Self Consistency: A General Recipe for Wavelet Estimation With
Irregularly-spaced and/or Incomplete Data*

Thomas C. M. Lee†
The Chinese University of Hong Kong/
Colorado State University

Xiao-Li Meng‡
Harvard University

January 6, 2006

Abstract

Inspired by the key principle behind the EM algorithm, we propose a general methodology for
conducting wavelet estimation with irregularly-spaced data by viewing the data as the observed
portion of an augmented regularly-spaced data set. We then invoke the self-consistency principle
to define our wavelet estimators in the presence of incomplete data. Major advantages of this
approach include: (i) it can be coupled with almost any wavelet shrinkage methods, (ii) it can
deal with non–Gaussian or correlated noise, and (iii) it can automatically handle other kinds of
missing or incomplete observations. We also develop a multiple-imputation algorithm and fast
EM-type algorithms for computing or approximating such estimates. Results from numerical
experiments suggest that our algorithms produce favorite results when comparing to several
common methods, and therefore we hope these empirical findings would motivate subsequent
theoretical investigations. To illustrate the flexibility of our approach, examples with Poisson
data smoothing and image denoising are also provided.

Key words and phrases: EM algorithm, image denoising, non-equispaced data, missing data,
multiscale methods, non-parametric regression, wavelet thresholding.

1 A Brief Literature Review and Overview

Since the early 1990s, wavelet techniques, especially for nonparametric regressions and signal
denoising, have attracted enormous attention from researchers across different fields. Two major
reasons for this are that wavelet estimators enjoy excellent theoretical properties and that they
are capable of adapting simultaneously to spatial and frequency inhomogeneities (e.g., see Donoho
& Johnstone 1994, Donoho & Johnstone 1995 and Donoho, Johnstone, Kerkyacharian & Picard
1995). Also, they are backed up by a fast algorithm (e.g., see Mallat 1989).

A restrictive vanilla setting is as follows. We have \( N = 2^J \) observations \( y_i \) satisfying

\[
y_i = f\left(\frac{i}{N}\right) + e_i, \quad e_i \sim \text{i.i.d. } \mathcal{N}(0, \sigma^2), \quad i = 0, \ldots, N - 1, \tag{1}
\]

and our goal is to estimate the regression function or signal \( f \) via a wavelet method. This usually
consists of three steps. The first step is to compute the empirical wavelet coefficient vector \( w \) by

*Abbreviated Title: Self Consistent Wavelet Estimation. AMS 1991 subject classifications: primary-65T60; secondary-62G08
†Department of Statistics, The Chinese University of Hong Kong, Shatin, New Territories, Hong Kong. Email: tlee@sta.cuhk.edu.hk
‡Department of Statistics, Harvard University, Science Center, 1 Oxford Street, Cambridge, MA 02138-2901, USA. Email: meng@stat.harvard.edu
applying a discrete wavelet transform (DWT) to \( y = (y_0, \ldots, y_{N-1})^T \); that is, if \( W \) denotes the DWT matrix, then \( w \) is given by \( w = Wy \). The second step is to apply a shrinkage operation (e.g., thresholding) to \( w \) to obtain an estimated wavelet coefficient vector \( \hat{w} \). Finally, \( \hat{f} = W^T \hat{w} \) is computed as the wavelet estimate or reconstruction of \( f = (f(0), \ldots, f(\frac{N-1}{2}))^T \). Details can be found in numerous monographs, such as Daubechies (1992), Ogden (1996), Mallet (1999), and Vidakovic (1999).

The most popular shrinkage operation in the statistical literature appears to be thresholding. Earliest examples include the “universal” thresholding method of Donoho & Johnstone (1994), the SURE thresholding method of Donoho & Johnstone (1995), and the method of Saito (1994) that uses the minimum description length (MDL) principle of Rissanen (1989). Since then many different thresholding methods have also been proposed, including the cross-validation method of Nason (1996), the refined MDL based methods of Moulin (1996) and of Lee (2002), the cross–validatory AIC method of Hurvich & Tsai (1998), and the method of Crouse, Nowak & Baraniuk (1998) that uses hidden Markov models. Moreover, various Bayesian and empirical Bayes methods have also been proposed; e.g., see Chipman, Kolaczyk & McCulloch (1997), Abramovich, Sapatinas & Silverman (1998), Clyde, Parmigiani & Vidakovic (1998), Vidakovic (1998), Clyde & George (1999, 2000), and Johnstone & Silverman (2005). In addition, some treatment has also been given to the issue of correlated noise, see, for example, Wang (1996), Johnstone & Silverman (1997), Jansen & Bultheel (1999) and Lee (2002). Lastly, robust wavelet smoothing has been considered for examples by Sardy, Tseng & Bruce (2001) and Oh, Nychka & Lee (2005), while methods for reducing boundary artifacts are studied by Lee & Oh (2004) and Oh & Lee (2005).

However, the applicability of many existing wavelet regression methods is limited by the assumptions that the data are not only complete but also equispaced and that the number of design points is an integer power of 2, as in (1). Different methods have been proposed to relax the equal-spaced assumption. Hall & Turlach (1997) proposed the uses of two interpolation rules for mapping the observed data into a regular grid. They provided a detailed theoretical analysis of their proposals and an asymptotic choice of the thresholding value. Kovac & Silverman (2000) also investigated the use of interpolation. They developed a fast algorithm for computing the noise covariance structure after the original observed data are mapped into an interpolation grid. With the knowledge of this new noise covariance structure, tailored thresholding values can be derived. Nason (2002) demonstrated that this fast covariance updating algorithm is extremely useful for speeding up cross–validation type calculations. Interpolation based methods were also studied in Sardy, Percival, Bruce, Gao & Stuetzle (1999). Cai & Brown (1998) took a different approach by invoking distributional assumptions for the design points. More recently, Antoniadis & Fan (2001) consider the use of penalized least squares for handling non–equally spaced data. In their procedure the best curve estimate is defined as the minimizer of a regularized least squares criterion. Starting with a regularized wavelet interpolation of the non–equally spaced observed data, a one–step iterative algorithm is applied to approximate the minimizer. Finally, the method of lifting can be applied to construct wavelets for irregularly spaced data (e.g., see Delouille, Simoens & von Sachs 2004 and Nunes, Knight & Nason 2006). However, thresholding methods for such lifting wavelet bases seem to be less developed.

Whereas most of these non-equispaced methods are effective for various applications, they impose some additional assumptions that are not used with regular designs; e.g., the implicit smoothness assumption in interpolation methods. In this article we demonstrate that it is possible to deal with the irregular design problem without making such assumptions. Our key idea is to
view an irregular design as a regular design with missing observations. This view allows us to utilize those well-known and widely tested methods in the extensive literature on estimation and computation for missing data, in particular EM-type algorithms (e.g., Dempster, Laird & Rubin 1977, Meng & Rubin 1993 and Meng & van Dyk 1997) and multiple imputation (e.g., Rubin 1987 and Meng 1994). We invoke a self-consistency criterion, in the same spirit as in Efron (1967) and as the self-consistency principle underlying the EM algorithm, to define the wavelet regression estimator with incomplete data.

Our approach is that, given only a complete-data procedure and the corresponding model on the missing-data mechanism, we first seek the most efficient wavelet estimator for the values of the regression function at the observed designed points; i.e., design points at which the response y is observed. We then incorporate into such “optimal” estimation procedure any additional assumptions that are not used with the complete procedure, provided that such assumptions can further improve the efficiency for any particular applications. This separation between the inherited information in the observed data and the information built into a procedure due to external assumptions helps to obtain more efficient wavelet regression estimates with irregular designs. This is not surprising as sensible self-consistent procedures often lead to the most efficient estimators, both in the parametric estimation (e.g., maximum likelihood estimation via the EM algorithm) and the nonparametric estimation (e.g., the Kaplan-Meier estimator) contexts. Indeed, as Tarpey & Flury (1996) put it, self-consistency is a fundamental concept in statistics, and is a general statistical principle for retaining as much as possible the information in the data. In addition, as demonstrated below, another advantage of this approach is that it can be straightforwardly extended to more complicated settings, such as image denoising and non-Gaussian errors, and it handles non-equispaced data simultaneously with other kind of incomplete data, such as in photo inpainting applications (see Section 5).

The rest of this article is organized as follows. Section 2 reviews the self-consistency principle. It also presents a self-consistency criterion for regression function estimation, from which our non-equispaced wavelet estimator is implicitly defined. In Section 3 three algorithms are constructed to compute or approximate this wavelet estimator. Further extensions including two-dimensional implementation are discussed in Section 4, and simulation results are reported in Section 5. Future work, especially regarding theoretical development, is discussed in Section 6.

2 Self Consistency: How Does It Work?

2.1 Self-consistency: An Intuitive Principle

To illustrate the self-consistency principle, imagine the following scenario. Mr. Littlestat was told by his boss to prepare a presentation on the growth in sales since the company’s inception in 1993, and to make a prediction for the next couple of years. Eye-balling the 13 years of the data he was given, he felt that he could draw a reasonably-looking line, but he also knew that it would not please his boss. He vaguely remembered something called “least-squares line” from his college days, but that was all he could remember. So he asked his teenage son, who was a member of a high-school math club. Indeed, his son not only knew about the method, but actually had just programmed it for a homework problem. However, as his son was dashing out for a movie date, he briefly showed the program on his laptop to his father and said “Dad, I’m really running late. Just type in your data as two columns here, click that little thing, and you will find a drawing at
the printer.”

Not thrilled by his son’s rushing but nevertheless pleased to have the program, Mr. Littlestat sat down and started to type in his data. Then he was really unhappy, as he found out that his son’s program was hardwired specifically for solving the homework that had 16 data points. It simply would not run for Mr. Littlestat data set with 13 data points. There were three more rows needed to be filled in.

“What should I do now?” Mr. Littlestat asked himself. He knew nothing about the method, nor how to modify his son’s program. Nor did he have the patience to wait for his son’s return, as he really needed to finish the preparation for his presentation tomorrow.

As the desperation set in, Mr. Littlestat thought “Well, what if I just make up some numbers for the next three years, and see what happens?” So he did, and clicked. Instantly, a printout came out from the printer with a drawing of a line and the 16 points displayed, including the three fake sales figures.

Excited, Mr. Littlestat examined the plot, and saw the line was visually a bad fit to the 13 years of the real data. “Well, that’s expected, since I put in some fake sales figures”, he murmured to himself. But then it hit him: “since I made up these three points anyway, why don’t I just make them to sit on this line and run the program again? That probably would be better ...”

So he reentered the three fake sales figures by reading off the sales figures from the line, clicked again. The new printout showed a line fitting better for the past 13 years, but the three fake sales were still off from the new line, though they were a bit closer. “Hmmm, this is interesting”. He was getting intrigued by his clever invention, “why not try this again?” So he reentered the three future sales by reading off the line again.

Mr. Littlestat’s excitement increased with the number of printouts, as the line fluctuated less and less, and the three future sales got closer and closer to the fitted line. Then everything stopped, and the three future figures sit on the line exactly, as far as Mr. Littlestat could tell. Visually inspecting all the plots he had, he exclaimed, “I guess this is it!” while holding the last plot.

Although Mr. Littlestat was far from sure whether the line in his hand had anything to do with the least-squares line he was after, he convinced himself that he had done his best given what he was given, because there was really nothing else he could do — the line simply stopped moving, and that had to be its best in some sense for it could not be improved further.

Mr. Littlestat was indeed right. His intuitive pursue actually had led to the correct answer. The limit of his iterative procedure is indeed the least-squares fit, a consequence of the “self-consistency” property, as we shall explain below. We provided this story to illustrate and emphasize that self-consistency is nothing more than good common sense. It does not always work, just as not all “common senses” would lead to good answers, but it is typically suggestive, and often leads to optimal solutions that can be justified mathematically.

In the least-squares example above, Mr. Littlestat’s method worked because the least-squares estimator is self-consistent in the following sense. Suppose we have a regression setting for which $x$ is univariate:

$$ y_i = \beta x_i + \epsilon_i, \quad i = 1, \ldots, n, \quad \epsilon_i \sim \text{i.i.d. } F(0, \sigma^2), $$

where $F(0, \sigma^2)$ denotes a distribution with mean zero and variance $\sigma^2$. The least-squares estimator of $\beta$ then is given by

$$ \hat{\beta}_n = \hat{\beta}_n(y_1, \ldots, y_n) = \frac{\sum_{i=1}^{n} y_i x_i}{\sum_{i=1}^{n} x_i^2}. \quad (2) $$
Then, for any \( m < n \), as long as \( \sum_{i=m+1}^{n} x_i^2 > 0 \),
\[
E \left( \hat{\beta}_n \left| y_1, \ldots, y_m; \beta = \hat{\beta}_m \right. \right) = \hat{\beta}_m.
\]
That is, the least-squares estimator has a Martingale-like property, and reaches a perfect equilibrium in its projective properties. Therefore, we can obtain \( \hat{\beta}_m \) from a procedure for computing \( \hat{\beta}_n \) with \( n > m \) by solving (3) for the “fixed point” \( \hat{\beta}_m \). And this can be solved iteratively without knowing the form of \( \hat{\beta}_n \) as long as we can compute the average of the values of \( \hat{\beta}_n \) over the conditional distribution as required by the left-hand side of (3). In this specific case, Mr. Littlestat’s “line fitting” method is correct because of the linearity of \( \hat{\beta}_n \) in the \( y_i \)'s. Specifically, starting with some initial guess, \( \beta^{(0)} \), say, at the \( t \)th iteration, we can impute the three “missing” \( y_i \)'s by their conditional expectations \( y_i^{(t)} = \hat{\beta}_i^{(t)} x_i \) (\( i = 14, 15, 16 \)), and then compute the next iterative estimate
\[
\hat{\beta}_i^{(t+1)} = \hat{\beta}_i^{(t)}(y_1, \ldots, y_{13}; y_{14}^{(t)}, y_{15}^{(t)}, y_{16}^{(t)}).
\]
Evidently, because of (2), the limit of (4), denoted by \( \hat{\beta}_{13} \), must satisfy
\[
\hat{\beta}_{13} = \frac{\sum_{i=1}^{13} y_i x_i + \hat{\beta}_{13} \sum_{i=14}^{16} x_i^2}{\sum_{i=1}^{16} x_i^2},
\]
which means that \( \hat{\beta}_{13} = \{ \sum_{i=1}^{13} y_i x_i / (\sum_{i=1}^{13} x_i^2) \} \), the correct least-squares estimate with 13 data points.

### 2.2 A Self–consistent Regression Estimator

Inspired by the least-squares estimator above, we propose to also invoke the self–consistency principle for regression function estimation in general when facing incomplete data. Note that although in this paper we focus on wavelet regression models, our proposal extends to more general non-parametric or semi-parametric regression setting. Specifically, let \( Z_{\text{obs}} \) and \( Z_{\text{com}} \) be, respectively, the observed and (imaginary) complete data. Suppose for the moment that, given \( Z_{\text{com}} \), we have a method for computing the “best” estimate \( \hat{f}_{\text{com}} \) for our regression function \( f \). We propose that \( \hat{f}_{\text{obs}} \), our estimate of \( f \) given \( Z_{\text{obs}} \), under squared loss, to be the solution of the following self–consistent equation:
\[
E \left\{ \hat{f}_{\text{com}}(\cdot) \left| Z_{\text{obs}}, f = \hat{f}_{\text{obs}}, \theta = \hat{\theta}_{\text{obs}} \right. \right\} = \hat{f}_{\text{obs}}(\cdot),
\]
where \( \theta \) collects all nuisance parameters (such as the variance parameter); for notation simplicity, for the rest of this paper we will suppress, but not ignore, this conditioning on \( \theta \). Since (5), as demonstrated in Section 3, can be solved numerically via iterations, it provides a way to obtain our “best” incomplete-data estimator \( \hat{f}_{\text{obs}} \) by simply using the corresponding complete-data procedure that computes \( \hat{f}_{\text{com}} \), much like the EM algorithm obtains the incomplete-data MLE via procedures for complete-data MLE. In doing so, no additional assumptions will be required, other then the necessary specification of the conditional distribution of the missing part of \( Z_{\text{com}} \) given the observed \( Z_{\text{obs}} \). We note that such specifications are necessary as otherwise what is missing can never be recovered.

The self–consistent equation (5) was also motivated by its success in estimating cumulative distribution function (CDF) with censored or truncated data, where \( f_{\text{com}}(\cdot) \) will be the empirical CDF, a topic that has been studied extensively in the literature (e.g., see Efron 1967). Indeed, it
is well known that the Kaplan–Meier estimator is the solution to (5) for the right censored data under non-informative censoring, and it is also the generalized maximal likelihood estimator (e.g., Johansen 1978). In addition, similar ideas have also been successfully adopted to construct various nonlinear data summaries; e.g., see Tarpey & Flury (1996) and references given therein. Of course, the most spectacular success of the self-consistency principle is the EM algorithm (Dempster et al. 1977) and its various generalizations (e.g., Meng & Rubin 1993 and Meng & van Dyk 1997) where the self-consistency principle, when applied to the complete-data score function, typically leads to incomplete-data maximum likelihood estimator. Indeed, our whole investigation as reported in this paper was guided closely by our knowledge and insights of the development of EM algorithm and its extensions, which turned out to be extremely fruitful for our current purposes.

2.3 Heuristics

In view of these great successes, it is natural to expect that our estimator $\hat{f}_{\text{obs}}$, the solution to (5), would possess excellent statistical properties. The following heuristics provides some insight and indication on why (5) can lead to efficient estimator under squared loss. Suppose $\hat{f}_{\text{com}}$ is the complete-data optimal estimator that minimizes $||f - \hat{f}_{\text{com}}||^2$. Then, for any $\hat{f}_{\text{obs}}$,

$$
||f - \hat{f}_{\text{obs}}||^2 = ||f - \hat{f}_{\text{com}}||^2 + ||\hat{f}_{\text{com}} - \hat{f}_{\text{obs}}||^2 \\
= ||f - \hat{f}_{\text{com}}||^2 + ||\hat{f}_{\text{com}} - E(\hat{f}_{\text{com}}|Z_{\text{obs}}, f)||^2 + ||E(\hat{f}_{\text{com}}|Z_{\text{obs}}, f) - \hat{f}_{\text{obs}}||^2.
$$

Thus, minimizing $||f - \hat{f}_{\text{obs}}||^2$ over $\hat{f}_{\text{obs}}$ is equivalent to minimizing $||E(\hat{f}_{\text{com}}|Z_{\text{obs}}, f) - \hat{f}_{\text{obs}}||^2$, because the first two terms in the right hand side of the last equality are constants with respect to the minimization. It is thus natural to suspect that the solution of (5) is the minimizer of $||f - \hat{f}_{\text{obs}}||^2$, at least asymptotically.

There is also a Bayesian heuristics for (5). From a Bayesian view point, if $\hat{f}_{\text{com}}$ is the Bayesian estimator for $f$ given the complete data, then the Bayesian estimator given the observed data, under the squared loss, is $\hat{f}_{\text{obs}}(\cdot) = E\{\hat{f}_{\text{com}}(\cdot)|Z_{\text{obs}}\}$. Although $\hat{f}_{\text{obs}}$ depends on the choice of prior, if it is an efficient estimate of the true $f$, it is reasonable to expect that asymptotically the ratio between the posterior expectation and the conditional sampling expectation evaluated at $f = \hat{f}_{\text{obs}}$, both of $\hat{f}_{\text{com}}$, converges (almost surely) to 1. That is,

$$
\frac{E\{\hat{f}_{\text{com}}(\cdot)|Z_{\text{obs}}, f = \hat{f}_{\text{obs}}\}}{\hat{f}_{\text{obs}}(\cdot)} = \frac{E\{\hat{f}_{\text{com}}(\cdot)|Z_{\text{obs}}, f = \hat{f}_{\text{obs}}\}}{E\{\hat{f}_{\text{com}}(\cdot)|Z_{\text{obs}}\}} \to 1, \quad (6)
$$

which implies that (5) should hold at least asymptotically.

We note that although the heuristics arguments above are appealing, we currently do not have rigorous theoretical results to establish the optimality of the self-consistent estimator. Indeed, even to prove the existence of the solution to the self-consistent equation (5) is a theoretical task that we have not been able to carry through. Our work, as reported in this paper, therefore has been more of an “engineering nature”, focusing on constructing algorithms to solve (5) and to demonstrate empirically the good performance, conceptual and implemental simplicity, as well as the flexibility of the self-consistent approach. We hope that these empirical demonstrations show the great promise of this self-consistent approach, and thereby stimulate investigation of the theoretical properties, including optimality, of the self-consistent estimator.
3 Three Algorithms

The self-consistent estimator \( \hat{f}_{\text{obs}} \) would not be of much practical value if (5) could not be solved with reasonable computational effort. To solve (5), two steps are involved. The first is to carry out the conditional expectation on the left-hand side, and the second is to solve the equation, in analogous to the E-step and M-step of the EM algorithm, respectively. However, unlike many common EM applications where the E-step is in closed form, in the wavelet applications, the exact E-step is typically analytically infeasible. This is because, due to the shrinkage operation, \( \hat{f}_{\text{com}} \) is a highly complicated non-linear function of the missing \( y' \)'s. There are two general approaches for dealing with such a problem. The first is to use a Monte Carlo E-step, as in Wei & Tanner (1990) and Meng & Schilling (1996), and the second is to trade the exactness for simplicity by making certain approximations to the conditional expectation. Below we propose three algorithms for computing or approximating \( \hat{f}_{\text{obs}} \), one of which is based on the first Monte Carlo approach, and the other two follow the second approximation approach.

3.1 A Multiple Imputation Self-Consistent (MISC) Algorithm

First we fix the notation. Define \( I_{\text{obs}} \) as the “observed data index set”: \( i \in I_{\text{obs}} \) if the \( i \)th data point \((x_i, y_i)\) in (1) is observed. Let \( y = (y_0, \ldots, y_{N-1})^T \) denote the complete responses, and let \( y_{\text{mis}} \) and \( y_{\text{obs}} \) denote respectively the missing and observed portions of \( y \). That is, \( y_{\text{mis}} = \{y_i : i \notin I_{\text{obs}}\} \) and \( y_{\text{obs}} = \{y_i : i \in I_{\text{obs}}\} \). Define \( x_{\text{obs}} = \{x_i : i \in I_{\text{obs}}\} \), and hence the observed data is \( Z_{\text{obs}} = \{x_{\text{obs}}, y_{\text{obs}}\} \). Also denote the covariance matrix of \( y \) as \( \text{Cov}(y) = \Sigma \). Thus we allow the error terms \( e_i \)'s to be correlated, as long as \( \Sigma \) can be identified and efficiently estimated from \( y_{\text{obs}} \).

Our first algorithm, termed the multiple imputation self-consistent (MISC) algorithm, assumes that a complete-data wavelet regression procedure has been chosen (e.g., the SURE method of Donoho & Johnstone 1995). Starting with initial estimates \( \hat{f}^{(0)} \) and \( \hat{\Sigma}^{(0)} \), the algorithm iterates the following three steps for \( t = 1, \ldots \):

**Step 1 Multiple Imputation:** For \( m = 1, \ldots, M \), simulate \( y_{\text{mis},m} \) independently from

\[
P(y_{\text{mis},m}|y_{\text{obs}}; f = \hat{f}^{(t-1)}, \Sigma = \hat{\Sigma}^{(t-1)}).
\]

**Step 2 Wavelet Shrinkage:** For \( m = 1, \ldots, M \), apply the chosen complete-data wavelet shrinkage procedure to the completed data \( y_{\text{m}} = \{y_{\text{obs}}, y_{\text{mis},m}\} \) and obtain \( \hat{f}_m(x_i), i = 0, \ldots, N - 1 \).

**Step 3 Combining Estimates:** Compute the \( t \)-th iterative estimate of \( f \) as

\[
\hat{f}^{(t)}(x_i) = \frac{1}{M} \sum_{m=1}^{M} \hat{f}_m(x_i), \quad i = 0, \ldots, N - 1.
\]

Also, use the residuals \( \{y_i - \hat{f}^{(t)}(x_i) : i \in I_{\text{obs}}\} \) to obtain an efficient estimate \( \hat{\Sigma}^{(t)} \), such as MLE, of \( \Sigma \).

In Step 1 above, the larger the \( M \), the better results one would expect, but at the expense of increased computational time. Our numerical experience indicates that, as long as \( M \) is larger than a minimum cutoff, the additional improvement on \( \hat{f} \) computed with a larger \( M \) is not largely
significant. In our numerical experiments, we typically used $M = 100$, though $M = 10$ is sometimes acceptable as well.

It is evident that the above is a generic algorithm, in the sense that it is not restricted by the specific form of the complete-data wavelet regression procedure, nor by the choice of the error distribution. On one hand, this is a great advantage as it is extremely flexible and the additional programming, relative to that for the complete-data procedure, is minimal as long as it is easy to draw from the conditional distribution in the first step (which typically is the case for Gaussian errors, independent or not). It also provides a benchmark and basis for developing more specialized and sophisticated algorithms. On the other hand, it is a “brute force” algorithm, and is thus quite inefficient as a numerical algorithm.

3.2 A Simple (Sim) Approximated Algorithm

To construct a faster algorithm for computing $\hat{f}_{\text{obs}}$, we started with replacing the costly multiple imputation step in the MISC algorithm by a very simple analytical approximation. We label the resulting algorithm as the simple (Sim) algorithm, and it is designed for a specific type of shrinkage methods, namely, for hard thresholding methods for which the thresholding value is a known function $g(\hat{\sigma})$ of $\hat{\sigma}$, where $\hat{\sigma}$ is an estimate of $\sigma$. A classical example for $g(\hat{\sigma})$ is the universal thresholding scheme of Donoho & Johnstone (1994), for which $g(\hat{\sigma}) = \hat{\sigma}\sqrt{2\log N}$.

Starting with $\hat{f}(0)$ and $\hat{\sigma}(0)$, the Sim algorithm iterates, at the $t$-th iteration, the following steps:

Step 1 For each $i$ such that $i \notin I_{\text{obs}}$, impute the corresponding missing $y_i$ by $y_i^{(t)} = \hat{f}^{(t-1)}(x_i)$, which creates a completed-data set: $y^{(t)} = \{y_i : i \in I_{\text{obs}}\} \cup \{y_i^{(t)} : i \notin I_{\text{obs}}\}$.

Step 2 Apply a DWT to $y^{(t)}$ to obtain the empirical wavelet coefficients $w^{(t)} = W y^{(t)}$.

Step 3 Obtain a robust estimate $\hat{\sigma}^{(t)}$ of $\sigma$ from $w^{(t)}$, for example, the median absolute deviation method used by Donoho & Johnstone (1994). We call $\hat{\sigma}^{(t)}$ the unadjusted estimate for $\sigma$.

Step 4 Use the following variance inflation formula to obtain an adjusted estimate $\hat{\sigma}^{(t)}$ for $\sigma$:

$$\hat{\sigma}^{(t)} = \sqrt{(\hat{\sigma}^{(t)})^2 + C_m (\hat{\sigma}^{(t-1)})^2}, \quad (8)$$

where $C_m = 1 - \frac{m}{N}$ is the fraction of missing observations.

Step 5 Compute $\hat{w}^{(t)}$ by thresholding $w^{(t)}$ with the thresholding value $g(\hat{\sigma}^{(t)})$.

Step 6 Apply the inverse DWT to $\hat{w}^{(t)}$ and obtain the $t$-th iterative estimate $\hat{f}^{(t)} = W^T \hat{w}^{(t)}$.

In all our numerical experiments, convergence was declared if $|\hat{\sigma}^{(t)} - \hat{\sigma}^{(t-1)}| / \hat{\sigma}^{(t)} < \epsilon$. Upon convergence, estimates of $f$, as well as $\sigma$, will be obtained. It is obvious that computationally this Sim algorithm is much less intensive than MISC, because it only requires one complete-data wavelet shrinkage computation within each iteration, in contrast to the $M$ sets of computation required by MISC.

A key component of the Sim algorithm is the variance inflation formula (8), which takes into account the effect of those imputed $y_i^{(t)}$’s on the estimation of $\sigma^2$. The formula was borrowed
from the EM algorithm for estimating \( \sigma^2 \) with normal regression under independent errors, which requires replacing each missing \( y_i^2 \) by its conditional expectation

\[
E \left[ y_i^2 \mid y_{\text{obs}}; f = \hat{f}^{(t-1)}, \sigma^2 = \{\alpha^{(t-1)}\}^2 \right] = \{y_i^{(t)}\}^2 + \{\alpha^{(t-1)}\}^2.
\]

Replacing the complete-data sufficient statistics \( \sum_i y_i \) and \( \sum_i y_i^2 \) in the complete-data MLE for \( \sigma^2 \) then lead to (8). Although we recognize that this adjustment may not be consistent with the method used for obtaining the unadjusted \( \hat{\sigma}^{(t)} \), which often is not MLE, we adopt (8) for its conceptual and implemental simplicity. As demonstrated in Section 5, it works quite well in the sense that not carrying out this variance inflation adjustment would lead to noticeably poorer wavelet estimates. However, this variance inflation adjustment does not account for all the uncertainty in the thresholding due to missing data, and hence it does not work well when the percentage of missing data is large. A better approximation therefore is needed.

### 3.3 A Refined (Ref) Fast Algorithm

To simplify presentation, we will use single-indexing \( w_t \) instead of the usual double-indexing \( w_{jk} \) notation to denote a wavelet coefficient. At the \( t \)-th step, we need to calculate

\[
\hat{f}^{(t)} = E \left\{ \hat{f}_{\text{com}} \mid y_{\text{obs}}, f = \hat{f}^{(t-1)} \right\},
\]

which amounts to calculating all

\[
\hat{w}_l^{(t)} = E \left\{ 1_{|w| \geq g(\hat{\sigma})} w_l \mid y_{\text{obs}}, f = \hat{f}^{(t-1)} \right\},
\]

where \( w_l \)'s and \( \hat{\sigma} \) are respectively the complete-data empirical wavelet coefficients and variance estimate. The simple algorithm in Section 3.2 took a very crude approximation of this conditional expectation, by thresholding the conditional expectation of \( w_l \) with \( g(\hat{\sigma}) \) using the adjusted \( \hat{\sigma} \). We can obtain better approximations if we are willing to give up some generality (but see Section 4.2).

For example, under the i.i.d. normal error setting of (1), we can obtain a much refined analytic approximation if we ignore the conditional variability in \( \hat{\sigma} \) when calculating the conditional expectation for (9). To proceed, we need to define the quantity \( \eta_l \): if \( \hat{w}_l^{(t)} \) is the empirical wavelet coefficient obtained from Step 2 of the Sim algorithm, then the conditional distribution of \( w_l \) given \( \{y_{\text{obs}}, \sigma^2\} \) is \( N(\hat{w}_l^{(t)}, \eta_l^2 \sigma^2) \). With this setup and denote \( g(\sigma) \) by a constant \( c \) to signify the fact that we assume it is known, we shown in Appendix A that (9) has the following simple analytic expression

\[
\hat{w}_l^{(t)} = \alpha(w_l^{(t)}, \eta_l) + \beta(w_l^{(t)}, \eta_l) \times w_l^{(t)},
\]

where the \( \alpha \) and \( \beta \) functions are given by

\[
\alpha(w, \eta) = \frac{\eta \sigma}{\sqrt{2\pi}} \left\{ e^{-\frac{1}{2} \left( \frac{w - \eta \sigma}{\eta \sigma} \right)^2} - e^{-\frac{1}{2} \left( \frac{w + \eta \sigma}{\eta \sigma} \right)^2} \right\} \quad \text{and} \quad \beta(w, \eta) = 2 - \Phi \left( \frac{c - w}{\eta \sigma} \right) - \Phi \left( \frac{c + w}{\eta \sigma} \right),
\]

with \( \Phi \) being the CDF function of \( N(0, 1) \).

The resulting algorithm is identical to Sim except that we replace its Step 5 by (10) and (11), where we use \( \sigma = \hat{\sigma}^{(t)} \) and \( c = g(\hat{\sigma}^{(t)}) \). Thus, computationally, this new refined (Ref) algorithm is almost as efficient as Sim, and it is also straightforward to program as only standard functions are
involved in (10) and (11). However, because it provides a much more refined E-step, the statistical efficiency of the resulting estimator is expected to be much closer to that of the MISC estimator with \( M = \infty \). Numerical results from Section 5 strongly support this expectation.

The analytic formula (10)-(11) deserves several important remarks. First, under the assumption of i.i.d. noise error, the value of \( \eta_l \) can be easily computed at the outset of iteration as the \( l \)th diagonal element of \( \mathbf{I} - \mathbf{W} \mathbf{R} \mathbf{W}^\top \), where \( \mathbf{R} \) is an \( N \times N \) matrix whose off-diagonal elements are all zero, and whose \( l \)th diagonal element is one if \( y_l \) is observed and zero otherwise; that is, the diagonal of \( \mathbf{R} \) forms a “response indicator” vector for \( y \). Note that the assumption of i.i.d. error is non-essential as long as the error covariance \( \Sigma \) is known, which we do assume for the current approximation. Intriguingly, approximating all \( \eta_l \)’s by their average, which is exactly the fraction of missing data \( C_m \) as used in the variance inflation formula (8), works surprisingly well – see Section 4.3 for more discussion.

Second, as a by-product, the quantity \( \eta_l \) can be seen as a measure of the percentage of missing information in \( w_l \) due to missing data. It is because \( 0 \leq \eta_l \leq 1 \), and that \( \eta_l \) is one when there is no information in the observed data about \( w_l \) and zero when \( w_l \) is fully observed. Since the missing data here are the result of irregular design points, we can also view \( \eta_l \) as a measure of irregularity in the data for the wavelet coefficient \( w_l \). Figure 1 provides some illustrative plots of \( \eta_l \), and demonstrates well that the effects of the missing data, as expected, are localized. Also, the plots indicate that these missing data seem to have a stronger impact on high frequency wavelet coefficients. We believe that this interesting by-product, that is, the “irregularity plot” (i.e., by displaying the \( \eta_l \)’s on a frequency-location plot, as we do with the empirical wavelet coefficients), is worthy of further exploring, for example, for the purpose of diagnosing and determining the suitability of a particular wavelet regression model for a particular irregular design.

Third, an intriguing insight suggested by (10) is that even when we choose to use hard thresholding with complete data, if we adopt the self-consistency recipe (5), we should use “soft” thresholding with incomplete data, as (10) is a form of soft thresholding. This can be seen from the fact that as long as \( \eta > 0 \), \( 0 < |\tilde{w}_l^{(t)}| < |w_l^{(t)}| \), as implied trivially by (9). It can also been seen partially from the fact that as long as \( \eta > 0 \), \( 0 < \beta(w, \eta) < 1 \), and therefore it provides an “regression/shrinkage” effect. When \( \eta \to 0 \), \( \alpha(w, \eta) \to 0 \) and \( \beta(w, \eta) \to \frac{1}{|w| \geq c} \), and thus (10) goes back to the original hard-thresholding, as it should be.

Fourth, although the updating expressions (10) and (11) for \( \tilde{w}_l^{(t)} \) were derived for the hard thresholding operation, it can be easily modified for soft thresholding: \( 1_{(|w| \geq c)} \text{sign}(w_l)\{|w_l| - c\} \). By similar calculations as in Appendix A, for soft-thresholding, we only need to add a simple term to \( \tilde{w}_l^{(t)} \) of (10), relabeled as \( \tilde{w}_l^{(t),\text{hard}} \). That is, we replace (10) by

\[
\tilde{w}_l^{(t),\text{soft}} = \tilde{w}_l^{(t),\text{hard}} + c \left\{ \Phi \left( \frac{c - w_l^{(t)}}{\eta_l \sigma} \right) - \Phi \left( \frac{c + w_l^{(t)}}{\eta_l \sigma} \right) \right\}.
\]

(12)

Our numerical experiments suggest that the uses of the hard and the soft thresholding operations inside the Ref algorithm give similar practical performances, perhaps a reflection that both are forms of “soft” thresholding after all in the presence of incomplete data. For this reason, for the rest of this paper we shall concentrate on the hard thresholding operation in our numerical experimentations.
4 Modifications and Extensions

4.1 Incorporating Smoothness Assumptions

As we shall see from the simulation results in Section 5, when comparing to the interpolation based methods, the self-consistency based procedures produce superior estimates for \( f(x_i) \) if \( i \in I_{\text{obs}} \), but tend to produce inferior estimates when \( i \notin I_{\text{obs}} \). The reason for this is as follows. Whenever an interpolation is employed to fill in the missing responses \( \{y_i : i \notin I_{\text{obs}}\} \), some kind of smoothness constraint is effectively imposed on \( \{f(x_i) : i \notin I_{\text{obs}}\} \). However, our basic self-consistency procedures do not impose any prior smoothness assumptions on \( \{f(x_i) : i \notin I_{\text{obs}}\} \). Since many regression functions are mostly smooth, it is reasonable to expect that methods that do not impose any smoothness constraints on \( \{f(x_i) : i \notin I_{\text{obs}}\} \) tend to produce inferior estimates for \( \{f(x_i) : i \notin I_{\text{obs}}\} \) than those methods that do impose such constraints. This is similar to comparing the maximum likelihood procedure with a less efficient estimation procedure but with a good constraint, or prior. The latter can be better than the former, not because it is better in
retaining the relevant information in the data, but because of the reasonable constraint or prior information.

This analogy also suggests that, if such a smoothness assumption is sensible, then this assumption should be included in the self-consistent procedures; that is, to take advantage of both the information available in the data and in the prior. For example, if linear interpolation is a good smoothing procedure to use, then the MISC, Sim, and Ref algorithms can be further improved by modifying $\hat{f}(t)$ at the end of each iteration:

$\star$ For each $i \not\in I_{\text{obs}}$, replace $\hat{f}(t)(x_i)$ by the following linearly interpolated value

$$\hat{f}(t)(x_{a}) + \frac{\hat{f}(t)(x_{b}) - \hat{f}(t)(x_{a})}{x_{b} - x_{a}}(x_{i} - x_{a}),$$

(13)

where $x_{a} \in x_{\text{obs}}$ is the largest observed design point that is less than $x_{i}$, and $x_{b} \in x_{\text{obs}}$ is the smallest observed design point that is larger than $x_{i}$.

The above linear interpolation rule, of course, can be replaced by many other interpolation rules if the corresponding smoothness assumptions are more appropriate. As demonstrated in Section 5, when such smoothness assumptions are appropriate, these hybrid algorithms outperform both the pure interpolation methods and the pure self-consistent methods in terms of the mean squared errors integrated over both $i \in I_{\text{obs}}$ and $i \not\in I_{\text{obs}}$.

### 4.2 Non–Gaussian Errors

As emphasized before, one of the main advantages of adopting the self-consistency equation (5) is that it is not restricted to any particular model or error distribution, much like the EM algorithm is not restricted to a particular class of models. Indeed, the MISC algorithm is completely general, and can be easily implemented to obtain estimators under non-Gaussian errors: simply replace the Gaussian noise wavelet shrinkage procedure in Step 2 of MISC with a suitable non–Gaussian shrinkage method. A numerical demonstration with the Poisson model will be given in Section 5.4.

Given specific non-Gaussian thresholding rules, we can also derive analytic approximations to the conditional expectation needed by (5), under a specified non-Gaussian model. That is, we can obtain refined algorithms for various other error structures, including correlated errors. It is not possible to give a general recipe here as how to perform such approximations, as they need to be worked out on a case by case basis. We emphasize that this should not be viewed as a disadvantage of the self-consistency method, because it is a general principle for constructing estimators, much like the implementation of the EM algorithm always calls for individual derivations of its E-step (and M-step) under the model of interest.

### 4.3 Two–Dimensional Settings

In many imaging applications, especially in remote sensing area, due to detector malfunction or some other reasons, the readings of a small fraction of image pixels are missing. These missing values forbid the direct use of wavelet techniques for image reconstruction. This is an ideal setting for the self-consistent procedures, as it is automatically an incomplete design problem and the percentage of missing data tends to be small.

The generalizations of our methods from 1D settings to 2D settings are in fact quite trivial. Indeed, for all of the above algorithms, the only modification needed is to replace the 1D DWT
with a 2D DWT. As for the 1D case, one would expect that the MISC algorithm would give the best results, followed by Ref and then Sim. However, due to its computational cost, the 2D MISC algorithm may not be practical. Even for the 2D Ref algorithm, the calculations of \( \eta_l \) in (10)-(11) can be lengthy, because one needs to calculate individual elements of a 2D DWT matrix \( W \). A very simple approximation is to replace all \( \eta_l \)'s by their average, which is

\[
\frac{1}{N} \text{trace}(I - WRW^T) = 1 - \frac{n}{N} \equiv C_m. \tag{14}
\]

This exceedingly simple approximation turned out working surprisingly well in all our simulation studies, for both 1D and 2D cases, and therefore we recommend its use whenever the more refined calculations or approximations of \( \eta_l \) are too expensive.

Lastly we remark that in the image restoration or similar contexts, the incorporation of an off-the-shelf interpolation step often is not a good idea because real images tend to contain a large amount of discontinuities (e.g., edges).

5 Numerical Experiments

This section reports results of five sets of numerical experiments that were conducted to study the empirical properties of the above methods. Throughout this section, the D5 wavelet of Daubechies (1992) was used as the mother wavelet, and the primary resolution was 3.

5.1 Visual Inspection

Our first numerical experiment was simply to see if the MISC (Section 3.1), the Sim (Section 3.2) and the Ref (Section 3.3) algorithms, without using interpolation, would work at all. We used the four well-known testing functions of Donoho & Johnstone (1994): Heavisine, Doppler, Blocks and Bumps. For each of the test function, we first simulated a regularly-spaced noisy data set as in (1), with \( N = 2048 \) and signal–to–noise ratio (snr) 7. We then randomly deleted 30% and 50% of the observations and applied the three algorithms to reconstruct the curve, using the universal thresholding.

The results for Heavisine are displayed in the first two columns of Figure 2 respectively. Each column in Figure 2 plots, from top to bottom, the noisy incomplete data set; the initial estimate \( \hat{f}^{(0)} \) obtained using the S-Plus function \texttt{lowess} with a 10\% smoothing span; the first and the third iteration estimates \( \hat{f}^{(1)} \) and \( \hat{f}^{(3)} \); the final estimate declared as soon as reaching \( |\hat{\sigma}^{(t)} - \hat{\sigma}^{(t-1)}|/\hat{\sigma}^{(t)} < 0.0001 \); and the plot of log(MRSS$_\text{obs}$) and log(MSE$_\text{obs}$) against the iteration step \( t \). Here MRSS$_\text{obs}$ is the mean residual sum of squares and MSE$_\text{obs}$ is the mean squared error calculated over all the observed design points:

\[
\text{MRSS}_\text{obs} = \frac{1}{n} \sum_{i \in I_\text{obs}} \{y_i - \hat{f}^{(t)}(x_i)\}^2 \quad \text{and} \quad \text{MSE}_\text{obs} = \frac{1}{n} \sum_{i \in I_\text{obs}} \{f(x_i) - \hat{f}^{(t)}(x_i)\}^2.
\]

Visually, we observe that the Sim algorithm produces good estimate when \( C_m = 30\% \) (first column), and slightly worse estimate when \( C_m = 50\% \) (second column). This worsening is expected because with 50\% missing data there is a lot more uncertainty that could be adequately captured by the variance inflation formula (8). However, it still produces a much better result than the
naive approach that directly uses the unadjusted $\hat{\sigma}^2$ in Step 4, namely, by setting $C_m = 0$ in (8), as displayed in the third column. Results of the MISC algorithm, with $M = 100$, is displayed in the fifth column for $C_m = 50\%$. This MISC algorithm does give a smaller MSE than the Sim algorithm, but at an expense of increased computational time. It is because the MISC algorithm takes more iterations and each iteration is roughly $M = 100$ times more expensive. A very exciting observation is that the Ref algorithm provides essentially identical results as MISC, as displayed in the fourth column, but with a computational load (especially when approximating all $\eta_l$’s by $C_m$) almost identical to that of the Sim algorithm.

All the observations above were almost identically replicated with the other three testing functions; due to space limitation we omit these displays. For all these combinations, the Sim and the Ref algorithms used 0.5 to 3.5 seconds user time on a Sun Ultra 60 machine, depending on the testing function. For the MISC algorithm, it took about $100 \times T$ times longer, where $T$ is the ratio of the number of iterations under MISC to that of Sim or Ref ($T$ typically varies from 2 to 5).

5.2 Effects of Interpolation and Approximations

Our second numerical experiment was conducted to study the effects of the different approximations used in the Sim and the Ref algorithms, using the MISC algorithm as a benchmark for comparison. The effects of interpolation is also studied. The same four testing functions were used. Other experimental factors are: $N = 512$, snr = (5, 7) and missing percentage $C_m = (10\%, 30\%, 50\%)$. For each combination of testing function, snr and missing percentage, 100 incomplete data sets were generated. Then for each of these 100 generated data sets, eight algorithms were applied to estimate $f$:

1. MISC,
2. MISCI — MISC with the interpolation step (13);
3. Sim,
4. SimI — Sim with (13);
5. Ref,
6. RefI — Ref with (13);
7. RefA — Ref with all $\eta_l$’s approximated by $C_m$, namely, by (14), and
8. RefAI — RefA with (13).

Instead of using the universal thresholding value $\hat{\sigma} \sqrt{2 \log N}$, in this experiment we follow Antoniadis & Fan (2001) and used $\hat{\sigma} \sqrt{2 \log N - \log(1 + 256 \log N)}$. Antoniadis & Fan (2001) showed that this latter thresholding value is superior to the universal thresholding value.

Figure 3 presents the boxplots, from top to bottom, of MSE$_{\text{com}}$, MSE$_{\text{obs}}$, and MSE$_{\text{mis}}$, where each column corresponds to a testing function. Here MSE$_{\text{obs}}$ is the same as in Section 5.1, and MSE$_{\text{mis}}$ and MSE$_{\text{com}}$ are its counterparts summing over respectively all the missing design points and all the design points. The fraction of missing data is $C_m = 30\%$ and snr = 7. Results for $C_m = (10\%, 50\%)$ and snr = 5 are similar and hence are omitted.

From the boxplots the following empirical observations can be made:
Figure 2: The performances of the Sim, Ref, and MISC algorithms for recovering *Heavisine*. For the second to the fourth rows, solid curves are regression estimates and the dotted curve is the true function. See text for further details.
• Algorithms with interpolation are superior to their counterparts. This illustrates the importance of employing suitable prior information in wavelet regression, and the above results helped to identify this importance by separating the efficiency inherited in the observed data from the prior information implicitly built into interpolation procedures.

• When comparing results from Ref and RefA, and from RefI and RefAI, the $\eta$ approximation (14) does not seem to have any adverse effects on Ref or RefI.

• The performance of those Ref-type algorithms are very similar to the two MISC algorithms. In fact for many experimental configurations, results from formal statistical tests (not reported here) suggest that the difference of the MSE values of RefAI and MISCI are statistically insignificant.

Given these observations, RefAI seems to be the best compromise, both in terms of statistical performance and computational speed.

Figure 3: Boxplots of $\text{MSE}_{\text{com}}$, $\text{MSE}_{\text{obs}}$ and $\text{MSE}_{\text{mis}}$ for the eight algorithms tested in Section 5.2.
5.3 Comparisons with Existing Methods

In this third experiment the algorithms SimI and RefAI are compared to two popular wavelet regression procedures for non-equispaced data. These two procedures are the IRREGSURE procedure of Kovac & Silverman (2000) and the ROSE procedure of Antoniadis & Fan (2001). The procedure IRREGSURE is an interpolation based procedure where the SURE method of Donoho & Johnstone (1995) is employed as the thresholding procedure. The ROSE procedure is a one-step iterative procedure that uses penalized least squares and hard thresholding. Again, for SimI, RefAI and ROSE, the thresholding value used was $\hat{\sigma} \sqrt{2 \log N - \log(1 + 256 \log N)}$.

We first present our results for those cases with $N = 512$. For each combination of testing functions, snr = (5, 7) and $C_m = (10\%, 30\%, 50\%)$, 200 noisy data sets were generated. For each noisy data set, the four procedures SimI, RefAI, IRREGSURE and ROSE were applied to obtain estimates for $f$, and their corresponding MSE$_{com}$, MSE$_{obs}$, and MSE$_{mis}$ were computed. In addition, for the reason of providing a benchmark comparison, the universal thresholding procedure with threshold $\hat{\sigma} \sqrt{2 \log N - \log(1 + 256 \log N)}$ was also applied to the complete noisy data set. We will label this procedure UniComp. As the complete data set was available to UniComp, it is expected that UniComp would produce smaller MSE values than the other four procedures. For the case snr = 7 and $C_m = 30\%$, boxplots of the MSE$_{com}$, MSE$_{obs}$ and MSE$_{mis}$ values for the five methods are displayed in Figure 4 in a similar fashion as before. Results under snr = 5 and $C_m = (10\%, 50\%)$ are similar and hence are omitted.

Pairwise Wilcoxon tests were applied to test if any two of the four procedures have significantly different median values for MSE$_{com}$, MSE$_{obs}$ and MSE$_{mis}$. The significance level used was 1.25% because of Bonferroni correction for multiple comparisons. Based on the testing results, we ranked a procedure first if its median MSE value is significantly less than those of the remaining three, we ranked it second if its median MSE value is significantly less than two but greater than the remaining one, and so on so forth. If the median MSE values of two procedures are not significantly different, they will share the same averaged rank. These rankings are listed in Table 1. While no ranking method is perfect, such a ranking does provide a good indicator of the relative merits of the methods being compared. Rankings under snr = 5 are almost identical, and thus are omitted.

From Figure 4 and Table 1 one may conclude that RefAI is generally superior to the other three procedures, SimI and IRREGSURE are roughly the same, while ROSE is inferior. A partial explanation for the relatively poorer performance of ROSE is that it does not employ interpolation, indicating potentially misleading comparative conclusions if we do not distinguish between the information from the data and that from (implicitly) imposed smoothness assumptions. By examining the boxplots, one can see that, when comparing the MSE$_{obs}$ values, the performance of RefAI is in fact very similar to UniComp. The above experiment was repeated for $N = 2048$, but without the ROSE procedure. The relative rankings of RefAI, SimI and IRREGSURE remain the same.

5.4 Poisson Model

Displayed in the top panel of Figure 5 are the Poisson photon counts, captured at 256 time intervals, from the collapsed star RXJ1856.5-3754, which is about 400 light years from Earth in the constellation Corona Australis. Note that in this plot the time index has been re-scaled to $[0, 1]$. One reason that astrophysics scientists are interested in this collapsed star is that they believe its
matter is even denser than nuclear matter, the most dense matter found on Earth. We refer interested readers to [http://chandra.harvard.edu/photo/2002/0211/index.html](http://chandra.harvard.edu/photo/2002/0211/index.html) for the scientific issues surrounding this data set.

From this Poisson data set the following two smoothed curve estimates are obtained. The first one was constructed from the complete data set while the second one was constructed with the presence of missing data. The complete data reconstructed curve was obtained by applying the Poisson wavelet thresholding rule of Kolaczyk (1999, Equation (13)) to the full data set, and it is displayed as the solid line in the bottom panel of Figure 5. The missing-data curve estimate was obtained as follows. Firstly 5% of the data points were removed. These missing data points are marked by “o” in the top panel and their locations are indicated by “x” in the bottom panel of Figure 5. Then the MISC algorithm was applied to this missing data set with the same Poisson thresholding rule and $M = 100$. The resulting curve estimate is the broken line in the bottom panel of Figure 5.

From these plots one could notice that in regions with no missing data (e.g., for $t$ in $[0, 0.15]$ and $[0.6, 0.75]$) or when the values of the missing data are not local extrema (e.g., at $t \approx 0.26$ and $t \approx 0.76$), the complete-data and the missing-data estimates are virtually the same. However,
the two estimates do have small differences at regions where missing data are clustered (e.g., at \( t \approx 0.44 \)) or when the values of the missing data are local extrema (e.g., \( t \approx 0.19 \) and \( t \approx 0.79 \)). This simple example therefore illustrates both the feasibility of the MISC algorithm for non-Gaussian data, and the fact that wavelets methods have the ability to localize the damage caused by the incomplete observations.

### 5.5 Image Denoising

In this last experiment we explore the performance of our algorithms in the context of image denoising. We are only aware of a very limited number of existing methods that are specifically designed to perform image denoising with missing data. One method was described by Naveau & Oh (2004), which although can be applied to handle missing values around the image edges, was primarily proposed to reduce boundary effects. Their updating algorithm is similar to our Sim algorithm, but without the variance inflation formula (8), and therefore inferior results are expected. The other is by Hirakawa & Meng (2006) using an EM-type approach for simultaneous denoising and denosing, which in fact was motivated by our current work.

Three 2D algorithms were studied: the MISC (with \( M = 10 \)), the Sim, the RefA procedures. Two testing images of size 256 × 256 were used: the well-known Lena image displayed in Figure 6(a) and the Airplane image displayed in Figure 6(b). Also, two snrs and three missing data percent-

![Table 1](image-url)

Table 1: Wilcoxon rankings for the four wavelet regression procedures compared in Section 5.2 when \( N = 512 \).
Figure 5: Top panel: photon counts from the collapsed star RXJ1856.5-3754. Circles indicate (artificial) missing observations. Bottom panel: reconstructed curves using the complete data (solid line) and missing data (broken line). The crosses mark the locations of the missing data.

ages were tested: snr = (5, 7) and $C_m = (10\%, 30\%, 50\%)$. Lastly, two missing data formation mechanisms were tested. The first one is missing at random, in which missing pixel locations were randomly selected from the image, while in the second mechanisms the missing pixels were clustered together. Note that because of the computational cost, it was too costly to run MISC with $M = 100$ for our simulation studies, which typically takes roughly 1 hour for one replicate on a Sun Ultra 60 machine.

For each of the above experimental factor combinations, 100 noisy images were generated, and the above three algorithms were applied to reconstruct the corresponding true images, using the adjusted universal thresholding value: $\hat{\sigma} = \sqrt{2\log N - \log(1 + 256\log N)}$. As to provide a benchmark for comparison, for each noisy image, we also applied the universal denoising method (Donoho & Johnstone 1994), with the same adjusted thresholding value, to reconstruct the corresponding true image using the complete data. As before, we refer this method as UniComp. As UniComp has the full information from $y$, it is expected that it would produce better reconstructed images than the other three algorithms.

For every reconstructed images, we calculated $\text{MSE}_{\text{com}}$, $\text{MSE}_{\text{obs}}$ and $\text{MSE}_{\text{mis}}$ as measures of reconstruction quality. (We are aware of the fact that MSE is not a good measure for visual quality, but in the absence of a commonly agreed measure for visual quality, the MSE still serves as a statistically useful criterion for comparisons). In addition we also computed the following MSE
Similar MSE ratios for the observed \( r_{\text{obs}}(\text{MISC}) \) and missing data \( r_{\text{mis}}(\text{MISC}) \), and for the Sim and RefA algorithms \( r_{\text{com}}(\text{Sim}), r_{\text{obs}}(\text{Sim}), r_{\text{mis}}(\text{Sim}), r_{\text{com}}(\text{RefA}), r_{\text{obs}}(\text{RefA}) \) and \( r_{\text{mis}}(\text{RefA}) \) were also calculated. Since UniComp reconstructed the images with the complete data, it is expected that all these MSE ratios are bigger than 1. For \( \text{snr} = 7 \) and \( C_m = 30\% \), boxplots of these MSE ratios are given in Figure 7. Boxplots for other experimental settings are similar and hence omitted. From Figure 7 some major empirical conclusions can be obtained.

First, as all \( r_{\text{obs}}(\text{MISC}) \) values are fairly close to 1, the easy-to-implement benchmark MISC algorithm performs reasonably well for those observed pixels. Secondly, the RefA algorithm is superior to the other two algorithms, as it does not require multiple imputation (as opposed to MISC) and it uses a better approximation than the Sim algorithm. Lastly, an unexpected observation is that, \( r_{\text{obs}}(\text{RefA}) \) is in fact less than 1 when the locations of the missing data are clustered together. We currently do not have an explanation for this phenomenon, other than noting that hidden biases resulting from model defects can be more pronounced with more data.

For the purpose of visual inspection, two degraded versions of Lena are displayed in Figures 8(a) and 9(a). Those black pixels represent the locations of the missing values. The snr is 7 and the missing percentage is 10\%. Figures 8(b) and 9(b) display the corresponding reconstructed images obtained from the RefA algorithm. The quality of the reconstructed ones is quite acceptable. The one with clustered missing data is particularly impressive, especially considering that the method we used did not take into account the cluster nature of the missing data. Reconstruction algorithms as such are particularly useful for image inpainting (e.g., see Criminisi, Perez & Toyama 2004 and Tschumperle & Deriche 2005).

6 Summary and Future Works

A main goal of this paper is to demonstrate that the self-consistency principle is a very versatile and fruitful method for dealing with wavelet modeling, and more generally with non-parametric and semi-parametric regressions, when facing incomplete data. By viewing irregularly-spaced data as a form of incomplete data, it also provides a rather general methodology for wavelet reconstructions with irregularly-spaced data by taking advantage of the existence of those well-studied methods developed for regularly-spaced data, much in the same spirit as with the EM algorithm or multiple imputation. Indeed, the specific algorithms we proposed here directly use either multiple imputation or steps very similar to the E-step (and M-step) of the EM algorithm.

Much remains to be done, of course. The most urgent ones are theoretical properties of the estimators and algorithms we propose. Simulations were crucial in our development of the estimators and algorithms given in this paper, but they are no substitute of rigorous theoretical investigations. It is therefore our hope that our empirical findings are convincing, or at least suggestive, enough that they would motivate a general theoretical investigation of the self-consistent wavelet estimators, or more generally self-consistent regression estimators. It should also be of great interest to investigate the theoretical connections between the self-consistent wavelet estimators and other constructions of wavelet estimators with irregularly-spaced data, such as via lifting (e.g., Delouille et al. 2004 and Nunes et al. 2006).
On the methodological side, besides developing even more refined algorithms, especially for non-Gaussian and correlated errors, a logical next step is to combined the self-consistent principle with Bayesian methods. Indeed, with Bayesian methods, the dealing with missing data is done jointly with the inference of parameters and regression functions. Preliminary work has shown great promise, as reported in Hirakawa & Meng (2006). We are currently investigate the self-consistent approach with a number of Bayesian methods for wavelet reconstructions, including over-complete expansions (e.g., Abramovich et al. 1998, Johnstone & Silverman 2005 and Donoho, Elad & Temlyakov 2006).

Acknowledgment

The authors thank the National Science Foundation for partial support (Grants No. 0203901 and No. 0204552) for this work, particularly for the funding to present an abbreviated and preliminary version (Lee & Meng 2005) at the 30th IEEE International Conference on Acoustics, Speech, and Signal Processing.

A Derivations of (10), (11) and (12)

To show (10) and (11), let $w_l \sim \mathcal{N}(d, \tau^2)$ and write $Z = \frac{w_l - d}{\tau}$; i.e., $Z$ is standard normal. We will also need the following fact. Let $Z \sim \mathcal{N}(0, 1)$, and denote its probability density function by $\phi(z)$.
Figure 7: Boxplots of the MSE ratios resulted from the image denoising experiment in Section 5. In each panel the left, middle and right boxplots correspond, respectively, to the MISC, Sim and RefA algorithms.

Then, for any constant $c$, $E\{1_{(Z>c)}Z\} = \phi(c)$ and $E\{1_{(Z<-c)}Z\} = -\phi(c)$.

With this setup, we have

$$E\{1_{(|w|\geq c)w}\} = E\{1_{(d+\tau Z \geq c)}(d + \tau Z)\} + E\{1_{(d+\tau Z \leq -c)}(d + \tau Z)\}$$

$$= dP\left(Z \geq \frac{c - d}{\tau}\right) + \tau E\{1_{(Z \geq \frac{c - d}{\tau})}Z\} + dP\left(Z \leq \frac{(c + d)}{\tau}\right) + \tau E\{1_{(Z \leq \frac{(c + d)}{\tau})}Z\}$$

$$= d\left\{1 - \Phi\left(\frac{c - d}{\tau}\right)\right\} + \tau\phi\left(\frac{c - d}{\tau}\right) + d\left\{1 - \Phi\left(\frac{c + d}{\tau}\right)\right\} - \tau\phi\left(\frac{c + d}{\tau}\right)$$

$$= d\left\{2 - \Phi\left(\frac{c - d}{\tau}\right) - \Phi\left(\frac{c + d}{\tau}\right)\right\} + \tau\left\{\phi\left(\frac{c - d}{\tau}\right) - \phi\left(\frac{c + d}{\tau}\right)\right\}.$$  

Now by substituting $d = w_{t}^{(l)}$ and $\tau = \eta \sigma$ into the above expression, we obtain (10) and (11).

Equation (12) is derived using essentially the same steps as above, except the whole calculation begins with $E\{1_{(|w|\geq c)\text{Sign}(w)}\{|w| - c\}]$, which only differs from the hard-thresholding formula by
the simple term $-cE\{1_{|w_l| \geq c}\}\text{sign}(w_l) = -c\{P(w_l \geq c) - P(w_l \leq -c)\}$. This can be easily seen to be corresponding to the second term on the right-hand side of (12).

References

Abramovich, F., Sapatinas, T. & Silverman, B. W. (1998), ‘Wavelet thresholding via a Bayesian approach’, *Journal of the Royal Statistical Society Series B* 60, 725–749.

Antoniadis, A. & Fan, J. (2001), ‘Regularization of wavelet approximations (with discussion)’, *Journal of the American Statistical Association* 96, 939–967.

Cai, T. & Brown, L. (1998), ‘Wavelet shrinkage for nonequispaced samples’, *The Annals of Statistics* 26, 1783–1799.

Chipman, H. A., Kolaczyk, E. D. & McCulloch, R. E. (1997), ‘Adaptive Bayesian wavelet shrinkage’, *Journal of the American Statistical Association* 92, 1413–1421.

Clyde, M. & George, E. I. (1999), Empirical Bayes estimation in wavelet nonparametric regression, *in* P. Muller & B. Vidakovic, eds, ‘Bayesian Inference in Wavelet-Based Models’, New York: Springer-Verlag, pp. 309–322.

Clyde, M. & George, E. I. (2000), ‘Flexible empirical Bayes estimation for wavelets’, *Journal of the Royal Statistical Society Series B* 62, 681–698.
Clyde, M., Parmigiani, G. & Vidakovic, B. (1998), ‘Multiple shrinkage and subset selection in wavelets’, *Biometrika* **85**, 391–402.

Criminisi, A., Perez, P. & Toyama, K. (2004), ‘Region filling and object removal by exemplar-based image inpainting’, *IEEE Transactions on Image Processing* **13**, 1200–1212.

Crouse, M. S., Nowak, R. D. & Baraniuk, R. G. (1998), ‘Wavelet-based statistical signal processing using hidden markov models’, *IEEE Transactions on Signal Processing* **46**, 886–902.

Daubechies, I. (1992), *Ten Lectures on Wavelets*, Philadelphia: SIAM.

Delouille, V., Simoens, J. & von Sachs, R. (2004), ‘Smooth design-adapted wavelets for nonparametric stochastic regression’, *Journal of the American Statistical Association* **99**, 643–658.

Dempster, A. P., Laird, N. M. & Rubin, D. B. (1977), ‘Maximum likelihood from incomplete data via the EM algorithm (with discussion)’, *Journal of the Royal Statistical Society, Series B, Methodological* **39**, 1–37.

Donoho, D. L. & Johnstone, I. M. (1994), ‘Ideal spatial adaptation by wavelet shrinkage’, *Biometrika* **81**, 425–455.

Donoho, D. L. & Johnstone, I. M. (1995), ‘Adapting to unknown smoothness via wavelet shrinkage’, *Journal of the American Statistical Association* **90**, 1200–1224.

Donoho, D. L., Elad, M. & Temlyakov, V. N. (2006), ‘Stable recovery of sparse overcomplete representations in the presence of noise’, *IEEE Transactions on Information Theory* **52**, 6–18.
Donoho, D. L., Johnstone, I. M., Kerkyacharian, G. & Picard, D. (1995), ‘Wavelet shrinkage: Asymptopia? (with discussion)’, *Journal of the Royal Statistical Society Series B* 57, 301–369.

Efron, B. (1967), The two sample problem with censored data, in ‘Proceedings of the Fifth Berkeley Symposium on Mathematical Statistics and Probability’, Vol. IV, Berkeley: University of California Press, pp. 831–851.

Hall, P. & Turlach, B. A. (1997), ‘Interpolation methods for adapting to sparse design in nonparametric regression’, *Journal of the American Statistical Association* 92, 466–476.

Hirakawa, K. & Meng, X.-L. (2006), An empirical Bayes EM-wavelet unification for simultaneous denoising, interpolation, and/or demosaicing, in ‘Proceedings of the 2006 IEEE International Conference on Image Processing’. To appear.

Hurvich, C. M. & Tsai, C.-L. (1998), ‘A crossvalidated AIC for hard wavelet thresholding in spatially adaptive function estimation’, *Biometrika* 85, 701–710.

Jansen, M. & Bultheel, A. (1999), ‘Multiple wavelet threshold estimation by generalized cross validation for images with correlated noise’, *IEEE Transactions on Image Processing* 8, 947–953.

Johansen, S. (1978), ‘The product limit estimator as maximum likelihood estimator’, *Scandinavian Journal of Statistics* 5, 195–199.

Johnstone, I. M. & Silverman, B. W. (1997), ‘Wavelet threshold estimators for data with correlated noise’, *Journal of the Royal Statistical Society Series B* 59, 319–351.

Johnstone, I. M. & Silverman, B. W. (2005), ‘Empirical Bayes selection of wavelet thresholds’, *The Annals of Statistics* 33, 1700–1752.

Kolaczyk, E. D. (1999), ‘Wavelet shrinkage estimation of certain Poisson intensity signals using corrected thresholds’, *Statistica Sinica* 9, 119–135.

Kovac, A. & Silverman, B. W. (2000), ‘Extending the scope of wavelet regression method by coefficient–dependent thresholding’, *Journal of the American Statistical Association* 95, 172–183.

Lee, T. C. M. (2002), ‘Tree–based wavelet regression for correlated data using the minimum description length principle’, *Australian & New Zealand Journal of Statistics* 44, 23–39.

Lee, T. C. M. & Meng, X.-L. (2005), A self-consistent wavelet method for denoising images with missing pixels, in ‘Proceedings of the 30th IEEE International Conference on Acoustics, Speech, and Signal Processing’, Vol. 2, pp. 41–44.

Lee, T. C. M. & Oh, H.-S. (2004), ‘Automatic polynomial wavelet regression’, *Statistics and Computing* 14, 337–341.

Mallat, S. G. (1989), ‘A theory for multiresolution signal decomposition: The wavelet representation’, *IEEE Transactions on Pattern Analysis and Machine Intelligence* 11, 674–693.
Mallet, S. (1999), *A Wavelet Tour of Signal Processing*, second edn, Academic Press, Cambridge.

Meng, X.-L. (1994), ‘Multiple-imputation inferences with uncongenial sources of input’, *Statistical Science* **9**, 538–558. (Discussion: p558-573).

Meng, X.-L. & Rubin, D. B. (1993), ‘Maximum likelihood estimation via the ECM algorithm: A general framework’, *Biometrika* **80**, 267–278.

Meng, X.-L. & Schilling, S. (1996), ‘Fitting full-information item factor models and an empirical investigation of bridge sampling’, *Journal of the American Statistical Association* **91**, 1254–1267.

Meng, X.-L. & van Dyk, D. A. (1997), ‘The EM algorithm – an old folk song sung to a fast new tune (with discussion)’, *Journal of the Royal Statistical Society, Series B, Methodological* **59**, 511–567.

Moulin, P. (1996), Signal estimation using adapted tree-structured bases and the MDL principle, in ‘Proceedings of the IEEE Signal Processing Symposium on Time–Frequency and Time–Scale Analysis’, Paris, pp. 141–143.

Nason, G. P. (1996), ‘Wavelet shrinkage using cross-validation’, *Journal of the Royal Statistical Society Series B* **58**, 463–479.

Nason, G. P. (2002), ‘Choice of wavelet smoothness, primary resolution and threshold in wavelet shrinkage’, *Statistics and Computing* **12**, 219–227.

Naveau, P. & Oh, H.-S. (2004), ‘Polynomial wavelet regression for images with irregular boundaries’, *IEEE Transactions on Image Processing* **13**, 773–781.

Nunes, M. A., Knight, M. I. & Nason, G. P. (2006), ‘Adaptive lifting for nonparametric regression’, *Statistics and Computing* **16**, 143–159.

Ogden, T. (1996), *Essential Wavelets for Statistical Applications and Data Analysis*, Birkhauser, Boston.

Oh, H.-S. & Lee, T. C. M. (2005), ‘Hybrid local polynomial wavelet shrinkage: Wavelet regression with automatic boundary adjustment’, *Computational Statistics and Data Analysis* **48**, 809–819.

Oh, H.-S., Nychka, D. & Lee, T. C. M. (2005), ‘The role of pseudo data for robust smoothing with application to wavelet regression’. Unpublished manuscript.

Rissanen, J. (1989), *Stochastic Complexity in Statistical Inquiry*, World Scientific, Singapore.

Rubin, D. B. (1987), *Multiple Imputation for Nonresponse in Surveys*, John Wiley & Sons, New York.

Saito, N. (1994), Simultaneous noise suppression and signal compression using a library of orthonormal bases and the minimum description length criterion, in E. Foufoula-Georgiou & P. Kumar, eds, ‘Wavelets in Geophysics’, New York: Academic Press, pp. 299–324.
Sardy, S., Percival, D. B., Bruce, A. G., Gao, H.-Y. & Stuetzel, W. (1999), ‘Wavelet shrinkage for unequally spaced data’, *Statistics and Computing* **9**, 65–75.

Sardy, S., Tseng, P. & Bruce, A. (2001), ‘Robust wavelet denoising’, *IEEE Transactions on Signal Processing* **49**, 1146–1152.

Tarpey, T. & Flury, B. (1996), ‘Self–Consistency: A fundamental concept in statistics’, *Statistical Science* **11**, 229–243.

Tschumperle, D. & Deriche, R. (2005), ‘Vector-valued image regularization with PDEs: A common framework for different applications’, *IEEE Transactions on Pattern Analysis and Machine Intelligence* **27**, 506–517.

Vidakovic, B. (1998), ‘Nonlinear wavelet shrinkage with Bayes rules and Bayes factors’, *Journal of the American Statistical Association* **93**, 173–179.

Vidakovic, B. (1999), *Statistical Modeling by Wavelets*, John Wiley & Sons, Inc., New York.

Wang, Y. (1996), ‘Function estimation via wavelet shrinkage for long–memory data’, *The Annals of Statistics* **24**, 466–484.

Wei, G. & Tanner, M. A. (1990), ‘A Monte Carlo implementation of the EM algorithm and the poor man’s data augmentation algorithm’, *Journal of the American Statistical Association* **85**, 699–704.