Superconformal Ward identities are derived for the the four point functions of chiral primary BPS operators for $\mathcal{N} = 2,4$ superconformal symmetry in four dimensions. Manipulations of arbitrary tensorial fields are simplified by introducing a null vector so that the four point functions depend on two internal $R$-symmetry invariants as well as two conformal invariants. The solutions of these identities are interpreted in terms of the operator product expansion and are shown to accommodate long supermultiplets with free scale dimensions and also short and semi-short multiplets with protected dimensions. The decomposition into $R$-symmetry representations is achieved by an expansion in terms of two variable harmonic polynomials which can be expressed also in terms of Legendre polynomials. Crossing symmetry conditions on the four point functions are also discussed.

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1. Introduction

Since the discovery of the AdS/CFT correspondence there has been a huge resurgence of interest in superconformal theories in four dimensions, for a review see [1]. In particular for the $\mathcal{N} = 4$ superconformal $SU(N)$ gauge theory, to which the AdS/CFT correspondence is most directly applicable, many new and exciting results have been obtained. Much has been discovered concerning the spectrum of operators and their scale dimensions both in the large $N$ limit, through the supergravity approximation to the AdS/CFT correspondence, and also perturbatively as an expansion in the coupling $g$.

Of the operators present in the theory the simplest are the chiral primary operators belonging to $SU(4)_R$ R-symmetry representations, with Dynkin labels $[0, p, 0]$. These are represented by symmetric traceless rank $p$ tensors formed by gauge invariant traces of the elementary scalar fields and satisfy BPS like constraints so that they belong to short supermultiplets of the superconformal group $PSU(2, 2|4)$. They are therefore protected against renormalisation effects and have scale dimension $\Delta = p$. Their three point functions have been fully analysed [2]. For the case of $p = 2$, when the supermultiplet contains the energy momentum tensor, the four point functions have been found both perturbatively [3] and in the large $N$ limit [4]. Such results have also been extended more recently to chiral primary operators with $p = 3, 4$ [5,6,7]. The explicit results for the four point correlation functions has then allowed an analysis of those operators which contribute to the operator product expansion for two chiral primary operators [8,9,10,11,12,13,14].

To take the analysis of the operator product expansion of correlation functions beyond the lowest scale dimension operators for each $SU(4)_R$ representation it is necessary to have an explicit form for the conformal partial waves which give the contribution of a quasi-primary operator of arbitrary scale dimension and spin and all its conformal descendants to conformally covariant four point functions. In four dimensions a simple expression was found in [15]. In addition since all operators in a superconformal multiplet must have the same anomalous dimensions it is desirable to have a procedure for analysing the operator product expansion for each supermultiplet as a single contribution. This depends on a solution of all superconformal Ward identities since this should allow all possible operator product expansion contributions to be found in a form compatible with the superconformal symmetry. For the simplest case of the four point function for $[0, 2, 0]$ chiral primary operators this was undertaken in [16] and applied to determine the one loop anomalous dimensions for all operators with lowest order twist two.

The procedure adopted in [16] is somewhat involved and does not simply generalise to correlation functions of more general chiral primary operators. As was shown in [16] the superconformal Ward identities are simplified if they are expressed in terms of new variables.
In terms of the standard correspondence for the space-time coordinates $x^a \to x = x^a \sigma_a$, where $x$ is a $2 \times 2$ spinorial matrix such that $\det x = -x^2$, then, for four points $x_1, x_2, x_3, x_4$ and $x_{ij} = x_i - x_j$, $x, \bar{x}$ may be defined, as shown in [7], as the eigenvalues of $x_{12} x_{42}^{-1} x_{43} x_{13}^{-1}$. By conformal transformations we may choose a frame such that $x_2 = 0, x_3 = \infty, x_4 = 1$ and $x_1 = \begin{pmatrix} x & 0 \\ 0 & \bar{x} \end{pmatrix}$. The two conformal invariants are then given in terms of $x, \bar{x}$ by

$$u = \det(x_{12} x_{42}^{-1} x_{43} x_{13}^{-1}) = x \bar{x},$$
$$v = \det(1 - x_{12} x_{42}^{-1} x_{43} x_{13}^{-1}) = \det(x_{14} x_{24}^{-1} x_{23} x_{13}^{-1}) = (1 - x)(1 - \bar{x}).$$

(1.1)

For a Euclidean metric on space-time $x, \bar{x}$ are complex conjugates. In our analysis we find that the linear equations which follow from superconformal invariance naturally separate into ones involving just $x$ and conjugate equations with $x \to \bar{x}$.

A technical complication in dealing with arbitrary chiral primary operators represented by symmetric traceless tensorial fields $\varphi_{r_1 \ldots r_p}$ is that the four point function for four chiral primary fields, for arbitrary $p_1, p_2, p_3, p_4$, involves in general a proliferation of independent tensorial invariants as the $p_i$ are increased. The construction of projection operators corresponding to different $R$-symmetry representations also becomes a non-trivial exercise. Such tensorial complications in the analysis of superconformal Ward identities and also in applying the operator product expansion are avoided here by taking

$$\varphi_{r_1 \ldots r_p}(x) \to \varphi^{(p)}(x, t) = \varphi_{r_1 \ldots r_p}(x) t_{r_1} \ldots t_{r_p},$$

(1.2)

where $t$ is an arbitrary complex vector satisfying

$$t^2 = 0. \quad (1.3)$$

(For a more mathematical discussion of using such vectors for the treatment of representations of $SO(n)$ see [18], see also appendix A in [19]). Clearly $\varphi_{r_1 \ldots r_p}$ can be recovered from $\varphi^{(p)}$. The four point function then becomes a homogeneous polynomial in $t_1, t_2, t_3, t_4$, of respective degree $p_1, p_2, p_3, p_4$, invariant under simultaneous rotations on all $t_i$'s. Due to

---

1 Since $1 + u - v = x + \bar{x}$ and $1 + u^2 + v^2 - 2uv - 2u - 2v = (x - \bar{x})^2$ it is easy to invert these results to obtain $x, \bar{x}$ in terms of $u, v$ up to the arbitrary sign of the square root of the square root $\sqrt{(x - \bar{x})^2}$. For any $f(u, v)$ there is a corresponding symmetric function $\hat{f}(x, \bar{x}) = \hat{f}(\bar{x}, x)$ such that $\hat{f}(x, \bar{x}) = f(u, v)$.

2 Such null vectors may also be motivated by considering the harmonic superspace approach and were used similarly for instance in [33]. Our application is independent of the harmonic superspace formalism and is essentially motivated just by the requirement of simplifying the treatment of arbitrary rank symmetric traceless tensors, we do not anywhere consider the conjugate of $t$.  

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2
the condition \[(1.3)\] for each \(t_i\) the conformally covariant four point function is reducible to an invariant function \(F(u, v; \sigma, \tau)\) with \(\sigma, \tau\) the two independent invariants, homogeneous of degree zero in each \(t_i\), which are analogous to the conformal invariants \(u, v\),

\[
\sigma = \frac{t_1 \cdot t_3 \cdot t_2 \cdot t_4}{t_1 \cdot t_2 \cdot t_3 \cdot t_4}, \quad \tau = \frac{t_1 \cdot t_4 \cdot t_2 \cdot t_3}{t_1 \cdot t_2 \cdot t_3 \cdot t_4}. \tag{1.4}
\]

In general \(F(u, v; \sigma, \tau)\) is a polynomial in \(\sigma, \tau\), with degree determined by the \(p_i\), where the number of independent terms match exactly the number of tensorial invariants necessary for the general decomposition of the four point function for the corresponding symmetric traceless tensorial fields, for \(p_i = p\) there are \(\frac{1}{2}(p + 1)(p + 2)\) terms.

Just as the invariants \(u, v\) are expressed in terms of \(x, \bar{x}\) it is convenient to write \(\sigma, \tau\) in a similar form involving new variables \(\alpha, \bar{\alpha}\),

\[
\sigma = \alpha \bar{\alpha}, \quad \tau = (1 - \alpha)(1 - \bar{\alpha}). \tag{1.5}
\]

For the \(N = 2\) case further restrictions impose \(\alpha = \bar{\alpha}\). The superconformal Ward identities are then simply expressed in terms of \(\hat{F}(x, \bar{x}; \alpha, \bar{\alpha}) = F(u, v; \sigma, \tau)\), with \(\hat{F}\) symmetric in \(x, \bar{x}\) and also a symmetric polynomial in \(\alpha, \bar{\alpha}\). The superconformal identities constrain \(\hat{F}(x, \bar{x}; \alpha, \bar{\alpha})\) for \(\alpha = 1/x\) to be expressible only in terms of a function involving \(\bar{x}\) and \(\bar{\alpha}\) (for \(N = 2\) just a single variable function of \(\bar{x}\) appears). Taking into account the symmetry conditions the unconstrained or dynamical part of \(F\) is therefore of the form

\[
F(u, v; \sigma, \tau)_{\text{dynamical}} = (\alpha x - 1)(\alpha \bar{x} - 1)(\bar{\alpha} x - 1)(\bar{\alpha} \bar{x} - 1) \mathcal{H}(u, v; \sigma, \tau), \tag{1.6}
\]

with \(\mathcal{H}(u, v; \sigma, \tau) = \hat{\mathcal{H}}(x, \bar{x}; \alpha, \bar{\alpha})\) a polynomial in \(\sigma, \tau\), or \(\alpha, \bar{\alpha}\), of reduced degree. In the \(N = 2\) the corresponding result is

\[
F(u, v; \sigma, \tau)_{\text{dynamical}} = (\alpha x - 1)(\alpha \bar{x} - 1) \mathcal{H}(u, v; \sigma, \tau), \quad \mathcal{H}(u, v; \sigma, \tau) = \hat{\mathcal{H}}(x, \bar{x}; \alpha). \tag{1.7}
\]

Similar results were previously obtained by Heslop and Howe \[17\] based on expansions in terms of Schur polynomials for SU(2, 2|2) and PSU(2, 2|4)\[18\]. Using the formalism of harmonic superspace \[20\] also provides a method for deriving superconformal identities which are equivalent to those obtained here.

These results have a natural interpretation in terms of the operator product expansion when the four point function is expanded in terms of conformal partial waves corresponding to operators with various scale dimensions \(\Delta\) and spins \(\ell\) belonging to the various

\[\text{In eq. (49) of \[17\] } S_{020}(Z) = (X_1 - Y_1)(X_1 - Y_2)(X_2 - Y_1)(X_2 - Y_2) \text{ which appears as an overall factor in the Schur polynomial for long representations.} \]
possible representations of the $R$-symmetry group. The conformal partial waves are explicit functions of $u, v$, more simply given in terms of $x, \bar{x}$ \cite{13}. To disentangle the different $R$-symmetry representations the correlation functions are also expanded in terms of two variable harmonic polynomials corresponding to the possible $R$-symmetry representations which may be formed. Explicit simple expressions are found here for these harmonic polynomials in the $\mathcal{N} = 4$ case using the variables $\alpha, \bar{\alpha}$ (for $\mathcal{N} = 2$ they reduce to a single variable Legendre polynomial). If $\mathcal{H}$ in (1.4) is simultaneously expanded in such harmonic polynomials and conformal partial waves then the factors multiplying $\mathcal{H}$, for each term in the expansion, through various recurrence relations generate contributions corresponding to all operators belonging to a single superconformal long multiplet. For such a long multiplet the scale dimension has only a lower bound due to unitarity and, in a perturbative expansion, $\mathcal{H}$ therefore includes dynamical renormalisation effects leading to anomalous scale dimensions. The remaining parts in the solution of the superconformal identities which involve functions of $x$ or $\bar{x}$ are also analysed here. For $\mathcal{N} = 2$ there is a single variable function $f(x)$ whereas for $\mathcal{N} = 4$ the superconformal identities allow for $f(x, \alpha)$, which is polynomial in $\alpha$ and satisfies the constraint that it is a constant $k$ when $\alpha = 1/x$. These functions correspond to semi-short multiplets with protected scale dimensions, determined by $\ell$ and the $R$-symmetry representation, and also in special cases to short multiplets. The full set of possible semi-short supermultiplets are obtained by decomposing long multiplets at the unitarity threshold and the short multiplet contributions are realised by extending the semi-short results to $\ell = -1, -2$.

The superconformal Ward identities are most powerful for four point functions which are extremal, so that there is only one possible $SU(4)_R$ invariant coupling, or next-to-extremal when there are just three invariant couplings. Various calculations \cite{21} have shown that the correlation functions are identical with the results obtained in free field theory. The superconformal Ward identities for the extremal case show that the correlation function depends only on the constant $k$ whereas the next-to-extremal case is given just by the function $f(x, \alpha)$, without any dynamical contribution of the form exhibited in (1.7).

For four point functions of identical primary operators there are further constraints arising from crossing symmetry which corresponds to the permutation group $S_3$. In such four point functions $S_3$ acts on $u, v$ and also $\sigma, \tau$ so that $\mathcal{F}(u, v; \sigma, \tau)$ is invariant up to an explicit overall factor. All invariant contributions to $\mathcal{F}$ may be formed by combining $S_3$ irreducible representations constructed from functions of $u, v$ and also $\sigma, \tau$. Crossing symmetry further constrains the single variable functions $f(x)$ or $f(x, \alpha)$ that arise in solving the superconformal Ward identities. We argue here, based on superconformal representation theory and analyticity requirements, that these can be extended to a fully crossing symmetric contribution to $\mathcal{F}(u, v; \sigma, \tau)$ in terms of two variable crossing symmetric...
polynomials which are constructed here. These essentially correspond to generalised free field contributions to the correlation function. A similar discussion is also applicable to the next-to-extremal correlation function in the case of the identical operators and there is still a $S_3$ symmetry.

In detail the structure of this paper is then as follows. In section 2 we derive the superconformal Ward identities for $\mathcal{N} = 2$ superconformal symmetry and these are applied in section 3 by analysing the contributions of different supermultiplets in the operator product expansion. The discussion is extended to the $\mathcal{N} = 4$ case in sections 4 and 5. For the operator product expansion it is shown how there are potential contributions from non-unitary semi-short supermultiplets although they may be cancelled so that only unitary multiplets remain. In section 6 we take into account the restrictions imposed by crossing symmetry making use of $S_3$ representations. In section 7 we summarise some results obtained previously for large $N$ in the framework of this paper and a few comments are made in a conclusion. Various technical issues are addressed in four appendices. In appendix A we discuss how derivatives involving the null vector $t_r$ are compatible with $t^2 = 0$. In appendix B we consider two variable harmonic polynomials, depending on $\sigma, \tau$ given in (1.4), which are used in the expansion of general four point correlation functions. Appendix C describes some differential operators which play an essential role in our analysis whereas in appendix D we consider non-unitary semi-short representations for $PSU(2,2|4)$ which are important in our operator product analysis.

2. Superconformal Ward Identities, $\mathcal{N} = 2$

The algebraic complications involved in the analysis of Ward identities are much simpler for $\mathcal{N} = 2$ superconformal symmetry. In this case the $R$-symmetry group is just $U(2)$ and discussion of the representations is much easier. In order to facilitate the comparison with the $\mathcal{N} = 4$ case later we consider BPS chiral primary operators which belong to representations of $SU(2)_R$ symmetry for $R = n$, an integer. The BPS condition requires that the scale dimension $\Delta = 2n$. Such fields form superconformal primary states for a short supermultiplet with necessarily unrenormalised scale dimensions. The fields in this case are represented by symmetric traceless tensors $\varphi_{r_1...r_n}$ with $r_i = 1, 2, 3$. To derive the Ward identities we need to consider just the superconformal transformations at the lowest levels of the multiplet. First

$$\delta \varphi_{r_1...r_n} = \hat{e} \tau(r_n \psi_{r_1...r_{n-1}} + \bar{\psi}(r_1...r_{n-1} \tau_{r_n}) \hat{\bar{e}}), \quad (2.1)$$

where $\psi_{r_1...r_{n-1} \alpha}, \bar{\psi}_{r_1...r_{n-1} \dot{\alpha}}$ are spinor fields, traceless and symmetric on the indices $r_1...r_{n-1}$, satisfying, with $i = 1, 2$ and $\tau_r$ the usual Pauli matrices,

$$\tau_r \psi_{rr_1...r_{n-2}} = 0, \quad \bar{\psi}_{r_1...r_{n-2}r} \tau_r = 0. \quad (2.2)$$
Thus both $\psi$ and $\bar{\psi}$ belong to $SU(2)_R$ representations with $R = n - \frac{1}{2}$. In (2.1) we have

$$\hat{\epsilon}^\alpha_i(x) = \epsilon^\alpha_i - i \bar{n}_i \hat{x}^\alpha \, , \quad \hat{\epsilon}^{i\dot{\alpha}}(x) = \bar{\epsilon}^{i\dot{\alpha}} + i \hat{x}^{\dot{\alpha}} \eta_i^\alpha . \quad (2.3)$$

where $\epsilon^\alpha_i, \bar{n}_i, \bar{\epsilon}^{i\dot{\alpha}}, \eta_i^\alpha$ are the $R = \frac{1}{2}$ anticommuting parameters for an $\mathcal{N} = 2$ superconformal transformation. In addition to (2.1) we use

$$\delta \psi_{r_1...r_{n-1}1} = i \partial_{\alpha \dot{\alpha}} \varphi_{r_1...r_{n-1}s} \tau_s \hat{\epsilon}^\alpha - \frac{n-1}{2n-1} \tau_{(r_1} J_{r_2...r_{n-1}s) \alpha \dot{\alpha}} \tau_s \hat{\epsilon}^\alpha , \quad (2.4)$$

where $J_{r_1...r_{n-1} \alpha \dot{\alpha}}$, a symmetric traceless rank $n-1$ tensor, is a $R = n-1$ current. Using (2.3) together with its conjugate we may verify closure of the superconformal algebra acting on $\varphi_{r_1...r_n}$,

$$[\delta_2, \delta_1] \varphi_{r_1...r_n} = -v \cdot \varphi_{r_1...r_n} - n(\sigma + \bar{\sigma}) \varphi_{r_1...r_n} + n t_{r|s} \varphi_{r_1...r_{n-1}s} , \quad (2.5)$$

where $v^a$, which is quadratic in $x$, and $\sigma, \bar{\sigma}, t_{rs} = -t_{sr}$, which are linear in $x$, are constructed from $\hat{\epsilon}_1, \hat{\epsilon}_1, \hat{\epsilon}_2, \hat{\epsilon}_2$.

For the general analysis here we define $\varphi^{(n)}(x,t)$ as in (1.2), where $t_r$ is here a 3-vector, and in a similar fashion also $\psi^{(n-1)}(x,t) = \psi_{r_1...r_{n-1}1}(x) t_1...t_{n-1}$ and $
abla^{(n-1)}(x,t) = \psi_{r_1...r_{n-1}1}(x) t_1...t_{n-1}$ while $J^{(n-1)}(x,t) = J_{r_1...r_{n-1}1}(x) t_1...t_{n-1}$. With this notation (2.1) may be rewritten as

$$\delta \varphi^{(n)}(t) = \hat{\epsilon} \cdot t \psi^{(n-1)}(t) + \bar{\psi}(n-1)(t) \cdot t \hat{\epsilon} , \quad (2.6)$$

and (2.4) becomes

$$\delta \psi^{(n-1)}(t) = \frac{1}{n} \tau \cdot \partial_t \left( i \partial_{\alpha \dot{\alpha}} \varphi^{(n)}(t) \hat{\epsilon}^\alpha + 4 \tau \cdot \partial_t \varphi^{(n)}(t) \eta_\alpha \right)$$

$$+ \left( 1 - \frac{1}{2n-1} \right) \tau \cdot t \tau \cdot \partial_t J_{\alpha \dot{\alpha}}^{(n-1)}(t) \hat{\epsilon}^\alpha . \quad (2.7)$$

A precise form for differentiation with respect to $t_r$ satisfying (1.3) is given in appendix A. The conditions (2.2) are now

$$\tau \cdot \partial_t \psi^{(n-1)}(t) = 0 , \quad \bar{\psi}^{(n-1)}(t) \tau \cdot \partial_t = 0 . \quad (2.8)$$

---

4 Thus 4-vectors are identified with $2 \times 2$ matrices using the hermitian $\sigma$-matrices $\sigma_a, \bar{\sigma}_a, \sigma_a \bar{\sigma}_b = -\eta_{ab} 1, \ x^a \rightarrow x_{\alpha \dot{\alpha}} = x^a (\sigma_a)_{\alpha \dot{\alpha}}, \ \tilde{x}^{\dot{\alpha} \alpha} = x^a (\bar{\sigma}_a)^{\dot{\alpha} \alpha} = \epsilon^{\alpha \beta} \epsilon_{\dot{\alpha} \dot{\beta}} x_{\beta \dot{\beta}}$, with inverse $x^a = -\frac{1}{2} \text{tr}(\sigma^a \tilde{x})$. We have $x \cdot y = x^a y_a = -\frac{1}{2} \text{tr}(\tilde{x} y)$, $\text{det} x = x^2$, $x^{-1} = -\tilde{x}/x^2$. 

6
The four point correlation functions of interest here then have the form

\[
\langle \varphi^{(n_1)}(x_1, t_1) \varphi^{(n_2)}(x_2, t_2) \varphi^{(n_3)}(x_3, t_3) \varphi^{(n_4)}(x_4, t_4) \rangle = \frac{r_{23}^{\Sigma - 2n_2 - 2n_3} r_{34}^{\Sigma - 2n_3 - 2n_4}}{r_{13}^{2n_1} r_{24}^{\Sigma - 2n_3}} F(u, v; t), \quad \Sigma = n_1 + n_2 + n_3 + n_4, \tag{2.9}
\]

where

\[ r_{ij} = (x_i - x_j)^2, \tag{2.10} \]

and \(u, v\) are the two independent conformal invariants,

\[
u = \frac{r_{12} r_{34}}{r_{13} r_{24}}, \quad v = \frac{r_{14} r_{23}}{r_{13} r_{24}}, \tag{2.11}
\]

which is equivalent to (1.1). \(F(u, v; t)\) is also a SU(2)\(_R\) scalar which is specified further later, clearly there is a freedom to modify it by suitable powers of \(u\) or \(v\) at the expense of changing the terms involving \(r_{ij}\) in (2.9). The choice made on (2.9) has some convenience in the later discussion.

The fundamental superconformal Ward identities arise from expanding

\[
\delta \langle \bar{\psi}_\alpha^{(n_1 - 1)}(x_1, t_1) \varphi^{(n_2)}(x_2, t_2) \varphi^{(n_3)}(x_3, t_3) \varphi^{(n_4)}(x_4, t_4) \rangle = 0, \tag{2.12}
\]

using (2.6) and (2.7). This gives, suppressing the arguments \(t_i\) for the time being,

\[
\frac{1}{n_1} i \partial_{\alpha \dot{\alpha}} \tau \cdot \frac{\partial}{\partial t_1} \langle \varphi^{(n_1)}(x_1) \varphi^{(n_2)}(x_2) \varphi^{(n_3)}(x_3) \varphi^{(n_4)}(x_4) \rangle \hat{\epsilon}^\dot{\alpha}(x_1) \\
+ 4 \tau \cdot \frac{\partial}{\partial t_1} \langle \varphi^{(n_1)}(x_1) \varphi^{(n_2)}(x_2) \varphi^{(n_3)}(x_3) \varphi^{(n_4)}(x_4) \rangle \eta_\alpha \\
+ \left( 1 - \frac{1}{2n_1 - 1} \tau \cdot t_1 \tau \cdot \frac{\partial}{\partial t_1} \right) \langle J^{(n_1 - 1)}_{\alpha \dot{\alpha}}(x_1) \varphi^{(n_2)}(x_2) \varphi^{(n_3)}(x_3) \varphi^{(n_4)}(x_4) \rangle \hat{\epsilon}^\dot{\alpha}(x_1) \tag{2.13}
\]

\[
+ \langle \bar{\psi}_\alpha^{(n_1 - 1)}(x_1) \bar{\psi}_\alpha^{(n_2 - 1)}(x_2) \varphi^{(n_3)}(x_3) \varphi^{(n_4)}(x_4) \rangle \tau \cdot t_2 \hat{\epsilon}^\dot{\alpha}(x_2) \\
+ \langle \bar{\psi}_\alpha^{(n_1 - 1)}(x_1) \varphi^{(n_2)}(x_2) \bar{\psi}_\alpha^{(n_3 - 1)}(x_3) \varphi^{(n_4)}(x_4) \rangle \tau \cdot t_3 \hat{\epsilon}^\dot{\alpha}(x_3) \\
+ \langle \bar{\psi}_\alpha^{(n_1 - 1)}(x_1) \varphi^{(n_2)}(x_2) \varphi^{(n_3)}(x_3) \bar{\psi}_\alpha^{(n_4 - 1)}(x_4) \rangle \tau \cdot t_4 \hat{\epsilon}^\dot{\alpha}(x_4) = 0.
\]

To apply this we make use of general expressions compatible with conformal invariance for
each four point function which appears. Thus

\[
\left\langle \psi^{(n_1-1)}_\alpha (x_1) \bar{\psi}^{(n_2-1)}_\alpha (x_2) \varphi^{(n_3)} (x_3) \varphi^{(n_4)} (x_4) \right\rangle
= 2i \frac{r_{23}^{\Sigma-2n_2-2n_3} r_{34}^{\Sigma-2n_3-2n_4}}{r_{12}^{2n_1} r_{24}^{\Sigma-2n_3}} \left( \frac{1}{r_{12}} x_{12\alpha \dot{\alpha}} R_2 + \frac{1}{r_{13} r_{24}} (x_{13} \bar{x}_{34} x_{42})_{\alpha \dot{\alpha}} S_2 \right),
\]

\[
\left\langle \psi^{(n_1-1)}_\alpha (x_1) \varphi^{(n_2)} (x_2) \bar{\psi}^{(n_3-1)}_\dot{\alpha} (x_3) \varphi^{(n_4)} (x_4) \right\rangle
= 2i \frac{r_{23}^{\Sigma-2n_2-2n_3} r_{34}^{\Sigma-2n_3-2n_4}}{r_{12}^{2n_1} r_{24}^{\Sigma-2n_3}} \left( \frac{1}{r_{13}} x_{13\alpha \dot{\alpha}} R_3 + \frac{1}{r_{14} r_{23}} (x_{14} \bar{x}_{42} x_{23})_{\alpha \dot{\alpha}} S_3 \right),
\]

\[
(2.14)
\]

\[
\left\langle \psi^{(n_1-1)}_\alpha (x_1) \varphi^{(n_2)} (x_2) \varphi^{(n_3)} (x_3) \bar{\psi}^{(n_4-1)}_\dot{\alpha} (x_4) \right\rangle
= 2i \frac{r_{23}^{\Sigma-2n_2-2n_3} r_{34}^{\Sigma-2n_3-2n_4}}{r_{12}^{2n_1} r_{24}^{\Sigma-2n_3}} \left( \frac{1}{r_{14}} x_{14\alpha \dot{\alpha}} R_4 + \frac{1}{r_{13} r_{24}} (x_{13} \bar{x}_{32} x_{24})_{\alpha \dot{\alpha}} S_4 \right),
\]

where \( R_n, S_n \) are functions of \( u, v \) and also scalars formed from \( t_i \) (to verify completeness of the basis chosen in (2.14) we use relations such as \( x_{13} \bar{x}_{34} x_{42} + x_{14} \bar{x}_{43} x_{32} = r_{34} x_{12} \)). In addition we have

\[
\left\langle J^{(n_1-1)}_{\alpha \dot{\alpha}} (x_1) \varphi^{(n_2)} (x_2) \varphi^{(n_3)} (x_3) \varphi^{(n_4)} (x_4) \right\rangle
= 2i \frac{r_{23}^{\Sigma-2n_2-2n_3} r_{34}^{\Sigma-2n_3-2n_4}}{r_{12}^{2n_1} r_{24}^{\Sigma-2n_3}} \left( X_{1234\alpha \dot{\alpha}} I + X_{1432\alpha \dot{\alpha}} J \right),
\]

(2.15)

for

\[
X_{i[jk]} = \frac{x_{ij} \bar{x}_{jk} x_{ki}}{r_{ij} r_{ik}} = \frac{1}{r_{ij}} x_{ij} - \frac{1}{r_{ik}} x_{ik},
\]

(2.16)

which transforms under conformal transformations as a vector at \( x_i \) and is antisymmetric in \( jk \).

Using (2.14) and (2.15) in (2.13), noting that

\[
i \partial_{1\alpha \dot{\alpha}} \frac{1}{r_{13}^{2n_1}} \dot{\bar{\psi}}_\alpha (x_1) + 4n_1 \frac{1}{r_{13}^{2n_1}} \eta_\alpha = -4n_1 i \frac{1}{r_{13}^{2n_1+1}} \dot{\bar{\psi}}_\alpha (x_3),
\]

(2.17)

and \( \partial_{1\alpha \dot{\alpha}} u = 2u X_{1234\alpha \dot{\alpha}}, \partial_{1\alpha \dot{\alpha}} v = 2v X_{1432\alpha \dot{\alpha}}, \) we may decompose (2.13) into independent contributions involving \( \dot{\bar{\psi}} (x_1) \) and \( \dot{\bar{\psi}} (x_3) \) (note that \( x_{12} \dot{\bar{\psi}} (x_2)/r_{12} = x_{13} \dot{\bar{\psi}} (x_2)/r_{13} + X_{1234} \dot{\bar{\psi}} (x_1) \)) and also for \( x_2 \to x_4 \) giving two linear relations,

\[
\begin{align*}
\frac{1}{n_1} \tau \cdot \partial_{t_1} & \left( X_{1234} u \partial_u + X_{1432} v \partial_v \right) F + \left( 1 - \frac{1}{2n_1 - 1} \tau \cdot t_1 \tau \cdot \partial_{t_1} \right) \left( X_{1234} I + X_{1432} J \right) \\
& + X_{1234} R_2 \tau \cdot t_2 + X_{1432} R_4 \tau \cdot t_4 + \left( u X_{1234} - v X_{1432} \right) (S_2 \tau \cdot t_2 - S_4 \tau \cdot t_4) = 0,
\end{align*}
\]

(2.18a)

\[
\begin{align*}
-2 \tau \cdot \partial_{t_1} F & + \left( R_2 + (u - v) S_2 \right) \tau \cdot t_2 + R_3 \tau \cdot t_3 + \left( R_4 + (1 - u + v) S_4 \right) \tau \cdot t_4 \frac{1}{r_{13}} x_{13} \\
& + \left( v S_2 \tau \cdot t_2 + S_3 \tau \cdot t_3 - v S_4 \tau \cdot t_4 \right) \frac{1}{r_{14} r_{23}} x_{14} \bar{x}_{42} x_{23} = 0.
\end{align*}
\]

(2.18b)
It is easy to decompose \((2.18a, b)\) into independent equations but crucial simplifications are obtained essentially by diagonalising each \(2 \times 2\) spinorial equation in terms of new variables \(x, \bar{x}\) which, as mentioned in the introduction, are the eigenvalues of \(x_{12} x_{42}^{-1} x_{43} x_{13}^{-1}\). These are related to the conformal invariants \(u, v\) defined in \((2.11)\) by \((1.1)\). In \((2.18b)\) the spinorial matrix \(\bar{x}_{41}^{-1} \bar{x}_{42} \bar{x}_{32}^{-1} \bar{x}_{31}\) may be replaced by \(1/(1 - x)\) and in \((2.18a)\) we may effectively replace \(X_{1[23]} \to 1/x\) and \(X_{1[43]} \to -1/(1 - x)\) and in each case also for \(x \to \bar{x}\).

Using
\[
\frac{\partial}{\partial x} = \bar{x} \frac{\partial}{\partial u} - \left(1 - \bar{x}\right) \frac{\partial}{\partial v},
\]
and the definitions
\[
T_2 = R_2 + x S_2, \quad T_3 = R_3 + \frac{1}{1-x} S_3, \quad T_4 = R_4 + (1-x) S_4, \quad K = \frac{1}{x} I - \frac{1}{1-x} J, \quad (2.20)
\]
we may then obtain from \((2.18a, b)\)
\[
\frac{1}{n_1} \tau \cdot \frac{\partial}{\partial t_1} \frac{\partial}{\partial x} F = - \left(1 - \frac{1}{2n_1 - 1}\right) \tau \cdot t_1 \tau \cdot \frac{\partial}{\partial t_1} K - \frac{1}{x} T_2 \tau \cdot t_2 + \frac{1}{1-x} T_4 \tau \cdot t_4, \quad (2.21a)
\]
\[
2 \tau \cdot \frac{\partial}{\partial t_1} F = T_2 \tau \cdot t_2 + T_3 \tau \cdot t_3 + T_4 \tau \cdot t_4. \quad (2.21b)
\]

Together with the corresponding equations obtained by \(x \to \bar{x}\) in \((2.21a, b)\) with also \(T_i \to \bar{T}_i, K \to \bar{K}\), which are defined just as in \((2.21)\) for \(x \to \bar{x}\).

The equations in \((2.21a, b)\) are equations for \(\partial F/\partial t_1\). The integrability conditions, which are required by virtue of \((\tau \cdot \partial_{t_i})^2 = 0\), are satisfied since we have, for \(i = 2, 3, 4\),
\[
\tau \cdot \frac{\partial}{\partial t_1} T_i = 0, \quad \bar{T}_i \tau \cdot \frac{\partial}{\partial t_i} = 0, \quad (2.22)
\]
as a consequence of \((2.8)\). To reduce \((2.21a, b)\) into separate equations we first write,
\[
T_i \tau \cdot t_i = \tau \cdot V_i + W_i \tau, \quad (2.23)
\]

where \(W_i\) and \(V_{i,\tau}\) are respectively a scalar and a vector. From the results of appendix A we further decompose \(V_i\) uniquely in the form
\[
V_i = \frac{1}{n_1} \frac{\partial}{\partial t_1} U_i + \dot{V}_i, \quad t_1 \cdot V_i = U_i, \quad t_1 \cdot \dot{V}_i = 0. \quad (2.24)
\]

The first equation in \((2.22)\) then gives
\[
\frac{\partial}{\partial t_1} \cdot V_i = 0, \quad i \frac{\partial}{\partial t_1} \times V_i + \frac{\partial}{\partial t_1} W_i = 0, \quad (2.25)
\]
where we may let $V_i \rightarrow \hat{V}_i$ without change. From (2.25) we may then find
\[ L_1 W_i = i n_1 \hat{V}_i , \tag{2.26} \]
where we define the $SU(2)_R$ generators by
\[ L_i = t_i \times \frac{\partial}{\partial t_i} . \tag{2.27} \]

Substituting (2.23) into (2.21a) gives
\[ \frac{\partial}{\partial x} F = -\frac{1}{x} U_2 + \frac{1}{1-x} U_4 , \tag{2.28} \]
and
\[ \frac{n_1}{2n_1 - 1} K = -\frac{1}{x} W_2 + \frac{1}{1-x} W_4 , \tag{2.29} \]
which is just an equation giving $K$, and also
\[ -\frac{1}{2n_1 - 1} i L_1 K = -\frac{1}{x} \hat{V}_2 + \frac{1}{1-x} \hat{V}_4 . \tag{2.30} \]
It is easy to see that this follows from (2.31) as a consequence of (2.26). Similarly substituting (2.23) into (2.21b) gives three equations
\[ 2n_1 F = \sum_{i=2}^{4} U_i , \tag{2.31} \]
and
\[ \sum_{i=2}^{4} \hat{V}_i = 0 , \tag{2.32} \]
as well as
\[ \sum_{i=2}^{4} W_i = 0 . \tag{2.33} \]
Clearly (2.32) follows from (2.33).

An essential constraint may also be obtained from the second equation in (2.22) which gives
\[ (n_i + 1) T_i \tau \cdot t_i = -(T_i \tau \cdot t_i) i \tau \cdot \hat{T}_i . \tag{2.34} \]
With the decomposition (2.23) this leads to
\[ (n_i + 1) W_i = -i L_i \cdot V_i , \tag{2.35a} \]
\[ (n_i + 1) V_i = -L_i \times V_i - i L_i W_i . \tag{2.35b} \]
Contracting \([2.35]\) with \(L_i\), and using \(L_i \times L_i = -L_i\), \(L_i^2 W_i = n_i (n_i+1) W_i\), gives \([2.35a]\). In addition we have from \(T_i (\tau \cdot t_i)^2 = 0\)

\[
t_i \cdot V_i = 0, \quad i t_i \times V_i = t_i W_i. \tag{2.36}
\]

With the aid of the results in appendix A we may obtain \((2n_i + 1) \partial_i (t_i W_i - i t_i \times V_i) = (2n_i + 3)((n_i+1) W_i + i L_i \cdot V_i)\) so that \([2.36]\) implies \([2.35a]\). Similarly, since \(\partial_i (t_i \times V_i) = (\partial_i \times t_i) \times V_i + \partial_i (t_i \cdot V_i) - \partial_i (t_i V_i)\), we have from appendix A \(2(n_i + 1) \partial_i \times (t_i \times V_i) = -(2n_i + 3)(L_i \times V_i + (n_i + 4)V_i)\) and \((2n_i + 1) \partial_i \times (t_i W_i) = -(2n_i + 3) L_i W_i\). Hence it is clear that \([2.36]\) also implies \([2.35a]\).

Using \([2.24]\) and \([2.26]\) for \(\hat{V}_i\) in \([2.35a]\) we obtain

\[
(L_1 \cdot L_i + n_1 (n_i+1)) W_i = \frac{1}{2} ((L_1 + L_i)^2 + (n_1 + n_i)(n_1 + n_i + 1)) W_i = -i \frac{\partial}{\partial t_1} \cdot L_i U_i. \tag{2.37}
\]

\(U_i(u, v; t)\), which is defined by \([2.23]\), is a homogeneous polynomial in \(t_1, t_i\) of \(O(t_1^{n_1}, t_i^{n_i})\) such that the \(SU(2)_R\) representation with \(R(i) = n_1 + n_i\) is absent. In consequence the operator \((L_1 + L_i)^2 + (n_1 + n_i)(n_1 + n_i + 1)\), which commutes with \(\partial_1 \cdot L_i\), in \([2.37]\) may be inverted to give \(W_i\) in terms of \(U_i\). Alternatively we may obtain from \([2.36]\)

\[
i t_i \times \partial_1 U_i = -t_1 \times L_1 W_i + n_1 t_i W_i. \tag{2.38}
\]

To analyse these equations further we now consider the decomposition of \(F\) and also \(U_i\) in terms of \(SU(2)_R\) scalars. We first assume the \(n_i\) are ordered so that

\[
n_1 \leq n_2 \leq n_3 \leq n_4, \tag{2.39}
\]

and further assume

\[
n_4 = n_1 + n_2 + n_3 - 2E, \tag{2.40}
\]

for integer \(E = 0, 1, 2, \ldots\), where \(E\) is a measure of how close the correlation function is to the extremal case. With \([2.39]\) and \([2.40]\) \(F\), which is \(O(t_1^{n_1}, t_2^{n_2}, t_3^{n_3}, t_4^{n_4})\), can in general be written in the form

\[
F(u, v; t) = (t_1 \cdot t_4)^{n_1 - E} (t_2 \cdot t_4)^{n_2 - E} (t_1 \cdot t_2)^E (t_3 \cdot t_4)^{n_3} \mathcal{F}(u, v; \sigma, \tau), \tag{2.41}
\]

where \(\mathcal{F}\) is a polynomial in \(\sigma, \tau\), defined in \([1.4]\), with all terms \(\sigma^p \tau^q\) satisfying \(p + q \leq E\). If \(E > n_1\) then all terms in \(\mathcal{F}\) must contain a factor \(\mu^{E - n_1}\) to cancel negative powers of

---

\(^5\) The converse follows using \(t_i \cdot L_i = 0, t_i \times L_i = t_i t_i \cdot \partial_i\) and \(t_i \times (L_i \times V_i) = -t_i \times V_i.\)
$t_1 \cdot t_4$ in (2.41). Since $t_i$ are three dimensional vectors $t_{1[t_2,t_3,t_4]} = 0$ so that $\sigma, \tau$ are not independent but obey the relation

$$\Lambda \equiv \sigma^2 + \tau^2 + 1 - 2\sigma\tau - 2\sigma - 2\tau = 0. \quad (2.42)$$

This may be solved in terms of a single variable $\alpha$ by

$$\sigma = \alpha^2, \quad \tau = (1 - \alpha)^2, \quad (2.43)$$

so that

$$\mathcal{F}(u,v;\sigma,\tau) = \hat{\mathcal{F}}(x,\bar{x};\alpha). \quad (2.44)$$

$\hat{\mathcal{F}}(x,\bar{x};\alpha)$ is symmetric in $x, \bar{x}$ and, for $E \leq n_1$, is a polynomial in $\alpha$ of degree $2E$, so that there are $2E + 1$ independent coefficients, while if $E > n_1$ then it must be of the form $(1 - \alpha)^{2(E-n_1)} p(\alpha)$ with $p$ a polynomial of degree $n_1$, so that the number of coefficients is $2n_1 + 1$. These results correspond exactly of course to the number of $SU(2)_R$ invariants which can be formed in the four point function, subject to (2.40), together with (2.39), that can be found using standard $SU(2)$ representation multiplication rules.

A similar expansion to (2.41) can be given for each $U_i$

$$U_i(x,\bar{x};t) = (t_1 \cdot t_4)^{n_1-E}(t_2 \cdot t_4)^{n_2-E}(t_1 \cdot t_2)^E(t_3 \cdot t_4)^{n_3} U_i(x,\bar{x};\sigma,\tau), \quad (2.45)$$

The analysis of (2.29) and (2.33) depends on using (2.37), or (2.38), as shown in appendix C, to relate $W_i$ and $U_i$. Defining

$$W_i = i t_2 \cdot (t_3 \times t_4) (t_1 \cdot t_4)^{n_1-E}(t_2 \cdot t_4)^{n_2-E}(t_1 \cdot t_2)^E(t_3 \cdot t_4)^{n_3-1} W_i, \quad (2.46)$$

then we obtain

$$2(2n_1 - 1)W_i = \hat{D}_i \hat{U}_i, \quad (2.47)$$

where $\hat{D}_i$ are linear operators given by

$$\hat{D}_2 = \frac{d}{d\alpha} + \frac{2(E-n_1)}{1-\alpha}, \quad \hat{D}_3 = \frac{d}{d\alpha} - \frac{2n_1}{\alpha} + \frac{2(E-n_1)}{1-\alpha}, \quad \hat{D}_4 = \frac{d}{d\alpha} + \frac{2E}{1-\alpha}. \quad (2.48)$$

The superconformal identities (2.28), (2.31) and (2.33) then become

$$\frac{\partial}{\partial x} \hat{\mathcal{F}} = -\frac{1}{x} \hat{U}_2 + \frac{1}{x-1} \hat{U}_4, \quad (2.49a)$$

$$2n_1 \hat{\mathcal{F}} = \hat{U}_2 + \hat{U}_3 + \hat{U}_4, \quad (2.49b)$$

$$\hat{D}_2 \hat{U}_2 + \hat{D}_3 \hat{U}_3 + \hat{D}_4 \hat{U}_4 = 0. \quad (2.49c)$$
By acting on (2.49) with \( \hat{D}_3 \) and using (2.49c) we may obtain
\[
\hat{D}_3 \hat{F} = -\frac{1}{\alpha} \hat{U}_2 - \frac{1}{\alpha(1-\alpha)} \hat{U}_4,
\] (2.50)
and substituting in (2.49a) gives
\[
\left( x \frac{\partial}{\partial x} - \alpha \hat{D}_3 \right) \hat{F} = \left( \frac{x}{1-x} + \frac{1}{1-\alpha} \right) \hat{U}_4.
\] (2.51)
The right hand side of (2.51) vanishes when \( \alpha = 1/x \) leaving an equation for \( \hat{F} \) alone. With the explicit form for \( \hat{D}_3 \) in (2.48) we have
\[
\frac{\partial}{\partial x} \left( x^{2n_1} \left( 1 - \frac{1}{x} \right)^{2(n_1-E)} \hat{F} \left( x, \bar{x}; \frac{1}{x} \right) \right) = 0.
\] (2.52)
Together with its partner or conjugate equation involving \( \partial/\partial \bar{x} \) (2.52) provides the final result for the constraints due to superconformal identities for the four point function when \( \mathcal{N} = 2 \).

For the \( \mathcal{N} = 2 \) case we may also require instead of (2.40)
\[
n_4 = n_1 + n_2 + n_3 - 2E - 1,
\] (2.53)
since \( F \) can then be written as
\[
F(u, v; t) = (t_1 \cdot t_4)^{n_1-E} (t_2 \cdot t_4)^{n_2-E-1} (t_1 \cdot t_2)^E (t_3 \cdot t_4)^{n_3-1} t_2 \cdot t_3 \times t_4 \hat{F}(x, \bar{x}; \alpha).
\] (2.54)
There is an essentially unique expression in (2.54), with a single function \( \hat{F} \) as a consequence of identities for the various possible vector cross products for null vectors which take the form
\[
\begin{align*}
t_1 \cdot t_2 \times t_3 t_2 \cdot t_4 &= \frac{1}{2} (\sigma - \tau + 1) t_2 \cdot t_3 \times t_4 t_1 \cdot t_2, \\
t_1 \cdot t_2 \times t_4 t_2 \cdot t_3 &= \frac{1}{2} (\sigma - \tau - 1) t_2 \cdot t_3 \times t_4 t_1 \cdot t_2, \\
t_1 \cdot t_3 \times t_4 t_2 \cdot t_4 \tau &= \frac{1}{2} (\sigma + \tau - 1) t_2 \cdot t_3 \times t_4 t_1 \cdot t_4.
\end{align*}
\] (2.55)
Since, as shown in appendix B, effectively \( t_1 \cdot t_4 t_2 \cdot t_3 \times t_4 = O(1-\alpha) \) we can take in (2.54), if \( n_1 - E \geq 1, (1-\alpha) \hat{F}(x, \bar{x}; \alpha) \) to be a polynomial of degree \( 2E + 1 \). If \( n_1 - E < 1 \) then \( \hat{F}(x, \bar{x}; \alpha) \) must contain a factor \( (1-\alpha)^{2(E-n_1)-1} \). It is easy to see that the number of independent coefficients matches with the number of independent terms in the four point function obtained by counting possible representations in each case.

There is a similar expansion as (2.54) for \( U_i \). Instead of (2.46) and (2.47) we now have
\[
W_i = \frac{i}{2n_1 - 1} (t_1 \cdot t_4)^{n_1-E-1} (t_2 \cdot t_4)^{n_2-E} (t_1 \cdot t_2)^E (t_3 \cdot t_4)^{n_3} \tau \hat{D}_i \hat{U}_i,
\] (2.56)
with \( \hat{D}_i \) exactly as in (2.48). In consequence the superconformal identities reduce to (2.49a, b, d) and we may derive the final result (2.52), albeit with \( E \) given by (2.53).
3. Solution of Identities, $\mathcal{N} = 2$

Although in the $\mathcal{N} = 2$ case the identities can be solved rather trivially we show here how they may be put in a form which makes the connection with the operator product expansion, and the possible supermultiplets which may contribute to it, rather obvious. For the purposes of analysing the operator product expansion for $x_1 \sim x_2$ an alternative form to (2.9) is more convenient so we write

$$\langle \varphi^{(n_1)}(x_1, t_1) \varphi^{(n_2)}(x_2, t_2) \varphi^{(n_3)}(x_3, t_3) \varphi^{(n_4)}(x_4, t_4) \rangle = \frac{1}{r_{12}^{n_1+n_2} r_{34}^{n_3+n_4}} \left( \frac{r_{24}}{r_{14}} \right)^{n_1-n_2} \left( \frac{r_{14}}{r_{13}} \right)^{n_3-n_4} G(u, v; t),$$

(3.1)

where

$$G(u, v; t) = u^{n_1+n_2} v^{n_1+n_4-n_2-n_3} F(u, v; t).$$

(3.2)

For application of the superconformal Ward identities here it is convenient here to replace the variable $\alpha$ by $y$ where

$$y = 2\alpha - 1,$$

(3.3)

and $x, \bar{x}$ by $z, \bar{z}$ given by

$$z = \frac{2}{x} - 1, \quad \bar{z} = \frac{2}{\bar{x}} - 1.$$

(3.4)

Assuming now

$$G(u, v; t) = (t_1 \cdot t_4)^{n_1-E} (t_2 \cdot t_4)^{n_2-E} (t_1 \cdot t_2)^{E} (t_3 \cdot t_4)^{n_3} G(u, v; y),$$

(3.5)

the solution of (2.52) and its conjugate equation, maintaining the symmetry under $z \leftrightarrow \bar{z}$, becomes

$$G(u, v; z) = u^{n_1+n_2-2E} f(z), \quad G(u, v; \bar{z}) = u^{n_1+n_2-2E} f(\bar{z}),$$

(3.6)

where $f$ is an unknown single variable function. Since $G(u, v; y)$ is just a polynomial in $y$ (3.6) requires

$$G(u, v; y) = u^{n_1+n_2-2E} \frac{(y - \bar{z}) f(\bar{z}) - (y - z) f(z)}{z - \bar{z}} + (y - z)(y - \bar{z}) K(u, v; y),$$

(3.7)

where $\mathcal{K}(u, v; y)$ is undetermined, if $G(u, v; y)$ is a polynomial of degree $2E$ in $y$ then clearly $\mathcal{K}$ is a polynomial of degree $2E - 2$.

The operator product expansion applied to this correlation function is realised by expanding it in terms of conformal partial waves $G_{\Delta}^{(\ell)}(u, v; \Delta_{21}, \Delta_{43})$, $\Delta_{ij} = \Delta_i - \Delta_j$, which represent the contribution to a four point function for four scalar fields, with scale dimensions $\Delta_i$, from an operator of scale dimension $\Delta$ and spin $\ell$, and all its conformal
In terms of the variables $x, \bar{x}$ defined in (1.1), which satisfy

$$G^{(\ell)}_{\Delta}(u, v; \Delta_{21}, \Delta_{43}) = (-1)^{\ell} v^{\frac{1}{2} \Delta_{43}} G^{(\ell)}_{\Delta}(u/v, 1/v; -\Delta_{21}, \Delta_{43})$$

$$= v^{\frac{1}{2}(\Delta_{43} - \Delta_{21})} G^{(\ell)}_{\Delta}(u, v; -\Delta_{21}, -\Delta_{43}).$$

For this case the expansion is also over the contributions for differing $SU(2)_R$ representations and has the form, if $n_1 \geq E$,

$$G(u, v; y) = \sum_{R=n_1-n_3}^{n_1+n_2} \sum_{\Delta, \ell} a_{R, \Delta, \ell} P^{(2n_1-2E, 2n_2-2E)}_{R+n_3-n_4}(y) G^{(\ell)}_{\Delta}(u, v; 2(n_2-n_1), 2(n_4-n_3)),

\text{(3.9)}$$

with $P^{(a,b)}_n(y)$ a Jacobi polynomial. For $a$ a negative integer $P^{(a,b)}_n(y) \propto (1-y)^{-a}$ and $n + a \geq 0$. Hence when $n_1 < E$ we require a similar expansion to (3.9) but with $R = n_2-n_1, \ldots, n_1+n_2$ and then $G(u, v; y) \propto y^{E-n_1}$ as required in (3.3) to avoid negative powers of $t_1 t_2$. The different terms appearing in the sum in (3.9) then determine the necessary spectrum of operators required by this correlation function. The symmetry properties of this operator product expansion follow from (3.9) and $P^{(a,b)}_n(y) = (-1)^n P^{(b,a)}_n(y) \propto (1-y)^{-a}$.

We first consider the case when $n_1 = n_2 = n_3 = n_4 = n$, so that $E = n$. To apply (3.6) we first consider the expansion in terms of Legendre polynomials (to which the Jacobi polynomial reduce in this case),

$$G(u, v; y) = \sum_{R=0}^{2n} a_R(u, v) P_R(y), \quad K(u, v; y) = \sum_{R=0}^{2n-2} A_R(u, v) P_R(y). \quad \text{(3.10)}$$

The $P_R(y)$ in (3.10) correspond to the $2n + 1$ possible $SU(2)_R$ invariants for the four point function (3.1) and, as a consequence of results in appendix B, the coefficients $a_R$ represent the contribution to the correlation function from operators belonging just to the $SU(2)_R$ $R$-representation in the operator product expansion for $\varphi^{(n)}(x_1, t_1) \varphi^{(n)}(x_2, t_2)$.

From (3.7) it is easy to see that the single variable function $f$ involves terms linear in $y$ and so contributes only for $R = 0, 1$ giving

$$a^f_0 = \frac{zf(z) \bar{z} f(\bar{z})}{z - \bar{z}}, \quad a^f_1 = \frac{f(z) - f(\bar{z})}{z - \bar{z}}. \quad \text{(3.11)}$$

$$G^{(\ell)}_{\Delta}(u, v; \Delta_{21}, \Delta_{43}) = \frac{x^{\frac{1}{2}(\Delta - \ell)}}{x - \bar{x}} \left( (-\frac{1}{2}x)^\ell F(\frac{1}{2}(\Delta + \Delta_{21} + \ell), \frac{1}{2}(\Delta - \Delta_{43} + \ell); \Delta + \ell; x) \right.$$

$$\times F(\frac{1}{2}(\Delta + \Delta_{21} - \ell - 2), \frac{1}{2}(\Delta - \Delta_{43} - \ell - 2); \Delta - \ell - 2; \bar{x}) - x \leftrightarrow \bar{x} \right).$$

15
Using the expansion in (3.10) for $K$ in (3.4) and standard recurrence relations for Legendre polynomials gives corresponding expressions for $a_R$. For the terms involving $A_R$ we have

$$a_{R+2}^R = \frac{(R+1)(R+2)}{(2R+1)(2R+3)} A_R, \quad a_{R-2}^R = \frac{(R-1)R}{(2R-1)(2R+1)} A_R,$$

$$a_{R+1}^R = -\frac{2(R+1)}{2R+1} \frac{1-v}{u} A_R, \quad a_{R-1}^R = -\frac{2R}{2R+1} \frac{1-v}{u} A_R,$$

$$a_R^R = \left(2 \frac{1+v}{u} - \frac{1}{2} + \frac{1}{2(2R-1)(2R+3)}\right) A_R. \quad (3.12)$$

For $R \geq 2$ $a_R$ is therefore given in terms $A_{R\pm 2}, A_{R\pm 1}, A_R$ while for $R = 0, 1$, with (3.11), we have

$$a_0 = a_0 + \left(2 \frac{1+v}{u} - \frac{2}{3}\right) A_0 - \frac{2}{3} \frac{1-v}{u} A_1 + \frac{2}{15} A_2,$$

$$a_1 = a_1 - \frac{2}{5} \frac{1-v}{u} A_0 + \left(2 \frac{1+v}{u} - \frac{2}{5}\right) A_1 - \frac{4}{5} \frac{1-v}{u} A_2 + \frac{6}{35} A_3. \quad (3.13)$$

In (3.12) and (3.13) any contributions involving $A_R$ for $R > 2n - 2$ should be dropped.

The significance of the results given by (3.12) and (3.13) is that they correspond exactly to the $\mathcal{N} = 2$ supermultiplet structure of operators appearing in the operator product expansion. Each $a_R(u, v)$ may then be expanded in terms of $G^{(\ell)}_\Delta(u, v) \equiv G^{(\ell)}_\Delta(u, v; 0, 0)$

$$a_R(u, v) = \sum_{\Delta, \ell} b_{R, \Delta, \ell} G^{(\ell)}_\Delta(u, v). \quad (3.14)$$

The conformal partial waves $G^{(\ell)}_\Delta(u, v)$ satisfy crucial recurrence relations [10],

$$-2 \frac{1-v}{u} G^{(\ell)}_\Delta(u, v) = 4 G^{(\ell+1)}_{\Delta-1}(u, v) + G^{(\ell-1)}_{\Delta-1}(u, v) + a_s G^{(\ell+1)}_{\Delta+1}(u, v) + \frac{1}{4} a_{t-1} G^{(\ell+1)}_{\Delta-1}(u, v),$$

$$\left(2 \frac{1+v}{u} - 1\right) G^{(\ell)}_\Delta(u, v) = 4 G^{(\ell-2)}_{\Delta-2}(u, v) + 4 a_s G^{(\ell+2)}_{\Delta}(u, v) + \frac{1}{4} a_{t-1} G^{(\ell-2)}_{\Delta}(u, v) + \frac{1}{4} a_s a_{t-1} G^{(\ell)}_{\Delta+2}(u, v), \quad (3.15)$$

where

$$s = \frac{1}{2}(\Delta + \ell), \quad t = \frac{1}{2}(\Delta - \ell), \quad a_s = \frac{s^2}{(2s-1)(2s+1)}. \quad (3.16)$$

In (3.13) $a_{t-1} > 0$ if $\Delta > \ell + 3$.

If $A_R$ is restricted to a single partial wave so that

$$A_R \rightarrow G^{(\ell)}_{\Delta+2}. \quad (3.17)$$

then, using (3.13) with (3.12),

$$a_{R'}^{A_{R'}} \rightarrow a_{R'}(A_{R, \ell}) = \sum_{(\Delta', \ell')} b_{(\Delta', \ell')} G^{(\ell')}_{\Delta'}, \quad (3.18)$$

$$|R' - R| = 2, \ (\Delta'; \ell') = (\Delta + 2; \ell), \ |R' - R| = 1, \ (\Delta'; \ell') = (\Delta + 3, \Delta + 1; \ell \pm 1),$$

$$R' - R = 0, \ (\Delta'; \ell') = (\Delta + 4, \Delta; \ell), \ (\Delta'; \ell') = (\Delta + 2; \ell \pm 2, \ell). \quad (3.18)$$
This gives exactly the expected contributions corresponding to those operators present in a long $\mathcal{N} = 2$ supermultiplet, which we may denote $A^\Delta_{R,\ell}$, whose lowest dimension operator has dimension $\Delta$, spin $\ell$ belonging to the $SU(2)_R$-representation. From (3.12) and the positivity constraints for (3.15) we may then easily see that in (3.14) $b(\Delta',\ell') > 0$ for $\Delta > \ell + 1$. For a unitary representation, so that all states in $A^\Delta_{R,\ell}$ have positive norm, (we consider here multiplets whose $U(1)_R$ charge is zero) the requirement is

$$\Delta \geq 2R + \ell + 2. \tag{3.19}$$

Since $G^\ell(u,v) = u^{\frac{3}{2}(\Delta - \ell)} F(u,v)$ with $F(u,v)$ expressible as a power series in $u, 1 - v$ we must have from (3.17) for $u \sim 0$,

$$A_R(u,v) \sim u^{R + 2 + \epsilon}, \quad \epsilon \geq 0. \tag{3.20}$$

The contribution of the single variable function $f$ (3.11) represents operators just with twist $\Delta - \ell = 2$. From the results in [15] we have

$$G_{\ell+2}^{(\ell)}(u,v) = u \frac{g_{\ell+1}(x) - g_{\ell+1}(\bar{x})}{x - \bar{x}} = -2 \frac{g_{\ell+1}(x) - g_{\ell+1}(\bar{x})}{z - \bar{z}}, \tag{3.21}$$

for

$$g_\ell(x) = (-\frac{1}{2}x)^{\ell-1} x F(\ell,\ell; 2\ell; x) = -\frac{2}{z^{\ell}} F \left( \frac{1}{2}, \frac{1}{2} \ell + \frac{1}{2}; \ell + \frac{1}{2}, \frac{1}{2} \ell + \frac{1}{2}; \frac{1}{z^{2}} \right), \tag{3.22}$$

where $F$ is just an ordinary hypergeometric function.\footnote{As shown in [16] $g_\ell$ satisfies

$$z g_\ell(x) = -g_{\ell-1}(x) - a_\ell g_{\ell+1}(x). \tag{3.23}$$

In general we therefore expand the single variable function $f$ in (3.11) in the form

$$f(z) = \sum_{\ell=0}^{\infty} b_\ell g_\ell(x). \tag{3.24}$$

For this to be possible $f(z)$ must be analytic in $1/z$, or equivalently in $x$. If we consider just $f \to 2g_{\ell+2}$ and use (3.23) in (3.11) then $a_\ell^f \to a_R(C_0,\ell)$ where

$$a_1(C_0,\ell) = G_{\ell+3}^{(\ell+1)}, \quad a_0(C_0,\ell) = G_{\ell+2}^{(\ell)} + a_{\ell+2} G_{\ell+4}^{(\ell+2)} \tag{3.25}.$$}

These results for $a_0, a_1$ then correspond to the contributions of operators belonging to a semi-short $\mathcal{N} = 2$ supermultiplet $C_0,\ell$ whose lowest dimension operator is a $SU(2)_R$ singlet with spin $\ell$ and $\Delta = \ell + 2$, i.e. at the unitarity threshold (3.19).

\footnote{\textit{g}_\ell(x) \propto Q_{\ell-1}(z)$ with $Q_{\nu}$ an associated Legendre function.}
In general we denote by \( C_{R, \ell} \) the semi-short multiplet whose lowest dimension operator has spin \( \ell \), belongs to the representation \( R \), and has \( \Delta = 2R + \ell \), so that the bound (3.19) is saturated. At the unitarity threshold given by (3.19) a long multiplet \( A^\Delta_{R, \ell} \) may be decomposed into two semi-short supermultiplets \( C_{R, \ell} \) and \( C_{R+1, \ell-1} \) \[22\]. This is reflected in the contributions to the four point function since, with \( a_{R'}(A^\Delta_{R, \ell}) \) defined by (3.17) and (3.18),

\[
a_{R'}(A^\Delta_{R, \ell}) = 4a_{R'}(C_{R, \ell}) + \frac{R + 1}{2R + 1} a_{R'}(C_{R+1, \ell-1}).
\]

(3.26)

where we take

\[
a_{R}(C_{R,\ell}) = \mathcal{G}^{(\ell)}_{2R+\ell+2} + \frac{1}{4} a_{R} \mathcal{G}^{(\ell)}_{2R+\ell+4} + a_{R+\ell+2} \mathcal{G}^{(\ell+2)}_{2R+\ell+4},
\]

\[
a_{R-1}(C_{R,\ell}) = \frac{R}{2R + 1} \left\{ \mathcal{G}^{(\ell+1)}_{2R+\ell+3} + \frac{1}{4} \mathcal{G}^{(\ell-1)}_{2R+\ell+3} + \frac{1}{4} a_{R+\ell+2} \mathcal{G}^{(\ell+1)}_{2R+\ell+5} \right\},
\]

(3.27)

\[
a_{R-2}(C_{R,\ell}) = \frac{(R - 1)R}{4(2R - 1)(2R + 1)} \mathcal{G}^{(\ell)}_{2R+\ell+4},
\]

\[
a_{R+1}(C_{R,\ell}) = \frac{R + 1}{2R + 1} \mathcal{G}^{(\ell+1)}_{2R+\ell+3}.
\]

For \( R = 0 \) (3.27) coincides with (3.25). Thus the contribution of any semi-short supermultiplet \( C_{R,\ell}, R = 0, 1, \ldots, 2n - 1 \), to the four point function may be obtained by combining the results for long supermultiplets at unitarity threshold with (3.25). There is no reason why any particular \( C_{R,\ell} \), except \( C_{0,0} \) which contains the energy momentum tensor and the conserved \( SU(2)_R \) current, should be present but if \( f(z) \) is non zero it is necessary for there to be at least one semi-short contribution involving operators with protected dimensions.

A special case arises if we set \( \ell = -1 \). Formally, as shown in \[24\], \( C_{R,-1} \simeq B_{R+1} \) where \( B_R \) denotes the short supermultiplet whose lowest dimension operator belongs to the \( R \)-representation with \( \Delta = 2R, \ell = 0 \), obeying the full \( \mathcal{N} = 2 \) shortening conditions. The conformal partial waves as shown in \[16\] satisfy

\[
(\frac{1}{4})^{\ell-1} \mathcal{G}^{(-\ell)}_{\Delta} = -\mathcal{G}^{(\ell-2)}_{\Delta}, \quad \mathcal{G}^{(-1)}_{\Delta} = 0,
\]

(3.28)

and hence from (3.27) we have

\[
a_{R'}(C_{R,-1}) = \frac{R + 1}{2R + 1} a_{R'}(B_{R+1}),
\]

(3.29)

where

\[
a_{R}(B_R) = \mathcal{G}^{(0)}_{2R}, \quad a_{R-1}(B_R) = \frac{R}{2R + 1} \mathcal{G}^{(1)}_{2R+1},
\]

\[
a_{R-2}(B_R) = \frac{(R - 1)R}{4(2R - 1)(2R + 1)} \mathcal{G}^{(0)}_{2R+2}.
\]

(3.30)
The operators whose contributions appear in (3.30) are just those expected for the short supermultiplet \( B_R \) and there are possible contributions to the four point function for \( R = 1, 2, \ldots, 2n \). Since
\[
G_0^{(0)}(u, v) = 1,
\]
then it is easy to see from (3.30) that
\[
a_R(B_0) = a_R(I) = \delta_{R0},
\]
where \( a_R(I) \) denotes the contribution of the identity in the operator product expansion. Besides (3.29) we may also note that
\[
a_R'(C_{R,-2}) = -4 a_R'(B_R).
\]
For \( R = 0 \) this is in accord with (3.32) since \( G_0^{(-2)} = -4 \).

Apart from the case of the correlation function for four identical operators as considered above there are other solutions of the superconformal Ward identities which are of interest corresponding to extremal and next-to-extremal correlation functions \[21\]. The extremal case corresponds to taking \( E = 0 \) in (2.40). There is then a unique \( SU(2)_R \) invariant coupling which also follows from the requirement that \( F \) in (2.45), or \( G \) where in (3.2) \( G(u, v; t) = (t_1 \cdot t_4)^{n_1} (t_2 \cdot t_4)^{n_2} (t_3 \cdot t_4)^{n_3} G(u, v) \), must be independent of \( \sigma, \tau \) and hence equivalently also of \( \alpha \). In this case the result (2.52) for \( \partial_x \) and its conjugate for \( \partial_{\bar{x}} \) simply imply
\[
G(u, v) = C u^{n_1+n_2},
\]
where \( C \) is independent of both \( x, \bar{x} \) and thus a constant. To interpret this in terms of the operator product expansion for \( \varphi^{(n_1)}(x_1, t_1) \varphi^{(n_2)}(x_2, t_2) \) we may use the result from \[15\],
\[
G_{\Delta_1+\Delta_2}^{(0)}(u, v; \Delta_{21}, \Delta_1 + \Delta_2) = u^{\frac{1}{2} (\Delta_1 + \Delta_2)}.
\]
The result (3.34) then shows that the only operators contributing to the operator product expansion in the extremal case have \( \Delta = 2(n_1 + n_2) \), \( \ell = 0 \) and necessarily \( R = n_1 + n_2 \). Such operators can only be found as the lowest dimension operator in the short supermultiplet \( B_{n_1+n_2} \).

For the next-to-extremal case we set \( E = 0 \) in (2.53). The solution of (2.52) can be conveniently expressed as
\[
(1 - z) G(u, v; z) = u^{n_1+n_2-1} f(z),
\]
where in (3.2) we have \( G(u, v; t) = (t_1 \cdot t_4)^{n_1} (t_2 \cdot t_4)^{n_2-1} (t_3 \cdot t_4)^{n_3-1} t_2 \cdot t_3 \times t_4 G(u, v; y) \). For \( E = 0 \), \( (1 - y)G(u, v; y) \) is linear in \( y \) and from (1.30) and its conjugate we may find
\[
(1 - y)G(u, v; y) = u^{n_1+n_2-1} \frac{(y - z) f(z) - (y - z) f(z)}{z - \bar{z}},
\]
so that $G$ is determined just by the single variable function $g$ in this case.

For the next-to-extremal correlation function there are just two independent $SU(2)_R$ invariant couplings. In a similar fashion to (3.10), we have an expansion, from appendix B, in terms of two Jacobi polynomials

$$(1 - y) G(u, v; y) = a_{n_1+n_2-1}(u, v) P_0^{(2n_1-1,2n_2-1)}(y) + a_{n_1+n_2}(u, v) P_1^{(2n_1-1,2n_2-1)}(y),$$

(3.38)

where $a_R$, $R = n_1 + n_2 - 1$, $n_1 + n_2$ represent the contribution of the two possible $R$-representations of $SU(2)_R$ in this case. From (3.37) we obtain

$$a_{n_1+n_2-1} = -\frac{1}{n_1+n_2} u^{n_1+n_2} \frac{f(z) - f(\bar{z})}{z - \bar{z}},$$

$$a_{n_1+n_2} = u^{n_1+n_2} \left( z f(z) - \bar{z} f(\bar{z}) + \frac{n_1 - n_2}{n_1 + n_2} \frac{f(z) - f(\bar{z})}{z - \bar{z}} \right).$$

(3.39)

To interpret this in terms of the operator product expansion we may use, extending (3.21),

$$G_{\Delta_1+\Delta_2+\ell}(u, v; \Delta_1, \Delta_2) = u^{\frac{1}{2}(\Delta_1+\Delta_2)} \frac{g_{\ell+1}(x; \Delta_1, \Delta_2) - g_{\ell+1}(\bar{x}; \Delta_1, \Delta_2)}{x - \bar{x}},$$

$$g_{\ell}(x; \Delta_1, \Delta_2) = (-\frac{1}{2}x)^{\ell-1} x F(\ell + \Delta_2 - 1, \ell; 2\ell + \Delta_1 + \Delta_2 - 2; x).$$

(3.40)

In consequence only operators with twist $\Delta_1 + \Delta_2$ can contribute for the solution for $a_R$ given by (3.39). If in (3.39) let we $f(z) \rightarrow 2g_{\ell+2}(x; 2n_1, 2n_2)$ then $a_R \rightarrow a_R(C_{n_1+n_2-1,\ell})$ where

$$a_{n_1+n_2}(C_{n_1+n_2-1,\ell}) = \frac{1}{n_1+n_2} G_{2n_1+2n_2+\ell+1}^{(\ell+1)},$$

$$a_{n_1+n_2-1}(C_{n_1+n_2-1,\ell}) = G_{2n_1+2n_2+\ell}^{(\ell)} + (n_2 - n_1) \frac{(\ell + 1)(2n_1 + 2n_2 + \ell)}{(n_1 + n_2 + \ell)(n_1 + n_2 + \ell + 1)(n_1 + n_2)} G_{2n_1+2n_2+\ell+1}^{(\ell+1)}$$

$$+ \frac{(\ell + 2)(2n_1 + \ell + 1)(2n_2 + \ell + 1)(2n_1 + 2n_2 + \ell)}{(n_1 + n_2 + \ell + 1)^2(n_1 + n_2 + \ell + 1)(2n_1 + 2n_2 + 2\ell + 3)} G_{2n_1+2n_2+\ell+2}^{(\ell+2)},$$

(3.41)

where $G_{2n_1+2n_2+\ell}$ are as in (3.40) with $\Delta_i \rightarrow 2n_i$. The contributions appearing in (3.41) correspond to those expected from the semi-short supermultiplet $C_{n_1+n_2-1,\ell}$. Using $C_{R,-1} \simeq B_{R+1}$ again we may obtain the contribution for the short multiplet $B_{n_1+n_2}$,

$$a_R(C_{n_1+n_2-1,-1}) = \frac{1}{n_1+n_2} a_R(B_{n_1+n_2}),$$

(3.42)

giving

$$a_{n_1+n_2}(B_{n_1+n_2}) = G_{2n_1+2n_2}^{(0)},$$

$$a_{n_1+n_2-1}(B_{n_1+n_2}) = \frac{4n_1n_2}{(n_1 + n_2)(2n_1 + 2n_2 + 1)} G_{2n_1+2n_2+1}^{(1)},$$

(3.43)
For the next-to-extremal correlation function therefore only the protected short and semi-short supermultiplets $B_R$ and $C_{R-1,\ell}$ can contribute to the operator product expansion.

By analysis \[23,24\] of three point functions the possible $\mathcal{N} = 2$ supermultiplets which may appear in the operator product expansion of two $\mathcal{N} = 2$ short supermultiplets is determined by the decomposition, for $n_2 \geq n_1$,

$$
B_{n_1} \otimes B_{n_2} \simeq \bigoplus_{n=n_2-n_1}^{n_2+n_1} B_n \oplus \bigoplus_{\ell \geq 0} \left( \bigoplus_{n=n_2-n_1}^{n_2+n_1-1} C_{n,\ell} \oplus \bigoplus_{n=n_2-n_1}^{n_2+n_1-2} A_{n,\ell}^\Delta \right),
$$

(3.44)

where for $A_{n,\ell}^\Delta$ all $\Delta > 2n + \ell + 2$ is allowed. By considering also the corresponding result for $B_{n_3} \otimes B_{n_4}$ in all cases discussed above the general solution of the $\mathcal{N} = 2$ superconformal identities accommodates all possible $\mathcal{N} = 2$ supermultiplets which may contribute to the four point function in the operator product expansion according to (3.44). In the extremal case it is clear that only $B_{n_1+n_2}$ contributes while for the next-to-extremal case long multiplets which undergo renormalisation are also excluded.

4. Superconformal Ward Identities, $\mathcal{N} = 4$

We here describe an analysis of the superconformal Ward identities for the four point function of $\mathcal{N} = 4$ chiral primary operators belonging to the $SU(4)_R [0, p, 0]$ representation with $\Delta = p$ represented by symmetric traceless fields $\varphi_r(x,t), r_i = 1, \ldots, 6$. As in (1.2) we define $\varphi^{(p)}(x,t)$, homogeneous of degree $p$ in $t$, in terms of a six dimensional null vector $t_r$. The superconformal transformation of $\varphi^{(p)}(x,t)$ is then expressible in the form

$$
\delta \varphi^{(p)}(x,t) = -\bar{\epsilon}(x) \gamma^i \partial_i \psi^{(p-1)}(x,t) + \overline{\psi^{(p-1)}(x,t)} \partial^i \bar{\epsilon}(x),
$$

(4.1)

where the conformal Killing spinors $\bar{\epsilon}^a_i(x), \bar{\epsilon}^{a\dot{a}}(x)$ are as in (2.3), with $i = 1, 2, 3, 4$ and $\gamma^{ij}_r = -\gamma^{ji}_r$, $\gamma_{rij} = \frac{1}{2}\varepsilon_{ijkl}r^{kl}_i$ are $SU(4)$ gamma matrices, $\gamma^a \gamma_s + \gamma_s \gamma^a = -2\delta^a_s$, $\gamma^+_r = -\gamma^a_r$. In (4.1) $\psi^{(p-1)}_{ia}(x,t), \psi^{(p-1)}_{i\dot{a}}(x,t)$ are homogeneous spinor fields of degree $p - 1$ in $t$ and satisfy constraints similar to (2.8)

$$
\gamma^i \frac{\partial}{\partial t} \psi^{(p-1)}_{i\alpha}(x,t) = 0, \quad \overline{\psi^{(p-1)}_{i\alpha}(x,t)} \gamma^i \frac{\partial}{\partial t} = 0,
$$

(4.2)

which are necessary for them to belong to $SU(4)_R$ representations $[0, p-1,1], [1, p-1,0]$. At the next level the superconformal transformations involve a current belonging to the $[1,p-1,1]$ representation which corresponds to a homogeneous field of degree $p - 1$ with one $SU(4)_R$ vector index $J^{(p-1)}_{r\alpha\dot{a}}(x,t)$ satisfying

$$
t_r J^{(p-1)}_{r\alpha\dot{a}}(x,t) = 0, \quad \frac{\partial}{\partial t_r} J^{(p-1)}_{r\alpha\dot{a}}(x,t) = 0.
$$

(4.3)
The superconformal transformation of $\psi^{(p-1)}(x,t)$, neglecting $\dot{\epsilon}$ terms, is then

$$
\delta \psi^{(p-1)}_\alpha(x,t) = \frac{1}{p} \bar{\gamma} \cdot \frac{\partial}{\partial t} \frac{i}{\partial \alpha} \bar{\varphi}^{(p)}(x,t) \dot{\gamma} \cdot (x) + 2 \bar{\gamma} \cdot \frac{\partial}{\partial t} \varphi^{(p)}(x,t) \eta_{\alpha} + \left(1 + \frac{1}{2p + 2} \bar{\gamma} \cdot t \gamma \cdot \frac{\partial}{\partial t} \right) J^{(p-1)} r_{\alpha \dot{\alpha}}(x,t) \bar{\gamma} \cdot \delta \psi^{(p-1)}(x,t).
$$

(4.4)

Superconformal transformations which generate the full BPS multiplet listed in [25] can be obtained similarly to [16] but the superconformal Ward identities depend only on (4.1) and (4.4).

The general four point function of chiral primary operators can be written in an identical form to (2.9),

$$
\langle \varphi^{(p_1)}(x_1, t_1) \varphi^{(p_2)}(x_2, t_2) \varphi^{(p_3)}(x_3, t_3) \varphi^{(p_4)}(x_4, t_4) \rangle
= \frac{r_{23}}{r_{13}} \frac{r_{34}}{r_{24}} \frac{\Sigma_{-p_2-p_3}}{r_{23}} \frac{\Sigma_{-p_3-p_4}}{r_{34}} F(u, v; t), \quad \Sigma = \frac{1}{2} (p_1 + p_2 + p_3 + p_4).
$$

(4.5)

The derivation of superconformal Ward identities initially follows an almost identical path as that in section 2 leading to (2.21a,b). With similar definitions to (2.14), (2.15), taking $2n_i \to p_i$, and (2.20) we find

$$
\frac{1}{p_1} \bar{\gamma} \cdot \frac{\partial}{\partial t_1} \frac{\partial}{\partial x} F = - \left(1 + \frac{1}{2p_1 + 2} \bar{\gamma} \cdot t_1 \gamma \cdot \frac{\partial}{\partial t_1} \right) \bar{\gamma} \cdot K - \frac{1}{x} T_2 \bar{\gamma} \cdot t_2 + \frac{1}{1 - x} T_4 \bar{\gamma} \cdot t_4, \quad (4.6a)
$$

$$
\bar{\gamma} \cdot \frac{\partial}{\partial t_1} F = T_2 \bar{\gamma} \cdot t_2 + T_3 \bar{\gamma} \cdot t_3 + T_4 \bar{\gamma} \cdot t_4. \quad (4.6b)
$$

Instead of (2.22) we have the constraints, which follow from (4.2) and (4.3),

$$
\gamma \cdot \frac{\partial}{\partial t_i} T_i = 0, \quad T_i \bar{\gamma} \cdot \frac{\partial}{\partial t_i} = 0, \quad t_1 \cdot K = \frac{\partial}{\partial t_1} \cdot K = 0. \quad (4.7)
$$

As with (2.23) we exhibit the dependence on $SU(4)$ gamma matrices by writing

$$
T_i \bar{\gamma} \cdot t_i = \bar{\gamma} \cdot V_i + \frac{1}{6} \bar{\gamma} \cdot [r_s \gamma_s \gamma_w] W_i,rsu. \quad (4.8)
$$

Since we take

$$
[\gamma_s \gamma_u \gamma_v \gamma_w \gamma_z] = i \varepsilon_{rsuvwz},
$$

$$
[\gamma_s \gamma_u \gamma_w] = - \frac{1}{6} i \varepsilon_{rsuvwz} \gamma_v \gamma_w \gamma_z, \quad (4.9)
$$

so that we must require the self-duality condition

$$
W_{i,rsu} = \frac{1}{6} i \varepsilon_{rsuvwz} W_{i,vuw}. \quad (4.10)
$$

\footnote{Note that $(\gamma_1 \gamma_2 \gamma_3 \gamma_4 \gamma_5 \gamma_6)^t = -\gamma_1 \gamma_2 \gamma_3 \gamma_4 \gamma_5 \gamma_6$.}
Imposing the first equation in (4.7) we have
\[ \frac{\partial}{\partial t_1} V_i = 0, \quad \partial_{1[r} V_{i,s]} = \partial_{1u} W_{i,rsu}. \] (4.11)

Just as in (2.24) we write,
\[ V_i = \frac{1}{p_1} \frac{\partial}{\partial t_1} U_i + \hat{V}_i, \quad t_1 \cdot V_i = U_i, \quad t_1 \cdot \hat{V}_i = 0. \] (4.12)

so that in (4.11) we may let \( V_i \to \hat{V}_i \).

Using (4.8) with (4.12), and \( \bar{\gamma} \cdot t_1 \gamma \cdot t_1 \partial \bar{\gamma} \cdot K = -p_1 \bar{\gamma} \cdot K + \frac{1}{2} \bar{\gamma} \gamma \bar{\gamma} \gamma_u L_{1,rs} K_u \), (1.6a, b) may be decomposed into three pairs of equations,
\[ \frac{\partial}{\partial x} F = -\frac{1}{x} U_2 + \frac{1}{1-x} U_4, \quad p_1 F = \sum_{i=2}^{4} U_i, \] (4.13)

and
\[ \frac{p_1 + 2}{2p_1 + 2} K_r = -\frac{1}{x} \hat{V}_{2,r} + \frac{1}{1-x} \hat{V}_{4,r}, \quad \sum_{i=2}^{4} \hat{V}_{i,r} = 0, \] (4.14)

and
\[ \frac{3}{2p_1 + 2} \left( L_{1,[rs} K_{u]} \right)_{sd} = -\frac{1}{x} W_{2,rsu} + \frac{1}{1-x} W_{4,rsu}, \quad \sum_{i=2}^{4} W_{i,rsu} = 0, \] (4.15)

where we define for \( i = 1, 2, 3, 4 \)
\[ L_{i,rs} = t_{ir} \partial_{is} - t_{is} \partial_{ir}, \] (4.16)

and for any \( X_{rsu} = X_{[rsu]} \) the self dual part, satisfying (1.10), is given by
\[ (X_{rsu})_{sd} = \frac{1}{2} X_{rsu} + \frac{1}{12} i \varepsilon_{rsuvwz} X_{uvw}. \] (4.17)

Since \( 2t_{1s} \partial_{1[r} V_{i,s]} = -p_1 \hat{V}_{i,r} \) we may obtain from (4.11)
\[ p_1 \hat{V}_{i,r} = -L_{1,su} W_{i,rsu}, \] (4.18)

which gives \( \hat{V}_{i,r} \) in terms of \( W_{i,rsu} \). Furthermore from (1.7) \( L_{1,rs} K_s = K_r \) and using also, as a consequence of the commutation relations for \( L_1 \), \( [L_{1,rs}, L_{1,ru}] = -4L_{1,sw} \) we have \( L_{1,rs} L_{1,ru} K_s = 3K_u \). With, in addition, \( \frac{1}{2} L_{1,rs} L_{1,rs} K_u = -(p_1 - 1)(p_1 + 3) K_u \) we may then obtain
\[ 3L_{1,rs} L_{1,[rs} K_{u]} = -2p_1 (p_1 + 2) K_u. \] (4.19)
Since also $\varepsilon_{rsuvwz} L_{1,rs} L_{1,vuw} K_z = 0$ it is clear from (4.18) and (4.19) that eqs. (4.15) imply (4.14). However, if we define
\[
\hat{W}_{i,rsu} = 3(L_{1,[rs} \hat{V}_{i,u]})_{sd} - (p_1 + 2) W_{i,rsu} ,
\]
with $\hat{V}_{i,u}$ determined by (4.18), then as a consequence of (4.14) and (4.15) we must also require
\[
\frac{1}{x} \hat{W}_{2,rsu} = \frac{1}{1 - x} \hat{W}_{4,rsu} .
\]

From the second equation in (4.7) we may obtain
\[
\frac{1}{2} L_{i,rs}(T_1 \vec{\gamma}_r \cdot t_i) \vec{\gamma}_s = (p_i + 4) T_1 \vec{\gamma}_r \cdot t_i
\]
which leads to the relations
\[
L_{i,rs} V_{i,s} - L_{i,sv} W_{i,rsu} = (p_i + 4) V_{i,r} ,
\]
\[
3( L_{i,[rs} V_{i,u]} )_{sd} + 3 L_{i,|rv} W_{i,svu} = (p_i + 4) W_{i,rsu} ,
\]
where $L_{i,|uv} W_{i,rs} v$ is self dual as a consequence of (4.10). We also have from $T_1 \vec{\gamma}_r \cdot t_i \vec{\gamma}_s = 0$
\[
t_i \cdot V_i = 0 , \quad t_i \cdot V_i + W_{i,rsu} t_{iu} = 0 .
\]
For consistency we note that $\partial_{is}(t_i V_{i,s} + W_{i,rsu} t_{iu}) = 0$ is identical with (4.22a). Furthermore using (4.10) we have $(\partial_{[i|r} W_{i,sv} t_{iv})_{sd} = \frac{1}{2} (\partial_{[i|r} W_{i,sv} t_{iv} - \partial_{iv} t_{[i|r} W_{i,sv]}) + \frac{1}{3} \partial_{iv} (W_{i,rsu} t_{iv})$ and, from appendix A, $(p_i + 2) \partial_{iv} (W_{i,rsu} t_{iv}) = (p_i + 3)(p_i + 4) W_{i,rsu}$ while acting on $W_{i,sv}$ similarly $(p_i + 2) \partial_{iv} (W_{i,sv} t_{iv}) = -(p_i + 3) \frac{1}{2} L_{i,rs}$. Hence we have demonstrated that $(\partial_{[i|r} (t_i V_{i,u} + W_{i,svu} t_{iv}))_{sd} = 0$ is identical to (4.22b) so that this equation is also implied by (4.23). Combining (4.23) with (4.12) gives the essential equation
\[
t_i [d_{1|r} U_i] + p_1 t_i [d_{1|r} \hat{V}_{i,s}] + p_1 W_{i,rsu} t_{iu} = 0 ,
\]
where $p_1 \hat{V}_{i,s}$ is determined by (1.18).

As in (2.45) we may expand the correlation function $F$, as defined in (4.3), in terms of $SU(4)$ invariants
\[
F(u, v; t) = (t_1 \cdot t_4)^{p_1-E} (t_2 \cdot t_4)^{p_2-E} (t_1 \cdot t_2)^E (t_3 \cdot t_4)^{p_3} F(u, v; \sigma, \tau) ,
\]
where we assume
\[
p_1 \leq p_2 \leq p_3 \leq p_4 , \quad 2E = p_1 + p_2 + p_3 - p_4 .
\]
In (4.23) $F(u, v; \sigma, \tau)$ is a polynomial in $\sigma, \tau$ consistent with $F(u, v; t) = O(t_1^{p_1}, t_2^{p_2}, t_3^{p_3}, t_4^{p_4})$ and hence $E \geq 0$ is an integer. For $p_1 \geq E$ then $F$ is expressible as a polynomial of degree
$E$ in $\sigma, \tau$, i.e. a linear expansion in the $\frac{1}{2}(E+1)(E+2)$ independent monomials $\sigma^p \tau^q$ with $p + q \leq E$. For $p_1 < E$ it is necessary also that $q \geq E - p_1$ giving only $\frac{1}{2}(p_1 + 1)(p_1 + 2)$ independent terms. It is easy to see that this matches the number of invariants that may be constructed by finding common representations in $[0, p_1, 0] \otimes [0, p_2, 0]$ and $[0, p_3, 0] \otimes [0, p_4, 0]$ using the tensor product result

$$[0, p_1, 0] \otimes [0, p_2, 0] \simeq \bigoplus_{r=0}^{p_1} \bigoplus_{s=0}^{p_1-r} [r, p_2 - p_1 + 2s, r].$$ \hspace{1cm} (4.27)

Hence representations $[r, p_2 - p_1 + 2s, r]$ may contribute for $s = 0, \ldots, n - r$, $r = 0, \ldots, n$ with $n = E$ if $p_1 \geq E$, otherwise $n = p_1$.

In an exactly similar fashion to (4.25) we may express $U_i(x, \bar{x}; t)$ in terms of $\mathcal{U}_i(x, \bar{x}; \sigma, \tau)$ so that (4.13) becomes

$$\frac{\partial}{\partial x} \mathcal{F} = -\frac{1}{x} \mathcal{U}_2 + \frac{1}{1 - x} \mathcal{U}_4, \quad p_1 \mathcal{F} = \mathcal{U}_2 + \mathcal{U}_3 + \mathcal{U}_4.$$ \hspace{1cm} (4.28)

Furthermore we may also decompose $W_{i, rsu}(x, \bar{x}; t)$ for $i = 2, 3, 4$ in terms of four independent self dual tensors,

$$W_{i, rsu} = -(t_1 \cdot t_4)^{p_1 - E} (t_2 \cdot t_4)^{p_2 - E} (t_1 \cdot t_2)^{E - 2} (t_3 \cdot t_4)^{p_3 - 1}$$

\[ \times \left( (t_1[r t_3 s t_4 u])_{sd} t_2 \cdot t_4 A_i + (t_1[t_3 s t_4 u])_{sd} t_2 \cdot t_3 B_i \right) \]

\[ + \left( (t_1[r t_3 s t_4 u])_{sd} t_2 \cdot t_3 t_2 \cdot t_4 \frac{1}{t_3 \cdot t_4} C_i + (t_2[r t_3 s t_4 u])_{sd} t_1 \cdot t_2 W_i \right), \]

with $A_i, B_i, C_i$ and $W_i$ polynomials in $\sigma, \tau$ of degree $E - 2$ and $E - 1$, if $p_1 \geq E$. From its definition in (4.28) we must have

$$C_2 = C_3 = A_4 = 0.$$ \hspace{1cm} (4.30)

The result (4.13) then requires

$$A_2 + A_3 = 0, \quad B_2 + B_4 = 0, \quad C_3 + C_4 = 0, \quad W_2 + W_3 + W_4 = 0.$$ \hspace{1cm} (4.31)

We may similarly decompose $\hat{V}_{i, r}$ in the form

$$\hat{V}_{i, r} = (t_1 \cdot t_4)^{p_1 - E - 1} (t_2 \cdot t_4)^{p_2 - E} (t_1 \cdot t_2)^{E - 1} (t_3 \cdot t_4)^{p_3 - 1}$$

\[ \times \left( (t_2[r t_1 s t_4 t_3])_{sd} t_3 \cdot t_4 \mathcal{I}_i + (t_3[r t_1 s t_4 t_3])_{sd} t_2 \cdot t_4 \mathcal{J}_i + t_1[r t_2 s t_3 t_4 t_4 \mathcal{V}_i \right), \]

(4.32)
where we impose $t_1 \dot{V}_i = 0$. The coefficient of $t_1 r$ is determined by the requirement
\[ \partial_1 \dot{V}_i = 0, \]
with differential operators
\[ O_\sigma = (\sigma + \tau - 1) \frac{\partial}{\partial \sigma} + 2\tau \frac{\partial}{\partial \tau} + p_1 - 2E + 1, \]
\[ O_\tau = 2\sigma \frac{\partial}{\partial \sigma} + (\sigma + \tau - 1) \left( \frac{\partial}{\partial \tau} + \frac{p_1 - E}{\tau} \right) - p_1 + 1. \] (4.34)

Using (4.18) we get
\[ 6p_1 I_i = (p_1 + 2)(\sigma A_i - \tau B_i) - (\sigma O_\sigma - \tau O_\tau) W_i, \]
\[ 6p_1 J_i = - (p_1 + 2)(A_i - \tau C_i) + O_\sigma W_i. \] (4.35)

From (4.33) we then obtain
\[ 6p_1 V_i = \tau \left( (\sigma + 1) B_i - (\sigma + 1) A_i - (\sigma (\sigma + 1) - \tau (\sigma + 1)) C_i \right). \] (4.36)

As a consequence of (4.24) the coefficients in (4.25) are not independent but we have
relations which determine $A_i, B_i, C_i$ for each $i$,
\[ (p_1 + 1) A_2 = 3 \frac{\partial}{\partial \sigma} U_2 + \frac{1}{2} (\sigma - p_1) W_2, \]
\[ (p_1 + 1) B_2 = -3 \left( \frac{\partial}{\partial \tau} + \frac{p_1 - E}{\tau} \right) U_2 + \frac{1}{2} (\sigma - p_1) W_2, \]
\[ (p_1 + 1) \sigma A_3 = 3 \left( \frac{\partial}{\partial \sigma} + \tau \frac{\partial}{\partial \tau} - E \right) U_3 + \frac{1}{2} (\sigma - p_1) W_3, \]
\[ (p_1 + 1) \sigma C_3 = 3 \left( \frac{\partial}{\partial \sigma} + \frac{p_1 - E}{\tau} \right) U_3 - \frac{1}{2} (\sigma + p_1) W_3, \]
\[ (p_1 + 1) \tau B_4 = -3 \left( \frac{\partial}{\partial \sigma} + \tau \frac{\partial}{\partial \tau} - E \right) U_4 - \frac{1}{2} (\sigma - p_1) W_4, \]
\[ (p_1 + 1) \tau C_4 = -3 \frac{\partial}{\partial \sigma} U_4 - \frac{1}{2} (\sigma + p_1) W_4. \] (4.37)

Combining this with (4.31) and also the result in (4.28) for $p_1 F$ leads to
\[ \left( \frac{\tau}{\partial \tau} + p_1 - E \right) F = U_4 + \frac{1}{6} (\sigma - \tau - 1) W_4 - \frac{1}{3} \tau W_2, \]
\[ \left( \frac{\sigma}{\partial \sigma} + \frac{\tau}{\partial \tau} - E \right) F = -U_2 + \frac{1}{6} (\sigma - \tau - 1) W_2 - \frac{1}{3} W_4. \] (4.38)
If $\tilde{W}_{i,rsu}$ given by (4.20) is defined in terms of $\tilde{A}_i, \tilde{B}_i, \tilde{C}_i$ and $\tilde{W}_i$ as in (4.29) then it is easy to see that $\tilde{W}_i = -(p_1 + 2)W_i$ and, as a consequence of (4.35) and (4.37),

$$2(p_1 + 1)\tilde{A}_i = \left( \frac{\partial}{\partial \sigma} (\sigma O_\sigma - \tau O_\tau + p_1 + 2) + (\sigma \frac{\partial}{\partial \sigma} + \tau \frac{\partial}{\partial \tau} - E + 1) (O_\sigma - p_1 - 2) \right) W_i,$$

$$\tau \tilde{C}_i - \tilde{A}_i = \sigma W_i, \quad \tau \tilde{B}_i - \sigma \tilde{A}_i = (\sigma O_\sigma - \tau O_\tau) W_i.$$  

Hence (4.21) reduces to just

$$\frac{1}{x} W_2 = \frac{1}{1 - x} W_4. \quad (4.40)$$

In terms of the variables $\alpha, \bar{\alpha}$, defined in (1.3), (4.38) becomes

$$\alpha (1 - \alpha) \frac{\partial}{\partial \alpha} F + E \alpha F - p_1 F = -(1 - \alpha) U_2 - U_4 + \frac{1}{6} (\alpha - \bar{\alpha}) ((1 - \alpha) W_2 + W_4), \quad (4.41)$$

together with the conjugate equation obtained for $\alpha \leftrightarrow \bar{\alpha}$. If this is used together with (4.28) for $\partial_x F$ we may eliminate $U_2$ to obtain

$$\left( x \frac{\partial}{\partial x} - \alpha \frac{\partial}{\partial \alpha} - E + \frac{\alpha}{1 - \alpha} + p_1 \frac{1}{1 - \alpha} \right) F = \left( \frac{x}{1 - x} + \frac{1}{1 - \alpha} \right) U_4 - \frac{1}{6} (\alpha - \bar{\alpha}) \left( W_2 + \frac{1}{1 - \alpha} W_4 \right) \left( W_4 + \frac{1}{1 - \alpha} W_2 \right), \quad (4.42)$$

where we have used (4.40). Writing $F(u, v; \sigma, \tau) = \hat{F}(x, \bar{x}; \alpha, \bar{\alpha})$ evidently

$$\left( x \frac{\partial}{\partial x} - \alpha \frac{\partial}{\partial \alpha} + E + (p_1 - E) \frac{1}{1 - \alpha} \right) \hat{F}(x, \bar{x}; \alpha, \bar{\alpha}) \bigg|_{\alpha = \frac{1}{x}} = 0, \quad (4.43)$$

which is solved by writing

$$u^E (1 - v)^{p_1 - E} \hat{F}(x, \bar{x}; \frac{1}{x}, \bar{\alpha}) = f(x, \bar{\alpha}). \quad (4.44)$$

Together with the conjugate equation in which $\alpha \rightarrow \bar{\alpha}$ (4.44) is the basic solution of the superconformal Ward identities in this context.

5. Solution of Identities, $\mathcal{N} = 4$

We here extend the results of section 3 to the $\mathcal{N} = 4$ case. As previously it is more convenient for consideration of the operator product expansion to change from $F(u, v; t)$
to $G(u, v; t)$, defined in a similar fashion to (3.1) with $2n_i \rightarrow p_i$. Writing $G(u, v; t)$ in a similar fashion to (4.25) then the corresponding function $G$ is given in terms of $F(u, v; \sigma, \tau)$ by

$$G(u, v; \sigma, \tau) = u^{\frac{1}{2}((p_1+p_2)-E} v^{p_1-E} F(u, v; \sigma, \tau).$$  \hspace{1cm} (5.1)

For the applications in the section it is convenient to write

$$G(u, v; \sigma, \tau) = \hat{G}(u, v; y, \bar{y}) = \hat{G}(u, v; \bar{y}, y),$$  \hspace{1cm} (5.2)

where $\hat{G}$ depends on the variables

$$y = 2\alpha - 1, \hspace{0.5cm} \bar{y} = 2\bar{\alpha} - 1.$$  \hspace{1cm} (5.3)

The solution (4.44) then gives, with $z, \bar{z}$ defined in (3.4),

$$\hat{G}(u, v; z, \bar{y}) = u^{\frac{1}{2}(p_1+p_2)-E} f(z, \bar{y}), \hspace{0.5cm} \hat{G}(u, v; \bar{z}, y) = u^{\frac{1}{2}(p_1+p_2)-E} f(z, \bar{y}).$$  \hspace{1cm} (5.4)

For consistency, since $f(z, z) = f(\bar{z}, \bar{z})$, we must have

$$f(z, z) = k.$$  \hspace{1cm} (5.5)

In general the conformal partial wave expansion and the decomposition into contributions for differing $SU(4)_R$ representations further into conformal partial waves is realised by writing for $p_1 \geq E$.

$$\hat{G}(u, v; y, \bar{y}) = \sum_{0 \leq m \leq n \leq E} a_{nm}(u, v) P^{(p_1-E, p_2-E)}_{nm}(y, \bar{y})$$

$$= \sum_{0 \leq m \leq n \leq E} \sum_{\Delta, \ell} a_{nm, \Delta, \ell} P^{(p_1-E, p_2-E)}_{nm}(y, \bar{y}) G^{(\ell)}_{\Delta}(u, v; p_2 - p_1, p_4 - p_3),$$  \hspace{1cm} (5.6)

where $G^{(\ell)}_{\Delta}$ are described in section 3 and $P^{(a,b)}_{nm}(y, \bar{y})$ are symmetric polynomials of degree $n$ (i.e. for an expansion in terms of the form $(y\bar{y})^s(y^t + \bar{y}^t)$, $s + t \leq n$) which are discussed in appendix B and which are given in terms of Jacobi polynomials

$$P^{(a,b)}_{nm}(y, \bar{y}) = \frac{P^{(a,b)}_{n+1}(y) P^{(a,b)}_{m}(\bar{y}) - P^{(a,b)}_{m}(y) P^{(a,b)}_{n+1}(\bar{y})}{y - \bar{y}} = -P^{(a,b)}_{m-1 n+1}(y, \bar{y}),$$  \hspace{1cm} (5.7)

In (5.6) $a_{nm, \Delta, \ell}$ then corresponds to the presence of an operator in the operator product expansion if for $\varphi^{(p_1)}$ and $\varphi^{(p_2)}$ belonging to the $SU(4)_R$ representation with Dynkin labels $[n - m, p_1 + p_2 - 2E + 2m, n - m]$ and with scale dimension $\Delta$, spin $\ell$. The expansion (5.6) also extends to $p_1 < E$ save that then $m \geq E - p_1$ and $P^{(a,b)}_{nm}(y, \bar{y}) \propto \tau^{E-p_1}$.  

28
We consider initially in detail the case with \( p_i = p = E \) for all \( i \) and \( G(u; v; y, \bar{y}) \) is a symmetric polynomial in \( y, \bar{y} \) with degree \( p \). Since it must also be symmetric in \( z, \bar{z} \) (5.4) implies

\[
\hat{G}(u; v; y, \bar{y}) = -k + \frac{(y - z)(\bar{y} - \bar{z})(f(z, \bar{y}) + f(\bar{z}, y)) - (y - \bar{z})(\bar{y} - z)(f(z, y) + f(\bar{z}, \bar{y}))}{(z - \bar{z})(y - \bar{y})}
\]

\[+ (y - z)(y - \bar{z})(\bar{y} - z)(\bar{y} - \bar{z}) K(u; v; \sigma, \tau), \]

with \( K(u; v; \sigma, \tau) = \hat{K}(u; v; y, \bar{y}) \) defining an undetermined symmetric polynomial in \( y, \bar{y} \) of degree \( p - 2 \). This term corresponds to the result (1.6) described in the introduction. To take account of the constraint (5.5) we write

\[
f(z, y) = k + (y - z) \hat{f}(z, y), \]

with \( \hat{f}(z, y) \) a free function, polynomial in \( y \) of degree \( p - 1 \).

The decomposition of \( \hat{G}(u; v; y, \bar{y}) \) into the contributions for different possible \( SU(4)_R \) representations is given by (5.6) where \( a_{nm} \) are for this case the coefficients corresponding to the representation with Dynkin labels \( [n - m, 2m, n - m] \). For this case in (5.7) \( P_n^{(0,0)}(y) = P_n(y) \), conventional Legendre polynomials.

We first consider the contribution resulting from the constant \( k \) in (5.8) and (5.9). It is easy to see that this gives only

\[
a_{00}^k = k. \]

To analyse the contributions arising from the function \( \hat{f}(z, y) \) this may be expanded as

\[
\hat{f}(z, y) = \sum_{n=0}^{p-1} f_n(z) P_n(y). \]

Using this in (5.9) and (5.8) then \( f_n \) gives rise to the following contributions to \( a_{nm} \) just for \( m = 0, 1 \),

\[
a_{n+1}^{f_n} = \frac{(n + 1)(n + 2)}{2n + 1}(2n + 3) F_{nm}(z, \bar{z}), \quad a_{n-3}^{f_n} = \frac{(n - 1)n}{(2n - 1)(2n + 1)} F_{nm}(z, \bar{z}),
\]

\[
a_{n+1}^{f_n} = -\frac{n + 1}{2n + 1} (z + \bar{z}) F_{nm}(z, \bar{z}), \quad a_{n-2}^{f_n} = -\frac{n}{2n + 1} (z + \bar{z}) F_{nm}(z, \bar{z}),
\]

\[
a_{n-1}^{f_n} = \left( \frac{1}{2} + \frac{1}{2(2n - 1)(2n + 3)} \right) F_{nm}(z, \bar{z}),
\]

where

\[
F_{n1}(z, \bar{z}) = -\frac{f_n(z) - f_n(\bar{z})}{z - \bar{z}}, \quad F_{n0}(z, \bar{x}) = \frac{zf_n(z) - \bar{x}f_n(\bar{z})}{z - \bar{z}}. \]

(5.13)
For low $n$ the results need to be modified but these can be obtained from (5.12) by taking into account the symmetry relation in (5.7). For $n = 0$, $a_{11}^{f_0}, a_{10}^{f_0}$ are as in (5.12) but for $a_{00}^{f_0}$ we need to take

$$a_{00}^{f_0} - a_{-11}^{f_0} + a_{00}^{f_0} = -\frac{(z^2 - \frac{1}{3})f_0(z) - (\bar{z}^2 - \frac{1}{3})f_0(\bar{z})}{z - \bar{z}},$$

(5.14)

while for $n = 1$, $a_{21}^{f_1}, a_{20}^{f_1}, a_{11}^{f_1}, a_{10}^{f_1}$ are given by (5.12) but

$$a_{00}^{f_1} = \frac{\bar{z}(z^2 - \frac{1}{3})f_1(z) - z(\bar{z}^2 - \frac{1}{3})f_1(\bar{z})}{z - \bar{z}} + \frac{4}{15} F_{11}.$$  

(5.15)

It remains to consider the contribution of the two variable function $K$ in (5.8) which is expanded, for $P_{nm}(y, \bar{y}) \equiv P_{nm}^{(0,0)}(y, \bar{y})$, as

$$\hat{K}(u, v; y, \bar{y}) = \sum_{0 \leq m \leq n \leq p-2} A_{nm}(u, v) P_{nm}(y, \bar{y}),$$

(5.16)

with $\frac{1}{2}(p - 1)p$ terms. In this case the Legendre recurrence relations give

$$a_{n-2 m-2}^{A_{nm}} = \frac{(m - 1)m n(n + 1)}{(2m - 1)(2m + 1)(2n + 1)(2n + 3)} A_{nm},$$

$$a_{n-2 m+2}^{A_{nm}} = \frac{(m + 1)(m + 2)n(n + 1)}{(2m + 1)(2m + 3)(2n + 1)(2n + 3)} A_{nm},$$

$$a_{n+2 m-2}^{A_{nm}} = \frac{(m - 1)m(n + 2)(n + 3)}{(2m - 1)(2m + 1)(2n + 3)(2n + 5)} A_{nm},$$

$$a_{n+2 m+2}^{A_{nm}} = \frac{(m + 1)(m + 2)(n + 2)(n + 3)}{(2m + 1)(2m + 3)(2n + 3)(2n + 5)} A_{nm},$$

$$a_{n-2 m-1}^{A_{nm}} = -\frac{2mn(n + 1)}{(2m + 1)(2n + 1)(2n + 3)} \frac{1 - v}{u} A_{nm},$$

$$a_{n-2 m+1}^{A_{nm}} = -\frac{2(m + 1)n(n + 1)}{(2m + 1)(2n + 1)(2n + 3)} \frac{1 - v}{u} A_{nm},$$

$$a_{n-1 m-2}^{A_{nm}} = -\frac{2(m - 1)m(n + 1)}{(2m - 1)(2m + 1)(2n + 3)} \frac{1 - v}{u} A_{nm},$$

$$a_{n-1 m+2}^{A_{nm}} = -\frac{2(m + 1)(m + 2)(n + 1)}{(2m + 1)(2m + 3)(2n + 3)} \frac{1 - v}{u} A_{nm},$$

$$a_{n+2 m-1}^{A_{nm}} = -\frac{2m(n + 2)(n + 3)}{(2m + 1)(2n + 3)(2n + 5)} \frac{1 - v}{u} A_{nm},$$

$$a_{n+2 m+1}^{A_{nm}} = -\frac{2(m + 1)(n + 2)(n + 3)}{(2m + 1)(2n + 3)(2n + 5)} \frac{1 - v}{u} A_{nm},$$

$$a_{n+1 m-2}^{A_{nm}} = -\frac{2(m - 1)m(n + 2)}{(2m - 1)(2m + 1)(2n + 3)} \frac{1 - v}{u} A_{nm},$$

30
\[
\begin{align*}
    a_{n+1m}^n &= -\frac{2(m+1)(m+2)(n+2)}{(2m+1)(2m+3)(2n+3)} \frac{1-v}{u} A_{nm}, \\
    a_{n-1m-1}^n &= \frac{4m(n+1)}{(2m+1)(2n+3)} (1-v)^2 A_{nm}, \\
    a_{n-1m+1}^n &= \frac{4(m+1)(n+1)}{(2m+1)(2n+3)} (1-v)^2 A_{nm}, \\
    a_{n+1m-1}^n &= \frac{4m(n+2)}{(2m+1)(2n+3)} (1-v)^2 A_{nm}, \\
    a_{n+1m+1}^n &= \frac{4(m+1)(n+2)}{(2m+1)(2n+3)} (1-v)^2 A_{nm}, \\
    a_{n-2m}^n &= \frac{2n(n+1)}{(2n+1)(2n+3)} B_m A_{nm}, \\
    a_{n-1m}^n &= -\frac{4(n+1)}{2n+3} B_m \frac{1-v}{u} A_{nm}, \\
    a_{n-1m}^n &= -\frac{4m}{2m+1} B_{n+1} \frac{1-v}{u} A_{nm}, \\
    a_{n}^n &= 4B_m B_{n+1} A_{nm}, \\
\end{align*}
\]

where
\[
B_m = \frac{1+v}{u} - \frac{m^2 + m - 1}{(2m-1)(2m+3)}. 
\]

For \(m = n, n-1, n-2, n-3\), and also if \(n = 0, 1, 2\), \((5.7)\) may be used to combine terms to ensure that we only have \(a_{n'm'}^n\) for \(0 \leq m' \leq n'\). For \(m = n = 0\) this prescription gives
\[
\begin{align*}
    a_{22} &= \frac{4}{15} A_{00}, & a_{21} &= -4 \frac{1-v}{u} A_{00}, & a_{20} &= \frac{4}{15} \left(3 \frac{1+v}{u} - 1\right) A_{00}, \\
    a_{11} &= \frac{4}{15} \left(10 \frac{(1-v)^2}{u^2} - 5 \frac{1+v}{u} + 1\right) A_{00}, & a_{10} &= -\frac{4}{3} \left(2 \frac{1+v}{u} - 1\right) \frac{1-v}{u} A_{00}, \\
    a_{00} &= \frac{4}{15} \left(15 \frac{(1+v)^2}{u^2} - 5 \frac{(1-v)^2}{u^2} - 8 \frac{1+v}{u} + 1\right) A_{00}, \\
\end{align*}
\]

which is equivalent to the results in [16]. Similarly for \(n = 1, m = 0, 1\) the resulting \(a_{n'm'}^n\) correspond to those in [3].

The solution of the superconformal identities given by \((5.10)\), \((5.12)\) and \((5.17)\) may now be naturally interpreted in terms of the operator product expansion. If in \((5.17)\) we consider a single conformal partial wave for \(A_{nm}\) by letting
\[
A_{nm} \to G_{\Delta+4}^{(l)},
\]

\[31\]
then, if \( A_{[y,p,q],\ell}^{\Delta} \) denotes a long superconformal multiplet whose lowest state has spin \( \ell \), scale dimension \( \Delta \) and which belongs to a \( SU(4)_R \) representation with Dynkin labels \( [q,p,q] \), we obtain

\[
a_{n'm'}^{A_{nm,\ell}} \to a_{n'm'}^{A_{nm,\ell}}(A_{nm,\ell}^{\Delta}) , \quad \Delta_{nm,\ell}^{A} \equiv A_{[n-m,2m,n-m],\ell}^{\Delta} . \tag{5.21}
\]

The non zero results obtained from (5.17) with (5.20) may be conveniently expressed in the form

\[
a_{n+i,m+j}^{A_{nm,\ell}}(A_{nm,\ell}^{\Delta}) = N_{n+1,1}N_{m,j}A_{[i][j]}^{A_{nm}} , \quad i,j = \pm 2, \pm 1, 0 \tag{5.22}
\]

for

\[
N_{m,2} = \frac{(m+1)(m+2)}{(2m+1)(2m+3)} , \quad N_{m,1} = \frac{m+1}{2m+1} , \quad N_{m,0} = 1 , \quad N_{m,-1} = \frac{m}{2m+1} , \quad N_{m,-2} = \frac{(m-1)m}{(2m-1)(2m+1)} . \tag{5.23}
\]

and using (3.15) we have

\[
a_{n+i,m+j}^{A_{nm,\ell}}(A_{nm,\ell}^{\Delta}) = \sum_{(\Delta';\ell')} b_{(\Delta',\ell')} G_{\Delta}^{(\ell')} , \quad |i| = |j| = 2 , \quad (\Delta';\ell') = (\Delta + 4; \ell) ,
\]

\[
|i| = 2 , \quad |j| = 1 , \quad |i| = 1 , \quad |j| = 2 , \quad (\Delta';\ell') = (\Delta + 5, \Delta + 3; \ell \pm 1) ,
\]

\[
|i| = |j| = 1 , \quad (\Delta';\ell') = (\Delta + 6, \Delta + 4, \Delta + 2; \ell \pm 2, \ell) ,
\]

\[
|i| = 1 , \quad |j| = 0 , \quad |i| = 0 , \quad |j| = 1 , \quad (\Delta';\ell') = (\Delta + 7, \Delta + 1; \ell \pm 1) , \quad (\Delta + 5, \Delta + 3; \ell \pm 3, \ell \pm 1) ,
\]

\[
i = j = 0 , \quad (\Delta';\ell') = (\Delta + 8, \Delta; \ell) , \quad (\Delta + 6, \Delta + 2; \ell \pm 2, \ell) , \quad (\Delta + 4; \ell \pm 4, \ell \pm 2, \ell) . \tag{5.24}
\]

In consequence \( a_{n'm'}(A_{nm,\ell}^{\Delta}) \) corresponds to the contribution in the operator product expansion applied to the correlation function for all expected operators belonging to \( A_{nm,\ell}^{\Delta} \). In (5.24) \( b_{(\Delta',\ell')} > 0 \) if \( \Delta > \ell + 1 \). If \( m \leq n \leq m + 3 \) the results are modified since we then obtain from (5.17) contributions with \( m' \geq m' \). In this case, for \( n' \geq m' \) and \( a_{m'-1,n'+1}(A_{nm,\ell}^{\Delta}) \) non zero, we should take

\[
a_{n'm'}(A_{nm,\ell}^{\Delta}) - a_{m'-1,n'+1}(A_{nm,\ell}^{\Delta}) \to a_{n'm'}(A_{nm,\ell}^{\Delta}) . \tag{5.25}
\]

Furthermore any contribution with \( m' = n' + 1 \) should be dropped. Using this result and (5.25) we may then easily show that

\[
a_{n'm'}(A_{nm,\ell}^{\Delta}) = 0 , \tag{5.26}
\]

and for later reference we also note the symmetry relation

\[
a_{n'm'}(A_{nm,\ell}^{\Delta}) = a_{m'-1,n'+1}(A_{m-1,n+1,\ell}^{\Delta}) . \tag{5.27}
\]
The unitarity condition for a long multiplet $A_{nm,\ell}^\Delta$ requires
\[ \Delta \geq 2n + \ell + 2, \] (5.28)
and so, as in (3.20), using (5.20) we must have for $u \sim 0$,
\[ A_{nm}(u, v) \sim u^{n+3+\epsilon}, \quad \epsilon \geq 0. \] (5.29)

We now consider the operator product expansion interpretation of the remaining terms in the solution of the superconformal identities given by (5.8) and (5.9). The constant $k$, whose contribution is just given by (5.10), clearly corresponds to the identity operator,
\[ a_{nm}(\mathcal{I}) = \delta_{n0}\delta_{m0}. \] (5.30)

To analyse the contribution of the single variable functions $f_n$ in (5.11) we use the result (3.21) for the conformal partial wave for twist two operators as well as
\[ G_{\ell}^{(\ell)}(u, v) \Delta_1 = \Delta_2 = \Delta_3 = \Delta_4 = \frac{1}{2} \frac{\bar{z} \ell_1(x) - z \ell_1(\bar{x})}{z - \bar{z}}, \] (5.31)
with $g_{\ell}$ as in (3.22), for twist zero. Taking
\[ f_{n+1}(z) \to \frac{1}{2} g_{\ell+2}(x), \quad n = 0, 1, 2, \ldots, \] (5.32)
in (5.13) and (5.11) then leads to results corresponding to only twist zero and twist two operators. These operators can be interpreted as belonging to a multiplet $\mathcal{D}_{n0,\ell}$, where in general we denote by $\mathcal{D}_{n,\ell,\ell} \equiv \mathcal{D}_{[n-m,2m,n-m],\ell}$ the semi-short supermultiplet in which the lowest dimension operator has $\Delta = 2m+\ell$, or twist $2m$, and belongs to the $[n-m,2m,n-m]$ $SU(4)_R$ representation. These non unitary super multiplets are discussed in appendix D. For $\mathcal{D}_{n,\ell}$ the conformal partial waves may be expressed in general in the form
\[ a_{n+i,m+j}(\mathcal{D}_{nm,\ell}) = N_{n+1,i}N_{m,j}D_{i|j}^{nm}, \quad D_{i|j}^{nm} = 0. \] (5.33)

Corresponding to (5.32) we then have
\[
D_{01}^{n0} = \frac{1}{4} G_{\ell+3}^{(\ell+1)}, \quad D_{10}^{n0} = \frac{1}{4} \left( G_{\ell+2}^{(\ell)} + a_{\ell+2} G_{\ell+4}^{(\ell+2)} \right), \\
D_{11}^{n0} = G_{\ell+2}^{(\ell+2)} + \frac{1}{4} \left( G_{\ell+2}^{(\ell)} + a_{\ell+2} G_{\ell+4}^{(\ell+2)} \right), \\
D_{10}^{n0} = G_{\ell+1}^{(\ell+1)} + a_{\ell+2} G_{\ell+3}^{(\ell+3)} + \frac{1}{4} \left( G_{\ell+1}^{(\ell)} + b_{\ell} G_{\ell+3}^{(\ell+1)} + a_{\ell+2} a_{\ell+3} G_{\ell+5}^{(\ell+3)} \right), \\
D_{01}^{n0} = G_{\ell+1}^{(\ell+1)} + a_{\ell+2} G_{\ell+3}^{(\ell+3)} + \frac{1}{4} b_{n} G_{\ell+3}^{(\ell+1)}, \\
D_{00}^{n0} = G_{\ell}^{(\ell)} + b_{\ell} G_{\ell+2}^{(\ell+2)} + a_{\ell+2} a_{\ell+3} G_{\ell+4}^{(\ell+4)} + \frac{1}{4} b_{n} \left( G_{\ell+2}^{(\ell)} + a_{\ell+2} G_{\ell+4}^{(\ell+2)} \right),
\] (5.34)
whereas \( a_\ell \) is as in (3.16) and

\[
b_\ell = a_{\ell+2} + a_{\ell+1} = \frac{2\ell^2 + 6\ell + 3}{(2\ell + 1)(2\ell + 5)}.
\]

(5.35)

A list of relevant representations for differing dimensions contained in \( \mathcal{D}_{n0,\ell} \equiv \mathcal{D}_{[n,0,n],\ell} \) is listed in appendix D, the twist zero and twist two representations correspond with those necessary for (5.34). For \( f_0 \) these results are modified. From (5.14) only twist two contributions are required since, taking now \( f_0(z) \to 2g_{\ell+3}(x) \),

\[
a_{00}(C_{00,\ell}) = g^{(\ell)}_{\ell+2} + \frac{2(\ell + 2)(\ell + 3)}{3(2\ell + 3)(2\ell + 7)} g^{(\ell+2)}_{\ell+4} + a_{\ell+3}a_{\ell+4}g^{(\ell+4)}_{\ell+6},
\]

(5.36)

\[
a_{10}(C_{00,\ell}) = \frac{2}{3} \left( g^{(\ell+1)}_{\ell+3} + a_{\ell+3}g^{(\ell+3)}_{\ell+5} \right),
\]

\[
a_{11}(C_{00,\ell}) = \frac{2}{3} g^{(\ell+2)}_{\ell+4}.
\]

Here we denote by \( C_{nm,\ell} \equiv C_{[n-m,2m,n-m],\ell} \) the semi-short supermultiplet in which the lowest dimension operator has \( \Delta = 2n + \ell + 2 \) and belongs to the \([n-m,2m,n-m]\) \( SU(4)_R \) representation.

The multiplets \( \mathcal{D}_{[q,p,q],\ell} \) fail to satisfy the unitarity condition (5.28) on \( \Delta \) and so their contributions as in (5.34) must be cancelled in a unitary theory. This may be achieved by a corresponding long multiplet contribution. When \( \Delta = 2m + \ell \) or \( \Delta = 2n + \ell + 2 \) the long multiplet \( \mathcal{A}_{nm,\ell}^\Delta \) can be decomposed into semi-short multiplets resulting in

\[
a_{n'm'}(\mathcal{A}_{nm,\ell}^{2m+\ell}) = 16 a_{n'm'}(\mathcal{D}_{nm,\ell}) + \frac{4(m + 1)}{2m + 1} a_{n'm'}(\mathcal{D}_{n+1m,\ell-1}),
\]

(5.37)

and, at the unitarity threshold (5.28),

\[
a_{n'm'}(\mathcal{A}_{nm,\ell}^{2n+\ell + 2}) = 16 a_{n'm'}(\mathcal{C}_{nm,\ell}) + \frac{4(n + 2)}{2n + 3} a_{n'm'}(\mathcal{C}_{n+1m,\ell-1}).
\]

(5.38)

When \( n = m \) we have the special case

\[
a_{n'm'}(\mathcal{A}_{nm,\ell}^{2n+\ell}) = 16 a_{n'm'}(\mathcal{D}_{n,n,\ell}) + \frac{(n + 1)(n + 2)}{(2n + 1)(2n + 3)} a_{n'm'}(\mathcal{C}_{n+1n,\ell-2}).
\]

(5.39)

The results (5.37), (5.38) and (5.39) reflect a decomposition of long multiplets at particular values of \( \Delta \) as described in appendix D. From (5.37) we may obtain \( a_{n'm'}(\mathcal{D}_{nm,\ell}) \) iteratively starting from (5.34). With the notation in (5.33) the results are

\[
D_{2-2}^{nm} = \frac{1}{16} g^{(\ell)}_{2m+\ell+4}, \quad D_{21}^{nm} = \frac{1}{4} g^{(\ell+1)}_{2m+\ell+3},
\]

\[
D_{2-1}^{nm} = \frac{1}{16} \left( g^{(\ell-1)}_{2m+\ell+3} + 4 g^{(\ell+1)}_{2m+\ell+3} + a_{m+\ell+2} g^{(\ell+1)}_{2m+\ell+5} \right),
\]

\[
D_{20}^{nm} = \frac{1}{16} \left( 4 g^{(\ell)}_{2m+\ell+2} + a_{m} g^{(\ell)}_{2m+\ell+4} + 4a_{m+\ell+2} g^{(\ell+2)}_{2m+\ell+4} \right),
\]

34
\[
D_{1-2}^{nm} = \frac{1}{16} (G_{2m+\ell+3}^{(\ell-1)} + 4G_{2m+\ell+3}^{(\ell+1)} + \frac{1}{4}a_{m+1}G_{2m+\ell+5}^{(\ell-1)} + a_{m+\ell+2}G_{2m+\ell+5}^{(\ell+1)}) ,
\]
\[
D_{11}^{nm} = \frac{1}{4} (G_{2m+\ell+2}^{(\ell+2)} + 4G_{2m+\ell+2}^{(\ell)} + 4a_{m}G_{2m+\ell+4}^{(\ell)} + a_{m+\ell+2}G_{2m+\ell+4}^{(\ell+2)} ,
\]
\[
D_{1-1}^{nm} = \frac{1}{16} (G_{2m+\ell+3}^{(\ell-2)} + 8G_{2m+\ell+2}^{(\ell)} + (b_{m+\ell} + a_{m})G_{2m+\ell+4}^{(\ell)} + \frac{1}{4}a_{m+1}a_{m+\ell+2}G_{2m+\ell+6}^{(\ell)}
+ 16G_{2m+\ell+2}^{(\ell+2)} + 8a_{m+\ell+2}G_{2m+\ell+4}^{(\ell+2)} + a_{m+\ell+2}a_{m+\ell+3}G_{2m+\ell+6}^{(\ell+2)} ,
\]
\[
D_{10}^{nm} = \frac{1}{16} (4G_{2m+\ell+1}^{(\ell-1)} + 2a_{m}G_{2m+\ell+3}^{(\ell-1)} + \frac{1}{4}a_{m+1}G_{2m+\ell+5}^{(\ell-1)}
+ 16G_{2m+\ell+1}^{(\ell+1)} + 4(b_{m+\ell} + a_{m})G_{2m+\ell+3}^{(\ell+1)} + 2a_{m}a_{m+\ell+2}G_{2m+\ell+5}^{(\ell+1)}
+ 16a_{m+\ell+2}G_{2m+\ell+3}^{(\ell+3)} + 4a_{m+\ell+2}a_{m+\ell+3}G_{2m+\ell+5}^{(\ell+3)} ,
\]
\[
D_{0-2}^{nm} = \frac{1}{16} (4G_{2m+\ell+2}^{(\ell)} + 2a_{m+1}G_{2m+\ell+4}^{(\ell-2)} + b_{n}G_{2m+\ell+4}^{(\ell)}
+ 4a_{m+\ell+2}G_{2m+\ell+4}^{(\ell+2)} + \frac{1}{4}a_{m+1}a_{m+\ell+2}G_{2m+\ell+6}^{(\ell)} ,
\]
\[
D_{01}^{nm} = \frac{1}{4} (G_{2m+\ell+1}^{(\ell+1)} + \frac{1}{4}a_{m}G_{2m+\ell+3}^{(\ell-1)} + b_{n}G_{2m+\ell+3}^{(\ell-1)}
+ 4a_{m+\ell+2}G_{2m+\ell+3}^{(\ell+1)} + \frac{1}{4}a_{m+1}a_{m+\ell+2}G_{2m+\ell+5}^{(\ell+1)} ,
\]
\[
D_{0-1}^{nm} = \frac{1}{16} (4a_{m+1}G_{2m+\ell+3}^{(\ell-3)} + 4G_{2m+\ell+1}^{(\ell-1)} + (a_{m} + b_{n})G_{2m+\ell+3}^{(\ell-1)}
+ \frac{1}{4}a_{m+1}b_{m+\ell}G_{2m+\ell+5}^{(\ell-1)}
+ 16G_{2m+\ell+1}^{(\ell+1)} + 4(b_{m+\ell} + b_{n})G_{2m+\ell+3}^{(\ell+1)} + a_{m+\ell+2}(a_{m} + b_{n})G_{2m+\ell+5}^{(\ell+1)}
+ \frac{1}{4}a_{m+1}a_{m+\ell+2}a_{m+\ell+3}G_{2m+\ell+7}^{(\ell+1)}
+ 16a_{m+\ell+2}G_{2m+\ell+3}^{(\ell+3)} + 4a_{m+\ell+2}a_{m+\ell+3}G_{2m+\ell+5}^{(\ell+3)} ,
\]
\[
D_{00}^{nm} = \frac{1}{16} (a_{m}G_{2m+\ell+2}^{(\ell-2)} + \frac{1}{4}a_{m}a_{m+1}G_{2m+\ell+4}^{(\ell-2)} + 16G_{2m+\ell+2}^{(\ell)} + 4(a_{m} + b_{n})G_{2m+\ell+4}^{(\ell)}
+ a_{m}(b_{m+\ell} + b_{n})G_{2m+\ell+4}^{(\ell)} + \frac{1}{4}a_{m}a_{m+1}a_{m+\ell+2}G_{2m+\ell+6}^{(\ell)}
+ 16b_{m+\ell}G_{2m+\ell+2}^{(\ell+2)} + 4a_{m+\ell+2}(a_{m} + b_{n})G_{2m+\ell+4}^{(\ell+2)} + a_{m}a_{m+\ell+2}a_{m+\ell+3}G_{2m+\ell+6}^{(\ell+2)}
+ 16a_{m+\ell+2}a_{m+\ell+3}G_{2m+\ell+4}^{(\ell+4)} .
\]

The corresponding results for the semi-short multiplet \(C_{n,m,\ell}\) may be obtained from those for \(D_{n,m,\ell}\) given above by taking
\[
a_{n'}m'(C_{n,m,\ell}) = a_{m'-1n'+1}(D_{m-1n+1,\ell}) .
\]

Using (5.27) then (5.38) easily follows from (5.37). We may also verify that (5.39) is satisfied. Combining (5.37) for \(m = n + 1\) with (5.26) we may then obtain
\[
a_{n'}m'(A_{n,n,\ell}^{2n+\ell}) - a_{m'-1n'+1}(A_{n,n,\ell}^{2n+\ell}) = 16(a_{n'm'}(D_{n,n,\ell}) - a_{m'-1n'+1}(D_{n,n,\ell}))
+ \frac{(n + 1)(n + 2)}{(2n + 1)(2n + 3)} (a_{m'-1n'+1}(C_{n+1n+1,\ell-2}) + a_{n'm'}(C_{n+1n+1,\ell-2}) ,
\]
which for \(n' \geq m'\), and noting the requirement (5.25), gives exactly (5.39).
In general the results from (5.41) can be expressed as

\[ a_{n+i \cdot m+j}(C_{n \cdot m, \ell}) = N_{n+1,i}N_{m,j} C_{i,j}^{nm}, \quad C_{2,j}^{nm} = 0. \] (5.43)

For general \( n, m \) the necessary operators are just those given in table 4 of [22]. For \( m = n \) the relation (5.41) combined with (5.40) in this case and applying the corresponding results to (5.23) gives

\[
C_{11}^{nm} = g^{(\ell+2)}_{2n+\ell+4},
\]
\[
C_{10}^{nm} = g^{(\ell+1)}_{2n+\ell+3} + \frac{1}{4} a_n g^{(\ell+1)}_{2n+\ell+5} + a_n + \ell + 3 g^{(\ell+3)}_{2n+\ell+5},
\]
\[
C_{00}^{nm} = g^{(\ell)}_{2n+\ell+2} + \frac{1}{4} a_n g^{(\ell)}_{2n+\ell+4} + (b_n + \ell + 1 - a_n + 1) g^{(\ell+2)}_{2n+\ell+4},
\]
\[
\quad + \frac{1}{16} a_n a_{n+1} g^{(\ell+2)}_{2n+\ell+6} + \frac{1}{4} a_n a_{n+1} a_{n+\ell + 3} g^{(\ell+2)}_{2n+\ell+6} + a_n + \ell + 3 a_n + \ell + 4 g^{(\ell+4)}_{2n+\ell+6},
\]
\[
C_{1-1}^{nm} = \frac{1}{4} g^{(\ell+2)}_{2n+\ell+4} + \frac{1}{4} a_n + \ell + 3 g^{(\ell+2)}_{2n+\ell+6},
\]
\[
C_{0-1}^{nm} = \frac{1}{4} g^{(\ell-1)}_{2n+\ell+3} + g^{(\ell+1)}_{2n+\ell+3} + \frac{1}{4} a_n + 1 g^{(\ell-1)}_{2n+\ell+5} + \frac{1}{4} b_n + \ell + 1 g^{(\ell+1)}_{2n+\ell+5},
\]
\[
\quad + a_n + \ell + 3 g^{(\ell+3)}_{2n+\ell+5} + \frac{1}{16} a_n + 1 a_n + \ell + 3 g^{(\ell+1)}_{2n+\ell+7} + \frac{1}{4} a_n + \ell + 3 a_n + \ell + 4 g^{(\ell+3)}_{2n+\ell+7},
\]
\[
C_{-1-1}^{nm} = \frac{1}{16} g^{(\ell-2)}_{2n+\ell+4} + \frac{1}{4} g^{(\ell+2)}_{2n+\ell+4} + \frac{1}{4} b_n + \ell + 1 - a_n + 2 g^{(\ell)}_{2n+\ell+6},
\]
\[
\quad + \frac{1}{4} a_n + \ell + 3 g^{(\ell+2)}_{2n+\ell+6} + \frac{1}{16} a_n + \ell + 3 a_n + \ell + 4 g^{(\ell+4)}_{2n+\ell+6}.
\]
\[
C_{1-2}^{nm} = \frac{1}{4} g^{(\ell+1)}_{2n+\ell+5},
\]
\[
C_{0-2}^{nm} = \frac{1}{4} g^{(\ell)}_{2n+\ell+4} + \frac{1}{16} a_n + 1 g^{(\ell)}_{2n+\ell+6} + \frac{1}{4} a_n + \ell + 3 g^{(\ell+2)}_{2n+\ell+6},
\]
\[
C_{-1-2}^{nm} = \frac{1}{16} g^{(\ell-1)}_{2n+\ell+5} + \frac{1}{4} g^{(\ell+1)}_{2n+\ell+5} + \frac{1}{16} a_n + \ell + 3 g^{(\ell+1)}_{2n+\ell+7},
\]
\[
C_{-2-2}^{nm} = \frac{1}{16} g^{(\ell)}_{2n+\ell+6}. \] (5.44)

The necessary operators correspond exactly to those listed in [22] (see table 3) as present in the semi-short supermultiplet for this case. For \( n = 0 \) (5.44) reproduces (3.21). We may also note that, since for \( m \geq 1, \frac{1}{4} < a_m \leq \frac{1}{3} \) and \( b_n \geq \frac{1}{2} \), all coefficients in (5.44) are positive as required by unitarity.

As in the \( N = 2 \) case the semi-short results also include the contributions for short BPS multiplets when extended to negative \( \ell \). Formally as shown in [22] \( C_{[q,p,q],-1} \simeq B_{[q+1,p,q+1]} \) where \( B_{[q,p,q]} \) denotes the BPS supermultiplet whose lowest state has spin zero, \( \Delta = 2q+p \), and belongs to the \( SU(4)R \) \([q,p,q]\) representation. For \( q > 0 \) the lowest state is annihilated by \( \frac{1}{4} \) of the \( Q \) and also \( \bar{Q} \) supercharges whereas when \( q = 0 \) we have a \( \frac{1}{2} \)-BPS multiplet with \( \frac{1}{2} \) the \( Q \) and \( \bar{Q} \) supercharges annihilating the lowest state. As earlier we identify, for \( n \geq m \), \( B_{n,m} \equiv B_{[n-m,2m,n-m]} \) and we then have

\[
a_{n\cdot m'}(C_{n\cdot m,-1}) = \frac{n + 1}{2n + 1} a_{n\cdot m'}(B_{n+1,m}), \] (5.45)
where
\[ a_{n+i,m+j}(B_{n,m}) = N_{n+1,i} N_{m,j} B_{i,j}^{nm}, \quad B_{2j}^{nm} = B_{1j}^{nm} = 0. \] (5.46)

For general \( n, m \) we have
\[ B_{i,j}^{nm} = B_{i[j]}^{nm}, \] (5.47)

and

\[ B_{02}^{nm} = \frac{1}{4} G_{2n+2}^{(0)}, \]
\[ B_{-22}^{nm} = \frac{1}{16} G_{2n+4}^{(0)}, \]
\[ B_{-12}^{nm} = \frac{1}{4} G_{2n+3}^{(1)}, \]
\[ B_{01}^{nm} = G_{2n+1}^{(1)} + \frac{1}{4} a_{n+1} G_{2n+3}^{(1)}, \]
\[ B_{-21}^{nm} = \frac{1}{4} G_{2n+3}^{(1)} + \frac{1}{16} a_{n+2} G_{2n+5}^{(1)}, \] (5.48)
\[ B_{-11}^{nm} = \frac{1}{4} G_{2n+2}^{(0)} + G_{2n+2}^{(2)} + \frac{1}{16} a_n G_{2n+4}^{(0)} + \frac{1}{4} a_{n+2} G_{2n+4}^{(2)}, \]
\[ B_{00}^{nm} = G_{2n}^{(0)} + \frac{1}{4} (b_{m-1} - a_n) G_{2n+2}^{(0)} + a_{n+1} G_{2n+2}^{(2)} + \frac{1}{16} a_n a_{n+1} G_{2n+4}^{(0)}, \]
\[ B_{-20}^{nm} = \frac{1}{4} G_{2n+2}^{(0)} + \frac{1}{16} (b_{m-1} - a_{n+1}) G_{2n+4}^{(0)} + \frac{1}{4} a_{n+2} G_{2n+4}^{(2)} + \frac{1}{64} a_n a_{n+1} a_{n+2} G_{2n+6}^{(0)}, \]
\[ B_{-10}^{nm} = G_{2n+1}^{(1)} + \frac{1}{4} b_{m-1} G_{2n+3}^{(1)} + a_{n+2} G_{2n+3}^{(3)} + \frac{1}{16} a_n a_{n+2} G_{2n+5}^{(1)}. \]

Again all coefficients are positive and the necessary operators are exactly as expected for this supermultiplet (see table 2 in [22]). For \( n = m + 1 \) the multiplet is truncated with, in (5.46), the following non zero,

\[ B_{01}^{m+1m} = G_{2m+3}^{(1)}, \]
\[ B_{00}^{m+1m} = G_{2m+2}^{(0)} + \frac{1}{4} a_m G_{2m+4}^{(0)} + a_{m+2} G_{2m+4}^{(2)}, \]
\[ B_{-10}^{m+1m} = G_{2m+3}^{(1)} + \frac{1}{4} a_m G_{2m+5}^{(1)} + a_{m+3} G_{2m+5}^{(3)}, \]
\[ B_{0-1}^{m+1m} = G_{2m+3}^{(1)} + \frac{1}{4} a_{m+2} G_{2m+5}^{(1)}, \] (5.49)
\[ B_{-1-1}^{m+1m} = \frac{1}{4} G_{2m+4}^{(0)} + G_{2m+4}^{(2)} + \frac{1}{16} a_{m+1} G_{2m+6}^{(0)} + \frac{1}{4} a_{m+3} G_{2m+6}^{(2)}, \]
\[ B_{-2-1}^{m+1m} = \frac{1}{4} G_{2m+5}^{(1)} + \frac{1}{16} a_{m+3} G_{2m+7}^{(1)}, \]
\[ B_{0-2}^{m+1m} = \frac{1}{4} G_{2m+4}^{(0)}, \quad B_{-1-2}^{m+1m} = \frac{1}{4} G_{2m+5}^{(1)}, \quad B_{-2-2}^{m+1m} = \frac{1}{16} G_{2m+6}^{(0)}. \]

The necessary operators correlate again with those expected for this \( \frac{1}{4} \)-BPS multiplet (see table 5 in [22]).

If we consider the semi-short multiplet for \( \ell = -2 \) we get
\[ a_{n,m'}(C_{n,m'-2}) = -4 a_{n',m'}(B_{n,m}), \] (5.50)

37
which allows results for \( a_{n'm'}(B_{n,m}) \) to be derived for \( m = n \) in addition to \( m < n \) as given by (5.43). However in this case there is a further decomposition into contributions corresponding to \( \frac{1}{2} \)-BPS multiplets. Such \( \frac{1}{2} \)-BPS contributions are obtained in (5.46) by letting \( B_{nm} \to \hat{B}_{nn} \) and \( B_{ij} \to \hat{B}_{ij} \) where

\[
\hat{B}_{00}^{nn} = g_{2n}^{(0)}, \quad \hat{B}_{0-1}^{nn} = g_{2n+1}^{(1)}, \quad \hat{B}_{-1-1}^{nn} = g_{2n+2}^{(0)}, \quad \hat{B}_{-2-2}^{nn} = \frac{1}{16} g_{2n+4}^{(0)},
\]

(5.51)

(the relevant operators here correspond to table 1 in [22]). With the result given in (5.51) we can then write in (5.50)

\[
a_{n'm'}(B_{n,n}) = a_{n'm'}(\hat{B}_{n,n}) - \frac{(n+1)(n+2)}{4(2n+1)(2n+3)} a_{n'm'}(\hat{B}_{n+1,n+1}).
\]

(5.52)

From (3.31) and (5.30) it is also easy to see that

\[
a_{nm}(\hat{B}_{00}) = a_{nm}(I).
\]

(5.53)

Any \( \frac{1}{2} \)-BPS contribution \( a_{n'm'}(\hat{B}_{n,n}) \) may then be isolated by considering appropriate linear combinations of \( a_{n'm'}(C_{n,n-2}) \) together with \( a_{n'm'}(I) \).

We also consider the extremal and next-to-extremal cases. When \( E = 0 \) \( G \) is independent of \( y, \bar{y} \) and so must also be the function \( f \) in (3.4). From (3.5) and (3.35) we then get the solution

\[
G(u,v) = u^{\frac{1}{2}p_+} k,
\]

(5.54)

where we define

\[
p_\pm = p_2 \pm p_1.
\]

(5.55)

Noting that

\[
P_{00}(p_1,p_2)(y,\bar{y}) = \frac{1}{2}(p_+ + 2), \quad g_{p_+}^{(0)}(u,v;p_-,p_+) = u^{\frac{1}{2}p_+},
\]

(5.56)

it is clear that the only operator which is necessary in the operator product expansion has \( \Delta = p_+ \) and is spinless belonging to the \([0,p_+,0]\) representation. This is of course may be identified with the contribution of just the \( \frac{1}{2} \)-BPS operator belonging to the short \( B_{[0,p_+]} \) supermultiplet so that for the extremal case, up to a constant factor,

\[
a_{nm}(B_{[0,p_+]}) = \delta_{n0}\delta_{m0} g_{p_+}^{(0)}.
\]

(5.57)

The correlation function in this case has the very simple form

\[
\left. \langle \varphi^{(p_1)}(x_1,t_1) \varphi^{(p_2)}(x_2,t_2) \varphi^{(p_3)}(x_3,t_3) \varphi^{(p_4)}(x_4,t_4) \rangle \right|_{p_4 = p_1 + p_2 + p_3} = \frac{(t_1 \cdot t_4)^{p_1} (t_2 \cdot t_4)^{p_2} (t_1 \cdot t_3)^{p_3}}{r_1^{p_1} r_2^{p_2} r_3^{p_3}} k.
\]

(5.58)
For the next-to-extremal case, $E = 1$, we have a similar solution to that given by (3.4) and (5.9), but with no arbitrary $K$ term and $\hat{f}$ a single variable function of $z$,

$$\hat{G}(u, v; y, \bar{y}) = u^{\frac{1}{2}p+1} \left( k - \frac{1}{z - \bar{z}} \left( (y - z) (\bar{y} - z) \hat{f}(z) - (y - \bar{z}) (\bar{y} - \bar{z}) \hat{f}(\bar{z}) \right) \right)$$

$$= \sum_{0 \leq m \leq n \leq 1} a_{nm}(u, v) P_{nm}^{(p_1 - 1, p_2 - 1)}(y, \bar{y}),$$

where we have expanded in terms of the different possible $SU(4)_R$ representations. From this we obtain

$$\frac{1}{16} p_+ (p_+ + 1) (p_+ + 2) a_{11} = \hat{a}_{11} = F_0,$$

$$\frac{1}{8} (p_+ + 1) (p_+ + 2) a_{10} = \hat{a}_{10} = F_1 + \frac{p_-}{p_+} F_0,$$

$$\frac{1}{2} p_+ \ a_{00} = \hat{a}_{00} = k u^{\frac{1}{2}p+1} + F_2 + \frac{2p_+ + 2}{p_+ + 2} F_1 + \frac{p_-^2 - (p_+ + 2)}{(p_+ + 1) (p_+ + 2)} F_0,$$

for

$$F_n(z, \bar{z}) = -( -1)^n \ u^{\frac{1}{2}p+1} \frac{z^n \hat{f}(z) - \bar{z}^n \hat{f}(\bar{z})}{z - \bar{z}} .$$

Keeping only the term in (5.60) involving $k$ we may easily from (5.56) see that this represents the contribution of just the $\frac{1}{2}$-BPS chiral primary operator belonging to the $B_{[0, p_+ - 2, 0]}$ supermultiplet so that in the next-to-extremal case we have

$$\hat{a}_{nm}(B_{[0, p_+ - 2, 0]}) = \delta_{n0} \delta_{m0} G_{p_+ - 2}^{(0)} .$$

If in (5.60) and (5.61) we let $\hat{f}(z) \to 2z^{\ell+1} \ (x; p_1, p_2)$ and use the definitions in (3.40) we obtain the contributions for the semi-short supermultiplet $C_{[0, p_+ - 2, 0], \ell}$,

$$\hat{a}_{11}(C_{[0, p_+ - 2, 0], \ell}) = G_{p_+ + \ell + 1}^{(\ell+2)},$$

$$\hat{a}_{10}(C_{[0, p_+ - 2, 0], \ell}) = G_{p_+ + \ell + 1}^{(\ell+1)} + b_{\ell+2} G_{p_+ + \ell + 3}^{(\ell+3)} + \frac{4(\ell + 2) p_-(p_+ + \ell + 1)}{(p_+ + 2\ell + 2)(p_+ + 2\ell + 4)} G_{p_+ + \ell + 2}^{(\ell+3)},$$

$$\hat{a}_{00}(C_{[0, p_+ - 2, 0], \ell}) = G_{p_+ + \ell}^{(\ell+1)} + b_{\ell+2} G_{p_+ + \ell + 2}^{(\ell+4)} + c_{\ell+2} G_{p_+ + \ell + 2}^{(\ell+3)}$$

$$+ \frac{8(\ell + 1) p_-(p_+ + \ell + 1)}{(p_+ + 2)(p_+ + 2\ell)(p_+ + 2\ell + 4)} G_{p_+ + \ell + 1}^{(\ell+1)}$$

$$+ \frac{8(\ell + 2) p_-(p_+ + \ell + 2)}{(p_+ + 2)(p_+ + 2\ell + 2)(p_+ + 2\ell + 6)} G_{p_+ + \ell + 3}^{(\ell+3)},$$

for

$$b_\ell = \frac{4(\ell + 1)(p_1 + \ell)(p_2 + \ell)(p_+ + \ell - 1)}{(p_+ + 2\ell - 1)(p_+ + 2\ell + 1)},$$

$$c_\ell = \frac{2(\ell + 1)(p_1 + \ell - 1)}{(p_+ + 1)(p_+ + 2\ell - 3)(p_+ + 2\ell + 1)} \left( p_+ - 1 + \frac{p_-^2 (8(\ell - 1)(p_+ + \ell) - p_+(p_+ - 1))}{(p_+ + 2)(p_+ + 2\ell - 2)(p_+ + 2\ell)} \right).$$

39
The necessary operators required for (5.63) correspond exactly with those in this semi-short supermultiplet (see table 3 in [22]).

Just as previously we may extend these (5.63) to \( \ell = -1, -2 \) to obtain results for short multiplets. Thus

\[
\hat{a}_{nm}\left(C_{[0,p_+ - 2,0],-1}\right) = \hat{a}_{nm}\left(B_{[1,p_+ - 2,1]}\right), \\
\hat{a}_{nm}\left(C_{[0,p_+ - 2,0],-2}\right) = \hat{a}_{nm}\left(B_{[0,p_+,0]} \right) - 4 \hat{a}_{nm}\left(B_{[0,p_+-2,0]} \right),
\]

where, together with (5.63),

\[
\hat{a}_{11}\left(B_{[1,p_+-2,1]}\right) = G^{(1)}_{p_++1}, \\
\hat{a}_{10}\left(B_{[1,p_+-2,1]}\right) = G^{(0)}_{p_+} + \frac{4p_-}{p_+(p_++2)}G^{(1)}_{p_++1} + b_1 G^{(2)}_{p_++2}, \\
\hat{a}_{00}\left(B_{[1,p_+-2,1]}\right) = b_1\left(G^{(1)}_{p_++1} + \frac{8p_-(p_++2)}{p_+(p_++2)(p_++4)}G^{(2)}_{p_++2} + b_2 G^{(3)}_{p_++3}\right),
\]

and

\[
\hat{a}_{11}\left(B_{[0,p_+,0]}\right) = G^{(0)}_{p_+}, \\
\hat{a}_{10}\left(B_{[0,p_+,0]}\right) = b_0 G^{(1)}_{p_++1}, \\
\hat{a}_{00}\left(B_{[0,p_+,0]}\right) = b_0 b_1 G^{(2)}_{p_++2}.
\]

The necessary operators here correspond to table 5 and table 1 in [22].

The results obtained above show that the operator product expansion for \( \frac{1}{2} \)-BPS operators can be decomposed into short, semi-short and long supermultiplets. For \( p_- = p_2 - p_1 \geq 0 \),

\[
B_{[0,p_1,0]} \otimes B_{[0,p_2,0]} \simeq \bigoplus_{0 \leq m \leq n \leq p_1} B_{[n-m,p_-+2m,n-m]} \\
\oplus \bigoplus_{\ell \geq 0} \bigoplus_{0 \leq m \leq n \leq p_1-1} C_{[n-m,p_-+2m,n-m],\ell} \\
\oplus \bigoplus_{\ell \geq 0} \bigoplus_{0 \leq m \leq n \leq p_1-2} A^\Delta_{[n-m,p_-+2m,n-m],\ell},
\]

in accordance with the results of Eden and Sokatchev [24]. In (5.68) we identify \( B_{[0,0,0]} \simeq I \), corresponding to the unit operator in the operator product expansion. It immediately follows from (5.68) that long supermultiplets, with non zero anomalous dimensions, cannot contribute to extremal and next-to-extremal correlation functions.

6. Crossing Symmetry

The operator product expansion provides the strongest constraints when combined with crossing symmetry. For a correlation function for four identical chiral primary operators the correlation function is invariant under permutations of all \( x_i, t_i \) for all \( i = 1, 2, 3, 4 \).
Permutations of the form \((ij)(kl)\) act trivially so we may restrict to permutations leaving \(x_4, t_4\) invariant so that crossing symmetry transformations correspond to the permutation group \(S_3\), which is of order 6. The action of each permutation on the essential conformal invariants \(u, v\) or \(x, \bar{x}\) or \(y, \bar{z}\) and also on the \(R\)-symmetry invariants \(\sigma, \tau\) or \(\alpha, \bar{\alpha}\) or \(y, \bar{y}\) is given in table 1, where the transformations of \(\bar{x}\) are identical to those of \(x\), and similarly for \(\bar{z}, \bar{\alpha}, \bar{y}\).

| \(e\) | (12) | (13) | (23) | (123) | (132) | \(e\) | (12) | (13) | (23) | (123) | (132) |
|------|------|------|------|------|------|------|------|------|------|------|------|
| \(u\) | \(\frac{u}{v}\) | \(v\) | \(\frac{1}{u}\) | \(\frac{v}{u}\) | \(\frac{1}{v}\) | \(\sigma\) | \(\tau\) | \(\frac{\sigma}{\tau}\) | \(\frac{1}{\sigma}\) | \(\frac{1}{\tau}\) | \(\frac{\tau}{\sigma}\) |
| \(v\) | \(\frac{1}{v}\) | \(u\) | \(\frac{v}{u}\) | \(\frac{1}{u}\) | \(\frac{u}{v}\) | \(\tau\) | \(\sigma\) | \(\frac{1}{\tau}\) | \(\frac{\tau}{\sigma}\) | \(\frac{1}{\sigma}\) | \(\frac{1}{\tau}\) |
| \(x\) | \(\frac{x}{x-1}\) | \(1-x\) | \(\frac{1}{x}\) | \(\frac{x-1}{x}\) | \(\frac{1}{1-x}\) | \(\alpha\) | \(1-\alpha\) | \(\frac{\alpha}{\alpha-1}\) | \(\frac{1}{\alpha}\) | \(\frac{1}{1-\alpha}\) | \(\frac{\alpha-1}{\alpha}\) |
| \(z\) | \(\frac{z+3}{z-1}\) | \(\frac{3z}{z+3}\) | \(\frac{3+z}{1-z}\) | \(\frac{z-3}{z+1}\) | \(y\) | \(-y\) | \(\frac{y+3}{y-1}\) | \(\frac{3-y}{1+y}\) | \(\frac{y+3}{y-1}\) | \(\frac{3-y}{1+y}\) |

Table 1. Symmetry transformations of variables under crossing.

For the \(\mathcal{N} = 4\) case with \(p_i = p\) the crossing symmetry conditions on the correlation function \(\mathcal{G}(u, v; \sigma, \tau)\) are generated by considering just (12) and (13) which give

\[
\mathcal{G}(u, v; \sigma, \tau) = \mathcal{G}(u/v, 1/v; \tau, \sigma) = \left(\frac{u}{v}\right)^p \mathcal{G}(v, u; \sigma/\tau, 1/\tau). \tag{6.1}
\]

The general construction of such invariant correlation functions follows by determining polynomials in \(\sigma, \tau\) which transform according to the irreducible representations of \(S_3\). We first consider symmetric polynomials satisfying

\[
S_p(\sigma, \tau) = S_p(\tau, \sigma) = \tau^p S_p(\sigma/\tau, 1/\tau). \tag{6.2}
\]

As described by Heslop and Howe [17], for any given \(p\), \(S_3\) acts on the \(\frac{1}{2}(p+1)(p+2)\) monomials \(\sigma^r \tau^s\), \(r + s \leq p\), giving chains of length 6 or 3 or 1 which may be added to give minimal polynomial solutions of (6.2). If the chain contains a monomial \((\sigma \tau)^r\), for \(0 \leq r \leq \lfloor \frac{p}{2} \rfloor\), where \(\lfloor x\rfloor\) denotes the integer part of \(x\), then this term is invariant under the action of the permutation (12) and the chain is of length 3, except if \(p\) is divisible by 3 then \((\sigma \tau)^{p/3}\) satisfies (6.2) by itself and so forms a chain of length 1. All other chains are of length 6. With this counting the number of independent such minimal symmetric polynomials is,

\[
N_p = \begin{cases} 
(n+1)3n+1, & p = 6n; \\
(n+1)(3n+q), & p = 6n + q, \quad q = 1, 2, 3, 4, 5.
\end{cases} \tag{6.3}
\]

We list the first few non trivial cases in table 2, of course \(S_0(\sigma, \tau) = 1\).
| $p$ | polynomial | $(i, j)$ |
|-----|------------|----------|
| 1   | $\sigma + \tau + 1$ | (0,0) |
| 2   | $\sigma^2 + \tau^2 + 1$ | (0,0),(1,0) |
| 3   | $\sigma^3 + \tau^3 + 1$ | (0,0),(1,0) |
|     | $\sigma^2 + \sigma^2 \tau + \sigma + \tau$ | (0,1) |
| 4   | $\sigma^4 + \tau^4 + 1$ | (0,0),(1,0),(2,0) |
|     | $\sigma^3 \tau + \sigma^3 + \tau^3 + \sigma + \tau$ | (0,1) |
|     | $\sigma^2 \tau + \sigma^2 \tau^2 + \sigma \tau$ | (0,1) |
| 5   | $\sigma^5 + \tau^5 + 1$ | (0,0),(1,0),(2,0) |
|     | $\sigma^4 \tau + \sigma^4 + \tau^4 + \sigma + \tau$ | (0,1) |
|     | $\sigma^3 \tau^2 + \sigma^3 \tau + \sigma^3 + \tau^3 + \sigma^2 \tau^2$ | (0,1) |
|     | $\sigma^2 \tau + \sigma^2 \tau^2 + \sigma \tau$ | (0,1) |

Table 2. Symmetric polynomials.

An alternative basis for $S_p$, valid for general $p$, may be obtained by constructing from $\sigma, \tau$ two invariants $I_1, I_2$ under $S_3$ and then introducing for any $p$ a factor to ensure that (5.2) holds. With suitable restrictions the result becomes a polynomial expressible in the form

$$ S_{p,(i,j)}(\sigma, \tau) = (\sigma + \tau + 1)^p I_1(\sigma, \tau)^i I_2(\sigma, \tau)^j, $$

$$ I_1(\sigma, \tau) = \frac{\sigma \tau + \sigma + \tau}{(\sigma + \tau + 1)^2}, \quad I_2(\sigma, \tau) = \frac{\sigma \tau}{(\sigma + \tau + 1)^3}, $$

$$ i = 0, 1, \ldots, \left[\frac{1}{2}p\right], \quad j = 0, 1, \ldots, \left[\frac{1}{3}(p - 2i)\right]. $$

Lists of possible $(i, j)$ for $p$ up to 7 are given in table 2. This result may also be easily expressed as symmetric polynomial in $y, \bar{y}$ by using

$$ \sigma + \tau + 1 = \frac{1}{2}(y \bar{y} + 3), \quad \sigma \tau = \frac{1}{16}(1 - y^2)(1 - \bar{y}^2), $$

$$ \Lambda = (\sigma + \tau + 1)^2 - 4(\sigma \tau + \sigma + \tau) = \frac{1}{4}(y - \bar{y})^2, $$

where $\Lambda$ is defined in (2.42). Completeness of the basis provided by (6.4) is straightforwardly demonstrated by showing that it gives the same number of independent polynomials $N_p$ as given in (5.3).

For the antisymmetric representation of $S_3$ we require,

$$ a \rightarrow -a, \quad a \rightarrow a. $$

(6.6)
while the two-dimensional mixed symmetry representation of $S_3$ is defined on a basis $(b,c)$ where
\[
\begin{pmatrix} b \\ c \end{pmatrix} \overset{(12)}{\mapsto} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix}, \quad \begin{pmatrix} b \\ c \end{pmatrix} \overset{(123)}{\mapsto} \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix}.
\]

It is easy to see that the tensor products formed by $aa'$ and $bb' + cc'$ are symmetric while $(bc' + c'b, bb' + cc')$ is a basis for a mixed symmetry representation and $bc' - c'b$ is antisymmetric.

For functions of $\sigma, \tau$ (6.6) is satisfied by
\[
a(\sigma, \tau) = \frac{(\sigma - \tau)(\sigma - 1)(\tau - 1)}{(\sigma + \tau + 1)^3}. \tag{6.8}
\]

For $p \geq 3$, $a(\sigma, \tau)S_{p, (i,j)}(\sigma, \tau)$ is a polynomial if we allow $i = 0, 1, \ldots, \lfloor \frac{1}{2}(p - 3) \rfloor$ and $j = 0, 1, \ldots, \lfloor \frac{1}{3}(p - 2i - 3) \rfloor$ giving $N_{p-3}$ antisymmetric polynomials. For the mixed symmetry transformations in (6.7) there essentially two independent possibilities
\[
b_1(\sigma, \tau) = \frac{\sigma - \tau}{\sigma + \tau + 1}, \quad c_1(\sigma, \tau) = \frac{\sigma + \tau - 2}{\sqrt{3}(\sigma + \tau + 1)}. \tag{6.9}
\]
and
\[
b_2(\sigma, \tau) = \frac{\sigma - \tau}{(\sigma + \tau + 1)^2}, \quad c_2(\sigma, \tau) = \frac{\sigma + \tau - 2 \sigma \tau}{\sqrt{3}(\sigma + \tau + 1)^2}. \tag{6.10}
\]

By considering $(b_r(\sigma, \tau), c_r(\sigma, \tau))S_{p, (i,j)}(\sigma, \tau)$ for $p \geq r$, $r = 1, 2$, for appropriate $i, j$ we obtain $N_{p-r}$ polynomial mixed symmetry representations of $S_3$. Together with the symmetric polynomials $S_{p, (i,j)}$ and $aS_{p, (i,j)}$ these provide a complete basis for two variable polynomials in $\sigma, \tau$ of order $p$ since $N_p = (N_{p-1} + N_{p-2}) + N_{p-3} = \frac{1}{2}(p + 1)(p + 2)$. We may also note that these polynomials form a closed set under multiplication since
\[
b_1^2 + c_1^2 = \frac{4}{3} - 4I_1, \quad \sqrt{3}(2b_1c_1, b_1^2 - c_1^2) = 2(b_1 - 3b_2, c_1 - 3c_2),
\]
\[
b_1b_2 + c_1c_2 = \frac{2}{3}I_1 - 6I_2, \quad \sqrt{3}(b_1c_2 + c_1b_2, b_1b_2 - c_1c_2) = 2(3b_1b_2 - I_1b_2, I_1c_1 - c_2),
\]
\[
b_2^2 + c_2^2 = \frac{4}{3}I_1^2 - 4I_2, \quad \sqrt{3}(2b_2c_2, b_2^2 - c_2^2) = 2(3b_1b_2 - I_1b_2, 3I_2c_1 - I_1c_2),
\]
\[
\sqrt{3}(b_1c_2 - c_1b_2) = 2a, \quad a^2 = I_1^2 - 4I_1^3 + 18I_1I_2 - 4I_2 - 27I_2^2. \tag{6.11}
\]

where $I_1, I_2$ are the invariants defined in (6.4).

For $N = 2$ there are further restrictions as a consequence of (6.2). Taking $p \to 2n$ we construct, instead of (6.2) since $\sigma, \tau$ are expressible in terms of just $\alpha$ by (2.43), the single variable polynomials $f_n$ of degree $2n$, satisfying under the action of $S_3$
\[
f_n(\alpha) = f_n(1 - \alpha) = (\alpha - 1)^{2n}f_n\left(\frac{\alpha}{\alpha - 1}\right) = \alpha^{2n}f_n\left(\frac{1}{\alpha}\right)
\]
\[
= (\alpha - 1)^{2n}f_n\left(\frac{1}{1 - \alpha}\right) = (\alpha - 1)^{2n}f_n\left(\frac{\alpha - 1}{\alpha}\right). \tag{6.12}
\]
As shown by Heslop and Howe, [17], the sum of terms produced by the action of \( S_3 \) as given by (6.12) starting from \( \alpha^r \) generates a linearly independent set of polynomials for \( r = 0, 1, \ldots, [\frac{1}{3} n] \), giving \([\frac{1}{3} n] + 1\) solutions for \( f_n \). Alternatively an equivalent basis is provided by

\[
S_{n,j}(\alpha) = (\alpha^2 - \alpha + 1)^n s(\alpha)^j, \quad j = 0, 1, \ldots, [\frac{1}{3} n],
\]

where \( s(\alpha) \) is the \( S_3 \) invariant

\[
s(\alpha) = \frac{\alpha^2 (1 - \alpha)^2}{(\alpha^2 - \alpha + 1)^3} = \frac{(1 - y^2)^2}{(y^2 + 3)^3}.
\]

The solutions given by (6.13) correspond to (6.4) for \( i = 0 \) since \( \Lambda = 0 \) in this case. A general polynomial solution of (6.12) is then given by

\[
f_n(\alpha) = (\alpha^2 - \alpha + 1)^n P(s(\alpha)),
\]

with \( P(s) \) a polynomial of degree \([\frac{1}{3} n]\).

We may also consider other representations of \( S_3 \). For the antisymmetric representation, as in (6.8), we may define

\[
a(\alpha) = (2\alpha - 1) \frac{(\alpha - 2)(\alpha - 1)\alpha(\alpha + 1)}{(\alpha^2 - \alpha + 1)^3} = 4 \frac{y(y^2 - 1)(y^2 - 9)}{(y^2 + 3)^3},
\]

so that \( a(\alpha)S_{n,j}(\alpha) \) is then a polynomial for \( n \geq 3 \) and \( j = 0, 1, \ldots, [\frac{1}{3} n] - 1 \). For the mixed symmetry representation there are two essential solutions which can be written in the form

\[
b_1(\alpha) = \frac{2\alpha - 1}{\alpha^2 - \alpha + 1} = 4 \frac{y}{y^2 + 3},
\]

\[
c_1(\alpha) = \frac{1}{\sqrt{3}} \frac{2\alpha^2 - 2\alpha - 1}{\alpha^2 - \alpha + 1} = 2 \frac{y^2 - 3}{\sqrt{3} y^2 + 3},
\]

and

\[
b_2(\alpha) = (2\alpha - 1) \frac{\alpha(\alpha - 1)}{(\alpha^2 - \alpha + 1)^2} = 4 \frac{y(y^2 - 1)}{(y^2 + 3)^2},
\]

\[
c_2(\alpha) = \sqrt{3} \frac{\alpha(\alpha - 1)}{(\alpha^2 - \alpha + 1)^2} = 4\sqrt{3} \frac{y^2 - 1}{(y^2 + 3)^2}.
\]

It is easy to see that \((b_r(\alpha), c_r(\alpha))S_{n,j}(\alpha)\) are polynomials for \( j = 0, 1, \ldots, [\frac{1}{3}(n-r)] \) if \( n \geq r \) for \( r = 1, 2 \). The basis provided by \( S_{n,j}(\alpha), (b_r(\alpha), c_r(\alpha))S_{n,j}(\alpha), \) \( r = 1, 2 \) and \( a(\alpha)S_{n,j}(\alpha) \) is then complete in that it gives \( 2n + 1 \) linearly independent polynomials, allowing for the expansion of any arbitrary polynomial of degree \( 2n, 2([\frac{1}{3} n] + [\frac{1}{3}(n-1)] + [\frac{1}{3}(n-2)]) + 5 = 2n + 1 \).
For \( \mathcal{N} = 4 \) the superconformal Ward identities require
\[
\mathcal{G}(u, v; \sigma, \tau) \bigg|_{\alpha=\frac{1}{2}} = f(x, \alpha), \tag{6.19}
\]
so that \((6.1)\) gives
\[
f(x, \alpha) = f\left(\frac{x}{x-1}, 1-\alpha\right) = \left(\frac{x(\alpha - 1)}{1-x}\right)^p f\left(1-x, \frac{\alpha}{\alpha-1}\right) = (x\alpha)^p f\left(\frac{1}{x}, \frac{1}{\alpha}\right). \tag{6.20}
\]

To obtain an extension to a fully crossing symmetric correlation function we may consider for any \( S_p \) satisfying \((6.2)\)
\[
\mathcal{G}(u, v; \sigma, \tau) = S_p\left(\frac{u}{v}, \frac{u}{\tau}\right), \tag{6.21}
\]
which obeys \((6.1)\) as a consequence of \((6.2)\). From \((6.19)\) we obtain
\[
f(x, \alpha) = S_p\left(x\alpha, \frac{x(1-\alpha)}{x-1}\right), \tag{6.22}
\]
which automatically satisfies \((6.20)\).

For the \( \mathcal{N} = 2 \) case instead of \((6.1)\) we have
\[
\mathcal{G}(u, v; \sigma, \tau) = \mathcal{G}\left(\frac{u}{v}, 1/\tau; \tau\right) = \left(\frac{u^2}{v^2} \frac{\sigma}{\tau}\right)^n \mathcal{G}(v, u; \sigma/\tau, 1/\tau), \tag{6.23}
\]
where, with \( \sigma, \tau \) constrained as in \((2.43)\), the superconformal Ward identities are
\[
\mathcal{G}(u, v; \sigma, \tau) \bigg|_{\alpha=\frac{1}{2}} = f(x) = f\left(\frac{x}{x-1}\right) = \left(\frac{x}{x-1}\right)^{2n} f(1-x) = x^{2n} f\left(\frac{1}{x}\right), \tag{6.24}
\]
where we also exhibit the crossing symmetry relations for the single variable function \( f \).

The corresponding solution to \((6.21)\) is given by
\[
\mathcal{G}(u, v; \sigma, \tau) = S_n\left(u^2 \sigma, \frac{u^2}{v^2} \tau\right), \tag{6.25}
\]
which implies
\[
f(x) = S_n\left(x^2, \frac{x^2}{(1-x)^2}\right). \tag{6.26}
\]

In this case if we consider the contribution of individual factors in the basis given by \((6.4)\) to \( f(x) \) as expected from \((6.23)\) and \((6.24)\) we have
\[
P = \left(\frac{u^2 \sigma + u^2}{v^2} \tau + 1\right) \bigg|_{\alpha=\frac{1}{2}} = p^2, \quad Q = \frac{u^4}{v^2} \sigma \tau \bigg|_{\alpha=\frac{1}{2}} = q^2, \tag{6.27}
\]
\[
R = \left(\frac{u^4}{v^2} \sigma \tau + u^2 \sigma + \frac{u^2}{v^2} \tau\right) \bigg|_{\alpha=\frac{1}{2}} = 2pq,
\]
where
\[ p(x) = \frac{x^2 - x + 1}{1 - x}, \quad q(x) = \frac{x^2}{1 - x}. \tag{6.28} \]
so that we have the relation \( R^2 = 4PQ \). In consequence we may restrict in (6.4) to those polynomials with \( i = 0, 1 \).

Conversely we may argue that for the \( \mathcal{N} = 2 \) case all single variable functions \( f(x) \) may be expressible in terms of \( S_n \) as in (6.26) and therefore may be extended to a fully crossing symmetric form for \( G(u,v;\sigma,\tau) \) as exhibited in (6.23). To demonstrate this we suppose all solutions of the crossing symmetry relations in (6.24) for \( f \) are solvable by writing
\[ f(x) = p(x)^{2n} g(s(x)), \quad s(x) = \frac{q(x)}{p(x)^3}, \tag{6.29} \]
for some function \( g \) of the crossing invariant \( s \) given by (6.14). Note that for \( x \to 0, s \sim x^2, x \to 1, s \sim (1 - x)^2 \) and for \( x \to \infty, s \sim 1/x^2 \). From the superconformal representation theory for the corresponding contributions to the operator product expansion \( f(x) \) should be analytic in the neighbourhood of \( x = 0 \) with singularities only at \( x = 1, \infty \). In consequence \( g(s) \) must be a polynomial which is then restricted to have maximal degree \( \left[ \frac{3}{2}n \right] \) to avoid singularities when \( x^2 - x + 1 = 0 \). It is then easy to see that \( f \) can be written as a polynomial in \( P, Q \) with terms also linear in \( R \), as defined in (6.27), which is consistent with (6.26) where \( S_n \) has an expansion in terms of \( S_{n,(i,j)} \) with \( i = 0, 1 \) and \( j \) restricted as in (6.4).

A similar discussion is possible for \( \mathcal{N} = 4 \). The function \( f(x,\alpha) \) is required to be a general solution of the crossing symmetry conditions given by (6.20) which is also a polynomial of degree \( p \) in \( \alpha \). It is also analytic in \( x \) in the neighbourhood of \( x = 0 \) with singularities only at \( x = 1, \infty \). If we write
\[ f(x,\alpha) = P(x,\alpha)^p g(x,\alpha), \quad P(x,\alpha) = \frac{x^2\alpha - 2x\alpha + 2x - 1}{x - 1}, \tag{6.30} \]
then \( g \) is an invariant under the action of \( S_3 \), as displayed in Table 1. Determining a general form for \( g \) is then reducible to finding a basis for all possible independent invariants which may be formed from \( x \) and \( \alpha \). Since the action of \( S_3 \) on any polynomial in \( \alpha \) may be decomposed, up to functions of the invariant \( s(\alpha) \), into contributions linear in \( 1, a(\alpha) \) and \( (b_r(\alpha), c_r(\alpha)), r = 1, 2, \) as given in (6.16) and (6.17), (6.18), then a basis for such invariants is obtained, in addition to the separate invariants \( s(x), s(\alpha) \), by combining these non trivial irreducible representations with corresponding representations involving \( x \) to give
\[ A(x,\alpha) = a(x^{-1}) a(\alpha), \tag{6.31} \]
where \(a(x^{-1}) = -a(x)\), and also

\[
S_1(x, \alpha) = b_1(x^{-1}) b_1(\alpha) + c_1(x^{-1}) c_1(\alpha) \\
= \frac{4}{3} - \frac{2(x\alpha - 1)^2}{(\alpha^2 - \alpha + 1)(x^2 - x + 1)},
\]

\[
S_2(x, \alpha) = b_2(x^{-1}) b_2(\alpha) + c_2(x^{-1}) c_2(\alpha) \\
= \frac{2\alpha(1-\alpha) x(1-x)}{(\alpha^2 - \alpha + 1)(x^2 - x + 1)^2} (x\alpha - 2\alpha - 2x + 1),
\]

\[
S_3(x, \alpha) = b_2(x^{-1}) b_1(\alpha) + c_2(x^{-1}) c_1(\alpha) \\
= \frac{2x(1-x)}{(\alpha^2 - \alpha + 1)(x^2 - x + 1)^2} (x\alpha^2 - 2x\alpha + 2\alpha - 1),
\]

\[
S_4(x, \alpha) = b_1(x^{-1}) b_2(\alpha) + c_1(x^{-1}) c_2(\alpha) \\
= \frac{2\alpha(1-\alpha)}{(\alpha^2 - \alpha + 1)(x^2 - x + 1)} (x^2\alpha - 2x\alpha + 2x - 1).
\]

These are not independent since

\[
A(x, \alpha) = \frac{4}{3} (S_1(x, \alpha) S_2(x, \alpha) - S_3(x, \alpha) S_4(x, \alpha)),
\]

\[
S_2(x, \alpha) - \frac{1}{2} S_1(x, \alpha) S_3(x, \alpha) - \frac{1}{2} S_3(x, \alpha) = -2s(x),
\]

\[
S_2(x, \alpha) - \frac{1}{2} S_1(x, \alpha) S_4(x, \alpha) - \frac{1}{2} S_4(x, \alpha) = -2s(x),
\]

\[
2(S_3(x, \alpha) + S_4(x, \alpha)) - 6S_2(x, \alpha) + S_1(x, \alpha)^2 - \frac{2}{3} S_1(x, \alpha) = \frac{8}{9}.
\]

A crucial further constraint arises from (5.3) which here requires that \(f(x, x^{-1})\) is a constant. Since \(P(x, x^{-1}) = 3\) we also require that \(g\) depends on invariants \(s_r(x, \alpha)\) such that \(s_r(x, x^{-1})\) are constants. Taking account of the relations in (6.33) there are then two independent solutions which we take as

\[
s_1(x, \alpha) = 2 \frac{S_3(x, \alpha) s(\alpha)}{S_4(x, \alpha)^2} = \frac{R(x, \alpha)}{P(x, \alpha)^2},
\]

\[
s_2(x, \alpha) = 8 \frac{s(x) s(\alpha)^2}{S_4(x, \alpha)^3} = \frac{Q(x, \alpha)}{P(x, \alpha)^3},
\]

\[
R(x, \alpha) = \frac{x(x\alpha^2 - 2x\alpha + 2\alpha - 1)}{1-x},
\]

\[
Q(x, \alpha) = \frac{x^2\alpha(1-\alpha)}{x-1},
\]

where \(R(x, x^{-1}) = 3, Q(x, x^{-1}) = 1\). It is then evident that \(g\) in (6.30) must be of the form

\[
g = \sum_{i,j \geq 0} c_{ij} s_1^i s_2^j.
\]

Noting that

\[
P(x, \alpha) = \left( u\sigma + \frac{u}{v} \tau + 1 \right) \Bigg|_{\tilde{\alpha} = \frac{1}{2}},
\]

\[
Q(x, \alpha) = \left. \frac{u^2}{v} \sigma \tau \right|_{\tilde{\alpha} = \frac{1}{2}},
\]

\[
R(x, \alpha) = \left. \left( \frac{u^2}{v} \sigma \tau + u\sigma + \frac{u}{v} \tau \right) \right|_{\tilde{\alpha} = \frac{1}{2}},
\]

\[
47
\]
it is easy to see, as a consequence of (6.4), that \( I_r(u\sigma, u\tau/v)|_{\bar{\sigma}=1/\bar{x}} = s_r(x, \alpha) \) and hence the expression given by (5.30) and (5.35) for the function \( f \) may always be extended to a fully crossing symmetric result for the full correlation function \( G \) of the form (6.21) with \( S_p(\sigma, \tau) = (\sigma + \tau + 1)^p \sum_{i,j} c_{ij} I_1(\sigma, \tau)^i I_2(\sigma, \tau)^j \) and where \( f \) satisfies (6.22). With appropriate coefficients for the independent terms in \( S_p \) (6.21) corresponds to the results of free field theory. In general, using the formalism of harmonic superspace, the Intriligator insertion technique \cite{26} demonstrates that only \( H \) as in (1.6) or (1.7), or \( K \) as in (3.7) or (5.8), can depend on the coupling \( g \), and so are dynamical. The functions \( f(x) \) or \( f(x, \alpha) \) are then identical with the free theory, or \( g = 0 \), results.

The remaining part of the correlation function may also be expressed in terms of \( S_3 \) representations. It is convenient to define from (6.9) and (6.10) \( (b_r(u, v), c_r(u, v)) = (b_r(1/u, v/u), c_r(1/u, v/u)) \). For the \( N = 2 \) case we may then write for the factor which appears in the solution of the superconformal identities in (1.7)

\[
(\alpha x - 1)(\alpha \bar{x} - 1) = (\alpha^2 - \alpha + 1)(u + v + 1)(\frac{1}{2} - \frac{1}{2}(b_1'(u, v) b_1(\alpha) + c_1'(u, v) c_1(\alpha))).
\] (6.37)

For \( N = 4 \) we may also note

\[
(\alpha x - 1)(\alpha \bar{x} - 1)(\alpha x - 1)(\alpha \bar{x} - 1) = \frac{1}{16} u^2 (y - z)(y - \bar{z})(\bar{y} - z)(\bar{y} - \bar{z})
\]

\[
= v + \sigma^2 u v + \tau^2 u + \sigma v(u - v) + \tau(u + v - 1) + \sigma \tau u(u - 1 - v)
\]

\[
= (\sigma + \tau + 1)^2(u + v + 1)^2 \left( \frac{1}{4} I_1(u, v) + \frac{1}{4} I_1(\sigma, \tau) - 2 I_1(u, v) I_1(\sigma, \tau)
\right.
\]

\[
- \frac{1}{2}(b_1'(u, v) b_2(\sigma, \tau) + c_1'(u, v) c_2(\sigma, \tau)
\]

\[
+ b_2'(u, v) b_1(\sigma, \tau) + c_2'(u, v) c_1(\sigma, \tau)
\]

\[
- 3 b_2'(u, v) b_2(\sigma, \tau) - 3 c_2'(u, v) c_2(\sigma, \tau) \right).\) (6.38)

The function \( K \) in (5.8) must then satisfy the crossing symmetry relations

\[
K(u, v; \sigma, \tau) = K(u/v, 1/v; \tau, \sigma) = \left( \frac{u}{v} \right)^{p+2} \tau^{p-2} K(v, u; \sigma/\tau, 1/\tau).\) (6.39)

It is also of interest to extend the considerations of crossing symmetry to the next-to-extremal case when \( p_1 = p_2 = p_3 = p, p_4 = 3p - 2 \). In this case \( G \), defined by (5.1), must satisfy for the permutations (12) and (23)

\[
G(u, v; \sigma, \tau) = u^{p-1} G\left( \frac{u}{v}, \frac{1}{v}; \tau, \sigma \right), \quad G(u, v; \sigma, \tau) = u^{2p-1} \sigma G\left( \frac{1}{u}, \frac{v}{u}; \frac{1}{\sigma}, \frac{\tau}{\sigma} \right).\) (6.40)

The solution (5.59) can be rewritten as

\[
G(u, v; \sigma, \tau) = u^{p-1} \left( k + \frac{x\bar{x}}{x - \bar{x}} (\alpha - 1/x)(\bar{\alpha} - 1/x)f(x) - (\alpha - 1/\bar{x})(\bar{\alpha} - 1/\bar{x})f(\bar{x}) \right),\) (6.41)
and then (6.40) requires
\[
f(x) = -f\left(\frac{x}{x-1}\right), \quad f(x) = -x^2 f\left(\frac{1}{x}\right) + k x. \tag{6.42}
\]
A particular solution of (6.42) is given by
\[
f(x) = \frac{k}{3} \left(x - \frac{x}{x-1}\right). \tag{6.43}
\]
To obtain a general solution of (6.42) it is then sufficient to seek the general solution \( f_0(x) \) of (6.42) with \( k = 0 \). Using results obtained above this is
\[
f_0(x) = \frac{(x-2)x(x+1)(2x-1)}{(x-1)(x^2-x+1)} h(s(x)), \tag{6.44}
\]
where \( s \) is the invariant defined by (6.29) and (6.28). This introduces unphysical singularities for \( x^2 - x + 1 = 0 \) unless cancelled by \( h \). However, for compatibility with semi-short representations, \( h(s) \) must be analytic in \( s \) for \( s \sim 0 \) (if \( h(s) = 1/s \), which cancels the singularity at \( x^2 - x + 1 = 0 \), then \( f_0(x) \sim 1/x \) for \( x \to 0 \)). Hence we conclude that there is no possible solution of the form (6.44) and hence we only have (6.43). In this case
\[
G(u, v; \sigma, \tau) = \frac{1}{3} k u^{p-1} \left(1 + \sigma u + \tau \frac{u}{v}\right). \tag{6.45}
\]

7. Large \( N \) Results

In this paper we have endeavoured to work out the consequences of superconformal symmetry for the four point correlation functions of BPS operators. As a result our considerations are lacking in dynamical input since we do not consider any details of \( \mathcal{N} = 2 \) or \( \mathcal{N} = 4 \) superconformal theories. The results of our analysis demonstrate that the details of the dynamics resides in the function \( H \), which appears in (1.6) and (1.7), or \( K \) as in (5.8). In particular cases results have been obtained using perturbation theory \cite{3} or with the AdS/CFT correspondence \cite{4,6,7}. We here summarise some of the results obtained in \cite{4,6,7} in the context of this paper.

For \( p_i = p \) in the large \( N \) limit the leading result is, with a suitable normalisation, simply obtained from free field theory
\[
G_0(u, v; \sigma, \tau) = 1 + (\sigma u)^p + \left(\frac{\tau}{v}\right)^p. \tag{7.1}
\]
The definitions (5.4) and (5.5) then give
\[
f_0(z, y) = 1 + \left(\frac{1+y}{1+z}\right)^p + \left(\frac{1-y}{1-z}\right)^p, \quad k = 3. \tag{7.2}
\]
Using (5.8) we can then determine, assuming $K(u, v; \sigma, \tau) = \frac{1}{16} u^2 H(u, v; \sigma, \tau)$, for $p = 2$

$$H_0(u, v; \sigma, \tau) = 1 + \frac{1}{v^2}, \quad (7.3)$$

and for $p = 3$

$$H_0(u, v; \sigma, \tau) = \frac{1}{v^3} \left( \frac{1}{2} (\sigma + \tau) u (1 + v^3) + \frac{1}{2} (\sigma - \tau) (-3u(1 - v^3) + 2(1 - v)(1 + v^3)) + u(1 + v^3) - 1 + 2v + 2v^3 - v^4 \right), \quad (7.4)$$

while for $p = 4$

$$H_0(u, v; \sigma, \tau) = \frac{1}{v^4} \left( \sigma \tau u^2 (1 + v^4) + \frac{1}{2} (\sigma - \tau)^2 (2(1 - v)^2 (1 + v^4) - 5u(1 + v^5) + 3uv(1 + v^3) + 4u^2(1 + v^4)) + \frac{1}{2} (\sigma^2 - \tau^2) (u(1 - v)(1 + v^4) - 2u^2(1 - v^4)) + \frac{1}{2} (\sigma + \tau) (- (1 - v)^2 + u(1 + v))(1 + v^4) + \frac{1}{2} (\sigma - \tau) ((1 - v)(-3(1 + v) + 7u)(1 + v^4) + 8v(1 - v)(1 + v^3) - 4u^2(1 - v^4)) + 1 + v^6 - 3v(1 - v)(1 - v^3) - 2u(1 - v)(1 - v^4) + u^2(1 + v^4) \right). \quad (7.5)$$

In each case the crossing symmetry relation $H_0(u, v; \sigma, \tau) = H_0(u/v, 1/v; \tau, \sigma)/v^2$ is satisfied but the corresponding one for $u \leftrightarrow v$ is not since it is necessary to take account of the function $f_0(z, y)$ then as well.

The large $N$ results obtained through the AdS/CFT correspondence are expressible in terms of functions $\overline{D}_{n_1n_2n_3n_4}(u, v)$ which satisfy various identities listed in [13] and [3]. When $p = 2$

$$H(u, v; \sigma, \tau) = - \frac{4}{N^2} u^2 \overline{D}_{2422}(u, v), \quad (7.6)$$

and for $p = 3$,

$$H(u, v; \sigma, \tau) = - \frac{9}{N^2} u^3 ((1 + \sigma + \tau) \overline{D}_{3533} + \overline{D}_{3522} + \sigma \overline{D}_{2523} + \tau \overline{D}_{2532}). \quad (7.7)$$

For $p = 4$ the results obtained in [3] can be rewritten as

$$H(u, v; \sigma, \tau) = - \frac{4}{N^2} u^4 \left( (1 + \sigma^2 + \tau^2 + 4\sigma + 4\tau + 4\sigma\tau) \overline{D}_{4644} + 2(\overline{D}_{4633} + \overline{D}_{4622}) + 2\sigma^2(\overline{D}_{3634} + \overline{D}_{2624}) + 2\tau^2(\overline{D}_{3643} + \overline{D}_{2642}) - 4\sigma(\overline{D}_{4624} - 2\overline{D}_{3623}) - 4\tau(\overline{D}_{4642} - 2\overline{D}_{3632}) - 4\sigma\tau(\overline{D}_{2644} - 2\overline{D}_{2533}) \right). \quad (7.8)$$
Since $K(u, v; \sigma, \tau) = \frac{1}{16} u^2 H(u, v; \sigma, \tau)$ it is easy to verify both the crossing symmetry conditions (6.39) using $D$ identities. Furthermore the results given by (7.6), (7.7) and (7.8), in which overall factors of $u^p$ are present, are manifestly compatible with the unitarity conditions flowing from (5.16) and (5.29) since the leading log. term $D_n(u, v)$ is log $u$ itself. When expressed in terms of conformal partial waves $G^{(\ell)}(\Delta+4)$ it is easy to see in each case that only contributions with minimum twist $\Delta - \ell = 2p$ are required. Hence (7.6), (7.7) and (7.8) require the presence of operators belonging to long multiplets which have anomalous dimensions with twist, at zeroth order in $1/N$, $\Delta - \ell = 2(p + t), t = 0, 1, 2, \ldots$ for the lowest scale dimension operators in each multiplet. The condition $\Delta - \ell = 2p$ is stronger than that required by unitarity (5.29), with $n \leq p - 2$, which shows that for any representation some low twist multiplets decouple (thus for the singlet case twist 2 is absent as it disappears in the large $N$ limit but twist 4 multiplets, which are necessary in the $p = 2$ correlation function, decouple from the correlation functions for $p = 3, 4$).

To obtain the anomalous scale dimensions in detail it is necessary to decompose both (7.4), (7.7), (7.8) and (7.3), (7.4), (7.5) in terms of different representations, as in (5.16), and then to expand each term in conformal partial waves. The expressions (7.3), (7.4) and (7.5) require contributions with twist zero and above but the corresponding low twist operators in long supermultiplets, for which there are no anomalous dimensions, are cancelled by semi-short multiplets which are required by the expansion of $f_0(z, y)$. For $p = 2$ and $p = 3$ a detailed discussion is contained in [16] and [8] (although some details are different the analysis is equivalent to the the results that would be obtained by expanding $H$ as given by (7.3) and (7.4)).

8. Conclusion

In this paper we have derived requirements arising from $\mathcal{N} = 2$ and $\mathcal{N} = 4$ superconformal symmetry for the four point correlation functions of BPS operators. The derived conditions are clearly necessary but not manifestly sufficient in that it is possible to imagine that there are further constraints arising from superconformal transformations involving higher levels, although we have no reason to suppose that there any such additional conditions. A related question is whether the four point correlations functions of all descendant operators are determined uniquely from the basic correlation function for the superconformal primary BPS operators. In the simplest three point function case the correlation function for various descendants, including the energy momentum tensor three point function, was calculated by hand in [16]. In a superspace formalism these questions are straightforward to address, the question of uniqueness depends on whether there are any nilpotent superconformal invariants which are formed from the anti-commuting $\theta$ co-
ordinates. Nevertheless it would be nice to show directly that the correlation function of all descendant operators could in principle be obtained by the action of differential operators acting on the basic correlation function, subject to the conditions for superconformal invariance derived here.

Another area of possible future investigation is whether the requirements of crossing symmetry and superconformal invariance might be extended further using constraints arising from factorisation and the operator product expansion, as in the classic bootstrap framework. In the above we showed how there were conditions on the single variable functions that arise in the solution of superconformal identities which dictate that they are essentially of free field form. Our arguments are restricted to the case where all but one of the operators in the correlation function are identical.

Finally we may mention that the use of null vectors $t$ to conveniently express arbitrary rank traceless symmetric tensor fields in the form $\varphi^{(p)}(x,t)$, homogeneous of degree $p$ in $t$, may for conformal fields be also written in terms of homogeneous coordinates $\eta^A$ on the null cone $\eta^2 = 0$ \cite{27} such that $\phi(p)(\lambda \eta, t)) = \lambda^{-\Delta} \phi^{(p)}(\eta, t)$. For $\mathcal{N} = 4$ both $\eta$ and $t$ are 6-vectors. The expansion in terms of harmonic polynomials as discussed in appendix B has a direct analogue to the conformal partial wave expansion which was explored in \cite{28}.

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Appendix A. Results for Null Vectors

We discuss here some results for null vectors $t_r$ which are useful in the text. For generality we allow $t$ to be $d$-dimensional. As a consequence of (1.3) differentiation requires some care but for any null vector $a$ we may define as usual

$$\frac{\partial}{\partial t}(a \cdot t)^n = n (a \cdot t)^{n-1} a. \quad (A.1)$$

More generally for a set of null vectors $a_1, a_2, \ldots, a_p$ we have

$$\frac{\partial}{\partial t} \prod_{i=1}^{p} (a_i \cdot t)^{n_i} = \sum_{i=1}^{p} n_i (a_1 \cdot t)^{n_1} \ldots (a_i \cdot t)^{n_i-1} \ldots (a_p \cdot t)^{n_p} a_i$$

$$- R \sum_{1 \leq i < j \leq p} n_i n_j a_i \cdot a_j (a_1 \cdot t)^{n_1} \ldots (a_i \cdot t)^{n_i-1} \ldots (a_j \cdot t)^{n_j-1} \ldots (a_p \cdot t)^{n_p} t, \quad (A.2)$$

where

$$R = \frac{2}{2N + d - 4}, \quad N = \sum_{i=1}^{p} n_i. \quad (A.3)$$

The right hand side of (A.2), with (A.3), may be represented in the form

$$\left( \frac{\partial}{\partial t} - t \frac{1}{2t \cdot \partial + d} \partial^2 \right) \prod_{i=1}^{p} (a_i \cdot t)^{n_i}, \quad (A.4)$$

where the action of the derivatives is as usual, without regard to the constraint $t^2 = 0$. The resulting operator is equivalent to a definition given in [18] for an interior differential operator on the complex null cone.

From (A.2) we may readily find

$$\left[ \frac{\partial}{\partial t}, \frac{\partial}{\partial t_s} \right] \prod_{i=1}^{p} (a_i \cdot t)^{n_i} = 0, \quad \frac{\partial}{\partial t} \frac{\partial}{\partial t} \prod_{i=1}^{p} (a_i \cdot t)^{n_i} = 0, \quad t \frac{\partial}{\partial t} \prod_{i=1}^{p} (a_i \cdot t)^{n_i} = N \prod_{i=1}^{p} (a_i \cdot t)^{n_i}. \quad (A.5)$$

We also have

$$\left[ \frac{\partial}{\partial t}, t_s \right] \prod_{i=1}^{p} (a_i \cdot t)^{n_i} = \left( \delta_{rs} - \frac{2}{2N + d - 2} t_r \frac{\partial}{\partial t_s} \right) \prod_{i=1}^{p} (a_i \cdot t)^{n_i}, \quad (A.6)$$

which implies

$$\frac{\partial}{\partial t_r} \left( t_r \prod_{i=1}^{p} (a_i \cdot t)^{n_i} \right) = \frac{(2N + d)(N + d - 2)}{2N + d - 2} \prod_{i=1}^{p} (a_i \cdot t)^{n_i}. \quad (A.7)$$

53
Defining the generators of $SO(d)$ by

$$L_{rs} = t_r \partial_s - t_s \partial_r ,$$  \hspace{1cm} (A.8)

then the above results give

$$L_{ru} L_{su} \prod_{i=1}^{p} (a_i \cdot t)^{n_i} = -((N + d - 3) t_r \partial_s + (N - 1) t_s \partial_r + N \delta_{rs}) \prod_{i=1}^{p} (a_i \cdot t)^{n_i} ,$$  \hspace{1cm} (A.9)

and

$$\frac{1}{2} L_{rs} L_{rs} \prod_{i=1}^{p} (a_i \cdot t)^{n_i} = -N(N + d - 2) \prod_{i=1}^{p} (a_i \cdot t)^{n_i} ,$$  \hspace{1cm} (A.10)

which reproduces the appropriate eigenvalue of the Casimir operator for the representation formed by traceless rank $N$ tensors.

If $V_r(t)$ is homogeneous of degree $N$ then in general

$$V_r = \dot{V}_r + \frac{1}{N + 1} \frac{\partial}{\partial t_r} (t_s V_s) , \hspace{1cm} t_r \dot{V}_r = 0 ,$$  \hspace{1cm} (A.11)

as used in (2.24) and (4.12). If $V_r$ also satisfies

$$\frac{\partial}{\partial t_r} V_s - \frac{\partial}{\partial t_s} V_r = 0 , \hspace{1cm} \frac{\partial}{\partial t_r} V_r = 0 ,$$  \hspace{1cm} (A.12)

then, by contracting with $t_s$ and using (A.5), (A.6), we easily see that $\dot{V}_r = 0$. As a further corollary if $V_r = \partial_s U_{rs}$, $U_{rs} = -U_{sr}$, $\partial_{[r} U_{su]} = 0$ then $(N + 1) V_r = \partial_r (\frac{1}{2} L_{su} U_{su})$ with $L_{su}$ as in (A.8). In general we have the decomposition

$$V_r = \frac{2N + d - 4}{(2N + d - 2)(N + d - 3)} t_r \partial_s V$$

$$- \frac{1}{(2N + d)(N + d - 3)} ((2N + d - 2) \partial_s (t_r V_s - t_s V_r) + 2 \partial_r (t \cdot V)) .$$  \hspace{1cm} (A.13)

**Appendix B. Two Variable Harmonic Polynomials**

For the expansion of four point functions in terms of $R$-symmetry representations we consider here the eigenfunctions of the $SO(d)$ Casimir operator

$$L^2 = \frac{1}{2} L_{rs} L_{rs} ,$$  \hspace{1cm} (B.1)

where the generators are

$$L_{rs} = t_{1r} \partial_{1s} - t_{1s} \partial_{1r} + t_{2r} \partial_{2s} - t_{2s} \partial_{2r} ,$$  \hspace{1cm} (B.2)
formed by homogeneous functions of the null vectors \( t_1, t_2, t_3, t_4 \). Obviously \( L_{rs} t_1 \cdot t_2 = 0 \) and hence \( L^2(t_1 \cdot t_2)^k (t_3 \cdot t_4)^l f(\sigma, \tau) = (t_1 \cdot t_2)^k (t_3 \cdot t_4)^l L^2 f(\sigma, \tau) \), where \( \sigma, \tau \) are given by (1.4). We therefore first consider eigenfunctions which are polynomials in \( \sigma, \tau \)

\[
Y(\sigma, \tau) = \sum_{t \geq 0} \sum_{q=0}^t c_{t,q} \sigma^{t-q} \tau^q,
\]

satisfying

\[
L^2 Y(\sigma, \tau) = -2C Y(\sigma, \tau).
\]

With the aid of the given in appendix A we may easily calculate the action of \( L^2 \) on a monomial formed from \( \sigma, \tau \),

\[
L^2(\sigma^p \tau^q) = -2((d-2)(p+q) + 4pq)\sigma^p \tau^q + 2(1-\sigma - \tau)(p^2\sigma^{p-1} \tau^q + q^2\sigma^p \tau^{q-1}),
\]

or

\[
\frac{1}{2} L^2 \rightarrow D_d = (1-\sigma - \tau) \left( \frac{\partial}{\partial \sigma} \sigma \frac{\partial}{\partial \sigma} + \frac{\partial}{\partial \tau} \tau \frac{\partial}{\partial \tau} \right) - 4\sigma \tau \frac{\partial^2}{\partial \sigma \partial \tau} - (d-2) \left( \sigma \frac{\partial}{\partial \sigma} + \tau \frac{\partial}{\partial \tau} \right).
\]

Alternatively \( D_d \) may be written in the form

\[
D_d = \frac{1}{w} \partial^T w G \partial,
\]

\[
G = \begin{pmatrix} \sigma(1-\sigma - \tau) & -2\sigma \tau \\ -2\sigma \tau & \tau(1-\sigma - \tau) \end{pmatrix},
\]

\[
\partial = \begin{pmatrix} \partial_{\sigma} \\ \partial_{\tau} \end{pmatrix},
\]

where, with \( \Lambda = (\sqrt{\sigma} + \sqrt{\tau} + 1)(\sqrt{\sigma} + \sqrt{\tau} - 1)(\sqrt{\sigma} - \sqrt{\tau} + 1)(\sqrt{\sigma} - \sqrt{\tau} - 1) \) as in (2.42),

\[
w = \Lambda \frac{1}{2}(d-5).
\]

In general, for a polynomial as in (B.3) with \( t_{max} = n \), we must have that \( c_{n,q} \) forms an eigenvector for an \((n+1) \times (n+1)\) matrix \( M_n \),

\[
M_{n,pq} c_{n,q} = C c_{n,p}, \quad M_{n,pq} = \delta_{p,q} (n(n+d-2) + 2p(n-p)) + \delta_{p,q-1} q^2 + \delta_{p,q+1} (n-q)^2.
\]

The coefficients \( c_{t,q} \) with \( t < n \) may then be obtained by solving recurrence relations. For given \( n \) there are \( n+1 \) eigenvectors solving (B.9) and the corresponding eigenfunctions are

\[
Y_{nm}(\sigma, \tau), \quad C_{nm} = n(n+d-3) + m(m+1), \quad n = 0, 1, 2, \ldots, m = 0, \ldots n.
\]

As a consequence of (B.7) and (B.8) the polynomials are orthogonal for \( d > 5 \) with respect to integration over \( \sigma, \tau \geq 0, \sqrt{\sigma} + \sqrt{\tau} \leq 1 \) with weight \( w \) (for a general discussion of such two variable orthogonal polynomials see [29,30]).
The polynomials $Y_{nm}$ are also eigenfunctions for higher order Casimir invariants. By hand. With an arbitrary normalisation, we find for 

\[ (d-2)(d-3) L^2 \rightarrow Q \]  

(B.11) then acting on any $Y(\sigma, \tau)$ we may express $Q$ in a form similar to (B.7),  

\[ Q = - \frac{1}{\Lambda^{\frac{1}{2}}(d-5)} (\partial_\sigma^2 - \partial_\tau^2) \Lambda^{\frac{1}{2}}(d-3) \left( \begin{array}{c} \sigma^2 - \sigma \tau \\ -\sigma \tau \\ \tau^2 \end{array} \right) \left( \begin{array}{c} \partial_\sigma^2 \\ -\sigma \tau \\ \partial_\tau^2 \end{array} \right) \]

\[ + (d-3) \frac{1}{\Lambda^{\frac{1}{2}}(d-5)} (\partial_\sigma - \partial_\tau) \Lambda^{\frac{1}{2}}(d-5) \left( \begin{array}{c} 2\sigma \\ \sigma + \tau - 1 \\ 2\tau \end{array} \right) \left( \begin{array}{c} \partial_\sigma \\ \partial_\tau \end{array} \right) \]  

(B.12)  

\[ + (d-2) \frac{1}{\Lambda^{\frac{1}{2}}(d-5)} (\partial_\sigma \Lambda^{\frac{1}{2}}(d-3) \partial_\tau + \partial_\tau \Lambda^{\frac{1}{2}}(d-3) \partial_\sigma) . \]

The harmonic polynomials then satisfy  

\[ Q Y_{nm} = -(n-m)(n+m+1)(n+m+d-3)(n-m+d-4) Y_{nm} . \]  

(B.13)  

Using (B.5) it is straightforward to construct the first few eigenfunctions satisfying (B.4) by hand. With an arbitrary normalisation, we find for $n = 0, 1, 2, 3,$  

\[ Y_{00}(\sigma, \tau) = 1, \]

\[ Y_{10}(\sigma, \tau) = \sigma - \tau, \]

\[ Y_{11}(\sigma, \tau) = \sigma + \tau - \frac{2}{d}, \]

\[ Y_{20}(\sigma, \tau) = \sigma^2 - \sigma \tau - \frac{2}{d-2} (\sigma + \tau) + \frac{2}{(d-2)(d-1)}, \]

\[ Y_{21}(\sigma, \tau) = \sigma^2 - \sigma \tau - \frac{4}{d+2} (\sigma - \tau), \]

\[ Y_{22}(\sigma, \tau) = \sigma^2 + \tau^2 + 4\sigma \tau - \frac{8}{d+4} (\sigma + \tau) + \frac{8}{(d+2)(d+4)}, \]

\[ Y_{30}(\sigma, \tau) = \sigma^3 - 3\sigma^2 \tau + 3\sigma \tau^2 - \tau^3 - \frac{6}{d} (\sigma^2 - \tau^2) + \frac{12}{d(d+1)} (\sigma - \tau), \]

\[ Y_{31}(\sigma, \tau) = \sigma^3 - \sigma^2 \tau - \sigma \tau^2 + \tau^3 - \frac{8(d-1)}{(d+1)(d+2)} (\sigma^2 + \tau^2) + \frac{8(d-6)}{(d+1)(d+2)} \sigma \tau 
\]

\[ + \frac{4(3d+2)}{(d+1)(d+4)(d-2)} (\sigma + \tau) - \frac{8}{(d+1)(d+4)(d-2)}, \]

\[ Y_{32}(\sigma, \tau) = \sigma^3 + 3\sigma^2 \tau - 3\sigma \tau^2 - \tau^3 - \frac{12}{d+6} (\sigma^2 - \tau^2) + \frac{24}{(d+1)(d+6)} (\sigma - \tau), \]

\[ Y_{33}(\sigma, \tau) = \sigma^3 + 9\sigma^2 \tau + 9\sigma \tau^2 + \tau^3 - \frac{18}{d+8} (\sigma^2 + \tau^2) - \frac{72}{d+8} \sigma \tau 
\]

\[ + \frac{72}{(d+6)(d+8)} (\sigma + \tau) - \frac{48}{(d+4)(d+6)(d+8)}, \]

Up to an overall normalisation for $d = 6$ each term may be identified with terms in the projection operators constructed in [3] where $Y_{nm}$ corresponds to the $SU(4) \simeq SO(6)$ representation with Dynkin labels $[n-m, 2m, n-m]$. For $m = n$ in (B.10) we have $c_{n,q} = \binom{n}{q}^2$ and the recurrence relations may be easily solved giving  

\[ Y_{nn}(\sigma, \tau) = A_n F_4(-n, n + 1; 1, 1; \sigma, \tau), \]  

(B.15)
where $F_4$ is one of Appell’s generalised hypergeometric functions and $A_n$ is some overall constant.

To obtain more general forms (see [31]) we used the variables $\alpha, \bar{\alpha}$ defined in (1.5).

Acting on $Y(\sigma, \tau) = \mathcal{P}(\alpha, \bar{\alpha}) = \mathcal{P}(\bar{\alpha}, \alpha)$

$$\frac{1}{2}L^2\mathcal{P}(\alpha, \bar{\alpha}) = \hat{\mathcal{D}}_d\mathcal{P}(\alpha, \bar{\alpha}), \tag{B.16}$$

where, using (B.5) or (B.6), we now have

$$\hat{\mathcal{D}}_d = \frac{\partial}{\partial \alpha} \alpha (1-\alpha) \frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \bar{\alpha}} \bar{\alpha} (1-\bar{\alpha}) \frac{\partial}{\partial \bar{\alpha}} + (d-4) \frac{1}{\alpha - \bar{\alpha}} \left( \alpha (1-\alpha) \frac{\partial}{\partial \alpha} - \bar{\alpha} (1-\bar{\alpha}) \frac{\partial}{\partial \bar{\alpha}} \right). \tag{B.17}$$

Corresponding to (B.4) and (B.10) we have

$$\hat{\mathcal{D}}_d \mathcal{P}_{nm}(\alpha, \bar{\alpha}) = - \left( n(n + d - 3) + m(m + 1) \right) \mathcal{P}_{nm}(\alpha, \bar{\alpha}), \tag{B.18}$$

where $\mathcal{P}_{nm}(\alpha, \bar{\alpha})$ are generalised symmetric Jacobi polynomials. For particular $d$ simplified formulae may be found in terms of well know single variable Legendre polynomials $P_n$. When $d = 4$ it is clear from (B.17) that $\hat{\mathcal{D}}_4$ is just the sum of two independent Legendre differential operators so that

$$\mathcal{P}_{nm}(\alpha, \bar{\alpha}) = \frac{1}{2} \left( P_n(y) P_m(\bar{y}) + P_m(y) P_n(\bar{y}) \right), \quad n \geq m, \tag{B.19}$$

with $y, \bar{y}$ defined in (7.3). For $d = 6$ we may use the result

$$\hat{\mathcal{D}}_6 \frac{1}{\alpha - \bar{\alpha}} = \frac{1}{\alpha - \bar{\alpha}} (\hat{\mathcal{D}}_4 + 2), \tag{B.20}$$

to see that we can take the eigenfunctions to be of the form

$$\mathcal{P}_{nm}(\alpha, \bar{\alpha}) = p_{n+1m}(y, \bar{y}), \quad n \geq m, \tag{B.21}$$

where

$$p_{nm}(y, \bar{y}) = p_{mn}(y, \bar{y}) = \frac{P_n(y) P_m(\bar{y}) - P_m(y) P_n(\bar{y})}{y - \bar{y}}. \tag{B.22}$$

It is also of interest to consider $d = 8$ when we take

$$\mathcal{P}(\alpha, \bar{\alpha}) = \frac{F(\alpha, \bar{\alpha})}{(\alpha - \bar{\alpha})^2}, \tag{B.23}$$

$$F_4(a, b; c, c'; x, y) = \sum_{m, n} \frac{(a)_{m+n} (b)_{m+n}}{(c)_{m+n} m! n!} x^m y^n.$$
and the eigenvalue equation becomes

\[
\mathcal{D}_6 F(\alpha, \bar{\alpha}) - \frac{2}{(\alpha - \bar{\alpha})^2} \left( \alpha(1 - \alpha) \frac{\partial}{\partial \alpha} ((\alpha - \bar{\alpha}) F(\alpha, \bar{\alpha})) - \bar{\alpha}(1 - \bar{\alpha}) \frac{\partial}{\partial \bar{\alpha}} ((\alpha - \bar{\alpha}) F(\alpha, \bar{\alpha})) \right) = -(C + 4) F(\alpha, \bar{\alpha}). \tag{B.24}
\]

If we assume

\[
F(\alpha, \bar{\alpha}) = \sum_{n,m} a_{nm} p_{nm}(y, \bar{y}), \tag{B.25}
\]

and use, from standard identities for Legendre polynomials,

\[
\frac{1}{y - \bar{y}} \left( (1 - y^2) \frac{\partial}{\partial y} - (1 - \bar{y}^2) \frac{\partial}{\partial \bar{y}} \right) ((y - \bar{y}) p_{nm}(y, \bar{y})) = \frac{m(m + 1)}{2m + 1} (p_{nm+1}(y, \bar{y}) - p_{nm-1}(y, \bar{y})) - \frac{n(n + 1)}{2n + 1} (p_{n+1m}(y, \bar{y}) - p_{n-1m}(y, \bar{y}))
\]

\[
(y - \bar{y}) p_{nm}(y, \bar{y}) = \frac{1}{2n + 1} ((n + 1)p_{n+1m}(y, \bar{y}) + np_{n-1m}(y, \bar{y})) - \frac{1}{2m + 1} ((m + 1)p_{nm+1}(y, \bar{y}) + mp_{nm-1}(y, \bar{y})) \tag{B.26}
\]

then we may set up recurrence relations for \(a_{nm}\) which for the appropriate value of \(C\) have just four terms. For \(C = n(n + 1) + m(m + 1) - 6 \tag{B.25}\) gives a solution

\[
q_{nm}(y, \bar{y}) = \frac{1}{(y - \bar{y})^2} \left\{ \frac{n + 1}{2n + 1} (n + m)(n - m - 1)p_{n+1m}(y, \bar{y}) + \frac{n}{2n + 1} (n + m + 2)(n - m + 1)p_{n-1m}(y, \bar{y}) - \frac{m + 1}{2m + 1} (n + m)(n - m + 1)p_{nm+1}(y, \bar{y}) - \frac{m}{2m + 1} (n + m + 2)(n - m - 1)p_{nm-1}(y, \bar{y}) \right\}, \tag{B.27}
\]

where \(q_{nm}(y, \bar{y}) = -q_{nm}(y, \bar{y}), q_{nn}(y, \bar{y}) = q_{n+1n}(y, \bar{y}) = 0\). Hence we can take

\[
\mathcal{P}_{nm}(\alpha, \bar{\alpha}) = q_{n+2m}(y, \bar{y}). \tag{B.28}
\]

The above results for harmonic polynomials in \(\sigma, \tau\) are relevant for discussing four point functions when each field belongs to the same \(SO(d)\) representation. For the more general case we also consider instead of \(\mathcal{P}_{nm}(\alpha, \bar{\alpha})\),

\[
L^2((t_1 \cdot t_4)^a(t_2 \cdot t_4)^b Y^{(a,b)}(\sigma, \tau)) = -2C((t_1 \cdot t_4)^a(t_2 \cdot t_4)^b Y^{(a,b)}(\sigma, \tau)), \tag{B.29}
\]

58
where now the action of $L^2$ is determined by

\[
L^2 \left( (t_1 \cdot t_4)^a (t_2 \cdot t_4)^b \sigma^p \tau^q \right) = (t_1 \cdot t_4)^a (t_2 \cdot t_4)^b \left( 2D_d - (a + b)(a + b + d - 2) - 4ap - 4bq \right) \left( \sigma^p \tau^q \right) + 2(1 - \sigma - \tau) \left( bp \sigma^{p-1} \tau^q + aq \sigma^p \tau^{q-1} \right),
\]

(B.30)

or

\[
\frac{1}{2} L^2 \left( (t_1 \cdot t_4)^a (t_2 \cdot t_4)^b f(\sigma, \tau) \right) = (t_1 \cdot t_4)^a (t_2 \cdot t_4)^b \left( D_d^{(a,b)} - \frac{1}{2}(a+b)(a+b+d-2) \right) f(\sigma, \tau),
\]

(B.31)

where

\[
D_d^{(a,b)} = D_d + (1 - \sigma - \tau) \left( a \frac{\partial}{\partial \tau} + b \frac{\partial}{\partial \sigma} \right) - 2a \sigma \frac{\partial}{\partial \sigma} - 2b \tau \frac{\partial}{\partial \tau}.
\]

(B.32)

This may also be written in the form (B.7) with $w = \sigma^b \tau^a \Lambda^\frac{1}{2}(d-5)$. The possible eigenvalues for polynomial eigenfunctions with maximum power $p + q = n$ are then determined by the matrix

\[
M_{n,pq} = \delta_{p,q} \left( n(n + d - 2 + a + b) + \frac{1}{2}(a + b)(a + b + d - 2) + 2p(n - p) + a(n - p) + bp \right) + \delta_{p,q-1} q(q + a) + \delta_{p,q+1} (n - q)(n - q + b).
\]

(B.33)

The eigenfunctions $Y_{nm}^{(a,b)}(\sigma, \tau)$ for $m = 0, 1, \ldots, n$ then have eigenvalues

\[
C_{nm} = (n + \frac{1}{2}(a + b))(n + \frac{1}{2}(a + b) + d - 3) + (m + \frac{1}{2}(a + b))(m + \frac{1}{2}(a + b) + 1).
\]

(B.34)

For $d = 6$ $Y_{nm}^{(a,b)}$ corresponds to the representation $[n-m, a+b+2m, n-m]$. The simplest non-trivial examples are

\[
Y_{10}^{(a,b)}(\sigma, \tau) = \sigma - \tau + \frac{a - b}{a + b + d - 2},
\]

\[
Y_{11}^{(a,b)}(\sigma, \tau) = \frac{1}{b+1} \sigma + \frac{1}{a+1} \tau - \frac{1}{a + b + \frac{1}{2}d}.
\]

(B.35)

Corresponding to (B.15) we have in general

\[
Y_{nm}^{(a,b)}(\sigma, \tau) = A_n F_4(-n, n + a + b + \frac{1}{2}d - 1; b + 1, a + 1; \sigma, \tau).
\]

(B.36)

Again more explicit results can be obtained by using the variables $\alpha, \bar{\alpha}$. In (B.32) the differential operator now becomes

\[
\hat{D}_d^{(a,b)} = \hat{D}_d - \left( a \alpha - b(1 - \alpha) \right) \frac{\partial}{\partial \alpha} - \left( a \bar{\alpha} - b(1 - \bar{\alpha}) \right) \frac{\partial}{\partial \bar{\alpha}},
\]

(B.37)
with \( \hat{D}_d \) given in (B.17). Denoting the eigenfunctions of \( \hat{D}_d^{(a,b)} \) by \( P_{nm}^{(a,b)}(\alpha, \bar{\alpha}) \) then previous results for \( d = 4, 6 \) for the eigenfunctions can be extended by using Jacobi polynomials \( D_n^{(a,b)} \). For \( d = 4 \) \( \hat{D}_4^{(a,b)} = D^{(a,b)} + \alpha^{(a,b)} \) where \( D^{(a,b)} \) is the ordinary differential operator defined by

\[
D^{(a,b)}(\alpha) = \frac{d}{d\alpha} \alpha(1-\alpha) \frac{d}{d\alpha} - a \frac{d}{d\alpha} + b(1-\alpha) \frac{d}{d\alpha} .
\] (B.38)

The eigenfunctions of \( D^{(a,b)} \) are just \( P_n^{(a,b)}(y) \), where \( y = 2\alpha - 1 \) and the eigenvalues are \(-n(n + a + b + 1)\). For \( d = 6 \) the generalisation of (B.21) and (B.22) is then

\[
P_{nm}^{(a,b)}(\alpha, \bar{\alpha}) = \frac{P_{n+1}^{(a,b)}(y) P_m^{(a,b)}(\bar{\alpha}) - P_m^{(a,b)}(y) P_{n+1}^{(a,b)}(\bar{\alpha})}{y - \bar{\alpha}}.
\] (B.39)

When \( d = 3 \) the above results need to be considered separately since \( \sigma, \tau \) are not independent and satisfy the constraint (2.42). The eigenfunctions \( Y_{nm}^{(a,b)}(\sigma, \tau) \) are also restricted since \( Y_{nm}^{(a,b)}(\sigma, \tau) = 0 \) for \( m < n - 1 \) as a consequence of (2.42). To obtain eigenfunctions of \( L^2 \) in general we make use of the solution (2.43) which amounts to setting \( \alpha = \bar{\alpha} \) in the above, so that we are restricted just to single variable functions. Instead of (B.31) we have

\[
L^2((t_1 \cdot t_4)^a(t_2 \cdot t_4)^b f(\alpha)) = (t_1 \cdot t_4)^a(t_2 \cdot t_4)^b(D^{(a,2b)}_{\alpha} - (a + b)(a + b + 1)) f(\alpha) ,
\] (B.40)

using the definition (B.38). In consequence

\[
L^2((t_1 \cdot t_4)^a(t_2 \cdot t_4)^b P_n^{(2a,2b)}(y)) = -(n + a + b)(n + a + b + 1) (t_1 \cdot t_4)^a(t_2 \cdot t_4)^b P_n^{(2a,2b)}(y) ,
\] (B.41)

corresponding to the \((n+a+b)\)-representation for \( SU(2) \cong SO(3) \). Hence for \( d = 3 \) we may then take

\[
P_{nm}^{(a,b)}(\alpha, \alpha) = P_{2n}^{(2a,2b)}(y) , \quad \mathcal{P}_{n n-1}^{(a,b)}(\alpha, \alpha) = P_{2n-1}^{(2a,2b)}(y) ,
\] (B.42)

with \( \mathcal{P}_{nm}^{(a,b)}(\alpha, \alpha) = 0 \) for \( m < n - 1 \).

For \( d = 3 \) there are also eigenfunctions involving cross products. To consider these we first define

\[
T_1 = t_1 \cdot t_3 \times t_4 (t_1 \cdot t_4)^{a-1}(t_2 \cdot t_4)^b , \quad T_2 = t_2 \cdot t_3 \times t_4 (t_1 \cdot t_4)^a(t_2 \cdot t_4)^{b-1} ,
\] (B.43)

and consider eigenfunctions of the form \( T_1 f_1(\alpha) + T_2 f_2(\alpha) \). The action of \( L^2 \) on such functions is given by

\[
(L^2 + (a + b - 1)(a + b)) (T_1 f_1 + T_2 f_2) = (T_1 \quad T_2) \begin{pmatrix} D^{(2a-1,2b+1)} - 2a & -2a \\ -2b & D^{(2a+1,2b-1)} - 2b \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} .
\] (B.44)
However for \( t_i \) three dimensional null vectors the basis given by (B.43) is not independent since we have from (2.55)

\[
T_1(1 - \alpha) + T_2\alpha = 0,
\]

so that \( f_1, f_2 \) are not unique. If we use this freedom to set \( f_2 = 0 \) the eigenvalue equation for \( L^2 \) reduces to

\[
\left(D^{(2a-1,2b+1)} - 2a + 2b \frac{1 - \alpha}{\alpha} - (a + b - 1)(a + b)\right)f_1
= \frac{1}{\alpha} \left(D^{(2a-1,2b-1)} - (a + b - 1)(a + b)\right)(\alpha f_1) = -Cf_1,
\]

which has solutions proportional to Jacobi polynomials,

\[
f_1(\alpha) = \frac{1}{\alpha} P_n^{(2a-1,2b-1)}(y), \quad C = (n + a + b - 1)(n + a + b).
\]

For \( n \geq 1 \) the apparent singularity for \( \alpha \to 0 \) may be removed by using (B.45) to give an appropriate non zero \( f_2 \). The eigenfunctions for the solution in (B.47) correspond to the \( SU(2) \) \( (n+a+b-1) \)-representation. Alternatively we may set \( f_1 = 0 \) and obtain the corresponding equation

\[
\frac{1}{1 - \alpha} \left(D^{(2a-1,2b-1)} - (a + b - 1)(a + b)\right)((1 - \alpha)f_2) = -Cf_2.
\]

### Appendix C. Calculation of Differential Operators

A non trivial aspect in the derivation of the superconformal identities is the determination of the differential operators (2.48) which appear in (2.47). To sketch how these were obtained we first obtain, for any dimension \( d \) and arbitrary \( f(\sigma, \tau) \),

\[
\frac{1}{2}(k + a + \frac{1}{2}d - 2) \left(L_{2[r}s\partial_{1u]}((t_1 \cdot t_2)^k(t_3 \cdot t_4)^l(t_1 \cdot t_4)^a(t_2 \cdot t_4)^b f)
= -(t_1 \cdot t_2)^{k-2}(t_3 \cdot t_4)^{l-1}(t_1 \cdot t_4)^a(t_2 \cdot t_4)^b(t_1[r]t_2s[t_3u] t_2 \cdot t_4 \mathcal{D}_1 + t_1[r]t_4s[t_2u] t_2 \cdot t_3 \mathcal{D}_2
+ (k + a + \frac{1}{2}d - 2) t_2[r]t_3s[t_4u] t_1 \cdot t_2 (\mathcal{D}_\sigma - \mathcal{D}_\tau)) f ,
\]

where

\[
\mathcal{D}_1 = \frac{\partial}{\partial\sigma} \mathcal{D}_\tau + \left(\sigma \frac{\partial}{\partial\sigma} + \tau \frac{\partial}{\partial\tau} + 1 - k\right)(\mathcal{D}_\sigma - \mathcal{D}_\tau) ,
\]

\[
\mathcal{D}_2 = - \left(\frac{\partial}{\partial\tau} + \frac{a}{\tau}\right)\mathcal{D}_\sigma + \left(\sigma \frac{\partial}{\partial\sigma} + \tau \frac{\partial}{\partial\tau} + 1 - k\right)(\mathcal{D}_\sigma - \mathcal{D}_\tau) ,
\]

61
For
\[ D_\sigma = \sigma (1 - \sigma) \frac{\partial^2}{\partial \sigma^2} - \tau^2 \frac{\partial^2}{\partial \tau^2} - 2 \sigma \tau \frac{\partial^2}{\partial \sigma \partial \tau} + (b + 1) \frac{\partial}{\partial \sigma} - (a + b + \frac{1}{2} d) \left( \sigma \frac{\partial}{\partial \sigma} + \tau \frac{\partial}{\partial \tau} \right) + k (k + a + b + \frac{1}{2} d - 1) , \] (C.3)

\[ D_\tau = \tau (1 - \tau) \frac{\partial^2}{\partial \tau^2} - \sigma^2 \frac{\partial^2}{\partial \sigma^2} - 2 \sigma \tau \frac{\partial^2}{\partial \sigma \partial \tau} + (a + 1) \frac{\partial}{\partial \sigma} - (a + b + \frac{1}{2} d) \left( \sigma \frac{\partial}{\partial \sigma} + \tau \frac{\partial}{\partial \tau} \right) + k (k + a + b + \frac{1}{2} d - 1) . \]

In terms of (B.32) and (B.6) we have
\[ \Delta^{(a,b)}_d \equiv D_\sigma + D_\tau + (\sigma - \tau) (D_\sigma - D_\tau) = D^{(a,b)}_d + 2 k (k + a + b + \frac{1}{2} d - 1) . \] (C.4)

The operators in (C.2) satisfy the identity
\[ 2 (\sigma - \tau + 1) D_1 + 2 (\tau - \sigma + 1) D_2 = D_2 \Delta^{(a,b)}_d - \Lambda \left( \frac{\partial}{\partial \sigma} + \frac{\partial}{\partial \tau} + a \frac{\partial}{\partial t} \right) (D_\sigma - D_\tau) - (2 k + 2 a - 1) (D_\sigma - D_\tau) , \] (C.5)

with \( \Lambda \) as in (2.42) and, as well as (2.4), defining
\[ D_2 = (\sigma - \tau + 1) \frac{\partial}{\partial \sigma} + (\sigma - \tau - 1) \left( \frac{\partial}{\partial \tau} + a \frac{\partial}{\partial t} \right) . \] (C.6)

When \( d = 3 \), and \( \Lambda = 0 \), this result with (2.53) leads to the simplified form for (C.1)
\[ 4 (k + a - \frac{1}{2}) \frac{\partial}{\partial t_1} \cdot L_2 ((t_1 \cdot t_2)^{k} (t_3 \cdot t_4)^{l} (t_1 \cdot t_4)^{a} (t_2 \cdot t_4)^{b} \hat{f}) = - t_2 \cdot t_3 \times t_4 (t_1 \cdot t_2)^{k-1} (t_3 \cdot t_4)^{l-1} (t_1 \cdot t_4)^{a} (t_2 \cdot t_4)^{b} \hat{D}_2 (D^{(2a,2b)}_2 + 2 k (2 k + 2 a + 2 b + 1)) \hat{f}, \] (C.7)

letting \( f(\sigma, \tau) = \hat{f}(\alpha) \) and \( D_2 \to \hat{D}_2 \) for
\[ \hat{D}_2 = \frac{d}{d \alpha} - \frac{2 a}{1 - \alpha} . \] (C.8)

From (B.40) the operator \( D^{(2a,2b)}_2 + 2 k (2 k + 2 a + 2 b + 1) \) acting on \( \hat{f}(\alpha) \) corresponds to \( L^2 + (2 k + a + b)(2 k + a + b + 1) \). We may also note that
\[ \hat{D}_2 D^{(2a,2b)}_2 = \frac{1}{1 - \alpha} D^{(2a-1,2b+1)} (1 - \alpha) \hat{D}_2 , \] (C.9)

and in (C.7) from (B.44)
\[ t_2 \cdot t_3 \times t_4 (t_1 \cdot t_2)^{k-1} (t_3 \cdot t_4)^{l-1} (t_1 \cdot t_4)^{a} (t_2 \cdot t_4)^{b} \times \frac{1}{1 - \alpha} \left( D^{(2a-1,2b+1)} + 2 k (2 k + 2 a + 2 b + 1) \right) ((1 - \alpha) f) \]
\[ = (L^2 + (2 k + a + b)(2 k + a + b + 1)) t_2 \cdot t_3 \times t_4 (t_1 \cdot t_2)^{k-1} (t_3 \cdot t_4)^{l-1} (t_1 \cdot t_4)^{a} (t_2 \cdot t_4)^{b} f , \]

62
The equivalent results to (C.4) for $L_2 \to L_3$ and $L_2 \to L_4$ can be found by using the permutations $2 \to 3 \to 4 \to 2$, along with $a \to a' = k - l$, $b \to -a$, $k \to a + l$, $l \to a + b + l$ and $\alpha \to \alpha' = -(1 - \alpha)/\alpha$, and also $2 \to 4 \to 3 \to 2$, along with in this case $a \to a'' = -b$, $b \to l - k$, $k \to a + b + l$, $l \to b + k$ and $\alpha \to \alpha'' = 1/(1 - \alpha)$. From (C.7) we then find

$$
\hat{D}_3 = \alpha^{2(a+l-1)}(1-\alpha)^{-2a} \hat{D}_2' \alpha^{-2(a+l)}(1-\alpha)^{2a} = \frac{d}{d\alpha} + \frac{2a}{1-\alpha} - \frac{2(k+a)}{\alpha},
$$

$$
\hat{D}_4 = \alpha^{-2b}(1-\alpha)^{2(b+k-1)} \hat{D}_2'' \alpha^{2b}(1-\alpha)^{-2(b+k)} = \frac{d}{d\alpha} + \frac{2k}{1-\alpha}.
$$

Together with (C.8), (C.11) is equivalent to (C.4).

For the analysis of the $\mathcal{N} = 4$ superconformal identities a particular solution of the constraints (C.7) is obtained by expressing $T_i$ in terms of scalar functions $Y_i(u, v; t)$

$$
T_i = -\bar{\gamma}_i \frac{\partial}{\partial t_1} Y_i \gamma \frac{\partial}{\partial t_i}.
$$

(4.8) and (4.12) then give

$$
U_i = (L_{i,rs} L_{i,rs} + p_1 p_i) Y_i, \quad W_{i,rsu} = 3 \left( \partial_{1, [r} L_{i, su]} Y_i \right)_{sd},
$$

$$
\hat{V}_{i,r} = \partial_{14} L_{i,rs} Y_i - \frac{1}{p_1} \partial_{14} \left( \frac{1}{2} L_{1, su} L_{i, su} Y_i \right).
$$

Writing

$$
Y_i(u, v; t) = (t_1 \cdot t_4)^{p_1-E} (t_2 \cdot t_4)^{p_2-E} (t_1 \cdot t_2)^{E} (t_3 \cdot t_4)^{p_3} Y_i(u, v; \sigma, \tau),
$$

then for $i = 2$, using (C.11) with $k = E$, $k + a = p_1$, $k + b = p_2$, we find

$$
U_2 = \Delta_0^{(p_1-E, p_2-E)} Y_2, \quad W_2 = 6(D_\sigma - D_\tau) Y_2.
$$

$A_2$ and $B_2$ are then given by (C.1) and (C.2) in terms of $U_2, W_2$ in accord with (4.37). The other results may be obtained by cyclic permutations. For $2 \to 3 \to 4 \to 2$, when $\sigma \to \tau/\sigma, \tau \to 1/\sigma$ and $E \to E - p_4 - p_2$, then $U_2 \to \tau^{p_1-E} \sigma^{p_2-p_4-E} U_3, W_2 \to \tau^{p_1-E} \sigma^{p_2-p_4-E+1} W_3, A_2 \to \tau^{p_1-E} \sigma^{p_2-p_4-E+2} A_3$ and $B_2 \to \tau^{p_1-E} \sigma^{p_2-p_4-E+2} . A_3$. For $2 \to 4 \to 3 \to 2$, so that $\sigma \to 1/\tau, \tau \to \sigma/\tau$ and $E \to -E + p_1 + p_2$, in this case $U_2 \to \tau^{-p_2} \sigma^{p_2-E} U_4, W_2 \to \tau^{1-p_2} \sigma^{p_2-E} W_4, A_2 \to \tau^{2-p_2} \sigma^{p_2-E} A_4$ and $B_2 \to \tau^{2-p_2} \sigma^{p_2-E} . C$. However the representation (C.12) is not valid in general since it excludes contributions involving the $\varepsilon$-tensor. Nevertheless equivalent results may be obtained by use of (4.24).
With the expansion
\[ t_2[r \partial_1 s] U_2 + p_1 t_2[r \hat{V}_{2,s}] + p_1 W_2,rsu t_{2u} \]
\[ = (t_1 \cdot t_4)^{p_1^{-E-1}} (t_2 \cdot t_4)^{p_2^{-E}} (t_1 \cdot t_2)^{-E-1} (t_3 \cdot t_4)^{p_3^{-1}} \]
\[ \times \left( t_2[r \partial_3 s] t_1 \cdot t_4 t_2 \cdot t_4 \left( \partial_\sigma U_2 + p_1 J_2 - \frac{1}{6} p_1 (A_2 + W_2) \right) \right. \]
\[ + t_2[r \partial_4 s] t_1 \cdot t_4 t_2 \cdot t_3 \left( \partial'_\tau U_2 - p_1 \frac{1}{\tau} (I_2 + \sigma J_2) + \frac{1}{6} p_1 (B_2 + W_2) \right) \]
\[ + t_1[r \partial_2 s] t_2 \cdot t_4 t_3 \cdot t_4 \left( \frac{\tau}{p_1 + 1} ((\partial_\sigma + \partial'_\tau)(E - \sigma \partial_\sigma - \tau \partial_\tau) + \partial_\sigma \partial'_\tau) U_2 \right. \]
\[ \left. \left. - p_1 V_2 - \frac{1}{6} p_1 \tau (A_2 - B_2) \right) \right), \tag{C.16} \]
where \( \partial'_\tau = \partial_\tau + (p_1 - E)/\tau \), then (4.24) requires
\[ 6 \partial_\sigma U_2 = -6 p_1 J_2 + p_1 (A_2 + W_2) = 2(p_1 + 1) A_2 - (O_\sigma - p_1) W_2, \tag{C.17} \]
\[ 6 \partial'_\tau U_2 = 6 p_1 \frac{1}{\tau} (I_2 + \sigma J_2) - p_1 (B_2 + W_2) = -2(p_1 + 1) B_2 + (O_\sigma - p_1) W_2, \]
using (4.35) for \( i = 2 \), which gives the first two equations in (4.37). The remaining results in (4.37) can be obtained by using permutations. In addition with (4.36) we also obtain
\[ (p_1 + 1) ((O_\sigma - p_1 + 1) B_2 - (O_\tau - p_1 + 1) A_2) \]
\[ = 6((E - 1 - \sigma \partial_\sigma - \tau \partial_\tau) (\partial_\sigma + \partial'_\tau) + \partial_\sigma \partial'_\tau) U_2 \tag{C.18} \]
\[ = -6 ((O_\sigma - p_1 + 1) \partial'_\tau + (O_\tau - p_1 + 1) \partial_\sigma) U_2. \]
It is then straightforward to see that (C.18) follows from (C.17) using \( [O_\sigma, O_\tau] = O_\sigma - O_\tau. \)

Appendix D. Non Unitary Semi-short Representations

In section 5 the analysis of the operator product expansion in general potentially required contributions below the unitarity threshold on the scale dimension \( \Delta \). We show here how such truncations of the full representation space arise for the superconformal algebra \( PSU(2, 2|4) \), following the approach in [2].

The essential results are found by considering the chiral subalgebra \( SU(2|4) \) (although no hermeticity conditions are imposed) which has generators \( Q^i_\alpha, S^\alpha_i, \alpha = 1, 2, i = 1, \ldots 4 \), where
\[ \{ Q^i_\alpha, S^\beta_j \} = 4(\delta^i_j (M^\alpha_\beta + \frac{1}{2} \delta^i_\alpha \hat{D}) - \delta^i_\alpha R^i_j) \tag{D.1} \]
\[ \]
as well as \(\{Q^i, Q^j\} = \{S^{i\alpha}, S^{j\beta}\} = 0\). In (D.1) \(M_{\alpha}\) are generators of \(SU(2)\) and \(R^i_j\) are generators of \(SU(4)\), \(\sum_i R^i_i = 0\), with standard commutation relations. \(\hat{D}\) is the dilation operator, with eigenvalues the scale dimension. The commutators with \(Q^i \) and \(S^{i\alpha}\) are then

\[
[M^\alpha_{\beta}, Q^i] = \delta^\beta_{\gamma} Q^i - \frac{i}{2} \delta^\beta_{\alpha} Q^i, \\
[M^\alpha_{\beta}, S^{i\gamma}] = -\delta^\alpha_{\gamma} S^{i\beta} + \frac{i}{2} \delta^\alpha_{\beta} S^{i\gamma}, \\
[R^i_j, Q^k] = \delta^k_j Q^i - \frac{i}{2} \delta^k_i Q^j, \\
[R^i_j, S^{k\alpha}] = -\delta^i_k S^{j\alpha} + \frac{i}{2} \delta^i_j S^{k\alpha}, \\
[\hat{D}, Q^i] = \frac{i}{2} Q^i, \\
[\hat{D}, S^{i\alpha}] = -\frac{i}{2} S^{i\alpha}.
\]

In terms of the usual \(J_3, J_{\pm}\)

\[
[M^\alpha_{\beta}] = \left( \begin{array}{cc} J_3 & J_+ \\ J_- & -J_3 \end{array} \right),
\]

and it clear then that \((Q^i_1, Q^i_2)\) and \((S^i_2, -S^i_1)\) form \(j = \frac{1}{2}\) doublets. In terms of a standard Chevalley basis \(E^\pm_r, H_r, r = 1, 2, 3\), where \(H^\dagger_r = H_r, E_r^{+\dagger} = E_r^{-}\) with commutators \([H_r, H_s] = 0, [E_r^+, E_s^-] = \delta_{rs} H_s, [H_r, E_s^\pm] = \pm K_{rs} E_s^\pm\), for \([K_{rs}]\) the \(SU(4)\) Cartan matrix,

\[
[K_{rs}] = \left( \begin{array}{ccc} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{array} \right),
\]

then we may take \(R^1_2 = E_1^+, R^2_3 = E_2^+, R^3_4 = E_3^+\) and \(R^i_i = \frac{1}{4}(3H_1 + 2H_2 + H_3) - \sum_{r=1}^{i-1} H_r\).

For \(SU(4) \otimes SU(2)\) highest weight states \(|p_1, p_2, p_3; j\rangle^{\text{hw}} \equiv |p; j\rangle^{\text{hw}}\) we have

\[
H_r|p; j\rangle^{\text{hw}} = p_r|p; j\rangle^{\text{hw}}, \qquad J_3|p; j\rangle^{\text{hw}} = j|p; j\rangle^{\text{hw}}, \qquad J_+|p; j\rangle^{\text{hw}} = E_r^+|p; j\rangle^{\text{hw}} = 0,
\]

from which states defining a representation with Dynkin labels \([p_1, p_2, p_3]j\) are constructed by the action of \(E_r^+, J_-\). The representations of \(SU(2|4)\) may then be formed from a highest weight state which is also superconformal primary,

\[
\hat{D}|p; j\rangle^{\text{hw}} = \Delta|p; j\rangle^{\text{hw}}, \qquad S^{i\alpha}|p; j\rangle^{\text{hw}} = 0.
\]

The states of a generic supermultiplet, labelled by \(a^\Delta_{[p_1, p_2, p_3]j}\), are obtained by the action of the supercharges, giving \(\prod_{i,\alpha} (Q^i_{\alpha})^{n_{i\alpha}}|p; j\rangle^{\text{hw}}\) with \(n_{i\alpha} = 0, 1\), together with the lowering operators \(E_r^-\). The possible \(SU(4) \otimes SU(2)\) representations \([p_1', p_2', p_3']j'\), with scale dimension \(\Delta'\), forming the supermultiplet \(a^\Delta_{[p_1, p_2, p_3]j}\) are obtained by adding the \(SU(4), SU(2)\) weights with \(n_{i\alpha} = 0, 1\) so that

\[
p_{r'} = p_r + \sum_{\alpha} (n_{r\alpha} - n_{r+1\alpha}), \quad j' = j + \frac{1}{2} \sum_i (n_{i1} - n_{i2}), \quad \Delta' = \Delta + \frac{1}{2} \sum_{i,\alpha} n_{i\alpha},
\]

65
It is easy to see that \( \dim a_{[p_1, p_2, p_3]j}^\Delta = 2^5 d(p_1, p_2, p_3)(2j + 1) \), where \( d(p_1, p_2, p_3) \) is the dimension of the \( SU(4) \) representation with Dynkin labels \([p_1, p_2, p_3]\). If in (D.7) any \( p_1', p_2', p_3' \) are negative the Racah-Speiser algorithm, described in [22], provides a precise prescription for removing such \([p_1', p_2', p_3']j'\).

Shortening conditions arise for suitable \( \Delta \) when descendant representations satisfy the conditions (D.6) to be superconformal primary. Since all \( S_1^\alpha \) are obtained by commutators of \( S_1^1 \) with \( E_r^+ \) and \( J_+ \) it is sufficient to impose only that \( S_1^1 \) annihilates the highest weight state of the representation. In such cases we may impose that the appropriate combinations of \( Q_1^i \) annihilate \([p; j]^{hw}\). For application here it is convenient to define, acting on states \(|\psi\rangle\) such that \( J_3|\psi\rangle = j|\psi\rangle \),

\[
\tilde{Q}^i = Q^i_2 - \frac{1}{2j'} Q^i_1 J_- . \tag{D.8}
\]

If \( J_+|\psi\rangle = 0 \) then \( J_+ \tilde{Q}^i|\psi\rangle = 0 \) and \( J_3 \tilde{Q}^i|\psi\rangle = (j - \frac{1}{2}) \tilde{Q}^i|\psi\rangle \). From (D.1) we have

\[
\frac{1}{2} j \{ S_1^1, \tilde{Q}^1 \} = (2j - J_3 - \frac{1}{2} \hat{D} + \frac{1}{4}(3H_1 + 2H_2 + H_3)) J_- , \quad \frac{1}{2} j \{ S_1^1, \tilde{Q}^2 \} = E_1^- J_- . \tag{D.9}
\]

It is straightforward to show that \( \tilde{Q}^i|p; j\rangle^{hw} \sim |p_1 + 1, p_2, p_3; j - \frac{1}{2}\rangle \). The shortening conditions considered in [22] and previously are obtained by imposing

\[
\tilde{Q}^i|p; j\rangle^{hw} = 0 \left\{ \begin{array}{ll}
  i = 1 , & \text{if } p_1 = 0 , \\
  i = 1, 2 , & \text{if } p_1 = p_2 = 0 , \\
  i = 1, 2, 3 , & \text{if } p_1 = p_2 = p_3 = 0 .
\end{array} \right. \tag{D.10}
\]

In each case there is a consistency condition on \( \Delta \) which can be found by using (D.9) and (D.6),

\[
\Delta = 2 + 2j + \frac{1}{2}(3p_1 + 2p_2 + p_3) . \tag{D.11}
\]

The corresponding supermultiplet is here denoted by \( c_{[p_1, p_2, p_3]j} \). Detailed results were given in [22], the \( SU(4) \times SU(2) \) representations present may be calculated as in (D.7) with the restriction \( n_{i2} = 0 \) for those \( i \) listed in (D.10) for each case.

There are also additional shortening conditions of the form

\[
(\tilde{Q}^2 - \frac{1}{p_1} \tilde{Q}^1 E_1^-)|p; j\rangle^{hw} = 0 , \quad J_3 > 0 , \quad \tilde{Q}^1 \tilde{Q}^2|0, p_2, p_3; j\rangle^{hw} = 0 , \tag{D.12}
\]

where the left hand sides correspond to highest weight states \(|p_1 - 1, p_2 + 1, p_3; j - \frac{1}{2}\rangle \) and \(|0, p_2 + 1, p_3; j - 1\rangle \) respectively. Using (D.9) these conditions then require

\[
\Delta = 2j + \frac{1}{2}(-p_1 + 2p_2 + p_3) . \tag{D.13}
\]
For $p_2 = 0$ the condition (D.12) extends also to $(\hat{Q}^3 - \frac{1}{p_1} \hat{Q}^1 [E_2^-, E_1^-]) |p; j\rangle^{\text{hw}} = 0$. The supermultiplet in each case is denoted by $d_{[p_1,p_2,p_3]j}$. The representations are obtained as in (D.7) with $n_{22} = 0$, or if $p_2 = 0$ then $n_{22} = n_{32} = 0$. For $p_1 = 0$ it is sufficient to exclude $n_{12} = n_{22} = 1$.

These semi-short representations lead to decompositions of the generic multiplet,

\[
\begin{align*}
& a_{[p_1,p_2,p_3]j}^{2+2j+\frac{1}{2}(3p_1+2p_2+p_3)} \simeq c_{[p_1,p_2,p_3]j} \oplus c_{[p_1+1,p_2,p_3]j-\frac{1}{2}}, \\
& a_{[p_1,p_2,p_3]j}^{2j+\frac{1}{2}(-p_1+2p_2+p_3)} \simeq d_{[p_1,p_2,p_3]j} \oplus d_{[p_1-1,p_2+1,p_3]j-\frac{1}{2}}, \\
& a_{[0,p_2,p_3]j}^{2j+\frac{1}{2}(2p_2+p_3)} \simeq d_{[0,p_2,p_3]j} \oplus c_{[0,p_2+1,p_3]j-1}.
\end{align*}
\]

Formally, as discussed in [22], we have

\[
c_{[p_1,p_2,p_3]-\frac{1}{2}} \simeq b_{[p_1+1,p_2,p_3]}, \quad c_{[p_1,p_2,p_3]-1} \simeq -b_{[p_1,p_2,p_3]},
\]

where $b_{[p_1,p_2,p_3]}$ is the short supermultiplet formed by imposing $Q^1_\alpha |p; 0\rangle^{\text{hw}} = 0$ where we require $\Delta = \frac{1}{2}(3p_1 + 2p_2 + p_3)$. Just as in (D.15), $d_{[p_1,p_2,p_3]-\frac{1}{2}}$ may be identified with a multiplet obtained from the highest weight state $|p_1 - 1, p_2 + 1, p_3; 0\rangle^{\text{hw}}$, with $\Delta = \frac{1}{2}(-p_1 + 2p_2 + p_3 + 1)$, annihilated by $Q^2_\alpha - \frac{1}{p_1} Q^1_\alpha E_1^-$. Formally we have

\[
d_{[p_1,p_2,p_3]j} \simeq c_{[-p_1-2, p_2+p_2+3, -p_3-2]j},
\]

where, in accord with the Racah-Speiser algorithm described in [22], we may identify $SU(4)$ representations $[p_1, p_2, p_3] \simeq [-p_1-2, p_1+p_2+p_3+2, -p_3-2]$ which are related by an even element of the Weyl group. This allows the detailed representation content and dimension for $d_{[p_1,p_2,p_3]j}$ to be determined from the results given in [22].

The generators of the superconformal group $PSU(2,2|4)$ are obtained by extending those of $SU(2|4)$ to include the hermitian conjugates, $\hat{Q}_{i\dot{\alpha}} = Q^i_\alpha \dagger, \hat{S}^{i\dot{\alpha}} = S^i_\alpha \dagger, \hat{M}^{\alpha}_{\dot{\beta}} = (M^\alpha_{\dot{\beta}})^\dagger$, with an algebra obtained by conjugation of that for $SU(2|4)$, assuming $\hat{D}^\dagger = -\hat{D}$ and $(R^i_j)^\dagger = R^j_i$. In addition $\{S^i_\alpha, Q_{j\dot{\alpha}}\} = \{Q^i_\alpha, S^{j\dot{\alpha}}\} = 0$ and $\{Q^i_\alpha, \hat{Q}_{j\dot{\alpha}}\} = 2\delta^i_j (\sigma^a)_{\alpha\dot{\alpha}} P_a, \{\hat{S}^{i\dot{\alpha}}, S^j_\alpha\} = 2\delta^j_i (\bar{\sigma}^\dot{a})_{\dot{\alpha}\alpha} K_a$ where $P_a$ is the momentum operator and $K_a$ the generator of special conformal transformations. The supermultiplets are generated from highest weight superconformal primary states $|p_1, p_2, p_3; j, j\rangle^{\text{hw}}$, where $\hat{J}$ is the $SU(2)$ quantum number for $\hat{J}_+, \hat{J}_- \hat{J}_3$ obtained from $\hat{M}^{\alpha}_{\dot{\beta}},$ which is annihilated by $S^i_\alpha, \hat{S}^{i\dot{\alpha}}$ and $K_a$. The representations are of course infinite dimensional, since they are generated by arbitrary powers of $P_a$, but they are formed by a finite set of conformal primary representations, annihilated by $K_a$, which are straightforwardly constructed from $SU(2|4)$ supermultiplet representations, as described above, combined with their conjugates formed by the action of $Q_{i\alpha}$. For these $j \rightarrow \hat{j}$ and $p_1 \leftrightarrow p_3$. The generic supermultiplet
is denoted $A_{p_1,p_2,p_3}^{\Delta}(j,\bar{j})$, where $[p_1, p_2, p_3](j, \bar{j})$ are the labels for the representation with lowest scale dimension $\Delta$. The conformal primary states form representations labelled by $[p_1', p_2', p_3'](j', \bar{j}')$, with scale dimension $\Delta'$, which are given by

$$p'_{r} = p_{r} + \sum_{\alpha}(n_{r\alpha} - n_{r+1\alpha}) + \sum_{\dot{\alpha}}(\bar{n}_{r\dot{\alpha}} - \bar{n}_{r+1\dot{\alpha}}),$$

$$j' = j + \frac{1}{2} \sum_{i}(n_{i1} - n_{i2}), \quad \bar{j}' = \bar{j} + \frac{1}{2} \sum_{i}(\bar{n}_{i2} - \bar{n}_{i1}),$$

$$\Delta' = \Delta + \frac{1}{2} \sum_{i,\alpha} n_{i\alpha} + \frac{1}{2} \sum_{i,\dot{\alpha}} \bar{n}_{i\dot{\alpha}}, \quad n_{i\alpha}, \bar{n}_{i\dot{\alpha}} = 0, 1.$$  \hspace{1cm} (D.17)

The total dimension is $2^{16}d(p_1, p_2, p_3)(2j + 1)(2\bar{j} + 1)$.

We here consider the case when shortening conditions are imposed for both the $Q$ and $\bar{Q}$ charges. Requiring (D.10) with (D.11) together with its conjugate we have the semi-short multiplets,

$$C_{p_1,p_2,p_3}(j,\bar{j}) : \quad p_1 - p_3 = 2(j - \bar{j}), \quad \Delta = 2 + j + \bar{j} + p_1 + p_2 + p_3,$$  \hspace{1cm} (D.18)

where we impose in (D.17) $n_{12} = \bar{n}_{41} = 0$, with further restrictions if $p_1$ or $p_3$ are zero. Requiring (D.12) and (D.13) in both cases gives

$$D_{p_1,p_2,p_3}(j,\bar{j}) : \quad p_1 - p_3 = 2(j - \bar{j}), \quad \Delta = j + \bar{j} + p_2,$$  \hspace{1cm} (D.19)

and we require in (D.17) $n_{22} = \bar{n}_{31} = 0$. If $p_2 = 1$ then we exclude $n_{32} = \bar{n}_{21} = 1$ while if $p_2 = 0$ then we require $n_{32} = \bar{n}_{21} = 0$ as well. Corresponding to (D.16) we have

$$D_{p_1,p_2,p_3}(j,\bar{j}) \simeq C_{-p_1-2,p_1+p_2+p_3+2,-p_3-2}(j,\bar{j}).$$  \hspace{1cm} (D.20)

This result has essentially been used in (5.41). We may also impose (D.10) with (D.11) for the $Q$ charges and (D.12) and (D.13) for $\bar{Q}$ giving

$$E_{p_1,p_2,p_3}(j,\bar{j}) : \quad p_1 + p_3 = 2(\bar{j} - j - 1), \quad \Delta = 2 + j + \bar{j} + p_1 + p_2,$$  \hspace{1cm} (D.21)

and we may also obtain a conjugate $\bar{E}_{p_1,p_2,p_3}(j,\bar{j})$. Only for (D.18), where $\Delta$ is at the unitarity threshold, is there a unitary representation.

For relevance in section 5 we list the self-conjugate representations, when $p_1 = p_2$, $j = \bar{j}$, arising in $D_{q,0,q}(j,\bar{j})$, obtained by the action of equal powers of the $Q$ and $\bar{Q}$ supercharges for each $\Delta$.

| $\ell$ | $\ell + 1$ |
|-------|---------|
| $[q, 0, q]_{\ell}$ | $[q-1,0,q-1]_{\ell+1}, [q-1,2,q-1]_{\ell+1}, 3[q,0,q]_{\ell+1}, [q+1,0,q+1]_{\ell+1}$ |
| $[q-1,0,q-1]_{\ell-1}, 2[q,0,q]_{\ell-1}, [q+1,0,q+1]_{\ell-1}$ |
Table 3. Diagonal representations for each $\Delta$ in $\mathcal{D}_{[q,0,q](\ell,\ell)}$.

For application in the text we have the decompositions of self conjugate multiplets

$$
\mathcal{A}^{2j+p+2q}_{[q,p,q](j,j)} \simeq C_{[q,p,q](j,j)} \oplus C_{[q+1,p,q](j-\frac{1}{2},j)} \oplus C_{[q,p,q+1](j,j-\frac{1}{2})} \\
\quad \oplus C_{[q+1,p,q+1](j-\frac{1}{2},j-\frac{1}{2})},
$$

$$
\mathcal{A}^{2j+p}_{[q,p,q](j,j)} \simeq \mathcal{D}_{[q,p,q](j,j)} \oplus \mathcal{D}_{[q-1,p+1,q](j-\frac{1}{2},j)} \oplus \mathcal{D}_{[q,p+1,q-1](j,j-\frac{1}{2})} \\
\quad \oplus \mathcal{D}_{[q-1,p+2,q-1](j-\frac{1}{2},j-\frac{1}{2})},
$$

$$
\mathcal{A}^{2j+p}_{[0,p,0](j,j)} \simeq \mathcal{D}_{[0,p,0](j,j)} \oplus \mathcal{E}_{[0,p+1,0](j-1,j)} \oplus \tilde{\mathcal{E}}_{[0,p+1,0](j,j-1)} \\
\quad \oplus C_{[0,p+2,0](j-1,j-1)}.
$$

(D.22)

The first case represents the decomposition of a long multiplet into semi-short multiplets at the unitarity threshold, the second plays a crucial role in section 5 in relating the solution of the superconformal Ward identities to the operator product expansion.
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