SOME CRITERIA FOR UNIFORM K-STABILITY

CHUYU ZHOU AND ZIQUAN ZHUANG

Abstract. We prove some criteria for uniform K-stability of log Fano pairs. In particular, we show that uniform K-stability is equivalent to \( \beta \)-invariant having a positive lower bound. Then we study the relation between optimal destabilization conjecture and the conjectural equivalence between uniform K-stability and K-stability in twisted setting.

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1. Introduction

K-stability is an important concept introduced in [Tia97] (and later algebraically reformulated in [Don02]) to test whether there is a KE metric on a projective Fano manifold (see in particular [Tia15, CDS15a, CDS15b, CDS15c]). However, it’s difficult to check K-stability of a Fano manifold and various equivalent but simpler criteria have been introduced in terms of special test configurations [LX14], valuations and filtrations [Fuj19, Li17] and stability thresholds (or \( \delta \)-invariants) [FO18, BJ17].

In this note, we give some more criteria for uniform K-stability from these perspectives. Since uniform K-stability has certain openness property, i.e. K-semistability is preserved after small perturbation of the boundary divisor (see [Fuj17b]), we first have the following criterion (note that the direction (1) \( \Rightarrow \) (2) has been known by [Fuj17b]).

Theorem 1.1 (\( = \)Theorem 3.1). Let \( (X, \Delta) \) be a log Fano pair. The following are equivalent:

1. \( (X, \Delta) \) is uniformly K-stable.
2. There exists a \( \epsilon > 0 \) such that \( (X, \Delta + \epsilon D) \) is K-semistable for any \( D \in | -K_X - \Delta|_\mathbb{R} \).

Our next criterion gives a way to test uniform K-stability using only \( \beta \)-invariant (see Section 2 for related definitions):

Theorem 1.2. Let \( (X, \Delta) \) be a log Fano pair. The following are equivalent:

1. \( (X, \Delta) \) is uniformly K-stable.
2. There exists \( \epsilon > 0 \) such that \( \beta_{X, \Delta}(E) \geq \epsilon \) for any divisor \( E \) over \( X \).
There exists $\epsilon > 0$ such that $\beta_{X,\Delta}(E) \geq \epsilon$ for any dreamy divisor $E$ over $X$.

(4) There exists $\epsilon > 0$ such that $\beta_{X,\Delta}(E) \geq \epsilon$ for any weakly special divisor $E$ over $X$.

It is well expected that K-stability is equivalent to uniform K-stability. This statement is proved to be equivalent to the existence of divisorsal valuation computing $\delta$-invariant when $\delta(X, \Delta) = 1$ (see Section 6). In [BLZ19], an algebraic twisted K-stability theory is developed to study $\mathbb{Q}$-Fano varieties that are not uniformly K-stable. We introduce concepts of twisted K-stability and twisted uniform K-stability and similarly expect they are equivalent. We then explore the relation between this equivalence and the existence of divisorial valuation computing $\delta$-invariant when $\delta(X, \Delta) < 1$.

In particular, we prove:

**Theorem 1.3** (Theorem 4.4). Let $(X, \Delta)$ be a log Fano pair with $\delta(X, \Delta) \leq 1$, then for any $0 < \mu < \delta(X, \Delta)$, $(X, \Delta)$ is $\mu$-twisted uniformly K-stable. Besides, $(X, \Delta)$ is $\delta(X, \Delta)$-twisted K-semistable but not $\delta(X, \Delta)$-twisted uniformly K-stable.

This is a refinement of the twisted valuative criterion established in [BLZ19]. Using this result, we establishes the equivalence between the existence of divisorial $\delta$-minimizer and the conjecture “K-stable = Uniformly K-stable” in the twisted setting.

**Theorem 1.4** (Theorem 5.4). For a log Fano pair $(X, \Delta)$ with $\delta(X, \Delta) \leq 1$, that twisted K-stable is equivalent to twisted uniformly K-stable, is equivalent to the existence of a divisorial valuation computing $\delta(X, \Delta)$.

The paper is organized as follows. In Section 2 we recall the notion and some preliminaries that will be used later. In Section 3 we prove the criteria for uniform K-stability, i.e. Theorems 1.1 and 1.2. In Section 4 we introduce the concept of twisted K-stability and twisted uniform K-stability and prove Theorem 1.3. In Section 5 we prove Theorem 1.4.

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## 2. Notion and preliminaries

We work over $\mathbb{C}$. We refer to [KM98, Kol13] for the definition of singularities of pairs. A projective normal variety $X$ is called $\mathbb{Q}$-Fano if $-K_X$ is an ample $\mathbb{Q}$-Cartier divisor and $X$ admits klt singularities. A pair $(X, \Delta)$ is called log Fano if $-K_X - \Delta$ is an ample $\mathbb{Q}$-Cartier divisor and $(X, \Delta)$ is klt. The $\mathbb{R}$-linear system of an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor $L$ is defined to be $|L|_\mathbb{R} = \{D \geq 0 \mid D \sim_\mathbb{R} L\}$. Similar one can define the $\mathbb{Q}$-linear system $|L|_\mathbb{Q}$ of a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor.

### 2.1. Test configurations.

Let $(X, \Delta)$ be a log Fano pair. A test configuration $(\mathcal{X}, \Delta_{tc}; \mathcal{L})$ of $(X, \Delta; -K_X - \Delta)$ consists of the following data:

1. A projective morphism $\pi : \mathcal{X} \to \mathbb{A}^1$ and an effective $\mathbb{Q}$-divisor $\Delta_{tc}$ on $\mathcal{X}$.
2. A relatively ample $\mathbb{Q}$-line bundle $\mathcal{L}$ on $\mathcal{X}$.
3. A $\mathbb{C}^*$-action on $(\mathcal{X}, \Delta_{tc}; r\mathcal{L})$ for some sufficiently divisible integer $r$ such that $(\mathcal{X}^*; \Delta_{tc}^*; r\mathcal{L}|_{\mathcal{X}^*})$ is $\mathbb{C}^*$-equivariantly isomorphic to $(X, \Delta; -r(K_X + \Delta)) \times (\mathbb{A}^1 \setminus 0)$ via the projection $\pi$, where $\mathcal{X}^* = \mathcal{X} \setminus \mathcal{X}_0$ and $\Delta_{tc}^* = \Delta_{tc}|_{\mathcal{X}^*}$.

Unless otherwise specified, all test configurations considered in this note are assumed to be normal, i.e. $\mathcal{X}$ is normal in the above definition. One can glue $(\mathcal{X}, \Delta_{tc})$ and $(X, \Delta) \times (\mathbb{P}^1 \setminus 0)$ along their common open subset $(X, \Delta) \times (\mathbb{A}^1 \setminus 0)$ to get a natural compactification $(\overline{\mathcal{X}}, \overline{\Delta_{tc}}; \overline{\mathcal{L}})$. A test
configuration \((X, \Delta_{tc}; L)\) is called special (resp. weakly special) if \((X, \Delta_{tc} + X_0)\) is plt (resp. lc) and \(L \sim_{Q} -K_{X/A^1} - \Delta_{tc}\).

A test configuration \((X, \Delta_{tc}; L)\) is trivial if it’s \(\mathbb{C}^*\)-equivariant to \(((X, \Delta) \times \mathbb{A}^1, -K_{X \times \mathbb{A}^1/A^1})\). It is said to be of product type if it’s induced by a diagonal \(\mathbb{C}^*\)-action on \((X, \Delta) \times \mathbb{A}^1\) given by a one parameter subgroup of \(\text{Aut}(X)\).

2.2. K-stability. Given a test configuration \((X, \Delta_{tc}; L)\) of an \(n\)-dimensional log Fano pair \((X, \Delta)\), its generalized Futaki invariant is defined as follows:

\[
\text{Fut}(X, \Delta_{tc}; L) := \frac{nL^{n+1}}{(n+1) (-K_X - \Delta)^n} + \frac{L^n \cdot (K_X \mathbb{P}^1 + \Delta_{tc})}{(-K_X - \Delta)^n}.
\]

We say \((X, \Delta)\) is K-semistable if \(\text{Fut}(X, \Delta_{tc}; L) \geq 0\) for any non trivial test configuration \((X, \Delta_{tc}; L)\). We say \((X, \Delta)\) is K-stable if \(\text{Fut}(X, \Delta_{tc}; L) > 0\) for any non-trivial test configuration \((X, \Delta_{tc}; L)\). We also say \((X, \Delta)\) is K-polystable if it is K-semistable and \(\text{Fut}(X, \Delta_{tc}; L) > 0\) for any non product type test configuration \((X, \Delta_{tc}; L)\).

To define uniform K-stability, we introduce J-invariant of a test configuration \((X, \Delta_{tc}; L)\) as follows [BHJ17, Fuj19]:

\[
J(X, \Delta_{tc}; L) := \frac{\Pi^*(-K_{X \mathbb{P}^1/A^1} - \Delta_{tc}) \cdot \Theta^*\mathbb{L}}{(-K_X - \Delta)^n} - \frac{\mathbb{L}^{n+1}}{(n+1)(-K_X - \Delta)^n},
\]

where \(\Pi : Z \to X \times \mathbb{P}^1\) and \(\Theta : Z \to X\) denote the normalization of the graph of \(X \times \mathbb{P}^1 \to X\).

We say \((X, \Delta)\) is uniformly K-stable if there is a positive number \(0 < \epsilon < 1\) such that \(\text{Fut}(X, \Delta_{tc}; L) \geq \epsilon J(X, \Delta_{tc}; L)\) for any non trivial test configuration.

2.3. Dreamy divisor and special divisor. In this subsection, we introduce two kinds of divisors which will appear frequently later.

Let \((X, \Delta)\) be a log Fano pair. We say \(E\) is a divisor over \(X\) if there is a birational model \(\sigma : Y \to X\) such that \(E\) is a prime divisor on \(Y\). If \(E \subset X\) we just let \(\sigma = \text{id}_X\).

Definition 2.1 ([Fuj19]). We say that \(E\) is a dreamy divisor if \(\text{ord}_E\) is a dreamy valuation over \(X\) if \(\bigoplus_{i,j \in \mathbb{N}} H^0(X, -ir\sigma^*(K_X + \Delta) - jE)\) is finitely generated, where \(r\) is a positive integer such that \(-r(K_X + \Delta)\) is Cartier.

Definition 2.2. We say that \(E\) is a (weakly) special divisor if \(\text{ord}_E\) is a (weakly) special valuation over \(X\) if it’s induced by a non-trivial (weakly) special test configuration \((X, \Delta_{tc}; L)\), i.e. \(\text{ord}_E\) is proportional to the restriction of \(\text{ord}_E\) (since \(\text{ord}_E\) is a divisorial valuation on the function field \(K(X) = K(X \times \mathbb{A}^1)\), we just restrict the valuation to \(K(X)\) to get a divisorial valuation over \(X\); see [BHJ17]).

We have the following characterization of dreamy divisors (see [Fuj19] Theorem 5.1 and [Fuj17a] Lemma 3.8).

Lemma 2.3. If \(E\) is a dreamy divisor over \(X\), then there is a test configuration \((X, \Delta_{tc}; L)\) whose central fiber is integral such that \(\text{ord}_E\) is proportional to the restriction of \(\text{ord}_{X_0}\). Conversely, if \((X, \Delta_{tc}; L)\) is a test configuration whose central fiber is integral, then the restriction of \(\text{ord}_{X_0}\) is a dreamy valuation over \(X\).

Remark 2.4. If \((X, \Delta_{tc}; L)\) is a test configuration whose central fiber is integral, then \(L \sim_{Q} -K_{X/A^1} - \Delta_{tc}\).
2.4. Various invariants. In this subsection, we recall the $\beta$-invariants and $\delta$-invariants of log Fano pairs.

**Definition 2.5** ([Fuj19]). Let $(X, \Delta)$ be a log Fano pair and $E$ a divisor over $X$. The $\beta$-invariant of $E$ (or ord$_E$) is defined as:

$$\beta_{X,\Delta}(E) := A_{X,\Delta}(E) - \frac{1}{(-K_X - \Delta)^n} \int_0^\infty \text{vol}(-K_X - \Delta - xE)dx.$$ 

Note that the above definition differs from Fujita's original definition by a multiple. We also write

$$S_{X,\Delta}(E) := \frac{1}{(-K_X - \Delta)^n} \int_0^\infty \text{vol}(-K_X - \Delta - xE)dx$$

and let $T_{X,\Delta}(E)$ be the pseudo-effective threshold of $-E$ with respect to $-K_X - \Delta$, i.e.

$$T_{X,\Delta}(E) = \sup\{x \in \mathbb{R}^+ | \text{vol}(-K_X - \Delta - xE) > 0\}.$$ 

Finally we let $j_{X,\Delta}(E) = T_{X,\Delta}(E) - S_{X,\Delta}(E)$.

**Remark 2.6.** We have the following relation between $S_{X,\Delta}(E)$ and $T_{X,\Delta}(E)$ (see e.g. [BJ17, Lemma 2.6]):

$$\frac{1}{n+1} T_{X,\Delta}(E) \leq S_{X,\Delta}(E) \leq \frac{n}{n+1} T_{X,\Delta}(E).$$

It then follows that

$$\frac{1}{n} S_{X,\Delta}(E) \leq j_{X,\Delta}(E) \leq n S_{X,\Delta}(E).$$

$\beta$-invariant has a close relation to K-stability, as discovered in [Fuj19] and [Li17] (see also [BX18] for part of the statement):

**Theorem 2.7.** Let $(X, \Delta)$ be a log Fano pair. The following are equivalent:

1. $(X, \Delta)$ is $K$-semistable (resp. $K$-stable, uniformly $K$-stable).
2. $\beta_{X,\Delta}(E) \geq 0$ (resp. $> 0$, $\geq \epsilon j_{X,\Delta}(E)$ for some fixed $\epsilon > 0$) for any divisorial valuation ord$_E$ over $X$.
3. $\beta_{X,\Delta}(E) \geq 0$ (resp. $> 0$, $\geq \epsilon j_{X,\Delta}(E)$ for some fixed $\epsilon > 0$) for any dreamy divisorial valuation ord$_E$ over $X$.
4. $\beta_{X,\Delta}(E) \geq 0$ (resp. $> 0$, $\geq \epsilon j_{X,\Delta}(E)$ for some fixed $\epsilon > 0$) for any special divisorial valuation ord$_E$ over $X$.

The following $\delta$-invariant is introduced by [FO18] to characterize K-stability.

**Definition 2.8.** Let $(X, \Delta)$ be a log Fano pair and $E$ a divisor over $X$. We set

$$\delta_m(X, \Delta) := \sup \{ a \in \mathbb{R}^+ | (X, \Delta + aD_m) \text{ is lc for any } m\text{-basis type divisor } D_m \},$$

and

$$\delta(X, \Delta) := \limsup_m \delta_m(X, \Delta).$$

By [BJ17], the above limsup is in fact a limit and we have

$$\delta(X, \Delta) = \inf_E \frac{A_{X,\Delta}(E)}{S_{X,\Delta}(E)},$$

where the infimum runs over all divisorial valuations $E$ over $X$.

Parallel to Theorem 2.7 we have ([FO18], [BJ17], [Fuj19]):

**Theorem 2.9.** Let $(X, \Delta)$ be a log Fano pair. The following are equivalent:

1. $(X, \Delta)$ is $K$-semistable (resp. $K$-stable, uniformly $K$-stable).
For any divisorial valuation \( \text{ord}_E \) over \( X \).

Proof. i.e. \( D \) defined as follows: where \( E \) runs through all divisors over \( X \).

Suppose \( \text{ord}_E \) over \( X \).

Assume (2) holds, then \( \beta_{X,\Delta+\epsilon D}(E) \geq 0 \) for all \( D \in |-K_X-\Delta|_R \) and all divisors \( E \) over \( X \). Taking the supremum over \( D \) we have

\[
A_{X,\Delta}(E) - \epsilon T_{X,\Delta}(E) - (1-\epsilon)S_{X,\Delta}(E) \geq 0,
\]

i.e.

\[
A_{X,\Delta}(E) - S_{X,\Delta}(E) \geq \epsilon \cdot (T_{X,\Delta}(E) - S_{X,\Delta}(E)) = \epsilon \cdot j_{X,\Delta}(E),
\]

which implies (1).

Conversely, suppose \( (X,\Delta) \) is uniformly K-stable, then by Theorem 2.7, there exists some \( \mu \) with \( 0 < \mu < 1 \), such that \( \frac{A_{X,\Delta}(E)}{S_{X,\Delta}(E)} \geq 1 + \mu \) for any divisorial valuation \( \text{ord}_E \) over \( X \). By Remark 2.6 we can choose a \( 0 < \epsilon < 1 \) such that

\[
(1 + \mu) S_{X,\Delta}(E) \geq \epsilon \cdot T_{X,\Delta}(E) + (1-\epsilon)S_{X,\Delta}(E),
\]

say \( \epsilon = \frac{\mu}{n+1} \). Thus

\[
\beta_{X,\Delta+\epsilon D}(E) = A_{X,\Delta}(E) - \epsilon \cdot \text{ord}_E(D) - (1-\epsilon)S_{X,\Delta}(E)
\geq A_{X,\Delta}(E) - \epsilon \cdot T_{X,\Delta}(E) - (1-\epsilon)S_{X,\Delta}(E) \geq 0
\]

for any \( D \in |-K_X-\Delta|_R \). So, \( (X,\Delta+\epsilon D) \) is K-semistable by Theorem 2.7(1). \( \square \)

Inspired by the above Theorem 2.7, we can define a new invariant for a log Fano pair \( (X,\Delta) \), the uniformity of \( (X,\Delta) \), which characterizes how uniformly K-stable \( (X,\Delta) \) is.

Definition 3.2. Suppose \( (X,\Delta) \) is a given K-semistable log Fano pair. The uniformity of \( (X,\Delta) \) is defined as follows:

\[
u(X,\Delta) := \sup \{ a \in \mathbb{R}_{\geq 0} | (X,\Delta + aD) \text{ is K-semistable, } \forall D \in |-K_X-\Delta|_R \}.
\]

We can give a precise characterization for \( \nu(X,\Delta) \).

Proposition 3.3. Let \( (X,\Delta) \) be a K-semistable klt log Fano pair, then

\[
u(X,\Delta) = \inf_{E} \frac{\beta_{X,\Delta}(E)}{j_{X,\Delta}(E)},
\]

where \( E \) runs through all divisors over \( X \).
Proof. Suppose \( a \) is a nonnegative real number such that \((X, \Delta + aD)\) is K-semistable for any \( D \in |-K_X - \Delta|_\mathbb{R} \), then we have
\[
\beta_{X, \Delta + aD}(E) = A_{X, \Delta}(E) - \text{ord}_E(aD) - (1 - a)S_{X, \Delta}(E) \geq 0,
\]
\( \forall D \in |-K_X - \Delta|_\mathbb{R} \) and \( \forall E \) over \( X \). This is equivalent to
\[
A_{X, \Delta}(E) - S_{X, \Delta}(E) \geq a(T_{X, \Delta}(E) - S_{X, \Delta}(E))
\]
for any \( E \) over \( X \), i.e.
\[
a \leq \inf_E \frac{\beta_{X, \Delta}(E)}{J_{X, \Delta}(E)}.
\]

By Theorem 3.1, we have the following corollary:

**Corollary 3.4.** Suppose \((X, \Delta)\) is a K-semistable log Fano pair. Then

1. \((X, \Delta)\) is uniformly K-stable if and only if \( u(X, \Delta) > 0 \).
2. \( \delta(X, \Delta) = 1 \) if and only if \( u(X, \Delta) = 0 \).

**Remark 3.5.** By Remark 2.6, we have the following relation between \( u(X, \Delta) \) and \( \delta(X, \Delta) - 1 \):
\[
\frac{1}{n}(\delta(X, \Delta) - 1) \leq u(X, \Delta) \leq n(\delta(X, \Delta) - 1).
\]

Theorem 2.7, Theorem 2.9 and Theorem 3.1 give three characterizations of uniform K-stability. We now give another criterion using only \( \beta \)-invariant.

**Theorem 3.6.** Let \((X, \Delta)\) be a log Fano pair. The following three are equivalent:

1. \((X, \Delta)\) is uniformly K-stable.
2. There exists \( \epsilon > 0 \) such that \( \beta_{X, \Delta}(E) \geq \epsilon \) for any divisor \( E \) over \( X \).
3. There exists \( \epsilon > 0 \) such that \( \beta_{X, \Delta}(E) \geq \epsilon \) for any dreamy divisor \( E \) over \( X \).

**Proof.** (1) \( \Rightarrow \) (2): If \((X, \Delta)\) is uniformly K-stable, then there exists some \( \delta > 1 \) such that \( A_{X, \Delta}(E) \geq \delta \cdot S_{X, \Delta}(E) \) for all divisor \( E \) over \( X \). Since \((X, \Delta)\) is log Fano, we have \( A_{X, \Delta}(E) \geq \frac{1}{r} \) where \( r \) is an integer such that \( r(K_X + \Delta) \) is Cartier. Thus
\[
\beta_{X, \Delta}(E) = A_{X, \Delta}(E) - S_{X, \Delta}(E) \geq (1 - \delta^{-1})A_{X, \Delta}(E) \geq \frac{1 - \delta^{-1}}{r}
\]
for any divisor \( E \) over \( X \) and we may simply take \( \epsilon = 1 - \frac{1}{\delta} \).

(2) \( \Rightarrow \) (1): Suppose that \( \beta_{X, \Delta}(E) \geq \epsilon > 0 \) for all divisor over \( X \). By [BJ17, Corollary 3.6], there exists a sequence \( c_m \) (\( m = 1, 2, \cdots \)) of numbers depending only on \((X, \Delta)\) such that \( \lim_{m \to \infty} c_m = 1 \) and \( c_m \cdot \text{ord}_E(D_m) \leq S_{X, \Delta}(E) \) for any \( m \in \mathbb{N} \), any divisor \( E \) over \( X \) and all \( m \)-basis type divisor \( D_m \sim \mathbb{Q} - (K_X + \Delta) \). It follows that
\[
A_{X, \Delta + c_m D_m}(E) = A_{X, \Delta}(E) - c_m \cdot \text{ord}_E(D_m) \geq A_{X, \Delta}(E) - S_{X, \Delta}(E) = \beta_{X, \Delta}(E) \geq \epsilon
\]
for all \( m, E \) and \( D_m \) as above. In other words, the pair \((X, \Delta + c_mD_m)\) is lc. By [Bir16b, Theorem 1.6] (applied to the pair \((X, B = \Delta + c_mD_m), M = D_m\) and the very ample divisor \( A = -r(K_X + \Delta) \)) for sufficiently large and divisible \( r \), there exists some \( t > 0 \) depending only on \((X, \Delta)\) such that \( \text{let}(X, B; D_m) \geq t \) for all \( m \) and \( D_m \). Hence \((X, \Delta + (c_m + t)D_m)\) is lc for all \( m \in \mathbb{N} \) and all \( m \)-basis type divisor \( D_m \), which implies \( \delta_m(X, \Delta) \geq c_m + t \). Letting \( m \to \infty \) we see that \( \delta(X, \Delta) \geq 1 + t > 1 \) and therefore \((X, \Delta)\) is uniformly K-stable.

(3) \( \Leftrightarrow \) (2): One direction is obvious. For the other direction, note that by Theorem 2.7, (3) implies that \((X, \Delta)\) is K-semistable, hence it suffices to show that if \((X, \Delta)\) is a K-semistable log Fano pair, then any divisor \( E \) over \( X \) for which \( \beta_{X, \Delta}(E) < 1 \) is dreamy. This is proved in Lemma 3.8. \( \Box \)
Lemma 3.7. Let \((X, \Delta)\) be a K-semistable log Fano pair and \(E\) a divisor over \(X\). Suppose that \(\beta_{X, \Delta}(E) < 1\). Then \(E\) is dreamy.

Proof. By [BL17] Lemma 3.5 and Corollary 3.6, there exists \(m\)-basis type divisors \(D_m \sim \mathbb{Q} - (K_X + \Delta)\) \((m \in \mathbb{N})\) such that \(\operatorname{ord}_E(D_m) \to S_X, \Delta(E)\) \((m \to \infty)\). Let \(\lambda_m = \min\{\delta_m(X, \Delta), 1\}\). Since \((X, \Delta)\) is K-semistable, we have \(\lim_{m \to \infty} \lambda_m = 1\) and \((X, \Delta + \lambda_m D_m)\) is lc for all \(m \in \mathbb{N}\). Then as

\[
A_{X, \Delta + \lambda_m D_m}(E) = A_{X, \Delta}(E) - \lambda_m \operatorname{ord}_E(D_m) \to \beta_{X, \Delta}(E) < 1 \quad (m \to \infty),
\]

we see that \(A_{X, \Delta + \lambda_m D_m}(E) < 1\) for \(m > 0\). By [BCHM10 Corollary 1.4.3], one can extract \(E\) as a prime divisor on a Fano type variety and in particular \(E\) is dreamy.

\[
\square
\]

In general, there are many dreamy divisors over a log Fano pair. We now show that those with small \(\beta\)-invariants are weakly special. In particular, combining with Theorem 3.6, this completes the proof of Theorem 1.2.

Theorem 3.8. Let \((X, \Delta)\) be a K-semistable log Fano pair. Then there exists some \(0 < \epsilon_0 \ll 1\) such that any dreamy divisor \(E\) over \(X\) with \(\beta_{X, \Delta}(E) < \epsilon_0\) induces a weakly special test configuration of \((X, \Delta)\) with integral central fiber.

Proof. Let \(\mathcal{R} \subset [0, 1]\) be a finite set of rational numbers containing 1 and all finite sums of the coefficients of \(\Delta\). Choose \(\epsilon_0 \in \mathbb{Q} \cap (0, 1)\) such that a pair \((Y, B + G)\) (where \(G\) is a reduced divisor and \(\dim Y \leq \dim X + 1\)) is lc as long as \((Y, B + (1 - \epsilon_0)G)\) is lc and the coefficients of \(B\) belongs to \(\mathcal{R}\). Such \(\epsilon_0\) exists by the ACC of log canonical threshold [HM14]. Suppose \(E\) is a divisor over \(X\) with \(\beta_{X, \Delta}(E) < \epsilon_0\), then similar to the proof of Lemma 3.7 we can find a \(D \in -K_X - \Delta|_X\) such that \((X, \Delta + D)\) is klt and \(A_{X, \Delta + D}(E) < \epsilon_0\). By [BCHM10 Corollary 1.4.3], one can extract \(E\) on a birational model of \(X\), say \(\mu : Y \to X\) and

\[
K_Y + \tilde{D} + \tilde{\Delta} + cE = \mu^*(K_X + \Delta + D),
\]

where \(\tilde{D}\) and \(\tilde{\Delta}\) are strict transformation of \(D\) and \(\Delta\) respectively and \(1 - \epsilon_0 < c < 1\). Note that \(Y\) is of Fano type. Consider the pair \((X_{\mathbb{A}^1}, \Delta_{\mathbb{A}^1} + D_{\mathbb{A}^1} + X_0)\) \((where \(X_{\mathbb{A}^1} = X \times \mathbb{A}^1\), etc. and \(X_0 = X \times \{0\}\)) which is a plt pair. Then there is an induced morphism \(\mu_{\mathbb{A}^1} : Y_{\mathbb{A}^1} \to X_{\mathbb{A}^1}\). Let \(v\) be a quasi-monodial valuation over \(X_{\mathbb{A}^1}\) with weight \((1, 1)\) along the divisors \(X_0\) and \(E_{\mathbb{A}^1}\). It’s clear that \(v\) is a divisorial valuation over \(X_{\mathbb{A}^1}\) whose center is contained in \(X_0\). Denote by \(\mathcal{E}\) the corresponding divisor over \(X_{\mathbb{A}^1}\), then \(A_{X_{\mathbb{A}^1}, \Delta_{\mathbb{A}^1} + D_{\mathbb{A}^1} + X_0}(\mathcal{E}) = A_{X, \Delta + D}(E) < \epsilon_0 < 1\), hence by [BCHM10 Corollary 1.4.3] we can extract \(\mathcal{E}\) on a projective birational model \(\pi : Y' \to X_{\mathbb{A}^1}\) of \(X_{\mathbb{A}^1}\). We have

\[
K_{Y'} + \pi_*^{-1} \Delta_{\mathbb{A}^1} + \pi_*^{-1} D_{\mathbb{A}^1} + \tilde{X}_0 + cE = \pi^*(K_{X_{\mathbb{A}^1}} + \Delta_{\mathbb{A}^1} + D_{\mathbb{A}^1} + X_0),
\]

where \(\tilde{X}_0\) is the strict transformation of \(X_0\), \(c > 1 - \epsilon_0\) and \(Y'\) is of Fano type over \(\mathbb{A}^1\). Run the \(\tilde{X}_0\)-MMP/\(\mathbb{A}^1\) on \(Y'\), we get a minimal model \(Y' \to Y'\), and \(\tilde{X}_0\) is contracted by the negativity lemma (see e.g. [KM98 Lemma 3.39]). Let \(\Delta'_{\mathbb{A}^1}, D'_{\mathbb{A}^1}\) and \(\mathcal{E}'\) be the pushforward of \(\pi_*^{-1} \Delta_{\mathbb{A}^1}, \pi_*^{-1} D_{\mathbb{A}^1}\) and \(\mathcal{E}\) on \(Y'\) respectively, then we know \(Y' \to \mathbb{A}^1\) has an integral central fiber and the restriction of \(\operatorname{ord}_{Y'}\) is exactly \(\operatorname{ord}_{E'}\). Let \((X', \Delta_{tc}, \mathcal{E})\) be the test configuration induced by the dreamy divisor \(E\), then \(X'\) is the ample model of \(Y'\) over \(\mathbb{A}^1\) with respect to \(- (K_{Y'} + \Delta'_{\mathbb{A}^1})\). As \((Y', \pi_*^{-1} \Delta_{\mathbb{A}^1} + \pi_*^{-1} D_{\mathbb{A}^1} + c\mathcal{E}')\) is a klt Calabi-Yau pair (pullback of \((X_{\mathbb{A}^1}, \Delta_{\mathbb{A}^1} + D_{\mathbb{A}^1})\)), we know that \((Y', \Delta'_{\mathbb{A}^1} + D'_{\mathbb{A}^1} + c\mathcal{E}')\) is also klt and the same holds for its strict transform on \(X'\). It follows that \((X', \Delta_{tc} + cX_0)\) is a klt pair. As \(c > 1 - \epsilon_0\), we see that \((X', \Delta_{tc} + X_0)\) is lc by our choice of \(\epsilon_0\).

\[
\square
\]

Remark 3.9. The above theorem says the following two statements are equivalent:

1. \((X, \Delta)\) is uniformly K-stable.
2. There is a \(\epsilon > 0\) such that \(\beta_{X, \Delta}(E) \geq \epsilon\) for any weakly special divisor \(E\) over \(X\).
Compared with Theorems 2.7 and 2.9, one would expect that for uniform K-stability it’s sufficient to check $\beta(E) \geq \epsilon$ for all special divisors $E$ over $X$, although this doesn’t seem to follow from our current proof.

It’s expected that uniformly K-stable and K-stable are the same for any given log Fano pair. One direction is clear. Assume $(X, \Delta)$ is K-stable, to confirm uniform K-stability, it suffices to show that there is an $\epsilon > 0$ such that $\beta_{X, \Delta}(E) > \epsilon$ for any weakly special divisor $E$ over $X$. Let $c_0$ be as in the proof of Theorem 3.8. Our next result (inspired by the recent work Xu19) shows that it suffices to consider those $E$ that are bounded in some sense (note that a more general version that applies to all weakly special divisor is independently proved in BLX19 Theorem A.2 using a somewhat different method):

**Theorem 3.10.** Let $(X, \Delta)$ be a K-semistable log Fano pair. If $E$ is a divisor over $X$ with $\beta_{X, \Delta}(E) < c_0$, then we can find a $G \in \frac{1}{N}| - N(K_X + \Delta)|$ such that $E$ is a lc place of $(X, \Delta + G)$. Here $N$ is a positive integer number which only depends on $(X, \Delta)$.

**Proof.** As in the proof of Theorem 3.8, we can find a $D \in | - K_X - \Delta|_\mathbb{Q}$ such that $(X, \Delta + D)$ is lc and $A_{X, \Delta + D}(E) < c_0$. In addition, we can extract $E$ to be a divisor on a projective birational model of $X$, say $\mu : Y \to X$ and

$$K_Y + \Delta + \tilde{D} + cE = \mu^*(K_X + \Delta + D),$$

where $\Delta$ and $\tilde{D}$ are the strict transformations and $1 - c_0 < c < 1$. Note that $Y$ is of Fano type, then we can run MMP for $-(K_Y + \Delta + E)$. Suppose we get a Mori fiber space $Y \dasharrow Y' \to T$ and write $\Delta'$ and $E'$ for the pushforward of $\Delta$ and $E$ on $Y'$, then we know $(K_Y' + \Delta' + E')_T$ is ample where $F$ is the general fiber of $Y' \to T$. As $(Y, \Delta + \tilde{D} + cE)$ is a klt Calabi-Yau pair, so is $(Y', \Delta' + \tilde{D}' + cE')$. It follows that $(K_Y' + \Delta' + cE')_F$ is anti-nef and $(Y', \Delta' + cE')$ is klt, thus $(Y', \Delta' + E')$ is lc by the choice of $c_0$. But this contradicts [HMX14 Theorem 1.5]. So the MMP above produces a minimal model $Y \to Y'$ and $-(K_{Y'} + \Delta' + E')$ is semiample.

Now by the boundedness of complement [Bir16a Theorem 1.7], there exists some integer $N > 0$ depending only on the dimension and the set $\mathcal{S}$ such that if $(Y', \Delta' + E')$ is an lc pair of dimension $n$ with coefficients in $\mathcal{S}$, $Y'$ is of Fano type and $-(K_{Y'} + \Delta' + E')$ is nef, then there exists some effective divisor $G' \in \frac{1}{N}| - N(K_{Y'} + \Delta' + E')|$ such that $(Y, \Delta' + E' + G')$ is lc. It follows that $E$ is a lc place of the lc pair $(X, \Delta + G)$ where $G \in \frac{1}{N}| - N(K_X + \Delta)|$ is the pushforward of $G'$ to $X$. □

It is therefore very natural to ask the following question:

**Question 3.11.** Given a set $S$ of lc log Calabi-Yau pairs $(X, \Delta + D)$ such that $(X, \Delta)$ is log Fano. Let $S'$ be the set of lc log Calabi-Yau pairs that can be realized as special degenerations of pairs in $S$. Assume that $S$ is bounded. Is $S'$ bounded?

In particular, a positive answer to this question will lead to a proof that K-stability is equivalent to uniform K-stability (since the Futaki invariants have a bounded denominator in a bounded family). We don’t know any proof or counterexample to the above question.

**Remark 3.12.** Theorem 3.10 also gives an approximation for $\delta(X, \Delta) = 1$ using lc places of bounded lc complements, i.e. if $\delta(X, \Delta) = 1$, then $\delta(X, \Delta) = \inf E \frac{A_{X, \Delta + E}}{\delta_{X, \Delta}(E)}$, where $E$ is a lc place of $(X, \Delta + G)$ for some lc $N$-complement $G$ of $(X, \Delta)$. See [BLX19 Corollary 3.6] for a more general statement when $\delta(X, \Delta) \leq 1$.

4. Twisted setting

In this section, we will define K-stability in the twisted setting. To make it simple, we leave out the boundary as it doesn’t play essential roles. $X$ always denotes a $\mathbb{Q}$-Fano variety with $\delta(X) \leq 1$.

We first recall the definition of twisted K-stability [Der16, BLZ19].
\textbf{Definition 4.1.} Let \((\mathcal{X}, \mathcal{L})\) be a given normal test configuration of \(X\), \(0 < \mu \leq 1\), then \(\mu\)-twisted generalized Futaki invariant is defined to be

\[ \text{Fut}_{1-\mu}(\mathcal{X}, \mathcal{L}) := \sup_{D \in |-K_X|_0} \text{Fut}(\mathcal{X}, (1 - \mu)D; \mu\mathcal{L}) \]

where \(D\) is closure of \(D \times (\mathbb{A}^1 \setminus 0)\) in \(\mathcal{X}\).

\textbf{Definition 4.2.} \begin{enumerate}
\item We say \(X\) is \(\mu\)-twisted K-semistable if \(\text{Fut}_{1-\mu}(\mathcal{X}, \mathcal{L}) \geq 0\) for every normal test configuration \((\mathcal{X}, \mathcal{L})\).
\item We say \(X\) is \(\mu\)-twisted K-stable if \(\text{Fut}_{1-\mu}(\mathcal{X}, \mathcal{L}) > 0\) for every non-trivial normal test configuration \((\mathcal{X}, \mathcal{L})\).
\item We say \(X\) is \(\mu\)-twisted uniformly K-stable if there exists a positive real number \(\epsilon > 0\) such that \(\text{Fut}_{1-\mu}(\mathcal{X}, \mathcal{L}) \geq \epsilon J(\mathcal{X}, \mathcal{L})\) for every normal test configuration \((\mathcal{X}, \mathcal{L})\).
\end{enumerate}

In the above definition, one should check all normal test configurations to test twisted K-stability. However, by a special test configuration theory in twisted setting that has been established in [BLZ19, Theorem 1.6] which is parallel to [LX14], we have the following theorem:

\textbf{Theorem 4.3.} To test \(\mu\)-twisted K-semistability (resp. K-stability and uniform K-stability), it suffices to check all special test configurations.

While \(X\) may not be K-semistable, it can still be K-stable in the twisted sense [BLZ19]. The following result is a refinement of the twisted valuative criterion established in [BLZ19, Theorem 1.5].

\textbf{Theorem 4.4.} Let \(X\) be a \(\mathbb{Q}\)-Fano variety with \(\delta(X) \leq 1\), then \(X\) is \(\mu\)-twisted uniformly K-stable for \(0 < \mu < \delta(X)\), and \(X\) is \(\mu\)-twisted K-semistable but not \(\mu\)-twisted uniformly K-stable for \(\mu = \delta(X)\).}

\textbf{Proof.} For \(\mu < \delta(X)\), by [BL18, Theorem C], there is a \(D \in |-K_X|_\mathbb{Q}\) such that \((X, (1 - \mu)D)\) is uniformly K-stable. Thus there is a positive real number \(0 < \epsilon < 1\) such that

\[ \text{Fut}(\mathcal{X}, (1 - \mu)D; \mathcal{L}) \geq \epsilon J(\mathcal{X}, \mathcal{L}) \]

for any normal test configuration, so one has

\[ \text{Fut}_{1-\mu}(\mathcal{X}, \mathcal{L}) \geq \epsilon J(\mathcal{X}, \mathcal{L}) \]

For \(\mu = \delta(X)\), we can choose a sequence of special test configurations \((\mathcal{X}_i, \mathcal{L}_i)\) such that \(\lim_{n \to \infty} A(v_{X_i,0}) = \delta(X)\) [BLZ19, Theorem 4.3]. We aim to prove that \(X\) is not \(\delta\)-twisted uniformly K-stable where \(\delta = \delta(X)\). If not, there is a positive real number \(0 < \epsilon < 1\) such that

\[ \text{Fut}_{1-\delta}(\mathcal{X}_i, \mathcal{L}_i) \geq \epsilon J(\mathcal{X}_i, \mathcal{L}_i) \]

One can choose a general \(D \in |-K_X|_\mathbb{Q}\) such that

\[ \text{Fut}_{1-\delta}(\mathcal{X}_i, \mathcal{L}_i) = \text{Fut}(\mathcal{X}_i, (1 - \delta)D; \mathcal{L}_i) \]

for any \(i\), where \(D\) doesn’t contain any center of \(v_{X_i,0}\) [BLZ19, Theorem 3.7]. Thus one obtain

\[ \text{Fut}_{1-\delta}(\mathcal{X}_i, \mathcal{L}_i) = A(v_{X_i,0}) - \delta S(v_{X_i,0}) \geq \epsilon J(\mathcal{X}_i, \mathcal{L}_i) = \epsilon(T(v_{X_i,0}) - S(v_{X_i,0})), \]

which contradicts \(\lim_{n \to \infty} \frac{A(v_{X_i,0})}{S(v_{X_i,0})} = \delta(X)\). \qed
5. Optimal Destabilization Conjecture

It has long been expected that uniform K-stability is equivalent to K-stability. In [BX18], they reduced the problem to the existence of divisorial δ-minimizer for δ(X) = 1, that is, the divisorial valuation computing δ-invariant. The algebraic twisted K-stability theory has been established to study K-unstable Fano varieties [BLZ19], then the case δ < 1 can be studied in parallel to the case δ = 1. In this section, we will explain the relation between the following two conjectures.

Conjecture 5.1. (Optimal Destabilization Conjecture) Let X be a Q-Fano variety with δ(X) ≤ 1, then there exists a divisor E over X computing δ(X), i.e. \( \frac{A(E)}{S(E)} = \delta(X) \).

Conjecture 5.2. Let X be a Q-Fano variety with δ(X) ≤ 1, and 0 < µ ≤ 1, then X is µ-twisted K-stable is equivalent to that X is µ-twisted uniformly K-stable.

By Theorem 5.4, we know they are also equivalent.

The above two conjectures are equivalent by following result:

Theorem 5.4. Conjecture 5.1 is equivalent to Conjecture 5.2.

Proof. We first assume Conjecture 5.1, i.e. there is a divisor E computing δ = δ(X), then by [BLZ19] Theorem 1.1, E is a dreamy divisor over X which naturally induces a non-trivial test configuration \((X, \mathcal{L})\) such that \( \text{Fut}_1(X, \mathcal{L}) = 0 \), thus X is not δ-twisted K-stable. Conversely, assume X is not δ-twisted K-stable, then there exists a non-trivial test configuration \((X, \mathcal{L})\) such that \( \text{Fut}_1(X, \mathcal{L}) = 0 \). By [BLZ19] Theorem 3.9, it must be a special test configuration whose central fiber induces a divisorial valuation computing δ(X).

We can also translate optimal destabilization conjecture into vanishing of δ-twisted generalized Futaki invariant [BLZ19].

Theorem 5.5. Suppose X is a klt Fano variety with δ(X) ≤ 1. If there is a divisor E over X computing δ(X), i.e \( \frac{A(E)}{S(E)} = \delta(X) \), then there is a test configuration \((X, \mathcal{L})\) such that \( \text{Fut}_1(X, \mathcal{L}) = 0 \). Conversely, if there is a test configuration \((X, \mathcal{L})\) such that \( \text{Fut}_1(X, \mathcal{L}) = 0 \), then there is a divisor E over X computing δ(X).

Proof. Suppose there is a divisor E computing δ(X), then E is a dreamy divisor which naturally induces a test configuration whose δ-twisted generalized Futaki is zero, by [BLZ19] Theorem 1.1. Conversely, if there is a test configuration whose δ-twisted generalized Futaki is zero, then it must be a special test configuration whose central fiber induces a divisorial valuation computing δ(X), by [BLZ19] Theorem 4.6.

Remark 5.6. The first two conjectures in this section for δ(X) = 1 correspond to the following two conjectures:

1. (Optimal Destabilization Conjecture for δ = 1) Suppose X is a Q-Fano variety with δ(X) = 1, then there is a divisorial valuation ord_E computing δ(X), i.e. \( \delta(X) = \frac{A(E)}{S(E)} = 1 \).

2. For Fano varieties, uniform K-stability is equivalent to K-stability.

By Theorem 5.4 we know they are also equivalent.

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BEIJING INTERNATIONAL CENTER FOR MATHEMATICAL RESEARCH, PEKING UNIVERSITY, BEIJING 100871, CHINA
E-mail address: chuyuzhou@pku.edu.cn

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, NJ 08544, USA.
E-mail address: zzhuang@math.princeton.edu