Camassa–Holm equations and vortexons for axisymmetric pipe flows

Francesco Fedele and Denys Dutykh

1 School of Civil and Environmental Engineering & School of Electrical and Computer Engineering, Georgia Institute of Technology, Atlanta GA, USA
2 University College Dublin, School of Mathematical Sciences, Belfield, Dublin 4, Ireland
3 LAMA, UMR 5127 CNRS, Université de Savoie, Campus Scientifique, F-73376 Le Bourget-du-Lac Cedex, France

E-mail: fedele@gatech.edu and Denys.Dutykh@ucd.ie

Received 28 November 2012, in final form 20 September 2013
Published 18 November 2013
Communicated by E Knobloch

Abstract
In this paper, we study the nonlinear dynamics of an axisymmetric disturbance to the laminar state in non-rotating Poiseuille pipe flows. In particular, we show that the associated Navier–Stokes equations can be reduced to a set of coupled dispersive Camassa–Holm type equations. These support inviscid and smooth localized travelling waves, which are numerically computed using the Petviashvili method. In physical space they correspond to localized toroidal vortices that concentrate near the pipe boundaries (wall vortexons) or wrap around the pipe axis (centre vortexons) in agreement with the analytical soliton solutions derived by Fedele (2012 Fluid Dyn. Res. 44 45509) for small and long-wave disturbances. Inviscid singular vortexons with discontinuous radial velocities are also numerically discovered as associated to special travelling waves with a wedge-type singularity, viz. peakons. Their existence is confirmed by an analytical solution of exponentially shaped peakons that is obtained for the particular case of the uncoupled Camassa–Holm equations. The evolution of a perturbation is also investigated using an accurate Fourier-type spectral scheme. We observe that an initial vortical patch splits into a centre vortexon radiating vorticity in the form of wall vortexons. These can under go further splitting before viscosity dissipates them, leading to a slug of centre vortexons. The splitting process originates from a radial flux of azimuthal vorticity from the wall to the pipe axis in agreement with Eyink (2008 Physica D 237 1956–68). The inviscid and smooth vortexon is similar to the nonlinear neutral structures derived by Walton (2011 J. Fluid Mech. 684 284–315) and it may be a precursor to puffs and slugs observed at transition, since most likely it is unstable to non-axisymmetric disturbances.
1. Introduction

Transition to turbulence in non-rotating pipe flows is triggered by finite-amplitude perturbations (Hof et al. 2003), and the coherent structures observed at the transitional stage are in the form of localized patches known as puffs and slug structures (Wyganski and Champagne 1973, Wyganski et al. 1975). Puffs are spots of vorticity localized near the pipe axis surrounded by laminar flow, whereas slugs expand through the entire cross-section of the pipe while developing along the streamwise direction. Recent theoretical studies related slug flows to quasi-inviscid solutions of the Navier–Stokes (NS) equations. In particular, for non-axisymmetric flows (Smith and Bodonyi 1982) revealed the existence of nonlinear neutral structures localized near the pipe axis (centre modes) that are unstable equilibrium states (Walton 2005). Walton (2011) found the axisymmetric analogue of these inviscid travelling waves (TW) by studying the nonlinear stability of impulsively started pipe flows to axisymmetric perturbations. Walton’s modes are similar to the inviscid axisymmetric slug structures proposed by Smith et al. (1990).

Recently, Fedele (2012) investigated the dynamics of non-rotating axisymmetric pipe flows in terms of TW of nonlinear soliton bearing equations. He showed that at high Reynolds numbers, the dynamics of small long-wave perturbations of the laminar flow obey a coupled system of nonlinear Korteweg–de Vries-type (KdV) equations. These set of equations generalize the one-component KdV model derived by Leibovich (1968) (see also Leibovich (1969, 1984)) to study propagation of waves along the core of concentrated vortex flows (see also Benney (1966)) and vortex breakdown (Leibovich 1984). Fedele’s coupled KdV equations support inviscid soliton and periodic wave solutions in the form of toroidal vortex tubes, hereafter referred to as vortexons, which are similar to the inviscid nonlinear neutral centre modes found by Walton (2011). Fedele’s vortical structures eventually slowly decay due to viscous dissipation on the time scale $t \sim O(Re^{6.25})$ (Fedele 2012). The vortexon, Walton’s neutral mode and the inviscid axisymmetric slug proposed by Smith et al. (1990) are similar to the slugs of vorticity that have been observed in both experiments (Wyganski and Champagne 1973) and numerical simulations (Willis and Kerswell 2009). As discussed by Walton (2011), these inviscid structures may play a role in pipe flow transition as precursors to puffs and slugs, since most likely they are unstable to non-axisymmetric disturbances (Walton 2005).

In this paper, we extend Fedele’s analysis and show that the axisymmetric NS equations for non-rotating pipe flows can be reduced to a set of soliton bearing equations of Camassa–Holm (CH) type (Camassa and Holm 1993, Dullin et al. 2003). These support smooth and inviscid solitary waves that are numerically computed using the Petviashvili method (Petviashvili 1976), see also (Pelinovsky and Stepanyants 2004, Lakoba and Yang 2007, Yang 2010) confirming the validity of the theoretical solutions derived by Fedele (2012) for long-wave disturbances. Moreover, inviscid singular solitary waves in the form of peakons are numerically discovered, and the interpretation of the associated vortical structures is discussed. Finally, the evolution of a perturbation to the laminar state is investigated within the framework of the proposed soliton equations.

2. Camassa–Holm type equations for pipe flows

Consider the axisymmetric flow of an incompressible fluid in a pipe of circular cross section of radius $R$ driven by an imposed uniform pressure gradient. Define a cylindrical coordinate system $(r, \theta, z)$ with the $z$-axis along the streamwise direction, and $(u, v, w)$ as the radial, azimuthal and streamwise velocity components. The time, radial and streamwise lengths as
well as velocities are rescaled with $T, R$ and $U_0$; respectively. Here, $T = R / U_0$ is a convective time scale and $U_0$ is the maximum laminar flow velocity. The Stokes stream function $\psi$ of a perturbation $\left( u = -r^{-1} \partial_r \psi, w = r^{-1} \partial_\psi \psi \right)$ to the laminar base flow $W_0(r) = 1 - r^2$ satisfies the nonlinear equation (Itoh 1977)

$$\partial_t \psi + W_0 \partial_r \psi - \frac{1}{Re} \partial_r^2 \psi = N(\psi),$$

where the nonlinear differential operator

$$N(\psi) = -\frac{1}{r} \partial_r \psi \partial_r \psi + \frac{1}{r} \partial_r \psi \partial_r \psi - \frac{2}{r^2} \partial_\psi \psi \partial_r \psi,$$

the linear operator

$$L = \mathcal{L} + \partial_{zz}, \quad \mathcal{L} = \partial_{rr} - \frac{1}{r} \partial_r = r \partial_r \left( \frac{1}{r} \partial_r \right),$$

and $Re$ is the Reynolds number based on $U_0$ and $R$. The boundary conditions for (1) reflect the boundedness of the flow at the centreline of the pipe and the no-slip condition at the wall, that is $\partial_r \psi = \partial_\psi \psi = 0$ at $r = 1$.

Drawing from Fedele (2012), the solution of (1) can be given in terms of a complete set of orthonormal basis $\{ \phi_j(r) \}$ as

$$\psi(r, z, t) = \sum_{j=1}^{\infty} \phi_j(r) B_j(z, t),$$

where $B_j$ is the amplitude of the radial eigenfunctions $\phi_j$, which satisfy the boundary value problem (BVP) (Fedele et al. 2005, Fedele 2012)

$$\mathcal{L}^2 \phi_j = -\lambda_j^2 \mathcal{L} \phi_j,$$

with $r^{-1} \phi_j$ and $r^{-1} \partial_r \phi_j$ bounded at $r = +0$, and $\phi_j = \partial_\psi \phi_j = 0$ at $r = 1$. Since $\phi_j$ satisfies the pipe flow boundary conditions a priori, so does $\psi$ of (2). Note that the vorticity of the velocity field associated to the truncated expansion for $\psi$ is divergence-free. The positive eigenvalues $\lambda_j$ are the roots of $J_2(\lambda_j) = 0$, where $J_2(r)$ are the Bessel functions of first kind of second-order (see Abramowitz and Stegun 1972). The corresponding eigenfunctions

$$\phi_n = \frac{\sqrt{2}}{\lambda_n} \left[ \frac{r^2 - r J_1(\lambda_n r)}{J_1(\lambda_n)} \right],$$

form a complete and orthonormal set with respect to the inner product

$$\langle \varphi_1, \varphi_2 \rangle = -\int_0^1 \varphi_1 \mathcal{L} \varphi_2 r^{-1} \, dr = \int_0^1 \partial_r \varphi_1 \partial_r \varphi_2 \, dr.$$

A Galerkin projection of (1) onto the vector space $\mathcal{S}$ spanned by the first $N$ least stable modes $\{ \phi_j \}_{j=1}^N$ yields a set of coupled CH type equations (Camassa and Holm 1993, Dullin et al. 2003, 2004)

$$\partial_t B_j + c_{jm} \partial_z B_m + \beta_{jm} \partial_{zz} B_m + \alpha_{jm} \partial_{zzz} B_m + N_{jm}(B_n, B_m) + \frac{\lambda_j^2 B_j}{Re} = 0,$$

where $j = 1, \ldots, N$, the nonlinear operator

$$N_{jm}(B_n, B_m) = F_{jm} B_n \partial_z B_m + G_{jm} \partial_z B_n \partial_z B_m + H_{jm} B_n \partial_{zz} B_m,$$

and $j = 1, \ldots, N$. The nonlinear operator

$$N_{jm}(B_n, B_m) = F_{jm} B_n \partial_z B_m + G_{jm} \partial_z B_n \partial_z B_m + H_{jm} B_n \partial_{zz} B_m,$$
the coefficients $c_{jm}$, $\beta_{jm}$, $\alpha_{jm}$, $F_{jnm}$, $G_{jnm}$ and $H_{jnm}$ are given in A and summation over repeated indices $n$ and $m$ is implicitly assumed. A physical interpretation of the CH equation (4) is as follows: the perturbation is given by a superposition of radial structures (the eigenmodes $\phi_j$) that nonlinearly interact while they are advected and dispersed by the laminar flow in the streamwise direction.

Note that CH type equations arise also as a regularized model of the three-dimensional NS equations (Chen et al 1999, Domaradzki et al 2001, Foias et al 2001, 2002), the so-called NS-alpha model.

3. Is there wave dispersion in axisymmetric Navier–Stokes flows?

The Galerkin projection described above yields the dispersive CH type equation (4) for the space–time evolution of the stream function $\psi$. The term $\partial_{txxx} \psi$ arises also in the Benjamin–Bona–Mahony (BBM) equation (Benjamin et al 1972). It has the property to suppress dispersion, attenuating the dispersive effects induced by the KdV term $\partial_{xxx} \psi$.

Indeed, consider the linear equation with both BBM and KdV dispersion

$$A_t - \alpha A_{xxx} + c A_x + \gamma A_{xxx} = 0. $$

The associated linear phase speed of a Fourier wave $e^{i(kx - \omega t)}$ is

$$C(k) = \frac{\omega}{k} = \frac{c - \gamma k^2}{1 + \alpha k^2} $$

and as $k \to \infty$, $C_0 \to -\gamma/\alpha$. This implies that Fourier waves with large wavenumbers tend to travel at the same speed, that is dispersion is suppressed at high $k$’s, if $\alpha \neq 0$. As a result, self-steepening induced by nonlinearities can become dominant and blow-up is possible in finite time, or the two contrasting effects can balance each other leading to a peakon solution (Dullin et al 2001, 2003). The extreme case of dispersion suppression is when there is no dispersion, that is $C(k) = c$, as in the dispersionless CH equation, which also admits peakons (Camassa and Holm 1993). Clearly, if one adds a fifth-order dispersion term $A_{xxxxx}$, then $C(k)$ will grow as $k \to \infty$, and peakons do not exist since dispersion is too strong.

The CH/KdV dispersion is associated to a hidden ‘elastic energy’ that has no counterpart in axisymmetric NS flows, which are essentially two-dimensional (2D) since vortex stretching is absent. To understand the physical origin of such wave dispersion, we consider the 2D Euler equations for an inviscid fluid over the domain $\Omega$ in cartesian coordinates. The divergent-free velocity field is given by

$$v = \left( -\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x} \right),$$

where $\psi$ is the stream function and the vorticity

$$\omega = \Delta \psi.$$

The equation of motion is

$$\frac{\partial \omega}{\partial t} = -v \cdot \nabla \omega = -[\psi, \omega], \quad (6)$$

where the commutator

$$[f, g] = \frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial g}{\partial x}.$$
It will be useful to consider the Hamiltonian formulation of (6). Following Morrison (1998), this is given by
\[
\frac{\partial \omega}{\partial t} = \{\omega, H\},
\]
where
\[
H = \frac{1}{2} \int_{\Omega} |\nabla \psi|^2 \, d\Omega = -\frac{1}{2} \int_{\Omega} \omega \psi \, d\Omega.
\]
is the kinetic energy of the system and the non-canonical Lie–Poisson bracket is defined as
\[
\{F, G\} = \int_{\Omega} \omega \left[ \frac{\delta F}{\delta \omega} \frac{\delta G}{\delta \omega} \right] \, d\Omega,
\]
where \(\delta\) denotes variational derivative. Clearly, \(H\) is an invariant of motion because of the anti-symmetry of the Poisson bracket, i.e. \(\{F, G\} = -\{G, F\}\). The Hamiltonian structure of (7) yields a physical interpretation of the fluid motion in terms of a deformation of a 2D membrane. Indeed, the Hamiltonian \(H\) can be interpreted as the elastic energy of a membrane subject to tensional forces. The surface \(\psi(x, y)\) represents the displacements of the deformed membrane and the vorticity \(\omega\) is proportional to the mean curvature \(\kappa\) of \(\psi\). This changes according to (7), while the elastic energy \(H\) is kept invariant. As the curvature \(\kappa\) evolves in space and time, viz. vorticity is swept around \(\Omega\) and changes in time, the surface \(\psi\) locally bends sharply if \(\kappa\) increases or flattens if \(\kappa\) decreases. Since the velocity streamlines are the contours of \(\psi\), this implies that the vortical flow intensifies (attenuates) in regions of high (low) curvature of \(\psi\).

The wave dispersion associated to the ‘elastic energy’ \(H\) can be revealed if we express (6) solely in terms of \(\psi\), that is
\[
\partial_t \Delta \psi = -\partial_y \psi \partial_x \Delta \psi + \partial_x \psi \partial_y \Delta \psi.
\]
Here, the left-hand side yields the terms \(\partial_{xx} \psi\) and \(\partial_{yy} \psi\) that are typical of the CH equation. They indicate that as the vorticity changes in time, so does the curvature \(\kappa\) of the surface \(\psi\), which elastically deforms while the ‘energy’ is conserved. If the velocity field is given by the sum of a base flow and a perturbation, then KdV type dispersive terms \(\partial_{xx} \psi\) and \(\partial_{yy} \psi\) arise from the convection of the perturbation by the mean flow.

The NS-alpha model can be interpreted in a similar manner (Foias et al 2001). This is given by
\[
\partial_t \mathbf{V} + \mathbf{U} \cdot \nabla \mathbf{V} + \nabla \mathbf{U}^T \cdot \mathbf{V} + \nabla p = \nu \Delta \mathbf{V}
\]
\[
\nabla \cdot \mathbf{U} = 0,
\]
and \(\mathbf{V} = (1 - \alpha^2 \Delta) \mathbf{U}\). The typical CH terms arise from \(\partial_t \Delta \mathbf{U}\). If \(\mathbf{U}\) is the sum of a base flow and a perturbation, then KdV type dispersive terms arise as well.

4. Long-wave limit and KdV vortexons

As \(Re \to \infty\), Fedele (2012) showed that the nonlinear dynamics of a small long-wave perturbation \(b_j = \varepsilon B_j\), with \(\varepsilon \sim O(Re^{-2/3})\), can be reduced to that on the slow manifold of the laminar state spanned by the first few \(N\) least stable modes, and higher damped modes are neglected. This is legitimate as long as the amplitudes \(B_j\) remain small for all time and
the non-resonant condition
\[ \lambda_{i_1}^2 + \lambda_{i_2}^2 + \cdots + \lambda_{i_k}^2 \neq \lambda_j^2 \] is satisfied for any permutation \( \{i_1, i_2, \ldots, i_k\} \) of size \( k \leq N \) drawn from the set \( j = 1, \ldots, N \) (de la Llave 1997). For the BVP of (3) the relation (8) is verified numerically to hold up to \( N \approx 10^4 \). For time scales much less than \( t \approx O(\varepsilon^{-2.5}) \approx O(Re^{0.25}) \), the nonlinear dynamics of (4) is primary inviscid and obeys a set of coupled KdV equations (Fedele 2012)

\[ \partial_t b_j + \tilde{p}_{jm} \partial_{\xi \xi} b_j + \tilde{F}_{jm} b_m \partial_j b_m = 0, \]

defined on the stretched reference frame
\[ \xi = \frac{\varepsilon^{1/2}}{2}(z - Vt), \quad \tau = \varepsilon^{3/2} t, \]
Figure 2. Inviscid regular wall vortexon: (top) streamlines of the three-component CH solution of figure 1 and (bottom) velocity profiles of the perturbed (solid) and laminar (dash) flows ($c = 0.65, q = 0$).

where the tensors $\tilde{P}_{jm}, \tilde{F}_{jnm}$ are given in Fedele (2012) and the celerity $V$ is, with good approximation, the average of the eigenvalues of $c_{jm}$. The nonlinear system (9) support analytical TW, for example,

$$b_j^{(w)}(\xi, \tau) = k^2 x_j \left[ -\frac{2M^2 - 1}{3M^2} + \frac{2}{\text{cn}(k\xi)} \right],$$

(10)
Figure 3. Inviscid regular wall vortexon: (top) streamlines of the three-component CH solution for $c = 0.78$, $q = 0$, and (bottom) velocity profiles of the perturbed (solid) and laminar (dash) flows.

where $cn(\zeta)$ is the Jacobi elliptic function with modulus $0 \leq M \leq 1$, $k$ and $M$ are free parameters and $\{x_j\} \in \mathbb{R}^J$ is the intersection point of $J$ hyperconics $\Gamma_j$ given by

$$-12M^2\beta_{jj}x_j + \tilde{F}_{jnm}x_nx_m = 0, \quad j = 1, \ldots, N.$$
For $M \to 1$, (10) reduces to the family of localized sech-type solitary waves
\[ b_j^{(s)}(\xi, \tau) = -\frac{1}{2}k^2 x_j + k^2 x_j \text{sech}^2(k\xi). \] (11)

In physical space, (10) and (11) represent, respectively, localized and periodic toroidal vortices, which travel slightly slower than the maximum laminar flow speed $U_0$, viz.
Inviscid singular vortexon: wave components $B_1$, $B_2$ and $B_3$ of the CH equations ($c = 0.90$, $q = 0.025$).

In the following we will compute numerically TWs of the inviscid CH-type equation (4) and discuss the vortical structure of the associated disturbances.

5. Regular and singular inviscid vortexons

Consider the inviscid three-component CH equation (4) with $N = 3$, and an ansatz for the wave amplitudes of the form $B_j = q + F_j(z - ct)$, where $q$ is a free parameter and $c$ is the velocity of the TW. The associated nonlinear steady problem for $F_j$ (in the moving frame
Figure 6. Inviscid singular vortexon: streamlines of the three-component CH solution of figure 5 for $c = 0.90$, $q = 0.025$.

$z - ct$ is solved using the Petviashvili method (Petviashvili 1976), see also (Pelinovsky and Stepanyants 2004, Lakoba and Yang 2007, Yang 2010). This numerical approach has been successfully applied to derive TWs of the spatial Dysthe equation (Fedele and Dutykh 2011) and the compact Zakharov equation for water waves (Fedele and Dutykh 2012). To initialize the iterative process, the initial guess for the wave components $B_j$ is set equal to the analytical cnoidal TW of the uncoupled KdV equations associated to (4), viz. $c_{jm} \approx c_{jj}$, $F_{jnm} \approx F_{jjj}$, and $\alpha_{jm} = G_{jnm} = H_{jnm} = 0$. Then, a converged solution is numerically continued by varying the parameters $c$ or $q$. Note that the parameter that controls the strength of the nonlinearity in the truncated CH equations is the travelling wave amplitude.

The numerical basin of attraction of the Petviashvili scheme to localized TWs (solitons or solitary waves) is very sparse over the parameter space $(c, q)$. The generic topology of the flow structure associated to converged smooth TWs is the same as that of the theoretical counterpart derived by Fedele (2012): toroidal tubes of vorticity localized near the pipe boundaries (wall vortexons) or that wrap around the pipe axis (centre vortexons). In particular, wall vortexons are found in parameter window $c \sim [0.58, 0.66]$ and $q = 0$, however, the Petviashvili scheme did not converge for $q > 0$. For example, for $c = 0.65$ the wave components $B_j$ are shown in figure 1 and the streamlines of the associated flow perturbation are reported in the top panel of figure 2. The perturbed flow (laminar plus vortexon) is shown in the bottom panel of the same figure. Note that wave components of higher modes have smaller amplitudes as an indication that their effects may vanish as $N$ increases, but a more systematic numerical study of this trend is required.
Convergence to inviscid wall vortexons also occurred in the range of $c \sim [0.762, 0.79]$ and $q = 0$ (it did not converge for $q > 0$). For $c = 0.78$ the corresponding vortical structure is shown in figure 3. Centre vortexons converged for $c \sim [0.82, 0.90]$ and $q = 0$ as depicted in figure 4 ($c = 0.86$). In this range of values of $c$ we note that as $q$ increases from zero,
the smooth centre vortexon bifurcates to a travelling wave with a wedge-type singularity, viz. peakon, as shown in figure 5 for $c = 0.90$, $q = 0.025$. In physical space the peakon corresponds to a localized vortical structure with discontinuous radial velocity $u$ across $z - ct = 0$ (see figure 6), but continuous streamwise velocity $w$ since the mass flux through the pipe is conserved. As a result, a sheet of azimuthal vorticity is advected at speed $c$.

The Petviashvili method also converged to singular wall vortexons in the window $c \sim [0.69, 0.71]$ and only $q = 0$ as shown in figure 7 for the case of $c = 0.70$. The existence of singular vortexons is confirmed by an analytical solution of peakons obtained for the uncoupled version of the CH equation (4), viz.

$$\partial_t B_j + c_j \partial_z B_j + \beta_{jj} \partial_{zzz} B_j + \alpha_{jjj} \partial_{zzt} B_j + N_j(B_j) = 0,$$

(12)

where

$$N_j(B_j) = F_{jjj} B_j \partial_z B_j + G_{jjj} \partial_z B_j \partial_{zz} B_j + H_{jjj} B_j \partial_{zzz} B_j,$$

and here no implicit summation over repeated indices is assumed. Note that equation (12) is the dispersive CH equation with KdV dispersion, which admits peakon solutions (Dullin et al. 2003). These are given by (see appendix B for derivation)

$$B_j(z, t) = a_j e^{-\gamma_j |z-V_j t|},$$

(13)

where

$$a_j = \frac{V_j \alpha_{jj} - \beta_{jj}}{H_{jjj}}, \quad V_j = \frac{c_{jj} + \beta_{jj} s_j^2}{1 + \alpha_{jj} s_j^2}, \quad \gamma_j^2 = -\frac{F_{jjj}}{G_{jjj} + H_{jjj}}.$$
Figure 9. Inviscid singular vortexon associated to the peakon of figure 8: streamlines of the perturbation.

Note that the peakon arises as a special balance between the linear dispersion terms $\partial_{zzz}B_j$, $\partial_{zt}B_j$ and their nonlinear counterpart $B_j\partial_{zzz}B_j$ in (12). These three terms are interpreted in distributional sense because they give rise to derivatives of Dirac delta functions that must vanish by properly choosing the amplitude $a_j$, thus satisfying the differential equation (12) in the sense of distributions. The associated stream function $\psi^{(p)}_j$ is given by

$$\psi^{(p)}_j(r, z, t) = a_j e^{-\gamma_{j}|z-\nu_j t|} \phi_j(r).$$

For the least stable eigenmode $B_1$, figure 8 shows the remarkable agreement between the theoretical peakon (13) and the associated numerical solution obtained via the Petviashvili
method. The associated vortical structure (streamlines) is shown in figure 9 and it is similar to that of the numerical vortexons of figures 6 and 7.

Finally, note that viscous dissipation precludes the existence of peakons and slowly decaying smooth vortexons appear in the CH dynamics as discussed below.

6. Vortexon slugs

Hereafter, we investigate the dynamical evolution of a localized disturbance under the two-component CH dynamics with dissipation. To do so, we exploit a highly accurate Fourier-type pseudo-spectral method to solve the CH equation (4) as described in Fedele and Dutykh (2012). For $Re = 8000$ figure 10 depicts snapshots of the two-component CH solution at different times and the streamlines of the associated vortical structures are shown in figure 11. As time evolves, the waveform of each component steepens up and then splits into solitons and radiative waves as a result of the competition between the laminar-flow-induced wave dispersion and the nonlinear energy cascade associated to the CH nonlinearities. In physical space the initial vortical structure first compresses as a result of wave steepening and
then splits into a centre vortexon and patches of vorticity in the form of wall vortexons. These may further split causing the formation of new centre and wall vortexons until viscous effects attenuate them and annihilate splitting on the time scale $t \sim \mathcal{O}(Re^{0.25})$ (Fedele 2012). The formation of a vortexon slug is clearly evident in figure 12, in which we report the space–time plot of the difference $\beta = |B_1 - B_2|$ of the two wave components. Here, centre vortexons correspond to larger values of $\beta$ ($B_1$ and $B_2$ have opposite sign), whereas smaller values of are associated to wall vortexons ($B_1$ and $B_2$ have the same sign). The centre vortexon arises due to a radial flux $F^{\omega_\theta}_{yr} \approx u_\omega \omega_\theta$ of azimuthal vorticity $\omega_\theta$ from the wall to the pipe axis. This is the mechanism of inverse cascade of cross-stream vorticity in channel flows identified by Eyink (2008). Similar dynamics is also observed for long-wave disturbances associated to the KdV equation (9) (Fedele and Dutykh 2013).

Note that a vortexon slug is similar to the spreading of puffs in pipe turbulence at transition (Avila et al 2011), but they originate from different physical mechanisms. In realistic flows, a turbulent slug arises when new puffs are produced faster than their decay in the competition between puff decay (death) and puff splitting (birth) processes. Instead, a
vortexon slug arises as an inviscid competition between dispersion and nonlinear steepening of radial structures that are advected in the streamwise direction by the laminar flow.

Clearly, vortexon slugs are not the realistic slugs observed in experiments, which also have a non-axisymmetric component. However, similarly to the inviscid neutral modes found by Walton (2011), centre vortexons most likely are unstable to non-axisymmetric disturbances, and may persist viscous attenuation as precursors to puffs and slugs.

Finally, we note that observed vortex compression/splitting is also evident in the numerical simulations of the propagation of nonlinear Kelvin waves and fronts on the equatorial thermocline (Fedorov and Melville 1995, 2000). This is expected since the geostrophic flow is two dimensional in nature and the associated dynamical equations can be reduced to KdV/CH-type models (Benney 1966).
7. Concluding remarks

We have shown that the axisymmetric NS equations for non-rotating Poiseuille pipe flows can be reduced to a set of coupled CH type wave equations. These support inviscid and regular TW that are computed numerically using the Petviashvili method. The associated flow structures are localized toroidal vortices or vortexons that travel slightly slower than the maximum laminar flow speed, in agreement with the theoretical predictions by Fedele (2012). The vortical disturbance can be localized near the wall (wall vortexon) or wrap around the pipe axis (centre vortexon). Moreover, we also discovered numerically special TW with wedge-type singularities, viz. peakons, which bifurcate from smooth solitary waves. In physical space they correspond to localized toroidal vortical structures with discontinuous radial velocities (singular vortexon). The existence of such singular solutions is confirmed by an analytical solution of exponentially shaped peakons of the uncoupled wave equations. Clearly, the inviscid singular vortexon could be an artefact of the Galerkin truncation of the axisymmetric Euler equations that are projected onto the function space spanned by the first few Stokes eigenmodes. Viscous dissipation rules out the existence of peakons and the CH dynamics involves only regular vortexons. Indeed, we found numerically that an initial perturbation evolves into a vortexon slug, viz. a solitonic sea state of centre vortexons that split from patches of near-wall vorticity due to an inverse radial flux of azimuthal vorticity from the wall to the pipe axis in agreement with the cross-stream vorticity cascade of Eyink (2008).

Finally, we wish to emphasize the relevance of this work to the understanding of transition to turbulence. For chaotic dynamical systems the periodic orbit theory (POT) in (Cvitanović and Eckhardt 1991, Cvitanović 1995) interpret the turbulent motion as an effective random walk in state space where chaotic (turbulent) trajectories visit the neighbourhoods of equilibria, TWs or periodic orbits of the NS equations, jumping from one saddle to the other through their stable and unstable manifolds (Wedin and Kerswell 2004, Kerswell 2005, Gibson et al 2008). Non-rotating axisymmetric pipe flows do not exhibit chaotic behaviour (see, e.g. Patera and Orszag 1981, Willis and Kerswell 2008), and so the associated KdV or CH equations (even with dissipation). However, forced and damped KdV/CH equations are chaotic and the attractor is of finite dimension (see, e.g. Cox and Mortell 1986, Grimshaw and Tian 1994). Thus, the study of the reduced KdV–CH equations associated to forced axisymmetric NS equations using POT may provide new insights into understanding the nature of slug flows and their formation.

Acknowledgments

F F acknowledges the travel support received by the Geophysical Fluid Dynamics (GFD) Program to attend part of the summer school on ‘Spatially Localized Structures: Theory and Applications’ at the Woods Hole Oceanographic Institution in August 2012. D D acknowledges the support from ERC under the research project ERC-2011-AdG 290562-MULTIWA VE.

Appendix A. Coefficients in Camassa–Holm equations

\[
c_{jm} = -\int_{0}^{1} W_0 \phi_j \mathcal{L} \phi_m r^{-1} \, dr, \quad \alpha_{jm} = -\int_{0}^{1} \phi_j \phi_m r^{-1} \, dr, \quad \beta_{jm} = -\int_{0}^{1} W_0 \phi_j \phi_m r^{-1} \, dr, \\
F_{jnm} = -\int_{0}^{1} \phi_j \left[ \partial_r (\mathcal{L} \phi_m) - \partial_r (\mathcal{L} \phi_n) \, \phi_m + 2r^{-1} \mathcal{L} \phi_n \phi_m \right] r^{-2} \, dr.
\]
\[ H_{jmn} = - \int_0^1 \phi_j \phi_m \phi_n r^{-2} \, dr, \quad G_{jmn} = - \int_0^1 \phi_j \left[ -\phi_m \phi_n + 2r^{-1} \phi_n \phi_m \right] r^{-2} \, dr. \]

**Appendix B. Peakons of the dispersive CH equation**

To simplify the analysis, we drop the subscripts in (13) and consider

\[ B_t + \alpha B_{xxx} + c B_x + \beta B_x + F B_x + G B_x + H A A_{xxx} = 0. \]  

(B.1)

The ansatz for a peakon is

\[
B = \begin{cases} 
  a e^{-\gamma s(x-Vt)}, & s = 1, \quad x > Vt, \\
  a e^{-\gamma s(x-Vt)}, & s = -1, \quad x < Vt,
\end{cases}
\]

Substituting this into (B.1) yields

\[ e^{-\gamma s(x-Vt)} W_1 + e^{-2\gamma s(x-Vt)} W_2 = 0, \quad s = \pm 1, \]

where the coefficients \( W_j \) do not depend on \( s \) and are given by

\[ W_1 = -c + V - \gamma^2 (\beta - \alpha V), \quad W_2 = F + \gamma^2 (G + H). \]

Imposing \( W_1 = 0 \) and \( W_2 = 0 \) yield

\[ V = \frac{c + \beta \gamma^2}{1 + \alpha \gamma^2}, \quad \gamma^2 = -\frac{F}{G + H}. \]

Peakons exist if \( \gamma^2 > 0 \), but we still need to find their amplitude \( a \). To do so, let us consider the general ansatz

\[ B = R(\xi) = R(x - Vt), \]

where \( R \) follows from (B.1) and it satisfies

\[ -V R_\xi - \alpha V R_{\xi\xi\xi} + c R_\xi + \beta R_{\xi\xi} + F R_\xi + G R_\xi + H R_{\xi\xi} + H R_{\xi\xi\xi} = 0, \]

and subscripts denote derivatives with respect to \( \xi \). This can be written as

\[ \left( (c - V) R + (\beta - \alpha V + H R) R_{\xi\xi\xi} + FR_\xi + FR_\xi + (G + H) R_{\xi\xi} / 2 \right)_{\xi} = 0. \]  

(B.2)

Clearly, if a peakon exists the term \((\beta - \alpha V + H R) R_{\xi\xi\xi}\) must vanish at \( \xi = 0 \), or \( x = Vt \), because it is the only distributional term in (B.2) that yields derivatives of Dirac functions. Thus, the peakon amplitude \( a = R(\xi = 0) = \frac{\gamma_0}{\gamma^2} \).

**References**

Abramowitz M and Stegun I A 1972 *Handbook of Mathematical Functions* (New York: Dover)

Avila K et al 2011 The onset of turbulence in pipe flow *Science* 333 192–6

Benjamin T B et al 1972 Model equations for long waves in nonlinear dispersive systems *Phil. Trans. R. Soc. Lond. A* 272 47–78

Benney D J 1966 Long nonlinear waves in fluid flows *J. Math. Phys.* 45 52–63
Camassa R and Holm D 1993 An integrable shallow water equation with peaked solitons Phys. Rev. Lett. 71 1661–4
Chen S et al 1999 The Camassa–Holm equations and turbulence in pipes and channels Physica D 133 49–65
Cox E A and Mortell M P 1986 The evolution of resonant water-wave oscillations J. Fluid Mech. 162 99–116
Cvitanović P 1995 Dynamical averaging in terms of periodic orbits Physica D 83 109–23
Cvitanović P and Eckhardt B 1991 Periodic orbit expansions for classical smooth flows J. Phys. A: Math. Gen. 24 L237–41
de la Llave R 1997 ‘Invariant manifolds associated to nonresonant spectral subspaces J. Stat. Phys. 87 211–49
Domaradzki J A and Holm D 2001 Navier–Stokes-alpha model: LES equations with nonlinear dispersion Modern Simulation Strategies for Turbulent Flow ed B Geurts pp 107–22
Dullin H et al 2001 An Integrable shallow water equation with linear and nonlinear dispersion Phys. Rev. Lett. 87 194501
Dullin H et al 2003 Camassa–Holm, Korteweg–de Vries-5 and other asymptotically equivalent equations for shallow water waves Fluid Dyn. Res. 33 73–95
Dullin H R et al 2004 On asymptotically equivalent shallow water wave equations Physica D 190 1–14
Eyink G L 2008 Dissipative anomalies in singular Euler flows Physica D 237 1956–68
Fedele F 2012 Travelling waves in axisymmetric pipe flows Fluid Dyn. Res. 44 45509
Fedele F et al 2005 Revisiting the stability of pulsatile pipe flow Eur. J. Mech. B Fluids 24 237–54
Fedele F and Dutykh D 2011 Hamiltonian form and solitary waves of the spatial Dysthe equations JETP Lett. 94 84–90
Fedele F and Dutykh D 2012 Special solutions to a compact equation for deep-water gravity waves J. Fluid Mech. 712 646–60
Fedele F and Dutykh D 2013 Vortexons in axisymmetric Poiseuille pipe flows Europhys. Lett. 101 34003
Fedorov A V and Melville W K 1995 Propagation and breaking of nonlinear Kelvin waves J. Phys. Oceanogr. 25 2518–31
Fedorov A V and Melville W K 2000 Kelvin fronts on the equatorial thermocline J. Phys. Oceanogr. 30 1692–705
Foias C et al 2001 The Navier–Stokes-alpha model of fluid turbulence Physica D 152–153 505–19
Foias C et al 2002 The three dimensional viscous Camassa–Holm equations and their relation to the Navier–Stokes equations and turbulence theory J. Dyn. Differ. Eqns. 14 1–35
Gibson J F et al 2008 Visualizing the geometry of state space in plane Couette flow J. Fluid Mech. 611 107–30
Grimshaw R and Tian X 1994 Periodic and chaotic behaviour in a reduction of the perturbed Korteweg–de Vries equation Proc. R. Soc. Lond. A 445 1–21
Hof B et al 2003 Scaling of the turbulence transition threshold in a pipe Phys. Rev. Lett. 91 244502
Itoh N 1977 Nonlinear stability of parallel flows with subcritical Reynolds numbers. Part 2. Stability of pipe Poiseuille flow to finite axisymmetric disturbances J. Fluid Mech. 82 469–79
Kerswell R R 2005 Recent progress in understanding the transition to turbulence in a pipe Nonlinearity 18 R17–44
Lakoba T I and Yang J 2007 A generalized Petviashvili iteration method for scalar and vector Hamiltonian equations with arbitrary form of nonlinearity J. Comput. Phys. 226 1668–92
Leibovich S 1968 Axially symmetric eddies embedded in a rotational stream J. Fluid Mech. 32 529–48
Leibovich S 1969 Vortex breakdown in rotating fluids associated with wave motion along axis of rotation, considering effects of nonzero wave amplitude Plasma Dynamics Conf. (San Francisco, CA. 16–18 June 1969) (AIAA Paper 69–645) (New York: AIAA) p 10
Leibovich S 1984 Vortex stability and breakdown—survey and extension AIAA J. 22 1192–206
Morrison P J 1998 Hamiltonian description of the ideal fluid Rev. Mod. Phys. 70 467–521
Patera A T and Orszag S A 1981 Finite-amplitude stability of axisymmetric pipe flow J. Fluid Mech. 112 467–74
Pelinovsky D and Stepanyants Y A 2004 Convergence of Petviashvili’s iteration method for numerical approximation of stationary solutions of nonlinear wave equations SIAM J. Numer. Anal. 42 1110–27

Petviashvili V I 1976 Equation of an extraordinary soliton Sov. J. Plasma Phys. 2 469–72

Smith F T et al 1990 On displacement-thickness, wall-layer and mid-flow scales in turbulent boundary layers and slugs of vorticity in channel and pipe flows Proc. R. Soc. Lond. A 428 255–81

Smith F T and Bodonyi R J 1982 Amplitude-dependent neutral modes in the Hagen–Poiseuille flow through a circular pipe Proc. R. Soc. Lond. A 384 463–89

Walton A G 2005 The stability of nonlinear neutral modes in Hagen–Poiseuille flow Proc. R. Soc. Lond. A 461 813–24

Walton A G 2011 The stability of developing pipe flow at high Reynolds number and the existence of nonlinear neutral centre modes J. Fluid Mech. 684 284–315

Wedin H and Kerswell R R 2004 Exact coherent structures in pipe flow: travelling wave solutions J. Fluid Mech. 508 333–71

Willis A and Kerswell R 2008 Coherent structures in localized and global pipe turbulence Phys. Rev. Lett. 100 124501

Willis A P and Kerswell R R 2009 Turbulent dynamics of pipe flow captured in a reduced model: puff relaminarization and localized ‘edge’ states J. Fluid Mech. 619 213–33

Wygnanski I et al 1975 On transition in a pipe: II. The equilibrium puff J. Fluid Mech. 69 283–304

Wygnanski I J and Champagne F H 1973 On transition in a pipe: I. The origin of puffs and slugs and the flow in a turbulent slug J. Fluid Mech. 59 281–335

Yang J 2010 Nonlinear Waves in Integrable and Nonintegrable Systems (Philadelphia, PA: SIAM)