On global well-posedness for Klein-Gordon equation with concentrated nonlinearity

ELENA KOPYLOVA *
Faculty of Mathematics of Vienna University and IIITP RAS

Abstract

We prove global well-posedness for the 3D Klein-Gordon equation with a concentrated nonlinearity.

1 Introduction

The paper concerns a nonlinear interaction of the Klein-Gordon field with point oscillators. The system is governed by the following equations

\[
\begin{align*}
\psi(x,t) &= (\Delta - m^2)\psi(x,t) + \sum_{1 \leq j \leq n} \zeta_j(t) \delta(x-y_j) \\
\lim_{x \to y_j} (\psi(x,t) - \zeta_j(t) g_j(x)) &= F_j(\zeta(t)), \quad 1 \leq j \leq n
\end{align*}
\]

(1.1)

with \( m > 0 \) and \( \zeta(t) = (\zeta_1(t),...,\zeta_n(t)) \in \mathbb{C}^n \). Here \( g_j(x) = g(x-y_j) \), and \( g(x) \) is the Green’s function of the operator \(-\Delta + m^2\) in \( \mathbb{R}^3 \), i.e.,

\[
g(x) = \frac{e^{-|x|}}{4\pi |x|}.
\]

(1.2)

The nonlinearity \( F(\zeta) = (F_1(\zeta),...,F_n(\zeta)) \) admits a real-valued potential:

\[
F(\zeta) = \partial_\zeta U(\zeta), \quad U \in C^2(\mathbb{C}^n),
\]

(1.3)

where \( \partial_\zeta := \frac{1}{2}(\frac{\partial U}{\partial \zeta_1} + i \frac{\partial U}{\partial \zeta_2}) \) with \( \zeta_1 := \text{Re} \zeta \) and \( \zeta_2 := \text{Im} \zeta \). Let \( G = \{g_{jk}\} \) be a matrix with the entries

\[
g_{jk} := \begin{cases} 
\frac{|\zeta_j|^2}{4\pi |\zeta_j|^2}, & \text{if } j \neq k \\
0, & \text{if } j = k
\end{cases}
\]

(1.4)

and let \( \mathcal{G}(\zeta) = (G\zeta,\overline{\zeta}) = \sum_{1 \leq k \leq n} g_{jk} \zeta_k \overline{\zeta_k} \). We assume that \( U(\zeta) \) is such that

\[
U(\zeta) - \mathcal{G}(\zeta) \geq b|\zeta|^2 - a \quad \text{for } \zeta \in \mathbb{C}^n, \quad \text{where } b > 0 \quad \text{and} \quad a \in \mathbb{R}.
\]

(1.5)

Our main result is the following. For the initial data of type

\[
\psi(x,0) = \psi_0(x) + \sum_{1 \leq j \leq n} \zeta_0 g_j(x), \quad \dot{\psi}(x,0) = 0(x) + \sum_{1 \leq j \leq n} \dot{\zeta}_0 g_j(x),
\]

(1.6)

where \( \psi_0 \in H^2(\mathbb{R}^3) \) and \( \psi_0 \in H^1(\mathbb{R}^3) \), we prove a global well-posedness of the Cauchy problem for (1.1) (see Theorem 2.2 below).

In the context of the 3D Schrödinger and wave equations the point interaction of type (1.1) was introduced in [1] - [3], [9] - [11], where the well-posedness, blow-up and asymptotic stability of solutions was studied. The first justification of

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the model with a nonlinear point interaction was given in the NLS case in the recent paper [5]. The nonlinear Schrödinger
dynamics with a nonlinearity concentrated at a point is obtained in [5] as a scaling limit of a regularized nonlinear Schrödinger
dynamics.

However, for the 3D Klein-Gordon equation with a point interaction the global well-posedness and a justification of
the model has not been obtained. In present paper we concentrate on the first problem only. We suppose that a justification
can be done by suitable modification of methods [5], but it still remains an open question.

Let us comment on our approach. We develop for the Klein-Gordon equation the approach suggested in [10] for wave
equation. First we consider the linear system (1.1) with

\[ \psi(x,t) = \psi_f(x,t) + \phi(x,t), \]

where \( \psi_f(x,t) \) is a solution to the Cauchy problem for the free Klein-Gordon equation with the initial data \( \text{1.6} \), and \( \phi(x,t) \) is a solution to the coupled system of i) the Klein-Gordon equation with delta-like source and with zero initial
data, and of ii) the first-order linear integro-differential equation which control the dynamics of the coefficients \( \xi_i(t) \). As
a consequence, we derive an important regularity property of \( \psi_f(y_1,t) \) (see Proposition 4.2, cf. also [10] Lemma 3.4).

We use this regularity property for proving the existence of a local solution to (1.1) in the case of nonlinear function \( F(\xi) \). Then we obtain the energy conservation for the dynamics (1.1). Finally, we use the energy conservation to obtain a
global existence theorem.

The Klein-Gordon equation differs from the wave equation, considered in [9]–[11], by the absence of strong Huygens
principle. This result in additional integral terms (convolutions with Bessel functions) in many calculations.

We expect that the result and methods of present paper will be useful for the theory of attractors for \( U(1) \)-invariant
Hamiltonian system (1.1) (cf. [2], [4], [7], [8]).

Our paper is organized as follows. In Section 2 we formulate the main theorem. In Section 3 we study the structure of
solutions to the Klein-Gordon equation with linear one point interaction. In Section 4 we prove the key regularity property
of solutions to the free Klein-Gordon equation with initial data \( \text{1.6} \). In section 5 we derive some preliminary formulas.
In Section 6 we consider the nonlinear equation and prove the main theorem.

## 2 Main result

We fix a nonlinear function \( F: \mathbb{C}^n \to \mathbb{C}^n \) and define the domain

\[ D_F = \{ \psi \in L^2(\mathbb{R}^3): \psi(x) = \psi_{\text{reg}}(x) + \sum_{\xi_j \leq n} \xi_j g_j(x), \quad \psi_{\text{reg}} \in H^2(\mathbb{R}^3), \quad \psi_{\text{reg}}(x) = F(\xi) \}, \quad (2.1) \]

which generally is not a linear space. Note that the last condition in (2.1) is equivalent to

\[ \psi_{\text{reg}}(y_j) + \sum_{\xi_j \leq n} g_{\xi_j} = F(\xi). \quad (2.2) \]

Let \( H_F \) be a nonlinear operator on the domain \( D_F \) defined by

\[ H_F \psi = (\Delta - m^2) \psi_{\text{reg}}, \quad \psi \in D_F. \quad (2.3) \]

The system (1.1) for \( \psi(x,t) \in D_F \) reads

\[ \dot{\psi}(x,t) = H_F \psi(x,t), \quad x \in \mathbb{R}^3, \quad t \in \mathbb{R}. \quad (2.4) \]

Let us introduce a phase space for equation (2.3). Denote

\[ \mathcal{D} = \{ \pi \in L^2(\mathbb{R}^3): \pi(x) = \pi_{\text{reg}}(x) + \sum_{\xi_j \leq n} \eta_{\xi_j} g_j(x), \quad \pi_{\text{reg}} \in H^1(\mathbb{R}^3), \quad \eta_j \in \mathbb{C} \}. \]

Obviously, \( D_F \subset \mathcal{D} \).

**Definition 2.1.** (i) \( \mathcal{D}_F \) is the Hilbert space of states \( \Psi = (\psi(x), \pi(x)) \in D_F \oplus \mathcal{D} \) equipped with the finite norm

\[ \| \Psi \|_{\mathcal{D}_F}^2 := \| \psi_{\text{reg}} \|^2_{H^2(\mathbb{R}^3)} + \| \pi_{\text{reg}} \|^2_{H^1(\mathbb{R}^3)} + \sum_{\xi_j \leq n} |\xi_j|^2 + \sum_{\xi_j \leq n} |\eta_j|^2. \quad (2.5) \]
(ii) \( \mathcal{H} \) is the Hilbert space of the states \( \Psi = (\psi(x), \pi(x)) \in H^2(\mathbb{R}^3) \oplus H^1(\mathbb{R}^3) \) equipped with the finite norm
\[
\|\Psi\|_{\mathcal{H}}^2 := \|\psi\|_{H^2(\mathbb{R}^3)}^2 + \|\pi\|_{H^1(\mathbb{R}^3)}^2.
\] (2.6)

Denote \( \|\cdot\| = \|\cdot\|_{L^2(\mathbb{R}^3)} \). Our main result is the following.

**Theorem 2.2.** Let conditions (1.2) and (1.3) hold. Then

(i) For every initial data \( \Psi_0 = (\psi_0, \pi_0) \in \mathcal{D}_F \) the equation (2.3) has a unique solution \( \psi(t) \) such that
\[
\Psi(t) = (\psi(t), \psi(t)) \in C(\mathbb{R}, \mathcal{D}_F).
\]

(ii) The map \( W(t) : \Psi_0 \mapsto \Psi(t) \) is continuous in \( \mathcal{D}_F \) for each \( t \in \mathbb{R} \).

(iii) The energy is conserved, i.e.,
\[
\mathcal{H}_F(\Psi(t)) := \|\psi(t)\|^2 + \|\nabla \psi_{reg}(t)\|^2 + \|\psi_{reg}(t)\|^2 + U(\zeta(t)) - \mathcal{F}(\zeta(t)) = \text{const}, \quad t \in \mathbb{R}.
\] (2.7)

(iv) The following a priori bound holds
\[
|\zeta(t)| \leq C(\Psi_0), \quad t \in \mathbb{R}.
\] (2.8)

Obviously, it suffices to prove Theorem 2.2 for \( t \geq 0 \).

### 3 Linear equation: structure of solution

Here we consider the Klein-Gordon equation with a linear one point interaction, i.e. \( n = 1 \). More precisely, denote
\[
D_y = \{ \psi \in L^2(\mathbb{R}^3) : \psi = \psi_{reg} + \xi g(x-y), \quad \psi_{reg} \in H^2(\mathbb{R}^3), \quad \psi_{reg}(y) = \xi \in \mathbb{C} \},
\]
\[
\dot{D}_y = \{ \pi \in L^2(\mathbb{R}^3) : \pi(x) = \pi_{reg}(x) + \chi g(x-y), \quad \pi_{reg} \in H^1(\mathbb{R}^3), \quad \chi \in \mathbb{C} \},
\]
and consider the operator
\[
H_y \psi := (\Delta - m^2) \psi_{reg}, \quad \psi \in D_y.
\] (3.1)

**Proposition 3.1.** Let \( \psi_0 = \psi_{0,reg} + \xi_0 g(x-y) \in D_y \) and \( \pi_0 = \pi_{0,reg} + \xi_0 g(x-y) \in \dot{D}_y \). Then the Cauchy problem
\[
\dot{\psi}(x,t) = H_y \psi(x,t), \quad \psi(x,0) = \psi_0(x), \quad \psi(x,0) = \pi_0(x),
\] (3.2)
has a unique strong solution \( \psi(t) \in C(\mathbb{R}, D_y) \cap C^1(\mathbb{R}, \dot{D}_y) \cap C^2(\mathbb{R}, L^2(\mathbb{R}^3)) \).

**Proof.** The operator \( H_y \) is a symmetric operator, since
\[
\langle H_y \psi, \phi \rangle = \langle (\Delta - m^2) \psi_{reg}, \phi_{reg} + \xi \phi(-y) \rangle = \langle \psi_{reg}, (\Delta - m^2) \phi_{reg} \rangle - \xi \int \frac{\phi(x+y) - \phi(x-y)}{2} \, dx.
\]
Moreover,
\[
\langle -H_y \psi, \phi \rangle = \|\nabla \psi_{reg}\|^2 + \|\psi_{reg}\|^2 + \xi \int \frac{\phi(x+y) - \phi(x-y)}{2} \, dx \geq 0.
\]
Hence, \( H_y \) admits a unique selfadjoint extension, and the corresponding Cauchy problem has a unique strong solution in \( C(\mathbb{R}, D_y) \cap C^1(\mathbb{R}, \dot{D}_y) \cap C^2(\mathbb{R}, L^2(\mathbb{R}^3)) \) by the theory of abstract wave equations in Hilbert spaces (see e.g. [6, Chapter 2, Section 7]).

Proposition 3.1 implies that the strong solution \( \psi(x,t) \) of (3.2) satisfies
\[
\psi(x,t) = \psi_{reg}(x,t) + \xi(t) g(x-y), \quad \psi_{reg}(y,t) = \xi(t)
\] (3.3)
with \( \psi_{reg}(t) \in C(\mathbb{R}, H^2(\mathbb{R}^3)) \) and \( \xi(t) \in C^1(\mathbb{R}) \). Now we obtain an integral representation for \( \psi(x,t) \) via a solution to integro-differential equation.
Lemma 3.2. (cf. [9] Theorem 3). The strong solution \( \psi(x,t) \in C(\mathbb{R}, D_x) \cap C^1(\mathbb{R}, \dot{D}_x) \cap C^2(\mathbb{R}, L^2(\mathbb{R}^3)) \) of the Cauchy problem (3.2) for \( t \geq 0 \) is given by

\[
\psi(x,t) = \psi_f(x,t) + \frac{\theta(t-|x-y|)}{4\pi |x-y|} \xi(t-|x-y|) - \frac{m}{4\pi} \int_0^t \frac{\theta(s-|x-y|)J_1(m\sqrt{s^2-|x-y|^2})}{\sqrt{s^2-|x-y|^2}} \xi(t-s)ds.
\] (3.4)

Here \( \psi_f(x,t) \in C([0,\infty), L^2(\mathbb{R}^3)) \) is the unique solution to the Cauchy problem for the free Klein-Gordon equation

\[
\psi_f(x,t) = (\Delta - m^2)\psi_f(x,t), \quad \psi_f(x,0) = \psi_0(x), \quad \psi_f(x,0) = \pi_0(x),
\] (3.5)

and \( \xi(t) \in C([0,\infty)) \) is the unique solution to the Cauchy problem for the following first-order linear integro-differential equation with delay

\[
\frac{1}{4\pi}(\xi(t) - m\xi(t)) + \xi(t) - \lambda(t) + \frac{m}{4\pi} \int_0^t \frac{J_1(m(t-s))}{t-s} \xi(s)ds = 0, \quad \xi(0) = \xi_0, \quad t \geq 0,
\] (3.6)

where \( \lambda(t) \in C([0,\infty)), \) and \( \lambda(t) := \lim_{x \rightarrow y} \psi_f(x,t) \) for \( t > 0 \).

Proof. In notation (3.4) define the function

\[
\varphi(x,t) := \frac{\theta(t-|x-y|)}{4\pi |x-y|} \xi(t-|x-y|) - \frac{m}{4\pi} \int_0^t \frac{\theta(s-|x-y|)J_1(m\sqrt{s^2-|x-y|^2})}{\sqrt{s^2-|x-y|^2}} \xi(t-s)ds.
\] (3.7)

It is easy to verify that \( \varphi(t) \in C([0,\infty), L^2(\mathbb{R}^3)) \) and satisfy the equation

\[
\dot{\varphi}(x,t) = (\Delta - m^2)\varphi(x,t) + \xi(t)\delta(x-y), \quad \varphi(x,0) = 0, \quad \varphi(x,0) = 0.
\] (3.8)

Hence, for \( \psi_f = \psi - \varphi \in C([0,\infty), L^2(\mathbb{R}^3)) \), we obtain

\[
\psi_f(x,t) = \psi(x,t) - \varphi(x,t) = (\Delta - m^2)\psi_{\text{reg}}(x,t) - \left( (\Delta - m^2)\varphi(x,t) + \xi(t)\delta(x-y) \right)
\]
\[
= (\Delta - m^2)\left( \psi(x,t) - \xi(t)g(x-y) \right) - \left( (\Delta - m^2)\varphi(x,t) + \xi(t)\delta(x) \right)
\]
\[
= (\Delta - m^2)\left( \psi(x,t) - \varphi(x,t) \right) = (\Delta - m^2)\psi_f(x,t),
\]

and thus \( \psi_f \) is the solution to the Cauchy problem (3.5).

Let us prove the existence and continuity of \( \lambda(t) = \lim_{x \rightarrow y} \psi_f(x,t) \). We can split \( \psi_f(x,t) \) as

\[
\psi_f(x,t) = \psi_{f,\text{reg}}(x,t) + \psi_{f,\text{y}}(x,t),
\]

where \( \psi_{f,\text{reg}} \) is the solution to the free Klein-Gordon equation with regular initial data \( \psi_{0,\text{reg}}, \pi_{0,\text{reg}} \), and \( \psi_{f,\text{y}} \) are the solutions to the free Klein-Gordon equation with initial data \( \xi_0 g(x-y), \xi_0 g(x-y) \). Evidently, \( \psi_{f,\text{reg}}(x,t) \in C([0,\infty), H^2(\mathbb{R}^3)) \) and there exists

\[
\lambda_{\text{reg}}(t) := \lim_{x \rightarrow y} \psi_{f,\text{reg}}(x,t) = \psi_{f,\text{reg}}(y,t) \in C([0,\infty)).
\] (3.9)

Now consider \( \psi_{f,\text{y}}(x,t) \). Note that the function

\[
\eta_y(x,t) := \psi_{f,\text{y}}(x,t) - (\xi_0 + t\xi_0)g(x-y)
\]
satisfies

\[
\dot{\eta}_y(x,t) = (\Delta - m^2)\eta_y(x,t) - (\xi_0 + t\xi_0)\delta(x-y)
\]

with zero initial data. Therefore, one obtains

\[
\eta_y(x,t) = - \int_0^t \left( \frac{\delta(t-s-|x-y|)}{4\pi(t-s)} - \frac{m\theta(t-s-|x-y|)J_1(m\sqrt{(t-s)^2-|x-y|^2})}{4\pi|t-s|} \right)(\xi_0 + s\xi_0)ds
\]
\[
= - \frac{\theta(t-|x-y|)(\xi_0 + (t-|x-y|)\xi_0)}{4\pi|t-s|} + \frac{m}{4\pi} \int_0^t \frac{\theta(t-s-|x-y|)J_1(m\sqrt{(t-s)^2-|x-y|^2})}{\sqrt{(t-s)^2-|x-y|^2}}(\xi_0 + s\xi_0)ds.
\]
Applying the Fourier transform, we obtain

\[ \psi_{f,j}(x,t) = -\frac{\Theta(t-|x-y|)(\xi_0 + (t-|x-y|)\hat{\xi}_0) + (\xi_0 + t\hat{\xi}_0)e^{-m|x-y|}}{4\pi|x-y|} + \frac{m}{4\pi} \int_0^t \frac{\Theta(t-s-|x-y|)J_1(m\sqrt{(t-s)^2-|x-y|^2})}{\sqrt{(t-s)^2-|x-y|^2}}(\xi_0 + s\hat{\xi}_0)ds, \quad t \geq 0. \]  

\[ (3.10) \]

Therefore, for any \( t > 0 \) there exists

\[ \lambda_\nu(t) = \lim_{s \to 0} \psi_{f,j}(x,t) = -\frac{m(\xi_0 + t\hat{\xi}_0) - \xi_0}{4\pi} + \frac{m}{4\pi} \int_0^t J_1(m(t-s)) \frac{(\xi_0 + s\hat{\xi}_0)ds}{(t-s)} \]

and we set

\[ \lambda_\nu(0) = \lim_{t \to 0} \lambda_\nu(t) = -\frac{m\xi_0 - \hat{\xi}_0}{4\pi}. \]  

\[ (3.12) \]

Thus, (3.9)–(3.12) imply that

\[ \lambda(t) = \lambda_{eg}(t) + \lambda_\nu(t) \in C([0,\infty)). \]

It remains to prove that \( \xi(t) \) is a solution to the Cauchy problem (3.6). Writing the second equation of (3.3) in terms of the decomposition (3.4), we obtain

\[ \xi(t) = \lim_{s \to 0} (\psi(t,x) - \xi(t)g(x-y)) = \lim_{s \to 0} (\psi_{f,j}(t,x) + \varphi(x,t) - \xi(t)g(x-y)) \]

\[ = \lambda(t) + \lim_{s \to 0} \left( \frac{\Theta(t-|x-y|)\xi(t-|x-y|) - \xi(t)e^{-m|x-y|}}{4\pi|x-y|} \right) \]

\[ - \lim_{s \to 0} \frac{m}{4\pi} \int_0^t \frac{\Theta(t-s-|x-y|)J_1(m\sqrt{(t-s)^2-|x-y|^2})}{\sqrt{(t-s)^2-|x-y|^2}}\xi(s)ds \]

\[ = \lambda(t) - \frac{1}{4\pi} (\xi(t) - m\xi(t)) - \frac{m}{4\pi} \int_0^t \frac{J_1(m(t-s))\xi(s)ds}{t-s}, \quad t \geq 0. \]  

\[ (3.13) \]

Hence \( \xi(t) \) satisfies (3.6). \( \Box \)

4 Regularity property

Here we establish the key regularity property of solution \( \psi_{f,j}(x,t) \) to the free Klein-Gordon equation with initial data from \( \mathcal{X} \). First we prove an auxiliary lemma.

**Lemma 4.1.** (cf. [10] Lemma 3.2). Let \( 0 < T < \infty, y \in \mathbb{R}^3, f(t) \in C([0,T]), \) and \( h(t) = \hat{f}(t) \) in the sense of distribution. Let \( u(x,t) \in C([0,T],L^2(\mathbb{R}^3)) \) be the solution to the Cauchy problem

\[ \ddot{u}(x,t) = (\Delta - m^2)u(x,t) + h(t)g(x-y), \quad u(x,0) = u_0(x), \quad \dot{u}(x,0) = \dot{u}_0(x) \]  

\[ (4.1) \]

with initial data \((u_0, \dot{u}_0) \in \mathcal{X} \). Then \( h \in L^2([0,T]) \) if and only if \((u(t), \dot{u}(t)) \in C([0,T], \mathcal{X}) \).

**Proof.** It suffices to consider \( y = 0 \).

i). Suppose that \( h \in L^2([0,T]) \). We represent \( u(x,t) \) as the sum \( u(x,t) = u_1(x,t) + u_2(x,t) \), where \( u_1(x,t) \) is a solution to the free Klein-Gordon equation with the initial data \( u_0, \dot{u}_0 \), and \( u_2(x,t) \) is a solution to (4.1) with zero initial data. Evidently, \((u_1(t), \dot{u}_1(t)) \in C([0,\infty), \mathcal{X}) \). Let us prove that

\[ (u_2(t), \dot{u}_2(t)) \in C([0,T], \mathcal{X}) \]  

\[ (4.2) \]

Applying the Fourier transform, we obtain

\[ (|k|^2 + m^2)\hat{u}_2(k,t) = \int_0^t \frac{\sin((t-s)\sqrt{|k|^2 + m^2})}{\sqrt{|k|^2 + m^2}}h(s)ds, \quad \sqrt{|k|^2 + m^2}\hat{u}_2(k,t) = \int_0^t \frac{\cos((t-s)\sqrt{|k|^2 + m^2})}{\sqrt{|k|^2 + m^2}}h(s)ds. \]
Hence, for (4.2) it suffices to verify that for any $t \in [0, T]$,
\[
\left\| \int_0^t \frac{\sin((t-s)\sqrt{|k|^2 + m^2})}{\sqrt{|k|^2 + m^2}} h(s) ds \right\| \leq C_1(t) \|h\|_{L^2([0,t])},
\]
\[
\left\| \int_0^t \frac{\cos((t-s)\sqrt{|k|^2 + m^2})}{\sqrt{|k|^2 + m^2}} h(s) ds \right\| \leq C_2(t) \|h\|_{L^2([0,t])}.
\]
(4.3)

The both integrals are estimated in the same way, and we consider the first integral only. We have
\[
\left\| \int_0^t \frac{\sin((t-s)\sqrt{|k|^2 + m^2})}{\sqrt{|k|^2 + m^2}} h(s) ds \right\|^2
\]
\[
= 4\pi \lim_{R \to \infty} \int_0^t \int_0^t ds ds' \int_0^R dr \sin((t-s' - s)\sqrt{r^2 + m^2}) \frac{r^2 - r'^2}{r^2 + m^2}
\]
\[
= 4\pi \lim_{R \to \infty} \int_0^t \int_0^t ds ds' \int_0^R dr \sin((t-s' - s)\sqrt{r^2 + m^2}) \frac{r^2 - r'^2}{r^2 + m^2}
\]
\[
= 4\pi m^2 \int_0^t \int_0^t ds ds' \int_0^\infty dr \sin((t-s' - s)\sqrt{r^2 + m^2}) \frac{r^2 - r'^2}{r^2 + m^2} = I_1(t) + I_2(t)
\]
(4.4)

It is easy to prove that
\[
|I_2(t)| \leq C_1 \|h\|_{L^2([0,t])}^2, \quad t \in [0, T].
\]
(4.5)

It remains to estimate the term $I_1$. We have
\[
I_1(t) = \pi \lim_{R \to \infty} \int_0^t \int_0^t ds ds' \int_0^R dr \left( e^{i(s-x')\sqrt{r^2 + m^2}} + e^{i(2t-s-x')\sqrt{r^2 + m^2}} \right)
\]
\[
= \pi \sqrt{2\pi} \int_0^t ds \int_0^\infty dr \left( e^{i(s-x)\sqrt{r^2 + m^2}} + e^{i(2t-s-x)\sqrt{r^2 + m^2}} \right)
\]
(4.6)

where
\[
F(r) = \int_{-\infty}^{\infty} dr (e^{ix\sqrt{r^2 + m^2}} - e^{ixr}).
\]

Note that
\[
e^{ix\sqrt{r^2 + m^2}} - e^{ixr} = e^{ixr} (e^{ix(\sqrt{1+\frac{m^2}{r^2}}-1)} - 1) = e^{ixr} \left( \frac{izm^2}{2r} + R(r,z) \right), \quad |r| \geq 2m^2 + 1,
\]
where $|R(r,z)| \leq \frac{1}{4} (1 + |z|) |m|^2 + 1$. Hence,
\[
|F(z)| \leq \int_{|r| \geq 2m^2 + 1} |dz| + \int_{|r| \geq 2m^2 + 1} \int_{|r| \geq 2m^2 + 1} \frac{m^2}{2r} |dz| \leq C_1(m)(1 + |z|)^2.
\]
(4.7)

From (4.6) and (4.7) it follows that
\[
|I_1(t)| \leq C_1(m)(1 + t^3) \|h\|_{L^2([0,t])}^2, \quad t \in [0, T].
\]
(4.8)

Finally, (4.3), (4.5) and (4.8) imply (4.2), and then (4.2).

ii) Suppose now that $(u(t), \dot{u}(t)) \in C([0,T], \mathcal{X})$ and prove that $h \in L^2([0,T])$. As the first step, we will estimate $\|h\|_{L^2([0,T])}$ via
\[
M_T := \max_{t \in [0,T]} \|\dot{u}(t), \dot{u}(t))\|_{\mathcal{X}},
\]
assuming that $\xi \in C([0,T])$. For any fixed $\xi \in [0,T)$ and $t \in [\xi, T]$, we split $u(x,t)$ as $u(x,t) = u_1(x,t) + u_2(x,t)$, where $u_1(x,t)$ is the solution to the free Klein-Gordon equation with initial data $u(x,\xi), \dot{u}(x,\xi)$, and $u_2(x,t)$ is the solution to (4.1) with zero initial data at $t = \xi$. By the energy conservation for the free Klein-Gordon equation
\[
\| (u_1(x,t), \dot{u}_1(x,t)) \|_{\mathcal{X}} \leq C \| (u(\xi), \dot{u}(\xi)) \|_{\mathcal{X}} \leq CM_T, \quad t \in [\xi, T].
\]
Therefore,
\[
\| (u_2(x,t), \dot{u}_2(x,t)) \|_{\mathcal{X}} \leq CM_T, \quad t \in [\xi, T].
\]
(4.9)
Finally, we should consider general case solution to (4.1) with $v(x,t) = u(x,t)$, where $u(x,t) \in C([\tau, T], L^2(\mathbb{R}^3))$ is the unique solution to
\[ \psi(x,t) = (\Delta - m^2) v(x,t) + h(t) \delta(x) \]
with zero initial data at $t = \tau$. Estimate (4.9) implies
\[ \|v_\tau(t)\| \leq CM_T, \quad t \in [\tau, T]. \tag{4.10} \]
Similarly to (3.7)–(3.8), we obtain
\[ v_\tau(x,t) = \frac{\theta(t - \tau - |x|)}{4\pi|x|} h(t - |x|) - \frac{m}{4\pi} p_\tau(|x|, t), \quad t \geq \tau, \tag{4.11} \]
where
\[ p_\tau(r, t) = \int_{\tau}^{t} \frac{\theta(t - s - r)}{(t - s)^2 - r^2} J_1(m \sqrt{(t - s)^2 - r^2}) h(s) ds, \quad r \geq 0. \tag{4.12} \]
We have
\[ \|v_\tau(t)\|^2 = \frac{1}{4\pi} \int_{\tau}^{t - \tau} |h(t - r)|^2 dr + \frac{m^2}{4\pi} \int_{\tau}^{t - \tau} r^2 |p_\tau(r, t)|^2 dr + \frac{m}{2\pi} \int_{\tau}^{t - \tau} h(t - r) p_\tau(r, t) r dr, \quad t \in [\tau, T]. \]
Therefore,
\[ |h|_{L^2([\tau, T])}^2 + 2m \int_{\tau}^{t - \tau} h(t - r) p_\tau(r, t) r dr \leq 4\pi \|v_\tau(t)\|^2 \leq C_1 M_T, \quad t \in [\tau, T]. \tag{4.13} \]
due to (4.10), where $C_1$ does not depend on $\tau$ and $t$. Applying the Cauchy-Schwarz inequality to (4.12), we obtain
\[ |p_\tau(r, t)| \leq C |h|_{L^2([\tau, T])} \left( \int_{r}^{\infty} \frac{J_1^2(m \sqrt{s^2 - r^2})}{\sqrt{s^2 - r^2}} ds \right)^{1/2}. \]
The properties of Bessel function $J_1$ (see [12]) imply the uniform estimates
\[ \int_{r}^{\infty} \frac{J_1^2(m \sqrt{s^2 - r^2})}{\sqrt{s^2 - r^2}} ds \leq C, \quad 0 \leq r. \]
Hence,
\[ |p_\tau(r, t)| \leq C |h|_{L^2([\tau, T])}, \quad 0 \leq r, \quad t \in [\tau, T], \]
and the Cauchy-Schwarz inequality imply
\[ 2m \int_{\tau}^{t - \tau} |h(t - r)| p_\tau(r, t) r dr \leq C_0 (t - \tau)^{3/2} |h|_{L^2([\tau, T])}^2, \]
where $C_0$ does not depend on $\tau$ and $t$. Substituting the last inequality into (4.13), we obtain
\[ (1 - C_0 (t - \tau)^{3/2}) |h|_{L^2([\tau, T])}^2 \leq C_1 M_T, \quad t \in [\tau, T]. \]
Therefore, $h \in L^2([\tau, \tau + t_0])$, where $t_0 = \min\{T - \tau, (2C_0)^{-2/3}\}$. Since $0 \leq \tau < T$ is arbitrary, we can split the interval $[0, T]$ as
\[ [0, T] = [0, t_0] \cup [t_0, 2t_0] \cup \ldots \cup [nt_0, T] \]
and obtain that
\[ |h|_{L^2([0, T])} \leq C \max_{t \in [0, T]} \|u(t), \dot{u}(t)\|_{\mathcal{H}}. \tag{4.14} \]
Finally, we should consider general case $\xi \in C([0, T])$. In this case we define smooths approximations $\xi_\varepsilon(t)$:
\[ \xi_\varepsilon(t) = \xi + \varepsilon \rho_\varepsilon(t) = \int_{0}^{T} \rho_\varepsilon(s) \xi(t - s) ds \in C^\infty([0, T]), \]
where $\rho_\varepsilon(s) = \frac{1}{\varepsilon} \rho(\frac{s}{\varepsilon})$, and $\rho(s)$ is a smooth function with support in $[-1, 0]$ such that $\int \rho(s) ds = 1$. Let $u_\varepsilon(x,t)$ be a solution to (4.1) with $h_\varepsilon(t)$ instead of $h(t)$. Then we have
\[ |h_\varepsilon|_{L^2([0, T])} \leq C \max_{t \in [0, T]} \|u(t), \dot{u}(t)\|_{\mathcal{H}}. \]
Taking the limit as $\varepsilon \to 0$, we obtain (4.14) in general case. \qed
Proposition 4.2. Let \( \psi_f(x,t) \in C([0,\infty), L^2(\mathbb{R}^3)) \) be the unique solution to (3.3) with initial data \( \psi_0 \in D_y \) and \( \pi_0 \in D_y \), and let \( \hat{\lambda}(t) = \lim_{x \to y} \psi_f(x,t) \). Then \( \hat{\lambda} \in L^2_{\text{loc}}([0,\infty)) \).

Proof. Substituting (3.3) into (3.2), we obtain

\[
\psi_{reg}(x,t) = (\Delta - m^2)\psi_{reg}(x,t) - \xi(t)g(x-y),
\]

\[
\psi_{reg}(x,0) = \psi_{0,reg}, \quad \psi_{reg}(x,0) = \pi_{0,reg}.
\]

Note that \( \xi \in C^1(\mathbb{R}) \) since \( \|\psi(t)\|_{D_y}^2 := \|\psi_{reg}(t)\|_{H^2(\mathbb{R}^3)}^2 + |\xi(t)|^2 \). Lemma 4.1 implies that \( \xi \in L^2_{\text{loc}}([0,\infty)) \). Then \( \hat{\lambda} \in L^2_{\text{loc}}([0,\infty)) \) by (3.6).

Corollary 4.3. (cf. [17] Lemma 3.4). Let \( u(x,t) \) be the solution to the free Klein-Gordon equation with regular initial data \( (u_0,\dot{u}_0) \in \mathcal{D} \). Then for all \( y \in \mathbb{R}^3 \)

\[
\dot{u}(y,t) \in L^2_{\text{loc}}([0,\infty)).
\]

Proof. Consider the Cauchy problem (3.2) with initial data \( \psi_0(x) = u_0(x) + \xi_0g(x-y), \pi_0(x) = \dot{u}_0(x) + \xi_0g(x-y) \), where \( \xi_0 = u_0(y) \) and \( \dot{\xi}_0 \) is arbitrary. Then in notation of Lemma 3.2 we have \( u(x,t) = \psi_{f,reg}(x,t) \) and \( u(y,t) = \lambda_{reg}(t) = \lambda(t) - \lambda_y(t) \), where \( \lambda_y(t) \) is defined by (3.11). Evidently, \( \lambda_y(t) \in L^2_{\text{loc}}([0,\infty)) \). Then Proposition 4.2 implies (4.15).

5 Free Klein-Gordon equation

Here we introduce some notations and obtain some formulas which we will use below. We consider the free Klein-Gordon equation

\[
\psi_f(x,t) = (\Delta - m^2)\psi_f(x,t)
\]

with initial data

\[
\psi_0 = \psi_{0,reg} + \sum_{1 \leq k \leq n} \xi_{0,k}g_k, \quad \pi_0 = \pi_{0,reg} + \sum_{1 \leq k \leq n} \dot{\xi}_{0,k}g_k.
\]

We split \( \psi_f(x,t) \) as

\[
\psi_f(x,t) = \psi_{f,reg}(x,t) + \sum_{1 \leq j \leq n} \psi_{f,j}(x,t),
\]

where \( \psi_{f,reg} \) is the solution to the free Klein-Gordon equation with regular initial data \( \psi_{0,reg}, \pi_{0,reg}, \) and \( \psi_{f,j} \) are the solutions to the free Klein-Gordon equation with initial data \( \xi_{0,k}g_k, \dot{\xi}_{0,k}g_k \). Denote

\[
\lambda_{j,reg} := \psi_{f,reg}(y,t) \in C([0,\infty)), \quad \lambda_{j,k}(t) = \lim_{x \to y} \psi_{f,j}(x,t)
\]

Due to Corollary 4.3

\[
\lambda_{j,reg}(t) \in L^2_{\text{loc}}([0,\infty)).
\]

Further, we get similarly to (3.10)

\[
\psi_{f,j}(x,t) = -\frac{\theta(t - |x - y_k|)(\xi_{0,k} + (t - |x - y_k|)\dot{\xi}_{0,k})}{4\pi|x - y_k|} + \frac{(\xi_{0,k} + t\dot{\xi}_{0,k})e^{-|x - y_k|}}{4\pi|x - y_k|}
\]

\[
+ \frac{m}{4\pi} \int_0^t \frac{\theta(t - s - |x - y_k|)J_1(m\sqrt{|t - s|^2 - |x - y_k|^2})}{\sqrt{|t - s|^2 - |x - y_k|^2}}(\xi_{0,k} + s\dot{\xi}_{0,k})ds, \quad t \geq 0.
\]

Therefore, for \( j \neq k \) there exist

\[
\lambda_{j,k}(t) = \lim_{x \to y_j} \psi_{f,j}(x,t) = \psi_{f,j}(y_j,t) = -\frac{\theta(t - |y_j - y_k|)(\xi_{0,k} + (t - |y_j - y_k|)\dot{\xi}_{0,k})}{4\pi|y_j - y_k|} + (\xi_{0,k} + t\dot{\xi}_{0,k})g_{jk}
\]

\[
+ \frac{m}{4\pi} \int_0^t \frac{\theta(t - s - |y_j - y_k|)J_1(m\sqrt{|t - s|^2 - |y_j - y_k|^2})}{\sqrt{|t - s|^2 - |y_j - y_k|^2}}(\xi_{0,k} + s\dot{\xi}_{0,k})ds \in C([0,\infty)).
\]

Moreover, for any \( t > 0 \) there exist (cf. (3.11))

\[
\lambda_{f,j}(t) = \lim_{x \to y_j} \psi_{f,j}(x,t) = -\frac{m(\xi_{0,j} + t\dot{\xi}_{0,j}) - \dot{\xi}_0}{4\pi} + \frac{m}{4\pi} \int_0^t \frac{J_1(m(t - s))}{(t - s)}(\xi_{0,j} + s\dot{\xi}_{0,j})ds.
\]
We set
\[ \lambda_{j,j}(0) = \lim_{t \to 0} \lambda_{j,j}(t) = -\frac{m\zeta_{0j} - \zeta_{0j}}{4\pi}. \] (5.5)
Thus, (3.9)–(5.5) imply that
\[ \lambda_j(t) := \lim_{x \to y_j} \psi_j(x,t) = \lambda_{j,reg}(t) + \sum_{1 \leq k \leq n} \lambda_{k,j}(t) \in C([0,\infty)). \] (5.6)

**Remark 5.1.** Evidently,
\[ \dot{\lambda}_{j,j}(t) \in L_{loc}^2([0,\infty)). \] (5.7)

Nevertheless, \( \dot{\lambda}_{k,j}(t) \not\in L_{loc}^2([0,\infty)) \) for \( k \neq j \), because \( \dot{\lambda}_{k,j}(t) \) contains the term
\[ -\frac{\dot{\theta}(t-|y_j-y_k|)(\zeta_{0k} + (t-|y_j-y_k|)\dot{\zeta}_{0k})}{4\pi|y_j-y_k|} = -\frac{\delta(t-|y_j-y_k|)}{4\pi}. \] (5.8)

### 6 Nonlinear point interaction

First we adjust the nonlinearity \( F \) so that it becomes Lipschitz continuous. Define
\[ \Lambda(\Psi_0) = \sqrt{\|\mathcal{N}_F(\Psi_0) + a\|}/b, \] (6.1)
where \( \Psi_0 \in D_F \) is the initial data from Theorem 2.2 and \( a, b \) are constants from 1.5. Then we may pick a modified potential function \( \tilde{U}(\zeta) \in C^2(\mathbb{C},\mathbb{R}) \), so that

i) the identity holds
\[ \tilde{U}(\zeta) = U(\zeta), \quad |\zeta| \leq \Lambda(\Psi_0), \] (6.2)

ii) \( \tilde{U}(\zeta) \) satisfies 1.5 with the same constant \( a, b \) as \( U(\zeta) \) does:
\[ \tilde{U}(\zeta) - \mathcal{G}(\zeta) \geq b|\zeta|^2 - a, \quad \zeta \in \mathbb{C}, \] (6.3)

iii) the function \( \tilde{F} = \frac{\partial}{\partial \zeta} \tilde{U}(\zeta) \) is Lipschitz continuous:
\[ |\tilde{F}(\zeta_1) - \tilde{F}(\zeta_2)| \leq C|\zeta_1 - \zeta_2|, \quad \zeta_1, \zeta_2 \in \mathbb{C}. \] (6.4)

We suppose that \( \Psi_0 = (\psi_0, \pi_0) \in D_F = D_F \oplus D \), and consider the Cauchy problem for (2.4) with the modified nonlinearity \( \tilde{F} \). As before we denote by \( \psi_j(x,t) \in C([0,\infty), L^2(\mathbb{R}^3)) \) the unique solution to (6.5), and \( \tilde{\lambda}_j(t), t \geq 0 \) is defined by 5.6.

The following lemma is proved by standard argument from the contraction mapping principle.

**Lemma 6.1.** Let conditions (6.2)–(6.4) be satisfies. Then there exists \( \tau > 0 \) such that the Cauchy problem

\[ \frac{1}{4\pi} \tilde{\zeta}_j(t) = \frac{m}{4\pi} \tilde{\zeta}_j(t) + \sum_{k \neq j} \frac{\theta(t-|y_j-y_k|)}{4\pi|y_j-y_k|} \tilde{\zeta}_k(t-|y_j-y_k|) + \lambda_j(t) \]
\[ -\sum_{1 \leq k \leq n} m \int_0^t \frac{\theta(t-s-|y_j-y_k|)J_1(m\sqrt{(t-s)^2-|y_j-y_k|^2})}{\sqrt{(t-s)^2-|y_j-y_k|^2}} \tilde{\zeta}_k(s)ds - F_j(\tilde{\zeta}_j(t)), \quad \tilde{\zeta}_j(0) = \zeta_{0j} \] (6.5)

has a unique solution \( \zeta \in C([0,\tau]) \).

Denote
\[ \psi_j(t,x) := \frac{\theta(t-|x-y_j|)}{4\pi|x-y_j|} \zeta_j(t-|x-y_j|) - \frac{m}{4\pi} \int_0^t \frac{\theta(t-s-|x-y_j|)J_1(m\sqrt{(t-s)^2-|x-y_j|^2})}{\sqrt{(t-s)^2-|x-y_j|^2}} \zeta_j(s)ds, \quad t \in [0,\tau], \]
with \( \zeta_j \) from Lemma 6.1 Now we establish the local well-posedeness for (1.4).
Proposition 6.2. (Local well-posedness). Let the conditions (5.2)–(6.4) hold. Then the function
\[
\psi(x,t) := \psi_f(x,t) + \sum_{1 \leq j \leq n} \psi_j(x,t) \in D_F, \quad t \in [0, \tau]
\]
is a unique strong solution to the Cauchy problem
\[
\dot{\psi}(t) = (\Delta - m^2) \psi(t) + \sum_{1 \leq j \leq n} \zeta_j(t) g_j, \quad \psi(0) = \psi_0, \quad \psi(0) = \pi_0,
\]
(6.6)
\[
\lim_{x \to y_j} (\psi(x,t) - \zeta_j(t) g_j) = F_j(\zeta_j(t)),
\]
(6.7)
and
\[
\psi(t) \in \mathcal{D}, \quad t \in [0, \tau].
\]
Proof. Since \(\zeta_j(t)\) solves (6.5) one has similarly to (3.13)
\[
\lim_{x \to y_j} (\psi(t,x) - \zeta_j(t) g_j(x)) = \lambda_j(t) + \lim_{x \to y_j} \left( \sum_{1 \leq k \leq n} \frac{\theta(t-|x-y_k|) \zeta_k(t-|x-y_k|)}{4\pi|x-y_k|} - \frac{\zeta_j(t)e^{-m|x-y_j|}}{4\pi|x-y_j|} \right)
\]
\[
- \lim_{x \to y_j} \sum_{1 \leq k \leq n} \frac{m}{4\pi} \int_0^t \frac{\theta(t-s-|x-y_k|)J_1(m\sqrt{(t-s)^2-|x-y_k|^2})}{\sqrt{(t-s)^2-|x-y_k|^2}} \zeta_k(s)ds
\]
\[
= \lambda_j(t) - \frac{1}{4\pi} \sum_{1 \leq k \leq n} \frac{m}{4\pi} \int_0^t \frac{\theta(t-s-|y-y_k|)J_1(m\sqrt{(t-s)^2-|y-y_k|^2})}{\sqrt{(t-s)^2-|y-y_k|^2}} \zeta_k(s)ds
\]
(6.8)
and hence (6.7) are satisfied. Further,
\[
\dot{\psi} = \dot{\psi}_f + \sum_{1 \leq j \leq n} \dot{\psi}_j = (\Delta - m^2) \psi_f + \sum_{1 \leq j \leq n} (\Delta - m^2) \psi_j + \sum_{1 \leq j \leq n} \zeta_j \delta(-y_j) = (\Delta - m^2) \psi + \sum_{1 \leq j \leq n} \zeta_j \delta(-y_j)
\]
and \(\psi\) solves (6.6) then. Finally, let us prove that
\[
(\psi_{reg}(t), \psi_{reg}(t)) \in \mathcal{X}, \quad t \in [0, \tau].
\]
(6.9)
The function \(\psi_{reg}(x,t) = \psi(x,t) - \sum_{1 \leq j \leq n} \zeta_j(t) g_j(x)\) is a solution to
\[
\psi_{reg}(x,t) = (\Delta - m^2) \psi_{reg}(x,t) - \sum_{1 \leq j \leq n} \zeta_j(t) g_j(x)
\]
with regular initial data \((\psi_{0,reg}, \pi_{0,reg}) \in \mathcal{X}\). Due to (5.2), (6.6), and (5.7) the derivative with respect to \(t\) of the RHS of (6.5) belong to \(L^2([0, \tau])\), since the terms with \(\delta\)-functions cancel each other. Hence, \(\dot{\psi} \in L^2([0, \tau])\), and (6.5) holds by Lemma 4.1.

Suppose now that \(\dot{\psi} = \psi_{reg} + \sum_{1 \leq j \leq n} \bar{\zeta}_j g_j\) is another strong solution of (6.6). Then, by reversing the above argument, the boundary conditions (6.7) imply that \(\bar{\zeta}_j\) solves the Cauchy problem (6.5). The uniqueness of the solution of (6.5) implies that \(\bar{\zeta}_j = \zeta_j\). Then, defining
\[
\psi_f(t,x) := \frac{\theta(t-|x-y_j|)}{4\pi|x-y_j|} \zeta_j(t-|x-y_j|) - \frac{m}{4\pi} \int_0^t \frac{\theta(t-s-|x-y_j|)J_1(m\sqrt{(t-s)^2-|x-y_j|^2})}{\sqrt{(t-s)^2-|x-y_j|^2}} \zeta_j(s)ds, \quad t \in [0, \tau],
\]
for \(\dot{\psi}_f = \dot{\psi} - \sum_{1 \leq j \leq n} \psi_f(t,x)\) one obtains
\[
\dot{\psi}_f = (\Delta - m^2) \psi_{reg} - \sum_{1 \leq j \leq n} ((\Delta - m^2) \psi_j + \zeta_j \delta(-y_j)) = (\Delta - m^2) (\psi_{reg} - \sum_{1 \leq j \leq n} (\psi_j - \zeta_j g_j)) = (\Delta - m^2) \dot{\psi}_f,
\]
i.e. \(\dot{\psi}_f\) solves the Cauchy problem (3.5). Thus, by the uniqueness of the solution (3.5), \(\dot{\psi}_f = \psi_f\) and then \(\dot{\psi} = \psi\). 

\[\square\]
Lemma 6.3. Let conditions (6.2)–(6.4) hold, and let $\Psi(t) = (\psi(t), \dot{\psi}(t)) \in \mathcal{D}_F$, $t \in [0,\tau]$, be a solution to (6.6)–(6.7). Then
\[
\mathcal{H}_F(\Psi(t)) = \|\Psi(t)\|^2 + \|\dot{\psi}_{\text{reg}}(t)\|^2 + a\|\psi_{\text{reg}}(t)\|^2 + \tilde{U}(\zeta(t)) - \mathcal{G}(\zeta(t)) = \text{const}, \quad t \in [0,\tau].
\] (6.10)

Proof. Definition (2.3) of operator $H_F$ implies
\[
\frac{d}{dt}\|\Psi(t)\|^2 = \langle H_F \psi, \dot{\psi} \rangle + \langle \dot{\psi}, H_F \psi \rangle = \langle (\Delta - m^2) \psi_{\text{reg}}, \dot{\psi}_{\text{reg}} \rangle + \sum_{1 \leq j \leq n} \hat{\psi}_j(\xi_j) + \sum_{1 \leq j \leq n} \tilde{\psi}_j(\xi_j, \Delta - m^2) \psi_{\text{reg}}
\]
\[
= \langle (\Delta - m^2) \psi_{\text{reg}}, \dot{\psi}_{\text{reg}} \rangle + \langle \dot{\psi}_{\text{reg}}, (\Delta - m^2) \psi_{\text{reg}} \rangle - \sum_{1 \leq j \leq n} \hat{\psi}_j \psi_{\text{reg}}(\xi_j) - \sum_{1 \leq j \leq n} \tilde{\psi}_j \psi_{\text{reg}}(\xi_j)
\]
\[
= \frac{d}{dt} \left( -\|\dot{\psi}_{\text{reg}}\|^2 - m^2 \|\psi_{\text{reg}}\|^2 \right) - \sum_{1 \leq j \leq n} \hat{\psi}_j \dot{F}_j(\xi_j) - \sum_{1 \leq j \leq n} \tilde{\psi}_j \dot{F}_j(\xi_j) + \sum_{j \neq k} g_{jk}(\xi_j \xi_k + \tilde{\psi}_j \tilde{\psi}_k)
\]
\[
= \frac{d}{dt} \left( -\|\dot{\psi}_{\text{reg}}\|^2 - m^2 \|\psi_{\text{reg}}\|^2 - \tilde{U}(\zeta) + \mathcal{G}(\zeta) \right).
\]

Then (6.10) follows. \[\square\]

Corollary 6.4. The following identity holds
\[
\tilde{U}(\zeta(t)) = U(\zeta(t)), \quad t \in [0,\tau].
\] (6.11)

Proof. First note that
\[
\mathcal{H}_F(\Psi_0) \geq U(\zeta_0) - \mathcal{G}(\zeta_0) \geq b|\zeta_0|^2 - a.
\]
Therefore, $|\zeta_0| \leq \Lambda(\Psi_0)$ and then $\tilde{U}(\zeta_0) = U(\zeta_0), \mathcal{H}_F(\Psi_0) = \mathcal{H}_F(\Psi_0)$. Further,
\[
\mathcal{H}_F(\Psi(t)) \geq \tilde{U}(\zeta(t)) - \mathcal{G}(\zeta(t)) \geq b|\zeta(t)|^2 - a, \quad t \in [0,\tau].
\]

Hence (6.10) implies that
\[
|\zeta(t)| \leq \sqrt{(\mathcal{H}_F(\Psi(t)) + a)/b} = \sqrt{(\mathcal{H}_F(\Psi_0) + a)/b} = \sqrt{(\mathcal{H}_F(\Psi_0) + a)/b} = \Lambda(\Psi_0), \quad t \in [0,\tau].
\] (6.12)

From the identity (6.11) it follows that we can replace $\tilde{F}$ by $F$ in Proposition 6.2 and in Lemma 6.3.

Proof of Theorem 2.2. The solution $\Psi(t) = (\psi(t), \dot{\psi}(t)) \in \mathcal{D}_F$ constructed in Proposition 6.2 exists for $0 \leq t \leq \tau$, where the time span $\tau$ in Lemma 6.1 depends only on $\Lambda(\Psi_0)$. Hence, the bound (6.12) at $t = \tau$ allows us to extend the solution $\Psi$ to the time interval $[\tau, 2\tau]$. We proceed by induction to obtain the solution for all $t \geq 0$.

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