REMARKS ON AUTOMORPHISMS OF $\mathbb{C}^* \times \mathbb{C}^*$ AND THEIR BASINS

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Abstract. We study basins of attraction of automorphisms of $\mathbb{C}^2$ tangent to the identity that fix both axes. Our main result is that, if a well known conjecture about automorphisms of $\mathbb{C}^* \times \mathbb{C}^*$ holds, then there are no basins of attraction associated to the non-degenerate characteristic directions (in the sense of Hakim), and therefore we cannot find a Fatou-Bieberbach domain that does not intersect both axis with this method.

1. INTRODUCTION

A Fatou-Bieberbach domain is a proper domain of $\mathbb{C}^k$ that is biholomorphic to $\mathbb{C}^k$. These domains have been extensively studied, but many questions about them are still open [RR].

One of these open question is the following: (RR, p. 79)

**Question 1.** Is there a biholomorphic map from $\mathbb{C}^2$ into the set $\{zw \neq 0\}$ i.e. into the complement of the union of two intersecting complex lines?

One classical way of constructing Fatou-Bieberbach domains (i.e. biholomorphic maps from $\mathbb{C}^k$ into $\mathbb{C}^k$) is to find basins of attractions of automorphisms of $\mathbb{C}^k$ with a fixed point, as follows:

For $F \in \text{Aut}(\mathbb{C}^k), F(p) = p, F'(p) = A$ we define

$$\Omega_{F,p} = \{ z \in \mathbb{C}^k | \lim_{n \to \infty} F^n(z) = p \}.$$  

We say $F$ has an attracting fixed point when $A$ is a matrix with eigenvalues of modulus less than 1. In this case $\Omega_{F,p}$ is biholomorphic to $\mathbb{C}^k$ [RR].

In the semi-attracting case (eigenvalues of modulus smaller or equal than 1), and for automorphisms tangent to the identity (i.e., $A = \text{Id}$) then $\Omega_{F,p}$ can also be biholomorphically equivalent to $\mathbb{C}^l$ (where $l \leq k$). See [Ve] for semi-attracting case and [Hak1],[Hak2],[We] for automorphisms tangent to the identity.

Although not all Fatou-Bieberbach domains are basins of attraction of an automorphism of $\mathbb{C}^k$, these are the natural source of examples (and counter-examples) for various conjectures.

One natural approach, in order to answer Question 1 positively by using these results, is the following (for the definitions see [Hak1] or Section 3 below):

Find an automorphism $F$ of $\mathbb{C}^2$ such that:

- $F$ is tangent to the identity i.e. $F(0) = 0$ and $DF(0) = \text{Id}$.
- $F$ fixes the coordinate axes i.e. $F(z,0) = (z',0)$ and $F(0,w) = (0,w')$.

Then:

(a) There exists an attracting fixed point for $F$.

or

(b) There exists $v$ non-degenerate characteristic direction of $F$ at the origin such that $\text{Re}A(v) > 0$, where $A(v)$ is the number associated to the direction $v$.  

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Then, in the case of (a) we would have a basin of attraction associated to this fixed point \( RR \), or in the case of (b) we would a basin associated to \( v \) (as in Hakim’s notation) by [Hak1, Theorem 5.1]. In either case, this basin \( \Omega \) would be a Fatou Bieberbach domain and \( \Omega \subset \{ zw \neq 0 \} \).

Our main result is that, assuming a well known conjecture about automorphisms of \( \mathbb{C}^* \times \mathbb{C}^* \), this approach is not possible.

More precisely, we assume the following:

**Conjecture 2.** If \( F \in \text{Aut}(\mathbb{C}^* \times \mathbb{C}^*) \), then \( F \) preserves the form:

\[
\frac{dz \wedge dw}{zw}
\]

Assuming this, we prove:

**Proposition 3.** If Conjecture 2 is valid, and \( F \) is an automorphism of \( \mathbb{C}^2 \) tangent to the identity that fixes the coordinate axes, then (a) and (b) are both false.

Note that this does not answer Question 1 in general: in principle, we could have basins associated to degenerate characteristic directions.

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### 2. Automorphisms of \( \mathbb{C}^* \times \mathbb{C}^* \)

If \( F \) is an automorphism of \( \mathbb{C}^2 \) that fixes the coordinate axes, then we have that \( F|_{\mathbb{C}^* \times \mathbb{C}^*} \) is an automorphism of \( \mathbb{C}^* \times \mathbb{C}^* \).

The automorphism group of \( \mathbb{C}^* \times \mathbb{C}^* \) has been studied before [Ni], [Var], [An], but remains mysterious.

**Proposition 4.** Assume Conjecture 2 holds and let \( F : \mathbb{C}^2 \to \mathbb{C}^2 \) be an automorphism of \( \mathbb{C}^2 \) such that:

(a) \( F(0,0) = (0,0) \), \( F'(0,0) = 1d \)

(b) \( F \) fixes the coordinate axes \( \{ zw = 0 \} \).

Then we can write \( F \) as follows:

\[
F(z, w) = (ze^{ug(z,w)}, we^{zh(z,w)})
\]

where as power series we have:

\[
g(z, w) = \sum_{\alpha+\beta \geq k} c_{\alpha,\beta} z^\alpha w^\beta
\]

\[
h(z, w) = \sum_{\alpha+\beta \geq k} d_{\alpha,\beta} z^\alpha w^\beta
\]

and

\[
a_{\alpha-1,\beta} = -\frac{\alpha}{\beta} b_{\alpha,\beta-1}
\]

for \( k \leq \alpha + \beta \leq 2k \).
Proof. An easy computation shows that \( F(z, 0) = (z, 0) \) since \( F|_{z=0} \) is an automorphism of \( \mathbb{C} \)
and \( F \) is tangent to the identity. The same argument implies \( F(0, w) = (0, w) \). If we write \( F(z, w) = (f_1(z, w), f_2(z, w)) \) then we have: \( f_1(z, 0) = z \) and \( f_2(z, 0) = 0 \). In the same way \( f_1(0, w) = 0 \) and \( f_2(0, w) = w \). So we get \( f_1(z, w) = z + zwr(z, w) \) and \( f_2(z, w) = w + zws(z, w) \). So, we can write:
\[
g(z, w) := \frac{\log(1 + wr(z, w))}{w}
\]
and
\[
h(z, w) := \frac{\log(1 + zs(z, w))}{z}
\]
where \( g \) and \( h \) will be well defined function in \( \mathbb{C}^2 \) and therefore we have (1). Using Conjecture 3, we will have the following relationship between \( g, h \) and their derivatives:
\[
g_z + h_w - gh - zgh_z - whgw - zwg_wh_z + zwg_z h_w = 0
\]
Writing \( g \) and \( h \) as power series and looking at the terms of degree less than \( 2d \) in the left term we will have (2). \( \square \)

3. Basins of attraction

We recall now Hakim’s notation and results for automorphisms of \( \mathbb{C}^2 \) tangent to the identity.
Let \( F : \mathbb{C}^2 \to \mathbb{C}^2 \) be an automorphism of \( \mathbb{C}^2 \) tangent to the identity. We can write \( F \) as power series around the origin as follows:
\[
F(z, w) = (z + P_k(z, w) + P_{k+1}(z, w) + ... , w + Q_k(z, w) + Q_{k+1}(z, w) + ...)
\]
where \( P_k \) and \( Q_k \) are homogeneous polynomials of degree \( l \) (or identically 0). The order of \( F \) is the smallest \( k \geq 2 \) such that \( (P_k(z, w), Q_k(z, w)) \) does not vanish identically.

**Definition 5.** A characteristic direction is a direction \( v \neq 0 \) in \( \mathbb{C}^2 \) such that
\[
(P_k(v), Q_k(v)) = \lambda v
\]
for some \( \lambda \in \mathbb{C} \), where \( k \) is the order of \( F \). When \( \lambda \neq 0 \) then we call \( v \) a non-degenerate characteristic direction. Similarly if \( \lambda = 0 \) then \( v \) is a degenerate characteristic direction.

For \( v \in \mathbb{C}^2 \) a non-degenerate characteristic direction we can assume without loss of generality \( v = (1, u_0) \) with \( P_k(1, u_0) \neq 0 \).

If we define:
\[
r(u) := Q_k(1, u) - uP_k(1, u)
\]
then the non-degenerate characteristic directions of \( F \) are the zeroes of the polynomial function \( r \).

To each non-degenerate characteristic direction we associate the number:
\[
A(v) := \frac{r'(u_0)}{P_k(1, u_0)}
\]

Then Hakim proves [Hak1 Thm. 5.1 + Remark 5.3]:

**Theorem 6.** Let \( F \) be an automorphism of \( \mathbb{C}^2 \) tangent to the identity. Let \( v \) be a non-degenerate characteristic direction. Assume that \( \text{Re} A(v) > 0 \). Then there exists an invariant attracting domain \( D \) in which every point is attracted to the origin along a trajectory tangent to \( v \). Then the open set:
\[
D = \bigcup_{n=0}^{\infty} F^{-n}(D)
\]
is attracted to 0 and \( D \) is biholomorphic to \( \mathbb{C}^2 \).

With this result in mind we prove Proposition 2.
Proof of Prop. 2. We will first prove that there are no attracting fixed points. If we assume Conjecture 2, an easy computation shows that the Jacobian of $F$ will be:

$$JF(z, w) = e^{wg(z, w) + zf(z, w)}$$  \(4\)

The fixed points for $F$ are:

- $(0, 0)$ where $DF(0, 0) = Id$, therefore the origin is not attracting.
- $(z_0, 0)$ and $(0, w_0)$. For these points an easy computation shows that they are semi-attracting (or semi-repelling) fixed points (i.e. the eigenvalues are 1 and $\lambda$); therefore not attracting either.

- $(z_0, w_0)$ not necessarily on the axes, where $e^{w_0g(z_0, w_0)} = e^{z_0h(z_0, w_0)} = 1$. Using equation $(4)$ we have: $JF(z_0, w_0) = 1$, therefore $(z_0, w_0)$ will not be an attracting fixed point either (in case $DF(z_0, w_0) = Id$ see Remark 7).

Now we prove that (b) is not possible. We have:

$$F(z, w) = (ze^{wg(z, w)}, we^{zh(z, w)})$$  \(5\)

We assume that the lowest degree in $g$ (and therefore in $h$) is $k$. Then we have:

$$g(z, w) = \sum_{\alpha + \beta = k} c_{\alpha, \beta}z^\alpha w^\beta + h.o.t.$$  \(6\)

and

$$h(z, w) = \sum_{\alpha + \beta = k} d_{\alpha, \beta}z^\alpha w^\beta + h.o.t.$$  \(7\)

where

$$d_{\alpha - 1, \beta} = -\frac{\alpha}{\beta}c_{\alpha, \beta - 1}$$

Therefore the order of $F$ is $k + 2$:

$$F(z, w) = (ze^{wg(z, w)}, we^{zh(z, w)})$$

$$= (z(1 + wg(z, w) + O(w^2g^2)), w(1 + zh(z, w) + O(z^2h^2)))$$

$$= \left( z + zw \sum_{\alpha + \beta = k} c_{\alpha, \beta}z^\alpha w^\beta + O(|(z, w)|^{k+3}), w + zw \sum_{\alpha + \beta = k} d_{\alpha, \beta}z^\alpha w^\beta + O(|(z, w)|^{k+3}) \right)$$

We will call the lowest degree homogeneous terms $P_{k+2}$ and $Q_{k+2}$ as in Hakim's notation:

$$P_{k+2}(z, w) = \sum_{\alpha + \beta = k} c_{\alpha, \beta}z^{\alpha+1}w^{\beta+1}$$  \(6\)

$$Q_{k+2}(z, w) = \sum_{\alpha + \beta = k} d_{\alpha, \beta}z^{\alpha+1}w^{\beta+1}$$  \(7\)

Now we want to compute characteristic directions and the numbers associated to the non-degenerate ones. (Note that we have: $(P_{k+2}(1, 0), Q_{k+2}(1, 0)) = (0, 0)$ and $(P_{k+2}(0, 1), Q_{k+2}(0, 1)) = (0, 0)$; therefore $(1, 0)$ and $(0, 1)$ are degenerate characteristic directions).

From now on we assume that $v = (1, \theta)$ is a non-degenerate characteristic direction, with $\theta \neq 0$. So we want to solve:

$$r(u) = Q_{k+2}(1, u) - uP_{k+2}(1, u) = 0$$
Putting this back in (5) and (6) we get:

\[ r(u) = u \left( \sum_{\beta=0}^{k} dk_{-\beta,\beta} u^\beta - \sum_{\beta=0}^{k} ck_{-\beta,\beta} u^{\beta+1} \right) \]

\[ = u \left( dk_{0,0} + \sum_{\beta=1}^{k} (dk_{-\beta,\beta} - c_{k-\beta+1,\beta-1}) u^\beta - c_{0,k} \lambda^{k+1} \right) \]

Therefore the characteristic directions will be \((1, \theta)\) where \(s(\theta) = 0\) for \(r(u) = us(u)\) i.e.

\[ s(u) = dk_{0,0} + \sum_{\beta=1}^{k} (dk_{-\beta,\beta} - c_{k-\beta+1,\beta-1}) \theta^\beta - c_{0,k} \theta^{k+1} \]

and so we have:

\[ s(\theta) = dk_{0,0} + \sum_{\beta=1}^{k} (dk_{-\beta,\beta} - c_{k-\beta+1,\beta-1}) \theta^\beta - c_{0,k} \theta^{k+1} = 0 \]

Now we can compute the numbers associated to each direction with the following formula:

\[ A(\theta) = \frac{r'(\theta)}{P_{k+2}(1, \theta)} \]

Since we know:

\[ r(u) = dk_{0,0}u + \sum_{\beta=1}^{k} (dk_{-\beta,\beta} - c_{k-\beta+1,\beta-1}) u^{\beta+1} - c_{0,k} u^{k+2} \]

then we can easily get:

\[ r'(u) = dk_{0,0} + \sum_{\beta=1}^{k} (\beta + 1)(dk_{-\beta,\beta} - c_{k-\beta+1,\beta-1}) \theta^\beta - (k + 2)c_{0,k} u^{k+1} \]

Putting this back in (9) together with (6), we have:

\[ A(\theta) = \frac{dk_{0,0} + \sum_{\beta=1}^{k} (\beta + 1)(dk_{-\beta,\beta} - c_{k-\beta+1,\beta-1}) \theta^\beta - (k + 2)c_{0,k} u^{k+1}}{\sum_{\beta=0}^{k} c_{k-\beta,\beta} \theta^{\beta+1}} \]

(When \(k = 0\) the sum from \(\beta = 1\) to \(k\) is the empty sum.)

By (8) we have:

\[ dk_{0,0} = - \sum_{\beta=1}^{k} (dk_{-\beta,\beta} - c_{k-\beta+1,\beta-1}) \theta^\beta + c_{0,k} \theta^{k+1} \]

and putting this in (10) we can have some cancelations:

\[ A(\theta) = \frac{- \sum_{\beta=1}^{k} (dk_{-\beta,\beta} - c_{k-\beta+1,\beta-1}) \theta^\beta + c_{0,k} \theta^{k+1} + \sum_{\beta=1}^{k} (\beta + 1)(dk_{-\beta,\beta} - c_{k-\beta+1,\beta-1}) \theta^\beta - (k + 2)c_{0,k} \theta^{k+1}}{\sum_{\beta=0}^{k} c_{k-\beta,\beta} \theta^{\beta+1}} \]
So after simplifying and canceling \( \theta \) (since \( \theta \) is not equal to 0), we have:

\[
A(\theta) = \sum_{k=0}^{k} \beta \theta^k \left( \sum_{\beta=0}^{k} c_{k-\beta,0} \right) - (k + 1)c_{0,k} \theta^{k+1}
\]

\[
= \theta \left( \sum_{\beta=1}^{k} \beta (d_{k-\beta,0} - c_{k-\beta+1,0}) \theta^{\beta-1} - (k + 1)c_{0,k} \theta^k \right)
\]

\[
\frac{\theta \left( \sum_{\beta=0}^{k} c_{k-\beta,0} \theta^\beta \right)}{\sum_{\beta=0}^{k} c_{k-\beta,0} \theta^\beta}
\]

\[
(12)
\]

If the conjecture holds true, then we have:

\[
d_{\alpha-1,\beta} = -\frac{\alpha}{\beta} c_{\alpha,\beta-1}
\]

for \( k \leq \alpha + \beta - 1 \leq 2k \). So, using \( \alpha = k + 1 - \beta \) we have:

\[
d_{k-\beta,\beta} = -\frac{k - \beta + 1}{\beta} c_{k-\beta+1,\beta-1}
\]

(For \( k = 0 \) the condition is empty, but the result still holds.) So, back in our equation (12):

\[
A(\theta) = \sum_{k=0}^{k} \beta \theta^k \left( \sum_{\beta=0}^{k} c_{k-\beta,0} \right) - (k + 1)c_{0,k} \theta^{k+1}
\]

\[
= \frac{-(k + 1) \sum_{\beta=0}^{k} c_{k-\beta,0} \theta^{\beta}}{\sum_{\beta=0}^{k} c_{k-\beta,0} \theta^{\beta}}
\]

\[
= -(k + 1)
\]

This finishes the proof of Proposition 2. \( \square \)

**Remark 7.** If \((z_0, w_0)\) is a fixed point different from the origin, where \(DF(z_0, w_0) = Id\), then we can use the same technique to show that the result above still holds i.e. non-degenerate characteristic directions at \((z_0, w_0)\) will have a negative number associated to them.

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