A Lexicographic Public Good Ranking

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Abstract

In this paper, we consider the consistency of the desirability relation with the ranking of the players in a simple game provided by some well-known solutions, in particular the Public Good Index [12] and the criticality-based ranking [1]. We define a new ranking solution, the lexicographic Public Good ranking (LPGR), strongly related to the Public Good Index being rooted in the minimal winning coalitions of the simple game, proving that it is monotonic with respect to the desirability relation [15], when it holds. A suitable characterization of the LPGR solution is provided. Finally, we investigate the relation among the LPGR solution and the criticality-based ranking, referring to the dual game.

keywords: Desirability, Simple Games, Public Good Index, Ranking, Dual Game

1 Introduction

The desirability relation for players in simple games [3, 10, 15, 21] has been widely studied also in connection with the property-driven analysis of power indices [7, 9, 14]. A player $i$ is in desirability relation with a player $j$ (meaning that $i$ is at least as desirable as $j$) if we can replace player $j$ with player $i$ in any winning coalition without changing the outcome, i.e. for any winning coalition $S$ with $j \in S$ and $i \notin S$, we have that $S \setminus \{j\} \cup \{i\}$ is still a winning coalition. So, the desirability relation between $i$ and $j$ suggests that player $i$ is at least as influential as player $j$, for it is never harmful for coalitions to replace $j$ by $i$. Clearly, the desirability relation is not necessarily a total relation on the set of players, as two players may not be in any desirability relation (see, for example, [11, 14] for an exhaustive analysis of properties of the desirability relation on simple games).
A classical property for power indices based on the desirability relation, is the monotonicity property: a power index is monotone if, whenever a player \( i \) is at least as desirable as player \( j \), then the power of \( i \) is at least as much as the power of player \( j \). For instance, the Shapley-Shubik index \([20]\), the Banzhaf index \([2]\), the Johnston index \([16]\), the nucleolus \([19]\) and many other power indices satisfy the monotonicity property (see, for instance, \([9]\)). Instead, it is easy to provide examples of simple games showing this property is not satisfied by other famous indices that take into account exclusively minimal winning coalitions, like the Deegan-Packel index (DPI) \([4, 6]\) or the Public Good Index (PGI) \([12, 13]\) (see Section 3 for some examples from the literature \([5, 9]\)).

The main objective of this paper is to show that it is possible to rank players consistently with the desirability relation and using exclusively minimal winning coalitions in an ordinal way. In fact, while classical power indices convert the information about coalitions into a numerical personal score representing players’ relevance in a simple game, in many practical situations, having a reliable ranking to select the top players is enough and the information provided by players’ score is only marginal. For instance, in the application of power indices to computational biology, the goal is short-listing the most relevant genes on complex networks with a huge number of nodes \([18]\). In a similar way, ranking players is essential for the analysis of centrality of network elements with the goal to select the most critical or sensible parts of a system \([17]\), or in studies aimed at establishing which agents are the strongest or the weakest in a voting system \([8]\).

To that purpose, we introduce a ranking solution (formally, a map that associates to any simple game with player-set \( N \), a total preorder on \( N \)) aimed at ranking players in a simple game according to their influence and in a way that is compatible with the desirability relation. Our ranking solution contains elements of both the DPI and the PGI, taking into account the minimal winning coalitions an individual belongs to. More exactly, given \( n = |N| \) players, we first compute for each player a vector of \( n \) real numbers, where the \( k \)-th component of each vector is the number of minimal winning coalitions of size \( k \), with \( k = 1, \ldots, n \); second, our ranking solution lexicographically compares those real-valued vectors. Due to the similarity with the PGI to compute vectors components, we called such a ranking solution the lexicographic Public Good ranking (shortly, the \( \text{LPGR} \)).

We show that the \( \text{LPGR} \) is monotonic, and we prove that it is the unique solution satisfying (strong) monotonicity with respect to the desirability relation together with two other axioms: 1) the coalitional anonymity property, saying that the relative ranking between two players \( i \) and \( j \) in two different simple games should be independent of the identity of other players in minimal winning coalitions, provided that the number of minimal winning coalitions to which they belong in the two games is the same; 2) the property of independence of larger minimal winning coalitions, saying that once a player \( i \) is considered more influential than a player \( j \) in a simple game, player \( i \) will continue to be considered more influential than \( j \) in any simple game obtained by adding new “larger” minimal winning coalitions to the original game.
As a side-product of our analysis, we also point out some connections between our ranking solution and the criticality-based ranking provided in [1] to compare, in an ordinal way, the blocking power of players and we explore some similarities with the axioms used to characterize the dual version of the criticality-based ranking.

The paper is structured as follows. In Section 2 we provide some basic notions and notation. The definition of the \( \text{LPGR} \) is then introduced in Section 3 together with some examples comparing it with the ranking defined by other solutions from the literature. An axiomatic characterization of the \( \text{LPGR} \) is then presented and discussed in Section 4. A connection between the criticality-based ranking and the \( \text{LPGR} \) are investigated in Section 5 using the desirability relation on dual games. Section 6 concludes.

2 Preliminaries and notation

Given a finite set \( N \), we denote by \( |N| \) its cardinality and by \( 2^N = \{S \subseteq N\} \) its power set. A simple game is a pair \( (N,v) \), where \( N = \{1,2,\ldots,n\} \) denotes a finite set of players and \( v : 2^N \to \{0,1\} \) is a characteristic function, with \( v(\emptyset) = 0 \), \( v(N) = 1 \) and \( v(S) \leq v(T) \) for all sets \( S,T \) such that \( S \subseteq T \subseteq N \). A coalition \( S \subseteq N \) such that \( v(S) = 0 \) is said a losing coalition, whereas a coalition \( S \subseteq N \) such that \( v(S) = 1 \) is said a winning coalition. The class of simple games with \( N \) as the set of players is denoted by \( \text{SG}_N \).

Let \( W^v \) be the set of winning coalitions in \( (N,v) \)

\[
W^v = \{ S \subseteq N : v(S) = 1 \}
\]

and let \( W^v_{\text{min}} \) be the set of minimal winning coalitions in \( (N,v) \)

\[
W^v_{\text{min}} = \text{Min} W^v
\]

where, for any family of sets \( F \), the \( \text{Min} \) operator on \( F \) removes all non-inclusion-minimal sets of \( F \):

\[
\text{Min} F = \{ F \in F \mid \nexists G \in F : G \subset F \}
\]

A simple game \( (N,v) \) is a weighted majority game if there exists a vector of non-negative real numbers \( w \in \mathbb{R}^N \) and a quota \( q \in \mathbb{R}_{\geq 0} \) such that a coalition \( S \subseteq N \) is winning if and only if \( \sum_{i \in S} w_i \geq q \).

In [12] the author introduced the Public Good index (PGI) of a player in a simple game, as the quotient between the number of minimal winning coalitions containing that player and the sum of cardinalities of all the minimal winning coalitions. Let \( (N,v) \) be a simple game, the PGI of player \( i \in N \):

\[
h^v(i) = \frac{|W^v_{\text{min}}(i)|}{\sum_{j \in N} |W^v_{\text{min}}(j)|}
\]

where \( W^v_{\text{min}}(i) = \{ W \in W^v_{\text{min}} : i \in W \} \).
Example 1. Let $N = \{1, 2, 3, 4, 5\}$, and let
$$W_{\min}^v = \{\{1, 2\}, \{1, 3\}, \{3, 4\}, \{2, 4, 5\}, \{1, 4, 5\}\}.$$ 
We have
$$h_v(1) = 3 \frac{1}{12}, \quad h_v(2) = 2 \frac{1}{12}, \quad h_v(3) = 2 \frac{1}{12}, \quad h_v(4) = 3 \frac{1}{12}, \quad h_v(5) = 2 \frac{1}{12}.$$ 

In Example 1 we can observe that player 3 and 5 have the same PGI, but they belong to minimal winning coalitions of different cardinality and whenever player 5 is winning together with some coalition $S \subseteq N \setminus \{3, 5\}$ then player 3 is winning together with the same coalition.

In [4] the authors measure the power of a player according to the size of the minimal winning coalitions she belongs to. So, the Deegan-Packel index (DPI) for player $i$ is defined as:
$$\delta_v(i) = \sum_{W \in W_{\min}^v(i)} \frac{1}{|W_{\min}^v|} \frac{1}{|W|}.$$ 

Example 2. Taking the same simple game $(N, v)$ of Example 1 we have:
$$\delta_v(1) = \frac{8}{36}, \quad \delta_v(2) = \frac{5}{36}, \quad \delta_v(3) = \frac{6}{36}, \quad \delta_v(4) = \frac{7}{36}, \quad \delta_v(5) = \frac{4}{36}.$$ 

According to the DPI, player 3, for instance, has more power than player 5 because it belongs to two minimal winning coalitions of size smaller than the two minimal winning coalitions containing player 5.

3 A ranking solution and the desirability relation

Let us start recalling that a binary relation on $N$ is a subset of $N \times N$. A reflexive, transitive and total binary relation on $N$ is a total preorder (also called, a ranking) on $N$. We denote by $T^N$ the set of all total preorders on $N$. For instance, consider the lexicographic total preorder among vectors of real numbers:
$$x \geq_L y \quad \text{if either } x = y \quad \text{or } \exists k : x_t = y_t, t = 1, \ldots, k - 1 \quad \text{and } x_k > y_k.$$ 

We define a ranking solution or, simply, a solution, as a map $R : \mathcal{SG}^N \to T^N$ that associates to each simple game $v \in \mathcal{SG}^N$ a total preorder on $N$. The value assumed by a map $R$ on a simple game $v$ is the ranking on $N$ denoted by $R^v$. We use the notation $iR^v j$ to say that $(i, j) \in R^v$, and it means that $i$ is at least as important as $j$ according to ranking $R^v$, for all $i, j \in N$. We denote by $I^v$ the symmetric part of $R^v$, i.e. $iI^v j$ means that $(i, j) \in R^v$ and $(j, i) \in R^v$ (i and j are equivalent), and by $P^v$ its asymmetric part, i.e. $iP^v j$ means that $(i, j) \in R^v$ and $(j, i) \notin R^v$ (i is strictly more important than j).
Clearly, any real-valued $N$-vector numerically represents a total preorder over the player set $N$. Consequently, any power index $\phi : SG^N \to \mathbb{R}^N$ underpins a ranking solution denoted by $R_\phi$ and such that $i R_\phi^v j \iff \phi_i(v) \geq \phi_j(v)$.

In this section, we introduce a new ranking solution for simple games based on minimal winning coalitions. The main idea of the new solution is that the smaller is the size of a minimal winning coalition, the larger is the power of its members. Therefore, the ranking of a player is positively correlated first to the size of minimal winning coalitions the player belongs to and, second, to their number.

To define the ranking solution, we need to introduce the notation $i^k$ representing the number of minimal winning coalitions of size $k$ containing $i$ in a simple game $(N, v)$: $i^k = |\{S \in W^v_{\text{min}} : i \in S, |S| = k\}|$ for all $k \in \{1, \ldots, n\}$. For each $i \in N$, let $\theta_i(v)$ be the $n$-dimensional vector $\theta_i(v) = (i^1, \ldots, i^n)$ associated to $v$.

Definition 1. [Lexicographic Public Good Ranking (LPGR)] The Lexicographic Public Good Ranking (LPGR) solution is the function $R^v_l : SG^N \to \mathcal{T}^N$ defined for any simple game $v \in SG^N$ as

$$i R^v_l j \iff \theta_i(v) \preceq \theta_j(v).$$

Let $I^v_l$ and $P^v_l$ be the symmetric part and the asymmetric part of $R^v_l$, respectively.

Example 3. Consider the simple game of Example 7. We have that

$$\theta_i(1) = (0, 2, 1, 0, 0), \quad \theta_i(2) = (0, 1, 1, 0, 0),$$
$$\theta_i(3) = (0, 2, 0, 0, 0), \quad \theta_i(4) = (0, 1, 2, 0, 0),$$
$$\theta_i(5) = (0, 0, 2, 0, 0).$$

So, the LPGR solution $R^v_l$ ranks the players as follows

$$1 P^v_l 3 P^v_l 4 P^v_l 2 P^v_l 5.$$

Notice that the ranking provided by the PGI and the DPI do not coincide with the ranking $R^v_l$ on this example. In fact, for instance, $h_i(v)(4) > h_i(v)(3)$ and $\delta_i(v)(4) > \delta_i(v)(3)$, while $3 P^v_l 4$.

The LPGR solution always provides a total preorder over the player set $N$ for any simple game $(N, v)$. Instead, given a simple game $(N, v)$, the desirability relation $\succeq^v$ is a preorder over the elements of $N$ and is defined as follows.

Definition 2. Let $(N, v)$ be a simple game. For any pair of players $i, j \in N$, the desirability relation $\succeq^v \subseteq N \times N$ is defined as follows:

$$i \succeq^v j \iff [S \cup \{j\} \in W^v \Rightarrow S \cup \{i\} \in W^v \text{ for all } S \subseteq N \setminus \{i, j\}].$$

In the following, if the game $v$ on which $\succeq^v$ is defined is clear from the context, we denote relation $\succeq^v$ simply by $\succeq$. For any $i, j \in N$, $i \succeq j$ is interpreted as player $i$ is at least as desirable as player $j$ (as a coalitional member); $i \succ j$ means that $i \succeq j$ and $i 

and there exists a coalition \( T \subseteq N \setminus \{i, j\} \) such that \( T \cup \{i\} \in \mathcal{W}^v \) but \( T \cup \{j\} \notin \mathcal{W}^v \), and it is interpreted as player \( i \) is (strictly) more desirable than player \( j \); \( i \sim j \) means that \( i \geq j \) and \( j \geq i \), i.e. it is true that \( S \cup \{j\} \in \mathcal{W}^v \Leftrightarrow S \cup \{i\} \in \mathcal{W}^v \) for all \( S \subseteq N \setminus \{i, j\} \), and it is interpreted as players \( i \) and \( j \) are equally desirable.

As discussed in Section 1, the desirability relation, when it holds, represents a criterion to select between two players the most influential one, that is the player winning the maximum number of times. So, it is interesting to require the following basic property for ranking solutions.

**Property 1 (Desirable Monotonicity (DM)).** Let \( i, j \in N \). For any \( v \in \mathcal{S}\mathcal{G}^N \), a solution \( R \) satisfies the desirable monotonicity property if

\[
i \sim^v j \Rightarrow i^v j,
\]

and

\[
i \succ^v j \Rightarrow i^v j.
\]

A solution satisfying the desirable monotonicity should strictly obey to the desirability relation: if the desirability relation between two players is strict (i.e., \( i \succ^v j \)), then a ranking solution should put such players in a strict relation too (i.e., \( i^v j \)); of course, if two players are indifferently desirable (i.e., \( i \sim^v j \)) then the ranking solution must define the same kind of relation (i.e., \( i^v j \)). Notice that this kind of “strong” monotonicity relation is not satisfied by the ranking over players represented by the nucleolus [19], as it is easy to find examples of simple games having players in the symmetric part of the desirable relation and such that the allocation provided by the nucleolus is different (see, for instance, [9] page 600).

The rankings over players represented by the DPI and the PGI do not satisfy the desirable monotonicity property, as shown by the following example.

**Example 4.** Consider a weighted majority game \((N, v), N = \{1, 2, 3, 4, 5\}\), with weight function \((4, 2, 1, 1, 1)\) and quota \(q = 6\). So, the minimal winning coalitions are

\[
\mathcal{W}^v_{\text{min}} = \{\{1, 2\}, \{1, 3, 4\}, \{1, 3, 5\}, \{1, 4, 5\}\}.
\]

We have that \(1 \succ^v 2 \succ^v 3 \sim^v 4 \sim^v 5\). However, according to the PGI \(h_2(v) = \frac{1}{11} < \frac{2}{11} = h_3(v)\), while according to the DPI we have \(\delta_2(v) = \frac{1}{8} < \frac{1}{6} = \delta_3(v)\). So, according to the rankings underpinned by both indices, player 3 is ranked strictly higher than player 2.

On the other hand,

\[
\theta_\nu(1) = (0, 1, 3, 0, 0), \quad \theta_\nu(2) = (0, 1, 0, 0, 0),
\]

\[
\theta_\nu(3) = (0, 0, 2, 0, 0), \quad \theta_\nu(4) = (0, 0, 2, 0, 0),
\]

\[
\theta_\nu(5) = (0, 0, 2, 0, 0).
\]

So, \(1 P^v 2 P^v 3 I^v 4 I^v 5: R^v\) and \(\succeq^v\) coincide.
In general, a total preorder provided by the LPGR solution coincides with the desirability relation on any simple game where the desirability relation is total. This fact is an immediate consequence of the following proposition.

**Proposition 1.** The criticality-based solution $R_l$ fulfils the desirable monotonicity property.

**Proof.** Let $(N, v)$ be a simple game. It is easy to verify that the condition
\[ S \cup \{j\} \in W^v \iff S \cup \{i\} \in W^v \text{ for all } S \subseteq N \setminus \{i, j\} \]  
(1)
is equivalent to the condition
\[ S \cup \{j\} \in W_{min}^v \iff S \cup \{i\} \in W_{min}^v \text{ for all } S \subseteq N \setminus \{i, j\}. \]  
(2)
We prove that $i \sim^v j \Rightarrow iP_l^v j$.

Since $i \sim^v j$, according to the equivalence between relations (1) and (2), we immediately have that $S \cup \{j\} \in W_{min}^v \iff S \cup \{i\} \in W_{min}^v$ for all $S \subseteq N \setminus \{i, j\}$. So, $\theta_v(i) = \theta_v(j)$, and therefore $iP_l^v j$.

Now, we prove that $i \succ^v j \Rightarrow iP_l^v j$.

Let be $i \succ^v j$ and define $T = \{T \subseteq N \setminus \{i, j\} : T \cup \{i\} \in W_{min}^v, T \cup \{j\} \notin W_{min}^v\}$. We first need to prove that $T \neq \emptyset$. Since $i \succ^v j$, it must exist $T \subseteq N \setminus \{i, j\}$ such that $T \cup \{i\} \in W^v$ and $T \cup \{j\} \notin W^v$ and, by the equivalence between relation (1) and (2), it is not possible that $S \cup \{j\} \in W_{min}^v \iff S \cup \{i\} \in W_{min}^v$ for all $S \subseteq N \setminus \{i, j\}$.

Moreover, again for $i \succ^v j$, it is not possible that there exists $S \subseteq N \setminus \{i, j\}$ such that $S \cup \{i\} \notin W_{min}^v$ and $S \cup \{j\} \in W_{min}^v$. So, it must exist $T \subseteq N \setminus \{i, j\}$ such that $T \cup \{i\} \in W_{min}^v$ and $T \cup \{j\} \notin W_{min}^v$.

Now, let $k = \min\{|T| : T \in T\}$. If $k = 0$, we immediately have that $\{i\} \in W_{min}^v$ and $\{j\} \notin W_{min}^v$, so $iP_l^v j$.

Consider the case $k > 0$. By the minimality of $k$ we have that $i_t = j_t$ for all $t = 0, \ldots, k - 1$ and $i_k > j_k$ and so $iP_l^v j$.

**Remark 1.** It is well known from the literature that the desirability relation on weighted majority games is a total preorder and that the following monotonicity condition w.r.t. weights holds for a weighted majority game $(N, v)$ with weights $(w_1, \ldots, w_n)$:
\[ w_i \geq w_j \Rightarrow i \geq^v j, \]
for all $i, j \in N$ (see for instance [9]). As a direct consequence of Proposition 1 we have that also the LPGR solution on weighted majority games is monotonic w.r.t. weights, that is $w_i \geq w_j \Rightarrow iR_l^v j$ for all $i, j \in N$. 

\[ \square \]
4  An axiomatic characterization of the LPGR solution

Now, we introduce two new properties for ranking solutions that are inspired by similar properties introduced in [1] on the sets of blocking coalitions.

The next property says that winning coalitions of the same size should have the same impact on the ranking, independently of their members.

**Property 2** (Anonymity of Minimal Winning Coalitions (AMWC)). Let \( i, j \in N \), \( v, v_\pi \in SG^N \) and let \( \pi \) be a bijection on \( 2^{N\setminus\{i,j\}} \) with \(|\pi(S)| = |S|\) and such that

\[
S \cup \{i\} \in W^w_{\min} \iff S \cup \{i\} \in W^w_{\min},
\]

and

\[
S \cup \{j\} \in W^w_{\min} \iff \pi(S) \cup \{j\} \in W^w_{\min},
\]

for all \( S \in 2^{N\setminus\{i,j\}} \). A solution \( R \) satisfies the anonymity of minimal winning coalitions property if

\[
i R^v j \iff i R^w j.
\]

**Example 5.** Consider the weighted majority game \((N, v)\) of Example 4 and the players 3 and 4 in the role of players \( i \) and \( j \) of the definition of Property 2. Define a bijection \( \pi \) on \( 2^{\{1,2,5\}} \) such that \( \pi(\{1,5\}) = \{2,5\} \). So the simple game \((N, v_\pi)\) is such that

\[
W^w_{\min} = \{\{1,2\}, \{1,3,4\}, \{1,3,5\}, \{2,4,5\}\}.
\]

Game \( v \) differs from \( v_\pi \) in terms of minimal winning coalitions just for coalition \( \{1,4,5\} \) which is replaced in \( v_\pi \) by the minimal winning coalition \( \{2,4,5\} \). Nevertheless, the number of minimal winning coalitions of each size containing player 4 in game in \( v_\pi \) is precisely as in game in \( v \), so her capacity to form minimal winning coalitions should not be affected (assuming that the other players are equally inclined to form minimal winning coalitions with 4). So, the property of Anonymity of Minimal Winning Coalitions says that the relative ranking between 3 and 4 in \( v \) should be the same as in \( v_\pi \).

Another property we consider in our analysis is the one of independence of larger minimal winning coalitions, saying that, once a solution exists, in which a player \( i \) is ranked strictly better than a player \( j \), adding “larger” minimal winning coalitions should not affect the relative ranking between \( i \) and \( j \).

**Property 3** (Independence of Larger Minimal Winning Coalitions (ILMWC)). Let \( i, j \in N \). For any \( v \in SG^N \), let \( h = \max\{|S| : S \in W^w_{\min} \text{ and } S \cap \{i,j\} \neq \emptyset\} \) be the highest cardinality of coalitions in the set \( W^w_{\min} \) containing either \( i \) or \( j \). Let \( S_h \) be a collection of (minimal) coalitions with cardinality strictly larger than \( h \), i.e., \( S_h = \{S_1, \ldots, S_r\} \) such that \( S_k \subseteq N, |S_k| > h \) for \( k = 1, \ldots, r \) and there is
no \( Q \in W'_\min \cup S_h \) with \( Q \subset S_k \), for all \( k \in \{1, \ldots, r\} \). A solution \( R \) satisfies the \textit{independence of larger minimal winning coalitions} property if

\[
i P^v j \Rightarrow i P^{v'} j,
\]

where \( v' \) is a simple game such that the set of minimal winning coalitions is obtained as \( W'_\min = W'_\min \cup S_h \).

**Example 6.** Consider again the weighted majority game \((N, v)\) of Example 4 and the player 1 and 2 in the role of players \( i \) and \( j \) of the definition of Property 2. Let \( S_h = \{2, 3, 4, 5\} \) and consider a new simple game \((N, v')\) such that

\[
W'_{\min} = \{1, 2\}, \{1, 3, 4\}, \{1, 3, 5\}, \{1, 4, 5\}, \{2, 3, 4, 5\}.
\]

Notice that the new simple game \( v' \) contains one more minimal winning coalition containing 2 but not 1, but the size of such a minimal winning coalition in \( v' \) is strictly larger than the size of any minimal winning coalition in \( v \), and therefore is considered less likely to form. If a solution satisfying the property of independence of larger minimal winning coalitions ranks 1 strictly better than 2 in the simple game \( v \), in \( v' \) the solution also must rank 1 strictly better than 2: the new (and larger) minimal winning coalition does not affect the strict ranking decided on the basis of smaller minimal winning coalitions.

**Proposition 2.** Let \( R \) be a solution satisfying Properties 1 (DM) and 2 (AMWC). Then for any simple game \( v \) and \( i, j \in N \) such that \( \theta_v(i) = \theta_v(j) \) we have that \( i P^v j \).

**Proof.** Since \( \theta_v(i) = \theta_v(j) \), we have that \( i_k = j_k \) for all \( k \in \{1, \ldots, n\} \). Define a bijection \( \pi \) on \( 2^{N \setminus \{i, j\}} \) such that for each \( k \in \{1, \ldots, n - 1\} \) and for each coalition \( S \in 2^{N \setminus \{i, j\}} \) of size \( k - 1 \) with \( S \cup \{j\} \in W'_\min \), \( \pi(S) = T \), where \( T \in 2^{N \setminus \{i, j\}} \) is a coalition of size \( k - 1 \), with \( T \cup \{i\} \in W'_\min \). Consider a game \( v_\pi \) such that \( S \cup \{i\} \in W'_\min \iff S \cup \{i\} \in W'_\min \) and \( S \cup \{j\} \in W'_\min \iff \pi(S) \cup \{j\} \in W'_\min \). So, we have that for all \( k \in \{1, \ldots, n\} \) and all coalitions \( T \in 2^{N \setminus \{i, j\}} \) of size \( k - 1 \) with \( T \cup \{i\} \in W'_\min \)

\[
T \cup \{i\} \in W'_\min \iff T \cup \{j\} \in W'_\min.
\]

Then, \( i \sim v_\pi j \) (players \( i \) and \( j \) are equally desirable in \( v_\pi \)) and, by Property 1

\[
i I^{v_\pi} j.
\]

(3)

Notice that \( i, j, \pi, v \) and \( v_\pi \) satisfy the conditions for bijections demanded in the statement of Property 2 with \( v_\pi \) such that the set of minimal winning coalitions of \( v_\pi \) is

\[
W'_{\min} = \bigcup_{T \in 2^{N \setminus \{i, j\}} \text{ s.t. } T \cup \{i\} \in W'_\min} \{T \cup \{i\}, T \cup \{j\}\}
\]

(notice that the minimality of the elements in \( W'_\min \) is guaranteed by the minimality of the elements in \( W'_\min \)). So, since \( R \) satisfies Property 2, we have that

\[
i P^v j \iff i P^{v_\pi} j.
\]

(4)

So, by relation 3, we have \( i P^v j \), which concludes the proof. \( \square \)
Theorem 1. The LPGR solution $R_l$ is the unique solution that fulfils Properties 1 (DM), 2 (AMWC) and 3 (ILMWC).

Proof. By Proposition 1 we have that $R_l$ fulfils Property 1 (DM). It is easy to check that it also fulfils Properties 2 (AMWC) and 3 (ILMWC) (it directly follows from Definition 1 and the lexicographic relation).

To show that $R_l$ is the unique index fulfilling Properties 1 (DM), 2 (AMWC) and 3 (ILMWC), we need to prove that, if a solution $R: S^N \rightarrow T^N$ satisfies Properties 1 (DM), 2 (AMWC) and 3 (ILMWC), then $iR_l j \Leftrightarrow iRv j$ or, equivalently, $iP_l j \Leftrightarrow iPv j$ and $iI_l j \Leftrightarrow iIv j$.

We first prove that $iP_l j \Leftrightarrow iPv j$:

(⇒)
Let $iP_l j$. By Definition 3, let $k'$ be the smallest integer in $\{1, \ldots, n\}$ with $i_{k'} > j_{k'}$.
Let $s = i_{k'} - j_{k'}$ and $S_{k'} = \{S \in W_{\text{min}} : |S| = k' \text{ and } S \cap \{i, j\} = \{i\}\}$ be a subset of coalitions in $W_{\text{min}}$ of size $k'$ containing $i$ but not $j$ such that $|S_{k'}| = s$. Moreover, let $\Sigma = \{S \in W_{\text{min}} : |S| > k'\}$ be the set of coalitions in $W_{\text{min}}$ with cardinality strictly larger than $k'$.

Consider a new simple game $v'$ such that $W_{\text{min}} = W_{\text{min}} \setminus \Sigma$, and the set of minimal winning coalitions containing $j$ (of size at most $k'$) in $W_{\text{min}}$:

$$S^j = \{S \cup \{j\} : S \in 2^{N \setminus \{i, j\}} \text{ with } S \cup \{j\} \in W_{\text{min}}'\}.$$

Define a bijection $\pi$ on $2^{N \setminus \{i, j\}}$ such that for each $t \in \{1, \ldots, k'\}$ and for each coalition $S \in 2^{N \setminus \{i, j\}}$ of size $t - 1$, $\pi(S) = T$, where $T \in 2^{N \setminus \{i, j\}}$ is a coalition of size $t - 1$, with $T \cup \{i\} \in W_{\text{min}}'$. So, the set of minimal winning coalitions contained in $S^j$ after the transformation via $\pi$ is:

$$T^j = \{\pi(S) \cup \{j\} : S \in 2^{N \setminus \{i, j\}} \text{ with } S \cup \{j\} \in S^j\}.$$

Consider a new game $\widehat{v}_\pi$ such that

$$W_{\text{min}}^e = \left(W_{\text{min}}' \setminus S^j\right) \cup T^j.$$

So, we have that for all coalitions $T \in 2^{N \setminus \{i, j\}}$ of size $t - 1$, $t \in \{1, \ldots, k'\}$,

$$T \cup \{j\} \in W_{\text{min}}^e \Rightarrow T \cup \{i\} \in W_{\text{min}}^e,$$

and, consequently, for all $S \in 2^{N \setminus \{i, j\}}$,

$$S \cup \{j\} \in W_{\text{min}}^e \Rightarrow S \cup \{i\} \in W_{\text{min}}^e,$$

which means that $i \succeq_{\text{min}} j$. So, by Property 1 (DM), we have that $iP^e j$.

On the other hand,
and 

\[ S \cup \{j\} \in \mathcal{W}^v_{\text{min}} \iff \pi(S) \cup \{j\} \in \mathcal{W}^{\hat{v}}_{\text{min}}, \]

for all \( S \in 2^{N \setminus \{i,j\}} \), and therefore, by Property 2 on \( R \) applied to \( v' \) and \( \hat{v}_\pi \), we also have \( iP^{v'}j \).

Finally, by Property 3 on \( R \) (with \( v' \) in the role of \( v \) in the statement of Property 3), we have that \( iP^vj \), as \( \mathcal{W}^v_{\text{min}} = \mathcal{W}^{v'}_{\text{min}} \cup \Sigma \).

(\( \Leftarrow \))

Let \( iP^vj \). Suppose that \( iI^v_{\Delta}j \). Then, by Definition 3, \( \theta_v(i) = \theta_v(j) \). So, by Proposition 2 and the fact that \( R \) satisfies Properties 1 and 2, \( iI^v_{\Delta}j \), which yields a contradiction with \( iP^vj \). Since it can’t even be \( jP^v_i \) (by the other implication proved above), and by the fact that \( P^v \) is a total relation, it must be \( iP^v_{\Delta}j \).

We now prove that \( iI^v_{\Delta}j \iff iI^v_{\Delta}j \):

(\( \Rightarrow \))

Let \( iI^v_{\Delta}j \). Then, by Definition 3, \( \theta_v(i) = \theta_v(j) \). So, by Proposition 2 and the fact that \( R \) satisfies Properties 1 and 2, \( iI^v_{\Delta}j \).

(\( \Leftarrow \))

Let \( iI^v_{\Delta}j \). As we have shown previously, \( iP^v_{\Delta}j \iff iP^v_{\Delta}j \). So it is not possible that \( iP^v_{\Delta}j \) or \( jP^v_{\Delta}i \). Since \( P^v \) is a total relation, it must be \( iI^v_{\Delta}j \), which concludes the proof.

We end this section showing the logical independence of Properties 1, 2 and 3.

**Example 7.** [No Property 1] Given \( i, j \in N \), consider the ranking solution \( R_{DM}^v \) defined by

\[ iR_{DM}^v j \iff v(\{i\}) \geq v(\{j\}). \]

This solution satisfies all the Properties but Property 1.

**Example 8.** [No Property 2] For any \( i \in N \), let \( B(i) \) the the largest player index within minimal winning coalitions containing player \( i \), i.e.

\[ B^v(i) = \max_{S \in \mathcal{W}^v_{\text{min}}, i \in S} \left( \min_{j \in S \setminus \{i\}} j \right). \]

Consider the ranking solution \( R_{AMWC}^v \) such that

\[
\begin{cases}
  iI_{AMWC}^v j & \text{if } i \sim^v j, \\
  iP_{AMWC}^v j & \text{if } \theta_v(i) > \theta_v(j), \\
  iP_{AMWC}^v j & \text{if } (i, j) \notin \succeq^v, (j, i) \notin \succeq^v, \theta_v(i) = \theta_v(j) \text{ and } B^v(i) > B^v(j),
\end{cases}
\]

This solution satisfies all the Properties but Property 2. [It is clear that \( R_{AMWC}^v \) satisfies properties 1 and 3. To see that \( R_{AMWC}^v \) does not satisfy Property 2, consider, for instance, games \( v \) and \( v_\pi \) of Example 5. As we noticed, a solution satisfying Property 2 should rank players 3 and 4 in the same way in both games \( v \) and \( v_\pi \). However, since \( B^v(3) = B^v(4) = 1 \) in \( v \) and \( B^{v_\pi}(3) = 1 \) and \( B^{v_\pi}(4) = 2 \) (and the two players are not in a desirable relation in both games) we have that \( 3I_{AMWC}^v 4 \), while \( 4P_{AMWC}^v 3 \).]
Example 9. [No Property 3] For each \( i \in N \), let \( \overline{\theta}_v(i) = (i_n, \ldots, i_1) \) associated to \( v \). Given \( i, j \in N \), consider the vector ranking solution \( R_{v}^L \text{LMWC} \) such that

\[
\begin{cases}
    i P_{v}^L \text{LMWC} j & \text{if } i \succ_v j, \\
    i I_{v}^L \text{LMWC} j & \text{if } \theta_v(i) = \theta_v(j), \\
    i P_{v}^L \text{LMWC} j & \text{if } (i, j) \not\in \succeq_v \text{ and } \theta_v(i) > L \theta_v(j),
\end{cases}
\]

This solution satisfies all the Properties but Property 3.

[It is easy to verify that \( R_{v}^L \text{LMWC} \) satisfies Properties 1 and 2. To see that \( R_{v}^L \text{LMWC} \) does not satisfy property 3, consider, for instance, games \( v \) and \( v' \) of Example 6. Notice that \( 1 \succ_v 2 \) and, so, \( 1 P_{v}^L \text{LMWC} 2 \). However, in game \( v' \), \( (1, 2) \not\in \succeq' \) and \( (2, 1) \not\in \succeq' \) (1 and 2 are not in desirable relation), while

\[
\overline{\theta}_{v'}(2) = (0, 1, 0, 1, 0) >_L (0, 0, 3, 1, 0) = \overline{\theta}_{v'}(1)
\]

and therefore \( 2 P_{v'}^L \text{LMWC} 1 \).]

5 Duality

In this section we investigate the connections between the LPGR solution and the criticality-based ranking introduced in [1] to rank players in a simple game. In [1] a ranking over players is defined according to the power of blocking the grand coalition to be winning. Given a simple game \( (N, v) \) a coalition \( B \subseteq N \) is called blocking coalition for \( N \) if \( v(N \setminus B) = 0 \). Let \( B^v \) be the set of all blocking coalitions in the game \( (N, v) \) and let \( B^v_{\text{min}} = \text{Min } B^v \). Denote by \( i^*_k \) the number of minimal blocking coalitions (for \( N \)) of size \( k \) containing player \( i \), so \( i^*_k = |\{B \in B^v_{\text{min}} : i \in B, |B| = k\}| \) for all \( k \in \{1, \ldots, n\} \). For each \( i \in N \), let \( \theta^*_v(i) \) be the \( n \)-dimensional vector \( \theta^*_v(i) = (i^*_1, \ldots, i^*_n) \) associated to \( v \).

The criticality-based ranking is based on the idea that the smaller is the size of a blocking coalition, the larger is the influence on the blocking power of its members; the ranking of a player in terms of blocking power is positively correlated first to the size of minimal blocking coalitions the player belongs to and second to their number.

Definition 3. The criticality-based solution is the function \( R_C : SG^N \rightarrow T^N \) defined for any simple game \( v \in SG^N \) as

\[
i R_C^v j \quad \text{if} \quad \theta^*_v(i) \succeq_L \theta^*_v(j).
\]

Let \( I_C^v \) and \( P_C^v \) be the symmetric part and the asymmetric part of \( R_C^v \), respectively.

Example 10. Consider the simple game of Example 1 then we have that

\[
B^v_{\text{min}} = \{\{1, 4\}, \{1, 3, 5\}, \{2, 3, 4\}, \{2, 3, 5\}\}.
\]
Therefore,
\[ \theta^*_v(1) = (0, 1, 1, 0, 0), \quad \theta^*_v(2) = (0, 0, 2, 0, 0), \]
\[ \theta^*_v(3) = (0, 0, 3, 0, 0), \quad \theta^*_v(4) = (0, 1, 1, 0, 0), \]
\[ \theta^*_v(5) = (0, 0, 2, 0, 0). \]

So, the criticality-based ranking is such that
\[ 1 \overset{I_c}{\sim} 4 \overset{P}{\sim} 3 \overset{P}{\sim} 2 \overset{I_c}{\sim} 5. \]

We first show that the LPGR coincides with the criticality-based ranking of the dual game.

**Proposition 3.** Let \((N, v)\) be a simple game. Then \(R^v_l = R^*_c\).

**Proof.** Given the simple game \((N, v)\) its dual \(v^*\) is defined by
\[ v^*(S) = v(N) - v(N \setminus S), \tag{5} \]
for each coalition \(S \in 2^N\). The proposition follows recalling that \(W^v_{\text{min}} = B^v_{\text{min}}\), as proved in Proposition 3 in [1], and then \(\theta_v = \theta^*_v\). \(\square\)

As an immediate consequence of Proposition 3 we have that the set of axioms proposed in [1] to characterize the criticality-based ranking can be easily adapted to provide an alternative axiomatic characterization of the LPGR solution where the axioms of dual monotonicity and of anonymity (defined over the set of minimal winning coalitions instead of the minimal blocking coalitions as done in [1]) replace the desirable monotonicity property used in this paper.

On the other hand, it is also interesting to study under which conditions the LPGR and the criticality-based ranking coincide. To this purpose, we analysis the behaviour of the desirability relation on a simple game \(v\) and its dual \(v^*\).

**Proposition 4.** Given a simple game \((N, v)\) and the dual game \((N, v^*)\) then, \(\forall i, j \in N, i \neq j\)
\[ i \succ^v j \iff i \succ^{v^*} j. \]

**Proof.** \(\Rightarrow\) By hypothesis, for all \(S \subseteq N \setminus \{i, j\}\), \(S \cup \{j\} \in W^v\) implies that \(S \cup \{i\} \in W^v\). We want to prove that for all \(S \subseteq N \setminus \{i, j\}\) such that \(N \setminus \{S \cup \{j\}\} \notin W^v\) implies that \(N \setminus \{S \cup \{i\}\} \notin W^v\). Suppose that \(N \setminus \{S \cup \{i\}\}\) is winning then define \(T := N \setminus \{S \cup \{i\}\}\). We observe that \(T \cup \{j\}\) is winning then \(T \cup \{i\} = N \setminus \{S \cup \{j\}\}\) is winning, i.e. a contradiction.

\(\Leftarrow\) By hypothesis, \(\forall S \subseteq N \setminus \{i, j\}, N \setminus \{S \cup \{j\}\} \notin W^v\) implies that \(N \setminus \{S \cup \{i\}\} \notin W^v\). We want to prove that for all \(S \subseteq N \setminus \{i, j\}, S \cup \{j\} \in W^v\) implies that \(S \cup \{i\} \in W^v\). Suppose \(S \cup \{i\} \notin W^v\) and let \(T = N \setminus \{S \cup \{i, j\}\}\). We observe that on the one hand \(N \setminus \{T \cup \{i\}\}\) is not winning, but on the other hand \(N \setminus \{T \cup \{i\}\} = S \cup \{j\}\) is winning, i.e. a contradiction, and the proof is complete. \(\square\)
Corollary 1. Let \((N, v)\) be a simple game such that the desirability relation is total. Then, \(R^w_i = R^\ell_i\).

Example 11. Consider the weighted majority game \((N, v)\) of Example 4. The minimal winning and blocking coalitions are

\[
\mathcal{W}^v_{\text{min}} = \{\{1, 2\}, \{1, 3, 4\}, \{1, 3, 5\}, \{1, 4, 5\}\},
\]
\[
\mathcal{B}^v_{\text{min}} = \{\{1\}, \{2, 3, 4\}, \{2, 3, 5\}, \{2, 4, 5\}\}.
\]

We have that

\[
\theta^*(1) = (1, 0, 0, 0, 0), \quad \theta^*(2) = (0, 0, 3, 0, 0),
\]
\[
\theta^*(3) = (0, 0, 2, 0, 0), \quad \theta^*(4) = (0, 0, 2, 0, 0),
\]
\[
\theta^*(5) = (0, 0, 2, 0, 0).
\]

So, the criticality-based ranking is such that

\[
1 \ P^c_1 \ 2 \ P^c_2 \ 3 \ I^c_3 \ 4 \ I^c_4 \ 5 .
\]

So, as expected for a weighted majority game in which the desirable relation is total, \(R^w_i = R^\ell_i\).

6 Conclusions

In this paper, using the minimal winning coalitions of a simple game, we introduced a new ranking among players that satisfies the desirability relation: the Lexicographic Public Good ranking. The players are ranked according to the size of the minimal winning coalitions they belong to and then to the number of such coalitions. The ranking solution satisfies the coalitional anonymity property and the independence of larger minimal winning coalitions property that together with a monotonicity property rooted on the desirability relation uniquely characterized it. Looking at the dual game, we prove that there is a relation between the Lexicographic Public Good ranking and the criticality-based ranking and, consequently, between ranking players according to their power to win and to their power to block the grand coalition. In particular, if the desirable relation is total the two rankings coincide.

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