Asymptotic approximation of degenerate fiber integrals.

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Abstract
We study asymptotics of fiber integrals depending on a large parameter. When the critical fiber is singular, full-asymptotic expansions are established in two different cases: local extremum and isolated real principal type singularities. The main coefficients are computed and invariantly expressed. In the most singular cases it is shown that the leading term of the expansion is related to invariant measures on the spherical blow-up of the singularity. The results can be applied to certain degenerate oscillatory integrals which occur in spectral analysis and quantum mechanics.

Key words: Asymptotic approximation; Fiber integrals; Degenerate oscillatory integrals.

1 Introduction and statement of the main result.

In [3] J. Brüning & R. Seeley have studied asymptotic expansions of integrals:

\[ H(z) = \int_0^\infty \sigma(xz, x)dx, \ z \to \infty, \quad (1) \]

where \( \sigma(x, \xi) \) is a singular symbol. The result of [3] is quite remarkable, in particular because it can be directly applied to spectral analysis. Many asymptotic questions can be reduced to the study of the previous problem but it is also

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interesting to consider a generalization:

\[ I(z) = \int_X g(zf(x), x)dx, \ z \to \infty. \quad (2) \]

where \( g : \mathbb{R} \times X \to \mathbb{R}, \ f : X \to \mathbb{R} \) are smooth and \( X \) is a smooth differentiable manifold equipped with the \( C^\infty \) density \( dx \). As a motivation, we remark that the trace formula for certain linear self-adjoint operators can be stated in the form of Eq. (2). In the semi-classical regime when \( f \) is a quadratic form this problem was involved in [2]. See also [5] for the case of \( f \) with an isolated and degenerate singularity associated to an homogeneous definite jet.

**General assumptions.** Throughout this work, we will assume that \( |f| \) is strictly positive outside of a compact set and that the Fourier transform \( \hat{g} \) w.r.t. \( t \) exists with \( \partial^k_t \hat{g}(t, x) \in L^1(\mathbb{R} \times X), \forall k \). For this reason \( g \) will be called a symbol. □

**Remark 1** This assumption on \( g \) is strong but can be weakened. More general conditions can be found in [7]. Mainly, this condition will be used to reach integrals with compact supports.

As \( z \to \infty \), the asymptotic behavior of \( I(z) \) is related to the critical fiber:

\[ \mathcal{S} = f^{-1}(\{0\}) = \{x \in X / f(x) = 0\}. \quad (3) \]

This can easily be viewed with the Fourier inversion formula:

\[ I(z) = \int_X \int_{\mathbb{R}} e^{itzf(x)} \hat{g}(t, x)dtdx, \ z \to \infty, \quad (4) \]

where \( \hat{g}(t, x) \) is the normalized Fourier transform of \( g \) w.r.t. \( t \):

\[ \hat{g}(t, x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\tau t} g(\tau, x)d\tau, \quad (5) \]

In Eq. (4), the stationary points w.r.t. \( t \) are precisely given by \( \mathcal{S} \), i.e. \( I(z) \) is asymptotically supported by \( \mathcal{S} \). Since \( f \) is smooth \( \mathcal{S} \) is closed and according to the general assumptions above we obtain:

\[ (H_0) \ The \ fiber \ \mathcal{S} \ is \ compact. \quad (6) \]

\( (H_0) \) is not absolutely necessary, e.g. if \( g \) decreases fast enough near the boundary of \( X \cap \mathcal{S} \). But, to simplify, we will only consider the compact case. To obtain an easier formulation of the problem we recall an elementary result.

**Lemma 2** If \( \partial^k_t \hat{g} \in L^1(\mathbb{R} \times X), \forall k \in \mathbb{N} \), modulo terms \( \mathcal{O}(z^{-\infty}) \), asymptotics of Eq. (2) are not changed by assuming that \( \hat{g} \) is compactly supported near \( \mathcal{S} \).
Proof. With $\mathcal{S}$ compact we choose a cut-off function $\Theta \in C^\infty_0(X)$ such that $\Psi = 1$ near $\mathcal{S}$ and $0 \leq \Psi \leq 1$. We shall estimate the error integral:

$$E(z) = \int_X \int_\mathbb{R} e^{izf(x)}\hat{g}(t,x)(1-\Psi(x))dt dx, \quad z \to \infty.$$  \hfill (7)

With $L = -(i/zf(x))\partial_t$, we have $L^k e^{izf(x)} = e^{izf(x)}$, $\forall k \in \mathbb{N}$. By integration by parts and since $|f(x)| \geq C$ on supp$(1-\Psi)$, we obtain:

$$|E(z)| \leq (Cz)^{-k}||\partial_t^k \hat{g}(t,x)||_{L^1(\mathbb{R} \times X)} = O(z^{-k}), \quad \forall k \in \mathbb{N}.$$ \hfill (8)

This gives the desired result, with our hypothesis on $g$.

Lemma 1 allows to consider only integrals with compact support w.r.t. $x$ which simplifies all questions of convergence. Notice that we can weaken the condition on $g$ to $\partial_t^k \hat{g} \in L^1(\mathbb{R} \times X)$, $\forall k \leq k_0$ with an error $O(z^{-k_0})$. We are mainly interested in the situation where $\mathcal{S}$ has an isolated singularity. If $x_0 \in \mathcal{S}$ is such a critical point, let $\Theta \in C^\infty_0(X)$ be a cut-off microlocally supported near $x_0$. We split-up our integral as $I(z) = I_r(z) + I_s(z)$, where:

$$I_r(z) = \int_X g(zf(x),x)(1-\Theta)(x)dx,$$ \hfill (9)

$$I_s(z) = \int_X g(zf(x)t,x)\Theta(x)dx.$$ \hfill (10)

This procedure can be extended with finitely many critical points on $\mathcal{S}$. The regular part $I_r$ can be treated by the generalized stationary phase method, with non-degenerate normal Hessian, which we recall in section 2. Since $I_s$ is a local object and the main contributions below concern invariant objects, there is no loss of generality to assume that supp$(\Theta)$ is an open of $\mathbb{R}^n$, $n = \dim(X)$. For $x_0 \in \mathcal{S}$ a singularity of finite order we can write the germ of $f$ as:

$$f(x) = f_k(x) + O(||(x-x_0)||^{k+1}),$$ \hfill (11)

where $f_k \neq 0$ is homogeneous of degree $k \geq 2$ w.r.t. $(x-x_0)$. The first elementary result concerns extremum attached to such homogeneous germs:

**Theorem 3** If $f$ has a local extremum $x_0$ on $\mathcal{S}$ whose jet is given by Eq. (11) (a fortiori $k$ is even), we obtain a full-asymptotic expansion:

$$I_s(z) \sim \sum_{j \in \mathbb{N}} c_j z^{-\frac{k}{2}}.$$ \hfill (12)

If $\dim(X) = n$, the leading term is given by:

$$I_s(z) = z^{-\frac{n}{2}} \left< t e^{\frac{k}{n}}, g(t,x_0) \right> \frac{1}{k} \int_{\mathbb{S}^{n-1}} |f_k(\theta)|^{-\frac{n}{2}} d\theta + O(z^{-\frac{n+1}{2}}),$$ \hfill (13)

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with \( t_e = \max(t, 0) \) if \( x_0 \) is a minimum and \( \max(-t, 0) \) for a maximum.

The reader can observe that Theorem 3 includes the case of a non-integrable singularity on \( \mathcal{G} \) for \( k > n \). Accordingly, always for \( k > n \), the contribution of the critical point is bigger than the regular contribution (see section 2).

**Remark 4** The result stated in Eq. (13) includes the case \( k = 2 \) and the integral on the sphere can be expressed in terms of special functions. Under ad-hoc conditions, Theorem 3 can be extended to a singularity which is a sum of positively homogeneous singularities, e.g. \( f(x_1, x_2) = ||x_1||^4 + ||x_2||^6 \).

The case of non-extremum degenerate critical points is more difficult. Since this problem can be very complicated in general position we impose:

\( (H_1) \) \( \mathcal{G} \) has a unique critical point \( x_0 \). Moreover, \( f_k \) defined in Eq. (11) is non-degenerate in the sense that:

\[
\nabla f_k \neq 0 \quad \text{on} \quad \mathbb{S}^{n-1} \cap \{ f_k = 0 \}.
\]

By homogeneity, \( (H_1) \) is equivalent to "\( f_k \) has an isolated singularity". Observe that this condition is very close to Hörmander’s real principal type condition for distributions. We define the integrated density of \( f_k \) on the sphere (or co-area) as:

\[
\text{LVol}(w) = \int_{\{ f_k(\theta) = w \}} |dL(\theta)|, \quad dL(\theta) \wedge df_k(\theta) = d\theta.
\]

Where \( dL \) is the \( n-2 \) dimensional Liouville measure induced by \( f_k \) on \( \mathbb{S}^{n-1} \), i.e. the Riemannian density induced by \( f_k \) on the standard density of \( \mathbb{S}^{n-1} \). Note that \( (H_1) \) insures that LVol\( (w) \) is well defined and smooth near the origin.

**Theorem 5** Under the previous assumptions and if \( x_0 \) satisfies \( (H_1) \), the singular part of our integral admits a full asymptotic expansion:

\[
I_s(z) \sim \sum_{j=0}^{\infty} c_j z^{-\frac{\pi}{k}} + \sum_{j=0}^{\infty} d_j z^{-j} \log(z).
\]

a) If \( k > n \) (non-integrable singularity), the leading term is:

\[
I_s(z) = C_0 z^{-\frac{\pi}{k}} + O(z^{-\frac{n+1}{k}} \log(z)),
\]

where the distributional coefficient \( C_0 \) is given by:

\[
\frac{1}{k} \left\langle t_{\frac{\pi}{k}}^{\frac{\pi}{k}-1}, g(t, x_0) \right\rangle \int_{\{ f_k \geq 0 \}} |f_k(\theta)|^{-\frac{\pi}{k}} d\theta + \left\langle t_{\frac{\pi}{k}}^{\frac{\pi}{k}-1}, g(t, x_0) \right\rangle \int_{\{ f_k \geq 0 \}} |f_k(\theta)|^{-\frac{\pi}{k}} d\theta.
\]
b) If $k$ divides $n$ the leading term is logarithmic:

\[ I_s(z) = D_0 z^{-\frac{n}{k}} \log(z) + \mathcal{O}(z^{-\frac{n}{k}}), \]  

with $n = kp$, we obtain:

\[ D_0 = \frac{1}{k} \left( \frac{d^{p-1}}{dw^{p-1}} \text{Lvol}(0) \right) \int_{\mathbb{R}} |t|^{p-1} g(t, x_0) dt. \]

c) If $n > k$ and $n/k \notin \mathbb{N}$ (integrable singularity) we obtain the same result as in a) but with the modified distributions:

\[
\left\langle t^n, g(t, x_0) \right\rangle \left\langle \frac{d^n}{dw^n} w^n z^{\frac{n}{k} \mp 1}, \text{Lvol} \right\rangle + \left\langle t^n, g(t, x_0) \right\rangle \left\langle \frac{d^n}{dw^n} w^n z^{\frac{n}{k} \mp 1}, \text{Lvol} \right\rangle,
\]

where the derivatives w.r.t. $w$ are normalized distributional derivatives.

The meaning of normalized derivative it that one choose the normalization:

\[
\left\langle \frac{d^n}{dw^n} w^n z^{\frac{n}{k} \pm 1}, f(w) \right\rangle := \left\langle w^n z^{\frac{n}{k} \pm 1}, f(w) \right\rangle,
\]  

for all $f \in C_0^\infty$ with $f = 0$ in a neighborhood of the origin. The distributional bracket involving Lvol is detailed in section 3. Results c) and b) for $p \geq 2$ are not intuitive and are certainly difficult to be reached without geometry. In particular, for applications to oscillatory integrals (see below) one has to work in the dual since both Fourier transforms w.r.t. $t$ in c) and b) are distributional. In c), the $n$-th derivative is arbitrary and the result is the same for any normalized derivative of order greater than $E(n/k)$.

**Remark 6** Results a) and b) for $p = 1$ are certainly interesting for spectral analysis since these contributions are bigger than $I_r(z) = \mathcal{O}(z^{-1})$. As in Theorem 3, non-integrable singularities have a dominant contribution. Hence, the leading term of $I(z)$ is always an invariant.

**Application to oscillatory integrals.**

A typical application of Theorems 3&5 can be the asymptotic expansion of distributional traces of quantum propagators. Hence, it is interesting to remark that our results can be extended to asymptotic integrals:

\[
\tilde{I}(z) = \int_X G(z, z f(x), x) dx, \ z \to +\infty,
\]
if $G$ admits an asymptotic expansion with a priori estimates, i.e.:

$$G(z, t, x) = \sum_{j=0}^{l} z^{-\alpha_j} g_j(t, x) + R_l(z, t, x),$$

$$\forall k \in \mathbb{N}^*: ||R_k(z, t, x)||_{L^1(\mathbb{R} \times \mathcal{X})} = \mathcal{O}(z^{-(\alpha_k + \varepsilon)}), \quad \varepsilon > 0,$$

where $(\alpha_j)_j$ is a strictly increasing sequence and $R$ controlled by uniform estimates. Similarly, we can consider expansions in term of $z^{-\alpha_j} \log(z)^m$. This notion of graduation w.r.t. $z$ allows to apply our results but, to simplify the exposition, in this work we will just consider the case of an integral of a symbol $g(t, x)$. We can treat degenerate oscillatory integrals:

$$O(z) = \int_{\mathbb{R} \times \mathcal{X}} e^{iztf(x)} a(t, x, z) dt dx, \quad z \to +\infty,$$

providing that $f$ satisfies the conditions of Theorem 3 or 5. An important application in quantum mechanics is the case $X = T^*\mathbb{R}^n$ where, after some technical modifications, the localized (distributional) trace of $h$-pseudors:

$$\text{Tr}_{h}(A_h - E) := \text{Tr} \int_{\mathbb{R}} \hat{u}(t)e^{\hat{h}t(A_h - E)} dt, \quad \hat{u} \in C_0^\infty(\mathbb{R}), \quad E \in \mathbb{R},$$

can be written as a locally finite sum of oscillatory integrals:

$$\int_{\mathbb{R} \times T^*\mathbb{R}^n} e^{\hat{h}(S(t,y,\eta) - (y,\eta) - tE)} b(h, t, y, \eta) dt dy d\eta,$$

where $b(h, \bullet) \sim \sum h^{-k} b_k$ satisfies a priori estimates as above and $S$ is the local generating function of the group of diffeomorphism of the principal symbol of $A_h$. Here $z = h^{-1}$ is the parameter and, after a discussion based on classical mechanics, Eq. (21) can be reformulated as in Eq. (20) where $\mathcal{S}$ is the energy surface of level $E$. For more details, we refer to [2,4,5].

We recall now basics on homogeneous transformations, some of them will also be used below. The Melin transform of a function $h$ is defined as:

$$M[h](\xi) = \int_0^\infty h(t) t^{\xi-1} dt.$$  

(22)

Generally $M[h](\xi)$ is analytic, for example, in the strip $\Re(\xi) \in ]a, b[ $ with:

$$a = \inf_{x \in \mathbb{R}} \int_0^\infty |h(t)| t^{x-1} dt < \infty, \quad b = \sup_{x \in \mathbb{R}} \int_0^\infty |h(t)| t^{x-1} dt < \infty,$$
and when $M[h](c + iy) \in L^1(\mathbb{R}, dy)$ we have the inversion formula:

$$h(t) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} t^{-\xi} M[h](\xi) d\xi. \quad (23)$$

By elementary changes of path we obtain that:

$$M[e^{\pm it}]\{\xi\} = e^{\pm i\pi \xi/2} \Gamma(\xi), \quad \text{Re}(\xi) \in ]0, 1[.\]$$

We write $O(z) = O_+(z) + O_-(z)$, where:

$$O_\pm(z) = \int_{\{\pm tf(x) > 0\}} e^{izf(x)} a(t, x) dt dx.$$

Melin’s inversion formula leads to the distributional formulation:

$$O_\pm(z) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} e^{\pm i\xi \pi/2} \Gamma(\xi) z^{-\xi} \int_{\{\pm tf(x) > 0\}} |tf(x)|^{-\xi} a(t, x) dt dx d\xi,$$

where $0 < c < \delta$ for $\delta > 0$ depending on $f$. If no elementary method works, one can construct explicit meromorphic extensions of distributions:

$$\langle T^+_f(\xi), a \rangle = e^{+i\xi \pi/2} \Gamma(\xi) z^{-\xi} \int_{\{tf(x) > 0\}} (tf(x))^{-\xi} a(t, x) dx dt,$$

$$\langle T^-_f(\xi), a \rangle = e^{-i\xi \pi/2} \Gamma(\xi) z^{-\xi} \int_{\{tf(x) < 0\}} |tf(x)|^{-\xi} a(t, x) dx dt,$$

and use Cauchy’s residue formula to get asymptotic w.r.t. $z \to \infty$. This method is described in [8] when $f$ is monomial, which is a generic situation when $f$ is analytic on $\text{supp}(a)$ by Hironaka’s theorem of resolution of singularities [6]. Also in many cases it is possible to reach this setting by a non-analytic change of coordinates. This approach is very interesting but a great disadvantage is that there is 4 domains, a sequence of poles (of order 3) which does not contribute at all in the expansion and the complex factors $\exp(\pm i\pi \xi/2) \Gamma(\xi)$ lead to long calculations in presence of multiple poles.

In fact we will reach Theorem 3&5 more directly and by a method which avoids Fourier analysis until the last step of the proof.

## 2 Case of $\mathcal{G}$ regular and non-degenerate critical points.

If $f$ is regular on $\mathcal{G}$, then $\mathcal{G}$ is a compact and smooth submanifold.
Proposition 1 Under the previous assumptions and if $\mathcal{S}$ is a regular surface, $I(z)$ admits a full asymptotic expansion in powers of $z^{-1}$ with:

$$I(z) = \frac{1}{z} \int_{x \in \mathcal{S}} \int g(t, x) dt d\mathcal{S}(x) + O(z^{-2}),$$

where $d\mathcal{S}$ is the invariant surface measure of $\mathcal{S}$. The same result holds for the integral $I_r(z)$ with insertion of the cut-off in the integral.

By the implicit functions theorem and compactness, we pick some coordinates $y$, defined near $\mathcal{S}$, and a finite partition of unity $\Omega_j$ covering $\mathcal{S}$, such that $f$ is diffeomorphic to the coordinate $y_1$ in $\text{supp}(\Omega_j)$. By Lemma 2 we obtain:

$$I(z) = \sum I_j(z) + O(z^{-\infty}),$$

$$I_j(z) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^{n-1}} e^{izy_1} y^*(\Omega_j(x) \hat{g}(t, x)|Jy|) dt dy_1 dy_2 ... dy_n,$$

where $y^*$ and $|Jy|$ are respectively the pullback by $y$ and the multiplication by the standard Jacobian. Hence, we have to estimate an oscillatory integral with quadratic phase. The stationary phase Lemma, see e.g. Lemma 7.7.3 of [7] vol.1, provides a full-asymptotic expansion:

$$\int_{\mathbb{R}^2} e^{izy_1} y^*(\Omega_j(x) \hat{g}(t, x)|Jy|) dt dy_1 = \sum_{l=0}^{N-1} C_l z^{-(l+1)} + O(z^{-(N+1)}).$$

(25)

Since $\text{sign}(ty_1) = 0$, the expansion is real with leading term:

$$I_j(z) = \frac{2\pi}{z} \int_{\mathbb{R}^{n-1}} (y^*(\Omega_j(x) \hat{g}(t, x)|Jy|))(0, 0, y_2, ..., y_n) dy_2 ... dy_n + O(z^{-2}).$$

(26)

Geometrically, since $\mathcal{S}$ is locally given by $y_1 = 0$, this corresponds to integration on $\mathcal{S}$ w.r.t. the canonical $n - 1$ dimensional measure $d\mathcal{S}$. By summation over the indices $j$ we obtain the integral on the whole surface, i.e. :

$$I(z) \sim \frac{2\pi}{z} \int_{x \in \mathcal{S}} \hat{g}(0, x) d\mathcal{S}(x) = \frac{1}{z} \int_{x \in \mathcal{S}} \int g(t, x) dt d\mathcal{S}(x).$$

(27)

Remark 7 Eq. (27) is a particular (flat) case of stationary phase formulas with a smooth compact manifold of critical points and non-degenerate transverse Hessian. $d\mathcal{S}$ is the Liouville-measure of classical mechanics or Guelfand-Leray-measure in theory of singularities. The oscillatory representation of delta-Dirac distributions, by mean of Schwartz kernels, provides a very natural definition of this object.
Now, assume that $f$ admits a single non-degenerate critical point on $\mathcal{S}$. As usually, non-degenerate means that $Q(\xi) = \frac{1}{2} \langle \xi, d^2 f(x_0) \xi \rangle$ is a non-degenerate quadratic form. After perhaps a reduction of the cut off $\Theta$, the Morse Lemma, with coordinates $w$, provides a simpler problem:

$$F(z) = \int_{\mathbb{R} \times \mathbb{R}^n} e^{izQ(w)} w^*(\hat{g}(t, x)\Theta(x)|Jw|(x))dw. \quad (28)$$

This kind of asymptotic problem was precisely studied in [2]. The result is:

**Proposition 2** If $Q$ has at least one even index of inertia then:

$$F(z) \sim \sum_{j=0}^{\infty} c_j z^{-1-\frac{j}{2}}. \quad (29)$$

But, if $Q$ has both indices of inertia odd then we have:

$$F(z) \sim \sum_{j=0}^{\infty} \sum_{m=0,1} c_{j,m} z^{-1-j} \log(z)^m. \quad (30)$$

The method used in section 3.2 allows to find this result. Our proof is not simpler and that's why we refer to Proposition 3.4 and Theorem 3.5 of [2] for a detailed proof and discussion of the coefficients. This discussion is necessary because the exponential distributions of Eq. (28) possess spherical symmetries.

### 3 Proof of the main results.

#### 3.1 Critical points attached to a local extremum.

We start by the case where $f$ admits a local minimum $x_0$ on the fiber $\mathcal{S}$. We split-up our integral $I(z)$ via Eqs. (9,10). $I_r$ can be treated as in section 2 and from now we concentrate our attention on $I_s$. With the extremum condition, $x_0$ is isolated on $\mathcal{S}$ and $\mathcal{S} \cap \text{supp}(\Theta) = \{x_0\}$ for $\text{supp}(\Theta)$ small enough. To simplify notations we identify $x_0$ with the origin and we recall that we consider:

$$f(x) = f_k(x) + \mathcal{O}(||x||^{k+1}), \quad (31)$$

where $f_k$ is homogeneous of degree $k$ and definite positive. Using polar coordinates, the Taylor formula shows that:

$$f(r\theta) = r^k(f_k(\theta) + R(r, \theta)), \quad (32)$$
with \( R(0, \theta) = 0 \) and where we note again \( f_k(\theta) \) the restriction of \( f_k \) to \( S^{n-1} \). If \( \text{supp(\Theta)} \subset B(0, r_0) \) is chosen small enough we have:

\[
f_k(\theta) + R(r, \theta) \neq 0, \forall \theta \in S^{n-1}, \forall r \in [0, r_0[.
\]

Hence, with the homogenous coordinates \( v = (u, \theta) \) where:

\[
u(r, \theta) = r(f_k(\theta) + R(r, \theta))^\frac{1}{k},
\]

we can express our integral as:

\[
I_s(z) = \int_{\mathbb{R}^+} G(z u^k, u) du.
\]

The new symbol \( G \) is obtained by pullback and integration:

\[
G(t, u) = \int_{S^{n-1}} v^*(g(t, r\theta)\Theta(r\theta)r^{n-1}|J(v)|)d\theta.
\]

The existence of the asymptotic expansion is a consequence of:

**Lemma 8** For \( a \) in \( C_0^\infty(\mathbb{R} \times \mathbb{R}_+) \), the following asymptotic expansion holds:

\[
J(z) = \int_{\mathbb{R}^+} a(z u^k, u) du \sim \sum_{j \geq 0} z^{-\frac{N+1}{k}} d_j(a), \ z \to \infty,
\]

where the \( d_j \) are universal distributions given by:

\[
d_j = \frac{1}{k^j} \frac{1}{j!} (\tau_+^{\frac{j+1-k}{k}} \otimes \delta_0^{(j)}(u)).
\]

**Proof.** The crucial point is that \( u^k \) is increasing on \( \mathbb{R}_+ \). We have:

\[
J(z) = z^{-\frac{1}{k}} \int_0^\infty a(u^k, \frac{u}{z^k}) du.
\]

A Taylor expansion w.r.t. the second argument at the origins gives:

\[
a(u^k, \frac{u}{z^k}) = \sum_{l=0}^N \frac{1}{l!} z^{-\frac{l}{k}} u^l \frac{\partial^l a}{\partial u^l}(u^k, 0) + z^{-\frac{N+1}{k}} R_{N+1}(u, z),
\]

where \( R_{N+1}(u, z) \) is integrable w.r.t. \( u \) with \( L^1 \) norm uniformly bounded in \( z \). By a new change of variable we obtain:

\[
J(z) = \frac{1}{k} \sum_{l=0}^N \frac{1}{l!} z^{-\frac{1+l}{k}} \int_0^\infty \frac{\partial^l a}{\partial u^l}(\tau, 0) \tau^{\frac{1+l-k}{k}} d\tau + O(z^{-\frac{N+1}{k}}).
\]
These coefficients are well defined since $|\tau|^{(l-k)/k} \in L^1_{\text{loc}}(\mathbb{R})$ for all $l \in \mathbb{N}$. ■

**Remark 9** This expansion holds also for a pullback by $-u^k$ if we replace $\tau^\alpha_+$ by $\tau^\alpha_-$. This allows to treat the case of a local maximum in Theorem 3. Also for an application to oscillatory integrals we obtain a nice formulation via the Fourier transform of the distributions $\tau^\alpha_\pm$, which avoids any ”regularization”.

**Proof of Theorem 3.** We treat the case of a local minimum. We apply Lemma 8 to the integral of Eq. (34) to prove the existence of the asymptotic expansion and it remains to express invariantly the leading term. With the polar coordinates $G$ vanishes up to the order $n - 1$. Consequently, we have:

$$I_s(z) = z^{-\frac{n}{k}} \frac{1}{k} \frac{1}{(n-1)!} \left\langle t^{\frac{n-k}{k}}_+ \otimes \delta^{n-1}_0, G \right\rangle + O(z^{-\frac{n+1}{k}}).$$

Starting from Eq. (35) and since:

$$|Jv|(0, \theta) = |f_k(\theta)|^{-\frac{1}{k}},$$

by elementary manipulations on delta-Dirac distributions, we obtain that:

$$I_s(z) = z^{-\frac{n}{k}} \left\langle t^{\frac{n-k}{k}}_+, g(t, 0) \right\rangle \frac{1}{k} \int_{\mathbb{S}^{n-1}} |f_k(\theta)|^{-\frac{n}{k}} d\theta + O(z^{-\frac{n+1}{k}}). \quad (40)$$

Finally, for a local maximum we replace the distributions $t^{\frac{n-k}{k}}_+$ by $t^{\frac{n-k}{k}}_-$. ■

**Remark 10** The leading term is invariantly defined since Eq. (40) involves the evaluation $g(t, z_0)$ and $f_k$ is an invariant since the $(k-1)$-jet in $z_0$ is flat.

**On the integral on the sphere.**

The integrals on $\mathbb{S}^{n-1}$ of Eq. (40) can be reformulated. For example in dimension 2 these are elliptic integrals. In higher dimensions we define:

$$I(f_k) = \int_{\mathbb{R}^n} e^{-|f_k(x)|} dx, \quad (41)$$

since $f_k$ is homogeneous we obtain:

$$I(f_k) = \int_{\mathbb{R}^+ \times \mathbb{S}^{n-1}} e^{-r^k f_k(\theta)} r^{n-1} dr d\theta = \int_0^\infty e^{-r^k u^{n-1}} du \int_{\mathbb{S}^{n-1}} |f_k(\theta)|^{-\frac{n}{k}} d\theta.$$

Hence our integral is given by:

$$\frac{1}{k} \int_{\mathbb{S}^{n-1}} |f_k(\theta)|^{-\frac{n}{k}} d\theta = \frac{I(f_k)}{\Gamma(n/k)}. \quad (42)$$
$I(f_k)$ can be computed, by elementary methods, as a product of gamma factors or hypergeometric functions. We give an elementary example in section 4.

### 3.2 Case of non-extremum critical points.

This problem is more complicated and we recall that we just treat here singularities given by $(H_1)$. As before, we just have to study $I_s(z)$ and, in local coordinates, we identify the critical point with the origin. We define $C(f_k)$ as the trace on the unit sphere of the conical set of the zeros of $f_k$, i.e.:

$$C(f_k) = \{ \theta \in \mathbb{S}^{n-1} / f_k(\theta) = 0 \} = \mathbb{S}^{n-1} \cap \{ x \in \mathbb{R}^n / f_k(x) = 0 \}. \quad (43)$$

With $(H_1)$, $C(f_k)$ is a compact smooth submanifold of $\mathbb{S}^{n-1}$ of codimension 1. To perform a blow-up of the singularity we use polar coordinates $(r, \theta)$ and the next lemma gives a resolution of the singularity w.r.t. $C(f_k)$.

**Lemma 11** In a micro-local neighborhood of the origin there exists local coordinates $y$, on the blow-up of the critical point, such that :

$$f(x) \simeq \begin{cases} y_1^k, \text{ in all directions where } f_k(\theta) > 0, \\ -y_1^k, \text{ in all directions where } f_k(\theta) < 0, \\ y_1^k y_2, \text{ locally near } C(f_k). \end{cases}$$

*Proof.* By Taylor, there exists $R$ continuous in $r = 0$ such that :

$$f(x) \simeq f(r\theta) = r^k(f_k(\theta) + R(r, \theta)). \quad (44)$$

If $\theta_0 \notin C(f_k)$ and $\theta$ is close to $\theta_0$ we simply choose :

$$(y_2, ..., y_n)(r, \theta) = (\theta_1, ..., \theta_{n-1}),$$

$$y_1(r, \theta) = r |f_k(\theta) + R(r, \theta)|^{\frac{1}{k}}.$$

In these coordinates the phase becomes $y_1^k$ if $f_k(\theta_0)$ is positive (resp. $-y_1^k$ for a negative value) and the Jacobian satisfies $|Jy|(0, \theta) = |f_k(\theta)|^{\frac{1}{k}} \neq 0$ locally. Now, let $\theta_0 \in C(f_k)$. Up to a permutation, we can suppose that $\partial_{\theta_1} f_k(\theta_0) \neq 0$. We accordingly choose the new local coordinates :

$$(y_1, y_3, ..., y_n)(r, \theta) = (r, \theta_2, ..., \theta_{n-1}),$$

$$y_2(r, \theta) = f_k(\theta) + R(r, \theta).$$

Since we have $|Jy|(0, \theta_0) = |\partial_{\theta_1} f_k(\theta_0)| \neq 0$, lemma follows. ■
**Remark 12** These coordinates are admissible and this leads to an adapted system of charts via a partition of unity on \( S^{n-1} \) introduced below. Also the condition \((H_1)\) insures the existence of a canonical measure on \( C(f_k) \).

To use Lemma 11 we introduce an adapted partition of unity on \( S^{n-1} \). We pick cut-off functions \( \Psi_j \in C_0^\infty(S^{n-1}), 0 \leq \Psi_j \leq 1, \sum \Psi_j = 1 \) in a tubular neighborhood of \( C(f_k) \), with supports chosen so that normal forms of Lemma 11 exist, for \( r \) small enough, in a conic neighborhood of \( \text{supp}(\Psi_j) \). By compactness this set of functions can be chosen finite and we obtain a partition of unity on \( S^{n-1} \) by adding \( \Psi_0 = 1 - \sum \Psi_j \) to our family. The support of \( \Psi_0 \) is not connected and we define \( \Psi_0^+ \) with \( f_k(\theta) > 0 \) on \( \text{supp}(\Psi_0^+) \), and similarly we define \( \Psi_0^- \) where \( f_k < 0 \), so that \( \Psi_0 = \Psi_0^+ + \Psi_0^- \). If we accordingly split up \( I_s(z) \) we obtain:

\[
I_s^\pm(z) = \int_{\mathbb{R} \times \mathbb{R}_+ \times S^{2n-1}} \Psi_0^\pm(\theta) g(zf(r\theta), r\theta)\Theta(r\theta)r^{n-1}drd\theta
\]

\[
= \int_{\mathbb{R}_+} G_0^\pm(\pm zy^k_1, y_1)dy_1,
\]

respectively for the directions where \( f_k(\theta) > 0 \) and \( f_k(\theta) < 0 \), also:

\[
I_s^{0,j}(z) = \int_{\mathbb{R} \times \mathbb{R}_+ \times S^{n-1}} \Psi_j(\theta) g(zf(r\theta), r\theta)\Theta(r\theta)r^{n-1}drd\theta
\]

\[
= \int_{\mathbb{R}_+ \times \mathbb{R}} G_j(zy^k_1y_2, y_1, y_2)dy_1dy_2,
\]

for the set \( C(f_k) \). The new symbols are respectively given by:

\[
G_0^\pm(t, y_1) = \int y^*(\Psi_0^\pm(\theta) g(t, r\theta)\Theta(r\theta)r^{n-1}|Jy|)dy_2...dy_n,
\]

\[
G_j(t, y_1, y_2) = \int y^*(\Psi_j(\theta) g(t, r\theta)\Theta(r\theta)r^{n-1}|Jy|)dy_3...dy_n.
\]

Hence, the singular part of our integral can be written as a finite sum:

\[
I_s(z) = I_s^-(z) + I_s^+(z) + \sum_j I_s^{0,j}(z),
\]

where each term of the r.h.s. will be treated by elementary methods. Note that \( I_s^-(z) \) and \( I_s^+(z) \) can be treated as in the previous section.

**Remark 13** Since \( y_1(r, \theta) = r \), our new symbols satisfy \( G_j(t, y_1, y_2) = O(y_1^{n-1}) \), near \( y_1 = 0 \). Since the asymptotic expansion involves delta-Dirac distributions w.r.t \( y_1 \), cf. Lemma 14 below, the dimension will cause a shift in the expansion.
For $a \in C_0^\infty(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R})$, we define the family of elementary fiber integrals:

$$I_{n,k}(z) = \int_0^\infty \left( \int_{\mathbb{R}} a(zy_1^k y_2, y_1, y_2) dy_2 \right) y_1^{n-1} dy_1.$$  

(48)

**Lemma 14** There exists a sequence of distributions $(D_{j,p})$ such that:

$$I_{n,k}(z) \sim \sum_{p=0,1} \sum_{j \geq n} D_{j,p}(a) z^{-\frac{j}{k}} \log(z)^p, \text{ as } z \to \infty,$$  

(49)

where the logarithms only occur when $(j/k)$ is an integer. As concerns the leading term, if $(n/k) \notin \mathbb{N}^*$ we obtain:

$$I_{n,k}(z) = z^{-\frac{n}{k}}d(a) + \mathcal{O}(z^{-\frac{n+1}{k}} \log(z)),$$  

(50)

with:

$$d(a) = C_{n,k} \int_0^\infty \int_{\mathbb{R}} t^{\frac{n}{k}-1} y_2^{n-\frac{n}{k}} \left( \partial^n_y a(t,0,y_2) + \partial^n_y a(-t,0,-y_2) \right) dy_2 dt.$$  

But when $n/k = p \in \mathbb{N}^*$, we have:

$$I_{n,k}(z) = \frac{1}{k} z^{-\frac{n}{k}} \log(z) \int_{\mathbb{R}} |t|^{p-1} \partial_{y_2}^{p-1} a(t,0,0) dt + \mathcal{O}(z^{-\frac{n}{k}}).$$

**Remark 15** The remainder of Eq. (50) can be optimized to $\mathcal{O}(z^{-\frac{n+1}{k}})$ when $(n+1)/k$ is not an integer, as shows the proof below.

**Proof.** By a standard density argument we can assume that the amplitude is of the form $a(s, y_1, y_2) = f(s)b(y_1, y_2)$. The justification is that our coefficients below are computed by continuous linear functionals, i.e. distributions. We define the Melin transforms of $f$ as:

$$M_\pm(\xi) = \int_0^\infty s^{\xi-1} f(\pm s) ds.$$  

(51)

We split up $I_{n,k}$ as $I_+$ and $I_-$ by separating integrations $y_2 > 0$ and $y_2 < 0$. Via Melin’s inversion formula, we accordingly obtain:

$$I_+(z) = \frac{1}{2i\pi} \int_{\gamma} M_+(\xi) z^{-\xi} \int_{\mathbb{R}_+^2} (y_1 y_2^k)^{-\xi} b(y_1, y_2) y_1^{n-1} dy_1 dy_2 d\xi,$$  

(52)

where $\gamma = c + i\mathbb{R}$ and $0 < c < k^{-1}$. Similarly we have:

$$I_-(z) = \frac{1}{2i\pi} \int_{\gamma} M_-(\xi) z^{-\xi} \int_{\mathbb{R}_+^2} (y_1 y_2^k)^{-\xi} b(y_1, -y_2) y_1^{n-1} dy_1 dy_2 d\xi,$$  

(53)
The existence of a full asymptotic expansion is a direct consequence of:

**Lemma 16** The family of distributions $\xi \mapsto (y_1 y_2^k)^{-\xi}$ on $C_0^\infty(\mathbb{R}^2_+)$ initially defined in the domain $\Re(\xi) < k^{-1}$ is meromorphic on $\mathbb{C}$ with poles : $\xi_{j,k} = j/k$, $j \in \mathbb{N}^*$. These poles are of order 2 when $\xi_{j,k} \in \mathbb{N}^*$ and of order 1 otherwise.

**Proof.** We form the Bernstein-Sato polynomial $b_k$ attached to our problem :

$$T(y_2 y_1^k)^{1-\xi} := \frac{\partial}{\partial y_2} \frac{\partial^k}{\partial y_1^k} (y_2 y_1^k)^{1-\xi} = b_k(\xi)(y_2 y_1^k)^{-\xi},$$

$$b_k(\xi) = (1 - \xi) \prod_{j=1}^k (j - k\xi).$$

If $\Re(\xi) < k^{-1}$, $(k + 1)$-integrations by parts yield :

$$\int_{\mathbb{R}^2_+} (y_1 y_2^k)^{-\xi} f(y_1, y_2) dy_1 dy_2 = \frac{(-1)^{k+1}}{b_k(\xi)} \int_{\mathbb{R}^2_+} (y_1 y_2^k)^{1-\xi}(T f)(y_1, y_2) dy_1 dy_2.$$

Now the integral in the r.h.s. is analytic in $\Re(\xi) < 1 + k^{-1}$. After $m$ iterations the poles, with their orders, can be read off the rational functions :

$$\mathcal{R}_m(\xi) = \prod_{p=1}^m \frac{1}{b_k(\xi - p)}.$$ (54)

This gives the result since $m$ can be chosen arbitrary large. ■

**Remark 17** Starting from Eqs. (52,53) it is possible to obtain an asymptotic expansion as $z \to 0^+$. By shifting the path $\gamma$ to the left, only the poles of the Melin transforms contribute and we obtain an expansion in power of $z$. This can be predicted by a Taylor expansion of the associated oscillatory integral but the improvement is that one can establish sharp integral remainders.

Hence, the following functions are meromorphic on $\mathbb{C}$ :

$$g^\pm(\xi) = \int_{\mathbb{R}^2_+} (y_1 y_2^k)^{-\xi} b(y_1, \pm y_2) dy_1 dy_2.$$ (55)

A classical result, see e.g. [1], is that $M_\pm(c + ix) \in \mathcal{S}(\mathbb{R}_x)$ when $c \notin -\mathbb{N}$. If we shift the path of integration $\gamma$ to the right in our integral representation, Cauchy’s residue method provides the asymptotic expansion. In fact for any $d > c$, outside of the poles, we have :

$$I_+(z) = \int_{c+i\mathbb{R}} z^{-\xi} M_+(\xi) g^+(\xi) d\xi$$

$$= \sum_{c < \xi_{j,k} < d} \text{res}(z^{-\xi} M_+(\xi) g^+(\xi)) + \int_{d+i\mathbb{R}} z^{-\xi} M_+(\xi) g^+(\xi) d\xi.$$
Since $d$ is not a pole the last integral can be estimated via:

$$
| \int_{d+i\mathbb{R}} z^{-\xi} M_+ (\xi) g^+ (\xi) d\xi | \leq C(f, b) z^{-d} = \mathcal{O}(z^{-d}),
$$

(56)

where, for each $d$, the constant $C$ involves the $L^1$-norm of a finite number derivatives of $b$. This will indeed lead to an asymptotic expansion with precise remainders. Applying this method to $I_+(z)$ and $I_-(z)$ we obtain the existence of a full asymptotic expansion of the form:

$$
I(z) \sim \sum_{p=0, 1} \sum_{j \in \mathbb{N}^*} C_{j, p} z^{-\frac{j}{k}} \log(z)^p.
$$

(57)

Moreover, by Lemma 16, these logarithms only occur when $j/k$ is integer.

Now, we compute the leading term of this expansion for our particular problem. To avoid unnecessary discussions and calculations below, we remark that we can commute the polynomial weight via:

$$
T((y_1 y_2)^{1-\xi} y_1^{n-1}) = b(\xi)(y_2 y_1^k)^{-\xi} y_1^{n-1},
$$

(58)

$$
b(\xi) = (1-\xi) \prod_{j=1}^{k} (j - k\xi + n - 1).
$$

(59)

By iteration, we obtain that the poles are the rational numbers:

$$
\xi_{p,j,k,n} = p + \frac{j + n - 1}{k}, \quad j \in [1, ..., k], \quad p \in \mathbb{N}.
$$

For all $\alpha - \beta > 1$, $\beta \in \mathbb{N}^*$, we have:

$$
\int_0^\infty \partial^\beta_r (r^\alpha f(r) dr) = 0, \quad \forall f \in C_0^\infty.
$$

If we apply this to the integral w.r.t. $y_1$, there is no contribution before:

$$
\xi_0 = \frac{n}{k}.
$$

(60)

To compute the first effective residue we must distinguish out the case where $\xi_0$ is an integer or not. The optimal number of iterations to reach $\xi_0$ is $E(n/k) + 1$ but, by analytic continuation, any number bigger than this one is acceptable. A fortiori we can use $n$ iterations and our starting point will be:

$$
z^{-\xi} M_+ (\xi) (-1)^{n(k+1)} \mathfrak{B}_n(\xi) \int_{\mathbb{R}^2_+} (y_1 y_2)^{n-\xi} y_1^{n-1} T^n b(y_1, y_2) dy_1 dy_2,
$$

(61)

with:

$$
\mathfrak{B}_n(\xi) = \prod_{l=0}^{n-1} \frac{1}{b(\xi - l)}.
$$

(62)
Case of $\xi_0$ simple pole.

In this case our residue is simply given by:

$$Cz^{-\frac{n}{k}}M_{+}(\frac{n}{k}) \int_{\mathbb{R}^2_{+}} (y_1^k y_2)^{n-\frac{n}{k}} y_2^{n-1} T^n b(y_1, y_2) dy_1 dy_2,$$

with:

$$C = \lim_{\xi \to \frac{n}{k}} (-1)^{n(k+1)}(\xi - \frac{n}{k}) \mathfrak{B}_n(\xi).$$

In particular we can compute the integral w.r.t. $y_1$ via:

$$\int_{0}^{\infty} \frac{y_1^{kn-1}}{y_1^n} \frac{\partial^{kn}}{\partial y_1^{kn}} (\partial^n_{y_2} b(0, y_2)) dy_1 = (-1)^{kn}(kn - 1)! \partial^n_{y_2} b(0, y_2).$$

A similar result holds for $I_-$ and we obtain:

$$I_+(z) = z^{-\frac{n}{k}} C_{n,k} M_{+}(\frac{n}{k}) \int_{0}^{\infty} y_2^{n-\frac{n}{k}} (\partial^n_{y_2} b)(0, y_2) dy_2 + R_1(z),$$

$$I_-(z) = z^{-\frac{n}{k}} C_{n,k} M_{-}(\frac{n}{k}) \int_{0}^{\infty} y_2^{n-\frac{n}{k}} (\partial^n_{y_2} b)(0, -y_2) dy_2 + R_2(z).$$

Here $C_{n,k}$ is the canonical constant:

$$C_{n,k} = \frac{1}{k} \prod_{j=1}^{n} \frac{-1}{j - \frac{n}{k}}.$$ 

Also, according to the analysis above, each remainder is of order $O(z^{-\frac{n+1}{k}})$ if $(n + 1)/k \notin \mathbb{N}$ and $O(z^{-\frac{n+1}{k}} \log(z))$ otherwise.

Case of $\xi_0$ double pole.

If $h$ is meromorphic with a pole of order 2 in $\xi_0$ we have:

$$\text{res}(h)(\xi_0) = \frac{1}{2} \lim_{\xi \to \xi_0} \frac{\partial}{\partial \xi} (\xi - \xi_0)^2 h(\xi).$$

Applying this principle to our residue we obtain, via Leibnitz’s rule, that:

$$I_+(z) = B \log(z) z^{-\frac{n}{k}} + O(z^{-\frac{n}{k}}).$$

We can compute the distribution $B$ as before and we find:

$$B = -\frac{1}{2} D_{n,k} M_{+}(\frac{n}{k}) \int_{0}^{\infty} y_2^{n-\frac{n}{k}} (\partial^n_{y_2} b)(0, y_2) dy_2,$$

$$D_{n,k} = (-1)^{n(k+1)} \lim_{\xi \to \frac{n}{k}} (\xi - \frac{n}{k})^2 \mathfrak{B}_n(\xi).$$
Since \( p = n/k \) is an integer, by integration by parts we obtain:

\[
\int_0^\infty y_2^{n-p}(\partial_{y_2}^n b)(0, y_2)dy_2 = (-1)^{n-p+1} (n-p)! \partial_{y_2}^{n-1} b(0,0). \tag{69}
\]

Since a similar result holds for \( I_- \), we obtain the desired result by gathering all the constants and summation. Finally, we can extend our formulas since all coefficients in the expansion are of the form:

\[
\langle T^j, f \otimes b \rangle = \langle T^j_1, f \rangle \langle T^j_2, b \rangle, \quad T^j_{1,2} \in \mathcal{D}'.
\]

By linearity and continuity, the result holds for a symbol \( a(t, y_1, y_2) \).

Taking Remark 13 into account, to avoid unnecessary calculations we define:

\[
G^+_{0}(t, y_1) = y_1^{n-1} \tilde{G}^+_{0}(t, y_1), \tag{70}
\]

\[
G_j(t, y_1, y_2) = y_1^{n-1} \tilde{G}_j(t, y_1, y_2). \tag{71}
\]

Directions where \( f_k(\theta) \neq 0 \).

By Lemma 8, the first non-zero coefficient, obtained for \( l = n-1 \), is:

\[
\frac{z^{-\frac{n}{k}}}{k(n-1)!} \left\langle t_+^{n-k} \otimes \delta_0^{(n-1)}, G^+_0(t, y_1) \right\rangle = \frac{z^{-\frac{n}{k}}}{k} \int_{\mathbb{R}} t_+^{n-k} G^+_0(t,0)dt.
\]

By construction, we have:

\[
\tilde{G}^+_0(t,0) = \int_{\mathbb{S}^{n-1}} g(t,0) \Psi^+_0(\theta) |f_k(\theta)|^{-\frac{n}{k}} d\theta.
\]

A similar computation gives the contribution of \( \text{supp}(\Psi^-_0) \), and we obtain:

\[
I^+_s(z) = z^{-\frac{n}{k}} \left\langle t_+^{n-k}, g(t,0) \right\rangle \frac{1}{k} \int_{\mathbb{S}^{n-1}} \Psi^+_0(\theta) |f_k(\theta)|^{-\frac{n}{k}} d\theta + O(z^{-\frac{n+1}{k}}), \tag{72}
\]

\[
I^-_s(z) = z^{-\frac{n}{k}} \left\langle t_-^{n-k}, g(t,0) \right\rangle \frac{1}{k} \int_{\mathbb{S}^{n-1}} \Psi^-_0(\theta) |f_k(\theta)|^{-\frac{n}{k}} d\theta + O(z^{-\frac{n+1}{k}}). \tag{73}
\]

Microlocal contribution of \( C(f_k) \).

Here we examine the contribution of terms \( I^{0,j}_s(z) \). According to the analysis above, we must distinguish out the case \( k \) divides \( n \).

a) Case of \( k > n \), non-integrable singularity on \( \mathfrak{S} \).

Here \( n/k \in [0,1[ \), so that the singularity on the blow-up is integrable. Via Lemma 14, the contribution of \( I^{0,j}_s(z) \) is given by:

\[
\frac{1}{k} z^{-\frac{n}{k}} \int_{\mathbb{R}^2_+} \left| t^{\frac{n}{k}-1} y_2 \right|^{-\frac{n}{k}} \left( \tilde{G}_j(t,0,y_2) + \tilde{G}_j(-t,0,-y_2) \right) dt dy_2 + O(z^{-\frac{n+1}{k}} \log(z)).
\]
Since $\Psi_{\pm}$, $\Psi_j$ is a partition of unity on $\mathbb{S}^{n-1}$, by summation of all local contributions $I_s(z)$ is asymptotically equivalent to:

\[
\frac{z^{-\frac{n}{k}}}{k} \left( \left\langle t_{\pm}^{\frac{n}{k}-1}, g(t,0) \right\rangle \int_{\{f_k(\theta) \geq 0\}} |f_k(\theta)|^{-\frac{n}{k}} d\theta + \left\langle t_{\pm}^{\frac{n}{k}}, g(t,0) \right\rangle \int_{\{f_k(\theta) \leq 0\}} |f_k(\theta)|^{-\frac{n}{k}} d\theta \right).
\]

Note that none of these coefficients are equal unless $g$ or $f_k$ are symmetric. ■

b) Case of $p = n/k$ integer.

Here the contribution of $I_s^{0,j}(z)$ is dominant and we obtain:

\[
I_s^{0,j}(z) \sim \frac{1}{k} \log(z) z^{-p} \int_{\mathbb{R}} |t|^{p-1} \partial_{y_2}^{p-1} \tilde{G}_j(t,0,0) dt + \mathcal{O}(z^{-p}).
\]

Unless $p = 1$, there is no way to take the limit directly, and the geometric properties are still hidden in the Jacobian. But we will reach the result by the Schwartz kernel technic. Clearly, it is enough to evaluate our derivative and to integrate w.r.t. $t$. With $s = (s_1, s_2) \in \mathbb{R}^2$, we write the evaluation as:

\[
\partial_{y_2}^{p-1} \tilde{G}_j(t,0,0) = \frac{1}{(2\pi)^2} \int e^{i(s,(y_1,y_2))} (is_2)^{p-1} \tilde{G}_j(t,y_1,y_2) dy_1 dy_2 ds.
\]

Here we have used an oscillatory Schwartz kernel for $\delta_{y_1} \otimes \delta_{y_2}^{p-1}$. This integral representation allows to inverse our diffeomorphism to obtain:

\[
\partial_{y_2}^{p-1} \tilde{G}_j(t,0,0) = \frac{1}{(2\pi)^2} \int e^{i(s,(r y_2(r \theta)))} (is_2)^{p-1} t^{p-1} g(t,r \theta) \Psi_j(\theta) dt dr d\theta ds.
\]

Extending the integrand by 0 for $r < 0$, the normalized integral w.r.t. $(r, s_1)$ provides $\delta_r$. By construction $y_2(0, \theta) = f_k(\theta)$, hence:

\[
\partial_{y_2}^{p-1} \tilde{G}_j(t,0,0) = g(t,0) \frac{1}{(2\pi)} \int_{\mathbb{R} \times \mathbb{S}^{n-1}} e^{iuf_k(\theta)} (iu)^{p-1} \Psi_j(\theta) d\theta du. \quad (74)
\]

This Fourier integral makes sense with $\mathbb{S}^{n-1}$ compact. We recall the density:

\[
J_j(w) = \int_{\{f_k(\theta) = w\}} \Psi_j(\theta) dL_w(\theta), \quad (75)
\]

where $dL_w$ is the density induced by the Leray-form $dL_{f_k} : df_k \wedge dL_{f_k}(\theta) = d\theta$. Note that all these objects can be constructed by mean of local coordinates.
under the only condition that \( \text{supp}(\Psi_j) \) is small enough near \( C(f_k) \). Since \( f_k \) is continuous on \( S^{n-1} \), each \( J_j(w) \) is compactly supported and smooth near the origin. The sum over all the \( \Psi_j \) gives the geometric contribution:

\[
\frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{iuw}(iu)^{p-1} \sum_j J_j(w) dw du = \frac{d^{p-1}\text{Lvol}}{dw^{p-1}}(0).
\]

(76)

By integration w.r.t. \( t \) we obtain the result for a double pole in general position.

**Remark 18** The case \( n = k \) is directly accessible. In this case we obtain :

\[
I_s(z) = \frac{1}{k} \log(z) \frac{\text{LVol}(0)}{z} \int_{\mathbb{R}} g(t, 0) dt + O(z^{-1}).
\]

Here \( \text{LVol}(0) \) is the Liouville volume of \( C(f_k) \). Note that \( I_s \) dominates \( I_r \).

c) \( k < n \) and simple pole, integrable singularity on \( S \).

Finally, we treat the case of a simple pole with a non-integrable singularity on \( S^{n-1} \). For the positive part, the distributional coefficients are :

\[
\langle \mu^+_+, g \rangle = C_{n,k} \int_0^\infty \int_0^\infty t^{\frac{k}{2}-1} y_2^{-\frac{n}{2}} \partial_{y_2}^n G_j(t, 0, y_2) dy_2 dt.
\]

(77)

Clearly, we can use the same globalization technic as above. The sum of all \( \mu^+_+ \), completed with the main term of \( J^+_+ \), provides :

\[
\langle T_+, g \rangle = C_{n,k} \int_0^\infty |t|^{\frac{k}{2}-1} g(t, 0) dt \int_0^\infty w^{n-\frac{k}{2}} \frac{\partial^n L(w)}{\partial w^n} dw,
\]

(78)

where the integral w.r.t. \( w \) is absolutely convergent since the measure \( L(w) \) is compactly supported. A similar result holds for the directions where \( f_k(\theta) < 0 \) and we obtain the result stated in Theorem 5. 

Now we detail the construction of the distributional bracket of the part c) of Theorem 5. Let be \( \chi \in C_0^\infty, 0 \leq \chi \leq 1 \) on \( \mathbb{R} \), chosen such that \( \chi = 1 \) near the origin and \( \chi(u) = 0 \) for \( |u| \geq \varepsilon \). If \( \varepsilon > 0 \) is small enough, \( \text{Lvol} \) is smooth on \( ] -\varepsilon, \varepsilon [ \). We write the geometric contribution as :

\[
\langle T, \text{Lvol} \rangle = \langle T, \chi \text{Lvol} \rangle + \langle T, (1 - \chi)\text{Lvol} \rangle
\]

Away from the origin, e.g. for \( u > 0 \), we obtain directly :

\[
C_{n,k} \left< \frac{d^n}{du^n} \mu^+_+, (1 - \chi)(u)\text{Lvol}(u) \right> = \frac{1}{k} \int_{\{f_k(\theta) > 0\}} (1 - \chi)(f_k(\theta))|f_k(\theta)|^{-\frac{k}{2}} d\theta.
\]

(79)
Note that Eq. (67) for $C_{n,k}$ justifies the normalization of Eq. (19). On $\text{supp}(\chi)$, we use the local regularity of $L\text{vol}(u)$ near $u = 0$ and integrations by parts to conclude. Finally, to obtain a totally rigorous treatment, we remark that since $C(f_k)$ is compact and $f_k$ is continuous we can choose our partition of unity such that $\sum \Psi_j = 1$ for $|f_k(\theta)| \leq 2\varepsilon$.

**Remark 19** The key point here is that we can put in duality the distributions $\partial_n^\alpha|u|^{n-\alpha}$ and $L\text{vol}(u)$ since their singular supports are disjoints.

4 Elementary examples.

**A family of extremum.** If $p$ is an even integer and $a = (a_1, \ldots, a_n) \in (\mathbb{R}_+^*)^n$, we consider a $n$ dimensional integral obtained by integration in the fiber of:

$$f_{a,p}(x) = \sum_{j=1}^{n} a_j x_j^p$$

(80)

For example:

$$\int_{\mathbb{R}^n} e^{-z f_{a,p}(x)} dx = \prod_{j=1}^{n} \left( \int_{\mathbb{R}} e^{-2a_j x_j^p} dx \right) = (2\Gamma(1 + \frac{1}{p}))^n z^{-\frac{n}{p}} \prod_{j=1}^{n} a_j^{-\frac{1}{p}},$$

with $g(t, x) = e^{-t}$ symbol of order 0 on $\mathbb{R}^n$. Since we have here a local minimum, the distributional factor is simply:

$$\int_{0}^{\infty} t^{\frac{n-p}{p}} e^{-t} dt = \Gamma(n/p).$$

(81)

The Jacobian of the standard polar coordinates gives the value of the generalized elliptic integrals:

$$\int_{S^{n-1}} (f_{a,p}(\theta))^{-\frac{n}{p}} d\theta = \frac{1}{p} \frac{(2\Gamma(1 + \frac{1}{p}))^n}{\Gamma(n/p)} \prod_{j=1}^{n} a_j^{-\frac{1}{p}}.$$  

(82)

Note that Eq. (82) can be analytically continued with a phase $\sum a_j |x_j|^\alpha$, $\alpha > 0$.

**Conical singularities.** In 2 dimensions let be:

$$g(t, u) = \frac{e^{-|u|^2}}{(1 + t^2)}.$$
With \( f(x, y) = x^2 - y^2 \) we have \( \mathcal{S} = \{(x, x)\} \cup \{(-x, x)\} \). \( \mathcal{S} \) is not compact but the exponential decrease of \( g \) will compensate. Accordingly let be:

\[
I(z) = \int_{\mathbb{R}^2} \frac{e^{-(x^2+y^2)}}{1+z^2(x^2-y^2)^2} \, dx \, dy. \tag{83}
\]

Since \( n/k = 1 \), we expect a logarithm for the leading term. Since:

\[
\int_0^{2\pi} \frac{d\theta}{1+\alpha \cos^2(2\theta)} = \frac{2\pi}{\sqrt{1+\alpha}}, \quad \forall \alpha > -1,
\]

by passage in polar coordinates we obtain:

\[
I(z) = 2\pi \int_0^\infty e^{-r^2} \frac{r \, dr}{\sqrt{1+z^2r^4}} = \pi \int_0^\infty e^{-u} \frac{du}{\sqrt{1+z^2u^4}} \sim \pi \frac{\log(z)}{z}.
\]

Always in dimension 2, a singularity of degree 4 will generally not give a logarithmic leading term. For example, we have:

\[
\tilde{I}(z) = \int_{\mathbb{R}^2} \frac{e^{-(x^2+y^2)}}{1+z^2(x^4-y^4)^2} \, dx \, dy = \pi \int_0^\infty e^{-u} \frac{du}{\sqrt{1+z^2u^4}} \sim \frac{\pi}{\sqrt{z}} \int_0^\infty \frac{du}{\sqrt{1+u^4}}.
\]

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