On the Capacity of the Diamond Half-Duplex Relay Channel

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Abstract

We consider a diamond-shaped dual-hop communication system consisting a source, two parallel half-duplex relays and a destination. In a single antenna configuration, it has been previously shown that a two-phase node-scheduling algorithm, along with the decode and forward strategy can achieve the capacity of the diamond channel for a certain symmetric channel gains [1]. In this paper, we obtain a more general condition for the optimality of the scheme in terms of power resources and channel gains. In particular, it is proved that if the product of the capacity of the simultaneously active links are equal in both transmission phases, the scheme achieves the capacity of the channel.

I. INTRODUCTION

Half-duplex relays i.e., relays that can not transmit and receive at the same time, have recently attracted enormous attention due to their simplicity and cost efficiency. Some capacity results for the case of a single half-duplex relay are presented in [2]. To realize multiple-antenna benefits without increasing the weight and size of the equipments, multiple relays come into play. A simple model for investigating the potential benefits of the multiple relays is depicted in Fig. 1. The end-to-end capacity of the relay channel has been studied in [1], [3]–[7] and is referred to as diamond relay channel in [1]. Schein in [3] and [4]

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established upper and lower bounds on the capacity of the full-duplex diamond channel. Half-duplex relays have been considered in [1], [5]–[7]. Xue, and Sandhu in [1] proposed several communication schemes including multihop with spatial reuse, scale-forward, broadcast-multiaccess with common message, compress-forward, as well as hybrid ones. They assumed that there is no interference between the transmitting and receiving relays. However [5]–[7] considered such interference and used different names of two-way [5], [6] and successive [7] relaying, respectively for the multihop with spatial reuse protocol. In this work, we follow the set-up of [1] and assume there is no such interference.

In this paper, we are interested in situations where the simple strategy of successive relaying achieves the capacity of the diamond channel. Surprisingly, we prove that when the product of the capacity of the simultaneously active links in both transmission phases are equal, the scheme achieves the capacity. Note that this condition includes the result indicated in [1] as a special case.

A. Notations

Throughout this paper, boldface letters are used to denote vectors. $A \rightarrow B$ represents the link from node $A$ to node $B$. The notation also means: approaches to when the right hand side of $\rightarrow$ is a number. A circularly symmetric complex Gaussian random variable $Z$ with mean 0 and variance $\sigma^2$, is represented by $Z \sim \mathcal{CN}(0, \sigma^2)$.

II. Transmission Model over Diamond Relay Channel

In this work, a dual-hop wireless system depicted in Fig. I is considered. The model consists of a source ($S$), two parallel half-duplex relays ($Re_1$, $Re_2$) and a destination ($D$). The corresponding index for the nodes are 0, 1, 2, 3, as shown in the Fig. I. We assume that all the nodes are equipped with a single antenna. Also, due to the long distance or strong shadowing, no link is assumed between the source and the destination, as well as between the relays. The background noise at each receiver is considered to be additive white Gaussian noise (AWGN). The channel gain between node $i$ and $j$ is assumed to be constant and is represented by $g_{ij}$ as shown in the Fig. I. In addition, the channel state information knowledge at different nodes is as follows.
• Source knows all the channel gains (i.e., $g_{01}$, $g_{02}$, $g_{13}$ and $g_{23}$).
• The relay $i$ knows its inward and outward channel gains ($g_{0i}$ and $g_{i3}$).
• The destination knows its inward channel gains ($g_{13}$ and $g_{23}$).

For this model, the successive relaying scheme is performed in two stages:

1) In the first $\lambda$ portion of the transmission time, $S$ and $R_{e2}$ transmit to $R_{e1}$ and $D$, respectively.
2) In the remaining $(1 - \lambda)$ portion of the transmission time, $S$ and $R_{e1}$ transmit to $R_{e2}$ and $D$, respectively.

Note that the relays decode and then re-encode what they have received prior to their transmission. The other possible scheduling algorithm for the half-duplex diamond channel is called simultaneous relaying [7] or transmission policy I [1] and is as follows:

1) In the first $\lambda$ portion of the transmission time, $S$ broadcasts its data to both relays.
2) In the remaining $(1 - \lambda)$ portion of the transmission time, relays cooperatively transmit to $D$.

It is possible to combine both scheduling methods, however in this work, successive relaying is considered only. The received discrete-time complex baseband equivalent signals at $R_{e1}$

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1Scheduling is assumed to be done in advance.
2Total transmission time is $T$ time slots.
Re₂, and D are respectively given by

\[
\begin{align*}
Y_1[m] &= h_{01}X_0[m] + N_1[m] \\
Y_2[n] &= h_{02}X_0[n] + N_2[n] \\
Y_3[m] &= h_{23}X_2[m] + N_3[m] \\
Y_3[n] &= h_{13}X_1[n] + N_3[n],
\end{align*}
\]

(1)

where \(X_i\) and \(Y_j\) are the transmitted and received signals from and to node \(i\) and \(j\), respectively (see Fig. 1). \(m \in \{1, ..., \lambda T\}\) and \(n \in \{\lambda T + 1, ..., T\}\) denote the transmission time index corresponding to the two stages. \(h_{ij}\) is the complex channel coefficient and is connected to \(g_{ij}\) by \(g_{ij} = |h_{ij}|^2\). \(N_j\) is the noise at node \(j\) and \(N_j \sim \mathcal{CN}(0, \sigma_j^2)\), for \(j = 1, 2, 3\).

We assume average power constraint for \(S\), \(Re₁\) and \(Re₂\) and denote the constraints by \(P_S\), \(P_{Re₁}\) and \(P_{Re₂}\), respectively, i.e. for \(\lambda T\) tuple and \((1 - \lambda)T\) tuple sub-codewords, we have

\[
\begin{align*}
\frac{1}{\lambda T} \sum_{k=1}^{\lambda T} |x_{01}[k]|^2 &\leq P_S \\
\frac{1}{(1 - \lambda)T} \sum_{k=\lambda T+1}^{T} |x_{02}[k]|^2 &\leq P_S \\
\frac{1}{\lambda T} \sum_{k=1}^{\lambda T} |x_{23}[k]|^2 &\leq P_{Re₂} \\
\frac{1}{(1 - \lambda)T} \sum_{k=\lambda T+1}^{T} |x_{13}[k]|^2 &\leq P_{Re₁},
\end{align*}
\]

(2)

where \(x_{ij}[k]\) is the symbol corresponding to the \(k^{th}\) index, in the sub-codeword transmitted by node \(i\) to node \(j\). Note that instead of allocating \(P_S\) to both \(S \rightarrow Re₁\) and \(S \rightarrow Re₂\) links, we could allocate different powers to them i.e., \(\rho P_S\) and \(\mu P_S\) with the condition \(\rho \lambda + \mu (1 - \lambda) = 1\) for \(\rho \geq 0\) and \(\mu \geq 0\). However due to the simplicity of implementation and also exposition, we choose \(\rho = \mu = 1\). In addition, power budget at the relay nodes assumed to be fixed over time.

A rate \(R\) is said to be achievable for this scheme, if for \(T \rightarrow \infty\), D can decode the message with error probability \(\epsilon \rightarrow 0\).

\(^3\)It is assumed that \(\lambda T\) is an integer.
| State | R₁ | R₂ | Time |
|-------|----|----|------|
| S₁    | Rx | Rx | t₁   |
| S₂    | Tx | Rx | t₂   |
| S₃    | Rx | Tx | t₃   |
| S₄    | Tx | Tx | t₄   |

In the next three sections, we first derive the cut-set upper bound and the achievable rate using the successive relaying protocol, and then we state the condition for optimality of the scheme.

III. Cut-Set Upper Bound

The upper bound for half-duplex networks is given in [8] based on writing the well known cut-set bounds for different transmission states shown in table I and then adding them up. Tx and Rx denote transmit and receive modes in the table, respectively.

Applying the procedure to the diamond relay channel, we have [1]

\[
R_{C₁} = t₁I(X₀; Y₁, Y₂) + t₂I(X₀; Y₂) + t₃I(X₀; Y₁) + t₄.0
\]

\[
R_{C₂} = t₁I(X₀; Y₂) + t₂(I(X₀; Y₂) + I(X₁; Y₃)) + t₃.0 + t₄I(X₁; Y₃)
\]

\[
R_{C₃} = t₁I(X₀; Y₁) + t₂.0 + t₃(I(X₀; Y₁) + I(X₂; Y₃)) + t₄I(X₂; Y₃)
\]

\[
R_{C₄} = t₁.0 + t₂I(X₁; Y₃) + t₃I(X₂; Y₃) + t₄I(X₁, X₂; Y₃),
\]

where \(C_j\) for \(j = 1, ..., 4\) represents the \(j^{th}\) cut-set. The time-sharing coefficients \(tᵢ\)s satisfy \(\sum_{i=1}^{4} tᵢ = 1\). Using Gaussian codebooks and defining \(C_{ij} = \log_2(1 + g_{ij} \frac{P_{ij}}{σ_j^2})\) with \(P_{ij}\) as the
power allocated to that link, we have the following optimization problem:

\[
\max_{t_i \geq 0} R \quad s.t. \quad R \leq t_1 C_{012} + t_2 C_{02} + t_3 C_{01} + t_4.0
\]

\[
R \leq t_1 C_{02} + t_2 (C_{02} + C_{13}) + t_3.0 + t_4 C_{13}
\]

\[
R \leq t_1 C_{01} + t_2.0 + t_3 (C_{01} + C_{23}) + t_4 C_{23}
\]

\[
R \leq t_1.0 + t_2 C_{13} + t_3 C_{23} + t_4 C_{123}
\]

\[
\sum_{i=1}^{4} t_i = 1,
\]

where \( P_{01} = P_S, P_{02} = P_S, P_{13} = P_{R1}, P_{23} = P_{R2}, C_{012} = \log_2[1 + (\frac{g_{01}}{\sigma_1^2} + \frac{g_{02}}{\sigma_2^2})P_S], \) and \( C_{123} = \log_2[1 + \frac{1}{\sigma_3^2}(\sqrt{g_{13}P_{Re1}} + \sqrt{g_{23}P_{Re2}})^2]. \)

IV. Achievable Rate under Successive Relaying

It has been shown in [1] (Theorem 4.1) that the maximum achievable rate under this transmission policy, is given by

\[
R_{SR} = \max\{R_1, R_2\}
\]

\[
R_1 = \lambda_1 C_{01} + \min\{\lambda_1 C_{23}, (1 - \lambda_1)C_{02}\}
\]

\[
R_2 = \lambda_2 C_{23} + \min\{(1 - \lambda_2)C_{13}, \lambda_2 C_{01}\}
\]

with

\[
\lambda_1 = \frac{C_{13}}{C_{13} + C_{01}}
\]

\[
\lambda_2 = \frac{C_{02}}{C_{23} + C_{02}}.
\]

It has also been proved that the decode and forward scheme is the best forwarding scheme under the successive relaying protocol (Theorem 4.2 in [1]).

In [1], it is stated that for the case of \( P_S = P_{Re1} = P_{Re2} \), the scheme achieves the cut-set upper bound if the channel gains are such that \( C_{02} = C_{13} \) and \( C_{01} = C_{23} \) (corollary 4.3). Here we generalize the condition of optimality in theorem 1.
Before proving the theorem, it is noteworthy to find some special cases in the successive relaying scheme. Lemma 1 states such cases. In particular, We are interested in the cases where \( R_1 = R_2 \) in (5) and (6).

**Lemma 1** Assuming all links have non-zero capacity, the conditions for \( R_1 = R_2 \) are one of the followings:

\[
C_{01}C_{02} = C_{13}C_{23} \quad (8)
\]
\[
C_{01} = C_{02} \quad \text{if} \quad C_{01}C_{02} \leq C_{13}C_{23} \quad (9)
\]
\[
C_{13} = C_{23} \quad \text{if} \quad C_{01}C_{02} \geq C_{13}C_{23}. \quad (10)
\]

*Proof:* See [9]. \( \square \)

The rate in (4) associated with each condition of equality (8)-(10) is:

\[
R = \frac{C_{01}}{C_{13} + C_{01}}(C_{13} + C_{02}) \quad (11)
\]
\[
R = C_{02} \quad (12)
\]
\[
R = C_{13}. \quad (13)
\]

Now we want to check whether the special cases stated in (8)-(10) can meet the cut-set bound and hence achieve the capacity. Our numerical analysis show that the rates obtained for conditions (9) and (10) can not meet the bound. Theorem 1 states that the first case indeed meets the cut-set bound and hence gives the capacity of the diamond channel for the specified channel gains and power resources.

**Theorem 1** Assuming a diamond relay channel with non-zero capacity links, the successive relaying scheme achieves the capacity of the diamond channel if \( C_{01}C_{02} = C_{13}C_{23} \).

*Proof:*

The cut-set bounds can be written as
\[ R_{C_1} = t_1C_{012} + t_2C_{02} + t_3C_{01} + t_4.0 \]
\[ R_{C_2} = t_1C_{02} + t_2(C_{02} + C_{13}) + t_3.0 + t_4C_{13} \]
\[ R_{C_3} = t_1C_{01} + t_2.0 + t_3(C_{01} + C_{23}) + t_4C_{23} \]
\[ R_{C_4} = t_1.0 + t_2C_{13} + t_3C_{23} + t_4C_{123}. \]

(14)

We consider cuts 2 and 3 and try to maximize the minimum of these cuts by finding the best time-sharing vector \( t^* \triangleq (t^*_1, ..., t^*_4) \).

Lemma 2 The time-sharing vector that maximizes the minimum of the cuts 2 and 3, is obtained by choosing \( t^* = \left[ 0, \frac{C_{01}}{C_{13} + C_{01}}, \frac{C_{02}}{C_{02} + C_{23}}, 0 \right] \), and is in such a way that the cuts give the same rate.

Lemma 3 The cut-set rate of cuts 2 and 3 with the time-sharing vector \( t^* \) is
\[ \frac{C_{01}(C_{13} + C_{02})}{C_{13} + C_{01}}, \]
the same as the rate obtained in (11).

Now assume that we increase \( t^*_1 \) and \( t^*_4 \) from 0 to \( \epsilon \geq 0 \) and \( \eta \geq 0 \), respectively. To satisfy the constraint of (3), \( t^*_2 \) and \( t^*_3 \) have to be decreased by \( \gamma \) and \( \delta \), respectively. Note that one of the \( \gamma \) and \( \delta \) can be negative. To hold the condition (3), the following relation should exists between the adjustments

\[ \gamma + \delta = \epsilon + \eta. \]

(15)

Now let us calculate the rate change of the cuts 2 and 3. By considering \( C_{01}C_{02} = C_{13}C_{23} \), we have

\[ \Delta R_{C_2} = (C_{02}\epsilon - (C_{02} + C_{13})\gamma + C_{13}\eta) \]
\[ \Delta R_{C_3} = (C_{13}\epsilon - (C_{02} + C_{13})\delta + \eta)\frac{C_{01}}{C_{13}}, \]

(16) (17)

where \( \Delta R_{C_i} \) \( i = 1, ..., 4 \) is the rate difference between the rate obtained using the modified time-sharing vector and the rate acquired by the time-sharing vector \( t^* \). Using (15) and substituting \( \delta \) in (17), we have
\[ \Delta R_{C3} = (-C_{02} \epsilon + (C_{02} + C_{13}) \gamma - C_{13} \eta) \frac{C_{01}}{C_{13}}. \] (18)

Note that the signs of the rate changes in (16) and (18) are different. Considering the fact that with the given time-sharing vector in lemma 2 i.e. \( t^* \), we had \( R_{C2} = R_{C3} \), but with the adjustments actually we decrease the minimum of the rates which concludes the proof. It is interesting to see that the optimum time-sharing vector \( t^* \) makes all the cut-set bounds to be equal.

Therefore the successive relaying scheme achieves the capacity. \( \blacksquare \)

V. CONCLUSION

In this report, the condition for optimality of the successive relaying scheme in a diamond-shaped relay channel, has been generalized from \( C_{01} = C_{23} \) and \( C_{02} = C_{13} \) to a more general form of \( C_{01}C_{02} = C_{13}C_{23} \).

VI. ACKNOWLEDGEMENT

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I. INTRODUCTION

Half-duplex relays i.e., relays that can not transmit and receive at the same time, have recently attracted enormous attention due to their simplicity and cost efficiency. Some capacity results for the case of a single half-duplex relay are presented in [2]. To realize multiple-antenna benefits without increasing the weight and size of the equipments, multiple relays come into play. A simple model for investigating the potential benefits of the multiple relays is depicted in Fig. 1. The end-to-end capacity of the relay channel has been studied in [1], [3]–[7] and is referred to as diamond relay channel in [1]. Schein in [3] and [4] financial support provided by Nortel and the corresponding matching funds by the Natural Sciences and Engineering Research Council of Canada (NSERC), and Ontario Centres of Excellence (OCE) are gratefully acknowledged.
established upper and lower bounds on the capacity of the full-duplex diamond channel. Half-duplex relays have been considered in [1], [5]–[7]. Xue, and Sandhu in [1] proposed several communication schemes including multihop with spatial reuse, scale-forward, broadcast-multiaccess with common message, compress-forward, as well as hybrid ones. They assumed that there is no interference between the transmitting and receiving relays. However [5]–[7] considered such interference and used different names of two-way [5], [6] and successive [7] relaying, respectively for the multihop with spatial reuse protocol. In this work, we follow the set-up of [1] and assume there is no such interference.

In this paper, we are interested in situations where the simple strategy of successive relaying achieves the capacity of the diamond channel. Surprisingly, we prove that when the product of the capacity of the simultaneously active links in both transmission phases are equal, the scheme achieves the capacity. Note that this condition includes the result indicated in [1] as a special case.

A. Notations

Throughout this paper, boldface letters are used to denote vectors. $A \rightarrow B$ represents the link from node $A$ to node $B$. The notation also means: approaches to when the right hand side of $\rightarrow$ is a number. A circularly symmetric complex Gaussian random variable $Z$ with mean 0 and variance $\sigma^2$, is represented by $Z \sim \mathcal{CN}(0, \sigma^2)$.

II. TRANSMISSION MODEL OVER DIAMOND RELAY CHANNEL

In this work, a dual-hop wireless system depicted in Fig.[I] is considered. The model consists of a source (S), two parallel half-duplex relays ($Re_1$, $Re_2$) and a destination (D). The corresponding index for the nodes are 0, 1, 2, 3, as shown in the Fig. [I]. We assume that all the nodes are equipped with a single antenna. Also, due to the long distance or strong shadowing, no link is assumed between the source and the destination, as well as between the relays. The background noise at each receiver is considered to be additive white Gaussian noise (AWGN). The channel gain between node $i$ and $j$ is assumed to be constant and is represented by $g_{ij}$ as shown in the Fig. [I]. In addition, the channel state information knowledge at different nodes is as follows.
Fig. 1. The diamond relay channel

- Source knows all the channel gains (i.e., $g_{01}$, $g_{02}$, $g_{13}$ and $g_{23}$).
- The relay $i$ knows its inward and outward channel gains ($g_{0i}$ and $g_{i3}$).
- The destination knows its inward channel gains ($g_{13}$ and $g_{23}$).

For this model, the successive relaying scheme is performed in two stages:

1) In the first $\lambda$ portion of the transmission time, $S$ and $R_{e_2}$ transmit to $R_{e_1}$ and $D$, respectively.

2) In the remaining $(1 - \lambda)$ portion of the transmission time, $S$ and $R_{e_1}$ transmit to $R_{e_2}$ and $D$, respectively.

Note that the relays decode and then re-encode what they have received prior to their transmission. The other possible scheduling algorithm for the half-duplex diamond channel is called simultaneous relaying [7] or transmission policy I [1] and is as follows:

1) In the first $\lambda$ portion of the transmission time, $S$ broadcasts its data to both relays.

2) In the remaining $(1 - \lambda)$ portion of the transmission time, relays cooperatively transmit to $D$.

It is possible to combine both scheduling methods, however in this work, successive relaying is considered only. The received discrete-time complex baseband equivalent signals at $R_{e_1}$

\[^1\text{Scheduling is assumed to be done in advance.}\]
\[^2\text{Total transmission time is } T \text{ time slots.}\]
$Re_2$, and $D$ are respectively given by

\[
\begin{align*}
Y_1[m] &= h_{01} X_0[m] + N_1[m] \\
Y_2[n] &= h_{02} X_0[n] + N_2[n] \\
Y_3[m] &= h_{23} X_2[m] + N_3[m] \\
Y_3[n] &= h_{13} X_1[n] + N_3[n],
\end{align*}
\]

(1)

where $X_i$ and $Y_j$ are the transmitted and received signals from and to node $i$ and $j$, respectively (see Fig. 1). $m \in \{1, ..., \lambda T\}$ and $n \in \{\lambda T + 1, ..., T\}$ denote the transmission time index corresponding to the two stages. $h_{ij}$ is the complex channel coefficient and is connected to $g_{ij}$ by $g_{ij} = |h_{ij}|^2$. $N_j$ is the noise at node $j$ and $N_j \sim CN(0, \sigma_j^2)$, for $j = 1, 2, 3$.

We assume average power constraint for $S$, $Re_1$ and $Re_2$ and denote the constraints by $P_S$, $P_{Re_1}$ and $P_{Re_2}$, respectively, i.e. for $\lambda T$ tuple and $(1 - \lambda)T$ tuple sub-codewords, we have

\[
\begin{align*}
\frac{1}{\lambda T} \sum_{k=1}^{\lambda T} |x_{01}[k]|^2 &\leq P_S \\
\frac{1}{(1 - \lambda)T} \sum_{k=\lambda T+1}^{T} |x_{02}[k]|^2 &\leq P_S \\
\frac{1}{\lambda T} \sum_{k=1}^{\lambda T} |x_{23}[k]|^2 &\leq P_{Re_2} \\
\frac{1}{(1 - \lambda)T} \sum_{k=\lambda T+1}^{T} |x_{13}[k]|^2 &\leq P_{Re_1},
\end{align*}
\]

(2)

where $x_{ij}[k]$ is the symbol corresponding to the $k^{th}$ index, in the sub-codeword transmitted by node $i$ to node $j$. Note that instead of allocating $P_S$ to both $S \rightarrow Re_1$ and $S \rightarrow Re_2$ links, we could allocate different powers to them i.e., $\rho P_S$ and $\mu P_S$ with the condition $\rho\lambda + \mu(1 - \lambda) = 1$ for $\rho \geq 0$ and $\mu \geq 0$. However due to the simplicity of implementation and also exposition, we choose $\rho = \mu = 1$. In addition, power budget at the relay nodes assumed to be fixed over time.

A rate $R$ is said to be achievable for this scheme, if for $T \rightarrow \infty$, $D$ can decode the message with error probability $\epsilon \rightarrow 0$.

\footnote{It is assumed that $\lambda T$ is an integer.}
TABLE I
TRANSMISSION STATES IN DIAMOND RELAY CHANNEL.

| State | R1 | R2 | Time |
|-------|----|----|------|
| S1    | Rx | Rx | t1   |
| S2    | Tx | Rx | t2   |
| S3    | Rx | Tx | t3   |
| S4    | Tx | Tx | t4   |

In the next three sections, we first derive the cut-set upper bound and the achievable rate using the successive relaying protocol, and then we state the condition for optimality of the scheme.

III. CUT-SET UPPER BOUND

The upper bound for half-duplex networks is given in [8] based on writing the well known cut-set bounds for different transmission states shown in table I and then adding them up. Tx and Rx denote transmit and receive modes in the table, respectively.

Applying the procedure to the diamond relay channel, we have [1]

\[
R_{C_1} = t_1 I(X_0; Y_1, Y_2) + t_2 I(X_0; Y_2) + t_3 I(X_0; Y_1) + t_4.0
\]

\[
R_{C_2} = t_1 I(X_0; Y_2) + t_2 (I(X_0; Y_2) + I(X_1; Y_3)) + t_3.0 + t_4 I(X_1; Y_3)
\]

\[
R_{C_3} = t_1 I(X_0; Y_1) + t_2.0 + t_3 (I(X_0; Y_1) + I(X_2; Y_3)) + t_4 I(X_2; Y_3)
\]

\[
R_{C_4} = t_1.0 + t_2 I(X_1; Y_3) + t_3 I(X_2; Y_3) + t_4 I(X_1, X_2; Y_3),
\]

where \( C_j \) for \( j = 1, \ldots, 4 \) represents the \( j^{th} \) cut-set. The time-sharing coefficients \( t_i s \) satisfy \( \sum_{i=1}^{4} t_i = 1 \). Using Gaussian codebooks and defining \( C_{ij} = \log_2(1 + g_{ij} \frac{P_i}{\sigma_j^2}) \) with \( P_i \) as the
power allocated to that link, we have the following optimization problem:

$$\max_{t_i \geq 0} R \quad \text{s.t.} \quad R \leq \sum_{i=1}^{4} t_i = 1,$$

(3)

where $P_{01} = P_S$, $P_{02} = P_S$, $P_{13} = P_{R_1}$, $P_{23} = P_{R_2}$, $C_{012} = \log_2[1 + \frac{g_{01}}{\sigma_1^2} + \frac{g_{02}}{\sigma_2^2} P_S]$, and $C_{123} = \log_2[1 + \frac{1}{\sigma_3^2} (\sqrt{g_{13} P_{R_{e1}}} + \sqrt{g_{23} P_{R_{e2}}})^2]$.

IV. ACHIEVABLE RATE UNDER SUCCESSIVE RELAYING

It has been shown in [1] (Theorem 4.1) that the maximum achievable rate under this transmission policy, is given by

$$R_{SR} = \max\{R_1, R_2\}$$

(4)

$$R_1 = \lambda_1 C_{01} + \min\{\lambda_1 C_{23}, (1 - \lambda_1) C_{02}\}$$

(5)

$$R_2 = \lambda_2 C_{23} + \min\{(1 - \lambda_2) C_{13}, \lambda_2 C_{01}\}$$

(6)

with

$$\lambda_1 = \frac{C_{13}}{C_{13} + C_{01}},$$

$$\lambda_2 = \frac{C_{02}}{C_{23} + C_{02}}.$$  

(7)

It has also been proved that the decode and forward scheme is the best forwarding scheme under the successive relaying protocol (Theorem 4.2 in [1]).

In [1], it is stated that for the case of $P_S = P_{R_{e1}} = P_{R_{e2}}$, the scheme achieves the cut-set upper bound if the channel gains are such that $C_{02} = C_{13}$ and $C_{01} = C_{23}$ (corollary 4.3). Here we generalize the condition of optimality in theorem 1.
Before proving the theorem, it is noteworthy to find some special cases in the successive relaying scheme. Lemma 1 states such cases. In particular, We are interested in the cases where \( R_1 = R_2 \) in (5) and (6).

**Lemma 1** Assuming all links have non-zero capacity, the conditions for \( R_1 = R_2 \) are one of the followings:

\[
\begin{align*}
C_{01}C_{02} & = C_{13}C_{23} \\
C_{01} & = C_{02} \quad \text{if} \quad C_{01}C_{02} \leq C_{13}C_{23} \\
C_{13} & = C_{23} \quad \text{if} \quad C_{01}C_{02} \geq C_{13}C_{23}
\end{align*}
\]  

(8) (9) (10)

*Proof:* See [9].

The rate in (4) associated with each condition of equality (8)-(10) is:

\[
\begin{align*}
R & = \frac{C_{01}}{C_{13} + C_{01}}(C_{13} + C_{02}) \\
R & = C_{02} \\
R & = C_{13}
\end{align*}
\]  

(11) (12) (13)

Now we want to check whether the special cases stated in (8)-(10) can meet the cut-set bound and hence achieve the capacity. Our numerical analysis show that the rates obtained for conditions (9) and (10) can not meet the bound. Theorem 1 states that the first case indeed meets the cut-set bound and hence gives the capacity of the diamond channel for the specified channel gains and power resources.

**Theorem 1** Assuming a diamond relay channel with non-zero capacity links, the successive relaying scheme achieves the capacity of the diamond channel if \( C_{01}C_{02} = C_{13}C_{23} \).

*Proof:*

The cut-set bounds can be written as
\begin{align*}
R_{C_1} &= t_1C_{012} + t_2C_{02} + t_3C_{01} + t_4.0 \\
R_{C_2} &= t_1C_{02} + t_2(C_{02} + C_{13}) + t_3.0 + t_4C_{13} \\
R_{C_3} &= t_1C_{01} + t_2.0 + t_3(C_{01} + C_{23}) + t_4C_{23} \\
R_{C_4} &= t_1.0 + t_2C_{13} + t_3C_{23} + t_4C_{123}
\end{align*}

(14)

We consider cuts 2 and 3 and try to maximize the minimum of these cuts by finding the best time-sharing vector \( t^* \triangleq (t^*_1, ..., t^*_4) \).

**Lemma 2** The time-sharing vector that maximizes the minimum of the cuts 2 and 3, is obtained by choosing \( t^* = [0, \frac{C_{01}}{C_{13} + C_{01}}, \frac{C_{02}}{C_{02} + C_{23}}, 0] \), and is in such a way that the cuts give the same rate.

**Lemma 3** The cut-set rate of cuts 2 and 3 with the time-sharing vector \( t^* \) is \( \frac{C_{01}(C_{13} + C_{02})}{C_{13} + C_{01}} \), the same as the rate obtained in (11).

Now assume that we increase \( t^*_1 \) and \( t^*_4 \) from 0 to \( \epsilon \geq 0 \) and \( \eta \geq 0 \), respectively. To satisfy the constraint of (3), \( t^*_2 \) and \( t^*_3 \) have to be decreased by \( \gamma \) and \( \delta \), respectively. Note that one of the \( \gamma \) and \( \delta \) can be negative. To hold the condition (3), the following relation should exists between the adjustments

\[ \gamma + \delta = \epsilon + \eta \]

(15)

Now let us calculate the rate change of the cuts 2 and 3. By considering \( C_{01}C_{02} = C_{13}C_{23} \), we have

\begin{align*}
\Delta R_{C_2} &= (C_{02}\epsilon - (C_{02} + C_{13})\gamma + C_{13}\eta) \quad \text{(16)} \\
\Delta R_{C_3} &= (C_{13}\epsilon - (C_{02} + C_{13})\delta + \eta)\frac{C_{01}}{C_{13}} \quad \text{(17)}
\end{align*}

where \( \Delta R_{C_i} i = 1, ..., 4 \) is the rate difference between the rate obtained using the modified time-sharing vector and the rate acquired by the time-sharing vector \( t^* \). Using (15) and substituting \( \delta \) in (17), we have
\[ \Delta R_{C3} = (-C_{02}\epsilon + (C_{02} + C_{13})\gamma - C_{13}\eta)\frac{C_{01}}{C_{13}}. \]  

(18)

Note that the signs of the rate changes in (16) and (18) are different. Considering the fact that with the given time-sharing vector in lemma 2, i.e. \( t^* \), we had \( R_{C2} = R_{C3} \), but with the adjustments actually we decrease the minimum of the rates which concludes the proof. It is interesting to see that the optimum time-sharing vector \( t^* \) makes all the cut-set bounds to be equal.

Therefore the successive relaying scheme achieves the capacity. \( \square \)

V. Conclusion

In this report, the condition for optimality of the successive relaying scheme in a diamond-shaped relay channel, has been generalized from \( C_{01} = C_{23} \) and \( C_{02} = C_{13} \) to a more general form of \( C_{01}C_{02} = C_{13}C_{23} \).

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