On the invariant measure for the Yang-Mills configuration space in (3+1) dimensions

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Abstract

We consider a gauge-invariant Hamiltonian analysis for Yang-Mills theories in three spatial dimensions. The gauge potentials are parametrized in terms of a matrix variable which facilitates the elimination of the gauge degrees of freedom. We develop an approximate calculation of the volume element on the gauge-invariant configuration space. We also make a rough estimate of the ratio of 0^{++} glueball mass and the square root of string tension by comparison with (2 + 1)-dimensional Yang-Mills theory.
1 Introduction

In a series of recent papers a Hamiltonian analysis of Yang-Mills theories in (2+1) dimensions was developed [1, 2, 3]. This was mainly motivated by the fact that, while it is true that gauge theories of direct physical interest are in (3+1) dimensions, the study of Yang-Mills gauge theories in two spatial dimensions can be useful for two reasons. It can be a guide to the more realistic case of three dimensions, and secondly, gauge theories in two spatial dimensions can be interpreted as an approximation to the high-temperature phase of QCD with the mass gap playing the role of the magnetic mass. (It should be pointed out that, precisely for these reasons, there have been many analyses of (2+1)-dimensional gauge theories starting from the early days [4].) In this paper, we shall start a similar Hamiltonian analysis of Yang-Mills theories in (3+1) dimensions, carrying over some of the lessons from the lower dimensional analysis.

In the (2+1) dimensional theory, the $A_0 = 0$ gauge was chosen and the complex components of the spatial gauge field, viz., $A_z, A_{\bar{z}}$ were parametrized as $A_z = -\partial_z M M^{-1}, A_{\bar{z}} = M^\dagger \partial_{\bar{z}} M^\dagger$, where $M, M^\dagger$ are $SL(N, C)$-matrices for an $SU(N)$ gauge theory. The basic gauge-invariant variable for the theory is then the hermitian matrix field $H = M^\dagger M$. This particular parametrization of the potentials proved to be very useful since the Jacobian for the transformation of variables and the volume element on space of gauge-invariant configurations could be exactly calculated. This invariant volume measure on the physical configuration space, which also determines the inner products for wavefunctions, is given in terms of the Wess-Zumino-Witten (WZW) action for the field $H$ [5, 6, 1]. Considerations of integrable representations of the WZW model then showed that normalizable wavefunctions are functions of the current $J = (N/\pi)\partial_z H H^{-1}$. In other words, the wavefunctions have to be more restricted than being just functions of $H$; they can only depend on $H$ via the specific combination in $J$. The regularized kinetic energy operator, which is the Laplacian on this infinite-dimensional configuration space, is given in terms of functional derivatives with respect to $J$; the potential energy can also be written in terms of $J$ [2]. The vacuum wavefunction $\Psi_0$ of the theory was obtained by solving the (functional) Schrödinger equation in the approximation of keeping all terms in $\log \Psi_0$ which are quadratic in $J$, with a systematic expansion for the higher order terms. The vacuum wavefunction agrees with perturbation theory for the high momentum modes. The expectation value of the Wilson loop operator and hence the string tension were calculated [3]. The values for the string tension agree within 3% of recent Monte Carlo evaluations [7]. Finally, the propagating particles in the perturbative regime can be shown to have a mass $m = e^2 N/2\pi$. This may be taken as a prediction for the magnetic mass of gluons in high temperature QCD [8]. This result compares favorably with resummation calculations of this quantity [9] and with lattice estimates, keeping in mind that this is a difficult lattice calculation as well [10]. Finally, these techniques can also be extended to the Yang-Mills-Chern-Simons theory [11].

While this Hamiltonian analysis still leaves many open questions, it is fair to claim that some progress in understanding the (2+1)-dimensional case has indeed been achieved. It is worth noting that the vacuum wavefunction which was obtained, irrespective of the calculations preceding it, has the desirable features of agreeing with the perturbative vacuum wavefunction in the high momentum limit and giving an area law for the Wilson loop with a string tension which agrees closely with the lattice calculations. Therefore, further study along these lines, in particular exploring a similar strategy in (3+1) dimensions, is warranted.
In section 2, we will introduce the parametrization of the gauge potentials in terms of the matrix variables. The calculation of the volume measure of the configuration space (and hence the inner product for wavefunctions) is taken up in section 3. Section 4 gives some remarks on this result and, by comparison with (2 + 1)-dimensional Yang-Mills theory, makes a rough estimate of the ratio of $M_{0++}/\sqrt{\sigma}$ where $M_{0++}$ is the mass of the $0^{++}$ glueball and $\sigma$ is the string tension.

2 The parametrization of the gauge potentials

We shall discuss an $SU(N)$-gauge theory and also choose the gauge $A_0 = 0$, as is convenient for a Hamiltonian formulation. The remaining gauge potentials can be written as $A_i = -i t^a A^a_i$, $i = 1, 2, 3$, where $t^a$ are hermitian $(N \times N)$-matrices which form a basis of the Lie algebra of $SU(N)$ with $[t^a, t^b] = i f^{abc} t^c$, $\text{Tr}(t^a t^b) = \frac{1}{2} \delta^{ab}$.

We start by recalling that the key ingredients of the (2+1) dimensional analysis were the following:

1. the parametrization of the potentials in terms of the matrix $M$ which allowed the realization of gauge transformations in a homogeneous way, $M^g = gM$.
2. the calculation of the gauge-invariant measure on the configuration space.
3. evaluation of the Hamiltonian in terms of these gauge-invariant variables.
4. solving the functional Schrödinger equation for the vacuum wave function.
5. calculating the string tension and other quantities of interest.

The study of the first two steps in (3+1) dimensions will be taken up in this paper. Let $\mathcal{A}$ denote the set of all gauge potentials $A^a_i$. Gauge transformations act on $A_i$ in the standard way, $A_i \rightarrow A^g_i$, where

$$A_{i}^{(g)} = g A_{i} g^{-1} - \partial_{i} g g^{-1}$$

and $g(\vec{x}) \in SU(N)$. The gauge group $\mathcal{G}$ is defined by

$$\mathcal{G} = \left\{ \text{set of all } g(\vec{x}) : \mathbb{R}^3 \rightarrow SU(N), \ g \rightarrow 1 \text{ as } |\vec{x}| \rightarrow \infty \right\}$$

The space of gauge-invariant field configurations is $C = \mathcal{A}/\mathcal{G}$. A parametrization of the gauge potentials is equivalent to choosing coordinates on the configurations space. Since the space $C$ has nontrivial topology, any parametrization is restricted to some open region. We use a parametrization in a region which includes $A = 0$ and calculate (approximately) the volume measure of $C$ for this region. (Not surprisingly, the geometry and topology of the Yang-Mills configuration space in three spatial dimensions have also been studied by a number of authors, see references \[12\] \[13\]. For a recent summary and new results on the metric, see \[14\].)

Going back to $YM_{2+1}$, we start by asking why it is possible to parametrize $A_z$ as $-\partial_z MM^{-1}$. Notice that this parametrization may be written as $(\partial_z + A_z)M = 0$ and one can convert it to an
integral equation

$$M(x) = 1 - \int_{x'} S(x, x') A_z(x') M(x')$$

$$\partial_z S(x, x') = \delta^{(2)}(x - x')$$ (3)

With this equation, we see that, at least iteratively, we can find an $M$ for each given $A_z$. This establishes a mapping $A_z \rightarrow M$. (There are much more elegant and more general ways to justify the parametrization $A_z = -\partial_z MM^{-1}$, but this simple argument is most suitable for what follows [1].) Notice that the key ingredient is the invertibility of $\partial_z$. The first term involving $A$ in a series expansion for $M$, namely,

$$\int \frac{1}{\sigma \cdot \partial} A(y) M(y)$$

is a complex matrix which is traceless since $A_z$ has no trace. It is thus an element of the Lie algebra of $SL(N, \mathbb{C})$, showing that $M$ can be taken to be in $SL(N, \mathbb{C})$. Conversely, $M$ contains $\dim[SL(N, \mathbb{C})] = 2 \times \dim[SU(N)]$ independent functions corresponding exactly to the number of independent functions needed for the potential, $A_i$, $i = 1, 2$, therefore one has the map $M \rightarrow A_z$ as well.

Since $\partial_z$ is the chiral Dirac operator in two dimensions, the invertibility of $\partial_z$ is equivalent to the existence of the propagator for the chiral Dirac theory. In three Euclidean dimensions, which is appropriate for the (3+1)-dimensional theory, there is no chirality, but we can use the Dirac operator $\sigma \cdot \partial$ where $\sigma_i$, $i = 1, 2, 3$, are the Pauli matrices. We then define a matrix $M$ by

$$(\sigma \cdot \partial + \sigma \cdot A) M = 0$$ (4)

On such a matrix $M$, gauge transformations act by $M \rightarrow M^g = gM$, where $g$ is an element of $SU(N)$. Equation (4) has the formal inversion

$$M(x) = 1 - \int_y \left( \frac{1}{\sigma \cdot \partial} \right)_{xy} \sigma \cdot A(y) M(y)$$ (5)

where

$$\left( \frac{1}{\sigma \cdot \partial} \right)_{xy} = -\int \frac{d^3p}{(2\pi)^3} \frac{i\sigma \cdot p}{p^2} e^{ip \cdot (x-y)}$$

$$= -\sigma \cdot \partial_y G(x, y)$$

$$G(x, y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{p^2} e^{ip \cdot (x-y)}$$ (6)

To first order in the $A$’s, the solution for $M$ is then

$$M \approx 1 - it^a \theta^a + \sigma_i t^a \int_y G(x, y) \epsilon_{ijk} \partial_j A^a_k(y)$$

$$\theta^a = \int_y G(x, y) \partial \cdot A^a(y)$$ (7)

The term $-it^a \theta^a$ on the right hand side of expression (7) for $M$ can be removed by a gauge transformation of the form $M \rightarrow \exp(it^a \theta^a) M$, consistent with the fact that $\partial \cdot A$ represents the gauge degree of freedom, to linear order in $A$. The last term shows that the infinitesimal generators, for whatever group $M$ belongs to, must include $\sigma_i t^a$, which are a subset of generators
of $SU(2N, \mathbb{C})$. Completion of the algebra under commutation rules shows that we need all of $SU(2N, \mathbb{C})$. Thus generally we must take $M$ to be an element of $SU(2N, \mathbb{C})$. Equation (4) thus gives a map $A \rightarrow M \in SU(2N, \mathbb{C})$.

An arbitrary element of $SU(2N, \mathbb{C})$ will contain $2 \times (4N^2 - 1)$ parameter functions. Thus arbitrary $SU(2N, \mathbb{C})$-matrix functions $M$ contain too many parameters to give a faithful coordinization of a region of $\mathcal{A}$, we will need to use constraints on $M$. We will now work out the required constraints. For most of what follows, it is convenient to use $U(2N)$ rather than $SU(2N)$. We define the set of hermitian matrices $\{t^A\}$, $A = 1, 2, \cdots, 4N^2$ as the set $\{1 \otimes t^a, \sigma_i \otimes t^a\}$, $a = 1, 2, \cdots, N^2$. $t^a$ are taken as $(N \times N)$ hermitian matrices normalized by $\text{Tr} t^a t^b = \frac{1}{2} \delta^{ab}$. $\{-it^a\}$ form an antihermitian basis for $U(N)$ embedded in $U(2N)$. The set of matrices $\{-it^A, t^A\}$ form a basis for the Lie algebra of $U(2N, \mathbb{C})$. The normalization condition for the $t^A$ is $\text{Tr} t^A t^B = \delta^{AB}$; they further obey the completeness relation $t^A_{im} t^A_{pq} = \delta_{np} \delta_{mq}$.

Now let $M$ be an arbitrary $U(2N, \mathbb{C})$ matrix. We can then expand

$$\sigma \cdot \partial M M^{-1} = i \phi^a t^a + i \sigma \cdot A^a t^a$$

where $\phi^a, A^a$ are in general complex functions. If we were to start from real $A^a$ and use equation (4), then $\phi^a$ in (3) would be zero. Thus, to eliminate unwanted degrees of freedom starting from an arbitrary $M$, we must impose the conditions $A^a = \bar{A}^a = 0$, $\phi^a = 0$. These are equivalent to

$$\text{Tr}(t^a \sigma \cdot \partial M M^{-1}) = 0$$

$$\partial_i (M^\dagger \sigma_i M) = 0$$

The only remaining degree of freedom in $M$ then corresponds to the real part of $A^a$ which is the $U(N)$-gauge potential.

It is instructive to work out these conditions for $M$ close to the identity. Writing $M \approx 1 + i t^a \varphi^a + i \sigma_i t^a \Theta^a_i$, we find

$$\sigma \cdot \partial M M^{-1} = it^a \partial_i \Theta^a_i + i \sigma_k t^a \left( \partial_k \varphi^a + i \epsilon_{ijk} \partial_j \Theta^a_k \right)$$

Imposing the constraint (10) on this, and separating out the real and imaginary parts of the functions, we get

$$\partial_i (\text{Im} \Theta^a_i) = 0$$

$$\partial_i (\text{Im} \varphi^a) + \epsilon_{ijk} \partial_j \text{Re} \Theta^a_k = 0$$

The second of these equations gives the Laplace equation for $\text{Im} \varphi^a$, namely, $\partial^2 \text{Im} \varphi^a = 0$, so that with proper boundary conditions, we can take $\text{Im} \varphi^a = 0$. Further, we find $\text{Re} \Theta^a_i = \partial_i \xi^a$ for some scalar functions $\xi^a$. Putting this back into (11) and comparing with (8) we find

$$A^a_i = \partial_i \varphi^a - \epsilon_{ijk} \partial_j \text{Im} \Theta^a_k$$

$$\phi^a = \partial^2 \xi^a$$

The constraint (10) eliminates $\phi^a$ (or $\xi^a$). The functions $\varphi^a$ (which are now real) represent the gauge degrees of freedom. The gauge invariant degrees of freedom are given by $\text{Im} \Theta^a_i$, which are
only two polarizations \((2 \times N^2\) functions) because of the condition \(\partial_i (\text{Im} \Theta_i^a) = 0\) in \(12\). It is rather well known that an Abelian gauge potential can be parametrized in the form given in \(13\), \(A_i = \partial_i \varphi - \epsilon_{ijk} \partial_j \Theta_k\) with \(\partial_i \Theta_i = 0\). Near the zero potential, a similar parametrization will apply to the \(U(N)\) potentials as well; equation \(13\) is just this, with the required \(N^2\) replication of the functions.

Since \(M = 1\) corresponds to the zero potential, the above analysis shows that an arbitrary \(U(2N, C)\)-matrix \(M\), subject to the conditions \(9, 10\), does give a faithful coordinatization of \(A\) for a region containing the zero potential. For, given any \(A\), we can generate a corresponding \(M\) by solving \(4\) and conversely, given any \(M\) subject to \(9, 10\), we get a general gauge potential with the correct number of degrees of freedom. How much of \(A\) or \(C\) can be covered by this parametrization is a very valid and interesting question; at this stage there is no clear answer to this. This is also evidently related to the question of Gribov ambiguities and other topological issues for \(C\) \([15, 12, 16]\).

In the \((2+1)\)-dimensional case, a similar question arises for the parametrization \(A_z = -\partial_z M M^{-1}\). In that case, there is an ambiguity in \(M\) for a given \(A\), namely, \(M\) and \(MV(z)\) give the same \(A\); by ensuring invariance under this holomorphic symmetry for all physical quantities, at least some of the difficulties of transitions from one coordinate patch to another could be circumvented \([1]\).

### 3 The volume measure

We now turn to the calculation of the volume measure on the configuration space. In terms of the fields \(\phi^a, A_i^a\) given in \(8\), introduce the Euclidean metric
\[
 ds^2 = \int d^3x \ (\delta \bar{A}_i^a \delta A_i^a + \delta \bar{\phi}^a \delta \phi^a) \tag{14}
\]
For the gauge potential of interest which is the real part of \(A_i^a\), this is the Euclidean metric which is precisely the metric of interest for the gauge theory. The Euclidean volume measure for the real part of \(A_i^a\) can be written as
\[
 [d \text{Re} A_i^a] = \int [dA] \ \delta (A_i^a - \bar{A}_i^a) \ \delta (\phi^a) \ \delta (\bar{\phi}^a) \tag{15}
\]
\([dA]\) involves all components, \(A_i^a, \bar{A}_i^a, \phi^a\) and \(\bar{\phi}^a\). The functional Dirac delta functions eliminate all except the real part of \(A_i^a\). The volume \([dA]\) corresponds to the metric \(14\). From the definition \(8\), we have
\[
 \delta A_i^a = -i \text{Tr} [\sigma_i t^a \sigma_j D_j (\delta MM^{-1})] \\
 \delta \phi^a = -i \text{Tr} [t^a \sigma_j D_j (\delta MM^{-1})] \tag{16}
\]
where \(D_j\) is defined by
\[
 D_j \chi = \partial_j \chi + [A_j, \chi], \quad A_j = -\partial_j MM^{-1} \tag{17}
\]
The equations in \(16\) may be combined as \(\delta A^A = -i \text{Tr} [t^A \sigma_j D_j \theta], \ \theta = \delta MM^{-1}\). Using the completeness of the \(t^A\), the metric \(14\) can then be simplified as
\[
 ds^2 = \int d^3x \ \text{Tr} \left( \overrightarrow{D_i \theta} \sigma_i \sigma_j D_j \theta \right) \\
 = \int d^3x \ \text{Tr} \left( t^A \sigma_i \sigma_j t^B \right) \left( \overrightarrow{D_i \theta} \right)^A (D_j \theta)^B \tag{18}
\]
where we use the fact that, in terms of components in the Lie algebra, \( D_j \theta = -i t^A (D_j \theta)^A \), \( (D_j \theta)^A = \partial_j \theta^A + f^{ABC} A^B \theta^C \). \( (T^A)_{BC} = -i f^{ABC} \) are the Lie algebra generators in the adjoint representation of \( U(2N, \mathbb{C}) \). We now define the \((4N^2 \times 4N^2)\)-matrices

\[
(S_i)^{AB} = \text{Tr}(t^A \sigma_i t^B)
\]

(19)

By the completeness relation for the \( t^A \), these are seen to obey the relation

\[
\Sigma_i^{AB} \Sigma_j^{BC} = \text{Tr}(t^A \sigma_i t^C) = \delta_{ij} \delta^{AC} + i \epsilon_{ijk} \Sigma_k^{AC}
\]

(20)

The \( \Sigma_i \) are a \((4N^2 \times 4N^2)\) representation of the algebra of \( \sigma_i \). The metric (18) can thus be written as

\[
ds^2 = \int d^3 x \bar{\theta}^A \theta^A = \int d^3 x \text{Tr}((\Sigma \cdot D)^\dagger (\Sigma \cdot D))^{AB} \theta^B
\]

(21)

A metric of the form

\[
ds^2 = \int d^3 x \bar{\theta}^A \theta^A = \int d^3 x \text{Tr}(M^{\dagger -1} \delta M \delta M M^{-1})
\]

(22)

is the Cartan-Killing metric for \( U(2N, \mathbb{C}) \) (for each spatial point) and leads to the Haar measure \( d\mu(M, M^\dagger) \) for \( U(2N, \mathbb{C}) \). By comparison with this we see that the volume measure for the metric (21) can be written as

\[
[dA] = \det \left[(\Sigma \cdot D)^\dagger (\Sigma \cdot D)\right] d\mu(M, M^\dagger)
\]

\[
= d\mu(M, M^\dagger) \exp \{\Gamma + \bar{\Gamma}\}
\]

(23)

\[
\exp(\Gamma) = [\det(\Sigma \cdot D)]_{\text{reg}}
\]

(24)

In equation (24), we have explicitly indicated that the determinant is to be evaluated with proper regularization. The regularization should be such that \( \Gamma + \bar{\Gamma} \) is gauge-invariant. The volume element for the real part of \( A^\alpha_i \) is then given as

\[
[d\text{Re}A^\alpha_i] = \int e^{\Gamma + \bar{\Gamma}} d\mu(M, M^\dagger) \delta[\sigma \cdot \partial M M^{-1} + \text{h.c.}] \delta[\text{Tr}(t^a \sigma \cdot \partial M M^{-1}) - \text{h.c.}]
\]

(25)

The calculation of the volume thus involves several distinct steps. The first is the calculation of the determinant \( \exp(\Gamma + \bar{\Gamma}) \); the second is the reduction of the Haar measure \( d\mu(M, M^\dagger) \) by the elimination of the set of gauge transformations and finally we have to address the question of the constraints given by the \( \delta \)-functions.

The full determinant can be calculated by computing the determinants of the Dirac-like operators \( \Sigma \cdot D \) and its adjoint and putting the results together in a gauge-invariant way. The regulated form of the determinant of \( \Sigma \cdot D \) can be written as

\[
\Gamma_{\text{reg}} = \text{Tr} \log \Sigma \cdot D - \frac{M_2}{M_2 - M_1} \text{Tr} \log(\Sigma \cdot D + M_1) + \frac{M_1}{M_2 - M_1} \text{Tr} \log(\Sigma \cdot D + M_2)
\]

(26)
where $M_1$ and $M_2$ are regulator masses. We will need to use two regulators of the Pauli-Villars type, with coefficients as given, to eliminate all the divergences.

We can calculate the determinant by a series expansion in powers of the gauge potential. The only unusual point is that the simplification of the traces are more involved because the $\Sigma$-matrices do not commute with the Lie algebra of the $A_i$'s. Indeed if this were not so, the determinant would be trivial, apart from possible anomalies, since $A$ has the form $-\partial M M^{-1}$.

The term quadratic in the potentials is given by

\[
\Gamma^{(2)} = \frac{1}{2} \int_{x,y} \text{Tr}(t^A \partial_i M M^{-1})(x) \text{Tr}(t^B \partial_j M M^{-1})(y) \int \frac{d^3k}{(2\pi)^3} e^{ik \cdot (x-y)} \Pi^{AB}_{ij}(k)
\]

\[
\Pi^{AB}_{ij}(k) = \frac{-i}{16\pi} \left[ \frac{M_2}{M_2 - M_1} (\text{sgn} M_1) - \frac{M_1}{M_2 - M_1} (\text{sgn} M_2) \right] k_i \text{Tr}([\Sigma_r, \Sigma_i T^A] \Sigma_j T^B)
\]

\[
- \frac{k}{64} \left( \delta_{rs} + \frac{k_r k_s}{k^2} \right) \text{Tr}(\Sigma_r \Sigma_i T^A \Sigma_s T^B)
\]

where $\text{sgn} M = \frac{M}{|M|}$. The first term in $\Pi^{AB}_{ij}$ corresponds to a Chern-Simons term. The second term will be seen to be similar to the one-loop vacuum polarization result in three dimensions; the factor $\delta_{rs} + k_r k_s / k^2$ is correct with the connecting plus sign, the usual projection operator will emerge once the traces are evaluated. The following two observations help to simplify these expressions. First, notice that $A_i$ obeys the identity

\[
\partial_i A_j - \partial_j A_i + [A_i, A_j] = 0
\]

so that, to the quadratic order in the potentials we have $\partial_i A_j - \partial_j A_i \approx 0$. Secondly, the traces are in the adjoint representation of $U(2N)$, but these can be converted to the fundamental representation. For example, using the completeness of the $t^A$'s, we can write

\[
[t^A, \Sigma_r]^{BC} = -if^{ABD} \Sigma^D r + \Sigma^{BD} if^{ADC} = -\text{Tr}([t^A, t^B] \Sigma_r t^C) - \text{Tr}(t^B \Sigma_r [t^A, t^C]) = \text{Tr}(t^B [t^A, \Sigma_r] t^C)
\]

Using the algebra of the $\Sigma$'s, we then get

\[
\text{Tr}([\Sigma_r, \Sigma_i T^A] \Sigma_j T^B) = 2i\epsilon_{rik} \text{Tr}(\Sigma_k T^A \Sigma_j T^B) - \text{Tr}([t^A, \Sigma_r] \Sigma_j t^C) (-if^{BMN} C_2) = 2i\epsilon_{rik} \text{Tr}(\Sigma_k T^A \Sigma_j T^B) - C_2 \text{Tr}([t^A, \Sigma_r] \Sigma_j t^B)
\]

where $C_2$ is the quadratic Casimir for the adjoint representation of $U(2N)$, $f^{AMN} f^{BMN} = C_2 \delta^{AB}$. With $\epsilon_{rik} k_r A_i \approx 0$, the first term gives zero for the Chern-Simons contribution, reducing the trace to the trace in the fundamental. Similar simplification can be done for all the other terms and the final result is

\[
\Gamma^{(2)} = \frac{1}{2} C_2 \left[ -\frac{i}{16\pi} \left( \frac{M_2}{M_2 - M_1} (\text{sgn} M_1) - \frac{M_1}{M_2 - M_1} (\text{sgn} M_2) \right) \int \epsilon^{ijk} \partial_i A_j^a A_k^a
\]

\[
- \frac{1}{128} \int F_{ij}^a \frac{1}{\sqrt{-\nabla^2}} F_{ij}^a + \frac{1}{32} \int \phi^a \sqrt{-\nabla^2} \phi^a\right]
\]
Here $F_{ij}^a \approx \partial_i A_j^a - \partial_j A_i^a$ to the order we have calculated. The terms involving $\phi$’s are eventually set to zero by the constraints $[19, \Sigma]$. The form of the $\phi$-terms could also change depending on the regulators, but the final answer is unambiguous since we can set them to zero anyway. The Chern-Simons term will cancel out when we take $\Gamma + \bar{\Gamma}$, as it should, since there is no parity violation in pure (3+1)-dimensional gauge theory. Using these simplifications, we get for the volume measure

$$[d\text{Re}A] = \int d\mu(M, M^\dagger) \exp (\Gamma + \bar{\Gamma}) \delta[\sigma \cdot \partial MM^{-1} + \text{h.c.}]$$

$$\times \delta[\text{Tr}(t^a \sigma \cdot \partial MM^{-1}) - \text{h.c.}]$$

$$\Gamma + \bar{\Gamma} = -\frac{C_2}{128} \int F_{ij}^a \frac{1}{2} F_{ij}^a + O(A^3)$$

Based on gauge invariance, we can say that part of the higher order terms will render the first term fully invariant, so that the result is of the form

$$\Gamma + \bar{\Gamma} = -\frac{C_2}{128} \int F_{ij}^a \left[ \frac{1}{\sqrt{-(\partial + A)^2}} \right]^{ab} F_{ij}^{ab} + O(A^3)$$

with $F_{ij}^a = \partial_i A_j^a - \partial_j A_i^a + f^{abc} A_i^b A_j^c$.

We now turn to the Haar measure $d\mu(M, M^\dagger)$. We are interested in factoring out the gauge transformations which act as $M^g = gM, g \in U(N)$. Out of $M$ we can construct the gauge-invariant quantities $H = M^\dagger M$ and $W_i = M^\dagger \sigma_i M$. We write a generic $M$ as

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

(34)

where $a, b, c, d$ are $(N \times N)$-matrices. We can take $a$ and $d$ to be invertible in general $[17]$. Elements of the combinations $H$ and $W_i$ give $a^\dagger a, d^\dagger d, a^\dagger d, c^\dagger d, b^\dagger d$, etc. They can thus be regarded as functions of $H, W_i$. The square roots of $a^\dagger a$ and $d^\dagger d$ can be defined by diagonalizing them. We then see that we can write

$$a = U \sqrt{a^\dagger a}, \quad b = U \beta$$

$$c = U \gamma, \quad d = U V \sqrt{d^\dagger d}$$

(35)

where $U$ and $V$ are unitary matrices; $V$ is determined from $a^\dagger d$ as a function of $H, W_i$. Likewise $\beta$ and $\gamma$ are given by $c^\dagger d$ and $b^\dagger d$. Thus the matrix $M$ can generally be parametrized as

$$M = \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix} N$$

(36)

where $N$ is a function of $H, W_i$. The Haar measure is given by the top rank differential form $dM M^{-1} \wedge dM M^{-1} \cdots M^{-1} dM^\dagger \wedge M^{-1} dM^\dagger \cdots$ where we substitute (36) for $M$. This brings out a factor $d\mu(U) = dUU^{-1} \wedge dUU^{-1} \cdots$ which is the volume of the gauge part, $U(N)$. The remainder is given entirely in terms of the gauge-invariant combinations $H, W_i$. In other words, we have

$$d\mu(M, M^\dagger) = d\mu(U) d\mu(H, W); d\mu(H, W)$$

is the volume on the coset space $U(2N, C)/U(N)$. By taking the product of this formally over all spatial points, we have

$$d\mu(M, M^\dagger) = \prod_x d\mu(U) \prod_x d\mu(H, W)$$

(37)
The elimination of the gauge part of the measure is now trivial, we just get \( \prod_x d\mu(H, W) \).

Finally, it is easy to see that the \( U \)-dependence of the constraints drops out from the \( \delta \)-functions in (26) or (32); they can be written in terms of \( N \) or \( H, W \). Combining results (32) and (37) and the arguments given above, we can write the gauge-invariant measure as

\[
d\mu(C) = \prod_x d\mu(H, W) \delta[\sigma \cdot \partial NN^{-1} + \text{h.c.}] \delta[\text{Tr}(t^a \sigma \cdot \partial NN^{-1}) - \text{h.c.}] \\
\times \exp(\Gamma + \bar{\Gamma})
\]

\[\equiv d\mu \exp(\Gamma + \bar{\Gamma}) \tag{38}\]

where \( \Gamma + \bar{\Gamma} \) has the expansion (33).

We can now write the inner product for states \(|1\rangle\) and \(|2\rangle\), with the gauge-invariant wavefunctions \(\Psi_1\) and \(\Psi_2\), as

\[
\langle 1|2 \rangle = \int d\mu \exp(\Gamma + \bar{\Gamma}) \Psi_1^* \Psi_2 \tag{39}\]

The key result of this paper is this formula for the inner product, along with (33, 38), which summarize our results on the gauge-invariant volume element for the configuration space \(\mathcal{C}\). Notice that, as in the \((2+1)\)-dimensional case, the term \(\Gamma + \bar{\Gamma}\) is proportional to the quadratic Casimir \(C_2\), which vanishes for the Abelian theory, once again indicating a significant difference between the Abelian and nonabelian cases.

4 Discussion

Equation (39) for the inner product shows that the matrix elements of the \((3+1)\)-dimensional theory can be reduced to the correlators of a three-dimensional Euclidean gauge theory with the action \(\Gamma + \bar{\Gamma}\) and functional measure \(d\mu\). We have obtained the quadratic terms in this action, but not yet calculated the terms which will involve gauge-invariant combinations which are cubic and higher order in the fields, although some of these higher terms can be inferred from gauge invariance. Nevertheless, it is still interesting to look ahead and see what implications our results may have for the physics of the gauge theory.

We can establish some properties of the \((3+1)\)-dimensional theory by comparison with the \((2+1)\)-dimensional theory. The vacuum wave function for the \((2+1)\)-dimensional theory was of the form \(\exp[-(\pi/2e^4N) \int B^2]\) for long wavelength modes. With such a wave function, for the Wilson loop \(W_F(C)\) in the fundamental representation of \(SU(N)\) we find

\[
\langle W_F(C) \rangle = \text{constant} \exp[-\sigma A_C] \\
\sqrt{\sigma} = e^2 \sqrt{\frac{N^2 - 1}{8\pi}} \tag{40}\]

This result was obtained in the Hamiltonian description; nevertheless, based on the full Euclidean invariance of the Wick rotated theory, this may be expressed as

\[
\int d\mu(C) \exp\left(-\int \frac{F^2}{4e^2}\right) W_F(C) = \langle W_F(C) \rangle = \text{constant} \exp[-\sigma A_C] \tag{41}\]

This version may in turn be interpreted as the equal time correlator in the (3+1)-dimensional theory with a vacuum wave function of the form \( \sim \exp(-\int F^2/8\Lambda) \). Thus if the (3+1)-dimensional theory has a vacuum wave function

\[
\Psi_0 \sim \exp\left(-\int \frac{F^2}{8\Lambda}\right)
\] (42)

then we get confinement and a string tension

\[
\sqrt{\sigma} = \Lambda \sqrt{\frac{N^2 - 1}{8\pi}}
\] (43)

We therefore assume that the wave function has the form (42) and ask what other implications it may have. The mass of a \( 0^{++} \) glueball in the lower dimensional theory is given by

\[
\langle B^2(x)B^2(0) \rangle = \int d\mu(C) \exp\left(-\int \frac{F^2}{4e^2}\right) \sim \exp(-M_{0^{++}}|x|)
\] (44)

for large separations \(|x|\). The mass \( M_{0^{++}} = \alpha e^2 N \), where \( \alpha \) is, in principle, calculable in the Hamiltonian formulation of the (2+1)-dimensional theory. An explicit calculation is difficult; lattice data show that \( \alpha \approx 0.808 \) [7] as \( N \to \infty \). We can also think of the result (44) as an equal time correlator in the (3+1)-dimensional theory, for the wave function (42) (with \( e^2 \to \Lambda \)), in which case the glueball mass is given by \( M_{0^{++}} = \alpha \Lambda N \). This means that, if the wave function (42) is a good description in the (3+1)-dimensional case, the ratio \( M_{0^{++}}/\sqrt{\sigma} \) is the same in the (3+1)- and (2+1)-dimensional theories. Collecting results

\[
\frac{M_{0^{++}}}{\sqrt{\sigma}} = \alpha \sqrt{\frac{8\pi}{N^2 - 1}}
\]

where we have used the lattice value for the (2+1)-dimensional theory. This is then a prediction, based on the premise of (42), for the (3+1)-dimensional theory. The lattice estimate of this quantity for the (3+1)-dimensional theory is approximately 3.37 as \( N \to \infty \) [18]; the discrepancy is about 20%. Thus equation (42) may be considered to be a reasonable ansatz for a first approximation to the wave function. The important question is whether we can we derive it by solving the Schrödinger equation; this is under study.

The approximate dimension-independence of the glueball masses has been noted before in the context of lattice values. In the context of using wave functions, an argument which has some similarity to ours has been given in [19]. In the context of a parton mass for gluons, a similar observation has been made by Philipsen [20].

There is another lesson from the (2+1)-dimensional case that we can use. In three Euclidean dimensions an action of the form \( \int F^2/4\Lambda \) can generate a mass gap. This is not yet the gap for the (3+1)-dimensional theory, but a cutoff on modes of low momenta when integrations are actually carried out using a wave function of the form (42). In turn this can generate a mass gap for the (3+1)-dimensional theory in much the same way as the cut-off on low momentum modes due to the measure factors in the (2+1)-dimensional analysis can lead to a gap [1].
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