Abstract

We describe how the methods of quantum field theory and the renormalisation group may be applied to classical stochastic particle systems which appear in non-equilibrium statistical mechanics. An emphasis is placed on the similarities and differences between these methods and more conventional applications of quantum field theory. Some simple applications are discussed.

1 Introduction

Reaction-diffusion problems are simple examples of non-equilibrium statistical systems. While it has long been recognised that the methods of quantum field theory extend far beyond their original domain of application in elementary particle physics, to, for example, many body quantum systems in condensed matter physics, and to the theory of both the statics and dynamics of critical behaviour in equilibrium statistical mechanics, it is less well appreciated that they also provide a powerful tool for analysing classical statistical systems far from equilibrium. This application is certainly not new.\cite{1, 2, 3}
but it is only relatively recently that the full formalism of the renormalisation group has been brought to bear \[4, 5, 6\] in a very similar manner to that employed in equilibrium critical behaviour.

Reaction-diffusion problems are examples of classical stochastic particle systems. These particles are in general labelled by their species \(A, B, \ldots\), which may be thought of as corresponding to different chemical reactants. Their dynamics consists of two elements: first, the particles diffuse according to some kind of Brownian motion, with diffusion constants \(D_A, D_B, \ldots\). In a simulation, this might be described by random walks on a lattice. Second, they undergo reactions, for example \(A + B \rightarrow C\), at prescribed rates \(\lambda_{AB}\), whenever they meet. It is important that these processes are diffusion-limited, that is, the particles have to diffuse around before they find each other and react. In a chemical system this would mean that the reactants should not be stirred. In practice this may be realised by allowing the reactants to diffuse in a gel, or on a substrate. It is important to realise that these reactions are not usually reversible, and indeed the most interesting cases occur when they are completely irreversible.

The potential applications of these ideas to systems in chemistry, biology and physics are limited only by the imagination of the reader, and this is not the place to discuss them. Rather, we shall focus on some general features as illustrated by some rather simple examples. In general, from some random initial state, these systems will evolve in time towards a steady state described by some stationary probability distribution. Since the dynamics does not satisfy detailed balance, this is not in general a Gibbs measure. In some very simple cases, the steady state is trivial. For example, in the simplest system of all, with a single species of particle \(A\) undergoing the pure annihilation reaction \(A + A \rightarrow \) inert (the inert particles not influencing the reaction further), the steady state has no particles, and hence no fluctuations since it is a classical vacuum. But it turns out that the approach to this steady is critical in the sense that it exhibits universal scaling behaviour, critical exponents, and so on. This may be understood, and the universal features computed, within the quantum field theoretic renormalisation group approach to be described.

A second class of critical phenomena corresponds to a non-equilibrium phase transition in the steady state, as some parameter of the dynamics is varied. For example, if to the annihilation reaction described above the branching process \(A \rightarrow (m + 1)A\) at rate \(\sigma\) is added, it turns out that under
some circumstances there can be a transition at some finite value of $\sigma$ to a non-trivial stationary state with a finite density of particles.\[3\] \[4\] \[8\]. The universality class of this transition turns out to depend on the parity of $m$. As with equilibrium critical behaviour, the symmetries of a system are seen to play an important role in determining its universality class.

The layout of this paper is as follows. In the next section we describe the general formalism and illustrate it with the annihilation reaction $A + A \rightarrow$ inert. We comment on the similarities and the differences with ordinary many-body quantum theory, and on the connections between this approach to non-equilibrium behaviour and others via the Fokker-Planck equation and the Langevin equation. In Sec. 3 we discuss the renormalisation group approach to the $A + A \rightarrow$ inert reaction in more detail, and finally in Sec. 4 we consider the more difficult problem of when the branching process is added.

2 Formalism

The dynamics of such a stochastic particle system is described by a master equation governing the time evolution of the probabilities $p(\alpha; t)$ that the system be in a given microstate $\alpha$. For a system of particles on a lattice, for example, the $\alpha$s might label the occupation number basis $(n_1, n_2, \ldots)$, corresponding to having $n_j$ particles at site $j$. The master equation takes the form

$$dp(\alpha; t)/dt = \sum_\beta R_{\beta \rightarrow \alpha} p(\beta; t) - \sum_\beta R_{\alpha \rightarrow \beta} p(\alpha; t).$$

(1)

Here $R_{\alpha \rightarrow \beta}$ is the rate for transitions from state $\alpha$ into $\beta$; for a reaction diffusion-problem these are determined by the diffusion constants and the reaction rates.

Such classical particle problems have two features in common with relativistic many-body problems which renders a ‘second-quantised’ formalism particularly useful. First, particle numbers are not, in general, conserved: they are created and destroyed by the dynamics. Second, and more important, the master equation and the Schrödinger equation share the properties of being linear and first order in time. It is therefore not surprising that such a formalism is similarly successful for these stochastic problems.

The first step is to construct a Fock space from annihilation and creation operators satisfying the usual commutation relationships $[a_i, a_j^\dagger] = \delta_{ij}$, and
define the state vector

$$|\Psi(t)\rangle \equiv \sum_{\alpha} p(n_1, n_2, \ldots; t) a_{1}^{\dagger n_1} a_{2}^{\dagger n_2} \ldots |0\rangle. \quad (2)$$

Note that this is not normalised in the conventional manner, and what plays the role of a probability amplitude in quantum mechanics now is a probability. We have also chosen a bosonic representation, corresponding to the case when multiple occupancy of the sites is allowed. In simulations it is often more convenient to restrict the values of the $n_j$ to 0 or 1, in which case a representation in terms of Pauli operators is more appropriate. This leads to quantum spin models rather than immediately to quantum field theories, which, in some one-dimensional cases, turn out to be integrable. In fact there is a some very elegant mathematics in this branch of the subject.\footnote{[9]} For example the quantum group symmetry of certain spin chains appears very naturally from this perspective. However, from the point of view of understanding the renormalisation group approach and generalising to non-integer dimensions, the bosonic formulation is more useful. In any case, as long as we are studying problems where the average particle density is low, the probability of multiple occupation should be small and there should be no difference between the physical results of the two approaches.

The statement is now that the master equation (1) is completely equivalent to a Schrödinger-like equation for the state vector

$$d|\Psi(t)\rangle/dt = -H|\Psi(t)\rangle, \quad (3)$$

where the ‘hamiltonian’ $H$ is simply expressed in terms of the $a$s and the $a^\dagger$s. For example, for the reaction-diffusion problem $A + A \rightarrow$ inert, one finds that

$$H = D \sum_{(i,j)} (a_{i}^{\dagger} - a_{j}^{\dagger})(a_{i} - a_{j}) - \lambda \sum_{i} (a_{i}^{2} - a_{i}^{\dagger 2}a_{i}^{2}). \quad (4)$$

The simple hopping form of the first term (where the sum is over nearest neighbour pairs $(i,j)$), which corresponds to pure diffusion, is not surprising, but the second term may require some explanation. To understand its form, consider the simpler problem of the reaction at a single site. If $p(n; t)$ is probability of finding $n$ particles at this site, the master equation is simply

$$dp(n; t)/dt = \lambda(n+2)(n+1)p(n+2; t) - \lambda n(n-1)p(n; t), \quad (5)$$

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where the factors of \((n + 2)(n + 1)\) reflect the number of ways of choosing the pair of reacting particles. Defining \(|\Psi\rangle = \sum_n p(n; t)a^{i_n}|0\rangle\) as above, its equation of motion is

\[
d|\Psi\rangle/dt = \lambda \sum_n \left( (n + 2)(n + 1)p(n + 2) - n(n - 1)p(n) \right)a^{i_n}|0\rangle \tag{6}
\]

\[
= \lambda \sum_n \left( a^2 p(n + 2)a^{i_{n+2}} - a^2 a^2 p(n)a^{i_n} \right)|0\rangle \tag{7}
\]

\[
= \lambda (a^2 - a^2 a^2) \sum_n p(n)a^{i_n}|0\rangle. \tag{8}
\]

The second term in the reaction part of (4) therefore corresponds to the second term in the master equation (1), and is required by the conservation of probability.

From the lattice Hamiltonian (4) we may, if interested in long wavelength properties, proceed to the formal continuum limit

\[
H = \int \left[ D(\nabla a^\dagger)(\nabla a) - \lambda(a^2 - a^2 a^2) \right]d^dx, \tag{9}
\]

and thence to a representation as a path integral over fields \(a(x, t)\) and \(a^*(x, t)\) with a weight \(\exp(-S[a, a^*])\), with an action

\[
S \equiv \int [a^* \partial_t a + D(\nabla a^*)(\nabla a) - \lambda(a^2 - a^2 a^2)]dtd^dx. \tag{10}
\]

### 2.1 Differences from quantum mechanics

There are two immediately apparent differences from ordinary quantum field theory: first, there is no factor of \(i\) in the Schrödinger equation (3) – but this is familiar from euclidean formulations of conventional quantum theories; second, the Hamiltonian is not Hermitian. In many cases it will turn out that, nevertheless, its eigenvalues are real. (Complex eigenvalues correspond to oscillating states which are known to occur in some chemical reactions.) However, the most important difference is one of interpretation: expectation values of observables \(\mathcal{O}\) are not given by \(<\Psi|\mathcal{O}|\Psi\rangle\), since this would be bilinear, rather than linear, in the probabilities \(p(\alpha; t)\). Instead, for an observable which is diagonal in the occupation number basis, its expectation value is of course

\[
\mathcal{O} = \sum_{\{n_j\}} \mathcal{O}(\{n_j\})p(\{n_j\}; t), \tag{11}
\]
and it is straightforward to show that this may be expressed as

\[ \Omega = \langle 0 | e^{\sum_j a_j} O e^{-Ht} | \Psi(0) \rangle, \]  

(12)

since the state \( \langle 0 | e^{\sum_j a_j} \) is a left eigenstate of all the \( a_j \), with unit eigenvalue.

Conservation of probability then requires that \( \langle 0 | e^{\sum_j a_j} H = 0 \). This is equivalent to the requirement that \( H \) should formally vanish when every \( a_j \) is set to unity. The appearance of the state \( \langle 0 | e^{\sum_j a_j} \) may complicate some of the subsequent calculations, since the interaction part of the hamiltonian is not normal ordered with respect to it, and therefore the usual formalism of time-dependent perturbation theory and Wick’s theorem do not immediately apply. This problem may be avoided by first commuting the factor of \( e^{\sum_j a_j} \) through the operators \( O \) and \( H \) in (12). This has the effect of shifting \( a_j \rightarrow 1 + \bar{a}_j \), since \( e^{aa} = (1 + a) e^a \). The factor \( e^{\sum_j a_j} \) acting on the initial state \( | \Psi(0) \rangle \) is usually something simple, and the operators are now normal ordered.

Note that such a shift is convenient only if we are interested in what (in the language of particle physics) may be termed ‘inclusive’ probabilities, for example the expectation value of the local density \( \bar{n}_j = \langle a_j \rangle \). After the shift, we see that in fact \( \bar{n}_j = \langle a_j \rangle \), where \( \langle \cdot \rangle \) denotes the usual QFT expectation value. For so-called exclusive quantities, for example the probability \( \delta_{n_1} \prod_{j \neq i} \delta_{n_j,0} \) that there is only one particle in the system, at site \( i \), the factor \( e^{\sum_j a_j} \) simply reduces to \( a_j \) in the correlation function, and no shift is necessary.

2.2 Relation to other formalisms

Of course, there are several other important ways of formulating stochastic processes, through either the Langevin equation or its related Fokker-Planck equation. It is interesting to see how these emerge in the present formalism for the simple example under consideration. If we make the shift \( a^* = 1 + \bar{a} \) in the path integral, we find an action

\[ S[a, \bar{a}] = \int [\bar{a} \partial_t a + D(\nabla \bar{a})(\nabla a) + 2\lambda \bar{a} a^2 + \lambda \bar{a}^2 a^2] dt dx. \]  

(13)
The non-linear terms in $\bar{a}$ may be disentangled in terms of a Gaussian transformation

$$e^{-\lambda \bar{a}^2 a^2} = \int e^{-\eta \bar{a}} P(\eta) d\eta,$$

where $P(\eta)$ is suitable Gaussian distribution. The path integral over $\bar{a}$ may now be performed, yielding a functional delta function equivalent to the equation

$$\partial_t a = D \nabla^2 a - 2\lambda a^2 + \eta(x,t).$$

Neglecting the last term, this is just the so-called rate equation which one might write down as a first approximation to the equation of motion for the density (note that we have argued above that the expectation value of $a$ is the density.) The rate equation approximation assumes that the annihilation rate, which is proportional to the probability of finding two particles at the same point, is simply given by the square of the density. This approximation clearly neglects the correlations between the particles, and is on the same footing as the mean field approximation in equilibrium critical behaviour. In this sense (15) looks very much like a Langevin equation, with a noise term $\eta$. However, such equations are usually derived from the master equation through some kind of approximate coarse-graining, and the exact form of the noise term is often unclear, especially when the dynamics does not constrain this through detailed balance. By contrast, the correlations of the noise here are completely explicit

$$\langle \eta(x,t)\eta(x',t') \rangle = -\lambda a^2 \delta(x-x')\delta(t-t').$$

That the noise should depend on $a$ is expected, since there can be no noise when the density is zero. But the minus sign is surprising. It implies that the noise $\eta$ is pure imaginary, so that the solution of (15) is complex!

This curious result may be traced to the fact that, although the ‘quantum mechanical’ average $\langle a \rangle$ is the mean density $\bar{n}$, this is not true of higher moments. For example, the mean square density $\bar{n}^2$ is given by the average of $(a^\dagger a)^2 = a^\dagger^2 a^2 + a^\dagger a$. The operators $a^\dagger$ give unity acting to the left on the state $\langle 0 | e^a$, so that in fact $\bar{n}^2 = \langle a^2 \rangle + \langle a \rangle$. In general, one may show that, if $a$ has a Gaussian distribution, as it would in the case of pure diffusion, then the density $n$ would have a pure Poisson distribution. This is to be expected, since a simple random walk will have Poissonian statistics. The effect of the reactions is to modify this. In a sense the ‘noise’ $\eta$ in (15) represents only
that part of the physical noise which originates in the discreteness of the reaction process. Since, however, this cannot truly be disentangled from the diffusion noise, there is no need for its correlations to be positive.

The hamiltonian approach we have described above should be distinguished from another based on the Fokker-Planck equation. The latter begins from a coarse-grained Langevin description of the problem, and describes the time evolution of the probability distribution of the solution of this equation. Like the master equation, it is linear and first order in time, so may be cast in a hamiltonian formalism (in this case more usually called the liouvillean.) But the Fokker-Planck equation describes the diffusion in phase space and cannot easily accommodate processes where particles are continually being created and destroyed. It is therefore less useful for these types of problem.

3 Renormalisation group analysis

The field theory described by the action $\mathcal{L}$ is extremely simple. The bare propagator $(-i\omega + Dq^2)^{-1}$ is simply the Green function for the diffusion equation, and it may be represented by a directed line moving forward in time (conventionally from right to left.) The vertices are shown in Fig. 1. Immediately we see that, since the number of particles in a given intermediate state cannot increase as we move from right to left through a diagram, there can be no loop corrections to the propagator, hence no wave function renormalisation for the fields $a$ or $\bar{a}$. The only non-trivial renormalisation is that of the coupling constant $\lambda$, which may be seen through the loop corrections either to the vertex $\Gamma^{2,1}$ (shown in Fig. 1) or to $\Gamma^{2,2}$. (The fact that
these renormalise in the same way is a consequence of probability conservation, which relates the untruncated Green functions $\langle 0|a(t, x)a^\dagger(0, 0)|0 \rangle$ and $\langle 0|a^2(t, x)a^\dagger(0, 0)|0 \rangle$. It is interesting to note that probability conservation is not expressed through a Noether symmetry in this formalism.) The diagrams in Fig. 1 correspond to a simple inclusion-exclusion argument for the probability of finding two particles at the same point given that they have not reacted in the past. They may be simply summed, with the result that the one-loop renormalisation group beta function is exact:

$$\beta(g_R) = -\epsilon g_R + bg_R^2, \quad (17)$$

where $g_R$ is the dimensionless renormalised coupling, $b$ is a positive constant, and $\epsilon = 2 - d$.

For $d > 2$, then, $g_R$ is irrelevant in the infrared, and the rate equation (15) with no noise term is asymptotically valid, while for $d < 2$ it flows to a non-trivial fixed point $g^* = O(\epsilon)$. The consequences of this may be explored by writing down and solving the Callan-Symanzik equation in the usual way. Consider, for example, the mean density $n(t)$, which depends in principle on the initial density $n_0$ and the rate $\lambda$, expressed through $g_R$ and the normalisation scale $t_0$. (We work in units where the diffusion constant $D = 1$.) Then

$$n(t, n_0, g_R, t_0) = (t/t_0)^{-d/2} n(t_0, n_0(t/t_0)^{d/2}, g_R(t/t_0), t_0) \quad (18)$$

where the running coupling $\tilde{g}_R \to g^*$ as $t/t_0 \to \infty$. The simple exponent in the prefactor on the right hand side reflects the lack of any anomalous dimensions for the density. The right hand side may now be evaluated as a power series in $g^*$ (the lowest order being simply the mean field result), which is converted into a power series in $\epsilon$. Term by term, one may show that the result is in fact independent of the initial density $n_0$, so that in fact the whole expression is a universal function of $\epsilon$ only. The result is $n(t) \sim A/t^{d/2}$ where

$$A = \frac{1}{4\pi\epsilon} + \frac{2 \ln 8\pi - 5}{16\pi} + O(\epsilon). \quad (19)$$

Universal forms for correlation functions may be derived in a similar manner. The result is that the whole probability distribution for fluctuations becomes universal in the late time regime.
4 Branching and annihilating random walks

A more interesting type of critical behaviour emerges if to the annihilation reaction $A + A \rightarrow$ inert we add the branching process $A \rightarrow (m + 1)A$, with rate $\sigma$. Here $m$ is a positive integer. In the rate equation approximation, the mean density now satisfies

$$dn/dt = \sigma mn - \lambda n^2,$$

which would suggest that, for any $\sigma > 0$, the steady state has a non-zero density of particles (it is ‘active’ in the language of catalysis.) In fact, simulations suggest that this is only true in sufficiently high dimensions $d > 2$. For $d \leq 2$ the fluctuations need to be taken into account, and this may be done using the formalism described above.

In one dimension, with $m = 2$, this model may be interpreted as a dynamic Ising model, where the particles play the role of domain walls. In that case the processes of diffusion and annihilation are generated by single spin flips (these are assumed to occur at zero temperature so that pair creation of domain walls is suppressed), and the branching process $A \rightarrow 3A$ is associated with spin exchange (assumed to occur at infinite effective temperature). Since this model has two ‘temperatures’, it does not satisfy detailed balance and the stationary state is not Gibbsian. For that reason the model may undergo a nontrivial ordering transition, even in one dimension.

The additional term in the hamiltonian has the form

$$H_b = \sigma \int [a^\dagger a^{m+1} a - a^\dagger a]d^d x.$$  \hspace{1cm} (21)

Note that there is a difference depending on the parity of $m$. If it is even, then the number of particles is locally conserved modulo 2 by both the annihilation and the branching process, while for $m$ odd the latter violates this. For $m$ even this is manifested in the formal symmetry of the hamiltonian under $(a, a^\dagger) \rightarrow (-a, -a^\dagger)$. Note that if we make the shift $a^\dagger \rightarrow 1 + \bar{a}$ in order to develop the perturbative expansion for ‘inclusive’ processes, this symmetry becomes hidden. If we further make an expansion of the hamiltonian in powers of $\bar{a}$, and drop higher orders on the grounds of irrelevance by power counting, the symmetry is completely lost. This is evidently a dangerous thing to do, since it is well known that symmetries play a very important role in influencing universality classes of critical behaviour. Fortunately
the renormalisation group behaviour of the theory should be independent of which type of correlation functions we choose to study, and therefore may be computed in the unshifted theory where the symmetry is manifest.

4.1 Case of even $m$

The first question to be addressed is whether the branching rate $\sigma$ is relevant at the pure annihilation fixed point, that is, in the beta function $\beta_\sigma = -y\sigma + O(\sigma^2)$, is $y > 0$? If so, then as soon as $\sigma \neq 0$ it will flow away in the infrared into what is presumably the fixed point controlling the active phase. For $d > 2$, the pure annihilation reaction is controlled by the gaussian fixed point, so simple power counting suffices. This yields $y = 2$, consistent with the simulation results. For $d < 2$, the annihilation fixed point is accessible within the $\epsilon = 2 - d$ expansion, and so $y$ may be computed only perturbatively. The result is that

$$y = 2 - \frac{1}{2}m(m + 1)\epsilon + O(\epsilon^2).$$

Notice that the $O(\epsilon)$ corrections are large in $d = 1$, but no conclusion may be drawn from this since the higher terms have been neglected.

Fortunately it is possible to compute $y$ exactly in $d = 1$. There are several ways of doing this, but the simplest is to realise that in this limit it becomes a kind of a free fermion problem. This is because going to the annihilation fixed point $g_R \rightarrow g^*$ corresponds to taking the limit of the bare coupling $\lambda \rightarrow \infty$. In that case the term $\lambda a^\dagger_i a^2$ in (9) corresponds to an infinite hard core repulsion, so that the particles behave in one dimension like free fermions, at least in those periods of time evolution during which the other terms in the hamiltonian do not play a role. (For this reason the problem is not completely equivalent to free fermions.) In that limit, it does not make sense to create the new particles at the same lattice site. The best one can do is to distribute them between $m$ neighbouring sites, so that the corresponding term in the lattice hamiltonian has the form

$$\sigma \sum_j \prod_{i=j-m}^{j+m} c^\dagger_i c_j,$$

where $c^\dagger_i$ and $c_j$ are now anticommuting operators. In the continuum limit, we may make a Taylor expansion of each $c^\dagger_i$ about $i = j$, in powers of the
lattice spacing $b$. The lowest surviving term has the form

$$\tilde{\sigma} \int c^\dagger (\partial c^\dagger)(\partial^2 c^\dagger) \cdots (\partial^m c^\dagger) c \, dx,$$

(24)

where $\tilde{\sigma} = \sigma b^{(m+1)/2}$ is now the effective expansion parameter in the continuum limit. This extra factor modifies the dimensional analysis, which then implies that

$$y = 2 - \frac{1}{2} m (m + 1).$$

(25)

Thus the $O(\epsilon)$ result in (22) appears to be exact for $d = 1$. We have no simple explanation of this, as we have also computed explicitly the $O(\epsilon^2)$ terms and find them to be non-zero.

However, this does imply that the branching is irrelevant at the pure annihilation fixed point for $d = 1$, and hence the infrared or late time behaviour for sufficiently small $\sigma$ should be that of the pure annihilation process, with finitely renormalised parameters (for example, the diffusion constant.) It may also be shown that, even if the original branching process does not allow for $m = 2$ processes, these will inevitably be generated under renormalisation, and, since this coupling is the most relevant, it controls the late time behaviour. Physically both these effects may be understood as follows. In pure annihilation, the surviving particles become strongly anticorrelated in space. This is because each sweeps out a region around itself: for $d < 2$, every test particle placed within that region has probability one of eventually annihilating with it. (For $d > 2$ the test particle may escape.) When a small branching rate is turned on, the single particles occasionally branch into bunches of 3, 5, ... particles but these stay close together, and almost always annihilate with their siblings before visiting other bunches. The effect is therefore of diffusing bunches, which behave in many ways like single particles with a reduced diffusion constant. Clearly even if branchings only with $m > 2$ are allowed, the pair annihilation process will generate an effective $m = 2$ branching rate.

For larger values of the branching rate $\sigma$ there should be a transition to the active state, which should correspond to some nontrivial fixed point of the renormalisation group. But it seems to be very difficult to analyse this within any perturbative renormalisation group scheme. This is because the problem has two critical dimensionalities: $d = 2$ associated with the nontrivial nature of the annihilation, and $d \approx \frac{4}{3}$ where the value of $y$ changes

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sign, and therefore no systematic $\epsilon$-expansion is possible. So far we have not been able to find another small parameter, and the best we can do is a truncated loop expansion in fixed number of dimensions. This leads to the expected fixed point, but the estimated values for the critical exponents are far from those measured in simulations.

4.2 Case of odd $m$

Although the above analysis might suggest that for $m = 1$ the branching rate is relevant even when $d = 1$, so that there is no nontrivial transition, this is not the case, since now there is no conservation law modulo 2, and the process $A \rightarrow 0$ is immediately generated under renormalisation. This has eigenvalue 2 and corresponds to the generation of a mass gap in the theory. In fact one may show that for small branching rates the mean density should decay exponentially to zero. This conclusion is valid even when $d = 2$: although in this case the annihilation rate which generates the new term is irrelevant, it is only logarithmically vanishing, and meanwhile the rate for the process $A \rightarrow 0$ is growing under the renormalisation group. Once again, for sufficiently large $\sigma$, there should be a transition to the active state. In this case it is rather easier to analyse. On including the effective term $\Delta(a - a^\dagger a)$ in the hamiltonian, corresponding to $A \rightarrow 0$, and making the shift $a^\dagger \rightarrow 1 + \bar{a}$, one finds an interaction hamiltonian of the form

$$H_{\text{int}} = \int \left[ \delta \bar{a} a + \mu_1 \bar{a} a^2 - \mu_2 \bar{a}^2 a + \cdots \right] d^d x,$$

where $\mu_1$ and $\mu_2$ are positive constants, and $\delta$ may change sign (as it does at the mean field transition.) The omitted terms are of higher order, and their neglect is, this time, justified, since there is no symmetry relating them to the lower order terms. This is a well-known theory[12] which describes the universality class known as directed percolation (DP), although it was first studied by particle physicists in the context of reggeon field theory. Generically, any dynamical phase transition from an inactive state, with no noise, to active one, is in the DP universality class, and this has been verified for a number of models. The branching and annihilating random walks for $m$ even and $d = 1$ are therefore an interesting exception to this general rule. They evade it because they posses an additional conservation law. This is of course quite a familiar idea from equilibrium critical behaviour.
5 Conclusions

These simple examples I hope illustrate the point that quantum field theory still has many unexplored applications, which are not limited to quantum systems nor to equilibrium critical behaviour. Perhaps we are not yet at the stage when the mathematical beauty of such applications is apparent, but the richness of the subject is such that I believe that this may well emerge in the years to come.

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