COMMENTS ON THE SHIFTS AND THE SIGNS IN $A_{\infty}$-CATEGORIES

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Abstract. We discuss that there exist at least two different choices in the signs of the induced $A_{\infty}$-structures in shifting the degree of objects in an $A_{\infty}$-category. We show that both of these choices are natural in the sense that they are compatible with the homological perturbation theory of $A_{\infty}$-categories. These different choices give different triangulated categories out of an $A_{\infty}$-category.

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1. Introduction

In order to formulate his homological mirror symmetry conjecture, Kontsevich proposes a way of constructing a triangulated category $Tr(C)$ from an $A_{\infty}$-category $C$ [11]. This is a natural generalization of the Bondal-Kapranov’s DG-enhancement [1] to $A_{\infty}$ setting, and consists of three steps. First, for a given $A_{\infty}$-category $C$, we add formally

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shifted objects of \( \mathcal{C} \) and their direct sums and construct an additive enlargement \( \mathcal{C} \). We then construct a further enlarged \( A_\infty \)-category \( Tw(\mathcal{C}) \) consisting of one-sided twisted complexes, and the zero-th cohomology \( Tr(\mathcal{C}) := H^0(Tw(\mathcal{C})) \) turns to form a triangulated category. \(^1\)

In this paper, we discuss that there exist at least two different natural choices of inducing \( A_\infty \)-structures on \( \mathcal{C} \) from that in \( \mathcal{C} \). In order to simplify the signs appearing the defining equations of \( A_\infty \)-structures, we usually express them in the suspended notation. In this suspended notation, a natural induced \( A_\infty \)-structure in \( \mathcal{C} \) from \( \mathcal{C} \) is given in [3]. Let \( \{ m_k \}_{k \geq 1} \) be the \( A_\infty \)-structure in \( \mathcal{C} \), and we denote by \( \{ b_k := sm_k(s^{-1})^{\otimes k} \}_{k \geq 1} \) the one in the suspended notation. For \( X_i, X_j \in \mathcal{C} \) and \( \alpha_{ij} \in \mathcal{C}(X_i, X_j) \), we denote \( \alpha_{ij} := s(\alpha_{ij}) \in s\mathcal{C}(X_i, X_j) \), and denote by \( \alpha'_{ij} \in s\mathcal{C}(X_i[r_i], X_j[r_j]) \) the corresponding morphism. Then, one can easily check that \( \{ \tilde{b}_k \} \) defined by

\[
\tilde{b}_k(\alpha'_{12}, \ldots, \alpha'_{k(k+1)}) := (-1)^{r_1} (b_k(\alpha_{12}, \ldots, \alpha_{k(k+1)}))',
\]

where \( \alpha_{i(i+1)} \in \mathcal{C}(X_i, X_{i+1}) \), preserves the defining relations of \( A_\infty \)-structure.

However, there are other natural choices of \( \{ \tilde{b}_k \} \), or equivalently, \( \{ \tilde{m}_k \} \). One such is defined by

\[
\tilde{m}_k(\alpha'_{12}, \ldots, \alpha'_{k(k+1)}) := (-1)^{r_1 \cdots r_k} (m_k(\alpha_{12}, \ldots, \alpha_{k(k+1)}))'.
\]

In the suspended notation, this turns out to be

\[
\tilde{b}_k(\alpha'_{12}, \ldots, \alpha'_{k(k+1)}) := (-1)^{r_1 + \cdots + r_k} (b_k(\alpha_{12}, \ldots, \alpha_{k(k+1)}))'.
\]

We denote the former \( A_\infty \)-category and the latter \( A_\infty \)-category by \( \mathcal{C}^{(1)} \) and \( \mathcal{C}^{(2)} \), respectively. In general, these two cannot be linear \( A_\infty \)-isomorphic to each other. We discuss that both of these two are natural by showing that both of these preserve \( A_\infty \)-morphisms in the sense in Proposition 3.3 and preserve the homological perturbation theory (HPT) in the sense of Theorem 3.4. It is a typical property of homotopy algebras such as \( A_\infty \)-algebras that the homological perturbation theory works well as developed in [7, 5, 6, 8], etc. In this sense, we believe that Theorem 3.4 guarantees that both of these two choices are natural enough.

Let us explain how the latter choice \( \mathcal{C}^{(2)} \) also arises naturally. Recall that an \( A_\infty \)-category \( \mathcal{C} \) with higher products all zero, \( m_3 = m_4 = \cdots = 0 \), is a DG category. In particular, for a given additive category \( \mathcal{A} \), the DG category \( DG(\mathcal{A}) \) of complexes in \( \mathcal{A} \) has its differential and

\(^1\)Precisely speaking, what we call \( A_\infty \)-categories here should be unital ones in some sense as in [15].
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its composition of the type $\tilde{C}^{(2)}$. 2 Namely, instead of considering the category $\text{Comp}(\mathcal{A})$ of complexes in $\mathcal{A}$, we can naturally construct a DG category $\text{DG}(\mathcal{A})$ whose objects are the same as those in $\text{Comp}(\mathcal{A})$. For two complexes $X^\bullet, Y^\bullet$ in $\mathcal{A}$, a map in

$$\text{DG}^r(\mathcal{A})(X^\bullet, Y^\bullet)$$

is a collection $\varphi = \{\varphi^i : X^i \to Y^{i+r}\}_{i \in \mathbb{Z}}$ of homomorphisms. Thus, $\text{DG}(\mathcal{A})$ has a natural associative composition of morphisms, which we denote by $\varphi \circ \psi$ for $\varphi \in \text{DG}^r(\mathcal{A})(X^\bullet, Y^\bullet)$ and $\psi \in \text{DG}^r(\mathcal{A})(Y^\bullet, Z^\bullet)$. The differential

$$d : \text{DG}^r(\mathcal{A})(X^\bullet, Y^\bullet) \to \text{DG}^{r+1}(\mathcal{A})(X^\bullet, Y^\bullet)$$

is then defined by

$$d(\varphi) := d_X \circ \varphi - (-1)^r \varphi \circ d_Y,$$

or more explicitly, by

$$(d(\varphi))^i := d^i_X \circ \varphi^{i+1} - (-1)^r \varphi^i \circ d^{i+r}_Y$$

for $\varphi \in \text{DG}^r(\mathcal{A})(X^\bullet, Y^\bullet)$. When we denote

$$Z^r(\text{DG}(\mathcal{A}))(X^\bullet, Y^\bullet) := \text{Ker}(d : \text{DG}^r(\mathcal{A})(X^\bullet, Y^\bullet) \to \text{DG}^{r+1}(\mathcal{A})(X^\bullet, Y^\bullet)),$$

$$B^r(\text{DG}(\mathcal{A}))(X^\bullet, Y^\bullet) := \text{Im}(d : \text{DG}^{r-1}(\mathcal{A})(X^\bullet, Y^\bullet) \to \text{DG}^r(\mathcal{A})(X^\bullet, Y^\bullet)),$$

the category $Z^0(\text{DG}(\mathcal{A}))$ is nothing but $\text{Comp}(\mathcal{A})$, whose homotopy category is then the zero-th cohomology $H^0(\text{DG}(\mathcal{A}))$.

Here, the shift functor $T : \text{DG}(\mathcal{A}) \to \text{DG}(\mathcal{A})$ is defined by $T(X^\bullet) = (X[1]^\bullet) = \{((X[1])^i, d^i_{X[1]})_{i \in \mathbb{Z}}\}$, where

$$X[1]^i = X^{i+1}, \quad d^i_{X[1]} = -d^{i+1}_X.$$

Then, we can identify an element $\varphi \in \text{DG}^r(\mathcal{A})(X^\bullet, Y^\bullet)$ with $\mu \in \text{DG}^{r-1}(\mathcal{A})(X^\bullet, Y[1]^\bullet)$ by

$$\mu^i = \varphi^i.$$ 

Similarly, we can identify an element $\varphi \in \text{DG}^r(\mathcal{A})(X^\bullet, Y^\bullet)$ with $\nu \in \text{DG}^{r+1}(\mathcal{A})(X[1]^\bullet, Y^\bullet)$ by

$$\nu^{i-1} = \varphi^i.$$ 

Let us discuss compatibility of these identifications with the structure of the DG category. For $\mu \in \text{DG}^{r-1}(\mathcal{A})(X^\bullet, Y[1]^\bullet)$, the differential acts

2More generally, we may begin with a full subcategory $\mathcal{A}$ of an additive category and then obtain the DG category $\text{DG}(\mathcal{A})$. 
as
\[
(d(\mu))^i = d_X^i \circ \mu^{i+1} - (-1)^{r-1} \mu^i \circ d_Y^{i+1} \\
= d_X^i \circ \varphi^{i+1} - (-1)^r \varphi^i \circ d_Y^r \\
= (d(\varphi))^i.
\]

On the other hand, for \( \nu \in DG^{r+1}(A)(X[1]^\bullet, Y^\bullet) \), the differential acts as
\[
(d(\nu))^{i-1} = d_{X[1]}^{i-1} \circ \nu^i - (-1)^{r+1} \nu^{i-1} \circ d_Y^{(i-1)+(r+1)} \\
= -d_X^i \circ \varphi^{i+1} + (-1)^r \varphi^i \circ d_Y^r \\
= -(d(\varphi))^{i}.
\]

The composition of morphisms is compatible with the identifications in the sense that we have no sign.

Let us denote \( d := \tilde{m}_1 \) and \( \varphi \circ \psi := \tilde{m}_2(\varphi, \psi) \) for \( \varphi \in DG(A)(X^\bullet, Y^\bullet) \) and \( \psi \in DG(A)(Y^\bullet, Z^\bullet) \). Furthermore, for \( r_1, r_2, r_3 \in \mathbb{Z} \), we denote by \( \varphi' \in DG(A)(X[r_1]^\bullet, Y[r_2]^\bullet) \) and \( \psi' \in DG(A)(Y[r_2]^\bullet, Z[r_3]^\bullet) \) the morphism identified with \( \varphi \) and \( \psi \), respectively. Then, the compatibilities are summarized as
\[
\tilde{m}_1(\varphi') = (-1)^{r_1} (\tilde{m}_1(\varphi))', \quad \tilde{m}_2(\varphi', \psi') = (\tilde{m}_2(\varphi, \psi))'.
\]

This implies that \( DG(A) = \tilde{C}^{(2)} \) by choosing a full subcategory \( C \subset DG(A) \) suitably.

We cannot know from this example how we should determine the compatibilities for higher products. However, applying the HPT to a DG category such as \( DG(A) \), it turns out that all the derived higher products are exactly of the \( \tilde{C}^{(2)} \) type. This fact corresponds to the case that \( C \) is a DG-category in Theorem 3.4. In this sense, we may say that the latter choice \( \tilde{C}^{(2)} \) is natural in the DG notation or the unsuspended notation.

In the next section, we recall terminologies of \( A_\infty \)-algebras and \( A_\infty \)-categories in order to fix the notations, and also recall the HPT. Then, in section 3 we define \( \tilde{C}^{(a)} \), \( a = 1, 2 \), and discuss these properties. In particular, we show Proposition 3.3 and Theorem 3.4 hold. Once \( \tilde{C}^{(a)} \), \( a = 1 \) or 2, is constructed, one can further construct the \( A_\infty \)-category \( Tw^{(a)}(C) \) of one-sided twisted complexes, and the triangulated category \( Tr^{(a)}(C) := H^0(Tw^{(a)}(C)) \). These two triangulated categories \( Tr^{(1)}(C) \) and \( Tr^{(2)}(C) \) are not isomorphic to each other in any natural way. In subsection 3.3, we present the shift functor \( T : Tr^{(a)}(C) \rightarrow Tr^{(a)}(C) \) since it may not be presented explicitly in literatures.
2. $A_\infty$-categories and their properties

In order to fix notations, we recall basic terminologies and properties of $A_\infty$-algebras and $A_\infty$-categories. Throughout this paper, all vector spaces are those over a fixed base field $K$.

2.1. $A_\infty$-algebras.

**Definition 2.1** ($A_\infty$-algebra [17, 18]). An $A_\infty$-algebra $(A, m)$ consists of a $\mathbb{Z}$-graded vector space $A := \oplus_{n \in \mathbb{Z}} A^n$ with a collection of multilinear maps $m := \{m_n : A^{\otimes n} \to A\}_{n \geq 1}$ of degree $(2 - n)$ satisfying

\[ 0 = \sum_{k+l=n+1} \sum_{j=0}^{k-1} (-1)^j m_k(a_1, \ldots, a_j, m_l(a_{j+1}, \ldots, a_{j+l}), a_{j+l+1}, \ldots, a_n) \]  

for $n \geq 1$ with homogeneous elements $a_i \in A^{[a_i]}$, $i = 1, \ldots, n$, where the sign is given by $\star = (j+1)(l+1) + l(|a_1| + \cdots + |a_j|)$.  

That the multilinear map $m_k$ has degree $(2 - k)$ indicates the degree of $m_k(a_1, \ldots, a_k)$ is $|a_1| + \cdots + |a_k| + (2 - k)$. The $A_\infty$-relations (1) imply $(m_1)^2 = 0$ for $n = 1$, the Leibniz rule of the differential $m_1$ with respect to the product $m_2$ for $n = 2$, and the associativity of $m_2$ up to homotopy for $n = 3$. These facts further imply that the cohomology $H(A) := H(A, m_1)$ has the structure of a (non-unital) graded algebra, where the product is induced from $m_2$. We denote this algebra by $H(A, m)$ and call the cohomology algebra of $(A, m)$.

Note that the product $m_2$ is strictly associative in $A$ if $m_3 = 0$.

**Definition 2.2.** An $A_\infty$-algebra $(A, m)$ with higher products all zero, $m_3 = m_4 = \cdots = 0$, is called a **differential graded (DG) algebra**.

Let $s : A^r \to (A[1])^{r-1}$ be the suspension. We denote the induced $A_\infty$-structure by $b = \{b_n : (A[1])^{\otimes k} \to A[1]\}_{n \geq 1}$. Explicitly, $b_k$ is defined by

\[ b_k(s(a_1), \ldots, s(a_k)) = (-1)^r s \cdot m_k(a_1, \ldots, a_k), \]

\[ \star = (k-1)|s(a_1)| + (k-2)|s(a_2)| + \cdots + 2|s(a_{k-2})| + |s(a_{k-1})|, \]  

where the degree of $b_k$ is one for any $k$. Hereafter, we denote $\bar{a} := s(a) \in A[1]$. For the $A_\infty$-algebra $(A[1], b)$ in this suspended notation, the sign in the $A_\infty$-relation (1) is simplified as

\[ 0 = \sum_{k+l=n+1} \sum_{j=0}^{k-1} (-1)^j b_k(\bar{a}_1, \ldots, \bar{a}_j, b_l(\bar{a}_{j+1}, \ldots, \bar{a}_{j+l}), \bar{a}_{j+l+1}, \ldots, \bar{a}_n) \]

\[ \star = |\bar{a}_1| + \cdots + |\bar{a}_j| \]  

(3)
(Getzler-Jones [4]).

Let $T^c(A[1]) := \bigoplus_{k \geq 0} (A[1])^\otimes k$ be the tensor coalgebra of $A[1]$, where $(A[1])^\otimes 0 := K$. Here, the coproduct $\Delta : T^c(A[1]) \to T^c(A[1]) \otimes T^c(A[1])$ is defined by

$$\Delta(\bar{a}_1 \otimes \cdots \otimes \bar{a}_n) := \bigoplus_{j=0}^n (\bar{a}_1 \otimes \cdots \otimes \bar{a}_j) \otimes (\bar{a}_{j+1} \otimes \cdots \otimes \bar{a}_n)$$

for $\bar{a}_1 \otimes \cdots \otimes \bar{a}_n \in (A[1])^\otimes n$, where, for instance, the term of $j = 0$ reads $1_K \otimes (\bar{a}_1 \otimes \cdots \otimes \bar{a}_n)$. The $A_{\infty}$-structure $b_k$ is lifted to a coderivation $b_k : T^c(A[1]) \to T^c(A[1])$, $\Delta \circ b_k = (b_k \otimes id_{T^c(A[1])}) + id_{T^c(A[1])} \otimes b_k \Delta$. For the degree one coderivation $b := b_1 + b_2 + \cdots$, $(b)^2 = 0$ follows from the $A_{\infty}$-relation (3). Thus, an $A_{\infty}$-algebra $(A, b)$ is equivalent to a DG coalgebra $(T^c(A[1]), b, \Delta)$. This construction is called the bar construction of $(A, m)$.

For two $\mathbb{Z}$-graded vector spaces $A, B$, coalgebra maps $f : T^c(A[1]) \to T^c(B[1])$, $\Delta f = (f \otimes f)\Delta$, $f(1_K) = 1_K$, are in one-to-one correspondence with collections of degree zero multilinear maps $\{f_i : (A[1])^\otimes i \to B[1]\}_{i=1,\ldots}$. In fact, for $\bar{a}_1 \otimes \cdots \otimes \bar{a}_n \in (A[1])^\otimes n$, $f$ is defined by

$$f(\bar{a}_1 \otimes \cdots \otimes \bar{a}_n) := \sum_{i \geq 1} \sum_{j_1 + \cdots + j_i = n} f_{j_1}(\bar{a}_1, \ldots, \bar{a}_{j_1}) \otimes f_{j_2}(\bar{a}_{j_1+1}, \ldots, \bar{a}_{j_1+j_2}) \otimes \cdots \otimes f_{j_i}(\ldots, \bar{a}_n).$$

**Definition 2.3** ($A_{\infty}$-morphism). For two $A_{\infty}$-algebras $(A, m^A)$, $(B, m^B)$, an $A_{\infty}$-morphism $\tilde{f} : (A, m^A) \to (A', m^{A'})$ is a collection $\{f_i : (A[1])^\otimes i \to B'[1]\}_{i=1,2,…}$ of degree zero multilinear maps whose lift $\tilde{f} : T^c(A[1]) \to T^c(B[1])$ to a coalgebra map gives a morphism $\tilde{f} : (T^c(A[1]), b^A) \to T^c(B[1]), b^B)$ between the complexes:

$$b^B \circ \tilde{f} = \tilde{f} \circ b^A.$$ 

In particular, we call an $A_{\infty}$-morphism $\tilde{f}$ with $f_2 = f_3 = \cdots = 0$ a linear $A_{\infty}$-morphism.

The condition $b^B \circ \tilde{f} = \tilde{f} \circ b^A$ is equivalent to that $\tilde{f}$ satisfies the following equations

$$\sum_{i \geq 1} \sum_{j_1 + \cdots + j_i = n} b_i^B(f_{j_1}(\bar{a}_1, \ldots, \bar{a}_{j_1}), f_{j_2}(\bar{a}_{j_1+1}, \ldots, \bar{a}_{j_1+j_2}), \ldots, f_{j_i}(\ldots, \bar{a}_n))$$

$$= \sum_{k+l=n+1} \sum_{j=0}^{k-1} (-1)^{|\bar{a}_1|+\cdots+|\bar{a}_j|} f_k(\bar{a}_1, \ldots, \bar{a}_j, b_l^A(\bar{a}_{j+1}, \ldots, \bar{a}_{j+l}) \ldots, \bar{a}_n)$$

(4)

for any $n \geq 1$ and homogeneous elements $\bar{a}_1, \ldots, \bar{a}_n \in A[1]$. This defining equation (4) of the $A_{\infty}$-morphism for $n = 1$ implies that $f_1 :
$A[1] \to B[1]$ forms a chain map $f_1 : (A[1], m^A_1) \to (B[1], m^B_1)$. This together with the defining equation (4) for $n = 2$ then implies that $f_1 : A[1] \to B[1]$ induces a (non-unital) graded algebra map from $H(A, m^A)$ to $H(B, m^B)$.

The composition of two $A_\infty$-morphisms is defined by the composition of the corresponding two coalgebra maps. Thus, the composition is associative.

**Definition 2.4.** An $A_\infty$-morphism $f : (A, m^A) \to (B, m^B)$ is called an $A_\infty$-quasi-isomorphism iff $f_1 : (A[1], b^A_1) \to (B[1], b^B_1)$ induces an isomorphism between the cohomologies of these two complexes. If in particular $f_1$ is an isomorphism of complexes, then $f$ is called an $A_\infty$-isomorphism.

### 2.2. $A_\infty$-categories.

**Definition 2.5** ($A_\infty$-category [2]). An $A_\infty$-category $C$ over a base field $K$ consists of a set of objects $\text{Ob}(C) = \{X, Y, \ldots \}$, a $\mathbb{Z}$-graded $K$-vector space $C(X, Y) = \oplus_{r \in \mathbb{Z}} C^r(X, Y)$ for each two objects $X, Y \in \text{Ob}(C)$ and a collection of multilinear maps

$$m := \{m_n : C(X_1, X_2) \otimes \cdots \otimes C(X_n, X_{n+1}) \to C(X_1, X_{n+1})\}_{n \geq 1}$$

of degree $(2 - n)$ satisfying the $A_\infty$-relations (1).

In particular, an $A_\infty$-category $C$ with higher products all zero, $m_3 = m_4 = \cdots = 0$, is called a DG category.

**Definition 2.6.** For an $A_\infty$-category $C$, a (non-unital) graded category $H(C)$ is defined by $\text{Ob}(H(C)) := \text{Ob}(C)$ and for any $X, Y \in C$ the space of morphisms is the cohomology of the chain complex $(C(X, Y), m_1)$:

$$H(C)(X, Y) := H(C(X, Y), m_1).$$

The composition in $H(C)$ is given by the one induced from $m_2$ in $C$. We call $H(C)$ the cohomology of $C$.

We also define a (non-unital) category $H^0(C)$ by $\text{Ob}(H^0(C)) := \text{Ob}(C)$ and for any $X, Y \in C$ the space of morphisms is the cohomology of degree zero only:

$$H^0(C)(X, Y) := H^0(C(X, Y), m_1).$$

The composition in $H^0(C)$ is given by the one induced from $m_2$ in $C$. We call $H^0(C)$ the zero-th cohomology of $C$.

The suspension $sC$ of an $A_\infty$-category $C$ is defined by the shift

$$s : C(X, Y) \to s(C(X, Y)) =: (sC)(X, Y)$$

for any $X, Y \in \text{Ob}(C) = \text{Ob}(sC)$, where the degree $|b_n|$ of the $A_\infty$-products becomes one for all $n \geq 1$ as it does for $A_\infty$-algebras.
Definition 2.7 ($A_\infty$-functor). Given two $A_\infty$-categories $\mathcal{C}$, $\mathcal{D}$, an $A_\infty$-functor $\mathfrak{f} := \{ f; f_1, f_2, \ldots \} : \mathcal{C} \to \mathcal{D}$ is a map $f : \text{Ob}(s(\mathcal{C})) \to \text{Ob}(s(\mathcal{D}))$ of objects with degree preserving multilinear maps

$$f_k : (s\mathcal{C})(X_1, X_2) \otimes \cdots \otimes (s\mathcal{C})(X_k, X_{k+1}) \to (s\mathcal{D})(f(X_1), f(X_{k+1}))$$

for $k \geq 1$ satisfying the defining relations of an $A_\infty$-morphism (4).

In particular, if $f : \text{Ob}(s\mathcal{C}) \to \text{Ob}(s\mathcal{D})$ is bijective and $f_1 : (s\mathcal{C})(X, Y) \to (s\mathcal{D})(f(X), f(Y))$ induces an isomorphism between the cohomologies for any $X, Y \in \text{Ob}(s\mathcal{C})$, we call the $A_\infty$-functor $\mathfrak{f}$ an $A_\infty$-quasi-isomorphism functor. If $f_1$ is itself an isomorphism, then $\mathfrak{f}$ is called an $A_\infty$-isomorphism functor.

2.3. Homological perturbation theory (HPT). For an $A_\infty$-algebra $A = (A, m^A)$, $(A, d^A := m^A)$ forms a complex. The HPT starts with what is called strong deformation retract (SDR) data $[7, 5, 6, 8]$:

$$\begin{array}{ccc}
B & \mathcal{I} \longrightarrow & A \underline{\times} h
\end{array}$$

(5)

where $(B, d^B := \pi \circ d^A \circ \iota)$ is a complex with chain maps $\iota$ and $\pi$ so that $\pi \circ \iota = \text{Id}_B$ and $h : A \to A$ is a contracting homotopy satisfying

$$d^A h + h d^A = \text{Id}_A - P, \quad P := \iota \circ \pi. \quad (6)$$

By this definition, $P$ is an idempotent in $A$ which commutes with $d^A$, $Pd^A = d^A P$. If $d^A P = 0$, then the SDR (6) gives a Hodge decomposition of the complex $(A, d^A)$, where $P(A) \simeq H(A)$ gives the cohomology.

For an $A_\infty$-algebra $(A[1], b^A)$ and an SDR data $(B, \underline{\pi} \mathcal{I} \longrightarrow A \underline{\times} h)$, there exists an explicit construction of an $A_\infty$-algebra $(B[1], b^B)$ and an $A_\infty$-quasi-isomorphism $\mathfrak{f} : (B[1], b^B) \to (A[1], b^A)$. First we set $f_1 := \iota : B[1] \to A[1]$. The $A_\infty$-quasi-isomorphism $\mathfrak{f}$ is then defined inductively by

$$f_{n+1} := -h \sum_{j \geq 2} \sum_{k_1 + \cdots + k_j = n+1} b_j^A(f_{k_1} \otimes \cdots \otimes f_{k_j})$$

for given $f_1, f_2, \ldots, f_n$. Then, the $A_\infty$-structure $b^B$ in $B[1]$ is given by $b_1^B = d^B$ and the formula

$$b_{n+1}^B := \pi \sum_{j \geq 2} \sum_{k_1 + \cdots + k_j = n+1} b_j^A(f_{k_1} \otimes \cdots \otimes f_{k_j}) \quad (7)$$

for $n = 1, 2, \ldots$.

Lemma 2.8. This $(B[1], b^B)$ actually forms an $A_\infty$-algebra and this $\mathfrak{f} : (B[1], b^B) \to (A[1], b^A)$ is an $A_\infty$-quasi-isomorphism. \qed
This formula is given in [6, 8] when \((A, m^A)\) is a DG algebra, and in [12], for instance, for the general case. The statement of Lemma 2.8 is also called the homotopy transfer lemma, see [14], which is actually a part of the HPT developed for instance by [7, 5, 6, 8].

As a corollary of Lemma 2.8, when we start from an SDR data defining the Hodge decomposition of the complex \((A, d^A)\) where \(B \cong H(A)\), one obtains the minimal model theorem by Kadeishvili [9]: for a given \(A_\infty\)-algebra \((A, m^A)\), there exists a minimal \(A_\infty\)-algebra structure \(m^{H(A)}\) in \(H(A)\) and an \(A_\infty\)-quasi-isomorphism \(\tilde{f} : (H(A), m^{H(A)}) \rightarrow (A, m^A)\). Here, that \(m^{H(A)}\) is minimal means \(m^{H(A)}_1 = 0\). Such a minimal \(A_\infty\)-algebra \((H(A), m^{H(A)})\) is called a minimal model of \((A, m^A)\). We see that the cohomology algebra \(H(A, m^A)\) is obtained from a minimal model \((H(A), m^{H(A)})\) of \((A, m^A)\) by forgetting the higher \(A_\infty\)-products \(m^{H(A)}_3, m^{H(A)}_4, \ldots\).

It is straightforward to generalize this HPT for \(A_\infty\)-algebras to that for \(A_\infty\)-category \(C\) [12]. We begin with an SDR

\[
\begin{array}{ccc}
\mathcal{D}(X, Y) & \xrightarrow{\ell_{XY}} & \mathcal{C}(X, Y), \\
\xrightarrow{\ast_{XY}} & & h_{XY},
\end{array}
\]

for any \(X, Y \in C\). Then the \(A_\infty\)-structure of \(\mathcal{D}\) is given by the straightforward generalization of the formula (7) and an \(A_\infty\)-quasi-isomorphism functor

\[
\tilde{f} : \mathcal{D} \rightarrow \mathcal{C}
\]

is also given as above.

### 3. The shifts and the signs in \(A_\infty\)-categories

Before defining \(\tilde{C}^{(1)}\) and \(\tilde{C}^{(2)}\) mentioned in the introduction, in subsection 3.1, we begin with constructing \(A_\infty\)-categories \(\mathcal{C}' = \mathcal{C}^{(a)}\), \(a = 1, 2\), from an \(A_\infty\)-category \(\mathcal{C}\). We extend this to \(\tilde{C}^{(a)}\), \(a = 1, 2\), in subsection 3.2, and obtain Proposition 3.3 and Theorem 3.4 by just rewriting Proposition 3.1 and Theorem 3.2 obtained in subsection 3.1.

#### 3.1. The shifts and the signs

For an \(A_\infty\)-category \(\mathcal{C}\) and an object \(X \in \mathcal{C}\), we would like to define another \(A_\infty\)-category \(\mathcal{C}'\) where \(X \in \mathcal{C}\) is replaced by \(X[1]\) in the sense that

\[
(C')^r(Y, X[1]) := C^{r+1}(Y, X), \quad (C')^r(X[1], Y) := C^{r-1}(X, Y).
\]

We of course set \(C'(Y, Z) := C(Y, Z)\) if \(Y \neq X\) and \(Z \neq X\). For a morphism \(\alpha\) in \(\mathcal{C}\), we denote by \(\alpha'\) the corresponding morphism in \(\mathcal{C}'\).
A way of defining an $A_\infty$-structure in $C'$ is presented explicitly in [3]. For
\[
P_k: \ sC(X_1, X_2) \times \cdots \times sC(X_k, X_{k+1}) \to sC(X_1, X_{k+1})
\]
\[\mapsto b_k(\bar{\alpha}_{12}, \ldots, \bar{\alpha}_{k(k+1)})\]
in $C$, we set $b'_k$ to be
\[b'_k(\bar{\alpha}_{12}, \ldots, \bar{\alpha}_{k(k+1)}) = \pm (b_k(\bar{\alpha}_{12}, \ldots, \bar{\alpha}_{k(k+1)}))',\]
where the sign $\pm$ is $-1$ when $X_1 = X$ and $+1$ otherwise. One can check directly that this $m'$ preserves the $A_\infty$-relations of $m$ [3]. We denote the resulting $A_\infty$-category by $C' = C^{(1)}$.

Though the definition of this $m' =: m^{(1)}$ is natural enough, there are other choices of $A_\infty$-structures. One such $A_\infty$-structure is given by
\[b'_k(\bar{\alpha}_{12}, \ldots, \bar{\alpha}_{k(k+1)}) = \pm (b_k(\bar{\alpha}_{12}, \ldots, \bar{\alpha}_{k(k+1)}))',\]
where the sign $\pm$ is $-1$ if the number of $X$ in $X_1, \ldots, X_k$ (not $X_1, \ldots, X_{k+1}$) is odd and $+1$ if it is even. We denote the resulting $A_\infty$-category by $C^{(2)}$. In general, these two $A_\infty$-categories are not linearly $A_\infty$-isomorphic to each other.

First, by direct calculations, we see that the following holds.

**Proposition 3.1.** Any $A_\infty$-functor $f: D \to C$ induces an $A_\infty$-functor $f^{(a)}: D^{(a)} \to C^{(a)}$, with $a = 1$ or $2$, given by
\[f^{(a)}_k(\bar{\alpha}_{12}, \ldots, \bar{\alpha}_{k(k+1)}) := (f_k(\bar{\alpha}_{12}, \ldots, \bar{\alpha}_{k(k+1)}))'.\]

*proof.* We can check directly that this satisfies the defining equations of $A_\infty$-morphisms (4). \hfill $\square$

Now, we discuss both $C^{(1)}$ and $C^{(2)}$ are natural enough by showing that the $A_\infty$-structures of both types are preserved by the HPT. First, assume that an SDR data for $C$ is given. Namely, each space $C(Y, Z)$ of morphisms has an SDR
\[
( D(Y, Z) \xrightarrow{\iota_{YZ}} C(Y, Z), h_{YZ}),
\]
so that
\[d_{YZ}h_{YZ} + h_{YZ}d_{YZ} = \text{Id} - \iota_{YZ} \circ \pi_{YZ}.\] (8)
Then, an SDR data is induced for $C^{(a)}$ as follows. In both cases $a = 1$ and $2$, the $b'_1 = b'^{(a)}_1$ above satisfies
\[b'^{(a)}_1(\bar{\alpha}) := \begin{cases} -(b_1(\bar{\alpha}))' & Y = X \\ (b_1(\bar{\alpha}))' & \text{otherwise} \end{cases}\] (9)
for $\bar{\alpha} \in sC(Y, Z)$. Let us denote by $Y'$ and $Z'$ the objects in $C^{(a)}$ corresponding to $Y$ and $Z$ in $C$. Namely, $Y' = X[1]$ if $Y = X$ and
Then, (9) is equivalent to that
\[ d_{Y'Z'}^{(a)} = m_1^{(a)} \]
satisfies
\[
\begin{cases}
-(d_{YZ}(\alpha))' & \text{if } Y = X \\
(d_{YZ}(\alpha))' & \text{otherwise}.
\end{cases}
\]
Therefore, setting
\[
h_{Y'Z'}^{(a)}(\alpha') :=
\begin{cases}
-(h_{YZ}(\alpha))' & \text{if } Y = X \\
(h_{YZ}(\alpha))' & \text{otherwise},
\end{cases}
\]
the formula (8) is preserved,
\[
d_{Y'Z'}^{(a)} h_{Y'Z'}^{(a)} + h_{Y'Z'}^{(a)} = \text{Id} - \iota_{Y'Z'} \circ \pi_{Y'Z'},
\]
where \( \iota_{Y'Z'} \) is defined as \( \iota_{Y'Z'}(\alpha') := (\iota_{YZ}(\alpha))' \) and \( \pi_{Y'Z'} \) is defined similarly. We call this the induced SDR data on \( C^{(a)} \).

**Theorem 3.2.** For a given \( A_\infty \)-category \( C \), we fix an SDR data on \( C \). By the HPT, we obtain an \( A_\infty \)-category \( D \) and an \( A_\infty \)-quasi-isomorphism functor \( \hat{f} : D \to C \). Similarly, we obtain an \( A_\infty \)-quasi-isomorphism functor \( g : E \to C^{(a)} \) with \( a = 1 \) or \( 2 \) by applying the HPT for \( C^{(a)} \) with the induced SDR data. Then, one has \( E = D^{(a)} \) and \( g = \hat{f}^{(a)} \).

**proof.** We prove this when \( a = 1 \).

We first show that \( g = \hat{f}^{(1)} \). Recall that \( \hat{f} \) is given inductively by
\[
f_1 = \iota \quad \text{and} \quad f_i(\cdots) = -h \sum_{k \geq 2} \sum_{i_1 + \cdots + i_k = i} b_k^c(f_{i_1}(\cdots), \ldots, f_{i_k}(\cdots))
\]
for \( i \geq 2 \). The \( A_\infty \)-quasi-isomorphism \( g \) is defined in the same way but with replacing \( b_k^c \) by \( b_k^{(a)} \). For
\[
f_k : sD(X_1, X_2) \times \cdots \times sD(X_k, X_{k+1}) \to C(X_1, X_{k+1})
\]
\[
(\bar{\alpha}_{12}, \ldots, \bar{\alpha}_{k(k+1)}) \mapsto f_k(\bar{\alpha}_{12}, \ldots, \bar{\alpha}_{k(k+1)}),
\]
we show by induction that
\[
g_k(\bar{\alpha}_{12}', \ldots, \bar{\alpha}_{k(k+1}') = (f_k(\bar{\alpha}_{12}, \ldots, \bar{\alpha}_{k(k+1)}))'
\]
holds.

Assuming this holds for \( k \leq n \), one has
\[
g_{n+1}(\cdots) = -h^{(a)} \sum_{k \geq 2} \sum_{i_1 + \cdots + i_k = n+1} b_k^{(a)}(f_{i_1}(\cdots), \ldots, f_{i_k}(\cdots)),
\]
where the sign coming from \( h^{(a)} \) and that from \( b_k^{(a)} \) cancel with each other. Namely, (10) holds true for \( k = n + 1 \). Thus, the induction is completed and we have \( g = \hat{f}^{(1)} \).
Next, we show that $b^\varepsilon_k = b_k^{D(1)}$. We assume this holds for $1 \leq k \leq n$. For $k = n + 1$,

$$b^\varepsilon_{n+1}: sE(X_1', X_2') \times \cdots \times sE(X_{n+1}', X_{n+2}') \to sE(X_1', X_{n+2}')$$

is given by

$$b^\varepsilon_{n+1} = \sum_{a \geq 1} \sum_{i_1+\cdots+i_k=n+1} \pi b^{C(a)}_k (g_1(\cdots), \ldots, g_k(\cdots)).$$

Thus, the minus sign appears from $b^{C(a)}_k (g_1(\cdots), \ldots, g_k(\cdots))$ if and only if $X_1 = X$, i.e., $X_1' = X[1]$. This shows that $b^\varepsilon_{n+1} = b_k^{D(1)}$.

The case for $a = 2$ can also be shown in the parallel strategy. \hfill \square

### 3.2. Additive $A_\infty$-categories

For a given $A_\infty$-category $C$ and an object $X \in C$, by replacing $X$ by $X[1]$ we constructed the $A_\infty$-category $C^{(a)}$ with $a = 1$ or $2$. We can instead construct an $A_\infty$-category by adding $X[1]$ or $X[-1]$ to $C$.

Repeating this procedure yields an additive $A_\infty$-category $\tilde{C}^{(a)}$ as follows.

In both cases $a = 1, 2$, an object in $\tilde{C}^{(a)}$ is a finite direct sum $X := X_1[1] \oplus \cdots \oplus X_l[1]$, where $X_i \in C$ for each $i = 1, \ldots, l$ and $[r_i]$ indicates the formal degree shift by $r_i \in \mathbb{Z}$. For $X, Y \in \tilde{C}$, the space $\tilde{C}(X, Y) = \tilde{C}^{(a)}(X, Y)$ of morphisms is given by

$$\tilde{C}^{(a)}(X[n], Y[m]) := C^{r+m-n}(X, Y).$$

The $A_\infty$-structure $\tilde{b}^{(a)}_k = \tilde{\tilde{b}}^{(a)}_k$ is determined by defining

$$\tilde{b}^{(1)}_k (\tilde{\alpha}_{12}, \ldots, \tilde{\alpha}_{k(k+1)}) = (\tilde{\alpha}_{12}, \ldots, \tilde{\alpha}_{k(k+1)})'$$

for $\tilde{\alpha}_{i(i+1)} \in sE(X_i, X_{i+1})$ and $\tilde{\alpha}_{i(i+1)}' \in sE(X_i[1], X_{i+1}[1])$. Repeating the construction of $C^{(a)}$ in the previous subsection, we set

$$\tilde{m}^{(1)}_k (\tilde{\alpha}_{12}, \ldots, \tilde{\alpha}_{k(k+1)}) = (\tilde{\alpha}_{12}, \ldots, \tilde{\alpha}_{k(k+1)})'$$

and

$$\tilde{m}^{(2)}_k (\tilde{\alpha}_{12}, \ldots, \tilde{\alpha}_{k(k+1)}) = (\tilde{\alpha}_{12}, \ldots, \tilde{\alpha}_{k(k+1)})'. \tag{13}$$

Then, both $\tilde{C}^{(1)}$ and $\tilde{C}^{(2)}$ form $A_\infty$-categories.

In the unsuspended notation, by (2) we have

$$\tilde{m}^{(1)}_k (\alpha_{12}, \ldots, \alpha_{k(k+1)}) = (\alpha_{12}, \ldots, \alpha_{k(k+1)})' \tag{14}$$

and

$$\tilde{m}^{(2)}_k (\alpha_{12}, \ldots, \alpha_{k(k+1)}) = (\alpha_{12}, \ldots, \alpha_{k(k+1)}). \tag{14.1}$$
where $\alpha_{i(i+1)} \in C(X_i, X_{i+1})$.

As corollaries of the previous subsection, we obtain the followings.

We let $a$ be 1 or 2.

**Proposition 3.3.** An $\mathbb{A}_\infty$-functor $f : \mathcal{D} \to \mathcal{C}$ induces an $\mathbb{A}_\infty$-functor $\tilde{f}^{(a)} : \tilde{\mathcal{D}}^{(a)} \to \tilde{\mathcal{C}}^{(a)}$ which is defined by

$$f_k^{(a)}(\bar{\alpha}_{12}, \ldots, \bar{\alpha}_{k(k+1)}) := (f_k(\bar{\alpha}_{12}, \ldots, \bar{\alpha}_{k(k+1)}))'.$$

□

**Theorem 3.4.** For a given $\mathbb{A}_\infty$-category $\mathcal{C}$, we fix an SDR data on $\mathcal{C}$. By HPT, we obtain an $\mathbb{A}_\infty$-category $\mathcal{D}$ and an $\mathbb{A}_\infty$-quasi-isomorphism functor $\tilde{f}^{(a)} : \tilde{\mathcal{D}}^{(a)} \to \tilde{\mathcal{C}}^{(a)}$. Similarly, we obtain an $\mathbb{A}_\infty$-quasi-isomorphism functor $\tilde{g} : \tilde{\mathcal{E}} \to \tilde{\mathcal{C}}^{(a)}$ by applying HPT for $\tilde{\mathcal{C}}^{(a)}$ with the induced SDR data. Then one has

$$\tilde{\mathcal{E}} = \tilde{\mathcal{D}}^{(a)}, \quad \tilde{g} = \tilde{f}^{(a)}.$$ □

This theorem indicates that the procedure of taking $\tilde{\mathcal{C}}^{(a)}$ from $\mathcal{C}$ is compatible with the HPT.

### 3.3. Triangulated $\mathbb{A}_\infty$-categories

Once we construct $\tilde{\mathcal{C}}^{(a)}$, we can further construct an $\mathbb{A}_\infty$-category $\text{Tw}^{(a)}(\mathcal{C})$ of one-sided twisted complexes. When $\mathcal{C}$ is strictly unital, then $\text{Tw}^{(a)}(\mathcal{C})$ is also strictly unital, and its zero-th cohomology $\text{Tr}^{(a)}(\mathcal{C}) := H^0(\text{Tw}^{(a)}(\mathcal{C}))$ forms a triangulated category as proposed in [11]. In this sense, $\text{Tw}^{(a)}(\mathcal{C})$ is called a triangulated $\mathbb{A}_\infty$-category. The construction of $\text{Tw}^{(a)}(\mathcal{C})$ is discussed explicitly in [3]. When the shift functor $T : \text{Tr}^{(a)}(\mathcal{C}) \to \text{Tr}^{(a)}(\mathcal{C})$ is given, the exact triangles are defined quite naturally as we explain later.

In this subsection, we present the shift functor $T : \text{Tr}^{(a)}(\mathcal{C}) \to \text{Tr}^{(a)}(\mathcal{C})$ explicitly.

First, we explain the triangulated $\mathbb{A}_\infty$-category $\text{Tw}^{(a)}(\mathcal{C})$ constructed explicitly in [3]. For the notations, we mainly follow [10].

A *one-sided twisted complex* $(\mathcal{X}, \Phi)$ is a pair of an object $\mathcal{X} := X_1[r_1] \oplus \cdots \oplus X_l[r_l] \in \tilde{\mathcal{C}}^{(a)}$ and a degree zero morphism $\Phi = \oplus_{i,j=1}^l \phi_{ij} \in \tilde{\mathcal{C}}^{(a)}(\mathcal{X}, \mathcal{X})$ which satisfies $\phi_{ij} = 0$ for $i \geq j$ and the $\mathbb{A}_\infty$-Maurer-Cartan equation:

$$\check{b}_1^{(a)}(\Phi) + \check{b}_2^{(a)}(\Phi, \Phi) + \cdots = 0. \quad (15)$$
The category $Tw^{(a)}(\mathcal{C})$ consists of one-sided twisted complexes, where the spaces of morphisms are defined by $Tw^{(a)}(\mathcal{C})((\mathcal{X}, \Phi), (\mathcal{Y}, \Psi)) := \tilde{\mathcal{C}}^{(a)}(\mathcal{X}, \mathcal{Y})$, and the $A_\infty$-structure $b^{Tw^{(a)}}$ is given by

\[
b_n^{Tw^{(a)}}(\varphi_{12}, \ldots, \varphi_{n(n+1)}) = \sum_{k_1, \ldots, k_{n+1} \geq 0} \tilde{b}_{n+k_1+\ldots+k_{n+1}}^{Tw^{(a)}}((\Phi_1)^{k_1}, \varphi_{12}, (\Phi_2)^{k_2}, \ldots, \varphi_{n(n+1)}, (\Phi_{n+1})^{k_{n+1}})
\]

for $(\mathcal{X}_i, \Phi_i) \in Tw^{(a)}(\mathcal{C})$, $i = 1, \ldots, n + 1$. Then, $Tw^{(a)}(\mathcal{C})$ again forms an $A_\infty$-category [3].

Next, we define an additive isomorphism $T : \tilde{\mathcal{C}}^{(a)} \to \tilde{\mathcal{C}}^{(a)}$ which satisfies

\[
T(X) = X[1], \quad X \in \mathcal{C} \subset \tilde{\mathcal{C}}^{(a)}
\]

and

\[
T \tilde{b}_k^{(a)} = (-1)^k \tilde{b}_k^{(a)} (T \otimes \cdots \otimes T). \tag{16}
\]

For $\alpha_{ij} \in \mathcal{C}(X_i, X_j)$, denote the corresponding morphisms by $\alpha'_{ij} \in \tilde{\mathcal{C}}^{(a)}(X_i[r], X_j[r])$ and $\alpha''_{ij} \in \tilde{\mathcal{C}}^{(a)}(X_i[r_i+1], X_j[r_j+1])$. Then, we set $T : \tilde{\mathcal{C}}^{(1)}(X_i[r], X_j[r]) \to \tilde{\mathcal{C}}^{(1)}(X_i[r_i], X_j[r_j])$ by

\[
T(\alpha'_{ij}) = -\alpha''_{ij},
\]

and $T : \tilde{\mathcal{C}}^{(2)}(X_i[r], X_j[r]) \to \tilde{\mathcal{C}}^{(2)}(X_i[r_i], X_j[r_j])$ by

\[
T(\alpha'_{ij}) = \alpha''_{ij}.
\]

These determine $T$ for all objects and morphisms in $\tilde{\mathcal{C}}^{(a)}$, and we see that (16) actually holds.

This $T$ naturally induces an additive isomorphism $T : Tw^{(a)}(\mathcal{C}) \to Tw^{(a)}(\mathcal{C})$ as follows. For objects, in both cases $a = 1$ and 2, we first observe that $-T(\Phi)$ satisfies the $A_\infty$-Maurer-Cartan equation if $\Phi$ does due to (16) and (11) or (12). Thus, we set

\[
T(\mathcal{X}, \Phi) := (T(\mathcal{X}), -T(\Phi)),
\]

where the $T$’s in the left hand side are those in $\tilde{\mathcal{C}}^{(a)}$. For morphisms, for $\varphi_{ij} \in Tw^{(a)}(\mathcal{C})((\mathcal{X}_i, \Phi_i), (\mathcal{X}_j, \Phi_j))$, we set $T(\varphi_{ij})$ just as $T(\varphi_{ij})$ in the sense in $\tilde{\mathcal{C}}^{(a)}$. Then, again by (16), we have

\[
T \tilde{b}_k^{Tw^{(a)}(\mathcal{C})} = (-1)^k \tilde{b}_k^{Tw^{(a)}(\mathcal{C})} (T \otimes \cdots \otimes T).
\]

In particular, $Tm_2^{Tw^{(a)}(\mathcal{C})} = m_2^{Tw^{(a)}(\mathcal{C})} (T \otimes T)$ holds for $k = 2$, so this induces an additive isomorphism functor $T : Tr^{(a)}(\mathcal{C}) \to Tr^{(a)}(\mathcal{C})$.

\[3\]Unfortunately, this $T$ does not form a linear $A_\infty$-automorphism $T : \tilde{\mathcal{C}}^{(a)} \to \tilde{\mathcal{C}}^{(a)}$ because of the sign $(-1)^k$ in (16).
We treat this $T$ as the shift functor. For an $m_1\text{Tw}(^aC)$-closed morphism $\varphi \in \text{Tw}(^aC)((X, \Phi_X), (Y, \Phi_Y))$ of degree zero, the mapping cone $C(\varphi)$ is defined by

$$C(\varphi) := \left( T(X) \oplus Y, \begin{pmatrix} -T(\Phi_X) & \varphi' \\ 0 & \Phi_Y \end{pmatrix} \right),$$

where $\varphi'$ is the morphism in $s\text{Tw}(^aC)(T(X, \Phi_X), (Y, \Phi_Y))$ induced from $\varphi$. Then, exact triangles in $\text{Tr}(^aC)$ are defined as sequences which are isomorphic to

$$\cdots (X, \Phi_X) \xrightarrow{\varphi'} (Y, \Phi_Y) \rightarrow C(\varphi) \rightarrow T(X, \Phi_X) \rightarrow \cdots$$

with some $\varphi$. As mentioned in [3], it is shown by direct calculations that these actually satisfy the axiom of exact triangles. Thus, $\text{Tr}(^aC)$ forms a triangulated category when $C$ is a strictly unital $A_\infty$-category.

3.4. **Concluding remarks.** For a given $A_\infty$-functor $\mathcal{F} : D \rightarrow C$ of strictly unital $A_\infty$-categories, the $A_\infty$-functor $\mathcal{F}^{(a)} : \text{Tw}(^aD) \rightarrow \text{Tw}(^aC)$ and then a functor $\text{Tr}(^a(D)) \rightarrow \text{Tr}(^a(C)$ of triangulated categories as explained in [3, 16], etc. In particular, if $\mathcal{F}$ is an $A_\infty$-equivalence functor, then the induced functor between the triangulated categories is an equivalence. See [16].

Note that $\tilde{\mathcal{C}}_1$ and $\tilde{\mathcal{C}}_2$ are not $A_\infty$-isomorphic to each other in general. Actually, there exist such examples of $\mathcal{C}$ even if $m_1 = m_3 = \cdots = 0$. This means, there does not exist any $A_\infty$-isomorphism functor between $\text{Tw}(^1(C)$ and $\text{Tw}(^2(C)$ so that it preserves $\tilde{\mathcal{C}}^{(a)} \subset \text{Tw}(^a(C)).$ In this sense, there does not exist any natural isomorphism functor of triangulated categories between $\text{Tr}(^1(C)$ and $\text{Tr}(^2(C)$.

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