Quantum Corrections to $Q$-Balls

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We extend calculational techniques for static solitons to the case of field configurations with simple time dependence in order to consider quantum effects on the stability of $Q$-balls. These nontopological solitons exist classically for any fixed value of an unbroken global charge $Q$. We show that one-loop quantum effects can destabilize very small $Q$-balls. We show how the properties of the soliton are reflected in the associated scattering problem, and find that a good approximation to the full one-loop quantum energy of a $Q$-ball is given by $\omega - E_0$, where $\omega$ is the frequency of the classical soliton’s time dependence, and $E_0$ is the energy of the lowest bound state in the associated scattering problem.

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INTRODUCTION

A pure scalar theory in three dimensions with a cubic coupling can support classically stable, time-dependent non-topological solutions to the equations of motion carrying an unbroken global charge $Q$, called $Q$-balls [1]. Supersymmetric extensions of the standard model generically contain such objects [2]. They become particularly interesting in cosmological applications at small values of $Q$, because then it is easier for them to form in the early universe [3]. In this regime, quantum corrections to the soliton’s energy become increasingly important in determining its stability. The methods of [4] (see also earlier related work in [5] and theoretical justification in the Appendices of [6]) provide an efficient, robust framework for computing quantum corrections to time-independent field configurations. In this Letter, we extend this approach to $Q$-balls. We show how to express the computation in terms of an effective time-independent problem. In this formalism, the full one-loop quantum correction can be computed efficiently. We also derive a very simple estimate for this result. The result is that we can compare the energy of the $Q$-ball in the quantum theory to the energy of free particles carrying the same charge $Q$, and determine if the $Q$-ball remains stable in the quantum theory.

Our starting point is the classical analysis of $Q$-balls carried out in [1,3]. We will take the same simple model,

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \varphi)(\partial^\mu \varphi) - U(\varphi) \quad (1)$$

where $\varphi$ is a complex field with unit charge under a global $U(1)$ symmetry that is unbroken at $\varphi = 0$. We will consider the potential

$$U(\varphi) = \frac{1}{2} M^2 |\varphi|^2 - A|\varphi|^3 + \lambda |\varphi|^4. \quad (2)$$

A particular configuration $\varphi(x,t)$ has charge

$$Q = \frac{1}{2i} \int d^3x \left(\varphi^* \partial_t \varphi - \varphi \partial_t \varphi^* \right). \quad (3)$$

Upon fixing the charge $Q$ of the configuration via a Lagrange multiplier $\omega$, we obtain the classical $Q$-ball solution as the minimum of the energy functional

$$\mathcal{E}_\omega[\varphi] = \int d^3x \left(\frac{1}{2} |\partial_t \varphi - i \omega \varphi|^2 + \int d^3x \left(\frac{1}{2} |\nabla \varphi|^2 + U(\varphi) \right) + \omega Q \right. \quad (4)$$
with respect to independent variations of $\varphi(x, t)$ and $\omega$, where

$$U_{\omega}(\varphi) = U(\varphi) - \frac{1}{2} \omega^2 \varphi^2. \quad (5)$$

The Q-ball solution then has simple time dependence

$$\varphi(x, t) = e^{i\omega t} \phi(x) \quad (6)$$

so we must simply minimize

$$E_{\omega}[\phi] = \int d^3 x \left( \frac{1}{2} |\nabla \phi|^2 + U_{\omega}(\phi) \right) + \omega Q \quad (7)$$

by varying $\omega$ and $\phi(x)$. As shown in [1], if the quantity $U(\phi)/\phi^2$ is minimized at $\phi_0 > 0$, then for $\omega_0 = \sqrt{2U(\phi_0)/\phi_0^2}$, the effective potential $U_{\omega_0}(\phi)$ will have degenerate minima. For $\omega > \omega_0$, a solution to the equations of motion is given by the bounce solution for tunneling in three Euclidean dimensions in the potential $U_{\omega}(\phi)$. The bounce is the solution to

$$\frac{d^2}{dr^2} \phi_0(r) + \frac{2}{r} \frac{d}{dr} \phi_0(r) = U''_{\omega}(\phi_0(r)) \quad (8)$$

with the boundary conditions

$$\lim_{r \to \infty} \phi_0(r) = 0 \quad \text{and} \quad \frac{d}{dr} \phi_0(r) \bigg|_{r=0} = 0. \quad (9)$$

We can then find the solution using the shooting method detailed in [7], and minimize the resulting energy over $\omega$. For large enough $Q$, the optimal value of $\omega$ approaches $\omega_0$, allowing [1] to use the thin-wall approximation to demonstrate the existence of a global minimum, which is the Q-ball solution. For small $Q$, the optimal value of $\omega$ approaches $M$, and [3] uses the thick-wall approximation to show that there exists a global minimum in this case as well. Thus classically bound solitons exist all the way down to $Q = 1$. For all $Q$, we have $\omega_0 < \omega < M$ at the minimum. We can thus consider the classical binding energy as a function of $Q$ by comparing the Q-ball’s energy to $QM$, the energy of a collection of free particles carrying charge $Q$.

Although Q-balls are classically stable even as $Q \to 1$, the binding energy per charge is going to zero in this limit. This case is of particular interest for cosmological applications, however, since Q-balls of large charge, while favored energetically, are disfavored as the temperature increases by their low entropy. To answer the question of whether Q-balls have a significant chance of being formed in the early universe, we must therefore verify that the classical conclusions are not invalidated by quantum corrections.

**QUANTUM CORRECTIONS**

To compute the leading quantum correction the Q-ball energy, we extend the method of [4]. We write the quantum field $\varphi$ as the classical solution plus a quantum correction, which we write in corotating coordinates

$$\varphi(x, t) = e^{i\omega t} (\phi_0(x) + \eta(x, t)) \quad (10)$$

where we can then expand the quantum field $\eta(x, t)$ in small oscillations, which are given by the solutions to

$$\left[ (\partial_t - i\omega)^2 - \nabla^2 + U''(\phi_0(x)) \right] \psi(x, t) = 0. \quad (11)$$

Parametrizing $\psi(x, t) = e^{i\omega t} e^{-iEt} \psi(x)$ gives the mode an energy $E - \omega$, where the time-independent wavefunction $\psi(x)$ solves

$$\left[ -\nabla^2 + U''(\phi_0(x)) \right] \psi(x) = E^2 \psi(x) \quad (12)$$

which is an ordinary Schrödinger equation. The Casimir energy is given formally by the sum over zero-point energies of these oscillations

$$E_{\text{Cas}}[\phi_0] \sim \frac{1}{2} \sum_j |E_j - \omega|. \quad (13)$$
Since the spectrum of eq. (12) is symmetric in $E \to -E$, we can sum over both signs of the energy and obtain
\[
\mathcal{E}_c^{\text{bare}}[\phi_0] \sim \frac{1}{2} \sum_{E_j \geq 0} (|E_j + \omega| + |E_j - \omega|) = \sum_{E_j \geq 0} \max(|\omega|, |E_j|) .
\] (14)

We will use the methods of [4] to extract the quantum correction to the energy in terms of the continuum scattering data for the reduced problem of eq. (13). Since the potential is spherically symmetric, we can decompose the spectrum into partial waves $\ell$. We have wavefunctions
\[
\psi_{\ell}(x) = \frac{Y_{\ell m}(\Omega)}{r} \eta_\ell(r)
\] (15)
for $m = -\ell, -\ell + 1, \ldots, \ell - 1, \ell$. The radial wavefunction $\eta_\ell(r)$ satisfies
\[
\left(-\frac{d^2}{dr^2} + \frac{\ell(\ell + 1)}{r^2} + U''(\phi_0(r))\right) \eta_\ell(r) = E^2 \eta_\ell(r)
\] (16)
with scattering boundary conditions.

In each partial wave, we will find a continuum starting at $E = M$ and possibly bound states with $0 \leq E_j \leq M$. (Since the spectrum is symmetric in $E$, we will only consider $E \geq 0$.) It is instructive to consider the properties of eq. (12) that betray its origin from a field theory soliton. The full oscillation spectrum should have a zero mode in the $\ell = 1$ channel, corresponding to the translation invariance of the $Q$-ball solution. The threefold degeneracy of this state corresponds to the three directions of translation. From eq. (13), we see that in the reduced problem, the zero mode appears as a bound state with energy $E = \omega$. Since this state appears in the $\ell = 1$ channel, there must exist an even more tightly bound state in the $\ell = 0$ channel. In the case of an ordinary static solution, this state would correspond to an instability of the full soliton. But from eq. (14), we see that the destabilizing effect of this mode is neutralized by the time dependence of the classical solution, which results in it making the same contribution to the mode sum as the zero modes do. All other modes have energies greater than $\omega$.

Having rewritten the Casimir energy in terms of the eigenmodes of the reduced scattering problem in eqs. (12) and (14), we are prepared to apply the methods of [4]. We obtain the renormalized Casimir energy as a sum over partial waves $\ell$. In each partial wave we have a sum over positive energy bound states $E_{j,\ell}$ and an integral over continuum states weighted by the density of states $\rho(k)$, where $k = \sqrt{E^2 - M^2}$. We subtract the corresponding integral in the free case, obtaining
\[
\mathcal{E}_c^{\text{bare}}[\phi_0] = \sum_{\ell=0}^{\infty} (2\ell + 1) \left[ \int_0^{\infty} \frac{dk}{\pi} E \left( \rho_\ell(k) - \rho_\ell^0(k) \right) + \sum_j \max(\omega, E_{j,\ell}) \right] .
\] (17)

This integral diverges in the unrenormalized theory, because we have not yet included the contribution of the counterterms. We will compute the continuum integral by relating the difference between the interacting and free densities of states to the phase shifts,
\[
\rho_\ell(k) - \rho_\ell^0(k) = \frac{1}{\pi} \frac{d}{dk} \delta_\ell(k) .
\] (18)

The Born expansion for the phase shift is then in exact correspondence with the expansion of the effective energy in one-loop diagrams with all possible insertions of the background field. Subtracting the first two Born approximations from the phase shift corresponds to subtracting the first two diagrams in this expansion, which are the only divergent terms. The remaining integral is then finite. We then add the divergent terms back in, together with counterterms, as ordinary Feynman diagrams. Full details are given in [4].

We thus obtain the renormalized Casimir energy
\[
\mathcal{E}_c[\phi_0] = \Gamma^{(2)}[\phi_0] + \sum_{\ell=0}^{\infty} (2\ell + 1) \left[ \int_0^{\infty} \frac{dk}{\pi} \frac{d}{dk} \left( \delta_\ell(k) - \delta_\ell^{(1)}(k) - \delta_\ell^{(2)}(k) \right) + \sum_j \max(\omega, E_{j,\ell}) \right] .
\] (19)

where $\delta_\ell(k)$ is the scattering phase shift in partial wave $\ell$, $\delta_\ell^{(1)}(k)$ and $\delta_\ell^{(2)}(k)$ are its first and second Born approximations, and $\Gamma^{(2)}[\phi_0]$ is the contribution to the energy from the two-point function, computed in ordinary Feynman perturbation theory. This piece includes the counterterms, which we fix using physical renormalization conditions.
We demand that the tadpole graph vanish, and that the mass of the free \( \phi \) particle is unchanged. In addition to holding the location of the pole in the propagator fixed at \( M \), we also perform wavefunction renormalization so that its residue is unchanged as well.

We can now compute eq. (10) directly. We simply require the scattering phase shifts, their Born approximations, and the bound states of eq. (12). Efficient algorithms for obtaining these are detailed in [1]. The contribution to the energy from the two-point function is computed using conventional techniques, giving

\[
\Gamma_2[\phi_0] = \int_0^\infty \frac{4q^2dq}{(4\pi)^3} \left[ (\frac{2\sqrt{q^2 + 4M^2}}{q}) \arctanh \frac{q}{\sqrt{q^2 + 4M^2}} - \frac{5\pi}{3\sqrt{3}} + \frac{1}{q} \right] |\tilde{\chi}(q)|^2 - 4q^2 \left( \frac{2\pi}{3\sqrt{3}} - 1 \right) |\tilde{\phi}(q)|^2 \tag{20}
\]

where \( \tilde{\chi}(q) \) and \( \tilde{\phi}(q) \) are the spatial Fourier transforms of \( U''(\phi_0(r)) - M^2 \) and \( \phi(r) \) respectively.

Examining this calculation in detail yields a very accurate estimate for the quantum correction to the energy, which is very easy to compute. Using the analysis of bound states above, we can separate eq. (19) into

\[
\mathcal{E}_c[\phi_0] = \Gamma^{(2)}[\phi_0] + \sum_{\ell=0}^{\infty} (2\ell + 1) \left[ \int_0^\infty \frac{dk}{\pi} E \frac{d}{dk} \left( \delta_{\ell}(k) - \delta_{\ell}^{(1)}(k) - \delta_{\ell}^{(2)}(k) \right) + \sum_j E_{j,\ell} \right] + (\omega - E_0) \tag{21}
\]

where \( E_0 \) is energy of the most tightly bound state, which appears in the \( \ell = 0 \) channel. It is the only state with energy less than \( \omega \). We define the reduced Casimir energy as

\[
\mathcal{E}_{c,\text{red}}[\phi_0] = \Gamma^{(2)}[\phi_0] + \sum_{\ell=0}^{\infty} (2\ell + 1) \left[ \int_0^\infty \frac{dk}{\pi} E \frac{d}{dk} \left( \delta_{\ell}(k) - \delta_{\ell}^{(1)}(k) - \delta_{\ell}^{(2)}(k) \right) + \sum_j E_{j,\ell} \right] = \Gamma^{(2)}[\phi_0] + \sum_{\ell=0}^{\infty} (2\ell + 1) \left[ \int_0^\infty \frac{dk}{\pi} (E - M) \frac{d}{dk} \left( \delta_{\ell}(k) - \delta_{\ell}^{(1)}(k) - \delta_{\ell}^{(2)}(k) \right) + \sum_j (E_{j,\ell} - M) \right] \tag{22}
\]

where we have used Levinson’s theorem in the second line. This quantity is simply the Casimir energy of a time-independent soliton giving rise to the reduced small oscillations of eq. (12). (Of course, such a soliton would not solve the field theory equations of motion, but we could imagine holding it in place with an external source). The reduced potential is shallow and slowly varying, especially in the limit of small \( Q \), which corresponds to \( \omega \) approaching \( M \). It causes only a slight deformation of the small oscillations spectrum — in particular, there is only one state bound more tightly than \( \omega \). For a generic potential of this kind, the contributions from the bound states and continuum will be opposite in sign; roughly, rearrangement of the continuum spectrum partially compensates for the effect of the states that become bound.

A direct application of [4] allows us to evaluate the full result of eq. (22). We can also estimate this result in the derivative expansion. To lowest order, we have simply the effective potential contribution

\[
\mathcal{E}_{c,\text{red,DE}}[\phi_0] = \int_0^\infty r^2dr \frac{M^4}{8\pi} \left( (1+z)^2 \log(1+z) - z - \frac{3}{2}z^2 \right) \tag{23}
\]

where

\[
z = \frac{U''(\phi_0(r)) - M^2}{M^2}. \tag{24}
\]

Using either technique, explicit computations show that this reduced Casimir energy is very small compared to the classical binding energy of the \( Q \)-ball (typically 5\% or less for small \( Q \)). Thus we lose very little accuracy by dropping this term, obtaining a very simple estimate for the Casimir energy:

\[
\mathcal{E}_c[\phi_0] \approx \mathcal{E}_{c,\text{red}}[\phi_0] = \omega - E_0. \tag{25}
\]

**APPLICATIONS**

To see whether the \( Q \)-ball is stable, we must compare its energy to the energy of a state with the same charge built on the trivial vacuum

\[
\mathcal{B}[\phi_0] = \mathcal{E}[\phi_0] -QM. \tag{26}
\]
FIG. 1: $Q$-ball binding as a function of $Q$, in units of $M$. Parameters are $A = 0.325M$ and $\lambda = 0.055$ (left panel), and $A = 0.425M$ and $\lambda = 0.095$ (right panel). Shown are three calculations of the difference $B$ between the energy of a $Q$-ball and the energy of a state with charge $Q$ built on the trivial vacuum: the classical approximation $B_{\text{class}}[\phi_0]$, the full one-loop calculation $B_{\text{full}}[\phi_0]$, and the estimated one-loop result $B_{\text{est}}[\phi_0]$.

Figure 1 shows the result of different calculations of $E$, each as a function of $Q$, for two choices of the coupling constants. In both cases, they are chosen so that $\phi = 0$ remains the global minimum of $U(\phi)$.

1. We work in units of $M$, which sets the scale of the problem. In the classical approximation,

$$B_{\text{class}}[\phi_0] = E_\omega[\phi_0] - QM$$ (27)

we see the result of 1: the $Q$-ball is stable for all $Q$, though the binding energy per charge is going to zero as $Q \to 0$.

In the full one-loop calculation,

$$B_{\text{full}}[\phi_0] = E_\omega[\phi_0] + E_C[\phi_0] - QM$$ (28)

we see that the quantum corrections overwhelm the weak classical binding up to $Q_{\text{min}} \approx 7$. Above this value, the $Q$-ball is stable. Finally, we see that using eq. (25) to approximate to the one-loop result by taking

$$B_{\text{est}}[\phi_0] = E_\omega[\phi_0] + E_{C_{\text{est}}}[\phi_0] - QM$$ (29)

yields a result that is very close the full one-loop result. It is also interesting to note that this approximation is particularly good near the value of $Q$ at which the $Q$-ball becomes bound in the full one-loop calculation.

CONCLUSIONS

We have computed one-loop quantum corrections to the energies of $Q$-balls. For small $Q$, these corrections can play an important role in determining the stability of these extended objects. Since $Q$-balls are genuine solutions to the classical equations of motion, and we are working in a regime where the coupling constants are small, the one-loop approximation should be very good. We have seen that the one-loop correction is indeed very small compared to the classical energy of the $Q$-ball, although can be significant when compared to its binding energy, the difference between the energy of the $Q$-ball and the energy of a collection of free particles carrying the same charge. The higher-loop corrections should be correspondingly smaller than the one-loop corrections, and therefore can safely be neglected. Of course, if the theory contains other particles coupled to the $\varphi$ field, we may need to include their contributions as well, using the same techniques as we have developed here. For small coupling constants, the one-loop approximation should continue to be reliable. Within the one-loop approximation, we have seen that the quantum correction can be very accurately estimated at small $Q$ by considering the difference between the Lagrange multiplier $\omega$, which gives

1 I thank M. Postma for reminding me of this requirement.
the frequency of the $Q$-ball’s time dependence, and the energy $E_0$ of the lowest bound state of the small oscillations potential $U''(\phi_0(x))$. The result is a prediction of the minimum value of $Q$ for which the $Q$-ball is stable in the quantum theory. For typical values of the coupling constants, we find $Q_{\text{min}} \approx 7$.

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