Conversion of Love waves in a forest of trees

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We inspect the propagation of shear polarized surface waves akin to Love waves through a forest of trees of same height atop a guiding layer on a soil substrate. We discover that the foliage of trees brings a radical change in the nature of the dispersion relation of these surface waves, which behave like spoof plasmons in the limit of a vanishing guiding layer, and like Love waves in the limit of trees with a vanishing height. When we consider a forest with trees of increasing or decreasing height, this hybrid "Spoof Love" wave is either reflected backwards or converted into a downward propagating bulk wave.

Research in engineered metasurfaces, which support a host of electromagnetic surface waves, has greatly benefited from the concept of spoof plasmon polaritons that opened new vistas in the microwave regime [1], and this inspired further studies in plasmonics [2] and flat optics [3]. A particularly appealing object in this area is the so-called rainbow. In their seminal work [4], Tsakmakidis and co-workers demonstrated that the optical properties of surface electromagnetic waves can be tailored, by varying the surface nanotopology, via surface dispersion engineering. The resulting graded metasurface allows for light localization and segregation of different light colors [5], a concept which found a counterpart for sound [6].

In the context of elasticity, previous studies focused on the case of polarized surface waves known as Rayleigh waves. Despite the differences between these mechanical waves and the electromagnetic waves, it has been shown that Rayleigh waves propagating over elastic crystals share common features with their electromagnetic counterparts, such as the existence of elastic Bragg bandgaps [7]. These bandgaps have been exploited to create a shielding effect for Rayleigh waves propagating at 50 hertz through a soil structured with an array of boreholes [8]. Recently, elastic resonant metasurfaces with subwavelength structurations have been considered and the concept of rainbow for optical waves has been translated to Rayleigh waves with the exciting application to the control of seismic surface waves [9–12]. The concept of seismic rainbow has been first demonstrated for ultrasonubs in experiments at the laboratory scale [9], and extended up to the geophysical scale [11]. A forest densely populated with trees represents a naturally occurring geophysical metasurface for Rayleigh waves and an experiment in an actual forest environment [11] confirmed filtering properties due to the presence of stop bands around 100 hertz. The dispersion relation derived in [12] revealed the existence of an effective wave that transitions from Rayleigh wave-like to shear wave-like behaviour. These works opened the door to the development of seismic metasurfaces with the first realization of the so-called meta-wedge that is capable of mode-converting destructive seismic surface waves into mainly harmless downward propagating bulk shear waves.

In this Letter, we show that such seismic metasurfaces can be designed for Love waves [13]. Love waves are shear polarized surface seismic waves, which produce a horizontal shaking particularly deleterious for the foundations of infrastructures. Unlike for Rayleigh waves, Love waves require a guiding layer to propagate at the air-soil surface and we shall see that these surface waves are particularly sensitive to the shape of structural elements above the soil, in the present case, a forest of trees with some foliage (Fig. 1). The effective dispersion relation reflects a cooperation between the guiding layer and the trees, resulting in effective bandgaps for a hybrid wave. As a result it is shown that a forest of trees with varying height can reflect, localize or convert a Love wave.

![FIG. 1: Periodic array of trees with spacing ℓ and total height H; the ground region is surmounted by a guiding layer able to support Love waves. The tree trunks have a diameter d and filling fraction ϕ and the foliage of height H_f a surface filling fraction ϕ_f.](image-url)
medium is governed by the Navier equation

\[ \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = \text{div} \sigma, \quad \sigma = C : \nabla \mathbf{u}, \]  

with \( t \) the time variable. The body force is assumed to be zero, \( C = (C_{ijkl}) \), is the 4-order elasticity tensor, which satisfies Hookes’ law \( C_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \), \( i, j, k, l = 1, 2, 3 \), with \( \lambda \) and \( \mu \) the Lamé coefficients, \( \rho \) is the density and \( (u, \sigma) \) the vector displacement field and the stress tensor, respectively. For an elastic medium which is invariant along one direction, say \( y \), the Navier equation splits into an in-plane equation on \( u = (u_x, u_z) \) and an out-of-plane equation on \( u = u_y \). We focus on the out-of-plane polarization which concerns Love waves; if one further assumes some time-harmonic dependence \( e^{-i\omega t} \), with \( \omega \) the angular wave frequency, this problem takes the simple form

\[ \text{div} \sigma + \rho \omega^2 u = 0, \quad \sigma = \mu \nabla u. \]  

Let us now dive into our problem. We consider the two-dimensional configuration of a forest of trees periodically spaced by a distance \( \ell \); the ground is composed of a layer with a lower velocity than that of the soil substrate (Fig. 1). In the absence of trees, this low velocity layer can support Love waves, which propagate within the layer and vanish when moving far from it in the substrate. We shall see that it is possible to describe how the guiding layer couples to the trees to produce a new type of guided waves, that we term ”Spoof Love” waves. Using homogenization tools, the region of the trees can be replaced by an equivalent slab filled with a homogeneous anisotropic medium [14][15], see also the [Supplemental Material]. In this medium, \( (u, \sigma) \) satisfy

\[ \sigma = \mu_3 \begin{pmatrix} 0 & 0 \\ 0 & \varphi \end{pmatrix} \nabla u, \quad \text{div} \sigma + \rho_3 \omega^2 \varphi u = 0, \]  

with \( \rho_3, \mu_3 \) the mass density and the shear modulus of the wood which composes the trees, and \( \varphi \) the filling fraction of tree trunks (denoting \( d \) the diameter of the trunk, \( \varphi = d/\ell \)). It is worth noting that 13 tells us that the propagation is allowed along the trees only, with the wavenumber \( \omega/c_3, c_3 = \sqrt{\mu_3/\rho_3} \), as in a single tree (since we have \( \partial_z u + (\omega^2/c_3^2) u = 0 \)). In the complete homogenized problem shown in Fig. 2, boundary conditions have to be applied at the interfaces of the effective slab at \( z = 0 \) and \( z = H \). At the interface \( z = 0 \) with the guiding layer, the boundary conditions are the usual continuity relations of \( u \) and \( \sigma_z \). At the interface \( z = H \) with the air above the trees, a non intuitive effective condition applies of the form

\[ \sigma_z = -L_{e} \frac{\partial \sigma_z}{\partial z}, \quad L_{e} = H_1 \left( \frac{\varphi}{\varphi} - 1 \right), \quad \text{at} \ z = H, \]  

with \( H_1 \), the foliage height and \( \varphi \), the foliage filling fraction, both of which characterize the foliage (inset in Fig. 1). Finally, the propagation is described by [2] in the guiding layer, for \( 0 > z > -e \), with the material parameters \( (\rho_2, \mu_2) \) and in the substrate, for \( z < -e \), with the material parameters \( (\rho_1, \mu_1) \).

We start our analysis in the absence of foliage, whence 14 reduces to \( \sigma_z|_{z=H} = 0 \). Looking for a guided wave solution, which is the solution of the problem in the absence of source, we can derive the dispersion relation. Specifically, we are looking for solutions of the form

\[ u(x, z) = \begin{cases} e^{i\alpha_1(z+e)} e^{i\beta x}, & z < -e, \\ [A \cos k_2 z + B \sin k_2 z] e^{i\beta x}, & -e < z < 0, \\ C \cos k_3 (z - H) e^{i\beta x}, & 0 < z < H. \end{cases} \]  

In 15, we have accounted for the condition \( \sigma_z|_{z=H} = 0 \), and we have defined the vertical wavenumbers

\[ \alpha_1 = \sqrt{\beta^2 - \omega^2/c_1^2}, \quad k_2 = \sqrt{\omega^2/c_2^2}, \quad k_3 = \frac{\omega}{c_3}. \]  

We are looking for \( \alpha_1 \) real positive and \( \beta \) real, which correspond to a guided wave in the ground, but \( k_2 \) can be a priori real or imaginary; if \( k_2 \) is imaginary, the wave is evanescent in the guiding layer as it is in the ground. Applying the continuity of \( u \) and \( \sigma_z \) at the interfaces at \( z = -e \) and 0, 16 leaves us with 4 equations on \( (A, B, C, \beta) \) for each frequency \( \omega \), from which the dispersion relation \( \beta(\omega) \) can be inferred. It reads as

\[ 1 - \frac{\mu_2 k_2}{\mu_1 \alpha_1} \tan k_2 e - \frac{\mu_3 k_3}{\mu_2 k_2} \varphi \tan k_3 H \left( \tan k_2 e + \frac{\mu_2 k_2}{\mu_1 \alpha_1} \right) = 0. \]  

The above dispersion relation describes a guided wave supported by the guiding layer coupled to the trees. Obviously for \( H = 0 \), we recover the dispersion relation of Love waves \( \alpha_1 = (\mu_2/\mu_1) k_2 \tan k_2 e \) [13]. An other interesting limit is \( e = 0 \) where we see that the trees alone are able to support guided waves, with \( \alpha_1 = (\mu_3/\mu_1) \varphi k_3 \tan k_3 H \); in the case \( \mu_3 = \mu_1 \), we recover the dispersion relation of the so-called spoof plasmon corresponding in acoustics and electromagnetism to guided waves propagating over a rough rigid surface [1]. To account for the foliage at the tree top, it is sufficient to
modify in \( \mathbf{5} \) the form of the solution for \( 0 < z < H \), specifically
\[
u(x, z) = C \left[ \cos k_3(z - H) + k_3 L_x \sin k_3(z - H) \right] e^{i \beta x},
\]
which satisfies \( \mathbf{4} \), and eventually the complete dispersion relation is obtained in the form
\[
F_1 \left[ 1 - \frac{\mu_2 k_2^2}{\mu_1 \alpha_1} \tan k_2 e \right] - F_2 \frac{\mu_3 k_3}{\mu_2 k_2} \left( \tan k_2 e + \frac{\mu_2 k_2}{\mu_1 \alpha_1} \right) = 0,
\]
with
\[
\begin{cases}
F_1 = 1 - k_3 L_x \tan k_3 H, \\
F_2 = \tan k_3 H + k_3 L_x.
\end{cases}
\]
Obviously, for \( L_x = 0 \), \( \mathbf{9} \) simplifies in \( \mathbf{7} \).

From now on, we use the following material parameters: \( \rho_1 = 1300 \text{ kg.m}^{-3}, c_1 = 495 \text{ m.s}^{-1} \) for the ground, \( \rho_2 = 2600 \text{ kg.m}^{-3}, c_2 = 350 \text{ ms}^{-1} \) for the guiding layer, and \( \rho_3 = 450 \text{ kg.m}^{-3}, c_3 = 1200 \text{ m.s}^{-1} \) for the wood. The dimensions are \( e = 2 \text{ m}, H = 10 \text{ m}, d = 0.3 \text{ m}, \ell = 2 \text{ m} \) (\( \varphi = 0.15 \)) when the foliage is considered, we use \( d_f = 1.5 \text{ m} \) (\( \varphi_f = 0.75 \)) and \( H_f = 1 \text{ m} \), thus the height of trunk is \( 9 \text{ m} \) in this case (note we assume same elastic parameters for trunk and foliage, but this assumption can be lifted [Supplemental Material]). We begin our physical discussion of Spoof Love waves with the inspection of their dispersion relation in the actual problem by computing numerically the reflection coefficient \( R \) for an incident evanescent wave (using a multimodal method based on eigenfunction expansions \( \mathbf{10} \)). Specifically we consider a solution for \( z < -e \) of the form
\[
u(x, z) = e^{i \beta x} \left( e^{-\alpha_1 z} + Re^{\alpha_1 (z+e)} \right),
\]
with \( \alpha_1 \) in \( \mathbf{10} \). Above the light line \( \beta < \omega/c_1, \alpha_1 \) is purely real whence \( |R| = 1 \); but below the light line \( \alpha_1 \) is purely imaginary and \( |R| \) is unbounded; when \( |R| = \infty \) we recover a guided wave as in \( \mathbf{5} \). Results on the reflection coefficient \( R \) are reported in Fig. \( \mathbf{3} \). For comparison, we report the dispersion relations of the Love waves in the layer on its own and those of the spoof plasmons in the trees on their own. The actual dispersion relations show that the guiding layer couples to the trees, resulting in a hybrid guided wave, which appears to be accurately described by our model \( \mathbf{7} \) and \( \mathbf{9} \). Roughly speaking, the Spoof Love wave remains close to the Love wave except in the vicinities of the cut-off frequencies of the spoof plasmons corresponding to the resonances of a single tree \( f_n = (2n+1)c_3/(4H) \), \( n \) integer (the first cut off frequency around 30 Hz is clearly seen). There, the trees dominate and the wave becomes evanescent not only in the soil substrate but also in the guiding layer, with \( \beta > \omega/c_2 \). When the foliage is accounted for, the cut-off frequencies \( f_n \) are significantly decreased, a fact that is already true for the spoof plasmons in the absence of guiding layer (\( \varepsilon = 0 \) in \( \mathbf{10} \)). It is worth noting that such a decrease in the resonance frequency is not attributable only to a larger inertia of the tree; for instance, increasing \( \varphi \) does not affect the asymptotes at \( f_n \), which remains dictated by \( H \) only.

We now turn to the influence of the tree height. The Figs. \( \mathbf{3} \) show the dispersion relations for \( H \in (2; 18) \text{ m} \) at the frequency 70 Hz. The main features observed in Figs. \( \mathbf{3} \) are recovered. At this frequency, resonances occur for trees of heights \( H \approx 4 \text{ m} \) and 13 \text{ m}, resulting in two bandgaps for \( H \in (3.5, 4) \text{ m} \) and \( H \in (12.3, 13) \text{ m} \) respectively. With the foliage, the first gap is shifted to \( H \in (10, 10.5) \text{ m} \) (and the second to \( H \in (1.5, 1.9) \text{ m} \)). We also reported in Fig. \( \mathbf{3} \)a) the displacement fields of the spoof Love waves at the upper limit of the bandgaps (\( \beta = 1.5 \text{ m}^{-1} \), close to \( \pi/\ell \)) for a forest of trees without foliage and \( H_1 = 12.2 \text{ m} \) (see black arrow \( H = H_1 \) in

\[
\begin{align*}
\varphi &\approx 0.15, \\
L_x &\approx 2 \text{ m}.
\end{align*}
\]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig3}
\caption{Dispersion relation of the hybrid "spoof Love" wave for \( H = 10 \text{ m} \), calculated numerically by means of the divergence of the reflection coefficient \( |R| \) (in log, color scale) and the dispersion relation \( \mathbf{7}, \mathbf{9} \) (dashed white lines). The hybrid wave results from a cooperation between the guiding layer supporting Love wave (dashed dark grey lines) and the trees supporting spoof plasmons (SPPs, light grey lines); the light lines \( \beta = \omega/c_1, \omega/c_2 \) are reported in plain grey lines.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig4}
\caption{Dispersion relation of the spoof Love waves in the plane (\( \beta, H \)) at \( f = 70 \text{ Hz} \); (a) without and (b) with foliage (same representation as in Figs. \( \mathbf{3} \)). \( \omega/c_1 = 0.89 \text{ m}^{-1}, \beta_{\text{Love}} = 1.15 \text{ m}^{-1}, \omega/c_2 = 1.25 \text{ m}^{-1}, \pi/\ell = 1.57 \text{ m}^{-1}. \)
\end{figure}

\begin{align*}
\beta &\approx 1.5 \text{ m}^{-1}, \\
H &\approx 10 \text{ m}.
\end{align*}
Fig. 4(a)). When the trees support a foliage, the guided wave changes significantly its shape, being now close to a classical Love wave with $\beta \approx 1 \text{ m}^{-1}$. Reversely, a forest of trees of height $H = H_3 = 10.2 \text{ m}$ with foliage supports a highly confined guided wave with $\beta = 1.5 \text{ m}^{-1}$ while suppressing the foliage produces a drastic change in the characteristics of the guided waves (black arrows $H = H_2$ in Figs. 4). In the 4 considered cases, we report for comparison the fields given by (5) along with (7) (and (8) with (9)), which confirms the capacity of the model to predict the main features of the Spoof Love wave.

Beyond the important effect of the foliage, another interesting feature is visible, which may impact significantly the propagation of guided waves through a forest of trees with increasing or decreasing heights. Let us assume that a local analysis can be done, which means that the wavenumber $\beta(H)$ in a forest of trees with varying heights can be estimated from our present analysis, and consider the cut-off frequency for $H = 4 \text{ m}$ in Fig. 4(a). When the wave propagates from shorter $H < 4 \text{ m}$ towards higher trees, the local wavenumber $\beta(H)$ increases along the branch $\beta(H < 4 \text{ m})$. Eventually, it reaches the value $\pi/\ell$ where the confinement in the trees is maximum (the wave is already evanescent in the guiding layer). On the contrary, if the wave propagates from higher to shorter height trees, $\beta(H)$ is decreasing along the branch $\beta(4 < H < 13 \text{ m})$. In this case, it reaches the wavenumber $\beta_1 = \omega/c_1$ of the shear wave in the bulk where the confinement vanishes. Owing to this analysis, it seems not too hazardous to state that a wave propagating along trees with decreasing height becomes more and more adapted to be converted into a shear wave in the bulk. Conversely, the wave propagating along trees with increasing heights is more and more confined being eventually supported by the trees only. When reaching the tree realizing the resonance, it is not adapted to be converted in a bulk wave, nor adapted to pass through the cut off frequency; as an alternative, it is simply reflected backwards. This strongly non symmet-
In this Letter, we have reported direct numerical observations of shear surface waves propagating in a forest of trees atop a guiding layer. Analytical dispersion relation shows that this wave shares common features with both Love waves and spoof plasmons, whose dispersion relations are recovered in two limit cases (trees of vanishing height, and guiding layer of vanishing thickness, respectively), hence the nickname ‘spoof Love wave’. The dispersion relation reveals bandgaps dictated by the local resonances of a single tree, with strongly asymmetric characteristics of the wave at the band edges. This finding shows that spoof Love waves propagate following the same scenario than the one analyzed for Rayleigh waves. When propagating in a forest with increasing height, they are easily converted into a downward bulk wave, while when propagating in a forest with decreasing tree height, they are eventually aspired within the trees resulting in a strong backscattering due to a rainbow effect. Finally, we have shown that the presence of foliage significantly affects the local resonances of the trees, hence the whole dispersion relation, a phenomenon which is neatly captured by our model. Interesting extensions include the study of other boundary layer effects due to heterogeneities as the tree roots or the presence of rocks in the guiding layer and the three dimensional analysis.

It is worth noting that our analysis applies mutatis mutandis to Love waves interacting with taller resonators resulting in lower frequency bandgaps. This opens potential applications in civil engineering where wind farms (with wind turbine of 50-100 m height) could be used for low frequency filtering (1 to 10 Hz), or in the study of site-city interaction involving tall buildings in sedimentary basins.

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