Zero-Knowledge for QMA from Locally Simulatable Proofs

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Abstract

We provide several advances to the understanding of the class of Quantum Merlin-Arthur proof systems (QMA), the quantum analogue of NP. First, we answer a longstanding open question by showing that the Consistency of Local Density Matrices problem is QMA-complete under Karp reductions. We also show for the first time a commit-and-open computational zero-knowledge proof system for all of QMA as a quantum analogue of a “sigma” protocol. We then define a Proof of Quantum Knowledge, which guarantees that a prover is effectively in possession of a quantum witness in an interactive proof, and show that our zero-knowledge proof system satisfies this definition. Finally, we show that our proof system can be used to establish that QMA has a quantum non-interactive zero-knowledge proof system in the secret parameters setting.

Our main technique consists in developing locally simulatable proofs for all of QMA: this is an encoding of a QMA witness such that it can be efficiently verified by probing only five qubits and, furthermore, the reduced density matrix of any five-qubit subsystem can be computed in polynomial time and is independent of the witness. This construction follows the techniques of Grilo, Slofstra, and Yuven [FOCS 2019].

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1 Introduction

The complexity class QMA is the quantum analogue of the classical class NP, the class of problems whose solutions can be verified in deterministic polynomial time. More precisely, in QMA, an all-powerful prover produces a quantum proof that is verified by a quantum polynomially-bounded verifier. Given the probabilistic nature of quantum computation, we require that for true statements, there exists a quantum proof that makes the verifier accept with high probability (this is called completeness), whereas all “proofs” for false statements are rejected with high probability (which is called soundness).

The class QMA was first defined by Kitaev [KSV02], who also showed that deciding if a \( k \)-local Hamiltonian problem has low-energy states is QMA-complete. The importance of this result is two-fold: first, from a theoretical computer science perspective, it is the quantum analogue of the Cook-Levin theorem, since it establishes the first non-trivial QMA-complete problem. Secondly, it shows deep links between physics and complexity theory, since the \( k \)-local Hamiltonian problem is an important problem in many-body physics. Thus, a better understanding of QMA would lead to a better understanding of the power of quantum resources in proof verification, as well as the role of quantum entanglement in low-energy states.

Follow-up work strengthened our understanding of this important complexity class, e.g., by showing that QMA is contained in the complexity class PP [KW00]\(^\text{1}\); that it is possible to reduce completeness and soundness errors without increasing the length of the witness [MW05]; understanding the difference between quantum and classical proofs [AK07, GKS16, FK18]; possibility of perfect completeness [Aar09]; and, more recently, the relation of QMA and with non-local games [NV17, NV18, CGJV19].

Also, much follow-up work focused on understanding the complete problems for QMA, mostly by improving the parameters of the QMA-hard Local Hamiltonian problem, or making it closer to models more physically relevant [KR03, Liu06, KKR06, OT08, CM14, HNN13, BC18]. In 2014, a survey of QMA-complete languages [Boo14] contained a list of 21 general problems that are known to be QMA-complete\(^\text{2}\), and since then, the situation has not drastically changed. This contrasts with the development of NP, where only a few years after the developments surrounding 3–SAT, Karp published a theory of reducibility, including a list of 21 NP-complete problems [Kar72]; while 8 years later, a celebrated book by Garey and Johnson surveyed over 300 NP-complete problems [GJ90].

Recently, the role of QMA in quantum cryptography has also been explored. For instance, several results used ideas of the QMA-completeness of the Local Hamiltonian problem in order to perform verifiable delegation of quantum computation [FHM18, Mah18, Gri19]. Furthermore, another line of work studies zero-knowledge protocols for QMA [BJSW16, VZ19]; which is extremely relevant, given the fundamental importance in cryptography of zero-knowledge protocols for NP.

Despite the multiple advances in our understanding of QMA and related techniques, a number of fundamental open questions remain. In this work, we solve some of these open problems by showing: (i) QMA-hardness of Consistency of Local Density Matrix (CLDM) problem under Karp reductions; (ii) “commit-and-open” Zero-Knowledge (ZK) proof of quantum knowledge (PoQ) protocols for QMA; and (iii) a non-interactive zero-knowledge (NIZK) protocol in the secret parameter scenario. Our main technical tool consists in showing a new characterization of QMA: we show that SimQMA = QMA, where SimQMA is the class of proofs that are locally simulatable. In order

\(^{1}\)PP is the complexity class of decision problems that can be solved by probabilistic polynomial-time algorithms with error strictly smaller than \(\frac{1}{2}\).

\(^{2}\)We remark that these problems can be clustered as variations of a handful of base problems.
to describe our results and appreciate their contribution to a better understanding of QMA, we first
give an overview of these areas and how they relate to these particular problems.

1.1 Background

In this section, we discuss the background on the topics that are relevant to this work, summarizing
their current state-of-the-art.

Consistency of Local Density Matrices (CLDM). The Consistency of Local Density matrices prob-
lem (CLDM) is as follows: given the classical description of local density matrices $\rho_1, ..., \rho_m$, each
on a set of at most $k$ qubits and for a global system of $n$ qubits, is there a state $\tau$ that is consistent
with such reduced states? Liu [Liu06] showed that this problem is in QMA and that it is QMA-hard
under Turing reductions, i.e., a deterministic polynomial time algorithm with access to an oracle
that solves CLDM in unit time can solve any problem in QMA.

We remark that this type of reduction is rather troublesome for QMA, since the class is not
known (nor expected) to be closed under complement, i.e., it is widely believed that QMA $\neq$
coQMA. If this is indeed the case, then Turing reductions do not allow a black-box generalization
of results regarding the CLDM problem to all problems in QMA. This highlights the open problem
of establishing the QMA-hardness of the CLDM problem under Karp reductions, i.e., to show an
efficient mapping between yes- and no-instances of any QMA problem to yes- and no-instances of
CLDM, respectively.

Zero-Knowledge (ZK) Proofs for QMA. In an interactive proof, a limited party, the Verifier, receives
the help of some untrusted powerful party, the Prover, in order to decide if some statement is true.
This is a generalization of a proof, where we allows multiple rounds of interaction. As usual, we
require that the completeness and soundness properties hold. For cryptographic applications, the
zero-knowledge (ZK) property is often desirable: here, we require that the Verifier learn nothing
from the interaction with the Prover. This property is formalized by showing the existence of an
efficient simulator, which is able to reproduce (i.e., simulate) the output of any given verifier on a
yes instance (without having direct access to the actual prover or witness).^3

As paradoxical as it sounds, statistical zero-knowledge interactive proofs are known to be possible
for a host of languages, including the Quadratic Non-Residuosity, Graph Isomorphism, and
Graph Non-Isomorphism problems [GMW91, GMR89]; furthermore, all languages that can be
proven by multiple provers (MIP) admits perfect zero-knowledge MIPs [BGKW88]. By introducing
computational assumptions, it was shown that all languages that admit an interactive proof system
also admit a zero-knowledge interactive proof system [BOGG+88]. Zero-knowledge interactive
proof systems have had a profound impact in multiple areas, including cryptography [GMW87]
and complexity theory [Vad07].

We now briefly review the zero-knowledge interactive proof system for the NP-complete problem
of Graph 3-colouring (3-COL). This is a 3-message proof system, and has the additional property
that, given a witness, the prover is efficient. As a first message, the prover commits to a
permutation of the given 3-colouring (meaning that the prover randomly permutes the colours to
obtain colouring $c$, and produces a list $(v_i, \text{commit}(c(v_i)))$, using a cryptographic primitive commit
which is a commitment scheme). In the second message, the verifier chooses uniformly at random
an edge $\{v_i, v_j\}$ of the graph. The prover responds with the information that allows the verifier to

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^3Different definitions of “reproduce” result in different definitions of zero-knowledge protocols. A protocol is perfect
zero-knowledge if the distribution of the output of the simulator is exactly the same as the distribution of output of
transcripts of the protocol. A protocol is statistical zero-knowledge if such distributions are statistically close. Finally, a
protocol is computational zero-knowledge if no efficient algorithm can distinguish both distributions. The convention is
that in the absence of such specification, we are considering the case of computational zero-knowledge.
open the commitments to the colouring of the vertices of this edge (and nothing more). The verifier accepts if and only if the revealed colours are different. It is easy to see that the protocol is complete and sound. For the zero-knowledge property, the simulator consists in a process that guesses which edge will be requested by the verifier and commits to a colouring that satisfies the prover in case this guess is correct. If the guess is incorrect, the technique of rewinding allows the simulator to re-initialize the interaction until it is eventually successful. Protocols that follow the commit-challenge-response structure of this proof system are called $\Sigma$-protocols and, due to their simplicity, they play a very important role, for instance in the celebrated Fiat-Shamir transformation [FS87].

In the cryptographic scenario, an important relaxation of zero-knowledge proof systems are zero-knowledge argument systems for NP. In this model, the Prover is also bounded to polynomial-time computation, and, for positive instances, the Prover is provided a witness to the NP instance. This model allows much more efficient protocols which enables it to be used in practice [BSCG+14, PHGR16, BSCR+19].

The foundations of zero-knowledge in the quantum world were established by Watrous, who showed a technique called quantum rewinding [Wat09] which is used to show the security of some classical zero-knowledge proofs (including the protocol for 3-COL described above), even against quantum adversaries. The importance of this technique is that quantum measurements typically disturb the measured state. When we consider quantum adversaries, such difficulties concern even classical proof systems, due to the rewinding technique that is ubiquitous (see example in the case of 3-COL above). Indeed, in the quantum setting, intermediate measurements (such as checking if the guess is correct) may compromise the success of future executions, since it is not possible a priori to “rewind” to a previous point in the execution in a black-box way.

Another dimension where quantum information poses new challenges is in the study of interactive proof systems for quantum languages. We point out that Liu [Liu06] observed very early on that the CLDM problem should admit a simple zero-knowledge proof system following the “commit-and-open” approach, as in the 3-COL protocol. Inspired by this observation, recent progress has established the existence of zero-knowledge protocols for all of QMA [BJSW16]. We note that although the proof system used in [BJSW16] is reminiscent of a $\Sigma$-protocol, there are a number of reasons why it is not a “natural” quantum analogue of a $\Sigma$ protocol. These include: (i) the use of a coin-flipping protocol, which makes the communication cost higher than 3 messages; (ii) the fact that the verifier’s message is not a random challenge; and (iii) the final answer from the Prover is not only the opening of some committed values.

Recently, Vidick and Zhang [VZ19] showed how to make classical all of the interaction between the Verifier and the Prover in [BJSW16], by considering argument systems instead of proof systems. In their protocol, they compose the result of Mahadev [Mah18] for verifiable delegation of quantum computation by classical clients with the zero-knowledge protocol of [BJSW16].

**Zero-Knowledge Proofs of Knowledge (PoK).** In a zero-knowledge proof, the Verifier becomes convinced of the existence of a witness, but this a priori has no bearing on the prover actually having in her possession such a witness. In some circumstances, it is important to guarantee that the Prover actually has a witness. This is the realm of a zero-knowledge proof of knowledge (PoK) [GMR89, BG93].

We give an example to depict this subtlety. Let us consider the task of anonymous credentials [Cha83]. In this setting, Alice wants to authenticate into some online service using her private credentials. In order to protect her credentials, she could engage in a zero-knowledge proof; this, however would be unsatisfactory, since the verifier in this scenario would be become convinced of the existence of accepting credentials, which does not necessarily translate to Alice actually being

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4The Greek letter $\Sigma$ visualizes the flow of the protocol.
in the possession of these credentials. To remedy this situation, the PoK property establishes an “if-and-only-if” situation: if the Verifier accepts, then we can guarantee that the Prover actually knows a witness. This notion is formally defined by requiring the existence of an extractor, which is polynomial-time process $K$ that outputs a valid witness when given oracle access to some Prover $P^*$ that makes the Verifier accept with high enough probability.

In the quantum case, there has been some positive results in terms of the security of classical proofs of knowledge for NP against quantum adversaries [Unr12]. However, in the fully quantum case (that is, proofs of quantum knowledge for QMA), no scheme has been proposed. One of the possible reasons why no such proof of quantum knowledge protocols was proposed is the lack of a simple zero-knowledge proof for QMA.

**Non-Interactive Zero-Knowledge Proofs (NIZK).** The interactive nature of zero knowledge proof systems (for instance, in $\Sigma$-protocols) means that in some situations they are not applicable since they require the parties to be simultaneously online. Therefore, another desired property of such proof systems is that they are non-interactive, which means the whole protocol consists in a single message from the Prover to the Verifier. Non-interactive zero-knowledge proofs (NIZK) is a fundamental construction in modern cryptography and has far-reaching applications, for instance to cryptocurrencies [BSCG+14].

We note that NIZK is known to be impossible in the standard model [GO94], i.e., without extra assumptions, and therefore NIZK has been considered in different models. In one of the models most relevant in cryptography, we assume a common reference string (CRS) [BFM88], which can be seen as a trusted party sending a random string to both the Prover and the Verifier. In another model, the trusted party is allowed to send different (but correlated) messages to the Prover and the Verifier; this is called the secret parameter setup [PS05]. Classically, this model has been shown to be very powerful, since even its statistical zero-knowledge version is equivalent to all of the problems in the complexity class AM (this is the class that contains problem that can be verified by public-coin polynomial-time verifiers). As mentioned in [PS05], this model encompasses another model for NIZK where the Prover and the Verifier perform an offline pre-processing phase (which is independent of the input) and then the Prover provides the ZK proof [KMO89]. This inclusion holds since the parties could perform secure multi-party computation to compute the trusted party’s operations.

In the quantum case, very little is known on non-interactive zero-knowledge. Chailloux, Ciocan, Kerenidis and Vadhan studied this problem in a setup where the message provided by the trusted party can depend on the instance of the problem [CCKV08]. Recently, some results also showed that the Fiat-Shamir transformation for classical protocols is still safe in the quantum setting, in the quantum random oracle model [LZ19, DFMS19, Cha19]. One particular and intriguing open question is the possibility of NIZKs for QMA.

### 1.2 Results

As we have shown so far, the state-of-the-art in the study of QMA is that the body of knowledge is still developing, and that there are some specific goals that, if achieved, would help us better understand QMA and devise new protocols for quantum cryptography. Given this context, we present now our results in more details.

Our first result (Section 3) is to show that the CLDM problem is QMA-hard under Karp reductions, solving the 13-year-old problem proposed by Liu [Liu06].

**Result 1.** The CLDM problem is QMA-complete under Karp reductions.
We capture the techniques used in establishing the above into a new characterization of QMA that provides the best-of-both worlds in terms of two proof systems for QMA in an abstract way: we define SimQMA as the complexity class with proof systems that are (i) locally verifiable (as in the Local Hamiltonian problem), and (ii) every reduced density matrix of the witness can be efficiently computed (as in the CLDM problem). We thus show (Section 3.2):

**Result 2.** SimQMA = QMA.

The above is the basis for the results in the remainder of the paper, which deal with applications to quantum cryptography.

Next, we define a quantum notion of a classical $\Sigma$-protocol, which we call a $\Xi$-protocol\(^5\) (please note, both a $\Sigma$ and $\Xi$ protocol is also referred throughout as “commit-and-open” protocols.) Using our characterization given in Result 2, we show a QMA-complete language that admits a $\Xi$-protocol. Taking into account the importance of $\Sigma$ protocols for zero-knowledge proofs, we are able to show (Section 4) a quantum analogue of the celebrated [GMW91] paper:

**Result 3.** All problems in QMA admit a computational zero-knowledge $\Xi$-proof system.

We are also able to show that simple changes in the construction achieving the result above allow us to achieve the first statistical zero-knowledge argument system for QMA (Section 4.3).

**Result 4.** All problems in QMA admit a statistical zero-knowledge $\Xi$-argument system.

Then we provide the definition of Proof of Quantum Knowledge (PoQ).\(^6\) In short, we say that a proof systems is a PoQ if there exists a quantum polynomial-time extractor $K$ that has oracle access to a quantum Prover which makes the verifier accept with high enough probability, and the extractor is able to output a sufficiently good witness for a “QMA-relation”. We note that this definition for a PoQ is not a straightforward adaptation of the classical definition; this is because NP has many properties such as perfect completeness, perfect soundness and even that proofs can be copied, that are not expected to hold in the QMA case. More details are given in Section 5. We are then able to show that our $\Xi$ protocols for QMA described in Results 3 and 4 are both PoQs. This is the first proof of knowledge for QMA.\(^7\)

**Result 5.** All problems in QMA admit a zero-knowledge proof of quantum knowledge proof system and a statistical zero-knowledge proof of quantum knowledge argument system.

We remark that using techniques for post-hoc delegation of quantum computation [FHM18], our PoQ for QMA may be understood as a proof-of-work for quantum computations, since it could be used to convince a verifier that the Prover has indeed created the history state of some pre-defined computation. This is very relevant in the scenario of testing small-scale quantum computers in the most adversarial model possible: the zero-knowledge property ensures that the verifier learns nothing but the truth of the statement, while the PoQ property means that the Prover has indeed prepared a ground state with the given properties. Comparatively, all currently known protocols either make assumptions on the devices, or certify only the answer of the computation, but not the knowledge of the prover.

Finally, using the techniques of Result 3, we show that every problem in QMA has a non-interactive statistical zero-knowledge proof in the secret parameter model. We are even able to

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\(^5\) Besides being an excellent symbolic reminder of the interaction in a 3-message proof system, $\Xi$ is chosen as a shorthand for what we might otherwise call a $q\Sigma$ protocol, due to the resemblance with the pronunciation as “c$sigma$”.

\(^6\) This definition is joint work with Coladangelo, Vidick and Zhang.

\(^7\) See also independent and concurrent work by Coladangelo, Vidick and Zhang (unpublished).
strengthen our result to the complexity class QAM (recall that in a QAM proof system, the verifier first sends a random string to the Prover, who answers with a quantum proof). Note that QAM trivially contains QMA.

**Result 6.** All problems in QAM have a non-interactive statistical zero-knowledge protocol in the secret parameter model.

Note that, as in the classical case [PS05], our result also implies a QNIZK protocol where the Prover and the Verifier run an offline (classical) pre-processing phase (independent of the witness) and then the Prover sends the quantum ZK proof to the Verifier. We notice also that even though these models are less relevant to the cryptographic applications of NIZK, we think that our result moves us towards a QNIZK protocol for QMA in a more standard model.

### 1.3 Techniques

The starting point for our results are *locally simulatable codes*, as defined in [GSY19]. We give now a rough intuition on the properties of such codes and leave the details to Section 2.5.1.

First, a quantum error correcting code is *k*-simulatable if there exists an efficient classical algorithm that outputs the reduced density matrices of codewords on every subset of at most \( k \) qubits. Importantly, this algorithm is oblivious of the logical state that is encoded. We note that it was already known that the reduced density matrices of codewords hide the encoded information, since quantum error correcting codes can be used in secret sharing protocols [CGL99], and in [GSY19] they show that there exist codes such that the classical description of the reduced density matrices of the codewords can be efficiently computed. Next, [GSY19] extends the notion of simulatability of *logical operations* on encoded data as follows. Recalling the theory of fault-tolerant quantum computation, according to which some quantum error-correcting codes allow computations over encoded data by using “transversal” gates and encoded magic states (see [GSY19] for an overview). The definition of *k*-simulatability is extended to require that the simulator also efficiently computes the reduced density matrix on at most \( k \) qubits of the *history state* (Equation (1) depicts a history state) of the operations that implement a logical gate on the encoded data (again, by transversal gates and magic states).

In [GSY19], the authors show that the concatenated Steane code is a locally simulatable code. With this tool, in [GSY19], it is shown that every MIP\(^*\) protocol\(^8\) can be made zero-knowledge, thus quantizing the celebrated result of [BGKW88]. In our work, for the first time, we apply the techniques of *k*-simulatability from [GSY19] to QMA, which enables us to solve many open problems as previously described.

In order to explain our approach to achieving Result 1, we first recall the quantum Cook-Levin theorem proved by Kitaev [KSV02]. In his proof, Kitaev uses the circuit-to-Hamiltonian construction [Fey82], mapping an arbitrary QMA verification circuit \( V = U_T \ldots U_1 \) to a local Hamiltonian \( H_V \) that enforces that low energy states are *history states* of the computation, i.e., a superposition of the snapshots of \( V \) for every timestep \( 0 \leq t \leq T \):

\[
|\text{hist}\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T+1} |t\rangle \otimes U_t \ldots U_1 |\psi_{\text{init}}\rangle,
\]

In the above, the first register is called the clock register, and it encodes the timestep of the computation, while the second register contains the snapshot of the computation at time \( t \), i.e., the quantum

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\( ^8\)MIP\(^*\) is the set of languages that admit a classical multi-prover interactive proof, where, in addition, the prover share entanglement
gates $U_1, ..., U_t$ applied to the initial state $|\psi_{init}\rangle = |\phi\rangle|0\rangle^\otimes A$, that consists of the quantum witness provided by the Prover and auxiliary qubits. The Hamiltonian $H_V$ also guarantees that $|\psi_{init}\rangle$ has the correct form at $t = 0$, and that the final step accepts, i.e. the output qubit is close to $|1\rangle$.

In [GSY19], they note that an important obstacle to making $|\text{hist}\rangle$ locally simulatable is its dependence on the witness state $|\phi\rangle$. Their solution is to consider a different QMA verification algorithm $V'$ that implements $V$ on encoded data, much like in the theory of fault-tolerant quantum computing. In more details, for a fixed locally simulatable code, $V'$ expects the encoding of the original witness $\text{Enc}(|\phi\rangle)$ and then, with her raw auxiliary states, she creates encodings of auxiliary states $\text{Enc}(|0\rangle)$ and magic states $\text{Enc}(|\text{MS}\rangle)$, and then performs the computation $V$ through transversal gates and magic state gadgets, and finally decodes the output qubit. This gives rise to a new history state:

$$|\text{hist}'\rangle = \frac{1}{\sqrt{T'+1}} \sum_{t=0..T'+1} |t\rangle \otimes U'_t \ldots U'_1 |\psi_{init}'\rangle,$$

where $|\psi_{init}'\rangle = \text{Enc}(|\phi\rangle)|0\rangle^\otimes A'$ and $U'_1, ..., U'_{T'}$ are the gates of $V'$ described above. In [GSY19], they show that from the properties of the locally simulatable codes, the reduced density matrix on every set of 5 qubits of $|\text{hist}'\rangle$ can be efficiently computed. In this work, we prove that these reduced density matrices are in fact QMA-hard instances of CLDM. More concretely, we show that these reduced density matrices of a hypothetical history state of an accepting QMA-verification can always be computed, and there exists a global state (namely the history state) consistent with these reduced density matrices if and only if the original QMA verification accepts with overwhelming probability (and therefore we are in the case of a yes-instance).

Given Result 1, this opens up a number of possible applications to cryptographic settings. But we face a tradeoff: in CLDM, we have the description of the local density matrices, which would appear to yield a $\Xi$ protocol which would be zero-knowledge. However, the QMA verification for such problem is rather complicated: we need multiple copies of the global state to perform tomography on the reduced states,\(^9\) instead of a single copy that is needed in the Local Hamiltonian problem.

In order to combine these two desired properties in a single object, we describe a powerful technique that we call **locally simulatable proofs**. In a locally simulatable proof system for some problem $A = (A_{yes}, A_{no})$, we require that: (i) the verification test performed by the verifier acts on at most $k$ out of the $n$ qubits of the proof, and (ii) for every $x \in A_{yes}$, there exists a locally simulatable witness $|\psi\rangle$, i.e., a state $|\psi\rangle$ that passes all the local tests and such that for every $S \subseteq [n]$ with $|S| \leq k$, it is possible to compute the reduced state of the $|\psi\rangle$ on $S$ efficiently (without the help of the Prover). Notice that we have no extra restrictions on $x \in A_{no}$, since any quantum witness should make this verifier reject.

We then show that all problems in QMA admit a locally simulatable proof system. In order to achieve this, we use the local tests on the encoded version of the QMA verification algorithm that come from the Local Hamiltonian problem, together with the fact that the history state of such computation is a low-energy state and it is simulatable (this is also used to establish the QMA-hardness of CLDM).

In the next section, we describe our applications of locally simulatable proofs. But first, we remark that a direct classical version of locally simulatable proofs as we define them is impossible. This is because, given the local values of a classical proof, it is always possible to reconstruct the full proof by gluing these pieces together. The fact that this operation is hard to perform quantumly is intrinsically related to entanglement: given the local density matrices, it is not a priori possible

\(^9\)See Lemma 3.3.
to know which parts are entangled in order to glue them together. As discussed in the next section, this allows us to achieve a type of simple zero-knowledge protocol that defies all classical intuition.

1.3.1 Locally Simulatable Proofs in Action

We now sketch how each of Result 3–Result 5 is obtained via the lens of locally simulatable proofs.

**Zero Knowledge.** We use the characterization QMA = SimQMA to give a new zero knowledge proof system for QMA. Our protocol is much simpler than previous results [BJSW16], and it follows the “commit-challenge-response” structure of a \( \Sigma \)-protocol. Since our commitment is a quantum state (the challenge and response are classical), we call this type of protocol a “\( \Xi \)-protocol” (see Section 1.2).

The main idea is to use the quantum one-time pad to split the first message in the protocol into a quantum and a classical part. More concretely, the prover sends \( X^a Z^b |\psi\rangle \) and commitments to each bit of \( a \) and \( b \) to the Verifier, where \( |\psi\rangle \) is a locally simulatable quantum witness for some instance \( x \) and \( a \) and \( b \) are uniformly random strings. The Verifier picks some \( c \in [m] \), which corresponds to one of the tests of the simulatable proof system, and asks the Prover to open the commitment of the encryption keys to the corresponding qubits. The honest Prover opens the commitment corresponding to the one-time pad keys of the qubits involved in test \( c \). The Verifier then checks if: (i) the openings are correct and, (ii) the decrypted reduced state passes test \( c \).

Assuming the existence of unconditionally binding and computationally hiding commitment schemes, we show that our protocol is a computational zero-knowledge proof system for QMA. Completeness and soundness follow trivially, whereas the zero-knowledge property is established by constructing a simulator that exploits the properties of the locally simulatable proof system and the rewinding technique of Watrous [Wat09].

We also show that if the commitment scheme is computationally binding and unconditionally hiding, then our protocol is the first statistical zero-knowledge argument for QMA. This is because the protocol is secure only against polynomial-time malicious Provers, since unbounded provers would be able to open the commitment to different values. On the other hand, we achieve statistical zero-knowledge since the commitments that are never opened effectively encrypt the witness in an information-theoretic sense.

To the best of our knowledge, this is the first time that quantum techniques are used in zero-knowledge to achieve a commit-and-open protocol that requires no randomization of the witness. Indeed, for reasons already discussed, all classical zero-knowledge \( \Sigma \) protocol require a mapping or randomization of the witness (e.g. in the 3 – COL protocol, this is the permutation that is applied to the colouring before the commitment is made). We thus conclude that quantum information enables a new level of encryption that is not possible classically: the “juicy” information is present in the global state, whose local parts are fully known [GSY19].

**Proof of Quantum Knowledge for QMA.** As discussed in Section 1.2, our first challenge here is to define a Proof of Quantum Knowledge (PoQ). We recall that in the classical setting, we require an extractor that outputs some witness that passes the NP verification with probability 1, whenever the Verifier accepts with probability greater than some parameter \( \kappa \), known as the knowledge error.

In the quantum case, given: (i) that we are not able to clone quantum states and (ii) QMA is not known to be closed under perfect completeness, the best that we can hope for is to extract some quantum state that would pass the QMA verification with some probability to be related to the acceptance probability in the interactive protocol, whenever this latter value is above some threshold \( \kappa \).

To define a PoQ, we first fix the verification algorithm \( V_x \) for some instance of a problem in
QMA. We also assume $P^*$ to be a Prover that makes the Verifier accept with probability at least $\epsilon > \kappa$ in the $\Xi$ protocol.\(^{10}\) We assume that $P^*$ only performs unitary operations on a private and message registers. We then define a quantum polynomial-time algorithm $K$ that has oracle access to $P^*$, meaning that $K$ can execute the unitary operations of $P^*$, their inverse operations and has access to the message register of $P^*$.\(^{11}\) The protocol is said to be a Proof of Quantum Knowledge if $K$ outputs, with non-negligible probability, some quantum state $\rho$ that would make $V_x$ accept with probability at least $q(\epsilon, n)$, where $q$ is known as the quality function, or aborts otherwise.

The difficulty in showing that our $\Xi$ protocols are PoQs lies in the fact that any measurement performed by the extractor disturbs the state held by $P^*$, and therefore when we rewind $P^*$ by applying the inverse of his operation, we do not come back to the original state. We overcome this difficulty in the following way. We set $\kappa$ to be some value very close to 1, namely $\kappa = 1 - \frac{1}{p(n)}$ for some large enough polynomial $p$. Our extractor starts by simulating $P^*$ on the first message of the $\Xi$ protocol, and then holds the (supposed) one-time-padded state and the commitments to the one-time-pad keys. $K$ follows by iterating over all possible challenges of the $\Xi$ protocol, runs $P^*$ on this challenge, perform the Verifier’s check and then rewinds $P^*$. By the assumption that $P^*$ has a very high acceptance probability, the measurements performed by $K$ do not disturb the state too much, and in this case, $K$ can retrieve the correct one-time pads for every qubit of the witness. If $K$ is successful, i.e. is able to open every committed bit, $K$ can decode the original one-time-padded state and it is a good witness for $V_x$ with high probability.

We then analyse the sequential repetition of the protocol, that allows us to have a PoQ with \textit{exponentially small} knowledge error $\kappa$, and extracts one good witness from $P^*$ (out of the polynomially many copies that $P^*$ should have in order to cause the verifier to accepted in the multiple runs of the protocol).

**Non-Interactive zero knowledge proof for QMA with secret parameters.** Finally, in Section 6, we achieve our non-interactive statistical zero-knowledge protocol for QMA in the secret parameter setting using techniques similar to our $\Xi$ protocol: the trusted party chooses the one-time pad key and a random (and small) subset of these values that are reported to the Verifier. Since the Prover does not know which are the values that were given to the Verifier, he should act as in the $\Xi$-protocol, but now the Verifier does not actually need to ask for the openings, since the trusted dealer has already sent them. Although this is a less natural model, we hope that this result will shed some light in developing QNIZK proofs for QMA in more commonly-used models.

### 1.4 Open problems

**Further QMA-complete languages.** We note that a number of problems are currently known to be QMA-complete under Turing reductions, including the $N$-representability [LCV07] and bosonic $N$-representability problems [WMN10] as well as the universal functional of density function theory (DFT) [SV09]. It is an open question if these problems can be shown to be QMA-complete under Karp reductions using the techniques presented in our work.

**Complexity of $k$ CLDM for $k < 5$.** We prove in this work that 5-CLDM is QMA-hard under Karp reductions. We leave as an open problem proving if the problem is still QMA-complete for $k < 5$.

**Marginal reconstruction problem.** We remark that the classical version of CLDM is defined as follows: given the description of $m$ marginal distributions on sets of bits $C_1, \ldots, C_m$, such that $|C_i| \leq k$, decide if there is a probability distribution that is close to those marginals, or such a distribution does not exist. This problem was proven NP-complete by Pitowsky [Pit91], and its

\(^{10}\)Note that we reserve the word “Verifier” here for the $\Xi$ protocol and refer to $V_x$ as the QMA verification algorithm.

\(^{11}\)This model is already considered by [Unr12] in his work of quantum proofs of knowledge for NP.
containment in NP is proved by using the fact that such distribution can be seen as a point \( p \) in the correlation polytope in a polynomial-size Hilbert space. In this case, by Caratheodory’s theorem, \( p \) is a convex combination of polynomially many vertices of such polytope, and therefore these vertices serve as the NP-proof and a linear program verifies if there is a convex combination of them that is consistent with the marginals of the problem’s instance.

The difference here is that the proof and the marginals are different (but connected) objects. We leave as an open problem if we can extract a notion of a locally simulatable classical proof from this (or any other) problem, and its applications to cryptography and complexity theory. In particular, we wonder if there is a natural zero-knowledge protocol for this problem.

**Applications of quantum ZK protocols.** In classical cryptography, ZK and PoK protocols are a fundamental primitive since they are crucial ingredients in a plethora of applications. We discussed in Section 1.2 that our quantum ZK PoQ for QMA could be used as a proof-of-work for quantum computations. An interesting open problem is finding other settings in which the benefits of our simple ZK protocols for QMA can be applied. We list now some possibilities that could be explored in future work: authentication with uncloneable credentials [CDS94]; proof of quantum ownership [BJM19]; or ZK PoQ verification for quantum money [AC12].

**Practical ZK protocols for QMA.** Even if we reach a conceptually much simpler ZK protocol for QMA, the resources needed for it are still very far from practical. We leave as an open problem if one could devise other protocols that are more feasible from a physical implementation viewpoint, which could include classical communication protocols based on the protocols proposed by Vidick and Zhang [VZ19], or device-independent ones based on the ideas of Grilo [Gri19].

**Non-interactive Zero-knowledge protocols for QMA in the CRS model.** In this work, we propose a QNIZK protocol where the information provided by the trusted dealer is asymmetric. We leave as an open problem if one could devise a protocol where the dealer distributes a common reference string (CRS)(or shared EPR pairs) to the Prover and the Verifier.

A possible way of achieving such non-interactive protocol would be to explore the properties of \( \Xi \)-protocols, as done classically with \( \Sigma \)-protocols. For instance, the well-known Fiat-Shamir transformation [FS87] allows us to make \( \Sigma \)-protocols non-interactive (in the Random Oracle model). We wonder if there is a version of this theorem when the first message can be quantum.

**Witness indistinguishable/hiding protocols for QMA.** Classically, there are two weaker notions that can substitute for ZK in different applications. In Witness Indistinguishable (WI) proofs, we require that the Verifier cannot distinguish if she is interacting with a Prover holding a witness \( w_1 \) or \( w_2 \), for any \( w_1 \neq w_2 \). In Witness Hiding (WH), we require that the Verifier is not able to cook-up a witness for the input herself. Notice that zero-knowledge implies both such definitions, and we leave as an open problem finding WI/WH protocols for QMA with more desirable properties than the known ZK protocols.

**Computational Zero-Knowledge proofs vs. Statistical Zero-Knowledge arguments.** In this work, we show that QMA admits quantum ZK proofs and statistical ZK arguments. We notice that classically, it is known that the class of problems with computational ZK proofs is closely related to the class of problems with statistical ZK arguments [OV07]. We wonder if this relation is also true in the quantum setting.

### 1.5 Structure

The remainder of this document is structured as follows: Section 2 presents Preliminaries and Notation. In Section 3, we prove our results on the CLDM problem and we present our framework
of simulatable proofs. Section 4 establishes the zero-knowledge \( \Xi \) protocol for QMA, while in Section 5, we define a proof of quantum knowledge and show that the interactive proof system satisfies the definition. Finally, in Section 6, we show a non-interactive zero-knowledge proof for QMA in the secret parameters model.

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2 Preliminaries

2.1 Notation

For \( n \in \mathbb{N} \), we define \([n] := \{0, \ldots, n-1\} \). For some finite set \( S \), we denote \( s \in_S S \) as an element \( s \) picked uniformly at random from \( S \). We say that a function \( f \) is negligible (\( f(n) = \text{negl}(n) \)), if for every constant \( c \), we have \( f(n) = o\left(\frac{1}{n^c}\right) \). Given two discrete probability distributions \( P \) and \( Q \) over the domain \( \mathcal{X} \), we define its statistical distance as \( d(P,Q) = \sum_{x \in \mathcal{X}} |P(x) - Q(x)| \).

2.2 Quantum computation

We assume familiarity with quantum computation, and refer to [NC00] for the definition of basic concepts such as qubits, quantum states (pure and mixed), unitary operators, quantum circuits and quantum channels.

For an \( n \)-qubit state \( \rho \) and an \( m \)-qubit state \( \sigma \), we define \( \rho^S \otimes \sigma^S \) as the \( m+n \)-qubit quantum state \( \tau \) that consists of the tensor product of \( \rho \) and \( \sigma \) where the qubits of \( \rho \) are in the positions indicated by \( S \subseteq [m+n] \), \( |S| = n \), and the qubits of \( \sigma \) are in the positions indicated by \( \overline{S} \), with the ordering of the qubits consistent with the ordering in \( \rho \) and \( \sigma \), as well as the integer ordering in \( S \) and \( \overline{S} \). We extend this notation and write \( A^S \otimes B^\overline{S} \) for operators \( A \) and \( B \) acting on \( |S| \) and \( |\overline{S}| \) qubits, respectively.

We define quantum gates with sans-serif font (\( X, Z, \ldots \)), and we define \( I, X, Y \) and \( Z \) to be the Pauli matrices, \( P_k = \{I, X, Y, Z\}^\otimes k \), \( H \) to be the Hadamard gate, \( \text{CNOT} \) to be the controlled-Not gate, \( P = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \) and \( T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \). A Clifford circuit is a quantum circuit composed of Clifford gates: \( I, X, Y, H, \text{CNOT} \) and \( P \). It is well-known that universal quantum computation can be achieved with Clifford and \( T \) gates.

For an operator \( A \), the trace norm is \( \|A\|_{tr} := \text{Tr} \left( \sqrt{A^\dagger A} \right) \), which is the sum of the singular values of \( A \). For two quantum states \( \rho \) and \( \sigma \), the trace distance between them is

\[
D(\rho, \sigma) = \frac{1}{2} \|\rho - \sigma\|_{tr} = \max_P \text{Tr} (P(\rho - \sigma)),
\]

where the maximization is taken over all possible projectors \( P \).

If \( D(\rho, \sigma) \leq \varepsilon \), we say that \( \rho \) and \( \sigma \) are \( \varepsilon \)-close. If \( \varepsilon = \text{negl}(n) \), then we say that \( \rho \) and \( \sigma \) are statistically indistinguishable, and we write \( \rho \approx_S \sigma \).
If for every polynomial time algorithm $A$, we have that
\[ |Pr[A(\rho) = 1] - Pr[A(\sigma) = 1]| \leq \negl(n), \]
then we say that $\rho$ and $\sigma$ are computationally indistinguishable, and we write $\rho \approx_c \sigma$.

For some $S \subseteq \{0, 1\}^*$, let $\{\Psi_x\}_{x \in S}$ and $\{\Phi_x\}_{x \in S}$ be two families of quantum channels from $q(|x|)$ qubits to $r(|x|)$ qubits, for some polynomials $q$ and $r$. We say that these two families are computationally indistinguishable, and denote it by $\Psi_x \approx_c \Phi_x$, if for every $x \in S$ and polynomial $s$ and $k$ and every state $\sigma$ on $q(|x|) + k(|x|)$ and every polynomial-size circuit acting on $r(|x|) + k(|x|)$ qubits, it follows that
\[ |Pr[Q((\Psi_x \otimes 1I)(\sigma)) = 1] - Pr[Q((\Phi_x \otimes 1I)(\sigma)) = 1]| \leq \negl(n). \]

Finally, we state a result on rewinding by [Wat09].

**Lemma 2.1** (Simplified version of Lemma 9 of [Wat09]). Let $Q$ be an $(n, k)$-quantum circuit that acts on an $n$-qubit state $|\psi\rangle$ and $m$ auxiliary systems $|0\rangle$. Let
\[ p(\psi) = \|(0| \otimes I)Q(|\psi\rangle \otimes |0\rangle^\otimes m)\|^2 \text{ and } |\phi(\psi)\rangle = \frac{1}{\sqrt{p(\psi)}}((0| \otimes I)Q(|\psi\rangle \otimes |0\rangle^\otimes m). \]

If there exists value $q \in (0, 1)$ and $\varepsilon \in (0, \frac{1}{2})$ such that for every $|\psi\rangle$, $p(\psi) - q \leq \varepsilon$, then for every $\varepsilon > 0$ there is a quantum circuit $R$ of size at most
\[ O\left(\frac{\log(1/\varepsilon) \cdot \text{size}(Q)}{q(1 - q)}\right), \]
such that on input $|\psi\rangle$, $R$ computes the quantum state $\rho(\psi)$ that satisfies
\[ \langle \phi(\psi)|\rho(\psi)|\phi(\psi)\rangle \geq 1 - 16\varepsilon \frac{\log^2 \frac{1}{\varepsilon}}{q^2(1 - q)^2}. \]

### 2.3 Complexity classes

In this section, we define several complexity classes that are considered in this work.

**Definition 2.2** (QMA). A promise problem $A = (A_{yes}, A_{no})$ is in QMA if there exist polynomials $p$, $q$ and a polynomial-time uniform family of quantum circuits $\{Q_n\}$, where $Q_n$ takes as input a string $x \in \Sigma^*$ with $|x| = n$, a $p(n)$-qubit quantum state $|\psi\rangle$, and $q(n)$ auxiliary qubits in state $|0\rangle^\otimes q(n)$, such that:

**Completeness:** If $x \in A_{yes}$, there exists some $|\psi\rangle$ such that $Q_n$ accepts $(x, |\psi\rangle)$ with probability at least $1 - \negl(n)$.

**Soundness:** If $x \in A_{no}$, for any state $|\psi\rangle$, $Q_n$ accepts $(x, |\psi\rangle)$ with probability at most $\negl(n)$.

We define now a quantum interactive protocol between two parties.

**Definition 2.3** (Quantum interactive protocol between $A$ and $B$ ($A \leftrightarrows B$)). Let $A$ and $B$ be the private registers of parties $A$ and $B$, respectively, and $M$ be the message register. A quantum interactive protocol between $A$ and $B$ is a sequence of unitaries $U_0, ..., U_m$ where $U_i$ acts on registers $A$ and $M$ for even $i$, and on registers $B$ and $M$ for odd $i$. The size of the register $A, B$ and $M$, the number $m$ of messages and the complexity of the allowed $U_i$ is defined by each instance of the protocol. We can also consider interactive protocols where $A$ outputs some value after interacting with $B$, and we also denote such output as ($A \Rightarrow B$).
**Definition 2.4.** A promise problem $A = (A_{yes}, A_{no})$ is in QZK if there is an interactive protocol $(V \leftrightarrow P)$, where $V$ is polynomial-time and is given some input $x \in A$ and outputs a classical bit indicating acceptance or rejection of $x$, $P$ is unbounded, and the following holds

**Completeness:** If $x \in A_{yes}$, $\Pr[(V \Leftarrow P) = 1] \geq 1 - \text{negl}(n)$.

**Soundness:** If $x \in A_{no}$ for all $P^*$, we have that $\Pr[(V \Leftarrow P^*) = 1] \leq \frac{1}{\text{poly}(n)}$.

**Computational zero-knowledge:** For any $x \in A_{yes}$ and any polynomial-time $V'$ that receives the inputs $x$ and some state $\zeta$, there exists a polynomial-time quantum channel $S_{V'}$ that also receives $x$ and $\zeta$ as input such that $(V' \Leftarrow P) \approx_c S_{V'}$.

Following the result by Pass and Shelat [PS05], we define now the notion of non-interactive zero-knowledge proofs in the secret parameters model.

**Definition 2.5** (Non-interactive statistical zero-knowledge proofs in the secret parameter model). A triple of algorithms $(D, P, V)$ is a non-interactive statistical zero-knowledge proof in the secret parameter model for a promise problem $A = (A_{yes}, A_{no})$ where $D$ is a probabilistic polynomial time algorithm, $V$ is a quantum polynomial time algorithm and $P$ is an unbounded quantum algorithm such that there exists a negligible function $\varepsilon$ such that the following conditions follow:

**Completeness:** for every $x \in A_{yes}$, there exists some $P$

$$\Pr[(r_p, r_V) \leftarrow D(1^{|x|}); \pi \leftarrow P(x, r_p); V(x, r_V, \pi) = 1] \geq 1 - \varepsilon(n).$$

**Soundness:** for every $x \in A_{no}$ and every $P$

$$\Pr[(r_p, r_V) \leftarrow D(1^{|x|}); \pi \leftarrow P(x, r_p); V(x, r_V, \pi) = 1] \leq \varepsilon(n).$$

**Statistical zero-knowledge:** there is a probabilistic polynomial time algorithm $S$ such that for every $x \in A_{yes}$, the statistical distance of the distribution of the output of $S(x)$ and the distribution of $(r_V, \pi)$ for $(r_p, r_V) \leftarrow D(1^{|x|})$ and $\pi \leftarrow P(x, r_p)$ is $\text{negl}(n)$.

### 2.4 Circuit-to-Hamiltonian construction

In Kitaev’s proof of QMA-hardness of the Local Hamiltonian problem, he uses the circuit-to-Hamiltonian proposed by Feynman [Fey82] in order to reduce arbitrary QMA verification procedures to time-independent Hamiltonians in a way that the Local Hamiltonian has low-energy states if and only if the QMA verification accepts with high probability.

More concretely, Kitaev shows a reduction from a quantum circuit $V$ consisting of $T$ gates $U_1, \ldots, U_T$ acting on a $p$-qubit state $|\psi\rangle$ provided by the Prover and an auxiliary register $|0\rangle^A$ to some Hamiltonian $H = \sum_{i \in [m]} H_i$, where the the terms $H_1, \ldots, H_m$ act on $T + p + A$ qubits, and they range between the following types:

**clock consistency** $H^\text{clock} = |01\rangle\langle 01|_{t,t+1}$, for $0 \leq t \leq T - 1$

**Initialization** For $j \in [A]$, $H^\text{init} = |0\rangle\langle 0|_0 \otimes |1\rangle\langle 1|_{T + p + j}$,
and in this case we call it an \( \text{Lemma 2.6} \) set of transversal gates \( T \) transversally and the operation \( H \) output \( n \gg k \).

Recently, Grilo, Slofstra and Yuen \cite{GSY19}, defined the notion of \textit{simulatable codes} which require that the \( s \)-qubit reduced density matrices of any codeword can be efficiently computed independently of the encoded state. In their definition, they also require that the reduced density state between the intermediate steps of computation on encoded data (either through transversal gates or computation by teleportation with magic states) can also be computed efficiently. We call these codes \textit{locally} simulatable codes in order to emphasize that only small parts of codewords can be simulated.

We present now the formal definition of Locally Simulatable codes.

\begin{align*}
H_0^\text{prop} &= \frac{1}{2} (|00\rangle_0 + |10\rangle_0|0\rangle_{0,1} - |10\rangle_0 \otimes (U_0)_{J_0} - |01\rangle_1 \otimes (U_1^T)_{J_0}) \\
H_T^\text{prop} &= \frac{1}{2} (|10\rangle_{T-1,T} + |11\rangle_T - |11\rangle_0 \otimes (U_0)_{J_T} - |0\rangle_1 \otimes (U_1^T)_{J_T})
\end{align*}

For \( 1 \leq t \leq T-1 \),
\[
H_t^\text{prop} = \frac{1}{2} (|100\rangle_{t-1,t,t+1} + |110\rangle_{c(t-1,t,t+1)} - |110\rangle_{t-1,t,t+1} \otimes (U_t)_{J_t} - |100\rangle_{t-1,t,t+1} \otimes (U_t^T)_{J_t})
\]

\textbf{output} \( H^\text{out} = |1\rangle_1 \otimes |0\rangle_{T+1} \)

The facts that we need from this reduction is summarized in the following lemma.

\textbf{Lemma 2.6} \cite{KSV02}. If there exists some state \( |\psi\rangle \) that makes \( V \) accept with probability \( 1 - \text{negl}(n) \), then the history state
\[
\frac{1}{\sqrt{T+1}} \sum_{t \in [T+1]} \text{unary}(t) \otimes U_t \ldots U_1 (|\psi\rangle \otimes A),
\]
has energy \( \text{negl}(n) \) according to \( H_V \). If every quantum state \( |\psi\rangle \) makes \( V \) reject with probability at least \( \varepsilon \), then the groundenergy of \( H_V \) is at least \( \Omega \left( \frac{\varepsilon}{T^2} \right) \).

\section{Quantum error correcting codes and locally simulatable codes}

For \( n > k \), an \( [[n,k]] \) quantum error correcting code (QECCs) is a mapping from a \( k \)-qubit state \( |\psi\rangle \) into an \( n \)-qubit state \( Enc(|\psi\rangle) \). The distance of an \( [[n,k]] \) QECC is \( d \) if for an arbitrary quantum operation \( E \) acting on \( (d-1)/2 \) qubits, the original state \( |\psi\rangle \) can be recovered from \( E(Enc(|\psi\rangle)) \), and in this case we call it an \( [[n,k,d]] \) QECC.

Given a fixed QECC, we say that some operation \( U \) can be applied \text{transversally}, if we apply \( U^\otimes n Enc(|\psi\rangle) = Enc(U|\psi\rangle) \) for every \( k \)-qubit state \( |\psi\rangle \). It is known that no code admits a universal set of transversal gates \cite{EK09}. In order to overcome this difficulty to achieve fault-tolerant computation, one can use tools from computation by teleportation. In this case, provided a resource called \textit{magic states}, one can simulate the non-transversal gate by applying some procedure on the target qubit and the magic state and such procedure contains only (classically controlled) transversal gates.

In this work, we consider the \( k \)-fold concatenation of the Steane code. We will just state the properties we need from it in this work and we refer to \cite{GSY19} for further details. The \( k \)-fold concatenation of the Steane code is \( [[7^k,1,3^k]] \) QECC such that all Clifford operations can be performed transversally and the \( T \) gates can be applied using the magic state \( T\mathbf{1} \).

\subsection{Locally simulatable codes}

Recently, Grilo, Slofstra and Yuen \cite{GSY19}, defined the notion of \textit{simulatable codes} which require that the \( s \)-qubit reduced density matrices of any codeword can be efficiently computed independently of the encoded state. In their definition, they also require that the reduced density state between the intermediate steps of computation on encoded data (either through transversal gates or computation by teleportation with magic states) can also be computed efficiently. We call these codes \textit{locally} simulatable codes in order to emphasize that only small parts of codewords can be simulated.

We present now the formal definition of Locally Simulatable codes.
**Definition 2.7.** Let $C$ be a QECC that allows universal quantum computation by applying transversal gates and using magic states. Let $\rho$ be a $n$ qubit state, $V$ be a logical gate acting on at most $k$ qubits out of an $n$ qubit system, $U_1,...,U_\ell$ be the transversal circuit that computes $V$, possibly with the help of a magic state $\sigma_V$. We say that $C$ is $s$-simulatable if for all integers $0 \leq t \leq \ell$ and subsets $S$ of the physical qubits of $Enc(\rho) \otimes \sigma_V$ with $|S| \leq s$, the partial trace

$$\text{Tr}_{\bar{S}}( (U_t \cdots U_1)Enc(\rho) \otimes \sigma_V(U_t \cdots U_1)^\dagger)$$

can be computed in $\text{poly}(2^k, n)$-time from $t$, $V$ and $S$. In particular, the partial trace is independent of the state $\rho$.

In [GSY19], it was shown that the concatenated Steane code is locally simulatable for some $s$.

**Lemma 2.8 ([GSY19]).** For every $k > \log(s+3)$, the $k$-fold concatenated Steane code is $s$-simulatable.

### 2.6 Commitment schemes

A commitment scheme is a two-phase protocol between two parties, the Sender and Receiver. In the first phase, the Sender, who holds some message $m$, unknown by the Receiver, sends a commitment $c$ to the Receiver. In the second phase, the Receiver will reveal the committed value $m$. We require two properties of the protocol: hiding, meaning that the Receiver cannot guess $m$ from $c$, and binding, which stays that the Receiver cannot decide to open value $m' \neq m$ when $c$ was committed for $m$.

It is well-known that there is no commitment schemes with unconditionally hiding and unconditionally binding properties, but such schemes can be achieved if either of the properties holds only computationally.\(^\text{12}\)

In this work, we consider both commitment schemes which are unconditionally binding but computationally hiding or computationally binding but unconditionally hiding. There are recent quantum-secure instantiations of these schemes, assuming the hardness of Learning Parity with Noise [JKPT12, XXW13, BKLH15].

We present now the formal definition of such schemes.

**Definition 2.9** (Computationally hiding and unconditionally binding commitment schemes). Let $\eta$ be some security parameter, $p$ be some polynomial, and $\mathcal{M}, \mathcal{C}, \mathcal{D} \subseteq \{0,1\}^{p(\eta)}$ be the message space, commitment space opening space, respectively. A computationally hiding and unconditional binding commitment scheme consist of a pair of algorithms $(\text{commit}, \text{verify})$, where

- `commit` takes as input a value from $\mathcal{M}$ and some value in $\mathcal{C} \times \mathcal{D}$,
- `verify` takes as input a value from $\mathcal{M} \times \mathcal{C} \times \mathcal{D}$ and outputs some value in $\{0,1\}$

with the following properties

**Correctness:** If $(c, d) \leftarrow \text{commit}(m)$, then $\text{verify}(m, c, d) = 1$

**Computationally hiding:** For any polynomial-time quantum adversary $A$ and $(c, d) \leftarrow \text{commit}(m)$

$$\Pr[A(c) = m] \leq \frac{1}{|\mathcal{M}|} + \text{negl}(\eta)$$

\(^{12}\text{Where unconditionally means that the property is guaranteed even against unbounded adversaries, in contrast to computational case, when the property is only guaranteed against quantum polynomial-time adversaries.}\)
Unconditionally binding: For any \((m, d)\) and \((m', d')\) such that \(m \neq m'\), it follows that

\[
\text{verify}(m, c, d) = 1 \implies \text{verify}(m', c, d') = 1 \leq \text{negl}(n)
\]

Definition 2.10 (Computationally hiding and unconditionally binding commitment schemes). Let \(\eta, p, M, C\) and \(D\) be defined as Definition 2.9. A unconditionally hiding and computationally binding commitment scheme consist of a pair of algorithms \((\text{commit}, \text{verify})\), where

Correctness: If \((c, d) \leftarrow \text{commit}(m)\), then \(\text{verify}(m, c, d) = 1\)

Unconditionally hiding: For any quantum adversary \(A\) and \((c, d) \leftarrow \text{commit}(m)\)

\[
\Pr[A(c) = m] \leq \frac{1}{|M|} + \text{negl}(\eta)
\]

Computationally binding: For every quantum polynomial time adversary \(A\)

\[
\Pr[(c, m, d, m', d') \leftarrow A(\cdot) \text{ and } m \neq m' \text{ and } \text{verify}(m, c, d) = 1 \text{ and } \text{verify}(m', c, d') = 1] \leq \text{negl}(n).
\]

3 Consistency of local density matrices and locally simulatable proofs

In this section, we prove the QMA-hardness of the CLDM problem (Section 3.1) and present our framework of Locally Simulatable proofs (Section 3.2).

3.1 Consistency of local density matrices is QMA-hard

Let us start by formally defining the CLDM problem.

Definition 3.1 (Consistency of local density matrices problem (CLDM) [Liu06]). Let \(n \in \mathbb{N}\). The input to the consistency of local density matrices problem consists of \(((C_1, \rho_1), \ldots, (C_m, \rho_m))\) where \(C_i \subseteq [n]\) and \(|C_i| \leq k\); and \(\rho_i\) is a density matrix on \(|C_i|\) qubits and each matrix entry of \(\rho_i\) has precision \(\text{poly}(n)\). Given two parameters \(\alpha\) and \(\beta\), assuming that one of the following conditions is true, we have to decide which of them holds.

Yes. There exists some \(n\)-qubit quantum state \(\tau\) such that for every \(i \in [m]\), \(\|\text{Tr}_{C_i}(\tau) - \rho_i\|_\text{tr} \leq \alpha\).

No. For every \(n\)-qubit quantum state \(\tau\), there exists some \(i \in [m]\) such that \(\|\text{Tr}_{C_i}(\tau) - \rho_i\|_\text{tr} \geq \beta\).

Remark 3.2. Notice that in [Liu06], the definition of the problem sets \(\alpha = 0\). In our case, we define the problem more generally, otherwise we would only achieve QMA\(_1\) hardness (the version of QMA with perfect completeness) rather than QMA-hardness.

In the proof of containment of CLDM in QMA, Liu uses a characterization of QMA called QMA\(_+\) [AR03]. For completeness, we start by showing the containment of CLDM in QMA, by presenting a standard verifier for the problem, which is a straightforward composition of the results from [AR03] and [Liu06].

Lemma 3.3. The consistency of local density matrices problem is in QMA for any \(k = O(\log n)\), and \(\alpha, \beta\) such that \(\varepsilon := \frac{\beta^4}{4} - \alpha \geq \frac{1}{\text{poly}(n)}\).
Proof. Let \(((C_1, \rho_1), ..., (C_m, \rho_m))\) be an instance for CLDM. Let \(p\) be a polynomial such that \(p(n)\varepsilon^2 = \Omega(n)\), the verification system expects some state \(\psi\) consisting of \(p(n)\) copies of the state \(\tau\) that is supposed to be consistent with the local density matrices. The verifier then picks \(i \in [m]\) and \(P \in \mathcal{P}_{|C_i|}\) uniformly at random. The verifier then measures each (supposed) one of the \(p(n)\) copies according to the observable \(P^C_i \otimes I^C_i\), and let \(\overline{p}\) be the average of its outcomes. The verifier accepts if and only if \(|\overline{p} - \text{Tr}(P\rho_i)| \leq \alpha + \frac{\varepsilon}{2}\).

In the completeness case, we have that \(\psi = \tau^\otimes \ell\) and in this case, each of the \(p(n)\) measurements is 1 with probability \(\text{Tr}\left((P^C_i \otimes I^C_i)\tau\right)\) for every \(i\). By Hoeffding’s inequality, we have that with probability at least \(1 - 2\exp(p(n)\varepsilon^2/8) = 1 - \text{negl}(n)\),
\[
|\text{Tr}(P^C_i \otimes I^C_i)\tau) - \overline{p}| \leq \frac{\varepsilon}{2}.
\]
Using the fact that \(\tau\) is consistent with \(\rho_i\), we also have that
\[
|\text{Tr}(P^C_i \otimes I^C_i)\tau) - \text{Tr}(P\rho_i)| \leq \alpha,
\]
and therefore by the triangle inequality, the verifier accepts with probability at least \(1 - \text{negl}(n)\).

For soundness, let \(\psi_j\) be the reduced density state considering the register of the \(j\)th copy of the state. Let \(\phi = \frac{1}{p(n)}\sum_j \psi_j\). Since we have a no-instance, there exists some \(i \in [m]\) such that \(\|\text{Tr}_{C_i}(\phi) - \rho_i\|_{tr} \geq \beta\). Let us write
\[
\text{Tr}_{C_i}(\phi) - \rho_i = \frac{1}{4|C_i|} \sum_{P \in \mathcal{P}_{|C_i|}} \gamma_P P.
\]
There must be some choice of \(P \in \mathcal{P}_{|C_i|}\) such that \(|\gamma_P| \geq \frac{\beta}{4|C_i|}\). Notice that by the definition of \(\phi\),
\[
\text{Tr}(P^C_i \otimes I^C_i)\phi) = \frac{1}{p(n)} \sum_j \text{Tr}(P\psi_j).
\]
In this case, if we measure each register corresponding to the \(j\)th copy of the state, then the expected value of its average is \(\text{Tr}(P^C_i \otimes I^C_i)\phi\). Let \(\overline{p}\) be the average of the outcomes of the performed measurements. Again using Hoeffding’s inequality, we have that with probability at least \(1 - 2\exp(p(n)\varepsilon^2/8) = 1 - \text{negl}(n)\),
\[
|\text{Tr}(P^C_i \otimes I^C_i)\phi) - \overline{p}| \leq \frac{\varepsilon}{2},
\]
and therefore, we have that for this fixed \(i\) and \(P\), the prover accepts with probability \(\text{negl}(n)\). Since such \(i\) is picked with probability \(\frac{1}{m}\) and such a \(P\) is picked with probability at least \(\frac{1}{p(n)}\), the overall acceptance is at most \(1 - O\left(\frac{1}{m4\pi}\right)\) (where we account for the negligible factors inside the \(O\)-notation).

We show now that CLDM is QMA-hard under standard Karp reductions.

Lemma 3.4. The consistency of local density matrices problem is QMA-hard under Karp reductions.

Proof. Let \(A = (A_{\text{yes}}, A_{\text{no}})\) be a promise problem in QMA. Then there exists a verification circuit \(V_x\) that acts on \(q(n)\) auxiliary qubits and a \(p(n)\)-qubit quantum state \(|\psi\rangle\), such that if \(x \in A_{\text{yes}}\), there exists some state \(|\psi\rangle\) such that \(V_x\) accepts with probability exponentially close to 1, whereas
if \( x \in A_{no} \), all quantum states \( |\psi\rangle \) make \( V_x \) accept with probability exponentially small. We assume, without loss of generality, that \( V_x \) is composed of gates in the set \( \{X, Z, H, \text{CNOT}, T\} \).

Let \( C \) be a \([N, 1, D]\) quantum error correcting code that is 3L-simulatable. As in [GSY19], we define the verification algorithm \( V'_x(C) \) acting on \( N(q(n) + p(n) + t) \) qubits as follows:

1. For each auxiliary qubit, encode \( |0\rangle \) under \( C \)
2. For every T-gate of \( V_x \), create \( |T\rangle \) and encode it under \( C \)
3. Check if the witness is encoded under \( C \), and reject if this is not the case
4. Simulate each gate of \( V \), either transversally, or using gadgets with the magic state
5. Decode the output bit, and accept or reject depending on its value.

Whenever \( C \) is clear from the context, we write \( V'_x(C) = V'_x \). If \( V_x \) accepts with probability at most \( \delta \), we have also that \( V'_x \) accepts with probability at most \( \delta \): the acceptance probability of \( V'_x \) on \( Enc(|\psi\rangle) \) is the same as the acceptance probability of \( V_x \) on \( |\psi\rangle \); also, if the Prover sends some witness \( |\phi\rangle \) that is orthogonal to the codespace of \( C \), then \( V'_x \) rejects with probability 1.

We assume that \( V'_x \) also contains dummy identity gates between each one of the steps discussed before, and also in between the encoding of different qubit or the logical computation of different gates. Again, this does not change the acceptance probability of \( V'_x \), but it will be necessary in the reduction to CLDM. Let \( T \) be the number of gates in \( V'_x \) (considering also the dummy identity gates) and \( U_1, ..., U_T \in \{\text{CNOT}, T, H, X, Z\} \) be the gates in \( V'_x \).

As discussed in Section 2.4, from \( V'_x \), we can define the Local Hamiltonian instance \( H_{V'_x} \) acting on \( W \) qubits and if there is a witness that makes \( V'_x \) accept with probability \( 1 - \text{negl}(n) \), then the history state of \( V'_x \) when given \( Enc(|\psi\rangle) \) as witness

\[
|\text{hist}\rangle = \frac{1}{\sqrt{T+1}} \sum_{t \in [T+1]} |\text{unary}(t)\rangle \otimes U_t \ldots U_1(Enc(|\psi\rangle)|0\rangle^{\otimes A}),
\]

has exponentially small energy. In this case, given the properties of our locally simulatable code \( C \), [GSY19] showed there is an efficient deterministic algorithm that outputs classical description of the reduced density matrix of \( |\text{hist}\rangle \) on at most \( L \) qubits. Let \( \rho_i \) be the output of such simulator on the set the qubits on which the local term \( H_i \) of \( H_{V'_x} \). By inspection, we have that \( \text{Tr}(H_i \rho_i) = 0 \), even for no-instances. We record this in the following lemma, and, for completeness, we prove it Appendix A.

**Lemma 3.5** ([GSY19]). Let \( C \) be a \([N, 1, D]\) quantum error correcting code that is a 3L-simulatable code, for some \( L \geq 5 \). Then there exists a polynomial-time deterministic algorithm \( \text{Sim}_{V'_x(C)} \) such that for any \( S \subseteq [W] \) with \( |S| \leq L \), \( \text{Sim}_{V'_x(C)}(S) \) outputs the classical description of an \(|S|\)-qubit density matrix \( \rho_S \) such that

1. if \( x \) is a yes-instance, then \( \|\rho_S - \text{Tr}_{\overline{S}}(|\text{hist}\rangle\langle\text{hist}|)\|_{tr} \leq \text{negl}(n) \)
2. Let \( S_i \) be the set of qubits on which \( H_i \) acts non-trivially. Then \( \text{Tr}(H_i \rho_{S_i}) = 0 \).

We fix \( C \) to be the 3-fold concatenation of the Steane code and we define the following instance of CLDM:

\[
\{(S, \text{Sim}_{V'_x(C)}(S))\}_{S \subseteq [W], |S| = 5}
\]

---

\(^{13}\)We remark that the T gates are only necessary to create the magic states \( |T\rangle \) from a \( |0\rangle \) auxiliary system.
By [GSY19], $C$ is 24-simulatable, and therefore we can use $\text{Sim}_{V_i(C)}$ of Lemma 3.5 to compute these density matrices.

We show now that $x \in A_{\text{yes}}$ if and only if there exists some state $\tau$ that is consistent with each of these reduced density matrices.

For some $x \in A_{\text{yes}}$, let $|\psi\rangle$ be a quantum state that makes $V_x$ accept with probability $1 - \text{negl}(n)$. Then by Lemma 3.5, we have that $|\text{hist}\rangle$ is consistent with the reduced density matrices defined in Equation (4).

We show now that if $x \in A_{\text{no}}$, there is no state $\tau$ such that for every $S \subseteq [W]$ with $|S| \leq 5$

$$|\text{Tr}_C(\tau) - \text{Sim}_{V_i(C)}(x, S)|_{tr} < \frac{1}{T}.$$ Let us assume now, by way of contradiction, that such $\tau$ exists.

We show that in this case $\tau$ has energy $O\left(\frac{1}{T^5}\right)$ with respect to the Local Hamiltonian $H_{V_i}$ defined above, which is a contradiction, since we know that $H_{V_i}$ has groundenergy $\Omega\left(\frac{1}{T^5}\right)$, and this finishes the proof.

Let $S_i$ be the set of at most 5-qubits on which the $i$-th term of $H_{V_i}$ acts and $\rho_i = \text{Sim}_{V_i(C)}(x, S_i)$. We have that the energy of such $\tau$ is at most

$$\text{Tr}(H\tau) = \sum_i \text{Tr}(H_i \text{Tr}_{\tau_i}(\tau)) \leq \sum_i \left(\text{Tr}(H_i \rho_i) + \frac{1}{T^5}\right) \leq O\left(\frac{1}{T^5}\right),$$

where the first inequality comes from the assumption that $|\text{Tr}_C(\tau) - \rho_i|_{tr} < \frac{1}{T}$ for all $i$, and the second inequality follows since there are $O(T)$ terms and from Lemma 3.5 we have that $\text{Tr}(H_i \rho_i) = 0$.

\[\square\]

3.2 Locally simulatable proofs

Notice that we show that the QMA verification algorithm for CLDM presented in Lemma 3.3 is only possible with multiple copies of the state $\tau$ that is supposed to be consistent with the local density matrices. On the other hand, in the Local Hamiltonian problem, one copy is sufficient to perform the energy estimation of the Hamiltonian, up to a good enough accuracy. However, in the latter case, we don’t have, a priori, any knowledge of the local parts of the low-energy state.

In order to put together the two properties of local verifiability and simulation in an abstract and self-contained way, we define here the notion of \textit{locally simulatable proofs}.

\begin{definition}[$k$-SimQMA] A promise problem $A = (A_{\text{yes}}, A_{\text{no}})$ is in the complexity class $k$-SimQMA with soundness $\beta(n) \geq \frac{1}{\text{poly}(n)}$ if there exist polynomials $m, p$ such that given $x \in A$, there is an efficient deterministic algorithm that computes $m(|x|)$ $k$-qubit POVMs $\{\Pi_1, I - \Pi_1\}, \ldots, \{\Pi_{m(|x|)}, I - \Pi_{m(|x|)}\}$, that act on some quantum state of size $p(|x|)$, such that:

\textbf{Simulatable completeness:} If $x \in A_{\text{yes}}$, there exists a set of $k$-qubit density matrices $\{\rho^x_S\}_{S \subseteq [p(n)]}$ that can be computed in polynomial time from $x$ and some $p(|x|)$-qubit state $\tau$, that we call a simulatable witness such that for all $c \in [m]$

$$\text{Tr}(\Pi_c \tau) \geq 1 - \text{negl}(|x|),$$

and for every $S \subseteq [p(n)]$ of size $k$

$$|\text{Tr}_S(\tau) - \rho^x_S|_{tr} \leq \text{negl}(|x|).$$

\textbf{Soundness:} If $x \in A_{\text{no}}$, for any $p(|x|)$-qubit state $\tau$ we have that

$$\frac{1}{m} \sum_{c \in [m]} \text{Tr}(\Pi_c \tau) \leq \beta(|x|).$$

\end{definition}
**Lemma 3.7.** Every problem in QMA is in 5-SimQMA.

*Proof (Sketch).* We can consider the POVMs that arise from the verification of the Local Hamiltonian problem and the density matrices that are the simulation of the history state of the computation. From [KSV02] and Lemma 3.4, the result follows.

4 Zero-knowledge $\Xi$-protocol for QMA

In this section, we show that simulatable proof systems lead to a zero-knowledge protocols with a very simple proof structure, which can be classified in the “commit-challenge-response” framework. As mentioned in Section 1.1, when all the messages are classical, such type of protocols are called $\Sigma$-protocols. We extend this definition to the quantum setting by allowing the first message to be a quantum state and in this case we call it a $\Xi$-protocol.

**Definition 4.1 ($\Xi$-protocol).** An $\Xi$-protocol consists of a three-round protocol between a Prover and a Verifier and it takes the following form:

- **Commitment:** In the first round, the prover sends some initial quantum state.
- **Challenge:** In the second round, the verifier sends a uniformly random challenge $c \in [m]$.
- **Open:** The Prover answers the challenge $c$ with some classical value.

For simplicity we denote $\Xi$-QZK as the class of problems that have a $\Xi$ computational quantum zero-knowledge proof.

4.1 Protocol

| Notation | Meaning |
|----------|---------|
| $n$      | Number of the qubits in the SimQMA proof |
| $k$      | Locality parameter |
| $\Pi_c$  | POVM corresponding to a check of SimQMA proof system |
| $S_c$    | Set of qubits on which $\Pi_c$ acts non-trivially |
| $m$      | Number of different SimQMA checks |
| $\rho_S$ | Reduced density matrix of the proof on set $S$ of qubits for $|S| = k$ |
| $\tau$  | Quantum state that is supposed to pass the checks and be consistent with all local density matrices up to negligible error |
| $\sigma(c)$ | $\rho_c^S \otimes |0\rangle\langle 0|^c$ |
| $\zeta$ | Side-information of a malicious verifier |
| $\tilde{\phi}_{a,b}$ | $X^aZ^b\phi X^aZ^b$, for a $q$-qubit quantum state $\phi$ and $a, b \in \{0, 1\}^q$ |

Figure 1: Notation reference

We describe in Figure 2 the zero knowledge $\Xi$ protocol for QMA, whose informal description was given in Section 1.3.
Let $A = (A_{yes}, A_{no})$ be a problem in $k$-SimQMA with soundness $\delta$, $x \in A$, $\{\Pi_c\}$ be the set of POVMs for $x$, and $\tau$ be a (supposed) simulatable witness for $x$.

1. Prover picks $a, b \in \{0, 1\}^n$ and $r \in \mathbb{R}$, $\mathcal{R}$ is all of the possible randomness needed to commit to $2n$ bits.
2. Prover sends $\tilde{\tau}_{a,b} \otimes |\text{comm}_{a,b}\rangle \langle \text{comm}_{a,b}|$, where $\text{comm}_{a,b}$ is the commitment to each bit of $a$ and $b$.
3. The Verifier sends $c \in \{m\}$.
4. The Prover opens the commitment for $a|S_c$ and $b|S_c$, where $S_c$ is the set of qubits on which $\Pi_c$ acts non-trivially.
5. If the commitments do not open the verifier rejects.
6. The Verifier measures $X^a|S_c, Z^a|S_c \tilde{\tau}_{a,b} Z^a|S_c X^a|S_c$ with POVMs $\{\Pi_c, I - \Pi_c\}$, and accepts if and only if the outcome is $\Pi_c$.

Figure 2: Zero-knowledge $\Xi$-protocol for SimQMA.

4.2 Computational zero-knowledge proof for QMA

The goal of the section is to prove that every language in QMA has a $\Xi$-protocol that is a quantum computational zero-knowledge proof system if we assume that the commitment used in Figure 2 is computationally hiding and unconditionally binding.

We first state two lemmas that will be proved in Sections 4.2.1 and 4.2.2, respectively.

**Lemma 4.2.** The protocol in Figure 2 has completeness $1 - \text{negl}(n)$ and soundness $\delta$.

**Lemma 4.3.** The protocol in Figure 2 is computational zero-knowledge.

We can then state the main theorem of this section.

**Theorem 4.4.** QMA $\subseteq \Xi$-QZK.

**Proof.** Direct from Lemmas 4.2 and 4.3.

### 4.2.1 Proof of Lemma 4.2

**Lemma 4.2 (restated).** The protocol in Figure 2 has completeness $1 - \text{negl}(n)$ and soundness $\delta$.

**Proof.** If $x \in A_{yes}$ and the Prover picks $\tau$ to be some state that is consistent with all the POVMs and the local density matrices, and then behaves honestly, then the acceptance probability is exponentially close to 1.

Let us now analyze the case for $x \in A_{no}$. Let $\psi \otimes |z\rangle \langle z|$ be the state sent by the Prover in the first message, where $\psi$ is supposed to be the copies of the one-time padded state that is consistent with the POVMs and the reduced density matrices, and $z$ is the commitment to the one-time pad keys. We assume, without loss of generality, that $|z\rangle$ is a classical value, since the Verifier can measure it as soon as she receives it, and the Prover can send the $z$ that maximizes the acceptance probability.

For challenge $c$, the Prover answers with $|w_c\rangle$, where again we assume to be a classical value for the same reasons as above. Since the commitment scheme is unconditionally binding, we can
define the strings $a, b \in \{0, 1\}^n$ to be the string containing the unique bits that could be open for the corrected committed bits, or 0 if the commitment is defective.

Let $S_c \subseteq [n]$ be defined as in Figure 2. Notice that $|w_c\rangle$ is supposed to be the opening of bits of $a$ and $b$ in the subset $S_c$. Let $D_c$ be the event that $w_c$ is the correct opening for all of such bits, and $1_{D_c}$ be the indicator variable for such event.

We have then that the acceptance probability is

$$\frac{1}{m} \sum_{c \in [m]} 1_{D_c} \Tr \left( \Pi_c X^a |s_c\rangle \otimes Z^b |s_c\rangle \psi Z^b |s_c\rangle X^a |s_c\rangle \right)$$

(5)

$$\leq \frac{1}{m} \sum_{c \in [m]} \Tr \left( \Pi_c X^a |s_c\rangle \otimes Z^b |s_c\rangle \psi Z^b |s_c\rangle X^a |s_c\rangle \right)$$

$$= \frac{1}{m} \sum_{c \in [m]} \Tr \left( \Pi_c X^a \psi Z^b X^a \right)$$

(6)

$$\leq \max_{\phi} \frac{1}{m} \sum_{c} \Tr (\Pi_c \phi)$$

$$\leq \delta.$$

where in the equality we use the fact that $\Pi_c$ only acts on the qubits in $S_c$, and the last inequality follows since $x \in A_{no}$ and the SimQMA protocol has soundness $\delta$.

4.2.2 Proof of Lemma 4.3

We prove now the zero knowledge property of the protocol.

Before presenting the simulator, let us analyze how the verification algorithm behaves. We can assume, without loss of generality that the Verifier is composed of two verification algorithms $\hat{V}_1$ and $\hat{V}_2$.

For $\hat{V}_1$, since the classical part of the message can be copied and the challenge sent by the Verifier is measured by the prover, we can assume $\hat{V}_1$ acts like the following

$$\sum_{a,b,r} \hat{V}_1 \left( \tilde{\rho}_{a,b} \otimes |\text{comm}^r_{a,b}\rangle \langle \text{comm}^r_{a,b} | \otimes \zeta \right) \hat{V}_1^\dagger$$

(7)

$$= \sum_{a,b,c,r} p_{a,b,c,r} \phi_{a,b,c,r} \otimes |\text{comm}^r_{a,b}\rangle \langle \text{comm}^r_{a,b} | \otimes |c\rangle \langle c|,$n

(8)

where $\sum_{a,b,c,r} p_{a,b,c,r} = 1$ and we have traced-out the copy of $|c\rangle \langle c|$ that was sent to the Prover (and measured).

The message $|c\rangle$ is sent to the Prover, who answers then with some value $|o_c\rangle$, i.e., the opening of the commitments corresponding to the challenge $c$.

The Verifier then outputs

$$\sum_{a,b,c,r} p_{a,b,c,r} \hat{V}_2 \left( \phi_{a,b,c,r} \otimes |\text{comm}^r_{a,b}\rangle \langle \text{comm}^r_{a,b} | \otimes |c\rangle \langle c| \otimes |o_c\rangle \langle o_c| \right) \hat{V}_2^\dagger.$$

(9)
Let $A = (A_{yes}, A_{no})$ to be a problem in $k$-SimQMA, $x \in A_{yes}$, and and $\{\rho^x_S\}_S$ be the set of local density matrices of a simulatable witness $\tau$ for $x$.

1. Pick $c \in \{0, 1\}^n$, $r \in \{0, 1\}$.
2. Create the state $\tilde{\sigma}(c) \otimes |\text{comm}_r\rangle \langle \text{comm}_r| \otimes \zeta$, where $\sigma(c) = p^c \otimes |0\rangle \langle 0|^R$.
3. Run $\hat{V}_1$ on $\tilde{\sigma}(c) \otimes |\text{comm}_r\rangle \langle \text{comm}_r| \otimes \zeta$.
4. Measure the last register in the computational basis and abort if it is not $|c\rangle$.
5. Otherwise, append the register $|o_c\rangle$, apply $\hat{V}_2$ and output the result.

![Figure 3: Simulator Sim for ZK $\Xi$-protocol for QMA](image)

In order to show zero-knowledge, we start by showing that for a fixed $c$, $\sum_{a,b,r} p_{\sigma,a,b,c,r}$ is independent of $\sigma$, up to negligible factors, if the commitment scheme is hiding. We denote $R = |\mathcal{R}|$.

**Lemma 4.5.** Let $p_c = \frac{1}{R} \sum_r p_{I,0,0,c,r}$. Then for any $\sigma$, we have that

$$\left| \frac{1}{2^{2n}R} \sum_{a,b,r} p_{\sigma,a,b,c,r} - p_c \right| \leq \text{negl}(n),$$

where the probabilities are defined as Equation (8) for the polynomial-time adversary $\hat{V}_1$.

**Proof.** Let us suppose that there exist some state $\rho$, a challenge $c$ and a polynomial $q$ such that

$$\left| p_c - \left( \frac{1}{2^{2n}R} \sum_{a,b,r} p_{\sigma,a,b,c,r} \right) \right| \geq q(n). \quad (10)$$

Then it is possible to distinguish the states

$$\frac{1}{2^{2n}R} \sum_{a,b,r} \tilde{\sigma}_{a,b} \otimes |\text{comm}_r\rangle \langle \text{comm}_r|\text{ and } \frac{1}{R} \sum_r |l\rangle \otimes |\text{comm}_{0,0}\rangle \langle \text{comm}_{0,0}|$$

by appending $\zeta$, applying $\hat{V}_1$ and measuring the challenge register in the computational basis.

However, since the commitment scheme is computationally hiding, we have that

$$\frac{1}{2^{2n}R} \sum_{a,b,r} \tilde{\sigma}_{a,b} \otimes |\text{comm}_r\rangle \langle \text{comm}_r| \approx_c \frac{1}{2^{2n}R} \sum_{a,b,r} |l\rangle \otimes |\text{comm}_{0,0}\rangle \langle \text{comm}_{0,0}|$$

and therefore these states are indistinguishable. We conclude that the assumption in Equation (10) is false. \qed
Lemma 4.6. The Simulator described in Figure 3 does not abort with probability at least \( \frac{1}{m} - \text{negl}(n) \). In this case, its output is \( \text{negl}(n) \)-close to
\[
\frac{1}{2^{2n}} \sum_{a,b,c,r} \hat{V}_2 \left( (I \otimes |c\rangle\langle c|) \hat{V}_1 \left( \sigma(c)_{a,b} \otimes |\text{comm}^r_{a,b}\rangle|\text{comm}^r_{a,b}\rangle \otimes \zeta \right) \hat{V}_1^\dagger (I \otimes |c\rangle\langle c|) \otimes |o_{c}\rangle\langle o_{c}| \right) \hat{V}_2^\dagger.
\]

Proof. The state of Sim after step 2 is
\[
\frac{1}{2^{2n}} \sum_{a,b,c,r} \sigma(c)_{a,b} \otimes |\text{comm}^r_{a,b}\rangle|\text{comm}^r_{a,b}\rangle \otimes \zeta \otimes |c\rangle\langle c|.
\]

Sim runs \( \hat{V}_1 \) on the first three register of the state in Equation (13), resulting in
\[
\frac{1}{2^{2n}} \sum_{a,b,c,r} \hat{V}_1 \left( \sigma(c)_{a,b} \otimes |\text{comm}^r_{a,b}\rangle|\text{comm}^r_{a,b}\rangle \otimes \zeta \right) \hat{V}_1^\dagger \otimes |c\rangle\langle c|.
\]

Notice that Sim does not abort when \( c = c' \), and this event happens with probability
\[
\frac{1}{m} \sum_{c} \frac{1}{2^{2n}} \sum_{a,b,r} p_{\sigma(c),a,b,c,r} \geq \frac{1}{m} \sum_{c} (p_{c} - \text{negl}(n)) = \frac{1}{m} - \text{negl}(n),
\]
where the inequality follows from Lemma 4.5 and the equality from the fact that \( \sum_{c} p_{c} = 1 \).

In order to provide the output of the simulator, conditioned that he did not abort, we post-select in Equation (14) the event that \( c = c' \), which gives us
\[
\frac{1}{2^{2n}} \sum_{a,b,c,r} \sum_{c} (I \otimes |c\rangle\langle c|) \hat{V}_1 \left( \sigma(c)_{a,b} \otimes |\text{comm}^r_{a,b}\rangle|\text{comm}^r_{a,b}\rangle \otimes \zeta \right) \hat{V}_1^\dagger (I \otimes |c\rangle\langle c|) \otimes |c\rangle\langle c|,
\]
where we set \( p_{\text{succ}} = \frac{1}{2^{2n}} \sum_{a,b,c,r} p_{\sigma(c),a,b,c,r} \) to be the probability that Sim does not abort and the approximation holds by Equation (15).

Finally, Sim only needs to append the last register with \( |o_{c}\rangle \), which can be performed efficiently given \( a, b \) and \( c \), and apply \( \hat{V}_2 \).

Lemma 4.7. The output of a simulator that does not abort is computationally indistinguishable from the output of the malicious verifier in the protocol in Figure 2.

Proof. In the real protocol, the first message sent by the Prover is
\[
\frac{1}{2^{2n}} \sum_{a,b,r} \tilde{r}_{a,b} \otimes |\text{comm}^r_{a,b}\rangle|\text{comm}^r_{a,b}\rangle,
\]
and the Verifier applies \( \hat{V}_1 \) on the message sent by the Prover and the side-information \( \zeta \), resulting in the state
\[
\frac{1}{2^{2n}} \sum_{a,b,c,r} (I \otimes |c\rangle\langle c|) \hat{V}_1 \left( \tilde{r}_{a,b} \otimes |\text{comm}^r_{a,b}\rangle|\text{comm}^r_{a,b}\rangle \otimes \zeta \right) \hat{V}_1^\dagger (I \otimes |c\rangle\langle c|).
\]
On challenge $|c\rangle$, the Prover then answers with $|o_c\rangle$, the openings of the corresponding commitments. The Verifier then applies $\tilde{V}_2$, and outputs

$$\frac{1}{2^{2nR}} \sum_{a,b,c,r} \tilde{V}_2 \left( (1 \otimes |c\rangle\langle c|) \tilde{V}_1 (\tau_{a,b} \otimes |\text{comm}_{a,b}^r\rangle\langle \text{comm}_{a,b}^r| \otimes \zeta) \tilde{V}_1^\dagger (1 \otimes |c\rangle\langle c|) \otimes |o_c\rangle\langle o_c| \right) \tilde{V}_2.$$ 

We show now that this state is indistinguishable from the state that is output by the simulator, proved in Lemma 4.6. For simplicity, let $\xi_{a,b,r} = |\text{comm}_{a,b}^r\rangle\langle \text{comm}_{a,b}^r| \otimes \zeta$. Up to normalization factors, we have that

$$\sum_{a,b,c,r} \tilde{V}_2 \left( (1 \otimes |c\rangle\langle c|) \tilde{V}_1 (\tau_{a,b} \otimes \xi_{a,b,r}) \tilde{V}_1^\dagger (1 \otimes |c\rangle\langle c|) \otimes |o_c\rangle\langle o_c| \right) \tilde{V}_2^\dagger$$

$$\approx_c \sum_{a,b,c,r} \tilde{V}_2 \left( (1 \otimes |c\rangle\langle c|) \tilde{V}_1 \left( \text{Tr}_{S_c}(\tau)_{S_c} \otimes I_{S_c} \right) \otimes \xi_{a,b,r} \right) \tilde{V}_1^\dagger (1 \otimes |c\rangle\langle c|) \otimes |o_c\rangle\langle o_c| \right) \tilde{V}_2^\dagger$$

$$\approx_s \sum_{a,b,c,r} \tilde{V}_2 \left( (1 \otimes |c\rangle\langle c|) \tilde{V}_1 \left( \sigma_{a,b} \otimes \xi_{a,b,r} \right) \tilde{V}_1^\dagger (1 \otimes |c\rangle\langle c|) \otimes |o_c\rangle\langle o_c| \right) \tilde{V}_2^\dagger$$

where in the first approximation we use the fact that the commitments of $a_{S_c}$ and $b_{S_c}$ are never revealed and that the commitment is computationally hiding, and in the second approximation holds since we assume that $\|\text{Tr}_{S_c}(\sigma) - \rho_c\|_{tr} \leq \text{negl}(n)$.

By the definition of $\xi_{a,b,r}$ and Lemma 4.6, the state of Equation (18) is $\text{negl}(n)$-close to the output of the simulator. □

We are finally ready to prove Lemma 4.3.

**Lemma 4.3 (referred).** The protocol in Figure 2 is computational zero-knowledge.

**Proof.** Notice that from Lemmas 2.1 and 4.5, there exists a quantum algorithm $\text{Sim}'$ that runs in time

$$O \left( m \text{poly}(n) \left( \text{time} \left( \tilde{V}_1 \right) + \text{time} \left( \tilde{V}_2 \right) \right) \right)$$

whose output is $\text{negl}(n)$-close to the output of $\text{Sim}$, conditioned on non-abortion. From Lemma 4.7, the output of $\text{Sim}'$ is computationally indistinguishable from the run of the real protocol, and therefore it can be used as the simulator, finishing our proof. □

### 4.3 Quantum statistical zero-knowledge arguments for QMA

We show now that if the commitment used in Figure 2 is unconditionally hiding but computationally binding, then the $\Xi$ protocol is a statistical zero-knowledge argument. We formally define these argument systems now.

**Definition 4.8** (Quantum statistical zero-knowledge argument (QSZKA)). A promise problem $A = (A_{yes}, A_{no})$ is in QSZKA if there is an interactive protocol $(V \rightleftharpoons P)$, where $V$ is polynomial-time and is given some input $x \in A$ and outputs a classical bit indicating acceptance or rejection of $x$, $P$ is polynomial time and is given some input $\psi_x$, and the following holds

- **Completeness:** If $x \in A_{yes}$, there exists some $\psi_x$ such that $\Pr_r[(V \rightleftharpoons P(\psi_x)) = 1] \geq 1 - \text{negl}(n)$.

- **Soundness:** If $x \in A_{no}$, for all polynomial-time $P^*$, we have that $\Pr_r[(V \rightleftharpoons P^*) = 1] \leq \frac{1}{\text{poly}(n)}$. 

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**Statistical zero-knowledge:** For any \( x \in A_{yes} \) and any polynomial-time \( V' \) that receives the inputs \( x \) and some state \( \zeta \), there exists a polynomial-time quantum channel \( S_{V'} \) that also receives \( x \) and \( \zeta \) as input such that \((V' \equiv P) \approx_s S_{V'}\).

Similarly to \( \Xi \)-QZK, we define \( \Xi \)-QSZKA as the class of problems that have a \( \Xi \) quantum statistical zero-knowledge argument. We argue now that our modified \( \Xi \) protocol is a quantum zero-knowledge argument.

Notice that the binding property is only used in the soundness of the protocol. In this case, if we assume that the malicious prover is polynomial-time, and therefore cannot break the computationally binding property of the commitment scheme, the proof of Lemma 4.2 still holds.

We argue now that the statistical zero-knowledge property holds for this variation of the protocol. Notice that if the commitment scheme is unconditionally binding, the approximations in Equations (11), (12), (16) and (17) are statistical and not computational. In this case, the output of the simulator is statistically indistinguishable from the output of the verifier in the real protocol. Therefore, with the corresponding changes of the previous proofs, we can achieve the following result.

**Corollary 4.9.** \( \text{QMA} \subseteq \Xi \)-QSZKA.

### 4.4 Decreasing the soundness error

We remark that unlike the protocol in [GSY19], we do not know how to show parallel repetition for our protocol. The problem here is that for an \( \ell \)-fold version of our protocol, the simulator, as in Figure 3, would need to correctly answer the question for each one of these \( \ell \) copies of the game, what would happen with probability \( \frac{1}{m^\ell} \), and therefore the rewinding technique would have an exponential cost in \( \ell \).

However, as in the classical case, we can show that our protocol accepts sequential repetition, since the guess for each of the iterations is performed independently.

**Lemma 4.10.** Consider the \( \ell \)-fold sequential repetition of the QZK \( \Xi \)-protocol, where \( 1 \leq \ell = \text{poly}(n) \) and the Verifier accepts if and only if each parallel run accepts. Then this is a quantum zero-knowledge protocol for \( \text{SimQMA} \) with completeness \( 1 - \text{negl}(|x|) \) and soundness \( O\left(\delta(|x|)^\ell\right)\).

**Proof.** The completeness and soundness properties hold trivially.

Let us now argue about the zero-knowledge property. Notice that an honest prover \( P \) has an \( \ell \)-fold tensor product of the honest witness, and uses a single copy per iteration. In this case, we can run the simulator \( \ell \) times sequentially, using the output of the \( i \)-th run as the side-information of the \( i + 1 \)-st run.

**Remark 4.11.** Concurrently to this work, Bitansky and Shmueli [BS19] have proposed the first quantum zero-knowledge argument system for QMA with constant rounds and negligible soundness. Their main building block is a non black-box quantum extractor for a post-quantum commitment scheme. We leave as future work trying to use this tool to perform parallel repetition of our zero-knowledge \( \Xi \)-protocols.

### 5 Proofs of quantum knowledge

In this section, we define a **Proof of Quantum Knowledge** (Section 5.1) and and then prove that the Zero-knowledge protocols presented in the previous sections (Section 5.2) satisfy this new definition.
5.1 Definition

The content of this subsection was written in collaboration with Andrea Coladangelo, Thomas Vidick and Tina Zhang and it also appears in their concurrent and independent work.

A Proof of Knowledge (PoK) is an interactive proof system for some relation $R$ such that if the Verifier accepts some input $x$ with high enough probability, then she is convinced that the Prover knows some witness $w$ such that $(x, w) \in R$. This notion is formalized by requiring the existence of an efficient extractor $K$ that is able to output a witness for $x$ when $K$ is given oracle access to the Prover (and is able to rewind his actions).

**Definition 5.1** (Classical Proof of Knowledge [BG93]). Let $R \subseteq X \times Y$ be a relation. A proof system $(P, V)$ for $R$ is a Proof of Knowledge for $R$ with knowledge error $\kappa$ if there exists a polynomial $p > 0$ and a polynomial-time machine $K$ such that for any classical interactive machine $P^*$ that makes $V$ accept some instance $x$ of size $n$ with probability at least $\varepsilon > \kappa(n)$, we have

$$Pr \left[ (x, K^{P^*(x,y)}(x)) \in R \right] \geq p \left( (\varepsilon - \kappa(n)), \frac{1}{n} \right).$$

In the definition, $y$ corresponds to the side-information that $P^*$ has, possibly including some $w$ such that $(x, w) \in R$.

PoKs were originally defined only considering classical adversaries, and this notion was first studied in the quantum setting by Unruh [Unr12]. The first issue that arises in the quantum setting is which type of query $K$ could be able to perform. To solve this, we assume that $P^*$ always performs some unitary operation $U$. Notice that this can be done without loss of generality since (i) we can consider the purification of the Prover, (ii) all the measurements can be performed coherently, and (iii) $P^*$ can keep track of the round of communication in some internal register and $U$ implicitly controls on this value. Then, the quantum extractor $K$ has oracle access to $P^*$ by performing $U$ and $U^\dagger$ on the message register and private register of $P^*$, but $K$ has no direct access to the latter. We denote the extractor $K$ with such an oracle access to $P^*$ as $K|^{P^*(x,\rho)}$, where here $\rho$ is the (quantum) side-information held by $P^*$.

**Definition 5.2** (Quantum Proof of Knowledge [Unr12]). Let $R \subseteq X \times Y$ be a relation. A proof system $(P, V)$ for $R$ is a Quantum Proof of Knowledge for $R$ with knowledge error $\kappa$ if there exists a polynomial $p > 0$ and a quantum polynomial-time machine $K$ such that for any quantum interactive machine $P^*$ that makes $V$ accept some instance $x$ of size $n$ with probability at least $\varepsilon > \kappa(n)$, then

$$Pr \left[ (x, K^{P^*(x,\rho)}(x)) \in R \right] \geq p \left( (\varepsilon - \kappa(n)), \frac{1}{n} \right).$$

**Remark 5.3.** In the fully classical case of Definition 5.1, the extractor could repeat the procedure $\text{poly}((\varepsilon - \kappa(n)))$ times in order to increase the success probability. We notice that this is not known to be possible for a general quantum $P^*$, since the final measurement to extract the witness would possibly disturb the internal state of $P^*$, making it impossible to simulate the side-information that $P^*$ had originally in the subsequent simulations.

We finally move on to the full quantum setting, where we want a Proof of Quantum Knowledge (PoQ). Here, at the end of the protocol, we want the Verifier to be convinced that the Prover has a quantum witness for the input $x$. 

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The first challenge is defining the notion a “relation” between the input \( x \) and some quantum state \( |\psi\rangle \). Classically, we implicitly consider relations for NP languages by fixing some verification algorithm \( V \) for it, and defining \( (x, w) \in R \) if and only if \( V \) accepts the input \( x \) with the witness \( w \).

Quantumly, the situation is a bit more delicate, since a witness \( |\psi\rangle \) leads to acceptance probability \( \Pr[Q(x, |\psi\rangle) = 1] \). This issue also appears with probabilistic complexity classes such as MA, and we can solve it by fixing some parameter \( \gamma \) and defining the relation to contain \( (x, |\psi\rangle) \) for all quantum states \( |\psi\rangle \) that lead to acceptance probability at least \( \gamma \). The difference here is that we need also to consider the mixture of such quantum states, since they are also valid witnesses in a QMA protocol. Therefore, fixing some quantum verifier \( Q \) and \( \alpha \), we define a quantum relation as follows

\[
R_{Q, \alpha} = \{(x, \sigma) : Q \text{ accepts } (x, \sigma) \text{ with probability at least } \alpha \}
\]

Notice that with \( R_{Q, \alpha} \), we implicitly define subspaces \( \{S_x\}_x \) such that \( (x, \sigma) \in R_{Q, \alpha} \) if and only if \( \sigma \in S_x \).

With this in hand, we can define a QMA-relation.

**Definition 5.4 (QMA-relation).** Let \( A = (A_{\text{yes}}, A_{\text{no}}) \) be a problem in QMA, and let \( Q \) be an associated quantum polynomial-time verification algorithm (which takes as input an instance and a witness), with completeness \( \alpha \) and soundness \( \beta \). Then, we say that \( R_{Q, \alpha} \) is a QMA-relation with completeness \( \alpha \) and soundness \( \beta \) for the problem \( A \). In particular, we have that for \( x \in A_{\text{yes}} \), there exists some \( |\psi\rangle \) such that \( (x, |\psi\rangle) \in R_{Q, \alpha} \) and for \( x \in A_{\text{no}} \), for every \( \rho \) and \( \varepsilon > 0 \) it holds that \( (x, \rho) \notin R_{Q, \beta + \varepsilon} \).

We can finally define Proof of Quantum Knowledge.

**Definition 5.5 (Proof of Quantum Knowledge).** Let \( R_{Q, \gamma} \) be a QMA relation. A proof system \((P, V)\) is a Proof of Quantum Knowledge for \( R_{Q, \gamma} \) with knowledge error \( \kappa(n) > 0 \) and quality \( q \), if there exists a polynomial \( p > 0 \) and a polynomial-time machine \( K \) such that for any quantum interactive machine \( P^* \) that makes \( V \) accept some instance \( x \) of size \( n \) with probability at least \( \varepsilon > \kappa(n) \), we have

\[
\Pr\left[ (x, K^{P^*(x, \rho)}) (x) \in R_{Q, q(\varepsilon, \frac{1}{n})} \right] \geq p\left( (\varepsilon - \kappa(n)), \frac{1}{n} \right).
\]

**5.2 Proof of quantum-knowledge for our \( \Xi \)-protocol**

We show now that the \( \Xi \)-protocol of Figure 2 is a Proof of Quantum Knowledge with knowledge error inverse polynomially close to 1.
Let $A = (A_{yes}, A_{no})$ to be a problem in $k$-SimQMA, $x \in A$, $\{\Pi_c\}$ be the set of POVMs for $x$ and $P^*$ be a prover that makes the Verifier accept with probability at least $\kappa(n) \geq 1 - \frac{1}{2m^2}$ in the $\Xi$-protocol of Figure 2.

1. Run $P^*$ and store the first message $\psi \otimes |z\rangle \langle z|$

2. For every challenge $c$
   2.1. Simulate $P^*$ on challenge $c$
   2.2. Check (coherently) if the answer correctly opens the committed value, if not abort
   2.3. Copy the opening of the committed values
   2.4. Run $P^*$ backwards on challenge $c$

3. Let $a, b \in \{0, 1\}^n$ be the opened strings

4. Output $X^aZ^b\psi Z^bX^a$

### Figure 4: Single-shot Knowledge extractor $K$

**Lemma 5.6.** Let $A$ and $\{\Pi_c\}$ be as defined in Figure 4 and $Q$ be the verification algorithm for $A$ that consists of picking $c \in [m]$ uniformly at random and measuring the provided witness with $\Pi_c$. Let $\kappa(n) = 1 - \frac{1}{2m^2}$ and $K$ be the poly$(n)$-time extractor defined in Figure 4. If a quantum interactive machine $P^*$ makes $V$ accept the instance $x$ of size $n$ with probability at least $\varepsilon := 1 - \delta > \kappa(n)$, then we have that

$$\Pr[(x, K^{P^*(x,\varphi)}(x)) \in R_{Q,1-\delta-m^2\delta}] \geq 1 - m^2\delta.$$

**Proof.** Let $a, b \in \{0, 1\}^{2n}$ be the unique values that can be opened by the classical value $|z\rangle$ sent by the Prover (or zeroes if the commitments are mal-formed). Doing the same calculations of Equations (5) to (6), and assuming that the acceptance probability of the original protocol is at least $1 - \delta$, it follows that

$$\left(x, X^aZ^b\psi Z^bX^a\right) \in A_{Q,1-\delta}.$$  \hspace{1cm} (19)

Our goal now is show how to retrieve the values $a$ and $b$, without damaging the quantum state $\psi$ too much.

Let $\mu_{VM,P}$ be the state shared by the Verifier and Prover after the commitment phase. Since the message (for honest verifiers) is always a classical value, we can model $P^*$’s behaviour with the unitary $U_c$ performed by him on challenge $c$. Let also $\Pi$ be the projection of $V$ onto the acceptance subspace, $\Pi_c = U_c^\dagger \Pi U_c$ be the operation that performs the Prover’s unitary for challenge $c$, performs the measurement of the verifier, and then undoes the Prover’s unitary.

Given that $P^*$ makes $V$ accept with probability at least $1 - \delta$ and each challenge is picked with probability $\frac{1}{m}$, it follows that for any challenge $c$, $V$ accepts with probability at least $1 - m\delta$, otherwise the acceptance probability would be strictly less than $1 - \delta$. In other words, it follows that for every $c$, we have that

$$\text{Tr}\left(\Pi_c\mu\Pi_c\right) \geq 1 - m\delta,$$

which implies that

$$\text{Tr}\left(\tilde{\mu}\right) \geq 1 - m^2\delta, \quad \text{for } \tilde{\mu} = \Pi_m...\hat{\Pi}_1\mu\hat{\Pi}_1...\Pi_m$$  \hspace{1cm} (20)

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Let $O$ be the register where the Verifier holds the original state $\psi$ and $\phi = \text{Tr}_O(\hat{\mu})$. We have that
\[ D(\psi, \phi) = D(\text{Tr}_O(\mu), \text{Tr}_O(\hat{\mu})) \leq D(\mu, \hat{\mu}) \leq m^2 \delta. \tag{21} \]
where in the equality we use the definition of $\psi$, $\phi$ and the register $O$, the first inequality follows since trace distance is contractive under CPTP maps and the last inequality holds by Equation (20).

Notice that the decision of an abort by $K$ is strictly less restrictive than the rejections of an honest verifier, since the verifier also tests that the commitment correctly opens. In this case, we can conclude that $K$ does not abort with probability at least $1 - m^2 \delta$. Then, if we condition on the event that $K$ does not abort, all the committed information is opened and since the commitment is binding, $K$ holds the unique values $a, b \in \{0, 1\}^n$ that can be opened from $z$. $K$ finishes by outputting $X^aZ^b\phi Z^bX^a$. It follows from Equation (21) and the fact that trace distance is preserved under unitary operations, we have
\[ D(X^aZ^b\psi X^aZ^b, X^aZ^b\phi X^aZ^b) \leq m^2 \delta, \]
and therefore
\[ (x, X^aZ^b\phi X^aZ^b) \in R_{Q, 1-\delta-m^2\delta}, \]
which finishes the proof. \hfill \qed

### 5.2.1 Sequential repetition

In the previous section, we have the quantum extractor that works if the knowledge error is very high, namely inverse polynomially close to 1. We show here how to decrease the knowledge leakage, by considering the sequential repetition of the $\Xi$-protocol.

Let $A = (A_{\text{yes}}, A_{\text{no}})$ to be a problem in $k$-SimQMA, $x \in A$, $\{\Pi_c\}$ be the set of POVMs for $x$ and $P^*$ be a prover that makes the Verifier accept with probability at least $\kappa(n) \geq (1 - \frac{1}{2m^2})^\ell$ in the $\ell$-fold sequential repetition of the $\Xi$-protocol of Figure 2.

1. $K$ chooses $i \in [\ell]$
2. For $0 < j < i$
   1. Pick $c_j$ uniformly at random and put it in the message register
   2. Run $P^*$ with $c_j$
   3. If $V$ would reject, abort
3. Run Single-shot extractor for the $i$-th game

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**Figure 5:** Knowledge extractor $K$

**Lemma 5.7.** Fix $\ell \geq 1$. If some quantum interactive machine $P^*$ that makes $V$ accept some instance $x$ of size $n$ in $\ell$ sequential repetitions of the $\Xi$-protocol with probability at least $\varepsilon$, then there exists $i \in [\ell]$, such that the probability that $P^*$ passes game $i$, conditioned on the event that $P^*$ passed the games $1, \ldots, i - 1$, is at least $\varepsilon^{-\ell}$. 

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Proof. Let us prove this by contradiction. Let $E_j$ be the event that $P^*$ passes the game $j$. So let us assume that for all $j$, $Pr[E_j|E_1 \land \ldots \land E_{j-1}] < \varepsilon^{-\ell}$.

Notice that we can bound the overall acceptance probability as

$$
\varepsilon = Pr[E_1 \land \ldots \land E_\ell] = Pr[E_1] Pr[E_2|E_1] \ldots Pr[E_\ell|E_1 \land \ldots \land E_{\ell-1}] < (\varepsilon^{-\ell})^\ell = \varepsilon,
$$

which is a contradiction. 

Lemma 5.8. Let $A$ and $\{\Pi_c\}$ be defined as in Figure 5 and $Q$ be the verification algorithm for $A$ that consists of picking $c \in [m]$ uniformly at random and measuring the provided witness with $\Pi_c$. Let $\kappa(n) = (1 - \frac{1}{2m^2})^\ell$ and $K$ be the poly$(n)$-time extractor defined in Figure 5. If a quantum interactive machine $P^*$ makes $V$ accept the instance $x$ of size $n$ with probability at least $\varepsilon := (1 - \delta)^\ell > \kappa(n)$, then we have that

$$
Pr\left[ \left(x, K^{(P^*(x,c))}(x) \right) \in R_{Q,1-\delta-m^2\delta} \right] \geq 1 - m^2\delta.
$$

and $K$ runs in time poly$(n)$.

Proof. From Lemma 5.7, we know that there exists at least one value of $i^* \in [k]$ such that the success of probability in the $i^*$-th round, conditioned on the event of success on rounds $1, \ldots, i^*$ is $1 - \delta$, and we recall that by definition $\delta < \frac{1}{2m^2}$. We have then that the value $i$ guessed by $K$ is equal to $i^*$ with probability at least $\frac{1}{\ell}$.

Let us assume now that $i = i^*$. Using a slightly different notation from Lemma 5.6, let $\mu$ be the initial state of $P^*$ and $\hat{\Pi}_j = \Pi_{ic}^{(j)}U_jV_j$ be the Verifier operation in round $j$ $V_j$, followed by the provers’ operation operation on round $j$, and finally the projection onto the acceptance subspace on the $j$-th round.

The probability that $K$ does not abort in the first $i - 1$ steps is

$$
Tr \left( \hat{\Pi}_{i-1} \ldots \hat{\Pi}_1 \mu \hat{\Pi}_1 \ldots \hat{\Pi}_{i-1} \right) \geq Tr \left( \hat{\Pi}_{\ell} \ldots \hat{\Pi}_1 \mu \hat{\Pi}_1 \ldots \hat{\Pi}_{\ell} \right) = \varepsilon,
$$

where the equality holds since we assume that $P^*$ makes the Verifier accept with probability $\varepsilon$.

Therefore, we have that with probability $\frac{\varepsilon}{\ell}$, $K$ made the correct guess $i = i^*$ and the simulation did not abort during the first $i - 1$ steps. In this case, we have that the probability that $P^*$ passes the $i$-th game is at least $1 - \delta > 1 - \frac{1}{2m^2}$ and the result then follows by Lemma 5.6. 

Remark 5.9. We notice that $P^*$ would need multiple copies of the witness in order to pass the sequential repetitions of the $\Xi$-protocol. However, the extractor of Figure 5 can only extract one such copy. It could be easily extended to output a constant number of copies and we leave as an open problem achieving better PoQ extractors for our protocol.

6 Non-interactive zero-knowledge protocol with secret parameters for QMA

In this section, using similar techniques of Section 4, we show that all problems in SimQMA have a QNISZK protocol in the secret parameters model, quantizing the result by Pass and Shelat [PS05].

We start by defining the model.
**Definition 6.1** (Quantum non-interactive proofs in the classical secret parameter model). A triple of algorithms \((D, P, V)\) is called a quantum non-interactive proof in the secret parameter model for a promise problem \(A = (A_{\text{yes}}, A_{\text{no}})\) where \(D\) is a probabilistic polynomial time algorithm, \(V\) is a quantum polynomial time algorithm and \(P\) is an unbounded quantum algorithm such that there exists a negligible function \(\varepsilon\) such that the following conditions follow:

**Completeness:** for every \(x \in A_{\text{yes}}\), there exists some \(P\)

\[
Pr[(r_P, r_V) \leftarrow D(1^{|x|}); \pi \leftarrow P(x, r_P); V(x, r_V, \pi) = 1] \geq 1 - \varepsilon(n).
\]

**Soundness:** for every \(x \in A_{\text{no}}\) and every \(P\)

\[
Pr[(r_P, r_V) \leftarrow D(1^{|x|}); \pi \leftarrow P(x, r_P); V(x, r_V, \pi) = 1] \leq \varepsilon(n).
\]

**Statistical zero-knowledge:** for every \(x \in A_{\text{yes}}\), there is a polynomial time algorithm \(S\) such that for the state \(\sigma = S(x)\) and \(\rho = \sum_{(r_V, s_P) \leftarrow D(1^{|x|}; r_P \leftarrow D(1^{|s|}; s_P \leftarrow D(1^{|s|}))} P(x, r_P, r_V) \otimes P(x, r_P) \otimes P(x, r_V)\) we have that \(\sigma \approx_s \rho\).

The non-interactive protocol is very similar to the \(\Xi\) protocol, with a small (but crucial) change: instead of using commitments for the one-time pad key, the trusted party picks these values and reveals just a constant number of these values to the Verifier (and all of them to the Prover). Let us be a bit more precise. The trusted party picks uniformly at random the one-time pad keys \(a, b\) and sends them to the Prover. For the Verifier, the trusted party sends \(S, a|_S\) and \(b|_S\), where \(S\) is a random subset of \(k\) indices of \(a, b\).

The Prover uses \(a\) and \(b\) to one-time pad the simulatable proof and sends this one-time padded state to the Verifier.

The Verifier picks one of the checking terms uniformly at random. If the qubits corresponding to the chosen term are not in \(S\), the Verifier accepts. Otherwise, the Verifier uses \(a|_S\) and \(b|_S\) to decrypt the qubits corresponding to such term, and finally performs the check. The completeness of the protocol is straightforward. For soundness, we have that with inverse polynomial probability the revealed bits will allow the verifier to check the desired term of the encoded history state. Finally, the zero-knowledge property holds since the quantum proof is simulatable.

We describe now the protocol more formally.

Let \(A = (A_{\text{yes}}, A_{\text{no}})\) be a problem in \(k\)-SimQMA with soundness \(\delta\), \(x \in A\), \(\{\Pi_c\}\) be the set of POVMs for \(x\), and \(\tau\) be a (supposed) simulatable witness for \(x\).

1. \(D\) picks \(a, b \in \{0, 1\}^n\) and \(S \subseteq [n]\), with \(|S| = k\), uniformly at random.
2. \(D\) sends \((a, b)\) to the Prover and \((S, a|_S, b|_S)\) and to the Verifier.
3. The Prover sends \(\tilde{\tau}_{a,b}\) to the Verifier
4. The Verifier picks \(c \in \mathbb{S}[m]\)
5. If the qubits corresponding to \(\Pi_c\) are not in \(S\), the Verifier accepts
6. Otherwise, the Verifier applies \(X|_S\) \(Z|_S\) to the qubits in \(\Pi_c\) and accepts according to its output.

Figure 6: QNIZK protocol in the secret parameter model for SimQMA
Remark 6.2. We make the verifier pick one of the terms and then check with the set \( S \) in order to simplify the proof of soundness. The verifier could instead check some \( \Pi_c \) that matches with \( S \) or accept when this is not possible and this protocol would still be sound.

Lemma 6.3. The protocol in Figure 6 has completeness \( 1 - \text{negl}(|x|) \). and soundness \( 1 - \frac{1-n^{-\delta}}{n^k} \).

Proof. If \( x \in A_{\text{yes}} \), then the Prover sends the honest one-time pad of the witness \(|\psi_{x,r}\rangle\) that makes the Verifier of the QMA protocol accept with probability exponentially close to 1.

In this case, if the qubits of the randomly chosen \( \Pi_c \) are not included in \( S \), the Verifier always accepts, otherwise \( V \) accepts with probability exponentially close to 1, by the properties of \(|\psi_{x,r}\rangle\).

Let \( x \in A_{\text{no}}, (a,b) \) and \((S,a|S,b|S)\) be the values sent by the trusted party to the Prover and Verifier, respectively. Let also \( \rho \) be the quantum state sent by the Prover and \( \sigma = X^aZ^b \rho X^aZ^b \).

By definition, the acceptance probability of the protocol is then

\[
\frac{1}{m} \sum_{c} \text{Pr}[S_c \not\subseteq S] \text{Pr}[S_c \subseteq S] \text{Tr} \left( \Pi_c X^{a|S_c} Z^{b|S_c} \rho Z^{b|S_c} X^{a|S_c} \right) = \left( 1 - \frac{1}{n^k} \right) + \frac{1}{n^k} \text{Tr} \left( \frac{1}{m} \sum_{c} \Pi_c X^{a|S_c} Z^{b|S_c} \rho Z^{b|S_c} X^{a|S_c} \right) \]

Notice that

\[
\text{Tr} \left( \frac{1}{m} \sum_{c} \Pi_c X^{a|S_c} Z^{b|S_c} \rho Z^{b|S_c} X^{a|S_c} \right) = \text{Tr} \left( \frac{1}{m} \sum_{c} \Pi_c \sigma \right) \leq \max_{\tau} \text{Tr} \left( \frac{1}{m} \sum_{c} \Pi_c \tau \right) \leq \delta
\]

where the first equality holds since \( \Pi_c \) is acting only on the decoded values and the last inequality holds since \( x \in A_{\text{no}} \).

Therefore, the overall acceptance probability is at most \( 1 - \frac{1-n^{-\delta}}{n^k} \).

\[\Box\]

Let \( A = (A_{\text{yes}},A_{\text{no}}) \) to be a problem in \( k\)-SimQMA, \( x \in A_{\text{yes}} \), and and \( \{\rho^x_S\}_S \) be the set of local density matrices of a simulatable witness \( \tau \) for \( x \).

1. The simulator picks random values \( a, b \in \{0,1\}^\ell \) and \( S \subseteq [\ell] \)
2. Simulator computes the (constant-size) reduced density matrix \( \sigma \) of the qubits in positions \( S \) and let \( \sigma(S) = \rho^S \otimes I^S \)
3. Output \(|S\rangle\langle S| \otimes |a|S\rangle\langle a|S| \otimes |b|S\rangle\langle b|S| \otimes \sigma\).

Figure 7: Simulator for the QNIZK protocol

Lemma 6.4. The protocol is statistical zero-knowledge.

Proof. We show that the protocol is statistical zero-knowledge by showing the density matrices of the output of the simulator and the real protocol are close.

In the real protocol, let \(|\psi\rangle\) be the simulatable proof in the QMA protocol for a yes-instance. In the honest run of the protocol, we have that the view of the Verifier after the Prover sends the message is

\[
\frac{1}{2^{2^k}} \sum_{a,b,S} |S\rangle\langle S| \otimes |a|S\rangle\langle a|S| \otimes |b|S\rangle\langle b|S| \otimes \tilde{\psi}_{a,b} \tag{24}
\]
Notice that since we are averaging over all possible values of $a_S$ and $b_S$, Equation (24) is equal to
\[
\rho_p = \frac{1}{2^{2|S|}} \sum_{a_S, b_S, S} |Sangle\langle S| \otimes |a_S\rangle \langle a_S| \otimes |b_S\rangle \langle b_S| \otimes \left( \tilde{\psi}_{a_S, b_S, S} \otimes I^S \right),
\]
where $\tilde{\psi}_S = Tr_S(|\psi\rangle\langle \psi|)$.

By definition, the output of the simulator is
\[
\rho_s = \frac{1}{2^{2|S|}} \sum_{S, a_S, b_S} |Sangle\langle S| \otimes |a_S\rangle \langle a_S| \otimes |b_S\rangle \langle b_S| \otimes \left( \tilde{\rho}_{a_S, b_S, S} \otimes I^S \right).
\]

To conclude the proof, we have that
\[
D(\rho_p, \rho_s) \leq D(\psi_S, \sigma_S) \leq \text{negl}(n),
\]
where the first inequality holds since the trace distance is subadditive under tensor product and preserved under unitary operations. The second inequality follows from Definition 3.6.

Remark 6.5. In the cryptography literature, there is a notion called adaptive soundness and zero-knowledge where the witness is chosen after the trusted party provides the secret parameters (or CRS). We notice that our protocols can also handle these stronger notions.

6.1 Extension to QAM

In [KLGN19], the authors generalize both the complexity classes QMA to allow interaction between the Verifier and Prover to allow public randomness, both classical (i.e., classical coins) and quantum (i.e., sharing EPR pairs). In this framework, we consider the class QAM, where the verifier sends random coins to the prover, who then answers with a quantum state. We notice that in [KLGN19], this complexity class is called $\text{cqQAM}$.

Definition 6.6 (QAM). A promise problem $A = (A_{\text{yes}}, A_{\text{no}})$ is in QAM if and only if there exist polynomials $r$, $p$, $q$ and a polynomial-time uniform family of quantum circuits $\{Q_{r,n}\}$, where $Q_{r,n}$ takes as input a string $x \in \Sigma^*$ with $|x| = n$, a $p(n)$-qubit quantum state $|\psi\rangle$, and $q(n)$ auxiliary qubits in state $|0\rangle^{\otimes q(n)}$, such that:

Completeness: If $x \in A_{\text{yes}}$, $Pr_r[\exists |\psi_r\rangle$ s.t. $Q_n$ accepts $(x, |\psi\rangle)] \geq 2/3$.

Soundness: If $x \in A_{\text{no}}$, $Pr_r[\forall |\psi_r\rangle$ $Q_n$ accepts $(x, |\psi\rangle)] \leq 1/3$.

It is straightforward to generalize Definition 3.6 and define SimQAM where the POVMs and the reduced density matrices depend also in the public random string $r$. It is not hard to see that we can also generalize Lemma 3.7 and show that $\text{QAM} = \text{SimQAM}$.

In this case, our QNIZK protocol can be adapted to this complexity class, by just making the trusted party pick also the $r$ uniformly at random and sending it to both the Prover and the Verifier. Given a fixed $r$, the same arguments as shown for SimQMA hold.

Theorem 6.7. Every problem in SimQAM has a QNISZK in the secret parameters model.
A Computing the reduced density matrix of a history state

In this section we present a simpler version of the proof of Lemmas 15, 16 and 17 of [GSY19] tailored to our needs. More concretely, we show that the history state of the encoded computation is simulatable. We also add an extra remark that connects the output of the simulator and the Hamiltonian resulting from the circuit-to-Hamiltonian construction.

Let us recall some notation defined in the proof of Lemma 3.4. For some problem $A = (A_{\text{yes}}, A_{\text{no}})$ in QMA, let us fix some instance $x \in A$ and its corresponding QMA verification circuit $V_x$. $V_x$ acts on some witness $|\psi\rangle$ and auxiliary qubits. We also consider an $[N, 1, D]$ quantum error correcting code $C$ that is $3L$-simulatable, for some $L \geq 5$. We define $V = V'_x(C)$ as follows

1. For each auxiliary qubit, encode $|0\rangle$ under $C$
2. For every $T$-gate of $V_x$, create $|T\rangle$ and encode it under $C$
3. Check if the witness is encoded under $C$, and reject if this is not the case
4. Simulate each gate of $V$, either transversally, or using gadgets with the magic state
5. Decode the output bit, and accept or reject depending on its value.

We assume that $V$ also contains dummy identity gates between each one of the steps discussed before, and also in between the encoding of different qubit or the logical computation of different gates.

Let us define $T$ as the number of (physical) gates performed by $V$, $U_1, ..., U_T$ be such gates,

$$|\text{hist}\rangle = \frac{1}{\sqrt{T+1}} \sum_{t \in [T+1]} |\text{unary}(t)\rangle \otimes U_t \ldots U_1(Enc(|\psi\rangle))|0\rangle \otimes A,$$

be the history state of the computation.

For some interval $I$, we can define the type of $I$:

**Idling phase:** For every $t \in I$, $U_t = I$.

**Encoding phase:** max$(I)$ is before Step 3

**Logical phase:** The value max$(I)$ is after Step 3 and before Step 5

**Decoding phase:** max$(I)$ is after Step 5.

We now prove Lemma 3.5.

**Lemma 3.5 (restatement) [GSY19].** Let $C$ be a $[N, 1, D]$ quantum error correcting code that is a $3L$-simulatable code, for some $L \geq 5$. Then there exists a polynomial-time determinisitc algorithm $\text{Sim}_{V'_x(C)}$ such that for any $S \subseteq [W]$ with $|S| \leq L$, $\text{Sim}_{V'_x(C)}(S)$ outputs the classical description of an $|S|$-qubit density matrix $\rho_S$ such that

1. if $x$ is a yes-instance, then $||\rho_S - \text{Tr}_S(|\text{hist}\rangle\langle\text{hist}|)||_{tr} \leq \text{negl}(n)$
2. Let $S_i$ be the set of qubits on which $H_i$ acts non-trivially. Then $\text{Tr}(H_i \rho_{S_i}) = 0$. 

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Proof. Let
\[
|\Phi\rangle = \frac{1}{\sqrt{T+1}} \sum_{t \in [T+1]} |\text{unary}(t)\rangle |\Delta_t\rangle,
\]
for \(\Delta_t = \begin{cases} U_t \cdots U_1 (Enc(|\psi\rangle) |0\rangle^{\otimes A}), & \text{if } t \leq B \\ U_t \cdots U_{B+1} (Enc(|1\rangle |\phi_1\rangle)), & \text{otherwise} \end{cases} \)
where \(B\) is the index of the middle dummy identity gate performed between steps 4 and 5. We show now how to efficiently compute the reduced density matrices of \(|\Phi\rangle\) and this implies the first part of the lemma, since for \(x \in A_{\text{yes}}, |||\Phi\rangle - |\text{hist}\rangle||_2^2 \leq \text{negl}(n)\).

Let \(C_{tr} = S \cap [T + 1]\) denote the set of clock qubits that are not in \(S\). Notice that the reduced state \(\text{Tr}_{C_{tr}}(|\text{unary}(t)\rangle \langle \text{unary}(t')|)\) is non-zero only when \(t = t'\), or for all \(i \in C_{tr}\), either both \(t, t' > i\), or both \(t, t' < i\). In this case, all cross-terms of \(\text{Tr}_{C_{tr}}(|\Phi\rangle)\) involving times \(t, t'\) vanish whenever \(t \neq t'\) and there exists a \(i \in C_{tr}\) such that \(t < i\) and \(t' \geq i\) or vice-versa. Thus the only cross-terms that remain are times \(t, t'\) that come from an interval \(I \subseteq \{0, 1, 2, \ldots, T\}\) of consecutive time-steps where there is no \(i \in C_{tr}\) such that \(\min(I) \leq i \leq \max(I)\). Let \(\{0, 1, 2, \ldots, T\} \setminus C_{tr}\) be the union of maximal intervals \(I_1, I_2, \ldots, I_t\) of consecutive time steps. We have then that

\[
\text{Tr}_{C_{tr}}(|\Phi\rangle) = \sum_{t \in C_{tr}} \frac{1}{T+1} \text{Tr}_{C_{tr}}(|\Phi_{(t)}\rangle \langle \Phi_{(t)}|) + \sum_{j=1}^\ell \frac{|I_j|}{T+1} \text{Tr}_{C_{tr}}(|\Phi_{I_j}\rangle \langle \Phi_{I_j}|),
\]
where \(|\Phi_I\rangle = \frac{1}{\sqrt{|I|}} \sum_{t \in I} |\text{unary}(t)\rangle \otimes |\Delta_t\rangle\) for some set \(I \subseteq [T]\) (we also define \(\Phi_I = |\Phi_I\rangle \langle \Phi_I|\)).

Let us assume that we can compute \(\text{Tr}_{S}(|\Phi_I\rangle)\) for every interval \(I\) such that \(|I| \leq L\). Then we can compute the reduced density matrix of \(|\Phi\rangle\) on \(S\) as follows:

1. Compute the set \(C_{tr}\);
2. Compute the intervals \(I_1, \ldots, I_j\);
3. For each interval \(I_j\), compute the density matrix \(\rho_{I_j} = \text{Tr}_{S}(|\Phi_{I_j}\rangle)\);
4. For every \(t \in C_{tr}\) compute the density matrix \(\rho_t = \text{Tr}_{S}(|\Phi_{(t)}\rangle)\);
5. Output \(\sum_{t \in C_{tr}} \frac{1}{T+1} \rho_t + \sum_{j=1}^\ell \frac{|I_j|}{T+1} \rho_{I_j}\).

We only need to show now how to compute \(\text{Tr}_{S}(|\Phi_I\rangle)\) for every interval \(I = \{t_1, t_1 + 1, \ldots, t_2\}\) such that \(0 \leq t_2 - t_1 \leq L\).

Notice that
\[
\text{Tr}_{S}(|\Phi_I\rangle) = \frac{1}{|I|} \sum_{t, t' \in I} \text{Tr}_{S}(|\Delta_t\rangle \langle \Delta_{t'}|) \otimes \text{Tr}_{S}(|\Delta_{t}\rangle \langle \Delta_{t'}|),
\]
so we only need to show how to compute efficiently \(\text{Tr}_{S}(|\Delta_{t}\rangle \langle \Delta_{t'}|)\) for every \(t, t' \in I\), since \(\text{Tr}_{S}(|\Delta_{t}\rangle \langle \Delta_{t'}|)\) can be trivially computed.

Assume without loss of generality that \(t \leq t'\) and let \(G\) denote the union of the qubits that are acted upon by the gates \(U_t, \ldots, U_{t'}\). Since \(t' - t \leq |I| \leq L\), and each gate acts on at most 2 qubits, we get that \(|G| \leq 2L\). Now, we can write
\[
\text{Tr}_{S}(|\Delta_{t}\rangle \langle \Delta_{t'}|) = \text{Tr}_{S}(|\Delta_{t}\rangle \langle \Delta_{t'}| U_{t}^\dagger \cdots U_{t'}^\dagger) = \text{Tr}_{S \cup G} \left( \text{Tr}_{S \cup G} (|\Delta_{t}\rangle \langle \Delta_{t'}| U_{t}^\dagger \cdots U_{t'}^\dagger) \right) .
\]
Therefore, we just need to compute \(\text{Tr}_{S}(|\Delta_{t}\rangle \langle \Delta_{t'}|)\), where \(Y = S \cup G\), and the result follows. We notice that the number of qubits of this density matrix, \(|S \cup G|\), is at most \(3L\).

We now argue that the description of \(\text{Tr}_{S}(|\Delta_{t}\rangle \langle \Delta_{t'}|)\) can be efficiently computed for all \(t\). Since dummy identities were added between each phase of \(V\), we can prove this for each phase separately. Let \(t_0\) be the smallest timestep between \(t\) and the first dummy identity before it.
Idling phase During an idling phase, all qubits are either fully encoded or unencoded in some fixed (and known) state. The reduced density matrix \( \text{Tr}_Y (|\Delta_t\rangle\langle\Delta_t|) \) thus consists of either at most \(|Y|\) unencoded auxiliary qubits (in a known state), and the reduced density matrix of some encoded blocks on at most \(|Y| \leq 3L\) qubits. By Theorem 2.8, the reduced density matrix of the encoded blocks is efficiently computable, and thus \( \text{Tr}_Y (|\Delta_t\rangle\langle\Delta_t|) \) is efficiently computable.

Resource encoding In a resource encoding phase, a constant number of unencoded auxiliary qubits (in a known state) in \(|\Delta_{t_0}\rangle\) will be transformed into an encoded resource state in \(|\Delta_{\text{end}(t)}\rangle\), and the rest of the qubits are either in fully encoded block or unencoded auxiliary qubits. Thus the reduced density matrix \( \text{Tr}_Y (|\Delta_t\rangle\langle\Delta_t|) \) is a tensor product of the reduced density matrix of some encoded blocks (which is efficiently computable by Theorem 2.8), unencoded auxiliary qubits, and the reduced density matrix of the intermediate state of a resource encoding circuit acting on a constant number of auxiliary qubits (which is efficiently computable). Thus \( \text{Tr}_Y (|\Delta_t\rangle\langle\Delta_t|) \) is efficiently computable.

Logical operation In a logical operation phase, some logical gate is being applied either transversally or through a gadget using some magic state. In either case, we have that
\[
|\Delta_t\rangle = U_t \cdots U_{t_0} |\Delta_{t_0}\rangle,
\]
and all qubits of \(|\Delta_{t_0}\rangle\) are fully encoded. This corresponds to the simulation in the middle of the application of a logical gate, and again by Lemma 2.8, \( \text{Tr}_Y (|\Delta_t\rangle\langle\Delta_t|) \) can be efficiently computed.

Decoding If \( t \geq B - L \), we have \( t_0 = B \) and the state \(|\Delta_{t_0}\rangle\) can be written as a tensor product
\[
|\Delta_{t_0}\rangle = \text{Enc}(|1\rangle) \otimes \text{Enc}(|\phi_1\rangle).
\]
Therefore, the reduced density matrix \( \text{Tr}_Y (|\Delta_t\rangle\langle\Delta_t|) \) is a tensor product of the reduced density matrix of a decoding circuit acting on \( \text{Enc}(|1\rangle) \) (which is efficiently computable) and the reduced density matrix of \(|\Sigma_{t_0}\rangle\) on at most \(|Y| \leq 3L\) qubits (which is efficiently computable by Theorem 2.8).

The second part of the lemma can be proved by inspection: all auxiliary qubits are initialized to 0 at step \( t = 0 \) and the output is always hardcoded to 1 at step \( t = T \). For the propagation steps, notice that all cross terms of the consecutive steps in the reduced density matrix are consistent, so the energy for the propagation terms is also 0. We notice that if \( x \) is a no-instance, there is no global history state consistent with the local density matrices, but the fake density matrices can be computed as well.

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