BUNDLES OVER CONNECTED SUMS

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Abstract. A principal bundle over the connected sum of two manifolds need not be diffeomorphic or even homotopy equivalent to a non-trivial connected sum of manifolds. We show however that the homology of the total space of a bundle formed as a pullback of a bundle over one of the summands is the same as if it had that bundle as a summand. See Theorem 3.3. An application appears in [2].

Examples are given, including one where the total space of the pullback is not homotopy equivalent to a connected sum with that as a summand and some in which it is.

Finally, we describe the homology of the total space of a principal $U(1)$ bundle over a 6-manifold of the type described by Wall’s theorem. It is a connected sum of an even number of copies of $S^3 \times S^4$ with a 7-manifold whose homology is $\mathbb{Z}/k$ in degree 4 (and $\mathbb{Z}$ in degrees 0 and 7, and zero in all other degrees).

1. Introduction

Let $A$ be a connected sum $A \cong B \# C$ of $n$-manifolds. See for example Hatcher [1] for the definition of connected sum. Let $F \rightarrow L \rightarrow C$ be a bundle over $C$ where $F$ is a manifold.

Using the definition we get a map $A \rightarrow C$. Let $F \rightarrow M \rightarrow A$ be the pullback of the bundle $F \rightarrow L \rightarrow C$ to $A$.

Letting $B'$ denote the complement of a chart in $B$ and setting $X' := (B' \times F)/(* \times F)$ we prove the following. There is a cofibration $X' \rightarrow M \rightarrow L$ for which the corresponding long exact homology/cohomology sequences split to give $H_*(M) \cong H_*(X') \oplus H_*(L)$ and $H^*(M) \cong H^*(X') \oplus H^*(L)$. (See Theorems 3.1 and 3.3.)

These results suggest the possibility that $M$ is the connected sum of $L$ and some manifold $X$ whose $(n-1)$ skeleton is homotopy equivalent to $X'$ but we give an example to show that this is not necessarily the case. (See Example 3.4.) As we shall see, if $M \simeq X \# L$ then the cofibration sequence $X' \rightarrow M \rightarrow L$ would have to split to give $M \simeq X' \vee L$, but this fails in Example 3.4.

In the final section, we consider bundles over some 6-manifolds including the case where $A$ is a symplectic manifold and the $M$ is the...
total space of its associated prequantum line bundle. We find that in
that case \( M \cong \#^2(S^3 \times S^4)\#L \) where \( L \) is a 7-manifold whose nonzero
cohomology groups are \( \mathbb{Z} \) in degrees 0, 7 and \( \mathbb{Z}/k \) in degree 4, where \( r \)
and \( k \) are determined by the cohomology of \( A \). (See Theorem 4.1.)

For topological spaces \( X \) and \( Y \), let \( X \cong Y \) denote “\( X \) is homeo-
morphic to \( Y \)” and let \( X \simeq Y \) denote “\( X \) is homotopy equivalent
to \( Y \”).

2. Connected Sums

Let \( D^n \) denote the closed disk \( D^n := \{ x \in \mathbb{R}^n \mid \|x\| = 1 \} \).

**Lemma 2.1.** For any points \( a, b \) in the interior of \( D^n \) there exists a
self-diffeomorphism \( f : D^n \to D^n \) such that \( f(a) = b \) and \( f|_{\partial D^n} \) is the
identity.

*Proof.* Set \( f(a) = b \). For \( x \neq a \), let \( X_x \) be the point at which the
production of the line segment joining \( a \) to \( x \) meets \( \partial D^n \). Then \( x =
ta + (1 - t)X_x \) for some \( t \). Set \( f(x) = tb + (1 - t)X_x \). \( \square \)

More generally, we have

**Lemma 2.2.** Let \( U_p, V_q \) be subcharts of \( D^n \). Then there exists a self-
diffeomorphism \( f : D^n \to D^n \) such that the restriction of \( f \) to \( U_p \) is the
standard diffeomorphism on open balls and such that \( f|_{\partial D^n} \) is the
identity.

For a point \( p \) in an \( n \)-manifold \( X \), define a subchart around \( p \) to be
an open neighbourhood \( U_p \) of \( p \) which is diffeomorphic to an open ball
in \( \mathbb{R}^n \) within some chart of \( X \).

For a connected \( n \)-manifold \( X \), let \( X' = X \setminus D^n \) denote the com-
plement of a subchart of \( X \).

**Lemma 2.3.** Up to diffeomorphism, \( X' \) is independent of the choice
of the subchart removed.

*Proof.* Let \( U_p, U_q \) be subcharts of \( X \). In the special case where there
exists a chart \( W \) containing both \( U_p \) and \( U_q \) this follows from the earlier
lemma. Then for arbitrary \( U_p, U_q \) find a finite (by compactness) chain
of charts connecting \( U_p \) to \( V_q \), using connectivity.

\( \square \)

After removal of the subchart there is a deformation retraction \( X' \simeq
X^{(n-1)} \) to the \( (n - 1) \)-skeleton of \( X \). Let \( f_X : S^{n-1} \to X' \) denote the
attaching map of the top cell of \( X \).

Suppose that \( X, Y \) are simply connected oriented \( n \)-manifolds.
Lemma 2.4. Let $A$ be a homeomorphism to its image. Composing with the inverse of this injection is an injective map from a compact Hausdorff space, so it is a section see pp. 56–61 of [3]. Similarly there is a left splitting of the inclusion $Y \hookrightarrow X$. The canonical projections $X\#Y \to X$ and $X\#Y \to Y$ preserve the orientation class. That is, they induce isomorphisms on $H_n(\ )$.

Collapsing the centre of the tube $S^{n-1} \times I$ within $X\#Y$ gives a map $X\#Y \to X' \vee Y'$. If we form $(X\#Y)'$ by choosing the subchart to be removed to be within the centre of the tube then collapsing to produce $X' \vee Y'$ has collapsed a contractible subset of $(X\#Y)'$ giving a homotopy equivalence $(X\#Y)' \xrightarrow{\simeq} X' \vee Y'$.

By writing $S^n = S^n\#S^n$ and considering naturality of the pinch we see that the homotopy class of the attaching map of the top cell in $X\#Y$ is given by $f_X+f_Y$ by $\pi_n(X) \oplus \pi_n(Y) \subset \pi_n(X'\vee Y')$.

Choosing the subchart to be removed from $X\#Y$ to be within $Y'$ gives a (non-canonical) inclusion $X' \hookrightarrow (X\#Y)'$ with $(X\#Y)'/X' \cong Y'$. The composite $X' \rightarrow (X\#Y)' \rightarrow X$ with the canonical projection is an injective map from a compact Hausdorff space, so it is a homeomorphism to its image. Composing with the inverse of this homeomorphism is a left splitting of the inclusion $X' \hookrightarrow (X\#Y)'$.

Similarly there is a left splitting of the inclusion $Y' \hookrightarrow (X\#Y)'$.

Lemma 2.4. Let $M$ be a closed $n$-manifold and let $A \subset M'$ be a closed $n$-dim subset of $M$ with $\partial A \cong S^{n-1}$. Then $M \cong N\#X$ for some manifolds $N$ and $X$ with $N' = A$. Furthermore the canonical projection $M' \to N' = A$ is a left splitting of the inclusion $A \hookrightarrow M'$.

Proof. Set $\hat{X} := M \setminus A$. Then $\hat{X}$ is a manifold-with-boundary with $\partial \hat{X} = \partial A$. Let $T \cong S^{n-1} \times I$ be a tubular neighbourhood of $\partial \hat{X}$ in $\hat{X}$ and set $X' := \hat{X} \setminus T$. Then

$$M = A \cup_{S^{n-1} \times \{0\}} T \cup_{S^{n-1} \times \{1\}} \overline{X}$$

so $M = N\#X$ where $N = A \cup_{(T \times \{0\})} D^n$ and $X = X' \cup_{(T \times \{1\})} D^n$. By construction $A \hookrightarrow M' \to N' = A$ is the identity on $A$. $\square$

3. The cofibration sequence associated to a bundle over a connected sum

For definitions and properties of principal cofibrations used in this section see pp. 56–61 of [3].

Let $B$, $C$ be closed $n$-manifolds and let $A := B\#C$. Suppose

$$F \to L \to C$$
is a (locally trivial) fibre bundle whose fibre $F$ is a manifold. Then $L$ is a manifold of dimension $n + \dim(F)$, which we will denote by $m$.

Let $F \to M \to A$ be the pullback of the bundle under the canonical projection $A \to C$. The total space $M$ is a manifold of dimension $m$.

Let $\hat{L}$ be the total space of the restriction of the bundle to $C' := C \setminus \text{chart}$.

By definition,

$$A = B' \cup_{S^{n-1} \times I} C'$$

where by construction, the restriction of the bundle to $B'$ is trivial.

Taking inverse images under the bundle projection $M \to A$ gives

$$M = (B' \times F) \cup_{(S^{n-1} \times I \times F)} \hat{L}.$$ 

In other words, we have

$$S^{n-1} \times I \times F \hookrightarrow (B' \times F) \to (B' \times F)/((S^{n-1} \times I \times F))$$

where the left square is a pushout.

The space

$$M/\hat{L} = (B' \times F)/((S^{n-1} \times I \times F)) = (B'/((S^{n-1} \times I) \times F))/(* \times F) = (B \times F)/(* \times F).$$

has the same homology as $B \lor (B \land F)$. In fact, if $F$ is a suspension then $(B \times F)/(* \times F) \simeq B \lor (B \land F)$. (Selick, [3] Prop 7.7.8)

Set $X' := (B' \times F)/(* \times F)$.

**Theorem 3.1.** There is a cofibration diagram

$$
\begin{array}{ccc}
X' & \rightarrow & M' \\
\downarrow & & \downarrow \\
X' & \rightarrow & M \\
\end{array}
\quad
\begin{array}{ccc}
& \rightarrow & \\
& & \downarrow \\
& & L \\
\end{array}
$$

(i.e. the rows are cofibrations and the right square is a pushout.)

**Proof.** We had

$$M/\hat{L} = (B' \times F)/((S^{n-1} \times I) \times F)$$
Also, since \( L = \hat{L} \cup_{S^{n-1} \times I \times F} F \) we have

\[
\frac{L}{\hat{L}} = \frac{(S^n \times F)}{(\ast \times F)}
\]

(which can be regarded as the special case of the preceding with \( B = S^n \)). Thus we have a diagram

\[
\begin{array}{ccc}
X' & \longrightarrow & X' \\
\downarrow & & \downarrow \\
\hat{L} & \longrightarrow & M & \longrightarrow & \frac{M}{\hat{L}} = \frac{(B \times F)}{(\ast \times F)} \\
\downarrow & & \downarrow & & \downarrow \\
\hat{L} & \longrightarrow & L & \longrightarrow & \frac{L}{\hat{L}} = \frac{(S^n \times F)}{(\ast \times F)}
\end{array}
\]

in which the bottom right square is a pushout, the rows and right columns are cofibrations and which yields the cofibration \( X' \to M \to L \). Deleting a chart from \( L \) and deleting its preimage from \( M \) gives the first row of the theorem. \( \square \)

From the long exact homology sequence of the cofibration we get

**Corollary 3.2.** The lift \( M \to L \) of the canonical projection preserves the orientation class. That is, it induces isomorphisms on \( H_m(\ ) \), where \( m \) was defined earlier.

This Corollary can be proved in other ways such as naturality of the Serre spectral sequence.

Let \( f : X \to Y \) be a differentiable map between compact oriented \( m \)-manifolds. Let \( D_X : H^k(X) \cong H_{n-k}(X) \) and \( D_Y : H^k(Y) \cong H_{n-k}(Y) \) be the Poincaré Duality isomorphisms. Suppose \( f \) has degree \( \lambda \) (multiplies by \( \lambda \) on \( H_n(\ ) \)). Then \( f_* \circ D_X \circ f^* = \lambda D_Y \). In particular, if \( f \) preserves the orientation class (that is, has degree 1) then \( f^* \) is injective and \( f_* \) is surjective. Applying this to \( M \to L \) shows

**Theorem 3.3.** *(Decomposition Theorem)*

In the long exact homology sequence of the cofibration, the connecting map \( \partial : H_q(L) \to H_{q-1}(X') \) is zero. Likewise, in the long exact cohomology sequence, the map \( \delta : H^{q-1}(X') \to H^q(L) \) is zero. Thus for \( 0 < q < m \) we have \( H_q(M) \cong H_q(X') \oplus H_q(L) \) and \( H^q(M) \cong H^q(X') \oplus H^q(L) \).
This suggests that perhaps there is a manifold $X$ such that $M \simeq X \# L$ where $X$ is homotopy equivalent to the one-point compactification of $X'$, but this is not necessarily true.

**Example 3.4.** Consider $A = \mathbb{C}P^2$ and write $A = B \# C$ where $B = \mathbb{C}P^2$ and $C = S^4$ Consider the trivial bundle $S^7 \times S^4 \to S^4$. Then $M = S^7 \times \mathbb{C}P^2$; $B' = S^2$; $C' = \ast$; $A' = B' \vee C' = S^2$ while

\[
M' = (F \times A)' = (F \times A') \cup_{F \times A'} (F' \times A)
\]

\[
= (S^7 \times S^2) \cup_{S^7 \times S^2} (\ast \times \mathbb{C}P^2) = \mathbb{C}P^2 \vee S^7 \vee S^9
\]

and $L = S^4 \times S^7$ so $L' = S^4 \vee S^7$. Our cofibration is

\[
(S^2 \times S^7)/(* \times S^7) \to M' \to S^4 \vee S^7
\]

which becomes $S^2 \vee S^9 \to \mathbb{C}P^2 \vee S^7 \vee S^9 \to S^4 \vee S^7$. This does not split so in this example $M$ does not become homotopy equivalent to $X \# L$ for any $X$.

**4. Bundles over 6-manifolds**

Let $A$ be a 6-manifold such that $H^*(A)$ is simply connected and torsion-free. Suppose $H^2(A) = \mathbb{Z}$.

Let $x \in H^2(A)$ be a generator and let $V \in H^6(A)$ be the volume form. Then $x^3 = kV$ for some integer $k$.

By Wall [4], we can write $A = B \# C$ where $B = (S^3 \times S^3)^\# r$ for some $r$ and $C$ is a simply connected torsion-free 6-manifold with $H^3(C) = 0$ and $H^2(C) = \mathbb{Z}$.

Although $M$ is a $S^1$ bundle over $A$, it does not immediately follow from Wall’s result that $M$ also admits a decomposition as a connected sum. We shall see that this is in fact true. This is the content of our Theorem 4.1 below.

Associated to $x$ there are complex line bundles over $A$ and $C$ classified by $x$. Let $M$ and $L$ denote the sphere bundles of these line bundles. Then there are $S^1$-bundles $S^1 \to M \to A$ and $S^1 \to L \to C$. Note that the long exact homotopy sequence tells us that $\pi_1(M) = \mathbb{Z}$ and $\pi_q(M) = \pi_q(A)$ for $q \neq 1$.

As in Ho-Jeffrey-Selick-Xia [2] we calculate that the cohomology of the 7-manifold $L$ is given by $H^q(L) = \begin{cases} \mathbb{Z} & q = 0, 7; \\ \mathbb{Z}/k & q = 4. \\ 0 & \text{otherwise}. \end{cases}$

**Theorem 4.1.** We have

\[ M \simeq \#^{2r}(S^3 \times S^4) \# L, \]

where the homology of the space $L$ is specified above.
Proof. In the notation of the preceding section applied to $S^1 \to M \to A$ we have $B' = \vee_{2r} S^3, L' = P^4(k)$ and

$$X' := (B' \times S^1)/(\ast \times S^1) \simeq B' \vee (B' \wedge S^1) \vee_{2r} (S^3 \vee \Sigma^3 S^1)$$

where $P^n(k)$ denotes the Moore space $S^{n-1} \cup_k e^n$. Thus our cofibration sequence becomes

$$\vee_{2r} (S^3 \vee \Sigma^3 S^1) \to M' \to P^4(k)$$

or equivalently

$$\vee_{2r} (S^3 \vee S^4) \to M' \to P^4(k).$$

The composition of the bundle map $M' \to A'$ with the canonical projection $A' \to B'$ provides a splitting of the restriction of

$$\vee_{2r} (S^3 \vee S^4) \to M'$$

to $\vee_{2r} S^3$.

For degree reasons, the cofibration

$$\vee_{2r} (S^3 \vee S^4) \to M' \to P^4(k)$$

is principal, induced from some attaching map $P^3(k) \to \vee_{2r} (S^3 \vee S^4)$ whose image (for degree reasons) lands in $\vee_{2r} S^3$. Since the restriction of $\vee_{2r} (S^3 \vee S^4) \to M'$ to $\vee_{2r} S^3$ splits, this implies that this attaching map is trivial. Thus the cofibration splits to give

$$M' \simeq \vee_{2r} (S^3 \vee S^4) \vee_{2r} P^4(k).$$

To obtain $M$ from $M'$ we attach the top cell giving

$$H^q(M) = H^q(M') \oplus H^q(S^7) = \begin{cases} 
\mathbb{Z}_r & q = 0, 7 \\
\mathbb{Z}^{2r} & q = 3 \\
\mathbb{Z}^{2r} \oplus \mathbb{Z}/k & q = 4 \\
0 & \text{otherwise}
\end{cases}$$

Letting $\tilde{V}$ denote the generator of $H^7(M)$, using Poincaré duality we can pair the generators $\langle u_1, u_2, \ldots, u_r \rangle$ of $\mathbb{Z}$ in degrees 3 with the generators $\langle v_1, v_2, \ldots, v_r \rangle$ of $\mathbb{Z}$ in degrees 4 so that $u_i v_j = \delta_{ij} \tilde{V}$. If we reduce to $\mathbb{Z}/k$ coefficients, there is also a nonzero cup product $ab$ where $a, b$ are generators of $H^3(M; \mathbb{Z}/k)$ and $H^4(M; \mathbb{Z}/k)$ respectively.

Examining the cohomology of $M$, we see that

$$H^q(M) = H^q(\#^{2r}(S^3 \times S^4)\# L)$$

where $H^q(L) = \begin{cases} 
\mathbb{Z} & q = 0, 7 \\
\mathbb{Z}/k & q = 4 \\
0 & \text{otherwise}
\end{cases} \square$
The attaching maps $f_M$ and $f_{\#^2r(S^3 \times S^4)\#L}$ are both

$$\left[\iota_1^3, \iota_1^4\right] + \left[\iota_2^3, \iota_2^4\right] + \ldots + \left[\iota_r^3, \iota_r^4\right] + f_L$$

where the Whitehead product $[\iota^3, \iota^4]$ is the attaching map

$$f_{S^3 \times S^4},$$

and so

$$M \simeq \#^2r(S^3 \times S^4)\#L.$$

References

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