Study of weak solutions for parabolic variational inequalities with nonstandard growth conditions

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Abstract
In this paper, we study the degenerate parabolic variational inequality problem in a bounded domain. First, the weak solutions of the variational inequality are defined. Second, the existence and uniqueness of the solutions in the weak sense are proved by using the penalty method and the reduction method.

MSC: 35B40; 35K35
Keywords: Parabolic variational inequality; Weak solution; Penalty method; Existence

1 Introduction
This article concerned with initial-boundary problem whose model is

\[
\begin{align*}
\min\{Lu, u(x, 0) - u_0(x)\} &= 0, \quad (x, t) \in Q_T, \\
u(x, t) &= 0, \quad (x, t) \in \Gamma_T, \\
u(x, 0) &= u_0(x), \quad x \in \Omega,
\end{align*}
\]

with

\[
Lu = u_t - \text{div}(a(u)|\nabla u|^{p(x,t)-2}\nabla u) - f(x, t), \quad a(u) = u^\sigma + d_0,
\]

where \(\Omega \subset \mathbb{R}^+\) is a bounded simply connected domain, \(Q_T = \Omega \times (0, T]\), and \(\Gamma_T\) denotes the lateral boundary of the cylinder \(Q_T\).

This type of variational inequality was studied initially by Chen and Yi [1], who proposed the equation

\[
\begin{align*}
\frac{\partial}{\partial t} V - \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial x^2} V - (r - \frac{1}{2} \sigma^2) \frac{\partial}{\partial x} V + rV &\geq 0 \quad \text{in } \Omega_T, \\
V &\geq g(x), \quad \text{in } \Omega_T, \\
(\frac{\partial}{\partial t} V - \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial x^2} V - (r - \frac{1}{2} \sigma^2) \frac{\partial}{\partial x} V + rV)(V - g(x)) &= 0 \quad \text{in } \Omega_T, \\
V(t, x) &= 0 \quad \text{on } \partial \Omega_T, \\
V(x, 0) &= g(x) \quad \text{in } \Omega,
\end{align*}
\]

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for modeling the American option. When \( r \) and \( \sigma \) are positive constants, the existence and uniqueness of solutions to problem (4) were also studied in [2–4].

In 2014, the authors in [5] discussed the problem

\[
\begin{aligned}
&u_t - Lu - F(x, t, u, \nabla u) \geq 0 \quad \text{in } Q_T, \\
u(x, t) \geq u_0(x) \quad \text{in } Q_T, \\
(u_t - Lu - F(x, t, u, \nabla u))(u - u_0(x)) = 0 \quad \text{in } Q_T, \\
u(x, 0) = u_0(x) \quad \text{on } \Omega, \\
u(x, t) = g(x) \quad \text{on } \delta \Omega \times (0, T)
\end{aligned}
\]

with second-order elliptic operator

\[
L(x, t) = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_j} \left( a^{ij}(x, t) \frac{\partial}{\partial x_i} \right) - \sum_{i,j=1}^{d} b^j(x, t) \frac{\partial}{\partial x_i} - c(x, t).
\]

They proved the existence and uniqueness of a solution to this problem with some conditions on \( u_0, F, \) and \( L \). Later, the authors in [6, 7] extended the relative conclusions with the assumption that \( a(u) \) and \( p(x) \) are two positive constants. The author discussed the existence and uniqueness of a solution by the penalty method.

The existence and uniqueness of such a problem with the assumption that \( p(x) \) and \( a(u) \) are variables were less studied.

The aim of this paper is to study the existence and uniqueness of solutions for a degenerate parabolic variational inequality problem. Throughout the paper, we assume that the exponent \( p(x, t) \) is continuous in \( Q = \overline{Q_T} \) with logarithmic module of continuity:

\[
1 < p^- = \inf_{(x,t) \in Q} p(x,t) \leq p(x,t) \leq p^+ = \sup_{(x,t) \in Q} p(x,t) < \infty,
\]

\[
\forall \zeta = (x, t) \in Q, \quad \xi = (y, s) \in Q_T, \quad |\zeta - \xi| < 1, \quad |p(\zeta) - p(\xi)| \leq \omega(|\zeta - \xi|),
\]

where

\[
\lim_{t \to 0^+} \sup_{t < \lambda} \omega(t) \ln \frac{1}{\lambda} = C < +\infty.
\]

The outline of this paper is as follows. In Section 2, we introduce the function spaces of Orlicz-Sobolev type, give the definition of a weak solution to the problem, and prove the existence and uniqueness. Section 3 is devoted to the proof of the existence and uniqueness of the solution obtained in Section 2.

## 2 Basic spaces and the main results

To study our problems, let us introduce the Banach spaces:

\[
L^{p(x, \cdot)}(Q_T) = \left\{ u(x, t) \mid u \text{ is measurable in } Q_T, A_{p(\cdot)}(u) = \int_{Q_T} |u|^{p(\cdot)} \, dx \, dt < \infty \right\},
\]

\[
\|u\|_{p(\cdot)} = \inf \left\{ \lambda > 0, A_{p(\cdot)}(u/\lambda) \leq 1 \right\},
\]

\[
V_{tau}(\Omega) = \left\{ u \mid u \in L^2(\Omega) \cap W^{1,1}_0(\Omega), |\nabla u| \in L^{p(x, \cdot)}(\Omega) \right\}, \quad \|u\|_{V_{tau}(\Omega)} = \|u\|_{2, \Omega} + |\nabla u|_{p(\cdot); \Omega},
\]

\[
V_{\tau}(\Omega) = \left\{ u \mid u \in L^2(\Omega) \right\}, \quad \|u\|_{V_{\tau}(\Omega)} = \|u\|_{2, \Omega},
\]

\[
L^{p(x, \cdot)}(\Omega) = \left\{ u \mid u \text{ is measurable in } \Omega, A_{p(\cdot)}(u) = \int_{\Omega} |u|^{p(\cdot)} \, dx < \infty \right\},
\]

\[
\|u\|_{p(\cdot)} = \inf \left\{ \lambda > 0, A_{p(\cdot)}(u/\lambda) \leq 1 \right\},
\]

\[
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\]

\[
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\]

\[
L^{p(x, \cdot)}(\Omega) = \left\{ u \mid u \text{ is measurable in } \Omega, A_{p(\cdot)}(u) = \int_{\Omega} |u|^{p(\cdot)} \, dx < \infty \right\},
\]

\[
\|u\|_{p(\cdot)} = \inf \left\{ \lambda > 0, A_{p(\cdot)}(u/\lambda) \leq 1 \right\},
\]

\[
V_{\tau}(\Omega) = \left\{ u \mid u \in L^2(\Omega) \cap W^{1,1}_0(\Omega), |\nabla u| \in L^{p(x, \cdot)}(\Omega) \right\}, \quad \|u\|_{V_{\tau}(\Omega)} = \|u\|_{2, \Omega} + |\nabla u|_{p(\cdot); \Omega},
\]

\[
V_{\tau}(\Omega) = \left\{ u \mid u \in L^2(\Omega) \right\}, \quad \|u\|_{V_{\tau}(\Omega)} = \|u\|_{2, \Omega},
\]

\[
L^{p(x, \cdot)}(\Omega) = \left\{ u \mid u \text{ is measurable in } \Omega, A_{p(\cdot)}(u) = \int_{\Omega} |u|^{p(\cdot)} \, dx < \infty \right\},
\]

\[
\|u\|_{p(\cdot)} = \inf \left\{ \lambda > 0, A_{p(\cdot)}(u/\lambda) \leq 1 \right\},
\]

\[
V_{\tau}(\Omega) = \left\{ u \mid u \in L^2(\Omega) \cap W^{1,1}_0(\Omega), |\nabla u| \in L^{p(x, \cdot)}(\Omega) \right\}, \quad \|u\|_{V_{\tau}(\Omega)} = \|u\|_{2, \Omega} + |\nabla u|_{p(\cdot); \Omega},
\]

\[
V_{\tau}(\Omega) = \left\{ u \mid u \in L^2(\Omega) \right\}, \quad \|u\|_{V_{\tau}(\Omega)} = \|u\|_{2, \Omega},
\]
\[ W(Q_T) = \{ u : [0, T] \mapsto V_1(\Omega)| u \in L^2(Q_T), |\nabla u| \in L^{p(x,t)}(Q_T), u = 0 \text{ on } \Gamma_T \}, \]
\[ \|u\|_{W(Q_T)} = \|u\|_{L^2(Q_T)} + \|\nabla u\|_{L^{p(x,t)}(Q_T)} \]

and denote by \( W'(Q_T) \) the dual of \( W(Q_T) \) with respect to the inner product in \( L^2(Q_T) \).

In spirit of [3] and [4], we introduce the following maximal monotone graph:

\[ G(\lambda) = \begin{cases} 
0, & \lambda > 0, \\
[0, +\infty), & \lambda = 0. 
\end{cases} \]

In addition, we define the following function class for the solution:

\[ B = \{ u \in W(Q_T) \cap L^\infty(0, T; L^\infty(\Omega)) \}. \]

**Definition 2.1** A pair \((\mu, \xi) \in B \times L^\infty(\Omega_T)\) is called a weak solution of problem (1) if (a) \( u(x, t) \leq u_0(x) \), (b) \( u(x, 0) = u_0(x) \), (c) \( \xi \in G(\mu - u_0) \), (d) for all \( t_1, t_2 \in [0, T] \), the following identity holds:

\[
\int_{t_1}^{t_2} \int_\Omega \left[ u\phi_t - (u^\sigma + d_0) |\nabla u|^{p(x,t)-2} \nabla u \cdot \nabla \phi + f(x,t)\phi - \xi \phi \right] \, dx \, dt = \int_{\Omega} u\phi \, dx \bigg|_{t_1}^{t_2}.
\]

The main theorem in this section is the following:

**Theorem 2.1** Let \( p(x, t) \) satisfy conditions (3)–(4). Suppose also that the following conditions hold:

- \((H_1)\) \( \max\{1, \frac{2N}{N+2}\} < p^- < N, \ 2 \leq \sigma < \frac{2p^-}{p^- - 1} \),
- \((H_2)\) \( u_0 \geq 0, \ f \geq 0, \ \|u_0\|_{\infty, \Omega} + \int_0^T \|f(x,t)\|_{\infty, \Omega} dt = K(T) < \infty \).

Then problem (1) has at least one weak solution in the sense of Definition 2.1.

**Theorem 2.2** Suppose that the conditions in Theorem 2.1 are fulfilled and \( p^* \geq 2 \). Then problem (1) admits a unique solution in the sense of Definition 2.1.

### 3 Proof of the main results

In this section, we consider the family of auxiliary parabolic problems

\[
\begin{align*}
L_{\varepsilon} u_\varepsilon + \beta_\varepsilon (u_\varepsilon - u_0) &= 0, \quad (x, t) \in Q_T, \\
u(x, t) &= \varepsilon, \quad (x, t) \in \Gamma_T, \\
u(x, 0) &= u_0(x) + \varepsilon, \quad x \in \Omega.
\end{align*}
\]

Here, \( M \) is a positive parameter to be chosen later. Moreover,

\[
L_{\varepsilon} u_\varepsilon = \partial_t u_\varepsilon - \text{div}(a_{\varepsilon,M}(u_\varepsilon)|\nabla u_\varepsilon|^{p(x,t)-2}\nabla u_\varepsilon) - f(x,t),
\]

\[
0 < d_0 \leq a_{\varepsilon,M}(u) = (\min\{|u|^2, M^2\} + \varepsilon^2)^{\frac{1}{2}} + d_0 \leq (M^2 + 1) + d_0, \quad 0 < \varepsilon < 1,
\]

\[
\|u_\varepsilon\|_{L^2(Q_T)} + \|\nabla u_\varepsilon\|_{L^{p(x,t)}(Q_T)} \leq M + d_0,
\]

\[
\|\nabla u_\varepsilon\|_{L^{p(x,t)}} \leq M + d_0.
\]

Since \( P(\varepsilon) \) is independent of \( \varepsilon \), it follows that there exists a weak solution \( v \) of problem (1) satisfying conditions (3)–(4).
Following a similar method as in [6], we can prove that the regularized problem has a unique weak solution

\begin{align*}
\beta_{\epsilon}(x) & \geq 0, \\
\beta_{\epsilon}^{\prime}(x) & \leq 0, \\
\lim_{\epsilon \to 0} \beta_{\epsilon}(x) & = \begin{cases} 
0, & x > 0, \\
-\infty, & x < 0. 
\end{cases}
\end{align*}

(6)

We start with two preliminary results that will be used several times.

\textbf{Lemma 3.1} Let \( M(s) = |s|^{p(x,t)-2}s \). Then for all \( \xi, \eta \in \mathbb{R}^N \),

\( (M(\xi) - M(\eta))(\xi - \eta) \geq \begin{cases} 
2^{-p(x,t)}|\xi - \eta|^{p(x,t)}, & 2 \leq p(x,t) < \infty, \\
(p(x,t) - 1)|\xi - \eta|^2(|\xi|^{p(x,t)} + |\eta|^{p(x,t)})^{\frac{p(x,t)-2}{p(x,t)}}, & 1 < p(x,t) < 2. 
\end{cases} \)

\textbf{Lemma 3.2} (Comparison principle) Assume that \( 2 < \sigma < \frac{2p'}{p'-2}, p^* \geq 2 \), and \( u \) and \( v \) are in \( W(Q_T) \cap L^\infty(Q_T) \). If \( L_s u \geq L_s v \) in \( Q_T \) and if \( u(x,t) \leq v(x,t) \) on \( \partial Q_T \), then \( u(x,t) \leq v(x,t) \) in \( Q_T \).

\textbf{Proof} We argue by contradiction. Suppose \( u(x,t) \) and \( v(x,t) \) satisfy \( L_s u \geq L_s v \) in \( Q_T \) and there is \( \delta > 0 \) such that for \( 0 < \tau \leq T, w = u - v > \delta \) on the set \( \Omega^\delta = \Omega \cap \{ x : w(x,t) > \delta \} \), and \( \mu(\Omega^\delta) > 0 \). Let

\( F_{\epsilon}(\xi) = \begin{cases} 
\frac{1}{\alpha^{-1}}\xi^{1-\alpha} - \frac{1}{\alpha^{-1}}\xi^{1-\alpha} & \text{if } \xi > \epsilon, \\
0 & \text{if } \xi \leq \epsilon,
\end{cases} \)

where \( \delta > 2\epsilon > 0 \) and \( \alpha = \frac{\sigma}{\tau} \). Let a test-function \( \xi = F_{\epsilon}(w) \in Z \) in (8). Then

\begin{align*}
0 & \geq \int_{Q_T} \left[ \xi w F_{\epsilon}(w) + (\sigma')^2 + d_0 \right] \left[ |\nabla u|^{p(x,t)-2} \nabla u - |\nabla v|^{p(x,t)-2} \nabla v \right] \nabla F_{\epsilon}(w) \, dx \, dt \\
& \quad + \int_{Q_T} (u^\sigma - v^\sigma)|\nabla u|^{p(x,t)-2} \nabla u \nabla F_{\epsilon}(w) \, dx \, dt = J_1 + J_2 + J_3,
\end{align*}

(9)
where $Q_{t,x} = Q_t \cap \{(x,t) \in Q_T | w > \varepsilon \},$

\[ J_1 = \int \int_{Q_{t,x}} w_t F_t(w) \, dx \, dt, \quad J_2 = \int \int_{Q_T} (u^\alpha - v^\alpha) w^{-\alpha} (|\nabla u|^{p(x,t)-2} \nabla u \nabla w) \, dx \, dt, \]

\[ J_3 = \int \int_{Q_T} (v^\sigma + d_0) w^{-\sigma} (|\nabla u|^{p(x,t)-2} \nabla u - |\nabla v|^{p(x,t)-2} \nabla v) \nabla w \, dx \, dt. \]

Now, let $t_0 = \inf\{t \in (0, \tau] : w > \varepsilon \}$. Then we estimate $J_1$, $J_2$, and $J_3$ as follows:

\[ J_1 = \int \int_{Q_{t,x}} w_t F_t(w) \, dx \, dt = \int_{\Omega} \left( \int_{0}^{t_0} w_t F_t(w) \, dt + \int_{t_0}^{\tau} w_t F_t(w) \, dt \right) dx \]

\[ \geq \int \int_{Q_{t,x}} F_t(s) \, ds \, dx \geq \int \int_{Q_{x,t}} F_t(s) \, ds \, dx \]

\[ \geq \int_{\Omega} (w - 2\varepsilon) F_t(2\varepsilon) \, dx \geq (\delta - 2\varepsilon) F_t(2\varepsilon) \mu(\Omega) \quad (10) \]

Let us first consider the case $p^- \geq 2$. By the first inequality of Lemma 3.1 we get

\[ J_2 = \int \int_{Q_{t,x}} (v^\sigma + d_0) w^{-\sigma} (|\nabla u|^{p(x,t)-2} \nabla u - |\nabla v|^{p(x,t)-2} \nabla v) \nabla w \, dx \, dt \]

\[ \geq \int \int_{Q_{t,x}} (v^\sigma + d_0) w^{-\sigma} 2^{-p(x,t)} |\nabla w|^{p(x,t)} \, dx \, dt \]

\[ \geq 2^{-p^-} \int \int_{Q_{t,x}} (v^\sigma + d_0) w^{-\sigma} |\nabla w|^{p(x,t)} \, dx \, dt \geq 0. \quad (11) \]

Noting that $\frac{p(x,t)}{p^- - 1} \geq \frac{p^-}{p^- - 1} \geq \frac{\sigma}{2} = \alpha > 1$ and applying Young’s inequality, we can estimate the integrand of $J_3$ in the following way:

\[ \left| (u^\alpha - v^\alpha) w^{-\alpha} |\nabla w|^{p(x,t)-2} \nabla u \nabla w \right| \]

\[ = \left| \sigma w \left( \int_{0}^{1} (\theta u + (1 - \theta)v)^{\sigma - 1} d\theta w^{-\sigma} |\nabla w|^{p(x,t)-2} \nabla u \nabla w \right) \right| \]

\[ \leq C \left[ \frac{v^\sigma + d_0}{C} \right] |\nabla w|^{p(x,t)} + C_1(\sigma, d_0, K(T), p^\pm) |\nabla w|^{p(x,t)-\alpha} |\nabla u|^{p(x,t)} ] \]

\[ = \left( \frac{v^\sigma + d_0}{2^{p^- + 2^{p^-}} w^{\sigma}} \right) |\nabla w|^{p(x,t)} + C_1(\sigma, d_0, K(T), p^\pm) |\nabla w|^{p(x,t)-\alpha} |\nabla u|^{p(x,t)} \]

\[ \leq \left( \frac{v^\sigma + d_0}{2^{p^- + 2^{p^-}} w^{\sigma}} \right) |\nabla w|^{p(x,t)} + C_1(\sigma, d_0, K(T), p^\pm) |\nabla w|^{p(x,t)}. \quad (12) \]

Substituting (12) into $J_3$, we get

\[ J_3 \leq \frac{1}{2} J_2 + C \int \int_{Q_{t,x}} |\nabla u|^{p(x,t)} \, dx \, dt. \quad (13) \]
Second, we consider the case $1 < p^- \leq p(x, t) < 2$, $p^+ \geq 2$. According to the second inequality of Lemma 3.1, it is easily seen that the following inequalities hold:

$$J_2 = \int \int_{Q_\varepsilon, \tau} (\nu^\alpha + d_0) w^{-\alpha} (|\nabla w|^{p(x,t)-2} \nabla U - |\nabla v|^{p(x,t)-2} \nabla v) \nabla w \, dx \, dt \geq (p^- - 1) \int \int_{Q_\varepsilon, \tau} (\nu^\alpha + d_0) w^{-\alpha} (|\nabla w| + |\nabla v|)^{p(x,t)-2} |\nabla w|^2 \, dx \, dt. \quad (14)$$

Using the conditions $1 < \alpha \leq p^- + p^+ \leq 2$ and Young’s inequality, we can evaluate the integrand of $J_3$ as follows:

$$|((\sigma^\alpha - \nu^\alpha) w^{-\alpha} |\nabla w|^{p(x,t)-2} \nabla u \nabla w| = |\sigma w \int_0^1 (\theta u + (1-\theta)\nu)^{\sigma-1} \, d\theta w^{-\alpha} |\nabla w|^{p(x,t)-2} \nabla u \nabla w| \leq \frac{(\nu^\alpha + d_0)(p^- - 1)}{2w^\alpha} (|\nabla w| + |\nabla v|)^{p(x,t)-2} |\nabla w|^2$$

$$+ C_1 (\sigma, d_0, K(T), p^+) |w|^{2-\alpha} |\nabla w| + |\nabla v|^{p(x,t)}. \quad (15)$$

Plugging (15) into $J_3$, we get

$$J_3 \leq \frac{1}{2} J_2 + C \int \int_{Q_\varepsilon, \tau} |\nabla w| + |\nabla v|^{p(x,t)} \, dx \, dt. \quad (16)$$

Plugging estimates (10), (11), (13) and (10), (14), (16) into (9) and dropping the nonnegative terms, we arrive at the inequality

$$(\delta - 2\varepsilon)(1 - 2^{1-\alpha})e^{1-\alpha} \mu(\Omega_\varepsilon) \leq \tilde{C} \quad (17)$$

with a constant $\tilde{C}$ independent of $\varepsilon$.

Notice that $\lim_{\varepsilon \to 0} (\delta - 2\varepsilon)(1 - 2^{1-\alpha})e^{1-\alpha} \mu(\Omega_\varepsilon) = +\infty$, a contradiction. This means that $\mu(\Omega_\varepsilon) = 0$ and $w \leq 0$ a.e. in $Q_T$. \hfill \square

**Lemma 3.3** Let $u_\varepsilon$ be weak solutions of (5). Then

$$u_0 \leq u_\varepsilon \leq |u_0|_{\infty} + \varepsilon, \quad (18)$$

$$u_{\varepsilon_1} \leq u_{\varepsilon_2} \quad \text{for} \quad \varepsilon_1 \leq \varepsilon_2. \quad (19)$$

**Proof** First, we prove $u_\varepsilon \geq u_0 \varepsilon$ by contradiction. Assume that $u_\varepsilon \leq u_0 \varepsilon$ in $Q^0_T$, $Q^0_T \subset Q_T$. Noting that $u_\varepsilon \geq u_0 \varepsilon$ on $\partial Q_T$, we may assume that $u_\varepsilon = u_0 \varepsilon$ on $\partial Q^0_T$. With (5) and letting $t = 0$, we deduce that

$$Lu_0 \varepsilon = \beta_\varepsilon (u_0 \varepsilon - u_0 \varepsilon) = -1, \quad (20)$$

$$Lu_\varepsilon = \beta_\varepsilon (u_\varepsilon - u_0 \varepsilon) \leq -1. \quad (21)$$
From Lemma 3.2 we conclude that
\[ u(x, t) \leq u_0(x) \quad \text{for any } (x, t) \in \Omega_T, \]

obtaining a contradiction.

Second, we pay attention to \( u_\varepsilon(t, x) \leq |u_0|_\infty + \varepsilon \). Applying the definition of \( \beta_\varepsilon(\cdot) \), we have
\[ L(|u_0|_\infty + \varepsilon) = 0, \quad Lu_\varepsilon \leq 0. \]

From (5) it is easy to prove that \( u_\varepsilon(x, t) \geq \varepsilon \) on \( \partial \Omega \times (0, T) \) and \( u_0 \varepsilon(x) \geq \varepsilon \) in \( \Omega \). Thus, combining (21) and (23) and repeating Lemma 3.3, we have
\[ u_\varepsilon(x, t) \geq \varepsilon \quad \text{in } \Omega \times (0, T). \]

Third, we aim to prove (19). Since
\[ Lu_1 - \beta_{\varepsilon_1}(u_0 - u_\varepsilon_1) = 0, \]
\[ Lu_2 - \beta_{\varepsilon_2}(u_0 - u_\varepsilon_2) = 0. \]

It follows by \( \varepsilon_1 \leq \varepsilon_2 \) and the definition of \( \beta_\varepsilon(\cdot) \) that
\[ \partial_t u_\varepsilon_2 - Lu_\varepsilon_2 - \beta_{\varepsilon_2}(u_0 - u_\varepsilon_2) \]
\[ = \beta_{\varepsilon_2}(u_0 - u_\varepsilon_2) - \beta_{\varepsilon_1}(u_0 - u_\varepsilon_1) \geq \beta_{\varepsilon_2}(u_0 - u_\varepsilon_2) - \beta_{\varepsilon_1}(u_0 - u_\varepsilon_1) \geq 0. \]

Thus, Lemma 3.3 can be proved by combining initial and boundary conditions in (5). \( \square \)

Moreover, with (18), we assert that there exists a subsequence \( \varepsilon \) (still denoted by \( \varepsilon \)) such that
\[ u_\varepsilon \rightarrow u \in L^p(0, T; W_{0, \Omega}^{1, p}(\Omega_T)) \quad \text{as } \varepsilon \rightarrow 0, \]
\[ u_\varepsilon \geq u \geq 0 \quad \text{for any } \varepsilon > 0. \]

\textbf{Lemma 3.4} Let \( u_\varepsilon \) be a solution of problem (5). For any \( \varepsilon > 0 \), we have
\[ ||u_\varepsilon||_{\infty, \Omega_T} \leq ||u_0||_{\infty, \Omega} + \int_0^T ||f(x, t)||_{\infty, \Omega} \, dt = K(T) < \infty. \]

\textbf{Proof} Define
\[ u_{\varepsilon, M} = \begin{cases} 
M & \text{if } u_\varepsilon > M, \\
u_\varepsilon & \text{if } |u_\varepsilon| \leq M, \\
-M & \text{if } u_\varepsilon < -M.
\end{cases} \]
Choosing $u_{t,M}^{2k-1}$ as a test-function in (8) and letting $t_1 = t$ and $t_2 = t + h$, we conclude that

$$
\frac{1}{2k} \int_t^{t+h} \frac{d}{dt} \left( \int_\Omega u_{t,M}^{2k} \right) dt + \int_t^{t+h} \int_\Omega (2k - 1) a_{t,M}(u_{t,M}) u_{t,M}^{2(k-1)} |\nabla u_{t,M}|^{p(x,t)} dx dt
$$

$$
= \int_t^{t+h} \int_\Omega f u_{t,M}^{2k-1} dx dt - \int_t^{t+h} \int_\Omega \beta_a u_{t,M}^{2k-1} dx.
$$

(31)

Letting $h \to 0$ and applying Lebesgue’s dominated convergence theorem, we have that, for all $t \in (0, T)$,

$$
\frac{1}{2k} \frac{d}{dt} \int_\Omega u_{t,M}^{2k} dx + \int_\Omega (2k - 1) a_{t,M}(u_{t,M}) u_{t,M}^{2(k-1)} |\nabla u_{t,M}|^{p(x,t)} dx
$$

$$
= \int_\Omega f u_{t,M}^{2k-1} dx - \int_\Omega \beta_a u_{t,M}^{2k-1} dx.
$$

(32)

Using Holder’s inequality, we obtain

$$
\int \int_\Omega \beta_a u_{t,M}^{2k-1} dx \leq \|u_{t,M}(\cdot,t)\|_{2k,\Omega}, \quad \|f(\cdot,t)\|_{2k,\Omega}, \quad k = 1, 2, \ldots,
$$

$$
\int \int_\Omega \beta_a u_{t,M}^{2k-1} dx \leq \|u_{t,M}(\cdot,t)\|_{2k,\Omega}, \quad \|u_{t,M}(\cdot,t)\|_{2k,\Omega}, \quad (33)
$$

By integration over $(0, t)$, for all $t$, we have

$$
\|u_{t,M}(\cdot,t)\|_{2k,\Omega} \leq \|u_{t,M}(\cdot,0)\|_{2k,\Omega} + \int_0^T \|f\|_{2k,\Omega} dt + C(T) \|u_{t,M}\|_{2k,\Omega}, \quad k = 1, 2, \ldots.
$$

Then, as $k \to \infty$,

$$
\|u_{t,M}(\cdot,t)\|_{\infty,\Omega} \leq \|u_{t,M}(\cdot,0)\|_{\infty,\Omega} + \int_0^T \|f\|_{\infty,\Omega} dt
$$

$$
\leq \|u_0\|_{\infty,\Omega} + \int_0^T \|f\|_{\infty,\Omega} dt + C(T) = K(T).
$$

If we chose $M > K(T)$ then $u_{t,M}(\cdot,t) \leq \sup |u_{t,M}(\cdot,t)| \leq K(T) < M$, and therefore $u_{t,M}(\cdot,t) = u_{t}(\cdot,t)$.

**Corollary 3.1** Choosing $M$ large enough, we have

$$
\min \{u_{t,M}^2, M^2\} = u_{t}^2 \quad \text{and} \quad a_{t,M}(u_{t,M}) = a_{t,M}(u_t) = (\varepsilon^2 + u_t^2)^{\sigma/2} + d_0.
$$

**Corollary 3.2** If $u_0 \geq 0$ and $f \geq 0$, then the solution $u(\cdot,t)$ is nonnegative in $Q_T$. 
Lemma 3.6 The solution of problem (5) satisfies the estimates

\[
\int \int_{Q_T} u_t^p |\nabla u_t|^{p(x,t)} \, dx \, dt \leq K(T)|\Omega|^\frac{1}{2}, \tag{34}
\]

\[
\varepsilon^{\sigma} \int \int_{Q_T} |\nabla u_t|^{p(x,t)} \, dx \, dt \leq K(T)|\Omega|^\frac{1}{2}, \tag{35}
\]

\[
d_0 \int \int_{Q_T} |\nabla u_t|^{p(x,t)} \, dx \, dt \leq K(T)|\Omega|^\frac{1}{2}. \tag{36}
\]

Proof Similarly as in Lemma 3.4, we take \( k = 1 \) in (32) to get

\[
\frac{d}{dt} \| u_k(\cdot,t) \|_{2,\Omega} + \int_\Omega a_{\varepsilon,M}(u_k) |\nabla u_k|^{p(x,t)} \, dx \leq \| f \|_{2,\Omega} + C(T), \quad \forall t \in (0,T).
\]

Clearly, integrating over \((0,t)\), we have

\[
\| u_k(\cdot,t) \|_{2,\Omega} + \int_0^t \int_\Omega a_{\varepsilon,M}(u_k) |\nabla u_k|^{p(x,t)} \, dx \, dt \leq \| u_k(\cdot,t) \|_{2,\Omega} + \int_0^T \| f \|_{2,\Omega} \, dt + C(T). \tag{37}
\]

Note that the first term on the left-hand side is nonnegative. We drop the nonpositive term in (37) to get

\[
\int_0^t \int_\Omega a_{\varepsilon,M}(u_k) |\nabla u_k|^{p(x,t)} \, dx \, dt \leq K(T)|\Omega|^\frac{1}{2}.
\]

If \( a_{\varepsilon,M}(u_k) \geq d_0 \), then we have inequality (36), and if \( a_{\varepsilon,M}(u_k) \geq \varepsilon^{\sigma} \), then we have inequality (35) such that \( M > K(T) \), and we have \( a_{\varepsilon,M}(u_k) \geq u_t^p \). Furthermore, we get inequality (34). \qed

Lemma 3.5 The solution of problem (5) satisfies the estimates

\[
\| u_t \|_{W^1(Q_T)} \leq C(\sigma, p^\pm, K(T), |\Omega|). \tag{38}
\]

Proof From identity (7) we get

\[
\int \int_{Q_T} u_t \xi \, dx \, dt = -\int \int_{Q_T} \left[ (u_t^p + \varepsilon^2)^{\sigma/2} + d_0 \right] |\nabla u_t|^{p(x,t)-2} \nabla u_t \nabla \xi \, dx \, dt + \int \int_{Q_T} f(x,t) \xi (x,t) \, dx \, dt - \int \int_{Q_T} \beta_c(x,t) \xi(x,t) \, dx \, dt.
\]
Applying the fact that $\beta_\varepsilon(x, t) \in (0, 1)$, we get

$$
\int_{Q_T} u_{\varepsilon t} \xi \, dx \, dt \leq \int_0^T \int_{\Omega} \left[ (u_{\varepsilon}^n + \varepsilon^2) \right] \frac{1}{\varepsilon} \big| \nabla u_{\varepsilon} \big|^{p(x,t)-1} \nabla u_{\varepsilon} \big| \nabla \xi \big| \, dx \, dt \\
+ \int_0^T \int_{\Omega} |f + 1| \cdot |\xi| \, dx \, dt.
$$

Using the Hölder inequality repeatedly, we have that

$$
\int_{Q_T} u_{\varepsilon t} \xi \, dx \, dt \leq 2 \left[ (u_{\varepsilon}^n + \varepsilon^2) \right] \frac{1}{\varepsilon} \big| \nabla u_{\varepsilon} \big|^{p(x,t)-1} \| \nabla \xi \|_{p(x,t)} \\
+ 2 \| f + 1 \|_{p'(x,t)} \cdot \| \xi \|_{p(x,t)} \\
\leq 2 \max\{F_1, F_2\} \| \nabla \xi \|_{p(x,t)} + 2 \max\{F_3, F_4\} \| \xi \|_{p(x,t)} \\
\leq \left( 2((K^2(T) + 1) \right) \frac{1}{p-1} K(T) |\Omega| + 2| f + 1|_{\infty} | T \right) \| \xi \|_{W(Q_T)},
$$

where

$$
F_1 = \left( \int_0^T \int_{\Omega} \left[ (u_{\varepsilon}^n + \varepsilon^2) \right] \frac{1}{\varepsilon} \big| \nabla u_{\varepsilon} \big|^{p(x,t)-1} \, dx \, dt \right) \frac{1}{p-1},
$$

$$
F_2 = \left( \int_0^T \int_{\Omega} \left[ (u_{\varepsilon}^n + \varepsilon^2) \right] \frac{1}{\varepsilon} \big| \nabla u_{\varepsilon} \big|^{p(x,t)-1} \, dx \, dt \right) \frac{1}{p-1},
$$

$$
F_3 = \left( \int_0^T \int_{\Omega} |f|^{p'(x,t)} \, dx \, dt \right) \frac{1}{p'},
$$

$$
F_4 = \left( \int_0^T \int_{\Omega} |f + 1|^{p'(x,t)} \, dx \, dt \right) \frac{1}{p'}.
$$

Then (38) follows from Lemma 3.5. \qed

From [6] we can get the following inclusions:

$$
u_{\varepsilon} \in W(Q_T) \subseteq L^p(0, T; W_0^{1,p}(\Omega)), \quad u_{\varepsilon t} \in W'(Q_T) \subseteq L^{p'}(0, T; V_t(\Omega)),
$$

$$
W_0^{1,p}(\Omega) \subset L^2(\Omega) \subset V_t(\Omega) \quad \text{with} \quad V_t(\Omega) = \{ u(x) | u \in L^2(\Omega) \cap W_0^{1,1}(\Omega), |\nabla u| \in L^{p'} \}.
$$

These conclusions, together with the uniform estimates in $\varepsilon$, allow us to extract from the sequence $\{u_{\varepsilon}\}$ a subsequence (for simplicity, we assume that it merely coincides with the whole sequence) such that

\begin{align*}
&u_{\varepsilon} \to u \quad \text{a.e. in } Q_T, \\
&\nabla u_{\varepsilon} \to \nabla u \quad \text{weakly in } L^{p(x,t)}(Q_T), \\
&u_{\varepsilon}^{n} \nabla u_{\varepsilon} |^{p(x,t)-2} \partial_{ij} u_{\varepsilon} \to A_j(x, t) \quad \text{weakly in } L^{p'(x,t)}(Q_T), \\
&|\nabla u_{\varepsilon} |^{p(x,t)-2} \partial_{ij} u_{\varepsilon} \to W_j(x, t) \quad \text{weakly in } L^{p'(x,t)}(Q_T)
\end{align*}

for some functions $u \in W(Q_T), A_j(x, t) \in L^{p'(x,t)}(Q_T),$ and $W_j(x, t) \in L^{p'(x,t)}(Q_T)$.

**Lemma 3.7** For almost all $(x, t) \in Q_T$,

$$
\lim_{\varepsilon \to 0^+} \int_{Q_T} \left( (u_{\varepsilon}^n + \varepsilon^2) \right) \frac{1}{\varepsilon} \big| \nabla u_{\varepsilon} \big|^{p(x,t)-2} \nabla u_{\varepsilon} \nabla \xi \, dx \, dt = 0, \quad \forall \xi = W(Q_T).
$$
Proof. We first compute

\[
\begin{align*}
I & \triangleq \int \int_{Q_T} \left( (u_e^2 + \varepsilon^2)^{\frac{p}{2}} - u_e^2 \right) |\nabla u_e|^{p(x,t)-2} \nabla u_e \cdot \nabla \xi \, dx \, dt \\
& = \frac{\sigma}{2} \varepsilon^2 \int \int_{Q_T} \left( \int_0^1 (u_e^2 + s \varepsilon^2)^{\frac{p-2}{2}} ds \right) |\nabla u_e|^{p(x,t)-2} \nabla u_e \cdot \nabla \xi \, dx \, dt \\
& \leq \sigma \varepsilon^2 (K^2(T) + 1) \frac{2^{p-1}}{p} \|\nabla u_e\|_{p(x,t)} \|\nabla \xi\|_{p(x,t)} \\
& \leq C \varepsilon^2 \max \left\{ \left( \int \int_{Q_T} |\nabla u_e|^{p(x,t)} \, dx \, dt \right)^{\frac{p-1}{p}}, \left( \int \int_{Q_T} |\nabla u_e|^{p(x,t)} \, dx \, dt \right)^{\frac{p-1}{p}} \right\} \|\nabla \xi\|_{p(x,t)}.
\end{align*}
\]

By (35) we get

\[
I \leq C \varepsilon^{2-\frac{2(p-1)}{p}} \|\nabla \xi\|_{p(x,t)}.
\]

Passing to the limit as \( \varepsilon \to 0 \), we obtain Lemma 3.7. \qed

Lemma 3.8 For almost all \((x,t) \in Q_T\), we have

\[
A_i(x,t) = u\sigma W_i(x,t), \quad i = 1, 2, \ldots, N.
\]

Proof. In (39), letting \( \varepsilon \to 0 \), we have

\[
\begin{align*}
\int \int_{Q_T} u_e^2 |\nabla u_e|^{p(x,t)-2} \nabla u_e \cdot \nabla \xi \, dx \, dt & \to \sum \int \int_{Q_T} A_i(x,t) D_i \xi \, dx \, dt, \\
\int \int_{Q_T} u_e^2 |\nabla u_e|^{p(x,t)-2} \nabla u_e \cdot \nabla \xi \, dx \, dt & \to \sum \int \int_{Q_T} W_i(x,t) D_i \xi \, dx \, dt.
\end{align*}
\]

By Lebesgue’s dominated convergence theorem we have

\[
\lim_{\varepsilon \to 0} \sum_{i=1}^N \int \int_{Q_T} (u_e^2 - u^2) A_i(x,t) D_i \xi \, dx \, dt = 0.
\]

So

\[
\lim_{\varepsilon \to 0} \sum_{i=1}^N \int \int_{Q_T} (u_e^2 - u^2) |\nabla u_e|^{p(x,t)-2} D_i u_e - u^2 W_i(x,t) D_i \xi \, dx \, dt = 0.
\]

By (40)–(42) and the previous inequalities, we complete the proof of Lemma 3.8. \qed

Lemma 3.9 For almost all \((x,t) \in Q_T\), we have

\[
W_i(x,t) = |\nabla u_e|^{p(x,t)-2} D_i u, \quad i = 1, 2, \ldots, N.
\]
On the other hand, from that if \( \xi \) we have

\[
\int \int_{Q_T} [u_x(u_x - u)\Phi + \Phi(u_x^2 + d_0)|\nabla u_x|^{p(x,t)-2}\nabla u_x \nabla (u_x - u)] \, dx \, dt \\
+ \int \int_{Q_T} [(u_x - u)(u_x^2 + d_0)|\nabla u_x|^{p(x,t)-2}\nabla u_x \nabla \Phi - f(x,t)(u_x - u)\Phi] \, dx \, dt \\
+ \int \int_{Q_T} (u_x^\sigma - \epsilon^\sigma)^2 - u_x^\sigma |\nabla u_x|^{p(x,t)-2}\nabla u_x \nabla \xi \, dx \, dt = 0.
\]

It follows that

\[
\lim_{\epsilon \to 0} \int \int_{Q_T} \Phi(u_x^\sigma + d_0)|\nabla u_x|^{p(x,t)-2}\nabla u_x \nabla (u_x - u) \, dx \, dt = 0. \tag{43}
\]

On the other hand, from \( u_x, u \in L^\infty(Q_T) \) and \( |\nabla u| \in L^{p(x,t)}(Q_T) \) we get

\[
\lim_{\epsilon \to 0} \int \int_{Q_T} \Phi(u_x^\sigma + d_0)|\nabla u|^{p(x,t)-2}\nabla u \nabla (u_x - u) \, dx \, dt = 0, \tag{44}
\]

\[
\lim_{\epsilon \to 0} \int \int_{Q_T} \Phi(u_x^\sigma + u^\sigma)|\nabla u_x|^{p(x,t)-2}\nabla u_x \nabla (u_x - u) \, dx \, dt = 0. \tag{45}
\]

Note that

\[
0 \leq \left( |\nabla u_x|^{p(x,t)-2}\nabla u_x - |\nabla u|^{p(x,t)-2}\nabla u \right) \nabla (u_x - u) \\
\leq \frac{1}{d_0} \left[ (u_x^\sigma + d_0)|\nabla u_x|^{p(x,t)-2}\nabla u_x - (u_x^\sigma - u^\sigma)|\nabla u_x|^{p(x,t)-2}\nabla u \right] \nabla (u_x - u) \\
- \frac{1}{d_0} (u_x^\sigma + d_0)|\nabla u_x|^{p(x,t)-2}\nabla u \nabla (u_x - u). \tag{46}
\]

By (44)–(46) we have

\[
\lim_{\epsilon \to 0} \int \int_{Q_T} \Phi |\nabla u_x|^{p(x,t)-2}\nabla u_x - |\nabla u|^{p(x,t)-2}\nabla u \nabla (u_x - u) \, dx \, dt = 0. \tag{47}
\]

**Lemma 3.10** As \( \epsilon \to 0 \), we have

\[
b_x(u_x - u_0) \to \xi \in G(u - u_0). \tag{48}
\]

**Proof** Using (7) and the definition of \( \beta_x \), we have

\[
b_x(u_x - u_0) \to \xi \quad \text{as} \quad \epsilon \to 0.
\]

Now we prove that \( \xi \in G(u - u_0) \). According to the definition of \( G(\cdot) \), we only need to prove that if \( u(x_0, t_0) > u_0(x_0) \), then \( \xi(x_0, t_0) = 0 \). In fact, if \( u(x_0, t_0) > u_0(x) \), there exist a constant \( \lambda > 0 \) and a \( \delta \) neighborhood \( B_\delta(x_0, t_0) \) such that if \( \epsilon \) is small enough, we have

\[
u_x(x, t) \geq u_0(x) + \lambda, \quad \forall (x, t) \in B_\delta(x_0, t_0).
\]
Thus, if $\varepsilon$ is small enough, then we have
\[
0 \geq \beta_{\varepsilon}(u_{\varepsilon} - u_0) \geq \beta_{\varepsilon}(\lambda) = 0, \quad \forall (x, t) \in B_{\delta}(x_0, t_0).
\]
Furthermore, it follows by $\varepsilon \to 0$ that
\[
\xi(x, t) = 0, \quad \forall (x, t) \in B_{\delta}(x_0, t_0).
\]
Hence, (48) holds, and the proof of Lemma 3.10 is completed. □

Applying (28), (29), and Lemma 3.10, it is clear that
\[
u(x, t) \leq u_0(x) \quad \text{in } \Omega_T, \quad u(x, 0) = u_0(x) \quad \text{in } \Omega, \xi \in G(u - u_0),
\]
and thus (a), (b), and (c) hold. The remaining arguments of the existence part are the same as those of Theorem 2.1 in [8], and we omit the details. Moreover, the uniqueness of solutions can be proved by repeating Lemma 3.1.

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Authors’ contributions
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References
1. Chen, X., Yi, F., Wang, L.: American lookback option with fixed strike price 2-D parabolic variational inequality. J. Differ. Equ. 251, 3063–3089 (2011)
2. Zhou, Y., Yi, F.: A free boundary problem arising from pricing convertible bond. Appl. Anal. 89, 307–323 (2010)
3. Chen, X., Yi, F.: Parabolic variational inequality with parameter and gradient constraints. J. Math. Anal. Appl. 385, 928–946 (2012)
4. Zhou, Y., Yi, F.: A variational inequality arising from American installment call options pricing. J. Math. Anal. Appl. 357, 54–68 (2009)
5. Sun, Y., Shi, Y., Gu, X.: An integro-differential parabolic variational inequality arising from the valuation of double barrier American option. J. Syst. Sci. Complex. 27, 276–288 (2014)
6. Sun, Y., Shi, Y., Wu, M.: Second-order integro-differential parabolic variational inequalities arising from the valuation of American option. J. Inequal. Appl. 2014, 8 (2014)
7. Sun, Y., Shi, Y.: The existence of solution to a class of degenerate parabolic variational inequality. J. Inequal. Appl. 2015, 204 (2015)
8. Gao, B., Gao, W.: Study of weak solutions for parabolic equations with nonstandard growth conditions. J. Math. Anal. Appl. 374, 374–384 (2011)