Whittaker Modules for Classical Lie Superalgebras

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Abstract: We classify simple Whittaker modules for classical Lie superalgebras in terms of their parabolic decompositions. We establish a type of Miličić–Soergel equivalence of a category of Whittaker modules and a category of Harish–Chandra bimodules. For classical Lie superalgebras of type I, we reduce the problem of composition factors of standard Whittaker modules to that of Verma modules in their BGG categories $\mathcal{O}$. As a consequence, the composition series of standard Whittaker modules over the general linear Lie superalgebras $\mathfrak{gl}(m|n)$ and the ortho-symplectic Lie superalgebras $\mathfrak{osp}(2|2n)$ can be computed via the Kazhdan–Lusztig combinatorics.

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1. Introduction

1.1. Classical Whittaker modules. In the classical paper [50], Kostant introduced and classified a family of simple modules $Y_{\xi,\eta}$ over finite-dimensional complex semisimple Lie algebras. Motivated by the study of Whittaker models, he found the condition of the existence of a Whittaker vector for a simple module of linear semisimple Lie group. Subsequently, a systematic construction of the Whittaker modules in the category $\mathcal{N}$ over finite-dimensional complex semisimple Lie algebras, containing Kostant’s simple modules and modules in the BGG category $\mathcal{O}$, was studied by McDowell in [56,57] and by Miličić and Soergel in [62,63].
It is known in [62] that \( \mathcal{N} \) has certain standard Whittaker modules parametrized by cosets in the Weyl group of a certain subgroup (see also [56]). In particular, an equivalence of certain categories of Whittaker modules and Harish--Chandra bimodules was established by Miličić and Soergel in [62, Theorem 5.1]. As an application, the problem of composition factors of the standard Whittaker modules was partially solved in [62, Section 5]. Namely, this solution follows from the composition factors of Verma modules in the BGG category \( \mathcal{O} \), which is reduced to the Kazhdan–Lusztig conjectures (see, e.g., [8,12,51]). Around the same time, Backelin in [3] developed a complete solution to the same problem using Whittaker functors. Consequently, this problem can be completely calculated by the Kazhdan–Lusztig combinatorics (see [3, Theorem 6.2]).

There have been numerous attempts to obtain results toward the study of Whittaker modules for Lie algebras and related algebras that possess a structure similar to triangular decomposition; see, e.g., [2,14,20,34,44,53,54,64,65,67,71] and references therein. Inspired by these activities, Batra and Mazorchuk developed in [13] a general framework for Whittaker modules. More recently, Coulembier and Mazorchuk studied in [25] the extension fullness of the Whittaker categories.

1.2. Whittaker modules for Lie superalgebras. While there are now complete solutions to the problem of composition factors in standard Whittaker modules for the semisimple Lie algebras, the Whittaker modules for Lie superalgebras were not investigated until recently. In a recent exposition [17], Bagci, Christodouloupoulou and Wiesner initiated the study of Whittaker modules over Lie superalgebras in a systematic fashion, where the induced Whittaker modules were constructed for the following basic classical Lie superalgebras

\[
\mathfrak{gl}(m|n), \quad \mathfrak{sl}(m|n), \quad \mathfrak{psl}(n|n) \quad \text{and} \quad \mathfrak{osp}(2|2n).
\]

(1.1)

See, e.g., [28, Section 1] for the definitions. In particular, the authors proved that these induced Whittaker modules have simple tops as well as have finite length; see [17, Corollary 4.10, Proposition 4.16]. Therefore it is natural to classify simple modules and study the composition series of modules in the category of Whittaker modules. There have also been a variety of work done to study Whittaker modules and \( \mathcal{W} \)-algebras over basic Lie superalgebras (see (1.2) for the definition of basic Lie superalgebras); see, e.g., [11,72,73].

We study several aspects of Whittaker modules over classical Lie superalgebras. Namely, the present paper attempts to classify simple Whittaker modules and to construct a type of Miličić–Soergel equivalence between a category of Whittaker modules and a corresponding category of Harish–Chandra bimodules and to compute composition series of standard Whittaker modules.

1.3. Classical Lie superalgebras. Recall that a finite-dimensional Lie superalgebra \( \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) is called classical (or quasi-reductive in [68]) if the restriction of the adjoint representation of \( \mathfrak{g} \) to the Lie algebra \( \mathfrak{g}_0 \) is semisimple. The Killing-Cartan type classification of finite-dimensional complex simple Lie superalgebras has been established by Kac in his celebrated paper [48,49]. One of the most interesting subclass of Kac’s list is the following series of classical Lie superalgebras:

\[
\mathfrak{gl}(m|n), \quad \mathfrak{sl}(m|n), \quad \mathfrak{psl}(n|n), \quad \mathfrak{osp}(m|2n), \quad D(2, 1|\alpha), \quad G(3), \quad F(4),
\]

\[
\mathfrak{p}(n), \quad [\mathfrak{p}(n), \mathfrak{p}(n)], \quad \mathfrak{q}(n), \quad \mathfrak{sq}(n), \quad \mathfrak{pq}(n), \quad \mathfrak{psq}(n).
\]

(1.2) (1.3)
We refer to [28,59] for more details of these Lie superalgebras. A classical Lie superalgebra in (1.2) is called basic; see [28, Section 1.1].

A classical Lie superalgebra $g$ is called type I, if it has a compatible $\mathbb{Z}$-gradation of the form $g = g_{-1} \oplus g_0 \oplus g_1$, with $g_0 = g_0, g_1 = g_{-1} \oplus g_1$. In particular, we have the following Lie superalgebras of type I from the list (1.2)–(1.3):

\[(\text{Type A}) : \mathfrak{gl}(m|n), \mathfrak{sl}(m|n), \mathfrak{psl}(n|n), \quad (1.4)\]
\[(\text{Type C}) : \mathfrak{osp}(2|2n), \quad (1.5)\]
\[(\text{Type P}) : \mathfrak{p}(n), [\mathfrak{p}(n), \mathfrak{p}(n)]. \quad (1.6)\]

1.4. Parabolic and triangular decompositions. To explain the contents of the present paper in more detail, we start by explaining our precise setup. Let $g$ be a classical Lie superalgebra in (1.2). Now we fix a Cartan subalgebra $h$ of $g$. To explain the contents of the present paper in more detail, we start by explaining our precise setup. Let $g$ be a classical Lie superalgebra in (1.2). Now we fix a Cartan subalgebra $h$ of $g$. In particular, we have the following Lie superalgebras of type I from the list (1.2)–(1.3):

\[(\text{Type A}) : \mathfrak{gl}(m|n), \mathfrak{sl}(m|n), \mathfrak{psl}(n|n), \quad (1.4)\]
\[(\text{Type C}) : \mathfrak{osp}(2|2n), \quad (1.5)\]
\[(\text{Type P}) : \mathfrak{p}(n), [\mathfrak{p}(n), \mathfrak{p}(n)]. \quad (1.6)\]

1.4. Parabolic and triangular decompositions. To explain the contents of the present paper in more detail, we start by explaining our precise setup. Let $g$ be a classical Lie superalgebra in (1.2). Now we fix a Cartan subalgebra $h_0$ of $g$. Then we have a weight space decomposition with the set $\Phi \subset h_0^*$ of roots $g = \bigoplus\alpha \in \Phi \cup \{0\} g^{\alpha}$, where $g^{\alpha} = \{X \in g | [h, X] = \alpha(h)X, \text{ for any } h \in h_0\}$. The subalgebra $h := g^0$ is usually referred to as the Cartan subalgebra of $g$; see also [31, Section 1.3].

We recall the notion of parabolic decomposition of classical Lie superalgebras from [55, §2.4] (see also [31,40,59] and references therein). For given $H \in h_0$ we can define subalgebras of $g$

$$l := \bigoplus_{\text{Re}(z) = 0} g^\alpha, \quad u := \bigoplus_{\text{Re}(z) > 0} g^\alpha, \quad u^- := \bigoplus_{\text{Re}(z) < 0} g^\alpha,$$

where $\text{Re}(z)$ denotes the real part of $z \in \mathbb{C}$. Such a decomposition $g = u^- \oplus l \oplus u$ gives rise to a corresponding parabolic subalgebra $p = l \oplus u$. We refer to $l$ as the Levi subalgebra of $g$. If $l = g^0 = h$, then (1.7) leads to a triangular decomposition. In this case, we write $n := u, n^- := u^-$ and define the Borel subalgebra $b := h \oplus n$ (see also [59, Section 3.3]). Unless mention otherwise, for a given Borel subalgebra $b$ we will choose the vector $H$ in (1.7) such that $p \supset b, l = l_0$ and $\alpha(H) \in \mathbb{R}$, for all $\alpha \in \Phi$. In this case, we may note that $h = h_0$.

Throughout the present paper, we fix a triangular decomposition of $g_0$

$$g_0 = n^-_0 \oplus h_0 \oplus n_0 \quad (1.8)$$

with Borel subalgebra $b_0 = h \oplus n_0$. We define

$$\mathcal{I} := \{\zeta \in n_0^* | \zeta([n_0, n_0]) = 0\}. \quad (1.9)$$

Namely, $\mathcal{I}$ is the set of homomorphisms from $n_0$ to $\mathbb{C}$. Denote by $\Phi(n_0)$ the set of roots in $n_0$. For any $\zeta \in \mathcal{I}$, we define a subset of simple roots of $g_0$:

$$\Phi_\zeta := \{\alpha \in \Phi(n_0) | \zeta(g_0^\alpha) \neq 0\}. \quad (1.10)$$

This gives rise to a parabolic decomposition of $g_0$ with corresponding Levi subalgebra $l_\zeta$ generated by $g_0^{\pm \alpha} (\pm \alpha \in \Phi_\zeta)$ and $h$.

For a given triangular decomposition of $g$

$$g = n^- \oplus h \oplus n, \quad (1.11)$$
that extends (1.8) (i.e., the even parts of $n^-$, $n$ in (1.11) are $n^0_0$, $n_0$ in (1.8), respectively), we define $b := \mathfrak{h} \oplus n$ and let $\mathcal{I}_{L;b}$ denote the set of elements $\zeta \in \mathcal{I}$ that gives a Levi subalgebra $l_\zeta$ in a parabolic decomposition $\mathfrak{g} = u_\zeta \oplus l_\zeta \oplus u_\zeta$ such that $\mathfrak{p}_\zeta := l_\zeta \oplus u_\zeta \supseteq b$. The $b$ can be viewed as the suitable Borel subalgebra for elements $\zeta \in \mathcal{I}_{L;b}$. For later use, we identify $\mathcal{I}$ with the set

$$\{\zeta \in n^* | \zeta([n_0, n_0]) = 0, \zeta(n_1) = 0\}.$$ 

We note that $\mathcal{I}$ coincide with the set of homomorphisms from $n$ to $C$ provided that $n_1$ is commutative.

1.5. Classification of simple Whittaker modules. We denote by $\mathfrak{g}$-Mod and $\mathfrak{g}_0$-Mod the category of all $\mathfrak{g}$-modules and $\mathfrak{g}_0$-modules, respectively. There is a category $\mathcal{W}(\mathfrak{g}, n)$ consisting of $\mathfrak{g}$-modules that are locally finite over $n$ as considered in [17]. To study simple modules in $\mathcal{W}(\mathfrak{g}, n)$, we propose a full subcategory $\mathcal{N}$ of $\mathcal{W}(\mathfrak{g}, n)$, which contains modules in the BGG category $\mathcal{O}$. That is, the category $\mathcal{N}$ consists of finitely-generated $\mathfrak{g}$-modules that are locally finite over $n$ and over the center $Z(\mathfrak{g}_0)$ of the universal enveloping algebra $U(\mathfrak{g}_0)$. This is thus the category of $\mathfrak{g}$-modules that are restricted to $\mathfrak{g}_0$-modules in the category $\mathcal{N}$ of [62]. We will prove that $\mathcal{N}$ and $\mathcal{W}(\mathfrak{g}, n)$ have the same collection of simple objects; see Proposition 1 and Remark 2. In the present paper, the modules in $\mathcal{N}$ are called Whittaker modules.

Let $\tilde{\mathcal{N}}(\zeta)$ be the full subcategory of $\mathcal{N}$ consisting of modules $M \in \mathcal{N}$ such that $x - \zeta(x)$ acts locally nilpotently on $M$, for any $x \in n_0$. In the case when $\mathfrak{g} = \mathfrak{g}_0$, we write $\mathcal{N}(\zeta)$ instead, which has been considered in [62, Section 1]. We then have a decomposition

$$\tilde{\mathcal{N}} = \bigoplus_{\zeta \in \mathcal{I}} \mathcal{N}(\zeta) \quad (1.12)$$

by [17, Theorem 3.2]. We remark that the decomposition (1.12) depends only on the triangular decomposition in (1.8), but $\mathcal{I}_{L;b}$ depends on the triangular decomposition (1.11) of $\mathfrak{g}$. To see this, we consider the Lie superalgebra $\mathfrak{g} = gl(1|2)$; see Sect. 5.3.1, where $\mathfrak{g}$ has a matrix realization with elementary matrices $E_{i,j}$ as its basis elements, for $1 \leq i, j \leq 3$. Let $b = \mathfrak{h} \oplus n$ be the Borel subalgebra such that $n$ is generated by $E_{12}$, $E_{13}$ and $E_{23}$. Then we have $\mathcal{I}_{L;b} = \mathcal{I}$. On the other hand, there is a Borel subalgebra $b^l = \mathfrak{h} \oplus n^l$ such that $n^l$ is generated by $E_{13}, E_{21}$ and $E_{23}$. Suppose that $\zeta \in \mathcal{I}_{L;b^l}$ with $\zeta \neq 0$. Then $l_{\zeta} = \mathfrak{g}_0$ is a Levi subalgebra in a parabolic decomposition (1.7) with the parabolic subalgebra $\mathfrak{p}_{\zeta}$ given by a certain vector $H = aE_{11} + bE_{22} + cE_{33}$. But $\mathfrak{p}_{\zeta} \supseteq b^l$ and so $Re(\epsilon_1 - \epsilon_3)(H), Re(\epsilon_2 - \epsilon_1)(H) > 0$ and $Re(\epsilon_2 - \epsilon_3)(H) = 0$ (see Sect. 5.3.1 for the definition) give inequalities $Re(c) = Re(b) > Re(a)$ and $Re(a) > Re(c)$, leading to a contradiction.

For a given $\zeta \in \mathcal{I}$, we will clarify in Sect. 3.3 the existence of a Borel subalgebra $b$ that is suitable for $\zeta$ (i.e., $\zeta \in \mathcal{I}_{L;b}$), for various classes of Lie superalgebras.

Following notations in [62], we denote the family of Kostant’s simple modules $Y_{\lambda, \eta}$ by $Y_{\xi}(\lambda, \zeta)$ (see (3.1) for the definition), where $\lambda \in \mathfrak{h}^*$ and $\zeta$ is a character on $n_0$. Following [62, Section 1], we define $M(\lambda, \zeta)$ as the $\mathfrak{g}_0$-module that is parabolically induced from $Y_{\xi}(\lambda, \zeta)$ along the parabolic decomposition (1.8).

For a Lie superalgebra $\mathfrak{g}$ in (1.1) with $\zeta \in \mathcal{I}$, Bagci, Christodouloupolou and Wiesner defined in [17, Section 4.2] the induced Whittaker modules as $\text{Ind}_{\mathfrak{g}_0 \oplus \mathfrak{g}_1}(M(\lambda, \zeta))$ (for
any $\lambda \in \mathfrak{h}^*$), which we will denote by $\widetilde{M}'(\lambda, \zeta)$ in the present paper for an arbitrary type-I Lie superalgebras; see (3.7). The authors studied the structure of the $\widetilde{M}'(\lambda, \zeta)$. More precisely, the authors established the following results (except for the case when \( g = \mathfrak{psl}(n|n) \) and \( \Phi_\zeta \) is the set of all simple roots in \( \mathfrak{n}_0^\perp \)):

1. The module $\widetilde{M}'(\lambda, \zeta)$ has finite length; see [17, Corollary 4.10].
2. The module $\widetilde{M}'(\lambda, \zeta)$ has a simple top; see [17, Proposition 4.16].
3. Let $\mu \in \mathfrak{h}^*$. Then $\widetilde{M}'(\lambda, \zeta) \cong \widetilde{M}'(\mu, \zeta)$ if and only if $Y_\zeta(\lambda, \zeta) \cong Y_\zeta(\mu, \zeta)$; see [17, Proposition 4.16].

In the present paper, we extend and generalize the work of [17] in the Theorem 9 for type-I Lie superalgebras and in the following theorem for arbitrary classical Lie superalgebras:

**Theorem A.** Let \( g \) be an arbitrary classical Lie superalgebra. Suppose that \( l_\zeta \) is a Levi subalgebra in a parabolic decomposition \( g = u^-_\zeta \oplus l_\zeta \oplus u_\zeta \) of \( g \). Then, for any \( \lambda \in \mathfrak{h}^* \), the \( g \)-module \( \widetilde{M}(\lambda, \zeta) \) that is parabolically induced from \( Y_\zeta(\lambda, \zeta) \) has simple top, denoted \( \widetilde{L}(\lambda, \zeta) \), and the correspondence gives rise to a bijection between the sets of isomorphism classes of simple \( l_\zeta \)-modules of the form \( Y_\zeta(\lambda, \zeta) \) and simple \( g \)-modules of \( \widetilde{N}(\zeta) \).

We note that \( \widetilde{M}(\lambda, \zeta) = M(\lambda, \zeta) \) in the case when \( g = \mathfrak{g}_0 \). Therefore, the modules \( \widetilde{M}(\lambda, \zeta) \) may be called standard Whittaker modules by analogy with the Whittaker modules of Lie algebras (see, e.g., [62, Section 1]). It turns out that the module \( \widetilde{M}'(\lambda, \zeta) \) in (3.7) can be viewed as a specific case of \( \widetilde{L}(\lambda, \zeta) \) given in Theorem A (see also (3.3)) in the case when \( g \) is one of \( \mathfrak{gl}(m|n), \mathfrak{osp}(2|2n) \) and \( \mathfrak{pe}(n) \) with (distinguished) Borel subalgebra \( b = \mathfrak{b}_0 \oplus \mathfrak{g}_1 \).

Suppose that \( g \) is an arbitrary classical Lie superalgebra of type I. It was established in [24, Theorem A] that the Kac induction functor \( K(-) := \text{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}}(-) \) gives rise to a bijection between simple \( \mathfrak{g} \)-modules and simple \( \mathfrak{g}_0 \)-modules. Naturally, it turns out that the functor \( K(-) \) provides a bijection between simple objects in \( \mathcal{N}(\zeta) \) and in \( \widetilde{N}(\zeta) \) without assuming that \( l_\zeta \) is a Levi subalgebra, leading to a classification of simple objects of the category \( \mathcal{W}(\mathfrak{g}, n) \) from [17] too; see Theorem 9.

### 1.6. Milićić-Soergel type equivalence.

In [62], Milićić and Soergel developed an equivalence between categories of Harish–Chandra bimodules and Whittaker modules to identify "singular" blocks of \( \mathcal{O} \) with "regular" blocks of \( \mathcal{N}(\zeta) \). Before giving the results, we recall several preparatory definitions and notions from [62, Section 5]. Let \( g \) be a reductive Lie algebra with \( \zeta \in \mathcal{T} \). Then \( l_\zeta \) is a Levi subalgebra of \( g \). Any module \( M \in \mathcal{N}(\zeta) \) decomposes into generalized eigenspaces \( M_{\chi^l_\mu} \) according to the central characters \( \chi^l_\mu \) of \( l := l_\zeta \) associated with the weight \( \mu \in \mathfrak{h}^* \). Let \( \Upsilon \subset \mathfrak{h}^* \) denote the set of integral weights, that is, weights appearing in finite-dimensional \( g \)-modules. For a given weight \( \lambda \in \mathfrak{h}^* \), define \( \Lambda := \lambda + \Upsilon \) and

\[
\mathcal{N}(\Lambda, \zeta) = \{ M \in \mathcal{N}(\zeta) | M_{\chi^l_\mu} = 0 \text{ unless } \chi^l_\mu = \chi^l_\mu, \text{ for some } \mu \in \Lambda \}. \tag{1.13}
\]

Then \( \mathcal{N}(\zeta) \) decomposes into blocks \( \mathcal{N}(\Lambda, \zeta) \), for cosets \( \Lambda \in \mathfrak{h}^*/\Upsilon \).

The study of Harish–Chandra bimodules for semisimple Lie algebras goes back at least to the work of Bernstein and Gelfand [10]. They established an equivalence of
the category $O$ and a category of Harish–Chandra bimodules; see [10, Theorem 5.9]. Miličić and Soergel established in [62, Theorem 5.1] an equivalence of $\mathcal{N}(\Lambda, \zeta)$ and a corresponding category of Harish–Chandra bimodules. An analogue of Miličić–Soergel equivalence was also given and used in [52]. We will provide a type of Miličić–Soergel equivalence for our setup of Lie superalgebras in the present paper.

The category $O$ can be defined in a natural way in the setup of Lie superalgebras; see [28,59]. In particular, we have $O \subset \tilde{\mathcal{N}}(0)$. In [60], Mazorchuk and Miemietz established an analogous equivalence of the category $O$ and a category of Harish–Chandra $(g, g)$-bimodules for Lie superalgebra $g$. More recently, a similar version of equivalence of the category $O$ and a category of Harish–Chandra $(g, g_0)$-bimodules has been established in [22, Theorem 3.1], which turns out to be a powerful tool in the study of primitive spectrum of the periplectic Lie superalgebras $pe(n)$. With the Miličić–Soergel type equivalence developed in the present paper, we realize the annihilator ideals of simple Whittaker modules as annihilators of certain modules coming from the category $O$; see Sect. 5.5.

Consider a classical Lie superalgebra $g$ with $\zeta \in I$. We define the Weyl group $W$ of $g$ to be the Weyl group of $g_0$. Let $\lambda \in h^*$ be dominant under the dot-action of elements in $W$. Denote by $M(\lambda)$ the Verma module over $g_0$ of highest weight $\lambda$ with respect to the triangular decomposition (1.8). We set $B_0$ to be the full subcategory of Harish–Chandra $(g, g_0)$-bimodules consisting of objects that are annihilated by some power of the annihilator of $M(\lambda)$. For a given dominant weight $\mu \in h^*$, the stabilizer of $\mu$ under the dot-action of $W$ is denoted by $W_\mu$. Also, we set $W_\zeta \subseteq W$ to be the Weyl group of $I_\zeta$, for any $\zeta \in I$.

There is a version of the category $\tilde{\mathcal{N}}(\lambda, \zeta)$ adapted to our situation of classical Lie superalgebras, which we denote by $\tilde{\mathcal{N}}(\lambda, \zeta)$; see Sect. 4.1.2. The following is our second main result:

**Theorem B.** Let $\zeta \in I$ be such that $W_\lambda = W_\zeta$. Then $\tilde{\mathcal{N}}(\lambda, \zeta)$ and $B_\lambda$ are equivalent.

We remark some consequences: if $\eta \in I$ satisfies that $W_\lambda = W_\zeta = W_\eta$ then $\tilde{\mathcal{N}}(\lambda + \gamma, \zeta) \cong \tilde{\mathcal{N}}(\lambda + \gamma, \eta)$. Also, if $s \in W$ is a simple reflection such that $s$ does not lie in the integral Weyl group of $\lambda$, then we have $\tilde{\mathcal{N}}(\lambda + \gamma, \zeta) \cong \tilde{\mathcal{N}}(s \cdot \lambda + \gamma, \eta)$, for any $\eta \in I$ with $W_\eta = W_{s \cdot \lambda}$.

### 1.7. Multiplicities of standard Whittaker modules

The study of simplicity of standard Whittaker modules for Lie superalgebras of type I was first initiated in [17, Section 5.1], where the case of the special linear Lie superalgebra $sl(2|1)$ with non-trivial $\zeta$ was considered. In particular, the authors concluded in [17, Proposition 5.16] that the corresponding standard Whittaker module $\tilde{M}(\lambda, \zeta)$ is simple for arbitrary weight $\lambda$, which turns out to be not true in the case when $\lambda$ is atypical (see (5.9) for the definition). In the present paper, we shall correct this claim and give the complete composition factors of $\tilde{M}(\lambda, \zeta)$, for any weights $\lambda$.

Recall that the category $O$ for classical Lie superalgebras admit structures of highest weight category (see, e.g., [15,31]) with Verma modules $\tilde{M}(\lambda)$ indexed by $\lambda \in h^*$ as standard objects. The following is our third main result:

**Theorem C.** Suppose that $g$ is a classical Lie superalgebra of type I. Then for any $\lambda, \mu \in h^*$ and $\zeta \in I$, we have

$$ [\tilde{M}(\lambda, \zeta) : \tilde{L}(\mu, \zeta)] = \sum_{\nu} [\tilde{M}(\lambda) : \tilde{L}(\nu)], $$

(1.14)

where the summation runs over all $n_\zeta$-antidominant weights $\nu$ such that $\mu \in W_\zeta \cdot \nu$. 
As a consequence, the composition factors of standard Whittaker modules over the
general linear Lie superalgebra $\mathfrak{gl}(m|n)$ and the ortho-symplectic Lie superalgebras
$\mathfrak{osp}(2|2n)$ can be computed by recent works on the irreducible characters of the BGG
category $\mathcal{O}$; see, e.g., [4, 5, 16, 35–37].

All results in the paper for Lie superalgebras of type I are valid for the (distinguished)
triangular decompositions (1.7) with commutative $\mathfrak{n}_1^-,\mathfrak{n}_1$.

1.8. Structure of the paper. The paper is organized as follows. In Sect. 2, we provide
some background materials on classical Lie superalgebras and Whittaker modules. In
Sect. 3, we obtain a classification of standard and simple Whittaker modules for classical
Lie superalgebras in terms of their parabolic decompositions. The proof of Theorem A
will be given in Sect. 3.1. For classical Lie superalgebras of type I, an alternative defi-
nition of standard Whittaker modules will be introduced in Sect. 3.2 that are to be used
in the sequel. In this case, we will classify simple objects of $\tilde{\mathcal{N}}$ in full generality.

In Sect. 4, we review Harish–Chandra bimodules, cokernel categories and Miličić–
Soergel equivalence, including several essential ingredients for our main results. Ap-
plying tools in [22] and generalizing [62], we will establish in Sect. 4.3 the equivalence
stated in Theorem B.

In Sect. 5, we focus on the multiplicity problem of standard Whittaker modules for
classical Lie superalgebras of type I. Section 5.2 is devoted to the proof of Theorem C.
One can also find a detailed example of the general linear Lie superalgebra
$\mathfrak{gl}_2$ given in Sects. 5.2, 5.3 and 5.4. Also, we obtain in Sect. 5.4 the composition factors of standard Whittaker modules $\tilde{M}(\lambda, \zeta)$ over
$\mathfrak{pe}(n)$, for typical weights $\lambda$ defined in (5.18).

In Sect. 5.5, we discuss some connections among modules in $\mathcal{O}, \tilde{\mathcal{N}}$ and Harish–
Chandra ($\mathfrak{g}, \mathfrak{g}_0$)-bimodules. We will describe the annihilators ideals of simple Whittaker
modules in terms of that of certain modules in $\mathcal{O}$. Also, for $\mathfrak{g} = \mathfrak{gl}(m|n), \mathfrak{osp}(2|2n)$, we
reduce the problem of composition factors in certain standard Harish–Chandra bimodu-
ules to Kazhdan–Lusztig combinatorics in Sect. 5.5.

2. Preliminaries

2.1. Notations. Let $\mathfrak{g}$ be a finite-dimensional complex classical Lie superalgebra. De-
note by $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$. Let $Z(\mathfrak{g})$ be the center of $U(\mathfrak{g})$. For
a given $\lambda \in \mathfrak{h}^*$, we set $\chi_\lambda^0 : Z(\mathfrak{g}) \to \bar{\mathbb{C}}$ to be the central character associated with $\lambda$.

We sometimes write $\chi_\lambda^0$ instead of $\chi_{\lambda,0}^0$. Recall that we fixed a triangular decomposition
$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$ with even Cartan subalgebra $\mathfrak{h} = \mathfrak{h}_0$.

The Weyl group $W$ of $\mathfrak{g}$ is by definition the Weyl group $W$ of $\mathfrak{g}_0$ with its defining
action on $\mathfrak{h}^*$. The usual dot-action of $W$ on $\mathfrak{h}^*$ is defined as $w \cdot \lambda := w(\lambda + \rho_0) - \rho_0$,
for any $w \in W$ and $\lambda \in \mathfrak{h}^*$, where $\rho_0$ is the half of the sum of all positive roots of
$\mathfrak{g}_0$. A weight is called integral, dominant or anti-dominant if it is integral, dominant
or anti-dominant as a $\mathfrak{g}_0$-weight, respectively. We recall that $W_\lambda$ denotes the stabilizer
of $\lambda$ under the dot-action of $W$, for any weight $\lambda \in \mathfrak{h}^*$. For any $\zeta \in \mathcal{I}$, we denote by
$W_\zeta \subseteq W$ the Weyl group of $I_\zeta$, which naturally acts on $\mathfrak{h}^*$ via consistent dot-action. For
any weight module $M$, we set $\mathcal{P}(M)$ to be the set of all weights in $M$. Finally, we denote
by $\Upsilon \subseteq \mathfrak{h}^*$ the set of integral weights. Then we have $\mathbb{Z}\Phi \subseteq \Upsilon$. 
2.2. Induction, coinduction and restriction functors. Let \( \mathfrak{g} \) be a Lie superalgebra. For a subalgebra \( \mathfrak{s} \subset \mathfrak{g} \), we denote by \( \text{Res}_{\mathfrak{s}}^{\mathfrak{g}} \) the restriction functor from \( \mathfrak{g} \) to \( \mathfrak{s} \). We have exact induction and coinduction functors

\[
\text{Ind}_{\mathfrak{s}}^{\mathfrak{g}}(-) = U(\mathfrak{g}) \otimes_{U(\mathfrak{s})} - \quad \text{and} \quad \text{Coind}_{\mathfrak{s}}^{\mathfrak{g}}(-) = \text{Hom}_{U(\mathfrak{s})}(U(\mathfrak{g}), -).
\]

They are left and right adjoint functors to \( \text{Res}_{\mathfrak{s}}^{\mathfrak{g}} \). If \( \mathfrak{s} \) contains \( \mathfrak{t}_0 \), then [9, Theorem 2.2] (see also [41]) implies that \( \text{Ind}_{\mathfrak{t}}^{\mathfrak{g}}(-) \cong \text{Coind}_{\mathfrak{t}}^{\mathfrak{g}}(\Lambda^{\text{top}}(\mathfrak{t}/\mathfrak{s}) \otimes -) \).

We will use the undecorated notations

\[
\text{Ind}, \text{Coind} : \mathfrak{g}_0\text{-Mod} \to \mathfrak{g}\text{-Mod}, \quad \text{Res} : \mathfrak{g}\text{-Mod} \to \mathfrak{g}_0\text{-Mod},
\]

to refer to these functors when \( \mathfrak{g} = \mathfrak{g} \) and \( \mathfrak{s} = \mathfrak{g}_0 \).

In particular, the functors \( \text{Ind} \) and \( \text{Coind} \) are isomorphic, up to the equivalence given by tensoring with the one-dimensional \( \mathfrak{g}_0 \)-module \( \Lambda^{\text{top}}\mathfrak{g}_1 \) on the top degree subspace of \( \Lambda\mathfrak{g}_1 \).

**Proposition 1.** Suppose that \( S \) is a simple \( \mathfrak{g} \)-module. Then \( \text{Res} S \) is locally finite over \( Z(\mathfrak{g}_0) \).

**Proof.** We adapt the argument in [24, Lemma 4.2] to complete the proof. Since \( U(\mathfrak{g}) \) is a finitely-generated \( U(\mathfrak{g}_0) \)-module, the \( \mathfrak{g}_0 \)-module \( \text{Res} S \) is finitely-generated. Therefore \( \text{Res} S \) has a simple quotient \( V \). By adjunction we have

\[
\text{Hom}_{\mathfrak{g}}(S, \text{Coind} V) = \text{Hom}_{\mathfrak{g}_0}(\text{Res} S, V) \neq 0,
\]

which implies that \( S \) is a submodule of \( \text{Ind} W \), where \( W := \Lambda^{\text{top}}\mathfrak{g}_1^* \otimes V \) is a simple \( \mathfrak{g}_0 \)-module. We note that the module \( \text{Res} \text{Ind} W \cong U(\mathfrak{g}_1) \otimes W \) is locally finite over \( Z(\mathfrak{g}_0) \) by [10, Section 2.6]. This completes the proof. \( \square \)

**Remark 2.** To compare simple objects of the category \( \mathcal{W}(\mathfrak{g}, n) \) from [17] to that of our category \( \tilde{\mathcal{N}} \), we note that any simple \( \mathfrak{g} \)-module that is locally finite over \( n \) lies in \( \tilde{\mathcal{N}} \) by Proposition 1. Namely, we remark that \( \mathcal{W}(\mathfrak{g}, n) \) and \( \tilde{\mathcal{N}} \) have the same collection of simple objects.

**Lemma 3.** The functors \( \text{Ind}, \text{Coind} \) and \( \text{Res} \) restrict to exact functors between \( \mathcal{N} \) and \( \tilde{\mathcal{N}} \):

\[
\text{Ind}, \text{Coind} : \mathcal{N} \to \tilde{\mathcal{N}}, \quad \text{Res} : \tilde{\mathcal{N}} \to \mathcal{N}.
\]

**Proof.** Let \( M \in \tilde{\mathcal{N}} \). Then \( M \) is a finite sum of cyclic \( U(\mathfrak{g}) \)-submodules. We note that \( U(\mathfrak{g}) \) is finitely-generated over \( U(\mathfrak{g}_0) \) since \( U(\mathfrak{g}_1) \) is finite-dimensional. This means that \( \text{Res} M \in \mathcal{N} \).

Conversely, assume that \( V \in \mathcal{N} \). Then \( \text{Res} \text{Ind} V \cong U(\mathfrak{g}_1) \otimes V \) is locally finite over \( Z(\mathfrak{g}_0) \) (see, e.g., [10, Section 2.6]). Also, \( U(\mathfrak{g}_1) \otimes V \) is locally finite over \( n_0 \), and so \( \text{Ind} V \in \tilde{\mathcal{N}} \). \( \square \)

As a consequence of Lemma 3, we have the following corollary.

**Corollary 4.** Every object in \( \tilde{\mathcal{N}} \) has finite length.

**Proof.** By [62, Theorem 2.6], every object in \( \mathcal{N} \) has finite length (see also [56] and [25, Corolloary 4]). The conclusion follows. \( \square \)

For any \( \zeta \in \mathcal{I} \), we define the following blocks (see also [62, Section 1]):

\[
\mathcal{N}(\zeta) := \{ N \in \mathcal{N} \mid x - \zeta(x) \text{ acts locally nilpotently on } N, \text{ for any } x \in n_0 \}, \quad (2.1)
\]

\[
\tilde{\mathcal{N}}(\tilde{\zeta}) := \{ M \in \tilde{\mathcal{N}} \mid x - \zeta(x) \text{ acts locally nilpotently on } M, \text{ for any } x \in n_0 \}. \quad (2.2)
\]

We may observe that \( \text{Ind}, \text{Coind} \) and \( \text{Res} \) restrict to well-defined functors between \( \mathcal{N}(\zeta) \) and \( \tilde{\mathcal{N}}(\tilde{\zeta}) \).
3. Simple and Standard Whittaker Modules

In this section, we define the various generalizations of standard Whittaker modules and study their fundamental properties in our setup. We will classify the simple Whittaker modules in terms of parabolic decompositions for an arbitrary classical Lie superalgebra. For Lie superalgebras of type I, we provide a complete classification of simple Whittaker modules using the Kac functors.

3.1. Simple Whittaker modules: arbitrary classical Lie superalgebras. Let \( g \) be an arbitrary finite-dimensional complex classical Lie superalgebra. We recall the set \( I := \{ \zeta \in n^* \mid \zeta([n_0, n_0]) = 0, \zeta(n_{\bar{1}}) = 0 \} \) as defined in Sect. 1.4. For each \( \zeta \in I \), we denote by \( l_\zeta = n_{\bar{1}} \oplus h \oplus n_\zeta \) the corresponding triangular decomposition of \( l_\zeta \).

For any \( \lambda \in h^* \), let \( \text{Ker}(\chi_{l_\zeta}^\zeta) \) be the kernel of the central character \( \chi_{l_\zeta}^\zeta \) of \( l_\zeta \). Since elements of \( \text{Ker}(\chi_{l_\zeta}^\zeta) \) are central in \( U(l_\zeta) \), it follows that \( \text{Ker}(\chi_{l_\zeta}^\zeta) \) is a \((l_\zeta, l_\zeta)\)-bimodule and thus a \((l_\zeta, n_\zeta)\)-bimodule. We recall that Kostant’s simple Whittaker modules are defined as follows

\[
Y_{\zeta}(\lambda, \zeta) := U(l_\zeta)/\text{Ker}(\chi_{l_\zeta}^\zeta) U(l_\zeta) \otimes U(n_\zeta) \mathbb{C}_\zeta,
\]

(3.1)

where \( \mathbb{C}_\zeta \) is the one-dimensional \( n_\zeta \)-module associated with \( \zeta \). The isomorphism \( Y_{\zeta}(\lambda, \zeta) \cong Y_{\zeta}(\mu, \zeta) \) holds if and only if \( W_\zeta \cdot \lambda = W_\zeta \cdot \mu \).

Suppose that \( l_\zeta \) is the Levi subalgebra in a parabolic decomposition \( g = u_{\bar{1}} \oplus l_\zeta \oplus u_\zeta \) of \( g \). The standard Whittaker modules over \( g_{\bar{1}} \) and \( g \) are respectively defined as

\[
M(\lambda, \zeta) := U(g_{\bar{1}}) \otimes_{p_{\bar{1}}} Y_{\zeta}(\lambda, \zeta),
\]

(3.2)

\[
\tilde{M}(\lambda, \zeta) := U(g) \otimes_{p} Y_{\zeta}(\lambda, \zeta),
\]

(3.3)

where \( p := p_{\bar{1}} = l_\zeta \oplus u_\zeta \) is the corresponding parabolic subalgebra. The module \( M(\lambda, \zeta) \in \mathcal{N} \) has been studied in [56, 62] (see also [3]).

The following result is proved in [62, Proposition 2.1].

**Lemma 5** (Miličić-Soergel). For any \( \lambda \in h^* \) and \( \zeta \in I \), the standard Whittaker \( g_{\bar{1}} \)-module \( M(\lambda, \zeta) \) has simple top \( L(\lambda, \zeta) \). Let \( \mu \in h^* \), then

\[
L(\lambda, \zeta) \cong L(\mu, \zeta) \Longleftrightarrow M(\lambda, \zeta) \cong M(\mu, \zeta) \Longleftrightarrow W_\zeta \cdot \lambda = W_\zeta \cdot \mu.
\]

Every simple module in \( \mathcal{N}(\zeta) \) is of the form \( L(\lambda, \zeta) \), for \( \lambda \in h^* \).

We are now in a position to prove our first main result. The conclusion of Theorem B is a consequence of the following theorem.

**Theorem 6.** Let \( \zeta \in I \). Suppose that \( l_\zeta \) is a Levi subalgebra in a parabolic decomposition \( g = u_{\bar{1}} \oplus l_\zeta \oplus u_\zeta \). Then we have

1. \( \tilde{M}(\lambda, \zeta) \) has simple top, which is denoted by \( \tilde{L}(\lambda, \zeta) \), for each \( \lambda \in h^* \).
2. \{\( \tilde{L}(\lambda, \zeta) \mid \lambda \in h^* \}\} is the complete list of simple modules in \( \tilde{\mathcal{N}}(\zeta) \).
3. For any \( \lambda, \mu \in h^* \), the following are equivalent:
   a. \( \tilde{M}(\lambda, \zeta) \cong M(\mu, \zeta) \).
   b. \( \tilde{L}(\lambda, \zeta) \cong \tilde{L}(\mu, \zeta) \).
   c. \( W_\zeta \cdot \lambda = W_\zeta \cdot \mu \).
Proof. We first claim that $\widehat{M}(\lambda, \xi) \in \widehat{N}$. To see this, we may observe that $\text{Res} \widehat{M}(\lambda, \xi)$ is an epimorphic image of the $g_0$-module $U(g) \otimes M(\lambda, \xi)$ by the Poincaré–Birkhoff–Witt basis theorem. Therefore $\widehat{M}(\lambda, \xi)$ is locally finite over $Z(g_0)$ (see, e.g., [10, Section 2.6]) and over $n$. We may conclude that $\widehat{M}(\lambda, \xi) \in \widehat{N}$ since it is generated by any non-zero vector of $Y_\xi(\lambda, \xi) \subset \text{Res} \widehat{M}(\lambda, \xi)$.

Next, we shall proceed with an argument similar to the proof of [62, Proposition 2.1]. We may note that $H \in \bigcap_{\alpha \in \Phi_\xi} \text{Ker}(\alpha)$ and so $H$ acts on $Y_\xi(\lambda, \xi)$ via $\lambda(H)$. Therefore $\widehat{M}(\lambda, \xi)$ decomposes into eigenspaces $\tilde{M}(\lambda, \xi)_k$ with $k \in \lambda(H) + \sum_{\alpha \in P(u^-)} \mathbb{Z}_{\geq 0} \alpha(H)$ according to the eigenvalues of the action $H$. Since $\alpha(H) < 0$ for any $\alpha \in P(u^-)$, it follows that $\tilde{M}(\lambda(\xi), \lambda(H)) = Y_\xi(\lambda, \xi)$. Also, all the $\tilde{M}(\lambda, \xi)_k$ are $l_\xi$-submodules since $\alpha(H) = 0$ for any $\alpha \in \Phi_\xi$.

Let $N$ be a proper submodule of $\tilde{M}(\lambda, \xi)$, then $N$ decomposes $N = \bigoplus_k N_k$ with $l_\xi$-submodules $N_k \subseteq \tilde{M}(\lambda, \xi)_k$ according to the eigenvalues $k$ of $H$ acting on $N$. Since $Y_\xi(\lambda, \xi)$ is simple, we may conclude that $N_{\lambda(H)} = 0$. Therefore $\tilde{M}(\lambda, \xi)$ has a unique maximal submodule. This proves Part (1). We denote the simple top of $\tilde{M}(\lambda, \xi)$ by $\tilde{L}(\lambda, \xi)$.

Next we prove Part (2). Recall that we put $p := p_\xi$. Let $S \in \widehat{N}(\xi)$ be a simple module. Since $S$ has finite length, there exists $\mu \in h^*$ such that $L(\mu, \xi) \hookrightarrow \text{Res} \ S$. Therefore we have
\[
\text{Hom}_g(\text{Ind} M(\mu, \xi), S) = \text{Hom}_{g_0}(M(\mu, \xi), \text{Res} \ S) \neq 0. \tag{3.4}
\]

The $l_\xi$-module $\text{Res}^p_{l_\xi} \text{Ind}^p_{g_0} Y_\xi(\mu, \xi) \cong U(p_1) \otimes Y_\xi(\lambda, \xi)$ have composition factors of the form $Y_\xi(\lambda + \nu, \xi)$, for $\nu \in P(U(p_1))$ (see, e.g., [50, Theorem 4.6]). Note that $\nu(H) > 0$, for any $\nu \in P(U(p_1))$. Therefore $\mu$ has a filtration with simple subquotients $Y_\xi(\lambda + \nu, \xi)$, which are annihilated by $u_\xi$. We may observe that
\[
\text{Ind} M(\mu, \xi) = \text{Ind}^g_{g_0} \text{Ind}^p_{g_0} Y_\xi(\mu, \xi) \cong \text{Ind}^p \text{Ind}^p_{g_0} Y_\xi(\mu, \xi), \tag{3.5}
\]

which implies that $\text{Ind} M(\mu, \xi)$ admits a filtration of standard Whittaker modules. Consequently, $S$ is a composition factor of $\tilde{M}(\lambda, \xi)$ for some $\lambda \in h^*$. It remains to show that every composition factor of $\tilde{M}(\lambda, \xi)$ is of the form $\tilde{L}(\lambda', \xi)$, for $\lambda' \in h^*$.

Let $S$ be a composition factor of $\tilde{M}(\lambda, \xi)$. Again, under the action of $H$ the $S$ decomposes into eigenspaces $S_k$ with $k \in \lambda(H) + \sum_{\alpha \in P(u^-)} \mathbb{Z}_{\geq 0} \alpha(H)$. Since $u_\xi S_k \subseteq S_{k'}$ with $k < k'$, we may conclude that there exists an eigenvalue $m$ such that $S_m \neq 0$ and $u_\xi S_m = 0$. Since $S_m$ is a $l_\xi$-submodule of $\tilde{M}(\lambda, \xi)$, it follows from [50, Theorem 4.6] that $S_m$ has a simple submodule $Y_\xi(\gamma, \xi)$, for some $\gamma \in h^*$. Consequently, we have
\[
\text{Hom}_g(\tilde{M}(\gamma, \xi), S) = \text{Hom}_{p_\xi}(Y_\xi(\gamma, \xi), \text{Res}^g_{p_\xi} S) \neq 0. \tag{3.6}
\]

Therefore $S \cong \tilde{L}(\gamma, \xi)$. This proves Part (2).

We have known $(c) \Rightarrow (a)$ already. We now prove $(a) \Rightarrow (b)$. If $\tilde{M}(\lambda, \xi) \cong \tilde{M}(\mu, \xi)$ then $\tilde{M}(\lambda(\xi), \lambda(H)) = Y_\xi(\lambda, \xi)$ and $\tilde{M}(\mu(\xi), \mu(H)) = Y_\xi(\mu, \xi)$ are isomorphic as $l_\xi$-modules, and so $W_\xi \cdot \lambda = W_\xi \cdot \mu$ by Lemma 5.

Finally, we prove the direction $(b) \Rightarrow (c)$. Again, $\tilde{L}(\lambda, \xi)$ decomposes into eigenspaces $\tilde{L}(\lambda, \xi)_k$ with $k \in \lambda(H) + \sum_{\alpha \in P(u^-)} \mathbb{Z}_{\geq 0} \alpha(H)$ according to the eigenvalues of the action of $H$ on $\tilde{L}(\lambda, \xi)$. Thus, we have $\tilde{L}(\lambda, \xi)_\lambda(H) = Y_\xi(\lambda, \xi)$ and $\tilde{L}(\mu, \xi)_\mu(H) = Y_\xi(\mu, \xi)$, which implies $W_\xi \cdot \lambda = W_\xi \cdot \mu$ by Lemma 5. This completes the proof. \(\Box\)
Remark 7. We will discuss why the assumption that \( l_\zeta \) is a Levi subalgebra of \( g \) is essential for results given in this section. The way this assumption come into play originates in the method of parabolic induction functors, which makes a connection between Kostant’s Whittaker modules \( Y_\zeta (\lambda, \zeta) \) over \( l_\zeta \) and simple Whittaker modules \( L(\lambda, \zeta) \) over \( g_0 \) as employed in [56,57] and [63]. The simple Whittaker modules \( L(\lambda, \zeta) \) are then realized and classified via taking the simple tops of the modules that are parabolically induced from \( Y_\zeta (\lambda, \zeta) \) (i.e. they are the standard modules \( M(\lambda, \zeta) \)). As has been proved in Theorem A, these results generalizes to Lie superalgebras. Such a method goes back at least to the work of Verma on the construction of Verma modules, which are used in the classification of simple modules in the category \( \mathcal{O} \).

Now, suppose that \( l_\zeta \) is not a Levi subalgebra of \( g \). It is not clear how to construct analogous functors. In this case, it is natural to make a connection between Whittaker modules over \( g_0 \) and over \( g \) directly. If we use the full induction functor \( \text{Ind}(\zeta) \), then we still can use the adjunction between \( \text{Ind}(\zeta) \) and \( \text{Res}(\zeta) \) to realize simple Whittaker modules as simple quotients of \( \text{Ind} M(\lambda, \zeta) \). However, the modules \( \text{Ind} M(\lambda, \zeta) \) does not necessarily have simple tops. For instance, \( \text{Ind} M(\lambda, 0) \approx \text{Ind} M(\lambda) \) are not necessarily indecomposable.

We note that, if the parabolic induction functor exists (i.e., \( l_\zeta \) is a Levi subalgebra of \( g \)), then the standard Whittaker module \( \tilde{M}(\lambda, \zeta) \) can be regarded as a quotient of \( \text{Ind} M(\lambda, \zeta) \approx \text{Ind}_{g_0}^{g} \text{Ind}_{p_\zeta}^{g_\zeta} Y_\zeta (\lambda, \zeta) \). For a Lie superalgebra \( g = g_{-1} \oplus g_0 \oplus g_1 \) of type I with an arbitrary \( \zeta \in \mathcal{I} \), the (Kac) induced module \( K(M(\lambda, \zeta)) = \text{Ind}_{g_0}^{g} \text{Ind}_{g_0 \oplus g_1}^{g} M(\lambda, \zeta) \) is a quotient of \( \text{Ind}(M(\lambda, \zeta)) \) as well. In Sect. 3.2, we will develop a method to obtain analogous results using the Kac induction functor instead of the parabolic induction functor.

In Sect. 3.3, we will discuss which cases of \( \zeta \in \mathcal{I} \) are left to address, for various Lie superalgebras given in (1.2)–(1.3)

### 3.2. Simple Whittaker modules: Lie superalgebras of type I

In this subsection, we let \( g = g_{-1} \oplus g_0 \oplus g_1 \) be a finite-dimensional complex classical Lie superalgebra of type I. We will redefine the standard Whittaker modules in this case, leading to a complete classification of simple Whittaker \( g \)-modules. The advantage is that we do not need to assume that \( l_\zeta \) is a Levi subalgebra in a parabolic decomposition of \( g \).

For a given \( g_0 \)-module \( V \), we can extend \( V \) trivially to a \( g_0 \oplus g_1 \)-module and define the Kac module of \( V \) as \( K(V) := \text{Ind}_{g_0}^{g} V \), where \( g_{\geq 0} := g_0 \oplus g_1 \). Then this defines an exact functor \( K(\cdot) : g_0^{-}\text{-Mod} \rightarrow g^{-}\text{-Mod} \), which we call Kac functor (see also [24, Sections 2.4, 3]). We observe that \( K(V) \cong \Lambda(g_{-1}) \otimes V \) as vector spaces.

Throughout this subsection, for any \( \lambda \in h_0^* \) and \( \zeta \in \mathcal{I} \) we define the standard Whittaker module for type I Lie superalgebra as

\[
\tilde{M}'(\lambda, \zeta) := K(M(\lambda, \zeta)),
\]

where \( M(\lambda, \zeta) \) is the standard Whittaker module over \( g_0 \) as in (3.2). We remark that all results in the present paper are valid for type-I Lie superalgebras for the (distinguished) triangular decomposition (1.7) with commutative \( n_1^0 \), \( n_1 \). In particular, the definition (3.7) can be viewed as special cases of (3.3) when \( g \) is one of \( gl(m|n) \), \( osp(2|2n) \) and \( pe(n) \) with \( (u_\zeta)_1 = g_{-1}, (u_\zeta)_{0-1} = g_1 = b_1 \) (because \( \tilde{M}(\lambda, \zeta) = U(g) \otimes_p Y_\zeta (\lambda, \zeta) \approx \text{Ind}_{g_0}^{g} \text{Ind}_{p_\zeta}^{g_\zeta} Y_\zeta (\lambda, \zeta) = \tilde{M}'(\lambda, \zeta) \)); see Sect. 3.3.1.

In general, the standard Whittaker modules \( \tilde{M}'(\lambda, \zeta) \) in (3.7) and \( \tilde{M}(\lambda, \zeta) \) in (3.3) are not necessarily isomorphic. For instance, we consider \( g = gl(1|2) \); see Sect. 5.3.2.
There are two distinguished Borel subalgebras $b$ and $b'$ and a (non-distinguished) Borel subalgebra $b^1$ such that they have the same even parts, $b_1 = g_1$, $b'_1 = g_{-1}$ and $b^1_1 \neq g_{\pm 1}$. Now, we consider $\zeta = 0$. Then $l_{\xi} = l_0 = h$ is a Levi subalgebra in the triangular decompositions given by $b$, $b'$ and $b^1$, respectively. If we set $p_{\xi} = p_0$ to be $b^1$, then $\tilde{M}(\lambda, 0)$ and $\tilde{M}'(\lambda, 0)$ are respectively the Verma modules of highest weight $\lambda$ with respect to $b^1$ and $b$, which are non-isomorphic. A similar argument can be used to show that these standard Whittaker modules are non-isomorphic when we switch $g_1$ to $g_{-1}$ from the definition 3.7.

**Lemma 8.** The $Kac$ functor $K(-)$ defines an exact functor from $\mathcal{N}$ to $\tilde{\mathcal{N}}$.

**Proof.** Let $M \in \mathcal{N}$. We note that Res $K(M) \cong \Lambda(g_{-1}) \otimes M$ as $g_0$-modules. Therefore $K(M)$ is locally finite over $n$. It follows from [10, Section 2.3, 2.6] that $K(M)$ is a finitely-generated $g$-module and is locally finite over $Z(g_0)$, proving the claim. \[ \square \]

The following theorem is an analog of Theorem 6, but we do not need to assume that $l_{\xi}$ is a Levi subalgebra of $g$.

**Theorem 9.** Let $\zeta \in \mathcal{I}$. Then we have

1. $\tilde{M}'(\lambda, \zeta)$ has simple top, which is denoted by $\tilde{L}(\lambda, \zeta)$, for each $\lambda \in h^*_0$.
2. $\{\tilde{L}(\lambda, \zeta) \mid \lambda \in h^*_0\}$ is the complete list of simple modules in $\tilde{\mathcal{N}}(\zeta)$.
3. For any $\lambda, \mu \in h^*_0$, the following are equivalent:
   (a) $\tilde{M}'(\lambda, \zeta) \cong M'(\mu, \zeta)$.
   (b) $L(\lambda, \zeta) \cong L(\mu, \zeta)$.
   (c) $W_{\xi} \cdot \lambda = W_{\xi} \cdot \mu$.

**Proof.** For any $M \in g$-Mod, we define $M_{\mathbb{Z}\pm 1}$ to be the subspaces of $g_{\pm 1}$-invariants, namely, $M_{\mathbb{Z}_{\pm 1}} := \{m \in M \mid g_{\pm 1}m = 0\}$. We shall first adapt the argument in [24, Lemma 4.4] to complete the proof of Part (1). Suppose that $\phi : M'(\lambda, \zeta) \rightarrow S$ is a simple quotient of $M'(\lambda, \zeta)$. By [24, Theorem 4.1 (ii)] there is a simple $g_0$-module $V$ such that

$$S \hookrightarrow \text{Ind}^{g_0}_{g_{-1}} V.$$  \[ (3.8) \]

By definition the $g_0$-module $M(\lambda, \zeta)$ can be regarded as a $g_0$-submodule of Res $\tilde{M}'(\lambda, \zeta)$. Note that $\phi(M(\lambda, \zeta)) \subseteq \Lambda_{\text{top}} g_1 \otimes V$ since $g_1 \cdot M(\lambda, \zeta) = 0$ and the subspace of $g_1$-invariants of $\text{Ind}^g_{g_{0} + g_{-1}} V$ is $\Lambda_{\text{top}} g_1 \otimes V$. But $\Lambda_{\text{top}} g_1 \otimes V$ is a simple $g_0$-module, we may conclude that $\Lambda_{\text{top}} g_1 \otimes V \cong L(\lambda, \zeta)$ by Lemma 5. Therefore we obtain that $V \cong \Lambda_{\text{top}} g_1 \otimes L(\lambda, \zeta)$.

Finally, using adjunction and Schur’s lemma, it follows that

$$\dim \text{Hom}_g(M'(\lambda, \zeta), \text{Ind}^g_{g_{0} + g_{-1}} V) = \dim \text{Hom}_g(M(\lambda, \zeta), \Lambda_{\text{top}} g_1 \otimes V) = 1,$$

by Lemma 5. This proves Part (1).

We are going to prove (2). Let $S \in \tilde{\mathcal{N}}(\zeta)$ be a simple module. We claim that the subspace $S^{g_1}$ of $g_1$-invariant elements is a simple $g_0$-module in $\mathcal{N}(\zeta)$. It suffices to show that $S^{g_1}$ is simple. To see this, we note that $S$ is isomorphic to the socle of $\text{Ind}^g_{g_{0} + g_{-1}} V$, for some simple $g_0$-module $V$ by [24, Corollary 4.3]. By [24, Lemma 3.2], the simple $g_0$-submodule $(\text{Ind}^g_{g_{0} + g_{-1}} V)^{g_1} = \Lambda_{\text{top}} g_1 \otimes V$ generates the simple socle $S \cong \text{soc}(\text{Ind}^g_{g_{0} + g_{-1}} V)$ of $\text{Ind}^g_{g_{0} + g_{-1}} V$ as a $g$-submodule. It follows that $S^{g_1}$ is simple. By Lemma 5, there exists $\lambda' \in h^*_0$ such that $S^{g_1} \cong L(\lambda', \zeta)$. It follows from [24, Theorem 4.1] that $S$ is the quotient of $\tilde{M}'(\lambda', \zeta)$, that is, $S \cong \tilde{L}(\lambda', \zeta)$ by definition.
Finally, it remains to show Part (3). For any $\lambda \in h^*$, since $\tilde{L}(\lambda, \zeta)$ is the simple top of $K(L(\lambda, \zeta))$, we know that $\tilde{L}(\lambda, \zeta) \cong \tilde{L}(\mu, \zeta)$ if and only if $L(\lambda, \zeta) \cong L(\mu, \zeta)$ by [24, Theorem 4.1]. Thus, Parts (b), (c) are equivalent by Lemma 5. It remains to show the direction $(a) \Rightarrow (b)$.

Suppose $(a)$ holds. Then we have the following isomorphisms of $\mathfrak{g}_0$-modules

$$\Lambda^{\text{top}} \mathfrak{g}_{-1} \otimes M(\lambda, \zeta) = \tilde{M}'(\lambda, \zeta)^{\mathfrak{g}_{-1}} \cong \tilde{M}'(\mu, \zeta)^{\mathfrak{g}_{-1}} = \Lambda^{\text{top}} \mathfrak{g}_{-1} \otimes M(\mu, \zeta).$$

Therefore the conclusion of Part (3) follows from Lemma 5. $\square$

We should mention that the Part (1) in Theorem 9 generalizes the construction of simple modules in [17, Section 4], where the cases of Lie superalgebras in (1.1) were considered.

3.3. Examples. The aim of this subsection is to clarify in which Lie superalgebras with $\zeta \in \mathcal{I}$ the results in Sects. 3.1 and 3.2 are applicable. In particular, we will discuss how frequently $\zeta \in \mathcal{I}$ satisfy the assumption of Theorem A, for various classes Lie superalgebras $\mathfrak{g}$ from (1.2) and (1.3).

Let $\Phi_0$ and $\Phi_\dag$ denote the sets of even and odd roots in $\Phi$, respectively. Throughout this entire subsection, for each Lie superalgebra $\mathfrak{g}$, we will fix a specific Borel subalgebra $\mathfrak{b}_0$. We denote by $\Phi^+_0$ and $\Pi_0$ the corresponding positive and simple systems, respectively. Then we have $\Phi_\zeta \subseteq \Pi_0 \subset \Phi^+_0 \subset \Phi_0$.

With the notations fixed as above, we call $\zeta \in \mathcal{I}$ admissible provided that $\zeta$ satisfies the assumption of Theorem A, that is, $l_\zeta$ is a Levi subalgebra in a parabolic decomposition (1.7) of $\mathfrak{g}$. Namely, $\zeta$ is admissible if and only if there is $H \in h_0$ such that $l_\zeta = \bigoplus_{\Re \alpha(H) = 0} \mathfrak{g}^\alpha$. The following lemma characterize admissible $\zeta$.

Lemma 10. Let $\zeta \in \mathcal{I}$. Then the following are equivalent:

(a) There exists a Borel subalgebra $\mathfrak{b}$ such that $\zeta \in \mathcal{I}_L; \mathfrak{b}$.

(b) $\zeta$ is admissible.

Proof. It remains to show that (b) implies (a). Suppose that $l_\zeta$ is a Levi subalgebra in the parabolic decomposition $\mathfrak{g} = u^-_\zeta \oplus l_\zeta \oplus u^+_\zeta$ with the corresponding parabolic subalgebra $\mathfrak{p} := l_\zeta \oplus u^+\zeta$. It was shown in [31, Lemma 1.3] that $\mathfrak{b}_0 \oplus \mathfrak{p}_\dag$ is a Borel subalgebra of $\mathfrak{g}$. Consequently, if we define $\mathfrak{b} := \mathfrak{b}_0 \oplus \mathfrak{p}_\dag$ then $\mathfrak{p} \supseteq \mathfrak{b}$, as desired. $\square$

The following examples give a classification of admissible $\zeta \in \mathcal{I}$, for various classical Lie superalgebras. We shall complete the details in the following first example of Sect. 3.3.1, and the remaining cases can be argued in similar fashion. We refer to [48] and [28, Chapter 1] for more details about these Lie superalgebras, their Cartan subalgebras $h$ and the sets $\Phi_0, \Phi_\dag$; see also Sects. 5.3.1 and 5.4. We observe that if $h^* = \bigoplus_{i=1}^\ell C_\gamma_i$, then $\{\text{Re}(-)(h)|_{\bigoplus_{i=1}^\ell R\gamma_i} : \bigoplus_{i=1}^\ell R\gamma_i \to R | h \in h\} = \text{Hom}_R (\bigoplus_{i=1}^\ell R\gamma_i, R)$.

3.3.1. Lie superalgebras of type I Although results in Sect. 3.2 apply to type-I Lie superalgebras, it is natural to classify their admissible characters $\zeta$.

(1) We first consider the general linear Lie superalgebras $\mathfrak{g} = \mathfrak{gl}(m|n)$. The dual space $h^*$ of the Cartan subalgebra $h$ and a positive system for $\mathfrak{g}_0 \cong \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$ are
We claim that every \( \zeta \in \mathcal{I} \) is admissible. To see this, we note that there is \( \varphi_\zeta \in \text{Hom}_R(\bigoplus_{i=1}^{m+n} R\epsilon_i, R) \) such that

\[
\varphi_\zeta(\epsilon_i) = \begin{cases} 
\varphi_\zeta(\epsilon_{i+1}), & \text{if } \epsilon_i - \epsilon_{i+1} \in \Phi_\zeta; \\
\varphi_\zeta(\epsilon_{i+1}) + 1, & \text{otherwise},
\end{cases}
\]

where \( 1 \leq i \leq m+n-1 \). Then for any \( \alpha \in \Pi_0 \), it follows that

\[
\varphi_\zeta(\alpha) = 0 \iff \alpha \in \Phi_\zeta,
\]

\[
\varphi_\zeta(\beta) \neq 0, \text{ for any } \beta \in \Phi_1.
\]

Therefore, \( \varphi_\zeta \) gives rise to a parabolic decomposition of \( g \) with the Levi subalgebra \( l_\zeta \), as desired.

(2). Consider the ortho-symplectic Lie superalgebras \( g = \mathfrak{osp}(2|2m) \). The dual space \( \mathfrak{h}^* \) of the Cartan subalgebra \( \mathfrak{h} \) and a positive system for \( g_\hbar \cong \mathfrak{so}(2) \oplus \mathfrak{sp}(2m) \) are given as follows:

\[
\mathfrak{h}^* = C\epsilon_1 \oplus \cdots \oplus C\epsilon_n,
\]

\[
\Phi_0^+ = \{ \epsilon_i - \epsilon_j | 1 \leq i < j \leq n \},
\]

\[
\Pi_0 = \{ \epsilon_1 - \epsilon_2, \ldots, \epsilon_{n-1} - \epsilon_n \},
\]

\[
\Phi_1 = \{ \pm(\epsilon_i + \epsilon_j), 2\epsilon_i | 1 \leq i < j \leq n \}.
\]

Then every \( \zeta \in \mathcal{I} \) is admissible by an argument similar to that given in (1).

(3). We now consider the periplectic Lie superalgebra \( g = \mathfrak{pe}(n) \); see Sect. 5.4 for more details. Then \( g_\hbar \cong \mathfrak{gl}(n) \) and we have

\[
\mathfrak{h}^* = C\epsilon_1 \oplus \cdots \oplus C\epsilon_n,
\]

\[
\Phi_0^+ = \{ \epsilon_i - \epsilon_j | 1 \leq i < j \leq n \},
\]

\[
\Pi_0 = \{ \epsilon_1 - \epsilon_2, \ldots, \epsilon_{n-1} - \epsilon_n \},
\]

\[
\Phi_1 = \{ \pm(\epsilon_i + \epsilon_j), 2\epsilon_i | 1 \leq i < j \leq n \}.
\]

Then every \( \zeta \in \mathcal{I} \) is admissible by a similar argument.

(4). Let \( g \) be a Takiff superalgebra, i.e., the odd subalgebra \( g_1 \) is commutative and \( g_1 \) is isomorphic to \( g_\hbar \) under the adjoint action of \( g_\hbar \). If the Borel superalgebra \( g_0 \) is fixed, then the \( b \) is fixed as well. We note that \( \mathcal{I}_L \cdot b = 0 \), namely, every \( \zeta \in \mathcal{I} \) is not admissible. Nevertheless, \( g \) is of type I by letting \( g_1 := g_\hbar \), and thus the results in Sect. 3.2 hold for this case. There are also analogous Lie superalgebras, see for instance the generalized Takiff superalgebra introduced in [45, Section 3.3].

Below we will consider the case of Lie superalgebras \( g \) beyond type I. Before discussing these examples, we observe that if \( g \) is not of type I and \( \zeta \in \mathcal{I} \) such that \( \Phi_\zeta = \Pi_0 \) so that \( l_\zeta = g_\hbar \), then \( \zeta \) is not admissible. To see this, we suppose on the contrary that
there is an element $H$ such that $g = u_\tau \oplus \iota_\tau \oplus u_\xi$ is the corresponding parabolic decomposition of $g$ as given in (1.7). Since $g$ is not of type I, there are vectors $x \in g^\alpha$, $y \in g^{\alpha'}$ in one of $u_\tau$, $u_\xi$ such that $[x, y] \neq 0$, for some roots $\alpha$, $\alpha'$ so that $\text{Re}(H), \text{Re}(H') > 0$. But $[x, y] \in g_0$ implies $\text{Re}(H) + \text{Re}(H') = 0$, leading to a contradiction.

3.3.2. Ortho-symplectic Lie superalgebras In this subsection, we consider ortho-symplectic Lie superalgebras beyond $\mathfrak{osp}(2|2m)$.

(5). Consider the ortho-symplectic Lie superalgebras $g = \mathfrak{osp}(2n + 1|2m)$. The dual space $h^*$ of the Cartan subalgebra $h$ and a positive system chosen for $g_0 \cong \mathfrak{so}(2n + 1) \oplus \mathfrak{sp}(2m)$ are given as follows:

$$h^* = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n \oplus \mathbb{C}\delta_1 \cdots \oplus \mathbb{C}\delta_m,$$

$$\Phi_0^+ = \{\epsilon_k \pm \epsilon_\ell \mid 1 \leq k < \ell \leq n, 1 \leq q \leq n\} \cup \{\delta_i \pm \delta_j \mid 1 \leq i < j \leq m, 1 \leq p \leq m\},$$

$$\Pi_0 = \{\epsilon_1 - \epsilon_2, \ldots, \epsilon_{n-1} - \epsilon_n, \epsilon_n\} \cup \{\delta_1 - \delta_2, \ldots, \delta_{m-1} - \delta_m, 2\delta_m\},$$

$$\Phi_1 = \{\pm(\delta_p \pm \epsilon_q) \mid 1 \leq q \leq n, 1 \leq p \leq m\}.$$

Then $\zeta$ is admissible if and only if $2\delta_m \notin \Phi_\zeta$.

(6). Consider the ortho-symplectic Lie superalgebras $g = \mathfrak{osp}(2n|2m)$. The dual space of Cartan subalgebra $h^*$ and a positive system chosen for $g_0 \cong \mathfrak{so}(2n) \oplus \mathfrak{sp}(2m)$ are given as follows:

$$h^* = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n \oplus \mathbb{C}\delta_1 \cdots \oplus \mathbb{C}\delta_m,$$

$$\Phi_0^+ = \{\epsilon_k \pm \epsilon_\ell \mid 1 \leq k < \ell \leq n, 1 \leq q \leq n\} \cup \{\delta_i \pm \delta_j \mid 1 \leq i < j \leq m, 1 \leq p \leq m\},$$

$$\Pi_0 = \{\epsilon_1 - \epsilon_2, \ldots, \epsilon_{n-1} - \epsilon_n, \epsilon_{n-1} + \epsilon_n\} \cup \{\delta_1 - \delta_2, \ldots, \delta_{m-1} - \delta_m, 2\delta_m\},$$

$$\Phi_1 = \{\pm(\delta_p \pm \epsilon_q) \mid 1 \leq q \leq n, 1 \leq p \leq m\}.$$

Then $\zeta$ is admissible if and only if one the following is true:

(a) $2\delta_m \notin \Phi_\zeta$.
(b) $2\delta_m \in \Phi_\zeta$ and $\{\epsilon_{n-1} - \epsilon_n, \epsilon_{n-1} + \epsilon_n\} \subseteq \Phi_\zeta$.

3.3.3. Exceptional Lie superalgebras In this subsection, we consider the exceptional Lie superalgebras $g := D(2|1; \alpha), G(3)$ and $F(4)$ in (1.2). We refer the reader to [48, Section 2.5.4] for more details; see also [29], [32] and [30].

(7). We first consider the exceptional Lie superalgebras $g := D(2|1; \alpha)$. This is a family of simple Lie superalgebras of dimension 17, depending on a parameter $\alpha \in \mathbb{C}\setminus\{0, -1\}$. We have $g_0 \cong \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$. The dual space of the Cartan subalgebra $h$ can be written

$$h^* = \mathbb{C}\delta \oplus \mathbb{C}e_1 \oplus \mathbb{C}e_2.$$

We fix a positive system $\Phi_0^+$ for $g_0$ and then give the sets $\Pi_0, \Phi_1$ as follows

$$\Phi_0^+ = \Pi_0 = \{2\delta, 2\epsilon_1, 2\epsilon_2\},$$

$$\Phi_1 = \{\pm(\delta \pm \epsilon_1 \pm \epsilon_2)\}.$$

In this case, $l_\zeta$ is admissible if and only if $l_\zeta \notin g_0$, that is, $\Phi_\zeta \neq \Pi_0$. 

(8). We next consider the exceptional Lie superalgebra \( g := G(3) \). We have \( g_0 \cong \text{sl}(2) \oplus G(2) \). Then the dual space of the Cartan subalgebra \( \mathfrak{h} \) can be written as

\[
\mathfrak{h}^* = \mathbb{C}\delta \oplus \mathbb{C}\epsilon_1 \oplus \mathbb{C}\epsilon_2. 
\]

We fix a positive system \( \Phi^+ \) for \( \mathfrak{g}_0 \) and then give the sets \( \Pi_0, \Phi_1 \) as follows

\[
\Pi_0 = \{2\delta, \epsilon_2 - \epsilon_1, \epsilon_1, \epsilon_2, 2\epsilon_1 + \epsilon_2, \epsilon_1 + 2\epsilon_2\},
\]

\[
\Phi_1 = \{\pm \delta, \pm (\delta \pm \epsilon_1), \pm (\delta \pm \epsilon_2), \pm (\delta \pm (\epsilon_1 + \epsilon_2))\}. 
\]

In this case, \( l_\zeta \) is admissible if and only if \( 2\delta \notin \Phi_\zeta \).

(9). We now consider the exceptional Lie superalgebra \( g := F(4) \). We then have \( g_0 \cong \text{sl}(2) \oplus \text{so}(7) \), and the dual space of the Cartan subalgebra \( \mathfrak{h} \) can be written as

\[
\mathfrak{h}^* = \mathbb{C}\delta \oplus \mathbb{C}\epsilon_1 \oplus \mathbb{C}\epsilon_2 \oplus \mathbb{C}\epsilon_3. 
\]

We choose the sets \( \Phi^+_0, \Pi_0, \Phi_1 \) of roots as follows.

\[
\Phi^+_0 = \{\delta, \epsilon_1 \pm \epsilon_2, \epsilon_2 \pm \epsilon_3, \epsilon_1, \epsilon_2, \epsilon_3\},
\]

\[
\Pi_0 = \{\delta, \epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \epsilon_3\},
\]

\[
\Phi_1 = \{\pm \frac{1}{2} (\delta \pm \epsilon_1 \pm \epsilon_2 \pm \epsilon_3)\}. 
\]

Then \( \zeta \) is admissible if and only if one of the following is true:

(a) \( \delta \notin \Phi_\zeta \).

(b) \( \delta \in \Phi_\zeta \) and \( \epsilon_3 \notin \Phi_\zeta \).

(c) \( \Phi_\zeta = \{\delta, \epsilon_3\} \).

3.3.4. Queer Lie superalgebras \( q(n) \) Finally, we consider the queer Lie superalgebras \( q(n) \); see [28, Section 1.2.6] for details. Then we have \( g_0 \cong \text{gl}(n) \), and \( g_1 \cong g_0 \) under the adjoint action of \( g_0 \). The bracket on \( g_1 \) is not trivial, and \( g \) is not of type I. In this case, every \( \zeta \in \mathcal{I} \) is not admissible. Consequently, neither of two methods given in Sects. 3.1, 3.2 is applicable.

4. Whittaker Modules and Harish–Chandra Bimodules

In this section, we continue to assume that \( g \) is a classical Lie superalgebra with \( \zeta \in \mathcal{I} \). Recall that \( l_\zeta = n^-_\zeta \oplus \mathfrak{h} \oplus n^+_\zeta \) denotes the corresponding triangular decomposition of \( l_\zeta \). For a given weight \( \lambda \in \mathfrak{h}^* \), we set \( \Lambda := \lambda + \Upsilon \).

4.1. Action of the ring \( \hat{S}^W \).

4.1.1. In this subsection, we recall the action of the ring \( \hat{S}^W \) from [62, Section 4, Section 5]: let \( \hat{S} \) denote the completion of the symmetry algebra \( S \) over \( \mathfrak{h} \) at the maximal ideal
generated by \( h \). Denote by \( \hat{S}^W \) its invariants under the action of the Weyl group \( W_\xi \). We will give a natural action of \( \hat{S}^W \) on modules in \( \hat{N} \) using results in [62, Sections 4, 5].

We note that modules in \( \hat{N}(\xi) \) restrict to \( l_\xi \)-modules that are locally finite over \( Z(l_\xi) \). Therefore for any \( M \in \hat{N}(\xi) \) there is a ring homomorphism

\[
\theta_M : \hat{S}^W \to \text{End}_{l_\xi} M,
\]

as constructed in [62, Section 4], which we recall as follows. Set \( l := l_\xi \). Recall that \( \chi^l_\lambda \) denotes the central character of \( l \) associated with \( \lambda \in h^* \). Then \( M \) decomposes into generalized eigenspaces \( M = \bigoplus_{\lambda \in h^*} M_{\chi^l_\lambda} \) according to the action of elements of \( Z(l) \).

For a given \( \lambda \in h^* \), we are now going to define the action \( \theta_M(s) \) on \( M_{\chi^l_\lambda} \), for each \( s \in \hat{S}^W \).

Let \( \hat{Z}(l)_\lambda \) and \( \hat{S}_\lambda \) denote the completion of \( Z(l) \) at \( \text{Ker} \chi^l_\lambda \) and the completion of \( S \) at the ideal generated by \( \{ h - \lambda(h) | h \in h \} \), respectively. Note that there is an isomorphism \( \mathbb{T}_\lambda : \hat{S}_\lambda \cong \hat{S}_0 = \hat{S} \). Then the Harish–Chandra homomorphism \( \varphi : Z(l) \to S \) induces an inclusion \( \mathbb{T}_\lambda \circ \varphi : \hat{Z}(l)_\lambda \to \hat{S}_\lambda \cong \hat{S} \) that has an image containing \( \hat{S}^W \). This gives us an inclusion \( \mathbb{V}_\lambda \circ \varphi : \hat{Z}(l)_\lambda \to \hat{S}_\lambda \). Note that elements of \( \hat{Z}(l)_\lambda \) have well-defined action on the summand \( M_{\chi^l_\lambda} \).

We then define the action of \( \theta_M(s) \) on \( M_{\chi^l_\lambda} \) via \( \varphi_{\lambda}(s) \), for each \( s \in \hat{S}^W \).

As has been proved in [62, Theorem 4.1] (see also [62, Section 5]), the action of elements of \( \hat{S}^W \) via \( \theta_M \) in (4.1) commute with the \( g_\xi \)-action on \( M \). As an analog for our setup, we remark the following corollary, but we will not use it.

**Proposition 11.** We have \( \theta_M(\hat{S}^W) \subset \text{End}_g M \), for any \( M \in \hat{N} \).

**Proof.** Let \( \text{Id}^\xi \) denote the identity functor on the category of \( l_\xi \)-modules that are locally finite over \( Z(l_\xi) \). By the construction in [62, Section 4], for each \( s \in \hat{S}^W \) the element \( \theta_M(s) \) is the evaluation of an endomorphism \( \theta(s) : \text{Id}^\xi \to \text{Id}^\xi \) at \( M \), where \( \theta \) is a ring homomorphism from \( \hat{S}^W \) to the endomorphism ring of functor \( \text{Id}^\xi \). We note that the adjoint representation \( \text{ad}_g \) is semisimple over \( l_\xi \). Let \( \pi_M : \text{ad}_g \otimes M \to M \) be the canonical epimorphism. By [62, Theorem 4.1] we have

\[
\theta_{\text{ad}_g \otimes M}(s) = \text{Id}_{\text{ad}_g}^\xi \otimes \theta_M(s),
\]

which gives rise to the identity \( \pi_M \circ (\text{Id}_{\text{ad}_g}^\xi \otimes \theta_M(s)) = \theta_M(s) \circ \pi_M \), for any \( s \in \hat{S}^W \).

This completes the proof. \( \square \)

4.1.2. We recall some results and notations in [62, Section 5] as follows. For any positive integer \( n \), we define \( \mathcal{N}(\xi)^n := \{ M \in \mathcal{N}(\xi) | \theta(m)^n M = 0 \} \), where \( m \) is the maximal ideal of \( \hat{S}^W \). Set \( l := l_\xi \). As has been mentioned in the previous section, any module \( M \in \mathcal{N}(\xi) \) decomposes into generalized eigenspaces \( M = \bigoplus_{\lambda \in h^*} M_{\chi^l_\lambda} \) according to the action of elements of \( Z(l) \).

Recall that we denote the set of integral weights by \( \Upsilon \). For any coset \( \Lambda = \lambda + \Upsilon \) in \( h^*/\Upsilon \) we put

\[
\mathcal{N}(\lambda, \xi) := \{ M \in \mathcal{N}(\xi) | M_{\chi^l_\lambda} = 0 \text{ unless } \chi^l_\mu = \chi^l_\lambda, \text{ for some } \mu \in \Lambda \},
\]

\[
\mathcal{N}(\lambda, \xi)^n := \mathcal{N}(\lambda, \xi) \cap \mathcal{N}(\xi)^n,
\]

as defined in [62, Section 5]. Then \( \mathcal{N}(\xi) \) decomposes into \( \mathcal{N}(\xi) = \bigoplus_{\lambda \in h^*/\Upsilon} \mathcal{N}(\lambda, \xi) \).
Similarly, we define \( \widetilde{\mathcal{N}}(\Lambda, \zeta) := \{ M \in \mathcal{N}(\zeta) \mid \text{Res } M \in \mathcal{N}(\Lambda, \zeta) \} \). Since \( \mathcal{N}(\Lambda, \zeta) \) is stable under tensoring with finite-dimensional \( \mathfrak{g}_0 \)-modules (see [62, Theorem 4.1, Lemma 4.3]) and every module in \( \mathcal{N}(\Lambda, \zeta) \) has finite length, we know that the family of objects in \( \widetilde{\mathcal{N}}(\Lambda, \zeta) \) consists of composition factors of \( \text{Ind} \ X, \) for \( X \in \mathcal{N}(\Lambda, \zeta) \). Therefore \( \text{Ind} \) and \( \text{Res} \) restrict to well-defined functors between \( \widetilde{\mathcal{N}}(\Lambda, \zeta) \) and \( \mathcal{N}(\Lambda, \zeta) \). Also, we have the decomposition \( \widetilde{\mathcal{N}}(\zeta) = \bigoplus_{\Lambda \in \mathfrak{h}^* / \mathcal{Y}} \mathcal{N}(\Lambda, \zeta) \) (see also [62, Lemma 4.3]).

We define

\[
\widetilde{\mathcal{N}}(\zeta)^n := \{ M \in \widetilde{\mathcal{N}}(\zeta) \mid \theta_M(m)^n M = 0 = \{ M \in \widetilde{\mathcal{N}}(\zeta) \mid \text{Res } M \in \mathcal{N}(\zeta)^n \},
\]

\[
\mathcal{N}(\Lambda, \zeta)^n = \{ M \in \mathcal{N}(\zeta)^n \mid \text{Res } M \in \mathcal{N}(\Lambda, \zeta)^n \}.
\]

We claim that \( \text{Ind} \) and \( \text{Res} \) restrict to well-defined functors between \( \widetilde{\mathcal{N}}(\zeta)^n \) and \( \mathcal{N}(\zeta)^n \). To see this, let \( V \in \mathcal{N}(\zeta)^n \) and \( s \in m^a \). Consider the canonical epimorphism \( U(\mathfrak{g}_1) \otimes V \to V \). By [62, Theorem 4.3] we have \( \theta_{U(\mathfrak{g}_1)} \otimes V(s) = \text{Id}_{U(\mathfrak{g}_1)} \otimes \theta_V(s) = 0 \), which implies that \( \theta_{\text{Res} \ V}(s) = 0 \), as desired. Therefore \( \text{Ind} \) and \( \text{Res} \) restrict to well-defined functors between \( \widetilde{\mathcal{N}}(\Lambda, \zeta)^n \) and \( \mathcal{N}(\Lambda, \zeta)^n \), for any \( n \geq 0 \).

4.2. Harish–Chandra bimodules. We recall some conventions of Harish–Chandra bimodules; see, e.g., [22, Section 3], or [62, Section 3] for more details. In the rest of the present paper, we set \( \tilde{U} := U(\mathfrak{g}) \) and \( U := U(\mathfrak{g}_0) \).

4.2.1. We denote by \( \mathcal{F} \) the category of finite-dimensional semisimple \( \mathfrak{g}_0 \)-modules. Set \( \widetilde{\mathcal{F}} \) to be the category of finite-dimensional \( \mathfrak{g} \)-modules which restrict to objects in \( \mathcal{F} \). We denote the full subcategory of projective modules in \( \widetilde{\mathcal{F}} \) by \( \widetilde{\mathcal{P}} \). Modules in \( \widetilde{\mathcal{P}} \) are precisely the direct summands of modules \( \text{Ind} \ V \), for arbitrary \( V \in \mathcal{F} \). For a \( \mathfrak{g} \)-module \( M \), we denote by \( \widetilde{\mathcal{F}} \otimes M \) the category of \( \mathfrak{g} \)-modules of the form \( V \otimes M \), with \( V \in \widetilde{\mathcal{F}} \). Similarly, we define \( \mathcal{P} \otimes M, \mathcal{F} \otimes N \) and \( \mathcal{P} \otimes N \), for \( N \in \mathfrak{g}_0 \)-Mod.

For a given full subcategory \( \mathcal{C} \) of either \( \mathfrak{g} \)-Mod or \( \mathfrak{g}_0 \)-Mod, we denote by \( \text{add}(\mathcal{C}) \) the category of all modules isomorphic to direct summands of objects in \( \mathcal{C} \). Also, we set \( \langle \mathcal{C} \rangle \) to be the full subcategory of all modules isomorphic to subquotients of modules in \( \mathcal{C} \). Let \( \text{Coker}(\widetilde{\mathcal{F}} \otimes M) \) denote the \( \text{coker-category} \) of \( M \) consisting of all \( \mathfrak{g} \)-modules \( X \) that have a presentation

\[
A \to B \to X \to 0,
\]

where \( A, B \in \text{add}(\widetilde{\mathcal{F}} \otimes M) \). Similarly we define the \( \text{coker-category} \) \( \text{Coker}(\mathcal{F} \otimes N) \) of \( \mathfrak{g}_0 \)-modules \( N \) (see [61]).

4.2.2. For a given \( (\tilde{U}, U) \)-bimodule \( Y \), we denote by \( Y^{\text{ad}} \) the restriction of \( Y \) to the adjoint action of \( \mathfrak{g}_0 \). This is the restriction via \( U \mapsto \tilde{U} \otimes U^{\text{op}}, X \mapsto X \otimes 1 - 1 \otimes X \). Let \( \mathcal{B} \) denote the category of finitely-generated \( (\tilde{U}, U) \)-bimodules \( N \) for which \( N^{\text{ad}} \) is a direct sum of modules in \( \mathcal{F} \). Let \( J \subset U \) be a two-sided ideal, denote by \( \mathcal{B}(J) \) the full subcategory of \( \mathcal{B} \) consisting of bimodules \( N \) such that \( N J = 0 \). Also, we set \( \mathcal{B}_J := \bigcup_{n \geq 1} \mathcal{B}(J^n) \).

For a given \( \mathfrak{g} \)-module \( M \) and a given \( \mathfrak{g}_0 \)-module \( N \), we set \( \mathcal{L}(N, M) \) to be the maximal \( (\tilde{U}, U) \)-submodule of \( \text{Hom}_\mathfrak{g}(N, M)^{\text{ad}} \) that belongs to \( \mathcal{B} \). Namely, \( \mathcal{L}(M, N) \) is the maximal submodule which is a direct sum of modules in \( \mathcal{F} \) under the usual adjoint action. We have a canonical monomorphism

\[
\tilde{\tau}_M : \tilde{U} / \text{Ann}_{\tilde{U}}(M) \hookrightarrow \text{Hom}_\mathfrak{g}(M, M)^{\text{ad}}, \text{ for } M \in \mathfrak{g} \mod . \tag{4.2}
\]
4.3. Equivalence. We are going to establish an equivalence between $\tilde{\mathcal{N}}(\Lambda, \zeta)$ and a category of Harish–Chandra $(U, U)$-bimodules. The following theorem established in [22, Theorem 3.1] is a variation of [62, Theorem 3.1].

**Theorem 12.** Let $M \in \mathfrak{g}_0$-$\text{Mod}$. Set $I$ to be the annihilator ideal of $M$. Suppose that the monomorphism $\iota_M$ in (4.2) is an isomorphism and $M$ is projective in $(\mathcal{F} \otimes M)$. Then

$$- \otimes_U M : B(I) \to \text{Coker}(\tilde{\mathcal{F}} \otimes \text{Ind}(M))$$

is an equivalence of categories with inverse $\mathcal{L}(M, -)$.

Theorem 12 will be the main tool in the proof of Theorem B. Before giving the proofs, we need several preparatory results. For a given $\mu \in \mathfrak{h}^*$, recall that $\chi^l_{\mu} : Z(l_\ell) \to \mathbb{C}$ denotes the central character of $l_\ell$ associated with $\mu$.

Recall that $l_\ell$ is a Levi subalgebra in a parabolic decomposition of $\mathfrak{g}_0$, giving a corresponding parabolic subalgebra $q_\ell$ of $\mathfrak{g}_0$. For $\mu \in \mathfrak{h}^*$, we define

$$M^n(\mu, \zeta) := U \otimes_{q_\ell} Y^n_{\zeta}(\mu, \zeta),$$

where $Y^n_{\zeta}(\mu, \zeta) := U(l_\ell)/(\text{Ker} \chi^l_{\mu})^nU(l_\ell) \otimes U(n_\ell) \mathbb{C}_\zeta$; see [62, Section 5]. We put $I_{\mu} := U/\text{Ker} \chi^l_{\mu}$. We set $\mathcal{H}(I_{\mu}^n)$ to be the category of Harish–Chandra $(U, U)$-bimodules $X$ such that $XI_{\mu}^n = 0$. The following lemma is established in [62, Theorem 5.3].

**Lemma 13.** (Miličić-Soergel). Let $\lambda \in \mathfrak{h}^*$ be dominant such that $W_{\zeta} = W_{\lambda}$. Then the functor $N \to N \otimes_U M^n(\lambda, \zeta)$ provides an equivalence $T_n$ from $\tilde{\mathcal{N}}(\Lambda, \zeta)^n$ to $\mathcal{H}(I_{\lambda}^n)$, for each $n \geq 1$. This gives rise to an equivalence $T$ from $\bigcup_{n \geq 1} \mathcal{H}(I_{\lambda}^n)$ to $\tilde{\mathcal{N}}(\Lambda, \zeta)$.

Before proving Theorem B, we collect some useful facts from [62] as follows.

**Lemma 14.** (Miličić-Soergel). Let $\lambda \in \mathfrak{h}^*$ be dominant with $W_{\lambda} = W_{\zeta}$. Set $\Lambda := \lambda + \Upsilon$. Then for any $n \geq 1$ we have

1. $\tilde{\mathcal{N}}(\Lambda, \zeta)^n$ is stable under tensoring with finite-dimensional $\mathfrak{g}_0$-modules.
2. $\tilde{\mathcal{N}}(\Lambda, \zeta)^n$ has enough projective modules, and $M^n(\lambda, \zeta)$ is projective in $\tilde{\mathcal{N}}(\zeta)^n$.
3. $\text{Ann}_U M^n(\lambda, \zeta) = I^n_{\lambda}$, and $\iota_M(\lambda, \zeta)$ is an isomorphism for $M = M^n(\lambda, \zeta)$ in (4.2).
4. For any $n > m$ the canonical epimorphism $M^n(\lambda, \zeta) \to M^n(\lambda, \zeta)$ has kernel $\text{Ker}(\chi^l_{\mu})^m M^n(\lambda, \zeta)$.

**Proof.** Part (1) is a consequence of [62, Theorem 4.1, Lemma 4.3]. As has been noted in the proof of [62, Theorem 5.3], $\mathcal{H}(I_{\lambda}^n)$ has enough projective modules. Therefore, conclusions in Part (2) and Part (3) follow from [62, Lemma 5.11, Proposition 5.5] and Lemma 13. Part (4) is taken from [62, Lemma 5.14].

**Remark 15.** The facts (1) – (3) are also given in the proof of [62, Theorem 5.3].

We set $\mathcal{B}_{\lambda} := \mathcal{B}_{l_{\ell}}$ to be the full subcategory consisting of objects

$$\{X \in \mathcal{B} | XI_{\lambda}^n = 0, \text{ for } n \gg 0\}.$$
Theorem 16. Suppose that $\lambda \in h^*$ is dominant such that $W_\zeta = W_\lambda$. Then the functor $X \mapsto \lim X \otimes_U M^n(\lambda, \zeta)$ gives an equivalence of categories $\tilde{T} : B_\lambda \rightarrow \tilde{N}(\Lambda, \zeta)$.

Proof. By definition $\tilde{N}(\Lambda, \zeta)^n$ is the full subcategory of $\tilde{N}(\zeta)$ consisting of modules $M$ such that $\text{Res} M \in N(\Lambda, \zeta)^n$, for any $n \geq 1$. By [21, Proposition 2.2.1], we know that $\tilde{N}(\Lambda, \zeta)^n$ has enough projective objects, and any projective module in $\tilde{N}(\Lambda, \zeta)^n$ is a direct summand of $\text{Ind}_X^{U}$, for projective object $X \in \tilde{N}(\Lambda, \zeta)^n$. We claim that the functor $- \otimes_U M^n(\lambda, \zeta)$ gives an equivalence $B(I^n_\lambda) \cong \tilde{N}(\Lambda, \zeta)^n$.

Using Part (2) of Lemma 14, we know $M^n(\lambda, \zeta)$ is projective in $(\mathcal{F} \otimes M^n(\lambda, \zeta)) \subseteq \tilde{N}(\zeta)^n$. Then by Part (3) of Lemma 14 and Theorem 12, we obtain an equivalence

$$- \otimes_U M^n(\lambda, \zeta) : B(I^n_\lambda) \rightarrow \text{Coker}(\tilde{F} \otimes \text{Ind}(M^n(\lambda, \zeta))). \quad (4.3)$$

To complete the proof, we shall show that $\text{Coker}(\tilde{F} \otimes \text{Ind}(M^n(\lambda, \zeta))) = \tilde{N}(\Lambda, \zeta)^n$. We may observe that $E \otimes \text{Ind} M^n(\lambda, \zeta) \cong \text{Ind}(\text{Res} E \otimes M^n(\lambda, \zeta))$ is projective in $\tilde{N}(\Lambda, \zeta)^n$, for any $E \in \mathcal{F}$. Since $\tilde{N}(\Lambda, \zeta)^n$ has enough projectives, we need just to show that $\text{add}(\tilde{F} \otimes \text{Ind}(M^n(\lambda, \zeta)))$ contains all projective modules in $\tilde{N}(\Lambda, \zeta)^n$. To see this, let $P \in N(\Lambda, \zeta)^n$ be a projective object. Since every projective module in $\mathcal{H}(I^n_\lambda)$ is a direct summand of $E \otimes U/I^n_\lambda$, for $E \in \mathcal{F}$, we may conclude that $P$ is a direct summand of a projective module of the form $E \otimes M^n(\lambda, \zeta)$ by Lemma 13. Therefore we have $\text{Ind} P \in \text{add}(\text{Ind}(\mathcal{F} \otimes M^n(\lambda, \zeta)))$. Now we calculate

$$\tilde{P} \subseteq \text{add}(\text{Ind}(\mathcal{F} \otimes M^n(\lambda, \zeta)))$$

$$\subseteq \text{add}(\text{Ind}(\mathcal{F} \otimes \text{Res} M^n(\lambda, \zeta)))$$

$$= \text{add}(\text{Ind} \mathcal{F} \otimes \text{Ind} M^n(\lambda, \zeta))$$

$$= \text{add}(\tilde{P} \otimes \text{Ind} M^n(\lambda, \zeta))$$

$$\subseteq \text{add}(\tilde{F} \otimes \text{Ind} M^n(\lambda, \zeta)).$$

Consequently, we have equivalence $- \otimes_U M^n(\lambda, \zeta) : B(I^n_\lambda) \rightarrow \tilde{N}(\Lambda, \zeta)^n$. By Part (4) of Lemma 14, the functor $X \mapsto \lim X \otimes_U M^n(\lambda, \zeta)$ determines an equivalence of categories $\tilde{T} : B_\lambda \rightarrow \tilde{N}(\Lambda, \zeta)$. This completes the proof. □

Corollary 17. Let $\lambda \in h^*$ be dominant and $\zeta, \eta \in \mathcal{I}$ such that $W_\zeta = W_\lambda = W_\eta$. Then we have

$$\tilde{N}(\lambda + \gamma, \eta) \cong \tilde{N}(\lambda + \gamma, \zeta).$$

Corollary 18. Let $\lambda \in h^*$ be dominant such that $W_\zeta = W_\lambda$. Suppose that $s$ is a simple reflection such that $s$ does not lie in the integral Weyl group of $\lambda$. Then we have

$$\tilde{N}(s \cdot \lambda + \gamma, \eta) \cong \tilde{N}(\lambda + \gamma, \zeta),$$

for any $\eta \in \mathcal{I}$ with $W_{s \cdot \lambda} = W_\eta$.

Proof. By Theorem 16, we have the following equivalences

$$\tilde{N}(s \cdot \lambda + \gamma, \eta) \cong \tilde{B}_{s \cdot \lambda} = \tilde{B}_\lambda \cong \tilde{N}(\lambda + \gamma, \zeta). \quad (4.4)$$

□
5. Multiplicities of Standard Whittaker Modules

In this section, we assume that \( \mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) is a classical Lie superalgebra of type I and \( \zeta \in \mathcal{I} \) is a Levi subalgebra in a parabolic decomposition \( \mathfrak{g} = \mathfrak{u}_\zeta^- \oplus \mathfrak{l}_\zeta \oplus \mathfrak{u}_\zeta \) satisfying

\[
(u^-)_\zeta)_\zeta = \mathfrak{g}_{-1}, \quad (u_\zeta)_\zeta = \mathfrak{g}_1 = \mathfrak{b}_1. \tag{5.1}
\]

The \( \mathcal{I} \) coincide with the set of homomorphisms from \( \mathfrak{n} \) to \( \mathbb{C} \), as has been in Sect. 1.5. In particular, we are mainly interested in the following concrete subset of the classical Lie superalgebras:

\[
\mathfrak{g} = \mathfrak{gl}(m|n), \mathfrak{osp}(2|2n), \mathfrak{pe}(n). \tag{5.2}
\]

In this case, for an arbitrary \( \zeta \in \mathcal{I} \), the \( \mathfrak{l}_\zeta \) is always a Levi subalgebra in a parabolic decomposition \( \mathfrak{g} = \mathfrak{u}_\zeta^- \oplus \mathfrak{l}_\zeta \oplus \mathfrak{u}_\zeta \) that satisfies (5.1); see Sect. 3.3.1. We refer to [59, Section 3] and [31, Section 5] for more details. As has been noted in Sect. 3.2, the definitions of standard Whittaker module \( \mathcal{M}'(\lambda, \zeta) \) in (3.7) from Sect. 3.2 and \( \tilde{M}(\lambda, \zeta) \) in (3.3) from Sect. 3.1 coincide. Also, we remark that the character \( \zeta \) on \( \mathfrak{n} \) is naturally extended to a homomorphism from \( U(\mathfrak{n}) \) to \( \mathbb{C} \).

5.1. The category \( \mathcal{O} \) and the Whittaker vectors. We recall that the BGG category \( \mathcal{O} \) consists of finitely-generated \( \mathfrak{g} \)-modules which are semisimple over \( \mathfrak{h} \) and locally finite over \( \mathfrak{n} \). Therefore \( \mathcal{O} \) is the category of \( \mathfrak{g} \)-modules that restrict by \( \text{Res} \) to \( \mathfrak{g}_0 \)-modules in the BGG category of [19], which we will denote by \( \mathcal{O}^0 \) in the present paper. We refer to [28,59] for a more complete treatment. In particular, we may note that \( \mathcal{O} \subset \tilde{\mathcal{N}}(0) \).

Both categories \( \mathcal{O} \) and \( \mathcal{O}^0 \) admit highest category structures (see, e.g., [31,38,55]). The partial order \( \leq \) on \( \mathfrak{h}^* \) is defined as the transitive closure of the following relations

\[
\begin{aligned}
\lambda - \alpha &\leq \lambda, \quad \text{for } \alpha \in \Phi(\mathfrak{n}), \\
\lambda + \alpha &\leq \lambda, \quad \text{for } \alpha \in \Phi(\mathfrak{n}^-).
\end{aligned}
\]

We define \( M(\lambda) \) and \( \tilde{M}(\lambda) \) to be the Verma modules of highest weight \( \lambda \) as follows

\[
M(\lambda) = U(\mathfrak{g}_0) \otimes_{\mathfrak{b}_0} \mathbb{C}_\lambda, \quad \tilde{M}(\lambda) := U(\mathfrak{g}) \otimes_{\mathfrak{b}} \mathbb{C}_\lambda \cong K(\mathcal{M}(\lambda)).
\]

Denote by \( L(\lambda) \) and \( \tilde{L}(\lambda) \) the simple quotients of \( M(\lambda) \) and \( \tilde{M}(\lambda) \), respectively. There are canonical epimorphisms \( \tilde{M}(\lambda) \rightarrow K(L(\lambda)) \), \( K(L(\lambda)) \rightarrow \tilde{L}(\lambda) \), for any \( \lambda \in \mathfrak{h}^* \).

For any \( \mathfrak{g} \)-module \( M \) and \( \mathfrak{g}_0 \)-module \( V \), we define

\[
\text{Wh}_\zeta(V) := \{ v \in V \mid xv = \zeta(x)v, \text{ for any } x \in \mathfrak{n}_0 \}, \quad \text{Wh}_\zeta(M) := \{ m \in M \mid xm = \zeta(x)m, \text{ for any } x \in \mathfrak{n} \}.
\]

Following [50], elements in \( \text{Wh}_\zeta(V) \) and \( \text{Wh}_\zeta(M) \) are called Whittaker vectors. The following useful lemma is taken from [17, Lemma 3.3].

**Lemma 19.** (Bagci-Christodouloupolou-Wiesner) The subspace \( \text{Wh}_\zeta(M) \) is non-zero, for any \( M \in \tilde{\mathcal{N}}(\zeta) \). In particular, \( M \) is simple if \( \dim \text{Wh}_\zeta(M) = 1 \).

**Proof.** Let \( v \in M \) be non-zero. By [28, Lemma 3.17] there exists one-dimensional \( \mathfrak{n} \)-module \( \mathbb{C}_\eta \) of \( U(\mathfrak{n})v \), for some \( \eta \in \mathcal{I} \). We may conclude that \( \eta = \zeta \) since \( M \in \tilde{\mathcal{N}}(\zeta) \).

\( \Box \)
5.2. The functors \( \tilde{\Gamma}_\xi \). Let \( \xi \in \mathcal{I} \). For a given \( M \in \mathcal{O} \) and \( \lambda \in \mathfrak{h}^* \), let \( M_\lambda \) denote the corresponding weight subspace, that is, \( M_\lambda := \{ m \in M \mid hm = \lambda(h)m \}, \) for any \( h \in \mathfrak{h}^* \). We may associate the completion \( \overline{M} := \prod_{\lambda \in \mathfrak{h}^*} M_\lambda \), which admits a structure of \( \mathfrak{g} \)-module in a natural way. Similarly, we define \( \overline{N} \) for \( N \in \mathcal{O}^\circ \).

Define

\[
\tilde{\Gamma}_\xi(M) := \{ m \in \overline{M} \mid x - \xi(x) \text{ acts nilpotently at } m, \text{ for any } x \in \mathfrak{n}_0 \}.
\]

We may note that \( \tilde{\Gamma}_\xi(M) \in \mathfrak{g} \)-Mod since

\[
(x - \xi(x))ym = (ad(x)y)m + y(x - \xi(x))m,
\]

for any \( y \in U(\mathfrak{g}), x \in \mathfrak{n}_0 \) and \( m \in \tilde{\Gamma}_\xi(M) \). Therefore \( \tilde{\Gamma}_\xi \) defines a functor from \( \mathcal{O} \) to \( \mathfrak{g} \)-Mod.

The \( \mathfrak{g} \)-module \( \tilde{\Gamma}_\xi(M) \) is locally finite over \( n \) (see, e.g., [1, Lemma 1]). We recall the functor \( \Gamma_\xi(-) : \mathcal{O}^\circ \to \mathfrak{g}_0 \)-Mod constructed by Backelin in [3, Section 3.1], namely, \( \Gamma_\xi(N) \) is the set of elements in \( \overline{N} \) that are annihilated by some power of the kernel of \( \xi|_{U(\mathfrak{n}_0)} : U(\mathfrak{n}_0) \to \mathbb{C} \), for any \( N \in \mathcal{O}^\circ \). We may also note that \( \tilde{\Gamma}_\xi(M) \) is the set of elements in \( \overline{M} \) that are annihilated by some power of \( \text{Ker}_\xi \), and hence \( \text{Res} \tilde{\Gamma}_\xi(M) = \overline{\Gamma}_\xi(\text{Res} M) \in \mathcal{N}^\circ(\xi) \) (cf. [3, Lemma 3.2]). We recall that the functor \( \Gamma_\xi \) sends Verma modules to standard Whittaker modules and sends simple modules to simple modules or zeros by [3, Proposition 6.9].

The conclusion of Theorem C is an immediate consequence of the following theorem:

**Theorem 20.** The functor \( \tilde{\Gamma}_\xi \) defines an exact functor from \( \mathcal{O} \) to \( \tilde{\mathcal{N}}(\xi) \). Furthermore, for any \( \lambda \in \mathfrak{h}^* \) we have

\[
\tilde{\Gamma}_\xi(\overline{M}(\lambda)) = \overline{M}(\lambda, \xi); \tag{5.4}
\]

\[
\tilde{\Gamma}_\xi(K(L(\lambda))) \cong \begin{cases} K(L(\lambda, \xi)), & \text{if } \lambda \text{ is } n_\xi\text{- antidominant;} \\ 0, & \text{otherwise;} \end{cases} \tag{5.5}
\]

\[
\tilde{\Gamma}_\xi(\tilde{L}(\lambda)) \cong \begin{cases} \tilde{L}(\lambda, \xi), & \text{if } \lambda \text{ is } n_\xi\text{- antidominant;} \\ 0, & \text{otherwise.} \end{cases} \tag{5.6}
\]

**Proof.** The first claim follows from the isomorphism \( \text{Res} \circ \tilde{\Gamma}_\xi \cong \tilde{\Gamma}_\xi \circ \text{Res} \) and [3, Lemma 3.2]. We shall show that \( \tilde{\Gamma}_\xi \circ K \cong K \circ \Gamma_\xi \). For \( M \in \mathcal{O}^\circ \), the \( \mathfrak{g} \)-module \( \tilde{\Gamma}_\xi(K(M)) \) can be considered as the set of elements of \( K(\overline{M}) \) that are annihilated by some powers of \( \text{Ker}_\xi \). Therefore we have natural isomorphisms \( \tilde{\Gamma}_\xi(K(M)) \cong K(\overline{\Gamma}_\xi(M)) \) by the proof of [1, Proposition 3]. The conclusions of (5.4)–(5.5) follow from [3, Proposition 6.9].

We note that \( \tilde{\Gamma}_\xi(\tilde{L}(\lambda)) = 0 \) if \( \lambda \) is not \( n_\xi\)-antidominant by (5.5). Now, suppose that \( \lambda \) is \( n_\xi\)-antidominant. We are going to show that \( \tilde{\Gamma}_\xi(\tilde{L}(\lambda)) \cong \tilde{L}(\lambda, \xi) \). To see this, we first note that there is a non-zero homomorphism \( \phi \) from \( M(\lambda) \) to \( \text{Coind}_{\mathfrak{h}^+ + \mathfrak{n}^-}^\theta \mathcal{L}_\lambda \), since \( \lambda \) is the highest weight in the weight subspace of \( \text{Coind}_{\mathfrak{h}^+ + \mathfrak{n}^-}^\theta \mathcal{L}_\lambda \). Let \( S \) denote the image of \( \phi \). We note that \( \tilde{\Gamma}_\xi(S) \) is a \( \mathfrak{g} \)-submodule of \( \text{Coind}_{\mathfrak{h}^+ + \mathfrak{n}^-}^\theta \mathcal{L}_\lambda \) (see also the proof of [13, Theorem 36]). Also, \( \tilde{\Gamma}_\xi(S) \) is non-zero since \( \text{Res} \tilde{\Gamma}_\xi(S) \to \text{Res} \tilde{\Gamma}_\xi(\tilde{L}(\lambda)) \cong \overline{\Gamma}_\xi(\text{Res} \tilde{L}(\lambda)) \supset \overline{\Gamma}_\xi(L(\lambda)) = L(\lambda, \xi) \) by [3, Proposition 6.9].
Since \( \tilde{\Gamma}_\xi(S) \) is non-zero, the subspace \( \tilde{\mathcal{W}}h_\xi(\tilde{\Gamma}_\xi(S)) \) of Whittaker vectors of \( \tilde{\Gamma}_\xi(S) \) is non-zero. By a similar argument as used in [13, Lemma 37], we also know that the subspace \( \tilde{\mathcal{W}}h_\xi(\text{Coind}_{\delta_{b+n-C}}^{\delta_{\Phi_1}}) \) of Whittaker vectors of \( \text{Coind}_{\delta_{b+n-C}}^{\delta_{\Phi_1}} \) is of one-dimensional, which implies that \( \tilde{\Gamma}_\xi(S) \) is simple. Since \( \tilde{\Gamma}_\xi(S) \) is the simple quotient of \( \tilde{\Gamma}_\xi(\tilde{M}(\lambda)) \cong \tilde{M}(\lambda, \zeta) \), we may conclude that \( \tilde{\Gamma}_\xi(S) \cong \tilde{L}(\lambda, \zeta) \). Also, since \( \tilde{\Gamma}_\xi(\tilde{L}(\lambda)) \) is a non-zero quotient of \( \tilde{\Gamma}_\xi(S) \) by the exactness of \( \tilde{\Gamma}_\xi \), we have \( \tilde{\Gamma}_\xi(\tilde{L}(\lambda)) \cong \tilde{L}(\lambda, \zeta) \). This completes the proof. \( \square \)

The following corollary is an analog of Kostant’s characterizations of simple Whittaker modules in [50, Theorem 3.6.1].

**Corollary 21.** Let \( M \in \tilde{\mathcal{N}} \). Then \( M \) is simple if and only if \( \dim \tilde{\mathcal{W}}h_\xi(M) = 1 \).

**Proof.** By the proof of Theorem 20, we have \( \dim \tilde{\mathcal{W}}h_\xi(\tilde{L}(\lambda, \zeta)) = 1 \), for any \( \lambda \in \mathfrak{h}^* \). The conclusion follows from Lemma 19. \( \square \)

### 5.3. Basic Lie superalgebras of type I

The series of Lie superalgebras \( \mathfrak{gl}(m|n) \) and \( \mathfrak{osp}(2|2n) \) from the list (1.2) belong to the series of basic Lie superalgebras. In this subsection, we will give a detailed example of the composition series for the standard Whittaker modules over \( \mathfrak{gl}(1|2) \). We will also study several criteria of simplicity and the annihilator for standard Whittaker modules over Lie superalgebras of types A, C and P. Following [50, 62], an element \( \zeta \in \mathcal{I} \) is called regular (or nonsingular) if \( W_\xi = W \).

#### 5.3.1. The general linear Lie superalgebras \( \mathfrak{gl}(m|n) \)

For positive integers \( m, n \), the general linear Lie superalgebra \( \mathfrak{gl}(m|n) \) can be realized as the space of \( (m+n) \times (m+n) \) complex matrices

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix},
\]

(5.7)

where \( A, B, C \) and \( D \) are \( m \times m, m \times n, n \times m, n \times n \) matrices, respectively. The bracket is given by the super commutator. For any \( 1 \leq a, b \leq m+n \), set \( E_{ab} \) to be the elementary matrix in \( \mathfrak{gl}(m|n) \), namely, the \((a,b)\)-entry of \( E_{ab} \) is equal to 1 and all other entries are 0.

The Cartan subalgebra \( \mathfrak{h} \subset \mathfrak{g}_0 \) consists of diagonal matrices above. We denote the dual basis of \( \mathfrak{h}^* \) by \( \{\epsilon_1, \epsilon_2, \ldots, \epsilon_{m+n}\} \) with respect to the following standard basis of \( \mathfrak{h} \)

\[
\{H_i := E_{i,i} | 1 \leq i \leq m+n\}.
\]

(5.8)

The space \( \mathfrak{h}^* \) is equipped with a natural bilinear form \( (\cdot, \cdot) : \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C} \) by letting \( (\epsilon_i, \epsilon_j) = \delta_{ij} \). We fix a triangular decomposition \( \mathfrak{g} = n^- \oplus \mathfrak{h} \oplus n^+ \), where \( n^- \) and \( n^+ \) consisting of all strict upper and lower triangular matrices in (5.7), respectively. The corresponding Borel subalgebra is \( \mathfrak{b} = \mathfrak{h} \oplus n \).

Recall that we denote by \( \Phi \) the set of roots and by \( \Phi^+ \) the set of positive roots in the Borel subalgebra \( \mathfrak{b} \). As have been defined in Sect. 3.3, the \( \Phi_0 ^+ \) and \( \Phi_1 ^+ \) denote the sets of even and odd roots in \( \Phi \), respectively. The Weyl vector \( \rho \) is defined as

\[
\rho = \frac{1}{2} \sum_{\alpha \in \Phi_0 ^+} \alpha - \frac{1}{2} \sum_{\beta \in \Phi_1 ^+} \beta,
\]

(5.9)

where \( \Phi_i ^+ := \Phi_i \cap \Phi^+ \), for \( i = 0, 1 \). We recall that a weight \( \lambda \) is typical if \((\lambda + \rho, \alpha) \neq 0\), for any \( \alpha \in \Phi_1 \) and is atypical otherwise; see, e.g., [28, Section 2.2.6].
5.3.2. Example: $\mathfrak{g} = \mathfrak{gl}(1\vert2)$ We now consider $\mathfrak{g} = \mathfrak{gl}(1\vert2)$. The sets $\Phi^+_0, \Phi^+_1$ are given as follows:

$$\Phi^+_0 = \{e_2 - e_3\}, \quad \Phi^+_1 = \{e_1 - e_2, \, e_1 - e_3\}. \quad (5.10)$$

Also, we set $F_{ij} := E_{ij}$, for $1 \leq j < i \leq 3$. Let $\zeta \in \mathcal{I}$. Note that standard Whittaker modules and Verma modules coincide in the case when $\zeta = 0$. Throughout this subsection, we let $\zeta$ be regular, namely, $\zeta(E_{23}) \neq 0$. In this case we have $M(\lambda, \zeta) = L(\lambda, \zeta)$ for any $\lambda \in \mathfrak{h}^*$ by [50, Theorem 3.6.1].

In this subsection, we will construct composition series of standard Whittaker modules of $\widehat{\mathcal{N}}(\zeta)$ explicitly by finding their Whittaker vectors. Similar computation was also given in [17, Section 5.1], where the authors concluded that all standard Whittaker modules and Verma modules coincide in the case when $\zeta$ is not simple. It is worth pointing out that the assumption of typical weight $\lambda$ is needed to be added in the calculation in [17, Section 5.1].

We define the Chevalley generators $f := E_{32}, \, e := E_{23}$ and $h := E_{22} - E_{33}$ for $[\mathfrak{g}_0, \mathfrak{g}_0] \cong \mathfrak{sl}(2)$. The Casimir element $\Omega$ of $\mathfrak{g}_0$ is given by $\Omega := 4fe + h^2 + 2h$. Let $z := E_{11} + \frac{1}{2}(E_{22} + E_{33}) \in Z(\mathfrak{g}_0)$. For $\lambda \in \mathfrak{h}^*$, let $\chi^0_\lambda : Z(\mathfrak{g}_0) \to \mathbb{C}$ be the central character associated with $\lambda$. We set $a := \zeta(e) \neq 0, \, b := \chi^0_\lambda(\Omega)$ and $c := \chi^0_\lambda(z)$. Let $\lambda = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 \in \mathfrak{h}^*$ with complex numbers $\lambda_i$'s. Recall that we have defined $\bar{U} := U(\mathfrak{g}_0), \, U := U(\mathfrak{g}_0)$.

The following lemma is a consequence of [24, Corollary 6.8] and [62, Proposition 2.1(3)]

**Lemma 22.** Consider $\mathfrak{g} = \mathfrak{gl}(1\vert2)$ with notations as above. Suppose that $\zeta(E_{23}) \neq 0$. Then the following are equivalent:

1. $\lambda$ is atypical.
2. $b = 4(c^2 - c)$.
3. $M(\lambda, \zeta)$ is not simple.

**Proof.** By [50, Theorem 3.9] (see also [62, Proposition 2.1(3)]) we know that the annihilators of $\mathfrak{g}_0$-modules $M(\lambda, \zeta)$ and $M(\lambda)$ coincide. The fact that Part (1) and Part (3) are indeed equivalent was established in [24, Corollary 6.8].

The equality $b = 4c(c - 1)$ holds if and only if

$$(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_3 + 2) = 4(\lambda_1 + \frac{1}{2}\lambda_2 + \frac{1}{2}\lambda_3)(\lambda_1 + \frac{1}{2}\lambda_2 + \frac{1}{2}\lambda_3 - 1),$$

which is equivalent to $(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3 - 1) = 0$ by a direct computation. This shows that Part (1) and Part (2) are equivalent. The conclusion follows. $\Box$

**Alternative proof of Lemma 22.** By a direct computation we have

$$E_{12}E_{13}F_{31}F_{21}x = (z^2 - z - \frac{1}{4}\Omega)x = (c^2 - c - \frac{1}{4}b)x,$$

for any $x \in M(\lambda, \zeta)$ (see also [24, Example 6.6]). The conclusion follows from [24, Theorem 6.7]. $\Box$

We will generalize Lemma 22 later; see Proposition 28. We now turn to the composition series of $\widehat{M}(\lambda, \zeta)$ for atypical weight $\lambda$. The following lemma will be useful.
Lemma 23. For each atypical weight \( \lambda \in \mathfrak{h}^* \), there are exactly two antidominant composition factors of \( \tilde{M}(\lambda) \). They are \( \tilde{L}(\lambda) \), \( \tilde{L}(\lambda - \alpha) \), where \( \lambda \in W \cdot \lambda \) is antidominant and \( \alpha \) is the unique positive odd root such that \( (\lambda + \rho, \alpha) = 0 \) and \( \lambda - \alpha \) is antidominant.

Proof. Suppose that \( \lambda \) is non-integral. In this case, it is known that the block \( \mathcal{O}_\lambda \) is equivalent to the principal block of \( \mathfrak{gl}(1|1) \) (see, e.g., [32, Section 4.2] for an argument), and every simple module in \( \mathcal{O}_\lambda \) is antidominant. Therefore \( \tilde{M}(\lambda) \) has exactly two desired composition factors \( L(\lambda) \) and \( L(\lambda - \alpha) \) that are all antidominant. 

Suppose that \( \lambda \) is integral. Then the conclusion follows from the BGG reciprocity and Lemma 36 in Sect. 6. \( \square \)

The \( g_0 \)-module \( M(\lambda, \zeta) \) can be regarded as a submodule of \( \text{Res} \tilde{M}(\lambda, \zeta) \). Let \( v \in \text{Wh}_\xi(M(\lambda, \zeta)) \) be a non-zero Whittaker vector. By [17, Lemma 5.6], the set
\[
\{ v_1 := v, v_2 := F_{21}v, v_3 := F_{21}F_{31}v, v_4 := 2aF_{31}v + F_{21}h v \},
\]
forms a basis for \( \text{Wh}_\xi(\tilde{M}(\lambda, \zeta)) \).

Proposition 24. Suppose that \( \zeta(E_{23}) \neq 0 \) and \( \lambda \) is atypical with \( \lambda \in W \cdot \lambda \) antidominant. Let \( \alpha \) be the unique positive odd root such that \( (\lambda + \rho, \alpha) = 0 \) and \( \lambda - \alpha \) is antidominant. Then there is a short exact sequence
\[
0 \to \tilde{U} w \to \tilde{M}(\lambda, \zeta) \to \tilde{L}(\lambda, \zeta) \to 0,
\]
where the submodule \( \tilde{U} w \cong \tilde{L}(\lambda - \alpha, \zeta) \) is generated by the Whittaker vector
\[
w = \begin{cases} 
v_2 + \frac{1}{2(1-c)}v_4, & \text{for } c \neq 1; \\
v_4, & \text{for } c = 1;
\end{cases}
\]
and \( \tilde{L}(\lambda, \zeta) \) generated by the image of \( v \) in the quotient \( \tilde{M}(\lambda, \zeta)/\tilde{U} w \).

Proof. Set \( w := Bv_2 + Cv_3 + Dv_4 \) with \( B, C, D \in \mathbb{C} \). By a direct computation we have
\[
E_{12}w = 0 \iff Bc = \frac{-1}{2}Db, \quad B = 2D(1-c), \quad C = 0;
E_{13}w = 0 \iff B = 2D(1-c), \quad C = 0.
\]

Consequently, the equality \( b = 4(c^2 - c) \) coming from Lemma 22 determines all relations between the coefficients \( B, C \) and \( D \) such that \( w \in \tilde{\text{Wh}}_\xi(\tilde{M}(\lambda, \zeta)) \):
\[
\begin{cases} 
B = 2(1-c)D, & \text{for } c \neq 1; \\
B = 0 \text{ and } D \text{ is arbitrary}, & \text{for } c = 1;
C = 0.
\end{cases}
\]

By Lemma 23 and Theorem B, the vector \( w \in \tilde{\text{Wh}}_\xi(\tilde{M}(\lambda, \zeta)) \) satisfying (5.13) generates the desired proper simple submodule \( \tilde{U} w \cong \tilde{L}(\lambda - \alpha, \zeta) \). This completes the proof. \( \square \)

We also give an alternative proof of Proposition 24 without using Theorem B and the character formulas of \( \mathfrak{gl}(2|1) \) in Sect. 6.
Alternative proof of Proposition 24. We claim that the length of a composition series of \( \text{Res} \tilde{L}(\lambda, \zeta) \) is always 2, for any atypical weight \( \lambda \in \mathfrak{h}^* \). To see this, we first note that the length of \( \text{Res} M(\lambda, \zeta) \cong \Lambda g_{-1} \otimes L(\lambda, \zeta) \) is the dimension \( \dim \Lambda g_{-1} = 4 \) by [17, Proposition 5.1] (see also [50, Theorem 4.6]). By Lemma 22, it suffices to show that the length of \( \text{Res} L(\lambda, \zeta) \) is equal to or greater than 2.

By [24, Theorem 4.1] there exists \( \mu \in \mathfrak{h}^* \) such that \( \tilde{L}(\lambda, \zeta) \) is the socle of \( \tilde{M}(\mu, \zeta) \), which is generated by \( \Lambda^{\text{top}} g_{-1} \otimes L(\mu, \zeta) \). If \( \tilde{L}(\lambda, \zeta) \) is of length one then \( \text{Res} L(\lambda, \zeta) = \Lambda^{\text{top}} g_{-1} \otimes L(\mu, \zeta) \subset \text{Res} \tilde{M}(\mu, \zeta) \). But for any nonzero \( v \in \text{Wh}_\zeta(L(\mu, \zeta)) \) we calculate

\[
E_{12} F_{21} F_{31} v = F_{31}(\frac{1}{2} h - 1) v - \frac{1}{4a} F_{21}((b - 2h - h^2)v),
\]

which is nonzero by [57, Lemma 2] (see also [17, Lemma 5.5]). Therefore we have shown that the length of \( \text{Res} \tilde{L}(\lambda, \zeta) \) is 2. Consequently, there is a short exact sequence

\[
0 \to X \to \tilde{M}(\lambda, \zeta) \to Y \to 0,
\]

where both \( X, Y \) are simple Whittaker modules such that \( X \) is generated by the Whittaker vector \( w \) from (5.13).

If \( c \neq 0, 1 \) and \( (\lambda + \rho, \beta) = 0 \), for some positive odd root \( \beta \). Then by a direct computation we have

\[
\Omega w = (4c^2 - 8c + 3)w, \quad zw = (\frac{-1}{2} + c)w,
\]

which implies that \( Uw \) admits the central character \( \lambda^\circ_{\lambda - \beta} \) associated with the weight \( \lambda - \beta \). Since \( Uw \) is a proper simple submodule, we may conclude that \( Uw \cong L(\lambda - \beta, \zeta) \cong \tilde{L}(\lambda - \alpha, \zeta) \), as desired. The remaining cases \( c = 0, 1 \) can be calculated by similar arguments. The conclusion follows. \( \square \)

The following corollary gives a description of block decomposition of \( \tilde{N}(\zeta) \) for \( g = \text{gl}(1|2) \).

**Corollary 25.** Consider \( g = \text{gl}(1|2) \). Let \( \lambda, \mu \in \mathfrak{h}^* \) be atypical and \( \zeta \in \mathcal{I} \) regular. Then \( L(\lambda, \zeta) \) and \( \tilde{L}(\mu, \zeta) \) lie in the same indecomposable block of \( \tilde{N}(\zeta) \) if and only if \( \lambda \in \mathfrak{w} \cdot (\mu + k\alpha) \), where \( \alpha \) is an odd root with \( (\mu + \rho, \alpha) = 0 \).

### 5.3.3 Criteria for simplicity of standard Whittaker modules

In this subsection, we study the simplicity of standard Whittaker modules for classical Lie superalgebras of types A and C from (1.4), (1.5). We refer to [28] for more details about the ortho-symplectic Lie superalgebras \( \mathfrak{osp}(m|2n) \). In particular, the notions of typical and atypical weights for \( \mathfrak{osp}(m|2n) \) are defined in a similar fashion; see [28, Section 2.2.6].

For any \( M \in \mathfrak{g}\text{-Mod} \), denote by \( \text{Ann}_M \) the annihilator of \( M \). Similarly, we define the annihilator \( \text{Ann}_{\mathfrak{U}(\mathfrak{t})} N \) for any subalgebra \( \mathfrak{t} \subseteq \mathfrak{g} \) and \( N \in \mathfrak{t}\text{-Mod} \). We first show that the annihilators of standard Whittaker modules and Verma modules coincide.

**Proposition 26.** Let \( g \) be a classical Lie superalgebra of type I. Then

\[
\text{Ann}_{\tilde{g}} \tilde{M}(\lambda, \zeta) = \text{Ann}_{\tilde{g}} \tilde{M}(\lambda).
\]

In particular, if \( g \) is basic (i.e., \( g \) is in (1.2)) then \( \text{Ann}_{\tilde{g}} \tilde{M}(\lambda, \zeta) \) is centrally generated for typical \( \lambda \).
Let $V$ be a simple

Lemma 27. The following lemma is taken from [24, Corollary 6.8].

For any $\lambda \in \mathfrak{h}^*$ and $\xi \in \mathcal{I}$, the following are equivalent.

(1) $\tilde{M}(\lambda, \xi)$ is simple.

(2) $\lambda$ is typical and $M(\lambda, \xi)$ is simple.

(3) $\lambda$ is typical and there is a unique $n_\xi$-antidominant weight $v \in W \cdot \lambda$ with $\lambda \geq v$.

In particular, if $\xi$ is regular then $\tilde{M}(\lambda, \xi)$ is simple if and only if $\lambda$ is typical.

Proof. The fact that Part (2) and Part (3) are indeed equivalent is a consequence of [3, Theorem 6.2]. Next, we recall that the socle $\text{soc}M(\lambda)$ of $M(\lambda)$ has a typical highest weight if and only if $\lambda$ is typical. By [62, Proposition 2.1(3)] it follows that

$$\text{Ann}_U M(\lambda, \xi) = \text{Ann}_U M(\lambda) = \text{Ann}_U \text{soc} M(\lambda). \quad (5.15)$$

Now, suppose that $\tilde{M}(\lambda, \xi)$ is simple. Then the simplicity of $M(\lambda, \xi)$ follows from the exactness of Kac functor $K(\cdot)$ and (3.7). By Lemma 27, the socle $\text{soc}M(\lambda)$ is a simple $\mathfrak{g}_0$-module of typical highest weight, and so $\lambda$ is typical. This shows the implication (1) $\Rightarrow$ (2).

Conversely, suppose that $M(\lambda, \xi)$ is simple and $\lambda$ is typical. Then the proof of direction (2) $\Rightarrow$ (1) follows by Lemma 27. Finally, by [50, Theorem 3.6.1], we know that $M(\lambda, \xi)$ is simple if $\xi$ is regular. This completes the proof. \qed

Alternative proof of Proposition 28. As has been mentioned, Parts (2) and (3) are equivalent by [3, Theorem 6.2]. It remains to show (1) $\Leftrightarrow$ (2). First, we suppose on the contrary that $\tilde{M}(\lambda, \xi)$ is simple with $\lambda$ atypical. We know that $M(\lambda, \xi)$ is simple by the exactness of the functor $K(\cdot)$. Let $\lambda \in W \cdot \lambda$ be antidominant. By [33, Theorem 51] and Lemma 27 the socle of $M(\lambda)$ is isomorphic to the socle of $K(\lambda)$, which is a simple module of antidominant highest weight $\gamma$ with $\lambda \neq \gamma$; see also [31, Theorem 4.4, Proposition 4.15]. Using the grading operator $d^0$ from [24, Sections 5.1, 5.2], we know $\gamma \notin W \cdot \lambda$, and so $\gamma \notin W \cdot \lambda$. By Theorem 5.2, $\tilde{\gamma}(\tilde{L}(\gamma)) \cong \tilde{L}(\gamma, \xi)$ is a composition factor that is not isomorphic to $\tilde{L}(\lambda, \xi)$ by Theorem 9, a contradiction.

Conversely, suppose that $\lambda$ is typical. Then we have $[\tilde{M}(\lambda) : \tilde{L}(\mu)] = [M(\lambda) : L(\mu)]$, for any $\mu \in \mathfrak{h}^*$ (see, e.g., [42, Theorem 1.3.1]). Therefore the simplicities of $M(\lambda, \xi)$ and $\tilde{M}(\lambda, \xi)$ are equivalent by Theorem B. \qed
Corollary 29. Let \( \lambda \in \mathfrak{h}^* \) and \( \zeta \in \mathcal{I} \) satisfy one of conditions (1)-(3). Then

\[
\text{Ann}_\mathfrak{g} \tilde{L}(\lambda, \zeta) = \text{Ann}_\mathfrak{g} \tilde{L}(\lambda),
\]
is centrally generated, where \( \lambda \in W \cdot \lambda \) is antidominant.

5.4. The periplectic Lie superalgebras. The standard matrix realization of the periplectic Lie superalgebra \( \mathfrak{pe}(n) \) is given by

\[
\mathfrak{pe}(n) := \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \mid A, B, C \in \mathbb{C}^{n \times n}, \ B' = B \text{ and } C' = -C \right\} \subset \mathfrak{gl}(n|n).
\]
(5.16)

The Cartan subalgebra \( \mathfrak{h} \subset \mathfrak{g}_\mathfrak{I} \) consists of diagonal matrices above. There is a standard basis of \( \mathfrak{h} \) defined as

\[
\{ H_i := E_{i,i} - E_{n+i,n+i} \mid 1 \leq i \leq n \},
\]
(5.17)

where \( E_{a,b} \in \mathfrak{gl}(n|n) \) denotes the \( (a, b) \)-matrix unit, for \( 1 \leq a, b \leq 2n \). We denote by \( \{ \epsilon_1, \epsilon_2, \ldots, \epsilon_n \} \) the dual basis for \( \mathfrak{h}^* \) with respect to the basis \( \{ H_i \mid 1 \leq i \leq n \} \).

We recall from [69, Section 5] that a weight \( \lambda = \sum_{1 \leq i \leq n} \lambda_i \epsilon_i \in \mathfrak{h}^* \) is called typical if

\[
\prod_{1 \leq i \neq j \leq n} (\lambda_i - \lambda_j + j - i - 1) \neq 0.
\]
(5.18)

The following proposition gives composition factors of the standard Whittaker module \( \tilde{M}(\lambda, \zeta) \) in terms of the Kazhdan–Lusztig combinatorics for typical weight \( \lambda \).

Proposition 30. Let \( \lambda \in \mathfrak{h}^* \) be typical. Then for any \( \zeta \in \mathcal{I} \) we have

\[
[\tilde{M}(\lambda, \zeta) : \tilde{L}(\mu, \zeta)] = [M(\lambda) : L(\nu)],
\]
(5.19)

where \( \nu \in W_\mathfrak{e} \cdot \mu \) is \( n_\mathfrak{e} \)-antidominant.

Proof. By [23, Corollary 4.4] we have \([\tilde{M}(\lambda) : \tilde{L}(\mu)] = [M(\lambda) : L(\mu)]\), for any \( \mu \in \mathfrak{h}^* \) (as also [69, Corollary 5.8]). The conclusion follows Theorem 20. □

We have the following sufficient condition for the simplicity of standard Whittaker modules over \( \mathfrak{pe}(n) \).

Corollary 31. Let \( \lambda \in \mathfrak{h}^* \) and \( \zeta \in \mathcal{I} \). Then \( \tilde{M}(\lambda, \zeta) = \tilde{L}(\lambda, \zeta) \) if either \( \lambda \) is antidominant or \( \zeta \in \mathcal{I} \) is regular. In particular, if \( \lambda \) is antidominant then we have

\[
\text{Ann}_\mathfrak{g} \tilde{L}(\lambda, \zeta) = \text{Ann}_\mathfrak{g} \tilde{L}(\lambda).
\]

Proof. It was shown in [22, Lemma 5.11] (see also [69, Lemma 3.2]) that \( \tilde{M}(\lambda) \) is simple if \( \lambda \) is antidominant. The conclusion follows from Theorem 20 and Proposition 26. □

Example 32. Consider \( \mathfrak{g} = \mathfrak{pe}(2) \). Set \( X_{12} := E_{14} + E_{23}, \ Y_{12} := E_{32} - E_{41} \). Suppose that \( \zeta \in \mathcal{I} \) is regular. Then \( M(\lambda, \zeta) \) is simple for any \( \lambda \in \mathfrak{h}^* \). We have \( \text{Wh}_\zeta (\text{Res} \tilde{M}(\lambda, \zeta)) = \mathbb{C}v \oplus \mathbb{C}Y_{12}v \), for any nonzero vector \( v \in \text{Wh}_\zeta (M(\lambda, \zeta)) \). Then the elements in the set \( \{(H_1 - H_2)^k v \mid k \geq 0 \} \) are linear independent by [57, Lemma 2]. By a direct computation we obtain that \( X_{12}Y_{12}v = (H_1 - H_2)v \), and so this verifies that \( \text{Wh}_\zeta (\tilde{M}(\lambda, \zeta)) = \mathbb{C}v \).
5.5. The equivalence $\tilde{T}$. In this subsection, we continue to work under the assumption that $\mathfrak{g}$ is a classical Lie superalgebra of type I. As has been mentioned, all results in the present paper for $\mathfrak{g}$ are valid for the (distinguished) triangular decompositions (1.7) with $n^-_1, n^+_1$ commutative. Let $\lambda \in \mathfrak{h}^*$ and $\zeta \in \mathbb{T}$ such that $\lambda$ is dominant with $W_\lambda = W_\zeta$.

We recall that the equivalence $\tilde{T}$ of the categories $\mathcal{B}_\lambda$ and $\tilde{\mathcal{N}}(\lambda + \gamma, \zeta)$ as constructed in Theorem 16. We are going to consider the effect of $\tilde{T}$ on standard and simple objects in $\mathcal{B}_\lambda$. Recall the bimodule $\mathcal{L}(N, M)$ from Sect. 4.2.2, for $M \in \mathfrak{g}$-mod, $N \in \mathfrak{g}_0$-mod. The following is an analog of [62, Proposition 5.15].

**Proposition 33.** For any $\mu \in \lambda + \gamma$, the module $\mathcal{L}(\lambda, \tilde{M}(\mu))$ has a simple top $S_{\lambda, \zeta}(\mu)$ such that

$$\tilde{T}(\mathcal{L}(\lambda, \tilde{M}(\mu))) = \tilde{M}(\mu, \zeta),$$

$$\tilde{T}(S_{\lambda, \zeta}(\mu)) = \tilde{L}(\mu, \zeta).$$

**Proof.** We first note that

$$\tilde{T}(\mathcal{L}(\lambda, \tilde{M}(\mu))) \cong \tilde{T}(\mathcal{L}(\lambda, K(\mu))),$$

where $T$ is the equivalence in Lemma 13. By [62, Proposition 5.15], we know that $T(\mathcal{L}(\lambda, K(\mu))) \cong M(\mu, \zeta)$. This claim follows. $\Box$

**Remark 34.** We recall that, for $\mathfrak{g} = \mathfrak{g}_0$, Milićić and Soergel used in [62, Section 5] the equivalence $\tilde{T}$ to compute the multiplicities of standard Whittaker modules $M(\chi \cdot \lambda, \zeta)$ ($\chi \in W$) in the case when $\lambda$ is regular and integral. Namely, there is an equivalence

$$\mathcal{N}^0(\mu) \cong \mathcal{H}_\lambda \mathcal{T} \mathcal{H}_\mu \cong \mathcal{N}^0(\zeta),$$

sending the Verma module $M(\chi^{-1} \cdot \mu)$ to the standard Whittaker module $M(\chi \cdot \lambda, \zeta)$, where $\mathcal{N}^0(\mu), \mathcal{N}^0(\zeta)$ respectively denote the (central) blocks of $\mathcal{N}^0(\mu), \mathcal{N}^0(\zeta)$ according to the central characters associated with $\mu, \lambda$, and $\mu \mathcal{H}_\lambda$ denotes the full subcategory of $\cup_{n \geq 1} \mathcal{H}(I_n^\mu)$ consisting of $(\mathfrak{g}_0, \mathfrak{g}_0)$-bimodules $X$ such that $XI_n^\mu = 0$ and $I_n^\mu X = 0$ for $n, m \gg 0$. The equivalence $\mathcal{T} \mathcal{H}_\lambda \mathcal{H}_\mu$ is given by interchanging the left and the right action via the Chevalley anti-automorphism of $\mathfrak{g} = \mathfrak{g}_0$. It is interesting to ask whether there are analogous methods for Lie superalgebras.

On the other hand, one can use Theorems 16, 20 and Proposition 33 to compute the composition factors of the "standard" Harish–Chandra bimodule $\mathcal{L}(\lambda, \tilde{M}(\mu))$ in Proposition 33 via Kazhdan–Lusztig combinatorics for $\mathfrak{g} = \mathfrak{gl}(m|n)$ and $\mathfrak{osp}(2|2n)$ established in [4,5,36].

**Remark 35.** Duflo proved that, for a semisimple Lie algebra, every primitive ideal is an annihilator ideal of a simple module in $\mathcal{O}$. The analogue of Duflo’s theorem for simple classical Lie superalgebras was established in [58, Section 2.2]; see also [22, Theorem 4.1]. It is natural to identify the annihilator ideals of simple Whittaker modules with primitive ideals from the category $\mathcal{O}$ (as considered in Corollaries 29 and 31), in the spirit of the results of Duflo and Musson.

There is an approach to the study of primitive ideals using Harish–Chandra ($\mathfrak{g}, \mathfrak{g}_0$)-bimodules as given in [22]. We are now going to give a description of the annihilator ideals of simple Whittaker modules in terms of annihilators of certain modules from the category $\mathcal{O}$ using Proposition 33.
Recall that $I_{\lambda} := \text{Ann}_U M(\lambda)$ as defined in Sect. 4.3. It follows from Theorem 12 that the subcategory $\text{Coker}(\mathcal{F} \otimes \text{Ind} M(\lambda))$ of $\mathcal{O}$ is equivalent to $\mathcal{B}(I_{\lambda})$ via the functor $\mathcal{L}(M(\lambda), -)$. Therefore we have equivalences

$$\text{Coker}(\mathcal{F} \otimes \text{Ind} M(\lambda)) \xrightarrow{\mathcal{L}(M(\lambda), -)} \mathcal{B}(I_{\lambda}) \xrightarrow{\mathcal{F}} \mathcal{N}(\Lambda, \xi),$$

sending $S_{\lambda, \xi}(\mu) \otimes_U M(\lambda)$ to $\mathcal{L}(\mu, \xi)$, with the same annihilator. We have the following natural question.

**Question.** Let $\mu' \in \mathfrak{h}^*$ such that $\text{Ann}_{U} S_{\lambda, \xi}(\mu) \otimes_U M(\lambda) = \text{Ann}_{U} \mathcal{L}(\mu')$. What is the relation between $\mu$ and $\mu'$?

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6. Appendix

6.1. Character formulae of $\mathfrak{gl}(2|1)$. The goal of this subsection is to give the list of character formulae of projective covers in the principal block of $\mathcal{O}$ of $\mathfrak{g} := \mathfrak{gl}(2|1)$ computed in [27]. With the canonical isomorphic $\mathfrak{gl}(1|2) \cong \mathfrak{gl}(2|1)$ and the BGG reciprocity, the composition factors of Verma modules over $\mathfrak{gl}(1|2)$ can be read off from character formulae in this subsection.

We choose the Borel subalgebra $\mathfrak{b} := \bigoplus_{1 \leq i, j \leq 3} \mathbb{C} E_{ij}$ and the standard Cartan subalgebra $\mathfrak{h}^* := \bigoplus_{i=1}^{3} \mathbb{C} E_{ii}$. Set $\rho \in \mathfrak{h}^*$ to be the corresponding Weyl vector. For $\mu \in \mathfrak{h}^*$, we let $M^a(\mu)$ denote Verma module over $\mathfrak{gl}(2|1)$ with highest weight $\mu - \rho$. Also, we let $P^a(\mu)$ denote the projective cover of the simple quotient of $M^a(\mu)$. For any $a, b, c \in \mathbb{C}$, we set $(a|b,c) := a e_1 + be_2 + ce_3$. We adopt the notation $P^a(\lambda) = \sum_{\mu \in \mathfrak{h}^*}(P^a(\lambda) : M^a(\mu))M^a(\mu)$ to record the Verma flag structure $P^a(\lambda)$.

**Lemma 36** [27, Section 9]. We have the following character formulae:

1. $P^a(0, 0|0) = M^a(0, 0|0) + M^a(0, 1|1) + M^a(1, 0|1)$.
2. $P^a(0, -1|1) = M^a(0, -1|1) + M^a(0, 0|0) + M^a(1, 0|1)$.
3. $P^a(-1, 0|1) = M^a(-1, 0|1) + M^a(0, -1|1) + M^a(00|0)$.
4. $P^a(0, -k|k) = M^a(0, -k|k) + M^a(0, -(k-1)|(k-1)), \text{ for } k > 1$.
5. $P^a(-k, 0|k) = M^a(-k, 0|k) + M^a(0, -k|k) + M^a(-(k-1), 0|(k-1)) + M^a(0, -(k-1)|(k-1)), \text{ for } k > 1$.
6. $P^a(k, 0| -k) = M^a(k, 0| -k) + M^a(k+1, 0| -k+1), \text{ for } k \geq 1$.
7. $P^a(0, k| -k) = M^a(0, k| -k) + M^a(k, 0| -k) + M^a(0, k+1| -k+1)$ + $M^a(k+1, 0| -k+1), \text{ for } k \geq 1$. 


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