Abstract. Let $G$ be a graph of order $n$ and size $m$ and let $k \geq 1$ be an integer. A $k$-tuple total dominating set in $G$ is called a $k$-tuple total restrained dominating set of $G$ if each vertex $x \in V(G) - S$ is adjacent to at least $k$ vertices of $V(G) - S$. The minimum number of vertices of a such sets in $G$ are the $k$-tuple total restrained domination number $\gamma_{k,t}^r(G)$ of $G$. The maximum number of classes of a partition of $V(G)$ such that its all classes are $k$-tuple total restrained dominating sets in $G$, is called the $k$-tuple total restrained domatic number of $G$.

In this manuscript, we first find $\gamma_{k,t}^r(G)$, when $G$ is complete graph, cycle, bipartite graph and the complement of path or cycle. Also we will find bounds for this number when $G$ is a complete multipartite graph. Then we will know the structure of graphs $G$ which $\gamma_{k,t}^r(G) = m$, for some $m \geq k + 1$ and give upper and lower bounds for $\gamma_{k,t}^r(G)$, when $G$ is an arbitrary graph. Next, we mainly present basic properties of the $k$-tuple total restrained domatic number of a graph and give bounds for it. Finally we give bounds for the $k$-tuple total restrained domination number of the complementary prism $G\overline{G}$ in terms on the similar number of $G$ and $\overline{G}$ when $G$ is a regular graph or an arbitrary graph. And then we calculate it when $G$ is cycle or path.

Keywords: $k$-tuple total (restrained) domination number, $k$-tuple total (restrained) domatic number.

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1. Introduction

The research of the domination in graphs has been an evergreen of the graph theory. Its basic concept is the dominating set and the domination number. A numerical invariant of a graph which is in a certain sense dual to it is the domatic number. And many variants of the dominating set were introduced and the corresponding numerical invariants were defined for them. Here, we initial to study of the $k$-tuple total restrained domination number and the $k$-tuple total restrained domatic number.

We start with definitions of various concepts concerning the domination in graphs. A subset $S \subseteq V(G)$ is called a $k$-tuple total dominating set, briefly kTDS, in $G$, if for each $x \in V(G)$, $|N(x) \cap S| \geq k$. Recall that 1-tuple total dominating set is known as total dominating set.

Let $k \geq 1$ be an integer. A $k$-tuple total dominating set in $G$ is called a $k$-tuple total restrained dominating set, briefly kTRDS, in $G$, if each vertex $x \in V(G) - S$ is adjacent to at least $k$ vertices of $V(G) - S$. The minimum number of vertices of a $k$-tuple total dominating set in a graph $G$ is the $k$-tuple total domination number of $G$ and denoted by $\gamma_{k,t}^r(G)$. Analogously the $k$-tuple total restrained domination number $\gamma^r_{x,k,t}(G)$ is defined. Obviously, $\gamma_{x,k,t}^r(G) \leq \gamma^r_{x,k,t}(G)$.

The domatic number of a graph was introduced in [1], and the total domatic number in [2]. Sheikholeslami and Volkmann extended the last definition to the $k$-tuple total domatic number $d_{x,k,t}(G)$ in [10]. In an analogous way we will define the $k$-tuple total restrained domatic number and then we will discuss the purpose of defining it. Let $D$ be a partition of the vertex set $V(G)$ of $G$. If all classes of $D$ are $k$-tuple total restrained dominating sets in $G$, then $D$ is called a $k$-tuple total...
restrained domatic partition, briefly kTRDP, of \( G \). The maximum number of classes of a \( k \)-tuple total restrained domatic partition of \( G \) is the \( k \)-tuple total restrained domatic number \( d_{r,k,t}(G) \) of \( G \).

Haynes et al. in [6] defined a new type of graph product that generalizes the concept of a cartesian product. Let \( G \) and \( H \) be two graphs with the vertex sets \( V(G) = \{u_i \mid 1 \leq i \leq n\} \) and \( V(H) = \{v_j \mid 1 \leq j \leq p\} \). Let \( R \) be a subset of \( V(G) \) and \( S \) be a subset of \( V(H) \). The complementary product \( G(R)\overline{H}(S) \) are defined as follows. The vertex set \( G(R)\overline{H}(S) \) is \( \{(u_i, v_j) : 1 \leq i \leq n, 1 \leq j \leq p\} \). And the edge \((u_i, v_j)(u_k, v_l)\) is in \( E(G(R)\overline{H}(S)) \)

1. if \( i = h, u_i \in R \) and \( v_jv_k \in E(H) \), or if \( i = h, u_i \notin R \) and \( v_jv_k \notin E(H) \), or
2. if \( j = k, v_j \in S \) and \( u_iu_k \in E(G) \), or if \( j = k, v_j \notin S \) and \( u_iu_k \notin E(G) \).

In other words, for each \( u_i \in V(G) \), we replace \( u_i \) with a copy of \( H \) if \( u_i \) is in \( R \) and with a copy of its complement \( \overline{H} \) if \( u_i \) is not in \( R \), and for each \( v_j \in V(H) \), we replace each \( v_j \) with a copy of \( G \) if \( v_j \in S \) and a copy of \( \overline{G} \) if \( v_j \notin S \). If \( R = V(G) \) (respectively, \( S = V(H) \)), we write simply \( G\overline{H}(S) \) (respectively, \( G(R)\overline{H} \)). Thus, \( G\overline{H}(S) \) is the graph obtained by replacing each vertex \( v \) of \( H \) by a copy of \( G \) if \( v \in S \) and by a copy of \( \overline{G} \) if \( v \notin S \), and replacing each \( u_i \) with a copy of \( H \). Therefore, the cartesian product of \( G \) and \( H \) is simply \( G(V(G))\overline{H}(V(H)) = G\overline{H} \).

The complementary prism \( \overline{G} \) of a graph \( G \) is the special complementary product \( G\overline{K}_2(S) \) where \( |S| = 1 \). In other words the complementary prism \( \overline{G} \) of \( G \) is the graph formed from the disjoint union \( G\overline{G} \) of \( G \) and \( \overline{G} \) by adding the edges of a perfect matching between the corresponding vertices (same label) of \( G \) and \( \overline{G} \). For example, the graph \( C_5\overline{C}_5 \) is the Petersen graph. Also, if \( G = K_n \), the graph \( K_n\overline{K}_1 \) is the corona \( K_n \circ K_1 \), where the corona \( G \circ K_1 \) of a graph \( G \) is the graph obtained from \( G \) by attaching a pendant edge to each vertex of \( G \).

The \( k \)-join \( G \circ_k H \) of a graph \( G \) to a graph \( H \) of order at least \( k \) is the graph obtained from the disjoint union of \( G \) and \( H \) by joining each vertex of \( G \) to at least \( k \) vertices of \( H \).

The notation we use is as follows. Let \( G \) be a simple graph with vertex set \( V = V(G) \) and edge set \( E = E(G) \). The order \( |V| \), and size \( |E| \), of \( G \) are respectively denoted by \( n = n(G) \) and \( m = m(G) \). For every vertex \( v \in V \), the open neighborhood \( N_G(v) \) is the set \( \{u \in V \mid uv \in E\} \) and its closed neighborhood is the set \( N_G[v] = N_G(v) \cup \{v\} \). The degree of a vertex \( v \in V \) is \( deg(v) = |N(v)| \). The minimum and maximum degree of a graph \( G \) are denoted by \( \delta = \delta(G) \) and \( \Delta = \Delta(G) \), respectively. If every vertex of \( G \) has degree \( k \), then \( G \) is said to be \( k \)-regular. The complement of a graph \( G \) is denoted by \( \overline{G} \) which is a graph with \( V(G) = V(G) \) and for every two vertices \( v \) and \( w \), \( vw \notin E(G) \) if and only if \( vw \in E(G) \). We write \( K_n \) for the complete graph of order \( n \) and \( K_{n_1, \ldots, n_p} \) for the complete \( p \)-partite graph.

Also we write \( C_n \) and \( P_n \), respectively, for a cycle and a path of order \( n \), in which \( C_n\overline{C}_n \) and \( P_n\overline{P}_n \) denote their complementary prisms. Here we assume that \( V(C_n) = V(P_n) = \{i \mid 1 \leq i \leq n\} \) and \( E(C_n) = E(P_n) \cup \{in\} = \{ij \mid 1 \leq i < j \leq n \text{ and } j \equiv i + 1 \pmod{n}\} \). We also assume \( V(G) = \{i \mid 1 \leq i \leq n\} \), where \( G \) is \( C_n \) or \( P_n \), and every vertex \( i \) in \( G \) is adjacent to its respective vertex \( i \) in \( \overline{G} \).

This paper is organized as follows. In section 2, we present the \( k \)-tuple total restrained domination number of the complete graphs, cycles, bipartite graphs and the complement of paths or cycles. Also we will present some bounds for the \( k \)-tuple total restrained domination number of the complete multipartite graph. Then, in section 3, we will show the structure of graphs \( G \) which satisfies \( \gamma'_{r,k,t}(G) = m \), for some \( m \geq k+1 \) and give upper and lower bounds for \( \gamma'_{r,k,t}(G) \), when \( G \) is an arbitrary graph. In the next section, we mainly present basic properties of the \( k \)-tuple total restrained domatic number of a graph and give bounds for it. Also we give some sufficient conditions for the \( k \)-tuple domination (resp. domatic) number of a graph is its \( k \)-tuple restrained domination (resp. domatic) number. Finally, in the last section, we give some bounds for the \( k \)-tuple total restrained domination number of the complementary prism \( \overline{G} \) in terms of the similar number of \( G \) and \( \overline{G} \) when \( G \) is a regular graph or an arbitrary graph. And then we calculate it for the complementary prism of a cycle or path.

The following observations and propositions are useful.

**Observation 1.** Let \( G \) be a graph of order \( n \) in which \( \delta(G) \geq k \). Then

1. every vertex of degree at most \( 2k - 1 \) of \( G \) and at least its \( k \) neighbors belong to every \( k \)TRDS,
If $\delta(G) \leq 2k - 1$, then $d^r_{x,k,t}(G) = 1$.

iii. If $\gamma^{r}_{x,k,t}(G) < n$, then $\Delta(G) \geq 2k$, and so $n \geq 2k + 2$.

**Observation 2.** Let $k < n$ be two positive integers. Then $d^r_{x,k,t}(K_n) = \lceil \frac{n}{k+1} \rceil$.

**Proposition A.** (Kazemi [9] 2011) Let $n \geq 4$. Then

$$\gamma_t(C_n \overline{C_n}) = \begin{cases} 2 \lceil \frac{n}{4} \rceil + 2 & \text{if } n \equiv 0 \pmod{4}, \\ 2 \lceil \frac{n}{4} \rceil + 1 & \text{if } n \equiv 3 \pmod{4}, \\ 2 \lceil \frac{n}{4} \rceil & \text{Otherwise}. \end{cases}$$

**Proposition B.** (Kazemi [9] 2011) If $n \geq 5$, then $\gamma_{x,2,t}(C_n \overline{C_n}) = n + 2$.

**Proposition C.** (Kazemi [9] 2011) Let $n \geq 4$. Then

$$\gamma_t(P_n \overline{P_n}) = \begin{cases} 2 \left\lceil \frac{n-2}{4} \right\rceil + 1 & \text{if } n \equiv 3 \pmod{4}, \\ 2 \left\lceil \frac{n-2}{4} \right\rceil + 2 & \text{otherwise}. \end{cases}$$

2. $k$-tuple total restrained domination number in some graphs

By Observation 1(iii), we have $\gamma^{r}_{x,k,t}(K_n) = n$ if $n \leq 2k + 1$. Since also every $(k + 1)$-subset of vertices is a kTRDS of $K_n$, when $n \geq 2k + 2$, then we have the next result.

**Proposition 3.** Let $k < n$ be positive integers. Then

$$\gamma^{r}_{x,k,t}(K_n) = \begin{cases} n & \text{if } n \leq 2k + 1, \\ k+1 & \text{otherwise.} \end{cases}$$

Next three propositions present $\gamma^{r}_{x,k,t}(\overline{C_n})$, $\gamma^{r}_{x,k,t}(\overline{P_n})$ and $\gamma^{r}_{x,k,t}(C_n)$.

**Proposition 4.** Let $n \geq k + 3 \geq 4$. Then

$$\gamma^{r}_{x,k,t}(\overline{C_n}) = \begin{cases} n & \text{if } n \leq 2k + 2, \\ k+2 & \text{if } 2k + 3 \leq n \leq 3k + 2, \\ k+1 & \text{if } n \geq 3k + 3. \end{cases}$$

**Proof.** We first prove that $\gamma^{r}_{x,k,t}(\overline{C_n}) = k+1$ if and only if $n \geq 3k+3$. Let $S$ be a kTRDS of $\overline{C_n}$ with cardinal $k+1$. Then for every two arbitrary vertices $i$ and $j$ in $S$, $|i - j| \geq 3$, and so $n \geq 3k + 3$. Since also $\{3i + \overline{T} \mid 0 \leq i \leq k\}$ is a kTRDS of $\overline{C_n}$, when $n \geq 3k + 3$, then $\gamma^{r}_{x,k,t}(\overline{C_n}) = k+1$.

Observation 1(iii) follows that $\gamma^{r}_{x,k,t}(\overline{C_n}) = n$ if and only if $k+3 \leq n \leq 2k + 1$. Let now $n = 2k + 2$. Then $\delta(G) = \Delta(G) = n - 3 = 2k - 1$. Let $S$ be a kTRDS of $\overline{C_n}$. Let $i \in V - S$. Since $|N(i) \cap S| \geq k$ and $|N(i) \cap (V - S)| \geq k$, then $\deg(i) \geq 2k$ that is not possible. Therefore $S = V$ and so $\gamma^{r}_{x,k,t}(\overline{C_n}) = n$. For the other cases, obviously $S = \{2i + 1 \mid 0 \leq i \leq k + 1\}$ is a kTRDS of $\overline{C_n}$ and so $\gamma^{r}_{x,k,t}(\overline{C_n}) = k+2$.

**Proposition 5.** Let $n \geq k + 3 \geq 4$. Then

$$\gamma^{r}_{t}(P_n) = \begin{cases} n & \text{if } n = 4, \\ 2 & \text{if } n \geq 5, \end{cases}$$

and if $k \geq 2$, then

$$\gamma^{r}_{x,k,t}(P_n) = \begin{cases} n & \text{if } n \leq 2k + 2, \\ k+2 & \text{if } 2k + 3 \leq n \leq 3k, \\ k+1 & \text{if } n \geq 3k + 1. \end{cases}$$

**Proof.** One can verify that $\gamma^{r}_{t}(P_n)$ is 2 if and only if $n \geq 5$, and otherwise is $n$. Let now $k \geq 2$. It can be easily verify that $\gamma^{r}_{x,k,t}(P_n) = k+1$ if and only if there exists a kTRDS of $P_n$ such that for every two disjoint vertices $\overline{i}$ and $\overline{j}$ in $S$, the difference between $\overline{i}$ and $\overline{j}$ to modulo $n$ is at least 3 or $\{\overline{i}, \overline{j}\} = \{\overline{1}, \overline{n}\}$. And this is equivalent to $n \geq 3k + 1$. Since $S = \{3i + 1 \mid 0 \leq i \leq k - 1\} \cup \{\overline{1}\}$ is a kTRDS of $P_n$, for $n \geq 3k + 1$, then $\gamma^{r}_{x,k,t}(P_n) = k+1$. Let now $n = k + i \leq 3k$, and let $S$ be a kTRDS of $P_n$. For every vertex $x$ in $V - S$, $\deg(x) \geq n - 1 - |S| \geq n - k - 3 = i - 3$. Since also $\deg(x) \geq k$, then $i \geq k + 3$. Hence $\gamma^{r}_{x,k,t}(P_n) = n$ if $n \leq 2k + 2$. Let now $2k + 3 \leq n \leq 3k$.2
Since $\overline{C_n}$ is a spanning subgraph of $\overline{T_n}$, then $\gamma^r_{x,k,t}(\overline{C_n}) \leq \gamma^r_{x,k,t}(\overline{T_n}) = k + 2$, by Proposition 3. Now $\gamma^r_{x,k,t}(T_n) > k + 1$ follows $\gamma^r_{x,k,t}(\overline{T_n}) = k + 2$. \hfill \Box

**Proposition 6.** Let $n \geq 4$. Then $\gamma^r_{x,2,t}(C_n) = n$ and $$\gamma^r_t(C_n) = \begin{cases} \left\lceil \frac{n}{4} \right\rceil & \text{if } n \equiv 1 \pmod{4}, \\ 2 & \text{if } n \equiv 3 \pmod{4}, \\ \left\lfloor \frac{n}{4} \right\rfloor & \text{otherwise.} \end{cases}$$

**Proof.** It is trivial that $\gamma^r_{x,2,t}(C_n) = n$. We note that $$\gamma^r_t(C_n) = \begin{cases} \left\lceil \frac{n}{4} \right\rceil - 1 & \text{if } n \equiv 1 \pmod{4}, \\ \left\lceil \frac{n}{4} \right\rceil & \text{otherwise}. \end{cases}$$

If $n \equiv 0, 1, 2 \pmod{4}$, since the corresponding sets $S_0 = \{2 + 4i, 3 + 4i \mid 0 \leq i \leq \lfloor n/4 \rfloor - 1\}$, $S_1 = S_0 \cup \{n - 1\}$ and $S_2 = S_0 \cup \{1, n - 2\}$ are total restrained dominating sets with cardinal $\gamma^r_t(C_n)$, then we have proved proposition, when $n \not\equiv 3 \pmod{4}$. Let now $n \equiv 3 \pmod{4}$. Then it can be easily verify that $\gamma^r_t(C_n) \geq \gamma^r_t(C_n) + 1$, and since $S_3 = S_0 \cup \{1, n - 3, n\}$ is a total restrained dominating set of $C_n$ with cardinal $\gamma^r_t(C_n) + 1$, then $\gamma^r_t(C_n) = 2 \left\lfloor \frac{n}{4} \right\rfloor + 1$. \hfill \Box

Now we present the $k$-tuple total restrained domination number of the bipartite graphs.

**Proposition 7.** Let $G$ be a bipartite graph with $\delta(G) \geq k \geq 1$. Then $2k \leq \gamma^r_{x,k,t}(G) \leq n$. Moreover, if $X$ and $Y$ are the bipartite sets of $G$, then $\gamma^r_{x,k,t}(G) = 2k$ if and only if there exist two $k$-subsets $S \subseteq X$ and $T \subseteq Y$ such that for each vertex $x \in X$, $N(x) \supseteq T$, and for each vertex $y \in Y$, $N(y) \supseteq S$ and the minimum degree of the induced subgraph $G[(X - S) \cup (Y - T)]$ is at least $k$.

**Proof.** Let $D$ be a $\gamma^r_{x,k,t}(G)$-set, and let $w \in X$ and $z \in Y$ be two arbitrary vertices. The definition implies that $|D \cap N(w)| \geq k$ and $|D \cap N(z)| \geq k$. Since $N(w) \cap N(z) = \emptyset$, we deduce that $|D| \geq 2k$ and thus $2k \leq \gamma^r_{x,k,t}(G) \leq n$. If there exist two $k$-subsets $S \subseteq X$ and $T \subseteq Y$ such that for each vertex $x \in X$, $N(x) \supseteq T$, and for each vertex $y \in Y$, $N(y) \supseteq S$ and also the minimum degree of the induced subgraph $G[(X - S) \cup (Y - T)]$ is at least $k$, then obviously $D = S \cup T$ is a $k$-tuple total restrained dominating set of $G$. This implies $\gamma^r_{x,k,t}(G) \leq 2k$ and so $\gamma^r_{x,k,t}(G) = 2k$.

Conversely, assume that $\gamma^r_{x,k,t}(G) = 2k$, and let $D$ be a $\gamma^r_{x,k,t}(G)$-set. It follows that $|D \cap X| = |D \cap Y| = k$.

Now let $S = D \cap X$ and $T = D \cap Y$. Then $T \subseteq N(x)$ for each vertex $x \in X$ and $S \subseteq N(y)$ for each vertex $y \in Y$. Now if $|X| > k$ and $|Y| > k$, then, by the definition, $\delta(G[(X - S) \cup (Y - T)]) \geq k$ and the proof is complete. \hfill \Box

**Corollary 8.** Let $G = K_{n,m}$ be a complete bipartite graph with $n \geq m \geq k \geq 1$. Then $$\gamma^r_{x,k,t}(G) = \begin{cases} 2k & \text{if } n \geq m \geq 2k, \\ n + m & \text{otherwise.} \end{cases}$$

Now we present some bounds for $\gamma^r_{x,k,t}(G)$, where $G = K_{n_1,\ldots,n_p}$ is a complete multipartite graph and $p \geq 3$.

**Proposition 9.** Let $G = K_{n_1,\ldots,n_p}$ be the complete $p$-partite graph of order $n$. If $\gamma^r_{x,k,t}(G) < n$, then $$\left\lceil \frac{kp}{p-1} \right\rceil \leq \gamma^r_{x,k,t}(G) \leq n - k.$$

**Proof.** We assume that $G$ has vertex partition $V = X_1 \cup \ldots \cup X_p$ such that $|X_i| = n_i$ and $n = n_1 + \ldots + n_p$. Let $S$ be an arbitrary kTRDS of $G$. Since every vertex of $X_i$ is adjacent to at least $k$ vertices of $S - X_i = \bigcup_{j=1, j \neq i}^p S_j$, then
$$\sum_{j=1}^p s_j - s_i \geq k.$$
for each $1 \leq i \leq p$, and hence $(p-1) \mid S \geq pk$ that follows $\mid S \geq \lceil \frac{pk}{p-1} \rceil$. Since $S$ was arbitrary, therefore $\gamma^r_{x,k,t}(G) \geq \lceil \frac{pk}{p-1} \rceil$.

For proving the another inequality, we use the following definitions and notations. Let $S$ be a kTRDS of $G$ and let $S_i = X_i \cap S$, $S_i' = X_i - S$ and $\mid S_i \mid = s_i$. Let also $t(S)$ be the number of $i$'s that $s_i < n_i$ and let

$$t_0 = \min \{t(S) \mid S \text{ is a kTRDS of } G\}.$$  

We may assume that $t(S) \geq 1$. Because $t_0 = 0$ if and only if $\gamma^r_{x,k,t}(K_{n_1,\ldots,n_p}) = n$. Then obviously $t(S) \geq 2$. Without less of generality, we may assume that $s_i < n_i$ if and only if $1 \leq i \leq t(S)$. Let $w_j \in X_j - S = X_j - S_j$, for each $1 \leq j \leq t(S)$. Then $\mid N(w_j) \cap (V - S) \mid \geq k$, since $S$ is a kTRDS.

Since also $N(w_j) \cap (V - S) = \bigcup_{i=1,i\neq j} N(w_j) \cap S_i'$, then for each $1 \leq j \leq t$ we have:

$$k \leq \mid N(w_j) \cap (V - S) \mid = \sum_{i=1,i\neq j} \mid N(w_j) \cap S_i' \mid = \sum_{i=1,i\neq j} \mid S_i' \mid = \sum_{i=1}^{t(S)} \mid S_i' \mid - \mid S_j' \mid.$$  

By summing the inequalities we have

$$t(S)k \leq (t(S) - 1) \sum_{i=1}^{t(S)} (n_i - s_i) = (t(S) - 1) \sum_{i=1}^{p} (n_i - s_i) = (t(S) - 1)(n - \mid S \mid)$$

and hence $\mid S \mid \leq n - k - \lceil \frac{k}{t(S) - 1} \rceil$. Since $S$ was arbitrary, then

$$\gamma^r_{x,k,t}(G) \leq n - k - \lceil \frac{k}{t_0 - 1} \rceil \leq n - k.$$  

If we look at closer to the proof of Proposition 9 we have the next result.

**Proposition 10.** Let $G = K_{n_1,\ldots,n_p}$ be the complete $p$-partite graph of order $n$. If $\gamma^r_{x,k,t}(G) < n$, then $\gamma^r_{x,k,t}(G) \leq n - k - \lceil \frac{k}{t_0 - 1} \rceil$.

3. **Bounds for k-tuple total restrained domination number**

In this section, we first give a necessary and sufficient condition for $\gamma^r_{x,k,t}(G) = m$, for some $m \geq k + 1$, and then present some lower and upper bounds for $\gamma^r_{x,k,t}(G)$ in terms on $k$, $n$ and $m$.

**Theorem 11.** Let $G$ be a graph with $\delta(G) \geq k$. Then for any integer $m \geq k + 1$, $\gamma^r_{x,k,t}(G) = m$ if and only if $G = K'_m$ or $G = F \circ K'_m$, for some graph $F$ and some spanning subgraph $K'_m$ of $K_m$ with $\delta(F) \geq k$ and $\delta(K'_m) \geq k$ such that $m$ is minimum in the set

$$\{t \mid G = F \circ K'_t, \text{ for some } F \text{ and some spanning subgraph } K'_t \text{ of } K_t \text{ with } \delta(F) \geq k, \delta(K'_t) \geq k\}.$$  

**Proof.** Let $S$ be a $\gamma^r_{x,k,t}(G)$-set and $\gamma^r_{x,k,t}(G) = m$, for some $m \geq k + 1$. Then, $\mid S \mid = m$, and every vertex has at least $k$ neighbors in $S$, and also every vertex in $V - S$ has at least $k$ neighbors in $V - S$. Then $G[S] = K'_m$, for some spanning subgraph $K'_m$ of $K_m$ with $\delta(K'_m) \geq k$. If $\mid V \mid = m$, then $G = K'_m$. If $\mid V \mid > m$, then let $F$ be the induced subgraph $G[V - S]$. Then $\delta(F) \geq k$ and $G = F \circ K'_m$. Also by the definition of $k$-tuple total restrained domination number, $m$ is the minimum of the set given in (1).

Conversely, let $G = K'_m$ or $G = F \circ K'_m$, for some graph $F$ with $\delta(F) \geq k$ and some spanning subgraph $K'_m$ of $K_m$ with $\delta(K'_m) \geq k$ such that $m$ is the minimum of the set given in (1). Then, since $V(K'_m)$ is a kTRDS of $G$ with cardinal $m$, $\gamma^r_{x,k,t}(G) \leq m$. If $\gamma^r_{x,k,t}(G) = m' < m$, then the previous paragraph concludes that for some graph $F'$ with $\delta(F') \geq k$ and some spanning subgraph
Corollary 12. Let $G$ be a graph with $\delta(G) \geq k$. Then $\gamma^r_{x,k,t}(G) = k + 1$ if and only if $G = K_{k+1}$ or $G = F\circ_k K_{k+1}$, for some graph $F$ with $\delta(F) \geq k$.

Theorem 13. If $G$ is a graph with minimum degree at least $k$ on $n$ vertices and with $m$ edges, then

$$\gamma^r_{x,k,t}(G) \geq \frac{3n}{2} - \frac{m}{k}.\]$$

Proof. Let $S$ be a minimum $k$TDS of $G = (V, E)$. Since $\delta(G[S]) \geq k$, $\delta(G[V - S]) \geq k$ and $S$ is $k$TDS, we have the following inequalities:

$$m_1 \geq \frac{k\gamma^r_{x,k,t}(G)}{k}$$
$$m_2 \geq \frac{k(n - \gamma^r_{x,k,t}(G))}{k}$$
$$m_3 \geq kn - \gamma^r_{x,k,t}(G),$$

where $m_1$ and $m_2$ are respectively the number of edges in induced subgraphs $G[S]$ and $G[V - S]$ and $m_3$ is the number of edges connecting vertices of $V - S$ to vertices of $S$. By summing the inequalities, we obtain

$$m = m_1 + m_2 + m_3 \geq \frac{3kn}{2} - k\gamma^r_{x,k,t}(G),$$

and thus $\gamma^r_{x,k,t}(G) \geq \frac{3n}{2} - \frac{m}{k}$. \hfill \Box

Corollary 14. If $G$ is a graph without isolated vertex on $n$ vertices and with $m$ edges, then

$$\gamma^r_1(G) \geq \frac{3}{2}n - m.$$  

Theorem 15. Let $G$ be a graph with minimum degree at least $k$. Let $\delta(G) \geq a + k$, for some finite number $a$. If $\gamma_{x,k,t}(G) \leq a$, then $\gamma^r_{x,k,t}(G) \leq a$.

Proof. Let us consider a $k$TDS $S$ such that $|S| \leq a$. For every $v \in V(G) - S$,

$$\deg(v) \geq \delta(G) \geq a + k \geq |S| + k.$$

Therefore $|N(v) \cap (V(G) - S)| \geq k$, that means $S$ is a $k$TDS of $G$ and so $\gamma^r_{x,k,t}(G) \leq a$. \hfill \Box

4. SOME PROPERTIES OF $k$-TUPLE TOTAL RESTRAINED DOMATIC NUMBER

In this section we mainly present basic properties of $d^r_{x,k,t}(G)$ and bounds on the $k$-tuple total restrained domatic number of a graph.

Theorem 16. If $G$ is a graph of order $n$ with $\delta(G) \geq k$, then

$$\gamma^r_{x,k,t}(G) \cdot d^r_{x,k,t}(G) \leq n.\]$$

Moreover, if $\gamma^r_{x,k,t}(G) \cdot d^r_{x,k,t}(G) = n$, then for each $k$TRDP $\{V_1, V_2, ..., V_d\}$ of $G$ with $d = d^r_{x,k,t}(G)$, each set $V_i$ is a $\gamma^r_{x,k,t}(G)$-set.

Proof. Let $\{V_1, V_2, ..., V_d\}$ be a kTRDP of $G$ such that $d = d^r_{x,k,t}(G)$. Then

$$d \cdot \gamma^r_{x,k,t}(G) \leq \sum_{i=1}^{d} \gamma^r_{x,k,t}(G) \leq \sum_{i=1}^{d} |V_i| = n.$$

If $\gamma^r_{x,k,t}(G) \cdot d^r_{x,k,t}(G) = n$, then the inequality occurring in the proof becomes equality. Hence for the kTRDP $\{V_1, V_2, ..., V_d\}$ of $G$ and for each $i$, $|V_i| = \gamma^r_{x,k,t}(G)$. Thus each set $V_i$ is a $\gamma^r_{x,k,t}(G)$-set. \hfill \Box

An immediate consequence of Theorem 16 and Corollary 12 now follows.
Corollary 17. If $G$ is a graph of order $n$ with $\delta(G) \geq k$, then
\[ d^r_{x,k,t}(G) \leq \frac{n}{k+1}, \]
with equality if and only if $G = K_{k+1}$ or $G = F \circ_k K_{k+1}$, for some graph $F$ with $\delta(F) \geq k$.

For bipartite graphs, we can improve the bound given in Corollary 17 by Proposition.[8]

Corollary 18. If $G$ is a bipartite graph of order $n$ with vertex partition $V(G) = X \cup Y$ and $\delta(G) \geq k$, then
\[ d^r_{x,k,t}(G) \leq \frac{n}{2k}, \]
with equality if and only there exist two $k$-subsets $S \subseteq X$ and $T \subseteq Y$ such that for each vertex $x \in X$, $N(x) \supseteq T$, and for each vertex $y \in Y$, $N(y) \supseteq S$ and the minimum degree of the induced subgraph $G[(X - S) \cup (Y - T)]$ is at least $k$.

Now, we show that the $k$-tuple total restrained domatic number of every graph is equal to its $k$-tuple total domatic number.

Theorem 19. Let $G$ be a graph with $\delta(G) \geq k \geq 1$. Then $d^r_{x,k,t}(G) = d_{x,k,t}(G)$.

Proof. Each $k$-tuple total restrained dominating set in $G$ is a $k$-tuple total dominating set in $G$, therefore each $k$-tuple total restrained domatic partition of $G$ is a $k$-tuple total domatic partition of $G$ and $d^r_{x,k,t}(G) \leq d_{x,k,t}(G)$. Now let $d = d_{x,k,t}(G) \geq 2$ and let $D = \{D_1, \ldots, D_d\}$ be a $k$-tuple total domatic partition of $G$. Choose $D_1$ as an arbitrary class of $D$. Let $x \in V(G)$. As $D_1$ is a $k$-tuple total dominating set in $G$, there exists $k$-set $S^t_1$ such that $S^t_1 \subseteq N(x) \cap D_1$. Now suppose $x \in V(G) - D_1$. Then $x \in D_i$ for some $2 \leq i \leq d$. The set $D_i$ is also a $k$-tuple total dominating set in $G$, therefore there exists $k$-set $S^t_i$ such that $S^t_i \subseteq N(x) \cap D_i$ and evidently $S^t_i \subseteq V(G) - D_1$, because $D_1 \cap D_i = \emptyset$. Therefore, we have proved that $D_1$ is a $k$-tuple total restrained dominating set in $G$. The set $D_1$ was chosen arbitrarily, therefore $D$ is a $k$-tuple total restrained domatic partition of $G$ and $d_{x,k,t}(G) \leq d^r_{x,k,t}(G)$, which together with the former inequality gives the required result. \(\square\)

Corollary 20. [12] Let $G$ be a graph without isolated vertices. Then $d^r_t(G) = d_t(G)$.

Now, we give a sufficient condition for $\gamma^r_{x,k,t}(G) = \gamma_{x,k,t}(G)$.

Theorem 21. Let $G$ be a graph with minimum degree at least $k$. If $d_{x,k,t}(G) \geq 2$, then
\[ \gamma^r_{x,k,t}(G) = \gamma_{x,k,t}(G). \]

Proof. Since every $k$-tuple total restrained dominating set in $G$ is also $k$-tuple total dominating set in $G$, therefore $\gamma_{x,k,t}(G) \leq \gamma^r_{x,k,t}(G)$. For the converse inequality, let $S$ be a minimum $k$-tuple total dominating set of $G$. Since $d_{x,k,t}(G) \geq 2$, then there exists another $k$-tuple total dominating set $S'$ in $G$ which is disjoint of $S$. Let $x \in V(G) - S$. Then $x$ is adjacent to at least $k$ vertices of $S'$, since $S'$ is a $k$-tuple total dominating set of $G$. This follows that $x$ is adjacent to at least $k$ vertices of $V(G) - S$. Therefore, $S$ is a $k$-tuple total restrained dominating set of $G$ and so $\gamma^r_{x,k,t}(G) \leq \gamma_{x,k,t}(G)$. The previous two inequalities follow $\gamma^r_{x,k,t}(G) = \gamma_{x,k,t}(G)$. \(\square\)

Corollary 22. Let $G$ be a graph without isolated vertex. If $d_t(G) \geq 2$, then $\gamma^r_t(G) = \gamma_t(G)$.

The converse of Theorem 21 does not hold. For example, if $G = K_{k+1}$, then $\gamma^r_{x,k,t}(G) = \gamma_{x,k,t}(G) = k + 1$ but $d_{x,k,t}(G) = 1$. Also as another example let $G = K_{n,m}$ be the complete bipartite graph with this conditions that $k \leq n \leq m < 2k$ and $(n,m) \neq (k,k)$. Then $\gamma_{x,k,t}(G) = 2k < \gamma^r_{x,k,t}(G) = n + m$, but $d_{x,k,t}(G) = 1$.

5. COMPLEMENTARY PRISMS

First we calculate the $k$-tuple total restrained domination number of the complementary prism of a regular graph for some integer $k$. 


Theorem 23. Let $k$ and $\ell$ be integers such that $1 \leq k - 1 \leq \ell \leq 2k - 2$. If $G$ is a $\ell$-regular graph of order $n$, then

$$\gamma_{\ell,k,t}(\overline{G}) \geq n + k,$$

with equality if and only if $n \geq \ell + 2k$ and $V(\overline{G})$ contains a $k$-subset $T$ such that for each vertex $i \in V(\overline{G})$, $|N(i) \cap T| \geq k - 1$ and also if $i \in V(\overline{G}) - T$, then $|N(i) \cap (V(\overline{G}) - T)| \geq k$.

Proof. Let $V(\overline{G}) = V(G) \cup V(\overline{G})$ such that $V(G) = \{i \mid 1 \leq i \leq n\}$ and $V(\overline{G}) = \{\overline{i} \mid 1 \leq i \leq n\}$. Let $n \geq 2k + \ell$, and let $S$ be an arbitrary kTRDS of $G$. Since each vertex $i$ has degree $\ell + 1 \leq 2k - 1$, then $V(G) \subseteq S$, by Observation $\square$. Let $i \notin S$. Then $|N(i) \cap V(\overline{G}) \cap S| \geq k - 1$. If $|N(i) \cap V(\overline{G}) \cap S| \geq k$, then we have nothing to prove. Thus let $N(i) \cap V(\overline{G}) \cap S = \{j | 1 \leq i \leq k - 1\}$. But this follows that there exists at least one vertex $T \in S - \{\overline{j} | 1 \leq i \leq k - 1\}$ such that its corresponding vertex $t$ in $G$ is adjacent to some vertex $j_i$, when $1 \leq i \leq k - 1$. So $|S| \geq n + k$, and since $S$ was arbitrary, then $\gamma_{\ell,k,t}(\overline{G}) \geq n + k$.

Obviously, it can be seen that $\gamma_{\ell,k,t}(\overline{G}) = n + k$ if and only if $n \geq \ell + 2k$ and $V(\overline{G})$ contains a $k$-subset $T$ such that for each vertex $i \in V(\overline{G})$, $|N(i) \cap T| \geq k - 1$ and also if $i \in V(\overline{G}) - T$, then $|N(i) \cap (V(\overline{G}) - T)| \geq k$.

Observation $\square$ follows the next result.

Corollary 24. Let $k$ and $\ell$ be integers such that $1 \leq k - 1 \leq \ell \leq 2k - 2$. If $G$ is a $\ell$-regular graph of order $n \leq \ell + 2k - 1$, then

$$\gamma_{\ell,k}(\overline{G}) = 2n.$$

Corollary 25. Let $n \geq 4$. Then

$$\gamma_{\ell,2,t}(C_n \overline{C_n}) = \begin{cases} 2n & \text{if } n = 4, 5, \\ n + 2 & \text{if } n \geq 6. \end{cases}$$

The next theorem state lower and upper bounds for $\gamma_{\ell,k,t}(\overline{G})$, when $G$ is an arbitrary graph.

Theorem 26. If $G$ is a graph of order $n$ with $k \leq \min\{\delta(G), \delta(\overline{G})\}$, then

$$\gamma_{\ell,k-1,t}(G) + \gamma_{\ell,k-1,t}(\overline{G}) \leq \gamma_{\ell,k,t}(\overline{G}) \leq \gamma_{\ell,k,t}(G) + \gamma_{\ell,k,t}(\overline{G}),$$

where $k \geq 2$ in the lower bound and $k \geq 1$ in the upper bound.

Proof. For proving $\gamma_{\ell,k-1,t}(G) + \gamma_{\ell,k-1,t}(\overline{G}) \leq \gamma_{\ell,k,t}(\overline{G})$, let $k \geq 1$ and let $D$ be a kTRDS of $G$. Since every vertex of $V(G)$ (resp. $V(\overline{G})$) is adjacent only one vertex of $V(G)$ (resp. $V(\overline{G})$), then we have a nontrivial partition $D = D' \cup D''$ such that $D'$ is a $(k-1)$TRDS of $G$ and $D''$ is a $(k-1)$TRDS of $\overline{G}$. Then

$$\gamma_{\ell,k-1,t}(G) + \gamma_{\ell,k-1,t}(\overline{G}) \leq |D'| + |D''| = |D| = \gamma_{\ell,k,t}(\overline{G}).$$

We now prove $\gamma_{\ell,k,t}(\overline{G}) \leq \gamma_{\ell,k,t}(G) + \gamma_{\ell,k,t}(\overline{G})$, let $k \geq 1$. Since for every kTRDS $S$ of $G$ and every kTRDS $S'$ of $\overline{G}$, the set $S \cup S'$ is a kTRDS of $\overline{G}$, then

$$\gamma_{\ell,k,t}(\overline{G}) \leq \gamma_{\ell,k,t}(G) + \gamma_{\ell,k,t}(\overline{G}).$$

In continues, we will determine $\gamma_{\ell,t}(C_n \overline{C_n})$, $\gamma_{\ell,2,t}(C_n \overline{C_n})$ and $\gamma_{\ell,t}(P_n \overline{P_n})$.

Proposition 27. Let $n \geq 4$. Then $d_t(C_n \overline{C_n}) \geq 2$.

Proof. We consider the following four cases.

Case 1. Let $n \equiv 0 \pmod{4}$. For $n = 4$, we choose $S = \{1, \overline{1}, 2, \overline{2}\}$ and $S' = \{3, \overline{3}, 4, \overline{4}\}$. If $n > 4$, then we choose $S = \{1, \overline{1}, 2, \overline{2}\} \cup \{5+4i, 6+4i | 0 \leq i \leq \lfloor n/4 \rfloor - 2\}$ and $S' = \{3, \overline{3}, 4, \overline{4}\} \cup \{7+4i, 8+4i | 0 \leq i \leq \lfloor n/4 \rfloor - 2\}$.

Case 2. Let $n \equiv 1 \pmod{4}$. For $n = 5$, we choose $S = \{1, \overline{1}, 4, \overline{4}\}$ and $S' = \{2, \overline{2}, 5, \overline{5}\}$. If $n = 9$, we choose $S = \{1, \overline{1}, 4, \overline{4}, 7, \overline{7}\}$ and $S' = \{2, \overline{2}, 5, \overline{5}, 8, \overline{8}\}$. If $n > 9$, then we choose $S = \{1, \overline{1}, 4, \overline{4}, 7, \overline{7}, 10, 11, 12, 13+4i | 0 \leq i \leq \lfloor n/4 \rfloor - 4\}$ and $S' = \{3, \overline{3}, 6, \overline{6}, 9, \overline{9}\} \cup \{12+4i, 13+4i | 0 \leq i \leq \lfloor n/4 \rfloor - 4\}$. 


Case 3. \( n \equiv 2 \pmod{4} \). For \( n = 6 \), we choose \( S = \{1, 4, 3, 4\} \) and \( S' = \{2, 3, 5, 5\} \). For \( n > 6 \), we choose \( S = \{1, 4, 3, 4\} \cup \{7 + 4i, 8 + 4i, 0 \leq i \leq \lfloor n/4 \rfloor - 3\} \) and \( S' = \{3, 3, 6, 6\} \cup \{9 + 4i, 10 + 4i, 0 \leq i \leq \lfloor n/4 \rfloor - 3\} \).

Case 4. \( n \equiv 3 \pmod{4} \). For \( n = 7 \), we choose \( S = \{1, 4, 3, 4\} \) and \( S' = \{2, 3, 5, 5, 7\} \). For \( n > 7 \), we choose \( S = \{1, 4, 3, 4\} \cup \{7 + 4i, 8 + 4i, 0 \leq i \leq \lfloor n/4 \rfloor - 3\} \) and \( S' = \{2, 3, 3, 6, 6\} \cup \{9 + 4i, 10 + 4i, 0 \leq i \leq \lfloor n/4 \rfloor - 3\} \).

Since in all cases, \( S \) and \( S' \) are disjoint \( \gamma_t(C_n \Gamma_n) \)-sets, then \( d_t(C_n \Gamma_n) \geq 2 \). \( \square \)

Propositions \( 27 \) and Theorem \( 21 \) imply the next result.

**Proposition 28.** Let \( n \geq 4 \). Then

\[
\gamma_t^r(C_n \Gamma_n) = \begin{cases} 
2 \lfloor n/4 \rfloor + 2 & \text{if } n \equiv 0 \pmod{4}, \\
2 \lfloor n/4 \rfloor + 1 & \text{if } n \equiv 3 \pmod{4}, \\
2 \lfloor n/4 \rfloor & \text{Otherwise.}
\end{cases}
\]

**Proposition 29.** Let \( n \geq 4 \). Then

\[
\gamma_t^r(P_n \Gamma_n) = \begin{cases} 
2 \lfloor n/4 \rfloor + 2 & \text{if } n \equiv 0 \pmod{4}, \\
2 \lfloor n/4 \rfloor + 1 & \text{if } n \equiv 3 \pmod{4}, \\
2 \lfloor n/4 \rfloor & \text{Otherwise.}
\end{cases}
\]

**Proof.** Proposition \( C \) with this fact that for every graph \( G \), \( \gamma_{t,k,t}(G) \leq \gamma_{t,k,t}(G) \), follow that

\[
\gamma_t^r(P_n \Gamma_n) \geq \gamma_t^r(P_n \Gamma_n) = \begin{cases} 
2 \lfloor (n - 2)/4 \rfloor + 1 & \text{if } n \equiv 3 \pmod{4}, \\
2 \lfloor (n - 2)/4 \rfloor + 2 & \text{otherwise.}
\end{cases}
\]

Let \( n \equiv 0 \pmod{4} \). For \( n = 8 \), set \( S = \{1, 8, 3, 4, 5, 6\} \) and for \( n > 8 \) set \( S = \{1, n - 6, n - 3, n - 2\} \cup \{3 + 4i, 4 + 4i, 0 \leq i \leq \lfloor n/4 \rfloor - 3\} \). If \( n \equiv 1, 2, 3 \pmod{4} \), then respectively set \( S = \{1, n - 2, n - 2\} \cup \{3 + 4i, 4 + 4i, 0 \leq i \leq \lfloor n/4 \rfloor - 2\} \), \( S = \{1, n, 3 + 4i, 4 + 4i, 0 \leq i \leq \lfloor n/4 \rfloor - 1\} \) and \( S = \{1, n - 1, n - 1\} \cup \{3 + 4i, 4 + 4i, 0 \leq i \leq \lfloor n/4 \rfloor - 1\} \). Since in all cases, \( S \) is a TRDS of \( P_n \Gamma_n \) with cardinal \( \gamma_t^r(P_n \Gamma_n) \), thus we have completed our proof. \( \square \)

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