A straightening algorithm for row-convex tableaux.*

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Abstract
We produce a new basis for the Schur and Weyl modules associated to a row-convex shape \( D \). The basis is indexed by new class of “straight” tableaux which we introduce by weakening the usual requirements for standard tableaux. Spanning is proved via a new straightening algorithm for expanding elements of the representation into this basis. For skew shapes, this algorithm specializes to the classical straightening law. The new straight basis is used to produce bases for flagged Schur and Weyl modules, to provide Groebner and sagbi bases for the homogeneous coordinate rings of some configuration varieties and to produce a flagged branching rule for row-convex representations. Systematic use of super-symmetric letterplace techniques enables the representation theoretic results to be applied to representations of the general linear Lie superalgebra as well as to the general linear group.

1 Introduction
Akin, Buchsbaum, and Weyman in \[ABW82\] give a construction that associates a \( GL_n \)-representation to any generalized shape like

\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 0 & 0
\end{bmatrix}
\quad \text{or} \quad
\begin{bmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
1 & 0 & 0
\end{bmatrix}
\quad \text{or} \quad
\begin{bmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
0 & 0 & 0
\end{bmatrix}
\quad \text{or} \quad
\begin{bmatrix}
1 & 1 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

(1)

Significant progress has been made by Reiner and Shimozono and by Lakshmibai and Magyar in describing bases for these \( GL_n \)-representations and for the associated flagged representations of the Borel subgroup of upper triangular matrices in \( GL_n \). As one expects, these bases are indexed by some subset of the generalized tableaux found by filling each cell in the generalized shape with a number from 1 to \( n \).

*http://www.math.wayne.edu/~bdt/straightening.ps
The present paper shows how to construct a well-behaved straight basis for the representations associated to any row-convex shape, such as \( \begin{array}{c|c|c}
1 & 2 & 3 \\
\hline
4 & 5 & 6
\end{array} \), with no gaps in any row. In particular, we give a local condition for testing whether a tableau is straight, we give a straightening law that modifies only two rows at a time, and the basis we present reduces immediately to a flagged basis. The straight basis is distinct from the bases produced by Reiner-Shimozono and Lakshmibai-Magyar, but see [T99] for some combinatorial and algebraic relationships between these bases. The straight bases provide a canonical choice of basis for certain row-convex and column-convex “almost-skew shapes.” These shapes were shown by Woodcock in [W94] to possess a class of easily flagged bases, but no method was presented for distinguishing a basis in this class or for straightening elements of the representation into a linear combination of basis elements.

Results on flagged tableaux are deduced in Section 6 from the main theorem on straight bases. As shown in Section 6, the straight basis and straightening algorithm may be applied to produce quadratic Groebner bases and sagbi bases for the homogeneous coordinate rings of certain configuration varieties. Further applications to commutative algebra may be found in [T97a] and [T99]. Applications to the representation theory of \( GL_n, B_n, S_n \), and the general linear Lie superalgebras are derived in Section 8 where a branching rule is produced for decomposing a row-convex \( GL_n \)-representation in terms of \( GL_{n-1} \)-representations.

This paper studies the Schur and Weyl modules as special cases of the super Schur modules which we construct as submodules of the letterplace superalgebra. All results in this paper are characteristic–free and the requisite background on superalgebras is detailed in Section 2. Much of the presentation in Section 2 is new and, we hope, accessible to the non-specialist. The construction proper is given in Section 3. Straight tableaux are introduced and independence is proved in Section 4. Section 5, the heart of the paper, presents the straightening algorithm.

2 Polynomial superalgebras

This section introduces the definitions required to make the main results of this paper characteristic free and applicable to Weyl modules. The reader concerned only with Schur modules in characteristic 0 may safely take \( \mathcal{L} \) and \( \mathcal{P} \) to be the positive integers, \( \mathbb{N} \), (or finite subsets of \( \mathbb{N} \).) The set \( \mathcal{L} \) may be thought of as the indexing the rows of a generic matrix \( (x_{i,j}) \) and \( \mathcal{P} \) indexes the columns. We may then take take \( \text{Super}([\mathcal{L} | \mathcal{P}]) \) to be the polynomial ring whose variables are matrix entries \( x_{i,j} \). The letterplace \( (i|j) \) is taken to be shorthand for \( x_{i,j} \) and the expression \([i_1, \ldots, i_k|j_1, \ldots, j_k]\) is taken to be the determinant of the \( k \times k \) minor \( (x_{i_r,j_s}) \) of the matrix \( (x_{i,j}) \).

The constructions used in this paper take place inside polynomial superalgebras over the integers, \( \mathbb{Z} \), that is inside tensor products of symmetric, exterior, and divided powers algebras. We construct the polynomial superalgebras over \( \mathbb{Z} \) as \( \mathbb{Z} \)-subalgebras of a symmetric algebra over the rationals, \( \mathbb{Q} \), tensored with an
exterior algebra over $\mathbb{Q}$. Write the symmetric and exterior $\mathbb{Z}$-algebras associated to a set $\mathcal{L}$ as $\text{Sym}(\mathcal{L})$ and $\Lambda(\mathcal{L})$. These are $\mathbb{Z}$-subalgebras of the symmetric and exterior $\mathbb{Q}$-algebras $\text{Sym}_\mathbb{Q}(\mathcal{L})$ and $\Lambda_\mathbb{Q}(\mathcal{L})$ associated to $\mathcal{L}$. The divided powers algebra, $\text{Div}(\mathcal{L})$ of a set $\mathcal{L}$ is the $\mathbb{Z}$-subalgebra of $\text{Sym}_\mathbb{Q}(\mathcal{L})$ generated by all $x^i_i$ for all $x \in \mathcal{L}$.

We define a signed set to be a set $\mathcal{L}$ together with a a function $|| : \mathcal{L} \to \mathbb{Z}_2$. We say that elements in the preimage of 0 are positively signed; we call this preimage $\mathcal{L}^+$. Elements in the preimage, $\mathcal{L}^-$, of 1 are said to be negatively signed. A signed set $\mathcal{L}$ endowed with a total order, $<$, is called a (signed) alphabet. For notational convenience, we define two new inequalities, $\ll$ and $\lll$ on $\mathcal{L}$. We say that $a \ll b$ (respectively $a \ll b$) when $a < b$ or when $a = b$ and $|a| = |b| = 0$ (respectively $a = b$ and $|a| = |b| = 1$.)

A superalgebra is simply an algebra with a $\mathbb{Z}_2$-grading. We construct a $\mathbb{Q}$-superalgebra with the elements of a signed set as generators and such that the grading on these generators is $||$. For any signed set $\mathcal{L}$, define $\text{Super}_\mathbb{Q}(\mathcal{L})$ to be $\text{Sym}_\mathbb{Q}(\mathcal{L}^+) \otimes \Lambda_\mathbb{Q}(\mathcal{L}^-)$. Likewise, we define $\text{Super}(\mathcal{L})$ to be $\text{Div}(\mathcal{L}^+) \otimes \Lambda(\mathcal{L}^-)$; as above, we may consider this to be a $\mathbb{Z}$-subalgebra of $\text{Super}_\mathbb{Q}(\mathcal{L})$. Given another signed set $\mathcal{P}$, we will define the “letter-place” algebra, $\text{Super}(\mathcal{L} \mid \mathcal{P})$, to be a $\mathbb{Z}$-subalgebra of $\text{Super}_\mathbb{Q}(\{x_{a,d}\}_{a \in \mathcal{L}, d \in \mathcal{P}})$ where $|x_{a,d}| = |a| + |d|$. In particular, $\text{Super}(\mathcal{L} \mid \mathcal{P})$ is the subalgebra generated by all $x_{a,d}$ and by all $\frac{x_{a,d}}{i!}$ with $a, d \in \mathcal{L}^+, i \in \mathbb{N}$. This algebra is naturally isomorphic to

$$\Lambda(\mathcal{L}^- \times \mathcal{P}^+) \cup \Lambda(\mathcal{L}^+ \times \mathcal{P}^-) \otimes \text{Sym}(\mathcal{L}^- \times \mathcal{P}^-) \otimes \text{Div}(\mathcal{L}^+ \times \mathcal{P}^+).$$

We extend $||$ to a $\mathbb{Z}_2$ grading of $\text{Super}(\mathcal{L} \mid \mathcal{P})$. Following [GRS87], we write the elements $x_{a,d}$ of $\text{Super}(\mathcal{L} \mid \mathcal{P})$ as the signed variables $(a,d)$, and we will define the biproduct, $(w_1, \ldots, w_k \mid v_1, \ldots, v_k)$, of a pair of sequences $w$ and $v$ in $\mathcal{L}$ and $\mathcal{P}$ respectively.

**Definition 2.1** Given sequences $w = w_1, \ldots, w_k \in \mathcal{L}$ and $v = v_1, \ldots, v_k \in \mathcal{P}$, define $(w \mid v) = (w_1, \ldots, w_k \mid v_1, \ldots, v_k) = \sum_{\sigma \in S_k} (-1)^{n_\sigma} (w_{\sigma(1)} \mid v_1) \cdots (w_{\sigma(k)} \mid v_k)$ where $n_\sigma = \# \{(i,j) : i < j, \sigma^{-1}(i) > \sigma^{-1}(j), w_i, w_j \text{ are negative}\}$

$$+ \# \{(i,j) : i > j \text{ and } w_{\sigma(i)}, v_j \text{ are negative}\}.$$  

The following definition/proposition indicates that the biproduct can be thought of as a bilinear map on $\text{Super}(\mathcal{L}) \times \text{Super}(\mathcal{P})$.

**Definition 2.2** Given sequences $w = w_1, \ldots, w_k \in \mathcal{L}$ and $v = v_1, \ldots, v_k \in \mathcal{P}$, define $(w_1 \cdots w_k \mid v_1 \cdots v_k) = (w_1, w_2, \ldots, w_k \mid v_1, v_2, \ldots, v_k).$ Extend this by bilinearity to a map $(| |) : \text{Super}(\mathcal{L}) \times \text{Super}(\mathcal{P}) \to \text{Super}(\mathcal{L} \mid \mathcal{P})$.

It is straightforward to to check that this map is well-defined.

In order to better handle divided powers as elements of a $\mathbb{Z}$-subalgebra contained in a $\mathbb{Q}$-algebra, we make the following definition. If $w$ is a sequence in $\mathcal{A}$ for some signed set $\mathcal{A}$ then we define $c(w)!$ to be $\prod_{i \in \mathcal{A}^+} (\# \text{ times } i \text{ appears in } w)!$. 

A straighteng algorithm for row-convex tableaux.
A straightening algorithm for row-convex tableaux.

Call a monomial \( M = \prod_{i} (l_{i}|p_{i}) \in \text{Super}_{Q}(\mathcal{L} | \mathcal{P}) \) sorted when \( p_{1} \leq p_{2} \leq \cdots \) and \( p_{i} = p_{i+1} \in \mathcal{P}^+ \) implies \( l_{i} < l_{i+1} \) and dually \( p_{i} = p_{i+1} \in \mathcal{P}^- \) implies \( l_{i} > l_{i+1} \). \( \text{Super}(\mathcal{L} | \mathcal{P}) \) is a free \( \mathbb{Z} \)-module with basis consisting of the divided powers monomials \( \{ \frac{1}{c(M)!} M \} \) for all sorted monomials \( M \in \text{Super}_{Q}(\mathcal{L} | \mathcal{P}) \).

Here \( c(M)! = c((l_{1}|p_{1}), (l_{2}|p_{2}), \ldots)! \). We will consider two monomials (respectively divided powers monomials) the same when they differ by a nonzero scalar multiple (respectively a multiple of \( \pm 1 \)).

Define a function \( \text{Tab}(w|v) \) when \( w \) (respectively \( v \)) is a \( k \)-tuple of letters in \( \mathcal{L} \) (respectively \( \mathcal{P} \)) by

\[
\text{Tab}(w_{1}, \ldots, w_{k}|v_{1}, \ldots, v_{k}) = \frac{1}{c(w)!c(v)!} (-1)^{\# \{(i,j): i > j, w_{i} \in \mathcal{L}^-, v_{j} \in \mathcal{P}^+\}} (w|v).
\]

Observe that the divided powers monomials occur with coefficient \( \pm 1 \) in the expansion of \( \text{Tab}(w|v) \) and that if \( w_{1} < w_{2} < \cdots < w_{k} \) and \( v_{1} < v_{2} < \cdots < v_{k} \), then the basis element \( \frac{1}{c(w)!c(v)!} \prod_{i} (w_{i}|v_{i}) \) appears with coefficient 1.

### 3 Schur modules, Weyl modules, generalizations

In this section, we define our primary object of study, the super-Schur module as a \( \mathbb{Z} \)-submodule of a letterplace algebra. The unsigned cases produce the Schur and Weyl modules of Akin-Buchsbaum-Weyman \([ABW82]\) when \( \mathcal{L} = \mathcal{L}^- \) and \( \mathcal{L} = \mathcal{L}^+ \) respectively.

We define two tableaux associated to a shape. The first is useful for referring to cells in the shape and the second plays a fundamental role in our construction of the super-Schur modules.

**Definition 3.1** Let \( D \) be a shape.

Define \( F(D) \) to be the tableau of shape \( D \) whose cells are labeled 1, 2, 3, \ldots starting with the northmost cell in the leftmost column and continuing down the column, then down the second leftmost column, etc. In this paper, the signs of the letters in \( F(D) \) are irrelevant.

A tableau of shape \( D \) is termed Deruyts if it is obtained by filling each cell in the diagram with the cell’s column index viewed as a negative variable. We denote such a tableau by \( \text{Der}^{-}(D) \).

Shapes appearing in this paper are assumed, unless otherwise noted, to have first coordinate 1 in their top rows and second coordinate 1 in their leftmost columns.

**Example 3.1**

\[
F\left(\begin{matrix} 3 & 6 \\ 2 & 5 \end{matrix}\right) = \begin{array}{c} 1 \\ 2 \end{array} \quad \text{Der}^{-}\left(\begin{matrix} 3 & 4 & 7 \\ 3 & 4 & 5 \end{matrix}\right) = \begin{array}{c} 1^- \\ 3^- 4^- \\ 3^- 4^- \end{array}
\]
Definition 3.2 Suppose $S$ and $T$ are tableaux of the same shape. Let the word $s_i$ be the $i$th row of $S$ and let $t_i$ be the $i$th row of $T$. Define $[S[T] = \prod_i \text{Tab}(s_i|t_i)$. Hence $[S[T] = \prod_i [s_i|t_i]$. 

Now suppose that $T$ is a tableau of shape $D$. Suppose that $L$ contains the set of letters present in $T$ and that $P^-$ contains the indices for all columns present in $D$. Define an element $[T] \in \text{Super}([L|P^-])$ indexed by $T$ by,

$$[T] = [T \mid \text{Der}^{-}(D)]$$

Example 3.2 Let $L = L^- = \{a, b, c, d, e, f, g\}$ and let $P^- = \{1, 2, 3, 4\}$.

Let $T = \begin{bmatrix} a & d \\ b & c \\ e \\ f & g \end{bmatrix}$. Then

$$[T] = \begin{bmatrix} a & d \\ b & c \\ c & e \\ f & g \end{bmatrix}.$$

In other words, $[T]$ is a scalar multiple of $(ad|23)(be|123)(f|2)(g|1)$. The scalar in this example being $1$, we have

$$[T] = \det \begin{pmatrix} (d|3) & (d|4) \\ (a|3) & (a|4) \end{pmatrix} \det \begin{pmatrix} (e|1) & (e|3) & (e|4) \\ (b|1) & (b|3) & (b|4) \end{pmatrix} (f|2)(g|1) .$$

Definition 3.3 Suppose that $D$ is a shape. Define the super-Schur module

$$S^D(L) = \text{span}_Z \{ [T] : \text{shape}(T) = D \text{ and } T \text{ is filled with letters from } L \}.$$

In the case that $L$ is negative (respectively positive) then $S^D(L)$ is called the Schur (respectively Weyl) module associated with the diagram $D$. These terms are justified by the following result. A proof may be found in [T97a].

Proposition 3.4 Let $R$ be a commutative ring. Let $F$ be a free $R$-module of rank $n$. Let $\alpha$ be the $0/1$-matrix having $1$’s precisely where $D$ has cells.

If $L = L^-$ has cardinality $n$, then $R \otimes_Z S^D(L^-) = L_\alpha(F)$, where $L_\alpha(F)$ is the Akin-Buchsbaum-Weyman “Schur functor” associated to the generalized shape matrix $\alpha$.

If $L = L^+$ has cardinality $n$, then $R \otimes_Z S^D(L^+) = K_\alpha(F)$, where $K_\alpha(F)$ is the Akin-Buchsbaum-Weyman “coSchur functor.”

Example 3.3 The Weyl module of shape $\bigcup$ on positive letters $a, b$ is spanned by

$$\begin{bmatrix} a & a & a \\ b & b & a \end{bmatrix}, \begin{bmatrix} a & a & a \\ b & b & b \end{bmatrix}, \begin{bmatrix} b & b & b \\ a & a & a \end{bmatrix}, \begin{bmatrix} a & b & b \\ a & a & b \end{bmatrix}, \begin{bmatrix} a & b & b \\ b & b & a \end{bmatrix}, \begin{bmatrix} a & a & b \\ a & a & b \end{bmatrix}, \begin{bmatrix} a & a & b \\ a & a & b \end{bmatrix}, \begin{bmatrix} b & b & b \\ b & b & b \end{bmatrix}.$$
in the exterior algebra generated by the anti-commuting variables \((a|1), (a|2), (a|3), (b|1), (b|2), (b|3)\). The last two of the above skew-polynomials are identically 0. In the next section we single out the first three elements as a basis.

4 Row-convex diagrams and straight tableaux

The usual bases for skew Weyl modules consist of the semistandard Young tableaux, namely all tableaux which weakly increase in their rows and strictly increase in their columns. Example 3.3 showed that this is not the case for more general shapes. Nevertheless, the basis of \([ABW82]\) for shape \(D\) skew Weyl modules indexed by standard Young tableaux of shape \(D\) has a number of properties we wish to preserve. In particular:

1. The rows of the tableaux in the indexing set weakly increase.
2. Knowing the number of times a letter appears in each column of a tableau in the indexing set determines that tableau.
3. It is combinatorially “obvious” when a tableau is in the indexing set.
4. The elements \([T]\) where \(T\) is in the index set form a basis for module.
5. There is an easy to describe algorithm for rewriting \([T]\) in terms of basis elements.

Property 2 underlies the sagbi-basis algorithms of \([Stu93]\); in \([W94]\) Woodcock shows that there must exist bases satisfying this property when \(D\) is “almost-skew.”

Only slightly more complicated shapes, \(\square\) for instance, fail to simultaneously possess properties 1, 3, and 5. To see this, examine the Specht module associated to this shape. Recall that this is the subspace of the associated Schur module spanned by all tableaux containing letters \(1^-, 2^-, 3^-, 4^-\) with no repeats; here the indexing shape is transposed from the indexing shape used in \([Sa91]\). This Specht module is isomorphic to the one indexed by \(\square\) hence has dimension 5. However, there are only 4 tableaux of shape \(\square\) satisfying conditions 1 and 3.

We define a class of “straight” tableaux satisfying the above properties. The elements \([T]\) where \(T\) is straight and of shape \(D\) will form a basis for the super-Schur module \(S^D\) for any “row-convex” shape \(D\).

**Definition 4.1** A row-convex shape, such as \(\square\), is a shape with no gaps in any row. I.E., if cells \((r,i)\) and \((r,k)\) are in a shape \(D\), then \((r,j)\) is in \(D\), for all \(i < j < k\). Since the constructions of section 3 are not sensitive to the order of rows in a diagram, we assume that the rows of a row convex diagram are sorted so that higher rows end at least as far to the right as lower rows.
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We can denote any row-convex shape by $\lambda/m$ where $\lambda$ is a partition and $m$ is a composition satisfying $m_i \leq \lambda_i$ for all $i$; a cell is in position $(i, j)$ of $\lambda/m$ iff $m_i \leq j \leq \lambda_i$.

Following [GRS87], and employing the notation for inequalities introduced on page 3, a tableau $T$ with entries in a signed set is standard when it ($\prec$-)increases across rows and ($\prec$-)increases down columns.

I introduce the notion of a straight tableau of row-convex shape by slightly relaxing the usual conditions for standardness of a tableau.

**Definition 4.2** A row-convex tableau is called straight when

1. The contents of any row $\prec$-increase from left to right, and
2. Given two cells in the same column, say $(i, k)$ and $(j, k)$ for $i < j$, the entry in the top cell, $(i, k)$, may be ($\succ$-)larger than the entry in $(j, k)$ (i.e. the cells form an inversion) only if cell $(i, k - 1)$ exists and its content is ($\succ$-)larger than the content of $(j, k)$.

This definition amounts to requiring that the columns are as close as possible to ($\prec$-)increasing, subject to the condition that the rows remain ($\prec$-)increasing. A more precise version of the preceding fact is implicit in the correctness Algorithm Straight-Filling in Figure 1. A tableau satisfying condition 1 is called row-standard and an inversion violating condition 2 is called a flippable inversion.

**Proposition 4.3** A skew tableau, $T$, is straight iff it is standard.

**Proof.** Since a standard tableau has no inversions, it suffices to prove the only-if part. We prove the contrapositive. We can assume that $T$ is row-standard. Suppose that the cells $(i, k), (j, k)$ with $i < j$ are an inversion. Let $k_0$ be the least (leftmost) column such that $(i, k_0), (j, k_0)$ is an inversion. If $(i, k_0 - 1)$ exists then by skewness so does $(j, k_0 - 1)$ and thus by assumption $T_{i, k_0 - 1} \prec T_{j, k_0 - 1} \prec T_{j, k_0}$ hence $T$ is not straight.

**Corollary 4.4** The straight tableaux of skew shape with only positively signed letters are the usual semistandard Young tableaux.

**Definition 4.5** Given a tableau $T$ its column word, $c_T$ is the word formed by reading the entries of $T$ from bottom to top and left to right. Its modified column word is the word $w_T$ formed by writing the entries of the first column in weakly decreasing order followed by the entries of the second column in decreasing order, etc.

We shall also occasionally require a reverse column word $w'_T$ of $T$ formed by writing the entries of the first column of $T$ in increasing order then those of the second column in increasing order etc.
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Input: A word $w'$ of length $n$, and an $n$-celled row-convex shape $D$.
Output: A straight tableau $T$ with $w'_T = w'$ or "IMPOSSIBLE" if no such tableau exists.

Let $c_j$ be the column index of the $j$ in $F(D)$.
Let $T$ be an empty tableau of shape $D$ for $k = 1 \ldots n$.
Let $i$ be the smallest (northmost) index such that $(i, c_k) \in D$ is still empty and either there is no cell in position $(i, c_k - 1)$ or $T_{i,c_k-1} < w'_k$. 
if there is no such $i$ then return "IMPOSSIBLE"
else $T_{i,c_k} \leftarrow w'_k$.

Figure 1: Algorithm Straight-Filling

Theorem 4.6 If $T$ and $T'$ are straight tableaux of the same shape, then $T \neq T'$ implies $w_T \neq w_{T'}$. More strongly, if there exists a straight tableau $T$ of shape $D$ with $w_T = w$ then the algorithm Straight-Filling in Figure 1 produces it.

Proof. A tableau, $T$, produced by this algorithm must be straight. If in a fixed column, $k$, the letter $y$ is inserted into row $i$ by the algorithm while $x < y$ was inserted into row $j > i$, then it must be that $T_{i,k-1} > x$ else cell $T_{i,k}$ would have been available to $x$ hence $x$ would have been placed there.

Now suppose that the algorithm produces a tableau $T$ with reverse column word $w'$. Let $\bar{c}$ be as in the algorithm. Any tableau with reverse column word $w'$ can be produced by a similar filling process. Define $i$ so that reading through $w'$ and inserting $w'_k$ into cell $(i_k, c_k)$ gives the desired tableau. Let us assume that if $w'_k$ appears in multiple cells in column $c_k$ that the first $w'_k$ in $w'$ is used to fill the northmost appearance in the column, the second is used to fill the second northmost appearance, etc.

Let $\bar{i}$ be the filling sequence corresponding to $T$, this is the sequence produced by the Algorithm Straight-Filling. Let $\bar{i}'$ be the filling sequence corresponding to some other tableau $T'$. Let $k_0$ be the smallest integer such that $i_{k_0} \neq i'_{k_0}$. So in filling $T'$, we have placed $w'_{k_0}$ into cell $(i'_{k_0}, c_{k_0})$ when according to Algorithm Straight-Filling, it could have been put into $(i_{k_0}, c_{k_0})$ where $i_{k_0} < i'_{k_0}$. By necessity, in filling $T'$, something $(\geq)$-larger than $w'_{k_0}$ must be placed in $(i_{k_0}, c_{k_0})$. By our assumptions about repeated letters in the definition of $\bar{c}$, this inequality is strict. But these facts guarantee that the inversion $\{(i_{k_0}, c_{k_0}), (i'_{k_0}, c_{k_0})\}$ of $T'$ violates condition 2 in the definition of straight tableaux.

The above argument says that if we try to create a straight tableau $T$ with $w'_T = w'$ by reading across $w'$ and sequentially filling its letters into a tableau then at each step the choice of where to insert the letters is forced on us. If at any point during execution of the algorithm there is no place to put a letter which preserves row-standardness, then it is in fact not possible to find a straight
tableau with the designated column content and shape. This is precisely the circumstance under which “IMPOSSIBLE” is returned.

We conclude that not only does \texttt{Straight-Filling} produce a straight tableau, but any other tableau, \( T' \) having the same modified (equivalently reverse) column word is not straight. \qed

\textbf{Corollary 4.7} The matrix expressing the super-polynomials \([T]\) indexed by straight tableaux as \( \mathbb{Z} \)-linear combinations of divided powers monomials in the polynomial superalgebra is in echelon form with \( \pm 1 \) at each pivot. Hence the straight basis elements are linearly independent.

We defer the proof in order to develop the appropriate orders on basis elements and monomials. Monomials are ordered according to a generalization of the “diagonal term order” in \cite{Stu93} which requires that the smallest monomial in \( \det(A) \), where \( A \) is a minor of \( (x_{i,j}) \), be the product of the elements on the diagonal. For compatibility with lexicographic order in Lemma \ref{lem:walk}, this is backwards from the convention in commutative algebra which has \( \prod_{i} (x_{i,i}) \) be the largest monomial in \( \det(A) \).

\textbf{Definition 4.8} A diagonal term order on \( \text{Super}(\mathcal{L} | \mathcal{P}) \) is

1. A total order, \( \prec \), on monomials in \( \text{Super}(\mathcal{L} | \mathcal{P}) \) such that for monomials \( m, m', n, n' \), the relations \( m \prec m' \) and \( n \prec n' \) imply that \( mn \prec m'n' \) or \( mn = 0 \) or \( m'n' = 0 \).

2. The smallest monomial in a nonzero biproduct \( (i_1, \ldots, i_k | j_1, \ldots, j_k) \) with \( i_1 < \cdots < i_k \) and \( j_1 < \cdots < j_k \) is \( \prod_l (i_l | j_l) \).

The default diagonal term order, \( \prec_{\text{diag}} \) that we utilize is characterized below. We order letterplaces \( (i | j) \) by \( (i | j) > (i' | j') \) when \( j < j' \) or when \( j = j' \) and \( i > i' \). Let \( M, N \in \text{Super}(\mathcal{L} | \mathcal{P}) \) be two nonzero monomials. Suppose \( (i | j) \) is the largest letterplace appearing to a different power in \( M \) and \( N \). Write \( N \prec_{\text{diag}} M \) when \( M \) is divisible by a higher power of \( (i | j) \) than is \( N \).

\textbf{Example 4.1} Suppose \( \mathcal{L} = \{1^-, 2^- \} \) and \( \mathcal{P} = \{a^+, x^- \} \), then \((2 | a) > (1 | a) > (2 | x) > (1 | x)\). Further, we have
\[
(1 | x)^6 < (2 | a)(1 | x)^2 (2 | x) < (2 | a)(2 | x)^2 (1 | x) < (2 | a)(1 | a)(1 | x)^2.
\]

The following lemma is immediate.

\textbf{Lemma 4.9} A normalized monomial \( \prod_{l=1}^{k} (i_l | j_l) \neq 0 \) in \( \text{Super}(\mathcal{L} | \mathcal{P}) \) is a monomial written so that \( (i_l | j_l) \geq (i_{l+1} | j_{l+1}) \) in the default diagonal term order. For two normalized monomials, \( M = \prod_{i=1}^{k} (i_i | j_i) \) and \( N = \prod_{i=1}^{k} (i_i' | j_i') \) differing only in their letters, \( M < N \) in the default diagonal term order iff \( i_1, \ldots, i_k \) is lexicographically less than \( i_1', \ldots, i_k' \). \qed
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**Definition 4.10** Let $\Psi$ be the function taking a normalized monomial $\prod_{i=1}^{k} (i|j) \in \text{Super}([\mathcal{L}|\mathcal{P}])$ to $i_1, \ldots, i_k$.

**Definition 4.11** Given $p \in \text{Super}([\mathcal{L}|\mathcal{P}])$ and an order $\prec$ on monomials, define the initial monomial $\text{init}_{\prec}(p)$ of $p$ to be the smallest (divided powers) monomial appearing in $p$.

Sometimes the phrase “initial term” will be used when the coefficient of the initial monomial is to be included.

The following result says that in most cases the modified column word of $T$ can be read directly from the smallest monomial appearing in $T$.

**Proposition 4.12** If $T$ is a tableau whose rows ($\prec$)-increase and whose columns contain no repeated positive letters, then

$$w_T = \Psi(\text{init}_{\prec_{\text{diag}}}(T)).$$

**Proof.** Suppose $[T] = \prod_i \text{Tab}(w_{i,c_{i,1}}, w_{i,c_{i,2}}, \ldots, w_{i,c_{i,l_i}}|c_{i,1}, c_{i,2}, \ldots, c_{i,l_i})$. The initial term (with coefficient) of the $i$th multiplicand is $\prod_j (w_{i,c_{i,j}}|c_{i,j})$, and since positive letters never repeat in a column the product of these initial terms is nonzero and hence equals $\text{init}_{\prec_{\text{diag}}}(T)$. \hfill \Box

Note that the initial term $\prod_i \prod_j (w_{i,c_{i,j}}|c_{i,j})$ appearing above is (up to sign) a basis element in the monomial $\mathbb{Z}$-basis for $\text{Super}([\mathcal{L}|\mathcal{P}])$ and we have proved the following.

**Proposition 4.13** If $T$ is straight of shape $D$, and if $c_l$ is the index of the column of $F(D)$ containing $l$ then

$$\prod_l \left( (w'_T)_l|c_l \right) = \text{init}_{\prec_{\text{diag}}}(T)$$

The coefficient of the initial monomial of $[T]$ is $\pm 1$. \hfill \Box

**Corollary 4.14** Suppose $T$ is a straight tableau, then $w_T = \Psi(\text{init}_{\prec_{\text{diag}}}(T))$. \hfill \Box

We now complete the proof of the independence result.

**Proof.** (of Corollary 4.7.) Since Theorem [4.6] says that distinct straight tableaux have distinct modified column words, we conclude from Corollary [4.14] that if monomials are ordered by $\prec_{\text{diag}}$ and the polynomials $[T]$ corresponding to straight tableaux are ordered lexicographically by their modified column words, then the matrix expressing the $[T]$ in terms of divided powers monomials is in echelon form with $\pm 1$’s as pivots. \hfill \Box
Corollary 4.15 Suppose $p = \sum_i \alpha_i T_i$ is a linear combination of row-standard tableaux such that $p = \sum_j \beta_j [S_j]$ where the $S_j$ are distinct straight tableaux and where all tableaux have the same row-convex shape $D$. The smallest modified column word of a tableaux in the $S_j$'s is weakly larger (lexicographically) than the smallest modified column word appearing in the $T'_i$.

Proof. Let $c_l$ be the column of $F(D)$ containing $l$. Suppose that $w_{T_i} \leq w_{T_i}$ for all $i$ and suppose $w_{S_{j_0}} < w_{S_j}$ for all $j \neq j_0$—recall by Theorem 4.6 that distinct straight tableaux have distinct modified column words. We want to show $w_{T_i} \leq w_{S_{j_0}}$. Now because straight tableaux have distinct modified column words $\prod_i (w_{S_{j_0}} | c_l)$ is the smallest monomial occurring in $p$. That means that it must appear in $\sum_i \alpha_i T_i$ if that expression is expanded out to a polynomial in $\text{Super}([L | P])$. But if $w_{T_i}$ is always larger than $w_{S_{j_0}}$ then no monomial as small as $\prod_i (w_{S_{j_0}} | c_l)$ can appear in $\sum_i \alpha_i T_i$. \qed

The next section shows that any $\sum_i \alpha_i [T_i]$ can be rewritten in the above fashion.

5 A straightening algorithm

We produce an explicit two-rowed straightening law for reducing any tableaux to a linear combination of straight tableau. This algorithm, straighten-tableau shown in Figure 2, starts with a tableau $T$ and returns a formal linear combination $\sum_i \alpha_i S_i$ of straight tableau with integer coefficients such that $[T] = \sum_i \alpha_i [S_i]$. In each step, the algorithm looks for a pair of rows containing a flippable inversion. If these exist, it applies the sub-algorithm row-straighten in Figure 3 to “straighten” these two rows via the Grosshans-Rota-Stein syzygies of Definition 5.1.

We provide an example of the straightening law below.

Example 5.1 Let $L = L^- = \{1, 2, \ldots, 8\}$. In each step we shall look for a non–straight tableaux $T$ and locate two rows (say $r_1$ above $r_2$) in $T$ containing a flippable inversion. In this example we will mark by a $\star$ every cell in row $r_1$ weakly right of the left most flippable inversion in those rows and every cell in row $r_2$ that is weakly left of this flippable inversion and weakly right of a cutoff column $c_1$. The cutoff $c_1$ indexes the leftmost column of row $r_2$ such that either $T_{r_1,c_1-1}$ does not exist or $T_{r_1,c_1-1} < T_{r_2,c_1}$. In this example, $c_1$ happens to always index the leftmost column in row $r_2$. We mark the remaining elements in row $r_1$ by $\bullet$'s.

The Grosshans-Rota-Stein syzygies (proved for the commutative case $L = L^-$ in [DRS76]) says that anti-symmetrizing all the $\star$’d elements in $T$, is the same (up to sign) as collecting all the $\bullet$’d elements into the row $r_1$, replacing those removed from row $r_2$ with these $\bullet$’d elements, and anti-symmetrizing the $\bullet$’d elements. We shall repeatedly apply this identity.
Input: A row-convex tableau $T$.
Output: $\sum_i \alpha_i S_i$ such that $|T| = \sum_i \alpha_i |S_i|$ where each $S_i$ is a straight tableau and $\alpha_i \in \mathbb{Z}$.

If $T$ is straight then output $T$.
else there exists a flippable inversion in some rows $i, j$.
Let $\sum_k \beta_k \cdot \frac{\upsilon_k}{\upsilon_k}$ be the output of row-straighten($\ldots T_i \ldots$);
Let $N_k$ be $\frac{\text{(\# pos. letters in } \upsilon_k + \text{\# pos. letters in } T_j) \cdot \text{(\# pos letters in } T_{i+1}\ldots T_{j-1})}{\upsilon_k}$.

Output $\sum_k (-1)^{N_k} \beta_k \cdot \text{straighten-tableau} \left( \begin{array}{c} \ldots T_i \ldots \\ \ldots T_{i-1} \ldots \\ \ldots \upsilon_k \ldots \\ \ldots T_{i+1} \ldots \\ \ldots T_{j-1} \ldots \\ \ldots \upsilon_k \ldots \\ \ldots T_{j+1} \ldots \end{array} \right)$.

Figure 2: Algorithm straighten-tableau. $T_i$, $T_j$, etc. are the $i$th, $j$th, etc. rows of the tableau $T$.

So, observing that the entries 4 and 2 form a flippable inversion, we first have,

$$
\begin{bmatrix}
1 & 3 & 4^* & 5^* \\
3 & 8 & 2^* & & \\
2 & 5 & & & \\
\end{bmatrix}
\begin{bmatrix}
1 & 3 & 5^* \\
3 & 8 & 4^* \\
\end{bmatrix}
= 
\begin{bmatrix}
1 & 3 & 5^* \\
3 & 8 & 4^* \\
\end{bmatrix} - 
\begin{bmatrix}
1 & 3 & 5^* \\
3 & 8 & 4^* \\
\end{bmatrix}.
$$

But the cells in column 3 and rows 2 and 3 of the first tableau on the right hand side now contain a flippable inversion. We straighten as follows,

$$
\begin{bmatrix}
1^* & 3 & 2^* & 5 \\
3 & 8 & 4^* & & \\
\end{bmatrix}
\begin{bmatrix}
1 & 3 & 5^* \\
3 & 8 & 4^* \\
2 & 5 & & & \\
\end{bmatrix}
= 
\begin{bmatrix}
1 & 3 & 5^* \\
3 & 8 & 4^* \\
2 & 5 & & & \\
\end{bmatrix} - 
\begin{bmatrix}
1 & 3 & 5^* \\
3 & 8 & 4^* \\
2 & 5 & & & \\
\end{bmatrix} + 
\begin{bmatrix}
1 & 3 & 5^* \\
3 & 8 & 4^* \\
2 & 5 & & & \\
\end{bmatrix} - 
\begin{bmatrix}
1 & 3 & 5^* \\
3 & 8 & 4^* \\
2 & 5 & & & \\
\end{bmatrix}.
$$

Now the first two tableaux above are straight, but the last two are not. We straighten the next to last tableau by,

$$
\begin{bmatrix}
1 & 3 & 4 & 7^* \\
3 & 8 & 4^* & & \\
\end{bmatrix}
\begin{bmatrix}
1 & 3 & 5^* \\
3 & 8 & 4^* \\
2 & 5 & & & \\
\end{bmatrix}
= 
\begin{bmatrix}
1 & 3 & 5^* \\
3 & 8 & 4^* \\
2 & 5 & & & \\
\end{bmatrix} - 
\begin{bmatrix}
1 & 3 & 5^* \\
3 & 8 & 4^* \\
2 & 5 & & & \\
\end{bmatrix} + 
\begin{bmatrix}
1 & 3 & 5^* \\
3 & 8 & 4^* \\
2 & 5 & & & \\
\end{bmatrix} + 
\begin{bmatrix}
1 & 3 & 5^* \\
3 & 8 & 4^* \\
2 & 5 & & & \\
\end{bmatrix}.
$$

and the last tableau by

$$
\begin{bmatrix}
1 & 3 & 4 & 7^* \\
3 & 8 & 4^* & & \\
\end{bmatrix}
\begin{bmatrix}
1 & 3 & 5^* \\
3 & 8 & 4^* \\
2 & 5 & & & \\
\end{bmatrix}
= 
\begin{bmatrix}
1 & 3 & 5^* \\
3 & 8 & 4^* \\
2 & 5 & & & \\
\end{bmatrix} - 
\begin{bmatrix}
1 & 3 & 5^* \\
3 & 8 & 4^* \\
2 & 5 & & & \\
\end{bmatrix} + 
\begin{bmatrix}
1 & 3 & 5^* \\
3 & 8 & 4^* \\
2 & 5 & & & \\
\end{bmatrix}.
$$

so
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Input: A two-rowed row-convex tableau $T = \begin{array}{c} v_{m_1} v_{m_1+1} \cdots v_{\lambda_1} \\ w_{m_2} w_{m_2+1} \cdots w_{\lambda_2} \end{array}$ which is row-standard but not straight.

Output: $\sum_{\kappa} \alpha_{\kappa} \cdot T_{\kappa}$ such that

Claim 1: $[T] = \sum_{\kappa} \alpha_{\kappa} [T_{\kappa}]$ where $\alpha_{\kappa} \in \mathbb{Z}$ and

Claim 2: the column word of $\cdots v_{c} \cdots$ is lexicographically larger than the column word of $T$.

Let $c_2$ be the index of the column containing the leftmost flippable inversion.

Let $c_1$ be the smallest column such that $c_1 \geq m_2$ and either $v_{c_1-1} < w_{c_1}$ or $c_1 - 1 < m_1$ (i.e. $v_{c_1-1}$ does not exist.)

Let $c_3$ be the rightmost column such that $w_{c_2} = w_{c_3}$.

if $c_1 < c_2$ then

Let $\sum_{i \in I} \beta_i T_i = Syz_{c_2, c_2+1, \cdots, \lambda_1; c_1, c_1+1, \cdots, c_3}(T)$.

$Expansion \leftarrow 0$.

for $i \in I$

if $w_{T_i} > w_T$ then $Expansion \leftarrow Expansion + \beta_i T_i$.

else $Expansion \leftarrow Expansion + \beta_i \cdot row-straighten(T_i)$.

Output $Expansion$.

else Comment: $c_1 = c_2$.

Let $c_0$ be the leftmost column such that $v_{c_0} \rightarrow w_{c_2}$.

Comment: $0 < c_2 \Rightarrow v_{c_0} = w_{c_2}$.

Let $\sum_{i \in I} \beta_i T_i = Syz_{c_0, c_0+1, \cdots, \lambda_1; c_1, c_1+1, \cdots, c_3}(T)$.

$Expansion \leftarrow 0$.

for $i \in I$

if $w_{T_i} > w_T$ then $Expansion \leftarrow Expansion + \beta_i T_i$.

else $Expansion \leftarrow Expansion + \beta_i \cdot row-straighten(T_i)$.

Output $Expansion$.

Figure 3: Algorithm row-straighten. If $L = L^-$, then we will always have $w_T > w_T$ so the algorithm will never recurse and instead could have directly output the expressions $Syz(T)$. The expression $Syz(T)$ is defined in Definition 5.1.
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The first step in verifying Algorithm \texttt{straighten-tableau} is to prove that when \( T \) is replaced with \( \sum_i \beta_i T_i \) by Algorithm \texttt{row-straighten} we have \( [T] = \sum_i \beta_i [T_i] \). The second step involves showing that each \( T_i \) is somehow closer to being straight than was \( T \). The first of these facts is an immediate consequence of the correctness of Algorithm \texttt{row-straighten}. This will come down to verifying the identities used in the preceding example. The second will follow from the correctness of \texttt{row-straighten} and the fact, proved in Proposition \ref{prop:row-straighten}, that given a tableau \( T \) and another tableau \( T' \) differing only in two rows \( i, j \), then the column word of the two-rowed subtableaux consisting of rows \( i, j \) of \( T \) is less than the corresponding column word determined by \( T' \) iff \( c_T < c_{T'} \).

The proof of Algorithm \texttt{straighten-tableau} thus depends solely on the correctness of Algorithm \texttt{row-straighten}. We will prove both claimed properties of Algorithm \texttt{row-straighten} for each of the two cases appearing in the algorithm. First we will produce the “determinantal” identities that will be used in Algorithm \texttt{row-straighten}.

Define a \textit{shuffle} of a word \( w = w_1, \ldots, w_n \) into parts of length \( k, k' = n - k \) to be an ordered pair of words \( w' \) and \( w'' \) of \( w \) having lengths \( k \) and \( k' \) respectively, such that \( w' \) and \( w'' \) can be found as a pair of disjoint subwords of \( w \). Neither \( w' \) nor \( w'' \) need be contiguous as a subword of \( w \). When \( w = 1, \ldots, n \) a shuffle amounts to a permutation \( \sigma \) of the index set \( 1, \ldots, n \) such that \( \sigma_1 < \cdots < \sigma_k \) and \( \sigma_{k+1} < \cdots < \sigma_{k+k'} \). Generalizing the length of a permutation we define the \textit{shuffle signature} \( \text{sign}(\omega_1, \ldots, \omega_k) \), of a word \( w \) to be the number of pairs \( 1 \leq i < j \leq k \) such that \( \omega_i > \omega_j \) and \(|\omega_i| = |\omega_j| = 1\).

\textbf{Definition 5.1} \textit{Let } \( i, j, k, l \) \textit{be nonnegative integers. Fix a two-rowed row-convex shape } \( D \) \textit{by specifying the starting and ending columns, } \( 1 \) \textit{through } \( i+j+l \) \textit{and } \( m \) \textit{through } \( m+l+k-1 \) \textit{of the top and bottom rows respectively. For convenience we have let the leftmost column index of the top row be } \( 1 \), \textit{but the column indices can of course be shifted left or right by any integer. With the above convention, we could have } \( m \leq 0 \), \textit{this produces a skew shape.}

Let \( T = \ldots \) \textit{be a two-rowed row-convex tableau of shape } \( D \). \textit{Fix two sequences of column indices, } \( c_1 < \cdots < c_l \) \textit{starting at column } \( i+l \) \textit{and ending at } \( i+l+j \) \textit{and } \( m \leq c'_1 < \cdots < c'_l \leq k+l \) \textit{starting at column } \( c_1 \) \textit{and ending at } \( c_1 + l - 1 \).

Define \( \text{Syz}_{c_1, \ldots, c_l; c'_1, \ldots, c'_l}(T) \) \textit{to be the formal linear combination}

\[
\sum_{\text{all nontrivial shuffles} \ y_\sigma(i_1) \cdots y_\sigma(i_l) \ y_\sigma(i_1+1) \cdots y_\sigma(i+j)} \frac{-\alpha_\sigma}{\alpha_e} \cdot T'_\sigma + \sum_{\text{all shuffles} \ x_{\tau(1)} \cdots x_{\tau(i+1)} \ x_{\tau(i+1)} \cdots x_{\tau(i+l)}} \frac{\beta_\tau}{\alpha_e} \cdot T''_\tau,
\] (2)
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where \( T'_\sigma, T''_\sigma, \alpha_\sigma \) and \( \beta_\sigma \) are defined as follows:

First, define words

\[
\begin{align*}
x' = v_1, \ldots, v_{c_i-1}, & \quad u_1, \ldots, u_{c'_i} = w_1', \ldots, w_{c'_i}, v_{c_i}, \ldots, v_c, \\
x'' = v_{r(i)}, \ldots, v_{r(i)}, & \quad u'_\sigma = u_{\sigma(1)}, \ldots, u_{\sigma(t)}, \\
\omega = v_{r(i+1)}, \ldots, v_{r(i+t)}, & \quad \omega' = u_{\sigma(l+1)}, \ldots, u_{\sigma(l+j)}, \quad \omega'' = w_{c'_i+1}, \ldots, w_{k+l-1} \\
\text{and} & \quad z_1 = w_{m_i}, \ldots, w_{c'_i-1}.
\end{align*}
\]

Define the tableau \( T'_\sigma \) to be the tableau obtained by sorting the rows of

\[
\begin{array}{cccc}
\cdots & x & \. & \\
\cdots & z_1 & \. & \\
\cdots & u_\sigma & \. & \\
\cdots & x'' & \. & \\
\cdots & z_2 & \\
\end{array}
\]

and \( T''_\sigma \) to be the result of sorting the rows of

\[
\begin{array}{cccc}
\cdots & u & \. & \\
\cdots & x' & \\
\cdots & z & \\
\end{array}
\]

We define

\[
\alpha_\sigma = (-1)^{N_1(\sigma)} \cdot \frac{c(z, u'')!}{c(x)! \cdot c(u''!)} \cdot \frac{c(z_1, u'_\sigma, z_2)!}{c(z_1) \cdot c(z_2)! \cdot c(u'!)!}
\]

with \( N_1(\sigma) = |z_1| \cdot |u'_\alpha| + |z| \cdot (i + j + l) + |u'_\sigma| \cdot (i + j + l) + \text{sign}(u''_\sigma, u'_\sigma) + m(z, u'') + m(z_1, u'_\sigma, z_2) \) where \( m(\omega) = \text{sign}(\omega) \cdot \left( \# \text{ neg. letters in } \omega \right) \)

and we define

\[
\beta_\sigma = (-1)^l \cdot (-1)^{N_2(\sigma)} \cdot \frac{c(u, x')!}{c(u) ! \cdot c(x'!)} \cdot \frac{c(x'', z)!}{c(x'')! \cdot c(z)!}
\]

with \( N_2(\tau) = |z| \cdot (i + j + l) + |x''| \cdot (i + j + l) - |u| \) \sign(x'', x'') + \( m(u, x') + m(x'', z) \).

Call the cells in columns \( c_1, \ldots, c_j \) of the top row and the cells in columns \( c'_1, \ldots, c'_i \) of the bottom row of \( T \) marked cells. If no positive letter appears in both marked and unmarked cells in the top row of \( T \) and no positive letter appears in both marked and unmarked cells in the bottom row of \( T \) then \( \alpha_e = \pm 1 \) so the identity holds over \( \mathbb{Z} \).

**Proposition 5.2** Let \( T \) be a tableau of two-rowed, row-convex shape \( D \) whose top row contains its bottom row. Without loss of generality assume the top row has leftmost column index 1. If column indices \( c_1, \ldots, c_j \) and \( c'_1, \ldots, c'_i \) are chosen as in Definition 5.1 then

\[
[T] = \sum_S a_S \cdot [S]
\]

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where \( \sum S \cdot \mathbf{S} = \text{Sy}_{x_1 \cdots \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot 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\cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot 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Let \( D \) be the row-convex shape \((\lambda_1, \lambda_2)/(m_1, m_2)\) with \( m_1 < m_2 \).

Given a non-straight, row standard, two rowed, row-convex tableau, \( T \), of shape \( D \) Algorithm \texttt{row-straighten} produces a formal linear combination of tableaux each of which has a lexicographically larger column word than \( c_T \).

\textbf{Proof.} The proof is by induction on \( c_T \).

Suppose that \( T \) is the tableau

\[
\begin{array}{cccccccc}
\text{Column} & m_1 & m_1+1 & m_2 & m_2+1 & \lambda_2 & \lambda_1 \\
\text{v} & v_{m_1} & v_{m_1+1} & \ldots & v_{\lambda_1} \\
\text{w} & w_{m_2} & w_{m_2+1} & \ldots & w_{\lambda_2}
\end{array}
\]

We set up a straightening syzygy that expresses \( T \) in terms of tableau \( T_i \) such that \( c_T \) is always lexicographically smaller than \( c_T \). We first describe the structure of \( T \) with respect to its leftmost flippable inversion. Let \( q \) be minimal such that if column \( q+1 \) were to contain an inversion, then that inversion would be flippable. Thus \( q = \min_{m_2 \leq i \leq \lambda_2} i - 1 \). The presence of a flippable inversion guarantees that \( q \) exists. The value of \( c_1 \) in Algorithm \texttt{row-straighten} is \( q+1 \). The first column strictly right of \( c_1 - 1 \) that actually has an inversion has index \( c_2 = \min_{\lambda_2 \leq j \leq \lambda_2} j \). This inversion must be flippable, thus \( c_2 \) indexes the column of the leftmost flippable inversion in the tableau.

Two cases arise in the algorithm namely \( c_1 < c_2 \) and \( c_1 = c_2 \). The pictures in Figures 4 and 5 outline these situations. The symbol, “•”, indicates a cell in the diagram. An arrow from one cell to another indicates that the contents of the first cell are larger than the contents of the second. The decoration of an arrow by “−” (respectively “+”) indicates that the contents of the cells at either end are may be equal if these contents are negatively (respectively positively) signed. Sequences of cells surrounded by parentheses or braces may be omitted.

If no positive letter appears multiple times in \( T \), then Cases I and II can be treated simultaneously. We begin with Case I.
Apply Corollary 5.3 to write

\[
\begin{bmatrix}
  v_{m_1} & \cdots & \cdots & v_{c_2} & \cdots & v_{\lambda_1} \\
  w_{m_2} & \cdots & w_{c_1} & \cdots & w_{c_2} & \cdots & w_{c_3} \\
\end{bmatrix} = S_{yz_{c_2, \ldots, \lambda_2; c_1, \ldots, c_3}} = B + A
\]

where \( B \) (respectively \( A \)) is the first (respectively second) summand in the \( S_{yz_{c_2, \ldots, \lambda_2; c_1, \ldots, c_3}}(T) \) as defined in expression 3. The over/underlines are visual aids which indicate the “marked” entries used to define \( S_{yz}(T) \).

It suffices to show that each tableau appearing in \( A \) or \( B \) has lexically larger column word than \( c_T \).

Suppose \( T' \) appears in \( A \). We can write

\[
T' = \begin{bmatrix}
x_{m_1} & \cdots & x_{t-1} & w_{c_1} & \cdots & w_{c_3} & v_{c_2} & \cdots & \cdots & v_{\lambda_1} \\
w_{m_2} & \cdots & w_{c_1-1} & y_1 & \cdots & y_{c_2-1} & w_{c_3+1} & \cdots & w_{c_2} & \cdots & w_{c_3} \end{bmatrix}
\]

where \( x_{m_1} \ldots x_{t-1}; y_1 \ldots y_{c_2-1} \) is a shuffle of \( v_{m_1} \ldots v_r \), where \( t = c_2 + c_1 - c_3 - 1 \), and where the boxed entries must be sorted in order to give a row-standard tableau. Denote the entries in the bottom row by \( z_{m_2}, \ldots, z_{\lambda_2} \) and the entries in the top row by \( w_{m_1}, \ldots, w_{\lambda_1} \).

To check that only the boxed entries need to be sorted in the top row, it suffices to observe that the \( x_i \) are taken from \( v_{m_1}, \ldots, v_r \) and that \( v_{c_2} \succ w_{c_2} = w_{c_3} \).

Checking the bottom row, it suffices to note that \( w_{c_3+1} \succ v_{c_2-1} \).

Now let \( k + 1 \) index the leftmost column in the top row in which \( T' \) differs from \( T \). In fact, \( k = \min_{m_1 \leq i \leq t} i = 1 \) which follows from being in Case I: Suppose \( w_i = v_i \) for \( m_1 \leq i < t = c_1 - 1 - c_3 + c_2 \). Since by construction \( v_{t-1} \prec v_{c_1-1} \prec w_{c_1} \), we have that \( x_i = v_i \) for all \( i \) as above and thus the boxed elements in the top row are already in order. But then \( w_i = w_{c_1} \neq v_i \), since by Case I \( v_{c_1-1} \prec w_{c_1} \). So \( k < t \).

Now we examine the column words. Our construction shows \( w \succ v \), so by the preceding paragraph \( w \succ v \). So if \( k + 1 < m_2 \) we conclude directly that \( c_T \) is lexically larger than \( c_T \).

Suppose that \( k + 1 \geq m_2 \). We show that \( v_{k+1} \leq y_1 \). Suppose to the contrary that \( v_{k+1} > y_1 \). Since \( y_1 \) comes from \( v_{m_1}, \ldots, v_r \) this says that \( y_1 = v_j \) for some \( j \leq k \) and \( y_1 \neq v_{j'} \) for \( j' > k \). Now the upper row of \( T' \) still contains \( v_1 \ldots v_{k} \) even though a \( y_1 \) has been removed to the bottom row. But this implies that \( y_1 \) also appears in \( w_{c_1}, \ldots, w_{c_2} \) which is impossible since \( w_{c_1} \succ v_{c_1-1} \succ v_{k+1} \succ y_1 \).

To check that \( k > 1 \leq t < c_1 \) the diagram for Case I shows that \( w_{k+1} < v_{k+1} \), hence \( w_{k+1} < y_1 \). So, after sorting, we find that \( z_{m_2} = w_{m_2}, z_{m_2+1} = w_{m_2+1}, \ldots, z_{k+1} = w_{k+1} \). So in tableaux \( T \) and \( T' \), the columns \( m_1, \ldots, k \) agree as does the bottom entry of column \( k + 1 \). But the top entry in column \( k + 1 \) is larger in \( T' \) than in \( T \). Hence \( c_T \) is lexically larger than \( c_T \).

At last we deal with tableaux appearing in \( B \) in equation 5. Recall that tableaux in \( B \) arise from nontrivially shuffling the over/underlined entries and then resorting the rows. Let \( \underline{w} = w_{c_1} \cdots w_{c_3} v_{c_2} \cdots v_{\lambda_1} \). Let \( \underline{w}', \underline{w}'' \) be a shuffle of \( \underline{w} \) into two parts of size \( \lambda_1 - c_2 + 1 \) and \( c_3 - c_1 + 1 \) respectively. Since \( w_{c_1} \succ v_{c_1-1} \),
such a tableau will look like
\[
T' = \begin{bmatrix}
v_{m_1} & \ldots & v_{c_{1} - 1} & v_{c_1} & \ldots & v_r & \ldots & u' & \ldots \\
w_{m_2} & \ldots & w_{c_{1} - 1} & \ldots & w_{c_3 + 1} & \ldots & w_{\lambda_1}
\end{bmatrix},
\]
where, as before, the boxed elements must be sorted so that \( T' \) will be row standard. Again denote the top and bottom rows of \( T' \) by \( \omega_{m_1}, \ldots, \omega_{\lambda_1} \) and \( z_{m_2}, \ldots, z_{\lambda_2} \).

Now let \( k + 1 \) be the leftmost column in which the bottom rows of \( T \) and \( T' \) disagree. We claim \( c_1 \leq k + 1 \leq c_3 \) and that \( z_{k+1} > w_{k+1} \). By construction \( u'_i \neq w_{c_1} \cdots w_{c_3} \). Write \( u'_i = u'_{i+1} \cdots u'_{c_3} \). Let \( j \) be minimal such that \( u'_j \neq w_j \). Because \( v_{c_2} \nRightarrow w_{c_2} = w_{c_3} \), this implies that \( u'_j > w_j \). But since also \( w_{c_3+1} > w_{c_3} \), we find \( z_j > w_j \), and \( z_i = w_i \) for all \( i < j \), so \( k + 1 = j \).

Subcase 1. Suppose \( k < c_2 - 1 \). Any letter appearing in the multiset difference \( u'' = \{ v_{c_2}, \ldots, v_{\lambda_1} \} \) is (\( \nRightarrow \))-greater than \( w_{k+1} \). But since \( k < c_2 - 1 \), the picture of case I shows that \( v_{k+1} \nRightarrow w_{k+1} \), this means that on resorting the boxed elements of the top row, every element in \( u'' \) stays in column \( k + 2 \) or higher. Hence columns \( m_1, \ldots, k \) agree in \( T \) and \( T' \). But the bottom of column \( k + 1 \) is larger in \( T' \) than \( T \). Thus \( c_{T'} \) is lexically larger than \( c_T \).

The above argument also generates the fact (unused in this proof, but see the comment after Corollary 5.7) that, \( T \) and \( T' \) agree in the top element of column \( k + 1 \).

Subcase 2: suppose that \( k \geq c_2 - 1 \). This says that the bottom rows of \( T, T' \) agree at least through column \( c_2 - 1 \). Since \( v_{c_2 - 1} \nRightarrow w_{c_2} \), we have immediately that the top rows of \( T, T' \) agree through column \( c_2 - 1 \). Now either the bottom of column \( c_2 \) changes (hence increases) so \( c_{T'} \) is lexically larger than \( c_T \) and we are done or the the number of positive letters in the bottom row that equal \( w_{c_2} \) decreases. In the latter case, not only do the tableaux \( T, T' \) agree up to column \( c_2 - 1 \) but \( T' \) still has a flippable inversion in column \( c_2 \) since the entry in the top of that column now equals the positive letter \( w_{c_2} \) that remains at the bottom.

We repeat the straightening law on \( T'' \), producing some tableaux with lexicographically larger modified column words and some tableaux that are unchanged in columns smaller than \( c_2 \) and unchanged at the bottom of column \( c_2 \) but which have fewer copies of \( w_{c_2} \) in their bottom rows. Eventually, we must run out of positive letters equal to \( w_{c_2} \) in the bottom row and so eventually the modified column word increases.

We now treat Case II. Here \( c_1 = c_2 \).

We replace \( [T] \) with \( Syz_{A_1; A_2; \ldots; A_{c_1}; \ldots; c_3} \) where \( c_0 \) is minimal such that \( v_{c_0} \nRightarrow w_{c_2} \). As before, Corollary 5.7 lets us write
\[
\begin{bmatrix}
v_{m_1} & \ldots & v_{c_0} & \ldots & v_{c_2} & \ldots & v_{\lambda} \\
w_{m_2} & \ldots & w_{c_1} & \ldots & w_{c_2} & \ldots & w_{\lambda_1}
\end{bmatrix} = B + A
\]
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Figure 5: Case II: \( c_1 = c_2 \): Relations between entries in a two-row tableau being straightened by Algorithm row-straighten.

All entries in the bottom row from \( c_2 \) through \( c_3 \) are equal but distinct from any entry in column \( c_3 + 1 \) of that row. If \( c_0 < c_2 \), then the entries in the top row that equal the bottom row entry in column \( c_2 \) must start at \( c_0 \) and extend at least as far as \( c_2 - 1 \).

Suppose \( T' \) appears in \( A \). To this purpose, let \( w = v_{m_1} \ldots v_r \) be the word being shuffled. It is easily verified that

\[
T' = \begin{vmatrix}
    m_1 & \ldots & x_{t-1} & v_{c_1} & x_{c_2} & \ldots & v_{c_3} & \ldots & w_{\lambda_1} \\
    w_{m_2} & \ldots & w_{c_2-1} & y_1 & \ldots & y_{c_3-1} & w_{c_3+1} & \ldots & w_{\lambda_2}
\end{vmatrix},
\]

where \( t = c_0 - c_3 + c_2 - 1 \), \( x_{m_1} \ldots x_{t-1} \), \( y_1 \ldots y_{c_3-1} \) is a shuffle of \( v_{m_1} \ldots v_{c_3} \) and where the boxed entries must be sorted in order to get a row-standard tableau. Maintain the notation \( T' = \ldots \Xi \ldots \).

Since \( t < c_0 \) we have \( v_{c_0} \neq v_1 \) and thus \( k = \min_{m_1 \leq i \leq t} i - 1 \) is well defined. As in Case I, if \( k + 1 < m_2 \) we conclude directly that \( c_{T'} \) is lexically larger than \( c_T \).

Suppose that \( k + 1 \geq m_2 \). We show that \( v_{k+1} \leq y_1 \). Suppose to the contrary that \( v_{k+1} > y_1 \). Since \( y_1 \) comes from \( v_{m_1} \ldots v_{c_3-1} \) this says that \( y_1 = v_j \) for some \( j \leq k \) and \( y_1 \neq v_{j'} \) for \( j' > k \). But this says that if the letter \( y_1 \) occurs in the \( y_1 \ldots y_{c_3-1} \) part of the shuffle, then the \( x_{m_1} \ldots x_k \) part cannot start with \( v_{m_1} \ldots v_k \)-contradiction.

Thus since the diagram for Case II shows that \( w_{k+1} < v_{k+1} \), we find \( w_{k+1} < y_1 \). So, after sorting, we discover that \( z_{m_2} = w_{m_2} \), \( z_{m_2+1} = w_{m_2+1} \), \( \ldots \), \( z_{k+1} = w_{k+1} \). So in tableaux \( T, T' \), the columns \( m_1, \ldots, k \) agree as does the bottom entry of column \( k + 1 \). But the top entry in column \( k + 1 \) is larger in \( T' \) than in \( T \). Hence \( c_{T'} \) is lexically larger than \( c_T \).

Suppose now that \( T' \) appears in \( B \) in equation 3. Define \( c_4 = \min_{w_{c_2} < v_i} i \).
Since \( w_{c_2} \rightarrow v_{c_2-1} = v_{c_0} \), we have

\[
T' = \begin{vmatrix}
    v_{m_1} & \ldots & v_{c_2-1} & w_{c_2} & \ldots & = w_{c_2} & \ldots & w_{\lambda_1} \\
    w_{m_2} & \ldots & w_{c_2-1} & w_{c_2+s} & \ldots & w_{c_3} & \ldots & W' \ldots & w_{c_3+1} & \ldots & w_{\lambda_2}
\end{vmatrix},
\]

where \( W', W'' \) is a shuffle of \( v_{c_4} \ldots v_{\lambda_1} \) into parts of size \( \lambda_1 - c_4 - s + 1 \) and \( 1 \leq s \leq c_3 - c_2 + 1 \) respectively and, as before, the boxed elements must be sorted so that \( T' \) will be row standard.
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The top rows of $T, T'$ agree through column $c_2 - 1$. Again either the bottom of column $c_2$ increases and we are done or the the number of positive letters in the bottom row that equal $w_{c_2}$ decreases. So iterating the straightening law on $T'$ eventually increases the modified column word.

The algorithm row-straighten specializes to the classical straightening for skew and partition shaped tableau when $m_1 \geq m_2$. The preceding result implies that the column word also increases in the skew case.

**Corollary 5.5** Proposition 5.4 also holds with $m_1 \geq m_2$.

**Proof.** We use what is often called the method of fake letters. Fill cells $m_2 - 1, \ldots, m_1 - 1$ in the the top row with new negative letters disjoint from and smaller than the letters in $L$, we will name these letters $f_{m_2 - 1}, \ldots, f_{m_1 - 1}$. Straighten this new tableau.

The tableaux appearing in expression $B$ in the preceding proof have the “fake” letters $f_{m_2 - 1}, \ldots, f_{m_1 - 1}$ in the same positions as does the original tableau $T$. If we apply the algebra homomorphism that sending $(f_{i|j})$ to $\delta_{i,j}$, $[T']$ is sent to 0 for all $T'$ in the expression $A$ in the preceding proof and the fake letters are erased from all other tableaux in the expression.

With these new tableaux, we have now established the correctness of Algorithm row-straighten.

**Theorem 5.6** The straight tableaux of shape $D$ form a $\mathbb{Z}$-basis for $S^D$ and Algorithm straighten-tableau expands any generator of $S^D$ in terms of this basis. Further, given a row-standard tableau $T$, the expansion of $[T]$ is in terms of tableaux with larger column words than $w_T$.

By Corollary 4.13, carefully analyzing the proof Proposition 5.4 we can extend the preceding result.

**Corollary 5.7** Algorithms row-straightening and straighten-tableau produce tableaux with weakly larger modified column words than that of the input tableau.

### 6 Flagged super-Schur modules

The flagged Schur modules $S^D_f$ have been the subject of considerable interest (see for instance [LS90, RS96, LM97]). We apply the preceding results to flagged super-Schur modules of row-convex shape. The fact that the initial terms in the straight basis are distinct allow the straight bases to descend to bases of the corresponding flagged module. I will start by formalizing the notion of a flagged super-Schur module.
Definition 6.1 Let $f$ be a weakly increasing sequence of letters in the alphabet $\mathcal{L}$. Regard this sequence as indexed by elements of $\mathcal{P}$. The flagged superSchur module $\mathcal{S}D^f(\mathcal{L})$ is the subquotient of $\text{Super}([\mathcal{L} | \mathcal{P}])$ equal to the image of the submodule $\mathcal{S}D^f(\mathcal{L})$ under the map $\phi_f$ which quotients $\text{Super}([\mathcal{L} | \mathcal{P}])$ by setting $(l|p) = 0$ whenever $l > f_p$.

A tableau $T$ is flagged if, in each column $i$, $T$ has no entry exceeding $f_i$.

If $T$ is row-standard (of any shape) and fails to be flagged, then $\phi_f([T]) = 0$ since each monomial in the expansion of any row in which the flagging condition is violated has some factor $(l|p)$ with $l > f_p$.

Classical results (see [Sta76]) tell us that if $D$ is a skew-tableau then a basis for $\mathcal{S}D^f$ is given by all $[T]$ such that $T$ is standard and flagged. This result carries over to flagged row-convex superSchur modules.

Theorem 6.2 Let $D$ be a row-convex shape. Fix a weakly increasing flag $f$. A basis for $\mathcal{S}D^f(\mathcal{L})$ is given by the elements $\phi_f([T])$ where $T$ runs over all flagged, straight tableaux of shape $D$ with entries chosen from $\mathcal{L}$.

Proof. It suffices to show that the basis elements are linearly independent, indeed that their initial terms under any diagonal term order are still distinct. This follows by Proposition 4.12, and the observation immediately following it since the tableaux $T$ are both straight and flagged.

This result has the following easy generalization. Let $f, g$ both be weakly increasing sequences of letters in $\mathcal{L}$ indexed by elements of $\mathcal{P}$ such that $f \leq g$ componentwise. Define the doubly flagged superSchur module $\mathcal{S}_{D}^{f, g}(\mathcal{L})$ to be the image of $\mathcal{S}D^f(\mathcal{L})$ under the map $\phi_{f, g}$ quotienting $\text{Super}([\mathcal{L} | \mathcal{P}])$ by the ideal generated by $\{(l|p) : l \notin f, \ldots, g\}$. Call a tableau $T$ doubly flagged with respect to $f, g$ if every entry in column $i$ is between $f_i$ and $g_i$. The same proof as above shows the following.

Theorem 6.3 A basis for $\mathcal{S}_{D}^{f, g}(\mathcal{L})$ is given by the elements $\phi_{f, g}([T])$ where $T$ runs over all doubly flagged, straight tableaux of shape $D$ with entries chosen from $\mathcal{L}$.

When $\mathcal{L} = \mathcal{L}^-$ and $D$ is skew, this results appears in [Sta76].

7 Groebner and SAGBI bases.

The basis theorems developed above have various ring-theoretic applications. For convenience, we state them in terms of commutative rings, i.e. we assume that the alphabet $\mathcal{L}$ consists entirely of negative letters. We can then take the subalgebra of the polynomial algebra $k[x_{i,j}]$ generated by all polynomials indexed by tableaux of some fixed shape.
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Definition 7.1 Let $D$ be a shape. Let $R^D$ be the subalgebra generated by all $[T]$ for all tableaux $T$ of shape $D$. Allowing that $[T]$ has degree 1, this is a graded algebra generated by its degree 1 part.

These algebras turn out to be the homogeneous coordinate rings of certain configuration varieties (see [M98] for example) embedded in projective space by a combination of Plucker embeddings, Segre products and Veronese maps. Configuration varieties parameterize a tuple of subspaces of $n$-space subject to certain lower bounds on the dimensions in which they may intersect. The results in this section are stated for the rings $R^D$ but they hold equally for the subquotient rings generated by the (doubly) flagged modules $S_{\Delta_2}^D(L)$.

Below we produce a Groebner basis for $R^D$. First we establish some notation.

Definition 7.2 Let $D$ be a shape. Let $T, T'$ be tableaux of shape $D$. Define $T \circ T'$ to be the tableau (no longer of shape $D$) formed by alternating rows of $T$ and $T'$, starting with the top row of $T$, then the top row of $T'$ etc.

Before stating the theorem we recall our convention that the initial term of a polynomial is the smallest term in the polynomial and define the degree of a Groebner basis be the highest degree of a polynomial appearing in Groebner basis.

Theorem 7.3 Let $D$ be an $n$-rowed, row-convex shape.

Consider the polynomial ring whose variables consist of all straight tableaux of shape $D$ on $L$. We define a graded term order on this ring as follows. We say $T' < T''$ when $c_T < c_{T''}$ and we define $\prod_{i=1}^k T'_i < \prod_{i=1}^k T''_i$ where $r < s$ implies $T'_r < T''_r$ and $T''_r < T''_s$ when $T'_1(T'_1), \ldots, T'_k(T'_k), \ldots, T''_1(T''_1), \ldots$ is smaller in lexicographic order than $T'_1(T''_1), \ldots, T'_k(T''_k), \ldots, T''_1(T''_1), \ldots$ where $T_i(T)$ is the $i$th column of $T$ read from bottom to top.

If $I$ is the kernel of the ring map sending a straight tableau $T$ to $[T]$, then $I$ has a degree 2 Groebner basis consisting of all polynomials

$$ T' \cdot T'' - \sum_{\kappa} \beta_{\kappa} \cdot \begin{array}{c} \cdots \cdot T'_1 \cdots \\ \cdots \cdots \\ \cdots \cdot T'_{i-1} \cdots \\ \cdots \cdot T'_i \cdots \\ \cdots \cdots \\ \cdots \cdot T'_{i+1} \cdots \\ \cdots \cdots \\ \cdots \cdot T'_n \cdots \\ \cdots \\ \cdots \cdot T''_1 \cdots \\ \cdots \cdots \\ \cdots \cdot T''_{j-1} \cdots \\ \cdots \cdots \\ \cdots \cdot T''_j \cdots \\ \cdots \cdots \\ \cdots \cdot T''_n \cdots \\ \cdots \cdots \\ \cdots \cdots \end{array} = \sum_{\kappa} \beta_{\kappa} \cdot \begin{array}{c} \cdots \cdot w_1 \cdots \\ \cdots \cdots \\ \cdots \cdot w_2 \cdots \\ \cdots \cdots \\ \cdots \cdot w_k \cdots \\ \cdots \cdots \\ \cdots \cdots \end{array} = \text{Syz}_{a_1, \ldots, a_r, b_1, \ldots, b_s} \left( \begin{array}{c} \cdots \cdot T_i \cdots \\ \cdots \cdots \\ \cdots \cdot T_j \cdots \\ \cdots \cdots \end{array} \right) ; $
A straightening algorithm for row-convex tableaux.

if \( \cdots \cdot t_i \cdots \cdot t_j \cdots \) is not skew, \( a \) and \( b \) are allowed to range over all sequences of column indices such that \( a \) ends in the last column of \( t_i' \) and if \( \cdots \cdot t_i \cdots \cdot t_j \cdots \) is skew, \( a \) and \( b \) range over all sequences of column indices such that \( r + s \) is large than the number of columns in \( \cdots \cdot t_i \cdot \).  

Proof. Let \( T \leq T' \) be straight tableaux of shape \( D \). Corollary 5.2 verified that the polynomials \( \Pi \) are in \( I \). It suffices to show that if \( T := T' \circ T'' \) is not straight then there exists a relation in the Groebner basis \( \Pi \) whose initial term is \( T' \circ T'' \). Since \( T \) is not straight, there exists \( i \leq j \) such that if \( t_i \) is the \( i \)th row of \( T \) and \( t_j \) is the \( j \)th row of \( T'' \) then the tableau \( \cdots \cdot t_i \cdot \cdots \cdot t_j \cdot \) is not straight. But then Proposition 5.4 and Corollary 5.5 show that choosing \( a \) and \( b \) as in algorithm row-straighten gives \( T' \cdot T'' \) as the initial term of the polynomial \( \Pi \).  

The Groebner basis of Theorem 7.3 is not reduced, nor are the initial terms of its elements polynomials necessarily distinct. The initial terms can be made distinct by choosing \( a \) and \( b \) as in algorithm row-straighten. The above theorem does not require that \( \mathcal{L} = L^- \), although one requires the notion of a non-commutative Groebner basis for general \( \mathcal{L} \). Specifically when \( \mathcal{L} = L^- \), it is possible to restrict \( r + s \) to be one more than the maximum of the number of columns in \( T_i \) and the number of columns in \( T_j \) while simultaneously eliminating any other restrictions on the indices \( a \) and \( b \); the underlying straightening law is presented in Chapter III of [T97a]. In general, that straightening law fails \( \mathcal{L} \) contains positive letters.

Porism 7.4 The monomial \( T_1 \cdot T_2 \ldots T_k \) is standard with respect to the Groebner basis in Theorem 7.3 iff \( T_1 \circ T_2 \circ \ldots \circ T_k \) is straight.

The existence of a degree 2 Groebner basis for an algebra is known to imply that there is an (infinite) linear free resolution of the ground field over the algebra.

A sagbi (Subalgebra Analogue of a Groebner Basis for Ideals) basis, see [KaMa89] and [RoSw90], is a generating set for a subalgebra such that the initial terms of the subalgebra are contained in the algebra generated by the initial terms of the generating set.

Theorem 7.5 Let \( D \) be a row-convex shape. The straight basis elements of shape \( D \) form a sagbi basis for \( R^D \) with respect to any diagonal term order.

Proof. It suffices to show that if \( p \) is in \( R^D \) then its initial term is the product of the initial terms of some multiset of straight tableaux. Define \( D^k \) to be the shape formed by replacing each row of \( D \) with \( k \) copies of itself. So init \( (p) \) is init \( (p_k) \) where \( p_k \) is the component of \( p \) lying in \( S^{D^k} \) with \( k \) maximal such that \( p_k \neq 0 \). Since the initial terms of straight tableaux of fixed shape are distinct,
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init \((p) = [T]\) for some straight tableaux \(T\) of shape \(D^{\circ k}\). But we can write \(T = T_1 \circ \cdots \circ T_k\). Each \(T_i\) must be straight and init ([\(T_i\)]) = \(\prod_{i=1}^k \)init ([\(T_i\)])

**Corollary 7.6** Let \(D\) be a row-convex shape. The row-standard tableaux of shape \(D\) form a SAGBI basis for \(R^D\) with respect to any diagonal term order.

By the usual results these SAGBI bases give algorithms for determining whether a polynomial in variables \(x_{i,j}\) belongs to \(R^D(L)\) or \((a\ for\ a\ posteriori)\ S^D(L)\) and, if so, writing it in terms of the generators \([T]\) where \(T\) is straight of shape \(D\). By results of \([Stu96]\), a SAGBI basis for an algebra allows that algebra to be deformed to an algebra generated by monomials. In \([T97a]\) and \([T99]\) this deformation is used to prove that the subalgebra generated by all tableaux of a fixed row-convex shape is Cohen-Macaulay.

**8 A branching rule and flagged corner-cell recurrence.**

Our final application concerns a branching rule for row-convex representations. The Schur and Weyl modules \(S^D(L^-)\) and \(S^D(L^+)\) are \(GL_n\) representations with \(GL_n\)-action induced by the algebra homomorphism \(g : Super([L | P]) \rightarrow Super([L | P])\) given by \(g((r|s)) = \sum g_{i,r}(i|s)\) where \(g \in GL_n\) equals \(g_{i,j}\). In order to handle sets \(L\) containing letters of both positive and negative sign, we will work with representations of the general linear Lie superalgebra, \(pl_{\mathcal{L}}\).

We express a \(pl_{\mathcal{L}}\)-representation, corresponding to a row-convex shape \(D\), in terms of \(pl_{\mathcal{L}\setminus\{a\}}\) representations (for some \(a \in \mathcal{L}\)) corresponding to subshapes of \(D\). The combinatorics for the case \(\mathcal{L} = L^+\) is identical to that of the branching rule in \([RS98]\) and new when \(\mathcal{L} = L^-\). We present a filtration that realizes this branching rule in a characteristic-free fashion. This provides the row-convex case of filtration conjectured in \([RS98]\) to exist for all comment deleted. It should be noted that the orientations of \([RS98]\) are at variance from those of \([RS95]\); we adhere to the orientation of the latter. Thus the term row-convex in \([RS98]\) should be read as “column-convex” in the context of both \([RS95]\) and the present paper. The branching rule presented below generalizes to the case of flagged super-Schur modules; branching rules for flagged Schur modules are not treated in \([RS98]\).

First we construct the general linear Lie superalgebras following Scheunert \([Sc73]\).

A free \(\mathbb{Z}\)-module \(F\) is signed when it has distinguished free submodules \(F_0\) and \(F_1\) whose direct sum is \(F\). Elements of \(F_0\) and \(F_1\) are called homogeneous and \(|x| = i\) for \(x \in F_i\).

A free signed \(\mathbb{Z}\)-module is a Lie superalgebra when it is endowed with a superbracket \(\[,\]\) satisfying the commutativity relation,

\([x, y] = -(-1)^{|x||y|}[y, x]\)
for homogeneous elements $x, y$ and the super-Jacobi identity
\[ (-1)^{|a||c|}[a, [b, c]] + (-1)^{|a||b|}[b, [c, a]] + (-1)^{|b||c|}[c, [a, b]] = 0 \]
for homogeneous elements $a, b, c$.

Following \[ \text{BR91} \], the general linear Lie superalgebra $pl_\mathcal{L}$, associated to the signed alphabet, $\mathcal{L}$, is the vector space (over $\mathbb{Q}$) with basis $E_{a,b}$ for $a, b \in \mathcal{L}$, where $|E_{a,b}| = |a| + |b|$ and the bracket is
\[ [E_{a,b}, E_{c,d}] = \delta_{b,c} E_{a,d} - (-1)^{(|a|+|b|)(|c|+|d|)} \delta_{a,c} E_{b,d}. \]

We next describe an action of $E_{a,b}$ on $Super([\mathcal{L} | \mathcal{P}])$.

A (left) superderivation $D$ on a superalgebra $A$ is a $\mathbb{Z}$-linear endomorphism of $A$ such that for $p, q$ homogeneous in the $\mathbb{Z}_2$ grading of $A$, the identity $D(pq) = (Dp)q + (-1)^{|p|}p(Dq)$ holds for some fixed $\epsilon \in \mathbb{Z}_2$. This $\epsilon$ is the sign of $D$, written $|D|$.

We define the letter polarization $D_{a,b} : Super([\mathcal{L} | \mathcal{P}]) \to Super([\mathcal{L} | \mathcal{P}])$ to be the superderivation with sign $|a| + |b|$ such that $D_{a,b}(c|p) = \delta_{b,c}(a|p)$ where $\delta$ is the Kronecker delta. It is easy to check that these superderivations are well-defined on the $\mathbb{Z}$-subalgebra $Super([\mathcal{L} | \mathcal{P}])$.

The next example describes the action of the polarization operators in the case that the biproduct is the determinant of a minor.

**Example 8.1** If $\mathcal{L} = \mathcal{L}^-$ and $\mathcal{P} = \mathcal{P}^-$, then $Super([\mathcal{L} | \mathcal{P}])$ is isomorphic to \( \mathbb{Z}[x_{i,j} : i \in \mathcal{L}, j \in \mathcal{P}] \). The action of $D_{i,j}$ (respectively $i,jR$ on this algebra is given by \( \sum_{p \in \mathcal{P}} x_{i,p} \frac{\partial}{\partial x_{i,p}} \) (respectively $\sum_{i \in \mathcal{L}} x_{i,i} \frac{\partial}{\partial x_{i,i}}$).

To make our results characteristic free, we work over $U(pl_\mathcal{L})$, the $\mathbb{Z}$-subalgebra of the universal enveloping superalgebra of $pl_\mathcal{L}$ generated by all $E_{a,b}$, by $\frac{E_{a,b}}{i}$ for all $i \in \mathbb{N}$ and all $a \neq b$ such that $|a| = |b|$, and by $\big( \frac{E_{a,a}}{i} \big)$, for $i \in \mathbb{N}$.

**Proposition 8.1** The map $E_{a,b} \mapsto D_{a,b}$, provides a representation of $U(pl_\mathcal{L})$ on $Super([\mathcal{L} | \mathcal{P}])$.

If $D$ is a shape such that any letter appearing in $Der^- (D)$ appears in $\mathcal{P}$, then this action descends to an action on $S^D(\mathcal{L})$.

**Proof Sketch.**

Defining the polarization operators to be superderivations on $Super(\mathcal{L})$, it can be shown that the polarization operators satisfy $D_{a,b}(p|q) = (D_{a,b}p|q)$ where $(|)$ is the bilinear form of Definition 2.2. It is then clear that $S^D(\mathcal{L})$ is closed under the action of the superderivations.

Details may be found in [\text{BR97a}].

The branching rule for $S^D$ involves removing vertical or horizontal strips from $D$.

**Definition 8.2** Let $D$ be a sorted row-convex shape. Define a horizontal strip, $E^+$, in $D$ to be any subset of the cells of $D$ such that there exists a shape $D$
straight tableau, \( T \), on some alphabet \( a^+ < b_1 < b_2 < \cdots \) where the cells in \( T \) that contain \( a^+ \) are precisely the cells of \( E \). Similarly, define a vertical strip, \( E^- \) as any set of cells containing all the negative letters \( a^- \) appearing in some straight tableau of shape \( D \) on some alphabet \( a^- < b_1 < b_2 < \cdots \).

Let \( g, f \) be two weakly increasing sequences of letters indexed by the elements of \( P \). A vertical or horizontal strip is \( a^- \)-flagged (with respect to \( g, f \)) if it contains cells only in columns \( i \) where \( g_i \leq a \leq f_i \).

Note that strips are allowed to be empty.

**Lemma 8.3** Let \( E \) be a vertical (respectively horizontal) strip in a row-convex shape \( D \). Let \( I_E \) be the multiset of column indices indicating in which columns the cells of the strip appear. If \( E' \) is another vertical (respectively horizontal) strip, then \( I_E = I_{E'} \) implies \( E = E' \).

**Proof.** We utilize an alternative to Algorithm Straight-Filling for producing straight tableau with specified column content and shape. Suppose \( D \) has \( n \) cells. Define the desired contents of the columns of a tableau by a biword \( u = (\cdots \hat{w} \cdots \check{w} \cdots) \) where \( \hat{w} = w_{D_{\text{lex}}^{-}(D)} \) and \( \check{w}_i < \hat{w}_{i+1} \) if \( \hat{w}_i = \hat{w}_{i+1} \). Define a biword \( (\cdots \check{w} \cdots \hat{w} \cdots) = (u_{\sigma(1)}, \ldots, u_{\sigma(n)}) \) by permuting the entries of \( u \) so that \( \check{w} \) weakly increases and so \( \check{w}_i = \check{w}_{i+1} \) implies \( \hat{w}_i \leq \hat{w}_{i+1} \); when \( |\check{w}_i| = 0 \), this inequality is strict.

Using this biword \( w \), we fill the tableau by starting with an empty tableau of shape \( D \) and adding successive letters reading left to right through the biword. At step \( j \) we place \( \check{w}_j \) in the northmost available cell (say row \( i \)) in column \( \hat{w}_j \) such that either \((i, \check{w}_j - 1)\) is not in the diagram or such that the cell \((i, \check{w}_j - 1)\) contains a letter \( x \) with \( x < \check{w}_j \). If no such cell exists, then the biword does not arise from a straight tableau of the given shape. To verify this algorithm, observe that if we put \( \check{w}_j \) into another row \( i' \), then either row-standardness is violated or we have created a flippable inversion in cells \((i, \check{w}_j)\) and \((i', \check{w}_j)\).

The lemma is an immediate consequence of the algorithm’s correctness.

**Definition 8.4** Suppose that \( D \) is a row-convex shape, \( \mathcal{L} \) is an alphabet, and \( \mathbb{Z}[t_l : l \in \mathcal{L}] \) is a polynomial ring. Let \( g, f \) be two weakly increasing sequences in \( \mathcal{L} \) indexed by the elements of \( P \). Define

\[
ch^{D}_{g,f}(\mathcal{L}) = \sum_{T \text{ straight}} \prod_{(i,j) \in D} t_{T(i,j)}
\]

where the sum runs over all \( g, f \)-doubly flagged straight tableaux of shape \( D \) on \( \mathcal{L} \) and where \( T(i,j) \) is the \((i,j)\)th entry of \( T \).

When \( f \) and \( g \) are trivial, that is they contain respectively only the largest and smallest elements of \( \mathcal{L} \), and when \( \mathcal{L} \) contains letters of only one sign, this is
the formal character of the \( GL(|L|) \)-representation \( S^D(L) \). If just one of \( f, g \) is trivial, we get the formal character of a representation of a Borel subgroup.

The following identity is immediate from the definition of a straight tableau, when \( L = L^+ \) it is due to \([RS98]\).

**Proposition 8.5** Fix two weakly increasing sequences \( g, f \) of letters, and choose \( a \in L \). If \( D \) is a sorted row-convex diagram, then

\[
ch^D_{g,f}(L) = \sum_E ch^{D/E}_{g,f}(L\{a\})
\]

where the sum runs over all \( a \)-flagged horizontal (respectively vertical) strips \( E \) in \( D \) when \( a \in L \) is positive (respectively negative), and where \( D/E \) is the diagram formed by removing \( E \) from \( D \).

Preparatory to establishing a filtration for \( pl_L \)-modules \( S^D(L) \) that realizes this identity we define some components of that filtration.

**Definition 8.6** If \( E, E' \) are two vertical (or two horizontal) strips in \( D \), define \( E < E' \) in dominance order when for all \( i \), the number (counted with multiplicity) of elements in \( \{1, \ldots, i\} \) in \( I_E \) is at least as large as the number of times these elements appear in \( I_{E'} \).

Let \( E \) be a vertical (respectively horizontal) strip and let \( a \in L \) be negatively (respectively positively) signed. Define

\[
S^{D; E}_g(L; a) = \text{span}_Z \{ [T] \}
\]

where \( T \) runs over all shape \( D \) tableaux on \( L \) in which \( a \) appears in a vertical strip \( E' \) weakly dominating \( E \).

Define \( S^{D; E}_g(L; a) \) identically except for the requirement that \( E' \) must strictly dominate \( E \).

It is immediate from the definition that \( S^{D; E}_g(L; a) \) and \( S^{D; E}_g(L; a) \) are \( pl_{L\{a\}} \)-representations.

**Theorem 8.7** Let \( D \) be a row-convex shape. If \( a \) is a negatively (respectively positively) signed letter in \( L \) and \( E \) is a horizontal (respectively vertical) strip in \( D \), then

\[
S^{D; E}_g(L; a) / S^{D; E}_g(L; a) \sim S^{D/E}(L\{a\})
\]

as a \( pl_{L\{a\}} \)-representation. Here \( D/E \) is the shape formed by removing \( E \) from \( D \).

**Proof.** It suffices to observe that given a row-standard tableau \( T \) such that the cells occupied by \( a \) comprise \( E \), then any tableaux appearing in the straightened form of \([T]\) has the cells occupied by \( a \) form a strip \( E' \) determined by a multiset
$I' \geq I$. This can be seen by directly examining the straightening relations. In particular, any straightening relation which moves the $a$'s produces a row-standard tableau in which the $a$'s form a horizontal (respectively vertical) strip indexed by some $I' > I$.

A more sophisticated result on the allowable contents of a tableau appearing in the straightening of $[T]$ is proved in [T97a] Chapter III, Section 6.

**Corollary 8.8** Let $D$ be a row-convex shape and let $a$ be a negatively (respectively positively) signed letter in $\mathcal{L}$. Let $E_1, \ldots, E_k$ be all vertical (respectively horizontal) strips in $D$ ordered compatibly with dominance, so that $i > j$ implies $E_j \not\geq E_k$. The filtration

$$S_D = \sum_{i=1}^{k} S_{D,E_i}^f(\mathcal{L}; a) \supset \cdots \supset \sum_{i=j}^{k} S_{D,E_i}^f(\mathcal{L}; a) \supset \cdots \supset S_{D,E_k}^f(\mathcal{L}; a) \supset 0$$

has $S_{D,E_i}^f(\mathcal{L}\setminus\{a\})$ as the quotient, up to $pl_{\mathcal{L}\setminus\{a\}}$-isomorphism, of its $j$th term by its $j + 1$st term.

If $\mathcal{L}$ is sufficiently large, then the containments in the above filtration are all strict.

The preceding results generalize immediately to the $S_n$-representations provided by the Specht modules.

If we define $S_{f,E_i}^f(\mathcal{L}; a)$ to be the flagged super-Schur module found by taking the image of $S_{f,E_i}^f(\mathcal{L}; a)$ under the map $(l|p) \mapsto 0$ when $l > f_p$, then we have the following.

**Proposition 8.9** Maintaining the notation of Corollary 8.8, the filtration

$$S_f^D = \sum_{i=1}^{k} S_{f,E_i}^f(\mathcal{L}; a) \supset \cdots \supset \sum_{i=j}^{k} S_{f,E_i}^f(\mathcal{L}; a) \supset \cdots \supset S_{f,E_k}^f(\mathcal{L}; a) \supset 0$$

of $S_f^D$ by $B$-modules has $S_{f,E_i}^f(\mathcal{L}\setminus\{a\})$ as the quotient, up to isomorphism, of its $j$th term by its $j + 1$st term. Here $B$ is the subalgebra of $U(pl_{\mathcal{L}})$ generated by all $E_{b,a}^i/i!$ for $b > a$ and all $(E_{a,a}^i)$.

When $\mathcal{L} = \mathcal{L}^-$, these results generalize to quantum Schur modules, details appear in [T97a] and [T99].

9 **Acknowledgments**

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