Hydra games for recursively Mahlo operations

Toshiyasu Arai

Graduate School of Science, Chiba University
1-33, Yayoi-cho, Inage-ku, Chiba, 263-8522, JAPAN
tosarai@faculty.chiba-u.jp

Abstract

Encouraged by W. Buchholz [7], a hydra game is proposed, and the fact that every hydra eventually die out is shown to be equivalent (over a weak arithmetic) to the 1-consistency of set theory KPM for recursively Mahlo universes.

1 Introduction

In M. Rathjen [9], W. Buchholz [6] and [2][3] the set theory KPM for recursively Mahlo universes has been analyzed proof-theoretically.

As to the proof-theoretic analyses on such strong impredicative theories, let us quote from Buchholz [7):

Contemporary ordinal-theoretic proof theory (i.e., the part of proof theory concerned with ordinal analyses of strong impredicative theories) suffers from the extreme (and as it seems unavoidable) complexity and opacity of its main tool, the ordinal notation systems. This is not only a technical stumbling block which prevents most proof-theorists from a closer engagement in that field, but it also calls the achieved results into question, at least as long as these results do not have interesting consequences, such as e.g., foundational reductions or intuitively graspable combinatorial independence results.

If proofs or constructions looks too complicated to grasp, and this makes us doubtful about what we have gained, I would reply that this defect are mainly due to the scarcity of our experiences of mathematics in the strength of strong impredicative theories $T$.

One thing we can do is to give alternative proofs, thereby could shed light on the same results from another angle, and gain an insight in mathematical reasoning and structures codified in $T$. Another thing to be done is to find combinatorial independence results. This line of research was suggested and

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1Indeed, it’s complicated as compared with those for predicative theories such as PA.
encouraged by W. Buchholz [7]. In a sense, such an (optimal) independence result might be viewed as a finitary essence of \( T \). One prototype of combinatorial independence results is the hydra games in Kirby-Paris [8] and Buchholz [3]. This says that given a hydra game, a theory such as \( \text{PA} \) or \( (\Pi_1^0-\text{CA})+\text{BI} \) proves that each hydra eventually die out, but the theory in question does not prove its universal closure, any hydra must die out.

In this paper a hydra game for recursively Mahlo ordinals is proposed, and a result of the same kind is shown for the games, and \( \text{KPM} \).

A tree \( (T, <, \ell) \) is said to be structured if the (finite) set of immediate successors \( \{s : t < s & \exists u(t < u < s)\} \) of each node \( t \) in \( T \) is linearly ordered.

A hydra is a triple \( (T, <, \ell) \) such that \( (T, <) \) is a finite and structured tree and \( \ell : T \rightarrow \{\ast\} \cup Lb_0 \cup Lb_1 \), where \( \ast \) is the label attached to the root of the tree \( T \). \( Lb_0 = \{1, n \cdot m, n \cdot A, n \cdot \ast, n \cdot \ast_\omega, n \cdot \ast_\mu : 0 < n, m, \omega, A \in L\} \) is the set of labels for leaves, and \( Lb_1 = \{\omega\} \cup \{\{A : A \in L \cup \{\ast_\mu, \mu\}\} \cup \{\varphi_{A+n}, D_A : A \in L^*, n < \omega\}\} \) is the set of labels for internal nodes. \( L \) is the set of labels defined below, and \( L^* \) the set of finite sets of labels in \( L \). Each hydra and each label in \( L \) is a term over symbols \( \{n : 0 < n < \omega\} \cup \{\omega, \mu, d, \{\}\} \cup \{\ast, \ast_\omega, \ast_\mu, \varphi, d\} \). The set \( L \) of labels ordered by a linear order \( < \) with the largest element \( \mu \). The set of hydrams \( \mathcal{H} \) and the set of labels \( L \) are defined simultaneously.

Hydras produce a finite set of labels, as the games go. In some limit cases of the hydra game, a hydra \( (T, <, \ell) \) freely chooses a label from the finite set of labels, which are available for the current hydra. The set of their labels might grow in some cases called (Production).

A free choice of labels means that, for a hydra \( H \), a finite set \( lb \) of labels and natural numbers \( \ell \), there are finitely many possible moves written as \((H, lb) \rightarrow_\ell (K, lb')\). Given a hydra \( H_0 \) and a finite set \( lb_0 \) of labels in \( L \), a finitely branching tree is obtained as follows. For \( t < ^\omega \omega \) we define moves \((H_0[t], lb[t]) \rightarrow_\ell (K, lb')\) in the hydra game. \( H_0[\epsilon] := H_0 \) and \( lb[\epsilon] := lb_0 \) for the empty sequence \( \epsilon \). \( \{(H_0[t \ast (i)], lb[t \ast (i)])\} \) is the set of the pairs \((K, lb')\) such that \((H_0[t], lb[t]) \rightarrow_\ell (K, lb')\) for the length \( \ell = |t| \) of the finite sequence \( t \). The finitely branching tree \( Tr(H_0, lb_0) = \{t < ^\omega \omega : (H_0[t], lb_0[t]) \text{ is defined}\} \) is thus obtained from \( H_0 \) and \( lb_0 \). The tree \( Tr(H_0, lb_0) \) is seen to be well founded for every hydra \( H_0 \) and every finite set \( lb_0 \) of labels. Let \( F[H_0, lb_0] \) denote the length of maximal runs in the game, i.e., the length of the tree:

\[
F[H_0, lb_0] = \max\{|t| : t \in ^\omega \omega, H_0[t] \text{ is defined with } lb[\epsilon] = lb_0\}.
\]

Let \( H + n := H + 1 + \cdots + 1 \) be the hydra obtained from the hydra \( H \) by adding the trivial hydra 1 \( n \)-times. Now our theorem runs as follows.

**Theorem 1.1**

1. Each provably total recursive function in \( \text{KPM} \) is dominated by a function \( n \mapsto F[H_0 + n, 0] \) for some hydra \( H_0 \).

2. Conversely for each hydra \( H_0 \) and each finite set \( lb_0 \subset L \), the function \( n \mapsto F[H_0 + n, lb_0] \) is provably total recursive function in \( \text{KPM} \).
3. The fact that for every hydra $H_0$, the hydra game eventually terminates, i.e., the tree $\{ t \in <\omega : H_0[t] \text{ is defined with } lb[t] = \emptyset \}$ is finite, or equivalently the $\Pi_2^0$-statement $\forall H_0 \forall F[H_0, 0] \downarrow$ is equivalent to the 1-consistency $\text{RFN}_{\Pi_2^0}(\text{KPM})$ of KPM over the elementary arithmetic $\text{EA}$.

Let us mention the contents of the paper. In section 2 the hydra game is defined through a linear ordering $A < B$ on labels, which is based on an assignment of ordinal diagrams $o(H), o(A) \in O(\mu)$ to hydras $H$ and labels $A \in L$. In section 3 we show that $d_\Omega (o(K)\#o(lb')) < d_\Omega (o(H)\#o(lb'))$ when $(H, lb) \to (K, lb')$ is a possible move. Thus Theorem 1.1.2 follows from the fact in [2] that the wellfoundedness up to each ordinal diagram $< \Omega$ is provable in KPM. In section 4 we introduce first a theory $[\Pi_0^0, \Pi_0^0]$-Fix for non-monotonic inductive definitions of $[\Pi_0^0, \Pi_0^0]$-operators in [10]. In [3] it is shown that the 1-consistency of the set theory KPM is reduced to one of the theory $[\Pi_0^0, \Pi_0^0]$-Fix. Second we assign hydras to proofs in the theory. In section 5 we define rewritings on proofs in such a way that each rewriting corresponds to a move on hydras attached to proofs. Theorems 1.1.1 and 1.1.3 are concluded. Finally the linearity of the relation $A < B$ on labels is briefly discussed.

2 Hydra game

In this section we introduce a hydra game for recursively Mahlo ordinals.

In the next Definition 2.1 the set $\mathcal{H} = \mathcal{H}_0 \cup \mathcal{H}_1$ of hydras and the set $L$ of labels are defined simultaneously. Also we define a subset $\mathcal{T}_i \subset \mathcal{H}_i$ for $i = 0, 1$.

Definition 2.1 (Hydras)

1. $L = \{ d_\mu(h_0) : h_0 \in \mathcal{H}_0 \} \cup \{ d_{d_\mu(h_0)}(h_1) : h_0 \in \mathcal{H}_0, h_1 \in \mathcal{H}_1 \}$.

$L^*$ denotes the set of all finite sets of labels in $L$. The singleton $\{A\}$ is identified with labels $A \in L$. $R(A)$ holds iff either $A = \mu$ or $A = d_\mu(h)$. 

2. $\mathcal{T}_i \subset \mathcal{H}_i$ for $i = 0, 1$.

3. $0 \in \mathcal{H}_0 \cap \mathcal{H}_1$ and $1 \in \mathcal{T}_0 \cap \mathcal{T}_1$.

4. If $h_0, \ldots, h_k \in \mathcal{T}_i (k > 0)$, then $(h_0 + \cdots + h_k) \in \mathcal{H}_i$ for $i = 0, 1$.

5. $\{ n \cdot *_\omega, n \cdot *_\mu \} \cup \{ n \cdot m, n \cdot A : 0 < n, m < \omega, A \in L \} \subset \mathcal{T}_0$.

6. If $h \in \mathcal{H}_0$, then $\omega(h) \in \mathcal{T}_0$.

7. If $h \in \mathcal{H}_0$, then $\{ \mu \}(h), \{ *_\mu \}(h) \in \mathcal{T}_0$. Also if $h \in \mathcal{H}_1$ and $A \in L$, then $\{ A \}(h) \in \mathcal{T}_1$ for $i = 0, 1$.

8. If $h \in \mathcal{H}_0$ and $C \in L^*$, then $D(C; h) \in \mathcal{T}_1$. Let $D_C(h) := D(C; h)$.

9. If $h \in \mathcal{H}_1, \emptyset \neq C \in L^*$ and $n < \omega$, then $\varphi(C + n; h) \in \mathcal{T}_1$.

Let $\varphi(\emptyset; h) := h$ when $C = \emptyset$ and $n = 0$, and $\varphi_{C+n}(h) := \varphi(C + n; h)$. 

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Definition 2.2 The set of labels \( Lb(h) \) and the fixed part \( (h)_f \subset Lb(h) \) for hydros \( h \in \mathcal{H} \) are defined recursively as follows.

1. \( Lb(0) = Lb(1) = Lb(n \cdot *\omega) = Lb(n \cdot *\mu) = Lb(n \cdot m) = \emptyset \), and \((0)_f = (1)_f = (n \cdot *\omega)_f = (n \cdot *\mu)_f = (n \cdot m)_f = \emptyset \).
2. \( Lb(h_0 + \cdots + h_k) = \bigcup_{i \leq k} Lb(h_i) \), and \((h_0 + \cdots + h_k)_f = \bigcup_{i \leq k} (h_i)_f \).
3. \( Lb(n \cdot C) = \{ C \} \) and \((n \cdot C)_f = \emptyset \) for \( C \in L \).
4. \( Lb(D(C; h)) = Lb(\varphi_{C+n}(h)) = C \cup Lb(h) \) and \((D(C; h))_f = (\varphi_{C+n}(h))_f = C \cup (h)_f \) for \( C \in L^* \).
5. \( \langle A \rangle(h) = \{ A \} \cup Lb(h) \) and \((\{ A \}(h))_f = (h)_f \) for \( A \in L \) and \( n < \omega \).
6. \( Lb(\omega(h)) = Lb(\mu(h)) = Lb(\mu_{\omega}(h)) = Lb(h) \) and \((\omega(h))_f = (\mu(h))_f = (\mu_{\omega}(h))_f = (h)_f \).

In [2] a system \((O(\mu), <)\) of ordinal diagram, a computable system of ordinal notations is defined, and it is shown that KPM proves the wellfoundedness up to each \( \alpha < \Omega \). Let us recall a slightly modified system \((O(\mu), <)\) briefly. The set \( O(\mu) \) is generated from \( 0 \) and \( \mu \) by the addition +, the fixed point free binary Veblen function \( \varphi_{\alpha\beta}(\alpha, \beta < \mu) \), the exponential above \( \mu \), \( \omega^\alpha (\alpha > \mu) \), and the collapsing function \( d : (\sigma, \alpha) \mapsto d_{\sigma \alpha} \) for the regular diagram \( \sigma \), i.e., either \( \sigma = \mu \) or \( \sigma = d_{\mu \beta} \) for a \( \beta \). \( R \) denotes the set of all regular diagrams, and \( \Omega := d_{\mu 0} \). \( \sigma, \tau, \kappa, \rho \ldots \) denote regular diagrams. Each \( d_{\sigma \alpha} \) is a strongly critical number. Crucial definitions are as follows. \( \alpha \prec \beta \) iff either \( \alpha = d_{\beta \gamma} \) or \( \alpha = d_{d_{\gamma \delta} \beta} \) for some \( \gamma, \delta \). \( \alpha \preceq \beta : \iff (\alpha \prec \beta \lor \alpha = \beta) \). The set \( K_{\sigma \alpha} \) of subdiagrams of \( \alpha \) is defined as follows.

1. \( K_{\sigma 0} = K_{\sigma \mu} = \emptyset \), \( K_{\sigma}((\alpha_1 + \cdots + \alpha_n)) = \bigcup\{ K_{\sigma \alpha_i} : 1 \leq i \leq n \} \), and \( K_{\sigma \omega} \) \( K_{\sigma \alpha} \cup K_{\sigma \beta} \).
2. \( K_{\sigma \mu} \) \( \{ d_{\sigma \alpha} \} \) \( d_{\tau \sigma} \) \( \sigma \preceq \tau \)
   \( K_{\sigma \tau} \cup K_{\sigma \alpha} \) \( \sigma < \tau \)
   \( K_{\sigma \tau} \) \( \tau < \sigma \lor \tau \not\preceq \sigma \)

For \( \sigma \neq \tau \), \( d_{\sigma \alpha} < d_{\tau \beta} \) iff one of the following conditions holds:

1. \( \sigma < \tau \lor (\sigma \preceq \tau \sigma) \lor K_{\sigma \mu} \).
2. \( \tau < \sigma \land d_{\sigma \alpha} < \tau < K_{\sigma \mu} \).

\( d_{\sigma \alpha} < d_{\tau \beta} \) iff one of the following conditions holds:

1. \( d_{\sigma \alpha} \preceq K_{\sigma \beta} \).
2. \( K_{\sigma \alpha} < \beta \lor d_{\tau \alpha} < d_{\tau \beta} \).

where \( \tau = \min \{ \tau \in R \cup \{ \infty \} : (\sigma < \tau < \infty \land K_{\tau} \{ \alpha, \beta \} \not= \emptyset) \lor \tau = \infty \} \), and, by definition, \( d_{\infty \alpha} := \alpha, \forall \alpha \in O(\mu)(\alpha < \infty) \) and \( \infty \not\in O(\mu) \).

We associate an ordinal diagram \( o(h) \in O(\mu) \) for hydros \( h \).
Definition 2.3 We associate $o(A), o(h) \in O(\mu)$ for labels $A \in L$ and hydras $h \in \mathcal{H}$ as follows.

1. $o(d_{\mu}(h)) = d_{\mu}(o(h))$ and $o(d_{d_{\mu}(h)}(h_1)) = d_{o(d_{\mu}(h))}(o(d_{\mu}(h)) \# o(h_1))$ with the natural (commutative) sum $\#$ in $O(\mu)$.

2. $o(0) = 0$ and $o(1) = 1 := \omega^0$.

3. $o(h_0 + \cdots + h_k) = o(h_0) \# \cdots \# o(h_k)$.

4. $o(n \cdot \omega) = \omega$, $o(n \cdot *_{\mu}) = \mu$, $o(n \cdot m) = n \cdot m$ and $o(n \cdot A) = o(A)$.

5. $o(\varphi_{C+n}(h)) = \varphi_{o(C)+n+1}(o(h))$, where $o\{C_1, \ldots, C_n\} = o(C_1) \# \cdots \# o(C_n)$.

6. $o(\omega(h)) = \omega^{o(h)}$.

7. $o(\{\mu\}(h)) = o(\{*_{\mu}\}(h)) = \mu \# o(h)$.

8. $o(\{A\}(h)) = \begin{cases} o(A) \# 1 \# o(h) & \text{if } h \in \mathcal{H}_0 \\ \varphi_{o(A)}(o(h)) & \text{if } h \in \mathcal{H}_1 \end{cases}$.

9. $o(D(C; h)) = d_{\mu}(o(C) \# o(h))$.

For $A, B \in L \cup \{\mu\} \cup \mathcal{H}, n, m < \omega$, and $A, B \in L^*$, let

$$
A + n < B + m :\Leftrightarrow o(A) + n < o(B) + m \quad (1)
$$

$$
A \leq B :\Leftrightarrow o(A) \leq o(B)
$$

$$
A \simeq B :\Leftrightarrow o(A) = o(B)
$$

$$
A < B :\Leftrightarrow \exists B \in B \forall A \in A \left( A < B \right)
$$

$$
A \leq B :\Leftrightarrow \forall A \in A \exists B \in B \left( A \leq B \right)
$$

where $<, \leq$ in the RHS denote the relations in $O(\mu)$.

We are going to define moves of hydras. For a pair $(H, lb_0)$ of a hydra $H$ and a finite set $lb_0$ of labels in $L$, there are some possible moves $(H, lb_0) \to_{\ell} (K, lb_1)$ depending on a number $\ell < \omega$. The finite sets of labels may grow in two cases (Production) in Definition 2.4.

Definition 2.4 (Moves)

Let $(H, lb)$ be a pair of a hydra $H$ and a finite set $lb$ of labels in $L$, and $\ell < \omega$. We define possible moves $(H, lb) \to_{\ell} (K, lb')$.

1. (Necrosis) $(H, lb) \to_{\ell} (0, lb)$ for $H \neq 0$, $(H, lb) \to_{\ell} (1, lb)$ for $H \notin \{0, 1\}$ and $(\{\mu\}(H), lb) \to_{\ell} (H, lb)$.

2. $(n \cdot *_{\mu}, lb) \to_{\ell} ((n \cdot A) + n, lb)$ for $A \in lb$.

$(n \cdot B, lb) \to_{\ell} ((n \cdot A) + n, lb)$ for $lb \ni A < B$, where $n = 1 + \cdots + 1$.

$(n \cdot *_{\omega}, lb) \to_{\ell} (n \cdot m, lb)$ for $0 < m \leq \ell$.

$((n + 1) \cdot (m + 1), lb) \to_{\ell} (((n + 1) \cdot m) + n, lb)$ for $n \geq 0$ and $m > 0$.

$((n + 1) \cdot 1, lb) \to_{\ell} (n, lb)$. 

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3. \((d(H+1), lb) \rightarrow_{\ell} (d(H) \cdot k, lb)\), where \(k \leq \ell + 1\) and \(d(H) \cdot k := d(H) + \cdots + d(H)\) with \(k\)’s \(d(H)\) for \(d \in \{\omega\} \cup \{D_C, \varphi_{C+n} : C \in L^*, C \in L, n \leq \ell\}\).

4. \((D_C(H+1), lb) \rightarrow_{\ell} (\varphi_{A+n}(D_C(H) \cdot 2), lb)\) where \(A \in lb\) with \(A \leq C\), and \(n \leq \ell\).

5. \((\varphi_{C+n}(H+1), lb) \rightarrow_{\ell} (\varphi_{A+m}(\varphi_{C+n}(H) + \varphi_B(H)), lb)\) and 
   \((\varphi_{C+n}(H+1), lb) \rightarrow_{\ell} (\varphi_{A+m}(\varphi_B(H) + \varphi_{C+n}(H)), lb)\),
   where \(A \in lb\), \(m \leq \ell\), \(A + m < C + n\), \(B \in lb\) and \(B < C\).

6. \((\{*\}, lb) \rightarrow_{\ell} (\{A\}, lb)\).

7. \((d(K + \{B\}(H)), lb) \rightarrow_{\ell} (\{B\}(d(K + H) \cdot 2), lb)\) where \(R(B)\), and \(d = \omega\) if \(B = \mu\). Otherwise \(d \in \{\omega\} \cup \{\varphi_{A+n} : B \leq A, 0 < n \leq \ell\}\).
   For \(B \in L\) with \(B < \mu\) and \(C \in L^*, (D_C(K + \{B\}(H)), lb) \rightarrow_{\ell} (\{B\}(D_C(K + H) \cdot 2), lb)\).

8. (Production) For \(A = d_\mu(K + \{B\}(H))\) with \(lb \cup \{0\} \ni B < D(C; K + \{\mu\}(H)), n \leq \ell\),
   \((D(C; K + \{\mu\}(H)), lb) \rightarrow_{\ell} (\varphi_{A+n}(D(C \cup \{A\}; K + H) \cdot 2), lb \cup \{A\})\).

9. (Production) Let \(R(B), H \in H_1\) and \(lb \cup \{0\} \ni C < B\). Also \(e(*)\) is a hydra with a hole \(*\) generated from the hole \(*\) by applying \(H(*), \varphi_{C+n}(H(*)), \varphi_{C}(H(*))\) for \(lb \supset \{C_0\} \cup C < B\). Then for \(A = d_B(\{C\}(e(H)))\) and \(n \leq \ell\), \((\{B\}(H)), lb) \rightarrow_{\ell} (\varphi_{A+n}(\varphi_A(e(H)) \cdot 2), lb \cup \{A\})\).

10. If \((H, lb) \rightarrow_{\ell} (K, lb)\) for \(H \in H_0\) and \(\forall A \in lb(A < D_C(H))\), then 
    \((D_C(H), lb) \rightarrow_{\ell} (D_C(K), lb)\).

11. If \((H, lb_0) \rightarrow_{\ell} (K, lb_1)\), then \((d(H), lb_0) \rightarrow_{\ell} (d(K), lb_1)\) for \(d \in \{\omega\} \cup \{\varphi_{A+n} \cdot \varphi_A : A \in L, n \leq \ell, A \in L^* \} \cup \{H_0 + : H_0 \in H\}\).

\((H, lb) \rightarrow_{\ell} (K, lb')\) denotes the reflexive and transitive closure of the relation \(\rightarrow_{\ell}\).

Definition \((2.3.1.3)\) means that \((\omega(H+1), lb) \rightarrow_{\ell} (\omega(H) \cdot 2, lb), (D_C(H+1), lb) \rightarrow_{\ell} (D_C(H) \cdot 2, lb)\) and \((\varphi_{C+n}(H+1), lb) \rightarrow_{\ell} (\varphi_{C+n}(H) \cdot 2, lb)\). Definition \((2.3.1.1)\) means that \((\omega(H), lb_0) \rightarrow_{\ell} (\omega(K), lb_1), (\varphi_{A+n}(H), lb_0) \rightarrow_{\ell} (\varphi_{A+n}(K), lb_1), (\varphi_A(H), lb_0) \rightarrow_{\ell} (\varphi_A(K), lb_1)\) and \((H_0 + H, lb_0) \rightarrow_{\ell} (H_0 + K, lb_1)\) if \((H, lb_0) \rightarrow_{\ell} (K, lb_1)\).

It is clear that both of the relations \((H, lb) \rightarrow_{\ell} (K, lb')\) and \(A < B\) elementary recursive on hydrams \(H, K\), finite sets \(lb, lb'\), labels \(A, B\) and numbers \(\ell\).
Moreover when \((H, lb) \rightarrow_{\ell} (K, lb')\), either \(lb' = lb\) or \(lb' = lb \cup \{A\}\) for a label \(A\).

Given a hydra \(H_0\) and a finite set \(lb_0\) of labels in \(L\), a finitely branching tree \(Tr(H_0, lb_0) = \{t \in \omega : (H_0[t], lb_0[t])\) is defined\} is obtained as follows. 
\(H_0[\epsilon] := H_0\) and \(lb[\epsilon] := lb_0\) for the empty sequence \(\epsilon\). \(\{(H_0[t * (i)], lb[t * (i)])\}\) is the set of the pairs \((K, lb')\) such that \((H_0[t], lb[t]) \rightarrow_{\ell} (K, lb')\) for the length \(\ell = |t|\) of \(t \in \omega\). We see that the tree \(Tr(H_0, lb_0)\) is elementary recursive.
3 Provable

We show the following holds as long as \( H_0(t \ast (k)) \) is defined for \( t \in \omega \omega \) and \( lb_0[\epsilon] = lb_0 \):

\[
d_\omega(o(H_0[t \ast (k)] \# o(lb_0[t \ast (k)])) < d_\omega(o(H_0[t]) \# o(lb_0[t]))
\]

Then Theorem 1.12 follows from [2].

**Lemma 3.1** Let \((H, lb) \to (K, lb')\), and \( A \) be the label defined as follows. If \( lb = lb \), then let \( A := 0 \). Otherwise \( lb' = lb \cup \{A\} \).

Then \( o(K) \# o(A) < o(H) \), \( \forall \sigma[K_\sigma(o(K)) \subseteq K_\sigma(o(H)) \cup K_\sigma(o(lb'))] \) and \( \forall \sigma \forall \alpha \in K_\sigma(o(A)) \in K_\sigma(o(H)) \cup K_\sigma(o(lb')) \forall \alpha < d_\sigma(o(H)) \), where \( K_\sigma(o(lb')) = \bigcup \{K_\sigma(o(B)) : B \in lb'\} \).

**Proof.** We show the lemma by main induction on the sum of the sizes \( \#(H) + \#(K) \) with subsidiary induction on the cardinality of the finite sets \( lb \).

Consider the case in Definition 2.410

First let \((d(K + \{B\}(H)), lb) \to ((\{B\}(d(K + H) \cdot 2), lb)) \) where \( R(B) \), and \( d = \omega \) if \( B = \mu \). Otherwise \( d \in \{\omega \} \cup \{\varphi_{A+n} : B \in A, 0 < n \leq \ell \} \). Let \( \alpha = o(K), \beta = o(H) \) and \( \sigma = o(B) \). First consider the case \( d = \omega \). Then \( H, K \in H_0 \) by Definition 2.410 and \( \xi := o(d(K + \{B\}(H))) = \omega^{\sigma + \# \alpha \# \beta} \), while \( \xi := o((\{B\}(d(K + H) \cdot 2)) = \sigma + \# \alpha \# \beta \cdot 2 \). It is clear that \( \xi < \eta \) and \( \forall \tau(K \xi < K, \eta) \).

Next let \( d = D_{C} \) and \( \sigma < \mu \). Then \( H, K \in H_0 \) by Definition 2.410 and \( \eta := o(d(K + \{B\}(H))) = d_\mu(\gamma \# \alpha \# \sigma \# \beta \# 1) \) with \( \gamma := o(C) \), while \( \xi := o((\{B\}(d(K + H) \cdot 2)) = \varphi_\sigma(d_\mu(\gamma \# \alpha \# \beta \# 1) \cdot 2) \). We see \( \xi < \eta \) from \( \sigma < \eta \), and \( \forall \tau < \mu(K, \xi \in K_\xi, \gamma, \alpha, \sigma, \beta \leq K, \eta) \). Also \( K_\mu \xi = \{\sigma, d_\mu(\gamma \# \alpha \# \beta \# 1) \} \). Finally let \( d = \varphi_{A+n} \) with \( B \leq A \) and \( \rho = o(\sigma) \). Then \( \eta := o(d(K + \{B\}(H))) = \varphi_{\rho + n + 1}(\alpha \# \varphi_\sigma(\beta)) \) and \( \xi := o((\{B\}(d(K + H) \cdot 2)) = \varphi_\sigma(\varphi_{\rho + n + 1}(\alpha \# \beta) \cdot 2) \). We have \( \sigma < \rho + n + 1 \). We see \( \xi < \eta \) from \( \sigma < \rho + n + 1 \) and \( \beta < \varphi_\sigma(\beta) \). It is clear that \( \forall \tau(K, \xi < K, \eta) \).

Second let for \( B \in L \) with \( B < \mu \) and \( C \in L^*, (D_{C}(C + \{B\}(H)), lb) \to ((\{B\}(D_{C \cup \{B\}}(H)), lb)) \).

Then \( o(D_{C}(C + \{B\}(H))) = d_\mu(\gamma \# \sigma \# \alpha \# \beta \# 1) \), and \( o((\{B\}(D_{C \cup \{B\}}(H))) \cdot 2) = \varphi_\sigma(d_\mu(\gamma \# \sigma \# \alpha \# \beta \# 1) \cdot 2) \).

Next consider the case in Definition 2.419

Then \( (D_{C}(C + \{B\}(H)), lb) \to ((\varphi_{A+n}(D_{C}(C \cup \{A\})(H)), lb \cup \{A\})) \).

Finally let \( (e(\{B\}(H)), lb) \to (\varphi_{A+n}(e(H)), lb \cup \{A\})) \), where \( R(B), H \in H_1, n \leq \).

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In [4] we show that the wellfoundedness is provable up to each ordinal diagram \( \alpha < 4 \). A theory assuming the fact that the hydra game eventually terminates.

\( KPM \) consistency of the set theory. For Theorem 1.1.3 it suffices to show, over \( EA \), the \( \Pi^0_1 \)-Fix for non-monotonic inductive definitions of \( \Pi^0_1 \)-operators in \([10]\). In \([4]\) it is shown that the 1-consistency of the set theory KPM is reduced to one of the theory \( \Pi^0_1 \)-Fix. For Theorem \([4,13]\) it suffices to show, over \( EA \), the 1-consistency of \( \Pi^0_1 \)-Fix assuming the fact that the hydra game eventually terminates.

4 Unprovability

In this section we introduce first a theory \( \Pi^0_1 \)-Fix for non-monotonic inductive definitions of \( \Pi^0_1 \)-operators in \([10]\). In \([4]\) it is shown that the 1-consistency of the set theory KPM is reduced to one of the theory \( \Pi^0_1 \)-Fix. For Theorem \([4,13]\) it suffices to show, over \( EA \), the 1-consistency of \( \Pi^0_1 \)-Fix assuming the fact that the hydra game eventually terminates.

4.1 A theory \( \Pi^0_1 \)-Fix

In \([4]\) we show that the wellfoundedness is provable up to each ordinal diagram \( \alpha < \Omega \) in a theory \( \Pi^0_1 \)-Fix for \( \Pi^0_1 \)-non-monotonic inductive definitions in \([10]\).

For a class of formulas \( \Phi \), the theory \( \Phi \)-Fix for non-monotonic inductive definitions are two-sorted: one sort \( x \) for natural numbers and the other \( \alpha \) for
ordinals. The binary predicate $x \in I^a$, then, denotes the $a$-th stage of inductive definition by a fixed operator $\Gamma : \mathcal{P}(\omega) \rightarrow \mathcal{P}(\omega)$, which is defined by a first order formula $\Gamma(X, x) \in \Phi$ in the language of the first order arithmetic $\mathcal{L}(\mathcal{P}A)$ with an extra unary predicate $X$. The axioms of the theories are:

1. Axioms of $\mathcal{P}A$ and equality axioms for either sort.
2. The defining axiom of $x \in I^a$: $x \in I^a \leftrightarrow \exists b < a[x \in \Gamma(I^b)]$.
3. Closure axiom: $\Gamma(I^\infty) \subset I^\infty$ for $I^\infty := \{x : \exists a(x \in I^a)\}$.
4. Axioms for the well ordering $<$ on ordinals:
   (a) $<$ is a linear ordering:
   i. $<$ is irreflexive and transitive.
   ii. (trichotomy) $x < y \lor x = y \lor y < x$  \hspace{1cm} (3)

(b) transfinite induction schema for any formula $F$:
   $\forall a[\forall b < aF(b) \rightarrow F(a)] \rightarrow \forall aF(a)$.

4.2 Hydras associated with proofs

In what follows assume that $[\Pi_1^0, \Pi_0^1]$-Fix is 1-inconsistent. This means that there exists a true $\Pi_0^1$-sentence $\forall x B(x)$ with a quantifier-free $B$ such that $[\Pi_0^0, \Pi_0^1]$-Fix+$\forall x B(x)$ is inconsistent.

Let $P_0$ be a proof in $[\Pi_1^0, \Pi_0^1]$-Fix+$\forall x B(x)$ of a contradiction. (Proofs are specified later.) We associate a hydra $H_0 = \Omega(P_0)$ to $P_0$, and define a rewriting step $r : P \rightarrow r(P)$ on proofs $P$ in $[\Pi_1^0, \Pi_0^1]$-Fix+$\forall x B(x)$. For each $P_\ell = r^{(\ell)}(P_0)$, associate a hydra $H[\ell] = \Omega(P_\ell)$ again so that $\{H[\ell]\}_\ell$ is a path through the tree $\mathcal{H}(H_0, \emptyset)$. $P_0$ tells the hydras which way to proceed. Namely $H[\ell + 1]$ is one of possible moves for the hydra $H[\ell]$, i.e., $(H[\ell], lb[\ell]) \rightarrow_\ell (H[\ell + 1], lb[\ell + 1])$. Assuming $P_0$ is a proof in $[\Pi_1^0, \Pi_0^1]$-Fix+$\forall x B(x)$ of a contradiction, we see that the path is infinite, i.e., the hydra game $\{H[\ell]\}_\ell$ goes forever. Moreover all of these are done in $\mathcal{E}A$.

$\mathcal{L}_2^2$ denotes the class of lower elementary recursive functions in $[\Pi_1^0]$. The class of functions containing the zero, successor, projection and modified subtraction functions and which is closed under composition and summation of functions. $\mathcal{L}_2^2$ denotes the class of lower elementary recursive relations. The arithmetical part of the language $\mathcal{L}$ of $[\Pi_0^0, \Pi_1^0]$-Fix is chosen to consist of predicate constants for lower elementary recursive relations $R \in \mathcal{L}_2^\ast$.

The language $\mathcal{L}$ of $[\Pi_0^0, \Pi_1^0]$-Fix consists of

1. two sorts of variables, one for (natural) numbers $\mathbb{N}$ and the other for ordinals, i.e., stages $O$ of inductive definitions. $x, y, \ldots$ are variables for natural numbers, and $a, b, \ldots$ are variables ranging over the domain of a well ordering $<$,
two binary predicate symbols $a = O b$ and $a < b$, and their negations $a \neq b$ and $a \not< b$ on $O$,

3. function constants $0^N$ and $x'$ (successor) on $N$,

4. arithmetic predicate constants on $N$ for lower elementary recursive relations $R \in \mathcal{L}_2^*$ and their negations $\neg R$,

5. the binary predicate symbol $I(a, x)$ and its negation $\neg I(a, x)$ denoting the stages $I^a = \{ x \in \omega : I(a, x) \}$ of a fixed $[\Pi_0^0, \Pi_1^0]$-formula $\mathcal{A}(X, x) \equiv \forall y B_0(X, x, y) \vee [\forall x \{ \forall y B_0(X, x, y) \rightarrow x \in X \} \wedge \forall z B_1(X, x, z)]$ where $B_i$ is a bounded formula in the arithmetic language $\mathcal{L}_2^* \cup \{0^N, t\}$ with a unary predicate $X$ for $i = 0, 1$, and

6. logical connectives $\land, \lor, \forall, \exists$.

The negation $\neg \varphi$ of a formula $\varphi$ is defined by using de Morgan’s law and the elimination of double negations. A prime formula $R(t_1, \ldots, t_n)$ or its negation $\neg R(t_1, \ldots, t_n)$ is an arithmetic predicate $R$ and a prime formula $t = s, t < s$ for stage terms $t, s$ and their negations are stage formulas.

There are four kinds of quantifications, bounded number quantifiers $\exists x \leq t, \forall x \leq t$, unbounded number quantifiers $\exists x, \forall x$, bounded stage quantifiers $3a < b, \forall a < b$ and unbounded stage quantifiers $\exists a, \forall a$. A formula is said to be unbounded if it contains an unbounded stage quantifier.

The axioms in $[\Pi_1^0, \Pi_1^0]$. Fix are axioms for function and arithmetic predicate constants, the axioms for the linear ordering $<$, the induction axioms $(VJ)$, $(TJ)$, the defining axiom $(I)$ of stages, and the closure axiom $(Cl)$: for arbitrary formula $F$,

$(VJ)$ $F(0) \land \forall x (F(x) \rightarrow F(x')) \rightarrow \forall x F(x)$.

$(TJ)$ $\forall a (\forall b < a F(b) \rightarrow F(a)) \rightarrow \forall a F(a)$.

$(I)$ $\forall x \forall a [x \in I^a \leftrightarrow (\mathcal{A}(I^{<a}, x) \lor x \in I^{<a})]$ where $(x \in I^a) \leftrightarrow I(a, x)$ and $(x \in I^{<a}) \leftrightarrow \exists b < a (x \in I^b)$ with $I^b = \{ x : I(b, x) \}$.

$(Cl)$ $\mathcal{A}(I^{<\infty}) \subseteq I^{<\infty}$, i.e., $\forall x (\mathcal{A}(I^{<\infty}, x) \rightarrow x \in I^{<\infty})$, where $(x \in I^{<\infty}) \leftrightarrow \exists a (x \in I^a)$. This is equivalent to $\forall y B_0(I^{<\infty}, y) \subseteq I^{<\infty} \land \forall z B_1(I^{<\infty}, z) \subseteq I^{<\infty}$, where $\forall y B_0(I^{<\infty}, y) = \{ x : \forall y B_0(I^{<\infty}, x, y) \}$ and $\forall z B_1(I^{<\infty}, z) = \{ x : \forall z B_1(I^{<\infty}, x, z) \}$.

Let us extend the language $\mathcal{L}$ to $\mathcal{L}_H$ by adding a unary predicate $R(a)$ of stage sort, and individual constants $A$ denoting labels $A \in L$ and a constant $0^\mathcal{O}$ for the hydra $0 \in \mathcal{H}$. By definition these constants $A$ is of stage sort. $A = O B$ is defined to be true iff $o(A) = o(B)$, and $A < B$ is true if $o(A) < o(B)$. $lb(\varphi)$ denotes the set of stage constants occurring in the formula $\varphi$. On the other side $R(t)$ and $\neg R(t)$ are s.p.f’s, and $R(A)$ is defined to be true iff $A = d_\mu(h)$ for some $h$. $R(A)$ is intended to denote the fact that $A$ is recursively regular. Then the axiom $(Cl)$, $\mathcal{A}(I^{<\infty}) \subseteq I^{<\infty}$ is proved from the following axioms.
∀x{∀yB_0(I^{<\infty},x,y) \rightarrow \exists a[R(a) \wedge \forall yB_0(I^{<a},x,y)]}
and
∀a (R(a) \rightarrow \forall x{∀yB_0(I^{<a},x,y) \rightarrow \exists b<a[\forall yB_0(I^{<b},x,y)]})

∀z{∀yB_1(I^{<\infty},z,y) \rightarrow \exists a[R(a) \wedge \forall yB_1(I^{<a},z,y)]}

In what follows by a formula we mean a formula in \( \mathcal{L}_H \).

A formula is said to be an \( \exists \)-formula if it is either an a.p.f. or a s.p.f. or a formula in one of the following shapes; \( \psi \vee \psi, \exists x \leq t\phi, \exists x \phi, \exists a < b \phi, \exists a \phi \) or \( t \in I^s \). A formula is a \( \forall \)-formula if its negation is an \( \exists \)-formula. If a formula is an \( \exists \)-formula and simultaneously a \( \forall \)-formula, then it is either an a.p.f. or a s.p.f.

\([\Pi^0_1, \Pi^0_1] \)-Fix is formulated in one sided sequent calculus. Finite sets of formulae are called a sequents. Sequents are denoted by \( \Gamma, \Delta, \) etc.

**Definition 4.1 Axioms** in \([\Pi^0_1, \Pi^0_1]\)-Fix are:

**logical axioms** \( \Gamma, \neg \phi, \phi \)
where \( \phi \) is an a.p.f. or a s.p.f. or a formula of the shape \( t \in I^s \).

**arithmetical axioms** 1. \( \Gamma, \Delta_R \)
where \( \Delta_R \) consists of a.p.f.‘s and corresponds to the definition of a lower elementary relation \( R \).

2. \( \Gamma, \phi \) for a true closed a.p.f. \( \phi \).

3. \( \Gamma, \Delta_0 \)
where there exists a sequent \( \Delta_1 \) so that \( \Delta = \Delta_0 \cup \Delta_1 \) is an instance of a defining axiom for \( R \) in 1 and \( \Delta_1 \) consists solely of false closed a.p.f.’s.

Any true closed a.p.f. in an arithmetical axiom is said to be a principal formula of the axiom.

**stage prime axioms** 1. \( \Gamma, t_0 \neq t_0, \Gamma, t_0 \neq t_1, t_1 \neq t_2, t_0 < t_2, \text{ and } \Gamma, t_0 < t_1, t_0 = t_1, t_1 < t_0 \text{ for terms } t_0, t_1, t_2 \text{ of stage sort.} \)

2. \( \Gamma, \phi \) for a true closed s.p.f. \( \phi \).

3. \( \Gamma, t \notin I^O \).

Each true closed s.p.f. in a stage prime axiom is said to be a principal formula of the axiom.

Observe that the relation 'a sequent \( \Gamma \) is an axiom in \([\Pi^0_1, \Pi^0_1]\)-Fix' is elementary recursive and hence so is the relation 'P is a proof in \([\Pi^0_1, \Pi^0_1]\)-Fix' with the inference rules defined below.

**Inference rules** in \([\Pi^0_1, \Pi^0_1]\)-Fix are:

\( (\wedge), (\vee), (b \forall)^N, (b \exists)^N, (\forall)^N, (\exists)^N, (b \forall)^O, (b \exists)^O, (\forall)^O, (\exists)^O, (I), (\neg I), (\text{cut}) \) and \( (VJ), (TJ), (Cl) \).
1. Basic rules \((\land), (\lor), (b\forall)^N, (b\exists)^N, (\forall)^N, (b\forall)^O, (b\exists)^O, (\forall)^O, (\exists)^O, (I), (\neg I)\):

In these rules the principal formula is contained in the upper sequent. For example

\[
\frac{\varphi_0 \lor \varphi_1, \Gamma}{\varphi_0 \lor \varphi_1, \Gamma} (\lor) \quad \frac{\exists x \leq t \varphi(x), u \leq t \land \varphi(u), \Gamma}{\exists x \leq t \varphi(x), \Gamma} (\exists)^N
\]

where \(i = 0, 1\), \(u\) is a number term. The minor formula of these rules are defined to be the formula \(\varphi_i\) in \((\lor)\), and \(\varphi(u)\) in \((b\exists)^N, (\exists)^N\), resp. The term \(u\) in \((b\exists)^N, (\exists)^N\) is the witnessing term of the rules.

\[
\frac{\exists a < t \varphi(a), s < t \land \varphi(s), \Gamma}{\exists a < t \varphi(a), \Gamma} (b\exists)^O \quad \frac{\exists a \varphi(a), \varphi(s), \Gamma}{\exists a \varphi(a), \Gamma} (\exists)^O
\]

where \(s, t\) are stage terms. The minor formula of these rules are defined to be the formula \(\varphi(s)\) both in \((b\exists)^O\) and in \((\exists)^O\).

For a number term \(t\) and a stage term \(s\),

\[
\frac{t \in I^s, A(I^{<s}, t), \Gamma}{t \in I^s, \Gamma} (I) \quad \frac{t \in I^s, t \in I^{<s}, \Gamma}{t \in I^s, \Gamma} (I)
\]

\(A(I^{<s}, t)\) and \(t \in I^{<s}\) are the minor formula of the rules \(I\).

\[
\frac{\Gamma, t \notin I^s}{\Gamma, t \notin I^s, t \notin I^{<s}} (\neg I)
\]

2. In the rule \((cut)\)

\[
\frac{\Gamma, \neg \varphi, \varphi, \Delta}{\Gamma, \Delta} (cut)
\]

the cut formula \(\varphi\) is an \(\exists\)-formula.

3. \(\varphi(0)\)

\[
\frac{\Gamma, \varphi(0), \neg \varphi(x), \Gamma, \varphi(x'), \neg \varphi(t), \Gamma}{\Gamma} (VJ)
\]

for any \(\forall\)-formulae \(\varphi\) and number terms \(t\), where \(x\) is the eigenvariable. \(t\) is said to be the induction term of the \((VJ)\).

4. \(\neg \varphi(t)\)

\[
\frac{\neg \varphi(t), \Gamma, \neg \forall b < a \varphi(b), \varphi(a)}{t \not\in s, \Gamma} (TJ)
\]

for any \(\forall\)-formulae \(\varphi\) and any stage terms \(t, s\), where \(a\) is the eigenvariable. \(\varphi(a)\) is said to be the induction formula and \(s\) the induction term of the \((TJ)\).
5. For an eigenvariable $a$,

$\Gamma, \forall y B_{0}(I^{<s}, t, y) \quad a \not\in s, \neg \forall y B_{0}(I^{<a}, t, y), \Gamma$

\[ (Cl.R) \]

where $s$ denotes either a stage variable or a constant $A \in L$.

$\Gamma, \forall y B_{i}(I^{<\mu}, t, y) \quad \neg R(a), \neg \forall y B_{i}(I^{<a}, t, y), \Gamma$

\[ (Cl,\mu) \]

where $i = 0, 1$.

For a stage variable or a stage constant $s$, let $R_{s}(a) :\equiv (0 \in s)$.

$(R(\mu)) :\equiv (0 \in \mu)$, and $(R_{\mu}(a)) :\equiv (R(a))$.

These two rules are then unified to the following rule:

$\Gamma, \forall y B_{i}(I^{<s}, t, y) \quad \neg R_{s}(a), a \not\in s, \neg \forall y B_{i}(I^{<a}, t, y), \Gamma$

\[ (Cl.s) \]

where $s$ denotes either a stage variable or a constant $A \in L \cup \{\mu\}$, and $i = 0$ when $s \neq \mu$.

**Definition 4.2** For each formula $\varphi$ and sequent $\Gamma$, let $P_{\Gamma, \varphi}$ denote a canonically constructed proof of $\Gamma, \neg \varphi, \varphi$ using logical axioms and rules ($\lor$, $\land$, $\exists$, $\forall$).

1. If $\varphi$ is an a.p.f. or a s.p.f. or a formula of the shape $t \in I^s$, then $P_{\Gamma, \varphi}$ denotes the logical axiom $\Gamma, \neg \varphi, \varphi$.

2. If $\varphi \equiv (\theta_{0} \lor \theta_{1})$, then for $\Delta = \Gamma, \neg \varphi$,

$P_{\Gamma, \varphi} = \Delta, \neg \theta_{0}, \theta_{0} \quad \Delta, \neg \theta_{1}, \theta_{1} \quad \Delta, \neg \theta_{0}, \theta_{1} \quad \Delta, \neg \theta_{1}, \theta_{0} \quad \Delta, \neg \theta_{0}, \theta_{0} \quad \Delta, \neg \theta_{1}, \theta_{1}$

\[ (\lor) \]

3. If $\varphi \equiv (\exists a < t \theta(a))$, then for $\Delta = \Gamma \cup \{\neg \varphi, \varphi, a \not\in t \lor \neg \theta(a), a < t \land \theta(a)\}$,

$P_{\Gamma, \varphi} = \Delta, a \not\in t, a < t \quad \Delta, a < t \quad \Delta, \neg \theta(a), \theta(a) \quad \Delta, \theta(a) \quad (\lor) \quad (\land) \quad (\lor)$

\[ (b\exists)^{\mathcal{O}} \]

4. And similarly for the cases $\varphi \equiv (\exists x < t \theta(x)), (\exists x \theta(x)), (\exists a \theta(a))$. 

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Definition 4.3 The rank \( \text{rk}(\varphi) \in \{A + n : A \in \text{lb}(\varphi) \cup \{0, \mu\}, n < \omega\} \) and the label complexity \( \text{lk}(\varphi) \in \text{lb}(\varphi) \cup \{0, \mu\} \) of a formula \( \varphi \) in \( \mathcal{L}_H \) are defined recursively. Let \( Q \in \{\forall, \exists\} \).

1. \( \text{rk}(\varphi) = \text{lk}(\varphi) = 0 \) for an a.p.f. or a s.p.f. \( \varphi \).

2. \( \text{rk}(Qx \leq t \varphi) = \text{rk}(Qx \varphi) = \text{rk}(\varphi) + 1 \) and \( \text{lk}(Qx \leq t \varphi) = \text{lk}(\varphi) \) for the number variable \( x \).

3. For \( \circ \in \{\land, \lor\} \), \( \text{rk}(\varphi_0 \circ \varphi_1) = \max\{\text{rk}(\varphi_0), \text{rk}(\varphi_1)\} + 1 \) and \( \text{lk}(\varphi_0 \circ \varphi_1) = \max\{\text{lk}(\varphi_0)\} \).

4. \( \text{rk}(Qa \varphi(a)) = \mu \) if \( \varphi(a) \) is bounded. Otherwise \( \text{rk}(Qa \varphi(a)) = \text{rk}(\varphi(0^O)) + 1 \).

5. \( \text{rk}(Qa < B \varphi(a)) = \max\{\text{rk}(\varphi(0^O)) + 1, B\} \).

6. \( \text{rk}(Qa < 0^O \varphi(a)) = \text{rk}(\varphi(0^O)) \).

7. For variables \( b \), \( \text{rk}(Qa < b \varphi) = \mu \) if \( Qa < B \varphi(a) \) is bounded. Otherwise \( \text{rk}(Qa < b \varphi) = \text{rk}(\varphi(0^O)) + 1 \).

8. \( \text{rk}(t \in I^B) = \text{rk}(t \notin I^B) = B + (d_A + 1) \), where \( d_A = \text{rk}(\mathcal{A}(I^O^A, t)) \) denotes the depth of (number) quantifiers and propositional connectives \( \land, \lor \) in \( \mathcal{A}(X, t) \).

9. \( \text{rk}(t \in I^O^A) = \text{rk}(t \notin I^O^A) = 0 \).

10. For variables \( b \), \( \text{rk}(t \in I^b) = \text{rk}(t \notin I^b) = \mu \) and \( \text{lk}(t \in I^b) = \text{lk}(t \notin I^b) = 0 \).

Observe that \( \text{rk}(t \in I^{<\infty}) = \text{rk}(\exists a(t \in I^a)) = \mu \), while \( \text{rk}(t \in I^{<A}) = \text{rk}(\exists a < A(t \in I^a)) = \max\{\text{rk}(t \in I^O^A) + 1, A\} = A \) for \( A \in L \).

Lemma 4.4 For any constants \( \mu \neq A, B \in L \), the following hold.

1. Let \( \varphi \) be a closed formula. Then \( \text{rk}(\varphi) < \mu \iff \varphi \) is bounded, and \( \text{rk}(\varphi) = \text{lk}(\varphi) + n \) for an \( n < \omega \).

2. \( \text{rk}(\varphi) < \mu \iff \varphi \) is bounded, and there occurs no subformulas \( Qa < b \theta \), \( t \in I^b, t \notin I^b \) with variables \( b \) in \( \varphi \).

3. \( \text{rk}(\neg \varphi) = \text{rk}(\varphi) \).

4. For each formula \( \varphi \), there exists a label \( A \in \text{lb}(\varphi) \cup \{0, \mu\} \) such that \( \text{rk}(\varphi) \leq A + \max\{d_A + 1, n\} \), where \( n \) denotes the number of occurrences of logical connectives \( \land, \lor, \forall, \exists \) in \( \varphi \).

5. \( \text{rk}(\varphi_i) < \text{rk}(\varphi_0 \lor \varphi_1) \) for \( i = 0, 1 \).
6. $\text{rk}(\varphi(\bar{n})) < \text{rk}(\exists x \varphi(x))$ for the $n$-th numeral $\bar{n}$.

7. $\text{rk}(\varphi(A)) < \text{rk}(\exists a \varphi(a))$ if $\varphi(A)$ is closed.

8. Assume that $A < B$. Then $\text{rk}(\varphi(A)) < \text{rk}(\exists a < B \varphi(a))$.

9. $\text{rk}(\mathcal{A}(I^<A, \bar{n})) < \text{rk}(\bar{n} \in I^A)$.

**Proof.** Lemma 4.4.7 and 4.4.8 follow from the facts that $\text{rk}(\varphi(0^0)) = \text{rk}(\varphi(A))$ for unbounded $\varphi$, and for bounded $\varphi$, $\text{rk}(\varphi(A)) \in \{\text{rk}(\varphi(0^0))\} \cup \{A + n : n < \omega\}$.

For Lemma 4.4.9 first observe that $\text{rk}(t \in I^<A) = A$. This yields $\text{rk}(\mathcal{A}(I^<A, \bar{n})) = A + d_A < \text{rk}(\bar{n} \in I^A)$. \qed

**Definition 4.5** We write $Qa < \mu$ for unbounded stage quantifier $Qa$. For stage constants $A$ and formulas $\varphi$, $\varphi^A$ denotes the result of restricting any unbounded stage quantifiers $Qa < \mu$ to $Qa < A$ in $\varphi$.

$\Gamma^A := \{\varphi^A : \varphi \in \Gamma\}$ for sequents $\Gamma$.

For example $(\forall y B_i(I^<\mu, x, y))^A \equiv (\forall y B_i(\{z : \exists a (z \in I^a)\}, x, y))^A \equiv (\forall y B_i(\{z : \exists a < A (z \in I^a)\}, x, y) \equiv (\forall y B_i(I^<A, x, y))$.

The following definition is needed to handle bounded number quantifiers and propositional connectives, cf. subsections 4.3 and 5.2.

**Definition 4.6** Resolvents of a (closed) formula $\varphi$ are defined recursively as follows.

1. $\Delta = \{\varphi\}$ is a resolvent of $\varphi$.

2. There is a resolvent $\Delta_1 \cup \{\theta_0 \lor \theta_1\}$ such that $\Delta = \Delta_1 \cup \{\theta_0, \theta_1\}$.

3. There is a resolvent $\Delta_1 \cup \{\theta_0 \land \theta_1\}$ such that $\Delta = \Delta_1 \cup \{\theta_i\}$ for an $i = 0, 1$.

4. There is a resolvent $\Delta_1 \cup \{\exists x \leq m \theta(x)\}$ such that $\Delta = \Delta_1 \cup \{\theta(k) : k \leq m\}$.

5. There is a resolvent $\Delta_1 \cup \{\forall x \leq m \theta(x)\}$ such that $\Delta = \Delta_1 \cup \{\theta(k)\}$ for a $k \leq m$.

Let $A \in L \cup \{\mu\}$. A bounded formula $\theta$ is a $\Delta^A$-formula if each stage constant $C$ occurring in $\theta$ is $C < A$. A formula $\varphi$ is a $\Sigma^A$-formula if either $\varphi$ is $\Delta^A$ or $\varphi \equiv (\exists a < A \theta(a))$ with a $\Delta^A$-formula $\theta$. A $\Pi^A$-formula is defined to be the dual of a $\Sigma^A$-formula.

**Definition 4.7** The system $[\Pi^0_1, \Pi^0_1]$-$\text{Fix} + \forall x B(x)$

Let $\forall x B(x)$ denote a fixed true $\Pi^0_1$-sentence with an a.p.f. $B$. The system $[\Pi^0_1, \Pi^0_1]$-$\text{Fix} + \forall x B(x)$ is obtained from $[\Pi^0_1, \Pi^0_1]$-$\text{Fix}$ by adding the axioms

$$\begin{align*}
(B) \Gamma, B(t)
\end{align*}$$
for arbitrary terms \( t \) of number sort, and five inference rules; the padding rule
\( H_0(\text{pad})H_1 \), the resolvent rule \((\text{res})H\), the rank rule \((\text{rank})C\) the height rule \((h)\), and the collapsing rule \((c)_{A(H)}^\Delta\)

\[
\frac{\Gamma, \Delta}{\Gamma, \Delta} 
\frac{\text{H}_0(\text{pad})H_1}{t \leq \bar{n} \land \varphi(t). \bigcup_{k \leq n} \Pi_k, \Delta_0} 
\frac{(\text{res})H}{\Gamma} 
\frac{(\text{rank})C, (h)}{\Gamma^A \frac{\Delta}{\Gamma^{dA(H)}} (c)_{A(H)}^\Delta} 
\]

where \( R(A) \) and \( \Gamma \) denotes a finite set of closed subformulas of \( \Pi_1^0 \)-formulae \( \forall y B_i(I^{< \omega}, n, y) \) with numerals \( n \). Each formula in \( \Gamma^A \) is obtained from \( \Sigma^A \)-formulas, \( \Pi^A \)-formulas by propositional connectives \( \lor, \land \) and bounded number quantifications \( \exists x \leq t, \forall x \leq t \). In \( (\text{res}) \) each \( \Pi_k \) is a resolvent of the formula \( \varphi(k) \), and \( t \equiv y, y' \) for a variable \( y \). The formula \( t \leq \bar{n} \land \varphi(t) \) is the minor formula of the \( (\text{res}) \).

In \( (\text{res}) \), \( \varphi \) is a subformula of one of \( B_i(I^{< B}, n, m) \) and \( -B_i(I^{< B}, n, m) \) for some numerals \( n, m \).

A proof in the system \([\Pi_1^0, \Pi_1^0]-\text{Fix} + \forall x B(x)\) is a finite labelled tree of sequents which is locally correct with respect to the axioms and inference rules in Definition 4.1.

**Definition 4.8** Let \( P \) be a proof.

1. For finite sequences \( t, s \in {< \omega} \omega \) of natural numbers, \( t \preceq s \) iff \( t \) is an initial segment of \( s \) in the sense that \( s = t \ast u \) for a \( u < \omega \).

2. \( \text{Tr}(P) \subset {< \omega} \omega \) denotes the underlying tree of \( P \), where the endsequent corresponds to the root \( \epsilon \) (the empty sequence), and if a lowersequent \( \Gamma \) of a rule \( t \ast (\varepsilon) : J \) corresponds to a node \( t \), then its uppersequents \( \Lambda_0, \ldots, \Lambda_n \) correspond to \( t \ast (0, 0), \ldots, t \ast (0, n) \), resp.

\[
\frac{t \ast (0, 0) : \Lambda_0 \quad \cdots \quad t \ast (0, n) : \Lambda_n}{t \ast (0) : J} 
\]

3. For a node \( t : \Gamma \) designates that the sequent \( \Gamma \) is situated at the node \( t \) in \( P \).

4. For each node \( t \in \text{Tr}(P) \), \( P \uparrow t \) denotes the subproof of \( P \) whose endsequent is the sequent corresponding to the node \( t \).

5. For each node \( t \in \text{Tr}(P) \), \( L(P \uparrow t) \) denotes the set of stage constants occurring in the subproof \( P \uparrow t \).

\( L(P) = L(P \uparrow \epsilon) \) denotes the set of stage constants occurring in \( P \).

**Definition 4.9** Let \( P \) be a proof (in \([\Pi_1^0, \Pi_1^0]-\text{Fix} + \forall x B(x)\)) and \( t : \Gamma \) a node in the proof tree \( \text{Tr}(P) \). We define the height \( h(t) = h(t; P) \in \omega \) in \( P \) as follows:

1. \( h(\epsilon; P) = h(\epsilon; P) = 0 \) if \( \epsilon : \Gamma \) is the endsequent of \( P \).

In what follows let \( t : \Gamma \) be an upper sequent of a rule \( J \) with its lower sequent \( s : \Delta \):

\[
\frac{\cdots t : \Gamma \quad \cdots}{s : \Delta} 
\]

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2. \( h(t) = h(s) + 1 \) if \( J \) is an (h).
3. \( h(t) = h(s) \) otherwise.

In a proof \( P \), each lowest rule \( h(t) \) with \( h(t; P) = 0 \) for its lowersequent \( t : \Gamma \) is denoted \( (D) \), cf. Definition 4.11.14.

**Definition 4.10** Let \( P \) be a proof (in \( [\Pi_1^0, \Pi_1^0]-\text{Fix} + \forall \forall B(x) \)). We define the label height \( lh(t) = lh(t; P) \in L \cup \{0, \mu\} \) of nodes \( t : \Gamma \) in \( P \) as follows.

1. \( lh(\epsilon; P) = 0 \) if \( \epsilon : \Gamma \) is the endsequent of \( P \).
2. \( lh(t) = \mu \) if \( h(t) > 0 \).

In what follows let \( t : \Gamma \) be an upper sequent of a rule \( J \) with its lower sequent \( s : \Delta \) such that \( h(t) = 0 \):

\[
\cdots t : \Gamma \quad \cdots \quad s : \Delta \quad J
\]

3. \( lh(t) = \max\{lh(s), lq(\varphi)\} \) if \( J \) is one of basic rules, \((res)\) and \((cut)\), where \( \varphi \) denotes the minor formula of \( J \) when \( J \) is one of the basic rules and \((res)\), and \( \varphi \) is the cut formula when \( J \) is a \((cut)\).
4. \( lh(t) = lh(s) \) otherwise.

**Definition 4.11** Let \( P \) be a proof in \( [\Pi_1^0, \Pi_1^0]-\text{Fix} + \forall \forall B(x) \). Let \( \Omega \) be an assignment of a hydra \( \Omega(t) = \Omega(t; P) \in H \) to each occurrence of a sequent \( t : \Gamma \) in \( P \). Let \( \Omega \) assigns a label \( \Omega(s) \in L \cup \{0\} \) to rules \( s : (D) \). From the assignment \( \Omega \), its fixed part \( \Omega_f(t) := (\Omega(t))_f \) is determined by Definition 2.2.

If the assignment \( \Omega \) enjoys the following conditions, then we say that \( \Omega \) is a hydra assignment for \( P \). For simplicity we write \( \Omega(t) \) for \( \Omega(t; P) \).

1. \( \Omega(t) = 1 \) for each axiom \( t : \Gamma \).

Assume that \( t : \Gamma \) is the lower sequent of a rule \( s : J \) and \( \{t_i : \Gamma_i\}_{i \leq m} (m = 1, 2, 3) \) denote the upper sequents of \( J \).

\[
\frac{t_0 : \Gamma_0 \quad t_1 : \Gamma_1 \quad t_2 : \Gamma_2}{t : \Gamma} \quad s : J
\]

2. \( \Omega(t) = \Omega(t_0) \) if \( J \) is one of rules \((b \forall)^N, (\forall)^N, (b \forall)^O, (\forall)^O, (\neg I), (c)_B^A, (\forall)\) and \((b \exists)^N\).
3. \( \Omega(t) = \Omega(t_0) + \Omega(t_1) \) if \( J \) is \((\land)\).
4. \( \Omega(t) = \Omega(t_0) + H \) for a non-zero hydra \( H \neq 0 \) if \( J \) is one of rules \((\exists)^N, (b \exists)^O, (\exists)^O, (I) \). In this case we write, e.g., \((I)_H \) for the rule \((I) \).
Let $P_{\vdash \phi}$ be a canonically constructed proof of $\Gamma, \neg \phi, \phi$ using logical axioms and rules ($\lor$, $(\land)$, $(b \exists \forall)^{N}$, $(b \forall \exists)^{N}$, $(b \forall \forall)^{O}$, $(b \forall \exists)^{O}$, $(\exists)O$, $(\forall)O$ in Definition 4.2. Then let $\alpha_{\phi}$ denote the (finite) ordinal canonically associated to $P_{\vdash \phi}$.

Namely $H_{\phi} = 1$ if $\phi$ is an a.p.f. or a s.p.f. or a formula of the shape $t \in I^{s}$. $H_{\phi} = H_{\theta_{0}} + H_{\theta_{1}}$ if $\phi \equiv (\theta_{0} \lor \theta_{1})$. $H_{\phi} = H_{\theta} + 4$ if $\phi \equiv (\exists a < t \theta(a))$. $H_{\phi} = H_{\theta} + 1$ if $\phi \equiv (\exists x < t \theta(x)), (\exists x \theta(x)), (\exists a \theta(a))$.

5. $\Omega(t) = H_{0} + \Omega(t_{0}) + H_{1}$ if $J$ is a $H_{0}(pad)_{H_{1}}$.

6. $\Omega(t) = \Omega(t_{0}) + H$ if $J$ is a $(res)_{H}$.

7. Let $J$ be a $(cut)$ with the cut formula $\theta$.

$$\Omega(t) = \begin{cases} \varphi(rk(\theta); \Omega(t_{0}) + \Omega(t_{1})) & \text{if } h(t) = 0 \& 0 \neq rk(\theta) < \mu \\ \Omega(t_{0}) + \Omega(t_{1}) & \text{otherwise} \end{cases}$$

8. Let $J$ be a $(rank)_{C}$.

$$\Omega(t) = \begin{cases} \varphi(C; \Omega(t_{0})) & \text{if } h(t) = 0 \\ \Omega(t_{0}) & \text{otherwise} \end{cases}$$

9. Let $J$ be a $(VJ)$ with the induction term $t$. $\Omega(t) = (\Omega(t_{1}) + 1) \cdot mj(t)$ where $\Omega(t_{1}) = \Omega(t_{0}) + \Omega(t_{2}) < \omega$, $mj(t) = *_{\omega}$ if $t$ is a variable. Otherwise $t$ is a numeral $\bar{n}$. Then $mj(t) \in \{1 + n, *_{\omega}\}$.

10. Let $J$ be a $(TJ)$ with the induction formula $\varphi(a)$ and the induction term $s$. $\Omega(t) = (\Omega(t_{0}) + \Omega(t_{1})) \cdot mj(s)$, where $\Omega(t_{1}) = H_{\phi}, mj(s) = *_{\mu}$ if $s$ is a variable. Otherwise $s$ is a constant $A$, and $mj(s) \in \{A, *_{\mu}\}$.

$$\frac{\Gamma, \neg \forall b < a \varphi(b), \varphi(a)}{\neg \varphi(t), \Gamma} \quad (TJ)$$

11. $\Omega(t) = \{\mu\}(\Omega(t_{0}) + \Omega(t_{1}))$ if $J$ is a $(Cl, \mu)$.

12. If $J$ is a $(Cl, B)$ with $B \neq \mu$, then $\Omega(t) = \{B^{*}\}(\Omega(t_{0}) + \Omega(t_{1}))$ where $B^{*} \in \{B, *_{\mu}\}$.

13. $\Omega(t) = \omega(\Omega(t_{0}))$ if $J$ is an $(h)$ with $h(t) > 0$.

14. $\Omega(t) = D(\Omega(s); \Omega(t_{0}))$ for $\Omega(s) \in L^{*}$ if $J$ is a $(D)$, i.e., an $(h)$ with $h(t) = 0$. In this case the rule $(D)$ is denoted by $(D_{A})$ or by $(D_{C})$ with $C = \Omega(s)$.

For a hydra assignment $o$ for a proof $P$ we set $\Omega(P) = \Omega(\epsilon : \Gamma_{end})$ with the endsequent $\epsilon : \Gamma_{end}$ of $P$. 


For hydras and labels $H, H_0$ and labels $B \in L \cup \{0, \mu\}$,

\[
H \ll_B H_0 \iff o(H_0) < o(H) \land \forall \tau \geq o(B)(K_\tau o(H) < d_\tau o(H_0))
\]

\[
H \ll_B H_0 \iff H = H_0 \lor H \ll_B H_0
\]

\[
H \ll_B H_0 \iff o(H_0) < o(H) \land \forall \tau > o(B)(K_\tau o(H) < d_\tau o(H_0))
\]

\[
H \ll_B H_0 \iff H = H_0 \lor H \ll_B H_0
\]

**Definition 4.12** Let $P$ be a proof in $[\Pi_1^0, \Pi_1^0]$-\text{Fix} $+ \forall x B(x)$ ending with the empty sequent, $\Omega$ a hydra assignment for $P$. Also $lb$ is a finite set of labels.

We say that the triple $(P, \Omega, lb)$ is a **regular proof** if the following conditions are fulfilled:

**(p0)** Let $t : \Gamma$ be an uppersequent of a (cut) with its cut formula $\theta$ in $P$. Then $rk(\theta) < \mu + h(t; P)$.

For a lower sequent $t : \Gamma$ of rules $(VJ), (TJ)$ in $P$, $h(t; P) > 0$.

For a lower sequent $t : \Gamma$ of rules $(Cl.B)$ with $B^* \in \{\mu, \ast \mu\}$, $h(t; P) > 0$.

**(p1)** Let $t$ be a node such that $h(t; P) = 0$ and $u$ a leaf (an axiom) in $P$ above $t$, i.e., $t \subset u$. Then there exists an $s : (D)$ between $t$ and $u$, $t \subset s \subset u$. In particular $\Omega(P) \in \mathcal{H}_0$.

$L(P) \subset lb$.

For a label $C \in L$, let $t$ be a lowest node such that $h(t; P) = 0$ and $lh(t; P) = C$, and $A \in L(P \uparrow t)$ be a label occurring above $t$ such that $C \leq A$. Then $A \in \Omega_f(t)$.

**(p2)** Let $\frac{t : \Gamma_1; H}{\Gamma_0} s : (c)_{H}^{\mathcal{R}}$ be a rule in $P$.

Then $L(P \uparrow s) \ll_B H_0$ and $H \ll_B H_0$ for $H = \Omega(t; P)$ and $A = d_B(H_0)$.

**(p3)** Let $\frac{t_0 : \Gamma}{\Gamma_1} s : (D)$ be a rule in $P$. Then $L(P \uparrow s) \ll_{\Omega} \Omega(s)$.

Note that by (p2) $L(P \uparrow s) \cap B < C$ holds for rules $s : (c)_{H}^{\mathcal{R}}$, i.e., any constant $C$ occurring above the rule $s$ is $C < A$ if $C < B$.

Observe again that the relation

\[
'x \text{ is a triple } (p, \Omega, lb) \text{ such that } p \text{ is a proof in the system } [\Pi_1^0, \Pi_1^0]-\text{Fix} + \forall x B(x) \text{ with an h.a } \Omega \& \Omega(p) = H''
\]

is elementary recursive.

**Proposition 4.13** Assume $[\Pi_1^0, \Pi_1^0]$-\text{Fix} $+ \forall x B(x)$ is inconsistent. Then there exists a regular proof $(P, \Omega, \emptyset)$. 

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Proof. Let $P_0$ be a proof in $[\Pi^1_1, \Pi^0_0]$-Fix + $\forall x B(x)$ ending with the empty sequent. Leaves for complete induction schema are replaced by the following:

\[
\begin{align*}
\Gamma, \neg A(0), A(0); d & \quad \Gamma, A(y), \neg A(y); d & \quad \Gamma, \neg A(y'), A(y'); d & \quad (\land), (\exists) & \quad \Gamma, \neg A(x), A(x); d & \quad (pad)_1 \\
\Gamma, \neg A(x), A(x); d & \quad \Gamma, \Delta, \neg A(y), A(y') \quad 2d + 1 & \quad (\forall), (\land) & \quad \Gamma, \Delta, A(x); (2d + 2) \cdot * \omega & \quad (V, J)
\end{align*}
\]

where $\Delta = \{ \neg A(0), \exists y (A(y) \land A(y')) \}$, $d = H_{A(y)}$, and $* \omega = mj(x)$.

Leaves for transfinite induction schema are replaced by

\[
\begin{align*}
\Gamma, \forall a < b A(a), \neg \forall a < b A(a); d + 4 & \quad \Gamma, \neg Prg, \forall a < b A(a), A(b); d_0 & \quad (\forall), (\land) & \quad \Gamma, \neg Prg, \forall a < b A(a), A(b); d_0 & \quad (T, J) \\
\Gamma, \Delta, A(a), \neg A(a); d & \quad \Gamma, \neg Prg, \forall a < b A(a); d_0 & \quad (b \forall) & \quad \Gamma, \neg Prg, \forall a < b A(a), A(b); d_0 & \quad (cut) \\
\Gamma, \neg Prg, A(b); d_1 + d_0 & \quad \Gamma, \neg Prg, \forall a < b A(a) \to A(b) \to \forall b A(b); (3d + 5) \cdot *_m + 2d + 5 & \quad (\forall), (\forall)
\end{align*}
\]

where $\Delta = \{ \neg Prg, A(a) \}$ with $Prg \equiv (\forall b(\forall a < b A(a) \to A(b)))$ and $d = H_{A(a)}$, $d + 4 = H_{\forall a < b A(a)}$, $d_0 = 2d + 5$, and $d_1 = 3d + 5$. Also *$_m = mj(b)$.

$P_0$ contains none of rules (c), (pad), (h), and no constant of stage sort occurs in $P_0$ besides $0^0$. Below the endsequent of $P_0$ attach some (h)'s to enjoy the condition (p1). A hydra assignment $\Omega$ for $P$ is chosen canonically, and $\Omega(s) = \emptyset$.

Namely the bottom of $P$ looks like

\[
P = \begin{array}{c}
\vdots \\
\emptyset \\
\emptyset \\
\end{array}
\]

$\epsilon : \emptyset; H \\
\tau : \emptyset; D_0(H)$

The resulting quadruple $(P, \Omega, \emptyset)$ is regular.

\[\square\]

4.3 Inversions

Let $\Omega$ be a hydra assignment for a proof $P$, and $\tau : \Gamma, \theta_1$ a node in $P$.

\[
P = \begin{array}{c}
\vdots \\
P_0 \\
\emptyset \\
\emptyset \\
\end{array}
\]

$\epsilon : \emptyset; D_0(H)$

Let us define a proof $P'_0$ of a $\Gamma, \theta_0$ by inversion so that $\Omega(\tau; P'_0) = H = \Omega(\tau; P)$ according to the formulas $\theta_1$. 

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1. \( \theta_1 \equiv (\forall x \leq \bar{n} \theta(x)) \) for the bounded number quantifier \( \forall x \leq \bar{n} \): For \( k \leq n \), let \( \theta_0 \equiv \theta(\bar{k}) \). To get a \( \Phi' \) by inversion, change \((b\forall)^N\) to \((pad)_0\), and eliminate the false \( k \leq \bar{n} \) if necessary:

\[
\Psi, \forall x \leq \bar{n} \theta(x), y \leq \bar{n} \theta(y); H_0 \sim (b\forall)^N \sim \Psi, \theta(\bar{k}); H_0 (pad)_0
\]

and

\[
\Phi, \varphi(0); H_1 \sim \varphi(x), \Phi, \varphi(x'); H_2 \sim \varphi(y), \Phi; H_3 (V.J)
\]

turns to the following with \( mj(\bar{k}) = *_\omega \).

\[
\Phi, \varphi(0); H_1 \sim \varphi(x), \Phi, \varphi(x'); H_2 \sim \varphi(\bar{k}), \Phi; H_3 (V.J)
\]

The resolvent \( \Pi_m = \Pi'_m \cup \{\forall x \leq \bar{n} \theta(x)\} \) of a \( \varphi \) turns to a resolvent \( \Pi'_m \cup \{\theta(\bar{k})\} \) of the same formula.

\[
t \leq p \land \varphi(t), \bigcup_{i \leq p} \Pi_i, \Delta_0 \sim (res) \sim \bigcup_{i \leq p} \Pi_i, \Delta_0 (res) \]

Moreover when the variable \( y \) in a \((res)\) is replaced by \( \bar{k} \leq \bar{m} \), one of the formulas \( \varphi(\bar{k}) \) and \( \varphi(\bar{k} + 1) \) is replaced by its resolvent \( \Pi_k \) and \( \Pi_{k+1} \) by inversions, resp.

\[
y \leq \bar{m} \land \varphi(y), \bigcup_{k \leq m} \Pi_k, \Delta_0; H_0 \sim \bigcup_{k \leq m} \Pi_k, \Delta_0; H_0 + K (pad) \]

2. \( \theta_1 \equiv (\forall a \theta(a)) \) for the unbounded number quantifier \( \forall a \): Similar to the case for bounded universal number quantifiers, but there is no concern with resolvents.

3. \( \theta_1 \equiv (\forall a \theta(a)) \) for the stage quantifier \( \forall a \): For a stage constant \( C \), let \( \theta_0 \equiv \theta(C) \). To get a \( \Phi' \) by inversion, change \( (\forall)^C \) to \((pad)_0\) if necessary:

\[
\Psi, \forall a \theta(a), H_0; H_0 \sim (\forall)^C \sim \Psi, \theta(A); \beta_0 (pad)_0
\]

and

\[
\Phi, \sim \varphi(s); H_1 \sim \varphi(c), \forall b < c \varphi(b), \Phi; H_2 (T.J)
\]

\( s \not\equiv a_0, \Phi; H_1 + (H_\varphi + H_2) \cdot *_\mu \)
turns to the following with \(mj(A) = *_{\mu}\).

\[
\Phi, \neg \varphi(s); H_1, \varphi(c), \neg \forall b < c \varphi(b), \Phi; H_2 \quad (TJ)
\]

\(s \not\leq A, \Phi; H_1 + (H_\varphi + H_2) \cdot *_{\mu}\)

4. \(\theta_1 \equiv (\forall a < A \theta(a))\) for the stage bounded quantifier \(\forall a\): Similar to the case for unbounded universal stage quantifiers.

5. \(\theta_1 \equiv (\theta_2 \land \theta_3)\): For \(i = 2, 3\), let \(\theta_0 \equiv \theta_i\). To get a \(P'_0\) by inversion, change \((\land)\) to \((pad)\) if necessary:

\[
\frac{\Psi, \theta_2 \land \theta_3, \theta_2; H_2}{\Psi, \theta_2 \land \theta_3; H_2} \quad \frac{\Psi, \theta_2 \land \theta_3, \theta_3; H_3}{\Psi, \theta_3; H_3} \quad \frac{\land}{\sim} \frac{\Psi, \theta_2; H_2 + H_3}{(pad)}
\]

Moreover the resolvent \(\Pi_m = \Pi'_m \cup \{\theta_2 \land \theta_3\}\) of a \(\varphi\) turns to a resolvent \(\Pi'_m \cup \{\theta_i\}\) of the same formula.

\[
t \leq \bar{p} \land \varphi(t), \bigcup_{i \leq p} \Pi_i, \Delta_0 \quad \frac{(res)_K}{\sim} \quad t \leq \bar{p} \land \varphi(t), \bigcup_{i \neq m} \Pi_i, \Pi'_m \cup \{\theta_i\}, \Delta_0 \quad (res)_K
\]

6. \(\theta_1 \equiv (\exists x < \bar{n} \theta(x))\): Let \(\theta_0 = \{\theta(k) : k \leq n\}\). To get a \(P'_0\) by inversion, change \((b3)^N\) to \((pad)\) if necessary: for \(k \leq n\),

\[
\frac{\bar{k} \leq \bar{n} \land \theta(\bar{k}), \exists x < \bar{n} \theta(x), \Delta_0; H_0}{\exists x < \bar{n} \theta(x), \Delta_0; H_0 + K} \quad \frac{(b3)_K}{\sim} \quad \frac{\bar{k} \leq \bar{n} \land \theta(\bar{k}), \exists x < \bar{n} \theta(x), \Delta_0; H_0}{\exists x < \bar{n} \theta(x), \Delta_0; H_0 + K} \quad (pad)_K
\]

where some rules \((\land)\) with the principal formula \(\bar{k} \leq \bar{n} \land \theta(\bar{k})\) is also replaced by paddings together with eliminating the left upper part of the \((\land)\) if \(k \leq n\), and eliminating the left upper part of the \((\land)\) if \(k > n\).

If the witnessing term \(t\) is a variable \(y\), then the rule becomes a \((res)_K\):

\[
\frac{y \leq \bar{n} \land \theta(y), \exists x \leq \bar{n} \theta(x), \Delta_0; H_0}{\exists x \leq \bar{n} \theta(x), \Delta_0; H_0 + K} \quad \frac{(b3)_K}{\sim} \quad \frac{y \leq \bar{n} \land \theta(y), \exists x \leq \bar{n} \theta(x), \Delta_0; H_0}{\exists x \leq \bar{n} \theta(x), \Delta_0; H_0 + K} \quad (res)_K
\]

The case when \(t \equiv y'\) is similar:

\[
\frac{y' \leq \bar{n} \land \theta(y'), \exists x \leq \bar{n} \theta(x), \Delta_0; H_0}{\exists x \leq \bar{n} \theta(x), \Delta_0; H_0 + K} \quad \frac{(b3)_K}{\sim} \quad \frac{y' \leq \bar{n} \land \theta(y'), \exists x \leq \bar{n} \theta(x), \Delta_0; H_0}{\exists x \leq \bar{n} \theta(x), \Delta_0; H_0 + K} \quad (res)_K
\]

Moreover the resolvent \(\Pi_m = \Pi'_m \cup \{\exists x \leq \bar{n} \theta(x)\}\) of a \(\varphi\) turns to a resolvent \(\Pi'_m \cup \{\theta(k) : k \leq n\}\) of the same formula.

\[
\frac{t \leq \bar{p} \land \varphi(t), \bigcup_{i \leq p} \Pi_i, \Delta_0}{\bigcup_{i \leq p} \Pi_i, \Delta_0} \quad \frac{(res)_K}{\sim} \quad \frac{t \leq \bar{p} \land \varphi(t), \bigcup_{i \neq m} \Pi_i, \Pi'_m \cup \{\theta(k)\}, \Delta_0}{\bigcup_{i \neq m} \Pi_i, \Pi'_m \cup \{\theta(k) : k \leq n\}, \Delta_0} \quad (res)_K
\]
7. $\theta_1 \equiv (\theta_2 \lor \theta_3)$: Let $\theta_0 = \{\theta_2, \theta_3\}$. To get a $P'_0$ by inversion, change $(\lor)$ to $(\text{pad})$ if necessary:

$$\frac{\theta_2 \lor \theta_1, \Delta_0; H_0}{\theta_2 \lor \theta_3, \Delta_0; H_0 + K} \quad (\lor)_K \quad \frac{\theta_2, \theta_3, \Delta_0; H_0}{\theta_2, \theta_3, \Delta_0; H_0 + K} \quad (\text{pad})_K$$

Moreover the resolvent $\Pi_m = \Pi'_m \cup \{\theta_2 \lor \theta_3\}$ of a $\varphi$ turns to a resolvent $\Pi'_m \cup \{\theta_2, \theta_3\}$ of the same formula.

$$t \leq \bar{p} \land \varphi(t), \bigcup_{i \leq 3} \Pi_i, \Delta_0 \quad (\text{res})_K \quad t \leq \bar{p} \land \varphi(t), \bigcup_{i \neq m} \Pi_i, \Pi_m \cup \{\theta(k)\}, \Delta_0 \quad (\text{res})_K$$

## 5 Rewritings

In this section we define rewritings on proofs in such a way that each rewriting corresponds to a move on hydras attached to proofs.

Let $P$ be a proof in $[[\Pi^0, \Pi^0]\text{-Fix} + \forall x B(x)]$. num($P$) denotes the set of numerals $\bar{n}$ occurring in $P$, and Fml($P$) denotes the set of formulas occurring in $P$. For formulas $\varphi$, let $q(\varphi)$ denote the number of occurrences of logical connectives $\land, \lor, \forall, \exists$ in $\varphi$. Then let $c(P) = \max(\{n + 1 : \bar{n} \in \text{num}(P)\}) \cup \{q(\varphi) : \varphi \in \text{Fml}(P)\}$ and $A = A + c(P)$ by Lemma 4.4.4.

Next for terms $t$ of number sort, $c'(t)$ denotes a natural number defined as follows. $c'(\bar{n}) = 0$ for numerals $\bar{n}$ and $c'(t') = c'(t) + 1$ if $t$ is not a numeral. Let $w(P)$ denote the set of witnessing terms of rules $(b3)^N, (\exists)^N$ and induction terms of rules $(V, J)$ in $P$. Then let $c'(P) = \max(\{0\} \cup \{c'(t) : t \in w(P)\})$.

Suppose $c'(P) \leq 1$, and let $P_0$ be a proof obtained from $P$ by substituting a numeral $\bar{n}$ for a variable of number sort with $n < c(P)$. We see then that $c'(P_0) \leq c'(P)$ and $c(P_0) \leq c(P) + 1$.

Let $(P_{-2}, \Omega_{-2}, \emptyset)$ be a regular proof in $[[\Pi^0, \Pi^0]\text{-Fix} + \forall x B(x)]$ in which no stage constant except $0^0$ occurs, cf. Proposition 4.13. Suppose $c'(P_{-2}) > 1$. Let us construct a regular proof $P_{-1}$ without stage constant except $0^0$ such that $c'(P_{-1}) \leq 1$. Such a proof $P_{-1}$ is obtained by replacing rules $(b3)^N, (\exists)^N, (V, J)$ with a ‘big’ term $t'$ with $c'(t') > 1$ repeatedly as follows. Replace

$$\frac{\exists x \varphi(x), \varphi(t'), \Gamma}{\exists x \varphi(x), \Gamma} \quad (\exists)^N$$
with a fresh variable $y$ by the following:

$$\exists x \varphi(x), \varphi(t'), \Gamma \quad \vdash \varphi(t'), \varphi(y'), y' = t', y \neq t \quad \text{(cut)}$$

$$\exists x \varphi(x), \varphi(y'), \Gamma, \forall y(y \neq t) \quad \vdash \varphi(t'), \varphi(y'), y \neq t \quad \text{(cut)}$$

$$\exists x \varphi(x), \Gamma \quad \vdash t = t \quad (\exists)^N$$

Then the resulting proof $P_{-1}$ such that $c'(P_{-1}) \leq 1$ can be assumed to be regular for some $\Omega_{-1}$. Otherwise insert some rules $(h)$ for newly arising $(cut)$'s.

Let $c = c(P_{-1})$, and for $k \leq c$, $P_i$ be a proof obtained from $P_{-1}$ by adding a rule $(pad)_{c-k}$ as the last rule:

$$(P_k, \Omega_k, \emptyset)$$

with $\Omega_k = \Omega_{-1}$ is a regular proof such that $c'(P_k) \leq 1$ and $c(P_k) \leq c$. Moreover $(D_0(H) + c - k, \emptyset) \rightarrow_k (D_0(H) + c - k - 1, \emptyset)$ for $0 \leq k < c$. This yields a regular proof $(P_c, \Omega_c, \emptyset)$ such that $c'(P_c) \leq 1$ and $c(P_c) \leq c$.

We construct regular proofs $(P[\ell], \Omega[\ell], lb[\ell])$ for $c \leq \ell < \omega$ in such a way that $(P[c], \Omega[c], lb[c]) = (P_c, \Omega_c, \emptyset)$, $L(P[\ell]) \subseteq lb[\ell]$, and $(P[\ell], lb[\ell]) \rightarrow_\ell (P[\ell + 1], lb[\ell + 1])$ for each $\ell \geq c$. Moreover for each $\ell \geq c$

$$c'(P[\ell]) \leq 1 \& c(P[\ell]) \leq \ell \quad (4)$$

This means that $\{t_\ell\}_\ell$ is an infinite path through the tree $Tr(H_0, lb_0)$, where $H_0 = \Omega_0(P_0)$, $lb_0 = \emptyset$, $t_0 = \epsilon$, and $t_{\ell+1} = t_\ell * (n_\ell)$ such that $(\Omega[\ell + 1])(P[\ell] + 1)$ is the $n_\ell$'s move from $(\Omega[\ell])(P[\ell])$. 

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Let $P = P[\ell], lb = lb[\ell], \Omega = \Omega[\ell]$ and $P' = P[\ell + 1], lb' = lb[\ell + 1], \Omega' = \Omega[\ell + 1]$. Except **Case 3.4** and **Case 3.5** in subsection 5.4 $lb' = lb[\ell + 1] = lb[\ell] = lb$ holds.

**Definition 5.1 Main branch**

Let $P$ be a proof ending with the empty sequent. The main branch of $P$ is a series \( \{t_i : \Gamma_i \}_{i \leq n} \) of occurrences of sequents in $P$ such that:

1. $t_0 : \Gamma_0$ is the endsequent of $P$, i.e., $t_0 = \epsilon$.
2. For each $i < n$ $t_{i+1} : \Gamma_{i+1}$ is the right upper sequent of a rule $J_i$ so that $t_i : \Gamma_i$ is the lower sequent of $J_i$ and $J_i$ is one of the rules $H \cdot (\text{pad})_0, (\text{res})_0, (h), (c)$ and (cut) with a cut formula in one of the shapes $\exists x \varphi, \exists a < s \varphi, \exists a \varphi, t \in I^s$.
3. Either $t_n : \Gamma_n$ is an axiom, or $t_n : \Gamma_n$ is the lower sequent of one of the rules $(\exists)a^N, (\exists)a^O, (\exists) (VJ), (TJ), (I), (Cl)$ and $H \cdot (\text{pad})_H$ and $(\text{res})_H$ with $H \neq 0$, or $t_n : \Gamma_n$ is the lower sequent of a (cut) with an unbounded cut formula in one of the shapes $\varphi_0 \lor \varphi_1$ or $\exists x \leq n \varphi$ for numerals $n$.

The sequent $t_n : \Gamma_n$ is said to be the top (of the main branch) of the proof $P$.

Let $\Phi$ denote the top of the proof $P$ with the hydra assignment $\Omega$. Observe that we can assume $\Phi$ contains no free variable for otherwise substitute $0^N$ for number variables, and $0^O$ for stage variables. The same hydra assignment works for the substituted proof.

In each case below the new hydra assignment $\Omega'$ for the new proof $P'$ is defined obviously from the hydra assignment $\Omega$ and the subscripts $H$ of the displayed padding rules $H \cdot (\text{pad})_H$.

### 5.1 Rewritings by necrosis

In this subsection we consider the cases when the top $\Phi$ is either the lower sequent of a padding $(p)_H = H \cdot (\text{pad})_H$ with $H \neq 0$ or the lower sequent of one of rules one of rules $(p)_H = (b\exists)^N_H, (\exists)^N_H, (b\exists)^O_H, (\exists)^O_H, (VJ)_H, (TJ)_H, (I)_H, (Cl)_H$ and $H \cdot (\text{pad})_H$ with $H \notin \{0, 1\}$ or an axiom $(ax)$.

**Case 1.** $\Phi$ is the lower sequent of a padding $(p)_H = H \cdot (\text{pad})_H$ with $H \neq 0$. Then kill the padding by (Necrosis) $(H, \emptyset) \rightarrow (0, \emptyset)$ in Definition 9.41

\[
P = \Phi; H_0 + H (p)_H \quad P' := \Phi; H_0 \cdot (p)_0\]
**Case 2.** \( \Phi \) is the lower sequent of one of rules one of rules \((p)_H = (b \exists \hat{N}^n_H, (\exists)_H^N, (b \exists)_H^N, (\exists)_H^N, (cut)_H \) with \( H \not\in \{0, 1\} \). Again kill the padding by \((\text{Necrosis})\) \((H, \emptyset) \rightarrow_\ell (1, \emptyset)\) in Definition 2.4.1.

\[
P = \frac{\cdots}{\Phi; H_0 + \hat{H}} (p)_H \quad P' := \frac{\cdots}{\Phi; H_0 + H} (p)_1
\]

**Case 3.** \( \Phi \) is a nonlogical axiom: Then, since \( \Phi \) contains no free variable and \( \forall x B(x) \) is assumed to be true, \( \Phi \) contains a \( \theta \) which is either a true a.p.f. or a true s.p.f. or \( \emptyset \not\in I^\kappa \). Let \( \Phi = \theta, \Delta_0 \) with \( \text{rk}(\theta) = lq(\theta) = 0 \). Eliminate the false prime formula \( \neg \theta \) and insert a \((\text{pad})_0\). \( \Omega(P') \) is obtained from \( \Omega(P) \) by \((\text{Necrosis})\), \((H + K, \emptyset) \rightarrow_\ell (H, \emptyset)\). Note that \( lh(t; P') = lh(t; P) \) since \( lq(\theta) = 0 \).

\[
P = \frac{\cdots}{\Gamma, \theta; H, \Delta_0; K} 1 \quad P' := \frac{\cdots}{\Gamma, \theta; H + K} (\text{pad})_0
\]

**Case 4.** \( \Phi \) is a logical axiom: \( \Phi = \neg \theta, \theta, \Delta_0 \), where \( \theta \) is an a.p.f. or a s.f. Note that the case when \( \theta \equiv (n \in I^A) \) is excluded since the endsequent is empty, and \( n \not\in I^A \) is not an \( \exists \)-formula. Consider a \((\text{cut})\) whose right upper sequent is a sequent \( \neg \theta, \Delta \) with \( \theta \in \Delta \), and \( \theta \) is its cut formula. \( \Omega(P') \) is obtained from \( \Omega(P) \) by \((\text{Necrosis})\).

\[
P = \frac{\cdots}{\Gamma, \theta; H, \neg \theta; \Delta; K} 1 \quad P' := \frac{\cdots}{\Gamma, \theta; H + K} (\text{pad})_0
\]

5.2 Rewritings on bounded logical rules

In this subsection we consider the cases when the top \( \Phi \) is a lower sequent of a \((\text{cut})\) with an unbounded cut formula in one of the shapes \( \varphi_0 \lor \varphi_1 \) or \( \exists x \leq \bar{n} \varphi \).
for numerals $\bar{n}$. Let us consider the latter case, and $P$ be the following.

$$
\begin{align*}
\Gamma, \forall x \leq \bar{n} \phi(x); H_0 \quad & \exists x \leq \bar{n} \phi(x), \Delta; H_1 \\
\vdots & \vdots \\
\tau : \Gamma, \Delta; H_2 & \vdots \\
\Gamma_3; H_3 + 1 & u : \Gamma_3; \omega(H_3 + 1) \\
\end{align*}
$$

where $H_2 = H_0 + H_1 + 1$, and $u$ denotes the uppermost rule ($h$) below the top $\tau$. We have $h(u) > 0$ by (p0) since $\phi$ is unbounded. We obtain $(\omega(H_3 + 1), lb) \rightarrow_\ell (\omega(H_3) \cdot n, lb)$ by Definition 2.4.3, where $n < \ell$ by 4.

For each $k \leq n$, let $P'_0(k)$ be a proof of $\Gamma, \neg \phi(k)$, which is obtained from $P_0$ by inversion. Let $P_i'$ be obtained from $P_1'$ by replacing the formula $\exists x \leq n \phi(x)$ by the set $\{\phi(k) : k \leq n\}$, cf. subsection 4.3 Let $P'$ be the following when $\phi$ is an unbounded $\exists$-formula.

$$
\begin{align*}
\Gamma, \neg \phi(0); H_0 & \vdots \\
\Gamma, \Delta, \neg \phi(0); H_0 + H_1 & \vdots \\
\Gamma_3, \neg \phi(0); H_3 & \vdots \\
\Gamma_3; \omega(H_3) & \vdots \\
\end{align*}
\begin{align*}
\Gamma, \neg \phi(n); H_0 & \vdots \\
\Gamma, \Delta, \neg \phi(n); H_0 + H_1 & \vdots \\
\Gamma_3, \neg \phi(n); H_3 & \vdots \\
\Gamma_3; \omega(H_3) & \vdots \\
\end{align*}
\begin{align*}
\{\phi(k) : k \leq n\}, \Delta; H_1 & \vdots \\
\{\phi(k) : k \leq n\}, \Gamma, \Delta; H_0 + H_1 & \vdots \\
\{\phi(k) : k \leq n\}, \Gamma_3; H_3 & \vdots \\
\{\phi(k) : k \leq n\}, \Gamma_3; \omega(H_3) & \vdots \\
\end{align*}
$$

where $J$ denotes several (cut)'s with unbounded cut formulas $\phi(k)$.

Note that due to the the rewritings in this subsection, we can assume the following. Let $\phi$ be an unbounded formula such that $\phi^B$ is in the upper sequent of a rule $(c)^B_A$ and $\phi^A$ is in its lower sequent. Then $\phi^B$ is one of the formulas $\forall y \exists x (I < B, \bar{n}, y), \Sigma^B$-formula $\bar{m} \in I < B$, or $\Pi^B$-formula $\bar{m} \notin I < B$ for some numerals $\bar{n}, \bar{m}$. This means that when the rule $(c)^B_A$ is on the main branch, $\phi^B$ is a $\Sigma^B$-formula $\bar{m} \in I < B$.

### 5.3 Rewritings on logical rules

In this subsection we consider the cases when the top $\Phi$ is a lower sequent of a rule $J_1$, which is one of basic rules $(\forall)_1$, $(\exists^*_0), (\exists^*_{\bar{n}}), (\exists^*_{\bar{x}}), (I)$ introducing an $\exists$-formula. Let $J$ denote a (cut) at which the descendant of the principal formula of the rule $J_1$ vanishes.
Case 1. Between the top $\Phi$ and $J$, either there is an $(h)$, or there is a (cut) with its height $= 0$. Let $s : J_0$ be the uppermost such rule:

$$P = \frac{\cdots; H_1}{\Phi; H_1 + 1} \frac{\cdots; H + 1}{\varepsilon : \Gamma; d(H + 1)} s : J_0$$

where $H = K + H_1$, $d(H + 1) = D_s(H + 1)$ if $s : J_0$ is a $(D)$, $d(H + 1) = \omega(H + 1)$ if $s : J_0$ is an $(h)$, and $d(H + 1) = \varphi(rk(\varphi); H + 1)$ for the bounded cut formula $\varphi$ of $s : J_0$.

For the minor formula $\theta$ of $J_1$, let

$$P' := \frac{\cdots; H}{\theta'; \Gamma; d(H)} J_0 \frac{\cdots; H}{\varepsilon : \Gamma; d(H) \cdot 2} (p)_{d(H)}$$

where $\theta'$ is a descendant of $\theta$, which may differ from $\theta$ due to rules (c). We have $(d(H + 1), \emptyset) \rightarrow (d(H) \cdot 2, \emptyset)$ by Definition 2.4.3.

It is easy to see that $P'$ is regular for the same ordinal assignment to rules $(D)$.

For example, when $(p)_1 = (b\exists) Q$ with a principal formula $\bar{n} \in I^{<C} :\equiv (\exists a < C(\bar{n} \in I^a)) (C \leq \mu)$ and a minor formula $\bar{n} \in I^B$ with $A < C$, we have $A < B$. 

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in the following figures by \((p2)\) in Definition 4.12

\[
\begin{array}{rll}
\frac{\bar{n} \in I^{A}, \bar{n} \in I^{<C}, \Delta_{0}: H_{1}}{\bar{n} \in I^{<C}, \Delta_{0}: H_{1} + 1} \quad & \text{(b\exists)_{I}^{C}} & \frac{\bar{n} \in I^{A}, \bar{n} \in I^{<C}, \Delta_{0}: H_{1}}{\bar{n} \in I^{A}, \bar{n} \in I^{<C}, \Delta_{0}: H_{1}} \quad & \text{(pad)_{0}}
\end{array}
\]

\[
\begin{array}{rll}
\frac{\bar{n} \in I^{A}, \bar{n} \in I^{<C}, \Delta_{1}: H_{0}}{\bar{n} \in I^{<B}, \Delta_{1}: H_{0} + 1} \quad & \text{u : (c)_{B}^{C}} & \frac{\bar{n} \in I^{A}, \bar{n} \in I^{<C}, \Delta_{1}: H_{0}}{\bar{n} \in I^{A}, \bar{n} \in I^{<B}, \Delta_{1}: H_{0}} \quad & \text{(c)_{B}^{C}}
\end{array}
\]

\[
\begin{array}{rll}
\frac{\bar{n} \in I^{<B}, \Delta_{2}: d(H + 1)}{\bar{n} \in I^{<B}, \Delta_{2}: d(H) \cdot 2} \quad & \text{J_{0}} & \frac{\bar{n} \in I^{<B}, \Delta_{2}: d(H)}{\bar{n} \in I^{<B}, \Delta_{2}: d(H \cdot 2)} \quad & \text{(b\exists)_{d(H)}}
\end{array}
\]

\textbf{Case 2.} The cut formula of \(J\) is unbounded.

Due to \textbf{Case 1} in this subsection, there is no \((h)\) nor \((D)\) between \(\Phi\) and \(J\). Let \(\theta_{1}\) be the unbounded cut formula. The principal formula of \(J_{1}\) is the formula \(\theta_{1}\). Let \(\theta_{0}\) be its minor formula.

\[
P = \frac{\theta_{0}, \theta_{1}, \Delta_{0}: H_{0}}{\theta_{1}, \Delta_{0}: H_{0} + 1} \quad J_{1}
\]

\[
P = \frac{\Gamma, -\theta_{1}: H_{1}}{\Gamma, \Delta: H_{3} + 1} \quad J
\]

\[
P = \frac{\Gamma_{4}: H_{4} + 1}{\tau: \Gamma_{4}: d(H_{4} + 1)} \quad s : J_{0}
\]

where \(H_{2} = H_{3} + H_{0}\) for some \(H_{3}, H_{4} = H_{1} + H_{2}\), and \(s : J_{0}\) is the uppermost rule \((h)\) or \((D)\) below \(J\). \(d = \omega\) if \(h(\tau) > 0\). Otherwise \(d(H) = D_{2}(H)\). Let \(\text{rk}(\theta_{1}) = \mu + n_{1}\) with \(n_{1} \leq h(\tau) + 1\) by \((p0)\), and \(\text{rk}(\theta_{0}) = A + n_{0}\). We have \(n_{0} < n_{1}\) if \(A = \mu\), and \(n_{0} < \ell\). We have \((\omega(H_{4} + 1), \emptyset) \rightarrow_{\ell} (\omega(H_{4}) \cdot 2, \emptyset)\) by Definition 2.4.3 and by Definition 2.4.4 \((D_{2}(H_{4} + 1), lb) \rightarrow (\varphi_{A+n_{0}+1(D_{2}(H_{4}) \cdot 2)}, lb)\) for \(A \in lb(\theta_{0}) \subset L(P) \subset lb\) and \(L(P \uparrow \tau) \ni A_{\infty}, s: (p3)\).

Assuming that \(\theta_{0}\) is a \(\forall\)-formula, let \(P'\) be the following with \(e(H) \equiv H\) if
where \( P'_0 \) is obtained from \( P_0 \) by inversion. When \( d = D_s \), let \( \Omega'_i(s_i) = \Omega(s) \) for \( i = 0, 1 \). We see that \( P' \) is regular as follows. Let \( \theta_1 \equiv \exists a \varphi(a) \) with a bounded formula \( \varphi \), and \( \theta_0 \equiv \varphi(C) \). Consider the condition (p1) for nodes \( t_i \), \( i = 0, 1 \). Assume \( \max\{\ell h(t_0),\ell q(\theta_0)\} = \ell h(t_0); P' \leq C \). Then \( \ell h(t) \leq C \in L(P \uparrow t) \). We obtain \( C \in \Omega_j(t) = \Omega(s) = \Omega'(s_i) = \Omega'_j(s_i) \) by (p1).

In what follows assume that the cut formula of \( J \) is bounded.

Due to Case 1 in this subsection, between \( \Phi \) and \( J \) there is no \((h)\), \((D)\) nor \((cut)\) with its height = 0.

Case 3. There is one of rules rule \((h)\) and \((D)\) below \( J \).

Let \( \theta_1 \) be the bounded cut formula. The principal formula of \( J_1 \) is the formula \( \theta_1 \). Let \( \theta_0 \) be its minor formula.

\[
P = \frac{\theta_0, \theta_1, \Delta_0; H_0}{\theta_1, \Delta_0; H_0 + 1} J_1 \\
\frac{P_0}{\Gamma, \theta_1; H_1} \\
\vdots \\
\frac{\Gamma, \theta_1; H_3}{\Gamma, \Delta; H_4 + 1} \\
\frac{\Gamma_4; H_4 + 1}{\tau: \Gamma_4; d(H_4 + 1)} s_0 \\
\frac{\Gamma_4, \theta_0; H_4}{\Gamma_4, \theta_0; d(H_4)} s_1 \\
\vdots \\
\frac{\Gamma_4, \theta_0; d(H_4)}{\tau: \Gamma_4; e(d(H_4) \cdot 2)}
\]

where \( H_2 = H_5 + H_0 \) for some \( H_5 \), \( H_3 = H_1 + H_2 \), and \( s: J_0 \) is the uppermost rule \((h)\) or \((D)\) below \( J \). \( d = \omega \) if \( h(\tau) > 0 \). Otherwise \( d(H) = D_s(H) \).
Assuming that $\theta_0$ is a $\forall$-formula, let $P'$ be the following.

\[
\begin{array}{c}
\theta_0, \theta_1, \Delta_0; H_0 \\
\theta_1, \theta_0, \Delta_0; H_0 \\
\vdots \\
P_0 \\
\Gamma, \neg \theta_1; H_1 \\
\theta_1, \theta_0, \Delta; H_2 \\
\vdots \\
\Gamma, \Delta, \theta_0; H_3 \\
\Gamma, \Delta, \theta_0; H_3 \\
\vdots \\
P_0' \\
\Gamma, \neg \theta_0; H_1 \\
\vdots \\
\Gamma, \Delta, \theta_0; H_2 \\
\Gamma, \Delta, \theta_0; H_3 \\
\vdots \\
\end{array}
\]

\[
\begin{array}{c}
\gamma : \Gamma, \neg \theta_0; H_1 \\
\vdots \\
\Gamma, \neg \theta_0; d(H_4) \\
\vdots \\
\Gamma, \theta_0; d(H_4) \\
\Gamma, \theta_0; d(H_4) \\
\vdots \\
\end{array}
\]

where $e(H) \equiv H$ when $d = \omega$ with $(\omega(H_4 + 1), 0) \rightarrow \ell (\omega(H_4) \cdot 2, \emptyset)$ by Definition 2.4.3. Otherwise $e(H) \equiv \varphi(\text{rk}(\theta_0); H)$ with $\text{rk}(\theta_0) = A + n < \mu$. In this case $(D_2(H_4 + 1), lb) \rightarrow (\varphi_{A+n+1}(D_2(H_4) \cdot 2), lb)$ by Definition 2.4.4 for $A \leq \Omega(a)$ and $n < \ell$, cf. [4]. $A_{\alpha, \xi, \Omega(a)}$ is seen from $A \in lb(\theta_0) \cup \{0\}$ and (p3). Under the same assignment of ordinals to rules (D), (c), we see that $P'$ is regular as in Case 2 of this subsection.

In what follows assume that there is no rule (h) nor (D) below $J$.

**Case 4.** $J_1$ is one of rules (b$\exists)^N$, (b$\exists)^O$ introducing bounded quantifier whose minor formula $\theta_0$ contains a false immediate subformula. This means for example, $\theta_1 \equiv (\exists a < B\theta(a))$, $\theta_0 \equiv (A < B \land \theta(A))$ with $A \not< B$. Then replace the rule $J_1$ by $(\text{pad})_0$ by Necrosis,

\[
\begin{array}{c}
P = \theta_0, \theta_1, \Delta_0; H_0 \\
\theta_1, \Delta_0; H_0 + 1 \\
\vdots \\
P_0 \\
\theta_1, \Delta_0; H_0 \\
\vdots \\
J_1 \\
\vdots \\
P_0' \\
\theta_0, \theta_1, \Delta_0; H_0 \\
\theta_1, \Delta_0; H_0 \\
\vdots \\
P_0' \\
\theta_1, \Delta_0; H_0 \\
\vdots \\
J_1 \\
\vdots \\
\end{array}
\]

$P_0'$ is obtained from $P_0$ by inversion, and eliminating the false prime formula $A < B$.

**Case 5.** Let $\theta_1'$ be the bounded cut formula. The principal formula $\theta_1$ of $J_1$ may differ from the formula $\theta_1'$ due to rules (c). Let $\theta_0$ be its minor formula.

\[
\begin{array}{c}
\theta_0, \theta_1, \Delta_0; H_0 \\
\theta_1, \Delta_0; H_0 + 1 \\
\vdots \\
P_0 \\
\Gamma, \neg \theta_1'; H_1 \\
\theta_1', \Delta; H_2 + 1 \\
\vdots \\
J \\
\end{array}
\]

where $H_2 = H_5 + H_0$ for some $H_5$, and $h(\tau) = 0$, $H_3 = H_1 + H_2$.  

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Assuming that $\theta_0$ is a $\forall$-formula, let $P'$ be the following with $B + m = \text{rk}(\theta_0') < \text{rk}(\theta_1') = A + n$ and $C := \{C \in L(P \uparrow u) \cup \text{lb}(\theta_0') : \max\{l_h(t), l_q(\theta_0')\} \leq C < l_q(\theta_1')\}$. We have $(\varphi_{A+n}(H_3 + 1), lb) \to_{\ell}(\varphi_{B+m}(\varphi_{A+n}(H_3) + \varphi_C(H_3)), lb)$ by Definition [4] where $B \in lb$, $m < \ell$ by [9], and either $B \leq A$, or $B = A$ and $m < n$.

\[
P' = \frac{\theta_0, \theta_1, \Delta_0; H_0}{\text{delay}} \frac{P_0}{\text{seed}} \frac{\theta_1, \Delta, \theta_0'; H_2}{\text{skew}} \frac{\theta_0'; H_1}{\text{pad}} \frac{\text{rank}_C \theta_0'; H_3}{\text{rank}_C} \frac{\varphi(\text{rk}(\theta_1'); H_3)}{\varphi(\text{rk}(\theta_0'); H_3)} \frac{\varphi(\text{rk}(\theta_1'); H_3)}{A} \frac{\varphi(\text{rk}(\theta_0'); H_3)}{\varphi(\text{rk}(\theta_1'); H_3)}
\]

where $P'_0$ is obtained from inversion. We see that $P'$ is regular as follows. Let us examine the case when $\theta_1'$ is a formula $\exists a < A \theta(a)$, where $\theta_0' \equiv \theta_1$ unless $\theta_1' \equiv (n \in I^{<A})$, $\theta_1 \equiv (n \in I^{<B})$ and there exists a rule $(c)_A^B$ between $J_1$ and $J$. In the latter case with $B = \mu$, $J_1$ is a rule $(\exists)^B$. Otherwise $J_1$ is a rule $(b)^B$. Assuming $\theta_1' \neq \theta_1$, the witnessing constant $A_0$ for $n \in I^{<B}$, i.e., $\theta_0 \equiv (A_0 < B \land n \in I^{A_0})$ is smaller than $A$ if $A_0 < B$, where we can assume that $A_0 < B$ due to Case 4.

First consider the condition (p1) for the node $u'$ in $P'$. Assume $l_h(u') = \max\{l_h(t), l_q(\theta_0')\} > l_h(t)$. Let $C \in L(P' \downarrow u')$ be a label such that $C \geq l_h(u')$. If $C \geq l_q(\theta_1') \geq A$, then we see $C \in L(P \uparrow u)$ since if $C \in l_h(\theta_0)$ and $C \notin l_q(\theta_1')$, then $C \leq l_q(\theta_1')$.

Therefore $C \in \Omega_f(u) \subset \Omega_f'(u')$ by (p1) for $P$. Let $\max\{l_h(t), l_q(\theta_0')\} = l_h(u') \leq C < l_q(\theta_1')$. Then $C \in C$, and hence $C \in \Omega_f'(u')$.

Next the condition (p2) for rules $(c)_A^C$, and (p3) for rules $(D)$ in $P'_0$. Let $v' : J' \equiv (\exists a < A \theta(a))$ be one of such rule, and $\theta_0' \equiv (\forall a < A \theta(a))$, and $\theta_0 \equiv (A_0 < A' \land \theta(A_0))$, where either $A' = A$ or $A = d_A(K)$ for a $K$ due to a rule $(c)_A^C$. The constant $A_0$ may occur in $P'_0$ when the variable $a$ occurs in $\theta(a)$, although it need to do so in $P_0$. We have $A_0 < C \leq l_q(\theta_1')$ if $A_0 < A$.

Consider first the case when $\neg \theta_0' \equiv (\forall a < A \neg \theta(a))$ is in the upper sequent of the corresponding rule $v : J$ in $P$. Then $L(P \uparrow v) \equiv A \not
earrow_{\empset} K$ if $J$ is a rule $(c)_A^C$, and $A \not
earrow_{\empset} \Omega(v)$ if $J$ is a $(D)$. In the former case we have $A < C$, and $A_0 \not
earrow_{\empset} A$, and in the latter $A_0 \not
earrow_{\empset} A < \mu$. Finally consider the case when $\neg \theta_0' \equiv (\forall a < A \neg \theta(a))$ is in the lower sequent of the corresponding rule $v : (c)_A^B$ in $P$, and $\forall a < B \neg \theta(a)$ is in its upper sequent. Let $A = d_B(K)$. We need to show $A_0 \not
earrow_{\empset} K$. We have $A_0 \not
earrow_{\empset} B$ for $L(P \uparrow v) \equiv B \not
earrow_{\empset} K$. On the other hand we have $A_0 < d_B(K) = A$. Hence $A_0 \not
earrow_{\empset} K$.

### 5.4 Rewritings on induction and reflection

In this subsection we consider the cases when the top $\Phi$ is a lower sequent of one of rules $(V, J), (T, J), (C, I)$. 

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Case 1. $\Phi$ is the lower sequent of a $(VJ)$.

\[ P = \frac{\Phi, \varphi(0); H_1 \ \neg \varphi(x), \Phi, \varphi(x'); H_2 \ \neg \varphi(n), \Phi; H_3}{t : \Phi; (H_2 + 1) \cdot mj(n)} \ (VJ) \]

where $H_2 = H_1 + H_3 < \omega$, $\varphi$ is a $\forall$-formula, and $h(t) > 0$ by (p0).

Case 1.1. $mj(n) = \ast \omega$. Let

\[ P' = \frac{\Phi, \varphi(0); H_1 \ \neg \varphi(x), \Phi, \varphi(x'); H_2 \ \neg \varphi(n), \Phi; H_3}{\Phi; (H_2 + 1) \cdot (1 + n)} \ (VJ) \]

where $\ell > n$ with $P = P[\ell]$ by (4). $((H_2 + 1) \cdot \ast \omega, \emptyset) \to_\ell ((H_2 + 1) \cdot (1 + n), \emptyset)$ by Definition 2.4.2.

Case 1.2. $mj(n) = 1 + n$.

When $n > 0$, let $P'$ be the following with $mj(n - 1) = n$. $((H_2 + 1) \cdot (1 + n), \emptyset) \to_\ell ((H_2 + 1) \cdot n + H_2, \emptyset)$ by Definition 2.4.2.

\[ \frac{\Phi, \varphi(0); H_1 \ \neg \varphi(x), \Phi, \varphi(x'); H_2 \ \neg \varphi(n - 1), \Phi, \varphi(n); H_2}{\Phi, \varphi(n); (H_2 + 1) \cdot n \ \Phi; (H_2 + 1) \cdot n + H_2} \ (VJ) \]

\[ \frac{\neg \varphi(n), \Phi; H_3}{\neg \varphi(n), \Phi; H_2} \ (pad)_{H_1} \]

\[ \frac{\Phi; (H_2 + 1) \cdot n + H_2}{(cut)} \]

If $n = 0$, then let $P'$ be the following. $((H_2 + 1) \cdot 1, \emptyset) \to_\ell (H_2, \emptyset)$ by Definition 2.4.2.

\[ \frac{\Phi, \varphi(0); H_1 \ \neg \varphi(0), \Phi; H_3}{\Phi; H_2} \ (cut) \]

Case 2. $\Phi$ is the lower sequent of a $(TJ)$.

\[ P = \frac{\Phi, \varphi(a), \forall b < a \varphi(b); H \ \neg \varphi(A), \Phi; H_\varphi}{t : A \neq B, \Phi; (H + H_\varphi) \cdot mj(B)} \ (TJ) \]

where $A, B$ are constants and $mj(B) \in \{ B, \ast B \}$, $\varphi$ is a $\forall$-formula, and $h(t) > 0$ by (p0).
Case 2.1. $A \not< B$: Then $A \not< B$ is a true s.p.f., and $A \not< B, \Gamma$ is a stage prime axiom. This case is reduced to the Case 2 in subsection 5.1 by (Necrosis).

Case 2.2. $A < B$: Then $A < o(mj(B))$ with $o(B) = B$ and $o(*_{\mu}) = \mu$.

We have $A \vDash \ln : = \ln[\ell]$ with $P = P[\ell]$.

Let with $A = mj(A)$, $(H + H_{\varphi}) \cdot mj(B), \ln) \rightarrow [\ell] ((H + H_{\varphi}) \cdot A + H + H_{\varphi}, \ln)$ by Definition 2.4.2. By (p2) we have $A \ll_{C} H_{0}$ for any rule $s : (c)_{d_{c}(H_{0})}$ occurring below the $(TJ), s \subset_{\ell} \tau$. Hence $\Omega(s_{0}, P^{*}) \ll_{C} H_{0}$ holds for the upper sequent $s_{0}$ of $s$. (p2) is enjoyed, and $P'$ is regular.

\[
\Phi(\varphi(a), \neg \forall a < A \varphi(b); H \varphi(a), \neg \varphi(a), \Phi; H_{\varphi}) \quad (TJ)\; a := A \quad \Phi, \forall a < A \varphi(a) \quad \Phi, \forall \varphi(A), \neg \forall a < A \varphi(b); H \neg \varphi(A), \Phi; H_{\varphi} \quad \text{cut}
\]

Case 3. The top $\tau : \Phi$ is the lower sequent of a $(Cl.B)$.

Let $P$ be the following with $\theta(a) := (\forall y B_{i}(I^{<a}, \infty, y))$:

\[
\Phi, \theta(B); H_{0} \quad \neg R_{B}(a), a \not< B, \neg \theta(a), \Phi; H_{1} \quad \tau : \Phi; \{B^{*}\} \{H_{0} + H_{1}\} \quad (Cl.B)
\]

where $B$ denotes a constant $B \leq \mu$, and $i = 0$ when $B \neq \mu$, $B^{*} = \mu$ when $B = \mu$, and $B^{*} \in \{*_{\mu}, B\}$ otherwise. $\neg R(B)$ is in $\Phi$.

Case 3.1. $B^{*} = *_{\mu}$: Then $h(\tau) > 0$ by (p0). Let $P'$ be the following. We have $(\{*_{\mu}\} \{H_{0} + H_{1}\}, \ln) \rightarrow [\ell] \{B\} \{H_{0} + H_{1}\}, \ln)$ by Definition 2.4.9 for $B \in \ln[\ell]$.

\[
\Phi, \theta(B); H_{0} \quad \neg R_{B}(a), a \not< B, \neg \theta(a), \Phi; H_{1} \quad P' \quad \Phi, \{B\} \{H_{0} + H_{1}\} \quad (Cl.B)
\]

Case 3.2. $B^{*} = B$ and $R(B)$ is false: $B$ is a constant $< \mu$, and $\Phi$ is an axiom with the true s.p.f. $\neg R(B)$. This case is reduced to the Case 2 in subsection 5.1 by (Necrosis).

In what follows suppose that $B^{*} = B$ and $R(B)$ is true.

Case 3.3. Either $B = \mu$ and there exists a rule $(h)$ below the top, or $B \neq \mu$ and below the top, there exists one of rules $(h), (D)$ and $(cut)$ with its cut rank $A + n \geq B$ and $n \leq \ell$ by [1].

Let $s : J$ denote the uppermost such rule, and $\tau : \Gamma$ its lower sequent. Let $d = \omega$ when $J$ is a rule $(h)$, $d = D_{s}$ when $J$ is a $(D)$, and $d = \varphi_{A+n}$ with $B \leq A, n \leq \ell$. 

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\[
\frac{\Phi, \theta(B); H_0 \quad \neg R_B(a), a \not\in B, \neg \theta(a), \Phi; H_1}{\Phi; \{B\}(H) \quad (C|B)}
\]

\[\mathfrak{s} : J\]

where \(H = H_0 + H_1\).

We have by Definition 5.2.4 that \((d(K + \{B\}(H)), lb) \rightarrow \ell (\{B\}(d(K + H) \cdot 2), lb)\).

Let \(P'\) be the following, where \(\Omega'(s_i) = \Omega(s)i = 0, 1\) when \(d = D_s\):

\[
\frac{\Phi, \theta(B); H_0 \quad (pad)H_1}{\Phi, \theta(B); H \quad -R_B(a), a \not\in B, \neg \theta(a), \Phi; H_1 \quad H_0(pad)}
\]

\[\mathfrak{s}_0 \quad \Gamma, \theta(B); d'(K + H) \quad \mathfrak{s}_1 \quad (C|B)\]

\[\mathfrak{t} : \Gamma; \{B\}(d(K + H) \cdot 2)\]

**Case 3.4.** \(B = \mu\) and \(h(t) = 1\) for the top \(\mathfrak{t} : \Phi\).

\[
\frac{\Phi, \theta(\mu); H_0 \quad \neg R(\mu), \neg \theta(\mu), \Phi; H_1}{\mathfrak{t} : \Phi; \{\mu\}(H) \quad (C|\mu)}
\]

\[\mathfrak{u} : \Gamma_0; K + \{\mu\}(H)\]

\[\Gamma; K + \{\mu\}(H)\]

\[\mathfrak{s} : \Gamma; D(C; K + \{\mu\}(H)) \quad (D)\]

where \(H = H_0 + H_1\) and \(\mathfrak{u} : \Gamma_0\) denotes the upper sequent of the uppermost rule \((c)^{\mu}\) below the top \(\mathfrak{t}\) if such a rule exists. Otherwise \(\mathfrak{u} : \Gamma_0 = \Gamma\) is the upper sequent of the rule \((D)\) below the top \(\mathfrak{t}\). We have \(\Gamma_0 \subseteq \Sigma^{\mu}\) due to subsection 5.2. Also \(C \subseteq lb\) such that \(L(P \uparrow \mathfrak{u}) \subseteq C\) for \(lb = lb[\ell]\) and \(P = P[\ell]\) by the condition (p3) for the rule \(\mathfrak{s}_D : (D)\).
Let $P' = P[\ell + 1]$ be the following:

\[
\begin{array}{c}
\Phi, \theta(\mu); H_0 \\
\Phi, \theta(\mu); H \quad \text{(pad)}
\end{array}
\]

\[
\begin{array}{c}
u_0: \Gamma_0, \theta(\mu); K + H \\
\Gamma_0, \theta(A); K + H \quad \text{(cut)}
\end{array}
\]

where $P'_0$ is obtained from the subproof $P_0$ of $P$ by substituting the constant $A$ for the eigenvariable $a$, and eliminating the false formula $\neg R(A)$. The label heights $lh'$ for $P'$ are defined by $lh'(s) = lh(s)$ and $lh'(s_1) = \max\{lh(s), lq(\theta(A))\}$. The condition (p1) is enjoyed for $P'$ since $A \in (D(C \cup \{A\}; K + H))$. Let $A = d_\mu(\{A_p\}(K + H)) \in L$ for $A_p = \max\{\emptyset \cup L(P \uparrow u)\}$. We have $rk(\theta(A)) = \max(i \in A, A_d \leq \ell)$ by (3). For the rules (D) in $P'$, let $\Sigma(\Sigma_D) = C \cup \{A\}$. Then we have $(D(C; K + \{\mu\}(H)), lb) \Rightarrow t \rightarrow (\varphi_{A+\delta_A}(D(C \cup \{A\}; K + H) \cdot 2), lb \cup \{A\})$ by (Production) in Definition 2.4.8.

The condition (p3) is fulfilled for rules $\Sigma_D : (D) (i = 0, 1)$. Let $lb[\ell + 1] = lb' = \{A\} \cup lb$. Consider the condition (p2) for the new rule $\Sigma_D : (D)$ in $P'$. (p2) is fulfilled since $A \leq \mu H_0$ from $L(P \uparrow v_0) \ni A_p \leq \mu H_0$ and $\{A_p\}(K + H) < K + \{\mu\}(H) \leq H_0$.

Third consider the condition (p2) for rules $\Sigma_D : (D)$ below $\Sigma$. We have $B < \mu$. Let $C_1 = d_B(K_1)$, $K_2 = \Omega(v)$ and $K_2' = \Omega'(v)$. We have $K_2 \leq B K_1$ and $L(P \uparrow v_0) \ni A_p \leq B K_1$. We need to show $A \leq B K_1$ and $K_2' \leq B K_2$. We see $\Omega'(s) = \varphi_{A+\delta_A}(D(C \cup \{A\}; K + H) \cdot 2) \leq \mu D(C; K + \{\mu\}(H)) = \Omega(s)$ from $A \leq \mu D(C; K + \{\mu\}(H))$, which in turn follows from $A \leq \mu D(C; K + \{\mu\}(H))$. Hence $A, K_3 \leq B K_2$. On the other hand we have $L(P \uparrow v_0) \ni A_p \leq B K_1$ and $K_2 \leq B K_1$. Hence we obtain $A \leq B K_1$ and $K_2' \leq B K_2$.

Therefore $(P', lb')$ is a regular proof.

**Case 3.5.** $B \neq \mu$ and below the top, there is no rule $(h)$, $(D)$ and (cut) with its cut rank $A + n \geq B$. 

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Let $H = H_0 + H_1$, and $B > lh(t) = lh(t_1)$ with the lowest $t_1 \subseteq t$, cf. Lemma 4.4.1. We have $\Phi_1 \subset \Sigma^B$ due to subsection 6.2, and $H \in H_1$. Let $A_p = \max((L(P \uparrow t_1) \cap B) \cup \{0\})$ and $A = d_B(\{A_p\}(e(H)))$. We have $L(P \uparrow t_1) \cap B \leq A_p \in lb \cup \{0\}$, and \(rk(\theta(A)) = rk(\forall y \forall b < AB_1, I^{<A}, n, y) = A + d_A\) with $d_A \leq \ell$ by \(\textit{H}\).

We have $(e(\{B\}(H)), lb) \rightarrow (\varphi_A + d_A(\varphi_A(e(H)) \cdot 2), lb \cup \{A\})$ by (Production) in Definition 2.4.4. Let $P'$ be the following:

\[
\begin{align*}
\Phi, \theta(B); H_0 & \quad \vdash \quad P_0 \\
\Phi, \theta(B); H & \quad (\text{pad}) \\
\Phi_1, \theta(B); e(H) & \quad \vdash \quad \Phi_1, \theta(A); e(H) \\
\Phi_1, \theta(A); e(H) & \quad \vdash \quad u_0 : \Phi_1, \theta(A); \varphi_A(e(H)) \\
\Phi_1; \varphi_A(e(H)) & \quad \vdash \quad \Phi_1; \varphi_A(e(H)) \cdot 2 \\
\Phi, \theta(B); H_0 & \quad \vdash \quad \neg \theta(A), \Phi; H_1 \\
\Phi, \theta(B); H & \quad (\text{pad}) \\
\Phi_1, \theta(A); e(H) & \quad \vdash \quad \Phi_1, \theta(A); e(H) \\
\Phi_1, \theta(A); e(H) & \quad \vdash \quad u_1 : \neg \theta(A), \Phi_1; e(H) \\
\Phi_1; \varphi_A(e(H)) & \quad \vdash \quad \Phi_1; \varphi_A(e(H)) \cdot 2 \\
\end{align*}
\]

where $P_0'$ is obtained from the subproof $P_0$ of $P$ by substituting the constant $A$ for the eigenvariable $a$, and eliminating the false formula $A \not\subset B$. The conditions (p1) and (p3) are enjoyed for $P'$.

Consider the condition (p2) for the new $t_0 : (c)^B_0$. Let $C \in L(P' \uparrow t_0)$. We have $C \in L(P' \uparrow t_0) \subset L(P \uparrow t)$. $L(P \uparrow t) \cap B \leq A_p$ yields $C \ll_B \{A_p\}(H)$ if $C < B$. Let $C \geq B > lh(t) = lh(t_1)$ for the lowest $t_1 \subseteq t$. Then $C \in \Omega_{t_1}$ by the condition (p1). Hence $C \ll_0 e(H)$, and $C \ll_B \{A_p\}(e(H))$.

Consider the condition (p2) for $s : (c)^B_{C_0}$ in $P'$ other than the new $t_0 : (c)^B_0$. Let $C_1 = d_{C_0}(K)$.

First let $t_1 \subseteq s$. Then $C_0 > B$. We can assume that the formula $a \not\subset B$ is in the upper sequent $s_0$ of $s$, i.e., $B \in L(P \uparrow s)$. We obtain $A \ll_{B^+} B \ll_{C_0} K$, and (p2) is enjoyed.

Second let $s \subset e t_1$ and $C_0 \leq B$. For $C_1 = d_{C_0}(K)$ it suffices to show $A = d_B(\{A_p\}(e(H))) \ll_{C_0} K$. We have by (p2) that $K' = \Omega(s_0; P)$. Obviously $\{A_p\}(e(H)) \ll_{B^+} e(\{B\}(H)) \ll_{C_0} K'$. On the other hand we have $B, A_p, H \ll_{C_0} K$ by (p2) and $B, A_p \in L(P \uparrow t)$. Hence $A = d_B(\{A_p\}(e(H))) <
\[ d_B(K) = C_1, \text{ and } A = d_B(\{A_p\}(e(H))) \ll_{C_0} K. \]

This completes a proof of Theorem 1.1.3. Theorem 1.1.1 is proved similarly. Given a proof of a \( \Sigma^0_1 \)-formula \( \exists x \theta(y, x) \), substitute a numeral \( \bar{n} \) for the variable \( y \), and add a \( \text{(cut)} \) with the cut formula \( \forall x B(x) \equiv \neg \exists y \theta(\bar{n}, y) \).

We obtain \( c = c(P) \geq n + 1 \) for a proof \( P \) of the empty sequent in \([\Pi^0_1, \Pi^0_1] \)-Fix + \( \forall x B(x) \). Let \( P_0 \) be a proof with a \( \langle \text{pad} \rangle \) as the last rule. Begin to rewrite proofs with \( P[\bar{c}] \) as in this section. Assuming that \( \exists y \theta(\bar{n}, y) \) is false, we obtain an infinite path through the tree.

## 5.5 Linearity and a theory \([\Pi^0_1, \Pi^0_1] \)-Fixp

Let \( \Phi \text{-Fixp} \) denote the theory obtained from \( \Phi \text{-Fix} \) by dropping the axiom (3) for the trichotomy. Namely in the weakened theory \(< \) is supposed to be a wellfounded partial order, but the linearity is not assumed.

It is easy to see \( |a| = |b| \Rightarrow I^a = I^b \) without assuming the linearity, where \( |a| \) denotes the rank \( \sup \{ |b| + 1 : b < a \} \). Therefore \( \Phi \text{-Fixp} \) is supposed to be equivalent to \( \Phi \text{-Fix} \). Indeed, the wellfoundedness proofs in \([3]\) are formalizable in \([\Pi^0_1, \Pi^0_1] \)-Fixp. Thus theories KPM, \([\Pi^0_1, \Pi^0_1] \)-Fix , \([\Pi^0_1, \Pi^0_1] \)-Fixp are proof-theoretically equivalent each other (, i.e., have the same \( \Pi^1_1 \)-theorems on \( \omega \)).

Specifically the linearity of the relation \( a < b \) was used only in the proof of Theorem 4.4 in \([3]\). Let \( \Gamma \) denote the operator \( \Gamma_2 \) in Definition 4.2 for \( Od(\mu) \) in \([3]\). The theorem can be restated as follows:

### Lemma 5.2

\[ \Gamma(I^a) \ni \alpha < \beta \in \Gamma(I^b) \Rightarrow \alpha \in I^b. \]

**Proof.** We show the theorem by induction on the natural sum \( |a| \# |b| \) of ordinals (or by main induction on \( a \) with subsidiary induction on \( b \), or vice versa) without assuming the linearity of \(< \).

Suppose \( \alpha < \beta \) and \( \alpha \in \Gamma(I^a) \), \( \beta \in \Gamma(I^b) \). Then \( \alpha \in \mathcal{G}(I^a) \) by the definition of the operator \( \Gamma \). The operator \( \mathcal{G} \) is defined in Definition 3.8.1 in \([3]\).

By IH we have

\[ I^a|\alpha = I^b|\alpha \quad (5) \]

Hence by the persistency of \( \mathcal{G} \), cf. Lemma 3.9 in \([3]\) \( \alpha \in \mathcal{G}(I^a)|\beta = \mathcal{G}(I^b)|\beta \). This suffices to see \( \alpha \in I^b \), cf. \([3]\). \( \square \)

By Lemma 5.2 we obtain the following equivalence.

### Corollary 5.3

The 1-consistency of KPM is equivalent to that of \([\Pi^0_1, \Pi^0_1] \)-Fixp over EA.

We don’t need to interpret the relation \( A < B \) for labels \( A, B \in L \) as in \([\Pi] \), and there is a chance to replace it by a partial order \( A \triangleleft B \), which enjoys \( A \triangleleft B \Rightarrow o(A) < o(B) \). Such a partial order \( A \triangleleft B \) could be defined through moves \( \rightarrow_\ell \) on hydras since labels \( A, B \) are essentially hydras. However the trichotomy seems to be indispensable in defining rewritings, e.g., in Case 3.3 of subsection 5.4. 

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References

[1] T. Arai, Consistency proof via pointwise induction, Arch. Math. Logic 37 (1998), pp. 149-165.

[2] T. Arai, Ordinal diagrams for recursively Mahlo universes, Arch. Math Logic 39 (2000), pp. 353-391.

[3] T. Arai, Proof theory for theories of ordinals I: recursively Mahlo ordinals, Ann. Pure Appl. Logic 122 (2003), pp. 1-85.

[4] T. Arai, Wellfoundedness proofs by means of non-monotonic inductive definitions I: $\Pi^0_2$-operators, Jour. Symb. Logic 69 (2004), pp. 830-850.

[5] W. Buchholz, An independence result for $(\Pi^1_1-C.A)+BI$, Ann. Pure Appl. Logic 33 (1987), pp. 131-155.

[6] W. Buchholz, A note on the ordinal analysis of KPM, in Logic Colloquium ’90, J. Oikkonen and J. Väänänen (eds.), Lect. Notes Logic 2, ASL, Cambridge UP, 1993, pp. 1-9.

[7] W. Buchholz, Relating ordinals to proofs in a perspicuous way, in Reflections on the foundations of mathematics (Stanford, CA, 1998), Lect. Notes Logic 15, ASL, Cambridge UP, 2002, pp. 37-59.

[8] L. Kirby and J. Paris, Accessible independence results for Peano arithmetic, Bull. Lon. Math. Soc. 14(1982), pp. 285-293.

[9] M. Rathjen, Proof-theoretic analysis of KPM, Arch. Math. Logic 30 (1991) pp. 377-403.

[10] W.H. Richter and P. Aczel, Inductive definitions and reflecting properties of admissible ordinals, Generalized Recursion Theory, Studies in Logic, vol.79, North-Holland, 1974, pp.301-381.

[11] Th. Skolem, Proof of some theorems on recursively enumerable sets. Notre Dame J. of Formal Logic 3 (1962), pp. 65-74.