BIHARMONIC SYSTEMS INVOLVING MULTIPLE RELlich–TYPE POTENTIALS AND CRITICAL RELlich–SOBoLEV NONLINEARITIES

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Abstract. In this paper, the minimizers of a Rellich–Sobolev constant are firstly investigated. Secondly, a system of biharmonic equations is investigated, which involves multiple Rellich–type terms and strongly coupled critical Rellich–Sobolev terms. The existence of nontrivial solutions to the system is established by variational arguments.

1. Introduction. In this paper, we study the system of biharmonic equations:

\[
\begin{align*}
\Delta^2 u - \sum_{j=1}^{k} \frac{\lambda_j u}{|x-a_j|^4} &= \sum_{j=1}^{k} \nu_j \left( \frac{|v|^{q_j} + \sigma_j |v|^{q_j}}{|x-a_j|^{t_j}} \right)^{\frac{p_j}{q_j} - 1} |u|^{q_j - 2} u, \quad x \in \Omega, \\
\Delta^2 v - \sum_{j=1}^{k} \frac{\mu_j v}{|x-a_j|^4} &= \sum_{j=1}^{k} \sigma_j \left( \frac{|v|^{q_j} + \sigma_j |v|^{q_j}}{|x-a_j|^{t_j}} \right)^{\frac{p_j}{q_j} - 1} |v|^{q_j - 2} v, \quad x \in \Omega, \\
u &= v = \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, \quad x \in \partial \Omega,
\end{align*}
\]

where \( \Omega \subset \mathbb{R}^N (N \geq 5) \) is a bounded domain with smooth boundary, \( \frac{\partial}{\partial n} \) is the outward normal derivative and the parameters satisfy the assumption:

\( (\mathcal{H}_1) \quad N \geq 5, \quad k \geq 2, \quad 0 \leq \lambda_j, \mu_j < \bar{\mu} := \left( \frac{N(N-4)}{4} \right)^2, \quad 0 < t_j < 4, \quad \nu_j, \sigma_j > 0, \)

\( 1 < q_j < p_j = 2^*(t_j) := \frac{2(N-t_j)}{N-4}, \quad a_j \in \Omega, \quad a_i \neq a_j, \quad i \neq j, \quad 1 \leq i, j \leq k. \)

For all \( t \in [0,4] \), note that \( 2^*(t) := \frac{2(N-t)}{N-4} \) is the critical Rellich–Sobolev exponent, \( 2^* := 2^*(0) = \frac{2N}{N-4} \) is the critical Sobolev exponent and \( \bar{\mu} \) is the best constant in the Rellich inequality (\([18]\)):

\[
\int_{\mathbb{R}^N} \frac{u^2}{|x-a|^4} dx \leq \frac{1}{\bar{\mu}} \int_{\mathbb{R}^N} |\Delta u|^2 dx, \quad \forall u \in D^{2,2}(\mathbb{R}^N), \quad a \in \mathbb{R}^N, \quad (2)
\]

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where $D := D^{2,2}(\mathbb{R}^N)$ is completion of $C_0^\infty(\mathbb{R}^N)$ with respect to $(\int_{\mathbb{R}^N} |\Delta u|^2 \, dx)^{1/2}$. Furthermore, there exists a constant $C(t) > 0$ such that the following Rellich–Sobolev inequality holds ([4, 9, 17]):

$$\left( \int_{\mathbb{R}^N} \frac{|u|^2(t)}{|x-a|^t} \, dx \right)^{2/t} \leq C(t) \int_{\mathbb{R}^N} |\Delta u|^2 \, dx, \quad \forall u \in D^{2,2}(\mathbb{R}^N), \ a \in \mathbb{R}^N. \quad (3)$$

In this paper, we use $H := H_0^2(\Omega)$ to denote the completion of $C_0^\infty(\Omega)$ with respect to the norm $(\int_\Omega |\Delta u|^2 \, dx)^{1/2}$. The functional of (1) is defined on $H \times H$ as follows

$$J(u, v) = \frac{1}{2} \int_\Omega \left( |\Delta u|^2 + |\Delta v|^2 - \sum_{j=1}^k \frac{\lambda_j u^2 + \mu_j v^2}{|x-a_j|^t} \right) \, dx$$

$$- \sum_{j=1}^k \left( \nu_j |u|^{q_j} + \sigma_j |v|^{q_j} \right) \frac{p_j}{p_j - 1} \, dx.$$ 

Then $J \in C^1(H \times H, \mathbb{R})$. A pair of functions $(u, v) \in H \times H$ is said to be a solution of (1) if $(u, v) \neq (0, 0)$ satisfies

$$\langle J'(u, v), (\phi, \psi) \rangle = 0, \quad \forall (\phi, \psi) \in H \times H,$$

where $J'(u, v)$ is the Fréchet derivative of $J$ at the point $(u, v)$ and

$$\langle J'(u, v), (\phi, \psi) \rangle = \int_\Omega \left( \Delta u \Delta \phi + \Delta v \Delta \psi - \sum_{j=1}^k \frac{\lambda_j u \phi + \mu_j v \psi}{|x-a_j|^t} \right) \, dx$$

$$- \int_\Omega \sum_{j=1}^k \frac{\nu_j (|u|^{q_j} + |v|^{q_j}) \frac{p_j}{p_j - 1}}{|x-a_j|^t} \, dx$$

$$- \int_\Omega \sum_{j=1}^k \frac{\sigma_j (|u|^{q_j} + |v|^{q_j}) \frac{p_j}{p_j - 1}}{|x-a_j|^t} \, dx.$$

$J$ is said to satisfy the $(PS)_c$ condition if any sequence $\{ (u_n, v_n) \} \subset H \times H$ satisfying $J'(u_n, v_n) \to c$, $J(u_n, v_n) \to 0$ in the dual space $(H \times H)^{-1}$, has a subsequence converging strongly in $H \times H$.

For all $\lambda, \mu < \bar{\mu}$, $\nu, \sigma > 0$, $0 \leq t < 4$, $1 < q \leq 2^*(t)$, $a \in \mathbb{R}^N$, by (2) and (3) the following best constants are well defined:

$$S(\mu, t) := \inf_{u \in D \setminus \{0\}} \frac{\int_{\mathbb{R}^N} \left( |\Delta u|^2 - \mu \frac{u^2}{|x-a|^t} \right) \, dx}{\left( \int_{\mathbb{R}^N} \frac{|u|^2(t)}{|x-a|^t} \, dx \right)^{2/t}}, \quad (4)$$

$$A(\lambda, \mu, t, \nu, \sigma, q) := \inf_{(u, v) \in D \times D \setminus \{(0, 0)\}} \frac{\int_{\mathbb{R}^N} \left( |\Delta u|^2 + |\Delta v|^2 - \lambda u^2 - \mu v^2 \right) \, dx}{\left( \int_{\mathbb{R}^N} \frac{|u|^{q_j} + |v|^{q_j}}{|x-a|^t} \, dx \right)^{2/t}}. \quad (5)$$

Note that $S(\mu, t)$ and $A(\lambda, \mu, t, \nu, \sigma, q)$ are independent of the singular point $a$. Furthermore, the minimizers of $S(\mu, t)$ was studied in [12] (see Lemma 2.1 of this
The minimizers of $A(\lambda, \mu, t, \nu, \sigma, q)$ are related to the problem:

$$
\begin{cases}
\Delta^2 u - \frac{\lambda u}{|x-a|^4} = \nu \left(\frac{v|u|^q + \sigma|v|^q}{|x-a|^t}\right)^{2^*(t)-1}|u|^{q-2}u, & x \in \mathbb{R}^N, \\
\Delta^2 v - \frac{\mu v}{|x-a|^4} = \sigma \left(\frac{v|u|^q + \sigma|v|^q}{|x-a|^t}\right)^{2^*(t)-1}|v|^{q-2}v, & x \in \mathbb{R}^N,
\end{cases}
(6)
$$

By a ground state solution $(u_0, v_0)$ to (6) we mean that $(u_0, v_0) \neq (0, 0)$ is a solution to (6) and has the least energy among all solutions. That is, the solution $(u_0, v_0)$ is also a minimizer of $A(\lambda, \mu, t, \nu, \sigma, q)$. For simplicity, we denote

$$
A_j := A(\lambda_j, \mu_j, t_j, \nu_j, \sigma_j, q_j), \quad 1 \leq j \leq k.
$$

Second order singular elliptic equations related to (1) have been studied and the existence of solutions to the problems have been established (e.g. [6, 10, 11, 13, 14, 19]). Biharmonic problems involving the Rellich inequality have been also studied (e.g. [2, 4, 5, 8, 9, 12]). However, the system (1) has not been investigated and is attractive for its multiple singularity and multiple Rellich–Sobolev critical nonlinearities. In this paper, we will study the minimizers of $A(\lambda, \mu, t, \nu, \sigma, q)$ and investigate the mountain–pass type solution to (1), which has the least energy among all solutions of (1).

Define the norm of $H \times H$:

$$
\|(u, v)\|^2 := \int_\Omega \left(\sum_{|\alpha| \leq 2} (|\Delta^\alpha u|^2 + |\Delta^\alpha v|^2)\right) \, dx, \quad \forall (u, v) \in H \times H.
$$

For all $N \geq 5$, $0 \leq \mu < \bar{\mu}$, $0 \leq t < 4$, the following constants are well defined:

$$
\mu^* := \frac{1}{16}(N^2 - 16)(N^2 - 8N), \\
\delta := \frac{N - 4}{2}, \quad \delta(t) := \frac{2(N - 4)}{4 - t}, \\
a(\mu) = \delta \varphi(\mu), \quad b(\mu) = \delta(2 - \varphi(\mu)),
$$

where $\varphi : [0, \bar{\mu}] \rightarrow [0, 1]$ is defined as

$$
\varphi(\mu) := 1 - \frac{1}{N - 4} \sqrt{N^2 - 4N + 8 - 4\sqrt{(N - 2)^2 + \mu}}, \quad \mu \in [0, \bar{\mu}].
$$

The following conditions are also needed:

$$(H_2) \quad \sum_{1 \leq j \leq k} \lambda_j < \bar{\mu}, \quad \sum_{1 \leq j \leq k} \mu_j < \bar{\mu}.
$$

$$(H_3) \quad \text{There exists } l, \ 1 \leq l \leq k, \text{ such that } 1 < q_l < 2, \ 0 \leq \lambda_l = \mu_l < \lambda^* \text{ and }
$$

$$
\frac{4 - t_l}{2(N - t_l)} \min \left\{ \frac{4 - t_j}{2(N - t_j)} A_j^{\frac{N-t_j}{4-t_j}}, \ 1 \leq j \leq k \right\}, \quad A_j := \frac{N-t_l}{N-t_j} A_j^{\frac{N-t_j}{4-t_j}}, \ 1 \leq j \leq k,
$$

$$
\lambda^* := \frac{1}{16} \left( N^2 - 4N + 8 - \left( \min \{t_j, 1 \leq j \leq k, j \neq l \} \right)^2 \right)^2 - (N - 2)^2.
$$
(H₄) There exists l, 1 ≤ l ≤ k, such that 2 ≤ q_l < 2⁺(t_l), 0 ≤ λ_l < λ⁺, and
\[ \nu_l^{-\frac{2}{q_l}} S(\lambda_l, t_l) \leq \sigma_l^{-\frac{2}{q_l}} S(\mu_l, t_l), \]
\[ \frac{4 - t_l}{2(N - t_l)} (\nu_l^{-\frac{2}{q_l}} S(\lambda_l, t_l))^{\frac{N - t_l}{4 - t_l}} = \min \left\{ \frac{4 - t_j}{2(N - t_j)} A_j^{\frac{N - t_j}{4 - t_j}}, 1 \leq j \leq k \right\}. \]

(\text{H}_5) There exists l, 1 ≤ l ≤ k, such that 2 ≤ q_l < 2⁺(t_l), 0 ≤ μ_l < λ⁺, and
\[ \sigma_l^{-\frac{2}{q_l}} S(\mu_l, t_l) \leq \nu_l^{-\frac{2}{q_l}} S(\lambda_l, t_l), \]
\[ \frac{4 - t_l}{2(N - t_l)} (\sigma_l^{-\frac{2}{q_l}} S(\mu_l, t_l))^{\frac{N - t_l}{4 - t_l}} = \min \left\{ \frac{4 - t_j}{2(N - t_j)} A_j^{\frac{N - t_j}{4 - t_j}}, 1 \leq j \leq k \right\}. \]

Let \( y^\circ_l(x - a) = y^\circ_l(|x - a|) \) be the family of minimizers to \( S(\mu, t) \) for all \( \mu \in [0, \bar{\mu}) \) (e.g. [12], see also Lemma 2.1 of this paper). For all \( \nu, \sigma > 0 \), define
\[ f(\tau) := \frac{1 + \tau^2}{(\nu + \sigma \tau^q)^{\frac{1}{q}}}, \quad \tau \geq 0, \quad \bar{\tau}_{\min} := \left( \frac{\sigma}{\nu} \right)^{\frac{q - 1}{q}}. \]

The main results of this paper are contained in the following theorems. We investigate the minimizers to \( \mathcal{A}(\lambda, \mu, t, \nu, \sigma, q) \) and prove the existence of mountain–pass type solution to (1). The result is new to the best of our knowledge.

**Theorem 1.1.** Assume that 0 ≤ \( \lambda, \mu < \bar{\mu} \), 0 < \( \nu, \sigma < \infty \), 0 ≤ t < 4, \( a \in \mathbb{R}^N \).

(i) Suppose furthermore that 1 < q < 2. Then
\[ \mathcal{A}(\lambda, \mu, t, \nu, \sigma, q) < \min \{ \nu^{-\frac{2}{q}} S(\lambda, t), \sigma^{-\frac{2}{q}} S(\mu, t) \}. \]
\( \mathcal{A}(\lambda, \mu, t, \nu, \sigma, q) \) has the positive minimizers \( \{ (C u_0, C v_0), C > 0 \} \subset D \times D, \) where \( (u_0, v_0) \) is radially symmetric and decreasing with respect to \( |x - a| \) and is a ground state solution to (6). In particular, \( \mathcal{A}(\lambda, \lambda, t, \nu, \sigma, q) = f(\bar{\tau}_{\min}) S(\lambda, t) \) and has the minimizers \( \{ (C y^\circ_l(x - a), C \bar{\tau}_{\min} y^\circ_l(x - a)), \varepsilon, C > 0 \} \).

(ii) Suppose furthermore that 2 ≤ q ≤ 2⁺(t). Then
\[ \mathcal{A}(\lambda, \mu, t, \nu, \sigma, q) = \min \{ \nu^{-\frac{2}{q}} S(\lambda, t), \sigma^{-\frac{2}{q}} S(\mu, t) \}, \]
and \( \mathcal{A}(\lambda, \mu, t, \nu, \sigma, q) \) has the semitrivial minimizers either \( \{ (C y^\circ_l(x - a), 0), C, \varepsilon > 0 \} \) or \( \{ (0, C y^\circ_l(x - a)), C, \varepsilon > 0 \} \).

**Theorem 1.2.** Assume that (\text{H}₁), (\text{H}₂) and one of (\text{H}₃)–(\text{H}₅) hold. Then the problem (1) has at least one solution \( (u_0, v_0) \in H \times H \setminus \{(0, 0)\} \). Furthermore, \( (u_0, v_0) \in (H \setminus \{0\})^2 \) if 1 < q_l < 2.

**Remark 1.3.** A direct calculation shows that \( \mu^* < \lambda^* < \bar{\mu} \). It can be verified that the conditions (\text{H}₃)–(\text{H}₅) can be satisfied. For example, we can take \( t_j = t \in (0, 4), q_j = q \in (1, 2⁺(t)), \nu_j = \nu > 0, \sigma_j = \sigma > 0, 1 \leq j \leq k \). Then (\text{H}₃) can be satisfied by choosing \( \lambda_j = \mu_j \in (0, \lambda^*) \) and \( \lambda_j, \mu_j \in [0, \lambda^*], 1 \leq j \leq k, j \neq l \). By Lemma 3.1 of this paper, (\text{H}₄) can be satisfied by choosing \( \lambda_l \in (0, \lambda^*), \nu_l \geq \sigma_l, \mu_l, \lambda_j, \mu_j \in [0, \lambda^*], 1 \leq j \leq k, j \neq l \). Similarly, (\text{H}₅) can be satisfied by choosing \( \mu_l \) and \( \sigma_l \) reasonably large.
This paper is organized as follows: Some preliminary results are established in Section 2, Theorem 1.1 is proved in Section 3 and Theorem 1.2 is verified in Section 4. In the following argument, $L^q(\Omega, |x-a_j|^{\alpha})$, $q > 1$, $\alpha \in \mathbb{R}$, denotes the usual weighted $L^q(\Omega)$ space with the weight $|x-a_j|^\alpha$. For all $\varepsilon > 0$, $t > 0$, $O(\varepsilon^t)$ denotes the quantity satisfying $|O(\varepsilon^t)|/\varepsilon^t \leq C$, $O_1(\varepsilon^t)$ means that there exist the constants $C_1, C_2 > 0$ such that $C_1 \varepsilon^t \leq O_1(\varepsilon^t) \leq C_2 \varepsilon^t$, $o(\varepsilon^t)$ means $|o(\varepsilon^t)|/\varepsilon^t \to 0$ as $\varepsilon \to \varepsilon_0$. We always denote positive constants as $C$ and omit $dx$ in integrals for convenience.

2. Preliminary results. First we need to establish some preliminary results.

Lemma 2.1 ([12]). Suppose that $N \geq 5$, $\mu \in \mathbb{R}^N$, $0 < t < 4$, $0 \leq \mu < \bar{\mu}$. Then, the best constant $S(\mu, t)$ defined in (4) has the unique minimizers, up to multiplicative constants,

$$
y_\mu^e(x-a) := \varepsilon^{\frac{4-N}{4-N-t}} U_\mu(\varepsilon^{-1}(x-a)), \varepsilon > 0,
$$

which are positive solutions to

$$
\Delta^2 u - \mu \frac{u}{|x-a|^4} = \frac{|u|^{2^*(t)-2}u}{|x-a|^4}, \quad x \in \mathbb{R}^N \setminus \{a\},
$$

and so, they satisfy

$$
\int_{\mathbb{R}^N} \left( |\Delta y_\mu^e(x-a)|^2 - \mu \frac{|y_\mu^e(x-a)|^2}{|x-a|^4} \right) = \int_{\mathbb{R}^N} \frac{|y_\mu^e(x-a)|^{2^*(t)}}{|x-a|^4} = S(\mu, t) \frac{N-4}{N-4-t}.
$$

Moreover, $U_\mu(x) = U_\mu(|x|) > 0$ is radially symmetric and decreasing. By setting $\rho = |x|$ there holds that

$$
U_\mu(\rho) = O_1(\rho^{-a(\mu)}), \quad \text{as } \rho \to 0,
$$

$$
U_\mu(\rho) = O_1(\rho^{-b(\mu)}), \quad U_\mu'(\rho) = O_1(\rho^{-b(\mu)-1}), \quad \text{as } \rho \to +\infty.
$$

Lemma 2.2. Assume that (H1) and (H2) hold. Then the functional $J$ satisfies the $(PS)_c$ condition for all $c < c^*$, where

$$
c^* := \frac{4-t_j}{2(N-t_j)} \mathcal{A}_j^{\frac{N-t_j}{N}} = \min \left\{ \frac{4-t_j}{2(N-t_j)} \mathcal{A}_j^{\frac{N-t_j}{N}}, 1 \leq j \leq k \right\}.
$$

Proof. As $n \to \infty$, let $\{(u_n, v_n)\} \subset H \times H$ satisfy

$$
J(u_n, v_n) \to c < c^*, \quad J'(u_n, v_n) \to 0 \quad \text{in } (H \times H)^{-1}.
$$

Then a standard argument shows that $\{(u_n, v_n)\}$ is bounded in $H \times H$ (e.g. [7]). Up to a subsequence we have that

$$
(u_n, v_n) \to (u, v) \text{ a.e. in } \Omega.
$$

$$
(u_n, v_n) \rightharpoonup (u, v) \text{ weakly in } H \times H.
$$

$$
(u_n, v_n) \rightharpoonup (u, v) \text{ weakly in } (L^2(\Omega, |x-a_j|^{-4}))^2, \quad 1 \leq j \leq k.
$$

$$
(u_n, v_n) \rightharpoonup (u, v) \text{ weakly in } (L^{p_j}(\Omega, |x-a_j|^{-t_j}))^2, \quad 1 \leq j \leq k.
$$

$$
(u_n, v_n) \rightharpoonup (u, v) \text{ strongly in } (L^q(\Omega, |x-a_j|^{-t_j}))^2, \quad \forall q \in [1, p_j).
$$

Since $0 < t_j < 4$, $2 < p_j < 2^*$, $1 \leq j \leq k$, by the concentration compactness principle ([10, 16, 17]), there exists a subsequence (still denoted by $\{(u_n, v_n)\}$).
and nonnegative real numbers $\tilde{\mu}_{a_j}$, $\tilde{\gamma}_{a_j}$, $\tilde{\nu}_{a_j}$, $1 \leq j \leq k$, such that the following convergences hold in the sense of measures:

$$
|\Delta u_n|^2 + |\Delta v_n|^2 \to d\tilde{\mu} \geq |\Delta u|^2 + |\Delta v|^2 + \sum_{j=1}^k \tilde{\mu}_{a_j} \delta_{a_j},
$$

(7)

$$
\frac{\lambda_j u_n^2 + \mu_j v_n^2}{|x-a_j|^4} \to d\tilde{\gamma} = \frac{\lambda_j u^2 + \mu_j v^2}{|x-a_j|^4} + \tilde{\gamma}_{a_j} \delta_{a_j},
$$

(8)

$$
\frac{(\nu_j|u_n|^{q_j} + \sigma_j|v_n|^{q_j})}{|x-a_j|^4} \to d\tilde{\nu} = \frac{(\nu_j|u|^{q_j} + \sigma_j|v|^{q_j})}{|x-a_j|^4} + \tilde{\nu}_{a_j} \delta_{a_j},
$$

(9)

where $\delta_x$ is the Dirac mass at $x$. From (5) it follows that

$$
\tilde{\mu}_{a_j} - \tilde{\gamma}_{a_j} \geq (\tilde{\nu}_{a_j})^{\frac{2}{p_j}} A_j, \quad 1 \leq j \leq k.
$$

(10)

Now we consider the possibility of concentration at the points $a_j$ ($1 \leq j \leq k$). For all $\epsilon > 0$ small enough, let $\varphi_j^\epsilon(x) \in C_0^\infty(\Omega)$ be a radial cut-off function centered at $a_j$ such that $\varphi_j^\epsilon(x) \equiv 1$ in the ball $B_\epsilon(a_j)$, $\varphi_j^\epsilon(x) = 0$ in $\mathbb{R}^N \setminus B_{2\epsilon}(a_j)$ and $0 \leq \varphi_j^\epsilon(x) \leq 1$. Then by (7)–(9) we deduce that

$\lim_{\epsilon \to 0} \lim_{n \to \infty} \int_\Omega (|\Delta u_n|^2 + |\Delta v_n|^2) \varphi_j^\epsilon = \lim_{\epsilon \to 0} \int_\Omega \varphi_j^\epsilon d\tilde{\mu} \geq \tilde{\mu}_{a_j},$

$\lim_{\epsilon \to 0} \lim_{n \to \infty} \int_\Omega \frac{\lambda_j u_n^2 + \mu_j v_n^2}{|x-a_j|^4} \varphi_j^\epsilon = \lim_{\epsilon \to 0} \int_\Omega \varphi_j^\epsilon d\tilde{\gamma} = \tilde{\gamma}_{a_j},$

$\lim_{\epsilon \to 0} \lim_{n \to \infty} \int_\Omega \frac{(\nu_j|u_n|^{q_j} + \sigma_j|v_n|^{q_j})}{|x-a_j|^4} \varphi_j^\epsilon = \lim_{\epsilon \to 0} \int_\Omega \varphi_j^\epsilon d\tilde{\nu} = \tilde{\nu}_{a_j},$

$\lim_{\epsilon \to 0} \lim_{n \to \infty} \int_\Omega \frac{\lambda_i u_n^2 + \mu_i v_n^2}{|x-a_i|^4} \varphi_j^\epsilon = 0, \quad i \neq j,$

$\lim_{\epsilon \to 0} \lim_{n \to \infty} \int_\Omega \frac{(\nu_i|u_n|^{q_i} + \sigma_i|v_n|^{q_i})}{|x-a_i|^4} \varphi_j^\epsilon = 0, \quad i \neq j,$

$\lim_{\epsilon \to 0} \lim_{n \to \infty} \int_\Omega (u_n \Delta u_n + v_n \Delta v_n) \Delta \varphi_j^\epsilon = 0.$

Consequently,

$$
0 = \lim_{\epsilon \to 0} \lim_{n \to \infty} \langle J'(u_n, v_n), (u_n \varphi_j^\epsilon, v_n \varphi_j^\epsilon) \rangle \geq \tilde{\mu}_{a_j} - \tilde{\gamma}_{a_j} - \tilde{\nu}_{a_j}.
$$

(11)

From (10) and (11) we derive that

$$
(\tilde{\nu}_{a_j})^{\frac{2}{p_j}} A_j \leq \tilde{\nu}_{a_j}, \quad 1 \leq j \leq k,
$$

which implies that

$$
either \tilde{\nu}_{a_j} = 0 or \tilde{\nu}_{a_j} \geq A_j^{\frac{N-1}{p_j-1}}.
$$

(12)

Furthermore, by (9) we have that

$$
c = J(u_n, v_n) - \frac{1}{2} \langle J'(u_n, v_n), (u_n, v_n) \rangle + o(1)
$$

$$
= \sum_{j=1}^k \left( \frac{1}{2} - \frac{1}{p_j} \right) \int_\Omega \frac{(\nu_j|u_n|^{q_j} + \sigma_j|v_n|^{q_j})^{\frac{p_j}{q_j}}}{|x-a_j|^{4\gamma_j}} + o(1)
$$
such that

Assume that Lemma 3.2.

3. Minimizers of the constant $A(\lambda, \mu, t, \nu, \sigma, q)$.

Lemma 3.1. Assume that $1 < q < 2^*(t), \ 0 < \lambda, \lambda < \bar{\mu}, \ 0 < \nu, \sigma < \infty, \ 0 \leq t < 4$.

(i) Suppose furthermore that $1 < q < 2$. Then

$$A(\lambda, \mu, t, \nu, \sigma, q) = \min \{ \nu^{-\frac{q}{2}} S(\lambda, t), \ \sigma^{-\frac{q}{2}} S(\mu, t) \}.$$ 

(ii) Suppose furthermore that $2 \leq q < 2^*(t)$. Then

$$A(\lambda, \mu, t, \nu, \sigma, q) = \min \{ \nu^{-\frac{q}{2}} S(\lambda, t), \ \sigma^{-\frac{q}{2}} S(\mu, t) \},$$

and $A(\lambda, \mu, t, \nu, \sigma, q)$ has the semitrivial minimizers

either $\{(C_y^\lambda(x-a), 0), C, \varepsilon > 0\}$ or $\{(0, C y^\lambda(x-a)), C, \varepsilon > 0\}$.

Proof. The argument is similar to that of [13, 14] and is thus omitted. \hfill \Box

Lemma 3.2. Assume that $1 < q < 2, \ 0 < \lambda < \bar{\mu}, \ 0 < \nu, \sigma < \infty, \ 0 \leq t < 4$.

Then

$$A(\lambda, \lambda, t, \nu, \sigma, q) = f(\bar{r}_{\min}) S(\lambda, t) = \left(\nu^{\frac{2}{2^*}} + \sigma^{\frac{2}{2^*}}\right)^{\frac{2}{2^*}} S(\lambda, t),$$

$A(\lambda, \lambda, t, \nu, \sigma, q)$ has the minimizers $\{(C y^\lambda(x-a), C \bar{r}_{\min} y^\lambda(x-a)), \varepsilon, C > 0\}$.

Proof. The argument is similar to that of Lemma 2.2 in [13] and is omitted. \hfill \Box
Lemma 3.3. Assume that $0 \leq \lambda, \mu < \bar{\mu}$, $0 < \nu, \sigma < \infty$, $0 < t < 4$, $a \in \mathbb{R}^N$, $1 < q < 2$. Then $\mathcal{A}(\lambda, \mu, t, \nu, \sigma, q)$ has a positive minimizer $(u_0, v_0) \in D \times D$, which is radially symmetric and decreasing with respect to $|x-a|$ and is a ground state solution to (6).

Proof. For simplicity we set $a = 0$ and define

$$\mathcal{A} := \mathcal{A}(\lambda, \mu, t, \nu, \sigma, q),$$

$$E(u, v) := |\Delta u|^2 + |\Delta v|^2 - \frac{\lambda u^2 + \mu v^2}{|x|^4}, \quad (u, v) \in D \times D,$$

$$F(u, v) := \frac{1}{|x|^t}(\nu|u|^q + \sigma|v|^q)^{\frac{t+1}{q}}, \quad (u, v) \in D \times D.$$  

By the Ekeland’s variational principle, we can choose a minimizing sequence $(\tilde{u}_n, \tilde{v}_n)$ for $E$ such that

$$\int_{\mathbb{R}^N} E(\tilde{u}_n, \tilde{v}_n) = \int_{\mathbb{R}^N} F(\tilde{u}_n, \tilde{v}_n) + o(1) = \mathcal{A}^{\frac{N-t}{4}} + o(1),$$

$$\Delta^2 \tilde{u}_n - \lambda \frac{\tilde{u}_n}{|x|^4} = \frac{\nu(|\tilde{u}_n|^q + |\tilde{v}_n|^q)^{\frac{q+1}{q}} - 1}{|x|^t} \tilde{u}_n |\tilde{v}_n|^2 \tilde{u}_n + f_n, \quad (13)$$

$$\Delta^2 \tilde{v}_n - \mu \frac{\tilde{v}_n}{|x|^4} = \frac{\sigma(|\tilde{u}_n|^q + |\tilde{v}_n|^q)^{\frac{q+1}{q}} - 1}{|x|^t} |\tilde{v}_n|^2 \tilde{v}_n + g_n,$$

where $f_n \to 0$ and $g_n \to 0$ as $n \to \infty$ in the dual space $D^{-1}$ of $D$.

Define the rescaling

$$(u_n(x), v_n(x)) := R_n^{\frac{N-4}{2}}(\tilde{u}_n(R_n x), \tilde{v}_n(R_n x)) \geq 0, \quad R_n > 0.$$  

Due to the invariance of $\int_{\mathbb{R}^N} E(\cdot, \cdot)$ and $\int_{\mathbb{R}^N} F(\cdot, \cdot)$ with respect to the rescaling, from (13) it follows that

$$\int_{\mathbb{R}^N} E(u_n, v_n) = \int_{\mathbb{R}^N} F(u_n, v_n) + o(1) = \mathcal{A}^{\frac{N-t}{4}} + o(1). \quad (14)$$

According to (13) and (14), by choosing suitable $R_n$ we can assume that

$$\int_{B_{1}(0)} F(u_n, v_n) = \int_{B_{R_n}(0)} F(\tilde{u}_n(x), \tilde{v}_n(x)) = \frac{1}{2} \mathcal{A}^{\frac{N-t}{4}} + o(1). \quad (15)$$

Then (2) and (14) imply that $\{(u_n, v_n)\}$ is bounded in $D \times D$. Arguing as in Lemma 2.2, up to a subsequence if necessary, for some $(u, v) \in D \times D$, there exist nonnegative real numbers $\bar{\rho}, \bar{v}_0, \gamma_0$, such that the following convergences hold in the sense of measures:

$$|\Delta u_n|^2 + |\Delta v_n|^2 \to d\bar{\rho} \geq |\Delta u|^2 + |\Delta v|^2 + \bar{\rho} \delta_0,$$

$$F(u_n, v_n) \to d\bar{v} = F(u, v) + \bar{v}_0 \delta_0,$$

$$\frac{\lambda u_n^2 + \mu v_n}{|x|^4} \to d\bar{\gamma} = \frac{\lambda u^2 + \mu v^2}{|x|^4} + \bar{\gamma}_0 \delta_0,$$  

(16)
where \( \delta_x \) is the Dirac mass at \( x \). To study the concentration at infinity, we set

\[
\tilde{\rho}_\infty = \lim_{R \to \infty} \limsup_{n \to \infty} \int_{|x| > R} (|\Delta u_n|^2 + |\Delta v_n|^2),
\]

\[
\tilde{\nu}_\infty = \lim_{R \to \infty} \limsup_{n \to \infty} \int_{|x| > R} F(u_n, v_n),
\]

\[
\tilde{\gamma}_\infty = \lim_{R \to \infty} \limsup_{n \to \infty} \int_{|x| > R} \frac{\lambda u_n^2 + \mu v_n^2}{|x|^4}.
\]

Then we have that

\[
A^{\frac{N-4}{2}} = \limsup_{n \to \infty} \int_{\mathbb{R}^N} E(u_n, v_n)
\]

\[
\geq A \limsup_{n \to \infty} \left( \int_{\mathbb{R}^N} F(u_n, v_n) \right)^{\frac{2}{N-4}}
\]

\[
= A \left( \int_{\mathbb{R}^N} F(u, v) + \tilde{\nu}_0 + \tilde{\nu}_\infty \right)^{\frac{2}{N-4}}
\]

\[
= A^{\frac{N}{N-4}}.
\]

From (18)–(20) it follows that \( \int_{\mathbb{R}^N} F(u, v), \tilde{\nu}_0, \tilde{\nu}_\infty \), are equal either to 0 or to \( A^{\frac{N}{N-4}} \). Furthermore, by (14) and (15) we have that

\[
\min\{\tilde{\nu}_0, \tilde{\nu}_\infty\} \leq \frac{1}{2} A^{\frac{N}{N-4}},
\]

which implies that \( \tilde{\nu}_0 = \tilde{\nu}_\infty = 0 \). Consequently,

\[
\int_{\mathbb{R}^N} F(u, v) = A^{\frac{N-4}{N-4}}.
\]

From (5) (14) and (16) it follows that

\[
A^{\frac{N}{N-4}} \geq \int_{\mathbb{R}^N} E(u, v) \geq A \left( \int_{\mathbb{R}^N} F(u, v) \right)^{\frac{2}{N-4}} = A^{\frac{N}{N-4}},
\]

which together with Lemma 3.1 implies that

\[
\int_{\mathbb{R}^N} E(u, v) = \int_{\mathbb{R}^N} F(u, v) = A^{\frac{N}{N-4}} < \min\{\nu^{-\frac{2}{N}} S(\lambda, t), \ \sigma^{-\frac{2}{N}} S(\mu, t)\}.
\]

Therefore \((u_n, v_n)\) converges strongly to \((u, v)\) and \((u, v)\) is a minimizer of \( A \) such that \( u \neq 0, v \neq 0 \). Furthermore, up to a multiplicative constant, \((u, v)\) is a solution to the limiting problem (6) with \( \alpha = 0 \).

Finally, we claim that \(((|u|, |v|) = (u^*, v^*))\), where \((u^*, v^*)\) is the Schwartz symmetrization of \((u, v)\). Arguing by contradiction, we assume that \(((|u|, |v|) \neq (u^*, v^*))\),
that is, either (i) $|u| \neq u^*$, $|v| = v^*$, or (ii) $|u| = u^*$, $|v| \neq v^*$, or (iii) $|u| \neq u^*$, $|v| \neq v^*$. Here we only study (i). (ii) and (iii) can be verified similarly.

Since $(u^2)^* = (u^*)^2$, $(v^2)^* = (v^*)^2$, we have that
\[
(u^2, v^2) \neq ((u^*)^2, (v^*)^2) = ((u^2)^*, (v^2)^*).
\]

Let $\tilde{u}, \tilde{v} \in L^2(\mathbb{R}^N)$ be respectively the solutions to the following equations:
\[
-\Delta \tilde{u} = (-\Delta u)^*, \quad -\Delta \tilde{v} = (-\Delta v)^*.
\]

As $\Delta u, \Delta v \in L^2(\mathbb{R}^N)$, we have that $\tilde{u}, \tilde{v} \in D^{2,2}(\mathbb{R}^N)$ and $\tilde{u} \geq u^*, \tilde{v} \geq v^*$ (e.g. [5]). Then by (21) we have that (e.g. [15, 20])
\[
\int_{\mathbb{R}^N} |\Delta \tilde{u}|^2 = \int_{\mathbb{R}^N} |(-\Delta u)^*|^2 = \int_{\mathbb{R}^N} |\Delta u|^2,
\]
\[
\int_{\mathbb{R}^N} |\Delta \tilde{v}|^2 = \int_{\mathbb{R}^N} |(-\Delta v)^*|^2 = \int_{\mathbb{R}^N} |\Delta v|^2,
\]

\[
\int_{\mathbb{R}^N} \frac{\tilde{u}^2}{|x|^4} \geq \int_{\mathbb{R}^N} \frac{(u^*)^2}{|x|^4} \geq \int_{\mathbb{R}^N} \frac{u^2}{|x|^4},
\]

\[
\int_{\mathbb{R}^N} \frac{\tilde{v}^2}{|x|^4} \geq \int_{\mathbb{R}^N} \frac{(v^*)^2}{|x|^4} \geq \int_{\mathbb{R}^N} \frac{v^2}{|x|^4},
\]

\[
\int_{\mathbb{R}^N} F(\tilde{u}, \tilde{v}) \geq \int_{\mathbb{R}^N} F(u^*, v^*) \geq \int_{\mathbb{R}^N} F(u, v).
\]

Furthermore, if $f, g$ are measurable such that $f > 0$ is radial and decreasing and $g \geq 0$ satisfies $g \neq g^*$, then we have that (e.g. [15])
\[
\int_{\mathbb{R}^N} fg^* > \int_{\mathbb{R}^N} fg.
\]

Taking $f = |x|^{-4}$ and $g = u^2$, we have that
\[
\int_{\mathbb{R}^N} \frac{(u^*)^2}{|x|^4} = \int_{\mathbb{R}^N} \frac{(u^2)^*}{|x|^4} > \int_{\mathbb{R}^N} \frac{u^2}{|x|^4}.
\]

From (22) and (23) it follows that
\[
A = \frac{\int_{\mathbb{R}^N} E(u, v)}{\left(\int_{\mathbb{R}^N} F(u, v)\right)^{\frac{2}{N-2}}} > \frac{\int_{\mathbb{R}^N} E(\tilde{u}, \tilde{v})}{\left(\int_{\mathbb{R}^N} F(\tilde{u}, \tilde{v})\right)^{\frac{2}{N-2}}} \geq A,
\]

which is a contradiction and implies that $(|u|, |v|) = (u^*, v^*)$.

Since $(|u|, |v|) = (u^*, v^*)$ is radially symmetric and non–increasing, we have that either $(u, v) = (u^*, v^*)$ or $(u, v) = -(u^*, v^*)$. Then we can take $(u, v) = (u^*, v^*)$ as a nonnegative minimizer of $A$ and therefore $(u, v) = (u^*, v^*)$ solves the following inequalities
\[
\begin{align*}
\Delta^2 u &\geq \frac{u^{2*(t)-1}}{|x|^t}, \quad u \geq 0 \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}, \\
\Delta^2 v &\geq \frac{v^{2*(t)-1}}{|x|^t}, \quad v \geq 0 \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}.
\end{align*}
\]

Arguing as in [5] and [12] we have that $(u, v) = (u^*, v^*)$ is strictly positive and radially decreasing. \[\square\]
Proof of Theorem 1.1. From Lemmas 3.1–3.3 the desired result follows. \qed

4. Mountain–pass type solutions to (1).

Lemma 4.1. Assume that \((\mathcal{H}_1)–(\mathcal{H}_3)\) hold. Set
\[
(u_\varepsilon, v_\varepsilon) := (w_\varepsilon^\lambda(x-a_l), \hat{\tau}_\min w_\varepsilon^\lambda(x-a_l)), \quad \hat{\tau}_\min := \left(\frac{\sigma_l}{p_l}\right)^{\frac{1}{p_l - q_l}}.
\]
Then \(\sup_{\tau \geq 0} J(\tau u_\varepsilon, \tau v_\varepsilon) < c^*\).

Proof. For all \(\tau \geq 0\), define the functions
\[
g(\tau) := \frac{\tau^2}{2} \int_\Omega \left(|\nabla u_\varepsilon|^2 + |\nabla v_\varepsilon|^2 - \lambda_1 \frac{u_\varepsilon^2 + v_\varepsilon^2}{|x-a_l|^4}\right) - \frac{\tau^{p_l}}{p_l} \int_\Omega \left(\frac{|\nu_l| u_\varepsilon^{q_l} + \sigma_l |v_\varepsilon|^{q_l}}{|x-a_l|^{t_l}}\right),
\]
\[
g(\tau) := J(\tau u_\varepsilon, \tau v_\varepsilon)
\]
\[
= \tilde{g}(\tau) - \frac{\tau^2}{2} \sum_{j \neq l, j = 1}^k \int_\Omega \frac{\lambda_j u_\varepsilon^2 + \mu_j v_\varepsilon^2}{|x-a_j|^4}
\]
\[
- \sum_{j \neq l, j = 1}^k \frac{\tau^{p_j}}{p_j} \int_\Omega \left(\frac{|\nu_j| u_\varepsilon^{q_l} + \sigma_j |v_\varepsilon|^{q_l}}{|x-a_j|^{t_j}}\right).
\]
Note that \(\sup_{\tau \geq 0} g(\tau)\) is attained at some finite \(\tau_0 > 0\). From \(g'(\tau_0) = 0\) it follows that
\[
\int_\Omega \left(|\nabla u_\varepsilon|^2 + |\nabla v_\varepsilon|^2 - \lambda_1 \frac{u_\varepsilon^2 + v_\varepsilon^2}{|x-a_l|^4}\right)
\]
\[
= (\tau_0)^{p_l - 2} \int_\Omega \left(\frac{|\nu_l| u_\varepsilon^{q_l} + \sigma_l |v_\varepsilon|^{q_l}}{|x-a_l|^{t_l}}\right)^{\frac{p_l}{q_l}} + \sum_{j \neq l, j = 1}^k \int_\Omega \frac{\lambda_j u_\varepsilon^2 + \mu_j v_\varepsilon^2}{|x-a_j|^4}
\]
\[
+ \sum_{j \neq l, j = 1}^k \frac{\tau_0^{p_j - 2}}{p_j} \int_\Omega \left(\frac{|\nu_j| u_\varepsilon^{q_l} + \sigma_j |v_\varepsilon|^{q_l}}{|x-a_j|^{t_j}}\right)^{\frac{p_j}{q_j}}
\]
\[
= (\tau_0)^{p_l - 2} \int_\Omega \left(\frac{|\nu_l| u_\varepsilon^{q_l} + \sigma_l |v_\varepsilon|^{q_l}}{|x-a_l|^{t_l}}\right)^{\frac{p_l}{q_l}} + O\left(\int u_\varepsilon^2\right)
\]
\[
+ \sum_{j \neq l, j = 1}^k \frac{\tau_0^{p_j - 2}}{p_j} O\left(\int u_\varepsilon^{p_j}\right).
\]
As \(\varepsilon \to 0^+\), from Lemma 2.3 it follows that there exist the positive constants \(C_1\) and \(C_2\) independent of \(\varepsilon\), such that
\[
C_1 < \tau_0 < C_2.
\]
For simplicity we set
\[
t_\min := \min\{t_j | 1 \leq j \leq k, j \neq l\} \in (0, 4).
\]
A direct calculation shows that
\[
\lambda_l < \lambda^* \iff 2(b(\lambda_l) - \delta) > t_\min.
\]
Furthermore, 

\[ 2^* (t_{\min}) = \frac{2(N - t_{\min})}{N - 4} = \frac{N - t_{\min}}{\delta} > \frac{N}{\delta + \frac{t_{\min}}{2}} > \frac{N}{b(\lambda^*_l)}, \]  

if \( \lambda_l < \lambda^* \), 

which together with (26) and Lemma 2.3 implies that 

\[ \int_\Omega |u_x|^2 (t_{\min}) = O_1 (\varepsilon^4_{t_{\min}}), \quad \text{if } \lambda_l < \lambda^*. \]  

(27)

Note that 

\[ \max_{\tau \geq 0} \left( \frac{\tau^2}{2} B_1 - \frac{\tau^p}{p_1} B_2 \right) = \frac{4 - t_l}{2(N - t_l)} B_1 \frac{N - t_l}{4} B_2 \frac{N - 4}{4}, \quad B_1, B_2 > 0. \]

From Lemmas 2.3 and 3.2 it follows that 

\[ \max_{\tau \geq 0} g(\tau) = \frac{4 - t_l}{2(N - t_l)} \left( \int_\Omega \left( |\Delta u_x|^2 + |\Delta v_x|^2 - \lambda_l \frac{u_x^2 + v_x^2}{|x - a_l|^2} \right) \right)^{N-4} - \frac{N - 4}{4} \]

\[ \times \left( \int_\Omega \left( |v_0| u_x |q_0| + \sigma_l |v_x| |q_0| \right) |x - a_l|^{\tau_l} \right)^{N-4} \]

\[ = \frac{4 - t_l}{2(N - t_l)} \left( 1 + (\tau_{\min})^2 \right) S(\lambda_l, t_l) \frac{N - t_l}{4} + O(\varepsilon^{2(b(\lambda)_l) - \delta}) \]

\[ \times \left( \nu_l + \sigma_l (\tau_{\min}) |q_0| \right)^{N-4} \frac{N - 1}{4} \]

\[ \times \left( \tau_{\min} \right)^{N-4} + O(\varepsilon^{2(b(\lambda)_l) - \delta}). \]  

(28)

From (H3), (24)–(28) and Lemma 2.3 it follows that 

\[ g(\tau_{\varepsilon}) \leq \tilde{g}(\tau_{\varepsilon}) - O_1 \left( \int_\Omega u_x^2 \right) - \sum_{j \neq l, j = 1}^k O_1 \left( \int_\Omega u_x^p \right) \]

\[ \leq \max_{\tau \geq 0} \tilde{g}(\tau) - O_1 \left( \int_\Omega u_x^2 \right) - O_1 \left( \int_\Omega |u_x|^{2^* (t_{\min})} \right) \]

\[ \leq \frac{4 - t_l}{2(N - t_l)} A_l \frac{N - t_l}{4} + O(\varepsilon^{2(b(\lambda)_l) - \delta}) - O_1 (\varepsilon_{t_{\min}}) \]

\[ < \frac{4 - t_l}{2(N - t_l)} A_l \frac{N - t_l}{4}. \]

The proof is complete. 

\[ \square \]

**Lemma 4.2.** Assume that (H1) and (H2) hold, 2 \( \leq q_l < 2^* (t_l) \). Set \((\hat{u}_x, \hat{v}_x) := (w_{\lambda_l}^0 (x - a_l), w_{\mu_l}^0 (x - a_l))\).

(i) Suppose furthermore that (H4) holds. Then \( \sup_{\tau \geq 0} J(\tau \hat{u}_x, 0) < c^* \).

(ii) Suppose furthermore that (H5) holds. Then \( \sup_{\tau \geq 0} J(0, \tau \hat{v}_x) < c^* \).

**Proof.** Since \( 2 \leq q_l < 2^* (t_l) \), from Lemma 3.1 it follows that 

\[ A_l = \min \left\{ \nu_l^{\frac{p}{2}} S(\lambda_l, t_l), \sigma_l^{\frac{p}{2}} S(\mu_l, t_l) \right\} \]

Then the argument is similar to that of Lemma 4.1 and is omitted. 

\[ \square \]
Proof of Theorem 1.2. Set \( c = \inf_{\gamma \in \Gamma} \max_{\tau \in [0,1]} J(\gamma(\tau)) \), where

\[
\Gamma = \left\{ \gamma \in C([0,1], H \times H) \mid \gamma(0) = 0, J(\gamma(1)) < 0 \right\}.
\]

Then \( c \neq 0 \). For all \((u, v) \in H \times H \setminus \{(0,0)\}\), from (2), (3) and \((H_1)-(H_2)\) it follows that

\[
J(u, v) \geq C\|u, v\|^2 - C \sum_{j=1}^{k} \|u, v\|^{p_j}.
\]

Then there exists a sufficiently small positive number \( \hat{\rho} \) such that

\[
b := \inf_{\|u, v\| = \hat{\rho}} J(u, v) > 0 = J(0,0).
\]

Suppose that \( 1 < q_l < 2 \). Since \( J(\tau u, \tau v) \to -\infty \) as \( \tau \to \infty \), thus there exists \( \tau_0 > 0 \) such that \( \|\tau_0 u, \tau_0 v\| > \hat{\rho} \) and \( J(\tau_0 u, \tau_0 v) < 0 \). By the Mountain Pass Theorem \([1, 3]\), there exists a sequence \( \{(u_n, v_n)\} \subset H \times H \) such that \( J(u_n, v_n) \to c \), \( J'(u_n, v_n) \to 0 \). From Lemma 4.1 it follows that

\[
c \leq \sup_{\tau \in [0,1]} J(\tau \tau_0 u, \tau \tau_0 v) \leq \sup_{\tau \geq 0} J(\tau u, \tau v) < c^*.
\]

If \( 2 \leq q_l < p_l \) then by Lemma 4.2 we also have that \( c < c^* \).

From Lemma 2.2, \( \{(u_n, v_n)\} \) has a subsequence, still denoted by \( \{(u_n, v_n)\} \), such that \( (u_n, v_n) \to (u_0, v_0) \) strongly in \( H \times H \). Then \( (u_0, v_0) \) is a solution to (1) such that \( J(u_0, v_0) = c \neq 0 \), which implies that \( (u_0, v_0) \neq (0,0) \).

To continue, we define the following Nehari manifolds:

\[
N_1 := \left\{ (u, 0) \mid u \in H \setminus \{0\}, J'(u, 0), (u, 0) \right\} = 0, \]

\[
N_2 := \left\{ (v, 0) \mid v \in H \setminus \{0\}, J'(0, v), (0, v) \right\} = 0, \]

\[
N := \left\{ (u, v) \in H \times H \setminus \{(0,0)\} \mid J(u, v), (u, v) \right\} = 0.
\]

Then \( N_1, N_2 \subset N \), and a direct calculation shows that (e.g. \([7]\))

\[
J(u_0, v_0) = c = \inf_{(u, v) \in N} J(u, v).
\]

Suppose that \( 1 < q_l < 2 \), \( \lambda_l = \mu_l \). We claim that \( u_0, v_0 \neq 0 \). In fact, if \( u_0 \neq 0, v_0 = 0 \), then \( (u_0, 0) \in N_1 \) and \( u_0 \) is a solution to the equation

\[
\begin{cases}
\Delta^2 u - \sum_{j=1}^{k} \frac{\lambda_j u}{|x - a_j|^4} = \sum_{j=1}^{k} \frac{\mu_j u^{p_j - 2} u}{|x - a_j|^{p_j}}, & x \in \Omega, \\
u = \frac{\partial u}{\partial n} = 0, & x \in \partial \Omega.
\end{cases}
\]

Furthermore,

\[
0 < \sum_{j=1}^{k} \left( 1 - \frac{1}{p_j} \right) \frac{\mu_j^{\frac{1}{p_j}}}{\lambda_j^{\frac{1}{p_j}}} \int_{\Omega} \frac{|u_0|^{p_j}}{|x - a_j|^{p_j}} = J(u_0, 0) = c = c_1 := \inf_{(u, v) \in N_1} J(u, v) < c^*.
\] (29)

In particular, \( c \) and \( c_1 \) are independent of \( \sigma_l > 0 \). From Lemma 3.2 it follows that

\[
c^* = \frac{4 - t_l}{2(N - t_l)} \left( \left( \nu_l^{\frac{2}{q_l}} + \sigma_l^{\frac{2}{q_l}} \right) \frac{2l - 2}{q_l} S(\lambda_l, t_l) \right)^{\frac{N - \lambda_l}{N - t_l}}.
\]

Therefore \( c^* \to 0 \) by taking \( \sigma_l \to \infty \), a contradiction with (29). Therefore, \( v_0 \neq 0 \). Similarly, \( u_0 \neq 0 \).
The proof is complete.

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