Attractor tempos for metrical structures

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Through new mathematical modelling based on experimentally-substantiated principles of cognitive science, this article provides robust principles for identifying “attractor tempos” which optimise the salience of metrical structures. This sheds light on core musical phenomena such as the inclination towards particular tempos in given metrical contexts, the use of (non-optimal) tempos as a source of expressive tension in music, and even what it means for music to be “fast” or “slow” in the first place. The study does not purport to set out a comprehensive model of metrical listening, but simply to identify the core principles which are necessarily involved in establishing attractor tempos. The (limited) dependence of those principles on the initial modelling parameters is discussed. Furthermore, there is no prescription of “correct” tempos here, instead the attractors model a set of defaults against which musicians may select tempos for expressive effect. An Online Supplement to this article, providing a subsidiary data table and further mathematical details, may be accessed at http://dx.doi.org/10.1080/17459737.2014.980343.

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1. Introduction/music theory

The basic hierarchies of common metres are well known to musicians and have been described in a number of ways by music theorists. Figure 1 reproduces an example from Lerdahl and Jackendoff (1983), in which metrical structure is represented by a stream of dots for each level.1 The importance of a metrical position (hereafter “metrical weight”) is summarised by the number of metrical streams which coincide at a given point. In the example, the start of each bar has at least four dots as the combination of bar, half-bar, crotchet, and quaver pulse streams.2 Lerdahl and Jackendoff extend this to the next (“hypermetrical”) level, adding a fifth

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1 The dot representation is not without precedent (London [2012], see page 78 and footnote 2 on page 202 for a summary) or successor (they become dashes for the otherwise similar representation in Kramer [1988]). Other representations used in seminal, twentieth-century accounts include: symbols for “strong–weak”-style alternation adopted from syllabic emphasis in poetic feet (Cooper and Meyer 1960); arrows to convey continuous accounts based on projection (Zuckerkandl 1956; Hasty 1997; Mirka 2009); and hierarchical tree-structures commonly used to map sentence structure in linguistics (Lerdahl and Jackendoff [1983] once again, and also Forth [2012]).

2 British musical terms are used throughout. North American equivalents are as follows: bar = measure; breve = double whole note; semi-breve = whole note; minim = half note; crotchet = quarter note; quaver = eighth note; semi-quaver = sixteenth note . . .
Figure 1. Lerdahl and Jackendoff’s account of metrical levels in the opening of Mozart’s great G minor symphony, K.550, © [The MIT Press]. Reproduced with permission. Lerdahl, Fred and Ray S. Jackendoff. (1983). A Generative Theory of Tonal Music. Cambridge, MA: The MIT Press, Figure 2.10.

dot for every second bar, and a sixth dot corresponding to a 4-bar unit (or phrase), the boundaries for which the authors consider to be at bars 3 and 7 in the passage given.\(^3\)

Viewed additively in this way, the metre might be summarised as follows:

- 8 quavers = 1, 1, 1, 1, 1, 1, 1, 1
- Grouped in 4 crotchets = 2, 1, 2, 1, 2, 1
- Grouped in 2 minims (beats 1 and 3 stronger than 2 and 4) = 3, 1, 2, 1, 3, 1, 2, 1
- Grouped into a bar (initial beat strongest) = 4, 1, 2, 1, 3, 1, 2, 1
- If considered in two bar phrases = 5, 1, 2, 1, 3, 1, 2, 1; 4, 1, 2, 1, 3, 1, 2, 1
- If considered in four bar phrases = 6 . . . ; 4 . . . ; 5 . . . ; 4 . . .

The advantages of this model include the separation of the metrical hierarchy into separate pulse streams, and the use of a visual representation which is compatible with pulse projection. The focus on metrical levels also implicitly accounts for some metrical commonalities which are obfuscated by musical notation. Consider the two moments from the third movement of Martinu’s Les Fresques de Piero della Francesca which are reproduced in Figure 2. There is a clear correspondence between the two moments: in the second version, the note lengths are twice as long, but the tempo is twice as fast and so the aural, musical result is nearly-identical to the original. Although they may look different, the two notational contexts are formally equivalent, distinguished only by the different associations they may invoke with respect to particular styles and epochs.\(^4\)

But what if one of these contexts used an extra metrical level at the fast-end (such as demi-semi-quavers in the latter)? We would then have a different starting point for counting dots, and so the equivalent metrical levels would be offset such that non-equivalent levels are related by the same number of dots. The addition of a metrical level is sure to have some effect, but not so great as to undermine the clear equivalence between corresponding levels in this case. Clearly this is a weakness.

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\(^3\) This is not a unanimously-held view. Bernstein considers the 4-bar metrical groupings to begin at bar 1: “the true shape turns out to be a symmetrical pyramid built of four pairs of bars right from the start” (Bernstein 1976, 103), and Kramer (1988) follows suit (at least for the passage given). For Lester, the “first true melodic downbeat” appears in bar 3 (Lester 1986, 70), though it is significant that he avoids the 4-bar level altogether, discussing only those levels unequivocally demonstrated in the musical design (the 2-bar level is articulated by the bass’s registral alternation, for instance). The clarity of the hierarchy at the tactus level as compared with the relative ambiguity of more remote, hypermetrical levels is a key consideration which the present article addresses. As Kramer’s diagrammatic analysis of a slightly longer excerpt (bars 1–22) shows, the metre is thoroughly consistent for levels up to that of the bar; there is just one deviation from consistency at the 2-bar level; and there is much greater variety at the 4-bar (Kramer 1988, 116). That said, Kramer’s inclination to discuss metre at all structural levels (see especially page 119 and subsequent pages) is strangely incompatible with his own attention to the perceptual present.

\(^4\) Other examples of this particular notational equivalence include the codas to the first and third movements of Prokofiev’s First Violin Concerto. Again, the “same” music is notated in 6/8 and 3/4, respectively.
Figure 2. Two notational contexts for the “same” material in the third movement of Martinu’s Les Fresques de Piero della Francesca. Left: the opening of the movement (in 6/8); right: the recapitulation of that opening material six bars before rehearsal mark 36 (in 3/4 at almost exactly twice the tempo).

The dot-wise system leads to strange correspondences of this kind in even the simplest examples. For instance, it attributes the same value to the bar-level in 3/4 as to the half-bar-level value in 4/4 (“3” in the examples given):

- 6 quavers = 1,1,1,1,1,1
- Grouped in 3 beats = 2,1,2,1,2,1
- Grouped into a bar (initial beat strongest) = 3,1,2,1,2,1.

This may allude to a valid way of viewing the relative structural weights – that metres with more internal levels require more definition than those with fewer – but it is unlikely that two clearly different metrical streams could be satisfactorily represented by the same value. This problem can be addressed in another direction by normalising the bar-level weightings at a certain value instead of the fastest pulse;\(^5\) however that approach merely re-locates the problem rather than solving it.

Ultimately, the integer values are too simplistic to deal with these core aspects of metre. This is not so much a criticism of Lerdahl and Jackendoff whose object is only to depict a hierarchy, but it is important to improve that model in line with these critiques because it is used as the basis for quantitative applications in which those values assume a significance.\(^6\)

To summarise, Lerdahl and Jackendoff’s model successfully incorporates accounts of metrical structure and the use of metrical levels. This is enough for it to predict the equivalence of the two Martinu moments. The main shortcoming is that it fails to address the other obvious commonality: tempo. A related weakness is Lerdahl and Jackendoff’s implicit assumption that the various metrical levels are equally salient and important to the metrical experience. Even their own Mozart analysis (shown above) hints at the fact that this is not valid: the highest hypermetrical level is equivocal, while the primary counting levels are perfectly clear.\(^7\) Again a consideration of tempo improves this situation.

This article addresses those problems by combining the viable aspects of the music-theoretic approach with core principles from the cognitive sciences to develop a new model of metrical weight and salience. This leads to the development of a heuristic for optimising the salience of any simple metre, and the identification of those optimised “attractor tempos” along with the metrical categorisation to which those tempos are linked. Concluding comments include a discussion of the limited extent to which altering the initial modelling parameters affects the final results: most of the core observations are relatively unconstrained by those specifics.

\(^5\) This is the approach taken by David Meredith in his 1996 thesis where “the metric strength of any location that is the initial location of a bar is 1” (Meredith 1996, 214–215).

\(^6\) See, for instance, Chapter 13 of Toussaint (2013). The “Generative Theory of Tonal Music” (in which this model appears) is a high-profile, oft-cited work, and this is obviously not the first refinement of the model to be proposed. For another refinement to a too-simplistic (“strongly reduced”) aspect of the metrical part, see London’s “weak reduction hypothesis” (London 1997).

\(^7\) This debate is taken up in other scholars’ analyses of this passage as discussed.
2. Individual pulse salience

One reason for the equivocal 4-bar hypermetre in the Mozart example may be that pulse streams are not equally discriminable across the spectrum. The literature suggests that:

- there is a preference for pulses around 100 beats per minute (hereafter “bpm”) which equates to an interonset interval (IOI) of 0.6 seconds;
- pulses shorter than 0.1 seconds cease to be metrically useful; and
- the upper limit for what can be grouped as a single (metrical) unit is approximately 6 seconds.

These three values give us a truer sense of the relative salience of pulses at different tempos, and the means to construct a perceptually-minded model of metrical weight.

The exact values for all three of these reference points are contested, especially that of the upper limit which varies considerably according to the information content: for our purposes, the musical context. This upper limit is closely related to the important categorical boundary of the “psychological present” – a time-span during which information is in current, active use rather than stored (and thus past) – and as John Michon has it, “Because the present is so highly adaptive, no fixed parameter values can be expected to describe it adequately” (Michon 1978, 89). However, although we may lack a definitive set of values to work with, we nevertheless have a sufficiently clear picture of the phenomenon to create a model which demonstrates its heuristic value by illuminating important, generalisable information about maximising cumulative pulse salience in metrical structures.

In developing a suitable model of pulse interonset interval \( x \) against salience \( S \), we require a smoothly continuous curve to account for all \( x \)-values; the \( x \)-value at which the curve peaks (indicating maximal salience) to be at 0.6 seconds; and the salience to be negligible in the range \( x < 0.1 \) and \( x > 6.0 \). We may achieve this using the Gaussian function given by equation (1), and illustrated in Figure 3.

\[
S = \exp \left\{ -\frac{[\log(x) - \log(\mu)]^2}{2\sigma^2} \right\} \tag{1}
\]

A logarithmic \( x \)-scale is used here to give a symmetrical decay from the peak at \( x = 0.6 \) to negligibly low values for 0.06 (below 0.1) and 6.0 seconds. The \( y \)-axis range is set from 0 to 1 arbitrarily, according to mathematical convention, and the width is given by \( \sigma = 0.3 \) to result in the \( y \)-values becoming negligibly low at the desired position. The value of \( \sigma \) is an important issue which is discussed further in Section 5.3. So, for \( \mu = 0.6 \) and \( \sigma = 0.3 \),

\[
S = \exp \left\{ -\frac{[\log(x/0.6)]^2}{0.18} \right\} \tag{2}
\]

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8 See London (2012, 27ff.) for a summary of the literature in support of these values.
9 James (1890) is frequently cited as the earliest exposition, though his first reference to the “specious present” is to cite a section from Clay (1882, 609) as the originator of the term and concept. James proceeds to cite empirical experimental data conducted by Wundt, Dietze, Estel, and Mehner concerning relevant temporal values before proceeding to a more speculative, original examination of the idea. Clay defines the “specious present” as distinct from the fleeting, ungraspable “real present” as well as from the “obvious past” (Clay 1882, 167–169). Subsequent landmarks include: Michon’s (1978) review of the existing psychological literature at that time; Clarke’s (1987) ground work for an analytical application; and the use of the “present” in prominent theories of metre such as Hasty (1997).
10 Further discussions in this article address the nature of existing data and models for pulse salience in the context of the approach taken here (Section 3), the generalisability of the principles which the present article deduces (Section 6.1), and some prospects for further empirical study (Section 6.2).
11 Despite the suggestive names “\( \mu \)” and “\( \sigma \),” this is not primarily intended as a probabilistic model.
12 The Gaussian constant is 1 and therefore omitted.
3. On the choice of distribution, the cognitive-scientific background, and the aims of the present heuristic

These equations do not constitute the first or only mathematical model of pulse perception, they merely outline the approach taken in this article as part of developing a new heuristic for understanding the factors involved in optimising metrical salience. Although developed independently, the logarithmic-Gaussian model of pulse salience used in this article is also used by (Parncook 1994, 438) to describe a “kind of band-pass filter, admitting only pulse sensations that lie within a given range of periods.” Alternative accounts include van Noorden and Moelants’s (1999) “effective resonance curve” which is shown in Figure 4, adapting to a range of data sets of potential relevance to the notion of pulse salience (including Parncook’s).¹³ Significant here is the relative agreement among the various data sets and models.

Van Noorden and Moelants’s alternative model appears to be based on the “Cauchy” (“Lorentz”) distribution (though nowhere is that quite stated).¹⁴ This distribution has assumed an importance in physics as the solution to the motion of a “driven damped harmonic oscillator” and van Noorden and Moelants hypothesise that the physiological basis of tempo perception works in an analogous way. That is not to say that there is necessarily any physical oscillator literally involved, but rather that the physics for this principle provides a useful model for the extent of the collective firing of neurons.

While we must be cautious about asserting connections between the separate discoveries of physiological and behavioural studies, the recent literature continues to strengthen the case for the eminently intuitive parallels among aspects relevant to the study of pulse preference.¹⁵ In the broadest terms, it is currently thought that oscillations in neural firing adapt in order to regularly preempt anticipated moments of interest.¹⁶ This leads to a distribution of neural resources that favours moments which are expected to contain important information. That distribution equates

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¹³ A version of this figure showing only van Noorden and Moelants’s data and model can be found in Leman (2003, Section 3.4.3).
¹⁴ The “resonance period” and “damping constant” variables conceptually correspond to some extent with the μ- and σ-values used as part of the Gaussian distribution here and in Parncook (1994). Incidentally, the normal and Cauchy distributions are related in that the latter is given by the x/y ratio of the former.
¹⁵ For summaries of relevant, recent literature, see London (2012, Chap. 3), Repp and Su (2013), especially Sections 1.5 and 4, and Calderone et al. (2014). Calderone et al. include an extremely useful, up-to-date overview of relevant, recent work including a tabular summary of both the behavioural and neurological effects that have been observed in connection with oscillatory entrainment.
¹⁶ For instance, Cravo et al. (2013) demonstrate the entrainment of alpha band oscillations in the visual cortex in response to a visual stimulus.
to a relative amplification of the signal which facilitates the processing of information received at that time.

However, while there are promising developments in understanding the neural correlates to pulse salience, and no shortage of excellent studies on its behavioural manifestations, it is not likely that so general a principle can ever be comprehensively modelled in the abstract, independent of context.\textsuperscript{17} How would a truly context-independent model of pulse salience be constructed? What data should be used? Should it be taken from physiological studies of ensemble neural firing, from behavioural studies, or from some combination of both? Among these, what kinds of experiments are the most generalisable? In order to ensure ecological validity for musical listening, should we limit ourselves to those few studies which make use of real musical examples?\textsuperscript{18} If so, then a first problem is that real music involves many metrical levels, mostly in fully-binary configurations such as 4/4. This makes them an interesting point of comparison for the final results of this model (models of “metrical salience,” particularly of simple binary metres),\textsuperscript{19} but inappropriate as a starting model of individual pulse preference. Secondly, musical genre has been shown to have an effect,\textsuperscript{20} so a representative balance of musical repertories would be needed.\textsuperscript{21}

One inevitably runs into large methodological questions of this kind, and (equally inevitably) falls short of a truly generalisable representation. However, as this article will show, only a very approximate handle on “pulse salience” is needed to make significant observations about the principles behind the emergence of certain (“attractor”) tempos which appear to optimise salience for whole categories of metres. Accordingly, this article does not attempt to address

\textsuperscript{17} Volk (2008), discussed below, represents a recent computational line of inquiry that is most promising for defining the phase and period of metrical levels of that musical context.
\textsuperscript{18} These include van Noorden and Moelants (1999), Toiviainen and Snyder (2003), and McKinney and Moelants (2006). See London (2012, 27ff.) for a discussion of ecological validity in this context.
\textsuperscript{19} The data would have to be separated or re-collected according to metrical structure and level usage.
\textsuperscript{20} As demonstrated in van Noorden and Moelants (1999) and McKinney and Moelants (2006).
\textsuperscript{21} See London (2013) for a discussion of what that might look like in practice.
those larger issues. It remains agnostic about what “pulse salience” exactly is, about the distribution which best accounts for it, and about the nature of the brain’s (and indeed the body’s) involvement. Rather, this article seeks merely to make musical use of what is – however approximately – known.

In keeping with the appropriate level of accuracy for modelling this phenomenon, and the theoretical-speculative approach accordingly taken here, the mathematical model used to represent pulse salience in this article is the simplest, most parsimonious available (the logarithmic Gaussian in equation 1). This allows for results to be expressed clearly, in basic categorical terms (particularly in relation to the $\mu$-value); in the context of this heuristic, categories are more useful than a large collection of apparently disassociated numbers. Section 6.1 sets out how changes to the initial model affect the numerical results generated, but (importantly) that the core principles advanced here remain intact.

In short, the hypothesis that humans prefer pulses in a certain range has been tested. The outcomes of those tests include various results which do not perfectly align with one another but which collectively point to a general agreement about the kind of shape that should be used for a model of pulse salience in the abstract. This article makes use of such a model to develop a new model of metrical salience which has explanatory value in terms of tempo choice. In turn, the new model also represents a testable hypothesis for the cognitive sciences (discussed in Section 6.2).

4. Combined periodicities (metre)

The model of individual pulse salience in Figure 3 outlines a central peak, but musical intuition and theoretical accounts of the metrical hierarchy tell us that metrical positions corresponding to longer durations have greater metrical weight. These principles are perfectly compatible as long as metre is modelled as the sum of several periodicities: a modelling assumption which this article shares with scholars like Lerdahl and Jackendoff and Parncutt alike. Modelling metre in this way necessarily preserves the positive correlation between duration and metrical weight while also leaving room to include a weighting of the constituent pulses’ importance. Here, the salience $S$ of each pulse $x$ is modelled by equation (1), and the weight of a metrical position is given by the sum of the saliences for the pulses which coincide there (as defined by the metrical hierarchy).

For instance, in a metrical structure consisting exclusively of groupings by 2 (a “binary metre”), the available pulse rates consist of all binary multiples and divisions of the tactus with an $x$-value between 0.1 and 6 seconds (the “metrical window” in London’s terminology). Where the crotchet pulse IOI = 0.6 seconds (100 bpm), this includes:

- 0.15 seconds (the semi-quaver),
- 0.3 seconds (the quaver),
- 0.6 seconds (the crotchet),
- 1.2 seconds (the minim),
- 2.4 seconds (the semi-breve: the length of the bar if in 4/4), and
- 4.8 seconds (the breve: a 2-bar unit in 4/4).

Assuming that all of these available levels are used, the metrical weight of any position is given by the sum of $S$-values for all pulses coinciding there. For the weight associated with

\[ \text{Footnotes:} \]
\[22\] See, for instance, Trainor et al. (2009).
\[23\] “The salience of a perceived meter may be estimated by summing the saliences of the consonant pulse sensations of which it is composed” (Parncutt 1994, 455).
\[24\] London uses the terms “temporal envelope” as well as “metrical window” for this range (London 2012, 27ff.), while Parncutt speaks of an “existence region” (Parncutt 1994, 436). “Metrical window” is the term used in this article.
the crotchet-level in this example, we sum the $S$-values corresponding to $x = 0.6$ (the crotchet level itself) as well as $x = 0.3$ and 0.15 (the faster levels); for the weight associated with the 4/4 bar-level, we add to that sum $S(1.2)$ and $S(2.4)$. This may continue to any further “hypermetrical” levels within the acceptable range (only one in this case). The values for this example are summarised in Table 1 (and also compared with three other important examples in the Online Supplement to this article – see the section Supplemental online material before the references list).

Table 1. Quantified pulse salience and metrical weight for the levels of an example binary metre. Individual pulse salience is given by equation (2), and metrical weightings are given by the sum of the longest pulse involved and all faster pulse constituents. This is given by equation (3) for full-represented binary metres and equation (4) for any other circumstance. In this example, note the symmetry of $S$ values about $S(0.6)$, the central peak of the (individual) pulse salience curve used.

| Pulse stream (IOI in seconds) | 0.15 | 0.3 | 0.6 | 1.2 | 2.4 | 4.8 |
|-------------------------------|------|-----|-----|-----|-----|-----|
| Individual pulse salience     | 0.133| 0.604| 1.000| 0.604| 0.133| 0.011|
| Cumulative (metrical) salience| 0.133| 0.738| 1.738| 2.342| 2.476| 2.487|

Equation (3) generalises this principle for any position in a binary metre. For any metrical position, let $x$ now stand for the duration it represents (the time that elapses before the next position of equal or higher metrical value). A binary metre’s metrical weight $B$ is given by the sum of the saliences $S$ of the constituent metrical levels: pulse IOIs of $x/2^n$ for natural numbers $n$. Because intervals outside the 0.1–6 second range have negligible salience, we can restrict the range of the sum to values of $n$ satisfying $0.1 < x/2^n < 6$ for $n \geq 0$. It can be shown that the maximum value of $n$ satisfying this restriction is given by $N = ⌊(\log x − \log 0.6)/\log 2⌋$. The metrical weight $B$ may therefore be expressed as:

$$B = \sum_{n=0}^{N} \exp \left\{ -\frac{|\log(x/2^n) − \log(0.6)|^2}{0.18} \right\}, \quad 0.1 < x < 6. \quad (3)$$

So this system adds a value for each metrical level (just as with Lerdahl and Jackendoff’s dots), but weights the relative importance of each constituent level according to its salience. The importance of the respective levels can be seen in the left-hand example of Figure 5, which sets out the metrical hierarchy corresponding to the example in Table 1. Strong-weak alterations can be easily observed for some levels, while for the others they are barely noticeable. For instance, the strong-weak distinction between $x = 0$ and $x = 4.8$ on the one hand ($S(0) = S(4.8) = 2.487$), and $x = 2.4$ on the other ($S(2.4) = 2.476$), can hardly be seen on Figure 5.

Once again, observations of this kind shed light on analytical matters such as the hypermetrical ambiguity in the Mozart example above. While strong and weak alternations are easily discerned at the most salient tactus level, it is often much harder to deduce relative metrical weight at higher levels. It may be that the Classical practice of varying phrase lengths while keeping the tactus levels constant is reliant on that fact.\(^{25}\)

Encouragingly, the new theoretical stem chart describes similar hierarchical patterns not only to traditional accounts of metre in a general sense, but also to corpus studies of metrical position

\(^{25}\) In the Mozart example, quaver, crotchet, minim, and semi-breve hierarchies are all evident from the notation and the musical cues. The breve (2-bar) level is absent from the notation and is almost consistently used; while the 4-bar level is considerably more varied still. Refer again to Kramer’s diagram for a diagrammatic analysis of bars 1–22 (Kramer 1988, 116).
usage. For comparison, Figure 5 also includes such an example from Prince and Schmuckler (2014), specifically for position usage in 4/4 contexts over a large sample of works. 26 Note the relatively large difference between the quaver and crotchet levels (compare positions 3, 7, 11, 15 with 5, 13), but the much lesser difference between those of the minim and semi-breve (positions 9 and 1). 27

5. Optimising metrical salience; identifying attractor tempos

We now turn to the primary goal of optimising “metrical salience” for all simple metres represented by any number of (consecutive) metrical levels, and thus identifying the “attractor tempos” for those metres (the x-value at which the metrical salience of that metre is optimised). Metrical salience is operationally defined here by the combined saliences of all the pulse streams present in the metrical structure (the highest single metrical weight value). If we represent the fastest pulse as $x$, and all higher levels by the relevant multipliers ($px, qx, rx, \ldots$), then the highest metrical weight value which can be obtained is given by

$$M = \sum_{n=1,p,q,r,\ldots} \exp \left\{ -\frac{[\log(nx/\mu)]^2}{2\sigma^2} \right\}. \quad (4)$$

The function $B(x)$ for binary metres is one case of $M(x)$. Optima are represented by local maxima of this sum which are located at zeros of the first derivative. 28

As Parn cott (1994, 438) briefly observes, this cumulative salience of a high metrical level (his “aggregate salience”) is not necessarily maximised by the tempo which optimises the salience of the individual pulse. Parncott suggests this as a reason why subjects in tapping studies are drawn

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26 Position usage provides another possible basis for quantifying metrical weight. See also (Volk 2008, 99) for a context-specific “quantification of the note’s [metrical position’s] metric importance [usage]” that is un-constrained by the notated metre, extending to hypermetrical levels, and dissonant configuration, for instance.

27 It would be particularly interesting to test the effect of tempo on position usage.

28 The necessary mathematics are supplied in the Online Supplement – see the section Supplemental online material before the references list.
to other tempos when tapping in specified groupings: to faster tempos for larger groupings.\textsuperscript{29} This appears to indicate a desire to balance the various pulse levels involved, maximising the combination rather than any individual pulse.

This section sets out a model for optimising metrical salience in the context of any simple metrical structure based on binary and ternary grouping at each consecutive, structural level. While it is possible to sustain up to six metrical levels within the metrical window, fewer are present in the majority of musical contexts (and certainly the tapping tasks that dominate the literature). This study therefore accounts for the number of levels represented as well as the proportional structure; level usage does indeed turn out to be an important factor.

5.1. \textit{An exhaustive list of metrical structures}

Table 2 provides a comprehensive list of every possible metrical structure based on consecutive levels of 2- and 3-groupings which fall within the metrical window defined. The metrical window constraint dictates that the single-unit level can be combined with further levels up to a limit of 60-units.\textsuperscript{30} Therefore, this table includes metres with 54-unit levels, but not those with 64 (the two values of $2^m \times 3^n$ which are closest to 60). Metres are defined by their proportional structure and level usage (not by time signatures). For instance, “binary metres” with grouping by 2 at each level comprise the whole first block; any binary time signature (“2/4,” “4/4,” \ldots) could refer to any one of these structures.

The first column lists examples of time signatures which express each of the metrical levels involved (along with the corresponding notational values in brackets). Some of those signatures representing many metrical levels are rare, but this does not necessarily mean that the metrical structure itself is also rare. For instance, “48/16” could be notated as a simple binary metre with triplets at the fastest level. However, it is worth noting that as the number of 3-levels increases the metrical structures do indeed become rarer. There is not even a standard symbol for a 9-unit value. We would have to invent a dotted(dotted(unit)) as distinct from a double-dotted one (7 units). Question marks in the table indicate this situation.

The second and third columns set out respectively the pulse length for each level (in units) and the proportional multipliers between consecutive levels (invariably 2 or 3). These columns also introduce notation used in the remainder of the article: sets of numbers in angular brackets denote the pulse length for each level (in ascending order starting with 1), and the proportional schemes between those consecutive levels are given in square brackets. For instance, [3, 2, 2] corresponds to the proportional scheme in \langle 1, 3, 6, 12 \rangle. If “1” refers to the quaver level, then “3” refers to that of the dotted crotchet, and so on.

The following “$x$-max” column deals with the $0.1 < nx < 6.0$ seconds constraint. The lower limit for $x$ is always set by 0.1, but the maximum depends on the metrical structure: the slowest pulse level must be no slower than 6 seconds. The $x$-max is therefore given by 6 seconds divided by the number of units in the slowest level. The $x$-max column thus gives a sense of how constrained each metre is by the values. Clearly, the longer the longest pulse level, the narrower the $x$-range. For instance, metres with a 54$x$ pulse level can only fit all of those levels within the

\textsuperscript{29} See Fraisse (1956, 1982) and Vos (1973). Fraisse tested the spontaneous tempo for tapping in metrical groups. His results indicate a preference for faster rates when tapping in 3s (IOI = 0.42 seconds), and faster still when in 4s (0.37 seconds) (Fraisse 1956, 15). This matches Vos (1973), a test of subjective grouping at different tempos in which participants were given pulses with IOIs of 0.15, 0.2, 0.3, 0.4, and 0.8 seconds and a choice of grouping in 2, 3, 4, 5, 6, 7, or 8. Grouping by 2 was most attractive for IOIs of 0.8 seconds, by 3 at 0.4 seconds, and by 4 at 0.3 seconds. That said, binary groupings (2, 4, 8) far outperformed any contender, accounting for 63/75 of the whole study. Both results indicate a correlation between the size of metrical grouping and the tempo preferred: the larger the group, the faster the tempo.

\textsuperscript{30} The lower limit, 0.1 seconds, multiplied by 60 units equals 6 seconds (the upper limit).
Table 2. All possible metrical structures based on grouping in 2s and 3s with details concerning the optimal $x$-value.

| Example | Pulse lengths | Proportions | $x$-max | Attractor ($x$) | Additional information |
|---------|---------------|-------------|---------|----------------|------------------------|
| $1/4$ (c only) | (1) | (n/a) | 6 | 0.6 | Sym.: $x = \mu$ |
| $2/4$ (c, m) | (1.2) | [2] | 3 | 0.424 | Sym.: $x = \mu/\sqrt{2}$ |
| $2/4$ (q, c, m) | (1,2,4) | [2,2] | 1.5 | 0.3 | Sym.: $2x = \mu$ |
| $2/4$ (sq, q, c, m) | (1,2,4,2,8) | [2,2,2,2] | 0.75 | 0.212 | Sym.: $2x = \mu/\sqrt{2}$ |
| $4/4$ (sq, q, c, m, sb) | (1,2,4,8,16) | [2,2,2,2] | 0.375 | 0.15 | Sym.: $4x = \mu$ |
| $4/2$ (sq, c, m, sb, b) | (1,2,4,8,16,32) | [2,2,2,2,2,2] | 0.1875 | 0.106 | Sym.: $4x = \mu/\sqrt{2}$ |

| Example | Pulse lengths | Proportions | $x$-max | Attractor ($x$) | Additional information |
|---------|---------------|-------------|---------|----------------|------------------------|
| $3/4$ (c, dm) | (1.3) | [3] | 2 | 0.346 | Sym.: $x = \mu/\sqrt{3}$ |
| $3/8$ (sq, q, dc) | (1.2,6) | [2,3] | 1 | 0.352 | |
| $6/8$ (q, dc, dm) | (1,3,6) | [3,2] | 1 | 0.170 | |
| $3/4$ (sq, q, c, dm) | (1,2,4,12) | [2,2,3] | 0.5 | 0.277 | |
| $6/8$ (sq, q, dc, dm) | (1,2,6,12) | [2,3,2] | 0.5 | 0.332; (0.0904) | Sym. about $2x = \mu/\sqrt{3}$ |
| $12/16$ (sq, dq, dc, dm) | (1,3,6,12) | [3,2,2] | 0.5 | 0.108 | |
| $3/2$ (sq, q, c, m, dsb) | (1,2,4,8,24) | [2,2,2,3] | 0.25 | 0.204 | |
| $6/4$ (sq, q, c, dsb, dm) | (1,2,4,12,24) | [2,2,3,2] | 0.25 | (0.274) | Secondary peak (0.454) also off |
| $12/8$ (sq, q, dc, dm, dsb) | (1,2,6,12,24) | [2,3,2,2] | 0.25 | (0.0546) | Secondary peak (0.331) also off |
| $24/16$ (sq, dq, dc, dm, dsb) | (1,3,6,12,24) | [3,2,2,2] | 0.25 | (0.0737) | As fast as possible |
| $3/1$ (sq, q, c, m, sb, db) | (1,2,4,8,16,32) | [2,2,2,3,3] | 0.125 | (0.147) | (48x = 7.03) |
| $6/2$ (sq, q, c, m, dsb, db) | (1,2,4,8,24,48) | [2,2,2,3,2] | 0.125 | (0.203) | Secondary peak (0.0227) also off |
| $12/4$ (sq, q, c, m, dsb, db) | (1,2,4,12,24,48) | [2,2,3,2,2] | 0.125 | (0.275; 0.0273) | Sym. about $4x = \mu/\sqrt{3}$ |
| $24/8$ (sq, q, dc, dm, dsb, db) | (1,3,6,12,24,48) | [3,2,2,2,2] | 0.125 | (0.0369) | Secondary peak (0.331) also off |
| $48/16$ (sq, dq, dc, dm, dsb, db) | (1,3,6,12,24,48) | [3,2,2,2,2,2] | 0.125 | (0.0512) | As fast as possible |

| Example | Pulse lengths | Proportions | $x$-max | Attractor ($x$) | Additional information |
|---------|---------------|-------------|---------|----------------|------------------------|
| $9/16$ (sq, dq, 2) | (1,3) | [3,3] | 0.6 | 0.2 | Sym.: $3x = \mu$ |
| $9/8$ (sq, q, dc) | (1,2,6,18) | [2,2,3] | 0.3 | (0.349) | (18x = 6.28) |
| $18/16$ (sq, dq, dq, 2) | (1,3,6,18) | [3,2,3] | 0.3 | 0.141 | Sym.: $3x = \mu/\sqrt{2}$ |
| $18/16$ (sq, dq, 2) | (1,3,9,18) | [3,2,3] | 0.3 | (0.0573) | As fast as possible |
| $9/4$ (sq, q, c, dm) | (1,2,4,12,36) | [2,2,3,3] | 0.16 | (0.277) | As fast as possible |
| $18/8$ (sq, q, dc, dm, ?) | (1,2,6,12,36) | [2,3,2,3,3] | 0.16 | (0.0733) | Secondary peak (0.332) also off |
| $18/8$ (sq, q, dc, ?, 2) | (1,2,6,18,36) | [2,3,3,2] | 0.16 | (0.349; 0.0287) | Sym. about $6x = \mu$ |
| $36/16$ (sq, dq, dq, dc, dm, 2) | (1,3,6,12,36) | [3,2,2,3,3] | 0.16 | 0.1 | Sym.: $6x = \mu$ |
| $36/16$ (sq, dq, ?, ?, 2) | (1,3,6,18,36) | [3,2,3,2,3] | 0.16 | 0.136 | Secondary peak off lower limit |
| $36/16$ (sq, dq, ?, ?, 2) | (1,3,9,18,36) | [3,3,2,2] | 0.16 | (0.0362) | As fast as possible |
| $27/16$ (sq, dq, ?, ?) | (1,3,9,27) | [3,3,3] | 0.12 | (0.0895) | Sym. about $3x = \mu/\sqrt{3}$ |
| $27/8$ (sq, q, dc, ?, ?) | (1,2,6,18,54) | [2,3,3,3] | 0.1 | (0.349) | As slow as possible |
| $54/16$ (sq, dq, q, dc, ?, 2) | (1,3,6,18,54) | [3,2,3,3] | 0.1 | (0.141) | As slow as possible |
| $54/16$ (sq, q, q, ?, 2) | (1,3,9,18,54) | [3,2,3,3] | 0.1 | (0.0474) | As fast as possible |
| $54/16$ (sq, dq, ?, ?, 2) | (1,3,9,27,54) | [3,3,3,2] | 0.1 | (0.0191) | Secondary plateau (c. 0.2) also off |

Note: In the “Example” column, the abbreviations used are: “c” for “semi-,” “q” for “dotted-,” “d” for “quaver,” “c” for “crotchet,” “m” for “minim,” and “b” for “breve.” The $x$-values are expressed in terms of the $\mu$ where possible, and given to three significant figures (3.s.f.) elsewhere. Rounding to three significant figures applies throughout this article. Parentheses enclose solutions which fall outside the acceptable range. In the last column, the abbreviation “Sym.” stands for “Symmetrical.”

metrical window if $0.1 < x < 0.1$. These constraints illustrate one of the difficulties associated with sustaining all the levels of such a metre.

The “Attractor” column lists the attractor tempos in the form of the $x$-values which optimise $\mathcal{M}(x)$ for each metre; and the final column provides some related information about that optimal value. This includes information about the categorisation of metres into groups which may be
optimised by the same $x$-value (a “categorical attractor”). This is the main result of the present article and the subject of the discussion which follows. A brief taxonomy of metrical types will facilitate that discussion.

- “Binary metres” consist exclusively of 2-unit groupings between consecutive levels;
- “ternary metres” include at least one level of 3-unit grouping (all non-binary metres);
- it will be useful to distinguish between metres employing an odd number of metrical levels (“odd metres”) and those with an even number (“even”); and finally
- “symmetrical” structures have a symmetrical pattern of ratios between consecutive levels, for instance $[3,2,3]$. All symmetrical metres are identified on Table 2.

5.2. Odd and even symmetrical metres

We begin with first principles and the simplest possible metres. The model of individual pulse salience in Figure 3 represents the optimising curve for a trivial metre of only one pulse level. Clearly, the optimal $x$-value for this “metre” is $\mu$, the same as for the pulse.

If we add a second metrical level to this pulse, then the optimum changes. Figure 6 shows the interaction of $x$ and $2x$ pulse streams: a binary metre represented by two metrical levels. The image on the left shows the individual pulses $x$ and $2x$ as two versions of Figure 3, as well as their combined curve. This combined $M(x)$ curve shows how the combination is optimised by a compromise between the two individual pulses such that $x = \mu/\sqrt{2} = 0.424$ seconds: a value which positions the two metrical levels at equal distance from the $\mu (0.6$ seconds) as shown by the right-hand image of Figure 6. The mathematics are explained in footnote 31.31 This equidistance generates an equal salience for $S(x)$ and $S(2x)$ which results in a higher combined $M(x)$ value than would have been achieved by optimising either pulse individually at the expense of the other.

If we add a third binary level to give the metrical structure $\langle 1, 2, 4 \rangle$, then we return to a situation where the optimal $M(x)$ value is given by optimising an individual pulse value at the $\mu$. Specifically, $\langle 1, 2, 4 \rangle$ is maximised by centring the metrical structure such that the middle $2x$ level is individually optimised at the $\mu (x = \mu/2)$ as shown in Figure 7.

31 Because we are dealing with a logarithmic scale, the distance between two $x$-values ($x_1$ and $x_2$) is measured by $\log(x_1/x_2)$. If $x_1$ and $x_2$ are consecutive metrical levels on either side of the $\mu$, then $x_2/\mu = \mu/x_1$. It is also the case here that $x_2 = 2x_1$ and so $2x_1/\mu = \mu/x_1$, so $x_1^2 = \mu^2/2$, and $x_1 = \mu/\sqrt{2}$. For a $\mu$ value of 0.6, this gives $x = 0.424$ seconds.
Figure 7. Left: the interaction of three pulse streams in a ratio of 1:2:4 with a peak at $x = \mu/2 = 0.3$. Right: this $x$-value along with the corresponding values for $2x$ and $4x$ on the curve for individual pulse salience, demonstrating their symmetrical centring on the mean.

The correspondence between $\langle 1 \rangle$ and $\langle 1, 2, 4 \rangle$ as distinct from $\langle 1, 2 \rangle$ represents the second main categorical distinction invoked above, between “odd” and “even” symmetrical metres. Odd and even metres are all centred and balanced in this way: odd metres by positioning the central level at the peak; even metres by positioning the central two levels equidistant from the peak on either side. For many symmetrical metres, these centred forms optimise the metrical salience, however this is conditional on the $\sigma$-value – an important parameter to which we now turn.

5.3. The $\sigma$-value

The measurement of distance from $\mu$ can be generalised for two metrical levels in any $n$-relation: $x$ and $nx$ are equidistant from $\mu$ when $x = \mu/\sqrt{n}$. However, that value cannot be generalised as an optimal solution because the shape of the combined curve is highly dependent on the $\sigma$-value: for larger values of $n$, and smaller values of $\sigma$, the model outputs a pair of optima nearer to the optima of the individual curves ($x = \mu$ and $nx = \mu$ respectively).

In Figure 8, the $x$ and $2x$ combination of Figure 6 is reproduced (left) along with an alternative form using a smaller $\sigma$-value (right) for comparison. As the $\sigma$-value is smaller, the individual $S(x)$ curves are narrower, sufficiently so in this case that their combined $M(x)$ curve separates into two distinct peaks, giving two equal maxima. For reference, the line at $x = 0.424$ is reproduced: this is the optimal $x$-value in the left-hand (large $\sigma$) condition of Figure 8, but is a local minimum in the right-hand (small $\sigma$) case. As the $\sigma$-value is increased, the two separate peaks converge, leading to the single, central maximum as in the left-hand case (and Figure 6).

At $\sigma = 0.3$, the combination of $x$ and $3x$ is optimised by $x = \mu/\sqrt{3}$, but the combination of $x$ and $4x$ is not optimised by $x = \mu/\sqrt{4}$. The $\sigma$-value has been set at 0.3 partly because this is a thoroughly appropriate benchmark for that categorical shift. At Parnicutt’s $\sigma$-value of “typically about 0.2” (Parnicutt 1994, 438), the combination of $x$ and $3x$ already separates into distinct regions, yielding the choice of high and low optima (0.152 and 0.590 respectively). That the categorical shift should occur here is inconsistent with intuition and both Fraisse’s and Vos’s results (discussed above). However, that the categorical shift should occur before the combination of $x$ and $4x$ is positively an asset as it gives meaning to the commonly invoked formal requirement that such a combination ought to include an intervening $2x$ level. Were the $\sigma$-value large enough to maximise $x$ and $4x$ at $x = \mu/\sqrt{4} = \mu/2$ then it would make no difference whether the $2x$ level is included or not ($\langle 1, 2, 4 \rangle$ is an odd, symmetrical binary metre also optimised by $x = \mu/2$). As $x = \mu/2$ is not the optimum for this $\sigma$-value, the $2x$ level is essential to the metrical structure.
Cases which lack a single optimum represent the prioritisation of one pulse over another. The effect is magnified when metrical levels are added as whole sets of pulses may group in “sub-metres” that are easier to handle than the whole. Frequently, the “optimal” solutions to these metres suggest an \( x \)-value for which peripheral pulse levels fall outside the metrical window. Once again, this models the difficulty of sustaining all the levels of a large metre and the concomitant tendency to concentrate on optimising a subset at the expense of the peripheral levels. Often, these subsets stand to generate a much higher total \( M(x) \) value than any legitimate solution to the larger metres could. While the \( \sigma \)-value has to be extremely low to affect the outcomes for any binary metres, ternary metres are more susceptible, as the greater ratio makes the peripheral levels concomitantly remote. We therefore turn specifically to consider ternary metres and larger combinations.

5.4. Ternary metres

For the present \( \sigma \)-value, all binary metres are optimised by the categorical \( x \)-values of \( nx = \mu \) for odds, and \( nx = \mu / \sqrt{2} \) for evens, where \( nx \) refers to the metrical level at or immediately below the \( \mu \). Symmetrical ternary metres may be optimised in the same categorical ways; but they are more susceptible to splitting into separate peaks.

For symmetrical odd ternary metres, the \( \sigma \)-value used here is such that \( nx = \mu \) maximises \( \langle 1, 3, 6, 12, 36 \rangle \{3, 2, 2, 3\} \) and \( \langle 1, 3, 9 \rangle \{3, 3\} \), but not \( \langle 1, 2, 6, 18, 36 \rangle \{2, 3, 3, 2\} \). The metre \( \{2,3,3,2\} \) splits into separate peaks not so much because of the number of levels (the same as for \( \{3,2,2,3\} \)), but more due to the preponderance of 3-grouping in the centre which thus separates two \( \langle 1,2 \rangle \) metres. This is an important consideration which applies to many ternary metres, symmetric or otherwise.

Like even binary meters, symmetrical even ternary metres may be optimised by \( x \)-values such that their central levels are equidistant from the \( \mu \). Here, we must distinguish between metres with a binary grouping at their centre, and those with a ternary grouping. The central levels are equidistant from \( \mu \) when the faster pulse stream satisfies \( nx = \mu / \sqrt{2} \) for binary groupings (as discussed) and \( nx = \mu / \sqrt{3} \) for ternary.\(^{32}\) Figures 6 and 9 represent the centre of these two metrical structures to which any symmetrical patterns of levels may be added. Figure 6 sets out the binary relationship (\( nx \) and 2\( nx \) equidistant from the \( \mu \)), while Figure 9 sets out the ternary equivalent for the combination of \( nx \) and 3\( nx \) including the optimum at \( nx = \mu / \sqrt{3} = 0.346 \) seconds. For the present \( \sigma \)-value, the metres \( \langle 1, 3 \rangle \{3\} \) and \( \langle 1, 3, 6, 18 \rangle \{3, 2, 3\} \) are optimised.

\(^{32}\) \( nx \) continues to refer to the metrical level at or immediately below the \( \mu \).
Figure 9. Left: the interaction of two pulse streams in a ratio of 1:3 with a peak at $x = \mu / \sqrt{3} = 0.346$ seconds. Right: this $x$-value along with the corresponding value for $3x$ on the curve for individual pulse salience, demonstrating their equal distance from the mean.

by the symmetrical centring,\(^3\) while $\{1, 2, 6, 12\}[2, 3, 2]$, $\{1, 2, 4, 12, 24, 48\}[2, 3, 2, 2]$, and $\{1, 3, 9, 27\}[3, 3, 3]$ are not. Once again, it is partly the use of many metrical levels, but especially the preponderance of 3-groupings in the centre that causes this.

Table 2 includes all relevant values for these split symmetrical metres: the two paired peak values are given in the “Attractor” column, and the symmetrical centre of these peaks (the categorical attractor) is discussed in “Additional information.” For instance, $\{1, 2, 6, 12\}$ is symmetrical and splits due to the ternary centre. Here, the optimum column shows these two solutions (0.332 and 0.0904), while the “additional information” is that they are equidistant from a categorical attractor at $2x = \mu / \sqrt{3}$.

It is useful to keep sight of the categorical attractors because for a larger $\sigma$-value the categorical attractor would be the optimum: the $M_{\text{max}}$ for split symmetrical metres is conditional on the $\sigma$-value, while the categorical attractors are not. It is also important not to rely unduly on the output optima because for large metres, the optimal tempo which the equation outputs is frequently such that the peripheral metrical levels would fall outside the metrical window. In Table 2, parentheses enclose solutions which fall outside the acceptable range. For instance, the metrical structure $\{1, 2, 4, 12, 24, 48\}$ outputs two solutions equidistant from $4x = \mu / \sqrt{3}$. The ternary separation of $\{1, 2, 4\}$ from $\{12, 24, 48\}$ means that the “optimum” is achieved by positioning one such binary trio much nearer the peak positions (with the central value of those three near the mean), leaving the other trio at unacceptably slow or fast rates.\(^3\) Clearly an “optimum” output of that kind is not a solution to the problem of optimising saliences for the full metrical structure. Rather, it highlights the difficulty of such metres, and the tendency for them to decompose to other, simpler metres with fewer levels. Sub-optimal solutions can be reached by tempos as close to the optimum as is allowable. Some of these are indicated in Table 2 by phrases such as “as fast as possible.”

5.5. Asymmetrical metres and a final kind of metrical category

Asymmetrical metres – those with asymmetrical proportional schemes, such as $\{1, 2, 6\}[2, 3]$ – do not group into these categories of metres optimised by the same $x$-value attractor. However, a kind of categorical pairing does exist. We approach this pairing by beginning once again from first principles. On the model of individual pulse salience, every $x$-value that we might consider

\(^3\) The metre $[3,2,3]$ is the only even ternary metre with a binary centre that can exist within the metrical window.

\(^3\) Note that for some of these metres, one of the optima does fall within the metrical window. The $x = 0.332$ solution to $\{1, 2, 6, 12\}$ generates a metrical structure within the window (as discussed), but the paired solution at $x = 0.0904$ does not.
Figure 10. The interaction of three pulse streams in a ratio of 1:2:6 (left) and 1:3:6 (right) along with their optima at $x = 0.352$ and $x = 0.170$ respectively which are equidistant from the line about which these curves are symmetrical to one another, $x = 0.245$.

(within the 0.1–6 second range) outputs an $S(x)$-value which is equal to the $S(x)$ for exactly one other $x$-value (a two-to-one mapping).\(^{35}\) On a symmetrical model such as that used here, pairs of $x$-values are equivalent when they are equidistant\(^ {36}\) from the $\mu$. By extension, every symmetrical metre has pairs of $x$-values which output the same net metrical weight.\(^ {37}\) So, for two levels of a binary metre, $x_1$ and $2x_1$, there exists a corresponding pair, $x_2$ and $2x_2$, such that $M(x_1) = M(x_2)$. This is because the individual pulses are paired by giving equivalent $S(x)$ values. Here, $S(x_1) = S(2x_2)$ and $S(2x_1) = S(x_2)$. Accordingly, $x_1$ and $x_2$ are in the relationship $x_2 = \mu^2/2x_1$, they are equidistant from the $\mu$.

Asymmetrical metres cannot group into simple categories of single $x$-optima as symmetrical metres may. However, in just the same way as symmetrical metres have (at least) pairs of $x$-values with equivalent output, so asymmetrical metres are paired with other asymmetrical metres. Specifically, asymmetrical metre pairs exist where the proportional schemes are symmetrical to one another. Once again, this is because the constituent pulses themselves output the same salience values.

Figure 10 shows one such pair of metres: $\langle 1, 2, 6 \rangle [2, 3]$ and $\langle 1, 3, 6 \rangle [3, 2]$. Consider the attractors for these metres: $x = 0.352$ for $[2, 3]$ and $x = 0.170$ for $[3, 2]$ (as shown on the two graphs). The $S$-value equivalences here are: $S(1 \times 0.352) = S(6 \times 0.170)$; $S(2 \times 0.352) = S(3 \times 0.170)$; and $S(6 \times 0.352) = S(1 \times 0.170)$. The same equivalence holds for any pair of $x$-values which are equidistant from their symmetrical centre\(^ {38}\) at 0.245.

6. Generalisability and improvements

6.1. The initial modelling parameters’ effect on the final results

It has been stressed that the shape of the pulse salience curve used here is not intended to be definitive, but only as a tool for the heuristic being developed. The initial modelling data provides only approximate boundaries and a sense of the shape. The attractor tempos are, of course, affected by changes to the core model of pulse salience and so it is necessary to understand the nature of these prospective changes. Principal among these are changes to the width $\sigma$ or centre

\(^{35}\) The single exception is $S(\mu)$ which is unique.

\(^{36}\) The distance is calculated exactly as discussed in relation to even binary metres. See the lower part of Figure 6 and surrounding text, for instance.

\(^{37}\) Symmetrical metres with split peaks have more than two solutions for certain $M(x)$ values.

\(^{38}\) The centre is equidistant from $x = 0.352$ and $x = 0.17$ at $\sqrt{0.352 \times 0.170} = 0.245$. 
μ of the current distribution, use of an alternative (asymmetrical) distribution, and the pursuit of a different definition of metrical salience.

The effect of the σ-value has been discussed amply in Section 5.3 above and needs no further comment here. Clearly, were the μ to be positioned at a faster or slower position, the optima it generates for metres would all follow in the same direction by logarithmic increments.\(^{39}\) One motivation for relocating the μ would be if the studies that have given rise to the idea of 0.6 are considered to be flawed as representatives of truly single-pulse metrical acts, and in fact indicative of a preferred solution to the \((1, 2)\) metre, for instance. Given the human tendency towards subjective rhythmisation, and the dominance of binary metres, this is not a trivial matter to unpick.

While some alternative distribution models for salience have also been discussed, their effect on the outcome merits further comment in terms of the model’s generalisability. Within reasonable bounds, the principal observations of this model apply to all related “bell-curve”-esque models. The nature of many pairings and equivalences will still hold, though the exact symmetrical values will clearly not. Any change to the distribution would have two main effects. Firstly, asymmetry would mean all optimal tempos move in the direction of the skew. For instance, an increased positive skewness – that is, a positive skew even on a logarithmic scale – would shift all optima to faster pulses (shorter IOIs).\(^{40}\) Secondly, any change to the shape of the distribution would have a similar effect to the width in that it would adjust the point at which metres decompose from single into separate peaks that may not best serve the metre as a whole. An asymmetric model would cause an asymmetric decomposition – affecting some metres more than others. Nevertheless, the principal categorical observations hold true including the fact of this decomposition and the variables which affect it: particularly the use of many metrical levels, and the preponderance of 3-groupings in the centre. The model thus remains widely applicable to more complex forms of the pulse salience model and provides a simple, useful handle on the principles of metrical salience.

Finally, we turn to the notion of “metrical salience” itself, here defined by the combined saliences of all the pulse streams present in the given structure: a value which also represents the metrical weight for the highest level present. This is taken to be a value worth optimising, though it is not the only conceivable measure of metrical salience. We could perhaps take the combined weights for all metrical positions, but this value would disproportionately favour optimising the lowest levels (which occur the most frequently). More plausibly, some manner of weighting could be given to the metrical levels in play according to the strength of their usage in the musical context. Computational models such as that of Volk (2008) are extremely promising as models of metrical level usage in real musical contexts (or at least digitised score representations thereof),\(^{41}\) though they tend only to account for a restricted set of musical parameters (excluding changes of harmony and orchestration, for instance). They also tend to remain agnostic on perceptual matters, thus neglecting to address the issues explored here. Therefore, a combination of context-specific level weighting with the present model of metrical salience is a very promising prospect (if used with analytical common sense). The fact that the model advanced here is already in broad agreement with the average frequency of metrical position usage (as shown in Figure 5 above) is most encouraging.

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\(^{39}\) For instance, increasing the μ from 0.6 to 0.7 would increase the \(\mu/\sqrt{2}\) value from 0.424 to 0.495.

\(^{40}\) This would be one effect of using the model from van Noorden and Moelants (1999), for instance.

\(^{41}\) See also the “pulse clarity measure” in Lartillot, Toiviainen, and Eerola’s (2008) “MIR toolbox.”
6.2. Refinement of the model by cognitive and corpus testing

The previous section (6.1) discussed the fact that many of the core observations of this model are relatively independent of the exact modelling values used. This justifies and gives strength to the heuristic observations made. Nevertheless, it is clearly desirable to test this model in order to hone its accuracy as far as realistically possible.

The model invites many kinds of empirical testing, particularly of the principal hypothesis that each metrical context may be made relatively easy or difficult to engage by the choice of tempo. Most simply, subjects may be presented with stimuli in different metrical configurations, and invited either to assess whether they consider the example to be “fast” or “slow,” or to select a preferred tempo for the extract independently. According to the hypothesis, preference ratings would reflect the shapes of the metrical salience curves for the metrical structure in question. Alternatively, subjects could be required to synchronise with given tempo–metrical contexts, or to observe timing perturbations contained therein; one would expect the performance to be best at tempos nearest the attractors.

It would also be desirable to assess the hypotheses in relation to real musical practice in addition to musically-impoverished experimental stimuli of questionable ecological validity. Corpora of works and performances are an attractive source of evidence, though that approach also comes with a number of methodological warnings. Firstly, as has been emphasised throughout, the “attractor” tempos are designed only to model a default, easiest practice, and certainly not the “correct” tempos for musical expression (which is often reliant on a deliberate distancing from easy, normative practice). In short, non-optimal “fast” and “slow” tempos are at least as common as “moderate” ones that align with their attractor. Even a very large sample may provide more evidence of congregation at the extremes than of the default, easiest practice modelled here. Secondly, the model advanced in this article places great store on the metrical structure and level usage as determinants of tempo choice. Clearly then, any corpus used to assess this model would need to include that information. This rules out studies like that of van Noorden and Moelants (1999) which include only tempo data.\footnote{Though those results may illuminate aspects of the binary metrical circumstances which dominate most of the genres studied there.} Instead, this necessitates the use of advanced music information retrieval techniques, such as in Volk (2008), discussed above.

7. Summary and exemplary musical applications

7.1. Summary

This article began by setting out a quantification of pulse salience (Section 2) to weight the significance of each metrical level according to its salience. That model was then used to develop a further model for metrical salience (Section 4) which enabled the deduction of general principles for optimising net pulse salience in simple metres (Section 5) and thus identifying “attractor” tempos to which those metres may be drawn. Metrical salience values depend on the tempo, the metrical structure, and the number of levels represented. All distinct metres have distinct metrical salience curves; however there are three main categories of structurally similar metres which may be optimised by the same attractor tempo. Figure 11 summarises the family structure of these categories.

This structure is intriguingly divergent from traditional accounts of metrical categories. Here, the primary criterion for metrical identity is not the customary distinction between binary and ternary metres (2/4 \textit{versus} 3/4, for instance), nor even the number of metrical levels represented,
Figure 11. A flowchart for metrical categories relevant to optimising. The level at or immediately below $\mu$ which optimises the metrical salience for metres in a category is given by $nx$; the multiplier $n$ depends on the number of levels in use. The grouping of the central level of symmetrical even metres is given by “[2]” and “[3]” at the lowest level of the chart.

but rather the presence or absence of a symmetrical proportional scheme among the levels. All three categorical attractors for maximising metrical salience exist within the class of symmetrical metres. Asymmetric metres do not group in metrical categories of that kind, though equivalent pairs of asymmetric metres are related by the shape of their metrical salience curves. The categorical attractors for the symmetrical metres are given at the bottom of Figure 11, and are shown in Figures 7, 6, and 9 respectively. The width of the original pulse salience curve dictates the limit of applicability for these categorical attractors (see Section 5.3). Metres with a large number of metrical levels, and especially those with ternary groupings in the centre are harder to sustain, and more liable to “optimise” at values which prioritise some sub-structure rather than the whole.

7.2. Musical illustrations: short, fundamental examples

The model is ultimately useful in so far as it is able to elucidate fundamental musical considerations. By way of conclusion, this section provides some brief illustrations of how the model appears to do so, and suggests avenues for future application.

At the heart of the model is the notion that adding a faster metrical level (further level of subdivision) generates a slower optimal tempo to accommodate (and vice versa – adding slow hypermetrical levels leads to a faster optimum). Figure 12 provides a simple illustration of this principle which is also suggestive of a possible stylistic application. In version (a), the fastest pulse level involved is the crotchet. If this is changed to a dotted rhythm as in version (b), the quaver level becomes involved, giving a metrical structure with a slower attractor tempo (assuming equal importance of the levels). In version (c) the rhythm is double-dotted, invoking the semi-quaver level, leading to a new, even slower optimal. This may contribute to an understanding of why (double-)dotted rhythms have come to be associated with grand (slow) styles such as the “French overture.” For instance, this example is taken from the opening of Mozart’s Così Fan Tutte in which version (c) is the form used (and in which that semi-quaver level is also used melodically, not just in these opening tutti chords).
Figure 12. Three versions of the same example, with additional metrical levels leading to slower attractor tempos.

Figure 13. A visual representation of what it means to be slow: the fastest plausible tempo for Barber’s Adagio.

This connects directly with some of the bases for tempo choice expressed by practising musicians since at least Quantz and Kirnberger in the eighteenth century. Throughout Quantz’s account of tempo (paragraph 55ff.), the suggested tempos are related to both the metrical structure, and the fastest pulse level used. Kirnberger ([1776–1779] 1982, [105ff.] 375ff.) similarly asserts that the tempo for each dance is “determined by the metre and the note values that are employed in it,” also focusing on the faster stream. These are direct precursors to the present, technical model of attractor tempos (which also considers the highest metrical level used as part of its formalisation).

The attractors enable even a principled definition of what it means for music to be “fast” or “slow.” For instance, what do we mean when we describe Barber’s Adagio for Strings as a “slow” piece? Well, the time signature is 4/2, implying a minim pulse and the metrical levels used include at least the crotchet (the shortest note-value used), minim, semi-breve, and breve. This combination is maximised by the $\mu/\sqrt{2}$ value, giving a minim value of 0.424 seconds (141 bpm). Yet, the tempo designation is *molto adagio*, presumably referring to that minim pulse. The fastest metronome mark that could still be considered *adagio* is about 75 bpm which equates to a 0.8 seconds IOI. This (fastest plausible) tempo is shown in Figure 13 in which it falls to the right side of centre.

Clearly even this is much slower than the (symmetrically centred) optimum. The use of longer bars in the piece (5/2, 6/2), the possible addition of hypermetrical levels, and acknowledgement of “molto” adagio (as distinct from merely adagio) all further this separation. The piece is necessarily slow for any realistic tempo within the range prescribed; Barber’s notation instructs

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43Quantz and Kirnberger’s major treatises date from 1752 (“Versuch einer Anweisung die Flöte traversiere zu spielen”) and 1776–1779 (“Die Kunst des reinen Satzes in der Musik”) respectively. See Quantz ([1752] 1966) and Kirnberger ([1776–1779] 1982) for the modern, translated editions with commentary that are referenced here.

44For instance, “each of the types of metre cited can be put into its proper tempo”; to do so “it is most important to consider both the word indicating the tempo at the beginning of the piece and the fastest notes used in the passage-work” (paragraph 51ff., my emphasis).

45Compare this with Kirnberger’s ([1776–1779] 1982, [107] 377) characterisation of tempo adjectives as a means to “add or take away from the fast or slow motion of the natural tempo” – the “attractor tempo” in this article’s terminology.
the performer to use a slow tempo, far from the attractor. In short, “slowness” has a lot to do with absolute tempo, of course, but just as much to do with the metrical structure: the Barber is slow exactly because it relies on metrical levels above the minim tactus. The attractor tempos provide useful benchmarks for assessing this.

This is one kind of insight concerning expressive deviation from a default (easiest) range that the present model makes possible (and note once again that we only need a very approximate understanding of the best distribution for the shape of pulse salience to make the observation). The example also emphasises for the last and final time the difference between the “attractor” tempos set out here (representing a kind of default, easiest practice) and the “correct” tempo for musical expression in a given context: clearly this music derives much of its expressive beauty from its slowness.

Even if one were deliberately to seek out these attractor tempos wherever possible as if they were correct, the vast majority of music changes metrical level usage – and often also the metrical structure itself – within sections governed by the same tempo. Pulse levels come and go and local changes such as hemiolas are used; each new structure brings its own attractor tempo which aligns to greater or lesser extent with whatever tempo is actually used in performance. This guarantees a play of tension–relaxation as one moves between relatively optimised and non-optimised tempo–metrical alignments – an expressive device which can now be articulated and assessed by means of the attractor tempo model advanced here.46

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Supplemental online material

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