Solution of Klein Gordon equation for trigonometric cotangent potential in the presence of a minimal length using Asymptotic Iteration Method

A Suparmi, C Cari, and Isnaini Lilis Elviyanti

Physics Department, Graduate Program, Sebelas Maret University

Email: soeparmi@staff.uns.ac.id

Abstract. Analysis of energy eigenvalue and wave function for Klein Gordon equation within minimal length formalism which was influenced by trigonometric cotangent potential has done. Asymptotic iteration method was used to solve the modified Klein Gordon equation to obtain energy eigenvalue and wave function. The energy eigenvalue was calculated numerically using Matlab software, and the general unnormalized wave function was expressed in hypergeometric terms. The results showed that the energy eigenvalue within the minimal length formalism was higher than energy eigenvalue without the minimal length formalism. The energy eigenvalue increased by the increase of minimal length parameters. The minimal length parameter caused the shape change in wave functions.

1. Introduction

The Klein Gordon equation was an equation to describe dynamic of spin-zero particles in relativistic quantum mechanics [1,2]. The vector potential (V) and scalar potential (S) of Klein Gordon equation have an exact solution in the case of exact symmetric spin condition when $S=V$ and $S=\pm V$ for exact symmetric pseudospin condition. In these two cases, Klein Gordon equation was reduced to Schrodinger like equation thus a state solution which was obtained using various methods in non-relativistic quantum mechanics [3].

The minimal length idea in quantum mechanics was based on the following deformed commutation relations between position and momentum operators in Generalized Uncertainty Principle (GUP) [4]. The minimal length may be viewed as an intrinsic scale characterizing the system [5]. The Klein Gordon equation in the presence of minimal length has been solved using Heun’s function for Coulomb potential [6], approach Algebraic for trigonometric potential [5] and approach Feynman for linear potential [7]. On the other hand, the Klein Gordon equation without the minimal length has been solved using Asymptotic Iteration Method (AIM) for Kratzer potential [8] and Trigonometric Poschl-Teller potential [9], approach Algebraic in Harmonic potential [10] and Nikiforov Uvarov method in Hulten Potential [11]. The solution of minimal length has been investigated by Alimohammadi and Hassanabadi using Bohr-Mottelson equation [12].

In this paper, we solved symmetric spin condition of Klein Gordon equation in the presence of a minimal length. The radial part solution of Klein-Gordon equation for trigonometric cotangent function potential in the presence of a minimal length has been studied [13]. The trigonometric cotangent potential was used to explain nucleon excitation [13]. The Asymptotic Iteration Method was applied to obtain the energy eigenvalues and radial wave functions in the presence of minimal length. AIM was used to solve equation because of its easy method [14] and the result was accurate.
where

\[ L = \frac{\hbar^2}{2m} \nabla^2 - (E + M_o V) \]

was obtained [15,16].

By setting \( \alpha_{\text{ML}} \) was a minimal length parameter that has very small positive values. By equation (1) the position and momentum operators [15-17] were defined as,

\[ \hat{X}_i = \hat{x}_i \]

\[ \hat{P}_i = \left( 1 + \alpha_{\text{ML}} \hat{p}^2 \right) \hat{p}_i \]

The general equation of Klein Gordon equation was given by [10,17]

\[ (E^2 - M_o^2 - (E + M_o V(r))) \psi(r) = \left( \Delta - (\Delta - 2\alpha_{\text{ML}} \Delta_0^2) \right) \psi(r) \]

From equation (5) was obtained \( \alpha_{\text{ML}} \) that had very small value, so \( \alpha_{\text{ML}} \) was ignored. In the case of Klein Gordon without the minimal length formalism with \( \alpha_{\text{ML}} = 0 \) [12], so square term of equation (5) was given as,

\[ \Delta_0^2 \left( E_o^2 - M_o^2 - (E_o + M_o V(r)) \right) \]

\( E_0 \) was energy eigenvalue at any level without the presence of minimal length. In equation (6) the expression of \( \Delta_0^2 \) depends only on the potential \( V(r) \) and it was not considered to be Laplacian.

By inserting equation (6) into equation (5) we get

\[ \left( E^2 - M_o^2 - (E + M_o V(r)) \right) \psi(r) = \left( \Delta - 2\alpha_{\text{ML}} \left( E_o^2 - M_o^2 - (E_o + M_o V(r)) \right) \right) \psi(r) \]

and the spherical Laplacian operator was given as,

\[ \Delta = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \]

And by setting \( \psi(r) = U(r) Y_{LM}(\theta, \phi) \), yields

\[ \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right) Y_{LM}(\theta, \phi) = -(L+1) Y_{LM}(\theta, \phi) \]

where \( L \) was angular momentum. Then equation (8) was substituted into equation (7), so we got
\[
\left[ 1 + \frac{\partial}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{L(L+1)}{r^2} \right] + \left( E^2 - M^2 - (E + M) V(r) \right) \left( E^2 - M^2 - 2(E^2 - M^2)(E^2 + M^2) V(r) + (E^2 + M^2)^2 V^2 \right) \right] \]

Klein Gordon equation in the presence of a minimal length for the case symmetric spin condition was shown by equation (10).

3. Asymptotic Iteration Method

Asymptotic Iteration Method is used to solve the second order differential equation which is expressed as[18-20],

\[
y_n^{*}(x) = \lambda_0(x) y_n^{*}(x) + s_0(x) y_n(x) \quad (11)
\]

where \( n \) is a quantum number, \( \lambda_0(x) \neq 0 \) and \( s_0(x) \) are the coefficient of a differential equation. By writing the \((k+1)th\) derivative of \( y \) which is obtained from equation (11), is given as,

\[
y_n^{k+1}(x) = \lambda_{k+1}(x) y_n^{*}(x) + s_{k+1}(x) y_n(x) \quad (12)
\]

where,

\[
\lambda_k(x) = \lambda_{k+1}(x) + s_{k+1}(x) + \lambda_0(x) \lambda_{k+1}(x) \quad (13)
\]

\[
s_k(x) = s_{k+1}(x) + s_0(x) \lambda_{k+1}(x) \quad (14)
\]

\[
k = 1, 2, 3...
\]

The eigenvalue can be obtained from the quantization condition which is given by

\[
\Delta_k(x) = \lambda_k(x) s_{k+1}(x) - \lambda_{k+1}(x) s_k(x) = 0 \quad (16)
\]

The one-dimensional Schrodinger like equation was reduced to a hypergeometric-type differential equation which is given as,

\[
y_n^{*}(x) = 2 \left( \frac{ax^{N+1}}{1-bx^{N+2}} - \frac{1}{t+1} \right) y_n^{*}(x) - \frac{wx^{N}}{1-bx^{N+2}} y_n(x) \quad (17)
\]

The eigenfunction which is obtained from equation (9) is given as,

\[
y_n(x) = (-1)^n C \left( \frac{N+2}{\sigma} \right)^{\frac{1}{N+2}} F_1(-n, t+n; \sigma; bx^{N+2}) \quad (18)
\]

where

\[
\left( \frac{\sigma}{\Gamma(\sigma+n)} \right) = \frac{\Gamma(\sigma+n)}{\Gamma(\sigma)} \quad (19)
\]

\[
\sigma = \frac{2t+N+3}{N+2} \quad (20)
\]

\[
\rho = \frac{(2t+1)b+2a}{(N+2)b} \quad (21)
\]

\( C \) is normalization constant and \( F_1 \) is a hypergeometric function. The wave function of Klein Gordon equation can be obtained by using equations(18-21) [18-20].
4. Result and Discussion

The function $1/r^2$ of equation (10) was approximated using the centrifugal approximation [21], was given as,

$$\frac{1}{r^2} = \frac{\sigma}{\sin^2(\sigma r)} = f(r) \quad (22)$$

Equation (22) was used for the small value of variable $r$ and the visual approximate was given in Figure 1.

![Figure 1](image)

Figure 1. The function $1/r^2$ and $\sigma/\sin^2(\sigma r)$

Figure 1 shows that line blue was function $1/r^2$ and line red was a function $\sigma/\sin^2(\sigma r)$. So, the function $1/r^2$ of equation (10) can be approximated using the centrifugal approximation.

The trigonometric cotangent potential [13] was given as,

$$V(r) = V_o \cot(\sigma r) + V_i \quad (23)$$

where $V_o$ and $V_i$ were potential constants, $\sigma$ was a range of potential. By applying equation (23) into equation (10) and by setting $U(r) = \frac{1}{r} \phi(r)$ and the centrifugal approximation, then equation (10) becomes,

$$\left[ \frac{d^2 \phi(r)}{dr^2} - \frac{\sigma L(L+1) + 2\alpha_{ML} \delta^2 V_o^2}{\sin^2(\sigma r)} \phi(r) + \left(4\alpha_{ML} \chi - (E + M_o)\right)V_o - 2\alpha_{ML} \delta^2 V_i \cot(\sigma r) \phi(r) \right] = 0 \quad (24)$$

where,

$$\delta = (E_0 + M_o), \quad \chi = \left(E_0^2 - M_o^2\right)\left(E_0 + M_o\right), \quad \eta = \left(E_0^2 - M_o^2\right) \quad (25)$$

Then equation (24) was reduced to,

$$\frac{d^2 \phi(r)}{dr^2} - \left[ \frac{v(v+1)}{\sin^2(\sigma r)} - 2q \cot(\sigma r) + \kappa^2 \right] \phi(r) = 0 \quad (26)$$

with,

$$v(v+1) = \left(\sigma L(L+1) + 2\alpha_{ML} \delta^2 V_o^2\right) \quad (27)$$

$$2q = \left(4\alpha_{ML} \chi - (E + M_o)\right)V_o - 2\alpha_{ML} \delta^2 V_i \quad (28)$$
\[-\kappa^2 = \left(4\alpha_M x - (E + M_o)\right)V_i + \left(E^2 - M_o^2 - 2\alpha_M \eta^2 + 2(\alpha_M \delta^2 V_o^2 - 2\alpha_M \delta^2 V_i^2)\right)\]  \hspace{1cm} (29)

Equation (26) was a differential equation which must be reduced to hypergeometric differential equation type. By using the suitable variable change \(\text{cot} (\varpi r) = i(1-2y)\), so we got,

\[y(1-y) \frac{d^2 \phi (r)}{dy^2} + (1-2y) \frac{d\phi (r)}{dy} + \left[ v' (v' + 1) - \frac{4\alpha^2}{4y} - \frac{4\beta^2}{4(1-y)}\right] \phi (r) = 0 \hspace{1cm} (30)\]

with,

\[\frac{2qi - \kappa^2}{\sigma^2} = 4\alpha^2\]

\[\frac{-2qi - \kappa^2}{\sigma^2} = 4\beta^2\]

\[v' (v' + 1) = \frac{v(v + 1)}{\sigma^2}\]  \hspace{1cm} (31)

Equation (30) was an intermediate of the hypergeometric differential equation. Equation (30) was reduced by using new wave function, \(\phi (r) = y^\alpha (1-y)^\beta f(y)\)  \hspace{1cm} (32)

So we get

\[y(1-y) \frac{f^2 (y)}{dy^2} + \left[ (2\alpha + 1) - (2\alpha + 2\beta + 2) y\right] \frac{f(y)}{dy} + \left[ v' (v' + 1) - (\alpha + \beta)(\alpha + \beta + 1)\right] f(y) = 0 \hspace{1cm} (33)\]

The equation (33) must be reduced to AIM type equation, by dividing it with \(y(1-y)\), then we got

\[f' (y) + \left[ \frac{(2\alpha + 1) - (2\alpha + 2\beta + 2) y}{y(1-y)}\right] f(y) + \left[ v'(v' + 1) - (\alpha + \beta)(\alpha + \beta + 1)\right] f(y) = 0 \hspace{1cm} (34)\]

By comparing equation (11) and equation (34), we had

\[\lambda_o = \frac{(2\alpha + 1) + (2\beta + 1)}{1-y}\]

\[s_o = \frac{(\alpha + \beta)(\alpha + \beta + 1) - v'(v' + 1) + (\alpha + \beta)(\alpha + \beta + 1) - v'(v' + 1)}{y (1-y)}\]  \hspace{1cm} (35)

\hspace{1cm} (36)

Using equations (18-21) and equation (34), we can obtain the eigenvalue, was given as

\[v'(v' + 1) = (\alpha + \beta + n)(\alpha + \beta + (n + 1))\]  \hspace{1cm} (37)

From the eigenvalue in equation (37), we obtained the energy eigenvalue equation by using equations (31) and (37). The energy eigenvalue equation of Klein Gordon equation for minimal length, was given by,

\[
\left( E^2 - M_o^2 \right) = \sigma^2 \left[ \left( 4\alpha_M x - (E + M_o) \right)^2 V_i + \gamma_{\alpha_M} \right]
\]  \hspace{1cm} (38)

where,

\[
\gamma_{\alpha_M} = 2\alpha_M \eta^2 + 2\alpha_M \delta^2 V_i^2 - 2\alpha_M \delta^2 V_o^2
\]  \hspace{1cm} (39)

\[
L_{\alpha_M} = \sqrt{\sigma L (L+1) + 2\alpha_M \delta^2 V_o^2} + \frac{1}{4} - n - \frac{1}{2}
\]  \hspace{1cm} (40)
and $n$ was a quantum number. Equation (38) was energy eigenvalue of Klein Gordon equation in the presence of a minimal length for trigonometric cotangent potential. The energy eigenvalue of equation (38) was calculated numerically by using the Matlab software. The results were shown in Table 1.

Table 1. The relativistic energy for $M_o=1$, $V_o=0.01$, $V_f=0.1$, $\vartheta=0.1$ and $L=0$ with various minimal length parameter and quantum number $n$.

| $n$ | $\alpha_{ML} = 0$ | $\alpha_{ML} = 0.02$ | $\alpha_{ML} = 0.04$ | $\alpha_{ML} = 0.06$ |
|-----|------------------|------------------|------------------|------------------|
| 1   | 1.0840           | 1.0841           | 1.0843           | 1.0844           |
| 2   | 1.1136           | 1.1137           | 1.1138           | 1.1139           |
| 3   | 1.1397           | 1.1398           | 1.1400           | 1.1401           |
| 4   | 1.1723           | 1.1727           | 1.1731           | 1.1735           |
| 5   | 1.2121           | 1.2131           | 1.2141           | 1.2152           |

Table 1 shows that the energy eigenvalue within the minimal length formalism was higher than energy eigenvalue without the minimal length formalism. The increasing of the minimal length parameter causes the increase of the energy eigenvalue. The energy eigenvalue increased by the increase of quantum number $n$.

We got the wave functions by inserting equations (26-35) into equation (37). The wave functions of Klein Gordon equation in the presence of minimal length for $n=0$, $n=1$ and $n=2$, were given as

$$\phi_0(r) = C\left(\frac{1+i\cot\vartheta r}{2}\right)^\alpha \left(\frac{1-i\cot\vartheta r}{2}\right)^\beta$$

$$\phi_1(r) = -C\left(\frac{1+i\cot\vartheta r}{2}\right)^\alpha \left(\frac{1-i\cot\vartheta r}{2}\right)^\beta \left(2\alpha + 1\right) \left[\left(-1\right)^{\left(2\alpha + 2\beta + 2\right)}\left(\frac{1+i\cot\vartheta r}{2}\right)^2\right]$$

$$\phi_2(r) = C\left(\frac{1+i\cot\vartheta r}{2}\right)^\alpha \left(\frac{1-i\cot\vartheta r}{2}\right)^\beta \left(2\alpha + 1\right) \left(2\alpha + 2\right) \left[\left(-2\right)^{\left(2\alpha + 2\beta + 3\right)}\left(\frac{1+i\cot\vartheta r}{2}\right)^3\right]$$

5. Conclusion

We investigated Klein Gordon equation in the presence of a minimal length for trigonometric cotangent potential using Asymptotic Iteration method to obtain the energy eigenvalues and radial wave functions. The energy eigenvalue within the minimal length formalism was higher than energy eigenvalue without the minimal length formalism. The increase of minimal length parameter causes the increase of the energy eigenvalue. The minimal length effect gives influence in increasing binding energy value. The binding energy eigenvalue increased by the increase of quantum number $n$. 
Acknowledgement
This research was partly supported by Sebelas Maret University Higher Education Project Grant 623/UN27.21/PP/2017.

References
[1] Ikhdair S M and Sever R 2008 arXiv:0808.1002v1 [quant-ph] 14.
[2] Momtazi E, Rajabi A A, and Yazarloo B H 2014 Turkish J. Phys 38 1 81–85.
[3] Greiner W 2000 Springer-Verlag Berlin Heidelberg, New York.
[4] Hassanabadi H, Maghsoodi E, Ikot N A, and Zarrinkamar S 2013 Hindawi Publishing Corporation.
[5] Jana K T and Roy P 2009 Physics Letters A 373 1239–41.
[6] Bouaziz D 2013 arXiv:1311.7405v1 [quant-ph].
[7] Merad M, Zeroual F and Benzair H 2010 Electronic Journal of Theoretical Physics 7 23 41-56.
[8] Nugraha D A, Suparmi A, Cari C, and Pratiwi B N 2017 J. Phys. Conf. Ser 795.
[9] Nugraha D A, Suparmi A, Cari C, and Pratiwi B N 2017 J. Phys. Conf. Ser 820.
[10] Poszwa A, 2014 Acta Physica Polonica A 126 6.
[11] A N Ikot, L E Akpabio and E J Uwah 2011 Electron. J. Theor. Phys 25 25 225-232.
[12] Alimohammadi M and Hassanabadi H 2017 Nuclear Physics A 957 439-449.
[13] Cari, Suparmi, Deta A U and Wediningingsih I S 2013 Makara J. Sci.
[14] Farisky N M, Suparmi A, Cari C and Yunianto M 2016 J. Phys. Conf. Ser.
[15] Sprenger M, Nicolini P and Bleicher M 2012 European Journal Of Physics 33 853-62.
[16] Garay J L 1994 International Journal of Modern Physics A 10 2 145-65.
[17] Chabab M, ElBatoul A, Labhas A and Oulne M 2016 Physics Letters B 212-16.
[18] H Ciftci, R L Hall and N Saad 2003 J. Phys. A. Math. Gen 36 47 11807-16.
[19] Pratiwi N B, Suparmi A, Cari C and Husein S A 2017 Pramana. J. Phys.
[20] Pramono S, Suparmi A, and Cari C 2016 Adv. High Energy Phys 2 016.
[21] Naderia L and Hassanabadi H 2016 The European Physical Journal Plus 131:133.