Nonlinear Second Order Ode’s

Factorizations and Particular Solutions

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(Received April 19, 2005)

We present particular solutions for the following important nonlinear second order differential equations: modified Emden, generalized Lienard, convective Fisher and generalized Burgers-Huxley. For the latter two equations these solutions are obtained in the travelling frame. All these particular solutions are the result of extending a simple and efficient factorization method that we developed in Phys. Rev. E 71 (2005), 046607.

§1. Introduction

The purpose of this paper is to obtain, through the factorization technique, particular solutions of the following type of differential equations:

\[ \ddot{u} + g(u)\dot{u} + F(u) = 0, \]  

(1.1)

where the dot means the derivative \( D = \frac{d}{d\tau} \), and \( g(u) \) and \( F(u) \) could in principle be arbitrary functions of \( u \). This is a generalization of what we did in a recent paper for the simpler equations with \( g(u) = \gamma \), where \( \gamma \) is a constant parameter.\(^1\) Factorizing Eq. (1.1) means to write it in the form

\[ [D - \phi_2(u)] [D - \phi_1(u)] u = 0. \]  

(1.2)

Performing the product of differential operators leads to the equation

\[ \ddot{u} - \frac{d\phi_1}{du} \dot{u}^2 - \phi_1 \dot{u} - \phi_2 \ddot{u} + \phi_1 \phi_2 u = 0, \]  

(1.3)

for which one very effective way of grouping the terms is\(^1\)

\[ \ddot{u} - \left( \phi_1 + \phi_2 + \frac{d\phi_1}{du} \right) \dot{u} + \phi_1 \phi_2 u = 0. \]  

(1.4)

Identifying Eqs. (1.1) and (1.4) leads to the conditions

\[ g(u) = - \left( \phi_1 + \phi_2 + \frac{d\phi_1}{du} \right), \]  

(1.5)

\[ F(u) = \phi_1 \phi_2 u. \]  

(1.6)

If \( F(u) \) is a polynomial function, then \( g(u) \) will have the same order as the bigger of the factorizing functions \( \phi_1(u) \) and \( \phi_2(u) \), and will also be a function of the constant parameters that enter in the expression of \( F(u) \).
In this research, we extend the method to the following cases: the modified Emden equation, the generalized Lienard equation, the convective Fisher equation, and the generalized Burgers-Huxley equation. All of them have significant applications in nonlinear physics and it is quite useful to know their explicit particular solutions. The present work is a detailed contribution to this issue.

§2. Modified Emden equation

We start with the modified Emden equation with cubic nonlinearity that has been most recently discussed by Chandrasekhar et al.,

\[
\ddot{u} + \alpha \dot{u} + \beta u^3 = 0 .
\]  

(2.1)

1) \(\phi_1(u) = a_1 \sqrt{\beta} u, \phi_2(u) = a_1^{-1} \sqrt{\beta} u,\) \((a_1 \neq 0\) is an arbitrary constant).

Then Eq. (1.5) leads to the following form of the function \(g(u)\)

\[
g_1(u) = -\sqrt{\beta} \left( \frac{2a_1^2 + 1}{a_1} \right) u .
\]

(2.2)

Thus we can identify \(\alpha = -\sqrt{\beta} \left( \frac{2a_1^2 + 1}{a_1} \right),\) or \(a_1^\pm = -\frac{\alpha \pm \sqrt{\alpha^2 - 8\beta}}{4\sqrt{\beta}},\) where we use \(a_1\) as a fitting parameter providing that \(a_1 < 0\) for \(\alpha > 0.\) Equation (2.1) is now rewritten as

\[
\ddot{u} - \sqrt{\beta} \left( 2a_1 + a_1^{-1} \right) \dot{u} + \beta u^3 \equiv \left( D - a_1^{-1} \sqrt{\beta} u \right) \left( D - a_1 \sqrt{\beta} u \right) u = 0 .
\]

(2.3)

Therefore, the compatible first order differential equation is \(\dot{u} - a_1 \sqrt{\beta} u^2 = 0,\) whose integration gives the particular solution of Eq. (2.3)

\[
u_1 = \frac{1}{a_1 \sqrt{\beta}(\tau - \tau_0)} \quad \text{or} \quad \nu_1 = \frac{4}{(\alpha \pm \sqrt{\alpha^2 - 8\beta})(\tau - \tau_0)} ,
\]

(2.4)

where \(\tau_0\) is an integration constant.

2) \(\phi_1(u) = a_1 \sqrt{\beta} u^2, \phi_2(u) = a_1^{-1} \sqrt{\beta}.\) Then, one gets

\[
g_2(u) = -\sqrt{\beta} \left( a_1^{-1} + 3a_1 u^2 \right) .
\]

(2.5)

Therefore, \(g_2\) is quadratic being higher in order than the linear \(g\) of the modified Emden equation. We thus get the particular case \(GE = 3\beta, A = 0\) of the Duffing-van der Pol equation (see case 3 of the next section)

\[
\ddot{u} - \sqrt{\beta} \left( a_1^{-1} + 3a_1 u^2 \right) \dot{u} + \beta u^3 \equiv \left( D - a_1^{-1} \sqrt{\beta} u \right) \left( D - a_1 \sqrt{\beta} u^2 \right) u = 0 ,
\]

(2.6)

which leads to the compatible first order differential equation \(\dot{u} - a_1 \sqrt{\beta} u^3 = 0\) with the solution

\[
u_2 = \frac{1}{\left[ -2a_1 \sqrt{\beta}(\tau - \tau_0) \right]^{1/2}} .
\]

(2.7)
§3. Generalized Lienard equation

Let us consider now the following generalized Lienard equation
\[ \ddot{u} + g(u)\dot{u} + F_3 = 0, \quad (3.1) \]
where \( F_3(u) = Au + Bu^2 + Cu^3 \). We introduce the notation \( \Delta = \sqrt{B^2 - 4AC} \), and assume that \( \Delta^2 > 0 \) holds. Then:

1) \( \phi_1(u) = a_1 \left( \frac{B+\Delta}{2} + Cu \right), \quad \phi_2(u) = a_1^{-1} \left( \frac{B-\Delta}{2C} + u \right) \); \( g(u) \) takes the form
\[
g_1(u) = - \left[ \frac{B+\Delta}{2} a_1 + \frac{B-\Delta}{2C} a_1^{-1} + (2Ca_1 + a_1^{-1}) u \right]. \quad (3.2)\]

For \( g(u) = g_1(u) \), we can factorize Eq. (3.1) in the form
\[
\left[ D - a_1^{-1} \left( \frac{B-\Delta}{2C} + u \right) \right] \left[ D - a_1 \left( \frac{B+\Delta}{2} + Cu \right) \right] u = 0. \quad (3.3)\]

Thus, from the compatible first order differential equation \( \dot{u} - a_1 \left( \frac{B+\Delta}{2} + Cu \right) = 0 \), the following solution is obtained
\[
u_1 = \frac{B+\Delta}{2} \left( \exp \left[ - a_1 \left( \frac{B+\Delta}{2} \right) \right], - C \right) \quad (3.4)\]

2) \( \phi_1(u) = a_1 (A + Bu + Cu^2), \quad \phi_2(u) = a_1^{-1} ; \quad g(u) \) is of the form
\[
g_2(u) = - \left[ (a_1 A + a_1^{-1}) + 2a_1 Bu + 3a_1 Cu^2 \right]. \quad (3.5)\]

Thus, the factorized form of the Lienard equation will be
\[
\left[ D - a_1^{-1} \right] \left[ D - a_1 \frac{F_3(u)}{u} \right] u = 0 \quad (3.6)\]
and therefore we have to solve the equation \( \dot{u} - a_1 F_3(u) = 0 \), whose solution can be found graphically from
\[
a_1 (\tau - \tau_0) = \ln \left( \frac{u^3}{F_3(u)} \right)^{\frac{1}{2}} - \ln \left( \frac{2Cu + B - \Delta}{2Cu + B + \Delta} \right)^{\frac{1}{2}} \quad (3.7)\]

3) The case \( B = 0 \) and \( C = 1 \): Duffing-van der Pol equation

The \( B = 0, C = 1 \) reduction of terms in Eq. (3.1) allows an analytic calculation of particular solutions for the so-called autonomous Duffing-van der Pol oscillator equation
\[
\ddot{u} + (G + E u^2) \dot{u} + Au + u^3 = 0, \quad (3.8)\]
where \( G \) and \( E \) are arbitrary constant parameters. Since we want to compare our solutions with those of Chandrasekhar et al., we use the second Lienard pair of factorizing functions \( \phi_1(u) = a_1 (A + u^2) \) and \( \phi_2(u) = a_1^{-1} \). Then
\[
g_2(u) = - (Aa_1 + a_1^{-1} + 3a_1 u^2). \quad (3.9)\]
Equation (3.8) is now rewritten
\[ \ddot{u} - (a_1 A + a_1^{-1} + 3a_1 u^2) \dot{u} + Au + u^3 \equiv [D - a_1^{-1}] [D - a_1 (A + u^2)] u = 0 . \] (3.10)

Therefore, the compatible first order equation \( \dot{u} - a_1 (A + u^2) u = 0 \) leads by integration to the particular solution of Eq. (3.10)
\[ u = \pm \left( \frac{A \exp[2a_1 A(\tau - \tau_0)]}{1 - \exp[2a_1 A(\tau - \tau_0)]} \right)^{1/2} = \pm \left( \frac{A \exp[-2AE(\tau - \eta_0)]}{1 - \exp[-2AE(\tau - \eta_0)]} \right)^{1/2} , \] (3.11)

where the last expression is obtained from the comparison of Eqs. (3.8) and (3.10) that gives \( a_1 = -\frac{E}{3} \) and \( G = \frac{AE^2 + 9}{3E} \).

This is a more general result for the particular solution than that obtained through other means by Chandrasekar et al.\(^3\) that corresponds to \( E = \beta \) and \( A = \frac{3}{\beta^2} \).

§4. Convective Fisher equation

Schönborn et al.\(^4\) discussed the following convective Fisher equation
\[ \frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + u(1 - u) - \mu u \frac{\partial u}{\partial x} , \quad \text{or} \quad \ddot{u} + 2(\nu - \mu u) \dot{u} + 2u(1 - u) = 0 \] (4.1)

where the transformation to the travelling variable \( \tau = x - \nu t \) was performed in the latter form. The positive parameter \( \mu \) serves to tune the relative strength of convection.

1) \( \phi_1(u) = \sqrt{2}a_1(1 - u), \quad \phi_2(u) = \sqrt{2}a_1^{-1} \). Then \( g(u) = -\sqrt{2} ([a_1 + a_1^{-1}] - 2a_1 u) \).

Therefore, for this \( g(u) \), we can rewrite the ordinary differential form in Eq. (4.1) as
\[ \ddot{u} + 2 \left( -\frac{1}{\sqrt{2}}(a_1 + a_1^{-1}) + \sqrt{2}a_1 u \right) \dot{u} + 2u(1 - u) = 0 . \] (4.2)

If we set the fitting parameter \( a_1 = -\frac{\mu}{\sqrt{2}} \), then we obtain \( \nu = \frac{\mu}{2} + \mu^{-1} \). Equation (4.2) is factorized in the following form:
\[ \left[ D - \sqrt{2}a_1^{-1} \right] \left[ D - \sqrt{2}a_1 (1 - u) \right] u = 0 , \] (4.3)

that provides the compatible first order equation \( \dot{u} + \mu u(1 - u) = 0 \), whose integration gives
\[ u_1 = (1 \pm \exp[\mu(\tau - \tau_0)])^{-1} . \] (4.4)

2) Since we are in the case of a quadratic polynomial, a second factorization means exchanging \( \phi_1(u) \) and \( \phi_2(u) \) between themselves. This leads to a convective Fisher equation with compatibility equation \( \ddot{u} - \sqrt{2}a_1^{-1} u = 0 \), where now \( a_1 = -\sqrt{2}\mu \), having exponential solutions of the type
\[ u_2 = \pm \exp[-\mu^{-1}(\tau - \tau_0)] . \] (4.5)
In this section we obtain particular solutions for the generalized Burgers-Huxley equation discussed by Wang et al.\textsuperscript{5)}

\[
\frac{\partial u}{\partial t} + \alpha u \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = \beta u(1 - u^\delta)(u^\delta - \gamma), \tag{5.1}
\]
or in the variable \(\tau = x - \nu t\)

\[
\ddot{u} + (\nu - \alpha u^\delta)\dot{u} + \beta u(1 - u^\delta)(u^\delta - \gamma) = 0. \tag{5.2}
\]

1) \(\phi_1(u) = \sqrt{\beta}a_1(1 - u^\delta), \phi_2(u) = \sqrt{\beta}a_1^{-1}(u^\delta - \gamma)\). Then, one gets

\[
g_1(u) = \sqrt{\beta} \left( \gamma a_1^{-1} - a_1 + [a_1(1 + \delta) - a_1^{-1}]u^\delta \right) \tag{5.3}
\]
and the following identifications of the constant parameters \(\nu = -\sqrt{\beta} \left( a_1 - \gamma a_1^{-1} \right)\), \(\alpha = -\sqrt{\beta} \left( a_1(1 + \delta) - a_1^{-1} \right)\). Writing Eq. (5.2) in factorized form

\[
\left[ D - \sqrt{\beta}a_1^{-1}(u^\delta - \gamma) \right] \left[ D - \sqrt{\beta}a_1(1 - u^\delta) \right] u = 0, \tag{5.4}
\]
the solution

\[
u_1 = \left( 1 \pm \exp[-a_1 \sqrt{\beta} \delta (\tau - \tau_0)] \right)^{-1/\delta} \tag{5.5}
\]
of the compatible first order equation \(\dot{u} - \sqrt{\beta}a_1 u(1 - u^\delta) = 0\) is also a particular kink solution of Eq. (5.2). It is easy to solve the second identification equation for \(a_1 = a_1(\alpha, \beta, \delta)\) leading to

\[
a_1 = \pm \frac{-\alpha \pm \sqrt{\alpha^2 + 4\beta(1 + \delta)}}{2\sqrt{\beta(1 + \delta)}}. \tag{5.6}
\]

Then Eq. (5.5) becomes a function \(u = u(\tau; \alpha, \beta, \delta)\) and \(\nu = \nu(\alpha, \beta, \gamma, \delta)\).

2) \(\phi_1(u) = \sqrt{\beta}e_1(u^\delta - \gamma), \phi_2(u) = \sqrt{\beta}e_1^{-1}(1 - u^\delta)\). This pair of factorizing functions lead to

\[
g_2(u) = \sqrt{\beta} \left( \gamma e_1 - e_1^{-1} + [e_1^{-1} - e_1(1 + \delta)]u^\delta \right) \tag{5.7}
\]
and the \(\nu\) and \(\alpha\) identifications: \(\nu = \sqrt{\beta} \left( e_1 \gamma - e_1^{-1} \right), \alpha = \sqrt{\beta} \left( e_1^{-1} - e_1(1 + \delta) \right)\). Equation (5.2) is then factorized in the different form

\[
\left[ D - \sqrt{\beta}e_1^{-1}(1 - u^\delta) \right] \left[ D - \sqrt{\beta}e_1(u^\delta - \gamma) \right] u = 0. \tag{5.8}
\]
The corresponding compatible first order equation is now \(\dot{u} - \sqrt{\beta}e_1 u(u^\delta - \gamma) = 0\), and its integration gives a different particular solution of Eq. (5.2) with respect to that obtained for the first choice of factorizing brackets:

\[
u_2 = \left( \frac{\gamma}{1 \pm \exp[e_1 \sqrt{\beta} \delta (\tau - \tau_0)]} \right)^{1/\delta}. \tag{5.9}
\]
$u_2$ is different from $u_1$ because the parameter $\alpha$ has changed for the second factorization. Solving the $\alpha$ identification for $e_1 = e_1(\alpha, \beta, \delta)$ allows to express the solution given by Eq. (5.9) in terms of the parameters of the equation, $u = u(\tau; \alpha, \beta, \gamma, \delta)$, and also one gets $\nu = \nu(\alpha, \beta, \gamma, \delta)$. If we set $\delta = 1$ in Eq. (5.9), then from $\alpha = \sqrt{\beta(e_1^{-1} - 2e_1)}$ one can get $e_{1\pm} = \frac{\alpha \pm \sqrt{\alpha^2 + 8\beta}}{4\sqrt{\beta}}$ that can be used to obtain $\nu_{\pm} = \nu(\alpha, \beta, \gamma)$. The solutions given by Eqs. (5.5) and (5.6) and in (5.9) have been obtained previously by Wang et al.\textsuperscript{5)} by a different procedure.

§6. Conclusion

In this paper, the efficient factorization scheme that we proposed in a previous study\textsuperscript{1)} has been applied to more complicated second order nonlinear differential equations. Exact particular solutions have been obtained for a number of important nonlinear differential equations with applications in physics and biology: the modified Emden equation, the generalized Lienard equation, the Duffing-van der Pol equation, the convective Fisher equation, and the generalized Burgers-Huxley equation.

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