Index theorem for the flat Andreev bound states at a dirty surface of a nodal superconductor

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Abstract. We discuss the stability of flat-band Andreev bound states appearing at a surface of a nodal unconventional superconductor. In the clean limit, the existence of the surface bound states is topologically characterized by a momentum-dependent topological invariant: one-dimensional winding number defined in the restricted Brillouin zone. Thus, such topological invariant is ill-defined in the presence of potential disorder which is inevitable in experiments. By paying attention to chiral symmetry of the Hamiltonian, we provide an alternative topological index $N_{ZES}$ that predicts the number of Andreev bound states at a dirty surface of an unconventional superconductor. Moreover, we demonstrate that the zero-bias differential conductance in a normal metal/unconventional superconductor junction is quantized at $(4e^2/h)|N_{ZES}|$ in the limit of strong impurity scattering in the normal metal.

1. Introduction
Exotic properties of unconventional superconductors have sparked a significant progress in physics of superconductivity. The essence of unconventional superconductivity is the internal phase of its pair potential. The sign changing of a pair potential on the Fermi surface causes the anomalous interference effect of a quasi-particle at the surface and results in the formation of the mid-gap Andreev bound states (ABSs) there [1, 2, 3, 4]. In a fully gapped unconventional superconductor, such as chiral and helical $p$-wave, the ABSs has the linear dispersion. On the other hand, a nodal unconventional superconductor, such as the $p_x$-wave, $d_{xy}$-wave and $d_{x^2-y^2}$-wave superconductor, hosts the dispersionless (flat) Andreev bound states. In the later case, the ABSs with the high degree of degeneracy at the Fermi level (zero energy) drastically affect the low-energy transport in the superconducting junctions. Actually, it has been shown that the flat ABSs causes the zero-bias anomaly in the differential conductance of a normal metal/superconductor junction [3, 4, 5, 6, 7] and the fractional Josephson effect in a superconductor/insulator/superconductor junction [8, 9, 10, 11, 12, 13].

Recently, the emergence of the flat ABSs has also been explained in terms of topology. Due to the presence of nodes in the superconducting gap, the well-known 10-fold topological classification [14] does not cover the nodal superconductors. Even though, a prescription called dimensional reduction enables us to topologically characterize the flat ABSs of a nodal superconductor [15, 16]. In a $d$-dimensional superconductor, it is possible to choose a one-dimensional Brillouin zone by fixing a $(d-1)$-dimensional wave number at a certain point (say $k$). When the momenta $k$ is away from the nodal points, it is possible to define the
one-dimensional winding number $w(k)$ for each restricted Brillouin zone [15]. From the bulk-boundary correspondence, the number of the ABSSs should be $|w(k)|$ for each $k$. Therefore the flat ABSSs are expected at a clear surface parallel to $k$.

In experiments, however, the potential disorder is inevitable in the vicinity of a surface or a junction interface of a superconductor. The one-dimensional Brillouin zone is ill-defined in the presence of disorder which breaks translational symmetry. Therefore, the winding number $w(k)$ no longer predict the number of ABSSs at a dirty surface [17]. In other words, the potential disorder may lift the high degeneracy at the flat ABSSs at a surface [18] and may wash out the unique transport properties to the flat ABSSs. Such a situation requires a theoretical tool that measures the stability of flat ABSSs in the presence of potential disorder. In this paper, we address this considerable issue.

We first study the stability of flat ABSSs at the dirty surface of a nodal superconductor in Sec. 2. We consider three types of pairing symmetries such as $p_x$-wave, $d_{xy}$-wave, and $f$-wave symmetry. By taking into account chiral symmetry of the Bogoliubov-de Gennes Hamiltonian [19, 20, 21, 22, 23], we prove that the index $N_{ZES} = \sum_k w(k)$ predicts the number of ABSSs at a dirty surface of a nodal superconductor. The index $N_{ZES}$ is a kind of Atiyah-Singer index, which relates a topological invariant and an invariant in terms of solutions of a differential equation. We also study the zero-bias differential conductance (ZBC) of a dirty normal-metal/superconductor (NS) junction in Sec. 3. As a consequence, we find that the ZBC decreases to the quantized value of $(4e^2/h)|N_{ZES}|$ with the decrease of the normal conductance.

2. Stability of flat Andreev bound states

2.1. Flat Andreev bound states at the clean surface

We start our discussion with a brief summary of the topological property of a nodal superconductor in the clean limit. Let us consider the three types of unconventional superconductors with $p_x$-wave, $d_{xy}$-wave, and $f$-wave pairing symmetry. We describe the present superconductor by the $2 \times 2$ Bogoliubov-de Gennes Hamiltonian

\[
\hat{H}_0(k) = \begin{bmatrix}
\xi(k) & \Delta(k) \\
\Delta(k) & -\xi(k)
\end{bmatrix}, \quad \Delta(k) = \begin{cases}
\Delta_0 k_x / k_F & \text{for } p_x\text{-wave,} \\
\Delta_0 k_x k_y / k_F^2 & \text{for } d_{xy}\text{-wave,} \\
\Delta_0 k_x (k_y^2 - 2k_y^2) / k_F^3 & \text{for } f\text{-wave,}
\end{cases}
\]

where $\xi(k) = (\hbar^2 k^2 / 2m) - \mu_F$, $m$ denotes the effective mass of an electron, $\mu_F$ is the chemical potential, $\Delta_0$ is the amplitude of the pair potential at zero temperature, and $k_F = \sqrt{2m\mu_F / \hbar}$ represents the Fermi wave number. The Hamiltonian satisfies

\[
\{\hat{\Gamma}, \hat{H}_0(k)\} = 0, \quad \hat{\Gamma} = \hat{\tau}_2,
\]

which represents chiral symmetry of the Hamiltonian. Pauli matrices in Nambu space is denoted by $\tau_i$ for $i = 1-3$. Since the pair potentials in Eq. (1) have nodes on the Fermi surface, it is impossible to topologically characterize such the superconductors in terms of wave functions of the whole Brillouin zone. Alternatively, we define a winding number in a restricted one-dimensional Brillouin zone as

\[
w(k_\|) = \frac{i}{4\pi} \int dk_\perp \text{Tr}[\hat{\Gamma} \hat{H}_0^{-1}(k) \partial_{k_\perp} \hat{H}_0(k)],
\]

where $k_\|$ ($k_\perp$) is momenta parallel (perpendicular) to the surface we consider. The winding number $w(k_\|$) is ill-defined when the integration path along $k_\perp$ intersects the nodal points. Therefore, we have to choose $k_\|$ so that $k_\perp$ can keep away from the nodal points. The winding
number \( w(k_y) \) can be nonzero in the finite region of \( k_y \). In what follows, we consider the surface perpendicular to the \( x \) direction. Namely, we chose \( k_x = k_x \) and \( k_y = k_y \), respectively. According to the bulk-boundary correspondence, the total number of topologically protected flat ABSs at the clean surface is calculated by

\[
N_{\text{clean}} = \sum_{k_y} |w(k_y)|, \tag{4}
\]

where \( \sum_{k_y}' \) denotes a summation over \( k_y \) excluding the nodal points.

### 2.2. Flat Andreev bound states at the dirty surface

Next, we study the degree of degeneracy in the flat ABSs at the dirty surface of an unconventional superconductor. The random impurity potentials in the bulk region strongly suppress the unconventional superconducting pair potential. Thus, we assume the potential disorder exists only near the surface. The BdG Hamiltonian in real space \( H_0(r) \) is obtained by replacing the momentum \( k \) by \( -i\nabla_r \) in Eq. (1). The non-magnetic random impurity potential in the vicinity of the surface is represented by \( \tilde{V}_{\text{imp}}(r) = v(r)\tilde{r}_3 \), where the random potential \( v(r) \) disappears rapidly with the decrease of distance from the surface. The total Hamiltonian is given by \( \tilde{H}(r) = H_0(r) + \tilde{V}_{\text{imp}}(r) \). The momentum \( k_y \) is no longer a good quantum number because the impurity potential breaks translational symmetry. Therefore, it is impossible to define the one-dimensional winding number \( w(k_y) \) in the presence of the potential disorder. However, the Hamiltonian \( \tilde{H}(r) \) preserves chiral symmetry in Eq. (2), which is the most important point in the argument below.

The central ingredient of our theory consists of the following two important properties of the eigenstates of a Hamiltonian \( \tilde{H} \) preserving chiral symmetry [15].

(i) First, the zero-energy states of \( \tilde{H}(r) \) are simultaneously the eigenstates of the chiral operator \( \Gamma \). Since \( \tilde{\Gamma}^2 = 1 \), the eigenvalue of \( \Gamma \) is either \( \gamma = +1 \) or \( \gamma = -1 \). Namely, a zero-energy states satisfying \( \tilde{H}(r)\varphi_{\gamma}(r) = 0 \) also satisfies \( \tilde{\Gamma}\varphi_{\gamma}(r) = \gamma\varphi_{\gamma}(r) \). We refer to \( \gamma \) as the chirality in the following.

(ii) Second, the nonzero energy states of \( \tilde{H}(r) \) are described by the linear combination of the two states: one has \( \gamma = +1 \) and the other has \( \gamma = -1 \). Namely, a nonzero energy state is described as \( \varphi_{E\neq 0}(r) = c_+\chi_+(r) + c_-\chi_-(r) \), where \( \Gamma\chi_{\pm}(r) = \pm\chi_{\pm}(r) \). Moreover, the relation \( |c_+| = |c_-| \) always holds.

Here, we define two integer numbers, \( N_+ \) and \( N_- \), in the clean limit. According to property (i), we can immediately conclude that each ABSs at a clean surface belongs to either the positive chiral state or the negative chiral state. The integer \( N_+(N_-) \) is the number of ABSs that have the positive (negative) chiral eigenvalue [See also Fig. 1(a)]. The total number of ABSs at the clean surface is represented by \( N_+ + N_- \), which must be identical to \( N_{\text{clean}} \) in Eq. (4).

The stability of the flat ABSs in the presence of impurities can be discussed by using property (ii). An ABS departs from the zero energy only when it can form a pair with its chiral partner. When \( N_+ > N_- \), for example, \( N_- \) negative chiral ABSs can couple to \( N_+ \) positive chiral ABSs under the potential disorder. As a result, they form nonzero energy states whose number is \( 2N_- \). However, \( N_+ - N_- \) positive chiral states remain at the zero energy even in the presence of impurities because their chiral partner is absent. The integer number defined by \( N_{\text{ZES}} = N_+ - N_- \), represents the number of ZESs that remain at a dirty surface. When \( N_+ < N_- \), the number of ZESs at a dirty surface is given by \( N_- - N_+ \). Therefore, in general, \( |N_{\text{ZES}}| \) is the degree of degeneracy at zero energy in the presence of potential disorder. The
Figure 1. (Color online) (a) Four-fold degenerate zero-energy states (ZESs) in the absence of random potential. Three of them belong to the positive chiral eigenvalue, (i.e., \( N_+ = 3 \)). One remaining ZES belongs to the negative chiral eigenvalue (i.e., \( N_- = 1 \)). The four-fold degeneracy is protected by translational symmetry. (b) In the presence of random potential, a positive and a negative chiral ZES form a pair and depart from zero energy. However, two positive chiral ZESs remain at zero energy. The index \( N_{ZES} = N_+ - N_- = 2 \) represents the number of ZESs remaining at zero energy in the presence of random potential.

The essence of this argument is illustrated in Figs. 1 (a) and (b) with \( N_+ = 3 \) and \( N_- = 1 \). In Fig. 1 (a), we consider four ABSs at a clean surface. In Fig. 1 (b), we introduce the impurity potential at the surface.

At the end of this section, we discuss the topological aspect of \( N_{ZES} \). As examined in Ref. [15], the index theorem relates to the winding number \( w(k_y) \) and the number of ABSs on a clean surface as follows

\[
w(k_y) = \pm [n_+(k_y) - n_-(k_y)],
\]

where \( n_+(k_y) [n_-(k_y)] \) denotes the number of positive (negative) chiral zero-energy states at \( k_y \). There are two possible choices for the sign on the right-hand side of Eq. (5). When we consider the surface of a semi-infinite superconductor occupying \( x_\parallel \leq 0 \), we should choose the positive sign. On the other hand, we should choose the negative sign at the surface of a semi-infinite superconductor occupying \( x_\parallel \geq 0 \) [15]. However, this sign has no physical meaning because the number of ABSs is always given by \( |n_+(k_y) - n_-(k_y)| \). As discussed above, the index \( N_{ZES} \) is represented by the difference between the total numbers of positive and negative chiral ABSs. Therefore, by taking Eq. (5) into account, we obtain an important relation

\[
N_{ZES} = \sum_{k_y} w(k_y) = \pm (N_+ - N_-),
\]

as a result of the index theorem. The index \( N_{ZES} \) is initially defined by using the solutions of the differential equation. From Eq. (6), we simultaneously find that the index \( N_{ZES} \) is a topological invariant. Mathematically, such an integer number is called the Atiyah-Singer index. Although the index \( N_{ZES} \) is defined in the presence of translational symmetry, it represents the degree of the degeneracy at the zero energy in the absence of translational symmetry. This is the first conclusion of this paper.

3. Quantization of conductance minimum

In this section, we numerically study the zero-bias differential conductance (ZBC) of a normal-metal/superconductor (NS) junction on the tight-binding model as shown in Fig. 2(a). The length of the normal metal segment is represented by \( L_N \). In the \( y \) direction, the width of the junction is represented by \( L_y \), and the periodic boundary condition is applied. In the normal segment, we introduce the random on-site potential \( V_N(r) \hat{\tau}_3 \), where the amplitude of \( \hat{V}_N(r) \) is
given randomly in the range of $-V_N/2 \leq V_N(r) \leq V_N/2$. We calculate the zero-bias differential conductance $G_{NS}$ of the NS junctions based on the Blonder-Tinkham-Klapwijk formula [24]. The normal and Andreev reflection coefficients are calculated by using the lattice Green’s function method [25, 26].

In Fig. 2(b), we plot the zero-bias conductance $G_{NS}$ versus the normal conductance $G_N$, where $G_N$ is calculated by setting $\Delta_0 = 0$. The parameters are chosen as $\mu_F/t = 1.0$, $\Delta_0/t = 0.01$, $L_N/a_0 = 30$, and $L_y/a_0 = 25$, where $t$ and $a_0$ denote the nearest neighbor hopping integral and the lattice constant, respectively. This parameter choice leads $N_{clean} = 9, 8, 9$ and $N_{ZES} = 9, 0, 5$ for the $p_x$-wave, $d_{xy}$-wave, and $f$-wave superconductor, respectively. The index $N_{clean}$ ($N_{ZES}$) is calculated by using Eq. (4) [Eq. (6)]. In the $p_x$-wave case, the ZBC is quantized at $(4e^2/h) \times 9$ irrespective of potential disorder in the normal segment [5]. In the $d_{xy}$-wave case, the ZBC goes to zero with the decrease of $G_N$ [10]. In the $f$-wave NS junction, we find that the ZBC decreases to the quantized value of $(4e^2/h) \times 5$ by decreasing $G_N$. By summarizing these numerical results, we can conclude that

$$
\lim_{G_N \to 0} G_{NS} \to \frac{4e^2}{h} |N_{ZES}|.
$$

The property of chiral symmetry enables us to explain such the characteristic transport property of Eq. (7). As shown in Ref. [7, 22], the zero-energy states in the normal segment become the eigenstates of $\Gamma$ due to the Andreev reflection. Namely, the Andreev reflection copies the chirality of the flat ABSs to the wave functions in the normal segment. Therefore, according to the chiral property (ii), the $|N_{ZES}|$ zero-energy states can retain their degeneracy in the dirty normal segment and can form the resonant transmission channels there. The quantization of conductance minimum at $(4e^2/h)|N_{ZES}|$ is the direct consequence from the existence of $|N_{ZES}|$ resonant transmission channels protected by chiral symmetry. This is the second conclusion of this paper.

4. Conclusion
In conclusion, we have discussed the stability of flat Andreev bound states appearing at the dirty surface of a nodal superconductor. In the clean limit, the flat ABSs at the clean surface are characterized by the momentum dependent topological invariant $w(k_0)$. In a real superconductor, however, the potential disorder near the surface or junction interface is inevitable and may lift the degeneracy of flat ABSs at the zero energy. By taking into account
chiral symmetry of the BdG Hamiltonian, we probe that the Atiyah-Singer index $N_{\text{ZES}}$ predicts the number of the flat ABSSs at the dirty surface. We have also discussed the zero-bias differential conductance $G_{\text{NS}}$ in a normal metal/superconductor (NS) junction. As a consequence, we find that the minimum value of $G_{\text{NS}}$ is quantized at $(4e^2/h)|N_{\text{ZES}}|$.

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