Quaternionic Kähler and hyperKähler manifolds with torsion and twistor spaces

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Abstract

The target space of a (4,0) supersymmetric two-dimensional sigma model with a Wess-Zumino term has a connection with a totally skew-symmetric torsion and holonomy contained in Sp(n)Sp(1) (resp. Sp(n)), QKT (resp. HKT)-spaces. We study the geometry of QKT and HKT manifolds and their twistor spaces. We show that the Swann bundle of a QKT manifold admits a HKT structure with special symmetry, if and only if the twistor space of the QKT manifold admits an almost hermitian structure with totally skew-symmetric Nijenhuis tensor. In this way we connect two structures arising from quantum field theories and supersymmetric sigma models with Wess-Zumino term.

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1 Introduction and statement of the results

An almost hyper complex structure on a 4n-dimensional manifold $M$ is a triple $H = (J_\alpha)$, $\alpha = 1, 2, 3$, of almost complex structures $J_\alpha : TM \to TM$ satisfying the quaternionic identities $J_\alpha^2 = -id$ and $J_1J_2 = -J_2J_1 = J_3$. When each $J_\alpha$ is a complex structure, $H$ is said to be a hyper complex structure on $M$.

An almost quaternionic structure on $M$ is a rank-3 subbundle $Q \subset \text{End}(TM)$ which is locally spanned by almost hypercomplex structure $H = (J_\alpha)$. Such a locally defined triple $H$ will be called an admissible basis of $Q$. A linear connection $\nabla$ on $TM$ is called a quaternionic connection if $\nabla$ preserves $Q$, i.e. $\nabla_X \sigma \in \Gamma(Q)$ for all vector fields $X$ and smooth sections $\sigma \in \Gamma(Q)$. An almost quaternionic structure is said to be quaternionic if there is a torsion-free quaternionic connection. A $Q$-hermitian metric is a Riemannian metric which is Hermitian with respect to each almost

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complex structure in $Q$. An almost quaternionic (resp. quaternionic) manifold with $Q$-hermitian metric is called an almost quaternionic Hermitian (resp. quaternionic hermitian) manifold.

For $n = 1$ an almost quaternionic structure is the same as an oriented conformal structure and it turns out to be always quaternionic. When $n \geq 2$, the existence of torsion-free quaternionic connection is a strong condition which is equivalent to the 1-integrability of the associated $GL(n, \mathbb{H})Sp(1)$ structure [9, 31, 40]. If the Levi-Civita connection of a quaternionic hermitian manifold $(M, g, Q)$ is a quaternionic connection then $(M, g, Q)$ is called a Quaternionic Kähler manifold (briefly QK manifold). This condition is equivalent to the statement that the holonomy group of $g$ is contained in $Sp(n)Sp(1)$ [1, 2, 37, 38, 24]. If on a QK manifold there exists an admissible basis $(H)$ such that each almost complex structure $(J_\alpha) \in (H), \alpha = 1, 2, 3$ is parallel with respect to the Levi-Civita connection then the manifold is called hyperKähler (briefly HK). In this case the holonomy group of $g$ is contained in Sp($n$).

The various notions of quaternionic manifolds arise in a natural way from the theory of supersymmetric sigma models as well as in string theory. The geometry of the target space of two-dimensional sigma models with extended supersymmetry is described by the properties of a metric connection with torsion [16, 21]. The geometry of (4,0) supersymmetric two-dimensional sigma models without Wess-Zumino term (torsion) is a hyperKähler manifold. In the presence of torsion the geometry of the target space becomes hyperKähler with torsion (briefly HKT) [22]. This means that the complex structures $J_\alpha, \alpha = 1, 2, 3$, are parallel with respect to a metric quaternionic connection with totally skew-symmetric torsion [22]. Local (4,0) supersymmetry requires that the target space of two dimensional sigma models with Wess-Zumino term (torsion) is either HKT or quaternionic Kähler with torsion (briefly QKT) [30] which means that the quaternionic subbundle is parallel with respect to a metric linear connection with totally skew-symmetric torsion and the torsion 3-form is of type (1,2)+(2,1) with respect to all almost complex structures in $Q$. The target space of two-dimensional (4,0) supersymmetric sigma models with torsion coupled to (4,0) supergravity is a QKT manifold [23].

HKT spaces with symmetry (homothety) arise in quantum field theories. The geometry coming from the Michelson and Strominger’s study of $N = 4B$ supersymmetric quantum mechanics with superconformal $D(2,1,\alpha)$-symmetry is a HKT geometry with a special homothety [29]. These special HKT spaces are studied recently in [35, 36]. It is shown in [36] that the special homothety generates an infinitesimal action of the non-zero quaternions and the quotient space carries a QKT structure which is of instanton type. Explicitly, this means that we can find a certain torsion-free quaternionic connection which induces on the real canonical bundle $\ell^R = \Lambda^R_0T^*M$ a connection whose curvature is of type (1,1) with respect to each $J_\alpha$. Conversely, for a QKT of instanton type, one can find (see [36]) a HKT structure with special homothety on the corresponding Swann bundle (a bundle constructed by A. Swann for QK manifold [43]) provided some nondegeneracy (positivity) conditions are fulfilled.

HKT manifolds are also important in string theory. The number of surviving supersymmetries in a compactification of a 10-dimensional string theory on $M$, depends on the number of spinors parallel with respect to a connection with totally skew-symmetric torsion. This imposes restrictions on the holonomy group: the spinor representation of the holonomy group should have a fixed non-trivial spinor. The HKT geometry is one of the possible models of such a compactification since the holonomy group of the HKT-connection is a subgroup of $Sp(n)$. For a more precise discussion concerning parallel spinors and holonomy of connection with torsion the reader may wish to consult [42, 27, 14, 26, 15].

The properties of HKT and QKT geometries resemble those of HK and QK ones, respectively. In particular, HKT [22, 18] and QKT [23] manifolds admit twistor constructions with twistor spaces.
which have similar properties to those of HK [20] and QK [37, 38, 39] assuming some conditions on the torsion [22, 23, 36]. It is shown in [39, 23] that the twistor space of a QKT manifold is always a complex manifold provided that its dimension is at least 8. Most of the known examples of QKT manifolds (e.g. the ones constructed in [32]) are homogeneous. However there is also a large class QKT spaces obtained by conformal transformations of QK or HK manifolds [25].

The main object of interest in this article are the differential geometric properties of QKT and HKT manifolds and their twistor spaces. We find relations between Riemannian scalar curvatures of a QKT space which allows us to express sufficient conditions for a compact 8-dimensional QKT manifold to be QK in terms of its Riemannian scalar curvatures (Theorem 3.7).

We consider two almost complex structures $I_1$, $I_2$ on the twistor space $Z$ over a QKT manifold. The structure $I_1$ was originally constructed in [6], while $I_2$ is constructed in [13] for the QK case. For QKT the integrability of $I_1$ was established in [23].

We define a family of Riemannian metrics $h_c$ on $Z$ depending on a parameter $c$, thus obtaining almost hermitian structures on $Z$. Investigating the corresponding almost hermitian geometry we prove that $I_2$ is never integrable and that the Swann bundle of a QKT manifold admits a HKT structure with special symmetry if and only if $(Z, h_c, I_2)$ is a $G_1$ manifold according to the Gray-Hervella classification [19] (Theorem 5.2). The class of $G_1$ manifolds can be viewed as a direct sum of Hermitian and Nearly Kähler manifolds and is characterized by the requirement that the Nijenhuis tensor should be a 3-form. These manifolds are of particular interest in physics since they arise as a target spaces of $(2,0)$- and $(2,2)$- supersymmetric sigma models [33]. The physical applications also require the existence of a linear connection $\nabla$ preserving the almost hermitian structure $(g, J)$ and having a totally skew-symmetric torsion. The $G_1$-manifolds are precisely the object of interest since this is the largest class where such a connection exists [14]. Using the integrability of $I_1$ we present new relations between the different Ricci forms (Theorem 5.1), i.e. between the different 2-forms which determine the curvature of the QKT connection.

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2 QKT manifolds

Let $H$ be the quaternions and identify $H^n = \mathbb{R}^{4n}$. To fix notation we assume that $H$ acts on $H^n$ by right multiplication. This defines an antihomomorphism $\lambda: \{\text{unit quaternions}\} \rightarrow SO(4n)$, where our convention is that $SO(4n)$ acts on $H^n$ on the left. Denote the image by $Sp(1)$ and let $I_0 = \lambda(i), J_0 = \lambda(j), K_0 = \lambda(k)$. The Lie algebra of $Sp(1)$ is $sp(1) = span\{I_0, J_0, K_0\}$. Define $Sp(n) = \{A \in SO(4n) : AB = BA\}$ for all $B \in Sp(1)$. The Lie algebra of $Sp(n)$ is $sp(n) = \{A \in so(4n) : AB = BA\}$ for all $B \in sp(1))$. Let $Sp(n)Sp(1)$ be the product of the two groups in $SO(4n)$. Abstractly, $Sp(n)Sp(1) = (Sp(n) \times Sp(1))/\mathbb{Z}_2$. The Lie algebra of the group $Sp(n)Sp(1)$ is isomorphic to $sp(n) \oplus sp(1)$.

Let $(M, g, (J_\alpha) \in Q, \alpha = 1, 2, 3)$ be a 4n-dimensional almost quaternionic manifold with Q-hermitian Riemannian metric $g$ and an admissible basis $(J_\alpha)$. The Kähler form $\Phi_\alpha$ of each $J_\alpha$ is defined by $\Phi_\alpha = g(\cdot, J_\alpha \cdot)$. Let $\nabla$ be a quaternionic connection i.e.

\begin{equation}
\nabla J_\alpha = -\omega_\beta \otimes J_\gamma + \omega_\gamma \otimes J_\beta,
\end{equation}
where the $\omega_\alpha, \alpha = 1, 2, 3$ are 1-forms.

Here $(\alpha, \beta, \gamma)$ stands for a cyclic permutation of $(1, 2, 3)$.

Let $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$ be the torsion tensor of type $(1,2)$ of $\nabla$. We denote by the same letter the torsion tensor of type $(0,3)$ given by $T(X,Y,Z) = g(T(X,Y),Z)$.

An almost quaternionic manifold $(M, (H_\alpha) \in Q)$ is a QKT manifold if it admits a hermitian quaternionic structure $(g, Q)$ and a metric quaternionic connection $\nabla$ (a QKT connection) with a totally skew symmetric torsion which is a $(1,2) + (2,1)$-form with respect to each $J_\alpha, \alpha = 1, 2, 3$.

Explicitly this means that
\[(2.2) \quad T(X,Y,Z) = T(J_\alpha X, J_\alpha Y, Z) + T(J_\alpha X, Y, J_\alpha Z) + T(X, J_\alpha Y, J_\alpha Z).\]

for all $\alpha = 1, 2, 3$.

It follows that the holonomy group of any QKT-connection is a subgroup of $Sp(n)Sp(1)$, i.e. the bundle $SO(M)$ of oriented orthonormal frames on a QKT manifold can be reduced to a principal $Sp(n)Sp(1)$-bundle $P(M)$ so that the QKT-connection 1-form on $P(M)$ is $sp(n) \oplus sp(1)$-valued.

Every QKT manifold is a quaternionic manifold [25]. Poon and Swann constructed explicitly a quaternionic torsion-free connection $\nabla$ on a QKT in [36]. Following [36] we will say that a QKT manifold is of instanton type if the curvature of $\nabla$ on the real canonical bundle is of type $(1,1)$ with respect to each $J_\alpha$. Conversely, any quaternionic manifold locally admits a QKT structure [36]. Globally it is not necessarily true that a quaternionic hermitian structure on a quaternionic manifold is a QKT structure. However, if a QKT structure exists then it is unique and the torsion 3-form is computed in terms of connection 1-forms $\omega_\alpha$ and the exterior derivative of the Kähler forms [25]. One consequence of this is the observation that QKT structures persist under conformal transformations of the metric [25]. For HKT, the local existence of a HKT structure on any hypercomplex manifold is proved in [18]. On a HKT manifold the torsion connection is unique. This fact is a consequence of the general results in [17] (see also [18]), which imply that on a hermitian manifold there exists a unique linear connection with totally skew-symmetric torsion preserving the metric and the complex structure. This connection is known as the Bismut connection. Bismut used this connection [8] to prove a local index theorem for the Dolbeault operator on non-Kähler manifold. In the physics literature, the geometry of this connection is referred to as KT-geometry. Several non-trivial obstructions to the existence of (non-trivial) Dolbeault cohomology groups on a compact KT-manifold were described in [5, 27].

Ivanov [25] introduced the torsion 1-form on a QKT manifold by the equality
\[(2.3) \quad t(X) = \frac{1}{2} \sum_{i=1}^{4n} T(J_\alpha X, e_i, J_\alpha e_i),\]

where $\{e_i\}, i = 1, \ldots, 4n$ is an orthonormal basis, and showed that it is independent of $J_\alpha$. It turns out that a QKT structure is of instanton type if and only if the exterior differential $dt$ of the torsion 1-form is of type $(1,1)$ with respect to each $J_\alpha$ [36]. We define a balanced QKT manifold to be a QKT manifold with a zero torsion 1-form. The first examples of (compact) balanced HKT manifolds were constructed by Dotti and Fino [12]. On a compact QKT manifold one can show [25] that a metric with a coclosed torsion 1-form exists in each conformal class which supports a QKT structure. Such metrics are called Gauduchon metrics.

### 3 Curvature of QKT manifold

Let $R = [\nabla, \nabla] - \nabla_{[\cdot, \cdot]}$ be the curvature tensor of type $(1,3)$ of $\nabla$. We denote the curvature tensor of type $(0,4)$ $R(X,Y,Z,V) = g(R(X,Y)Z,V)$ by the same letter. There are three Ricci forms and
identities hold.

\[ \rho(X,Y) = \frac{1}{2} \sum_{i=1}^{4n} R(X,Y,e_i,J_\alpha e_i), \]

The Ricci forms satisfy \( u[R(X,Y),J_\alpha] = \rho_\gamma(X,Y)J_\beta - \rho_\beta(X,Y)J_\gamma \). The Ricci tensor \( \text{Ric} \), the scalar curvatures \( \text{Scal} \) and \( \text{Scal}_\alpha \) of \( \nabla \) are defined by \( \text{Ric}(X,Y) = \sum_{i=1}^{4n} R(e_i,X,Y,e_i), \text{Scal} = \sum_{i=1}^{4n} \text{Ric}(e_i,e_i), \text{Scal}_\alpha = -\sum_{i=1}^{4n} \text{Ric}(e_i,J_\alpha e_i) \). We shall denote by \( \rho^9, \text{Ric}^9, \rho^\alpha, \) etc. the corresponding objects for the metric \( g \), i.e. the same objects taken with respect to the Levi-Civita connection \( \nabla^g \). We may consider \( \rho^g, \text{Ric}^g, \rho^\alpha \) as an almost hermitian structure. Then the tensor \( \rho^g(X,Y) = \rho^g(X,J_\alpha Y) \) is known as the *-Ricci tensor of the almost hermitian structure. It is equal to \( \rho^\alpha(X,Y) = -\sum_{i=1}^{4n} \text{Ric}^g(e_i,X,J_\alpha Y,J_\alpha e_i) \) by the Bianchi identity. The function \( \text{Scal}^g \) is known also as the *-scalar curvature. If the *-Ricci tensor is a scalar multiple of the metric then the manifold is said to be *-Einstein. In general, the *-Ricci tensor is not symmetric and the *-Einstein condition is a strong condition. We shall see in the last section that the *-Ricci tensors of a HKT manifold are always symmetric.

The property of a QKT structure to be of instanton type can be expressed in terms of the Ricci forms. Namely, a QKT structure is of instanton type if and only if each Ricci form \( \rho_\alpha \) is of type (1,1) with respect to \( J_\alpha [36] \).

We show in this section that the scalar curvature functions are not independent and define a new scalar invariant, the 'quaternionic *-scalar curvature' of a QKT space. We begin with

**Proposition 3.1** Let \((M,g, (J_\alpha) \in \mathcal{Q}) \) be a 4n-dimensional QKT manifold. Then the following identities hold

\[(3.4) \quad \sum_{i=1}^{4n} (\nabla_X T)(J_\alpha Y,e_i,J_\alpha e_i) = 2(\nabla_X t)Y; \]

\[(3.5) \quad \sum_{i,j=1}^{4n} dT(e_j,J_\alpha e_j,e_i,J_\alpha e_i) = -8\delta t + 8||t||^2 - \frac{4}{3}||T||^2, \quad \sum_{i,j=1}^{4n} dT(e_j,J_\beta e_j,e_i,J_\gamma e_i) = 0. \]

**Proof.** The formula (3.4) follows from (2.1) and the definition (2.3) of the torsion 1-form by straightforward calculations. To prove (3.5) we need the following algebraic

**Lemma 3.2** For a three form \( T \) of type (1,2)+(2,1) with respect to each \( J_\alpha \) one has

\[ \sum_{i,j=1}^{4n} g(T(e_i,e_j),T(J_\gamma e_i,J_\beta e_j)) = 0, \quad \sum_{i,j=1}^{4n} g(T(e_i,e_j),T(J_\beta e_i,J_\gamma e_j)) = \frac{1}{3}||T||^2 \]

**Proof of the Lemma.** Put \( A = \sum_{i,j=1}^{4n} g(T(e_i,e_j),T(J_\gamma e_i,J_\beta e_j)) \). Use (2.2) three times to get

\[ 2A = \sum_{i,j,k=1}^{4n} T(e_i,e_j,e_k)(T(J_\gamma e_i,J_\gamma e_j,J_\alpha e_k) - T(J_\beta e_i,J_\beta e_j,J_\alpha e_k) \]

\[ 2A = \sum_{i,j,k=1}^{4n} T(e_i,e_j,e_k)(-T(J_\alpha e_i,e_j,e_k) + T(J_\alpha e_i,J_\beta e_j,J_\beta e_k) \]

\[ 2A = \sum_{i,j,k=1}^{4n} T(e_i,e_j,e_k)(T(J_\alpha e_i,e_j,e_k) - T(J_\gamma e_i,J_\gamma e_j,J_\alpha e_k) \]

Adding these up yields \( 6A = 0 \). The proof of the second identity in the statement of Lemma 3.2 is similar and we omit it.
We need also the expression of $dT$ in terms of $\nabla$ (see e.g. [25, 27, 14]),

\[
dT(X,Y,Z,U) = \sigma_{XYZ} \left\{ (\nabla_X T)(Y,Z,U) + g(T(X,Y),T(Z,U)) \right\}
- (\nabla_U T)(X,Y,Z) + \sigma_{XYZ} \left\{ g(T(X,Y),T(Z,U)) \right\},
\]

where $\sigma_{XYZ}$ denote the cyclic sum of $X,Y,Z$. Taking the appropriate trace in (3.6) and applying Lemma 3.2, we obtain the first equality in (3.5). Finally from (3.6) combined with (3.4) and Lemma 3.2 we get that

\[
\sum_{i,j=1}^{4n} dT(e_j,J_\alpha e_i,e_i) = -4 \sum_{i,j=1}^{4n} g(T(e_i,e_j),T(J_\gamma e_i,J_\beta e_j)) = 0.
\]

Q.E.D.

**Proposition 3.3** On a $4n$-dimensional ($n > 1$) QKT-manifold we have the equalities:

\[
\text{Scal}_{\alpha,\alpha} = \text{Scal}_{\beta,\beta} = \text{Scal}_{\gamma,\gamma}, \quad \text{Scal}_{\alpha,\beta} = 0, \quad \text{Scal}_{\alpha} = \frac{1}{2}(dt,\Phi_\alpha)
\]

**Proof.** Applying (3.4) consequently to (3.45), (3.33) in [25], we obtain

\[
\frac{2(n-1)}{n} \left\{ \rho_\alpha(X,J_\alpha Y) - \rho_\beta(X,J_\beta Y) \right\} = \sum_{i=1}^{4n} \left\{ dT(e_i,J_\alpha e_i,X,J_\alpha Y) - dT(e_i,J_\beta e_i,X,J_\beta Y) \right\};
\]

\[
(n-1)\rho_\alpha(X,J_\alpha Y) = -\frac{n(n-1)}{n+2} \text{Ric}(X,Y) + \frac{n(n-1)}{n+2} (\nabla_X t)Y
+ \frac{n}{4(n+2)} \sum_{i=1}^{4n} \left\{ (n+1)dT(X,J_\alpha Y,e_i,J_\alpha e_i) - dT(X,J_\beta Y,e_i,J_\beta e_i) - dT(X,J_\gamma Y,e_i,J_\gamma e_i) \right\}.
\]

Now take the appropriate trace in (3.8), and use (3.5) to get $\text{Scal}_{\alpha,\alpha} = \text{Scal}_{\beta,\beta}, \quad \text{Scal}_{\alpha,\beta} = 0$. The last equality in (3.7) is a direct consequence of (3.4), $\text{Scal}_{\alpha,\beta} = 0$ and (3.9).

Q.E.D.

**Definition.** The three coinciding traces of the Ricci forms on a $4n$ dimensional QKT manifold ($n > 1$), give a well-defined global function. We call this function the **quaternionic scalar curvature** of the QKT connection and denote it by $\text{Scal}_Q := \text{Scal}_{\alpha,\alpha}$.

**Proposition 3.4** On a $4n$-dimensional ($n > 1$) QKT manifold we have

\[
\text{Scal}^g_\alpha = \text{Scal}^g_\beta = \text{Scal}^g_\gamma = \text{Scal}_Q - \delta t + ||T||^2 - \frac{1}{12} ||T||^2, \quad \text{Scal}^g_{\alpha,\beta} = \text{Scal}_Q = \frac{1}{2}(dt,\Phi_\gamma)
\]

**Proof.** We follow [25, 27]. The curvature $R^g$ of the Levi-Civita connection is related to $R$ via

\[
R^g(X,Y,Z,U) = R(X,Y,Z,U) - \frac{1}{2}(\nabla_X T)(Y,Z,U) + \frac{1}{2}(\nabla_Y T)(X,Z,U)
- \frac{1}{2}g(T(X,Y),T(Z,U)) - \frac{1}{4}g(T(Y,Z),T(X,U)) - \frac{1}{4}g(T(Z,X),T(Y,U)).
\]
Taking the traces in (3.11) and using (2.3) we obtain

\begin{equation}
(\rho_\alpha^g(X,J_\alpha Y) = \rho_\alpha(X,J_\alpha Y) - \frac{1}{2}(\nabla_X t)Y - \frac{1}{2}(\nabla_{J_\alpha Y} t)J_\alpha X
+ \frac{1}{2}t(J_\alpha T(X,J_\alpha Y)) + \frac{1}{4}\sum_{i=1}^{4n} g(T(X,e_i),T(J_\alpha Y,J_\alpha e_i)),
\end{equation}

To finish take the appropriate traces in (3.12) and apply Lemma 3.2 and Proposition 3.3. \textbf{Q.E.D.}

**Definition.** The three coinciding traces of the Riemannian Ricci forms on a 4n dimensional QKT manifold \((n > 1)\), give a well-defined global function. We call this function the quaternionic \(*\)-scalar curvature and denote it by \(\text{Scal}^g = \text{Scal}_\alpha^g\).

**Proposition 3.5** On a 4n-dimensional \((n > 1)\) QKT manifold \((M,g,Q)\) the scalar curvatures are related by

\[
\begin{align*}
\text{Scal}^g &= \frac{n+2}{n}\text{Scal}_Q - 3\delta t + 2||t||^2 - \frac{1}{12}||T||^2, \\
\text{Scal}^g_Q &= \text{Scal}_Q - \delta t + ||t||^2 - \frac{1}{12}||T||^2, \\
\text{Scal} &= \frac{n+2}{n}\text{Scal}_Q - 3\delta t + 2||t||^2 - \frac{1}{3}||T||^2.
\end{align*}
\]

**Proof.** We derive from (3.11) that

\begin{equation}
(3.13) \quad \text{Ric}^g(X,Y) = \text{Ric}(X,Y) + \frac{1}{2}\delta T(X,Y) + \frac{1}{4}\sum_{i=1}^{2n} g(T(X,e_i),T(Y,e_i)),
\end{equation}

\[
\text{Scal}^g = \text{Scal} + \frac{1}{4}||T||^2.
\]

Take the trace in (3.9) and use Lemma 3.2 to get the first equality of the proposition. The second equality is already proved in Proposition 3.10. The last one is a consequence of (3.13) and the already proven first equality in the proposition. \textbf{Q.E.D.}

As a consequence of the above result, we get

**Theorem 3.6** Let \((M,g,Q)\) be a compact 4n-dimensional \((n > 1)\) QKT manifold. Then

\begin{equation}
(3.14) \quad \int_M (\text{Scal}^g - \text{Scal}^g_Q - \frac{2}{n}\text{Scal}_Q) dV \geq 0.
\end{equation}

The equality in (3.14) is attained if and only if the QKT structure is balanced.

\begin{equation}
(3.15) \quad \int_M (\text{Scal}^g - 2\text{Scal}^g_Q - \frac{2-n}{n}\text{Scal}_Q) dV \geq 0.
\end{equation}

The equality in (3.15) is attained if and only if the QKT structure is quaternionic Kähler.
Proof. Proposition 3.5 implies
\[ \text{Scal}^g - \frac{2}{n} \text{Scal}_Q = -2\delta t + 2\|t\|^2, \]
(3.16)
\[ \text{Scal}^g - 2\text{Scal}^g_Q - \frac{2-n}{n} \text{Scal}_Q = -\delta t + \frac{1}{12} \|T\|^2. \]
Integrating the last two equalities over \( M \) we get the proof. Q.E.D.

**Remark 1.** From the proof of Theorem 3.6, it is clear that the statement of the theorem will still hold for a non-compact QKT, provided that \( t \) is a coclosed 1-form.

Applying Theorem 3.6 to an 8-dimensional QKT manifold, i.e. take \( n = 2 \) in (3.16), we get the main result of this section

**Theorem 3.7** Let \((M, g, Q)\) be an 8-dimensional compact connected QKT manifold. Then

a) \((M, g, Q)\) is a quaternionic Kähler manifold if and only if
\[ \int_M (\text{Scal}^g - 2\text{Scal}^g_Q) dV = 0. \]

b) \((M, g, Q)\) is a locally hyperKähler manifold if and only if the Riemannian scalar curvature and the quaternionic \( * \)-scalar curvature both vanish.

In particular, any compact 8-dimensional QKT manifold with a flat metric is flat locally hyperKähler and therefore is covered by a hyperKähler torus.

We finish this section with the following

**Theorem 3.8** A \(4n\)-dimensional QKT manifold is of instanton type if and only if each \( * \)-Ricci tensor is symmetric.

**Proof.** First, we observe that on a QKT manifold \((M, g, Q)\) the \((2,0)+(0,2)\)-parts of \( \rho^g, \rho, dt \) with respect to \( J_\alpha \) are related by the equality
\[ (3.17) \rho^g_\alpha(X, J_\alpha Y) + \rho^g_\alpha(J_\alpha X, Y) = \rho_\alpha(X, J_\alpha Y) + \rho_\alpha(J_\alpha X, Y) - \frac{1}{2} (dt(X, Y) - dt(J_\alpha X, J_\alpha Y)). \]

Indeed, put \( B(X, Y) = \sum_{i=1}^4 g(T(X, e_i), T(J_\alpha Y, J_\alpha e_i)) \). The tensor \( B \) is symmetric since the \((1,2)+(2,1)\)-type property of \( T \) leads to the expression
\[ 2B(X, Y) = \sum_{i,j=1}^4 T(X, e_i, e_j)T(Y, e_i, e_j) - T(X, e_i, e_j)T(J_\alpha e_i, J_\alpha e_j) \]
which is clearly symmetric. Then the skew-symmetric part of (3.12) gives (3.17), where we used (2.2) and the equality \( d^\nabla t(X, Y) := (\nabla_X t)Y - (\nabla_Y t)X = dt(X, Y) - t(T(X, Y)) \). Computations in [36] show the identity \( \rho_\alpha(X, J_\alpha Y) + \rho_\alpha(J_\alpha X, Y) = \frac{n}{2} (dt(X, Y) - dt(J_\alpha X, J_\alpha Y)) \). Consequently, (3.17) gives
\[ \rho^g_\alpha(X, J_\alpha Y) + \rho^g_\alpha(J_\alpha X, Y) = \frac{n+1}{2} (dt(X, Y) - dt(J_\alpha X, J_\alpha Y)). \] Hence, \( M \) is of instanton type if and only if \( \rho^g_\alpha \) is of type \((1,1)\) with respect to \( J_\alpha \). The latter property is equivalent to the condition that the corresponding \( * \)-Ricci tensor \( \rho^*_\alpha \) is symmetric. Q.E.D.
4 Twistor space of QKT manifolds

In this section we adapt the setup from [41, 10] to incorporate a totally skew-symmetric torsion. Our discussion is very close to that of [4].

Let \((M, g)\) be a 4n-dimensional QKT manifold and let \(\pi : P(M) \longrightarrow M\) be the natural projection. For each \(u \in P(M)\) we consider the linear isomorphism \(j(u)\) on \(T_{\pi(u)}M\) defined by \(j(u) = uJ_0u^{-1}\). It is easy to see that \(j(u)^2 = -id\) and \(g(j(u)X, j(u)Y) = g(X, Y)\) for all \(X, Y \in T_{\pi(u)}M\), i.e. \(j(u)\) is an orthogonal complex structure at \(\pi(u)\). For each point \(p \in M\) we define \(Z_p(M) = \{j(u) : u \in P(M), \pi(u) = p\}\). In other words, \(Z_p(M)\) is the space of all orthogonal complex structures in the tangent space \(T_pM\) which are compatible with the QKT structure.

We put \(Z = \bigcup_{p \in M} Z_p(M)\). Let \(H = Sp(n)Sp(1) \cap U(2n)\). There is a bijective correspondence between the symmetric space \(Sp(n)Sp(1)/H = Sp(1)/U(1) = \mathbb{CP}^1 = \mathbb{S}^2\) and \(Z_p(M)\) for every \(p \in M\). So we can consider \(Z\) as the associated fibre bundle of \(P(M)\) with standard fibre \(Sp(n)Sp(1)/H = \mathbb{CP}^1\). Hence, \(P(M)\) is a principal fibre bundle over \(Z\) with structure group \(H\) and projection \(j\). If \(\pi_1 : Z \longrightarrow M\) is the projection, we have that \(\pi_1 \circ j = \pi\). We consider the symmetric space \(Sp(n)Sp(1)/H\). We have the following Cartan decomposition \(sp(n) \oplus sp(1) = h \oplus m\), where \(h = \{A \in sp(n) \oplus sp(1) : A(J_0 = J_0A) = (sp(n) \oplus sp(1)) \cap u(2n)\}\) is the Lie algebra of \(H\) and \(m = \{A \in sp(n) \oplus sp(1) : A(J_0 = -J_0A)\}\). It is clear that \(m\) is generated by \(I_0\) and \(K_0\), i.e. \(m = \text{span}\{I_0, K_0\}\). Hence, if \(A \in m\) then \(J_0A \in m\). Let \((,\) be the inner product in \(gl(4n, \mathbb{R})\) defined by \((A, B) = \text{trace}(AB^T) = \sum_{i=1}^{4n} < Ae_i, Be_i>\) for \(A, B \in gl(4n, \mathbb{R})\), where \(<,>\) is the canonical inner product in \(\mathbb{R}^{4n}\). It is clear that \(sp(n) \perp sp(1)\) and \(I_0, J_0, K_0\) form an orthogonal basis of \(sp(1)\) with \((I_0, I_0) = (J_0, J_0) = (K_0, K_0) = 4n\). Hence, \(h \perp m\).

Let \(u \in P(M)\) and \(Q_u\) is the horizontal subspace of the tangent space \(T_uP(M)\) induced by the QKT-connection on \(M\) ([28]). The vertical space is \(h_u^* \oplus m_u^*\), where \(h_u^* = \{A_u^* : A \in h\}, m_u^* = \{A_u^* : A \in m\}\). Hence, \(T_uP(M) = h_u^* \oplus m_u^* \oplus Q_u\). For each \(u \in P(M)\) we put \(V_{j(u)} = j_u(h_u^* \oplus m_u^*), H_{j(u)} = j_uQ_u\). Thus we obtain the vertical and horizontal distributions \(V\) and \(H\) on \(Z\). Since \(P(M)\) is a principal fibre bundle over \(Z\) with structure group \(H\) we have \(\text{Ker} j_{su} = h_u^*\). Hence \(V_{j(u)} = j_{su}m_u^*\) and \(j_{su}m_u^* \oplus Q_u : m_u^* \oplus Q_u \longrightarrow T_{j(u)}Z\) is an isomorphism. We define almost complex structures \(I_1\) and \(I_2\) on \(Z\) by

\[
I_1j_{su}A^* = j_{su}(J_0A)^*, \quad I_2j_{su}A^* = -j_{su}(J_0A)^*
\]

\[
I_1j_{su}B(\xi) = j_{su}B(J_0\xi), \quad i = 1, 2,
\]

for \(u \in P(M), A, m, \xi \in \mathbb{R}^{4n}\). For twistor bundles of 4-manifolds the almost complex structure \(I_1\) is introduced in [6] and the almost complex structure \(I_2\) is introduced in [13] in terms of the horizontal spaces of the Levi-Civita connection. The almost complex structure \(I_1\) for QK, HKT and QKT manifolds was constructed in [37, 22, 23], respectively, where it is proved that it is actually integrable. For every \(c > 0\) a Riemannian metric \(h_c\) on \(Z\) is defined by

\[
h_c(j_{su}A^*, j_{su}B^*) = c^2(A, B), \quad h_c(j_{su}A^*, j_{su}B(\xi)) = 0
\]

\[
h_c(j_{su}B(\xi), j_{su}B(\eta)) = <\xi, \eta>.
\]

for \(u \in P(M), A, B, m, \xi, \eta \in \mathbb{R}^{4n}\). It is clear that \((I_i, h_c), i = 1, 2\), determine two families of almost hermitian structures on \(Z\).

In the QK case the properties of the almost hermitian geometry of \((I_i, h_c), i = 1, 2\) are considered in [11, 4]. Below we follow [4] making the necessary modifications required by the presence of torsion.

We split the curvature of a QKT-connection into \(sp(n)\)-valued part \(R^t\) and \(sp(1)\)-valued part \(R^\eta\) following the classical scheme (see e.g. [3, 24, 7])
Proposition 4.1. The curvature of a QKT manifold splits as follows

\[ R(X, Y) = R'(X, Y) + \frac{1}{2n}(\rho_1(X, Y)J_1 + \rho_2(X, Y)J_2 + \rho_3(X, Y)J_3), \]

\[ [R'(X, Y), J_\alpha] = 0, \quad \alpha = 1, 2, 3. \]

We denote by \( A^* \) (resp. \( B(\xi) \)) the fundamental vector field (resp. the standard horizontal vector field) on \( P(M) \) corresponding to \( A \in \text{sp}(n) \oplus \text{sp}(1) \) (resp. \( \xi \in \mathbb{R}^{4n} \)). Let \( \Omega, \Theta \) be the curvature 2-form and the torsion 2-form for the QKT-connection on \( P(M) \), respectively ([28]).

We shall denote the splitting of the \( \text{sp}(n) \oplus \text{sp}(1) \)-valued curvature 2-form \( \Omega \) on \( P(M) \), corresponding to Proposition 4.1, by \( \Omega = \Omega' + \Omega'' \), where \( \Omega' \) is a \( \text{sp}(n) \)-valued 2-form and \( \Omega'' \) is a \( \text{sp}(1) \)-valued form. Explicitly, we have \( \Omega'' = \Omega''_1 J_0 + \Omega''_2 J_0 + \Omega''_3 K_0 \), where \( \Omega''_\alpha, \alpha = 1, 2, 3 \), are 2-forms. If \( \xi, \eta, \zeta \in \mathbb{R}^{4n} \), then the 2-forms \( \Omega''_\alpha, \alpha = 1, 2, 3 \), are given by

\[ \Omega''_\alpha(B(\xi), B(\eta)) = \frac{1}{2n}\rho_\alpha(X, Y), \quad X = u(\xi), Y = u(\eta). \]

Since \( T \) is 3-form of type \((1,2)+(2,1)\), the torsion 2-form \( \Theta \) has the properties

\[ < \Theta_u(B(\xi), B(\eta)), \zeta > = -< \Theta_u(B(\xi), B(\zeta)), \eta >, \]

\[ < \Theta_u(B(\xi), B(\eta)), \zeta > = -< \Theta_u(B(\eta), B(\zeta)), \xi > + < \Theta_u(B(\xi), B(J_0\eta)), J_0\zeta > \]

Let \( F_i(X, Y, Z) = h_c((DX)_I Y, Z), i = 1, 2 \), where \( D \) is the covariant derivative of the Levi-Civita connection of \( h_c \). We denote by \( K \) the curvature tensor of \( h_c \).

In the rest of the paper, \( A, B, C, D \in m, \xi, \eta, \zeta, \tau \in \mathbb{R}^{4n} \).

The calculations made in [4] for the twistor space over QK manifold can be performed in our case by taking into account the torsion and their properties. In this way, using (4.18), (4.19) and (4.21), we obtain our technical tools, namely

Proposition 4.2. The next equalities hold at \( u \in P(M) \):

\[ F_i(j_{su}A^*, j_{su}B^*, j_{su}C^*) = 0, \quad F_i(j_{su}A^*, j_{su}B^*, j_{su}B(\xi)) = 0, \quad i = 1, 2, \]

\[ F_i(j_{su}A^*, j_{su}B(\xi), j_{su}B(\eta)) = \frac{c^2}{2}(A, \Omega(B(J_0\xi), B(\eta)) + \]

\[ + \frac{c^2}{2}(A, \Omega(B(B(\xi), B(J_0\eta)) + 2 < AJ_0\xi, \eta >, \quad i = 1, 2, \]

\[ F_i(j_{su}B(\xi), j_{su}A^*, j_{su}B^*) = 0, \quad i = 1, 2, \]

\[ F_2(j_{su}B(\xi), j_{su}A^*, j_{su}B(\eta)) = \frac{c^2}{2}(A, J_0\Omega(B(\xi), B(\eta)) + \]

\[ + \frac{c^2}{2}(A, \Omega(B(\xi), B(J_0\eta)), \]

\[ F_1(j_{su}B(\xi), j_{su}A^*, j_{su}B(\eta)) = \frac{c^2}{2}(J_0A, \Omega(B(\xi), B(\eta)) + \]

\[ + \frac{c^2}{2}(A, \Omega(B(\xi), B(J_0\eta)), \]

\[ F_i(j_{su}B(\xi), j_{su}B(\eta), j_{su}B(\zeta)) = -\frac{1}{2} < \Theta(B(\xi), B(J_0\eta)), \xi > - \]

\[ - \frac{1}{2} < \Theta(B(\xi), B(\eta)), J_0\zeta >, \quad i = 1, 2. \]

For the curvature tensor \( K \) of \( h_c \) we have
Proposition 4.3 The following equalities hold at any \( u \in P(M) \):

\[
K(j_u A^*, j_u B^*, j_u C^*, j_u D^*) = -c^2([A, B], [C, D]),
\]
\[
K(j_u A^*, j_u B^*, j_u C^*, j_u B(\xi)) = 0,
\]
\[
K(j_u A^*, j_u B(\xi), j_u B^*, j_u B(\eta)) = \frac{c^2}{2} ([A, B], \Omega(B(\xi), B(\eta)) - \frac{c^4}{4} (B, \Omega(B(\xi), B(e_i))(u))(A, \Omega(B(\eta), B(e_i)),
\]
\[
K(j_u B(\xi), j_u B(\eta), j_u B(\zeta), j_u A^*) = \frac{c^2}{2} (A, B(\xi)\Omega(B(\xi), B(\eta)))
\]
(4.23)

\[
\frac{c^2}{4} (A, \Omega([B(\eta), B(\zeta)], B(\xi))) - \frac{c^2}{4} (A, \Omega([B(\zeta), B(\xi)], B(\eta))),
\]
\[
K(j_u B(\xi), j_u B(\eta), j_u B(\zeta), j_u B(\tau)) = < \Omega(B(\xi), B(\eta))\zeta, \tau >
\]
\[
- \frac{c^2}{4} (\Omega_m(B(\xi), B(\tau)), \Omega_m(B(\eta), B(\zeta))) + \frac{c^2}{4} (\Omega_m(B(\xi), B(\zeta)), \Omega_m(B(\eta), B(\tau)))
\]
\[
+ \frac{1}{4} < \Theta(B(\xi), B(\zeta)), \Theta(B(\eta), B(\tau)) > + \frac{1}{2} < \Theta(B(\xi), B(\eta)), \Theta(B(\zeta), B(\tau)) >
\]
\[
- \frac{1}{2} < B(\xi)\Theta(B(\eta), B(\zeta)), \tau > + \frac{1}{2} < B(\eta)\Theta(B(\xi), B(\zeta)), \tau >,
\]

where \( \Omega_m \) denotes the \( m \)-component of \( \Omega \).

5 Almost Hermitian geometry of \((Z, h_c, I_1)\)

Let \((M, g, J)\) be a \(2n\)-dimensional almost Hermitian manifold and let \(F(X, Y, Z) = g((\nabla^g_X J)Y, Z)\), where \(\nabla^g\) is the covariant differentiation of the Levi-Civita connection on \(M\). We recall the definition of some classes according to the Gray-Hervella classification [19] in terms of the notations we use: \((M, g, J)\) is Kähler if \(F = 0\); Hermitian if \(H(X, Y, Z) = F(X, Y, Z) - F(JX, JY, Z) = 0\); semi-Kähler if \(trF = 0\), quasi-Kähler if \(F(X, Y, Z) + F(JX, JY, Z) = 0\); nearly Kähler if \(F(X, Y, Z) + F(Y, X, Z) = 0\); almost Kähler if \(F(X, Y, Z) + F(Y, Z, X) + F(Z, X, Y) = 0\); \(G_1\) manifold if \(\psi(X, Y, Z) = F(X, Y, Z) + F(Y, X, Z) - F(JX, JY, Z) - F(JY, JX, Z) = 0\).

Theorem 5.1 On a QKT manifold \((M^{4n}, g, (J_\alpha) \in \mathcal{Q})\) the \((2,0)+(0,2)\)-parts of the Ricci forms \(\rho_\alpha, \rho_\beta\) with respect to \(J_\gamma\) coincide.

Proof. We claim the following identities hold

\[
\rho_\alpha(J_\beta X, J_\beta Y) - \rho_\alpha(X, Y) = \rho_\gamma(J_\beta X, Y) + \rho_\gamma(X, J_\beta Y).
\]

Indeed, consider the almost hermitian structure \((h_c, I_1)\) on the twistor space \(Z\). The almost complex structure \(I_1\) is integrable [23]. Therefore \((Z, h_c, I_1)\) is a Hermitian manifold. We calculate taking into account (4.22) that

\[
0 = \frac{2}{c^2} (F_1(j_u A^*, j_u B(\xi), j_u B(\eta)) - F_1(I_1(j_u A^*), I_1(j_u B(\xi)), j_u B(\eta)))
\]
\[
= (J_0 A, \Omega(B(\xi), B(\eta))) + (A, \Omega(B(J_0 \xi), B(\eta))) + (A, \Omega(B(\xi), B(J_0 \eta))) - (J_0 A, \Omega(B(J_0 \xi), B(J_0 \eta))).
\]
The $m$ component $\Omega_m$ is given by $2n\Omega_m(B(\xi), B(\eta)) = \rho_1(B(\xi), B(\eta))I_0 + \rho_3(B(\xi), B(\eta))K_0$. The last two equalities imply
\[
\rho_1(B(J_0\xi), B(\eta)) + \rho_1(B(\xi), B(J_0\eta)) = \rho_3(B(J_0\xi), B(J_0\eta)) - \rho_3(B(\xi), B(\eta))
\]
The proof is completed by putting $J_1 = I_0, J_2 = J_0, J_3 = K_0$. Q.E.D.

We recall the notion of a Swann bundle [43, 34, 36]. On a 4n dimensional QKT manifold $M$ it is defined by $U(M) = (P \times_{Sp(n)} Sp(1)} H^*)/\{\pm\}$, where $H^*$ are the nonzero quaternions. It carries a hypercomplex structure [43, 34, 36]. If $M$ is of instanton type and the condition
\[
(5.25) \quad \rho_\alpha(J_\alpha X, Y) + \rho_\alpha(J_\gamma X, J_\beta Y) = \frac{1}{c^2}g(X, Y)
\]
holds then $U(M)$ has a HKT structure with special homothety [36]. This generalizes the Swann result stating that if $M$ is QK then $U(M)$ carries a HK structure [43]. An alternative construction [23] of a HK structure on the Swann bundle over QKT utilizes the assumption that $dT$ is a (2,2)-form with respect to each $J_\alpha$.

**Theorem 5.2** Let $(M^{4n}, g, (J_\alpha) \in Q)$ be a QKT manifold with twistor space $Z$.

1. The almost complex structure $I_2$ on $Z$ is never integrable.
2. The space $(Z, h_c, I_2)$ is a $G_1$ manifold if and only if the Swann bundle admits a HKT structure with special homothety.
3. The spaces $(Z, h_c, I_i), \quad i = 1, 2$ are semi-Kähler manifolds if and only if the QKT structure is balanced, i.e. if $t = 0$.
4. If $(Z, h_c, I_2)$ is quasiKähler, almost Kähler, nearly Kähler or $(Z, h_c, I_1)$ is Kähler then the torsion is zero and $M$ is a QK manifold.

**Proof.** The almost complex structure $I_2$ is integrable if and only if $H = 0$. We obtain using (4.22) that
\[
H(j_{su}B(\xi), j_{su}A^*, j_{su}B(\eta)) = \frac{c^2}{2}(J_0A, \Omega(B(J_0\xi), B(J_0\eta))) - \frac{c^2}{2}(J_0A, \Omega(B(\xi), B(\eta))) + c^2(A, \Omega(B(J_0\xi), B(\eta))) + c^2(A, \Omega(B(\xi), B(J_0\eta)))
\]
\[
H(j_{su}A^*, j_{su}B(\xi), j_{su}B(\eta)) = 4 < AJ_0\xi, \eta > + H(j_{su}B(\xi), j_{su}A^*, j_{su}B(\eta))
\]
Hence, $< AJ_0\xi, \eta >= 0$ which is impossible.

For b), $(Z, h_c, I_2)$ is a $G_1$-manifold if and only if $\psi = 0$. We get by (4.22) that the only non-zero term of $\psi$ is
\[
\psi(j_{su}A^*, j_{su}B(\xi), j_{su}B(\eta)) = 4 < AJ_0\xi, \eta > - c^2(J_0A, \Omega(B(\xi), B(\eta))) + c^2(A, \Omega(B(J_0\xi), B(\eta))) + c^2(A, \Omega(B(\xi), B(J_0\eta))) + c^2(J_0A, \Omega(B(J_0\xi), B(J_0\eta)))
\]
The last equality is equivalent to
\[
(5.26) \quad c^2\rho_1(B(J_0\xi), B(\eta)) + c^2\rho_1(B(\xi), B(J_0\eta)) = -2 < K_0\xi, \eta >
\]
\[
c^2\rho_3(B(J_0\xi), B(\eta)) + c^2\rho_3(B(\xi), B(J_0\eta)) = 2 < I_0\xi, \eta >
\]
Put $J_1 = I_0, J_2 = J_0, J_3 = K_0$ in (5.26) we derive (5.25). Combining the already proved (5.25) with (5.24) we get that $\rho_\alpha$ is of type $(1,1)$ with respect to $J_\alpha$. Hence, the QKT structure is of instanton type. The rest of b) follows by Theorem 6.1 and Remark 6.3 in [36].

We use (4.22) again to prove c). We have 0 = $tr F_i(jw,\rho^*) = -\varepsilon^{2} \sum_{k=1}^{4n}(A, \Omega(\rho(B(e_{k}),B(J_0 \eta_{k}))))$ which is equivalent to $\sum_{k=1}^{4n} \rho_\alpha(e_{k},J_3 \eta_{k}) = 0$ by Proposition 3.1. Further, 0 = $tr F_i(jw,B(\xi)) = -\frac{1}{2} \sum_{k=1}^{4n} \langle \Theta(B(e_{k}),B(J \eta_{k})),\xi \rangle \geq f(u(\xi))$.

The proof of d) is a direct consequence of the last equality in (4.22) and (4.21). Q.E.D.

Remark 2. We may consider the twistor space of an almost quaternionic hermitian manifold and construct the almost complex structure $I_1$ using horizontal spaces of a quaternionic connection with skew-symmetric torsion. It follows that $I_1$ is integrable if and only if the torsion is $(1,2)+(2,1)$-form, i.e. it is a QKT manifold.

Examples. The twistor space $(Z, h_c, I_i), i = 1, 2$ of balanced HKT structures on the nilpotent Lie groups constructed in [12] is semi-Kähler for $I_2$ and hermitian semi-Kähler (balance) for $I_1$.

6 Geometry of HKT manifold and twistor construction

We recall some notations. The Lee form $\theta$ of a $2n$-dimensional almost Hermitian manifold $(M, g, J)$ with Kähler form $\Phi = g(J, J\bullet)$ is defined by $\theta = -\delta \Phi \circ J$. On a HKT manifold there are three Lee forms corresponding to $J_\alpha, \alpha = 1, 2, 3$, which are all equal. The common Lee form $\theta$ is called the Lee form of the HKT structure. It turns out that the Lee form of a HKT manifold is equal to the torsion 1-form, $\theta = t$ [27, 25].

On a HKT manifold all the Ricci forms vanish and the exterior differential of the Lee form is of type $(1,1)$ with respect to each $J_\alpha$ [5, 27]. The latter property can be easily seen by comparing the curvatures of the Bismut and Chern connection taken with respect to any hermitian structure $J_\alpha$ on the corresponding canonical bundle. They differ by $d(J_\alpha \theta)$, where $J_\alpha \theta(X) = -\theta(J_\alpha X)$. The curvature of the Chern connection is of type $(1,1)$, the curvature of Bismut connection vanishes and the $(2,0)+(0,2)$-parts of $d\theta$ and $d(J_\alpha \theta)$ coincide. Thus, Theorem 3.8 implies that on a HKT manifold the Riemannian Ricci form $\rho_\alpha^*$ is of type $(1,1)$ with respect to the complex structure $J_\alpha$ and therefore the $*$-Ricci tensors are symmetric. We have proved

**Theorem 6.1** The $*$-Ricci tensors on a HKT manifold are symmetric.

**Proposition 6.2** Let $(Z, h_c, I_i, i = 1, 2)$ be a twistor space of a $4n$ dimensional $(n > 1)$ HKT manifold $(M, g, (J_\alpha))$ and $X = u(\xi), Y = u(\eta) \in T_{\pi(u)}M$. The Ricci tensor $\rho$ and the $*$-Ricci tensor $\rho_\alpha^*$ for $(Z, h_c, I_i, i = 1, 2)$ are given by

$$\text{Ric}(jw,\rho^*) = \rho_\alpha^*(jw,\rho^*) = \frac{1}{n}c^2 h_c(jw,\rho^*) = \frac{1}{n}c^2 h_c(jw,\rho^*) = \frac{1}{n}c^2 h_c(jw,\rho^*), \; \; i = 1, 2;$$

$$\text{Ric}(jw,B(\xi),J_\alpha A^*) = \rho_\alpha^*(jw,B(\xi),J_\alpha A^*) = 0, \; \; i = 1, 2;$$

$$\text{Ric}(jw,B(\xi),J_\alpha B(\eta)) = \text{Ric}(X,Y);$$

$$\rho_\alpha^*(jw,B(\xi),J_\alpha B(\eta)) = \rho_\alpha^*(X,Y), \; \; J = j(u), \; \; i = 1, 2.$$ 

In particular, the $*$-Ricci tensor $\rho_\alpha^*$ for $(Z, h_c, I_i)$ is symmetric and $I_i$-invariant, $i = 1, 2$. 
If the HKT space is Einstein (resp. ∗-Einstein with respect to each $J_\alpha$) with positive scalar curvature $\text{Scal}^g$ (resp. $\text{Scal}^g_Q$) then there exists an Einstein hermitian structure $(Z, h_c, I_1), c^2 = \frac{4}{\text{Scal}^g}$ (resp. ∗-Einstein almost hermitian structure $(Z, h_c, I_2), c^2 = \frac{4}{\text{Scal}^g_Q}$).

**Proof.** Take the trace into (4.23) and compare the result with (3.13), (3.12) to get the formulas in the theorem. The last one and Theorem 6.1 imply that the ∗-Ricci tensors on the twistor space are symmetric. The formula for the constant $c^2$ is a consequence of the fact that the ∗-Einstein curvature is exactly equal to the quaternionic curvature by Proposition 3.4. **Q.E.D.**

**Remark 3.** In view of the above results, the ∗-Einstein condition on a HKT manifold does not impose restrictions on the $(2,0)+(0,2)$-part of the ∗-Ricci tensor.

On a HKT manifold the quaternionic scalar curvature $\text{Scal}_Q = 0$. In this case Theorem 3.6 leads to the following

**Theorem 6.3** Let $M$ be a compact $4n$-dimensional $(n > 1)$ HKT manifold. Then

a) $\int_M (\text{Scal}^g - \text{Scal}^g_Q) \, dV \geq 0$ with the equality if and only if the HKT structure is balanced;

b) $\int_M (\text{Scal}^g - 2\text{Scal}^g_Q) \, dV \geq 0$ with the equality if and only if the HKT structure is hyperKähler;

In particular, any compact HKT manifold with flat metric is hyperKähler.

On a HKT manifold $dT$ is $(2,2)$-form with respect to each complex structure $J_\alpha$ and therefore all terms in the second line in (3.9) are equal (see [25]). Thus, (3.9) gives

(6.27) \[ \text{Ric}(X, Y) = (\nabla_X \theta) Y + \frac{1}{4} \sum_{i=1}^{4n} dT(X, J_\alpha Y, e_i, J_\alpha e_i). \]

As a consequence of (6.27) we get that on a $4n$ dimensional $(n \geq 2)$ HKT manifold the Ricci tensor is symmetric if and only if $d^X \theta(X, Y) = 0$. In particular, on a balanced $4n$ dimensional $(n \geq 2)$ HKT manifold the Ricci tensor is $J_\alpha$-invariant and symmetric. Therefore, the torsion 3-form is coclosed, $\delta T = 0$. We note that this is true in more general situation on any balanced Hermitian manifold which Bismut connection has holonomy contained in the special unitary group $SU(n)$ [27].

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