Discontinuous Galerkin approximations for near-incompressible and near-inextensible transversely isotropic bodies

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Abstract

This work studies discontinuous Galerkin (DG) approximations of the boundary value problem for homogeneous transversely isotropic linear elastic bodies. Low-order approximations on triangles are adopted, with the use of three interior penalty DG methods, viz. non-symmetric, symmetric and incomplete. It is known that these methods are uniformly convergent in the incompressible limit. This work focuses on behaviour in the inextensible limit. An error estimate suggests the possibility of extensional locking, a feature that is confirmed by numerical experiments. Under-integration of the extensional edge terms is proposed as a remedy. This modification is shown to lead to an error estimate that is consistent with locking-free behaviour. Numerical tests confirm the uniformly convergent behaviour, at an optimal rate, of the under-integrated scheme.

Keywords: discontinuous Galerkin methods, elasticity, nearly incompressible, nearly inextensible, transverse isotropy, interior penalty

1 Introduction

Anisotropic materials have a wide range of applications, e.g. in the geological domain, or in biomechanical systems such as the myocardium, brain stem, ligaments, and tendons [10, 21, 22]. Different types of anisotropy along with their mechanical restrictions are presented in [17, 24, 30]. In this work, we are particularly interested in transversely isotropic materials, which play a central role in theories describing the behaviour of fibre-reinforced composite materials [14, 15, 18].
When elastic materials are internally constrained, the associated displacement-based finite element approximations exhibit poor performance in the form of poor coarse-mesh approximations, as well as locking, in which they do not converge uniformly with respect to the constraint parameters. The incompressibility constraint has been widely studied in this context (see, for example, [8]), while behaviour in the inextensible limit has received less attention.

It has been shown that locking can be avoided using high-order elements [5], though low-order approximations remain of high interest. Low-order discontinuous Galerkin approximations with linear approximations on triangles are uniformly convergent in the incompressible limit [25]. The corresponding problem using bi- or trilinear approximations on quadrilaterals and hexahedra displays locking, which may be overcome by selective under-integration of edge terms involving the relevant Lamé parameter [11].

A range of mixed methods have been shown to be uniformly convergent for near-incompressibility (see for example [6] and the references therein), while the works [9, 16] provide a unified treatment of convergent approaches using two- or three-field approximations. Near-inextensibility is studied computationally in the work [4], using Lagrange multiplier and perturbed Lagrangian approaches. There have also been a number of computational investigations of transversely isotropic and inextensible behaviour for large-displacement problems [26, 29, 27, 28].

Theoretical studies of anisotropic elastic behaviour include the work [3], in which conditions for well-posedness are established for a Hellinger-Reissner formulation, and [14, 15], in which conditions for uniqueness and stability are established.

In recent work [19], the authors have studied the well-posedness of boundary value problems involving transversely isotropic elastic materials. They have also investigated theoretically and computationally the use of conforming finite element approximations, paying attention to both near-incompressibility and near-inextensibility. It is found in that work that, for low-order quadrilaterals, selective under-integration of volumetric and extensional terms serves to render the schemes locking-free. Further related work has recently been reported in [20], on a virtual element formulation for transverse isotropy: this formulation is shown to be robust and locking-free in the inextensional limit, for both constant and variable fibre directions.

The subject of this work is a study of DG approximations for transversely isotropic linear elasticity. The focus of the work is on three interior penalty methods: symmetric nonsymmetric, and incomplete. We
draw on earlier work on DG formulations for elliptic problems in [2] and elasticity in [11, 12, 13, 25], in addressing the problem of developing discrete formulations that are uniformly convergent in the incompressible and inextensible limits.

With the use of low-order triangles, the problem for isotropic elasticity is uniformly convergent in the incompressible limit [12, 25]. For bi- or trilinear approximations on quadrilaterals and hexahedra, however, it is known [11] that uniform convergence requires the use of under-integration of the edge terms in the formulation. Both of the corresponding error estimates rely ultimately on an a priori estimate presented in [7]. For the problem studied here, it is shown numerically that the DG methods in their original formulation result in locking behaviour in the inextensional limit, a problem that is resolved by underintegrating the relevant edge terms. There does not exist an a priori estimate for transverse isotropy analogous to that in [7] for the isotropic problem, but the error estimate for the formulation with underintegration has a structure similar to that of the isotropic a priori estimate, and therefore consistent with the uniformly convergent behaviour that is observed numerically.

The outline of the rest of this work is as follows. In Sections 2 and 3, we present the governing equations and weak formulation for problems of transversely isotropic linear elasticity, with a summary of conditions for well-posedness. The DG formulations are introduced in Section 4, their well-posedness established, and an a priori error bound derived. The likelihood of extensional locking is deduced from the error estimate, and an alternative formulation, based on selective under-integration, is introduced and analyzed in Section 5. The resulting error bound has a structure similar to that for the bound corresponding to isotropic elasticity, suggesting the locking-free behaviour of this formulation. Such behaviour is confirmed in Section 6, in which numerical results are presented for two model problems. The work concludes with a summary of results and a discussion of possible future work.

2 Transversely isotropic materials

For a transversely isotropic linearly elastic material with fibre direction given by the unit vector $a$, the elasticity tensor is given by [17, 23]

$$\mathbb{C} = \lambda \mathbb{I} \otimes \mathbb{I} + 2\mu \mathbb{I} + \beta \mathbb{M} \otimes \mathbb{M} + \alpha (\mathbb{I} \otimes \mathbb{M} + \mathbb{M} \otimes \mathbb{I}) + \gamma \mathbb{M}.$$  \hspace{1cm} (1)
Here \( I \) is the second-order identity tensor, \( \mathbb{I} \) is the fourth-order identity tensor, \( M = a \otimes a \), and \( \mathbb{M} \) is the fourth-order tensor defined by

\[
\mathbb{M}R = MR + RM \quad \text{for any second-order tensor } R. \tag{2}
\]

\( \lambda \) denotes the first Lamé parameter, the shear modulus in the plane of isotropy is \( \mu_t \), \( \mu_l \) is the shear modulus along the fibre direction, and

\[
\gamma = 2(\mu_l - \mu_t). \tag{3}
\]

The further material constants \( \alpha \) and \( \beta \) do not have a direct interpretation, though it will be seen that \( \beta \to \infty \) in the inextensible limit.

The corresponding linear stress-strain relation for small deformations is then

\[
\sigma = C\varepsilon = \lambda(\text{tr}\varepsilon)I + 2\mu_t\varepsilon + \beta(M : \varepsilon)M + \alpha((M : \varepsilon)I + (\text{tr}\varepsilon)M) + \gamma(\varepsilon M + M\varepsilon), \tag{4}
\]

in which \( \sigma \) and \( \varepsilon \) denote the stress and the infinitesimal strain tensors; \( \text{tr}\varepsilon \) denotes the trace of \( \varepsilon \), and \( M : \varepsilon = \varepsilon a \cdot a \), obtained from the definition of \( M \), gives the strain in the direction of \( a \). The special case of an isotropic material is recovered by setting \( \alpha = \beta = 0 \) and \( \mu_l = \mu_t \).

We can write the expressions of these five material parameters in terms of five physically meaningful constants, viz. \( E_t \): Young’s modulus in the transverse direction; \( E_l \): Young’s modulus in the fibre direction; and \( \nu_t \) and \( \nu_l \): Poisson’s ratios for the transverse strain with respect to the fibre direction and the plane normal to it respectively. The remaining constants are the two shear moduli \( \mu_t \) and \( \mu_l \), and one may further define \( \mu_l \) by

\[
\mu_l = \frac{E_l}{2(1 + \nu_l)}. \tag{5}
\]

Henceforth, we set

\[
E_l = pE_t \quad \text{and} \quad \mu_l = q\mu_t. \tag{6}
\]

Thus \( p \) measures the stiffness in the fibre direction relative to that in the plane of isotropy. We then
have

\[ \mu_l = \frac{qE_l}{2(1 + \nu_l)}, \]

(7a)

\[ \lambda = \frac{(\nu_l + \nu_l^2)}{(1 + \nu_l)((1 - \nu_l)p - 2\nu_l^2)}E_l, \]

(7b)

\[ \alpha = \frac{(\nu_l - \nu_l + \nu_l^2)p - \nu_l^2}{((1 + \nu_l)((1 - \nu_l)p - 2\nu_l^2))E_l}, \]

(7c)

\[ \beta = \frac{(1 - \nu_l^2)p^2 + (-2\nu_l\nu_l + 2q\nu_l - 2\nu_l + 1 - 2q)p - (1 - 4q)\nu_l^2}{(1 + \nu_l)((1 - \nu_l)p - 2\nu_l^2)}E_l. \]

(7d)

As shown in [19], sufficient conditions on the material constants for the elasticity tensor to be pointwise stable are

\[ \mu_l \geq \mu_l > 0, \quad p > 0, \quad \nu_l > -1, \]

(8a)

\[ (2\nu_l + 1)p - (2\nu_l + 1) > 0, \]

(8b)

and \((1 - \nu_l)p - 2\nu_l^2 > 0.\)

(8c)

### 3 Governing equations and weak formulation

Consider a transversely isotropic elastic body occupying a bounded domain \(\Omega \subset \mathbb{R}^d, \quad d = \{2, 3\}\), with boundary \(\Gamma = \Gamma_D \cup \Gamma_N\) having exterior unit normal \(n\). Here \(\Gamma_D\) is the Dirichlet boundary, \(\Gamma_N\) the Neumann boundary, and \(\Gamma_D \cap \Gamma_N = \emptyset\). The equilibrium equation is

\[ -\text{div} \, \sigma(u) = f \]

(9)

and the boundary conditions are

\[ u = g \text{ on } \Gamma_D, \]

(10a)

\[ \sigma(u)n = h \text{ on } \Gamma_N. \]

(10b)

Here \(\sigma\) is the Cauchy stress tensor defined by equation (4), \(u\) is the displacement vector, \(f\) is the body force, \(g\) a prescribed displacement, and \(h\) a prescribed surface traction.
We denote by $H^1(\Omega)$ the Sobolev space of functions which, together with their generalized first derivatives, are square-integrable, and set

$$V = \{ \mathbf{u} \in [H^1(\Omega)]^d : \mathbf{u} = \mathbf{0} \text{ on } \Gamma_D \},$$

which is endowed with the norm

$$\| \cdot \|_V = \| \cdot \|_{[H^1(\Omega)]^d}.$$

To take account of the non-homogeneous boundary condition (10a), we define the function $\mathbf{u}_g \in [H^1(\Omega)]^d$ such that $\mathbf{u}_g = \mathbf{g}$ on $\Gamma_D$, and the bilinear form $a(\cdot, \cdot)$ and linear functional $l(\cdot)$ by

$$a : [H^1(\Omega)]^d \times [H^1(\Omega)]^d \to \mathbb{R}, \quad a(\mathbf{u}, \mathbf{v}) = \int_{\Omega} \sigma(\mathbf{u}) : \varepsilon(\mathbf{v}) \, dx,$$

$$l : [H^1(\Omega)]^d \to \mathbb{R}, \quad l(\mathbf{v}) = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, dx + \int_{\Gamma_N} \mathbf{h} \cdot \mathbf{v} \, ds - a(\mathbf{u}_g, \mathbf{v}).$$

The weak form of the problem is then as follows: given $\mathbf{f} \in [L^2(\Omega)]^d$ and $\mathbf{h} \in [L^2(\Gamma_N)]^d$, find $\mathbf{U} \in [H^1(\Omega)]^d$ such that $\mathbf{U} = \mathbf{u} + \mathbf{u}_g, \mathbf{u} \in V$, and

$$a(\mathbf{u}, \mathbf{v}) = l(\mathbf{v}) \quad \forall \, \mathbf{v} \in V.$$

We write the bilinear form as

$$a(\mathbf{u}, \mathbf{v}) = a^{iso}(\mathbf{u}, \mathbf{v}) + a^{ti}(\mathbf{u}, \mathbf{v}),$$

where

$$a^{iso}(\mathbf{u}, \mathbf{v}) = \lambda \int_{\Omega} (\nabla \cdot \mathbf{u})(\nabla \cdot \mathbf{v}) \, dx + 2\mu_t \int_{\Omega} \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) \, dx,$$

$$a^{ti}(\mathbf{u}, \mathbf{v}) = \alpha \int_{\Omega} ((\mathbf{M} : \varepsilon(\mathbf{u}))(\nabla \cdot \mathbf{v}) + (\nabla \cdot \mathbf{u})(\mathbf{M} : \varepsilon(\mathbf{v}))) \, dx + \beta \int_{\Omega} (\mathbf{M} : \varepsilon(\mathbf{u}))(\mathbf{M} : \varepsilon(\mathbf{v})) \, dx$$

$$+ \gamma \int_{\Omega} (\varepsilon(\mathbf{u})\mathbf{M} : \varepsilon(\mathbf{v}) + \mathbf{M}\varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v})) \, dx.$$

Note that $a(\cdot, \cdot)$ is symmetric; furthermore, the weak problem is well-posed, as shown in [19].

We define the following notation for relevant norm and seminorms:

$$\| \cdot \|_{0,*} = \| \cdot \|_{[L^2(\star)]^d}, \quad \| \cdot \|_{1,*} = \| \cdot \|_{[H^1(\star)]^d}, \quad \text{and} \quad \| \cdot \|_{2,*} = \| \cdot \|_{[H^2(\star)]^d}.$$
4 Discontinuous Galerkin finite element approximations

Suppose that \( \Omega \) is polygonal (in \( \mathbb{R}^2 \)) or polyhedral (in \( \mathbb{R}^3 \)), partitioned into a triangular/tetrahedral conforming mesh, comprising \( n_e \) disjoint subdomains \( \Omega_e \) with boundary \( \partial \Omega_e \) consisting of edges/faces \( E \), and outward unit normal \( n_e \). Denote by \( \mathcal{T}_h := \{ \Omega_e \}_e \) the set of all elements. Define \( h_E := \text{diam}(E) \), \( h_e := \text{diam}(\Omega_e) \), and \( h := \max_{\Omega_e \in \mathcal{T}_h} \{ h_e \} \). We define the discrete space \( \mathcal{V}_h^{DG} \subset \mathcal{V} \) by

\[
\mathcal{V}_h^{DG} := \{ v \in [L^2(\Omega)]^d : v|_{\Omega_e} \in [P_1(\Omega_e)]^d, \ \forall \ \Omega_e \in \mathcal{T}_h \},
\]

where \( P_1(\Omega) \) is the space of polynomials on \( \Omega \) of maximum total degree 1.

Let \( \Omega_i, \Omega_e \in \mathcal{T}_h \) be two elements sharing an interior edge \( E = \partial \Omega_i \cap \partial \Omega_e \), and let \( n \) be the outward normal to \( \Omega_i \). Then for any vector quantity \( v \in [T(\Gamma)]^d := \prod_{\Omega_e \in \mathcal{T}_h} [L^2(\partial \Omega_e)]^d \) and any second order tensor \( \tau \in [T(\Gamma)]^{d \times d} \), we define the jumps

\[
[v] = (v_i - v_e) \otimes n, \quad [\tau] = (\tau_i - \tau_e)n, \quad [v] = (v_i - v_e) \cdot n,
\]

and the averages

\[
\{v\} = \frac{1}{2}(v_i + v_e), \quad \{\tau\} = \frac{1}{2}(\tau_i + \tau_e),
\]

where subscripts \( i \) and \( e \) denote values on the elements \( \Omega_i \) and \( \Omega_e \), respectively. The space \( \mathcal{V}_h^{DG} \) is endowed with the norm (see for example [11, 25])

\[
||u||_{DG}^2 := \sum_{\Omega_e \in \mathcal{T}_h} \varepsilon(u)|_{\Omega_e}^2 + \frac{1}{2} \sum_{E \in \Gamma_{iD}} \frac{1}{h_E} ||[u]||_{0,E}^2,
\]

where \( \Gamma_{iD} \) is the union of all interior edges and all Dirichlet boundary edges.

The general Interior Penalty Discontinuous Galerkin (IPDG) formulation is as follows [11, 25]: for all \( v \in \mathcal{V}_h^{DG} \), find \( u_h \in \mathcal{V}_h^{DG} \) such that

\[
a_h(u_h, v) = l_h(v),
\]

where
where

\[
\begin{align*}
    a_h(u, v) &= \sum_{\Omega_e \in \mathcal{T}_h} \int_{\Omega_e} \sigma(u) : \varepsilon(v) \, dx - \sum_{E \in \Gamma_{ID}} \frac{1}{h_E} \int_E [v] : \{\sigma(u)\} \, ds + \theta \sum_{E \in \Gamma_{ID}} \frac{1}{h_E} \int_E [u] : \{\sigma(v)\} \, ds \\
    &+ k_{\mu} \mu_t \sum_{E \in \Gamma_{ID}} \frac{1}{h_E} \int_E [u] : [u] \, ds + k_{\lambda} \lambda \sum_{E \in \Gamma_{ID}} \frac{1}{h_E} \int_E [v][u] \, ds \\
    &+ k_{\alpha} \alpha \sum_{E \in \Gamma_{ID}} \frac{1}{h_E} \int_E \left( [\sigma(u)](M : [u]) + ([v] : M)[u] \right) \, ds \\
    &+ k_{\beta} \beta \sum_{E \in \Gamma_{ID}} \frac{1}{h_E} \int_E (M : [v])(M : [u]) \, ds \\
    &+ k_{\gamma} \gamma \sum_{E \in \Gamma_{ID}} \frac{1}{h_E} \int_E [v] : ([u]M + M[u]) \, ds,
\end{align*}
\]

and

\[
\begin{align*}
    l_h(v) &= \sum_{\Omega_e \in \mathcal{T}_h} \int_{\Omega_e} f : v \, dx + \sum_{E \in \Gamma_N} \frac{1}{h_E} \int_E h : v \, ds + \theta \sum_{E \in \Gamma_D} \frac{1}{h_E} \int_E (g \otimes \mathbf{n}) : \sigma(v) \, ds \\
    &+ k_{\mu} \mu_t \sum_{E \in \Gamma_{ID}} \frac{1}{h_E} \int_E g : v \, ds + k_{\lambda} \lambda \sum_{E \in \Gamma_{ID}} \frac{1}{h_E} \int_E (v \cdot \mathbf{n})(g \cdot \mathbf{n}) \, ds \\
    &+ k_{\alpha} \alpha \sum_{E \in \Gamma_{ID}} \frac{1}{h_E} \int_E \left( \mathbf{v} \cdot \mathbf{n} \right)(M : g \otimes \mathbf{n}) + \left( \mathbf{v} \otimes \mathbf{n} : M \right)(g \cdot \mathbf{n}) \, ds \\
    &+ k_{\beta} \beta \sum_{E \in \Gamma_{ID}} \frac{1}{h_E} \int_E (M : \mathbf{v} \otimes \mathbf{n})(M : g \otimes \mathbf{n}) \, ds \\
    &+ k_{\gamma} \gamma \sum_{E \in \Gamma_{ID}} \frac{1}{h_E} \int_E \mathbf{v} \otimes \mathbf{n} : \left( (g \otimes \mathbf{n})M + M(g \otimes \mathbf{n}) \right) \, ds.
\end{align*}
\]

Here \(k_{\mu}, k_{\lambda}, k_{\alpha}, k_{\beta},\) and \(k_\gamma\) are non-negative stabilization parameters, \(\theta\) is a switch that distinguishes the three methods, \((\theta = 1\) for the Nonsymmetric IPG (NIPG) method, \(\theta = -1\) for Symmetric IPG (SIPG), and \(\theta = 0\) for Incomplete IPG (IIPG)), and the stress tensor \(\sigma\) is as defined in (4).

We confine attention to homogeneous bodies, so that the fibre direction \(\alpha\) is constant.

### 4.1 Consistency

Given the exact solution \(u \in [H^2(\Omega)]^d\), the problem is consistent if, for any \(v \in \mathcal{V}_{DG}^h\)

\[
a_h(u, v) - l_h(v) = 0.
\]
Using the fact that \( |u|_E = |u|_{E'} = |\sigma(u)|_E = 0 \) \( \forall E \in \Gamma_i \) (set of all interior edges), and \( |u| = g \otimes n \) and \( [u] = g \cdot n \) on \( \Gamma_D \), the proof of consistency is straightforward and may be carried out in a single argument for all three cases.

### 4.2 Coercivity

The bilinear form is coercive if, for any \( v \in V_h \),

\[
a_h(v, v) \geq C \|v\|_{DG},
\]

where \( C > 0 \) is a constant. To prove coercivity, each IP method will be investigated separately, as different approaches are used for each of them.

**NIPG (\( \theta = 1 \))** We have

\[
a_h(v, v) = \sum_{\Omega_e \in T_h} \int_{\Omega_e} \sigma(v) : \varepsilon(v) \, dx + \sum_{E \in \Gamma_{iD}} \frac{1}{h_E} \int_E (k_\mu \mu t |v|^2 + k_\lambda \lambda [v]^2 \\
+ 2k_\alpha [v](M : [v]) + k_\beta \beta (M : [v])^2 + k_\gamma [v] : (M [v] + M [v]) \) \, ds. \tag{17}
\]

For ease, we denote \( m = v_i - v_e \) and \( n \) the outward unit normal vector; then we have

\[
|v|^2 = |(v_i - v_e) \otimes n|^2 = m \cdot m,
\]

\[
[v] = (v_i - v_e) \cdot n = m \cdot n,
\]

\[
M : [v] = (a \otimes a) : (v_i - v_e) \otimes n = (a \cdot m)(a \cdot n),
\]

\[
[v] : [v] M = (v_i - v_e) \otimes n : (a \otimes a)((v_i - v_e) \otimes n) = (a \cdot m)^2 (m \cdot m),
\]

\[
M [v] = (v_i - v_e) \otimes n : (a \otimes a)(v_i - v_e) \otimes (a \otimes a) = (a \cdot m)^2.
\]

It is shown in Section 3 of [19] that

\[
\int_{\Omega_e} \sigma(v) : \varepsilon(v) \, dx \geq C_1 \mu t \|\varepsilon(v)\|^2_{0, \Omega_e}. \tag{18}
\]
The remaining terms in (17) can be rewritten as follows:

\[
\mathcal{R} := \sum_{E \in \Gamma_{1D}} \frac{1}{h_E} \int_E \left( k_\lambda \lambda (n \cdot m)^2 + k_\mu \mu_t (m \cdot m) + 2k_\alpha \alpha (n \cdot m)(a \cdot n)(a \cdot m)
\right.
\]

\[
+ k_\beta \beta (a \cdot n)^2 (a \cdot m)^2 + k_\gamma \gamma ((a \cdot m)^2 + (a \cdot n)^2 (m \cdot m)) \right) ds
\]

\[
\geq k_{\text{min}} \sum_{E \in \Gamma_{1D}} \frac{1}{h_E} \int_E \left( \lambda + 2\mu_t (n \cdot m)^2 + 2\mu_t ((m \cdot m) - (n \cdot m)^2) + 2\alpha (n \cdot m)(a \cdot n)(a \cdot m)
\right.
\]

\[
+ \beta (a \cdot n)^2 (a \cdot m)^2 + \gamma ((a \cdot m)^2 + (a \cdot n)^2 (m \cdot m)) \right) ds
\]

with \( k_{\text{min}} = \min\{ \frac{k}{2}, k_\lambda, k_\alpha, k_\beta, k_\gamma \} > 0 \). From the pointwise stability proof in [19], we have, for sufficiently small positive \( r \),

\[
A(r) = \left( \lambda + \frac{2r}{3} \mu_t \right) (n \cdot m)^2 + (\beta + 2\gamma)(\varepsilon a \cdot a)^2 + 2\alpha (\varepsilon a \cdot a) > 0. \quad (19)
\]

Choosing \( \varepsilon = \frac{1}{2} (m \otimes n + n \otimes m) \) and noting that \( m \cdot m \geq (a \cdot m)^2 \), this becomes

\[
A(r) = \left( \lambda + \frac{2r}{3} \mu_t \right) (n \cdot m)^2 + (\beta + 2\gamma)(a \cdot n)^2 (a \cdot m)^2 + 2\alpha (n \cdot m)(a \cdot n)(a \cdot m) > 0.
\]

Therefore,

\[
\mathcal{R} \geq k_{\text{min}} \sum_{E \in \Gamma_{1D}} \frac{1}{h_E} \int_E \left( A(r) + 2\mu_t \left( 1 - \frac{r}{3} \right) (n \cdot m)^2 + 2\mu_t ((m \cdot m) - (n \cdot m)^2)
\right.
\]

\[
+ \gamma ((a \cdot m)^2 + (a \cdot n)^2 (m \cdot m) - 2(a \cdot n)^2 (a \cdot m)^2) \right) ds
\]

\[
\geq k_{\text{min}} \sum_{E \in \Gamma_{1D}} \frac{1}{h_E} \int_E \left( 2\mu_t \left( 1 - \frac{r}{3} \right) (n \cdot m)^2 + 2\mu_t ((m \cdot m) - (n \cdot m)^2) + \gamma (a \cdot m)^2 (1 - (a \cdot n)^2) \right) ds
\]

\[
\geq 2\mu_t k_{\text{min}} \left( 1 - \frac{r}{3} \right) \sum_{E \in \Gamma_{1D}} \frac{1}{h_E} \int_E (m \cdot m) ds
\]

\[
\geq C_2 \mu_t \sum_{E \in \Gamma_{1D}} \frac{1}{h_E} ||| v |||_{0,E}^2,
\]

with \( C_2 = 2k_{\text{min}} \left( 1 - \frac{r}{3} \right) > 0 \), and \( k_{\text{min}} > 0 \).

Together with (18) and the definition (15), we see that the bilinear form is coercive for the NIPG case.
**SIPG (θ = −1)** To prove coercivity for the SIPG method, the same approach as presented in [11] for the isotropic case is used. The bilinear form can be written as

\[
a_h(v, v) = \sum_{Ω_e ∈ T_h} \int_{Ω_e} \sigma(v) : ε(v) \, dx - 2 \sum_{E ∈ Γ_{ID}} \int_E |v| : \{σ(v)\} \, ds
+ \sum_{E ∈ Γ_{ID}} \frac{1}{h_E} \int_E \left( k_λ \lambda |v|^2 + k_µ |v| + + 2k_α |v| (M : |v|) + k_β M : |v| \right)^2
+ k_γ |v| : (|v| M + M |v|) \, ds
= T_µ + T'_γ + T''_γ + T_e + T_i + T_D
\]

where

\[
T_µ := 2µ \left( \sum_{Ω_e ∈ T_h} ||ε(v)||^2_{0,Ω_e} - 2 \sum_{E ∈ Γ_{ID}} \int_E |v| : \{ε(v)\} \, ds + \sum_{E ∈ Γ_{ID}} \frac{k_µ}{2h_E} ||v||^2_{0,E} \right) ; \tag{20a}
T'_γ := γ \left( \sum_{Ω_e ∈ T_h} ||ε(v)a||^2_{0,Ω_e} - 2 \sum_{E ∈ Γ_{ID}} \int_E |v| a : \{ε(v)\}a \, ds + \sum_{E ∈ Γ_{ID}} \frac{k_γ}{h_E} ||v||^2_{0,E} \right) ; \tag{20b}
T''_γ := γ \left( \sum_{Ω_e ∈ T_h} ||ε(v)a||^2_{0,Ω_e} - 2 \sum_{E ∈ Γ_{ID}} \int_E |v| a : \{ε(v)\}a \, ds + \sum_{E ∈ Γ_{ID}} \frac{k_γ}{h_E} ||v||^2_{0,E} \right) ; \tag{20c}
T_e := λ \sum_{Ω_e ∈ T_h} ||∇ · v||^2_{0,Ω_e} + 2α \sum_{Ω_e ∈ T_h} \int_{Ω_e} (M : ε(v))(∇ · v) \, dx + β \sum_{Ω_e ∈ T_h} ||M : ε(v)||^2_{0,E} ; \tag{20d}
T_i := - \sum_{Ω_e ∈ T_h} \sum_{E ∈ Ω_e} \int_E \left( λ \{∇ · v\}[v] + α \{v\}((M : ε(v))I) + |v| : \{(∇ · v)M\} \right)
+ β |v| : \{M(M : ε(v))\} \, ds
+ \sum_{Ω_e ∈ T_h} \sum_{E ∈ Ω_e} \left( \frac{k_γ}{h_E} ||v||^2_{0,E} + \frac{k_α}{h_E} \int_E (M : |v|) \, |v| \, ds + \frac{k_β}{2h_E} ||M : |v||^2_{0,E} \right) ; \tag{20e}
T_D := - 2 \sum_{Ω_e ∈ T_h} \sum_{E ∈ Ω_e} \int_E \left( λ (v · n)(v · n) + α ((v · n)(M : ε(v))) + v ⊙ n : (∇ · v)M \right)
+ β (v ⊙ n) : \{(M : ε(v))M\} \, ds
+ \sum_{Ω_e ∈ T_h} \sum_{E ∈ Ω_e} \left( \frac{k_γ}{h_E} ||v · n||^2_{0,E} + \frac{k_β}{h_E} ||M · v ⊙ n||^2_{0,E} + \frac{2k_α}{h_E} \int_E (v · n)(M : v ⊙ n) \, ds \right). \tag{20f}
\]
It can be shown, as in [11], that for $k_\mu \geq 4 \left( m + \frac{C^2}{1 - m} \right)$, with $0 < m < 1$ and $C > 0$ independent of $h_E$, $\mu_t$ and $\Omega$,

$$T_\mu \geq C\|v\|_{DG}^2.$$ 

All that is required for coercivity is then to prove that the remaining terms are positive.

Following the same procedure in Section 3.1.1 of [11], we find that

$$T'_\gamma \geq \gamma \sum_{\Omega_e \in \mathcal{T}_h} \left( (1 - C'_\gamma \varepsilon'_\gamma)\|\varepsilon(v)a\|^2_{0,\Omega_e} + \left( \frac{k_\gamma}{2} - \frac{C'_\gamma}{\varepsilon'_\gamma} \right) \|h_E^{-1/2}[v]a\|^2_{0,\partial\Omega_e} \right),$$

for some $\varepsilon'_\gamma > 0$ and a constant $C'_\gamma > 0$ arising from bounding the interior and Dirichlet terms of $T'_\gamma$, independent of $h$ and $\gamma$. By setting $\varepsilon'_\gamma = \frac{1}{C'_\gamma}$, the first term vanishes. The choice $k_\gamma \geq 2C''_\gamma$ then gives $T'_\gamma \geq 0$.

An identical procedure is followed to obtain $T''_\gamma \geq 0$ by choosing $k_\gamma \geq 2C''_\gamma$ for some positive constant $C''_\gamma$ independent of $h$ and $\gamma$.

Next, considering the term $T_e$, and noting that $2(M : \varepsilon(v)) (\nabla \cdot v) \geq - \left( (M : \varepsilon(v))^2 + (\nabla \cdot v)^2 \right)$, we have

$$T_e \geq \sum_{\Omega_e \in \mathcal{T}_h} \left( (\lambda - \alpha)\|\nabla \cdot v\|^2_{0,\Omega_e} + (\beta - \alpha)\|M : \varepsilon(v)\|^2_{0,\Omega_e} \right).$$

The terms in $T_1$ can be bounded as follows:

$$\sum_{\Omega_e \in \mathcal{T}_h} \sum_{E \in \partial\Omega_e} \int_E \lambda(\nabla \cdot v)[v] \, ds \leq \lambda \sum_{\Omega_e \in \mathcal{T}_h} \sum_{E \in \partial\Omega_e} \|h_E^{1/2} \nabla \cdot v\|_{0,E} \|h_E^{-1/2}[v]\|_{0,E},$$

$$\leq \lambda \sum_{\Omega_e \in \mathcal{T}_h} \|h_E^{1/2} \nabla \cdot v\|_{0,\partial\Omega_e} \|h_E^{-1/2}[v]\|_{0,\partial\Omega_e}.$$ 

In a similar way, using the fact that $|M|^2 = 1$,

$$\sum_{\Omega_e \in \mathcal{T}_h} \sum_{E \in \partial\Omega_e} \int_E \left( \alpha[v] \{(M : \varepsilon(v))I\} + \alpha[v] : \{(\nabla \cdot v)M\} + \beta[v] : \{M(M : \varepsilon(v))\} \right) \, ds$$

$$\leq \sum_{\Omega_e \in \mathcal{T}_h} \left( \alpha\|h_E^{1/2} M : \varepsilon(v)\|_{0,\partial\Omega_e} \|h_E^{-1/2}[v]\|_{0,\partial\Omega_e} + \alpha\|h_E^{1/2} \nabla \cdot v\|_{0,\partial\Omega_e} \|h_E^{-1/2} M : [v]\|_{0,\partial\Omega_e} + \beta\|h_E^{1/2} M : \varepsilon(v)\|_{0,\partial\Omega_e} \|h_E^{-1/2} M : [v]\|_{0,\partial\Omega_e} \right).$$
With \((M : [v])|v| \geq \frac{1}{2}((M : [v])^2 + |v|^2)\), we have

\[
T_i \geq - \sum_{\Omega_c \in \mathcal{T}_h} \left( \lambda \|h_E^{1/2} \nabla \cdot v\|_{0, \partial \Omega_c} \|h_E^{-1/2}[v]\|_{0, \partial \Omega_c} + \alpha \|h_E^{1/2} M : \varepsilon(v)\|_{0, \partial \Omega_c} \|h_E^{-1/2}[v]\|_{0, \partial \Omega_c} \right) + \alpha \|h_E^{1/2} \nabla \cdot v\|_{0, \partial \Omega_c} \|h_E^{-1/2} M : [v]\|_{0, \partial \Omega_c} + \beta \|h_E^{1/2} M : \varepsilon(v)\|_{0, \partial \Omega_c} \|h_E^{-1/2}[v]\|_{0, \partial \Omega_c} \geq \frac{1}{2} \sum_{\Omega_c \in \mathcal{T}_h} \left( (\lambda k_\alpha - \alpha k_\alpha) \|h_E^{-1/2}[v]\|_{0, \partial \Omega_c}^2 + (\beta k_\beta - \alpha k_\alpha) \|h_E^{-1/2} M : [v]\|_{0, \partial \Omega_c}^2 \right).
\]

The terms of \(T_D\) can be bounded similarly, and using \(2(v \cdot n)(M : v \otimes n) \geq -((v \cdot n)^2 + (M : v \otimes n)^2)\), we obtain

\[
T_D \geq -2 \sum_{\Omega_c \in \mathcal{T}_h} \left( \lambda \|h_E^{1/2} \nabla \cdot v\|_{0, \partial \Omega_c} \|h_E^{-1/2} v \cdot n\|_{0, \partial \Omega_c} + \alpha \|h_E^{1/2} M : \varepsilon(v)\|_{0, \partial \Omega_c} \|h_E^{-1/2} v \cdot n\|_{0, \partial \Omega_c} \right) + \alpha \|h_E^{1/2} \nabla \cdot v\|_{0, \partial \Omega_c} \|h_E^{-1/2} M : [v] \otimes n\|_{0, \partial \Omega_c} + \beta \|h_E^{1/2} M : \varepsilon(v)\|_{0, \partial \Omega_c} \|h_E^{-1/2} M : [v] \otimes n\|_{0, \partial \Omega_c} \geq \frac{1}{2} \sum_{\Omega_c \in \mathcal{T}_h} \left( (\lambda k_\lambda - \alpha k_\alpha) \|h_E^{-1/2} v \cdot n\|_{0, \partial \Omega_c}^2 + (\beta k_\beta - \alpha k_\alpha) \|h_E^{-1/2} M : [v] \otimes n\|_{0, \partial \Omega_c}^2 \right).
\]

Combining the bounds of \(T_i\) and \(T_D\) gives

\[
T_i + T_D \geq -\sum_{\Omega_c \in \mathcal{T}_h} \left( \lambda \left( \frac{\epsilon_\lambda}{2} \|\nabla \cdot v\|_{0, \Omega_c}^2 + \frac{1}{2\epsilon_\lambda} \|h_E^{-1/2}[v]\|_{0, \partial \Omega_c}^2 \right) + \alpha \left( \frac{\epsilon_\alpha}{2} \|M : \varepsilon(v)\|_{0, \Omega_c}^2 + \frac{1}{2\epsilon_\alpha} \|h_E^{-1/2}[v]\|_{0, \partial \Omega_c}^2 \right) + \alpha \left( \frac{\epsilon_\alpha}{2} \|\nabla \cdot v\|_{0, \Omega_c}^2 + \frac{1}{2\epsilon_\alpha} \|h_E^{-1/2} M : [v]\|_{0, \partial \Omega_c}^2 \right) + \beta \left( \frac{\epsilon_\beta}{2} \|M : \varepsilon(v)\|_{0, \Omega_c}^2 + \frac{1}{2\epsilon_\beta} \|h_E^{-1/2} M : [v]\|_{0, \partial \Omega_c}^2 \right) \right) + \frac{1}{2} \sum_{\Omega_c \in \mathcal{T}_h} \left( (\lambda k_\lambda - \alpha k_\alpha) \|h_E^{-1/2}[v]\|_{0, \partial \Omega_c}^2 + (\beta k_\beta - \alpha k_\alpha) \|h_E^{-1/2} M : [v]\|_{0, \partial \Omega_c}^2 \right),
\]

for some \(\epsilon_\lambda, \epsilon_\alpha, \epsilon_\beta > 0\).
Adding (21) and (22) together,

\[ T := T_e + T_i + T_D \]

\[ \geq \sum_{\Omega_e \in \mathcal{T}_h} \left( \left( \lambda - \alpha - \frac{\lambda \varepsilon \lambda}{2} - \frac{\alpha \varepsilon \alpha}{2} \right) \| \nabla \cdot \mathbf{v} \|^2_{0, \Omega_e} + \left( \beta - \alpha - \frac{\beta \varepsilon}{2} - \frac{\alpha \varepsilon}{2} \right) \| M : \varepsilon(\mathbf{v}) \|^2_{0, \Omega_e} \right) + \left( \lambda k_\alpha - \frac{\alpha k_\alpha}{2} - \frac{\lambda}{2 \varepsilon} \right) \| h_E^{-1/2}[\mathbf{v}] \|^2_{0, \partial \Omega_i D} + \left( \beta k_\beta - \frac{\alpha k_\beta}{2} - \frac{\beta}{2 \varepsilon} \right) \| h_E^{-1/2} \mathcal{M} : |\mathbf{v}| \|^2_{0, \partial \Omega_i D} \right) \].

(23)

If

\[ k_\alpha > 0, \]

\[ k_\beta \geq \frac{2 - \varepsilon}{2 + \varepsilon} \left( k_\alpha + \frac{1}{\varepsilon \alpha} \right) + \frac{1}{\varepsilon \beta}, \]

and \( k_\lambda \geq \frac{\alpha}{\lambda} \left( k_\alpha + \frac{1}{\varepsilon \alpha} \right) + \frac{\lambda}{\lambda \varepsilon - \alpha \varepsilon \alpha} \),

each of the coefficients of the terms in (23) will be non-negative, and thus \( T \geq 0 \). Therefore we can conclude that for these choices of the five stabilization parameter values, the bilinear form \( a_h \) is coercive for SIPG.

**IIPG (\( \theta = 0 \))** For this case

\[ a_h(\mathbf{v}, \mathbf{v}) = \sum_{\Omega_e \in \mathcal{T}_h} \int_{\Omega_e} \sigma(\mathbf{v}) : \varepsilon(\mathbf{v}) - \sum_{E \in \Gamma_{iD}} \frac{1}{h_E} \int_E [\mathbf{v}] : \{ \sigma(\mathbf{v}) \} \, ds \]

\[ + \sum_{E \in \Gamma_{iD}} \frac{1}{h_E} \int_E \left( k_\mu \mu_t |[\mathbf{v}]|^2 + k_\lambda \lambda |[\mathbf{v}]|^2 + 2 k_\alpha \alpha (\mathcal{M} : [\mathbf{v}]) + k_\beta \beta (\mathcal{M} : [\mathbf{v}])^2 \right. 
\]

\[ \left. + k_\gamma [\mathbf{v}] : ([\mathbf{v}] \mathcal{M} + \mathcal{M} [\mathbf{v}]) \right) \, ds. \]

The only difference between this form and the corresponding SIPG bilinear form is the coefficient in the second term; thus the proof of coercivity for IIPG case is identical to that for the SIPG case up to a constant.

We summarize these results.

**Theorem 1.** The bilinear functional \( a_h(\cdot, \cdot) \) defined in (16) is coercive if:
(a) When \( \theta = 1 \), \( k_{\text{min}} = \min\left\{ \frac{k_\mu}{2}, k_\lambda, k_\alpha, k_\beta, k_\gamma \right\} > \frac{3}{2(3-r)} \) for some small \( r > 0 \);

(b) When \( \theta \in \{0, -1\} \),

\[
\begin{align*}
    k_\mu &\geq C_\mu, \\
    k_\gamma &\geq C_\gamma, \\
    k_\alpha &> 0, \\
    k_\beta &\geq \frac{2 - \varepsilon_\beta}{2 + \varepsilon_\alpha} \left( k_\alpha + \frac{1}{\varepsilon_\alpha} \right) + \frac{1}{\varepsilon_\beta}, \\
    \text{and } k_\lambda &\geq \frac{\alpha}{\lambda} \left( k_\alpha + \frac{1}{\varepsilon_\alpha} \right) + \frac{\lambda}{\lambda \varepsilon_\lambda - \alpha \varepsilon_\alpha},
\end{align*}
\]

where \( C_\mu, C_\gamma, \varepsilon_\alpha, \varepsilon_\beta, \) and \( \varepsilon_\lambda \) are positive constants to be calculated.

### 4.3 Error bound

As shown in [25], one has uniform (\( \lambda \)-independent) convergence for the isotropic problem when linear triangles are used. We present here a corresponding bound for transversely isotropic materials, assuming a constant fibre direction \( \alpha \). To establish the bound, we adopt the same approach as in [25]; that is, splitting the error using a linear interpolant \( \Pi_e u \in [P_1(\Omega_e)]^d \), for \( u \in [\mathcal{H}^2(\Omega_e)]^d \), which is defined by

\[
\Pi_e u(\bar{x}_E) := \frac{1}{h_E} \int_E u \ ds \quad \forall \ E \in \partial \Omega_e,
\]

where \( \bar{x}_E \) is the midpoint of edge \( E \).

The corresponding global interpolant \( \Pi : [\mathcal{H}^2(\Omega_e)]^d \rightarrow \mathcal{V}^h_{DG} \) is defined by

\[
\Pi u|_{\Omega_e} = \Pi_e u \quad \forall \ \Omega_e \in \mathcal{T}_h.
\]

**Proposition 4.1.** The interpolant has the following properties:

\[
\begin{align*}
    \int_E (u - \Pi_e u) \ ds &= 0, \\
    \int_E (u - \Pi_e u) \cdot n_e \ ds &= 0, \\
    \int_{\Omega_e} \nabla \cdot (u - \Pi_e u) \ dx &= 0, \\
    \int_{\Omega_e} M : \varepsilon(u - \Pi_e u) \ dx &= 0.
\end{align*}
\]

**Proof.** The proofs of (26a), (26b) and (26c) are given in [25].
For (26d), using integration by parts, we have
\[
\int_{\Omega_e} M : \epsilon(u - \Pi_e u) \, dx = \int_{\partial\Omega_e} (u - \Pi_e u) \otimes n_e : M \, ds - \int_{\Omega_e} (\nabla \cdot M) \cdot (u - \Pi_e u) = 0.
\]
\[
\Box
\]

Proposition 4.2. The following interpolation error estimates hold:

\[
\|u - \Pi_e u\|_{0,\Omega_e} + h_e|u - \Pi_e u|_{1,\Omega_e} \leq Ch_e^2|u|_{2,\Omega_e}, \quad (27a)
\]
\[
|u - \Pi_e u|_{2,\Omega_e} = |u|_{2,\Omega_e}, \quad (27b)
\]
\[
\|\nabla \cdot (u - \Pi_e u)\|_{0,\Omega_e} \leq Ch_e|\nabla \cdot u|_{1,\Omega_e}, \quad (27c)
\]
\[
|\nabla \cdot (u - \Pi_e u)|_{1,\Omega_e} = |\nabla \cdot u|_{1,\Omega_e}, \quad (27d)
\]
\[
\|M : \epsilon(u - \Pi_e u)\|_{0,\Omega_e} \leq Ch_e|M : \epsilon(u)|_{1,\Omega_e}, \quad (27e)
\]
\[
|M : \epsilon(u - \Pi_e u)|_{1,\Omega_e} = |M : \epsilon(u)|_{1,\Omega_e} \quad (27f)
\]

where \(C > 0\) is in each case a constant independent of \(h_e\) and \(u\).

**Proof.** The proofs for (27a)-(27d) are given in [25].

Proof of (27f). Setting \(U = u - \Pi_e u\), then since \(\Pi_e u \in [P_1(\Omega_e)]^d\), it follows that
\[
|M : \epsilon(U)|_{1,\Omega_e} = |M : \epsilon(u)|_{1,\Omega_e}. \quad (28)
\]

Proof of (27e). Lemma A.3 in [25] applied to \(M : \epsilon(U)\) gives
\[
\left\|M : \epsilon(U) - \frac{1}{|\Omega_e|} \int_{\Omega_e} M : \epsilon(U) \, dx\right\|_{0,\Omega_e} \leq Ch_e|M : \epsilon(U)|_{1,\Omega_e}.
\]

With (26d) and (28), we obtain (27e). \(\Box\)

Let \(u \in \mathcal{H}^2(\Omega_e)\) be the exact solution to the problem and \(u_h \in V_{DG}^h\) the corresponding finite element
approximation; then the approximation error is

\[ e = u - u_h = u - \Pi u + \Pi u - u_h. \]

In particular \( \xi_{|\Omega_e} \in [P_1(\Omega_e)]^d \). The DG-norm of the error is

\[ \|e\|_{DG}^2 \leq \|\eta\|_{DG}^2 + \|\xi\|_{DG}^2. \]

**Useful bounds:** For \( v \in [H^1(\Omega_e)]^d \) and \( \phi \in [\mathcal{C}^2(\Omega_e)]^d \), we have

\[
\begin{align*}
\|v\|_{0,\partial \Omega_e} & \leq C \left( h_e^{-1/2} \|v\|_{0,\Omega_e} + h_e^{1/2} |v|_{1,\Omega_e} \right), \\
\sum_{E \in \Gamma_{1D}} \frac{1}{h_E} \|v\|_{0,E}^2 & \leq \sum_{\Omega_e \in T_h} \sum_{E \in \partial \Omega_e} \frac{2}{h_E} \|v\|_{0,E}^2, \\
\sum_{E \in \Gamma_{1D}} h_E \|\phi\|_{0,E}^2 & \leq C \sum_{\Omega_e \in T_h} h_e \|\phi\|_{0,\partial \Omega_e}^2, \\
\sum_{E \in \Gamma_{1D}} \frac{1}{h_E} \|\phi\|_{0,E}^2 & \leq \sum_{\Omega_e \in T_h} \sum_{E \in \partial \Omega_e} \frac{1}{h_E} \|\phi\|_{0,E}^2.
\end{align*}
\]

Starting with \( \|\eta\|_{DG}^2 \) we have, from (15) and using (29a) and (29c),

\[
\|\eta\|_{DG}^2 = \sum_{\Omega_e \in T_h} \|\varepsilon(\eta)\|_{0,\Omega_e}^2 + \frac{1}{2} \sum_{E \in \Gamma_{1D}} \frac{1}{h_E} \|\eta\|_{0,E}^2 \\
\leq C \left( \sum_{\Omega_e \in T_h} \|\nabla \eta\|_{0,\Omega_e}^2 + \sum_{\Omega_e \in T_h} \sum_{E \in \partial \Omega_e} \frac{1}{h_E} \|\eta\|_{0,E}^2 \right) \\
\leq C \left( \sum_{\Omega_e \in T_h} |\eta|_{2,\Omega_e}^2 + \sum_{\Omega_e \in T_h} \left( h_e^{-2} \|\eta\|_{0,\Omega_e}^2 + |\eta|_{1,\Omega_e}^2 \right) \right) \\
\leq C \sum_{\Omega_e \in T_h} h_e^{-2} |\eta|_{2,\Omega_e}^2. \tag{30}
\]

To bound \( \|\xi\|_{DG}^2 \), we have \( \|\xi\|_{DG}^2 \leq |a_h(\eta, \xi)| \) from coercivity of the bilinear form. To bound \( |a_h(\eta, \xi)| \), the technique is to extract a factor of \( \|\xi\|_{DG} \), leaving some function in terms of \( \eta \) which will be bounded by norms of the exact solution \( u \) from each term. The fact that \( \xi \in [P_1(\Omega)]^d \) so that \( e(\xi), \nabla \cdot \xi, \) and \( \nabla \xi \) are constants, is also useful.
We have

\[ |a_h(\eta, \xi)| \leq |a_h^{iso}(\eta, \xi)| + |a_h^{ti}(\eta, \xi)|, \]

where the isotropic part is bounded as follows (see [11]):

\[ |a_h^{iso}(\eta, \xi)| \leq C ||\xi||_{DG} \left( \sum_{\Omega_e \in T_h} h_e^2 \left( \mu_t^2 |u_{2,\Omega_e}^2 + \lambda^2 \right) \right)^{1/2}. \]  

(31)

For the remaining part, we have

\[ |a_h^{ti}(\eta, \xi)| \leq \]

\[ + \alpha \left| \sum_{\Omega_e \in T_h} \int_{\Omega_e} (M : \varepsilon(\eta)) \nabla \cdot \xi + (\nabla \cdot \eta)(M : \varepsilon(\xi)) \, dx \right| + \beta \left| \sum_{\Omega_e \in T_h} \int_{\Omega_e} (M : \varepsilon(\eta))(M : \varepsilon(\xi)) \, dx \right| \]

\[ + 2\gamma \left| \sum_{\Omega_e \in T_h} \int_{\Omega_e} \varepsilon(\eta)M : \varepsilon(\xi) \, dx \right| + \left| \sum_{E \in \Gamma_{iD}} \int_{E} \left| \sigma^{ti}(\xi) \right| \, ds \right| \]

\[ + \beta \left| \sum_{E \in \Gamma_{iD}} \int_{E} \left| \xi \right| \{ (M : \varepsilon(\eta)) \} \, ds \right| + \alpha \left| \sum_{E \in \Gamma_{iD}} \int_{E} \left| \xi \right| \{ (M : \varepsilon(\eta))I + (\nabla \cdot \eta)M \} \, ds \right| \]

\[ + \gamma \left| \sum_{E \in \Gamma_{iD}} \int_{E} \left| \xi \right| \{ \varepsilon(\eta)M + M\varepsilon(\eta) \} \, ds \right| + k_\beta \left| \sum_{E \in \Gamma_{iD}} \frac{1}{h_E} \int_{E} (M : [\eta])(M : [\xi]) \, ds \right| \]

\[ + k_\alpha \left| \sum_{E \in \Gamma_{iD}} \frac{1}{h_E} \int_{E} [\eta](M : [\xi]) + [\xi](M : [\eta]) \, ds \right| + k_\gamma \left| \sum_{E \in \Gamma_{iD}} \frac{1}{h_E} \int_{E} \left| \xi \right| \{ [\eta]M + M[\eta] \} \, ds \right|. \]

(32)

The struck-through terms are zero from the properties of the interpolant. We now bound each of the
remaining terms in (32):

\[ I = 2\gamma \left| \sum_{\Omega_e \in T_h} \varepsilon(\eta) M : \varepsilon(\xi) \; dx \right| \]

\[ \leq 2\gamma \left( \sum_{\Omega_e \in T_h} \|\varepsilon(\eta) M\|_{0,\Omega_e}^2 \right)^{1/2} \left( \sum_{\Omega_e \in T_h} \|\varepsilon(\xi)\|_{0,\Omega_e}^2 \right)^{1/2} \]

\[ \leq C\gamma \left( \sum_{\Omega_e \in T_h} \|\varepsilon(\eta)\|_{0,\Omega_e}^2 \right)^{1/2} \left( \sum_{\Omega_e \in T_h} \|\varepsilon(\xi)\|_{0,\Omega_e}^2 \right)^{1/2} \]

(since \( |\varepsilon M|^2 \leq C |\varepsilon|^2 \))

\[ \leq C\gamma \|\xi\|_{DG} \left( \sum_{\Omega_e \in T_h} \|\nabla \eta\|_{0,\Omega_e}^2 \right)^{1/2} \]

\[ \leq C\gamma \|\xi\|_{DG} \left( \sum_{\Omega_e \in T_h} \|\eta\|_{1,\Omega_e}^2 \right)^{1/2} \]

\[ \leq C\gamma \|\xi\|_{DG} \left( \sum_{\Omega_e \in T_h} h_{e}^2 |u|_{2,\Omega_e}^2 \right)^{1/2} \quad (\text{using (27a)}. \]

\[ II = \beta \left| \sum_{E \in \Gamma_{\text{id}}} \int_E |\xi| : \{M(M : \varepsilon(\eta))\} \; ds \right| \]

\[ \leq \beta \left( \sum_{E \in \Gamma_{\text{id}}} \frac{1}{h_E^2} \|\xi\|_{0,E}^2 \right)^{1/2} \left( \sum_{E \in \Gamma_{\text{id}}} h_E \|\{M(M : \varepsilon(\eta))\}\|_{0,E}^2 \right)^{1/2} \]

(since \(|M : \varepsilon)|^2 \leq |M : \varepsilon|^2 \))

\[ \leq \beta \|\xi\|_{DG} \left( \sum_{E \in \Gamma_{\text{id}}} h_E \|\{M(M : \varepsilon(\eta))\}\|_{0,E}^2 \right)^{1/2} \]

\[ \leq C\beta \|\xi\|_{DG} \left( \sum_{\Omega_e \in T_h} h_{e} \|M : \varepsilon(\eta)\|_{0,\partial\Omega_e}^2 \right)^{1/2} \quad (\text{using (29a)).} \]

\[ \leq C\beta \|\xi\|_{DG} \left( \sum_{\Omega_e \in T_h} \|M : \varepsilon(\eta)\|_{0,\Omega_e}^2 + h_{e}^2 |M : \varepsilon(\eta)|_{1,\Omega_e} \right)^{1/2} \quad (\text{using (29a)).} \]

\[ \leq C\beta \|\xi\|_{DG} \left( \sum_{\Omega_e \in T_h} h_{e}^2 |M : \varepsilon(u)|_{1,\Omega_e}^2 \right)^{1/2} \quad (\text{using (27e) and (27f)).} \]
\[ II_{Ia} := \alpha \left| \sum_{E \in \Gamma_{ID}} \int_{E} \xi : \{(M : \epsilon(\eta))I\} \, ds \right| \leq C\alpha \|\xi\|_{DG} \left( \sum_{\Omega_c \in T_h} h_c^2 \|M : \epsilon(u)\|_{1,\Omega_c}^2 \right)^{1/2} \]  
(similar to II).

\[ II_{Ib} := \alpha \left| \sum_{E \in \Gamma_{ID}} \int_{E} \xi : \{(\nabla \cdot \eta)M\} \, ds \right| \]

\[ \leq \alpha \left( \sum_{E \in \Gamma_{ID}} \frac{1}{h_E} \|\xi\|^2_{L^2,\partial\Omega_E} \right)^{1/2} \left( \sum_{E \in \Gamma_{ID}} h_E \|\{(\nabla \cdot \eta)M\}\|^2_{L^2,\Omega_E} \right)^{1/2} \]

\[ \leq C\alpha \|\xi\|_{DG} \left( \sum_{\Omega_c \in T_h} h_c \|\nabla \cdot \eta\|^2_{L^2,\partial\Omega_c} + h_c^2 \|\nabla \cdot \eta\|^2_{L^2,\Omega_c} \right)^{1/2} \]  
(using (29c) and \((\nabla \cdot \eta)M)^2 \leq |\nabla \cdot \eta|^2\).

\[ \leq C\alpha \|\xi\|_{DG} \left( \sum_{\Omega_c \in T_h} h_c^2 \|\nabla \cdot u\|^2_{L^2,\Omega_c} \right)^{1/2} \]  
(using (27a)).

\[ IV = \gamma \left| \sum_{E \in \Gamma_{ID}} \int_{E} \xi : \{\epsilon(\eta)M\} \, ds \right| \]

\[ \leq \gamma \left( \sum_{E \in \Gamma_{ID}} \frac{1}{h_E} \|\xi\|^2_{L^2,\partial\Omega_E} \right)^{1/2} \left( \sum_{E \in \Gamma_{ID}} h_E \|\{\epsilon(\eta)M\}\|^2_{L^2,\Omega_E} \right)^{1/2} \]

\[ \leq C\gamma \|\xi\|_{DG} \left( \sum_{\Omega_c \in T_h} h_c^2 \|u\|^2_{L^2,\Omega_c} \right)^{1/2}. \]
following steps similar to those in \( I \).

\[
V = k_\beta \left[ \sum_{E \in \Gamma_{id}} \frac{1}{h_E} \int_E (M : [\eta])(M : [\xi]) \, ds \right]
\]

\[
\leq k_\beta \left( \sum_{E \in \Gamma_{id}} \frac{1}{h_E} \|M : [\xi]\|^{2}_{0,E} \right)^{1/2} \left( \sum_{E \in \Gamma_{id}} \frac{1}{h_E} \|M : [\eta]\|^{2}_{0,E} \right)^{1/2}
\]

\[
\leq C_\beta \|\xi\|_{DG} \left( \sum_{E \in \Gamma_{id}} \frac{1}{h_E} \|\eta\|^{2}_{0,E} \right)^{1/2} \quad \text{(since } |M : [\eta]|^{2} < C |[\eta]|^{2})
\]

\[
\leq C_\beta \|\xi\|_{DG} \left( \sum_{E \in \Gamma_{id}} \frac{2}{h_E} \|\eta\|_{0,E} \right)^{1/2} \quad \text{(using (29b))}
\]

\[
\leq C_\beta \|\xi\|_{DG} \left( \sum_{E \in \Gamma_{id}} \frac{1}{h_E} \|\eta\|_{0,E} + |\eta|_{1,\Omega_e} \right)^{1/2} \quad \text{(using (29a))}
\]

\[
\leq C_\beta \|\xi\|_{DG} \left( \sum_{E \in \Gamma_{id}} \frac{1}{h_E} \|\eta\|_{0,E} \right)^{1/2} \quad \text{(using (27a)).}
\]

\[
VI_a := k_\alpha \left[ \sum_{E \in \Gamma_{id}} \frac{1}{h_E} \int_E [\eta](M : [\xi]) \, ds \right] \leq C_\alpha \|\xi\|_{DG} \left( \sum_{E \in \Gamma_{id}} \frac{1}{h_E} \|\eta\|^{2}_{0,E} \right)^{1/2} \quad \text{(similar to } V).\]

\[
VI_b := k_\alpha \left[ \sum_{E \in \Gamma_{id}} \frac{1}{h_E} \int_E [\xi](M : [\eta]) \, ds \right] \leq C_\alpha \|\xi\|_{DG} \left( \sum_{E \in \Gamma_{id}} \frac{1}{h_E} \|\eta\|^{2}_{0,E} \right)^{1/2} \quad \text{(similar to } V).\]

\[
VII = k_\gamma \left[ \sum_{E \in \Gamma_{id}} \frac{1}{h_E} \int_E [\xi] : [\eta]M \, ds \right] \leq C_\alpha \|\xi\|_{DG} \left( \sum_{E \in \Gamma_{id}} \frac{1}{h_E} \|\eta\|^{2}_{0,E} \right)^{1/2} \quad \text{(similar to } V).\]

We use these results to bound each term of \( |a_h^{\ell_i}(\eta, \xi)| \), which leads us to

\[
|a_h^{\ell_i}(\eta, \xi)| \leq C \|\xi\|_{DG} \left( \sum_{E \in \Gamma_{id}} \frac{1}{h_E} \left( (\alpha^2 + \beta^2 + \gamma^2) \|\eta\|^{2}_{2,\Omega_e} + \alpha^2 \|\nabla \cdot u\|^{2}_{1,\Omega_e} + (\alpha^2 + \beta^2) \|M : \varepsilon(u)\|^{2}_{1,\Omega_e} \right) \right)^{1/2}
\]
Thus,
\[
\|\xi\|_{DG}^2 \leq C \sum_{\Omega_e \in \mathcal{T}_h} h_e^2 \left( (\mu_t^2 + \alpha^2 + \beta^2 + \gamma^2) |u|_{2,\Omega_e}^2 + (\lambda^2 + \alpha^2) |\nabla \cdot u|_{1,\Omega_e}^2 + (\alpha^2 + \beta^2) |M : \varepsilon(u)|_{1,\Omega_e}^2 \right). \tag{33}
\]

With (30) and (33), the full DG error bound is
\[
\|e\|_{DG}^2 \leq Ch^2 \left( (\mu_t^2 + \alpha^2 + \beta^2 + \gamma^2) |u|_{2,\Omega}^2 + (\lambda^2 + \alpha^2) |\nabla \cdot u|_{1,\Omega}^2 + (\alpha^2 + \beta^2) |M : \varepsilon(u)|_{1,\Omega}^2 \right). \tag{34}
\]

**Remark.** For the case of isotropy, the error estimate is (see [11])
\[
\|e\|_{DG}^2 \leq Ch^2 \left( \mu_t^2 |u|_{2,\Omega}^2 + \lambda^2 |\nabla \cdot u|_{1,\Omega}^2 \right). \tag{35}
\]

Brenner & Sung in [7] have derived the following uniform estimate for the case of problems on polygonal domains $\Omega \subset \mathbb{R}^2$
\[
\|u\|_{2,\Omega} + \lambda \|\nabla \cdot u\|_{1,\Omega} \leq C (\|f\|_{0,\Omega} + \|g\|_{0,\Gamma_D}). \tag{36}
\]
This allows the right-hand side of (35) to be bounded independent of $\lambda$, thus confirming the locking-free behaviour of the DG formulation in the incompressible limit.

A similar estimate for the transversely isotropic problem is not available; however, one would expect that an analogous estimate would allow the terms of the form
\[
(\mu_t^2 + \alpha^2 + \beta^2) |u|_{2,\Omega}^2 + (\lambda^2 + \alpha^2) |\nabla \cdot u|_{1,\Omega}^2 + (\alpha^2 + \beta^2) |M : \varepsilon(u)|_{1,\Omega}^2 \tag{37}
\]

to be bounded independent of $\lambda$ and $\beta$. The presence of $\beta$ in the first term of (34) suggests that locking may occur in the inextensible limit. Numerical experiments discussed in Section 6 will explore these features.

The term that leads to the undesirable $\beta$-dependence in the error bound is term $V$ in (32). To circumvent the $\beta$-dependence, one would need to find a way to modify the formulation in such a way that this term is eliminated.
5 Under-integration

It has been shown in [1] that projecting an integrand in a formulation onto the space of constants is equivalent to the use of under-integration in the numerical implementation. For the case of isotropy, using bilinear elements, the undesirable $\lambda$-dependency of the error bound in the incompressible limit may be circumvented by under-integrating the problematic terms [11]. The same approach is used here in order to overcome locking in the extensible limit: the $\beta$-stabilization term $V$ will be under-integrated. An additional term, the $\alpha$-stabilization term $VI$, will also be under-integrated, as this will be necessary to maintain coercivity.

If we define by $\Pi_0$ the $L^2$-orthogonal projection onto the space of constants, the new DG formulation with under-integration is:

$$a_h^{UI}(u, v) = l_h^{UI}(v)$$

where

$$a_h^{UI}(u, v) = a_h(u, v) + k_\alpha \sum_{E \in \Gamma_{ID}} \frac{1}{h_E} \int_E \left( (\Pi_0 - I)[v](M : [u]) + (\Pi_0 - I)(M : [v])[u] \right) ds + k_\beta \sum_{E \in \Gamma_{ID}} \frac{1}{h_E} \int_E (\Pi_0 - I)(M : [v])(M : [u]) ds,$$

and

$$l_h^{UI}(v) = l_h(v) + k_\alpha \sum_{E \in \Gamma_D} \frac{1}{h_E} \int_E \left( (\Pi_0 - I)(v \cdot n)(M : g \otimes n) + (\Pi_0 - I)(v \otimes n : M)(g \cdot n) \right) ds + k_\beta \sum_{E \in \Gamma_D} \frac{1}{h_E} \int_E (\Pi_0 - I)(M : v \otimes n)(M : g \otimes n) ds.$$

Note that under-integration of the edge term $\int_E [\cdot] : \{\sigma(\cdot)\} ds$ is not necessary since for any $u \in [P_1(\Omega_e)]^d$, the integrand is linear, so that one-point integration is exact. The integrands in the terms $V$ and $VI$ in (32) are then replaced with their projected quantities, and are easily shown to be zero. However, this has involved modification of the DG formulation itself, so that it is necessary to show coercivity and consistency of the modified bilinear form.
5.1 Coercivity

Each IP method will be investigated separately, using the same approach as that in Section 4.2.

NIPG \( (\theta = 1) \)  We have

\[
a_{h}^{ij}(v, v) = \sum_{\Omega_e \in T_h} \int_{\Omega_e} \sigma(v) : \varepsilon(v) \, dx + k_{\mu} \mu_{\ell} \sum_{E \in \Gamma_{iD}} \frac{1}{h_{E}} \|v\|_{0,E}^{2} + k_{\lambda} \lambda \sum_{E \in \Gamma_{iD}} \frac{1}{h_{E}} \|v\|_{0,E}^{2} \\
+ k_{\alpha} \alpha \sum_{E \in \Gamma_{iD}} \frac{2}{h_{E}} \int_{E} \Pi_{0}(v) : M \Pi_{0}(v) \, ds + k_{\beta} \beta \sum_{E \in \Gamma_{iD}} \frac{1}{h_{E}} \|\Pi_{0}(M : \varepsilon(v))\|_{0,E}^{2} \\
+ k_{\gamma} \gamma \sum_{E \in \Gamma_{iD}} \frac{1}{h_{E}} \int_{E} |v| : (\Pi_{0}(M + M|v|)) \, ds.
\]

We proceed exactly as in Section 4.2 for NIPG, but now choosing \( \varepsilon = \frac{1}{2} \Pi_{0}(m \otimes n + n \otimes m) \); for some small \( r > 0 \), thus

\[
A(r) := \left( \lambda + \frac{2r}{3} \mu_{\ell} \right) \Pi_{0}(n \cdot m)^{2} + (\beta + 2\gamma)(a \cdot n)^{2}\Pi_{0}(a \cdot m)^{2} + 2\alpha(a \cdot n)\Pi_{0}(n \cdot m)\Pi_{0}(a \cdot m) > 0.
\]

Therefore,

\[
a_{h}^{ij}(v, v) \geq C_{1} \mu \sum_{\Omega_e \in T_h} \|\varepsilon(v)\|_{0,\Omega_e}^{2} + k_{\min} \sum_{E \in \Gamma_{iD}} \frac{1}{h_{E}} \int_{E} \left( A(r) + 2\mu_{\ell} \left( 1 - \frac{r}{3} \right) \Pi_{0}(n \cdot m)^{2} \\
+ 2\mu_{\ell}(m \cdot m) - \Pi_{0}(n \cdot m)^{2} \right) \, ds \\
\geq C_{1} \mu \sum_{\Omega_e \in T_h} \|\varepsilon(v)\|_{0,\Omega_e}^{2} + k_{\min} \sum_{E \in \Gamma_{iD}} \frac{1}{h_{E}} \int_{E} \left( 2\mu_{\ell} \left( 1 - \frac{r}{3} \right) \Pi_{0}(n \cdot m)^{2} + 2\mu_{\ell}(m \cdot m) - \Pi_{0}(n \cdot m)^{2} \right) \, ds \\
= C_{1} \mu \sum_{\Omega_e \in T_h} \|\varepsilon(v)\|_{0,\Omega_e}^{2} + 2k_{\min} \left( 1 - \frac{r}{3} \right) \mu_{\ell} \sum_{E \in \Gamma_{iD}} \frac{1}{h_{E}} \|v\|_{0,E}^{2} \\
\geq C_{3} \mu \|v\|_{DG},
\]

with

\[
C_{3} = \min \left\{ C_{1}, 2k_{\min} \left( 1 - \frac{r}{3} \right) \right\} \quad \forall 0 \leq r < 1.
\]
SIPG ($\theta = -1$) We have

\[
\begin{align*}
    a_h^{\text{UI}}(v, v) &= \sum_{\Omega_e \in T_h} \int_{\Omega_e} \sigma(v) : \varepsilon(v) \, dx - 2 \sum_{E \in \Gamma_{1D}} \int_E \{\sigma(v)\} : \{\varepsilon(v)\} \, ds + k_\mu \lambda \sum_{E \in \Gamma_{1D}} \frac{1}{h_E} ||v||_{0,E}^2 \\
    &\quad + k_\lambda \alpha \sum_{E \in \Gamma_{1D}} \frac{1}{h_E} ||v||_{0,E}^2 + k_\alpha \alpha \sum_{E \in \Gamma_{1D}} \frac{2}{h_E} \int_E \Pi_0(M : [v]) \Pi_0[v] \, ds \\
    &\quad + k_\beta \beta \sum_{E \in \Gamma_{1D}} \frac{1}{h_E} ||\Pi_0(M : [v])||_{0,E}^2 + k_\gamma \gamma \sum_{E \in \Gamma_{1D}} \frac{1}{h_E} \int_E \sigma(v) : \varepsilon(v) \, dx - 2 \sum_{E \in \Gamma_{1D}} \int_E \{v\} : (\varepsilon(v)) \, dx \\
    &\quad + k_\mu \mu \sum_{E \in \Gamma_{1D}} \frac{1}{h_E} \Pi_0(M : [v]) \Pi_0(v) \, ds
\end{align*}
\]

We proceed exactly as in Section 4.2 for SIPG. In (20e) and (20f), the $\beta$-stabilization terms are replaced with

\[
\begin{align*}
    k_\beta \beta \frac{1}{2h_E} ||\Pi_0([v] : M)||_{0,E}^2
    \quad \text{and} \quad
    k_\beta \beta \frac{1}{h_E} ||\Pi_0(M : v \otimes n)||_{0,E}^2
\end{align*}
\]

respectively, and the $\alpha$-stabilization terms in (20e) and (20f) are replaced with

\[
\begin{align*}
    k_\alpha \alpha \frac{1}{h_E} \int_E \Pi_0([v] : M) \Pi_0[v] \, ds
    \quad \text{and} \quad
    2k_\alpha \alpha \frac{1}{h_E} \int_E \Pi_0((v \cdot n) \Pi_0(M : v \otimes n) \, ds
\end{align*}
\]

respectively, which lead us to the following bound for $T^{UI}$ (the under-integrated expression of $T$ from equation (23)):

\[
\begin{align*}
    T^{UI} &\geq \sum_{\Omega_e \in T_h} \left[ \left( \lambda - \alpha - \frac{\lambda \varepsilon_\lambda}{2} - \frac{\alpha \varepsilon_\alpha}{2} \right) ||\nabla \cdot v||_{0,\Omega_e}^2 + \left( \beta - \alpha - \frac{\beta \varepsilon_\beta}{2} - \frac{\alpha \varepsilon_\alpha}{2} \right) ||M : \varepsilon(v)||_{0,\Omega_e}^2 \\
    &\quad + \left( \frac{\lambda k_\lambda}{2} - \frac{\alpha k_\alpha}{2} - \frac{\lambda}{2} \varepsilon_\lambda - \frac{\alpha}{2} \varepsilon_\alpha \right) ||h^{-1/2}_E [v]||_{0,\partial \Omega_e}^2 \\
    &\quad + \left( \frac{\beta k_\beta}{2} - \frac{\alpha k_\alpha}{2} - \frac{\beta}{2} \varepsilon_\beta - \frac{\alpha}{2} \varepsilon_\alpha \right) ||h^{-1/2}_E \Pi_0(M : [v])||_{0,\partial \Omega_e}^2 \right],
\end{align*}
\]

which is non-negative using the same parameters choices as in (24).

IIPG ($\theta = 0$) The proof of coercivity for IIPG with under-integration case is identical to that for the SIPG with under-integration case up to a constant.

**Theorem 2.** Under the conditions stated, the nonsymmetric, symmetric and incomplete interior penalty formulations (38) are coercive.
5.2 Consistency

With the continuous exact solution \( u \in H^2(\Omega) \) satisfying properties given in Section 4.1, we have

\[
a^{h,i}(u, v) - l^{h,i}(v) = \sum_{\Omega_e \in T_h} \int_{\Omega_e} \sigma(u) : \varepsilon(v) \, dx - \sum_{E \in \Gamma} \int_{E} \{\sigma(u)\} : \{v\} \, ds - \sum_{\Omega_e \in T_h} \int_{\Omega_e} f \cdot v \, dx
- \sum_{E \in \Gamma_N} \int_{E} h \cdot v \, ds.
\]

Since \( u \) satisfies the weak form,

\[
\sum_{\Omega_e \in T_h} \int_{\Omega_e} \sigma(u) : \varepsilon(v) \, dx = \sum_{\Omega_e \in T_h} \int_{\Omega_e} f \cdot v \, dx + \sum_{\Omega_e \in T_h} \int_{\partial \Omega_e} v \otimes n : \sigma(u)
= \sum_{\Omega_e \in T_h} \int_{\Omega_e} f \cdot v \, dx + \sum_{E \in \Gamma} \int_{E} \{\sigma(u)\} : \{v\} + \sum_{E \in \Gamma_N} \int_{E} h \cdot v \, ds.
\]

Therefore,

\[
a^{h,i}(u, v) - l^{h,i}(v) = 0
\]

as desired.

Error bound The approximation error is then bounded by

\[
||e||^2_{DG} \leq Ch^2\left( (\mu^2 + \gamma^2)|u|^2_{L^2,\Omega} + (\lambda^2 + \alpha^2)|\nabla \cdot u|^2_{L^2,\Omega} + (\alpha^2 + \beta^2)|M : \varepsilon(u)|^2_{L^2,\Omega} \right). \quad (40)
\]

The first term on the right-hand side is now independent of \( \beta \). The bound (40) is then in a form that would be expected to lead to a uniform estimate, by analogy with the bound (36). The behaviour of the under-integrated DG formulation will be explored further in the next section.

6 Numerical tests

In this section, we present the results of numerical simulations of two model problems to illustrate the formulations discussed in the preceding sections. All examples are under conditions of plane strain and based on three- and six-noded triangular elements with standard linear and quadratic interpolations.
of the displacement field. Unless otherwise stated, all examples are presented for the case of near-incompressibility by choosing \( \nu_t = 0.49995 \). We fix the value of the two Poisson’s ratios to be equal, and also set \( \mu_l = \mu_t \). We consider values of \( p > 1 \), so that the conditions (8) for pointwise stability are satisfied.

Within the constraints of the coercivity requirements, the following values for the stabilization parameters are used for all the methods: \( k_\mu = k_\alpha = k_\gamma = 10 \), and \( k_\lambda = k_\beta = 100 \).

Define \( \hat{a} := (\hat{O}x, \hat{a}) \), the angle between the \( x \)-axis and the fibre direction \( a \). For each problem, we consider values for \( \hat{a} \) in the range \( 0 \leq \hat{a} \leq \pi \).

The results in the examples that follow are for the following element choices:

- \( P_1 \text{-CG} \): The standard displacement formulation of order 1
- \( P_2 \text{-CG} \): The standard displacement formulation of order 2
- \( P_1 \text{-NIPG} \): The nonsymmetric interior penalty method of order 1
- \( P_1 \text{-SIPG} \): The symmetric interior penalty method of order 1
- \( P_1 \text{-IIPG} \): The incomplete interior penalty method of order 1
- \( P_1 \text{-NIPG}_{\alpha \beta} \): The nonsymmetric interior penalty method of order 1 with under-integration of the \( \alpha \)- and \( \beta \)-stabilization terms
- \( P_1 \text{-SIPG}_{\alpha \beta} \): The symmetric interior penalty method of order 1 with under-integration of the \( \alpha \)- and \( \beta \)-stabilization terms
- \( P_1 \text{-IIPG}_{\alpha \beta} \): The incomplete interior penalty method of order 1 with under-integration of the \( \alpha \)- and \( \beta \)-stabilization terms

6.1 Cook’s membrane

![Image](image1.png)

Figure 1: Cook’s membrane geometry and boundary conditions

The Cook’s membrane test consists of a tapered panel fixed along one edge and subject to a shearing load at the opposite edge as depicted in Figure 1. The applied load is \( f = 100 \) and \( E_t = 250 \). This test
problem has no analytical solution. A mesh of $32 \times 32$ elements is used. The vertical tip displacement at corner $C$ is measured.

Figure 2 shows semilog plots of the tip displacement vs $p$ for angles $\pi/3$ and $3\pi/4$ for the various element choices, and for moderate and high values of $p$. To investigate locking of the proposed formulation, we compare the results with the results obtained using the standard $P_2$-element.

In Figure 2(a), for moderate values of $p$ ($1 \leq p \leq 5$) the $P_{1\,CG}$ formulation behaves well away from $p = 1$, with evidence of locking behaviour as $p \to 1$. All three IPDG methods show no locking. Under-integrated IPDG methods are not necessary here since they give same results. In Figure 2(b) for higher
values of $p$ ($10 \leq p \leq 10^5$), the $P_1\_CG$ and all three IPDG methods show locking behaviour. Locking is avoided for the under-integrated formulations for larger values of $p$.

Figure 3 shows tip displacements for various fibre orientations, where the degree of anisotropy is fixed at $p = 10^5$. Some deterioration in accuracy is observed for the conforming $P_2$-element, for angles in the range $\hat{a} > \pi/2$. Whether this indicates mild locking would depend on further parameter-explicit analysis, which is currently absent for the transversely isotropic problem at near-inextensibility.

### 6.2 Bending of a beam

We consider the beam shown in Figure 4 subject to a linearly varying load along the edge CD. The horizontal displacement $u$ is constrained at node B, while the node A is constrained in both directions. The beam has length $L = 10$ and height $H = 2$ and the linearly varying load has a maximum value
Here, \( f = 3000 \). The boundary conditions are

\[
\begin{align*}
    u(0, y) &= g(y), \\
    v(0, -\frac{H}{2}) &= 0,
\end{align*}
\]

where

\[
g(y) = -\frac{f}{H} S_{31} \left(y^2 - \frac{H^2}{4}\right). \]

The compliance coefficients \( S_{ij} \) are given in [19], as is the analytical solution. We measure the vertical tip displacement at corner C with meshing \( 80 \times 16 \) elements.

\[\hat{a} = \frac{\pi}{6}\] for moderate values of \( p \).

\[\hat{a} = \frac{7\pi}{8}\] for high values of \( p \).

Figure 5: Tip displacement vs \( p \) for the beam problem.
In Figure 5, which shows semilog plots of tip displacement for different values of $p$, with the angle of the fibre direction $\pi/6$ and $7\pi/8$, locking behaviour is investigated by comparison with the analytical solution. The same behaviour as appears for the Cook’s example is seen, i.e. for moderate values of $p$ away from $p = 1$ (approximately, $1.1 \leq p \leq 5$), there is locking-free behaviour with $P_1_{-CG}$, while locking occurs as $p$ approaches 1. This is overcome by using IPDG methods (Figure 5(a)). For high values of $p$ there is purely extensional locking with $P_1_{-CG}$ and all IPDG methods, which is overcome by using under-integration of the $\alpha$- and $\beta$-stabilization terms (Figure 5(b)). Figure 6 shows tip displacements for various fibre orientations where the degree of anisotropy is fixed at $p = 10^7$. We recall the same behaviour as stated in [19], that is, extensional locking for $P_1_{-CG}$ except for the angles 0, where the material is very stiff, and $\pi/2$, where the extensional term is bounded.

The following set of results shows behaviour for various fibre orientations, and for values of $p = 1.0001$, 3 and $10^4$.

Figure 7 shows the $H^1$ relative error convergence plots for all three IPDG formulations and $P_1_{-CG}$, for $p = 1.0001$. Here all three IPDG formulations show slightly better than optimal (linear) convergence for any fibre direction. $P_1_{-CG}$ shows poor convergence, indicative of volumetric locking.

Figure 8 shows the $H^1$ relative error convergence plots for all three IPDG formulations and $P_1_{-CG}$, for $p = 3$. Here, all formulations at any fibre direction are linearly convergent.

Figure 9 shows the $H^1$ relative error convergence plots for all three IPDG formulations and $P_1_{-CG}$, for $p = 10^4$. $P_1_{-CG}$ and the full IPDG methods show poor convergence, indicative of extensional locking.
Figure 7: Comparison of $\mathcal{H}^1$ relative errors for conforming and full IPDG formulations, for different fibre orientations, and for $p = 1.0001$.
Figure 8: Comparison of $\mathcal{H}^1$ relative errors for conforming and full IPDG formulations, for different fibre orientations, and for $p = 3$.
All three IPDG methods with under-integration show optimal convergence at rate 1.6.

**General remark.** It was found that under-integration of only the $\beta$-edge term gave results almost indistinguishable from those obtained by under-integrating both the $\alpha$- and $\beta$-edge terms, despite the analysis requiring that both be under-integrated.
7 Conclusions

This paper has presented new IPDG formulations for transversely isotropic linear elasticity, and their analyses. Numerical investigations, as in the case of isotropy, have shown that these methods are uniformly convergent at the near-incompressible limit. However, locking has been shown to manifest at the near-inextensible limit, and a theoretical error bound has been presented to support why this might reasonably be expected. The use of under-integration in the extensional stabilization edge terms of the IPDG formulations has been proposed to circumvent this locking behaviour. The analysis presented, assuming an a priori estimate analogous to that of Brenner and Sung [7] for the isotropic case, proves that the modified formulations are uniformly convergent with respect to the extensibility parameter. Computational results for a range of measures of anisotropy and of fibre directions support the conclusions of the theoretical analysis.

A corresponding study using conforming finite element approximations was presented in previous work [19]. This work is presented as an extension to that by using discontinuous Galerkin approaches. Both works are intended to enhance current understanding of the cases of near-incompressibility and near-inextensibility. Further investigations will involve analysis and implementation of a mixed formulation.

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