NEW VARIABLE SEPARATION APPROACH: APPLICATION TO NONLINEAR DIFFUSION EQUATIONS

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ABSTRACT. The concept of the derivative-dependent functional separable solution, as a generalization to the functional separable solution, is proposed. As an application, it is used to discuss the generalized nonlinear diffusion equations based on the generalized conditional symmetry approach. As a consequence, a complete list of canonical forms for such equations which admit the derivative-dependent functional separable solutions is obtained and some exact solutions to the resulting equations are described.

1. Introduction

A number of methods have been proved to be effective for finding symmetry reduction and constructing exact solutions to nonlinear diffusion equations. These include the Lie’s classical approach [1], the nonclassical approach [2], the direct method [3], the modified direct method [4], the generalized conditional symmetry (GCS) method [5], the nonlocal symmetry method [6], the truncated Painlevé approach [7], the sign-invariant and invariant space methods [8], the transformation method [9] and the ansatz-based methods [10-13] etc. There are many different directions of the mathematical and physical theory to concern their exact solutions and various properties.

It is well known that the method of variable separation is one of the most universal and efficient means for study of linear partial differential equations (PDEs). Several methods of variable separation for nonlinear partial differential equations (PDEs) such as the classical method [14], the differential Stäckel matrix approach [15], the ansatz-based method [12-16], the geometrical method [17], the formal variable separation approach (nonlinearization of the Lax pairs or symmetry constraints) [18] and the informal variable separation methods [19] have been suggested. From the point of view of symmetry group and the ansatz of the solution form, we now emphasize two of those ansatzs. One is the ordinary additive or product separable solution. The other is the functional separable solution which is a generalization of the former, where the compatibility of the symmetry constraint with the considered equations is concerned [11-13]. In [19], exact solutions depending on arbitrary functions and their derivatives to many (2 + 1)-dimensional nonlinear integrable models have emerged through the variable separation process, and abundant localized excitations and their rich interaction behaviors have been revealed. All these prompt us to extend our results in [12, 13] to be more general.

In [12, 13], we have discussed the functional separable solution

\[ f(u) = a(x) + b(t), \quad (1.1) \]
to the generalized porous medium equation

\[ u_t = (D(u)u^n)_x + F(u), \quad (1.2) \]

for \( n = 1 \) and \( n \neq 1 \) respectively by using the GCS method.

To obtain more abundant exact solutions to nonlinear PDEs, our basic idea is to weaken
the general symmetry constraint condition and to include much more solutions. In this paper,
we extend the concept of functional separable solution (1.1) to that of derivative-dependent
functional separable solution (DDFSS)

\[ f(u, u_x) = a(x) + b(t), \quad (1.3) \]

and apply it to the nonlinear diffusion equation

\[ u_t = A(u, u_x)u_{xx} + B(u, u_x). \quad (1.4) \]

It is clear that when \( f_{u_x}(u, u_x) = 0 \), (1.3) becomes (1.1), so we assume that \( f_{u_x}(u, u_x) \neq 0 \) hereafter.

The compatibility of (1.3) and (1.4) can be described in terms of the GCS method. The
GCS is a natural generalization of both the generalized symmetry and the conditional symmetry [5].

Consider the general \( m \)-th order \((1 + 1)\)-dimensional evolution equation

\[ u_t = E(t, x, u, u_1, u_2, \cdots, u_m), \quad (1.5) \]

where \( u_k = \frac{\partial^k u}{\partial x^k}, 1 \leq k \leq m \), and \( E \) is a smooth function of the indicated variables. Let

\[ V = \eta(t, x, u, u_1, u_2, \cdots, u_j) \frac{\partial}{\partial u}, \quad (1.6) \]

be an evolutionary vector field and \( \eta \) its characteristic.

**Definition 1.** The evolutionary vector field (1.6) is said to be a generalized symmetry of
(1.3) if and only if

\[ V^{(m)}(u_t - E)|_L = 0. \]

where \( L \) is the solution set of (1.3), and \( V^{(m)} \) is the \( m \)-th prolongation of \( V \).

**Definition 2.** The evolutionary vector field (1.6) is said to be a GCS of (1.3) if and only if

\[ V^{(m)}(u_t - E)|_L \cap W = 0. \quad (1.7) \]

where \( W \) is the set of equations \( D^i \eta = 0, i = 0, 1, 2, \cdots \).

It follows from (1.7) that (1.3) admits the GCS (1.6) if and only if

\[ D_t \eta = 0. \quad (1.8) \]

where \( D_t \) denotes the total derivative in \( t \). Moreover, if \( \eta \) does not depend on time \( t \) explicitly, then

\[ \eta E|_L \cap W = 0, \]

where

\[ \eta(u)E = \lim_{\epsilon \to 0} \frac{d}{d\epsilon} \eta(u + \epsilon E) \]

denotes the Fréchet derivative of \( \eta \) along the direction \( E \).
The outline of this paper is as follows. In Section 2, we will classify Eq. (1.4) which admit derivative-dependent functional separable solutions. Some exact solutions to the resulting equations are presented in Section 3. Section 4 is a summary and discussion.

2. Equations with DDFSSs

In [12], we have proved the following theorem:

Theorem 1. Eq. (1.5) possesses the additive separable solution

\[ u = a(x) + b(t), \]

if and only if it admits the GCS

\[ V = u_x \frac{\partial}{\partial u}. \]  

(2.1)

Theorem 2. Eq. (1.5) possesses the derivative-dependent functional separable solution (1.3) if and only if it admits the GCS

\[ V = \left( f_{uu} u_t + f_{uu} u_{xt} \right) u_x + \left( f_{uu} u_t + f_{uu} u_{xt} \right) u_{xx} + f_{ux} u_{xx} + \frac{f_u u_{xt}}{f_u} \frac{\partial}{\partial u}. \]  

(2.2)

Proof: Let \( v = f(u, u_x) = a(x) + b(t) \), in Theorem 1, after replacing \( u \) by \( v = f(u, u_x) \), and simplifying (2.1), we get

\[ v = f(u, u_x) = a(x) + b(t) \]

if and only if

\[ V = v_x \frac{\partial}{\partial v} = \left( f_{uu} u_t + f_{uu} u_{xt} \right) u_x + \left( f_{uu} u_t + f_{uu} u_{xt} \right) u_{xx} + f_{ux} u_{xx} + \frac{f_u u_{xt}}{f_u} \frac{\partial}{\partial u}. \]  

(2.3)

And then the assertion holds.

From Theorem 2, we know that equation (1.4) admits the DDFSSs (1.3) if and only if it admits the GCS (2.3).

By using the Leibnitz rule on \( n \)-th order differentiation of product functions, we arrive at the following lemma:

Lemma 1. Assume \( G(r) \neq 0, F(r) \) and \( G(r) \) are arbitrary smooth functions, then \( D_i^x F(r) = 0, i = 0, 1, \ldots, N \), if and only if \( D_i^x \left( \frac{F(r)}{G(r)} \right) = 0, i = 0, 1, \ldots, N \).

In order to cover the special case \( f_u = 0 \) in (1.3), on the basis of Lemma 1, we can take away the denominator \( f_u \) in (2.2) and choose \( \eta \) and \( V \) as the following form,

\[ V = \left( f_{uu} u_t + f_{uu} u_{xt} \right) u_x + \left( f_{uu} u_t + f_{uu} u_{xt} \right) u_{xx} + f_{ux} u_{xx} + \frac{f_u u_{xt}}{f_u} \frac{\partial}{\partial u}. \]  

(2.4)

The invariant condition for (2.4) reads

\[ V^{(2)}(u_t - A(u, u_x) u_{xx} - B(u, u_x)) = D_i \eta - \left( A_u \eta + A_{ux} D_x \eta \right) u_{xx} - A(u, u_x) D_x^2 \eta - \left( B_u \eta + B_{ux} D_x \eta \right) \]

\[ = D_i \eta = 0, \]

(2.5)

whenever \( D_i x \eta = 0, (i = 0, 1, 2, \ldots) \) and \( u_t = A(u, u_x) u_{xx} + B(u, u_x) \), where

\[ \eta \equiv \left( f_{uu} u_t + f_{uu} u_{xt} \right) u_x + \left( f_{uu} u_t + f_{uu} u_{xt} \right) u_{xx} + f_{ux} u_{xx} + f_u u_{xt}. \]  

(2.6)
Substituting (1.4) into (2.6) gives

\[ \eta = (u_{xx} f_{uu} + u_x f_u)(A_{xx} + B) + (u_{xx} f_{ux} u_x + u_x f_u u_x + f_u)(A_{ux} + B) \\
+ f_u A_{xx} + B) \]

\[ = f_{ux} A_{uxxx} + (f_{ux} A_{ux} u_x + f_{ux} A_{uxx} u_x^3 + ((f_{ux} A_{ux} + f_{ux} A_{uxx} A_u + 2 f_{ux} A_{uxu}) u_x \\
+ f_{ux} A_u + f_{ux} A + f_u A_{uxx} + f_{ux} B_{ux} u_x + f_{ux} B_{uxx} + f_{ux} B_{uxx} u_x) u_{xx}^2 + ((f_{ux} B_{ux} + 2 f_{ux} B_{uxx} \\
+ f_{ux} B + f_{ux} B_u + f_{ux} A_{uxx} u_x) u_{xxx} \\
+ (f_{ux} B + f_{ux} B_u) u_x = 0. \tag{2.7} \]

In order to determine all the possible \( f, A \) and \( B \) from (2.5), a straightforward substitution leads to

\[ D_t \eta = \frac{\partial}{\partial t} [f_{ux} A_{uxxx} + (f_{ux} A_{ux} u_x + f_{ux} A_{uxx} u_x^3 + ((f_{ux} A_{ux} + f_{ux} A_{uxx} A_u + 2 f_{ux} A_{uxu}) u_x \\
+ f_{ux} A_u + f_{ux} A + f_u A_{uxx} + f_{ux} B_{ux} u_x + f_{ux} B_{uxx} + f_{ux} B_{uxx} u_x) u_{xx}^2 + ((f_{ux} B_{ux} + 2 f_{ux} B_{uxx} \\
+ f_{ux} B + f_{ux} B_u + f_{ux} A_{uxx} u_x) u_{xxx} \\
+ (f_{ux} B + f_{ux} B_u) u_x = 0. \tag{2.8} \]

Using the integrable condition between (1.4) and \( \eta = 0 \), we can express \( u_{xxxxx}, u_{xxxxx}, \) and \( u_{xxx} \) in terms of \( u, u_x, u_{xx} \) and \( u_{xxx} \), and substituting these expressions into (2.8), we have

\[ D_t \eta = (h_1 u_{xx} + h_2)u_{xxx}^2 + (h_3 u_{xx}^3 + h_4 u_{xx}^2 + h_5 u_{xx} + h_6)u_{xxx} \\
+ h_7 u_{xx}^5 + h_8 u_{xx}^4 + h_9 u_{xx}^3 + h_{10} u_{xx}^2 + h_{11} u_{xx} + h_{12} = 0, \tag{2.9} \]

or equivalently

\[ h_i = h_i(u, u_x) = 0, \ i = 1, 2, \cdots, 12. \tag{2.10} \]

where the expressions for \( h_i \) are complicated, and are given in the appendix A. Eq. (1.4) possesses the DDFSSs (1.3) if only (2.9) holds, or equivalently, the system of PDEs (2.10) holds. From the system (2.10), one obtains the following relations among \( A, B \) and \( f \):

\[ 0 = \left(3 + 5 \sqrt{1 + g_0(u)} \right) \ln A - \sqrt{1 + g_0(u)} \ln[-3 A_{ux}^2 g_1(u)(2 + g_0(u)) - g_2(u) \\
-2 g_0(u) A^3 + 2 \sqrt{3} A_{ux}^2 g_1(u)(1 + g_0(u)) + 2 \sqrt{3} g_0(u) A^3 \sqrt{g_1(u) A_{ux}} \\
-2 \ln[A_{ux} \sqrt{3} g_1(u)(1 + g_0(u)) + \sqrt{3} A_{ux}^2 g_1(u)(1 + g_0(u)) + 2 g_0(u) A^3] \right] \tag{2.11} \]

\[ f_{ux} = f_0(u) \frac{A_{ux}}{A} \exp \left( -\frac{2}{3 g_1(u)} \int \frac{A^2}{A_{ux}} du_x \right), \tag{2.12} \]
\[
B = \int \left\{ \int \left[ (-4f_{ux}u_x f_{ux}u_x A^2 - 2f_{ux}u_x f_u A^2 - 2(f_{ux})^2 A_u A + 2(f_{ux})^2 A_u A_{ux} u_x \\
+ 3f_{ux} A f_u A_{ux} - 2(f_{ux})^2 A_u A_{ux} u_x + 3f_{ux} f_{ux} A u_x A_{ux} A_{ux} + 2f_{ux} f_{ux} A u_x A_{ux} u_x + 2u_x f_{ux} u_x f_{ux} A^2 \right] \frac{1}{A^2 f_{ux}^2} \right] du_x + b_0(u) \right\} A du_x + b_1(u),
\]

where \( f_0(u), g_0(u), g_1(u), g_2(u), b_0(u) \) and \( b_1(u) \) are arbitrary functions of \( u \).

It seems that it is impossible to obtain the general solution \( A(u, u_x) \) from the transcendental equation \((2.11)\) for arbitrary \( g_0(u), g_1(u) \) and \( g_2(u) \). It is clear that in order to find explicit solution of \( A(u, u_x) \) from \((2.11)\), the only possible cases are, (i) the factor \( A_{ux} \) appears only in one of two logarithmic functions and (ii) the ratio of the coefficients of the two logarithmic functions of \((2.11)\) are integers. After a lengthy computation and tedious analysis, we finally attain the following results:

**Theorem 3.** The equation

\[
u_t = A(u, u_x) u_{xx} + B(u, u_x)
\]

admits nontrivial DDFSSs of the form \((1.3)\) with \( f_{ux}(u, u_x) \neq 0 \), if it is locally equivalent to one of the following equations, up to equivalence under translation and dilatation of \( u \):

1. \( u_t = \exp[c_3 \phi + \phi_u u_x][u_{xx} + \frac{u_{xx}}{u_x} u_x^2 + c_3 u_x + c_1 \phi_u^{-1}] + c_2 \phi_u^{-1}, \) \( (2.14) \)
   \[
f(u, u_x) = \phi_u u_x + c_3 \phi + c_4;
\]
2. \( u_t = u_{xx} + c_1 u_x + \frac{\phi_u u_x}{\phi_u} u_x^2 + (c_4 + c_5 \phi) \phi_u^{-1}, \) \( (2.16) \)
   \[
f(u, u_x) = \phi_u u_x + c_3 \phi + c_2;
\]
3. \( u_t = (c_1 u + c_2) (-u_x)^{\alpha - 1} u_{xx} - \frac{2c_2}{1 + \alpha} (-u_x)^{\alpha + 1} + c_4 u + c_3, \) \( (2.18) \)
   \[
f(u, u_x) = \ln(-u_x); \]
4. \( u_t = (-u_x)^{\alpha - 1} u_{xx} + c_2 (-u_x)^{\alpha} + c_3 u + c_4, \) \( (2.20) \)
   \[
f(u, u_x) = \ln(-u_x); \]
5. \( u_t = u_x u_{xx} + c_3 u_x^2 + c_4 u^2 + c_2 u + c_1, \) \( (2.22) \)
   \[
f(u, u_x) = \ln(-u_x); \]
6. \( u_t = \frac{1}{c_3 (c_1 u + c_2) - u_x} \left[ (c_1 u + c_2) u_{xx} - 2c_1 u_x^2 + \left( \frac{2c_3 c_4^2 - c_1 c_4}{c_2} \right) u + 2c_3 c_1 c_2 - c_4 \right] u_x \\
+ c_3 c_1 \left( c_3 c_2^2 + \frac{c_1 c_4}{c_2} \right) u^2 + 2c_1 c_3 (c_4 - c_3 c_1 c_2) u + c_2 c_3 (c_4 - c_3 c_1 c_2), \) \( (2.24) \)
   \[
f(u, u_x) = \ln \left[ c_3 (c_1 u + c_2) - u_x \right]; \) \( (2.25) \)
(7) \[ u_t = (c_1u + c_3 - u_x)^{-1}[u_{xx} - (c_4u + c_2)u_x + c_1c_4u^2 + (c_1c_2 - c_1^2 + c_3c_4)u + c_3(c_2 - c_1)], \]
\[ f(u, u_x) = \ln(c_1u + c_3 - u_x); \]

(8) \[ u_t = (c_1 - u_x)^{-1}u_{xx} - c_3\ln(c_1 - u_x) + c_4u + c_2, \]
\[ f(u, u_x) = \ln(c_1 - u_x); \]

(9) \[ u_t = [c_1(c_2u + c_3) - u_x]^2[u_{xx} + (c_2u + c_3)u_x^2 - (2c_1c_2u_x^2 + 4c_1c_2c_3u + c_4 + 2c_1c_3^2)u_x + c_2c_3u + 2c_1c_3^2u^2 + c_1c_2(c_4 + 3c_3^2 - c_1c_2)u + c_1^2c_3^2 + c_1c_3c_4 - c_1^2c_3c_4], \]
\[ f(u, u_x) = \ln[c_1(c_2u + c_3) - u_x]; \]

(10) \[ u_t = (c_1 - u_x)^\alpha u_{xx} + c_2, \quad \alpha \neq -1, \alpha \neq -2, c_1 \neq 0, \]
\[ f(u, u_x) = \ln(c_1 - u_x); \]

(11) \[ u_t = u_{xx} + c_4, \]
\[ f(u, u_x) = c_2\text{arcsinh}[\tan(u_x + c_1)] + c_3; \]

(12) \[ u_t = \frac{6c_3^2}{(\phi u_x - c_3\phi - c_3c_4)^2}[-u_{xx} - \frac{\phi u_{uu}}{\phi u}u_x^2 + 2c_3u_x - c_3\phi u_x^{-1} - c_3^2c_4\phi u_x^{-1}], \]
\[ f(u, u_x) = \frac{1}{(\phi u_x - c_3\phi - c_3c_4)^2}[c_1\phi u_x^2 - 2c_1c_3(\phi + c_4)\phi u_x + c_1c_3^2(\phi + c_4)^2 + c_2c_3^2]; \]

(13) \[ u_t = \frac{6\phi^2}{(u_x - c_3\phi)^2}[-u_{xx} + \frac{6\phi u + c_4}{6\phi}u_x^2 - \frac{c_3c_4}{3}u_x + \frac{1}{6}c_3^2c_4\phi], \]
\[ f(u, u_x) = \frac{c_1u_x^2 - 2c_1c_3\phi u_x + (c_2 + c_1c_3^2)\phi^2}{(u_x - c_3\phi)^2}; \]

(14) \[ u_t = \frac{-3(c_4\phi - c_3)^2}{2(\phi u_x - 1)^2}[u_{xx} + \frac{\phi u_{uu}}{\phi u}u_x^2], \]
\[ f(u, u_x) = \frac{1}{4(\phi u_x - 1)^2}[4c_1\phi u_x^2 - 8c_1\phi u_x + c_2c_3^2\phi^2 - 2c_2c_3c_4\phi + 4c_1 + c_2c_3^2]; \]
\[ u_t = -\frac{-6\phi^2}{(u_x - c_4\phi)^2 - \phi^2} [u_{xx} - (\frac{\phi u}{\phi} + c_3)u_x^2 + 2c_3c_4u_x + c_3(1 - c_4^2)\phi], \quad (2.42) \]

\[ f(u, u_x) = c_1 \ln\left[ \frac{(u_x - c_4\phi)^2}{(u_x - c_4\phi)^2 - \phi^2} \right] + c_2; \quad (2.43) \]

\[ u_t = \frac{\sqrt{6}}{3} g_u u_x [u_{xx} + \frac{g_u}{g} u_x^3 + \frac{\sqrt{6}}{2} c_1 u_x + \frac{c_5}{3c_2 g_u} (g + c_4), \quad (2.44) \]

\[ f(u, u_x) = c_2 \ln\left[ \frac{g_u u_x}{\sqrt{c_3}} \right]; \quad (2.45) \]

\[ u_t = \frac{\sqrt{6}}{3} u_x \phi u_x [u_{xx} + \frac{\sqrt{6}}{18(c_2\phi - 2c_3)} (6(c_2\phi - 2c_3)\phi_{uu} - 5c_2\phi_u^2)u_x^3, \quad (2.46) \]

\[ f(u, u_x) = c_1 \ln\left[ \frac{3c_1}{c_2\phi - 2c_3} \phi u_x \right]; \quad (2.47) \]

\[ u_t = \left( \sqrt{\frac{2}{3}} \phi u_x + c_2 \right) u_x + \sqrt{\frac{2}{3}} \phi^2 u_x^3 + \frac{c_2}{c_5} \left( \frac{c_4\phi - (4c_3 - 3c_5)\phi u}{\phi} \right) u_x^2 \]

\[ + \frac{6c_2^2}{c_5} \left( c_1 - \frac{2(2c_3 - c_5)\phi u}{\phi^2} \right) u_x - \frac{3c_2^3(2c_3 - c_5)\phi u}{c_5\phi^3} \]

\[ + \frac{c_2(3c_4c_2^2 + 2c_1c_5)}{2c_5} \phi^{-1}, \quad (2.48) \]

\[ f(u, u_x) = \ln(\sqrt{6}\phi u_x + 3c_2); \quad (2.49) \]

where \( \phi(u) \) and \( g(u) \) are arbitrary functions of \( u \), and \( \alpha_i, \mu, c_i, i + 1, 2, \cdots \), are arbitrary constants.

**3. Explicit exact DDFSSs**

In this section, we deduce exact solutions of the equations obtained in the last section by means of the DDFSS ansatz (1.4). We affirm that the resulting equations enjoy abundant exact solutions due to their inclusive arbitrary functions and constants. Now we just give some of which resulting from the derivative-dependent functional separable procedure for all the models listed in the theorem 3.

**Example 1.** For the equation (2.14), to obtain exact solutions via derivative-dependent functional separable procedure, one solves the DDFSS ansatz (1.3) with (2.15) first and then substitute the result to the original equation (2.14) to fix the concrete functions \( a(x), b(t) \).
and the integration function. Finally, we find that (2.14) has an implicit separable solution
\[
\phi(u) = t c_2 + \frac{c_1}{c_3^2} + \ln(c_2 c_3) - \ln(1 - \lambda \exp[c_2 c_3(t + a_2)]) + a_2 c_2 - \frac{c_1 x}{c_3} \left[ a_1 - \frac{\exp(c_3 x) (c_3 x - 1 + \ln(c_3 c_1(c_1 - c_3)))}{c_3} \right] \exp(-c_3 x)
\]
for \( c_3 \neq 0 \), where and hereafter \( \lambda, \mu, a_i, c_i, i \in \mathbb{Z} \) are arbitrary constants.

For \( c_3 = 0 \), the equation (2.14) has two derivative-dependent functional separable solutions given implicitly by
\[
\phi(u) = \frac{1}{2c_1} \ln\left( \frac{c_1}{1 - e^{c_1(x+a_1)}} \right) \ln\left( \frac{c_1 (1 - e^{c_1(x+a_1)\lambda})}{e^{c_1(x+a_1)\lambda}} \right) - \frac{1}{2} c_1 x^2 - \frac{1}{2} \left( 2a_1 c_1^2 - 2 a_0 c_1 + 2 c_1 c_4 \right) x - \left( \lambda e^{c_1(-c_1+a_0) - c_2} \right) t - \frac{1}{c_1} \left( \text{dilog}\left( \frac{e^{c_1(x+a_1)\lambda}}{e^{c_1(x+a_1)\lambda} - 1} - a_2 c_1 \right) \right),
\]
and
\[
\phi(u) = \left( \int \ln\left( \frac{\mu c_1 x - \mu - c_1 \lambda}{c_1^2 e^{c_1 x}} - e^{a_1} \right) dx - x \ln(\mu(t + c_3) + c_2 t + a_2) \right),
\]
where the function \( \text{dilog}(x) \) is the usual dilogarithm function defined by:
\[
\text{dilog}(x) = \int_1^x \frac{\ln(t)}{1-t} \, dt.
\]

**Example 2.** In the same way as for the last example, we can find that the equation (2.16) has the implicit separable solution
\[
\phi(u) = \frac{1}{-2c_3 c_5(c_5 + c_3^2 - c_1 c_3)} \left\{ \left[ 2\left( c_4 c_3 \exp(c_3 x) - a_1 c_5 c_3 \exp((c_5 + c_3^2 - c_1 c_3) t) \right) + a_2 c_5 (c_5 + c_3^2 - c_1 c_3) \exp(c_3 x + c_5 t) \right] - c_3 a_4 c_5 (2 c_3 - c_1 - \sqrt{c_1^2 - 4c_5}) \exp\left( -\frac{1}{2} (c_1 - 2 c_3 + \sqrt{c_1^2 - 4c_5}) x \right) \right\} e^{-c_3 x}
\]
(3.1)
for \( c_5 \neq 0 \).

If \( c_5 = 0 \), the equation has the DDFSS
\[
\phi(u) = -\frac{c_4 x}{c_1} - \frac{a_1 e^{-c_3 x}}{c_1 (c_3 - c_1)} + \frac{c_3 [c_4 + \mu + c_1 (\mu t - c_2 + a_2 + a_4)] - \mu}{c_3^2 c_1} + a_3 \exp[-c_3 (x + (c_1 - c_3) t)].
\]

Notice that for \( c_1 = 0, c_4 = 0, c_5 = 0, \phi(u) = e^u \), the equation turns into the potential Burgers equation
\[
u_t = u_{xx} + u_x^2.
\]
It is related to the usual Burgers equation
\[v_t = v_{xx} + 2vv_x\]
by \(v = u_x\).

In this case, the DDFSS (2.11) becomes
\[v = u_x = \{\ln[(a_4 + a_2 + a_3)c_3 - 1 + a_1 \exp(c_3^2t - c_3x)]\}_x.\]

Usually, if the inverse function of \(\phi(u)\) in (2.10) is well-defined then the solution (3.1) denotes a multiple soliton resonant solution. For instance, setting
\[\phi(u) = \tan u,\]
then the nonlinear diffusion equation (2.16) becomes
\[u_t = u_{xx} + c_1u_x + 2u_x^2 \tan(u) + c_4 \cos^2(u) + \frac{1}{2}c_5 \sin(2u)\]
and the corresponding solution (3.1) becomes
\[u = \arctan \left\{ \frac{2a_3 \exp \left[ \frac{1}{2} \left( \sqrt{c_1^2 - 4c_5} + c_1 \right) x \right]}{2c_3 - c_1 - \sqrt{c_1^2 - 4c_5}} + \frac{2a_4 \exp \left[ \frac{1}{2} \left( \sqrt{c_1^2 - 4c_5} - c_1 \right) x \right]}{2c_3 - c_1 + \sqrt{c_1^2 - 4c_5}} + \frac{a_2 \exp(c_5t) - c_6}{c_5} + a_1 \exp[(c_3^2 - c_3c_1 + c_5)t - c_3x] \right\}.\]

When \(a_2 = a_3 = a_4 = 0\), \(a_1 \neq 0\) the solution (3.3) denotes a travelling kink solution. (3.3) is a static kink solution for \(a_2 = a_1 = a_4 = 0\), \(a_3 \neq 0\) or \(a_2 = a_3 = a_1 = 0\), \(a_4 \neq 0\). (3.3) becomes an instanton solution for \(a_1 = a_2 = a_4 = 0\), \(a_3 \neq 0\). Generally, the solution (3.3) is a resonant solution of the travelling kink, static kink and the instanton excitations. Fig. 1 is the evolution plot of the single kink solution (3.3) with
\[a_1 = 1, \ a_2 = 0, \ a_3 = 0, \ a_4 = 0, \ c_4 = 4, \ c_5 = 4, \ c_1 = 5, \ c_3 = 3.\] (3.4)

Fig. 2 is the evolution plot of the resonant solution of the travelling kink and the static kink while the corresponding parameters are taken as follows
\[a_1 = 1, \ a_2 = 0, \ a_3 = 1, \ a_4 = 0, \ c_4 = 4, \ c_5 = 4, \ c_1 = 5, \ c_3 = 3.\] (3.5)

The resonant solution shown by Fig. 2 denotes the fusion interaction between kink and anti-kink. Before the interaction, there are one large travelling kink and one small travelling kink. After the interaction, the large kink and the small anti-kink degenerate to a single smaller static kink. The soliton fusion and fission phenomena can be found in many (1+1)-dimensional integrable models [20] and have been observed in some real physical systems.

**Example 3.** The equation (2.18) possesses an explicit separable solution
\[u = \int \left[ -\gamma (c_1c_3 - c_2c_4 + 2(-1)^{1+2\alpha}c_2c_3)\exp(a_2\alpha c_4 + (\alpha - 1)c_4t - c_3\exp(-c_4t)) \right] \times \left[ (-c_1 + 2c_2(-1)^{2\alpha})\gamma \exp(c_4\alpha (t + a_2)) - 1 \right] \frac{\exp \left[ \frac{c_1 - c_1 \alpha + 2c_2(-1)^{2\alpha}}{\alpha (-c_1 + 2c_2(-1)^{2\alpha})} dt \right] + a_3}{\sqrt{c_4} \exp \left[ \frac{c_1 - c_1 \alpha + 2c_2(-1)^{2\alpha}}{\alpha (-c_1 + 2c_2(-1)^{2\alpha})} \exp(c_4t) \right] \times \left[ (-c_1 + 2c_2(-1)^{2\alpha})\gamma \exp(c_4\alpha (t + a_2)) - 1 \right] \frac{\exp \left[ \frac{c_1 - c_1 \alpha + 2c_2(-1)^{2\alpha}}{\alpha (-c_1 + 2c_2(-1)^{2\alpha})} dt \right] + a_3}{\sqrt{c_4} \exp \left[ \frac{c_1 - c_1 \alpha + 2c_2(-1)^{2\alpha}}{\alpha (-c_1 + 2c_2(-1)^{2\alpha})} \exp(c_4t) \right] \times \left[ (-c_1 + 2c_2(-1)^{2\alpha})\gamma \exp(c_4\alpha (t + a_2)) - 1 \right] \frac{\exp \left[ \frac{c_1 - c_1 \alpha + 2c_2(-1)^{2\alpha}}{\alpha (-c_1 + 2c_2(-1)^{2\alpha})} dt \right] + a_3}{\sqrt{c_4} \exp \left[ \frac{c_1 - c_1 \alpha + 2c_2(-1)^{2\alpha}}{\alpha (-c_1 + 2c_2(-1)^{2\alpha})} \exp(c_4t) \right] \times \left[ (-c_1 + 2c_2(-1)^{2\alpha})\gamma \exp(c_4\alpha (t + a_2)) - 1 \right] \frac{\exp \left[ \frac{c_1 - c_1 \alpha + 2c_2(-1)^{2\alpha}}{\alpha (-c_1 + 2c_2(-1)^{2\alpha})} dt \right] + a_3}}.\]
Figure 1. Evolution plot of the single kink solution (3.1) with (3.4).

Figure 2. Kink fusion interaction expressed by (3.1) with (3.5).

If \( c_1 = 0, c_2 = (-1)^n, c_3 = 0, \alpha = n, \) the above equation becomes
\[
 u_t = -u_x^{n-1}u_{xx} + \frac{2}{1+n}u_x^{n+1} + c_4u
\]
which is just the equation (A.2) of Theorem 2 in [13] for
\[
 g_1 = 0, b_1 = -1, b_2 = 0, \beta = -2, \gamma = -c_4.
\]

Here we obtain its new variable separation solution, the DDFSS, which is given explicitly by
\[
 u = \left\{ \int \gamma c_4 \exp[c_4(a_2n + t(n-1))] \frac{2 \gamma \exp[c_4n(t+a_2)] - (-1)^n}{2\gamma} dt + a_3 \right\} e^{c_4t}
\]
\[
 - \frac{1}{1+n} \left[ n(x+a_1) \right]^{\frac{1+n}{2}} \left\{ \frac{2(-1)^{n+1}\gamma \exp[c_4n(t+a_2)] + 1}{c_4\gamma} \right\}^{\frac{1}{n}} \exp(c_4t + a_2c_4).
\]
Example 4. An explicit separable solution of (2.20) reads
\[
\begin{align*}
    u &= - \left( \int \exp(a(x) + c_3(a_3 + t))dx - \frac{\mu \exp(c_3 t)}{\lambda} \right) \left( \frac{c_3 a}{1 - \lambda \exp(c_3(\alpha - 1)(t + a_3))} \right)^\frac{1}{\alpha - 1} \\
    & \quad + a_2 \exp(c_3 t) - \frac{c_4}{c_3},
\end{align*}
\]
for \( c_3 \neq 0 \), where \( a(x) \) satisfies
\[
a''(x) + aa'^2 + c_2a' - \frac{\lambda}{\alpha} e^{(1-\alpha)a(x)} = 0.
\]
If \( c_3 = 0 \), an DDFSS is given by
\[
\begin{align*}
    u &= -e^{a_0} \int \left( \frac{1}{c_2} \left( a_1 c_1^{2/3} e^{-\frac{\alpha a_0 a_1^{2/3} + a_2 b}{c_1^{2/3}}} + a_2 b_2 \right) \right)^\frac{1}{\alpha} dx + (c_4 + \alpha \sqrt{c_1 a_2 c_2}) t + a_3.
\end{align*}
\]
Example 5. The equation (2.22) has an explicit separable solution
\[
u = -e^{b(t)} \int e^{a(x)} dx + s(t),
\]
for \( c_4 \neq 0 \), where \( a(x) \), \( b(t) \) and \( s(t) \) satisfy
\[
\begin{align*}
    a'''(x) + (5a'(x) + 2c_3) a''(x) + 2a'^3 + c_3 a'^2 + 2c_4 &= 0, \\
b(t) &= \ln \frac{4a_1(c_5^2 - 4c_1 c_4)}{4(c_5^2 - 4c_1 c_4)(\mu^2 + 4\lambda c_4) - a_1(\mu - \sqrt{c_5^2 - 4c_1 c_4})^2}, \\
s(t) &= \frac{-c_5 + b'(t) + \mu e^{b(t)}}{2c_4}.
\end{align*}
\]
If \( c_4 = 0 \), the DDFSS of (2.22) becomes
\[
\begin{align*}
u &= \left( -\lambda^2 c_1 - \mu c_3^2 - \frac{\alpha}{c_3^2} \int e^{a(x)} dx \right) e^{c_5(t+a_0)} + \lambda \left( -a_1 c_5 \left( e^{c_5 t} + c_1 + \lambda e^{c_5(2t+a_0)} \right) \right),
\end{align*}
\]
where \( a(x) \) satisfies
\[
a'' - 2a'^2 + c_3 a' - \lambda e^{-a(x)} = 0.
\]
Example 6. An explicit variable separable solution of Eq. (2.24) has the form
\[
\begin{align*}
u &= \frac{1}{\mu c_1} \left[ 2 c_1^2 \exp \left( \frac{(\mu c_2 - 2 c_1^2 c_4) t + c_2 \mu x - 4 c_2 a_1 c_1}{2 c_1 c_2} \right) - c_2 \mu e^{(-c_3 c_1 x)} \\
& \quad + \frac{a_2 \mu c_1 (\mu c_2 + 2 c_1 c_4)}{2 c_2} \right] e^{c_1 c_3 x}.
\end{align*}
\]
If \( c_1 = 0 \), the DDFSS of (2.24) is given explicitly by
\[
\begin{align*}
u &= -\sqrt{c_2 a_1} \left( t + a_3 \right) \tan \left( \frac{1}{2} \sqrt{c_2 a_1 (x + a_2)} \right) + c_4 t + c_2 c_3 x + a_4 + \frac{1}{2} a_1 a_2 a_3.
\end{align*}
\]
If \( c_4 = 0, c_3 = 0 \), the equation (2.24) becomes
\[
\begin{align*}
u_t &= - \left( \frac{c_1 u + c_2}{u_x} + 2 c_1 u_x \right) u_x
\end{align*}
\]
which is equivalent to the equation (4.3) of Theorem 2 in [13] for
\[ g_1 = 0, \beta = 0, n = 0, b_1 = -c_2, b_2 = -c_1, g_2 = 0. \]

It has a new solution
\[ u = \frac{1}{c_1\mu} \left[ 2c_1^2 \exp \left( -\frac{c_1\mu t + x\mu - 4a_1c_1}{2c_1} \right) - \mu c_2 + a_2\mu c_1 \exp \left( \frac{1}{2} \mu t \right) \right]. \]

**Example 7.** The equation (2.28) possesses an explicit variable separation solution
\[ u = \frac{(\mu c_1 - c_1 e^{c_1(t+a_2)}) \int e^{-c_1 x + a(x)} dx + a_1 c_1 c_4 e^{c_4 t} - c_3 c_4 e^{-c_1 x} + \lambda c_1}{c_1 c_4 e^{-c_1 x}} \]
for \( c_1 c_4 \neq 0 \), where \( a(x) \) satisfies
\[ a''(x) - c_1 a'(x) - \mu e^{a(x)} - c_3 c_4 - c_1^2 + c_1 c_2 = 0. \]

If \( c_1 = 0, c_4 = 0 \), its DDFSS is given explicitly by
\[ u = \frac{(e^{c_4(t+a_1)} - \lambda) \int e^{a(x)} dx}{c_4} + c_3 x + a_2 e^{c_4 t} - \frac{c_2 + \mu}{c_4}, \]
where \( a(x) \) satisfies
\[ \pm \int \frac{a(x)}{\sqrt{-2\lambda e^x + 2c_4 c_3 s + a_3}} ds = x + a_4. \]

If \( c_1 \neq 0, c_4 = 0 \), its DDFSS is given explicitly by
\[ u = \lambda (t + a_1) e^{c_1 x} \int e^{a(x) - c_1 x} dx + (\mu t + a_2) e^{c_1 x} - \frac{c_3}{c_1}, \]
where \( a(x) \) satisfies
\[ a'' - c_1 a' + \lambda e^{a(x)} + c_1 c_2 - c_1^2 = 0. \]

If both \( c_1 \) and \( c_4 \) are zero, the derivative-dependent functional separable solution of (2.28) should be
\[ u = \frac{b_1(c_3 x + b_2 t + b_0) \exp[b_3 (x + b_4)] + c_3 b_1 x + b_1 (b_2 - 2b_3 c_2) t - 2b_3 c_2 t_0 + b_0 b_1}{b_1 \exp[(x + b_4)b_3] + 1}, \]
with \( b_0, b_2, b_3, b_4 \) and \( t_0 \) being all arbitrary constants.

**Example 8.** For \( c_4 \neq 0 \), the derivative-dependent functional separable solution of Eq. (2.28) reads
\[ u = \frac{1}{c_4 \lambda} \left( \exp[c_4 (t + a_3)] - \lambda \right) \left[ \lambda \int \exp[a(x)] dx - c_3 \ln(-\frac{\exp[a_3 c_4 + c_4 t]}{c_4} - \lambda) \right] + c_1 x \]
\[ + a_2 e^{c_4 t} + \frac{c_3 e^{c_4(t+a_3)} t}{\lambda} - \frac{\mu + c_2}{c_4}, \]
where \( a(x) \) satisfies
\[ a''(x) + c_3 a'(x) + \lambda e^{a(x)} - c_1 c_4 = 0. \]

If \( c_4 = 0 \), the DDFSS of (2.28) is changed to
\[ u = -\lambda (t + a_2) \int e^{a(x)} dx - c_3 (t + a_2) \ln[\lambda (t + a_2)] \]
\[ + (\mu + c_3 + c_2) t + c_1 x + c_3 a_2 + a_1, \]
\[ n = 0, b_1 = -c_2, b_2 = -c_1, g_2 = 0. \]
where \( a(x) \) satisfies

\[
a''(x) + c_3a'(x) - \lambda e^{a(x)} = 0.
\]

**Example 9.**

\[
u = \left[ \frac{-\sqrt{e^2c_2(t+a_2)} - \lambda \left( \lambda \int e^{-c_1c_2x+a(x)}dx + \mu \right) + a_1e^{c_2t}}{\sqrt{c_2}} \right] e^{c_1c_2x} - \frac{c_3}{c_2},
\]

with \( a(x) \) being a solution of

\[
a''(x) - a'(x)^2 + (c_4 - 2c_1c_2)a'(x) - \lambda e^{2a(x)} + c_1c_2(c_4 - c_1c_2) = 0,
\]
is an exact solution of Eq. \((2.30)\) via the variable separation formula \((1.3)\) with \((2.31)\).

When \( c_2 = 0 \), the DDFSS is given explicitly by

\[
u = -\sqrt{\frac{\alpha}{(t + a_2)}} \left( \int e^{a(x)}dx - \mu \lambda \right) + a_1 + c_1c_3x + c_3t,
\]

where \( a(x) \) satisfies

\[
a''(x) - a'(x)^2 + c_4a'(x) - \lambda e^{2a(x)} = 0.
\]

**Example 10.** The equation \((2.32)\) has the following special variable separable solution

\[
u = -\left( \frac{1}{\mu \alpha (t + a_4)} \right)^\frac{1}{2} \int e^{a(x)}dx + c_1x - \left( \frac{1}{\mu \alpha (t + a_4)} \right)^\frac{1}{2} \lambda \alpha (t + a_4) + c_2t + a_3,
\]

where \( a(x) \) satisfies

\[
\pm \sqrt{\alpha + 2} \int e^{(a+1)x} \sqrt{a_1 - 2\mu e^{(a+2)x}} d\xi - x - a_2 = 0.
\]

**Example 11.** The equation \((2.34)\) is a trivial linear diffusion equation which allows of course infinitely many product variable separation solutions. It is easy to see that it allows a DDFSS which is simply equivalent to a trivial special additive separation solution

\[
u = a_1x + c_4t.
\]

**Example 12.** The DDFSS of \((2.36)\) with \( c_2 \neq 0 \) is given by

\[
\phi(u) = -i\sqrt{c_2}e^x \left[ \int \frac{e^x}{\sqrt{-a_3e^{-2x} - a_2e^{2x} + 2\sqrt{a_2a_3} \tanh \left( \frac{24\sqrt{a_2a_3}(t + a_1)}{c_2} \right)}} dx \\
+ \frac{1}{a_2a_3} \arctan \left( \frac{4\sqrt{a_3a_2} \tanh \left( \frac{48\sqrt{a_3a_2}(t + a_1)}{c_2} \right)}{c_2} - a_3 - a_2 \right) \\
- a_4e^x + c_4. \right.
\]

**Example 13.** The equation \((2.38)\) has the following special solution

\[
\int u (\phi(s))^{-1} ds = -2 \sqrt{\frac{3\mu^2 t + \sqrt{3\mu c_2x + 3a_1c_2 + 3a_2c_2 - 3c_1c_2}}{\mu}} + c_3x + c_4t.
\]

There are three special cases of \((2.38)\), which are known in literature:

(i) If \( c_3 = 0 \), \( \phi(u) = c_0 \), \((2.38)\) becomes

\[
u_t = -6 \frac{u_{xx}}{u_x^2} + c_4,
\]

\[(3.6)\]
which has a DDFSS given explicitly by

\[ u = \left( -2 \sqrt{\frac{3\mu c_2 x + 3\mu^2 t + 3(a_1 c_2 + a_2 c_2 - c_1 c_2)}{\mu}} + c_4 t \right). \]

This equation is equivalent to the equation (19) of Example 3.1 in [13] for \( n = -1, \alpha = 0 \);
(ii) If \( c_3 = 0, \phi(u) = u \), (2.38) is simplified to

\[ u_t = -6 \frac{u^2 u_{xx}}{u_x^2} + (6 + c_4) u, \]

which has the DDFSS

\[ u = \exp \left( -2 \sqrt{\frac{3\mu c_2 x + 3\mu^2 t + 3(a_1 c_2 + a_2 c_2 - c_1 c_2)}{\mu}} + c_4 t \right). \]

Eq. (3.7) is equivalent to the equation (26) of Example 3.2 in [13] for \( n = -1, \alpha = 0 \), the equation (26) is a generalization of the curve shortening equation

\[ w_t = w^2 w_{xx} + w^3. \]

(iii) If \( c_4 = 0, c_3 = 0, \phi(u) = e^u \), (2.38) becomes

\[ u_t = -6 e^{2u} u_{xx} \frac{1}{u_x^2} + 6e^2 u, \]

which has the DDFSS

\[ u = \ln \left( -2 \sqrt{\frac{3\mu c_2 x + 3\mu^2 t + 3(a_1 c_2 + a_2 c_2 - c_1 c_2) - a_3 \mu}{\mu}} \right). \]

After transformation \( u = w/2 \), (3.8) is transformed to

\[ w_t = -24 e^{w} w_{xx} \frac{1}{w_x^2} + 12e^w, \]

which is equivalent to the equation (23) in the example 3.4 in [13] for \( \beta = 0, \sigma = -\frac{1}{2}, n = -1, \alpha = 0 \).

**Example 14.** The equation (2.40) admits a special separable solution given implicitly by

\[ \phi(u) = -\frac{1}{c_4^2 c_2} \left[ 2c_4 \sqrt{a_0 c_2 x + 3a_0^2 t - c_1 c_2 + b_0 c_2 + 2a_0 - c_3 c_2 c_4} \right. \]

\[ - \left. a_1 c_4^2 c_2 \exp \left( c_4 \left( 2 \sqrt{a_0 c_2 x + 3a_0^2 t - c_1 c_2 + b_0 c_2 - 3c_4 a_0 t} \right) \right) \right]. \]
Example 15. The DDFSS of (2.42) has the form
\[
\int_0^u (\phi(s))^{-1} ds = \frac{c_4c_1}{a_2} - \frac{c_1}{a_2} \ln \left[ \exp \left( \frac{3a_2^2 t + a_3c_1 + a_1c_1 + c_1c_2}{c_1^2} \right) \right] \\
-2\exp \left( \frac{c_1(2a_3 + 2a_1 + a_2x) + 6a_2^2 t}{c_1^2} \right) \\
-2i\sqrt{-\exp \left( \frac{c_1(a_2x + a_3 + a_1) + 3a_2^2 t}{c_1^2} \right) + \exp \left( \frac{c_2}{c_1} \right) } \\
\times \exp \left( \frac{c_1(a_3 + 3a_1 + a_2x) + 9a_2^2 t}{2c_1^2} \right) \right] - \frac{(\ln 2)c_1}{a_2} \\
+ \frac{c_4c_2c_1x + (-3a_2^2 + 6c_3a_2c_1)t + ic_1^2\pi + (a_4a_2 - a_3 - a_1 - c_2)c_1}{c_1c_2}
\]

Example 16.
\[
g(u) = \frac{\lambda}{\mu} \exp \left( \frac{2\mu t + 6a_2}{3c_2} \right) + \left[ -3c_2a_1 e^{\frac{\mu + 3a_2}{3c_2}} \mu^{-1} \\
+ \frac{1}{c_1^{3/2}\sqrt{c_2\mu}} \left( -\sqrt{2\mu} \sqrt{-e^{-\sqrt{6c_1}(x+a_3)} + \lambda + \sqrt{2(\ln 2)}\mu \sqrt{\lambda} } + \sqrt{2\mu} \sqrt{\lambda} \ln(\lambda + \sqrt{\lambda}) \left( e^{-\sqrt{6c_1}(x+a_3)} + \lambda \right) + \sqrt{3}\mu \sqrt{c_1(x+a_3)} \\
+ 3a_4 \sqrt{c_3\mu \frac{c_1^{3/2}}{\sqrt{c_2}}} \right) e^{\frac{\mu + 3a_2}{3c_2}} \right]
\]
is a special DDFSS of (2.44).

Example 17. The equation (2.46) for \( c_2 \neq 0 \) has a DDFSS determined implicitly by
\[
\phi(u) = \frac{1}{2916c_2c_1^3} \left[ 243c_2^2c_1^2 e^{2a_1} \left( e^{c_1 t} x + a_3 \right)^2 - 18\sqrt{6c_2^3c_1^4} e^{\frac{4a_1}{c_1}} \left( e^{\frac{4a_1}{c_1}} x + a_3 \right)^2 t \\
+ 486 e^{\frac{2a_1+x}{c_1}} c_2^2 a_3 x + 2 e^{\frac{2a_1+x}{c_1}} c_2^2 a_3 x + 5832 c_3 c_2 \right].
\]
For \( c_2 = 0 \), the equation (2.46) enjoys an exact DDFSS in the form
\[
\phi(u) = -\frac{\sqrt{6}}{3c_1 \lambda^2(t + a_2)} \left( -i\lambda c_1^{3/2} \sqrt{c_3} \int e^{\frac{a(x)}{c_1}} dx - \mu c_1 + a_1 \lambda^2(t + a_2) \right),
\]
where \( a(x) \) satisfies
\[
\sqrt{c_3} \int e^{a(x)} \frac{1}{e^{\frac{3a_3}{c_1} + ic_1 \frac{3}{2} \sqrt{c_3} \lambda e^{\frac{a_3}{c_1}}}} ds = x + a_4.
\]

Example 18. For the equation (2.48), a special DDFSS is given by
\[
\int_0^u \phi(s) ds = \frac{1}{\sqrt{6}} \left[ e^{b(t)} \int e^{a(x)} dx - 3c_2 x - s(t) \right],
\]
with
\[
\frac{\phi u}{\phi^2} = \frac{1}{6c_2(2c_3 - c_5)e^{2a(x)}e^{2b(t)}} \left[ -3\sqrt{6c_5} b'(t) \left( e^{a(x)} dx e^{b(t)} + 3\sqrt{6c_5} s'(t) \\
+ \sqrt{6c_5} a'(x) e^{2a(x)} e^{2b(t)} + 3c_4 c_2 e^{2a(x)} e^{2b(t)} + 18c_1 c_2 c_5 \right).}
\]

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Given any \( \phi(u) \), then \( a(x), b(t) \) and \( s(t) \) are determined by the above two relations, thus the separable solution \( u \) is determined.

4. Summary and discussion

In summary, we have brought forward a new conception of DDFSS to nonlinear evolution equations. Taking the generalized nonlinear diffusion equation as a concrete example and using the GCS approach, we have obtained a complete list of explicit canonical forms for such equations which admit the DDFSSs. As the consequence, some exact explicit solutions to the resulting equations have been obtained via solving the DDFSS ansatz (1.3). The approach also provides a symmetry group interpretation to the DDFSSs. Our approach is more general than the others due to the involvement of the derivative-dependent functional separable function in the ansatz (1.3). Subsequently, we can obtain a good many new nonlinear models which can be solved by means of generalized nonlinear variable separation procedures. Some new exact solutions of some known models are given explicitly. Several different types of localized excitations of some complicated nonlinear diffusion equations have been found via the DDFSS approach.

Though the variable separation approach has been developed in several different directions [12-13], it is still far beyond of perfect. There are some important problems should be studied further. One of the most important problem may be how to unify all the known informal variable separation approaches? Perhaps, we can propose a unified variable separation ansatz in a most general way \( u = u(x_1, x_2, \ldots, x_n), G_j \equiv G_j(\xi_1, \ldots, \xi_m), \xi_k = \xi_k(x_1, \ldots, x_n), k = 1, \ldots, m, m_j < n \)

\[
f(x_1, x_2, \ldots, u, u_{x_1}, u_{x_1x_2}, \ldots) = g(x_1, x_2, \ldots, G_j, G_j\xi_i, G_j\xi_i\xi_k, \ldots).
\]

(4.1)

Though all the ansatszs of the known informal variable separation approaches are the special cases of (4.1), the concrete realization procedures for different known approaches are quite different. Can we find a universal method, say, the GCS method, to realize the generalized variable separation ansatz (4.1)?

APPENDIX A. Expressions of \( h_i \) of (2.10)

It takes a dozen pages to write down \( h_i, i = 1, 2, \ldots, 12 \), explicitly in terms of \( f \), \( A \) and \( B \). For simplicity, we display them by introducing the following notations:

\[
\Gamma_0 = (f_{ux}B)u_xu_x^2 + (f_uB)u_x, \quad (A.1)
\]
\[
\Gamma_1 = (f_{ux}A_{ux})u_x, \quad (A.2)
\]
\[
\Gamma_2 = (f_{ux}A)u + [(f_{ux}A_{ux})u + (f_{ux}A_{ux})u_x]u_x + (f_{ux}B_{ux})u_x + f_uA_{ux}, \quad (A.3)
\]
\[
\Gamma_3 = f_{ux}A_{ux} + 3f_{ux}A_{ux}, \quad (A.4)
\]
\[
\Gamma_4 = (f_{ux}A)u_u^2 + [(f_{ux}B_{ux})u + (f_{ux}B_{ux})u_x + (f_uA)u_x + (f_uB)u_x + f_{ux}B_u, \quad (A.5)
\]
\[
\Gamma_5 = (2f_{ux}A_{ux}u + f_{ux}A)u_x + f_uA + f_uA_{ux}, \quad (A.6)
\]
\[
F_i = \Gamma_i f_{ux}^{-1}A^{-1}, \quad i = 0, 1, \ldots, 5, \quad (A.7)
\]
\[
G_1 = F_3F_1 - F_{1ux}, \quad (A.8)
\]
\[
G_2 = -F_{1ux}u_x - F_{2ux} + F_5F_1 + F_3F_2, \quad (A.9)
\]
\[
G_3 = -3F_1 + F_3^2 - F_{3ux}, \quad (A.10)
\]
\[
G_4 = F_3F_2 - F_{2ux}u_x + F_4F_1 - F_{4ux}, \quad (A.11)
\]
\[
G_5 = -F_{5ux} + 2F_3F_5 - F_{3ux}u_x - 2F_2, \quad (A.12)
\]
With the help of the above notations, $h_i$, $i = 1, 2, ..., 12$ in (2.10) read

$$h_1 = -9F_3A_{u} + 2AG_3 + 15A_{u}u_xu_x,$$  \hspace{1cm} (A.16)

$$h_2 = (-2F_3A_u - F_{3u}A + 12A_{uuu}u_x)u_x - 7F_5A_{u} + 4A_u + 3B_{uux} + F_{5ux}A + AG_5,$$  \hspace{1cm} (A.17)

$$h_3 = (5G_3 + F_{3uu} - 4F_3^2 - F_1)A_{u} - 3F_3A_{u}u_x + 10A_{u}u_xu_xu_x,$$  \hspace{1cm} (A.18)

$$h_4 = AG_{5ux} + F_{5ux}A_{u} - 4F_2A_{u} - F_3^2B_{u} - 2F_3A_{u} + 3F_1B_{u} - 4F_5A_{u}u_xu_x$$
$$+ 4A_{u}u_xG_3 + AG_{3uu}u_x + F_{3uu}A_{u}u_x + F_2u_xA - 3F_1AF_5 - 8F_3A_{u}F_5$$
$$+ B_{ux}G_3 - 5F_2u_xA_{ux} - 2F_3A_{u}F_3 + 5A_{u}u_xG_3 + 6F_1A_{u}u_x - 3F_3^2A_{u}u_x$$
$$+ 24u_xA_{u}u_xu_x + 3AG_2 + F_{3u}A + F_{3u}B_{u} + 22A_{u}u_x + 6B_{uux}u_xu_x,$$  \hspace{1cm} (A.19)

$$h_5 = (18A_{uuu}u_x - 2F_3A_{u}u_x^2 + ((B_{uu} - 6F_5A_u)F_3 - 7F_5A_{u}u_x + 4A_uG_5 + F_{3ux}B_{u}$$
$$+ AG_{5u} + F_{5ux}A_{u} + 12B_{uux}u_x + 4F_2A_{u} + 16A_{uu}u_x)u_x + (-2F_3B_{u} + B_{u})F_3$$
$$- 7F_4A_{u} - 4F_5^2A_{u} + (-2F_3A - 3A_{u} - B_{uux})F_5 + F_{3u}B + 10B_{uux} + B_{u}G_5$$
$$+ F_{5u}B_{u} + F_5uA + AG_{7ux} + 5A_{u}G_7 + 2AG_4 + 2F_2B_{u} + AF_{4ux},$$  \hspace{1cm} (A.20)

$$h_6 = 4A_{uuu}u_x^3 + (F_3B_{u} - 3F_5A_{u} + 6B_{uux})u_x^2 + (4A_uG_7 + F_{5ux}B_{u}$$
$$- F_5B_{uux} + AG_{7u} + 4B_{uuxe} + (2F_4 - 3F_5^2)A_{u}u_x + (B_{ux} - F_3)A_{g}$$
$$- 10A_{ux}F_0 + (F_{0ux} + F_0F_3 + G_6)A + B_{ux}(G_7 - F_3^2) + F_5uB,$$  \hspace{1cm} (A.21)

$$h_7 = A_{u}u_xu_xu_x + A(G_{1ux} - 3F_3^2) + (-7A_{ux}u_x - AG_3 - 4F_3A_{u})F_1$$
$$+ (A_{u}u_xu_x + AG_1)F_3 + 4A_{ux}(F_{1ux} + 5G_1),$$  \hspace{1cm} (A.22)

$$h_8 = |(-3F_3A_u - 10A_{uuu})F_1 + 4A_{uuu}u_xu_x + AG_{1u} + 3F_3A_{u}u_xu_x + F_{1ux}A_u + 4A_uG_1|u_x$$
$$+ (4A_u - 5F_3A - B_{ux}u_x - F_3B_{u} - 4F_5A_{u})F_1 + (-4F_3A_{ux} - 8A_{u}u_x - AG_3)F_2$$
$$+ (3A_{ux} + B_{ux}u_xu_x + AG_2)F_3 + (A_{ux}u_xu_x + AG_1)F_5 + B_{ux}u_xu_xu_x + 5A_{ux}G_2$$
$$- AG_5F_1 + 6A_{ux}u_xu_x + AG_{2u}u_x + F_{1ux}B_{u} + B_{ux}G_1 + F_{2ux}A_{u} + F_{1ux}A,$$  \hspace{1cm} (A.23)

$$h_9 = (-3F_1A_u + 3F_3A_{u}u_x + 6A_{uuu}u_x)u_x^2 + [(2B_{uu} - 3F_5A_u)F_1 - 3(F_3A_u + 4A_{ux})F_2$$
$$+ (3A_{ux} + 3B_{ux}u_x)F_3 + F_{1ux}B_{u} + AG_{2u} + 12A_{u}u_xu_x + F_{2ux}A_u + 3F_5A_{u}u_xu_x$$
$$+ 4A_uG_2 + 4B_{uux}u_x)u_x - AG_5F_3 + (2B_{u} - AG_7 - F_5B_{u} - 4F_4A)F_1 - 2F_2^2A$$
$$- (F_3B_{u} + 2B_{ux}u_x + 4F_5A_{u} + 5A_{ux}F_2 + (AG_4 + 3B_{ux}u_x - 4F_4A_{u})F_3$$
$$- (9A_{ux}u_x + AG_3)F_4 + (A_{ux} + B_{ux}u_xu_x + AG_2)F_5 + 5A_{ux}G_4 + AF_{2u}$$
$$+ F_{4ux}u_x + B_{ux}G_2 + 3A_{ux} + F_{1u}B + 6B_{uux}u_x + F_{2ux}B_{u} + AG_{4ux},$$  \hspace{1cm} (A.24)

$$h_{10} = (5A_{ux} + F_3A)G_6 + [4A_{uuuu}u_x + F_3A_{uuu}]u_x^3 + (3F_1B_{uu} + 3F_7A_{uuu} + 6A_{uuu} + 3F_3B_{uuu}$$
$$- 4F_2A_u + 6B_{uuu}u_x)u_x^2 + [3F_5A_{uu} + (14A_{uuu} + 3F_3A_{u})F_4 + 3F_5A_{u}F_2$$
$$+ AG_{4u} + 3F_3B_{uuxu_x} + 3F_3B_{uu} + 4A_uG_4 + 12B_{uuu}u_x + F_{2ux}B_{u} + F_{4ux}A_{u}]u_x$$
$$- (3F_2A + 6A_u + AG_5 + F_3B_{u} + 3B_{uuxu_x} + 4F_5A_{u})F_4 - 10A_{ux}u_xu_x + 3F_1A$$
$$+ AG_3 + 4F_3A_{ux}F_0 + 3F_5B_{uuxu_x} + F_4A + F_2B_{u} + F_{4ux}B_{u} + B_{ux}G_4$$
$$- AG_7F_2 + 3B_{ux}u_x + F_5AG_4 + AG_{6ux} + F_{0ux}A_{ux} + F_{2u}B - F_2B_{ux}F_2,$$  \hspace{1cm} (A.25)

$$h_{11} = (F_3A + 4A_uu_x + B_{ux})G_6 + (5A_{ux} + F_3A)G_8 + A_{uuuu}u_x^4 + (F_5A_{uuu} + 4B_{uuu}u_x + F_3B_{uuu})u_x^3$$
\[\begin{align*}
+ &(6B_{uuu} - 5F_4A_{uu} + 2F_2B_{uu} + 3F_5B_{uuu})u_x^2 + [F_{4uu}B_u + AG_{6u} + 3F_5B_{uu} + F_{0ux}A_u \\
- & (2B_{ux} + 3F_3A_u)F_4 - (3F_3A_u + 16A_{ux})F_0]u_x - F^2_4A - (F_5B_{ux} + AG_7)F_4 - (4B_{ux}u_x \\
+ & A(G_5 + 2F_2) + F_3B_{ux} + 4F_5A_{ux} + 7A_u)F_0 + (F_{0u} + G_{8ux})A + F_{0ux}B_{ux} + F_{ux}B,
\end{align*}\]

\(h_{12} = (F_5A + 4A_uu_x + B_{ux})G_8 + B_{uuuu}u_x^4 + F_5u_x^3B_{uuu} - (6A_{uu}F_0 - F_4B_{uu})u_x^2 +
\]

\[+(AG_{8u} - 4B_{ux}F_0 - 3F_5A_uF_0 + F_{0ux}B_u)u_x - F_4AF_0 - F_5B_{ux}F_0 + (-AG_7 - B_u)F_0 + F_{0u}B.\]

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