Third boundary-value problem of the heat conduction equation
for a system with plane-parallel boundaries

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(November 20, 2018)

We obtained a new representation of a solution of the heat conduction
equation with boundary condition of the third kind for a layer. The result
is presented as a superposition of fundamental solutions for an unbounded
system with variable coefficients, the explicit form of which is given. We
consider the well-known problem of the evolution of the temperature field
initially uniformly distributed in a layer. The distribution of the temperature
field is represented in terms of the obtained functions.

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I. INTRODUCTION

Theoretical description of the distribution of heat in bounded solids is based on the solution of the heat conduction equation with the corresponding initial and boundary conditions. In the case where the heat transfer between a heated body occupying the region \(0 \leq z \leq L, \ -\infty < x, y < \infty\) and the external medium with the zero temperature proceeds by the Newton law, the problem is reduced to the solution the heat conduction equation [1–3] for \(t > 0\)

\[
\frac{\partial T(\vec{r}, t)}{\partial t} - \kappa \Delta T(\vec{r}, t) = f(\vec{r}, t) \quad (1)
\]

with the boundary conditions of the third kind

\[
\frac{\partial T(\vec{r}, t)}{\partial z} - \lambda_1 T(\vec{r}, t) = 0 \quad \bigg|_{z=0}, \quad (2)
\]

\[
\frac{\partial T(\vec{r}, t)}{\partial z} + \lambda_2 T(\vec{r}, t) = 0 \quad \bigg|_{z=L} \quad (3)
\]

and the initial condition

\[
T(\vec{r}, 0) = T_0(\vec{r}) \quad (4)
\]

Here, \(\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\) is the Laplace operator, \(\vec{r} = (x, y, z)\), \(\kappa\) is the thermal diffusivity, \(\lambda_1\) and \(\lambda_2\) are the relative heat transfer coefficients between the body and the external medium at \(z = 0\) and \(z = L\), respectively, and \(f(\vec{r}, t)\) is the density of thermal sources normalized to \(c\rho\), where \(c\) and \(\rho\) are, respectively, the specific heat capacity and the density of the substance.

The methods for solving this boundary-value problem are well-known and widely described in the literature both for the theory of heat conduction [1–3] and for equations of mathematical physics [4–7]. It is worth noting that a solution of this problem is presented as a series in which the summation is carried out over an infinite set of eigenvalues \(\alpha\), where \(\alpha\) are positive roots of the transcendental equation
\[
\tan \alpha L = \alpha \frac{\lambda_1 + \lambda_2}{\alpha^2 - \lambda_1 \lambda_2},
\]
which, in the general case, is solved numerically. In particular cases where the temperature of the boundary surfaces is equal to zero (\( \lambda = \infty \), the first boundary-value problem) or the heat flow through the boundaries is absent (\( \lambda = 0 \), the second boundary-value problem), the roots of Eq. (5) are known. In these cases, the solution of the heat conduction equation is represented in terms of normal distributions created by instantaneous heat sources situated at the corresponding points of the space. This enables one, without recourse to numerical calculations, to carry out analytical estimations of solutions of the heat conduction equation for the first and second boundary-value problems for various special cases (thick layer, near the boundary surfaces and at the center of the layer, time temperature asymptotics, etc.).

For arbitrary values of \( \lambda \), Eq. (5) was numerically investigated in detail and the results are presented in the form of tables and plots (see, e.g., [3]).

As far as we know, for any \( \lambda > 0 \), in the literature, there is no representation of a solution of problem (1)–(3) in terms of fundamental solutions for an unbounded system similar to those for the first and second boundary-value problems for which it is not necessary to find eigenvalues \( \alpha \) of the transcendental equation (5). The possibility of this representation for arbitrary \( \alpha \) is shown in the second section of the paper for Green’s function. Unlike the known solutions of boundary-value problems of the first and second kinds, in this case, the fundamental solutions are in the sum with the corresponding weights that are functions of coordinates and time.

As an example, in the third section, we use the obtained Green function for studying the evolution of an initially uniform temperature field in a layer.

II. GREEN’S FUNCTION OF THE THIRD BOUNDARY-VALUE PROBLEM

We consider the homogeneous heat conduction equation (1) with boundary conditions (2)–(3) for \( \lambda_1 = \lambda_2 = \lambda \) and the initial condition
\[ T(\vec{r}, 0) \equiv T_0(\vec{r}) = \delta(\vec{r} - \vec{r}'), \] (6)

which corresponds to a point instantaneous heat source situated at the point \( \vec{r}' = (x', y', z') \), where \( 0 \leq z' \leq L \), and find a solution continuous in the region \( 0 \leq z \leq L, \ -\infty < x, y < \infty \) for \( t > 0 \). Note that in the general case of different \( \lambda_1 \) and \( \lambda_2 \), the expression for Green’s function for the third boundary-value problem \( (1) - (3), (6) \) expressed in terms of fundamental solutions for an unbounded system was obtained in \[8\], where the Brownian motion of particles in a layer was investigated for various values of adsorption coefficients of particles by the boundary surfaces. We also noted there that the specific case \( \lambda_1 = \lambda_2 \) should be considered in its own right because in going from the general expressions presented in \[8\] to the case \( \lambda_1 = \lambda_2 \), we have to sum expressions each term of which is divergent.

By using methods of the operational calculus, we reduce Eq. \((3)\) with initial condition \((6)\) to an ordinary differential equation of the second order, the solution of which \( T(\vec{r}, t) \equiv G(\vec{r}, \vec{r}', t) \) (Green’s function of the heat conduction equation) can be written as follows:

\[
G(\vec{r}, \vec{r}', t) = \frac{\exp\left(-\frac{\vec{R}_\perp^2}{4\kappa t}\right)}{8\pi^2\kappa t} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} dk_z \left\{ \exp(-k_z^2 \kappa t + ik_z Z_-(n)) \left(\frac{k_z - i\lambda}{k_z + i\lambda}\right)^{2n} + \exp(-k_z^2 \kappa t + ik_z Z_+(n)) \left(\frac{k_z - i\lambda}{k_z + i\lambda}\right)^{2n+1} \right\},
\] (7)

where \( \vec{R}_\perp = \vec{r}_\perp - \vec{r}'_\perp, \vec{r}_\perp = (x, y, 0), \vec{r}'_\perp = (x', y', 0), Z_\pm(n) = Z_\pm + 2nL, \) where \( n = 0, \pm 1, \pm 2, \ldots, Z_\pm = z \pm z' \).

Then we use the integral representation

\[
\exp(-k_z^2 \kappa t) = \frac{1}{(4\pi \kappa t)^{1/2}} \int_{-\infty}^{\infty} d\xi \exp\left(-\frac{\xi^2}{4\kappa t} + ik_z \xi\right),
\] (8)

change the order of integration, and take integrals over \( k_z \) and \( \xi \). We present the final result in the following form:

\[
G(\vec{r}, \vec{r}', t) = \frac{\exp\left(-\frac{\vec{R}_\perp^2}{4\kappa t}\right)}{(4\pi \kappa t)^{3/2}} \sum_{n=-\infty}^{\infty} \left\{ P_n^-(Z_-(n), \lambda) \exp\left(-\frac{Z_-(n)}{4\kappa t}\right) + P_n^+(Z_+(n), \lambda) \exp\left(-\frac{Z_+(n)}{4\kappa t}\right) \right\},
\] (9)
where

\[ P_n^(-)(z, \lambda) = 1 - (1 - \delta_{n0}) \sum_{k=0}^{2|n| - 1} C_{2|n|}^k \beta_k(|z|, \lambda), \quad n = 0, \pm 1, \pm 2, \ldots, \]

\[ P_n^+(z, \lambda) = 1 - \sum_{k=0}^{2n} C_{2n+1}^k \beta_k(z, \lambda), \quad n \geq 0, \]

\[ P_n^+(z, \lambda) = 1 - \sum_{k=0}^{2|n|-2} C_{2|n|-1}^k \beta_k(|z|, \lambda), \quad n < 0, \] (10)

\[ C_m^n \] is the binomial coefficient, \( \delta_{n0} \) is the Kronecker symbol, \( \tilde{\lambda} = \lambda(\kappa t)^{1/2} \),

\[ \beta_k(z, \lambda) = \pi^{1/2}(2\tilde{\lambda})^{k+1} F_k \left( \frac{z}{2(\kappa t)^{1/2}} + \tilde{\lambda} \right), \quad k = 0, 1, 2, \ldots, \] (11)

\[ F_k(x) = \frac{1}{k!} \frac{d^k}{dx^k} \left( \exp(x^2) \operatorname{erfc} x \right), \quad k = 0, 1, 2, \ldots, \] (12)

\[ \operatorname{erfc} x = \frac{2}{\pi^{1/2}} \int_x^\infty dy \exp(-y^2). \] (13)

By the direct substitution of solution (9) into Eqs. (1)–(3), we can make sure that this representation of Green’s function satisfies both the homogeneous equation (1) and boundary conditions (2) and (3) (for details, see Appendix). To carry out calculations one should use the following recurrence relationship

\[ F_k(x) = \frac{2}{k} \left[ F_{k-2}(x) + x F_{k-1}(x) \right], \quad k \geq 2 \] (14)

for the function \( F_k(x) \). To obtain relation (14), we use the known recurrence relationships for multiply probability integrals \[ \] and representation (12) for the function \( F_k(x) \). As the subscript \( k \) of the function \( F_k(x) \) increases, the function \( F_k(x) \) decreases and, for all values of the argument \( x \), remains either positive (for even \( k \) and \( k = 0 \)) or negative (for odd \( k \)). The behavior of the first six functions \( F_k(x) \) is shown in Fig. 1 \( [k = 0, 1, 2, 3, 4, 5, \text{ the number near a curve corresponds to the subscript } k \text{ of the function } F_k(x)] \).
In the limiting case $\lambda \to \infty$, with regard for properties of binomial coefficients, it follows from Eq. (10) that all quantities $P_n^(-)$ and $P_n^(+)$ are the same and constant

$$\lim_{\lambda \to \infty} P_n^\pm(Z_\pm(n), \lambda) = \pm 1$$

(15)

and Eq. (9) is reduced to the well-known solution of the first boundary-value problem for Green’s function in a layer [1,10]

$$G(\vec{r}, \vec{r}', t) = \frac{\exp\left(-\frac{R^2}{4\kappa t}\right)}{(4\pi\kappa t)^{3/2}} \sum_{n=-\infty}^{\infty} \left\{ \exp\left(-\frac{Z_\pm^2(n)}{4\kappa t}\right) - \exp\left(-\frac{Z_\pm^2(n)}{4\kappa t}\right) \right\},$$

(16)

As $\lambda \to 0$, $[\lambda(\kappa t)/|Z_\pm| \ll 1]$, in view of Eqs. (10) and (11), we have

$$\beta_k(|Z_\pm(n)|, \lambda) \approx \begin{cases} 2\tilde{\lambda}^{1/2}F_0\left(\frac{|Z_\pm(n)|}{2(\kappa t)^{1/2}}\right), & k = 0, \\ o(\tilde{\lambda}^k), & k = 1, 2, \ldots \end{cases}$$

(17)
and

\[ P_n^{(-)}(Z_-(n), \lambda) \approx 1 - 4\lambda \left| Z_-(n) \right| F_0 \left( \frac{Z_-(n)}{2(\kappa t)^{1/2}} \right), \quad n = 0, \pm 1, \pm 2, \ldots \]

\[ P_n^{(+)}(Z_+(n), \lambda) \approx 1 - 2(2n + 1)\lambda \left| Z_+(n) \right| F_0 \left( \frac{Z_+(n)}{2(\kappa t)^{1/2}} \right), \quad n \geq 0, \]

\[ P_n^{(+)}(Z_+(n), \lambda) \approx 1 - 2(2|n| - 1)\lambda \left| Z_+(n) \right| F_0 \left( \frac{Z_+(n)}{2(\kappa t)^{1/2}} \right), \quad n < 0, \] (18)

whence, for \( \lambda = 0 \) \[ P_n^{(\pm)}(Z_\pm(n), 0) = 1 \], it follows the well-known solution of the second boundary-value problem [1,10]

\[ G(\vec{r}, \vec{r}', t) = \exp \left( -\frac{R^2}{4\kappa t} - \frac{1}{4\kappa t} \right) \sum_{n=-\infty}^{\infty} \left\{ \exp \left( -\frac{Z_-^2(n)}{4\kappa t} \right) + \exp \left( -\frac{Z_+^2(n)}{4\kappa t} \right) \right\} (19) \]

In the limiting case of an infinitely thick layer \((L \to \infty)\), we have

\[ P_n^{(-)}(Z_-(n), \lambda) = \delta_{n0}, \]

\[ P_n^{(+)}(Z_+(n), \lambda) = \delta_{n0} [1 - \beta_0(Z_+, \lambda)] \] (20)

and expression (19) is reduced to the known solution of the third boundary-value problem for a half space \((z \geq 0, -\infty < x, y < \infty)\) [3,7,10]

\[ G(\vec{r}, \vec{r}', t) = \frac{\exp \left( -\frac{R^2}{4\kappa t} \right)}{(4\pi \kappa t)^{3/2}} \left\{ \exp \left( -\frac{Z_-^2}{4\kappa t} \right) + \exp \left( -\frac{Z_+^2}{4\kappa t} \right) - 2\lambda \pi^{1/2} \exp(\bar{\lambda}^2 + \lambda Z_+) \text{erfc} \left( \frac{Z_+}{2(\kappa t)^{1/2}} + \bar{\lambda} \right) \right\}. \] (21)

### III. EVOLUTION OF A UNIFORM TEMPERATURE DISTRIBUTION

The obtained expression (19) for Green’s function can be used for the solution of various boundary-value problems for a layer. As an example, we consider the simplest well-known problem on the evolution of the temperature in an initially uniformly heated layer \([T_0(\vec{r}) = T_0]\) with boundary conditions (2) and (3) for \( \lambda_1 = \lambda_2 \equiv \lambda \). The required distribution \( T(\vec{r}, t) \) is defined by the integral
\[ T(\vec{r}, t) = T_0 \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dy' \int_{0}^{\infty} G(\vec{r}, \vec{r}', t). \] (22)

By virtue of Eqs. (9)–(13), relation (22) can be represented in the following form:

\[ T(\vec{r}, t) \equiv T(z, t, \lambda) = T(z, t, \lambda = \infty) + \delta T(z, t, \lambda), \] (23)

where \( T(z, t, \lambda = \infty) \) is the known solution of the first boundary-value problem \[1,3\]

\[ T(z, t, \lambda = \infty) = T_0 \left\{ \sum_{n=0}^{\infty} (-1)^n \left[ \text{erfc}(nl - \xi) + \text{erfc}((n + 1)l + \xi) \right] - 1 \right\}. \] (24)

and the additional term \( \delta T(z, t, \lambda) \) caused by nonzero value of the quantity \( \lambda^{-1} \) has the form

\[ \delta T(z, t, \lambda) = T_0 \left\{ \sum_{n=1}^{\infty} \sum_{k=0}^{2n-1} C_{2n}^{k+1} \left[ R_{k,n}(z) + R_{k,n}(L - z) \right] \\
+ \sum_{n=0}^{\infty} \sum_{k=0}^{2n} C_{2n+1}^{k+1} \left[ R_{k,n}(-z) + R_{k,n}(z - L) \right] \right\}. \] (25)

Here, \( l = L/(4\kappa t)^{1/2} \), \( \xi = z/(4\kappa t)^{1/2} \), and

\[ R_{k,n} = \frac{1}{\lambda \pi^{1/2}} \sum_{m=0}^{k} (-2)^{k-m-1} \left[ \exp \left( -((2n + 1)l - \xi)^2 \right) \beta_m((2n + 1)L - z, \lambda) \\
- \exp \left( -(2nl - \xi)^2 \right) \beta_m(2nL - z, \lambda) \right]. \] (26)

It is worth to note that expressions (23)–(25) can be written at once by using Eq. (72) in [8] for the density of Brownian particles in a layer with absorbing boundaries and carrying out changes in the corresponding notation in it.

As is seen from Eqs. (23)–(26), distribution of the temperature in the layer remains symmetric about the middle of the layer at any time \( t > 0 \):

\[ T(L - z, t, \lambda) = T(z, t, \lambda), \]

which is quite natural due to the symmetry of the system under investigation about the plane \( z = L/2 \) and uniformity of the initial distribution of temperature.

By collecting terms with the same indices \( m \) in Eq. (25), we can represent the quantity \( \delta T(z, t, \lambda) \) as follows:

\[ \delta T(z, t, \lambda) = \frac{T_0}{\lambda \pi^{1/2}} \left\{ \tilde{V}_0(z, \lambda) + \sum_{n=1}^{\infty} \left[ V_n(z, \lambda) + \tilde{V}_n(z, \lambda) \right] \right\}, \] (27)
where
\[
V_n(z, \lambda) = \sum_{m=1}^{2n-1} l_{m,n} V_{m,n}(z, \lambda), \quad n = 1, 2, \ldots,
\]
\[
\tilde{V}_n(z, \lambda) = \sum_{m=0}^{2n} \tilde{l}_{m,n} \tilde{V}_{m,n}(z, \lambda), \quad n = 0, 1, 2, \ldots,
\]
\[
V_{m,n}(z, \lambda) = \exp \left( -z_n^{(\pm)} \right) \eta_m(z_n^{(\pm)}, \lambda) - \exp \left( -z_n^{(\pm)} \right) \eta_m(z_n^{(\pm)}, \lambda)
+ \exp \left( -\tilde{z}_n^{(\pm)} \right) \eta_m(\tilde{z}_n^{(\pm)}, \lambda) - \exp \left( -\tilde{z}_n^{(\pm)} \right) \eta_m(\tilde{z}_n^{(\pm)}, \lambda),
\]
\[
\tilde{V}_{m,n}(z, \lambda) = \exp \left( -z_n^{(\pm)} \right) \eta_m(z_n^{(\pm)}, \lambda) - \exp \left( -z_n^{(\pm)} \right) \eta_m(z_n^{(\pm)}, \lambda)
+ \exp \left( -\tilde{z}_n^{(\pm)} \right) \eta_m(\tilde{z}_n^{(\pm)}, \lambda) - \exp \left( -\tilde{z}_n^{(\pm)} \right) \eta_m(\tilde{z}_n^{(\pm)}, \lambda),
\]
\[
\eta_n(z, \lambda) = (-1)^n 2^{-(n+1)} \beta_n(|z|, \lambda), \quad n = 0, 1, 2, \ldots, \tag{28}
\]
\[
z_n^{(\pm)} = 2nL \pm z, \quad \tilde{z}_n^{(\pm)} = (2n - 1)L \pm z, \quad \xi_n^{(\pm)} = z_n^{(\pm)}/(4\kappa t)^{1/2}, \quad \tilde{\xi}_n^{(\pm)} = \tilde{z}_n^{(\pm)}/(4\kappa t)^{1/2}.
\]

The quantities \( l_{m,n} \) and \( \tilde{l}_{m,n} \) are defined by the recurrence relations
\[
l_{m,n} = l_{m-1,n} + (-1)^m 2^{m-1} C_{2n}^m, \quad n = 2, 3, \ldots; \quad m = 1, 2, \ldots, 2n - 1, \]
\[
l_{1,n} = -2n,
\]
\[
\tilde{l}_{m,n} = \tilde{l}_{m-1,n} + (-1)^m 2^{m-1} C_{2n+1}^m, \quad n = 1, 2, \ldots; \quad m = 1, 2, \ldots, 2n,
\]
\[
\tilde{l}_{0,n} = 1, \quad \tilde{l}_{1,n} = -2n, \tag{29}
\]

which are convenient for \( m \leq n \). For \( m > n \), the following relations are more convenient:
\[
l_{m,n} = l_{m+1,n} + (-2)^m C_{2n-1}^{2n-m-1}, \quad m < 2n - 1, \]
\[
l_{2n-1,n} = -2^{2n-1},
\]
\[
\tilde{l}_{m,n} = \tilde{l}_{m+1,n} + (2)^m C_{2n+1}^{2n-m}, \quad m \leq 2n - 1,
\]
\[
\tilde{l}_{2n,n} = 2^{2n}. \tag{30}
\]

Relation (27) is valid for any \( \lambda > 0 \). In particular case of great values of \( \tilde{\lambda} \), which corresponds holding the first correction terms caused by a finite value of the relative heat transfer coefficient of the layer with the environment, relation (27) is simplified to the form
\[ \delta T(z, t, \lambda) = \frac{T_0}{\lambda^{1/2}} \left\{ \sum_{n=-\infty}^{\infty} \frac{1 - (1 + |\gamma(n)|)(1 + 2|\gamma(n)|)^k}{\gamma(n)(1 + |\gamma(n)|)} \exp \left( -(2nl - \xi)^2 \right) \right. \\
- \left. \frac{1 - (1 + |\alpha(n)|)(1 + 2|\alpha(n)|)^m}{\alpha(n)(1 + |\alpha(n)|)} \exp \left( -((2n + 1)l - \xi)^2 \right) \right\}, \tag{31} \]

where \( \alpha(n) = ((2n + 1)l - \xi)/\tilde{\lambda}, \quad \gamma(n) = (2nl - \xi)/\tilde{\lambda}, \quad n = 0, \pm 1, \pm 2, \ldots, \)

\[
m = \begin{cases} 
2n, & n \geq 0, \\
-(2n + 1), & n < 0,
\end{cases} \quad k = \begin{cases} 
2n - 1, & n > 0, \\
-2n, & n \leq 0.
\end{cases}
\]

For a thick layer \((l \gg 1)\), relations (23), (24), and (31) for distribution of the temperature can be simplified. In this case, it is sufficient to retain only terms with \(n = 0, 1\) in sum (24) and the term with \(n = 0\) in sum (31). This yields

\[ T(z, t, \lambda) \approx T^{sb}(z, t, \lambda) + T^{sb}(L - z, t, \lambda) - T_0, \tag{32} \]

where \( T^{sb}(z, t, \lambda) \) and \( T^{sb}(L - z, t, \lambda) \) are the known distributions of the temperature in semibounded bodies \([7]\) that occupy the \(z \geq 0\) and \(z \leq L\) regions, respectively. In the case \(\tilde{\lambda} \gg 1\), they have the form

\[ T^{sb}(z, t, \lambda) \approx T_0 \left[ \text{erf} \frac{\xi}{\lambda} + \frac{\exp(-\xi^2)}{\pi^{1/2}(\xi + \lambda)} \right], \tag{33} \]

where

\[ \text{erf} \xi = 1 - \text{erfc} \xi = \frac{2}{\pi^{1/2}} \int_0^\xi dx \exp(-x^2). \]
FIG. 2. Space and time distribution of the temperature in the layer: Bi = ∞ (a), 100 (b), 10 (c), 1 (d), and 0.1 (e).
The space and time distribution of the normalized temperature \( T(z, t, Bi)/T_0 \) in the layer calculated by the general formulae (23)–(24), (27)–(30) are displayed in Fig. 2 for various Biot numbers \( Bi = L\alpha/2 \). With regard for the symmetry of the system at hand about the \( z = L/2 \) plane, the distribution of temperature is shown only for the \( 0 \leq z \leq L/2 \). It is quite natural that the layer cools down with maximum rate in the case corresponding to the first-boundary problem \((Bi = \infty)\) when the boundaries of the layer are kept at zero temperature for the entire time (Fig. 2a). Note that a noninstantaneous drop in the temperature shown in Fig. 2a from the initial value \( T_0 \) to zero at the initial time \((t = 0)\) as the observation point approaches the boundary of the layer is not the result of calculations but appears due to the used uniformity of the space and time grid for the temperature field.

For finite great values of the Biot number (Fig. 2b), the distribution of temperature inside the layer, in fact, does not change, whereas it takes place the smooth but fast cooling of the boundary of the layer. For the time of order of \( 0.1\tau \), where \( \tau = L^2/4\kappa \), the surface temperature is as small as 6% of its initial value. As the Biot number decreases \((Bi = 10, \text{Fig. 2c})\), the cooling of the surface in noticeably slows down and the surface temperature tends to zero even for \( t \approx \tau \). In this case, the finite value of \( \lambda \) influences the distribution of temperature deep into the layer, e.g., \( T(L/2, \tau, Bi = 10)/T(L/2, \tau, Bi = \infty) \approx 1.52 \). For intermediate values of the Biot number (Fig. 2d), heat removal from the surface does not lead to a radical redistribution of the temperature in the central and peripheral regions of the layer as for \( Bi \gg 1 \). For example, for \( Bi = 1 \) and \( t = \tau \), the ratio \( T(0, \tau, Bi)/T(L/2, \tau, Bi) \) is only 0.65. For small Biot numbers \((Bi = 0.1, \text{Fig. 2d})\), the temperature equalization inside the layer proceeds much faster than heat removal from the surface. This case is characterized by a slightly uniform distribution of temperature across the section of the layer \([T(0, \tau, Bi)/T(L/2, \tau, Bi) \approx 0.95]\) and a slow decrease in the temperature with time \([T(L/2, \tau, Bi)/T_0 \approx 0.92]\).
IV. CONCLUSIONS

In the present paper, we gave a new representation for a solution of the third boundary-value problem of the heat conduction equation for a layer which needs no calculations of eigenvalues on the basis of the transcendental equation. This solution is represented in the form of a superposition of the fundamental solutions for an unbounded system caused by instantaneous heat sources situated at the points of location of sources–images of the layer. It is the form of representation of the solution that is widely used in the literature for representation of solutions of the first and second boundary-value problems.

Since new representation (9) enables one to carry out analytic investigations not using graphic and table data, it can be useful for various physical problems for which, in parallel with numerical calculations, analytic estimations of solutions must be performed. In particular, we can find the first additional terms caused by the account of the heat transfer of the layer with the environment to the known results determined under the assumption of a given temperature of the surface or the absence of the thermal flow through it. In the paper, we illustrate the possibility of finding correction terms in powers of $\lambda^{-1}$, where $\lambda > 1$, for a special problem of the evolution of the initially uniform temperature field in the layer.

In addition to analytic investigations, Green’s function (9) can be also useful for the numerical analysis of various boundary-value problems of the third order. Using recurrence relations (14), we significantly simplify this analysis by reducing it to the calculation of probability integrals and the majorant estimation of the remainder terms of the series given in the Appendix.
APPENDIX:

We prove that solution (9) satisfies the homogeneous heat conduction equation (1) with boundary conditions (2) and (3). For this purpose, first, we prove the uniform convergence of the series in solution (9) at $0 \leq z \leq L$ and $t > 0$. In view of the following representation of the quantity $\beta_k(|Z_\pm(n)|, \lambda)$:

$$
\beta_k(|Z_\pm(n)|, \lambda) \approx (-1)^k \left( \frac{4\lambda\kappa t}{(|Z_\pm(n)| + 2\lambda\kappa t)} \right)^{k+1},
$$

(A1)

which is valid for $|n| > (\kappa t)^{1/2}/L$, we obtain that for finite values of $\lambda$, for all $|n| \geq N$, where $N \gg \lambda\kappa t/L$, the coefficients $P_n^{(\pm)}(Z_\pm(n), \lambda) \equiv P_n^{(\pm)}(\lambda)$ can be represented as follows:

$$
P_n^{(-)}(\lambda) \approx \left( 1 - \frac{2\lambda\kappa t}{L|n|} \right)^{2|n|},
$$

$$
P_n^{(+)}(\lambda) \approx \left( 1 - \frac{2\lambda\kappa t}{L|n|} \right)^{2|n|+\text{sign} n}.
$$

(A2)

As $|n| \to \infty$, the following estimate for the coefficients holds:

$$
\lim_{|n| \to \infty} P_n^{(\pm)}(\lambda) = \exp \left( -\frac{4\lambda\kappa t}{L} \right) \leq 1.
$$

(A3)

Thus, starting from $|n| \geq N$, the series $\sum_{|n|=N}^{\infty} P_n^{(\pm)}(\lambda) \exp \left( -Z_\pm^2(n)/4\kappa t \right)$ in solution (9) can be majorized by the uniformly convergent series $\sum_{|n|=N}^{\infty} \exp \left( -Z_\pm^2(n)/4\kappa t \right)$. If follows from here that series in solution (9) are also uniformly convergent. Therefore, we can integrate solution (9) term by term and use the majorant estimate

$$
R(z, z', t) = \frac{1}{(4\pi\kappa t)^{1/2}} \left| \sum_{|n|=N}^{\infty} \left[ P_n^{(-)}(\lambda) \exp \left( -\frac{Z_\pm^2(n)}{4\kappa t} \right) + P_n^{(+)}(\lambda) \exp \left( -\frac{Z_\pm^2(n)}{4\kappa t} \right) \right] \right|
$$

$$
\leq \frac{1}{t} \text{erfc} \left( \frac{2N-1}{2\kappa t} L \right)
$$

(A4)

for the remainder of the series which determines Green’s function (9).

It is easy to show that

$$
\lim_{|n| \to \infty} \frac{\partial P_n^{(\pm)}(Z_\pm(n), \lambda)}{\partial z} = 0.
$$
Therefore, the series of derivatives
\[
\sum_{n=-\infty}^{\infty} \left[ \tilde{P}_n^-(Z_-(n), \lambda) \exp \left( -\frac{Z_-(n)}{4\kappa t} \right) + \tilde{P}_n^+(Z_+(n), \lambda) \exp \left( -\frac{Z_+^2(n)}{4\kappa t} \right) \right],
\]

where
\[
\tilde{P}_n^\pm(Z_\pm(n), \lambda) = \frac{\partial P_n^\pm(Z_\pm(n), \lambda)}{\partial z} - \frac{Z_\pm(n)}{2\kappa t} P_n^\pm(Z_\pm(n), \lambda),
\]
converges uniformly as well because the series \( \sum_{|n|=N}^{\infty} \tilde{P}_n^\pm(Z_\pm(n), \lambda) \exp \left( -Z_\pm^2(n)/4\kappa t \right) \) are majorized by the uniformly convergent series \( \sum_{|n|=N}^{\infty} (nL/\kappa t) \exp \left( -Z_\pm^2(n)/4\kappa t \right) \). Thus, solution (9) is term-by-term differentiable with respect to \( z \). The possibility of double term-by-term differentiation with respect to \( z \) and differentiation with respect to \( t \) is proved in a similar way.

We represent Green’s function (9) in the form
\[
G(\vec{r}, \vec{r}', t) = \sum_{\alpha=\pm} \sum_{n=-\infty}^{\infty} G_n^{(\alpha)}(\vec{r}, \vec{r}', t),
\]

where
\[
G_n^{(\pm)}(\vec{r}, \vec{r}', t) = P_n^{(\pm)}(Z_\pm(n), \lambda) \exp \left( -\frac{R_\perp^2 + Z_\pm^2(n)}{4\pi\kappa t} \right), \quad n = 0, \pm 1, \pm 2, \ldots,
\]

substitute expression (A5) into homogeneous equation (1), and carry out term-by-term differentiation of the series. By using recurrence relation (14) for the functions \( F_k(x) \), we obtain that expression (A5) satisfies the required equations, moreover, each term \( G_n^{(\alpha)}(\vec{r}, \vec{r}', t) \) of series (A5) satisfies this equation. By substituting solution (A5) into initial (6) and boundary (2) and (3) conditions, we make sure that the solution satisfies these conditions. Note that, as opposed to the heat conduction equation, boundary conditions (2) and (3) are valid not separately for each term \( G_n^{(\alpha)}(\vec{r}, \vec{r}', t) \) of series (A5) but for pairs, namely:
\[
\left( \frac{\partial}{\partial z} - \lambda \right) \left( G_n^-(\vec{r}, \vec{r}', t) + G_{-n}^+(\vec{r}, \vec{r}', t) \right) = 0 \bigg|_{z=0}, \quad n = 0, \pm 1, \pm 2, \ldots,
\]
\[
\left( \frac{\partial}{\partial z} + \lambda \right) \left( G_n^-(\vec{r}, \vec{r}', t) + G_{-(n+1)}^+(\vec{r}, \vec{r}', t) \right) = 0 \bigg|_{z=L}, \quad n = 0, \pm 1, \pm 2, \ldots.
\]

Thus, relation (9) is the solution of the third boundary-value problem of the heat conduction equation.
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