Rainbow Ramsey Problems for the Boolean Lattice

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Abstract

We address the following rainbow Ramsey problem: For posets $P$, $Q$ what is the smallest number $n$ such that any coloring of the elements of the Boolean lattice $B_n$ either admits a monochromatic copy of $P$ or a rainbow copy of $Q$. We consider both weak and strong (non-induced and induced) versions of this problem.

Keywords Extremal set systems · Forbidden subposet problem · Ramsey theory

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1 Introduction

In this paper we consider rainbow Ramsey-type problems for posets. Given posets \( P \) and \( Q \), we say that \( X \subseteq Q \) is a weak copy of \( P \), if there is a bijection \( \alpha: P \to X \) such that \( p \leq_P p' \) implies \( \alpha(p) \leq_Q \alpha(p') \). If \( \alpha \) has the stronger property that \( p \leq_P p' \) holds if and only if \( \alpha(p) \leq_Q \alpha(p') \), then \( X \) is a strong or induced copy of \( P \). A copy \( X \) of \( P \) is monochromatic with respect to a coloring \( \phi: Q \to \mathbb{Z}^+ \), if \( \phi(q) = \phi(q') \) for all \( q, q' \in X \) and rainbow if \( \phi(q) \neq \phi(q') \) for all \( q \neq q' \in X \). We will be looking for monochromatic and/or rainbow copies of some posets in the Boolean lattice \( B_n \), the subsets of an \( n \)-element set ordered by inclusion. The set of elements of \( B_n \) corresponding to sets of the same size is called a level of \( B_n \).

**Definition 1.1** The weak Ramsey number \( R(P_1, P_2, \ldots, P_k) \) is the smallest number \( n \) such that for any coloring of the elements of \( B_n \) with \( k \) colors, say \( 1, 2, \ldots, k \) there is a monochromatic copy of the poset \( P_i \) in color \( i \) for some \( 1 \leq i \leq k \). We simply write \( R_k(P) \) for \( R(P_1, P_2, \ldots, P_k) \), if \( P_1 = \ldots = P_k = P \). We define the strong Ramsey number \( R^*(P_1, P_2, \ldots, P_k) \) and \( R^*_k(P) \) for strong copies of posets analogously.

Ramsey theory of posets is an old and well investigated topic, see e.g., [11, 15]. However, the study of Ramsey problems in the Boolean lattice was initiated only recently: weak Ramsey numbers were studied by Cox and Stolee [3] and strong Ramsey numbers were investigated by Axenovich and Walzer [1]. In addition, some results in the latter one were improved by Lu and Thompson [12].

In this article, we study rainbow Ramsey numbers for the Boolean lattice.

**Definition 1.2** For two posets \( P, Q \) the weak (or not necessarily induced) rainbow Ramsey number \( RR(P, Q) \) is the minimum number \( n \) such that any coloring (using an arbitrary number of colors) of \( B_n \) admits either a monochromatic weak copy of \( P \) or a rainbow weak copy of \( Q \). The strong (or induced) rainbow Ramsey number can be defined analogously and is denoted by \( RR^*(P, Q) \).

Rainbow Ramsey numbers for graphs have been intensively studied (they are sometimes called constrained Ramsey numbers or Gallai–Ramsey numbers), for a recent survey see [4]. The results on the rainbow Ramsey number for Boolean posets are sporadic [2, 10]. Nevertheless, the following easy observation connects (usual) Ramsey numbers to rainbow Ramsey numbers.

**Proposition 1.3** For any pair \( P \) and \( Q \) of posets we have

(i) \( RR(P, Q) \geq R_{|Q|-1}(P) \), and

(ii) \( RR^*(P, Q) \geq R^*_{|Q|-1}(P) \).

**Proof** To see (i) observe that if a coloring \( \phi \) uses at most \( |Q|-1 \) colors, then clearly it cannot contain a rainbow weak copy of \( Q \). Therefore any such coloring showing \( R_{|Q|-1}(P) > n \) also shows \( RR(P, Q) > n \). An identical proof with strong copies implies (ii). □

In this paper, we show many examples of posets \( P, Q \) for which the inequality in (i) of Proposition 1.3 holds with equality, while in Section 3, we show another example of posets \( P, Q \) for which (ii) of Proposition 1.3 holds with strict inequality. Unfortunately, we do not know whether there exist posets \( P, Q \) for which (i) holds with strict inequality.
Many of the tools used in [1, 3] come from the related Turán-type problem, the so-called forbidden subposet problem. Let us introduce some terminology. For a poset $P$, a family $\mathcal{F} \subseteq B_n$ of sets is called (induced) $P$-free if $\mathcal{F}$ does not contain a weak (strong) copy of $P$. The size of the largest (induced) $P$-free family in $B_n$ is denoted by $La(n, P)$ (resp. $La^*(n, P)$). For a poset $P$, we denote by $e(P)$ the maximum number $m$ such that for any $n$ the union of any consecutive $m$ levels of $B_n$ is $P$-free. The analogous strong parameter is denoted by $e^*(P)$. The most widely believed conjecture [5] in the area of forbidden subposet problems states that for any poset $P$ we have

$$\lim_{n \to \infty} \frac{La(n, P)}{\binom{n}{\lfloor n/2 \rfloor}} = e(P) \text{ and } \lim_{n \to \infty} \frac{La^*(n, P)}{\binom{n}{\lfloor n/2 \rfloor}} = e^*(P).$$

It is worth noting that this conjecture is already wide open for a very simple poset called the diamond poset $D_2$ (defined on four elements $a, b, c, d$ with relations $a < b, c, d$ and $b, c < d$). See [9] for the best known bounds in this direction.

For a family $\mathcal{F} \subseteq B_n$ of sets, its Lubell-mass is $\lambda_n(\mathcal{F}) = \sum_{F \in \mathcal{F}} \frac{1}{|F|}$. For a poset $P$, we define $\lambda_n(P)$ to be the maximum value of $\lambda_n(\mathcal{F})$ over all $P$-free families $\mathcal{F} \subseteq B_n$ and $\lambda_{\text{max}}(P)$ is defined to be $\sup_n \lambda_n(P)$. Its finiteness follows from the fact that every poset $P$ is a weak subposet of $C_{|P|}$ (where $C_l$ denotes the $l$-chain, the totally ordered set of size $l$) and the $k$-LYM-inequality stating that $\lambda_n(\mathcal{F}) \leq k$ for any $C_{k+1}$-free family $\mathcal{F} \subseteq B_n$. Analogously, $\lambda_{n}^*(P)$ is the maximum value of $\lambda_n(\mathcal{F})$ over all induced $P$-free families $\mathcal{F} \subseteq B_n$ and $\lambda_{\text{max}}^*(P)$ is defined to be $\sup_n \lambda_{n}^*(P)$. It was proved to be finite by Méroueh [13].

Observe that, by definition of $e(P)$ and $e^*(P)$, we have $e(P) \leq \lambda_n(P)$ and $e^*(P) \leq \lambda_{n}^*(P)$ for every poset $P$ and integer $n \geq e(P)$ or $n \geq e^*(P)$. We say that a poset is uniformly Lubell-bounded if $e(P) \leq \lambda_n(P)$ holds for all positive integers $n$. Similarly, a poset is uniformly induced Lubell-bounded if $e^*(P) \leq \lambda_{n}^*(P)$ holds for all positive integers $n$. An instance of posets equipped with this property is the class of chain posets $C_l$. For $k \geq 2$ the generalized diamond poset $D_k$ consists of $k+2$ elements $a, b_1, b_2, \ldots, b_k, c$ with relations $a < b_i < c$ for $1 \leq i \leq k$. Griggs, Li and Lu [6] proved that infinitely many of the $D_k$’s are uniformly Lubell-bounded and Patkós [14] proved that an overlapping but distinct and infinite subset of the $D_k$’s is uniformly induced Lubell-bounded. For more uniformly Lubell-bounded posets, see [8].

In [1] and [3], it was observed that if $P$ is uniformly Lubell-bounded or uniformly induced Lubell-bounded, then $R_k(P) = k \cdot e(P)$ or $R_{k}^*(P) = k \cdot e^*(P)$ holds, respectively.

Our main result concerning weak rainbow Ramsey numbers extends the above observation.

**Theorem 1.4** Let $P$ be a uniformly Lubell-bounded poset and $\mathcal{F} \subseteq B_n$ be a family of sets with $\lambda_n(\mathcal{F}) > e(P)(k - 1)$. Then any coloring of $\phi: \mathcal{F} \to \mathbb{Z}_+^*$ admits either a monochromatic weak copy of $P$ or a rainbow copy of $C_k$.

**Corollary 1.5** If $P$ is uniformly Lubell-bounded, then $RR(P, Q) = e(P)(|Q| - 1)$ holds for any poset $Q$.

**Proof** As $\lambda_n(B_n) = n + 1$, the inequality $RR(P, Q) \leq e(P)(|Q| - 1)$ is a direct consequence of Theorem 1.4 as any poset $Q$ is a weak subposet of $C_{|Q|}$.

Let $n = (|Q| - 1)e(P) - 1$. The lower bound $RR(P, Q) > n$ follows from coloring $B_n$ so that the color classes form a partition of the levels of $B_n$ into $|Q| - 1$ intervals, each of size $e(P)$. As we use only $|Q| - 1$ colors, we avoid rainbow copies of $Q$ and by definition of $e(P)$ we avoid monochromatic copies of $P$. \qed
For strong copies of posets, the coloring from the proof of Corollary 1.5 yields the same lower bound $RR^*(P, Q) \geq e^*(P)(|Q| - 1)$, but one can easily observe that in most cases this trivial lower bound can be improved by slightly modifying the above coloring: If $Q$ does not have a unique smallest element, then one can color $\emptyset$ with an otherwise unused color $i$. Since no other sets are colored $i$, it does not help to create a strong monochromatic copy of $P$, and since $Q$ does not have a unique smallest element, it does not help to create a strong rainbow copy of $Q$. Therefore one can introduce the following function. For any poset $Q$, let $f(Q) = 0$, if $Q$ has both a unique largest and a unique smallest element, let $f(Q) = 2$, if $Q$ has neither largest nor smallest element, and define $f(Q) = 1$ otherwise. One obtains $RR^*(P, Q) \geq e^*(P)(|Q| - 1) + f(Q)$ for all posets $P$ and $Q$. For this lower bound, the strong version of Corollary 1.5 would be expected for $P$ being uniformly induced Lubell-bounded. Nonetheless, we will show the above inequality is strict when $P$ is strongly induced. For which uniformly induced Lubell-bounded posets $P$, does one have

$$RR^*(P, Q) = e^*(P)(|Q| - 1) + f(Q)$$

(1)

for every poset $Q$?

Despite the above counterexample to Eq. 1, we prove that it holds for most uniformly induced Lubell-bounded posets $P$ and $Q = A_3$. Indeed, we have a general upper bound for $RR^*(P, A_k)$ for any poset $P$ and $k \geq 2$.

**Theorem 1.7** Given an integer $k \geq 2$, let $m_k = \min\{m : (\frac{m}{\lfloor m/2 \rfloor}) \geq k\}$. For any poset $P$ we have

$$RR^*(P, A_k) \leq [(k - 1)\lambda^*_{max}(P)] + m_k.$$ 

Moreover, if $P$ is not $C_1$ or $C_2$, then we have

$$RR^*(P, A_3) \leq [2\lambda^*_{max}(P)] + 2.$$ 

Since $\lambda^*_{max}(P) = e^*(P)$ for every uniformly induced Lubell-bounded poset $P$, we have the next corollary immediately from the latter part of Theorem 1.7.

**Corollary 1.8** For every uniformly induced Lubell-bounded poset $P$ other than $C_1$ or $C_2$ we have

$$RR^*(P, A_3) = 2 + 2e^*(P).$$

**Structure of the paper** The remainder of the paper is organized as follows: Theorem 1.4 and other results on weak copies are proved in Section 2. Section 3 contains the proofs of the counterexample to Eq. 1 and Theorem 1.7.

**Notation** For $n \in \mathbb{Z}^+$ we denote by $[n]$ the set $\{1, 2, \ldots, n\}$. For a set $F$, we write $U_F = U_{n,F} = \{G \subseteq [n] : F \subseteq G\}$, $D_F = D_{n,F} = \{G \subseteq [n] : G \subseteq F\}$, and $I_F = I_{n,F} = U_{n,F} \cup D_{n,F}$. For sets $F \subseteq H$, we write $B_{F,H} = \{G : F \subseteq G \subseteq H\}$. For integers $0 \leq a \leq b \leq n$, we write $\lambda_n(B_{a,b}) = \lambda_n(B_{F,H})$ for some $F \subseteq H \subseteq [n]$ with $|F| = a$, $|H| = b$. Let $B_n^+$ and $B_n^−$ denote the truncated Boolean lattices obtained by removing the smallest and the largest element of the cubes $B_n$ and $B_{F,H}$, respectively. For a coloring $\phi : B_n \rightarrow \mathbb{Z}^+$, let $\|\phi\|$ denote the number of colors used by $\phi$. For a coloring $\phi : B_n \rightarrow \mathbb{Z}^+$ and a positive
integer $i$, let $\mathcal{H}_i = \mathcal{H}_{\phi,i} = \{F \subseteq [n] : \phi(F) = i\}$. We use $\binom{n}{\leq k}$ to denote $\sum_{j=0}^{k} \binom{n}{j}$. All logarithms are of base 2 in this paper.

2 Weak Copies

In this section, we prove Theorem 1.4 and some other results on weak Ramsey and weak rainbow Ramsey numbers. We start with a couple of definitions.

We denote by $C_n$ the set of all maximal chains in $B_n$. For a family $\mathcal{F} \subseteq B_n$ and set $F \in \mathcal{F}$, we define $C_{n,F} = C_{n,F,\mathcal{F}}$ to be the set of those maximal chains $C \in C_n$ for which the largest set of $\mathcal{F} \cap C$ is $F$. Then the max-partition of $C_n$ consists of the blocks $C_{n,F}$ for each $F \in \mathcal{F}$ and $C_{n,-}$ which contains all maximal chains $C$ with $\mathcal{F} \cap C = \emptyset$.

The Lubell mass $\lambda_n(\mathcal{F}) = \sum_{F \in \mathcal{F}} \binom{|C_{n,F}|}{n!} \lambda_{|F|}(D_F \cap \mathcal{F})$ is the average number of sets of $\mathcal{F}$ in a maximal chain $C$ chosen uniformly at random from $C_n$. As observed by Griggs and Li [7], if we condition on the largest set $F$ in $\mathcal{F} \cap C$, then we obtain

$$\lambda_n(\mathcal{F}) = \sum_{F \in \mathcal{F}} \frac{|C_{n,F}|}{n!} \lambda_{|F|}(D_F \cap \mathcal{F}).$$

Proof of Theorem 1.4 We proceed by induction on $k$. The base case $k = 1$ is trivial as any colored set forms a “rainbow” copy of $C_1$. Let $k \geq 2$ and suppose the statement is proven for $k - 1$ and let $\mathcal{F} \subseteq B_n$ be a family of sets with $\lambda_n(\mathcal{F}) > e(\mathcal{F})(k - 1)$. Fix a coloring $\phi : \mathcal{F} \to \mathbb{Z}^+$ and consider the max-partition $\{C_{n,F} : F \in \mathcal{F}\} \cup \{C_{n,-}\}$. Using

$$\lambda_n(\mathcal{F}) = \sum_{F \in \mathcal{F}} \frac{|C_{n,F}|}{n!} \lambda_{|F|}(D_F \cap \mathcal{F}),$$

we obtain a set $F \in \mathcal{F}$ with $\lambda_{|F|}(D_F \cap \mathcal{F}) > e(\mathcal{F})(k - 1)$. Let $\mathcal{F}_1 = \{G \in D_F : \phi(G) = \phi(F)\}$. If $\mathcal{F}_1$ contains a weak copy of $P$, then we are done as, by definition, $\mathcal{F}_1$ is monochromatic. Otherwise, as $P$ is uniformly Lubell-bounded, we have $\lambda_{|F|}(\mathcal{F}_1) \leq e(\mathcal{F})$ and thus

$$\lambda_{|F|}(D_F \cap \mathcal{F}) > e(\mathcal{F})(k - 1) - e(\mathcal{F}) = e(\mathcal{F})(k - 2).$$

Applying our inductive hypothesis to $(D_F \cap \mathcal{F}) \setminus \mathcal{F}_1$ we either obtain a monochromatic weak copy of $P$ or a rainbow copy of $C_{k-1}$. As all sets in $(D_F \cap \mathcal{F}) \setminus \mathcal{F}_1$ are colored differently than $F$, we can extend the rainbow copy of $C_{k-1}$ to a rainbow copy of $C_k$ by adding $F$. \(\square\)

Remark Note that a simple modification of the above proof shows that if $P$ is a uniformly induced Lubell-bounded poset and $\mathcal{F} \subseteq B_n$ is a family of sets with $\lambda_n(\mathcal{F}) > e^*(P)(k - 1)$, then any coloring of $\phi : \mathcal{F} \to \mathbb{Z}^+$ admits either a monochromatic strong copy of $P$ or a rainbow copy of $C_k$, and therefore $RR^*(P, C_k) = e^*(P)(k - 1)$ holds.

The equality in Proposition 1.3 (i) holds for uniformly Lubell-bounded posets $P$ and any posets $Q$. To find posets $P$ and $Q$ with $RR(P, Q) > R_{[Q] - 1}(P)$, we have to choose a non-uniformly Lubell-bounded poset as $P$. However, regardless of $P$, Proposition 1.3 (i) still holds with equality if $Q$ is one of the following posets: for $r \geq 2$ the $r$-fork poset $V_r$ consists of a minimum element and $r$ other elements that form an antichain. Similarly, for $s \geq 2$ the $s$-broom poset $\Lambda_s$ consists of a maximum element and $s$ other elements that form an antichain.
Proposition 2.1 For any poset $P$, we have

(i) $RR(P, V_r) = R_r(P)$, and
(ii) $RR(P, \Lambda_i) = R_i(P)$.

Proof By Proposition 1.3, $RR(P, V_r) \geq R_r(P)$. Let $n = R_r(P)$. Any coloring $\phi : B_n \to \mathbb{Z}^+$ with $\|\phi\| \geq r + 1$ contains a rainbow weak copy of $V_r$: the empty set and one representative from each of any other $r$ color classes.

The proof of (ii) is similar by taking the universal set $[n]$ and one representative from each of any $s$ other color classes if $\|\phi\| \geq s + 1$. \qed

If $P$ and $Q$ are both fork posets, then we have $RR(V_r, V_k) = R_k(V_r)$ for any $r, k \geq 1$. In our next result, we manage to determine this value asymptotically for fixed $k$. We write $f_k(r) = R_k(V_r)$ for simplicity. A simple way to define a $k$-coloring of $B_n$ is to color sets of the same size with the same color such that color classes consist of consecutive levels. Formally, let $i_1, i_2, \ldots, i_k$ be positive integers with $\sum_{j=1}^{k} i_j = n + 1$ and consider the coloring $\phi(F) = h$ if and only if $\sum_{j=1}^{h-1} i_j \leq |F| < \sum_{j=1}^{h} i_j$. (The empty sum equals 0, so $\phi(F) = 1$ if and only if $|F| < i_1$ holds.) We call such a coloring $\phi$ a consecutive level $k$-coloring and define $g_k(r)$ to be the smallest integer $n$ such that any consecutive level $k$-coloring of $B_n$ admits a monochromatic weak copy of $V_r$. By definition, we have $g_k(r) \leq f_k(r)$.

For $c \in (0, 1)$ let $h(c) = -c \log c - (1-c) \log(1-c)$, the binary entropy function. Note that for $c \in (0, 1)$ and $n$ large enough we have

$$\frac{1}{\sqrt{n}} 2^{nh(c)} \leq \left(\frac{n}{cn}\right) \leq 2^{nh(c)}.$$  

We will use the fact that for $0 < \varepsilon \leq 1/2$ and $k \leq (1/2 - \varepsilon)n$ we have $\frac{(k-1)}{q} = \frac{k}{n-k} \leq 1/2 - \varepsilon$ $\Rightarrow c$. It implies

$$\left(\begin{array}{c} n \\ \leq k \end{array}\right) = \sum_{i=0}^{k} \left(\begin{array}{c} n \\ i \end{array}\right) \leq \left(\begin{array}{c} n \\ k \end{array}\right) \sum_{i=0}^{k} c^{-i} \leq \frac{n}{1-c} \left(\begin{array}{c} n \\ k \end{array}\right).$$  

(2)

In the proof we omit floor and ceiling signs for simplicity.

Theorem 2.2 For any positive integer $k$ there exists a constant $c_k$ such that

$$\lim_{r \to \infty} \frac{g_k(r)}{\log r} = \lim_{r \to \infty} \frac{f_k(r)}{\log r} = c_k.$$  

Moreover, $c_1 = 1$ and the sequence $\{c_k\}_{k=1}^{\infty}$ satisfies the equality $c_{k+1} h\left(\frac{c_{k+1} - c_k}{c_{k+1}}\right) = 1$ for any $k \geq 1$.

Proof The proof is based on the recursive inequalities contained in the following claim. In part (i) of Claim 2.3, the term $\min\{a : \left(\frac{a+f_k(2r-1)}{\leq a}\right) > r\}$ ensures that in $B_{g_k(r)+a}$ the levels 0, 1, \ldots, $a$ contain together more than $r$ sets. Similarly, in part (ii) of Claim 2.3 the term $\max\{a : \left(\frac{a+g_k(r)}{\leq a}\right) \leq r\}$ ensures that in $B_{g_k(r)+a}$ the levels 0, 1, \ldots, $a$ contain together at most $r$ sets.

Claim 2.3 For any $k \geq 1$ and $r \geq 1$ we have

(i) $f_{k+1}(r) \leq f_k(2r - 1) + \min\{a : \left(\frac{a+f_k(2r-1)}{\leq a}\right) > r\}$,
**Proof of the claim** Let $N = f_k(2r - 1) + \min\{a : \left(\frac{a + g_k(r)}{a} \right) \leq r\} + 1$ and let us consider a coloring $\phi : B_N \to [k+1]$. Without loss of generality we may assume $\phi(\emptyset) = k + 1$ for the empty set $\emptyset$. Assume first that there exists a set $F \in B_N$ with $|F| \leq \min\{a : \left(\frac{a + g_k(r)}{a} \right) > r\}$ and $\phi(F) \neq k + 1$. Then consider the $k$-coloring $\phi' : B_{F,N} \to [k]$ defined by $\phi'(G) = \phi(G)$, if $\phi(G) \in [k]$ and $\phi'(G) = \phi(F)$ otherwise. As $N - |F| = f_k(2r - 1)$, $\phi'$ admits a monochromatic weak copy $C$ of $V_{2r-1}$ in $B_{F,N}$. If its color is not $\phi(F)$, then its elements have the same color in $\phi$, thus $C$ is a monochromatic weak copy of $V_{2r-1}$ with respect to $\phi$. If the color of $C$ is $\phi(F)$ and $C$ contains at least $r$ sets that were colored $k + 1$ in the coloring $\phi$, then together with the empty set, they form a monochromatic weak copy of $V_r$ with respect to $\phi$. Otherwise $C$ contains at least $r$ sets, including $F$, that were colored $\phi(F)$. Then $F$ together with $r$ other such sets form a monochromatic weak copy of $V_r$ with respect to $\phi$.

Assume next that all sets of size at most $\min\{a : \left(\frac{a + g_k(r)}{a} \right) > r\}$ are colored $k + 1$. Then the empty set and $r$ other such sets form a monochromatic weak copy of $V_r$. This proves (i).

To prove (ii), let us consider a consecutive level $k$-coloring $\psi : B_{g_k(r) - 1} \to [k]$ defined by the positive integers $i_1, i_2, \ldots, i_k$ such that $\psi$ does not admit a monochromatic weak copy of $V_r$. We “add max\{a : \left(\frac{a + g_k(r)}{a} \right) \leq r\} + 1$ extra levels”, i.e., we let $j_1 := \max\{a : \left(\frac{a + g_k(r)}{a} \right) \leq r\} + 1$, and $j_{h+1} := i_h$ for all $1 \leq h \leq k$ and set $N' := \left(\sum_{h=1}^{k+1} j_h\right) - 1$. We claim that the corresponding consecutive level $(k + 1)$-coloring $\psi'$ does not admit a monochromatic weak copy of $V_r$, which proves (ii). Indeed, by definition the union of the first $j_1$ layers does not contain $r + 1$ sets, so no monochromatic $V_r$ exists in this color. To see the $V_r$-free property of the other color classes, observe that for any set $F$ of size $j_1$, the cube $B_{F,N'}$ has dimension $g_k(r) - 1$, and the consecutive level $k$-coloring that we obtain by restricting $\psi'$ to $B_{F,N'}$ is isomorphic to $\psi$. If $G$ is the set corresponding to the bottom element of a copy $C$ of $V_r$, then for a $j_1$-subset $F$ of $G$, the copy $C$ belongs to $B_{F,N'}$, so it cannot be monochromatic. 

To prove the theorem we proceed by induction on $k$. If one can use only one color, then all colorings are consecutive level 1-colorings and $B_N$ does not admit a monochromatic $V_r$ if and only if $2^N \leq r$, so $g_1(r) = f_1(r) = \lceil \log r \rceil + 1$ and $c_1 = 1$.

Assume now that the statement of the theorem is proved for some $k \geq 1$ and let us fix $\varepsilon > 0$. Observe that using Claim 2.3 (ii) and the inductive hypothesis we obtain that for $r$ large enough we have

$$g_{k+1}(r) \geq g_k(r) + \max\left\{a : \left(\frac{a + g_k(r)}{a} \right) \leq r\right\} + 1,$$

and $(c_k - \varepsilon) \log r \leq g_k(r) \leq (c_k + \varepsilon) \log r$. We claim that if $d_k$ is the constant that satisfies

$$(d_k + c_k)h\left(\frac{d_k}{d_k + c_k}\right) = 1,$$

then the maximum $a$ in Inequality Eq. 3 is at least $(d_k - \varepsilon) \log r$.

Indeed, there exist positive constants $c_0$ and $\delta$ such that

$$ \left(\frac{(d_k - \varepsilon) \log r + g_k(r)}{(d_k - \varepsilon) \log r}\right) \leq \left(\frac{(d_k + c_k) \log r}{(d_k - \varepsilon) \log r}\right) \leq \left(\frac{(d_k + c_k) \log r}{(d_k - \varepsilon) \log r}\right) \leq c_0 \left(\frac{(d_k + c_k) \log r}{(d_k - \varepsilon) \log r}\right) \leq c_0 r^{1-\delta} < r$$
holds, where for the second inequality we used $d_k < c_k$ and Inequality Eq. 2 and for the penultimate inequality we used that the entropy function is strictly increasing in $(0, 1/2)$. Therefore, we have $g_{k+1}(r) \geq (c_k + d_k - 2\varepsilon) \log r$.

On the other hand, according to Claim 2.3 (i), we have

$$f_{k+1}(r) \leq f_k(2r - 1) + \min \left\{ a : \left( a + f_k(2r - 1) \right) > r \right\}. \quad (4)$$

By the inductive hypothesis, for sufficiently large $r$ we have

$$(c_k - \varepsilon) \log r \leq f_k(r) \leq f_k(2r - 1) \leq (c_k + \varepsilon) \log(2r - 1) \leq (c_k + 2\varepsilon) \log r.$$ 

We claim that the minimum $a$ in Inequality Eq. 4 is at most $(d_k + \varepsilon) \log r$. Indeed, for some positive $\delta'$ and large enough $r$ we have

$$\left( (d_k + \varepsilon) \log r + f_k(2r - 1) \right) \leq (d_k + \varepsilon) \log r \leq \left( d_k + c_k \right) \log r \geq \frac{1}{\log r} 2^h \left( \frac{d_k + c_k}{\log r} \right) (d_k + c_k) \log r$$

$$= \frac{1}{\log r} \log r \left( \frac{d_k + c_k}{\log r} \right) (d_k + c_k) \geq \frac{r^{1+\delta'}}{\log r} > r.$$ 

Therefore, we have $f_{k+1}(r) \leq (c_k + d_k + 3\varepsilon) \log r$ and consequently

$$(c_k + d_k - 2\varepsilon) \log r \leq g_{k+1}(r) \leq f_{k+1}(r) \leq (c_k + d_k + 3\varepsilon) \log r,$$

showing $c_{k+1} = c_k + d_k$. Plugging back to the defining equation $(d_k + c_k)h \left( \frac{d_k}{d_k + c_k} \right) = 1$ we obtain $c_{k+1}h \left( \frac{c_{k+1} - c_k}{c_{k+1}} \right) = 1$ as claimed.

Note that Cox and Steele [3] obtained general but not tight upper bounds on the Ramsey number $R(V_{r_1}, \ldots, V_{r_s}, \Lambda_{r+1}, \ldots, \Lambda_{r})$. Theorem 2.2 is an improvement on their result in case all target posets are the same.

### 3 Strong Copies

The lower bounds in most of our theorems are obtained via trivial colorings where sets of the same size receive the same color. We introduce the following parameters: let $m(P) = \max\{m : B_m \text{ does not contain a weak copy of } P\}$ and $m^*(P) = \max\{m : B_m \text{ does not contain a strong copy of } P\}$. We say that $Q \subseteq B_n$ is thin if $Q$ contains at most one set from each level. Also, let $r^*(P) = \max\{r : B_r \text{ does not contain a thin, strong copy of } P\}$. Note that the corresponding weak parameter $r(P) = \max\{r : B_r \text{ does not contain a thin, weak copy of } P\}$ trivially equals $|P| - 2$ as $B_{|P|-1}$ contains a chain of length $|P|$ and thus a weak copy of $P$. Also, it is not hard to see that $r^*(P) \leq 2|P| - 2$. This is certainly true for all one and two-element posets. Then we proceed by induction on $|P|$. Fix a maximal element $p \in P$. By induction, there exists a thin, strong copy of $P \setminus \{p\}$ in $B_N$ with $N = 2|P| - 4$. Denote the embeddig by $\phi$. Set $A := \cup_{p' < p} \phi(p')$ and partition $P \setminus \{p\}$ into $R_1 = \{p' : |\phi(p')| \leq |A| \}$ and $R_2 = \{p' : |\phi(p')| > |A| \}$. Then it is easy to check that the embedding $\phi'$ defined as $\phi'(p') = \phi(p')$ if $p' \in R_1$, $\phi'(p') = \phi(p') \cup \{N + 2\}$ if $p' \in R_2$ and $\phi'(p) = A \cup \{N + 1\}$ creates a thin, strong copy of $P$ into $B_{N+2}$.

In the next proposition, we prove some lower bounds using non-trivial colorings. A poset $P$ is said to be connected if for any pair $p, q \in P$ there exists a sequence $r_1, r_2, \ldots, r_k$ such that $r_1 = p$, $r_k = q$ and $r_i, r_{i+1}$ are comparable for any $i = 1, 2, \ldots, k - 1$. 

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Proposition 3.1 If \( P \) is a connected poset with \(|P| \geq 2\) and \( Q \) is an arbitrary poset, then we have

(i) \( RR(P, Q) > m(P) + |Q| - 2\),
(ii) \( RR^*(P, Q) > m^*(P) + |Q| - 2\),
(iii) \( RR^*(P, Q) > r^*(Q)\).

Proof Set \( N = m(P) + |Q| - 2, N^* = m^*(P) + |Q| - 2 \) and \( R = [|Q| - 2] \). Consider the colorings \( \phi: B_N \rightarrow \{1, \ldots, |Q| - 1\} \) and \( \phi^*: B_N^* \rightarrow \{1, \ldots, |Q| - 1\} \) defined by \( \phi(F) = |F \cap R| + 1 \) and \( \phi^*(G) = |G \cap R| + 1 \). Observe that \( \phi \) and \( \phi^* \) do not admit a rainbow copy of \( Q \) as only \(|Q| - 1\) colors are used.

By definition of \( m(P) \), for any set \( T \subseteq R \) the family \( \mathcal{F}_T = \{ F \subseteq [N] : F \cap R = T \} \) cannot contain a weak copy of \( P \). Thus a monochromatic weak copy of \( P \) (admitted by \( \phi \)) must contain two sets \( F, F' \) with \( F \in \mathcal{F}_T \) and \( F' \in \mathcal{F}_{T'} \) such that \(|T| = |T'| \) and \( T \neq T' \). As \( P \) is connected, we can choose \( F, F' \) to be comparable. However, since each \( F \in \mathcal{F}_T \) is incomparable to each \( F' \in \mathcal{F}_{T'} \), as \( T \) is incomparable to \( T' \), this is a contradiction. So the coloring \( \phi \) does not admit a monochromatic weak copy of \( P \). This proves (i), and one can prove (ii) in a similar way.

To see (iii) let us consider the trivial coloring \( \phi: B_{r^*(Q)} \rightarrow \{1, \ldots, r^*(Q) + 1\} \) defined by \( \phi(F) = |F| + 1 \). As \( P \) is connected with \(|P| \geq 2\), \( \phi \) does not admit a monochromatic copy of \( P \) and by definition of \( r^*(Q) \), \( \phi \) does not admit a rainbow strong copy of \( Q \).

Proposition 3.2 If \( n \geq 4 \), then \( r^*(A_n) = n + 1 \) holds.

Proof Let \( \mathcal{F} \subseteq B_n \) be a thin antichain. Then we claim \(|\mathcal{F}| \leq n - 2 \) holds, which shows \( r^*(A_n) \geq n + 1 \). Indeed, if \( \emptyset \in \mathcal{F} \) or \([n] \in \mathcal{F} \), then \( \mathcal{F} = \{\emptyset\} \) or \( \mathcal{F} = \{[n]\} \). Also, if both a 1-element and an \((n - 1)\)-element sets are in \( \mathcal{F} \), they have to be complements, and then no other sets can be in \( \mathcal{F} \).

For the upper bound we prove the stronger statement that \( B_n \) contains a thin antichain of size \( n - 2 \) with set sizes \( 1, 2, \ldots, n - 2 \). We proceed by induction on \( n \). The statement is trivial for \( n = 4 \) and \( n = 5 \). Assume the statement holds for some \( n \geq 4 \), and we prove it for \( n + 2 \). Hence we can find a thin antichain \( \mathcal{F} \) in \( B_n \) that has cardinality \( n - 2 \) with set sizes \( 1, 2, \ldots, n - 2 \). Then let \( \mathcal{F}' = \{ F \cup [n + 1] : F \in \mathcal{F} \} \cup \{[n], [n + 2]\} \). It is easy to see that \( \mathcal{F}' \subseteq B_{n+2} \) is a thin antichain of size \( n \) with set sizes \( 1, 2, \ldots, n \).

Propositions 3.1 and 3.2 together yield \( RR^*(C_2, A_k) \geq k + 2 \), which is larger than both \( e^*(C_2)(|A_k| - 1) + f(A_k) = k + 1 \) and \( R_{k-1}^*(C_2) = k - 1 \), showing that \( C_2 \) does not possess the property of Question 1.6 and that there exists a pair of posets for which Proposition 1.3 (ii) holds with a strict inequality.

Definition 3.3 We say that the families \( \mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_j \) are mutually comparable if for any \( F_i \in \mathcal{F}_i \) and \( F_j \in \mathcal{F}_j \) with \( 1 \leq i < j \leq l \) we have \( F_i \subseteq F_j \) or \( F_j \subseteq F_i \), and they are mutually incomparable if for any \( F_i \in \mathcal{F}_i \) and \( F_j \in \mathcal{F}_j \) with \( 1 \leq i < j \leq l \) we have \( F_i \not\subseteq F_j \) and \( F_j \not\subseteq F_i \).

Proof of Theorem 1.7 Set \( N = \left\lfloor \lambda_{\text{max}}^*(P)(k - 1) \right\rfloor + m_k \) and consider a coloring \( \phi: B_N \rightarrow \mathbb{Z}^+ \). Observe that if \( \phi \) does not admit a monochromatic induced copy of \( P \), then for any set \( S \subseteq [m_k] \), \( \phi \) must admit at least \( k \) colors on the family \( \mathcal{Q}_S = \{ S \cup T : T \subseteq [N]\setminus[m_k] \} \). Indeed, if there are at most \( k - 1 \) colors on some \( \mathcal{Q}_S \), then consider the corresponding coloring \( \phi' \) of \( B_{N-m_k} \) such that \( \phi'(\{i_1, i_2, \ldots, i_\ell\}) = \phi(S \cup \{i_1 + m_k, i_2 + m_k, \ldots, i_\ell + m_k\}) \) for every set \( \{i_1, i_2, \ldots, i_\ell\} \in B_{N-m_k} \). Then \( \phi' \) is a \((k - 1)\)-coloring of \( B_{N-m_k} \), \( \square \).
and one of the color classes has Lubell-mass strictly larger than $\lambda^{*}_{\text{max}}(P)$. So $\phi'$ admits a monochromatic induced copy of $P$ in $B_{N-m_k}$. This implies that $\phi$ admits a monochromatic induced copy of $P$ in $Q_S$.

By the definition of $m_k$, we can pick $k$ subsets $S_1, S_2, \ldots, S_k$ of $[m_k]$ of size $[m_k/2]$. As the $S_i$’s form an antichain, the families $Q_{S_1}, Q_{S_2}, \ldots, Q_{S_k}$ are mutually incomparable. By the above paragraph, on each of these families $\phi$ admits at least $k$ colors otherwise we find a monochromatic induced copy of $P$. But then we can pick a rainbow antichain from the $Q_{S_i}$’s greedily: a set $F_1$ from $Q_{S_1}$, then $F_2$ from $Q_{S_2}$ and so on with $\phi(F_i) \neq \phi(F_j)$ for all $i < j$. This completes the proof of the first part of Theorem 1.7.

Now we prove the second part. For any $P$ other than $C_1$ or $C_2$, $F = \{\emptyset, [n]\} \subset B_n$ is induced $P$-free for all $n \geq 2$. Hence $\lambda^{*}_{\text{max}}(P) = \sup \lambda^{*}_n(P) \geq 2$. Let $N = \lfloor 2\lambda^{*}_{\text{max}}(P) \rfloor + 2$.

For any coloring $\psi$ of $B_N$, we show that it admits either a monochromatic induced copy of $P$ or a rainbow copy of $A_3$. If $\|\psi\| \leq 2$, then $\lambda^{*}_N(B_N) = N - 1$ hence one of the color classes has Lubell-mass strictly larger than $\lambda^{*}_{\text{max}}(P)$, so by the definition of $\lambda^{*}_{\text{max}}$, $\psi$ admits a monochromatic induced copy of $P$.

Therefore, we can assume that $\|\psi\| \geq 3$. Let $Q_i = \{i\} \cup T : T \subseteq [N] \setminus \{2\}$ for $i = 1, 2$. Note that $Q_1$ and $Q_2$ are mutually incomparable. By the same reasoning as in the previous case, if $\psi$ admits only 2 colors on some $Q_i$, then we can find a corresponding 2-coloring $\psi'$ of $B_{N-2}$ and a monochromatic copy of $P$ in $B_{N-2}$ with respect to $\psi'$. As before, this implies that there is a monochromatic copy of $P$ in $Q_i$ with respect to $\psi$. Hence we consider the case that $\psi$ admits at least three colors on each $Q_i$. If there are two sets $F_1, F_2 \in Q_1$ of the same size with distinct colors, then a set of third color in $Q_2$ together with $F_1$ and $F_2$ form a rainbow $A_3$. So we may assume that all subsets of the same size in $Q_1$ have the same color. Now if all sets in $Q_1 \setminus \{\emptyset\}$ are of the same color, then the corresponding coloring $\psi'$ admits only one color on $B_{N-2}$. Since $\lambda^{*}_{\text{max}}(P) \geq 2$, we have $\lambda^{*}_{N-2}(B_{N-2}) = N - 3 = \lfloor 2\lambda^{*}_{\text{max}}(P) \rfloor - 1 > \lambda^{*}_{\text{max}}(P)$. Thus, $\psi'$ admits a monochromatic $P$ in $B_{N-2}$ and then $\psi$ admits a monochromatic $P$ in $Q_1$ as well. If there are at least two colors on $Q_1 \setminus \{\emptyset\}$ and sets of the same size have the same color, then we can easily find two incomparable sets from two levels of distinct colors. The two sets together with a set of third color in $Q_2$ form a rainbow $A_3$. This completes the proof.

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Declarations

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