Existence and uniqueness of functional differential equations with n delay

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Abstract

In this paper we give a necessary and sufficient conditions for the existence and uniqueness of periodic solutions of functional differential equations with n delay \( \frac{d}{dt}x(t) = Ax(t) + \sum_{j=1}^{n} Bx(t - r_j) + f(t) \). The conditions are obtained in terms of R-boundedness of operator valued Fourier multipliers.

Mathematics Subject Classification: xxxxx

Keywords: functional differential equations with n delay, R-bounded.

1 Introduction

Let A and B be two closed linear operators defined on a Banach space X with domains D(A) and D(B), respectively such that \( D(A) \subset D(B) \). In this paper we show existence and uniqueness of solutions for the following differential equation with n delay

\[
\begin{cases}
\frac{d}{dt} x(t) = Ax(t) + \sum_{j=1}^{n} Bx(t - r_j) + f(t) \\
x(0) = x(2\pi).
\end{cases}
\] (1)

where \( f \in L^p([-r_{2\pi}, 0], X) \) for some \( 1 \leq p < \infty, r_{2\pi} = 2\pi N (N \in \mathbb{N}) \) and we suppose B is bounded. The theory of operator-valued Fourier multipliers has attracted the attention of many papers in recent years. For example, this theory was used in [1] to obtain results about equations \( \frac{dx(t)}{dt} = Ax(t) + f(t) \), and in [11] to obtain results about delay equation \( \frac{dx(t)}{dt} = Ax(t) + F(x_t) + f(t) \). In [6], S.Bu studied \( L^p \)-Maximal Regularity of Degenerate delay Equations with Periodic Conditions. We note that in the special case when \( B = 0 \), maximal regularity of Eq. (1) has been studied by Arendt and Bu in \( L^p \)-spaces case and Besov spaces case [[1], [2]], Bu and Kim in TriebelLizorkin spaces case [8]. The corresponding integro-differential equations were treated by Keyantuo
and Lizama [[17], [18]], Bu and Fang [7]. In this paper, we characterize the existence and uniqueness for the n delay equation (1) under the condition that X is a UMD space. Here the operator A is not necessarily the generator of a $C_0$-semigroup. We use the operator valued multiplier Fourier method. The organisation of this work is as follows: In section 2, we present preliminary results on UMD spaces and $L^p$-multiplier. In section 3, we study the existence of periodic strong solution for Eq. (1) with finite delay. In section 4, we give the main abstract result ( theorem [4.2] ) of this work.

1) for every $f \in L^p(T; X); 1 < p < \infty$, there exists a unique $2\pi$-periodic strong $L^p$-solution of Eq. (1).

2) $(ikI - A - \sum_{j=1}^{n} B_{j,k})$ has bounded invertible for all $k \in \mathbb{Z}$ and $\{ik(ikI - A - \sum_{j=1}^{n} B_{j,k})^{-1}\}_{k \in \mathbb{Z}}$ is R-bounded.

2 Preliminary Notes

Let X be a Banach Space. Firstly, we denote By $\mathbb{T}$ the group defined as the quotient $\mathbb{R}/2\pi \mathbb{Z}$. There is an identification between functions on $\mathbb{T}$ and $2\pi$-periodic functions on $\mathbb{R}$. We consider the interval $[0, 2\pi)$ as a model for $\mathbb{T}$.

**Definition 2.1.** A Banach space X is said to be UMD space if the Hilbert transform is bounded on $L^p(\mathbb{R}, X)$ for all $1 < p < \infty$.

**Example 2.2.** [9]
1. Any Hilbert space is an UMD space.
2. $L^p(0, 1)$ are UMD spaces for every $1 < p < \infty$.
3. Any closed subspace of a UMD space is a UMD space.

**Definition 2.3.** [1]
A family of operators $T = (T_j)_{j \in \mathbb{N}^*} \subset B(X, Y)$ is called R-bounded (Rademacher bounded or randomized bounded), if there is a constant $C > 0$ and $p \in [1, \infty)$ such that for each $n \in \mathbb{N}, T_j \in T, x_j \in X$ and for all independent, symmetric, $\{-1, 1\}$-valued random variables $r_j$ on a probability space $(\Omega, M, \mu)$ the inequality

$$\left\| \sum_{j=1}^{n} r_j T_j x_j \right\|_{L^p(0, 1; Y)} \leq C \left\| \sum_{j=1}^{n} r_j x_j \right\|_{L^p(0, 1; X)}$$

is valid. The smallest $C$ is called R-bounded of $(T_j)_{j \in \mathbb{N}^*}$ and it is denoted by $R_p(T)$.

**Definition 2.4.** [11]
For $1 \leq p < \infty$, a sequence $\{M_k\}_{k \in \mathbb{Z}} \subset B(X, Y)$ is said to be an $L^p$-multiplier if for each $f \in L^p(\mathbb{T}, X)$, there exists $u \in L^p(\mathbb{T}, Y)$ such that $\hat{u}(k) = M_k \hat{f}(k)$ for all $k \in \mathbb{Z}$. 


Proposition 2.5. [1, Proposition 1.11] Let $X$ be a Banach space and \( \{ M_k \}_{k \in \mathbb{Z}} \) be an \( L^p \)-multiplier, where \( 1 \leq p < \infty \). Then the set \( \{ M_k \}_{k \in \mathbb{Z}} \) is R-bounded.

Theorem 2.6. (Marcinkiewicz operator-valued multiplier Theorem). Let \( X, Y \) be UMD spaces and \( \{ M_k \}_{k \in \mathbb{Z}} \subset \mathcal{B}(X,Y) \). If the sets \( \{ M_k \}_{k \in \mathbb{Z}} \) and \( \{ k(M_{k+1} - M_k) \}_{k \in \mathbb{Z}} \) are R-bounded, then \( \{ M_k \}_{k \in \mathbb{Z}} \) is an \( L^p \)-multiplier for \( 1 < p < \infty \).

We observe that the condition of R-boundedness for \( (M_k)_{k \in \mathbb{Z}} \) is necessary.

Remark 2.7. [13] Let \( f \in L^1(\mathbb{T}; X) \). If \( g(t) = \int_0^t f(s)ds \) and \( k \in \mathbb{Z}, k \neq 0 \), then
\[
\hat{g}(k) = \frac{i}{k} \hat{f}(0) - \frac{1}{k} \hat{f}(k).
\]

3 A criterion for periodic solutions

Definition 3.2. Let \( f \in L^p(\mathbb{T}; X) \). A function \( x \in H^1_p(\mathbb{T}; X) \) is said to be a \( 2\pi \)-periodic strong \( L^p \)-solution of Eq. (1) if \( x(t) \in D(A) \) for all \( t \geq 0 \) and Eq. (1) holds almost everywhere.

Lemma 3.3. [1, Lemme 2.1] Let \( 1 \leq p < \infty \) and \( u, v \in L^p(\mathbb{T}; X) \). Then the following assertions are equivalent:
(i) \( \int_0^{2\pi} v(s)ds = 0 \) and there exists \( x \in X \) such that \( u(t) = x + \int_0^t v(s)ds \).
(ii) \( \hat{v}(k) = ik \hat{u}(k) \) for any \( k \in \mathbb{Z} \).

Definition 3.4. For \( 1 \leq p < \infty \), we say that a sequence \( \{ M_k \}_{k \in \mathbb{Z}} \subset \mathcal{B}(X,Y) \) is an \( (L^p, H^1_p) \)-multiplier, if for each \( f \in L^p(\mathbb{T}, X) \) there exists \( u \in H^1_p(\mathbb{T}, Y) \) such that \( \hat{u}(k) = M_k \hat{f}(k) \) for all \( k \in \mathbb{Z} \).

Lemma 3.5. Let \( 1 \leq p < \infty \) and \( (M_k)_{k \in \mathbb{Z}} \subset \mathcal{B}(X) \) (\( \mathcal{B}(X) \) is the set of all bounded linear operators from \( X \) to \( X \)). Then the following assertions are equivalent:
(i) \( (M_k)_{k \in \mathbb{Z}} \) is an \( (L^p, H^1_p) \)-multiplier.
(ii) \( (ikM_k)_{k \in \mathbb{Z}} \) is an \( (L^p, L^p) \)-multiplier.

Proposition 3.6. Let \( A \) be a closed linear operator defined on an UMD space \( X \). Suppose that \( \sigma_Z(\Delta) = \phi \). Then the following assertions are equivalent:
(i) \( \{ ik(ikI - A - \sum_{j=1}^n B_{j,k})^{-1} \}_{k \in \mathbb{Z}} \) is an \( L^p \)-multiplier for \( 1 < p < \infty \).
(ii) \( \{ ik(ikI - A - \sum_{j=1}^n B_{j,k})^{-1} \}_{k \in \mathbb{Z}} \) is R-bounded.
Proof. By [1, Proposition 1.11] it follows that (i) implies (ii). Conversely, define \( M_k = ik(C_k - A)^{-1} \), where \( C_k = ik - \sum_{j=1}^{n} B_{j,k} \). By Theorem 2.6 it is sufficient to prove that the set \( \{k(M_{k+1} - M_k)\}_{k \in \mathbb{Z}} \) is R-bounded. We claim first that the set \( \{\sum_{j=1}^{n} B_{j,k}\}_{k \in \mathbb{Z}} \) is R-bounded.

since given \( x_j \in D(A) \) we have :

\[
\left\| \sum_{l=1}^{m} r_l(\sum_{j=1}^{n} B_{j,l})x_l \right\|_{L^p(0,1;X)}^p = \int_0^1 \left\| \sum_{l=1}^{m} r_l(t)B(\sum_{j=1}^{n} e^{-ir_jx_l}) \right\|_{X}^p dt
\]

\[
= \int_0^1 \left\| B(\sum_{l=1}^{m} r_l(t) \sum_{j=1}^{n} e^{-ir_jx_l}) \right\|_{X}^p dt
\]

\[
\leq ||B||^p \int_0^1 \left\| \sum_{l=1}^{m} r_l(t) \sum_{j=1}^{n} e^{-ir_jx_l} \right\|_{X}^p dt
\]

By (Lemma 1.7, [1]) we obtain that

\[
\left\| \sum_{l=1}^{m} r_l(\sum_{j=1}^{n} B_{j,l})x_l \right\|_{L^p(0,1;X)}^p \leq 2n^p||B||^p \int_0^1 \left\| \sum_{l=1}^{m} r_l(t)x_l \right\|_{X}^p dt
\]

We conclude that

\[
\left\| \sum_{l=1}^{m} r_l(\sum_{j=1}^{n} B_{j,l})x_l \right\|_{L^p(0,1;X)} \leq 2^{1/p}n||B||.
\]

and the claim is proved. Next. We note the following identities

\[
k [M_{k+1} - M_k] = k [i(k+1)(C_{k+1} - AD)^{-1} - ik(C_k - AD)^{-1}]
\]

\[
= k(C_{k+1} - AD)^{-1}[i(k+1)(C_k - AD) - ik(C_{k+1} - AD)](C_k - AD)^{-1}
\]

\[
= k(C_{k+1} - AD)^{-1}[ik(C_k - C_{k+1}) + i(C_k - A)][(C_k - AD)^{-1}
\]

\[
= k(C_{k+1} - AD)^{-1}[ik(C_k - C_{k+1})(C_k - AD)^{-1} + iI]
\]

\[
= \frac{-ik}{k+1} M_{k+1}(C_k - C_{k+1}) M_k + \frac{k}{k+1} M_{k+1}.
\]

We have

\[
C_k - C_{k+1} = -iI + \sum_{j=1}^{n} B e^{-ir_j}(1 - e^{-ir_j}).
\]

Since products and sums of R-bounded sequences is R-bounded [11, Remark 2.2]. Then \( \{k(M_{k+1} - M_k)\}_{k \in \mathbb{Z}} \) is R-bounded and by theorem 2.6, \( \{M_k\}_{k \in \mathbb{Z}} \) is an \( L^p \)-multiplier. \( \square \)

**Theorem 3.7.** Let \( X \) be a Banach space. Suppose that for every \( f \in L^p(\mathbb{T}; X) \) there exists a unique strong solution of Eq. (1) for \( 1 \leq p < \infty \). Then
1. for every \( k \in \mathbb{Z} \) the operator \( \Delta_k = (ikI - A - \sum_{j=1}^{n} B_{j,k}) \) has bounded inverse

2. \( \{ik\Delta_k^{-1}\}_{k \in \mathbb{Z}} \) is R-bounded.

Before to give the proof of Theorem (3.7), we need the following Lemma.

**Lemma 3.8.** if \( (ikI - A - \sum_{j=1}^{n} B_{j,k}(x)) = 0 \) for all \( k \in \mathbb{Z} \), then \( u(t) = e^{ikt}x \) is a \( 2\pi \)-periodic strong \( L^p \)-solution of the following equation (1) corresponding to the function \( f = 0 \).

**Proof.** \( (ikI - A - \sum_{j=1}^{n} B_{j,k}(x)) = 0 \Rightarrow ikx = Ax + \sum_{j=1}^{n} B_{j,k}x. \)

We have \( u(t) = e^{ikt}x \) then

\[
u'(t) = ike^{ikt}x = e^{ikt}(ikx) = e^{ikt}[Ax + \sum_{j=1}^{n} B_{j,k}x] = Au(t) + \sum_{j=1}^{n} Bu(t - r_j).
\]

**Proof of Theorem 3.7** 1) Let \( k \in \mathbb{Z} \) and \( y \in X \). Then for \( f(t) = e^{ikt}y \), there exists \( x \in H^{1,p}(\mathbb{T}; X) \) such that:

\[
\frac{d}{dt} x(t) = Ax(t) + \sum_{j=1}^{n} Bx(t - r_j) + f(t)
\]

Taking Fourier transform, by Lemma 3.3 we have:

\[
\hat{x}'(k) = ik\hat{x}(k) = A\hat{x}(k) + \sum_{j=1}^{n} B_{j,k}\hat{x}(k) + \hat{f}(k).
\]

Then we obtain: \( (ikI - A - \sum_{j=1}^{n} B_{j,k})\hat{x}(k) = \hat{f}(k) = y \Rightarrow (ikI - A - \sum_{j=1}^{n} B_{j,k}) \) is surjective.

If \( (ikI - A - \sum_{j=1}^{n} B_{j,k})u = 0 \), then by Lemma 3.8 \( x(t) = e^{ikt}u \) is a \( 2\pi \)-periodic strong \( L^p \)-solution of Eq. (1) corresponding to the function \( f = 0 \) Hence \( x(t) = 0 \) and \( u = 0 \) then \( (ikI - A - \sum_{j=1}^{n} B_{j,k}) \) is injective.

2) Let \( f \in L^p(\mathbb{T}, X) \). By hypothesis, there exists a unique \( x \in H^{1,p}(\mathbb{T}, X) \) such that the Eq. (1) is valid. Taking Fourier transforms, we deduce that \( \hat{x}(k) = (ikI - A - \sum_{j=1}^{n} B_{j,k})^{-1} \hat{f}(k) \) for all \( k \in \mathbb{Z} \). Hence

\[
 ik\hat{x}(k) = ik(ikI - A - \sum_{j=1}^{n} B_{j,k})^{-1} \hat{f}(k)
\]

On the other hand, since \( x \in H^{1,p}(\mathbb{T}, X) \), there exists \( v \in L^p(\mathbb{T}, X) \) such that \( \hat{v}(k) = ik\hat{x}(k) = ik(ikI - A - \sum_{j=1}^{n} B_{j,k})^{-1} \hat{f}(k) \) i.e \( \{ik\Delta_k^{-1}\}_{k \in \mathbb{Z}} \) is an \( L^p \)-multiplier. Then \( \{ik\Delta_k^{-1}\}_{k \in \mathbb{Z}} \) is R-bounded. \( \square \)
4 Existence of mild solutions of Eq. (1)

It is well known that in many important applications the operator $A$ is the infinitesimal generator of $C_0$-semigroup $(T(t))_{t \geq 0}$ on the space $X$. Let $A$ be a generator of a $C_0$-semigroup $(T(t))_{t \geq 0}$.

**Definition 4.1.** Assume that $A$ generates a $C_0$-semigroup $(T(t))_{t \geq 0}$ on $X$. A function $x$ is called a mild solution of Eq. (1) if:

$$x(t) = T(t)\varphi + \int_0^t T(t-s)\left(\sum_{j=1}^n Bx(s-r_j) + f(s)\right)ds \text{ for } 0 \leq t \leq 2\pi.$$ 

**Remark 4.2.** [14, Remark 4.2]

Let $(T(t))_{t \geq 0}$ be the $C_0$-semigroup generated by $A$. If $g : [0, a] \to X$ is a continuous function, then $\int_0^t \int_0^s T(t-\xi)g(\xi)d\xi ds \in D(A)$ and

$$A \int_0^t \int_0^s T(t-\xi)g(\xi)d\xi ds = \int_0^t (T(t-s) - I)g(s)ds \text{ for all } 0 \leq t \leq a.$$ 

**Lemma 4.3.** [10]

Assume that $A$ generates a $C_0$-semigroup $(T(t))_{t \geq 0}$ on $X$, if $x$ is a mild solution of Eq. (1) then

$$x(t) = \varphi + A \int_0^t x(s)ds + \int_0^t \left(\sum_{j=1}^n Bx(s-r_j) + f(s)\right)ds \text{ for } 0 \leq t \leq 2\pi.$$ 

**Theorem 4.4.** Assume that $A$ generates a $C_0$-semigroup $(T(t))_{t \geq 0}$ on $X$ and $f \in L^p(\mathbb{T}, X)$ for some $1 \leq p < \infty$, if $x$ is a mild solution of Eq. (1). Then

$$(ikI - A - \sum_{j=1}^n B_{j,k})\hat{x}(k) = \hat{f}(k) \text{ for all } k \in \mathbb{Z}.$$ 

**Proof.** Let $x$ be a mild solution of Eq. (1). Then by Lemma 4.3, we have

$$x(t) = \varphi + A \int_0^t x(s)ds + \int_0^t \left(\sum_{j=1}^n Bx(s-r_j) + f(s)\right)ds$$

For $t = 2\pi$, we have

$$x(2\pi) = \varphi + A \int_0^{2\pi} x(s)ds + \int_0^{2\pi} \left(\sum_{j=1}^n Bx(s-r_j) + f(s)\right)ds;$$
Since: \( x(2\pi) = \varphi \), then

\[
A \int_0^{2\pi} x(s)ds + \int_0^{2\pi} \left( \sum_{j=1}^n Bx(s - r_j) + f(s) \right) ds = 0
\]

\[
\Rightarrow \frac{1}{2\pi} A \int_0^{2\pi} x(s)ds + \frac{1}{2\pi} \int_0^{2\pi} \left( \sum_{j=1}^n Bx(s - r_j) + f(s) \right) ds = 0
\]

\[
\Rightarrow \frac{1}{2\pi} A \int_0^{2\pi} x(s)ds + \frac{1}{2\pi} \int_0^{2\pi} \sum_{j=1}^n Bx(s - r_j) ds + \frac{1}{2\pi} \int_0^{2\pi} f(s) ds = 0
\]

\[
\Rightarrow \frac{1}{2\pi} A \int_0^{2\pi} e^{-i0s} x(s) ds + \frac{1}{2\pi} \int_0^{2\pi} e^{-i0s} \sum_{j=1}^n Bx(s - r_j) ds + \frac{1}{2\pi} \int_0^{2\pi} e^{-i0s} f(s) ds = 0
\]

\[
\Rightarrow (0 - A - \sum_{j=1}^n B_{j,0}) \hat{x}(0) = \hat{f}(0),
\]

which shows that the assertion holds for \( k = 0 \).

Now, define

\[
v(t) = \int_0^t x(s) ds
\]

and

\[
g(t) = x(t) - \varphi - \int_0^t \left( \sum_{j=1}^n Bx(s - r_j) + f(s) \right) ds
\]

by Remark 2.7 We have:

\[
\hat{v}(k) = \frac{i}{k} \hat{x}(0) - \frac{i}{k} \hat{x}(k)
\]

\[
A\hat{v}(k) = \frac{i}{k} A\hat{x}(0) - \frac{i}{k} A\hat{x}(k)
\]

and

\[
\hat{g}(k) = \hat{x}(k) - \left[ \frac{i}{k} G_0 \hat{x}(0) - \frac{i}{k} G_k \hat{x}(k) \right] - \left[ \frac{i}{k} \hat{f}(0) - \frac{i}{k} \hat{f}(k) \right]
\]

\[
= \hat{x}(k) - \frac{i}{k} G_0 \hat{x}(0) + \frac{i}{k} G_k \hat{x}(k) - \frac{i}{k} \hat{f}(0) + \frac{i}{k} \hat{f}(k)
\]

\[
\square
\]

**Corollary 4.5.** Assume that \( A \) generates a \( C_0 \)-semigroup \((T(t))_{t \geq 0}\) on \( X \) and let \( f \in L^p(\mathbb{T}, X) : 1 \leq p < \infty \) and \( x \) be a mild solution of Eq. (1). If \((ikI - A - \sum_{j=1}^n B_{j,k})\) has a bounded inverse. Then \((ikI - A - \sum_{j=1}^n B_{j,k})\) is an \( L^p \)-multiplier.

**Proof.** Let \( f \in L^p(\mathbb{T}, X) \) then from Theorem (4.4) we have:

\[
\hat{x}(k) = (ikD_k - AD_k - G_k)^{-1} \hat{f}(k)
\]

for all \( f \in L^p(\mathbb{T}, X) \), then \((ikI - A - \sum_{j=1}^n B_{j,k})^{-1}\) is an \( L^p \)-multiplier. \( \square \)
5 Main Result

Our main result in this work is to establish that the converse of theorem (3.7) and corollary (4.5) is true, provided $X$ is an UMD space.

**Theorem 5.1.** *(Fejer Theorem)*: Let $f \in L^p(\mathbb{T}, X)$. Then

$$f = \lim_{n \to +\infty} \sigma_n(f)$$

where $\sigma_n(f) = \frac{1}{n+1} \sum_{m=0}^{n} \sum_{k=-m}^{m} e_k \hat{f}(k)$, with $e_k(t) = e^{ikt}$.

**Theorem 5.2.** Let $X$ be an UMD space and $A : D(A) \subset X \to X$ be a closed linear operator. Then the following assertions are equivalent for $1 < p < \infty$.

1) for every $f \in L^p(\mathbb{T}, X)$ there exists a unique strong $L^p$-solution of Eq.(1).

2) $\sigma_{Z}(\Delta) = \phi$ and $\{ik\Delta_k^{-1}\}_{k \in \mathbb{Z}}$ is $R$-bounded.

**Proof.** $1 \Rightarrow 2)$ see Theorem 3.7.

$1 \Leftarrow 2)$ Let $f \in L^p(\mathbb{T}; X)$. Define $\Delta_k = (ikI - A - \sum_{j=1}^{n} B_j) e_k \hat{f}(k)$,

By Proposition 3.6, the family $\{ik\Delta_k^{-1}\}_{k \in \mathbb{Z}}$ is an $L^p$-multiplier it is equivalent to the family $\{\Delta_k^{-1}\}_{k \in \mathbb{Z}}$ is an $L^p$-multiplier that maps $L^p(\mathbb{T}; X)$ into $H^{1,p}(\mathbb{T}; X)$, namely there exists $x \in H^{1,p}(\mathbb{T}, X)$ such that

$$\hat{x}(k) = \Delta_k^{-1} \hat{f}(k) = (ikI - A - \sum_{j=1}^{n} B_j) \hat{f}(k)$$ (2)

In particular, $x \in L^p(\mathbb{T}; X)$ and there exists $v \in L^p(\mathbb{T}; X)$ such that $\hat{v}(k) = ik\hat{x}(k)$

$$\hat{x}'(k) := \hat{v}(k) = ik\hat{x}(k)$$ (3)

By Theorem 5.1 we have for $j \in \{1...n\}$

$$x(t - r_j) = \lim_{l \to +\infty} \frac{1}{l+1} \sum_{m=0}^{l} \sum_{k=-m}^{m} e^{ikt} e^{-ikr_j} \hat{x}(k)$$

Then, since $B$ is bounded linear

$$\sum_{j=1}^{n} Bx(t - r_j) = \lim_{l \to +\infty} \frac{1}{l+1} \sum_{m=0}^{l} \sum_{k=-m}^{m} e^{ikt} (\sum_{j=1}^{n} B_j \hat{x}(k))$$

By (2) and (3) we have:

$$\hat{x}'(k) = ik\hat{x}(k) = A\hat{x}(k) + \sum_{j=1}^{n} B_{j,k}\hat{x}(k) + \hat{f}(k), \text{ for all } k \in \mathbb{Z}$$
Then using that \( A \) and \( B \) are closed we conclude that \( x(t) \in D(A) \) [[1], Lemma 3.1] and from the uniqueness theorem of Fourier coefficients that

\[
x'(t) = Ax(t) + \sum_{j=1}^{n} B x(t - r_j) + f(t).
\]

We have \( x \in H^{1,p}(\mathbb{T}, X) \) then by lemma 3.3, \( x(0) = x(2\pi) \), then the Eq. (1) has a unique \( 2\pi \)-periodic strong \( L^p \)-solution. \( \square \)

**Theorem 5.3.** Let \( 1 \leq p < \infty \). Assume that \( A \) generates a \( C_0 \)-semigroup \( (T(t))_{t \geq 0} \) on \( X \). If \( \sigma_Z(\Delta) = \emptyset \) and \( (ikI - A - \sum_{j=1}^{n} B_{j,k})^{-1} \) is an \( L^p \)-multiplier then there exists a unique mild solution periodic of Eq. (1).

**Proof.** For \( f \in L^p(\mathbb{T}; X) \) we define

\[
f_l(t) = \frac{1}{l+1} \sum_{m=0}^{l} \sum_{k=-m}^{m} e^{ikt} \hat{f}(k)
\]

By the Fejér Theorem we can assert that \( f_l \to f \) as \( l \to \infty \) for the norm in \( L^p(\mathbb{T}; X) \). We have \( (ikI - A - \sum_{j=1}^{n} B_{j,k})^{-1} \) is an \( L^p \)-multiplier then there exists \( x \in L^p(\mathbb{T}; X) \) such that \( \hat{x}(k) = (ikI - A - \sum_{j=1}^{n} B_{j,k})^{-1} \hat{f}(k) \)

put

\[
x_l(t) = \frac{1}{l+1} \sum_{m=0}^{l} \sum_{k=-m}^{m} e^{ikt} (ikI - A - \sum_{j=1}^{n} B_{j,k})^{-1} \hat{f}(k)
\]

Using again the Fejér Theorem we obtain that \( x_n(t) \to x(t) \) (as \( n \to \infty \)) and \( x_n(t) \) is strong \( L^p \)-solution of Eq. (1) and \( x_n(t) \) verified

\[
x_l(t) = T(t) \varphi_l + \int_{0}^{t} T(t-s) \left( \sum_{j=1}^{n} B x_l(s - r_j) + f_l(s) \right) ds \quad (4)
\]

With \( t = 2\pi \) we obtain

\[
x_l(2\pi) = T(2\pi) \varphi_l + \int_{0}^{2\pi} T(2\pi-s) \left( \sum_{j=1}^{n} B x_l(s - r_j) + f_l(s) \right) ds.
\]

from which we infer that the sequence \( (\varphi_l)_n \) is convergent to some element \( \varphi \) as \( l \to \infty \)( \( \varphi_l = x_l(0) = x_l(2\pi) \)). Moreover, \( \varphi \) satisfies the condition

\[
\varphi = T(2\pi) \varphi + \int_{0}^{2\pi} T(2\pi-s) \left( \sum_{j=1}^{n} B x(s - r_j) + f(s) \right) ds. \quad (5)
\]
Taking the limit as \( l \) goes to infinity in (4), we can write

\[
x(t) = T(t)\varphi + \int_0^t T(t-s)\left(\sum_{j=1}^n Bx(s-r_j) + f(s)\right)ds := g(t)
\]

\[
g(2\pi) = T(2\pi)y + \int_0^{2\pi} T(2\pi-s)\left(\sum_{j=1}^n Bx(s-r_j) + f(s)\right)ds \rightarrow \varphi = g(0) \tag{5}
\]

Then \( x(2\pi) = \varphi \Rightarrow x(2\pi) = x(0) \), we conclude that \( x \) is a \( 2\pi \)-periodic mild solution of Eq. (1).

Acknowledgements. This is a text of acknowledgements.

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Received: Month xx, 20xx