A renormalisation-group treatment of two-body scattering

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Nonrelativistic two-body scattering by a short-ranged potential is studied using the renormalisation group. Two fixed points are identified: a trivial one and one describing systems with a bound state at zero energy. The eigenvalues of the linearised renormalisation group are used to assign a systematic power-counting to terms in the potential near each of these fixed points. The expansion around the nontrivial fixed point is shown to be equivalent to the effective-range expansion.

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Recently there has been much interest in applying the techniques of effective field theory (EFT) to the scattering of massive particles interacting via short-ranged forces. This has been spurred by Weinberg’s power-counting rules for the low-momentum expansion of the nucleon-nucleon potential [1], which raised the possibility of applying the techniques of chiral perturbation theory to nuclear physics [2]. These would provide a systematic method for expanding few-nucleon bound-state properties and scattering observables in powers of nucleon momenta and the pion mass.

But focussing on the potential, one avoids contributions where the two intermediate nucleons are almost on-shell, giving small denominators. However, this is the physics responsible for nuclear binding, and so to describe nuclei with an EFT it is not enough to write down a potential; one needs to solve the corresponding Schrödinger or Lippmann-Schwinger equation. At this point one encounters a problem. The EFT is based on a Lagrangian or Lippmann-Schwinger equation. At this point one encounters a problem. The EFT is based on a Lagrangian with local couplings between the particles and these include contact interactions between the nucleons. Such interactions correspond to δ-function potentials, and the resulting scattering equations only make sense after a further regularisation and renormalisation.

A variety of approaches has been explored for renormalising two-body scattering by such potentials [2, 3]. (For reviews of the various approaches and further references, see Ref. [4].) These have shown that it is difficult to set up a useful and systematic EFT for two-body scattering when the scattering length is unnaturally large, as indeed noted by Weinberg [1]. More recently an alternative to Weinberg’s power counting has been suggested by Kaplan, Savage and Wise (KSW) [13], based on dimensional regularisation with a “power divergence subtraction” scheme (PDS). The same counting has also been proposed by van Kolck [10, 15] within the framework of a general subtractive renormalisation scheme, using a momentum cut-off set at a large scale, typical of the underlying physics responsible for the short-distance interactions.

To examine some of the questions raised by these approaches, we have studied nonrelativistic two-body scattering from the viewpoint of Wilson’s continuous (or the “exact”) renormalisation group (RG) [17]. In this approach, one imposes a cut-off on the momenta of virtual states at some scale Λ and demands that physical quantities be independent of Λ. A rescaled Hamiltonian is introduced in which all dimensioned variables are expressed in units of Λ. The RG then describes the variation with Λ of this rescaled Hamiltonian.

Of particular interest in the present context is the behaviour as the cut-off is taken to zero. In this limit, more and more physics is integrated out and its effects are included implicitly in the effective Hamiltonian (in the present case, two-body potential). Eventually only Λ is left to set an energy scale and as a result the rescaled potential becomes independent of Λ. This is an infra-red fixed point of the RG. In the case of two-body scattering we find two types of fixed point, which correspond to two possible scale-independent forms for the zero-energy scattering amplitude, expressed in units of Λ. One is a trivial fixed point with a vanishing scattering amplitude. The others are a set of fixed points which give bound states at exactly zero energy.

Low-energy scattering (in physical units) can then be described in terms of perturbations of the potential around one of the fixed points. These perturbations can be expanded in terms of eigenfunctions of the linearised RG equations, which scale with definite powers of Λ. The counting of powers of Λ in these eigenfunctions provides a systematic way to organise the terms in the potential according to how rapidly they head towards, or away from, the fixed point. The eigenvalues also show whether the fixed point is stable or not.

For two-body scattering we find that the trivial fixed point is stable and that perturbations around it have scaling dimensions given by Weinberg’s power counting rules [1]. In contrast the fixed-points with zero-energy bound states are unstable. We find that each (energy-dependent) perturbation around the least unstable of these points corresponds to a term in the effective-range expansion [18]. The power counting for these perturbations agrees with that suggested by KSW [13] and van Kolck [15]. For a purely short-range potential, there is thus an equivalence between the effective field theory and the effective-range expansion [18, 19].

We consider here s-wave two-body scattering by a potential that consists of contact interactions only. In general this potential depends on energy as well as on the initial and final relative momenta. To second order in
the momentum expansion it has the form

$$V(k', k, p) = C_{00} + C_{20}(k^2 + k'^2) + C_{02} p^2 \cdots ,$$

(1)

where, as throughout this paper, we use $k$ and $k'$ to denote relative momenta and the energy-dependence is expressed in terms of $p = \sqrt{M E}$, the on-shell momentum corresponding to the energy $E$. Unlike previous applications of RG ideas to two-body scattering, we do not restrict our potential to only leading or next-to-leading terms in the momentum expansion.

In treating the scattering non-perturbatively, it is convenient to work with the reactance matrix, $K$. The off-shell $K$-matrix satisfies a Lippmann-Schwinger (LS) equation that is very similar to that for the scattering matrix $T$, except that the Green’s function satisfies standing-wave boundary conditions:

$$K(k', k, p) = V(k', k, p) + \frac{M}{2\pi^2} P \int_0^\Lambda q^2 dq \frac{V(k', q, p)K(q, k, p)}{p^2 - q^2},$$

(2)

where $P$ denotes the principal value of the integral. We have chosen to regulate the integral here by imposing a sharp cut-off at $q = \Lambda$. The inverse of the off-shell $K$-matrix differs from that of the on-shell $T$-matrix by a term $iMP/4\pi$, which ensures that $T$ is unitary if $K$ is Hermitian. This allows the effective-range expansion to be written as an expansion of $1/K$:

$$\frac{1}{K(p, p, p)} = -\frac{M}{4\pi} \left( \frac{1}{a} + \frac{1}{2} r_e p^2 + \cdots \right),$$

(3)

where $a$ is the scattering length and $r_e$ is the effective range.

The RG equation for the potential is obtained by making $V$ dependent on $\Lambda$ and demanding that the off-shell $K$-matrix be independent of $\Lambda$. This is obviously sufficient to ensure that all scattering observables do not depend on $\Lambda$. By differentiating the LS equation (2) with respect to $\Lambda$, setting $\partial K/\partial \Lambda = 0$ and then operating from the right with $(1 + G_0 K)^{-1}$ (where $G_0$ is the free Green’s function in Eq. (3)), one obtains the equation

$$\frac{\partial V}{\partial \Lambda} = \frac{M}{2\pi^2} V(k', \Lambda, p, \Lambda) \frac{\Lambda^2}{\Lambda^2 - p^2} V(\Lambda, k, p, \Lambda).$$

(4)

If this effective potential is to describe scattering by short-ranged interactions, it should be an analytic function of $k^2$ and $k'^2$ for small $k$ and $k'$. Also, if the energy lies below all thresholds for production of other particles then the potential should be an analytic function of the energy, $E = p^2/M$. Under these restrictions, the boundary conditions that we impose on $V$ are that it should have an expansion in non-negative, integer powers of $k^2$, $k'^2$ and $p^2$.

We now introduce dimensionless momentum variables, $\hat{k} = k/\Lambda$ etc., and define a rescaled potential,

$$\hat{V}(\hat{k}', \hat{k}, \hat{p}, \Lambda) = \frac{M\Lambda}{2\pi^2} V(\Lambda \hat{k}', \Lambda \hat{k}, \Lambda \hat{p}, \Lambda).$$

(5)

In terms of these quantities, the RG equation takes the form

$$\Lambda \frac{\partial \hat{V}}{\partial \Lambda} = \hat{k}' \frac{\partial \hat{V}}{\partial \hat{k}'} + \hat{k} \frac{\partial \hat{V}}{\partial \hat{k}} + \hat{p} \frac{\partial \hat{V}}{\partial \hat{p}} + \hat{V}$$

$$+ \hat{V}(\hat{k}', 1, \hat{p}, \Lambda) - \frac{1}{1 - \hat{p}^2} \hat{V}(1, \hat{k}, \hat{p}, \Lambda).$$

(6)

In what follows, the idea of a fixed point of the RG will be crucial. As the cut-off $\Lambda$ is taken to zero, more and more physics is integrated out until $\Lambda$ itself is the only remaining scale. In the limit $\Lambda \to 0$, the rescaled potential, being dimensionless, must become independent of $\Lambda$. This means that as $\Lambda$ varies the rescaled potential must flow towards an infra-red fixed point of the RG equation (6). The fixed points are described by solutions of this equation that satisfy

$$\frac{\partial \hat{V}}{\partial \Lambda} = 0.$$

(7)

An obvious example is the trivial fixed point,

$$\hat{V}(\hat{k}', \hat{k}, \hat{p}, \Lambda) = 0.$$  

(8)

The $K$-matrix for this potential is also zero, corresponding to no scattering.

If, for a particular system, we find that the rescaled potential tends towards this fixed point as we lower the cut-off towards zero, then we can describe the low-energy behaviour in terms of the perturbations that scale with definite powers of $\Lambda$. To find these we linearise the RG equation (6) about the trivial fixed point and look for solutions of the form

$$\hat{V}(\hat{k}', \hat{k}, \hat{p}, \Lambda) = C \Lambda^\nu \phi(\hat{k}', \hat{k}, \hat{p}),$$

(9)

where the functions $\phi$ satisfy the eigenvalue equation

$$\hat{k}' \frac{\partial \phi}{\partial \hat{k}'} + \hat{k} \frac{\partial \phi}{\partial \hat{k}} + \hat{p} \frac{\partial \phi}{\partial \hat{p}} + \phi = \nu \phi.$$  

(10)

It is not hard to see that the solutions to this that are well-behaved as the momenta and energy tend to zero are

$$\phi(\hat{k}', \hat{k}, \hat{p}) = \hat{k}'^{\nu} \hat{k}^m \hat{p}^n,$$

(11)

with RG eigenvalues $\nu = l + m + n + 1$, where $l$, $m$ and $n$ are non-negative even integers. The RG eigenvalues are all positive and so the fixed point is a stable one: starting
from any potential in the vicinity of \( \hat{V} = 0 \) the RG flow will take the potential to the fixed point as \( \Lambda \to 0 \).

The momentum expansion of the full potential near the trivial fixed point can be written

\[
\hat{V}(k', \hat{k}, \hat{p}, \Lambda) = \sum_{l,n,m} \hat{C}_{lmn} \left( \frac{\Lambda}{\Lambda_0} \right) \nu \hat{k}^l \hat{k}^m \hat{p}^n.
\]  

(12)

For a Hermitian potential one can take real coefficients with \( \hat{C}_{lmn} = \hat{C}_{mnl} \). We have chosen to write the coefficients in this expansion in a dimensionless form by taking out a factor of \( \Lambda_0^{-\nu} \), where \( \Lambda_0 \) is some scale associated with the underlying physics that determines where our momentum expansion of the potential breaks down. This can be seen more clearly from the form of the corresponding unscaled potential,

\[
V(k', k, p, \Lambda) = \frac{2\pi^2}{M \Lambda_0} \sum_{l,n,m} \hat{C}_{lmn} \frac{k^l k^m p^n}{\Lambda_0}.
\]  

(13)

In a “natural” theory it is possible to choose the scale \( \Lambda_0 \) in such a way that all of the dimensionless coefficients are of order unity. The power counting in this expansion of the potential (12) is just the one proposed by Weinberg [1] if we assign an order \( d = \nu - 1 \) to each term.

The behaviour near the trivial fixed point gives weak scattering at low energies which can be treated perturbatively. It can be used to describe systems where the scattering length is small. For such systems Weinberg’s power counting has already been shown to provide a systematic treatment in the context of both dimensional regularisation with minimal subtraction [2] and cut-off approaches [3] where the cut-off is chosen to be well below \( 1/a \) and \( 1/r_e \). In these cases the \( K \)-matrix is given by the first Born approximation and so is equal to the unscaled potential [4]. That unscaled potential is independent of \( \Lambda \) as it must be since \( K \) does not depend on \( \Lambda \).

Of more interest for nuclear physics are nontrivial fixed points that can describe systems with very strong scattering at low energies. The simplest of these can be found by considering a potential that depends only on energy, \( \hat{V} = \hat{V}_0(\hat{p}) \). This will be a fixed-point solution to the full RG equation (11) if it satisfies

\[
\hat{p} \frac{\partial \hat{V}_0}{\partial \hat{p}} + \hat{V}_0(\hat{p}) + \frac{\hat{V}_0(\hat{p})^2}{1 - \hat{p}^2} = 0.
\]  

(14)

Solving this equation subject to the boundary condition that the potential be analytic in \( \hat{p}^2 \) as \( \hat{p}^2 \to 0 \) we obtain

\[
\hat{V}_0(\hat{p}) = -\left[ 1 - \frac{\hat{p}}{2} \ln \frac{1 + \hat{p}}{1 - \hat{p}} \right]^{-1}.
\]  

(15)

Note that, although the detailed form of the energy-dependence of this potential is determined by our particular choice of cut-off, the fact that it tends to a constant as \( \hat{p} \to 0 \) is a generic feature. The corresponding unscaled potential is

\[
V_0(p, \Lambda) = \frac{2\pi^2}{M} \left[ \Lambda - \frac{p}{2}\ln \left( \frac{\Lambda + p}{\Lambda - p} \right) \right]^{-1}.
\]  

(16)

At \( p = 0 \), this potential is inversely proportional to \( \Lambda \), a property which holds for any form of cut-off.

For a momentum-independent potential like \( V_0 \) the LS equation takes a particularly simple form. For \( V_0 \) we find that its solution for \( K \) is infinite, or rather \( 1/K = 0 \). This corresponds to a system with infinite scattering length, or equivalently a bound state at exactly zero energy.

To study the behaviour near this fixed point we consider small perturbations about it that scale with definite powers of \( \Lambda \):

\[
\hat{V}(k', \hat{k}, \hat{p}, \Lambda) = \hat{V}_0(\hat{p}) + C \Lambda^\nu \phi(\hat{k}', \hat{k}, \hat{p}).
\]  

(17)

These satisfy the linearised RG equation

\[
\hat{k}' \frac{\partial \phi}{\partial k'} + \hat{k} \frac{\partial \phi}{\partial k} + \hat{p} \frac{\partial \phi}{\partial \hat{p}} + \phi + \frac{V_0(\hat{p})}{1 - \hat{p}^2} \left[ \phi(\hat{k}', 1, \hat{p}) + \phi(1, \hat{k}, \hat{p}) \right] = \nu \phi.
\]  

(18)

Let us first look for solutions to Eq. (18) that depend only on energy \( (\hat{p}^2) \). The equation can be integrated straightforwardly, making use of the fixed point equation for \( V_0 \) (15). The solutions are

\[
\phi(\hat{p}) = \hat{p}^{\nu + 1} V_0(\hat{p})^2.
\]  

(19)

If we demand that these be well-behaved functions of \( \hat{p}^2 \) as \( \hat{p}^2 \to 0 \), then we find that the allowed RG eigenvalues are \( \nu = -1, 1, 3, \ldots \). From this we see that the fixed point is unstable: it has one negative eigenvalue.

The RG flow corresponding to Eq. (11) is shown in Fig. 1, projected into the plane corresponding to the first two terms in the expansion of the potential in powers of energy, \( \hat{V}(\hat{p}) = b_0 + b_2 \hat{p}^2 + \cdots \). This shows the two fixed points discussed above: the trivial one at the origin and the nontrivial one at \( b_0 = b_2 = -1 \). The precise position of the latter point depends on our particular choice of cut-off, but the pattern of the RG flow is general. Potentials that lie exactly on the “critical surface” \( (b_0 = -1) \) for our sharp cut-off flow towards the nontrivial fixed point as \( \Lambda \to 0 \). For a system with a small perturbation away from this surface, the potential initially flows towards the nontrivial fixed point for large \( \Lambda \). Eventually, however, the unstable perturbation causes these potentials to flow either to the trivial fixed point or to infinity.

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1Terms with forms like \( i(k^2 - k'^2) \) need not be included since they can be shown to vanish when integrated by parts in coordinate space.
To first order in the coefficients \( \hat{b} \) are in one-to-one correspondence with the terms in the expansion of the potential. We see that the terms in the expansion of the potential have the form

\[ V(p, \mu) = 4\pi \frac{M \Lambda_{0}}{\mu^{2}} \left[ -1 - \frac{1}{\mu a} \frac{r_{e}}{2\mu^{2}} p^{2} + \cdots \right]. \tag{23} \]

Remembering that the scale \( \mu \) in a subtractive renormalisation scheme acts like a resolution scale and so plays an analogous role to the cut-off \( \Lambda \), we see that the \( 1/\mu \) dependence of the first term in Eq. (23) agrees with the \( 1/\Lambda \) dependence of the fixed-point potential [14]. Similarly the factors of \( 1/\mu^{2} \) in the second and third terms agree with the \( 1/\Lambda^{2} \) factors in the energy-dependent perturbations in Eq. (20). If, as for perturbations around the trivial fixed point, we assign an order \( d = \nu - 1 \) to each term in the potential, then we see that the power counting for (energy-dependent) perturbations around the nontrivial fixed point agrees with that of KSW.

For systems with finite but large scattering lengths, it is still possible to build a description based on Weinberg’s expansion around the trivial fixed point. To do so one must solve the RG equation to all order in the scattering length, resumming terms involving powers of \( \Delta_{0} \). This is approach adopted by van Kolck [10,15]. It can be pictured as following one the flow lines in Fig. 1 that approach the trivial fixed point close to the line that connects the two fixed points. For values of \( \Lambda \) that are large compared with \( 1/a \) such a flow line lies close to the critical surface for the nontrivial fixed point and so the behaviour of the system can be organised according to the power counting associated with that fixed point.

So far we have considered perturbations of the potential that depend only on energy, but there are also ones that depend on momentum as well. To find these, we look for solutions (14) that have the form

\[ \phi(\hat{k}', \hat{k}, \hat{p}) = \hat{k}'^{\nu} \phi_{1}(\hat{p}) + \phi_{2}(\hat{p}). \tag{24} \]

(A Hermitian potential can be obtained by adding a similar term with \( k \to k' \).) In this case the solutions are

\[ \phi(\hat{k}', \hat{k}, \hat{p}) = \left[ \hat{k}'^{n} \hat{p}^{n} + \sum_{m=0}^{n/2-1} \frac{\hat{p}^{m}}{n-2m+1} V_{0}(\hat{p}) \right] V_{0}(\hat{p}), \tag{25} \]

with RG eigenvalues \( \nu = n = 2, 4, 6 \ldots \). Multiplying any of these functions by \( p^{m} \) where \( m \) is a positive even integer also gives an eigenfunction, with \( \nu = n + m \). An important point to note is that the momentum-dependent eigenfunctions have different eigenvalues from the corresponding purely energy-dependent ones. This is quite unlike the more familiar case of perturbations around the trivial fixed point where, for example, the \( p^{2} \) and \( k^{2} \) terms in the potential are both of the same order, \( \nu = 3 \). It means that, in the vicinity of the nontrivial fixed point, one cannot make a field transformation to eliminate energy dependence from the potential in favour of momentum dependence without introducing a much more complicated cut-off dependence into the effective potential.

To complete the picture, we note that it is possible to find solutions to Eq. (13) that depend on both \( k \) and \( k' \). These are products of two factors of the form of the

FIG. 1. The RG flow of the first two terms in the expansion of the rescaled potential in powers of energy. The two fixed points are indicated by the black dots. The solid lines are flow lines that approach one of the fixed points along a direction corresponding to an RG eigenfunction; the dashed lines are more general flow lines. The arrows indicate the direction of flow as \( \Lambda \to 0 \).

The unscaled potential including the perturbations [14] is

\[ V(k', k, p, \Lambda) = V_{0}(p, \Lambda) + \frac{M \Delta_{0}}{2\pi^{2}} \sum_{n=0}^{\infty} \tilde{C}_{2n-1} \left( \frac{p}{\Lambda_{0}} \right)^{2n} V_{0}(p, \Lambda)^{2}. \tag{20} \]

The (on-shell) \( K \)-matrix for this potential has the effective-range expansion

\[ \frac{1}{K(p, p, p)} = -\frac{M \Delta_{0}}{2\pi^{2}} \sum_{n=0}^{\infty} \tilde{C}_{2n-1} \left( \frac{p}{\Lambda_{0}} \right)^{2n} + \cdots. \tag{21} \]

To first order in the coefficients \( \tilde{C}_{\nu} \) of the eigenfunctions, we see that the terms in the expansion of the potential are in one-to-one correspondence with the terms in the effective-range expansion. In particular, \( \tilde{C}_{-1} \) and \( \tilde{C}_{1} \) are given in terms of the scattering length and effective range by

\[ \tilde{C}_{-1} = \frac{-\pi}{2\Delta_{0} a}, \quad \tilde{C}_{1} = \frac{\pi \Delta_{0} r_{e}}{4}. \tag{22} \]

At this point we can compare our potential (20) with that found by KSW in the PDS scheme [13]. To first order in \( 1/a \) and \( p^{2} \) their potential can be written in the form

\[ V(p, \mu) = \frac{4\pi}{M \mu} \left[ -1 - \frac{1}{\mu a} \frac{r_{e}}{2\mu^{2}} p^{2} + \cdots \right]. \tag{23} \]
expression in square brackets in Eq. (25); one depending on $k$ and one on $k'$, and they have RG eigenvalues $\nu = n + n' + 1 = 5, 7, 9, \ldots$. Each of these can also be multiplied by even powers of $p$ to yield further eigenfunctions. Together, the eigenfunctions described above contain all possible products of powers of $k^2, k'^2$ and $p^2$. They thus form a complete set that can be used to expand any perturbation about the fixed-point potential that is well-behaved as $k^2, k'^2, p^2 \to 0$ and so has a power-series expansion in these quantities.

When the $\nu = 2$ momentum-dependent perturbation is included, the term

$$\hat{C}_2 \left\{ \left( k^2 - p^2 + \frac{1}{3} \frac{M \Lambda^3}{2\pi^2} V_0 \right) \frac{V_0(p, \Lambda)}{\Lambda_0} + (k \to k') \right\}$$

(26)

must be added to the potential (20). The resulting potential has a two-term separable structure and so the LS equation can be solved using the techniques in Refs. [6,9].

To first order in the $\hat{C}_\nu$, we find that the effective-range expansion is again given by Eq. (21); the momentum-dependent perturbation does not contribute to the on-shell scattering. This can be understood from the fact that its coefficient involves an even power of the underlying scale $\Lambda_0$, whereas all of the terms in the effective range expansion contain odd powers of that scale.

The identification of the terms in the potential and the effective-range expansion is straightforward at first order in the coefficients $\hat{C}_\nu$, because only energy dependent perturbations contribute to the scattering. To check whether this equivalence persists to higher order in the $\hat{C}_\nu$, we have solved the RG equation to second order in these coefficients. To illustrate the behaviour, we consider here corrections to the potential (20) up to order $\hat{C}_{-1}\hat{C}_2$. We find the following additional pieces

$$\frac{M}{2\pi^2} \hat{C}_{-1}\hat{C}_2 \left( k^2 + k'^2 + A p^2 + \frac{4}{3} \frac{M \Lambda^3}{2\pi^2} V_0 \right) \frac{V_0(p, \Lambda)^2}{\Lambda_0},$$

(27)

where $A$ is a constant of integration which is not fixed by the boundary conditions. This undetermined piece arises from the solution of the homogeneous part of the linearised RG equations, and has the exactly same structure as the $\nu = 1$ term in (24).

This second-order piece (27) will in general contribute to the effective range, along with the $\nu = 1$ term. The direct correspondence between scattering observables and terms in the potential can be maintained by choosing $A = -2$, which ensures that the contribution of $\hat{C}_{-1}\hat{C}_2$ to the effective range vanishes.

Alternatively, one could set $\hat{C}_1 = 0$ (and $A = 0$) and generate the effective range entirely from the $\nu = 2$ momentum-dependent perturbation. However the required coefficient in this case is

$$\hat{C}_2 = \frac{\Lambda^2 a r_\epsilon}{4},$$

(28)

implying the existence of large factors of $\Lambda_0$ that need to be resummed in the potential. This can be done by starting from the trivial fixed point, where energy dependence can be eliminated in favour of momentum dependence, and then using the approach of van Kolck [10,15] to follow the RG flow back to the vicinity of the critical surface for the nontrivial fixed point. However the resummation procedure tends to mask the simple nature of the RG flow close to the nontrivial fixed point.

We have seen that, to second order in the coefficients $\hat{C}_\nu$ in the expansion about the nontrivial fixed point, only the energy-dependent RG eigenfunctions contribute to the on-shell scattering. So long as analogous procedures can be carried out to all orders, the effective theory defined by an expansion around the nontrivial fixed point is systematic and the terms in the potential are in one-to-one correspondence with those of the effective-range expansion. The resulting potential is determined entirely by on-shell scattering observables. Indeed, as has long been known from the effective-range expansion, the scattering length and effective range are also sufficient to determine the asymptotic $s$-wave part of the deuteron wave function [3].

Our RG analysis thus demonstrates that there is an equivalence between the effective field theory based on the nontrivial fixed point and the effective-range expansion [18], as previously suggested by van Kolck [8,15].

The effective-range expansion is based on

$$p \cot \delta(p) = -\frac{4\pi}{M K(p, p, p)} \frac{1}{p},$$

(29)

which is the logarithmic derivative at the origin of the asymptotic wave function. It can thus be regarded as an energy-dependent boundary condition on the wave function at the origin. As noted by van Kolck [8,15], this corresponds to the fact that the parts of the effective field theory which contribute to observables act like a quasipotential [20].

The fixed point is, as already noted, an unstable one. Only potentials that lie on the “critical surface” defined by $1/a = 0$ (corresponding to a bound state at zero energy) will flow to the fixed point as $\Lambda \to 0$. For small but nonzero values of $1/a$ the potential will eventually either tend to the trivial fixed point or diverge to infinity, depending on the sign of $a$. Nonetheless for cut-offs in the range $1/a \ll \Lambda \ll \Lambda_0$ the behaviour of the potential may be dominated by the flow towards the nontrivial fixed point. In such a regime we can still use the eigenfunctions found above to define a systematic expansion of the potential, as noted in Ref. [13].

To summarise: by applying the renormalisation group to two-body scattering we have identified two fixed points. One is the trivial fixed point describing perturbative scattering. The other describes systems with bound states at zero energy and is directly related to the effective-range expansion [18]. By studying the eigenfunctions of the linearised RG we can assign a systematic
power-counting to terms in the expansion of the potential around each of these fixed points. In the case of the nontrivial fixed point, each term in this expansion corresponds to a term in the effective-range expansion. The potential to next-to-leading order depends only on energy and is determined entirely by on-shell scattering observables, namely the scattering length and effective range. The success of the effective-range expansion [18,19] can be therefore be understood in terms of an effective field theory based on this nontrivial fixed point [11–13]. It also suggests that there should be ways to extend this expansion to treat long-ranged forces such as pion-exchange [13], as well as three-body systems [11], both of which are being actively pursued.

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[1] S. Weinberg, Phys. Lett. B251, 288 (1990); Nucl. Phys. B363, 3 (1991).
[2] C. Ordonez, L. Ray and U. van Kolck, Phys. Rev. Lett. 72, 1982 (1994); Phys. Rev. C53, 2086 (1996).
[3] S. K. Adhikari and T. Frederico, Phys. Rev. Lett. 74, 4572 (1995); S. K. Adhikari and A. Ghosh, J. Phys. A: Math. Gen. 30, 6553 (1997).
[4] D. B. Kaplan, M. J. Savage and M. B. Wise, Nucl. Phys. B478, 629 (1996).
[5] T. D. Cohen, Phys. Rev. C55, 67 (1997); D. R. Phillips and T. D. Cohen, Phys. Lett. B390, 7 (1997).
[6] S. R. Beane, T. D. Cohen and D. R. Phillips, Nucl. Phys. A632, 445 (1998).
[7] G. P. Lepage, nucl-th/9706029.
[8] U. van Kolck, Nucl. Phys. A631, 56c (1998).
[9] K. G. Richardson, M. C. Birse and J. A. McGovern, hep-ph/9708435.
[10] U. van Kolck, hep-ph/9711222.
[11] P. F. Bedaque and U. van Kolck, Phys. Lett. B428, 221 (1998); P. F. Bedaque, H.-W. Hammer and U. van Kolck, Phys. Rev. C58, R641 (1998); Nucl. Phys. A646, 444 (1999).
[12] T.-S. Park, K. Kubodera, D.-P. Min and M. Rho, hep-ph/9711463.
[13] D. B. Kaplan, M. J. Savage, and M. B. Wise, Phys. Lett. B424, 390 (1998); Nucl. Phys. B534, 329 (1998); Phys. Rev. C59, 617 (1999).
[14] J. Geselia, Phys. Lett. B429, 227 (1998); nucl-th/9805008.
[15] U. van Kolck, Nucl. Phys. A645, 273 (1999).
[16] R. Seki, U. van Kolck and M. J. Savage (editors), Nuclear Physics with Effective Field Theory, (World Scientific, Singapore, 1998).
[17] K. G. Wilson and J. G. Kogut, Phys. Rep. 12, 75 (1974); J. Polchinski, Nucl. Phys. B231, 269 (1984); R. D. Ball and R. S. Thorne, Ann. Phys. 236, 117 (1994); T. R. Morris, Prog. Theor. Phys. Suppl. 131, 395 (1998).
[18] J. Schwinger, Phys. Rev. 72, 742A (1949); J. M. Blatt and J. D. Jackson, Phys. Rev. 76, 18 (1949); H. A. Bethe, Phys. Rev. 76, 38 (1949).
[19] H. A. Bethe and C. Longmire, Phys. Rev. 77, 647 (1950).
[20] G. Breit, Phys. Rev. 71, 215 (1947); S. Albeverio, F. Gesztesy, R. Hoegh-Krohn and H. Holden, Solvable Models in Quantum Mechanics, (Springer Verlag, New York, 1988); R. Jackiw, in M. A. B. Bégu Memorial Volume, edited by A. Ali and P. Hoodbhoy, (World Scientific, Singapore, 1991).