INVERSE MONOIDS AND IMMERSIONS OF ∆-COMPLEXES

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Abstract. An immersion \( f : D \to C \) between ∆-complexes is a ∆-map that induces injections from star sets of \( D \) to star sets of \( C \). We study immersions between finite-dimensional connected ∆-complexes by replacing the fundamental group of the base space by an appropriate inverse monoid. We show how conjugacy classes of the closed inverse submonoids of this inverse monoid may be used to classify connected immersions into the complex. This extends earlier results of Margolis and Meakin for immersions between graphs and of Meakin and Szakács on immersions into 2-dimensional CW-complexes.

1. Introduction

The notion of immersion arises from differential geometry: it is a differentiable function between differentiable manifolds whose derivative is everywhere injective. An immersion is essentially a local smooth embedding: a typical example is the immersion of the Klein bottle into 3-space — it is not an embedding, but it is a local embedding, which suffices for the purpose of visualization.

In the absence of a differentiable structure, one can define a topological notion of immersion called a topological immersion, that is, a continuous map which is a local homeomorphism onto its image. Every immersion between differentiable structures is a topological immersion. In our paper, we consider a combinatorial notion of immersions between connected, finite-dimensional ∆-complexes, which are, as usual in this setting, also assumed to take \( k \)-cells to \( k \)-cells and commute with the characteristic maps. This definition is the higher-dimensional generalization of graph immersions in the sense of Stallings [12], which have played a significant role in the study of subgroups of free groups. Every topological immersion is an immersion in this sense and for ∆-maps between locally compact ∆-complexes, the converse is true, which we prove in the Appendix.

Covering maps are also immersions, and unlike immersions, for “nice enough” topological spaces, they are very well understood by means of the fundamental group of the base space. This classification makes use of the fact that every path in the base space lifts uniquely to any given point in the preimage of its initial point; furthermore, homotopic paths lift to homotopic paths. It follows that in order to characterize covering maps over a space, it is sufficient to know which closed paths lift to closed paths under the covering map, and this is encoded by the fundamental groups of the two spaces.

In the case of immersions, there is no unique path lifting in the sense described above; there is however a unique partial path lifting, that is, paths lift partially, and this partial lifting is unique at every point. Furthermore, even when they do lift, homotopic paths do not necessarily lift to homotopic paths, so homotopy is not the correct equivalence relation to encode immersions. In order to obtain a similar characterization to the classification of covers, we need an algebraic structure rich enough to differentiate between non-homotopic paths, and to encode not just when closed paths lift to closed paths, but also when paths lift at all. This seems difficult in general, but manageable at least for certain classes of cell complexes that have a strong enough combinatorial structure, in particular for ∆-complexes and for immersions between these complexes that respect the cell structure.

In the cell complex setting, there is a unique partial lifting of cells of arbitrary dimension and there are complex interrelationships between liftings of cells of different dimensions. We need to encode...
this information algebraically. For cell complexes of dimensions 1 and 2 this has been accomplished by introducing an inverse monoid that serves the role of the fundamental group (see the papers [5] for graphs and [6] for 2-dimensional complexes). However, in the case of higher dimensional cell complexes this requires significant additional care.

In this paper, we introduce the notion of a generalized path, which is essentially a sequence of connecting cells of any dimension. We then define an equivalence relation on generalized paths that serves the role of homotopy in the theory of immersions. We construct an inverse monoid of equivalence classes of generalized paths which we call a loop monoid, and prove that loop monoids encode immersions between $\Delta$-complexes in the way the fundamental group encodes coverings.

Loop monoids are defined in Section 3: the loop monoid of a complex $\Delta$ is a disjoint set in one-one correspondence with the set of edges of $\Delta$. One of the basic notions of inverse monoids and immersions between $\Delta$-complexes that we will need is listed in the following proposition.

**Proposition 2.1.** Let $M$ be an inverse monoid. An inverse monoid is a monoid $M$ with the property that for each $a \in M$ there is a unique element $a^{-1}$ (the inverse of $a$) in $M$ such that $a = aa^{-1}a$ and $a^{-1} = a^{-1}aa^{-1}$.

Inverse monoids arise naturally in the study of partial symmetry in mathematics in much the same way as groups arise in the study of symmetry. In fact the Wagner-Preston Theorem states that every inverse monoid embeds in an appropriate symmetric inverse monoid SIM($Q$), i.e. the monoid of all bijections between subsets of the set $Q$ under the usual composition of partial maps. For this and many additional properties of inverse monoids and their connections with other fields of mathematics, we refer the reader to the book of Lawson [5]. Some of the most basic properties of inverse monoids that we will need are listed in the following proposition.

**Proposition 2.2.** Let $M$ be an inverse monoid with set $E(M) = \{e \in M : e = e^2\}$ of idempotents. Then $E(M)$ is nonempty, and the following hold:

- The idempotents of $M$ commute, i.e. $ef = fe$ for all $e, f \in E(M)$. Thus the set $E(M)$ of idempotents forms a lower semilattice with respect to $e \wedge f = ef$. In particular, $g \leq ef$ if and only if $g \leq e$ and $g \leq f$ for any $e, f, g \in E(M)$.
- The relation defined on $M$ by $a \leq b$ iff there exists $e \in E(M)$ is a partial order on $M$, called the natural partial order on $M$. The natural partial order is compatible with the multiplication and inversion operations in $M$.
- Let $\sigma_M$ be a relation on $M$ such that $a \sigma_M b$ iff there exists $c \in M$ with $c \leq a$ and $c \leq b$. This is a congruence on $M$, called the minimum group congruence on $M$. The quotient $M/\sigma_M$ is a group, the greatest group homomorphic image of $M$. Equivalently, $\sigma_M$ is generated as a congruence by pairs of the form $(aa^{-1}, 1)$, where $a \in M$.

Inverse monoids also arise naturally as transition monoids of inverse automata, which are automata whose underlying graphs are edge labeled over an alphabet $X \cup X^{-1}$ in the sense described below.

Let $X$ be a set and $X^{-1}$ a disjoint set in one-one correspondence with $X$ via a map $x \rightarrow x^{-1}$ and define $(x^{-1})^{-1} = x$. We extend this to a map on $(X \cup X^{-1})^*$ by defining $(x_1x_2 \cdots x_n)^{-1} = x_n^{-1} \cdots x_2^{-1}x_1^{-1}$, giving $(X \cup X^{-1})^*$ the structure of the free monoid with involution on $X$. Throughout this paper by an $X$-graph (or just an edge-labeled graph if the labeling set $X$ is understood) we mean a strongly connected digraph $\Gamma$ with edges labeled over the set $X \cup X^{-1}$ such that the labeling is consistent with an involution: that is, there is an edge labeled $x \in X \cup X^{-1}$ from vertex $v_1$ to vertex $v_2$ if and only if there is an inverse edge labeled $x^{-1}$ from $v_2$ to $v_1$. The initial vertex of an edge $e$...
will be denoted by $\alpha(e)$ and the terminal vertex by $\omega(e)$. The $X$-graph $\Gamma$ with one vertex and one positively labeled edge labeled by $x$ for each $x \in X$ is referred to as the bouquet of $|X|$ circles and is denoted by $B_X$. If $X = \emptyset$, then we view $\Gamma$ as the graph with one vertex and no edges.

The label on an edge $e$ is denoted by $\ell(e) \in X \cup X^{-1}$. There is an evident notion of path in an $X$-graph. The initial (resp. terminal) vertex of a path $p$ will be denoted by $\alpha(p)$ (resp. $\omega(p)$). The label on the path $p = e_1 e_2 \ldots e_k$ is the word $\ell(p) = \ell(e_1) \ell(e_2) \ldots \ell(e_k) \in (X \cup X^{-1})^*$. $X$-graphs occur frequently in the literature. The Cayley graph $\Gamma(G, X)$ of a group $G$ relative to a set $X$ of generators is an $X$-graph: its vertices are the elements of $G$ and it has an edge labeled by $x$ from $g$ to $gx$ for each $x \in X \cup X^{-1}$.

If we designate an initial vertex (state) $\alpha$ and a terminal vertex (state) $\beta$ of $\Gamma$, then the birooted $X$-graph $\mathcal{A} = (\alpha, \Gamma, \beta)$ may be viewed as an automaton. See for example the book of Hopcroft and Ullman [2] for basic information about automata theory. The language accepted by this automaton is the subset $L(A)$ of $(X \cup X^{-1})^*$ consisting of the words in $(X \cup X^{-1})^*$ that label paths in $\Gamma$ starting at $\alpha$ and ending at $\beta$. This automaton is called an inverse automaton if it is deterministic (and hence co-deterministic), i.e. if for each vertex $v$ of $\Gamma$ there is at most one edge with a given label starting at $v$ or ending at $v$. This also implies that any path is uniquely determined by its initial vertex and its label. A deterministic $X$-graph can also be defined as an edge-labeled graph $\Gamma$ that admits a label-preserving graph morphism to $B_X$ which, for any vertex $v$ of $\Gamma$, is injective on the set of edges $e$ with $\alpha(e) = v$. Such maps are called graph immersions in [12].

For each subset $N$ of an inverse monoid $M$, we denote by $N^\omega$ the set of all elements $m \in M$ such that $m \geq n$ for some $n \in N$. The subset $N$ of $M$ is called closed if $N = N^\omega$.

Closed inverse submonoids of an inverse monoid $M$ arise naturally in the representation theory of $M$ by partial injections on a set [11]. An inverse monoid $M$ acts (on the right) by injective partial functions on a set $Q$ if there is a homomorphism from $M$ to $\text{SIM}(Q)$. Denote by $q.m$ the image of $q$ under the action of $m$ if $q$ is in the domain of the action by $m$. The following basic fact is well known (see [11]).

**Proposition 2.2.** If an inverse monoid $M$ acts on $Q$ by injective partial functions, then for every $q \in Q$, $\text{Stab}(q) = \{m \in M : q.m = q\}$ is a closed inverse submonoid of $M$.

Conversely, given a closed inverse submonoid $H$ of $M$, we can construct a transitive representation of $M$ as follows. A subset of $M$ of the form $(Hm)^\omega$ where $mm^{-1} \in H$ is called a right $\omega$-coset of $H$. Let $X_H$ denote the set of right $\omega$-cosets of $H$. If $m \in M$, define an action on $X_H$ by $Y.m = (Ym)^\omega$ if $(Ym)^\omega \in X_H$ and undefined otherwise. This defines a transitive action of $M$ on $X_H$ with $\text{Stab}(H) = H$. Conversely, if $M$ acts transitively on $Q$, then this action is equivalent in the obvious sense to the action of $M$ on the right $\omega$-cosets of $\text{Stab}(q)$ in $M$ for any $q \in Q$. See [11] or [10] for details.

The $\omega$-coset graph $\Gamma(H, X)$ (or just $\Gamma_H$ if $X$ is understood) of a closed inverse submonoid $H$ of an $X$-generated inverse monoid $M$ is constructed as follows. The set of vertices of $\Gamma_H$ is $X_H$ and there is an edge labeled by $x \in X \cup X^{-1}$ from $(Ha)^\omega$ to $(Hb)^\omega$ if $(Hb)^\omega = (Hax)^\omega$. Then $\Gamma_H$ is a deterministic $X$-graph. The birooted $X$-graph $(H, \Gamma_H, H)$ is called the $\omega$-coset automaton of $H$. The language accepted by this automaton is $H$ (or more precisely the set of words $w \in (X \cup X^{-1})^*$ whose natural image in $M$ is in $H$). Clearly, if $G$ is a group generated by $X$, then $\Gamma_H$ coincides with the coset graph of the subgroup $H$ of $G$.

We call two closed inverse submonoids $H_1, H_2$ of an inverse monoid $M$ conjugate if there exists $m \in M$ such that $mH_1m^{-1} \subseteq H_2$ and $m^{-1}H_2m \subseteq H_1$. It is clear that conjugacy is an equivalence relation on the set of closed inverse submonoids of $M$: however, conjugate closed inverse submonoids of an inverse monoid are not necessarily isomorphic. For example, the closed inverse submonoids $\{1, aa^{-1}, a^2a^{-2}\}$ and $\{1, aa^{-1}, a^{-1}a, aa^{-2}a\}$ of the free inverse monoid on the set $\{a\}$ are conjugate by $a^{-1}$ but not isomorphic.

Here we note that since inverse monoids form a variety of algebras (in the sense of universal algebra — i.e. an equationally defined class of algebras), free inverse monoids exist. We will denote the free inverse monoid on a set $X$ by $\text{FIM}(X)$. This is the quotient of $(X \cup X^{-1})^*$, the free monoid with involution, by the congruence that identifies $ww^{-1}w$ with $w$ and $ww^{-1}ww^{-1}$ with $uu^{-1}uu^{-1}$ for
all words $u, w \in (X \cup X^{-1})^*$. See [11] or [3] for much information about FIM($X$). In particular, [11] and [3] provide an exposition of Munn’s solution [9] to the word problem for FIM($X$) via birooted edge-labeled trees called Munn trees.

In his thesis [13] and paper [14], Stephen initiated the combinatorial theory of presentations of inverse monoids by extending Munn’s results about free inverse monoids to arbitrary presentations of inverse monoids. We refer the reader to [14] or our paper [6] for details of Stephen’s construction of Schützenberger graphs and Schützenberger automata and their use in the study of presentations of inverse monoids.

We recall that an inverse category is a category $C$ with the property that for every morphism $p$ in $C$ there is a unique inverse morphism $p^{-1}$ such that $p = pp^{-1}p$ and $p^{-1} = p^{-1}pp^{-1}$. Note this implies $\alpha(p) = \omega(p^{-1})$ and $(p^{-1})^{-1} = p$. In this paper, the inverse categories we consider will consist of certain equivalence classes of paths in graphs, thus the domain of $p$ is denoted by $\alpha(p)$, the codomain by $\omega(p)$, and we multiply morphisms from left to right. The loop monoids $L(C, v)$ of an inverse category, that is, the set of all morphisms from $v$ to $v$, where $v$ is an arbitrary vertex, are inverse monoids.

Analogously to inverse semigroups, a natural partial order can be defined on the morphisms of each inverse category $C$ by putting $p \leq q$ if $p = qe$ for some idempotent $e$. Note that in this case the morphisms $p$ and $q$ are necessarily coterminal. Each inverse category $C$ has a greatest groupoid homomorphic image, obtained by identifying two morphisms $p$ and $q$ if and only if there exists a morphism $t$ in $C$ such that $t \leq p$ and $t \leq q$ in the natural partial order on $C$, and in this case $p$ and $q$ are both coterminal with $t$ and thus with each other. Alternatively, the morphisms identified are precisely the pairs in the congruence generated by $pp^{-1} \approx 1_{\alpha(p)}$, where $p$ is any morphism.

**Proposition 2.3.** Let $C$ be an inverse category and $v$ a vertex in $C$. Then the maximal group homomorphic image of $L(C, v)$ is isomorphic to the vertex group of $v$ in the maximal groupoid image of $C$.

**Proof.** Two loops $p$ and $q$ at $v$ in $C$ are identified in the maximal groupoid image of $C$ if and only if they have a common lower bound $cp = cq$ in the natural partial order, and that lower bound must be in $L(C, v)$ since idempotents in $C$ are loops. Thus $p$ and $q$ are identified in the maximal groupoid image of $C$ if and only if they are identified in the maximal group image of the loop monoid $L(C, v)$.

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2.2. **$CW$-complexes and $\Delta$-complexes.** As our results build on previous, analogous results for $CW$-complexes [6], we regard $\Delta$-complexes as $CW$-complexes with special attaching maps, following the definition in [4]. Recall the following definition of a finite dimensional $CW$-complex $C$:

1. Start with a discrete set $C^0$, the 0-cells of $C$.
2. Inductively, form the $n$-skeleton $C^n$ from $C^{n-1}$ by attaching $n$-cells $C^n_\tau$ via attaching maps $\varphi_\tau: S^{n-1} \to C^{n-1}$. This means that $C^n_\tau$ is the quotient space of $C^{n-1} \sqcup B^n_\tau$ under the identifications $x \sim \varphi_\tau(x)$ for $x \in \partial B^n_\tau$. The cell $C^n_\tau$ is a homeomorphic image of $B^n_\tau - \partial B^n_\tau$ under the quotient map. (Here $B^n$ is the unit ball in $\mathbb{R}^n$ and $S^{n-1} = \partial B^n$ is its boundary).
3. Stop the inductive process after a finite number of steps to obtain a finite dimensional $CW$-complex $C$.

The dimension of the complex is the largest dimension of one of its cells. Note that a 1-dimensional $CW$-complex is just an undirected graph, with the usual topology. We denote the set of $n$-cells of $C$ by $C^{(n)}$. Each cell $C^n_\tau$ is open in the topology of the $CW$-complex $C$. A subset $A \subseteq C$ is open iff $A \cap C^n$ is open in $C^n$ for each $n$. We emphasize that each cell of the complex is an open cell by definition.

Each cell $C^n_\tau$ has a characteristic map $\sigma_\tau$, which is defined to be the composition $B^n_\tau \hookrightarrow (C^{n-1} \sqcup B^n_\tau) \rightarrow C^n \rightarrow C$. This is a continuous map whose restriction to the interior of $B^n_\tau$ is a homeomorphism onto $C^n_\tau$ (equipped with a subspace topology) and whose restriction to the boundary of $B^n_\tau$ is the corresponding attaching map $\varphi_\tau$. An alternative way to describe the topology on $C$ is to note that a subset $A \subseteq C$ is open iff $\sigma^{-1}_\tau(A)$ is open in $B^n_\tau$ for each characteristic map $\sigma_\tau$.

The standard $n$-simplex is the set...
\[ \Delta^n = \{(t_0, ..., t_n) \in \mathbb{R}^{n+1} : \Sigma t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i\}. \]

We denote the \( n+1 \) vertices of \( \Delta^n \) by \( v_i = (0, \ldots, 0, 1, 0, \ldots, 0) \) (1 in \( i \)-th position). We order vertices by \( v_i < v_j \) if \( i < j \). The faces of the simplex are the subsimplices with vertices any non-empty subset of the \( v_i \)'s. There are \( n+1 \) faces of dimension \( n-1 \), namely the faces \( \Delta_i^{n-1} = [v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n] \) for \( i = 0, 1, \ldots, n \) spanned by omitting one vertex.

A \( \Delta \)-complex is a quotient space of a collection of disjoint simplices obtained by identifying certain of their faces via the canonical linear homeomorphisms that preserve the ordering of vertices.

Equivalently, a \( \Delta \)-complex is a \( CW \)-complex \( X \) in which each \( n \)-cell \( e^n_\alpha \) has a characteristic map \( \sigma_\alpha : \Delta^n \to X \) such that the restriction of \( \sigma_\alpha \) to each \( (n-1) \)-dimensional face of \( \Delta^n \) is the characteristic map for an \((n-1)\)-cell of \( X \).

The order on the vertices of the simplex makes it naturally possible to regard each \( k \)-cell \( C_k^\sigma \) of a \( \Delta \)-complex \( C \) as a rooted cell, with distinguished root the image under the characteristic map \( \sigma_k^\tau \) of the minimal 0-cell in the order on 0-cells in \( \Delta^k \). We will denote the root of the cell \( C \) by \( \alpha(C) \).

Thus we may regard the 1-skeleton as a digraph with each 1-cell (edge) \( e \) directed from its initial vertex (the root of the cell) to its terminal vertex \( \omega(e) \) (the image of the maximal 0-cell of \( \Delta^1 \) under the characteristic map). It is also convenient to introduce a ghost edge \( e^{-1} \) for each 1-cell (edge) \( e \) of the 1-skeleton of \( C \). Here \( \alpha(e^{-1}) = \omega(e), \omega(e^{-1}) = \alpha(e) \) and the set \( \{e^{-1} : e \in C^{(1)}\} \) is assumed to be disjoint from and in one-one correspondence with \( C^{(1)} \). We extend the notation by defining \( (e^{-1})^{-1} = e \) for each \( e \in C^{(1)} \). For technical reasons, if \( C \) is a cell of at least 2 dimensions, we define \( \omega(C) = \alpha(C) \).

In the sequel, we will further assume all complexes to be connected.

2.3. Immersions between \( \Delta \)-complexes. We call a map \( f : D \to C \) between \( \Delta \)-complexes a \( \Delta \)-map if for each \( k \)-cell \( D_k^k \) of \( D \), \( f(D_k^k) \) is a \( k \)-cell of \( C \) and \( f \) commutes with the characteristic maps; i.e., \( f \circ \sigma_k^D = \gamma_k^C \) where \( \sigma_k^D \) is the characteristic map of \( D_k^k \) and \( \gamma_k^C \) is the characteristic map of \( f(D_k^k) \). The notion of a \( \Delta \)-map comes from regarding \( \Delta \)-complexes as geometric realizations of \( \Delta \)-sets (also called semi-simplicial sets). Such maps are also known to be continuous [7]. Also note that their restrictions to (open) cells are homeomorphisms onto their images and they map the root of a cell to the root of its image.

For a vertex \( v \) of a \( \Delta \)-complex \( C \) we define \( \text{star}_C(v) = \{e : e \text{ is an edge or a ghost edge or a cell of dimension } k \geq 2 \text{ in } C \text{ with } \alpha(e) = v\} \). Then a \( \Delta \)-map \( f : D \to C \) induces a map \( f_v : \text{star}_D(v) \to \text{star}_C(f(v)) \) in the obvious way.

**Definition 2.4.** An immersion from a \( \Delta \)-complex \( D \) to a \( \Delta \)-complex \( C \) is a \( \Delta \)-map \( f : D \to C \) for which each induced map \( f_v : \text{star}_D(v) \to \text{star}_C(f(v)) \) is injective.

We remark that for graphs (1-dimensional complexes) this concept coincides with the definition of immersion between graphs in the sense of Stallings [12]. In the Appendix, we show that an immersion is precisely a locally injective \( \Delta \)-map. Hence topological immersions are immersions in the sense of Definition 2.4 and in the locally compact case, the converse also holds.

3. Generalized paths

A path in a \( \Delta \)-complex \( C \) is a path on its 1-skeleton in the graph theoretic sense, where we allow edges to be read in either direction. Recall that the 1-skeleton can be regarded as a digraph since the edges of a simplex are oriented from vertices with smaller indices to vertices with larger indices. Also each edge \( e \) comes equipped with a ghost edge \( e^{-1} \). A path then is a sequence of directed edges and ghost edges \( e_1 e_2 ... e_n \) with \( \omega(e_i) = \alpha(e_{i+1}) \). We also consider empty paths \( 1_v \) around every 0-cell \( v \).

A generalized path is a sequence \( e_1 e_2 ... e_n \) where each \( e_i \) is either a \( k \)-cell with \( k \geq 1 \) or a ghost edge, and \( \omega(e_{i-1}) = \alpha(e_i) \) for \( i = 2, ..., n-1 \). (Recall that \( \omega(e) = \alpha(e) \) if \( e \) is a \( k \)-cell with \( k \geq 2 \).) If \( p = e_1 ... e_n \), then we define \( \alpha(p) = \alpha(e_1) \) and \( \omega(p) = \omega(e_n) \). We define \( p^{-1} \) to be \( e_n^{-1} ... e_2^{-1} \), where

\[
\epsilon_i = \begin{cases} 
-1, & \text{if } e_i \text{ is a 1-cell or a ghost edge} \\
1, & \text{otherwise.}
\end{cases}
\]
Note that $p^{-1}$ is a generalized path from $\omega(p)$ to $\alpha(p)$, and $(p^{-1})^{-1} = p$. The length of a generalized path $p = e_1e_2...e_n$ is $n$, the length of $p = 1_v$ is 0. We will denote the length of $p$ by $|p|$. In order to define the equivalence relation on generalized paths that will give rise to loop monoids, we need the notion of boundary paths. If $\Delta^k = [v_0, v_1, ..., v_k]$, then $\Delta^k$ has $k + 1$ faces of dimension $k - 1$, namely the $(k - 1)$-simplices $\Delta^{k-1}_i$ for $i = 0, ..., k$. All of these faces except $\Delta^{k-1}_0$ contain the vertex $v_0$. The smallest vertex of $\Delta^{k-1}_0$ under the order on vertices is $v_1$.

If $C$ is a $\Delta$-complex and $C_k$ is a $k$-cell of $C$, there is a corresponding characteristic map $\sigma^k: \Delta^k \to C$. The restriction of $\sigma^k$ to $\Delta^{k-1}_i$ is a characteristic map $\sigma^{k-1}_i$ of some $(k - 1)$-dimensional cell $C^{k-1}_i$ of $C$, by definition of a $\Delta$-complex. The root of $C_k$ is $\alpha(C_k) = \sigma^k(v_0)$, and the root of $C^{k-1}_i$ is also $\sigma^k(v_0)$ if $i \neq 0$, but the root of $C^{k-1}_0$ is $\sigma^k(v_1)$. Thus the 1-cell $\sigma^k([v_0, v_1])$ is a directed edge in the 1-skeleton of $C$ from the root of $C_k$ to the root of $C^{k-1}_0$.

For a 2-cell $C^2$ of a $\Delta$-complex $C$, the boundary path\footnote{This is called boundary walk in \cite{Meakin:1994}} $bp(C^2)$ of $C^2$ is the image in $C$ of the path $[v_0, v_1][v_1, v_2][v_2, v_0]^{-1}$ in the 1-skeleton of $\Delta^2 = [v_0, v_1, v_2]$ under the corresponding characteristic map from $\Delta^2$ to $C$. For a $k$-cell $C^k$ of $C$, denote the image under $\sigma^k$ of the 1-cell $[v_0, v_1]$ of $\Delta^k$ by $e(C^k)$. For $k \geq 3$, the boundary path $bp(C^k)$ of $C^k$ is the generalized path $C^{k-1}_kC^{k-1}_{k-1}...C^{k-1}_1e(C^2)C^{k-1}_0(e(C^2))^{-1}$.

3.1. The loop monoid $L(C, v)$. We define a category $FC(C)$ with objects $C^0$ and morphisms generalized paths of $C$ — this is the analogue of the free category on a graph. The domain of $p$ is its initial vertex and its codomain its terminal vertex, thus, as mentioned, $\alpha(p)$ and $\omega(p)$ are unambiguous. Let $\sim$ denote the congruence on $FC(C)$ generated by the relations $p \sim pp^{-1}p$, and $pp^{-1}qq^{-1} \sim qq^{-1}pp^{-1}$ for all generalized paths $p = (p, q)$, and the additional relations $CC \sim C$ and $C \sim Cbp(C)$ for all $C \in C^{(k)}, k \geq 2$. Let $IC(C)$ denote the category $FC(C)/\sim$. Note that $IC(C)$ is an inverse category. We define the loop monoid of $C$ at $v \in C^0$ to be the inverse monoid $L(IC(C), v)$. We denote it by $L(C, v)$. This consists of the $\sim$-classes of generalized paths around $v$.

A $\Delta$-map $f: \mathcal{D} \to C$ induces a map $f: FC(\mathcal{D}) \to FC(C)$ defined on the objects as before, and on the morphisms by $f(e_1...e_n) = f(e_1)...f(e_n)$ for any non-empty generalized path $e_1...e_n$, and $f(1_v) = 1_{f(v)}$ for $v \in \mathcal{D}^{(0)}$. Note that the induced map is a functor, i.e. $\alpha(f(p)) = f(\alpha(p))$, $\omega(f(p)) = f(\omega(p))$, and $f(p)f(q) = f(pq)$ for any connecting generalized paths $p$ and $q$ in $\mathcal{D}$. Notice furthermore that $f(p)^{-1} = f(p^{-1})$ and $f(bp(C)) = bp(f(C))$ holds for any generalized path $p$ and any cell $C$ of dimension greater than 1. This implies that $f$ respects the relations generating $\sim$, and hence whenever $p \sim q$ we have $f(p) \sim f(q)$. Thus $f$ induces a functor $f: IC(\mathcal{D}) \to IC(C)$, a fact we will use frequently.

The following lemma formalizes the statement that when paths lift along immersions, they lift uniquely.

**Lemma 3.1.** If $f: \mathcal{D} \to C$ is an immersion and $p, q$ are generalized paths in $\mathcal{D}$ with $f(p) = f(q)$ and $\alpha(p) = \alpha(q)$, then $p = q$.

**Proof.** Note that $f(p) = f(q)$ implies $|p| = |q| = k$. We prove the lemma by induction on $k$. If $k = 0$, then by $\alpha(p) = \alpha(q)$ we have $p = q$. Assume that $k \geq 1$, and let $p = p'p_e, q = q'q_e$ where $p_e, q_e$ are cells or ghost edges. Note that we have $f(p') = f(q')$ and $\alpha(p') = \alpha(q')$ by the assumption that $f(p) = f(q)$ and $\alpha(p) = \alpha(q)$, and thus by induction $p' = q'$. Therefore $\alpha(p_e) = \omega(p') = \omega(q') = \alpha(q_e)$, and denoting this common vertex by $v$ we have $e_p, e_q \in star_\mathcal{D}(v)$. It follows from the assumption we have $f(e_p) = f(e_q)$ and so we obtain $e_p = e_q$ from star-injectivity. This proves $p = q$. \hfill $\square$

**Proposition 3.2.** For any vertex $v$ in a (connected) $\Delta$-complex $C$, the greatest group homomorphic image of $L(C, v)$ is the fundamental group of $C$ at $v$.

**Proof.** Recall that the fundamental groupoid of $C$ is obtained by factoring the category of paths (on the 1-skeleton) with the congruence generated by relations of the form $pp^{-1} \approx 1_{\alpha(p)}$ for any path $p$, and $bp(C) \approx 1$ for any 2-cell $C$. Observe that this is the greatest groupoid homomorphic image of $IC(C)$, that is, it is obtained by factoring $IC(C)$ with the congruence generated by relations of the
form $qq^{-1} \approx 1_{\alpha(q)}$ for any generalized path $q$ (as this implies $C \approx 1_{\alpha(C)}$ when $C \in C^{(k)}, k \geq 2$). It follows by Proposition 3.2 that the fundamental group $\pi_1(C, v)$ is the greatest group homomorphic image of the corresponding loop monoid $L(C, v)$.

4. Labeled $\Delta$-complexes

In this section, we introduce a way to assign labels to all cells of a $\Delta$-complex in a way that cells with the same root have different labels, and boundary paths of cells with the same label also have the same label. This will allow us to identify generalized paths with their initial vertex and label, and to think of elements $L(C, v)$ as equivalence classes of labels. Labeled $\Delta$-complexes are higher dimensional analogues of $X$-graphs, and, like $X$-graphs, are defined via an immersion into a $\Delta$-complex with one 0-cell.

**Lemma 4.1.** Every $\Delta$-complex $C$ admits an immersion into a $\Delta$-complex with one 0-cell.

**Proof.** If we identify all 0-cells of $C$, then the quotient cell complex is also a $\Delta$-complex $B$ with one 0-cell. The corresponding map $f: C \rightarrow B$ is an immersion (in fact a topological immersion) since it is injective on all $k$-cells with $k > 0$.

Let $B$ be a $\Delta$-complex of dimension $n$ with one 0-cell. Let $\{e^1_\rho : \rho \in X\}$ be its set of 1-cells, $\{e^k_\rho : \rho \in P_k\}$ its set of $k$-cells for $2 \leq k \leq n$, and let $\beta^k_\rho : \Delta^k \rightarrow B$ be the characteristic map of $e^k_\rho$ for $k \geq 1$. Here we assume that the sets $X, P_k$ are all mutually disjoint. We denote this $\Delta$-complex $B$ by $B(X, P_2, \ldots, P_n, \{\beta^k_\rho\})$, or more briefly by $B(X, P)$ where $P = P_2 \cup \cdots \cup P_n$. We view $X$ as a set of labels for the 1-cells of $B(X, P)$ and $P_k$ as a set of labels of the $k$-cells of $B(X, P)$ for $2 \leq k \leq n$.

That is, the label on the $k$-cell $e^k_\rho$ is $\ell(e^k_\rho) = \rho$.

If $B$ is one dimensional, we obtain a bouquet of circles, usually denoted by $B_X$. One should think of $B(X, P)$ as a generalization. The 1-skeleton of $B(X, P)$ is $B_X$, which we regard as an $X$-graph as usual; i.e. each edge labeled by $x \in X$ is equipped with an inverse (ghost) edge labeled by $x^{-1}$. The labeling on the 1-cells of $B(X, P)$ extends to a labeling on paths in the 1-skeleton of $B(X, P)$ in the obvious way. The label on a path $p$ in $B(X, P)$ will be denoted by $\ell(p)$: thus $\ell(p) \in (X \cup X^{-1})^*$. More generally we may extend the labeling on cells to a labeling on generalized paths in the obvious way: if $e_1e_2\cdots e_t$ is a generalized path, then $\ell(e_1e_2\cdots e_t) = \ell(e_1)\ell(e_2)\cdots\ell(e_t) \in (X \cup X^{-1})^*$. The label of an empty path is the empty word.

**Definition 4.2.** Let $f: C \rightarrow B(X, P)$ be an immersion between $\Delta$-complexes and define the label of a cell $C^k_\rho \in C^{(k)}$ with $k \geq 1$ by $\ell(C^k_\rho) = \ell(f(C^k_\rho))$. We say that $C$ is labeled over the complex $B(X, P)$ via the immersion $f$.

Note that $\ell(C^1_\rho) \in X$ and $\ell(C^k_\rho) \in P_k$ if $2 \leq k \leq n$, and cells of $C$ have the same label if and only if they have the same image under $f$. If the underlying complex $B(X, P)$ is understood, we just say that $C$ is a labeled complex. In particular every complex $B(X, P)$ is labeled via the identity map on itself.

By Lemma 4.1, every $\Delta$-complex admits some labeling. The immersion $f$ constructed in the proof of that lemma assigns different labels to all cells of $C$. Of course we would usually choose smaller sets $X, P_k$ as sets of labels for the cells of $C$ if possible.

**Remark 4.3.** Notice that if $p$ and $q$ are generalized path of a labeled $\Delta$-complex $C$ with $\alpha(p) = \alpha(q)$ and $\ell(p) = \ell(q)$, by Lemma 3.4 applied to the immersion inducing the labeling, we have $p = q$. In particular if $C^k_\rho$ and $C^*_{\tau}$ are distinct cells of a labeled $\Delta$-complex $C$ with the same root $v$ then $\ell(C^k_\rho) \neq \ell(C^*_{\tau})$. Furthermore, if $e_1$ and $e_2$ are distinct 1-cells of a $\Delta$-complex with the same terminal vertex, then $\ell(e_1) \neq \ell(e_2)$.

**Definition 4.4.** An immersion $g: D \rightarrow C$ of $\Delta$-complexes is said to commute with the labeling (or to respect the labeling) if $C$ and $D$ are labeled over the same complex $B(X, P)$ by immersions $f_C: C \rightarrow B(X, P)$ and $f_D: D \rightarrow B(X, P)$, and $g$ commutes with these labeling maps, that is, $f_C \circ g = f_D$. 
Note that if \( g : D \to C \) is an immersion of \( \Delta \)-complexes, then a labeling \( f_C \) of \( C \) over \( B(X, P) \) induces a labeling \( f_D = f_C \circ g \) of \( D \) over \( B(X, P) \) such that \( g \) respects the labeling.

For any cell \( C \in C^{(k)} \), \( k \geq 2 \), define the boundary label \( bl(C) \) of \( C \) as \( \ell(bp(C)) \). We then have the following lemma:

**Lemma 4.5.** Let \( g : D \to C \) be an immersion of a \( \Delta \)-complexes that commutes with the labeling and let \( D^k \) be a \( k \)-cell of \( D \) with \( k \geq 1 \). Then

1. \( \ell(D^k) = \ell(g(D^k)) \),
2. \( g(\alpha(D^k)) = \alpha(g(D^k)) \), and if \( k \geq 2 \), then
3. \( bl(D^k) = bl(g(D^k)) \).

In particular, if \( f \) is an immersion of \( D \) into \( B(X, P) \) that defines a labeling of \( D \), then \( bl(D^k) = bl(f(D^k)) \) for every \( k \)-cell \( D^k \) of \( D \) where \( k \geq 2 \).

**Proof.** The statement is immediate from the definition and the fact that immersions commute with characteristic maps. \( \square \)

For the remainder of this paper, we will assume that all \( \Delta \)-complexes are labeled and that all immersions between \( \Delta \)-complexes respect the labeling, as described above.

## 5. Properties of the loop monoid

Note that it follows from Lemma 4.5 that if \( C \) is a labeled \( \Delta \)-complex, then any two \( k \)-cells \( (k \geq 2) \) that have the same label also have the same boundary label. This allows us to define relations on \( FIM(X \cup P) \) analogous to those defining the inverse category \( IC(C) \). For \( \rho \in P_k \) we define \( bl(\rho) \) to be the common boundary label of all the cells with label \( \rho \).

**Definition 5.1.** Consider a \( \Delta \)-complex \( B(X, P) \). We define \( M(X, P) \) as the inverse monoid with generators \( X \cup P \) and relations

- \( \rho^2 = \rho \) for each \( \rho \in P \) and
- \( \rho = \rho bl(\rho) \) for each \( \rho \in P \).

We remark that the conditions \( \rho = \rho^2 \) and \( \rho = \rho bl(\rho) \) are equivalent to \( \rho = \rho^2 \) and \( \rho \leq bl(\rho) \). It is sometimes more convenient to use this characterization of the defining relations for \( M(X, P) \).

**Proposition 5.2.** The inverse monoid \( M(X, P) \) is the loop monoid of \( B(X, P) \) based at its unique 0-cell \( v \).

**Proof.** Observe that the inverse monoid \( M(X, P) \) can be defined as the quotient of the free monoid \( (X \cup X^{-1} \cup P \cup P^{-1})^* \) by the congruence generated by pairs of the form \((w, ww^{-1}w)\) and \((ww^{-1}uu^{-1}, uu^{-1}ww^{-1})\) where \( u, w \) are arbitrary words, and the additional relations \( \rho^2 = \rho \) and \( \rho = \rho bl(\rho) \) for each \( \rho \in P \). Note that these relations imply \( \rho = \rho^{-1} \), therefore \( P^{-1} \) can be omitted from the generating set.

In \( B(X, P) \), generalized paths are exactly words in \( X \cup X^{-1} \cup P \), and the relation \( \sim \) on these words is generated by exactly the above pairs. Hence \( L(B(X, P), v) \) is indeed equal to \( M(X, P) \). \( \square \)

The following two lemmas are crucial to prove that immersions into a \( \Delta \)-complex are encoded by its loop monoids.

**Lemma 5.3.** Let \( C \) be any \( k \)-cell with \( k \geq 2 \) in a \( \Delta \)-complex labeled by an immersion into \( B(X, P) \). If \( p \) is a path on \( \partial C \) obtained as the image under the attaching map of a closed path around \( v_0 \) in the 1-skeleton of \( \Delta^k \), then \( \ell(C) \leq \ell(p) \) in \( M(X, P) \).

**Proof.** Let \( q \) be a path on \( \Delta^k \) around \( v_0 \), and let \( p \) be its image under the attaching map. We work by induction on \( |p| \). If \( |p| = 0 \), then \( \ell(p) = 1 \geq \ell(C) \) holds as \( \ell(C) \) is an idempotent. Otherwise put \( q = f_1 \cdots f_t \) and \( p = e_1 e_2 \cdots e_t \). Note we cannot have \( |p| = 1 \). If \( |p| = 2, 3 \), then \( q \) lies in a 2-face \( F \) of \( \Delta^k \) containing \( v_0 \). Let \( D \) be the cell corresponding to \( F \) under the characteristic map of \( C \). We claim that \( \ell(D) \geq \ell(C) \). This can be seen by induction on \( k \): if \( k = 2 \), then \( D = C \) so it is clear; if \( k > 2 \), let \( \Delta_i \) be a \((k - 1)\)-face of \( C \) containing \( F \) and hence \( v_0 \). For the corresponding cell
Lemma 5.4. Let $C$ be any $k$-cell with $k \geq 2$ in a $\Delta$-complex labeled over $B(X, P)$. If $p$ is the image under the corresponding attaching map of any closed generalized path around $v_0$ on the boundary of $\Delta^k$, then $\ell(C) \leq \ell(p)$ in $M(X, P)$.

Proof. Consider a closed generalized path $q = q_1 \Delta q_2 \Delta q_3 \ldots q_t \Delta q_{t+1}$ around $v_0$ on $\partial \Delta^k$, where $q_i$ is a path, and $\Delta$ is a face of $\Delta^k$ of at least 2-dimensions, and let $p = p_1 D_1 p_2 D_2 \ldots p_t D_{t+1}$ be the image of $q$ under the attaching map. Let $s_i$ be the image of the shortest path from $v_0$ to $\alpha(\Delta_i)$ on $\partial \Delta^k$ (this is either an edge or the empty path). We claim that $\ell(C) \leq \ell(s_i) \ell(D_i) \ell(s_i^{-1})$. This can be proved by induction on $n = k - \dim(D_i)$. For $n = 0$ we have $C = D_1$ and $s_i$ the empty path, so the claim is immediate. If $n \geq 1$, then let $\Delta'$ be a dim$(D_i) + 1$-dimensional face of $\Delta^k$ containing $v_0$, and let $D = \sigma C(\Delta')$. Then $\ell(D)$ is the product of $\ell(s_i) \ell(D_i) \ell(s_i^{-1})$ and some idempotents, so $\ell(D) \leq \ell(D) \leq \ell(s_i) \ell(D_i) \ell(s_i^{-1})$ from the presentation, and $\ell(C) \leq \ell(D)$ by induction.

By Lemma 5.3 we have $\ell(C) \leq \ell(p_1 s_i^{-1})$, $\ell(C) \leq \ell(s_1 p_2 s_2^{-1})$, \ldots, $\ell(C) \leq \ell(s_t p_{t+1})$. Hence, by multiplying all of these inequalities, we obtain

$$\ell(C) \leq \ell(p_1 s_1^{-1}) \ell(D_1) \ell(s_1^{-1}) \ell(s_1 p_2 s_2^{-1}) \ell(D_2) \ell(s_2^{-1}) \ldots$$

$$\ell(s_{t-1} p_t s_t^{-1}) \ell(D_t) \ell(p_{t+1}) \leq \ell(p_1) \ell(D_1) \ell(p_2) \ell(D_2) \ldots \ell(D_t) \ell(p_{t+1}) = \ell(p)$$

as required. □

A small, but useful observation:

Proposition 5.5. The inverse submonoid of $M(X, P)$ generated by $X$ is isomorphic to $\text{FIM}(X)$.

Proof. Recall that $M(X, P)$ is obtained as a factor of the free monoid on $X \cup X^{-1} \cup P$ by the defining relations of inverse monoids and those introduced in the presentation; these are all equations which, if they contain a letter in $P$, then they contain it on both sides. Therefore, to a word in $(X \cup X^{-1})^*$, one can only apply those relations which define the free inverse monoid on $X$. □

5.1. $L(C, v)$ as a stabilizer. Now let $C$ be a $\Delta$-complex labeled over $B(X, P)$. Then we may define a natural action of the inverse monoid $M(X, P)$ by partial one-to-one maps on the set $C^{(0)}$ of 0-cells of $C$ as follows. For $x \in X \cup X^{-1}$ and $v \in C^{(0)}$ define $v.x = w$ if there is an edge (or a ghost edge) labeled by $x$ from $v$ to $w$ in the 1-skeleton of $C$, and $v.x$ is undefined if there is no such edge. For $\rho \in P_k$ with $k \geq 2$ and $v \in C^{(0)}$, define $v.\rho = v$ if $v = \alpha(C^k)$ for some $k$-cell $C^k$ with $\alpha(C^k) = v$, $\ell(C^k) = \rho$, and $v.\rho$ is undefined otherwise.

We begin by a simple, but useful lemma about the above action.

Lemma 5.6. Let $C$ be a $\Delta$-complex labeled over $B(X, P)$.

1. For any generalized path $p$ in $C$, $\alpha(p).\ell(p)$ is defined and is equal to $\omega(p)$.
2. For any $w \in (X \cup X^{-1} \cup P)^*$ if $v.w$ is defined for some $v \in C^{(0)}$, then there exists a generalized path in $C$ with $\alpha(p) = v$, $\ell(p) = w$, and $\omega(p) = v.w$.

Proof. For (1) we prove the statement by induction on the length of $p$. If $p$ is an empty path, then $\ell(p) = 1$ and $\omega(p) = \alpha(p)$, so the statement holds. Otherwise put $p = p'e$ with $e$ a cell or a ghost edge. Then by induction $v.\ell(p')$ is defined and is equal to $\omega(p')$, and
notice that $\alpha(e) = \omega(p')$. Then by definition we have $\omega(p').\ell(e) = \omega(e) = \omega(p)$, and so $v.\ell(p) = v.\ell(p').\ell(e) = \omega(p)$ as needed.

(2) The proof is by induction on the length of $w$. If $w$ is the empty word, then $v.w = v$ and $p = 1_v$ suffices. Otherwise put $w = w'y$ with $y \in X \cup X^{-1} \cup P$. As $v.w = v.w'y$, we have that $v.w'$ is defined and thus there exists by induction a generalized path $p'$ in $C$ with $\alpha(p') = v$, $\ell(p') = w'$, $\omega(p') = v.w'$. Then $v.w = \omega(p'), y$, thus $\omega(p'). y$ is defined. Since $y \in X \cup X^{-1} \cup P$, by the definition of the action, there is a cell or ghost edge $e$ with $\alpha(e) = \omega(p')$, $\ell(e) = y$, $\omega(e) = v.w$. Put $p = p'e$, this is indeed a generalized path with $\alpha(p) = v$, $\omega(p) = v.w$ and $\ell(p) = \ell(p').\ell(e) = w'y = w$.

\[ \square \]

Lemma 5.7. The above action of the generators $X \cup X^{-1} \cup P$ of $M(X, P)$ on $C^{(0)}$ extends to a well-defined action of $M(X, P)$ on $C^{(0)}$.

Proof. By the Remark 4.3, the labeling we introduced ensures that letters in $X \cup X^{-1} \cup P$ act by partial one-to-one maps on $C^{(0)}$, and the action of $x \in X$ is the inverse of the action of $x^{-1}$. Thus the action of the generators extends to a well-defined action of the free inverse monoid $\text{FIM}(X \cup P)$ on $C^{(0)}$. We need to show that the action respects the defining relations of $M(X, P)$. The actions of letters in $P$ are, by definition, identity maps on their domains, hence idempotent maps. If $p \in P_2$ fixes $v$ and $\ell(C^2) = \rho$ with $\alpha(C^2) = v$ then necessarily $bp(C^2)$ determines a closed path at $v$ and so $\text{bl}(p)$ fixes $v$. So the action by $\rho$ is the same as the action by $\rho \text{bl}(p)$ for $p \in P_2$. Finally, for $\rho \in P_k, k \geq 3$, the action by $\rho$ is also a restriction of the action by $\text{bl}(p)$. This is because each of the elements $\ell(C_{i}^{-1})$ (for $i = 1, \ldots, k$) and $\ell(e(C^k))C_{0}^{-1}(e(C^k))^{-1}$ that arise in the definition of the boundary label $\text{bl}(C^k)$ of a cell $C^k$ stabilize $\alpha(C^k)$. So again the action by $\rho$ is the same as the action by $\rho \text{bl}(p)$. Hence the action respects all defining relations in $M(X, P)$, as required. \[ \square \]

The stabilizer $\text{Stab}(v)$ of a 0-cell $v \in C^{(0)}$ under the action by $M(X, P)$ is a closed inverse submonoid of $M(X, P)$, and stabilizers of different vertices in $C^{(0)}$ are conjugate closed inverse submonoids of $M(X, P)$. Hence the immersion $f : C \to B(X, P)$ that defines the labeling of $C$ gives rise to a conjugacy class of closed inverse submonoids of $M(X, P)$.

Proposition 5.8. For any labeled $\Delta$-complex $C$ and any $v \in C^{(0)}$, we have $\text{Stab}(v) \cong L(C, v)$.

Proof. We show that the map between $L(C, v)$ and $\text{Stab}(v)$ taking the $\sim$-class of a generalized path $p$ to the value of $\ell(p)$ in $M(X, P)$ is an isomorphism.

Denote the $\sim$-class of a generalized path $p$ in $C$ by $[p]$ and the equivalence class of a word $w \in (X \cup X^{-1} \cup P)^*$ under the congruence $\equiv$ on $(X \cup X^{-1} \cup P)^*$ defining the inverse monoid $M(X, P)$ by $[[w]]$. We claim that for any pair of coterminal generalized paths $p, q$, we have $p \sim q$ if and only if $\ell(p) \approx \ell(q)$. Note that for $w_1, w_2 \in (X \cup X^{-1} \cup P)^*$, $w_1 \approx w_2$ if and only if it is possible to pass from $w_1$ to $w_2$ by a sequence of elementary transitions involving the relations used to define the free inverse monoid on $X \cup P$ together with the additional relations used to define $M(X, P)$ as a quotient of this free inverse monoid. Similarly, $p \sim q$ if it is possible to pass from $p$ to $q$ by a sequence of elementary transitions involving paths, using the relations defining the category $\text{IC}(C)$.

It is clear that any sequence of such elementary transitions of paths defines a sequence of elementary transitions of their labels, so $p \sim q$ implies $\ell(p) \approx \ell(q)$.

For the converse, let $p$ be any generalized path, and $w$ a word in $X^*$ such that $\ell(p)$ and $w$ are related by an elementary transition. We claim that then $w$ labels a generalized path $q$ coterminal with $p$, and hence there is a corresponding path transition from $p$ to $q$. If the transition from $\ell(p)$ to $w$ uses an identity defining inverse monoids, then this is easy to see. The same is true if $w$ uses a relation of the form $\ell(C)^2 = \ell(C)$ as $\ell(C)$ always labels a closed path. For the last type of relation $\ell(C) \text{bl}(C) = \ell(C)$, notice that whenever we have a (closed) path labeled by $\ell(C)$ at some vertex $u$, we also have a closed path labeled by $\text{bl}(C)$ at $u$. Thus we can freely remove or add $\text{bl}(C)$ after $\ell(C)$ and obtain a generalized path.

This implies that for any coterminal paths $p, q$, if we have any sequence of elementary transitions from $\ell(p)$ to $\ell(q)$, there must be a sequence of corresponding elementary path transitions from $p$ to
a path which is coterminal with \(p\) and has label \(\ell(q)\), but this must be \(q\). Thus \(\ell(p) \approx \ell(q)\) implies \(p \sim q\).

We claim that the map \(\phi: [p] \mapsto ([\ell(p)])\) is an isomorphism from from \(L(C, v)\) to \(\text{Stab}(v)\). First note that by the previous assertion, for any pair of closed generalized paths \(p, q\) around \(v\), we have \([p] = [q]\) if and only if \([\ell(p)] = [\ell(q)]\). This shows that \(\phi\) is injective, and a well-defined homomorphism into \(M(X, P)\). Furthermore \([p] \in L(C, v)\) if and only if \(p\) is a closed generalized path in \(C\) at \(v\), which by Lemma \ref{lem:closedgeneralizedpath} implies \(v.\ell(p) = v\), that is \([\ell(p)] \in \text{Stab}(v)\). Conversely if \([w]\) \(\in \text{Stab}(v)\) then by Lemma \ref{lem:closedgeneralizedpath} there is a generalized path \(p\) with \(\ell(p) = w\) and \(v.\ell(p) = v\), thus \(\phi([p]) = [w]\). This shows that the image of \(\phi\) is \(\text{Stab}(v)\) indeed. This completes the proof. \hfill \(\Box\)

In the sequel, we identify \(\sim\)-classes of \((v, v)\)-paths with their common label in \(M(X, P)\), and regard \(L(C, v)\) equal to \(\text{Stab}(v)\).

6. Classification of immersions

The previous section has shown how an immersion into \(B(X, P)\) gives rise to a conjugacy class of closed inverse submonoids of its (unique) loop monoid \(M(X, P)\). This section shows that this is true for any immersion, moreover, the converse also holds: immersions are in one-to-one correspondence with conjugacy classes of closed inverse submonoids of loop monoids.

**Proposition 6.1.** Let \(C\) and \(D\) be \(\Delta\)-complexes labeled over a common complex \(B(X, P)\), and suppose \(g: D \to C\) is an immersion, and let \(v \in \Omega^0\). Then \(L(D, v)\) is a closed inverse submonoid of \(L(C, f(v))\).

**Proof.** There are immersions \(f_C: C \to B(X, P)\) and \(f_D: D \to B(X, P)\) defining the labeling, and \(f_C \circ g = f_D\). Then \(L(D, v) = \text{Stab}(D, v)\) and \(L(C, f(v)) = \text{Stab}(C, f(v))\) are both closed inverse submonoids of the corresponding \(M(X, P)\). Therefore it is enough to show that \(\text{Stab}(D, v) \subseteq \text{Stab}(C, f(v))\). Indeed, suppose \(w\) labels a closed path \(p\) around \(v\) in \(D\). Then \(g(p)\) is a closed path around \(f(v)\) with the same label \(w\), which proves the statement. \hfill \(\Box\)

We proceed to develop the theory of the converse part of the correspondence. Fix a \(\Delta\)-complex \(C\) labeled over some \(B(X, P)\) by an immersion \(g: C \to B(X, P)\). We define a graph \(\Gamma_C\) associated with \(C\) as follows:

\[
V(\Gamma_C) = C^0 \quad \text{and} \quad E(\Gamma_C) = C^{(1)} \cup \{f_{C^k} : C^k \text{ is a } k\text{-cell in } C^{(k)}, k \geq 2\},
\]

where \(f_{C^k}\) denotes a loop based at \(\alpha(C^k)\) and labeled by \(\ell(C^k)\). Thus the edges in \(C^{(1)}\) are labeled over \(X \cup X^{-1}\) and the edges of the form \(f_{C^k}\) (for \(C^k\) a \(k\)-cell) are labeled over \(P_k\). Since an edge labeled by \(\rho \in P_k\) is always a loop labeled by an idempotent in \(M(X, P)\), we may identify \(P_k\) with \(P_k^{-1}\) and regard \(\Gamma_C\) as an (\(X\cup P\))-graph in the sense defined in the introduction. If we abuse notation slightly by identifying the loop \(f_{C^k}\) in \(\Gamma_C\) with the \(k\)-cell \(C^k\), then paths in the graph \(\Gamma_C\) are just identified with generalized paths in the \(\Delta\)-complex \(C\).

Let \(u \in \Omega^0\). Given a closed inverse submonoid \(H\) of \(L(C, u)\), we construct a complex \(C_H\) that immerses into \(C\) with \(H = L(C_H, \bar{u})\) for some \(\bar{u} \in \Omega^0_H\) in the preimage of \(u\).

The complex \(C_H\) is defined with the help of the \(\omega\)-coset graph \(\Gamma_H\) of \(H\). The inverse monoid \(M(X, P)\) acts on the vertices of \(\Gamma_H\). The idempotents, of course, all label loops.

We build a \(\Delta\)-complex \(C_H\) such that \(\Gamma_C = \Gamma_H\). This complex has the following sets of cells:

\[
C_H^0 = V(\Gamma_H),
\]

\[
C_H^{(1)} = \{e \in E(\Gamma_H) : \ell(e) \in X\},
\]

if \(k \geq 2\), \(C_H^{(k)} = \{C_e : e \in E(\Gamma_H) \text{ and } \ell(e) \in P_k\}\).

The attaching maps of 1-cells are the ones inherited from the graph \(\Gamma_H\). Note that the 1-skeleton then immerses into the 1-skeleton of \(C\) via some map \(f\): \(C_H^1 \to C^1\) by the results of \[\[\], as Proposition \ref{prop:embeddedconjugacy} ensures that our definitions and assumptions reduce to those of \[\[\] when applied to graphs. Moreover, the vertex \(H\) is mapped to \(u\).
We build the rest of the attaching maps of $C_H$ inductively. Let $k \geq 2$ and denote the set of edges of $\Gamma_H$ with labels in $P_k$ by $E_k$. Suppose $C_H^{k-1}$ is a $\Delta$-complex with 1-skeleton $C_H^1$ such that there exists an immersion $f_{k-1}: C_H^{k-1} \to C$ with $H \to u$, and $\Gamma_H^{k-1}$ is the subgraph of $\Gamma_H$ induced by the edges labeled by $X \cup P_2 \cup \ldots \cup P_{k-1}$. Note we then have $V(\Gamma_{C_H^{k-1}}) = V(\Gamma_H)$, because $\Gamma_H$ is connected and $P_2 \cup \ldots \cup P_{k-1}$ all label loops.

Let $e \in E_k$, then $e$ is a loop. Put $\rho = \ell(e)$ and $v = \alpha(e) = \omega(e)$. Let $p$ be a path in $\Gamma_H$ (and so in $C_H^{k-1}$) from $H$ to $v$. Then $\ell(p)\rho(p)^{-1} \in H \subseteq L(C, u)$, and so $\rho$ labels a $k$-cell $C_e$ in $C$ based at $\omega(f_{k-1}(p)) = f_{k-1}(v)$. Denote its attaching map by $\varphi_e$.

**Lemma 6.2.** There exists a unique attaching map $\varphi_e : \partial \Delta^k \to C_H^{k-1}$ of a $k$-cell $C_e$ based at $v$ such that $f_{k-1} \circ \varphi_e = \varphi_e$.

**Proof.** We can write the boundary of a $k$-simplex as a (non-disjoint) union of $(k - 1)$-dimensional simplices:

$$\partial \Delta^k = \bigcup_{i=1}^{k+1} \Delta^i_{k-1}.$$  

By the definition of a $\Delta$-complex, $\varphi_e = \bigcup_{i=1}^{k+1} \sigma_i^{k-1}$ for the characteristic maps $\sigma_i^{k-1}: \Delta^i_{k-1} \to C$ of some cells $C_i^{k-1}$, and likewise, if such a $\varphi_e$ exists, it has to be the form of $\varphi_e = \bigcup_{i=1}^{k+1} \tilde{\sigma}_i^{k-1}$ for the characteristic maps $\tilde{\sigma}_i^{k-1}: \Delta^i_{k-1} \to C^{k-1}$ of some cells $\tilde{C}_i^{k-1}$. Since $f_{k-1} \circ \varphi_e = \varphi_e$, we have that $\tilde{C}_i^{k-1}$ is a preimage of $C_i^{k-1}$ under $f_{k-1}$, in particular, $\ell(\tilde{\sigma}_i^{k-1}) = \ell(\sigma_i^{k-1})$.

For every $i$ between 1 and $k+1$, let $p_i$ be a path on the one-dimensional faces (edges) of $\Delta^k$ from $v_0$ to the root of $\Delta^k_{k-1}$. Put $q_i = \varphi_e(p_i)$, these are paths on the one-cells of $C$. Since $p_i\Delta^i_{k-1}(\Delta^k_{k-1})^{-1}p_i^{-1}$ is a closed generalized path on $\partial \Delta^k$ around $v_0$, for the path $\varphi_e(p_i\Delta^i_{k-1}(\Delta^k_{k-1})^{-1}p_i^{-1}) = q_i\tilde{C}_i^{k-1}(\tilde{C}_i^{k-1})^{-1}q_i^{-1}$ in $C$ around $f_{k-1}(v)$, by Lemma [5.4] we have

$$\rho = \ell(C^k) \leq \ell(q_i)\ell(C^{k-1})\ell(C^{k-1})^{-1}\ell(q_i)^{-1},$$  

therefore $\ell(q_i)\ell(C^{k-1})\ell(C^{k-1})^{-1}\ell(q_i)^{-1}$ labels a closed path in $\Gamma_H$ around $v$, and therefore in $C^H$ also. The cell $\tilde{C}_i^{k-1}$ thus can only be the unique $(k-1)$-cell with the label $\ell(C_i^{k-1})$ occurring in the previous path by Remark [1.3] and $\tilde{\sigma}_i^{k-1}$ is the characteristic map corresponding to $\tilde{C}_i^{k-1}$.

Note that for this cell we have $f_{k-1}(\alpha(\tilde{C}_i^{k-1})) = \alpha(C_i^{k-1})$ and so $f_{k-1}(\tilde{C}_i^{k-1}) = C_i^{k-1}$ indeed, thus $f_{k-1} \circ \tilde{\sigma}_i^{k-1} = \sigma_i^{k-1}$.

It remains to be shown that the map $\tilde{\varphi}_e = \bigcup_{i=1}^{k+1} \tilde{\sigma}_i^{k-1}$, given as a union of maps on non-disjoint domains, is well-defined; that is, for any intersection $\Delta_{i,j} = \Delta_i \cap \Delta_j$, we have $\tilde{\sigma}_i^{k-1}|_{\Delta_{i,j}} = \tilde{\sigma}_j^{k-1}|_{\Delta_{i,j}}$.

Since $\Delta_{i,j}$ is a face of both $\Delta_i$ and $\Delta_j$, both maps $\tilde{\sigma}_i^{k-1}|_{\Delta_{i,j}}$ and $\tilde{\sigma}_j^{k-1}|_{\Delta_{i,j}}$ are characteristic maps $\tilde{\sigma}_i^{k-2}$ and $\tilde{\sigma}_j^{k-2}$ for some $k-2$ cells $\tilde{C}_i^{k-2}$ and $\tilde{C}_j^{k-2}$. Of course, since $\varphi_e = \bigcup_{i=1}^{k+1} \sigma_i^{k-1}$, we have $\sigma_i^{k-1}|_{\Delta_{i,j}} = \sigma_j^{k-1}|_{\Delta_{i,j}} = \varphi_e|_{\Delta_{i,j}}$, hence

$$\tilde{\sigma}_i^{k-2} = \sigma_i^{k-1}|_{\Delta_{i,j}} = \sigma_j^{k-1}|_{\Delta_{i,j}} = f_{k-1} \circ \tilde{\sigma}_j^{k-2}.  \tag{1}$$

Now take a path $s_i$ on the edges of $\Delta_i$ from the root of $\Delta_i$ to the root of $\Delta_{i,j}$, and similarly a path $s_j$ on $\Delta_j$. Let $t_i = \sigma_i^{k-1}(s_i)$, $t_j = \sigma_j^{k-1}(s_j)$, and $\tilde{t}_i = \tilde{\sigma}_i^{k-1}(s_i)$, $\tilde{t}_j = \tilde{\sigma}_j^{k-1}(s_j)$. Since $p_i s_i s_j^{-1} p_j^{-1}$ is a closed path on $\Delta^k$ around $v_0$, for the path $\varphi(p_i s_i s_j^{-1} p_j^{-1}) = q_i t_i t_j q_j^{-1}$ in $C$, we have $\ell(q_i t_i t_j q_j^{-1}) \geq \rho$ by Lemma [5.3] and there is a closed path at $v$ labeled by $\ell(q_i t_i t_j q_j^{-1})$ in $C^H$. But, the unique path from $v$ labeled by $\ell(q_i)$ ends in $\alpha(\tilde{C}_i^{k-1})$ — that is how $\tilde{C}_i^{k-1}$ was defined — and the unique path from $\alpha(\tilde{C}_i^{k-1})$ labeled by $\ell(t_i)$ is $\tilde{t}_i$. The same can be said about $\alpha(\tilde{C}_j^{k-1})$ and $\tilde{t}_j$, which implies that $\omega(\tilde{t}_i) = \omega(\tilde{t}_j)$, hence $\alpha(\tilde{C}_i^{k-2}) = \alpha(\tilde{C}_j^{k-2})$. Since $f_{k-1}$ is an immersion which maps $\tilde{C}_i^{k-2}$ and $\tilde{C}_j^{k-2}$ to the same cell by [11], this immediately implies $\tilde{C}_i^{k-2} = \tilde{C}_j^{k-2} = \tilde{C}^{k-2}$, and hence $\tilde{\sigma}_i^{k-2} = \tilde{\sigma}_j^{k-2}$. That proves that $\varphi_e$ is well-defined, and by nature of the construction, unique.
Note that \( \tilde{\varphi}_e \) is continuous, since it is a union of continuous maps defined on closed sets, completing the proof.

We now define \( C^k_H \) as the \( \Delta \)-complex with \((k-1)\)-skeleton \( c^{k-1}_H \) and \( k \)-cells \( c^k_H = \{ \tilde{C}_e : e \in E_k \} \), where the attaching map of a cell \( \tilde{C}_e \) is \( \varphi_e \), as defined in the previous lemma. Notice that we by construction have that \( \Gamma_{C^k_H} \) is the subgraph of \( \Gamma_H \) induced by the edges labeled by the \( X \cup P_2 \cup \ldots \cup P_k \).

**Lemma 6.3.** There is a unique immersion \( f_k : C^k_H \to C \) for which \( f_k|_{c^{k-1}_H} = f_{k-1} \).

**Proof.** Let \( \tilde{\sigma}_e \) denote the characteristic map of the cell \( \tilde{C}_e \) of \( C^k_H \) for any \( e \in E_k \), and let \( \sigma_e \) denote the characteristic map of respective cell \( C_e \) of \( C \). Let \( f_k : C^k_H \to C \) be the map defined by

\[
\begin{align*}
  f_k|_{c^{k-1}_H} &= f_{k-1} \\
  f_k|_{\tilde{C}_e} &= \sigma_e|_{|\text{int } \Delta^k} \circ (\tilde{\sigma}_e|_{|\text{int } \Delta^k})^{-1}
\end{align*}
\]

for \( e \in E_k \). We show that \( f_k \) is an immersion: it suffices to show that \( f_k \) commutes with the characteristic maps, and induces injections between star sets.

Since \( f_{k-1} \) is an immersion, \( f_k \) clearly commutes with the characteristic map of any cell contained in \( C^{k-1}_H \). To see that \( f_k \) commutes with the characteristic map of a \( k \)-cell \( \tilde{C}_e \) where \( e \in E_k \), let \( x \in \Delta^k \) be an arbitrary point, and consider \( f_k \circ \tilde{\sigma}_e(x) \). If \( x \in |\text{int } \Delta^k| \), then \( f_k \circ \tilde{\sigma}_e(x) = \sigma_e(x) \) by (2).

If \( x \in \partial \Delta^k \), then \( x \) lies in a simplex \( \Delta^j \) on \( \partial \Delta^k \), and \( \tilde{\sigma}_e|_{|\Delta^j} \) is a characteristic map that commutes with \( f_k \), therefore we again obtain \( f_k \circ \tilde{\sigma}_e(x) = \sigma_e(x) \), as desired.

Also, since by construction \( \Gamma_{C^k_H} \) is a subgraph of \( \Gamma_H \), thus whenever \( C_1, C_2 \) are cells of \( C^k_H \) with \( \ell(C_1) = \ell(C_2) \) and \( \alpha(C_1) = \alpha(C_2) \), we have \( C_1 = C_2 \). In particular if \( C_1, C_2 \in \star(C^k_H, v) \) for some \( v \), then \( f(C_1) = f(C_2) \) implies \( C_1 = C_2 \), thus \( f \) is an immersion.

The uniqueness of \( f \) follows from the fact that any map satisfying the conditions of the lemma must satisfy (2). □

We are ready to state and prove the main theorems of the paper. If \( C \) is a \( \Delta \)-complex with \( u \in C \), we call the pair \((C, u)\) a pointed \( \Delta \)-complex. A pointed \( \Delta \)-map between the pointed \( \Delta \)-complexes \((C, u) \to (D, v)\) is a \( \Delta \)-map \( f : D \to C \) with \( f(u) = v \). In particular we can talk about pointed immersions and pointed isomorphisms.

Our first main result says that the pointed immersions into a \( \Delta \)-complex are up to pointed isomorphisms classified by the closed inverse submonoids of its loop monoid.

**Theorem 6.4.** Let \( C \) be a \( \Delta \)-complex labeled over some \( B(X, P) \), let \( u \in C^{(0)} \), and let \( H \) be any closed inverse submonoid of \( L(C, u) \). Then there exists a pointed \( \Delta \)-complex \((C_H, v)\) and a pointed immersion \( f : (C_H, v) \to (C, u) \) with \( H = L(C_H, v) \). Furthermore, if \((C'_H, v')\) is another pointed \( \Delta \)-complex with an immersion \( f' : (C'_H, v') \to (C, u) \), then there is a pointed isomorphism \( g : (C'_H, v') \to (C_H, v) \) such that \( f' = f \circ g \).

**Proof.** The existence part of the theorem is clear from the previous construction. For uniqueness up to pointed isomorphism, note that \( H = L(C_H, v) = L(C'_H, v') \) dictates that there is a pointed isomorphism between both \((\Gamma_{C_H}, v), (\Gamma_{C'_H}, v')\) and the \( \omega \)-coset graph \((\Gamma_H, H)\), which in particular implies there is a pointed isomorphism \( g_1 \) between their subgraphs induced by the \( X \)-labeled edges, and hence between \((C_H, v)\) and \((C'_H, v')\). Note that if \( p \) is an path in \( C^0_H \) with \( \alpha(p) = v' \), then \( \alpha(g_1(p)) = v \) and hence \( f'(p) \) and \( f \circ g_1(p) \) are both paths in \( C \) starting at \( f(v) = f'(v') = u \) with label \( \ell(p) \), thus they are equal. As \( C^0_H \) is connected, any of its 1-cells lies on such a path \( p \), which shows \( f \circ g_1 = f' \).

It follows by induction from the uniqueness of the construction in Lemma 6.2 that \( g_1 \) extends to an isomorphism \( g : (C'_H, v') \to (C_H, v) \). Indeed, notice that if \( g_{k-1} : (C^{k-1}_H, v) \to (C^{k-1}_H, v') \) is an isomorphism with \( f \circ g_{k-1} = f' \) and \( e \in E_k \), denoting the constructed cells of \( C^k_H \) by \( \tilde{C}_e \), \( \tilde{C}'_e \) with attaching maps \( \tilde{\varphi}_e, \tilde{\varphi}'_e \) respectively, then \( g_{k-1} \circ \tilde{\varphi}_e \) and \( \tilde{\varphi}'_e \) both satisfy the conditions of Lemma 6.2.
and are thus equal. Hence putting $g_k(\tilde{C}_v') = \tilde{C}_v$ for $e \in E_k$, $g|_{C^{k-1}_H} = g_{k-1}$ defines an isomorphism $g_k: (C^{k}_H, v') \to (C^{k}_H, v)$ with $f' = f \circ g_k$. 

An immediate consequence of Proposition 6.1 and Theorem 6.4 is our first main theorem, characterizing pointed, connected immersions over finite-dimensional, connected $\Delta$-complexes.

**Theorem 6.5.** Let $(C, u)$ and $(D, v)$ be connected $\Delta$-complexes labeled over a common $\Delta$-complex $B(X, P)$, and suppose $f: (D, v) \to (C, u)$ is a connected, pointed immersion that commutes with the labeling maps, so $f$ induces an embedding of $L(D, v)$ into $L(C, u)$.

Conversely, let $C$ be a $\Delta$-complex labeled over a $\Delta$-complex $B(X, P)$, and let $H$ be a closed inverse submonoid of the corresponding inverse monoid $M(X, P)$ such that $H \subseteq L(C, u)$ for some $u \in C^0$. Then there exists a pointed $\Delta$-complex $(D, v)$ labeled over the same $B(X, P)$, with $L(D, v) = H$, and there is a pointed immersion $f: (D, v) \to (C, u)$.

Furthermore, if $(D', v')$ is another pointed $\Delta$-complex with an immersion $f': (D', v') \to (C, u)$, then there is a pointed isomorphism $g: (D', v') \to (D, v)$ such that $f' = f \circ g$.

The following theorem states that immersions into a complex $C$ up to isomorphism are characterized by the conjugacy classes of closed inverse submonoids of any of its loop monoids.

**Theorem 6.6.** Take a pointed $\Delta$-complex $(C, u)$ labeled over $B(X, P)$. If $(D, v)$, $(D', v')$ are pointed $\Delta$-complexes with pointed immersions $f, f'$ into $(C, u)$ respectively, then there is an isomorphism $g: D' \to D$ with $f' = f \circ g$ if and only if $L(D, v)$ is conjugate to $L(D', v')$ in $L(C, u)$.

**Proof.** First, suppose such an isomorphism $g$ exists. Let $m$ label a path from $v$ to $g(v')$ in $D$, then $m$ labels a path from $f(v) = u$ to $f(g(v')) = f'(v') = u$ in $C$, hence $m \in L(C, u)$. If $w$ labels a closed generalized path in $D'$ at $v'$, then it also labels a closed generalized path at $g(v')$ in $D$, so $mm^{-1}$ labels a closed generalized path in $D$ at $v$, and $mL(D', v')m^{-1} \subseteq L(D, v)$. Similarly $m^{-1}L(D, v)m \subseteq L(D', v')$, hence they are conjugate in $L(C, u)$ indeed.

For the converse, suppose $L(D, v)$ is conjugate to $L(D', v')$ in $L(C, u)$. Then there exists some $m \in L(C, u)$ such that

$$m^{-1}L(D, v)m \subseteq L(D', v')$$

in particular, $mm^{-1} \in L(D, v)$. Therefore $m$ labels a generalized path from $v$ to some 0-cell $z$ in $D$, and note that $f(z) = u$ by Remark 4.3 since $f(v) = u$ and $m \in L(C, u)$. We will show that $L(D', v') = L(D, z)$. If $k \in L(D', v')$, then $mm^{-1} \in L(D, v)$ so $mm^{-1}$ labels a closed generalized path around $v$ in $D$, hence $k$ labels closed generalized path around $z$. Thus we have $L(D', v') \subseteq L(D, z)$. On the other hand, if $n \in L(D, z)$, then $mm^{-1}$ labels a closed generalized path around $v$, so $mm^{-1} \in L(D, v)$, and $m^{-1}mm^{-1}m \subseteq L(D', v')$. Since $L(D', v')$ is closed and $m^{-1}mm^{-1}m \subseteq n$, this yields $n \in L(D', v')$, therefore $L(D', v') = L(D, z)$. The existence of $g$ then follows from the last statement of Theorem 6.5 applied to $(D', v')$ and $(D, z)$. 

□

7. Closing remarks

We remark that the constructions of the inverse monoid $M(X, P)$ and of the complex associated with a closed inverse submonoid of $M(X, P)$ are effective. The proof of the following theorem makes use of Stephen’s construction of Schützenberger graphs [14] and an extension of this developed in [6]. We note that the result is somewhat surprising in view of the fact that the maximal group image of $M(X, P)$ is the fundamental group of $B(X, P)$, which may have undecidable word problem. However, the fact that $M(X, P)$ is not $E$-unitary enables $M(X, P)$ to have decidable word problem while its maximal group image may not necessarily have decidable word problem. The proof follows closely along the lines of the proof of Theorem 5.7 of [6], so we will omit it.

**Theorem 7.1.** (a) If $X$ and $P$ are finite sets, then the word problem for $M(X, P)$ is decidable.

(b) If $X$ and $P$ are finite sets and $H$ is a finitely generated closed inverse submonoid of $M(X, P)$, then the associated $\Delta$-complex $C_H$ is finite and effectively constructible.
Similarly, one may obtain the following characterization of the covering maps. Again the proof closely follows the proof of Theorem 6.1 of [9].

**Theorem 7.2.** Let $C, D$ be $\Delta$-complexes labeled by an immersion over some complex $B(X,P)$, let $f : C \to D$ be an immersion that respects the labeling, and let $v \in C_0$ be an arbitrary 0-cell. Then $f$ is a covering map if and only if $L(C,v)$ is a full closed inverse submonoid of $L(D,f(v))$, that is, it contains all idempotents of $L(D,f(v))$.

We conclude by raising the question as to whether an extension of some of the ideas contained in this paper may be developed to provide a classification of immersions between more general topological spaces (for example for arbitrary CW-complexes). It would also be of interest to provide a “presentation-free” characterization of the inverse category $\text{IC}(C)$ that serves the role of the fundamental groupoid in covering space theory.

8. Appendix

In this section we discuss the relationship between our definition of immersions between $\Delta$-complexes and the more classical concept of topological immersions. Recall that a $\Delta$-map $f : D \to C$ is locally injective if every point in $D$ has a neighborhood $U$ with $f|_U : U \to f(U)$ injective. It is a topological immersion if we furthermore require that $f|_U$ is a homeomorphism between $U$ and $f(U)$, where both sets are equipped with the subspace topology.

We begin with a rather technical, but useful lemma.

**Lemma 8.1.** Let $f : D \to C$ be a $\Delta$-map, and $u$ be any point in $D$. Then $f$ is locally injective at $u$ if and only if for any cells $D_1, D_2$ of $D$ with $f(D_1) = f(D_2)$ and $\sigma_{D_1}(u) \cap \sigma_{D_2}(u) \neq \emptyset$, we have $D_1 = D_2$.

**Proof.** Let $D_1, D_2$ be cells as in the statement. Notice that $f(D_1) = f(D_2)$ implies $D_1, D_2$ are of the same dimension $n$. Assume $D_1 \neq D_2$ for contradiction. Let $N$ be an arbitrary neighborhood of $u$ in $D$, and take the set $U := \sigma_{D_1}^{-1}(N) \cap \sigma_{D_2}^{-1}(N) \subseteq \Delta^n$. This is an open set in $\Delta^n$, and it is also nonempty, as it contains $\sigma_{D_1}^{-1}(u) \cap \sigma_{D_2}^{-1}(u)$. Take a point $x \in U \setminus \partial \Delta^n$ – there certainly exists such a point, as $U$ is nonempty and open. Then as $x_i \in D_i$, we have $x_1 \neq x_2$, and $f \circ \sigma_{D_1} = f \circ \sigma_{D_2}$ implies $f(x_1) = f(x_2)$. As $x_i \in N$, we obtain that $f$ is not locally injective at $u$.

For the converse, assume that $f$ is not locally injective at $u$; we need to show that cells $D_1$ and $D_2$ exist as above. Note we must have $\dim(D) \geq 1$ for local injectivity to fail. Let $D$ be the unique cell containing $u$, and let $\sigma_D : \Delta^l \to D$ be its characteristic map. (If $u$ is a 0-cell, its characteristic map just maps the 0-simplex to $u$.) Consider the preimage $\sigma_D^{-1}(u)$ of $u$, which is a point in $\text{int}(\Delta^l)$. Regard simplices as metric spaces with the Euclidean metric they inherit from $\mathbb{R}^n$, and if $l > 0$, let

$$M = d(\sigma_D^{-1}(u), \partial \Delta^l).$$

Notice $M > 0$. If $l = 0$, put $M = 0$. For the l-simplex $S$, define

$$S^M = \{ x \in S : d(x, \partial S) \geq M \}.$$ 

This is a closed subset of $\text{int}(S)$, containing $\sigma_D^{-1}(u)$. For any $k \in \mathbb{N}$, let

$$m_k = \min \{ d(F^M, H) : F, H \text{ are faces of } \Delta^k \text{ with } \dim(F) = l, F \nsubseteq H \},$$

note $m_k \in \mathbb{R}^+ \cup \{ \infty \}$. Of course if $k \geq 1, l$, then $\Delta^k$ has an l-dimensional face $F$, for which $F^M \neq \emptyset$, and a face $H$ with $F \nsubseteq H$, so in this case $m_k \in \mathbb{R}^+$. Let

$$m = \min \{ m_k : k \leq \dim(D) \},$$

where $\dim(D)$ denotes the dimension of $D$. Notice that $m \in \mathbb{R}^+$.  

**An important observation.** Let $C$ be any $k$-cell of $D$ whose closure contains $u$, and denote its characteristic map by $\sigma_C$. For any $x \in \sigma_C^{-1}(u)$, let $S_x$ be the unique face of the simplex $\Delta^k$ containing $x$ in its interior. Then $S_x$ is the $l$-dimensional simplex $\Delta^l$ and $\sigma_C|_{S_x} = \sigma_D$, so $x \in \sigma_D^{-1}(u) \subseteq S_x^M$. Suppose $d(x,y) < m$ for some $y \in \Delta^k$, and let $H$ be any face of $\Delta^k$ with $y \in H$. Then of course $d(S_x^M, H) < m \leq m_k$, so the definition of $m_k$ implies $S_x \subseteq H$.  

```
Let $N_u$ be the neighborhood of $u$ defined as follows. For each cell $C$ of $\mathcal{D}$, consider its characteristic map $\sigma_C : \Delta^k \to \mathcal{D}$, and let $U_C$ be the open subset of $\Delta^k$ defined by

$$U_C = \bigcup \{ B_{m/2}(x) : x \in \sigma_C^{-1}(u) \},$$

where $B_{m/2}(x)$ is the open ball of radius $m/2$ at $x$ in $\Delta^k$. Notice that if $u \notin \overline{C}$, then $U_C = \emptyset$.

Let

$$N_u = \bigcup \{ \sigma_C(U_C) : C \text{ is a cell of } \mathcal{D} \}.$$

Clearly $u \in N_u$. We also claim that for any $k$-cell $C$ in $\mathcal{D}$, $\sigma^{-1}(N_u) = U_C$. The containment $\sigma^{-1}(N_u) \supseteq U_C$ is immediate from the definition.

We need to show that $\sigma^{-1}(N_u) \subseteq U_C$. Assume $z \in \sigma^{-1}(N_u)$, and let $S_z$ be the unique face of $\Delta^k$ containing $z$ in its interior. Then $\sigma_C|_{S_z}$ is a characteristic map of some cell $B$. Let $y = \sigma_B(z) \in B \cap N_u$. Then $y \in \sigma_C(U_C)$ for some $k'$-cell $C'$ with $B \subseteq \overline{C'}$. We can rewrite $y \in \sigma_C(U_C)$ as $d(\sigma_C^{-1}(y), \sigma_C^{-1}(u)) < \frac{m}{2}$, that is, $d(\tilde{y}, \tilde{u}) < \frac{m}{2}$ for some $\tilde{y} \in \sigma_C^{-1}(y)$, $\tilde{u} \in \sigma_C^{-1}(u)$. Let $S_u$ be the unique face of $\Delta^{k'}$ containing $\tilde{u}$ in its interior, and $S_{\tilde{y}}$ the unique face containing $\tilde{y}$ in its interior. Then by the important observation, $d(\tilde{y}, \tilde{u}) < \frac{m}{2}$ yields that $S_u \subseteq S_{\tilde{y}}$, in particular $\tilde{u} \in S_{\tilde{y}}$. Notice that $\sigma_C(S_{\tilde{y}}) = B$, so $\sigma_C^{-1}, \sigma_B$, and $\sigma_C^{-1}(y) = \sigma_B^{-1}(y) = \{ z \}$, hence $\tilde{y} = \tilde{z}$. By $\sigma_C|_{S_u} = \sigma_B$ we also have $\tilde{u} \in \sigma_C^{-1}(u) = \sigma_B^{-1}(u) = \sigma_C^{-1}(u)$, so $d(z, \sigma_C^{-1}(u)) \leq d(z, \tilde{u}) < \frac{m}{2}$, so $z \in U_C$ indeed.

This shows that $\sigma^{-1}(N_u) = U_C$, which also immediately yields that $N_u$ is open. It also shows that if $C$ is a cell such that $u \notin \overline{C}$, then $N_u \cap \overline{C} = \emptyset$.

Let $w_1$ and $w_2$ be distinct points of $\mathcal{D}$ with $f(w_1) = f(w_2)$. For each $w_i$, there is exactly one cell $D_i$ such that $w_i \in D_i$. Since $f$ restricted to any cell of $\mathcal{D}$ is a homeomorphism, $f(w_1) = f(w_2)$ implies $D_1 \neq D_2$, and $f(D_1) = f(D_2)$; in particular, $D_1$ and $D_2$ must be of the same dimension, say $n$. Notice that since $u \in \overline{D_1 \cap D_2}$, $D \subseteq \overline{D_1 \cap D_2}$.

Denote the characteristic map of $D_i$ by $\sigma_{D_i}$, the common cell $f(D_1) = f(D_2)$ by $C$, the characteristic map of $C$ by $\sigma_C$. Since $\sigma_{D_i}$ restricted to $\text{int}(\Delta^n)$ is a homeomorphism, $\sigma_{D_i}^{-1}(w_i)$ is a single point $\tilde{w}_i$, which satisfies $\sigma_C(\tilde{w}_i) = f \circ \sigma_{D_1}(\tilde{w}_1) = f \circ \sigma_{D_2}(\tilde{w}_2) = \sigma_C(\tilde{w}_2)$.

But as $\sigma_C$ restricted to $\text{int}(\Delta^n)$ is again a homeomorphism, the above implies $\tilde{w}_1 = \tilde{w}_2$, which we will denote by just $\tilde{w}$.

Now consider $\sigma_{D_1}^{-1}(N_u) = U_{D_1}$. Notice that $\tilde{w} \in U_{D_1}$, so there exists $\tilde{u}_i$ with $\sigma_{D_i}(\tilde{u}_i) = u_i$, and $d(\tilde{u}_i, \tilde{w}) < \frac{m}{2}$. We have $\tilde{u}_i \in \partial \Delta^n$ as $u \in \partial D_i$. Let $F_i$ be the face of $\Delta^n$ with $\tilde{u}_i \in \text{int}(F_i)$. Then

$$d(\tilde{u}_1, F_2) \leq d(\tilde{u}_1, \tilde{w}) + d(\tilde{w}, \tilde{u}_2) < m,$$

so by our important observation, we have $F_1 \subseteq F_2$, that is, $F_1 = F_2 = : F$. From $\sigma_{D_1}(F) = D$ we obtain $\sigma_{D_1}|_F = \sigma_D = \sigma_{D_2}|_F$. Denote the preimage of $u$ under the common map $\sigma_D|_F$ by $\tilde{u}$. Then $\tilde{u} \in \sigma_{D_1}^{-1}(u) \cap \sigma_{D_2}^{-1}(u)$, thus $D_1$ and $D_2$ are such that $f(D_1) = f(D_2)$, $\sigma_{D_1}^{-1}(u) \cap \sigma_{D_2}^{-1}(u) \neq \emptyset$, $D_1 \neq D_2$, as required.

**Proposition 8.2.** A $\Delta$-map $f : \mathcal{D} \to \mathcal{C}$ is an immersion if and only if it is locally injective at each 0-cell of $\mathcal{D}$.

**Proof.** Assume first that $f$ is locally injective at each 0-cell, and let $v \in \mathcal{D}^0$. Let $D_1, D_2 \in \text{star}_D(v)$ and assume $f(D_1) = f(D_2)$. If $D_1, D_2$ are cells, then by $v = \alpha(D_1) = \alpha(D_2)$ we have $v_0 \in \sigma_{D_1}^{-1}(v) \cap \sigma_{D_2}^{-1}(v)$; if they are ghost edges with $D_1 = e_1^{-1}$, then similarly $v_1 \in \sigma_{D_1}^{-1}(v) \cap \sigma_{D_2}^{-1}(v)$. In both cases, applying Lemma 8.1, we obtain $D_1 = D_2$. Hence $f$ is injective on the star set of $v$ for any $v \in \mathcal{D}^0$, thus it is an immersion.

For the converse, assume $f$ is an immersion, and assume for contradiction that $f$ is not locally injective at some 0-cell $v$. Then by Lemma 8.1, there exist distinct cells $D_1, D_2$ with $f(D_1) = f(D_2)$ and $\sigma_{D_1}^{-1}(v) \cap \sigma_{D_2}^{-1}(v) \neq \emptyset$. Let $D_1, D_2$ be such examples of minimal dimension $n$. If $\alpha(D_0) = \alpha(D_1)$,
then $D_1, D_2 \in \text{star}_D(v)$ and so $f$ is not an immersion, a contradiction. So assume this is not the case.

Take $\tilde{v} \in \sigma_{D_1}^{-1}(v) \cap \sigma_{D_2}^{-1}(v)$, then $\tilde{v} \neq v_0$. If $n = 1$, that is, $D_1$ and $D_2$ are edges, then $\tilde{v} = v_1$. In this case $D_1', D_2' \in \text{star}_D(v)$, and so $f$ is not injective on $\text{star}_D(v)$, a contradiction.

Now assume $n \geq 2$. As $f \circ \sigma_{D_1} = f \circ \sigma_{D_2}$, note that for each proper face $F$ of $\Delta^k$ with $\tilde{v} \in F$, the cells $D_1' = \sigma_{D_1}(F)$ and $D_2' = \sigma_{D_2}(F)$ also satisfy

$$f(D_1') = f \circ \sigma_{D_1}(F) = f \circ \sigma_{D_2}(F) = f(D_2'),$$

and $\tilde{v} \in \sigma_{D_1}^{-1}(v) \cap \sigma_{D_2}^{-1}(v)$, hence $D_1' = D_2'$ by the minimality of $n$. Take $F$ to be the edge $[v_0, \tilde{v}]$ of $\Delta^n$. Then the common cell $\sigma_{D_1}(F)$ is an edge of $D$ from $\alpha(D_1)$ to $v$, so $\alpha(D_1) = \alpha(D_2)$. Denoting the latter vertex by $u$, we obtain that $D_1, D_2 \in \text{star}_D(u)$, which is, again, a contradiction. \( \square \)

**Proposition 8.3.** A $\Delta$-map $f: D \rightarrow \mathcal{C}$ is locally injective at 0-cells if and only if it is locally injective.

**Proof.** For the nontrivial implication, let $u \in D$ be any point. Take any cells $D_1, D_2$ with $f(D_1) = f(D_2)$, $\sigma_{D_1}^{-1}(u) \cap \sigma_{D_2}^{-1}(u) \neq \emptyset$: by Lemma 8.1 we only need to show that $D_1 = D_2$. Let $\tilde{u} \in \sigma_{D_1}^{-1}(u) \cap \sigma_{D_2}^{-1}(u)$, and let $F$ be the unique face of $\Delta^n$ containing $\tilde{u}$ in its interior. Then $\sigma_{D_1}|_F$ is the characteristic map of the unique cell containing $\sigma_{D_1}(\tilde{u}) = u$, so $\sigma_{D_1}|_F = \sigma_{D_2}|_F$. Let $\tilde{v}$ be an arbitrary 0-cell in $F$, and let $v$ be the common 0-cell $\sigma_{D_1}(\tilde{v})$. By assumption, $f$ is locally injective at $v$, and as $\tilde{v} \in \sigma_{D_1}^{-1}(v) \cap \sigma_{D_2}^{-1}(v)$, applying Lemma 8.1 we obtain that $D_1 = D_2$ indeed. \( \square \)

**Theorem 8.4.** A $\Delta$-map $f: D \rightarrow \mathcal{C}$ is an immersion if and only if it is locally injective. Furthermore, if $\mathcal{C}$ and $\mathcal{D}$ are locally compact, then $f$ is an immersion if and only if it is a topological immersion.

**Proof.** The first statement is an immediate consequence of Propositions 8.2 and 8.3. In the second statement, it is clear topological immersions are locally injective and hence immersions. For the converse, let $u \in D$ be any point, by local injectivity there is a neighborhood $N_u$ of $u$ such that $f|_{N_u}: N_u \rightarrow f(N_u)$ is a continuous bijection. As $D$ is locally compact, there exists a compact neighborhood $K_u$ of $u$. Let $N$ be an open neighborhood of $u$ such that $\overline{N} \subseteq K_u \cap N_u$. Then $f|_{\overline{N}}: \overline{N} \rightarrow f(\overline{N})$ is a continuous bijection with $\overline{N}$ compact and $f(\overline{N})$ Hausdorff, therefore it is a homeomorphism. Consequently, $f|_N: N \rightarrow f(N)$ is also a homeomorphism. This proves that $f$ is a topological immersion. \( \square \)

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