Large $N$ Field Theory Of $\mathcal{N} = 2$ Strings and Self-Dual Gravity

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Abstract

We review some aspects of the construction of self-dual gravity and the associated field theory of $\mathcal{N} = 2$ strings in terms of two-dimensional sigma models at large $N$. The theory is defined through a large $N$ Wess-Zumino-Witten model in a nontrivial background and in a particular double scaling limit. We examine the canonical structure of the theory and describe an infinite-dimensional Poisson algebra of currents.

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1 Introduction

String theories exhibit remarkable physical and mathematical properties and their description in terms of fields in the target space-time where the string propagates is interesting and relevant for both the study of non-perturbative aspects of the theory and for the understanding of its fundamental meaning. Of recent major interest is the nature and origin of gravitational interactions in string theory and its possible partonic constituent structure. The $\mathcal{N} = 2$ string\textsuperscript{2} is one of the simplest string theories and possesses remarkable properties. In the case of the closed string the spectrum consists essentially of one real massless scalar field which for the critical theory lives in a target space-time that is a two-dimensional complex manifold of signature $(2, 2)$ and equipped with a (pseudo) Kahler metric \cite{1}-\cite{9}. This theory shares some similarities with the two-dimensional theory of non-critical strings where the only bulk degree of freedom was a massless scalar \cite{10}. In the present case the scalar field $\Omega = \Omega(y, \bar{y}, z, \bar{z})$ parametrizes possible metric deformations of the free $\mathcal{N} = 2$ supersymmetric sigma model action

$$S_0 = \int d^2x d^2\theta d^2\bar{\theta} K_0(y, \bar{y}, z, \bar{z})$$

where $y$ and $z$ are complex coordinates on the target space and where the flat Kahler potential is given by $K_0 = y\bar{y} - z\bar{z}$. One has the deformations $K_0 \rightarrow K = K_0 + \Omega$ describing a non-flat Kahler metric $g_{ij} = \partial_i\partial_j K$ \cite{1}. This geometrical interpretation of the field $\Omega$ follows from an examination of its equation of motion which is determined by the string theory amplitudes. Due to the strong kinematical constraints that exist in a space-time of signature $(2, 2)$ the $\mathcal{N} = 2$ string amplitudes have peculiar properties which make them particularly simple. In fact, all tree level amplitudes beyond and including the 4-point function vanish and this result is expected to hold at loop level as well. This is an indication of some underlying topological content of

\textsuperscript{2}For an excellent review see \cite{6}.
the theory. The tree level 3-point string amplitude, as well as the vanishing of the 4-point and higher tree level correlation functions as demonstrated by Ooguri and Vafa, is reproduced by the cubic action for \( \Omega \)

\[
S_{\text{cubic}} = \int d^2y d^2z \exp \left( \frac{1}{2} (\partial_y \Omega \partial_{\bar{y}} \Omega - \partial_z \Omega \partial_{\bar{z}} \Omega) + \frac{1}{3} \Omega \{ \partial_y \Omega, \partial_z \Omega \} \right),
\]

which gives as equation of motion the Plebański first heavenly equation for self-dual gravity \([17]\)

\[
\det(g_{ij}) = -1 \quad \longleftrightarrow \quad \partial^2_{yy} \Omega - \partial^2_{zz} \Omega - \{ \partial_y \Omega, \partial_z \Omega \} = 0,
\]

where \( \{ , \} \) denotes the Poisson bracket \( \partial_y \partial_{\bar{z}} - \partial_z \partial_{\bar{y}} \) \([18]\). Since the determinant of the Kahler metric is constant, the target space is a Ricci flat four-dimensional Kahler manifold, and therefore it is also hyperkahler and the Riemann tensor is self-dual (or anti-self-dual). Thus, as we should expect of a theory of closed strings, the closed \( \mathcal{N} = 2 \) string also contains gravity and we may consider it as a definition of what a quantum theory of self-dual gravity should be. A proper understanding of the theory would also have implications to other subjects of recent interest, for example the quantum theory of membranes \([11, 12, 13]\).

Although the cubic action in \((1)\) describes the string tree level amplitudes quite well, at one-loop level some problems arise \([3, 5, 6]\). For example, the string one-loop partition function behaves as \( \sim \tau_2^{-1} \) for small \( \tau_2 \) where \( \tau_2 \) is the imaginary part of the modular parameter describing the moduli space of tori, while the field theory one-loop partition function for a massless scalar in four real dimensions would be expected to behave as \( \sim s^{-2} \) for small \( s \) where \( s \) is a Schwinger parameter which is the field theory analog of \( \tau_2 \). The behaviour of the string one-loop partition function corresponds instead to a massless scalar field living in \( \text{two} \) real dimensions. A similar mismatch occurs for the one-loop 3-point function where the string gives an infrared divergence of the form \( \sim \tau_2^2 \) for large \( \tau_2 \), which would again correspond to a two-dimensional field theory, while on the other hand the cubic action \((1)\) gives instead
a divergence of type $\sim s$. Therefore, although the critical target space dimension is four real dimensions it looks like that at the quantum level the string would “prefer” to live in only two real dimensions \[3, 3, 3\]!

For the open string the analogous situation arises where now self-dual gravity is replaced by self-dual Yang-Mills theory and the massless scalar field takes values in the Lie algebra of the gauge group $G$ (which for this type of strings can be chosen arbitrarily) corresponding to the Chan-Paton factors attached to the ends of the strings.

The problem of trying to match the amplitudes in the field theory and in the string theory at the quantum level, indicates that some type of two-dimensional description would provide a better formulation of the field theory of $\mathcal{N} = 2$ strings. Self-dual gravity as shown at classical level by Q.Park can indeed be interpreted as a two-dimensional sigma model at large $N$. The basic idea \[18\] is to use the fact that as $N \to \infty$ the Lie algebra of $SU(N)$ “approaches” the Poisson algebra of functions on some two-dimensional surface $\Sigma$ with the Lie bracket at finite $N$ being replaced by the Poisson bracket when $N \to \infty$ \[3\] \[14\]. One starts with a two-dimensional equation of motion for some Lie algebra valued fields and when the Lie bracket is replaced by the Poisson bracket one effectively generates an equation of motion in a four-dimensional space-time with the two extra variables coming from the large $N$ color following the original proposal of \[16\] made in the context of Yang-Mills theory.

In \[30\] (hereafter refered to as (I)) we have explored the approach of \[18\] and considered a large $N$ Wess-Zumino-Witten model to establish a field theory of $\mathcal{N} = 2$ strings, \textit{i.e.} self-dual gravity. We have introduced a nonlinear formulation of the theory which when taken in a particular classical background and a in a certain double scaling limit was seen to give self-dual gravity. We have studied the corresponding

\footnote{One should be careful about the way in which the limit is taken as several different large $N$ limits are possible. See for example \[15\].}
scattering amplitudes up to one-loop level. In this paper we review and continue to further examine the model formulated in (I). In section 2 we discuss some general questions on the formulation of self-dual gravity as a large \( N \) sigma model and in section 3 we describe our construction. Section 4 explains the relation of the model with self-dual Yang-Mills theory and in section 5 we briefly comment on the field theory amplitudes obtained from our model. In Section 6 we establish the Hamiltonian structure of the theory and construct an infinite-dimensional Poisson algebra of currents. We close in section 7 with some conclusions.

2 Self-Dual Gravity as a Large \( N \) 2D Sigma Model

We will consider a family of two-dimensional sigma models defined in a space-time with Lorentzian signature \((1,1)\), light-cone coordinates \( x^+ = x^1 + x^2 \), \( x^- = x^1 - x^2 \) and metric \( ds_{2d}^2 = dx^+ dx^- \). The models will consist of a chiral model term together with a Wess-Zumino term, where the coefficient of the chiral term will be perturbed away from the conformal fixed point value, so that the models can be seen as perturbed WZW models at level \( K \),

\[
S(g) = \frac{(1 + \epsilon)K}{4\pi} \int dx^+ dx^- \text{Tr}(\partial_+ g \partial_- g^{-1}) + \frac{K}{12\pi} \Gamma_{WZW}(g),
\]

where \( g \) is the group valued field, \( \epsilon \) is a real parameter and \( \Gamma(g) \) is the Wess-Zumino term. When \( \epsilon = 0 \) we obtain the usual WZW model fixed point. The group \( G \) remains unspecified for now, but shortly we will take the Lie algebra to be some large \( N \) limit of \( su(N) \).

The classical equations of motion will be given by

\[
\frac{\epsilon}{1 + \epsilon} \partial_+(g^{-1} \partial_- g) + \frac{2 + \epsilon}{1 + \epsilon} \partial_-(g^{-1} \partial_+ g) = 0.
\]

Let us consider the case where there’s only the WZW term, that is the limit
$\epsilon \to -1$. The equation of motion becomes,

$$\partial_+(I_-) - \partial_-(I_+) = 0.$$  \hspace{1cm} (5)

where the Lie algebra valued currents are given by $I_\pm = g^{-1}\partial_\pm g$. Of course the currents also satisfy the flatness condition,

$$\partial_+(I_-) - \partial_-(I_+) + [I_+, I_-] = 0.$$  \hspace{1cm} (6)

One now takes the large $N$ limit and replaces the Lie bracket $[ , ]$ by the Poisson bracket $\{ , \} = \partial_q \partial_p - \partial_p \partial_q$ of functions of $q$ and $p$ where $q, p$ are coordinates on some two-dimensional surface $\Sigma$, and one solves the equation of motion (5) for the currents by setting $I_+=\partial_+\Omega$ and $I_- = \partial_-\Omega$ in terms of a scalar field $\Omega(x^+, q, x^-, p)$. Then the flatness condition (6) becomes an equation of motion for $\Omega(x^+, q, x^-, p)$,

$$\{\partial_+\Omega, \partial_-\Omega\} = 0.$$  \hspace{1cm} (7)

On the other hand, if one now sets $\Omega \to \Omega - x^+q+x^-p$ in the first heavenly equation (2) and let $(x^+, q, x^-, p) \leftrightarrow (y, \bar{y}, z, \bar{z})$ one obtains $\{\partial_+\Omega, \partial_-\Omega\} = 1$. This is essentially the same as equation (7) if one replaces the functions $\Omega(x^+, q, x^-, p)$ by the corresponding symplectic vector fields $\xi_{\Omega} = \partial_q\Omega \partial_p - \partial_p\Omega \partial_q$ such that the Lie bracket of two such vector fields satisfies $\xi_{\{f,g\}} = [\xi_f, \xi_g]$ and such that constant functions give the vector field zero [18]. Therefore, classical self-dual gravity is obtained as the large $N$ limit of a two-dimensional sigma model. The equation of motion (2) for $\Omega$ can now be obtained from the cubic action (1) of the previous section. However, at the quantum level we may expect that the large $N$ two-dimensional sigma model and the cubic action, which is a “pseudo-dual” version of the original sigma model behave differently [22]. Indeed, although the family of two-dimensional sigma models (3) has a well known infinite-dimensional Poisson algebra of currents, that desirable feature is far from obvious in the canonical structure of the cubic action (1). If we canonically quantized (1), in the usual time-like quantization, we would obtain the momenta $\Pi = \partial_0\Omega$ and we would set $\{\Pi, \Pi\}_{P.B.} = 0 = \{\partial_0\Omega, \partial_0\Omega\}_{P.B.}$ while we know that the currents $I_0 = \partial_0\Omega$ in
the two-dimensional sigma model generate a rich Kac-Moody type Poisson algebra with the Poisson bracket given schematically by
\[
\{I_0^\alpha, I_0^\beta\}_{P.B.} = f_{\gamma}^{\alpha \beta} I_0^\gamma
\]
where the \( f_{\gamma}^{\alpha \beta} \) are the Lie algebra structure constants. It is clear that the canonical structure of the pseudo-dual cubic action is different from the canonical structure of the original two-dimensional sigma model. This is unfortunate since we certainly would like to keep the original current algebras of the two-dimensional sigma models in our string field theory.

One would therefore like to define a field theory of \( \mathcal{N} = 2 \) strings formulated as a two-dimensional nonlinear sigma model in such a form which preserves its canonical structure. In addition, we would prefer to work with functions of \( q \) and \( p \) rather than with symplectic vector fields. To achieve this and to generate the four-dimensional space-time out of the two dimensions \( x^+, x^- \) together with the large \( N \) “color” gauge algebra variables \( q, p \), we went back to the nonlinear version of models in (3).

3 Large \( \mathcal{N} \) WZW Field Theory of \( \mathcal{N} = 2 \) Strings

Let us consider the general family of two-dimensional sigma models in (3). As long as \( \epsilon \neq -2, -1, 0 \) in (3), one can define \( Q \) by
\[
\frac{2 + \epsilon}{1 + \epsilon} g^{-1} \partial_+ g = Q^{-1} \partial_+ Q \quad \text{and} \quad \frac{\epsilon}{1 + \epsilon} g^{-1} \partial_- g = Q^{-1} \partial_- Q
\]
where \( Q \) is a classical solution of the pure chiral model equation obtained by setting \( \epsilon \to \infty \), see for example\(^4\) [28],
\[
\partial_+ (Q^{-1} \partial_- Q) + \partial_- (Q^{-1} \partial_+ Q) = 0.
\]

To make the connection between the two-dimensional sigma model (3) and four-dimensional self-dual gravity, we start by expanding the field \( g \) arround a specific

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\(^4\)One can easily check that this is consistent with the flatness conditions for both the \( g \) and \( Q \) currents.
classical configuration. For reasons that will become clear shortly, this classical background will be given by
\[ Q^{-1} \partial_+ Q = \frac{\hat{q}}{\Lambda} + \frac{x^-}{2\Lambda^2}; \quad Q^{-1} \partial_- Q = \frac{\hat{p}}{\Lambda} - \frac{x^+}{2\Lambda^2}, \tag{8} \]
where the (infinite) matrices \( \hat{q} \) and \( \hat{p} \) satisfy \([\hat{q}, \hat{p}] = 1\) and \( \Lambda \) is an arbitrary parameter.

In the context of our approach, what is important about \( \hat{q} \) and \( \hat{p} \) is that in the large \( N \) limit they become canonical variables \( q \) and \( p \), possibly after some rescaling by factors of \( N \). The corresponding \( g \) will be given by
\[ g_0^{-1} \partial_+ g_0 = \frac{1 + \epsilon}{2 + \epsilon} \left( \frac{\hat{q}}{\Lambda} + \frac{x^-}{2\Lambda^2} \right); \quad g_0^{-1} \partial_- g_0 = \frac{1 + \epsilon}{\epsilon} \left( \frac{\hat{p}}{\Lambda} - \frac{x^+}{2\Lambda^2} \right). \tag{9} \]

We will now expand the action around this classical background. We take \( N \to \infty \) such that the fields will take values in the infinite dimensional Poisson algebra of functions on the surface \( \Sigma \), \( sdiff(\Sigma) \). The trace becomes the integration over the \( su(\infty) \) “color” variables \( q \) and \( p \). The quantum field fluctuations will be described by a field \( \omega \), which will take values in the Poisson algebra, where we define
\[ g = g_0 \exp(\omega). \tag{10} \]

From the Polyakov-Wigman formula \[25, 27\] and after rescaling the field \( \omega \) to get a normalized kinetic term we obtain\[5\]
\[ S(\omega) = \frac{1}{2} \int dx^+ dx^- \text{Tr}(\partial_+ \omega \partial_- \omega - \partial_- \omega \partial_+ \omega) + \frac{1}{3!} g_st \int dx^+ dx^- \text{Tr}(\omega \{\partial_+ \omega, \partial_- \omega\} - \omega \{\partial_\omega, \partial_+ \omega\}) + \frac{1}{4!} g_st^2 \int dx^+ dx^- \text{Tr}(\omega \{\partial_+ \omega, \partial_+ \omega\} \{\partial_- \omega, \partial_- \omega\} - \partial_\omega \{\partial_- \omega, \partial_\omega\}) + \cdots \tag{11} \]
where the coupling constant \( g_st = \sqrt{\frac{4\pi\Lambda}{K(1+\epsilon)}} \) will be identified with the string coupling constant when we look at the string amplitudes as in \[\text{[3]}\]. We see that expanding

\[\text{The } x^+ \text{ and } x^- \text{ dependent terms in the classical background currents (8) give vanishing contributions to } S(\omega).\]
around the background generates cubic, quartic and an infinite number of higher point vertices for the field $\omega$, and this is to be compared with the purely cubic action (1). Moreover, the two-dimensional propagator $\partial_+ \omega \partial_- \omega$, together with other terms without derivatives in the “color directions”, is multiplied by factor of $\Lambda$. In the limit $\Lambda \to 0$ these terms disappear and the remaining quadratic terms in $\omega$ define a four-dimensional-looking propagator for a metric of $(2, 2)$ signature

$$ ds_{4d}^2 = dx^+ dq - dx^- dp. $$(12)

We will take $\Lambda \to 0$ for the remainder of the paper, defining a double-scaling limit where the level $K$ and the parameter $\epsilon$ are such that $g_{st}$ is finite and nonzero. The cubic vertex of order $g_{st}$ appearing in (11) represents an SO$(2, 2)$ Lorentz transformation of the cubic vertex $\Omega\{\partial_+ \Omega, \partial_- \Omega\}$ of (1), whose equation of motion is the Plebański equation (2)\(^7\). Its amplitudes \(^{19}\) are closely related to the ones of the closed $\mathcal{N} = 2$ string \([4] - [9]\). However, we note that in the present model there is in addition an infinite series of higher point vertices. At this point we can analytically continue in $x^+, q, x^-, p$ so that we get the flat Kähler metric

$$ ds_{4d}^2 = dy d\bar{y} - dz d\bar{z} $$

where $y, \bar{y}, z, \bar{z}$ are complex coordinates corresponding to $x^+, q, x^-, p$ respectively. The field $\omega$ is then related to deformations of the flat Kähler potential \([4]\).

In the limit when $\Lambda \to 0$ with $g_{st}$ kept fixed, the terms that remain in (11) can be written more elegantly in terms of a group valued field $g = \exp(\omega)$ as

$$ S = \frac{1}{g_{st}^2} \int dx^+ dx^- \text{Tr}(q g \partial_- g^{-1} + p g \partial_+ g^{-1}) $$

\(^6\)Notice that the fact that we obtain signature $(2, 2)$ is closely related with the choice of sign in the classical background \([3]\).

\(^7\)We note that the Plebański equation is not SO$(2, 2)$ invariant, at least manifestly. For a discussion of the Lorentz symmetries of self-dual gravity see \([20]\).
This is the Lagrangian that we would like to test as a field theory of closed $\mathcal{N} = 2$ strings. The corresponding equation of motion is easily obtained by varying $g$ inside the trace and gives

$$\{g\partial_-g^{-1}, q\} + \{g\partial_+g^{-1}, p\} = 0 \leftrightarrow \partial_q J_+ - \partial_p J_- = 0$$

(15)

where $J_\pm = g\partial_\pm g^{-1}$. As before we also have the flatness condition $\partial_+ J_- - \partial_- J_+ + \{J_+, J_-\} = 0$. If we solve (15) by setting $J_- = \partial_q \Omega$ and $J_+ = \partial_p \Omega$ the flatness condition becomes

$$\partial_+ \partial_q \Omega - \partial_- \partial_p \Omega + \{\partial_p \Omega, \partial_q \Omega\} = 0$$

and in complex coordinate notation this is

$$\partial_y \partial_\bar{y} \Omega - \partial_z \partial_\bar{z} \Omega - \{\partial_\bar{y} \Omega, \partial_\bar{z} \Omega\} = 0$$

which is the second form of the heavenly equation for self-dual gravity [17, 18]. In this way we formulate self-dual gravity in terms of the Lagrangian (14) for a large $N$ group valued field $g$. Notice that expanding in $\omega$ where $g = \exp(\omega)$ produces the complicated infinite series of terms that we already described above. Moreover, the relation between the field $\Omega$ whose derivatives give the currents $J_\pm$ and the field $\omega$ is highly non-linear and quite complicated. While such a field $\Omega$ could be described by a purely cubic action analogous to (1) we would expect that action to be physically different, at the quantum level, from the more “complete” action (14) for $g = \exp(\omega)$.

4 A Reduction of Large $N$ Self-Dual Yang-Mills

As is well known, the self-dual Yang-Mills equations generate many known integrable systems by reduction. If one takes the self-dual Yang-Mills equations of motion at large $N$, that is if one takes the Lie algebra to be the Poisson algebra of functions on a surface $\Sigma$, one obtains a six-dimensional equation of motion where as before the
two extra variables come from the coordinates $q, p$ on $\Sigma$. In [21] it was shown that dimensionally reducing to four dimensions along two specific directions which “mix” space and “color” variables, produces self-dual gravity. Along the same lines it is possible to show that the model in (11) in the limit $\Lambda \rightarrow 0$ is a reduction of the large $N$ Donaldson-Nair-Schiff [23] action for self-dual Yang-Mills theory.

Consider the self-dual Yang-Mills equations

$$F_{\mu\nu} = F_{\mu\nu} = 0,$$  \hspace{1cm} (16)

in the gauge where $A_{\bar{\mu}} = 0, \mu, \nu = 1, 2$. The first equation in (16) is solved by $A_\mu = g^{-1} \partial_\mu g$ so that the equation of motion, Yang’s equation, reads

$$\eta^{\mu\bar{\nu}} \partial_{\bar{\nu}}(g^{-1} \partial_\mu g) = 0.$$  \hspace{1cm} (17)

We now consider the Donaldson-Nair-Schiff action [23, 24]

$$S = \frac{i}{4\pi} \int d^4 x \text{Tr}(g^{-1} \partial^\mu g g^{-1} \partial_\mu g) + \frac{i}{12\pi} \int_{M_5} \omega \wedge \text{Tr}(g^{-1} dg)^3,$$  \hspace{1cm} (18)

where $\text{Tr}$ denotes the trace on the Lie algebra of the gauge group, $G(N)$. Parametrizing $g(x^\mu, x^\bar{\mu}) = \exp(\hat{\phi})$, one expands the action in powers of the field $\hat{\phi}$. In the limit $N \rightarrow \infty$, we have a six-dimensional non-linear scalar field theory where the matrix field $\hat{\phi}(x^\mu, x^\bar{\mu})$ becomes a scalar field $\phi(x^\mu, x^\bar{\mu}, q, p)$, where $q, p$ are the large $N$ color variables [16]. If we reduce by identifying $x^1$ with $q$ and $x^2$ with $p$, that is if we impose

$$(\partial_1 - \partial_q)\phi = (\partial_2 - \partial_p)\phi = 0,$$  \hspace{1cm} (19)

we obtain a four-dimensional theory. In the Leznov-Parkes gauge this was seen to lead to the second heavenly equation of self-dual gravity [21]. In our case, we have a non-linear theory since the vertices follow from the four-dimensional WZW action evaluated in the large $N$ limit. Expanding (18), we have after some algebra

$$S = \int d^4 x \left\{ \frac{1}{2} \phi \partial^\mu \partial_\mu \phi + \frac{1}{3!} \eta^{\mu\bar{\nu}\bar{\rho}} \varepsilon_{\rho\sigma} \partial_\mu \phi \partial_\nu \phi \partial_\sigma \phi + \frac{1}{4!} \eta^{\mu\bar{\nu}\bar{\rho}\bar{\lambda}} \varepsilon_{\rho\sigma} \varepsilon_{\bar{\lambda}\bar{\xi}} \partial_\mu \phi \partial_\nu \phi \partial_\rho \phi \partial_\sigma \phi \partial_\bar{\lambda} \phi \partial_\bar{\xi} \phi + \cdots \right\}$$  \hspace{1cm} (20)

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These terms agree, to this order, with the vertices of the double scaling limit (14) of the two-dimensional model (11) as in (I). That is, the Lagrangian (14) also describes self-dual gravity as reduced large \( N \) self-dual Yang-Mills theory.

5 The Amplitudes

In this section we will make some brief comments about the relation of the amplitudes in our model and the closed string amplitudes up to one-loop. The results have appeared in (I). Since in our approach we generate an infinite series of higher point vertices, we have to check that the contributions to the amplitudes coming from these vertices are identical with the \( \mathcal{N} = 2 \) closed string amplitudes. The fact that this infinite series of higher point vertices is present, and the close relation of this theory with the WZW model, leads us to believe that it will have nice properties beyond the one-loop level. Moreover, the action (14) provides a systematic expansion for determining these higher point vertices. It is hoped that the fact that two of the four dimensions appear as large \( N \) color variables in disguise may shed some light into the problem of matching string and field theory amplitudes at one-loop level [4, 5, 6, 9]. In fact, the large \( N \) approach suggests a possible infra-red regulator for the momenta along \( \bar{y}, \bar{z} \). There is also the possibility that some specific choice of \( \Sigma \), for example a torus \( T^2 \), will regulate these momenta even at infinite \( N \).

The amplitudes will be \( U(1,1) \) invariant functions of the components of the external momenta \( k_i \). The metric is given by (13) and the corresponding components of the momenta \( k \) will be \( k_y, k_{\bar{y}}, k_z, k_{\bar{z}} \). The inner product will then be given by \( k_i \cdot k_j = \overline{k_j \cdot k_i} = k_{iy}k_{j\bar{y}} - k_{iz}k_{j\bar{z}} \). Following [1, 19] we introduce the kinematical
The propagator then becomes
\[ \Delta(k, -k) = \frac{1}{k \cdot k} = \frac{1}{2s_{kk}}. \] (22)

The tree level 3-point function receives contributions only from the cubic vertex of (3) and, on-shell, it is simply \( V_3 = g_{st} c_{13} \bar{a}_{13}. \) In [19], Parkes shows that this can be obtained from the usual cubic vertex in (3), which is \( a_{13} \bar{a}_{13}, \) by an \( SO(2,2) \) transformation. In our approach, since the \( \bar{y} \) and \( \bar{z} \) coordinates are "color" variables, Lorentz transformations are related to Lie algebra redefinitions of the field \( \omega, \) for example through commutators with \( q \) and \( p. \) For example, if we were using exactly the WZW model, the classical background would be a product of one anti-holomorphic term on the left and one holomorphic term on the right. We could choose to define the quantum fluctuations either on the left or right, or even in between these two terms. The resulting four-dimensional field theory would have looked different, and the different quantum fields would be related by Lie algebra operations which would be connected with Lorentz transformations. Of course, these quantum theories would be essentially equivalent.

We could show directly that the tree level on shell 4-point function vanishes as a consequence of the identity \([\bar{a}_{12}\bar{a}_{13}s_{23} + \bar{a}_{13}\bar{a}_{23}s_{12} - \bar{a}_{12}\bar{a}_{23}s_{13}] = 0\) valid on shell [3, 19]. However, we can instead use the results of the last section where we have shown that our model is a dimensional reduction of large \( N \) self-dual Yang-Mills theory. This property can be used to conclude about the tree level on-shell amplitudes of our model from those of self-dual Yang-Mills theory. The later represents (at finite \( N \)) a field

\[ c_{ij}^2 \]

The quantity \( c_{ij}^2 \) of [3] is given by \( a_{ij} \bar{a}_{ij}. \) When \( k_i, k_j \) and \( (k_i + k_j) \) are on-shell this becomes \( c_{ij}^2. \)
theory of open $\mathcal{N} = 2$ strings with Chan-Paton factors for the gauge group, say $U(N)$ \cite{footnote1}. These amplitudes are given in a factorized sum over non-cyclic permutations
\[ \sum_{\sigma} \text{Tr}(T_{\sigma_1}T_{\sigma_2}\cdots T_{\sigma_n})S_n(k_1,\ldots,k_n). \]

In momentum space, the reduction (19) is performed by identifying the conjugate momenta $k_q = k_1$ and $k_p = k_2$. At tree level, by momentum conservation at the vertices, if these relations are imposed on the external momenta they will be preserved throughout the Feynman graphs. Then, the vanishing of the open $\mathcal{N} = 2$ string amplitudes for $n \geq 4$, implies the vanishing of the corresponding $S_n$'s and also of the amplitudes for the model (3), indicating that it is indeed an appropriate field theory of the closed $\mathcal{N} = 2$ string, at least at tree level.

At loop level however, there are integrations of the momenta along the loops, and this argument does not apply. We therefore need a direct calculation. In the pure cubic action for the field theory of the closed $\mathcal{N} = 2$ string, the one-loop 3-point amplitude \cite{footnote2} is less infra-red divergent than the corresponding string amplitude, while it is also ultra-violet finite. In our case, with the action in (11), we will obtain similar results. However, as we mentioned before, the underlying $2 + 2$ structure, with two dimensions coming from color, may if further explored solve this problem.

At one-loop, the 3-point amplitude may receive contributions from several types of graphs. This is to be contrasted with the cubic theory where only one type of graph enters. Although, as we will see below, only this graph contributes also in our case, it is tempting to conjecture that the higher point vertices in (11) will be important to ensure good properties of the theory at more than one-loop level. Indeed, that is the case for the usual two-dimensional WZW model.

In our case, all the diagrams that could potentially give ultra-violet divergent contributions turn out to vanish due to various symmetries of the integrands. The only surviving term is the infra-red divergent one in the diagram containing just cubic
vertices, which is proportional to

$$g_{st}^3 (c_{13} \bar{a}_{13})^3 \int_0^\infty d\alpha_1 d\alpha_2 d\alpha_3 \frac{\alpha_1^2 \alpha_2^2 \alpha_3^2}{(\alpha_1 + \alpha_2 + \alpha_3)^8} \exp(- (\alpha_1 + \alpha_2 + \alpha_3) \beta) (\int d^4 p \exp(ip^2)).$$

(23)

where $\beta$ is a regulating parameter. We stress that one should be careful in interpreting the integrals in (23), since it is not clear which regularizing prescription to use, because of the peculiarities of the $(2,2)$ signature.

The Schwinger parameter integration in (23) gives an infra-red divergence of the form

$$\int_\varepsilon ds \frac{1}{s^3} \sim \frac{1}{\varepsilon^2}$$

where $\varepsilon$ is some infra-red cut-off. This is indeed equivalent to the result of [5] so that at one-loop this theory taken strictly at infinite $N$ gives the same result as the pseudodual cubic theory. However, one also has to note an infra-red divergence associated with the singularity due to the $(2,2)$ metric. In fact, the gaussian momentum integral in (23) is also divergent. In the present approach, two of the momentum components come from large $N$ color, $(k_q, k_p) = (2\pi n_q/N, 2\pi n_p/N)$. This defines a natural infra-red regulator $\varepsilon = 2\pi/N$. The transition from the sum over $n_q, n_p$ to the integral $\int dq dp$ then involves a factor of $N^2$ which is equivalent to $1/\varepsilon^2$. The total dependence on the infra-red regulator would then be $1/\varepsilon^2 \cdot 1/\varepsilon^2 = 1/\varepsilon^4$ which would agree with the $\mathcal{N} = 2$ string one. This suggests that a further study of the finite $N$ theory could be relevant for solving the puzzle of matching string and field theory amplitudes.

6 The Canonical Structure and the Current Algebra

In this section we want to find the canonical structure defined by the Lagrangian (14). We will obtain an infinite-dimensional current algebra of Poisson brackets sim-
ilar to the current algebra of two-dimensional sigma models. The action (14) has a
global symmetry given by $g \rightarrow gU$ where $U$ is an element of the group “$SU(\infty)$”.
The corresponding conserved currents are $Q = g^{-1}qg$ and $P = g^{-1}pg$ and obey the
conservation law
\[
\partial_+ P + \partial_- Q = 0 \tag{24}
\]
which is equivalent to the equation of motion (13). The currents $Q$ and $P$ also satisfy
a constraint $\{Q, P\} = 1$ where we use the ad-invariance of the Lie-Poisson bracket.
The equation (24) and the constraint can be directly related with those appearing in
$(2,2)$ self-dual gravity. Namely the Kahler form corresponding to the metric :
\[
\Omega = \partial_i \partial_j K dx^i \wedge dx^j
\]
where the $x^i$ are $y$ or $z$ ($(x^+, q, x^-, p) \leftrightarrow (\bar{y}, \bar{z})$) and $K$ is the Kahler potential
can be written locally as:
\[
\Omega = dP \wedge dz - dQ \wedge dy \tag{25}
\]
where $P, Q$ are functions of $(y, \bar{y}, z, \bar{z})$. The hermiticity of the form will imply that
the $dy \wedge dz$ term should cancel giving equation (24). The Ricci flatness of the Kahler
metric comes from the condition $\text{det}(g_{ij}) = -1$, where $g_{ij} = \partial_i \partial_j K$ and from (24) this
is exactly the constraint we have above.

We now wish to find an Hamiltonian formulation for (14) and the corres-
ponding Poisson brackets. The Poisson brackets of the theory can be obtained following
Witten’s method for finding the current algebra in the WZW model. We choose
light-cone quantization with $x^+$ playing the role of time. The only piece of the La-
grangian that contributes to the Poisson structure is the one containing derivatives
in $x^+$ which when varied with respect to $g$ gives
\[
\frac{1}{g_{st}^2} \int dx^+ dx^- \text{Tr}(\delta g g^{-1}\{p_+, \partial_+ gg^{-1}\}) = \frac{1}{g_{st}^2} \int dx^+ dx^- f_{abc} p^a (\delta g g^{-1})^c (\partial_+ gg^{-1})^b \tag{26}
\]
\[\text{We will denote these Poisson brackets for the field theory by } \{\ , \}_P \text{ to avoid confusion with the } su(\infty) \text{ Lie-Poisson bracket } \{\ , \}.\]
where $f_{abc}$ are the $sdiff(\Sigma)$ structure constants in some particular basis. The symplectic form will then be given by

$$F_{ab} = \frac{1}{g_{st}} f_{abc} p^c$$

(27)

where we raise and lower Lie algebra indices using the Killing metric as usual. The Poisson brackets of functions of $g$ will be given in terms of the inverse $F^{ab}$ of the matrix $F_{ab}$. In our case one just takes $\delta \Phi^a = (\delta g g^{-1})^a$ as coordinates of tangent vectors to the space of the fields $g$ and set [26]

$$\{X, Y\}_{PB} = \sum_{a,b} F^{ab} \frac{\delta X}{\delta \Phi^a} \frac{\delta Y}{\delta \Phi^b}. \quad (28)$$

We assume that $\Sigma$ is the plane$^{10}$ and take a basis of plane waves for $sdiff(\Sigma)$, that is for the Poisson algebra of functions on $\Sigma$, with $L_{zw} = \exp(izq + iwp)$ where $z, w$ are real indices$^{11}$ and $q, p$ are coordinates on $\Sigma$. The Poisson algebra becomes

$$\{L_a, L_b\} = -(a \times b)L_{a+b} \quad (29)$$

where the indices $a, b$ run over all pairs of real numbers and if $a = (z, w), b = (z', w')$ then $(a \times b) = zw' - z'w$. The Killing metric is given by $K_{ab} = \text{Tr}(L_a L_b) = (2\pi)^2 \delta^2(a + b)$ and the structure constants are

$$f_{abc} = -(2\pi)^2(a \times b)\delta^2(a + b + c)$$

which is totally antisymmetric as usual. The components of $p$ are given by Fourier transforming $p$ with the result $p^a = p^{(z,w)} \equiv p(z, w) = i\delta(z)\delta'(w)$. Then we have the Hamiltonian

$$H = -\frac{1}{g_{st}} \int dx - \text{Tr}(q g \partial_g^{-1}) \quad (30)$$

$^{10}$We expect this choice to be appropriate for the field theory of the uncompactified string. It would be interesting to discuss other possible choices for $\Sigma$ and to relate them to specific compactifications of the $\mathcal{N} = 2$ string.

$^{11}$Presumably to study the string compactified on $T^2$ we would take $\Sigma = T^2$ and would take only integer indices corresponding to periodic $q$ and $p$. 

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and the symplectic form (27) is given by
\[ F(z, w)(z', w') = (2\pi)^2 \frac{i}{g_{st}^2} (zw' - z'w)\delta(z + z')\delta(w + w') \]  
(31)
with inverse
\[ F(z, w)(z', w') = -i \frac{g_{st}^2}{(2\pi)^2} \frac{1}{zw' - z'w} \delta(z + z')\theta(w + w'). \]  
(32)

We can now easily compute the Poisson brackets of the current \( P \) with itself. We have the variation
\[ \delta P(z, w) = \delta \text{Tr}(P \exp(izq + iwp)) = g_{st}^2 \delta \Phi^a F_{ab}(g \cdot \exp(iqz + ipw) \cdot g^{-1})^b \]  
(33)
and inserting \( \delta P/\delta \Phi^a \) in the Poisson bracket (28) \( F_{ab} \) cancels against its inverse and we obtain at equal \( x^+ \)
\[ \{ P(z, w)(x^{-}), P(z', w')(x'^{-}) \}_PB = -g_{st}^2(zw' - z'w)P_{(z+z', w+w')}\delta(x^- - x'^-), \]  
(34)
which gives an infinite-dimensional current algebra that is the large \( N \) limit of the current algebra of the two-dimensional sigma model. In fact, we can also easily compute the Poisson brackets \( P \) with a function of the form \( O = g^{-1} \cdot f(q, p) \cdot g \) since the same cancellation between \( F_{ab} \) and \( F^{ab} \) will occur with the result
\[ \{ P(z, w)(x^{-}), O(z', w')(x'^{-}) \}_PB = -g_{st}^2(zw' - z'w)O_{(z+z', w+w')}\delta(x^- - x'^-). \]  
(35)

We can also insert the basis elements \( L_{zw}(q, p) \) in (35) and obtain the Poisson algebra for the matrix elements of the current with the operators \( O(x^{-}, q, p) \),
\[ \{ P(x^{-}, q, p), O(x'^{-}, q', p') \}_PB = \]
\[ = -g_{st}^2(\partial_q O \partial_p \delta(p - p')\delta(q - q') - \partial_p O \partial_q \delta(q - q')\delta(p - p')) \delta(x^- - x'^-) = \]
\[ = -g_{st}^2(O, \delta(q - q')\delta(p - p')) \delta(x^- - x'^-). \]  
(36)

We now introduce the functions \( M = g^{-1} \cdot ((q^2 + p^2)/2) \cdot g \) and observe that
\[ \{ Q, M \} = P \]
\[ \{ M, P \} = Q. \]  
(37)
From this one has $M = 1/2(Q^2 + P^2)$ and the action (14) can be written

$$S = \int dx^+ dx^- \frac{1}{2} \text{Tr}((Q^2 + P^2)(\partial_+ Q - \partial_- P))$$

(38)

which gives an Hamiltonian of the form

$$H = \int dx^- \text{Tr}(\partial_- M\{M, Q\})$$

(39)

and the Poisson structure

$$\{Q(x^-, q, p), M(x^-, q', p')\}_{PB} = \delta(q - q')\delta(p - p')\delta(x^- - x^-')$$

$$\{Q(x^-, q, p), Q(x^-, q', p')\}_{PB} = 0$$

$$\{M(x^-, q, p), M(x^-, q', p')\}_{PB} = 0$$

(40)

which together with the constraints (37) does produce the equations of motion

$$\partial_+ M = \{M, \partial_- M\} \quad \text{and} \quad \partial_+ Q = \{\partial_- Q, M\} + 2\{Q, \partial_- M\}$$

(41)

which are equivalent to (15). In this way we obtain a canonical formulation for self-dual gravity which is similar to the large $N$ limit of the canonical structure of the two-dimensional models.

7 Conclusions

In this paper we reviewed and further explored the formulation of the field theory of $\mathcal{N} = 2$ strings, i.e. self-dual gravity, as a large $N$ two-dimensional sigma model, both at the classical and one-loop quantum levels.

We have seen that in contrast with the usual purely cubic action for the theory, we obtained a field theory for an $Sdiff(\Sigma)$ group valued field $g = \exp(\omega)$ which contains an infinite series of higher point vertices for the Lie algebra valued field $\omega$. At the quantum level, and in particular beyond one-loop, we expect this theory to
have better properties than the cubic theory which is a pseudo-dual version \[22\]. This model can also be obtained as a dimensional reduction of large \(N\) self-dual Yang-Mills theory at large \(N\) and as such it is closely related to a conjecture of Ooguri and Vafa \[4\].

The mismatch between string and field theory one-loop amplitudes remains an open problem although we believe and have given indications that a complete treatment of the large \(N\) limit together with a proper definition of the finite \(N\) cutoff will help in solving this problem. As we mentioned in section 5, the structure of \(4 = 2 + 2\) with two of the four dimensions coming from large \(N\) color suggests by naive arguments that the carefully defined (in terms of powers of \(N\)) field theory amplitudes may coincide with the string amplitudes as desired. It is exciting to speculate that the partonic structure \[31\] present in the large \(N\) formulation of self-dual gravity ultimately manages to improve its UV properties.

In the second part of the paper we found an Hamiltonian formulation of the theory which gives a close analog of the Hamiltonian structure of two-dimensional sigma models. This Poisson structure results in an infinite-dimensional algebra of currents and represents a large \(N\) limit of the usual current algebras of two-dimensional sigma models. This construction gives an Hamiltonian approach to self-dual gravity which will be interesting to explore further in connection with matrix string theory, Hamiltonian and dimensional reductions and the study of Lorentz symmetries.

Other aspects of the problem that should be further studied are the inclusion of worldsheet \(U(1)\) instantons in the amplitudes \[4-9\] and the consideration of the compactified string which we expect will lead us to more interesting choices for \(\Sigma\).

Finally, we mention that the field theory of open \(\mathcal{N} = 2\) strings \[6\] should in a similar way be very naturally expressable as a large \(N\) two-dimensional sigma model. Indeed, Chan-Paton factors in the group \(G\) can be incorporated by using an
extension of $sdiff(\Sigma)$ by the Lie algebra of $G$ as in [18]. We also expect that related considerations, but in a some more non-trivial way, extend this type of construction to the heterotic string theory of [4, 12].

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