Towards the $C^0$ flux conjecture

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Abstract

In this note, we generalise a result of Lalonde, McDuff and Polterovich concerning the $C^0$ flux conjecture, thus confirming the conjecture in new cases of a symplectic manifold. Also, we prove the continuity of the flux homomorphism on the space of smooth symplectic isotopies endowed with the $C^0$ topology, which implies the $C^0$ rigidity of Hamiltonian paths, conjectured by Seyfaddini.

1 Introduction and main results

The celebrated Eliashberg-Gromov rigidity theorem [E1, E2, G] states that on any closed symplectic manifold $(M,\omega)$, the group $\text{Symp}(M,\omega)$ of symplectomorphisms of $M$ is $C^0$-closed inside the group $\text{Diff}(M)$ of diffeomorphisms of $M$. A related natural conjecture (called the $C^0$ flux conjecture) was raised in Banyaga’s foundational paper [B]: is the group $\text{Ham}(M,\omega)$ of Hamiltonian diffeomorphisms of $M$ $C^0$-closed inside $\text{Symp}_0(M,\omega)$, the connected component of identity in $\text{Symp}(M,\omega)$?

The reader may wonder why it is asked if $\text{Ham}(M,\omega)$ is $C^0$-closed in $\text{Symp}_0(M,\omega)$ rather than $\text{Symp}(M,\omega)$. The difficulty in addressing the latter question is that, although the Eliashberg-Gromov rigidity theorem tells us that $\text{Symp}(M,\omega)$ is $C^0$-closed in the group $\text{Diff}(M)$ of diffeomorphisms of $M$, it is not known if $\text{Symp}_0(M,\omega)$ is $C^0$-closed in $\text{Symp}(M,\omega)$. To avoid this difficulty the $C^0$ flux conjecture is usually stated for $\text{Symp}_0(M,\omega)$.

A weak form of the $C^0$ flux conjecture is the $C^1$ flux conjecture, which states that $\text{Ham}(M,\omega)$ is $C^1$-closed in $\text{Symp}_0(M,\omega)$. This statement is equivalent to that the flux group $\Gamma \subset H^1(M,\mathbb{R})$ is discrete. Some cases of the $C^1$ flux conjecture were proven in [B], [L-M-P], [M]; it was finally confirmed in full generality by Ono [O-1]. However, the $C^0$ flux conjecture still remains open in case of a general symplectic manifold.

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1The author also uses the spelling “Buhovski” for his family name.
It has been confirmed by Lalonde, McDuff and Polterovich in certain cases \cite{L-M-P} (these cases are described below). Also, Humilière and Vichery established more cases of the $C^0$ flux conjecture in their joint work \cite{Hu-V}.

A different weak form of the $C^0$ flux conjecture (the “$C^0$ rigidity of Hamiltonian paths”) was proposed by Seyfaddini \cite{Sey}: Is it true that on any closed symplectic manifold, the space of smooth Hamiltonian isotopies is $C^0$ close in the space of smooth symplectic isotopies? In \cite{Sey}, Seyfaddini showed that a symplectic isotopy which is a $C^0$ limit of a sequence of Hamiltonian isotopies is itself Hamiltonian, provided that the corresponding sequence of generating Hamiltonians is a Cauchy sequence in the $L^{(1,\infty)}$ topology.

The results of this note are concerned with the $C^0$ flux conjecture, and with the mentioned conjecture of Seyfaddini (the $C^0$ rigidity of Hamiltonian paths).

### 1.1 The $C^0$ flux conjecture

We denote $H = \text{Ham}(M,\omega) \subseteq G = \text{Symp}_0(M,\omega)$, by $\tilde{G} = \widetilde{\text{Symp}}_0(M,\omega)$ we denote the universal cover of $G = \text{Symp}_0(M,\omega)$, and by $H_0 \subseteq \tilde{G}$ we denote those elements of $\tilde{G}$ whose endpoint belongs to $H$. Next, by $H_0 \subseteq G$ we denote the $C^0$ closure of $H$ inside $G$, and by $\tilde{H}_0 \subseteq \tilde{G}$ we denote the lift of $H_0$ to $\tilde{G}$. Also, we use the notation $\text{Map}_0(M)$ for the connected component of the identity in the space of all smooth maps $M \to M$.

Denote by $\Gamma \subset H^1(M,\mathbb{R})$ the flux group, i.e. the image of $\tilde{H}$ (or, equivalently, of $\pi_1(G)$) under the flux homomorphism, and by $\Gamma_0 \subset H^1(M,\mathbb{R})$ the image of $H_0$ under the flux homomorphism. It is not hard to see that the $C^0$ flux conjecture is equivalent to the equality $\Gamma_0 = \Gamma$ (this follows from the well-known fact that for a path $\phi^t$, $t \in [0,1]$ of symplectic diffeomorphisms, its endpoint $\phi^1$ belongs to $H$ if and only if its flux belongs to $\Gamma$). The restriction of the flux homomorphism to $\pi_1(G)$ admits a natural extension to a homomorphism (which we again call flux homomorphism) from $\pi_1(\text{Map}_0(M))$ to the $H^1(M,\mathbb{R})$. Following \cite{L-M-P}, we denote by $\Gamma_{top}$ the image of $\pi_1(\text{Map}_0(M))$ under the flux homomorphism. Consider the evaluation homomorphism $\text{ev} : \pi_1(\text{Map}_0(M)) \to \pi_1(M)$. For any $a \in \pi_1(M)$ we denote by $\Gamma_{top}^a \subseteq \Gamma$ the image of $\text{ev}^{-1}(a) \subseteq \pi_1(\text{Map}_0(M))$ under the flux homomorphism.

The following result was proved in \cite{L-M-P}:

**Theorem 1.1.** If $M$ is Lefschetz, then $\Gamma_0 \subseteq \Gamma_{top}$.

As a consequence, Lalonde, McDuff and Polterovich conclude:

**Corollary 1.2.** Assume that $M$ is Lefschetz and that $\Gamma_{top} = \Gamma$. Then the $C^0$ flux conjecture holds for $M$. 

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As an example, one can take $M$ to be a closed Kähler manifold of nonpositive curvature such that its fundamental group has a trivial center. As another example, one can take the $2n$-dimensional torus with a translation invariant symplectic structure. See [L-M-P] for more details.

Now we turn to our results. Our main result is:

**Theorem 1.3.** Let $(M, \omega)$ be a closed symplectic manifold. Then $\Gamma_0 \subseteq \Gamma_{\text{top}} + \overline{\Gamma_{\text{top}}^e}$, where $e \in \pi_1(M)$ is the identity, and $\overline{\Gamma_{\text{top}}^e} \subseteq H^1(M, \mathbb{R})$ is the closure of $\Gamma_{\text{top}}^e$ inside $H^1(M, \mathbb{R})$.

As a result, we obtain the following corollary:

**Corollary 1.4.** Let $(M, \omega)$ be a closed symplectic manifold such that $\Gamma_{\text{top}} = \Gamma$. Then the $C^0$ flux conjecture holds for $M$.

Indeed, if $\Gamma_{\text{top}} = \Gamma$, then since $\Gamma$ is closed (the closeness of $\Gamma$ is exactly the statement of the $C^1$ flux conjecture proven in [O-1]), it follows that $\Gamma_{\text{top}} + \overline{\Gamma_{\text{top}}^e} = \Gamma$ and hence $\Gamma_0 = \Gamma$ by Theorem 1.3.

In particular, the $C^0$ flux conjecture holds for a closed symplectically aspherical symplectic manifold $(M, \omega)$ such that the fundamental group $\pi_1(M)$ has a trivial center. Indeed, in this case, since the center of $\pi_1(M)$ is trivial, we get $ev(\pi_1(\text{Map}_0(M))) = \{e\}$ and so $\Gamma_{\text{top}} = \overline{\Gamma_{\text{top}}^e}$, and moreover, since $M$ is symplectically aspherical, we conclude that $\Gamma_{\text{top}}^e = \{0\}$. Therefore $\Gamma_{\text{top}} = 0$ and hence $\Gamma_{\text{top}} = \Gamma = \{0\}$.

As another example, we get that the $C^0$ flux conjecture holds for any product $(M, \omega) = (\mathbb{T}^{2k} \times N, \sigma \oplus \tau)$, where $(\mathbb{T}^{2k}, \sigma)$ is a symplectic torus with a translation invariant $\sigma$, and $(N, \tau)$ is a closed symplectically atoroidal symplectic manifold. Indeed, since $\mathbb{T}^{2k}$ is symplectically aspherical, and $N$ is symplectically atoroidal, it follows that for any $a = (b, c) \in \pi_1(M) \cong \pi_1(\mathbb{T}^{2k}) \times \pi_1(N)$ and any $a \in \pi_1(\text{Map}_0(M))$ with $ev(a) = a$, the flux of $a$ is uniquely determined by $b$. Moreover, since translations of the torus $\mathbb{T}^{2k}$ generate a large enough subgroup of symplectomorphisms of $(\mathbb{T}^{2k}, \sigma)$, for any $b \in \pi_1(\mathbb{T}^{2k})$ we can find an element $b \in \pi_1(G) = \pi_1(\text{Symp}_0(M, \omega))$ such that $ev(b) = (b, 0)$. Therefore we conclude $\Gamma_{\text{top}} = \Gamma$.

**Remark 1.5.** The reader may ask if there exist examples of closed symplectic manifolds for which $\Gamma_{\text{top}} \neq \Gamma$. The following construction is due to Seidel [Se-1, Se-2]. Let $(N, \omega_N)$ be a closed symplectic manifold with $H^1(N, \mathbb{R}) = 0$, let $\psi : N \to N$ be a symplectic diffeomorphism which is smoothly isotopic to the identity, but which is not isotopic to the identity via a smooth path of symplectic diffeomorphisms. Look at the symplectic mapping torus $E = E_\psi$ of $\psi$, which is the total space of the fibration over the two-torus with fibre $N$ and monodromy $\psi$ in one direction, or explicitly,

$$E = \mathbb{R}^2 \times N / (p, q, x) \sim (p - 1, q, x) \sim (p, q - 1, \psi(x)),$$
\[ \omega_E = dp \wedge dq + \omega_N. \]

Because \( \psi \) is smoothly isotopic to the identity, the fibration \( E \to \mathbb{T}^2 \) is trivial as a smooth one, and it is easy to see that for \( E \) we have \( \Gamma_{\text{top}} = H^1(E, \mathbb{Z}) \). However, it is possible that for \( E \) we have \( \Gamma \neq H^1(E, \mathbb{Z}) \). The closed 1-form \( dp \) on \( E \) generates the symplectic vector field \( \frac{\partial}{\partial q} \), whose time-1 map is \( \phi(p, q, x) = (p, q + 1, x) = (p, q, \psi(x)) \). If \( \phi \) turns out to be a non-Hamiltonian diffeomorphism, then we get that \( [dp] \notin \Gamma \), so in particular \( \Gamma \neq \Gamma_{\text{top}} \). One way of detecting this is by looking at the Floer cohomology \( HF^*(\psi) \). That is, if we are in a situation when \( HF^*(\psi) \) has total rank different from that of \( H^*(N) \), then \( HF^*(\phi) \cong H^*(\mathbb{T}^2) \otimes HF^*(\psi) \) is not isomorphic to \( H^*(E) \), and hence in particular \( \phi \) is a non-Hamiltonian diffeomorphism (here we consider cohomologies with coefficients in the corresponding Novikov ring).

1.2 The \( C^0 \) rigidity of Hamiltonian paths

Our next result is:

**Theorem 1.6.** Let \((M, \omega)\) be a closed symplectic manifold. Fix a Riemannian metric \( g \) on \( M \), which induces a distance function \( d: M \times M \to \mathbb{R} \), which in turn, induces a distance \( d \) between maps \( M \to M \): for any \( f, h : M \to M \) we set \( d(f, h) = \sup_{x \in M} d(f(x), h(x)) \). Fix a norm \(| \cdot |\) on \( H^1(M, \mathbb{R}) \). Then there exist constants \( c = c(M, \omega, g), C = C(M, \omega, g, | \cdot |) \), such that for any path \( \phi^t, t \in [0, 1] \) of symplectomorphisms of \( M \), \( \phi^0 = id_M, \phi^1 = \phi \), with \( \max_{t \in [0,1]} d(id_M, \phi^t) < c \), we have \(|\text{Flux}(\{\phi^t\})| \leq Cd(id_M, \phi)\).

Theorem 1.6 has a direct corollary:

**Corollary 1.7.** 1) On any closed symplectic manifold, the flux homomorphism is continuous with respect to the \( C^0 \) distance between smooth paths of symplectomorphisms.

2) \( C^0 \) rigidity of Hamiltonian paths: on any closed symplectic manifold, the space of smooth Hamiltonian isotopies of \( M \) is \( C^0 \)-closed in the space of smooth symplectic isotopies of \( M \). This confirms the mentioned above conjecture of Seyfaddini.

Let us remark, that there is another weak version of the \( C^0 \) flux conjecture, which is due to Seyfaddini, and it concerns with the topological (continuous) Hamiltonian dynamics initially introduced by Oh and Müller [Oh-M]: Is it true that any Hamiltonian homeomorphism (in the sense of [Oh-M]) which belongs to \( \text{Symp}_0(M, \omega) \), is in fact a Hamiltonian diffeomorphism?
Notations

Let $A > 0$. We denote by $B(A) \subset \mathbb{R}^2$ the open euclidean disc centered at the origin having area $A$, i.e. $B(A) = \{ z \in \mathbb{R}^2 \mid \pi |z|^2 < A \}$. We denote $S(A) = \partial B(A) = \{ z \in \mathbb{R}^2 \mid \pi |z|^2 = A \}$, the euclidean circle centered at the origin enclosing a disc of area $A$. Also, we use the notation $B'(A) = B(A) \setminus \{0\} \subset \mathbb{R}^2$ for the punctured disc. On $T^*S^1$ with canonical coordinates $(q,p)$, where $q \in \mathbb{R}/\mathbb{Z}$, $p \in \mathbb{R}$, and with the standard symplectic form $dp \wedge dq$, we will use the notation $S^1 \subset T^*S^1$ for the zero-section, and we denote $W(A) = \{(q,p) \mid |p| < A\} \subset T^*S^1$, so that $W(A)$ is a neighbourhood of the zero-section in $T^*S^1$ having area $2A$.

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2 Proofs

Consider the evaluation homomorphism $ev : \pi_1(\text{Map}_0(M)) \to \pi_1(M)$. In the next lemma we show that its restriction to $\pi_1(G)$ can be naturally extended to a homomorphism $ev' : \tilde{H}_0 \to \pi_1(M)$:

Lemma 2.1. The homomorphism $ev|_{\pi_1(G)} : \pi_1(G) \to \pi_1(M)$ admits a natural extension to a homomorphism $ev' : \tilde{H}_0 \to \pi_1(M)$.

Proof. Let us first present a construction of $ev'$. Fix a Riemannian metric $g$ on $M$. Let $\tilde{\phi} \in \tilde{H}$, let $\phi^t, t \in [0, 1]$ be a path of symplectomorphisms representing $\tilde{\phi}$, and denote $\phi = \phi^1$. Consider a Hamiltonian diffeomorphism $\psi \in H$ such that $\psi$ is sufficiently $C^0$-close to $\phi$ (it is possible to find such a Hamiltonian diffeomorphism, since by our assumption $\phi$ lies in the $C^0$ closure of $H$ inside $G$). Let $\psi^t, t \in [0, 1]$ be a Hamiltonian isotopy of $M$, such that $\psi^1 = \psi$. Define a continuous loop $f^t, t \in [0, 3]$ in $\text{Map}_0(M)$, such that $f^t = \phi^t$ for $t \in [0, 1]$, such that for any $x \in M$, the path $f^t(x), t \in [1, 2]$ is the shortest $g$-geodesic connecting $\phi(x)$ and $\psi(x)$, and such that $f^t = \psi^{3-t}$ for $t \in [2, 3]$. We now define $ev'(\phi)$ to be the value of the evaluation map $ev$ at the loop $f^t, t \in [0, 3]$.
Let us show that the definition does not depend on the choice of \( \psi \) and of the path \( \psi' \), \( t \in [0, 1] \). Let \( \psi_1, \psi_2, t \in [0, 1] \) be two Hamiltonian isotopies of \( M \), such that \( \psi_1 = \psi_1^t \) and \( \psi_2 = \psi_2^t \) are sufficiently \( C^0 \)-close to \( \phi \). Define the corresponding loops \( f_1^t, f_2^t, t \in [0, 3] \) as above. Define the loop \( h_1^t, t \in [0, 6] \) in \( \text{Map}_0(M) \) by \( h_1^t = f_1^{3-t}, t \in [0, 3] \) and \( h_1^t = f_2^{2-3}, t \in [3, 6] \). It is enough to show that the value of \( ev \) at the loop \( h_1^t, t \in [0, 6] \) equals \( e \in \pi_1(M) \). Clearly, the loop \( h_1^t, t \in [0, 6] \) is homotopic to the loop \( h_2^t, t \in [0, 4] \), where \( h_2^t = \psi_1^t \) for \( t \in [0, 1] \), where for any \( x \in M \) the path \( h_2^t(x), t \in [1, 2] \) is the shortest \( g \)-geodesic between \( \psi_1(x) \) and \( \phi(x) \), and the path \( h_2^t(x), t \in [2, 3] \) is the shortest \( g \)-geodesic between \( \phi(x) \) and \( \psi_2(x) \), and finally \( h_2^t = \psi_2^{3-t} \) for \( t \in [3, 4] \). Also, since \( \psi_1 \) and \( \psi_2 \) are \( C^0 \) close to \( \phi \), it follows that the loop \( h_2^t, t \in [0, 4] \) is homotopic to the loop \( h_3^t, t \in [0, 3] \), where \( h_3^t = \psi_1^t \) for \( t \in [0, 1] \), where for any \( x \in M \) the path \( h_3^t(x), t \in [1, 2] \) is the shortest \( g \)-geodesic between \( \psi_1(x) \) and \( \psi_2(x) \), and \( h_3^t = \psi_2^{3-t} \) for \( t \in [2, 3] \). It is enough to show that the value of the evaluation map \( ev \) at \( h_3^t, t \in [0, 3] \) equals \( e \in \pi_1(M) \). Now pick some increasing bijective smooth function \( \nu : [0, 1] \to [0, 1] \), such that the derivatives of \( \nu \) of all orders vanish at \( 1 \in [0, 1] \), and look at the smooth Hamiltonian flow \( h', t \in [0, 2] \) defined by \( h' = \psi_1^{\nu(t)} \) for \( t \in [0, 1] \) and \( h' = \psi_2^{\nu(2-t)} \) for \( t \in [1, 2] \). Then by the solution of the Arnold’s conjecture [C-Z, F1, F2, O-2, H-S, Li-T, Fu-O-1, R, Fu-O-2], the time-2 map \( h^2 \) of the Hamiltonian flow \( h', t \in [0, 2] \) has a fixed point \( p \in M \), such that its trajectory under the flow is a contractible loop. Hence we get \( \psi_1(p) = \psi_2(p) \), and as a result, the loop \( t \mapsto h_3^t(p), t \in [0, 3] \) is contractible. Therefore the value of the evaluation map \( ev \) at the loop \( h_3^t, t \in [0, 3] \) equals \( e \in \pi_1(M) \).

Finally, it is easy to see the independence of \( ev' \) of the choice of metric \( g \), and that \( ev' \) is a homomorphism.

\[ \square \]

The main step in the proof of Theorem 1.3 is the following proposition:

**Proposition 2.2.** Let \((M, \omega)\) be a closed symplectic manifold, and let \( \tilde{\phi} \in \tilde{H}_0 \). Then \( \text{Flux}(\tilde{\phi}) \in \Gamma_{\text{top}}, \) where \( a = ev'(\tilde{\phi}) \in \pi_1(M) \).

*Proof.\* Choose a Riemannian metric \( g \) on \( M \). Denote by \( \phi \in G \) the endpoint of \( \tilde{\phi} \), and let \( \phi', t \in [0, 1] \) be a symplectic isotopy of \( M \) representing \( \tilde{\phi} \), such that \( \phi^0 = \text{id}_M \), \( \phi^1 = \phi \). Choose a smooth closed loop \( \gamma : [0, 1] \to M \), and define the loop \( \alpha = \phi \circ \gamma \). There exists a neighbourhood \( U \) of \( \alpha([0, 1]) \) which is symplectomorphic to the product of a neighbourhood of the zero-section in \( T^*S^1 \) and \( n - 1 \) small 2-dimensional discs, i.e. there exists \( \epsilon > 0 \), such that for \( W(\epsilon) = \{(q, p) \in T^*S^1 \mid |p| < \epsilon \} \subset T^*S^1 \) (here \( S^1 \cong \mathbb{R}/\mathbb{Z} \), so that the symplectic area of \( W(\epsilon) \) in \( T^*S^1 \) is \( 2\epsilon \)) and for the standard

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1Strictly speaking, except for the case of a semi-positive symplectic manifold, the existing proofs of the Arnold’s conjecture are conditional to the virtual cycle techniques which are not yet accepted by all the experts.
2-dimensional disc $B(\epsilon) \subset \mathbb{R}^2$ of area $\epsilon$ centered at the origin, we have a symplectic embedding $\iota : W(\epsilon) \times B(\epsilon)^{n-1} \to M$, such that $S^1 \times \{0\} \times \ldots \times \{0\}$ is mapped onto $\alpha([0,1])$, where $S^1 \subset T^* S^1$ is the zero-section. We set $U = \iota(W(\epsilon) \times B(\epsilon)^{n-1})$.

Now let $\psi \in H$ be sufficiently $C^0$-close to $\phi$, and let $\psi^t, t \in [0,1]$ be a Hamiltonian isotopy on $M$ such that $\psi^1 = \psi$. Define, as in the proof of Lemma 2.1, a continuous loop $f^t$, $t \in [0,3]$ in $\text{Map}_0(M)$, such that $f^t = \phi^t$ for $t \in [0,1]$, such that for any $x \in M$, the path $f^t(x)$, $t \in [1,2]$ is the shortest $g$-geodesic connecting $\phi(x)$ and $\psi(x)$, and such that $f^t = \psi^{3-t}$ for $t \in [2,3]$. Then since $\psi$ is sufficiently $C^0$-close to $\phi$, Lemma 2.1 tells us that the value of $ev$ at the loop $f^t$, $t \in [0,3]$ equals to $ev'(\phi)$. Define smooth cylinders $w, u, v : [0,1] \times [0,1] \to M$ by $w(s,t) = f^t(\gamma(s)) = \phi^t(\gamma(s))$, $u(s,t) = f^{\epsilon + 1}(\gamma(s))$, $v(s,t) = f^{\epsilon + 2}(\gamma(s)) = \psi^{1-t}(\gamma(s))$, for $(s,t) \in [0,1] \times [0,1]$. The loop $L := \psi \circ \gamma$ is $C^0$-close to the loop $\alpha = \phi \circ \gamma$, hence the image $\beta([0,1])$ lies inside $U$, and moreover the image $u([0,1] \times [0,1])$ lies inside $U$. The union of the images of $w, u, v$ is a torus, which is the isotopy of the loop $\gamma$ via the path $f^t$, $t \in [0,3]$. We therefore have the equality $\omega(w) + \omega(u) + \omega(v) = \text{Flux}(f^t)(\gamma)$. We have $\omega(w) = \text{Flux}(\phi^t)(\gamma)$, and $\omega(v) = 0$ since the isotopy $\psi^t$, $t \in [0,1]$ is Hamiltonian. Thus we get $\text{Flux}(f^t)(\gamma) - \text{Flux}(\phi^t)(\gamma) = \omega(u)$. Hence it is enough to show that for any initially chosen loop $\gamma : [0,1] \to M$ as above, the symplectic area $\omega(u)$ is arbitrarily small, provided that $\psi$ is sufficiently $C^0$-close to $\phi$.

Let us show this by a contradiction. Assume the contrary, i.e. that there exists some $\epsilon > 0$, such that one can find $\psi$ arbitrarily $C^0$-close to $\phi$ for which we have $|\omega(u)| \geq \epsilon$. WLOG we may assume that $\epsilon < \epsilon$. Now pick $\psi \in H$ which is sufficiently $C^0$-close to $\phi$ and such that $|\omega(u)| \geq \epsilon$. Recall that we have a symplectic embedding $\iota : W(\epsilon) \times B(\epsilon)^{n-1} \to M$, such that $S^1 \times \{0\} \times \ldots \times \{0\}$ is mapped onto $\alpha([0,1])$, and we have $U = \iota(W(\epsilon) \times B(\epsilon)^{n-1})$. Put $\delta = \epsilon/2$, and consider the Lagrangian $L = S^1 \times S(\delta) \times \ldots \times S(\delta) \subset W(\epsilon') \times B(\epsilon')^{n-1} \subset W(\epsilon) \times B(\epsilon)^{n-1}$, where $S(\delta) = \{ z \in \mathbb{R}^2 | \pi |z|^2 = \delta \}$ is the circle on $\mathbb{R}^2$ centered at the origin enclosing a disc of area $\delta$. Since $\psi$ is sufficiently $C^0$-close to $\phi$, it follows that $\psi \circ \phi^{-1}$ is sufficiently $C^0$-close to $\iota$ and in particular $\psi \circ \phi^{-1}((\iota(W(\epsilon') \times B(\epsilon')^{n-1})) \subset U = \iota(W(\epsilon) \times B(\epsilon)^{n-1})$, and $\psi \circ \phi^{-1}(\iota(L)) \subset \iota(W(\epsilon')/4 \times B(\epsilon)^{n-1})$ (recall that $B(\epsilon') = B(\epsilon) \setminus \{0\} \subset \mathbb{R}^2$ is the open punctured euclidean disc centered at the origin having area $\epsilon$). Now, if we denote $\bar{L} := \iota^{-1}(\iota(L)) \subset U$, and $\pi_1(\bar{L})$ is generated by the loops $\tilde{\beta}_1, \ldots, \tilde{\beta}_n$, which are the push-forwards of the loops $\bar{\beta}_1, \ldots, \bar{\beta}_n$ on $L = S^1 \times S(\delta) \times \ldots \times S(\delta)$, such that the homotopy classes of $\beta_1, \ldots, \beta_n$ in $\pi_1(L)$ correspond to the factors $S^1, S(\delta), \ldots, S(\delta)$. Consider the 1-form $\lambda = p_0 dq_0 + \frac{1}{2}(x_1 dy_1 - y_1 dx_1) + \ldots + \frac{1}{2}(x_{n-1} dy_{n-1} - y_{n-1} dx_{n-1}) = p_0 dq_0 + \frac{1}{2}x_1^2 d\theta_1 + \ldots + \frac{1}{2}x_{n-1}^2 d\theta_{n-1}$ on $W(\epsilon) \times B(\epsilon)^{n-1}$. Then $d\lambda = dp_0 \wedge dq_0 + dx_1 \wedge dy_1 + \ldots + dx_{n-1} \wedge dy_{n-1}$ is the standard symplectic form on $W(\epsilon) \times B(\epsilon)^{n-1}$. Since the map $\iota^{-1} \circ \psi \circ \phi^{-1} \circ \iota$ is well defined on $W(\epsilon') \times B(\epsilon')^{n-1}$, and is symplectic, the 1-form $(\iota^{-1} \circ \psi \circ \phi^{-1} \circ \iota)\ast \lambda - \lambda$ on $W(\epsilon') \times B(\epsilon')^{n-1}$ is closed, and its evaluation at
the loop \( S^1 \times \{0\} \times \ldots \times \{0\} \subset W(\epsilon') \times B'(\epsilon')^{\times n-1} \) equals to the symplectic area \( \omega(u) \).

Hence at the level of cohomology we have \( [(i^{-1} \circ \psi \circ \phi^{-1} \circ i)^{\star} \lambda - \lambda] = \omega(u)[dq] \). In particular, we have \( \lambda(\tilde{\beta}_1) = (i^{-1} \circ \psi \circ \phi^{-1} \circ i)^{\star} \lambda(\beta_1) = \omega(u) + \lambda(\beta_1) = \omega(u) \), and for \( 2 \leq j \leq n \) we have \( \lambda(\tilde{\beta}_j) = (i^{-1} \circ \psi \circ \phi^{-1} \circ i)^{\star} \lambda(\beta_j) = \lambda(\beta_j) = \delta \). Now let us present two possible ways of finishing the proof via arriving to a contradiction. The second way is easier and it was suggested by Seyfaddini.

**First way:** Consider the 1-form \( \lambda' = \frac{1}{2}(x_0 dy_0 - y_0 dx_0) + \frac{1}{2}(x_1 dy_1 - y_1 dx_1) + \ldots + \frac{1}{2}(x_{n-1} dy_{n-1} - y_{n-1} dx_{n-1}) = \frac{1}{2}r_0^2 d\theta_0 + \frac{1}{2}r_1^2 d\theta_1 + \ldots + \frac{1}{2}r_{n-1}^2 d\theta_{n-1} \) on \( B(\epsilon'/2) \times B'(\epsilon')^{\times n-1} \) endowed with coordinates \((x_0, y_0, \ldots, x_{n-1}, y_{n-1})\), where \( x_i = r_i \cos \theta_i, y_i = r_i \sin \theta_i \) for \( i = 0, 1, \ldots, n-1 \). We have that \( d\lambda' = dx_0 \wedge dy_0 + dx_1 \wedge dy_1 + \ldots + dx_{n-1} \wedge dy_{n-1} \) is the standard symplectic form on \( B(\epsilon'/2) \times B'(\epsilon')^{\times n-1} \).

Consider the embedding \( \iota': W(\epsilon'/4) \times B'(\epsilon')^{\times n-1} \hookrightarrow B(\epsilon'/2) \times B'(\epsilon')^{\times n-1} \), given by \( \iota'(q_0, p_0, x_1, y_1, \ldots, x_{n-1}, y_{n-1}) = (r_0, \theta_0, x_1, y_1, \ldots, x_{n-1}, y_{n-1}), \pi r_0^2 = p_0 + \epsilon'/4, \theta_0 = 2\pi q_0 \). Then we have \( (\iota')^{\star} \lambda' = \lambda + \frac{\epsilon'}{4} dq_0 \). Hence for the Lagrangian \( \tilde{L} = \iota'(\tilde{L}) \subset B(\epsilon'/2) \times B'(\epsilon')^{\times n-1} \subset \mathbb{R}^2 \times B'(\epsilon')^{\times n-1} \), and the loops \( \tilde{\beta}_j := \iota' \circ \tilde{\beta}_j, j = 1, \ldots, n \), generating \( \pi_1(\tilde{L}) \), we have \( \lambda'(\tilde{\beta}_1) = \omega(u) + \epsilon'/4 \).

Therefore, if we consider \( \tilde{L} \) as a Lagrangian inside \( \mathbb{R}^2 \times B'(\epsilon')^{\times n-1} \) endowed with the standard symplectic form, then it follows that the symplectic area of any disc in \( \mathbb{R}^2 \times B'(\epsilon')^{\times n-1} \) with boundary on \( \tilde{L} \), is an integer multiple of \( \omega(u) + \epsilon'/4 \), and hence its absolute value is \( \geq |\omega(u)| - \epsilon'/4 \geq \epsilon' - \epsilon'/4 = 3\epsilon'/4 > \epsilon'/2 \). By the Chekanov’s theorem [Ch], the displacement energy \( e(\tilde{L}) \) of \( \tilde{L} \) inside \( \mathbb{R}^2 \times B'(\epsilon')^{\times n-1} \) is greater than or equal to the minimal area of a non-constant holomorphic disc with boundary on \( \tilde{L} \). Hence we conclude that \( e(\tilde{L}) > \epsilon'/2 \). But on the other hand, since \( \tilde{L} \subset B(\epsilon'/2) \times B'(\epsilon')^{\times n-1} \), one can clearly displace \( \tilde{L} \) with a Hamiltonian isotopy of energy \( \epsilon'/2 \). Contradiction.

**Second way:** Consider the Liouville form \( \lambda' = p_0 dq_0 + p_1 dq_1 + \ldots + p_{n-1} dq_{n-1} \) on \( T^*\mathbb{T}^n \). Let \( \iota': W(\epsilon'/4) \times B'(\epsilon')^{\times n-1} \rightarrow T^*\mathbb{T}^n \) be the symplectic embedding given by

\[
\iota'(q_0, p_0, r_1, \theta_1, \ldots, r_{n-1}, \theta_{n-1}) = (q_0, p_0, q_1, p_1, \ldots, q_{n-1}, p_{n-1}), \quad \pi r_i^2 - \delta = p_i, \quad \theta_i = 2\pi q_i \text{ for } i = 1, 2, \ldots, n-1.
\]

The image of \( \iota' \) lies inside \( W(\epsilon'/4) \times (T^*S^1)^{\times n-1} = W(\epsilon'/4) \times T^*\mathbb{T}^{n-1} \). We have \( (\iota')^{\star} \lambda' = \lambda - \frac{\delta}{2\pi} d\theta_1 - \ldots - \frac{\delta}{2\pi} d\theta_{n-1} \). Hence for the Lagrangian \( \tilde{L} = \iota'(\tilde{L}) \subset W(\epsilon'/4) \times T^*\mathbb{T}^{n-1} \subset T^*\mathbb{T}^n \) and the loops \( \tilde{\beta}_j := \iota' \circ \tilde{\beta}_j, j = 1, \ldots, n \), generating \( \pi_1(\tilde{L}) \), we have \( \lambda'(\tilde{\beta}_1) = \omega(u) \), and \( \lambda'(\tilde{\beta}_j) = 0, 2 \leq j \leq n \). Now consider the symplectic shift \( \Phi: T^*\mathbb{T}^n \rightarrow T^*\mathbb{T}^n \) given by \( \Phi(q_0, p_0, q_1, p_1, \ldots, q_{n-1}, p_{n-1}) = (q_0, p_0 - \omega(u), q_1, p_1, \ldots, q_{n-1}, p_{n-1}) \). Then it follows that the shifted Lagrangian \( \Phi(\tilde{L}) \subset T^*\mathbb{T}^n \) is exact, and moreover it does not intersect the zero-section, since \( \tilde{L} \subset W(\epsilon'/4) \times T^*\mathbb{T}^{n-1} \) and \( |\omega(u)| \geq \epsilon' > \epsilon'/4 \). However, by the theorem of Gromov [C] (see section 2.3.B' in [C]), a closed exact Lagrangian submanifold of a cotangent bundle must intersect the zero-section. Contradiction. \( \square \)

Now, Theorem [1.3] is a straightforward consequence of Proposition [2.2]
Proof of Theorem 1.3. By Proposition 2.2, for any $\tilde{\phi} \in \tilde{H}_0$ we have $\text{Flux}(\tilde{\phi}) \in \Gamma^a_{\text{top}} = \Gamma^a_{\text{top}} + \Gamma^e_{\text{top}} \subseteq \Gamma_{\text{top}} + \Gamma^e_{\text{top}}$, where $a = ev'(\tilde{\phi}) \in \pi_1(M)$. Hence $\Gamma_0 \subseteq \Gamma_{\text{top}} + \Gamma^e_{\text{top}}$. 

Now we turn to the proof of Theorem 1.6.

Proof of Theorem 1.6. It is clearly enough to prove that for any smooth embedded loop $\gamma : [0,1] \to M$, there exist constants $c = c(M, \omega, g, \gamma), C = C(M, \omega, g, \gamma)$, such that for any path $\phi^t, t \in [0,1]$ of symplectomorphisms of $M$, $\phi^0 = \text{id}_M, \phi^1 = \phi$, with $\text{max}_{t \in [0,1]} d(\text{id}_M, \phi^t) < c$, we have $|\text{Flux}(\phi^t)(\gamma)| \leq C d(\text{id}_M, \phi)$. Now fix a smooth embedded loop $\gamma : [0,1] \to M$. Then a neighbourhood of $\gamma([0,1])$ is standard, and hence for some $\epsilon > 0$ one can find a symplectic embedding $\iota : W(\epsilon) \times B(\epsilon)^{n-1} \to M$, such that $\iota(S^1 \times \{0\} \times \ldots \times \{0\}) = \gamma([0,1])$. We set

$$c_1 = d(\iota(W(\epsilon/2) \times S(\epsilon/3)^{n-1}), M \setminus \iota(W(\epsilon) \times B'(\epsilon)^{n-1}))$$

(recall that the notation $B'(\epsilon) = B(\epsilon) \setminus \{0\} \subset \mathbb{R}^2$ stands for the punctured disc). Now let $\phi^t, t \in [0,1]$ be a path of symplectomorphisms of $M$, $\phi^0 = \text{id}_M, \phi^1 = \phi$, with $\text{max}_{t \in [0,1]} d(\text{id}_M, \phi^t) < c_1$. Define the “flux function” $\kappa : [0,1] \to \mathbb{R}$ by $\kappa(t) = \text{Flux}(\phi^t)(\gamma)$. We assume that $\kappa(1) = \text{Flux}(\phi)(\gamma) \neq 0$. If $\text{max}_{t \in [0,1]} |\kappa(t)| > \epsilon/3$, then we define $T = 0$ to be minimal such that $|\kappa(T)| = \epsilon/3$, otherwise we set $T = 1$. We have $|\kappa(t)| \leq \epsilon/3$ for all $t \in [0, T]$. Now, on $\iota(W(\epsilon) \times B(\epsilon)^{n-1})$, consider the time dependent vector field $X^t_\kappa = \kappa(t) \frac{\partial}{\partial p_0}$, where $(q_0, p_0, x_1, y_1, ..., x_{n-1}, y_{n-1})$ are the standard coordinates on $W(\epsilon) \times B(\epsilon)^{n-1}$. Denote by $X^t$ the time dependent symplectic vector field of the flow $\phi^t$. The difference $Y^t = X^t - X^t_\kappa, t \in [0, T]$, is a time dependent Hamiltonian vector field on $\iota(W(\epsilon) \times B(\epsilon)^{n-1})$. With help of a cut-off, we can find a time dependent Hamiltonian vector field $Z^t, t \in [0, T]$ on $M$, such that $Y^t(x) = Z^t(x)$ for any $x \in \iota(W(\epsilon/2) \times B(\epsilon/2)^{n-1})$ and $t \in [0, T]$. Now look at the time dependent symplectic vector field $\tilde{X}^t = X^t - Z^t, t \in [0, T]$, on $M$, and denote by $\tilde{\phi}^t, t \in [0, T]$ its symplectic flow on $M$. Then $\psi^t := (\phi^t)^{-1} \circ \tilde{\phi}^t$, $t \in [0, T]$ is a Hamiltonian flow on $M$ since it has zero flux at all times. Consider the Lagrangian $L = \iota(S^1 \times S(\epsilon/3) \times \ldots \times S(\epsilon/3)) \subset \iota(W(\epsilon) \times B(\epsilon)^{n-1}) \subset M$. We have that $\tilde{X}^t(x) = \tilde{X}^t_\kappa(x)$ for any $x \in \iota(W(\epsilon/2) \times B(\epsilon/2)^{n-1})$ and $t \in [0, T]$, and hence $\tilde{\phi}^t(\iota(q_0, p_0, x_1, y_1, ..., x_{n-1}, y_{n-1})) = \iota(q_0, p_0 + \kappa(t), x_1, y_1, ..., x_{n-1}, y_{n-1})$ whenever $\iota(q_0, p_0, x_1, y_1, ..., x_{n-1}, y_{n-1}) \in L$ and $t \in [0, T]$, and so for any $t \in [0, T]$, $\tilde{\phi}^t(L)$ is obtained from $L$ by shifting it by $\kappa(t)$ in the “$p_0$ direction”. Clearly there exists a constant $c_2 = c_2(M, \omega, g, \gamma, \iota)$, such that the distance $d(\tilde{\phi}^t(L), L)$ is greater than or equal to $c_2|\kappa(T)|$. Also note that since $d(\text{id}_M, \phi^t) < c_1$ for any $t$, we get that $\psi^t(L) = (\phi^t)^{-1} \circ \tilde{\phi}^t(L) \subset \iota(W(\epsilon) \times B'(\epsilon)^{n-1})$ for any $t \in [0, T]$. Now, $L$ cannot be Hamiltonianly displaced inside $\iota(W(\epsilon) \times B'(\epsilon)^{n-1})$, and so we must have $\psi^T(L) \cap L = (\phi^T)^{-1} \circ \tilde{\phi}^T(L) \cap L \neq \emptyset$. Since in addition we have $d(\tilde{\phi}^T(L), L) \geq c_2|\kappa(T)|$, we conclude that $d(\text{id}_M, \phi^T) \geq c_2|\kappa(T)|$. 

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We have shown that for any path \( \phi^t, \ t \in [0, 1] \) of symplectomorphisms of \( M \), \( \phi^0 = \text{id}_M, \phi^1 = \phi \), with \( \max_{t \in [0,1]} d(\text{id}_M, \phi^t) < c_1 \), we must have \( d(\text{id}_M, \phi^T) \geq c_2|\kappa(T)| \). Thus, if we set \( c = \min(c_1, c_2 \epsilon/3) \) and \( C = 1/c_2 \), then for any path \( \phi^t, \ t \in [0, 1] \) of symplectomorphisms of \( M \), \( \phi^0 = \text{id}_M, \phi^1 = \phi \), with \( \max_{t \in [0,1]} d(\text{id}_M, \phi^t) < c \), we have \( c_2 \epsilon/3 > c \geq d(\text{id}_M, \phi^T) \geq c_2|\kappa(T)| \), hence \( |\kappa(T)| \neq \epsilon/3 \), which means that \( T = 1 \), and we therefore get \( |\text{Flux} \{ \{ \phi^t \} \}(\gamma)| = |\kappa(1)| = |\kappa(T)| \leq C d(\text{id}_M, \phi^T) = C d(\text{id}_M, \phi) \).

\[ \square \]

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