \odot (x \oplus x_m)^{a_m}}{(x \oplus y_1)^{b_1} \odot \cdots \odot (x \oplus y_n)^{b_n}}$$

where $a_i$’s and $b_i$’s are the absolute value of the slope change of $\varphi$ around each non-differentiable point. Then $\psi$ and $\varphi$ differ by a linear function, which can be determined by computing two values of $\varphi$.

To solve minimal representation problems in higher dimensions, it is easier to think about minimal balancing problems. As we turn to now. Let $X \subset \mathbb{R}^d$ be a weighted polyhedral complex pure of codimension 1. We call $X$ completely unbalanced if $X$ does not contain any nontrivial balanced subcomplex pure of codimension 1. Later on, we will focus on completely unbalanced complexes. This is a harmless assumption.

Proposition 16. Let $X$ be an unbalanced polyhedral complex.

1. If $X$ is balanced minimally by $f_1 \odot \cdots \odot f_m$ w.r.t. the factorization length, then any factor $f_n_1 \odot \cdots \odot f_n_k$ balances $X \cap V(f_n_1 \odot \cdots \odot f_n_k)$ minimally.

2. Suppose $X$ is a polyhedral complex in $\mathbb{R}^2$. Let $R$ be a ray with the base point at any position and has general orientation. Let $A_R$ be the canonical arrangement of $R$. Then $V \cup A_R$ is a minimal balancing of $X \cup R$.

To see how the monomial length fails to have the above properties, consider the example shown in Figure 3. Suppose the blue fan is already balanced. By Theorem 8, the number of regions in the above arrangement is a linear function in the number of intersections. To minimize the monomial length is the same as minimizing
the intersection of the balancing for the black fan and the blue fan. There are two consequences. First, the minimal balancing fails to satisfy property 1 in Proposition 16. Second, the balancing of the black fan is sensitive to the geometry of blue fan, which is not natural at all.

![Figure 3: Minimal balancing with respect to monomial length](image)

**Proof.** Suppose for a contradiction that some factor, say, $f_1 \odot \cdots \odot f_k$ does not balance $X \cap V(f_{m_1} \odot \cdots \odot f_{n_k})$ minimally. Then there is some other balancing $g_1 \odot \cdots \odot g_s$ having smaller factorization length. Note that $g_1 \odot \cdots \odot g_s \odot f_{k+1} \odot \cdots \odot f_m$ is also a balancing of $X$, such that

\[
\text{flen}(g_1 \odot \cdots \odot g_s \odot f_{k+1} \odot \cdots \odot f_m) = \text{flen}(g_1 \odot \cdots \odot g_s) + \text{flen}(f_{k+1} \odot \cdots \odot f_m) - 1 < \text{flen}(f_1 \odot \cdots \odot f_k) + \text{flen}(f_{k+1} \odot \cdots \odot f_m) - 1 = \text{flen}(f_1 \odot \cdots \odot f_m)
\]

This contradicts to the minimality of $f_1 \odot \cdots \odot f_m$. This proves the first property.

Note that $\text{flen}(V \cup A_R) = \text{flen}(V) + 1$. Let $W$ be a balancing of $X \cup R$. Then $\text{flen}(W) \geq \text{flen}(V)$, as $V$ is a minimal balancing of $X$. Since $R$ has general orientation, $\text{flen}(W) > \text{flen}(V)$. This means $\text{flen}(W) \geq \text{flen}(V \cup A_R)$. Hence $V \cup A_R$ is minimal. This proves the second.

To prove Theorem 1, we need the following several lemmas concerning the uniqueness of minimal balancing for completely unbalanced fans. Throughout the next three lemmas, $X$ is a completely unbalanced fan of dimension 1 in $\mathbb{R}^2$ consisting of $m$ rays.

**Lemma 17.** $X$ has a unique minimal balancing $Y$ w.r.t. the monomial length and $\text{flen}(Y) = m + 1$

**Proof.** Without loss of generality, suppose $X$ is based at the origin. Let $v_1,...,v_m$ be the vectors that span $X$, whose lengths are their corresponding weights. Let $v' = -\sum_{i=1}^{m} v_i$. Since $X$ is completely unbalanced, $v'$ is not parallel to any existing rays. Let $Y$ be the fan spanned by $v_1,...,v_m,v'$. Then $Y$ is a balancing of $X$ and it has monomial length $m + 1$. Since $X$ is completely unbalanced, any balancing of $X$ must have monomial length at least $m + 1$. This means $Y$ is a minimal balancing of $X$ w.r.t. the monomial length. Any other balancing with monomial length $m + 1$ must contain the ray spanned by $v'$ and thus must be $Y$. Hence, $Y$ is the unique minimal balancing.

**Lemma 18.** If $Y_1 + \cdots + Y_k$ is a minimal balancing of $X$ w.r.t. the factorization length, then $Y_i \cap Y_j$ is a point.

**Proof.** When $k = 2$, suppose $Y_1 \cap Y_2$ contains a ray. By Proposition 16, $Y_1$ is the minimal balancing of $Y_1 \cap X$ and $Y_2$ is the minimal balancing of $Y_2 \cap X$. By Lemma 17, the factorization length of $Y_1 + Y_2$ is at least $m + 2$, which is not minimal. For $k$ larger than 2, we apply the above argument to $(Y_i + Y_j) \cap X$ and use Proposition 16.

**Lemma 19.** $X$ consisting of $m$ rays has $B_m$ many minimal balancings w.r.t. the factorization length, where $B_m$ is the $m$-th Bell number. The minimal balancing w.r.t. the monomial length is one of them.

**Proof.** By Lemma 17, the lower bound for the factorization length of any balancing of $X$ is no larger than $m+1$. We show that it is equal to $m+1$. Let $Y_1 + \cdots Y_k$ be a minimal balancing of $X$. Then by Proposition 16, $Y_i$ is a minimal balancing for $Y_i \cap X$, which is again a completely unbalanced fan. By the previous paragraph, there is a unique balancing for $Y_i \cap X$. By Lemma 18, the rays in $Y_i \cap X$ form a partition for the rays in $X$. 

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Any such partition gives a minimal balancing of factorization length \( m + 1 \), hence minimal. Therefore, the number of minimal balancing of \( X \) w.r.t. the factorization length corresponds bijectively to the partition of a set of size \( m \), which is \( B_m \).

**Remark 3.** In Lemma 19, it is possible that two different minimal balancings are different factorizations of the same fan. For example, consider a completely unbalanced fan consisting of four rays spanned by \( v_1, v_2, v_3, v_4 \). If \( v_1 + v_2 = v_3 + v_4 \), then balancing the fan spanned by \( \{v_1, v_2\} \) and the fan spanned by \( \{v_3, v_4\} \) gives the same fan as the minimal balancing of \( \{v_1, v_2, v_3, v_4\} \).

Figure 4 shows an example where \( m = 4 \). Both are minimal balancings w.r.t. the factorization length. The first one corresponds to the partition of the rays of \( X \) consisting of the whole set, while the second one corresponds the partition where each ray is a single set. The first balancing is the unique minimal balancing w.r.t. the monomial length.

![Figure 4: Two balancings of a fan (black) in \( \mathbb{R}^2 \)](Image 216x482 to 396x561)

Now we prove Theorem 1. A *decomposition* of \( X \) is two sign-balanced polyhedral complexes \( X_1 \) and \( X_2 \) such that \( X = X_1 + X_2 \), \( X_1 \) and \( X_2 \) are both sign-balanced, and neither of \( X_1 \) nor \( X_2 \) is a proportion of the other. \( X \) is called irreducible if it does not have such a decomposition. The motivation for this definition is that we want to define a “minimal unit” for solving the minimal representation problem. If \( X \) has a decomposition \( X_1 + X_2 \), then one should be able to study the minimal representation for \( X_1 \) and \( X_2 \) separately. It is easy to check that if \( X \) is an irreducible sign-balanced polyhedral complex with non-empty positive part \( X_+ \) and negative part \( X_- \), then \( X_+ \) and \( X_- \) are completely unbalanced.

**Proof of Theorem 1.** Suppose \( V_+ \) has \( m_1 \) rays and \( V_- \) has \( m_2 \) rays. By Lemma 17, there is a unique minimal balancing for \( V_+ \), by adding a single ray \( \mathbb{R}^r \). This construction also balances \( V_- \) minimally. Hence it gives a minimal representation for \( \varphi \) w.r.t. the monomial length. This minimal representation has monomial length \( (m_1 + 1, m_2 + 1) \). Let \( (r, s) \) be the monomial length of some other minimal representation. Then \( r \leq m_1 + 1 \) or \( s \leq m_2 + 1 \). Note that a representation of length \( (r, s) \) induces a balancing for \( V_+ \) of length \( r \) and a balancing for \( V_- \) of length \( s \). This means \( r \geq m_1 + 1 \) and \( s \geq m_2 + 1 \). Hence, any minimal balancing has length \( (m_1 + 1, m_2 + 1) \) and induces minimal balancing for \( V_+ \) and \( V_- \). Since the minimal balancing is unique, so is the minimal representation.

Let \( g \ast h \) be the representation we obtain from above. Let \( g_1 \circ \cdots \circ g_k \circ (h_1 \circ \cdots \circ h_l) \) be a minimal representation w.r.t. the factorization length. We show that \( g = g_1 \circ \cdots \circ g_k \) up to a linear shift. Suppose this is not true. Using a argument similar to above, we know \( g_1 \circ \cdots \circ g_k \circ (h_1 \circ \cdots \circ h_l) \) must have factorization length \( (m_1 + 1, m_2 + 1) \). This means \( g_1 \circ \cdots \circ g_k \) must be a minimal balancing of \( V_+ \). Since \( V(g) \neq V(g_1 \circ \cdots \circ g_k) \), \( V(g_1 \circ \cdots \circ g_k) - V_+ \) must contain at least two rays. Moreover, \( V(g_1 \circ \cdots \circ g_k) - V_+ \) does not contain any nontrivial balanced subcomplex, otherwise \( V(\varphi) \) would contain a nontrivial sign-balanced subcomplex, a contradiction to the assumption that \( V(\varphi) \) is irreducible. Note that \( h_1 \circ \cdots \circ h_l \) is a minimal balancing of \( V(g_1 \circ \cdots \circ g_k) - V_+ + V_- \) a completely unbalanced fan with more than \( m_2 \) rays, so it has factorization length at least \( m_2 + 2 \), which means it is not a minimal balancing of \( V_- \). This contradicts to the assumption that \( g_1 \circ \cdots \circ g_k \circ (h_1 \circ \cdots \circ h_l) \) is a minimal representation.

Notice that, in the above construction, we obtain minimal balancings for \( V_+ \) and \( V_- \) simultaneously, which constitute the minimal representation for \( \varphi \). In general, the relation between the minimal balancing problem and the minimal representation problem can be demonstrated by Figure 5. The gray area covers the lengths of all possible representations of \( \varphi \). The vertical axis is the length of the numerator and the horizontal axis the length of the denominator. The points on the southwestern corner locate the minimal
for can think of them as vectors $v \in \mathbb{R}^X$ with weight function $\min$ for a unique such polygon. Let's call it the $v$ such that $R$ whose length is its weight. Choose an orientation for $P,Q$ pairs of polytopes represent the same piecewise linear function, up to a linear shift. Hence, the minimal representation problem for polyhedral fans. Given two polytopes $P,Q$ such that whenever $\tau$ is dual to a tetrahedron. In this simple example, it is easy to check that it is the unique minimal balancing w.r.t. the monomial length.

Theorem 1. Let $V$ be a locus of non-linearity of some sign-balanced polyhedral fan of dimension $d$ such that has codimension 1 in $X$. Suppose $\sigma_1,...,\sigma_n$ is the maximal cells of $X_+$ containing $\tau_1$, with weight $w_X(\sigma_1),...,w_X(\sigma_n)$. The positive spans of $\sigma_1,...,\sigma_n$ form a 1-dimensional fan in $\mathbb{R}^d/\mathbb{R}^\tau$. Construct its minimal polygon $P$. Fix any vertex of $P$ at the origin.

Step 2: suppose we have constructed the minimal polygons $P_1,...,P_k$ for codimension-1 cells $\tau_1,...,\tau_k$. Choose the next codimension-1 cell $\tau_{k+1}$ and construct its minimal polygon $P_{k+1}$. Place $P_{k+1}$ in $\mathbb{R}^d$ such that whenever $\tau_{k+1}$ and $\tau_i$ are the faces of the same maximal cell $\sigma$, $P_{k+1}$ and $P_i$ share the same edge that is dual to $\sigma$.

Step 3: construct the minimal polygons for every codimension-1 cell. Take the convex hull $P$ of all the minimal polygons $P_1,...,P_m$. Construct the polytope $Q$ the same way for $X_-$.

**Conjecture 1.** Suppose $X$ is irreducible. W.r.t. either the monomial length or the factorization length, the polytope $P$ constructed above is the unique minimal balancing for $X_+$, $Q$ is the unique minimal balancing for $X_-$, and $P - Q$ is the unique minimal representation for $X$.

Figure 6 shows an example of the above construction. $F$ is an unbalanced weighted fan consisting of three 2-dimensional simplicial cones $A,B$ and $C$. Around each 1-cell of $F$, we solve a minimal balancing problem given two rays. We get a triangle for each of the 1-cell. The resulting balancing is given by (the dual complex of) a tetrahedron. In this simple example, it is easy to check that it is the unique minimal balancing w.r.t. the monomial length.
If Conjecture 1 were true, then for each weighted pure polyhedral fan $F$, one could define its "span" in the space of all balanced polyhedral fans containing $F$ to be this minimal balancing. For arbitrary minimal balancing problem, the special phenomena in Theorem 1 no longer sustain. First, the monomial length and the factorization length may induce different minimal balancing. This already comes up in Figure 2. Consider the minimal balancing problem given by the middle triangle in the locus of non-linearity. Then $\mathcal{V}(g)$ and $\mathcal{V}(h)$ are both its balancings. $\mathcal{V}(g)$ is a minimal balancing w.r.t. the monomial length but not the factorization length, whereas $\mathcal{V}(h)$ is a minimal balancing w.r.t. the factorization length but not the monomial length, which is the proof to Theorem 2.

As the dimension and the input data (i.e. some unbalanced weighted complex) grow, so does the distinction between the monomial length and the factorization length. This is revealed in Proposition 16. Our evidence shows that the factorization length is a more flexible notion to work with in the development of an algorithm to solve the minimal balancing problem. It has the following advantages. First, it is shorter in many cases, as is shown by Theorem 3. Second, by the second property in Proposition 16, it allows one to find the minimal balancing locally, using backward induction. For an example, consider the minimal balancing problem given by the leftmost black complex in the Figure 7.

![Figure 7: Balance a complex (leftmost) inductively](image)

Starting at the unbalanced point $A$. Given that $AB$ is in general position relative to the rest, then one just needs to balance the part excluding $AB$. Given that $AC$ is in general position, then one proceeds to balances the rest, and so forth. Finally, one results in the balancing given by the union of three lines and a fan with three rays. This idea of solving a minimal balancing problem locally can be made precise in the following special case.

**Proposition 20.** Suppose $X_1, \ldots, X_m \subset \mathbb{R}^2$ are completely unbalanced polyhedral fans in general position, $\mathcal{V}_1, \ldots, \mathcal{V}_m$ balance $X_1, \ldots, X_m$ minimally, respectively. Then $\mathcal{V}_1 + \cdots + \mathcal{V}_m$ balances $X_1 + \cdots + X_m$ minimally, with respect to the factorization length.

**Proof.** Suppose $X_k$ contains $r_k$ rays. Suppose $W_1 \cup \cdots \cup W_n$ balances $X_1 \cup \cdots \cup X_m$. Then $\text{rec}(W_1 \cup \cdots \cup W_n) = \text{rec}(W_1) \cup \cdots \cup \text{rec}(W_n)$ balances $X_1 \cup \cdots \cup X_m$. Since each $\text{rec}(W_i)$ balances $\text{rec}(X_1) \cap \text{rec}(X_1 \cup \cdots \cup X_m)$, the number of rays in $\text{rec}(W_i)$ is at least the number of rays in $\text{rec}(W_1) \cap \text{rec}(X_1 \cup \cdots \cup X_m)$ plus 1. Therefore,

$$\text{flen}(\text{rec}(W_1) \cup \cdots \cup W_n)) \geq \sum_{k=1}^{m} r_k + 1 = \text{flen}(\mathcal{V}_1 \cup \cdots \cup \mathcal{V}_m)$$

Since $\text{flen}(W_1 \cup \cdots \cup W_n) \geq \text{flen}(\text{rec}(W_1 \cup \cdots \cup W_n))$, we have

$$\text{flen}(W_1 \cup \cdots \cup W_n) \geq \text{flen}(\mathcal{V}_1 \cup \cdots \cup \mathcal{V}_m)$$
Therefore, $V_1 \cup \cdots \cup V_m$ is a minimal balancing for $X_1 \cup \cdots \cup X_m$.

In general, the behavior of monomial length is much more complicated. As is illustrated in Figure 3, the minimal balancing is sensitive to global information. Suppose $X$ is given by two connected components $X_1$ and $X_2$, then it’s possible that slight perturbation in the position of $X_2$ changes how $X_1$ is balanced dramatically. The greedy algorithm does not work very well, either.

Now we turn to another perspective of the balancing problem. For any polyhedral complex $X$, the canonical arrangement $\mathcal{A}_X$ is a very natural way to balance $X$, even without considering minimality of any kind. Proposition 4 shows that another benefit from considering the factorization length: the canonical arrangement is a good balancing, if not the best.

Proof of Proposition 4. We first consider the case when the minimal balancing of $X$ is $V(g)$ for some irreducible tropical signomial $g$. In that case, the number of hyperplanes in $\mathcal{A}_X$ is no more than the 1-cells in $V(g)$. The number of 1-cells in $V(g)$ is the number of edges in the regular subdivision of $\Delta(g)$ induced by $\Delta^\uparrow(g)$ and $\text{mlen}(g)$ is the number of vertices. Let $l$ be the number of 2-cells in this subdivision. By the Euler-Poincaré relation,

$$\text{flen}(\mathcal{A}_X) = \text{mlen}(g) + l$$

Since every 2-cell has at least three 1-cells in its boundary, and every 1-cell is in the boundary of at most two 2-cells, we have

$$2(\text{flen}(\mathcal{A}_X) - 1) \geq 3l$$

From there we get

$$\text{flen}(\mathcal{A}_X) \leq 3\text{mlen}(g) - 2$$

Suppose the minimal balancing of $X$ is given by a factorization $g_1 \odot \cdots \odot g_m$. Each $g_i$, by Proposition 16, balances $V(g_i) \cap X$ minimally. Set $X_i = V(g_i) \cap X$. Then for each $i$ we have

$$\text{flen}(\mathcal{A}_{X_i}) \leq 3\text{mlen}(g_i) - 2$$

Therefore,

$$\frac{\text{flen}(\mathcal{A}_X)}{\text{flen}(g_1 \odot \cdots \odot g_m)} \leq \frac{\sum_i \text{flen}(\mathcal{A}_{X_i}) - (m - 1)}{\sum_i \text{mlen}(g_i) - (m - 1)} \leq \frac{3\sum_i \text{mlen}(g_i) - 3m + 1}{\sum_i \text{mlen}(g_i) - (m - 1)} \leq 3$$

\hfill \square

5 Conclusion

We conclude this paper with several open questions we think worth exploring. Recall that we give a dual description of the minimal representation problem for weighted polyhedral fans in Section 4 as finding an equivalent pair of polytopes that is minimal. The dual description for the minimal balancing problem for weighted fans is to find a polytope with minimal vertices whose associated tropical hypersurface is a balancing of the given weighted fan. This can be regarded as a variation of the Minkowski reconstruction problem\cite{GH}. In the classical Minkowski reconstruction problem, one is given a collection of vectors summing up to zero, and one wants to find a polytope whose facets are dual to the given vectors. In our setting, we are given data in dimension 1, i.e. the weights and orientations of the dual cells of the edges of $P$, instead of codimension 1, and we pursue a minimal solution. We expect the minimal polytope to simultaneously minimize the volumes in all dimensions, which is true when $d = 2$.

Open Question 1. Study the minimal balancing problem and the minimal representation problem for weighted polyhedral fans in higher dimensions. In particular, prove or disprove Conjecture 1.

Another question not entirely clear to us is whether or not tropical signomials have unique irreducible factorizations with minimal factorization length. Tropical signomials don’t have unique irreducible factorizations. However, do we get the uniqueness if we only consider those with minimal factorization length? Note that the answer is negative if we drop the word “irreducible”.

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Open Question 2. Do tropical signomials have unique irreducible factorization w.r.t. the factorization length?

Finally, in both the minimal representation problem and the minimal balancing problem, the input data is a piecewise linear function. In many settings, one is given a tropical rational signomial and needs to find another tropical rational signomial representing the same piecewise linear function. In this situation, the problem is more or less symbolic, because one may not even need to know what the underlying real-valued function is. This amounts to studying the structure of $F(\varphi)$. Reducing a representation to a minimal one could be thought of as a word problem. In a classic word problem, one is given letters and rules for converting the letters. The word problem in our setting is a rather geometric one.

Open Question 3. Study the structure of $F(\varphi)$. In particular, how does one reduce a representation to a minimal one?

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