On some spectral spaces associated to tensor triangulated categories

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Abstract

We consider a closure operator $c$ of finite type on the space $SMod(M)$ of thick $K$-submodules of a triangulated category $M$ that is a module over a tensor triangulated category $(K, \otimes, 1)$. Our purpose is to show that the space $SMod^c(M)$ of fixed points of the operator $c$ is a spectral space that also carries the structure of a topological monoid.

MSC (2010) Subject Classification: 13B22, 18E30.

Keywords: Spectral spaces, tensor triangulated categories.

1 Introduction

The study of tensor triangular geometry was begun by Paul Balmer in [1], where he associated to a tensor triangulated category $(K, \otimes, 1)$ a spectrum $Spec(K)$ of “prime thick tensor ideals” of $K$. The classification of thick subcategories in tensor triangular geometry (see Balmer [1], [2], [3], [4] and also Balmer and Favi [5], [6]) unites ideas from far and wide in mathematics: from that of Benson, Carlson and Rickard [9] in modular representation theory, that of Devinatz, Hopkins and Smith [10] in homotopy theory to that of Thomason [25] in algebraic geometry. As such, tensor triangular geometry over the years has emerged as a field of study in itself (see Klein [17], [18], Peter [20], Sanders [21] and Stevenson [22], [23], [24]).

Given a tensor triangulated category $(K, \otimes, 1)$, its spectrum $Spec(K)$ defined by Balmer [1] is a spectral space, i.e., it must be homeomorphic to the Zariski spectrum of a commutative ring. In [16], Hochster famously characterized spectral spaces in purely topological terms. More recently, Finocchiaro [14, Corollary 3.3] has obtained a new criterion for a topological space to be spectral, using ultrafilters to give if and only if conditions for a collection of subsets to be a subbasis of quasi-compact opens of a spectral space (see also related work by Finocchiaro, Fontana and Loper in [13]). Further, Finocchiaro’s criterion has recently been used by Finocchiaro, Fontana and Spirito [15] to give several natural examples of spectral spaces appearing in commutative algebra. More precisely, if $M$ is a module over a commutative ring $R$ and $c$ is a closure operator of finite type (see [15, § 3] for definitions) on submodules of $M$, we know from [15, Proposition 3.4] that the space
of submodules of $M$ fixed by $c$ is a spectral space. In [8], we have shown that these methods can be adapted more generally to abelian categories. In this paper, we use closure operators to create spectral spaces associated to a module over a tensor triangulated category. This also fits in well with the general philosophy that notions in abelian categories should have parallels in triangulated categories (see, for instance, Krause [19]). For more on closure operators in commutative algebra, see, for instance, Epstein [11], [12].

In this paper, we begin in Section 2 by considering a triangulated category $M$ that is a module over a tensor triangulated category $(K, \otimes, 1)$ in the sense of Stevenson [22]. We consider a closure operator and more generally, an operator $c$ on the space $SMod(M)$ of thick $K$-submodules of $M$ that is extensive, order-preserving and of finite type (see Definition [2.4]). Then, our first main result is that the space $SMod^c(M)$ of fixed points of the operator $c$ is a spectral space. For instance, if $K = M$, then the radical is an example of a closure operator of finite type on the thick tensor ideals of $K$. More generally, we characterize closure operators of finite type in terms of families of submodules satisfying certain conditions. In particular, this implies that any family of thick submodules of $M$ closed under intersections and filtered directed unions is an example of a spectral space.

Thereafter, in Section 3, we study thick $K$-submodules of $M$ generated by a given set $X$ of objects of $M$. In particular, we show that each object in the submodule generated by $X$ can be obtained starting from only finitely many objects of $X$. We then use this to show that if $c : SMod(M) \rightarrow SMod(M)$ is an operator that is extensive, order-preserving and of finite type, the space $SMod^c(M)$ of fixed points of $c$ actually becomes a topological monoid. Finally, we mention here that in order to avoid certain set theoretical complications, we will assume that all categories in this paper are essentially small, i.e., the isomorphism classes of their objects form a set.

Acknowledgements: I am grateful for the hospitality of the Stefan Banach Center at the IMPAN in Warsaw, where part of this paper was written.

2 Spectral spaces and tensor triangulated actions

Throughout this section and the rest of this paper, $(K, \otimes, 1)$ will be a tensor triangulated category. In other words, $K$ is a triangulated category equipped with a symmetric monoidal product $\otimes : K \times K \rightarrow K$ that is exact in each variable. The unit object in $K$ is denoted by $1 \in K$. A thick tensor ideal $I$ in $(K, \otimes, 1)$ is a thick triangulated and full subcategory of $K$ such that $b \otimes a \in I$ for any object $a \in I$ and any $b \in K$. Then, following Balmer [1], we say that a thick tensor ideal $P \subseteq K$ is prime if $x \otimes y \in P$ for some $x, y \in K$ means that at least one of $x$ and $y$ is in $P$. In [1], Balmer began the study of tensor triangular geometry by constructing the spectral space $Spec(K)$ of prime ideals in $(K, \otimes, 1)$.

We now consider a triangulated category $M$ that is a “module” over $(K, \otimes, 1)$ in the sense of Stevenson [22]. In other words, we have an action:

$$*: K \times M \rightarrow M$$

(2.1)
that is exact in both variables, satisfies appropriate associativity, distributivity and unit properties and is well behaved with respect to the translation functor on both $\mathcal{K}$ and $\mathcal{M}$ (see [22, Definition 3.2]). The translation functor on $\mathcal{K}$ (resp. on $\mathcal{M}$) will be denoted by $T_\mathcal{K}$ (resp. by $T_\mathcal{M}$), but whenever there is no danger of confusion, we will drop the subscripts and refer to both of these translation functors simply as $T$.

**Definition 2.1.** (see [22, Definition 3.4]) Let $\mathcal{M}$ be a triangulated category that is a module over a tensor triangulated category $(\mathcal{K}, \otimes, 1)$ via an action $\ast : \mathcal{K} \times \mathcal{M} \rightarrow \mathcal{M}$. A full subcategory $\mathcal{L} \subseteq \mathcal{M}$ containing $0$ is said to be a thick $\mathcal{K}$-submodule of $\mathcal{M}$ if it satisfies the following three conditions:

(a) The composition of functors

$$\mathcal{K} \times \mathcal{L} \hookrightarrow \mathcal{K} \times \mathcal{M} \xrightarrow{\ast} \mathcal{M} \quad (2.2)$$

factors through $\mathcal{L}$.

(b) Given objects $m, m' \in \mathcal{M}$, then $m \oplus m' \in \mathcal{L}$ if and only if both $m, m' \in \mathcal{L}$.

(c) For any distinguished triangle $m' \rightarrow m \rightarrow m''$ in $\mathcal{M}$, if two out of the three objects $m', m, m''$ lie in $\mathcal{L}$, so does the third.

The collection of all thick $\mathcal{K}$-submodules of $\mathcal{M}$ will be denoted by $SMod(\mathcal{M})$.

When $(\mathcal{K}, \otimes, 1)$ is considered as a module over itself via the action $\otimes : \mathcal{K} \times \mathcal{K} \rightarrow \mathcal{K}$, Definition 2.1 reduces to the notion of a thick tensor ideal in [1, Definition 1.1].

We also notice that condition (c) in Definition 2.1 implies that any $\mathcal{K}$-submodule $\mathcal{L}$ of $\mathcal{M}$ must be replete. In other words, if $m \in \mathcal{L}$ and $m' \in \mathcal{M}$ is any other object such that we have an isomorphism $m \cong m'$, we must have $m' \in \mathcal{L}$. Now, since $\mathcal{M}$ is assumed to be essentially small, it follows that the collection $SMod(\mathcal{M})$ is always a set.

We now define a topology on $SMod(\mathcal{M})$ by declaring the following as a subbasis for open sets:

$$U(m) := \{ N \in SMod(\mathcal{M}) \mid m \in N \} \subseteq SMod(\mathcal{M}) \quad \forall m \in \mathcal{M} \quad (2.3)$$

We should remark here that the definition in (2.3) is rather the “opposite” of what we would expect from looking at [15, § 2] and more generally at the usual constructions in commutative algebra. However, this reversal is actually common in tensor triangular geometry (see [11, § 2]).

**Proposition 2.2.** Let $\mathcal{M}$ be a triangulated category that is a module over a tensor triangulated category $(\mathcal{K}, \otimes, 1)$. Then, the collection $\{U(m)\}_{m \in \mathcal{M}}$ satisfies the following properties:

(a) $U(0) = SMod(\mathcal{M})$.

(b) For any distinguished triangle $m' \rightarrow m \rightarrow m''$ in $\mathcal{M}$, we have $U(m) \supseteq U(m') \cap U(m'')$.

(c) For objects $m, m' \in \mathcal{M}$, we have $U(m \oplus m') = U(m) \cap U(m')$.

(d) For any $a \in \mathcal{K}$ and any $m \in \mathcal{M}$, we have $U(m) \subseteq U(a \ast m)$.

(e) If $T_\mathcal{M}$ is the translation functor on $\mathcal{M}$, we have $U(T_\mathcal{M}(m)) = U(m)$ for each $m \in \mathcal{M}$. 


Proof. The properties (a), (b), (c) and (d) follow directly from Definition 2.1. For part (e), we notice the isomorphism $T_M(m) \cong K(1) \ast m$ for any $m \in \mathcal{M}$ (see [22, Definition 3.1]) which gives us $U(m) \subseteq U(T_M(m))$. However, we also have the isomorphism $T^{-1}_M(T_M(m)) \cong T^{-1}_K(1) \ast T_M(m)$ which gives $U(T_M(m)) \subseteq U(m)$.

\[ \square \]

Corollary 2.3. Consider the set $SMod(\mathcal{M})$ along with the topology given by $\{U(m)\}_{m \in \mathcal{M}}$ forming a subsbasis of open sets. Then, the collection $\{U(m)\}_{m \in \mathcal{M}}$ forms a basis of open sets in the topology on $SMod(\mathcal{M})$.

Proof. It suffices to show that the collection $\{U(m)\}_{m \in \mathcal{M}}$ is closed under finite intersections. This follows from part (c) of Proposition 2.2 above.

Given any $m \in \mathcal{M}$, we denote by $K(m)$ the smallest thick $K$-submodule of $\mathcal{M}$ containing $m$. From condition (b) in Definition 2.1, it is immediate that a thick $K$-submodule generated by a finite set $\{m_1, ..., m_k\}$ of objects of $\mathcal{M}$ coincides with the submodule generated by the single object $m_1 \oplus m_2 \oplus \ldots \oplus m_k$. The following notions should be compared to the definition of closure operators in module categories (see [11, Definition 7.0.1]).

Definition 2.4. Let $\mathcal{M}$ be a triangulated category that is a module over $(K, \otimes, 1)$. An operator

$$c : SMod(\mathcal{M}) \rightarrow SMod(\mathcal{M})$$

will be said to be:

(a) Extensive if $N \subseteq c(N)$ for each $N \in SMod(\mathcal{M})$.

(b) Order-preserving if $N \subseteq N'$ implies that $c(N) \subseteq c(N')$.

(c) Idempotent if $c(N) = c(c(N))$ for each $N \in SMod(\mathcal{M})$.

(d) Finite type if $c(N) = \bigcup_{n \in N} c(K(n))$.

We will refer to an operator satisfying (a), (b) and (c) as a closure operator on $SMod(\mathcal{M})$. A closure operator $c$ that also satisfies (d) will be called a closure operator of finite type.

Given an operator $c$ as in Definition 2.4 we set $SMod^c(\mathcal{M})$ to be the collection of submodules of $\mathcal{M}$ that are fixed by $c$. Our aim is to give conditions for $SMod^c(\mathcal{M})$ to be a spectral space.

First, we recall (see, for instance, [14, § 1]) that a filter $\mathcal{U}$ on a set $X$ is a collection of subsets of $X$ such that: (a) $\emptyset \notin \mathcal{U}$, (b) $Y, Z \in \mathcal{U} \Rightarrow Y \cap Z \in \mathcal{U}$ and (c) $Y \subseteq Z \subseteq X$ and $Y \in \mathcal{U}$ implies that $Z \in \mathcal{U}$. An ultrafilter $\mathcal{U}$ is a maximal element in the collection of filters on $X$.

Proposition 2.5. Let $\mathcal{M}$ be a triangulated category that is a module over a tensor triangulated category $(K, \otimes, 1)$. Let $c : SMod(\mathcal{M}) \rightarrow SMod(\mathcal{M})$ be an operator that is extensive, order-preserving and of finite type. Then, $SMod^c(\mathcal{M})$ is a spectral space having the collection $\{U(m) \cap SMod^c(\mathcal{M})\}_{m \in \mathcal{M}}$ as a basis of quasi-compact open subspaces.
Proof. We first verify that $S\text{Mod}^c(\mathcal{M})$ satisfies the $T_0$-axiom. If $\mathcal{L}$, $\mathcal{L}'$ are two distinct points of $S\text{Mod}^c(\mathcal{M})$, there exists an object $m \in \mathcal{M}$ that lies in exactly one of $\mathcal{L}$ and $\mathcal{L}'$. Then, $U(m) \cap S\text{Mod}^c(\mathcal{M})$ is an open set that contains exactly one of the two points $\mathcal{L}$ and $\mathcal{L}'$.

Now suppose that $\mathcal{U}$ is an ultrafilter on the space $S\text{Mod}^c(\mathcal{M})$. We now set

$$\mathcal{L}_\mathcal{U} := \{ m \in \mathcal{M} \mid U(m) \cap S\text{Mod}^c(\mathcal{M}) \in \mathcal{U} \}$$

(2.5)

First, we show that $\mathcal{L}_\mathcal{U} \in S\text{Mod}(\mathcal{M})$, i.e., $\mathcal{L}_\mathcal{U}$ is a submodule. If $m \in \mathcal{L}_\mathcal{U}$, then for any object $a \in \mathcal{K}$, $U(a \cdot m) \cap S\text{Mod}^c(\mathcal{M}) \supseteq U(m) \cap S\text{Mod}^c(\mathcal{M}) \in \mathcal{U}$ and hence $U(a \cdot m) \cap S\text{Mod}^c(\mathcal{M}) \in \mathcal{U}$, i.e., $a \cdot m \in \mathcal{L}_\mathcal{U}$.

On the other hand, since $U(m \oplus m') = U(m) \cap U(m')$ for any objects $m$, $m' \in \mathcal{M}$, it follows that $m \oplus m' \in \mathcal{L}_\mathcal{U}$ if and only if both $m$, $m' \in \mathcal{L}_\mathcal{U}$. Also, given a distinguished triangle $m' \rightarrowtail m \twoheadrightarrow m''$ with two of $m'$, $m$ and $m'' \in \mathcal{L}_\mathcal{U}$, it follows from parts (b) and (e) of Proposition 2.2 that the third object also lies in $\mathcal{L}_\mathcal{U}$.

Next, we claim that $\mathcal{L}_\mathcal{U} \in S\text{Mod}^c(\mathcal{M})$. For this, we choose some $m \in c(\mathcal{L}_\mathcal{U})$. Since $c$ is of finite type, we can find some object $n \in \mathcal{L}_\mathcal{U}$ such that $m \in c(\mathcal{K}(n))$. We now consider some submodule $N \subseteq U(n) \cap S\text{Mod}^c(\mathcal{M})$. Since $c$ is order-preserving, we have $c(\mathcal{K}(n)) \subseteq c(N) = N$ and hence $m \in N$. It follows that $U(n) \cap S\text{Mod}^c(\mathcal{M}) \subseteq U(m) \cap S\text{Mod}^c(\mathcal{M})$ and $\mathcal{U}$ being an ultrafilter, we get that $m \in \mathcal{L}_\mathcal{U}$, i.e., $c(\mathcal{L}_\mathcal{U}) \subseteq \mathcal{L}_\mathcal{U}$. Further since $c$ is extensive, we get $\mathcal{L}_\mathcal{U} = c(\mathcal{L}_\mathcal{U})$.

Finally, suppose that for some object $m \in \mathcal{M}$, the subset $U(m) \cap S\text{Mod}^c(\mathcal{M})$ lies in the ultrafilter $\mathcal{U}$. Then, from (2.5), it follows that $m \in \mathcal{L}_\mathcal{U}$ and hence $\mathcal{L}_\mathcal{U} \subseteq U(m) \cap S\text{Mod}^c(\mathcal{M})$. Conversely, if $\mathcal{L}_\mathcal{U} \subseteq U(m) \cap S\text{Mod}^c(\mathcal{M})$ for some $m \in \mathcal{M}$, then $m \in \mathcal{L}_\mathcal{U}$ and hence $U(m) \cap S\text{Mod}^c(\mathcal{M}) \in \mathcal{U}$. The result now follows by applying Finocchiaro’s criterion [13, Corollary 3.3] to the subbasis $\{U(m) \cap S\text{Mod}^c(\mathcal{M})\}_{m \in \mathcal{M}}$ of the space $S\text{Mod}^c(\mathcal{M})$. Additionally, from Corollary 2.3, it follows that $\{U(m) \cap S\text{Mod}^c(\mathcal{M})\}_{m \in \mathcal{M}}$ is actually a basis for the spectral space $S\text{Mod}^c(\mathcal{M})$. 

For example, if we take $\mathcal{K}$ as a module over itself, the radical defines a closure operator on the thick tensor ideals of $(\mathcal{K}, \otimes, 1)$. We recall here (see [1, Definition 4.1]) that the radical $\text{rad}(\mathcal{I})$ of a thick tensor ideal $\mathcal{I} \subseteq \mathcal{K}$ is defined as follows:

$$\text{rad}(\mathcal{I}) := \{ a \in \mathcal{K} \mid \exists n \geq 1 \text{ such that } a^\otimes n \in \mathcal{I} \}$$

(2.6)

From (2.6) it is also clear that the radical is a closure operator of finite type. We can give another example of an extensive and order-preserving operator of finite type as follows: let $\mathcal{S}$ be a multiplicatively closed family of objects of $\mathcal{K}$. Then, to any thick tensor ideal $\mathcal{I}$ in $(\mathcal{K}, \otimes, 1)$, we associate the ideal (see [7, § 2]):

$$\mathcal{I} \div \mathcal{S} := \{ a \in \mathcal{K} \mid \exists s \in \mathcal{S} \text{ such that } a \otimes s \in \mathcal{I} \}$$

(2.7)

From Proposition 2.5 it follows that the collection of thick tensor ideals fixed by these operators are spectral spaces. We will conclude this section by describing a more explicit method for obtaining closure operators of finite type in terms of families of submodules. For closure operators in abelian categories, we have made similar constructions in [8].
Proposition 2.6. Let $\mathcal{M}$ be a triangulated category that is a module over a tensor triangulated category $(K, \otimes, 1)$. Let $\mathfrak{F} = \{F_i\}_{i \in I}$ be a family of submodules of $\mathcal{M}$ such that $\mathcal{M} \in \mathfrak{F}$. Then, the following statements are equivalent:

1. The family $\mathfrak{F}$ satisfies the following two conditions:
   (a) Given any non-empty subset $J \subseteq I$, we have $\bigcap_{j \in J} F_j \in \mathfrak{F}$.
   (b) Let $\{F_k\}_{k \in K}$ be a filtered directed collection of objects from the family $\mathfrak{F}$. Then, the filtered directed union $\bigcup_{k \in K} F_k$ also lies in $\mathfrak{F}$.

2. There exists a closure operator $c : \text{SMod}(\mathcal{M}) \to \text{SMod}(\mathcal{M})$ of finite type such that $\mathfrak{F}$ is the collection of fixed points of $c$. In particular, $\mathfrak{F}$ is a spectral space with $\{U(m) \cap \mathfrak{F}\}_{m \in \mathcal{M}}$ being a basis of quasi-compact open sets.

Proof. (1) $\Rightarrow$ (2) : For each object $m \in \mathcal{M}$, we begin by setting:

$$c(K(m)) := \bigcap_{m \in F, F \in \mathfrak{F}} F \quad (2.8)$$

Now, for any submodule $N \in \text{SMod}(\mathcal{M})$, we define the operator $c : \text{SMod}(\mathcal{M}) \to \text{SMod}(\mathcal{M})$ as follows:

$$c(N) := \bigcup_{n \in N} c(K(n)) \quad (2.9)$$

From (2.8) and (2.9), it is immediate that $c$ is extensive, order-preserving and of finite type. We now choose some $n' \in c(N)$. It follows from (2.9) that we can find some $n'' \in N$ such that $n' \subseteq c(K(n''))$. Now, if $F \in \mathfrak{F}$ is such that $n'' \subseteq F$, we have $n' \subseteq c(K(n'')) \subseteq F$. Then, $c(K(n')) \subseteq F$ and hence $c(K(n')) \subseteq c(K(n''))$. Since $c(K(n)) = \bigcup_{n' \in c(N)} c(K(n'))$, it now follows that $c(K(n)) \subseteq c(N)$ and hence $c(K(n)) = c(N)$.

We now pick some $F \in \mathfrak{F}$. For each object $f \in F$, it follows from (2.8) that $c(K(f)) \subseteq F$. As we go over all objects in $F$, the expression in (2.9) shows that $c(F) \subseteq F$ and hence $c(F) = F$ for each $F \in \mathfrak{F}$. Conversely, we notice that the right hand sides of (2.8) and (2.9) always lie in $\mathfrak{F}$ and hence any fixed point of the operator $c$ must lie in $\mathfrak{F}$.

(2) $\Rightarrow$ (1) : Given a non-empty subset $J \subseteq I$, we have:

$$c(\bigcap_{j \in J} F_j) \subseteq \bigcap_{j \in J} c(F_j) = \bigcap_{j \in J} F_j \quad (2.10)$$

Combining with the fact that $c$ is extensive, it follows that $\bigcap_{j \in J} F_j$ is a fixed point of $c$. On the other hand, let $\{F_k\}_{k \in K}$ be a filtered directed family of objects from $\mathfrak{F}$ and consider $F = \bigcup_{k \in K} F_k$. We now pick some object $f \in F$. Then, we can find some $k_0 \in K$ such that $f \in F_{k_0}$. But then, $c(K(f)) \subseteq c(F_{k_0}) = F_{k_0}$. Since $c$ is of finite type, this now gives $c(F) = \bigcup_{f \in F} c(K(f)) \subseteq \bigcup_{k \in K} F_k = F$ and hence $c(F) = F$. \qed
3 Topological monoids and \( \mathcal{K} \)-modules

Suppose that \( c : SMod(\mathcal{M}) \rightarrow SMod(\mathcal{M}) \) is an operator that is extensive, order-preserving and of finite type. In this section, our aim is to show that the spectral space \( SMod^f(\mathcal{M}) \) is a topological monoid. For this, we will now obtain a more explicit description for the smallest thick submodule \( \mathcal{K}(X) \in SMod(\mathcal{M}) \) containing a generating set \( X \) of objects of \( \mathcal{M} \). We will do this by extending from [7, Proposition 3.3] our methods on the generation of thick tensor ideals. Given a set \( X \) of objects of \( \mathcal{M} \), we now consider:

\[
\bar{X} := \{ n \in \mathcal{M} \mid \exists m \in X, a \in \mathcal{K} \text{ and } n' \in \mathcal{M} \text{ s.t. } n \oplus n' \cong a \ast m \}
\] (3.1)

We notice that 0 \( \in \bar{X} \) and that \( \bar{X} = X \). On the other hand, we let \( \Delta(X) \) denote the collection of all objects \( m \in \mathcal{M} \) such that there exist \( m', m'' \in X \) with \( m, m' \) and \( m'' \) forming a distinguished triangle in \( \mathcal{M} \) (in some order). Now if \( 0 \in X \), the fact that \( m \xrightarrow{1} m \rightarrow 0 \) forms a distinguished triangle for any \( m \in \mathcal{M} \) shows that \( X \subseteq \Delta(X) \).

**Proposition 3.1.** Let \( X \) be a set of objects in \( \mathcal{M} \). Put \( X_0 := X \) and inductively define:

\[
X_{i+1} := \Delta(\bar{X}_i) \quad \forall i \geq 0
\] (3.2)

Then, the thick \( \mathcal{K} \)-submodule \( \mathcal{K}(X) \) of \( \mathcal{M} \) generated by \( X \) is given by the union:

\[
\mathcal{K}(X) = \bigcup_{i=0}^{\infty} X_i
\] (3.3)

**Proof.** From the construction, it is clear that the submodule \( \mathcal{K}(X) \) generated by \( X \) contains each \( X_i \). In order to prove (3.3), it therefore suffices to show that the union \( \bigcup_{i=0}^{\infty} X_i \) is a thick submodule. For the sake of convenience, we set \( X' := \bigcup_{i=0}^{\infty} X_i \). We now choose some \( m \in X' \). Then, we can find some \( i \geq 0 \) such that \( m \in X_i \). Now, for any \( a \in \mathcal{K} \), it is clear that \( a \ast m \in \bar{X}_i \subseteq \Delta(\bar{X}_i) = X_{i+1} \subseteq X' \). Similarly, if \( m \) splits as a direct sum \( m \cong m_1 \oplus m_2 \), both \( m_1 \) and \( m_2 \) lie in \( \bar{X}_i \subseteq X_{i+1} \subseteq X' \). Finally, suppose that we have a distinguished triangle \( m' \rightarrow m \rightarrow m'' \) in \( \mathcal{M} \) such that two of \( m, m' \) and \( m'' \) \( \in X' \). For the sake of definiteness, suppose that \( m', m'' \in X' \). Then we can choose some \( j \geq 0 \) large enough so that both \( m', m'' \in X_j \). But then, \( m \in \Delta(X_j) \subseteq \Delta(\bar{X}_j) = X_{j+1} \subseteq X' \). It follows that \( X' \) is a thick \( \mathcal{K} \)-submodule of \( \mathcal{M} \).

\[\Box\]

We note that one of the simple consequences of Proposition 3.1 is the fact that if \( \{a_i\}_{i \in I} \) is a family of objects of \( \mathcal{K} \) and \( \{m_j\}_{j \in J} \) is a family of objects of \( \mathcal{M} \), the submodule generated by \( \{a_i \ast m_j\}_{i \in I, j \in J} \) contains all the objects \( a \ast m \), where \( a \) (resp. \( m \)) lies in the ideal of \( \mathcal{K} \) (resp. the submodule of \( \mathcal{M} \)) generated by \( \{a_i\}_{i \in I} \) (resp. by \( \{m_j\}_{j \in J} \)). In the case of thick tensor ideals with \( \mathcal{K} = \mathcal{M} \), we have noted this consequence in [7, Lemma 3.4]. However, we should mention that this fact
has been previously established by a different approach (in the slightly different case of localizing submodules) by Stevenson in [22, Lemma 3.11].

The following result gives us a better understanding of generating sets of thick $\mathcal{K}$-submodules: we show that any element $m$ in the submodule $\mathcal{K}(X)$ can be obtained starting from only finitely many objects in the generating set $X$.

**Proposition 3.2.** Let $X$ be a set of objects in $\mathcal{M}$ and let $\mathcal{K}(X)$ be the thick submodule generated by $X$. Then, given an object $m \in \mathcal{K}(X)$, there exist finitely many objects $m_1, m_2, \ldots, m_k \in X$ such that $m$ lies in the submodule generated by the set $\{m_1, m_2, \ldots, m_k\}$.

**Proof.** We maintain the notation from the proof of Proposition 3.1. We know that $\mathcal{K}(X) = \bigcup_{i=0}^{\infty} X_i$.

We now suppose that for any $0 \leq j \leq N$ and any object $m \in X_j$, we can find finitely many objects $m_1, m_2, \ldots, m_k \in X$ such that $m$ lies in the submodule generated by the set $\{m_1, m_2, \ldots, m_k\}$. This is already true for $N = 0$. We now pick an object $m \in X_{N+1}$.

By definition, $X_{N+1} = \Delta(X_N)$ and hence we can find elements $n', n'' \in X_N$ such that $m, n'$ and $n''$ form a distinguished triangle (in some order). Now, applying the definition of $X_N$, it follows that we can find objects $a', a'' \in \mathcal{K}$ and $m', m'' \in X_N$ such that $n'$ (resp. $n''$) is a direct summand of $a' * m'$ (resp. $a'' * m''$). It follows that $m \in X_{N+1}$ lies in the submodule generated by $m'$ and $m''$.

However, since $m'$ and $m''$ lie in $X_N$, we can find a finite set $\{m_1, \ldots, m_k\}$ (resp. $\{m_{k+1}, \ldots, m_l\}$) of objects in $X$ such that $m'$ (resp. $m''$) lies in the submodule generated by $\{m_1, \ldots, m_k\}$ (resp. $\{m_{k+1}, \ldots, m_l\}$). Then, $m \in X_{N+1}$ must lie in the thick $\mathcal{K}$-submodule generated by the finite set $\{m_1, \ldots, m_k, m_{k+1}, \ldots, m_l\} \subseteq X$. This proves the result.

Given submodules $\mathcal{N}, \mathcal{N}'\in SMod(\mathcal{M})$, we denote by $\mathcal{N} + \mathcal{N}'$ the smallest thick submodule of $\mathcal{M}$ containing both $\mathcal{N}$ and $\mathcal{N}'$. It is clear that addition of submodules makes $SMod(\mathcal{M})$ into a commutative monoid. However, in order to make $SMod^c(\mathcal{M})$ into a monoid, we will need the following result.

**Lemma 3.3.** (a) Let $c : SMod(\mathcal{M}) \rightarrow SMod(\mathcal{M})$ be an operator that is extensive, order-preserving and of finite type. Given a submodule $\mathcal{N} \subseteq \mathcal{M}$, we set $c^\infty(\mathcal{N}) := \bigcup_{i \geq 0} c^i(\mathcal{N})$. Then, for any $\mathcal{N} \in SMod(\mathcal{M})$, the object $c^\infty(\mathcal{N})$ lies in $SMod^c(\mathcal{M})$.

(b) The operator $c^\infty : SMod(\mathcal{M}) \rightarrow SMod(\mathcal{M})$ is of finite type, i.e., for any $\mathcal{N} \in SMod(\mathcal{M})$, we have $c^\infty(\mathcal{N}) = \bigcup_{n \in \mathcal{N}} c^\infty(\mathcal{K}(n))$.

**Proof.** (a) We choose some $\mathcal{N} \in SMod(\mathcal{M})$ and some $n \in c^\infty(\mathcal{N})$. Since $c$ is of finite type, there exists some $n_0 \in c^\infty(\mathcal{N})$ such that $n \in c(\mathcal{K}(n_0))$. Then, we can choose $i_0 \geq 1$ such that $n_0 \in c^{i_0}(\mathcal{N})$, i.e., $\mathcal{K}(n_0) \subseteq c^{i_0}(\mathcal{N})$. But then, $c(\mathcal{K}(n_0)) \subseteq c^{i_0+1}(\mathcal{N}) \subseteq c^\infty(\mathcal{N})$ which shows that $n \in c^\infty(\mathcal{N})$, i.e., $c(c^\infty(\mathcal{N})) \subseteq c^\infty(\mathcal{N})$. Since $c$ is extensive, it follows that $c^\infty(\mathcal{N}) \in SMod^c(\mathcal{N})$.

(b) For the sake of convenience, we set $\mathcal{N}' := \bigcup_{n \in \mathcal{N}} c^\infty(\mathcal{K}(n))$. For any $n_1, n_2 \in \mathcal{N}$, it is clear that $c^\infty(\mathcal{K}(n_1)), c^\infty(\mathcal{K}(n_2))$ both lie inside $c^\infty(\mathcal{K}(n_1 \oplus n_2))$, which shows that $\mathcal{N}'$ is a filtered union of
submodules and hence $N' \in \text{SM}_\mathcal{M}$. We now choose some $n' \in N'$. Then, $n' \in c^\infty(\mathcal{K}(n))$ for some $n \in \mathcal{N}$, i.e., $\mathcal{K}(n') \subseteq c^\infty(\mathcal{K}(n))$. From part (a), we know that $c^\infty(\mathcal{K}(n))$ is fixed by $c$ and hence $c(\mathcal{K}(n')) \subseteq c(c^\infty(\mathcal{K}(n))) = c^\infty(\mathcal{K}(n)) \subseteq N'$. Since $c$ is an operator of finite type, we know that $c(N') = \bigcup_{n' \in N'} c(\mathcal{K}(n'))$ and hence $c(N') = N'$. On the other hand, it is clear from the definitions that $\mathcal{N} \subseteq N' \subseteq c^\infty(\mathcal{N})$. It follows that $c^i(\mathcal{N}) \subseteq N' \subseteq c^\infty(\mathcal{N})$ for each $i \geq 0$. Hence, $c^\infty(\mathcal{N}) = N'$, which proves the result.

\[ \square \]

**Proposition 3.4.** Let $c : \text{SM}_\mathcal{M} \to \text{SM}_\mathcal{M}$ be an operator that is extensive, order-preserving and of finite type. Let $\mathcal{N} \in \text{SM}_\mathcal{M}$ be a submodule of $\mathcal{M}$ that is fixed by $c$. Then, the function $f$ defined as follows:

\[ f : \text{SM}_\mathcal{M} \to \text{SM}_\mathcal{M} \quad N' \mapsto c^\infty(\mathcal{N} + N') \tag{3.4} \]

is a continuous function on the spectral space $\text{SM}_\mathcal{M}$. In other words, the spectral space $\text{SM}_\mathcal{M}$ equipped with the operation $(\mathcal{N}', N') \mapsto c^\infty(\mathcal{N} + N')$ is a topological monoid.

**Proof.** Since the collection $\{U(m) \cap \text{SM}_\mathcal{M}\}_{m \in \mathcal{M}}$ forms a basis of open sets, it suffices to check that each $f^{-1}(U(m) \cap \text{SM}_\mathcal{M})$ is open in $\text{SM}_\mathcal{M}$. For each object $m \in \mathcal{M}$, we denote by $\mathcal{N}^c(m)$ the set of isomorphism classes of objects $m' \in \mathcal{M}$ such that $m \in c^\infty(\mathcal{N} + \mathcal{K}(m'))$. We claim that:

\[ f^{-1}(U(m) \cap \text{SM}_\mathcal{M}) = \bigcup_{m' \in \mathcal{N}^c(m)} (U(m') \cap \text{SM}_\mathcal{M}) \tag{3.5} \]

On the one hand, if we have any $N' \in U(m') \cap \text{SM}_\mathcal{M}$ for some $m' \in \mathcal{N}^c(m)$, then $m' \in \mathcal{N}'$ and hence $m \in c^\infty(\mathcal{N} + \mathcal{N}')$, i.e., $f(N') = c^\infty(\mathcal{N} + \mathcal{N}') \in U(m) \cap \text{SM}_\mathcal{M}$.

Conversely, suppose that we choose some $N' \in f^{-1}(U(m) \cap \text{SM}_\mathcal{M})$. Then, $m \in c^\infty(\mathcal{N} + \mathcal{N}')$. From Lemma 3.3(b), we know that $c^\infty$ is of finite type and we can choose $n \in \mathcal{N} + \mathcal{N}'$ such that $m \in c^\infty(\mathcal{K}(n))$. We let $X$ (resp. $X'$) denote the set of isomorphism classes of objects in $\mathcal{N}$ (resp. $\mathcal{N}'$). Then, $X \cup X'$ is a generating set for the submodule $\mathcal{N} + \mathcal{N}'$. We know that $n \in \mathcal{N} + \mathcal{N}'$ and it follows from Proposition 3.2 that we can choose a finite set $\{x_1, ..., x_{k-1}, x_k, x_{k+1}, ..., x_l\} \subseteq X \cup X'$ such that \{x_1, ..., x_k\} $\subseteq X$, \{x_{k+1}, ..., x_l\} $\subseteq X'$ and $n \in \mathcal{N} + \mathcal{N}'$ lies in the submodule generated by \{x_1, ..., x_k, x_{k+1}, ..., x_l\}. We now consider the object $n_0 := x_{k+1} \oplus x_{k+2} \oplus ... \oplus x_l$ which lies in $\mathcal{N}'$, i.e., $\mathcal{N}' \in U(n_0)$. Also $m$ lies in $c^\infty(\mathcal{N} + \mathcal{K}(n_0))$ and hence we can find some $n' \in \mathcal{N}^c(m)$ such that $n_0 \cong n'$. It follows that $\mathcal{N}' \in U(n_0) = U(n')$. Hence, the result follows.

\[ \square \]

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