A SIMPLE PROOF OF THE GAUSS-BONNET-CHERN FORMULA FOR FINSLER MANIFOLDS

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ABSTRACT. From the point of view of index theory, we give a simple proof of a Gauss-Bonnet-Chern formula for all Finsler manifolds by the Cartan connection. Based on this, we establish a Gauss-Bonnet-Chern formula for any metric-compatible connection.

1. Introduction

In [7, 8], S. S. Chern gave a intrinsic proof of the Gauss-Bonnet-Chern (GBC) formula for all oriented closed \(n\)-dimensional Riemannian manifolds \((M, g)\)

\[-\int_M \Omega = \chi(M),\]

(1.1)

where

\[
\Omega = \begin{cases} 
\frac{(-1)^{p-1}}{2^{p}p!} \varepsilon_{i_1\ldots i_{2p}} \Omega^i_{i_1} \wedge \cdots \wedge \Omega^i_{i_{2p-1}}, & n = 2p, \\
0, & n = 2p + 1.
\end{cases}
\]

and \((\Omega^i_j)\) is the local curvature form of the Levi-Civita connection on \(M\). Let \(\pi : SM \to M\) be the sphere bundle of \(M\) and \(S_x M\) be the fibre on \(\pi^{-1}(x)\). Chern’s idea is to show that \(\pi^* \Omega\) is the derivative of a ”total curvature” \(\Pi\) on \(SM\) satisfying \(\Pi|_{S_x M}\) is the normalized volume form of the fibre. To realize this, Chern constructed two polynomials \(\Phi_k, \Psi_k\) miraculously and found the important relationship

\[
d\Phi_k = \Psi_k - 1 + \frac{n - 2k - 1}{2(k + 1)} \Psi_k.
\]

(1.2)

By (1.2), he succeed in constructing the total curvature

\[
\Pi = \begin{cases} 
\frac{1}{(2\pi)^p} \sum_{k=0}^{p-1} \frac{(-1)^k}{(2p-2k-1)!k!2^k} \Phi_k, & n = 2p, \\
\frac{1}{\pi^{2p+1}} \sum_{k=0}^{p} (-1)^{k+1} \binom{p}{k} \Phi_k, & n = 2p + 1.
\end{cases}
\]

such that

\[
\pi^* \Omega = d\Pi.
\]

(1.3)

Thus, pulling back (1.3) by a unit vector with isolated zeros and using the Poincaré-Hopf theorem, one can obtain the formula (1.1).

2010 Mathematics Subject Classification. Primary 53B40, Secondary 47A53.

Key words and phrases. Gauss-Bonnet-Chern formula, Finsler manifold, index theory, Thom form.
Let \((M, F)\) be a \(n\)-dimensional Finsler manifold. Denote by \(\pi : \pi^*TM \rightarrow M\) the projective sphere bundle and \(\pi^*TM\) the pull-back bundle. \(F\) induces a natural Riemannian metric on \(\pi^*TM\). There are many important linear connection on \(\pi^*TM\), but none of them is both "torsion-free" and "metric-compatible". For example, the Cartan connection is metric-compatible while the Chern connection is torsion-free. Refer to [1, 3] for other interesting connections.

It is natural to ask whether an analogue of (1.1) still holds for Finsler manifolds. And it begins with a work of Lichnerowicz’s[10]. In that paper, Lichnerowicz obtained a GBC formula for Cartan-Berwald spaces by the Cartan connection. One interesting feature of a \(n\)-dimensional Cartan-Berwald manifold is \(V(x) = \text{vol}(S_{n-1})\), where \(V(x)\) is the Riemannian volume of \(S_xM\) induced by \(F\) (see [2, 4] or Section 2 below). Fifty years later, Bao and Chern[2] reconsidered this problem and established a GBC formula for Finsler manifolds with \(V(x) = \text{constant}\) by using the Chern connection. In fact, Lackey[9] obtained several formulas of GBC type by the Cartan connection for a certain class of Finsler manifolds. Recently, Shen[13] established a GBC formula for any torsion-free connection. The same technique also appeared in an unpublished work of Shen[12] and a GBC formula for any metric-compatible connection was established.

All the methods used in References[10, 2, 13, 9, 12, 9] are inspired by Chern’s original idea presented before. It should be noticeable that \(\Phi_k\) and \(\Psi_k\) constructed by the Chern connection don’t satisfy (1.2) while \(\Pi\) constructed by the Cartan connection is not a "total curvature". Hence, whichever connection is chosen, one needs a lot of techniques to calculate and correct (1.2) and (1.3), not to mention an arbitrary torsion-free or metric-compatible connection.

Note that the GBC formula (1.1) is the simplest case of the Atiyah-Singer index theorem[5, 11]. The purpose of this paper is to give a simple proof of a GBC formula for any metric-compatible connection and for all Finsler manifold from the point of view of index theory. Given any metric-compatible connection \(D\) on \(\pi^*TM\), we define

\[
\Omega^D = \begin{cases} 
\frac{(-1)^{p-1}}{2^{n-1}p!} \epsilon_{i_1\ldots i_{2p}} \Omega^{i_1}_{i_1} \wedge \cdots \wedge \Omega_{i_{2p-1}}^{i_{2p}}, & n = 2p, \\
0, & n = 2p + 1.
\end{cases}
\]

where \((\Omega^D_i)\) is the curvature of \(D\). First, for the Cartan connection \(\nabla\), we have

**Theorem 1.1.** Let \((M, F)\) be a closed Finsler \(n\)-manifold and \(X\) be a vector field with isolated zeros \(\{x_i\}\). Then we have

\[
- \int_M [X]^* \left( \frac{\Omega^\nabla + \mathcal{D}}{V(x)} \right) = \frac{\chi(M)}{\text{vol}(S_{n-1})},
\]

where \([X] : M \setminus \{x_i\} \rightarrow SM\) is the section induced by \(X\), \(V(x)\) is the Riemannian volume of \(S_xM\), and \(\mathcal{D}\) is some \(n\)-form on \(SM\).

It is noticeable that \((1 - s)D + s\nabla\) is a metric-compatible connection, for any metric-compatible connection \(D\). Thus, by Theorem 1.1 and a transgression formula (see Section 4), we obtain the following
Theorem 1.2. Let \((M, F)\) be a closed Finsler \(n\)-manifold and \(X\) be a vector field with isolated zeros. Given any metric-compatible connection \(D\), we have

\[-\int_M [X]^* \left( \frac{\Omega^D + \mathcal{E}}{V(x)} \right) = \frac{\chi(M)}{\text{vol}(S^{n-1})},\]

where \((\mathcal{E} - D)\) is some exact \(n\)-form on \(SM\).

See Section 4 below for the precise formulas of \(D\) and \(\mathcal{E}\). It is remarkable that Theorem 1.1 and Theorem 1.2 are independent of the choice of the vector field \(X\).

In the Riemannian case, \(\Omega^\nabla = \pi^* \Omega\), \(V(x) = \text{vol}(S^n)\), \(\int_M [X]^* \mathcal{D} = 0\) and \((\mathcal{E} - D)\) is an exact \(n\)-form pulled back from \(M\) (see Remark 3-4 below). Since \([X]^* \pi^* = \text{id}\), both Theorem 1.1 and Theorem 1.2 imply the GBC formula for Riemannian manifolds\(^3\)\(^7\)\(^8\).

We remark that Shen\(^12\) also established a GBC formula for any metric-compatible connection and for all Finsler manifolds. But the formula and the method in \(^{12}\) are different from ours here. Refer to \(^{12}\) for more details.

Acknowledgements. The authors wish to thank Professor Y-B. Shen for his advice and encouragement. This work was supported partially by National Natural Science Foundation of China (Grant No. 11171297).

2. Preliminaries

In this paper, the rules that govern our index gymnastics are as follows: Latin indices run from 1 to \(n\); Greek indices run from 1 to \(n - 1\).

A Finsler \(n\)-manifold \((M, F)\) is an \(n\)-dimensional differential manifold \(M\) equipped with a Finsler metric \(F\) which is a nonnegative function on \(TM\) satisfying the following two conditions:

1. \(F\) is positively homogeneous, i.e., \(F(\lambda y) = \lambda F(y)\), for any \(\lambda > 0\) and \(y \in TM\);
2. \(F\) is smooth on \(TM\setminus\{0\}\) and the Hessian \(\frac{1}{2} [F^2]_{ij'y'}(x, y)\) is positive definite, where \(F(x, y) := F(y^i \frac{\partial}{\partial x^i}|_x)\).

Let \(\pi : SM \to M\) and \(\pi^* TM\) be the projective sphere bundle and the pullback bundle, respectively. For each \((x, [y]) \in SM\), the distinguished section \(\ell\) of \(\pi^* TM\) is defined by

\[\ell_{(x, [y])} = \frac{y^i}{F(y)} \partial_i,\]

where \(\partial_i := (x, [y], \frac{\partial}{\partial x^i}|_x),\ i = 1, \ldots, n\) denote the local natural frame of \(\pi^* TM\).

The Finsler metric \(F\) induces a natural Riemannian metric \(g := g_{ij} dx^i \otimes dx^j\) and the Cartan tensor \(A := A_{ijk} dx^i \otimes dx^j \otimes dx^k\) on \(\pi^* TM\), where

\[g_{ij}(x, [y]) := \frac{1}{2} \frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j},\quad A_{ijk}(x, [y]) := \frac{F}{4} \frac{\partial^3 F^2(x, y)}{\partial y^i \partial y^j \partial y^k}.\]

In general, there is no linear connection in \(\pi^* TM\) such that it is not only "torsion-free" but also "metric-compatible". There are two important connections in \(\pi^* TM\), which are the Cartan connection\(^6\) and the Chern connection\(^3\). The former is compatible with \(g\) while the latter is torsion-free. From now on, we denote by \(\nabla\) the Cartan connection and \(\nabla\) the Chern connection.

Throughout this paper, we assume that \(\{e_i\}\) is a local orthonormal frame field for \(\pi^* TM\), where \(e_n = \ell\), and \(\{\omega^i\}\) is the dual frame field. Let \(\nabla e_i : = \omega^j_i \otimes e_j\) and
Cartan connection \( \nabla \): \[
\begin{align*}
d\omega^i - \omega^j \wedge \omega_j^i &= -A_{j\alpha}^i \omega^j \wedge \omega_n^\alpha, \\
\varpi^i_j + \varpi^j_i &= 0.
\end{align*}
\]

Chern connection \( \nabla \): \[
\begin{align*}
d\omega^i - \omega^j \wedge \omega_j^i &= 0, \\
\varpi^i_j + \varpi^j_i &= -2A_{j\alpha}^i \omega_n^\alpha.
\end{align*}
\]

In fact, \( \varpi^i_j = \varpi_i^j + A^i_{j\alpha} \varpi^n_\alpha \) (see [3, p.39]). Since \( A_{j\alpha}^i = 0 \), \( \varpi^i_j = \varpi_i^j \). Given any \( x \in M \), let \( S_x M := \pi^{-1}(x) \) and \( i_x : S_x M \rightarrow SM \) be the injective map. It follows from [4] that the Riemannian volume form \( dV(x) \) of \( S_x M \) induced by \( g \) satisfies
\[
dV(x) = i_x^* (\varpi^n_1 \wedge \cdots \wedge \varpi^n_{n-1}) = i_x^* (\varpi^n_1 \wedge \cdots \wedge \varpi^n_{n-1}).
\]

3. A transgression formula for the Cartan connection

For convenience, let \( \mathcal{A}^i j := \Gamma(SM, \wedge^i T^*SM \otimes \wedge^j \pi^*TM) \) and \( \mathcal{A} := \sum_{i,j} \mathcal{A}^i j \). In this section, we will investigate \( \mathcal{A} \) and derive a transgression formula for the Cartan connection.

Clearly, \( \mathcal{A} \) is a bigraded algebra (cf. [11]). Hence, for \( a \otimes b \in \mathcal{A}^{i,j} \) and \( c \otimes d \in \mathcal{A}^{k,l} \), the product of \( a \otimes b \) and \( c \otimes d \) is defined by
\[
(a \otimes b) \cdot (c \otimes d) = (-1)^{ik}(a \otimes c) \otimes (b \wedge d).
\]

For each \( s \in \Gamma(SM, \pi^*TM) \), the contraction \( \iota(s) : \mathcal{A}^{i,j} \rightarrow \mathcal{A}^{i,j-1} \) is defined by
\[
\iota(s)(\alpha \otimes s_1 \wedge \cdots \wedge s_j) = \sum_k (-1)^{i+k-1} g(s, s_k) \alpha \otimes (s_1 \wedge \cdots \wedge s_k \wedge \cdots \wedge s_j).
\]

We always identify \( \mathfrak{so}(\pi^*TM) \) with \( \bigwedge^2 \pi^*TM \) by the map
\[
B \in \mathfrak{so}(\pi^*TM) \rightarrow \frac{1}{2} \sum_{i,j} g(\mathcal{B}e_i, e_j) e_i \wedge e_j.
\]

Note that the Cartan connection is a operator from \( \mathcal{A}^{i,j} \) to \( \mathcal{A}^{i+1,j} \), i.e.,
\[
\nabla(a \otimes b) = da \otimes b + (-1)^i a \wedge \nabla b, \quad \forall a \otimes b \in \mathcal{A}^{i,j}.
\]

In particular, \( \nabla \Omega = 0 \). Here, we view the curvature of the Cartan connection \( \Omega \) as an element of \( \mathcal{A}^{2,2} \).

An easy calculation yields the following proposition, which is useful in this paper.

**Proposition 3.1.** For each \( s \in \mathcal{A}^{0,1}, \alpha \in \mathcal{A}^{i,j} \) and \( \beta \in \mathcal{A}^{k,l} \), we have
\[
\begin{align*}
(1) \quad & \alpha \cdot \beta = \beta \cdot \alpha; \\
(2) \quad & \iota(s)(\alpha \cdot \beta) = (\iota(s)\alpha) \cdot \beta + (-1)^{i+j} \alpha \cdot (\iota(s)\beta); \\
(3) \quad & \nabla(\alpha \cdot \beta) = (\nabla \alpha) \cdot \beta + (-1)^{i+j} \alpha \cdot (\nabla \beta).
\end{align*}
\]

**Remark 1.** In fact, Proposition 3.1 holds for any connection on \( \pi^*TM \).

It is not hard to see that
\[
0 = \iota(\ell) \nabla \ell, \quad \nabla(\nabla \ell) = \iota(\ell) \Omega, \quad \iota(\ell) \ell = 1.
\]

(3.1) together with Proposition 3.1 yields immediately the following proposition.
Proposition 3.2. Define $\Theta_t := t^2/2 + it\nabla \ell + \Omega$. Then
\[
(\nabla - it(\ell))f(\Theta_t) = 0,
\]
where $f : \mathbb{R} \to \mathbb{R}$ is a smooth function and
\[
f(\Theta_t) := \sum_{k=0}^{\infty} \frac{f^{(k)}(t^2/2)}{k!} (it\nabla \ell + \Omega)^k.
\]

Let $\mathcal{A}(SM) := \bigoplus i \mathcal{A}^{i,0}$. Since $\pi^*TM$ is an oriented bundle, we can induce the Berezin integral $\mathcal{B}$ (cf. [5]) to $\mathcal{A}$ such that
\[
\mathcal{B} : a \otimes \eta \in \mathcal{A} \mapsto \mathcal{B}(a \eta) \in \mathcal{A}(SM),
\]
with
\[
\mathcal{B}(e_I) = \begin{cases} 
\epsilon_I, & |I| = n, \\
0, & \text{otherwise.}
\end{cases}
\]
Moreover, it follows from [5 Proposition 1.50] that
\[
d\mathcal{B}(\xi) = \mathcal{B}(\nabla \xi), \text{ for any } \xi \in \mathcal{A}. \tag{3.2}
\]

Combining Proposition 3.2 and (3.2), we have the following

Lemma 3.3. $U_t := \mathcal{B}(e^{-\Theta_t})$ is a closed $n$-form on $SM$.

Proof. Since $e^{-\Theta_t} \in \mathcal{A}^{n,n}$, $U_t$ is a $n$-form on $SM$. Now it follows from (3.2) and Proposition 3.2 that
\[
dU_t = \mathcal{B}[(\nabla - it(\ell))(e^{-\Theta_t})] = 0.
\]
\[\Box\]

Remark 2. Mathai-Quillen’s proof of the formula (1.1) was carried out by constructing a Thom form on $TM$, which pulled back by the zero-section is exactly the Euler form. Although $U_t$ is similar to the Mathai-Quillen’s Thom form restricted to $SM$, the argument in [11] cannot be applied in Finsler manifolds directly, since the connection form of any metric-compatible (or torsion-free) connection on $\pi^*TM$ cannot be extended to the zero-section for a general Finsler metric.

From above, we obtain the following transgression formula.

Lemma 3.4.
\[
\frac{d}{dt}U_t = -i d \left[ \mathcal{B}(\ell \cdot e^{-\Theta_t}) \right].
\]

Proof. In view of (3.1), we have $\frac{d}{dt}\Theta_t = i(\nabla - it(\ell))\ell$. This together with Lemma 3.3 and (3.2) yields
\[
\frac{d}{dt}U_t = - \mathcal{B} \left( (i(\nabla - it(\ell))\ell) \cdot e^{-\Theta_t} \right)
= -i \mathcal{B} \left( (\nabla - it(\ell)) (\ell \cdot e^{-\Theta_t}) \right)
= -i d[\mathcal{B}(\ell \cdot e^{-\Theta_t})].
\]
\[\Box\]
4. Proofs of Theorem 1.1 and Corollary 1.2

Recall that $\Omega \in \mathcal{A}^{2,2}$. According to [5] Definition 1.35, the Pfaffian of $-\Omega$ is defined by

$$\text{Pf}(-\Omega) := \mathcal{B}(\exp(-\Omega)).$$ \hspace{1cm} (4.1)

Note that $U_t = e^{-t^2/2} \mathcal{B}(e^{-(it\nabla + \Omega)}) \to 0$ (as $t \to \infty$). Hence, Lemma 3.4 yields

$$\text{Pf}(-\Omega) = U_0 = id \left[ \int_0^\infty \mathcal{B}(\ell \cdot e^{-\Omega t}) dt \right] \in \mathcal{A}^n(SM).$$

Denote by $\Xi$ the component of $e^{-(it\nabla + \Omega)}$ in $\mathcal{A}^{n-1,n-1}$. Thus,

$$\text{Pf}(-\Omega) = id \left[ \int_0^\infty e^{-t} \mathcal{B}(\ell \cdot \Xi) dt \right].$$ \hspace{1cm} (4.2)

Using this simple observation, we easily get the following formula.

**Lemma 4.1.**

$$\text{Pf}(-\Omega) = (-1)^{n-1}d \left( \sum_{k=0}^{\left[ \frac{n-1}{2} \right]} \frac{(-1)^{k+2 \frac{n}{2}} \Phi_k}{k!(n-1-2k)!2^{2k+1}} \Gamma \left( \frac{n-2k-2}{2} \right) \right),$$

where $\Gamma(s)$ is the gamma function and

$$\Phi_k := \sum_{\alpha_1 \cdots \alpha_n-1} \Omega_{\alpha_1}^{\alpha_2} \wedge \cdots \wedge \Omega_{\alpha_{2k-1}}^{\alpha_{2k}} \wedge \Omega_{\alpha_{2k+1}}^{n} \wedge \cdots \wedge \Omega_{\alpha_{n-1}}^{n}.$$

**Proof.** It is not hard to see that

$$\Xi = \sum_{k=0}^{\infty} \frac{(-1)^{k}}{k!} \left( \sum_{\{s:(k-s)+2s=n-1, 0 \leq s \leq k\}} \binom{k}{s} (t\nabla \ell)^{k-s} \cdot \Omega^s \right)$$

$$= \sum_{k=0}^{\left[ \frac{n-1}{2} \right]} \frac{(-1)^{k}}{(n-1-k)!2^{k}(n-1)!} (t\nabla \ell)^{2k-(n-1)} \cdot \Omega^{n-1-k}$$

$$= (-i)^{n-1} \sum_{k=0}^{\left[ \frac{n-1}{2} \right]} \frac{t\nabla \ell)^{n-1-2k} \cdot \Omega^k}{k!(n-1-2k)!}.$$

Clearly,

$$(\nabla \ell)^k = (-1)^{k+2 \frac{n}{2}} \omega_n^{\alpha_1} \wedge \cdots \wedge \omega_n^{\alpha_k} \otimes e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_k},$$

$$\Omega^k = \frac{1}{2^k} \Omega_{j_1}^{j_2} \wedge \cdots \wedge \Omega_{j_{2k-1}}^{j_{2k}} \otimes e_{j_1} \wedge \cdots \wedge e_{j_{2k}}.$$

Combining $\ell = e_n$ and all the equalities above, we have

$$\ell \cdot \Xi = \frac{(-1)^{n-1}}{e(n+1)} \sum_{k=0}^{\left[ \frac{n-1}{2} \right]} \frac{t^{n-1-2k}(-1)^k}{k!(n-1-2k)!2^{2k}} \Phi_k \otimes e_1 \wedge \cdots \wedge e_n,$$

where

$$e(n) := \begin{cases} 
1, & n = 2p, \\
i, & n = 2p+1.
\end{cases}$$
Now it follows from (4.1) that
\[ \text{Pf}(-\Omega) = (-1)^{n-1} \epsilon(n)d \left( \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k 2^\frac{k}{2} \Phi_k}{k!(n-1-2k)!2^{k+1}} \Gamma \left( \frac{n-2k}{2} \right) \right). \]

However, if \( n = 2p + 1 \), then (4.1) implies \( \text{Pf}(-\Omega) = 0 \) and (therefore)
\[ (-1)^{n-1}d \left( \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k 2^\frac{k}{2} \Phi_k}{k!(n-1-2k)!2^{k+1}} \Gamma \left( \frac{n-2k}{2} \right) \right) = 0 = \text{Pf}(-\Omega). \]

\[ \square \]

Define \( \Omega^\nabla := \frac{1}{(2\pi)^{\frac{n}{2}}} \text{Pf}(-\Omega) \) and
\[ \Pi := \frac{(-1)^n}{\pi^{\frac{n}{2}}} \left( \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k \Phi_k}{k!(n-1-2k)!2^{k+1}} \Gamma \left( \frac{n-2k}{2} \right) \right). \]

An easy calculation yields that
\[ \Omega^\nabla = \begin{cases} 
\frac{(-1)^{p-1}}{2^{p-1}} \epsilon_{i_1 \cdots i_{2p}} \Omega^i_{i_1} \cdots \Omega^i_{i_{2p-1}} & n = 2p, \\
0 & n = 2p + 1.
\end{cases} \]

\[ \Pi = \begin{cases} 
\frac{1}{(2\pi)^p} \sum_{k=0}^{p-1} \frac{(-1)^k}{(2p-2k-1)!!2^p} \Phi_k & n = 2p, \\
\frac{1}{\pi^{2p}} \sum_{k=0}^{p} (-1)^{k+1} \frac{p}{k} \Phi_k & n = 2p + 1.
\end{cases} \]

Hence, \( \Omega^\nabla \) and \( \Pi \) are of the same form as the ones defined in \([7, 8, 2, 13]\). And Lemma 4.1 implies that \( \Omega^\nabla = d\Pi \). Clearly, this formula holds for any metric-compatible connection. For simplicity, set
\[ \Upsilon_1 := \frac{(-1)^n \Phi_0}{2\pi^{\frac{n}{2}}} \Gamma \left( \frac{n}{2} \right), \]
\[ \Upsilon_2 := \frac{(-1)^n}{\pi^{\frac{n}{2}}} \left( \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k \Phi_k}{k!(n-1-2k)!2^{k+1}} \Gamma \left( \frac{n-2k}{2} \right) \right), \]
\[ \mathcal{D} := -d\Upsilon_2 - d\log V(x) \wedge \Upsilon_1. \]

Applying the same technique as in \([13]\), we obtain the following lemma.

**Lemma 4.2.** Given any vector filed \( X \) on \( M \) with isolated zeros \( \{x\} \), let \( [X] : M \setminus \{x\} \to SM \setminus \Sigma_x M \) denote the section induced by \( X \). Then, for each isolate zero \( x \) and each small \( \epsilon > 0 \),
\[ \int_{\partial B^+_\epsilon(x)} [X]^* \left( \frac{\Upsilon_1}{V(x)} \right) = (-1)^{\text{index}(X; x)} \frac{\text{vol}(S^{n-1})}{\text{vol}(S^{n-1})}. \]
Proof. Define a map \( \varphi : S^2 M \rightarrow SM \) by
\[
\varphi(x) = [X] \circ \kappa(|y|),
\]
where \( \kappa(|y|) := \exp_x \left( \frac{\epsilon y}{F(y)} \right) \). (\( \exp_x \)) is a diffeomorphism that \( \kappa \) is a diffeomorphism between \( S^2 M \) and \( \partial B^+_\epsilon(\gamma) \) and (therefore) \( \operatorname{deg}(\kappa) = 1 \). Hence,
\[
\operatorname{deg}(\varphi) = \operatorname{deg}(\kappa) = \operatorname{deg}(\kappa(\epsilon)) = \operatorname{index}(X; x).
\]
Recall that \( \operatorname{vol}(\mathbb{S}^{n-1}) = 2\pi^{n/2}/(\Gamma(n/2)) \) and \( \Phi_0 |_{S^2 M} = (n - 1)! \operatorname{dV}(x) \).
Thus,
\[
\int_{\partial B^+_\epsilon(\gamma)} [X]^* \left( \frac{\Upsilon(y)}{V(x)} \right) = \int_{\kappa(S^2 M)} [X]^* \left( \frac{\Upsilon(y)}{V(x)} \right) = \int_{\partial B^+_\epsilon(\gamma)} [X]^* \left( \frac{\Upsilon(y)}{V(x)} \right)
\]
\[
= \int_{S^2 M} \varphi^* \left( \frac{\Upsilon(y)}{V(x)} \right) = \operatorname{deg}(\varphi) \int_{S^2 M} \frac{\Upsilon(y)}{V(x)} = (-1)^{n+1} \frac{\operatorname{index}(X; x)}{\operatorname{vol}(\mathbb{S}^{n-1})}.
\]
\( \square \)

From above, we now prove Theorem 1.1.

Proof of Theorem 1.1. It is easy to see that
\[
\frac{\Omega^V}{V(x)} = \frac{d\Pi}{V(x)} = d \left( \frac{\Pi}{V(x)} \right) = -d \left( \frac{1}{V(x)} \right) \wedge \Pi = d \left( \frac{\Upsilon(y)}{V(x)} \right) = \frac{\Upsilon(y)}{V(x)}. \tag{4.3}
\]
Let \( \{x_i\} \) be the isolated zeros of \( X \). Choose a small \( \epsilon > 0 \) such that the forward balls \( B^+_\epsilon(x_i) \) are disjoint from each other. (4.3) together with Lemma 4.2 now yields
\[
\int_{M \cup B^+_\epsilon(\gamma)} [X]^* \left( \frac{\Omega^V + D}{V(x)} \right) = \int_{M \cup B^+_\epsilon(\gamma)} [X]^* d \left( \frac{\Upsilon(y)}{V(x)} \right)
\]
\[
= -\sum_i \int_{\partial B^+_\epsilon(\gamma)} [X]^* \left( \frac{\Upsilon(y)}{V(x)} \right) = \frac{(-1)^{n+1}}{\operatorname{vol}(\mathbb{S}^{n-1})} \sum_i \operatorname{index}(X; x_i) = (-1)^{n+1} \frac{\chi(M)}{\operatorname{vol}(\mathbb{S}^{n-1})}.
\]
Note that \( \chi(M) = 0 \) when \( n = 2p + 1 \). We finish the proof by letting \( \epsilon \to 0^+ \). \( \square \)

Remark 3. In the Riemannian case, \( \operatorname{V}(x) = \operatorname{vol}(\mathbb{S}^{n-1}) \) and \( \nabla \) is the pull-back connection induced by the Levi-Civita connection. Hence, \( \Upsilon_2 \) has no pure-\( dy \) part and (therefore) \( \int_M [X]^* \mathcal{D} = \lim_{\epsilon \to 0} \int_{M \cup B^+_\epsilon(\gamma)} [X]^* \mathcal{D} = 0 \). Thus, the theorem above implies the Gauss-Bonet-Chern theorem\(^7, 8\). However, in the non-Riemannian case, \( \Upsilon_2 \) may have the pure-\( dy \) part even when \( \operatorname{V}(x) = \text{const} \). Refer to [13] for more interesting results in this case.

Let \( D_s \) denote a family of connections compatible with \( g \). Set \( D_s := d + \omega_s \), where \( \omega_s \in \mathfrak{a}^1 (SM) \otimes \mathfrak{so}(\pi^* TM) \). Thus, \( \frac{\partial D_s}{\partial s} = \frac{1}{2} \sum \omega_s e_i \wedge e_j \in \mathfrak{a}^{1,2} \) and
\[
\frac{\partial}{\partial s} \Omega_s = D_s \left( \frac{\partial D_s}{\partial s} \right) \in \mathfrak{a}^{2,2},
\]
where \( \Omega_s \) is the curvature of \( D_s \). Define \( \Omega^{D_s} := \frac{\exp(\Omega_s)}{(2\pi)^{n/2}} \). \( D_s \Omega_s = 0 \) together with (3.2) now yields
\[
\frac{\partial}{\partial s} \Omega^{D_s} = \frac{-1}{(2\pi)^{n/2}} \frac{\partial}{\partial s} \int (\exp(-\Omega_s)) = \frac{1}{(2\pi)^{n/2}} \int \left( D_s \left( \exp(-\Omega_s) \cdot \frac{\partial D_s}{\partial s} \right) \right)
\]
\[
= \frac{1}{(2\pi)^{n/2}} \int \left( \exp(-\Omega_s) \cdot \frac{\partial D_s}{\partial s} \right).
\]
By the argument above, we obtain Corollary 1.2 directly.

**Proof of Theorem 1.2.** Set \( D_s = s\nabla + (1-s)D \). The transgression formula above implies that \( \Omega^Y = \Omega^D + dY_3 \), where \( Y_3 := (2\pi)^{-1} \int_0^1 \beta \left( \exp(-\Omega_s \cdot \frac{\partial D}{\partial s}) \right) ds \). Set \( \mathcal{E} = \nabla + dY_3 \). Corollary 1.2 now follows from Theorem 1.1. \( \square \)

**Remark 4.** In the Riemannian case, Theorem 1.2 implies that, for any metric-compatible connection \( D \), \( \int_M \Omega^D = \chi(M) \), that is, \( \Omega^D \) is a Euler form (cf. [5, Theorem 1.56]).

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