DISTRIBUTIONALLY ROBUST FRONT DISTRIBUTION CENTER INVENTORY OPTIMIZATION WITH UNCERTAIN MULTI-ITEM ORDERS

YULI ZHANG\textsuperscript{1,*}, LIN HAN\textsuperscript{1} AND XIAOTIAN ZHUANG\textsuperscript{2,*}

\textsuperscript{1}School of Management and Economics, Beijing Institute of Technology
Beijing 100081, China

\textsuperscript{2}Department of Intelligent Supply Chain
Beijing Jingdong Zhenshi Information Technology Co., Ltd.
Beijing 100176, China

Abstract. As a new retail model, the front distribution center (FDC) has been recognized as an effective instrument for timely order delivery. However, the high customer demand uncertainty, multi-item order pattern, and limited inventory capacity pose a challenging task for FDC managers to determine the optimal inventory level. To this end, this paper proposes a two-stage distributionally robust (DR) FDC inventory model and an efficient row-and-column generation (RCG) algorithm. The proposed DR model uses a Wasserstein distance-based distributional set to describe the uncertain demand and utilizes a robust conditional value at risk decision criterion to mitigate the risk of distribution ambiguity. The proposed RCG is able to solve the complex max-min-max DR model exactly by repeatedly solving relaxed master problems and feasibility subproblems. We show that the optimal solution of the non-convex feasibility subproblem can be obtained by solving two linear programming problems. Numerical experiments based on real-world data highlight the superior out-of-sample performance of the proposed DR model in comparison with an existing benchmark approach and validate the computational efficiency of the proposed algorithm.

1. Introduction. The timeliness of order delivery has become the key competitiveness for E-commerce companies and platforms. To rapidly respond to customer demand, industry bellwethers are rushing in increasing the warehouse layout density by introducing front distribution centers (FDCs). As a new warehouse model, FDC integrates retail and distribution functions, and aims at providing timely delivery service for the bestselling products. For example, JD.com has built a three-level warehouse network of “central distribution centers + regional distribution centers + FDCs”. By virtue of the wide deployment of FDCs, JD.com is capable of offering the “211” delivery service. That is, for goods in stock at the FDCs, any orders received by the morning deadline (11:00 a.m. in most of the cities) will be delivered on the same day, and any orders received by the evening deadline (11:00 p.m.) will be delivered by 3:00 p.m. on the following day [7].

To offer same-day or even faster delivery to consumers, the following challenges encountered by FDCs need to be addressed. (1) The inventory level for each product

2020 Mathematics Subject Classification. Primary: 58F15, 58F17; Secondary: 53C35.
Key words and phrases. Distributionally robust, inventory optimization, conditional value at risk, front distribution center, row-and-column generation.

*Corresponding authors: Yuli Zhang and Xiaotian Zhuang.
stock keeping unit (SKU) needs to be determined before customer orders have been placed. Furthermore, the probability distribution of customer demand is usually difficult to estimate accurately since it may fluctuate over different periods. We analyze the customer demand faced by a FDC of JD Logistics over 42 weeks from June 8, 2020 to April 7, 2021. To demonstrate the uncertainty of customers’ demand probability distribution, these 42 weeks are divided into two periods where each period has 21 weeks. Figure 1 shows the histogram of random customer demand for a product over the two periods. (2) A large proportion of customer orders contain more than one product SKU, i.e, the so-call “multi-item orders”. It is common that customers usually buy multiple items in a single order for convenience or a free shipment [18]. Figure 2 shows the proportion of single-item order and multi-item orders in two FDCs of JD Logistics. More detailed data are given in Subsection 5.1. As Figure 2 illustrates, about 40% to 60% of orders are multi-item orders. The multi-item orders lead to the order-splitting and lost sales. For FDCs, if one of the ordered SKUs is out of stock, customers usually transfer to other channels, which thus leads to lost sales. (3) Located in urban areas or even residential areas, FDCs are typically very small due to the expensive rent, and thus can only accommodate a very limited number of products. In practice, the scale of most FDCs is about 300 to 500 square meters, which is far smaller than the required scale to meet the demand. Thus, the inventory level for each SKU is a critical decision for the FDC manager that determines both the profit and order fulfillment rate.

To this end, this paper proposes a distributionally robust (DR) inventory optimization approach for FDCs. Specifically, we propose a Wasserstein distance-based distributional set to describe the uncertain multi-item orders, and a robust conditional value at risk (CVaR) decision criterion to mitigate the risk of probability distribution ambiguity. The proposed two-stage DR inventory optimization model aims at finding an optimal inventory level in face of high uncertain multi-item orders. We further provide an equivalent reformulation for the proposed model and design an efficient row-and-column generation (RCG) algorithm. Numerical experiments based on real-world data are conducted to validate the effectiveness of the proposed model and algorithm.

The contributions of this paper are summarized as follows.

- To the best of our knowledge, it is the first work to investigate the FDC inventory optimization problem considering the uncertain multi-item orders. A novel two-stage DR FDC inventory model is proposed with the following unique features. First, we propose a Wasserstein distance-based distributional set using historical data to describe the high uncertain multi-item orders. Second, the model adopts a robust CVaR decision criterion to find robust inventory decision. Third, we provide an equivalent deterministic reformulation for the proposed model by exploiting the duality form of the inner optimization problem.

- This paper proposes an efficient and exact RCG algorithm for the three layer max-min-max DR model. Different from existing RCG algorithms (e.g., [15]), which need to solve a series of mixed 0-1 linear programming problems to generate variables and constraints for the master problem, we show that the proposed RCG algorithm only needs to solve linear programming problems by exploiting the structure properties of the feasibility subproblem.
This paper is organized as follows. Section 2 reviews related literature. Sections 3 and 4 give the proposed model and algorithm. Section 5 reports experimental results and Section 6 concludes this paper.

2. Literature review. The multi-item order leads to split-order fulfillment and lost sales. To reduce its adverse effect, researchers have developed different methods, including allocation assortment, order allocation, and inventory optimization.

Allocation assortment aims at reducing the number of splitting orders from the strategic perspective by allocating closely related SKUs to the same warehouse. [4] identify two main reasons for order splitting: shortage and multiple SKUs stored in the different warehouses. They analyze the problem of allocating SKUs to warehouses to reduce total shipment cost and propose four heuristics. [3] discuss which products should be sold in multiple distribution centers and propose a model to maximize the expected profit. [12] propose a prediction model that combines exponential smoothing and community detection to forecast future orders. To enhance robustness of the solution, they propose a robust optimization model and develop
Figure 2. Proportion of single-item order and multi-item orders in two FDCs of JD.com.

an efficient solution approach. [18] propose a K-links heuristic clustering algorithm to optimize product category allocation among multiple warehouses based on the distribution of multi-item orders.

Order allocation is to allocate real-time orders to warehouses with different SKU inventory levels with the goal of minimizing transportation costs by reducing order splits. [13] and [14] construct a neighborhood search heuristic algorithm to periodically determine real-time order allocation. [1] study the issue of order fulfillment by considering the opportunity cost of future orders. They establish an asymptotic optimal method to estimate the expected transportation cost in the future. [14] incorporate demand forecasting into the multi-item order problem, approximately construct a stochastic control formula, and propose two heuristic algorithms based on linear programming.
Inventory optimization method aims at reducing the operational cost by seeking the optimal inventory level. [2] study how to allocate inventory to multiple fulfillment centers under the joint replenishment policy to minimize the outbound transportation cost. [5] consider the inventory management of Alibaba’s two echelon distribution system. [6] consider a joint pricing and inventory problem with promotion constrains over a finite planning horizon for a single fast-moving good under the monopolistic environment. [8] study a fuzzy stock replenishment policy for inventory items that follow linear demand and Weibull deterioration.

This paper differs from existing studies in the following two aspects. First, it is the first to study the FDC inventory optimization problem by considering the effect of uncertain multi-item orders. Second, this paper proposes a novel DR inventory model under the robust CVaR decision criterion and designs an efficient row-and-column generation algorithm.

3. Problem formulation. Motivated by the practical FDC inventory optimization problem, this paper studies how to optimally determine the inventory levels for a set of product SKUs in a FDC to maximize the total profit. Before introducing the proposed model, we first give some notations; please see Table 1 for the notation list. Given a set of product SKUs $I = \{1, \cdots, n\}$ and a set of possible multi-item orders $J = \{1, \cdots, m\}$, let $a_{ij}$ denote the number of SKU $i \in I$ contained in order $j \in J$. Let $p'_i$ and $h'_i$ be the net profit and holding cost for each SKU $i \in I$, respectively. Then, $p_j = \sum_{i \in I} a_{ij} p'_i$ and $h_j = \sum_{i \in I} a_{ij} h'_i$ denote the net profit and holding cost for each multi-item order $j \in J$, respectively. Let $c_i$ and $C$ be the occupied capacity for each SKU $i \in I$ and the capacity of the FDC, respectively. We use $d_j$ to denote the random demand for each multi-item order $j \in J$.

At the beginning of every ordering period (e.g., one week), the inventory level of each SKU $i \in I$ is replenished to $s_i$. Due to the limited FDC capacity, the total inventory level should be less than or equal to the inventory capacity $C$, i.e., $s \in S \triangleq \{s \geq 0 : \sum_{i \in I} c_i s_i \leq C\}$. Then the inventory at hand is used to satisfy the uncertain demand. We use $y_j$ to denote the number of satisfied multi-item order $j \in J$ during an ordering period. Here we assume the unsatisfied orders are lost since the FDC aims at a timely delivery for bestselling products, and customers usually transfer to other channels if there is stockout. The FDC manager needs to decide the inventory level $s$ before the realization of the uncertain demand $d$, while the number of satisfied multi-item order $y$ is determined after the demand is revealed. Thus, the considered problem is modeled by a two-stage model, where $g(s, d)$ denotes the second-stage net profit function for given $s$ and $d$.

As pointed in Section 1, there are several issues needed to be addressed. First, the FDC faces uncertain customer demand, and a large proportion of customer orders contain multiple SKUs. Using historical data, we proposes a Wasserstein distance-based distributional set to describe the uncertain multi-item orders in Subsection 3.1. Second, to mitigate the risk of demand uncertainty, the worst-case CVaR of the random profit is used as the objective function in Subsection 3.2. Finally, in Subsection 3.3, we give the proposed DR inventory optimization model and its equivalent reformulation.

3.1. Distributional set for uncertain demand. Since the exact probability distribution function (PDF) of customer demand $P$ is difficult to estimate, this subsection proposes a Wasserstein distance-based distributional set to describe the uncertain multi-item orders using historical sale data.
of customer demand

3.2. Robust CVaR objective function. Although the expected profit is a reasonable decision criterion, it may lead to over optimistic decisions that suffer from large profit fluctuation. In practice, the FDC manager is usually risk-averse, and thus we adopt a robust CVaR decision criterion as the objective function.

For a random loss $X$, its $\alpha$-level ($\alpha \in (0, 1]$) CVaR is defined as the expected value of $X$ conditioning on $X \geq \text{VaR}_\alpha(X)$, i.e., $\text{CVaR}_\alpha(X) = E[X|X \geq \text{VaR}_\alpha(X)]$, where $\text{VaR}_\alpha(X)$ denotes $1 - \alpha$ quantile of $X$, i.e., $\text{VaR}_\alpha(X) = \min\{z : F_X(z) \geq 1 - \alpha\}$.

### Table 1. Notation list.

| Sets          | Description                                           |
|---------------|-------------------------------------------------------|
| $I$           | Set of product SKUs. $I = \{1, \ldots, n\}$.         |
| $J$           | Set of multi-item orders. $J = \{1, \ldots, m\}$.     |

| Parameters    | Description                                           |
|---------------|-------------------------------------------------------|
| $a_{ij} \in \mathbb{Z}_+$ | Number of SKU $i \in I$ contained in order $j \in J$. |
| $p'_i \in \mathbb{R}_+$  | Net profit for each SKU $i \in I$.                    |
| $p_j \in \mathbb{R}_+$   | Net profit for each multi-item order $j \in J$, i.e., $p_j = \sum_{i \in I} a_{ij} p'_i$. |
| $h'_i \in \mathbb{R}_+$  | Holding cost for each SKU $i \in I$.                  |
| $h_j \in \mathbb{R}_+$   | Holding cost for each multi-item order $j \in J$, i.e., $h_j = \sum_{i \in I} a_{ij} h'_i$. |
| $c_i \in \mathbb{R}_+$   | Occupied capacity for each SKU $i \in I$.             |
| $C \in \mathbb{R}_+$     | Capacity of the FDC.                                  |

| Random Parameters | Description                                      |
|-------------------|--------------------------------------------------|
| $d_j$             | Random demand for the multi-item order $j \in J$. |

| Decision Variables and Functions | Description                                      |
|----------------------------------|--------------------------------------------------|
| $s_i \in \mathbb{R}_+$          | First-stage decision; Inventory level for SKU $i \in I$. |
| $y_j \in \mathbb{R}_+$          | Second-stage decision; Number of satisfied multi-item order $j \in J$. |
| $g(s, d)$                       | Second-stage net profit function for given $s$ and $d$. |

Suppose we have a set of $T$ periods historical data $\{d^t : t = 1, \ldots, T\}$, where $d^t_j$ denotes the number of multi-item order $j$ in period $t$. Then the empirical PDF of customer demand $\mathbb{P}_0$ satisfies $\mathbb{P}_0(d = d^t) = \frac{1}{T}$. Here we assume that the true PDF of customer demand $\mathbb{P}$ is unknown, but lies in the following Wasserstein distance-based distributional set with a size parameter $\theta \geq 0$:

$$\mathcal{D} = \{\mathbb{P} \in \mathcal{M}(D) : d_W(\mathbb{P}, \mathbb{P}_0) \leq \theta\},$$

where $\mathcal{M}(D)$ is the set of PDFs over some support set $D$. We assume $d^t \in D$ holds for any $t = 1, \ldots, T$. The Wasserstein distance $d_W(\cdot)$ over the $l_{\infty}$ metric between two PDFs $\mathbb{F}_1$ and $\mathbb{F}_2$ is defined as

$$d_W(\mathbb{F}_1, \mathbb{F}_2) = \inf \left\{ \int_{\xi_1, \xi_2} \|\xi_1 - \xi_2\|_{l_{\infty}} K(d\xi_1, d\xi_2) : \int_{\xi_1} K(\xi_1, d\xi_2) = \mathbb{F}_1(\xi_1), \int_{\xi_1} K(d\xi_1, \xi_2) = \mathbb{F}_2(\xi_2) \right\},$$

where $K$ denotes the joint PDF of $(\xi_1, \xi_2)$.

The parameter $\theta$ in $\mathcal{D}$ determines the maximum deviation of the true PDF $\mathbb{P}$ from the empirical PDF $\mathbb{P}_0$, and thus controls the size of $\mathcal{D}$. A larger value of $\theta$ leads to more choices for the true PDF $\mathbb{P}$. In particular, when $\theta = 0$, we have $\mathcal{D} = \{\mathbb{P}_0\}$. That is, we assume the true PDF $\mathbb{P}$ is identical to the empirical PDF $\mathbb{P}_0$ in this case.
Compared with VaR\(_\alpha(X)\), CVaR is a convex consistent risk measure and is easier to handle due to the following property [9]:

\[
\text{CVaR}_\alpha(X) = \min_z \left\{ z + \frac{1}{\alpha}E[|X - z|^+] \right\},
\]

where \([x]^+ = x\) if \(x \geq 0\); otherwise, \([x]^+ = 0\).

In our problem, the FDC manager aims at minimizing the \(\alpha\)-level CVaR of the random loss \(-g(s, d)\), i.e., \(\text{CVaR}_\alpha(-g(s, d))\). Equivalently, we can maximize the negative \(\alpha\)-level CVaR of \(-g(s, d)\) defined as follows:

\[
\text{CVaR}'_\alpha(g(s, d)) = -\text{CVaR}_\alpha(-g(s, d))
\]

\[
= \max_z \left\{ -z - \frac{1}{\alpha}E[-g(s, d) - z]^+ \right\}
\]

\[
= \max_z \left\{ z - \frac{1}{\alpha}E[z - g(s, d)]^+ \right\}
\]

\[
= \max_z \left\{ z + \frac{1}{\alpha}E[\min\{g(s, d) - z, 0\}] \right\}.
\]

For any random variable \(X\), when \(\alpha = 1\), \(\text{CVaR}'_1(X)\) reduces to the expectation of \(X\). To see this, without loss of generality, suppose \(X\) is a random variable with the support set \([a, b]\) and probability density function \(f_X\). Let \(\phi(z) = z + E[\min\{X - z, 0\}]\). Then, \(\phi(z) = z\), if \(z \leq a\); \(\phi(z) = z + \int_a^z (x - z)f_X(x)dx\) and \(\phi'(z) = 1 - \text{Prob}(X \leq z) \geq 0\), if \(a \leq z \leq b\); and \(\phi(z) = E[X]\), if \(z \geq b\). Thus, \(\phi(z)\) is non-decreasing in \(z\), and we have \(\text{CVaR}'(X) = \max_z \phi(z) = E[X]\).

Since the true PDF \(P\) is unknown but lies in \(\mathcal{D}\), we consider the robust CVaR (RCVaR), which is the worst-case value of \(\text{CVaR}'(g(s, d))\) when \(P\) varies over \(\mathcal{D}\), i.e.,

\[
\text{RCVaR}_\alpha(g(s, d)) \triangleq \inf_{P \in \mathcal{D}} \text{CVaR}'_\alpha(g(s, d)).
\]

The \(\text{RCVaR}_\alpha(g(s, d))\) gives a lower bound of the expected profit conditioning on \(g(s, d) \leq \text{VaR}_{1-\alpha}(g(s, d))\) for given inventory level \(s\). For example, when \(\alpha = 1\), \(\text{CVaR}'(g(s, d)) = E[g(s, d)]\). The RCVaR method has also been used in other fields [17]. Thus, the RCVaR reduces to the worse-case expected profit, i.e., \(\text{RCVaR}(g(s, d)) \triangleq \inf_{P \in \mathcal{D}} E[g(s, d)]\).

### 3.3. DR inventory optimization model

To mitigate the risk of demand uncertainty, we adopt the distributionally robust optimization method [16, 11] and propose a DR inventory optimization model. Specifically, using the RCVaR as the decision criterion, the FDC inventory optimization problem is to find an optimal inventory level to maximize the robust conditional expected profit while meeting the capacity constraint. Specifically, we propose the following DR inventory optimization model:

\[
(P) \quad \max_{s \in S} \text{RCVaR}_\alpha(g(s, d)) = \max_{s \in S} \min_{P \in \mathcal{D}} \text{CVaR}'_\alpha(g(s, d)),
\]

where \(S = \{ s \geq 0 : \sum_{i=1}^m a_i s_i \leq C \}\) and the second-stage net profit function \(g(s, d)\) is given as

\[
g(s, d) = \max_y \sum_{j \in J} p_j y_j - \sum_{i \in I} h'_i (s_i) - \sum_{j=1}^m a_{ij} y_j = \sum_{j \in J} (p_j + h_j) y_j - \sum_{i \in I} h'_i s_i
\]
\[
\begin{align*}
\text{s.t.} & \quad \sum_{j=1}^{m} a_{ij} y_j \leq s_i, & \forall i \in I, & (3) \\
y_j & \leq d_j, & \forall j \in J, & (4) \\
y_j & \geq 0, & \forall j \in J. & (5)
\end{align*}
\]

Constraint (3) ensures the number of satisfied multi-item orders is less than or equal to the inventory level. Constraint (4) ensures that the number of satisfied multi-item orders is also no more than the customer demand. (P) is a three layer max-min-max problem and difficult to solve directly.

To this end, we provide an equivalent reformulation of (P). We first analyze the RCVaR for given first-stage decision \( s \). The following Lemma 3.1 gives an equivalent dual reformulation for RCVaR.

**Lemma 3.1.** For given first-stage decision \( s \), we have

\[
\text{RCVaR}_\alpha (g(s,d)) = \max_{z, \lambda, w} \left\{ z + \frac{1}{\alpha T} \left( \sum_{t=1}^{T} w_t - \theta T \lambda \right) \right\}
\]

\[
\text{s.t.} \quad w_t \leq \lambda \|d - d^t\|_\infty + g(s, d) - z, \quad \forall d \in D, \quad t = 1, \cdots, T, \\
\lambda \geq 0, \quad z \in \mathbb{R}.
\]

**Proof.** From equation (1) and definition (2), we have

\[
\text{RCVaR}_\alpha (g(s,d)) = \inf_{P \in \mathcal{D}} \max_{z} \left\{ z + \frac{1}{\alpha} \mathbb{E}_P[\min\{g(s, d) - z, 0\}] \right\}
\]

\[
= \max_{z} \left\{ z + \frac{1}{\alpha} \inf_{P \in \mathcal{D}} \mathbb{E}_P[\min\{g(s, d) - z, 0\}] \right\}
\]

where the second equality is due to the Mini-Max Theory [10].

From the definition of Wasserstein distance-based distributional set, we have

\[
\inf_{P \in \mathcal{D}} \mathbb{E}_P[\min\{g(s, d) - z, 0\}] = \inf_{\mathcal{D}} \int_{d \in D} \min\{g(s, d) - z, 0\} P(dd)
\]

\[
\text{s.t.} \quad \int_{d \in D} \|d - d^t\|_\infty K(dd, dd^t) \leq \theta, \\
K(\cdot, dd^t) = P(\cdot), \\
K(\cdot, \cdot) = P_0(\cdot), \\
K \in \mathcal{M}(D \times D).
\]

Let \( P^t \) be the conditional PDF of \( P \) conditioning on \( d^t = d^t \). That is, \( P = \frac{1}{T} \sum_{t=1}^{T} P^t \). Then from the law of total probability, we have

\[
\inf_{P \in \mathcal{D}} \mathbb{E}_P[\min\{g(s, d) - z, 0\}] = \inf_{P \in \mathcal{D}} \frac{1}{T} \sum_{t=1}^{T} \int_{d \in D} \min\{g(s, d) - z, 0\} P^t(dd)
\]

\[
\text{s.t.} \quad \frac{1}{T} \sum_{t=1}^{T} \int_{d \in D} \|d - d^t\|_\infty P^t(dd) \leq \theta, \\
\int_{d \in D} P^t(dd) = 1, \quad \forall t = 1, \cdots, T, \\
P^t(dd) \geq 0, \quad \forall t = 1, \cdots, T.
\]
By introducing dual variables $\lambda \geq 0$ and $\gamma_t \in R$ for the first two constraints of (8), we have the following dual problem
\[
\max_{\lambda, \gamma} \sum_{t=1}^{T} \gamma_t - \theta \lambda \\
\text{s.t.} \quad \gamma_t - \frac{\lambda}{T} ||d - d^t||_{\infty} \leq \frac{1}{T} \left[ \min \{ g(s, d) - z, 0 \} \right], \quad \forall d \in D, t = 1, \cdots, T, \\
\lambda \geq 0. \quad (9)
\]
Note that the problem (8) is a convex optimization problem with relative interior points, i.e., $P_t(d = d^t) = 1 \ (t = 1, \cdots, T)$, the strong duality theorem holds.

Let $w_t = T \gamma_t$. The dual problem can be reformulated as
\[
\max_{\lambda, w} \sum_{t=1}^{T} w_t - \theta \lambda \\
\text{s.t.} \quad w_t - \lambda ||d - d^t||_{\infty} \leq \min \{ g(s, d) - z, 0 \}, \quad \forall d \in D, t = 1, \cdots, T, \\
\lambda \geq 0. \quad (10)
\]
Finally, note that for any $t = 1, \cdots, T$, the constraint $w_t - ||d - d^t||_{\infty} \leq \min \{ g(s, d) - z, 0 \}$ is equivalent to the following constraints:
\[
\begin{cases}
    w_t - \lambda ||d - d^t||_{\infty} \leq g(s, d) - z, & \forall d \in D, \\
    w_t - \lambda ||d - d^t||_{\infty} \leq 0, & \forall d \in D.
\end{cases}
\]
Furthermore, the second constraint is equivalent to $w_t \leq \min_{d \in D} \lambda ||d - d^t||_{\infty} = 0$ since $d^t \in D$. We complete the proof.

Using Lemma 3.1, the proposed DR inventory optimization model can be equivalently reformulated into a deterministic optimization problem.

**Proposition 1.** The DR inventory optimization model (P) is equivalent to the following problem:

\[
\max_{s, z, \lambda, w} z + \frac{1}{\alpha T} \sum_{t=1}^{T} w_t - \frac{\theta}{\alpha} \lambda \\
\text{s.t.} \quad w_t \leq \lambda ||d - d^t||_{\infty} + g(s, d) - z, \quad \forall d \in D, t = 1, \cdots, T, \quad (12) \\
\quad w_t \leq 0, \quad \forall t = 1, \cdots, T, \quad (13) \\
\quad \sum_{i \in I} c_i s_i \leq C, \quad (14) \\
\quad s \geq 0, \lambda \geq 0, z \in R. \quad (15)
\]

To solve the equivalent problem (P'), there are two challenges that need to be addressed. First, there are an infinite number of constraints since (12) holds for any $d \in D$. Second, even for a given $d = d$, this constraint is nonlinear as $g(s, d)$ is nonlinear in $s$. We propose a novel row-and-column generation (RCG) algorithm in Section 4 to address these challenges.

4. **Row-and-column generation algorithm.** The proposed RCG algorithm includes two key steps, i.e., a relaxed master problem (RMP) and a feasibility check and generation procedure (FCGP). The RMP only considers a finite number of constraint (12). In this way, we can reformulate (P') into a solvable mixed integer linear programming problem. After solving a RMP, the FCGP is called to check whether the constraint (12) is satisfied. If so, we obtain the optimal solution of
4.2. Feasibility check and generation procedure. To check the feasibility of $(\hat{s}, \hat{w}, \hat{y}, \hat{z}, \hat{\lambda})$, it is sufficient to check the following condition for any $t = 1, \cdots, T$:

\[
(FP_t) \quad \min_{d \in D} \left\{ \tilde{\lambda} \|d - d^t\|_\infty + g(\hat{s}, d) \right\} \geq \hat{w}_t + \hat{z}.
\]
Generally, \((FP_t)\) is NP-hard. For a general support set \(D\), \((FP_t)\) can be reformulated into a mixed integer linear programming problem based on KKT conditions [15].

To simplify the computation, we consider a conditional support set \(D^t\) for each \(t = 1, \cdots, T\): 
\[
D^t = \{d|d_j = d^*_j + \delta \xi_j, \xi_j \in [-1, 1], \forall j \in M\},
\]
where \(\delta \geq 0\) is a size parameter that controls the variation range of \(d_j\). That is, we assume that for each \(t\), the true demand lies in a cube \(D^t\) centered at \(d^*_t\), and each demand component \(d_j\) lies in \([d^*_j - \delta, d^*_j + \delta]\). The following Lemma 4.1 reformulates \((FP_t)\) into a bilinear optimization problem.

**Lemma 4.1.** For any \(t = 1, \cdots, T\), \((FP_t)\) with \(D = D^t\) can be equivalently reformulated into the following bilinear optimization problem:

\[
\begin{align*}
\min_{\eta, \xi, u, v} \quad & \lambda \delta \eta + \hat{s}^T (u - h') + (d^t)^T v + \delta \xi^T v \\
\text{s.t.} \quad & \sum_{i=1}^{n} a_{ij} u_i + v_j \geq p_j + h_j, \quad \forall j \in J, \\
& |\xi_j| \leq 1, \quad \forall j \in J, \\
& |\xi_j| \leq \eta, \quad \forall j \in J, \\
& u_i, v_j \geq 0, \quad \forall i \in I, j \in J.
\end{align*}
\]

**Proof.** By introducing dual variables \(u \geq 0\) and \(v \geq 0\) for the constraints (3) and (4), we obtain the following dual form of \(g(\hat{s}, d)\) as follows
\[
g(\hat{s}, d) = \min_{u, v} \hat{s}^T (u - h') + d^T v
\]

\[
\text{s.t.} \quad \sum_{i=1}^{n} a_{ij} u_i + v_j \geq p_j + h_j, \quad \forall j \in J, \\
\quad u_i, v_j \geq 0, \quad \forall i \in I, j \in J.
\]
Thus, \((FP_t)\) is equivalent to

\[
\begin{align*}
\min_{d, u, v} \quad & \lambda \|d - d^t\|_{+\infty} + \hat{s}^T (u - h') + d^T v \\
\text{s.t.} \quad & \sum_{i=1}^{n} a_{ij} u_i + v_j \geq p_j + h_j, \quad \forall j \in J, \\
& d \in D^t, \\
& u_i, v_j \geq 0, \quad \forall i \in I, j \in J.
\end{align*}
\]

From the definition of \(D^t\), let \(d_j = d^*_j + \delta \xi_j\). We complete the proof.

Based on Lemma 4.1, we analyze the optimal condition of \((FP_t)\) and reformulate \((FP_t)\) into linear programming problems by eliminating the bilinear term \(\xi^T v\).

**Proposition 2.** For any \(t = 1, \cdots, T\), let \((\eta^*, \xi^*, u^*, v^*)\) be the optimal solution of \((FP_t)\) with \(D = D^t\).

(i) We have either \(\eta^* = 0\) or \(\eta^* = 1\).

(ii) When \(\eta^* = 0\), we have \(\xi^* = 0\), and \((FP_t)\) reduces to a linear programming problem.

(iii) When \(\eta^* = 1\), we have \(\xi_j^* = -1\), and \((FP_t)\) also reduces to a linear programming problem.
Therefore, in the optimal solution, we have $\eta^* = 0$. For given $\eta \in [0, 1]$ and $(u, v)$, it is easy to see that the optimal $\xi_j = -\eta$. Thus, (FP$_t$) is equivalent to the following minimization problem:

$$\min_{\eta, u, v} \lambda \delta \eta + \delta^T(u - h') + (d^t)^Tv - \delta \sum_{j \in M} v_j$$

s.t. $\sum_{i=1}^n a_{ij}u_i + v_j \geq p_j + h_j, \quad \forall j \in J,$

$0 \leq \eta \leq 1, \quad \forall j \in J,$

$u_i, v_j \geq 0, \quad \forall i \in I, j \in J.$

Therefore, in the optimal solution, we have $\eta^* = 0$ if $\hat{\lambda} - \sum_{j \in M} v_j^* \geq 0$; otherwise, $\eta^* = 1$.

(ii) When $\eta^* = 0$, we have $\xi_j^* = 0$. Furthermore, (FP$_t$) reduces to the following linear programming problem:

$$\min_{u, v} \delta^T(u - h') + (d^t)^Tv$$

s.t. $\sum_{i=1}^n a_{ij}u_i + v_j \geq p_j + h_j, \quad \forall j \in J,$

$u_i, v_j \geq 0, \quad \forall i \in I, j \in J.$

(iii) When $\eta^* = 1$, we have $\xi_j^* = -1$. (FP$_t$) reduces to the following linear programming problem:

$$\min_{\eta, u, v} \lambda \delta + \delta^T(u - h') + (d^t)^Tv - \delta \sum_{j \in M} v_j$$

s.t. $\sum_{i=1}^n a_{ij}u_i + v_j \geq p_j + h_j, \quad \forall j \in J,$

$u_i, v_j \geq 0, \quad \forall i \in I, j \in J.$

Using Proposition 2, (FP$_t$) can be solved easily by solving two linear programming problems. Let $\varphi_j^*$ be the optimal value of (FP$_t$). If we have $\varphi_j^* \geq \hat{w}_t + \hat{z}, \quad \forall t = 1, \cdots, T,$

then $(\hat{s}, \hat{w}, \hat{y}, \hat{z}, \hat{\lambda})$ is feasible for the original problem (P), and thus it is optimal for (P). Otherwise, if $\varphi_j^* < \hat{w}_t + \hat{z}$, then we identify a new demand scenario $\bar{d}^{t, k_t+1} = d^t + \delta \xi^*|$ and expand $D^t = D^t \cup \{d^{t, k_t+1}\}$.

Proposition 2 shows that the worst-case demand scenario in $D^t$ is either $d^t$ (corresponding to $\xi^* = 0$) or $d^t - \delta e$ (corresponding to $\xi^* = -e$) where $e = (1, \cdots, 1)^T$. From this observation, we show that (P) can be further simplified when $D = D^t$ for $t = 1, \cdots, T$.

**Proposition 3.** If $D = D^t$ for $t = 1, \cdots, T$, then (P) can be equivalently reformulated into the following optimization problem:

$$\max_{s, z, \alpha, w, \theta} \frac{z + 1}{\alpha T} \sum_{t=1}^T w_t - \theta \lambda$$

s.t. $w_t \leq \sum_{j \in J} (p_j + h_j) \overline{y}_{j}^{t,1} - \sum_{i \in I} h_i^t s_i - z, \quad \forall t = 1, \cdots, T,$

$w_t \leq \overline{\delta} \lambda + \sum_{j \in J} (p_j + h_j) \overline{y}_{j}^{t,2} - \sum_{i \in I} h_i^t s_i - z, \quad \forall t = 1, \cdots, T,$
\[
\sum_{j=1}^{m} a_{ij} y_{j}^{l,k} \leq s_i, \quad \forall t = 1, \ldots , T, i \in I, k = 1, 2, \\
0 \leq y_{j}^{l,1} \leq d_{j}, \quad \forall t = 1, \ldots , T, j \in J, \\
0 \leq y_{j}^{l,2} \leq d_{j} - \delta, \quad \forall t = 1, \ldots , T, j \in J, \\
w_t \leq 0, \\
\sum_{i \in I} c_i s_i \leq C, \\
s \geq 0, \lambda \geq 0, z \in R.
\]

Proof. From Proposition 2, constraint (12) can be equivalently reduced to
\[
\begin{cases}
w_t \leq g(s, d^t) - z, & \forall t = 1, \ldots , T, \\
w_t \leq \delta \lambda + g(s, d^t - \delta \epsilon_e) - z, & \forall t = 1, \ldots , T.
\end{cases}
\]
From the definition of \( g \), for any \( t \), the first constraint is equivalent to
\[
\begin{cases}
w_t \leq \sum_{j \in J} (p_j + h_j) y_{j}^{l,1} - \sum_{i \in I} h_i^t s_i - z \\
\sum_{j=1}^{m} a_{ij} y_{j}^{l,1} \leq s_i, & \forall i \in I, \\
0 \leq y_{j}^{l,1} \leq d_{j}, & \forall j \in J.
\end{cases}
\]
Similarly, for any \( t \), the second constraint is equivalent to
\[
\begin{cases}
w_t \leq \delta \lambda + \sum_{j \in J} (p_j + h_j) y_{j}^{l,2} - \sum_{i \in I} h_i^t s_i - z \\
\sum_{j=1}^{m} a_{ij} y_{j}^{l,2} \leq s_i, & \forall i \in I, \\
0 \leq y_{j}^{l,2} \leq d_{j} - \delta, & \forall j \in J.
\end{cases}
\]

4.3. RCG framework. The RCG algorithm framework is given in Algorithm 1. The RCG algorithm initializes the demand scenario set as \( \bar{D}^t = \{d^t\} \) for \( t = 1, \ldots , T \). Then, it repeatedly solves the RMP and calls the FCGP until the stop criterion \( \varphi_t^* \geq \bar{w}_t + \bar{z}_t, \forall t = 1, \ldots , T \) is satisfied.

5. Numerical experiments. This section conducts a numerical study based on historical data from FDCs of JD.com to validate the effectiveness of the proposed model and RCG algorithm. Subsection 5.1 gives the experimental setup. Subsection 5.2 presents a benchmark approach. Subsection 5.3 reports the computational results.

5.1. Experimental setup. By collaboration with the FDC manager, we collect customer demand data of 27 consecutive months (2019.3.8 - 2021.5.7) in three FDCs of JD.com. The statistical summary is given in Table 2. Consistent with the 80/20 principle, we select the top 50 product SKUs which meet nearly 80% of the total customer orders. The number of order types for these 50 SKUs is 46765. Among the 46765 order types, the top 300 order types are selected, which account for about 75% of the total number of 491269 orders. Thus we consider the inventory optimization problem with 50 SKUs and 300 order types, i.e., \( n = 50 \) and \( m = 300 \). To avoid
Algorithm 1  
Row-and-Column Generation Algorithm

**Input:** $a, p, h, h', c, C, d^t, \alpha$ and $\delta$.

**Output:** $(s^*, w^*, y^*, z^*, \lambda^*)$.

1. **Initialization:** For $t = 1, \ldots, T$, initialize demand scenario set as $ar{D}^t = \{d^t\}$.

2. **Solve Relaxed Master Problem:**
   For given $ar{D}^t (t = 1, \ldots, T)$, solve RMP to obtain $(\hat{s}, \hat{w}, \hat{y}, \hat{z}, \hat{\lambda})$.

3. **Feasibility Check and Generation Procedure:**
   For $t = 1, \ldots, T$, solve (FP$_t$) to obtain $(\eta^*, \xi^*, u^*, v^*)$ and $\varphi_t^*$.

4. **Stop Criterion:**
   If $\varphi_t^* \geq \hat{w}_t + \hat{z}$ holds for all $t = 1, \ldots, T$, then return $(s^*, w^*, y^*, z^*, \lambda^*) = (\hat{s}, \hat{w}, \hat{y}, \hat{z}, \hat{\lambda})$;
   Otherwise, for $t$ such that $\varphi_t^* > \hat{w}_t + \hat{z}$,
   update $\bar{D}^t = \bar{D}^t \cup \{\bar{d}^{t,k_{t+1}}\}$, and goto Step 2.

the promotion effect, we remove promotion weeks, including Nov. 11th, Dec. 12nd, Mar. 8th, and Jun 18th. A total number of $T = 110$ historical week sale data are used.

| Table 2. Statistical Summary. |
|------------------------------|
|                              | FDC A     | FDC B     | FDC C     |
| Number of orders             | 2632408   | 669161    | 1094012   |
| Number of order types        | 588994    | 226448    | 319329    |
| Number of product SKUs       | 684       | 659       | 684       |

The profit for each product SKU $i$ is uniformly generated in $[1, 100]$, the occupied capacity for each SKU $i$ is uniformly generated in $[0.1, 2]$ and the holding cost for each SKU $i$ is uniformly generated in $[1, 3]$. We use the historical data from JD.com as the training set. For the test set, we randomly generate 500 demand samples based on historical data such that $d_j$ is uniformly generated over $[d_j - \sigma_j, d_j + \sigma_j]$, where $\sigma_j$ is the standard deviation of $d_j$.

All the computations are performed on a work station with Xeon E-2286G CPU and 12 GB memory. The optimization problems are solved by Gurobi 9.0.3 solver coded in Matlab 2016a.

5.2. **Benchmark model: SKU-based model.** Traditionally the FDC manager neglects the effect of multi-item orders, and determines the inventory level based on empirical PDF of the random demand. Specifically, the FDC manager solves the following SKU-based model (SM) to determine the optimal inventory level:

$$(SM) \quad \max_{s} \quad \sum_{i \in I} \text{CVaR}_\alpha(f_i(s_i, b_i))$$

\[ \text{s.t.} \]

\[ \sum_{i \in I} c_i s_i \leq C, \]

\[ s_i \geq 0, \quad \forall i \in I. \]
where \( f_i(s_i, b_i) = p_i \min\{s_i, b_i\} - h_i[s_i - b_i]^+ = \max\{p_i y_i - h_i(s_i - y_i) : y_i \leq s_i, y_i \leq b_i\} = \max\{p_i + h_i y_i : y_i \leq s_i, y_i \leq b_i\} - h_i s_i \) and \( b_i \) is the demand for SKU \( i \), i.e., \( b_i = \sum_{j \in J} a_{ij} d_j \).

**Proposition 4.** Given the empirical PDF \( \mathbb{P}_0 \), (SM) can be equivalently reformulated into the following linear programming problem:

\[
\begin{align*}
\text{(SM)} \quad \max_{s, z, y, w} & \quad \sum_{i \in I} \left\{ z_i + \frac{1}{\alpha T} \sum_{t=1}^{T} w_i^t \right\} \\
\text{s.t.} & \quad \sum_{i \in I} c_i s_i \leq C, \\
& \quad (p_i^t + h_i^t) y_i^t \geq w_i^t + h_i^t s_i + z_i, \quad \forall i \in I, t = 1, \cdots, T, \\
& \quad y_i^t \leq b_i^t, \quad \forall i \in I, t = 1, \cdots, T, \\
& \quad y_i^t \leq s_i, \quad \forall i \in I, t = 1, \cdots, T, \\
& \quad w_i^t \leq 0, s_i \geq 0, z_i \in R, \quad \forall i \in I, t = 1, \cdots, T.
\end{align*}
\]

**Proof.** From the definition of CVaR, we have

\[
\text{CVaR}_\alpha (f_i(s_i, b_i)) = \max_{z_i \in R} \left\{ z_i + \frac{1}{\alpha} \mathbb{E}_{\mathbb{P}_0}[\min\{f_i(s_i, b_i) - z_i, 0\}] \right\}
\]

Therefore, the objective function can be reformulated as

\[
\max_{s, z, R} \sum_{i \in I} \left\{ z_i + \frac{1}{\alpha T} \mathbb{E}_{\mathbb{P}_0}[\min\{f_i(s_i, b_i) - z_i, 0\}] \right\}
\]

\[
= \max_{s, z, R} \sum_{i \in I} \left\{ z_i + \frac{1}{\alpha T} \sum_{t=1}^{T} \min\{f_i(s_i, b_i^t) - z_i, 0\} \right\}
\]

Let \( w_i^t = \min\{f_i(s_i, b_i^t) - z_i, 0\} \), SM can be further reformulated as

\[
\max_{s, z, w} \quad \sum_{i \in I} \left\{ z_i + \frac{1}{\alpha T} \sum_{t=1}^{T} w_i^t \right\} \\
\text{s.t.} & \quad \sum_{i \in I} c_i s_i \leq C, \\
& \quad w_i^t \leq f_i(s_i, b_i^t) - z_i, \quad \forall i \in I, t = 1, \cdots, T, \\
& \quad w_i^t \leq 0, \quad \forall i \in I, t = 1, \cdots, T, \\
& \quad s_i \geq 0, z_i \in R, \quad \forall i \in I.
\]

From the definition of function \( f_i \), for any \( t \) and \( i \), the constraint \( w_i^t \leq f_i(s_i, b_i^t) - z_i \) is equivalent to

\[
\begin{align*}
(p_i^t + h_i^t) y_i^t \geq w_i^t + h_i^t s_i + z_i, \quad \forall i \in I, \\
y_i^t \leq b_i^t, \quad \forall i \in I, \\
y_i^t \leq s_i, \quad \forall i \in I.
\end{align*}
\]

5.3. **Experimental results.** We use the following metrics to measure the performance of the proposed DR model (DRM) with the benchmark SM:

- CVaR: the \( \alpha \)-level CVaR of the profit over the test set;
- Mean: the mean profit over the test set;
- Std: the standard deviation of profit over the test set;
- SL: the service level, i.e., the proportion of satisfied orders.
In the following, we first compare the performance of DRM and SM under different risk averse \( \alpha \)-level and inventory capacity \( C \). Then, we report the computational efficiency of the proposed RCG algorithm.

### Table 3. Performance of the proposed DRM in comparison with SM under different \( \alpha \)-levels.

| \( \alpha \) | CVaR | Mean |
|---|---|---|
| DRM | SM | \( \Delta(\%) \) | DRM | SM | \( \Delta(\%) \) |
| 0.85 | 44003.3 | 425087.0 | 3.5 | 46365.2 | 451409.4 | 2.7 |
| 0.9 | 44835.1 | 429470.7 | 4.4 | 46504.4 | 448640.9 | 3.7 |
| 0.95 | 45680.1 | 435469.0 | 4.9 | 46572.6 | 445833.8 | 4.5 |
| 1 | 46567.3 | 44738.3 | 4.7 | 46567.3 | 44738.4 | 4.7 |

| \( \alpha \) | Std | SL |
|---|---|---|
| DRM | SM | \( \Delta(\%) \) | DRM | SM | \( \Delta(\%) \) |
| 0.85 | 96361.7 | 100889.8 | 4.5 | 0.855 | 0.831 | 2.9 |
| 0.9 | 100830.4 | 106131.3 | 5.0 | 0.851 | 0.819 | 3.9 |
| 0.95 | 104735.4 | 109443.4 | 4.3 | 0.845 | 0.811 | 4.2 |
| 1 | 106993.8 | 112404.5 | 4.8 | 0.841 | 0.803 | 4.7 |

**Effect of Risk Averse \( \alpha \)-level.** Table 3 gives the performance of the DRM and SM under different risk averse \( \alpha \)-level when the inventory capacity \( C = 8000 \). For each performance metric, we report the values of both models and the improvement \( \Delta \) introduced by the proposed DRM. For example, for the Mean metric, \( \Delta \) is calculated as \( \frac{\text{Mean}_{\text{DRM}} - \text{Mean}_{\text{SM}}}{\text{Mean}_{\text{SM}}} \) where \( \text{Mean}_{\text{DRM}} \) and \( \text{Mean}_{\text{SM}} \) denote the expected profit of the optimal inventory level decision given by DRM and SM, respectively.

From Table 3, we have the following observations. First, the proposed model not only increases CVaR and the expected profit, but also decreases the profit fluctuation measured by Std. This effect validates the robustness of the proposed model to mitigate the risk of distribution ambiguity. Second, the proposed model also improves the service level. The reason for this phenomenon is that our DRM considers multi-item orders while SM neglects the demand correlation between different SKUs. Finally, as \( \alpha \) approaches to 1, the value of CVaR metric increases and converges to the value of Mean metric. This is consistent with the definition of CVaR since \( \text{CVaR}_1(X) = \mathbb{E}[X] \).

**Effect of Inventory Capacity \( C \).** Table 4 gives the performance of the DRM and SM under different inventory capacity \( C \) when the risk averse level \( \alpha = 0.95 \). From Table 4, we find that the proposed model also outperforms SM in terms of CVaR, Mean, Std and SL metrics. This is consistent with the findings given by Table 3. Furthermore, in general when the inventory capacity is small, the improvement of DRM over SM is more significant. For example, when \( C = 5000 \), the proposed model is able to increase the CVaR, expected profit and service level by 5.5%, 4.3% and 3.8%, respectively, and decrease the profit fluctuation by 36.1% in comparison with SM.

**Computational Efficiency.** Table 5 reports the computational efficiency of the proposed RCG algorithm when \( \alpha = 0.85 \). The columns labeled “T”, “T\(_R\)” and
Table 4. Performance of the proposed DRM in comparison with SM under different $C$.

| $C$  | CVaR         | Mean          |
|------|--------------|---------------|
|      | DRM          | SM            | DRM          | SM            |                  |
| 5000 | 382737.4     | 362668.6      | 5.5          | 385246.1      | 369465.5        | 4.3            |
| 6000 | 416751.5     | 389990.2      | 6.9          | 422162.8      | 398222.2        | 6.0            |
| 7000 | 440279.7     | 409294.0      | 7.6          | 447352.9      | 419146.7        | 6.7            |
| 8000 | 456801.3     | 435469.0      | 4.9          | 465726.2      | 445833.8        | 4.5            |
| 9000 | 468414.9     | 452002.1      | 3.6          | 478903.7      | 463654.4        | 3.3            |
| 10000| 475970.4     | 463618.8      | 2.7          | 488072.9      | 476586.1        | 2.4            |

Table 5. Computational efficiency of the proposed RCG algorithm.

| $C$  | T(s)         | T_R(s)        | Iter_R | Var.   | Cons. | T_F(s) | Iter_F |
|------|--------------|---------------|--------|--------|-------|--------|--------|
| 5000 | 57.897       | 32.835        | 3      | 66606  | 36421 | 22.107 | 330    |
| 6000 | 67.840       | 32.131        | 3      | 67259  | 37625 | 34.127 | 330    |
| 7000 | 51.998       | 20.851        | 2      | 67259  | 31003 | 29.824 | 220    |
| 8000 | 59.025       | 20.980        | 2      | 67259  | 31003 | 36.657 | 220    |
| 9000 | 66.092       | 21.044        | 2      | 67259  | 31003 | 43.591 | 220    |
| 10000| 72.998       | 21.054        | 2      | 67259  | 31003 | 50.402 | 220    |

Average 62.642 24.816 2.33 67150 33010 36.118 256.667

"T_F" report the CPU time for solving problem (P), the RMP, and the feasibility subproblem using the FCGP in seconds, respectively. The columns labeled “Iter_R” and “Iter_F” report the number of RMP solved, and the number of feasibility subproblems solved by the FCGP. The columns labeled “Var.” and “Cons.” report the number of variables and constraints generated. The last row reports the average of the performance metrics. Table 5 shows that the proposed RCG algorithm is very efficient to solve the proposed model. For all test instances, no more than three RMPs are solved, and each feasibility subproblem can be solved efficiently by the linear programming solver within 0.2 seconds on average. It takes about one minute on average to solve the inventory optimization problem, and thus the proposed model and algorithm are suitable as effective practical decision support tools.
6. Conclusions. This paper investigates an inventory optimization problem motivated by practical FDC operations. To satisfy the uncertain multi-item orders with limited inventory capacity, the FDC manager strives for the optimal inventory level for the bestselling products. To this end, this paper proposes a distributionally robust inventory optimization approach with the following features. First, to mitigate the risk of probability distribution ambiguity, we propose a Wasserstein distance-based distributional set to describe the uncertain demand using historical data. Second, we propose a distributionally robust inventory model which aims at maximizing the robust CVaR decision criterion. Finally, a novel row-and-column generation algorithm is designed to solve the model efficiently. Experimental results based on real-world data are conducted to highlight the superior out-of-sample performance of the proposed model in terms of CVaR, mean, stand deviation and service level in comparison with an existing benchmark approach, and the computational efficiency of the proposed algorithm.

The effectiveness of the proposed robust model depends on the constructed distributional set. Incorporating more accurate demand forecasting into the construction of the distributional set can improve the performance of the robust model and reduce its conservativeness. Thus, one possible future research direction is to combine the DR inventory optimization model with demand forecasting based on advanced machine learning methods. Another extension of this paper is to extend the proposed model to joint inventory optimization of multiple FDCs to reduce the operational cost due to split-order fulfillment among multiple FDCs.

Acknowledgments. We thank the editor and the three anonymous reviewers for their valuable comments that have greatly improved the manuscript. This work is supported by the National Key Research and Development Program of China under Grant 2018AAA0101602, National Nature Science Foundation under Grant 71871023, 72061127001, Science and Technology Innovation Project of Beijing Institute of Technology, and Dongguan Innovative Research Team Program 201807202007.

REFERENCES

[1] J. Acimovic and S. C. Graves, Making better fulfillment decisions on the fly in an online retail environment, Manufacturing & Service Operations Management, 17 (2015), 34–51.
[2] J. Acimovic and S. C. Graves, Mitigating spillover in online retailing via replenishment, Manufacturing & Service Operations Management, 19 (2017), 419–436.
[3] B. Bebitoglu, A. Şen and P. Kaminsky, Multi-location assortment optimization under capacity constraints, Available at SSRN 3249175, 2018.
[4] A. Catalán and M. Fisher, Assortment allocation to distribution centers to minimize split customer orders, Available at SSRN 2166687.
[5] B. Dai, H. Chen, Y. Li, Y. Zhang, X. Wang and Y. Deng, Inventory replenishment planning in a distribution system with safety stock policy and minimum and maximum joint replenishment quantity constraints, In 2019 International Conference on Industrial Engineering and Systems Management (IESM), (2019), 1–6.
[6] L. Deng, W. Bi, H. Liu and K. L. Teo, A multi-stage method for joint pricing and inventory model with promotion constrain, Discrete Contin. Dyn. Syst. Ser. S, 13 (2020), 1653–1682.
[7] JD.com, https://ir.jd.com/static-files/8bc55c1e-93de-4b87-80f9-f6649375ff2f, 2021.
[8] D. K. Nayak, S. S. Routray, S. K. Paikray and H. Dutta, A fuzzy inventory model for weibull deteriorating items under completely backlogged shortages, Discrete Contin. Dyn. Syst. Ser. S, 14 (2021), 2435–2453.
[9] R. T. Rockafellar, S. Uryasev et al., Optimization of conditional value-at-risk, Stochastic Optimization: Algorithms and Applications, 54 (2001), 411–435.
[10] M. Sion, On general minimax theorems, Pacific J. Math., 8 (1958), 171–176.
[11] Z. Wang, K. You, S. Song and Y. Zhang, Wasserstein distributionally robust shortest path problem, European J. Oper. Res., 284 (2020), 31–43.
[12] T. Wu, H. Mao, Y. Li and D. Chen, Assortment selection for a frontend warehouse: A robust data-driven approach, In 49th International Conference on Computers and Industrial Engineering (CIE 2019), (2019), 56–64.
[13] P. J. Xu, Order Fulfillment in Online Retailing: What Goes Where, PhD thesis, Massachusetts Institute of Technology, 2005.
[14] P. J. Xu, R. Allgor and S. C. Graves, Benefits of reevaluating real-time order fulfillment decisions, Manufacturing & Service Operations Management, 11 (2009), 340–355.
[15] B. Zeng and L. Zhao, Solving two-stage robust optimization problems using a column-and-constraint generation method, Oper. Res. Lett., 41 (2013), 457–461.
[16] Y. Zhang, Z.-J. M. Shen and S. Song, Exact algorithms for distributionally β-robust machine scheduling with uncertain processing times, INFORMS J. Comput., 30 (2018), 662–676.
[17] Y. Zhang, S. Song, Z.-J. M. Shen and C. Wu, Robust shortest path problem with distributional uncertainty, IEEE Transactions on Intelligent Transportation Systems, 19 (2017), 1080–1090.
[18] S. Zhu, X. Hu, K. Huang and Y. Yuan, Optimization of product category allocation in multiple warehouses to minimize splitting of online supermarket customer orders, European J. Oper. Res., 290 (2021), 556–571.

Received November 2021; revised December 2021; early access January 2022.

E-mail address: zhangyuli@bit.edu.cn
E-mail address: 3120201585@bit.edu.cn
E-mail address: zhuangxiaotian@jd.com