Natural Cutoffs and Quantum Tunneling from Black Hole Horizon

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Abstract.
We study tunneling of massless particles through quantum horizon of a Schwarzschild black hole where quantum gravity effects are taken into account. These effects are encoded in the existence of natural cutoffs as a minimal length, a minimal momentum and a maximal momentum through a generalized uncertainty principle. We study possible correlations between emitted particles to address the information loss problem. We focus also on the role played by these natural cutoffs on the tunneling rate through quantum horizon.

Key Words: Quantum Tunneling, Hawking Radiation, Black Hole Entropy, Information Loss Paradox, Generalized Uncertainty Principle

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1. Introduction

Incorporation of gravity in quantum field theory leads naturally to an effective cutoff; a minimal measurable length in the ultraviolet regime. In fact, the high energies used to probe small distances significantly disturb the spacetime structure by their powerful gravitational effects. Some approaches to quantum gravity proposal such as string theory \[1\]-\[7\], loop quantum gravity \[8\] and quantum geometry \[9\] all indicate the existence of a minimal measurable length of the order of the Planck length, \(\ell_p \sim 10^{-35}\text{ m}\) (see also \[10\]-\[13\]). Moreover, some Gedanken experiments in the spirit of black hole physics have also supported the idea of existence of a minimal measurable length \[14\]. So, the existence of a minimal observable length is a common feature of all promising quantum gravity candidates. The existence of a minimal measurable length modifies the Heisenberg uncertainty principle (HUP) to the so called Generalized (Gravitational) uncertainty principle (GUP) represented for instance as

\[
\Delta x \Delta p \geq \hbar \left[ 1 + \alpha^2 \ell_p^2 (\Delta p)^2 \right].
\]

In the HUP framework there is essentially no restriction on the measurement precision of particles’ position, so that \(\Delta x\) as the minimal position uncertainty could be made arbitrarily small toward zero. But, this is not essentially the case in the GUP framework due to existence of a minimal uncertainty in position measurement. In this respect, there is a finite resolution of spacetime points in quantum gravity regime.

On the other hand, Doubly Special Relativity theories (for review see for instance \[15\]) suggest that existence of a minimal measurable length would restrict a test particle’s momentum uncertainty to take any arbitrary values leading nontrivially to an upper bound, \(P_{\text{max}}\), on a test particle’s momentum. This means that there is a maximal particle’s momentum due to the fundamental structure of spacetime at the Planck scale \[16\]-\[19\]. Such a smallest length and maximal momentum cause an alteration of the position-momentum uncertainty relation in such a way that

\[
\Delta x \Delta p \geq \hbar \left[ 1 - \alpha \ell_p (\Delta p) + \alpha^2 \ell_p^2 (\Delta p)^2 \right]
\]

where \(\alpha\) is a dimensionless constant of the order of unity that depends on the details of the quantum gravity hypothesis. In the standard limit, \(\Delta x \gg \ell_p\), it yields the standard Heisenberg uncertainty principle, \(\Delta x \Delta p \geq \hbar\).

It has been shown that for a spherical black hole the thermodynamic quantities can be obtained by using the standard uncertainty principle \[20\]. In the same manner, incorporation of quantum gravity effects in black hole physics and thermodynamics through a GUP with the mentioned natural cutoffs brings several new and interesting implications and modifies the result dramatically. Specially, the final stages of black hole evaporation obtains a very rich phenomenology in this framework. Application of GUP with minimal length to black hole physics has been widely studied in recent years (see for instance \[21\]-\[28\]). In an elegant work, Parikh and Wilczek constructed a procedure to describe the Hawking radiation emitted from a Schwarzschild black hole.
as a tunneling through its quantum horizon \[29\]. This procedure provides a leading correction to the tunneling probability (emission rate) arising from the reduction of the black hole mass because of the energy carried by the emitted quantum. However, because of the lack of correlation between different emitted modes in the black hole radiation spectrum, the form of the correction is not adequate by itself to recover information. The Parikh-Wilczek tunneling in the presence of a minimal measurable length and possible resolution of the information loss problem in this framework has been studied recently in \[24\] (see also \[25\] for treating the problem in noncommutative framework). Here we generalize this work in two steps: firstly we consider a GUP that admits both a minimal measurable length and a maximal momentum (see \[30\] and references therein) and secondly the case that there are a minimal length, a minimal momentum and a maximal momentum (see \[30\] and also \[31\]). The later case is the most general treatment of the problem with incorporation of all natural cutoffs. The crucial difference of our study in this paper with our previous study reported in \[24\] is in the fact that \[24\] ignores the existence of a minimal and maximal momentum for emitted particles. As we shall see, the minimal and maximal momentum arise respectively by addition of terms proportional to \((\Delta x)^2\) and \(\Delta p\) in the right hand side of relation (1). Existence of a minimal and a maximal momentum for emitted particles have important implications on the tunneling rate of particles through quantum horizon. We show that in this framework, correlations between different modes of radiation evolve, which reflects the fact that information emerges continuously during the evaporation process at the quantum gravity level. This feature has the potential to answer some questions regarding the black hole information loss problem and provides a more realistic background for treating the black hole evaporation in its final stage of evaporation.

With these motivations, we apply back-reaction and quantum gravity effects to study black hole evaporation process and thermodynamics. We use suitable GUPs in each case to drive a modified black hole temperature. Then, by using the first law of black hole thermodynamics, we obtain modified black hole entropy. We show that in the presence of new natural cutoffs as minimal and maximal momentum, in the final stage of the black hole evaporation, when the black hole mass is of the order of the Planck mass, the temperature is more than the cases that we consider just the minimal length or minimal length-maximal momentum schemes. We also show that the GUP prevents black holes to evaporate totally, resulting a stable remnant.

2. Minimal Length, Maximal Momentum and the Parikh-Wilczek Tunneling Mechanism

The first step to discuss the quantum tunneling through the black hole horizon is to find a proper coordinate system for the black hole metric where all the constant lines are flat, and the tunneling path is free of singularities. Painlevé coordinates are suitable
choices in this respect. In these coordinates, the schwarzschild metric is given by
\[ ds^2 = -(1 - \frac{2m}{r})dt^2 + \left(2\sqrt{\frac{2m}{r}}\right)dt\,dr + dr^2 + r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right), \] (2)
which is stationary, non-static, and non-singular at the horizon. The radial null geodesics are given by
\[ \dot{r} \equiv \frac{dr}{dt} = \pm 1 - \sqrt{\frac{2M}{r}}, \] (3)
where the plus (minus) sign corresponds to outgoing (ingoing) geodesics respectively. Now we incorporate quantum gravity effects encoded in the presence of the minimal length and maximal momentum via the generalized uncertainty principle (1) which motivates modification of the standard dispersion relation. If the GUP is a fundamental outcome of quantum gravity proposal, it should appear in the de Broglie relation as follows (with \( \hbar = 1 \))
\[ \lambda \simeq \frac{1}{p}\left(1 - \alpha \ell_p p + \alpha^2 \ell_p^2 p^2\right), \] (4)
or equivalently
\[ E \simeq E\left(1 - \alpha \ell_p E + \alpha^2 \ell_p^2 E^2\right). \] (5)

With these preliminaries, we consider a massless particle ejected from a black hole in the form of a massless shell, and take into consideration the response of background geometry to radiated quantum of energy \( E \) with GUP correction, i.e. \( \mathcal{E} \). The particle moves on the geodesics of a spacetime with central mass \( M - E \) substituted for \( M \). The description of the motion of particles in the s-wave as spherical massless shells in a dynamical geometry and the analysis of self-gravitating shells in Hamiltonian gravity have been reported in Refs. [35, 36]. If one assumes that time increases in the direction of the future, then the metric should be modified due to back-reaction effects. We set the total ADM mass, \( M \), of the spacetime to be fixed but allow the hole mass to fluctuate and replace \( M \) by \( M - \mathcal{E} \) both in the metric and the geodesic equation. Since the characteristic wavelength of the radiation is always arbitrarily small near the horizon due to the infinite blue-shift, the wavenumber approaches infinity. Therefore the WKB approximation is valid in the vicinity of the horizon. In the WKB approximation, the tunneling probability for the classically inhibited area as a function of the imaginary part of the particle action at the stationary phase takes the form
\[ \Gamma \sim \exp(-2\text{Im}\,\mathcal{I}) \approx \exp(-\beta E). \] (6)
As we will see later, to first order in \( E \) the right hand side of this expression substitutes the Boltzman factor in the canonical ensemble characterized by the inverse temperature \( \beta \). In the s-wave picture, particles are spherical massless shells traveling on the radial null geodesics and transfer across the horizon as outgoing positive-energy particles from \( r_{in} \) to \( r_{out} \). The imaginary part of the action is then given by
\[ \text{Im}\,\mathcal{I} = \text{Im}\int_{r_{in}}^{r_{out}} p_r\,dr = \text{Im}\int_{r_{in}}^{r_{out}} \int_0^{p_r} dp_r\,dr \] (7)
As is clear from the GUP expression, the commutation relation between the radial coordinate and the conjugate momentum should be modified as follows
\[
[r, p_r] = i \left(1 - \alpha \ell_p p_r + \alpha^2 \ell_p^2 p_r^2\right).
\] (8)

In the classical limit it is replaced by the following poisson bracket
\[
\{r, p_r\} = 1 - \alpha \ell_p p_r + \alpha^2 \ell_p^2 p_r^2.
\] (9)

Now, using the deformed Hamiltonian equation,
\[
\dot{r} = \{r, H\} = \{r, p_r\} \frac{dH}{dr} \bigg|_{r},
\] (10)
since the Hamiltonian is
\[
H = M - E',
\]
we assume
\[
p^2 \simeq E'^2, \quad p \simeq E'
\]
and eliminate the momentum in the favor of the energy in integral (7)
\[
\text{Im} \mathcal{I} = \text{Im} \int_{0}^{\mathcal{E}} \int_{r_{\text{in}}}^{r_{\text{out}}} \left(1 - \alpha \ell_p E' + \alpha^2 \ell_p^2 E'^2\right) dE' \, dr \left( - dE' \right).
\] (11)

The \( r \) integral can be performed by deforming the contour around the pole at the horizon, where it lies along the line of integration and gives \((-\pi i)\) times the residue
\[
\text{Im} \mathcal{I} = \text{Im} \int_{0}^{\mathcal{E}} 4(-\pi i) \left(1 - \alpha \ell_p E' + \alpha^2 \ell_p^2 E'^2\right) (M - E')( - dE').
\] (12)

This allows us to consider the leading-order correction to be just proportional to \((\alpha^2 \ell_p^2)\) for simplicity and without loss of generality. Now the imaginary part of the action takes the following form
\[
\text{Im} \mathcal{I} = 4\pi ME - 2\pi E^2 \left(3M \alpha \ell_p + 1\right) + 4\pi E^3 \left(\frac{7}{3}M \alpha^2 \ell_p^2 + \frac{4}{3} \alpha \ell_p\right) - 10\pi \alpha^2 \ell_p^2 E^4 + \mathcal{O}\left(\alpha^2 \ell_p^4\right).
\] (13)

The tunneling rate is therefore
\[
\Gamma \sim \exp \left[ -8\pi ME + 4\pi E^2 \left(3M \alpha \ell_p + 1\right) - 8\pi E^3 \left(\frac{7}{3}M \alpha^2 \ell_p^2 + \frac{4}{3} \alpha \ell_p\right) + 20\pi \alpha^2 \ell_p^2 E^4 + \mathcal{O}\left(\alpha^2 \ell_p^4\right) \right] = \exp(\Delta S),
\] (14)

where \(\Delta S\) is the difference in black hole entropies before and after emission \cite{33, 36, 37}. In the string theory, it is anticipated that the tunneling rates from excited \(D\)-branes in the microcanonical ensemble depends on the final and initial number of microstates available to the system. In a more precise expression, it was shown that the emission rates on the high-energy scales corresponds to differences between the counting of states in the microcanonical and in the canonical ensembles \cite{36}. The first term in the exponential gives a thermal, Boltzmannian spectrum. However, existence of extra terms shows that the radiation is not completely thermal. We note that the anticipation of the effect of a natural cutoff as maximal momentum for emitted particle leads extra terms in comparison to the results of \cite{24}. These extra terms enhance the non-thermal behavior of radiation in our setup. In other words, existence of a maximal momentum
enhances the non-thermal character of the radiation. As equation (14) shows, only to first order of $E$ the tunneling rate results in the Boltzmann factor in the canonical ensemble characterized by the inverse temperature $\beta$. So, we see that in the presence of quantum gravity effects, the emission spectrum cannot be strictly thermal. On the other hand, if we take the limit of equation (13) in which back reaction can be neglected ($\frac{E}{M} \ll 1$), we find

$$\text{Im} I = 4\pi ME - 6\pi \alpha\ell p ME^2 + \frac{28}{3}\pi\alpha^2\ell_p^2 ME^3.$$

This result shows the pure quantum gravity effects in the absence of back reaction.

We note that with a GUP that admits just a minimal length, the emission spectrum takes the following form [24]

$$\Gamma \sim \exp\left[-8\pi ME + 4\pi E^2 - 2\pi\alpha\ell_p^2 E^3\left(\frac{16}{3}M - 5E\right) + O\left(\alpha^2\ell_p^4\right)\right].$$ (15)

In comparison between this tunneling rate and the one that we obtained in this paper as equation (14), it is obvious that when we consider both maximal momentum and minimal length in the GUP relation, an additional term of $3M\alpha\ell_p$ appears in the term that is second order in $E$, which depends on the GUP through $(\alpha\ell_p)$. This reflects the fact that in the presence of the maximal momentum, the tunneling rate deviates from ordinary result more considerably than the case that we consider just the effect of minimal length. As we know, in the original Parikh-Wilczek tunneling framework, these additional terms are not included since they have not considered quantum gravitational effects. Since the emitted particle cannot carry an arbitrary amount of momentum, (in accordance with Doubly Special Relativity which predicts the existence of maximal momentum), maximal energy of emitted particles should be of the order of the Planck energy. Also there are other additional terms depending on the GUP parameter in second order that can not be neglected once the black hole mass becomes comparable to the Planck mass.

We now illustrate that the emission rates for the different modes of radiation during the evaporation are mutually correlated from a statistical viewpoint. It means that, the total tunneling probabilities of two particles with energies $E_1$ and $E_2$ are not similar to tunneling probability of a particle with the total energy $E_1 + E_2$, since there are some correlations between them. Utilizing (14), the emission rate for a first quantum with energy $E_1$, gives

$$\ln \Gamma_{E_1} = -8\pi ME_1 + 4\pi E_1^2(3M\alpha\ell_p + 1) - 8\pi E_1^3\left(\frac{7}{3}M\alpha^2\ell_p^2 + \frac{4}{3}\alpha\ell_p\right) + 20\pi\alpha^2\ell_p^3 E_1^4.$$ (16)

Similarly, the emission rate for a single quantum $E_2$, takes the form

$$\ln \Gamma_{E_2} = -8\pi(M - E_1)E_2 + 4\pi E_2^2\left(3(M - E_1)\alpha\ell_p + 1\right)$$

$$- 8\pi E_2^3\left(\frac{7}{3}(M - E_1)\alpha^2\ell_p^2 + \frac{4}{3}\alpha\ell_p\right) + 20\pi\alpha^2\ell_p^3 E_2^4.$$ (17)
Correspondingly, the emission rate for a single quantum with total energy, \( E = E_1 + E_2 \), yields
\[
\ln \Gamma_{(E_1 + E_2)} = -8\pi M(E_1 + E_2) + 4\pi(E_1 + E_2)^2(3M\alpha\ell_p + 1) \\
- 8\pi(E_1 + E_2)^3\left(\frac{7}{3}M\alpha^2\ell_p^2 + \frac{4}{3}\alpha\ell_p\right) + 20\pi\alpha^2\ell_p^2(E_1 + E_2)^4.
\]

(18)

It can be proved that these probabilities are actually correlated. The non-zero statistical correlation function is
\[
\chi(E_1 + E_2; E_1, E_2) = 8\pi E_1 E_2 \left(1 + 12M\alpha\ell_p\right) \\
- 8\pi E_1 E_2^2 \left(10\alpha\ell_p + 7M\alpha^2\ell_p^2\right) - 8\pi E_2 E_1^2 \left(7M\alpha^2\ell_p^2 + 4\alpha\ell_p\right).
\]

(19)

As we see the statistical correlation function has the terms depending on the \( M \) and \( \ell_p \). Also in comparison with the statistical correlation function obtained in Ref. [24], the correlation in our case is enhanced due to incorporation of maximal momentum for emitted particles. In fact, whenever one quantum of emission is radiated from the surface of the black hole horizon, the aberrations are created on the Planck scale that influence the second quantum of emission, and these aberrations cannot be neglected, particularly once the black hole mass becomes comparable with the Planck mass. Therefore as is mentioned in Ref. [24], in this way the form of modifications as back reaction effects by incorporation of GUP with minimal length and maximal momentum are sufficient by themselves to recover information. Information leaks out from the black holes as the non-thermal GUP correlations within the Hawking radiation.

3. Thermodynamics

In this section we study thermodynamics of an evaporating black hole in the presence of minimal length and maximal momentum. We mainly focus on the effect of maximal particles’ momentum in thermodynamical quantities. In the standard viewpoint of black hole thermodynamics, the temperature and entropy can be obtained easily by using the standard uncertainty principle [20]. Hawking considers a vacuum quantum state near the horizon and predicts that there is a fluctuating sea of virtual pair particles. For instance, a photon with energy \(-E\) is absorbed by the black hole and a photon with energy \(+E\) is emitted to infinity. The energy of the emitted particle can be obtained by using the uncertainty in momentum as follows
\[
\Delta x \Delta p \sim \hbar. \quad \Delta x \simeq r_s = \frac{2GM}{c^2}.
\]

(20)

The uncertainty in the energy of the emitted particle can be obtained by using the uncertainty in momentum as follows
\[
\Delta E = c\Delta p \approx \frac{\hbar c^3}{2GM}.
\]

(21)

This is the energy of the emitted particle. With calibration factor \( 4\pi \), and setting \( k_B = 1 \), the temperature takes the following form
\[
T_H = \frac{c\Delta p}{4\pi} = \frac{\hbar c^3}{8\pi GM}.
\]

(22)
As we see, in this approach there is no limit on lowering black hole mass in the final stage of evaporation. So, in the framework of Heisenberg uncertainty principle the black hole evaporates totally in the final stage of its lifetime. According to (22), when the mass of black hole approaches zero, its temperature becomes infinite. The entropy of black hole in this case is given by

$$S_B = \frac{4\pi GM^2}{\hbar c} = \frac{A}{4\ell_p^2}. \quad (23)$$

Now we extend this argument to the case that quantum gravity effects which are encoded in the GUP are taken into account. Starting with

$$\Delta x \Delta p \geq \hbar \left(1 - \alpha \ell_p \Delta p + \alpha^2 \ell_p^2 (\Delta P)^2\right) \quad (24)$$

we solve this equation for the momentum uncertainty in terms of position uncertainty, which we again takes to be the Schwarzschild radius $r_s$. This gives the following momentum and temperature for radiated photons in the presence of maximal particle’s momentum and minimal length

$$\Delta p = \left(\frac{2M}{M_p^2} + \alpha \ell_p c\right) \left(1 \pm \sqrt{1 - \frac{4\alpha^2 \ell_p^2}{\left(\frac{2M}{M_p^2} + \alpha \ell_p\right)^2}}\right) \quad (25)$$

where we have used $G = \frac{\hbar c}{M_p^2}$. So we find

$$T_{(GUP)} = \frac{c \Delta p}{4\pi} = \left(\frac{2M}{M_p^2} + \alpha \ell_p c\right) \left(1 \pm \sqrt{1 - \frac{4\alpha^2 \ell_p^2}{\left(\frac{2M}{M_p^2} + \alpha \ell_p\right)^2}}\right), \quad (26)$$

where we have again used the calibration factor $4\pi$. This result agrees with the standard result for large masses if the negative sign is chosen. Whereas the positive sign has no evident physical meaning. This temperature as a function of mass is shown in Fig. 1.

The entropy of black hole is obtained by integration of $dS = \frac{c^2 dM}{4\pi}$. In this way we obtain the following GUP-corrected entropy

$$S_{(GUP)} = \frac{2\pi M^2}{M_p^2} + 2\pi \alpha \ell_p M + \pi \sqrt{\frac{4M^2}{M_p^4} + \frac{4M \alpha \ell_p}{M_p^2} - 3\alpha^2 \ell_p^2 M}$$

$$+ 8\pi \alpha^2 \ell_p^2 M_p^2 \left[\sqrt{\frac{4M^2}{M_p^4} + \frac{4M \alpha \ell_p}{M_p^2} - 3\alpha^2 \ell_p^2} - \frac{M_p^2}{2} \left(\frac{2M}{M_p^2} + \frac{4M}{M_p^4}\right) + \sqrt{\frac{4M^2}{M_p^4} + \frac{4M \alpha \ell_p}{M_p^2} - 3\alpha^2 \ell_p^2}\right]. \quad (27)$$

Note that we have normalized the modified entropy to zero at $M_p$ as shown in Fig. (2).

This normalization is viable since GUP prevents total evaporation of black hole (see for instance [21, 22]). In this picture, by incorporating maximal momentum and minimal length, we obtained a modified temperature and entropy. In fact in this approach, a black hole should radiate until it approaches the Planck mass and size. At the Planck mass it ceases to radiate and its entropy vanishes, while its effective temperature reaches a maximum value. In this phase it cannot radiate further and becomes an inert remanent. We note that the maximum temperature at the final stage of evaporation when we consider both maximal momentum and minimal length is smaller than the case that we consider just the minimal length hypothesis. We note also that as usual the leading order correction term to the black hole entropy has a logarithmic nature which is consistent with loop quantum gravity considerations (see for instance [35, 36]).
Figure 1. Temperature of black hole versus its mass. Mass is in units of the Planck mass and temperature is in units of Planck energy. The upper curve is temperature that is obtained by using a GUP admitting just a minimal length and the lower one is obtained by using a GUP that admits both a minimal length and a maximal momentum.

Figure 2. Entropy of black hole versus its mass. Mass is in the units of the Planck mass and entropy is in units of the Planck energy. The lower curve is obtained by using a GUP admitting just a minimal length but the upper curve is obtained by using a GUP that admits both a minimal length and a maximal momentum.
4. A more general framework

In Ref. [31] the authors have shown that particles’ momentum cannot be zero if curvature effects are taken into account. In fact, minimal length and minimal momentum are respectively ultra-violet and infra-red cutoffs of quantum field theory with gravitational effect. We note that a framework with a finite minimal uncertainty $\Delta x_0$ can be understood to describe effectively nonpointlike particles in a fuzzy space. In the same manner, on large scales a minimal uncertainty $\Delta p_0$ may offer new possibilities to describe situations where momentum cannot be precisely determined, in particular on curved space. Using the path integral formulation it has been shown in [31] that such noncommutative background geometries can ultraviolet and infrared regularise quantum field theories in arbitrary dimensions through minimal uncertainties $\Delta x_0$ and $\Delta p_0$ (for more details, see [31]. Taking these facts into account, now we can write a GUP that admits a minimal length, a minimal momentum and a maximal momentum. This form of GUP is the most general form that contains all possible natural cutoffs. This GUP can be represented as (see [31,32,33,34])

$$\Delta x \Delta p \geq \hbar \left( 1 - \alpha \ell_p \Delta p + \alpha^2 \ell_p^2 (\Delta p)^2 + \beta^2 \ell_p^2 (\Delta x)^2 \right)$$

(28)

This relation has a minimal length which is $(\Delta x)_{min} = \alpha \ell_p$, a minimal momentum that is $(\Delta p)_{min} = 2 \beta \ell_p$ and also a maximal momentum as $(\Delta p)_{max} = \frac{1}{\alpha \ell_p} \left( 1 + \sqrt{1 - (1 + \beta^2 \alpha^2 \ell_p^4)} \right)$.

Now we want to use this more general GUP to find the tunneling rate of emitted particles through quantum horizon of black hole like the way we used in previous sections. By using the Parikh-Wilczek quantum tunneling mechanism, and quantum gravity effects, we want to calculate the tunneling rate of emitted particles from Schwarzschild black hole by incorporation of the above GUP. First, we note that with this new GUP, the de Broglie relation is represented as

$$\lambda_{\pm} = \frac{p}{2 \beta^2 \ell_p^2} \left( 1 \pm \sqrt{1 - \frac{4 \beta^2 \ell_p^2 (1 - \alpha \ell_p p + \alpha^2 \ell_p^2 p^2)}{p^2}} \right).$$

It is easy to see that positive sign does not recover ordinary relation in the limit $\alpha \to 0$ and $\beta \to 0$. So, after expansion of square root we consider the minus sign as

$$\lambda_- = \frac{1}{p} \left( 1 - \alpha \ell_p p + \alpha^2 \ell_p^2 p^2 \right) \left( 1 + \frac{\beta^2 \ell_p}{p^2} \left( 1 - \alpha \ell_p p + \alpha^2 \ell_p^2 p^2 \right) \right),$$

(29)

or equivalently

$$\mathcal{E} = E \left( 1 - \alpha \ell_p E + \alpha^2 \ell_p^2 E^2 \right) \left( 1 + \frac{\beta^2 \ell_p}{E^2} \left( 1 - \alpha \ell_p E + \alpha^2 \ell_p^2 E^2 \right) \right).$$

(30)

Also as it is clear from the above GUP that the commutation relation between the radial coordinate and its conjugate momentum should be modified as

$$[r, p_r] = i \left( 1 - \alpha \ell_p p + \alpha^2 \ell_p^2 p^2 + \beta^2 \ell_p^2 r^2 \right)$$

(31)

So, in the classical limit, it is replaced by the following poisson bracket

$$\{r, p_r\} = \left( 1 - \alpha \ell_p p + \alpha^2 \ell_p^2 p^2 + \beta^2 \ell_p^2 r^2 \right).$$

(32)

With these relations and by using the Parikh-Wilczek method, the imaginary part of the action takes the following form

$$\text{Im} \mathcal{I} = \frac{4 \pi M \beta^2 \ell_p^2}{E} - 4 \pi \beta^2 \ell_p^2 - 8 \pi M \alpha \beta^2 \ell_p^3 - 4 \pi \beta^2 \ell_p^2 M^4 + 4 \pi E \left( M - M \alpha \ell_p + 4 \alpha \beta^2 \ell_p^3 \right)$$
Now the tunneling rate can be calculated by using relation (6). It is easy to see that by incorporation of these natural cutoffs, there are more new terms with origin on quantum gravity effects than previous cases, even up to the first order of $E$. These extra terms enhance the non-thermal character of the radiation. In comparison between this tunneling rate and the one that we obtained for instance in section 2 and also the tunneling rate which is calculated in [24], it is obvious that by considering all natural cutoffs in GUP relation, many additional terms are appeared that shows strong deviation from ordinary thermal radiation. Then, we can calculate the non-zero statistical correlation function $\chi(E_1 + E_2; E_1, E_2)$ to see that there are more correlation terms in this case and therefore more possibility to recover information as non-thermal correlations.

Now we study the thermodynamics of an evaporating black hole in the presence of minimal length, minimal momentum and also maximal momentum that are encoded in the GUP as non-thermal correlations.

\[
\Delta p = \frac{1}{2 \alpha^2\ell_p^2} \left( (\Delta x + \alpha \ell_p) \pm \sqrt{(\Delta x + \alpha \ell_p)^2 - 4 \alpha^2 \ell_p^2 (1 + \beta^2 \ell_p^2 (\Delta x)^2)} \right) \tag{34}
\]

Since
\[
T = \frac{c \Delta p}{4 \pi}
\]
we find
\[
T = \frac{c}{8 \pi \alpha^2 \ell_p^2} \left( \frac{2GM}{c^2} + \alpha \ell_p \right) \pm \sqrt{\left( \frac{2GM}{c^2} + \alpha \ell_p \right)^2 - 4 \alpha^2 \ell_p^2 (1 + \beta^2 \ell_p^2 G^2 M^2 / c^4)} \tag{35}
\]

The negative sign agrees with the standard result for large masses. The entropy of black hole is derived by integration of $dS = \frac{c^2 dM}{T}$, that is,

\[
S = \frac{\pi}{2} M^2 G \frac{c}{G} + \frac{\pi}{2} M c \alpha \ell_P + \pi c \sqrt{\frac{4M^2 G^2}{c^4} + \frac{4M \alpha \ell_P G}{c^2}} - 3M \alpha^2 \ell_p^2
\]

\[
+ \frac{\pi}{2} G \alpha \ell_P \sqrt{\frac{4M^2 G^2}{c^4} + \frac{4M \alpha \ell_P G}{c^2}} - 3\alpha^2 \ell_p^2
\]

\[
- \frac{\pi}{2} \frac{\alpha^2 \ell_p^2 \ell_P^3}{G} \ln \left( \alpha \ell_P + \frac{2MG}{c^2} \right) + \sqrt{\frac{4M^2 G^2}{c^4} + \frac{4M \alpha \ell_P G}{c^2}} - 3\alpha^2 \ell_p^2 \tag{36}
\]

We have normalized this modified entropy to zero at $M_P$. In this way, black hole radiates until it approaches the Planck mass. At this stage it ceases to radiate and its entropy vanishes leading to an stable remnant. In Fig. 3 we have compared entropies calculated in three different situations: the upper curve is for the case that the applied GUP admits a minimal length and a maximal momentum. The middle curve is obtained by a GUP that admits just a minimal length, and finally the lower curve contains the effects of all natural cutoffs as
minimal length, minimal momentum and maximal momentum. Since $\frac{dS}{dM} = \frac{c^2}{T}$, we see from slopes of these curves that in the presence of all natural cutoffs, temperature of black hole for a given mass increases in comparison with the cases that some of these cutoffs are not taken into account.

![Figure 3. Entropy of black hole versus its mass for three different GUPs. Mass is in the units of Planck mass and entropy is in the units of Planck energy. The upper curve is obtained by using a GUP that admits a minimal length and a maximal momentum, the middle one with a GUP that admits just a minimal length and the lower one is obtained with GUP (28) that admits all natural cutoffs. The slopes of curves reflect the role of natural cutoffs in each case.](image)

5. Radiation rate of micro black holes with natural cutoffs

Any black hole which has a temperature more than the CMB temperature (2.725K), should radiate its energy as photons and other ordinary particles. So, the mass of a black hole decreases and its temperature increases as its radiates. If we assume that dominant radiation is photons, we can use the Stefan-Boltzmann law to estimate the mass output as a function of time which is the emission rate equation. This emission rate was studied for a Schwarzschild black hole for the standard case and also in the presence of quantum gravity effects as a GUP with minimal length in [21,38]. In this section we calculate emission rate of a black hole in the presence of both a minimal length and maximal momentum. By using the Stefan-Boltzmann law and the temperature which was obtained by GUP relation (1), we have

$$\frac{d}{dt} \left( \frac{M}{M_p} \right) = -\frac{16}{t_{ch}} \left( \frac{M}{M_p} \right)^6 \left( 1 - \sqrt{1 - \frac{4t_p^2}{(2t_p(M/M_p) + \ell_p)^2}} \right). \quad (37)$$

where $t_{ch} = 60(16)^2 \pi T_p$ is a characteristic time and is about $4.8 \times 10^4$ times the Planck time. The emission rate for a black hole by using the GUP with just a minimal length is [21]

$$\frac{d}{dt} \left( \frac{M}{M_p} \right) = -\frac{16}{t_{ch}} \left( \frac{M}{M_p} \right)^6 \left( 1 - \sqrt{1 - \frac{1}{(M/M_p)^2}} \right). \quad (38)$$
These emission rates as a function of mass are shown in Fig. 4. We see that at the final stage of evaporation, when the mass of black hole is of the order of the Planck mass, the emission rate reaches a maximum as black hole temperature increases. Note also that the emission rate at the final stage of evaporation with a GUP that admits just a minimal length is larger than the corresponding quantity in the presence of both minimal length and maximal momentum. This feature coincides with the result that we obtained in section 3, which tells us that temperature of black hole in the final stage of its evaporation in the presence of both minimal length and maximal momentum is less than the case that we consider just the effect of minimal length. On the other hand, the GUP relation (1) with minimal length and maximal momentum leads to further reduction of black hole lifetime. Black hole in the presence of natural cutoffs as minimal length and maximal momentum should evaporate sooner than the case that we consider just a minimal length.

**Figure 4.** Radiation rate versus the mass. The mass is in the units of Planck mass. The lower curve is obtained by relation (51), and the upper one is obtained by relation (52).

### 6. Comparison and Conclusion

In this paper we considered possible effects of natural cutoffs as a minimal length, a maximal momentum, and a minimal momentum on the tunneling rate and thermodynamics of TeV scale black holes. In this respect we considered two different GUPs. While previous studies in this field have considered just the effect of minimal length and ignored possible cutoffs on emitted particles’ momentum, here we considered the effects of all possible cutoffs on length and momentum measurement. We considered two types of GUPs: one with a minimal length and maximal momentum, and the other with all natural cutoffs including minimal length, minimal momentum, and maximal momentum. The later one is the most general form of GUP adopted to study TeV black hole thermodynamics. We have shown that in the presence of all natural cutoffs, the tunneling rate is strongly deviated from thermal emission. Even in the first order of $E$, extra terms that depend on the GUP parameters are appeared in the tunneling rate relation, leading to more deviation from thermal emission in comparison with
the case that one considers just the effect of minimal length. Also we have shown that a part of information can be recovered as correlations between emitted modes in the presence of all natural cutoffs. In the final stage of black hole evaporation, both adopted GUPs predict a minimal mass remnant for black hole. Part of information can be supplied in this Planck size remnant. We have shown also that, at the final stage of evaporation black hole reaches a maximum temperature and its entropy becomes zero, so it turns into a stable remanent. As both GUPs have a minimal length $\Delta x_{\text{min}} = \alpha\ell_p$, the minimum mass would be $M_{\text{min}} = \frac{\alpha\ell_p c^2}{2G}$.

By using this minimum mass we found the final temperature of black hole in the presence of both GUPs. With a GUP as given by relation (1) that admits a minimal length and a maximal momentum, we have $T_{\text{final}} = \frac{c}{4\pi \alpha \ell_p}$. On the other hand, with most general GUP (28) admitting all natural cutoffs, we obtain $T_{\text{final}} = \frac{c(1+\beta\alpha \ell_p)}{4\pi \alpha \ell_p}$. With these results in mind, we see that the GUP with minimal length and maximal momentum predicts that the final state temperature is less than the case that we consider just a minimal length GUP. Nevertheless, when we consider all natural cutoffs through GUP (28), the maximum temperature is larger than the case that one ignores the existence of minimal momentum or considers just the effect of minimal length. In fact, the GUP with all natural cutoffs shows that at the final stage of evaporation of black hole, its temperature increases as it is clear from the slopes of the curves in Fig 3. Another important impact of these natural cutoffs on black hole thermodynamics can be seen in black hole radiation rate.

At the final stage of evaporation, when the mass of black hole is of the order of the Planck mass, the emission rate reaches a maximum as black hole temperature increases. The emission rate at the final stage of evaporation with a GUP that admits just a minimal length is larger than the corresponding quantity in the presence of both minimal length and maximal momentum. We have shown also that in the presence of both minimal length and maximal momentum the lifetime of black hole decreases relative to the case that we consider just the effect of minimal length.

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