A MIRROR SYMMETRIC CONSTRUCTION OF $qH^*_T(G/P)_{(q)}$

KONSTANZE RIETSCH

Abstract. Let $G$ be a simple simply connected complex algebraic group. We give a Lie-theoretic construction of a conjectural mirror family associated to a general flag variety $G/P$, and show that it recovers the Peterson variety presentation for the $T$-equivariant quantum cohomology rings $qH^*_T(G/P)_{(q)}$ with quantum parameters inverted. For $SL_n/B$ we relate our construction to the mirror family defined by Givental and its $T$-equivariant analogue due to Joe and Kim.

1. Introduction

According to Givental [17] and Eguchi, Hori and Xiong [11] mirror symmetry should have an extension to Fano manifolds $X$, where it means in essence a ‘mirror side’ representation of the quantum cohomology $D$-module, or quantum differential equations, of $X$ by complex oscillatory integrals. Such mirror models have previously been constructed for toric Fano manifolds and the flag variety $SL_n/B$ by Givental [17, 18]. Moreover for $SL_n/B$ Joe and Kim proved also a $T$-equivariant version of Givental’s mirror theorem [20].

In this paper we are interested in the case where $X$ is a general flag variety $G/P$. Explicitly the ingredients for the mirror symmetric model associated to $X$ should be the following.

1. A $k$-parameter family $Z$ of (affine) varieties $Z_s$ of dimension $d = \dim\mathbb{C}(X)$.
   Here $k = \dim H^2(X, \mathbb{C})$.

2. A family of holomorphic $d$-forms $\omega_s$ on the fibers $Z_s$ in the family.

3. A holomorphic function $F : Z \rightarrow \mathbb{C}$, which will play the role of the phase.

From these data one can write down complex oscillatory integrals

\[
S_F(s) = \int_{\Gamma_s} e^{F/\hbar} \omega_s,
\]

where the $\Gamma$ are certain continuous families of (possibly non-compact) cycles $\Gamma_s$ in $Z_s$, for example associated to $F$ via Morse theory of $Re(F)$, see [17], [1].

In our case $X = G/P$ and has an action of a maximal torus $T$. Let $\mathfrak{h}$ be the Lie algebra of $T$. To obtain a $T$-equivariant analogue we need to add one more item to the data (1-3).

4. A multi-valued holomorphic function $\phi : Z \times \mathfrak{h} \rightarrow \mathbb{C}$. Or more precisely, a holomorphic function $\tilde{\phi}$ on a covering $\tilde{Z} \times \mathfrak{h}$ of $Z \times \mathfrak{h}$.
Using (4) one can write down the more general integrals
\[(1.2) \tilde{\mathcal{S}}(\Gamma, s, h) = \int_{\Gamma_s} e^{\tilde{\mathcal{F}}/h} \tilde{\phi}(\cdot, h) \tilde{\omega}_s,\]
where \(\Gamma_s\) now lies in the covering \(\tilde{Z}\) of \(Z\) and we have denoted the pullbacks of \(\mathcal{F}\) and \(\omega_s\) to \(\tilde{Z}\) by \(\tilde{\mathcal{F}}\) and \(\tilde{\omega}_s\), respectively.

Mirror symmetry for \(G/P\) should involve a presentation of the set of solutions to the \((T\text{-equivariant})\) quantum differential equations associated to \(G/P\), see [16, 20, 9], via integrals of the form (1.1), respectively (1.2).

We now turn our attention to quantum cohomology. There is a remarkable, unified Lie-theoretic presentation for the \((T\text{-equivariant})\) quantum cohomology rings \(qH^*_T(G/P)\) which was discovered by Dale Peterson [33]. From his point of view the quantum cohomology rings arise as (possibly non-reduced) coordinate rings,
\[qH^*_T(G/P) \cong \mathbb{C}[\mathcal{Y}_P],\]
where \(\mathcal{Y}_P\) is a particular affine stratum, of the so-called ‘Peterson variety’ \(\mathcal{Y}\) in \(G' / B' \times h\). We will review Peterson’s results in Section 3.2.

Following Givental [16] on the other hand, relations for the small quantum cohomology ring are obtained as equations for the characteristic variety of the quantum cohomology \(D\)-module. And for this variety there is a somewhat corresponding construction on the mirror side, which is to look at what is ‘swept out’ by the critical points of \(\mathcal{F}\) along the fibers of the family \(Z\) (or the critical points of \(\mathcal{F} + \ln \phi(\cdot, h)\), in the \(T\)-equivariant case).

In this paper we will give a Lie theoretic construction associating to any \(G/P\) a family \(Z = Z_P\) with associated data (1-4). The fibers \(Z_s\) of the family turn out to have natural compactifications to \(G' / P'\), and the base is the algebraic torus \(H^2(G/P, \mathbb{C}) / 2\pi i H^2(G/P, \mathbb{Z})\), or is \(H^2(G/P, \mathbb{C})\) if we pull back along the exponential map. The main result, Theorem 4.1, says that the critical points of \(\mathcal{F} + \ln \phi(\cdot, h)\) along the fibers \(Z_s\) and for varying \(h\) indeed recover the Peterson variety stratum \(\mathcal{Y}_P\) (or, more precisely, the open dense part in \(\mathcal{Y}_P\) where the quantum parameters are nonzero).

This result supports the mirror conjectures, stated in Section 8, that the integrals (1.1) and (1.2) defined in terms of our data \((Z_P, \omega, \mathcal{F}_P, \phi_P)\) should give solutions to the quantum differential equations associated to \(G/P\) and their \(T\)-equivariant analogues, respectively.

In the final section we verify these mirror conjectures in the special case of \(SL_n/B\) by comparing our mirror construction with Givental’s [17], and, in the equivariant case, with the construction of Joe and Kim, [20]. Explicitly, Givental’s mirror family is shown to appear as an open subset inside our \(Z_B\), and \(\mathcal{F}\) and the \(\omega_s\) are related by restriction. The relationship with Joe and Kim’s integrals is via a comparison map which is a covering of an open inclusion, but 1-1 on any of Joe and Kim’s integration contours.

We plan to discuss the mirror conjecture for the general \(G/B\) case in a future paper. In that setting the quantum differential equations were determined by Kim, and the mirror conjecture can be interpreted as saying that the integrals define Whittaker functions obeying the quantum Toda lattice associated to the Langlands dual root system. In this direction, but still confined to type \(A\), there has already been some interesting independent work of Gerasimov, Kharchev, Lebedev and Oblezin [14], who reproved Givental’s mirror theorem using representation theory.
This work was motivated on the one hand by a desire to put the papers [17, 20, 3], concerning mirror constructions for classical flag varieties, into a Lie theoretic context. And on the other hand it is an attempt to better understand Dale Peterson’s powerful point of view about quantum cohomology, [33]. Morally speaking, it says that Peterson’s presentation of \( qH^*_T(G/P) \) via the variety \( Y_P \) might be considered a mirror symmetry phenomenon for \( G/P \).

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2. Background and notation

Let \( G \) be a simple simply connected algebraic group over \( \mathbb{C} \) of rank \( n \). We fix opposite Borel subgroups \( B = B_- \) and \( B_+ \) with unipotent radicals \( U_- \) and \( U_+ \), respectively. Let \( T \) be the maximal torus \( T = B_+ \cap B_- \), and \( W = N_G(T)/T \) the Weyl group.

Let \( \mathfrak{g} \) be the Lie algebra of \( G \) and \( \mathfrak{b}_-, \mathfrak{b}_+, \mathfrak{u}_- \), \( \mathfrak{u}_+ \), \( \mathfrak{h} \) the Lie algebras of \( B_- \), \( B_+, U_- \), \( U_+ \), and \( T \), respectively. The adjoint action of \( G \) is denoted by a dot for simplicity. So \( g \cdot X := \text{Ad}(g)X \), for \( g \in G \) and \( X \in \mathfrak{g} \). Similarly for the coadjoint action, so when \( X \in \mathfrak{g}^* \).

Let \( X^*(T) \) be the character group of \( T \) and \( Q \subset X^*(T) \) the root lattice. We will sometimes view these as lying in \( \mathfrak{h}^* \). Let \( \Delta_+ \) be the set of positive roots corresponding to \( B_+ \), so that the Lie algebra of \( B_+ \) written as sum of weight spaces with respect to the adjoint action of \( T \) is

\[
\mathfrak{b}_+ = \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha.
\]

We set \( I = \{1, \ldots, n\} \), where \( n \) is the rank of \( G \), and use \( I \) to enumerate the simple roots \( \{\alpha_i \mid i \in I\} \) in \( \Delta_+ \). Corresponding to the simple roots (and their negatives), we have the Chevalley generators \( e_i \) and \( f_i \) in \( \mathfrak{g}_{\alpha_i} \) and \( \mathfrak{g}_{-\alpha_i} \), respectively. These define the one parameter subgroups

\[
x_i(t) := \exp(te_i), \quad y_i(t) := \exp(tf_i),
\]

where \( t \in \mathbb{C} \). Let

\[
(2.1) \quad \hat{s}_i = x_i(1)y_i(-1)x_i(1).
\]

Then \( \hat{s}_i \) represents a simple reflection in \( W \) which we denote by \( s_i \). For general \( w \in W \), a representative \( \hat{w} \in G \) is defined by \( \hat{w} = \hat{s}_{i_1}\hat{s}_{i_2}\cdots\hat{s}_{i_m} \), where \( s_{i_1}s_{i_2}\cdots s_{i_m} \) is a (any) reduced expression for \( w \). The length \( m \) of a reduced expression for \( w \) is denoted by \( \ell(w) \).

Let \( P \supseteq B \) be a (fixed) parabolic subgroup of \( G \). Define \( I_P = \{i \in I \mid \hat{s}_i \in P\} \) and let \( I^P \) be its complement in \( I \). We will usually denote the elements of \( I^P \) by

\[
I^P = \{n_1, \ldots, n_k\},
\]

for \( 1 \leq n_1 < n_2 < \cdots < n_k \). We denote by \( W_P \) the parabolic subgroup of \( W \) associated to \( P \), and by \( W^P \) the set of minimal length coset representatives in
Let $w_P$ be the longest element in the parabolic subgroup $W_P$. For example $w_B = 1$ and $w_G$ is the longest element in $W$, also denoted $w_0$.

Let $G^\vee$ be the Langlands dual group to $G$. Note that $G^\vee$ is adjoint since $G$ was simply connected. We will use all the same notation for $G^\vee$ as for $G$, but with an added superscript where required. For example the Chevalley generators of $g^\vee$ are denoted by $e_i^\vee$ and $f_i^\vee$, where $i \in I$. The Weyl group for $G^\vee$ is again $W$. For simplicity we will write $\check{w}$ again for the representative of $w$ in $G^\vee$ obtained as above. Identify $\mathfrak{h}^\vee$ with $\mathfrak{h}^*$, the dual of the Lie algebra of $T$. In particular we may view the weight and root lattices of $G$ as lying inside $\mathfrak{h}^\vee$. The dual pairing between $\mathfrak{h}$ and $\mathfrak{h}^\vee$ is denoted by $\langle , \rangle$.

We will also consider the universal covering group $\tilde{G}^\vee$ of $G^\vee$. Let $\pi : \tilde{G}^\vee \rightarrow G^\vee$ be the covering map. The group $\tilde{G}^\vee$ has maximal torus $\tilde{T}^\vee = \pi^{-1}(T^\vee)$ and Borel subgroups $\tilde{B}_-^\vee = \pi^{-1}(B_-^\vee)$ and $\tilde{B}_+^\vee = \pi^{-1}(B_+^\vee)$. The unipotent radicals of $\tilde{B}_-^\vee$ and $\tilde{B}_+^\vee$ can be identified with $U_-^\vee$ and $U_+^\vee$, respectively, via $\pi$. Also the Weyl group representatives in $\tilde{G}^\vee$ defined via (2.1) are identified via $\pi$ with those in $G^\vee$, and we will suppress the difference in our notation.

For any dominant coweight $\lambda^\vee$ we have an irreducible representation $V(\lambda^\vee)$ of $\tilde{G}^\vee$. In each $V(\lambda^\vee)$ let us fix a lowest weight vector $v^\vee_{\lambda^\vee}$. Then for any $v \in V(\lambda^\vee)$ and extremal weight vector $\check{w} \cdot v^\vee_{\lambda^\vee}$ we have the coefficient $\langle v, \check{w} \cdot v^\vee_{\lambda^\vee} \rangle \in \mathbb{C}$ defined by

$$v = \langle v, \check{w} \cdot v^\vee_{\lambda^\vee} \rangle \check{w} \cdot v^\vee_{\lambda^\vee} + \text{other weight space summands}.$$ 

Let $v^\vee_{\lambda^\vee} := \check{w}_0 \cdot v^\vee_{\lambda^\vee}$. The most important choices for $\lambda^\vee$ are the fundamental coweights $\omega_i^\vee$, where $i \in I$, and $\rho^\vee := \sum_{i \in I} \omega_i^\vee$. If $\lambda^\vee \in Q^\vee$ then also $G^\vee$ acts on $V(\lambda^\vee)$.

In the Langlands dual context we will consider the flag variety $G^\vee/B_-^\vee$. Then for two elements $v, w \in W$ with $v \leq w$ we have the intersection of opposed Bruhat cells

$$\mathcal{R}^\vee_{v, w} := (B_-^\vee \check{w}B_-^\vee \cap B_-^\vee \check{w}B_-^\vee)/B_-^\vee$$

in $G^\vee/B_-^\vee$. It is known that $\mathcal{R}^\vee_{v, w}$ is smooth and irreducible of dimension $\ell(w) - \ell(v)$, [21] [30].

3. Equivariant quantum cohomology of $G/P$ and Peterson’s presentations.

3.1. The (small) quantum cohomology ring of $G/P$ is a deformation of the usual cohomology ring with $k = \dim H^2(G/P)$ parameters,

$$qH^*(G/P) \cong H^*(G/P) \otimes \mathbb{C}[q_1, \ldots, q_k],$$

where the deformed cup product has structure constants given by genus 0, 3-point Gromov-Witten invariants. We refer the reader to [9] [12] for definitions and background. Note that we will always take coefficients to be in $\mathbb{C}$. For an equivariant
version of quantum cohomology see the papers [4, 19, 23]. The $T$-equivariant quantum cohomology $qH^*_T(G/P)$ is a module over $\mathbb{C}[q_1, \ldots, q_k]$ and $\mathbb{C}[h]$, and is a simultaneous deformation of the quantum cohomology and the equivariant cohomology rings.

In the literature there are many special cases of flag varieties where presentations of quantum cohomology rings have been explicitly determined. See for example [22, 2, 8, 23] in type $A$, [23] for general $G/B$, and [38] for Grassmannians in other types.

The structure of (non-equivariant) quantum cohomology for general $G/P$ is described in [13, 39, 33]. For $qH^*_T(G/P)$ Mihalcea has given a quantum Chevalley formula [32], and thereby completely determined the ring structure.

The only general construction of presentations for quantum cohomology rings of flag varieties $G/P$ is due to Dale Peterson [33], unpublished so far. It involves the remarkable ‘Peterson variety’ $\mathcal{Y}$ which we now introduce.

3.2. Following [33] we define a closed $2n$-dimensional subvariety $\mathcal{Y}$ of $G^\vee/B^\vee \times \mathfrak{h}$. Let us canonically identify $\mathfrak{h}$ with the zero weight space $(g^\vee)^*$ via $\mathfrak{h} \cong (g^\vee)^*$. Define

$$F := \sum_{i \in I} (e^\vee_i)^* \in (g^\vee)^*,$$

where $(e^\vee_i)^*$ denotes the linear functional which is one on $e^\vee_i$ and zero along all other weight spaces. We write $g \cdot \eta$ for the coadjoint action of $g \in G^\vee$ on $\eta \in (g^\vee)^*$. The (equivariant) Peterson variety is the subvariety of $G^\vee/B^\vee \times \mathfrak{h}$ defined by

$$\mathcal{Y} := \{ (gB^\vee, h) \in (G^\vee/B^\vee) \times \mathfrak{h} \mid g^{-1} \cdot F \text{ vanishes on } [u^\vee, u^\vee] \}.$$

Its fiber over $0 \in \mathfrak{h}$ is

$$Y := \{ gB^\vee \mid g^{-1} \cdot F \text{ vanishes on } [u^\vee, u^\vee] \},$$

and may also be called the Peterson variety, see [27].

To a parabolic $P \supseteq B$ associate strata $\mathcal{Y}_P$ and $Y_P$, in $\mathcal{Y}$ and $Y$, respectively, which arise from (possibly non-reduced) intersections with Bruhat cells for $B^\vee$,

$$\mathcal{Y}_P := \mathcal{Y} \times_{(G^\vee/B^\vee) \times \mathfrak{h}} (B^\vee_w \check{w}_PB^\vee/B^\vee \times \mathfrak{h}),$$

$$Y_P := Y \times_{G^\vee/B^\vee} (B^\vee_w \check{w}_PB^\vee/B^\vee).$$

Moreover we consider the following open subvarieties obtained by intersection with the big Bruhat cell for $B^\vee$,

$$\mathcal{Y}^* := \mathcal{Y} \times_{(G^\vee/B^\vee) \times \mathfrak{h}} (B^\vee_w \check{w}_0B^\vee/B^\vee \times \mathfrak{h}),$$

$$Y^* := Y \times_{G^\vee/B^\vee} (B^\vee_w \check{w}_0B^\vee/B^\vee),$$

and their strata

$$\mathcal{Y}_P^* := \mathcal{Y} \times_{(G^\vee/B^\vee) \times \mathfrak{h}} (R^\vee_{w_P, w_0} \times \mathfrak{h}),$$

$$Y_P^* := Y \times_{G^\vee/B^\vee} R^\vee_{w_P, w_0}.$$
3.3.1. First of all, \( \mathcal{Y}_P \) and \( Y_P \) are (possibly non-reduced) affine varieties of pure dimension \(|I^P| + n\) and \(|I^P|\), respectively, and one has the decomposition

\[
\mathcal{Y}(\mathbb{C}) = \bigcup_P \mathcal{Y}_P(\mathbb{C})
\]

for the \( \mathbb{C} \)-valued points. Here \( P \) runs over the set of all parabolic subgroups of \( G \) containing \( B \). See [27, 28] for a treatment of the non-equivariant case, in particular Kostant proved that \( Y_B \) is irreducible.

3.3.2. There is an isomorphism,

\[
\mathcal{Y}^* \cong \{ (b, h) \in B^\vee \times \mathfrak{h} \mid b \cdot (F - h) = F - h \}
\]

given by the map \( (b, h) \mapsto (b, hB^\vee, h) \) from right to left. Setting \( h = 0 \) we have that \( \mathcal{Y}^* \) is isomorphic to the stabilizer in \( B^\vee \) of \( F \).

3.3.3. There is an explicit isomorphism

\[
qH^*(G/P) \xrightarrow{\sim} \mathbb{C}[\mathcal{Y}_P],
\]

from the quantum cohomology ring of \( G/P \) to the coordinate ring of \( Y_P \). See [31, 35] for a description and proof of Peterson’s isomorphism in the case of \( qH^*(SL_n/P) \).

Replacing \( Y \) by \( \mathcal{Y} \) and \( qH^*(G/P) \) by \( qH_q^*(G/P) \) in (3.1) gives an equivariant version of this result,

\[
qH_q^*(G/P)[q_1^{-1}, \ldots, q_k^{-1}] \xrightarrow{\sim} \mathbb{C}[\mathcal{Y}_P],
\]

which was formulated in [33], and follows from [33] and [32].

3.3.4. The varieties \( \mathcal{Y}_P^*, Y_P^* \) are open dense in \( \mathcal{Y}_P \) and \( Y_P \), respectively, and the map (3.2) induces an isomorphism

\[
qH_q^*(G/P)[q_1^{-1}, \ldots, q_k^{-1}] \xrightarrow{\sim} \mathbb{C}[\mathcal{Y}_P^*],
\]

3.3.5. In the case of \( G/B \) the isomorphism

\[
qH_q^*(G/B) \xrightarrow{\sim} \mathbb{C}[\mathcal{Y}_B],
\]

is related to Kim’s presentation [21] of \( qH^*(G/B) \) as follows. Kim described the relations of the \( T \)-equivariant small quantum cohomology ring of \( G/B \) in terms of integrals of motion of the Toda lattice associated to the Langlands dual group. The phase space \( T^*(T^\vee) = T^\vee \times \mathfrak{h} \) of the Toda lattice for \( G^\vee \) may be embedded into \( (\mathfrak{g}^\vee)^* \) by

\[
(t, h') \mapsto F - h' - \sum_{i \in I} \alpha_i(t)(f_i^\vee)^*,
\]

where \( (f_i^\vee)^* \) is defined analogously to \( (e_i^\vee)^* \) and \( \mathfrak{h} \) is identified with \( (\mathfrak{h}^\vee)^* \) viewed as a subspace of \( (\mathfrak{g}^\vee)^* \). This is Kostant’s construction [20]. The image of the embedding is the translate by \( F \) of a \( B^\vee \)-coadjoint orbit in \( (\mathfrak{h}^\vee)^* \subset (\mathfrak{g}^\vee)^* \), and the integrals of motion of the Toda lattice are given by restrictions of \( G^\vee \)-invariant polynomials on \( (\mathfrak{g}^\vee)^* \). By Chevalley’s restriction theorem \( \mathbb{C}[(\mathfrak{g}^\vee)^*]^{G^\vee} = \mathbb{C}[\mathfrak{h}]^W \) and the latter is a polynomial ring with \( n \) homogeneous generators \( \Sigma_1, \ldots, \Sigma_n \). Now let \( \mathcal{A} := F + \mathfrak{h} \oplus \sum \mathbb{C}(f_i^\vee)^* \subset (\mathfrak{g}^\vee)^* \), and consider the map

\[
\Sigma : \mathcal{A} \to \mathfrak{h}/W
\]
obtained from $\mathbb{C}[h]^W \hookrightarrow \mathbb{C}[(q^\vee)^\ast]$. Then Kim’s presentation of $qH_T^*(G/B)$ can be restated as an isomorphism

$$qH_T^*(G/B) \approx \mathbb{C}[A \times h/W h].$$

Finally, we have a map

$$\mu : Y_B \rightarrow A \times h/W h$$

$$(uB, h) \mapsto (u^{-1} \cdot (F - h), h),$$

where $u \in U_T$. This is an isomorphism by another result of Kostant’s, see [25]. Peterson’s map (3.4) is given by the composition of Kim’s presentation with $\mu \ast$.  

3.3.6. For $w \in W_P$ let

$$\sigma_{G/P}^w \in qH_T^*(G/P)$$

denote the corresponding (quantum equivariant) Schubert class. The Schubert classes $\sigma_{G/B}^w$ for $G/B$ may be viewed as rational functions on $Y$. Peterson’s theory implies that if $w \in W_P$, then the restriction of $\sigma_{G/B}^w$ to $Y_P$ is a regular function and under (3.2) represents the Schubert class $\sigma_{G/P}^w$. In particular it follows that all of the isomorphisms (3.2) for varying $P$ are explicitly determined by (3.4). In the special case of $qH^*(SL_n/P)$ an ad hoc proof of this relationship between the Schubert classes is given in [34].

3.3.7. Let $j \in \{1, \ldots, k\}$. The map (3.2) identifies $q_j$ with the regular function on $Y_P$ given by

$$(uw_P B^\vee, h) \mapsto -(F - h)(uw_P \cdot f^\vee_{n_j}).$$

4. A MIRROR SYMMETRIC CONSTRUCTION FOR $H_T^*(G/P)_{(q)}$

In this section we will introduce the ingredients (1), (3) and (4) of mirror symmetry described in the introduction for a general flag variety $G/P$. Then we will state the main theorem, which gives a mirror symmetric construction of the strata $Y_P^\ast$ in the equivariant Peterson variety.

4.1. Let

$$(1) \quad Z = Z_P := \{(t, b) \in (T^\vee)^{WP} \times B^\vee \mid b \in U_T^\vee t\bar{w}_P\bar{w}_0^{-1}U_T^\vee\}.$$

We view $Z_P$ as a family of varieties via the map $pr_1 : Z_P \rightarrow (T^\vee)^{WP}$ projecting onto the first factor. For $t \in (T^\vee)^{WP}$ let us write

$$(2) \quad Z_P^t := B^\vee \cap U_T^\vee \bar{t}\bar{w}_P\bar{w}_0^{-1}U_T^\vee,$$

which we may identify with the fiber $pr_1^{-1}(t)$ in $Z_P$. We record the following basic properties of the family $Z_P$.

(1) Projection onto the second factor in $Z_P$ restricts to an isomorphism

$$pr_2 : Z_P \stackrel{\sim}{\longrightarrow} B^\vee \cap U_T^\vee (T^\vee)^{WP} \bar{w}_P\bar{w}_0^{-1}U_T^\vee.$$

(2) Fix $t \in (T^\vee)^{WP}$. Then the fiber $Z_P^t$ is smooth of dimension $n_P = \dim G/P$, and may be identified with $\mathcal{R}_{\bar{w}_P, \bar{w}_0}$ by

$$Z_P^t \rightarrow \mathcal{R}_{\bar{w}_P, \bar{w}_0}$$

$$b \mapsto b\bar{w}_0 B^\vee / B^\vee.$$
(3) Using the isomorphism from (2) to identify all the fibers we obtain a trivialization
\[ Z_P \xrightarrow{\sim} (T^\vee)^{W_P} \times \mathcal{R}^\vee_{w_P,w_0}. \]

The map \( \psi_P : Z_P \to \mathcal{R}^\vee_{w_P,w_0} \) obtained by composing with the projection onto the second factor in the trivialization will be important later on.

The properties (1-3) are straightforward to verify. We note that the fibers can be naturally compactified to give the Langlands dual flag variety \( G^\vee/P^\vee \).

4.2. Let
\[ f^\vee = \sum_{i \in I} f^\vee_i, \]
\[ f^\vee_{(P)} = \sum_{i \in I} \frac{1}{<\rho^\vee, w_P \cdot \alpha_i>} f^\vee_i. \]

In particular \( f^\vee_{(P)} = f^\vee \). We define a function \( \mathcal{F} = \mathcal{F}_P : Z_P \to \mathbb{C} \) in terms of the representation \( \mathcal{V}(\rho^\vee) \) of \( \tilde{G}^\vee \) as follows.
\[ \mathcal{F}_P(t,b) = \frac{\left< \tilde{b} \cdot v^+_{\rho^\vee}, w_P \cdot v^-_{\rho^\vee} \right> + \left< \tilde{b} f^\vee \cdot v^+_{\rho^\vee}, w_P \cdot v^-_{\rho^\vee} \right>}{\left< \tilde{b} \cdot v^+_{\rho^\vee}, w_P \cdot v^-_{\rho^\vee} \right> \left< \tilde{b} \cdot v^+_{\rho^\vee}, w_P \cdot v^-_{\rho^\vee} \right>}, \]

where \( \tilde{b} \in \tilde{G}^\vee \) with \( \pi(\tilde{b}) = b \). Note that the denominator insures that \( \mathcal{F}_P \) is well defined, that is, independent of the choice of \( v^-_{\rho^\vee} \) or lift \( \tilde{b} \). We also denote by \( \mathcal{F}_P \) the restriction to any \( Z^P_t \).

4.3. Consider the fundamental representations \( \mathcal{V}(\omega^\vee_i) \). In terms similar to (4.4) the multi-valued function \( \phi = \phi_P : Z_P \times \mathfrak{h} \to \mathbb{C} \) we will define can be thought of as taking the form
\[ \phi(t,b;h) = \prod_{i \in I} \left< \tilde{b} \cdot v^+_{\omega^\vee_i}, v^+_{\omega^\vee_i} \right>^{\alpha_i(h)} \]

This is only a well-defined function if \( h \), after identifying \( \mathfrak{h} \) with \( (\mathfrak{h}^\vee)^* \), is in the root lattice for \( G^\vee \). If \( h \) is also dominant we can simply look at the representation \( \mathcal{V}(h) \), and \( \phi \) becomes the highest weight coefficient
\[ \phi(t,b;h) = \left< b \cdot v^+_{h^\vee}, v^+_{h^\vee} \right>. \]

To give the definition more generally we consider the covering space
\[ \tilde{Z}_P := Z_P \times_{T^\vee} \mathfrak{h}^\vee = \{(t,b,h^\vee) \in Z_P \times \mathfrak{h}^\vee | b \exp(-h^\vee_R) \in U_-^\vee \} \]
of \( Z_P \). If \( t \in (T^\vee)^{W_P} \), let us also write
\[ \tilde{Z}_P^t := \{(b,h^\vee) \in Z_P^t \times \mathfrak{h}^\vee | b \exp(-h^\vee_R) \in U_-^\vee \}, \]
in correspondence with (4.2). Then we have two families of varieties related by a covering map \( c_P \),
\[ \tilde{Z}_P \xrightarrow{c_P} Z_P \]
\[ \xymatrix{ \text{pr}_1 \downarrow \ar[d] & \downarrow \text{pr}_1 \ar[d] \ar[r] & \} \]
\[ (T^\vee)^{W_P} = (T^\vee)^{W_P}, \]
where $\tilde{Z}_P^t$ naturally identifies with a fiber on the left hand side, and such that each of these fibers is also a covering of the corresponding fiber of $Z_P$,

$$\tilde{Z}_P^t \rightarrow Z_P^t : (b, h_{R}^v) \mapsto b.$$ 

We define a holomorphic function $\tilde{\phi}$ on $Z_P \times \mathfrak{h}$ by

$$\tilde{\phi} : Z_P \times \mathfrak{h} \rightarrow \mathbb{C},$$

$$(t, b, h_{R}^v, h) \mapsto e^{<h, h_{R}^v>},$$

where $<,>$ is the dual pairing between $\mathfrak{h}$ and $\mathfrak{h}^v$. It is clear that this agrees with the matrix coefficient $\langle b \cdot v^+_h, v^+_h \rangle$ if $h$ is a dominant weight in the root lattice of $G^v$. We denote the restriction of $\tilde{\phi}$ to any $Z_P^t$ again by $\tilde{\phi}$.

We now take (4.6) to be our definition of the multi-valued function $\phi$. While $\phi$ is multi-valued on $Z_P$, note that it follows immediately from the definition that the logarithmic derivative of $\tilde{\phi}$ along any $Z_P$ direction is independent of the chosen branch (i.e. depends only on $(t, b)$ and not on $h_{R}^v$). In particular for fixed $h \in \mathfrak{h}$ it makes sense to talk about critical points of $\ln(\phi( ; h))$ in a fiber $Z_P^t$ of the original mirror family $Z_P$, as we will do below.

4.4. We can now formulate our main result connecting the mirror data constructed above with the quantum cohomology rings of the homogeneous spaces $G/P$. Let

$$Z_{P,T}^{crit} := \{(t, b; h) \in Z_P \times \mathfrak{h} | b \text{ is a critical point for } (\mathcal{F}_P + \ln \phi( ; h)) | Z_P^t \}.$$ 

Note that the quantum parameters $q_1, \ldots, q_k$ in $\mathbb{Q}[G/P]$ can be naturally thought of as functions $e^{t_j}$ on $H^2(G/P, \mathbb{C})/2\pi i H^2(G/P, \mathbb{Z})$, where the $t_j$ run through a certain basis in $H_2(G/P)$ (dual to the Schubert basis of $H^2(G/P)$). If we identify

$$H^2(G/P, \mathbb{C}) = (\mathfrak{h}^v)^{WP}$$

by the Borel-Weil homomorphism then the $t_j$ are represented by the roots $\alpha_{n_j}$ (of $G^v$) associated to $I^P$. Therefore the $q_j$ are identified with the corresponding functions on $(\mathfrak{h}^v)^{WP}$, which we again denote by $\alpha_{n_j}$. This is precisely how the quantum parameters will appear below.

**Theorem 4.1.** The map $\psi_P : Z_P \rightarrow \mathcal{R}_{wp,w_0}^\nu$ from (3) in Section 4.1 induces an isomorphism

$$\psi_P \times id_\mathfrak{h} : Z_{P,T}^{crit} \xrightarrow{\sim} \mathcal{Y}_P^r,$$

such that the following diagram commutes

$$\begin{array}{ccc}
Z_{P,T}^{crit} & \xrightarrow{\sim} & \mathcal{Y}_P^r \\
\downarrow pr_1 & & \downarrow (q_i)_{i=1}^k \\
(T^{\nu})^{WP} & \xrightarrow{\sim} & (\mathbb{C}^*)^k.
\end{array}$$

Here the isomorphism $(T^{\nu})^{WP} \xrightarrow{\sim} (\mathbb{C}^*)^k$ is given by $(\alpha_{n_j})_{j=1}^k$. Moreover we have

$$Z_{P,T}^{crit} = \{(t, b; h) \in Z_P | b \cdot (F - h) = F - h \}.$$ 

**Corollary 4.2.** Combining (4.7) with the isomorphism (3.3) one obtains

$$\mathbb{Q}[H^*_T(G/P)\{q_1^{-1}, \ldots, q_k^{-1}\}] \xrightarrow{\sim} \mathbb{C}[Z_{P,T}^{crit}].$$
5. Proof of Theorem 3.1

We prove first some preparatory lemmas.

**Lemma 5.1.** Let \( i \in I \) and \( i^* \) be such that \( w_0 \cdot \alpha_i = -\alpha_{i^*} \).

1. Then
   \[ \dot{w}_0 \cdot f_i^\gamma = \dot{w}_0^{-1} \cdot f_i^\gamma = -e_i^\gamma. \]

2. For any parabolic \( P, \)
   \[ \dot{w}_P \dot{w}_P \in (T^\gamma)^W_P. \]

3. If \( i^* \) lies in \( I_P, \) and \( \tilde{i} \in I_P \) is defined by \( w_0 w_i^{-1} \cdot \alpha_i = \alpha_{\tilde{i}}, \) then
   \[ \dot{w}_P \dot{w}_0^{-1} \cdot f_i^\gamma = f_i^\gamma. \]

**Proof.** Note that we have \( \dot{s}_i^{-1} \cdot f_i^\gamma = -e_i^\gamma, \) which can be checked by a direct calculation. Similarly \( \dot{s}_i^{-1} \cdot e_i^\gamma = -f_i^\gamma. \) Now consider the fundamental representation \( V(\omega^\gamma) \) of \( \overline{G^\gamma}. \) We have

\[
\left< (\dot{w}_0^{-1} \cdot f_i^\gamma) \cdot v_{\omega^\gamma}, e_i^\gamma \cdot v_{\omega^\gamma} \right> = \left< \dot{w}_0^{-1} f_i^\gamma \cdot v_{\omega^\gamma}, e_i^\gamma \cdot v_{\omega^\gamma} \right> = - \left< \dot{w}_0^{-1} s_i e_i^\gamma \dot{s}_i^{-1} \cdot v_{\omega^\gamma}^+, e_i^\gamma \cdot v_{\omega^\gamma} \right> = - \left< \dot{s}_i^{-1} \cdot e_i^\gamma f_i^\gamma \cdot v_{\omega^\gamma}, e_i^\gamma \cdot v_{\omega^\gamma} \right> = - \left< \dot{s}_i^{-1} \cdot v_{\omega^\gamma}^+, e_i^\gamma \cdot v_{\omega^\gamma} \right> = -1.
\]

This implies the second equality in (1). Analogously we can show the identity

\[
\left< (\dot{w}_0 \cdot f_i^\gamma) \cdot v_{\omega^\gamma}, e_i^\gamma \cdot v_{\omega^\gamma} \right> = -1,
\]

and this implies also the first equality.

For (2) let \( \epsilon = \dot{w}_P \dot{w}_P. \) Then \( \epsilon \in T^\gamma, \) and we need to show that \( \alpha^\gamma(\epsilon) = 1 \) whenever \( i \in I_P. \) This holds since by (1) we have

\[ \dot{w}_P \dot{w}_P \cdot e_i^\gamma = e_i^\gamma, \quad \text{for } i \in I_P. \]

Applying (1) twice as follows,

\[ \dot{w}_P^{-1} \cdot f_i^\gamma = -e_i^\gamma = \dot{w}_0^{-1} \cdot f_i^\gamma, \quad \text{for } i^* \in I_P, \]

implies (3). \( \square \)

**Lemma 5.2.** Let \( b \in B^\gamma_+ \cap U^\gamma_+ (T^\gamma)^W_P \) \( \dot{w}_P \dot{w}_0^{-1} U^\gamma_+ \)

with factorization \( b = u_1 \tilde{t} \dot{w}_P \dot{w}_0^{-1} u_2^{-1} \) for \( u_1, u_2 \in U^\gamma_+ \) and \( t \in (T^\gamma)^W_P. \) Then

\[ \mathcal{F}_P(b) = F(u_2 \cdot \rho) - F(u_1 \cdot \rho). \]

**Proof.** Let \( \tilde{i} \in (T^\gamma)^W_P \) with \( \pi(\tilde{i}) = t \) and \( \tilde{b} = u_1 \tilde{t} \dot{w}_P \dot{w}_0^{-1} u_2^{-1} \in \overline{G^\gamma} \) covering \( b. \) Note that

\[ \left< \tilde{b} \cdot v_{\rho^\gamma}, \dot{w}_P \cdot v_{\overline{\rho}} \right> = \left< u_1 \tilde{t} \dot{w}_P \dot{w}_0^{-1} u_2^{-1} f^\gamma \cdot v_{\rho^\gamma}, \dot{w}_P \cdot v_{\overline{\rho}} \right> = \rho(\tilde{i}^{-1}) \cdot \rho^\gamma(\tilde{i})^{-1} \cdot \rho^\gamma(\tilde{i}). \] Then we have

\[
\begin{align*}
\mathcal{F}_P(b) &= \frac{1}{\rho(\tilde{i})^{-1}} \left( \left< f_{P}^\gamma \tilde{b} \cdot v_{\rho^\gamma}, \dot{w}_P \cdot v_{\overline{\rho}} \right> + \left< \tilde{b} f^\gamma \cdot v_{\rho^\gamma}, \dot{w}_P \cdot v_{\overline{\rho}} \right> \right) \\
&= \rho^\gamma(\tilde{i}) \left( \left< f_{P}^\gamma u_1 \tilde{t} \dot{w}_P \dot{w}_0^{-1} \cdot v_{\rho^\gamma}, \dot{w}_P \cdot v_{\overline{\rho}} \right> + \left< u_1 \tilde{t} \dot{w}_P \dot{w}_0^{-1} u_2^{-1} f^\gamma \cdot v_{\rho^\gamma}, \dot{w}_P \cdot v_{\overline{\rho}} \right> \right) \\
&= \left< f_{P}^\gamma u_1 \tilde{t} \dot{w}_P \dot{w}_0^{-1} u_2^{-1} f^\gamma \cdot v_{\rho^\gamma}, \dot{w}_P \cdot v_{\overline{\rho}} \right> + \left< u_1 \tilde{t} \dot{w}_P \dot{w}_0^{-1} u_2^{-1} f^\gamma \cdot v_{\rho^\gamma}, \dot{w}_P \cdot v_{\overline{\rho}} \right>.
\end{align*}
\]
Here the $\rho^\vee(\tilde{t})$ was cancelled against the $\tilde{t}$ factors in both summands. Note that

\begin{equation}
(5.2) \quad s_i w_P \cdot (-\rho^\vee) - w_P \cdot (-\rho^\vee) \in \begin{cases}
\mathbb{Z}_{<0} \alpha_i^\vee & \text{for } i \in I_P, \\
\mathbb{Z}_{>0} \alpha_i^\vee & \text{for } i \in I_P^c.
\end{cases}
\end{equation}

Therefore if $i \in I_P$ then $w_P \cdot v_{\rho^\vee}$ is annihilated by $e_i^\vee$, and if $i \in I_P^c$ then it is annihilated by $f_i^\vee$. Now the left hand summand of (5.1) simplifies to

$$\left\langle f_i^\vee u_1 \dot{w}_P \cdot v_{\rho^\vee} \cdot \dot{w}_P \cdot v_{\rho^\vee} \right\rangle = \sum_{i \in I} \frac{1}{<w_P \cdot \rho^\vee, \alpha_i>} \left\langle f_i^\vee u_1 \dot{w}_P \cdot v_{\rho^\vee} \cdot \dot{w}_P \cdot v_{\rho^\vee} \right\rangle$$

$$= - \sum_{i \in I_P} \frac{1}{<w_P \cdot \rho^\vee, \alpha_i>} (e_i^\vee)^* (u_1 \cdot \rho) \left\langle f_i^\vee u_1 \dot{w}_P \cdot v_{\rho^\vee} \cdot \dot{w}_P \cdot v_{\rho^\vee} \right\rangle = - \sum_{i \in I_P} (e_i^\vee)^* (u_1 \cdot \rho),$$

using also that $[f_i, e_i]$ acts on $\dot{w}_P \cdot v_{\rho^\vee}$ by a factor of $<w_P \cdot \rho^\vee, \alpha_i>$. For the right hand summand from (5.1) we obtain

$$\left\langle u_1 \dot{w}_P \dot{w}_0^{-1} u_2^{-1} f_i^\vee \cdot v_{\rho^\vee}^+, \dot{w}_P \cdot v_{\rho^\vee} \right\rangle$$

$$= \left\langle u_1 \dot{w}_P \dot{w}_0^{-1} f_i^\vee \cdot v_{\rho^\vee}^+, \dot{w}_P \cdot v_{\rho^\vee} \right\rangle + \sum_{i \in I} (e_i^\vee)^* (u_2 \cdot \rho) \left\langle u_1 \dot{w}_P \dot{w}_0^{-1} [e_i^\vee, f_i^\vee] \cdot v_{\rho^\vee}^+, \dot{w}_P \cdot v_{\rho^\vee} \right\rangle$$

$$= \left\langle u_1 \dot{w}_P \dot{w}_0^{-1} f_i^\vee \cdot v_{\rho^\vee}^+, \dot{w}_P \cdot v_{\rho^\vee} \right\rangle + F(u_2 \cdot \rho) \left\langle u_1 \dot{w}_P \cdot v_{\rho^\vee}, \dot{w}_P \cdot v_{\rho^\vee} \right\rangle$$

$$= \sum_{i \in I_P} \left\langle u_1 \dot{w}_P \dot{w}_0^{-1} f_i^\vee \cdot v_{\rho^\vee}^+, \dot{w}_P \cdot v_{\rho^\vee} \right\rangle + F(u_2 \cdot \rho),$$

by similar weight space considerations as above. Finally, using also Lemma 5.1 (3), the right hand summand of (5.1) simplifies further to

$$\sum_{i \in I_P} \left\langle u_1 \dot{w}_P \dot{w}_0^{-1} f_i^\vee \cdot v_{\rho^\vee}^+, \dot{w}_P \cdot v_{\rho^\vee} \right\rangle + F(u_2 \cdot \rho)$$

$$= \sum_{i \in I_P} \left\langle u_1 f_i^\vee \dot{w}_P \cdot v_{\rho^\vee}, \dot{w}_P \cdot v_{\rho^\vee} \right\rangle + F(u_2 \cdot \rho)$$

$$= - \sum_{i \in I_P} (e_i^\vee)^* (u_1 \cdot \rho) \left\langle [e_i^\vee, f_i^\vee] \dot{w}_P \cdot v_{\rho^\vee}, \dot{w}_P \cdot v_{\rho^\vee} \right\rangle + F(u_2 \cdot \rho)$$

$$= - \sum_{i \in I_P} (e_i^\vee)^* (u_1 \cdot \rho) + F(u_2 \cdot \rho),$$

noting that $<w_P \cdot \rho^\vee, \alpha_i> = 1$ for $i \in I_P$. Combining the two summands gives

$$F_P(b) = - \sum_{i \in I_P} (e_i^\vee)^* (u_1 \cdot \rho) + \left( - \sum_{i \in I_P} (e_i^\vee)^* (u_1 \cdot \rho) + F(u_2 \cdot \rho) \right)$$

$$= F(u_2 \cdot \rho) - F(u_1 \cdot \rho).$$
Lemma 5.3. Let $Q \supset B$ be the parabolic subgroup determined by $W_{Q} = w_{0}W_{P}w_{0}^{-1}$. If $b \in B'_{\nu} \cap U'_{\nu}(T'_{W'})w_{0}^{-1}W_{P}^{+}$ then

$$b^{-1} \in B'_{\nu} \cap U'_{\nu}(T'_{W'})w_{Q}w_{0}^{-1}U'_{\nu},$$

and the map $b \mapsto b^{-1}$ induces an isomorphism $\sigma_{P} : Z_{P} \rightarrow Z_{Q}$. Moreover we have

$$\mathcal{F}_{P}(b) = -\mathcal{F}_{Q}(b^{-1}).$$

Proof. Let us write $b = u_{1}t\dot{w}_{P}\dot{w}_{0}^{-1}u_{2}^{-1}$ in the usual way. Then

$$b^{-1} = u_{2}(\dot{w}_{0}t^{-1}\dot{w}_{0}^{-1})\dot{w}_{0}\dot{w}_{P}^{-1}u_{1}^{-1} = u_{2}(\dot{w}_{0}t^{-1}\dot{w}_{0}^{-1})(\dot{w}_{0}\dot{w}_{P}^{-1}\dot{w}_{Q}w_{0}^{-1}u_{1}^{-1}.$$

Now let $\epsilon = \dot{w}_{0}\dot{w}_{P}^{-1}\dot{w}_{0}w_{0}^{-1} = \dot{w}_{Q}w_{0}^{-1}w_{Q}^{-1}$, By Lemma 5.1 (2) we have $\dot{w}_{0}w_{0}^{-1} = 1$, since $G'$ is adjoint, and $\epsilon = (\dot{w}_{Q}w_{Q})^{-1} \in (T'_{W'})w_{Q}$. The isomorphism $\sigma_{P} : Z_{P} \rightarrow Z_{Q}$ is given explicitly by

$$\sigma_{P}(t,b) := (\dot{w}_{0}t^{-1}\dot{w}_{0}^{-1}\epsilon, b^{-1}).$$

Its inverse is $\sigma_{Q}$. The identity 5.4 follows from Lemma 5.2.

Recall that by 5.1 (2) we had an isomorphism

$$B'_{\nu} \cap U'_{\nu}t\dot{w}_{P}\dot{w}_{0}^{-1}U'_{\nu} \rightarrow R_{w_{P},w_{0}}.$$ In particular $Z_{P} = B'_{\nu} \cap U'_{\nu}t\dot{w}_{P}\dot{w}_{0}^{-1}U'_{\nu}$ is smooth of dimension $n_{P}$. We now determine its tangent space at a point $b_{0}$.

Lemma 5.4. Fix $t \in (T'_{W'})W_{P}$ and consider $b_{0} \in B'_{\nu} \cap U'_{\nu}t\dot{w}_{P}\dot{w}_{0}^{-1}U'_{\nu}$ with factorization $b_{0} = u_{1}t\dot{w}_{P}\dot{w}_{0}^{-1}u_{2}^{-1}$, for $u_{1}, u_{2} \in U'_{\nu}$. We view elements of $b'_{\nu}$ as right invariant vector fields on $B'_{\nu}$. Then the map

$$\eta : u'_{\nu} \cap \dot{w}_{P} \cdot u'_{\nu} \rightarrow b'_{\nu},$$

$$\zeta \mapsto \eta_{\zeta} := \text{pr}_{b'_{\nu}}(u_{1} \cdot \zeta),$$

gives rise to an isomorphism

$$u'_{\nu} \cap \dot{w}_{P} \cdot u'_{\nu} \rightarrow T_{b_{0}}(B'_{\nu} \cap U'_{\nu}t\dot{w}_{P}\dot{w}_{0}^{-1}U'_{\nu}),$$

$$\zeta \mapsto (\eta_{\zeta})_{b_{0}}.$$ Proof. Let $\lambda \gg 0$ in $Q'_{\nu}$. We consider the representations $V(\lambda)$ and $V(\lambda + \alpha'_{\nu})$ of $G'_{\nu}$. Then $B'_{\nu} \cap U'_{\nu}t\dot{w}_{P}\dot{w}_{0}^{-1}U'_{\nu}$ inside $B'_{\nu}$ is described by the equations

$$\frac{\langle b \cdot v_{\lambda}^{+}, \dot{w}_{P} \cdot v_{\lambda}^{-} \rangle}{\langle b \cdot v_{\lambda + \alpha'_{\nu}}^{+}, \dot{w}_{P} \cdot v_{\lambda + \alpha'_{\nu}}^{-} \rangle} = \alpha'_{\nu}(t),$$

$$\langle b \cdot v_{\lambda}^{+}, \dot{w} \cdot v_{\lambda}^{-} \rangle = 0$$

for $w \in W$ with $w \not\geq w_{P}$, where $b \in B'_{\nu}$, and keeping in mind that $G'_{\nu}$ is of adjoint type.

Let $\zeta \in u'_{\nu} \cap \dot{w}_{P} \cdot u'_{\nu}$. We apply the vector field $\eta_{\zeta}$ to the defining equations from above. So

$$\eta_{\zeta}(\dot{w}_{\lambda}^{+}, \dot{w} \cdot v_{\lambda}^{+}) (b_{0}) = \langle \text{pr}_{b'_{\nu}}(u_{1} \cdot \zeta)b_{0} \cdot v_{\lambda}^{+}, \dot{w} \cdot v_{\lambda}^{+} \rangle$$

$$= \langle u_{1} \zeta u_{1}^{-1}b_{0} \cdot v_{\lambda}^{+}, \dot{w} \cdot v_{\lambda}^{+} \rangle - \langle \text{pr}_{u'_{\nu}}(u_{1} \cdot \zeta)b_{0} \cdot v_{\lambda}^{+}, \dot{w} \cdot v_{\lambda}^{+} \rangle$$

$$= \langle u_{1} \zeta t\dot{w}_{P} \cdot v_{\lambda}^{+}, \dot{w} \cdot v_{\lambda}^{+} \rangle - \langle \text{pr}_{u'_{\nu}}(u_{1} \cdot \zeta)u_{1}t\dot{w}_{P} \cdot v_{\lambda}^{+}, \dot{w} \cdot v_{\lambda}^{+} \rangle.$$
Since $\zeta \in w_P \cdot u^\vee$ it follows that the first summand vanishes. The second summand is zero whenever $w \not\preceq w_P$, by weight space considerations. So

$$\eta_\zeta \left( \langle \cdot v^+_\lambda, \bar{w} \cdot v^-_\lambda \rangle \right) (b_0) = 0$$

if $w = w_P$ or $w \not\preceq w_P$, and for any $\lambda$. In particular also

$$\eta_\zeta \left( \langle \cdot v^+_\lambda, \bar{w}_P \cdot v^-_\lambda \rangle \right) (b_0) = 0.$$  

It follows that $(\eta_\zeta)_b_0$ is tangent to $B^\vee_- \cap U^\vee_+ \bar{w}_P \bar{w}_0^{-1} U^\vee_+$.

Suppose $\zeta \in u^\vee$ is homogeneous of weight $-\alpha^v$. Then we have

$$\eta_\zeta = \zeta + \bigoplus_{\beta^v \in \Delta^v_+} g^\vee \cdot \alpha^v + \beta^v,$$

and therefore $\eta$ is injective. Comparing dimensions this implies that the map from (5.6) is an isomorphism. \hfill \square

Proof of Theorem 4.1 Consider a fixed $b = u_1 t \bar{w}_P \bar{w}_0^{-1} u_2^{-1}$ in $B^\vee_-$, with $t \in (T^\vee)^{W_P}$ and $u_1, u_2 \in U^\vee_+$. 

Derivatives of $F_P$. Let $\zeta \in u^\vee \cap \bar{w}_P \cdot u^\vee$. We may assume $\zeta$ is homogeneous. We want to compute the derivative of $F_P$ in the $(\eta_\zeta)_b$ direction. Let us write $(\eta_\zeta)_b = -pr_{u^\vee}(u_1 \cdot \zeta) + u_1 \cdot \zeta$. Note that as for the adjoint action of $G^\vee$ on $g^\vee$ we also denote below the conjugation action of the group on itself by a dot. So $g \cdot h := ghg^{-1}$ for $g, h \in G^\vee$. Then we have

$$\eta_\zeta(F_P)(b) = \frac{d}{ds} \bigg|_{s=0} F_P \left( e^{-s \text{pr}_{u^\vee}(u_1 \cdot \zeta)} e^{s(u_1 \cdot \zeta)} b \right)$$

$$= \frac{d}{ds} \bigg|_{s=0} F_P \left( e^{-s \text{pr}_{u^\vee}(u_1 \cdot \zeta)} u_1 e^{s \zeta} t \bar{w}_P \bar{w}_0^{-1} u_2^{-1} \right)$$

$$= \frac{d}{ds} \bigg|_{s=0} F_P \left( e^{-s \text{pr}_{u^\vee}(u_1 \cdot \zeta)} u_1 t \bar{w}_P \bar{w}_0^{-1} (\bar{w}_0 \bar{w}_P^{-1} t^{-1} \cdot e^{s \zeta}) u_2^{-1} \right)$$

$$= \frac{d}{ds} \bigg|_{s=0} F \left( u_2 (\bar{w}_0 \bar{w}_P^{-1} t^{-1} \cdot e^{-s \zeta}) \cdot \rho \right) - \frac{d}{ds} \bigg|_{s=0} F \left( e^{-s \text{pr}_{u^\vee}(u_1 \cdot \zeta)} u_1 \cdot \rho \right),$$

using Lemma 5.2 for the last equality, and the fact that $\bar{w}_0 \bar{w}_P^{-1} \cdot \zeta \in u^\vee_+$. The right hand summand now simplifies as follows,

$$\frac{d}{ds} \bigg|_{s=0} F \left( e^{-s \text{pr}_{u^\vee}(u_1 \cdot \zeta)} u_1 \cdot \rho \right) = F \left( \left[ \text{pr}_{u^\vee}(u_1 \cdot \zeta), u_1 \cdot \rho \right] \right)$$

$$= F \left( \left[ \text{pr}_{u^\vee}(u_1 \cdot \zeta), \rho \right] \right) = -F \left( \text{pr}_{u^\vee}(u_1 \cdot \zeta) \right) = -F \left( u_1 \cdot \zeta \right).$$

For the left hand summand we have

$$\frac{d}{ds} \bigg|_{s=0} F \left( u_2 (\bar{w}_0 \bar{w}_P^{-1} t^{-1} \cdot e^{-s \zeta}) \cdot \rho \right) = -F \left( u_2 \cdot [\bar{w}_0 \bar{w}_P^{-1} t^{-1} \cdot \zeta, \rho] \right)$$

$$= -F \left( [\bar{w}_0 \bar{w}_P^{-1} t^{-1} \cdot \zeta, \rho] \right) = F \left( \bar{w}_0 \bar{w}_P^{-1} t^{-1} \cdot \zeta \right).$$
We now write \( u^\vee = [u^\vee, u^\vee] \oplus \bigoplus_{i \in I} g^\vee_{-\alpha_i} \) and distinguish between two cases, corresponding to whether \( w_{-\alpha_i}^{\vee} \cdot \zeta \) lies in the one summand, \([u^\vee, u^\vee]\), or the other, \( \bigoplus_{i \in I} g^\vee_{-\alpha_i} \).

**Case 1.** Suppose \( w_{-\alpha_i}^{\vee} \cdot \zeta \in [u^\vee, u^\vee] \). Then we have
\[
\dot{w} \dot{w}_{-\alpha_i}^{\vee} \cdot \zeta \in [u^\vee, u^\vee]
\]
and therefore \( F(\dot{w} \dot{w}_{-\alpha_i}^{\vee} t^{-1} \cdot \zeta) = 0 \).

**Case 2:** In this case, since \( w_{-\alpha_i}^{\vee} \cdot \zeta \) must also lie in \( w_{-\alpha_i}^{\vee} \cdot u^\vee \), and
\[
\left( \bigoplus_{i \in I} g^\vee_{-\alpha_i} \right) \cap w_{-\alpha_i}^{\vee} \cdot u^\vee = \bigoplus_{i \in I^P} g^\vee_{-\alpha_i},
\]
we have \( \zeta \in \dot{w} \cdot g^\vee_{-\alpha_i} \) for some \( i \in I^P \). Suppose therefore \( \zeta = \dot{w} \cdot f_i^\vee \) for \( i \in I^P \). Then \( (5.9) \) simplifies further to
\[
F(\dot{w} \dot{w}_{-\alpha_i}^{\vee} t^{-1} \cdot \zeta) = F(\dot{w} t^{-1} \cdot f_i^\vee) = \alpha_i^\vee(t) F(\dot{w} \cdot f_i^\vee) = -\alpha_i^\vee(t),
\]
using also Lemma \( 5.11 \). (1).

Combining \( (5.8) \) with the above two cases for \( (5.9) \) we get
\[
(5.10) \quad \eta(\mathcal{F}_P)(b) = \begin{cases} -F(u_1 \cdot \zeta) & \text{if } \dot{w}_{-\alpha_i}^{\vee} \cdot \zeta \in [u^\vee, u^\vee], \\ -\alpha_i^\vee(t) - F(u_1 \dot{w} \cdot f_i^\vee) & \text{if } \zeta = \dot{w} \cdot f_i^\vee \text{ and } i \in I^P. \end{cases}
\]

Note that, if \( \eta(\mathcal{F}_P)(b) = 0 \) for all \( \zeta \in u^\vee \cap \dot{w} \cdot u^\vee \), then by Case 1 above we have
\[
\dot{w}_{-\alpha_i}^{\vee} t^{-1} \cdot F \bigg|_{[u^\vee, u^\vee]} = 0.
\]
This implies that \( u_1 \dot{w} \cdot B^\vee / B^- \), or equivalently \( b \dot{w} B^- / B^- \), lies in the non-equivariant Peterson variety \( Y_P \).

**Logarithmic derivative of \( \phi \).** Let \( \zeta \in u^\vee \cap \dot{w} \cdot u^\vee \). Decomposing
\[
(\eta)_h = \text{pr}_{u^\vee}(u_1 \cdot \zeta) + \text{pr}_{h^\vee}(u_1 \cdot \zeta)
\]
we see that \( (\eta)_h \) lifts to the tangent vector
\[
(\dot{\eta})(b, h^\vee) = (\text{pr}_{h^\vee}(u_1 \cdot \zeta), \text{pr}_{h^\vee}(u_1 \cdot \zeta))
\]
in \( T(b, h^\vee)(Z^\vee_P) \subset b^\vee \oplus h^\vee \). The logarithmic derivative of \( \tilde{\phi} \) in this direction is therefore given by
\[
(5.11) \quad \dot{\eta}(\ln \tilde{\phi}(; h))(b, h^\vee) = \frac{d}{ds} \bigg|_{s=0} < h, s \text{pr}_{h^\vee}(u_1 \cdot \zeta) + h^\vee > = < h, \text{pr}_{h^\vee}(u_1 \cdot \zeta) >.
\]
Note again that \( \dot{\eta}(\ln \tilde{\phi}(; h))(b, h^\vee) \) no longer depends on the choice of lift \( (b, h^\vee) \in Z^\vee_P \) of \( b \), and is a well-defined function on \( Z^\vee_P \). We view this as the derivative of the multi-valued function \( \ln \tilde{\phi}(; h) \) at the point \( b \) in \( Z^\vee_P \) in the direction \( (\eta)_h \), and may also denote it by \( \eta(\ln \phi)(b, h) \).

**The critical points of \( \mathcal{F}_P + \ln \phi( ; h) \) along fibers.** By definition
\[
Z^\text{crit}_{P,T} = \{(t, b ; h) \in Z_P \times h \mid \eta(\mathcal{F}_P)(b) + \eta(\ln \phi)(b ; h) = 0 \text{ for all } \zeta \in [u^\vee, \dot{w} \cdot u^\vee] \}.
\]
As before we have two cases for $\zeta$.

(1) If $\zeta \in \dot{w}_P \cdot [u^\vee, u^\vee] \cap u^\vee$, then, by (5.11) and (5.10),

$$\eta(\mathcal{F}_P)(b) + \eta(\ln \phi(b; h)) = u_1^{-1} \cdot (F + h) (\zeta),$$

where $h \in \mathfrak{h} = (g^\vee)^*$ is considered as an element of $(g^\vee)^*$. Let us replace $\zeta$ by $\check{\zeta} := \dot{w}_P^{-1} \cdot \zeta$, so $\check{\zeta} \in [u^\vee, u^\vee] \cap \dot{w}_P^{-1} \cdot u^\vee$. The critical point condition $\eta(\mathcal{F}_P + \ln \phi(b; h))(b) = 0$ in this case reads

$$\dot{w}_P^{-1} u_1^{-1} \cdot (F + h)(\check{\zeta}) = 0, \quad \text{for all } \check{\zeta} \in [u^\vee, u^\vee] \cap \dot{w}_P^{-1} \cdot u^\vee,$$

in terms of $\check{\zeta}$.

(2) If $\zeta = \dot{w}_P \cdot f^\vee_i$ for $i \in I_P$, then by (5.11) and (5.10) we have

$$\eta_{\dot{w}_P \cdot f^\vee_i}(\mathcal{F}_P)(b) + \eta_{\dot{w}_P \cdot f^\vee_i}(\ln \phi)(b; h) = -\alpha^\vee_i(t) - (u_1^{-1} \cdot (F - h))(\dot{w}_P \cdot f^\vee_i).$$

Finally, note that if $\check{\zeta} \in [u^\vee, u^\vee] \cap \dot{w}_P^{-1} \cdot u^\vee$, then $\dot{w}_P \cdot \check{\zeta} \in [u^\vee, u^\vee]$, so

$$\dot{w}_P^{-1} u_1^{-1} \cdot (F + h)(\check{\zeta}) = 0$$

automatically. Therefore the critical point condition (1) implies

$$\dot{w}_P^{-1} u_1^{-1} \cdot (F + h)[u^\vee, u^\vee] = 0.$$

Combining (1) and (2) above, we find that the critical point locus $Z_{P,T}^{crit}$ is given by

$$Z_{P,T}^{crit} = \left\{ (t, b; h) \in Z_P \times \mathfrak{h} \mid \begin{array}{l}
\dot{w}_P^{-1} u_1^{-1} \cdot (F - h)[u^\vee, u^\vee] = 0 \\
(F - h)(u_1 \dot{w}_P \cdot f^\vee_i) = -\alpha^\vee_i(t), \\
\text{for } b = u_1 t \dot{w}_P \dot{w}_0^{-1} u_2^{-1} \text{ and } i \in I_P
\end{array} \right\}.$$

This implies that $b \mapsto b \dot{w}_0 B^\vee_i / B^\vee$ is a map

$$Z_{P,T}^{crit} \rightarrow \mathcal{Y}_P^*.$$

Comparing with Peterson's description of the quantum parameters, Section 3.3.4, we see that the diagram (4.8) commutes.

To show that (5.13) is an isomorphism, consider $(u_1 \dot{w}_P B^\vee_i / B^\vee, h) \in \mathcal{Y}_P^*$, where $u \in U^\vee$. Define $t \in (T^\vee)^{W_P}$ by the condition

$$(F - h)(u_1 \dot{w}_P \cdot f^\vee_i) = -\alpha^\vee_i(t), \quad \text{for all } i \in I_P.$$

Then since $u_1 \dot{w}_P B^\vee_i / B^\vee \in \mathcal{R}_{u_1 \dot{w}_P \cdot w_0}$, there is a unique $b = u_1 t \dot{w}_P \dot{w}_0^{-1} u_2^{-1} \in Z_P$ with

$$b \dot{w}_0 B^\vee_i / B^\vee = u_1 \dot{w}_P B^\vee_i / B^\vee,$$

as in (2) of Section 6.11. It is clear from (5.12) that $(t, b; h) \in Z_{P,T}^{crit}$ and so we have defined an inverse to (5.13).

The description (4.9) of $Z_{P,T}^{crit}$ is an immediate consequence of (5.13) being an isomorphism, together with the analogous result for the Peterson variety from Section 3.3.2, which is due to Dale Peterson and originates from a description of Kostant's for the leaves of the Toda lattice. We explain the proof here for completeness. It starts with the observation that the condition

$$\dot{w}_P^{-1} u_1^{-1} \cdot (F - h)[u^\vee, u^\vee] = 0$$

implies the element

$$b^{-1} \cdot (F - h) = u_2 \dot{w}_0 \dot{w}_P^{-1} t^{-1} u_1^{-1} \cdot (F - h)$$
vanishes on \([u^\vee_1, u^\vee_2]\), and therefore lies in \(\bigoplus_{i \in J} (\mathfrak{g}_\alpha^\vee)^* \oplus (\mathfrak{h}^\vee)^*\). Now the fact that \(b \in B^\vee\) implies immediately that
\[
b^{-1} \cdot (F - h) (h^\vee) = (F - h) (b \cdot h^\vee) = - < h, h^\vee >
\]
for any \(h^\vee \in \mathfrak{h}^\vee\). And direct calculation using the second condition from (5.12), along with the fact that \(\alpha_i(t) = 1\) if \(i \in I_P\), shows that
\[
b^{-1} \cdot (F - h) (e_i^\vee) = 1
\]
for any \(i \in I\). Therefore in fact \(b^{-1} \cdot (F - h) = F - h\) and
\[
(5.14) \quad Z_{P,i}^{\ sexist} \subseteq \{(t, b, h) \in Z_P \mid b \cdot (F - h) = F - h\}.
\]

The opposite inclusion follows using the identity
\[
t w_0^{-1} u_2^{-1} \cdot (F - h) = t w_0^{-1} u_2^{-1} b^{-1} \cdot (F - h) = w_0^{-1} u_2^{-1} \cdot (F - h),
\]
for any \(b \) in the right hand side of (5.14). \(\Box\)

6. Deodhar Stratifications and Standard Coordinates

In this section we introduce coordinate systems on intersections of opposite Bruhat cells. These will be used in the subsequent sections firstly to define the holomorphic \(nP\)-forms we need to state the mirror conjecture for \(G/P\), and secondly to compare our mirror construction with the one from \(\[17, 20\] \) in type \(A\).

6.1. Intersections of opposed Bruhat cells \(R_{v,w}\) were decomposed into strata isomorphic to products of the form \(\mathbb{C}^p \times (\mathbb{C}^*)^q\) by Deodhar \(\[10\]\). We will give a practical definition of these strata following \(\[34\]\). This latter description has the advantage of providing for every stratum natural coordinates to work with.

Let \(w \in W\) and \(s_{i_1} s_{i_2} \ldots s_{i_m} = w\) be a fixed reduced expression which we denote by \(i = (i_1, \ldots, i_m)\). We consider a sequence of integers \(1 \leq j_1 < \ldots < j_t \leq m\) as giving a subexpression \(s_{i_{j_1}} \ldots s_{i_{j_t}}\) of \(s_{i_1} \ldots s_{i_m}\). We say it is a subexpression for \(v\) if \(s_{i_{j_1}} \ldots s_{i_{j_t}} = v\). Note that \((i_{j_1}, \ldots, i_{j_t})\) need not be a reduced expression of \(v\).

A subexpression \(j = (j_1, \ldots, j_t)\) of \(i\) is called distinguished if
\[
(s_{i_{j_1}} \ldots s_{i_{j_t}}) s_k > s_{i_{j_1}} \ldots s_{i_{j_t}}\text{ for all } j_1 < k < j_{t+1},
\]
where \(1 \leq l \leq t\). There is a unique subexpression for \(v\) with the stronger property that
\[
(s_{i_{j_1}} \ldots s_{i_{j_t}}) s_k > s_{i_{j_1}} \ldots s_{i_{j_t}}\text{ for all } j_1 < k \leq j_{t+1},
\]
where \(1 \leq l \leq t\). We may set \(j_{t+1} = m + 1\) everywhere above. We call this subexpression the positive subexpression for \(v\). It is the unique distinguished subexpression that gives a reduced expression for \(v\).

Deodhar’s construction associates to any reduced expression of \(w\) a stratification of \(R_{v,w}\) which has a stratum for every distinguished subexpression for \(v\). And the positive subexpression for \(v\) corresponds to the unique open stratum.

For a reduced expression \(i\) and subexpression \(j\) let
\[
J_0(j) = \{1, \ldots, m\} \setminus \{j_1, \ldots, j_t\},
\]
\[
J_+(j) = \left\{l \mid l = j_r \text{ some } r = 1, \ldots, t, \text{ and } s_{i_{j_1}} \ldots s_{i_{j_r}} > s_{i_{j_1}} \ldots s_{i_{j_{r-1}}} \right\},
\]
\[
J_-(j) = \left\{l \mid l = j_r \text{ some } r = 1, \ldots, t, \text{ and } s_{i_{j_1}} \ldots s_{i_{j_r}} < s_{i_{j_1}} \ldots s_{i_{j_{r-1}}} \right\},
\]

where \(1 \leq l \leq t\). These sets are all nonempty.
where we suppress the i in the notation since it is usually clear from context. If j is distinguished, define a subset $\mathcal{R}_{j,i}$ of the flag variety $G/B_-$ by

$$
\mathcal{R}_{j,i} := \left\{ g_1 \ldots g_m B_- / B_- \mid g_i = \begin{cases} x_i(t_i), & \text{if } l \in J_0(j) \\ \delta_i, & \text{if } l \in J_+(j) \\ y_i(m_l) \delta_i^{-1}, & \text{if } l \in J_-(j) \end{cases}, \quad m_l \in \mathbb{C}, \quad m_l \in \mathbb{C} \right\}.
$$

Here the parameters $t_i \in \mathbb{C}^*$ and $m_l \in \mathbb{C}$ can also be used as coordinates on $\mathcal{R}_{j,i}$ giving an isomorphism $\mathcal{R}_{j,i} \sim (\mathbb{C}^*)^{\mid J_0(j) \mid} \times \mathbb{C}^{\mid J_-(j) \mid}$. We will refer to these coordinates as the standard coordinates on $\mathcal{R}_{j,i}$. If $j_+$ is the positive subexpression for $v$ in $i$ then $\mathcal{R}_{j_+,i} \sim (\mathbb{C}^*)^{\ell(w)-\ell(v)}$.

By [31, Proposition 5.2] the $\mathcal{R}_{j,i}$ agree precisely with Deodhar’s strata in $\mathcal{R}_{v,w}$. So fixing $i$ we have

$$
\mathcal{R}_{v,w} = \bigcup_j \mathcal{R}_{j,i},
$$

where the union is over all distinguished subexpressions $j$ of $i$. Note only that our conventions differ from [31] in that $B_+$ and $B_-$ are interchanged.

7. A holomorphic nP-form on $\mathcal{R}^\vee_{w_P,w_0}$

To define the oscillatory integrals and state the mirror conjecture for $G/P$ we require holomorphic nP-forms on the fibers of the proposed mirror family. Therefore we want to define a holomorphic nP-form on an intersection of opposed Bruhat cells $\mathcal{R}^\vee_{w_P,w_0}$. This holomorphic differential form will be defined by writing it down explicitly on a large enough open subset of $\mathcal{R}^\vee_{w_P,w_0}$.

Let $i$ be a reduced expression of $w_0$ and $j = j_+(i)$ the corresponding positive subexpression for $w_P$. Consider the open Deodhar stratum

$$(7.1) \quad \mathcal{R}^\vee_{j_+(i),i} := \left\{ g_1 \ldots g_N B_-^\vee / B_-^\vee \mid g_i = \begin{cases} x_i(t_i) & \text{for } t_i \in \mathbb{C}^* \\ \delta_i & \text{otherwise.} \end{cases}, \quad m_l \in \mathbb{C} \right\},$$

in $\mathcal{R}^\vee_{w_P,w_0}$. Let $\mathcal{U}$ be the union of these open sets. So

$$
\mathcal{U} = \bigcup_i \mathcal{R}^\vee_{j_+(i),i},
$$

where $i$ ranges over all the reduced expressions of $w_0$.

**Lemma 7.1** (essentially Lemma 3.6 in [40]). $\mathcal{U}$ is an open dense subset of $\mathcal{R}^\vee_{w_P,w_0}$ with complement of codimension greater than or equal to 2.

**Proof.** Since $\mathcal{R}^\vee_{w_P,w_0}$ is irreducible it is clear that $\mathcal{U}$ is open dense. We have an isomorphism

$$
U_-^\vee \cap B_-^\vee \bar{w}_P \bar{w}_0 B_+^\vee \sim \mathcal{R}^\vee_{w_P,w_0} : \quad u \mapsto u \bar{w}_0 B_-^\vee,
$$

whereby the double Bruhat cell $B_-^\vee \cap B_+^\vee \bar{w}_P \bar{w}_0 B_+^\vee$ in the group can be identified with $\mathcal{R}^\vee_{w_P,w_0} \times T^\vee$. Now Lemma 3.6 in [40], which is about $B_-^\vee \cap B_+^\vee \bar{w}_P \bar{w}_0 B_+^\vee$, implies the lemma. \hfill □

\footnote{M. Brion pointed out to us that by [47] Theorem 4.2.1(i) the canonical bundle of the closure of any $\mathcal{R}_{v,w}$, the so-called Richardson variety $X_{v,w} := \overline{\mathcal{R}_{v,w}}$, is $\mathcal{O}_{X_{v,w}}(\partial X_{v,w})$, and he conjectured that our form might come from there by restriction, [6]. If so, this would give a more intrinsic definition of our form, at least up to scalar.
Proposition 7.2. Fix a reduced expression \( l_0 \) of \( w_0 \). There is a unique holomorphic \( n_P \)-form \( \omega \) on \( R_{w_P,w_0}^\vee \) such that the restriction of \( \omega \) to \( R_{j_i,i}^\vee \) is given by

\[
\omega|_{R_{j_i,i}^\vee} = \epsilon_i \prod_{t \in J_0(j_i,i)} \frac{dt_t}{t_t},
\]

in terms of the standard coordinates \( t_t \) on \( R_{j_i,i}^\vee \), where \( \epsilon_1 \in \{ \pm 1 \} \) and \( \epsilon_0 = 1 \).

Here we use the obvious order on \( J_0(j_i,i) \) for defining the wedge product.

Proof. By Lemma 7.1 and Hartog’s theorem if \( \omega \) is well defined on \( U \) then it extends holomorphically to all of \( R_{w_P,w_0}^\vee \).

Let \( i \) and \( i' \) be reduced expressions of \( w_0 \) such that \( i \) is obtained from \( i' \) by a single braid relation. It suffices to show that the rational transformation \( (t_1, \ldots, t_{n_P}) \mapsto (t'_1, \ldots, t'_{n_P}) \) from the standard coordinates on \( R_{j_i,i}^\vee \) to those of \( R_{j_i,i'}^\vee \) gives

\[
\frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_{n_P}}{t_{n_P}} = \pm \frac{dt'_1}{t'_1} \wedge \cdots \wedge \frac{dt'_{n_P}}{t'_{n_P}}.
\]

The remainder of the proof consists of checking the possible coordinate transformations that can occur.

**Simply laced case :**

1. If \( s_i s_j = s_j s_i \) then
   \[
   x_i(a)x_j(b) = x_j(b)x_i(a)
   \]
   giving the simplest change of coordinates \( C_0(a,b) = (b,a) \).

2. If \( s_i s_j s_i = s_j s_i s_j \) then it is easy to check that
   \[
   x_i(a)x_j(b)x_i(c) = x_j\left(\frac{bc}{a+c}\right)x_i(a+c)x_j\left(\frac{ab}{a+c}\right)
   \]
   and
   \[
   x_i(a)x_j(b)s_i = x_j(b)s_i x_j(ab)y_i(-a).
   \]
   We may record these two changes of coordinates as
   \[
   C_1(a,b,c) = \left(\frac{bc}{a+c}, a+c, \frac{ab}{a+c}\right)
   \]
   \[
   C_2(a,b) = (b,ab)
   \]
   Note also that
   \[
   x_i(a)s_j s_i = s_j s_i x_j(a).
   \]

**Type \( B_2 \) braid relations :**

If \( s_i s_j s_i = s_j s_i s_j \) and \( \alpha_i \) is the long root, then the following relation holds (see Section 3.1).

\[
x_i(a)x_i(b)x_i(c)x_i(d) = x_i(d')x_j(c')x_i(b')x_j(a')
\]
where \( a' = \frac{abc}{y}, b' = \frac{y^2}{x}, c' = \frac{x}{y}, d' = \frac{bc^2d}{x} \)
and \( x = a^2b + d(a+c)^2, y = ab + d(a+c) \).

Let us denote this change of coordinates by
\[
C_3(a,b,c,d) = (d', c', b', a').
\]
Its inverse is \((C_3)^{-1}(d, c, b, a) = (a', b', c', d')\), where \(a', b', c', d'\) are given by the same formulas as above.

Furthermore it is easy to check the pairs of (inverse) identities

\[
\begin{align*}
    x_i(a)x_j(b)x_i(c)s_j &= x_j \left( \frac{bc}{a + c} \right) x_i(a + c)s_jx_i \left( \frac{ab^2c}{a + c} \right) y_j \left( \frac{-ab}{a + c} \right), \\
    x_j(a)x_i(b)s_jx_i(c) &= x_i \left( \frac{bc}{c + a^2b} \right) x_j \left( \frac{c + a^2b}{ab} \right) x_i \left( \frac{a^2b^2}{c + a^2b} \right) s_jy_i \left( \frac{c}{ab} \right),
\end{align*}
\]

and

\[
\begin{align*}
    x_j(a)x_i(b)s_js_i &= x_i(b)s_js_ix_i(ab)u, \\
    x_i(a)s_js_ix_j(b) &= x_j \left( \frac{b}{a} \right) x_i(a)s_js_iu',
\end{align*}
\]

for some \(u\) and \(u'\) in \(U_-,\) and

\[
\begin{align*}
    x_j(a)x_i(b)x_j(c)s_i &= x_i \left( \frac{c^2b}{a + c} \right) x_j(a + c)s_ix_j \left( \frac{abc}{a + c} \right) y_i \left( \frac{-a^2b - 2abc}{(a + c)^2} \right), \\
    x_i(a)x_j(b)s_ix_j(c) &= x_j \left( \frac{bc}{c + ab} \right) x_i \left( \frac{(c + ab)^2}{ab^2} \right) x_j \left( \frac{ab^2}{c + ab} \right) s_ix_i \left( \frac{2abc + c^2}{ab^2} \right),
\end{align*}
\]

and

\[
\begin{align*}
    x_i(a)x_j(b)s_is_j &= x_j(b)s_is_jx_i(ab^2)y, \\
    x_j(a)s_is_jx_i(b) &= x_i \left( \frac{b}{a^2} \right) x_j(a)s_is_jy',
\end{align*}
\]

for some \(y\) and \(y'\) in \(U_-\). Finally

\[
\begin{align*}
    x_i(a)s_js_is_j &= s_js_is_jx_i(a), \\
    x_j(a)s_is_js_i &= s_is_js_is_jx_j(a).
\end{align*}
\]

We record the remaining four nontrivial changes of coordinates

\[
\begin{align*}
    C_4(a, b, c) &= \left( \frac{bc}{a + c}, a + c, \frac{ab^2c}{a + c} \right), \\
    C_5(a, b) &= (b, ab), \\
    C_6(a, b, c) &= \left( \frac{c^2b}{(a + c)^2}, a + c, \frac{abc}{a + c} \right), \\
    C_7(a, b) &= (b, ab^2).
\end{align*}
\]

**Type \(G_2\) braid relations:**

1. If \(s_1s_2s_3s_4s_2s_3s_2s_3 = s_2s_3s_2s_3s_2s_3s_2\) and \(\alpha_i\) is the long root, then we have (see \[5\] Section 3.1)

\[
x_i(a)x_j(b)x_i(c)x_j(d)x_i(e)x_j(f) = x_j(f')x_i(e')x_j(d')x_i(c')x_j(b')x_i(a')
\]

for

\[
\begin{align*}
    a' &= \frac{ab^2de}{\pi_1}, & b' &= \frac{\pi_3^3}{\pi_4}, & c' &= \frac{\pi_4}{\pi_1\pi_2}, \\
    d' &= \frac{\pi_3^3}{\pi_3\pi_4}, & e' &= \frac{\pi_3}{\pi_2}, & f' &= \frac{bc^3d^2e^3f}{\pi_3}.
\end{align*}
\]
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\[ \pi_1 = abcd + abc + e \]
\[ \pi_2 = ab^2c^2d + ab^2c + e \]
\[ \pi_3 = ab^2c^2 + ab^2c + e \]
\[ \pi_4 = ab^2c^2d + 2abc + e \]

This gives rise to the eighth relevant change of coordinates

\[ C_8(a, b, c, d, e, f) = (f', e', c', b', a') \]

(2) Next we have the relation

\[ x_i(a)x_j(b)x_i(c)x_j(d)\delta_jx_i(e) = \]
\[ = x_j \left( \frac{bc^2d}{x} \right) x_i \left( \frac{y}{x} \right) x_j \left( \frac{z}{x^2} \right) x_i \left( \frac{w}{abc^2d} \right) x_j \left( \frac{y^3}{z} \right) \delta_ju, \]

where \( u \in U_+ \) and

\[ x = 3acde + ac + ad + c^3d + e + a^2(2bde + d^2 + b^2(c^3d + e)), \]
\[ y = cde + ad + ac + ad + c^2 + 2b^2(c^3d + e)), \]
\[ z = e + ac + ad + cd + a^2b^2 + e + abc^2, \]
\[ v = a^2b^2 + c^2d + e + cd + ab + ad^2. \]

So we may take

\[ C_9(a, b, c, d, e) = \left( \frac{bc^2d}{x}, \frac{y}{z}, \frac{z}{abc^2d} \right) \]

Similarly,

\[ x_j(a)x_i(b)x_j(c)x_i(d)\delta_jx_i(e) = \]
\[ = x_i \left( \frac{bcd}{x'} \right) x_j \left( \frac{y'}{y} \right) x_i \left( \frac{z'}{z} \right) x_j \left( \frac{w'}{abc^2d} \right) x_i \left( \frac{y'^3}{y'} \right) \delta_ju, \]

where \( u \in U_+ \) and

\[ x' = cbde + a(2bde + d^2 + b^2(c^3d + e)), \]
\[ y' = cd + 2bc^2 + 2bc^3 + c^3d + e + 2c^2 + b^2c^2 + c^3d + e + 2bc^2 + d^2c^2 + b^2c^2 + 2cd + 3e). \]

This gives

\[ C_{10}(a, b, c, d, e) = \left( \frac{bcd}{y'}, \frac{y'}{y'}, \frac{z'}{abc^2d} \right) \]

(3) Next we have

\[ x_j(a)x_i(b)x_j(c)x_i(d)\delta_jx_i = \]
\[ = x_i \left( \frac{bc^2d}{z_1x + cdz_2} \right) x_j \left( \frac{az_1^2 + cdz_2}{z_1} \right) x_i \left( \frac{z_1^3}{az_1^2 + cdz_2} \right) \delta_jx_i \left( \frac{abc^2d}{z_1} \right) u \]
where \( z_1 = cd + a(b + d) \), \( z_2 = a^2 b + (a + c)^2 d \) and \( u \in U_\pm \), and also

\[
x_j(a)x_i(b)x_j(c)\dot{s}_i\dot{s}_j x_i(d) = x_i \left( \frac{bc^3}{z_4} \right) x_j \left( \frac{z_4}{z_3} \right) \dot{s}_i\dot{s}_j x_i \left( \frac{z_3^3}{a^3 b^2 c^6 z_4} \right) x_j \left( \frac{a^2 b^2 c^4}{z_3} \right) u'
\]

where

\[
z_3 = a^2 b^2 c^3 + (a + c)^2 d, \quad z_4 = a^3 b^2 c^3 + (a + c)^3 d,
\]

and \( u' \in U^- \). So we set

\[
C_{11}(a, b, c, d) = \left( \frac{bc^3 d^2}{az_1^2 + cdz_2}, \frac{az_1^2 + cdz_2}{z_1^2}, \frac{z_3}{az_1^2 + cdz_2}, \frac{abc^2 d}{z_1} \right),
\]

\[
C_{12}(a, b, c, d) = \left( \frac{bc^3 d^2}{z_4}, \frac{z_4}{z_3}, \frac{z_3}{a^3 b^2 c^6 z_4}, \frac{a^2 b^2 c^4}{z_3} \right).
\]

(4) Moreover

\[
x_j(a)x_i(b)x_j(c)\dot{s}_i\dot{s}_j x_i = x_i \left( \frac{bc^3}{(a + c)^3} \right) x_j(a + c) \dot{s}_i\dot{s}_j x_j \left( \frac{abc^2}{a + c} \right) u
\]

and similarly

\[
x_j(a)x_i(b)\dot{s}_j\dot{s}_j x_i(c) = x_i \left( \frac{bc}{a^2 b^2 + c} \right) x_j \left( \frac{a^3 b^2 + c}{a^2 b^2} \right) x_i \left( \frac{a^3 b^3}{a^2 b^2 + c} \right) \dot{s}_j\dot{s}_j u'
\]

for some \( u, u' \in U^- \). So

\[
C_{13}(a, b, c) = \left( \frac{bc^3}{(a + c)^3}, a + c, \frac{abc^2}{a + c} \right),
\]

\[
C_{14}(a, b, c) = \left( \frac{bc}{a^2 b^2 + c}, \frac{a^3 b^2 + c}{a^2 b^2}, \frac{a^3 b^3}{a^2 b^2 + c} \right).
\]

(5) Finally,

\[
x_i(a) x_j(b) \dot{s}_i\dot{s}_j \dot{s}_j = x_j(b) \dot{s}_i\dot{s}_j \dot{s}_j x_i(ab^3) u
\]

and

\[
x_j(a)x_i(b)\dot{s}_j\dot{s}_j \dot{s}_j = x_i(b) \dot{s}_j\dot{s}_j \dot{s}_j x_i(ab) u'
\]

up to \( u, u' \in U_\pm^- \). Also we have

\[
x_i(a) \dot{s}_j\dot{s}_j \dot{s}_j \dot{s}_j = \dot{s}_j\dot{s}_j \dot{s}_j \dot{s}_j x_i(a)
\]

\[
\dot{s}_j\dot{s}_j \dot{s}_j \dot{s}_j x_j(a) = x_j(a) \dot{s}_j\dot{s}_j \dot{s}_j \dot{s}_i
\]

So the last new coordinate transformation is

\[
C_{15}(a, b) = (b, ab^3).
\]

Here the transformations in (2-5) immediately above were computed with the help of Mathematica, realizing \( G_2 \) inside a group of type \( B_3 \) and using all of the relations from types \( A_2 \) and \( B_2 \).

Now let

\[
L : t = (t_1, \ldots, t_m) \mapsto (L^1(t), \ldots, L^m(t))
\]

be one of the changes of coordinates \( C_j \) with \( j = 0, \ldots, 15 \). The form given by

\[
\bigwedge_{i=1}^l \frac{dt_i}{t_i}
\]
is invariant up to sign under these changes of coordinates if for each of the $L = C_j$
\[
\frac{\text{Jac}(L)}{L^{1 \cdots m}} = \pm \frac{1}{t_1 \cdots t_m},
\]
where $\text{Jac}(L) = \det \left( \frac{\partial L^j}{\partial X_k} \right)_{i,k=1,\ldots,m}$ is the Jacobian. This is the case as can easily be checked e.g. using Mathematica. The sign is minus precisely in the cases
\[C_0, C_2, C_3, C_5, C_7, C_8, C_{11}, C_{12}, C_{15},\]
where there is an even number of coordinates involved in the coordinate transformation.

\[ \Box \]

Remark 7.3. Notice that all of the coordinate transformations $C_0, \ldots, C_{15}$ are subtraction-free rational functions. The well-defined subset in the real points of an intersection of Bruhat cells $R_{v,w}$, consisting of those points in an (any) open Deodhar stratum $R_{j+1(i)} \cup$ all of whose canonical coordinates take values in $\mathbb{R}_{>0}$, coincides with the totally positive part of $R_{v,w}$ defined by Lusztig \[29\], see \[31\] Theorem 11.3.

8. The mirror conjecture for $G/P$

Let $Z_P^\vee \rightarrow (\mathfrak{h}^\vee)^{W_P}$ be the pullback of the family $pr_1 : Z_P \rightarrow (T^\vee)^{W_P}$ under the exponential map $\exp : (\mathfrak{h}^\vee)^{W_P} \rightarrow (T^\vee)^{W_P}$. So, explicitly,
\[Z_P^\vee = \{(h^\vee, b) \in (\mathfrak{h}^\vee)^{W_P} \times B_P^\vee \mid b \in U_+ \exp(h^\vee) \hat{w}_P \hat{w}_0^{-1} U_+^\vee \}.
\]
For $h^\vee$ in $\mathfrak{h}^\vee$ we write $Z_P^\vee$ for the fiber over $h^\vee$ in $Z_P^\vee$. We may identify this fiber with
\[B_\vee \cap U_+^\vee \exp(h^\vee) \hat{w}_P \hat{w}_0^{-1} U_+^\vee.
\]

Note that as in Section 4.4,
\[Z_P^\vee \xrightarrow{\sim} (\mathfrak{h}^\vee)^{W_P} \times R_{w_P,\omega_0}^\vee,
\]
\[(h^\vee, b) \mapsto (h^\vee, bw_0 B_\vee^\vee).
\]

The phase function $F_P$ pulled back to $Z_P^\vee$ will be again denoted by $F_P$.

Now let $i_0$ be a reduced expression of $w_0$ and $\omega$ the $n_P$-form on $R_{w_P,\omega_0}^\vee$ defined in Proposition\[72\]. Let us pull this $n_P$-form back to $Z_P^\vee$ by the map
\[Z_P^\vee \rightarrow R_{w_P,\omega_0}^\vee,
\]
\[(h^\vee, b) \mapsto bw_0 B_\vee^\vee / B_\vee^\vee,
\]
and denote the resulting form again by $\omega$. Note that $\omega$ depends on the reduced expression $i_0$ only for its sign. We write $\omega_{h^\vee}$ for the restriction of $\omega$ to the fiber $Z_P^\vee$.

Conjecture 8.1. The integrals \[13\] defined in terms of the mirror datum $(Z_P^\vee, \omega, F_P)$ give solutions to the quantum differential equations \[16\] of $G/P$.

We now want to state a $T$-equivariant version of the above conjecture. For this we need to integrate over functions defined on the covering $\tilde{Z}_P$ of $Z_P$. We therefore pull back also this covering family $pr_1 : \tilde{Z}_P \rightarrow (T^\vee)^{W_P}$ to $(\mathfrak{h}^\vee)^{W_P}$, to get
\[\tilde{Z}_P^\vee = \{(h^\vee, b, h_R) \mid (\exp(h^\vee), b, h_R) \in \tilde{Z}_P \}.
\]
The pullbacks of $\tilde{\phi}$ and $\mathcal{F}_P$ to $\tilde{Z}^h_{\nu}$ will again be denoted by $\tilde{\phi}$ and $\mathcal{F}_P$, respectively. Moreover the map $\tilde{Z}^h_{\nu} \to Z^h_{\nu}$ which forgets $h_R$ is again a covering, and the $n_P$-form $\omega$ on $Z^h_{\nu}$ pulls back to an $n_P$-form on $\tilde{Z}^h_{\nu}$ which we denote by $\tilde{\omega}$. The restriction of $\tilde{\omega}$ to a fiber $\tilde{Z}^h_{\nu}$ of the family $pr_1 : \tilde{Z}^h_{\nu} \to (\mathfrak{h}^\vee)^W$ is denoted by $\tilde{\omega}_{h^\vee}$.

**Conjecture 8.2.** A full set of solutions to the $T$-equivariant quantum differential equations [16, 9] of $G/P$ is given by integrals

$$\tilde{S}_{\tilde{\Gamma}}(h^\vee, h) = \int_{\tilde{\Gamma}_{h^\vee}} e^{\mathcal{F}_P/h^\vee} \tilde{\phi} \tilde{\omega}_{h^\vee}, \tag{8.1}$$

where $\tilde{\Gamma} = (\tilde{\Gamma}_{h^\vee})_{h^\vee \in \mathfrak{h}^\vee}$ is a continuous family of suitable integration contours $\tilde{\Gamma}_{h^\vee}$ in $\tilde{Z}^h_{\nu}$ such that $\tilde{S}_{\tilde{\Gamma}}$ converges.

We note here that the equivariant quantum cohomology ring of $G/P$ is semisimple. (This follows from the fact that equivariant cohomology is semisimple.) Correspondingly, in a generic fiber determined by an $h^\vee \in (\mathfrak{h}^\vee)^W$ and for generic $h \in \mathfrak{h}$, the function $\mathcal{F}_P + \ln \phi(\cdot, h)$ has the correct number, $\dim H^*(G/P)$, of non-degenerate critical points counted in $Z^h_{\nu}$. This suggests that one could be able to construct the right number of suitable integration contours using Morse theory, as asserted in the $SL_{n+1}/B$ case by Givental, and Joe and Kim, which would hopefully give a basis of solutions. The same need not be true for general $G/P$ in the non-equivariant setting, that is for $h = 0$ above. However, the non-equivariant quantum cohomology ring is also known to be always semisimple for the full flag variety $G/B$ [27], and for Grassmannians, see [15, 37].

**9. The mirror constructions for $SL_{n+1}/B$ of Givental, Joe and Kim**

Givental constructed a mirror family for $SL_{n+1}/B$ in [17]. In this section we recall Givental’s construction and identify his mirror family with a restriction of ours to an open subset. We will show in that case that the oscillatory integrals (1.1) arising from our mirror construction agree with those of Givental, proving Conjecture 8.1 in that case. In the $T$-equivariant setting an analogue of Givental’s mirror theorem was given by Joe and Kim [20]. We go on to review their construction and compare it with ours for the equivariant case, showing that the integrals constructed by Joe and Kim can indeed be written in the form (8.1). This supports our Conjecture 8.2.

We note also that Batyrev, Ciocan-Fontanine, Kim and van Straten [3] proposed a mirror family for $SL_{n+1}/P$ in the style of Givental’s. For the direct relationship between their construction and the type $A$ Peterson variety see [36].

**9.1.** Let $G = SL_{n+1}$, so $G^\vee = PSL_{n+1}$. We use the standard choice of Chevalley generators $e_i = e_i^\vee = E_{i,i+1}$ and $f_i = f_i^\vee = E_{i+1,i}$, where $E_{j,k}$ is the matrix with 1 in position $(j, k)$ and zeros elsewhere. Correspondingly we have the simple root subgroups $x_i(t) = 1_{n+1} + t E_{i,i+1}$ which we may consider to be lying in $SL_{n+1}$ or $PSL_{n+1}$, depending on the context. We now recall the type $A$ mirror construction from [17].
9.1.1. Consider the quiver \((\mathcal{V}, \mathcal{A})\) which looks as follows:

\[
\begin{array}{c}
\bullet \\
\uparrow \\
\bullet \leftarrow \bullet \\
\uparrow \uparrow \\
\bullet \leftarrow \bullet \leftarrow \bullet \\
\vdots \vdots \ddots \\
\bullet \leftarrow \bullet \ldots \bullet \\
\uparrow \uparrow \uparrow \ddots \\
\bullet \leftarrow \bullet \\
\end{array}
\]

We divide the set of vertices \(\mathcal{V}\) up into the vertices along the diagonal, \(\mathcal{V}_\circ = \{v_{11}, \ldots, v_{n+1,n+1}\}\), and the vertices below the diagonal, \(\mathcal{V}_- = \{v_{ij} \mid 1 \leq j < i \leq n+1\}\). The labeling is as for the entries of a matrix. Let \(\mathcal{A} = \mathcal{A}_c \sqcup \mathcal{A}_d\) be the set of arrows, divided into vertical and horizontal arrows, respectively. For any arrow \(a\) denote by \(h_a, t_a \in \mathcal{V}\) the head and tail of \(a\). In fact let us label the arrow \(a\) by \(c_{ij}\) if \(a\) is a vertical arrow and \(d_{ij}\) if \(a\) is horizontal with \(t_a = v_{ij}\). Let

\[
Z := \{\sigma = (\sigma_a)_{a \in \mathcal{A}} \in (\mathbb{C}^\times)^{\mathcal{A}} \mid \sigma_d \sigma_c = \sigma_{d'} \sigma_{c'} \text{ for all configurations (9.1)}\},
\]

where

\[
\begin{array}{c}
\bullet \\
\downarrow \\
\bullet \\
\end{array}
\quad \overset{d}{\underset{c}{\leftrightarrow}}
\quad \begin{array}{c}
\bullet \\
\downarrow \\
\bullet \\
\end{array}
\]

is a square in the quiver.

For simplicity of notation we identify the arrows with coordinate functions on \(Z\). In other words we may think of \(\mathcal{A} \subset \mathbb{C}[Z]\) as invertible generators for the coordinate ring of \(Z\). Define

\[
\tilde{q}_i = c_{ii} d_{i+1,i+1} \in \mathbb{C}[Z],
\]

for \(i = 1, \ldots, n\). Then we have a family of varieties

\[
(9.2) \quad \tilde{q} = (\tilde{q}_1, \ldots, \tilde{q}_n) : Z \longrightarrow (\mathbb{C}^\times)^n.
\]

Let the fiber over \(Q \in (\mathbb{C}^\times)^n\) be denoted by \(Z_Q\). This map is a trivial fibration with fiber isomorphic to \((\mathbb{C}^\times)^{n+1}\). Explicitly, consider the isomorphism

\[
(9.3) \quad (\mathbb{C}^\times)^{V_-} \times (\mathbb{C}^\times)^n \overset{\sim}{\longrightarrow} Z
\]

given by \((z_{v_i})_{v \in \mathcal{V}_-} \times (z_{v_{ij}})_{i=1}^n) \mapsto \sigma := (z_{h_a} z_{t_a}^{-1})_{a \in \mathcal{A}}, \text{ where we set } z_{v_{a+1,n+1}} = 1.\text{ In particular we have vertex coordinates given by } t_v(\sigma) = z_v. \text{ We will denote } t_{v_{ij}} \text{ also by } t_{ij} \text{ for convenience. Note that } t_{i,i+1,i+1} = \tilde{q}_i.\]

9.1.2. The phase function on this family is defined to be

\[
\tilde{F}(\sigma) = \sum_{a \in \mathcal{A}} \sigma_a.
\]
One has the following simple description of the critical points of \( \tilde{F} \) along the fibers of \( \tilde{q} \),

\[
\mathcal{Z}^{crit} = \left\{ \sigma \in \mathbb{Z} \mid \sum_{a \in A, \ h_a = v} \sigma_a = \sum_{a \in A, \ t_a = v} \sigma_a, \ \text{for all} \ v \in \mathbb{V}_- \right\}.
\]

9.1.3. We will now recall the construction of the quantum Toda lattice in type A and state Givental’s mirror theorem. Consider the map

\[
\varepsilon : \mathbb{C}^{n+1} \rightarrow (\mathbb{C}^*)^n
\]

\[
(T_i)_{i=1}^{n+1} \mapsto (\exp(T_i - T_i+1))_{i=1}^n.
\]

Let \( \tilde{Z} \rightarrow \mathbb{C}^{n+1} \) be the pullback of the bundle \( \tilde{q} : Z \rightarrow (\mathbb{C}^*)^n \) by \( \varepsilon \). We denote \( \varepsilon^* (\tilde{q}) \) and \( \varepsilon^* (\tilde{F}) \) again by \( \tilde{q} \) and \( \tilde{F} \), respectively.

Solving the \( \mathfrak{g}l_{n+1} \) quantum Toda lattice means finding smooth functions \( S = S(T_1, \ldots, T_{n+1}) \) satisfying

\[
(9.6) \quad \det \left( x_{1_{n+1}} + \begin{pmatrix}
\hbar \frac{\partial}{\partial T_1} & e^{T_1-T_2} & & \\
-1 & \hbar \frac{\partial}{\partial T_2} & e^{T_2-T_3} & \\
& -1 & \ddots & \ddots \\
& & \ddots & \hbar \frac{\partial}{\partial T_{n+1}}
\end{pmatrix} \right) S = x^{n+1} S,
\]

where \( \hbar \in \mathbb{R}_{>0} \). Note that the coefficients of the polynomial in \( x \) on the left hand side are well-defined differential operators. By [17, 24] the quantum differential equations for \( SL_{n+1}/B \) make up the quantum Toda lattice for \( \mathfrak{g}l_{n+1} \), whose solutions are obtained by restricting the solutions \( S \) of (9.6) to the subspace of \( \mathbb{C}^{n+1} \) defined by \( \sum_{i=1}^{n+1} T_i = 0 \).

**Theorem 9.1** (Givental [17]). Let \( \Gamma = (\Gamma_{T_i})_{T_i \in \mathbb{C}^{n+1}} \) be a continuous family of possibly non-compact \( (n+1) \)-cycles

\[
\Gamma_{T_i} \subset \mathcal{Z}_{\varepsilon(T_1, \ldots, T_{n+1})}
\]

obtained from descending Morse cycles for \( \text{Re}(\tilde{F}) \). The integrals

\[
\tilde{S}_T(T_1, \ldots, T_{n+1}) = \int_{\Gamma_{T_{n+1}}} e^{\tilde{F}/\hbar} \bigwedge_{v \in \mathbb{V}_-} \frac{dt_v}{t_v}
\]

solve the system of differential equations (9.6).

9.2. Let us now consider \( G^\vee = PSL_{n+1}(\mathbb{C}) \). We have \( n_B = \frac{n(n+1)}{2} \) and let \( i_0 = (i_{1}, \ldots, i_{n_B}) \) be the reduced expression of \( w_0 \) obtained by successively concatenating the sequences \( l_k = (n, \ldots, k) \) for \( k = 1, \ldots, n \). To \( \sigma \in \mathcal{Z} \) we associate two unipotent upper-triangular matrices,

\[
(9.7) \quad x_c(\sigma) = \prod_{k=1}^{n} \left( \prod_{j=n}^{k} x_j(\sigma_{c_{j,k}}) \right),
\]

\[
(9.8) \quad x_d(\sigma) = \prod_{k=n}^{1} \left( \prod_{j=1}^{k} x_{n-j+1}(\sigma_{d_{k+1,j+1}}) \right).
\]
We also associate to $\sigma$ an element $\tau(\sigma)$ of the maximal torus $T^\vee$ of $PSL_{n+1}$, which is given by
\[
\begin{pmatrix}
t_{11}(\sigma) & \cdots & t_{22}(\sigma) \\
\vdots & \ddots & \vdots \\
t_{nn}(\sigma) & \cdots & 1
\end{pmatrix}
= \begin{pmatrix}
\tilde{q}_1 \cdots \tilde{q}_n(\sigma) \\
\vdots \\
\tilde{q}_{n-1} \tilde{q}_n(\sigma) & 1
\end{pmatrix}.
\]

The definition of a matrix $x_c(\sigma)$ associated to a point in Givental’s $Z$ can already be found in [14] and [36]. Its combination with $x_d(\sigma)$ and $\tau(\sigma)$ required for the full Lie theoretic interpretation of $Z$, see the theorem below, appears here for the first time.

**Theorem 9.2.** Let $Z_B$ and $F_B$ be as defined in Section 4.7
(1) The map
\[
\beta : Z \rightarrow G^\vee, \quad \sigma \mapsto x_c(\sigma)\tau(\sigma)\tilde{w}_0^{-1}x_d(-\sigma)^{-1}
\]
has image in $B^\vee_\lor$ and, taken together with $\tau$, defines an open embedding
\[(\tau, \beta) : Z \hookrightarrow Z_B.\]

(2) We have $(\tau, \beta)^*(F_B) = \tilde{F}$. In particular the map $\sigma \mapsto x_c(\sigma)B^\vee_\lor$ identifies $Z^{crit}$ with the intersection of $Y_B$ and the open Deodhar stratum in $R^\vee_{1,0}$ corresponding to $i_0$.

(3) For $Q \in (\mathbb{C}^*)^n$ let $\gamma_Q : Z_Q \rightarrow R^\vee_{1,0}$ be the embedding of a fiber given by $\gamma_Q(\sigma) = x_c(\sigma)B^\vee_\lor$. Suppose $\omega$ is a holomorphic $n_B$-form on $R^\vee_{1,0}$ as in Proposition 7.2. Then
\[
\gamma_Q^*(\omega) = \epsilon \bigwedge_{v \in V_-} dt_v/t_v.
\]
where $\epsilon \in \{\pm 1\}$ and is independent of $Q$. Here the $t_v$ are the vertex coordinates defined in 9.1.1.

Note that Theorem 9.2 together with Givental’s Theorem 9.1 implies the Conjecture 9.1 for $SL_{n+1}/B$.

**Lemma 9.3.** Let $t = (\tilde{t}_v)_{v \in V} \in (\mathbb{C}^*)^V$, where we may write $\tilde{t}_{ij}$ for $\tilde{t}_{v_{ij}}$ for short. Let $b(t) \in GL_{n+1}$ be defined by
\[
b(t) := x_c((\tilde{t}_{ha} \tilde{t}_{ta}^{-1})_{a \in A}) \begin{pmatrix}
\tilde{t}_{11} & \cdots & \tilde{t}_{22} \\
\vdots & \ddots & \vdots \\
\tilde{t}_{n+1,n+1}
\end{pmatrix} \tilde{w}_0^{-1}x_d((-\tilde{t}_{ha} \tilde{t}_{ta}^{-1})_{a \in A})^{-1}.
\]

Then for the fundamental representation $V(\omega_k)$ we have
\[
\langle b(t) \cdot v^+_w, v^+_w \rangle = \left( \prod_{i=1}^{n+1} \tilde{t}_{ii} \right) \left( \prod_{i-j=k} \tilde{t}_{ij}^{-1} \right).
\]
Proof. Let \( \{ v_1, \ldots, v_{n+1} \} \) be the standard basis of \( \mathbb{C}^{n+1} \), and choose the standard highest weight vector \( v^+_w := v_1 \land \ldots \land v_k \) in \( V(\omega_k) = \bigwedge^k \mathbb{C}^{n+1} \). Then we have the lowest weight vector

\[
v^-_w := \omega_0^{-1} \cdot (v_1 \land \ldots \land v_k) = v_{n-k+2} \land \ldots \land v_{n+1},
\]

and

\[(9.10)\]

\[
b(t) \cdot v^+_w = x_e \left( (\tilde{t}_a \tilde{t}^{-1}_a)_{a \in A} \right) \begin{pmatrix}
\tilde{t}_{11} & \tilde{t}_{22} & \cdots \\
\cdots & \cdots & \cdots \\
\tilde{t}_{n+1,n+1}
\end{pmatrix} \cdot v_{n-k+2} \land \ldots \land v_{n+1}
\]

Now note that, written out,

\[
x_e(\{ \sigma_a \}_{a \in A} ) = x_n(\sigma_{c_{1,1}}) x_{n-1}(\sigma_{c_{n-1,1}}) \cdots \cdot x_{k+1}(\sigma_{c_{k+1,1}}) x_k(\sigma_{c_{k,1}}) \cdots \cdot x_2(\sigma_{c_{2,1}}) x_1(\sigma_{c_{1,1}})
\]

\[
\vdots
\]

\[
x_n(\sigma_{c_{n,n-1}}) x_{n-1}(\sigma_{c_{n-1,n-1}}) \cdots \cdot x_{n-k}(\sigma_{c_{n-k,n-k}})
\]

\[
x_n(\sigma_{c_{n,n-k-1}}) x_{n-1}(\sigma_{c_{n-1,n-k-1}}) \cdots \cdot x_{n-k+1}(\sigma_{c_{n-k+1,n-k+1}})
\]

\[
x_n(\sigma_{c_{n,n-k-2}}) x_{n-1}(\sigma_{c_{n-1,n-k-2}}) \cdots \cdot x_{n-k+2}(\sigma_{c_{n-k+2,n-k+2}})
\]

\[
\vdots
\]

\[
x_n(\sigma_{c_{n,n-1}}) x_{n-1}(\sigma_{c_{n-1,n-1}})
\]

\[
x_n(\sigma_{c_{n,n}}).
\]

Each \( x_j(a) = \exp(\alpha e_j) \) simply acts by \( 1 + \alpha e_j \) on \( \bigwedge^k \mathbb{C}^{n+1} \), and it is not hard to check that in order to get from the lowest to the highest weight space via \( x_e(\{ \sigma_a \}_{a \in A} ) \) we need to take the \( e_j \)-summand precisely from each of the underlined \( x_j(\sigma_{c_{n}}) \) factors. Since in our case \( \sigma_a = \tilde{t}_a \tilde{t}^{-1}_a \), we have

\[
\sigma_{c_{j+k-1-j}} \sigma_{c_{j+k-2-j}} \ldots \sigma_{c_{j,j}} = \tilde{t}_{jj} \tilde{t}^{-1}_{j+k,j},
\]

for the resulting contribution of the \( j \)-th row above, and so we find that in total

\[
\langle x_e( (\tilde{t}_a \tilde{t}^{-1}_a)_{a \in A} ) \cdot v^-_w, v^+_w \rangle = \prod_{j=1}^{n-k+1} \tilde{t}_{jj} \tilde{t}^{-1}_{j+k,j} = \left( \prod_{j=1}^{n-k+1} \tilde{t}_{jj} \right) \left( \prod_{i-j=k} \tilde{t}^{-1}_{ij} \right).
\]

Combining this with \(9.10\) we see that

\[
\langle b \cdot v^+_w, v^+_w \rangle = \left( \prod_{j=1}^{n+1} \tilde{t}_{jj} \right) \left( \prod_{i-j=k} \tilde{t}^{-1}_{ij} \right)
\]

and the lemma is proved. \( \square \)
Proof of Theorem 9.2. We assume for the moment that we have proved that $\beta$ has image in $B^\vee$, and observe how the rest of the theorem follows from this assertion. If $\beta$ has image in $B^\vee$, then $(\tau, \beta)$ defines a map $Z \to Z_B$. From this point of view the $c_{ij}$ correspond precisely the standard coordinates for $x_c(\sigma)B^\vee/B^\vee_\tau = \beta(\sigma)w_0B^\veew_0/B^\vee_\tau$ in the open Deodhar stratum $R^\vee_{1 + (i_0), i_1}$. Moreover the values of the $c_{ij}$ together with those of the $\tilde{q}_i$ suffice to determine a point in $Z$ uniquely. Therefore we see that $(\tau, \beta)$ is injective, and its image is equal to the preimage of $T^\vee \times R^\vee_{1 + (i_0), i_1}$ under the trivialization $\bigl(1, 3\bigr)$ of $Z_B$. This proves (1). Part (2) then follows from (1) and Lemma 5.2 combined with Theorem 4.1. See also $[36]$. The third part of the theorem is an easy consequence of the definition of $x_c$.

It remains to prove that $\beta$ has image in $B^\vee$. We may work in $GL_{n+1}$, rather than $PSL_{n+1}$, choosing the representative for $\tau(\sigma)$ as the one from its definition. Multiplying out the product $x_c$ we obtain a matrix

$$x_c = \begin{pmatrix}
1 & G^{(1)}_1 & G^{(2)}_1 & \ldots & G^{(n)}_1 \\
1 & G^{(2)}_1 & G^{(3)}_1 & \ldots & G^{(n-1)}_1 \\
1 & \vdots & \ddots & \ddots & \ddots \\
1 & \ddots & \ddots & \ddots & \ddots \\
1 & \ddots & \ddots & \ddots & 1
\end{pmatrix}$$

with entries given by

$$G^{(j)}_k = \sum_{1 \leq m_1 < \ldots < m_k \leq j} \left( \prod_{i=1}^{k} c_{j-k+i, m_i} \right).$$

Similarly let $\tilde{x}(\sigma) := x_d(\sigma)^{-1}$. Then

$$\tilde{x} = \begin{pmatrix}
1 & \tilde{G}^{(1)}_1 & \tilde{G}^{(2)}_1 & \ldots & \tilde{G}^{(n)}_1 \\
1 & \tilde{G}^{(2)}_1 & \tilde{G}^{(3)}_1 & \ldots & \tilde{G}^{(n-1)}_1 \\
1 & \vdots & \ddots & \ddots & \ddots \\
1 & \ddots & \ddots & \ddots & \ddots \\
1 & \ddots & \ddots & \ddots & 1
\end{pmatrix}$$

where

$$\tilde{G}^{(j)}_k = \sum_{n-j+1 \leq m_1 \leq \ldots \leq m_k \leq n+1} \left( \prod_{i=1}^{k} d_{m_{k-i+1}, n+1-j+i} \right).$$

The $(j, r + 1)$ entry of the matrix $\beta(\sigma) = x_c(\sigma)\tau(\sigma)\tilde{w}_0^{-1}x_d(\sigma)^{-1} \cdot \beta_\tau(\sigma)$ is

$$\beta_{j, r+1} := (0, \ldots, 0, 1, G^{(j)}_1, G^{(j+1)}_2, \ldots, G^{(n)}_{n-j+1})$$

(9.11)
evaluated at $\sigma$. We want to show that this expression is zero when $j \leq r$.

In the rank 1 case we have for $j = r = 1$

$$(1, G_1^{(1)}) \cdot \left( -\tilde{q}_1 \tilde{G}_1^{(1)} \right) = -\tilde{q}_1 + c_{11}d_{22} = 0.$$  

We will prove the general case by induction.

Consider the two embeddings of $GL_n$ into $GL_{n+1}$ corresponding to the subsets $I_L = \{1, \ldots, n-1\}$ and $I_R = \{2, \ldots, n\}$ of $I$. The first gives the subgraph $(\mathcal{V}_L, \mathcal{A}_L)$ of $(\mathcal{V}, \mathcal{A})$ obtained by erasing the last row of vertices. And the second gives the subgraph $(\mathcal{V}_R, \mathcal{A}_R)$ where the first column has been removed.

We add superscripts $L$ and $R$ to any of the matrices $\tilde{x}, x, \tau, \beta$ if we are referring to their analogues defined in terms of the graphs $(\mathcal{V}_L, \mathcal{A}_L)$ or $(\mathcal{V}_R, \mathcal{A}_R)$, respectively. We denote by $\tilde{G}_k^{(r,L)}$ and $\tilde{G}_k^{(r,R)}$ the matrix coefficients of $\tilde{x}^L$ and $\tilde{x}^R$, each viewed inside its respective copy of $GL_n$. Similarly for $x^L_c$ and $x^R_c$ and their entries.

It is easy to check that

$$\tilde{G}_k^{(r)} = \tilde{G}_k^{(r-1,L)} + d_{n+1,n-r+2}\tilde{G}_k^{(r-1,R)}.$$  

So we have

$$(9.12) \quad \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \pm \tilde{q}_n - r + 1 \cdots \tilde{q}_1 1 \\ \mp \tilde{q}_n - r + 2 \cdots \tilde{q}_1 \tilde{G}_1^{(r)} \\ \vdots \\ \tilde{q}_n - 1 \tilde{q}_n \tilde{G}_r^{(r)} \\ -\tilde{q}_n \tilde{G}_r^{(r)} \\ \tilde{G}_r^{(r)} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \pm \tilde{q}_n - r + 1 \cdots \tilde{q}_1 1 \\ \mp \tilde{q}_n - r + 2 \cdots \tilde{q}_1 \tilde{G}_1^{(r-1,L)} \\ \vdots \\ \tilde{q}_n - 1 \tilde{q}_n \tilde{G}_r^{(r-1,L)} \\ -\tilde{q}_n \tilde{G}_r^{(r-1,L)} \\ 0 \end{pmatrix} + d_{n+1,n-r+2} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \pm \tilde{q}_n - r + 2 \cdots \tilde{q}_1 1 \end{pmatrix}.$$  

We now want to evaluate (9.11) using (9.12). The first summand gives a contribution of

$$(0, \ldots, 0, 1, G_1^{(j)}, G_2^{(j+1)}, \ldots, G_{n-j+1}^{(n)}) \cdot \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \pm \tilde{q}_n - r + 1 \cdots \tilde{q}_1 1 \\ \mp \tilde{q}_n - r + 2 \cdots \tilde{q}_1 \tilde{G}_1^{(r-1,L)} \\ \vdots \\ \tilde{q}_n - 1 \tilde{q}_n \tilde{G}_r^{(r-1,L)} \\ -\tilde{q}_n \tilde{G}_r^{(r-1,L)} \\ 0 \end{pmatrix}.$$  

$$(9.11) \quad \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \pm \tilde{q}_n - r + 1 \cdots \tilde{q}_1 1 \\ \mp \tilde{q}_n - r + 2 \cdots \tilde{q}_1 \tilde{G}_1^{(r-1,L)} \\ \vdots \\ \tilde{q}_n - 1 \tilde{q}_n \tilde{G}_r^{(r-1,L)} \\ -\tilde{q}_n \tilde{G}_r^{(r-1,L)} \\ 0 \end{pmatrix}.$$  

This equals $-\beta_{j,r}^L$ and is therefore zero if $j < r$ by the induction hypothesis, where we are reducing to the graph with the last row removed. Note that in this induction step we are also removing the last row of $\tau$, which we had normalized to 1 in the rank $n$ case. This accounts for the apparent factor of $\tilde{q}_n$ in the above formula for $-\beta_{j,r}^L$. 


For the second summand notice that we can decompose the entries of the row vector in (9.11) using
\[ G_{k+1}^{(j+k)} = c_{j+1}G_k^{(j+k,R)} + G_{k+1}^{(j+k,R)}. \]
Then we get \( d_{n+1,n-r+2}(c_{j+1}\beta_{j+r}^R + \beta_{j+r+1}^R) \), which is also zero whenever \( j < r \), by
the induction hypothesis this time applied to the graph with the left most column
removed.

Thus we have seen that (9.11) vanishes whenever \( j < r \). If \( j = r \) we are left with
two nonzero summands, giving
\[ \beta_{r,r+1} = -\beta_{r,r}^L + d_{n+1,n-r+2}c_{r,1}\beta_{r,r}^R. \]
(9.13)
It remains to show that this matrix coefficient vanishes.

By induction assumption \( \beta^L \) and \( \beta^R \) are upper-triangular, so we have
\[ \langle \beta^X \cdot v_{\omega_k}^+, v_{\omega_k}^+ \rangle = \beta_{1,1}^{X,2} \beta_{2,2}^{X,3} \ldots \beta_{r,r}^{X}, \]
for \( X = L \) or \( R \). Now let \( \sigma \in \mathcal{Z} \) and consider the vertex coordinates \( \tilde{t}_v = t_v(\sigma) \).
The corresponding ‘truncated’ elements are \( \sigma_L = (\tilde{t}_{h_n}^{-1})_{a \in A_L} \) and \( \sigma_R = (\tilde{t}_{h_n}^{-1})_{a \in A_R} \), and Lemma 9.3 says that
\[ \langle \beta^L(\sigma_L) \cdot v_{\omega_k}^+, v_{\omega_k}^+ \rangle = \left( \prod_{i=1}^{n} \tilde{t}_{i} \right) \left( \prod_{j=1}^{n-r+1} \tilde{t}_{j+r-1,j}^{-1} \right), \]
\[ \langle \beta^R(\sigma_R) \cdot v_{\omega_k}^+, v_{\omega_k}^+ \rangle = \left( \prod_{i=2}^{n+1} \tilde{t}_{i} \right) \left( \prod_{j=2}^{n-r+1} \tilde{t}_{j+r-1,j}^{-1} \right). \]
Combining these formulas with (9.13) we find that
\[ \beta^L_{r,r}(\sigma_L) = \left( \prod_{j=1}^{n-r+1} \tilde{t}_{j+r-1,j}^{-1} \right), \]
\[ \beta^R_{r,r}(\sigma_R) = \left( \prod_{j=2}^{n-r+2} \tilde{t}_{j+r-1,j}^{-1} \right). \]
Now substituting also \( c_{r,1}(\sigma) = \tilde{t}_{r,1}^{-1} \tilde{t}_{r+1,1}^{-1} \) and \( d_{n+1,n-r+2}(\sigma) = \tilde{t}_{n+1,n-r+1}^{-1} \tilde{t}_{n+1,n-r+2}^{-1} \) it follows directly that
\[ d_{n+1,n-r+2}c_{r,1}\beta_{r,r}^R = \beta^L_{r,r}. \]
This shows that \( \beta_{r,r+1} = 0 \), by (9.13), and finishes the proof.

9.3. The T-equivariant case. In Joe and Kim’s work [20], mirror symmetric solutions to the T-equivariant quantum differential equations of \( SL_{n+1}/B \) are given as integrals over a function defined on a universal cover of \( \mathcal{Z} \). We briefly review this construction here and compare it with our definitions applied to the equivariant \( SL_{n+1}/B \) case.
The T-equivariant quantum differential equations are deformations of the usual quantum differential equations by the ring
\[ H_T(pt) = \mathbb{C}[h] = \mathbb{C}[\lambda_1, \ldots, \lambda_{n+1}]/(\sum \lambda_i). \]
Namely in the $T$-equivariant case the differential equations to solve are obtained by replacing (9.16) by

\begin{equation}
\det \left( x1_{n+1} + \begin{pmatrix} \ell \frac{\partial}{\partial T_1} & e^{T_1-T_2} & e^{T_2-T_3} & \cdots & e^{T_n-T_{n+1}} \\ -1 & \ell \frac{\partial}{\partial T_2} & e^{T_2-T_3} & \cdots & e^{T_n-T_{n+1}} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ -1 & \cdots & -1 & \ell \frac{\partial}{\partial T_n} & e^{T_n-T_{n+1}} \\ \end{pmatrix} \right) \tilde{S} = \prod_{i=1}^{n+1} (x + \lambda_i) \tilde{S}.
\end{equation}

To generalize Givental’s mirror theorem to solve (9.15) Joe and Kim deform the phase function $\tilde{F}$ of Givental, or more precisely they first pull it back to a universal cover of $\mathcal{Z}$ and then deform it there.

**Definition 9.4.** Recall the notations from Section 9.1.1 We let $\tilde{\mathcal{Z}} := \{(T_v)_{v \in \mathcal{V}} \in \mathbb{C}^\mathcal{V} \mid \sum T_{v_i} = 0\}$. For given $(T_1, \ldots, T_{n+1}) \in \mathbb{C}^{n+1}$ with $\sum_i T_i = 0$ we define $\tilde{\mathcal{Z}}_{(T_1, \ldots, T_{n+1})} := \{(T_v)_{v \in \mathcal{V}} \mid T_{v_i} = T_i \text{ for } i = 1, \ldots, n+1\}$.

The map $c : (T_v)_{v \in \mathcal{V}} \mapsto (e^{T_{ha} - T_{ha}})_{a \in \mathcal{A}}$ makes $\tilde{\mathcal{Z}}$ into a universal covering space for $\mathcal{Z}$. We may think of the $T_v$ as logarithmic vertex variables, although the $\exp(T_v)$ recover the vertex variables $t_v$ from Section 9.1.1 only up to a common scalar multiple, as we are working with a different normalization now: We have $\sum T_{v_i} = 0$ rather than $T_{v_{n+1}, n+1} = 0$.

To deform Givental’s phase function Joe and Kim attach ‘weights’ depending on the parameters $\lambda_i$ to the edges of the graph $(\mathcal{V}, \mathcal{A})$ as follows,

\begin{align*}
\lambda_{e_{1i}} &:= \lambda_1 + \frac{1}{2}(\lambda_1 + \cdots + \lambda_{i-1}), \\
\lambda_{e_{ij}} &:= \frac{1}{2}\lambda_{i-j+1}, \quad \text{if } j > 1, \\
\lambda_{d_{n+1, i}} &:= -\lambda_{n+1-j} - \frac{1}{2}(\lambda_1 + \cdots + \lambda_{n-j}), \\
\lambda_{d_{ij}} &:= -\frac{1}{2}\lambda_{i-j+1}, \quad \text{if } i < n+1,
\end{align*}

and set

\begin{equation}
\tilde{F}_{JK}( (T_v)_{v \in \mathcal{V}}; (\lambda_i)_{i} ) := \tilde{F}( (e^{T_{ha} - T_{ha}})_{a \in \mathcal{A}} ) + \sum_{a \in \mathcal{A}} \lambda_a (T_{ha} - T_{ha}).
\end{equation}

**Theorem 9.5** (Joe and Kim [20]). Let $T_\ast$ run through the $(T_1, \ldots, T_{n+1}) \in \mathbb{C}^{n+1}$ with $\sum T_i = 0$, and let $\Gamma = (\Gamma_T)_T \ast$ be a continuous family of possibly non-compact $(n+1)$-cycles,

\[ \Gamma_T \subset \tilde{\mathcal{Z}}_{(T_1, \ldots, T_{n+1})}, \]

obtained as descending Morse cycles for $\text{Re}(\tilde{F}_{JK})$. The integrals

\[ \tilde{S}_T(T_1, \ldots, T_{n+1}; \lambda_1, \ldots, \lambda_{n+1}) = \int_{\Gamma_T} e^{\frac{\partial}{\partial T_v}} \wedge dT_v \]

are obtained as descending Morse cycles for $\text{Re}(\tilde{F}_{JK})$. The integrals

\[ \tilde{S}_T(T_1, \ldots, T_{n+1}; \lambda_1, \ldots, \lambda_{n+1}) = \int_{\Gamma_T} e^{\frac{\partial}{\partial T_v}} \wedge dT_v \]

are obtained as descending Morse cycles for $\text{Re}(\tilde{F}_{JK})$. The integrals

\[ \tilde{S}_T(T_1, \ldots, T_{n+1}; \lambda_1, \ldots, \lambda_{n+1}) = \int_{\Gamma_T} e^{\frac{\partial}{\partial T_v}} \wedge dT_v \]

are obtained as descending Morse cycles for $\text{Re}(\tilde{F}_{JK})$. The integrals

\[ \tilde{S}_T(T_1, \ldots, T_{n+1}; \lambda_1, \ldots, \lambda_{n+1}) = \int_{\Gamma_T} e^{\frac{\partial}{\partial T_v}} \wedge dT_v \]
solve the $T$-equivariant quantum differential equations associated to $SL_{n+1}/B$.

We want to now give an explicit lift of the comparison map $(\tau, \beta) : \mathcal{Z} \to Z_B$ and extend our Theorem 9.2 about comparing phase functions to the equivariant case.

**Definition 9.6.** Let $\gamma_R : \mathcal{Z} \to \mathfrak{h}^\vee$ be defined by

$$\omega_k(\gamma_R((T_v)_{v \in V})) = -\sum_{i-j=k} T_{v_{ij}}.$$

Written out explicitly, $\gamma_R((T_v)_{v \in V})$ is

$$\begin{pmatrix}
  -\sum_{i-j=1} T_{v_{ij}} \\
  \left( \sum_{i-j=1} T_{v_{ij}} \right) - \left( \sum_{i-j=2} T_{v_{ij}} \right) \\
  \vdots \\
  (T_{v_{n1}} + T_{v_{n+1,1}}) - T_{v_{n+1,1}}
\end{pmatrix}.$$  

We also define $\tilde{\beta} := \beta \circ c : \mathcal{Z} \to B_{\mathfrak{h}^\vee}$,

and a map $\gamma : \mathcal{Z} \to \mathfrak{h}^\vee$,

$$\gamma((T_v)_{v \in V}) = \begin{pmatrix}
  T_{v_{11}} \\
  T_{v_{22}} \\
  \vdots \\
  T_{v_{n+1,n+1}}
\end{pmatrix}.$$

**Theorem 9.7.**

1. The maps $\gamma, \tilde{\beta}$ and $\gamma_R$ define a map

$$(\gamma, \tilde{\beta}, \gamma_R) : \mathcal{Z} \to \mathcal{Z}_B^{\mathfrak{h}^\vee},$$

which is a covering composed with an open embedding. Moreover $(\gamma, \tilde{\beta}, \gamma_R)$ takes the fiber $\mathcal{Z}(T_1, \ldots, T_{n+1})$ to the fiber $\mathcal{Z}_{B_{\mathfrak{h}^\vee}}$, where $\mathfrak{h}^\vee$ is the diagonal matrix with entries $(T_1, \ldots, T_{n+1})$.

2. We have

$$(\gamma, \tilde{\beta}, \gamma_R)^*(\bar{\omega}) = \pm \bigwedge_{v \in V} dT_v$$

for the pullback of our form $\bar{\omega}$ from Section 8 to $\mathcal{Z}$.

3. The integrand $e^{\tilde{F}_{JK}}$ of Joe and Kim is obtained by pullback from our integrand,

$$e^{\tilde{F}_{JK}} = (\gamma, \tilde{\beta}, \gamma_R; h)^*(e^{\tilde{F}_{\Phi}}),$$

where $h(\lambda_1, \ldots, \lambda_{n+1})$ is the diagonal matrix with entries $\lambda_1, \ldots, \lambda_{n+1}$.

**Proof.**

1. For $(T_v)_{v \in V} \in \mathcal{Z}$ we have that $\tilde{\beta}((T_v)_{v \in V})$ is given explicitly by (9.17)

$$x_c \left( (e^{T_{v_{a-a}}})_{a \in A} \right) \left[ \begin{array}{cccc}
  e^{T_{v_{11}}} & e^{T_{v_{22}}} & \cdots & e^{T_{v_{n+1,n+1}}} \\
\end{array} \right] \bar{w}_0^{-1} x_d \left( ((-e^{T_{v_{a-a}}})_{a \in A})^{-1} \right).$$
Then by Lemma 9.3 we have the lower-triangular matrix \( \tilde{\omega} \) given by equation (9.18).

Here we have substituted \( \sigma \) using also that \( \sum \tau \) need to compare the effect of Joe and Kim’s correction term with our factor \( \tilde{\omega} \) as we are working in \( PSL_n+1 \), we can clear the denominators. From Theorem 9.2 together with (9.17) it is now immediate that \( (\gamma, \tilde{\beta})(T_v)_{v \in V} \in \mathbb{Z}_B^h \).

To show that \( (\gamma, \tilde{\beta}, \gamma_R)((T_v)_{v \in V}) \) lies in the covering space \( \tilde{\mathbb{Z}}_B^h \), it remains to prove that the diagonal part of \( \tilde{\beta} \) \((T_v)_{v \in V}\) is equal to \( \exp(\gamma_R((T_v)_{v \in V})) \). For this let us consider the lower-triangular matrix \( \tilde{\beta} \) in \( SL_n+1 \) which covers \( \tilde{\beta} \) \((T_v)_{v \in V}\) and is given by (9.18)

\[
\tilde{\omega}^{-1} x_d \left( (e^{T_{v_{i+1}}} )_{a \in A} \right) \begin{pmatrix}
 e^{T_{v_{11}}} & \cdots & \cdots \\
 \vdots & \ddots & \vdots \\
 e^{T_{v_{n+1,n+1}}} & \cdots & \cdots 
\end{pmatrix} \tilde{\omega}^{-1} x_d \left( (e^{T_{v_{i+1}}} )_{a \in A} \right)^{-1}.
\]

Then by Lemma 9.3 we have (9.19)

\[
\langle \tilde{\beta} \cdot v_{i-k}^+, v_{j-k}^+ \rangle = e^{-\sum_{i-j=k} T_{v_{ij}}} ,
\]

using also that \( \sum T_{v_{ii}} = 0 \). This implies the rest of (1), comparing also with Definition 9.6.

Part (2) of the theorem is a consequence of Theorem 9.2 (3), using that \( dT_v \) is the pullback to \( \tilde{\mathbb{Z}} \) of \( dt_v/t_v \).

It now remains to show (3), namely that

\[
(e^{\tilde{H}} \tilde{\phi}) \circ (\gamma, \tilde{\beta}, \gamma_R; h) = e^{\tilde{F}_J K}.
\]

By Theorem 9.2 we already know that \( e^{\tilde{H}} \circ (\gamma, \tilde{\beta}, \gamma_R) = e^{\tilde{F}_J 0} \). Therefore we only need to compare the effect of Joe and Kim’s correction term with our factor \( \tilde{\phi} \). By definition

\[
\tilde{\phi} \circ (\gamma, \tilde{\beta}, \gamma_R; h) \ ((T_v)_{v \in V}; (\lambda_i)_i) = e^{<h((\lambda_i)_i), \gamma_R((T_v)_{v \in V})>},
\]

and the exponent evaluates to (9.20)

\[
< h((\lambda_i)_i), \gamma_R((T_v)_{v \in V}) > = \sum_{k=1}^{n} (\lambda_{k+1} - \lambda_k) \left( \sum_{i-j=k} T_{ij} \right).
\]

However, the weight factors of Joe and Kim are chosen precisely so that for every vertex \( v = v_{ij} \) with \( i - j = k \),

\[
\sum_{a, h_a = v} \lambda_a - \sum_{a, t_a = v} \lambda_a = \lambda_{k+1} - \lambda_k.
\]

Therefore Joe and Kim’s correction term \( \sum_{a \in A} \lambda_a (T_{h_a} - T_{t_a}) \), reordered as a sum of \( T_v \)'s with \( \lambda_i \) coefficients, gives precisely (9.20), and we are done.

9.4. To show that the solutions to the equivariant quantum differential equations constructed by Joe and Kim can be put in the form of our Conjecture 8.2, we need finally to argue that our comparison map \( (\gamma, \tilde{\beta}, \gamma_R) \) is one-to-one when restricted to the integration contours put forward by Joe and Kim.
Recall that $(\gamma, \tilde{\beta}, \gamma_R) : \tilde{Z} \to \tilde{Z}_R$ defines a covering onto its image. Let us choose compatible Riemann metrics on $\tilde{Z}$ and $(\gamma, \tilde{\beta}, \gamma_R)(\tilde{Z})$, so that one is pulled back from the other. Suppose $p_0 \in \tilde{Z}_T$ is a critical point of $\tilde{F}_{JK}$ with its corresponding ‘descending Morse cycle’ for $Re(\tilde{F}_{JK})$, denoted $\Gamma_{T^*}$. The gradient flow of $Re(\tilde{F}_{JK})$ starting at $p \in \Gamma_T$, should therefore approach $p_0$ in the positive limit. This gradient flow maps out a curve which can also be obtained as the unique lifting through $p$ of the gradient flow curve of $Re(\tilde{F}_{\tilde{B}\tilde{\phi}})$ starting at $\tilde{p} := (\gamma, \tilde{\beta}, \gamma_R)(p)$ in the base. Suppose now there was another point $p' \in \Gamma_{T^*}$ with the same image $\tilde{p}' := (\gamma, \tilde{\beta}, \gamma_R)(p') = \tilde{p}$. Then the gradient flow curve below, connecting $\tilde{p} = \tilde{p}'$ with the image $p_0$ of the critical $p_0$, would have a lift through $p$ which ends up at $p_0$, and another lift through $p'$ which also ends at $p_0$. This, however, is in contradiction with the unique lifting of curves property of our covering.

So we have seen that no two points in $\Gamma_{T^*}$ can map to the same point under $(\gamma, \tilde{\beta}, \gamma_R)$. Therefore the map $(\gamma, \tilde{\beta}, \gamma_R)$ amounts to a change of coordinates on the integration contour, and moreover by Theorem 9.7, a change of coordinates under which Joe and Kim’s integrals transform to ones of the form (8.1).

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E-mail address: konstanze.rietsch@kcl.ac.uk

King’s College London, UK