THE INTERLACE POLYNOMIAL OF A GRAPH

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Abstract. Motivated by circle graphs, and the enumeration of Euler circuits, we define a one-variable “interlace polynomial” for any graph. The polynomial satisfies a beautiful and unexpected reduction relation, quite different from the cut and fuse reduction characterizing the Tutte polynomial.

It emerges that the interlace graph polynomial may be viewed as a special case of the Martin polynomial of an isotropic system, which underlies its connections with the circuit partition polynomial and the Kauffman brackets of a link diagram. The graph polynomial, in addition to being perhaps more broadly accessible than the Martin polynomial for isotropic systems, also has a two-variable generalization that is unknown for the Martin polynomial. We consider extremal properties of the interlace polynomial, its values for various special graphs, and evaluations which relate to basic graph properties such as the component and independence numbers.

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1. Introduction

1.1. Motivation. This work was originally motivated by a problem relating to DNA sequencing by hybridization [ABCS00]. At the mathematical heart of the problem was to count the number of 2-in, 2-out digraphs (Eulerian, directed graphs in which each vertex has in-degree two and out-degree two) having a given number of Euler circuits, formulas for which are given in [ABCS00].

This may be thought of as a sort of inverse of a standard problem, counting the Euler circuits in a directed graph. That can be done (in polynomial time) with a combination of the matrix-tree theorem [Kir47, Tut48] (see also [Bol98, p. 58]) and the so-called BEST theorem [ST41, DBvAE51] (see also [Bol98, p. 18]). However, this approach does not give the structural information that was needed to solve the inverse problem.

The successful approach led to the graph polynomial discussed here. In addition to counting Euler circuits in a 2-in, 2-out digraph, the interlace polynomial (like the Martin polynomial) can count, for any k, the number of k-component circuit partitions of the graph. In this paper we define the interlace polynomial; treat

Date: 25 October 2003 (Journal submission 15 December 2000).

Key words and phrases. pairing; interlace graph; circle graph; Euler circuit; circuit partition; matrix-tree theorem; BEST theorem; Martin polynomial; Tutte polynomial; Kauffman bracket; extremal.

† Bollobás was supported by NSF grant DMS-9971788.
its connection to the Martin polynomial and the Kauffman brackets of a link diagram; and look into other apparently unrelated properties of the polynomial, some of them only conjectural. Basic questions about the polynomial remain open, but an understanding is developing. The connection with the Kauffman brackets was clarified, and that with the Martin polynomial discovered, since the time of our first publication [ABS00].

1.2. **Looking backward.** The interlace graph polynomial has antecedents, notably in the work of Bouchet. Specifically, starting from the study of Euler circuits of undirected 4-regular graphs and the transformations of Kotzig [Kot68], Bouchet [Bou87] and Bouchet [Bou99] include the pivot for graphs given by our \( I_D \), and define a generalized Tutte-Martin polynomial on isotropic systems (and on multimatroids), of which as Bouchet [Bou99] pointed out our interlace polynomial may be viewed as a special case. This is made explicit by Aigner and van der Holst [AvdH04]. If a graph \( G \) has vertex set \( V \) and adjacency matrix \( A \), \( I \) is the identity matrix of the same dimension as \( A \), and \( S_G = (V, L_G) \) is the isotropic system given by the row space \( L_G \) of \((A | I)\), then our polynomial is equal to the Martin polynomial \( m(S_G; x) \).

1.3. **Looking forward.** In contrast to the reduction formula of Theorem 12, the expression of the interlace polynomial as the Martin polynomial of an isotropic system gives an explicit expansion, as in Corollary 1 of [AvdH04]. Such an expansion was also produced, without reference to isotropic systems, in [ABS01], where a two-variable generalization of the interlace polynomial is shown to have a formal similarity to the Tutte polynomial.

1.4. **Back to the present.** In this paper we introduce the one-variable interlace polynomial in the circuit-enumeration context that led us to discover it, and consider extremal properties of the polynomial, its values for various special graphs, and evaluations which relate to basic graph properties such as the component and independence numbers.

1.5. **Outline.** The paper is structured as follows. In Section 2 we define an interlace graph \( H \) for an Euler circuit \( C \) of a 2-in, 2-out directed graph \( D \). Section 3 defines a pivoting function on any undirected graph; applied to an interlace graph \( H \) the pivot is consistent with transpositions on \( C \), in a sense specified by Lemma 7. In Section 4 this consistency is exploited to show how \( H \) determines the number of Euler circuits of \( D \), giving a sort of preview of Sections 5 and 6.

Section 5 contains essential identities for the graph pivot operator, and these are used in Section 6 to prove that the interlace graph polynomial \( q \) is well-defined on any undirected graph \( G \), not just on interlace graphs.

In the interlace-graph setting, Section 4 connects \( q(H) \) with the circuit partition polynomial and the Martin polynomial \( m(D) \). There is a further connection to the Kauffman brackets of a link diagram.

Section 5 computes \( q(G) \) for some natural graphs, and some general observations about the interlace polynomial are drawn in Section 9. “Substituted” and “rotated” graphs are considered in Section 10. Calculating \( q(G) \) on such graphs is of mild interest in itself, and is essential machinery for the computation in Section 11 of extremal properties of the graph polynomial. Section 12 considers some open problems.

2. Interlacings

Our approach to counting circuits is based on “interlacings”, as in Read and Rosenstiehl [RR78].

**Definition 1.** Given an Euler circuit \( C \) of a 2-in, 2-out directed graph \( D \), two vertices \( a \) and \( b \) of \( D \) are interlaced if \( C \) visits them in the sequence \( a \ldots b \ldots a \ldots b \ldots \). The interlace graph \( H = H(C) \) corresponding to \( C \) has the same vertex set as \( D \), with an edge \( ab \) in \( H \) if \( a \) and \( b \) are interlaced in \( C \).

The class of graphs arising as interlace graphs coincides with the class of circle graphs, as in [GGR89, HSS71, SSS89]. Trees constitute a particularly tractable set of interlace graphs.

**Definition 2.** Given an Euler circuit \( C \) with \( a \) and \( b \) interlaced, a transposition on the pair \( ab \) is the circuit \( C_{ab} \) resulting from the edge sequences from \( a \) to \( b \) with the other.

See Figure 1 for an example. Note that if \( C \) traverses edge \( e_1 \) into \( a \) and \( e_2 \) out on one visit, and \( e_3 \) in and \( e_4 \) out on the other visit, then \( C_{ab} \) reverses this “resolution”, following \( e_1 \) with \( e_4 \) and \( e_2 \) with \( e_3 \).

The result stated below as Lemma 7 is due to Ukkonen and Pevzner [UK92, Pev93]. Their result is set in a more general context, allowing “rotations” and “3-way repeats”; the relation between their context and
2-in, 2-out digraphs is described in detail in Section 1.4 of [ABC00]. In the context of 2-in, 2-out digraphs, we provide a simple proof of Lemma 4 starting with the case of a circuit having no interlaced pair.

**Lemma 3.** If a 2-in, 2-out digraph $D$ has an Euler circuit $C$ with no interlaced pair, then $D$ has only one Euler circuit.

**Proof.** Use induction on the order $n$ of $D$. For $n = 1$ the only case is $D$ consisting of two directed loops on a common vertex, for which there is a single Euler circuit. For $n > 1$ the given circuit $C$ visits the vertices in a sequence $s_1s_2 \cdots s_{2n}$ in which some vertex $a$ must appear twice consecutively. (The two occurrences of each vertex can be viewed as a pair of matched parentheses, and the absence of interlacings is equivalent to the parentheses being balanced. A vertex $a$ we seek corresponds to an innermost pair of parentheses.) At $a$ there is a loop; contracting out the loop, and then the vertex $a$, gives a graph $D'$ with the same number of Euler circuits as $D$. $D'$ also has no interlaced pair, and is of order $n - 1$, so by induction $D'$ has only one Euler circuit, and thus so does $D$. \hfill $\square$

**Lemma 4.** For a 2-in, 2-out digraph $D$, all Euler circuits form a single orbit under transpositions on interlaced pairs.

**Proof.** Induction on the order $n$ of $D$. The unique graph $D$ for the base case $n = 1$ has only one Euler circuit, so the assertion is trivial. Otherwise, let two circuits $C, C'$ of $D$ be given. If there is a vertex $a$ which the two circuits resolve alike — each of $a$’s two in-edges is followed by the same out-edge in $C'$ as it is in $C$ — then use this resolution to contract $a$ out of $D$, $C$, and $C'$. In the contracted case, by induction, one circuit can be converted to the other by a sequence of transpositions, and this sequence lifts to the original case.

If there is no vertex $a$ resolved alike in $C$ and $C'$, then $C$ and $C'$ are different, so by Lemma 3 $D$ must have had some interlaced pair $ab$. Transposing $C'$ on $ab$ gives a circuit $C''$ which resolves vertex $a$ oppositely to $C'$, and therefore the same as $C$. By the previous case, $C$ can be converted to $C''$, and in turn to $C'$, by a sequence of transpositions. \hfill $\square$

Lemma 3 says that if $H(C)$ has no edges, then the Euler circuit is unique. Also, if $C$ has only one interlaced pair $ab$, transposing on it gives a circuit $C'$ in which $ab$ is still the sole interlaced pair, so these two circuits constitute the full orbit; that is, if $H(C)$ has one edge, then there are exactly two Euler circuits. Such observations, studied in [ABC00], suggest a question: does the interlace graph determine the number of Euler circuits? We shall show that it does. (In [ABC00] a more elementary fact is exploited, that if a pairing generates $k$ Euler circuits, it has at most $k - 1$ interlaced pairs, which define a sort of skeleton. Thus the enumeration of pairings with $k$ Euler circuits is reduced to an enumeration over a finite number of skeletons, and a use of generating functions to count pairings with a given skeleton. While closely related, the interlace polynomial does not simplify the enumeration.)

## 3. The Pivot Operator on Graphs

We now define a pivot operator $G \mapsto G^{ab}$ on any graph $G$. It is connected with interlacings by Lemma 4 if $H = H(C)$, and vertices $a$ and $b$ are interlaced in $C$, then the circuit transposition $C \mapsto C^{ab}$ and the pivot operator $H \mapsto H^{ab}$ commute, modulo a vertex relabelling.

Figure 2 illustrates the following definition.
Definition 5 (Pivot). Let $G$ be any undirected graph, and $ab$ a pair of distinct vertices in $G$. Construct the pivot $G^{ab}$ as follows. Partition the vertices other than $a$ and $b$ into four classes:

1. vertices adjacent to $a$ alone;
2. vertices adjacent to $b$ alone;
3. vertices adjacent to both $a$ and $b$; and
4. vertices adjacent to neither $a$ nor $b$.

Begin by setting $G^{ab} = G$. For any vertex pair $xy$ where $x$ is in one of the classes (1–3) and $y$ is in a different class (1–3), “toggle” the pair $xy$: if it was an edge, make it a non-edge, and if it was a non-edge, make it an edge.

Note that $G^{ab} = G^{ba}$. Although the pivot operation is defined for any vertex pair $ab$, we shall only ever pivot about an edge $ab$.

![Figure 2](image-url) Pivoting a graph $G$ on a pair $ab$ (always an edge, in practice). Vertices are divided into 4 classes: those joined to $a$; to $b$; to both; or to neither. If vertices $x$ and $y$ lie in different classes among the first three classes named, then if $xy$ was an edge in $G$, it becomes a non-edge in $G^{ab}$, and similarly if $xy$ was a non-edge, it becomes an edge.

We will need one other, trivial, operation.

Definition 6 (label-swapping). For any directed or undirected labelled graph $G$ (or circuit $C$), and any pair of distinct vertices $a$ and $b$, let $G_{ab}$ (or $C_{ab}$) denote the same object with the labels of $a$ and $b$ swapped.

The pivot function defined above satisfies the following lemma.

Lemma 7. If $H = H(C)$, and $H$ has an edge $ab$ (i.e., $a$ and $b$ are interlaced in $C$), then $H_{ab} = (H(C^{ab}))_{ab}$.

Proof. A transposition on an Euler circuit $C$ may be understood through Figure 1. Vertices are written around the circumference of a circle in the order they are visited by $C$, and vertices $a$ and $b$ are interlaced if the chord joining the two occurrences of $a$ intersects the chord joining the two occurrences of $b$.

First, note that after transposition, $a$ and $b$ remain interlaced. Next, look at single vertices other than $a$ and $b$. A vertex like 4, interlaced with neither $a$ nor $b$ before transposition, remains interlaced with neither. A vertex like 3, interlaced with both $a$ and $b$, remains interlaced with both. A vertex like 1, interlaced with just $a$ in $C$, becomes interlaced with just $b$ in $C^{ab}$, and therefore in $(C^{ab})_{ab}$ is interlaced with just $a$, as it was in $C$. After label-swapping, then, incidences between $a$ and $b$ and other vertices are unaffected by transposition.

Finally, consider incidences between pairs of vertices other than $a$ and $b$. In the example of Figure 1, 4 remains uninterlaced with 1 and 3, but 1 and 3 become interlaced, matching Definition 5. All other cases can be worked out with similar figures, and all accord with Definition 6; this completes the lemma’s proof.

4. Counting circuits in 2-in, 2-out digraphs

The results in this section are a special case of those presented in sections 5–7, in particular Theorems 12 and 24. However, the special case is relatively concise, and reveals the interlace polynomial’s relationship with Euler circuits (its motivation) and the nature of the reduction it obeys.
Definition 8. For any directed graph \( D \), let \( r_1(D) \) be the number of Euler circuits of \( D \), and, more generally, let \( r_k(D) \) be the number of partitions of the edges of \( D \) into \( k \) circuits.

The fact that all Euler circuits of a 2-in, 2-out digraph can be generated through transpositions starting with any Euler circuit \( C \), and transpositions are mirrored by pivots of the interlace graph, will mean that the number of Euler circuits of \( D \) can be computed from the interlace graph of any Euler circuit \( C \) of \( D \). This is the basis of the following theorem.

Theorem 9. There exists a function \( q_1 \), from the set of interlace graphs to the integers, such that for any 2-in, 2-out digraph \( D \) with Euler circuit \( C \), the number of Euler circuits of \( D \) is equal to \( q_1(H(C)) \). Moreover, \( q_1 \) is uniquely defined by the following recursion:

\[
q_1(H) = \begin{cases} 
q_1(H - a) + q_1(H^{ab} - b) & \text{if } ab \text{ is an edge of } H \\
1 & \text{if } H \text{ has no edges.}
\end{cases}
\]

Proof. In the graph \( D \), vertex \( a \) has two in-edges, \( e_1 \) and \( e_2 \), and two out-edges, \( e_3 \) and \( e_4 \). (Since we allow loops, the in-edges are not necessarily distinct from the out-edges.) The set of all Euler circuits in \( D \) can be partitioned into those that follow \( e_1 \) with \( e_3 \) (and therefore follow \( e_2 \) with \( e_4 \)), and those that follow \( e_1 \) with \( e_4 \) (and therefore follow \( e_2 \) with \( e_3 \)). (We do not presume that either class is non-empty.)

It follows that \( r_1(D) = r_1(D') + r_1(D'') \). Here \( D' \) is derived from \( D \) by merging edge \( e_1 \) with \( e_3 \) (replacing the two with a single new edge going from the tail of \( e_1 \) to the head of \( e_3 \)), merging \( e_2 \) with \( e_4 \), and deleting the now-isolated vertex \( a \); and \( D'' \) is similarly formed by merging \( e_1 \) with \( e_4 \), merging \( e_2 \) with \( e_3 \), and deleting \( a \).

Define an analogous deletion operator on Euler circuits, so that \( C - a \) means merging the edge preceding and the edge following each occurrence of \( a \), and removing \( a \). Note that an Euler circuit \( C \) contains more information than its parent graph \( D \), and in particular \( D = D(C) \) is determined by \( C \). In the present case, we have \( D = D(C) \); letting \( C' = C - a \) and \( C'' = C^{ab} - a \), we also have \( D' = D(C') \) and (because the transposition \( C^{ab} \) switches the allegiances of the two edges into \( a \) with the two out) \( D'' = D(C'') \). Defining \( r_1(C) = r_1(D(C)) \), it is immediate that \( r_1(C) = r_1(C - a) + r_1(C^{ab} - a) \).

The interlace graphs of these circuits are given by \( H' = H(C - a) = H(C) - a \), and (by Lemma 17) \( H'' = H(C^{ab} - a) = (H(C))^{ab} - b \). (In the last expression \( b \) rather than \( a \) is deleted because of label-swapping; see Lemma 17.) Since \( C - a \) and \( C^{ab} - a \) each have one vertex less than \( C \), suppose inductively that their Euler circuits are counted as per the Theorem, that is, \( r_1(D') = q_1(H') \) and \( r_1(D'') = q_1(H'') \). Then

\[
r_1(D) = r_1(D') + r_1(D'') \\
= q_1(H - a) + q_1(H^{ab} - b) \\
= q_1(H),
\]

the last equality following from the recursive definition of \( q_1(H) \). This shows both the independence of the quantity \( q_1(H) \) from the first pivot \( ab \) used in its recursive calculation, and the desired equality \( r_1(D) = q_1(H) \).

The key point of this lies in equation \( 1 \), showing the connection between two in–out “resolutions” at a vertex \( a \) and the interlace graph pivoted and not pivoted on an edge \( ab \).

5. Pivoting about an edge

Consider the pivoting operation of Definition 8 and recall that \( G^{ab} = G^{ba} \). Also note that pivoting is an involution: for any pair \( ab \), \( G^{(ab)(ab)} = G \), where the notation \( G^{(ab)(cd)} \) represents an iterated pivot \( (G^{ab})^{cd} \). We shall need the following properties of pivoting, which are considerably more subtle.

Lemma 10. Let \( a, b, \) and \( c \) be distinct vertices of a graph \( G \) with \( ab, ac \in E(G) \). Then

(i) \( G^{(ab)(ac)}(ab) = G_{bc} \), and
(ii) \( G^{(ab)(ac)}(ac) = (G^{ac})_{bc} \).

Two remarks before the proof. The identities need not hold unless \( ab \) and \( ac \) are both edges. And, the identities are needed to prove Lemma 11 which in turn is needed for Theorem 12.
Proof. Focus first on identity (i), in the case where $bc \notin E(G)$. Proof of (i) consists of checking three types of edges.

First, inspection of Definition 5 shows that the presence or absence in $G^{(ab)(ac)(bc)}$ of any edge in $\{a, b, c\} \times \{a, b, c\}$ is determined by $G\uppeq_{\{a, b, c\}}$, the subgraph of $G$ induced by $a$, $b$, and $c$. Checking (i) here merely requires carrying out the pivot operations on the three-vertex graph with edges $ab$ and $ac$. (This and the subsequent checks described can certainly be carried out by hand, but we did it by computer instead, with a small amount of manual spot-checking.)

Second, for any vertex $u \notin \{a, b, c\}$, the presence or absence in $G^{(ab)(ac)(bc)}$ of any edge in $\{u\} \times \{a, b, c\}$ is determined by $G\uppeq_{\{a, b, c, u\}}$. Checking (i) here requires checking 8 cases, one for each setting (present/absent) of the three edges $ua$, $ub$, and $uc$. They may be checked “in parallel” by computing $G^{(ab)(ac)(bc)}$ for a single graph $G$ with vertices $a$, $b$, $c$, and 8 more vertices belonging to different equivalence classes as defined by adjacency to $a$, $b$, and $c$. Let us define $G$ not to include any edges except those incident on $\{a, b, c\}$, as such edges are irrelevant to this calculation.

Finally, for any two vertices $u, v \notin \{a, b, c\}$, the number of times the presence/absence of the edge $uv$ is toggled depends on the number of times $u$ and $v$ lie in different classes (1–3) of Definition 5. But the class of $u$ (respectively $v$) is determined by $G\uppeq_{\{a, b, c, u\}}$. Note then that the number of times $uv$ is toggled is independent of the initial presence or absence of $uv$ or any other edge not incident on $\{a, b, c\}$. Thus here too it suffices to compute $G^{(ab)(ac)(bc)}$ for the same 11-vertex graph $G$ as above.

In short, in the case $bc \notin E(G)$, identity (i) can be verified using $G^{(ab)(ac)(bc)}$ for a single 11-vertex graph with vertices $a$, $b$, and $c$, and 8 other vertices representing the 8 equivalence classes defined by adjacency to $a$, $b$, and $c$.

In the case $bc \in E(G)$ the identity is checked the same way, using the same 11-vertex graph as above but with the edge $bc$ added.

Identity (ii) could be proved using the same pair of graphs, but in fact, (i) and (ii) are equivalent. Since $G \mapsto G^{ab}$ is an involution, $G^{(ab)(ac)(ab)} = G_{bc}^{(ab)}$ if $G^{(ab)(ac)} = (G_{bc})^{ab}$. But $(G_{bc})^{ab} = (G^{ac})_{bc}$, so we are done.

\[\square\]

Remark 11. If $G$ is connected, then for any edge $ab$ of $G$, $G^{ab}$ is connected.

Proof. Edge relations change only between vertices “distinguished” by $a$ and $b$, but in $G^{ab}$ these vertices remain joined to $a$ or $b$ (or both), and $a$ and $b$ remain joined, so $G^{ab}$ is connected. \[\square\]

6. The interlace polynomial

We will now show that the integer $q_1(G)$ defined by Theorem 12 can be generalized to a one-variable graph polynomial $q(G; x)$ which is defined for all graphs $G$, not just interlace graphs, and which for interlace graphs $G$ satisfies $q(G; 1) = q_1(G)$.

We shall call $q(G)$ the interlace polynomial of $G$, and shall write $q(G; x)$, $q(G)(x)$, and $q_{10}(x)$ interchangeably. Later we will show that when $H$ is an interlace graph of $D$, $q_{10}(x)$ can be used to count partitions of $D$ into circuits.

Historically, we began with the graph function $q_1$, whose well-definedness — on interlace graphs — followed from its construction. Generalizing to a graph polynomial $q$ was a leap of faith, but computer experiments quickly convinced us that the polynomial of Theorem 12 was indeed well-defined, on arbitrary graphs — that the sequence of pivot elements chosen was irrelevant. We expected that to prove it we would need to prove something like Lemma 4, i.e., that we would have to prove the equivalence of any two first pivot elements, and that this could be boiled down to the equivalence of $q_{ab}$ to $q_{ba}$, and that of $q_{ab}$ to $q_{ac}$ (or perhaps to $q_{ca}$). Then, we guessed that showing for example that $q_{ab}(G) = q_{ac}(G)$ would mean showing that following the $ab$ pivot with one on $ac$ or $ca$ gives the same set of graphs as following the $ac$ (or $ca$) pivot with one on $ab$ or $ba$. Computer experiments then indicated the correct identities, leading to the rather obscure but easy to prove identities of Lemmas 10 and 11.

6.1. Definition and well-definedness. As usual, let us write $G$ for the class of finite undirected graphs having no loops nor multiple edges.

Theorem 12 (Interlace polynomial). There is a unique map $q : G \rightarrow \mathbb{Z}[x]$, $G \mapsto q(G)$, such that the following two conditions hold.
Hence, we may assume that
\[ G \]
and similarly the fourth term of (3) is equal to the second of (4), so
\[ q^{ab}(G) = q^{ba}(G). \]

By the inductive hypothesis, both the latter terms are well defined, and we may apply any sequence of pivot edges we like. Pivot the first term on
\[ bd \]
edges we like. Pivot the first term on
\[ bd \]
by induction on the order of
\[ G \]
and similarly the fourth term of (3) is equal to the second of (4), so
\[ q^{ab}(G) = q^{ba}(G). \]

\[ q^{ba}(G) = q(G - a) + q(G^{ab} - b) \].

(ii) If \( ab \) and \( ac \) are edges of \( G \) then
\[ q^{ab}(G) = q^{ac}(G). \]

Proof. Both parts of the proof will be by induction on the order of \( G \).
(i) If \( d(a) = 1 \) or \( d(b) = 1 \) then \( G^{ab} = G \), so
\[ q^{ab}(G) = q(G - a) + q(G^{ab} - b) = q^{ba}(G). \]

\[ q^{ab}(G) = q(G - a) + q(G^{ab} - b) \].

By the inductive hypothesis, both the latter terms are well defined, and we may apply any sequence of pivot edges we like. Pivot the first term on \( bd \), and the second on \( ac \), which is an edge of \( G^{ab} \) as it was of \( G \):
\[ = \{q(G - a - b) + q((G - a)^{bd} - d)\} + \]
\[ \{q(G^{ab} - b - a) + q((G^{ab} - b)^{ac} - c)\} \]
\[ = q(G - a - b) + q(G^{bd} - a - d) + \]
\[ q(G^{ab} - a - b) + q(G^{(ab)(ac)} - b - c). \]

Symmetrically, swapping the roles of \( a \) and \( b \), and those of \( c \) and \( d \),
\[ q^{ba}(G) = q(G - a - b) + q(G^{ac} - b - c) \]
\[ + q(G^{ab} - a - b) + q(G^{(ab)(bd)} - a - d). \]

In these expansions of \( q^{ab}(G) \) and \( q^{ba}(G) \), the first terms are identical, as are the third terms. Also, by Lemma 14, (ii), the second term of \( q^{ab}(G) \) is equal to the fourth of \( q^{ba}(G) \), because \( G^{bd} - a - d \equiv G^{(ab)(bd)} - a - d \), and similarly the fourth term of \( q^{ab}(G) \) is equal to the second of \( q^{ba}(G) \), so \( q^{ab}(G) = q^{ba}(G) \).

(ii) This identity is also an immediate consequence of Lemma 14 and is proved similarly, but here the pivot sequences needed do not respect symmetry.
\[ q^{ab}(G) = q(G - a) + q(G^{ab} - b); \]
both these terms are well-defined by induction, and pivoting the second on \( ac \),
\[ = q(G - a) + \{q(G^{ab} - b - a) + q((G^{ab} - b)^{ac} - c)\} \]
\[ = q(G - a) + q(G^{ab} - a - b) + q(G^{(ab)(ac)} - b - c). \]

And,
\[ q^{ac}(G) = q(G - a) + q(G^{ac} - c); \]
pivoting the second term on $ba$ (not on $ab$ as symmetry would suggest),

$$q(G - a) + \{q(G^{ac} - c - b) + q((G^{ac} - c)^{ba} - a)\} = q(G - a) + q(G^{ac} - b - c) + q(G^{(ac)(ab)} - a - c).$$

By Lemma 14, $(G^{ab} - a - b)_{bc} = G^{(ac)(ab)} - a - c$ and $G^{(ab)(ac)} - b - c = G^{ac} - b - c$, so $q^{ab}(G) = q^{ac}(G)$. □

Now we are ready to prove the main theorem.

Proof of Theorem 12. If $G$ is a graph of order $n$ containing an edge $ab$ then the pivot-reduction formula reduces the computation of $q(G)$ to the computation of $q$ on two graphs of order $n - 1$, namely $G - a$ and $G^{ab} - b$. Repeating this process, we reduce $q(G)$ to a linear combination $q(G) = \sum_{k=1}^{n-1} a_k q(E_k)$, where each $a_k$ is a nonnegative integer. By making use of the boundary conditions $q(E_k) = x^k$, we find $q(G)$. Hence, if there is a function $q$ then it is unique.

However, it is not clear that there is a function $q$ satisfying (i) and (ii). To prove this, we have to show that the same result arises irrespective of the pivot edges chosen in the recursive reduction of $G$ to edgeless graphs. The proof will be by induction on the order $n$ of $G$. At $n = 1$ there is nothing to prove, so assume that $n > 1$. By the inductive hypothesis, $q(G - a)$ and $q(G^{ab} - b)$ are both well defined, so we need only show that $q(G - a) + q(G^{ab} - b)$ is independent of the first pivot edge $ab$ chosen. (Note that $q^{ab} \text{ def} = q(G - a) + q(G^{ab} - b)$, unlike $G^{ab} = G^{ba}$, depends on the ordered pair $(a, b)$, that is, on the oriented edge $ab$.) To restate, we must show that for any two oriented edges $ab$ and $a'b'$, $q(G - a) + q(G^{ab} - b) = q(G - a') + q(G^{a'b'} - b')$.

We consider two cases. The first comes if $ab$ and $a'b'$ lie in two different components of $G$: say $a, b \in G_1$ and $a', b' \in G_2$, with $G = G_1 \cup G_2$. Pivoting $G$ on $ab$ does not affect the component $G_2$ at all, since all $G_2$'s vertices fall into function $\tilde{q}$'s case (ii). Thus

$$q(G - a) + q(G^{ab} - b) = q((G_1 - a) \cup G_2) + q((G_1^{ab} - b) \cup G_2).$$

By the inductive hypothesis, each of the latter terms is well defined, and we may continue as we like. Pivoting on $a'b'$,

$$= q((G_1 - a) \cup (G_2 - a')) + q((G_1^{ab} - b) \cup (G_2^{a'b'} - b')).$$

By symmetry, this is clearly equal to pivoting first on $a'b'$ and then on $ab$; thus $q(G - a) + q(G^{ab} - b) = q(G - a') + q(G^{a'b'} - b')$ as was required. Note that isomorphism of $G$ and $G'$ is our only tool for proving equality of $q(G)$ and $q(G')$, so our method was (and always will be) to apply pivot-reduction to each of $q(G)$ and $q(G')$ until the graphs comprising their sums are in one-to-one correspondence.

The more interesting case comes if $ab$ and $a'b'$ lie in the same component of $G$. If so, $G$ contains a walk $a, b, \ldots, a', b'$. Lemma 14 shows that for any sub-walk $v_i, v_{i+1}, v_{i+2}$, we have $q^{v_i,v_{i+1}}(G) = q^{v_{i+1},v_{i+2}}(G)$, whether $v_{i+2}$ is distinct from $v_i$ (the usual case), or $v_{i+2} = v_i$ (the case we use to re-orient a terminal edge if necessary). Given this, $q^{ab}(G) = \cdots = q^{a'b'}(G)$ as desired. □

While we will not exploit the following mild generalization of Theorem 12, it demonstrates that the structure of polynomial multiplication is not required: the powers $x, x^2, \ldots$ could be replaced by indeterminates $x_1, x_2, \ldots$.

**Theorem 15.** There is a unique map $\tilde{q} : \mathcal{G} \rightarrow \mathbb{Z}[x_1, x_2, \ldots]$, the $\mathbb{Z}$-module with basis $x_1, x_2, \ldots$, such that the following two conditions hold.

(i) If $G$ contains an edge $ab$ then

$$\tilde{q}(G) = \tilde{q}(G - a) + \tilde{q} (G^{ab} - b).$$

(ii) On the edgeless graph $E_n$, $\tilde{q}(E_n) = x_n$.

**Proof.** Identical to the proof of Theorem 12 □

On the other hand, polynomial multiplication does have a natural role for the interlace polynomial, as shown by Remark 16, the first of several properties we now consider.
6.2. **Simple properties.** We should first remark that \( q(G; x) \) does not appear to be a specialization of the Tutte polynomial \( T(G; x, y) \). At least, for any tree \( G \) of order \( n \) the Tutte polynomial is always \( T(G) = x^{n-1} \), while the interlace polynomial varies widely over trees.

With \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \), and \( V_1 \cap V_2 = \emptyset \), let \( G = G_1 \cup G_2 \) denote the disjoint union of the two graphs, so \( G = (V_1 \cup V_2, E_1 \cup E_2) \). The following remark is trivial but frequently employed.

**Remark 16.** For any graphs \( G_1 \) and \( G_2 \) on disjoint vertex sets, \( q(G_1 \cup G_2) = q(G_1) q(G_2) \).

For the Tutte polynomial, the corresponding equality \( T(G_1 \cup G_2) = T(G_1) T(G_2) \) holds even if \( G_1 \) and \( G_2 \) share a single vertex, but the path \( P_2 \) suffices to demonstrate that this generalization does not hold for the interlace polynomial.

**Remark 17.** For any graph \( G \) of order \( n \), \( q_G(2) = 2^n \).

**Proof.** If \( G = E_n \) the conclusion is immediate. Otherwise it follows from a trivial induction on \( n \): \( q(G; 2) = q(G - a; 2) + q(G^{ab} - b; 2) = 2^{n-1} + 2^{n-1} = 2^n \). □

**Remark 18.** For any graph \( G \) and any edge \( ab \), \( q(G^{ab}) = q(G) \).

**Proof.** Pivoting \( G \) on \( ab \), \( q(G) = q(G - a) + q(G^{ab} - b) \). Pivoting \( G^{ab} \) on \( ba \), \( q(G^{ab}) = q(G^{ab} - b) + q(G^{ab}(ba) - a) = q(G^{ab} - b) + q(G - a) \). □

**Remark 19.** Let \( G \) be a graph and let \( H \) be any induced subgraph of \( G \). Then \( \deg(q(H)) \geq \deg(q(G)) \). In particular, \( \deg(q(G)) \geq \alpha(G) \) where \( \alpha(G) \) is the independence number, i.e., the maximum size of an independent set.

**Proof.** For the main result, all one has to show is that \( \deg(q(G)) \geq \deg(q(G - a)) \). This holds because if \( a \) is an isolated vertex, then \( q(G) = x q(G - a) \); otherwise, for any edge \( ab \), \( q(G) = q(G - a) + q(G^{ab} - b) \), and the non-negativity of all terms implies \( q(G) \geq q(G - a) \).

The corollary follows by choosing \( H \) to be a maximum independent set in \( G \). □

We do not know if there exist graphs for which \( \deg(q(G)) \gg \alpha(G) \).

**Remark 20.** The degree of the lowest-degree term of \( q(G) \) is precisely \( k(G) \), the number of components of \( G \).

**Proof.** From Remark 18 it suffices to show that for any connected graph \( G \), \( q(G) \) has a linear term and no constant term. That the constant term of \( q \) is 0 whenever \( |G| \geq 1 \) is trivial by induction. We also prove by induction that \( q(G) \) has a linear term. Any connected graph \( G \) has some non-cut vertex \( a \). Pivoting on any edge \( ab \), \( q(G) = q(G - a) + q(G^{ab} - b) \), \( G - a \) is connected and is subject to the inductive hypothesis, \( q(G^{ab} - b) \) adds no constant term, and so \( q(G) \)'s lowest-order term is linear. □

**Remark 21.** Let \( \mu(G) \) denote size of a maximum matching (maximum set of independent edges) in a graph \( G \), and let \( \deg(q) \) denote the degree of the polynomial \( q \). If \( G \) is a forest with \( n \) vertices, then \( \deg(q(G)) = n - \mu(G) \).

The straightforward inductive proof is omitted.

Taken together, Remarks 17, 19, 20, and 21 mean that from the interlace polynomial \( q(G) \) we can read off the graph’s order, an upper bound on its independence number, its component number, and (if \( G \) is a forest) its matching number.

7. **Circuit partitions and the Martin and interlace polynomials**

In this section we relate the interlace polynomial to the enumeration of circuit partitions of a 2-in, 2-out digraph. Various circuit-counting polynomials were defined by Martin [Mar77], and the topic has been explored by Martin [Mar78a], Las Vergnas [LV81, LV83, LV88], Jaeger [Jae88], Ellis-Monaghan [EM98, EM99], and Bollobás [Bo92].

If \( H = H(C) \) is the interlace graph of an Euler circuit \( C \) of a 2-in, 2-out digraph \( D \) of order \( n \), then by Theorem 12 \( q_H(1) \) is the number of Euler circuits of \( D \). Since every pairing of the in- and out-edges at each vertex of \( D \) leads to a circuit partition of \( D \), there are \( 2^n \) such circuit partitions; also, it can be seen immediately from Theorem 12 that \( q_H(2) = 2^n \). In fact, the entire polynomial \( q_H(x) \) can be interpreted in terms of decompositions of \( D \) into circuits.
Let us define a somewhat unusual class of objects. We call a digraph weakly Eulerian if it is an edge-disjoint union of oriented circuits and "free loops", that is loops without vertices. Equivalently, a weakly Eulerian digraph is a digraph in which every vertex has the same in-degree as out-degree, together with some free loops. These graphs may have multiple edges (and multiple loops), all of which are distinguishable.

Definition 22 of polynomial \( r_k(D) \) counting the partitions of a digraph \( D \) into \( k \) circuits, extends naturally to weakly Eulerian digraphs. Note that \( \sum_k r_k(D) > 0 \) if and only if \( D \) is weakly Eulerian.

**Definition 22.** For any weakly Eulerian digraph \( D \), the directed circuit partition polynomial is

\[
r(D)(x) = \sum_k r_k(D)x^k.
\]

For example if \( D \) consists of 3 loops on a single vertex, \( r_1(D) = 2 \), \( r_2(D) = 3 \), and \( r_3(D) = 1 \). As observed by Las Vergnas [LV83], if \( D \) consists of \( m \) loops on a single vertex then \( r_k(D) = s(m, k) \), an unsigned Stirling number of the first kind (see for example [Wil90] and [PTW83]), and \( r(D)(x) = x(x+1) \cdots (x+m-1) \). Note that \( r_0(D) = 0 \) unless \( D \) has no edges, in which case \( r(D) = 1 \).

It is immediate that if \( D_1, \ldots, D_k \) are the components of \( D \) then \( r(D) = \prod_{i=1}^{k} r(D_i) \). To compute \( r(D) \), choose any vertex \( a \) of \( D \), and let \( D_\sigma \) be the various resolutions at \( a \) (matchings of in-edges to out-edges, followed by contraction of the vertex \( a \)); then \( r(D) = \sum_\sigma r(D_\sigma) \).

The circuit partition polynomial is simply a transformation of the Martin polynomial. Introduced by Martin in 1977 [Mar77], this polynomial was studied extensively by Las Vergnas [LV83], who is responsible for its generalization to all Eulerian graphs.

**Definition 23.** For any weakly Eulerian directed graph \( D \), the Martin polynomial is

\[
m(D; x) = \frac{1}{x-1} r(D; x-1).
\]

Recall that \( q(G) \) is defined for any undirected graph \( G \), and we have just defined the circuit partition polynomial \( r(D) \) and the Martin polynomial \( m(D; x) \) for any weakly Eulerian digraph \( D \). In the event that \( D \) is an Eulerian (connected) 2-in, 2-out digraph, and \( G \) an associated interlace graph, the following theorem describes an interesting connection between them.

**Theorem 24.** Let \( D \) be an Eulerian 2-in, 2-out digraph, \( C \) any Euler circuit of \( D \), and \( H = H(C) \) its interlace graph. Then

\[
q(H; x) = \frac{1}{x-1} r(D; x-1) = m(D; x).
\] (5)

**Proof.** The second equality is just Definition 22. For the first, we prove the transformation \( xq(H; 1+x) = r(D; x) \). Since \( C \) conveys more information than \( D \), let us define \( r(C) = r(D) \). First, suppose some \( a \) and \( b \) are interlaced in \( C \) (so \( ab \) is an edge in \( H \)). Recall that, as in the proof of Theorem 9 \( C \) and \( C^{ab} \) "resolve" the vertex \( a \) in the two different ways. Thus,

\[
r(D; x) = r(C; x)
\]

\[
= r(C - a; x) + r(C^{ab} - a; x)
\]

which by induction on the order of \( D \) (and of \( C \) and \( H \))

\[
= xq(H - a; 1+x) + xq(H^{ab} - b; 1+x)
\]

\[
= xq(H; 1+x).
\]

If there is no interlaced pair \( a, b \) in \( C \), i.e., if \( H \) is an edgeless graph, then \( C \) must have a loop on some vertex, and this loop may either be traversed as a separate circuit, giving \( xr(C - a; x) \), or may combine with the rest, giving \( r(C - a; x) \); in all, this gives \( r(C; x) = (1+x)r(C - a; x) \), and induction completes the proof. \( \Box \)

There is a trivial generalization to disconnected 2-in, 2-out graphs. Such a graph \( D \) consists of vertex-disjoint 2-in, 2-out components, \( D = D_1 \cup \cdots \cup D_n \). Each \( D_i \) has an Euler circuit \( C_i \), and if we define an interlace graph \( H = \bigcup H_i \) to be the union of the components’ interlace graphs, then \( r(D) = \prod r(D_i) \) and \( q(H) = \prod q(H_i) \), and it is immediate from Theorem 22 that \( r(D) = xq(H; 1+x) \).
Theorem 24 has the following consequence for the coefficients of the circuit-counting and interlace polynomials.

**Corollary 25.** Let $D$ be an Eulerian 2-in, 2-out digraph, $C$ any Euler circuit of $D$, and $H = H(C)$ its interlace graph. If $D$ has $r_k$ partitions into $k$ circuits, and $q(G; x) = \sum_k a_k x^k$, then
\[
    r_k = \sum_{\ell} a\ell \binom{\ell}{k-1} \quad \text{and} \quad a_k = \sum_{\ell} r_{\ell+1} (-1)^{\ell-k} \binom{\ell}{k}.
\]

**Proof.** The first part is immediate from the preceding theorem and the binomial formula. The second follows from the inclusion-exclusion principle (see for example [PTW83]), in particular, from the fact that the matrix $M$ with entries $m_{ij} = \binom{i}{j}$, $i$ and $j$ ranging from 0 to $n$, has as its inverse a similar matrix with entries $(m^{-1})_{ij} = (-1)^{i+j} \binom{i}{j}$. \qed

Corollary 27 below makes another connection between the Martin polynomial, and the interlace polynomial of an interlace graph. To introduce it we need a couple of simple preliminaries and a result of Martin’s. Let $a$ be a vertex of a 2-in, 2-out digraph $D$ of order $n$, with arcs $e_1, e_2$ entering a (say from $u_1$ and $u_2$) and arcs $f_1, f_2$ leaving a (say to $v_1$ and $v_2$). Let $D'_a$ and $D''_a$ be the digraphs obtained from $D$ by resolving $a$ in the two possible ways by uniting an incoming edge with an outgoing edge. For example, $D'_a$ is obtained from $D - a$ by adding to it an arc from $u_1$ to $v_1$ and an arc from $u_2$ to $v_2$, and $D''_a$ is obtained from $D - a$ by adding an arc from $u_1$ to $v_2$ and one from $u_2$ to $v_1$. If there is a loop at $a$, so that $u_1 = v_1 = a$, say, then $D'_a$ consists of a 2-in, 2-out digraph with $n - 1$ vertices and a vertexless loop, and in $D''_a$ all arcs incident with $a$ are replaced by a single arc. Note that
\[
    r(D) = r(D'_a) + r(D''_a).
\]

An **anti-circuit** is a circuit whose consecutive arcs have opposite orientations. Thus a cycle (cycle meaning circuit with no repeated vertex) is an anti-circuit if every other vertex has in-degree 2 and out-degree 2, the remaining vertices having in-degree 2 and out-degree 0. Every 2-in, 2-out digraph $D$ has a unique decomposition into anti-circuits; we write $a(D)$ for the number of anti-circuits in this decomposition. Martin made the following connection between anti-circuits and the value of $m(D)$ at $-1$; our proof is much simpler than that in Martin’s thesis [Mar77].

**Theorem 26 (Martin).** Let $D$ be a 2-in, 2-out digraph with $n$ vertices. Then
\[
    -2m(D; -1) = r(D; -2) = (-1)^{n + a(D)} 2^{a(D)}.
\]

**Proof.** The first equality is immediate from (5). For the second, we apply induction on $n$. For $n = 1$ our graph $D$ consists of two loops on a vertex, so $a(D) = 1$, $r(D; x) = x + x^2$ and so $r(D; -2) = (-1)^{1 + 1 + 2} 1$, as claimed. Suppose then that $n > 1$ and the theorem holds for smaller values of $n$. Let $D'_a$ and $D''_a$ be as in (6). If $a$ is contained in two alternating circuits then $a(D'_a) = a(D''_a) = a(D) - 1$, so
\[
    r(D; -2) = r(D'_a; -2) + r(D''_a; -2)
    = 2 \cdot (-1)^{(n-1)+(a(D)-1)} 2^{a(D)-1} = (-1)^{n+a(D)} 2^{a(D)}.
\]
If on the other hand $a$ is contained in a single alternating oriented circuit, then \{a(D'_a), a(D''_a)\} = \{a(D), a(D) + 1\}, so
\[
    r(D; -2) = (-1)^{(n-1)+a(D)} 2^{a(D)} + (-1)^{(n-1)+(a(D)+1)} 2^{a(D)+1} = (-1)^{n+a(D)} 2^{a(D)},
\]
completing the proof. \qed

**Corollary 27.** Let $H$ be an interlace graph. Then
\[
    q_H(-1) = (-1)^{|H|+1} (-2)^k
\]
for some non-negative integer $k$.

This led us to conjecture in [ABS00] that for all undirected graphs $G$, not just interlace graphs, $q(G; -1) = (-1)^{|G|} (-2)^k$ for some value $k$. This has recently been proved by Balister, Bollobás, Cutler, and Pebody [BBCP02], which for the interlace polynomial extends a result of Rosenstiehl and Read on the bicycle dimension of a graphical matroid [RR78]. The conjecture’s implications may be at least as interesting as the conjecture itself. If $ab$ is an edge of $H$ and $G = H - a$, then $q(H; -1) = q(G; -1) + q(H^{ab} - b; -1)$. For
all three to be powers of 2, it must be that $q(H; -1) \in \{\frac{1}{2} q(G; -1), -q(G; -1), 2q(G; -1)\}$. That is, adding a vertex to any graph $G$ in any way multiplies $q(-1)$ by a factor of $1/2$, $-1$, or $2$; symmetrically, deleting any vertex from any graph $H$ multiplies $q(-1)$ by a factor of $1/2$, $-1$, or $2$.

In conclusion, let us point out a connection between the interlace polynomial and the Kauffman bracket. Given an alternating link diagram $L$ with set of crossings $V$, let $D$ be the 2-in, 2-out digraph with vertex set $V$ whose edges are the strands of $L$, with each strand oriented from its over-crossing to its under-crossing. Writing $[L](A, B, d)$ for the Kauffman square bracket, it follows from results of Martin [Mar78] and Kauffman [Kau87] that

$$[L](1, 1, x) = \frac{1}{x} r_D(x) = m_D(x + 1) = q_H(x + 1);$$

these identities are also immediate consequences of the definitions.

An anonymous referee points out another connection, in the same setting. If $G$ is a planar graph and $G_m$ is its medial graph, oriented by going counterclockwise around the “black” faces, then the Tutte polynomial is related to the Martin polynomial by $T(G; x, y) = m(G_m; x)$ [Mar77, LV81], and in turn to the interlace polynomial through Theorem 24.

8. Interlace polynomials of some simple graphs

The following results are all quite simple and so the proofs are omitted for brevity.

Example 28. The interlace polynomials of edgeless graphs, complete graphs, stars, and complete bipartite graphs are given by

$$q(E_n) = x^n$$
$$q(K_n) = 2^{n-1}x$$
$$q(K_{1n}) = 2x + x^2 + x^3 + \cdots + x^n$$
$$q(K_{mn}) = (1 + \cdots + x^{m-1})(1 + \cdots + x^{n-1}) + x^m + x^n - 1$$

Example 29. For $n \geq 2$, the interlace polynomial of the path $P_n$ (with $n + 1$ vertices and $n$ edges) satisfies

$$q(P_n) = q(P_{n-1}) + xq(P_{n-2}),$$

and, for $n \geq 0$ and with $y = \sqrt{1 + 4x}$,

$$q(P_n)(x) = \frac{(3 + y)(y - 1)}{4y} \left(\frac{1 + y}{2}\right)^{n+1} + \frac{(3 - y)(y + 1)}{4y} \left(\frac{1 - y}{2}\right)^{n+1}.$$  (10)

Corollary 30. For the path $P_n$, $q(P_n)(1) = F_{n+2}$, the $(n + 2)$’nd Fibonacci number (with the convention $F_0 = 0, F_1 = 1$).

Example 31. The interlace polynomials of the cycles $C_n$ are $q(C_3) = 4x$ and, for $n \geq 4$ and with $y = \sqrt{1 + 4x}$,

$$q(C_n)(x) = \left(\frac{1 - y}{2}\right)^n + \left(\frac{1 + y}{2}\right)^n + \frac{y^4 - 10y^2 - 7}{16}$$

for $n$ even,

$$q(C_n)(x) = \left(\frac{1 - y}{2}\right)^n + \left(\frac{1 + y}{2}\right)^n + \frac{y^2 - 5}{4}$$

for $n$ odd.

9. Partitions, pairings, digraphs, and the polynomial

In this section we touch lightly on how the polynomial partitions the set of graphs, and related issues.
9.1. Partition of graphs by the interlace polynomial. As one would expect, the interlace polynomial does not resolve graph isomorphism. A variety of counterexamples is provided by taking two Euler circuits $C_1$ and $C_2$ in a 2-in, 2-out digraph $D$, and producing their corresponding interlace graphs $H_1$ and $H_2$; generally $H_1$ and $H_2$ will not be isomorphic, but $q(H_1) = q(H_2)$ since both count circuit partitions of $D$.

Another set of counterexamples is provided by Remark 16, noting that $q(K_m \cup K_{n-m}) = 2^{n-2}x^2$ regardless of $m$. Another counterexample is provided by the 5-cycle, and the 5-cycle with one chord-edge added: both have the polynomial $6x + 5x^2$.

In fact, the polynomial does not always even distinguish trees. For example, the path 1–2–3–4–5–6–7 with branches 3–8 and 5–9 has polynomial $2x + 9x^2 + 17x^3 + 13x^4 + 4x^5$, and so does the path 1–2–3–4–5–6 with branches 4–7–8 and 5–9.

We have not seriously explored either question, but we imagine that the number of different polynomials given by graphs (respectively, trees) of order $n$ is exponentially large in $n$, but that the chance that a graph (tree) is uniquely identified by its polynomial is exponentially small.

9.2. (Lack of) monotonicity. Inspection of a few examples initially suggests that if $G$ and $G'$ are graphs of the same order, with edge sets $E \subseteq E'$, then $q_G(1) \leq q_{G'}(1)$. In fact this is not the case. A small example is when $G'$ is the 4-spoke wheel (with edges consisting of the cycle 1, 2, 3, 4 united with the star from 5 to 1, 2, 3, and 4), and $G$ is the same graph with a “circumference edge” deleted (say the edge from 1 to 2). Here $q_G(x) = 6x + 5x^2$ and $q_G(x) = 4x^2 + 4x^3 + x^4$, so $q_G(1) = 11$ and $q_{G'}(1) = 9$.

9.3. Pairings and partitions by digraphs and interlace graphs. A “pairing” may be defined as a list of vertices 1 to $n$, in which each vertex appears twice: we think of this as the order in which a circuit visits these vertices. From a pairing we can construct a unique 2-in, 2-out digraph, and also a unique interlace graph. Since several pairings may lead to the same 2-in, 2-out digraph or interlace graph, the 2-in, 2-out digraphs and the interlace graphs both partition the pairings.

Neither of these partitions on the pairings refines the other. In a smallest example, the pairings 1 1 2 2 3 3 and 1 1 2 3 3 2 have isomorphic interlace graphs (both are edgeless), but non-isomorphic 2-in, 2-out digraphs. Also in a smallest example, the pairings 1 2 3 1 3 4 2 4 and 1 2 4 1 3 4 2 3 have non-isomorphic interlace graphs, but the same 2-in, 2-out digraph.

However, as shown by Propositions 32 and 33, the interlace graphs and 2-in, 2-out graphs partition each other: having pairings with the same interlace graph is an equivalence relation on 2-in, 2-out graphs, and having pairings with the same 2-in, 2-out graph is an equivalence relation on interlace graphs.

Note first that Lemmas 11 and 12 have the following immediate consequence. For a graph $G$, let $P(G)$ be the orbit of $G$ under pivoting. Then for every $C \in C(D)$, $H(C(D)) = P(H(C))$.

**Proposition 32.** Let $\mathcal{H}(D)$ denote the set of interlace graphs of all Euler circuits of $D$. Let $D$ and $D'$ be 2-in, 2-out digraphs such that $\mathcal{H}(D) \cap \mathcal{H}(D') \neq \emptyset$. Then $\mathcal{H}(D) = \mathcal{H}(D')$.

**Proof.** Choose $C \in C(D)$ and $C' \in C(D')$ such that $H(C) = H(C')$. Then $\mathcal{H}(D) = H(C(D)) = P(H(C)) = \mathcal{H}(D')$. □

Let $\mathcal{D}(H)$ denote the set of 2-in, 2-out digraphs obtained from all pairings with interlace graph $H$. Note that for every interlace graph $H$ and edge $ab \in E(H)$, $\mathcal{D}(H) = \mathcal{D}(H^{ab})$. (Any $D \in \mathcal{D}(H)$ is witnessed by a circuit $C$ with $D = D(C)$ and $H = H(C)$; $C^{ab}$ is a witness that $D \in \mathcal{D}(H^{ab})$. Thus $\mathcal{D}(H) \subseteq \mathcal{D}(H^{ab})$; by symmetry, then, $\mathcal{D}(H) = \mathcal{D}(H^{ab})$.) Putting it slightly differently, $\mathcal{D}(H) = \mathcal{D}(P(H))$.

**Proposition 33.** If $H$ and $H'$ are interlace graphs such that $\mathcal{D}(H) \cap \mathcal{D}(H') \neq \emptyset$, then $\mathcal{D}(H) = \mathcal{D}(H')$.

**Proof.** Let $D \in \mathcal{D}(H) \cap \mathcal{D}(H')$ and choose $C, C' \in C(D)$ such that $H(C) = H$ and $H(C') = H'$. $C(D)$ is a single orbit under transpositions, and contains $C$ and $C'$, so $H$ and $H'$ belong to the same orbit under pivoting: $P(H) = P(H')$. Hence $\mathcal{D}(H) = \mathcal{D}(P(H)) = \mathcal{D}(H')$, as claimed. □

10. Interlace polynomials of substituted and rotated graphs

We now compute interlace polynomials for “substituted” and “rotated” graphs. These are of mild interest in themselves, and are needed for the proofs (and occasionally the statements) of the results in Section 11 on extremal properties of the interlace polynomial.
10.1. Substituted graphs. The following definition of graph substitution is recalled from [Bol98].

**Definition 34.** If $G$ is a graph with vertices $v_1, \ldots, v_n$, and $G_1, \ldots, G_n$ are arbitrary graphs on disjoint vertex sets, then $G^* = G[G_1, \ldots, G_n]$ is obtained from $\bigcup_{i=1}^n G_i$ by joining all vertices of $G_i$ to all vertices of $G_j$ whenever $ij \in E(G)$.

We say that $G^*$ is obtained from $G$ by substituting $G_1, \ldots, G_n$ for the vertices or by replacing the vertices with $G_1, \ldots, G_n$. Note that if we replace the vertices of $G$ one by one with the graphs $G_1, \ldots, G_n$, we get the same graph $G^*$. If each $G_i$ is a complete graph then we call $G^*$ a solid graph with template $G$, or a solid $G$-graph. Similarly, if each $G_i$ is an edgeless graph then $G^*$ is a thick graph with template $G$ or a thick $G$-graph.\(^1\)

Clearly, a graph $G$ is a solid graph with a template of order $k$ if and only if $V(G) = V_1 \cup \ldots \cup V_k$ such that each induced subgraph $G |_{V_i}$ is complete and for every pair $i, j$, $1 \leq i < j \leq k$, either $G$ contain all $V_i - V_j$ edges, or it contains none of them. Thick graphs have a similar description, except $G |_{V_i}$ is not complete but edgeless. In particular, the complement $\overline{G}$ of a solid graph $G$ with template $H$ is a thick graph with template $\overline{H}$. Specializing further, the complement of a solid path of length $3$ is a thick path of length $3$.

Immediately from the definition, substitution is transitive: if $G^*$ has template $G$ and $G$ has template $H$, then $G^*$ has template $H$. Specifically, we have the following.

**Proposition 35.** If $G^* = G[G_1, \ldots, G_n]$ and $G = H[H_1, \ldots, H_m]$ then there exist graphs $H'_1, \ldots, H'_n$ such that $G^* = H[H'_1, \ldots, H'_n]$.

Proof. The result has no mathematical content, but is confusing. Each vertex $v_i$ in $H$ is replaced by a graph $H_i$ to form $G$. Then each vertex in $G$ — write it as $u'_i$ to indicate that it is the $j$'th vertex within graph $H_i$ — is itself substituted with a graph $G'_i$. Writing $n_i = |H_i|$, the net result is, specifically,

$$G^* = H[H_1[G'_1, \ldots, G'_1], \ldots, H_m[G'_m, \ldots, G'_m]].$$

□

In the case of greatest interest, we have the following.

**Lemma 36.** If $G^*$ is a solid $G$-graph and $G$ is a solid $H$-graph, then $G^*$ is a solid $H$-graph.

Proof. This is a specialization of the previous proposition in which each $H_i$ is a complete graph and each $G'_i$ is a complete graph, and thus each $H_j[G'_1, \ldots, G'_m]$ is a complete graph, making $G^*$ a solid $H$-graph. □

**Lemma 37.** Let $a$ be a vertex of a graph $H$ such that $G = H - a$ is a solid graph with template of order $k$. Then $H$ is a solid graph with template of order at most $2k + 1$.

Also, if $ab$ is an edge of a solid graph $G$ with template of order $k$, then $G_{ab}$ is also a solid graph with template of order $k$.

Proof. First, let $V_1, \ldots, V_k$ be the vertex classes showing that $G = H - a$ is a solid graph with template of order $k$. For $1 \leq i \leq k$, let $U_i = V_i \cap \Gamma(a)$ and $W_i = V_i - \Gamma(a)$. Then the classes $\{a\}, U_1, W_1, \ldots, U_k, W_k$ (some of which may be empty), show that $H$ is a solid graph with template of order $k$.

Second, if $a$ and $b$ belong to the same class $V_i$ then $G_{ab} = G$. If, on the other hand, $a \in V_1$ and $b \in V_2$, say, then $V'_1 = V_1 \cup \{b\} - \{a\}$, $V'_2 = V_2 \cup \{a\} - \{b\}$, $V_3, \ldots, V_k$ show that $G_{ab}$ is a solid graph with template of order $k$.

□

**Proposition 38.** Let $G^* = G[K_{m_1}, \ldots, K_{m_n}]$. Then $q(G^*) = q(G) \cdot 2^{|G^*| - |G|}$.

Proof. For any $m_i > 1$, let $H = G[K_{m_1}, \ldots, K_{m_i-1}, \ldots, K_{m_n}]$. Let $a, b$ denote two of the vertices in the $K_{m_i}$, and pivot-reduce on edge $ab$. Note that $(G^*)^{ab} = G^*$ and that $G^* - a = (G^*)^{ab} - b = H$, so that $q(G^*) = 2q(H)$. The proposition follows inductively. □

The simplest case of graph substitution is duplicating a single vertex $a$, i.e., replacing $a$ with $E_2$; we will write this $G^* = G \circ a$. For the case of multiplying the $i$'th vertex $k_i$-fold, i.e., taking $G[E_{k_1}, \ldots, E_{k_n}]$, we will write $G^* = G[k_1, \ldots, k_n]$. As far as the interlace polynomial is concerned, general graph substitution can be expressed in terms of vertex multiplication, via the following theorem.

\(^1\)The case that a substituted graph $G_i$ is the null graph ($|G_i| = 0$) is unnatural but sometimes convenient, for example in the proof of Proposition 35.
Proposition 39. For the general graph substitution \( G^* = G[G_1, \ldots, G_n] \), with \( q(G_i) = \sum a(i, k)x^k \), we have
\[
q(G^*) = \sum_{k_1, \ldots, k_n} a(1, k_1) \cdots a(n, k_n) q(G[k_1, \ldots, k_n]).
\]

Proof. In \( G[G_1, \ldots, G_n] \), the vertices in \( G_i \) cannot distinguish among any vertices in \( G_j \), regardless of whether \( ij \) is an edge in \( G \). Thus, pivots can be performed within each \( G_i \) separately, with no interactions. \( \square \)

The interlace polynomials \( q(G[k_1, \ldots, k_n]) \) occurring above can all be expressed as combinations of the the \( 2^n \) interlace polynomials \( q(H) \), as \( H \) runs over the induced subgraphs of \( G \). We carry this out in the next three propositions, first calculating the effect of a single vertex duplication, then calculating the effect of duplicating an arbitrary subset of the vertices, and finally handling the general case.

Proposition 40. If \( a \) is a vertex of a graph \( G \), then
\[
q(G \circ a) = (1 + x)q(G) - xq(G - a).
\]

Proof. If \( a \) is an isolated vertex then \( q(G \circ a) = xq(G) \) and \( xq(G - a) = q(G) \), proving the assertion. Otherwise, pivot on an edge \( ab \). Consider the “duplicate” vertex \( a' \) in the context of the four classes of Definition \( 4 \) neighbors \( N(a) \) of \( a \) alone, neighbors \( N(b) \) of \( b \) alone, neighbors \( N(ab) \) of both, and vertices \( N_0 \) that are neighbors of neither. We know that before pivoting, like \( a' \in N(b) \), \( a' \) is adjacent to every vertex in \( N(a) \) and \( N(ab) \), and \( a' \) is adjacent to no vertex \( N(b) \) nor \( N_0 \). After pivoting, then, \( a' \in N(b) \), \( a' \) is now adjacent to no vertex in \( N(a) \) nor in \( N(ab) \), and \( a' \) remains adjacent to no vertex in \( N(b) \) nor \( N_0 \). So in \( G^{ab} \), \( a' \) is adjacent only to \( b \), and in \( G^{ab} \circ a' \) is an isolated vertex: \( (G \circ a)^{ab} - b = (G^{ab} - b) \cup E_1 \). Thus \( q(G \circ a) = q(G \circ a - a) + q((G \circ a)^{ab} - b) = q(G) + xq(G^{ab} - b) = q(G) + x[q(G) - q(G - a)] = (1 + x)q(G) - xq(G - a). \) \( \square \)

Proposition 41. For a graph \( G \) with vertices \( v_1, \ldots, v_m \), for any \( 1 \leq m \leq n \), the interlace polynomial for the graph \( G^* \) formed by duplicating \( v_1, \ldots, v_m \) can be expressed in terms of the interlace polynomials of the \( 2^m \) graphs formed by deleting some subset of \( \{v_1, \ldots, v_m\} \), as follows. Let \( k_1 = \cdots = k_m = 2 \), \( k_{m+1} = \cdots = k_n = 1 \), and \( \ell_{m+1} = \cdots = \ell_n = 1 \). Then with \( p_1(x) = (1 + x) \) and \( p_0(x) = -x \),
\[
q(G[k_1, \ldots, k_n]) = \sum_{\ell_1=0}^1 \cdots \sum_{\ell_n=0}^1 \{ q(G[\ell_1, \ldots, \ell_n]) \prod_{i=1}^m p_{\ell_i}(x) \}, \tag{11}
\]

Proof. This proposition is proved by repeatedly applying Proposition \( 10 \). Note first that for \( m = 1 \), with \( v_1 = a \), \( 11 \) is precisely Proposition \( 10 \). Similarly, for \( m = 2 \), writing \( v_1 = a, v_2 = b \), \( 11 \) simply says
\[
q(G \circ a \circ b) = (1 + x)^2(q(G) - x(1 + x)(q(G - a) + q(G - b)) + x^2q(G - a - b)). \tag{12}
\]

Let \( H = G \circ a \), so that \( H \circ b = G \circ a \circ b \) has \( q(H \circ b) = (1 + x)q(H) - xq(H - b) \). Note that \( H - b = (G - b) \circ a \) has \( q(H - b) = q((G - b) \circ a) = (1 + x)q(G - b) - xq(G - a - b) \). Thus
\[
q(G \circ a \circ b) = q(H \circ b)
= (1 + x)[q(H)] - x[q(H - b)]
= (1 + x)[(1 + x)q(G) - xq(G - a)] - x[(1 + x)q(G - b) - xq(G - a - b)],
\]
which shows \( 12 \). The general case follows by induction on \( m \), applying Proposition \( 10 \) to compute the effect of duplicating the \( m + 1 \)st, in the graph where the first \( m \) vertices have already been duplicated. \( \square \)

Proposition 42. Given a graph \( G \) on vertices \( v_1, \ldots, v_n \), with \( n \geq 1 \), define polynomials \( q(k_1, \ldots, k_n)(x) \) by
\[
q(k_1, \ldots, k_n)(x) = \sum_{\ell_1=0}^1 \cdots \sum_{\ell_n=0}^1 (-1)^{n+\ell} q(G[\ell_1, \ldots, \ell_n])(x) \prod_{i=1}^n \sum_{j=1-\ell_i}^{k_i-1} x^j
\]
where \( \ell = \sum_{i=1}^n \ell_i \). Then for \( k_1, \ldots, k_n \geq 1 \),
\[
q(G[k_1, \ldots, k_n])(x) = q(k_1, \ldots, k_n)(x). \tag{13}
\]
Proof. The proof is by induction on \(k_1 + \cdots + k_n\), and the basis consists of \(2^n\) cases — those with all \(k_i \leq 2\). (The case of \(k_i \geq 3\) with all \(k_i = 1\) is trivial, and the other basis cases are given by Proposition 11.) Assume then that some \(k_i \geq 3\); without loss of generality we may take \(i = 1\). Write \(a = v_1\) and \(H = G[k_1 - 1, k_2, \ldots, k_n]\), so that by Proposition 10 \(q(G[k_1, k_2, \ldots, k_n]) = q(H \circ a) = (1 + x)q(H) - qx(H - a)\). By induction, \(q(H)\) and \(q(H - a)\) may be expanded as \(q(k_1 - 1, \ldots, k_n)\) and \(q(k_2, \ldots, k_n)\), in which, for each choice of \(\ell_1, \ldots, \ell_n\), all factors are equal, except for the factor indexed by \(i = 1\). For this factor, the contribution to \((1 + x)q(H) - qx(H - a)\) is \((-1)^{1-\ell_i}\) times
\[
(1 + x) \left( \sum_{j=1}^{k_1-1} x^j \right) - (x) \left( \sum_{j=1}^{k_1-2} x^j \right) = \left( \sum_{j=1}^{k_1-1} x^j \right),
\]
which is the coefficient of \((-1)^{1-\ell_i}\) in the corresponding factor in the expression for \(q(k_1, k_2, \ldots, k_n)\). This shows \(q(G[k_1, k_2, \ldots, k_n]) = q(k_1, k_2, \ldots, k_n)(x)\), completing our proof by induction.

Corollary 43. For \(r \geq 1\) and \(k_1, \ldots, k_r \geq 1\), the complete \(r\)-partite graph \(G^* = K_r(k_1, \ldots, k_r)\) has
\[
q(G^*)(x) = \frac{x}{2} \prod_{i=1}^{r} (2 + x + \cdots + x^{k_i - 1}) - (-1)^r (1 - \frac{x}{2}) \prod_{i=1}^{r} (x + \cdots + x^{k_i - 1}).
\]

Proof. With \(n \equiv r\), this is the special case \(G = K_n\) in Proposition 11. In detail, for \(l_1, \ldots, l_n \in \{0, 1\}\) with \(s = l_1 + \cdots + l_n\), note that \(G(l_1, \ldots, l_n) = K_s\), with \(q(K_s)(x) = 2^{s-1}x\) provided that \(s > 0\), but \(q(K_s)(x) = 1 = 2^{s-1}x + (1 - x/2)\) for \(s = 0\). Adding this “correction” to the term with \(l_1 = \cdots = l_n = 0\) gives a sum which factors:
\[
q(G^*) + (1 - \frac{x}{2}) \prod_{i=1}^{n} \left( \sum_{j=1}^{k_i-1} x^j \right) = \sum_{\ell_i = 0}^{1} \cdots \sum_{\ell_n = 0}^{1} \left\{ \frac{x}{2} \prod_{i=1}^{n} \left[ \left( \sum_{j=1}^{k_i-1} x^j \right) \right] \right\}
\]
\[
= \frac{x}{2} \prod_{i=1}^{r} \left[ 2^1 (1 + x + \cdots + x^{k_i - 1}) - 2^0 (x + \cdots + x^{k_i - 1}) \right]
\]
\[
= \frac{x}{2} \prod_{i=1}^{r} (2 + x + \cdots + x^{k_i - 1}).
\]

10.2. Rotated graphs. Let \(H\) be a graph including distinct vertices \(u, v, w\), in which \(uw\) is an edge and no other edges on \(w\) are allowed. Let \(G\) be obtained from \(H - w\) by toggling the \(vw\) relation, so that \(uw\) is an edge of \(G\) if and only if it is not an edge of \(H\). We say that \((G, H)\) is a rotation.

The operation’s name comes from the case where \(uw\) was an edge in \(G\), and this edge is “rotated” around the pivot point \(u\) to become the new edge \(uw\).

Theorem 44. Let \((G, H)\) be a rotation and \(1 \leq x\). Then \(q_G(x) \leq q_H(x)\).

Proof. We prove the inequality by induction on the order of \(H\). For \(|H| = 3\) there are two cases to check, according to whether \(uw\) is or is not an edge in \(H\). When \(uw\) is an edge, \(q_H(x) = 2x + x^2 \geq x^2 = q_G(x)\), as long as \(x \geq 0\). When \(uw\) is not an edge, \(q_H(x) = 2x^2 \geq 2x = q_G(x)\) as long as \(x \geq 1\).

In proving the induction step, fix an \(x \geq 1\), and assume that \(|H| > 3\). We consider the following cases. Other than \(u\) and \(v\), \(G\) has: (1) only isolated vertices; (2) an edge \(ab\), \(a\) and \(b\) distinct from \(u\) and \(v\); (3) no edge as above, and some vertex \(c\) adjacent to \(v\) but not \(w\); (4) none of the preceding, and some vertex \(d\) adjacent to \(u\) but not \(v\); (5) none of the preceding, and some vertex \(e\) adjacent to both \(u\) and \(v\).

If there is an isolated vertex \(a\) of \(G\), then \(a\) is also isolated in \(H\), and \(q(G; x) = qx(G - a; x) \leq qx(H - a; x) = q(H; x)\), where the inequality follows from the inductive hypothesis.

If there is an edge \(ab\) independent of \(u\) and \(v\), pivot-reduce both \(G\) and \(H\) on \(ab\). The \(ab\)-pivot either toggles the edge \(uv\) in both \(G\) and in \(H\), or leaves it unchanged in both \(G\) and \(H\); also it does not affect the
neighborhood of $w$. Thus $(G^{ab}, H^{ab})$ is a rotation, and so is $(G^{ab} - b, H^{ab} - b)$. Trivially, $(G - a, H - a)$ is also a rotation, and, by the inductive hypothesis, $q(G; x) = q(G-a; x) + q(G^{ab} - b; x) \leq q(H - a; x) + q(H^{ab} - b; x) = q(H; x)$.

Failing the above, it must be that every other vertex in $G$ is adjacent to $u$ or $v$ or both, but to no other vertices. Suppose there is a vertex $w$ adjacent only to $v$. If $q(G - w; x) = q(G - v; x)$, then $q(G; x) = q(G - w; x) + q(G - v; x)$. By the inductive hypothesis, $q(G - w; x) \leq q(H - w; x) + q(H - v; x)$ and $q(G - v; x) \leq q(H - v; x)$. Summing the two inequalities, $q(G; x) \leq q(H; x)$.

The next case is that there is a vertex $d$ adjacent only to $u$. Then $q(G; x) = q(G - d; x) + q(G - u; x) \leq q(H - d; x) + q(H - u; x) = q(H; x)$. If $q(G - d; x) \leq q(H - d; x)$ is by the inductive hypothesis, while $q(H - d; x) = xq(H - v - w; x) = xq(G - u; x) \geq q(G - u; x)$ because $H - u$ and $G - u$ differ just in the isolated vertex $w$.

The final case is that every vertex in $G$, other than $u$ and $v$, is adjacent to both $u$ and $v$. Pick one such vertex $e$ and pivot-reduce $G$ and $H$ on $ev$. By the inductive hypothesis, $q(G - e; x) \leq q(H - e; x)$. On the other hand, $H^{ev}$ consists of a star on $v$, an edge $uw$, and possibly an edge $uv$; $G^{ev}$ is the same but without the vertex $w$. Pivot-reducing $H^{ev}$ on $uw$ gives $q(H^{ev}) = q(H^{ev} - w) + q(H^{ev} - u) = q(G^{ev}) + xq(G^{ev} - u) \geq q(G^{ev})$. Summing the two inequalities, here too $q(G; x) \leq q(H; x)$, and we are done.

11. Extremal properties of the interlace polynomial

In this section we consider extremal values of the interlace polynomial’s degree, its evaluation at 1, and its number of nonzero terms, in terms of a graph’s order $|G|$ (the number of vertices) and its size $e(G)$ (the number of edges).

11.1. Extremal values of the degree of $q(G)$.

**Proposition 45.** For any graph $G$ of order $n$, $\deg(q(G)) \leq n$, with equality only for $G = E_n$. Also, $q_G(1) \geq 1$, with equality only for $G = E_n$.

**Proof.** The first assertion follows from Remark 19: $\deg(q(G)) = n$ means the independence number $\alpha(G) = n$, and $G$ must be an edgeless graph.

The second assertion is proved by induction on the order of $G$. If $G = E_n$, then $q_G(1) = 1$ and we are done. Otherwise there is an edge $ab$ in $G$, and $q(G) = q(G - a) + q(G^{ab} - b) \geq 2$ by the inductive hypothesis.

**Remark 46.** If $q(G)$ is purely linear, then $G$ is a complete graph.

**Proof.** By Remark 19, $\deg(q(G)) = 1$ means $\alpha(G) = 1$, and $G$ is complete.

11.2. Extremal values of $q_G(1)$.

**Propositions 48, 49, 50, and 51** provide upper and lower bounds for $q_G(1)$ in terms of $G$’s size and order. The proof of Proposition 48 will need the following lemma.

**Lemma 47.** For any tree $G$ of order $n$, $q_G(1) \leq F_{n+1}$, with equality achieved precisely by a path $P_{n+1}$.

**Proof.** One direction is immediate from Corollary 30 if $G = P_{n-1}$ then $q_G(1) = F_{n+1}$. For the other direction we apply induction on $n$ to show that any tree $G$ achieving the bound must be a path. The base case with $n = 1$ is trivial, as is that $G \neq E_n$.

If $n > 1$, then pick any leaf $a$, with parent $b$. The pivot reduction on $ab$ gives $q(G) = q(G - a) + q(G - b) = q(G - a) + xq(G - b - a)$. Evaluating at $x = 1$, $q(G; 1) = q(G - a; 1) + q(G - b - a; 1) \leq F_n + F_{n-1}$ by the inductive hypothesis, with equality only if $G - a$ and $G - a - b$ are both paths. For this to be the case, $b$ must be a terminal of the path $G - a$, and therefore $G$ is also a path. Hence the only tree achieving the upper bound is the path.

For convenience, we subsume complete bipartite graphs and edgeless graphs into complete tripartite graphs, by allowing one or two of the three vertex classes to be empty.

**Proposition 48** (Size, lower bound). For every graph $G$, $q_G(1) \geq e(G) + 1$, with equality if and only if $G$ consists of a complete tripartite graph and isolated vertices.

**Proof.** Apply induction on the order $n$ of $G$. For $n \leq 3$ or $e(G) = 0$ the result holds by inspection, so assume that $n \geq 4$, $e(G) > 0$, and the assertion holds for all smaller values of $n$. 
Let \( ab \in E(G) \). Then

\[
q(G; 1) = q(G - a; 1) + q(G^{ab} - b; 1)
\]

\[
\geq [e(G) - d(a)] + 1 + [e(G^{ab} - b)] + 1
\]

\[
\geq [e(G) - d(a)] + 1 + [(d(a) - 1) + 1]
\]

\[
= e(G) + 1,
\]

where the first inequality follows from the inductive hypothesis. This proves the inequality and — since both inequalities hold with equality when \( G \) consists of a tripartite graph and isolated vertices — also proves the “if” direction for equality.

If equality holds, then \( e(G^{ab} - b) = d(a) - 1 \), thus all edges of \( G^{ab} - b \) are incident to \( a \) or \( b \). Write \( \Gamma \) for the neighborhood in \( G^{ab} \). Letting \( V_a = \Gamma(a) \setminus \Gamma(b) \), \( V_b = \Gamma(b) \setminus \Gamma(a) \), and \( V_{ab} = \Gamma(a) \cap \Gamma(b) \), pivoting \( G^{ab} \) on \( ab \) to recover \( G \), we find that \( G \) is the complete tripartite graph with vertex classes \( V_a, V_b, \) and \( V_{ab} \). (Note that \( V_a \) and \( V_b \) are both non-empty, since \( b \in V_a \) and \( a \in V_b \) )

The next extremal proposition, bounding \( q_G(1) \) by a Fibonacci number, uses a property of graph rotation from Section 10.2.

**Proposition 49** (size, upper bound). For any connected graph \( G \) with \( m \) edges, \( q_G(1) \leq F_{m+2} \), with equality achieved precisely by a path \( P_m \). If \( G \) consists of components \( G_i \) with sizes \( m_i \), then \( q_G(1) \leq \prod F_{m_i+2} \). Always, if \( G \) has \( m \) edges then \( q_G(1) \leq 2^m \), with equality if and only if \( G \) consists of independent edges, and isolated vertices.

**Proof.** The proposition’s essence is its first assertion. One direction is immediate from Corollary 30 if \( G = P_m \) then \( q_G(1) = F_{m+2} \).

For the other direction, suppose that \( q_G(1) \geq F_{m+2} \). If \( G \) is a tree, by Lemma 17 it can only achieve the bound \( q(G; q) = F_{n+1} = F_{m+2} \) if it is a path. Otherwise, by the explicit calculations in Example 31 \( G \) cannot be a simple cycle, and so it has some vertex \( v \) of degree at least 3. Choose any edge \( ab \) in a cycle of \( G \) and not incident to \( v \), and “rotate” the edge out; that is, delete it, and add a new vertex incident to \( a \) alone. Repeat until no cycles remain. As we only deleted cycle edges, the graph remains connected. By Theorem 34 each rotation can only increase \( q(1) \), yet the final tree, whose vertex \( v \) shows it not to be a path, has \( q(1) < F_{m+2} \). Thus \( q_G(1) < F_{m+2} \), completing the proof of the main assertion.

The result for graphs with several components follows immediately by Remark 16. Also, \( F_{m+2} \leq 2^m \), with equality if \( m = 0 \) or 1, proving the final statement.

**Proposition 50** (order, lower bound). Let \( G \) be a graph of order \( n \), having no isolated vertices. Then \( q_G(1) \geq n \), with equality if and only if \( G \) is a star, or \( n = 4 \) and \( G \) consists of two independent edges.

**Proof.** The “if” direction is trivial. To prove the “only if”, we apply induction on \( n \). For \( n \leq 4 \) the result is easily checked, so let us assume that \( n > 4 \) and the result holds for all smaller values of \( n \). If \( G \) is disconnected, say \( G = G_1 \cup G_2 \) with \( |G_1| = n_1 \geq 2 \) and \( |G_2| = n_2 \geq 2 \), then, by the inductive hypothesis, \( q_G(1) \geq n_1n_2 > n \). If on the other hand \( G \) is connected, then either \( G \) is a tree with \( n - 1 \) edges, or \( e(G) \geq n \). Hence, by Proposition 48, \( q_G(1) \geq n \), with equality if \( G \) is a tree and is an extremal graph of that Theorem.

A graph that is tree and is a complete tripartite graph is a star, completing the proof.

This simple proposition was used by Balister, Bollobás, Riordan and Scott [BBRS01] to give a particularly simple proof of the classical theorem of Bankwitz [Ban30] that if a knot has a nontrivial reduced alternating diagram then it is nontrivial.

Note that the final proposition in this set of four has a nice parallel to Lemma 47.

**Proposition 51** (order, upper bound). For any graph \( G \) of order \( n \), \( q_G(1) \leq 2^{n-1} \), with equality if and only if \( G = K_n \).

**Proof.** The proof is by induction on \( n \), with trivial base case \( n = 1 \). For \( n > 1 \), \( G \) cannot be the edgeless graph, so choose an edge \( ab \) in \( G \). Then \( 2^n = q(G) = q(G - a) + q(G^{ab} - b) \leq 2^{n-1} + 2^{n-1} \), so both \( q(G - a) \) and \( q(G^{ab} - b) \) must achieve their extremal values, and by the inductive hypothesis each must be isomorphic \( K_{n-1} \). Since \( G^{ab} - b \) is complete, the neighborhood of \( a \) in \( G^{ab} \) consists of all vertices (\( a \) is connected to \( b \) by definition). The neighborhoods of \( a \) in \( G \) and in \( G^{ab} \) are always the same by definition of the pivot operator, thus \( a \) is connected to every other vertex in \( G \). Since \( G - a \cong K_{n-1} \), \( G \cong K_n \).
11.3. The several largest values of $q_G(1)$. Proposition \ref{prop:solid} allows an extension, given by Proposition \ref{prop:largest}
It and the following results rely on properties of graph substitution from Section \ref{sec:substitution}.

**Proposition 52.** Over graphs $G$ of order $n$, the second-maximum value of $q_G(1)$ is $\frac{3}{2}2^{n-1}$, and is achieved exactly when $G$ is a solid $P_2$-graph but not a complete graph. (That is, $G$ is composed of a pair of complete graphs sharing some but not all their vertices.)

**Proof.** Observing that a complete graph is a special case of a solid $P_2$-graph, and is covered by Proposition \ref{prop:solid}, the present proposition is equivalent to: $q_G(1) \geq \frac{3}{2}2^n$ precisely when $G$ is a solid $P_2$-graph (where one of the three components may be of size 0).

With a little work, we can check that the assertion holds for $n \leq 7$, so assume that $n > 7$ and the assertion holds for graphs of orders up to $n - 1$. Let $G$ be a graph of order $n > 7$ with $q_G(1) \geq \frac{3}{2}2^n$. Then $G$ is not edgeless; let $ab \in E(G)$. Since $q(G)(1) = q(G - a)(1) + q(G^{ab} - b)(1)$, either $q(G - a)(1) \geq \frac{3}{2}2^{n-1}$ or $q(G^{ab} - b)(1) \geq \frac{3}{2}2^{n-1}$. But then, by the inductive hypothesis, either $G - a$ or $G^{ab} - b$ is a solid $P_2$-graph. By the first part of Lemma \ref{lem:substitution}, either $G$ or $G^{ab}$ is a solid graph of order $n$ with a template $H$ of order $h \leq 7$. By Proposition \ref{prop:solid} we have $\frac{3}{2}2^n \leq q(H)(1) \cdot 2^{(n-1) - h}$, so $q(H)(1) \geq \frac{3}{2}2^h$ and thus $H$ is a solid $P_2$-graph. But then so is at least one of $G$ and $G^{ab}$. In the second case, $G$ too is a solid $P_2$-graph, by the second part of Lemma \ref{lem:substitution}. (If the template of $G$ were any other-3 graph other than $P_2$, the template would still have to be connected — since $G^{ab}$ is and therefore $G$ is — so it could only be $K_3$, which as a template may be regarded as a special case of $P_2$ two of whose three groups have size 0.) Hence $G$ is a solid $P_2$-graph, as claimed. \hfill \Box

In fact, the proof above gives the following assertion. Suppose that every graph of order $n \leq 2k + 1$ with $q_G(1) \geq c2^n$ is a solid $G_i$-graph for some $i = 1, \ldots, \ell$, where $G_1, \ldots, G_\ell$ are graphs of order at most $k$. Let now $G$ be a graph of order $n$ with $q_G(1) \geq c2^n$. Then $G$ is a solid $G_i$-graph for some $i, 1 \leq i \leq \ell$.

Just as there is a large gap between $q_G(1)$’s maximum value of $2^{n-1}$ and its second-maximum value of $\frac{3}{2}2^{n-1}$, there seem to be other such gaps. (Table 6 in \cite{interlace} exhibits some, albeit only for interlace graphs.) The assertion above might be used to prove the following conjecture.

**Conjecture 53.** There are constants $c_1 = \frac{1}{2} > c_2 > \ldots$ such that for every $k \geq 1$ and $n$ sufficiently large, then, first, there are graphs $G_1, \ldots, G_k$ of order $n$ such that $q(G_i)(1) = c_i2^n$, and, second, if $G$ is any graph of order $n$ with $q_G(1) \geq c_i2^n$, then $q_G(1) = c_i2^n$ for some $i \leq k$.

11.4. Zero and nonzero terms of $q(G)$.

**Lemma 54.** Let $G$ be a connected graph of order at least $\ell + 4$ such that every connected induced subgraph of it with $\ell + 3$ vertices is a solid path of length at most $\ell$. Then $G$ is a solid path of length at most $\ell$.

**Proof.** We may clearly assume that $\ell \geq 3$, for some two vertices $a$ and $b$ the graph $G - a - b$ is a solid $\ell$-path, and the graphs $G - a$, $G - b$ and $G - a - b$ are connected. Let $P$ be an induced $\ell$-path in $G - a - b$. Since the induced subgraph of $G$ with vertex set $V(P) \cup \{a\} \cup \{b\}$ is a solid $\ell$-path, so is the entire graph $G$. \hfill \Box

**Proposition 55.** Let $G$ be a graph such that $q(G)$ has precisely two non-zero terms. Then one of the components of $G$ is a solid path of length 2 or 3, and all other components are complete graphs.

**Proof.** It is easily seen that in proving this we may assume that $G$ is an incomplete connected graph of order $n \geq 6$, and the assertion holds for graphs of smaller order. Then every connected incomplete induced subgraph $H$ of $G$ has precisely two non-zero terms, so $H$ is a solid path of length at most 3. Writing $\ell$ for the maximal length of an induced path in $G$, Lemma \ref{lem:substitution} implies that $G$ itself is a solid path of length $\ell$. \hfill \Box

**Proposition 56.** If $|G| = n \geq 3$ and $q(G)$ has $n - 1$ nonzero terms, then $q(G) = 2x + x^2 + \cdots x^{n-1}$, and $G$ is an $n$-star.

**Proof.** Since by Remark \ref{rem:degree}, $2^n = q_G(2) = \sum_{i=1}^{\infty} a_i 2^i$ with each $a_i$ a non-negative integer, we can only have $q_G(x) = 2x + x^2 + \cdots x^{n-1}$. By Remark \ref{rem:linear}, since $q(G)$ has nonzero linear term, $G$ is connected. Choose an edge $ab$ of $G$ such that $G - a$ is connected. In particular, for $n \geq 3, G - a$ has some edge, so (by Proposition \ref{prop:degree}) $\deg(q(G - a)) \leq n - 2$. Thus the $x^{n-1}$ term of $q(G)$ must belong to $q(G^{ab} - b)$. As $q(G^{ab} - b)(2) = x^{n-1}$, we can only have $q(G^{ab} - b) = x^{n-1}$, and so (again by Proposition \ref{prop:degree}) $G^{ab} - b = E_{n-1}$. Since $G^{ab}$ is connected (see Remark \ref{rem:degree}), it must be the star with center $b$. We conclude that $G = G^{(ab)(ab)}$ is also the star with center $b$. \hfill \Box
12. Open problems

The foregoing was a snapshot of our knowledge of the interlace polynomial at one time. As pointed out in the introduction, the polynomial’s understanding can be furthered by its interpretation as a special case of the Martin polynomial of an isotropic system \[ \text{Bon90, AvdH10}. \] Also, a two-variable generalization of the polynomial and a clarified view of the one-variable polynomial, without reference to isotropic systems, are given in \[ \text{ABS01}. \]

It is clear that the interlace polynomial is not a special case of the Tutte polynomial, for one thing because the interlace polynomial varies over trees of order \( n \), where the Tutte polynomial is always \( x^n - 1 \). In \[ \text{ABS01} \] we show that a 2-variable extension of the interlace polynomial has an expansion formula structurally identical to the Tutte polynomial’s, but with a sum taken over vertex subsets rather than edge subsets, and with ranks computed over \( \mathbb{F}_2 \) rather than \( \mathbb{R} \). That is, the two polynomials have a close structural relationship but are completely different objects.

There remain many unanswered questions about the interlace polynomial. First, we mention a pair of related conjectures.

For the first conjecture, we have numerical evidence compiled from thousands of random graphs of orders up to 13, as well as all graphs of orders up to 8. To interpret its second assertion, recall from Theorem \[ \text{24} \] that when \( G \) is an interlace graph with an associated digraph \( D \), \( xq(G; 1 + x) = r(D; x) \) is the circuit counting polynomial of \( D \).

**Conjecture 57.** For any graph \( G \), the sequence of coefficients of \( q_G(x) \) is unimodal: non-decreasing up to some point, then non-increasing. The associated polynomial \( xq_G(1 + x) \) likewise has unimodal coefficients.

This conjecture is reminiscent of an outstanding pair of conjectures about the chromatic polynomial, namely that its coefficient sequence is always unimodal \[ \text{Rea05 and in fact log-concave [Hog74] (see [Wel93]).} \] There are two reasons for caution. First, a similar conjecture for the Tutte polynomial was falsified by counterexample in 1993 \[ \text{Sch93} \].

Second, the interlace polynomial’s coefficients are certainly not log-concave; the smallest counterexample is \( q(K_{1,3}) = 2x + x^2 + x^3 \). (We know of no case where the coefficients of the associated polynomial \( r(x) \) fail to be log-concave.) This means that, unlike all our other statements about the interlace polynomial, it does not suffice to prove Conjecture \[ \text{57} \] for connected graphs: while a product of log-concave polynomials is log-concave, a product of unimodal polynomials is not necessarily unimodal. (Even squaring a unimodal polynomial can result in a polynomial with an arbitrarily large number of modes, as shown in \[ \text{Sat02}. \])

We have made little headway in proving the conjecture; for instance we have not even shown that coefficient sequences have no internal zeros, \( i.e. \), that the nonzero coefficients form a single block. And, let us note that the second part of the conjecture is not a trivial consequence of the first: it is easy to construct a unimodal polynomial \( p(x) \) for which \( p(1 + x) \) is not unimodal, for example \( p(x) = 1000x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12} \).

If \( G \) is not the interlace graph of any 2-in, 2-out digraph, then \( xq_G(1 + x) \) has no interpretation as a circuit partition polynomial. However, Conjecture \[ \text{57} \] would imply that for any 2-in, 2-out digraph \( D \), the number of partitions \( r_k \) into \( k \) directed circuits is unimodal in \( k \). This leads us to speculate the following.

**Conjecture 58.** For any weakly Eulerian graph or digraph, the number of partitions into \( k \) circuits is unimodal in \( k \).

Beyond Conjectures \[ \text{57 and 58} \], specifically, there are general questions about circuit counting. For Eulerian graphs and digraphs, to what degree are the interesting algebraic properties of the interlace graph mirrored in such properties of the circuit partition and Martin polynomials? (For example, Ellis-Monaghan \[ \text{Pan05} \] shows that the Martin polynomial is a translation of a universal skein-type graph polynomial which is a Hopf map.) For \( k > 1 \), is there a polynomial-time method for counting \( k \)-circuit partitions of graphs, directed graphs, or even 2-in, 2-out directed graphs?

There are also basic, unanswered questions about the interlace polynomial. Can \( q(G) \) be generalized from graphs to matroids? Can it be computed efficiently?

An exponential-time computation of \( q(G) \) is immediate from the definition, and we suspect that no polynomial-time computation is possible. We have explored the issue only very briefly, but we remark that it may not be immediate to derive the hardness of the interlace polynomial from its connection with the Martin polynomial (Theorem \[ \text{24} \]) or the further connection to the diagonal Tutte polynomial, because the
connections apply only for limited classes of graphs where the latter polynomial may not be hard. At least some evaluations are easy: For interlace class graphs $H$, $q_H(1)$ can be computed in polynomial time: find a circle arrangement $C$ for $H$ [Sp91] to obtain a 2-in, 2-out graph $D$, and count its Euler circuits with a combination of the directed matrix-tree theorem, [Kir47] [Lun48] (see also [Bol98] p. 58) and the so-called BEST theorem [St41, dBvAE51] (see also [Bol98] p. 18). By the same token, if the full circuit-counting polynomial $r(D)$ for a 2-in, 2-out digraph $D$ cannot be computed in polynomial time, then $q(G)$ cannot be computed in polynomial time.

How many different polynomials $q_G$ are there, say for graphs of order $n$? How many graphs may share a single polynomial? Is there a way to recognize a polynomial obtainable as $q(G)$? What can we say about $q(G)$ when $G$ is a random graph?

We think that the real question is what $q(G)$ computes about $G$ itself (say when $G$ is not an interlace graph). One referee suggests that “if $S = (L, V)$ is the isotropic system defined by a graphic presentation using $G$, say $(G, A, B)$, then $q(G; x)$ is equal to the restricted Tutte-Martin polynomial $m(G, A + B; x)$” and

$$m(G, A + B, 1) \text{ counts the number } n \text{ of Eulerian vectors of the isotropic system } S \text{ that are supplementary of the vector } A + B.$$ 

While this is surely an answer to the question, we continue to hope for more elementary combinatorial interpretations. (For example in [ABS01] it is shown that a specialization of the two-variable interlace polynomial is the independence polynomial, but that specialization is not the $q(G)$ considered here.)

ACKNOWLEDGMENTS

We are grateful to Bill Jackson for bringing the work of André Bouchet to our attention, and to Bouchet and the anonymous referees for their elucidation of this polynomial’s debt to the Tutte-Martin polynomial, and many helpful detailed comments. Thanks also to Hein van der Holst for discussing [AvdH04] with us.

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