Spectral extremal graphs for intersecting cliques

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Abstract

The \((k, r)\)-fan is the graph consisting of \(k\) copies of the complete graph \(K_r\) which intersect in a single vertex, and is denoted by \(F_{k,r}\). Erdős, Füredi, Gould and Gunderson [J. Combin. Theory Ser. B 64 (1995) 89–100] determined the maximum number of edges in an \(n\)-vertex graph that does not contain \(F_{k,r}\) as a subgraph. Furthermore, Chen, Gould, Pfender and Wei [J. Combin. Theory Ser. B 89 (2003) 159–171] proved the analogous result on \(F_{k,r}\) for the general case \(r \geq 3\). In this paper, we show that for sufficiently large \(n\), the graphs of order \(n\) that contain no copy of \(F_{k,r}\) and attain the maximum spectral radius are also edge-extremal. That is, such graphs must have \(\text{ex}(n, F_{k,r})\) edges.

Key words: Spectral radius; Intersecting cliques; Extremal graph; Stability method.

1 Introduction

In this paper, we consider only simple and undirected graphs. Let \(G\) be a simple connected graph with vertex set \(V(G) = \{v_1, \ldots, v_n\}\) and edge set \(E(G) = \{e_1, \ldots, e_m\}\). For a vertex \(v \in V(G)\), we write \(N(v)\) for the set of neighbors of \(v\). Let \(d(v)\) be the degree of a vertex \(v\) in \(G\). That is, \(d(v) = |N(v)|\). Let \(S\) be a set of vertices. We write \(N_S(v)\) for the set of neighbors of \(v\) in the set \(S\), and \(d_S(v)\) for the number of neighbors of \(v\) in the set \(S\), that is, \(d_S(v) = |N_S(v)| = |N(v) \cap S|\). And we denote by \(e(S)\) the number of edges contained in \(S\).

The main tasks in extremal graph theory are to maximize or minimize a graph parameter over a specific family of graphs. The Turán number of a graph \(F\) is the maximum number of edges that may be in an \(n\)-vertex graph without a subgraph isomorphic to \(F\), and this quantity is usually denoted by \(\text{ex}(n, F)\). We say that a graph \(G\) is \(F\)-free if it does not contain a subgraph isomorphic to \(F\), i.e., \(G\) contains no copy of \(F\). A graph on \(n\) vertices with no subgraph \(F\) and with \(\text{ex}(n, F)\) edges is called an extremal graph for \(F\) and we denote by \(\text{Ex}(n, F)\) the set of all extremal graphs on \(n\) vertices for \(F\). It is a cornerstone of extremal graph theory to investigate both \(\text{ex}(n, F)\) and \(\text{Ex}(n, F)\) for various graphs \(F\); see [20] [21] [33] for related surveys.

Dating back to 1941, Turán [35] first raised the natural question of determining \(\text{ex}(n, K_{r+1})\) where \(K_{r+1}\) is the complete graph on \(r + 1\) vertices. Let \(T_r(n)\) denote the complete \(r\)-partite graph on \(n\) vertices whose part sizes are as equal as possible, i.e., each part has size \([n/r]\) or \([n/r] + 1\). Turán [35] extended a result of Mantel [24] and obtained that if \(G\) is an \(n\)-vertex graph containing no \(K_{r+1}\), then \(e(G) \leq e(T_r(n))\), equality holds if and only if \(G = T_r(n)\). There are many extensions and generalizations of Turán’s result; see, e.g., [4] p. 294. After this result, the problem of determining \(\text{ex}(n, F)\) is usually called the Turán-type extremal problem. The most celebrated result is a theorem of Erdős, Stone and Simonovits [13] [12], which states that

\[
\text{ex}(n, F) = \left(1 - \frac{1}{\chi(H) - 1}\right) \frac{n^2}{2} + o(n^2),
\]

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where \( \chi(F) \) is the vertex-chromatic number of \( H \). This provides good asymptotic estimates for the extremal numbers of non-bipartite graphs. However, for bipartite graphs, where \( \chi(F) = 2 \), it only gives the bound \( \text{ex}(n, F) = o(n^2) \).

The history of studying bipartite graphs began in 1954 with the Kővári–Sós–Turán theorem [22], which asserts that if \( K_{s,t} \) is the complete bipartite graph with vertex classes of size \( s \geq t \), then \( \text{ex}(n, K_{s,t}) = O(n^{2-1/4}) \); see [17, 18] for more details. Although there have been numerous attempts to find better bounds of \( \text{ex}(n, F) \) for various bipartite graphs \( F \), we know very little in this case. We refer the interested reader to the comprehensive survey by Füredi and Simonovits [20].

1.1 Background and motivation

In this section, we shall review the exact value of \( \text{ex}(n, F) \) for some special graphs \( F \), instead of the asymptotic estimation. A graph on \( 2k+1 \) vertices consisting of \( k \) triangles which intersect in exactly one common vertex is called a \( k \)-fan (also known as the friendship graph) and denoted by \( F_k \). Since \( \chi(F_k) = 3 \), the Erdős–Stone–Simonovits theorem in [1] implies that \( \text{ex}(n, F_k) = n^2/4 + o(n^2) \). In 1995, Erdős, Füredi, Gould and Gunderson [14] proved the following exact result.

**Theorem 1.1** (Erdős et al. [14]). For every \( k \geq 1 \), and for every \( n \geq 50k^2 \), we have

\[
\text{ex}(n, F_k) = \left\lfloor \frac{n^2}{4} \right\rfloor + \begin{cases} 
    k^2 - k, & \text{if } k \text{ is odd,} \\
    k^2 - \frac{3}{2}k, & \text{if } k \text{ is even.}
\end{cases}
\]

A graph on \( (r-1)k + 1 \) vertices consisting of \( k \) cliques each with \( r \) vertices, which intersect in exactly one common vertex, is called a \( (k, r) \)-fan and denoted by \( F_{k,r} \). Clearly, when \( r = 3 \), \( F_{k,3} \) reduces to the general \( k \)-fan graph \( F_k \). Note that \( \chi(F_{k,r}) = r \). Similarly, the Erdős–Stone–Simonovits theorem also implies that \( \text{ex}(n, F_{k,r}) = (1 - \frac{1}{r})n^2 + o(n^2) = t_{r-1}(n) + o(n^2) \). In 2003, Chen, Gould, Pfender and Wei [5] proved an exact answer and generalized Theorem 1.1 as follows.

**Theorem 1.2** (Chen et al. [5]). For every \( k \geq 1 \) and \( r \geq 2 \), if \( n \geq 16k^3r^3 \), then

\[
\text{ex}(n, F_{k,r}) = t_{r-1}(n) + \begin{cases} 
    k^2 - k, & \text{if } k \text{ is odd,} \\
    k^2 - \frac{3}{2}k, & \text{if } k \text{ is even.}
\end{cases}
\]

The extremal graphs of Theorem 1.2, denoted by \( G_{n,k,r} \), are constructed by taking the \( (r-1) \)-partite Turán graph \( T_{r-1}(n) \) and embedding a graph \( G_0 \) in one vertex part. If \( k \) is odd, \( G_0 \) is isomorphic to two vertex disjoint copies of \( K_k \). If \( k \) is even, \( G_0 \) may be isomorphic to any graph with \( 2k-1 \) vertices, \( k^2 - \frac{3}{2}k \) edges with maximum degree \( k-1 \).

1.2 Spectral extremal problem

The adjacency matrix of \( G \) is defined as \( A(G) = (a_{ij})_{n \times n} \) with \( a_{ij} = 1 \) if two vertices \( v_i \) and \( v_j \) are adjacent in \( G \), and \( a_{ij} = 0 \) otherwise. The spectral radius of \( A(G) \) is defined as the largest value among the absolute values of eigenvalues of \( A(G) \). Note that the spectral radius is not necessarily an eigenvalue. The celebrated Perron–Frobenius theorem implies that the spectral radius of \( A(G) \) is a largest eigenvalue since \( A(G) \) is nonnegative matrix. The eigenvalues of a graph \( G \) are defined as the eigenvalues of adjacency matrix \( A(G) \). We write \( \lambda(G) \) or \( \lambda_1(G) \) for the spectral radius of \( G \). The spectral radius of a graph may at times give some information about the structure of graphs. For example, it is well-known that \( \lambda(G) \) is located between the average degree and the maximum degree of \( G \), and the vertex-chromatic number is at most \( \lambda(G) + 1 \); see [3] p. 34] for more details.

In this paper we consider spectral analogues of Turán-type problems for graphs, that is, determining the maximum value of eigenvalues instead of the number of edges among all \( n \)-vertex \( F \)-free graphs. We denote

\[
\text{ex}_{sp}(n, F) = \max \{ \lambda(G) : |G| = n, G \text{ is } F\text{-free} \}.
\]

These problems are commonly based on the techniques applying the eigenvalues or eigenvectors of a graph. The fundamental inequality \( 2e(G)/n \leq \lambda(G) \) yields the following relation:

\[
\text{ex}(n, F) \leq \frac{n}{2}\text{ex}_{sp}(n, F).
\]
The problems of studying \( \text{ex}_{sp}(n, F) \) has a rapid development in spectral extremal graph theory recently. For most graphs, this study is again fairly complete due to large part to a longstanding work of Nikiforov \[31\]. For example, he extended the classical theorem of Turán, by determining the maximum spectral radius of any \( K_{r+1} \)-free graph \( G \) on \( n \) vertices.

The following problem regarding the adjacency spectral radius was proposed in \[24\]: What is the maximum spectral radius of a graph \( G \) on \( n \) vertices without a subgraph isomorphic to a given graph \( F \)? Wilf \[36\] and Nikiforov \[24\] obtained spectral strengthening of Turán’s theorem when the forbidden substructure is the complete graph. Soon after, Nikiforov \[25\] showed that if \( G \) is a \( K_{r+1} \)-free graph on \( n \) vertices, then \( \lambda(G) \leq \lambda(T_r(n)) \), with equality if and only if \( G = T_r(n) \). Moreover, Nikiforov \[25\] and Zhai and Wang \[38\] determined the maximum spectral radius of \( K_{2,2} \)-free graphs. Furthermore, Nikiforov \[27\], Babai and Guiduli \[2\] independently obtained the spectral generalization of the Kővari-Sós-Turán theorem when the forbidden graph is the complete bipartite graph \( K_{s,t} \). Finally, Nikiforov \[28\] characterized the spectral radius of graphs without paths and cycles of specified length. In addition, Fiedler and Nikiforov \[16\] obtained tight sufficient conditions for graphs to be Hamiltonian or traceable. For many other spectral analogues of results in extremal graph theory we refer the reader to the survey \[31\]. It is worth mentioning that a corresponding spectral strengthening \[29\] of the Erdős–Stone–Simonovits theorem states that

\[
\text{ex}_{sp}(n, F) = \left( 1 - \frac{1}{\chi(F) - 1} \right) n + o(n).
\]

From this result, we know that \( \text{ex}_{sp}(n, F_k) = \frac{n}{2} + o(n) \) where \( F_k \) is the \( k \)-fan graph. Recently, Cioabă, Feng, Tait and Zhang \[8\] generalized this bound by improving the error term \( o(n) \) to \( O(1) \), and obtained a spectral counterpart of Theorem \[1.1\]. More precisely, they showed that the extremal graphs that attain the maximum spectral radius in a graph on \( n \) vertices containing no copy of \( k \)-fan must be in \( \text{Ex}(n, F_k) \) for \( n \) sufficiently large.

**Theorem 1.3** (Cioabă et al. \[8\]). Let \( G \) be a graph of order \( n \) that does not contain a copy of a \( k \)-fan, \( k \geq 2 \). For sufficiently large \( n \), if \( G \) has the maximal spectral radius, then

\[
G \in \text{Ex}(n, F_k).
\]

Recall that \( F_{k,r} \) is the graph consisting of \( k \) cliques of order \( r \) which intersect in exactly one common vertex. In this paper, we shall prove the following theorem, which is an extension of Theorem \[1.3\].

**Theorem 1.4** (Main result). Let \( G \) be a graph of order \( n \) that does not contain a copy of \( F_{k,r} \), where \( k \geq 1 \) and \( r \geq 2 \). For sufficiently large \( n \), if \( G \) has the maximal spectral radius, then

\[
G \in \text{Ex}(n, F_{k,r}).
\]

Our theorem is a spectral result of the Turán extremal problem for \( F_{k,r} \), it not only can be viewed as an extension of Theorem \[1.3\], but also a spectral analogue of Theorem \[1.2\]. Our treatment strategy of the proof is mainly based on the stability method. To some extent, this paper could be regarded as a continuation and development of \[8\]. However, we highlight that there are some differences in the approach compared from \[8\]. In \[8\], the extremal graph is constant edit distance from a bipartite graph. One of the key steps is to show that the extremal graph has a large bipartite subgraph. To do so, the authors prove a lemma \((8\) Lemma 7\) that relates the number of edges to the spectral radius and the number of triangles in the graph and then use the triangle removal lemma and a stability theorem of Füredi \[19\]. Unfortunately, for this problem the extremal graph is constant edit distance from an \((r-1)\)-partite graph and for \( r \geq 3 \) the same approach fails. Instead we used a spectral stability theorem of Nikiforov (See Section \[3\]).

**Remark.** The ideas of this paper were developed independently and simultaneously by two groups. Since the arguments of our two papers were similar, we present them as a joint work.

## 2 Some Lemmas

In this section, we state some lemmas which are needed in our proof.

**Lemma 2.1** (Nikiforov \[30\]). Let \( r \geq 2, 1/\ln n < c < r^{-8(r+21)(r+1)}, 0 < \varepsilon < 2^{-36r^{-24}} \) and \( G \) be a graph on \( n \) vertices. If \( \lambda(G) > (1 - \frac{1}{r} - \varepsilon)n \), then one of the following statements holds:

(a) \( G \) contains a \( K_{r+1}([c \ln n], \ldots, [c \ln n], [n^{1-\sqrt{r}}]) \);

(b) \( G \) differs from \( T_r(n) \) in fewer than \((\varepsilon^{1/4} + c^{1/(8r+8)})n^2\) edges.
From the above theorem, one can easily get the following spectral analogue of the classical Erdős-Simonovits stability theorem \[32, 19\].

**Corollary 2.2.** Let $F$ be a graph with chromatic number $\chi(F) = r + 1$. For every $\varepsilon > 0$, there exist $\delta > 0$ and $n_0$ such that if $G$ is an $F$-free graph on $n \geq n_0$ vertices with $\lambda(G) \geq (1 - \frac{1}{r} - \delta)n$, then $G$ can be obtained from $T_r(n)$ by adding and deleting at most $\varepsilon n^2$ edges.

Let $G$ be a simple graph with matching number $\beta(G)$ and maximum degree $\Delta(G)$. For given two integers $\beta$ and $\Delta$, define $f(\beta, \Delta) = \max\{e(G) : \beta(G) \leq \beta, \Delta(G) \leq \Delta\}$.

In 1976, Chvátal and Hanson \[7\] obtained the following result.

**Lemma 2.3** (Chvátal-Hanson \[7\]). For every two integers $\beta \geq 1$ and $\Delta \geq 1$, we have

$$f(\beta, \Delta) = \Delta \beta + \left\lfloor \frac{\Delta}{2} \right\rfloor + \left\lfloor \frac{\beta}{\Delta/2} \right\rfloor \leq \Delta \beta + \beta.$$

We will frequently use a special case proved by Abbott, Hanson and Sauer \[1\]:

$$f(k - 1, k - 1) = \begin{cases} k^2 - k, & \text{if } k \text{ is odd}, \\ k^2 - \frac{k^2}{2}, & \text{if } k \text{ is even}. \end{cases}$$

Furthermore, the extremal graphs attaining the equality case are exactly those we embedded into the Turán graph $T_{r - 1}(n)$ to obtain the extremal $F_{k,r}$-free graph.

Denote by $K_{n_1, n_2, \ldots, n_{r - 1}}$ the complete $(r - 1)$-partite graph on $n = \sum_{i=1}^{r-1} n_i$ vertices. For convenience, we assume that $n_1 \geq n_2 \geq \ldots \geq n_{r - 1} > 0$. It is well-known \[10, p. 74\] or \[11\] that the characteristic polynomial of $K_{n_1, n_2, \ldots, n_{r - 1}}$ is given as

$$\phi(K_{n_1, n_2, \ldots, n_{r - 1}}) = x^{n - r + 1} - \frac{\sum_{i=1}^{r-1} n_i}{x + n_i} \prod_{j=1}^{r-1} (x + n_j).$$

So the spectral radius $\lambda(K_{n_1, n_2, \ldots, n_{r - 1}})$ satisfies the following equation:

$$\sum_{i=1}^{r-1} \frac{n_i}{\lambda(K_{n_1, n_2, \ldots, n_{r - 1}}) + n_i} = 1 \quad \text{for all } i \in [r]. \tag{2}$$

Feng, Li and Zhang \[15\] Theorem 2.1] proved implicitly the following lemma, which can also be seen in Stevanović, Gutnam and Rehman \[34\].

**Lemma 2.4** (Feng et al. \[15\], Stevanović et al. \[34\]). If $n_i - n_j \geq 2$, then

$$\lambda(K_{n_1, \ldots, n_1 - 1, \ldots, n_j + 1, \ldots, n_{r - 1}}) > \lambda(K_{n_1, \ldots, n_j, \ldots, n_{r - 1}}).$$

For a connected graph $G$ on $n$ vertices, let $x = (x_1, \ldots, x_n)^T$ be an eigenvector of $A(G)$ corresponding to $\lambda(G)$. By the celebrated Perron–Frobenius theorem, we can choose $x$ as a positive real vector.

$$\lambda(G)x_i = \sum_{j=1}^{n} a_{ij}x_j = \sum_{j \in N_G(i)} x_j \text{ for any } i \in [n]. \tag{3}$$

Another useful result concerns the Rayleigh quotient:

$$\lambda(G) = \max_{x \in \mathbb{R}^n} \frac{x^T A(G)x}{x^T x} = \max_{x \in \mathbb{R}^n} \frac{2 \sum_{\{i,j\} \in E(G)} x_i x_j}{x^T x}. \tag{4}$$

Let $G$ be a graph with a partition of the vertices into $r - 1$ non-empty parts $V(G) = V_1 \cup V_2 \cup \ldots \cup V_{r - 1}$. Let $E_{cr}(G) = \bigcup_{1 \leq i < j \leq r - 1} E(V_i, V_j)$ be the crossing edges of $G$. The following lemma was proved in Chen et al. \[5\].
Lemma 2.5 (Chen et al. [5]). Suppose \( G \) is partitioned as above so that the following conditions are satisfied

\[
\sum_{j \neq i} \beta(G[V_j]) \leq k - 1 \quad \text{and} \quad \Delta(G[V_i]) \leq k - 1, \tag{5}
\]

\[
d_{G[V_i]}(v) + \sum_{j \neq i} \beta(G[N(v) \cap V_j]) \leq k - 1, \tag{6}
\]

for any \( i \in [r-1] \) and \( v \in V_i \). If \( G \) is \( F_{k,r} \)-free, then

\[
\sum_{i=1}^{r-1} |E(G[V_i])| - \left( \sum_{1 \leq i < j \leq r-1} |V_i||V_j| - |E_{rr}(G)| \right) \leq f(k-1, k-1).
\]

3 Proof of Theorem 1.4

In the sequel, we always assume that \( G \) is a graph on \( n \) vertices containing no \( F_{k,r} \) as a subgraph and attaining the maximum spectral radius. The aim of this section is to prove that \( e(G) = \text{ex}(n, F_{k,r}) \) for \( n \) large enough.

First of all, we note that \( G \) must be connected since adding an edge between different components will increase the spectral radius and also keep \( G \) being \( F_{k,r} \)-free. Let \( \lambda(G) \) be the spectral radius of \( G \). By the Perron–Frobenius Theorem, we know that \( \lambda(G) \) has an eigenvector with all entries being positive, we denote such an eigenvector by \( \mathbf{x} \).

For a vertex \( v \in V(G) \), we will write \( x_v \) for the eigenvector entry of \( \mathbf{x} \) corresponding to \( v \). We may normalize \( \mathbf{x} \) so that it has maximum entry equal to 1, and let \( z \) be a vertex such that \( x_z = 1 \). If there are multiple such vertices, we choose and fix \( z \) arbitrarily among them.

In the sequel, we shall prove Theorem 1.4 iteratively, giving successively more precise estimates on both the structure of \( G \) and the eigenvector entries of the vertices, until finally we can show that \( e(G) = \text{ex}(n, F_{k,r}) \).

The proof of Theorem 1.4 is outlined as follows.

- We apply Corollary 2.2 to give a lower bound \( e(G) \geq t_{r-1}(n) - o(n^2) \). Moreover, \( G \) has a very large multipartite subgraph on parts \( V_1, \ldots, V_{r-1} \) such that \( \frac{n}{r} - o(n) \leq |V_i| \leq \frac{n}{r} + o(n) \); see Lemma 3.2.

- We show that the number of vertices that have \( \Omega(n) \) neighbors in their own part is bounded by \( o(n) \), and the number of vertices that have degree less than \( (\frac{n}{r} - o(1))n \) is also bounded by \( o(n) \); see Lemmas 3.3 and 3.4 respectively. Furthermore, we will prove that such vertices do not exist, and each \( G[V_i] \) is \( K_{1,k} \)-free and \( M_k \)-free; see Lemmas 3.5 and 3.6.

- Based on the previous lemmas, we shall refine the structure of \( G \), and show that almost all vertices in \( V_i \) are adjacent to every vertex in \( V_i \)\( ^c \), implying the presence of a large complete \((r-1)\)-partite subgraph in \( G \); see Lemma 3.9. Moreover, we shall prove that \( x_u = 1 - o(1) \) for every \( u \in V(G) \); see Lemma 3.10.

- Once we know that all vertices have eigenvector entry close to 1, we can show that the \((r-1)\)-partition is balanced; see Lemma 3.11. Invoking these facts, we finally show that \( e(G) = \text{ex}(n, F_{k,r}) \).

Lemma 3.1. Let \( G \) be an \( F_{k,r} \)-free graph on \( n \) vertices with maximum spectral radius. Then

\[
\lambda(G) \geq \left( 1 - \frac{1}{r-1} \right)n - \frac{r-1}{4n}.
\]

Proof. Let \( H \) be an \( F_{k,r} \)-free graph on \( n \) vertices with maximum number of edges. Since \( G \) is the graph maximizing the spectral radius over all \( F_{k,r} \)-free graphs, in view of Theorem 1.2, we can see by the Rayleigh quotient that

\[
\lambda(G) \geq \lambda(H) \geq \frac{1^T A(H) 1}{1^T 1} = \frac{2(t_{r-1}(n) + f(k-1, k-1))}{n}.
\]

Note that \( t_{r-1}(n) \geq (1 - \frac{1}{r-1})n^2 - \frac{r-1}{r}n \), so we have \( \lambda(G) \geq (1 - \frac{1}{r-1})n - \frac{r-1}{4n} \). \( \square \)

Applying Lemma 3.1 and Corollary 2.2, we obtain the asymptotic structure of \( G \). Roughly speaking, we can find a large \((r-1)\)-partite subgraph in \( G \).
Lemma 3.2 (Approximate structure). Let $G$ be an $F_{k,r}$-free graph on $n$ vertices with maximum spectral radius. For every $\epsilon > 0$, there is an integer $n_0$ such that if $n \geq n_0$, then

$$e(G) \geq t_{r-1}(n) - \epsilon n^2.$$  

Furthermore, there exists $\epsilon_1 = \sqrt{\epsilon}b$ such that $G$ has a maximum $(r-1)$-cut $V = V_1 \cup \ldots \cup V_{r-1}$ with

$$\sum_{1 \leq i < j \leq r-1} e(V_i, V_j) \geq t_{r-1}(n) - \epsilon n^2,$$

and for each $i \in [r-1]$,

$$\left(\frac{1}{r-1} - \epsilon_1\right) n \leq |V_i| \leq \left(\frac{1}{r-1} + \epsilon_1\right) n.$$

Proof. As suggested above, it follows from Lemma 2.1 and Corollary 2.2 that for any given $\epsilon > 0$, we can take a large enough $n$ such that $e(G) \geq t_{r-1}(n) - \epsilon n^2$. The same results also provide that there is a partition of $V(G) = U_1 \cup \ldots \cup U_{r-1}$ with $\sum_{i=1}^{r-1} e(U_i) \leq \epsilon n^2$, $\sum_{1 \leq i < j \leq r-1} e(U_i, U_j) \geq t_{r-1}(n) - \epsilon n^2$ and $\sum_{i=1}^{r-1} |e(U_i)| \leq \left| \frac{\epsilon}{r-1} \right|$ for each $i \in [r-1]$. Thus, any maximum $(r-1)$-cut of $V = V_1 \cup \ldots \cup V_{r-1}$ must have $\sum_{i=1}^{r-1} e(V_i) \leq \sum_{i=1}^{r-1} e(U_i) \leq \epsilon n^2$ and $\sum_{1 \leq i < j \leq r-1} e(V_i, V_j) \geq \sum_{1 \leq i < j \leq r-1} e(U_i, U_j) \geq t_{r-1}(n) - \epsilon n^2$.

Furthermore, since $G$ has edit distance at most $\epsilon n^2$ from some graph isomorphic to $T_{r-1}(n)$, we may let $a = \max \left\{ \left\| V_j \right\| - \frac{n}{r-1}, j \in [r-1] \right\}$. Without loss of generality, we assume $\left\| V_1 \right\| - a = a$. Then

$$e(G) \leq \sum_{1 \leq i < j \leq r-1} |V_i| |V_j| + \sum_{i=1}^{r-1} e(V_i)
\leq |V_1|(n - |V_1|) + \sum_{2 \leq i < j \leq r-1} |V_i| |V_j| + \epsilon n^2
\leq |V_1|(n - |V_1|) + \frac{1}{2} \left( \sum_{j=2}^{r-1} |V_j|^2 - \sum_{j=2}^{r-1} |V_j|^2 \right) + \epsilon n^2
\leq |V_1|(n - |V_1|) + \frac{1}{2} (n - |V_1|)^2 - \frac{1}{2(r-2)} (n - |V_1|)^2 + \epsilon n^2
\leq \frac{r-1}{2(r-2)} a^2 + \frac{r-2}{2(r-1)} n^2 + \epsilon n^2,$$

where the last second inequality holds by Hölder’s inequality, and the last inequality holds since $\left\| V_1 \right\| - \frac{n}{r-1} = a$. On the other hand,

$$e(G) \geq t_{r-1}(n) - \epsilon n^2 \geq (1 - \frac{1}{r-1}) n^2 - \frac{r-1}{8} n^2 - \epsilon n^2 \geq \frac{r-2}{2(r-1)} n^2 - 2\epsilon n^2,$$

as $n$ is large enough. Therefore, $\frac{r-1}{2(r-2)} a^2 < 3\epsilon n^2$, which implies that $a < \sqrt{\frac{3r-2}{r-1} n^2} < \sqrt{6\epsilon n} = \epsilon_1 n$. The proof is completed.

Lemma 3.3. Let $\epsilon$ and $\theta$ be two sufficiently small constants with $\epsilon < \theta^2/3$. We denote

$$W := \cup_{i=1}^{r-1} \{ v \in V_i : |N_G(v) \cap V_i| \geq \theta n \}. \quad (7)$$

For sufficiently large $n$, we have

$$|W| \leq \frac{2\theta}{3} n + \frac{2k^2}{\theta n} < \theta n.$$
Proof. We obtain from Lemma 3.2 that \( \sum_{1 \leq i < j \leq r-1} e(V_i, V_j) \geq t_{r-1}(n) - en^2 \). Hence,

\[
\sum_{i=1}^{r-1} e(V_i) = e(G) - \sum_{1 \leq i < j \leq r-1} e(V_i, V_j) \leq t_{r-1}(n) + k^2 - t_{r-1}(n) + en^2 \leq en^2 + k^2.
\]

On the other hand, invoking the fact that \( e(G) \) exceeds the upper bound on the number of edges in any \( F \)-free graph on \( n \) vertices. Thus, we can get \( d(V_i, V_j) \geq |W_i| = \frac{|W|}{2} \theta n \).

Therefore, we have that \( \frac{|W|}{2} \theta n \leq en^2 + k^2 \). This proves that \( |W| \leq \frac{2\theta}{\theta n} + \frac{2k^2}{\theta n} < \theta n \). □

Lemma 3.4. Let \( k \geq 2 \) and \( \frac{2(r-2)}{r-1} \epsilon < \epsilon_2^2 \). We denote

\[
L := \left\{ v \in V(G) : d(v) \geq \left( 1 - \frac{1}{r-1} - \epsilon_2 \right) n \right\}.
\]

Then \( |L| \leq \epsilon_3 n \), where \( \epsilon_3 \ll \epsilon_2 \) is a sufficiently small constant satisfying \( \frac{r-2}{r-1} \epsilon_3^2 - \epsilon_2 \epsilon_3 + \epsilon < 0 \).

Proof. To prove this, assume to the contrary that the cardinality of \( L \) is greater than \( \epsilon_3 n \). Then there exists a subset \( L' \subseteq L \) with \( |L'| = |\epsilon_3 n| \). Therefore,

\[
e[G \setminus L'] \geq e(G) - \sum_{v \in L'} d(v) \geq t_{r-1}(n) - \epsilon_3 n \left( 1 - \frac{1}{r-1} - \epsilon_2 \right)
\]

\[
> \frac{(n - |L'|)^2}{2} \left( 1 - \frac{1}{r-1} \right) + k^2
\]

\[
\geq t_{r-1}(n - |L'|) + k^2.
\]

However, this is a contradiction as the above lower bound for \( e[G \setminus L] \) exceeds the upper bound on the number of edges in any \( F_{k,r} \)-free graph on \( n - |L'| \) vertices. □

The following lemma was given in [8].

Lemma 3.5 (Cioabă et al. [8]). If \( A_1, \ldots, A_p \) be finite sets, then

\[
|A_1 \cap \ldots \cap A_p| \geq \sum_{i=1}^{p} |A_i| - (p-1) \left| \bigcup_{i=1}^{p} A_i \right|.
\]

Lemma 3.6. Let \( W \) and \( L \) be the sets of vertices defined in (7) and (8). Then \( W \subseteq L \).

Proof. Suppose on the contrary that there exists a vertex \( u_0 \in W \) and \( u_0 \notin L \). Without loss of generality, we may assume that \( u_0 \notin V_1 \). Since \( V_1, \ldots, V_{r-1} \) form a maximum \((r-1)\)-partite subgraph, we have \( d_{V_1}(u_0) \leq d_{V_i}(u_0) \) for each \( i \in [2, r-1] \). Indeed, otherwise, we can move the vertex \( u_0 \) into some part \( V_i \) and strictly increase the number of edges between \( V_1 \) and \( V_i \). Thus, we can get \( d(u_0) \geq (r-1)d_{V_1}(u_0) \), which implies \( d_{V_2}(u_0) \geq d(u_0) - d_{V_1}(u_0) - (r-3)n \left( \frac{1}{r-1} + \epsilon_1 \right) \).

On the other hand, invoking the fact that \( u_0 \notin L \), we get \( d(u_0) > (1 - \frac{1}{r-1} - \epsilon_2)n \). So

\[
d_{V_2}(u_0) \geq \left( 1 - \frac{1}{r-1} \right) d(u_0) - (r-3)n \left( \frac{1}{r-1} + \epsilon_1 \right)
\]

\[
\geq \frac{n}{(r-1)^2} - \frac{r-2}{r-1} \epsilon_2 n - (r-3)\epsilon_1 n.
\]

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Recall from Lemmas 3.3 and 3.4 that \(|W| < \theta n + \epsilon_3 n\). Hence, for fixed \(\theta, \epsilon_3\) and sufficiently large \(n\), we have
\[
|V_i \setminus (W \cup L)| \geq \left(\frac{1}{r-1} - \epsilon_1\right)n - \theta n - \epsilon_3 n \geq k.
\]

**Claim.** \(u_0\) is adjacent to at most \(k - 1\) vertices in \(V_i \setminus (W \cup L)\).

Suppose that \(u_0\) is adjacent to \(k\) vertices \(u_1^{(1)}, u_2^{(1)}, \ldots, u_k^{(1)}\) in \(V_i \setminus (W \cup L)\). Since \(u_j^{(1)} \not\in L\), we have
\[
d(u_j^{(1)}) > \left(1 - \frac{1}{r-1} - \epsilon_2\right)n.
\]
On the other hand, we have \(d_{V_i}(u_j^{(1)}) < \theta n\) because \(u_j^{(1)} \not\in W\). So for each \(j \in [k]\),
\[
d_{V_2}(u_j^{(1)}) \geq d(u_j^{(1)}) - d_{V_i}(u_j^{(1)}) - (r - 3)\left(\frac{1}{r-1} + \epsilon_1\right)n
\geq \frac{n}{r-1} - \epsilon_2 n - \theta n - (r - 3)\epsilon_1 n.
\]
By Lemma 3.5, we consider the common neighbors of \(u_0, u_1^{(1)}, \ldots, u_k^{(1)}\) in \(V_2\),
\[
|N_{V_2}(u_0) \cap N_{V_2}(u_1^{(1)}) \cap \cdots \cap N_{V_2}(u_k^{(1)}) \setminus (W \cup L)|
\geq d_{V_2}(u_0) + \sum_{j=1}^{k} d_{V_2}(u_j^{(1)}) - k|V_2| - |W| - |L|
\geq \frac{n}{(r-1)^2} - \frac{r^2}{r-1}\epsilon_2 n - (r - 3)\epsilon_1 n + k\left(\frac{n}{r-1} - \epsilon_2 n - \theta n - (r - 3)\epsilon_1 n\right) - k\left(\frac{1}{r-1} + \epsilon_1\right)n - \theta n - \epsilon_3 n
\geq \frac{n}{(r-1)^2} - o(n) > k
\]
for sufficiently large \(n\). So there exist \(k\) vertices \(u_1^{(2)}, u_2^{(2)}, \ldots, u_k^{(2)}\) in \(V_2 \setminus (W \cup L)\) such that the subgraph formed by two partitions \(\{u_1^{(1)}, \ldots, u_k^{(1)}\}\) and \(\{u_1^{(2)}, \ldots, u_k^{(2)}\}\) is a complete bipartite graph. It is easy to see that the subgraph of \(G\) formed by the vertex \(u_0\) together with such a complete bipartite graph can contain a copy of \(F_{k,3}\) centered at the vertex \(u_0\). In the sequel, we shall extend this copy to the intersecting cliques \(F_{k,r}\). Let \(s \in [2, r - 2]\) be a positive integer. Assume that we have found the vertices \(u_1^{(i)}, u_2^{(i)}, \ldots, u_k^{(i)}\) in \(V_i \setminus (W \cup L)\), \((i = 1, 2, \ldots, s)\) such that these vertices form a complete \(s\)-partite subgraph in \(G\). We next consider the common neighbors of these vertices in \(V_{s+1}\). Similarly, we get that for each \(i \in [s]\) and \(j \in [k]\),
\[
d_{V_{s+1}}(u_j^{(i)}) \geq d(u_j^{(i)}) - d_{V_i}(u_j^{(i)}) - (r - 3)\left(\frac{1}{r-1} + \epsilon_1\right)n
\geq \frac{n}{r-1} - \epsilon_2 n - \theta n - (r - 3)\epsilon_1 n.
\]
By Lemma 3.5 again, we can obtain
\[
|N_{V_{s+1}}(u_0) \cap \bigcap_{i \in [s], j \in [k]} N_{V_{s+1}}(u_j^{(i)}) \setminus (W \cup L)|
\geq d_{V_{s+1}}(u_0) + \sum_{i \in [s], j \in [k]} d_{V_{s+1}}(u_j^{(i)}) - ks|V_{s+1}| - |W| - |L|
\geq \frac{n}{(r-1)^2} - \frac{r^2}{r-1}\epsilon_2 n - (r - 3)\epsilon_1 n + ks\left(\frac{n}{r-1} - \epsilon_2 n - \theta n - (r - 3)\epsilon_1 n\right) - ks\left(\frac{1}{r-1} + \epsilon_1\right)n - \theta n - \epsilon_3 n
\geq \frac{n}{(r-1)^2} - o(n) > k
\]
Thus we can find \(k\) vertices \(u_1^{(s+1)}, u_2^{(s+1)}, \ldots, u_k^{(s+1)}\) in \(V_{s+1} \setminus (W \cup L)\), which together with the previous vertices \(u_j^{(i)} \in V_i \setminus (W \cup L), (i \in [s], j \in [k])\) form a complete \((s + 1)\)-partite subgraph in \(G\). Thus, for each \(i \in [r - 1]\), we
can find \( k \) vertices from every vertex part \( V_i \setminus (W \cup L) \) such that these vertices together with \( u_0 \) form a copy of \( F_{k,r} \) centered at \( u_0 \), this is a contradiction. Therefore \( u_0 \) is adjacent to at most \( k - 1 \) vertices in \( V_i \setminus (W \cup L) \).

Hence, applying Lemmas 3.3 and 3.4 again, we have

\[
d_{V_i}(u_0) \leq |W| + |L| + k - 1
< 2\theta n + 2k^2 n + \epsilon n + k - 1
< \theta n
\]

for sufficiently large \( n \). This is a contradiction to the fact that \( u_0 \in W \). Hence \( W \subseteq L \).

\[\square\]

**Lemma 3.7.** For each \( i \), there exists an independent set \( I_i \subseteq V_i \) such that

\[|I_i| \geq |V_i| - \epsilon n - k^2.\]

**Proof.** Since \( V_i \setminus L \) is large enough by Lemma 3.4, we first prove that there exists a large complete multipartite subgraph between \( V_1, V_2, \ldots, V_{r-1} \). Let \( u_1^{(1)}, u_2^{(1)}, \ldots, u_{2k}^{(1)} \) be \( 2k \) vertices chosen arbitrarily from \( V_1 \setminus L \). Then \( u_j^{(1)} \not\in L \) which implies that \( d(u_j^{(1)}) > \left(1 - \frac{1}{r-1} - \epsilon_2\right)n \). Note that \( W \subseteq L \) by Lemma 3.6 so \( u_j^{(1)} \not\in W \), then

\[
d_{V_1}(u_j^{(1)}) < \theta n.
\]

Hence

\[
d_{V_2}(u_j^{(1)}) \geq \frac{n}{r-1} - \epsilon_2 n - \theta n - (r-3)\epsilon_1 n.
\]

Furthermore, by Lemma 3.5 we have

\[
\left| N_{V_2}(u_1^{(1)}) \cap N_{V_2}(u_2^{(1)}) \cap \cdots \cap N_{V_2}(u_{2k}^{(1)}) \setminus L \right|
\geq \sum_{j=1}^{2k} d_{V_2}(u_j^{(1)}) - (2k-1)|V_2| - |L|
\geq 2k \left(\frac{n}{r-1} - \epsilon_2 n - \theta n - (r-3)\epsilon_1 n\right) - (2k-1) \left(\frac{1}{r-1} + \epsilon_1\right)n - \epsilon n
\geq \frac{n}{r-1} - o(n) > 2k
\]

for sufficiently large \( n \). Hence there exist \( 2k \) vertices \( u_1^{(2)}, u_2^{(2)}, \ldots, u_{2k}^{(2)} \in V_2 \) such that the subgraph formed between the two parts \( \{u_1^{(1)}, \ldots, u_{2k}^{(1)}\} \) and \( \{u_1^{(2)}, \ldots, u_{2k}^{(2)}\} \) is a complete bipartite graph. Let \( s \in \{2, r-2\} \) be a positive integer. Assume that we have found the vertices \( u_1^{(s)}, u_2^{(s)}, \ldots, u_{2k}^{(s)} \in V_i \setminus L, (i = 1, 2, \ldots, s) \) such that these vertices form a complete \( s \)-partite subgraph in \( G \). We next consider the common neighbors of these vertices in \( V_{s+1} \). Similarly, we get that for each \( i \in [s] \) and \( j \in [2k] \),

\[
d_{V_{s+1}}(u_j^{(i)}) \geq \frac{n}{r-1} - \epsilon_2 n - \theta n - (r-3)\epsilon_1 n.
\]

By Lemma 3.5 again, we can obtain

\[
\left| \left( \cap_{i \in [s], j \in [2k]} N_{V_{s+1}}(u_j^{(i)}) \right) \setminus L \right|
\geq \sum_{i \in [s], j \in [2k]} d_{V_{s+1}}(u_j^{(i)}) - (2ks-1)|V_{s+1}| - |L|
\geq 2ks \left(\frac{n}{r-1} - \epsilon_2 n - (r-3)\epsilon_1 n\right) - (2ks-1) \left(\frac{1}{r-1} + \epsilon_1\right)n - \epsilon n
\geq \frac{n}{r-1} - o(n) > 2k
\]

Thus we can find \( 2k \) vertices \( u_1^{(s+1)}, u_2^{(s+1)}, \ldots, u_{2k}^{(s+1)} \in V_{s+1} \setminus L \), which together with the vertices \( u_j^{(i)} \in V_i \setminus L, (i \in [s], j \in [2k]) \) form a complete \((s+1)\)-partite subgraph in \( G \). Thus, for any \( 2k \) vertices in \( V_1 \setminus L \), we can find \( 2k \) vertices from \( V_i \setminus L \) for each \( i \in [2, r-1] \) such that all these vertices form a complete \((r-1)\)-partite subgraph in \( G \).
Claim. $G[V_1 \setminus L]$ is both $K_{1,k}$-free and $M_k$-free.

Recall that $G$ contains a large complete $(r - 1)$-partite subgraph with each part in $V_i \setminus L$. If $G[V_1 \setminus L]$ contains a copy of $K_{1,k}$ centered at a vertex $u_0 \in V_1$ with leaves $u_1^{(1)}, u_2^{(1)}, \ldots, u_k^{(1)}$, then by the discussion above, we can embed the $F_{k,r}$ into $G$. Therefore, $G[V_1 \setminus L]$ is $K_{1,k}$-free. Now, we assume that $\{u_1^{(1)}, u_2^{(1)}, u_3^{(1)}, u_4^{(1)}, \ldots, u_{2k-1}^{(1)}, u_{2k}^{(1)}\}$ is a matching of size $k$. Then for each $j \in [k]$, the vertices $u_{2j-1}^{(1)}, u_{2j}^{(1)}, u_{j}^{(2)}, u_{j}^{(3)}, \ldots, u_{j}^{(r-1)}$ form a clique of order $r$, and these $r$ cliques intersect at the vertex $u_1^{(2)}$. So $G[V_1 \setminus L]$ is $M_k$-free.

Hence both the maximum degree and the maximum matching number of $G[V_1 \setminus L]$ are at most $k - 1$, respectively. By Theorem 3.3

$$e(G[V_1 \setminus L]) \leq f(k - 1, k - 1).$$

The same argument gives that for each $j \in [2, r - 1]$,

$$e(G[V_j \setminus L]) \leq f(k - 1, k - 1).$$

For each $i \in [r - 1]$, since $G[V_1 \setminus L]$ has at most $f(k - 1, k - 1)$ edges, then the subgraph obtained from $G[V_1 \setminus L]$ by deleting one vertex of each edge in $G[V_1 \setminus L]$ contains no edges, which is an independent set of $G[V_1 \setminus L]$. By Lemma 3.4 there exists an independent set $I_i \subseteq V_i$ such that

$$|I_i| \geq |V_i \setminus L| - f(k - 1, k - 1) \geq |V_i| - \epsilon_3 n - k^2.$$

This completes the proof.

Lemma 3.8. $L$ is empty, and each $G[V_i]$ is $K_{1,k}$-free and $M_k$-free.

Proof. Recall that $Ax = \lambda(G)x$ and $z$ is defined as a vertex with maximum eigenvector entry and satisfies $x_z = 1$. So we have

$$d(z) \geq \sum_{w \sim z} x_w = \lambda(G)x_z = \lambda(G) \geq \left(1 - \frac{1}{r - 1} - \frac{r - 1}{4n^2}\right)n > \left(1 - \frac{1}{r - 1} - \epsilon_2\right)n,$$

as $n$ is large enough. Hence $z \notin L$. Without loss of generality, we may assume that $z \in V_1$. Since the maximum degree in the induced subgraph $G[V_1 \setminus L]$ is at most $k - 1$ (containing no $K_{1,k}$), from Lemma 3.4 we have $|L| \leq \epsilon_3 n$ and

$$d_{V_1}(z) = d_{V_1 \cap L}(z) + d_{V_1 \setminus L}(z) \leq \epsilon_3 n + k - 1.$$

Therefore, by Lemma 3.7 we have

$$\lambda(G) = \lambda(G)x_z = \sum_{v \sim z} x_v = \sum_{v \in V_1} x_v + \sum_{v \in T_2} x_v + \sum_{v \in V_1 \setminus L} x_v$$

$$= \sum_{v \in V_1} x_v + \sum_{v \in T_2} x_v + \sum_{v \in V_1 \setminus L} x_v$$

$$\leq d_{V_1}(z) + \sum_{v \in T_2} x_v + |\bigcup_{i=2}^{r-1} V_i \setminus I_i|$$

$$\leq \epsilon_3 n + k - 1 + \sum_{v \in T_2} x_v + (r - 2)(\epsilon_3 n + k^2).$$

By Lemma 3.1 we can get

$$\sum_{v \in T_2} x_v \geq \left(1 - \frac{1}{r - 1} - \frac{r - 1}{4n^2}\right)n - (r - 1)\epsilon_3 n - (r - 2)k^2 - k + 1. \quad (9)$$

Next we are going to prove $L = \emptyset$.

By way of contradiction, assume that there is a vertex $v \in L$, so $d_G(v) \leq (1 - \frac{1}{r - 1} - \epsilon_2)n$. Consider the graph $G^+$ with vertex set $V(G)$ and edge set $E(G^+) = E(G \setminus \{v\}) \cup \{vw : w \in \bigcup_{i=2}^{r-1} I_i\}$. Roughly speaking, in this process, the
number of added edges is greater than the number of deleted edges. Note that adding a vertex incident with vertices in $I_i$ does not create any cliques, and so $G^+$ is $F_{k,r}$-free. Note that $x$ is a vector such that $\lambda(G) = \frac{x^T A(G) x}{x^T x}$, and the Rayleigh theorem implies $\lambda(G^+) \geq \frac{x^T (A(G^+) - A(G)) x}{x^T x}$. Furthermore,

$$\lambda(G^+) - \lambda(G) \geq \frac{x^T (A(G^+) - A(G)) x}{x^T x} = 2x_v^T x \left( \frac{\sum_{w \in I_0 \cup \cdots \cup I_{r-1}} x_{wv} - \sum_{w \in E(G)} x_w}{x^T x} \right) \geq \frac{2x_v}{x^T x} \left( e_2 n - \frac{r - 1}{4n} - (r - 1)\epsilon_3 n - (r - 2)^2 - k + 1 \right) > 0,$$

where the last inequality holds for $n$ large enough and $\epsilon_3 \ll \epsilon_2$. This contradicts $G$ having the largest spectral radius over all $F_{k,r}$-free graphs, so $L$ must be empty. Furthermore, the claim in the proof of Lemma 3.8 implies that each $G[V_i]$ is $K_{1,k}$-free and $M_k$-free. \[ \square \]

**Lemma 3.9.** For any $i \in [r - 1]$, let $B_i = \{ u \in V_i : d_{V_i}(u) \geq 1 \}$ and $C_i = V_i \setminus B_i$. Then

1. $|B_i| \leq 2k^2 + 1$;
2. For every vertex $u \in C_i$, $u$ is adjacent to all vertices of $V \setminus V_i$.

**Proof.** We prove the assertions by contradiction.

1. If there exists a $j \in [r - 1]$ such that $|B_j| > 2k^2 + 1$, then $\sum_{u \in B_j} d_{V_j}(u) > 2k^2 + 1$. Since $G[V_j]$ is both $K_{1,k}$-free and $M_k$-free, $e(G[V_j]) \leq f(k - 1, k - 1) < k^2$. Therefore,

$$2k^2 + 1 < \sum_{u \in B_j} d_{V_j}(u) = \sum_{u \in V_j} d_{V_j}(u) = 2e(G[V_j]) < k^2,$$

which is a contradiction.

2. If there exists a vertex $v \in C_1$ such that there is a vertex $w_{1,1} \notin V_1$ and $vw_{1,1} \notin E(G)$. Let $G'$ be the graph with $V(G') = V(G)$ and $E(G') = E(G) \cup \{vw_{1,1}\}$. We claim that $G'$ is $F_{k,r}$-free. Otherwise, $G'$ contains a copy of $F_{k,r}$, say $F_0$, as a subgraph, then $vw_{1,1} \in E(F_0)$. We may assume that $v$ is the center of $F_0$ (the case that $v$ is not the center of $F_0$ can be proved similarly). As $v$ is the center of $F_0$, there exist vertices $w_{1,1}, w_{1,2}, \cdots, w_{1,r-1}, w_{2,1}, \cdots, w_{2,r-1}, \cdots, w_{k,1}, \cdots, w_{k,r-1} \notin V_1$ such that for any $i \in [k]$, the vertex set $\{w_{i,1}, w_{i,2}, \cdots, w_{i,r-1}\}$ induces a copy of $K_{r-1}$ in $G$. Therefore, for any $i \in [k]$ and $j \in [r - 1]$, we have

$$d_{V_i}(w_{i,j}) = d(w_{i,j}) - d_{V \setminus V_i}(w_{i,j}) \geq \delta(G) - (k - 1) - (r - 3) \left( \frac{n}{r - 1} + \epsilon_1 n \right),$$

where the last inequality holds as $G[V_i]$ is $K_{1,k}$-free, $|V_i| \leq \frac{n}{r - 1} + \epsilon_1 n$ for any $s \in [r - 1]$. Since $L$ is empty by Lemma 3.8 we have $\delta(G) > (\frac{r - 3}{r - 1} - \epsilon_2) n$. It follows that

$$d_{V_i}(w_{i,j}) > \frac{n}{r - 1} - o(n).$$

Using Lemma 3.5 we get

$$\left| \bigcap_{i=1}^k \bigcap_{j=1}^{r-1} N_{V_i}(w_{i,j}) \setminus B_1 \right|$$

$$\geq \sum_{i=1}^k \sum_{j=1}^{r-1} \left| (N_{V_i}(w_{i,j})) \setminus (k(r - 1) - 1) \cup \bigcup_{i=1}^k \bigcup_{j=1}^{r-1} N_{V_i}(w_{i,j}) \right| - |B_1|$$

$$\geq \sum_{i=1}^k \sum_{j=1}^{r-1} d_{V_i}(w_{i,j}) - (kr - k - 1)|V_i| - |B_1|$$

$$> kr - 1 - o(n) - (kr - k - 1) \left( \frac{n}{r - 1} + o(n) \right) - (2k^2 + 1)$$

$$\geq \frac{n}{r - 1} - o(n) > 1.$$
Then there exists $v' \in C_1$ such that $v'$ is adjacent to $w_{1,1}, \ldots, w_{1,r-1}, \ldots, w_{k,1}, \ldots, w_{k,r-1}$. Then $(F_0 \setminus \{v\}) \cup \{v'\}$ is a copy of $F_{k,r}$ in $G$, which is a contradiction. Thus $G'$ is $F_{k,r}$-free. From the construction of $G'$, we see that $\lambda(G') > \lambda(G)$, which contradicts the assumption that $G$ has the maximum spectral radius among all $F_{k,r}$-free graphs on $n$ vertices. \hfill \blacksquare

Lemma 3.10. For any $u \in V(G)$, $x_u \geq 1 - \frac{20k^2r^2}{n}$.

Proof. Recall that $x_z = \max\{x_i : i \in V(G)\} = 1$. Without loss of generality, we may assume that $z \in V_1$. Then

$$\lambda(G)x_z = \sum_{v \sim z} x_v = \sum_{v \sim z, w \in V_1} x_w + \sum_{v \sim z, w \in V_i} x_w = \sum_{v \sim z, w \in V_1} x_w + \sum_{v \sim z, w \in V_i} x_w,$$

which implies that

$$\sum_{v \sim z, w \in C_1} x_w = \lambda(G) - \sum_{v \sim z, w \in V_1} x_w - \sum_{v \sim z, w \in V_i} x_w \geq \lambda(G) - d_{V_1}(z) - \sum_{v \in B_i} \left( \sum_{w \sim z, w \in V_1} x_w \right) \geq \lambda(G) - (k - 1) - (r - 2)(2k^2 + 1),$$

where (10) holds as $G[V_1]$ is $K_{1,k}$-free, and $|B_i| \leq 2k^2 + 1$ for any $i \in [r - 1]$.

We will prove this lemma by contradiction. Suppose that there is a vertex $v \in V(G)$ with $x_v < 1 - \frac{20k^2r^2}{n}$. Let $G'$ be the graph with $V(G') = V(G)$ and $E(G') = E(G \setminus \{v\}) \cup \{vw : w \in N(z) \cap (\cup_{i=2}^{r-1} C_i)\}$. Since $C_i$ is an independent set for any $i \in [r - 1]$, one may observe that $G'$ is $F_{k,r}$-free. By (10), we have

$$\lambda(G') - \lambda(G) \geq \frac{x^T(A(G') - A(G))x}{x^T x} \geq \frac{2x_v}{x^T x} \left( \sum_{i=2}^{r-1} \left( \sum_{v \sim z, w \in C_1} x_w \right) - \sum_{v \in E(G)} x_v \right) \geq \frac{2x_v}{x^T x} \left( \sum_{v \sim z, w \in C_1} x_w - \lambda(G)x_v \right) \geq \frac{2x_v}{x^T x} \left( \lambda(G) - (k - 1) - (r - 2)(2k^2 + 1) - \lambda(G) \left( 1 - \frac{20k^2r^2}{n} \right) \right) \geq \frac{2x_v}{x^T x} \left( \frac{r - 2}{r - 1} \cdot 20k^2r^2 - \frac{r - 1}{4n} \cdot 20k^2r^2 - k + 1 - (r - 2)(2k^2 + 1) \right) > 0,$$

where the last inequality follows by $l(G) \geq (1 - \frac{1}{r-1})n - \frac{1}{4n}$ by Lemma 3.1. This contradicts the assumption that $G$ has the maximum spectral radius among all $F_{k,r}$-free graphs on $n$ vertices. Thus $x_u \geq 1 - \frac{20k^2r^2}{n}$ for any $u \in V(G)$. \hfill \blacksquare

Let $G_{in} = \cup_{i=1}^{r-1} G[V_i]$. For any $i \in [r - 1]$, let $|V_i| = n_i$ and $F = K_{n_1,n_2,\ldots,n_{r-1}}$ be the complete $(r - 1)$-partite graph on $V_1, V_2, \ldots, V_{r-1}$. Let $G_{out}$ be the graph with $V(G_{out}) = V(G)$ and $E(G_{out}) = E(F) \setminus E(G)$.

Lemma 3.11. For any $1 \leq i < j \leq r - 1$, $|V_i| - |V_j| \leq 1$. 

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Proof. Suppose \( n_1 \geq n_2 \geq \ldots \geq n_{r-1} \). We prove the assertion by contradiction. Assume that there exist \( i_0, j_0 \) with 
\[ 1 \leq i_0 < j_0 \leq r - 1 \] such that \( n_{i_0} - n_{j_0} \geq 2 \).

Claim 1. There exists a constant \( c_1 > 0 \) such that \( \lambda(T_{r-1}(n)) - \lambda(F) \geq \frac{c_1}{n} \).

Proof. Let \( F' = K_{n_1, \ldots, n_0-1, \ldots, n_{j_0}+1, \ldots, n_{r-1}} \). Assume \( F' \cong K_{n'_1, n'_2, \ldots, n'_{r-1}} \), where \( n'_1 \geq n'_2 \geq \ldots \geq n'_{r-1} \). By (2), we have

\[
1 = \sum_{i=1}^{r-1} \frac{n_i}{\lambda(F) + n_i} = \sum_{i=1}^{r-1} \frac{n_{i_0}}{\lambda(F) + n_{i_0}} + \frac{n_{j_0}}{\lambda(F) + n_{j_0}} + \sum_{i \in \{r-1\} \setminus \{i_0, j_0\}} \frac{n_i}{\lambda(F) + n_i},
\]

and

\[
1 = \sum_{i=1}^{r-1} \frac{n'_i}{\lambda(F') + n'_i} = \sum_{i=1}^{r-1} \frac{n_{i_0} - 1}{\lambda(F') + n_{i_0} - 1} + \frac{n_{j_0} + 1}{\lambda(F') + n_{j_0} + 1} + \sum_{i \in \{r-1\} \setminus \{i_0, j_0\}} \frac{n_i}{\lambda(F') + n_i}.
\]

Subtracting (12) from (11), we get

\[
\begin{align*}
\frac{2(n_{i_0} - n_{j_0} - 1)\lambda^2(F) + (n_{i_0} + n_{j_0})(n_{i_0} - n_{j_0} - 1)\lambda(F)}{(\lambda(F) + n_{i_0} - 1)(\lambda(F) + n_{i_0})(\lambda(F) + n_{j_0} + 1)(\lambda(F) + n_{j_0})} \\
= \sum_{i \in \{r-1\} \setminus \{i_0, j_0\}} \frac{n_i(\lambda(F') - \lambda(F))}{(\lambda(F) + n_i)(\lambda(F') + n_i)} + \frac{(n_{i_0} - 1)(\lambda(F') - \lambda(F))}{(\lambda(F) + n_{i_0} - 1)(\lambda(F) + n_{i_0} - 1)} \\
+ \frac{(n_{j_0} + 1)(\lambda(F') - \lambda(F))}{(\lambda(F) + n_{j_0} + 1)(\lambda(F') + n_{j_0} + 1)} \\
\leq \frac{\lambda(F') - \lambda(F)}{\lambda(F) + n'_{r-1}} \left( \sum_{i \in \{r-1\} \setminus \{i_0, j_0\}} \frac{n_i}{\lambda(F') + n_i} + \frac{n_{i_0} - 1}{\lambda(F') + n_{i_0} - 1} + \frac{n_{j_0} + 1}{\lambda(F') + n_{j_0} + 1} \right) \\
= \frac{\lambda(F') - \lambda(F)}{\lambda(F) + n'_{r-1}} 
\end{align*}
\]

where the inequality holds as \( n'_{r-1} \leq \min\{n_1, \ldots, n_{i_0} - 1, \ldots, n_{j_0} + 1, \ldots, n_{r-1}\} \), and the last equality is by (12). Combining with the assumption \( n_{i_0} - n_{j_0} \geq 2 \), we obtain

\[
\frac{2\lambda^2(F) + (n_{i_0} + n_{j_0})\lambda(F)}{(\lambda(F) + n_{i_0} - 1)(\lambda(F) + n_{i_0})(\lambda(F) + n_{j_0} + 1)(\lambda(F) + n_{j_0})} \leq \frac{\lambda(F') - \lambda(F)}{\lambda(F) + n'_{r-1}}. \tag{13}
\]

In view of the construction of \( F \), we see that

\[
n - \left( n - \frac{n}{r - 1} + e_1 n \right) \leq \delta(F) \leq \lambda(F) \leq \Delta(F) \leq n - \left( n - \frac{n}{r - 1} - e_1 n \right),
\]

thus \( \lambda(F) = \Theta(n) \). From (13), it follows that there exists a constant \( c_1 > 0 \) such that \( \lambda(F') - \lambda(F) \geq \frac{c_1}{n} \). Therefore, by Lemma 2.4, \( \lambda(T_{r-1}(n)) - \lambda(F) \geq \lambda(F') - \lambda(F) \geq \frac{c_1}{n} \).

Claim 2.

\[ \lambda(G) \geq \lambda(T_{r-1}(n)) + \frac{2f(k-1, k-1)}{n} \left( 1 - \frac{2}{n} \right). \]

Proof. Let \( y \) be an eigenvector of \( T_{r-1}(n) \) corresponding to \( \lambda(T_{r-1}(n)) \), \( a = n - (r - 1)\left\lfloor \frac{n}{r-1} \right\rfloor \). Since \( T_{r-1}(n) \) is a complete \((r-1)\)-partite graph on \( n \) vertices where each partite set has either \( \lfloor \frac{n}{r-1} \rfloor \) or \( \lceil \frac{n}{r-1} \rceil \) vertices, we may assume \( y = (y_1, \ldots, y_{n-1}, y_{n-1+1}, \ldots, y_{n})^T \). Thus by (3), we have

\[
\lambda(T_{r-1}(n))y_1 = (r - a - 1)\left\lfloor \frac{n}{r-1} \right\rfloor y_2 + (a - 1)\left\lceil \frac{n}{r-1} \right\rceil y_1
\]
and
\[
\lambda(T_{r-1}(n))y_2 = (r - a - 2)\left\lfloor \frac{n}{r - 1} \right\rfloor y_2 + a\left\lfloor \frac{n}{r - 1} \right\rfloor y_1.
\] (15)

Combining (14) and (15), we obtain
\[
\lambda(T_{r-1}(n)) + \left\lfloor \frac{n}{r - 1} \right\rfloor \frac{y_1}{y^T y} \geq \lambda(T_{r-1}(n)) + \frac{2\sum_{ij \in E(G_0)} y_i y_j}{y^T y} \geq \lambda(T_{r-1}(n)) + \frac{2f(k-1, k-1)}{n} \left(1 - \frac{2}{n}\right).
\] (16)

**Claim 3.** \(e(G_{in}) - e(G_{out}) \leq f(k-1, k-1)\).

**Proof.** It follows from the definitions of \(G_{in}\) and \(G_{out}\), that we have \(e(G_{in}) = \sum_{i=1}^{r-1} |E(G[V_i])|\) and \(e(G_{out}) = \sum_{1 \leq i < j \leq r-1} |V_i| |V_j| - |E(G'[G])|\). To get the claim, we need to prove (3) and (6) by Lemma 2.5. Obviously (6) implies (5), so it is sufficient to prove (6). We prove (6) by contradiction. Without loss of generality, suppose that there exists a vertex \(u \in V_1\) such that
\[
d_{G[V_1]}(u) + \sum_{j=2}^{r-1} \beta(G[N(u) \cap V_j]) \geq k.
\]

Let \(\{w_1w_2, \ldots, w_{2k-1}w_{2k}\}\) be an \(\ell\)-matching of \(\bigcup_{j=2}^{r-1} G[N(u) \cap V_j]\) and \(u_1, \ldots, u_{k-\ell} \in V_1\) be in the neighborhood of \(u\). By Lemma 3.9, there exist \(v_1, \ldots, v_{k-\ell} \in C_2\) such that \(\{u, u_1, \ldots, u_{k-\ell}, v_1, \ldots, v_{k-\ell}, w_1, \ldots, w_{2k}\}\) induce an \(F_k\) of \(G\). For each \(u_i v_i (1 \leq i \leq k - \ell)\), there exist \(r - 3\) vertices \(t_3 \in C_3, t_4 \in C_4, \ldots, t_{r-1} \in C_{r-1}\) such that \(u, u_i, v_i, t_3, t_4, \ldots, t_{r-1}\) induce a \(K_r\) of \(G\). For any \(w_{i-1}w_i \in \{w_1w_2, \ldots, w_{2k-1}w_{2k}\}\), without loss of generality, suppose that \(w_{i-1}w_i \leq E(G[V_2])\), then there exist \(r - 3\) vertices \(z_3 \in C_3, z_4 \in C_4, \ldots, z_{r-1} \in C_{r-1}\) such that \(u, w_{i-1}, w_i, z_3, z_4, \ldots, z_{r-1}\) induce a \(K_r\) of \(G\). Thus we find a copy of \(F_{k, r}\) from the above \(F_k\), a contradiction. □

According to the definitions of \(G_{in}, G_{out}\) and \(F\), we have \(e(G) = e(G_{in}) + e(F) - e(G_{out})\). By Lemma 3.9 for any \(i \in [r-1]\), and every vertex \(u \in C_i\), \(u\) is adjacent to all vertices of \(V \setminus V_i\). Thus
\[
e(G_{out}) \leq \sum_{1 \leq i < j \leq r-1} |B_i||B_j| \leq \left(\frac{r-1}{2}\right)(2k^2 + 1)^2 \leq 9k^2r^2.
\]
Then

\[
\lambda(G) = \frac{x^T A(G)x}{x^T x} = \frac{2 \sum_{ij \in E(F)} x_i x_j + 2 \sum_{ij \in E(G_{in})} x_i x_j - 2 \sum_{ij \in E(G_{out})} x_i x_j}{x^T x} \leq \lambda(F) + \frac{2e(G_{in})}{x^T x} - \frac{2e(G_{out})(1 - \frac{20k^2r^2}{n})^2}{x^T x} \leq \lambda(F) + \frac{2(e(G_{in}) - e(G_{out}))}{x^T x} + \frac{2e(G_{out})40k^2r^2}{x^T x} \leq \lambda(F) + \frac{2f(k-1, k-1)}{x^T x} + \frac{720k^6r^4}{x^T x},
\]

Using (16), (17) and \(x^T x \geq n(1 - \frac{20k^2r^2}{n})^2 \geq n - 40k^2r^2\), we get

\[
\lambda(T_{r-1}(n)) - \lambda(F) \leq \frac{2f(k-1, k-1)}{n} - \frac{2f(k-1, k-1)}{n^2} + \frac{720k^6r^4}{n^2} \leq \frac{2f(k-1, k-1)}{n - 40k^2r^2} - \frac{2f(k-1, k-1)}{n} + \frac{4f(k-1, k-1)}{n^2} + \frac{720k^6r^4}{n(n - 40k^2r^2)} \leq \frac{c_2}{n^2},
\]

where \(c_2\) is a positive constant.

Combining with Claim 1, we have

\[
\frac{c_1}{n} \leq \lambda(T_{r-1}(n)) - \lambda(F) \leq \frac{c_2}{n^2},
\]

which is a contradiction when \(n\) is sufficiently large. Thus \(||V_i| - |V_j|| \leq 1\) for any \(1 \leq i < j \leq r - 1\).

\[\square\]

**Proof of Theorem 1.4.** Now we prove that \(e(G) = \text{ex}(n, F_{k,r})\). Otherwise, we assume that \(e(G) \leq \text{ex}(n, F_{k,r}) - 1\). Let \(H\) be an \(F_{k,r}\)-free graph with \(e(H) = \text{ex}(n, F_{k,r})\) and \(V(H) = V(G)\). By Lemma 3.11, we may assume that \(V_1, \ldots, V_{r-1}\) induce a complete \((r-1)\)-partite graph in \(H\). Let \(E_1 = E(G) \setminus E(H), E_2 = E(H) \setminus E(G)\), then \(E(H) = (E(G) \cup E_2) \setminus E_1\), and

\[
|E(G) \cap E(H)| + |E_1| = e(G) < e(H) = |E(G) \cap E(H)| + |E_2|,
\]

which implies that \(|E_2| \geq |E_1| + 1\). Furthermore, by Lemma 3.3, we have

\[
|E_2| \leq f(k-1, k-1) + \sum_{1 \leq i < j \leq r-1} |B_i||B_j| \leq k^2 + \binom{r-1}{2}(2k^2 + 1)^2 \leq 10k^4r^2.
\]
According to [4] and (18), we deduce, for sufficiently large $n$, that

$$
\lambda(H) \geq \frac{x^T A(H)x}{x^T x} = \frac{x^T A(G)x}{x^T x} + \frac{2 \sum_{ij \in E_2} x_i x_j}{x^T x} - \frac{2 \sum_{ij \in E_1} x_i x_j}{x^T x}
$$

$$
= \lambda(G) + \frac{2}{x^T x} \left( \sum_{ij \in E_2} x_i x_j - \sum_{ij \in E_1} x_i x_j \right)
$$

$$
\geq \lambda(G) + \frac{2}{x^T x} \left( |E_2| (1 - \frac{20k^2 r^2}{n})^2 - |E_1| \right)
$$

$$
\geq \lambda(G) + \frac{2}{x^T x} \left( |E_2| - \frac{40k^2 r^2}{n} |E_2| - |E_1| \right)
$$

$$
\geq \lambda(G) + \frac{2}{x^T x} \left( 1 - \frac{40k^2 r^2}{n} |E_2| \right)
$$

$$
\geq \lambda(G) + \frac{2}{x^T x} \left( 1 - \frac{40k^2 r^2}{n} 10k^4 r^2 \right)
$$

$$
> \lambda(G),
$$

which contradicting the assumption that $G$ has the maximum spectral radius among all $F_{k,r}$-free graphs on $n$ vertices. Hence $e(G) = ex(n, F_{k,r})$. \qed

4 Concluding remarks

To avoid unnecessary calculations, we did not attempt to get the best bound on the order of graphs in the proof. It would be interesting to determine how large $n$ needs to be for our result.

Recently, Cioabă, Desai and Tait [9] investigated the largest spectral radius of an $n$-vertex graph that does not contain the odd-wheel graph $W_{2k+1}$, which is the graph obtained by joining a vertex to all vertices of a cycle of length $2k$. Moreover, they raised the following more general conjecture.

Conjecture 4.1. Let $F$ be any graph such that the graphs in $\text{Ex}(n, F)$ are Turán graphs plus $O(1)$ edges. Then for sufficiently large $n$, a graph attaining the maximum spectral radius among all $F$-free graphs on $n$ vertices is a member of $\text{Ex}(n, F)$.

We say that $F$ is edge-color-critical if there exists an edge $e$ of $F$ such that $\chi(F - e) < \chi(F)$. Let $F$ be an edge-color-critical graph with $\chi(F) = r + 1$. By a result of Simonovits [32] and a result of Nikiforov [26], we know that $\text{Ex}(n, F) = \text{Ex}_{sp}(n, F) = \{T_r(n)\}$ for sufficiently large $n$, where $\text{Ex}_{sp}(n, F)$ denotes the set of $F$-free graphs on $n$ vertices, attaining the maximum spectral radius. This shows that Conjecture 4.1 is true for all edge-color-critical graphs. As we mentioned before, Theorem 1.3 says that Conjecture 4.1 holds for the $k$-fan graph $F_k$. Moreover, the result in [37] implies that Conjecture 4.1 also holds for the flower graph $H_{s,k}$, the graph defined by intersecting $s$ triangles and $k$ odd cycles of length at least 5 in exactly one common vertex. In addition, our main result (Theorem 1.4) tells us that Conjecture 4.1 also holds for the intersecting cliques $F_{k,r}$. Note that $F_k$, $H_{s,k}$ and $F_{k,r}$ are not edge-color-critical.

Let $S_{n,k}$ be the graph consisting of a clique on $k$ vertices and an independent set on $n - k$ vertices in which each vertex of the clique is adjacent to each vertex of the independent set. Clearly, we can see that $S_{n,k}$ does not contain $F_k$ as a subgraph. Recently, Zhao, Huang and Guo [39] proved that $S_{n,k}$ is the unique graph attaining the maximum signless Laplacian spectral radius among all graphs of order $n$ containing no $F_k$ for $n \geq 3k^2 - k - 2$. Soon after, Chen, Liu and Zhang [6] solved the corresponding case for $H_{s,k}$-free graphs. So it is a natural question to consider the maximum signless Laplacian spectral radius among all graphs containing no $F_{k,r}$. We write $q(G)$ for the signless Laplacian spectral radius, i.e., the largest eigenvalue of the signless Laplacian matrix $Q(G) = D(G) + A(G)$, where $D(G) = \text{diag}(d_1, \ldots, d_n)$ is the degree diagonal matrix and $A(G)$ is the adjacency matrix. We end our paper with
the following problem, and leave it for the interested readers. Clearly, when \( r = 3 \), this problem reduces to the result of Zhao et al. \[39\].

**Problem.** For integers \( k \geq 1 \) and \( r \geq 3 \), there exists an integer \( n_0(k, r) \) such that if \( n \geq n_0(k, r) \) and \( G \) is an \( F_{k,r} \)-free graph on \( n \) vertices, then \( q(G) \leq q(S_{n,k(r−2)}) \), equality holds if and only if \( G = S_{n,k(r−2)} \).

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