Using long-term transit timing to detect terrestrial planets

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ABSTRACT

We propose that the presence of additional planets in extrasolar planetary systems can be detected by long-term transit timing studies. If a transiting planet is on an eccentric orbit then the presence of another planet causes a secular advance of the transiting planet’s pericentre over and above the effect of general relativity. Although this secular effect is impractical to detect over a small number of orbits, it causes long-term differences when future transits occur, much like the long-term decay observed in pulsars. Measuring this transit-timing delay would thus allow the detection of either one or more additional planets in the system or the first measurements of non-zero oblateness \((J_2)\) of the central stars.

Key words: gravitation – celestial mechanics – planetary systems.

1 INTRODUCTION

The study of long-term orbital precession was one of the triumphs of celestial mechanics, when the planetary theories of Laplace and Lagrange showed that essentially all the known long-term precessions of the planetary orbits could be explained by their mutual gravitational interaction. The perturbation caused by the small planetary masses ‘breaks’ the perfect central force character of the Sun’s gravitational field, causing the planetary orbital nodes to regress and their perihelia to slowly advance, with typical periods of \(10^5–10^6\) yr. It was this advanced understanding of celestial mechanics that permitted LeVerrier and Adams to detect a new planet in our Solar System (Neptune) by inverting its observable effect on the known planets to predict Neptune’s mass and position.

The only known exception to these predictions at the start of the 20th century was the mystery that Mercury’s perihelion longitude advanced 0.42 arcsec yr\(^{-1}\) faster than the predicted rate of \(\approx 5.31\) arcsec yr\(^{-1}\) produced by the perturbative effects of all the other planets. The obvious possibility was that the Sun’s mass distribution was not spherically symmetric, but explaining the Mercurian advance would require a solar oblateness (measured by the parameter \(J_2\)) of several percent, which was uncomfortably large (e.g. Hall 1900). General relativity provided the solution, explaining essentially the entire discrepancy. In fact the Sun’s oblateness is left with only an empirical upper bound of \(J_2 < 1 \times 10^{-6}\), with a current theoretical estimate for its true value of \(J_2 \sim 1 \times 10^{-7}\) (Pireaux & Rozelot 2003).

As of the beginning of 2006, nearly two hundred extrasolar planets have been discovered and several of them exhibit transits (as catalogued by Butler et al. 2006, and references therein, especially Mayor et al. 2004; Vogt et al. 2005). The longitude of periastron of eccentric hot Jupiters (3-d periods) will show a secular advance of the instant of transit. If the orbital period can be well established (via transit timing or radial velocity) then the long-term drift of the transit centres will allow one to measure the slow advance of periastron. Observations on 10-yr baselines should certainly show the relativistic advance, which is much more important for close-in hot Jupiters since they are closer to their parent stars than Mercury is to our Sun. In this paper we discuss the possibility of using the periastron advance rate to measure either a host-star oblateness or the presence (and thus discovery) of additional planets in the system.

Miralda-Escudé (2002) provided the first estimates of the various contributions to the periastron advance of extrasolar planets. Agol et al. (2005) and Holman & Murray (2005) build upon this work by including the effects of resonances and including other contributions to the timing noise of planetary transits. Although these numerical studies automatically include the secular advance of periastron, they focus more on the stochastic variation of the interval between transits. Our work, as with Miralda-Escudé (2002), focuses specifically on the secular periastron advance and builds upon that work by including a more accurate calculation of the advance for planets whose orbits have similar semimajor axes and by outlining several techniques to measure the periastron advance and their associated precision (including the effects of general relativity).

In Section 2 we calculate the secular advance of periastron of the orbit of a planet around a star due to other planets in the system under reasonable approximations. We also present the relativistic and quadrupole contributions to the periastron advance. Section 3 places these calculations in the context of extrasolar planetary systems. Section 4 presents analytic and numerical estimates of how well we can determine the advance of periastron using various techniques (timing of the primary and secondary transits with or without radial velocity information). The special relativistic corrections are outlined in Section 5, and Section 6 outlines the prospects of this technique using the planets discovered so far as a guide.

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2 SECULAR ADVANCE

In the Newtonian two-body problem the Laplace Runge–Lenz vector (also known as Hamilton’s vector, or the eccentricity vector), which points from the star to the planetary orbit’s pericentre, is stationary, so the location of periastron is constant. Several effects can cause the periastron to advance. In our Solar System in increasing order of importance we have:

(i) stellar oblateness,
(ii) general relativity and
(iii) other planets.

The amplitude of these effects depends on several parameters, and thus in extrasolar planetary systems the order of importance may differ. In the following sections we provide expressions for the rate of periastron advance for these effects.

2.1 Stellar contributions

The central star can cause periastron advance by either being non-spherical or due to general relativistic effects caused by its mass. These effects cause a periastron advance (Misner, Thorne & Wheeler 1973) of

$$\delta \sigma = \frac{6 \pi G M_s}{c^2 a(1 - e^2)} + J_2 \frac{3 \pi R_s^2}{a^2(1 - e^2)^2}$$

per radial period due to the star itself. Here, $c$ is the speed of light, $G$ Newton’s gravitational constant, $a$ the semimajor axis of the planet’s orbit, $e$ its eccentricity, and $M_*$, $R_*$ and $J_2$ are the mass, radius and oblateness parameter of the star, respectively.

Since $a = (P^2 G M_*/4 \pi^2)^{1/3}$, we get a time rate of change for the relativistic component of

$$\sigma_{GR} \frac{\delta \sigma}{P} = \frac{121 \text{arcsec yr}^{-1}}{1 - e^2} \left( \frac{M_*}{M_\odot} \right)^{2/3} \left( \frac{P}{3 \text{d}} \right)^{ -5/3}$$

and for the stellar oblateness

$$\sigma_{J_2} = \frac{3.1 \text{arcsec yr}^{-1}}{(1 - e^2)^2} J_2 \left( \frac{R_*}{R_\odot} \right)^2 \left( \frac{M_*}{M_\odot} \right)^{-2/3} \left( \frac{P}{3 \text{d}} \right)^{-7/3}$$

where we have scaled these effects to typical values appropriate for a hot Jupiter. For Mercury the relativistic effect is 0.43 arcsec yr$^{-1}$. This drops as $a^{-5/2}$ for more distant orbits, being 0.086 arcsec yr$^{-1}$ for Venus, 0.038 arcsec yr$^{-1}$ for Earth and 0.013 arcsec yr$^{-1}$ for Mars. The scaling value of $J_2 = 10^{-6}$ used in equation (3) is essentially a firm upper limit for the oblateness of our Sun that can be obtained from theoretical and observational considerations (Pireaux & Rozelot 2003).

2.2 Planetary contributions

Just as Mercury’s pericentre advance is affected by the other planets, an exoplanet’s orbit will precess due to the perturbations of an unseen planet. We will generally assume that a second planet in the system is an external one, although the theory is almost identical if the perturber is interior.

A simple way to estimate the contribution to the periastron advance from another planet in the system is to assume that the mass of the second planet is smeared out over a circular ring coplanar with the first planet’s orbit and calculate the contribution of this ring to the potential. If the orbit of the second planet is elliptical or not coplanar with the observed planet, the eccentricity, inclination and longitude of the ascending node will also change on a secular time-scale.

An additional planet in the system causes the periastron of an observed planet to advance by

$$\delta \sigma \approx \frac{\pi m_2 a^3 (3a_2^2 - a^2)}{2 M_* a_2 (a_2^2 - a^2)^{3/2}} \approx \frac{3\pi m_2 a^3}{2 M_* a_2^2}$$

during each radial orbit (Price & Rush 1979). The approximation holds if $a \ll a_2$ and $a_2$ is the radius of the second planet’s orbit. The mass of second planet is $m_2$.

We can do a bit better if we relax the assumption of $a \ll a_2$. In this case we find that the potential due to the second planet is

$$V_2 = -\frac{G m_2}{a_2^2} \frac{2}{\pi(\alpha + 1)} K \left( \frac{2a_1^{1/2}}{\alpha + 1} \right),$$

where $K$ is the complete elliptic integral of the first kind and $\alpha \equiv a_1/a_2$. From Price & Rush (1979), the advance of periastron per radial orbit is

$$\delta \sigma = 2\psi - 2\pi$$

where the increase in azimuthal angle for half a radial orbit is

$$\psi = \pi \left\{ 3 + a \left[ \frac{V''(a)}{V(a)} \right]^{-1/2} \right\}$$

and $V(a)$ is the potential due to the star and the perturbing planet. The limit $V(a) = 1/a$ for no perturbing planet yields $\psi = \pi$ as expected. For $m_2 \ll M_*$ we have

$$\delta \sigma = \frac{m_2}{2 M_*} \frac{\alpha}{(\alpha + 1)^2} \left[ (\alpha^2 + 1) E \left( \frac{2a_1^{1/2}}{\alpha + 1} \right) - (\alpha - 1)^2 K \left( \frac{2a_1^{1/2}}{\alpha + 1} \right) \right]$$

for $\alpha < 1$,

where $E$ is the complete elliptic integral of the second kind. The function $b^{(1)}_1(\alpha)$ is a Laplace coefficient from classical perturbation theory; it is equal to $3\alpha$ for $\alpha \ll 1$, is of the order of unity for $\alpha = 0.1$–0.5, and then increases rapidly to $>100$ for $\alpha > 0.9$. In the limit of a large ratio between the two semimajor axes we obtain

$$\delta \sigma \approx \frac{m_2}{2 M_*} \times \left\{ \frac{3\pi a^3}{2} \alpha \ll 1, \text{ exterior perturber,} \right\} \left\{ \frac{3\pi a^3}{2} \alpha \gg 1, \text{ interior perturber.} \right\}$$

This is perhaps more transparently expressed by noticing that $N_0 = 2\pi/\delta \sigma$ is the number of inner planet revolutions required for a full precession of its orbit:

$$N_0 = \frac{2\pi}{\delta \sigma \delta \sigma} = \frac{4}{a^2 b^{(1)}_1(\alpha)} \frac{M_*}{m_2}$$

for $a < a_2$. Fig. 1 gives the exact value of $N_0 \times (m_2/M_*)$ and its asymptotic form. As examples, an unseen Jupiter-mass planet ($m_2/M_\star \sim 10^{-3}$) at 10 times the semimajor axis of an interior hot Jupiter (3-d period) will cause the hot Jupiter’s orbit to precess completely in about $10^5$ orbits = $3 \times 10^6$ d, or about 8000 yr; a 3-Earth-mass planet ($m_2/M_\star \sim 10^{-5}$) 1.5 times more distant than a hot Jupiter would cause precession in only about $10^5$ orbits (800 yr).

The analysis above greatly underestimates the effect if resonant or near-resonant terms are important to the dynamics. It is unclear whether resonant configurations are ubiquitous or happenstance;
some extrasolar planetary systems are already known to exhibit near-resonant behaviour (Rasio et al. 1992; Ford, Lystad & Rasio 2005). Agol et al. (2005) and Holman & Murray (2005) include the effects of orbital resonances and the eccentricity of the perturber’s orbit; however, Holman & Murray (2005) focus on the stochastic variation of the intertransit interval and Agol et al. (2005) do not consider secular terms in their calculations, which is the focus of the analysis here.

An explicit expression for the precession rate of the longitude of periastron, if the semimajor axis of the outer planet is much larger than that of the inner observed planet ($a_2 \ll a_1$), is

$$\dot{\alpha} = 355 \text{ arcsec yr}^{-1} \frac{m_2}{M_*} \frac{M_\odot}{M_*} \left(\frac{a}{a_2}\right)^3 \frac{3 \text{ d}}{P},$$

where $P$ is the period of the inner planet. This asymptotic estimate of the rate of periastron advance (shown in Fig. 1) significantly underestimates the rate for $\alpha > 1/4$. In fact for $\alpha = 0.5$ the estimate falls short by a factor of 2 (cf. Miralda-Escudé 2002).

3 WHAT IS THE SENSITIVITY TO OTHER PLANETS IN THE SYSTEM?

Fig. 2 shows the precession contribution from the three effects on an observed transiting planet. The most interesting component is the contribution from other planets in the system, so we would like to have firm upper limits on the contributions from the other two effects in order to estimate the excess that might be due to an unseen planet.

The relativistic contribution is the most certain. The mass of the star can generally be estimated to a few percent and the period of the orbit of the planet can be measured to a part in $10^5$ or better; therefore, the value in equation (2) can be estimated to a high degree of precision.

Since the GR-induced precession rate is simply proportional to the mass of the star, the fractional precision of this mass estimate will set the sensitivity limit of the periastron advance method. If we take 3 per cent as a nominal $M_*$ precision then, if one observes a transiting hot Jupiter with $P \approx 3 \text{ d}$, a precession rate different from the GR rate by less than 3 per cent will not provide a reliable detection of a planet or an oblate star; the 3-d period used for Fig. 2 gives by chance that the induced rate from the host star with $J_2 = 10^{-6}$ falls close to the detection limit.

The value of the oblateness ($J_2$) for the host stars of extrasolar planets is unknown – we do not even know it well for our Sun. However, we can estimate an upper limit to its value from the solar estimates and a scaling of the rotation rate of our Sun to that of the host star if it is known. From observations of solar oscillations and theoretical considerations, the value of $J_2$ for the Sun is probably around a few times $10^{-7}$ (Pireaux & Rozelot 2003). Winn et al. (2005) recently determined the spin rate of the star in the transiting planetary system HD 209458. The value that they obtain $v \sin i = (4.70 \pm 0.16) \text{ km s}^{-1}$ is not much larger than the typical values for the Sun of 1.4–2.0 km s$^{-1}$, so it is unlikely that the value of $J_2$ for at least this system is much larger than that of the Sun. The value $10^{-6}$ probably provides an upper limit; this yields a oblateness contribution of about 3 arcsec yr$^{-1}$, an order of magnitude less that the contribution from a nearby Earth-mass planet. It is unlikely that stars have values of $J_2 > 10^{-6}$; detected precession rates more than $\sim 3$ arcsec yr$^{-1}$ faster than the GR-induced rate would be strong evidence for the presence of an unseen planet.

The detection of a periastron advance rate for an extrasolar planet with a 3-d period that exceeded the relativistic amount by More than that of the inner observed planet ($\alpha \ll a_2$), is

$$\dot{\alpha} = 355 \text{ arcsec yr}^{-1} \frac{m_2}{M_*} \frac{M_\odot}{M_*} \left(\frac{a}{a_2}\right)^3 \frac{3 \text{ d}}{P},$$

where $P$ is the period of the inner planet. This asymptotic estimate of the rate of periastron advance (shown in Fig. 1) significantly underestimates the rate for $\alpha > 1/4$. In fact for $\alpha = 0.5$ the estimate falls short by a factor of 2 (cf. Miralda-Escudé 2002).
where we have assumed that the periastron advance per orbit is small. As the orbit precesses, the time between transits will change, according to

$$\frac{d\Delta t}{dt} = -\left(1 + \frac{dM}{dv} \frac{\delta M}{2\pi} \right)^{-2} (\delta M)^2 \frac{d^2 M}{dv^2} \frac{1}{2\pi}$$

(16)

At face value this gradual change of the timing of the transits appears hopeless to detect because the change is proportional to the very small square of the advance of periastron per orbit. However, the difference in the timing accumulates from orbit to orbit as the orbit precesses, so in practice one would predict the timing of the transits from a few observations and look for a difference from that prediction after many hundreds of orbits had passed. Essentially we are interested in the integral of $\Delta t$ over many periods.

If one observes several initial transits and determines a value of $\Delta t$, one can predict the timing of future transits. These predictions will be incorrect for a precessing orbit by the following amount

$$t_{\text{pred}} - t_{\text{actual}} = -\left(M(v) - M(v_0) - \frac{dM}{dv} \int_0^v (v - v_0) \right) \frac{P}{2\pi}$$

(18)

where $v_0$ is the true anomaly during the transit at the initial epoch and $v$ is the true anomaly during the transit at the later epoch.

One can calculate

$$\frac{d^2 M}{dv^2} = \frac{1 - e^2 \sin v (1 - e \cos E)}{(1 + e \cos v)^2}$$

(19)

or expanding for small eccentricities

$$\frac{d^2 M}{dv^2} = 2e \sin v - 3e^2 \sin 2v + 3e^3 \sin 3v + O(e^4).$$

(20)

From equation (17) one can see that for an error in the predictions to accumulate the second derivative of the mean anomaly with respect to the true anomaly must not vanish. Thus, how quickly the error accumulates depends on the initial epoch of the observations. If the transit is initially occurring at periastron ($v = 0$) or apastron ($v = \pi$), the second derivative vanishes so it will take significantly longer for the time delay to become observable.

To lowest order in the change in the periastron advance (or the true anomaly at transit), we have

$$t_{\text{pred}} - t_{\text{actual}} \approx \frac{P}{2\pi} (v - v_0)^2 \frac{d^2 M}{dv^2} \bigg|_{v_0}$$

(21)

$$\approx \frac{P}{2\pi} \left[\frac{\delta M (t - t_0)}{2} \right] \frac{d^2 M}{dv^2} \bigg|_{v_0}.$$

(22)

so the delay accumulates quadratically in time. Using reasonable values for the various numbers we have

$$t_{\text{pred}} - t_{\text{actual}} \approx 1 \text{ mas} \frac{e \sin v_0}{0.1} \frac{P}{3 \text{ d}} \left(\frac{t - t_0}{100 \text{ arcsec yr}^{-1}} \right)^2.$$

(23)

### 4.1 Error analysis

Equation (23) makes it seem hopeless to detect the timing delay because one can determine the time of a particular transit to possibly 10 s; therefore, naively one would expect to have to wait 100 yr before detecting an advance with $t_{\text{pred}} - t_{\text{actual}} = 10$ s. Fortunately, one can detect the periastron advance in the series of transit
times long before one could detect it in the timing of an individual transit. In practice one characterizes the timing of the transits with a formula of the following form

\[ t_n = A + Bn + Cn^2 = t_0 + \Delta t_0 \pi + \frac{P}{4\pi} (\delta \sigma)^2 \left. \frac{d^2 M}{dv^2} \right|_0 n^2, \]  

(24)

where \( n \) is the number of the transit, \( t_0 \) is the time of an initial reference transit, \( \Delta t_0 \) is the initial time between transits and the quadratic term contains the periastron advance.

Using the standard results for \( \chi^2 \) fitting, we obtain

\[ \sigma_c = \sigma_0 \left( \frac{180s_0^3}{(r_0 N)((r_0 N)^2 - 5(r_0 N)^2 + 4)} \right)^{1/2} \]  

(25)

\[ \approx 13.4\sigma_0 P_0^{-1/2} N^{-5/2}(1 + O(N^{-2})), \]  

(26)

where \( N \) is the number of the last transit sampled, \( r_0 \) is the fraction of transits with times and \( \sigma_0 \) is the timing error on each transit.

The upper limit obtained for the value of the advance per orbit \( \delta \sigma \) will be

\[ \sigma_{\sigma \sigma} = \sigma_c^{1/2} \left( \frac{P}{4\pi} \frac{d^2 M}{dv^2} \right)^{-1/2}, \]  

(27)

where we have ignored the fractional error in the values of \( P \) and \( d^2 M/dv^2 \). This yields

\[ \sigma_{\sigma \sigma} \approx 1.6 \times 10^{-3} \left( \frac{N}{1000} \right)^{-5/4} r_0^{-1/4} \left( \sigma_0 \frac{3d0.1|\sin v_0|}{10\pi P} \right)^{1/2} \]  

(28)

as an upper limit on the advance per orbit. This is not much larger than the expected relativistic contribution of

\[ (\delta \sigma)_{GR} \approx 5 \times 10^{-6} \frac{1}{1 - e^2} \left( \frac{M}{M_{\odot}} \right)^{2/3} \left( \frac{P}{3\text{ d}} \right)^{-2/3}. \]  

(29)

So for \( N \) greater than a few thousand the relativistic term will dominate over the statistical errors in the timing. These estimates agree with the results of Miralda-Escudé (2002) who considered the effects of periastron advance on the timing of the primary transit and the duration of the primary transit.

One thousand transits of a planet with a 3-d orbit takes just a shade under eight and a quarter years. The upper limit on \( \delta \sigma \) decreases with time as \( t^{-5/4} \) until a reliable periastron advance is detected. After this time, the errors on this detection decrease as \( t^{-5/2} \). The time to achieve the desired sensitivity scales as the \( P^{1/2} \) so this technique is also applicable to planets with longer orbital periods. These estimates assume that every transit is timed (\( r_0 = 1 \)). The error analysis assumes that the observed transits are evenly spaced in time; it should be possible to devise an observing strategy that achieves errors similar to the \( r_0 = 1 \) case with many fewer observations – this is beyond the scope of this paper.

### 4.1.2 Why do not observations of the primary transit tell us more?

Each time the planet orbits the star is takes a bit less than an orbital period for it to reach the point of primary transit, because the orbit is shifting a bit. However, since we do not know the radial orbital period itself, this time is essentially unobserved. The time for the planet to cover the missing angular distance is related to the distance from the star to the planet at transit. As the orbit precesses, this distance will change which in turn will change the time between transits. It is this change in the time between transits that we try to observe. The correction in the time between transits is proportional to the periastron advance. The change in the distance between the star and the planet from orbit to orbit is also proportional to the periastron advance. Combining these facts indicates that the change in the time between transits is second order in the small periastron advance; consequently, it takes a relatively many orbits to detect the periastron shift, if one times only one type of transit.

### 4.2 Secondary transit

Looking at the secondary transit (when the planet goes behind the star) does not just provide a new set of times to fit but also provides new information and possibly a faster way of detecting unseen planets in the system. The secondary transit occurs when the true anomaly is \( 180^\circ \) away from where the primary transit occurs; therefore, we will denote quantities that describe the secondary transit with the subscript \( \pi \). The primary transit is given by the subscript \( 0 \). Let us examine at the time between two successive primary and two successive secondary transits. Using the earlier formulae we have to lowest order in the advance of periastron

\[ \Delta t_0 = P \left( 1 - \frac{dM}{dv} \right) \frac{(\delta \sigma)^2}{2\pi} \]  

and

\[ \Delta t_\pi = P \left( 1 - \frac{dM}{dv} \right) \frac{(\delta \sigma)^2}{2\pi}, \]  

(30)

(31)

where

\[ \frac{dM}{dv} = \sqrt{1 - e^2} - e \cos E \]  

\[ = 1 - 2e \cos v + \frac{3}{2} e^2 \cos 2v - e^3 \cos 3v + O(e^4). \]  

(32)

If we take the difference between these two values we get

\[ \Delta t_0 - \Delta t_\pi = \left( \frac{dM}{dv} \right)_0 - \left( \frac{dM}{dv} \right)_\pi \frac{(\delta \sigma)^2}{2\pi}. \]  

(33)

So the interval between two successive primary and two successive secondary transits differs by an amount proportional to the advance of periastron per orbit. This should be compared with observations of the primary transit alone in which the advance of periastron only enters at second order.

Furthermore, the difference in the time between the primary and secondary transits also contains some valuable information. We have

\[ t_n - t_0 = [M(v_0 + \pi) - M(v_0)] P \]  

(34)

If the orbit is eccentric this will differ from half of the orbital period. Expanding in the eccentricity we have

\[ t_\pi - t_0 = \frac{P}{2} + \frac{P}{2\pi} \left[ 4e \sin v_0 + \frac{2}{3} e^2 \sin 3v_0 + O(e^3) \right]. \]  

(35)

The first term in the series is twice the value of the first term in the series for \( d^2 M/dv^2 \), so the interval between the primary and secondary transits helps to calculate the periastron advance when one uses the timing of the primary transits.

### 4.2.1 Error analysis

How well can we determine the time of the transits and the time between successive transits? Fitting the transit times to a timing model

\[ t_{0,n} = t_0 + \Delta t_0 n \]  

(36)
and
\[ t_{s,n} = t_r + \Delta t_n n. \]  
(37)

Because the periastron advance now enters in the difference between
the interval between the successive transits, we only need to fit the
times to first order in the number of the transit \( n \). From the \( \chi^2 \)
analysis we obtain the following error estimates
\[ \sigma_{t_n} = \sigma_t \left( \frac{2(2r_s N) + 1}{(r_s N)(N - 1)} \right)^{1/2} \]  
(38)

and
\[ \sigma_{\Delta t_n} = \sigma_t \left( \frac{12r_s^2}{(r_s N)(N - 1)} \right)^{1/2}, \]  
(39)

where \( r_s \) and \( \sigma_t \) are the fraction of secondary transits with times
and the error in the timing of the secondary transit. The error in
the time of the initial transit scales as \( N^{-1/2} \) where \( N \) is the number of
orbits that have elapsed between the first and last one observed.

We are interested in the differences
\[ \Delta t_0 - \Delta t_n = \left[ -4e \cos \nu_0 - 2e^3 \cos 3\nu_0 + \mathcal{O}(e^5) \right] \delta \sigma \frac{P}{2\pi} \]  
(40)

and
\[ \sigma_{\Delta t_0 - \Delta t_n} \approx \left[ \left( \frac{\sigma_t^2}{r_0^2} + \frac{\sigma_\nu^2}{r_\nu^2} \right) \frac{12}{N^2} \right]^{1/2}, \]  
(41)

for large \( N \), and especially the error in
\[ \sigma_{\delta \nu} \approx 7 \times 10^{-8} \frac{1}{e^2} \cos \nu_0 \left( \frac{N}{1000} \right)^{3/2} \frac{3}{P} \frac{d}{d^2} \]  
\[ \times \left( \sigma_\nu^2 + \frac{\sigma_\nu^2}{10^3 r_0^2} + \frac{\sigma_\nu^2}{10^3 r_\nu^2} \right)^{1/2}. \]  
(42)

Combining results for the secondary transit with those from the
primary transit yields an increase in sensitivity of a factor of \( \sim 200 \).
If we could time the secondary transit to the same precision of 10 s it
would take only \( N \sim 100 \) to detect an Earth-like planet within twice
the semimajor axis of the observed planet. With current instruments
and the brightest targets, the secondary transits can be timed to a
precision of about 100 s (J. Matthews, private communication),
yielding an estimate of about \( N \sim 400 \) orbits for a similar detection.
The time to achieve the desired sensitivity scales as \( P^{1/2} \), so this
method is also applicable to planets with longer orbital periods;
nevertheless, one is sensitive to smaller planets in systems with larger
eccentricities.

4.2.2 Why does the secondary transit help so much?

After analysing the primary transit, one saw how difficult it was to
disentangle the change in the angle of periastron from the obser-
vations of the orbital period of the system. The timing of the
secondary transit breaks this degeneracy, and it is straightforward to
understand why. Unless the orbit is perfectly circular (or if we are
so unlucky as to have \( \nu_0 \approx \pi/2 \) or \( 3\pi/2 \) in the epoch of observa-
tions), the planet is at different distance from the star at the primary
and secondary transit, so according to Kepler’s second law (conserva-
tion of angular momentum) its angular velocity along the orbit is
different at these two times. If the periastron shifts as the planet
orbits, it takes a different amount of time to cover the missing angle
at the primary than at the secondary transit; consequently, the time
between secondary transits differs from that between the primary
transits – if one can detect this time difference one can detect the
advance of periastron.

4.3 Radial velocity information

The radial velocity information is arguably the most difficult to ob-
tain. It turns out that it is essentially the least useful (at least in
quantity) for the purposes of characterizing the periastron shift.
It is difficult to imagine obtaining timing of the radial velocity
data with a precision of tens of seconds, so it is not directly use-
ful in getting additional timing points, as we did with the sec-
ondary transit. In principle, one would find that the time interval
between when the star passed through a particular radial velocity
and when it repeated itself would depend on the radial velocity
in question.

However, the period found by fitting the radial velocity curve
is typically precise to about 1 s. This time interval would only
differ from the interval between transits by a tiny amount, of the
order of the periastron advance. Determining accurately the rela-
tionship between the time between periastrons (what we have
called the period) and the period found by fitting the radial ve-
clocity requires Monte Carlo simulations of the observed data.
One can also gain some insight into what time interval emerges
from fitting radial velocities by considering orbits that are nearly
circular.

When one fits the radial velocity measurements one is most sen-
tive to parts of the orbit with large accelerations to or from the
observer. The acceleration along the line of sight reaches an extreme
when the jerk vanishes. To first order in the orbital eccentricity
\[ \frac{d^3 v_{\text{los}}}{dt^3} \approx \left( \frac{2\pi}{P} \right)^3 \sin(v - \nu_0) \]  
\[ + 2e \left( 11 \cos v \sin(v - \nu_0) + 5 \sin \nu_0 \right); \]  
(43)

so the jerk vanishes where
\[ \sin(v - \nu_0) = -10e \sin \nu_0 + \mathcal{O}(e^2). \]  
(44)

To lowest order in the eccentricity, the radial velocity measure-
ments are equally sensitive to the timing at the primary \( (v - \nu_0 = 0) \)
and secondary transits \( (v - \nu_0 = \pi) \), so we assume that the time interval
determined by fitting the radial velocity is given by the average of
the two intervals discussed earlier
\[ \Delta t_{\text{RV}} = \frac{1}{2} (\Delta t_0 + \Delta t_n) + \mathcal{O}(e^2) = P \left( 1 - \frac{\delta \sigma}{2\pi} \right) + \mathcal{O}(e^2). \]  
(45)

The ‘period’ obtained by fitting the radial velocity data differs
slightly by the period between periastrons or the period between
primary transits. We confirmed this by generating radial velocity
data with an advancing periastron and fitting these data with purely
Keplerian radial velocity curves. These simulations gave equa-
tion (45) for small eccentricities.

If we take the difference between the two observable quantities
we get
\[ \Delta t_{\text{RV}} - \Delta t_0 = 2e \cos \nu_0 \frac{\delta \sigma}{2\pi} \frac{P}{2\pi}. \]  
(46)

The error in this quantity is given by
\[ \sigma_{\delta \sigma} \approx \left( \frac{\sigma_{\delta \sigma}}{(100 \text{ ms})^2} + \frac{\sigma_\nu^2}{10^4 r_0^2} \right)^{1/2} \frac{2\pi}{P} \frac{1}{2e \cos \nu_0}, \]  
(47)

\[ \approx 10^{-5} \left[ \frac{\sigma_{\delta \sigma}}{(100 \text{ ms})^2} + 0.12 \frac{\sigma_\nu^2}{(10^4 s)^2} \right]^{1/2} \]  
\[ \times \left( \frac{3}{P} \frac{0.1}{2e \cos \nu_0} \right). \]  
(48)
We see that the timing errors in the radial velocity measurements dominate over the transit timing for a quoted precision in the radial-velocity period of 100 ms. Currently the best period estimates for hot Jupiters without transit information are precise to 800 ms (Butler et al. 2006), but analysis of radial velocity measurements over longer baselines would provide a more precise estimate of this period.

Even without a detailed understanding of the relationship between the velocity and transit timing, the radial velocity data are crucial to convert a observed timing solution into a periastron shift by determining the values of the eccentricity and the true anomaly at transit, and in combination with the timing of either the primary or secondary transit could yield hints of the periastron advance due to other planets in the system.

As the orbit precesses, the radial velocities observed over an orbit will also shift. However, over the time required to detect an Earth-like planet the orbit will only precess about an arcminute. It is difficult to imagine that radial velocity measurements will become so sensitive as to characterize an orbit to the required precision of $10^{-4}$. If they did, one could probably detect the planet causing the precession in the radial velocity data already.

4.4 Rapid precession

If the transiting planet’s orbital eccentricity goes to zero, the precession rate of its pericentre longitude will formally go to infinity, and equation (23) misleadingly indicates that the pericentre rate will be trivial to detect. However, equation (15) is not correct in the case of rapid precession.

If the rate of the precession is a constant ($\delta \sigma$) per radial orbit we have following equation for the true anomalies of the transits

$$\nu \mod 2\pi = \left[\nu_0 - \frac{\delta \sigma}{P} \frac{t}{\pi}\right] \mod 2\pi. \quad (49)$$

If we assume that the orbit is a precessing ellipse and that the angular momentum of the observed planet is conserved, we can use Kepler’s equations to determine the time corresponding to each true anomaly. To lowest order in the eccentricity we have

$$t = P\frac{\nu}{2\pi} - \frac{P}{2\pi} \frac{\nu^2}{4} \sin 2\nu. \quad (50)$$

If we first ignore the eccentricity, we find that the time between two successive transits is given by the angular period

$$\Delta t^{(0)} = P \left(1 + \frac{\delta \sigma}{2\pi}\right)^{-1} \quad (51)$$

and

$$\Delta \nu = \nu_2 - \nu_1 = -\delta \sigma \left(1 + \frac{\delta \sigma}{2\pi}\right)^{-1}. \quad (52)$$

Looking at Kepler’s equation we find that the correction to this quantity introduced by the eccentricity of the orbit is limited by $e^2 P/(4\pi)$. We have

$$\Delta t^{(2)} = \Delta t^{(0)} - \frac{P}{2\pi} \frac{e^2}{4} \left(\sin 2\nu_2 - \sin 2\nu_1\right) \quad (53)$$

$$= \Delta t^{(0)} \left\{1 - \left(1 + \frac{\delta \sigma}{2\pi}\right) \frac{e^2}{4\pi} \left[\sin \Delta \nu \cos (2\nu_1 + \Delta \nu)\right]\right\}. \quad (54)$$

For a perturber in an elliptical orbit $\delta \sigma$ is inversely proportional to the eccentricity of the observed planet (Murray & Dermott 2000); thus equation (54) indicates that the observable correction to the time interval is proportional to the eccentricity, proving that the timing error becomes unobservable as $e$ tends to zero rather than diverging.

5 SPECIAL RELATIVISTIC CORRECTIONS

The foregoing analysis focused on the angles necessary for a transit to occur. It was essentially geometry with any kinematics. Specifically it neglected the time for light to travel across the system. The variation in the distance of the planet and the star from transit to transit as the orbit precesses would affect the times that we observe the transits to occur.

The light travel time will affect the observed time difference between the primary transit and the secondary transit that immediately follows it (Loeb 2005),

$$[t_{\nu} - t_0]_{\text{obs}} = [\mathcal{M}(\nu_1 + \pi) - \mathcal{M}(\nu_0)] \frac{P}{2\pi} + \frac{1}{c} \left[r(\nu_0) + r(\nu_0 + \pi)\right], \quad (55)$$

where

$$r(t) = \frac{a(1 - e^2)}{1 + e \cos \nu}. \quad (56)$$

The light traveltime will cause the primary transit to appear to occur about $20 \text{s}(P/2)^{3/2}(M_P/M_\odot)^{1/3}$ earlier. The appearance of the secondary transit will be delayed by similar interval. This timing signature may be used to constrain the physical size of the orbit; however, the difference in the intertransit interval due to geometry is of the order of the period of the orbit, a factor of $10^4$ larger, so the eccentricity of the orbit must be known accurately for the relativistic corrections to be useful.

Other quantities that we have examined are the interval from primary to primary transit and from secondary to secondary transit, and the time derivative of these quantities. If the orbit did not precess the light travel time would not affect either of these intervals, so we know that the relativistic correction to these intervals will be proportional to the change in an angle of periastron during an orbit.

We have

$$[\Delta t_{\nu}]_{\text{obs}} = P \left[1 - \left(\frac{P}{2\pi} \frac{d \mathcal{M}}{d \nu}\right|_0 - \frac{1}{c} \frac{d r}{d \nu}\right|_0 \right] \frac{\sigma}{\pi}. \quad (57)$$

and

$$[\Delta t_{\nu}]_{\text{obs}} = P \left[1 - \left(\frac{P}{2\pi} \frac{d \mathcal{M}}{d \nu}\right|_0 + \frac{1}{c} \frac{d r}{d \nu}\right|_0 \right] \frac{\sigma}{\pi}. \quad (58)$$

The ratio of the two corrections to the interval between primary transits is

$$\frac{\Delta t'_{\nu,\text{rel}}}{\Delta t'_{\nu,\text{ang}}} = \frac{\tan \frac{\nu_0}{\pi} \frac{a}{c}}{P}\frac{1}{\pi} \left(\frac{M_\star 3.4 M_\odot}{P}\right)^{1/3}. \quad (59)$$

$$\approx 8 \times 10^{-5} \tan \frac{\nu_0}{\pi} \left(\frac{M_\star 3.4 M_\odot}{P}\right)^{1/3}. \quad (60)$$

The ratio is similar for the secondary transits; therefore, as found by Agol et al. (2005) it is safe to ignore the relativistic corrections to the intervals between two similar transits.
J. S. Heyl and B. J. Gladman

Figure 3. The duration of the time-series of primary and secondary transits (or primary transits and radial velocity timing) to yield a detectable $\delta \sigma$ of $10^{-7}$ for the planets in the catalogue of Butler et al. (2006), assuming a 10-s error in the timing. This is the sensitivity required to detect an Earth-mass planet at the $3\sigma$ level within a factor of 4 in semimajor axis of the transiting planet, orbiting a solar mass star. The required length of the time-series increases as $\sigma^{-2/3}$.

6 OUTLOOK

6.1 Systems with a transiting planet

Nearly two hundred extrasolar planets have been detected as of the beginning of Butler et al. (Butler et al. 2006). The eccentricity and the longitude of periastron have been measured for most of these planets; unfortunately, for the few transiting planets, this information is lacking, so we will use the orbital elements for all the nearby exoplanets to calculate the sensitivity of timing measurements to periastron advance and more importantly how long of a timing series would be required to achieve a precision of $10^{-7}$ in the measurement of $\delta \sigma$; this is sufficient to detect an Earth-mass planet orbiting within a factor of 4 of the semimajor axis of the transiting planet for a 1 M$_\odot$ host star. Fig. 3 shows that for about 10 systems this level of sensitivity could be achieved within a decade (these results have assumed that the sampling rate $r$ is unity; determining the optimal sampling rate and schedule considering observational constraints is beyond the scope of this paper).

To be certain of the presence of an unseen planet one has to be sure that the periastron precession exceeds that expected from the star. For this discussion we shall assume that the contribution due to stellar oblateness is small (according to Fig. 2 even for a 3-d period, the expected contribution from oblateness is a factor of 30–300 below the relativistic value; this ratio increases as $P^{2/3}$ so oblateness does not make the dominant contribution for any of the systems in Fig. 3). On the other hand, Fig. 4 shows the expected relativistic precession is greater than $10^{-7}$ for most of the systems in the catalogue; consequently, the mass of most of the host stars for these systems must be determined to better than about a tenth of a solar mass. Otherwise, the error in the determination of the mass of the host star will dominate the statistical error in the timing.

Figure 4. The advance of the longitude of periastron per orbit for the planets of Butler et al. (2006) from general relativity. The four systems for which the relativistic advance is most rapid are HD 86081, HD 149026, tau Boo, HD 73256.

Figure 5. The sensitivity to the advance of the longitude of periastron caused by additional planets (or stellar oblateness) in the planetary systems of Butler et al. (2006) assuming that the mass of the host star is known to 0.03 M$_\odot$ with a 10-yr baseline of transit timing. To detect an Earth-mass planet with an orbit up to a factor of 4 larger than the observed planet the sensitivity to the planetary contribution to the precession must be less than $10^{-7}$.

Fig. 5 shows the sensitivity to finding additional planets in the transiting systems with a 10-yr time-series assuming that the mass of the host star is known with a precision of 0.03 M$_\odot$. The properties of the transit itself, especially the shape of the ingress and egress (Loeb 2005) and duration (e.g. Winn et al. 2007), may constrain the mass of the star further. Four objects stand out in Figs 3–5; GJ 436 b, HD 33283 b, HD 74156 b, and HD 74156 b require the shortest time-series to achieve a sensitivity of $10^{-7}$ in $\delta \sigma$. These systems have moderate eccentricities between 0.2 and 0.6. Because of its short 2.6-d orbit, the orbital precession of GJ 436 b would be dominated by relativistic effects with $(\delta \sigma)_{\text{GR}} \approx 2.3 \times 10^{-6}$ assuming a mass of 0.41 M$_\odot$ for GJ 436 (Butler et al. 2006). There are several
planets in the catalogue whose orbits will precess at a rate similar to GJ 436b due to GR, but the geometry and eccentricity of the orbits are not as favourable for detecting the precession by transit timing. On the other hand, the maximum sensitivity to unseen planets is found in systems like HD 74156 and HD 168443 because their relatively long orbits (52 and 58 d) reduce the expected contribution due to GR and stellar oblateness. The minimum detectable value of $(\delta\sigma)_{\text{max}}$ is about $5 \times 10^{-8}$, sufficient to detect an Earth-mass planet with a period less than about 460 d after a decade of observation.

Unfortunately, those planets that require the least time to constrain additional bodies in the system have relatively long periods, of the order of 50–100 d. It is a factor of 7–10 times less likely that planets with such long periods transit their star than the hot Jupiters with 3-d periods. However, as more transiting planets are discovered one would expect to find both planets with longer periods and higher eccentricities than the current cohort of transiting planets. Indeed Butler et al. (2006) list several planets that orbit their stars with periods of a few days in eccentric orbits ($e \gtrsim 0.1$). None of these systems have exhibited transits, but it is only a matter of time before a transiting planet with an eccentric orbit is found and these techniques may be brought to bear.

6.2 Systems with more than one observed planet

A system that already has more than one planet detected provides a chance to characterize an additional planet and verify that the stellar oblateness does not dominate over the planetary signal. Of course, each observed planet will induce a periastron precession in the other, thus giving the planetary masses directly. This must be subtracted from the observed shifts along with the relativistic shifts. Because both planets orbit the same star the contribution due to the stellar oblateness is proportional to $P^{-7/3}$; consequently, unless the residual periastron precession rate in each planet that remains is proportional to $P^{-7/3}$, there must be other planets in the system.

If one can argue from other data that the value of $J_2$ is small, one can use the residual precession rate to find the mass of the unseen planet and its semimajor axis. With a single observed planet one can only constrain the combination that determines $\delta \sigma$. With three observed planets, one could unravel $J_2$ and the properties of an unseen planet or alternatively the properties of an unseen planet and the possibility of further planets!

6.3 Systems with fewer than one observed planet

It may seem odd to suggest measuring the periastron advance in systems without any planets yet discovered. However, the techniques outlined here could be used in eclipsing binary systems. In single-lined systems it could be used to constrain the total mass of the system through the relativistic term. In double-lined systems or if the total mass of the system is known, measurements of the periastron advance through timing of the radial velocity data and one or both transits could be used to constrain the values of $J_2$ for the stars or to look for planets in the systems. One could search for planets with the long-term timing measurements of eclipsing binaries that have already been taken.

7 CONCLUSIONS

Accurate timing data (of pulsars) allowed the discovery of the first extrasolar Earth-mass planets (Wolszczan & Frail 1992). Careful and accurate timing of planet transits and radial velocity data is sensitive to additional planets down to the mass of Earth and below. The combination of two sets of timing data provides much stronger constraints on the presence of additional bodies in the system than looking at the primary transits alone (cf. Miralda-Escudé 2002; Schneider 2003; Agol et al. 2005; Holman & Murray 2005). The timing signature of an Earth-mass planet is an induced shift in the periastron of the orbits of the known planets. There are generally two dominant contributions to this shift: general relativity and other planets. The stellar oblateness can also contribute but only competes with the other effects if the oblateness is nearly two orders of magnitude larger than that of the Sun. Consequently, if the observed periastron shift exceeds the relativistic expectation, either the system has additional planets or the parent star has an unusually large oblateness. Even the less likely possibility of a large oblateness would give tantalizing hints to the origins of these close-in extrasolar planets. More likely would be the presence of an Earth-mass or even a Mars-mass planet in a nearby orbit to the known planet.

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