GENERIC HECKE ALGEBRA AND THETA CORRESPONDENCE OVER FINITE FIELDS

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Abstract. We study the Hecke algebra modules arising from theta correspondence between certain Harish-Chandra series for type I dual pairs over finite fields. For the product of the pair of Hecke algebras under consideration, we show that there is a generic Hecke algebra module whose specializations at prime powers give the Hecke algebra modules and whose specialization at 1 can be explicitly described. As an application, we prove the conservation relation on the first occurrence indices for all irreducible representations. As another application, we generalize the results of Aubert-Michel-Rouquier and Pan on theta correspondence between the Harish-Chandra series.

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1. INTRODUCTION

1.1. Let $F$ be a finite field of characteristic $p$ and let $q$ be the cardinality of $F$. Let $\tau$ be an element in $\text{Gal}(F/F_0)$ such that $\tau^2$ is the identity and $F_0$ be the fixed field of $\tau$. We fix a non-trivial additive character $\psi: F_0 \to \mathbb{C}^\times$ throughout the paper. When $p$ is odd, let $\xi$ denote the unique quadratic character of $F_0^\times$, i.e.,

$$\xi: F_0^\times \to \{ \pm 1 \}, \quad a \mapsto a^{\frac{q-1}{2}}. \tag{1.1}$$

1.2. For $\epsilon \in \{ \pm 1 \}$, an $\epsilon$-Hermitian space $V$ (over $F$) is an $F$-vector space equipped with a non-degenerate form $\langle \cdot, \cdot \rangle_V : V \times V \to F$ such that

$$\langle av_1 + v_2, v_3 \rangle_V = a \langle v_1, v_3 \rangle_V + \epsilon \langle v_3, v_2 \rangle_V^\tau, \quad \forall a \in F \text{ and } v_1, v_2, v_3 \in V.$$ Let $n = \dim V$ and let $U(V) := \{ g \in \text{End}_F(V) \mid \langle gv_1, gv_2 \rangle_V = \langle v_1, v_2 \rangle_V, \quad \forall v_1, v_2 \in V \}$ be the isometry group of $\langle \cdot, \cdot \rangle_V$. If $F = F_0$ and $\epsilon = 1$, we take a basis $\{ e_1, \ldots, e_n \}$ of $V$ and define the discriminant of $V$ by

$$\text{disc}(V) := (-1)^n a_{i,j} \text{det} ((\langle e_i, e_j \rangle_V)_{i,j}) \in F^\times / F^\times 2. \tag{1.2}$$

Note that $\text{disc}(V)$ is independent of the choice of the basis $\{ e_1, \ldots, e_n \}$ (cf. [22, Chapter 2, §2.1]).
1.3. A dual pair \((V, V')\) consists of an \(\epsilon\)-Hermitian space \(V\) and an \(\epsilon'\)-Hermitian space \(V'\) where \(\epsilon, \epsilon' \in \{\pm 1\}\) such that \(\epsilon \epsilon' = -1\). Given such a dual pair \((V, V')\), \(V \otimes F V'\) is naturally a \((-1)\)-Hermitian space with
\[
\langle v_1 \otimes v'_1, v_2 \otimes v'_2 \rangle_{V \otimes F V'} := \langle v_1, v_2 \rangle_V \langle v'_1, v'_2 \rangle_{V'},
\]
and there is a natural map
\[
(1.3) \quad U(V) \times U(V') \longrightarrow U(V \otimes F V').
\]
According to different choices of \(F\) and \(\epsilon\), we have the following five cases:

(A) \(F \neq F_0\). In this case, \((U(V), U(V'))\) is a unitary dual pair;

(B) \(F = F_0, \epsilon = 1\) and \(\dim V\) is odd. In this case, \((U(V), U(V'))\) is an odd orthogonal-symplectic dual pair;

(C) \(F = F_0, \epsilon = -1\) and \(\dim V'\) is even. In this case, \((U(V), U(V'))\) is a symplectic-even orthogonal dual pair;

(\(\tilde{C}\)) \(F = F_0, \epsilon = -1\) and \(\dim V'\) is odd. In this case, \((U(V), U(V'))\) is a symplectic-odd orthogonal dual pair. We view \(U(V)\) as a “metaplectic group” which is a notion borrowed from the local theta correspondence;

(D) \(F = F_0, \epsilon = 1\) and \(\dim V\) is even. In this case, \((U(V), U(V'))\) is an even orthogonal-symplectic dual pair.

In this paper, we assume that \(p \neq 2\) in Case \(B, C, \tilde{C}\) and \(D\), where \(p\) is the characteristic of \(F\).

1.4. Let \(\omega_{V \otimes F V'}\) be the Weil representation of \(U(V \otimes F V')\) attached to the additive character \(\psi\) constructed in Gérardin [10, Theorem 3.3] for Case (A) and in Gérardin [10, Theorem 2.4] for other cases \((p \neq 2\) here). The (modified) Weil representation \(\omega_{V, V'}\) of \(U(V) \times U(V')\) is defined by
\[
(1.4) \quad \omega_{V, V'} := \begin{cases} 
\omega_{V \otimes F V'} & \text{in case (A)}, \\
\begin{pmatrix} \omega_{V \otimes F V'} \otimes \mathbb{1}_{U(V')} \otimes \det_{U(V)} \otimes \mathbb{1}_{U(V')}, & \text{in case (B) (D)}, \\
\mathbb{1}_{U(V)} \otimes (\xi \circ \det_{U(V)} \otimes \mathbb{1}_{U(V)}), & \text{in case (C) (\(\tilde{C}\)),}
\end{pmatrix}
\end{cases}
\]
where \(\omega_{V \otimes F V'}\) is viewed as a \((U(V) \times U(V'))\)-representation via (1.3), and \(\det_{U(V)}\) (resp. \(\mathbb{1}_{U(V)}\)) denotes the determinant (resp. trivial) character of \(U(V)\).

1.5. Let \(\text{Rep}(G)\) denote the category of representations of \(G\) over \(\mathbb{C}\) and \(\text{Irr}(G)\) the set of irreducible representations of \(G\). For each dual pair \((V, V')\), we define a functor \(\Theta_{V, V'}\) from \(\text{Rep}(U(V))\) to \(\text{Rep}(U(V'))\) by
\[
\Theta_{V, V'}(\pi) := \text{Hom}_{U(V)}(\pi, \omega_{V, V'}).
\]
When \(\pi \in \text{Irr}(U(V))\), the \(\pi\)-isotypic component of \(\omega_{V, V'}\) is canonically isomorphic to
\[
\pi \otimes \Theta_{V, V'}(\pi).
\]
Therefore, our definition agrees with the conventional definition of the big theta lift map.

1.6. The hyperbolic \(\epsilon'\)-Hermitian plane \(H_{\epsilon'}\) is the unique (up to isomorphism) 2-dimensional \(\epsilon'\)-Hermitian space containing a non-zero isotropic vector. A set \(\mathcal{V}'\) of \(\epsilon'\)-Hermitian spaces is called a Witt tower if there is an anisotropic \(\epsilon'\)-Hermitian space \(V'_{an}\) and \(\mathcal{V}'\) consists of all \(\epsilon'\)-Hermitian spaces \(V'^{\dagger}\) such that
\[
V'^{\dagger} \cong V'_{an} \oplus H_{\epsilon'} \oplus \cdots \oplus H_{\epsilon'}.\]
The integer \(r\) above is called the split rank of \(V'^{\dagger}\) and the parity of \(\dim V'_{an}\) is called the parity of \(\mathcal{V}'\). If \(F = F_0\) and \(\epsilon' = 1\), we have
\[
\text{disc}(V'^{\dagger}) = \text{disc}(V'_{an}) \quad \text{for all } V'^{\dagger} \in \mathcal{V}'.
\]
Therefore, we define \(\text{disc}(\mathcal{V}') := \text{disc}(V'_{an})\) and call it the discriminant of \(\mathcal{V}'\) in this case.
For a dual pair \((V, V')\), let \(\mathcal{V}'\) be the Witt tower containing \(V'\) and we define the generalized Witt tower \(\mathcal{V}'_V\) to be the collection
\[
\mathcal{V}'_V = \left\{ V'^{\eta}, \omega_{V, V'^{\eta}} \mid V'^{\eta} \in \mathcal{V}' \right\},
\]
where \(\omega_{V, V'^{\eta}}\) is the modified Weil representation defined in (1.4) (cf. [20, § 3]).

We define the companion generalized Witt tower \(\mathcal{V}_V\) as follows:
- In Case (A), there exists a unique Witt tower \(\mathcal{V}'\) such that parity of \(\mathcal{V}'\) and \(\mathcal{V}'_V\) are different. We define
  \[
  \mathcal{V}_V := \left\{ \left( V'^{\eta}, \omega_{V, V'^{\eta}} \right) \mid \mathcal{V}' \in \mathcal{V}'_V \right\}.
  \]
- In Case (C) and (\(\tilde{C}\)), there exists a unique Witt tower \(\mathcal{V}'\) such that \(\mathcal{V}'\) and \(\mathcal{V}'_V\) have the same parity and different discriminants. We define
  \[
  \tilde{\mathcal{V}}_V := \left\{ \left( \tilde{V}'^{\eta}, \omega_{\tilde{V}, \tilde{V}'^{\eta}} \right) \mid \tilde{V}'^{\eta} \in \tilde{\mathcal{V}}' \right\}.
  \]
- In Case (B) and (D), there is only one Witt tower of symplectic spaces. Let \(\tilde{\mathcal{V}} := \mathcal{V}'\) and
  \[
  \tilde{\mathcal{V}}_V := \left\{ \left( \tilde{V}'^{\eta}, \omega_{\tilde{V}, \tilde{V}'^{\eta}} \otimes (\det_{U(V)} \otimes \mathbb{1}_{U(V'^{\eta})}) \right) \mid \tilde{V}'^{\eta} \in \tilde{\mathcal{V}}' \right\}.
  \]

1.7. For a generalized Witt tower \(\mathcal{V}'_V \in \{ \mathcal{V}'_V, \tilde{\mathcal{V}}_V \}\) of \(c'\)-Hermitian spaces, we define a function \(n_{\mathcal{V}'_V}\) from \(\operatorname{Irr}(U(V))\) to the set \(\mathbb{N}\) of non-negative integers by sending \(\pi \in \operatorname{Irr}(U(V))\) to
\[
n_{\mathcal{V}'_V}(\pi) := \min \left\{ \dim V'^{\eta} \mid \left( V'^{\eta}, \omega \right) \in \mathcal{V}'_V \text{ and } \operatorname{Hom}_{U(V)}(\pi, \omega) \neq 0 \right\}.
\]
The number \(n_{\mathcal{V}'_V}(\pi)\) is called the first occurrence index of \(\pi\) with respect to \(\mathcal{V}'_V\), which is well-defined by the non-vanishing of theta lift when the split rank of \(V'^{\eta} \in \mathcal{V}'_V\) is greater than \(\dim V\), see [16, 3.IV].

In the rest of the introduction, we will retain the notations \(V, \mathcal{V}_V\) etc. defined above. All definitions and notations extend naturally to \(V'\) by switching the roles of \(V\) and \(V'\).

1.8. Suppose that \(V\) has split rank \(r\). We fix a chain of \(c'\)-Hermitian spaces \(\{ V_i \mid -r \leq i \in \mathbb{Z} \}\) in the Witt tower containing \(V\) such that \(V_i \subset V_{i+1}\) and \(\dim V_i = \dim V + 2i\) for each integer \(i \geq -r\). Note that \(V_0 = V\). For each integer \(0 \leq k \leq r\), let \(Q_k = M_k N_k\) be a parabolic subgroup of \(U(V)\) with Levi component
\[
M_k \cong \operatorname{GL}_k(F) \times U(V_{-k})
\]unipotent radical \(N_k\). Let \(J_{U(V)}^{M_k}\) be the Jacquet functor from \(U(V)\) to \(M_k\) by taking \(N_k\)-coinvariants. Set
\[
\chi_{V, V'} := \begin{cases} 
\xi & \text{in Case (\(\tilde{C}\))}, \\
1 & \text{otherwise}.
\end{cases}
\]
We abbreviate \(\chi_{V, V'}\) to be \(\chi\) if there is no confusion caused. For each \(\pi \in \operatorname{Irr}(U(V))\), define
\[
c(\pi) := \max \left\{ k \mid \operatorname{Hom}_{\operatorname{GL}_k}(J_{U(V)}^{M_k}(\pi), \chi \circ \det) \neq 0 \right\}.
\]
Clearly if \(\sigma\) is a cuspidal representation of \(U(V)\), then \(c(\sigma) = 0\). In general, we call a representation \(\sigma \in \operatorname{Irr}(U(V))\) \(\text{theta cuspidal}\) (with respect to the cases listed in Section 1.2) when \(c(\sigma) = 0\).

The behavior of theta cuspidal representations are similar to cuspidal representations under theta correspondence. Indeed, by the argument in [12] (see also [16, 3.IV]) adapted to our situation, one can see that, when \(\sigma\) is theta cuspidal, \(\Theta_{V, V'}(\sigma)\) is irreducible or zero for every \(V' \in \mathcal{V}'_V\). Moreover, \(\Theta_{V, V'}(\sigma)\) is theta cuspidal if and only if \(\dim \tilde{V}' = n_{\mathcal{V}_V}(\sigma)\).
1.9. We prove the following conservation relation on the first occurrence indices. Let
\begin{equation}
\delta = \begin{cases} 
1 - \epsilon & \text{if } F = F_0, \\
1 & \text{if } F \neq F_0,
\end{cases}
\text{ and } \delta' = \begin{cases} 
1 + \epsilon & \text{if } F = F_0, \\
1 & \text{if } F \neq F_0.
\end{cases}
\end{equation}

**Theorem 1.1.** For \( \pi \in \text{Irr}(U(V)) \), we have
\begin{equation}
n_{V_\pi}(\pi) + n_{V_{\sigma}}(\pi) + c(\pi) = 2 \dim V + \delta.
\end{equation}

We will first prove Theorem 1.1 for theta-cuspidal representations (Proposition 3.7). The general case will be proved in Section 5.4, using Theorem 1.3 below (whose proof relies on the theta-cuspidal case of Theorem 1.1).

For cuspidal representations, Pan obtained (1.8) by a reduction to the unipotent case [19, Theorem 12.3]. Recently Pan also obtained the general case by a reduction to the unipotent case [19].

1.10. Now we introduce another important player, the generic Hecke algebra. Fix an integer \( l \geq 0 \). Let \( W_l \) be the Coxeter group of type \( BC_l \) with simple reflections
\[ \Delta_l := \{ s_1, s_2, \cdots, s_{l-1}, t_l \}, \]
where the subgroup \( S_l \) of \( W_l \) generated by \( \{ s_1, \cdots, s_{l-1} \} \) is isomorphic to the symmetric group on \( l \) letters. Let \( l : W_l \to \mathbb{N} \) be the length function on \( W_l \).

Let \( \nu \) be an indeterminate and \( R := \mathbb{Z}[\nu^\frac{1}{2}, \nu^{-\frac{1}{2}}] \) the ring of polynomials with coefficients in \( \mathbb{Z} \) generated by \( \nu^\frac{1}{2} \). For \( \mu \in \frac{1}{2}\mathbb{Z} \) and a non-negative integer \( l \), the generic Hecke algebra \( H_{l, \mu} \) of \( W_l \) is the unique associative algebra over \( R \) with \( R \)-basis \( \{ T_{w, \mu} \mid w \in W_l \} \) (we abbreviate \( T_w := T_{w, \mu} \)), such that
\begin{enumerate}
\item[(a)] \( (T_{s_i} + 1)(T_{s_i} - \nu) = 0, \quad i = 1, \cdots, l - 1, \)
\item[(b)] \( (T_{t_l} + 1)(T_{t_l} - \nu^\mu) = 0, \)
\item[(c)] \( T_{w_1}T_{w_2} = T_{w_1w_2} \) if \( l(w_1w_2) = l(w_1) + l(w_2). \)
\end{enumerate}

Let \( C_q \) be the \( R \)-module whose underlying \( Z \)-module is \( C \) and \( \nu^\frac{1}{2} \) acts by \( \sqrt{q} \). Tensoring of an \( R \)-algebra or \( R \)-module with \( C_q \) is called the specialization at \( \nu = q \). We write \( H_{l, \mu, \nu = q} := H_{l, \mu} \otimes_R C_q \) for the specialization of an \( R \)-algebra \( H_{l, \mu} \) at \( \nu = q \) and \( M_{\nu = q} := M \otimes_R C_q \) for the \( H_{l, \mu, \nu = q} \)-module obtained by specialization of an \( H_{l, \mu} \)-module \( M \) at \( \nu = q \).

1.11. Let \( P_l = L_lU_l \) be a parabolic subgroup of \( U(V_l) \) with unipotent radical \( U_l \) and Levi component
\[ L_l \triangleq \text{GL}_1(F) \times \cdots \times \text{GL}_1(F) \times U(V_0). \]

Let \( \sigma \) be an irreducible theta cuspidal representation of \( U(V_0) \) and let
\begin{equation}
\sigma_l := \chi^{\otimes l} \otimes \sigma
\end{equation}
be an irreducible representation of \( L_l \). By abuse of notation, we also use \( \sigma_l \) to denote its inflation to \( P_l \). Consider the intertwining algebra
\begin{equation}
H_l := \text{End}_{U(V_l)} \left( \text{Ind}_{P_l}^{U(V_l)}(\sigma_l) \right),
\end{equation}
where \( \sigma_l^\vee \) denotes the contragredient of \( \sigma_l \).

**Theorem 1.2.** There is an isomorphism \( H_l \cong H_{l, \mu(\sigma), \nu = q} \) with
\begin{equation}
\mu(\sigma) := \frac{1}{2} (n_{V_\pi}(\sigma) - n_{V_{\sigma}}(\sigma)).
\end{equation}

The isomorphism in Theorem 1.2 will be explicitly constructed in Section 3.7. Note that we also have \( H_l \cong H_{l, -\mu(\sigma), \nu = q} \) by switching the role of \( V_\pi \) and \( V_{\sigma} \) in (1.11). These two isomorphisms are related by a natural isomorphism \( \kappa : H_{l, -\mu(\sigma)} \cong H_{l, \mu(\sigma)} \), see Remark 3.8.

In the case that \( \sigma \) is cuspidal, by Howlett-Lehrer [11, Theorem 3.14], Lusztig [14, 8.6] and Geck [8, Corollary 2], it was already known that \( H_l = H_{l, \mu, \nu = q} \) for some \( \mu \in \frac{1}{2}\mathbb{Z} \). Thus the content of Theorem 1.2 is to relate the parameter of the generic Hecke algebra with the first occurrence indexes of \( \sigma \).
1.12. Let \((V, V', \sigma, \sigma')\) be a quadruple such that \((V, V')\) is a dual pair and \((\sigma, \sigma') \in \text{Irr}(U(V)) \times \text{Irr}(U(V'))\) is a pair of theta cuspidal representations that satisfy \(\Theta_{V, V'}(\sigma) = \sigma'\). By Theorem 1.1 in the theta-cuspidal case (Proposition 3.7) and Theorem 1.2, we have

\[
\mu(\sigma) = \dim V' - \dim V - \frac{1}{2} \delta
\]

and

\[
\mu(\sigma') = \dim V - \dim V' - \frac{1}{2} \delta' = -1 - \mu(\sigma).
\]

In view of (1.7) and (1.12), let

\[
\mu \in \begin{cases} 
\mathbb{Z} + \frac{1}{2} & \text{in Case (A)}, \\
2\mathbb{Z} + 1 & \text{in Case (B) and (C)}, \\
2\mathbb{Z} & \text{in Case (C') and (D)}.
\end{cases}
\]

We say that \(\mu\) and a quadruple \((V, V', \sigma, \sigma')\) as above are relevant if

\[
\mu = \dim V' - \dim V - \frac{1}{2} \delta.
\]

We prove that every \(\mu\) in (1.13) is relevant to some quadruple as above (Proposition 3.11).

1.13. Below in this introduction, we use Hecke algebras to analyze theta lifts. Fix \(\mu\) in (1.13) and integers \(l, l' \geq 0\). Let \((V, V', \sigma, \sigma')\) be a quadruple relevant to \(\mu\). By [16, §3, IV.4], we know that the irreducible components of the theta lifts of elements in the Harish-Chandra series

\[
\mathcal{E}(U(V)_l, \sigma) := \{ \pi \in \text{Irr}(U(V)_l) \mid \pi \text{ occurs in } \text{Ind}_{P_\mu}^{U(V)_l}(\sigma) \}
\]

to \(U(V'_l)\) lie in the Harish-Chandra series \(\mathcal{E}(U(V'_l), \sigma')\). From the definition (1.10) and Theorem 1.2, \(\mathcal{E}(U(V'_l), \sigma')\) is parameterized by simple modules of \(\mathcal{H}_l\). Likewise \(\mathcal{E}(U(V'_l), \sigma')\) is parameterized by simple modules of \(\mathcal{H}'_{l'}\) which is defined as in (1.10). Thus the decomposition of the multiplicity space

\[
\mathcal{M} := \text{Hom}_{U(V)_l \times U(V'_{l'})} \left( \text{Ind}_{P_\mu}^{U(V)_l}(\sigma) \otimes \text{Ind}_{P_\mu}^{U(V'_{l'})}(\sigma'), \omega_{V, V'} \right)
\]

as a left \(\mathcal{H}_l \otimes \mathcal{H}'_{l'}\)-module will give rise to the theta correspondence between \(\mathcal{E}(U(V)_l, \sigma)\) and \(\mathcal{E}(U(V'_l), \sigma')\). We show that \(\mathcal{M}\) lifts to a generic Hecke algebra module.

Let \(H = H_{l, \mu}\) and \(H' = H_{l'-1, -\mu}\). Fix the isomorphisms \(\mathcal{H}_l \cong H_v = q \mathbb{Z}\) and \(\mathcal{H}'_{l'} \cong H'_{v'} = q \mathbb{Z}\) from Theorem 1.2. Fix the natural isomorphism \(\mathbb{C}[W_l] \cong H_{v=1}\) by sending \(w \in W_l\) to \(T_w \otimes R 1\), and likewise \(\mathbb{C}[W_{l'}] \cong H'_{v'=1}\). Define the “signature” representation of \(H\) by

\[
\varepsilon_l(T_w) = \nu \quad \text{for } i = 1, \ldots, l - 1, \quad \text{and } \varepsilon_l(T_{w_l}) = -1.
\]

To ease the notation, we still denote the specialization of \(\varepsilon_l\) at \(\nu = q\) or \(\nu = 1\) by the same symbol. The same definition works for \(H'\) and its specializations as well.

For each non-negative integer \(k\), we have natural subgroups \(W_{l-k} \times W_k \subset W_l\) and \(W_k \times W_{l'-k} \subset W_{l'}\). Let \(\Delta W_k \subset (W_{l-k} \times W_k) \times (W_k \times W_{l'-k}) \subset W_l \times W_{l'}\) be the diagonal embedding.

**Theorem 1.3.** There is an \(H \otimes H'\)-module \(M\), free over \(R\) such that

(a) as \(H_{v=q} \otimes H'_{v'=q} \cong \mathcal{H}_l \otimes \mathcal{H}'_{l'}\)-modules, \(M_{v=q} \cong \mathcal{M}\);

(b) as \(H_{v=1} \otimes H'_{v'=1} \cong \mathbb{C}[W_l] \otimes \mathbb{C}[W_{l'}]\)-modules,

\[
M_{v=1} \cong \sum_{k=0}^{\min(l, l')} \text{Ind}_{W_{l-k} \times \Delta W_k \times W_{l'-k}}^{W_l \times W_{l'}} (\varepsilon_{l-k} \otimes \varepsilon_k \otimes \varepsilon_{l'-k}).
\]

We will explicitly construct the module \(M\) in Section 5.1.

**Remark.** The generic Hecke algebras \(H, H'\), and the module \(M\) only depend on the triple \((\mu, l, l')\).
1.14. By Tits’ deformation [6, 10.11], we have a natural bijection between simple modules of $H_l$ and $\text{Irr}(W_l)$ under our fixed isomorphisms $H_l \cong H_{0=q}$ and $\mathbb{C}[W_l] \cong H_{0=1}$. Thus we can identify $E(U(V_l), \sigma)$ with $\text{Irr}(W_l)$. Likewise, we identify $E(U(V'_l), \sigma')$ with $\text{Irr}(W'_l)$. We now recall some basic facts in the representation theory of $S$ and $W_l$. Let $S_a$ be the Grothendieck group of finite dimensional representations of $S_a$. Then $S := \bigoplus_{a \in \mathbb{N}} S_a$ is a graded commutative unital ring under the multiplication

\begin{equation}
\alpha \cdot \beta := \text{Ind}_{S_a \times S_b}^{S_{a+b}} (\alpha \otimes \beta), \quad \forall \alpha \in S_a, \beta \in S_b.
\end{equation}

Let $1_l$ be the trivial representation of $S_l$. Define $X_l: S \to S$ by $\alpha \mapsto 1_l \cdot \alpha$ and let $X'_l: S \to S$ be the adjoint of $X_l$, under the non-degenerate pairing that assigns $(\alpha, \beta) = \delta_{\alpha, \beta}$ (Kronecker delta) if both $\alpha$ and $\beta$ are irreducible representations. The operators $X_l$ and $X'_l$ can be computed by Pieri’s rule (cf. [7, 4.44]). Similarly, we define $W := \bigoplus_{a \in \mathbb{N}} W_a$ with $W_a$ being the Grothendieck group of finite dimensional representations of $W$. Then $W$ is also a commutative unital ring under the obvious analog of (1.17). For $\alpha \in \text{Irr}(S_a)$, let $\tilde{\alpha}$ denote the inflation of $\alpha$ to $W_a$ via the natural quotient map $W_l \to S_l$. It is a classical result that the map

\[ \text{Irr}(S_a) \times \text{Irr}(S_b) \ni (\alpha, \beta) \mapsto \alpha \times \beta := \text{Ind}_{W_a \times W_b}^{W_{a+b}} (\tilde{\alpha} \otimes (\tilde{\beta} \otimes \varepsilon_b)) \in W_{a+b}. \]

extends to a graded ring isomorphism $S \otimes S \to W$, see [24, Section 7.3].

**Theorem 1.4.** Suppose $\pi \in E(U(V_l), \sigma)$ and corresponds to $\alpha \times \beta \in \text{Irr}(W_l)$. Then $\Theta(\pi)$ is a multiplicity-free combination of irreducible representations in $E(U(V'_l), \sigma')$ corresponding to

\begin{equation}
\min \{ l, l' \} \sum_{k=0}^{\min \{ l, l' \} } X'_{l-k}(\beta) \times X_{l'-k}(\alpha) \in W_{l'}.
\end{equation}

Theorem 1.4 will be proved in Section 5.3 using Theorem 1.3. This theorem covers the results of Aubert-Michel-Rouquier [2, Theorem 3.10, Conjecture 3.11] and Pan [18, Theorem 3.4] for theta lifts between unipotent representations and quadratic-unipotent representations (called $\theta$-representations in [15, Theorem 3.3]), see Section 5.5.

**Acknowledgment.** The first author thanks Anne-Marie Aubert, Tomasz Przebinda, and Binyong Sun for helpful discussions. He is supported in part by the National Natural Science Foundation of China (Grant No. 11701364 and Grant No. 11971305) and Xiamen University Malaysia Research Fund (Grant No. XMUMRF/2022-C9/IMAT/0019). The second author is partially supported by the NSF grant DMS-2000533. The third author thanks Yifeng Liu and Zhiwei Yun for encouraging discussions. He is partially supported by a Singapore government MOE Tier 1 grant R-146-000-320-114.

2. **Preliminary: Parabolic subgroup, Weyl group, and Weil representation**

We explicate some parabolic subgroups of $U(V_l)$ and the corresponding relative Weyl groups. We also recall the mixed model of the Weil representation and Kudla’s filtration.

2.1. **Parabolic subgroup and Weyl group.** Recall that for $l \geq 0$, $V_l = V \oplus H'_l$. Fix an isotropic basis $v_i, v_{-i}$ of the $i$-th $H_k$ such that $\langle v_i, v_{-i} \rangle_{V_l} = 1$. For each $0 \leq k \leq l$, we define

\[ V_k^+ = \text{Span} \{ v_1, \ldots, v_k \}, \quad V_k^- = \text{Span} \{ v_{-1}, \ldots, v_{-k} \}, \]

\[ \hat{V}_{l-k}^+ = \text{Span} \{ v_{k+1}, \ldots, v_l \}, \quad \hat{V}_{l-k}^- = \text{Span} \{ v_{-(k+1)}, \ldots, v_{-l} \}. \]

Note that $V_l^+ = V_k^+ \oplus \hat{V}_{l-k}^+$. We also define

\begin{equation}
\hat{V}_{l-k} = \hat{V}_{l-k}^+ \oplus V \oplus \hat{V}_{l-k}^-.
\end{equation}

Then

\begin{equation}
V_l = V_k^+ \oplus \hat{V}_{l-k} \oplus V_k^-.
\end{equation}

The discussion and notations here and below apply to $V$ and $V'$ in obviously parallel ways.
Using these subspaces, we describe the parabolic subgroup of $U(V_l)$ that appeared in 1.11. For $v \in V_l$, let $\langle v \rangle$ be its span. We identify $GL_1(\langle v_i \rangle) \subset GL(V_l^+) \subset U(V_l^+ \oplus V_l^-)$ and $U(V_0)$ as subgroups of $U(V_l)$ via the above choices of bases and decompositions. Specify the parabolic subgroup $P_l = L_l U_l$ of $U(V_l)$ in 1.11 to be the one stabilizing the flag $0 \subset V_l^+ \subset \cdots \subset V_l^+$, and specify its Levi subgroup $L_l$ to be the one stabilizing the flag $0 \subset V_l^- \subset \cdots \subset V_l^-$. Let

$$T_l = GL_1(\langle v_1 \rangle) \times \cdots GL_1(\langle v_l \rangle) \subset GL(V_l^+).$$

It is direct to check the following equality

$$\text{Norm}_{U(V_l)}(L_l) = \text{Norm}_{U(V_l^+ \oplus V_l^-)}(T_l) \times U(V_0)$$

between the normalizers. In particular, the natural embedding yields the following isomorphism

$$\text{Norm}_{U(V_l^+ \oplus V_l^-)}(T_l)/T_l \cong \text{Norm}_{U(V_l)}(L_l)/L_l.$$ 

Now let us consider the Weyl group and mark the simple reflections of $W_l$ on the following Dynkin diagram of type $BC_l$:

$$s_1 \cdots s_{l-1} t_l s_{l-1} \cdots s_1.$$

For $k = 1, \ldots, l - 1$, let

$$(2.3) \quad t_k := s_k \cdots s_{l-1} t_l s_{l-1} \cdots s_k.$$ 

Recall that $S_l$ is the subgroup of $W_l$ generated by $\{s_1, \ldots, s_{l-1}\}$ which is isomorphic to the symmetric group on $l$ letters. Let $D_l$ be the subgroup of $W_l$ generated by $\{t_1, \ldots, t_l\}$ which is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^l$. We have

$$(2.4) \quad W_l = D_l \rtimes S_l.$$

We make the following choices of lifts of elements in $W_l$ to $\text{Norm}_{U(V_l^+ \oplus V_l^-)}(T_l)$ according to Tits [23]:

- For $i = 1, \ldots, l - 1$, the lift of $s_i$ is

$$(2.5) \quad s_i : \begin{cases} v_i & \mapsto -\epsilon v_{i+1}, \\ v_{i+1} & \mapsto \epsilon v_i, \\ v_{-(i+1)} & \mapsto \epsilon v_{-i}, \\ v_{-i} & \mapsto -\epsilon v_{-(i+1)}, \\ v_j & \mapsto v_j \text{ if } -l < j \leq l \text{ and } j \neq \{i, i+1, -i, -(i+1)\}. \end{cases}$$

- The lift of $t_l$ is

$$(2.6) \quad t_l : \begin{cases} v_l & \mapsto -v_l, \\ v_{-l} & \mapsto -\epsilon v_l, \\ v_j & \mapsto v_j \text{ if } -l < j < l. \end{cases}$$

- If $w = w_1 \cdots w_j$ is an reduced expression of $w \in W_l$ with $w_i \in \Delta_i$, take the lift of $w$ to be $w = w_1 \cdots w_j$, where $v_i, 1 \leq i \leq j$ is the lift of $w_i$ in (2.5) and (2.6).

The definition $w \mapsto w$ above does not depend on the choice of reduced expression of $w$ since the lifts in (2.5) and (2.6) satisfy the following braid relation

$$(2.7) \quad s_is_j = s_js_i, \quad \text{for all } i \neq j \pm 1 \text{ and } 1 \leq i, j \leq l - 1,$$ 

$$s_is_{i+1}s_i = s_{i+1}s_is_{i+1}, \quad \text{for all } 1 \leq i \leq l - 1,$$ 

$$s_it_l = t_ls_i, \quad \text{for } 1 \leq i \leq l - 2,$$ 

$$s_{l-1}t_ls_{l-1}t_l = t_ls_{l-1}t_ls_{l-1}.$$ 

The lifts above induce the natural identification

$$W_l = \text{Norm}_{U(V_l^+ \oplus V_l^-)}(T_l)/T_l = \text{Norm}_{U(V_l)}(L_l)/L_l.$$
By (2.3), for $k = 1, \ldots, l - 1$, the lift of $t_k$ equals

$$
\begin{cases}
  v_k & \mapsto (-1)^{l-k+1}v_k, \\
v_{-k} & \mapsto c(-1)^{l-k+1}v_k, \\
v_j & \mapsto -v_j \quad \text{if } j > k \text{ or } j < -k, \\
v_j & \mapsto v_j \quad \text{if } -k < j < k.
\end{cases}
$$

We end this subsection by introducing some maximal parabolic subgroups. For $1 \leq k \leq l$, by (2.2), we identify $\GL(V^+_{k})$ and $\tilde{U}(\V_{k}, \V)$ as subgroups of $U(V_l)$ naturally. Let $P(V^+_{k}, \V_l) = L(V^+_{k}, \V_l)N(V^+_{k}, \V_l)$ be the parabolic subgroup of $U(V_l)$ stabilizing $V^+_{k}$, where $L(V^+_{k}, \V_l) = \GL(V^+_{k}) \times U(V_l)$ is the Levi component of $P(V^+_{k}, \V_l)$ stabilizing $V^+_{k}$.

Now we specify the unipotent $N(V^+_{k}, \V_l)$. For $T \in \Hom(V_l, V_l)$, we define $T^* \in \Hom(V_l, V_l)$ such that

$$
\langle Tv, v' \rangle_{V_l} = \langle v, T^* v' \rangle_{V_l}, \quad \text{for every } v, v' \in V_l.
$$

Consider $\Hom(\tilde{V}_{l-k}, V^+_{k}) \subset \Hom(V_l, V_l)$ and $\Hom(V^+_{k}, \tilde{V}_{l-k}) \subset \Hom(V_l, V_l)$ via the decomposition (2.2). For each $b \in \Hom(\tilde{V}_{l-k}, V^+_{k})$ and $c \in \Hom(V^+_{k}, \tilde{V}_{l-k})$, we define

$$
u(b, c) = 1 + b - b^* + c.
$$

Then

$$
N(V^+_{k}, \V_l) = \{ \nu(b, c) \mid b \in \Hom(\tilde{V}_{l-k}, V^+_{k}), c \in \Hom(V^+_{k}, \tilde{V}_{l-k}) \}.
$$

In particular, the center of $N(V^+_{k}, \V_l)$ is

$$
Z(V^+_{k}, \V_l) := \{ \nu(0, c) \mid \Herm(V^+_{k}, \V_l) \},
$$

where

$$
\Herm(V^+_{k}, \V_l) := \{ c \in \Hom(V^+_{k}, \V_l) \mid c^* + c = 0 \}.
$$

2.2. The Mix Model for Weil representation. We recall some explicit formulas for the Weil representation by Gérardin [10, §2, §3].

Let $\V$ be a $(-1)$-Hermitian space over $F$ with Hermitian form $\langle , \rangle_{\V}$. For all $v_1, v_2 \in \V$, we define

$$
\langle v_1, v_2 \rangle_{\V} := \begin{cases}
  \frac{1}{2} \langle v_1, v_2 \rangle_{\V}, & \text{if } F = F_0, \\
  \langle v_1, v_2 \rangle_{\V}, & \text{otherwise},
\end{cases} \quad \text{for all } v_1, v_2 \in \V.
$$

Note that we assumed $p \neq 2$ if $F = F_0$. The Heisenberg group attached to $\V$ is given by

$$
H(\V) := \{ (v, t) \in \V \oplus F \mid t - t^\sigma = \langle v, v \rangle_{\V} \}
$$

with the group law

$$(v_1, t_1) \cdot (v_2, t_2) = (v_1 + v_2, t_1 + t_2 + \langle v_1, v_2 \rangle_{\V}).$$

The additive group $F_0$ is identified with the center of $H(\V)$ under the following exact sequence:

$$
0 \longrightarrow F_0 \xrightarrow{t \mapsto (0, t)} H(\V) \xrightarrow{(v, t) \mapsto v} \V \longrightarrow 0.
$$

We denote by $\rho_{\psi}$ the unique irreducible representation of $H(\V)$ with central character $\psi$, which is called the Heisenberg representation. In particular, if we have a polarization $\V = \V^+ \oplus \V_0 \oplus \V^-$, then $\rho_{\psi}$ can be realized on the space $\mathbb{C}[\V^-] \otimes \rho_{\psi_0}$ of $\rho_{\psi_0}$-valued functions on $\V^-$. For $f \in \mathbb{C}[\V^-] \otimes \rho_{\psi_0}$ and $x \in \V^-$, the action is given as follows:

$$
\begin{align*}
(\rho_{\psi}(y^+, 0)f)(x) &= \psi(\tr_{F/F_0}(\langle x, y^+ \rangle_{\V}))f(x) & y^+ & \in \V^+ \\
(\rho_{\psi}(y^-, 0)f)(x) &= f(x + y^-) & y^- & \in \V^- \\
(\rho_{\psi}(v_0, t)f)(x) &= \rho_{\psi_0}(v_0, t)f(x) & (v_0, t) & \in H(\V_0).
\end{align*}
$$

Here $\tr_{F/F_0} : F \to F_0$ is the trace map.
Recall that \( V_i \otimes F V'_i \) is a \(-1\)-Hermitian vector space. We define \( \text{Hom}(V_i, V'_i) \) to be the set of all \( F \)-conjugate linear maps from \( V_i \) to \( V'_i \), i.e.,

\[
\text{Hom}(V_i, V'_i) = \{ T : V_i \to V'_i | T(av) = a^\tau T(v), \text{ for all } a \in F, v \in V_i \}.
\]

We can make \( \text{Hom}(V_i, V'_i) \) an \( F \)-vector space by defining \((a \cdot T)(v) = aT(v)\). Then the map

\[
V_i \otimes V'_i \longrightarrow \text{Hom}(V_i, V'_i)
\]

\[
v \otimes v' \mapsto (\cdot \mapsto \langle v, \cdot \rangle_{V} v')
\]

(2.11)

is a linear isomorphism. Transporting via this isomorphism, the \((-1\)-Hermitian form on \( \text{Hom}(V_i, V'_i) \) is given by

\[
\langle T_1, T_2 \rangle_{\text{Hom}(V_i, V'_i)} = \text{tr}(T_1^\dagger T_2) \quad \text{for all } T_1, T_2 \in \text{Hom}(V_i, V'_i).
\]

Here for \( T \in \text{Hom}(V_i, V'_i) \), we define \( T^\dagger \in \text{Hom}(V'_i, V_i) \) by requiring

\[
\langle Tv, v' \rangle_{V'_i} = \left\langle v, T^\dagger v' \right\rangle_{V_i} \quad \text{for all } v \in V_i \text{ and } v' \in V'_i.
\]

Taking the partial polarization

\[
\text{Hom}(V_i, V'_i) = \text{Hom}(V_i^+, V'_i) \oplus \text{Hom}(V_0, V'_i) \oplus \text{Hom}(V_i^-, V'_i),
\]

the mixed model of \( \omega_{V_i, V'_i} \) can be realized on the space

\[
\omega_{V_i, V'_i} = \mathbb{C}[\text{Hom}(V_i^+, V'_i)] \otimes \omega_{V_0, V'_i}
\]

of \( \omega_{V_0, V'_i} \)-valued functions on \( \text{Hom}(V_i^+, V'_i) \). For \( f \in \omega_{V_i, V'_i} \) and \( x \in \text{Hom}(V_i^+, V'_i) \), the action of \( P(V_i^+, V_i) \times U(V'_i) \) is given as follows:

\[
\begin{align*}
(\omega_{V_i, V'_i}(g^0)f)(x) &= \omega_{V_i, V'_i}(g^0)f((g^0)^{-1} x) & g^0 \in U(V'_i), \\
(\omega_{V_i, V'_i}(a)f)(x) &= \chi(\det_{V_i^+}(a))f(xa) & a \in \text{GL}(V_i^+), \\
(\omega_{V_i, V'_i}(g_0)f)(x) &= \omega_{V_i, V'_i}(g_0)f(x) & g_0 \in U(V_0), \\
(\omega_{V_i, V'_i}(u(b, c))f)(x) &= \rho_{V_0, V'_i}(x b, -\langle x c, x \rangle_{\text{Hom}(V_i, V'_i)})f(x) & b \in \text{Hom}(V_0, V_i^+), \\
& & c \in \text{Hom}(V_i^-, V_i^+).
\end{align*}
\]

- \( \chi \) is a quadratic character of \( F^\times \) defined in (1.6);
- \( u(b, c) \) is defined in (2.9);
- \( \rho_{V_0, V'_i} \) is the Heisenberg representation of the group \( H(\text{Hom}(V_0, V'_i)) \) with central character \( \psi \), see (2.10);

Next, we describe the action of \( t_k \) for \( 1 \leq k \leq l \). Write \( x = (x_1, \cdots, x_l) \) for \( x \in \text{Hom}(V_i^+, V'_i) \) and \( x_i \in \text{Hom}(\langle v_i \rangle, V'_i) \). Then the action of \( t_k \) is given by the partial Fourier transform on \( \text{Hom}(\langle v_k \rangle, V'_i) \):

\[
(\omega_{V_i, V'_i}(t_k)f)(x) := \gamma_{V'_i}^{-1} \int_{\text{Hom}(\langle v_k \rangle, V'_i)} f(x_1, \cdots, x_{k-1}, y, x_{k+1}, \cdots, x_l) \psi \left( \text{tr}_{F/F_0} \left( \langle y, x_k t_k \rangle_{\text{Hom}(V_i, V'_i)} \right) \right) dy.
\]

- Here and below, we choose the Haar measure on any vector space over \( F \) so that the volume of the vector space is the square root of its cardinality.
- \( \gamma_{V'_i} \) is defined by

\[
\gamma_{V'_i} := \begin{cases} 
(-1)^{\dim V'_i} & \text{in Case } A, \\
1 & \text{in Case } B \text{ and } D, \\
\xi(\text{disc}(V'_i)) & \text{in Case } C, \\
\gamma_{\psi}(1)\xi(\text{disc}(V'_i)) & \text{in Case } \tilde{C}, 
\end{cases}
\]
where $\xi$ is the unique quadratic character of $F^\times$ as in (1.1), and in case $\tilde{C}$

$$\gamma_\psi(1) = \int_F \psi \left( \frac{1}{2} x^2 \right) dx$$

is the usual Weil index.

Finally we describe the action of $w$ for $w \in W_i$ where $w$ is the fixed lift of $w$ in Section 2.1. Write $w = ds$ for $d \in D_i, s \in S_i$ by (2.4). Since $D_i$ is the set of distinguished representatives of the right coset $W_i/S_i$, we have $w = ds$, where $d$ and $s$ are lifts of $d$ and $s$. We may further write $d = t_{i_1} \cdots t_{i_k}$ uniquely for $1 \leq i_1 < \cdots < i_k \leq l$. It is routine to check that

$$l(d) = l(t_{i_1}) + \cdots + l(t_{i_k}).$$

Therefore, we have $s \in \text{GL}(V_i^+), d = t_{i_1} \cdots t_{i_k}$, and the action of $w$ can be deduced from (2.13) and (2.14). We describe it explicitly below. Let

$$V_w^+ = w^{-1}V_i^+ \cap V_i^+, \quad V_w^- = w^{-1}V_i^- \cap V_i^+$$

and

$$X_w^+ := \text{Hom}(V_w^+, V_i^+), \quad X_w^- := \text{Hom}(V_w^-, V_i^-).$$

Write an element in $\text{Hom}(V_i^+, V_i^+)$ as $(x^+, x^-)$ with $x^+ \in X_w^+$ and $x^- \in X_w^-$. Then the action of $w$ is given by:

$$(\omega_{V_i, V_i^+}(w)f)(x^+, x^-) = (\gamma_{V_i^+})^{(w)} \int_{X_w^-} f(x^+ s, y) \psi \left( \text{tr}_{F/F_0} \left( \langle y, x^- w \rangle_{\text{Hom}(V_i, V_i^+)} \right) \right) dy,$$

where

$$\iota(w) := \dim V_w^-.$$  

2.3. **Kudla's filtration.** Using the mixed model (2.12) of $\omega_{V_i, V_i^+}$, we compute the Jacquet module of $\left( \omega_{V_i, V_i^+} \right)$ with respect to the parabolic subgroup $P(V_i^+, V_i) = L(V_i^+, V_i)V_i$. Let

$$Z = \{ x \in \text{Hom}(V_i^+, V_i^+), V_i \mid \text{Im}(x) \text{ is an isotropic subspace of } V_i^+ \}.$$  

For $0 \leq k \leq l$, let

$$Z_k := \{ x \in Z \mid x \text{ has rank } k \}.$$  

It is non-zero only if $k \leq \text{split-rank of } V_i^+$.

**Proposition 2.1** (Kudla's filtration). (1) We have the following decomposition as $(P(V_i^+, V_i)/Z(V_i^+, V_i)) \times U(V_i^+)$-representations:

$$\omega_{V_i, V_i^+}^{Z(V_i^+, V_i)} = \bigoplus_k \mathbb{C}[Z_k] \otimes \omega_{V_0, V_i^+},$$

where the sum is over $0 \leq k \leq \text{min} \{ l, \text{split-rank of } V_i^+ \}$.

(2) For $0 \leq k \leq \text{min} \{ l, \text{split-rank of } V_i^+ \}$, we have

$$\left( \mathbb{C}[Z_k] \otimes \omega_{V_i, V_i^+} \right)^{N(V_i^+, V_i)/Z(V_i^+, V_i)} \cong \text{Ind}_{ \text{GL}(V_i^+) \times U(V_0)^{k} \times U(V_i^+)}^{\text{GL}(V_i^+)} \left( \chi \otimes \det(V_i^+) \otimes \mathbb{C}[Z_k] \otimes \omega_{V_0, V_i^+} \right),$$

as a $\text{GL}(V_i^+)$-representation, where

- $\hat{V}_k^+$, $V_i^+$ are defined as in Section 2.1 if $k \leq l$, and if $k > l$, $V_i^+$ is the direct sum of $V_i^+$ and a maximal totally isotropic subspace of the orthogonal complement of $V_i^+ - \text{in } V_0^+$;
- $P(V_i^+, V_i^+)$ is the maximal parabolic subgroup of $\text{GL}(V_i^+)$ stabilizing $V_i^+$;
- $P(V_i^+, V_i^+)$ is the parabolic subgroup of $U(V_i^+)$ defined in the same way as in the end of Section 2.1;
- $\text{Isom}(\hat{V}_k^+, V_i^+)$ is the set of invertible $F$-conjugate linear maps from $\hat{V}_k^+$ to $V_i^+$.
• GL(\(^\hat{V}_k^+\)) and GL(V\(^{\ell+}\)) acts on \(\mathbb{C}[\text{Isom}(\hat{V}_k^+, V_k^{\ell+})]\) as
\[
((g, h) \cdot f)(x) = \chi(\det \hat{V}_k^+(g)) f(h^{-1} x g)
\]
for \((g, h) \in \text{GL}(\hat{V}_k^+) \times \text{GL}(V_k^{\ell+}), f \in \mathbb{C}[\text{Isom}(\hat{V}_k^+, V_k^{\ell+})]\) and \(x \in \text{Isom}(\hat{V}_k^+, V_k^{\ell+});\)

**Proof.** The proof goes in the same way as [12, Theorem 2.8], see also [16, 3.IV]. □

Let \(\sigma\) and \(\sigma'\) be irreducible theta cuspidal representations of \(U(V_0)\) and \(U(V_0')\) such that \(\Theta_{V_0, V_0'}(\sigma) = \sigma'\). Then \((V_0, \sigma')\) is the first occurrence of \(\sigma\) (see 1.8). Let \(P'_1\) be the parabolic subgroup of \(U(V_0')\) defined in the same way as \(P_1\) in Section 2.1. For \(0 \leq k \leq \min\{l, \text{split-rank of } V_0'\}\), let
(2.20) \[\mathcal{F}_k := \text{Hom}_{P_1 \times P'_1}\left(\sigma_1 \otimes \sigma'_1, \mathbb{C}[\mathbb{Z}_k] \otimes \omega_{V_0, V_0'}\right).\]

By Frobenius reciprocity, \(\mathcal{M}\) defined in (1.15) is naturally identified with
(2.21) \[
\left(\omega_{V_1, V_1'}\right)_{\sigma_1 \otimes \sigma'_1} := \text{Hom}_{P_1 \times P'_1}(\sigma_1 \otimes \sigma'_1, \omega_{V_1, V_1'}).
\]

**Corollary 2.2.** We have
\[
\left(\omega_{V_1, V_1'}\right)_{\sigma_1 \otimes \sigma'_1} = \bigoplus_{k=0}^{\min\{l, l'\}} \mathcal{F}_k.
\]

**Proof.** Under the assumption of the first occurrence of \(\sigma\), we have \(\mathcal{F}_k = 0\) for \(k > l'\) and the equation follows. □

### 3. Hecke algebra and first occurrence index

We related the structure of the Hecke algebra for a theta cuspidal representation to its first occurrence index. This allows us to prove the conservation relation for theta cuspidal representations.

#### 3.1. Hecke algebra and induction

Let \(G\) be a reductive group over \(F_0\). Let \(P\) be a parabolic subgroup of \(G\) and \(\sigma_P\) be an irreducible representation of \(P\). Following [4], we define the Hecke algebra
\[
\mathcal{H}(G, P, \sigma_P)
\]
\[
= \text{End}_{G}(\text{Ind}_P^G(\sigma_P^\vee))
\]
\[
= \{ F : G \rightarrow \text{End}_C(\sigma_P^\vee) | F(p_1gp_2) = \sigma_P^\vee(p_1) F(g) \sigma_P^\vee(p_2), \forall p_1, p_2 \in P \}
\]
with the product given by the convolution
\[
F_1 \ast F_2(g) = \int_G F_1(h)F_2(h^{-1}g)dh
\]
for \(F_1, F_2 \in \mathcal{H}(G, P, \sigma_P)\). Here we normalize the Haar measure on \(G\) so that \(P\) has volume 1. We define the Hecke algebra \(\mathcal{H}(G, P, \sigma_P^\vee)\) in the same way by identifying \((\sigma_P^\vee)^\vee\) with \(\sigma_P\).

For \(T \in \text{End}_C(\sigma_P^\vee)\), we denote by \(T^\vee \in \text{End}_C(\sigma_P)\) the adjoint of \(T\). Then we have an anti-isomorphism
(3.1) \[
\mathcal{H}(G, P, \sigma_P) \cong \mathcal{H}(G, P, \sigma_P^\vee)
\]
\[
F \mapsto F^\vee(g) = F(g^{-1})^\vee.
\]

For a representation \(\pi\) of \(G\), let \(\pi_{\sigma_P} = \text{Hom}_P(\sigma_P, \pi)\). Following [5, p 591], we know that \(\pi_{\sigma_P}\) is a left \(\mathcal{H}(G, P, \sigma_P)\)-module via the action
(3.2) \[
F \cdot \phi : v \mapsto \int_G \pi(g)\phi(F^\vee(g^{-1})v)dg, \text{ for } F \in \mathcal{H}(G, P, \sigma_P), \phi \in \pi_{\sigma_P}, v \in \sigma_P.
\]

The following compatibility result of Hecke algebra modules and parabolic induction is well known, see [5, (8.4)].
Proposition 3.1. Let $P \leq Q$ be two parabolic subgroups in $G$ with corresponding Levi subgroups $L, M$. Assume that $L \subset M$ and $P_M = P \cap M$ is a parabolic subgroup of $M$ with Levi component $L$. In particular, we have a natural inclusion of $N_M(L)/L$ to $N_G(L)/L$. Assume that $\sigma$ is an irreducible representation of $L$. Let $\sigma_P$ and $\sigma_{P_M}$ be the inflations of $\sigma$ to $P$ and $P_M$ respectively.

(1) The natural map

$$t : \mathcal{H}(M, P_M, \sigma_{P_M}) \to \mathcal{H}(G, P, \sigma_P)$$

that sends $F \in \mathcal{H}(M, P_M, \sigma_{P_M})$ supported on $P_M$ to $s \in N_M(L)$ to the unique element in $\mathcal{H}(G, P, \sigma_P)$ supported on $PsP$ such that $(t(F))(s) = F(s)$ is an injective ring homomorphism.

(2) For a representation $\pi$ of $M$, let $\pi_Q$ be the inflation of $\pi$ to $Q$. Identify $\pi$ with the sections in $\text{Ind}_Q^G \pi$ supported on $Q$. This identification is $P_M$-equivariant and so induces a natural embedding $\sigma_{P_M} \to (\text{Ind}_Q^G \pi_Q)_{\sigma_P}$. Then the naturally induced map

$$\mathcal{H}(G, P, \sigma_P) \otimes_{\mathcal{H}(M, P_M, \sigma_{P_M})} \sigma_{P_M} \to (\text{Ind}_Q^G \pi_Q)_{\sigma_P}$$

is an isomorphism of $\mathcal{H}(G, P, \sigma_P)$-modules.

3.2. Hecke algebras of type $BC_l$. Let $\sigma \in \text{Irr}(U(V_0))$ be theta cuspidal and $\sigma_l$ be as in (1.9). Let

$$\mathcal{H}_l := \mathcal{H}(U(V_l), P_l, \sigma_l).$$

We study the structure of $\mathcal{H}_l$. For $w \in \mathcal{W}_l$, let $w$ be the lift of $w$ fixed in Section 2.1 and define the unnormalized Hecke operator $T_w \in \mathcal{H}_l$ by

$$T_w(g) := \begin{cases} \sigma_l^\vee(p_1)\sigma_l^\vee(p_2) & \text{if } g = p_1wp_2 \text{ for some } p_1, p_2 \in P_l, \\ 0 & \text{if } g \notin P_lwp_l. \end{cases}$$

Here, since $\sigma_l^\vee(wp_1^{-1}) = \sigma_l^\vee(p)$ for $p \in P_l \cap w^{-1}P_lw$, it is easy to check that $T_w(g)$ is well defined and does not depend on the choice of $p_1, p_2$ if $g \in P_lwp_l$.

Lemma 3.2. The set $\{T_w \mid w \in \mathcal{W}_l\}$ forms a basis of $\mathcal{H}_l$ (as a vector space over $\mathbb{C}$).

Proof. This follows by the argument [21, pp. 635-636] adapted to our situation, together with the analog of “basic geometric lemma” in [3, §3, 1.2]. \qed

It is easy to check that

$$T_{w_1}T_{w_2} = T_{w_1w_2} \text{ if } l(w_1w_2) = l(w_1) + l(w_2).$$

Moreover, for every $w \in \Delta_l$, $T_w$ is a linear combination of $T_w$ and $T_e$, i.e., we have a quadratic relation

$$T_w^2 = C_wT_w + A_wT_e$$

for some $C_w, A_w \in \mathbb{C}$. The following lemma gives the quadratic relation for $T_w$ for $i = 1, \ldots, l-1$.

Lemma 3.3. For $1 \leq i < l - 1$, we have

$$(T_{s_i} + 1)(T_{s_i} - g) = 0.$$ \qed

Proof. This follows by a GL$_2$-calculation, see [13, Proposition 5.8].

We will compute the quadratic relation for $T_l$ using theta lifts in Section 3.5.

3.3. Explicit formula of $T_w$ actions. Fix a $BN$-pair of $U(V_l)$ such that $B \subseteq P_l$. Let $\Sigma$ be the root system and $\Sigma^+ \subseteq \Sigma$ the set of positive roots. For $w \in \mathcal{W}_l$, let

$$U_w := \prod_{\alpha \in \Sigma^+, \ w^{-1}\alpha < 0, \ U_{\alpha} \in U_l} U_{\alpha}$$

where $U_{\alpha}$ is the one parameter subgroup of $U(V_l)$ corresponding to $\alpha$. Then the map

$$U_w \times P_l \to P_lwp_l, \quad (u, p) \mapsto uwp$$

is an isomorphism of $H_{\Sigma^+\Sigma^+\Sigma^+}$-modules.
is a bijection. For a representation $\pi$ of $U(V_l)$ and $\phi \in \pi_{\sigma_l} = \text{Hom}_{P_l}(\sigma_l, \pi)$, it follows directly by (3.2), (3.3) and (3.5) that

$$T_w \cdot \phi = \sum_{u \in U_w} \pi(uw)\phi.$$  

(3.6)

3.4. First occurrence. Assume that the first occurrence of $\sigma$ with respect to generalized Witt tower $\mathcal{V}'$ (see 1.6 and recall that $V_0 = V$) is achieved on $V_0'$. Let $\sigma' = \Theta_{V_0',V_0'}(\sigma)$. By the argument in [16, 3.IV] adapted to our situation, one can see that $\sigma'$ is irreducible and theta cuspidal. In particular, the multiplicity space

$$\omega_{V_0, V_0'} = \text{Hom}_{U_{V_0} \times U_0(V_0')}(\sigma \otimes \sigma', \omega_{V_0, V_0'})$$

is 1-dimensional and we fix a non-zero element $u_0$ inside it. We use the mixed model (2.12) of $\omega_{V_0, V_0'}$ and view the multiplicity space

$$\left(\omega_{V_0, V_0'}\right)_{\sigma \otimes \sigma'} = \text{Hom}_{P_l \times U_0(V_0')}(\sigma_l \otimes \sigma', \mathbb{C}[\text{Hom}(V_l^+, V_0')] \otimes \omega_{V_0, V_0'})$$

as a subspace of $\mathbb{C}[\text{Hom}(V_l^+, V_0')] \otimes \omega_{V_0, V_0'}$. For a subset $\mathcal{S} \subset \text{Hom}(V_l^+, V_0')$, let $I_{\mathcal{S}}$ be the characteristic function on $\mathcal{S}$. If $\mathcal{S} = \{v\}$ is a singleton, we abbreviate $I_{\mathcal{S}}$ to $I_v$.

Lemma 3.4. The multiplicity space $(\omega_{V_0, V_0'})_{\sigma \otimes \sigma'}$ is spanned by $I_0 \otimes u_0$. Moreover,

$$T_{s_i} \cdot (I_0 \otimes u_0) = q(I_0 \otimes u_0) \quad i = 1, \cdots, l - 1,$$

$$T_{t_i} \cdot (I_0 \otimes u_0) = \gamma_{V_0'} q^{\dim V_0 - \frac{1}{2} \dim V_0 + \frac{1}{2} s} (I_0 \otimes u_0),$$

(3.8)

where $\gamma_{V_0'}$ is defined in (2.15).

Proof. By Corollary 2.2, $(\omega_{V_0, V_0'})_{\sigma \otimes \sigma'} = \mathcal{F}_0$. Then the first part of the lemma is a consequence of the fact that $u_0$ spans $(\omega_{V_0, V_0'})_{\sigma \otimes \sigma'}$.

Next, we compute the action of $T_{s_i}$ on $I_0 \otimes u_0$.

$$T_{s_i} \cdot (I_0 \otimes u_0) = \sum_{u \in U_{s_i}} \omega_{V_0, V_0'}(us_i)(I_0 \otimes u_0) \quad \text{by (3.6)}$$

$$= \sum_{u \in U_{s_i}} \chi(\det V_l^+(us_i))(I_0 \otimes u_0) \quad \text{by (2.12) and } U_{s_i} \subseteq \text{GL}(V_l^+)$$

$$= q(I_0 \otimes u_0) \quad \text{by } |U_{s_i}| = q \text{ and } \det V_l^+(us_i) = 1.$$

Let $\lambda_{t_i} \in \mathbb{C}$ be the scalar such that $T_{s_i} \cdot (I_0 \otimes u_0) = \lambda_{t_i}(I_0 \otimes u_0)$. To compute $\lambda_{t_i}$, it is sufficient to compute $(T_{t_i} \cdot (I_0 \otimes u_0))(0)$. Note that $U_{t_i} \subseteq N(V_l^+, V_i)$ and we have

$$U_{t_i} = \{u(b, c) \mid b \in \text{Hom}(V_0, \langle q \rangle), c \in \text{Hom}(\langle e \rangle), c + c^* + bb^* = 0\},$$

and $|U_{t_i}| = q^{\dim V_0 + \frac{1}{2} s}$ by (2.9). Then it follows from (2.13), (2.14), (3.6) and (3.9) that

$$T_{t_i} \cdot (I_0 \otimes u_0)(0)$$

$$= \sum_{u(b, c) \in U_{t_i}} \left(\omega_{V_0, V_0'}(u(b, c) t_i)(I_0 \otimes u_0)\right)(0)$$

$$= \gamma_{V_0'} q^{-\frac{1}{2} \dim V_0} \sum_{u(b, c) \in U_{t_i}} \left(\omega_{V_0, V_0'}(u(b, c))(I_{\text{Hom}(V_0, V_0')} \otimes t_0)\right)(0)$$

$$= \gamma_{V_0'} q^{-\frac{1}{2} \dim V_0} \sum_{u(b, c) \in U_{t_i}} \rho_{V_0, V_0'}((0 b, \langle 0 c, c \rangle)_{\text{Hom}(V_0, V_0')})t_0$$

$$= \gamma_{V_0'} q^{\dim V_0 - \frac{1}{2} \dim V_0 + \frac{1}{2} s} t_0$$

Then the equation for the action of $T_{t_i}$ follows. \qed
3.5. The quadratic relation of \( T_{\tilde{u}} \). Suppose that \((\tilde{V}_0', \tilde{\omega})\) is the first occurrence of \( \sigma \) in the companion generalized Witt tower \( \tilde{V}'_0 \) (see Section 1.6). Then \( \tilde{\sigma}' := \text{Hom}_{U(V)}(\sigma, \tilde{\omega}) \) is theta cuspidal. In case \( B \) and \( D \), let \( \gamma_{\tilde{V}_0'} = -\gamma_{\tilde{V}_0'} \). Otherwise, let \( \gamma_{\tilde{V}_0'} \) be defined as in (2.15). By the definition of companion generalized Witt tower, a straightforward computation shows that

\[
\gamma_{\tilde{V}_0'}/\gamma_{\tilde{V}_0'} = -1.
\]

Set

\[
\lambda_{\tilde{u}} := \gamma_{\tilde{V}_0'} q^\dim V_0 - \frac{1}{2} \dim V_0 + \frac{1}{2} \delta \quad \text{and} \quad \tilde{\lambda}_{\tilde{u}} := \gamma_{\tilde{V}_0'} q^\dim V_0 - \frac{1}{2} \dim V_0 + \frac{1}{2} \delta.
\]

The same proof of Lemma 3.4 gives the following result.

Lemma 3.5. The multiplicity space \( \tilde{\omega}_{\tilde{\sigma}, \tilde{\sigma}} \) is one-dimensional and \( T_{\tilde{u}} \) acts on it by the scalar \( \tilde{\lambda}_{\tilde{u}} \).

By (3.10), we have

\[
\tilde{\lambda}_{\tilde{u}}/\lambda_{\tilde{u}} = -q^{\frac{1}{2}(\dim V_0 - \dim \tilde{V}_0')}.
\]

In particular, \( \lambda_{\tilde{u}} \neq \tilde{\lambda}_{\tilde{u}} \). Combining Lemma 3.4 and Lemma 3.5, we have the following proposition.

Proposition 3.6. The quadratic relation of \( T_{V} \) is

\[
(T_{\tilde{u}} - \tilde{\lambda}_{\tilde{u}})(T_{\tilde{u}} - \tilde{\lambda}_{\tilde{u}}) = 0.
\]

3.6. Conservation relation for theta cuspidal representations. By (3.11), we have

\[
\lambda_{\tilde{u}} \tilde{\lambda}_{\tilde{u}} = \gamma_{\tilde{V}_0'} q^{2\dim V_0 - \frac{1}{2}(\dim V_0 + \dim \tilde{V}_0') + \delta}.
\]

On the other hand, by evaluating (3.13) at the identity \( e \in U(V) \) and using (3.3), we deduce

\[
-\lambda_{\tilde{u}} \tilde{\lambda}_{\tilde{u}} \text{id}_{\gamma'} = T_{\tilde{u}}^2(e) = \int_{U(V)} T_{\tilde{u}}(h) T_{\tilde{u}}(h^{-1})dh = \int_{U_{\tilde{u}}} \int_{P_{\tilde{u}}} T_{\tilde{u}}(ut)p) T_{\tilde{u}}(p^{-1}t^{-1}u^{-1})du dp = [U_{\tilde{u}} | \sigma_{\tilde{V}}^{\gamma'}(t_{\tilde{u}}^{-2})].
\]

Note that

\[
t_{\tilde{u}}^2 = \begin{cases} 
\epsilon & \text{if } \epsilon = 1, \\
\text{diag}(1, \cdots, 1, -1, -1, \cdots, 1) & \text{if } \epsilon = -1. 
\end{cases}
\]

Therefore, \( \sigma_{\tilde{V}}^{\gamma'}(t_{\tilde{u}}^{-2}) = \chi(-1)\text{id}_{\gamma'} \). By the above calculations and (3.9), we deduce

\[
\lambda_{\tilde{u}} \tilde{\lambda}_{\tilde{u}} = -\chi(-1) |U_{\tilde{u}}| = -\chi(-1) q^{\dim V_0 + \frac{1}{2} \delta}.
\]

Note that \( \gamma_{\tilde{V}_0'} \) and \( \gamma_{\tilde{V}_0'} \) are roots of unity. Combining (3.14) and (3.15), we get

\[
\gamma_{\tilde{V}_0'}/\gamma_{\tilde{V}_0'} = -\chi(-1).
\]

and the conservation for theta cuspidal representations:

Proposition 3.7. If \( \sigma \) is theta cuspidal, then Theorem 1.1 holds, i.e.,

\[
n_{V'}(\sigma) + n_{V'_0}(\sigma) = 2 \dim V + \delta.
\]
3.7. Normalization of $H_l$. Define the normalized Hecke operators

$$T_{s_i} := T_{s_i} \quad \text{for } i = 1, \ldots, l-1, \quad \text{and}$$

$$T_{s_l} := -\lambda_{s_l}^{-1}T_{s_l} = -\gamma_{V'_0}^{-1}q^{-\dim V_0 + \frac{1}{2}\dim V'_0 - \frac{1}{2}\delta}T_{s_l}.$$  

(3.17)

For a general $w \in W_l$, let $w = w_1 \cdots w_r$ be a reduced expression of $w \in W_l$ ($w_1, \ldots, w_r \in \Delta_l$) and we define the normalized Hecke operator

$$T_w := T_{w_1} \cdots T_{w_r}.$$  

(3.18)

By (3.4), we know that the definition does not depend on the choice of the reduced expression of $w$ and

$$T_{w_1w_2} = T_{w_1}T_{w_2} \quad \text{if } l(w_1w_2) = l(w_1) + l(w_2).$$

(3.19)

Proof of Theorem 1.2. By Lemma 3.3, Proposition 3.6, (3.12) and (3.17), we have $(T_{s_i} + 1)(T_{s_i} - q) = 0$ for $i = 1, \ldots, l-1$, and $(T_{s_l} + 1)(T_{s_l} - q^{\mu(\sigma)}) = 0$. Combining with (3.19), the isomorphism

$$\mathcal{J}: H_l \xrightarrow{\cong} H_{l,\mu,\nu=q}$$

is given by $T_w \mapsto T_w \otimes_R 1$ for $w \in W_l$. 

\begin{proof}

Remark 3.8. (1) There is a natural isomorphism $\kappa$ from $H_{l,\mu}$ to $H_{l,\mu}$ defined by

(a) $\kappa(T_{s_i,\mu}) = T_{s_i,\mu}$ for $1 \leq i \leq l-1$, and

(b) $\kappa(T_{s_l,\mu}) = -\nu^{-\mu}T_{s_l,\mu}$.

(2) Switching the role of $V'_0$ and $V_0'$, we get another normalization of $T_{s_l}$ by $\tilde{T}_{s_l} = -\tilde{\lambda}_{s_l}^{-1}T_{s_l}$ and the quadratic relation

$$(\tilde{T}_{s_l} + 1)(\tilde{T}_{s_l} - q^{-\mu(\sigma)}) = 0.$$  

Then we have an isomorphism $\tilde{\mathcal{J}}: H_l \xrightarrow{\cong} H_{l,\mu,\nu=q}$ by sending $T_{s_i}$ to the specialization of $T_{s_i,\mu}$ for $i = 1, \ldots, l-1$ and $\tilde{T}_{s_l}$ to the specialization of $T_{s_l,\mu}$ at $\nu = q$.

(3) Clearly, we have the following commutative diagram:

\begin{center}
\begin{tikzcd}
H_l \ar{r}{\kappa} \ar{dr}{\tilde{\mathcal{J}}} & H_{l,\mu,\nu=q} \\
H_{l,\mu,\nu=q} \ar{r}{\kappa} & H_{l,\mu,\nu=q}
\end{tikzcd}
\end{center}

The rest of this section consists of several results which will be used later. By Lemma 3.4 and (3.17), we have the following:

Lemma 3.9. The action of $H_l$ on $(\omega_{V_i,V_0'})_{\sigma,\sigma'}$ is by the character $\varepsilon_{l,i}$ defined in (1.16). 

\begin{proof}

By (2.3), (3.17) and (3.19), we have the normalized Hecke operators

$$T_{s_k} = -\gamma_{V_0'}^{-1}q^{-\dim V_0 + \frac{1}{2}\dim V_0' - \frac{1}{2}\delta}T_{s_k}, \quad k = 1, \ldots, l.$$  

(3.20)

Changing the role of $(V_0, \sigma, l)$ and $(V'_0, \sigma', l')$, we have the normalized Hecke operators

$$T'_{s_k} = -\gamma_{V_0'}^{-1}q^{-\dim V_0' + \frac{1}{2}\dim V_0' - \frac{1}{2}\delta}T'_{s_k}, \quad k = 1, \ldots, l,$$

(3.21)

where $\delta' = 2 - \delta$, see (1.7).

3.8. Abundance of theta cuspidal dual pairs. In this section, we fix a case listed in Section 1.3 but allow the spaces $V$ and $V'$ to vary. We say that $\mu$ in (1.13) is relevant (with respect to our fixed case) if it is relevant to some quadruple $(V, V', \sigma, \sigma')$ (see (1.14)).

Lemma 3.10. If $\mu$ is relevant, $-\mu$ and $-\mu - 2$ are also relevant (in the same case).
Proposition 3.7, we have Proposition 3.11. Combining the above equation with (1.14), we have Therefore, \( (\tilde{\sigma}, V', \tilde{\sigma}, \sigma') \) is relevant to \(-\mu - 2\) which proves the lemma.

**Proposition 3.11.** Every \( \mu \) in (1.13) is relevant.

**Proof.** We only prove Case (A) and other cases are similar. By Lemma 3.10, it is enough to show that \( \frac{1}{2} \) is relevant in Case (A). Consider the dual pair \( (V, V') \) for \( \dim V = 0 \) and \( \dim V' = 1 \). Then \( \Theta_{V, V'}(1_U(V)) = 1_U(V') \). In other word, \( (V, V', 1_U(V), 1_U(V')) \) is relevant to \( \dim V' - \dim V - \frac{1}{2} \delta = \frac{1}{2} \). This finishes the proof.

## 4. Multiplicity space as a Hecke module

We have related Hecke algebras and theta lifts in the last section. Retrieve the set-up in 1.13. In the rest of the paper, we study the \( H_l \times H_{l'} \)-module \( M = \left( \Theta_{V, V'}^I \right)_{\sigma_1 \otimes \sigma'_1} \) (see (2.21)).

In this section, we first show that \( \left( \Theta_{V, V'}^I \right)_{\sigma_1 \otimes \sigma'_1} \) is an induced module for the product of \( H_{l'} \), and the parabolic subalgebra of \( H_l \) generated by \( T_k \)'s, \( i = 1, \cdots, l - 1 \), using Kudla’s filtration in Proposition 2.1. Then we use the induced structure to find a basis \( \mathcal{S} \) of \( \left( \Theta_{V, V'}^I \right)_{\sigma_1 \otimes \sigma'_1} \) and realize the basis explicitly in the mixed model. Finally, we compute the action of \( \mathcal{T}_k \) on \( \mathcal{S} \), and thus we get a complete description of \( \left( \Theta_{V, V'}^I \right)_{\sigma_1 \otimes \sigma'_1} \) as an \( H_l \times H_{l'} \)-module.

### 4.1. Action of \( H_{V^+_l} \otimes H_{l'} \)

Retrieve the notations in Section 2.1. For each \( 1 \leq k \leq l \), let \( B(V^+_l) \) and \( B(V^+_k) \) be the Borel subgroups of \( GL(V^+_l) \) and \( GL(V^+_k) \) stabilizing the flag

\[
V^+_1 \subseteq \cdots \subseteq V^+_l \quad \text{and} \quad \tilde{V}^+_1 \subseteq \cdots \subseteq \tilde{V}^+_k
\]

respectively.

Let \( T_{l-k} = GL(V^+_l) \times \cdots \times GL(V^+_{l-k}) \) and \( \tilde{T}_k = GL(V^+_1 \times \cdots \times GL(V^+_k) \) be the maximal tori inside \( B(V^+_l) \) and \( B(V^+_k) \). We denote by \( \chi_{B(V^+_l)} \) and \( \chi_{B(V^+_k)} \) the inflations of \( \chi^{\otimes l-k} \) and \( \chi^{\otimes k} \) on \( T_{l-k} \) and \( \tilde{T}_k \). Let

\[
H_{V^+_l} := \mathcal{H}(GL(V^+_l), B(V^+_l), \chi_{B(V^+_l)})
\]

and

\[
H_{\tilde{V}^+_k} := \mathcal{H}(GL(V^+_k), B(V^+_k), \chi_{B(V^+_k)}).
\]

Then \( H_{V^+_l} \) and \( H_{\tilde{V}^+_k} \) are Hecke algebras of type \( A_{l-k} \) and \( A_k \) respectively with parameter \( q \) and we identify them as sub-algebras of \( H_l \) by Proposition 3.1. We define the relevant Hecke algebra \( H_{V^+_k} \) for \( GL(V^+_k) \) similarly. Using the basis \( \{ v_{l-k+1}, \cdots, v_l \} \) of \( \tilde{V}^+_k \) and the basis \( \{ v'_1, \cdots, v'_k \} \) of \( V^+_k \) (see 2.1), we identify \( GL(V^+_k) \), GL(\( V^+_k \)) with \( GL_k(F) \). It induces unique isomorphisms

\[
H_{\tilde{V}^+_k} \cong H_{V^+_k} \cong H_{GL_k} \cong \mathcal{H}(GL_k(F), B_k(F), \mathbb{1})
\]

(4.1)
preserving the support. Here $B_k(F)$ is the Borel subgroup of $GL_k(F)$ and $\mathbb{1}$ is the trivial character of $B_k(F)$. Via the isomorphisms (4.1), we define an action of $H_{\hat{V}^+_k} \times H_{\hat{V}'_{l-k}}$ on $H_{GL_k}$ by

\[(4.2) \quad (T_{w_1}, T_{w_2}') \cdot T = T_{w_2} T T_{w_1}^{-1},\]

where $T \in H_{GL_k}, T_{w_1} \in H_{\hat{V}^+_k}, T_{w_2} \in H_{\hat{V}'_{l-k}}$. For each $0 \leq k \leq \min \{l, l'\}$, recall that we have defined $F_k$ in (2.20). It is easy to check that each $F_k$ is stable under the action of $H_{\hat{V}^+_k} \otimes H_{\hat{V}'_{l-k}}$. 

**Proposition 4.1.** The $H_{\hat{V}^+_k} \otimes H_{\hat{V}'_{l-k}}$-module $F_k$ is isomorphic to

\[(4.3) \quad \left( H_{\hat{V}^+_k} \otimes H_{\hat{V}'_{l-k}} \right) \otimes \left( H_{\hat{V}^+_k} \otimes H_{\hat{V}'_{l-k}} \right) \left( \mathbb{1}_{V^+_k} \otimes H_{GL_k} \otimes \varepsilon_{l-k} \right),\]

where $\mathbb{1}_{V^+_k}$ is the trivial character of $H_{\hat{V}^+_k}$ and $\varepsilon_{l-k}$ is the character of $H_{\hat{V}'_{l-k}}$ defined in (1.16).

**Proof.** It follows from Proposition 2.1 that $F_k \cong \left( \text{Ind}_{P(V^+_k \times V'_{l-k})}^{GL(V^+_k \times V'_{l-k})} \left( (\chi \circ \det_{V^+_k}) \otimes \mathbb{C}[\text{Isom}(\hat{V}^+_k, V'_{l-k})] \right) \otimes \omega_{V^+_k, V'_{l-k}} \right) \chi_{B(V^+_k)} \otimes \sigma \otimes \sigma'. $

We now analyze the $\left( H_{\hat{V}^+_k} \otimes H_{\hat{V}'_{l-k}} \right) \otimes \left( H_{\hat{V}^+_k} \otimes H_{\hat{V}'_{l-k}} \right)$-module

\[\left( (\chi \circ \det_{V^+_k}) \otimes \mathbb{C}[\text{Isom}(\hat{V}^+_k, V'_{l-k})] \right) \otimes \omega_{V^+_k, V'_{l-k}} \chi_{B(V^+_k)} \otimes \sigma \otimes \sigma'. \]

Note that

- $\left( (\chi \circ \det_{V^+_k}) \chi_{B(V^+_k)} \right)$ corresponds to the trivial character $\mathbb{1}_{V^+_k}$ of $H_{\hat{V}^+_k}$;

- $\left( \mathbb{C}[\text{Isom}(\hat{V}^+_k, V'_{l-k})] \right) \chi_{B(V^+_k)} \otimes \chi_{B(V'_{l-k})}$ corresponds to $H(GL_k)$ as a module of $H_{\hat{V}^+_k} \otimes H_{\hat{V}'_{l-k}}$ defined in (4.2);

- Since $(V', \sigma')$ is the first occurrence of $\sigma$, we know that $\left( \omega_{V^+_k, V'_{l-k}} \right)_{\sigma \otimes \sigma'_{l-k}}$ is nonzero if and only if $k \leq l'$. Moreover, for $k \leq \min \{l, l'\}$, it follows from Lemma 3.9 that $\left( \omega_{V^+_k, V'_{l-k}} \right)_{\sigma \otimes \sigma'_{l-k}}$ corresponds to the character $\varepsilon_{l-k}$ of $H_{\hat{V}'_{l-k}}$. The lemma follows by Proposition 3.1. \[\square\]

Fix an nonzero vector $I^k_{1,1,1} \in F_k$ in the following one dimensional subspace of (4.3)

\[(4.4) \quad \left( \mathbb{1}_{V^+_k} \otimes \mathbb{1}_{l} \right) \otimes \left( \mathbb{1}_{\hat{V}^+_k} \otimes \mathbb{1}_{GL_k} \otimes \varepsilon_{l-k} \right), \]

where $\mathbb{1}_{V^+_k}, \mathbb{1}_{l}$ and $\mathbb{1}_{GL_k}$ are the vector spaces generated by the identity operators in $H_{\hat{V}^+_k}, H_{\hat{V}'_{l-k}}$ and $H_{GL_k}$ respectively. Let $D_k$ and $D'_k$ be the set of distinguished representatives of the left cosets $S_l/(S_{l-k} \times S_k)$ and $W_l/(S_k \times W_{l-k})$. For $(d_1, d_2, x) \in D_k \times D'_k \times S_k$, let $T_{d_1}, T_{d_2}, T_{x}$ and $T'_{x}$ be the normalized Hecke operators in $H_{\hat{V}^+_k}, H_{\hat{V}'_{l-k}}, H_{\hat{V}^+_k}$ and $H_{\hat{V}'_{l-k}}$ respectively. Here we identify $H_{\hat{V}^+_k}$ and $H_{\hat{V}^+_k}$ with $H_{GL_k}$ via (4.1). For each $(d_1, d_2, x) \in D_k \times D'_k \times S_k$, we define

\[(4.5) \quad I^k_{d_1, d_2, x} := T_{d_1} T_{d_2} T_{x} I^k_{1,1,1} = T_{d_2} T_{d_1} T_{x} I^k_{1,1,1}. \]

**Corollary 4.2.** The following set is a basis of $F_k$

\[\{I^k_{d_1, d_2, x}, (d_1, d_2, x) \in D_k \times D'_k \times S_k\}\]
In particular, we get a basis of \(\left(\omega_{V_l'\nu_\nu'}\right)_{\sigma_l\otimes\sigma'_\nu}:
\)

\[
\mathfrak{G} := \bigcup_{k \leq \min\{l,l'\}} \mathfrak{G}_k, \quad \text{where} \quad \mathfrak{G}_k := \left\{ \mathcal{T}_{d_1,d_2,x}^k \mid (d_1,d_2,x) \in D_k \times D'_k \times S_k \right\}.
\]

**Proposition 4.3.** The action of \(\mathcal{H}_{V_l^+} \otimes \mathcal{H}_{\nu'}\) on \(\mathfrak{G}\) is given as follows:

1. For \(i = 1, \ldots, l - 1\), we have
   \[
   \mathcal{T}_s \cdot \mathcal{T}_{d_1,d_2,x}^k = \begin{cases} 
   \mathcal{T}_{s,d_1,d_2,x}^k & \text{if } s d_1 \in D_k \text{ and } l(s d_1) > l(d_1), \\
   q\mathcal{T}_{d_1,d_2,x}^k + (q - 1)\mathcal{T}_{d_1,d_2,x}^k & \text{if } s d_1 \in D_k \text{ and } l(s d_1) = l(d_1), \\
   q\mathcal{T}_{d_1,d_2,x}^k & \text{otherwise}.
   \end{cases}
   \]

2. For \(i = 1, \ldots, l' - 1\), we have
   \[
   \mathcal{T}_{s'} \cdot \mathcal{T}_{d_1,d_2,x}^k = \begin{cases} 
   \mathcal{T}_{d_1,s'd_2,x}^k & \text{if } s' d_2 \in D'_k \text{ and } l(s' d_2) > l(d_2), \\
   q\mathcal{T}_{d_1,d_2,x}^k + (q - 1)\mathcal{T}_{d_1,d_2,x}^k & \text{if } s' d_2 \in D'_k \text{ and } l(s' d_2) = l(d_2), \\
   q\mathcal{T}_{d_1,d_2,x}^k & \text{otherwise}.
   \end{cases}
   \]

3. Finally
   \[
   \mathcal{T}_{t'} \cdot \mathcal{T}_{d_1,d_2,x}^k = \begin{cases} 
   \mathcal{T}_{d_1,t'd_2,x}^k & \text{if } t' d_2 \in D' \text{ and } l(t' d_2) > l(d_2), \\
   q^{-1} \mathcal{T}_{d_1,t'd_2,x}^k + (q^{-1} - 1)\mathcal{T}_{d_1,d_2,x}^k & \text{if } t' d_2 \in D' \text{ and } l(t' d_2) = l(d_2), \\
   -\mathcal{T}_{d_1,d_2,x}^k & \text{otherwise}.
   \end{cases}
   \]

**Proof.** According to Proposition 4.1, the proposition follows from the quadratic and braided relations of the Hecke algebras and Deodhar’s lemma stated below. \(\square\)

**Lemma 4.4** ([9, Lemma 2.1.2]).

1. Let \(d_1 \in D_k\). For \(i = 1, \ldots, l - 1\), either \(s d_1 \in D_k\) or \(s d_1 = d_i s_j\) for some \(1 \leq j \leq l - 1, j \neq l - k\).

2. Let \(d_2 \in D'_k\).
   (a) For \(i = 1, \ldots, l' - 1\), either \(s' d_2 \in D_k\) or \(s' d_2 = d_j s' \) for some \(1 \leq j \leq l - 1, j \neq k\); or either \(t' d_2 \in D_k\) or \(t' d_2 = d_j t'\).

4.2. **Realization of \(\mathcal{T}_{d_1,d_2,x}^k\) in the mixed model.** Via the mixed model \(\omega_{V_l'\nu_\nu'} = \mathbb{C}[\text{Hom}(V_l^+, V_l')] \otimes \omega_{V_0',\nu_\nu'}\) (see (2.12)), we view \(\left(\omega_{V_l'\nu_\nu'}\right)_{\sigma_l\otimes\sigma'_\nu}\) as a subspace of

\[
\mathbb{C}[\text{Hom}(V_l^+, V_l')] \otimes \left(\omega_{V_0',\nu_\nu'}\right)_{\sigma_l\otimes\sigma'_\nu}.
\]

By (2.13), we have the following lemma.

**Lemma 4.5.** Assume that \(f \in \left(\omega_{V_l'\nu_\nu'}\right)_{\sigma_l\otimes\sigma'_\nu}\) is supported on a single \(B(V_l^+) \times P'_\nu\)-orbit \(\mathcal{O}\) in \(\text{Hom}(V_l^+, V_l')\) via (4.7). Then \(f\) is determined by its value on any point \(A \in \mathcal{O}\).

Applying the mixed model twice, we identify \(\omega_{V_l'\nu_\nu'}\) with

\[
\mathbb{C}[\text{Hom}(V_l^+, V_l')] \otimes \mathbb{C}[\text{Hom}(V_l'^+, V_0)] \otimes \omega_{V_0',\nu_\nu'}.
\]

and view \(\left(\omega_{V_l'\nu_\nu'}\right)_{\sigma_l\otimes\sigma'_\nu}\) as a subspace of

\[
\mathbb{C}[\text{Hom}(V_l^+, V_l')] \otimes \mathbb{C}[\text{Hom}(V_l'^+, V_0)] \otimes \left(\omega_{V_0',\nu_\nu'}\right)_{\sigma_l\otimes\sigma'_\nu}.
\]

For \((d_1, d_2, x) \in D_k \times D'_k \times S_k\), we denote by \(d_1, d_2\) and \(x\) the lifts of \(d_1, d_2\) and \(x\) in \(\text{GL}(V_l^+) \subset U(V_l)\), \(U(V_l')\) and \(\text{GL}(V_k'^+) \subset U(V_l')\) respectively, as fixed in Section 2.1. From now, we identify

\[
\text{Hom}(V_l^+, V_l') \cong V_l^- \otimes V_l'.
\]
Then by Lemma 4.8 and (2.17),

\[ (4.16) \]

\[ \mathcal{O}_{d_1, d_2, x}^k := P_{\nu}^{\dagger} (d_2 x A_k d_1^{-1}) B(V_i^{+}) \subset \text{Hom}(V_i^{+}, V_i^{+}). \]

For \( w \in W_f \), we define

\[ V_w^{\dagger} := w^{-1} V_{\nu}^{\dagger} \cap V_i^{+}, \]

\[ \iota(w) := \dim V_w^{\dagger}, \quad \text{and} \]

\[ Y_w^{-} := \text{Hom}(V_w^{\dagger}, V_0) \subseteq \text{Hom}(V_i^{+}, V_0) \]

similarly to (2.16) and (2.18).

Let

\[ (4.13) \]

\[ A_k := \mathcal{O}_{1, 1, 1}^k = P_{\nu}^{\dagger} A_k B(V_i^{+}), \]

and let \( I_{A_k}, I_0 \) be characteristic functions as in Section 3.4. Keeping track of the proof of Proposition 2.1 and Proposition 4.1, it follows that the non-zero vector in (4.4) can be set to

\[ (4.14) \]

\[ \mathcal{I}_{1, 1, 1}^k := I_{A_k} \otimes I_0 \otimes I_0. \]

Here \( I_0 \) is a fixed non-zero vector in the one-dimensional space \( (\omega_{V_0, V_0})_{\sigma \oplus \sigma}, \) see (3.7). Recall the definition of \( T_{d_1, d_2, x}^k \) by (4.5). The rest of this subsection is devoted to proving the following proposition.

**Proposition 4.6.** Via (4.9), \( T_{d_1, d_2, x}^k \) is supported on \( \mathcal{O}_{d_1, d_2, x} \) and determined by

\[ (4.15) \]

\[ T_{d_1, d_2, x}(d_2 x A_k d_1^{-1}) = \left( -q^{- \dim V_0^{\dagger} - 1 + \frac{1}{2} \delta} \right)^{\iota(d_2)} I_{Y_w^{-} \otimes I_0}, \]

according to Lemma 4.5.

**Remark 4.7.** Note that \( \mathcal{O}_{d_1, d_2, x}^k \subset Z_k \subset \text{Hom}(V_i^{+}, V_i^{+}) \) as predicted by Proposition 2.1.

In the proof of Proposition 4.6, we need the following description of \( \mathcal{O}_{d_1, d_2, x}^k \):

**Lemma 4.8.** The following map is an injection with image \( \mathcal{O}_{d_1, d_2, x}^k \):

\[ (u_2, u_1, A) \mapsto \text{Hom}(V_i^{+}, V_i^{+}) \]

\[ (u_2, u_1, A) \mapsto u_2 d_2 x A(u_1 d_1)^{-1}. \]

Here \( d_1, d_2 \) and \( x \) are lifts of \( d_1, d_2 \) and \( x \).

**Proof.** The proof is routine by using Bruhat decomposition and the fact that \( D_k \) and \( D_k' \) are the set of distinguished representatives. We omit the details. \( \square \)

**Proof of Proposition 4.6.** The key is to compute \( T_{d_1, d_2, x} \cdot T_{1, 1, 1}^k \). Since \( d_2 \in D_k' \), \( l(d_2) = l(d_2) + l(x) \) and \( d_2 x \) is the lift of \( d_2 x \), it follows from the mixed model formula (2.13) and (3.6) that

\[ T_{d_1, d_2, x} \cdot T_{1, 1, 1}^k = T_{d_1, d_2, x} \cdot (I_{A_k} \otimes I_0 \otimes I_0) \]

\[ = \sum_{u_2 \in U_{d_2}'} \sum_{u_1 \in U_{d_1}} \sum_{A \in A_k} \omega_{V_i^{\dagger}, V_i^{\dagger}}(u_1 d_1 u_2 d_2 x) (I_A \otimes I_0 \otimes I_0) \]

\[ = \sum_{u_2 \in U_{d_2}'} \sum_{u_1 \in U_{d_1}} \sum_{A \in A_k} I_{u_2 d_2 x, A(u_1 d_1)^{-1}} \omega_{V_0, V_0'}(u_2 d_2 x) (I_0 \otimes I_0). \]

Then by Lemma 4.8 and (2.17), \( T_{d_1, d_2, x} \cdot T_{1, 1, 1}^k \) is supported on \( \mathcal{O}_{d_1, d_2, x}^k \) and

\[ \left( T_{d_1, d_2, x} \cdot T_{1, 1, 1}^k \right)(d_2 x A_k d_1^{-1}) = \omega_{V_0, V_0'}(d_2 x) (I_0 \otimes I_0) \]

\[ = \left( \gamma^{V_0 q^{- \frac{1}{2} \dim V_0}} \right)^{\iota(d_2)} I_{Y_w^{-} \otimes I_0}. \]
Here we also use the fact that \( \iota(d_2 x) = \iota(d_2) \) and \( \mathcal{V}_d x = \mathcal{V}_d^x \). Taking care of the normalizations
\[
T_d = T_{d_1}, \quad T'_d = \left( -\gamma V_0^{-1} g \cdot \dim V_0^l + b \cdot \dim V_0^{-1} + q \right)^{\iota(d_2)} T'_d, \quad T'_x = T'_x
\]
given by (3.17) and (3.21), we deduce the proposition from (4.5).

\[\square\]

4.3. The action of \( T_i \). For \( (d_1, d_2, x) \in D_k \times D'_k \times S_k \), by (4.5) and (4.14), we have
\[
T_t \cdot T_{d_1, d_2, x} = T_t \bigg( T_{d_1} T_{d_2} T'_x T_{1,1,1} \bigg) = T_{d_2} T'_x \bigg( T_t T_{d_1} T'_{1,1,1} \bigg).
\]
We compute \( T_t T_{d_1} \) as follows. Note that we have the following double coset decomposition
\[
S_t = \coprod_{i=1,2} (S_{l-1} \times S_1) w_i (S_{l-k} \times S_k),
\]
where \( w_1 := 1 \) and \( w_2 := s_{l-1} \cdots s_{l-k} \) are two distinguished double coset representatives. For each \( i \), let \( D^i_k \) be the set of distinguished left coset representatives of \( w_i (S_{l-k} \times S_k) w_i^{-1} \cap (S_{l-1} \times S_1) \) in \( S_{l-1} \times S_1 \).

**Lemma 4.9.** We have \( D_k = \coprod_{i=1,2} D^i_k w_i \). In particular, each \( d_1 \in D_k \) has a unique decomposition
\[
d_1 = y_i w_i, \quad \text{where} \ i \in \{ 1, 2 \}, y_i \in D^i_k \text{ and } l(d_1) = l(y_i) + l(w_i).
\]

**Proof.** See [9, Lemma 2.1.9]. \[\square\]

1. Suppose \( d_1 \in D^1_k w_1 \subseteq S_{l-1} \times S_1 \). Then \( d_1 T_t = t_1 d_1 \) and \( l(t_1 d_1) = l(d_1 t_1) = l(d_1) + 1 \).

Therefore,
\[
T_{t_1} T_{d_1} = T_{d_1} T_{t_1}.
\]

2. Suppose \( d_1 \in D^2_k w_2 \). Write \( d_1 = y w_2 \) with \( y \in D^2_k \subseteq S_{l-1} \times S_1 \). Then \( l(d_1) = l(y) + l(w_2) \) and \( T_{d_1} = T_y T_{w_2} \). We have
\[
T_{w_2} T_{d_1} = T_{w_2} T_t T_{w_2} = T_y T_{w_2} T_t = T_y T_{w_2}^{-1} (T_{w_2}^{-1} T_{t} T_{w_2}).
\]

Note that \( w_2^{-1} t w_2 = t_{l-k} \), and \( l(t_{l-k}) = 2l(w_2) + 1 \). Hence \( T_{w_2}^{-1} T_{t} T_{w_2} = T_{t_{l-k}} \) and we deduce
\[
T_{t_1} T_{d_1} = T_{t} T_{w_2}^{-1} T_{t_{l-k}}.
\]

By (4.17), (4.18), (4.19) and Proposition 4.3, the computation of \( T_t \cdot T_{d_1, d_2, x} \) can be reduced to the computations of \( T_t \cdot T'_{1,1,1} \) and \( T_{t_{l-k}} \cdot T'_{1,1,1} \). We present the formulas, whose proofs will be given in Appendix A and Appendix B. For each \( 1 \leq i \leq j \leq l \), we define \( s_{i,j} \in W_l \) by
\[
s_{i,j} := s_j \cdots s_i.
\]
Here \( s_{i,i} = 1 \) by convention. The notation naturally extends to \( W_{l'} \).

**Proposition 4.10.** We have
\[
T_t \cdot T'_{1,1,1} = -q^{k-l'+l} T_{1,1,1}^{k-l'}
\]
\[
- (1 - q) q^{k-l'} \left( q^{l'} \sum_{i=k+1}^{l'} T_{l_{i-1} s_{i-1,1}} - q^{-1} T_{l_{k+1} s_{k+1,1} - l_{1,1,1}} - q^{-1} \sum_{i=k}^{l'} T_{l_{i-1} s_{i-1,1}} \right),
\]
\[
T_{t_{l-k}} \cdot T'_{1,1,1} = -q^{2k-l} T_{1,1,1}^{k-l}
\]
\[
- q^{k-l'} \left( \sum_{i=k+1}^{l'} T_{l_{i-1} s_{i-1,k}} - q^{l+1} \sum_{i=k+1}^{l'} T_{l_{i-1} s_{i-1,k} - l_{1,1,1}} + (q - 1) \sum_{i=1}^{k} q^{k-l} T_{l_{k-1} s_{k-1,k} - l_{1,1,1}} \right).
\]
Combining (4.18), (4.19) and Proposition 4.10, we deduce the following corollary.
Corollary 4.11. For each \((d_1, d_2, x) \in D_k \times D'_k \times S_k\), there are Laurent polynomials \(h_{d_1, d_2, x}\) with coefficients in \(\mathbb{Z}\) and in the indeterminate \(\nu^{\frac{1}{2}}\), i.e., \(h_{d_1, d_2, x}(\nu) \in \mathbb{Z}[\nu^{\frac{1}{2}}, \nu^{-\frac{1}{2}}]\), such that
\[
\mathcal{T}_i \cdot \mathcal{T}_{d_1, d_2, x} = \sum_{(d_1', d_2', x')} h_{d_1, d_2, x}' (q) \mathcal{T}_{d_1', d_2', x'},
\]
where \((d_1', d_2', x')\) runs over \(\min\{(l', l')\} \bigcup_{i=0}^{\min\{(l', l')\}} D_i \times D'_i \times S_i\).

5. The generic Hecke algebra module and deformation

In this section, we prove Theorem 1.3 by explicitly constructing the generic Hecke algebra module \(M\), using the explicit formulas in the last section.

5.1. The construction of \(M\). Retrieve the set-up in 1.13. Let \(H^\circ\) be the subalgebra of \(H := H_{I, \mu}\) generated by \(\{T_i \mid i \in 1, \cdots, l - 1\}\) so that \(H\) is generated by \(H^\circ\) and \(T_i\). Then \(H^\circ\) is isomorphic to a generic Hecke algebra of type \(A_t\) over \(R\) with parameter \(\nu\) and the specialization of \(H^\circ\) at \(\nu = q\) is isomorphic to \(H_{V^\circ}^\circ\). For \(1 \leq k \leq \min\{l', l\}\), Let \(M_k\) be generic version of the induced module \(\mathcal{F}_k\) in (4.3) with \(H_{l'}^t\) replaced by \(H^t\) and so on. Explicitly, \(M_k\) is a free \(R\)-module with basis
\[
\{\text{symbol } l_k^{d_1, d_2, x}(d_1, d_2, x) \in D_k \times D'_k \times S_k\},
\]
where the action of \(H^\circ \times H^t\) is given by the formulas in Proposition 4.3 with \(\"T\"\), \(\"T^t\"\) and \(\"q\"\) replaced by \(\"T\", \"T^t\"\) and \(\"\nu\"\) respectively. By the compatibility of specialization with induction (see [9, §9.1.5]), we know that the specialization of \(M_k\) at \(\nu = q\) is identified with \(\mathcal{F}_k\) as an \(H_{V^\circ}^\circ \times H_{l'}^t\)-module.

The module \(M_k\) leads to an \(R\)-algebra homomorphism:
\[
\Psi_k : H^\circ \times H^t \to \text{End}_R(M_k).
\]

Let
\[
M := \bigoplus_{k=1}^{\min\{l', l\}} M_k, \quad \text{and} \quad \Psi := \bigoplus_{k=1}^{\min\{l', l\}} \Psi_k.
\]

We now extend \(\Psi\) to \(H \times H^t\) by defining the image of the \(T_{i'}\). Recall there are the Laurent polynomials \(h_{d_1, d_2, x}' \in R\) determined by Corollary 4.11 for all \((d_1', d_2', x'), (d_1, d_2, x) \in \prod_{k=1}^{\min\{l', l\}} D_k \times D'_k \times S_k\). Let \(\mathcal{F}_i \in \text{End}_R(M)\) be the matrix with entries \(h_{d_1, d_2, x}'\) under the basis (5.1). We define
\[
\Psi(T_{i'}) := \mathcal{F}_i.
\]

Proposition 5.1. Under the definition (5.2), \(\Psi\) extends to an \(R\)-algebra homomorphism:
\[
H \times H^t \to \text{End}_R(M).
\]

Proof. Since the Hecke algebras can be equivalently defined using the quadratic relations and the braid relations for generators, it is enough to prove \((\mathcal{F}_i + 1)(\mathcal{F}_i - \nu^{\frac{1}{2}}) = 0, (\mathcal{F}_i \Psi(T_{i-1}))^4 = 1\) and \((\mathcal{F}_i \Psi(T_i))^2 = 1 (i \in \{1, \cdots, l - 2\})\) in \(\text{End}_R(M)\). These equations form a system of finite many equations of Laurent polynomials in the indeterminate \(\nu^{\frac{1}{2}}\). By Proposition 3.11 and Corollary 4.11, the equations hold at \(\nu = q^n\) for every positive integer \(n\). Therefore, the equations hold. \(\square\)
5.2. Proof of Theorem 1.3. The desired property (a) on the specialization of $M$ at $\nu = q$ follows from the construction. Now we study the specialization of $M$ at $\nu = 1$.

Lemma 5.2. For $1 \leq k \leq \min\{l, l'\}$, we have
\[
\begin{align*}
\left( T_{l_k} \cdot I_{1,1,1} \right)_{\nu=1} &= \left( T_{l'_k} \cdot I_{1,1,1} \right)_{\nu=1}, \\
\left( T_{l_{k-1}} \cdot I_{1,1,1} \right)_{\nu=1} &= \left( I_{1,1,1} \right)_{\nu=1} \mod \bigoplus_{\nu > k} M_{l,\nu=1}.
\end{align*}
\]

Proof. These follow from Proposition 4.3 and Proposition 4.10. \qed

By the construction of $M_{k,\nu=1}$ as an $H_{\nu=1} \otimes H'_{\nu=1} = \mathbb{C}[S] \times \mathbb{C}[W_{l'}]$-module, $M_{k,\nu=1}$ is naturally identified with $\text{Ind}_{S_{l-k} \times \Delta S_k \times W_{l'-k}}^{S_{l-k} \times W_{l}-k} (1_{l-k} \otimes 1_k \otimes \varepsilon_{l'-k})$, which is the restriction of the desired module. Lemma 5.2 implies that
\[M_{k,\nu=1}/M_{k+1,\nu=1} \cong \text{Ind}_{W_{l-k} \times \Delta W_k \times W_{l'-k}}^{W_{l-k} \times W_{l}-k} (\varepsilon_{l-k} \otimes \varepsilon_k \otimes \varepsilon_{l'-k}).\]
Since $M_{\nu=1}$ is a semisimple $\mathbb{C}[W_{l}] \times \mathbb{C}[W_{l'}]$-module, property (b) follows.

Remark 5.3. We emphasize that $\bigoplus_{i \geq k} M_i$ is not stable under the action of $H \times H'$ since $\bigoplus_{i \geq k} M_{i,\nu=\pi} = \bigoplus_{i \geq k} F_i$ coming from Kudla’s filtrations is not stable under the action of $H_{\pi} \otimes H'_{\pi}$ (see Proposition 4.10).

5.3. Proof of Theorem 1.4. Retrieve the notations in Section 1.14. Note that
\[\text{Ind}_{\Lambda W_k}^{W_{l-k} \times W_k}(\varepsilon_k) = \sum_{\pi \in \text{Irr}(W_k)} \pi \otimes (\pi \varepsilon_k \otimes \varepsilon_k),\]
and $\pi \cong \pi \varepsilon_k$ for any representation of $W_k$. We have
\[
\begin{align*}
M_{\nu=1} \cong \sum_{k=0}^{\min\{l,l'\}} \text{Ind}_{W_{l-k} \times \Delta W_k \times W_{l'-k}}^{W_{l-k} \times W_{l-k} \times W_{l'-k}} (\varepsilon_{l-k} \otimes \varepsilon_k \otimes \varepsilon_{l'-k})
\cong \sum_{k=0}^{\min\{l,l'\}} \sum_{\pi \in \text{Irr}(W_k)} \text{Ind}_{W_{l-k} \times W_k}(\varepsilon_{l-k} \otimes \pi \varepsilon_k) \otimes \text{Ind}_{W_{l-k} \times W_{l'} \times \Delta W_k}^{W_{l-k} \times W_{l-k} \times W_{l'-k}}((\pi \varepsilon_k) \otimes \varepsilon_{l'-k})
\cong \sum_{k=0}^{\min\{l,l'\}} \sum_{\pi \in \text{Irr}(W_k)} \text{Ind}_{W_{l-k} \times W_k}(\varepsilon_{l-k} \otimes \pi \varepsilon_k) \otimes \text{Ind}_{W_{l-k} \times W_{l'-k}}^{W_{l-k} \times W_{l'-k}}((\pi \varepsilon_k \otimes \varepsilon_{l'-k}) \otimes \varepsilon_{l'}),
\end{align*}
\]
where $1_{l'-k}$ is the trivial character of $W_{l'-k}$.

Lemma 5.4. Assume that $\gamma \times \eta \in \text{Irr}(W_k)$. Then
\[\text{Ind}_{W_{l-k} \times W_k}(\varepsilon_{l-k} \otimes (\gamma \times \eta)) = \gamma \times X_{l-k}(\eta),\]
\[\text{Ind}_{W_{l-k} \times W_{l'-k}}^{W_{l-k} \times W_{l-k} \times W_{l'-k}}((\gamma \times \eta) \otimes 1_{l'-k}) = X_{l'-k}(\gamma) \times \eta,\]
and
\[(5.3) \quad (\gamma \times \eta) \otimes \varepsilon_k \cong \eta \times \gamma.\]

Proof. See [2, Proposition 3.5]. \qed

Now
\[
\begin{align*}
M_{\nu=1} &\cong \sum_{k=0}^{\min\{l,l'\}} \sum_{\gamma \times \eta \in \text{Irr}(W_k)} \text{Ind}_{W_{l-k} \times W_k}^{W_{l-k} \times W_{l-k} \times W_{l'-k}}(\varepsilon_{l-k} \otimes (\gamma \times \eta))
\otimes \left( \text{Ind}_{W_{l-k} \times W_{l'-k}}^{W_{l-k} \times W_{l-k} \times W_{l'-k}}((\gamma \times \eta) \otimes 1_{l'-k}) \otimes \varepsilon_{l'} \right)
\cong \sum_{k=0}^{\min\{l,l'\}} \sum_{\gamma \times \eta \in \text{Irr}(W_k)} (\gamma \times X_{l-k}(\eta)) \otimes (\eta \times X_{l'-k}(\gamma)).
\end{align*}
\]
Note that $\alpha \times \beta$ is an irreducible component of $\gamma \times X_{l-k}(\eta)$ if and only if $\gamma = \alpha$ and
$$\langle \beta, X_{l-k}(\eta) \rangle = \langle X'_{l-k}(\beta), \eta \rangle \neq 0,$$
which must equal to 1 by Pieri’s rule (cf. [7, 4.44]). Then the theorem follows.

5.4. Conservation relation: general cases. Recall that we proved Theorem 1.1 for theta cuspidal representations in Proposition 3.7. In this subsection, we prove Theorem 1.1 in the general cases.

The following lemma is easy to verify and will be used in the proof of Theorem 1.1.

Lemma 5.5. We have the following commutative diagram

$$
\begin{array}{ccc}
\text{Rep}(H_{\mu}) & \xrightarrow{\kappa^*} & \text{Rep}(H_{\mu}) \\
\downarrow & & \downarrow \\
\text{Rep}(W_l) & \xrightarrow{\otimes \varepsilon_l} & \text{Rep}(W_l).
\end{array}
$$

Here $\kappa$ is defined in Remark 3.8 and $\kappa^*$ denotes the pull-back via $\kappa$.

Retrieve the notations in Section 1.8 and Section 1.14. For each $\gamma \in \text{Irr}(S_k)$, we define
$$r_1(\gamma) := \max \{ \text{non-negative integer } i \mid X^i(\gamma) \neq 0 \}.$$  

Proof of Theorem 1.1. We may assume $\pi \in \mathcal{E}(U(V_l), \sigma)$ for a theta cuspidal representation $\sigma$ of $U(V_0)$. We want to show
$$n_{V_{\tilde{V}_l}}(\pi) + n_{\tilde{V}_{\tilde{V}_l}}(\pi) + c(\pi) = 2 \dim V_l + \delta.$$  

Let $(V'_0, \sigma')$ and $(\tilde{V}'_0, \tilde{\sigma}')$ be the first occurrences of $\sigma$ in the generalized Witt towers $V'_0$ and $\tilde{V}'_0$ respectively. Then both $\sigma'$ and $\tilde{\sigma}'$ are theta cuspidal. Suppose $\pi$ corresponds to $\alpha \times \beta \in W_l$ for some $c \in \mathbb{N}$, $\sigma \in \text{Irr} S_c$, and $\beta \in \text{Irr} S_{l-c}$ under the isomorphism $H_l \cong H_{l-c}(\dim V'_0 - \dim \tilde{V}'_0)$, $\nu = q$. Then
$$n_{V_{\tilde{V}_l}}(\pi) = \dim V'_{l-r_1(\beta)}$$  

by Theorem 1.4. As for the generalized Witt tower $\tilde{V}'_0$, by Lemma 5.5 and (5.3), $\pi$ corresponds to $\beta \times \alpha \in W_l$ under the isomorphism $H_l \cong H_{l-c}(\dim V'_0 - \dim \tilde{V}'_0)$. Then
$$n_{\tilde{V}_{\tilde{V}_l}}(\pi) = \dim \tilde{V}'_{l-r_1(\alpha)}$$  

by Theorem 1.4. Since we have
$$\dim V'_0 + \dim \tilde{V}'_0 = 2 \dim V_0 + \delta$$  

by the theta cuspidal case of Theorem 1.1, the proof is completed by the following lemma. □

Lemma 5.6. We have $r_1(\alpha) + r_1(\beta) = c(\pi)$.

Proof. Suppose that $d \in \mathbb{N}$ and $\pi_0 \in \text{Irr}(U(V_{l-d}))$ such that
$$\text{Hom}_{U(V_l)}(\pi, \text{Ind}^{U(V_l)}_{Q_d}(\chi \circ \text{det} \otimes \pi_0)) \neq 0,$$
then we have $\pi_0 \in \mathcal{E}(U(V_{l-d}), \sigma)$. Assume that $\pi_0$ corresponds to $\alpha_0 \times \beta_0 \in \text{Irr}(W_{l-d})$ under the isomorphism $H_{l-d} \cong H_{l-d-c}(\dim V_{l-d} - \dim \tilde{V}_{l-d})$, $\nu = q$. By Proposition 3.1 and Tits’ deformation, we have
$$\text{Hom}_{\text{Ind}^{W_l}_{S_d \times W_{l-d}}}(\alpha \times \beta, \text{Ind}^{W_l}_{S_d \times W_{l-d}}(1_d \otimes (\alpha_0 \times \beta_0))) \neq 0.$$  

By the Frobenius reciprocity and dimension counting, we have the following branching formula
$$\text{Ind}^{W_l}_{S_d \times W_{l-d}}(1_d) \cong \bigoplus_{a+b=d} \text{Ind}^{W_{a \times W_b}}(1_a \otimes 1_b) = \bigoplus_{a+b=d} 1_a \times 1_b.$$  

From this, we deduce that
$$\text{Ind}^{W_l}_{S_d \times W_{l-d}}(1_d \otimes (\alpha_0 \times \beta_0)))$$  

$$\cong \bigoplus_{a+b=d} \text{Ind}^{W_{a \times W_b \times W_{l-d}}}(1_a \times 1_b \otimes (\alpha_0 \times \beta_0)))$$  

$$\cong \bigoplus_{a+b=d} X_a(\alpha_0) \times X_b(\beta_0).$$
The rest then follows. ∎

5.5. The comparison with Aubert-Michel-Rouquier [2] and Pan [18]. Theorem 1.4 covers the results of Aubert-Michel-Rouquier [2, Theorem 3.10] and Pan [18, Corollary 3.5] for theta lifts between unipotent representations and quadratic-unipotent representations (called $\theta$-representations in [15, Theorem 3.3]). The formulation of Theorem 1.4 is slightly different from loc. cit. We will explain these differences in Case $A$ and other cases are similar.

Let us first recall the results in [1] on theta lifts of cuspidal unipotent representations in Case $A$. By Lusztig (see [13, Section 8] and [1, Theorem 5.1]), if $G = U(V)$ is a unitary group, then $G$ has a cuspidal unipotent representation if and only if $\dim V = \frac{m(m+1)}{2}$ for some non-negative integers $m$. In this case, there is a unique irreducible cuspidal unipotent representation and we denote it by $\sigma(m)$. In [1], it is proved that the first occurrence of $\sigma(m)$ in two different Witt towers are $\sigma(m-1)$ and $\sigma(m+1)$ respectively. Combining these with Theorem 1.2, we know that $|\mu(\sigma(m))| = m + \frac{1}{2}$. This coincides with Lusztig’s computation of $|\mu(\sigma(m))|$ in [13, §4.6].

Let $\pi$ be a unipotent representation of a unitary group. By the classification of unipotent representations ([14], also see [2, Section 3]), there are some non-negative integers $l$ and $m$ such that $\pi \in \mathcal{E}(U(V_l), \sigma(m))$ with $\dim V_0 = \frac{m(m+1)}{2}$. Let $V'$ be an $\epsilon'$-Hermitian spaces of dimension $\frac{(m+1)(m+2)}{2}$ and set $(\sigma, \sigma') := (\sigma(m), \sigma(m+1))$. Consider the theta lifts of $\pi$ to the generalized Witt tower $Y'_{\pi}$ containing $V'$. We know that $\Theta_{V_l, V'_l}(\pi)$ lies in the Harish-Chandra series $\mathcal{E}(U(V'_l), \sigma(m+1))$. Applying Theorem 1.4, Lemma 5.5 and (5.3), we have the following theorem due to Aubert-Michel-Rouquier [2, Theorem 3.10] (cf. [18, Corollary 3.5]).

**Theorem 5.7.** Adopt Lusztig’s normalization (see [13, §4.6]): $H_{l} \cong H_{l,m+1}^{\frac{1}{2},\nu=q}$ and $H_{l} \cong H_{l,m+1}^{\frac{1}{2},\nu=q}$. Suppose $\pi \in \mathcal{E}(U(V_l), \sigma(m))$ and $\pi$ corresponds to $\alpha \times \beta \in \text{Irr}(W_l)$. Then $\Theta_{V_l, V'_l}(\pi)$ is a multiplicity free combination of representations in $\mathcal{E}(U(V'_l), \sigma(m+1))$, and corresponds to

$$
\sum_{k=0}^{\min\{l',\}} X_{l'-k}(\alpha) \times X_{l-k}(\beta).
$$

**Appendix A. Explicit Computation of $T_{t_1} \cdot T_{I_{1,1,1}}^k$**

We prove the first equation in Proposition 4.10. It suffices to compute the action of unnormalized Hecke operator $T_{t_1}$ on $T_{I_{1,1,1}}^k$. By (3.6), we have

$$
T_{t_1} \cdot T_{I_{1,1,1}} = T_{t_1} \cdot (I_{A_k} \otimes I_0 \otimes \iota_0) = \sum_{u \in U_t^l} \omega_{V_l \otimes V'_l}^{\iota_0} (ut_1) (I_{A_k} \otimes I_0 \otimes \iota_0).
$$

By (2.9) and (3.9), we have an exact sequence

$$
1 \longrightarrow \text{Herm}(\langle v_{-l} \rangle, \langle v_l \rangle)^{\otimes u(0,c)} \longrightarrow U_t \longrightarrow \text{Hom}(V_0, \langle v_l \rangle) \longrightarrow 1.
$$

For each $b \in \text{Hom}(V_0, \langle v_l \rangle)$, we choose an element $u(b, A_k) \in U_t$. Then we may write

$$
U_t = \{ u(b, A_k) u(0,c) | b \in \text{Hom}(V_0, \langle v_l \rangle), c \in \text{Herm}(\langle v_{-l} \rangle, \langle v_l \rangle) \}.
$$

It follows by (A.1) and (A.2) that

$$
T_{t_1} \cdot T_{I_{1,1,1}}^k = \sum_{b \in \text{Hom}(V_0, \langle v_l \rangle)} \sum_{c \in \text{Herm}(\langle v_{-l} \rangle, \langle v_l \rangle)} \omega_{V_l \otimes V'_l} (u(b, A_k) u(0,c) t_1) (I_{A_k} \otimes I_0 \otimes \iota_0).
$$

A.1. First, we compute $\omega_{V_l \otimes V'_l} (t_1) (I_{A_k} \otimes I_0 \otimes \iota_0)$. Since $\omega_{V_l \otimes V'_l} (t_1)$ acts by the partial Fourier transform on $\text{Hom}(\langle v_l \rangle, \langle v'_l \rangle)$ as in (2.14), to simplify the computation, we write $A_k$ as the difference of two “hyperplanes” as follows. Via the isomorphism (4.10), define

$$
A_{k-1}' := \sum_{1 \leq i \leq k-1} v_{-(l-k+i)} \otimes v_i' \in \text{Hom}(V_{l}^+, V'_{l}).
$$
(Compare with $A_k$ defined in (4.11)). Let $A_{k-1}' := P'_{k'} A_{k-1}' B(V'_1)$. Define
\[
\overline{A}_k := A_{k-1}' + \langle v \rangle \otimes V'_k, \quad \partial A_k := A_{k-1}' + \langle v \rangle \otimes V'_{k-1},
\]
\[
\overline{A}_k := A_{k-1}' + \langle v \rangle \otimes (V'_{k})^\perp, \quad \partial A_k := A_{k-1}' + \langle v \rangle \otimes (V'_{k-1})^\perp.
\]
Then $A_k = \overline{A}_k - \partial A_k$ and $I_{A_k} = I_{A_k} - I_{\partial A_k}$. Applying (2.14), we have
\[
\omega_{V'_i} (I_{A_k} \otimes I_0 \otimes I_0) = \gamma V'_i q^{\text{dim} V'_i + \frac{1}{2} \text{dim} V'_i - \frac{1}{2} \text{dim} V'_i + \text{dim} V'_i + \text{dim} V'_i} (I_{A_{k-1}} \otimes I_0 \otimes I_0).
\]

A.2. We now consider the action of $u(0, c)$. Let
\[
\mathcal{N}' := \{ v' \in V'_1 \mid \langle v', v' \rangle V'_1 = 0 \}.
\]
It follows from (2.13) that
\[
\sum_{c \in \text{Herm}((v_i), (v_i))} u(0, c) (I_{A_k^\perp} \otimes I_0 \otimes I_0) = q^{\frac{1}{2} \delta} (I_{A_{k-1}' + \langle v \rangle \otimes (V'_{k})^\perp} \otimes I_0 \otimes I_0),
\]
and
\[
\sum_{c \in \text{Herm}((v_i), (v_i))} u(0, c) (I_{\partial A_k^\perp} \otimes I_0 \otimes I_0) = q^{\frac{1}{2} \delta} (I_{A_{k-1}' + \langle v \rangle \otimes (V'_{k})^\perp} \otimes I_0 \otimes I_0).
\]

A.3. Note that the action of $u(b, \lambda_b)$ on a general $f \in (\omega_{V'_i}, V'_i)_{\sigma_i \times \sigma'_i'}$ will not enlarge the support of $f$ as a function on $\text{Hom}(V'_i, V'_i)$ by (2.13). In view of Lemma 4.5, we now classify the $B(V'_i) \times P'_{k'}$-orbits in $A_{k-1}' + \langle v \rangle \otimes (V'_{k})^\perp \cap \mathcal{N}'$. Recall that we have a partial flag
\[
0 = V'_0^+ \subseteq V'_1^+ \subseteq \cdots \subseteq V'_l^+ \subseteq (V'_{l'})^\perp \subseteq \cdots \subseteq (V'_1)^\perp \subseteq (V'_0)^\perp = V'_0.
\]
For each $i = 1, \cdots, l'$, let
\[
\mathcal{N}'_i := (V'_i)^\perp \cap \mathcal{N}', \quad \mathcal{N}'_i := (V'_i)^\perp \cap \mathcal{N}'.
\]
For each $i = k, \cdots, l'$, define
\[
A_{k,i} := A_{k-1}' + \langle v \rangle \otimes \mathcal{N}'_i, \quad A_{k-1,i} := A_{k-1}' + \langle v \rangle \otimes \mathcal{N}'_i.
\]
Note that $A_{k,k} = A_k$ by (4.13). We also define
\[
\mathcal{N}'_0 := (V'_0)^\perp \cap \mathcal{N}', \quad A_{k,0} := A_{k-1}' + \langle v \rangle \otimes \mathcal{N}'_0.
\]

The following lemma is routine to check.

**Lemma A.1.** (1) We have $A_{k-1}' = C_{k-1}^{k-1}$ for $i = k, \cdots, l'$.

(2) The set $A_{k,0} \neq \emptyset$ if and only if $\text{dim} V'_0 \neq 0$, in which case $A_{k,0}$ is a single $B(V'_i) \times P'_{k'}$-orbit. Moreover $A_{k,0}$ is not of the form $O_{d_1, d_2, x}$ for any $0 \leq i \leq \min \{ l, l' \}$ and $(d_1, d_2, x) \in D_i \times D'_i \times S_i$.\]
(3) We have the following decompositions:

\[ (A.7) \quad A'_{k-1} + \langle v_{-1} \rangle \otimes \left( (V'_{k-1})^\perp \cap \mathcal{N}' \right) = \left( \bigsqcup_{i=k}^{l'} A_{k,i} \right) \sqcup \left( \bigsqcup_{i=k}^{l'} A_{k,-i} \right) \sqcup A'_{k-1} \sqcup A_{k,0}; \]

\[ (A.8) \quad A'_{k-1} + \langle v_{-1} \rangle \otimes \left( (V'_{k+1})^\perp \cap \mathcal{N}' \right) = \left( \bigsqcup_{i=k}^{l'} A_{k,i} \right) \sqcup \left( \bigsqcup_{i=k}^{l'} A_{k,-i} \right) \sqcup A'_{k-1} \sqcup A_{k,0}. \]

A.4. The right-hand sides of (A.4) and (A.5) decomposes as constant functions supported on \( B_{V_{-1}^+} \times P_{l'}^0 \)-orbits appearing in (A.7) and (A.8). The following lemma computes the average over \( \sum u(b, \lambda_b) \) on each term in the decomposition.

**Lemma A.2.** We have

\[ \sum_{b \in \text{Hom}(V_0, V_{l'})} \omega_{V_1, V_{l'}}(u(b, \lambda_b)) (I_* \otimes I_0 \otimes \iota_0) = \begin{cases} q^{\dim V_0} \mathcal{T}_{1, s_{k,i}, 1}^k & \text{if } * = A_{k,i} \text{ and } i \in \{ k, \ldots, l' \}, \\ -q^{\dim V_0 - 1 + \frac{1}{2} \delta} \mathcal{T}_{1, s'_{k,i}, 1}^k & \text{if } * = A_{k,-i} \text{ and } i \in \{ k, \ldots, l' \}, \\ q^{\dim V_0} \mathcal{T}_{V_0, V_{l'}, k-1} & \text{if } * = A'_{k-1}, \\ 0 & \text{if } * = A_{k,0}. \end{cases} \]

**Proof.** We only prove the first case and the proofs for the rest are similar. Since the action does not enlarge the support, it follows from Lemma 4.5 and Proposition 4.6 that the left-hand side of (A.9) must be a scalar multiple of \( \mathcal{T}_{1, s_{k,i}, 1}^k \). To find the scalar, we evaluate both sides of the equation at the representative element \( A_{k,i} := s_{k,i} A_k \). By (2.13), we have

\[ \omega_{V_1, V_{l'}}(u(b, \lambda_b)) (I_{A_{k,i}} \otimes I_0 \otimes \iota_0) (A_{k,i}) = \rho_{V_0, V_{l'}}(\left( A_{k,i} b, -\langle A_{k,i} \lambda_b, A_{k,i} \rangle_{\text{Hom}(V_1, V_{l'})} \right)) (I_0 \otimes \iota_0) = \rho_{V_0, V_{l'}}(\left( A_{k,i} b, 0 \right)) (I_0 \otimes \iota_0). \]

Note that \( A_{k,i} b \in \text{Hom}(V_0, \langle v_{l'} \rangle) \cong V_0 \otimes \langle v_{l'} \rangle \). Then by (2.10), we have

\[ \rho_{V_0, V_{l'}}(\langle A_{k,i} b, 0 \rangle) (I_0 \otimes \iota_0) = \psi \left( \text{tr}_{F/F_0} \left( \langle A_{k,i} b, 0 \rangle_{V_0 \otimes V_{l'}} \right) \right) (I_0 \otimes \iota_0) = I_0 \otimes \iota_0. \]

So the value of the left-hand side of (A.9) at \( A_{k,i} \) is \( q^{\dim V_0} (I_0 \otimes \iota_0) \). For the right-hand side of (A.9), it follows from Proposition 4.6 that

\[ \mathcal{T}_{1, s_{k,i}, 1}^k(A_{k,i}) = \mathcal{T}_0 \otimes \iota_0 \]

since \( \iota(s_{k,i}') = 0 \) and \( V_{-s_{k,i}}^- = 0 \). This finished the proof of the first case. \( \square \)

A.5. Finally, the previous discussion and our normalization of \( \mathcal{T}_{l_i} \) in (3.20) give the formula of \( \mathcal{T}_{l_i} \cdot \mathcal{T}_{1,1,1}^k \) in Proposition 4.10.

**Appendix B. The Explicit Computation of \( \mathcal{T}_{l_i-k} \cdot \mathcal{T}_{1,1,1}^k \)**

We prove the second equation in Proposition 4.10. It suffices to compute the action of the unnormalized Hecke operator \( T_{l_i-k} \). By (3.6), we have

\[ T_{l_i-k} \cdot \mathcal{T}_{1,1,1}^k = T_{l_i-k} \cdot (I_{A_k} \otimes I_0 \otimes \iota_0) = \sum_{u \in U_{l_i-k}} \omega_{V_1, V_{l'}}(u t_{l_i-k}) (I_{A_k} \otimes I_0 \otimes \iota_0). \]
Note that \( U_{t_{l-k}} \subseteq N(V_{l-k}^+ \otimes V_l) \) and we can write

\[
U_{t_{l-k}} = \left\{ u(b, c) \right\} = \left\{ \begin{array}{l}
\begin{array}{l}
b \in \text{Hom}(\hat{V}_k, \langle v_{l-k} \rangle), \\
c \in \text{Hom}(\langle v_{l-k} \rangle, \langle v_{l-k} \rangle), \\
c + c^* + bb^* = 0
\end{array}
\end{array} \right. 
\]

according to (2.9). Using the decomposition

\[
\hat{V}_k = \hat{V}_k^+ \oplus V_0 \oplus \hat{V}_k^-
\]

as in (2.1), we may also write

\[
U_{t_{l-k}} = \left\{ u(b_2, \lambda b_2)u(b_1, 0)u(b_3, 0)u(0, c) \right\} = \left\{ \begin{array}{l}
\begin{array}{l}
b_1 \in \text{Hom}(\hat{V}_k^+, \langle v_{l-k} \rangle), \\
b_2 \in \text{Hom}(V_0, \langle v_{l-k} \rangle), \\
b_3 \in \text{Hom}(\hat{V}_k^-, \langle v_{l-k} \rangle), \\
c \in \text{Hom}(\langle v_{l-k} \rangle, \langle v_{l-k} \rangle)
\end{array}
\end{array} \right.
\]

where \( u(b_2, \lambda b_2) \) is a fixed element in \( U_{t_{l-k}} \) for each \( b_2 \in \text{Hom}(V_0, \langle v_{l-k} \rangle) \). Then we have

\[
T_{t_{l-k}} \cdot T_{t_{l-k}}^{1,1,1} = \sum_{b_2 \in \text{Hom}(V_0, \langle v_{l-k} \rangle)} \sum_{b_1 \in \text{Hom}(\hat{V}_k^+, \langle v_{l-k} \rangle)} \sum_{b_3 \in \text{Hom}(\hat{V}_k^-, \langle v_{l-k} \rangle)} \sum_{c \in \text{Hom}(\langle v_{l-k} \rangle, \langle v_{l-k} \rangle)} \omega_{V_l \otimes V_l^+(t_{l-k})} \left( u(b_2, \lambda b_2) u(b_1, 0) u(b_3, 0) u(0, c) t_{l-k} \right) \left( I_{A_k} \otimes I_0 \otimes \iota_0 \right).
\]

B.1. First, we compute \( \omega_{V_l \otimes V_l^+(t_{l-k})} \left( I_{A_k} \otimes I_0 \otimes \iota_0 \right) \). Since \( I_{A_k} \) is supported on 0 when restricted to \( v_{l-k} \otimes V_l^+ \), it follows from (2.14) that

\[
\omega_{V_l \otimes V_l^+(t_{l-k})} \left( I_{A_k} \otimes I_0 \otimes \iota_0 \right) = \gamma_{V_l^+} q^{-\frac{1}{2} \dim V_l^+} \left( I_{A_k+(v_{l-k})} \otimes I_0 \otimes \iota_0 \right).
\]

B.2. It follows from (2.13) that

\[
\sum_{c \in \text{Hom}(\langle v_{l-k} \rangle, \langle v_{l-k} \rangle)} \omega_{V_l \otimes V_l^+(u(0, c))} \left( I_{A_k+(v_{l-k})} \otimes I_0 \otimes \iota_0 \right) = q^{\frac{1}{2} k} \left( I_{A_k+(v_{l-k})} \otimes I_0 \otimes \iota_0 \right).
\]

Here \( \mathcal{N}' \) is defined in (A.3).

B.3. By a routine computation, we deduce from the fourth equation of (2.13) that

\[
\sum_{b_3 \in \text{Hom}(\hat{V}_k^-, \langle v_{l-k} \rangle)} \omega_{V_l \otimes V_l^+(u(b_3, 0))} \left( I_{A_k+(v_{l-k})} \otimes I_0 \otimes \iota_0 \right) = q^k \left( I_{A_k+(v_{l-k})} \otimes I_0 \otimes \iota_0 \right).
\]

B.4. The action of \( u(b_1, 0) \) is more complicated. It follows from the second equation of (2.13) that

\[
\sum_{b_1 \in \text{Hom}(\hat{V}_k^+, \langle v_{l-k} \rangle)} \omega_{V_l \otimes V_l^+(u(b_1, 0))} \left( I_{A_k+(v_{l-k})} \otimes I_0 \otimes \iota_0 \right) = \sum_{b_1 \in \text{Hom}(\hat{V}_k^+, \langle v_{l-k} \rangle)} \sum_{b \in A_k+(v_{l-k})} \sum_{(V_k^+)^{l-k} \cap \mathcal{N}'} \left( I_{B \cdot u(b_1, 0)} \right) \otimes I_0 \otimes \iota_0.
\]

To compute (B.1), we consider \( B(V_l^+) \times P_l \)-orbits in the image of

\[
\text{Hom}(\hat{V}_k^+, \langle v_{l-k} \rangle) \times \left( A_k + \langle v_{l-k} \rangle \right) \otimes (V_k^+)^{l-k} \cap \mathcal{N}' \rightarrow \mathcal{Z}
\]

\[
(b_1, B) \mapsto B \cdot u(b_1)^{-1}.
\]
For each \( i = 1, \cdots, l' \), define
\[
B_{k,i}^o := A_k + \langle v_{-(l-k)} \rangle \otimes N_i^o \quad \text{and} \quad B_{k,-i}^o := A_k + \langle v_{-(l-k)} \rangle \otimes N_{-i}^o,
\]
where \( N_i^o \) and \( N_{-i}^o \) are defined in (A.6). We also define \( B_{k,0}^o := A_k \) and
\[
B_{k,0}^o := A_k + \langle v_{-(l-k)} \rangle \otimes N_0^o.
\]

Note that \( B_{k,0}^o \) is non-empty if and only if \( \dim V_0^o \neq 0 \). Then we have
\[
A_k + \langle v_{-(l-k)} \rangle \otimes \left( (V_k^o)^o \cap N_i^o \right) = \left( \bigsqcup_{i=0}^l B_{k,i}^o \right) \sqcup \left( \bigsqcup_{i=k+1}^{l'} B_{k,-i}^o \right) \sqcup B_{k,0}^o.
\]

The following proposition is easy to check.

**Proposition B.1.** The restriction of \((B.2)\) to \( \text{Hom}(\hat{V}_k^+, \langle v_{l-k} \rangle) \times B_{k,i}^o \) is

\[
\begin{align*}
\{ a q^{k-i}(q - 1) \text{ to } 1 \text{ map } & \text{ if } i = 1, \cdots, k, \\
\{ a q^k \text{ to } 1 \text{ map } & \text{ if } i = 0, \\
\{ \text{an injection } & \text{ if } i = \pm(k + 1), \cdots, \pm l' \text{ or } 0.
\end{align*}
\]

For each \( i \) above, we define \( B_{k,i} \) to be the image of \( \text{Hom}(\hat{V}_k^+, \langle v_{l-k} \rangle) \times B_{k,i}^o \). Then a standard calculation shows that
\[
B_{k,i} = \begin{cases} 
O_{s_{l-k,l-k+i+1},s_{1,i}}^k & \text{if } i = 1, \cdots, k, \\
O_{s_{l-k+1},s_{1,k}}^{k+1} & \text{if } i = k + 1, \cdots, l', \\
O_{s_{1,k},s_{1,k}}^{k-1} & \text{if } i = -(k + 1), \cdots, -l', \\
O_{1,1,1} & \text{if } i = 0.
\end{cases}
\]

Note that \( B_{k,0} \) is not of the form \( O_{d_1,d_2,x} \) for any \( 1 \leq i \leq \min \{ l, l' \} \) and \((d_1, d_2, x) \in D_i \times D_i \times S_i \).

Now (B.1) equals
\[
q^k (I_{A_k} \otimes I_0 \otimes \iota_0) + \sum_{i=1}^k q^{k-i}(q - 1) \left( I_{B_{k,i}} \otimes I_0 \otimes \iota_0 \right) \\
+ \sum_{i=k+1}^l (I_{B_{k,i}} \otimes I_0 \otimes \iota_0) + \sum_{i=k+1}^{l'} (I_{B_{k,-i}} \otimes I_0 \otimes \iota_0) + (I_{B_{k,0}} \otimes I_0 \otimes \iota_0)
\]

(B.3)

**B.5.** The final computation is to average the action of \( u(b_2, \lambda_{b_2}) \) to each term in (B.3). Similar to Lemma A.2, we have
\[
\sum_{b_2 \in \text{Hom}(V_0^o, \langle v_{l-k} \rangle)} \omega_{V_0^o \otimes V_i^o}(u(b_2, \lambda_{b_2})) \left( I_0 \otimes I_0 \otimes \iota_0 \right)
\]

\[
= \begin{cases} 
q^{\dim V_0^o} \frac{1}{T_{s_{l-k,l-k+i+1},s_{1,i}}^k} & \text{if } \ast = B_{k,i} \text{ and } i \in \{ 1, \cdots, k \}, \\
q^{\dim V_0^o} \frac{1}{T_{s_{l-k+1},s_{1,k}}^{k+1}} & \text{if } \ast = B_{k,i} \text{ and } i \in \{ k + 1, \cdots, l' \}, \\
-q^{\dim V_0^o} \frac{1}{T_{s_{1,k},s_{1,k}}^{k+1}} & \text{if } \ast = B_{k,-i} \text{ and } i \in \{ k + 1, \cdots, l' \}, \\
q^{\dim V_0^o} \frac{1}{T_{1,1,1}} & \text{if } \ast = B_{k,0} = A_k, \\
0 & \text{if } \ast = B_{k,0}.
\end{cases}
\]

**B.6.** Combining all these computations and our normalization of \( T_{s_{l-k}} \) in (3.20), we deduce the formula of \( T_{u} \cdot 2_{1,1,1} \) in Proposition 4.10 by keeping track of the scalars.
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