SEMICONVOLUTIONAL BASIS GENERATORS OF THE CLUSTER
ALGEBRA OF TYPE $A_1^{(1)}$

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1. Introduction

The (coefficient-free) cluster algebra $A$ of type $A_1^{(1)}$ is a subring of the field $\mathbb{Q}(x_1, x_2)$ generated by the elements $x_m$ for $m \in \mathbb{Z}$ satisfying the recurrence relations

$$x_{m-1}x_{m+1} = x_m^2 + 1 \quad (m \in \mathbb{Z}).$$

This is the simplest cluster algebra of infinite type; it was studied in detail in [2, 6].

Besides the generators $x_m$ (called cluster variables), $A$ contains another important family of elements $s_0, s_1, \ldots$ defined recursively by

$$s_0 = 1, \quad s_1 = x_0x_3 - x_1x_2, \quad s_n = s_1s_{n-1} - s_{n-2} \quad (n \geq 2).$$

As shown in [2, 6], the elements $s_1, s_2, \ldots$ together with the cluster monomials $x_m^p x_{m+1}^q$ for all $m \in \mathbb{Z}$ and $p, q \geq 0$, form a $\mathbb{Z}$-basis of $A$ referred to as the semicanonical basis.

As a special case of the Laurent phenomenon established in [3], $A$ is contained in the Laurent polynomial ring $\mathbb{Z}[x_1^\pm, x_2^\pm]$. In particular, all $x_m$ and $s_n$ can be expressed as integer Laurent polynomials in $x_1$ and $x_2$. These Laurent polynomials were explicitly computed in [2] using their geometric interpretation due to P. Caldero and F. Chapoton [1]. As a by-product, there was given a combinatorial interpretation of these Laurent polynomials, which can be easily seen to be equivalent to the one previously obtained by G. Musiker and J. Propp [5].

The purpose of this note is to give short, self-contained and completely elementary proofs of the combinatorial interpretation and closed formulas for the Laurent polynomial expressions of the elements $x_m$ and $s_n$.

2. Results

We start by giving an explicit combinatorial expression for each $x_m$ and $s_n$, in particular proving that they are Laurent polynomials in $x_1$ and $x_2$ with positive integer coefficients. By an obvious symmetry of relations (II), each element $x_m$ is obtained from $x_{3-m}$ by the automorphism of the ambient field $\mathbb{Q}(x_1, x_2)$ interchanging $x_1$ and $x_2$. Thus, we restrict our attention to the elements $x_{n+3}$ for $n \geq 0$.  

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Following [2, Remark 5.7] and [4, Example 2.15], we introduce a family of Fibonacci polynomials $F(w_1, \ldots, w_N)$ given by
\begin{equation}
F(w_1, \ldots, w_N) = \sum_{D} \prod_{k \in D} w_k,
\end{equation}
where $D$ runs over all totally disconnected subsets of $\{1, \ldots, N\}$, i.e., those containing no two consecutive integers. In particular, we have
\begin{equation}
F(\emptyset) = 1, \quad F(w_1) = w_1 + 1, \quad F(w_1, w_2) = w_1 + w_2 + 1.
\end{equation}
We also set
\begin{equation}
f_N = x_1^\left\lfloor \frac{N+1}{2} \right\rfloor x_2^\left\lfloor \frac{N}{2} \right\rfloor F(w_1, \ldots, w_N)|_{w_k=x_{\langle k \rangle}^2},
\end{equation}
where $\langle k \rangle$ stands for the element of $\{1, 2\}$ congruent to $k$ modulo 2. In view of (3), each $f_N$ is a Laurent polynomial in $x_1$ and $x_2$ with positive integer coefficients. In particular, an easy check shows that
\begin{equation}
f_0 = 1, \quad f_1 = \frac{x_2^2 + 1}{x_1} = x_3, \quad f_2 = \frac{x_1^2 + x_2^2 + 1}{x_1x_2} = s_1.
\end{equation}

**Theorem 2.1.** [2, Formula (5.16)] For every $n \geq 0$, we have
\begin{equation}
s_n = f_{2n}, \quad x_{n+3} = f_{2n+1}.
\end{equation}
In particular, all $x_m$ and $s_n$ are Laurent polynomials in $x_1$ and $x_2$ with positive integer coefficients.

Using the proof of Theorem 2.1 we derive the explicit formulas for the elements $x_m$ and $s_n$.

**Theorem 2.2.** [2, Theorems 4.1, 5.2] For every $n \geq 0$, we have
\begin{equation}
x_{n+3} = x_1^{n-1}x_2^{-n}(x_2^{2(n+1)} + \sum_{q+r \leq n} \binom{n-r}{q} \binom{n+1-q}{r} x_1^{2q}x_2^{2r});
\end{equation}
\begin{equation}
s_n = x_1^{-n}x_2^{-n} \sum_{q+r \leq n} \binom{n-r}{q} \binom{n-q}{r} x_1^{2q}x_2^{2r}.
\end{equation}

3. **Proof of Theorem 2.1**

In view of (3), the Fibonacci polynomials satisfy the recursion
\begin{equation}
F(w_1, \ldots, w_N) = F(w_1, \ldots, w_{N-1}) + w_N F(w_1, \ldots, w_{N-2}) \quad (N \geq 2).
\end{equation}
Substituting this into (4) and clearing the denominators, we obtain
\begin{equation}
x_{\langle N \rangle} f_N = f_{N-1} + x_{\langle N-1 \rangle} f_{N-2} \quad (N \geq 2).
\end{equation}
Thus, to prove (6) by induction on $n$, it suffices to prove the following identities for all $n \geq 0$ (with the convention $s_{-1} = 0$):
\begin{equation}
x_1x_{n+3} = s_n + x_2x_{n+2};
\end{equation}
\begin{equation}
x_2s_n = x_{n+2} + x_1s_{n-1}.
\end{equation}
We deduce (11) and (12) from (2) and its analogue established in [6, formula (5.13)]:

\[ x_{m+1} = s_1 x_m - x_{m-1} \quad (m \in \mathbb{Z}). \]  

(For the convenience of the reader, here is the proof of (13). In view of (2) and (1), we have

\[ s_1 = x_1^2 + x_2^2 + 1 = x_1 + x_3 = x_0 + 2 \quad \text{if } m = 0, \]

By the symmetry of the relations (1), this implies that

\[ s_1 = (x_{m-1} + x_{m+1})/x_m \quad \text{for all } m \in \mathbb{Z}, \]

proving (13).

We prove (11) and (12) by induction on \( n \). Since both equalities hold for \( n = 0 \) and \( n = 1 \), we can assume that they hold for all \( n < p \) for some \( p \geq 2 \), and it suffices to prove them for \( n = p \). Combining the inductive assumption with (2) and (13), we obtain

\[
x_1 x_{p+3} = x_1 (s_1 x_{p+2} - x_{p+1}) = s_1 (s_{p-1} + x_2 x_{p+1}) - (s_{p-2} + x_2 x_{p}) = (s_1 s_{p-1} - s_{p-2}) + x_2 (s_1 x_{p+1} - x_p) = s_p + x_2 x_{p+2},
\]

and

\[
x_2 s_p = x_2 (s_1 s_{p-1} - s_{p-2}) = s_1 (x_{p+1} + x_1 s_{p-2}) - (x_p + x_1 s_{p-3}) = (s_1 x_{p+1} - x_p) + x_1 (s_1 s_{p-2} - s_{p-3}) = x_{p+2} + x_1 s_{p-1},
\]

finishing the proof of Theorem 2.1.

4. **Proof of Theorem 2.2**

Formulas (7) and (8) follow from (11) and (12) by induction on \( n \). Indeed, assuming that, for some \( n \geq 1 \), formulas (7) and (8) hold for all the terms on the right hand side of (11) and (12), we obtain

\[
x_{n+3} = x_1^{-1} (s_n + x_2 x_{n+2}) = x_1^{-n} x_2^{-n} \left( \sum_{q+r \leq n} \binom{n-r}{q} \binom{n-q}{r} x_1^{2q} x_2^{2r} \right) + \binom{2(n+1)}{2} \sum_{q+r \leq n-1} \left( \binom{n-1-r}{q} \binom{n-q}{r} x_1^{2q} x_2^{2(r+1)} \right)
\]

\[
= x_1^{-n} x_2^{-n} \binom{2(n+1)}{2} + \binom{2(n+1)}{2} \sum_{q+r \leq n} \left( \binom{n-r}{q} \binom{n-1-q}{r} + \binom{n-1-q}{r} \right) x_1^{2q} x_2^{2r}.
\]
and
\[ s_n = x_2^{-1}(x_{n+2} + x_1 s_{n-1}) \]
\[ = x_1^{-n} x_2^{-n} \left( x_2^{2n} + \sum_{q+r \leq n-1} \binom{n-1-r}{q} \binom{n-q}{r} x_1 x_2^{2q} x_2^{2r} \right. \]
\[ + \sum_{q+r \leq n-1} \binom{n-1-r}{q} \binom{n-1-q}{r} x_1^{2(q+1)} x_2^{2r} \right) \]
\[ = x_1^{-n} x_2^{-n} \sum_{q+r \leq n} \left( \binom{n-1-r}{q} + \binom{n-1-r}{q-1} \right) \binom{n-q}{r} x_1^{2q} x_2^{2r} \]
\[ = x_1^{-n} x_2^{-n} \sum_{q+r \leq n} \binom{n-r}{q} \binom{n-q}{r} x_1^{2q} x_2^{2r}, \]
as desired.

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