Article

K-Groups of Trivial Extensions and Gluings of Abelian Categories

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Abstract: This paper focuses on the $K_i$-groups of two types of extensions of abelian categories, which are the trivial extension and the gluing of abelian categories. We prove that, under some conditions, $K_i$-groups of a certain subcategory of the trivial extension category is isomorphic to $K_i$-groups of the similar subcategory of the original category. Moreover, under some conditions, we show that the $K_i$-groups of a left (right) gluing of two abelian categories are isomorphic to the direct sum of $K_i$-groups of two abelian categories. As their applications, we obtain some results of the $K_i$-groups of the trivial extension of a ring by a bimodule ($i \in \mathbb{N}$).

Keywords: abelian category; gluing of categories; $K_i$-group; trivial extension of category; trivial extension ring

1. Introduction

K-theory plays an important role in many fields of mathematics and physics, such as algebraic topology, algebraic geometry, operator algebra, and Type II string theory. It also can be used in classical mathematics [1–3]. In 1973, Quillen [4] introduced the notion of $K_i$-groups of an exact category. Since the category whose objects are all the finite generated projective modules over a ring is an exact category, $K_i$-groups of a ring is defined by $K_i$-groups of this exact category, which is one of the useful tools for the study of rings.

In studies of the category of perverse sheaves on a singular space, Beilinson, Bernstein and Deligne [5] introduced the gluing (recollement) of triangulated categories. Given a gluing (recollement) of triangulated categories and t-structures of the triangulated categories in the gluing, they found that the hearts of the t-structures also had the relation similar to the gluing of the triangulated categories under some conditions [5]. It is known that the heart of a t-structure is an abelian category. Since the gluing (recollement) of triangulated categories is widely used, it is interesting to define and study a gluing of abelian categories. Then, Franjou and Pirashvili defined and studied the gluing of abelian categories in detail [6]. A representative construction of a gluing of abelian categories was given by Macpherson and Vilonen [7], called the MV-construction for short in this paper.

Fossum and Griffith [8] defined and studied the right (left) trivial extension of an abelian category by an endofunctor. Applying the research conclusions, they get some conclusions of a trivial extension of a ring by a bimodule.

We realized that the study of categories can be applied in the study of some specific objects. For example, the modules over a ring form an abelian category, of which the research conclusions can be used in the research of the rings and modules. The main difficulty in dealing with certain objects is finding the proper categorical way to view them [9–11].

In this paper, we intend to study $K_i$-groups of two kinds of the extension categories, the trivial extension category and the gluing of categories. Compared with the original category, the structure of extension categories is more complex. If we can find out the relation between $K_i$-groups of the extension categories and $K_i$-groups of the original categories, it
will be helpful for applying the research of $K_i$-groups of the original categories to study $K_i$-groups of the extension categories. Inspired by the above statement, we will study the relation between $K_i$-groups of the two extension categories and $K_i$-groups of the original category. It is known that comma category is an example of the trivial extension category (the gluing of categories). Hence, applying the research conclusions, we can get some more complete conclusions about the $K_i$-groups of comma category.

This paper is organized as follows. Let $i$ be a natural number, $\mathcal{C}$ be an abelian category, $F$ be a right exact endofunctor of $\mathcal{C}$, and $\mathcal{C} \times F$ be the right trivial extension of $\mathcal{C}$ by $F$ (see Definition 1). We denote by $\mathcal{P}(\mathcal{C})$ the full subcategory of $\mathcal{C}$ whose objects are projective and finitely generated by the set $\mathcal{U}$ in $\mathcal{C}$. In Section 2, we recall the definitions and some properties of the trivial extension category and the gluing of categories. In Section 3, we obtain that $K_0(\mathcal{P}(\mathcal{C} \times F))$ is isomorphic to $K_0(\mathcal{P}(\mathcal{C}))$. In Theorem 2, we show that $K_i(\mathcal{P}(\mathcal{C} \times F))$ are isomorphic to $K_i(\mathcal{P}(\mathcal{C}))$ under some conditions ($i \geq 1$). Then we apply our results to deduce the conclusions about the $K_i$-groups of comma category. In Section 4, under some conditions, we obtain that the $K_i$-groups of a left (right) gluing of two abelian categories are isomorphic to the direct sum of the $K_i$-groups of the two abelian categories. Finally, in Section 5, we apply our results to deduce the following results about the $K_i$-groups of a trivial extension of a ring by a bimodule. For a ring $R$ and an $R$-$R$-bimodule $M$, we prove that $K_0(R) \cong K_0(R \times M)$. If $M$ is a finitely generated left $R \times M$-module, $R \times M$ is a noetherian ring, $M \otimes_R M = 0$ and $pd_{R \times M} M < \infty$, then $K_i(R) \cong K_i(R \times M)$.

2. Preliminaries

In this section, we recall the definitions and some results of a right trivial extension category and a gluing of categories.

Without special introduction, we use Quillen’s definitions of the $K_i$-groups [4], and the exact categories considered are all assumed to be small, thereby making the $K_i$-groups defined (in fact, if the exact category can be equivalent to a small subcategory, the $K_i$-groups are also defined [4] Page 103).

First, we introduce the definition of a right trivial extension category, which is called the trivial extension category for short in this paper.

**Definition 1** ([8]). Let $\mathcal{C}$ be an additive category and $F : \mathcal{C} \longrightarrow \mathcal{C}$ be an additive functor. The right trivial extension of $\mathcal{C}$ by $F$, denoted by $\mathcal{C} \ltimes F$, is defined as follows.

An object in $\mathcal{C} \times F$ is a morphism $\alpha : F(A) \longrightarrow A$ for an object $A$ in $\mathcal{C}$ such that $\alpha F(\alpha) = 0$.

For two objects $A, B$ in $\mathcal{C} \times F$, a morphism $\gamma : \alpha \longrightarrow \beta$ is a morphism $\gamma : A \longrightarrow B$ in $\mathcal{C}$ such that $\gamma \alpha = \beta F(\gamma)$.

**Remark 1** ([8]).

1. If $\mathcal{C}$ is an abelian category and $F$ is a right exact endofunctor of $\mathcal{C}$, then $\mathcal{C} \ltimes F$ is an abelian category.

2. Let $R$ be a ring and $M$ be an $R$-$R$-bimodule. Let $F$ be the tensor product functor $M \otimes_R -$. It is known that $(R \text{-Mod}) \ltimes F$ and $(R \times M \text{-Mod})$ are isomorphic.

The following lemma is crucial to study the $K_i$-groups of the trivial extension category.

**Lemma 1** ([8]). Let $\mathcal{C}$ be an abelian category and $F$ be a right exact endofunctor of $\mathcal{C}$. The tensor functor $T : \mathcal{C} \longrightarrow \mathcal{C} \ltimes F$ is defined by $T(A) = \begin{pmatrix} 0 & F(\alpha) \\ 1 & 0 \end{pmatrix} : F(A) \oplus F^2(A) \longrightarrow A \oplus F(A)$ and $T(\alpha) = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}$. The cokernel functor $C : \mathcal{C} \ltimes F \longrightarrow \mathcal{C}$ is defined by $C(F(A) \longrightarrow A) = \text{coker} \alpha$ and $C(\gamma) \alpha$ is the induced map for $\gamma : \alpha \longrightarrow \beta$. The underlying functor $U : \mathcal{C} \ltimes F \longrightarrow \mathcal{C}$ is defined by $U(F(A) \longrightarrow A) = A$ and $U(\gamma) = \gamma$. The zero functor $Z : \mathcal{C} \longrightarrow \mathcal{C} \ltimes F$ is defined by $Z(A) = F(A) \longrightarrow A$ and $Z(\gamma) = \gamma$. Then $CT = \text{id}_{\mathcal{C}}$, $UZ = \text{id}_{\mathcal{C}}$, and $(T, U), (C, Z)$ are adjoint pairs. $T$ and $C$ are right exact, $U$ and $Z$ are exact.
Next we recall the definition of a comma category as follows [8].

**Definition 2.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be abelian categories and \( F : \mathcal{A} \to \mathcal{B} \) be an additive functor. The comma category \((F, \mathcal{B})\) is the category whose objects are triples \((A, B, f)\) where \( f : F(A) \to B \) is a morphism in \( \mathcal{B} \) and whose morphisms are pairs

\[
(a, \beta) : (A, B, f) \to (A', B', f')
\]

where \( \alpha : A \to A' \), \( \beta : B \to B' \) and \( \beta f = f' \alpha \).

**Remark 2 ([8]).** A comma category \((F, \mathcal{B})\) is isomorphic to the right trivial extension category \((\mathcal{A} \times \mathcal{B}) \to \mathcal{B}\), where \( \tilde{F} : \mathcal{A} \times \mathcal{B} \to \mathcal{A} \times \mathcal{B} \) is the functor given by \( \tilde{F}(A, B) = (0, F(A)) \) and \( \tilde{F}(\alpha, \beta) = (0, F(\alpha)) \).

Finally, we recall the definition and some results of a gluing of categories.

**Definition 3 ([6]).** Let \( \mathcal{D}, \mathcal{D}' \) and \( \mathcal{D}'' \) be abelian categories. We call \( \mathcal{D} \) a gluing of \( \mathcal{D}' \) and \( \mathcal{D}'' \), if there exist six additive functors

\[
\begin{array}{ccc}
\mathcal{D}' & \xrightarrow{i^*} & \mathcal{D} & \xleftarrow{i_*} & \mathcal{D}'' \\
i^* & \xleftarrow{j_*} & j^* & \xrightarrow{j^*} & i^*
\end{array}
\]

which satisfy the following conditions:

(g1) \((i^*, i_*) = (i, j_*)\) and \((j_*, j^*) = (j^*, j_*)\) are adjoint triples, i.e., \( i^* \) is left adjoint to \( i_* \) which is also left adjoint to \( i \), etc.;

(g2) the functors \( i_* \), \( j_* \), and \( j^* \) are fully faithful;

(g3) \( \text{Im}(i_*) = \text{Ker}(j^*) \).

These notations will be kept throughout this paper.

**Remark 3 ([6]).** Let \( F : \mathcal{D}'' \to \mathcal{D}' \) be a right exact functor. Take \( \xi : F \to 0 \) to be the natural transformation to the trivial functor. It was denoted by \( \mathcal{D}' \times_F \mathcal{D}'' \) the gluing of \( \mathcal{D}' \) and \( \mathcal{D}'' \) obtained through MV-construction by \( F \) and \( \xi \). Obviously, \( \mathcal{D}' \times_F \mathcal{D}'' \) is a comma category. Dually, let \( G : \mathcal{D}'' \to \mathcal{D}' \) be a left exact functor. Take \( \xi' : 0 \to G \) to be a natural transformation. It was denoted by \( \mathcal{D}' \times_G \mathcal{D}'' \) the gluing of \( \mathcal{D}' \) and \( \mathcal{D}'' \) obtained through MV-construction by \( 0, G \) and \( \xi' \). Obviously, \( \mathcal{D}' \times_G \mathcal{D}'' \) is also a comma category.

### 3. The K-Groups of the Trivial Extension Category

In this section, we discuss the relation between the \( K_i \)-groups of \( \mathcal{D}(\mathcal{C} \times F) \) and the \( K_i \)-groups of \( \mathcal{D}(\mathcal{C}) \), for \( i \geq 0 \). Before proving the theorem, we show a lemma.

**Lemma 2.** Let \( A \) and \( B \) be objects in \( \mathcal{C} \). Then \( A \cong B \) if and only if \( T(A) \cong T(B) \).

**Proof.** Obviously, if \( A \cong B \) then \( T(A) \cong T(B) \). Conversely, if \( T(A) \cong T(B) \), then \( CT(A) \cong CT(B) \). By Lemma 1, \( A \cong B \).

Next, we recall two results of the trivial extension category [8].

**Lemma 3.** Let \( \mathcal{C} \) be an abelian category and \( F \) be a right exact endofunctor of \( \mathcal{C} \). For objects \( \alpha : F(A) \to A \), \( \beta : F(B) \to B \) and \( \gamma : F(C) \to C \) in \( \mathcal{C} \times F \), a sequence of objects

\[
0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0
\]

is exact in \( \mathcal{C} \times F \) if and only if \( 0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0 \) is exact in \( \mathcal{C} \).
Lemma 4. If an object $P$ is projective in $\mathcal{C}$ (resp. $\mathcal{C} \times F$), then the object $T(P)$ (resp. $C(P)$) is projective in $\mathcal{C} \times F$ (resp. $\mathcal{C}$).

Consequently, an object $P$ is projective in $\mathcal{C} \times F$ if and only if the following conclusions both are true: (1) the object $C(P)$ is projective in $\mathcal{C}$ and (2) $P \cong T(C(P))$.

Let $\mathcal{U}$ be a set of some objects in $\mathcal{C}$ and $\mathcal{T}$ be the set $\{T(U) \in \mathcal{C} \times F|U \in \mathcal{U}\}$. We denote by $\mathcal{P}(\mathcal{C})$ the full subcategory of $\mathcal{C}$ whose objects are projective and finitely generated by $\mathcal{U}$ in $\mathcal{C}$, and by $\mathcal{P}(\mathcal{C} \times F)$ the full subcategory of $\mathcal{C} \times F$ whose objects are projective and finitely generated by $\mathcal{T}$ in $\mathcal{C} \times F$. Obviously, $\mathcal{P}(\mathcal{C})$ and $\mathcal{P}(\mathcal{C} \times F)$ are exact categories.

Now, we study the $K_0$-group of the category $\mathcal{P}(\mathcal{C} \times F)$.

Theorem 1. Let $\mathcal{C}$ be an abelian category and $F: \mathcal{C} \rightarrow \mathcal{C}$ be a right exact functor. Then

$$K_0(\mathcal{P}(\mathcal{C})) \cong K_0(\mathcal{P}(\mathcal{C} \times F)).$$

Proof. Firstly, for $P \in \text{obj } \mathcal{P}(\mathcal{C})$, we denote by $< P >$ the isomorphism class $\{P'|P' \in \text{obj } \mathcal{P}(\mathcal{C}), P' \cong P\}$, where $P \in \text{obj } \mathcal{P}(\mathcal{C})$. Let $L$ be the free abelian group with the basis $\{< P > | P \in \text{obj } \mathcal{P}(\mathcal{C})\}$ and $L_0$ be the subgroup of $L$ spanned by the set $\{< X > + < Y > - < Z > | Z \cong X \oplus Y, \text{where } X, Y, Z \in \text{obj } \mathcal{P}(\mathcal{C})\}$. For objects $P, P', P$ in $\mathcal{P}(\mathcal{C})$, any short exact sequence $0 \rightarrow P' \rightarrow P \rightarrow P'' \rightarrow 0$ in $\mathcal{C}$ is split. So $K_0(\mathcal{P}(\mathcal{C})) \cong L/L_0$.

Denote by $H$ the free abelian group with the basis $\{< P > | P \in \text{obj } \mathcal{P}(\mathcal{C} \times F)\}$. By Lemmas 2 and 4, since $T$ and $C$ are right exact and $CT = \text{id }$, every isomorphism class of objects in $\mathcal{P}(\mathcal{C} \times F)$ can be denoted by $< T(P) >$, where $P \in \text{obj } \mathcal{P}(\mathcal{C})$. It yields that

$$\{< T(X) > + < T(Y) > - < T(Z) > | T(Z) \cong T(X) \oplus T(Y), \text{where } X, Y, Z \in \text{obj } \mathcal{P}(\mathcal{C})\} = \{< T(X) > + < T(Y) > - < T(Z) > | Z \cong X \oplus Y, \text{where } X, Y, Z \in \text{obj } \mathcal{P}(\mathcal{C})\}. \quad (3)$$

We denote by $H_0$ the subgroup of $H$ spanned by the set

$$\{< T(X) > + < T(Y) > - < T(Z) > | Z \cong X \oplus Y, \text{where } X, Y, Z \in \text{obj } \mathcal{P}(\mathcal{C})\}. \quad (4)$$

It is obvious that $K_0(\mathcal{P}(\mathcal{C} \times F)) \cong H/H_0$.

Secondly, by Lemma 2, we define a free abelian group isomorphism

$$\varphi: L \rightarrow H: \Sigma_i < P_i > \rightarrow \Sigma_i < T(P_i) >. \quad (5)$$

It induces an epimorphism $\overline{\varphi}: L \rightarrow \overline{H}: \Sigma_i < P_i > \rightarrow \Sigma_i \varphi(\varphi(\varphi)).$

Finally, we claim that $L_0 = \text{Ker}(\overline{\varphi})$. In fact, for each $\rho \in \text{Ker}(\overline{\varphi})$, $\varphi(\rho) = 0$. Hence $\varphi(\rho) \in H_0$. It follows that $\varphi(\rho) = \Sigma_i n_i(< X_i > + < Y_i > - < Z_i >)$ where $Z_i \cong X_i \oplus Y_i$ and $n_i < Z_i >$. Since $\varphi$ is an isomorphism, $\rho = \Sigma_i n_i(< X_i > + < Y_i > - < Z_i >)$. Thus $\rho \in L_0$. Hence $\text{Ker}(\overline{\varphi}) \subseteq L_0$.

Obviously, $L_0 \subseteq \text{Ker}(\overline{\varphi})$. So $L_0 = \text{Ker}(\overline{\varphi})$. Then $K_0(\mathcal{P}(\mathcal{C})) \cong K_0(\mathcal{P}(\mathcal{C} \times F))$. □

Next, we consider the $K_i$-groups of $P(\mathcal{C} \times F)$ by another method ($i \geq 1$).

We denote by $\mathcal{P}_\text{iso}(\mathcal{C} \times F)$ the full subcategory of $\mathcal{C} \times F$ whose objects have a finite resolution whose terms are in $\mathcal{P}(\mathcal{C} \times F)$. Obviously, $\mathcal{P}_\text{iso}(\mathcal{C} \times F)$ is an exact category.

Theorem 2. Let $\mathcal{C}$ be an abelian category and $F: \mathcal{C} \rightarrow \mathcal{C}$ be a right exact functor. If the following conditions are satisfied:

(i) $\mathcal{P}(\mathcal{C}) \cong \mathcal{C}_\text{iso}$

(ii) If $F(P) \in \text{obj } \mathcal{P}(\mathcal{C} \times F)$ for all $P \in \mathcal{C}$, then $K_i(T): K_i(\mathcal{P}(\mathcal{C})) \rightarrow K_i(\mathcal{P}(\mathcal{C} \times F))$ is an isomorphism.

Proof. Firstly, consider $P \in \mathcal{P}(\mathcal{C})$. There is an epimorphism $\bigoplus_{1 \leq i \leq n} U_i \rightarrow P$ where $U_i \in \mathcal{U}$. Since $T$ is a right functor, $\bigoplus_{1 \leq i \leq n} T(U_i) \rightarrow T(P)$ is an epimorphism. By Lemma 4,
Consider $a \in \mathcal{P}(\mathcal{C} \times F)$. There is an epimorphism $\bigoplus_{1 \leq i \leq n} T(U_i) \rightarrow a$ where $U_i \in \mathcal{U}$. According to that $C$ is a right functor and $CT = id$, we obtain that $\bigoplus_{1 \leq i \leq n} U_i \rightarrow C(a)$ is an epimorphism. By Lemma 4, $C(a) \in \mathcal{P}(\mathcal{C})$. So two functors $T : \mathcal{P}(\mathcal{C}) \rightarrow \mathcal{P}(\mathcal{C} \times F)$ and $C : \mathcal{P}(\mathcal{C} \times F) \rightarrow \mathcal{P}(\mathcal{C})$ are exact.

Then we have two group homomorphisms

$$T_a \triangleq K_t(T) : K_t(\mathcal{P}(\mathcal{C})) \rightarrow K_t(\mathcal{P}(\mathcal{C} \times F))$$

and

$$C_a \triangleq K_t(C) : K_t(\mathcal{P}(\mathcal{C} \times F)) \rightarrow K_t(\mathcal{P}(\mathcal{C}))$$

such that $C_a T_a = id$ on $K_t(\mathcal{P}(\mathcal{C}))$. Thus $K_t(\mathcal{P}(\mathcal{C}))$ is a direct summand of $K_t(\mathcal{P}(\mathcal{C} \times F))$.

Secondly, for any object $a : F(X) \rightarrow X$ in $\mathcal{P}(\mathcal{C} \times F)$, since $F^2|_{\mathcal{P}(\mathcal{C})} = 0$, $\ker (T(C(a))) = F^2(C(a)) = 0$ in $\mathcal{C}$. By Lemma 4, $a$ is isomorphic to $T(C(a))$. So $\ker a = 0$ in $\mathcal{C}$.

We observe that there is a commutative diagram

$$
\begin{array}{ccc}
F^2(X) & \rightarrow & F(X) \rightarrow FC(a) \\
0 & \downarrow & \alpha \downarrow & 0 \downarrow \\
0 & \rightarrow & F(X) \rightarrow X \rightarrow C(a) \rightarrow 0
\end{array}
$$

where $\pi$ is the coker of $a$ in $\mathcal{C}$. Then, by Lemma 3, there is an exact sequence in $\mathcal{C} \times F$

$$0 \rightarrow ZFU(a) \rightarrow a \rightarrow ZC(a) \rightarrow 0$$

Note that $ZFU(\mathcal{C}) \triangleq ZFU(T(C(a))) = ZF(C(a) \oplus F(C(a))) = ZF(C(a)) \oplus ZF^2(C(a)) = ZF(C(a))$. Since $ZF(C(a)) \in \obj \mathcal{P}_\infty(\mathcal{C} \times F)$, $ZFU(a) \in \mathcal{P}_\infty(\mathcal{C} \times F)$. Hence we obtain that $ZC(a) \in \obj \mathcal{P}_\infty(\mathcal{C} \times F)$.

For any $P \in \obj \mathcal{P}(\mathcal{C})$, we show that $Z(P) = Z(C(T(P)) \in \obj \mathcal{P}_\infty(\mathcal{C} \times F)$. So, restricted to $\mathcal{P}(\mathcal{C})$, the exact functor $Z$ induces an exact functor $J \triangleq Z|_{\mathcal{P}(\mathcal{C})} : \mathcal{P}(\mathcal{C}) \rightarrow \mathcal{P}_\infty(\mathcal{C} \times F)$. Then we get a group homomorphism

$$J_a \triangleq K_t(J) : K_t(\mathcal{P}(\mathcal{C})) \rightarrow K_t(\mathcal{P}_\infty(\mathcal{C} \times F)).$$

Finally, we define a group homomorphism

$$\zeta_\ast = (I_a)^{-1}J_a : K_t(\mathcal{P}(\mathcal{C})) \rightarrow K_t(\mathcal{P}(\mathcal{C} \times F)).$$

Let $(ZFU)_*$ be $K_t(ZFU)$ and $J,C_a$ be $K_t(JC)$. Using the exact sequence (9) and functor $J$, we obtain an exact sequence of exact functors

$$0 \rightarrow ZFU \rightarrow I \rightarrow J C \rightarrow 0$$

from $\mathcal{P}(\mathcal{C} \times F)$ to $\mathcal{P}_\infty(\mathcal{C} \times F)$. By [4], § 3, Corollary 1, the sequence yields that $I_\ast = (ZFU)_* + J_\ast C_a$. Denote $\xi_\ast = I_\ast^{-1}(ZFU)_*$. Then $I_\ast^{-1}I_\ast = I_\ast^{-1}(ZFU)_* + I_\ast^{-1}J_\ast C_a$. So $id = \xi_\ast + \xi_\ast C_a$. Hence $\xi_\ast = (id - \xi_\ast)T_\ast$ and $\xi_\ast C_a = T_\ast C_a - \xi_\ast T_\ast C_a$.

Finally, we show that the functor $ZFU_{\mathcal{C}}|_{\mathcal{P}(\mathcal{C} \times F)}$ is equivalent to $ZFU|_{\mathcal{P}(\mathcal{C} \times F)}$. In fact, for any object $a : F(A) \rightarrow A$ in $\mathcal{P}(\mathcal{C} \times F)$, since $h : a \rightarrow TC(a)$, there is a commutative diagram

$$
\begin{array}{ccc}
F^2(A) & \overset{F(a)}{\rightarrow} & F(A) \\
F^2(h) & \downarrow & F(h) \downarrow \\
F(FC(a) \oplus F^2(C(a)) & \rightarrow & F(C(a) \oplus FC(a))
\end{array}
$$

\[\text{Mathematics 2021, 9, 1864} \]
Note that \( F(FC(\alpha) \oplus F^2C(\alpha)) \cong F(FC(\alpha)) = 0 \). Hence \( F^2(A) = 0 \). For any objects \( \alpha : F(A) \to A \) and \( \beta : F(B) \to B \) in \( P(\mathcal{C} \times F) \), \( \alpha \to F(\beta) \in \text{mor} \mathcal{C} \times F \), applying the right exact functor \( F \), we obtain a commutative diagram

\[
\begin{array}{ccc}
F^2(A) & \xrightarrow{F(\alpha)} & F(A) & \xrightarrow{F(\pi)} & FC(\alpha) & \to & 0 \\
F^2(f) & \downarrow & F(f) & \downarrow & FC(f) & \downarrow & 0 \\
F^2(B) & \xrightarrow{F(\beta)} & F(B) & \xrightarrow{F(\pi')} & FC(\beta) & \to & 0,
\end{array}
\]

where \( \pi (\pi') \) is the coker of \( \alpha (\beta) \) in \( \mathcal{C} \). Since \( F^2(A) = 0 = F^2(B) \), there is a commutative diagram

\[
\begin{array}{ccc}
0 & \to & FU(\alpha) & \xrightarrow{F(\pi)} & FC(\alpha) & \to & 0 \\
& & FU(f) & \downarrow & FC(f) & \downarrow & 0 \\
0 & \to & FU(\beta) & \xrightarrow{F(\pi')} & FC(\beta) & \to & 0.
\end{array}
\]

Then the functor \( FU|_{\mathcal{P}(\mathcal{C} \times F)} \) is equivalent to \( FC|_{\mathcal{P}(\mathcal{C} \times F)} \). Hence \( ZFU|_{\mathcal{P}(\mathcal{C} \times F)} \) is equivalent to \( ZFC|_{\mathcal{P}(\mathcal{C} \times F)} \).

Therefore, \( \xi_i, C_i = \xi_0 \). So \( \xi_i, C_i = T_iC_i - \xi_i \) and \( T_iC_i = \xi_0 \).

We prove that \( K_i(\mathcal{P}(\mathcal{C} \times F)) \cong K_i(\mathcal{P}(\mathcal{C} \times F)) \).

Next, let \( \mathcal{A} \) and \( \mathcal{B} \) be abelian categories and \( F : \mathcal{A} \to \mathcal{B} \) be a right exact functor. Let \( \mathcal{U}_\mathcal{A} \) and \( \mathcal{U}_\mathcal{B} \) be the sets of some objects in \( \mathcal{A} \) and \( \mathcal{B} \) respectively, \( \mathcal{F} \) be the set \( \{T((A, B))|(A, B) \in \mathcal{U}_\mathcal{A} \times \mathcal{U}_\mathcal{B}\} \). We denote by \( \mathcal{P}(\mathcal{A} \times \mathcal{B}) \) the full subcategory of \( \mathcal{A} \times \mathcal{B} \) whose objects are projective and finitely generated by \( \mathcal{U}_\mathcal{A} \times \mathcal{U}_\mathcal{B} \) in \( \mathcal{A} \times \mathcal{B} \) and by \( \mathcal{P}(\mathcal{F} \times \mathcal{B}) \) the subcategory of \( \mathcal{F} \times \mathcal{B} \) whose objects are projective and finitely generated by \( \mathcal{F} \times \mathcal{B} \). Obviously, \( \mathcal{P}(\mathcal{A} \times \mathcal{B}) = \mathcal{P}(\mathcal{A}) \times \mathcal{P}(\mathcal{B}) \).

Now , by Remark 2 and Theorems 1 and 2, we get some results about the comma category.

**Corollary 1.** \( K_0(\mathcal{P}(\mathcal{F} \times \mathcal{B})) \cong K_0(\mathcal{P}(\mathcal{A})) \oplus K_0(\mathcal{P}(\mathcal{B})) \).

**Corollary 2.** If \( F(P) \) has a finite resolution whose terms are in \( \mathcal{P}(\mathcal{B}) \) for every \( P \in \text{obj} \mathcal{P}(\mathcal{A}) \), then \( K_i(\mathcal{P}(\mathcal{A})) \oplus K_i(\mathcal{P}(\mathcal{B})) \) is isomorphic to \( K_i(\mathcal{P}(\mathcal{F} \times \mathcal{B}))(i \geq 1) \).

**Proof.** It is known that \( (F, \mathcal{B}) \) is isomorphic to \( (\mathcal{A} \times \mathcal{B}) \times \mathcal{F} \), where \( \mathcal{F} : \mathcal{A} \times \mathcal{B} \to \mathcal{A} \times \mathcal{B} \) is a functor by \( \mathcal{F}((A, B)) = (F(A), 0) \) and \( \mathcal{F}((\alpha, \beta)) = (0, F(\alpha)) \). Obviously, \( \mathcal{F}^2 = 0 \).

We show that \( ZF((P, Q)) \) is an object in \( \mathcal{P}(\mathcal{A} \times \mathcal{B}) \times \mathcal{F} \) for every object \( (P, Q) \in \mathcal{P}(\mathcal{A} \times \mathcal{B}) \). In fact, \( ZF((P, Q)) = Z((0, F(P))) \).

Since \( F(P) \) has a finite resolution whose terms are in \( \mathcal{P}(\mathcal{B}) \), there is a long exact sequence in \( \mathcal{P}(\mathcal{B}) \)

\[
0 \to P_n \to \cdots \to P_1 \to F(P) \to 0,
\]

where \( P_i \in \mathcal{P}(\mathcal{B}) \) and \( i = 1 \ldots n \). We obtain a long exact sequence in \( \mathcal{A} \times \mathcal{B} \)

\[
\begin{array}{ccc}
(0, 0) & \to & \cdots & (0, 0) & \to & (0, 0) \\
\downarrow & & & \downarrow & & \downarrow \\
0 & \to & (0, P_n) & \to & \cdots & (0, P_1) & \to & (0, F(P)) & \to & 0.
\end{array}
\]

So a long sequence

\[
0 \to T((0, P_n)) \to \cdots \to T((0, P_2)) \to T((0, P_1)) \to Z((0, F(P))) \to 0
\]

is exact in \( \mathcal{A} \times \mathcal{B} \times \mathcal{F} \).

Hence \( ZF((P, Q)) \in \text{obj} \mathcal{P}(\mathcal{A} \times \mathcal{B}) \times \mathcal{F} \).

By Theorem 2, \( K_i(\mathcal{P}(\mathcal{A})) \oplus K_i(\mathcal{P}(\mathcal{B})) \cong K_i(\mathcal{P}(\mathcal{A} \times \mathcal{B})) \cong K_i(\mathcal{P}(\mathcal{A} \times \mathcal{B}) \times \mathcal{F}) \cong K_i(\mathcal{P}(\mathcal{F} \times \mathcal{B})) \).
4. The \( K \)-Groups of a Gluing of Abelian Categories

It is known that the comma category is also an example of a gluing of abelian categories. In this section, we will study the relation between the \( K_l \)-groups of a gluing of abelian categories and the \( K_r \)-groups of the abelian categories. Then we give the applications to the comma category.

For convenience, throughout this section we assume that categories are abelian and small. In order to describe the relation among the \( K_r \)-groups of abelian categories in the gluing more precisely, we introduce here two weaker forms of the gluing of abelian categories given as follows:

A left gluing of abelian categories consists of three categories \( \mathcal{D}, \mathcal{D}', \mathcal{D}'' \) and four functors \( i^*, i_*, j^*, j_* \) in (2), (denoted by \( \mathcal{D}' \rightrightarrows \mathcal{D} \rightrightarrows \mathcal{D}'') \), satisfying the conditions (lg1) \( (i^*, i_*) \) and \( (j^*, j_*) \) are adjoint pairs, (lg2) \( i_1 \) and \( j_1 \) are fully faithful functors and (lg3) \( \text{Im}(i_1) = \text{Ker}(j_2) \).

Now let us discuss the relation between the \( K_l \)-groups of a left (right) gluing of abelian categories and the \( K_r \)-groups of the abelian categories (\( i \geq 0 \)).

**Theorem 3.** If a left gluing of abelian categories \( \mathcal{D}' \rightrightarrows \mathcal{D} \rightrightarrows \mathcal{D}'' \) makes \( i^* \) and \( j_1 \), left exact, \( j^* \) and \( i_1 \) right exact, then \( K_l(\mathcal{D}') \oplus K_l(\mathcal{D}'') \) is a direct summand of \( K_l(\mathcal{D}) \).

Moreover, if the sequence of functors \( 0 \rightarrow j_1 \rightarrow id_{\mathcal{D}} \rightarrow i_1i^* \rightarrow 0 \) is exact, where \( j_1 \rightarrow id_{\mathcal{D}} \) is the back adjunction and \( id_{\mathcal{D}} \rightarrow i_1i^* \) is the front adjunction, then \( K_l(\mathcal{D}') \oplus K_l(\mathcal{D}'') \cong K_l(\mathcal{D}) \).

Dually, if a right gluing of abelian categories \( \mathcal{D}' \rightrightarrows \mathcal{D} \rightrightarrows \mathcal{D}'' \) makes \( j^* \) and \( i_1 \), left exact, \( i^* \) and \( j_1 \) right exact, then \( K_r(\mathcal{D}') \oplus K_r(\mathcal{D}'') \) is a direct summand of \( K_r(\mathcal{D}) \).

Moreover, if the sequence of functors \( 0 \rightarrow i_1i^* \rightarrow id_{\mathcal{D}} \rightarrow j_1j^* \rightarrow 0 \) is exact, where \( i_1i^* \rightarrow id_{\mathcal{D}} \) is the back adjunction and \( id_{\mathcal{D}} \rightarrow j_1j^* \) is the front adjunction, then \( K_r(\mathcal{D}') \oplus K_r(\mathcal{D}'') \cong K_r(\mathcal{D}) \).

**Proof.** For a left gluing of abelian categories \( \mathcal{D}' \rightrightarrows \mathcal{D} \rightrightarrows \mathcal{D}'' \), since \( (i^*, i_*) \) and \( (j^*, j_*) \) are adjoint pairs with the functor \( i_1 \) and \( j_1 \) left exact, \( j^* \) and \( i_1 \) right exact, then \( i^*, j^* \) and \( i_* \) are all exact. Thus we define two abelian group homomorphisms

\[
\varphi = \begin{pmatrix} K_l(i^*) & K_l(j^*) \end{pmatrix} : K_l(\mathcal{D}) \rightarrow K_l(\mathcal{D}') \oplus K_l(\mathcal{D}'') \tag{19}
\]

and

\[
\overline{\varphi} = (K_l(i_*), K_l(j_*)) : K_r(\mathcal{D}') \oplus K_r(\mathcal{D}'') \rightarrow K_r(\mathcal{D}). \tag{20}
\]

Recall that \( i_* \) and \( j_* \) are fully faithful functors. So \( j^* j_* \cong id_{\mathcal{D}''} \) and \( i^* i_* \cong id_{\mathcal{D}'} \). According to \( j^* j_* = 0 \), it is easy to prove that \( i^* j_1 = 0 \). Since \( j^* j_1 \cong id_{\mathcal{D}''} \), \( i^* i_* \cong id_{\mathcal{D}'} \), and \( j^* j_* = 0 = i^* j_1 \), we prove that

\[
\overline{\varphi} \varphi = \begin{pmatrix} K_l(i^*) & K_l(j^*) \end{pmatrix} (K_l(i_*), K_l(j_*)) = \begin{pmatrix} id_{K_l(\mathcal{D}')} & 0 \\ 0 & id_{K_l(\mathcal{D}'')} \end{pmatrix}.
\tag{21}
\]

So \( K_l(\mathcal{D}') \oplus K_l(\mathcal{D}'') \) is a direct summand of \( K_l(\mathcal{D}) \).

Moreover, since there is an exact sequence of functors \( 0 \rightarrow j_1 \rightarrow id_{\mathcal{D}} \rightarrow i_1i^* \rightarrow 0 \), by [4], Page 106, Corollary 1 we obtain that \( K_l(i_*) K_l(i^*) + K_l(j_*) K_l(j^*) = id \). Hence

\[
\overline{\varphi} \varphi = (K_l(i_*), K_l(j_*)) \begin{pmatrix} K_l(i^*) & K_l(j^*) \end{pmatrix} = id.
\tag{22}
\]

It follows that \( K_l(\mathcal{D}') \oplus K_l(\mathcal{D}'') \cong K_l(\mathcal{D}) \). The dual conclusion is similarly proved. \( \Box \)
Let’s give an example to show that the isomorphism doesn’t necessarily hold if there are no conditions.

**Example 1.** Let \( k \) be an algebraically closed field. Consider the path algebras ([12], Page 43). \( A = kQ_3, B = kQ_2 \) and \( C = kQ_1 \) where the quiver \( Q_3 \) is \( 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \), the quiver \( Q_2 \) is \( 1 \xrightarrow{\alpha} 2 \) and the quiver \( Q_1 \) is \( 1 \). Let \( A', B' \) and \( C' \) respectively be the Auslander algebras ([12], Page 419) of \( A, B \) and \( C \). Since \( A \) can be seen as a upper triangular matrix algebra \( \begin{pmatrix} B & P \\ 0 & C \end{pmatrix} \) where \( P \) is the projective \( B \)-module, \( A\mod \) is a gluing of \( C\mod \) and \( B\mod \). We note [13] that \( A\mod \) of \( A, B \) and \( C \). Since \( A \) can be seen as an algebraically closed field. Consider the path algebras ([12], Page 43). \[ \text{Example 1.} \]

\( \text{Let } (i^*, i_*) = (i_*, i^*) \) and \( (j^*, j_*) = (j_*, j^*) \) be adjoint triples in the gluing \( \mathcal{D}' \times_{\mathcal{F}} \mathcal{D}'' \) of abelian categories \( \mathcal{D}' \) and \( \mathcal{D}'' \). Then

1. \( i^* \) and \( i_* \) are exact functors;
2. for any object \( (X, V, a) \) in \( \mathcal{D}' \times_{\mathcal{F}} \mathcal{D}'' \), where \( a : F(X) \rightarrow V \) is a morphism of \( \mathcal{D}' \), there exists an exact sequence in \( \mathcal{D}' \times_{\mathcal{F}} \mathcal{D}'' \)

\[ 0 \rightarrow i^*((X, V, a)) \xrightarrow{(0, id_Y)} (X, V, a) \xrightarrow{(id_X, 0)} j_*j^*((X, V, a)) \rightarrow 0. \quad (23) \]

**Proof.** (1) is obvious. By the MV-construction it is easy to prove (2). In fact, for each \( (X, V, a) \in \text{obj}\ \mathcal{D}' \times_{\mathcal{F}} \mathcal{D}'' \), it is easy to check that the diagram

\[
\begin{array}{ccc}
(0, 0) & \rightarrow & (0, F(X)) \\
\downarrow & & \downarrow \\
(0, V) & \xrightarrow{(0, id_Y)} & (X, V) \\
& & \xrightarrow{(id_X, 0)} (X, 0)
\end{array}
\]

is commutative with the bottom row \( 0 \rightarrow (0, V) \xrightarrow{(0, id_Y)} (X, V) \xrightarrow{(id_X, 0)} (X, 0) \rightarrow 0 \) exact in \( \mathcal{D}' \times_{\mathcal{F}} \mathcal{D}'' \). Hence by Lemma 3 we get that

\[ 0 \rightarrow i^*((X, V, a)) \xrightarrow{(0, id_Y)} (X, V, a) \xrightarrow{(id_X, 0)} j_*j^*((X, V, a)) \rightarrow 0 \quad (24) \]

is an exact sequence in \( \mathcal{D}' \times_{\mathcal{F}} \mathcal{D}'' \). \[\square\]

Dually, there are similar results about the gluing \( \mathcal{D}' \times_{\mathcal{C}} \mathcal{D}'' \).

By Theorem 3 and Lemma 5, we obtain the following result.

**Corollary 3.** There exists \( K_i(\mathcal{D}') \oplus K_i(\mathcal{D}'') \cong K_i(\mathcal{D}' \times_{\mathcal{F}} \mathcal{D}'')(\text{resp. } K_i(\mathcal{D}') \oplus K_i(\mathcal{D}'') \cong K_i(\mathcal{D}' \times_{\mathcal{C}} \mathcal{D}'')) \).

Moreover, by Corollary 3 and Remark 3, we apply the above conclusion to get the following result about the comma category.

**Corollary 4.** Let \( \mathcal{A} \) and \( \mathcal{B} \) be abelian categories and \( F : \mathcal{A} \rightarrow \mathcal{B} \) be a right exact functor. Then \( K_i(\mathcal{A'}) \oplus K_i(\mathcal{B}) \cong K_i((F, \mathcal{B})). \)

If \( \mathcal{D} \) has enough injective objects and satisfies that \( i^*j_* = 0 \), by Proposition 8.9 in [6] there is an equivalence of abelian categories \( \mathcal{D} \simeq \mathcal{D}' \times_{i^*} \mathcal{D}'' \). So we can get the following conclusion.
Corollary 5. Let $\mathcal{D}$ be a gluing of $\mathcal{D}'$ and $\mathcal{D}''$. If $\mathcal{D}$ has enough injective objects and satisfies that $\mathcal{D} \simeq \mathcal{D}' \times_{i_{ij}} \mathcal{D}''$, then $K_i(\mathcal{D}') \oplus K_i(\mathcal{D}'') \cong K_i(\mathcal{D})$. Dually, if $\mathcal{D}$ has enough projective objects and satisfies that $\mathcal{D} \simeq \mathcal{D}' \times_{i_{ij}} \mathcal{D}''$, then $K_i(\mathcal{D}') \oplus K_i(\mathcal{D}'') \cong K_i(\mathcal{D})$.

Proof. Since there is an equivalence of abelian categories $\mathcal{D} \simeq \mathcal{D}' \times_{i_{ij}} \mathcal{D}''$, by Corollary 3, we obtain that $K_i(\mathcal{D}') \oplus K_i(\mathcal{D}'') \cong K_i(\mathcal{D}' \times_{i_{ij}} \mathcal{D}'')$. Hence $K_i(\mathcal{D}') \cong K_i(\mathcal{D}' \times_{i_{ij}} \mathcal{D}'') \cong K_i(\mathcal{D}'') \oplus K_i(\mathcal{D}'')$. The dual conclusion is similarly proved.

5. Examples and Applications

In this section, we give some applications about the $K_i$-groups and $G_i$-groups of the trivial extension ring ($i \geq 0$).

Throughout this section, we assume that a ring $R$ is associative and with an identity. All the modules will be left modules. We denote by $R$-Mod the category of left modules over a ring $R$ and by $R$-mod the full subcategory of $R$-Mod whose objects are finitely generated left $R$-modules. Let $M$ be an $R$-$R$-bimodule and $R \ltimes M$ be a trivial extension ring of a ring $R$ by a bimodule $M$. Obviously, there are ring homomorphisms $i : R \to R \ltimes M$ and $p : R \ltimes M \to R$ where $i(r) = (r, 0)$ and $p(r, m) = r$. Hence $M$ can be seen as a left $R \ltimes M$-module.

It is known that there is an isomorphism of categories $\varphi : R \ltimes M$-Mod $\cong (R$-Mod $) \ltimes (M \otimes R -)$ and $\varphi(R \ltimes M) = T(R)$ [8, Page 6]. Let $\mathcal{F} = \{R\}$. Then $\mathcal{F} = \{T(R)\}$, $\mathcal{P}(R$-Mod$)$ is the full subcategory of $R$-Mod whose objects are finitely generated projective $R$-modules and $\mathcal{P}(R \ltimes M$-Mod$)$ is the full subcategory of $R \ltimes M$-Mod whose objects are finitely generated projective $R \ltimes M$-modules. $\mathcal{P}_\infty(R \ltimes M$-Mod$)$ is the full subcategory of $R \ltimes M$-Mod whose objects have a finite resolution whose terms are in $\mathcal{P}(R \ltimes M$-Mod$)$. Note that $K_i(R) \cong K_i(\mathcal{P}(R$-Mod$))$ and $G_i(R) \cong K_i(R$-Mod$)$. Then by the Theorems 1 and 2, we get the following conclusions.

Corollary 6. There is an abelian group isomorphism $K_0(R) \cong K_0(R \ltimes M)$.

Corollary 7. Let $M$ be an $R$-$R$-bimodule and $i \geq 1$. Assume that $M \in R \ltimes M$-mod. If $R \ltimes M$ is a noetherian ring, $M \otimes_R M = 0$ and $pd_{R \ltimes M}M < \infty$, then $K_i(R) \cong K_i(R \ltimes M)$.

Proof. Let $\Lambda \triangleq R \ltimes M$ and $F \triangleq M \otimes_R -$.

For any $P \in \mathcal{P}(R$-Mod$)$, we show that $ZF(P) = \Lambda_M \otimes_R P \in \mathcal{P}_\infty(\Lambda$-Mod$)$. In fact, for each $C \in \Lambda$-Mod, since $pd_\Lambda M < \infty$ and $Ext^1_\Lambda(\Lambda_M \otimes_R P, \Lambda C) \cong Hom_R(P, Ext^1_\Lambda(\Lambda_M, \Lambda C))$, we prove that $pd_\Lambda(\Lambda_M \otimes P) < \infty$. Applying the finitely generated left $\Lambda$-module $M$ yields $\Lambda_M \otimes_R P$ is a finitely generated left $\Lambda$-module. Note that $\Lambda$ is a noetherian ring. Hence, $\Lambda_M \otimes_R P \in \mathcal{P}_\infty(\Lambda$-Mod$)$. Since $M \otimes_R M = 0$, by the Theorem 2 we check that $K_i(R) \cong K_i(R \ltimes M)$.

The $R \ltimes M$-structure of $M$ is determined by its $R$-structure, but it cannot specify conditions which are equivalent to the condition of Corollary 7. This question is discussed in [8,14,15]. Next, let’s give an example to show that the isomorphism doesn’t necessarily hold if there are no conditions.

Example 2. It is seen in [16] that $K_1(Z/3Z) \cong \{1, -1\} \cong Z/2Z$ while $K_1((Z/3Z) \ltimes (Z/3Z)) \cong \{(1, 0), (1, 1), (1, -1), (-1, 0), (-1, 1), (-1, -1)\} = <(-1, 1) \cong Z/6Z \cong (Z/2Z) \oplus (Z/3Z)$. So $K_1(Z/3Z)$ is a proper direct summand of $K_1((Z/3Z) \ltimes (Z/3Z))$.

Finally, we give an application about the $G_i$-groups of the triangular matrix ring.

Corollary 8. Let $\Lambda = \begin{pmatrix} R & 0 \\ SM_R & S \end{pmatrix}$ be a triangular matrix ring. If $R$ and $S$ are noetherian rings and bimodule $SM_R$ is a finitely generated left $S$-module, then $G_i(\Lambda) \cong G_i(R) \oplus G_i(S)$. 

Proof. Let $F$ be the functor $sM_R \otimes -$ . Applying the finitely generated left $S$-module $sM_R$ yields that $F$ preserves the finitely generated modules. Then $F : R\text{-mod} \longrightarrow S\text{-mod}$ is defined.

Furthermore, since $R$ and $S$ are noetherian rings and bimodule $sM_R$ is a finitely generated left $S$-module, $\Lambda$ is also a noetherian ring. Hence $R\text{-mod}$, $S\text{-mod}$ and $\Lambda\text{-mod}$ are abelian and $\Lambda\text{-mod} \cong (F, (S\text{-mod}))$. So, by Corollary 4 we prove that $K_i(R\text{-mod}) \oplus K_i(S\text{-mod}) \cong K_i((F, (S\text{-mod}))) \cong K_i(\Lambda\text{-mod})$. So $G_i(\Lambda) \cong G_i(R) \oplus G_i(S)$. □

6. Conclusions

The study of higher $K$-groups is a hot issue, but it is difficult to perform. In this paper, we mainly use the category theory to study the relation between $K_i$-groups of the two extension categories and $K_i$-groups of the original category. We show the invariance under some conditions. Since a comma category is an example of the trivial extension category (the gluing of categories), we also show the invariance about $K_i$-groups of the comma category. Finally, we apply to deduce the results about the $K_i$-groups of a trivial extension ring. Are there any other applications? We know that there are many other extension categories, so how about their $K_i$-groups? In future work, we will aim to focus on them.

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