Summation Formulas for the product of the q-Kummer Functions from $E_q(2)$

H. Ahmedov$^1$ and I. H. Duru$^{2,1}$

1. Feza Gürsey Institute, P.O. Box 6, 81220, Çengelköy, Istanbul, Turkey
2. Trakya University, Mathematics Department, P.O. Box 126, Edirne, Turkey.

Abstract

Using the representation of $E_q(2)$ on the non-commutative space $zz^* - qz^*z = \sigma; \ q < 1, \ \sigma > 0$ summation formulas for the product of two, three and four q-Kummer functions are derived.

1. Introduction

Properties of manifolds can be investigated by means of their automorphism groups. Non-commutative spaces too are studied similarly. For example the quantum groups $E_q(2)$ and $SU_q(2)$ are the symmetry groups of the quantum plane and the quantum sphere respectively \cite{1}-\cite{3}. The group representation theory gives the possibility to construct the complete set of orthogonal functions on these deformed spaces. For example the Hahn-Exton q-Bessel and q-Legendre functions appear as the matrix elements of the unitary representations of $E_q(2)$ \cite{4}-\cite{7} and $SU_q(2)$ \cite{8} \cite{9} which are the complete set of orthogonal functions on the quantum plane and the quantum sphere respectively. Using group theoretical methods the invariant distance and the Green functions have also been written on the quantum sphere \cite{10} and the quantum plane \cite{11}.

In recent works we have studied the non-commutative space $[z, z^*] = \sigma$ (i.e. the space generated by the Heisenberg algebra) by means of its automorphism groups $E(2)$ and $SU(1,1)$ \cite{12}, \cite{13}. The basis in this non-commutative space where irreducible representations of $E(2)$ are realized were found to be the Kummer functions which involves the coordinates $z, \ z^*$ not as their arguments but as indices. That study enable us to obtain generic summation formulas involving Kummer and Bessel functions. For $SU(1,1)$ case the basis is given in terms of the hypergeometric functions having the non-commutative coordinates $z$ and $z^*$ as the parameters. Again we derived generic summation formulas involving hypergeometric and Jacobi functions. This analysis enable us to construct different complete sets of orthogonal functions on the non-commutative space. Both studies also provide new group theoretical interpretations for the already known relations involving special functions.

Motivated by the outcomes of the above mentioned studies, in the present work we consider the two parametric deformation of the plane which is the *-algebra $P^\sigma_q$ generated by $z$ and $z^*$ with

\begin{equation}
zz^* - qz^*z = \sigma, \quad q < 1, \ \sigma > 0
\end{equation}

$^1$E-mail : hagi@gursey.gov.tr and duru@gursey.gov.tr
which possesses the symmetry of the group $E_q(2)$. In $\sigma \to 0$ limit it becomes the usual quantum plane. In $q \to 1$ limit it turns to the algebra of functions on the Heisenberg algebra. This study allows us to obtain many identities involving several Hahn-Exton q-Bessel and Moak q-Laguerre functions which are the special forms of the q-Kummer functions. Note that previously some formulas involving q-Laguerre functions were derived by making use of the representation theory of the q-oscillator algebra [14]-[19]. Some relations involving the basic Bessel and Laguerre functions were also considered in [20].

In Section 2 we realize $E_q(2)$ as the automorphism group of the noncommutative space $P_\sigma^\sigma$.

In Section 3 we construct the basis in $P_\sigma^\sigma$ where the irreducible representations of $E_q(2)$ are realized.

Section 4 is devoted to the generic summation formulas for the product of two, three and four q-Kummer functions.

In Section 5 some simple examples are presented.

2. $E_q(2)$ as the symmetry group of $P_\sigma^\sigma$

Euclidean group transformations of $z$, $z^*$ plane are given by

$$\delta(z) = B + Az,$$
$$\delta(z^*) = B^* + A^*z^*.$$  \hspace{1cm} (2)

The conditions

$$\delta(z)\delta(z^*) - q\delta(z^*)\delta(z) = 1$$  \hspace{1cm} (4)

and

$$(\delta(z))^* = \delta(z^*)$$  \hspace{1cm} (5)

imply the $E_q(2)$ relations

$$BB^* = qB^*B, \ AB = qBA, \ AB^* = qB^*A, \ A^* = A^{-1}.$$  \hspace{1cm} (6)

Note that $z$, $z^*$ commute with $B$, $B^*$ and $A$. (Starting with (1) we employ $\sigma = 1$. When we need to calculate $\sigma \to 0$ limit we replace $z, z^*$ by $\frac{z}{\sigma}, \frac{z^*}{\sigma}$).

Formulas

$$z|n, j\rangle = \sqrt{(n)_q}|n-1, j\rangle,$$
$$z|n, j\rangle = \sqrt{(n+1)_q}|n+1, j\rangle,$$
$$B|n, j\rangle = q^{\frac{j}{2}}|n, j-1\rangle,$$
$$B^*|n, j\rangle = q^{\frac{j+1}{2}}|n, j+1\rangle,$$
$$A|n, j\rangle = |n, j-2\rangle,$$  \hspace{1cm} (7)

where

$$(n)_q = \frac{1 - q^n}{1 - q}$$

give the solution of (1) and (4) in some suitable domain of the Hilbert space $X$ with the basis $\{|n, j\rangle\}, \ n = 0, 1, 2, \ldots$ and $0, j \in \mathbb{Z}$. Let us define in $X$ a
new basis such that

\[ \delta(z)|n, j \rangle' = \sqrt{(n)_q} |n-1, j \rangle' \]  (8)

\[ \delta(z)|n, j \rangle' = \sqrt{(n)_q} |n+1, j \rangle'. \]  (9)

Due to

\[ ze^{-xz} = -xe^{xz} + e^{-qxz}z, \]  (10)

with

\[ e^x_q = \sum_{k=0}^{\infty} \frac{x^k}{(k)_q}! \]  (11)

being the q-deformed exponential function, we conclude that

\[ |0, j \rangle' = e^{-A^*B} \sqrt{e^{-B^*B}} |0, j \rangle \]  (12)

is the ground state of the new basis:

\[ \delta(z)|0, j \rangle' = 0. \]  (13)

Applying the creation operator \((\delta(z))^*\) on this state we can generate the desired basis in \(X\):

\[ |n, j \rangle' = \frac{(\delta(z)^*)^n}{\sqrt{(n)_q!}} |0, j \rangle'. \]  (14)

We also have

\[ \delta(z) = UzU^*, \]  (15)

where \(U\) is the unitary operator in

\[ |n, j \rangle' = U|n, j \rangle. \]  (16)

Thus \(\delta\) defines the homomorphic map of \(P^\sigma_q\) into the \(*\)-algebra generated by \(A, B, z\) and their adjoints. Applying twice this map we get

\[ \delta'(\delta(z)) = B' + A'\delta(z) = B'' + A''z, \]  (17)

where

\[ B'' = B' + A'B, \quad A'' = A'A, \]  (18)

is the group multiplication in \(E_q(2)\) (the operator \(B''\) has the same analytic properties as \(B \) \([21]\)). \(\delta\) defines the representation of \(E_q(2)\) in \(P^\sigma_q\).

Before closing this section we give the explicit formula for the matrix representation of \(U\):

\[ U_{(mi)(nj)} = \langle mi|n, j \rangle'. \]  (19)

For \(|n, j\rangle = |n\rangle|j\rangle\) we first define

\[ U_{mn} = \langle n|\delta(z^*)|^n e^{-A^*Bz^*} |0\rangle \sqrt{\frac{e^{-B^*B}}{(n)_q!}} \]  (20)

which is the function of \(B, B^*, A\) and \(A^*\). Then

\[ U_{(mi)(nj)} = \langle i|U_{mn}|j\rangle. \]  (21)
After some algebra we get
\[ U_{mn} = A^{-m} B^{n-m} \Phi_{mn}(\eta) \quad \text{for } n \geq m \] (22)
and
\[ U_{mn} = q^{(m-n)(m-n-1)/2} A^{-m} (-B)^{m-n} \Phi_{nm}(\eta) \quad \text{for } m \geq n, \] (23)
where \( \eta^2 \equiv B^* B \) and
\[ \Phi_{mn}(\eta) = \sqrt{\frac{(n)_q}{(m)_q}} \left( \frac{e^{-\eta^2}}{(n-m)_q} \right) \Phi(q^{-m}, q^{1+n-m}; q^{n+1}\eta^2). \] (24)

Here
\[ \Phi^q(a, b; x) = \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2} (a; q)_k (1-q)x^k}{(q; q)_k (b; q)_k} \] (25)
which in \( q \to 1 \) limit reduces to the Kummer function:
\[ \lim_{q \to 1} \Phi^q(c, d; x) = \Phi(c, d; x). \] (26)

We call it the q-Kummer function. The functions \( \Phi_{nm} \) can also be expressed in terms of Moak’s q-Laguerre polynomials [22]
\[ L_n^q(\alpha)(x) = \frac{(q^{1+\alpha}, q)_n}{(q; q)_n} \Phi(q^{-n}, q^{1+\alpha}; q^{1+\alpha+n}x) \] (27)
as
\[ \Phi_{mn}(\eta) = \sqrt{\frac{e^{-\eta^2}}{(n)_q}} L_{m-n}^q(\eta^2), \quad \text{for } n \geq m. \] (28)

3. Irreducible representations of \( E_q(2) \) in \( P_q^\sigma \)
The irreducible representation of the deformed enveloping algebra \( U_q(e(2)) \)
\[ P^* P = qPP^*, \quad KP = qPK, \quad P^* K = qKP^* \] (29)
defined by the weight \( \lambda \in \mathbb{R} \) in the non-commutative space \( P_q^\sigma \) is given by
\[ R(K) D_j^\lambda(z, z^*) = q^j D_j^\lambda(z, z^*) \] (30)
\[ R(P) D_j^\lambda(z, z^*) = \lambda q^{\frac{j}{2}} D_{j-1}^\lambda(z, z^*), \] (31)
\[ R(P^*) D_j^\lambda(z, z^*) = \lambda q^{\frac{j+1}{2}} D_{j+1}^\lambda(z, z^*), \] (32)
where
\[ R(K) z^n z^m = q^{m-n} z^n z^m \]
\[ R(P) z^n z^m = iq^{-n}(m)_q z^n z^{m-1} \] (33)
\[ R(P^*) z^n z^m = iq^{-n+1}(n)_q z^{n-1} z^m \]
defines the right realization of \( U_q(e(2)) \). [30] implies
\[ D_j^\lambda(z, z^*) = \begin{cases} f_j^\lambda(\zeta) z^j & \text{for } j \geq 0 \\ z^{s-j} f_{-j}^\lambda(\zeta), & \text{for } j \leq 0 \end{cases} \] (34)
where
\[ \zeta \equiv 1 - (1 - q)z^*z. \]  \hfill (35)

Inserting the ansatz \( \zeta \) in (31) and (32) we get
\[
f^\lambda_j(\zeta) = \frac{q^j}{(j)!} \Phi^q(\zeta^{-1}, q^{j+1}; q^{j+1} \lambda^2 \zeta). \quad \hfill (36)
\]

In the derivation of (36) we used
\[
z^*n z^n = (-)^n q^{-\frac{n(n+1)}{2}} (\zeta^{-1}; q)_n \zeta^n. \quad \hfill (37)
\]

By means of the universal T-matrix we can exponentiate (30) and get
\[
\delta(D^\lambda_j) = \sum_{i=-\infty}^{\infty} t^\lambda_{ji} D^\lambda_i, \quad \hfill (38)
\]
where
\[
t^\lambda_{ij} = q^{\frac{(i\lambda B)^{i-j} A^j j!}{(i-j)! q^j}} \Phi^q(0, q^{1+i-j}; (q-1)q^{1-j} (\lambda \eta)^2) \quad \text{for } i \geq j \hfill (39)
\]
and
\[
t^\lambda_{ij} = q^{\frac{A^i (i\lambda B^*)^{-i-j} \Phi^q(0, q^{1+j-i}; (q-1)q^{-i} (\lambda \eta)^2)}{(j-i)! q^i}} \quad \text{for } j \geq i, \hfill (40)
\]
are the matrix elements of the irreducible representations of \( E_q(2) \) [6], [7].

We can express them in terms of the Hahn-Exton q-Bessel functions [23]
\[
J^q_k(x) = \frac{x^k}{(k)!} \Phi^q(0; q^{1+k} | q; (q-1)qx^2) \hfill (41)
\]

as
\[
t^\lambda_{ij} = \left( \sqrt{-1} q^* \right)^{i-j} V^{i-j} A^j J^q_{i-j}(q^{-\frac{1}{2}} \lambda \eta) \quad \text{for } i \geq j \hfill (42)
\]
and
\[
t^\lambda_{ij} = \left( \sqrt{-1} q^{-\frac{1}{2}} \right)^{j-i} V^{j-i} A^j J^q_{j-i}(q^{-\frac{1}{2}} \lambda \eta) \quad \text{for } j \geq i, \hfill (43)
\]

where \( V \) is the unitary operator defined by \( B = V \eta \).

In \( \sigma \to 0 \) limit the non-commutative space \( P^\sigma_q \) becomes the quantum plane \( E_q(2)/U(1) \) generated by \( B, B^* \):

\[
\lim_{\sigma \to 0} D^\lambda_j(\sqrt{\sigma} \sqrt{i \psi}, \sqrt{i \psi}) = t^\lambda_{j0}. \hfill (44)
\]

In \( q \to 1 \) limit \( P^\sigma_q \) becomes the noncommutative space generated by the Heisenberg algebra [12]:

\[
\lim_{q \to 1} D^\lambda_j(z, z^*) = \Phi(-zz^*; 1 + j; \lambda^2 \frac{(i \lambda z)^j}{j!}). \hfill (45)
\]

In \( \sigma \to 0 \) and \( q \to 1 \) limit we arrive at the complex plane \( E(2)/U(1) \):

\[
\lim_{\sigma \to 0} \lim_{q \to 1} D^\lambda_j(\frac{r e^{i \psi}}{\sqrt{\sigma}}, \frac{r e^{-i \psi}}{\sqrt{\sigma}}) = \tilde{t}^\lambda_{j0} J^\lambda_j(\lambda r). \hfill (46)
\]
4. Summation formulas for the q-Kummer functions

(15) and (38) imply

\[ UD_j^λ U^* = \sum_{i=-\infty}^{\infty} t^λ_{ji} D^λ_i \]  

(47)
or

\[ UD_j^λ = \sum_{i=-\infty}^{\infty} t^λ_{ji} D^λ_i U, \]  

(48)

\[ D_j^λ = \sum_{i=-\infty}^{\infty} t^λ_{ji} U^* D^λ_i U. \]  

(49)

The above formulas define the summation of products of two, three and four q-Kummer functions. Sandwiching (47), (48) and (49) between the states \langle m \vert \text{ and } \vert n \rangle and using

\[ (D_j^λ)_{mn} = \left[ \frac{(n)_q!}{(m)_q!} f^λ_j(q^m) \delta_{jn,m} \right] \text{ for } j \geq 0, \]  

(50)

\[ (D_{-j}^λ)_{mn} = (D_j^λ)_{nm} \]  

(51)
we get

\[ \sum_{s=0}^{\infty} (D_j^λ)_{s+j} U_{ms} U^*_{s+jn} = (D_{n-m}^λ)_{mn} t^λ_{jn,m} \text{ for } j \geq 0 \]  

(52)

\[ \sum_{s=0}^{\infty} (D_j^λ)_{s-j} U_{ms-j} U^*_{sn} = (D_{n-m}^λ)_{mn} t^λ_{jn,m} \text{ for } j < 0, \]  

(53)

\[ \sum_{s=0}^{\infty} t^λ_{js-m} (D_j^λ)_{s-m} U_{sn} = (D_j^λ)_{n-j} U_{mn-j} \text{ for } n \geq j \]  

(54)

\[ \sum_{s=0}^{\infty} t^λ_{js-m} (D_j^λ)_{s-m} U_{sn} = 0 \text{ for } n < j \]  

(55)

and

\[ \sum_{s,l=0}^{\infty} t^λ_{jl-s} (D_j^λ)_{sl} U_{ms} U^*_{ln} = (D_j^λ)_{mn} \delta_{jn,m}. \]  

(56)

In the above formulas \( U' \)'s and \( t' \)'s are given in terms of the q-Kummer functions of the operator \( \eta = B^*B \) (see (22), (23) and (39).

In the coming section we give some simple examples.

5. Examples

A. For \( n = m = 0, \ j \geq 0 \) (22) implies

\[ \sum_{s=0}^{\infty} q^{s(s+1)/2} \eta^{-2s} (s)_q! \Phi(q^{-s}, q^{1+j}; q^{1+j+s}) = \frac{(i\lambda\eta)^{-j}(j)q!}{\sqrt{e^{-\eta^2\lambda^2} - e^{-q^2\eta^2}}} J_j^q(\lambda\eta) \]  

(57)

which in \( q \to 1 \) limit gives [24] (page 1038, Eq. 3 of 8.975)

\[ \sum_{s=0}^{\infty} \frac{\eta^{2s}}{s!} \Phi(-s, 1+j; \lambda^2) = j!(\eta\lambda)^{-j} e^{r^2} J_j(\lambda\eta) \]  

(58)
B. For $n = 0$ and $k \equiv -j \geq m$ (54) implies
\[ \sum_{s=0}^{\infty} q^{(s+m-2k)\frac{2}{2}} C_{sm} q^s J^q_{s+k-m} \left( q^{\frac{s+k}{2}} \lambda \eta \right) = \frac{q^{k(k-m)} \lambda^k \eta^{k-m}}{(m)_q!(k-m)_q!} \Phi^q(q^{-m}, q^{1+k-m}; q^{1+k} \eta^2), \]
where
\[ C_{sm} = \frac{(-)^s \lambda^{m-s}}{(s)_q!} \Phi_q(q^{-s}, q^{1+m-s}; q^{1+m} \lambda^2), \quad \text{for } m \geq s. \] (60)

For $s \geq m$ one has to replace $m, s$ with $s, m$ in the right hand side of the above expression. When $m = 0$ we have
\[ \sum_{s=0}^{\infty} q^{s^2 + sk} \frac{(\lambda \eta)^s}{(s)_q!} J^q_{s+k} (q^{\frac{s+k}{2}} \lambda \eta) = q^{k^2} \frac{\lambda \eta}{(k)_q!}, \]
which is the quantum analogue of a known formula [23] (page 974, Eq 1 of 8.515).

C. For $j = \lambda = 0$ (56) implies the unitarity condition for the operator $U$:
\[ \sum_{s=0}^{\infty} (U_{sm})^* U_{sm} = \delta_{nm}, \] (62)
where we used
\[ U_{ms}^* = (U_{sm})^*. \] (63)

For $n = m$ with $x = \eta^2$ we have
\[ \sum_{s=0}^{n-1} q^{\frac{(n-s)(n-s+1)}{2}} (n)_q! x^{n-s} \left( L^q_{s-n} (x) \right)^2 + \sum_{s=n}^{\infty} q^{\frac{(n-s)(n-s+1)}{2}} (n)_q! x^{n-s} \left( L^q_{n-s} (x) \right)^2 = e^{x_{q^{-1}}} \] (64)

In deriving the above examples one frequently uses the identities
\[ B^k B^{*k} = q^{\frac{k(k+1)}{2}} \eta^{2k}, \quad B^{*k} B^k = q^{\frac{k(1-k)}{2}} \eta^{2k}. \] (65)

References

[1] P. Podleś, Lett. Math. Phys., 14, 193 (1987).

[2] Yu. I. Manin, Quantum groups and noncommutative geometry. Publications of C.R.M. 1561, University of Montreal (1988).

[3] S. L. Woronowicz, Comm. Math. Phys., 111, 613 (1987).

[4] S. L. Woronowicz, Comm. Math. Phys., 144, 417 (1992), Comm. Math. Phys., 149, 637 (1992), Lett. Math. Phys., 23, 251 (1991);

[5] L. L. Vaksman and L. I. Korogodski, Dokl. Akad. Nauk SSSR, 304, 1036 (1989).

[6] F. Bonechi, N. Ciccoli, R. Giaretti, E. Sorace, M. Tarlini, Comm. Math. Phys., 175, 161 (1996),
[7] H. T. Koelink, *Duke Math. J.*, **76**, 483 (1994),

[8] T. H. Koornwinder, *Nederl. Akad. Wetensch. Proc.*, **A76**, 97 (1989); *SIAM J. Math. Anal.*, **22**, 295 (1991).

[9] N. Ya. Vilenkin and A. U. Klimyk, *Representations of Lie Groups and Special Functions*, vol 3, Kluwer Academic Press, The Netherlands (1991).

[10] H. Ahmedov and I. H. Duru, *J. Phys. A: Math. Gen*, **31**, 5741 (1998).

[11] H. Ahmedov and I. H. Duru, *J. Phys. A: Math. Gen*, **32**, 6255 (1999).

[12] H. Ahmedov and I. H. Duru, *J. Phys. A: Math. Gen*, **33**, 4277 (2000).

[13] H. Ahmedov and I. H. Duru, *Representation of SU(1,1) in the Non-commutative Space Generated by the Heisenberg Algebra*, arXiv: math. QA/0003078 (2000).

[14] R. Askey and S. K. Suslov, *J. Phys. A: Math. Gen*, **26**, L693 (1993).

[15] R. Askey and S. K. Suslov, *Lett. Math. Phys.*, **29**, 123 (1993).

[16] R. Floreanini and L. Vinet, *Phys. Lett. A*, **180**, 393 (1993).

[17] E. G. Kalnins, H. L. Manocha and W. Miller, *J. Math. Phys.*, **33**, 2365 (1992).

[18] E. G. Kalnins and W. Miller, *J. Math. Phys.*, **35**, 1951 (1994).

[19] E. G. Kalnins, W. Miller and S. Mukherjee, *J. Math. Phys.*, **34**, 5333 (1993).

[20] H. T. Koelink, *J. Comp. Apll. Math.*, **68**, 209 (1996),

[21] S. L. Woronowicz, *Comm. Math. Phys.*, **136**, 399 (1991).

[22] D. S. Moak, *J. Math. Anal. Appl.*, **81**, 20 (1981).

[23] T. H. Koornwinder and R. F. Swarttouw, *Tran. Amer. Math. Soc*, **333**, 445 (1992).

[24] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series and Products*. Academic Press, New-York (1980).