Chapter 5: Exercises

1. As we will discuss later in the course, a superconducting quantum dot separated from a bulk superconductor by a Josephson tunnel barrier can be modeled as a $O(2)$ quantum rotor coupled to an external field

$$H = \frac{g}{2} \hat{L}^2 - h\hat{n}_x$$

where $g$ is a measure of the Coulomb gap of the dot, and $h$ is the Josephson coupling. Determine the first two terms in the series for the ground state energy in limit of small and large $g$.

2. Derive the result (5.6) for the dispersion of the triplet quasiparticle excitation in the large $g$ limit of a $O(3)$ quantum rotor model.

3. Provide the missing steps leading to the last equation in (5.15). For this you simply have to find the normal modes of the “spin-wave” Hamiltonian discussed in class, and then quantize them. You may find the discussion in Section 3-1-1 of Itzykson and Zuber helpful.

4. Compute the value of

$$F_\alpha(\theta) = \exp \left( i\theta n_\beta \hat{S}_\beta \right) \hat{S}_\alpha \exp \left( -i\theta n_\gamma \hat{S}_\gamma \right)$$

where $\hat{S}_\alpha$ are quantum spin operators of angular momentum $S$ [$\hat{S}_\alpha \hat{S}_\alpha = S(S + 1)$], $n_\alpha$ is an arbitrary vector of unit length, and $\theta$ is an angle of rotation. First show that all the $d^n F_\alpha/d\theta^n$ can be written solely in terms of the commutators of $\hat{S}_\alpha$ at $\theta = 0$. Hence argue that $F_\alpha(\theta)$ can be written as

$$F_\alpha(\theta) = f_{\alpha\beta}(\theta) \hat{S}_\beta$$

where the functions $f_{\alpha\beta}(\theta)$ are independent of the value of $S$. Finally, determine the $f_{\alpha\beta}(\theta)$ by explicitly evaluating everything using the Pauli matrix representation valid for $S = 1/2$. 

