FINITELY \( \mathcal{F} \)-AMENABLE ACTIONS AND DECOMPOSITION COMPLEXITY OF GROUPS

ANDREW NICAS AND DAVID ROSENTHAL

ABSTRACT. In his work on the Farrell-Jones Conjecture, Arthur Bartels introduced the concept of a “finitely \( \mathcal{F} \)-amenable” group action, where \( \mathcal{F} \) is a family of subgroups. We show how a finitely \( \mathcal{F} \)-amenable action of a finitely generated group \( G \) on a compact metric space, where the asymptotic dimensions of the elements of \( \mathcal{F} \) are bounded from above, gives an upper bound for the asymptotic dimension of \( G \). We generalize this to families \( \mathcal{F} \) whose elements are contained in a collection, \( \mathcal{C} \), of metric families that satisfies some basic permanence properties: If \( G \) is a finitely generated group and each element of \( \mathcal{F} \) belongs to \( \mathcal{C} \) and there exists a finitely \( \mathcal{F} \)-amenable action of \( G \) on a compact metrizable space, then \( G \) is in \( \mathcal{C} \). Examples of such collections of metric families include: metric families with weak finite decomposition complexity, exact metric families, and metric families that coarsely embed into Hilbert space.

1. Introduction

The celebrated Farrell-Jones Conjecture asserts that certain “assembly maps” are isomorphisms. This conjecture is central to the modern study of high dimensional topology, see [Lüc10] and [Bar] for an overview. Building on his approach in [Bar16], Bartels formulated the following concept of a “finitely \( \mathcal{F} \)-amenable action” in order to relate some of the key geometric conditions used to establish the Farrell-Jones Conjecture for many classes of groups to various notions of amenability arising in geometric group theory and analysis ([Bar17, Remark 0.4]).

Definition ([Bar17, Definition 0.1]). Let \( \mathcal{F} \) be a family of subgroups of \( G \) and let \( N \) be a non-negative integer. A \( G \)-action on a space \( X \) is \( N \)-\( \mathcal{F} \)-amenable if for any finite subset \( S \) of \( G \) there exists an open \( \mathcal{F} \)-cover \( \mathcal{U} \) of \( G \times X \) (equipped with the diagonal \( G \)-action) such that:

1. the dimension of \( \mathcal{U} \) is at most \( N \); and

Date: April 26, 2018.
2010 Mathematics Subject Classification. Primary 20F69; Secondary 20F65.
Key words and phrases. Asymptotic dimension, finite decomposition complexity, amenable action.
(2) for all \( x \in X \) there is a \( U \in \mathcal{U} \) with \( S \times \{ x \} \subseteq U \).

A \( G \)-action is called finitely \( \mathcal{F} \)-amenable if it is \( N\mathcal{F} \)-amenable for some \( N \).

Bartels employed finitely \( \mathcal{F} \)-amenable actions to elucidate the conditions used by Bartels, Lück and Reich in [BLR08] and by Bartels and Lück in [BL12a] to prove the Farrell-Jones Conjecture for word hyperbolic groups and, respectively, for CAT(0) groups. In particular, he showed that if \( G \) is a group that admits a finitely \( \mathcal{F} \)-amenable action on a compact, finite-dimensional, contractible ANR, then \( G \) satisfies the \( K \)-theoretic Farrell-Jones Conjecture relative to \( \mathcal{F} \), [Bar17]. A similar statement holds for the \( L \)-theoretic Farrell-Jones Conjecture. Bartels and Bestvina established the Farrell-Jones Conjecture for mapping class groups by showing that the action of a mapping class group on the Thurston compactification of Teichmüller space is finitely \( \mathcal{F} \)-amenable, where \( \mathcal{F} \) is the family of virtual point stabilizers, [BB]. The notion of finite \( \mathcal{F} \)-amenity has also been studied by Sawicki [Saw17] in the guise of equivariant asymptotic dimension.

In this paper, we study the coarse geometric applications of finitely \( \mathcal{F} \)-amenable actions. Willet and Yu observed (see [Bar16, Remark 1.3.5]) that if there is a uniform bound on the asymptotic dimension of the groups in \( \mathcal{F} \), then a group \( G \) that admits a finitely \( \mathcal{F} \)-amenable action on a compact metrizable space \( X \) must have finite asymptotic dimension. Motivated by this observation, we establish the following theorem using methods that allow us to extend it in a natural manner to a more general setting.

**Theorem** (Theorem 4.4). Let \( G \) be a finitely generated group and \( \mathcal{F} \) be a family of subgroups of \( G \). If there exists an \( N\mathcal{F} \)-amenable action of \( G \) on a compact metrizable space \( X \) and \( \text{asdim}(F) \leq k \) for each \( F \in \mathcal{F} \), then \( \text{asdim}(G) \leq N + k \).

A metric family is a set whose elements are metric spaces. A permanence property of a collection \( \mathfrak{C} \) of metric families is an operation that when applied to members of \( \mathfrak{C} \) yields another member of \( \mathfrak{C} \). Our proof of the above theorem generalizes to families of subgroups whose elements are contained in a collection of metric families that satisfies some basic permanence properties.

**Theorem** (Theorem 4.5). Let \( G \) be a finitely generated group, \( \mathcal{F} \) be a family of subgroups of \( G \), and \( \mathfrak{C} \) be a collection of metric families satisfying Coarse Permanence, Finite Amalgamation Permanence, and Finite Union Permanence.
If there exists an \(N\)-\(\mathcal{F}\)-amenable action of \(G\) on a compact metrizable space and each \(F \in \mathcal{F}\) belongs to \(\mathcal{C}\), then \(G\) is \(N\)-decomposable over \(\mathcal{C}\).

In the case \(\mathcal{C}\) is the collection of metric families with asymptotic dimension at most \(k\), Theorem 4.5 reduces to Theorem 4.4. This depends on the fact that if a metric family \(\mathcal{X}\) \(N\)-decomposes over the collection of metric families with asymptotic dimension at most \(k\), then \(\text{asdim}(\mathcal{X})\) is at most \(N + k\), which we prove in Theorem 2.16.

Theorem 4.5 also applies to the collection \(\mathcal{D}\), of metric families with finite decomposition complexity (abbreviated to “FDC”), and to the collection \(w\mathcal{D}\), of metric families with weak finite decomposition complexity (abbreviated to “weak FDC”), concepts introduced by Guentner, Tessera and Yu in [GTY12]. A metric family \(\mathcal{X}\) is said to be weakly decomposable over a collection \(\mathcal{C}\) if, for some non-negative integer \(n\), \(\mathcal{X}\) is \(n\)-decomposable over \(\mathcal{C}\) (see Definition 2.1) and \(\mathcal{X}\) is strongly decomposable over \(\mathcal{C}\) if \(\mathcal{X}\) is 1-decomposable over \(\mathcal{C}\). The collection \(\mathcal{D}\) is the smallest collection of metric families that contains all bounded metric families (that is, metric families whose elements have uniformly bounded diameters) and is stable under strong decomposition. The collection \(w\mathcal{D}\) is the smallest collection of metric families that contains all bounded metric families and is stable under weak decomposition. FDC and weak FDC are interesting conditions because they have important topological consequences. A finitely generated group with weak FDC satisfies the Novikov Conjecture, and a metric space with FDC and bounded geometry satisfies the Bounded Borel Conjecture, [GTY12, GTY13]. These results were obtained via an analysis of assembly maps in \(L\)-theory and topological \(K\)-theory.

Two other collections of metric families of importance for the Novikov Conjecture are:

- \(\mathcal{E}\), the collection of exact metric families ([GTY13 Definition 4.0.8]), and
- \(\mathcal{H}\), the collection of metric families that are coarsely embeddable into Hilbert space (see Definition 2.11).

It follows from [STY02, Theorem 6.1] that a countable group, equipped with a proper left-invariant metric, in \(\mathcal{H}\) satisfies the Novikov Conjecture. Note that \(w\mathcal{D} \subset \mathcal{E} \subset \mathcal{H}\). Moreover, both \(\mathcal{E}\) and \(\mathcal{H}\) satisfy Coarse Permanence, Finite Amalgamation Permanence, and Finite Union Permanence.

Since \(w\mathcal{D}\), \(\mathcal{E}\) and \(\mathcal{H}\) are each stable under weak decomposition, we get the following corollary to Theorem 4.5.
Corollary (Corollary 4.7). Let \( \mathcal{C} \) be equal to \( w\mathcal{D}, \mathcal{E} \) or \( \mathcal{F} \). Let \( G \) be a finitely generated group and \( \mathcal{F} \) be a family of subgroups of \( G \) such that each \( F \in \mathcal{F} \) belongs to \( \mathcal{C} \). If there exists a finitely \( \mathcal{F} \)-amenable action of \( G \) on a compact metrizable space, then \( G \) is in \( \mathcal{C} \).

In [Bar17], Bartels proved that if a countable group \( G \) is relatively hyperbolic with respect to peripheral subgroups \( P_1, \ldots, P_n \), then the action of \( G \) on its boundary is finitely \( \mathcal{P} \)-amenable, where \( \mathcal{P} \) is the family of subgroups of \( G \) that are either virtually cyclic or subconjugated to one of the \( P_i \)'s. Thus, we obtain the following application of Theorem 4.5 to relatively hyperbolic groups.

Theorem (Theorem 4.8). Let \( G \) be a finitely generated group that is relatively hyperbolic with respect to peripheral subgroups \( P_1, \ldots, P_n \), and let \( \mathcal{C} \) be a collection of metric families satisfying Coarse Permanence, Finite Amalgamation Permanence, and Finite Union Permanence. If \( \mathcal{C} \) contains \( P_1, \ldots, P_n \) and the infinite cyclic group \( \mathbb{Z} \), then \( G \) is \( N \)-decomposable over \( \mathcal{C} \) for some \( N \).

While Bartels and Lück succeeded in verifying the Farrell-Jones Conjecture for CAT(0) groups [BL12a, BLR08], it is still unknown if CAT(0) groups always have finite asymptotic dimension (Question 5.1). Theorem 4.4 suggests a possible approach to this question. Let \( Y \) be a finite dimensional CAT(0) space on which the CAT(0) group \( G \) acts geometrically. Let \( \mathcal{F} \) be the family of virtual abelian subgroups of \( G \). If it is true that Caprace’s refined boundary, \( \partial^\text{fine}_\infty Y \), as defined in [Cap09] has a compact metrizable topology for which the \( G \)-action on it is finitely \( \mathcal{F} \)-amenable (Question 5.2) then \( G \) has finite asymptotic dimension (Proposition 5.3).

For the reader’s convenience, in the Appendix (§6) we collect some facts that we need about uniform simplicial complexes.

Acknowledgements. Andrew Nicas was partially supported by a grant from the Natural Sciences and Engineering Research Council of Canada. David Rosenthal was partially supported by a grant from the Simons Foundation (#524141). Rosenthal would also like to thank Wolfgang Lück for his hospitality in Bonn, where portions of this project were completed, and for support from Lück’s European Research Council Advanced Grant “KL2MG-interactions” (#662400).
2. Decomposition Over a Collection of Metric Families

In this section, we treat aspects of the coarse geometry of metric families needed for the proofs of our main technical results in Section 3. We recall the notion of $n$-decomposition, introduced by Guentner, Tessera and Yu as a generalization of finite asymptotic dimension, \cite{GTY12}. Permanence properties of certain important collections of metric families are discussed. We show that if a metric family $\mathcal{X}$ is $n$-decomposable over the collection of metric families with asymptotic dimension at most $k$, then $\text{asdim}(\mathcal{X})$ is at most $n + k$ (Theorem 2.16). This result plays an important role in our proof of Theorem 3.4.

**Definition 2.1.** Let $r > 0$ and $n$ be a non-negative integer. The metric family $\mathcal{X}$ is \((r, n)\)-decomposable over the metric family $\mathcal{Y}$, denoted $\mathcal{X} \xrightarrow{(r, n)} \mathcal{Y}$, if for every $X$ in $\mathcal{X}$, $X = X_0 \cup X_1 \cup \cdots \cup X_n$ such that for each $i$

$$X_i = \bigsqcup_{r\text{-disjoint}} X_{ij}$$

where each $X_{ij}$ is in $\mathcal{Y}$.

**Definition 2.2.** Let $n$ be a non-negative integer, and let $\mathcal{C}$ be a collection of metric families. The metric family $\mathcal{X}$ is $n$-decomposable over $\mathcal{C}$ if for every $r > 0$ $\mathcal{X}$ is $(r, n)$-decomposable over some metric family $\mathcal{Y}$ in $\mathcal{C}$.

Following \cite{GTY12}, we say that $\mathcal{X}$ is weakly decomposable over $\mathcal{C}$ if $\mathcal{X}$ is $n$-decomposable over $\mathcal{C}$ for some non-negative integer $n$, and $\mathcal{X}$ is strongly decomposable over $\mathcal{C}$ if $\mathcal{X}$ is 1-decomposable over $\mathcal{C}$.

**Definition 2.3.** A metric family $\mathcal{Z}$ is bounded if the diameters of the elements of $\mathcal{Z}$ are uniformly bounded, that is, if $\text{diam}(\mathcal{Z}) = \sup\{\text{diam}(Z) \mid Z \in \mathcal{Z}\} < \infty$. The collection of all bounded metric families is denoted by $\mathcal{B}$.

**Example 2.4.** Let $X$ be a metric space. The statement that the metric family $\{X\}$ is $n$-decomposable over $\mathcal{B}$ is equivalent to the statement that $\text{asdim}(X) \leq n$.

The following definition is equivalent to Bell and Dranishnikov’s definition of a collection of metric spaces having finite asymptotic dimension “uniformly” \cite[Section 1]{BD04}.

**Definition 2.5.** Let $n$ be a non-negative integer. The metric family $\mathcal{X}$ has asymptotic dimension at most $n$, denoted $\text{asdim}(\mathcal{X}) \leq n$, if $\mathcal{X}$ is $n$-decomposable over $\mathcal{B}$.
Definition 2.6. Let $\mathcal{D}$ be the smallest collection of metric families containing $\mathcal{B}$ that is closed under strong decomposition, and let $w\mathcal{D}$ be the smallest collection of metric families containing $\mathcal{B}$ that is closed under weak decomposition. A metric family in $\mathcal{D}$ is said to have finite decomposition complexity (abbreviated to “FDC”), and a metric family in $w\mathcal{D}$ is said to have weak finite decomposition complexity (abbreviated to “weak FDC”).

Clearly, finite decomposition complexity implies weak finite decomposition complexity. The converse is unknown.

Next, we recall some terminology introduced in [GTY13] that generalizes basic notions from the coarse geometry of metric spaces to metric families.

Let $\mathcal{X}$ and $\mathcal{Y}$ be metric families. A map of families, $F : \mathcal{X} \to \mathcal{Y}$, is a collection of functions $F = \{f : X \to Y\}$, where $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$, such that every $X \in \mathcal{X}$ is the domain of at least one $f$ in $F$. The inverse image of $Z$ under $F$ is the subspace of $\mathcal{X}$ given by $F^{-1}(Z) = \{f^{-1}(Z) \mid Z \in Z, f \in F\}$.

Definition 2.7. (1) A map of metric families, $F : \mathcal{X} \to \mathcal{Y}$, is a coarse embedding if there exist non-decreasing functions $\delta, \rho : [0, \infty) \to [0, \infty)$, with $\lim_{t \to \infty} \delta(t) = \infty = \lim_{t \to \infty} \rho(t)$, such that for every $f : X \to Y$ in $F$ and every $x, y \in X$,

$$\delta(d_X(x, y)) \leq d_Y(f(x), f(y)) \leq \rho(d_X(x, y)).$$

(2) A map of metric families, $F : \mathcal{X} \to \mathcal{Y}$, is a coarse equivalence if for each $f : X \to Y$ in $F$ there is a map $g_f : Y \to X$ such that:

(i) the collection $G = \{g_f\}$ is a coarse embedding from $\mathcal{Y}$ to $\mathcal{X}$; and

(ii) the composites $f \circ g_f$ and $g_f \circ f$ are uniformly close to the identity maps $\text{id}_Y$ and $\text{id}_X$, respectively, in the sense that there is a constant $C > 0$ with

$$d_Y(y, f \circ g_f(y)) \leq C$$

and

$$d_X(x, g_f \circ f(y)) \leq C,$$

for every $f : X \to Y$ in $F$, $x \in X$, and $y \in Y$.

Definition 2.8. A collection of metric families, $\mathcal{C}$, satisfies Coarse Permanence if whenever $Y \in \mathcal{C}$ and $F : \mathcal{X} \to \mathcal{Y}$ is a coarse embedding, then $\mathcal{X} \in \mathcal{C}$.

Guentner, Tessera and Yu proved that both $\mathcal{D}$ and $w\mathcal{D}$ satisfy Coarse Permanence [GTY13, Coarse Invariance 3.1.3]. It is straightforward to check that the following collections of metric families also satisfy Coarse Permanence.
Example 2.9. Collections of metric families that satisfy Coarse Permanence:

1. \( \mathcal{B} \), the collection of bounded metric families.
2. \( \mathcal{A} \), the collection of metric families with finite asymptotic dimension.
3. \( \mathcal{A}_n \), the collection of metric families with asymptotic dimension at most \( n \).
4. \( \mathcal{E} \), the collection of exact metric families (see Definition 2.10).
5. \( \mathcal{H} \), the collection of metric families that are coarsely embeddable into Hilbert space (see Definition 2.11).

Definition 2.10. A metric family \( \mathcal{X} = \{ X_\alpha | \alpha \in I \} \) is exact if for every \( R > 0 \) and every \( \varepsilon > 0 \), each \( X_\alpha \in \mathcal{X} \) admits a partition of unity \( \{ \phi_{U_\alpha} \} \) subordinate to a cover \( U_\alpha \) such that:

1. \( \forall \alpha \in I, \forall x, x' \in X_\alpha, d_\alpha(x, x') \leq R \Rightarrow |\phi_{U_\alpha}(x) - \phi_{U_\alpha}(x')| \leq \varepsilon; \text{ and} \)
2. \( \bigcup_{\alpha \in I} U_\alpha \) is a bounded metric family.

Definition 2.11. A metric family \( \mathcal{X} = \{ X_\alpha | \alpha \in I \} \) is coarsely embeddable into Hilbert space if there is a family of Hilbert spaces \( \mathcal{H} = \{ H_\alpha | \alpha \in I \} \) and a map of metric families \( F = \{ F_\alpha : X_\alpha \to H_\alpha | \alpha \in I \} \) such that \( F : \mathcal{X} \to \mathcal{H} \) is a coarse embedding. The collection of all metric families that are coarsely embeddable into Hilbert space is denoted by \( \mathcal{H} \).

Exactness was introduced by Guentner and is closely related to Yu’s Property A (see [Gue14, Section 5.2] for a discussion). One of the goals of Yu’s definition was to obtain a property that would imply coarse embeddability into Hilbert space and that is relatively easy to verify. In particular, \( \mathcal{E} \subset \mathcal{H} \). Note that there are examples of discrete groups that lie in \( \mathcal{H} \) but not \( \mathcal{E} \). These concepts arose in conjunction with Yu’s highly impactful work on the Novikov Conjecture, [Yu98, Yu00, STY02].

Definition 2.12. A collection of metric families, \( \mathcal{C} \), satisfies Finite Amalgamation Permanence if the following holds. If \( \mathcal{X} = \bigcup_{i=1}^{n} \mathcal{X}_i \) and each \( \mathcal{X}_i \in \mathcal{C} \), then \( \mathcal{X} \in \mathcal{C} \).

It follows from [GTY13, Fibering Theorem 3.1.4] that the collection of metric families with finite decomposition complexity, \( \mathcal{D} \), and the collection of metric families with weak finite decomposition complexity, \( w\mathcal{D} \), satisfy Finite Amalgamation Permanence (also see [KNR, Theorem 5.6] for a proof).

1 The notion of a metric family that is coarsely embeddable into Hilbert space was introduced by Dadarlat and Guentner in [DG03], although they called it a “family of metric spaces that is equi-uniformly embeddable.”
In the absence of Finite Amalgamation Permanence, we can still obtain the following elementary fact.

**Lemma 2.13.** Let $Z$ be a metric space and let $\mathcal{X} = \{X_{\alpha} \mid \alpha \in I\}$ be a collection of metric subspaces of $Z$. Then, for a fixed natural number $m$, the metric family $\{X_{\alpha_1} \cup \cdots \cup X_{\alpha_m} \mid \alpha_1, \ldots, \alpha_m \in I\}$ is $(r,m)$-decomposable over $\mathcal{X}$ for every $r > 0$.

**Proof.** Immediate from Definition 2.1. Indeed, for every $r > 0$, each $Y = X_{\alpha_1} \cup \cdots \cup X_{\alpha_m}$ is $(r,m)$-decomposable over $\mathcal{X}$. □

A basic property of asymptotic dimension is that if $A$ and $B$ are metric subspaces of some larger metric space, then $\text{asdim}(A \cup B) = \max\{\text{asdim}(A), \text{asdim}(B)\}$. This is also true for metric families, a fact known as **Finite Union Permanence**.

**Definition 2.14.** A collection of metric families, $\mathcal{C}$, satisfies **Finite Union Permanence** if the following holds. Let $n \in \mathbb{N}$ and let $\mathcal{X}, \mathcal{X}_1, \ldots, \mathcal{X}_n$ be metric families. If every $\mathcal{X}_i \in \mathcal{C}$ and for each $X \in \mathcal{X}$ there exist $X_i \in \mathcal{X}_i$, $1 \leq i \leq n$, such that $X = \bigcup_{i=1}^{n} X_i$, then $\mathcal{X} \in \mathcal{C}$.

**Theorem 2.15.** Let $G$ be a finitely generated group with finite symmetric generating set $S$, and let $m$ be a natural number. Let $H_1, \ldots, H_m$ be subgroups of $G$ and $\mathcal{H}_m$ be the metric family $\{\bigcup_{i=1}^{m} g_i H_i \mid g_1, \ldots, g_m \in G\}$, where each $\bigcup_{i=1}^{m} g_i H_i$ is considered as a metric subspace of $G$. Then $\text{asdim}(\mathcal{H}_m) = \max\{\text{asdim}(H_i) \mid 1 \leq i \leq m\}$.

**Proof.** This follows from the fact that the collection of metric families with asymptotic dimension at most $k$ satisfies Finite Union Permanence [Gue14, Theorem 6.3]. □

The next result is needed to establish Theorem 3.4.

**Theorem 2.16.** If $\mathcal{X}$ is a metric family that is $n$-decomposable over $\mathfrak{A}_m$ (the collection of metric families with asymptotic dimension at most $m$) then $\text{asdim}(\mathcal{X}) \leq m + n$.

The proof of this theorem, which begins with following lemma, is an adaptation of the Kolmogorov trick used in [BDLM08] to study the asymptotic dimension of metric spaces.

We denote the number of elements of a finite set $F$ by $\#F$.

**Lemma 2.17.** Let $\mathcal{X}$ be a metric family that is $n$-decomposable over $\mathfrak{A}_m$ and let $k$ be a non-negative integer. Given $r > 0$, for each $X \in \mathcal{X}$ there is a decomposition $X = X_0 \cup \cdots \cup X_{n+k}$ such that,
(i) for every $x \in X$, $#T_x \geq k + 1$ where $T_x = \{i \mid x \in X_i\} \subset \{0, 1, \ldots, n + k\}$;
(ii) for each $i$, $X_i = \bigcup_{r\text{-disjoint}} X_{ij}$; and
(iii) $\mathcal{X}^* = \{X_{ij} \mid \text{all } i, j \text{ and } X \in \mathcal{X}\} \in \mathfrak{A}_m$.

Proof. The proof proceeds by induction on $k$ (our method is adapted from the proof of [BDLM08, Theorem 2.4]). The base case of the induction, $k = 0$, is the given assertion that the metric family $\mathcal{X}$ is $n$-decomposable over $\mathfrak{A}_m$. Assume that the Claim is valid for the integer $k$. Let $r > 0$. For each $X \in \mathcal{X}$, there is a decomposition $X = X_0 \cup \cdots \cup X_{n+k}$ such that, for every $x \in X$, $#T_x \geq k + 1$ and $X_i = \bigcup_{3r\text{-disjoint}} X_{ij}$ and

$$
\mathcal{X}^* = \{X_{ij} \mid \text{all } i, j \text{ and } X \in \mathcal{X}\} \in \mathfrak{A}_m.
$$

Let $X'_{ij}$ be the $r$-neighborhood of $X_{ij}$, that is,

$$
X'_{ij} = \{y \mid \text{there exists } x \in X_{ij} \text{ such that } d(x, y) < r\}.
$$

For $0 \leq i \leq n + k$, let $\mathcal{U}'_i = \{X'_{ij} \mid j\}$ and $X'_i = \bigcup \mathcal{U}'_i$. Observe that $\mathcal{U}'_i$ is $r$-disjoint. Let $\mathcal{U}'_{n+k+1}$ be the collection of subspaces of the form

$$
X_{I,J} = X_{i_1,j_1} \cap \cdots \cap X_{i_{k+1},j_{k+1}} \setminus \bigcup_{\ell \in \{i_1, \ldots, i_{k+1}\}} X'_{i_\ell}
$$

where $I = \{i_1, \ldots, i_{k+1}\}$ consists of $k + 1$ distinct elements of $\{0, \ldots, n + k\}$ and $J = \{j_1, \ldots, j_{k+1}\}$. Let $I' = \{i'_1, \ldots, i'_{k+1}\}$ be another set of $k + 1$ distinct elements of $\{0, \ldots, n + k\}$. If $I = I'$ then clearly any two distinct sets of the form $X_{I,J}$ and $X_{I',J'}$ are $r$-disjoint. Suppose $I \neq I'$, $a \in X_{I,J}$, $b \in X_{I',J'}$ and $d(a, b) < r$. Since $I \neq I'$ there exists $i_\ell \notin \{i_1, \ldots, i_{k+1}\}$. Note that $b \notin X'_{i_\ell}$. However, by definition of $a \in X_{I,J}$, $a \notin X'_{i_\ell}$. This implies that $d(a, b) \geq r$, a contradiction. Hence $X_{I,J}$ and $X_{I',J'}$ are $r$-disjoint. This shows that $\mathcal{U}'_{n+k+1}$ is $r$-disjoint. Also note that $\mathcal{U}'_{n+k+1} \in \mathfrak{A}_m$.

Let $X'_{n+k+1} = \bigcup \mathcal{U}'_{n+k+1}$. Observe that $X = X'_0 \cup \cdots \cup X'_{n+k+1}$. For $x \in X$, let $S_x = \{\ell \mid x \in X'_\ell \text{ and } 0 \leq \ell \leq n + k\}$ and $T'_x = \{\ell \mid x \in X'_\ell, 0 \leq \ell \leq n + k + 1\}$. If $\#S_x \geq k + 2$, then $\#T'_x \geq k + 2$, so we are done. Suppose $\#S_x = k + 1$. Then for some $J$, $x \in X_{i_1,j_1} \cap \cdots \cap X_{i_{k+1},j_{k+1}}$, where $i_1 \in T_x \subset S_x$ and $j_1 \in J$. If $x \notin X_{S_x,J}$, then $x \in X'_\ell$ for some $\ell \notin S_x$, a contradiction. Hence, $x \in X_{S_x,J} \subset X'_{n+k+1}$ and so $\#T'_x \geq k + 2$. This completes the induction step. \hfill $\square$

Proof of Theorem [2.10]. By Lemma [2.17] given $r > 0$, for each $X \in \mathcal{X}$ there is a decomposition $X = X_0 \cup \cdots \cup X_{n+m}$ such that,
For every Condition (C).

(i) for every $x \in X$, \( \#T_x \geq k + 1 \) where $T_x = \{ i \mid x \in X_i \} \subset \{0, 1, \ldots, n + k\}$;
(ii) for each $i$, $X_i = \bigcup_{r \text{-disjoint}} X_{ij}$; and
(iii) $\mathcal{X}^* = \{ X_{ij} \mid \text{all } i, j \text{ and } X \in \mathcal{X} \} \in \mathcal{A}_m$.

By [BDLM08, Theorem 2.4], there is an $(m, m + n + 1)$-dimensional control function, $\mathcal{D}$, for $\mathcal{X}^*$, that is, for $r > 0$ as above there are covers $\mathcal{U}^{ij}$ of $X_{ij} \in \mathcal{X}^*$ such that,

1. $\mathcal{U}^{ij} = \mathcal{U}_0^{ij} \cup \cdots \cup \mathcal{U}_m^{ij}$,
2. $\mathcal{U}_k^{ij}$ is $r$-disjoint and $\mathcal{D}(r)$ bounded,
3. For every $x \in X_{ij}$, $\#S_x \geq n + 1$ where $S_x = \{ \ell \mid x \text{ belongs to an element of } \mathcal{U}^{ij} \}$.

Let $\mathcal{V}_i = \bigcup_{j} \mathcal{U}^{ij}$ for $i = 0, \ldots, m + n$. Note that $\mathcal{V}_i$ is $r$-disjoint and $\mathcal{D}(r)$-bounded. For $x \in X \in \mathcal{X}$, let $S'_x = \{ \ell \mid x \text{ belongs to } \mathcal{U}^{ij} \text{ for some } i, j \}$. Recall that $\#T_x \geq m + 1$, where $T_x = \{ i \mid x \in X_i \}$. Note that $S_x \subset S'_x$ and so $\#S'_x \geq n + 1$. Hence $S'_x \cap T_x \neq \emptyset$ because $S'_x$ and $T_x$ are subsets of $\{0, \ldots, m + n\}$, a set with $m + n + 1$ elements. Since $x \in X_{ij}$ implies that $S_x \subset S'_x$, it follows that for $\mu \in S'_x \cap T_x$, there exists $j$ and $U \in \mathcal{U}_\mu^{ij}$ with $x \in U$. This shows that $\mathcal{V} = \mathcal{V}_0 \cup \cdots \cup \mathcal{V}_{m+n}$ is a cover of $\mathcal{X}$. It follows that $\text{asdim}(\mathcal{X}) \leq m + n$. \(\square\)

We conclude this section by stating three alternative definitions (established in [NR]) for a metric family $\mathcal{X}$ to be $n$-decomposable over a collection of metric families $\mathcal{C}$. Condition (C) is a key technical tool needed for the proofs of Theorems 3.4, 3.5 and 3.7.

In what follows, let $\mathcal{X} = \{ X_\alpha \mid \alpha \in I \}$ be a metric family, where $I$ is a countable indexing set, and let $\mathcal{C}$ be a collection of metric families. Let $n$ be a non-negative integer.

**Condition (A).** For every $d > 0$, there exists a cover $\mathcal{V}_\alpha$ of $X_\alpha$, for each $\alpha \in I$, such that:

(i) the $d$-multiplicity of $\mathcal{V}_\alpha$ is at most $n + 1$ for every $\alpha \in I$; and
(ii) $\bigcup_{\alpha \in I} \mathcal{V}_\alpha$ is a metric family in $\mathcal{C}$.

**Condition (B).** For every $\lambda > 0$, there exists a cover $\mathcal{U}_\alpha$ of $X_\alpha$, for each $\alpha \in I$, such that:

(i) the multiplicity of $\mathcal{U}_\alpha$ is at most $n + 1$ for every $\alpha \in I$;
(ii) the Lebesgue number $L(\mathcal{U}_\alpha) \geq \lambda$ for every $\alpha \in I$; and
(iii) $\bigcup_{\alpha \in I} \mathcal{U}_\alpha$ is a metric family in $\mathcal{C}$.

**Condition (C).** For every $\varepsilon > 0$, there exists a uniform simplicial complex $K_\alpha$ and an $\varepsilon$-Lipschitz map $\varphi_\alpha : X_\alpha \to K_\alpha$, for each $\alpha \in I$, such that:

(i) $\dim(K_\alpha) \leq n$ for every $\alpha \in I$; and
(ii) $\bigcup_{\alpha \in I} \{ \varphi_\alpha^{-1}(\text{star}(v)) \mid v \in K_\alpha^{(0)} \}$ is a metric family in $\mathcal{C}$. 
The following is proved in [NR, Propositions 3.1 and 3.3].

**Proposition 2.18.** Let $\mathcal{X}$ be a metric family and $\mathcal{C}$ be a collection of metric families that satisfies Coarse Permanence. Then Conditions (A), (B) and (C) are each equivalent to Definition 2.2.

3. Decomposition of a Group Over a Collection of Metric Families

In [Bar16], Bartels reformulated, in geometric group theoretic terms, the conditions that he used with Lück and Reich in [BLR08] and with Lück in [BL12a] to prove the Farrell-Jones Conjecture for word hyperbolic groups and, respectively, for CAT(0) groups. Willet and Yu observed (see [Bar16, Remark 1.3.5]) that a group satisfying these conditions must have finite asymptotic dimension. In this section, we approach this fact from the viewpoint of metric families in Theorem 3.5 allowing us to give a generalization that applies to more general coarse geometric notions (Theorem 3.7). The conditions used by Bartels in [Bar16], which are of a technical nature, evolved into his notion of finitely $\mathcal{F}$-amenable actions in [Bar17]. We formulate our results using this language in Section 4.

**Definition 3.1 (Bar16).** Let $G$ be a finitely generated group with a finite symmetric generating set $S$. Let $\varepsilon \geq 0$. A map $f: X \to Y$ between $G$-spaces, where $Y$ is equipped with a left invariant metric $d$, is $G$-equivariant up to $\varepsilon$ if $d(f(sx), sf(x)) \leq \varepsilon$ for every $s \in S$ and every $x \in X$.

Recall that given a group $G$ together with a finite symmetric generating set $S \subset G$, the length of $g \in G$ with respect to $S$ is the non-negative integer

$$\|g\|_S = \min\{n \mid g = s_1s_2 \cdots s_n, s_j \in S\}.$$  

The corresponding left-invariant *word length metric* on $G$ is given by $d_S(g, h) = \|g^{-1}h\|_S$. Any two such finite generating sets for $G$ yield quasi-isometric metric spaces. An equivalent characterization of “$G$-equivariant up to $\varepsilon$” is given by the next Lemma.

**Lemma 3.2.** A map $f: X \to Y$ as in Definition 3.1 is $G$-equivariant up to $\varepsilon$ if and only if for all $g \in G$ and $x \in X$, $d(f(gx), gf(x)) \leq \varepsilon \|g\|_S$.

**Proof.** Assume $d(f(gx), gf(x)) \leq \varepsilon \|g\|_S$ for all $g \in G$ and $x \in X$. If $s \in S$ then $\|s\|_S = 1$ and so $d(f(sx), sf(x)) \leq \varepsilon$, that is, $f$ is $G$-equivariant up to $\varepsilon$. 

Assume $d(f(sx), sf(x)) \leq \varepsilon$ for all $s \in S$ and $x \in X$. We use induction on the word length, $\|g\|_S$, of $g \in G$ to show that $d(f(gx), gf(x)) \leq \varepsilon \|g\|_S$. If $\|g\|_S = 1$ then $g \in S$ and the conclusion is clear. Assume that the induction hypothesis is true for all elements of length $n$. Let $h \in G$ have length $n + 1$. Then, $h = sg$, where $s \in S$ and $\|g\|_S = n$. Thus,

$$d(f(hx), hf(x)) = d(f((sg)x), (sg)f(x))$$
$$\leq d(f((sg)x), sf(gx)) + d(sf(gx), s(gf(x)))$$
$$= d(f(sgx), sf(gx)) + d(f(gx), gf(x))$$
$$\leq \varepsilon + \varepsilon n = \varepsilon \|h\|_S,$$

verifying the induction step. \hfill \Box

**Lemma 3.3.** Let $G$ be a finitely generated group with finite symmetric generating set $S$. Let $X$ and $Y$ be left $G$-spaces, and let $d$ be a left-invariant metric on $Y$. Let $f : X \to Y$ be a map that is $G$-equivariant up to $\varepsilon$. Then, for any $x \in X$, the map $\varphi_x : G \to Y$ defined by $\varphi_x(g) = gf(g^{-1}x)$ is $\varepsilon$-Lipschitz.

**Proof.** Making use Lemma 3.2 we have,

$$d(\varphi_x(g), \varphi_x(h)) = d(gf(g^{-1}x), hf(h^{-1}x))$$
$$= d(f(g^{-1}x), (g^{-1}h)f(h^{-1}x))$$
$$= d(f((g^{-1}h)(h^{-1}x)), (g^{-1}h)f(h^{-1}x))$$
$$\leq \varepsilon \|g^{-1}h\|_S = \varepsilon d_S(g, h),$$

that is, $\varphi_x$ is $\varepsilon$-Lipschitz. \hfill \Box

A **uniform simplicial complex** is the geometric realization of an abstract simplicial complex endowed the the $\ell^1$-metric, denoted $d^1$ (see the Appendix, §6).

**Theorem 3.4.** Let $G$ be a group with finite symmetric generating set $S$, and let $k$ and $n$ be non-negative integers. Assume that for every $\varepsilon > 0$ there is a compact $G$-space $X$, a uniform simplicial complex $E$ equipped with a simplicial $G$-action, and a map $f : X \to E$ such that

(i) $\dim(E) \leq n$;

(ii) $f$ is $G$-equivariant up to $\varepsilon$ (Definition 3.1),
(iii) for each vertex \( v \in E \), \( \operatorname{asdim}(G_v) \leq k \), where \( G_v \) is the stabilizer subgroup of \( v \) and is viewed as a metric subspace of \( G \).

Then \( \operatorname{asdim}(G) \leq n + k \).

Proof. We will show that \( G \) is \( n \)-decomposable over \( \mathfrak{A}_k \), the collection of metric families with asymptotic dimension at most \( k \). Then, by Theorem 2.16, \( \operatorname{asdim}(G) \leq n + k \).

We proceed by showing that \( G \) satisfies Condition (C) with respect to \( n \) and \( \mathfrak{A}_k \). Let \( \varepsilon > 0 \) be given. Then there is a compact \( G \)-space \( X \), a uniform simplicial complex \( E \) equipped with a simplicial \( G \)-action, and a map \( f : X \to E \) that satisfy assumptions (i), (ii) and (iii). By Lemma 6.10, we can assume that \( f \) factors through the identity map \( \tilde{\text{id}} : E_w \to E \), where \( E_w \) denotes the underlying set of \( E \) topologized with the weak topology determined by the collection of closed simplices of \( E \) (note that the weak topology and the metric topology on \( E \) need not coincide, see Proposition 6.8). The space \( E_w \) is a CW-complex whose \( n \)-skeleton is the union of all the closed simplices of \( E \) of dimension at most \( n \). Since \( X \) is compact and \( f \) is continuous (as a map into \( E_w \)), \( f(X) \) is a compact subset of \( E_w \) and, thus, is contained in the union of finitely many simplices of \( E_w \).

Let \( E_f \) be a finite subcomplex of \( E \) with \( f(X) \subset E_f \), and let \( \{v_1, \ldots, v_m\} \) be the vertex set of \( E_f \). Let \( G_{v_i} = \{ g \in G \mid gv_i = v_i \} \) be the stabilizer of \( v_i \). For each vertex \( w \) of \( E \), let \( Q_i(w) = \{ g \in G \mid gw = w \} \). If \( Q_i(w) \) is non-empty, choose \( g_{w,i} \in Q_i(w) \). Then \( Q_i(w) = g_{w,i}G_{v_i} \).

Fix \( x \in X \) and define \( \varphi_x : G \to E \) by \( \varphi_x(g) = gf(g^{-1}x) \); it is \( \varepsilon \)-Lipschitz by Lemma 3.3. If \( v \) is a vertex of \( E \), then

\[
\varphi_x^{-1}(\{v\}) = \{ g \in G \mid \varphi_x(g) = v \} = \{ g \in G \mid gf(g^{-1}x) = v \} = \{ g \in G \mid f(g^{-1}x) = g^{-1}v \} \subset \{ g \in G \mid g^{-1}v \in \{v_1, \ldots, v_m\} \} = \bigcup_{i=1}^{m} \{ g \in G \mid g^{-1}v = v_i \},
\]

since \( f(X) \subset E_f \). Thus,

\[
\varphi_x^{-1}(\{v\}) \subset \bigcup_{i=1}^{m} \{ g \in G \mid g^{-1}v = v_i \} = \bigcup_{i=1}^{m} Q_i(v) = \bigcup_{i=1}^{m} g_{v_i}G_{v_i}.
\]
More generally, we have

$$\varphi^{-1}(\text{star}(v)) \subset \bigcup_{i=1}^{m} g_{v,i}G_{v,i},$$

since

$$g \in \varphi^{-1}(\text{star}(v)) \iff (\varphi_x)_v \neq 0 \iff f(g^{-1}x)_{g^{-1}v} \neq 0 \iff g^{-1}v \in \{v_1, \ldots, v_m\}.$$ 

Therefore, \(\{\varphi^{-1}(\text{star}(v)) \mid v \in \mathcal{E}(0)\}\subset \bigcup_{i=1}^{m} g_{v,i}G_{v,i}\). By Theorem 2.15, asdim \(\bigcup_{i=1}^{m} g_{v,i}G_{v,i}\) \(\leq k\). Thus, asdim \(\{\varphi^{-1}(\text{star}(v)) \mid v \in \mathcal{E}(0)\}\) \(\leq k\). Therefore, \(G\) is \(n\)-decomposable over \(\mathcal{A}_k\).

Guentner, Willet and Yu showed that when \(\mathcal{F}\) is the family of finite groups, then the action of \(G\) on \(X\) has finite dynamic asymptotic dimension [GWY17, Theorem 4.11].

We wish to generalize Theorem 3.4 to allow for isotropy groups that are contained in a collection of metric families with sufficiently nice properties.

**Theorem 3.5.** Let \(\mathcal{C}\) be a collection of metric families that satisfies Coarse Permanence, Finite Amalgamation Permanence, and Finite Union Permanence. Let \(G\) be a group with finite symmetric generating set \(S\), and let \(n\) be a non-negative integer. Assume that for every \(\varepsilon > 0\) there is a compact \(G\)-space \(X\), a uniform simplicial complex \(E\) equipped with a simplicial \(G\)-action, and a map \(f : X \to E\) such that

(i) \(\dim(E) \leq n\);

(ii) \(f\) is \(G\)-equivariant up to \(\varepsilon\) (Definition 3.1);

(iii) for each vertex \(v \in E\), the stabilizer subgroup \(G_v = \{g \in G \mid gv = v\}\), considered as a metric subspace of \(G\), is in \(\mathcal{C}\).

Then \(G\) is \(n\)-decomposable over \(\mathcal{C}\).

In particular, if \(\mathcal{C}\) is also stable under weak decomposition, then \(G\) is in \(\mathcal{C}\).

**Proof.** By Proposition 2.18, the result will follow from showing that \(G\) satisfies Condition (C) with respect to \(n\) and \(\mathcal{C}\).

Let \(\varepsilon > 0\) be given. Then there is a compact \(G\)-space \(X\), a uniform simplicial complex \(E\) equipped with a simplicial \(G\)-action, and a map \(f : X \to E\) that satisfy assumptions (i), (ii) and (iii). As in the proof of Theorem 3.4 we can assume, by Lemma 6.10 that there is a finite subcomplex \(E_f\) of \(E\) with \(f(X) \subset E_f\). Let \(\{v_1, \ldots, v_m\}\) be the vertex set of \(E_f\), and let \(G_{v_i} = \{g \in G \mid gv_i = v_i\}\) be the stabilizer of \(v_i\). Then, as established in the proof of Theorem 3.4 \(\{\varphi^{-1}(\text{star}(v)) \mid v \in \mathcal{E}(0)\}\subset \bigcup_{i=1}^{m} g_{v,i}G_{v,i} \mid v \in \mathcal{E}(0)\}\).
Since the metric on $G$ is left-invariant, each $g_{v,i}G_{v_i}$ is isometric to $G_{v_i}$. Therefore, 
$$\{g_{v,i}G_{v_i} \mid 1 \leq i \leq m, v \in E^{(0)}\}$$
is coarsely equivalent to the metric family 
$$\{G_{v_i} \mid 1 \leq i \leq m\},$$
which is in $\mathcal{C}$ by Fine Amalgamation Permanence. Thus, 
$$\{g_{v,i}G_{v_i} \mid 1 \leq i \leq m, v \in E^{(0)}\}$$
is in $\mathcal{C}$ by Coarse Permanence, and so 
$$\bigcup_{i=1}^{m} g_{v,i}G_{v_i} \mid v \in E^{(0)}\}$$
is in $\mathcal{C}$ by Finite Union Permanence. Since inclusions are a special case of Coarse Permanence, 
$$\{g^{-1}(\text{star}(v)) \mid v \in E^{(0)}\}$$
is in $\mathcal{C}$. Thus, $G$ satisfies Condition (C) with respect to $n$ and $\mathcal{C}$, as desired.

Theorem 3.4 is a special case of Theorem 3.5 since the collection $\mathcal{A}_k$ of metric families with asymptotic dimension less than or equal to $k$ satisfies Coarse Permanence, Finite Amalgamation Permanence, and Finite Union Permanence. Only $\mathcal{B}, \mathcal{W}, \mathcal{E}$, and $\mathcal{F}$ are stable under weak decomposition ([GTY13, Proof of Theorem 4.3], [NR, Theorem 3.6]).
4. FINITELY $\mathcal{F}$-AMENABLE ACTIONS

In his work on relatively hyperbolic groups and the Farrell-Jones Conjecture, Bartels introduced the notion of a finitely $\mathcal{F}$-amenable action [Bar17], where $\mathcal{F}$ is a family of subgroups of a given group that is closed under conjugation and taking subgroups. Such actions provide examples to which the results of Section 3 can be applied, for example relatively hyperbolic groups (see Theorems 4.8 and 4.9 below).

**Definition 4.1.** Let $X$ be a $G$-space and $\mathcal{F}$ be a family of subgroups of $G$.

1. An open set $U$ in $X$ is an $\mathcal{F}$-subset if there is an $F \in \mathcal{F}$ such that $gU = U$ for every $g \in F$ and $gU \cap U = \emptyset$ for every $g \notin F$.
2. An open cover $\mathcal{U}$ of $X$ is $G$-invariant if $gU \in \mathcal{U}$ for all $g \in G$ and all $U \in \mathcal{U}$.
3. A $G$-invariant cover $\mathcal{U}$ of $X$ is an $\mathcal{F}$-cover if all of the members of $\mathcal{U}$ are $\mathcal{F}$-subsets.

**Definition 4.2.** [Bar17, Definition 0.1] Let $\mathcal{F}$ be a family of subgroups of $G$ and let $N$ be a non-negative integer. A $G$-action on a space $X$ is $N$-$\mathcal{F}$-amenable if for any finite subset $S$ of $G$ there exists an open $\mathcal{F}$-cover $\mathcal{U}$ of $G \times X$ (equipped with the diagonal $G$-action) such that:

1. the dimension of $\mathcal{U}$ is at most $N$; and
2. for all $x \in X$ there is a $U \in \mathcal{U}$ with $S \times \{x\} \subseteq U$.

A $G$-action is called finitely $\mathcal{F}$-amenable if it is $N$-$\mathcal{F}$-amenable for some $N$.

**Proposition 4.3.** Let $G$ be a group with finite symmetric generating set $S$ and $\mathcal{F}$ be a family of subgroups of $G$. If there exists an $N$-$\mathcal{F}$-amenable action of $G$ on $X$, where $X$ is compact and metrizable, then for every $\varepsilon > 0$ there exists a uniform simplicial complex $E$ equipped with a simplicial $G$-action and a map $f : X \to E$ such that:

1. $\text{dim}(E) \leq N$;
2. $f$ is $G$-equivariant up to $\varepsilon$ (Definition 3.1),
3. the stabilizer subgroup of each vertex in $E$ is an element of $\mathcal{F}$.

**Proof.** The proposition follows from [Bar17, Remarks 0.2-0.4], but we include a proof here for the reader’s convenience.

Since $X$ is compact and metrizable, there is a metric $d$ on $G \times X$ that is $G$-invariant with respect to the diagonal action [BLR08, Proposition 4.3]. Furthermore, it has the property that $d((g,x),(h,x)) = d_G(g,h)$ for every $g,h \in G$ and $x \in X$. Let $\varepsilon > 0$ be given. Since
the action of $G$ on $X$ is $N$-$\mathcal{F}$-amenable, there exists an open $\mathcal{F}$-cover $\mathcal{U}$ of $G \times X$ of dimension at most $N$ such that for each $x \in X$ there is a $U_x \in \mathcal{U}$ with $B_R(e) \times \{x\} \subseteq U_x$, where $B_R(e)$ is the ball of radius $R = \frac{(2N + 2)(2N + 3)}{\varepsilon}$ in $G$ around the identity of $G$. Let $E = \text{Nerve}(\mathcal{U})$ equipped with the uniform metric. Then $\dim(E) = \dim(\mathcal{U}) \leq N$. It follows from the definition of an $\mathcal{F}$-cover that the stabilizer subgroup of a vertex in $E$ is an element of the family $\mathcal{F}$.

It remains to define a map $f : X \to E$ that satisfies item (ii). Recall the following standard construction. For each $U \in \mathcal{U}$, define $\psi_U : G \times X \to [0,1]$ by
\[
\psi_U(y) = \frac{d(y, U^c)}{\sum_{V \in U} d(y, V^c)}
\]
where $U^c$ is the complement of $U$ in $G \times X$. Define $\psi : G \times X \to E$ to be the map
\[
\psi(y) = \sum_{U \in \mathcal{U}} \psi_U(y) \cdot [U]
\]
where $[U]$ denotes the vertex of $E$ corresponding to $U$. Let $f : X \to E$ be the map defined by $f(x) = \psi(e, x)$.

Note that $\psi$ is $G$-equivariant. Since $d(gy, U^c) = d(y, g^{-1}U^c)$, it follows that $\psi_U(gy) = \psi_{g^{-1}U}(y)$ and so
\[
g\psi(y) = \sum_{U \in \mathcal{U}} \psi_U(y) \cdot [gU] = \sum_{U \in \mathcal{U}} \psi_{g^{-1}U}(y) \cdot [U] = \sum_{U \in \mathcal{U}} \psi_U(gy) \cdot [U] = \psi(gy).
\]
Also note that since $B_R(e) \times \{g^{-1}x\} \subseteq U_{g^{-1}x}$,
\[
d((g, x), (gU_{g^{-1}x})^c) = d((e, g^{-1}x), (U_{g^{-1}x})^c) \geq R.
\]

For every $y, y' \in G \times X$ and $U \in \mathcal{U}$, the triangle inequality implies
\[
|d(y, U^c) - d(y', U^c)| \leq d(y, y').
\]
Therefore,
\[
|\psi_U(y) - \psi_U(y')| = \left| \frac{d(y, U^c)}{\sum_{V \in U} d(y, V^c)} - \frac{d(y', U^c)}{\sum_{V \in U} d(y', V^c)} \right|
\]
\[
\leq \frac{|d(y, U^c) - d(y', U^c)|}{\sum_{V \in U} d(y, V^c)} + \frac{d(y', U^c)}{\sum_{V \in U} d(y', V^c)} - \frac{d(y', U^c)}{\sum_{V \in U} d(y', V^c)}
\]
which is less than or equal to
\[
\frac{d(y,y')}{\sum_{V \in U} d(y,V^c)} + \frac{d(y',U^c)}{\left(\sum_{V \in U} d(y,V^c)\right) \left(\sum_{V \in U} d(y',V^c)\right)} \cdot \sum_{V \in U} |d(y,V^c) - d(y',V^c)|.
\]

which is less than or equal to
\[
\frac{1}{\sum_{V \in U} d(y,V^c)} \left( d(y,y') + \sum_{V \in U} |d(y,V^c) - d(y',V^c)| \right).
\]

Thus, for each \( x \in X \),
\[
|\psi_U(g,x) - \psi_U(h,x)| \leq \frac{1}{d((g,x),(gU_{g^{-1}x}))} \left( d((g,x),(h,x)) + \sum_{V \in U} |d(y,V^c) - d(y',V^c)| \right)
\]
\[
\leq \frac{1}{R} \left( d((g,x),(h,x)) + 2(N+1) d((g,x),(h,x)) \right)
\]
\[
= \frac{1}{R} (2N+3) d_G(g,h)
\]

and so,
\[
d^1(\psi(g,x),\psi(h,x)) = \sum_{U \in U} |\psi_U(g,x) - \psi_U(h,x)| \leq \frac{2(N+1)(2N+3)}{R} d_G(g,h).
\]

Hence, if \( x \in X \) and \( s \in S \)
\[
d^1(f(sx),sf(x)) = d^1(\psi(e,sx),s\psi(e,x))
\]
\[
= d^1(\psi(e,sx),\psi(s,sx))
\]
\[
\leq \frac{1}{R} (2N+2)(2N+3) d_G(e,s) = \varepsilon
\]

This completes the proof. \( \square \)

Proposition 4.3 yields the following corollaries to Theorems 3.4, 3.5, and 3.7.

**Theorem 4.4.** Let \( G \) be a finitely generated group and \( \mathcal{F} \) be a family of subgroups of \( G \). If there exists an \( N \)-\( \mathcal{F} \)-amenable action of \( G \) on a compact metrizable space \( X \) and \( \operatorname{asdim}(F) \leq k \) for each \( F \in \mathcal{F} \), then \( \operatorname{asdim}(G) \leq N + k \).

**Theorem 4.5.** Let \( G \) be a finitely generated group, \( \mathcal{F} \) be a family of subgroups of \( G \), and \( \mathcal{C} \) be a collection of metric families satisfying Coarse Permanence, Finite Amalgamation Permanence, and Finite Union Permanence.

If there exists an \( N \)-\( \mathcal{F} \)-amenable action of \( G \) on a compact metrizable space \( X \) and each \( F \in \mathcal{F} \) belongs to \( \mathcal{C} \), then \( G \) is \( N \)-decomposable over \( \mathcal{C} \).
Theorem 4.6. Let $G$ be a finitely generated group, $\mathcal{F}$ be a family of subgroups of $G$, and $\mathcal{C}$ be a collection of metric families that satisfies Coarse Permanence, Finite Amalgamation Permanence, and is stable under weak decomposition.

If there is a finitely $\mathcal{F}$-amenable action of $G$ on a compact metrizable space $X$ and each $F \in \mathcal{F}$ belongs to $\mathcal{C}$, then $G$ is in $\mathcal{C}$.

Since the collections $w\mathcal{D}$, $\mathcal{E}$ and $\mathcal{H}$ all satisfy the assumptions of Theorem 4.6, we have the following corollary.

Corollary 4.7. Let $\mathcal{C}$ be equal to $w\mathcal{D}$, $\mathcal{E}$ or $\mathcal{H}$. Let $G$ be a finitely generated group and $\mathcal{F}$ be a family of subgroups of $G$ such that each $F \in \mathcal{F}$ belongs to $\mathcal{C}$. If there exists a finitely $\mathcal{F}$-amenable action of $G$ on a compact metrizable space, then $G$ is in $\mathcal{C}$.

The main theorem of [Bar17] tells us that if a countable group $G$ is relatively hyperbolic with respect to peripheral subgroups $P_1, \ldots, P_n$, then the action of $G$ on its boundary is finitely $\mathcal{P}$-amenable, where $\mathcal{P}$ is the family of subgroups of $G$ that are either virtually cyclic or subconjugated to one of the $P_i$’s. Combining this with Theorems 4.5 and 4.6 we get the following results.

Theorem 4.8. Let $G$ be a finitely generated group that is relatively hyperbolic with respect to peripheral subgroups $P_1, \ldots, P_n$, and let $\mathcal{C}$ be a collection of metric families satisfying Coarse Permanence, Finite Amalgamation Permanence, and Finite Union Permanence. If $\mathcal{C}$ contains $P_1, \ldots, P_n$ and the infinite cyclic group $\mathbb{Z}$, then $G$ is $N$-decomposable over $\mathcal{C}$ for some $N$.

Proof. Let $\mathcal{P}$ be the family of subgroups of $G$ whose members are either virtually cyclic or subconjugated to one of the $P_i$’s. Note that a virtually cyclic group is coarsely equivalent to either $\mathbb{Z}$ or the trivial group and that any two conjugate subgroups of $G$ are coarsely equivalent. Therefore, since $\mathcal{C}$ satisfies Coarse Permanence, every element of $\mathcal{P}$ is in $\mathcal{C}$. The theorem now follows from Theorem 4.5 and the above mentioned result of Bartels, [Bar17], that the action of such a group on its boundary is finitely $\mathcal{P}$-amenable. □

Theorem 4.9. Let $G$ be a finitely generated group that is relatively hyperbolic with respect to peripheral subgroups $P_1, \ldots, P_n$, and let $\mathcal{C}$ be a collection of metric families that satisfies Coarse Permanence, Finite Amalgamation Permanence, and is stable under weak decomposition. If $\mathcal{C}$ contains $P_1, \ldots, P_n$ and the infinite cyclic group $\mathbb{Z}$, then $G$ is in $\mathcal{C}$. 
Proof. The proof is the same as the one for Theorem 4.8 except that Theorem 4.6 is needed instead of Theorem 4.5. □

The above theorems are generalizations of Osin’s well-known result that a finitely generated relatively hyperbolic group has finite asymptotic dimension if its peripheral subgroups do [Osi05, Theorem 1.2]. In particular, Theorem 4.9 applies to the collections $\mathfrak{D}$, $\mathfrak{E}$ and $\mathfrak{J}$. Other authors have also found generalizations of Osin’s result. It is interesting to compare Theorems 4.8 and 4.9 above to Ramras-Ramsey’s [RR, Theorem 3.9] where a different combination of permanence properties was employed to extend properties of metric spaces to relatively hyperbolic groups. In [Bar17], Bartels obtained the remarkable result that a countable relatively hyperbolic group satisfies the Farrell-Jones Conjecture if its peripheral subgroups do.

5. CAT(0) groups

Recall that a discrete group $G$ is said to act geometrically on a metric space $Y$ if it acts by isometries, the action is properly discontinuous and the quotient $Y/G$ is compact. A CAT(0) group is a countable discrete group that admits a geometric action on some finite dimensional CAT(0) space (see [BH99, II.1.1, page 158] for the definition of a CAT(0) space).

Question 5.1. Is the asymptotic dimension of a CAT(0) group finite?

Nick Wright showed that the asymptotic dimension of a finite dimensional CAT(0) cube complex is bounded above by its geometric dimension, [Wri12]. Hence any group that acts geometrically on such a complex has finite asymptotic dimension thereby providing an abundance of examples of CAT(0) groups for which the answer to Question 5.1 is affirmative.

Let $G$ be a CAT(0) group. Any amenable subgroup of $G$ is virtually abelian [AB98, Corollaries B and C]; furthermore, there is an upper bound, $r$, on the rank of all such subgroups, [CM13, Theorem C]. Thus if $\mathcal{F}$ is the collection of amenable subgroups of $G$ then for all $H \in \mathcal{F}$, $\text{asdim } H \leq r$. One approach to answer Question 5.1 affirmatively would be to attempt to apply our Theorem 4.4 by constructing a finitely $\mathcal{F}$-amenable action of $G$ on a suitable compact metrizable space $X$. Let $Y$ be a finite dimensional CAT(0) space on which $G$ acts geometrically and let $\partial_\infty Y$ denote the visual boundary of $Y$ (see [BH99].
While $\partial_\infty Y$ would, at first glance, appear to be a natural candidate for the sought after compact space $X$, it cannot fulfill this role as the $G$ action on $\partial_\infty Y$ may have fixed points. Nevertheless, Caprace constructed a set $\partial^\text{fine}_\infty Y$, which he calls a refinement of $\partial_\infty Y$, with the property that for each $x \in \partial^\text{fine}_\infty Y$ the stabilizer $G_x$ is amenable, that is, $G_x \in \mathcal{F}$, \cite{Cap09}.

**Question 5.2.** Does Caprace’s refined boundary, $\partial^\text{fine}_\infty Y$, have a compact metrizable topology for which the $G$ action on it is finitely $\mathcal{F}$-amenable?

An application of Theorem 4.4 yields the following.

**Proposition 5.3.** An affirmative answer to Question 5.2 implies an affirmative answer to Question 5.1, that is, all CAT(0) groups have finite asymptotic dimension. \qed

A theme of \cite{BL12a, BLR08}, which was further emphasized by Bartels in \cite{Bar17} (see his Theorem 4.3), is that the Farrell-Jones Conjecture for a group $G$ holds relative to a family $\mathcal{F}$ if there exists an action of $G$ on a compact, finite dimensional, contractible ANR (absolute neighborhood retract) satisfying a suitable regularity condition relative to $\mathcal{F}$. In particular, the Farrell-Jones Conjecture for CAT(0) groups could be approached by an affirmative answer to a stronger version of Question 5.2 where the condition that $Y \cup \partial^\text{fine}_\infty Y$ is a compact, finite dimensional, contractible ANR is also required. However, the issue of finding a suitable boundary for $Y$ was cleverly bypassed by Bartels and Lück in their proof of the Farrell-Jones Conjecture for CAT(0) groups, \cite{BL12a, BL12b}, by making use of homotopy actions and large balls in $Y$.

6. **Appendix. Uniform simplicial complexes**

In this appendix we gather some facts about uniform simplicial complexes.

Given a set $S$ and a real valued function $f : S \to \mathbb{R}$, the support of $f$ is the set $\text{supp}(f) = \{ s \in S \mid f(s) \neq 0 \}$. The real vector space with basis $S$ is the set $\mathbb{R}^S = \{ f : S \to \mathbb{R} \mid \text{supp}(f) \text{ is finite} \}$ together with the familiar addition and scalar multiplication operations: $(f + g)(s) = f(s) + g(s)$ for $f, g \in \mathbb{R}^S$ and $(\lambda f)(s) = \lambda f(s)$ for $f \in \mathbb{R}^S$ and $\lambda \in \mathbb{R}$. For $s \in S$ define $e_s \in \mathbb{R}^S$ by $e_s(t) = 1$ if $t = s$ and $e_s(t) = 0$ otherwise. The set $\{ e_s \mid s \in S \}$ is the standard basis for $\mathbb{R}^S$ and can be identified with $S$ via the bijection $s \mapsto e_s$. Observe that $f = \sum_{s \in S} f(s)e_s$ for any $f \in \mathbb{R}^S$. 
Let $1 \leq p < \infty$. The $\ell^p$-norm on $\mathbb{R}^S$, denoted $\| \cdot \|_p$, is given by $\| f \|_p = \left( \sum_{s \in S} |f(s)|^p \right)^{1/p}$. The $\ell^\infty$-norm on $\mathbb{R}^S$, denoted $\| \cdot \|_\infty$, is given by $\| f \|_\infty = \max_{s \in S} |f(s)|$.

**Proposition 6.1.** Let $1 \leq p \leq q \leq \infty$. For $f \in \mathbb{R}^S$,

$$\| f \|_q \leq \| f \|_p \leq (\# \text{supp}(f))^{1/p-1/q} \| f \|_q$$

where $\# \text{supp}(f)$ is the cardinality of $\text{supp}(f)$ and, by convention, $1/\infty = 0$.

**Proof.** The conclusion is consequence of the standard inequality

$$\| x \|_q \leq \| x \|_p \leq n^{1/p-1/q} \| x \|_q,$$

applied to $x = (f(s_1), \ldots, f(s_n))$ where $\text{supp}(f) = \{s_1, \ldots, s_n\}$.

For $1 \leq p \leq \infty$, the $\ell^p$-metric on $\mathbb{R}^S$ is given by $d^p(f, g) = \| f - g \|_p$.

Let $S$ be an infinite set. The **infinite simplex with vertex set $S$**, denoted $\Delta(S)$, is the convex hull of the standard basis of $\mathbb{R}^S$, that is,

$$\Delta(S) = \{ f \in \mathbb{R}^S \mid f \geq 0 \text{ and } \sum_{s \in S} f(s) = 1 \}.$$ 

Since the image of $f \in \Delta(S)$ lies in the the unit interval $I = [0, 1]$, we can view $\Delta(S)$ as a subset of the product $\prod_{s \in S} I$. Recall that the **product topology** on $\prod_{s \in S} I$ is the smallest topology such that for each $t \in S$ the evaluation map $\text{ev}_t \colon \prod_{s \in S} I \to I$, given by $\text{ev}_t(f) = f(t)$, is continuous. Hence $\Delta(S)$ inherits a topology, called the **strong topology**, as a subspace of $\prod_{s \in S} I$. Each of the $\ell^p$-metrics restricts to a metric on $\Delta(S)$ and so determines a topology, namely the corresponding metric topology.

**Proposition 6.2.** For $1 \leq p \leq \infty$, the $\ell^p$-metric topology on $\Delta(S)$ coincides with the strong topology.

**Proof.** Let $f \in \Delta(S)$. For $r > 0$ define $U(f, r) \subset \Delta(S)$ by

$$U(f, r) = \{ g \in \Delta(S) \mid \text{ for all } s \in \text{supp}(f), \ |g(s) - f(s)| < r \}.$$ 

Observe that $U(f, r) = \bigcap_{s \in \text{supp}(f)} \text{ev}_s^{-1}((f(s) - r, f(s) + r))$. Hence $U(f, r)$ is open in the strong topology and indeed sets of this form give a basis for the strong topology.

Consider the open ball of radius $r$ centered at $f$ with respect to the $\ell^p$-metric, $B_p(f, r) = \{ g \in \Delta(S) \mid d^p(g, f) < r \}$. Let $N = \# \text{supp}(f)$. Let $N_p = N$ if $p = \infty$ and $N_p = (N + N^p)^{1/p}$ if $1 \leq p < \infty$. 

Claim. $U(f, r/N_p) \subset B_p(f, r) \subset U(f, r)$.

The Claim implies that the strong topology and the $\ell^p$-metric topology coincide. The inclusion $B_p(f, r) \subset U(f, r)$ is straightforward. We give a proof of $U(f, r/N_p) \subset B_p(f, r)$. Let $g \in U(f, r/N_p)$. Then

$$\sum_{s \notin \text{supp}(f)} g(s) = 1 - \sum_{s \in \text{supp}(f)} g(s) \quad \text{because} \quad \sum_{s \in S} g(s) = 1$$

$$= \sum_{s \in \text{supp}(f)} f(s) - g(s) \quad \text{because} \quad \sum_{s \in \text{supp}(f)} f(s) = 1$$

$$\leq \sum_{s \in \text{supp}(f)} |f(s) - g(s)| < N(r/N_p).$$

Assume $1 \leq p < \infty$. Note that $\left(\sum_{s \notin \text{supp}(f)} g(s)^p\right)^{1/p} \leq \sum_{s \notin \text{supp}(f)} g(s) < N(r/N_p)$. We have,

$$(d^p(f, g))^p = \sum_{s \in \text{supp}(f)} |f(s) - g(s)|^p + \sum_{s \notin \text{supp}(f)} |g(s)|^p$$

$$< N(r/N_p)^p + (N(r/N_p))^p = r^p.$$

Hence $d^p(f, g) < r$ and so $g \in B_p(f, r)$.

Assume $p = \infty$. We have, $\max_{s \notin \text{supp}(f)} g(s) \leq \sum_{s \notin \text{supp}(f)} g(s) < N(r/N_\infty) = r$. Hence $d^\infty(f, g) = \max\left(\max_{s \in \text{supp}(f)} |f(s) - g(s)|, \max_{s \notin \text{supp}(f)} g(s)\right) < r$ and so $g \in B_\infty(f, r)$. □

We observe that $\Delta(S)$ is contained in the closed unit ball, $\bar{B}_p = \{f \in \mathbb{R}^S \mid \|f\|_p \leq 1\}$, for any of the $\ell^p$-metrics.

**Proposition 6.3.** For $1 \leq p \leq q \leq \infty$, $\Delta(S) \subset \bar{B}_p \subset \bar{B}_q$.

**Proof.** The inequality $\|f\|_q \leq \|f\|_p$ implies $\bar{B}_p \subset \bar{B}_q$. Note that $\|e_s\|_1 = 1$ for all $s \in S$. If $f \in \Delta(S)$ then

$$\|f\|_1 = \|\sum_{s \in S} f(s)e_s\|_1 \leq \sum_{s \in S} |f(s)||e_s\|_1 = \sum_{s \in S} f(s) = 1$$

and so $f \in \bar{B}_1$. □

We recall the definition of an abstract simplicial complex and the associated terminology.

**Definition 6.4.** A simplicial complex consists of vertex set $K_0$ together with a collection $K$ of non-empty finite subsets of $K_0$ such that:
1. For every \( x \in K_0, \{x\} \in K \),
2. if \( \sigma \in K \) and \( \tau \subseteq \sigma \) and \( \tau \) is non-empty then \( \tau \in K \).

For brevity, we sometimes write \( K \) for \((K_0, K)\). The dimension of \( \sigma \in K \), denoted by \( \dim(\sigma) \), is \( \dim(\sigma) = \#\sigma - 1 \). An element \( \sigma \in K \) with \( \dim(\sigma) = k \) is called a \( k \)-simplex of \( K \). The dimension of \( K \), denoted by \( \dim(K) \), is \( \dim(K) = \sup\{\dim(\sigma) \mid \sigma \in K\} \) (the value \( \infty \) is allowed).

The geometric realization of a simplicial complex \((K_0, K)\), denoted by \( |K| \), is the subset of \( \Delta(K_0) \) given by
\[
|K| = \{f \in \Delta(K_0) \mid \text{supp}(f) \subseteq K\}.
\]
The set \( |K| \) inherits the strong topology from \( \Delta(K_0) \) which by Proposition 6.2 is the same as the metric topology determined by any of the \( \ell^p \)-metrics. A space of the form \( |K| \) is also known as a uniform simplicial complex.

The following proposition is a direct consequence of Proposition 6.1.

**Proposition 6.5.** Let \( K \) be a simplicial complex such that \( \dim(K) = N < \infty \). Then for all \( f, g \in |K| \) and \( 1 \leq p \leq q \leq \infty \),
\[
d^q(f, g) \leq d^p(f, g) \leq (2N + 2)^{1/p - 1/q} d^q(f, g).
\]
In particular, all of the \( \ell^p \)-metrics restricted to \( |K| \) are Lipschitz equivalent. \( \square \)

**Remark 6.6.** If \( K \) is infinite dimensional and \( 1 \leq p < q \leq \infty \) then the \( \ell^p \)-metric on \( |K| \) is not Lipschitz equivalent to the \( \ell^q \)-metric. Let \( \sigma^n = \{s_0^n, \ldots, s_n^n\} \in K, n \geq 0 \), be a sequence of simplices in \( K \) with \( \dim(\sigma^n) = n \). For \( n \geq 0 \), define \( f_n \in |K| \) by \( f_n = \sum_{i=0}^{n} (n+1)^{-1} e_{s_i^n} \). Then \( \| f_n \|_p / \| f_n \|_q = (n+1)^{1/p - 1/q} \to \infty \) as \( n \to \infty \). Hence for any given \( h \in |K| \) there cannot be a constant \( C \) such that \( d^p(f_n, h) = \| f_n - h \|_p \leq C \| f_n - h \|_q = C d^q(f_n, h) \) for all \( n \).

Let \( T \subset S \) be an inclusion of sets. Given \( f \in \mathbb{R}^T \), define \( \bar{f} \in \mathbb{R}^S \) by \( \bar{f}(s) = 0 \) if \( s \in S - T \) and \( \bar{f}(s) = f(s) \) if \( s \in T \). It is straightforward to show:
1. For all \( f \in \mathbb{R}^T \), \( \text{supp}(f) = \text{supp}(\bar{f}) \),
2. the map \( i: \mathbb{R}^T \to \mathbb{R}^S, i(f) = \bar{f} \) is linear,
3. for \( 1 \leq p \leq \infty \), \( \| \bar{f} \|_p = \| f \|_p \) and so \( i: \mathbb{R}^T \to \mathbb{R}^S \) is isometric embedding for the \( \ell^p \)-metric,
4. \( i(\Delta(T)) \subset \Delta(S) \).

Let \((K_0, K)\) be a simplicial complex. Another simplicial complex \((L_0, L)\) is a subcomplex of \((K_0, K)\) if \(L_0 \subset K_0\) and \(L \subset K\). If \((L_0, L)\) is a subcomplex of \((K_0, K)\) then the linear map \(i: \mathbb{R}^{L_0} \to \mathbb{R}^{K_0}\) defined above restricts to an isometric embedding \(i: |L| \to |K|\) for any of the \(\ell^p\)-metrics. The following elementary lemma is useful.

**Lemma 6.7.** Let \((L_0, L)\) be a subcomplex of \((K_0, K)\). Let \(w \in K_0\). If \(f \in |L|\) and \(\bar{f}(w) \neq 0\) then \(w \in L_0\).

**Proof.** The hypothesis \(\bar{f}(w) \neq 0\) implies \(w \in \text{supp}(\bar{f}) = \text{supp}(f) \subset L_0\). \(\Box\)

Let \(G\) be a group and \((K_0, K)\) a simplicial complex. A simplicial (left) \(G\)-action on \(K\) is a (left) \(G\)-action on the vertex set \(K_0\) such that if \(\{s_0, \ldots, s_n\}\) is an \(n\)-simplex of \(K\) then for any \(g \in G\), \(\{g \cdot s_0, \ldots, g \cdot s_n\}\) is an \(n\)-simplex of \(K\). A simplicial \(G\)-action on \(K\) yields a left \(G\)-action on \(|K|\) via the formula \((g \cdot f)(s) = f(g^{-1} \cdot s)\) for \(g \in G\) and \(s \in K_0\). Note that for a given \(1 \leq p \leq \infty\), and for all \(f, h \in |K|\) we have \(d^p(g \cdot f, g \cdot h) = d^p(f, h)\), that is, \(G\) acts by isometries on \(|K|\).

There is another useful topology on the geometric realization of a simplicial complex. The weak topology, also known as the Whitehead topology, on the underlying set of \(|K|\) is characterized as follows: a subset \(A \subset |K|\) is closed in the weak topology if and only if for every simplex \(\sigma \in K\), the set \(A \cap |\sigma|\) is closed in \(|\sigma|\). We denote the corresponding topological space by \(|K|_w\). With this topology, \(|K|_w\) is a CW complex with \(n\)-skeleton \((|K|_w)^n = \bigcup\{|\sigma| \mid \sigma \in K \text{ and } \dim(\sigma) \leq n\}\). The weak topology is finer than the strong topology, that is, the identity map \(\tilde{id}: |K|_w \to |K|\) is continuous. A simplicial complex \(K\) is locally finite if each vertex of \(K\) belongs to only finitely many simplicies of \(K\).

**Proposition 6.8.** \([FP11\text{ Proposition 3.3.4]}\) Let \(K\) be a simplicial complex. The weak topology on the underlying set of \(|K|\) coincides with the strong topology if and only if \(K\) is locally finite.

Although \(\tilde{id}: |K|_w \to |K|\) is not a homeomorphism if \(|K|\) is not locally finite, it is always a homotopy equivalence by \([Dow52\text{ §16, Theorem 1]}\). The following proposition is a consequence of Dowker's theory of metric complexes (see \([Dow52\text{ §14 and (15.2)]}\) and note that \(|K|\) with any of the \(\ell^p\)-metrics is a metric complex in the sense of Dowker).
Proposition 6.9. Let \( 1 \leq p \leq \infty \) and \( \varepsilon > 0 \) be given. There exists a continuous map \( h: |K| \to |K|_w \) such that \( \tilde{h} = \tilde{id} \circ h \) is \( \varepsilon \)-homotopic, with respect to the \( \ell^p \)-metric on \( |K| \), to the identity map \( \text{id}_{|K|}: |K| \to |K| \).

A pre-action of a group \( G \) on a space \( X \) is a continuous map \( G \times X \to X \), written \( g \cdot x \) for \( g \in G \) and \( x \in X \). The following technical lemma, which makes use of Proposition 6.9, is invoked in the proofs of Theorems 3.7 and 3.4.

Lemma 6.10. Let \( 1 \leq p \leq \infty \) and \( \varepsilon > 0 \) be given. Let \( G \) be a group, \( S \subset G \) a subset, \( X \) a space with a \( G \)-pre-action, \( K \) a simplicial complex equipped with a simplicial \( G \)-action and \( f: X \to |K| \) a map such that \( d^p(f(s \cdot x), s \cdot f(x)) \leq \varepsilon \) for all \( s \in S \) and all \( x \in X \). Then \( f \) is \( \varepsilon \)-homotopic to a map \( \tilde{f}: X \to |K| \) that factors through \( \tilde{id}: |K|_w \to |K| \) and satisfies \( d^p(\tilde{f}(s \cdot x), s \cdot \tilde{f}(x)) \leq 3\varepsilon \) for all \( s \in S \) and all \( x \in X \).

Proof. By Proposition 6.9, there is a map \( h: |K| \to |K|_w \) such that \( \tilde{h} = \tilde{id} \circ h \) is \( \varepsilon \)-homotopic to \( \text{id}_{|K|} \). Let \( H: |K| \times [0, 1] \to |K| \) be a homotopy with \( H_0 = \text{id}_{|K|}, H_1 = \tilde{h} \) and \( d^p(H_t(y), y) \leq \varepsilon \) for all \( t \in [0, 1] \) and \( y \in |K| \). Let \( \tilde{f}_t = H_t \circ f \) and let \( \tilde{f} = \tilde{f}_1 = \tilde{h} \circ f \) yielding an \( \varepsilon \)-homotopy from \( f \) to \( \tilde{f} \). For all \( t \in [0, 1], s \in S \) and \( x \in X \) we have

\[
d^p(\tilde{f}_t(s \cdot x), s \cdot \tilde{f}_t(x)) = d^p(H_t(f(s \cdot x)), s \cdot H_t(f(x))) \\
\leq d^p(H_t(f(s \cdot x)), f(s \cdot x)) + d^p(f(s \cdot x), s \cdot H_t(f(x))) \\
\leq \varepsilon + d^p(f(s \cdot x), s \cdot H_t(f(x))) \\
\leq \varepsilon + d^p(f(s \cdot x), s \cdot f(x)) + d^p(s \cdot f(x), s \cdot H_t(f(x))) \\
= \varepsilon + d^p(f(s \cdot x), s \cdot f(x)) + d^p(f(x), H_t(f(x))) \leq 3\varepsilon.
\]

Hence \( \tilde{f} \) satisfies the conclusion of the lemma. \( \square \)

References

[AB98] Scot Adams and Werner Ballmann, Amenable isometry groups of Hadamard spaces, Math. Ann. 312 (1998), no. 1, 183–195. MR 1645958

[Bar] Arthur Bartels, K-theory and actions on euclidean retracts, Submitted to Proceedings of the ICM 2018, preprint version: arXiv:1801.00020.

[Bar16] ———, On proofs of the Farrell-Jones conjecture, Topology and geometric group theory, Springer Proc. Math. Stat., vol. 184, Springer, [Cham], 2016, pp. 1–31. MR 3598160

[Bar17] ———, Coarse flow spaces for relatively hyperbolic groups, Compos. Math. 153 (2017), no. 4, 745–779. MR 3631229

[BB] Arthur Bartels and Mladen Bestvina, The Farrell-Jones conjecture for mapping class groups, arXiv:1606.02844.
1. **[BD04]** G. Bell and A. Dranishnikov, *On asymptotic dimension of groups acting on trees*, Geom. Dedicata **103** (2004), 89–101. MR 2034954 (2005b:20078)

2. **[BDLM08]** N. Brodskiy, J. Dydak, M. Levin, and A. Mitra, *A Hurewicz theorem for the Assouad-Nagata dimension*, J. Lond. Math. Soc. (2) **77** (2008), no. 3, 741–756. MR 2418302

3. **[BH99]** Martin R. Bridson and André Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319, Springer-Verlag, Berlin, 1999. MR 1744486

4. **[BL12a]** Arthur Bartels and Wolfgang Lück, *The Borel conjecture for hyperbolic and CAT(0)-groups*, Ann. of Math. (2) **175** (2012), no. 2, 631–689. MR 2993750

5. **[BL12b]** _____, *Geodesic flow for CAT(0)-groups*, Geom. Topol. **16** (2012), no. 3, 1345–1391. MR 2967054

6. **[BLR08]** Arthur Bartels, Wolfgang Lück, and Holger Reich, *The K-theoretic Farrell-Jones conjecture for hyperbolic groups*, Invent. Math. **172** (2008), no. 1, 29–70. MR 2385666

7. **[Cap09]** Pierre-Emmanuel Caprace, *Amenable groups and Hadamard spaces with a totally disconnected isometry group*, Comment. Math. Helv. **84** (2009), no. 2, 437–455. MR 2495801

8. **[CM13]** Pierre-Emmanuel Caprace and Nicolas Monod, *Fixed points and amenability in non-positive curvature*, Math. Ann. **356** (2013), no. 4, 1303–1337. MR 3072802

9. **[DG03]** Marius Dadarlat and Erik Guentner, *Constructions preserving Hilbert space uniform embeddability of discrete groups*, Trans. Amer. Math. Soc. **355** (2003), no. 8, 3253–3275. MR 1974686

10. **[Dow52]** C. H. Dowker, *Topology of metric complexes*, Amer. J. Math. **74** (1952), 555–577. MR 0048020 (13,965h)

11. **[FP11]** Davide L. Ferrario and Renzo A. Piccinini, *Simplicial structures in topology*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, Springer, New York, 2011, Translated from the 2009 Italian original by Maria Nair Piccinini. MR 2663748 (2012a:55001)

12. **[GTY12]** Erik Guentner, Romain Tessera, and Guoliang Yu, *A notion of geometric complexity and its application to topological rigidity*, Invent. Math. **189** (2012), no. 2, 315–357. MR 2947546

13. **[GTY13]** _____, *Discrete groups with finite decomposition complexity*, Groups Geom. Dyn. **7** (2013), no. 2, 377–402. MR 3054574

14. **[Gue14]** Erik Guentner, *Permanence in coarse geometry*, Recent progress in general topology. III, Atlantis Press, Paris, 2014, pp. 507–533. MR 3205491

15. **[GWY17]** Erik Guentner, Rufus Willett, and Guoliang Yu, *Dynamic asymptotic dimension: relation to dynamics, topology, coarse geometry, and C*-algebras*, Math. Ann. **367** (2017), no. 1-2, 785–829. MR 3606454

16. **[KNR]** Daniel Kasprowski, Andrew Nicas, and David Rosenthal, *Regular finite decomposition complexity*, To appear in J. Topol. Anal., preprint version: arXiv:1608.0415.

17. **[Lüc10]** Wolfgang Lück, *K- and L-theory of group rings*, Proceedings of the International Congress of Mathematicians. Volume II, Hindustan Book Agency, New Delhi, 2010, pp. 1071–1098. MR 2827832

18. **[NR]** Andrew Nicas and David Rosenthal, *Hyperbolic dimension and decomposition complexity*, To appear in a volume of the LMS Lecture Note Series, preprint version: arXiv:1509.06437.

19. **[Osi05]** D. Osin, *Asymptotic dimension of relatively hyperbolic groups*, Int. Math. Res. Not. (2005), no. 35, 2143–2161. MR 2181790

20. **[RR]** Daniel A. Ramras and Bobby W. Ramsey, *Extending Properties to Relatively Hyperbolic Groups*, To appear in Kyoto J. Math., preprint version: arXiv:1410.0060.

21. **[Saw17]** Damian Sawicki, *On equivariant asymptotic dimension*, Groups Geom. Dyn. **11** (2017), no. 3, 977–1002. MR 3692903

22. **[STY02]** G. Skandalis, J. L. Tu, and G. Yu, *The coarse Baum-Connes conjecture and groupoids*, Topology **41** (2002), no. 4, 807–834. MR 1905840
Nick Wright, *Finite asymptotic dimension for CAT(0) cube complexes*, Geom. Topol. 16 (2012), no. 1, 527–554. MR 2916293

Guoliang Yu, *The Novikov conjecture for groups with finite asymptotic dimension*, Ann. of Math. (2) 147 (1998), no. 2, 325–355. MR 1626745 (99k:57072)

, *The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space*, Invent. Math. 139 (2000), no. 1, 201–240. MR 1728880

Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario, Canada L8S 4K1

E-mail address: nicas@mcmaster.ca

Department of Mathematics and Computer Science, St. John’s University, 8000 Utopia Pkwy, Queens, NY 11439, USA

E-mail address: rosenthd@stjohns.edu