ON A QUESTION OF KAZHDAN AND YOM DIN
WITH AN APPENDIX BY NICOLAS MONOD*

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Dedicated to Benjamin Weiss, a teacher and a friend

ABSTRACT

Studying a question of Kazhdan and Yom Din we first prove some fixed point theorems for linear actions of a discrete countable group $G$ on a dual Banach space $V^*$, and then show that the answer to the question in full generality is in the negative. In an appendix it is shown that the answer is negative also for the Banach space $\mathcal{B}(H)$ of bounded linear operators on an infinite-dimensional Hilbert space.

Introduction

In an effort to prove “an approximate Schur’s lemma” David Kazhdan and Alexander Yom Din arrived at the following question, which they communicated to us (Benjamin Weiss and the author).

0.1 Question: Is the following assertion true?

Let $\delta > 0$ be given and let $V$ be a Banach space, equipped with a linear and isometric action of a discrete group $G$. Let $\alpha \in V^*$ be such that $\|\alpha\| = 1$ and $\|g\alpha - \alpha\| \leq \frac{\delta}{10}$ for all $g \in G$. Then there exists a $G$-invariant $\beta \in V^*$ such that $\|\alpha - \beta\| \leq \delta$.

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N. Monod thanks Eli Glasner for his comments and for welcoming his appendix.

** This research was supported by a grant of the Israel Science Foundation (ISF 1194/19).

Received March 2, 2021 and in revised form December 23, 2021
The most interesting and relevant Banach space for Kazhdan and Yom Din is the space $B(H)$, comprising the bounded linear operators on a Hilbert space and equipped with its weak operator topology. (On norm bounded sets of $B(H)$, the weak operator topology and the weak* (= ultraweak) topology coincide.)

As a motivation one can consider the following easy example.

0.2 Example: Consider the Banach space $V = l_1(G)$ (over $\mathbb{R}$) with the dual $V^* = l_\infty(G)$, where the latter is equipped with its sup norm. Clearly then the assertion in Question 0.1 holds in this case.

In fact, the action of $G$ on $l_\infty(G)$ is by translations:

$$(g \cdot \alpha)(h) = \alpha(hg), \quad \forall g, h \in G, \quad \alpha \in l_\infty(G),$$

and a function $\alpha \in l_\infty(G)$ is $G$-invariant iff it is a constant function. Now given $\alpha \in V^* = l_\infty(G)$ satisfying (i) $\|\alpha\| = 1$ and (ii) $\|g\alpha - \alpha\| \leq \frac{\delta}{10}$ for all $g \in G$, either the function $1 \in \ell_\infty(G)$ is a $G$-fixed point with $\|1 - \alpha\| \leq \delta$, or the function $-1 \in \ell_\infty(G)$ is a $G$-fixed point with $\|1 + \alpha\| \leq \delta$.

Since $\|\alpha\| = 1$, for some $h \in G$ we have either $|\alpha(h) - 1| \leq \frac{\delta}{10}$ or $|\alpha(h) + 1| \leq \frac{\delta}{10}$. Assuming the first case, if $\|1 - \alpha\| > \delta$, then for some $g \in G$ we have $|\alpha(g) - 1| = |\alpha(h) - 1| > \delta$. Now,

$$\delta < |\alpha(g) - 1|$$
$$\leq |\alpha(g) - \alpha(h)| + |\alpha(h) - 1|$$
$$= |((h^{-1}g)\alpha)(h) - \alpha(h)| + |\alpha(h) - 1|$$
$$\leq \|(h^{-1}g)\alpha - \alpha\| + |\alpha(h) - 1| \leq 2\frac{\delta}{10},$$

a contradiction. The second case is treated similarly.

The assertion of Question 0.1 certainly holds whenever the acting group $G$ is amenable. However, there are many general fixed point theorems which can be applied also in cases where the acting group is any topological group. Most notable are the classical Ryll Nardzewski’s fixed point theorem and its many generalizations (see, e.g., [15]).

The question 0.1 though introduces a new perspective on the question of the existence of fixed points for affine actions on compact convex sets, namely the $\delta$-norm assumption.
Our main results in this work are as follows.

In Section 1 we present a short proof of a Ryll Nardzewski type theorem for norm separable, convex weak*-compact sets, Theorem 1.2. We also show how results of Losert [23] and Bader, Gelander and Monod [2] provide a positive answer to question 0.1 for the Banach space $V = C(X)$, the space of continuous complex valued functions on a compact Hausdorff space $X$. In Section 2 we show how Theorem 1.2 may fail when the norm separability assumption is removed. In particular, these examples show that a general fixed point theorem for convex weak* compact sets fails for $V = \mathcal{B}(H)$. In Section 3 we provide a new proof of the fact that Question 0.1 has an affirmative answer for the Banach space $V = C(X)$, Theorem 3.7.

Finally, in Section 4 we show that for $G = F_2$, the free group on two generators, given $0 < \delta < 1$, there exists a separable Rosenthal Banach space $V$,\textsuperscript{1} specially contrived for that purpose, and an affine representation of a minimal and strongly proximal $G$ dynamical system $(X, G)$ on $V^*$, such that the resulting action of $G$ on $V^*$ has the following properties. The only $G$ fixed point in $V^*$ is 0, but there exists an element $\xi \in V^*$, $\|\xi\| = 1$ such that $\|g\xi - \xi\| \leq \delta$ for every $g \in G$. This shows that, in full generality, the assertion of Question 0.1 fails.

An important aspect of the present work is the application of the newly developed theory of representations of dynamical systems on Banach spaces. This theory was developed by Michael Megrelishvili and the author in a series of works. We refer to the review article [17] for more details.

ACKNOWLEDGEMENTS. I would like to thank David Kazhdan and Alexander Yom Din for addressing the question to us. I thank Benjamin Weiss and Michael Megrelishvili whose invaluable advice greatly improved this work. Also, thanks are due to the referee for many very helpful remarks.

When this work was accepted for publication I posted it on the arXiv [12]. After reading my work there Nicolas Monod obtained a negative answer to the question of Kazhdan and Yom Din in the case of the space $\mathcal{B}(H)$. He then kindly agreed to publish it as an appendix to this paper.

\textsuperscript{1} A Banach space is called Rosenthal if it does not contain an isomorphic copy of $\ell_1(\mathbb{N})$. 
1. Ryff Nardzewski theorem for norm separable $w^*$-compact convex sets

1.1 Definition: A dynamical system $(X, G)$ is called **proximal** if for every pair of points $x, y \in X$ there is a net $g_i \in G$ and a point $z \in X$ such that
\[
\lim g_i x = \lim g_i y = z;
\]
it is called **strongly proximal** if the induced action on the space $M(X)$ of Borel probability measures, equipped with its weak* topology, is proximal, or equivalently, when for every $\mu \in M(G)$ there is a net $g_i \in G$ and a point $x \in X$ such that $\lim g_i \mu = \delta_x$. Both proximality and strong proximality are preserved under surjective homomorphisms of dynamical systems. For more details see [11].

The following theorem is not new (see [29, Theorem 2.4], [25, Theorem 15] and also [15]); however, our proof, adapted from [11, Chapter III, Theorem 5.2], is more transparent and much shorter.

1.2 Theorem: Let $V$ be a Banach space and $K$ a subset of $V^*$ which is (i) norm separable, (ii) convex, (iii) closed in the weak* topology. Let $G < \text{Iso}(V)$ be a group of linear isometries such that $gx \in K$ for every $g \in G$ and $x \in K$ (with respect to the natural action of $G$ on $V^*$). Then there is a point $x_0 \in K$ such that $gx_0 = x_0$ for every $g \in G$.\footnote{Nicolas Monod brought to my attention the fact that this result can be found in Bourbaki’s TVS book [3, EVT IV.41, case (c)].}

Proof. We can assume that the affine dynamical system $(K, G)$ is irreducible (i.e., if $\emptyset \neq Q \subset K$ is $G$-invariant, convex and weak* closed, then $Q = K$).

Let $B$ be a norm closed ball of radius 1 centered at $0 \in V^*$. Denote $X = \text{ext}(K)$ (the $w^*$-closure of the set of extreme points).

By the separability of $K$, given $\epsilon > 0$ there exists a set $\{x_i\}_{i=1}^\infty$ of points in $X$ such that $\{x_i + \epsilon B\}_{i=1}^\infty$ is a denumerable cover of $X$. We claim that $B$ is weak* closed. In fact, if $y \not\in B$ there is a vector $v \in V$ such that $r = y(v) > 1$ and then
\[
\left\{ z \in V^* : z(v) > \frac{r + 1}{2} \right\}
\]
is a weak* open neighborhood of $y$ which is disjoint from $K$.\footnote{Nicolas Monod brought to my attention the fact that this result can be found in Bourbaki’s TVS book [3, EVT IV.41, case (c)].}
By Baire’s category theorem there is an $i$ and a non-empty weak* open set $W$ such that

$$W \cap X \subset (x_i + \epsilon B) \cap X.$$  

Since $K$ is irreducible, $X$ is the unique minimal set of the system $(K,G)$ and it is strongly proximal ([11, Chapter III, Theorem 2.3], see Definition 1.1 below). If $x,y \in X$ we can find $g \in G$ such that $gx$ and $gy$ are in $X \cap W \subset x_i + \epsilon B$. Since $\epsilon$ was arbitrary, this contradicts the fact that the elements of $G$ act by norm isometries, unless $X$ and hence also $K$ are trivial one point sets.

Recall that a Banach space $V$ is an Asplund space if the dual of every separable linear subspace of $V$ is separable (iff $V^*$ has the Radon–Nikodým property). Reflexive Banach spaces and spaces of the type $c_0(\Gamma)$ are Asplund. For more details see, e.g., [8].

1.3 COROLLARY: Let $V$ be an Asplund Banach space and $K$ a subset of $V^*$ which is (i) convex, (ii) closed in the weak* topology. Let $G < \text{Iso}(V)$ be a countable group of linear isometries such that $gx \in K$ for every $g \in G$ and $x \in K$ (with respect to the natural action of $G$ on $V^*$). Then there is a point $x_0 \in K$ such that $gx_0 = x_0$ for every $g \in G$.

Proof. In the case that $V$ is a separable Banach space the assumption (i) in Theorem 1.2 is redundant. In the general case we argue as follows. First observe that, as we assume that $G$ is a countable group, every element of $V$ is contained in a $G$-invariant separable subspace of $V$. Let $V_i$, $i \in I$, be the collection of such subspaces, partially ordered by inclusion. We then have for each $i \in I$, that

$$V_i^* \cong V^*/V_i^\perp$$  

(see, e.g., [9, Proposition 2.7]). We let $K_i$ be the canonical image of $K$ in $V_i^*$, and observe that

$$K \cong \lim_{\leftarrow} K_i.$$  

Let $K_0 \subset K$ be an irreducible subset (i.e., a $G$-invariant, weak*-closed, convex subset which is minimal with respect to these properties; such a subset always exists by Zorn’s lemma, see [11, Section III.2]). Now observe that by the Asplund property the image of $K_0$ in each $V_i^*$ is a single point and thus we finally conclude that $K_0$ itself is a singleton whose unique element is the required fixed point.
1.4 Theorem: Let $H$ be a separable Hilbert space and $K$ a subset of $\mathcal{B}(H)$ which is (i) norm separable, (ii) convex, (iii) closed in the weak operator topology. Let $G < \mathcal{U}(H)$ be a group of unitary operators such that $gx \in K$ for every $g \in G$ and $x \in K$. Then there is a $G$ fixed point in $K$.

Another application of the Ryll Nardzewski theorem is the following theorem which is a special case of [2, Theorem A] and [23].

1.5 Definition: A Banach space $V$ is said to be $L$-embedded if its bidual space $V^{**}$ can be decomposed as $V^{**} = V \oplus_1 V_0$ for some $V_0 \subset V^{**}$ (where $\oplus_1$ indicates that the norm is the sum of the norms on $V$ and $V_0$).

As mentioned in [2], by the Yosida–Hewitt decomposition [30], every $L_1$ space and more generally every predual of any von Neumann algebra [28, Theorem III, 2.14] is $L$-embedded; in particular, this is the case for the dual of any $C^*$-algebra (see [27, Theorem 1.17.2]).

1.6 Theorem ([2, Theorem A]): Let $A$ be a non-empty bounded subset of an $L$-embedded Banach space $V$. Then there is a point in $V$ fixed by every linear isometry of $V$ preserving $A$. Moreover, one can choose a fixed point which minimises $\sup_{a \in A} \|v - a\|$ over all $v \in V$.

Thus if $V$ is an $L$-embedded Banach space, $G$ is a group of linear isometries of $G$, and for some $a_0 \in V$ and $\delta > 0$ we have $\|ga_0 - a_0\| \leq \delta$, $\forall g \in G$, then with $A = \{ga_0 : g \in G\}$, by the theorem above, there is a $G$-fixed point $b \in V$ with

$$\|b - a_0\| \leq \sup_{a \in A} \|b - a\| \leq \sup_{a \in A} \|a_0 - a\| \leq \delta.$$ 

This confirms the assertion of Question 0.1 for $L$-embedded Banach spaces.

1.7 Theorem: The assertion of Question 0.1 holds for every $L$-embedded Banach space $V$.

In particular, since $C(X)^*$ is the dual of the (commutative) $C^*$-algebra $C(X)$ and hence the predual of a von Neumann algebra, this confirms the assertion of Question 0.1 for $V = C(X)$. In Section 3 we will provide a new proof of this theorem.
2. Examples where $K$ is not norm separable

As we will now see, the assumption that the set $K$ (or $X$) is norm separable in Theorems 1.2 and 1.4 is essential.

When one considers $\ell_\infty(G)$ as a subspace of $B(H)$, with $H = \ell_2(G)$, and elements $x \in \ell_\infty(G)$ are viewed as multiplication operators, it is easy to check that a subset $X \subset [0,1]^G$ is closed in $[0,1]^G$ iff $X$ is closed in the weak$^*$ topology on $\ell_\infty(G)$ (induced by $\ell_1(G)$), iff it is closed in the weak operator topology in $B(H)$.

When $X$ is a Bebutov subshift, i.e., a closed subset of $[0,1]^G$ invariant under left translations, the induced Koopman unitary representation $\pi : G \to U(H)$ defines an associated representation $\rho : G \to B(H)$,

$$\rho(g)T := \pi(g) \circ T \circ \pi(g)^{-1}.$$ 

Note that, when $X \subset \{0,1\}^G$, then, viewed as a subset of $B(H)$, it consists of projections.

Hereditarily almost equicontinuous systems (HAE for short) were defined and studied in [13]

2.1 Definition: Given a (compact metric) dynamical system $(X,G)$, a point $x \in X$ is called an equicontinuity point if for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$d(x,x') < \delta \Rightarrow d(gx,gx') < \epsilon$$

for every $g \in G$. When the set $Eq(X)$ of equicontinuity points is dense in $X$ we say that the system $(X,T)$ is almost equicontinuous and we say that it is **hereditarily almost equicontinuous** (HAE for short) if every subsystem $Y \subset X$ is almost equicontinuous.

2.2 Remark: A (not necessarily metric) dynamical system $(X,G)$ is called **not sensitive** when there exists an entourage $\epsilon$ (= a neighborhood of the diagonal in $X \times X$) such that for every $x \in X$ and any neighborhood $U$ of $x$ there exists $y \in U$ and $g \in G$ such that $(gx,gy) \notin \epsilon$. It is called **hereditarily non-sensitive** (HNS for short) when every subsystem $Y$ of $X$ is not sensitive. For metrizable systems the properties HNS and HAE coincide. For more details see [13].

The next theorem is a special case of [29], and [13, Theorem 9.14 and Corollary 10.4].
2.3 Theorem: Let $G$ be a discrete countable group and let $X \subset [0,1]^G$ be a closed invariant set. The system $(X,G)$ is HAE iff the set $X$, considered as a subset of $\ell_\infty(G)$, is norm separable. If $X \subset \{0,1\}^G$ then it is HAE iff it is countable.

Thus it follows that whenever the dynamical system $(X,G)$ (with $X \subset \{0,1\}^G$) is not HAE (e.g., this is the case when it is weakly mixing or strongly proximal), the set $X \subset \ell_\infty(G)$ is not norm separable.

2.4 Theorem: Let $G$ be a discrete countable non-amenable group. Then there is an infinite closed invariant $X \subset \{0,1\}^G$ with the following properties:

1. The system $(X,G)$ is minimal and strongly proximal (hence in particular not HAE).
2. The space $X$ considered as a subspace of $\mathcal{B}(H)$ is not norm separable.
3. The weak operator closed, convex, $G$-invariant set

$$Q := WO\text{-}\text{cls}(\text{conv}(X)) \subset \mathcal{B}(H)$$

is a strongly proximal affine $G$-system and therefore admits no fixed point.

Proof. It is well known that $G$ is non-amenable iff there is an infinite minimal strongly proximal $G$-system [11, Chapter III, Theorem 3.1]. Since $G$ is discrete, general well-known considerations show that such a system can be found with $X \subset \{0,1\}^G$. There are several ways to see this; one short proof is as follows. Consider the universal minimal strongly proximal $G$-system $(\Pi_s(G),G)$. Since $G$ is discrete the space $\Pi(G)$ is totally disconnected (in fact an extremely disconnected space; see, e.g., [19]). Now let $O \subsetneq \Pi_s(G)$ be a nonempty clopen subset and let $\pi : \Pi_s(G) \to \{0,1\}^G$ be the corresponding “name map”:

$$\pi(z)(g) = 1_O(gz), \quad z \in \Pi_s(G), \quad g \in G.$$ 

The image system $(X,G)$, where $X = \pi((\Pi_s(G)))$, is the required system. The rest of the claims follow from the above discussion (see Example 2.6 below for a concrete example of such a system for $F_2$).

2.5 Question: Which weak operator topology closed convex subsets of $\mathcal{B}(H)$ are norm separable?
2.6 Example: The free group $G = F_2$ on two generators, say $a$ and $b$, is a hyperbolic group and its boundary (the boundary of its Cayley graph) can be identified with the compact metric space $\Omega$ (a Cantor set) of all the one-sided infinite reduced words $\omega$ on the symbols $a, b, a^{-1}, b^{-1}$. The group action on $\Omega$ is given by

\[ F_2 \times \Omega \to \Omega, \quad (g, \omega) = g \cdot \omega, \]

where $g \cdot \omega$ is obtained by concatenation of $g$ (written in its reduced form) and $\omega$ and then performing the needed cancelations. The resulting dynamical system is minimal and strongly proximal. Taking any nontrivial clopen partition $\Omega = P_0 \sqcup P_1$, and defining the corresponding name map

\[ \phi: \Omega \to \{0, 1\}^G, \quad \phi(\omega)(g) = \epsilon \text{ when } g \cdot \omega \in P_\epsilon, \]

we obtain an infinite subshift $X = \phi(\Omega) \subset \{0, 1\}^G$ which is minimal and strongly proximal.

It turns out that the systems $(\Omega, G)$ and $(X, G)$ are tame (see Section 4 below and [18, Example 6.7]).

3. A fixed point theorem for measures

In this section we provide a direct proof of the assertion of Question 0.1 for the case $V = C(X)$, the Banach space of continuous real valued functions on a compact metric space $X$ equipped with the sup norm, and actions on $C(X)$ coming from actions by homeomorphisms on $X$ (see Theorem 1.7 above and the remark which follows it).

Let $(X, G)$ be a compact metric dynamical system. Let $M(X) \subset C(X)^*$ denote the convex $w^*$-compact set of Borel probability measures on $X$. We use $\|\cdot\|$ to denote both the sup norm on $C(X)$ and the total variation norm on its dual $C(X)^*$ (viewed as the space of signed measures).

We will use the following well-known lemma.

3.1 Lemma: For $\alpha, \beta \in M(X)$ we have the identity

\[ \frac{1}{2} \|\alpha - \beta\| + \|\alpha \wedge \beta\| = 1. \]

Thus $\alpha \perp \beta$ iff $\|\alpha - \beta\| = 2$. 

3.2 Theorem: Suppose $\alpha \in M(X)$ is such that, for a positive $\delta < 1$,
\[ \|g\alpha - \alpha\| \leq \frac{\delta}{10} \quad \text{for all } g \in G. \]
Then there exists a non zero $G$-invariant measure $\mu_0$ such that
\[ \|\alpha - \mu_0\| \leq \delta. \]

Proof. Let $Q_{\delta/2} = \{ \beta \in C(X) : \|\alpha - \beta\| \leq \delta/2 \}$, the ball of radius $\delta/2$ centered at $\alpha$. Let $Q = w^*\text{-cls}(\text{conv}(\{g\alpha : g \in G\})) \subset M(X) \cap Q_{\delta/2}$.

Note that $\|\theta - \eta\| \leq \delta$ for any pair $\theta, \eta \in Q$. Clearly $Q$ is a nonempty, $w^*$-closed, convex, $G$-invariant subset of $M(X)$.

Let $m$ be a symmetric probability measure on $G$ with full support. Let $P_m : M(X) \to M(X)$ be the corresponding Markov operator, defined by convolution with $m$:
\[ P_m(\theta) = m * \theta, \quad \theta \in M(X). \]

Clearly $P_m(Q) \subset Q$. Let $\lambda \in M(M(X))$ be any weak* limit point of the sequence $\frac{1}{N} \sum_{j=0}^{N-1} P_m^j \delta_\alpha$, say
\[ \lambda = \lim_{k \to \infty} \frac{1}{N_k} \sum_{j=0}^{N_k-1} P_m^j \delta_\alpha. \]

It follows that $\lambda \in M(Q)$ and that the measure $\lambda$ is $m$-stationary, i.e., $m*\lambda = \lambda$.

Let $\text{bary} : M(M(X)) \to M(X)$ denote the weak*-continuous, affine, barycenter map and set
\[ \text{bary}(\lambda) = \int_Q \theta d\lambda(\theta) := \mu \in Q. \]

The map $\text{bary}$ is an affine homomorphism of dynamical systems and we have $m*\mu = \mu$. Thus the non-singular dynamical systems $(Q, \lambda; G, m)$ and $(X, \mu; G, m)$ are $m$-stationary systems (see [10]).

For each $\theta \in Q$ we let
\[ \theta = \theta_a + \theta_d, \quad \theta_a \ll \mu, \quad \theta_d \perp \mu, \]
be the unique Lebesgue decomposition of $\theta$ with respect to $\mu$.

By Lemma 3.1, for every $\theta \in Q$,
\[ 0 < 1 - \frac{1}{2} \delta \leq 1 - \frac{1}{2} \|\theta - \mu\| = \|\theta \wedge \mu\| = \int \min \left( 1_X, \frac{d\theta_a}{d\mu} \right) d\mu \leq \int \frac{d\theta_a}{d\mu} d\mu = \|\theta_a\|. \]
Now for \( g \in G \) we have
\[
g_{\theta} = g_{\theta_a} + g_{\theta_d} = (g_{\theta})_a + (g_{\theta})_d,
\]
and
\[
g_{\theta_a} \ll g_{\mu} \cong \mu, \quad g_{\theta_d} \perp g_{\mu} \cong \mu.
\]
Hence, by the uniqueness of the decomposition, we have
\[
(1) \quad g_{\theta_a} = (g_{\theta})_a, \quad g_{\theta_d} = (g_{\theta})_d.
\]
We write
\[
F_{\theta} = \frac{d\theta_a}{d\mu} \quad \text{and} \quad H_{\theta} = \sqrt{F_{\theta}}.
\]
Note that \( F_{\mu} = H_{\mu} = 1_X \).
We consider the usual unitary representation of \( G \) on \( L_2(X, \mu) \) given by
\[
U_g f(x) = f(g^{-1}x)u(g^{-1}, x), \quad \text{with} \quad u(g, x) = \sqrt{\frac{dg^{-1}\mu}{d\mu}}.
\]
Then, for \( g \in G, \theta \in Q \) and \( f \in L_2(X, \mu) \), denoting \( v(g, x) = \frac{dg^{-1}\mu}{d\mu} \), we get
\[
\int f(x)d(g_{\theta_a}) = \int f(x)F_{g\theta}(x)d\mu = \int f(x)dg_{\theta_a} = \int f(gx)F_{\theta}(x)d\mu = \int f(x)F_{\theta}(g^{-1}x)dg_{\mu} = \int f(x)F_{\theta}(g^{-1}x)v(g^{-1}, x)d\mu.
\]
Hence \( F_{g\theta} = (F_{\theta} \circ g^{-1}) \cdot v(g^{-1}, \cdot) \) and
\[
(2) \quad H_{g\theta} = (H_{\theta} \circ g^{-1}) \cdot u(g^{-1}, \cdot) = U_g H_{\theta}.
\]
Let \( \phi : Q \to C(X)^* \) be defined by
\[
\phi(\theta) = \theta_a = F_{\theta}d\mu.
\]
Let
\[
Q_a := \phi(Q), \quad \lambda_a := \phi_*(\lambda).
\]
By equation (1), \( \phi \) intertwines the actions of \( G \) on \( Q \) and \( Q_a \).
Let $J : Q_a \to L_2(X, \mu)$ be defined by $J(\theta_a) = H_\theta$, and set

$$\tilde{Q} := J(Q_a), \quad \tilde{\mu} := J_*(\lambda_a).$$

Clearly the map $J : (Q_a, G) \to (\tilde{Q}, G)$ is a measurable isomorphism and by equation (2)

$$J(g \theta_a) = H_{g \theta} = U_g H_\theta = U_g J(\theta_a).$$

Thus the map

$$J : (Q_a, \lambda_a; G, m) \to (\tilde{Q}, \tilde{\mu}; G, m)$$

is an isomorphism of $m$-stationary systems, where again the action of $G$ on $\tilde{Q} \subset L_2(X, \mu)$ is via the unitary representation $g \mapsto U_g$.

Now the dynamical system $(\tilde{Q}, \tilde{\mu}; G, m)$ is WAP (weakly almost periodic) and $m$-stationary. By [10, Theorem 7.4], such a system is $m$-stiff, that is, every $m$-stationary measure is actually invariant. We therefore conclude that the $m$-stationary measure $\tilde{\mu}$ is $G$-invariant.

As the map $J$ is an isomorphism, this implies that also $\lambda_a$ is $G$-invariant. Finally, applying the barycenter map

$$\text{bary} : M(Q_a) \to Q_{\delta/2}$$

and denoting $\mu_0 := \text{bary}(\lambda_a) \in Q_{\delta/2}$, we conclude that $\mu_0$ is $G$-invariant with

$$\|\alpha - \mu_0\| \leq \delta,$$

and our proof is complete. \hfill \blacksquare

3.3 Remark: Note that it may happen that we start with $\mu = \mu_0 + \theta$, where $\theta \perp \mu_0$, $\|\theta\| < \delta$, $\mu_0$ is $G$-invariant, and $\theta$ is such that $w^*$-cls $(G\theta)$ admits an $m$-stationary measure which is not $G$-invariant.

3.4 Remark: Normalizing $\mu_0$ we obtain the existence of a $G$-invariant probability measure.

A slight modification of the proof will yield the following.

3.5 Theorem: If $\alpha \in M(X)$ is such that for some $\delta > 0$, $\|g \alpha - \alpha\| \leq 2 - \delta$ for all $g \in G$, then there exists a $G$-invariant probability measure.

3.6 Remark: Note the apparent similarity between Theorem 3.5 and [1, Proposition 4.14.(1)].
3.7 Theorem: Let \((X, G)\) be a compact \(G\)-space and suppose that \(\mu \in C(X)^*\) is a signed measure such that, for a positive \(0 < \delta < 1\), \(\| g \mu - \mu \| \leq \frac{\delta}{10}\) for all \(g \in G\). Then there exists a \(G\)-invariant signed measure \(\lambda\) such that \(\|\mu - \lambda\| \leq 2\delta\).

Proof. Let \(\mu = \mu^+ - \mu^-\) be the Jordan decomposition of \(\mu\). It is not hard to see that then also \(\| g \mu^\pm - \mu^\pm \| \leq \frac{\delta}{10}\) for all \(g \in G\). Denoting \(\mu^\pm_1 = \frac{\mu^\pm}{\|\mu^\pm\|} \in M(X)\) we have

\[
\| g \mu^\pm_1 - \mu^\pm_1 \| \leq \frac{\delta}{10\|\mu^\pm\|}
\]

and, by Theorem 3.2, there are \(G\)-invariant positive measures \(\lambda^\pm_1\) with

\[
\|\lambda^\pm_1 - \mu^\pm_1\| \leq \frac{\delta}{\|\mu^\pm\|}.
\]

Finally, with \(\lambda^\pm = \|\mu^\pm\|\lambda^\pm_1\) and \(\lambda = \lambda^+ - \lambda^-\), we conclude that \(\lambda\) is a \(G\)-invariant signed measure satisfying

\[
\|\lambda - \mu\| \leq 2\delta.
\]

4. A counter example to the \(\delta\) question

In this section we will show how to construct a counter example to the assertion of Question 0.1 for \(G = F_2\), the free group on two generators. In fact, the same construction will work for any discrete countable group that admits an effective minimal, strongly proximal, tame dynamical system. The idea is to start with such a minimal metric dynamical system \((X, G)\) and then, as in [14], by way of the Davis–Figiel–Johnson–Peczynski (DFJP) construction [6], to modify its natural representation on \(C(X)\) in order to create a Rosenthal Banach space \(V\) and a representation of \((X, G)\) on \(V^*\), so that in this representation the question 0.1 is refuted. For concreteness we will consider the \(F_2\) dynamical system from Example 2.6.

We start with some background on enveloping semigroups, on tame dynamical systems and on representations of dynamical systems on Banach spaces. For simplicity we will assume that our dynamical systems are metrizable. For more details see e.g. [14].

4.1 Definition: The enveloping semigroup of the system \((X, G)\), denoted by \(E(X)\), is defined as the pointwise closure in \(X^X\) of the set of \(g\)-translations, \(g \in G\).
4.2 Definition:

(1) A compact space $K$ is called **Rosenthal compact** if, for some Polish space $X$, it can be homeomorphically embedded in the space $B_1(X)$ of real valued Baire class 1 functions on $X$, endowed with the pointwise convergence topology.

(2) Let $X$ be a compact topological space. We say that a subset $F \subset C(X)$ is a **Rosenthal family** (for $X$) if $F$ is norm bounded and the pointwise closure $\text{cls}_p(F)$ of $F$ in $\mathbb{R}^X$ consists of Baire class 1 functions.

(3) A dynamical system $(X, G)$ is called **tame** if for every $f \in C(X)$ the orbit $\{f \circ g : g \in G\}$ is a Rosenthal family.

The dynamical Bourgain–Fremlin–Talagrand dichotomy (see [4]) marks a sharp division in the domain of dynamical systems into two classes: “tame” and “wild”. It was first introduced by Köhler [22], and then was further developed is a series of papers by Glasner and Megrelishvili, Kerr and Li, and many other authors (see [13], [20], [17]). The following theorem is from [13, theorem 3.2].

4.3 Theorem (The dynamical BFT dichotomy): Let $X$ be a compact metric dynamical $G$-system and let $E(X)$ be its enveloping semigroup. We have the following dichotomy: either

(1) $E(X)$ is a separable Rosenthal compactum, hence with cardinality

$$\text{card } E(X) \leq 2^{\aleph_0};$$

or

(2) $E(X)$ contains a homeomorphic copy of $\beta \mathbb{N}$ (the Stone–Čech compactification of $\mathbb{N}$), hence $\text{card } E(X) = 2^{2^{\aleph_0}}$.

The first case occurs if and only if the system $(X, G)$ is tame.

4.4 Definition: A Banach space $V$ is called **Rosenthal** if it does not contain an isomorphic copy of $\ell_1(\mathbb{N})$. For a separable Banach space $V$ an equivalent condition is that $\text{card } V = \text{card } V^{**} = 2^{\aleph_0}$ (see, e.g., [26] and [7]). Every Asplund space is Rosenthal.

Let $V$ be a Banach space. Denote by $\text{Iso}(V)$ the topological group of all linear isometries of $V$ onto itself equipped with the pointwise convergence topology.
4.5 Definition ([24]): Let $X$ be a $G$-space. A representation of $(X, G)$ on a Banach space $V$ is a pair

$$h: G \to \text{Iso}(V), \quad \alpha: X \to V^*$$

where $h: G \to \text{Iso}(V)$ is a continuous co-homomorphism and $\alpha: X \to V^*$ is a weak* continuous bounded $G$-map with respect to the dual action

$$G \times V^* \to V^*, \quad (g\varphi)(v) := \varphi(h(g)(v)).$$

We say that a representation $(h, \alpha)$ is faithful when $\alpha$ is a topological embedding.

Every compact $G$-space $X$ admits a canonical faithful representation on the Banach space $V = C(X)$ via the map

$$x \mapsto \delta_x \in C(X)^*.$$ 

A natural program is then to classify dynamical systems according to their representability properties on “nice” Banach spaces. In Table 1 we encapsulate some features of the trinity: a dynamical system $(X, G)$, its enveloping semigroup $E(X)$, and a class of Banach spaces on at least one of its members the dynamical system $(X, G)$ can be faithfully represented.

| Dynamical characterization | Enveloping semigroup | Banach representation |
|---------------------------|----------------------|-----------------------|
| WAP                       | cls$(fG)$ is a subset of $C(X)$ | Every element is continuous | Reflexive |
| HNS                       | cls$(fG)$ is metrizable   | $E(X)$ is metrizable     | Asplund   |
| Tame                      | cls$(fG)$ is Fréchet     | Every element is Baire 1 | Rosenthal |

Let $X$ be a compact metrizable $G$-space and $E(X)$ denote the corresponding enveloping semigroup. The symbol $f$ stands for an arbitrary function in $C(X)$ and

$$fG = \{f \circ g : g \in G\}$$

denotes its orbit. Finally, cls$(fG)$ is the pointwise closure of $fG$ in $\mathbb{R}^X$. For more details on this classification see e.g. the review [17]. WAP stands for weakly almost periodic and HNS for hereditarily not sensitive.

We have the following theorem ([14, Theorems 6.3 and Theorem 6.9]):
4.6 Theorem: Let $X$ be a compact $G$-space, $F \subset C(X)$ a Rosenthal family for $X$ such that $F$ is $G$-invariant (that is, $f \circ g \in F$, $\forall f \in F, \forall g \in G$).

(1) There exist: a Rosenthal Banach space $V$, an injective mapping $\nu : F \to B_V$ into the unit ball $B_V$ of $V$ and a continuous representation $h : G \to \text{Iso}(V)$, $\alpha : X \to V^*$ of $(X, G)$ on $V$ ($\alpha$ is a topological embedding if $F$ separates points of $X$) and

$$f(x) = \langle \nu(f), \alpha(x) \rangle \forall f \in F \forall x \in X.$$ 

Thus the following diagram commutes:

$$\begin{array}{ccc}
F \times X & \to & \mathbb{R} \\
\downarrow \nu & & \downarrow \text{id}_\mathbb{R} \\
V \times V^* & \to & \mathbb{R} \\
\downarrow \alpha & & \\
\end{array}$$

(2) If $X$ is metrizable then $V$ is separable.

In order to see that this theorem can produce the required counter example we will have to look into some details of these constructions, as follows.

For brevity of notation let $\mathcal{A} := C(X)$ denote the Banach space $C(X)$ where, $\| \cdot \|$ will denote the sup-norm on $C(X)$, $B$ will denote its unit ball, and $B^*$ will denote the weak* compact unit ball of the dual space $\mathcal{A}^* = C(X)^*$.

Let $W$ be the symmetrized convex hull of $F$, that is,

$$W := \text{conv}(F \cup -F).$$

It is shown in [14] that $W$ is a Rosenthal family for $X$, and also a Rosenthal family for the larger space $B^*$ (where $W$ is considered as a set of functions on $B^*$).

Consider the sequence of sets

$$M_n := 2^nW + 2^{-n}B.$$ 

Since $W$ is convex and symmetric we can apply the construction of Davis–Figiel–Johnson–Pelczyński [6] as follows. Let $\| \cdot \|_n$ be the Minkowski functional of the set $M_n$, that is,

$$\|v\|_n = \inf \{ \lambda > 0 : v \in \lambda M_n \}.$$
Then $\| \|_n$ is a norm on $\mathcal{A}$ equivalent to the given norm of $\mathcal{A}$. For $v \in \mathcal{A}$, set

$$N(v) := \left( \sum_{n=1}^{\infty} \|v\|_n^2 \right)^{1/2},$$

and let

$$V := \{ v \in \mathcal{A} : N(v) < \infty \}.$$ 

Also let

$$B_V = \{ v \in V : N(v) \leq 1 \} \quad \text{and} \quad S_V = \{ v \in V : N(v) = 1 \},$$

and denote by $j : V \hookrightarrow \mathcal{A}$ the inclusion map.

**4.7 Claim:** $(V, N)$ is a Banach space and $j : V \rightarrow \mathcal{A}$ is a continuous linear injection, with $W \subset j(B_V) = B_V$.

**Proof.** This is proved in the original DFJP paper. To see the last assertion note that if $v \in W$ then $2^n v \in M_n$, hence $\|v\|_n \leq 2^{-n}$ and

$$N(v)^2 \leq \sum_{n=1}^{\infty} 2^{-2n} = 1.$$ 

The given action $G \times X \rightarrow X$ induces a natural linear norm preserving continuous right action $C(X) \times G \rightarrow C(X)$ on the Banach space $\mathcal{A} = C(X)$. It follows by the above construction that $W$ and $B$ are $G$-invariant subsets in $\mathcal{A}$. This implies that $V$ is a $G$-invariant subset of $\mathcal{A}$ and the restricted natural linear action $V \times G \rightarrow V$, $(v, g) \mapsto vg$ is norm preserving, that is, $N(vg) = N(v)$. Therefore, the co-homomorphism

$$h : G \rightarrow \text{Iso}(V), \quad h(g)(v) := vg$$

is well defined.

Let $j^* : \mathcal{A}^* \rightarrow V^*$ be the adjoint map of $j : V \rightarrow \mathcal{A}$. Define $\alpha : X \rightarrow V^*$ as follows. For every $x \in X \subset C(X)^*$ set $\alpha(x) = j^*(\delta_x)$. Then $(h, \alpha)$ is a representation of $(X, G)$ on the Banach space $V$.

By the construction $F \subset W \subset B_V$. Define $\nu : F \hookrightarrow B_V$ as the natural inclusion. Then

$$f(x) = \langle \nu(f), \alpha(x) \rangle \quad \forall f \in F, \forall x \in X.$$ 

(We will write this more simply as $f(x) = \alpha(x)(f)$.)

It follows in particular that if $F$ separates points of $X$ then $\alpha$ is an embedding.
4.8 Claim: \( B_V \subset \bigcap_{n \in \mathbb{N}} M_n = \bigcap_{n \in \mathbb{N}} (2^n W + 2^{-n} B) \).

Proof. If \( \|v\| < 1 \) then \( \|v\|_n < 1 \) for all \( n \in \mathbb{N} \) and, as the sets \( M_n \) are convex, this implies that \( v \in M_n \) for every \( n \). If \( \|v\| = 1 \) we must have \( \|v\|_n < 1 \) for all \( n \in \mathbb{N} \) and again we conclude that \( v \in M_n \) for every \( n \).

One more important ingredient we will need is the following refinement of the construction of \( V \) [16, Lemma 17.(2) (Lemma 4.4.(2) in the arXiv version)], which in turn relies on [8, Lemma 1.2.2].

4.9 Lemma: Consider the injective map \( j: V \to C(X) \) and let

\[
\alpha := j^* \circ \delta: X \to V^*
\]

(where \( \delta(x) = \delta_x \)). Then:

1. The image of \( j^* \) is norm dense in \( V^* \):

\[
\overline{j^*(C(X)^*)} = V^*.
\]

2. The image \( \alpha(X) \) is a \( w^* \)-generating subset of \( V^* \), i.e.,

\[
\text{span} \left( \text{conv}_{w^*} (\alpha(X)) \right)
\]

is norm dense in \( V^* \).

For the reader’s convenience we reproduce the proof of Lemma 4.9 in Appendix A below.

With this background at hand we now proceed with our construction as follows: Given a positive \( \delta < 1 \), let \( n_0 \) satisfy

\[
2^{-n_0} < \frac{\delta}{4}.
\]

Let \( \epsilon = \frac{1}{10} \delta 2^{-n_0} \).

We will next construct a suitable Rosenthal family \( F \subset X \). Choose a function \( f \in C(X) \) such that \( 1 - \epsilon \leq f(x) \leq 1 \) for every \( x \in X \) and such that the values 1 and \( 1 - \epsilon \) are attained by \( f \). Let \( F_0 = \{ f \circ g : g \in G \} \). We can assume that the set \( F_0 \) separates points on \( X \) (otherwise we will replace the system \( (X, G) \) by the factor which \( f_0 \) generates). Let \( F := \text{norm-cl} F_0 \). Since our system \( (X, T) \) is tame, the family \( F_0 \) is a \( G \)-invariant Rosenthal family and therefore so is \( F \).

We now use the set \( F \) to create \( W := \text{conv} (F \cup -F) \) and \( V \) as above.
4.10 Lemma: There is a number $t > 0$ such that

$$\|\alpha(x)\|_{V^*} = t, \quad \forall x \in X.$$ 

Proof. Let $x_0 \in X$ and let $\|\alpha(x_0)\|_{V^*} = t$. Then, by minimality of $(X, G)$, the set $\{gx_0 : g \in G\}$ is dense in $X$ and it follows that $\|\alpha(x)\|_{V^*} \leq t$ for every $x \in X$. Since the same argument applies to any $x \in X$, we conclude that indeed $\|\alpha(x)\|_{V^*} = t$ for every $x \in X$. Since the set $\alpha(X)$ generates $V^*$ we cannot have $t = 0$. 

4.11 Lemma:

$$\max \{|w(x) - w(y)| : x, y \in X\} \leq \epsilon$$

for every $w \in W$.

Proof. It suffices to show that this inequality holds for functions of the form $w = \sum_{j=1}^{N} p_j f_{n_j}$, where $\sum_{j=1}^{N} p_j = 1$, $0 < p_j$, and $f_{n_j} \in \pm F_0$, for $j = 1, 2, \ldots, N$. Now for such $w$ and $x, y \in X$,

$$|w(x) - w(y)| = \left| \sum_{j=1}^{N} p_j f_{n_j}(x) - \sum_{j=1}^{N} p_j f_{n_j}(y) \right|$$

$$= \left| \sum_{j=1}^{N} p_j (f_{n_j}(x) - f_{n_j}(y)) \right|$$

$$\leq \sum_{j=1}^{N} p_j |f_{n_j}(x) - f_{n_j}(y)|$$

$$\leq \sum_{j=1}^{N} p_j \epsilon = \epsilon. \quad \blacksquare$$

4.12 Lemma:

$$\|\alpha(x) - \alpha(y)\|_{V^*} \leq \delta$$

for every $x, y \in X$.

Proof. By definition

$$\|\alpha(x) - \alpha(y)\|_{V^*} = \sup_{v \in S_V} |(\alpha(x) - \alpha(y))(v)|$$

$$= \sup_{v \in S_V} |\alpha(x)(v) - \alpha(y)(v)|$$

$$= \sup_{v \in S_V} |v(x) - v(y)|.$$
Now, \( S_V \subset \bigcap_{n \in \mathbb{N}} (2^n W + 2^{-n} B) \) hence, a fortiori, \( S_V \subset 2^{n_0} W + 2^{-n_0} B \) (see equation (3) above). Writing \( v \in S_V \) as \( v = 2^{n_0} w + 2^{-n_0} b \), with \( w \in W \) and \( b \in B \), we have
\[
|v(x) - v(y)| = |2^{n_0} w(x) + 2^{-n_0} b(x) - (2^{n_0} w(y) + 2^{-n_0} b(y))| \\
\leq 2^{n_0} |w(x) - w(y)| + 2^{-n_0} |b(x) - b(y)| \\
\leq 2^{n_0} \epsilon + 2^{-n_0} \cdot 2 \\
\leq 2^{n_0} \left( \frac{1}{10} 2^{-n_0} \right) + 2\delta 4 \leq \delta.
\]

We are now ready to present our counter example to Question 0.1.

4.13 Theorem: For \( G = F_2 \), the free group on two generators and for every \( 0 < \delta < 1 \) there exists a separable Rosenthal Banach space \( V \) and a representation
\[
h: G \to \text{Iso}(V), \quad \alpha: X \to V^*
\]
such that:

1. The only \( G \) fixed point in \( V^* \) is 0.
2. There exists an element \( \xi \in V^* \), \( \|\xi\| = 1 \) such that \( \|g\xi - \xi\| \leq \delta \) for every \( g \in G \).

Proof. We consider the representation of the minimal strongly proximal and tame system \((X, G)\)
\[
h: G \to \text{Iso}(V), \quad \alpha: X \to V^*
\]
on the Rosenthal Banach space \( V \) described above. (Again we note that, since \((X, G)\) is tame, our family \( F \subset C(X) \) is a Rosenthal family, although this fact plays no part in the proof; see Remark 4.14 below.)

1. The continuous map \( j^*: C(X)^* \to V^* \) intertwines the \( G \)-actions on these spaces. Let \( Q := \text{conv}^{w^*}(\alpha(X)) \). As a homomorphic image of \((\text{conv}^{w^*}(X), G)\) the system \((Q, G)\) is a strongly proximal affine system. Moreover, the restriction of \( j^* \) to \( \{\delta_x : x \in X\} \), namely the function \( \alpha \), is an isomorphism
\[
\alpha: (X, G) \to (\alpha(X), G).
\]

Suppose now that \( \xi \in V^* \) is a nonzero \( G \) fixed point. Normalizing we can assume that \( \|\xi\| = 1 = t \) (see Lemma 4.10).
By Lemma 4.9, i.e., by the norm density of span($Q$), given $\epsilon > 0$ there are real numbers $a_1, a_2, \ldots, a_k$ and elements $\theta_1, \theta_2, \ldots, \theta_k$ in $Q$ such that

$$\left\| \sum_{i=1}^{k} a_i \theta_i - \xi \right\| \leq \epsilon.$$ 

By strong proximality there is a sequence $g_n$ in $G$, and $z$ a point in $X$, such that

$$\lim g_n \theta_i = \alpha(z), \quad \forall 1 \leq i \leq k$$

(to see this use induction on $k$ and the fact that $X$ is the unique minimal subset of $Q$). As $G$ acts by norm isometries we also have

$$\left\| \sum_{i=1}^{k} a_i g_n \theta_i - \xi \right\| \leq \epsilon, \quad \forall g_n.$$ 

As the closed ball of radius $\epsilon$ is $w^*$-compact, passing to the limit, we conclude that

$$\left\| \left( \sum_{i=1}^{k} a_i \right) \alpha(z) - \xi \right\| \leq \epsilon.$$ 

Note that, as both $\|\xi\| = 1$ and $\|\alpha(z)\| = 1$, this implies that $|\sum_{i=1}^{k} a_i| \leq 1 + \epsilon$. Finally, as $\epsilon$ was arbitrary we conclude, by compactness, that for some $z \in Z$ we have $\alpha(z) = \pm \xi$, contradicting the fact that the system $(\alpha(X), G)$ is not trivial.

(2) By Lemma 4.12 we have $\|\alpha(x) - \alpha(y)\| \leq \delta$, for every $x, y \in X$. Thus for any $x \in X$ we get, with $\xi = \alpha(x)$,

$$\|g \xi - \xi\| = \|g \alpha(x) - \alpha(x)\| = \|\alpha(gx) - \alpha(x)\| \leq \delta, \quad \forall g \in G.$$ 

4.14 Remark: Our proof does not rely on the full force of Theorem 4.6, namely the fact that the resulting Banach space $V$ is Rosenthal is not needed. All we need from the DFJP construction are the Claims 4.7 and 4.8.

4.15 Remark: A variant of the question 0.1, also suggested by Kazhdan and Yom Din, is as follows:

4.16 Question: For any Banach space $V$, equipped with a linear and isometric action of a discrete group $G$, there exists a positive function $\epsilon(\delta)$, $0 < \delta < 1$, such that for any $\alpha \in V^*$, $\|\alpha\| = 1$, such that $\|g(\alpha) - \alpha\| \leq \epsilon(\delta)$, $\forall g \in G$, there exists a $G$-invariant $\beta \in V^*$ such that $\|\beta - \alpha\| \leq \delta$. 

\[\square\]
Now we can easily tweak our construction to refute this latter question as well. Applying Theorem 4.13 with $\delta = 1/n$, let us denote by $V_n^*$ the resulting Banach space with its $F_2$ action. Let $W = \bigoplus_{n \in \mathbb{N}} V_n^*$ be the $\ell_2$-sum of these Banach spaces. This is a Rosenthal space and it is also true that

$$\left( \bigoplus_{n \in \mathbb{N}} V_n \right)^* = \bigoplus_{n \in \mathbb{N}} V_n^*$$

(see [16, Lemma 3 (Lemma 1.14 in the arXiv version)]). Let $G$ act diagonally on $W$. It is then easily checked that for this action of $G$ on $W$, 0 is the unique fixed point, and that the assertion of Question 4.16 does not hold.

Appendix A. A proof of Lemma 4.9

Consider a Banach space $(W, \| \cdot \|)$ and a sequence of equivalent norms $\{\| \cdot \|_n\}_{n=1}^\infty$ on $W$. Let $Z$ be the Banach space

$$Z = \left( \sum_{n=1}^\infty W, \| \cdot \|_n \right)_{\ell_2},$$

that is

$$Z = \left\{ (w_1, w_2, \ldots) \in W^\mathbb{N} : \sum_{n=1}^\infty \|w_n\|_n^2 < \infty \right\},$$

and norm $\| \cdot \|$ on $Z$ defined by

$$\|(w_1, w_2, \ldots)\| = \left( \sum_{n=1}^\infty \|w_n\|_n^2 \right)^{1/2}.$$

Let $Y \subset Z$ be the subspace

$$\{(w, w, \ldots) \in Z : w \in W\}$$

and define $T : Y \to W$ by $T((w, w, \ldots)) = w$.

A.1 LEMMA ([8, Lemma 1.2.2]): The map $T$ is linear, injective and continuous and $T^*W^*$ is dense in $Y^*$.

Proof. Clearly $T$ is linear and injective. The continuity of $T$ follows from the fact that $\| \cdot \|_1$ is equivalent to $\| \cdot \|$. Let $y^* \in Y^*$ be given. By the Hahn–Banach theorem there is $z^* \in Z^*$ such that $z^* \restriction Y = y^*$. Using the definition of $Z$ we can find elements $\xi_n \in W^*$ such that $\sum_{n=1}^\infty \|\xi_n\|_n^2 < \infty$ (here we
use the same symbol for a norm and its dual norm), and such that for every $y = (w, w, \ldots) \in B_Y$,

$$\langle y^*, y \rangle = \langle z^*, y \rangle = \langle z^*, (w, w, \ldots) \rangle = \sum_{n=1}^{\infty} \langle \xi_n, w \rangle.$$ 

It follows that for $m = 1, 2, \ldots$,

$$\langle y^* - T^* \left( \sum_{n=1}^{m} \xi_n \right), y \rangle = \langle y^*, y \rangle - \sum_{n=1}^{m} \langle \xi_n, w \rangle$$

$$\leq \sum_{n=m+1}^{\infty} \|\xi_n\||w||_n$$

$$\leq \left( \sum_{n=m+1}^{\infty} \|\xi_n\|^2_n \right)^{1/2} \left( \sum_{n=m+1}^{\infty} ||w||^2_n \right)^{1/2}.$$ 

Thus, as $m \to \infty$

$$\left\| y^* - T^* \left( \sum_{n=1}^{m} \xi_n \right) \right\| \leq \left( \sum_{n=m+1}^{\infty} \|\xi_n\|^2_n \right)^{1/2} \to 0.$$ 

But $T^* \left( \sum_{n=1}^{m} \xi_n \right)$ belongs to $T^*W^*$. Therefore $\overline{T^*W^*} = Y^*$. ■

Now, applying Lemma A.1 to the situation in Lemma 4.9, with $W = C(X)$, we clearly have $Y = V$ and $T = j$. Thus we have that

$$j^*(C(X)^*) = V^*$$

and the assertion of Lemma 4.9(2) follows easily.

**Appendix B. by N. Monod: A negative answer in the case of $\mathcal{B}(H)$**

In the above Eli Glasner proved (among other things) that Question 0.1 has a negative answer in general, including for some weak-* closed subspaces of $\mathcal{B}(H)$, suggesting that 0.1 also has a negative answer for $\mathcal{B}(H) = V^*$.\(^3\)

\(^3\) The predual $V$ for $\mathcal{B}(H)$ is the space of trace-class operators. See the remark following Question 0.1.
B.1 Theorem: Question 0.1 has a negative answer also in the case where \( V^* = \mathcal{B}(H) \).

Proof. We shall use ideas introduced by Bożejko–Fendler [5] in the context of Herz–Schur multipliers and uniformly bounded representations. This leads to a concrete and self-contained construction as follows.

Let \( G \) be the free group on an infinite set \( S \) of generators. For any non-trivial \( x \in G \), denote by \( x_- \) the element obtained by erasing the right-most letter in the reduced word for \( x \). The Hilbert space \( H \) for Question 0.1 will be \( H = \ell^2(G) \) with Hilbert basis \((\delta_x)_{x \in G}\). The \( G \)-action on \( \mathcal{B}(H) \) is the adjoint representation associated to the left regular representation \( \lambda \) on \( \ell^2(G) \). That is, for \( T \in \mathcal{B}(H) \) and \( g \in G \), we define

\[
g.T = \lambda(g) \circ T \circ \lambda(g)^{-1}.
\]

This is indeed isometric and induced by an isometric representation on the predual \( V \). Below, all norms are understood in \( V^* = \mathcal{B}(H) \) unless specified otherwise with a subscript such as at \( \| \cdot \|_{\ell^2} \).

Given \( n > 0 \), we choose a subset \( S_n \subset S \) of \( n \) generators. We define \( T_n \in \mathcal{B}(H) \) by requiring \( T_n\delta_x = \delta_{x_-} \) if \( x^{-1}x \in S_n^{\pm 1} \) (with \( x \neq e \)) and \( T_n\delta_x = 0 \) in all other cases; this defines a bounded operator.

Fix \( g \in G \). Then \( T_n \) almost commutes with \( \lambda(g) \) because \( (gx)_- = g(x_-) \) holds unless \( x \) is completely cancelled by \( g \) (or \( x \) is already trivial). Let us compute the commutator when this complete cancellation occurs. Write \( g \) in reduced form as \( g = s_1^{\epsilon_1} \cdots s_k^{\epsilon_k} \) with \( k \in \mathbb{N} \), \( s_i \in S \) and \( \epsilon_i = \pm 1 \) (we can assume \( k > 0 \)). Then \( x = s_k^{-\epsilon_k} \cdots s_{h+1}^{\epsilon_{h+1}} \) for some \( h < k \). Thus \( gx = s_1^{\epsilon_1} \cdots s_h^{\epsilon_h} \). Now we have

\[
(T_n \circ \lambda(g) - \lambda(g) \circ T_n)\delta_x = \delta_{s_1^{\epsilon_1} \cdots s_{h-1}^{\epsilon_{h-1}}} - \delta_{s_1^{\epsilon_1} \cdots s_{h+1}^{\epsilon_{h+1}}}
\]

provided \( s_h \) and \( s_{h+1} \) are in \( S_n \) (otherwise the corresponding term simply does not occur). This implies the estimate

\[
\|T_n \circ \lambda(g) - \lambda(g) \circ T_n\| \leq 2 \ \forall \ g \in G \ \forall \ n
\]

because, \( g \) being fixed, only one \( x \in G \) gives rise to the pair of elements \( s_1^{\epsilon_1} \cdots s_{h-1}^{\epsilon_{h-1}} \) and \( s_1^{\epsilon_1} \cdots s_{h+1}^{\epsilon_{h+1}} \). In other words, this commutator is a sum of two partial isometries. (A more high-tech argument with Riesz–Thorin interpolation can be found in [5, §2].)
At this point we have elements \( T_n \in V^* = \mathcal{B}(H) \) with \( \| g.T_n - T_n \| \leq 2 \) for all \( g \in G \). We now examine the distance

\[
d(T_n, V^*G) := \inf \{ \| T_n - \beta \| : \beta \text{ is } G\text{-fixed} \}
\]

from \( T_n \) to any \( G \)-fixed point in \( \mathcal{B}(H) \).

We have \( T_n \mathbf{1}_{S_n^\pm} = 2n\delta_e \) and use that \( \beta \) commutes with \( \lambda \) to compute

\[
2n = \| T_n \mathbf{1}_{S_n^\pm} \|_{\ell^2} \leq \| \beta \mathbf{1}_{S_n^\pm} \|_{\ell^2} + \| T_n - \beta \| \cdot \| \mathbf{1}_{S_n^\pm} \|_{\ell^2}
\]

\[
= \| \lambda(\mathbf{1}_{S_n^\pm}) \beta \delta_e \|_{\ell^2} + \sqrt{2n} \| T_n - \beta \|
\]

\[
\leq \| \lambda(\mathbf{1}_{S_n^\pm}) \| \cdot \| \beta \delta_e \|_{\ell^2} + \sqrt{2n} \| T_n - \beta \|
\]

\[
\leq \| \lambda(\mathbf{1}_{S_n^\pm}) \| \cdot \| T_n - \beta \| + \sqrt{2n} \| T_n - \beta \|,
\]

where at the end we used \( T_n \delta_e = 0 \). The norm \( \| \lambda(\mathbf{1}_{S_n^\pm}) \| \) is well-known to be \( 2\sqrt{2n-1} \), since it is a spectral radius that can be computed by enumerating paths in a tree, see [21, Theorem 3]. To be very precise: we can consider \( H = \ell^2(G) \) as a multiple of the regular representation of the subgroup \( F_n \) generated by \( S_n \) and Kesten’s computation takes place in this \( \ell^2(F_n) \); the resulting norm coincides with \( \| \lambda(\mathbf{1}_{S_n^\pm}) \| \) in \( \ell^2(G) \). Now the above inequalities imply

\[
d(T_n, V^*G) > \frac{\sqrt{2n}}{3} \quad \forall n.
\]

We have therefore our answer to the original question by defining \( \alpha \in V^* = \mathcal{B}(H) \) to be the unit vector corresponding to \( T_n \) and letting \( n \to \infty \). ■

References

[1] U. Bader, R. Boutonnet, C. Houdayer and J. Peterson, Charmenability of arithmetic groups of product type, https://arxiv.org/abs/2009.09952.
[2] U. Bader, T. Gelander and N. Monod, A fixed point theorem for \( L^1 \) spaces, Inventiones Mathematicae 189 (2012), 143–148.
[3] N. Bourbaki, Espaces vectoriels topologiques. Chapitres 1 à 5, Éléments de mathématique, Masson, Paris, 1981.
[4] J. Bourgain, D. H. Fremlin and M. Talagrand, Pointwise compact sets in Baire-measurable functions, American Journal of Mathematics 100:4 (1977), 845–886.
[5] M. Bożejko and G. Fendler, Herz–Schur multipliers and uniformly bounded representations of discrete groups, Archiv der Mathematik 57 (1991), 290–298.
[6] W. J. Davis, T. Figiel, W. B. Johnson and A. Pelczyński, Factoring weakly compact operators, Journal of Functional Analysis 17 (1974), 311–327.
[7] D. van Dulst, Characterization of Banach Spaces not Containing $l^1$, CWI Tract, Vol. 59, Stichting Mathematisch Centrum, Centrum voor Wiskunde en Informatica, Amsterdam, 1989.

[8] M. Fabian, Gâteaux Differentiability of Convex Functions and Topology, Canadian Mathematical Society Series of Monographs and Advanced Texts, John Wiley & Sons, New York, 1997.

[9] M. Fabian, P. Habala, P. Hájek, V. Montesinos Santalucía, J. Pelant and V. Zizler, Functional Analysis and Infinite-Dimensional Geometry, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, Vol. 8, Springer, New York, 2001.

[10] H. Furstenberg and E. Glasner, Stationary dynamical systems, in Dynamical Numbers—Interplay Between Dynamical Systems and Number Theory, Contemporary Mathematics, Vol. 532, American Mathematical Society, Providence, RI, 2010, pp. 1–18.

[11] S. Glasner, Proximal Flows, Lecture Notes in Mathematics, Vol. 517, Springer, Berlin–New York, 1976.

[12] E. Glasner, On a question of Kazhdan and Yom Din, https://arxiv.org/abs/2111.07312v1.

[13] E. Glasner and M. Megrelishvili, Hereditarily non-sensitive dynamical systems and linear representations, Colloquium Mathematicum 104 (2006), 223–283.

[14] E. Glasner and M. Megrelishvili, Representations of dynamical systems on Banach spaces not containing $\ell_1$, Transactions of the American Mathematical Society 364 (2012), 6395–6424.

[15] E. Glasner and M. Megrelishvili, On fixed point theorems and nonsensitivity, Israel Journal of Mathematics 190 (2012), 289–305.

[16] E. Glasner and M. Megrelishvili, Banach representations and affine compactifications of dynamical systems, in Asymptotic Geometric Analysis, Fields Institute Communications, Vol. 68, Springer, New York, 2013, pp. 75–144.

[17] E. Glasner and M. Megrelishvili, Representations of dynamical systems on Banach spaces, in Recent Progress in General Topology. III, Atlantis Press, Paris, 2014, pp. 399–470.

[18] E. Glasner and M. Megrelishvili, Todorcević' trichotomy and a hierarchy in the class of tame dynamical systems, Transactions of the American Mathematical Society, to appear, https://arxiv.org/abs/2011.04376.

[19] E. Glasner, T. Tsankov, B. Weiss and A. Zucker, Bernoulli disjointness, Duke Mathematical Journal 170 (2021), 615–651.

[20] D. Kerr and H. Li, Independence in topological and $C^*$-dynamics, Mathematische Annalen 338 (2007), 869–926.

[21] H. Kesten, Symmetric random walks on groups, Transactions of the American Mathematical Society 92 (1959), 336–354.

[22] A. Kühler, Enveloping semigroups for flows, Proceedings of the Royal Irish Academy. Section A. Mathematical and Physical Sciences 95, (1995), 179–191.

[23] V. Losert, The derivation problem for group algebras, Annals of Mathematics 168 (2008), 221–246.
[24] M. Megrelishvili, *Fragmentability and representations of flows*, Topology Proceedings 27 (2003), 497–544.

[25] I. Namioka and R. R. Phelps, *Banach spaces which are Asplund spaces*, Duke Mathematical Journal 42 (1975), 735–750.

[26] E. Odell and H. P. Rosenthal, *A double-dual characterization of separable Banach spaces containing \( l^1 \)*, Israel Journal of Mathematics 20 (1975), 375–384.

[27] S. Sakai, *C*-algebras and \( W^* \)-algebras*, Classics in Mathematics, Springer, Berlin, 1998.

[28] M. Takesaki, *Theory of Operator Algebras. I*, Encyclopaedia of Mathematical Sciences, Vol. 124, Springer, Berlin, 2002.

[29] W. A. Veech, *A fixed point theorem-free approach to weak almost periodicity*, Transactions of the American Mathematical Society 177 (1973), 353–362.

[30] K. Yosida and E. Hewitt, *Finitely additive measures*, Transactions of the American Mathematical Society 72 (1952), 46–66.