GLOBAL REGULAR SOLUTIONS TO TWO-DIMENSIONAL THERMOVISCOELASTICITY

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Abstract. A two-dimensional thermoviscoelastic system of Kelvin-Voigt type with strong dependence on temperature is considered. The existence and uniqueness of a global regular solution is proved without small data assumptions. The global existence is proved in two steps. First global a priori estimate is derived applying the theory of anisotropic Sobolev spaces with a mixed norm. Then local existence, proved by the method of successive approximations for a sufficiently small time interval, is extended step by step in time. By two-dimensional solution we mean that all its quantities depend on two space variables only.

1. Introduction. This article is devoted to the problem of global existence and uniqueness of regular solutions to a two-dimensional (2d) thermoviscoelasticity system for small strains. The system describes homogeneous isotropic linearly-responding viscoelastic materials in the Kelvin-Voigt rheology at small strains. We consider such thermoviscoelasticity system that the specific heat and the elasticity tensor depend on temperature in a very special relation.

Lately in [15] global existence of regular solutions to three-dimensional thermoviscoelasticity with specific heat linearly increasing with temperature and with constant heat conductivity is proved. Such setting is a particular case of systems presented in [2, 18]. Existence of weak solutions for generalized thermoviscoelastic materials with various kinds of boundary conditions can be found in [17, 19]. Moreover, papers of Roubíček [18, 19, 20] and Rossi-Roubíček [17] present a deep physical background of thermoviscoelastic materials.

Pioneering papers on global regular solutions to one-dimensional thermoviscoelasticity are [4, 5, 21] and the spherical case is considered in [7]. Lately, global existence of large solutions to spherically symmetric nonlinear viscoelasticity is proved in [8, 9].

In this paper we consider a two-dimensional thermoviscoelastic system with the temperature dependent specific heat of the form $c_\alpha = c_v \theta^\sigma$ with $\frac{1}{2} < \sigma < 1$, $c_v$ positive constant and with constant heat conductivity. Such setting is a particular case

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of systems addressed in [18]. Moreover, the stress tensor is given by a linear ther-
moviscoelastic law of the Kelvin-Voigt type (cf. [6, Chapter 5.4]). The aim of this paper is to prove existence of global regular solutions to the 2d-thermoviscoelastic system without smallness assumptions on data and for $\sigma$ as small as possible. In
3d-case (see [15]) global regular solutions for large data is only proved for $\sigma = 1$.
Restricting in this paper our considerations to the 2d case we were able to use the
specific heat $c_* = c_0t^{\sigma}$, $\sigma \in (1/2, 1)$.

The proof of global existence basis on two main steps. First we prove global a
priori estimate is Sobolev spaces $W^{1,1}_p(\Omega^T)$ with the mixed norm. This is possible
because equations for displacement and temperature are parabolic. The idea was
strongly developed in [15]. Since $c_* = c_0t^{\sigma}$ is the coefficient heat near $\theta_t$, we need
continuity of $\theta$ to apply the theory for parabolic equations, so $p$ and $p_0$ must be
sufficiently large. Next, we prove local existence in $W^{2,1}_p(\Omega^T)$-spaces by the method
of successive approximations. Combining these two steps we prove the main result:
global existence of regular solutions with large data.

Then we consider the following thermoviscoelasticity system

$$u_{tt} - \div((A_1\varepsilon_t) + (A_2\varepsilon) + A\theta) = b \quad \text{in} \quad \Omega^T = \Omega \times (0,T),$$

$$c_0t^{\sigma}\theta_{tt} - x\Delta \theta = \theta A\varepsilon_t + (A_1\varepsilon_t) \cdot \varepsilon_t + g \quad \text{in} \quad \Omega^T,$$

where $\Omega \subset \mathbb{R}^2$, with a boundary $S$, is bounded, $\sigma$ is a positive constant. We add
the boundary conditions

$$u = 0, \quad \bar{n} \cdot \nabla \theta = 0 \quad \text{on} \quad S^T = S \times (0,T),$$

where $\bar{n}$ is the unit outward normal vector to $S$, and the initial conditions

$$u|_{t=0} = u_0, \quad u_t|_{t=0} = u_1, \quad \theta|_{t=0} = \theta_0 \quad \text{in} \quad \Omega.$$
We assume that tensors $A_m$ satisfy the following symmetry conditions
\[(A_m)_{ijkl} = (A_m)_{jikl} = (A_m)_{klij}, \quad m = 1, 2,\] (1.11)
coeckivity and boundedness
\[a_m \varepsilon \leq (A_m \varepsilon) \cdot \varepsilon \leq a^*_m \varepsilon, \quad m = 1, 2,\] (1.12)
where $a_m$, $a^*_m$ are positive constants.

**Main Theorem.** Let (1.7), (1.8), $g > 0$ and the assumptions of Lemma 4.1 with $A = \text{const}$ hold. Let $u_0 \in W^2_p(\Omega)$, $u_1 \in B^{2-2/p_0}_p(\Omega)$, $\theta_0 \in B^{2-2/q_0}_q(\Omega)$, $p, p_0, q, q_0 \geq 4, 1/2 < \sigma \leq 1$. Let $S \in C^2$. Then there exists a global solution to problem (1.8)–(1.11) such that
\[u, t \in W^{2,1}_p(\Omega_T), \quad \theta \in W^{2,1}_q(\Omega_T), \quad \theta \geq \theta^*_s > 0\] with $T$ finite.

The paper is organized in the following way. In Section 1 the considered problem is formulated and its properties are listed. In [15] we showed that the property $g \geq 0$ implies that the second law of thermodynamics holds. In Section 3 we define used in this paper spaces with corresponding imbeddings and interpolations and also solvability results for some parabolic initial-boundary value problems (3.1) and (3.4) are presented. Section 4 is devoted to show a positive infimum of temperature. In Section 5 we derive some global a priori estimates. The main estimate is the Hölder continuity of temperature which implies that $W^{2,1}_{q,q_0}$-theory can be applied to equation (1.2). This is compatible with results of Section 6. Applying the method of successive approximations we prove in Section 6 local existence of solutions to problem (1.1)–(1.4). Finally, in Section 7 we show global existence of solutions to (1.1)–(1.4) by comparison results from Sections 5 and 6.

2. **Physical and thermodynamical background.** In the case of (1.1), (1.2) the free energy is specified by
\[f(\varepsilon, \theta) = f_*(\theta) + W(\varepsilon, \theta),\] (2.1)
where
\[f_*(\theta) = \frac{c_v}{\sigma(\sigma + 1)} \theta^{\sigma+1}, \quad c_v = \text{const} > 0,\] (2.2)
is the caloric energy, and
\[W(\varepsilon, \theta) = \frac{1}{2} (\varepsilon - \theta \alpha) \cdot A_2 (\varepsilon - \theta \alpha) - \frac{\theta^2}{2} \alpha \cdot (A_2 \alpha)\]
\[= \frac{1}{2} \varepsilon \cdot (A_2 \varepsilon) - \theta \varepsilon \cdot (A_2 \varepsilon), \quad A = -A_2 \alpha.\] (2.3)
From (2.2) the specific heat takes the form
\[c_* = -\theta f''_*(\theta) = c_v \theta^\sigma.\] (2.4)
The dissipation potential corresponding to (1.1), (1.2) is given by
\[D = \frac{1}{2\theta} \varepsilon, t \cdot (A_1 \varepsilon, t) + k \nabla \theta^2,\] (2.5)
where $k > 0$ and the Fourier law for a heat flux is used.
3. Notation and auxiliary results.

3.1. Notation. Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a domain in $\mathbb{R}^n$ with boundary $S$. Let $\Omega^T = \Omega \times (0,T)$, $S^T = S \times (0,T)$ with $T$ finite. By $W^k_p(\Omega)$, $k \in \mathbb{N} \cup \{0\} \equiv \mathbb{N}_0$, $p \in [1,\infty)$, we denote the Sobolev space with the finite norm

$$\|u\|_{W^k_p(\Omega)} = \left( \sum_{|\alpha| \leq k} \int_{\Omega} |D^\alpha u|^p dx \right)^{1/p},$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index, $\alpha_i \in \mathbb{N}_0$, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, $D^\alpha_x = \partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$. Let $H^k(\Omega) = W^k_{2,2}(\Omega)$.

Next, we introduce anisotropic Lebesgue spaces $L_{p,p_0}(\Omega^T) = L_{p_0}(0,T; L_p(\Omega))$, $p, p_0 \in [1,\infty]$, with the finite norm

$$\|u\|_{L_{p,p_0}(\Omega^T)} = \left( \int_0^T \|u(t)\|_{L_{p_0}(\Omega)}^{p_0} dt \right)^{1/p_0}.$$ Moreover, $W^{k,k/2}_{p,p_0}(\Omega^T)$, $k, k/2 \in \mathbb{N}_0$, $p,p_0 \in [1,\infty]$ are Sobolev spaces with a mixed norm, which are completion of $C^\infty(\Omega^T)$-functions under the finite norm

$$\|u\|_{W^{k,k/2}_{p,p_0}(\Omega^T)} = \left( \int_0^T \left( \sum_{|\alpha|+2a \leq k} \int_\Omega |D^\alpha x^{\alpha} u|^p dx \right)^{p_0/p} dt \right)^{1/p_0}.$$ By $W^{s,s/2}_{p,p_0}(\Omega^T)$, $s \in \mathbb{R}_+$, $p,p_0 \in [1,\infty]$, we denote the Sobolev-Slobodetskii space with the finite norm

$$\|u\|_{W^{s,s/2}_{p,p_0}(\Omega^T)} = \sum_{|\alpha|+2a \leq [s]} \|D^\alpha x^{\alpha} u\|_{L_{p,p_0}(\Omega^T)} + \left[ \int_0^T \left( \int_\Omega \int_{|\alpha|+2a = [s]} \frac{|D^\alpha x^{\alpha} u(x,t) - D^\alpha x^{\alpha} u(x',t)|^p}{|x-x'|^{n+p([s]-[s])}} dx dx' \right)^{p_0/p} dt \right]^{1/p_0} + \left[ \int_0^T \int_0^T \left( \int_\Omega \int_{|\alpha|+2a = [s]} \frac{|D^\alpha x^{\alpha} u(x,t) - D^\alpha x^{\alpha} u(x,t')|^p}{|t-t'|^{1+p([s/2]-[s/2])}} dx \right)^{p_0/p} dt dt' \right]^{1/p_0},$$

where $a \in \mathbb{N}_0$ and $[s]$ is the integer part of $s$.

For $s$ odd the middle term in the above norm vanishes whereas for $s$ even the two last terms vanish. We use also the notation $L_p(\Omega^T) = L_{p,p}(\Omega^T)$, $W^{s,s/2}_{p,p}(\Omega^T) = W^{s,s/2}_{p,p}(\Omega^T)$, and so on.

$B^{l,l}_{p,p_0}(\Omega)$, $l \in \mathbb{R}_+$, $p,p_0 \in [1,\infty]$ is the Besov space of functions making the following norm finite

$$\|u\|_{B^{l,l}_{p,p_0}(\Omega)} = \|u\|_{L_p(\Omega)} + \left( \sum_{i=1}^\infty \int_0^\infty \frac{\|\Delta_i^m(h,\Omega) \partial x^{k} u\|_{L_{p_0}(\Omega)}^{p_0}}{h^{1+l-k} p_0} dh \right)^{1/p_0},$$

where $k \in \mathbb{N}_0$, $m \in \mathbb{N}$, $m > l - k > 0$, $\Delta^j_i(h,\Omega) u$, $j \in \mathbb{N}$, $h \in \mathbb{R}_+$, is the finite difference of the order $j$ of the function $u(x)$ with respect to $x_i$ with $\Delta^j_i(h,\Omega) u = \Delta_i(h,\Omega) u = u(x_1, \ldots, x_i - 1, x_{i+1}, \ldots, x_n) - u(x_1, \ldots, x_n)$, $\Delta^j_i(h,\Omega) u = \Delta_i(h,\Omega) \Delta^{j-1}_i(h,\Omega) u$ and $\Delta^j_i(h,\Omega) u = 0$ for $x + jh \notin \Omega$.

From [11] it is known that the norms of the Besov space $B^{l,l}_{p,p_0}(\Omega)$ are equivalent for different $m$ and $k$ satisfying the condition $m > l - k > 0$. 

\[11\]
By $V_2(\Omega^T)$ we denote the space $L_\infty (0, T; L_2(\Omega)) \cap L_2(0, T; H^1(\Omega))$ of functions making the following norm finite

$$\|u\|_{V_2(\Omega^T)} = \operatorname{esssup}_{t\in[0, T]} \|u(t)\|_{L_2(\Omega)} + \|\nabla u\|_{L_2(\Omega^T)}.$$ 

Next $V_2^{1,0}(\Omega^T) = V_2(\Omega^T) \cap C([0, T]; L_2(\Omega)).$

By $C^{\alpha,\alpha/2}(\Omega^T)$, $\alpha \in (0, 1)$, we denote the anisotropic Hölder space of functions making the following norm finite

$$\|u\|_{C^{\alpha,\alpha/2}(\Omega^T)} = \sup_{\Omega^T} |u(x, t)| + \sup_{x',x''} \frac{|u(x', t) - u(x'', t)|}{|x' - x''|^\alpha} + \sup_{x,t,t'} \frac{|u(x, t') - u(x, t'')|}{|t' - t''|^\alpha/2}.$$ 

By $c$ we denote a generic positive constant which changes its value from formula to formula and depends at most on the imbedding constants, constants of the considered problem, and the regularity of the boundary.

By $\varphi = \varphi(\sigma_1, \ldots, \sigma_k), k \in \mathbb{N}$, we denote a generic function which is a positive increasing function of its arguments $\sigma_1, \ldots, \sigma_k$, and may change its form from formula to formula.

3.2. Auxiliary results. We need the following interpolation lemma

**Lemma 3.1.** (see [1, Ch. 4, Sect. 18]). Let $u \in W_{p,q;0}^{s/s',2}(\Omega^T), s \in \mathbb{R}^+, p, p_0 \in [1, \infty], \Omega \subset \mathbb{R}^2$. Let $\sigma \in \mathbb{R}_+ \cup \{0\}$, and

$$\kappa = \frac{2}{p} + \frac{2}{p_0} - \frac{2}{q} - \frac{2}{q_0} + \sigma + 2a \sigma < s.$$ 

Then $D_2^\sigma \partial_2^\sigma u \in W_{q,q_0;0}^{\sigma,\sigma/2}(\Omega^T), q \geq p, q_0 \geq p_0$, and there exists $\varepsilon \in (0, 1)$ such that

$$\|D_2^\sigma \partial_2^\sigma u\|_{W_{q,q_0;0}^{\sigma,\sigma/2}(\Omega^T)} \leq \varepsilon^{s-\kappa} \|u\|_{W_{p,p_0;0}^{s/s',2}(\Omega^T)} + c \varepsilon^{-\kappa} \|u\|_{L_{p,p_0}(\Omega^T)}.$$ 

As a special case of Lemma 3.1 we need

**Lemma 3.2.** (see [1, Ch. 4, Sect. 18]). Let $u \in W_{p}^s(\Omega), s \in \mathbb{R}^+, p \in [1, \infty], \Omega \subset \mathbb{R}^2$. Let $\sigma \in \mathbb{R}_+ \cup \{0\}$ and

$$\kappa = \frac{2}{p} - \frac{2}{q} + |\sigma| + \sigma < s.$$ 

Then $D_2^\sigma u \in W_{q}^\sigma(\Omega), q \geq p$, and there exists $\varepsilon \in (0, 1)$ such that

$$\|D_2^\sigma u\|_{W_q^\sigma(\Omega)} \leq \varepsilon^{s-\kappa} \|u\|_{W_q^s(\Omega)} + c \varepsilon^{-\kappa} \|u\|_{L_p(\Omega)}.$$ 

We also need the following interpolation result

**Lemma 3.3.** (see [1, Ch. 3, Sect. 15]). Assume that $u \in W_{p_2}^l(\Omega) \cap L_{p_1}(\Omega), \Omega \subset \mathbb{R}^2$ and

$$\frac{2}{p} - r = (1 - \theta) \frac{2}{p_1} + \theta \left( \frac{2}{p_2} - l \right).$$ 

Then there exists a constant $c$ such that

$$\sum_{|\alpha| = r} \|D_2^\alpha u\|_{L_p(\Omega)} \leq c \|u\|_{W_{p_2}^l(\Omega)} \|u\|_{L_{p_1}(\Omega)}^{1-\theta}.$$ 

We recall from [3] the trace and the inverse trace theorems for Sobolev-Slobodetskii spaces with a mixed norm
Lemma 3.4. (i) Let $u \in W^{r,s/2}_{p,p_0}(\Omega^T)$, $s \in \mathbb{R}^+, s > 2/p_0$, $p,p_0 \in (1,\infty)$. Then $u(x,t_0) = u(x,t)|_{t=t_0}$ for $t_0 \in [0,T]$ belongs to $B_{p,p_0}^{r-2/p_0}(\Omega)$ and
\[ ||u(\cdot,t_0)||_{B_{p,p_0}^{r-2/p_0}(\Omega)} \leq c||u||_{W^{r,s/2}_{p,p_0}(\Omega^T)}, \]
where constant $c$ does not depend on $u$.
(ii) For a given $\tilde{u} \in B_{p,p_0}^{r-2/p_0}(\Omega)$, $s \in \mathbb{R}^+, s > 2/p_0$, $p,p_0 \in (1,\infty)$, there exists a function $u \in W^{r,s/2}_{p,p_0}(\Omega^T)$ such that $u|_{t=t_0} = \tilde{u}$ for $t_0 \in [0,T]$ and
\[ ||u||_{W^{r,s/2}_{p,p_0}(\Omega^T)} \leq c||\tilde{u}||_{B_{p,p_0}^{r-2/p_0}(\Omega)}, \]
where constant $c$ does not depend on $u$.

Lemma 3.5. (see [1, Ch. 3, Sect. 10.4 and Ch. 4, Sect. 18]). Let $u \in W^{r,s/2}_{p,p_0}(\Omega^T)$, $s \in \mathbb{R}^+, p,p_0 \in (1,\infty)$, $\Omega \subset \mathbb{R}^2$. Let $\sigma \in \mathbb{R}^+$ and
\[ \kappa = \frac{2}{p} + \frac{2}{p_0} + \sigma < s. \]
Then $u \in C^{\sigma,s/2}(\Omega^T)$ and
\[ ||u||_{C^{\sigma,s/2}(\Omega^T)} \leq \epsilon^{s-\kappa}||u||_{W^{r,s/2}_{p,p_0}(\Omega^T)} + c\epsilon^{-\kappa}||u||_{L_{p,p_0}(\Omega^T)}. \]

Let us consider the problem
\begin{align*}
    u_t - Qu &= f & \text{in } \Omega^T, \\
    u &= 0 & \text{on } S^T, \\
    u|_{t=0} &= u_0 & \text{in } \Omega,
\end{align*}
(3.1)
where $\Omega \subset \mathbb{R}^2$ and
\[ Qu = \mu\Delta u + \nu\nabla(\nabla \cdot u) \]
with $\mu > 0$, $\nu > 0$. Let us notice that $Q$ replaces $Q_i$, so $\mu = \mu_i$, $\nu = \lambda_i + \mu_i$, $i = 1,2$. Hence assumption (1.8) implies that $\mu > 0$ and $\nu > 0$.

Lemma 3.6 (parabolic system in $W^{2,1}_{p,p_0}(\Omega^T)$ [12, 16, 22, 23]). (i) Assume that $f \in L_{p,p_0}(\Omega^T)$, $u_0 \in B_{p,p_0}^{2-2/p_0}(\Omega)$, $p,p_0 \in (1,\infty)$ and $S \in C^2$. If $2 - 2/p_0 - 1/p > 0$ the compatibility condition $u_0|_S = 0$ is assumed. Then there exists a unique solution to problem (3.1) such that $u \in W^{2,1}_{p,p_0}(\Omega^T)$ and
\[ ||u||_{W^{2,1}_{p,p_0}(\Omega^T)} \leq c(||f||_{L_{p,p_0}(\Omega^T)} + ||u_0||_{B_{p,p_0}^{2-2/p_0}(\Omega)}), \]
(3.2)
with constant $c$ depending on $\Omega$, $S$, $p$, $p_0$.
(ii) Assume that $f = \nabla \cdot g + b$, $g = \{g_{ij}\}$, $g,b \in L_{p,p_0}(\Omega^T)$, and $u_0 \in B_{p,p_0}^{2-2/p_0}(\Omega)$. Assume the compatibility condition
\[ u_0|_S = 0 \quad \text{if} \quad 1 - 2/p_0 - 1/p > 0. \]
Then there exists a unique solution to (3.1) such that $u \in W^{1,1/2}_{p,p_0}(\Omega^T)$ and
\[ ||u||_{W^{1,1/2}_{p,p_0}(\Omega^T)} \leq c(||g||_{L_{p,p_0}(\Omega^T)} + ||b||_{L_{p,p_0}(\Omega^T)} + ||u_0||_{B_{p,p_0}^{2-2/p_0}(\Omega)}), \]
(3.3)
with a constant $c$ depending on $\Omega$, $S$, $p$, $p_0$.

Let us consider the problem
\begin{align*}
    \alpha(x,t)\theta_{tt} - \Delta \theta &= f & \text{in } \Omega^T, \\
    \bar{n} \cdot \nabla \theta &= 0 & \text{on } S^T, \\
    \theta|_{t=0} &= \theta_0 & \text{in } \Omega.
\end{align*}
(3.4)
Lemma 3.7. Assume that \( f \in L_{p,p_0}^2(\Omega^T) \), \( \theta_0 \in B_{p,p_0}^{2-2/p_0} (\Omega) \), \( p,p_0 \in (1,\infty) \), \( S \subset C^2 \). Assume that \( \alpha \geq \alpha_0 > 0 \), \( \alpha_0 \) is a constant, \( \alpha \leq \alpha_* < \infty \), \( \alpha_* \) is constant, \( \alpha \in C^{\delta,\delta/2}(\Omega^T) \), \( \alpha \in L_{1/\mu,1/\mu}(\Omega^T) \), \( \mu \in (0,1) \). Then there exists a solution to problem (3.4) such that \( \theta \in W_{p,p_0}^{2,1}(\Omega^T) \) and the estimate holds
\[
\|\theta\|_{W_{p,p_0}^{2,1}(\Omega^T)} \leq \varphi \left( \frac{1}{\alpha_0}, \alpha_*, \|\alpha\|_{C^{\delta,\delta/2}}, \|\alpha\|_{L_{1/\mu,1/\mu}}\right) \cdot \left( \|f\|_{L_{p,p_0}(\Omega^T)} + \|\theta_0\|_{B_{p,p_0}^{2-2/p_0}(\Omega)} \right) .
\] (3.5)

4. Lower bound for temperature. The existence of the lower positive bound on temperature is very important for getting an a priori global estimate in this paper. However, we follow the arguments from the proof of Lemma 4.1 in [15], but our proof is essentially different.

Lemma 4.1. Assume that (1.2) holds, \( g > 0 \), \( \theta_0 > 0 \) and \( A = A(\varepsilon) \). Let
\[
\sigma = 1 \quad \text{and let} \quad a_1(t) = \frac{1}{2a_{1*}} \sup_{\Omega} |A(\varepsilon)|^2 ,
\] (4.1)
where \( a_{1*} \) is introduced in (1.12). Then for sufficiently regular solutions to problem (1.1)–(1.4) we have
\[
\theta(t) \geq \theta_0 \exp \left( - \int_0^t \frac{a_1(t')}{c_v} dt' \right) = \theta_* .
\] (4.2)

Let \( \sigma < 1 \) and let \( a_2(t) = \frac{1}{2a_{1*}} \lim_{\varepsilon \to \infty} ||A||_{L_{2(\varepsilon^{-1}(\varepsilon+1))}(\Omega)}^2 \). Then for a sufficiently regular solution to (1.1)–(1.4) it follows
\[
\theta(t) \geq \left[ \int_0^t \frac{a_2(t')}{c_v} dt' + \left( \frac{1}{\theta_0} \right)^{1-\sigma} \right]^{-\frac{1}{1-\sigma}} = \theta_* .
\] (4.3)

Proof. Multiplying (1.2) by \(-\theta^{-\varepsilon}\) and integrating over \( \Omega \) yields
\[
-c_v \int_{\Omega} \theta^{-\varepsilon} \partial_t \theta dx + \varv \int_{\Omega} \theta^{-\varepsilon} \Delta \theta \, dx + \int_{\Omega} (A_1 \varepsilon_1) \cdot \varepsilon_1 \theta^{-\varepsilon} \, dx \\
+ \int_{\Omega} g \theta^{-\varepsilon} \, dx + \int_{\Omega} A(\varepsilon) \cdot \varepsilon \theta^{-\varepsilon} \, dx = 0 .
\] (4.4)
Now we examine the particular terms in (4.4). The first term is equal to
\[
\frac{c_v}{\varv - (\sigma + 1)} \frac{d}{dt} \int_{\Omega} \theta \, dx .
\]
The second term equals
\[
\frac{4\varv \psi}{(\varv - 1)^2} \int_{\Omega} \left| \nabla \frac{\psi}{\theta} \right|^2 \, dx .
\]
In view of (1.12) the third term is bounded from below by
\[
a_{1*} \int_{\Omega} \frac{|\varepsilon_1|}{\theta^\varepsilon} \, dx .
\]
The fourth term is positive because \( g > 0 \).

Applying the Cauchy inequality the last term in (4.4) is bounded by
\[
\frac{a_{1*}}{2} \int_{\Omega} \frac{|\varepsilon_1|}{\theta^\varepsilon} \, dx + \frac{1}{2a_{1*}} \int_{\Omega} |A(\varepsilon)|^2 \theta^{2-\varepsilon} \, dx .
\]
In view of the above considerations (4.4) takes the form
\[
\frac{c_v}{\varrho - (\sigma + 1)} \frac{d}{dt} \int_\Omega \frac{1}{\varrho e^{-(\sigma + 1)}} dx + \frac{4\varrho \varphi}{(\varrho - 1)^2} \int_\Omega \left| \nabla \frac{1}{\varrho e^{\varepsilon}} \right|^2 dx \\
+ \frac{a_1}{2} \int_\Omega \left| \varepsilon_{t,1} \right|^2 dx + \int_\Omega \frac{g}{\varrho e} dx \leq \frac{1}{2a_1} \int_\Omega |A(\varepsilon)|^2 \varrho^{2-\varepsilon} dx,
\]

where \( \varrho \) is assumed to be large. Let us introduce the notation
\[
X = \left( \int_\Omega \frac{1}{\varrho e^{-(\sigma + 1)}} dx \right)^{\frac{1}{\varrho - (\sigma + 1)}}.
\]

Using (4.6) and assuming that \( a_1(t) = \frac{1}{2a_1} \sup_\Omega |A(\varepsilon)|^2 \) we obtain
\[
\frac{c_v}{\varrho - (\sigma + 1)} \frac{d}{dt} X^{\varrho - (\sigma + 1)} \leq a_1(t) \int_\Omega \frac{1}{\varrho e^{2}} dx.
\]

Setting \( \sigma = 1 \) we obtain from (4.7) the inequality
\[
\frac{c_v}{\varrho - (\sigma + 1)} \frac{d}{dt} X \leq a_1(t)X \quad \text{so} \quad X(t) \leq X(0) \exp \int_0^t a_1(t') \frac{c_v}{c_v} dt.
\]

Hence passing with \( \varrho \to \infty \) yields
\[
\theta(t) \geq \theta(0) \exp \left[ - \int_0^t a_1(t') \frac{c_v}{c_v} dt' \right],
\]

which yields (4.2). Let us consider the case \( \sigma < 1 \). Then (4.5) takes the form
\[
\frac{c_v}{\varrho - (\sigma + 1)} \frac{d}{dt} X^{\varrho - (\sigma + 1)} \\
\leq \frac{1}{2a_1} \| A \|_{L_2(\varrho - (\sigma + 1))} \left( \int_\Omega \left| \frac{1}{\varrho} \right|^{\varrho - (\sigma + 1)} dx \right)^{\frac{\varrho-2}{\varrho-(\sigma+1)}} \equiv a_2(\varrho, \varphi) X^{\varrho-2}.
\]

Hence
\[
\frac{d}{dt} X \leq \frac{a_2(t, \varrho)}{c_v} X^\sigma
\]

and
\[
\frac{1}{1 - \sigma} \frac{d}{dt} X^{1-\sigma} \leq \frac{a_2(t, \varrho)}{c_v}
\]

so
\[
X^{1-\sigma}(t) \leq \int_0^t \frac{a_2(t, \varrho)}{c_v} dt' + X^{1-\sigma}(0).
\]

Passing with \( \varrho \to \infty \) we get
\[
\theta(t) \geq \left[ \int_0^t \frac{a_2}{c_v} dt' + \left( \frac{1}{\varrho_0} \right)^{1-\sigma} \right]^{\frac{1}{1-\sigma}}.
\]

This implies (4.3) and concludes the proof.
5. A priori estimates. We prove estimates in this section under the existence of the lower bound on temperature proved in Lemma 4.1 such that
\[ \theta(x, t) \geq \theta_s, \quad t \leq T, \quad \theta_s > 0, \] (5.1)
where \( T \) is the time of local existence.

**Lemma 5.1.** Assume that \( u_1 \in L_2(\Omega), u_0 \in H^1(\Omega), \theta_0 \in L_{\sigma+1}(\Omega), b \in L_2(\Omega'), 0 < g \in L_1(\Omega'), t \leq T \). Then solutions to problem (1.1)-(1.4) satisfy
\[
\begin{align*}
\|u_t(t)\|_{L^2(\Omega)}^2 + \|u(t)\|_{H^1(\Omega)}^2 + \|	heta(t)\|_{L^{\sigma+1}(\Omega)}^{\sigma+1} & \leq c(t)(\|b\|_{L^2(\Omega')}^2 + \|g\|_{L_1(\Omega')} + \|u_1\|_{L^2(\Omega)}^2 + \|u_0\|_{H^1(\Omega)}^2) \\
& \quad + \|\theta_0\|_{L^{\sigma+1}(\Omega)}^{\sigma+1} = c_1(t), \quad t \leq T,
\end{align*}
\] (5.2)
where \( c(t) \) is an increasing function.

**Proof.** Multiplying (1.1) by \( u_t \) and integrating over \( \Omega \) yields
\[
\frac{1}{2} \frac{d}{dt} \|u_t\|_{L^2(\Omega)}^2 + \int_{\Omega} (A_1 \varepsilon, \varepsilon) \cdot \varepsilon_t dx + \int_{\Omega} (A_2 \varepsilon) \cdot \varepsilon_t dx + \int_{\Omega} A(\varepsilon) \theta \cdot \varepsilon_t dx = \int_{\Omega} b \cdot u_t dx.
\] (5.3)
Integrating (1.2) over \( \Omega \) implies
\[
\frac{c_v}{\sigma + 1} \frac{d}{dt} \int_{\Omega} \theta^{\sigma+1} dx = \int_{\Omega} A(\varepsilon) \theta \cdot \varepsilon_t dx + \int_{\Omega} (A_1 \varepsilon, \varepsilon) \cdot \varepsilon_t dx + \int_{\Omega} g dx.
\] (5.4)
In view of (1.9) we have
\[
- \int_{\Omega} \nabla \cdot (A_2 \varepsilon) \cdot u_t dx = - \int_{\Omega} [\mu_2 \Delta u \cdot u_t + (\lambda_2 + \mu_2) \nabla \text{div}u \cdot u_t] dx
\] (5.5)
where the boundary condition (1.3) is used. Adding (5.3) and (5.4) with applying (5.5) we get
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \left( \frac{2c_v}{\sigma + 1} \theta^{\sigma+1} + |u_t|^2 + \mu_2 |\nabla u|^2 + (\lambda_2 + \mu_2) |\text{div}u|^2 \right) dx = \int_{\Omega} b \cdot u_t dx + \int_{\Omega} g dx.
\] (5.6)
Integrating (5.6) with respect to time with applying the Gronwall inequality and the Poincaré inequality yields (5.2). This concludes the proof. \( \square \)

**Lemma 5.2.** Assume that \( u_1 \in H^1(\Omega), b \in L_{2,2}(\Omega'), \theta_0 \in L_{\sigma+2}(\Omega), g \in L_1(0, t; L_{\frac{\sigma+1}{\sigma}}(\Omega)), u_0 \in W^2_2(\Omega), \sigma > 1/2 \). Then
\[
\int_{\Omega} \theta^{\sigma+2} dx + \|\nabla \theta\|_{L^2(\Omega')}^2 \leq c(t, c_4),
\] (5.7)
where
\[
c_4 = \|u_1\|_{L_2(\Omega)} + \|b\|_{L_2(\Omega')} + \|\theta_0\|_{L_{\sigma+2}(\Omega)} + \|g\|_{L_1(0, t; L_{\frac{\sigma+1}{\sigma}}(\Omega))} + \|u_0\|_{W^2_2(\Omega)}.
\]
Applying Lemma 3.6 (ii) to problem (1.1)–(1.3) yields
\[
\|\varepsilon, t\|_{L_{p,r}(\Omega')} \leq c \|u, t\|_{W_{p,r}^{1,1/2}(\Omega')} \leq c\|\varepsilon\|_{L_{p,r}(\Omega)} + \|\theta\|_{L_{p,r}(\Omega')} + \|b\|_{L_{p,r}(\Omega')} + \|u_1\|_{B^{1-2r}_{p,r}(\Omega)} + \|u_0\|_{W^2_p(\Omega)}.
\] (5.8)
Hence, the Gronwall inequality implies
\[
\|\varepsilon, t\|_{L_{p,r}(\Omega')} \leq c(t)\|\varepsilon\|_{L_{p,r}(\Omega)} + \|u_1\|_{L_{p,r}(\Omega')} + \|u_0\|_{W^2_p(\Omega)}.
\] (5.9)
In view of (5.2) we have that \( \theta \in L_{\infty}(0, t; L_{\sigma+1}(\Omega)) \). Using that \( \sigma + 1 > 1 \) we obtain for \( p \leq \sigma + 1 \) the estimate
\[
\|\varepsilon, t\|_{L_{p+1,r}(\Omega')} \leq c(t)\|\varepsilon\|_{L_{p+1}(\Omega')} + \|u_1\|_{B^{1-2r}_{p+1,r}(\Omega)} + \|u_0\|_{W^2_{p+1}(\Omega)}.
\] (5.10)
Then for \( r = 2 \) we derive from (5.11) the inequality
\[
\|\varepsilon, t\|_{W_{2,r}^{1,1/2}(\Omega')} \leq c(t, c_1)\|\nabla \theta\|_{L_2(\Omega')} + c_3,
\] (5.12)
where
\[
c_3 = \|b\|_{L_2(\Omega')} + \|u_1\|_{L_2(\Omega)} + \|u_0\|_{W^2_2(\Omega)}.
\] (5.13)
Multiplying (1.2) by \( \theta \) and integrating the result over \( \Omega' \) we get
\[
\frac{1}{\sigma + 2} \int_{\Omega} \varepsilon^{\sigma+2} dx + \int_{\Omega'} |\nabla \theta|^2 dx dt' \leq c \int_{\Omega'} \varepsilon^{\sigma+2} dx dt' + \int_{\Omega'} \varepsilon^{\sigma+2} dx + c \int_{\Omega'} g \varepsilon dx dt' + \frac{1}{\sigma + 2} \int_{\Omega} \varepsilon^{\sigma+2} dx.
\] (5.14)
Now we estimate the integrals from the r.h.s. of (5.14). The second integral implies
\[
J \equiv \int_{\Omega'} \varepsilon^{\sigma+2} dx dt' \leq \int_{\Omega'} \varepsilon, t^2 dx dt' \leq \int_{\Omega'} \varepsilon, t^2 dx dt' + \int_{\Omega'} g \varepsilon dx dt' + \frac{1}{\sigma + 2} \int_{\Omega} \varepsilon^{\sigma+2} dx.
\]
where \( 1/\lambda_1 + 1/\lambda_2 = 1 \). Setting \( \lambda_2 = \sigma + 1 \) we have \( \lambda_1 = \frac{\sigma + 1}{\sigma} \).
Hence by (5.2) we have
\[
J_1 \leq c(c_1)\|\varepsilon, t\|_{L_{2(\sigma+1),2}(\Omega')}^2.
\]
By the interpolation
\[
\|\varepsilon, t\|_{L_{2(\sigma+1),2}(\Omega')} \leq \delta_1^{1/2}\|\varepsilon, t\|_{W_{2,2}^{1,1/2}(\Omega')} + c(1/\delta_1)\|\varepsilon, t\|_{L_{\sigma+1,2}(\Omega')} (5.15)
\] which holds for \( \sigma \) satisfying the relation
\[
\sigma > 0
\] (5.16)
and by (5.10) and (5.12), we have
\[
J \leq \delta_1\|\nabla \theta\|_{L_2(\Omega')}^2 + c(1/\delta_1, t, c_2, c_3),
\]
where \( \delta_1 \in (0, 1) \). Next we examine the first integral on the r.h.s. of (5.14),
\[
K = \int_{\Omega} \varepsilon^{\sigma+2} dx dt' \leq \sup_t \left( \int_{\Omega} \varepsilon^{2\lambda} dx \right)^{1/\lambda'} \int_0^t \varepsilon, t' \|\nabla \theta\|_{L_2(\Omega')} dx dt' = K_1K_2,
\]
where $1/\lambda + 1/\lambda' = 1$, $\lambda < \infty$ is arbitrary finite and $\lambda'$ is arbitrary close to 1. Hence, using Lemma 3.1 and (5.12), we have

$$K_2 \leq t^{1/2} \|\varepsilon, t\|_{L_{\lambda,2}(\Omega')} \leq c t^{1/2} \|\varepsilon, t\|_{W^{1,1/2,2}_2(\Omega')} \leq c t^{1/2} \|\nabla \theta\|_{L_2(\Omega')} + c_3.$$

Next,

$$K_1 \leq \sup_t \left( \int_{\Omega} \theta^{2\lambda/\mu_1} \lambda_1' \, dx \right)^{1/\lambda_1'} \sup_t \left( \int_{\Omega} \theta^{2\lambda/\mu_2} \lambda_2' \, dx \right)^{1/\lambda_2'} \equiv K_3,$$

where $\mu_1 + \mu_2 = 1$, $1/\lambda_1 + 1/\lambda_2 = 1$, $2\lambda/\mu_1 \lambda_1 = \sigma + 1$, $2\lambda/\mu_2 \lambda_2 = \sigma + 2$. Then

$$\frac{\mu_1}{\sigma + 1} + \frac{\mu_2}{\sigma + 2} = \frac{1}{2\lambda'}.$$

Continuing $\sigma + 1 + \mu_1 = \frac{(\sigma+1)(\sigma+2)}{2\lambda}$ so $\mu_1 = \frac{\sigma+1}{2\lambda} [\sigma - 2(\lambda' - 1)]$. Then $\mu_2 = 1 - \mu_1 = 1 - \frac{\sigma}{2\lambda} + (\sigma + 1) \frac{\lambda' - 1}{\lambda}$. In view of (5.2) we derive

$$K_3 \leq c(c_1) \left( \int_{\Omega} \theta^{\sigma+2} \, dx \right)^{2\mu_2/\sigma + 2} = K_4,$$

where

$$\frac{2\mu_2}{\sigma + 2} = 2 - \frac{\sigma + 1}{\lambda'}$$

and $2(\lambda' - 1)$ is as close to zero as we want. Therefore,

$$K_4 = c(c_1) \sup_t \left( \int_{\Omega} \theta^{\sigma+2} \, dx \right)^{2 - \frac{\sigma+1}{\lambda'}}$$

In view of the above estimations and bound of $K$, we have

$$K \leq c(c_1, t) \sup_t \left( \int_{\Omega} \theta^{\sigma+2} \, dx \right)^{2 - \frac{\sigma+1}{\lambda'}} \|\nabla \theta\|_{L_2(\Omega')} + c_3) \equiv K_5,$$

where since $\lambda'$ is close to 1 we get that $2 - \frac{\sigma+1}{\lambda'} < 1$ for $\sigma > 0$.

By the Young inequality, we derive

$$K_5 \leq \delta_1 \sup_t \left( \int_{\Omega} \theta^{\sigma+2} \, dx \right)^{(2 - \frac{\sigma+1}{\lambda'})\delta_1} + \frac{c}{\delta_2\delta_2} \|\nabla \theta\|_{L_2(\Omega')} + c_3)^{\delta_2} \equiv K_6,$$

where $\delta \in (0, 1)$, $1/\delta_1 + 1/\delta_2 = 1$. Choosing $\delta_1 = 1/(2 - \frac{\sigma+1}{\lambda'})$ we obtain that $\delta_2 = 1/(\frac{\sigma+1}{\lambda'} - 1)$. Then there exists constant $\alpha_1 \in (0, 1)$ such that

$$K \leq K_6 \leq \alpha_1 \sup_t \left( \int_{\Omega} \theta^{\sigma+2} \, dx + c(1/\alpha_1)(\|\nabla \theta\|_{L_2(\Omega')} + c_3) \right)^{\delta_2}.$$

Using the estimate in (5.14) we see that $\|\nabla \theta\|_{L_2(\Omega')}$ will be absorbed by the second term on the l.h.s. if $1/(\frac{\sigma+1}{\lambda'} - 1) < 2$ so $\frac{1}{2} \lambda' < 1 < \sigma$. Hence $\sigma > 1/2$.

Finally, the third integral on the r.h.s. of (5.14) is bounded by

$$\int_{\Omega} g^\delta dx dt' \leq \int_0^t \|g\|_{L^{\sigma+1}(\Omega)} \|\theta\|_{L_{\sigma+1}(\Omega)} dt' \leq c(c_1) \|g\|_{L_1(0, t; L^{\sigma+1}(\Omega))},$$

where (5.2) is used. Using the above estimates in (5.14) and assuming that $\delta_1$ and $\alpha_1$ are sufficiently small we obtain (5.7). This concludes the proof. \qed
From Lemma 5.2 we have
\[ \theta \in L_{\infty}(0, t; L_{\sigma+2}(\Omega)), \]
\[ \theta \in L_2(0, t; L_q(\Omega)), \quad q \in (1, \infty). \]  
In view of (5.17) inequality (5.8) implies
\[ \|\varepsilon, t\|_{L_{\sigma, r}(\Omega)} \leq c|c(c_4) + \|b\|_{L_{\sigma, r}(\Omega)} + \|u_1\|_{B_{\sigma,1}^{1/r}(\Omega)} + \|u_0\|_{W_{\infty, 3}^1(\Omega)} \equiv c_5, \quad r \in (1, \infty). \]  

**Lemma 5.3.** Assume that \( b \in L_{2,2}(\Omega') \), \( u_1 \in B_{2,2}^{1/2}(\Omega) \). Then
\[ \|\varepsilon, t\|_{L_{p_1, r_1}(\Omega')} \leq c(1 + \|b\|_{L_{2,2}(\Omega')} + \|u_1\|_{B_{2,2}^{1/2}(\Omega)} + \|u_0\|_{H^2(\Omega)}) \equiv c_5, \]
where \( r_1, p_1 \) are such that (5.22) holds.

**Proof.** From (5.7) and (5.11) we have
\[ \|\varepsilon, t\|_{W_{2,2}^{1/2}(\Omega')} \leq c(1 + \|b\|_{L_{2,2}(\Omega')} + \|u_1\|_{B_{2,2}^{1/2}(\Omega)} + \|u_0\|_{H^2(\Omega)}). \]
Hence (5.20) holds for finite \( p_1 \) and \( r_1 \) such that
\[ 1 \leq 2 \left( \frac{1}{p_1} + \frac{1}{r_1} \right). \]
For at least one of \( p_1, r_1 \) equal to infinity we have the strong inequality in (5.21). This concludes the proof. \( \square \)

Setting \( r_1 = 2 \) we obtain from (5.21) the estimate
\[ \|\varepsilon, t\|_{L_{p_1, 2}(\Omega')} \leq c_5, \]
where \( p \) is an arbitrary finite number.

**Lemma 5.4.** Assume that \( \theta_0 \in L_{s+\sigma+1}(\Omega), \ \theta \in L_{1}(0, t; L_{\frac{s+\sigma+1}{\sigma+1}}(\Omega)), \ s \in (1, \infty). \) Then
\[ \|\theta\|_{L_{s+\sigma+1}(\Omega)} \leq c|c(c_4, c_5) + c\|g\|_{L_1(0, t; L_{\frac{s+\sigma+1}{\sigma+1}}(\Omega))} + c\|\theta_0\|_{L_{s+\sigma+1}(\Omega)} \equiv c_6. \]

**Proof.** Multiplying (1.2) by \( \theta^s \) and integrating over \( \Omega \) we get
\[ \int \frac{c_v}{s + \sigma + 1} \frac{d}{dt} \int \theta^{s+\sigma+1} dx + \int \frac{4\kappa s}{(s+1)^2} \int |\nabla \theta^{s+\sigma+1}|^2 dx \leq c \int \theta^{1+\sigma}|\varepsilon, t|dx + c \int \theta^s|\varepsilon, t|^2 dx + \int \theta^s dx. \]  
In view of (5.23) and (5.18) we estimate the terms on the r.h.s. of (5.25). By the Hölder inequality the first term is bounded by
\[ \|\theta\|_{L_{p'}(\Omega)} \|\theta^s\|_{L_{p'_2}(\Omega)} \|\varepsilon, t\|_{L_{p'_1}(\Omega)} \leq \|\theta\|_{L_{p'_1}(\Omega)} \|\theta\|_{L_{p'_2}(\Omega)} \|\varepsilon, t\|_{L_{p'_1}(\Omega)}, \]
where \( 1/p'_1 + 1/p'_2 + 1/p'_2 = 1 \). Since \( p'_1 \) can be chosen arbitrary large, \( p'_2 \) is arbitrary close to 1. To estimate the term in view of the first term on the l.h.s. of (5.25) we choose \( p'_2 \) such that \( p'_2 s < s + 1 + \sigma \). Hence \( p'_2 < 1 + \frac{1+\sigma}{s} \). This means that \( \frac{1+\sigma}{s} > 0 \) so \( s \) can be chosen arbitrary large but finite. The second term on the r.h.s. of (5.25) is bounded by
\[ \|\theta^s\|_{L_{p'}(\Omega)} \|\varepsilon, t\|_{L_{p'_2}(\Omega)} \leq \|\theta\|_{L_{p'}(\Omega)} \|\varepsilon, t\|_{L_{p'_2}(\Omega)}, \]
where $1/p + 1/p' = 1$, $p's = s + \sigma + 1$ and the last by
\[ \|\theta^s\|_{L^r(\Omega)} \|g\|_{L^s(\Omega)} \leq \|\theta\|_{L^{s+\sigma}_t(\Omega)} \|g\|_{L^{s+\sigma}_t(\Omega)}, \]
where we used that $1/r + 1/r' = 1$, $r's = 1 + s + \sigma$, $r' = \frac{1 + s + \sigma}{s}$.

Employing the above estimates in (5.25) and omitting the second term on the l.h.s.
we get
\[ \|\theta\|_{L^{s+\sigma}_t(\Omega)} \leq \epsilon \|\theta\|_{L^{s+\sigma}_t(\Omega)} \|g\|_{L^{s+\sigma}_t(\Omega)}, \]
(5.26)

Since $\theta \geq \theta_\ast > 0$ and $s < \infty$ we obtain from (5.26) the inequality
\[ \frac{d}{dt}\|\theta\|_{L^{s+\sigma}_t(\Omega)} \leq c \|\theta\|_{L^{s+\sigma}_t(\Omega)} \|\varepsilon_t\|_{L^p_t(\Omega)} + \|\varepsilon_t\|_{L^p_t(\Omega)} + \|g\|_{L^{s+\sigma}_t(\Omega)}, \]
(5.27)

Integrating (5.27) with respect to time and using (5.18) and (5.23) we obtain (5.24).
This completes the proof. \[ \square \]

**Lemma 5.5.** Assume that $b \in L_4(\Omega^t)$, $u_1 \in B^{3/2}_{4,t}(\Omega)$, $u_0 \in W^2_4(\Omega)$,
$g \in L_2(0,t; L_4(\Omega))$, $\theta_0 \in L_4(\Omega)$, $\theta \geq \theta_\ast > 0$. Then
\[ \|\varepsilon, t\|_{L^2(0,t; L_\infty(\Omega))} \leq c_8, \]
where $c_8$ depends on all norms from the assumption.

**Proof.** In view of (5.24) inequality (5.9) implies
\[ \|\varepsilon, t\|_{L^{s+\sigma}_t(\Omega)} \leq c \|b\|_{L^{s+\sigma}_t(\Omega)} + \|u_1\|_{L^{s+\sigma}_t(\Omega)} + \|\theta_0\|_{L^{s+\sigma}_t(\Omega)} \equiv c_7, \]
(5.29)

where $\sigma_1, \sigma_2$ are arbitrary finite numbers.

Multiplying (1.2) by $\theta, t$ and integrating over $\Omega$ yields
\[ c_0 \int \theta^2 \theta_t^2 + \int \nabla \theta_t^2 \leq \int (\theta |\varepsilon_t|^2 + \|g\| |\theta_t|_t) dx \]
(5.30)

In view of (5.24), (5.29) and that $\theta \geq \theta_\ast > 0$ we obtain from (5.30) the estimate
\[ \int \theta |\varepsilon_t|^2 dx + \int \nabla \theta_t^2 dx \leq c(c_5, c_7) + c \int \nabla \theta_0^2 dx \]
(5.31)

Having (5.31) we can consider the following elliptic problem which follows from
(1.2), (1.3)
\[ -\nabla \Delta u = -c_0 \theta^2 \theta_t + \theta A \cdot \varepsilon_t + (A_1 \varepsilon_t) \cdot \varepsilon_t + g \quad \text{in} \quad \Omega, \]
\[ \tilde{n} \cdot \nabla \theta = 0. \]
(5.32)

By (5.24), (5.29) and (5.30) we obtain for (5.32) the estimate
\[ \|\theta\|_{L_2(0,t; L^2(\Omega))} \leq c(c_0, c_7) + c \|g\|_{L_2(\Omega^t)} + c(c_0) \|\nabla \theta_0\|_{L_2(\Omega)} \equiv c_8 \]
(5.33)

where $\delta < 2$ but arbitrary close to 2 and we used that $\varepsilon_t \in L_4(\Omega^t)$. Then (5.11) implies the estimate
\[ \|\varepsilon_t\|_{L^{p'}_t(\Omega^t)} \leq c(c_8 + \|b\|_{L^{p'}(\Omega)} + \|u_1\|_{L^{p'}(\Omega)} + \|u_0\|_{L^{p'}(\Omega) + c_1}), \]
(5.34)

where $p < \infty$ and $r = 2$. From (5.34) we have
\[ \|\varepsilon, t\|_{L^2(0,t; L_\infty(\Omega))} \leq c \|\varepsilon, t\|_{W^{1,1/2}_p(\Omega^t)} \]
(5.35)
for $p > 2$. To get the above estimate we only need that $\theta, \varepsilon, t \in L_{s}(\Omega^{t})$, so (5.24) and (5.29) imply the restrictions: $b \in L_{2}(\Omega^{t}), u_{1} \in B_{s}(\Omega^{t}), u_{0} \in W_{2}^{s}(\Omega), g \in L_{2}(0; t; L_{4}(\Omega)), \theta_{0} \in L_{4}(\Omega)$. This concludes the proof.

**Lemma 5.6.** Assume that $g \in L_{1}(0, T; L_{\infty}(\Omega)), \varepsilon, t \in L_{2}(0, T; L_{\infty}(\Omega)), \theta_{0} \in L_{\infty}(\Omega), \theta \geq \theta_{*} > 0, \sigma \in (1/2, 1)$. Then

$$
\|\theta\|_{L_{\infty}(\Omega)} \leq c((1/\theta_{*})|\Omega|[c_{s} + c_{3}^{2} + \|g\|_{L_{1}(0, T; L_{\infty}(\Omega))}]^{1/\sigma} + \|\theta_{0}\|_{L_{\infty}(\Omega)} \equiv c_{9}.
$$

**Proof.** Multiplying (1.2) by $\theta^{r}$, $r > 1$, and integrating over $\Omega$ gives

$$
\frac{c_{v}}{r + \sigma + 1} \frac{d}{dt}\|\theta\|_{L_{r+\sigma+1}(\Omega)} + \frac{4\sigma r}{(r + 1)^{2}} \int_{\Omega} |\nabla \theta^{r+\sigma}|^{2} dx \leq c \left[ \int_{\Omega} \theta^{1+r}\varepsilon_{t} dx + \int_{\Omega} \theta^{r}|\varepsilon_{t}|^{2} dx + \int_{\Omega} g^{r} \theta dx \right].
$$

From (5.37) we have

$$
\frac{c_{v}}{r + \sigma + 1} \frac{d}{dt}\|\theta\|_{L_{r+\sigma+1}(\Omega)} \leq c(1/\theta_{*})|\varepsilon_{t}|\|L_{r+\sigma+1}(\Omega) + \|g\|_{L_{\infty}(\Omega)}\|\theta\|_{L_{r+\sigma+1}(\Omega)}.
$$

Since $\theta \geq \theta_{*} > 0$, and $\sigma > 1/2$ we derive from (5.38) the inequality

$$
\frac{d}{dt}\|\theta\|_{L_{r+\sigma+1}(\Omega)} \leq c(1/\theta_{*})|\varepsilon_{t}|\|L_{r+\sigma+1}(\Omega) + \|g\|_{L_{\infty}(\Omega)}\|\theta\|_{L_{r+\sigma+1}(\Omega)}.
$$

Using

$$
\|\theta\|_{L_{r+\sigma+1}(\Omega)} \leq |\Omega|^{1/r + \sigma} \|\varepsilon_{t}\||\theta\|_{L_{r+\sigma+1}(\Omega)}
$$

in (5.39) yields

$$
\frac{d}{dt}\|\theta\|_{L_{r+\sigma+1}(\Omega)} \leq c(1/\theta_{*})|\Omega|^{1/r + \sigma} \|\varepsilon_{t}\|_{L_{\infty}(\Omega)} + \|g\|_{L_{\infty}(\Omega)}\|\theta\|_{L_{r+\sigma+1}(\Omega)}.
$$

Integrating (5.40) with respect to time, applying the Young inequality and passing with $r \rightarrow \infty$ we derive (5.36). This concludes the proof.

To prove the Hölder continuity of temperature we follow the method presented in [13, Ch. 2, Sect. 7]. For this purpose we recall the space $B_{2}(\Omega^{T}, M, \gamma, r, \delta, \varkappa), \Omega^{T} = \Omega \times (0, T), \Omega \subset \mathbb{R}^{n}$ and $M, \gamma, r, \delta, \varkappa$ are positive constants.

**Definition 5.7.** We say that $u \in B_{2}(\Omega^{T}, M, \gamma, r, \delta, \varkappa)$ if and only if

1. $u \in V_{2}^{1,0}(\Omega^{T})$,
2. $\sup_{\Omega^{T}}|u| \leq M$
3. the function $w(x, t) = \mp u(x, t)$ satisfies the inequalities

$$
\max_{t_{0} \leq t \leq t_{0} + \tau} \|\omega - k\|_{L_{2}(B_{\gamma}(x_{0}))} \leq \|\omega - k\|_{L_{2}(B_{\gamma}(x_{0}))} + \gamma[(\sigma_{1}\gamma)^{-2} + (\sigma_{2}\gamma)^{-1}]\|\omega - k\|_{L_{2}(Q(\gamma, \tau))} + \mu^{(1+\gamma)}\|k, \gamma, \tau\|
$$

and

$$
\|\omega - k\|_{L_{2}(Q(\gamma, \sigma_{1}\gamma, \tau - \sigma_{2}\gamma))} \leq \gamma[(\sigma_{1}\gamma)^{-2} + (\sigma_{2}\gamma)^{-1}]\|\omega - k\|_{L_{2}(Q(\gamma, \tau))} + \mu^{(1+\gamma)}\|k, \gamma, \tau\|.
$$
Here the following notation is used:

\[(\omega - k)_+ = \max\{\omega - k, 0\}\] - the truncation of \(\omega - k, k > 0\),

\[B_\theta(x_0) = \{x \in \Omega : |x - x_0| < \theta\},\]

\[Q(\varrho, \tau) = B_\theta(x_0) \times (t_0, t_0 + \tau),\]

and \(\varrho, \tau\) are arbitrary positive numbers, \(\sigma_1, \sigma_2 \in (0, 1)\) and \(k\) is a positive number such that

\[\text{esssup}_{Q(\varrho, \tau)} \omega(x, t) - k < \delta.\]

Moreover, \(V_2^{1,0}(\Omega_T)\) is defined in Section 3,

\[\mu(k, \varrho, \tau) = \int_{t_0}^{t_0 + \tau} \text{meas}^{r/q} A_{k, \varrho}(t) dt,\]

where \(A_{k, \varrho}(t) = \{x \in B_\theta(x_0) : \omega(x, t) > k\}\) and positive numbers \(q, r\) are linked by the relation \(\frac{1}{r} + \frac{n}{2r} = \frac{n}{2}\).

**Lemma 5.8.** Assume \(0 < \theta_* \leq \theta\), where \(\theta_*\) is defined by (4.1) and (4.2), respectively. Let \(M \equiv \|\theta\|_{L_\infty(\Omega_T)} \leq c_9\) (see (5.36)). Let \(\sup_{\Omega_1} \theta_0(x) < k\) and \(M - k < \delta\) with some \(\delta > 0\). Let \(\varepsilon_1 \in L_{2,1}(\Omega_T)\), \(g \in L_{1,1}(\Omega_T)\), \(\lambda = \frac{1}{1 - \kappa(1 + \kappa)}\), \(\frac{1}{2} + \frac{1}{q} = \frac{n}{2}\), \(\kappa > 0\). Then

\[\theta \in B_2(\Omega_T, M, \gamma, r, \delta, \kappa).\]  

**Proof.** Conditions (1) and (2) of Definition 5.7 are satisfied by (5.28) and Lemma 5.6. Now we satisfy condition (3). Since \(\theta \geq \theta_* > 0\) we can express (1.2) in the form

\[c_v \frac{d}{dt} \int_\Omega (\theta - k)_+ \zeta^2 dx = (\frac{C_3 + C_4}{2} + C_5) \int_\Omega (\theta - k)_+ \zeta^2 dx - c_v \int_\Omega (\theta - k)_+ \zeta \partial_t \zeta dx\]

Multiplying (5.42) by \((\theta - k)_+ \zeta^2(x, t)\) and integrating over \(\Omega\) gives

\[\int_\Omega (\theta - k)_+ \zeta^2 dx + \kappa \int_\Omega \nabla \theta \nabla \left( \frac{1}{\theta_*^2} (\theta - k)_+ \zeta^2 \right) dx\]

\[= \int_\Omega \frac{1}{\theta_*^2} [\theta (\zeta_t) + (A_1 \zeta_t) \cdot \zeta + g] (\theta - k)_+ \zeta^2 dx.\]  

The first term in (5.43) equals

\[\frac{c_v}{2} \int_\Omega (\theta - k)_+ \zeta^2 dx - c_v \int_\Omega (\theta - k)_+ \zeta \partial_t \zeta dx\]

and the second is transformed into

\[\kappa \int_\Omega \left| \nabla (\theta - k)_+ \right|^2 \left( \frac{1}{\theta_*^2} \right) dx - \kappa \sigma \int_\Omega \left| \nabla \theta \right|^2 (\theta - k)_+ \zeta^2 dx\]

\[+ 2 \kappa \int_\Omega \frac{1}{\theta_*^2} \nabla \theta (\theta - k)_+ \zeta \nabla \zeta dx.\]

The r.h.s. of (5.43) can be expressed in the short form

\[\int_\Omega G(\theta - k)_+ \zeta^2 dx.\]
Therefore, (5.43) takes the form
\[\frac{c_v}{2} \frac{d}{dt} \int_\Omega (\theta - k)_+^2 \zeta^2 dx + \kappa \int_\Omega |\nabla (\theta - k)_+^2 \frac{1}{\theta^\sigma} \zeta^2 dx \]
\[= c_v \int_\Omega (\theta - k)_+^2 \zeta^2 dx + \kappa \sigma \int_\Omega \frac{1}{\theta^\sigma + 1} |\nabla (\theta - k)_+^2 (\theta - k)_+ \zeta^2 dx \]
\[- 2\kappa \int_\Omega \frac{1}{\theta^\sigma} \nabla (\theta - k)_+ \theta \zeta dx + G(\theta - k)_+ \zeta^2 dx. \]
(5.44)

Since \(\theta_* \leq \theta \leq M\) and \(k\) satisfies \(M - k \leq \delta\), where \(\delta\) will be chosen as small as we wish, the second term on the r.h.s. of (5.44) is bounded by

\[\frac{\kappa \sigma \delta}{\theta_*} \int_\Omega \frac{1}{\theta^\sigma} |\nabla (\theta - k)_+^2 \zeta^2 dx \]

By the Hölder and Young inequalities the third term on the r.h.s. of (5.44) is bounded by

\[\frac{\kappa}{2} \delta_1 \int_\Omega \frac{1}{\theta^\sigma} |\nabla (\theta - k)_+^2 \zeta^2 dx + \frac{2\kappa}{\delta_1} \int_\Omega \frac{1}{\theta^\sigma} (\theta - k)_+^2 |\nabla \zeta|^2 dx \]

Setting \(\delta = \frac{\theta_0}{4\sigma}, \delta_1 = \frac{1}{2}\) we obtain from (5.44) the inequality

\[\frac{c_v}{2} \frac{d}{dt} \int_\Omega (\theta - k)_+^2 \zeta^2 dx + \frac{\kappa}{2M^\sigma} \int_\Omega |\nabla (\theta - k)_+^2 \zeta^2 dx \]
\[\leq c_v \int_\Omega (\theta - k)_+^2 \zeta^2 dx + \frac{4\kappa}{\sigma} \int_\Omega (\theta - k)_+^2 |\nabla \zeta|^2 dx + \int_\Omega |G(\theta - k)_+ \zeta^2 dx. \]
(5.45)

Integrating (5.45) with respect to time, using that \(\theta_0 < k\), setting \(c_* = \min \left\{ \frac{c_v}{2M^\sigma}, \frac{\kappa}{\sigma} \right\}\) and employing properties of function \(\zeta(x, t)\) we obtain

\[\|(\theta - k)_+(t)\|_{L_2(B_{\rho - \sigma_1\rho}(x_0))} + \|\nabla (\theta - k)_+\|_{L_2(Q(\rho - \sigma_1\rho, \tau - \sigma_2\rho))} \]
\[\leq \frac{1}{c_*} \left[ c_v (\sigma_2 \tau)^{-1} + \frac{4\kappa}{\sigma} (\sigma_1 \tau)^{-2} \right] \|(\theta - k)_+\|_{L_2(Q(\rho, \tau))} \]
\[+ \frac{1}{c_*} \int_{Q(\rho, \tau)} |G(\theta - k)_+ \zeta^2 dxdt. \]
(5.46)

The last term is bounded by

\[\delta \left( \int_{t_0}^{t_0 + \tau} \int_\Omega |G|^\lambda_1 dx dt \right)^{1/\lambda_1} \left( \int_{t_0}^{t_0 + \tau} \text{meas} A_{k, \theta}(t) dt \right)^{1/\lambda_2} \equiv I. \]

Setting \(1/\lambda_2 = \frac{2}{r}(1 + \sigma)\) with \(\frac{2}{r} + \frac{2}{q} = 1, \sigma > 0\) we obtain \(\frac{1}{\lambda_1} = 1 - 1/\lambda_2 = 1 - \frac{2}{r}(1 + \sigma)\) so \(\lambda_1 = \frac{1}{1 - \frac{2}{r}(1 + \sigma)}\). Since \(G = \frac{1}{\rho^r} \left[ \theta^a (A_{\rho, \tau}) + (A_1 \rho, \tau) \cdot \rho + \rho \right]\) we have

\[\|G\|_{L_{\lambda_1}(\Omega \times (t_0, t_0 + \tau))} \leq c(\theta_*, M) \|\rho\|_{L_{\lambda_1}(\Omega \times (t_0, t_0 + \tau))} \]
\[+ \|\rho\|_{C^{1,1}}^2 L_{\lambda_1}(\Omega \times (t_0, t_0 + \tau)) + \|\rho\|_{L_{\lambda_1}(\Omega \times (t_0, t_0 + \tau))}. \]

In view of (5.29) we have, for \(\sigma\) so small that \(1 - \frac{2}{r}(1 + \sigma) > 0\), the bound

\[I \leq c \rho^{\frac{2}{r}(1 + \sigma)}(\rho, \tau). \]

Hence, we proved the second inequality in part (3) of Definition 5.7. The first inequality in (3) follows in the case, where \(\zeta = \zeta(x), \text{supp}\zeta \subset B_\rho(x_0)\) and \(\zeta(x) = 1\) for \(x \in B_{\rho - \sigma_1\rho}(x_0)\). This concludes the proof. \(\square\)
Corollary 5.9. By virtue of the imbedding (see Theorem 7.1 from [13, Ch. 2])
\[ \mathcal{B}_2(\Omega^T, M, \gamma, r, \delta, k) \subset C^{\alpha,\alpha/2}(\Omega^T), \quad \alpha \in (0,1) \]
it follows from (5.41) that
\[ \theta \in C^{\alpha,\alpha/2}(\Omega^T), \quad (5.47) \]
where \( \alpha \) depends on \( M, \gamma, r, \delta, x \).

Corollary 5.10. In view of (5.24) and (5.29) we have that \( \theta, \varepsilon \) belong to \( L_{p,r}(\Omega^T) \), \( p, r \in (1, \infty) \). Using this and the Hölder continuity of \( \theta \) we get for solutions to problem (1.2), (1.3), (1.3) the estimate
\[ \|\theta\|_{W^{2,1}_{q,q_0}(\Omega^T)} \leq \varphi(c_4, c_6, c_0), \quad (5.48) \]
where \( q, q_0 \in (1, \infty) \). Hence
\[ \nabla \theta \in L_{r,r_0}(\Omega^T), \quad (5.49) \]
where
\[ \frac{2}{q} + \frac{2}{q_0} - \frac{2}{r} - \frac{2}{r_0} \leq 1 \quad (5.50) \]
For \( q = q_0 \) and \( r = r_0 \) condition (5.50) implies
\[ \frac{4}{q} - \frac{4}{r} \leq 1. \quad (5.51) \]
Since \( q \) can be chosen as an arbitrary number from \( (1, \infty) \) we can say the same about \( r \).

Similarly, for solutions to problem (1.1), (1.3), (1.4) we have
\[ \|u\|_{W^{2,1}_{p,p_0}(\Omega^T)} \leq \varphi(c_0, c_8) \quad (5.52) \]
where \( p, p_0 \in (1, \infty) \).

We are not interested to increase regularity of solutions to problem (1.1)–(1.6) as much as possible. We want to have only such regularity that the existence of local solutions holds and that the local solution might be extended in time to get the global existence.

6. Local existence. To prove local existence of solutions to problem (1.1)–(1.4) we use the following Banach successive approximations method:
\[ u_{(n+1),tt} - \nabla \cdot (A_1 \varepsilon(u_{(n+1),t})) = \nabla \cdot (A_2 \varepsilon(u_n) + A\varepsilon(u_n) + b) \text{ in } \Omega^T, \quad (6.1) \]
\[ c_0 A\varepsilon(u_{(n+1),t}) - \kappa A\varepsilon(u_{n+1}) = \varepsilon(u_n) A\varepsilon(u_n, t) + (A_1 \varepsilon(u_{(n),t})) \cdot \varepsilon(u_{(n),t}) + g \text{ in } \Omega^T, \quad (6.2) \]
\[ u_{(n+1)}|_{t=0} = u_0, \quad u_{(n+1),t}|_{t=0} = u_1, \quad \theta_{(n+1)}|_{t=0} = \theta_0 \text{ in } \Omega, \quad (6.3) \]
where \( u_n, \theta_{(n)} \) are treated as given.

Moreover, the zero approximations \( (u_{(0)}, \theta_{(0)}) \) are constructed by an extension of the initial data in such a way that
\[ u_{(0)}|_{t=0} = u_0, \quad u_{(0),t}|_{t=0} = u_1, \quad \theta_{(0)}|_{t=0} = \theta_0 \text{ in } \Omega, \quad (6.5) \]
and
\[ u(0) = 0, \quad \bar{n} \cdot \nabla \theta(0) = 0 \text{ on } \partial T. \quad (6.6) \]
First we formulate a result on uniform boundedness of the sequence \( \{u_{(n)}, \theta_{(n)}\} \).
Lemma 6.1. Let $X_0(t) = \|u(t)\|_{W^{2,1}_{p,p_0}(\Omega)} + \|\theta(t)\|_{W^{2,1}_{q,q_0}(\Omega')}$, where $u(0), \theta(0)$ are introduced by (6.5), be finite. Let $D(t) = \|u(t)\|_{W^{2,1}_{p,p_0}(\Omega)} + \|\theta(t)\|_{W^{2,1}_{q,q_0}(\Omega')}$. Assume that there exists a constant $\bar{A}$ and time $T$ such that $X_0(t) \leq \bar{A}, \varphi_1(t^n \bar{A}, D(t)) \leq \bar{A}, \alpha > 0$ and $\varphi_1(t^n, \bar{A}, D(t))$ follows from the proof of an uniform bound for the considered sequence, and $\alpha^{\tau} A \leq \theta_\alpha, \sigma > 0, \theta_\alpha = \min_{\Omega} \theta_0$. Then

\[ X_n(t) = \|u(n,t)\|_{W^{2,1}_{p,p_0}(\Omega)} + \|\theta(n,t)\|_{W^{2,1}_{q,q_0}(\Omega')} \leq \bar{A}, \text{ for any } n \in \mathbb{N}. \quad (6.7) \]

To show convergence of the sequence $\{u(n), \theta(n)\}$ we introduce the differences

\[ U_n(t) = u(n,t) - u(n-1,t), \quad \theta_n(t) = \theta(n,t) - \theta(n-1,t), \quad (6.8) \]

$n \in \mathbb{N}$, which are solutions to the problems

\[ U_{n+1,t,t} - \nabla \cdot (A_1 \varepsilon(U_{n+1,t})) = \nabla \cdot (A_2 \varepsilon(U_n)) \]

\[ + \nabla \cdot (A(\theta(n) - \theta(n-1))) \quad \text{in } \Omega^T, \]

\[ U_{n+1} = 0 \quad \text{on } S^T, \]

\[ U_{n+1}|_{t=0} = 0, \quad U_{n+1,t}|_{t=0} = 0 \quad \text{in } \Omega, \]

and

\[ c_v \theta^\tau_{(n)} \theta_{n+1} - \varkappa \Delta \theta_{n+1} = -c_v(\theta^\tau_{(n)} - \theta^\tau_{(n-1)})\theta_{(n),t} \]

\[ + (\theta(n) - \theta(n-1))\varepsilon(u(n),t) + \theta_{(n-1)}\varepsilon(U_{n,t}) \]

\[ + A_1 \varepsilon(U_{n,t}) \cdot \varepsilon(u(n),t) + A_1 \varepsilon(u(n-1),t) \cdot \varepsilon(U_{n,t}) \quad \text{in } \Omega^T, \]

\[ \bar{n} \cdot \nabla \theta_{n+1} = 0 \quad \text{on } S^T, \]

\[ \theta_{n+1}|_{t=0} = 0 \quad \text{in } \Omega. \]

Let

\[ Y_n(t) = \|U_n,t\|_{W^{2,1}_{p',p_0'}(\Omega')} + \|\theta_n\|_{W^{2,1}_{q',q_0'}(\Omega')} \quad (6.11) \]

Lemma 6.2. Let the assumptions of Lemma 6.1 hold. Then there exists a positive constant $d$ depending on $\bar{A}$ such that

\[ Y_{n+1}(t) \leq dt^\alpha Y_n(t), \quad (6.12) \]

$\alpha > 0$.

Theorem 6.3 (local existence). Let the assumptions of Lemma 6.1 hold. Then for a sufficiently small time $T$ there exists a solution to problem (1.1)–(1.6) such that there exists a constant $\bar{A}$ depending on $D$ and $T$ and

\[ X(t) = \|u,t\|_{W^{2,1}_{p,p_0}(\Omega')} + \|\theta\|_{W^{2,1}_{q,q_0}(\Omega')} \leq \bar{A}, \quad \text{where } t \leq T. \]

Remark 6.4. Applying Lemmas 3.6, 3.7 and the properties of the anisotropic Sobolev spaces the proofs of Lemmas 6.1, 6.2 is standard.
7. Proof of the Main Theorem. To prove global existence of solutions to problem (1.1)–(1.5) we need the existence of local solutions showed in Section 6 and the global a priori estimates derived in Section 5. Then global existence is proved step by step in time. In Section 6 we proved local existence of such solutions that
\[ u, t \in W^{2,1}_{p, p_0} (\Omega^T), \quad \theta \in W^{2,1}_{q, q_0} (\Omega^T), \quad \frac{1}{p} + \frac{1}{p_0} < 1, \quad \frac{1}{q} + \frac{1}{q_0} < 1, \quad q_0 > 2. \] (7.1)

To prove (7.1) we need the following regularity of data
\[ b \in L^{p, p_0} (\Omega^T), \quad u_0 \in W^2_p (\Omega), \quad u_1 \in B^{2-2/p_0}_{p, p_0} (\Omega), \quad \theta_0 \in B^{2-2/q_0}_{q, q_0} (\Omega), \quad g \in L^{q, q_0} (\Omega^T). \] (7.2)

To show global estimates in Section 5 we need (see Lemmas 5.5, 5.6, 5.8, Corollary 5.9)
\[ g \in L^1 (0, T; L^\infty (\Omega)) \cap L^2 (0, T; L^4 (\Omega)), \quad b \in L^4 (\Omega^T), \quad u_1 \in B^{3/2}_{4, 4} (\Omega), \quad u_0 \in W^2_4 (\Omega), \quad \theta_0 \in C^\alpha (\Omega), \quad \alpha > 0, \quad \theta \geq \theta_* > 0. \] (7.3)

In view of the imbedding
\[ \| \theta_0 \|_{C^\alpha (\Omega)} \leq c \| \theta_0 \|_{B^{2-2/q_0}_{q, q_0} (\Omega)}, \quad \frac{2}{q} + \frac{2}{q_0} + \alpha < 1, \]
we see that (7.2) and (7.3) are compatible if
\[ g \in L^{q, q_0} (\Omega^T) \cap L^1 (0, T; L^\infty (\Omega)). \] (7.4)

This concludes the proof.

Uniqueness can be proved in the standard way.

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