CIRCULAR AVERAGE RELATIVE TO FRACTAL MEASURES

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Abstract. We prove new $L^p$–$L^q$ estimates for averages over dilates of the circle with respect to fractal measures, which unify different types of maximal estimates for the circular average. Our results are consequences of $L^p$–$L^q$ smoothing estimates for the wave operator relative to fractal measures. We also discuss similar results concerning the spherical averages.

1. Introduction

Let $d \geq 2$. We consider the average

$$A f(x,t) = \int_{S^{d-1}} f(x-ty) d\sigma(y), \quad x \in \mathbb{R}^d.$$ 

Here $d\sigma$ denotes the normalized measure on the unit sphere $S^{d-1}$. We study estimates for $A$ relative to fractal measures. To motivate our study, we briefly review some of the previous results.

Stein [26] ($d \geq 3$) and Bourgain [4] ($d = 2$) proved that the maximal operator $f \mapsto \sup_{t>0} |A f(\cdot, t)|$ is bounded on $L^p$ if and only if $d^{d-1} < p \leq \infty$. If the supremum is taken over a compact interval $I$ contained in $(0, \infty)$, the consequent maximal function has the $L^p$ improving property ([24, 25, 14]). That is to say, the estimate

$$\| \sup_{t \in I} |A f(\cdot, t)| \|_{L^q(I)} \leq C_d \| f \|_{L^p(\mathbb{R}^d)}$$

holds for some $p < q$. Except for a few endpoint cases, we now have a complete characterization of $p,q$ for which (1.1) holds. Let $P_d \subset [0,1] \times [0,1]$ be the convex hull of the set consisting of the points $O := (0,0)$, $Q_1 := \left(\frac{d-1}{d^{d-1}}, \frac{d-1}{d^{d-1}}\right)$, and $Q_2 := \left(\frac{d-1}{d^{d-1}}, \frac{d-1}{d^{d-1}}\right)$. Schlag [24] ($d = 2$), and Schlag and Sogge [25] ($d \geq 3$) showed that the estimate (1.1) holds for $(p^{-1}, q^{-1}) \in P_d \setminus ((O, Q_1) \cup (Q_1, Q_2) \cup (Q_2, Q_3))$ and it fails if $(p^{-1}, q^{-1}) \notin (P_d \setminus \{Q_3\})$. The third named author [14] proved the borderline case $(p^{-1}, q^{-1}) \in (O, Q_1) \cup (Q_1, Q_2) \cup (Q_2, Q_3)$, however the case $(p^{-1}, q^{-1}) = Q_1, Q_2$ remain open. See also [23, 14] for recent developments regarding the circular and spherical maximal functions.

Different forms of maximal estimates have also been of interest. In particular, the estimate

$$\| \sup_{x \in \mathbb{R}^d} |A f(x, \cdot)| \|_{L^q(I)} \leq C_d \| f \|_{L^p(\mathbb{R}^d)}$$

holds for $p < q$. Except for a few endpoint cases, we now have a complete characterization of $p,q$ for which (1.2) holds. Let $P_d \subset [0,1] \times [0,1]$ be the convex hull of the set consisting of the points $O := (0,0)$, $Q_1 := \left(\frac{d-1}{d^{d-1}}, \frac{d-1}{d^{d-1}}\right)$, and $Q_2 := \left(\frac{d-1}{d^{d-1}}, \frac{d-1}{d^{d-1}}\right)$. Schlag [24] ($d = 2$), and Schlag and Sogge [25] ($d \geq 3$) showed that the estimate (1.2) holds for $(p^{-1}, q^{-1}) \in P_d \setminus ((O, Q_1) \cup (Q_1, Q_2) \cup (Q_2, Q_3))$ and it fails if $(p^{-1}, q^{-1}) \notin (P_d \setminus \{Q_3\})$. The third named author [14] proved the borderline case $(p^{-1}, q^{-1}) \in (O, Q_1) \cup (Q_1, Q_2) \cup (Q_2, Q_3)$, however the case $(p^{-1}, q^{-1}) = Q_1, Q_2$ remain open. See also [23, 14] for recent developments regarding the circular and spherical maximal functions.
for some $\epsilon > 0$ was used to study packing problems for the circle and sphere $\mathbb{S}^d$ [18, 20]. Here $\|f\|_{L^p(N)} := \|(1 - \Delta)^{\frac{\epsilon}{2}} f\|_{L^p(N)}$. When $d \geq 3$, Kolasa and Wolff [13] showed (1.2) with $p = q = 2$ for any $\epsilon > 0$. However, a similar estimate in $\mathbb{R}^2$ turned out to be more difficult. Wolff [27] obtained (1.2) with $p = q = 3$ for $\epsilon > 0$. Precisely, results in [27, 13] are given in a different form in which averages over the sphere (circle) were replaced by the average over annuli of thickness $\delta$. However, it is easy to see that those estimates are equivalent to the abovementioned estimates in the form of (1.2). It is also known that (1.2) fails if $\epsilon$ is removed (see [3, 12] or [13, Proposition 2.2] for example). Wolff’s result was extended to a variable coefficient setting by Zahl [29].

**Definition 1.1.** For $\alpha \in (0, d + 1]$, we say a non-negative Borel measure $\nu$ on $\mathbb{R}^{d+1}$ is $\alpha$-dimensional if there is a constant $C_{\nu}$ such that

$$
\nu(\mathbb{B}^{d+1}(z, \rho)) \leq C_{\nu} \rho^{\alpha}, \quad \forall (z, \rho) \in \mathbb{R}^{d+1} \times \mathbb{R}^+,
$$

where $\mathbb{B}^{d+1}(z, \rho) = \{z' \in \mathbb{R}^{d+1} : |z - z'| < \rho\}$. By $\mathcal{C}^{d+1}(\alpha)$ we denote the class of $\alpha$-dimensional measure $\nu$. For $\nu \in \mathcal{C}^{d+1}(\alpha)$, we also define

$$
\langle \nu \rangle_{\alpha} := \sup_{z \in \mathbb{R}^{d+1}, \rho > 0} \rho^{-\alpha} \nu(\mathbb{B}^{d+1}(z, \rho)).
$$

For the rest of the paper, we fix $I = (1, 2)$. We consider the estimate

$$
(1.4) \quad \|\mathcal{A}f\|_{L^p(I^d \times \mathbb{R}^{d+d})} \leq C(\nu)^{\frac{1}{\alpha}} \|f\|_{L^p(I^d)}.
$$

Remarkably, the estimate (1.4) unifies the maximal estimates (1.1) and (1.2) in a single framework. Indeed, it is not difficult to see that (1.4) implies the seemingly unrelated estimates (1.1) and (1.2). One can deduce the estimate (1.4) from (1.4) if the latter holds with a uniform $C$ for all $d$-dimensional measures with $\langle \nu \rangle_{d} \leq 1$. This can be shown by the Kolmogorov-Seliverstov-Plessner linearization argument and the Riesz representation theorem (see Lemma 3.2). Similarly, (1.2) follows if $1.4$ with $L^p$ replaced by $L^p$ holds uniformly for all 1-dimensional measures.

Since the averaging operator $\mathcal{A}$ is translation-invariant in $x$, it is not possible to have (1.4) unless $1 \leq p \leq q \leq \infty$. Also, taking a specific $\alpha$-dimensional measure, one can show (1.4) holds only if $\alpha p \geq q$ (see (i) in Proposition 4.1). Thus it is natural for (1.4) to assume $\alpha > 1$ and $p \leq q$.

When $d \geq 3$ the estimate (1.4) is relatively easier to obtain. There are various estimates which are straightforward consequences of the fractal Strichartz estimates for the wave equation (1.10). We discuss the matter in Section 3. However, when $d = 2$, such estimates are not enough to prove the estimate (1.4). In this paper we focus on the more interesting case $d = 2$.

**The circular average.** Let $1 < \alpha \leq 3$. We define $\mathcal{P}_2(\alpha) \subset [0, 1]^2$. If $1 < \alpha \leq 2$, we set

$$
\mathcal{P}_2(\alpha) := \left\{\left(\frac{1}{p}, \frac{1}{q}\right) \in [0, 1]^2 : \frac{1}{q} \leq \frac{1}{p} < \frac{\alpha}{q}, \quad \frac{3}{p} < 1 + \frac{\alpha - 1}{q}\right\},
$$

and for $2 < \alpha \leq 3$,

$$
\mathcal{P}_2(\alpha) := \left\{\left(\frac{1}{p}, \frac{1}{q}\right) \in [0, 1]^2 : \frac{1}{q} \leq \frac{1}{p} < \frac{\alpha}{q}, \quad \frac{3}{p} < 1 + \frac{2\alpha - 3}{q}, \quad \frac{2}{p} < 1 + \frac{\alpha - 2}{q}\right\}.
$$
In Section 4 we show that (1.4) fails if \((p^{-1}, q^{-1}) \notin \mathcal{P}_2(\alpha)\) when \(d = 2\). It seems to be reasonable to conjecture that \(\mathcal{P}_2(\alpha)\) determines, possibly except for the borderline cases, the optimal range of \(p, q\) on which (1.4) holds.

**Conjecture 1.2.** Let \(d = 2\) and \(1 < \alpha \leq 3\). The estimate (1.4) holds for \(\nu \in \mathcal{C}^3(\alpha)\) if \((p^{-1}, q^{-1}) \in \mathcal{P}_2(\alpha)\).

The following is our main result, which verifies the conjecture for \(\alpha \in [3 - \sqrt{3}, 3]\).

**Theorem 1.3.** Let \(d = 2\), \(1 < \alpha \leq 3\), and \(\nu \in \mathcal{C}^3(\alpha)\). Then, the estimate (1.4) holds if

\begin{equation}
(1.5) \quad (p^{-1}, q^{-1}) \in \mathcal{P}_2(\alpha) \quad \text{and} \quad p > (6 - 2\alpha)/\alpha.
\end{equation}

Note that \(4 - \alpha \geq (6 - 2\alpha)/\alpha\) if \(\alpha \geq 3 - \sqrt{3}\) and \(p > 4 - \alpha\) whenever \((1/p, 1/q) \in \mathcal{P}_2(\alpha)\). As a corollary we obtain \(L^p(\mathbb{R}^2) - L^q(d\mu)\) estimate for the circular maximal function relative to fractal measures.

**Corollary 1.4.** Let \(1 < \alpha \leq 2\) and \(\mu \in \mathcal{C}^2(\alpha)\). For \(p, q\) satisfying (1.5), we have

\begin{equation}
(1.6) \quad \left\| \sup_{t \in I} |A f(\cdot, t)| \right\|_{L^q(d\mu)} \leq C(\mu)^{\frac{1}{\alpha}} \|f\|_{L^p(\mathbb{R}^2)}.
\end{equation}

A modification of the examples which give the necessary conditions of (1.4) (Proposition 4.1) shows that (1.6) holds only if \((p^{-1}, q^{-1}) \in \mathcal{P}_2(\alpha)\). Consequently, Corollary 1.4 establishes boundedness on sharp range for \(\alpha \in (3 - \sqrt{3}, 2]\).

\(L^p(\mathbb{R}^d) - L^q(d\mu)\) estimate for \(M f := \sup_{t \in I} |A f(\cdot, t)|\) has been utilized to study problems in geometric measure theory. \(L^2(\mathbb{R}^d) - L^2(d\mu)\) estimate was studied by Mitsis [18] for \(1 < \alpha \leq d\), \(d \geq 3\). D. Oberlin and R. Oberlin [21] obtained \(L^2_{(1-\alpha)/2+}(\mathbb{R}^d) - L^q(d\mu)\) bound on \(M f\) when \(1 < q < 2\) and \(0 < \alpha < (d - 1)/2\). Also, for \(d \geq 3\), \(L^p(d\mu_1) - L^p(d\mu_2)\) estimate was discussed in Iosevich et al. [14] with \(\alpha_i\)-dimensional measure \(\mu_i\), \(1 < \alpha_i \leq d\), \(i = 1, 2\). Those estimates were used to study the sphere packing problem, Hausdorff dimension of the pinned distance set, and the divergence set of the solution to the wave equation (see also [9]).

Compared with the previous results, it is remarkable that Corollary 1.4 establishes the sharp \(L^p\) improving property for the circular maximal function relative to some \(\alpha\)-dimensional measures. As far as the authors are aware, no such results have appeared before. In [14] non-sharp local smoothing estimates were combined with a duality argument. However, the estimates in Corollary 1.4 cannot be obtained by the approach in [14] even if combined with the sharp local smoothing estimate [8]. Nevertheless, such estimates are easier to prove (see Corollary 4.5) in higher dimensions.

Furthermore, one can use the estimate (1.6) (with \(p = q\)) to prove the circle packing theorem which was shown in [28]: Suppose \(F \subset \mathbb{R}^2\) is a Borel set of Hausdorff dimension exceeding 1 and \(E\) is a compact set such that \(E \subset \mathbb{R}^2\) contains circles centered at each point in \(F\). Then \(E\) has positive Lebesgue measure.

Corollary 1.4 with \(\alpha = 2\) recovers the results on the circular maximal function except for the endpoint cases ([24], [25], [14]).

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1As seen in the above, some of the borderline cases are actually not true.
Local smoothing estimate relative to $\nu \in \mathcal{C}^3(\alpha)$. In order to prove Theorem 1.3, we consider $L^p_\nu - L^q_\nu(\nu)\nu$ estimate for the wave operator

$$e^{it\sqrt{-\Delta}}f(x) = \int e^{i(x\xi + t|\xi|)}\hat{f}(\xi)\,d\xi.$$  

For $2 < \alpha \leq 3$, we obtain the estimate (1.3) by relying mainly on $L^2_\nu - L^q_\nu(\nu)$ estimate for $e^{it\sqrt{-\Delta}}$, which is to be discussed in Section 3. However, for $1 < \alpha \leq 2$, such $L^2_\nu - L^q_\nu(\nu)$ estimates are not enough for our purpose. Instead, we exploit estimates for the wave operator relative to $\alpha$-dimensional measure $\nu$:

$$\left(\int_{B^2(0,1)\times I} |e^{it\sqrt{-\Delta}}f(x)|^q\,d\nu(x,t)\right)^{1/q} \leq C(\nu)^{\frac{1}{2}}\|f\|_{L^p_\nu(\mathbb{R}^2)}.$$  

Conjecture 1.2. When $2 < \alpha \leq 3$, the regularity assumption $\gamma > \frac{d}{2} + \frac{1}{p} - \frac{\alpha}{q}$ is sharp in general if $\gamma < \frac{1}{2} + \frac{1}{p} - \frac{\alpha}{q}$ (see Section 2). It seems to be natural to conjecture that the same remains valid provided that $\frac{1}{q} \leq \frac{1}{p}$ and $\frac{1}{p} + \frac{\kappa(\alpha)}{q} \leq 1$. This in turn implies Conjecture 1.2. When $2 < \alpha \leq 3$, $\frac{1}{q} \leq \frac{1}{p}$, and $\frac{1}{p} + \frac{\alpha}{q} \leq 1$, the estimate (1.7) is a consequence of the sharp local smoothing estimate [8]. In higher dimensions ($d \geq 3$), the local smoothing estimate in [5] and Lemma 2.7 yield an analogue of (1.7) for $d \leq \alpha \leq d + 1$ whenever $\gamma > \frac{d - 1}{2} + \frac{1}{p} - \frac{\alpha}{q}$, $\frac{1}{p} + \frac{\alpha}{q} \leq 1$, and $p \leq q$.

Estimate for the spherical average. In higher dimensions one can obtain results similar to Theorem 1.3 for the spherical average in $\mathbb{R}^{d+1}$, $d \geq 3$ (see Theorem 3.4). Such results can be shown by making use of the Strichartz estimates for the wave equation relative to fractal measure (see Theorem 3.3). The spherical average estimates in Theorem 3.4 seem to be sharp for $d = 3$, while we only manage to verify the optimality for integer $\alpha$. See Section 4.
Organization of the paper. In Section 2, we obtain the local smoothing estimates relative to $\alpha$-dimensional measure (Proposition 2.1 and Proposition 2.12), which we use to prove Theorem 1.5 and Theorem 1.6. In Section 3, we prove Theorem 1.6, Theorem 1.3, and Corollary 1.4 and obtain the estimates in higher dimensions. We discuss the sharpness of the estimate (1.4) in Section 4.

2. Weighted local smoothing estimates

In this section, we obtain some preparatory results which we use in proving Theorem 1.5 and Theorem 1.6. The main object is to prove Proposition 2.1 and Proposition 2.12. To this end, we adapt the argument due to Vargas and one of the authors [16] which is based on the trilinear restriction estimate.

We decompose $e^{it\sqrt{-\Delta}}f$ into two parts: a sum of operators with small frequency supports and a product of operators with angularly separated frequency supports. For the latter we apply the multilinear restriction estimate due to Bennett, Carbery, and Tao [2] by making use of the fact that angular separation implies transversality on the trilinear operator. For the first part, we rescale the operator and apply an induction argument (see (2.5)). However, the support of a rescaled measure is no longer contained in a bounded set, so the induction cannot be carried out on a bounded set. To get around this issue, we consider a class of weight functions which are not necessarily supported in a bounded set.

2.1. Weighted local smoothing estimates in $\mathbb{R}^{2+1}$. For $0 < \alpha < 3$, we denote by $\Omega(\alpha)$ the class of nonnegative measurable functions $\omega : \mathbb{R}^2 \times \mathbb{R} \mapsto \mathbb{R}$ which satisfy

\begin{equation}
\int_{\mathbb{R}^3} \omega(y)dy \leq r^\alpha, \quad \forall (z, r) \in \mathbb{R}^3 \times \mathbb{R}_+.
\end{equation}

By Littlewood-Paley decomposition, it is enough to consider the estimate

\begin{equation}
\|e^{it\sqrt{-\Delta}}P_\lambda f\|_{L^q(\mathbb{R}^2 \times I; \omega)} \leq C\lambda^\gamma \|f\|_p.
\end{equation}

Here $P_\lambda$ is the standard Littlewood-Paley projection operator given by $\hat{P_\lambda f}(\xi) = \beta(\lambda^{-1}|\xi|)\hat{f}(\xi)$ for $\beta \in C^\infty([1/2, 2])$ and $\|f\|_{L^q(\mathbb{R}^2 \times I; \omega)} := \int_{\mathbb{R}^2} |f(x)|^q \omega(x)dx$.

Instead of proving (2.2) directly, we consider a modified operator to facilitate rescaling. We consider

\begin{equation}
\psi_0(\theta) := \sqrt{1 + \theta^2} - 1.
\end{equation}

Note that $\psi_0$ can be regarded as a small perturbation of $\theta^2/2$.

For $\lambda \geq 1$ and $J \subset [-1, 1]$, let

\[ A_\lambda(J) = \{(\eta, \rho) : \lambda/4 \leq \eta \leq 4\lambda, \ \theta := \rho/\eta \in J\}. \]

We define

\begin{equation}
T_\lambda^{\psi_0}g(x, t) = \int_{A_\lambda([-1, 1])} e^{i(x, t) \cdot (\eta, \rho, \eta \phi(\rho/\eta))} \beta(\lambda^{-1}\eta)\beta_0(|\rho|/\eta)\tilde{g}(\eta, \rho)d\eta d\rho,
\end{equation}

where $\beta_0$ is a smooth function supported in $[0, 2]$.

Proposition 2.1. Let $1 < \alpha < 2$ and $\omega \in \Omega(\alpha)$. If $p, q \geq 1$ satisfy (1.6) and $\lambda \gg 1$, then for some $\gamma < 1/2$ we have

\begin{equation}
\|T_\lambda^{\psi_0}g\|_{L^q(\mathbb{R}^2 \times I; \omega)} \leq C\lambda^\gamma \|g\|_p.
\end{equation}

We deduce the estimate (2.2) from (2.4) via a change of variable (see the proof of Theorem 1.5 in Section 3).
Normalization of the phase function. For our induction argument, we begin with normalizing the phase function of $T_\lambda g$. Let

$$\phi_\epsilon(\theta) = \frac{1}{2}\theta^2.$$  

For a given $\epsilon > 0$, we define a class $\mathcal{S}(\epsilon)$ of $C^3$ functions by

$$\mathcal{S}(\epsilon) = \{ \phi \in C^3([-1,1]) : \phi(0) = \phi'(0) = 0, \| \phi - \phi_\epsilon \|_{C^3([-1,1])} \leq \epsilon \},$$

where $\| \phi \|_{C^3([-1,1])} = \sum_{j\leq n} \sup_{\theta \in [-1,1]} |\partial_\theta^j \phi(\theta)|$. For $\theta \in (-1,1)$, let $h$ be a real number such that $[\theta - h, \theta + h] \subset [-1,1]$. For $\phi \in C^3([-1,1])$ satisfying $\phi''(\theta) \neq 0$, we define

$$\phi_{\theta, h}(\theta) = (\phi''(\theta) h^2)^{-1} (\phi(h\theta + \theta) - \phi(h\theta) - \phi'(\theta)h\theta).$$

**Lemma 2.2.** Suppose $\phi \in \mathcal{S}(\epsilon)$ for some $0 < \epsilon_0 < 1/8$. Then, there exists $0 < h_0 = h_0(\epsilon_0)$ such that $\phi_{\theta, h} \in \mathcal{S}(\epsilon_0)$ whenever $0 < h \leq h_0$ and $[\theta - h, \theta + h] \subset [-1,1]$.

Throughout the paper, $\epsilon_0$ denotes a positive constant such that $0 < \epsilon_0 < 1/8$.

**Proof.** By Taylor series expansion, we have

$$\phi(h\theta + \theta) - \phi(h\theta) - \phi'(\theta)h\theta = \frac{\phi''(\theta) h^2 \theta^2}{2} + \mathcal{E}(\theta, h, \theta),$$

where $\| \mathcal{E}(\theta, h, \cdot) \|_{C^3([-1,1])} \leq C h^3$ with a constant $C$ independent of $\theta \in [-1,1]$. Since $\phi \in \mathcal{S}(\epsilon_0)$, $\phi''(\theta) \in [3/4, 5/4]$. So, we see

$$\| \phi_{\theta, h} - \phi_\epsilon \|_{C^3([-1,1])} \leq (\phi''(\theta) h^2)^{-1} \| \mathcal{E}(\theta, h, \cdot) \|_{C^3([-1,1])} \leq Ch.$$  

Thus, there exists $h_0 \in (0, \epsilon_0]$ such that $\phi_{\theta, h} \in \mathcal{S}(\epsilon_0)$ for $0 < h \leq h_0$. \qed

**Induction quantity.** As mentioned before, our proof is based on an induction on scale argument. We introduce a quantity which makes it possible to carry out the induction argument.

For $1 \leq p \leq q \leq \infty$, we define $Q(\lambda) = Q(\lambda, \epsilon_0, p, q, \alpha)$ by

$$Q(\lambda) = \sup \left\{ \| T^\phi g \|_{L^p(\mathbb{R}^2 \times I, \omega)} : \operatorname{supp} \widehat{g} \subset A_\lambda([-1,1]), \| g \|_p \leq 1, \phi \in \mathcal{S}(\epsilon_0), \omega \in \Omega(\alpha) \right\}.$$

(2.5)

It is easy to see $Q(\lambda) < \infty$. Indeed, one can show a rough bound

$$Q(\lambda) \lesssim \lambda^2, \quad \lambda \geq 1$$  

(2.6)

for $1 \leq p \leq q \leq \infty$. We write $T^\phi g(x, t) = K_\lambda(\cdot, t) \ast g$, where the kernel is given by

$$K_\lambda(x, t) = \int e^{i(x,\cdot) \cdot (\eta, \rho)} \eta^{\phi(\rho/\eta)} \beta(\lambda^{-1} \eta) \beta_0(|\rho/\eta|) d\eta d\rho.$$  

(2.7)

In order to show (2.6), we use an easy estimate:

$$|K_\lambda(x, t)| \lesssim K(x) := \lambda^2 (1 + |x|)^{-M}, \quad t \in I$$  

(2.8)

for any $M \geq 1$, which follows by routine integration by parts (for example, see (3.6) below). Using (2.1), it is easy to see

$$\| \int K(x - y) g(y) dy \|_{L^q(\mathbb{R}^2 \times I, \omega)} \lesssim \lambda^2 \| g \|_p$$  

(2.9)
for $1 \leq p \leq q$. Indeed, since $\int J \int K(x-y) \omega(x,t)\,dx\,dt \lesssim \lambda^2$ and $\int K(x-y)\,dy \lesssim \lambda^2$, (2.4) follows for $p = q$ by Schur’s test. Also, by Hölder’s inequality, we get (2.3) for $q = \infty$ and $1 \leq p \leq \infty$. Interpolating those estimates, we have (2.9) and therefore (2.6) for $1 \leq p \leq q$.

In what follows, we show $Q(\lambda) \leq C_\epsilon \lambda^{4-\epsilon}$ for some $\epsilon > 0$.

**Rescaling.** We first observe that $L^p-L^q$ bound on $T^\phi_\lambda$ is improved on a certain range of $p,q$ if $\hat{g}$ is supported in a narrow conic neighborhood. More precisely, we have the following.

**Lemma 2.3.** Let $0 < \alpha \leq 3$, $\omega \in \Omega(\alpha)$, and $\phi \in \mathcal{S}(\epsilon_0)$. For $h \geq \lambda^{-\frac{1}{2}}$, let $J_h = [\theta - h, \theta + h] \subseteq [-1,1]$. Suppose $\text{supp } \hat{g} \subseteq A_\lambda(J_h)$. Then, there is an $h_o = h_o(\epsilon_0)$ such that

$$||T^\phi_\lambda g||_{L^q(\mathbb{R}^2 \times \omega)} \leq Ch^{\frac{2\alpha - \nu(\alpha)}{q}} \frac{3}{\nu} Q(\phi''(\theta)h^2 \lambda)||g||_p$$

whenever $h \leq h_o$.

We note $\nu(\alpha) = \min\{3, \alpha + 1, 2\alpha\}$ (see (1.8)).

**Proof.** Since $\hat{g}(\eta,\rho)$ is supported in $A_\lambda(J_h)$, we have $\lambda/4 \leq \eta \leq 4\lambda$ and $\theta = \rho/\eta \in J_h$ i.e., $|\theta - \theta_o| \leq h$. We set

$$D_{\theta_o,h} = \frac{1}{\phi''(\theta_o)} \begin{bmatrix} h^{-2} & h^{-2} \phi(\theta_o) & h^{-2} \phi'(\theta_o) \\ 0 & h^{-1} \phi'(\theta_o) & 0 \\ 0 & 0 & \phi''(\theta_o) \end{bmatrix},$$

and

$$L_{\theta_o,h}(\eta,\rho) = \frac{1}{\phi''(\theta_o)}(h^{-2} \eta, h^{-1} \rho + h^{-2} \theta_o, \eta).$$

Let $z = (x,t)$. Note that $z \cdot (\eta,\rho,\eta \phi(\rho/\eta)) \mapsto D_{\theta_o,h} z \cdot (\eta,\rho,\eta \phi_{\theta_o,h}(\rho/\eta))$ under the transformation $(\eta,\rho) \mapsto D_{\theta_o,h}(\eta,\rho)$. Changing variables $(\eta,\rho) \mapsto L_{\theta_o,h}(\eta,\rho)$, we obtain

$$T^\phi_\lambda g(z) = T^\phi_{\lambda_1} g_{\theta_o,h}(D_{\theta_o,h}z), \quad \lambda_1 = \phi''(\theta_o)h^2 \lambda,$$

where

$$\phi_{\theta_o,h}(\eta,\rho) = \det(L_{\theta_o,h}) \hat{g}(L_{\theta_o,h}(\eta,\rho)).$$

Clearly, $\phi_{\theta_o,h}$ is supported in $A_{\lambda_1}([-1,1])$. By Lemma 2.2 there exists $h_o = h_o(\epsilon_0)$ such that $\phi_{\theta_o,h} \in \mathcal{S}(\epsilon_0)$ for $h \leq h_o$.

Let us define

$$\omega_{\theta_o,h}(z) = (1 + \tilde{C})^{-1} h^{\nu(\alpha) - 2\alpha + 3} \omega(D_{\theta_o,h}^{-1}z),$$

where $\tilde{C} > 0$ is a constant to be chosen later. We claim $\omega_{\theta_o,h} \in \Omega(\alpha)$ if $\tilde{C} > 0$ is large enough. To show this, we first observe

$$\int_{\mathbb{R}^2 \times (y_o,r)} \omega(D_{\theta_o,h}^{-1}z) \,dz = h^{-3} \int_{D_{\theta_o,h}^{-1} \mathbb{B}^3(y_o,r)} \omega(z) \,dz.$$

So, it is sufficient to show

$$\int_{D_{\theta_o,h}^{-1} \mathbb{B}^3(y_o,r)} \omega(z) \,dz \leq Cr^{\alpha} h^{2\alpha - \nu(\alpha)}.$$  

Note that $D_{\theta_o,h}^{-1} \mathbb{B}^3(y_o,r)$ is contained in a box of dimensions $ch^2 r \times ch r \times cr$ for a constant $c > 0$. We can cover $D_{\theta_o,h}^{-1} \mathbb{B}^3(y_o,r)$ with as many as $Ch^{-3}, Ch^{-1},$ and $C$
Proposition 2.5. We get the following.

Changing variables $z \mapsto D_{\theta_z,h}^{-1} z$, we have

$$\|T^\phi_T \|_{L^q(I \times \mathbb{R}^2; L^1(\Omega \times I: \omega))}^q = (1 + \overline{C}) h^{2\alpha - \kappa(\alpha)} \int_{\mathbb{R}^2 \times I} |T^\phi_{\lambda} g_{\theta_z,h}(z)|^q \omega_{\theta_z,h}(z) dz. \tag{2.12}$$

Since $\phi_{\theta_z,h} \in \mathcal{S}(\epsilon_\alpha)$, $\omega_{\theta_z,h} \in \Omega(\alpha)$, and $\overline{g_{\theta_z,h}}$ is supported in $A_{\lambda,1}([-1,1])$, by the definition of $Q$ we have

$$\int_{\mathbb{R}^2 \times I} |T^\phi_{\lambda} g_{\theta_z,h}(z)|^q \omega_{\theta_z,h}(z) dz \leq \left( Q(\phi''(\theta_z) h^2 \lambda) \|g_{\theta_z,h}\|_p \right)^q. \tag{2.13}$$

Therefore, we get (2.10) since $\|g_{\theta_z,h}\|_p = h^{-\frac{q}{2}} \|g\|_p$. \ \square

2.2. Trilinear estimate. In this section, we obtain a weighted local $L^p - L^q$ estimate for the trilinear operator $\prod_{i=1}^3 T_{\lambda}^\phi g_i$ by interpolating the multilinear restriction estimate due to Bennett, Carbery and Tao [2], the local smoothing estimate in [8], and an easy $L^1 - L^\infty$ estimate. Subsequently, we extend the local estimate to a global one by using decay of the kernel $K_\lambda$. For simplicity, we write

$$T_\lambda = T_{\lambda}^\phi.$$

For $\phi \in \mathcal{S}(\epsilon_\alpha)$ and an interval $J \subset [-1,1]$, we consider a truncated conic surface

$$\Gamma_{\lambda}(J) := \{ (\eta, \rho, \tau) \in \mathbb{R}^3 : \tau = \eta \phi(\rho/\eta), (\eta, \rho) \in A_{\lambda,1}(J) \}.$$

Let $N(\theta)$ be the normal vector to $\Gamma_{\lambda}(J)$ at $\eta(1, \theta, \phi(\theta))$. A computation gives

$$N(\theta) = (\phi(\theta) - \theta \phi'(\theta), \phi'(\theta), -1). \tag{2.14}$$

The following is a consequence of the multilinear restriction estimate due to Bennett-Carbery-Tao [2]. We denote by $d\sigma_{\lambda}$ the surface measure on $\Gamma_{\lambda}([-1,1])$.

**Theorem 2.4.** Let $\phi \in \mathcal{S}(\epsilon_\alpha)$ and $J_1, J_2, J_3 \subset [-1,1]$ be intervals. Suppose

$$\det( N(\theta_1), N(\theta_2), N(\theta_3)) \geq \delta$$

holds for some $\delta > 0$ whenever $\theta_i \in J_i, i = 1, 2, 3$. If $\epsilon_\alpha$ is sufficiently small and $\lambda \gg \delta^{-1}$, then for $\epsilon > 0$

$$\left\| \prod_{i=1}^3 f_i d\sigma_{\lambda} \right\|_{L^1(\mathbb{R}^2 \times I)} \leq C \lambda^\chi \prod_{i=1}^3 \|f_i\|_{L^2(\Gamma_{\lambda}(J_i))}$$

whenever $\text{supp } f_i \subset \Gamma_{\lambda}(J_i), i = 1, 2, 3$.

By interpolation with the local smoothing estimate and an easy $L^1 - L^\infty$ estimate, we get the following.

**Proposition 2.5.** Let $\phi \in \mathcal{S}(\epsilon_\alpha)$, and let $J_1, J_2, J_3 \subset [-1,1]$ be intervals. Also let

$$\frac{1}{q} \leq \frac{1}{r} = \min \left( \frac{1}{p}, \frac{1}{3p} + \frac{2}{\gamma_0}, \frac{2}{3p} \right) \text{ for } 1 \leq p \leq \infty. \tag{2.15}$$

Suppose (2.15) holds whenever $\theta_i \in J_i, i = 1, 2, 3$. If $\epsilon_\alpha$ is sufficiently small and $\lambda \gg \delta^{-1}$, then we have

$$\left\| \prod_{i=1}^3 T_{\lambda} g_i \right\|_{L^q(I \times \mathbb{R}^2 \times I)} \leq C \lambda^\gamma_0 \prod_{i=1}^3 \|g_i\|_p,$$

for $\gamma_0 > \frac{1}{2} + \frac{1}{p} - \frac{2}{q}$ whenever $\text{supp } g_i \subset A_{\lambda}(J_i), i = 1, 2, 3$. 
Proof. By interpolation it is enough to show (2.15) for \((\frac{1}{p}, \frac{1}{q}) = (0, 0), (\frac{1}{p}, \frac{1}{2}), (\frac{1}{p}, \frac{1}{4})\), and \((1, 0)\) if \(\gamma_0 > \frac{1}{2} + \frac{1}{p} - \frac{3}{q}\). For the first two cases, (2.15) follows from the local smoothing estimate (1.4) with \(dv = dx\, dt\) which holds for \(4 \leq p = q \leq \infty\) and \(\gamma_0 > \frac{1}{2} - \frac{1}{p}\). By Theorem 2.3 and Plancherel’s theorem, we have (2.15) with \((\frac{1}{p}, \frac{1}{q}) = (\frac{1}{2}, \frac{1}{2})\) for \(\gamma_0 > 0\). Also, (2.17) and van der Corput’s lemma yield \(\|K_{\lambda}\|_\infty \lesssim \lambda^{3/2}\) (e.g. (3.8)), which gives \(\|T_A g_i\|_{L^\infty(B^2(0, 1) \times I)} \lesssim \lambda^{3/2}\|g_i\|_1\). Thus, we have (2.15) with \((\frac{1}{p}, \frac{1}{q}) = (1, 0)\) for \(\gamma_0 \geq 3/2\). \(\square\)

From (2.8), we see that \(K_{\lambda}(\cdot, t)\) decays rapidly away from \(B^2(0, 1)\). Using this, we can extend the local estimate (2.15) to a global one.

**Lemma 2.6.** Under the same assumptions as in Proposition 2.5, we have

\[
(2.16) \quad \left\| \prod_{i=1}^3 T_{\lambda} g_i \right\|_{L^\frac{1}{2}(\mathbb{R}^2 \times I)} \leq C \lambda^{\gamma_0} \prod_{i=1}^3 \|g_i\|_p
\]

for \(\gamma_0 > \frac{1}{2} + \frac{1}{p} - \frac{3}{q}\).

**Proof.** We follow a standard localization argument (e.g., see [15] Proposition 2.10). For a constant \(r\) satisfying \(\lambda^{-1} < r < \delta\), let \(N_r(A(B_i))\) denote the \(r\)-neighborhood of \(A(B_i)\), \(i = 1, 2, 3\). Let \(\varphi_i\) be a smooth function supported in \(N_r(A(B_i))\) such that \(\varphi_i = 1\) on \(A(B_i)\) and \(|\partial_{\xi}^m \varphi_i(\xi)| \lesssim |\xi|^{-|m|}\).

Let us denote by \(\mathcal{F}_x\) the Fourier transform in \(x\) and define \(\mathcal{F}_x(\tilde{K}_{\lambda,i}(\cdot, t)) = \mathcal{F}_x(K_{\lambda,i}(\cdot, t)) \varphi_i\). Then \(T_{\lambda} g_i(x, t) = \tilde{K}_{\lambda,i}(\cdot, t) \ast g_i(x)\). We consider a collection \(\{B\}\) of finitely overlapping unit balls which cover \(\mathbb{R}^2\). Fixing \(\varepsilon > 0\), let \(B\) be a ball of radius \(\lambda\) with the same center as \(B\). For each \(i\), we decompose \(g_i = g_i,B + g_i,B^c\) where \(g_i,B := g_i \chi_B\) and \(g_i,B^c := g_i \chi_{B^c}\). We get

\[
\left\| \prod_{i=1}^3 T_{\lambda} g_i \right\|_{L^\frac{1}{2}(\mathbb{R}^2 \times I)} \lesssim \sum_B \left( \sum_{m_1, m_2, m_3: m_j, B_j^c} \int_B \int_B \prod_{i=1}^3 |\tilde{K}_{\lambda,i}(\cdot, t) \ast (g_{i,m_i})(x)|^\frac{2}{3} \, dx \, dt \right).
\]

We consider the case \(m_1 = m_2 = m_3 = B\) first. Since the Fourier transform of \(\tilde{K}_{\lambda,i}(\cdot, t) \ast (g_i,B)\) is supported in \(N_r(A(B_i))\), the transversality condition (2.14) holds for \(2^{-1}\delta\) replacing \(\delta\) if we take a sufficiently small \(r\). Applying Proposition 2.5, we get

\[
\sum_B \int_B \prod_{i=1}^3 |\tilde{K}_{\lambda,i}(\cdot, t) \ast (g_i,B)(x)|^\frac{2}{3} \, dx \, dt \lesssim \lambda^{\gamma_0} \sum_B \prod_{i=1}^3 \|g_i\|_p \lesssim \lambda^{\gamma_0 + \varepsilon} \prod_{i=1}^3 \|g_i\|_p
\]

for \(\gamma_0 > \frac{1}{2} + \frac{1}{3} - \frac{3}{q}\) and \(\varepsilon > 0\). The second inequality follows by Hölder’s inequality and the inclusion \(L^p \subset \ell^q\) for \(p < \infty\) since the balls \(B\) overlap at most \(C\lambda^{2\varepsilon}\).

If \(m_i = B^c\) for some \(i\), we use decay of \(\tilde{K}_{\lambda,i}\). As before, it is easy to show \(|\tilde{K}_{\lambda,i}(x, t)| \lesssim \lambda^2 (1 + |x|)^{-M}\) for any \(M \geq 1\) (c.f. (2.3)). Setting \(\mathcal{K}(x) = (1 + |x|)^{-3}\), we see \(|\tilde{K}_{\lambda,i}(\cdot, t) \ast g_i,B(x)| \lesssim \lambda^2 \mathcal{K} \ast |g_i|(x)\) and \(|\tilde{K}_{\lambda,i}(\cdot, t) \ast g_i,B^c(x)| \lesssim \lambda^2 \mathcal{K} \ast |g_i|(x)\) if \(x \in B\). Thus we have

\[
\prod_{i=1}^3 |\tilde{K}_{\lambda,i}(\cdot, t) \ast (g_{i,m_i})(x)| \lesssim \lambda^{c_1 - c_2 \varepsilon} \prod_{i=1}^3 \mathcal{K} \ast |g_i|(x), \quad (x, t) \in B \times I,
\]
for some constants $c_1, c_2 > 0$. By Hölder’s inequality and Young’s convolution inequality, there are positive constants $c_1, c_2$ such that

$$\sum_B \sum_{(m_1, m_2, m_3) \neq (B, B, B)} \int_{B \times I} \int_0^1 \int_1^3 \left| \hat{K}_{\lambda_i}(\cdot, t) \ast (g_{i, m_i})(x) \right|^q dx dt \lesssim \lambda^{c_1 - c_2 \epsilon M} \prod_{i=1}^3 \|g_i\|_p^q$$

for $p \leq q$. Combining the estimates above, we obtain

$$\left\| \sum_{i=1}^3 T_{\lambda_i} g_i \right\|_{L^q_x L^p_t (\mathbb{R}^2 \times I)} \lesssim (\lambda^{\gamma_0 + \epsilon} + \lambda^{c_1 - c_2 \epsilon M}) \prod_{i=1}^3 \|g_i\|_p^q$$

for $\gamma_0 > \frac{1}{4} + \frac{1}{\lambda^3} - \frac{3}{4}$. Taking $\epsilon > 0$ small enough and then a large $M$, we obtain the desired estimate \((2.16)\). \(\square\)

We now obtain \((2.16)\) with weights $\omega \in \Omega(\alpha)$ via the following lemma.

**Lemma 2.7.** Let $\omega \in \Omega(\alpha), 0 < \alpha \leq 3$. Suppose $\tilde{F}$ is supported in $\mathbb{B}^3(0, \lambda)$. Then, we have $\|F\|_{L^q_x L^p_t (\mathbb{R}^2 \times I)} \lesssim \lambda^{(3-\alpha)/q} \|F\|_{L^q_x (\mathbb{R}^2)}$ for $q \geq 1$.

**Proof.** Let $\varphi$ be a Schwartz function such that $\tilde{\varphi} = 1$ on $\mathbb{B}^3(0, 1)$, and set $\varphi_\lambda = \lambda^3 \varphi(\lambda \cdot)$. Since $F = F \ast \varphi_\lambda$, we have $\|F\|_{L^q_x (\mathbb{R}^2)} \lesssim \|\varphi_\lambda \ast \omega\|_{L^q_x (\mathbb{R}^2)}$. By rapid decay of $\varphi$, we see

$$|\varphi_\lambda \ast \omega(y) \lesssim C_N \lambda^3 \sum_{\ell \geq 0} 2^{-N\ell} \int_{\mathbb{B}^3(y, 2^\ell \lambda^{-1})} \omega(z) dz$$

for any $N > 3$. Since $\omega \in \Omega(\alpha)$, it follows by \((3.1)\) that $\|\varphi_\lambda \ast \omega\|_{L^q_x (\mathbb{R}^2)} \lesssim \lambda^{3-\alpha}$. Hence, we get the desired bound. \(\square\)

**Proposition 2.8.** Let $\phi \in \mathcal{S}(\mathbb{R}^3), \omega \in \Omega(\alpha), 0 < \alpha \leq 3$, and $\lambda^{-1} \ll \delta < 1$. Suppose \((2.14)\) holds whenever $\theta_i \in J_i, i = 1, 2, 3$. If \((2.16)\) holds for some $\gamma_0$, then

\[(2.17) \quad \left\| \sum_{i=1}^3 T_{\lambda_i} g_i \right\|_{L^q_x L^p_t (\mathbb{R}^2 \times I ; \omega)} \leq \lambda^{3\gamma} \prod_{i=1}^3 \|g_i\|_p\]

holds for $\gamma = \gamma_0 + \frac{3-\alpha}{q}$ whenever $\text{supp} g_i \subset A_\lambda(J_i)$.

**Proof.** Let $\tilde{x} \in C_0^\infty((1/2, 4))$ such that $\tilde{x} = 1$ on $I = [1, 2]$. We only have to show \((2.17)\) with $T_\lambda$ replaced by $\tilde{x} T_{\lambda}$. Since $\tilde{x}$ is compactly supported, the support of the space-time Fourier transform of $\tilde{x}(t) T_{\lambda} g_i(x, t)$ is unbounded. So, in order to apply Lemma 2.7, we decompose $\tilde{x} T_{\lambda} f$ in such a way that the Fourier supports of the consequent operators are contained in either a bounded set or its complement. Let us define a frequency localized operator $T_{\lambda}^1$ by

$$\mathcal{F}_{x,t}(T_{\lambda}^1 g)(\eta, \rho, \tau) = \mathcal{F}_{x,t} (\tilde{x} T_{\lambda} g)(\eta, \rho, \tau) \beta_0((c\lambda)^{-1}|\tau|),$$

where $c > 0$ is a constant to be chosen later. We also set $T_{\lambda}^2 g = (\tilde{x} T_{\lambda} - T_{\lambda}^1) g$. Then, we have

$$\left\| \sum_{i=1}^3 \tilde{x} T_{\lambda_i} g_i \right\|_{L^q_x L^p_t (\mathbb{R}^2 \times I ; \omega)} \leq I + II,$$
where

\[ I = \left\| \prod_{i=1}^{3} T_{\lambda_{i}} g \right\|_{L^{2}(\mathbb{R}^{3}; \omega)}, \quad \mathcal{II} = \sum_{(m_{1}, m_{2}, m_{3}) \notin \{(1,1,1)\}; m_{j} \in \{1,2\}} \left\| \prod_{i=1}^{3} T_{\lambda_{i}} g \right\|_{L^{2}(\mathbb{R}^{3}; \omega)}. \]

First, we show \( \mathcal{II} \lesssim \lambda^{-M} \prod_{i=1}^{3} \| g_{i} \|_{p} \) for any \( M > 1 \). It is enough to show

(2.18) \[ \| T_{\lambda} \|_{L^{p}(\mathbb{R}^{3}; \omega)} \lesssim \lambda^{-M} \| g \|_{p} \]

for any \( M > 0 \) if \( p \leq q \). Once we have (2.18), it follows that \( \| T_{\lambda} \|_{L^{p}(\mathbb{R}^{3}; \omega)} \lesssim \lambda^{-M} \| g \|_{p} \) since \( |T_{\lambda} g| \leq |\tilde{T}_{\lambda} g| + |T_{\lambda}^{2} g| \) and \( \| \tilde{T}_{\lambda} g \|_{L^{p}(\mathbb{R}^{3}; \omega)} \lesssim \lambda^{2} \| g \|_{p} \) for \( p \leq q \) (see (2.3) and (2.4)). Thus, the desired bound on \( \mathcal{II} \) follows by Hölder’s inequality.

To prove (2.18), let us set

\[ \Phi(\eta, \rho, \tau) = \frac{1}{2\pi} \int e^{-i(t - \eta)\phi(\rho/\eta)} \tilde{\chi}(t) dt \beta(\lambda^{-1}\eta)\beta_{0}(|\rho/\eta|). \]

Recalling the definition of \( T_{\lambda} \) (2.8), we have \( F_{x,t}(\tilde{T}_{\lambda} g)(\eta, \rho, \tau) = \Phi(\eta, \rho, \tau)\hat{g}(\eta, \rho) \). Note \( T_{\lambda}^{2} = (\tilde{T}_{\lambda} - T_{\lambda}^{1}) \), so \( T_{\lambda}^{2} g(x, t) = \mathcal{R}(\cdot, t) * g(x) \), where

\[ \mathcal{R}(x, t) = \int \int (1 - \beta_{0}((e\lambda)^{-1}|\tau|)) \Phi(\eta, \rho, \tau) e^{i(x \cdot (\eta, \rho) + t\tau)} d\eta d\rho d\tau. \]

Taking a sufficiently large \( c > 0 \), by integration by parts, we get

(2.19) \[ |\mathcal{R}(x, t)| \lesssim \lambda^{-M} (1 + |x|)^{-M} (1 + |t|)^{-M} \]

for \( M \geq 1 \). We see \( \partial_{\eta, \rho, \tau}^{m}((1 - \beta_{0}((e\lambda)^{-1}|\tau|))\Phi(\eta, \rho, \tau)) \) are bounded by \( C_{M} \lambda^{-M} (1 + |\tau|)^{-M} \) for any \( M \geq 1 \) since \( \partial_{\eta, \rho, \tau}^{m} \phi \) are bounded by a constant depending only on the multi index \( m \). Repeated integration by parts in \( \eta, \rho, \tau \) gives (2.19). Using (2.19), we get (2.18) with \( p = q \) by Schur’s test. Also, (2.18) holds with \( q = \infty \) by Young’s convolution inequality. By interpolation, we get (2.18) for any \( 1 \leq p \leq q \leq \infty \).

We now turn to the term \( I \). Since the Fourier transform of \( T_{\lambda} g_{i} \) is supported in a ball of radius \( 2e\lambda \), we can apply Lemma 2.7 to \( I \). So, we get

\[ I \lesssim \lambda^{3(\gamma_{0} + \frac{3}{2}\gamma)} \prod_{i=1}^{3} \left\| T_{\lambda_{i}} g_{i} \right\|_{L^{2}(\mathbb{R}^{3})}. \]

Since \( \| T_{\lambda_{i}} g \| \leq |\tilde{T}_{\lambda_{i}} g| + |T_{\lambda_{i}}^{2} g| \), using \( \| \tilde{T}_{\lambda_{i}} g \|_{L^{p}(\mathbb{R}^{3})} \lesssim \lambda^{2} \| g \|_{p} \) and (2.18), we see that \( \| \prod_{i=1}^{3} T_{\lambda_{i}} g_{i} \|_{q/3} \) is bounded by \( \| \prod_{i=1}^{3} \tilde{T}_{\lambda_{i}} g_{i} \|_{q/3} + C \lambda^{-N} \prod_{i=1}^{3} \| g_{i} \|_{p} \) for any \( N > 0 \). Now we apply Lemma 2.6 to \( \| \prod_{i=1}^{3} \tilde{T}_{\lambda_{i}} g_{i} \|_{q/3} \) expanding the interval \( I \) slightly. Choosing \( N \) large enough, we obtain

\[ I \lesssim \lambda^{3(\gamma_{0} + \frac{3}{2}\gamma)} \prod_{i=1}^{3} \| g_{i} \|_{p}. \]

Combining all the bounds on \( I \) and \( \mathcal{II} \), we get the desired estimate (2.17) by taking \( M \) sufficiently large.

The estimate (2.17) plays a crucial role in proving Theorem 1.6. However, in order to prove Theorem 1.6 (or Proposition 2.1), it suffices to obtain the following estimate which is not necessarily sharp.
Corollary 2.9. Let $\phi \in \mathcal{S}(\varepsilon)$, $\omega \in \Omega(\alpha)$ for $1 < \alpha \leq 2$, and $\lambda^{-1} \ll \delta \ll 1$. Suppose (2.14) holds whenever $\theta_i \in J_i$, $i = 1, 2, 3$, and let

$$\frac{1}{q} \leq \frac{1}{p} < \frac{\alpha}{q} \quad \text{and} \quad p > \begin{cases} \frac{2\alpha+3}{2\alpha}, & \frac{4}{3} \leq \alpha \leq 2; \\ \frac{6-2\alpha}{\alpha}, & 1 < \alpha \leq \frac{3}{2}. \end{cases}$$

Then we have (2.17) for some $\gamma < \frac{1}{2}$ whenever $\text{supp} \hat{g}_i \subset A_\lambda(J_i)$.

Proof. By Proposition 2.5 and Proposition 2.8, we have Corollary 2.9.

2.3. Proof of Proposition 2.1 In this subsection, we prove Proposition 2.1 by combining Lemma 2.3 and Corollary 2.9. To this end, we decompose the frequency support of $T^\lambda_\phi g$ independent of particular choice of $\phi$.

Decomposition. Let $K_i$, $i = 0, 1, 2$ be dyadic numbers such that

$$\lambda^{-1/3} \ll K_2 \ll K_1 \ll K_0 = 1.$$ 

For each $K_i$, we consider a collection $\mathcal{J}^i$ of dyadic intervals $J^i_j = [(j-1)K_i, (j+1)K_i]$ for $j \in \mathbb{Z}$, $|j| \leq K_i^{-1}$ such that union of $J^i_j$ covers $[-1, 1]$.

Let $g \in C^\infty_0((-1, 1))$ satisfying $\sum_{j \in \mathbb{Z}} g(-j) = 1$. For an interval $J = [\theta - K, \theta + K]$, we denote $g_J = g(K^{-1}(\cdot - \theta))$. Note that $\sum_{J^i_j \in \mathcal{J}^i} g_{J^i_j} = 1$ on $[-1, 1]$. We set $\hat{g}_J(\eta, \rho) = \hat{g}(\eta, \rho) g_J(\rho/\eta)$. Then, we have

$$T^\lambda_\phi g(x, t) = \sum_{J^i_j \in \mathcal{J}^i} T^\phi_\lambda g_{J^i_j}(x, t), \quad i = 0, 1, 2.$$ 

Note that $\hat{g}_{J^i_j}$ is supported in a rectangle of dimensions $c\lambda \times c\lambda K_i$ for a constant $c > 0$.

Following the argument in [16, Section 3], we have the next lemma.

Lemma 2.10. For each $(x, t) \in \mathbb{R}^2 \times I$, there exists a constant $C > 0$, independent of $K_i$ and $(x, t)$, such that

$$|T^\phi_\lambda g(x, t)| \leq \sum_{i=1}^2 CK_{i-1}^{-2} \max_{J^i_j \in \mathcal{J}^i} |T^\phi_\lambda g_{J^i_j}(x, t)|$$

$$+ CK_2^{-4} \max_{(J^1_k, J^2_k, J^3_k) \in \mathcal{J}^2(K_2)} \prod_{i=1}^3 |T^\phi_\lambda g_{J^i_k}(x, t)|^{1/4}.$$ 

where $\mathcal{J}^2(K_2) := \{(J^1_k, J^2_k, J^3_k) : J^1_k, J^2_k, J^3_k \in \mathcal{J}^2, \min_{k \neq l} \text{dist}(J^i_k, J^j_l) \geq K_2 \}$.

If $(J^1_k, J^2_k, J^3_k) \in \mathcal{J}^2(K_2)$, then (2.14) holds for $\theta_i \in J^i_k$, $i = 1, 2, 3$. More precisely, we have the following.
Lemma 2.11. Let $J_1, J_2, J_3$ be subintervals of $[-1,1]$ such that $\min_{k \neq i} \text{dist} (J_k, J_i) \gtrsim \delta^\frac{1}{2}$. Suppose that $\phi \in \mathcal{S}(\epsilon_0)$. If $\epsilon_0 \in (0,1/8)$, then $|\det (N(\theta_1), N(\theta_2), N(\theta_3))| \gtrsim \delta$ for $\theta_i \in J_i$, $i=1,2,3$.

Proof. By the generalized mean value theorem (see [22, Part V, Problem 95]), there are $u_j$ for $j = 1, 2, 3$ such that $\min_{1 \leq i \leq 3} \theta_i < u_j < \max_{1 \leq i \leq 3} \theta_i$ and

$$
\det (N(\theta_1), N(\theta_2), N(\theta_3)) = \det (N(u_1), N(u_2), N''(u_3)) \prod_{1 \leq k < \ell \leq 3} |\theta_k - \theta_\ell|.
$$

From (2.13) we have

$$
\det (N(u_1), N(u_2), N''(u_3)) = \det \begin{pmatrix} u_2 \phi''(u_2) & \phi''(u_3) + u_3 \phi^{(3)}(u_4) \\ \phi''(u_2) & \phi^{(3)}(u_3) \end{pmatrix}.
$$

Since $\phi \in \mathcal{S}(\epsilon_0)$ and $\epsilon_0 \in (0,1/8)$, $7/8 \leq |\phi''(\theta)| \leq 9/8$ and $|\phi^{(3)}(\theta)| \leq 1/8$ for $\theta \in [-1,1]$. Thus, it follows that $|\det (N(u_1), N(u_2), N''(u_3))| \geq 27/8^2$. So we obtain the desired bound since $\prod_{1 \leq k < \ell \leq 3} |\theta_k - \theta_\ell| \gtrsim \delta$. \hfill \square

We are now ready to prove Proposition 2.1.

Proof of Proposition 2.1. We begin by making a primary decomposition by which we reduce the matter to obtaining estimate for $T^\phi_\lambda$, $\phi \in \mathcal{S}(\epsilon_0)$ relative to $\omega \in \Omega(\alpha)$.

Let $\epsilon_0 \in (0,1/8)$. Following the argument in the proof of Lemma 2.2, we can fix a small positive dyadic number $h_\ast < h_\ast(\epsilon_0)$ such that $(\psi')_{\theta,h_\ast} \in \mathcal{S}(\epsilon_0)$ for $\theta \in [-1,1]$. Let $J_j = [(j-1)h_\ast, (j+1)h_\ast]$, $j \in \mathbb{Z}$ such that $|j| \leq h_\ast^{-1} + 2$. Then $\sum_{j} q_{J_j} = 1$ on $[-1,1]$. Denoting $f_j(\eta, \rho) = \tilde{g}(\eta, \rho) q_{J_j}(\rho/\eta)$, we get

$$
\|T^\phi_\lambda \eta \|_{L^p(\mathbb{R}^2 \times I; \omega)} \leq \sum_j \|T^\phi_\lambda f_j \|_{L^p(\mathbb{R}^2 \times I; \omega)}.
$$

Let $\theta_j = jh_\ast$. As before (cf. (2.12)), the change of variable $(\eta, \rho) \rightarrow L_{\theta_j, h_\ast}(\eta, \rho)$ gives

$$
\|T^\phi_\lambda f_j \|_{L^p(\mathbb{R}^2 \times I; \omega)} \leq C h_\ast^{2\alpha - \alpha(\alpha)} \|T^\psi(\theta'_{\theta_j, h_\ast}) g_j \|_{L^q(\mathbb{R}^2 \times I; \omega')},
$$

Note $\|g_j \|_{L^p(\mathbb{R}^2 \times I; \omega)} = h_\ast^{-3/p} \|g_j \|_p$ and $\omega_{\theta_j, h_\ast} \in \Omega(\alpha)$. Since $|\psi''(\theta_\ast) - 1| \leq \epsilon_0$ and $h_\ast$ is a fixed constant depending only on $\epsilon_0$, in order to prove Proposition 2.1 it suffices to show

$$
\|T^\phi_\lambda \|_{L^q(\mathbb{R}^2 \times I; \omega)} \leq C \lambda^\gamma \|g \|_p
$$

for some $\gamma < 1/2$ provided that $\phi \in \mathcal{S}(\epsilon_0)$ and $\omega \in \Omega(\alpha)$.

Since $1 < \alpha \leq 2$, we have $\frac{\alpha - 1}{q} - \frac{3}{p} + 1 > 0$ from the hypothesis. So we can choose $\gamma < \frac{1}{2}$ such that

$$
\frac{\alpha - 1}{q} - \frac{3}{p} + 2\gamma > 0.
$$

For such $\gamma$ and for $\lambda' \geq 1$ we set

$$
\Omega(\lambda') = \Omega(\lambda', p, q, \alpha) := \sup_{1 \leq \lambda \leq \lambda'} \lambda^{-\gamma} Q(\lambda),
$$

where $Q(\lambda)$ is defined by (2.5). The estimate (2.24) follows if we show $\Omega(\lambda') \leq C$. 

Let $h_0$ be a small positive number so that Lemma 2.2 and Lemma 2.3 hold. Let $K_1, K_2$ be positive dyadic numbers such that $0 < K_2 \ll K_1 \leq h_0$. We also set $K_0 = 1$ as before. Using the decomposition (2.22), we have
\[
\|T_\alpha^\phi g\|_{L^n(\mathbb{R}^2 \times I; \omega)} \leq S_1 + S_2 + S_3,
\]
where
\[
S_i = CK_i^{-2} \max_{j_i \in J_i} |T^\phi_\lambda g_{j_i}| \bigg\|_{L^n(\mathbb{R}^2 \times I; \omega)}, \quad i = 1, 2
\]
and
\[
S_3 = CK_3^{-4} \max_{(J_1^i, J_2^i, J_3^i) \in J_3^i(K_2)} \prod_{i=1}^3 \|T^\phi_\lambda g_{j_i}\|_{L^n(\mathbb{R}^2 \times I; \omega)}.
\]

Note that $S_i^q \leq C_i^n K_i^{-2q} \sum_{j_i \in J_i} \|T^\phi_\lambda g_{j_i}\|_{L^n(\mathbb{R}^2 \times I; \omega)}^q$. By Lemma 2.2 with $\kappa(\alpha) = \alpha + 1$, for $i = 1, 2$ we have
\[
S_i^q \leq C_i^n K_i^{-2q} (K_i^{\alpha - \frac{1}{p}} + \sup_{\theta \in [-1, 0]} Q(\phi''(\theta)K_i^2 \lambda)) \sum_{j_i \in J_i} \|g_{j_i}\|_{p}^q.
\]

By the embedding $\ell^p \subset \ell^q (p \leq q)$, it follows that $\sum_{j_i \in J_i} \|g_{j_i}\|_{p}^q \leq \|g\|_{p}^q$. Here, we use the estimate $\left( \sum_{j_i \in J_i} \|g_{j_i}\|_{p}^q \right)^{1/p} \leq \|g\|_{p}$ for $2 \leq p \leq \infty$, which can be shown by interpolation between trivial $L^\infty - \ell^\infty L^\infty$ estimate and $L^2 - \ell^2 L^2$ estimate via Plancherel’s theorem.

By Lemma 2.11 the transversality condition (2.14) with $\delta = (K_2)^3$ holds if $(J_1^i, J_2^i, J_3^i) \in J_3^i(K_2)$. By Corollary 2.9 we have $S_i \leq CK_i^{-3} |g|_{p}$ for $\lambda \gg K_2^{-3}$ and $p, q$ satisfying (2.20). For the case of $\lambda \ll K_2^{-3}$, we have $S_3 \leq CK_2^{-C} \lambda^3 \|g\|_p$ for some constant $C > 0$ from the easy estimate (e.g. 2.4) and Hölder’s inequality. So we get
\[
S_3 \leq CK_2^{-C} \lambda^3 \|g\|_p.
\]

Combining all the estimates for $S_1$, $S_2$, and $S_3$, which hold uniformly for $\phi \in \mathcal{E}(\epsilon_0)$ and $\omega \in \Omega(\alpha)$, we obtain
\[
Q(\lambda) \leq \sum_{i=1}^2 CK_i^{-2} K_i^{\alpha - \frac{1}{q} - \frac{3}{p} + 2\gamma} \sup_{\theta \in [-1, 1]} Q(\phi''(\theta)K_i^2 \lambda) + CK_i^{-C} \lambda^q.
\]
Since $|\phi''(\theta) - 1| \leq \epsilon_0$, note that $(\phi''(\theta)K_i^2 \lambda)^{-\gamma} Q(\phi''(\theta)K_i^2 \lambda) \leq \Omega(\lambda')$ if $\lambda' \geq 2^{-1}K_2^2 \lambda \geq 1$. Otherwise, i.e., if $1 \leq \lambda \leq 2K_2^{-2}$, we have $Q(\phi''(\theta)K_i^2 \lambda) \leq CK_i^{-C}$ for some $C > 0$. Thus, we have $(\phi''(\theta)K_i^2 \lambda)^{-\gamma} Q(\phi''(\theta)K_i^2 \lambda) \leq \Omega(\lambda') + CK_i^{-C}$.

We now multiply $\lambda^{-\gamma}$ to both sides of (2.27). Then, using the above observation we obtain
\[
\lambda^{-\gamma}Q(\lambda) \leq \sum_{i=1}^2 CK_i^{-2} K_i^{\alpha - \frac{1}{q} - \frac{3}{p} + 2\gamma} \Omega(\lambda') + CK_i^{-C}
\]
for $\lambda \leq \lambda'$. Taking supremum over $\lambda \geq \lambda'$, we get
\[
\Omega(\lambda') \leq \sum_{i=1}^2 CK_i^{-2} K_i^{\alpha - \frac{1}{q} - \frac{3}{p} + 2\gamma} \Omega(\lambda') + CK_i^{-C}.
\]
Since (2.20) holds, we can successively choose dyadic numbers $K_1, K_2$ such that
\[
CK_i^{-2} K_i^{\alpha - \frac{1}{q} - \frac{3}{p} + 2\gamma} \leq 3^{-1}, \quad i = 1, 2.
\]
Therefore, we obtain $\Omega(\lambda') \leq \frac{2}{3} \Omega(\lambda) + CK_2^{-C}$, which clearly gives $\Omega(\lambda') \leq CK_2^{-C}$ for $\lambda' > 1$ and $p, q$ satisfying (2.20) and $\frac{1}{p} \leq 1 + \frac{2}{3q}$, i.e., (1.5). This completes the proof.

\section*{2.4. Weighted local smoothing estimate with sharp regularity.}

In this section, modifying the proof of Proposition 2.1, we show (2.4) with $\gamma > \frac{1}{2} + \frac{1}{p} - \frac{\omega}{q}$, which implies Theorem 1.6. We also discuss sharpness of the exponent $\gamma$.

\textbf{Proposition 2.12.} Let $0 < \alpha \leq 3$ and $\omega \in \Omega(\alpha)$. If $\frac{1}{q} \leq \frac{1}{p} \leq \min\left(\frac{1}{3p}, \frac{1}{3q}, \frac{2}{3p}\right)$ and $\frac{1}{p} \geq \frac{\omega}{q}$, then $(2.4)$ holds for $\gamma > \frac{1}{2} + \frac{1}{p} - \frac{\omega}{q}$.

\textbf{Proof.} The proof is essentially identical to that of Proposition 2.1 so we shall be brief. The main difference is that we use Proposition 2.5 and Proposition 2.8 instead of Corollary 2.9. For a fixed $\gamma > \frac{1}{2} + \frac{1}{p} - \frac{\omega}{q}$, we need to show $\Omega(\lambda') \leq C$.

As in the proof of Proposition 2.1, we use the decomposition (2.22). Since $\sum_{J} \|g_J\|_p \leq \|g\|_p$, by Lemma 2.3 we have

\[ \max_{J \in \mathcal{J}} \|T_\lambda g_J\|_{L^q(\mathbb{R}^2 \times \mathcal{I}_\omega)} \leq CK_1^{2a-\varepsilon(\alpha)} q^{-\frac{3}{p}} Q(K_2^2 \lambda) \|g\|_p. \]

By Proposition 2.5 and Proposition 2.8 we also have

\[ \left\| \max_{(J_1, J_2, J_3) \in \mathcal{J}(\mathcal{K})} \prod_{i=1}^3 \|T_\lambda g_{J_i}\|_{L^q(\mathbb{R}^2 \times \mathcal{I}_\omega)} \right\|_{L^p(\mathbb{R}^2 \times \mathcal{I}_\omega)} \leq CK_2^{-\gamma} \|g\|_p \]

for $\gamma > \frac{1}{2} + \frac{1}{p} - \frac{\alpha}{q}$ if $\frac{1}{q} \leq \min\left(\frac{1}{3p}, \frac{1}{3q}, \frac{2}{3p}\right)$ and $1 \leq p < \infty$.

By following the same lines of argument as in the proof of Proposition 2.1 we obtain

$\Omega(\lambda') \leq C \sum_{i=1}^2 K_{i-1}^{-2} K_i^{2a-\varepsilon(\alpha)} - \frac{3}{p} \gamma + 2 \gamma \Omega(\lambda') + C K_2^{-C}$

for $\lambda' \geq 1$. We take $\gamma$ sufficiently close to $\frac{1}{2} + \frac{1}{p} - \frac{\omega}{q}$ such that $\frac{2a-\varepsilon(\alpha)}{q} - \frac{3}{p} \gamma + 2 \gamma > 1 - \frac{1}{p} - \frac{\varepsilon(\alpha)}{q}$, and then, we can choose $K_1, K_2$ such that $CK_1^{-2} K_2^{-\frac{2a-\varepsilon(\alpha)}{q}} - \frac{3}{p} \gamma + 2 \gamma \leq 1$ for $i = 1, 2$. Therefore, $\Omega(\lambda') \leq C$ holds provided that $\frac{1}{q} \leq \min\left(\frac{1}{3p}, \frac{1}{3q}, \frac{2}{3p}\right)$ and $1 - \frac{1}{p} - \frac{\varepsilon(\alpha)}{q} \geq 0$, as desired.

We discuss sharpness of the regularity exponent in Theorem 1.6. We show (1.7) holds only if

\begin{align}
(2.28) &\quad \gamma \geq \frac{1}{2} + \frac{1}{p} - \frac{\alpha}{q}, \\
(2.29) &\quad \gamma \geq \frac{3}{2p} + \frac{\varepsilon(\alpha) - 2\alpha}{2q}.
\end{align}

Note the two lower bounds on $\gamma$ coincide if $\frac{1}{p} + \frac{\varepsilon(\alpha)}{q} = 1$. So, (1.7) holds for $\gamma > \frac{1}{2} + \frac{1}{p} - \frac{\omega}{q}$ only if $\frac{1}{p} + \frac{\varepsilon(\alpha)}{q} \leq 1$.

\textbf{Proof of (2.28).} For $\lambda \gg 1$, we consider $f$ given by $\hat{f}(\xi) = e^{-i(\xi|\xi| - \xi \xi^\perp)}$ for a Schwartz function $\xi$ supported on $1 \leq |\xi| \leq 2$. Using the polar coordinates (see (3.10)), we have

$|f(x)| \lesssim \lambda^{\frac{3}{2}} (1 + \lambda |x| - 1)^{-N}$.
for any $N \geq 1$. Since $\text{supp} \hat{f}$ is supported in $\{|\xi| \sim \lambda\}$, we see $\|f\|_{L^p_{\xi}} \lesssim \lambda^\gamma \lambda^{\frac{3}{2} - \frac{1}{p}}$. Note that $|e^{it\sqrt{-\Delta}}f(x)| \gtrsim \lambda^2$ if $|x|, |t - 1| \leq c\lambda^{-1}$ for a sufficiently small $c > 0$. If we take

\[ \text{d} \nu(z) = \chi_{B^3(0,3)}(z)|z|^{p-3} \text{d}z, \quad z = (x, t) \in \mathbb{R}^3, \]

then it is easy to see $(\nu)_{\alpha} \lesssim C$ for some $C$. So, (1.7) gives $\lambda^{\frac{3}{2} - \frac{1}{p}} \lesssim \lambda^\gamma \lambda^{\frac{3}{2} - \frac{1}{p}}$. Taking $\lambda \to \infty$, we get (2.29). \hfill \Box

\section{Proofs of Theorem 1.3, Theorem 1.5 and Theorem 1.6}

In this section, we first prove Theorem 1.5 and Theorem 1.6 which we deduce from Proposition 2.1 and Proposition 2.12 respectively. Combining Littlewood-Paley decomposition and Theorem 1.5 we prove Theorem 1.3 and then Corollary 1.4. We also discuss results for the spherical average in higher dimensions.

\subsection{Proof of Theorem 1.5 and Theorem 1.6}

By Littlewood-Paley decomposition, it suffices to consider the operator $e^{it\sqrt{-\Delta}}P_\lambda f$ with $\lambda \geq 1$. We first observe that (2.2) is a simple consequence of the estimate (2.4) via a change of variable. Let $U(x, t) = (x_1 - t, x_2, t)$ where $x = (x_1, x_2)$. Setting $\xi = (\eta, \rho)$, we write $|\xi| - \eta = \eta \sqrt{1 + \rho^2/\eta^2} - \eta = \eta \psi_0(\rho/\eta)$. Then,

\[ U(x, t) \cdot (\xi, |\xi|) = (x_1, t) \cdot (\eta, \rho, \eta \psi_0(\rho/\eta)). \]

By decomposition into finite angular sectors and rotation, we may assume $\hat{f}$ is supported in a small conic neighborhood of $e_1$. Changing variables $(x, t) \to U(x, t)$, we have

\begin{equation}
\|e^{it\sqrt{-\Delta}}P_\lambda f\|_{L^p(\mathbb{R}^2 \times I; \omega)} \leq C\|T^\psi_{\lambda} f\|_{L^p(\mathbb{R}^2 \times I; \omega')},
\end{equation}

where $\omega'(x, t) := \omega(U(x, t))$ and $\hat{f}$ is supported on $4^{-1}\lambda \leq \eta \leq 4\lambda$ and $|\rho/\eta| \leq 1$. Since $U$ is an invertible linear map, it is clear that $C^{-1}\omega' \in \Omega(\alpha)$ for some $C > 0$. Therefore, (2.2) holds for the same $\gamma$ and $p, q$ as in Proposition 2.1.

The following lemma shows that estimates relative to $\alpha$-dimensional measures can be deduced from weighted estimates. See [17] Section 3 for example.

\begin{lemma}
Let $0 < \alpha \leq 3$ and $\nu \in C^3(\alpha)$. Set $\varphi_\lambda = \lambda^3\varphi(\lambda \cdot)$ for a Schwartz function $\varphi$ such that $\mathcal{F}^\prime = 1$ on $B^3(0, 4)$ and $\text{supp} \varphi \subset B^3(0, 8)$. There is a constant $C_1 = C_1(\nu) > 0$ such that $C_1^{-1}|\varphi_\lambda|*\nu \in \Omega(\alpha)$.
\end{lemma}
Proof of Lemma 3.2. Let us define $\nu_{\lambda} := |\hat{\varphi}| * \nu$. By rapid decay of $\varphi$, we have

$$\nu_{\lambda}(z) \leq C_N \lambda^3 \sum_{\ell \geq 0} 2^{-\ell N} \nu(B^3(z, 2^\ell \lambda^{-1}))$$

for any $N > 0$. It suffices to show $\int_{B^3(z,r)} \nu_{\lambda}(y) dy \leq C_1 r^\alpha$ for a constant $C_1$.

Let us consider two cases $r \leq \lambda^{-1}$ and $r > \lambda^{-1}$, separately. If $r \leq \lambda^{-1}$, using $\|\nu_{\lambda}\|_\infty \leq C<\nu> \lambda^{-\alpha}$, we see $\int_{B^3(z,r)} \nu_{\lambda}(z) dz \lesssim \langle \nu \rangle \lambda^{3-\alpha} r^3 \leq C_N \langle \nu \rangle r^\alpha$. To handle the case $r > \lambda^{-1}$, we use

$$\int_{B^3(z,r)} \nu_{\lambda}(B^3(z, 2^\ell \lambda^{-1})) dz \leq \int_{B^3(z,(2^\ell+1)r)} \lambda_{B^3(2^{\ell-1}z, \lambda^{-1})}(z) dz d\nu'(z').$$

Taking $N$ sufficiently large, we get

$$\int_{B^3(z,r)} \nu_{\lambda}(z) dz \leq C_N \lambda^3 \sum_{\ell \geq 0} 2^{-\ell N} (2^\ell \lambda^{-1})^3 (2^\ell + 1)^\alpha r^\alpha \leq C_N r^\alpha.$$ 

Therefore, taking $C_1 = \max\{C_N, C_N', \ldots\}$, we have $C_1^{-1} \nu_{\lambda} \in \Omega(\alpha)$. 

We are ready to prove Theorem 1.5 and Theorem 1.6.

Proof of Theorem 1.3 and Theorem 1.6. By Littlewood-Paley decomposition, it is enough to show

$$\|e^{it\sqrt{-\Delta}} P_{\lambda} f\|_{L^q(\mathbb{R}^d; I; \nu)} \leq C \lambda^\gamma \|f\|_p, \quad \lambda \geq 1.$$ 

Let $\varphi$ be the Schwartz function given in Lemma 3.2. Since $F_{\lambda}(e^{it\sqrt{-\Delta}} P_{\lambda} f) = F_{\lambda}(e^{it\sqrt{-\Delta}} P_{\lambda} f) \hat{\varphi}_{\lambda}$, we have $|e^{it\sqrt{-\Delta}} P_{\lambda} f|^q \lesssim |e^{it\sqrt{-\Delta}} P_{\lambda} f|^q \|\varphi_{\lambda}\|$ by Hölder’s inequality. Thus it follows that

$$\|e^{it\sqrt{-\Delta}} P_{\lambda} f\|_{L^q(\mathbb{R}^d; I; \nu)} \leq C \|e^{it\sqrt{-\Delta}} P_{\lambda} f\|_{L^q(\mathbb{R}^d; I; |\varphi| \ast \nu)}.$$ 

By Lemma 3.2, $C^{-1}|\varphi| \ast \nu \in \Omega(\alpha)$ for some $C > 0$. Since we are assuming $\hat{f}$ is supported in a small conic neighborhood of $e_1$, using equation (3.3) and (3.1), we see that the estimate (2.4) implies (3.2) for the same $p, q$ and $\gamma$ as in Proposition 2.1 and Proposition 2.12. Therefore, Proposition 2.1 and Proposition 2.12 give Theorem 1.5 and Theorem 1.6, respectively.

3.2. Estimates for the circular average. Recall the average $A$ in $\mathbb{R}^d$, $d \geq 2$. Let us set $P_0 = \sum_{j \geq 0} P_{2^j}$. The kernel $K(\cdot,t)$ of $f \rightarrow AP_0 f(\cdot,t)$ is rapidly decaying, i.e., $|K(\cdot,t)| \leq C(1 + |x|)^{-N}$ for any $N$. By Schur’s test, one can easily see $\|AP_0 f\|_{L^p(\mathbb{R}^2 \times I; \sigma_t)} \leq C \|f\|_p$ for $1 \leq p \leq q$. Therefore, it is sufficient to consider the contribution from $\sum_{j \geq 1} AP_{2^j} f$.

Let $d\sigma_t$ denote the normalized spherical measure. Then it is well known that

$$\widehat{\sigma_t}(\xi) = e^{i|x|} a_+ (t \xi) + e^{-i|x|} a_- (t \xi),$$

where $a_\pm$ are smooth functions satisfying $|\partial^m a_\pm (\xi)| \leq C_m (1 + |\xi|)^{-\frac{d-1}{m}}$ for any $m$. So, it is sufficient to consider $e^{i|\xi|} a_+ (t \xi)$ since the contribution from $e^{-i|\xi|} a_- (t \xi)$ can be handled similarly by reflection $t \rightarrow -t$. We set

$$A f(x,t) = \int e^{i|x|} (1 + |\xi|^2)^{-\frac{d-1}{2}} \hat{f}(\xi) d\xi$$

where $\hat{f}$ is the Fourier transform of $f$.
Proof of Theorem 1.3. As discussed above, we need only to show (3.5) for some \( \epsilon > 0 \) if \( \sum_{j \geq 1} \| A_j f \|_{L^q(\mathbb{R}^d \times I; dv)} \lesssim \langle \nu \rangle^{1/4} \| f \|_p \). Thus, the matter is reduced to showing

\[
\| A_j f \|_{L^q(\mathbb{R}^d \times I; dv)} \lesssim \langle \nu \rangle^{1/2} 2^{-\epsilon j} \| f \|_p, \quad j \geq 1,
\]

for some \( \epsilon > 0 \).

Let \( K_j(\cdot, t) \) denote the kernel of \( f \to A_j f(\cdot, t) \). It is easy to see

\[
| K_j(x, t) | \lesssim 2^j \left( 1 + 2^j |x| - |t| \right)^{-N}
\]

for any \( N \geq 1 \) (see, for example, [14]). Using the estimate we obtain

\[
\| A_j f \|_{L^\infty(\mathbb{R}^d \times I; dv)} \lesssim 2^j \| f \|_1,
\]

\[
\| A_j f \|_{L^1(\mathbb{R}^d \times I; dv)} \lesssim \langle \nu \rangle \min \left\{ 2^j, 2^{(d+1)j} \right\} \| f \|_1.
\]

In fact, (3.7) is clear from (3.6). To show (3.8), we use the fact that \( K_j \) is essentially supported in \( O(2^{-j}) \) neighborhood of the truncated cone with height \( \sim 1 \). Since the neighborhood can be covered by \( O(2^j) \) balls of radius \( 2^{-j} \), using (2.1) we see

\[
\| K_j \|_{L^1(\mathbb{R}^d \times I; dv)} \lesssim \langle \nu \rangle 2^{j(d+1)-j}. \quad \text{Besides, we have } | K_j | \lesssim 2^j (1 + |x|)^{-N} \text{ since } t \in I.
\]

So, \( \| K_j \|_{L^1(\mathbb{R}^d \times I; dv)} \lesssim \langle \nu \rangle \min \{ 2^j, 2^{(d+1)j} \} \). We get (3.8) by Fubini’s theorem.

**Proof of Theorem 1.5.** As discussed above, we need only to show (3.5) for some \( \epsilon > 0 \) if \( p, q \) satisfy (1.5). When \( 1 < \alpha \leq 2 \), the desired estimate (1.4) follows from Theorem 1.3 since

\[
\| A_j f \|_{L^q(\mathbb{R}^d \times I; dv)} \lesssim 2^{-\frac{j}{2}} e^{i t \sqrt{-\Delta} - \frac{\pi}{4}} P_{2j} f
\]

for some \( \tilde{f} \) satisfying \( \| \tilde{f} \|_p \lesssim \| f \|_p \).

For \( 2 < \alpha \leq 3 \), we use (3.7), (3.8), and

\[
\| A_j f \|_{L^r(\mathbb{R}^d \times I; dv)} \lesssim \langle \nu \rangle^{\frac{1}{2}} 2^{\left( \frac{d+1}{r} - \frac{1}{2} \right)j} \| f \|_r, \quad 2 \leq r \leq 4, \quad \epsilon > 0.
\]

To show (3.10), we use the estimate \( \| A_j f \|_{L^r(\mathbb{R}^2 \times I)} \lesssim 2^{\left( \frac{d+1}{r} + \epsilon \right)j} \| f \|_r \) for \( \epsilon > 0 \) and \( 2 \leq r \leq 4 \), which follows from interpolation between the estimates for \( r = 2 \) and \( r = 4 \). Indeed, the first is clear from Plancherel’s theorem and (3.9). The second follows by the sharp local smoothing estimate (1.7) and (3.9) combined with the globalization lemma (see Lemma 2.4). Then, modifying the argument in the proof of Proposition 2.8 we get the estimate (3.10).

When \( 2 < \alpha \leq 3 \), \( P_2(\alpha) \) is the closed quadrangle with vertices \((0, 0)\),

\[
P_1 := \left( \frac{\alpha}{\alpha + 3}, \frac{1}{\alpha + 3} \right), \quad P_2 := \left( \frac{\alpha - 1}{\alpha}, \frac{1}{\alpha} \right), \quad P_3 := \left( \frac{1}{4 - \alpha}, \frac{1}{4 - \alpha} \right).
\]

Interpolating the estimates (3.8) and (3.10) with \( r = 2 \), we have

\[
\| A_j f \|_{L^q(\mathbb{R}^d \times I; dv)} \lesssim \langle \nu \rangle^{\frac{1}{2}} 2^{\left( \frac{d+1}{r} - \frac{1}{2} \right)j} \| f \|_q
\]

for \( 1 \leq q \leq 2 \). Since \( \| A_j f \|_{L^\infty(\mathbb{R}^d \times I; dv)} \leq C \| f \|_\infty \), this gives (3.5) for \( 4 - \alpha < p = q < \infty \). Therefore, to complete the proof, by interpolation it is enough to show (3.3) for \((1/p, 1/q)\) arbitrarily close to \( P_1 \) and \( P_2 \). Interpolating (3.7) and (3.10) yields

\[
\| A_j f \|_{L^q(\mathbb{R}^d \times I; dv)} \lesssim \langle \nu \rangle^{\frac{1}{2}} 2^{\left( \frac{d+1}{r} - \frac{1}{2} \right)j} \| f \|_p
\]
for $\epsilon > 0$ if $1 - \frac{3}{q} \leq \frac{1}{p} \leq 1 - \frac{1}{q}$ and $p \leq q$. A simple computation shows that this gives (3.11) for $(1/p, 1/q)$ contained in the closed quadrangle with vertices $(1/4, 1/4), P_1, P_2,$ and $(1/2, 1/2)$ excluding the closed line segment $[P_1, P_2]$. □

Corollary 3.4 is a direct consequence of Theorem 3.3 and the following lemma combined with the globalization argument (e.g., Lemma 2.6). For $0 < k \leq d + 1$, let $V$ be a $k$-dimensional subspace in $R^{d + 1}$. We write $(x, t) = (z_1, z_2)$ where $z_1 \in V$, $z_2 \in V^\perp$.

**Lemma 3.2.** Let $V$ be a $k$-dimensional subspace in $R^{d + 1}$ for $k = 1, \ldots, d + 1$. Let $\mu$ be an $\alpha$-dimensional measure on $V$ for some $0 < \alpha \leq k$. If (1.4) holds with a uniform constant $C$, independent of $\nu \in C^{d + 1}(\alpha)$, then

$$\left\| \sup_{z_2 \in V^\perp} |\chi_{B^q(0,1) \times I} A f(\cdot, z_2)| \right\|_{L^p(V, d \mu)} \leq C^\alpha \langle \mu \rangle^{\frac{1}{\alpha}} \| f \|_p.$$  

By Lemma 3.2 and the globalization argument, $L^p - L^q$ estimate for $f \rightarrow \sup_{t \in \ell} |A f(\cdot, t)|$ can be deduced from (1.4) by taking $V = R^d \times \{0\}$ and $\nu \in C^{d + 1}(d)$.

**Proof.** By the Kolmogorov-Seliverstov-Plessner linearization it suffices to show

$$\left\| \sup_{z_2 \in V^\perp} |\chi_{B^q(0,1) \times I} A f(\cdot, z_2)| \right\|_{L^p(V, d \mu)} \leq C^\alpha \langle \mu \rangle^{\frac{1}{\alpha}} \| f \|_p$$

for any measurable function $z_2 : V \rightarrow V^\perp$ with $C$ independent of $z_2$. For any compactly supported continuous function $F$, we define a linear functional $\ell$ on $C_c(R^{d + 1})$ by $\ell(F) = \int F(z_1, z_2(z_1)) d \mu(z_1)$. Then, by the Riesz representation theorem, there is a unique Radon measure $\nu$ on $R^{d + 1}$ such that

$$\ell(F) = \int F(x, t) d \nu(x, t) = \int F(z_1, z_2(z_1)) d \mu(z_1).$$

Since $\nu(B^{d + 1}((y_1, y_2), r)) \leq \int \chi_{B^k(y_1, r)}(z_1) d \mu(z_1) \leq \langle \mu \rangle^{\alpha} r^\alpha$, the measure $\nu$ belongs to $C^{d + 1}(\alpha)$ and $\langle \nu \rangle^\alpha \leq \langle \mu \rangle^{\alpha}$. Therefore, $\| \chi_{B^q(0,1) \times I} A f(\cdot, z_2(\cdot)) \|_{L^q(d \mu_V)} = \| \chi_{B^q(0,1) \times I} A f \|_{L^q(d \nu)}$ and (3.11) follows by (1.4). □

### 3.3. Estimate for the spherical average

We close this section with results in higher dimensions. As mentioned in Section 4, we can obtain some estimates for the spherical average from the fractal Strichartz estimates for the wave equation in $R^{d + 1}$.

**Theorem 3.3 (Theorem 3.1).** Let $d \geq 3$, $2 \leq q < \infty$, and $1 < \alpha \leq d + 1$. Suppose $\nu \in C^{d + 1}(\alpha)$. Then, we have

$$\| e^{it\sqrt{-\Delta}} f \|_{L^q(R^d \times I; d \nu)} \lesssim \langle \nu \rangle^\frac{d}{q} \| f \|_{H^s}$$

holds with $s > s(\alpha, q, d)$, where

$$s(\alpha, q, d) = \begin{cases} \max \left\{ \frac{2}{q} - \frac{\alpha}{q}, \frac{d + 1}{4} + \frac{1 - \alpha}{2q}, \frac{3d + 1 - 2\alpha}{8} \right\}, & 1 < \alpha \leq d, \\ \max \left\{ \frac{2}{q} - \frac{\alpha}{q}, \frac{d + 1}{4} + \frac{d + 1 - 2\alpha}{2q}, \frac{d + 1 - \alpha}{2q} \right\}, & d < \alpha \leq d + 1. \end{cases}$$

In fact, the estimate in (7) Theorem 3.1 is a local one. However, it can be easily extended to a global estimate (3.12) by the argument in Lemma 2.6. The regularity exponents in Theorem 3.3 are sharp when $d = 3$ or $\alpha \in [1, \frac{d - 1}{2}] \cup [d, d + 1]$ in higher dimensions.
Using (3.7) and (3.8), we obtain an analogue of Theorem 1.3 for the spherical averages. To state the result, let us set

\[ p_d(\alpha) = \begin{cases} \frac{\alpha + 1}{\alpha}, & 1 < \alpha \leq \frac{d-1}{2}, \\ \frac{d + 3 + 2\alpha}{d - 1 + 2\alpha}, & \frac{d-1}{2} < \alpha \leq d. \end{cases} \]

**Theorem 3.4.** Let \( d \geq 3 \) and \( 1 < \alpha \leq d + 1 \). Suppose \( \nu \in \mathcal{C}^{d+1}(\alpha) \).

(i) If \( 1 < \alpha \leq d \), then (1.4) holds for \( p, q \) satisfying \( p > p_d(\alpha) \),

\[ q^{-1} \leq p^{-1} < \alpha q^{-1}, \]

\[ (d + 1)p^{-1} < d - 1 + (\alpha - 1)q^{-1}. \]  

(ii) If \( d < \alpha \leq d + 1 \), then (1.4) holds for \( p, q \) satisfying

\[ q^{-1} \leq p^{-1} < \alpha q^{-1}, \]

\[ (d + 1)p^{-1} < d - 1 + (2\alpha - d - 1)q^{-1}, \]

\[ dp^{-1} < d - 1 + (\alpha - d)q^{-1}. \]

When \( d < \alpha \leq d + 1 \), the results are sharp (see Proposition 4.1). We also expect the result to be sharp when \( d = 3 \), but the necessity of the condition \( p > p_d(\alpha) \) for \( 1 < \alpha \leq 3 \) is verified only for \( \alpha = 2, 3 \) (see (iv) and (v) in Proposition 4.1). Using the recent result due to Harris [10], which improves the exponent \( s \) in (3.12) for \( q = 2 \) and \( d + 1 \leq \alpha < d \), \( d \geq 4 \), one can obtain a slightly better result.

**Proof of Theorem 3.4.** As in the proof of Theorem 1.3, it suffices to show (3.5) for some \( \epsilon > 0 \). Note \( \mathfrak{A}_j f \equiv 2^{-\frac{2d+1}{2}} e^{2\sqrt{d-1}} \mathfrak{M}_2 j f \) for some \( j \) satisfying \( \|f\|_p \lesssim \|f\|_p \) (cf. (3.9)). By (3.12), we have

\[ \|\mathfrak{A}_j f\|_{L^s(R^d \times I; dv)} \lesssim \langle \nu \rangle \delta^\frac{s}{2} 2^{j(s - \frac{d - 1}{2})} \|f\|_2, \quad q \geq 2 \]

for \( s > s(\alpha, q, d) \).

We first consider the case \( 1 < \alpha \leq d \). We interpolate (3.14) and (3.7), and then (3.14) and the trivial estimate \( \|\mathfrak{A}_j f\|_{L^\infty(R^d \times I; dv)} \lesssim \|f\|_\infty \). Consequently, for \( p, q \) satisfying \( q^{-1} \leq p^{-1} \leq 1 - q^{-1} \), we have

\[ \|\mathfrak{A}_j f\|_{L^p(R^d \times I; dv)} \lesssim \langle \nu \rangle \delta^\frac{s}{2} 2^{j\frac{s}{2}} \|f\|_p \]

if

\[ \hat{s} > \max \left\{ \frac{1}{\alpha} - \frac{\alpha}{q} \frac{d + 1}{2p} + \frac{1 - \alpha}{2q} - \frac{d - 1}{2}, \frac{d + 3 + 2\alpha}{4p} - \frac{d - 1 + 2\alpha}{4} \right\}. \]

Therefore, (3.5) holds for some \( \epsilon > 0 \) if \( \hat{s} < 0 \) i.e., \( p, q \) satisfy (3.13), \( p > (d + 3 + 2\alpha)/(d - 1 + 2\alpha) \), and \( p^{-1} \leq 1 - q^{-1} \).

On the other hand, if \( p, q \) satisfy \( 1 - q^{-1} \leq p^{-1} \leq q^{-1} \), then we have (3.15) for

\[ \hat{s} > \max \left\{ \frac{\alpha + 1}{p} - \frac{\alpha}{q} \frac{d + 3 + 2\alpha}{4p} - \frac{d - 1 + 2\alpha}{4} \right\}, \]

which is a consequence of interpolation between (3.7), (3.8), and (3.14) with \( q = 2 \) and \( s > s(\alpha, 2, d) = \max \left\{ \frac{d + 3 + 2\alpha}{d - 1 + 2\alpha}, \frac{d + 3 + 2\alpha}{4p} - \frac{d - 1 + 2\alpha}{4} \right\} \). So, (3.5) holds for \( p, q \) satisfying \( 1 - q^{-1} \leq p^{-1} \leq q^{-1} \) and \( p > p_d(\alpha) \). Combining the results for the cases \( q^{-1} \leq p^{-1} \leq 1 - q^{-1} \) and \( 1 - q^{-1} \leq p^{-1} \leq q^{-1} \), we have (3.5) for \( p, q \) satisfying (3.13) and \( p > p_d(\alpha) \). This proves the first part of Theorem 3.4 (when \( 1 < \alpha \leq d \)).
Proposition 4.1. Let \( C(4.1) \) following are necessary for
Furthermore, if \( d < \alpha \) estimate (3.16) (also see (1.6)) when \( 1 < p^{-1} \leq 1 - q^{-1} \) if
\[ s > \max \left\{ \frac{1}{p} - \frac{\alpha}{q}, \frac{d + 1}{2p} + \frac{d + 1 - 2\alpha}{2q} - \frac{d - 1}{2}, \frac{\alpha}{p} - (\alpha - 1) \right\}. \]
Thus, (3.5) holds for some \( \epsilon > 0 \) if \( s < 0 \) i.e., \((p^{-1}, q^{-1})\) is contained in the interior of the quadrangle with vertices \((0, 0), Q_1, Q_2, (1/2, 1/2)\).

Finally, by interpolation we need only to obtain (3.5) for \((1/p, 1/q) \in [(1/2, 1/2), Q_3)\). To see this, we interpolate two estimates (3.8) and (3.14) with
\[ d + 1 - \frac{2d - \alpha}{p} \leq (\alpha - 1) \]
Thus, (3.5) holds for \((\frac{1}{p}, \frac{1}{q}) \in [(1/2, 1/2), Q_3), as desired.

Similarly as in Corollary 1.4, \( L^p-L^q(\mu) \) estimate for the spherical maximal function holds for the range of \( p, q \) as in Theorem 3.4.

Corollary 3.5. Let \( d \geq 3, 1 < \alpha \leq d \) and \( \mu \in C^d(\alpha) \). For \( p, q \) satisfying \( p > p_d(\alpha) \) and \( 3.13 \), we have
\[ \| \sup_{t \in I} |A f(\cdot, t)| \|_{L^q(\mu)} \leq C(\mu)^{\frac{1}{\alpha}} \| f \|_{L^p(\mathbb{R}^d)}. \]

4. Necessary conditions on \( p, q, \alpha \) for (4.1)

In this section, we obtain necessary conditions on \( p, q, \alpha \) for (4.1) to hold. Obviously, it is enough to consider the local estimate
\[ \| A f \|_{L^q(\mathbb{R}^{d-1} \times I; d\nu)} \leq C(\nu)^{\frac{1}{\alpha}} \| f \|_{L^p(\mathbb{R}^d)}. \]

This is done by taking specific \( \alpha \) dimensional measures and particular functions. A modification of these examples also gives necessary conditions for the maximal estimate (3.10) (also see (1.6)) when \( 1 < \alpha \leq d \).

Proposition 4.1. Let \( d \geq 2 \) and \( 1 < \alpha \leq d + 1 \). Suppose (4.1) holds for \( \nu \in C^{d+1}(\alpha) \). Then,
\[ p^{-1} \leq \alpha q^{-1}, \]
\[ (d + 1) p^{-1} \leq \begin{cases} \frac{d - 1 + (\alpha - 1) q^{-1}}{1 < \alpha \leq d}, \\ \frac{d - 1 + (2\alpha - d - 1) q^{-1}}{d < \alpha \leq d + 1}, \end{cases} \]
\[ d^{-1} p^{-1} \leq \begin{cases} \frac{d - 1}{1 < \alpha \leq d}, \\ \frac{d - 1 + (\alpha - d) q^{-1}}{d < \alpha \leq d + 1}. \end{cases} \]
Furthermore, if \( d \geq 3 \) and \( k \leq \alpha \leq k + 1 \) for an integer \( k \in [1, d - 1] \), then the following are necessary for (4.1) to hold:
\[ (d + k) p^{-1} \leq d + k - 2 + (\alpha - k) q^{-1}, \]
\[ p^{-1} \leq \frac{d + k - 1}{d + k + 1}. \]
\[(k+1)p^{-1} \leq k + (\alpha - k)q^{-1},\]

\[(v)\]

\[p^{-1} \leq \frac{k + 1}{k + 2}.\]

When \(d = 2\) and \(1 < \alpha \leq 3\), (4.1) holds only if

\[(vi)\]

\[p \geq 4 - \alpha.\]

It seems to be natural to expect that (4.1) holds for \(\nu \in \mathcal{C}^{d+1}(\alpha)\) only if, instead of (iv) and (v)

\[(4.2)\]

\[p^{-1} \leq \begin{cases} \frac{d+\alpha-2}{d+\alpha}, & d - 2 < \alpha \leq d, \\ \frac{\alpha}{\alpha+1}, & 1 < \alpha \leq d - 2. \end{cases}\]

When \(\alpha = k + 1\) \((k = 1, \ldots, d - 1)\), (4.2) matches with the second conditions in (iv) and (v). In view of the global estimate (1.4) with \(d = 2\), the condition (vi) is redundant since (1.4) holds only if \(p \leq q\). In fact, the conditions (i) (ii) (iii) (iv) combined with \(p \leq q\) show that (1.4) with \(d = 2\) fails unless (vi) holds.

**Proof of Proposition 4.1** Let us assume that (4.1) holds for all \(\nu \in \mathcal{C}^{d+1}(\alpha)\) with \(1 < \alpha \leq d + 1\). We show the necessity of the conditions (i) (vi) by considering specific measures contained in \(\mathcal{C}^{d+1}(\alpha)\) and functions which show the failure of (4.1) unless the conditions hold.

In what follows, \(0 < \delta \ll 1\) and \(c\) is a small positive constant.

**Proof of (i)** Let \(N_\delta = \{x : ||x| - 1| < \delta\}\) and set \(f = \chi_{N_\delta}\). Clearly, \(A f(x,t) \gtrsim 1\) if \((x,t) \in \mathbb{B}^d(0,c\delta) \times [1, 1 + c\delta]\). Let \(d\nu(x,t) = \psi(x,t)|x|^{\alpha-d-1}dxdt\) for \(\psi \in C_0^\infty(\mathbb{B}^{d+1}(0,3))\). It is easy to see \(\langle \nu \rangle_\alpha \lesssim 1\). Then (4.1) implies \(\delta^{\frac{\alpha}{d}} \lesssim \delta^\gamma\). Thus, letting \(\delta \to 0\), we get \(1/p \leq \alpha/q\).

**Proof of (ii)** Let us set

\[V_\delta = \{(x_1,x',t) \in \mathbb{R} \times \mathbb{R}^{d-1} \times [1,2] : |x_1 - t| \leq c\delta^2, |x'| \leq c\delta\}.

We also define a measure \(\nu\) by

\[d\nu(x,t) = \begin{cases} \delta^{\alpha-d-2}\chi_{V_\delta} dxdt, & 1 < \alpha \leq d, \\ \delta^{2(\alpha-d-1)}\chi_{V_\delta} dxdt, & d < \alpha \leq d + 1. \end{cases}\]

One can easily show \(\langle \nu \rangle_\alpha \lesssim 1\) for \(1 < \alpha \leq d + 1\). Indeed, when \(1 < \alpha \leq d\), \(\nu(\mathbb{B}^{d+1}(z, \delta)) \lesssim \delta^{\alpha-d-2}\delta^{d+1}\) for \(0 < r < \delta^2\), \(\nu(\mathbb{B}^{d+1}(z, r)) \lesssim \delta^{\alpha-d}\delta^d\) for \(\delta^2 < r < \delta\), and \(\nu(\mathbb{B}^{d+1}(z, r)) \lesssim \delta^{\alpha-1}\delta^r\) for \(r < \delta\). So we have \(\nu(\mathbb{B}^{d+1}(z, \delta)) \lesssim \delta^\alpha\) in each case. The case \(d < \alpha \leq d + 1\) can be handled similarly.

We consider \(f = \chi_{U_\delta}\) for \(U_\delta = [0, \delta^2] \times [0, \delta]^{d-1}\). Then, we have \(A f(x,t) \gtrsim \delta^{d-1}\) if \((x,t) \in V_\delta\). Note \(\nu(V_\delta) \sim \delta^{\alpha-1}\) for \(1 < \alpha \leq d\), and \(\nu(V_\delta) \sim \delta^{2\alpha-d-1}\) for \(d < \alpha \leq d + 1\). Since \(\|f\|_p = \delta^{\frac{\alpha}{d}}\), (4.1) implies \(\delta^{d-1}\delta^{\frac{\alpha}{d}} \lesssim \delta^{\frac{d}{d} + \frac{\alpha}{d}}\) for \(1 < \alpha \leq d\), which gives (ii) for \(1 < \alpha \leq d\). When \(d < \alpha \leq d + 1\), we get (ii) in the same manner.

**Proof of (iii)** Let us take \(f = \chi_{S_\delta}\), then \(\|f\|_p \lesssim \delta^{d/p}\). Then we have \(A f \gtrsim \delta^{d-1}\chi_{S_\delta}\), where \(S_\delta = \{(x,t) : ||x_1 - t| \leq c\delta, t \in [1,2]\}\) for a small \(c > 0\). Let

\[d\nu(x,t) = \begin{cases} \delta^{-1}\chi_{S_\delta} dxdt, & 1 < \alpha \leq d, \\ \delta^{\alpha-d-1}\chi_{S_\delta} dxdt, & d < \alpha \leq d + 1. \end{cases}\]
It is not difficult to see \( (\nu)_\alpha \lesssim 1 \). When \( d < \alpha \leq d + 1 \), \( \nu(B^{d+1}(z,r)) \lesssim \delta^{\alpha - d - 1} r^{d+1} \) for \( 0 < r \leq \delta \), and \( \nu(B^{d+1}(z,r)) \lesssim \delta^{\alpha - d} r^d \) for \( \delta < r \). The case \( 1 < \alpha \leq d \) can be treated similarly and we omit the detail. Since \( |S_\alpha| \sim \delta \), we have \( \nu(S_\alpha) \sim 1 \) for \( 1 < \alpha \leq d \), and \( \nu(S_\alpha) \sim \delta^{\alpha - d} \) for \( \alpha \leq d + 1 \). So, (4.1) implies \( \delta^{d-1} \lesssim \delta^{d/p} \) for \( 1 < \alpha \leq d \), and \( \delta^{d-1+(\alpha-d)/q} \lesssim \delta^{d/p} \) for \( d < \alpha \leq d + 1 \). Letting \( \delta \to 0 \) yields (4.3).

Proof of (iv) We modify the example in the proof of (iii) For a given \( \alpha \in (1, d] \), let \( k \in [1, d-1] \) be an integer such that \( k < \alpha \leq k+1 \). Let \( U^d_k = [0, \delta^k]^k \times [0, \delta]^{d-k} \). We have

\[
A \chi_{U^d_k} \gtrsim \delta^{2(k+1)} \chi_{V^d_k},
\]

where

\[
V^d_k := \{(x', x'', t) \in \mathbb{R}^k \times \mathbb{R}^{d-k} \times [1, 2] : ||x''| - t| \leq c\delta^2, |x'| \leq c\delta \}.
\]

Let us set \( d\nu = \delta^{\alpha - d - 2} \chi_{k} dxdtdt \). In the same manner as before it is easy to show \( (\nu)_\alpha \lesssim 1 \). Since \( \nu(V^d_k) \sim \delta^{\alpha - k} \) and \( ||\chi_{U^d_k}||_p \lesssim \delta^{(d+k)/p} \), (4.1) and (4.3) yield \( \delta^{d+k-2(\alpha-k)/q} \lesssim \delta^{d/(d+k)} \). This gives the first part of (iv).

For the second part of (iv) taking \( f = \chi_{U^d_{k+1}} \) instead of \( \chi_{U^d_k} \), we have \( A f \gtrsim \delta^{d+k-1} \chi_{V^d_{k+1}} \). We consider \( d\nu = \delta^{k-d-1} \chi_{V^d_{k+1}} dxdtdt \). Similarly, as before, one can easily see \( (\nu)_\alpha \lesssim 1 \) and \( \nu(V^d_{k+1}) \gtrsim 1 \). Since \( ||\chi_{U^d_{k+1}}||_p \lesssim \delta^{(d+k+1)/p} \), (4.1) implies \( \delta^{d+k-1} \lesssim \delta^{(d+k+1)/p} \). This shows the second part of (iv).

Proof of (v) We show (v) by combining the examples considered while proving (ii) and (iii). Let \( U_k \) be the \( \delta \)-neighborhood of the set \( \{(0, \bar{z}) \in \mathbb{R}^k \times \mathbb{R}^{d-k} : |\bar{z}| = 1 \} \). We set \( f = \chi_{U_k} \) so that \( ||f||_p \lesssim \delta^{1/k} \). Parametrizing \( S^{d-1} \), we have

\[
A f(x, t) \gtrsim \int_{|y| \leq 1/2} \chi_{U_k} \left( x - t(y, \sqrt{1 - |y|^2}) \right) dy.
\]

Denoting \( x = (x', x'', x_d) \in \mathbb{R}^k \times \mathbb{R}^{d-k-1} \times \mathbb{R} \), we set

\[
V_k = \{(x, t) : ||x'| - \sqrt{t^2 - 1} | \leq c\delta, |x''|, |x_d| \leq c\delta, t \in [3/2, 2] \}.
\]

We write \( y = (y', y'') \in \mathbb{R}^k \times \mathbb{R}^{d-k-1} \). Then, we claim

\[
x - t(y', y'', \sqrt{1 - |y''|^2}) \in U_k
\]

for \( t \in [3/2, 2] \) and \( |y''| \leq 1/2 \) if \( (x, t) \in V_k \) and \( |x' - ty'| \leq c\delta \). Since \( |t^{-1}x' - y'| \leq c\delta \) and \( ||x'| - \sqrt{t^2 - 1} | \leq c\delta \), it is clear that \( ||y''| - t^{-1} \sqrt{t^2 - 1} | \leq 3c\delta \). This gives \( t|y''| \sqrt{1 - |y''|^2} = t\sqrt{1 - |y''|^2} = 1 + O(c\delta) \). Thus, for \( |x''|, |x_d| \leq c\delta \) we have

\[
$$
||x', x_d| - t(y'', \sqrt{1 - |y''|^2})|| \leq (1 - 2^{-1} \delta, 1 + 2^{-1} \delta)
$$
\]

if \( c \) is small enough. This proves (4.4). Since \( y' = x'/t + O(c\delta) \) and \( |y''| \lesssim 1 \), (4.4) implies

\[
A f \gtrsim \delta^{k} \chi_{V_k}.
\]

Let \( d\nu = \delta^{\alpha - d - 1} \chi_{V_k} dxdtdt \). Then we have \( \nu(V_k) \gtrsim \delta^{\alpha - k} \), and \( (\nu)_\alpha \lesssim 1 \) since \( \nu(B^{d+1}(z,r)) \lesssim \delta^{\alpha - d - 1} r^{d+1} \) if \( 0 < r \leq \delta \) and \( \nu(B^{d+1}(z,r)) \lesssim \delta^{\alpha - k} r^k \) if \( \delta < r \). So (4.1) gives \( \delta^{d-\alpha/k} \lesssim \delta^{d+1} \). Hence the first part of (v) follows if we take \( \delta \to 0 \).

To prove the second part of (v) taking \( f = \chi_{U_{k+1}} \), we have \( ||f||_p \lesssim \delta^{2/k} \), and \( A \chi_{U_{k+1}} \gtrsim \delta^{k+1} \) whenever \( (x, t) \in V_{k+1} \). Considering \( d\nu = \delta^{k-d} \chi_{V_{k+1}} dxdtdt \), we have
\[\langle \nu \rangle_\alpha \lesssim 1 \] and \[\nu(V_{k+1}) \gtrsim 1 \] since \[\nu(B^{d+1}(z,r)) \lesssim \delta^{k-d_r} \] for \(0 < r \leq \delta\), and \[\nu(B^{d+1}(z,r)) \gtrsim \delta^k \] for \(\delta < r\). Therefore, (4.1) implies \(\delta^{k+1} \lesssim \frac{\delta^{1+}}{r^2}\) for \(0 < \delta \ll 1\). This gives the second part of (6).

**Proof of (7)** Let us consider the case \(1 < \alpha \leq 2\) first. Let \(\{\theta_t\}\) be a collection of \(\sim C^{\delta^{1-\alpha}}\) points on \(S^1\) which are separated by \(\delta^{\alpha-1}\). For each \(\theta_t\), we set

\[ T_\delta(\delta) = \{(x,t) \in \mathbb{R}^2 \times [1,2] : ||x| - t| \leq c\delta^2, \ |x/|x| - \theta_t| \leq c\delta\}. \]

Let \(T_\delta = \cup_T T_\delta(\delta)\). We consider \(dv(x,t) = \delta^{\alpha-4} \chi_{T_\delta}(x,t) dx dt\), then \(\langle \nu \rangle_\alpha \lesssim 1\). In fact, one can easily see

\[ \nu(B^3(z,r)) \leq C \begin{cases} 
\delta^{\alpha-4}r^3, & 0 < r \leq \delta^2, \\
\delta^{\alpha-2}r^2, & \delta^2 < r \leq \delta, \\
\delta^{\alpha-1}r, & \delta < r \leq \delta^{1-\alpha}, \\
\delta^{\alpha-1} - (r/\delta^{\alpha-1}), & \delta^{\alpha-1} < r < 1, \\
\delta^{\alpha-1} - (r/\delta^{\alpha-1}), & 1 < r.
\end{cases} \]

Therefore, we have \(\nu(B^3(z,r)) \lesssim r^\alpha\).

Let \(R(\theta, \delta)\) be a rectangle of side length \(\delta^2 \times \delta\) centered at the origin such that the short side is parallel to \(\theta \in S^1\). Setting \(R_\delta := \cup_{R(\theta, \delta)}\), we have \(A_{R_\delta} \gtrsim \delta^{\chi_{T_\delta}}\).

Since \(|R_{\delta}| \lesssim \delta^{(1-\alpha)/p}\) and \(\nu(T_\delta) \gtrsim 1\), we get \(\delta \lesssim \delta^{(1-\alpha)/p}\) which implies \(p \geq 4 - \alpha\).

Now, we consider the case \(2 < \alpha \leq 3\). Let \(\{\omega_j\} \subset \mathbb{R}^2\) be a collection of \(\sim C^{\delta^{2-\alpha}}\) points on the \(x_1\)-axis which are separated by \(\delta^{\alpha-2}\). For each \(\omega_j\), we denote

\[ C_j(\delta) = \{(x,t) \in \mathbb{R}^2 \times [1,2] : ||x - \omega_j| - t| \leq c\delta, \ |x_2| \leq 2^{-\alpha}\}. \]

We set \(T_\delta = \cup_{C_j(\delta)}\) and \(dv(x,t) = \delta^{\alpha-3} \chi_{T_\delta}(x,t) dx dt\). Then, it is easy to see \(\langle \nu \rangle_\alpha \lesssim 1\) and \(\nu(T_\delta) \sim 1\). Indeed,

\[ \nu(B^3(z,r)) \leq C \begin{cases} 
\delta^{\alpha-3}r^3, & 0 < r \leq \delta, \\
\delta^{\alpha-2}r^2, & \delta < r < \delta^{1-\alpha}, \\
\delta^{\alpha-2}r^2 \times (r/\delta^{\alpha-2}), & \delta^{\alpha-2} < r < 1, \\
\delta^{\alpha-2} \times (r/\delta^{\alpha-2}), & 1 < r.
\end{cases} \]

So, it follows that \(\nu(B^3(z,r)) \lesssim r^\alpha\). Also we have

\[ A_{\chi_{\cup_j B^2(\omega_j, c\delta)}}(x,t) \gtrsim \delta, \quad (x,t) \in T_\delta. \]

Since \(|\cup_j B^2(\omega_j, c\delta)| \lesssim \delta^{4-\alpha}\), (4.1) implies \(\delta \lesssim \delta^{\frac{1}{2}}\). Thus, we see \(p \geq 4 - \alpha\). □

**Acknowledgement.** This work was supported by the NRF (Republic of Korea) grants No. 2017R1C1B2002959 (Ham), No. 2019R1A6A3A01092525 (Ko), and No. 2021R1A2B5B02001786 (Lee).

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