A NEW APPROACH FOR SOLVING NONLINEAR VOLterra
INTEGRO-DIFFERENTIAL EQUATIONS WITH
MITTAG-LEFFLER KERNEL

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Abstract. In this work, we consider a general class of nonlinear Volterra integro-differential equations with Atangana–Baleanu derivative. We use the operational matrices based on the shifted Legendre polynomials to obtain numerical solution of the considered equations. By approximating the unknown function and its derivative in terms of the shifted Legendre polynomials and substituting these approximations into the original equation and using the collocation points, the original equation is reduced to a system of nonlinear algebraic equations. An error estimate of the numerical solution is proved. Finally, some examples are included to show the accuracy and validity of the proposed method.

1. Introduction

In recent years, several definitions for fixed and variable order fractional integral and derivative operators have been introduced, such as Riemann-Liouville \cite{7, 15, 30}, Gr"{u}nwald-Letnikov \cite{30}, Caputo \cite{30}, Caputo and Fabrizio \cite{3}, Atangana and Baleanu (AB) \cite{1} (to see more, refer to \cite{6, 15, 30, 27, 31}).

Many problems in different fields of science and engineering such as viscoelasticity and damping, diffusion and wave propagation and chaos can be modeled by fractional derivative and integral operators \cite{4, 5, 13, 15, 19, 26, 28, 29, 31}. For example, Pedro et al. \cite{23} have investigated the drag force acting on a particle due to the oscillatory flow of a viscous fluid.

Fractional integro-differential equations are used in different applications of physics, chemistry, biology, engineering and applied mathematics. For example, Zakes and Sniady in \cite{32} have studied the dynamic behavior of multispans uniform continuous beam arbitrarily supported on its edges subjected to various types of moving noninertial loads. Li et al. have used differential equations to model a memory behavior of shape-memory polymer \cite{16}.

In most cases, it is not easy to obtain exact solution of most these equations. Therefore, some researchers have proposed several approximation and numerical methods for solving such equations. For example, Rakshan and Effati applied a generalized Legendre–Gauss collocation method for solving nonlinear fractional

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differential equations [24]. Ganji et al. applied the fifth-kind Chebyshev polynomials to obtain solution variable orders differential equations [12]. Zhao and Wang solved space-time fractional partial differential equations using fast finite difference methods [33]. Liu et al. used a preconditioned fast quadratic spline collocation method for partial differential equations [17]. In [25], authors solved Fredholm integral equations by the reproducing kernel Hilbert space method. A numerical scheme based on the Bernstein polynomials for solving diffusion-wave equations has been proposed in [11] (to see more, refer to [7, 8, 14, 21]).

In recent years, by increasing number of applications of fractional integro-differential equations, researchers attention have been given to use of orthogonal basis functions. The concept of operational matrices of these basis functions can be employed for the solution of the problems [20]. Recently, many researchers applied orthogonal basis functions for approximating. For example, Liu et al. solved variable order fractional differential-integral equations using Chebyshev polynomials [18]. Nemati et al. applied Legendre polynomials for solving two-dimensional nonlinear Volterra integral equations [22]. In [9] and [10], Jacobi polynomials for solving differential equations of variable order have been proposed. By using these basis functions and their operational matrices, the main problem reduces to an algebraic system, which can be solved. Then, the approximate solution can be calculated.

In this work, we study the following type of the nonlinear Volterra integro-differential equations

\[ ABCD_t^\omega u(t) = f(t, u(t)) + \lambda T u(t), \quad t \in [0, 1], \]

where

\[ Tu(t) = \int_0^t K(t, s) \phi(s, u(s)) \, ds, \]

and \( 0 < \omega \leq 1 \), the parameter \( \lambda \) is a real constant, \( K, \phi \) and \( f \) are given known functions, \( u(t) \) is an unknown function. \( ABCD_t^\omega \) denotes the Caputo AB–derivative operator.

The organization of this paper is as follows: in Section 2, we present basic definitions of fractional calculus. Section 3 is introduced the main properties of the shifted Legendre polynomials (SLPs) and operational matrices based on the SLPs. Section 4 is devoted to proposing a new numerical method for solving problem (1.1). In Section 5, an error estimate is proved for the numerical solution. Section 6 includes some illustrative examples. Finally, we conclude the paper in Section 7.

2. Basic definitions and theorems

**Definition 2.1** (See [30]). Let \( 0 < \omega \leq 1 \). The RL–integral is defined as

\[ RLI_t^\omega u(t) = \frac{1}{\Gamma(\omega)} \int_0^t (t - s)^{\omega - 1} u(s) \, ds. \]
The RL-integral of order $\omega$ satisfies the following property
\[
RL_t^\omega t^v = \frac{\Gamma(v + 1)}{\Gamma(v + 1 + \omega)} t^{v+\omega}, \quad v \geq 0.
\]

**Definition 2.2** (See [30]). Let $0 < \omega \leq 1$, $u \in H^1(0, 1)$ and $AB(\omega)$ be a normalization function such that $AB(0) = AB(1) = 1$ and $AB(\omega) = 1 - \omega + \frac{\omega}{\Gamma(\omega)}$. Then

1. The Riemann AB-derivative is given as
\[
ABR_t^\omega u(t) = \frac{AB(\omega)}{1 - \omega} \frac{d}{dt} \int_0^t E_\omega(-\frac{\omega}{1-\omega}(t-s)^\omega)u(s) \, ds,
\]
where $E_\omega(t) = \sum_{i=0}^{\infty} \frac{t^i}{\Gamma(\omega i + 1)}$ is the Mittag-Leffler function.

2. The Caputo AB-derivative is defined as
\[
ABC_t^\omega u(t) = \frac{AB(\omega)}{1 - \omega} \int_0^t E_\omega(-\frac{\omega}{1-\omega}(t-s)^\omega)u'(s) \, ds.
\]

3. The AB-integral is given as
\[
ABI_t^\omega u(t) = \frac{1 - \omega}{AB(\omega)} u(t) + \frac{\omega}{AB(\omega)\Gamma(\omega)} \int_0^t (t-s)^{\omega-1} u(s) \, ds. \tag{2.1}
\]
Let $\alpha_\omega = \frac{1 - \omega}{AB(\omega)}$ and $\beta_\omega = \frac{1}{AB(\omega)\Gamma(\omega)}$, then we can rewrite (2.1) as
\[
ABI_t^\omega u(t) = \alpha_\omega u(t) + \beta_\omega \Gamma(\omega + 1) RL_t^\omega u(t). \tag{2.2}
\]

It is easy to report that the AB-derivative and AB-integral satisfy the following properties:

1. $ABC_t^\omega C = 0$, $C \in \mathbb{R}$,

2. $ABC_t^\omega t^v = \frac{AB(\omega)}{1 - \omega} \sum_{i=0}^{\infty} (-1)^i \omega^i \Gamma(v + 1) \frac{1 - \omega}{1 - \omega} (\omega i + v + 1) t^{\omega i + v}$, $v \geq 0$,

3. $ABC_t^\omega u(t) = ABR_t^\omega u(t) - \frac{AB(\omega)}{1 - \omega} u(0) E_\omega(-\frac{\omega}{1-\omega} t^\omega)$,

4. $ABI_t^\omega C = C(\alpha_\omega + \beta_\omega t^\omega)$, $C \in \mathbb{R}$,

5. $ABI_t^\omega t^v = t^v (\alpha_\omega + \beta_\omega (v + \omega + 1) B(v + 1, \omega + 1) t^\omega)$, where $B(\cdot, \cdot)$ is the Beta function,

6. $ABI_t^\omega (ABI_t^v u(t)) = ABI_t^v (ABI_t^\omega u(t))$,

7. $ABI_t^\omega (ABC_t^\omega u(t)) = u(t) - u(0)$.

**Theorem 2.1** (See [1]). Let $C[0, 1]$ be the space of all continuous functions defined on $[0, 1]$ and $f, g \in C[0, 1]$. Then the following inequalities can be established
\[
\|ABR_t^\omega f(t) - ABR_t^\omega g(t)\|_{\infty} \leq \delta \|f(t) - g(t)\|_{\infty},
\]
\[
\|ABC_t^\omega f(t) - ABC_t^\omega g(t)\|_{\infty} \leq \delta \|f(t) - g(t)\|_{\infty},
\]
where $\delta$ is a constant number.

**Theorem 2.2.** Suppose that $f$ and $g$ satisfy the assumptions of Theorem 2.1, then we have

$$\|ABI_t^\omega f(t) - ABI_t^\omega g(t)\|_\infty \leq \varepsilon\|f(t) - g(t)\|_\infty,$$

where $\varepsilon = \alpha_\omega + \beta_\omega$.

**Proof.** According to definition of the AB-integral, we have

$$\|ABI_t^\omega f(t) - ABI_t^\omega g(t)\|_\infty = \|ABI_t^\omega (f(t) - g(t))\|_\infty$$

$$\leq \alpha_\omega\|f(t) - g(t)\|_\infty + \beta_\omega\Gamma(\omega + 1)^{\|RLI_t^\omega (f(t) - g(t))\|_\infty}$$

$$\leq (\alpha_\omega + \beta_\omega)\|f(t) - g(t)\|_\infty.$$

(2.3)

Taking $\varepsilon = \alpha_\omega + \beta_\omega$, the proof is complete. $\square$

**Theorem 2.3.** Let $0 < \omega \leq 1$ and $g : [0,1] \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function such that there exits $\zeta$ satisfying

$$|f(g,u_1(t)) - f(g,u_2(t))| \leq \zeta|u_1(t) - u_2(t)|,$$

(2.4)

for all $u_1, u_2 \in \mathbb{R}$. Then the initial value problem given by equation (1.1) has a unique solution on $C[0,1]$

(1) If $(\alpha_\omega + \beta_\omega)\zeta_1 < 1$, when $\lambda = 0$,

(2) If $(\alpha_\omega + \beta_\omega)(\zeta_1 + |\zeta_2\zeta_3|) < 1$, when $\lambda \neq 0$.

**Proof.** Taking the AB-integral of both sides of equation (1.1) gets

$$u(t) = ABI_t^\omega f(t,u(t)) + \lambda^{ABI_t^\omega Tu(t) + u_0}$$

$$= \alpha_\omega f(t,u(t)) + \beta_\omega\Gamma(\omega + 1)^{RLI_t^\omega f(t,u(t)) + \lambda\alpha_\omega Tu(t) + \beta_\omega\Gamma(\omega + 1)^{RLI_t^\omega Tu(t) + u_0}$$

(2.5)

We define the operator $\mathfrak{N} : C[0,1] \rightarrow C[0,1]$ by

$$\mathfrak{N}u(t) = \alpha_\omega f(t,u(t)) + \beta_\omega\Gamma(\omega + 1)^{RLI_t^\omega f(t,u(t)) + \lambda\beta_\omega\Gamma(\omega + 1)^{RLI_t^\omega Tu(t) + \lambda\alpha_\omega Tu(t) + C},$$

where $c = u_0 - \alpha_\omega f(0,u_0)$. By (2.5), finding a solution of equation (1.1) in $C[0,1]$ is equivalent to finding a fixed point of the operator $\mathfrak{N}$. For $u_1, u_2 \in C[0,1]$ and all $t \in [0,1]$, we have

$$|\mathfrak{N}u_1(t) - \mathfrak{N}u_2(t)| = |\alpha_\omega (f(t,u_1(t)) - f(t,u_2(t))) + \lambda\alpha_\omega T(u_1(t) - u_2(t))$$

$$+ \beta_\omega\Gamma(\omega + 1)^{RLI_t^\omega (f(t,u_1(t)) - f(t,u_2(t))) + \beta_\omega\Gamma(\omega + 1)^{RLI_t^\omega Tu(t) + u_2(t))|$$

$$\leq |\alpha_\omega f(t,u_1(t)) - f(t,u_2(t))| + \lambda(|\alpha_\omega T(u_1(t) - u_2(t))|$$

$$+ \beta_\omega\Gamma(\omega + 1)^{RLI_t^\omega (f(t,u_1(t)) - f(t,u_2(t))) + \beta_\omega\Gamma(\omega + 1)^{RLI_t^\omega Tu(t) - u_2(t))|.$$
Let
\[ \varsigma_3 = \sup_{t,s \in [0,1]} K(t,s), \] (2.6)
and according to (2.4), we have
\[ |f(t, u_1(t)) - f(t, u_2(t))| \leq \varsigma_1 |u_1(t) - u_2(t)|, \]
\[ |\phi(t, u_1(t)) - \phi(t, u_2(t))| \leq \varsigma_2 |u_1(t) - u_2(t)|. \] (2.7)

In view of (2.6), (2.7) and according to \( T u(t) \), we have
\[ |\mathcal{N} u_1(t) - \mathcal{N} u_2(t)| \leq \alpha \omega_1 \varsigma_1 |u_1(t) - u_2(t)| + \beta \omega_1 \varsigma_2 |u_1(t) - u_2(t)| + \lambda \omega \varsigma_2 \varsigma_3 |u_1(t) - u_2(t)|.
\]
Therefore the operator \( \mathcal{N} \) is a contraction. The statement follows from Banach’s Fixed Point Theorem. \( \square \)

3. The SLPs and operational matrices based on the SLPs

3.1. The SLPs. The SLPs on \([0,1]\) consist the set of orthogonal functions. The analytic form of the SLP of degree \( n \) is defined by
\[ L_0(t) = 1, \quad L_1(t) = 2t - 1, \quad L_n(t) = \sum_{k=0}^{n} a_{n,k} t^k, \] (3.1)
where
\[ a_{n,k} = \frac{(-1)^{n+k} (n+k)!}{(n-k)! (k!)^2}. \]

For two arbitrary functions \( f, g \in L^2[0,1] \) the inner product and norm are defined, respectively, by
\[ \langle f,g \rangle = \int_0^1 f(t)g(t) dt, \]
\[ \|f\|_{L^2(0,1)}^2 = \sqrt{\langle f,f \rangle}. \]

Any arbitrary function \( u \in L^2[0,1] \) can be expanded using the SLPs as follows
\[ u(t) = \sum_{i=0}^{\infty} u_i L_i(t) = U^T \varphi(t), \] (3.2)
where
\[ u_i = Q^{-1} \int_0^1 u(t) L_i(t) dt, \]
and
\[ Q = [q_{ij}] \quad i,j = 0,1, \cdots, M, \]
and the elements \( q_{ij} \) given by
\[ q_{ij} = \begin{cases} \frac{1}{2i+1}, & i = j, \\ 0, & i \neq j. \end{cases} \]
The function \( u(t) \) can be approximated by taking only the first \( M + 1 \) terms in (3.2) as
\[
  u(t) \simeq u_M(t) = \sum_{i=0}^{M} u_i L_i(t) = U^T \mathcal{L}(t),
\]
where \( U = [u_0, u_1, \ldots, u_M]^T \) and
\[
  \mathcal{L}(t) = [L_0(t), L_1(t), \ldots, L_M(t)]^T.
\]

Similarly, any function \( K(t, s) \) in \( L^2([0, 1] \times [0, 1]) \) can be expanded in terms of the SLPs as
\[
  K(t, s) \simeq \mathcal{L}^T(t) K \mathcal{L}(s),
\]
where \( K \) is an \( (M + 1) \times (M + 1) \) matrix as
\[
  K = [k_{i,j}], \quad i, j = 0, 1, \ldots, M,
\]
and the shifted Legendre coefficients \( k_{i,j} \) are given by
\[
  k_{i,j} = \frac{\langle (K(t, s), \mathcal{L}(t)), \mathcal{L}(s) \rangle}{\|L_i(t)\|_2^2 \|L_j(s)\|_2^2}, \quad i, j = 0, 1, \ldots, M.
\]

3.2. Operational matrices of the SLPs.

1. The integration of the vectors \( \mathcal{L}(t) \) defined by (3.4) can be approximately given by
\[
  \int_0^t \mathcal{L}(s) ds \simeq P \mathcal{L}(t),
\]
where \( P \) is an \( (M + 1) \times (M + 1) \) matrix which is called the operational matrix of integration based on the SLPs and is given by
\[
  P = \frac{1}{2}
  \begin{bmatrix}
    1 & 1 & 0 & \cdots & 0 & 0 & 0 \\
    \frac{1}{3} & 0 & \frac{1}{3} & \cdots & 0 & 0 & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
    0 & 0 & 0 & \cdots & \frac{-1}{2M-1} & 0 & \frac{1}{2M-1} \\
    0 & 0 & 0 & \cdots & 0 & \frac{-1}{2M+1} & 0
  \end{bmatrix}.
\]

2. It is needed to evaluate the product of \( \mathcal{L}(t) \) and \( \mathcal{L}^T(t) \), that will give the product operational matrix of the shifted Legendre basis. First, we write
\[
  \mathcal{L}_i(t) \mathcal{L}_j(t) = \sum_{r=0}^{i+j} a_r \mathcal{L}_r(t).
\]

The coefficients \( a_r, r = 0, 1, \ldots, i+j \), are computed in the following manner. Multiplying both sides of equation (3.6) by \( \mathcal{L}_m(t) \), \( m = 0, 1, \ldots, i+j \), and integrating the result yields
\[
  \int_0^1 \mathcal{L}_i(t) \mathcal{L}_j(t) \mathcal{L}_m(t) dt = \sum_{r=0}^{i+j} a_r \int_0^1 \mathcal{L}_r(t) \mathcal{L}_m(t) dt = \frac{1}{2m+1} a_m.
\]
Using (3.7) and the analytic form of the SLPs given by (3.1), we obtain

\[ a_m = (2m + 1) \int_0^1 L_i(t) L_j(t) L_m(t) \, dt \]

\[ = (2m + 1) \sum_{k=0}^i \sum_{l=0}^j \sum_{s=0}^m c_{i,k} c_{j,l} c_{m,s} \int_0^1 t^{k+l+s} \, dt = (2m + 1) \Delta_{i,j,m}, \]

where

\[ \Delta_{i,j,m} = \sum_{k=0}^i \sum_{l=0}^j \sum_{s=0}^m c_{i,k} c_{j,l} c_{m,s} \frac{1}{k+l+s+1}. \]

By substituting \( a_m \) into equation (3.6), we have

\[ L_i(t) L_j(t) = \sum_{m=0}^{i+j} (2m + 1) \Delta_{i,j,m}. \]  \hfill (3.8)

Suppose that

\[ \mathbf{C} = [c_0, c_1, \ldots, c_M]^T, \]

then using (3.8), we can write

\[ \mathbf{L}(t) \mathbf{L}^T(t) \mathbf{C} \simeq \overline{\mathbf{C}} \mathbf{L}(t), \]  \hfill (3.9)

where \( \overline{\mathbf{C}} \) is an \((M + 1) \times (M + 1)\) matrix which is called the product operational matrix of the SLPs basis vector, and is given by

\[ \overline{\mathbf{C}} = [\overline{c}_{i,j}], \quad i, j = 0, 1, \ldots, M, \]

where

\[ \overline{c}_{i,j} = \frac{2j + 1}{T} \sum_{m=0}^M \Delta_{i,j,m} c_m. \]

Also, for an \((M + 1) \times (M + 1)\) matrix \( D = [d_{i,j}], \quad i, j = 0, 1, \ldots, M, \) we have

\[ \mathbf{L}^T(t) D \mathbf{L}(t) \simeq D^* \mathbf{L}(t), \]  \hfill (3.10)

where \( D^* = [d^*_{i}], \quad i = 0, 1, \ldots, M \) is an \( 1 \times (M + 1) \) vector given by

\[ d^*_{i} = (2i + 1) \sum_{m=0}^M \sum_{n=0}^M \Delta_{m,n,i} d_{m,n}. \]

(3) The operational matrix of RI–integral of order \( \omega \) for vector \( \mathbf{L}(t) \) is as

\[ \mathbf{R}^{1/\Gamma} \mathbf{L}(t) \simeq \mathbf{Y}^\omega \mathbf{L}(t), \]  \hfill (3.11)

where

\[ \mathbf{Y}^\omega = [\sigma_{i,j}], \quad \sigma_{i,j} = \sum_{p=0}^i \rho_{p,j}^{a_{i,p}} \frac{\Gamma(p + 1)}{\Gamma(p + \omega + 1)}, \quad i, j = 0, 1, \ldots, M, \]

with

\[ \rho_{p,j} = (2j + 1) \sum_{l=0}^j \frac{(-1)^{j+l}(j+l)!}{(j-l)!(l!)^2(p+l+\omega+1)}. \]
The matrix \( \Upsilon^\omega \) is called the operational matrix of RL–integral based on the SLPs [2].

(4) The operational matrix of AB–integral of order \( \omega \) for vector \( \mathcal{L}(t) \) is obtained as

\[
ABI^\omega_t \mathcal{L}(t) = \alpha^\omega \mathcal{L}(t) + \beta^\omega \Gamma(\omega + 1)RLI^\omega_t \mathcal{L}(t).
\]

According to (3.11), we have

\[
ABI^\omega_t \mathcal{L}(t) \simeq \alpha^\omega \mathcal{L}(t) + \beta^\omega \Gamma(\omega + 1)\Upsilon^\omega \mathcal{L}(t) = (\alpha^\omega I + \beta^\omega \Gamma(\omega + 1)\Upsilon^\omega) \mathcal{L}(t) = J^\omega \mathcal{L}(t),
\]

where \( I \) is an \((M + 1) \times (M + 1)\) identity matrix and \( J^\omega = \alpha^\omega I + \beta^\omega \Gamma(\omega + 1)\Upsilon^\omega \). The matrix \( J^\omega \) is called the operational matrix of AB–integral based on the SLPs.

4. Numerical method

In this section, we introduce a numerical method for the solution of nonlinear Volterra integro-differential equations of the form equation (1.1). For this suppose, first we approximate \( ABCD^\omega_t u(t) \) as

\[
ABC^\omega_t u(t) \simeq U^T \mathcal{L}(t).
\]

By taking the AB-integral of (4.1) and using initial condition, we have

\[
u(t) \simeq U^T J^\omega \mathcal{L}(t) + u_0.
\]

We approximate \( u_0 \) as

\[
u_0 \simeq U^T \mathcal{L}(t).
\]

By putting (4.3) in (4.2), we can write (4.2) as

\[
u(t) \simeq \gamma \mathcal{L}(t),
\]

where \( \gamma = U^T J^\omega + U^T \). Let \( g(t) = f(t, u(t)) \) and \( h(t) = \phi(t, u(t)) \). Then we approximate \( g(t) \) as

\[
g(t) \simeq G^T \mathcal{L}(t).
\]

For approximating \( Tu(t) \), first \( K(t, s) \) and \( h(t) \) are approximated as

\[
K(t, s) \simeq \mathcal{L}^T(t) K \mathcal{L}(s),
\]

\[
h(t) \simeq H^T \mathcal{L}(t).
\]

By using (4.6) and (4.7), \( Tu(t) \) is approximated as

\[
Tu(t) \simeq \int_0^t \mathcal{L}^T(t) K \mathcal{L}(s) \mathcal{L}^T(s) H ds
\]

\[
= \mathcal{L}^T(t) K \int_0^t \mathcal{L}(s) \mathcal{L}^T(s) H ds
\]

\[
= \mathcal{L}^T(t) K \mathcal{P} \int_0^t \mathcal{L}(s) ds = \mathcal{L}^T(t) K \mathcal{P} P \mathcal{L}(s).
\]
Substituting (4.1), (4.5) and (4.8) in equation (1.1), we have
\[ U^T L(t) = G^T L(t) + \mathcal{K} \mathcal{P} L(s). \] (4.9)

Let \( \Lambda = K \mathcal{P} L \). Using (3.10) to define the vector \( \Lambda^* \), and using this approximation in (4.9) we obtain the following system
\[ U^T - G^T - \Lambda^* = 0. \] (4.10)

In view of (4.4), we rewrite (4.5) and (4.7) as
\[ f(t, \gamma L(t)) = G^T L(t), \]
\[ \phi(t, \gamma L(t)) = H^T L(t). \] (4.11)

By putting \( M + 1 \) points \( t_i, i = 1, 2, \ldots, M + 1 \) in equation (4.11), gives
\[ f(t_i, \gamma L(t_i)) - G^T L(t_i) = 0, \]
\[ \phi(t_i, \gamma L(t_i)) - H^T L(t_i) = 0, \] (4.12)

where \( t_i = \frac{i}{M + 2} \).

Equations (4.10) and (4.12) form a system of \( 3(M + 1) \) nonlinear equations of the vectors of \( U, G \) and \( H \). By solving this system, the unknown parameters of the vectors of \( U, G \) and \( H \) are obtained. Finally the approximate solution can be computed by (4.4).

5. Error estimation

In this section, we give an estimate for the error of the numerical solution of equation (1.1) with initial condition obtained by the proposed method in Section 4.

**Theorem 5.1.** Suppose that \( u \) and \( u_M \) satisfy the assumptions of Theorems 2.1 and 2.2. Assume that \( u \in C^{M+1}[0,1] \) is the exact solution of equation (1.1) and \( u_M(t) = \gamma^T L(t) \) is its approximation given by the method proposed in Section 4. Then, we have
\[ \|u(t) - u_M(t)\|_2 \leq \frac{\rho}{(M + 1)!2^{2M+1}}, \]
\[ \|ABCD_t^\omega u(t) - ABD_I^\omega u_M(t)\|_\infty \leq \frac{\delta \rho}{\sqrt{M + 1}M!2^{2M+1}}, \]
\[ \|ABG_J^\omega u(t) - ABJ_I^\omega u_M(t)\|_\infty \leq \frac{\varepsilon \rho}{\sqrt{M + 1}M!2^{2M+1}}, \]
where \( \rho = \sup_{\theta \in [0,1]} |u^{(M+1)}(\theta)| \).

**Proof.** Let that \( u^*(t) \) is the interpolating polynomials to \( u(t) \) at points \( t_i, i = 0, 1, \ldots, M \) are the roots of the shifted Chebyshev polynomials of degree \( M + 1 \), then we can write
\[ u(t) - u^*(t) = \frac{u^{(M+1)}(\theta)}{(M + 1)!} \prod_{i=0}^{M} (t - t_i), \]
where \( \theta \in [0,1] \).
According to Chebyshev interpolation nodes, for \( t \in [0, 1] \) it gives

\[
|u(t) - u^*(t)| = \frac{\rho}{(M + 1)!2^{2M+1}},
\]

where \( \rho = \sup_{\theta \in [0,1]} |u^{(M+1)}(\theta)| \).

Since the best approximation of a given function \( u \in C^{M+1}[0,1] \) in the finite subspace \( S_M = \{ L_0(t), L_1(t), \ldots, L_M(t) \} \) is unique, we can write

\[
\|u(t) - u_M(t)\|_2^2 \leq \|u(t) - u^*(t)\|_2^2
\]

\[
= \int_0^1 |u(t) - u^*(t)|^2 dt
\]

\[
= \int_0^1 \left( \frac{\rho}{(M + 1)!2^{2M+1}} \right)^2 dt = \left( \frac{\rho}{(M + 1)!2^{2M+1}} \right)^2.
\]

Taking the squared root of both sides of (5.1) gets

\[
\|u(t) - u_M(t)\|_2 \leq \frac{\rho}{(M + 1)!2^{2M+1}}.
\]

We know that \( \|u\|_\infty \leq \sqrt{M+1}\|u\|_2 \). Then, in view of Theorems 2.1 and 2.2, the proof is complete. \( \square \)

6. Test examples

In this section, we consider several examples to confirm the accuracy and applicability of the proposed method.

**Example 6.1.** Let \( f(t, u(t)) = 2t - 1 - \frac{1}{4}t \left( -2e^t + 2e^{-t+2} + \sqrt{\pi}e^{\frac{3}{4}}(Erfi[\frac{1}{2}] - Erfi[\frac{1}{2} - t]) \right), \) \( \lambda = 1, \) \( u(0) = 1 \) and \( Tu(t) \) is given as

\[
Tu(t) = \int_0^t tse^{u(s)} ds.
\]

\( Erfi(\cdot) \) is the imaginary error function. The analytical solution of this example is \( u(t) = 1 - t + t^2 \) when \( \omega = 1 \). We solved this example by the proposed method. By considering \( M = 5 \), the numerical results are shown in Figure 1 by different values of \( \omega \). As seen in Figure 1, when the value of \( \omega \) tends to 1, the approximate solution converges to the analytical solution.

**Example 6.2.** Let \( f(t, u(t)) = e^t(1 + e^2) - \frac{1}{4}t^2 (1 + e^{2t}(-1 + 2t)) - u^2(t) \), \( \lambda = 1, \) \( u(0) = 1 \) and \( Tu(t) \) is given as

\[
Tu(t) = \int_0^t t^2su^2(s) ds.
\]

The analytical solution of this example is \( u(t) = e^t \) when \( \omega = 1 \). This example is solved by the proposed method with different values of \( \omega \). By considering \( M = 5 \), the numerical results are plotted in Figure 2.

**Example 6.3.** Let \( f(t, u(t)) = 1 - u^2(t), \) \( \lambda = 0 \) and \( u(0) = 0 \). The analytical solution of this example is \( u(t) = \frac{e^{2t} - 1}{e^{2t} + 1} \) when \( \omega = 1 \). By applying the proposed method, we solved this example. By considering \( M = 5 \), the numerical results are
Figure 1. (Example 6.1) The exact and approximate solutions given by different values of $\omega$ and $M = 5$.

Figure 2. (Example 6.2) The exact and approximate solutions given by different values of $\omega$ and $M = 5$.

Example 6.4. Consider the following system of nonlinear Volterra integro-differential equation

$$\begin{align*}
ABC^\omega u(t) &= f_1(t, u(t), v(t)) + \lambda_1 T_1(u, v)(t), \\
ABC^\omega v(t) &= f_2(t, u(t), v(t)) + \lambda_2 T_2(u, v)(t),
\end{align*}$$

plotted in Figure 3 by different values of $\omega$. As seen in Figure 3, when the value of $\omega$ tends to 1, the approximate solution converges to the analytical solution.
where
\[
\begin{align*}
\lambda_1 &= -1, \quad \lambda_2 = 1, \\
 f_1(t, u(t), v(t)) &= 1 + t^3 + t^4, \\
 f_2(t, u(t), v(t)) &= 1 - \frac{t^3}{2} - \frac{t^4}{3}, \\
 T_1(u, v)(t) &= \int_0^t t^2 (u(s) + v(s)) \, ds, \\
 T_2(u, v)(t) &= \int_0^t tu(s)v(s) \, ds, \\
u(0) &= 0, \quad v(0) = 1.
\end{align*}
\]

The analytical solutions of this example are \( u(t) = t \) and \( v(t) = 1 + t \) when \( \omega = 1 \). By using the proposed method, we solved this example. By considering \( M = 5 \), the numerical results are reported in Figure 4, Table 1 and 2 by different values of \( \omega \). The exact and numerical solutions obtained by different values of \( \omega \) are plotted in Figure 4. Furthermore, in Table 1 and 2, the exact and numerical solutions at some grid points with different values of \( \omega \) are seen.

Figure 3. (Example 6.3) The exact and approximate solutions given by different values of \( \omega \) and \( M = 5 \).

Figure 4. (Example 6.4) (a) The exact and approximate solutions \( (u(t)) \) (b) The exact and approximate solutions \( (v(t)) \) and \( M = 5 \).
Table 1. (Example 6.4) Numerical results $u(t)$ by different values of $\omega$ and $M = 5$.

| $t$ | $\omega = 0.85$ | $\omega = 0.9$ | $\omega = 0.99$ | $\omega = 0.999$ | $\omega = 1$ |
|-----|-----------------|-----------------|-----------------|-----------------|-------------|
| 0.1 | 0.30231         | 0.23076         | 0.11235         | 0.10123         | 0.10000     |
| 0.3 | 0.51480         | 0.44093         | 0.31363         | 0.30136         | 0.30000     |
| 0.5 | 0.69220         | 0.62946         | 0.51305         | 0.50131         | 0.50000     |
| 0.7 | 0.83586         | 0.79789         | 0.71085         | 0.70109         | 0.70000     |
| 0.9 | 0.93217         | 0.93812         | 0.90640         | 0.90066         | 0.90000     |

Table 2. (Example 6.4) Numerical results $v(t)$ by different values of $\omega$ and $M = 5$.

| $t$ | $\omega = 0.85$ | $\omega = 0.9$ | $\omega = 0.99$ | $\omega = 0.999$ | $\omega = 1$ |
|-----|-----------------|-----------------|-----------------|-----------------|-------------|
| 0.1 | 1.30285         | 1.23099         | 1.11236         | 1.10123         | 1.10000     |
| 0.3 | 1.52586         | 1.44615         | 1.31384         | 1.30138         | 1.30000     |
| 0.5 | 1.74109         | 1.65396         | 1.51429         | 1.50142         | 1.50000     |
| 0.7 | 1.97079         | 1.86860         | 1.71497         | 1.70148         | 1.70000     |
| 0.9 | 2.22022         | 2.09479         | 1.91644         | 1.90162         | 1.90000     |

7. Conclusion

In this work, we presented a numerical method based on operational matrices to solve Volterra integro-differential equations. For this do, we used the operational matrices of integral, produce, fractional integral of Riemann-Liouville type and obtained the operational matrix of fractional integral of Atangana and Baleanu type. Then, by substituting these matrices and collocation points, we converted the Volterra integro-differential equations to an algebraic system. By solving this system, we obtained the approximate solution. Finally, some examples are considered to confirm the accuracy of the proposed method.

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