Non-Gaussian Velocity Distribution Function in a Vibrating Granular Bed

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The simulation of granular particles in a quasi-two-dimensional container under vertical vibration as an experimental accessible model for granular gases is performed. The velocity distribution function obeys an exponential-like function during the vibration and deviates from the exponential function in free-cooling states. It is confirmed that this exponential-like distribution function is produced by Coulomb friction force. A Langevin equation with Coulomb friction is proposed to describe the motion of such a system.

KEYWORDS: granular gas, velocity distribution function, non-Gaussian, flatness

Granular physics has been a challenging field in statistical physics since the rediscovery of significant nature in granular materials in the late ’80s or early ’90s.1 An assembly of grains has strong fluctuations in the configuration and the motion such that mean-field theories cannot be used in most situations. A typical example of the strong fluctuation appears in the force distribution for a static granular assembly piled by gravity.2–4 The force propagates along force chains and the distribution function of the magnitude of the force on the bottom of a container does not obey a Gaussian function but an exponential function.2–4

Such a strong fluctuation originating from non-Gaussian properties should be relevant even in the dynamics of granular assemblies. However, there have not been many systematic research studies that focus on the statistical distribution functions in the steady state, because real and numerical experiments report no unified results, i.e., velocity distribution functions (VDFs) obey Gaussian-like functions with exponential tails,5–12 from the Gaussian to the exponential depending on density,13,14 stretched exponential15–17 and even power-law functions.18–21

As in standard statistical mechanics, an assembly of grains in a gas phase is an idealistic system for studying statistical weight. Approximate granular gases can be obtained by rapid granular flows on inclined slopes,22,23 gas-solid mixtures24,25 and grains under the external vibration.11–18 However, these systems cannot be regarded as idealistic granular gases, because (i) the boundary and gravity effects are strong in the rapid granular flows, (ii) the hydrodynamic interaction between particles in gas-solid mixtures are complicated, and (iii) a dense cluster appears in the vibrating experiment. Therefore, it is difficult to achieve free-cooling

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In this letter, to remove such problems, at first, we propose an experimental accessible situation to produce granular gases. Second, we demonstrate that VDFs in both dense and dilute granular gases under vertical vibration can obey exponential-like functions when Coulomb friction is important through our simulation based on the distinct element method (DEM). This exponential VDF disappears immediately after vibration has stopped, i.e., the grains are in a free-cooling process. Third, we introduce a phenomenological Langevin equation eq. (5) to describe the motion of particles and explain the mechanism that causes the exponential VDF to appear. The detailed analysis of experimental accessible granular gases will be reported elsewhere.

We use a three-dimensional DEM for monodispersed spherical particles. Instead of using the Hertzian contact force, we adopt the linear spring model to represent the repulsion of contacted spheres. We also include the rotation of spheres and Coulomb slip for tangential contact. The equations of motion of the sphere with the diameter $d$, the mass $m$ and the momentum of inertia $I = md^2/10$ at contact are given by

$$m \ddot{x}_i = \sum_{<ij>} \{ F_{nij} n_{ij} + \tilde{F}_{tij} t_{ij} \} - mg,$$

$$I \dot{\omega} = \frac{d}{2} \sum_{<ij>} \tilde{F}_{tij} \{ n_{ij} \times t_{ij} \},$$

where $n_{ij}, t_{ij}, \sum_{<ij>}$ and $g$ represent the normal and the tangential unit vectors at the contact point of the $j$ and $i$ particles, the summation over $j$ particles which are in contact with $i$ particle, and the gravitational acceleration, respectively. The contact forces $F_{nij}$ and $\tilde{F}_{tij}$ are respectively given by

$$F_{nij} = -k_n (d - |x_i - x_j|) - m\eta_n n_{ij} \cdot (\dot{x}_i - \dot{x}_j),$$

$$\tilde{F}_{tij} = \left\{ \begin{array}{ll} F_{tij} & \text{(if } |F_{tij}| < \mu|F_{nij}|) \\ \mu F_{nij} & \text{(otherwise)} \end{array} \right.,$$

where

$$F_{tij} = -k_t (d - |x_i - x_j|) - m\eta_t t_{ij} \cdot (\dot{x}_i - \dot{x}_j + \frac{d}{2}(\omega_i + \omega_j)).$$

Here, the friction coefficient $\mu$ describing Coulomb slip is assumed to be $\mu = 0.8$. In addition, we introduce four parameters $k_n, k_t, \eta_n$ and $\eta_t$ which are assumed to be $k_n = 5.0 \times 10^4 mg/d$, $k_t = 2.5 \times 10^4 mg/d$, $\eta_n = 4.47 \times 10^{\sqrt{d/g}}$ and $\eta_t = \eta_n/2$ in our simulation. This setup corresponds to the normal restitution coefficient $e = 0.638$. We adopt the second-order Adams-Bashforth method for the time-integration with the time interval $\Delta t = 1.0 \times 10^{-4}\sqrt{d/g}$.

We focus on the following situation. Particles are confined in a quasi-two-dimensional
Fig. 1. Schematic side view (left) and top view (right) of our setup. The random scatters are fixed spherical particles on the top board. The diameter of fixed particles on the top plate (white ones) is uniformly random between 0.6$d$ and 0.8$d$, and their centers are located 0.15$d$ above the top board. To avoid the stacking of particles in the corners, we introduce four fixed particles (black ones in the figure) at the corners.

container in which the height is 1.8$d$ and the horizontal plane is a square (Fig. 1). In this system, particles cannot form a multilayer configuration in the vertical direction, but the particles have vertical velocities in their motion. The mobile particles are randomly scattered by fixed particles on the top of the container if the container is vertically vibrated. The area fractions of fixed scatters are 0.21, 0.23 and 0.25 for 1,000, 3,000 and 10,000 mobile particles, respectively. The vibration is driven by a sinusoidal force whose amplitude and frequency are given by $A = 1.2d$ and $f = 0.5\sqrt{g/d}$. Thus, the acceleration amplitude becomes $\Gamma = A(2\pi f)^2/g = 11.5$. If the vibration is stopped, the particles move with the rotation on the bottom plate, then the system can simulate a two-dimensional free-cooling granular gas. We emphasize that the rotational particles on a flat plane can be regarded as an approximate assembly of gas particles in a two-dimensional system, because the energy loss in the rolling friction is very small in many cases.

The simulation starts from a stationary state of particles on the bottom plate. The particles gain the kinetic energy from the external oscillation and the system reaches a ‘steady state’ in the balance between collisional dissipations and the gain of the energy from the external force. Typically, the system reaches the statistical ‘steady state’ after 25 oscillation cycles in which hydrodynamic quantities are independent of how many cycles proceed from the initial condition but depend only on the phase of oscillation.

In the ‘steady state’, we confirm that the density is uniform and there is no long-range correlation in contrast with granular gases in free-cooling states. Thus, the system does not have any systematic flows nor any definite clusters. We also check that the effects of side boundaries can be neglected. This is because particles always hit the top wall or the bottom wall during the vibration. After we stop the vertical oscillation, the correlation grows with time as the free-cooling process proceeds.

The most significant quantity for charactering this gas system is VDF. In the ‘steady
Fig. 2.  Scaled VDF $\tilde{f}(c)$ in ‘steady states’ under the vertical vibration. Here, $\Psi$ represents the area fraction. For $\Psi = 7.2\%$ we use 10,000 particles, and use 3,000 particles for other cases.

Table I. Flatnesses (FN) under several conditions. See text for details.

| Condition           | Number of particles | Fractions(%) | FN   |
|---------------------|---------------------|--------------|------|
|                     |                     | Area  | Volume |      |
| steady state 1      | $1.0 \times 10^3$   | 6.5   | 2.4    | 8.00 |
|                     |                     | 10    | 3.7    | 7.63 |
|                     |                     | 20    | 7.4    | 6.90 |
|                     |                     | 30    | 11     | 6.26 |
|                     |                     | 40    | 15     | 5.67 |
|                     |                     | 50    | 19     | 5.36 |
|                     | $3.0 \times 10^3$   | 6.0   | 2.2    | 9.57 |
|                     |                     | 10    | 3.7    | 8.13 |
|                     |                     | 20    | 7.4    | 7.03 |
|                     |                     | 30    | 11     | 6.34 |
|                     |                     | 40    | 14     | 5.64 |
|                     |                     | 50    | 18     | 5.27 |
|                     | $1.0 \times 10^4$   | 7.2   | 2.6    | 6.85 |
| free-cooling 1      | $1.0 \times 10^4$   | 7.2   | 2.6    | 4.20 |
| no tangential 1     | $1.0 \times 10^3$   | 6.5   | 2.4    | 3.26 |
|                     | $3.0 \times 10^3$   | 6.0   | 2.2    | 3.43 |
| no friction 1       | $1.0 \times 10^3$   | 6.5   | 2.4    | 3.27 |
|                     | $3.0 \times 10^3$   | 6.0   | 2.2    | 3.43 |
| undulation 1        | $1.3 \times 10^3$   |       |        | 4.10 |
state’, the vertical component of VDF has double peaks where each peak corresponds to a lifting process or a falling process, while the horizontal VDF has a single peak. For later discussion, we are only interested in the horizontal VDFs for analysis. We plot the scaled horizontal VDF \( \tilde{f}(c) \) in Fig. 2 using
\[
f(v,t) = n v_0(t)^{-2} \tilde{f}(v/v_0)
\] with the density \( n \), the average speed \( v_0 = \sqrt{2T/m} \) with the granular temperature \( T \), and \( \int dc \tilde{f}(c) = \int dcc^2 \tilde{f}(c) = 1 \), where \( \tilde{f}(c) \) is averaged over the cycles. As in Fig. 2, the scaled VDF can be approximated by an exponential function. In fact, the flatness defined by \( < c^4 > / < c^2 >^2 = < c^4 > \) with \( < c^2 > = \int dcc^2 \tilde{f}(c) \) is not far from 6. It should be noted that the flatness with the Gaussian VDF is 3 and that with the exponential VDF is 6. In our simulation, the flatnesses are summarized in Table I for 1,000- and 3,000-particle simulations as functions of area fractions projected into the horizontal plane. In addition, the dependence of the flatness on the density is relatively weaker in our situation than that by Murayama and Sano.\(^{13} \) A large system with 10,000 particles with an area fraction of 7.2% has a smaller flatness of 6.85. If the external oscillation stops, the flatness decreases quickly and is saturated at 4.20 for 10,000 particles. As can be seen in Fig. 3, VDF in the cooling process is nearly Gaussian for low-energy particles but has an exponential tail for high-energy particles as in the usual gas systems.\(^{5–7} \)

Let us consider the origin of the exponential-like VDF. It is easy to verify that the exponential VDF cannot be reproduced without the existence of Coulomb friction in eq. (2). Namely, if we eliminate the tangential component of the contact force or the slip rule for \( |F_{ti}^j| < \mu|F_{nj}^j| \), VDF becomes a Gaussian-like distribution (Fig. 3). In fact, if all the components of tangential contact force are omitted, the flatness becomes 3.43 for 3,000 particles with the area fraction of 6.0%. While if we only neglect the effect of Coulomb friction in eq. (2), the flatness becomes 3.43 in the same situation (Table I). In our system, particles experience an strong shear force when they hit the fixed scatters on the top wall, and their direction of motion changes drastically. Thus, the tangential slips between particles and the fixed scatters are the dominant dissipative processes in the ‘steady state’. Since we specify the origin of exponential-like VDF, we can understand the weak dependence of VDF on the density. Namely, particles directly hit the fixed scatters for dilute case, while dense particles collide with each other and rotate without slips besides the collisions with the walls.

Therefore, the essence to produce the large flatness in VDF is apparently Coulomb friction. Thus, we propose the following Langevin equation to describe the horizontal motion of particles.
\[
\frac{du}{dt} = -\gamma u - \nabla \Phi + \eta
\]
Here, \( u, u = |u| \) and \( \Phi \) are respectively the velocity, the speed and the potential exerted
among particles. The friction coefficient $\gamma$ may be proportional to $\mu g$. The $\alpha$ component of the random force $\eta$ satisfies the fluctuation-dissipation relation.

$$< \eta_\alpha(t) >= 0, \quad < \eta_\alpha(t)\eta_\beta(t') >= 2D\delta_\alpha\delta(t-t')$$

Here, $D = \gamma\sqrt{T/3m}$ is the diffusion coefficient in the velocity space. The Langevin equation eq. (5) with eq. (6) can be converted into the Fokker-Planck or Kramers equation for the probability distribution function $P(x, u, t)$.

$$\frac{\partial P(x, u, t)}{\partial t} = \{-\nabla \cdot u + \frac{1}{m} \frac{\partial}{\partial u} \cdot \nabla \Phi + \gamma \frac{\partial}{\partial u} \frac{u}{u} + D \frac{\partial^2}{\partial u^2}\} P(x, u, t)$$

For spatial homogeneous ‘steady states’, the equation for $P \to P_{st}$ is reduced to $(u/u)P_{st} + \sqrt{T/3m}(d/du)P_{st} = 0$ for $u \neq 0$. Its solution can be obtained easily

$$P_{st}(u) = 2\sqrt{\frac{m}{3T}} \exp[-\sqrt{\frac{3m}{T}}u].$$

Thus, the Langevin equation with Colomrb friction law obeys the exponential VDF.

To check the validity of the new Langevin equation for the motion of particles, we evaluate both the diffusion coefficient $D$ and the friction coefficient $\gamma$. At first, we have confirmed that the motion of particles in the horizontal plane is diffusive. Then, the diffusion coefficient is evaluated from the relation $<(r(t) - r(t_0))^2> = 4D(t - t_0)$, where we choose $t_0$ as 25 cycles of the oscillation and simulate the motion of particles until 75 cycles. For each parameter of the oscillation we determine $T$ from the second moment of VDF. Thus, we obtain the relation between $D$ and $T$ for simulations of 1000 and 3000 particles as in Fig. 4. The best
Fig. 4. Relationship between \( \tilde{D} \) and \( \tilde{T} \). The solid line represents \( \tilde{D} = 0.0744\sqrt{\tilde{T}} - 0.0054 \). (a) and (b) represent the data for 3,000 particles and 1,000 particles, respectively.

The fitted relation is \( \tilde{D} = 0.0735\tilde{T}^{0.452} - 0.0078 \), where \( \tilde{D} = D/(a^{1/2}g^{3/2}) \) and \( \tilde{T} = T/(mgd) \), but this can be replaced by \( \tilde{D} = 0.0744\sqrt{\tilde{T}} - 0.0054 \). This result seems to be independent of the system size. We also note that the collective vertical motion suppresses the diffusion for larger \( T \), and no diffusion takes place because of the insufficient kinetic energy for smaller \( T \). From the relation between \( \tilde{D} \) and \( \tilde{T} \), we evaluate \( \gamma \) as \( \gamma = 0.129g \). Note that \( \gamma \) is independent of \( T \), because the dominant dissipative process involves collisions between particles and the horizontal walls caused by the collective motion of particles in the vertical direction. Thus, we believe that the Langevin equation eq. (5) can be used to describe the motion of particles.

Figure 3 also contains the VDF obtained from the simulation of granular particles confined in a dense cluster which exhibits undulations. In the simulation, we use \( k_n = 2.0 \times 10^4 mg/d \), \( k_t = k_n/2 \), \( \eta_n = \eta_t = 3.8 \times 10^3 \sqrt{g/d} \) and \( \mu = 0.4 \) with polydispersed particles whose diameters are uniformly distributed between 0.8\( d \) and \( d \). The number of particles is 1,300 and confined in a thin box whose depth is 4.5\( d \) and width is 60\( d \). The initial accumulated height of particles is approximately 6 layers. In this situation, the relative motion of particles is suppressed and particles are confined in a lattice-like cage. In this case, VDF is deviated from the exponential function where the flatness is approximately 4.10(Table I), but the high-energy particles apparently obey an exponential law as in the free-cooling case(Fig. 3). It is easy to imagine that particles with a high energy can slip on the cage and particles with a low energy oscillate within cages. This picture may be valid for the gas system in the free-cooling process. Thus, VDF can be approximated by the combination of the Gaussian part for low-energy particles and the exponential part for high-energy particles. The VDFs for the undulation and the free-cooling process are similar to those in experiments under vertical vibration in which a dense cluster exists and each VDF can be approximated by a single stretched exponential function.11,12,14–18 We consider, however, that the stretched exponential VDF can be understood
by the combination of the Gaussian part and the exponential part as stated here.

We also recall that VDF obeys a Gaussian-like function for dense flows on an inclined slope.\textsuperscript{28} This is because particles can move without slips because the lattice-like cage is very weak. We also compare our results with those of Murayama and Sano.\textsuperscript{13} Because of the nonexistence of random scatters in their simulation, the slip processes can occur as results of collisions among particles. Thus it is reasonable to get the transition from the Gaussian to the exponential as the density increases. On the other hand, in our case, the slips take place mainly in collisions between particles and fixed random scatters. Therefore the tendency to obey the exponential is emphasized for dilute gases.

This mechanism to obtain the exponential distribution function may be used for other situations, because Coulomb friction plays roles in sliding frictions.\textsuperscript{29} To look for possibilities of the universal feature of our scenario will be our future task.

In conclusion, we propose an experimental accessible system for granular gases. The VDF in a ‘steady state’ obeys an exponential-like function but changes to Gaussian-like distribution function when free-cooling starts. This exponential VDF is caused by Coulomb friction force. Thus, we propose the Langevin equation with Coulomb friction to reproduce the results of our simulation.

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