Improved Bounds on the Stretch Factor of $Y_4$ *

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Abstract. We establish an upper bound of $13 + 8\sqrt{2} \lesssim 4.931$ on the stretch factor of the Yao graph $Y_4^\infty$ defined in the $L_\infty$-metric, improving upon the best previously known upper bound of 6.31. We also establish an upper bound of $(11 + 7\sqrt{2})\sqrt{4 + 2\sqrt{2}} \lesssim 54.62$ on the stretch factor of the Yao graph $Y_4$ defined in the $L_2$-metric, improving upon the best previously known upper bound of 662.16.

1 Introduction

Let $V$ be a finite set of points in the plane. The directed Yao graph $Y_k$ with integer parameter $k > 0$, denoted $\overrightarrow{Y}_k$, is defined as follows. At each point $u \in V$, any $k$ equally-separated rays originating at $u$ define $k$ cones. In each cone, pick a shortest edge $(u, v)$, if there is one, and add to $\overrightarrow{Y}_k$ the directed edge from $u$ to $v$. Ties are broken arbitrarily. The undirected Yao graph $Y_k$ includes all edges of $\overrightarrow{Y}_k$ but ignores their directions. Most of the time we ignore the direction of an edge $(u, v)$. We refer to the directed version $(u, v)$ of $(u, v)$ only when its origin $(u)$ is important and unclear from the context. We will distinguish between $Y_k$, the Yao graph in the Euclidean $L_2$ metric, and $Y_k^\infty$, the Yao graph in the $L_\infty$ metric. Unlike $Y_k$ however, in constructing $Y_k^\infty$ ties are broken by always selecting the most counterclockwise edge. This tie breaking rule was first mentioned in [5], where it was required in order to maintain the planarity of $Y_4^\infty$. Throughout the rest of the paper we will refer to the points in $V$ as vertices, to distinguish them from other points in the plane.

For a given graph $G$ with vertex set $V$, we say that $H$ is a $t$-spanner of $G$ if, for any pair of vertices $u, v \in V$, a shortest path in $H$ from $u$ to $v$ is no longer than $t$ times the length of a shortest path in $G$ between $u$ and $v$. A graph $H$ is a $t$-spanner of $V$ if $H$ is a $t$-spanner of the complete graph on $V$. The value $t$ is called the stretch factor of $H$. If $t$ is constant, then $H$ is called a length spanner, or simply a spanner.

The spanning properties of Yao graphs have been extensively studied. Table 1 summarizes some results that are relevant to this paper.

| Reference | Graph | Stretch Factor |
|-----------|-------|----------------|
| [6]       | $Y_2$, $Y_3$ | $\infty$ |
| [5]       | $Y_4$ | $8\sqrt{2}(26 + 23\sqrt{2}) \lesssim 662.16$ |
| [1]       | $Y_5$ | $2 + \sqrt{3} \lesssim 3.74$ |
| [1]       | $Y_6$ | $5.8$ |
| [4]       | $Y_k$, $k \geq 7$ | $(1 + \sqrt{2} - 2\cos\theta)/(2\cos\theta - 1)$, where $\theta = 2\pi/k$ |
| [3]       | $Y_k^\infty$ | $6.31$ |
| [this paper] | $Y_4^\infty$ | $13 + 8\sqrt{2} \lesssim 4.94$ |
| [this paper] | $Y_4$ | $(11 + 7\sqrt{2})\sqrt{4 + 2\sqrt{2}} \lesssim 54.62$ |

Table 1. Upper bounds on the stretch factor of Yao graphs.

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Our contributions. We show that the stretch factor of $Y_4$ is at most $(11 + 7\sqrt{2})\sqrt{4 + 2\sqrt{2}} \lesssim 54.62$, which is a significant improvement upon the best previously known upper bound of 662.16 from [5]. We also show that the stretch factor of $Y_4^\infty$ is at most $13 + 8\sqrt{2} \lesssim 4.931$, improving the 6.31 bound from [3]. The graph $Y_4^\infty$ is of particular interest due to its planarity property (as a subgraph of the $L_\infty$-Delaunay triangulation [3]) and its applications in scheduling problems [4].

2 Definitions

Let $V$ be a set of vertices in the plane. For each vertex $u \in V$, let $x_u$ and $y_u$ denote the $x$-coordinate and the $y$-coordinate of $u$, respectively. For every pair of vertices $u, v \in V$, the horizontal distance between $u$ and $v$ is $d_x(u, v) = |x_u - x_v|$; the vertical distance is $d_y(u, v) = |y_u - y_v|$; the Euclidean distance is $d_2(u, v) = \sqrt{d_x(u, v)^2 + d_y(u, v)^2}$; and the $L_\infty$-distance is $d_\infty(u, v) = \max\{d_x(u, v), d_y(u, v)\}$. For any plane graph $G$ with vertex set $V$, the weight of an edge in $G$ is the Euclidean distance between its endpoints; the length of a path in $G$ is the sum of the weights of its constituent edges; and the distance in $G$ between $u, v \in V$, denoted $d_G(u, v)$, is the length of a shortest path in $G$ between $u$ and $v$. We denote by $(u, v)$ the edge or the line segment connecting $u$ and $v$, and the distinction between the two will become clear from the context.

$$Q_2(u) \quad Q_1(u) \equiv Q(u, v) \quad Q_3(u) \quad Q_4(u)$$

Fig. 1. (a) Definitions: quadrants $Q_i(u)$, $i = 1, 2, 3, 4$, and $Q(u, v)$.

A cone is the region in the plane between two rays that radiate from the same point. With each vertex $u \in V$ we associate four cones of angles $\pi/2$ delimited by two lines parallel to the coordinate axes passing through $u$. We label the cones $Q_1(u), Q_2(u), Q_3(u)$ and $Q_4(u)$ in counterclockwise order, starting to the first quadrant. Refer to Figure 1. To avoid overlapping boundaries, we assume that each cone is half-open and half-closed, meaning that a cone includes its clockwise bounding ray but excludes its counterclockwise bounding ray. For any $u, v \in V$, let $Q(u, v)$ denote the quadrant with apex $u$ that contains $v$.

The directed Yao graph $Y_4^\rightarrow$ with vertex set $V$ is constructed as follows. For each vertex $u \in V$ and each cone $Q_i(u)$, for $i = 1 \ldots 4$, extend a directed edge $(u, v)$ from $u$ to a vertex $v \in V$ that lies in $Q_i(u)$ and minimizes the Euclidean distance $d_2(u, v)$. Ties are broken arbitrarily. The Yao graph $Y_4^\rightarrow$ is defined similarly to $Y_4$, with two differences: (i) it uses the $L_\infty$-distance $d_\infty(u, v)$ rather than the Euclidean distance $d_2(u, v)$, and (ii) ties are broken by selecting the most counterclockwise edge in each quadrant. The undirected Yao graph $Y_4$ includes all edges of $Y_4^\rightarrow$ but ignores their directions, and similarly for $Y_4^\infty$. We are interested in the stretch factors of $Y_4$ and $Y_4^\infty$.

Let Del$^\infty$ denote the Delaunay triangulation on $V$ in the $L_\infty$-metric, defined as follows. For any pair of vertices $u, v \in V$, an edge $(u, v)$ is in Del$^\infty$ if and only if there is an axis-aligned square with $u$ and $v$ on its boundary that contains no other vertices in its interior. A well-known property of Del$^\infty$ is that, for each triangle $T$ in Del$^\infty$, the square whose sides pass through the three vertices of $T$ (the circumsquare of $T$) has no vertices of $V$ in its interior.

For any polygon $P$, let $\partial P$ denote the boundary of $P$. For any two vertices $u$ and $v$, let $R(u, v)$ denote the rectangle with sides parallel to the coordinate axes having $u$ and $v$ as opposite corners. (See Figure 2.)
We say that two edges intersect (cross) if they share a point (an interior point). Note that by this definition, two intersecting edges may share an endvertex. Throughout the paper, we use the symbol $\oplus$ to denote the concatenation operator.

3 $Y_4^\infty$ in the $L_\infty$ Metric

In this section we show that $Y_4^\infty$ has stretch factor at most $\sqrt{13 + 8\sqrt{2}} \lesssim 4.931$. This improves upon the best previously known stretch factor of $(1 + \sqrt{2})\sqrt{4 + 2\sqrt{2}} \lesssim 6.31$ from [3]. We begin with the following result established in [3].

Lemma 1. The graph $Y_4^\infty$ is a subgraph of $Del^\infty$, a $(1 + \sqrt{2})$-spanner of $Del^\infty$ and also a $(1 + \sqrt{2})\sqrt{4 + 2\sqrt{2}}$-spanner of $V$.

Although not explicitly stated, the proof of Lemma 1 from [3] implies the following result.

Lemma 2. For each triangle $\triangle uvw \in Del^\infty$, at least two of its edges are in $Y_4^\infty$. If $(u,v)$ is not in $Y_4^\infty$, then $u$ and $v$ lie on opposite sides of the circumsquare of $\triangle uvw$.

An immediate consequence of Lemma 2 is the following.

Corollary 3. For each triangle $\triangle uvw \in Del^\infty$, if $(u,v)$ is not in $Y_4^\infty$, then $$d_\infty(u,v) \geq \max\{d_\infty(u,w), d_\infty(w,v)\}.$$ These together yield the following result.

Lemma 4. For each edge $(u,v) \in Del^\infty$, there is a path in $Y_4^\infty$ of length $$d_{Y_4^\infty}(u,v) \leq (1 + \sqrt{2}) \cdot d_\infty(u,v)$$

Proof: If $(u,v)$ is in $Y_4^\infty$, then the theorem clearly holds. So assume that $(u,v) \not\in Y_4^\infty$. Let $T = \triangle uvw$ be a triangle in $Del^\infty$ with side $(u,v)$. By Lemma 2 both $(u,w)$ and $(v,w)$ are in $Y_4^\infty$. Thus $(u,w) \oplus (w,v)$ is a path in $Y_4^\infty$ between $u$ and $v$ of length $d_2(u,w) + d_2(w,v)$. Also by Lemma 2 $u$ and $v$ lie on opposite sides of $T$’s circumsquare. This implies that $d_2(u,w) + d_2(w,v)$ is bounded above by $(1 + \sqrt{2})d_\infty(u,v)$, which is achieved when one of $(u,w)$ and $(v,w)$ is a side, and the other is a diagonal of $T$’s circumsquare.

The following theorem is key in establishing an upper bound on the stretch factor of $Y_4^\infty$.

Theorem 5. Let $a$ and $b$ be arbitrary vertices in $V$. If $x = d_\infty(a,b) = \max\{d_x(a,b), d_y(a,b)\}$ and $y = \min\{d_x(a,b), d_y(a,b)\}$, then $$d_{Y_4^\infty}(a,b) \leq 2(1 + \sqrt{2})x + y.$$ We delay the proof of Theorem 5 until we establish some essential ingredients. The main result of this section, stated in Theorem 6 below, is an immediate consequence of Theorem 5.

Theorem 6. The stretch factor of $Y_4^\infty$ on a set of points $V$ is at most $$\sqrt{13 + 8\sqrt{2}} \lesssim 4.931$$
Fig. 2. **Lemma 7.** Squares $S_1, S_2, \ldots, S_{j-1}$ are not inductive. Square $S_j$ is inductive. Vertex $\ell_i$ is on the east side of square $S_i$.

**Proof.** By **Theorem 5** the stretch factor of $Y_4^\infty$ is no greater than the maximum of the function

$$\frac{2(1 + \sqrt{2})x + y}{\sqrt{x^2 + y^2}}$$

which is equal to $\sqrt{13 + 8\sqrt{2}}$ when $x/y = 2(1 + \sqrt{2})$.

Our approach in proving **Theorem 5** mimics the approach used in [2] to establish a stretch factor of $\sqrt{4 + 2\sqrt{2}}$ for $Del^\infty$. Before describing this approach, we need to introduce some definitions. To make it easy for the interested reader, most of the terminology in this section is similar to the one used in [2]. We assume without loss of generality that $a$ has coordinates $(0,0)$. In this case, the definitions used in the statement of **Theorem 5** imply that $b$ has coordinates $(x,y)$. Let $T_1, T_2, \ldots, T_r$ be the sequence of triangles in $Del^\infty$ that intersect the line segment $ab$ when moving from $a$ to $b$. For each triangle $T_i$, let $(h_i, \ell_i)$ be the rightmost edge of $T_i$ that intersects $ab$, with $h_i$ above $ab$ and $\ell_i$ below $ab$. We also let $h_0 = \ell_0 = a$, $h_r = b$ and $\ell_{r-1} = \ell_r$. Note that some vertices coincide: either $h_i = h_{i-1}$ and $T_i = \triangle h_i \ell_{i-1} \ell_i$, or $\ell_i = \ell_{i-1}$ and $T_i = \triangle h_{i-1} h_i \ell_i$. Let $S_i$ be the circumsquare of $T_i$. We call the square $S_i$ **inductive** if $d_\infty(h_i, \ell_i) = d_\infty(h_i, \ell_i)$.

The vertex $h_i$ or $\ell_i$ with the larger $x$-coordinate is the **inductive point** of $S_i$. In Figure 2 for example, $h_j$ is the inductive point of $S_j$.

One key ingredient in proving **Theorem 5** is the following lemma.

**Lemma 7.** Assume that $R(a,b)$ is empty. If no square $S_1, \ldots, S_r$ is inductive, then

$$d_{Y_4^\infty}(a,b) \leq 2(1 + \sqrt{2})x + y.$$  

Otherwise, let $S_j$ be the first inductive square in the sequence $S_1, \ldots, S_r$. If $h_j$ is the inductive point of $S_j$, then

$$d_{Y_4^\infty}(a,h_j) + (y_{h_j} - y) \leq 2(1 + \sqrt{2})x_{h_j}.$$  

If $\ell_j$ is the inductive point of $S_j$, then

$$d_{Y_4^\infty}(a,\ell_j) - y_{\ell_j} \leq 2(1 + \sqrt{2})x_{\ell_j}.$$
Proof. Because $S_j$ is inductive, $d_\infty(\ell_j, h_j) = |x_{\ell_j} - x_{h_j}|$ (by definition). This along with Lemma 4 implies

$$d_{Y^\infty}(\ell_j, h_j) \leq (1 + \sqrt{2})|x_{\ell_j} - x_{h_j}|$$

(1)

Assume first that $h_j$ is the inductive point of $S_j$, meaning that $x_{\ell_j} > x_{h_j}$. In this case $h_j$ lies on the east side of $S_j$ and $\ell_j$ lies on the west or south side of $S_j$. Let $T_1$ be the first triangle encountered when moving from $T_j$ leftward toward $T_1$, such that either $i = 0$ or $\ell_j$ lies on the east side of $T_i$. Refer to Figure 2. Note that $T_1 \neq T_j$, since $\ell_j$ does not lie on the east side of $T_j$. Then all edges in Del$^{\infty}$ on the path $p_{ij} = \ell_i, \ell_{i+1}, \ldots, \ell_j$ span between the west and south sides of their enclosing square, and by Lemma 2 they are also in $Y_4^\infty$.

Also note that the path $p_{ij}$ descends vertically, therefore $y_{\ell_j} > y_{h_j}$. These together with the triangle inequality applied on each edge of $p_{ij}$ imply

$$d_{Y^\infty}(\ell_j, \ell_j) < (x_{\ell_j} - x_{h_j}) + (y_{\ell_j} - y_{h_j}).$$

(2)

We now use the combined results from Lemmas 9 and 11 from [2] showing that

$$d_{Del^{\infty}}(a, \ell_i) \leq 2x_{\ell_i}.$$  
This along with the fact that $Y_4^\infty$ is a $(1 + \sqrt{2})$-spanner of Del$^{\infty}$ implies that

$$d_{Y^\infty}(a, \ell_i) \leq 2(1 + \sqrt{2})x_{\ell_i}.$$  

(3)

We are now ready to evaluate

$$d_{Y^\infty}(a, h_j) + (y_{h_j} - y) \leq d_{Y^\infty}(a, \ell_i) + d_{Y^\infty}(\ell_i, \ell_j) + d_{Y^\infty}(\ell_j, h_j) + y_{h_j}.$$  

Substituting inequalities (1), (2) and (3) in the right hand side above yields

$$d_{Y^\infty}(a, h_j) + (y_{h_j} - y) < 2(1 + \sqrt{2})x_{\ell_i} + (x_{\ell_j} - x_{\ell_i}) + (y_{\ell_j} - y_{\ell_i}) + (1 + \sqrt{2})(x_{h_j} - x_{\ell_i}) + y_{h_j}$$

$$< (2 + 2\sqrt{2} - 1)x_{\ell_i} - (1 + \sqrt{2})x_{\ell_j} + (x_{\ell_j} + y_{h_j} - y_{\ell_j}) + (1 + \sqrt{2})x_{h_j}$$

(4)

We safely ignored the quantity $y_{\ell_i} < 0$ in the right hand side of the inequality above. Recall that $d_\infty(\ell_j, h_j) = x_{h_j} - x_{\ell_j}$ (since $S_j$ is inductive), therefore $x_{\ell_i} + y_{h_j} - y_{\ell_i} \leq x_{h_j}$. Also note that $x_{\ell_j} > x_{\ell_i} \geq 0$, therefore $-x_{\ell_j} < -x_{\ell_i}$. Substituting these inequalities in (4) yields

$$d_{Y^\infty}(a, h_j) + (y_{h_j} - y) < (1 + 2\sqrt{2} - 1 - \sqrt{2})x_{\ell_i} + x_{\ell_j} + (1 + \sqrt{2})x_{h_j}$$

$$\leq \sqrt{2}x_{\ell_i} + x_{\ell_j} + (1 + \sqrt{2})x_{h_j}$$

$$< 2(1 + \sqrt{2})x_{h_j}.$$  

This latter inequality follows from the fact that $x_{\ell_j} < x_{h_j}$.

Assume now that $\ell_j$ is the inductive point of $S_j$, so $\ell_j$ lies on the east side of $S_j$ and $h_j$ lies on the west or north side of $S_j$. The analysis for this case is symmetric to the one used for the previous case. Redefine $T_i$ to be the first triangle encountered when moving from $T_j$ leftward toward $T_i$, such that either $i = 0$ or $h_i$ lies on the east side of $T_i$. Refer to Figure 3. Arguments similar to the ones used for the previous case show that

$$d_{Y^\infty}(a, h_i) - y_{\ell_j} \leq d_{Y^\infty}(a, h_i) + d_{Y^\infty}(h_i, h_j) + d_{Y^\infty}(h_j, \ell_j) - y_{\ell_j}$$

$$\leq 2(1 + \sqrt{2})x_{h_j} + (x_{h_j} - x_{\ell_j}) + (y_{h_j} - y_{\ell_j}) + (1 + \sqrt{2})(x_{\ell_j} - x_{h_j}) - y_{\ell_j}$$

$$< (2 + 2\sqrt{2} - 1)x_{h_j} - (1 + \sqrt{2})x_{\ell_j} + (x_{h_j} + y_{h_j} - y_{\ell_j}) + (1 + \sqrt{2})x_{\ell_j}$$

$$\leq (1 + 2\sqrt{2} - 1 - \sqrt{2})x_{h_j} + x_{\ell_j} + (1 + \sqrt{2})x_{\ell_j}$$

$$\leq \sqrt{2}x_{h_j} + x_{\ell_j} + (1 + \sqrt{2})x_{\ell_j}$$

$$< 2(1 + \sqrt{2})x_{\ell_j}.$$  

In deriving these inequalities we ignored the term $-y_{\ell_j} < 0$ and used the fact that $x_{h_j} < x_{\ell_j} < x_{\ell_i}$ and $x_{h_j} + y_{h_j} - y_{\ell_j} < x_{h_j} + y_{h_j} - y_{\ell_j} \leq x_{\ell_j}$ (by the lemma statement that $S_j$ is inductive).
We are now ready to prove Theorem 5. Our proof follows closely the proof from [2] used to establish a similar result in the context of Del$^\infty$, with some changes necessary to handle edges in Del$^\infty$ that do not exist in Y$^\infty$.

**Theorem 5** Let $a$ and $b$ be arbitrary vertices in $V$. If $x = d_\infty(a,b) = \max\{d_x(a,b), d_y(a,b)\}$ and $y = \min\{d_x(a,b), d_y(a,b)\}$, then

$$d_{Y^\infty}(a,b) \leq 2(1 + \sqrt{2})x + y$$

**Proof.** By the theorem statement, $a$ and $b$ are two arbitrary points in $V$ of coordinates $(0,0)$ and $(x,y)$ respectively, with $x = d_\infty(a,b) \geq y$. Our goal is to prove that $d_{Y^\infty}(a,b) \leq 2(1 + \sqrt{2})x + y$. The proof is by induction on the $L_\infty$-distance between pairs of points in $V$.

For the base case, assume that $a$ and $b$ are a closest pair of vertices in the $L_\infty$-metric. In this case $ab \in$ Del$^\infty$ and, by Lemma 4, $d_{Y^\infty}(a,b) \leq (1 + \sqrt{2}) \cdot d_\infty(a,b) = (1 + \sqrt{2})x$. Thus the theorem holds for the base case.

For the induction step, assume that $a,b \in V$ are arbitrary, and that the theorem holds for all pairs of vertices in $V$ strictly closer than $d_\infty(a,b)$ in the $L_\infty$-metric. We discuss two cases, depending on whether the interior of $R(a,b)$ is empty or not.

**Case 1.** Assume first that the interior of $R(a,b)$ is not empty. Partition the interior of $R(a,b)$ into three regions (call them $A$, $B$ and $C$ left to right) with two lines of slope one passing through $a$ and $b$. Any point $c$ in the mid-region $B$ (shaded in Figure 4) satisfies $x_c \geq y_c$ and $x - x_c \geq y - y_c$. If there is such a point,
then we can apply induction on the vertex pairs \((a, c)\) and \((c, b)\) to obtain
\[
d_Y^∞(a, c) \leq 2(1 + \sqrt{2})x_c + y_c\]
and \(d_Y^∞(c, b) \leq 2(1 + \sqrt{2})(x - x_c) + (y - y_c)\). Summing up these inequalities yields
\[
d_Y^∞(a, b) \leq d_Y^∞(a, c) + d_Y^∞(c, b) \leq 2(1 + \sqrt{2})x + y,\]
so the theorem holds for this case.

Let \(S_a\) (or \(S_b\)) be the largest empty square with bottom left corner \(a\) (top right corner \(b\)) that fits inside \(R(a, b)\). If region \(B\) is empty, then there must be a vertex \(c \in V\), with \(c \notin \{a, b\}\), that lies on the boundary of either \(S_a\) or \(S_b\). Assume without loss of generality that there is such a vertex on the boundary of \(S_a\), and let \(c\) be the most counterclockwise such vertex (relative to \(a\)). In this case \((a, c) \in Y_4^∞\) (by definition) and therefore
\[
d_Y^∞(a, c) = d_2(a, c) < x_c + y_c\]  
\[\text{(5)}\]

If \(c\) lies in region \(A\) (as in \(\text{Figure 4a}\)), then \(y_c > x_c\) and \(x - x_c > y - y_c\). We apply induction on the vertex pair \((c, b)\) to derive
\[
d_Y^∞(c, b) \leq 2(1 + \sqrt{2})(x - x_c) + (y - y_c).\]
This along with \((5)\) yields
\[
d_Y^∞(a, b) \leq d_Y^∞(a, c) + d_Y^∞(c, b) \]
\[
< x_c + y_c + 2(1 + \sqrt{2})(x - x_c) + y - y_c\]
\[
\leq 2(1 + \sqrt{2})x + y - (1 + 2\sqrt{2})x_c\]
\[
< 2(1 + \sqrt{2})x + y\]

If \(c\) lies in region \(C\) (as in \(\text{Figure 4c}\)), then \(x_c > y_c\) and \(y - y_c > x - x_c\). We apply induction on the vertex pair \((c, b)\) to derive
\[
d_Y^∞(c, b) \leq 2(1 + \sqrt{2})(y - y_c) + (x - x_c)\]
This along with \((5)\) yields
\[
d_Y^∞(a, b) \leq d_Y^∞(a, c) + d_Y^∞(c, b) \]
\[
< x_c + y_c + 2(1 + \sqrt{2})(y - y_c) + x - x_c\]
\[
\leq 2(1 + \sqrt{2})y + x - (1 + 2\sqrt{2})y_c\]
\[
< (1 + 2\sqrt{2})y + x + y\]
\[
< 2(1 + \sqrt{2})x + y\]
This latter inequality follows immediately from the fact that \(y < x\).

**Case 2.** Assume now that the interior of \(R(a, b)\) is empty. If no square in the sequence \(S_1, S_2, \ldots, S_r\) is inductive, then by \(\text{Lemma 7}\) we have
\[
d_Y^∞(a, b) \leq 2(1 + \sqrt{2})x + y\]
and the theorem holds. Otherwise, let \(S_j\) be the first inductive square in the sequence \(S_1, S_2, \ldots, S_r\).

Assume first that \(h_j\) is the inductive point of \(S_j\) (so \(h_j\) lies on the east side of \(S_j\)). Let \(h_k\) be the first vertex in the sequence \(h_j, h_{j+1}, \ldots, h_r = b\) such that
\[
x - x_{h_k} \geq y_{h_k} - y > 0\]
\[\text{(6)}\]

Refer to \(\text{Figure 2}\). Inequality \((3)\) implies that \(h_k\) is closer to \(b\) in the \(L_∞\) metric than \(a\) is. This enables us to use induction to determine an upper bound on \(d_Y^∞(h_k, b)\). Before we do so, note that each edge \((h_p, h_{p+1})\), for any \(j \leq p < k\), has its endpoints on the north and east sides of its enclosing squares \(S_{p+1}\). (The only other alternatives would be for \((h_p, h_{p+1})\) to span between the west and north sides, or between the west and east sides of \(S_{p+1}\). In each of these cases \(S_{p+1}\), which must pass through a vertex \(\ell_{p+1}\) below \(b\), would extend too far to the right and include the endpoint \(b\), since the horizontal distance from \(h_p\) to \(b\) is no longer than the vertical distance from \(h_p\) to \(b\). This contradicts the fact that \(S_{p+1}\) is empty.) This further implies that the path \(h_j, h_{j+1}, \ldots, h_k\) is in \(Y_4^∞\) (by \(\text{Lemma 2}\)). This along with the triangle inequality applied on each edge along this path yields
\[
p_Y^∞(h_j, h_k) < (x_{h_k} - x_{h_j}) + (y_{h_i} - y_{h_k})\]
This observation together with Lemma 7 used to bound $d_{Y_4^\infty}(a,h_j)$ and the inductive hypothesis used to bound $d_{Y_4^\infty}(h_k,b)$ yields
\[
d_{Y_4^\infty}(a,b) \leq d_{Y_4^\infty}(a,h_j) + d_{Y_4^\infty}(h_j,h_k) + d_{Y_4^\infty}(h_k,b)
< 2(1 + \sqrt{2})x_{h_j} - (y_{h_j} - y) + (x_{h_k} - x_{h_j}) + (y_{h_k} - y_{h_j}) + 2(1 + \sqrt{2})(x - x_{h_k}) + (y - y_{h_k})
= 2(1 + \sqrt{2})x + (2(1 + \sqrt{2}) - 1)x_{h_j} + (1 - 2(1 + \sqrt{2}))x_{h_k}
= 2(1 + \sqrt{2})x + (1 + 2\sqrt{2})x_{h_j} - (1 + 2\sqrt{2})x_{h_k}
\leq 2(1 + \sqrt{2})x.
\]

The last inequality follows immediately from the fact that $x_{h_j} \leq x_{h_k}$. Assume now that $\ell_j$ is the inductive point of $S_j$ (so $\ell_j$ lies on the east side of $S_j$). Let $\ell_k$ be the first vertex in the sequence $\ell_j, \ell_{j+1}, \ldots, \ell_r$ such that
\[
x - x_{\ell_k} \geq y - y_{\ell_k} > 0.
\]

Refer to Figure 3. Arguments similar to the ones used in the previous case show that
\[
p_{Y_4^\infty}(\ell_j, \ell_k) < (x_{\ell_k} - x_{\ell_j}) + (y_{\ell_k} - y_{\ell_j}).
\]

This along with Lemma 7 used to bound $d_{Y_4^\infty}(a,\ell_j)$ and the inductive hypothesis used to bound $d_{Y_4^\infty}(\ell_k,b)$ yields
\[
d_{Y_4^\infty}(a,b) \leq d_{Y_4^\infty}(a,\ell_j) + d_{Y_4^\infty}(\ell_j,\ell_k) + d_{Y_4^\infty}(\ell_k,b)
< 2(1 + \sqrt{2})x_{\ell_j} + y_{\ell_j} + (x_{\ell_k} - x_{\ell_j}) + (y_{\ell_k} - y_{\ell_j}) + 2(1 + \sqrt{2})(x - x_{\ell_k}) + (y - y_{\ell_k})
= 2(1 + \sqrt{2})x + y + (2(1 + \sqrt{2}) - 1)x_{\ell_j} + (1 - 2(1 + \sqrt{2}))x_{\ell_k}
= 2(1 + \sqrt{2})x + y + (1 + 2\sqrt{2})x_{\ell_j} - (1 + 2\sqrt{2})x_{\ell_k}
\leq 2(1 + \sqrt{2})x + y.
\]

This concludes the proof of Theorem 5.

4 \textbf{Y}_4 \textbf{ in the } L_2 \textbf{ Metric}

In this section we turn to the Yao graph $Y_4$ defined in the Euclidean metric space. It has been shown that, corresponding to each edge $(a,b) \in Y_4^\infty$, there is a path in $Y_4$ of length $d_{Y_4}(a,b) \leq (26 + 23\sqrt{2}) \cdot d_2(a,b)$ (Lemma 9 from [5]). Combined with the result of Theorem 6, which shows that $Y_4^\infty$ is a $\sqrt{13} + 8\sqrt{2}$-spanner, this yields a stretch factor of $(26 + 23\sqrt{2})\sqrt{13} + 8\sqrt{2} \leq 288.59$ for $Y_4$. This improves upon the best currently known stretch factor of $8\sqrt{2}(26 + 23\sqrt{2}) \leq 662.16$ for $Y_4$ established in [5]. In this section we further reduce the stretch factor of $Y_4$ to $(11 + 7\sqrt{2})\sqrt{4} + 2\sqrt{2} \leq 54.62$.

For ease of presentation, we introduce a few definitions. Let $p_R(a,b)$ denote the greedy path that begins at $a$, follows the $Y_4$ edges pointing in the direction of $b$, and ends at the first vertex exterior to, or on the boundary of, $R(a,b)$. Figure 5a illustrates this definition. Let $d_{R}(a,b)$ denote the length of $p_R(a,b)$. In our proofs we use the following preliminary results from [5].

Proposition 8 ([5]). For any triangle $\triangle abc$, $d_{4}(a,c)^2 < d_{2}(a,b)^2 + d_{2}(b,c)^2$, if $\angle bac < \pi/2$.

Lemma 9 ([5]). $d_{R}(a,b) \leq d_{2}(a,b)\sqrt{2}$, and each edge on $p_{R}(a,b)$ is no longer than $(a,b)$.

Lemma 10. Let $\overrightarrow{(a, b)}$ and $\overrightarrow{(c, d)}$ be two edges in $Y_4$ that intersect. If $\overrightarrow{(a, b)}$ and $\overrightarrow{(c, d)}$ share an interior point, let $(x,y)$ be a shortest side of the quadrilateral with vertices $a$, $b$, $c$ and $d$; otherwise, let $x = y$ be the common endpoint. In either case, $d_{Y_4}(x,y) \leq 3(2 + \sqrt{2}) \cdot \max\{d_{2}(a,b),d_{2}(c,d)\}$. 

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Proof. This result follows immediately from two intermediate results established in [3]. If $x = y$, then $d_{Y_4}(x, y) = 0$ and the lemma clearly holds. Otherwise, Lemma 4 from [5] shows that

$$d_2(x, y) \leq \max\{d_2(a, b), d_2(e, d)\}/\sqrt{2}$$

Lemma 8 from [5] shows that $d_{Y_4}(x, y) \leq \frac{6}{\sqrt{2} - 1} \cdot d_2(x, y)$. These together yield the inequality stated by the lemma.

We need one more lemma before we turn to the main result of this section.

**Lemma 11.** Let $\triangle abc \in \text{Del}^\infty$ and let $S$ be its circumsquare. Assume that $(a, b) \notin Y_4$ and $a, b$ lie on adjacent sides that meet at corner $w$ of $S$. Let $(a, a') \in Y_4$ and $(a', e) \in Y_4$ be such that $a' \in Q(a, b)$ and $e \in Q(a', b)$. If $(a, a')$ crosses the line segment $(w, b)$, then $(a', e)$ may not cross $(a, b)$.

Proof. Assume to the contrary that $(a, a')$ crosses $(w, b)$ and $(a', e)$ crosses $(a, b)$. Refer to Figure 5b. By definition $S$ is empty of vertices, therefore both $a'$ and $e$ lie outside of $S$. It follows that $(w, e)$ is longer than the side length of $S$, so the inequality $d_2(w, e) > d_2(w, b)$ holds. Let $o$ be the intersection point between $(w, b)$ and $(a', e)$. Summing up the triangle inequalities for $\triangle woe$ and $\triangle a'ob$ yields $d_2(w, e) + d_2(a', b) < d_2(w, b) + d_2(a', e)$. This along with $d_2(w, e) > d_2(w, b)$ yields $d_2(a', b) < d_2(a', e)$, contradicting the fact that $(a', e) \in Y_4$. It follows that $(a', e)$ may not cross $(a, b)$ and the lemma holds.

We are now ready to establish the main result of this section, showing that there is a short path in $Y_4$ between the endpoints of each edge in $\text{Del}^\infty$.

**Theorem 12.** For each edge $(a, b) \in \text{Del}^\infty$, $d_{Y_4}(a, b) \leq (11 + 7\sqrt{2}) \cdot d_2(a, b)$.

Proof: If $(a, b) \in Y_4$, then $d_{Y_4}(a, b) = d_2(a, b)$ and the theorem holds. So assume that $(a, b) \notin Y_4$, and let $(a, a') \in Y_4$, with $a' \in Q(a, b)$. By definition,

$$d_2(a, a') \leq d_2(a, b) \tag{7}$$

This along with Proposition 8 implies that $d_2(a', b)^2 < d_2(a, a')^2 + d_2(a, b)^2 \leq 2 \cdot d_2(a, b)^2$, so

$$d_2(a', b) < \sqrt{2} \cdot d_2(a, b) \tag{8}$$

Since $(a, b) \in \text{Del}^\infty$, there is a triangle $\triangle abc \in \text{Del}^\infty$ whose circumsquare $S$ contains no vertices in its interior. We discuss two cases, depending on whether $a$ and $b$ lie on adjacent sides or on opposite sides of $S$. 

---

**Fig. 5.** (a) Greedy path $p_R(a, b)$ (b) If $(a', e)$ crosses $(a, b)$, then $(a', e) \notin Y_4$. 


Consider first the simpler case when \( a \) and \( b \) lie on adjacent sides of \( S \). Assume without loss of generality that \( a \) and \( b \) lie on the west and north sides of \( S \) respectively, so \( b \in Q_1(a) \).

Assume first that \((a, a')\) lies counterclockwise from \((a, b)\), since \( S \) is empty of vertices, \( a' \) must be above \( b \).

Refer to Figure 6(a). Inequality (8) together with Lemma 9 implies

\[ d_R(a', b) \leq \sqrt{2} \cdot d_2(a', b) < 2 \cdot d_2(a, b), \]

and similarly for \( d_R(b, a') \).

By Lemma 11, \( p_R(a', b) \) may not cross \((a, b)\), therefore \( p_R(a', b) \) exits \( R(a', b) \) through its right side. This implies that the paths \( p_a = (a, a') \oplus p_R(a', b) \) and \( p_b = p_R(b, a') \) intersect. If \( p_a \) and \( p_b \) share a vertex, define \( x = y \) to be the common vertex; otherwise, let \((x, y)\) be a shortest side of the quadrilateral formed by the endpoints of the two crossing edges. Lemma 9 tells us that the two crossing edges are no longer than \( \max\{d_2(a, a'), d_2(a', b)\} \), and by inequalities (7) and (8) this quantity is no greater than \( \sqrt{2} \cdot d_2(a, b) \). This along with Lemma 10 implies that \( d_Y(x, y) \leq 3(2 + \sqrt{2}) \cdot d_2(a, b) = 6(1 + \sqrt{2}) \cdot d_2(a, b) \).

These together show that

\[
\begin{align*}
d_Y(a, b) & \leq d_2(a, a') + d_R(a', b) + d_R(b, a') + d_Y(x, y) \\
& \leq d_2(a, b) + 2 \cdot d_2(a, b) + 2 \cdot d_2(a, b) + 6(1 + \sqrt{2}) \cdot d_2(a, b) \\
& = (11 + 6\sqrt{2}) \cdot d_2(a, b)
\end{align*}
\]

Thus the theorem holds for this case.

Assume now that \((a, a')\) lies clockwise from \((a, b)\). Refer to Figure 6(b). Let \((b, b') \in Y_4 \), with \( b' \in Q(b, a) \).

By definition,

\[ d_2(b, b') \leq d_2(a, b) \quad (9) \]

If \((b, b')\) lies clockwise from \((b, a)\), we find ourselves in a situation similar to the one depicted in Figure 6(a), with \( a \) and \( b \) switching roles. An analysis similar to the one above shows that the theorem holds for this case. So assume that \((b, b')\) lies counterclockwise from \((b, a)\), as depicted in Figure 6(b). In this case \((b, b')\) and \((a, a')\) cross in an interior point. Let \((x, y)\) be a shortest side of the quadrilateral with vertices \( a, b', a' \) and \( b \). Lemma 10 along with inequalities (7) and (9), implies that \( d_Y(x, y) \leq 3(2 + \sqrt{2}) \cdot d_2(a, b) \).

Thus we have that

\[
\begin{align*}
d_Y(a, b) & \leq d_2(a, a') + d_2(b, b') + d_Y(x, y) \\
& \leq d_2(a, b) + d_2(a, b) + 3(2 + \sqrt{2}) \cdot d_2(a, b) \\
& = (8 + 3\sqrt{2}) \cdot d_2(a, b)
\end{align*}
\]

So the theorem holds for this case as well.
Thus we have identified two intersecting paths, \( p \)
the roles of \( a \)
Substituting the inequalities from (11) in the inequality above yields
We use these inequalities, along with Lemma 9, to establish the following upper bounds: each edge on
Also by Lemma 9 we have that
Let \( (b, b') \in Y_4 \), with \( b' \in Q(b, a') \).
Also by the definition of
Let \( e \) be the upper right corner of
We further assume that \( a \), \( c \) share a vertex, define \( x \) to be the common vertex; otherwise, let \( (x, y) \) be a shortest side of the quadrilateral formed by the endpoints of the two edges on \( p_a' \) and \( p_b \) that cross. Next we determine an upper bound on the length of these crossing edges, which together with Lemma 10 will help determine an upper bound on the distance in \( Y_4 \) between \( x \) and \( y \).
Let \( p \) be the upper right corner of \( S \). Let \( q \) be the intersection between the horizontal through \( p \) and the circle with center \( b \) and radius \( (b, c) \). Refer to Figure 6. Similarly, let \( v \) be the lower right corner of \( S \) and let \( v \) be the intersection between the horizontal through \( u \) and the circle with center \( a \) and radius \( (a, c) \). First observe that \( d_2(p, q) < d_2(p, c) \) (this follows immediately from the fact that \( \angle pcq < \angle bcq = \angle bqc \)), and similarly \( d_2(u, v) < d_2(u, c) \).
We use these inequalities, along with Lemma 9, to establish the following upper bounds: each edge on
Inequality (10) shows that the same upper bound of \( \sqrt{2} \cdot d_2(a, b) \) holds for \( d_2(b, b') \) as well. We conclude that each of the two crossing edges on \( p_a' \) and \( p_b \) is no longer than \( \sqrt{2} \cdot d_2(a, b) \).
Let \( p_{a'} = p_{R}(a', c) \) extend above and to the right of \( c \). This along with Lemma 10 implies that
We further assume that \( \angle e,b \) lies clockwise from \( \angle e,b' \) and that \( p_e \) may not cross \( b,b' \).
We seek to identify two intersecting paths in \( Y_4 \), one that begins at \( a' \) and extends toward \( b' \), and one that begins with \( (b, b') \) and then heads toward \( a' \).
Consider now the case where \( (a, a') \) and \( (a, a') \) intersect, we found our two intersecting paths. Otherwise, consider the more general case where \( e \) lies left of \( d \) (a similar analysis applies to the case where \( e \) lies right of \( d \)). In this case, note that \( p_{R}(e, b') \) is trapped underneath the path \( p_a = (b, b') \oplus p_{R}(b', c) \), therefore \( p_{R}(e, b') \) must intersect \( p_a \).
This along with Lemma 10 implies that
We use these inequalities, along with Lemma 9, to establish the following upper bounds: each edge on
Also by Lemma 9 we have that
The situation where \( (a, a') \) lies counterclockwise from \( (a, b) \) is similar to the one depicted in Figure 6(a) and the same analysis applies here as well. So assume that \( (a, a') \) lies clockwise from \( (a, b) \), as depicted in Figure 6(c). Let \( (b, b') \in Y_4 \), with \( b' \in Q(b, a') \).

Case 2. Consider now the case where \( a \) and \( b \) lie on opposite sides of \( S \). Recall that \( (a, b) \) is one side of the triangle \( \triangle abc \) enclosed in \( S \). Assume without loss of generality that \( a \) and \( b \) lie on the south and north sides of \( S \) respectively, and that \( b \in Q_1(a) \). We further assume that \( c \in Q_1(a) \); if this is not the case, we reverse the roles of \( a \) and \( b \) and rotate the vertex set \( V \) by \( \pi \) to make this assumption hold.

Also by Lemma 9 we have that
Let \( (a, a') \) and then heads toward \( a' \).

Also by the definition of
The situation where \( (a, b) \) lies counterclockwise from \( (a, c) \) is similar to the one depicted in Figure 6(a) and the same analysis applies here as well. So assume that \( (a, b) \) lies clockwise from \( (a, c) \), as depicted in Figure 6(c). Let \( (b, b') \in Y_4 \), with \( b' \in Q(b, a') \).



\[
d_2(b, b') \leq d_2(b, a') < \sqrt{2} \cdot d_2(a, b) \quad \text{(cf. inequality [8])} \tag{10}
\]



\[
d_2(c, q) < \sqrt{2} \cdot d_2(p, c) < \sqrt{2} \cdot d_2(a, b) \\
d_2(v, c) < \sqrt{2} \cdot d_2(u, c) < \sqrt{2} \cdot d_2(a, b) \tag{11}
\]


\[
d_2(a, b) \]


\[
Y_4(x, y) \leq 3(2 + \sqrt{2}) \sqrt{2} \cdot d_2(a, b) = 6(1 + \sqrt{2}) \cdot d_2(a, b) \tag{12}
\]
This latter inequality follows from the fact that each of $d_2(u, p)$ and $d_2(p, c)$ is bounded above by $d_2(a, b)$. This together with inequalities (7), (10) and (12) yields

$$d_Y(a, b) \leq d_2(a, a') + d_2(b, b') + (d_R(a', c) + d_R(e, b') + d_R(b', c)) + d_Y(x, y)$$

$$\leq d_2(a, b) + \sqrt{2} \cdot d_2(a, b) + 4 \cdot d_2(a, b) + 6(1 + \sqrt{2}) \cdot d_2(a, b)$$

$$= (11 + 7\sqrt{2}) \cdot d_2(a, b)$$

Thus the theorem holds for this case. The case where $(b, b')$ intersects $(a, a')$ is a special instance of the one discussed above, with the paths $p_{a'}$ and $p_R(b', c)$ reduced to null. This concludes the proof.

5 Conclusion

In this paper we improve the upper bounds on the stretch factors of $Y_4^\infty$ and $Y_4$. The best known lower bound on the stretch factor of $Y_4^\infty$ is the one established in [2] for $Del^\infty$, which is $\sqrt{4 + 2\sqrt{2}} \lesssim 2.62$. Narrowing the gap between this lower bound and the upper bound of 4.94 established in this paper remains open.

The second result of this paper reduces the upper bound on the stretch factor of $Y_4$ from 662.16 to 54.62. This bound might be improved with a more careful analysis that does not rely on the intermediate results from [5] employed by Lemma 10. We believe that the real stretch factor is much lower, and leave open reducing the upper bound further.

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