Incremental Sampling-based Algorithms for Optimal Motion Planning

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Abstract—During the last decade, incremental sampling-based motion planning algorithms, such as the Rapidly-exploring Random Trees (RRTs) have been shown to work well in practice and to possess theoretical guarantees such as probabilistic completeness. However, no theoretical bounds on the quality of the solution obtained by these algorithms have been established so far. The first contribution of this paper is a negative result: it is proven that, under mild technical conditions, the cost of the best path in the RRT converges almost surely to a non-optimal value. Second, a new algorithm is considered, called the Rapidly-exploring Random Graph (RRG), and it is shown that the cost of the best path in the RRG converges to the optimum almost surely. Third, a tree version of RRG is introduced, called the RRT*, which preserves the asymptotic optimality of RRG while maintaining a tree structure like RRT. The analysis of the new algorithms hinges on novel connections between sampling-based motion planning algorithms and the theory of random geometric graphs. In terms of computational complexity, it is shown that the number of simple operations required by both the RRG and RRT* algorithms is asymptotically within a constant factor of that required by RRT.

I. INTRODUCTION

The robotic motion planning problem has received a considerable amount of attention, especially over the last decade, as robots started becoming a vital part of modern industry as well as our daily life. Even though modern robots may possess significant differences in sensing, actuation, size, workspace, application, etc., the problem of navigating through a complex environment is embedded and essential in almost all robotics applications. Moreover, this problem has several applications in other disciplines such as verification, computational biology, and computer animation [1]–[6].

Informally speaking, given a robot with a description of its dynamics, a description of the environment, an initial state, and a set of goal states, the motion planning problem is to find a sequence of control inputs so as the drive the robot from its initial state to one of the goal states while obeying the rules of the environment, e.g., not colliding with the surrounding obstacles. An algorithm that solves this problem is said to be complete if it returns a solution when one exists and returns failure otherwise.

Early algorithmic approaches to the motion planning problem mainly focused on developing complete planners for, e.g., polygonal robots moving among polygonal obstacles [7], and later for more general cases using algebraic techniques [8]. These algorithms, however, suffered severely from computational complexity, which prohibited their practical implementations; for instance, the algorithm in [8] had doubly exponential time complexity. Regarding the computational complexity of the problem, a remarkable result was proven as early as 1979: Reif showed that a most basic version of the motion planning problem, called the piano movers problem, was PSPACE-hard [9], which strongly suggested that complete planners are would be unsuitable for practical applications.

Practical planners came around with the development of cell decomposition methods [10], [11], potential fields [12], and roadmap methods [13]. These approaches relaxed the completeness requirement to, for instance, resolution completeness, which guarantees completeness only when the resolution parameter of the algorithm is set fine enough. These planners demonstrated remarkable performance in accomplishing various tasks in complex environments within reasonable time bounds [14]. However, their practical applications were mostly limited to state spaces with up to five dimensions, since decomposition-based methods suffered from large number of cells, and potential field methods from local minima [1], [15].

Arguably, some of the most influential recent advances in robotic motion planning can be found in sampling-based algorithms [16]–[18], which attracted a large amount of attention over the last decade, including very recent work (see, e.g., [19]–[25]). In summary, sampling-based algorithms connect a set of points that are chosen randomly from the obstacle-free space in order to build a roadmap, which is used to form a strong hypothesis of the connectivity of the environment, in particular the connectivity of the initial state and the set of goal states. Informally speaking, sampling-based methods provide large amounts of computational savings by avoiding explicit construction of obstacles in the state space, as opposed to most complete motion planning algorithms. Even though these algorithms do not achieve completeness, they provide probabilistic completeness guarantees in the sense that the probability that the planner fails to return a solution, if one exists, decays to zero as the number of samples approaches infinity [26]. Moreover, the rate of decay of the probability of failure is exponential, under the assumption that the environment has good “visibility” properties [26]. More recently, the empirical success of sampling-based algorithms was argued to be strongly tied to the hypothesis that most practical robotic applications, even though involving robots with many degrees of freedom, feature environments with such good visibility properties [27].

A. Sampling-Based Algorithms

In practical applications, the class of sampling-based motion planning algorithms, including, for instance, Probabilistic Roadmaps (PRMs) [17], [28], and Rapidly-exploring Random

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In [43], the authors have shown that the RRG algorithm to generate feasible trajectories for satisfying specifications (RRG) algorithm [43]. The RRG algorithm was proposed as an incremental sampling-based method in order to extend RRTs to generate feasible trajectories for satisfying specifications other than the usual "avoid obstacles and reach the goal". In [43], the authors have shown that the RRG algorithm can handle any specification given in the form of $\mu$-calculus, which includes the widely-used specification language Linear Temporal Logic (LTL) as a strict subclass. The RRG algorithm incrementally builds a graph, instead of a tree, since several specifications in LTL (e.g., $\omega$-regular properties [44]) require cyclic trajectories, which are not included in trees.

### B. Optimal Motion Planning

Generally, the standard robotic motion planning problem does not take the “quality” of the solution into account. That is, the focus of the problem is to find a trajectory that starts from the initial state and reaches one of the goal states while staying inside the obstacle-free portion of the state space. Let us note that, as mentioned earlier, even this form of the problem is computationally challenging [9]. Consequently, there has been a significant amount of research effort devoted to constructing feasible solutions in a computationally effective manner. Hence, most of the existing algorithms pay almost no attention to the other properties of solution, e.g., its length, or its cost. As a result, the analysis of the quality of the solution returned by the algorithm, or algorithms tailored to provably improve the quality of the solution if kept running, have received relatively little attention, even though their importance is mentioned in early seminal papers [18]. In fact, in many field implementations of such algorithms (see, e.g., [39]), it is often the case that a feasible path is found quickly, and additional computation time is devoted to improving the solution with heuristics until the solution is executed.

Yet, the importance of the quality of the solution returned by the planners have been noticed, in particular, from the point of view of incremental sampling-based motion planning. In [45], Urmson and Simmons have proposed heuristics to bias the tree growth in the RRT towards those regions that result in low-cost solutions. They have also shown experimental results evaluating the performance of different heuristics in terms of the quality of the solution returned. In [46], Ferguson and Stentz have considered running the RRT algorithm multiple times in order to progressively improve the quality of the solution. They showed that each run of the algorithm results in a path with smaller cost, even though the procedure is not guaranteed to converge to an optimum solution. (Criteria for restarting multiple RRT runs, in a different context, were also proposed in [47].)

A different approach that also offers optimality guarantees is based on graph search algorithms (such as $A^*$) applied over a grid that is generated offline. Recently, these algorithms received a large amount of research attention. In particular, they were extended to run in an anytime fashion [48], [49], deal with dynamic environments [49], [50], and handle systems with differential constraints [51]. The have also been successfully demonstrated on various robotic platforms [51], [52]. However, optimality guarantees of these algorithms are only ensured up to the grid resolution. Moreover, since the number of grid points grows exponentially with the dimensionality of the state-space, so does the (worst-case) running time of these algorithms.
C. Statement of Contributions

To the best of our knowledge, this paper provides the first thorough analysis of optimality properties of incremental sampling-based motion planning algorithms. In particular, we show that the probability that the RRT converges to an optimum solution, as the number of samples approaches infinity, is zero under some reasonable technical assumptions. In fact, we give a formal proof of the fact that the RRT algorithm almost always converges to a suboptimal solution. Second, we show that the probability of the same event for the RRG algorithm is one. That is, the RRG algorithm is asymptotically optimal in the sense that it converges to an optimal solution almost surely as the number of samples approaches infinity. Third, we propose a novel variant of the RRG algorithm, called RRT*, which inherits the asymptotic optimality of the RRG algorithm while maintaining a tree structure. To do so, the RRT* algorithm essentially “rewires” the tree structure as it discovers new lower-cost paths reaching the nodes that are already in the tree. Finally, we show that the asymptotic computational complexity of the RRG and RRT* algorithms is essentially the same as that of RRTs. We also provide experimental evidence supporting our theoretical computational complexity results. We discuss the relation between the proposed algorithms and PRM-based methods, and propose a PRM-like algorithm that improves the efficiency of PRM while maintaining the same feasibility and optimality properties.

To our knowledge, the algorithms considered in this paper are the first computationally efficient incremental sampling-based motion planning algorithms with asymptotic optimality guarantees. Indeed, our results imply that these algorithms are optimal also from an asymptotic computational complexity point of view, since they closely match lower bounds for computing nearest neighbors. The key insight is that connections between vertices in the graph should be sought within balls whose radius vanishes with a certain rate as the size of the graph increases, and is based on new connections between motion planning and the theory of random geometric graphs [53], [54].

In this paper, we consider the problem of navigating through a connected bounded subset of a d-dimensional Euclidean space. As in the early seminal papers on incremental sampling-based motion planning algorithms such as [29], we assume no differential constraints, but our methods can be easily extended to planning in configuration spaces and applied to several practical problems of interest. We defer the extension to systems with differential constraints to another paper.

D. Paper Organization

This paper is organized as follows. In Section II, lays the ground in terms of notation and problem formulation. Section III is devoted to the introduction of the RRT and RRG algorithms. In Section IV, these algorithms are analyzed in terms of probabilistic completeness, asymptotic optimality, and computational complexity. The RRT* algorithm is presented in Section V where it is shown that RRT* inherits the theoretical guarantees of the RRG algorithm. Experimental results are presented and discussed in Section VI. Concluding remarks are given in Section VII. In order not to disrupt the flow of the presentation, proofs of important results are presented in the Appendix.

II. Preliminary Material

A. Notation

Let $\mathbb{N}$ denote the set of non-negative integers and $\mathbb{R}$ denote the set of reals. Let $\mathbb{R}_{>0}$ and $\mathbb{R}_{\geq 0}$ denote the sets of positive and non-negative reals, respectively. The set $\mathbb{N}_{>0}$ is defined similarly. A sequence on a set $A$ is a mapping from $\mathbb{N}$ to $A$, denoted as $\{a_i\}_{i \in \mathbb{N}}$, where $a_i \in A$ is the element that $i \in \mathbb{N}$ is mapped to. Given $a, b \in \mathbb{R}$, closed and open intervals between $a$ and $b$ are denoted by $[a, b]$ and $(a, b)$, respectively. More generally, if $a, b \in \mathbb{R}^d$, then $[a, b]$ is used to denote the line segment that connects $a$ with $b$, i.e., $[a, b] := \{r(a + (1 - r)b) \mid r \in \mathbb{R}, 0 \leq r \leq 1\}$. The Euclidean norm is denoted by $\|\cdot\|$. Given a set $X \subset \mathbb{R}^d$, the closure of $X$ is denoted by $\overline{X}(X)$. The closed ball of radius $r > 0$ centered at $x \in \mathbb{R}^d$ is defined as $B_{x,r} := \{y \in \mathbb{R}^d \mid \|y - x\| \leq r\}; B_{x,r}$ is also called the $r$-ball centered at $x$. Given a set $X \subset \mathbb{R}^d$, the Lebesgue measure of $X$, i.e., its volume, is denoted by $\mu(X)$. The volume of the unit ball in $\mathbb{R}^d$ is denoted by $\zeta_d$.

Given a set $X \subset \mathbb{R}^d$ and a scalar $s \geq 0$, a path in $X$ is a continuous function $\sigma : [0, s] \rightarrow X$, where $s$ is the length of the path defined in the usual way as $\sup\{n \in \mathbb{N}_0 \mid \tau_0 < \tau_1 < \cdots < \tau_n = s\} \sum_{i=1}^{n} \|\sigma(\tau_i) - \sigma(\tau_{i-1})\|$. Given two paths in $X$, $\sigma_1 : [0, s_1] \rightarrow X$, and $\sigma_2 : [0, s_2] \rightarrow X$, with $\sigma_1(0) = \sigma_2(0)$, their concatenation is denoted by $\sigma_1 \sigma_2$. More precisely, $\sigma = \sigma_1 \sigma_2 : [0, s_1 + s_2] \rightarrow X$ is defined as $\sigma(s) = \sigma_1(s)$ for all $s \in [0, s_1]$, and $\sigma(s) = \sigma_2(s - s_1)$ for all $s \in [s_1, s_1 + s_2]$. More generally, a concatenation of multiple paths $\sigma_1 \sigma_2 \cdots \sigma_n$ is defined as $\cdots((\sigma_1 \sigma_2) \sigma_3) \cdots \sigma_n$. The set of all paths in $X$ with nonzero length is denoted by $\Sigma_X$. The straight continuous path between two points $x_1, x_2 \in \mathbb{R}^d$ is denoted by $\text{Line}(x_1, x_2)$.

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega$ is a sample space, $\mathcal{F} \subseteq 2^\Omega$ is a $\sigma$-algebra, and $\mathbb{P}$ is a probability measure, an event $A$ is an element of $\mathcal{F}$. The complement of an event $A$ is denoted by $A^c$. Given a sequence of events $\{A_i\}_{i \in \mathbb{N}}$, the event $\bigcap_{j=1}^{\infty} \bigcup_{i \geq j} A_i$ is denoted by $\limsup_{i \rightarrow \infty} A_i$ (also called the event that $A_i$ occurs infinitely often). A (real) random variable is a measurable function that maps $\Omega$ into $\mathbb{R}$. An extended (real) random variable can also take the values $\pm \infty$. A sequence of random variables $\{Y_i\}_{i \in \mathbb{N}}$ is said to converge surely to a random variable $Y$ if $\lim_{i \rightarrow \infty} Y_i(\omega) = Y(\omega)$ for all $\omega \in \Omega$; the sequence is said to converge almost surely if $\mathbb{P}(\lim_{i \rightarrow \infty} Y_i = Y) = 1$. Finally, if $\varphi(\omega)$ is a property that is either true or false for a given $\omega \in \Omega$, the event that denotes the set of all samples $\omega$ for which $\varphi(\omega)$ holds, i.e., $\{\omega \in \Omega \mid \varphi(\omega) \text{ holds}\}$, is written as $\{\varphi\}$, e.g., $\{\omega \in \Omega \mid \lim_{i \rightarrow \infty} Y_i(\omega) = Y(\omega)\}$ is simply written as $\{\lim_{i \rightarrow \infty} Y_i = Y\}$. Let $f(n)$ and $g(n)$ be two functions with domain and range $\mathbb{N}$. The function $f(n)$ is said to be $O(g(n))$, denoted as $f(n) \in O(g(n))$, if there exists two constants $M$ and $n_0$ such that $f(n) \leq Mg(n)$ for all $n \geq n_0$. The function $f(n)$ is said to be $\Omega(g(n))$, denoted as $f(n) \in \Omega(g(n))$, if there exists
constants $M$ and $n_0$ such that $f(n) \geq Mg(n)$ for all $n \geq n_0$. The function $f(n)$ is said to be $\Theta(g(n))$, denoted as $f(n) \in \Theta(g(n))$, if $f(n) \in O(g(n))$ and $f(n) \in \Omega(g(n))$.

Let $X$ be a subset of $\mathbb{R}^d$. A (directed) graph $G = (V, E)$ on $X$ is composed of a vertex set $V$ and an edge set $E$, such that $V$ is a finite subset of $X$, and $E$ is a subset of $V \times V$. The set of all directed graphs on $X$ is denoted by $\mathcal{G}_X$. A directed path on $G$ is a sequence $(v_1, v_2, \ldots, v_n)$ of vertices such that $(v_i, v_{i+1}) \in E$ for all $1 \leq i \leq n - 1$. Given a vertex $x \in V$, the sets \{ $v \in V | (x, v) \in E$ $\}$ and \{ $v \in V | (v, x) \in E$ $\}$ are said to be its incoming neighbors and outgoing neighbors, respectively. A (directed) tree is a directed graph, in which each vertex but one has a unique incoming neighbor; the vertex with no incoming neighbor is called the root vertex. Vertices of a tree are often also called nodes.

B. Problem Formulation

In this section, two variants of the path planning problem are presented. First, the feasibility problem in path planning is formalized, then the optimality problem is introduced.

Let $X$ be a bounded connected open subset of $\mathbb{R}^d$, where $d \in \mathbb{N}$, $d \geq 2$. Let $X_{obs}$ and $X_{goal}$, called the obstacle region and the goal region, respectively, be open subsets of $X$. Let us denote the obstacle-free space, i.e., $X \setminus X_{obs}$ as $X_{free}$. Let the initial state, $x_{init}$, be an element of $X_{free}$. In the sequel, a path in $X_{free}$ is said to be a collision-free path. A collision-free path that starts at $x_{init}$ and ends in the goal region is said to be a feasible path, i.e., a collision-free path $\sigma : [0, s] \to X_{free}$ is feasible if and only if $\sigma(0) = x_{init}$ and $\sigma(s) \in X_{goal}$. The feasibility problem of path planning is to find a feasible path, if one exists, and report failure otherwise. The problem can be formalized as follows.

Problem 1 (Feasible planning) Given a bounded connected open set $X \subset \mathbb{R}^d$, an obstacle space $X_{obs} \subset X$, an initial state $x_{init} \in X_{free}$, and a goal region $X_{goal} \subset X_{free}$, find a path $\sigma : [0, s] \to X_{free}$ such that $\sigma(0) = x_{init}$ and $\sigma(s) \in X_{goal}$, if one exists. If no such path exists, then report failure.

Let $c : \Sigma_{X_{free}} \to \mathbb{R}_{>0}$ be a function, called the cost function, which assigns a non-negative cost to all nontrivial collision-free paths. The optimality problem of path planning asks for finding a feasible path with minimal cost, formalized as follows.

Problem 2 (Optimal planning) Given a bounded connected open set $X$, an obstacle space $X_{obs}$, an initial state $x_{init}$, and a goal region $X_{goal}$, find a path $\sigma^* : [0, s] \to cl(X_{free})$ such that (i) $\sigma^*(0) = x_{init}$ and $\sigma^*(s) \in X_{goal}$, and (ii) $c(\sigma^*) = \min_{\sigma \in \Sigma_{cl(X_{free})}} c(\sigma)$. If no such path exists, then report failure.

III. ALGORITHMS

In this section, two incremental sampling-based motion planning algorithms, namely the RRT and the RRG algorithms, are introduced. Before formalizing the algorithms, let us note the primitive procedures that they rely on.

Sampling: The function $\text{Sample} : \mathbb{N} \to X_{free}$ returns independent identically distributed (i.i.d.) samples from $X_{free}$.

Steering: Given two points $x, y \in X$, the function $\text{Steer} : \langle x, y \rangle \to z$ returns a point $z \in \mathbb{R}^d$ such that $z$ is “closer” to $y$ than $x$ is. Throughout the paper, the point $z$ returned by the function $\text{Steer}$ will be such that $z$ minimizes $\|z - y\|$ while at the same time maintaining $\|z - x\| \leq \eta$, for a prespecified $\eta > 0$.

$$\text{Steer}(x, y) = \arg\min_{z \in \mathbb{R}^d, \|z - x\| \leq \eta} \|z - y\|.$$ 

Nearest Neighbor: Given a graph $G = (V, E) \in \mathcal{G}_{X_{free}}$ and a point $x \in X_{free}$, the function $\text{Nearest} : (G, x) \to v$ returns a vertex $v \in V$ that is “closest” to $x$ in terms of a given distance function. In this paper, we will use Euclidean distance (see, e.g., [18] for alternative choices), i.e.,

$$\text{Nearest}(G = (V, E), x) = \arg\min_{v \in V} \|x - v\|.$$ 

Near Vertices: Given a graph $G = (V, E) \in \mathcal{G}_{X_{free}}$ and a point $x \in X_{free}$, and a number $n \in \mathbb{N}$, the function $\text{Near} : (G, x, n) \to V'$ returns a set $V'$ of vertices such that $V' \subset V$. The $\text{Near}$ procedure can be thought of as a generalization of the nearest neighbor procedure in the sense that the former returns a collection of vertices that are close to $x$, whereas the latter returns only one such vertex that is the closest. Just like the $\text{Nearest}$ procedure, there are many ways to define the $\text{Near}$ procedure, each of which leads to different algorithmic properties. For technical reasons to become clear later, we define $\text{Near}(G, x, n)$ to be the set of all vertices within the closed ball of radius $r_n$ centered at $x$, where

$$r_n = \min \left\{ \left( \frac{\gamma \log n}{\zeta d} \right)^{1/d}, \eta \right\},$$

and $\gamma$ is a constant. Hence, the volume of this ball is $\min(\gamma \log n, \zeta d \eta^d)$.

Collision Test: Given two points $x, x' \in X_{free}$, the Boolean function $\text{ObstacleFree}(x, x')$ returns $\text{true}$ if the line segment between $x$ and $x'$ lies in $X_{free}$, i.e., $[x, x'] \subset X_{free}$.

Both the RRT and the RRG algorithms are similar to most other incremental sampling-based planning algorithms (see Algorithm 1). Initially, the algorithms start with the graph that includes the initial state as its single vertex and no edges; then, they incrementally grow a graph on $X_{free}$ by sampling a state $x_{rand}$ from $X_{free}$ at random and extending the graph towards $x_{rand}$. In the sequel, every such step of sampling followed by extensions (Lines 2-5 of Algorithm 1) is called a single iteration of the incremental sampling-based algorithm.

Hence, the body of both algorithms, given in Algorithm 1, is the same. However, RRGs and RRTs differ in the choice of the vertices to be extended. The $\text{Extend}$ procedures for the RRT and the RRG algorithms are provided in Algorithms 2 and 3 respectively. Informally speaking, the RRT algorithm extends the nearest vertex towards the sample. The RRG algorithm first extends the nearest vertex, and if such extension is successful, it also extends all vertices returned by the $\text{Near}$ procedure. In both cases, all extensions resulting in collision-free trajectories

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1This steering procedure is used widely in the robotics literature, since its introduction in [29]. Our results also extend to the Rapidly-exploring Random Dense Trees (see, e.g., [30]), which are slightly modified versions of the RRTs that do not require tuning any prespecified parameters such as $\eta$ in this case.
are added to the graph as edges, and their terminal points as new vertices.

Algorithm 1: Body of RRT and RRG Algorithms
1 \( V \leftarrow \{x_{\text{init}}\}; E \leftarrow \emptyset; i \leftarrow 0; \)
2 while \( i < N \) do
3 \( G \leftarrow (V, E); \)
4 \( x_{\text{rand}} \leftarrow \text{Sample}(i); i \leftarrow i + 1; \)
5 \( (V, E) \leftarrow \text{Extend}(G, x_{\text{rand}}); \)

Algorithm 2: Extend \(_{\text{RRT}}\)
1 \( V' \leftarrow V; E' \leftarrow E; \)
2 \( x_{\text{nearest}} \leftarrow \text{Nearest}(G, x); \)
3 \( x_{\text{new}} \leftarrow \text{Steer}(x_{\text{nearest}}, x); \)
4 if \( \text{ObstacleFree}(x_{\text{nearest}}, x_{\text{new}}) \) then
5 \( V' \leftarrow V' \cup \{x_{\text{new}}\}; \)
6 \( E' \leftarrow E' \cup \{(x_{\text{nearest}}, x_{\text{new}})\}; \)
7 return \( G' = (V', E') \)

Algorithm 3: Extend \(_{\text{RRG}}\)
1 \( V' \leftarrow V; E' \leftarrow E; \)
2 \( x_{\text{nearest}} \leftarrow \text{Nearest}(G, x); \)
3 \( x_{\text{new}} \leftarrow \text{Steer}(x_{\text{nearest}}, x); \)
4 if \( \text{ObstacleFree}(x_{\text{nearest}}, x_{\text{new}}) \) then
5 \( V' \leftarrow V' \cup \{x_{\text{new}}\}; \)
6 \( E' \leftarrow E' \cup \{(x_{\text{nearest}}, x_{\text{new}}), (x_{\text{new}}, x_{\text{nearest}})\}; \)
7 \( x_{\text{near}} \leftarrow \text{Near}(G, x_{\text{new}}, V); \)
8 for all \( x_{\text{near}} \in X_{\text{near}} \) do
9 if \( \text{ObstacleFree}(x_{\text{new}}, x_{\text{near}}) \) then
10 \( E' \leftarrow E' \cup \{(x_{\text{near}}, x_{\text{new}}), (x_{\text{new}}, x_{\text{near}})\}; \)
11 return \( G' = (V', E') \)

Notice that the graph \( G \) maintained by the RRT algorithm is a tree, whereas that maintained by the RRG can, in general, include vertices with more than one incoming neighbor.

IV. ANALYSIS

A. Convergence to a Feasible Solution

In this section, the feasibility problem is considered. It is proven that the RRG algorithm inherits the probabilistic completeness as well as the exponential decay of the probability of failure (as the number of samples increase) from the RRT. These results imply that the RRT and RRG algorithms have the same performance in producing a solution to the feasibility problem as the number of samples increase. Before formalizing these claims, let us introduce some notation and preliminaries.

Note that the only randomness in Algorithms 1–3 may arise from the sampling procedure denoted as \text{Sample}.\footnote{We will not address the case in which the sampling procedure is deterministic, but refer the reader to \cite{55}, which contains an in-depth discussion of the relative merits of randomness and determinism in sampling-based motion planning algorithms.} To model this randomness, we define the sample space \( \Omega \) to be the infinite cartesian product \( X_{\text{free}} \). Intuitively, a single element \( \omega \) of \( \Omega \) denotes a sequence \( \{\text{Sample}(i)\}_{i \in \mathbb{N}} \) of infinitely many samples drawn from \( X_{\text{free}} \), and \( \Omega \) denotes the set of all such sequences.

Sets of vertices and edges of the graphs maintained by the RRT and the RRG algorithms, then, can be defined as functions from the sample space \( \Omega \) to appropriate sets. More precisely, let \( \{V_{\text{RRT}}^{i}\}_{i \in \mathbb{N}} \) and \( \{V_{\text{RRG}}^{i}\}_{i \in \mathbb{N}} \), sequences of functions defined from \( \Omega \) into finite subsets of \( X_{\text{free}} \), be the sets of vertices in the RRT and the RRG, respectively, at the end of iteration \( i \). By convention, we define \( V_{0}^{\text{RRT}} = V_{0}^{\text{RRG}} = \{x_{\text{init}}\} \). Similarly, let \( \mathcal{E}_{\text{RRT}}^{i} \) and \( \mathcal{E}_{\text{RRG}}^{i} \), defined for all \( i \in \mathbb{N} \), denote the set of edges in the RRT and the RRG, respectively, at the end of iteration \( i \). Clearly, \( \mathcal{E}_{0}^{\text{RRT}} = \mathcal{E}_{0}^{\text{RRG}} = \emptyset \).

An important lemma used for proving the equivalency between the RRT and the RRG algorithms is the following.

Lemma 3 For all \( i \in \mathbb{N} \) and all \( \omega \in \Omega \), \( V_{i}^{\text{RRT}}(\omega) = V_{i}^{\text{RRG}}(\omega) \) and \( \mathcal{E}_{i}^{\text{RRT}}(\omega) \subseteq \mathcal{E}_{i}^{\text{RRG}}(\omega) \).

Lemma 3 implies that the paths discovered by the RRT algorithm by the end of iteration \( i \) is, essentially, a subset of those discovered by the RRG by the end of the same iteration.

An algorithm addressing Problem 1 is said to be probabilistically complete if it finds a feasible path with probability approaching one as the number of iterations approaches infinity. Note that there exists a collision-free path starting from \( x_{\text{init}} \) to any vertex in the tree maintained by the RRT, since the RRT maintains a connected graph on \( X_{\text{free}} \) that necessarily includes \( x_{\text{init}} \). Using this fact, the probabilistic completeness property of the RRT is stated alternatively as follows.

Theorem 4 (see \cite{18}) If there exists a feasible solution to Problem 1, then \( \lim_{i \to \infty} \mathbb{P} \left( \{V_{i}^{\text{RRT}} \cap X_{\text{goal}} = \emptyset\} \right) = 1 \).

Moreover, under certain assumptions on the environment, the probability that the RRT fails to find a feasible path, even though one exists, decays to zero exponentially (see, e.g., \cite{18, 34}). We state these assumptions here and then state the theorem in our notation.

An attraction sequence \cite{18} is defined as a finite sequence \( A = \{A_{1}, A_{2}, \ldots, A_{k}\} \) of sets as follows: (i) \( A_{0} = \{x_{\text{init}}\} \), and (ii) for each set \( A_{i} \), there exists a set \( B_{i} \), called the basin such that for any \( x \in A_{i−1}, y \in A_{i}, \) and \( z \in X \setminus B_{i}, \) there holds \( \|x − y\| \leq \|x − z\|^{2} \). Given an attraction sequence \( A \) of length \( k \), let \( p_{k} \) denote \( \min_{i \in \{1, 2, \ldots, k\}} \left( \frac{\mu(A_{i})}{\mu(X_{\text{free}})} \right) \).

The following theorem states that the probability that the RRT algorithm fails to return a solution, when one exists, decays to zero exponentially fast.

Theorem 5 (see \cite{18}) If there exists an attraction sequence \( A \) of length \( k \), then \( \mathbb{P} \left( \{V_{i}^{\text{RRT}} \cap X_{\text{goal}} = \emptyset\} \right) \leq e^{-\frac{1}{2}(i(p_{k}-2k)).} \)

As noted in \cite{18}, the convergence gets faster as the length of the attraction sequence gets smaller and the volume of
the minimum volume set in the attraction sequence gets larger. Such properties are achieved when the environment does not involve narrow passages and, essentially, has good “visibility” properties. (See, e.g., [27] for dependence of the performance of probabilistic planners such as PRMs on the visibility properties of the environment).

With Lemma 3 and Theorems 4 and 5, the following theorem is immediate.

**Theorem 6** If there exists a feasible solution to Problem 1, then \( \lim_{k \to \infty} \mathbb{P} \left( \left\{ v^i_{\text{RRT}} \cap X_{\text{goal}} \neq \emptyset \right\} \right) = 1. \) Moreover, if an attraction sequence \( A \) of length \( k \) exists, then \( \mathbb{P} \left( \left\{ v^i_{\text{RRT}} \cap X_{\text{goal}} = \emptyset \right\} \right) \leq e^{-\frac{1}{2}(1 - e^{-2k})}. \)

**B. Asymptotic Optimality**

This section is devoted to the investigation of optimality properties of the RRT and the RRG algorithms. First, under some mild technical assumptions, we show that the probability that the RRT converges to an optimal solution is zero. However, the convergence of this random variable is guaranteed, which implies that the RRT converges to a non-optimum solution with probability one. On the contrary, we subsequently show that the RRG algorithm converges to an optimum solution almost-surely.

Let \( \{ v^i_{\text{RRT}} \}_{i \in \mathbb{N}} \) be a sequence of extended random variables that denote the cost of a minimum-cost path contained within the tree maintained by the RRT algorithm at the end of iteration \( i \). The extended random variable \( v^i_{\text{RRT}} \) is defined similarly. Let \( c^* \) denote the cost of a minimum-cost path in \( \text{cl}(X_{\text{free}}) \), i.e., the cost of a path that solves Problem 2.

Let us note that the limits of these two extended random variable sequences as \( i \) approaches infinity exist. More formally, notice that \( v^i_{\text{RRT}}(\omega) \leq v^i_{\text{RRT}}(\omega) \) holds for all \( i \in \mathbb{N} \) and all \( \omega \in \Omega \). Moreover, we have that \( v^i_{\text{RRT}}(\omega) \geq c^* \) for all \( i \in \mathbb{N} \) and all \( \omega \in \Omega \), by optimality of \( c^* \). Hence, \( \{ v^i_{\text{RRT}} \}_{i \in \mathbb{N}} \) is a sure non-increasing sequence of random variables that is surely lower-bounded by \( c^* \). Thus, for all \( \omega \in \Omega \), the limit \( \lim_{i \to \infty} v^i_{\text{RRT}}(\omega) \) exists. The same argument also holds for the sequence \( \{ v^i_{\text{RRG}} \}_{i \in \mathbb{N}} \).

1) **Almost Sure Suboptimality of the RRT:** Let us note the following assumptions, which will be required to show the almost-sure sub-optimality of the RRT. Let \( \Sigma^* \) denote the set of all optimal paths, i.e., the set of all paths that solve Problem 2. and \( X_{\text{opt}} \) denote the set of states that an optimal path in \( \Sigma^* \) passes through, i.e.,

\[
X_{\text{opt}} = \bigcup_{\sigma^* \in \Sigma^*} \bigcup_{\tau \in [0,s^*]} \{ \sigma^*(\tau) \}
\]

**Assumption 7 (Zero-measure Optimal Paths)** The set of all points in the state-space that an optimal trajectory passes through has measure zero, i.e., \( \mu(X_{\text{opt}}) = 0. \)

**Assumption 8 (Sampling Procedure)** The sampling procedure is such that the samples \( \{ \text{Sample}(i) \}_{i \in \mathbb{N}} \) are drawn from an absolutely continuous distribution with a continuous density function \( f(x) \) bounded away from zero on \( X_{\text{free}} \).

**Assumption 9 (Monotonicity of the Cost Function)** For all \( \sigma_1, \sigma_2 \in \Sigma_{X_{\text{free}}} \), the cost function \( c \) satisfies the following: \( c(\sigma_1) \leq c(\sigma_1|\sigma_2). \)

Assumptions 7 rules out trivial cases, in which the RRT algorithm can sample exactly an optimal path with non-zero probability. Note that this assumption is placed on the problem instance rather than the algorithm. Most cost functions and problem instances of interest satisfy this assumption, including, e.g., the Euclidean length of the path. Let us note that this assumption does not imply that there is a single optimal path; indeed, there are problem instances with uncountably many optimal paths, for which Assumption 7 holds. Assumption 8 also ensures that the sampling procedure cannot be tuned to construct the optimal path exactly. Finally, Assumption 9 merely states that extending a path to produce a longer path may not decrease its cost.

Recall that \( d \) denotes the dimensionality of the state space. The negative result of this section is formalized as follows.

**Theorem 10** Let Assumptions 7, 8, and 9 hold. Then, the probability that the cost of the minimum-cost path in the RRT converges to the optimal cost is zero, i.e.,

\[
\mathbb{P} \left( \lim_{i \to \infty} v^i_{\text{RRT}} = c^* \right) = 0,
\]

whenever \( d \geq 2. \)

As noted before, the limit \( \lim_{i \to \infty} v^i_{\text{RRT}}(\omega) \) exists and is a random variable. However, Theorem 10 directly implies that this limit is strictly greater than \( c^* \) with probability one, i.e., \( \mathbb{P} \left( \left\{ \lim_{i \to \infty} v^i_{\text{RRT}} > c^* \right\} \right) = 1. \) In other words, we establish, as a corollary, that the RRT algorithm converges to a suboptimal solution with probability one.

**Remark 11 (Effectiveness of Multiple RRTs)** Since the cost of the best path returned by the RRT algorithm converges to a random variable, say \( v_{\text{RRT}}^{\infty} \), Theorem 10 provides new insight explaining the effectiveness of approaches as in [46]. In fact, running multiple instances of the RRT algorithm amounts to drawing multiple samples of \( v_{\text{RRT}}^{\infty} \).

2) **Almost Sure Optimality of the RRG:** Let us note the following set of assumptions, which will be required to show the asymptotic optimality of the RRG.

**Assumption 12 (Additivity of the Cost Function)** For all \( \sigma_1, \sigma_2 \in \Sigma_{X_{\text{free}}} \), the cost function \( c \) satisfies the following: \( c(\sigma_1|\sigma_2) = c(\sigma_1) + c(\sigma_2). \)

**Assumption 13 (Continuity of the Cost Function)** The cost function \( c \) is Lipschitz continuous in the following sense: there exists some constant \( \kappa \) such that for any two paths \( \sigma_1 : [0,s_1] \to X_{\text{free}} \) and \( \sigma_2 : [0,s_2] \to X_{\text{free}}, \)

\[
|c(\sigma_1) - c(\sigma_2)| \leq \kappa \sup_{\tau \in [0,1]} \| \sigma_1(\tau s_1) - \sigma_2(\tau s_2) \|.
\]

**Assumption 14 (Obstacle Spacing)** There exists a constant \( \delta \in \mathbb{R}_+ \) such that for any point \( x \in X_{\text{free}} \), there exists \( x' \in \)
Assumption 13 ensures that two paths that are very close to each other have similar costs. Let us note that several cost functions of practical interest satisfy Assumptions 12 and 13. An example that is widely used, e.g., in optimal control, is the line integral of a continuous function $g : X_{\text{free}} \rightarrow \mathbb{R}_{>0}$ over trajectories of bounded length, i.e., $c(\sigma) = \int_0^1 g(\sigma(t)) \, dt$. Assumption 14 is a rather technical assumption, which ensures existence of some free space around the optimal trajectories to allow convergence. We also assume for simplicity that the sampling is uniform, although our results can be easily extended to more general sampling procedures.

Recall that $d$ is the dimensionality of the state-space $X$, and $\gamma$ is the constant defined in the $\text{Near}$ procedure. The positive result that states the asymptotic optimality of the RRG algorithm can be formalized as follows.

**Theorem 15** Let Assumptions 12, 13, and 14 hold, and assume that Problem 1 admits a feasible solution. Then, the cost of the minimum-cost path in the RRG converges to $c^*$ almost surely, i.e.,

$$\mathbb{P} \left( \lim_{i \to \infty} \gamma_i^{\text{RRG}} = c^* \right) = 1,$$

whenever $d \geq 2$ and $\gamma > \gamma_L := 2^{d(1 + 1/d)}\mu(X_{\text{free}})$.

### C. Computational Complexity

The objective of this section is to compare the computational complexity of RRTs and RRGs. It is shown that these algorithms share essentially the same asymptotic computational complexity in terms of the number of calls to simple operations such as comparisons, additions, and multiplications. Simple operations should not be confused with primitive procedures outlined in Section III, which are higher level functions that may have different asymptotic computational complexities in terms of the number of simple operations that they perform. Note that the objective of this section is not the evaluation of the computational complexity in absolute terms, but to compare the relative complexity of the two algorithms, hence the input of interest is the number of iterations, as opposed to parameters describing the problem instance.

Let us first investigate the computational complexity of the RRT and the RRG algorithms in terms of the number of calls to the primitive procedures. Notice that, in every iteration, the number of calls to Sample, Steer, and Nearest procedures are the same in both algorithms. However, number of calls to Nearest and ObstacleFree procedures differ: the former is never called by the RRT and is called at most once by the RRG, whereas the latter is called exactly once by the RRT and at least once by the RRG. Informally speaking, we first show that the expected number of calls to the ObstacleFree procedure by the RRG algorithm is $O(\log n)$, where $n$ is the number of vertices in the graph.

Let $\gamma_i^{\text{RRG}}$ be a random variable that denotes the number of calls to the ObstacleFree procedure by the RRG algorithm in iteration $i$. Notice that, as an immediate corollary of Lemma 3 the number of vertices in the RRT and RRG algorithms is the same at any given iteration. Let $N_i$ be the number of vertices in these algorithms at the end of iteration $i$. The following theorem establishes that the expected number of calls to the ObstacleFree procedure in iteration $i$ by the RRG algorithm scales logarithmically with the number of vertices in the graph as $i$ approaches infinity.

**Lemma 16** In the limit as $i$ approaches infinity, the random variable $\gamma_i^{\text{RRG}} / \log(N_i)$ is no more than a constant in expectation, i.e.,

$$\lim_{i \to \infty} \sup \mathbb{E} \left( \frac{\gamma_i^{\text{RRG}}}{\log(N_i)} \right) \leq \phi,$$

where $\phi \in \mathbb{R}_{>0}$ is a constant that depends only on the problem instance.

Hence, informally speaking, we have established that the RRG algorithm has an overhead of order $\log n$ calls to the ObstacleFree procedure and one call to the Nearest procedure, where $n$ is the number of vertices in the RRG; otherwise, both algorithms scale the same in terms of number of calls to the primitive procedures.

However, some primitive procedures clearly take more computation time to execute than others. For a more fair comparison, we also evaluate the time required to execute each primitive procedure in terms of the number of simple operations (also called steps in this paper) that they perform. This analysis shows that the expected number of simple operations performed by the RRG is asymptotically within a constant factor of that performed by the RRT, which establishes that the RRT and the RRG algorithms have the same asymptotic computational complexity in terms of number of calls to the simple operations.

First, notice that Sample, Steer, and ObstacleFree procedures can be performed in a constant number of steps, i.e., independent of the number of vertices in the graph. (Although all these procedures clearly depend on parameters such as the dimensionality of the state-space and the number of obstacles, these are fixed for each problem instance.)

Second, let us consider the computational complexity of the Nearest procedure. Indeed, the nearest neighbor search problem has been widely studied in the literature, since it has many applications in, e.g., computer graphics, database systems, image processing, data mining, pattern recognition, etc. [56], [57]. Let us assume that the distance computation between two points can be done in $O(d)$ time (note that for the purposes of comparing distances, the Euclidean distance can be evaluated without computing any roots). Clearly, a brute-force algorithm that examines every vertex runs in $O(dn)$ time and requires $O(1)$ space. However, in many online real-time applications such as robotics, it is highly desirable to reduce the computation time of each iteration under sublinear bounds, especially for anytime algorithms, which provide better solutions as the number of iterations increase.

Fortunately, it is possible to design nearest neighbor computation algorithms that run in sublinear time. Generally, such efficient computations of nearest neighbors are based on constructing a tree structure online so as to reduce the
query time [57]. Indeed, using k-d trees, nearest neighbor search can be performed within sublinear time bounds, in the worst case [57]. In fact, let us note that the k-d tree algorithm was extended to arbitrary topological manifolds and used for nearest neighbor computation in RRTs [58].

Worst case logarithmic time, however, is hard to achieve in many cases of practical interest. Note that when $d = 1$, a binary tree [59] will allow answering nearest neighbor queries in $O(\log n)$ time using $O(n)$ space, in the worst case. When $d = 2$, similar logarithmic time bounds can also be achieved by, for instance, computing a Voronoi diagram [60]. However, the time complexity of computing Voronoi diagrams is known to be $O(n[d/2])$ when $d > 2$. Several algorithms that achieve a worst-case query time that is logarthmic in the number of vertices and polynomial in the number of dimensions were designed (see, e.g., [61]). However, these algorithms use space that is exponential in $d$, which may be impractical. Unfortunately, no exact nearest neighbor algorithm that does not scale exponentially with the number of dimensions and runs queries in logarithmic time using roughly linear space is known [62].

Fortunately, computing an “approximate” nearest neighbor instead of an exact one is computationally easier. In the sequel, a vertex $y$ is said to be an $\varepsilon$-approximate nearest neighbor of a point $x$ if $\|y - x\| \leq (1 + \varepsilon)\|z - x\|$, where $z$ is the true nearest neighbor of $x$. An approximate nearest neighbor can be computed using balanced-box decomposition (BBD) trees, which achieves $O(c_{d,\varepsilon}\log n)$ query time using $O(dn)$ space [62], where $c_{d,\varepsilon} \leq d[1 + 6d/\varepsilon]^d$. This algorithm is computationally optimal in fixed dimensions, since it closely matches a lower bound for algorithms that use a tree structure stored in roughly linear space [62]. Let us note that using approximate nearest neighbor computation in the context of both PRMs and RRTs was also discussed very recently [63].

Assuming that the Nearest procedure computes an approximate nearest neighbor using the algorithm given in [62], in fixed dimensions the NearestNeighbor algorithm has to run in $\Omega(\log n)$ time as formalized in the following lemma. Let $M_{i}^{\text{RRT}}$ be the random variable that denotes the number of steps taken by the RRT algorithm in iteration $i$.

**Lemma 17** Assuming that Nearest is implemented using the algorithm given in [62], which is optimal in fixed dimensions, the number of steps executed by the RRT algorithm at each iteration is at least order $\log(N_i)$ in expectation in the limit, i.e., there exists a constant $\Phi_{\text{RRT}} \in \mathbb{R}_{>0}$ such that

$$\liminf_{i \to \infty} \mathbb{E} \left[ \frac{M_{i}^{\text{RRT}}}{\log(N_i)} \right] \geq \Phi_{\text{RRT}}.$$ 

Likewise, problems similar to that solved by the Nearest procedure are also widely-studied in the literature, generally under the name of **range search problems**, as they have many applications in, for instance, computer graphics and spatial database systems [57]. In the worst case and in fixed dimensions, computing the exact set of vertices that reside in a ball of radius $r_n$ centered at a query point $x$ takes $O(n^{1-1/d} + m)$ time using k-d trees [64], where $m$ is the number of vertices returned by the search. In [65], a thorough analysis of the average case performance is also provided under certain conditions.

Similar to the nearest neighbor search, computing approximate solutions to the range search problem is computationally easier. A range search algorithm is said to be $\varepsilon$-approximate if it returns all vertices that reside in the ball of size $r_n$ and no vertices outside a ball of radius $(1 + \varepsilon)r_n$, but may or may not return the vertices that lie outside the former ball and inside the latter ball. Computing $\varepsilon$-approximate solutions using BBD-trees requires $O(2^d \log n + d^2(3\sqrt{d}/\varepsilon)^{d-1})$ time when using $O(dn)$ space, in the worst case [66]. Thus, in fixed dimensions, the complexity of this algorithm is $O(\log n + (1/\varepsilon)^{d-1})$, which is known to be optimal, closely matching a lower bound [66]. More recently, algorithms that can provide trade-offs between time and space were also proposed [67].

Note that the Nearest procedure can be implemented as an approximate range search while maintaining the asymptotic optimality guarantee. Notice that the expected number of vertices returned by the Nearest procedure also does not change, except by a constant factor. Hence, the Nearest procedure can be implemented to run in order $\log n$ expected time in the limit and linear space in fixed dimensions.

Let $M_{i}^{\text{RRG}}$ denote the number of steps performed by the RRG algorithm in iteration $i$. Then, together with Lemma 16, the discussion above implies the following lemma.

**Lemma 18** The number of steps executed by the RRG algorithm at each iteration is at most order $\log(N_i)$ in expectation in the limit, i.e., there exists a constant $\Phi_{\text{RRG}} \in \mathbb{R}_{>0}$ such that

$$\limsup_{i \to \infty} \mathbb{E} \left[ \frac{M_{i}^{\text{RRG}}}{\log(N_i)} \right] \leq \Phi_{\text{RRG}}.$$ 

Finally, by Lemmas 17 and 18, we conclude that the RRT and the RRG algorithms have the same asymptotic computational complexity as stated in the following theorem.

**Theorem 19** Under the assumptions of Lemmas 17 and 18 there exists a constant $\phi \in \mathbb{R}_{>0}$ such that

$$\limsup_{i \to \infty} \mathbb{E} \left[ \frac{M_{i}^{\text{RRG}}}{M_{i}^{\text{RRT}}} \right] \leq \phi.$$ 

In section VI, we also provide some experimental evidence supporting Theorem 19.

D. On the efficient construction of Probabilistic RoadMaps

At this point, let us note that our results also imply an efficient PRM planning algorithm. The PRM algorithm samples a set of $n$ vertices from the state-space, and checks connectivity of each pair of samples via the Steer and ObstacleFree procedures to build a graph (called the roadmap) as follows. Any two vertices $x_1$ and $x_2$ are connected by an edge, if the path $\text{Steer}(x_1, x_2)$ that connects these two samples lies within the obstacle-free space. It is known that this algorithm

4In practice, some variations of the PRM algorithm attempt to connect every vertex to the vertices that lie within a fixed distance. However, this variation does not change the theoretical properties such as probabilistic completeness or the asymptotic computational complexity.
is probabilistically complete and, under certain assumptions, the probability of failure, if a solution exists, decays to zero exponentially. However, notice that, from an asymptotic computational complexity point of view, creating this graph requires $O(n^2)$ calls to the Steer and ObstacleFree procedures. On the contrary, our results imply that in a modified version of this algorithm, if each vertex is attempted connection to vertices within a ball of volume $\gamma \frac{\log n}{n}$, then, theoretically speaking, probabilistic completeness can be achieved, while incurring $O(n \log n)$ expected computational cost. Pursuing this direction is outside the scope of this work, and is left to future work.

V. A Tree Version of the RRG Algorithm

Maintaining a tree structure rather than a graph may be advantageous in some applications, due to, for instance, relatively easy extensions to motion planning problems with differential constraints, or to cope with modeling errors. The RRG algorithm can also be slightly modified to maintain a tree structure, while preserving the asymptotic optimality properties as well the computational efficiency. In this section a tree version of the RRG algorithm, called RRT*, is introduced and analyzed.

A. The RRT* Algorithm

Given two points $x, x' \in X_\text{free}$, recall that $\text{Line}(x, x') : [0, s] \to X_\text{free}$ denotes the path defined by $\sigma(t) = tx + (s - t)x'$ for all $t \in [0, s]$, where $s = \|x' - x\|$. Given a tree $G = (V, E)$ and a vertex $v \in V$, let Parent be a function that maps $v$ to the unique vertex $v' \in V$ such that $(v', v) \in E$.

The RRT* algorithm differs from the RRT and the RRG algorithms only in the way that it handles the Steer procedure. The body of the RRT* algorithm is presented in Algorithm 1 and the Extend procedure for the RRT* is given in Algorithm 4. In the description of the RRT* algorithm, the cost of the unique path from $x_{\text{init}}$ to a vertex $v \in V$ is denoted by $\text{Cost}(v)$. Initially, $\text{Cost}(x_{\text{init}})$ is set to zero.

Similarly to the RRT and RRG, the RRT* algorithm first extends the nearest neighbor towards the sample (Lines 2-3). However, it connects the new vertex, $x_{\text{new}}$, to the vertex that incurs the minimum accumulated cost up until $x_{\text{new}}$ and lies within the set $X_{\text{near}}$ of vertices returned by the Near procedure (Lines 6-13). RRT* also extends the new vertex to the vertices in $X_{\text{near}}$ in order to "rewire" the vertices that can be accessed through $x_{\text{new}}$ with smaller cost (Lines 14-17).

B. Convergence to a Feasible Solution

For all $i \in \mathbb{N}$, let $\mathcal{V}_i^{\text{RRT}*}$ and $\mathcal{E}_i^{\text{RRT}*}$ denote the set of vertices and the set of edges of the graph maintained by the RRT* algorithm, at the end of iteration $i$. Let $\mathcal{V}_0^{\text{RRT}*}(\omega) = \{x_{\text{init}}\}$ and $\mathcal{E}_0^{\text{RRT}*}(\omega) = \emptyset$ for all $\omega \in \Omega$.

The following lemma is the equivalent of Lemma 3:

**Lemma 20** For all $i \in \mathbb{N}$ and all $\omega \in \Omega$, $\mathcal{V}_i^{\text{RRT}*}(\omega) = \mathcal{V}_i^{\text{RGT}}(\omega)$, and $\mathcal{E}_i^{\text{RRT}*}(\omega) \subseteq \mathcal{E}_i^{\text{RGT}}(\omega)$.

Algorithm 4: Extend$_{\text{RRT}*}$

1. $V' \leftarrow V; E' \leftarrow E$;
2. $x_{\text{nearest}} \leftarrow \text{Nearest}(G, x)$;
3. $x_{\text{new}} \leftarrow \text{Steer}(x_{\text{nearest}}, x)$;
4. if ObstacleFree($x_{\text{nearest}}, x_{\text{new}}$) then
   5. $V' \leftarrow V' \cup \{x_{\text{new}}\}$;
   6. $x_{\text{min}} \leftarrow x_{\text{nearest}}$;
   7. $x_{\text{near}} \leftarrow \text{Near}(G, x_{\text{new}}, |V|)$;
   8. for all $x_{\text{near}} \in X_{\text{near}}$ do
      9. if ObstacleFree($x_{\text{near}}, x_{\text{new}}$) then
         10. $c' \leftarrow \text{Cost}(x_{\text{near}}) + c(\text{Line}(x_{\text{near}}, x_{\text{new}}))$;
         11. if $c' < \text{Cost}(x_{\text{new}})$ then
            12. $x_{\text{min}} \leftarrow x_{\text{near}}$;
            13. $E' \leftarrow E' \cup \{(x_{\text{parent}}, x_{\text{near}})\}$;
            14. $E' \leftarrow E' \cup \{(x_{\text{new}}, x_{\text{near}})\}$;
      15. else if ObstacleFree($x_{\text{new}}, x_{\text{near}}$) and
         16. Cost($x_{\text{near}}$) > Cost($x_{\text{new}}$) + $c(\text{Line}(x_{\text{new}}, x_{\text{near}}))$
         17. then $x_{\text{parent}} \leftarrow \text{Parent}(x_{\text{stream}})$;
         18. $E' \leftarrow E' \cup \{(x_{\text{parent}}, x_{\text{new}})\}$;
   19. return $G' = (V', E')$.

From Lemma 20 and Theorem 6, the following theorem, which asserts the probabilistic completeness and the exponential decay of failure probability of the RRT* algorithm, is immediate.

**Theorem 21** If there exists a feasible solution to Problem 1 then $\lim_{i \to \infty} P\left(\{V_i^{\text{RRT}*} \cap X_{\text{goal}} \neq \emptyset\}\right) = 1$. Moreover, if an attraction sequence $A$ of length $k$ exists, then $P\left(\{V_i^{\text{RRT}*} \cap X_{\text{goal}} = \emptyset\}\right) \leq e^{-\frac{k}{2}(1/P_k - 2k)}$.

C. Asymptotic Optimality

Let $\mathcal{V}_i^{\text{RRT}*}$ be a random variable that denotes the cost of a minimum cost path in the tree maintained by the RRT* algorithm, at the end of iteration $i$. The following theorem ensures the asymptotic optimality of the RRT* algorithm.

**Theorem 22** Let Assumptions 12, 13, and 14 hold. Then, the cost of the minimum cost path in the RRT* algorithm converges to $c^*$ almost surely, i.e., $\mathbb{P}\left(\{\lim_{i \to \infty} \mathcal{V}_i^{\text{RRT}*} = c^*\}\right) = 1$.

D. Computational Complexity

Let $M_i^{\text{RRT}*}$ be the number of steps performed by the RRT* algorithm in iteration $i$. The following theorem follows from Lemma 20 and Theorem 19.

**Theorem 23** Under the assumptions of Theorem 19, there exists a constant $\phi \in \mathbb{R}_{>0}$ such that

$$\limsup_{i \to \infty} \mathbb{E}\left[\frac{M_i^{\text{RRT}*}}{M_i^{\text{RGT}}}\right] \leq \phi.$$
VI. SIMULATIONS

This section is devoted to an experimental study of the algorithms. Three different problem instances are considered and the RRT and RRT\(^*\) algorithms are compared with respect to their running time and cost of the solution achieved. Both algorithms were implemented in C and run on a computer with 2.66 GHz processor and 4GB RAM running the Linux operating system.

We consider three problem instances. In the first two, the cost function is the Euclidean path length. The first scenario includes no obstacles. Both algorithms are run in a square environment. The trees maintained by the algorithms are shown in Figure 1 at several stages. The figure illustrates that, in this case, the RRT algorithm does not improve the feasible solution to converge to an optimum solution. On the other hand, running the RRT\(^*\) algorithm further improves the paths in the tree to lower cost ones. The convergence properties of the two algorithms are also investigated in Monte-Carlo runs. Both algorithms were run for 20,000 iterations 500 times and the cost of the best path in the trees were averaged for each iteration. The results are shown in Figure 2, which shows that in the limit the RRT algorithm has cost very close to a \(\sqrt{2}\) factor the optimal solution (see [68] for a similar result in a deterministic setting), whereas the RRT\(^*\) converges to the optimal solution. Moreover, the variance over different RRT runs approaches 2.5, while that of the RRT\(^*\) approaches zero. Hence, almost all RRT\(^*\) runs have the property of convergence to an optimal solution, as expected.

In the second scenario, both algorithms are run in an environment in presence of obstacles. In Figure 3 the trees maintained by the algorithms are shown after 20,000 iterations. The tree maintained by the RRT\(^*\) algorithm is also shown in Figure 4 in different stages. It can be observed that the RRT\(^*\) first rapidly explores the state space just like the RRT. Moreover, as the number of samples increase, the RRT\(^*\) improves its tree to include paths with smaller cost and eventually discovers a path in a different homotopy class, which reduces the cost of reaching the target considerably. Results of a Monte-Carlo study for this scenario is presented in Figure 5. Both algorithms were run alongside up until 20,000 iterations 500 times and cost of the best path in the trees were averaged for each iteration. The figures illustrate that all runs of the RRT\(^*\) algorithm converges to the optimum, whereas the RRT algorithm is about 1.5 of the optimal solution on average. The high variance in solutions returned by the RRT algorithm stems from the fact that there are two different homotopy classes of paths that reach the goal. If the RRTLuckily converges to a path of the homotopy class that contains an optimum solution, then the resulting path is relatively closer to the optimum than it is on average. If, on the other hand, the RRT first explores a path of the second homotopy class, which is often the case for this particular scenario, then the solution that RRT converges is generally around twice the optimum.

Finally, in the third scenario, where no obstacles are present, the cost function is selected to be the line integral of a function, which evaluates to 2 in the high cost region, 1/2 in the low cost region, and 1 everywhere else. The tree maintained by the RRT\(^*\) algorithm is shown after 20,000 iterations in Figure 6. Notice that the tree either avoids the high cost region or crosses it quickly, and vice-versa for the low-cost region. (Incidentally, this behavior corresponds to the well known Snell-Descartes law for refraction of light, see [69] for a path-planning application.)

To compare the running time, both algorithms were run alongside in an environment with no obstacles up until one million iterations. Figure 7. shows the ratio of the running time of RRT\(^*\) and that of RRT versus the number of iterations averaged over 50 runs. As expected from the complexity analysis of Section IV-C, this ratio converges to a constant value. The same figure is produced for the second scenario and provided in Figure 8.

VII. CONCLUSION

This paper presented the results of a thorough analysis of the RRT and RRG algorithms for optimal motion planning. It is shown that, as the number of samples increases, the RRT algorithm converges to a sub-optimal solution almost surely. On the other hand, it is proven that the RRG algorithm has the asymptotic optimality property, i.e., almost sure convergence to an optimum solution, which the RRT algorithm lacked. The paper also proposed a novel algorithm called the RRT\(^*\), which inherits the asymptotic optimality property of the RRG, while maintaining a tree structure rather than a graph. The RRG and the RRT\(^*\) were shown to have no significant overhead when compared to the RRT algorithm in terms of asymptotic computational complexity. Experimental evidence that demonstrate the effectiveness of the algorithms proposed and support the theoretical claims were also provided.

The results reported in this paper can be extended in a number of directions, and applied to other sampling-based algorithms other than RRT. First of all, the proposed approach, building on the theory of random graphs to adjust the length
of new connections can enhance the computational efficiency of PRM-based algorithms. Second, the algorithms and the analysis should be modified to address motion planning problems in the presence of differential constraints, also known as kino-dynamic planning problems. A third direction is the optimal planning problem in the presence of temporal/logic constraints on the trajectories, e.g., expressed using formal specification languages such as Linear Temporal Logic, or...
Finally, it is noted that the proposed algorithms may have applications outside of the robotic motion planning domain. In fact, the class of incremental sampling algorithm described in this paper can be readily extended to deal with problems described by partial differential equations, such as the eikonal equation and the Hamilton-Jacobi-Bellman equation.

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Fig. 4. RRT\textsuperscript{*} algorithm shown after 500 (a), 1,500 (b), 2,500 (c), 5,000 (d), 10,000 (e), 15,000 (f) iterations.

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directly implies a necessary condition for asymptotic optimality: with probability one, the initial state, $x_{init}$, must be chosen for extension infinity often.

**Lemma 25** The following inequality holds:

$$\mathbb{P}\left(\left\{ \lim_{i \to \infty} \psi_i^{RRT} = \epsilon^* \right\}\right) \leq \mathbb{P}\left(\lim_{i \to \infty} \sup C_i\right).$$

**Proof:** Denote by $\psi(x)$ a minimum-cost path starting from a given state $x \in X_{free}$ and reaching one of the states in the goal region $g$ and define a sequence $\{c_i\}_{i \in \mathbb{N}}$ of random variables as follows:

$$c_i = \min \{ \text{Cost}(\text{Line}(x_{init}, x) \mid \psi(x)) \mid x \in \mathcal{E}_i^{RRT}, (x_{init}, x) \in \mathcal{E}_{i+1}^{RRT} \}.$$  

Essentially, $c_i$ denotes the minimum cost that is incurred by following the line segment that connects $x_{init}$ to one of its children and following an optimal path afterwards. Notice that $c_i$ is strictly greater than the optimal cost $c^*$ for all $i \in \mathbb{N}$, unless the root node, $x_{init}$, has a child node that is a part of an optimal path.

Recall, from the proof of Lemma 24, that $B$ denotes the event $\cup_{i \in \mathbb{N}} \psi_i^{RRT} = c^*$. Let $A_i$ denote the event that $c_i$ decreases at iteration $i$, i.e., $A_i = \{ c_i < c_{i-1} \}$. First, notice that, conditioning on the event $B^c$, we have that the convergence of the cost of the best path in the RRT to the optimum cost (i.e., the event $\{ \lim_{i \to \infty} \psi_i^{RRT} = c^* \}$) implies that $c_i$ must decrease infinitely often, i.e., $A_i$ must hold infinitely often. More precisely, we have that $\{ \lim_{i \to \infty} \psi_i^{RRT} = c^* \} \cap B^c \subseteq \limsup_{i \to \infty} A_i$.

Noting that $\mathbb{P}(B^c) = 1$ (by Lemma 24), it follows that

$$\mathbb{P}\left(\left\{ \lim_{i \to \infty} \psi_i^{RRT} = c^* \right\}\right) = \mathbb{P}\left(\left\{ \lim_{i \to \infty} \psi_i^{RRT} = c^* \right\} \cap B^c \right) \leq \mathbb{P}\left(\limsup_{i \to \infty} A_i\right).$$

Notice that, by Assumption 9, for $A_i$ to occur infinitely often, $C_i$ must also occur infinitely often, i.e., $\limsup_{i \to \infty} A_i \subseteq \limsup_{i \to \infty} C_i$. Thus, $\mathbb{P}(\limsup_{i \to \infty} A_i) \leq \mathbb{P}(\limsup_{i \to \infty} C_i)$, which implies the lemma.

To prove the theorem, it is shown that RRT fails to satisfy this necessary condition:

**Lemma 26** The probability that the RRT algorithm extends its root node infinitely many times is zero, i.e.,

$$\mathbb{P}\left(\limsup_{i \to \infty} C_i\right) = 0.$$

**Proof:** Let $\{C_1, C_2, \ldots, C_K\}$ be a finite set of cones, such that (i) each cone is centered at $x_{init}$ and has aperture at most $\pi/3$, and (ii) collectively the cones cover the ball of radius $\eta$ around $x_{init}$. Defined for all $k \in \{1, 2, \ldots, K\}$ and all $j \in \mathbb{N}$, let $D_{k,j}$ be the event that a node from inside cone $C_k$ is connected to $x_{init}$ for the $j$th time. Clearly, for

**APPENDIX**

**Proof of Theorem 10**

Since the feasibility problem is assumed to admit a solution, the cost of the optimal path, $c^*$, is a finite real number. First, consider the following technical lemma.

**Lemma 24** The probability that the RRT constructs an optimal path at a finite iteration $i \in \mathbb{N}$ is zero, i.e.,

$$\mathbb{P}\left(\cup_{i \in \mathbb{N}} \psi_i^{RRT} = c^* \right) = 0.$$

**Proof:** Let $B_i$ denote the event the RRT has a path that has cost exactly equal to $c^*$ at the end of iteration $i$, i.e., $B_i = \{ \psi_i^{RRT} = c^* \}$. Let $B$ denote the event that the RRT finds a path that costs exactly $c^*$ at some finite iteration $i$. Then, $B$ can be written as $B = \cup_{i \in \mathbb{N}} B_i$. Notice that $B_i \subseteq B_{i+1}$; thus, we have that $\lim_{i \to \infty} \mathbb{P}(B_i) = \mathbb{P}(B)$, by monotonicity of measures. Also notice that by Assumptions 7 and 8, $\mathbb{P}(B_i) = 0$ for all $i \in \mathbb{N}$, since the probability that the set $\bigcup_{j=1}^{\infty} \text{Sample}(j)$ of points contains a point from a zero-measure set is zero. Hence, $\mathbb{P}(B) = 0$.
by the Borel-Cantelli Lemma \cite{71}, one can conclude that the event is denoted by $\mathcal{C}_k \setminus \mathcal{D}_{x_{\text{init}},y}$, which implies the lemma. Let $k$ be any index from \{1, 2, \ldots, $K$\}. Let $V_{k,j}$ denote the set of vertices inside cone $C_k$ right after the connection of $j$th node to $x_{\text{init}}$ from inside the cone $C_k$. Let $\min_{k,j}$ be the distance of the node that is inside $V_{k,j}$ and has minimum distance to $x_{\text{init}}$ among all the vertices in $V_{k,j}$, i.e.,

$$r_{\min,k,j} = \min_{v \in V_{k,j}} \|v - x_{\text{init}}\|.$$  

This distance is a random variable for fixed values of $k$ and $j$. Let $f_{r_{\min,k,j}}$ denote its probability density function. Also, given two points $x, y \in X$, let $D_{x,y}$ denote the set of all points in $X$ that are closer to $y$ than they are to $x$, i.e.,

$$D_{x,y} = \{z \in X \mid \|z - x\| \geq \|z - y\|\}.$$  

For any $j \in \mathbb{N}_0$, conditioning on $r_{\min,k,j} = z$, the probability that a node from $C_k$ is connected to $x_{\text{init}}$ for the $j$th time is upper bounded by the measure of the region $C_k \setminus D_{x_{\text{init}},y}$, where $y$ is any point with distance $z$ to $x_{\text{init}}$ (see Figure 9). Hence, \(P(D_{k,j} \mid r_{\min,k,j} = z) \leq \alpha \mu(C_k \setminus D_{x_{\text{init}},y})\) for some constant $\alpha \in \mathbb{R}_{>0}$ by Assumption 8. Furthermore, since $C_k$ has aperture at most $\pi/3$, the region $C_k \setminus D_{x_{\text{init}},y}$ is included within the ball of radius $z$ centered at $x_{\text{init}}$ (also illustrated in Figure 9). Hence,

$$P(D_{k,j}) = \int_0^\infty P(D_{k,j} \mid r_{\min,k,j} = z) f_{r_{\min,k,j}}(z)dz \leq \int_0^\infty \alpha \mu(C_k \setminus D_{x_{\text{init}},y}) f_{r_{\min,k,j}}(z)dz \leq \int_0^\infty \alpha \mu(B_{x_{\text{init}},z}) f_{r_{\min,k,j}}(z)dz.$$  

Note that $\mu(B_{x_{\text{init}},z}) \leq \zeta d z^d$, where $d$ is, recall, the dimensionality of the smallest Euclidean space containing $X$, and $\zeta d$ is the volume of the unit ball in this space. Hence,

$$P(D_{k,j}) \leq \alpha \zeta d \int_0^\infty z^d f_{r_{\min,k,j}}(z)dz = \alpha \zeta d \mathbb{E}[r_{\min,k,j}]^d,$$

where the last equality follows merely by the definition of expectation. By the order statistics of the minimum distance, we have that $\mathbb{E}[r_{\min,k,j}]^d$ evaluates to $\frac{\beta^d}{d}$ for some constant $\beta \in \mathbb{R}_{>0}$, under Assumption 8 (see, e.g., \cite{70}). Hence, we have that, for all $k$, $\sum_{j=1}^\infty P(D_{k,j}) < \infty$, since $d \geq 2$. Then, by the Borel-Cantelli Lemma \cite{71}, one can conclude that the probability that $D_{k,j}$ occurs for infinitely many $j$ is zero, i.e.,

$$P(\limsup_{j \to \infty} D_{k,j}) = 0,$$  

which implies the claim.

The theorem is immediate from Lemmas 25 and 26.

**Proof of Theorem 15**

In order to prove that $Y_{\text{RRG}}$ converges to $c^*$ almost surely, it will be shown that $\sum_{i=0}^\infty \mathbb{P}(Y_{i,\text{RRG}} > c^* + \varepsilon)$ is finite for all $\varepsilon > 0$ (see, for instance, \cite{71} for justification of this implication).

Recall that the volume of the unit ball in $d$ dimensions is denoted by $\zeta d$. Recall also that $r_i$ denotes the radius of the ball with volume $\frac{\log i}{\varepsilon}$ used in the $	ext{Near}$ procedure and that we have $r_i = \left(\frac{\gamma \log i}{\varepsilon}\right)^{1/d}$. Let $\sigma^* : [0, s^*] \to \text{cl}(X_{\text{free}})$ be an optimal path with length $s^*$. Recall that $i$ denotes the iteration number (see Algorithm 1). The proof is first outlined in the following section, and then described in detail.

**Outline of the proof:** First, construct a sequence $\{X_i\}_{i \in \mathbb{N}}$ of subsets of $X_{\text{free}}$ such that (i) the sequence is monotonically non-decreasing in the partial subset ordering, i.e., $X_i \subseteq X_{i+1}$ for all $i \in \mathbb{N}$, (ii) for all large enough $i \in \mathbb{N}$, any point in $X_i$ is at least a certain fraction of $r_i$ away from obstacles, i.e., $\|x - y\| \geq \lambda_i$ for all $x \in X_i$ and all $y \in X_{\text{obs}}$, where $\lambda_i = \alpha r_i$ for some constant $\alpha$ (see Figures 10(a) and 10(b)), in particular, the set $X_i$ is included in $X_{\text{free}}$, and (iii) the sequence converges to $X_{\text{free}}$ in the sense that $\cup_{i \in \mathbb{N}} X_i = X_{\text{free}}$.

Second, construct a sequence $\{\sigma_i\}_{i \in \mathbb{N}}$ of paths such that (i) for all $i \in \mathbb{N}$, the path $\sigma_i$ lies completely inside $X_i$, and (ii) the sequence of paths converges to the optimal path $\sigma^*$, i.e., $\lim_{i \to \infty} \|\sigma_i - \sigma^*\| = 0$. This can be done by dividing $\sigma^*$ into segments and constructing an approximation to any segment that lies outside $X_i$ (see Figure 10(c)). Note that, since for all large enough $i$, any point in $X_i$ is at least a certain fraction of $r_i$ away from the obstacles, in particular, any point on the path $\sigma_i$ is at least a certain fraction of $r_i$ away from the obstacles, for all such $i \in \mathbb{N}$.

Third, for any $i \in \mathbb{N}$, construct a set $B_i$ of overlapping balls, each with radius $q_i$ and centered at a point on the path $\sigma_i$, such that the balls in $B_i$ collectively “cover” the path $\sigma_i$ (see Figure 10(d)). Moreover, the radius $q_i$ is chosen in such a way that for all large enough $i$, we have that $q_i$ is less than $r_i$. Hence, for all such large enough $i$, all the balls in $B_i$ completely lie inside the obstacle-free region.

Finally, compute the probability of the event that at least one ball in $B_i$ contains no node of the RRG in iteration $i$; this event is denoted by $A_i$. Subsequently, one can show that $A_i$ can not occur infinitely often as $i$ approaches infinity, which implies that its complement, $\overline{A_i}$, must occur infinitely often.
Denote the boundary of $X_i$ by $\partial X_i$. The precise value of the constant $\theta_1$ will become clear shortly.

Construction of the sequence $\{\sigma_i\}_{i \in \mathbb{N}}$: Assume, without loss of any generality, that the beginning and the end of the optimal path $\sigma^*$ are at least $\delta$ away from the obstacles, i.e., for $\tau \in [0, s_\ast]$, it is the case that $||\sigma^*(\tau) - x|| \geq \delta$ for all $x \in X_{\text{obs}}$.

Construct a sequence $\{\sigma_i\}_{i \in \mathbb{N}}$ of paths, where $\sigma_i : [0, s_i] \rightarrow X_i$, as follows (see Figure 10(c) for an illustration). First, break up the optimal path $\sigma^*$ into a minimum number of segments such that each segment completely lies either inside $X_i$ or inside $c(X_{\text{free}} \setminus X_i)$. For this purpose, define a sequence $\{\tau_0, \tau_1, \ldots, \tau_m\}$ of scalars as follows:

- $\tau_0 = 0$ and $\tau_m = s_\ast$,
- for all $j \in \{1, 2, \ldots, m\}$, $\tau_j \in \partial X_i$, and
- for all $\{0, 1, \ldots, m\}$, either $\sigma^*(\tau') \in X_i$ for all $\tau \in (\tau_j, \tau_{j+1})$ or $\sigma^*(\tau') \notin X_i$ for all $\tau \in (\tau_j, \tau_{j+1})$.

Second, for each segment of $\sigma^*$ defined by its end points $\sigma^*(\tau_j)$ and $\sigma^*(\tau_{j+1})$, construct a path that lies in $X_i$ and agrees with this segment of $\sigma^*$ in its endpoints. More precisely, define a set $\{\rho_0, \rho_1, \ldots, \rho_m\}$ of paths as follows: (i) if $\sigma^*(\tau') \in X_i$ for $\tau' \in (\tau_j, \tau_{j+1})$, then let $\rho_i$ be the $i$th segment of $\sigma^*$, i.e., $\sigma^*(\tau') \in X_i$ for all $\tau' \in [\tau_j, \tau_{j+1}]$. (ii) if, on the other hand, $\sigma^*(\tau') \notin X_i$, then let $\rho_i$ be a minimum-cost continuous path in $X_i$ starting from $\sigma^*(\tau_j)$ and ending in $\sigma^*(\tau_{j+1})$. Such a path always exists by the construction of $X_i$ and Assumption 14. Finally, define $\sigma_i$ as the concatenation of path segments $\rho_1, \rho_2, \ldots, \rho_m$, i.e., $\sigma_i := \rho_0 \rho_1 \rho_2 \cdots \rho_m$.

Two important properties of the sequence of paths $\{\sigma_i\}_{i \in \mathbb{N}}$ follow. First, each path $\sigma_i$ is at least $\lambda_i$ away from the obstacles. More precisely, $||\sigma_i(\tau) - x|| \geq \lambda_i$ for all $i \in \mathbb{N}$, $\tau \in [0, s_i]$, and $x \in X_{\text{obs}}$. This claim follows from Assumption 14 and the fact that $\lambda_i \leq \delta$ for all $i$. The second property, regarding the costs of the paths in $\{\sigma_i\}_{i \in \mathbb{N}}$, is stated in the following lemma:

**Lemma 27** Under Assumptions 12 and 13, $\lim_{i \rightarrow \infty} c(\sigma_i) = c(\sigma^*)$ holds.

**Proof:** First, for all $i \in \mathbb{N}$, $\sigma_i$ is continuous by construction. Second, note that $X_i \subseteq X_{i+1}$ for all $i \in \mathbb{N}$, and that $\cup_{i \in \mathbb{N}} X_i = X_{\text{free}}$. Let $m_i$ be the number of segments of $\sigma^*$ that lie in $X_{\text{free}} \setminus X_i$. By Assumption 13 the cost of each such segment is bounded by $\kappa \lambda_i$. Then, by Assumption 12 $c(\sigma_i) - c(\sigma^*) \leq m_i \kappa \lambda_i$. Finally, since $m_i$ is a monotonically non-increasing function of $i$, and $\lambda_i$ converges to zero as $i$ approaches infinity, the lemma follows.

Construction of the sequence $\{B_i\}_{i \in \mathbb{N}}$ of sets of balls: Next, for each $i \in \mathbb{N}$, construct a set $B_i$ of equal-radius overlapping balls that collectively “cover” the path $\sigma_i$.

Recall the constant $\theta_1 \in (0, 1)$, introduced above. The amount of overlap between two consecutive balls will be parametrized by this constant $\theta_1$. Denote the radius of each ball in $B_i$ by $q_i$, which we define as $q_i := \frac{\lambda_i}{(1+\theta_1)}$ for reasons to become clear shortly.

Construct the set $B_i = \{B_{i,1}, B_{i,2}, \ldots, B_{k_{i,K+1}}\}$ of balls recursively as follows (see Figures 10(d) and 10(e)).
Let $B_{i,1}$ be centered at $\sigma_i(0)$.

For all $k > 1$, let $B_{i,k}$ centered at $\sigma_i(\tau)$, where $\tau$ is such that the centers of $B_{i,k}$ and $B_{i,k-1}$ are exactly $\theta_i q_i$ apart from each other.

Let $K_i$ be the number of balls that can possibly be generated in this manner and, finally, let $B_{i,K_i+1}$ be the ball centered at $\sigma_i(s_i)$.

Now, consider paths that can be constructed by connecting points from balls in $B_i$ in a certain way. More precisely, for all $k \in \{1, 2, \ldots, K_i + 1\}$, let $q_k$ be a point from the ball $B_{i,k}$. Consider a path $\sigma_i^k$ that is constructed by joining the line segments connecting two consecutive points in $\{x_1, x_2, \ldots, x_{K_i+1}\}$, i.e., $\sigma_i^k = \text{Line}(x_1, x_2)\text{Line}(x_2, x_3)\cdots\text{Line}(x_{K_i}, x_{K_i+1})$. With an abuse of notation, let $\Sigma_i'$ denote the set of all such paths.

First, notice that, by the choice of the radius $q_i$ of the balls in $B_i$, any path $\sigma_i^k \in \Sigma_i'$ is collision-free.

**Lemma 28** Let Assumption 14 hold. For any path $\sigma_i^k \in \Sigma_i'$, where $\sigma_i^k : [0, s_i'] \to X$, $\sigma_i^k(\tau) \in X_{\text{free}}$ for all $\tau \in [0, s_i']$.

**Proof:** Let $\{x_1, x_2, \ldots, x_{K_i+1}\}$ be the set of points that forms $\sigma_i^k = \text{Line}(x_1, x_2)\text{Line}(x_2, x_3)\cdots\text{Line}(x_{K_i}, x_{K_i+1})$. It will be shown that each segment $\text{Line}(x_k, x_{k+1})$ is obstacle-free, i.e., lies in $X_{\text{free}}$, which implies the lemma. Let $y_k$ denote the center of the ball $B_{i,k}$. Note that $x_k$ and $y_k$ have distance at most $q_i$. Similarly, noting that $x_{k+1}$ has distance at most $q_i$ to $y_{k+1}$ and that $y_k$ and $y_{k+1}$ are at most $\theta_i q_i$ apart from each other, one can conclude that $x_{k+1}$ has distance at most $(1 + \theta_i)q_i$ to $y_k$. Indeed, $y_k$ has distance at most $(1 + \theta_i)q_i = \lambda_i$ to any point in the convex combination $[x_k, x_{k+1}]$. Finally, recall that, as a consequence of Assumption 14 any point with distance at most $\lambda_i$ from $y_k$ lies in $X_{\text{free}}$, which completes the proof.

Second, notice that the cost of $\sigma_i^k$ “close” to that of $\sigma_i$. Indeed, the following lemma holds.

**Lemma 29** Let Assumptions 12 and 13 hold. Let $\{\sigma_i^k\}_{i \in \mathbb{N}}$ be any sequence of paths such that $\sigma_i^k \in \Sigma_i'$ for all $i \in \mathbb{N}$, then $\lim_{i \to \infty} |c(\sigma_i^k) - c(\sigma^*)| = 0$.

**Proof:** The distance between a path $\sigma_i^k \in \Sigma_i'$ and the path $\sigma_i$ approaches zero as $i$ approaches infinity. Moreover, by Assumptions 12 and 13 $|c(\sigma_i^k) - c(\sigma_i)|$ converges to zero. Hence, the result follows from Lemma 27.

Let us define $\epsilon_i = \sup_{\sigma_i^k \in \Sigma_i'} |c(\sigma_i^k) - c(\sigma_i)|$. Hence, Lemma 29 establishes that $\lim_{i \to \infty} \epsilon_i = 0$.

Third, according to our choice of $q_i$, any path in $\Sigma_i'$ is “constructable” by the RRG in the following sense.

**Lemma 30** For all $i \in \mathbb{N}$ and for all $k \in \{1, 2, \ldots, K_i\}$, $\|x_{k+1} - x_k\| \leq r_i$, where $x_k \in B_{i,k}$.

**Proof:** Recall that the radius of each ball in $B_i$ is $q_i = \frac{\lambda_i}{(1 + \theta_i)}$. Given any two points $x_k$ and $x_{k+1}$, both $x_k$ and $x_{k+1}$ have distances at most $q_i$ to the centers of the balls $B_{i,k}$ and $B_{i,k+1}$, respectively. Note also that the centers of these balls have distance at most $\theta_i q_i$ to each other. Using the triangle inequality together with the definitions of $\lambda_i$ and $q_i$, one obtains $\|x_{k+1} - x_k\| \leq (2 + \theta_i)q_i = \frac{2q_i + \theta_i}{1 + \theta_i} \leq r_i$.

Intuitively, Lemma 30 establishes that if each ball in $B_i$ contains at least one node of the RRG by the end of iteration $i$, then the RRG algorithm will indeed connect these vertices with edges and construct a path $\sigma_i^k$ from $\Sigma_i'$.

The probability that a path from $\Sigma_i'$ is constructed: Let $A_{i,k}$ be the event that the RRG has no node inside the ball $B_{i,k}$ at the end of iteration $i$. Moreover, let $A_i$ be the event that at least one of the balls in $B_i$ contains no node of the RRG at the end of iteration $i$, i.e., $A_i = \bigcup_{k=1}^{K_i+1} A_{i,k}$.

It is appropriate to explain how the main result of this theorem is related to the sequence $\{A_i\}_{i \in \mathbb{N}}$ of events. Recall that $\epsilon_i$ was defined as $\epsilon_i = \sup_{\sigma_i^k \in \Sigma_i'} |c(\sigma_i^k) - c(\sigma^*)|$ and that it was already established that $\epsilon_i$ converges to zero as $i$ approaches infinity. Hence, given any $\epsilon$, one can find a number $i_1 \in \mathbb{N}$ such that for all $i \geq i_1$, $\epsilon_i < \epsilon$. Recall also that $\epsilon_i$ converges to zero as $i$ approaches infinity. Hence, there exists some $i_2 \in \mathbb{N}$ such that $\epsilon_i \leq \delta$ for all $i \geq i_2$. Let $i_0 = \max\{i_1, i_2\}$ and note the following upper bound:

$$
\sum_{i=0}^{\infty} \mathbb{P}(\{\gamma_i^{\text{RRG}} > c^* + \epsilon\})
= \sum_{i=0}^{i_0-1} \mathbb{P}(\{\gamma_i^{\text{RRG}} > c^* + \epsilon\}) + \sum_{i=i_0}^{\infty} \mathbb{P}(\{\gamma_i^{\text{RRG}} > c^* + \epsilon\})
\leq i_0 + \sum_{i=i_0}^{\infty} \mathbb{P}(\{\gamma_i^{\text{RRG}} > c^* + \epsilon_i\})
\leq i_0 + \sum_{i=i_0}^{\infty} \mathbb{P}(A_i).
$$

To complete the proof, it will be established that $\sum_{i=0}^{\infty} \mathbb{P}(A_i) < \infty$, which implies that $\sum_{i=0}^{\infty} \mathbb{P}(\{\gamma_i^{\text{RRG}} > c^* + \epsilon_i\}) < \infty$ for all $\epsilon > 0$, which in turn implies the result.

To formally show this claim, define the following sequence of events. For all $i \in \mathbb{N}$, let $C_i$ be the event that for any point $x \in X_{\text{free}}$ the RRG algorithm includes a node $v$ such that $\|x - v\| \leq \eta$ and that the line segment $\text{Line}(x, v)$ joining $v$ and $x$ is obstacle-free. Under the assumptions of Theorem 6 the following lemma holds.

**Lemma 31** Let the assumptions of Theorem 6 and Assumption 14 hold, then there exist constants $a, b \in \mathbb{R}_{>0}$ such that $\mathbb{P}(C_i^c) \leq ae^{-bi}$ for all $i \in \mathbb{N}$.

**Proof:** The set $X_{\text{free}}$ can be partitioned into finitely many convex sets such that each partition is bounded by a ball of radius $\eta$, since $X_{\text{free}}$ is a bounded subset of $\mathbb{R}^d$. Since, by Theorem 6 the probability that each such ball does not include a node of the RRG decays to zero exponentially as $i$ approaches infinity, the probability that at least one such ball does not contain a node of the RRG also decays to zero exponentially (this follows from the union bound). Moreover, if each partition contains a node of the RRG, then any point inside the partition can be connected to the associated node of the RRG with an obstacle-free line segment of length less than $\eta$, since each partition is convex and is bounded by a ball of radius $\eta$.

The following lemma constitutes another important step in proving the main result of this theorem.
Lemma 32. Let $\gamma > 2^d(1 + 1/d)\mu(X_{\text{free}})$, and $\theta_2 \in (0, 1)$ be a constant. Then, $\sum_{i=1}^{\infty} \mathbb{P}(A_i | \bigcap_{j=\theta_2 i}^i C_j) < \infty$.

Proof: It is desired to compute $\mathbb{P}(A_i | \bigcap_{j=\theta_2 i}^i C_j)$ under the theorem’s assumptions. Since $\bigcap_{j=\theta_2 i}^i C_j$ is given, it can be assumed that, from iteration $\theta_2 i$ to iteration $i$, the RRG has the following coverage property: for every point $x \in X_{\text{free}}$, the RRG includes a node $v$ that is at most $\eta$ away from $x$ and the line segment between $x$ and $v$ lies in the obstacle-free region.

Recall that the length of $\sigma_i$ was $s_i$. A bound on the number of balls in $B_i$ with respect to $i$, $s_i$, and other constants of the problem can be derived as follows. Note that the segment of $\sigma_i$ that starts at the center of $B_{i,k}$ and ends at the center of $B_{i,k+1}$ has length at least $\theta_1 q_i$. (Recall that distance between the centers of two consecutive balls in $B_i$ is $\theta_1 q_i$ by construction), except for the last segment, which has length less than $\theta_1 q_i$. Thus, for all $i \geq i_2$,

$$|B_i| = K_i + 1 \leq \frac{s_i}{\theta_1 q_i} = \frac{(2 + \theta_1)s_i}{\theta_1 r_i} = 2 + \theta_1 \frac{\gamma}{\log i} \left(\frac{i}{\theta_1} \right)^{1/d}$$

Also, given $\bigcap_{j=\theta_2 i}^i C_j$, the probability that a single ball, say $B_{i,1}$, does not contain a node of the RRG at the end of iteration $i$ can be upper-bounded as follows:

$$\mathbb{P}(A_{i,1} | \bigcap_{j=\theta_2 i}^i C_j) \leq \left(1 - \frac{\mu(B_{i,1})}{\mu(X_{\text{free}})}\right)^{i - \theta_2 i} = \left(1 - \frac{\gamma}{(2 + \theta_1)^d \mu(X_{\text{free}})} \log i \right)^{(i - \theta_2 i)}$$

Using the fact that $(1 - 1/f(i))^r \leq e^{-r/f(i)}$, the right-hand side can further be bounded to obtain the following inequality:

$$\mathbb{P}(A_{i,1} | \bigcap_{j=\theta_2 i}^i C_j) \leq e^{-\frac{(i - \theta_2 i)}{(2 + \theta_1)^d \mu(X_{\text{free}})} \log i} \leq i^{-\frac{(i - \theta_2 i)}{(2 + \theta_1)^d \mu(X_{\text{free}})}}$$

As a result of the discussion above, we have that

$$\mathbb{P}(A_i | \bigcap_{j=\theta_2 i}^i C_j) \leq \mathbb{P} \left( \bigcup_{i=1}^{B_i} A_{i,k} | \bigcap_{j=\theta_2 i}^i C_j \right) \leq \sum_{i=1}^{B_i} \mathbb{P}(A_i | \bigcap_{j=\theta_2 i}^i C_j) = |B_i| \mathbb{P}(A_{i,1} | \bigcap_{j=\theta_2 i}^i C_j) \leq 2 s_i \mu(X_{\text{free}})^{1/d} \log i \left(\frac{i}{\theta_1} \right)^{1/d} \left(1 - \frac{(i - \theta_2 i)}{(2 + \theta_1)^d \mu(X_{\text{free}})}\right)$$

Note that $\sum_{i=1}^{\infty} \mathbb{P}(A_i | \bigcap_{j=\theta_2 i}^i C_j) < \infty$, for all choices of $\gamma$ such that $\frac{(i - \theta_2 i)}{(2 + \theta_1)^d \mu(X_{\text{free}})} \leq 1$, holds, i.e., whenever $\gamma > 2^d(1 + 1/d)\mu(X_{\text{free}})$, noting that for any such choice of $\gamma$, there exists $\theta_1 > 0$ and $\theta_2 > 0$ satisfying the former inequality. Hence, the lemma follows.

Lemma 33. Let $\theta_2 \in (0, 1)$ be a constant. Then, $\sum_{i=1}^{\infty} \mathbb{P}(\bigcap_{j=\theta_2 i}^i C_j) < \infty$.

Proof: Note the following inequalities:

$$\sum_{i=1}^{\infty} \mathbb{P}(\bigcap_{j=\theta_2 i}^i C_j) = \sum_{i=1}^{\infty} \mathbb{P}(\bigcup_{j=\theta_2 i}^i C_j) \leq \sum_{i=1}^{\infty} \sum_{j=\theta_2 i}^i \mathbb{P}(C_j) \leq \sum_{i=1}^{\infty} \sum_{j=\theta_2 i}^i a^j e^{-b j}$$

in which the right-hand side is finite for all $\theta_2 \in (0, 1)$, and where the second inequality follows from Lemma 31.

Finally, the relationship of Lemmas 32 and 33 to the main result can be pointed out. Note the following inequality:

$$\mathbb{P}(A_i | \bigcap_{j=\theta_2 i}^i C_j) = \mathbb{P}(A_i \cap (\bigcap_{j=\theta_2 i}^i C_j)) = \mathbb{P}(\bigcap_{j=\theta_2 i}^i C_j) \leq 1 - \mathbb{P}(A_i) - \mathbb{P}(\bigcap_{j=\theta_2 i}^i C_j)$$

Hence, taking infinite sums of both sides with respect to $\eta$ and rearranging, the following inequality is established:

$$\sum_{i=1}^{\infty} \mathbb{P}(A_i) \leq \sum_{i=1}^{\infty} \mathbb{P}(A_i | \bigcap_{j=\theta_2 i}^i C_j) \leq \sum_{i=1}^{\infty} \mathbb{P}(\bigcap_{j=\theta_2 i}^i C_j)$$

By Lemmas 32 and 33, the right-hand side is finite, whenever $\gamma > 2^d(1 + 1/d)\mu(X_{\text{free}})$, which implies that, under the same conditions, $\sum_{i=1}^{\infty} \mathbb{P}(A_i) < \infty$, which in turn implies the result.

Proof of Lemma 36

The proof of this lemma employs results from the Random Geometric Graph (RGG) theory (see, e.g., [53]). First, some of the relevant results in the theory of RGGs will be introduced, and then the proof of this lemma will be given.

An RGG $G(V_n, r_n) = (V_n, E_n)$ in $X_{\text{free}}$ with $n$ vertices is formed as follows: (i) $V_n$ is a set of $n$ vertices that are sampled identically and independently from $X_{\text{free}}$ according to a distribution with a continuous density function $f(x)$, (ii) two vertices $v, v' \in V_n$ are connected with an edge if and only if the distance between them is less than $r_n$, i.e., $\|v - v'\| < r_n$.

The following lemma is a special case of Proposition 3.1 in [53].

Lemma 34 (see [53]). If $r_n$ satisfies $\lim_{n \to \infty} r_n = 0$, then

$$\lim_{n \to \infty} \frac{\mathbb{E}[|E_n|]}{n^2 (r_n)^d} = \frac{1}{\gamma} \frac{\zeta_d}{d} \int_{x \in \mathbb{R}^d} f(x)^2 \, dx.$$
Lemma 35. If $\zeta_d(r_n)^d = \gamma \log(n)$ for some $\gamma \in \mathbb{R}_{>0}$, then the number of isolated vertices, $|V^c_n|$, satisfies the following:

$$\mathbb{E}[|V^c_n|] / n \leq \left(1 - \gamma \log(n) / n\right)^{n-1}.$$ 

Thus, $\lim_{n \to \infty} \mathbb{E}[|V^c_n|] / n = 0$; moreover, $\sum_{n=1}^{\infty} \mathbb{E}[|V^c_n|] / n$ is finite whenever $\gamma > 1$.

Proof: Reference [53] does not consider the case when $n(r_n)^d \to \infty$ as $n \to \infty$. However, following the reasoning in the proof for the case when $n(r_n)^d \to \rho \in \mathbb{R}_{>0}$ (see the proof of Proposition 3.3 in [53]), it is possible construct a proof for this lemma. Let us sketch this proof.

Following [53] (note that the the notation in [53] is slightly different from that in this paper), and after appropriate simplifications, one gets

$$\frac{\mathbb{E}[|V^c_n|]}{n} = \int_{x \in X_{\text{free}}} \left(1 - I_n(x)\right)^{n-1} f(x)dx,$$

where $f(x)$ is the density function, which is uniform over $X_{\text{free}}$ in the case under consideration, and $I_n(x) = \int_{x' \in B_{\rho} - r_n} x'dx'$, which evaluates to $I_n = \gamma \log(n)$ whenever $(r_n)^d = \gamma \log(n)$ (the reader is referred to [53] for details). Hence,

$$\frac{\mathbb{E}[|V^c_n|]}{n} = \int_{x \in X_{\text{free}}} \left(1 - \gamma \log(n) / n\right)^{n-1} \frac{1}{\mu(X_{\text{free}})}dx = \left(1 - \gamma \log(n) / n\right)^{n-1},$$

which converges to zero as $n$ approaches infinity. Moreover, the sum of these terms is finite whenever $\gamma > 1$.

Consider the random geometric graph with the vertex set $V_i$ that is formed with the Sample procedure, i.e., $V_i = \bigcup_{j=1}^i \{\text{Sample}(j)\}$. Let $V^0_i$ denote the set of isolated vertices and let $V^1_i$ denote the set of the remaining vertices, i.e., $V^1_i = V_i \setminus V^0_i$. We will denote by $E^0_i$ and $E^1_i$ the set of all edges that connect the vertices in $V^1_i$ and those in $V^0_i$, respectively. Hence, we have that $E^0_i = \emptyset$ and $E^1_i = E_n$ for all $i \in \mathbb{N}$.

Consider the RRG algorithm. Let $V^2_i$ denote the set of all vertices that result from extending the graph when the corresponding sample is isolated, i.e., no vertices were present within the ball of volume $\gamma \log(N_i)$ centered at the sample, $\text{Sample}(i)$, in iteration $i$. Let $E^2_i$ denote the set of all edges that are connected to the vertices in $V^2_i$. Notice that the number of calls to ObstacleFree in iteration $i$ is exactly the number of edges created at that iteration, hence is less than or equal to $|E^1_i \cup E^2_i| = |E^1_i| + |E^2_i|$. The following lemmas lead immediately to the result:

Lemma 36

$$\limsup_{i \to \infty} \mathbb{E} \left[ \frac{|E^1_i|}{N_i \log(N_i)} \right] \leq \frac{1}{2} \frac{\gamma}{\mu(X_{\text{free}})}.$$ 

Proof: Consider the expected value of $|E^1_i| / (n^2 \rho_n)^d$, when $n = N_i$. Substituting $\zeta_d(r_n)^d = \gamma \log(N_i)$ and using Lemma 34, the result follows.

Lemma 37

$$\limsup_{i \to \infty} \mathbb{E} \left[ \frac{|E^2_i|}{N_i \log(N_i)} \right]$$

is finite.

Proof: Let $M^2_i$ denote the number of edges added to $E^2_i$ at iteration $i$. Note that

$$\mathbb{E}[M^2_i] \leq \mathbb{E} \left[ \gamma \frac{\log(N_i)}{\mu(X_{\text{free}})} |V^2_i|^2 \right] = \gamma \frac{\log(N_i)}{\mu(X_{\text{free}})} \mathbb{E} \left[ \frac{\log(N_i)}{N_i} |V^2_i|^2 \right].$$

Hence, noting $E^2_i = \sum_{j=1}^i M^2_i$,

$$\limsup_{i \to \infty} \mathbb{E} \left[ \frac{|E^2_i|}{N_i \log(N_i)} \right] \leq \limsup_{i \to \infty} \mathbb{E} \left[ \frac{\sum_{j=1}^i \log(N_j)}{N_i \log(N_i)} |V^2_j|^2 / N_j \right].$$

Consider the expectation in the right hand side. Noting that, surely, $\log(N_j) \leq \log(N_i)$ for all $j \leq i$, the following inequalities hold:

$$\mathbb{E} \left[ \sum_{j=1}^i \log(N_j) \frac{|V^2_j|^2}{N_j} / N_i \log(N_i) \right] \leq \mathbb{E} \left[ \frac{1}{N_i} \sum_{j=1}^i \frac{|V^2_j|^2}{N_j} \right] \leq \mathbb{E} \left[ \frac{1}{N_i} \right]^{1/2} \left( \sum_{j=1}^i \frac{|V^2_j|^2}{N_j} \right)^{1/2}$$

where the second inequality follows from the Cauchy-Schwarz inequality. Notice that the first term approaches zero as $i \to \infty$, whereas, by Lemma 35, the second term is finite. Hence, it is concluded that

$$\limsup_{i \to \infty} \mathbb{E} \left[ \frac{1}{N_i} \right]^{1/2} \left( \sum_{j=1}^i \frac{|V^2_j|^2}{N_j} \right)^{1/2} < \infty,$$

which implies the lemma.