PARAMETRIZATIONS OF CANONICAL BASES AND IRREDUCIBLE COMPONENTS OF NILPOTENT VARIETIES

BASED ON THE TALK BY YONG JIANG

Throughout the talk $Q$ is a fixed quiver with the set of vertices $I$.
Let $\mathfrak{g}$ be the Kac–Moody Lie algebra associated with $Q$. By the $U_q^- (\mathfrak{g})$ we denote the subalgebra of the quantized enveloping algebra $U_q (\mathfrak{g})$ of $\mathfrak{g}$ generated by the elements $f_i, i \in I$. By the crystal basis of $U_q^- (\mathfrak{g})$ we mean a pair $(\mathcal{L}, \mathcal{B})$ such that $\mathcal{L}$ is an $\mathcal{A}_0$-lattice of $U_q^- (\mathfrak{g})$, i.e. $\mathcal{L} \otimes \mathbb{Q}(q) \simeq U_q^- (\mathfrak{g})$, where

$$\mathcal{A}_0 := \left\{ \frac{f}{g} \in \mathbb{Q}(q) : g(0) \neq 0 \right\},$$

and $\mathcal{B}$ is a $\mathbb{Q}$-basis of $\mathcal{L} / q \cdot \mathcal{L}$ (observe that $\mathcal{A}_0 / q \cdot \mathcal{A}_0 \simeq \mathbb{Q}$).

Now let $w$ be an element of the Weyl group $W$ associated with $Q$. We also fix a sequence $i = (i_1, \ldots, i_r)$ of vertices of $Q$ inducing a reduced expression of $w$. For each $k \in [1, r]$ we put

$$F_{i,k} := (T_{i_1} \circ \ldots \circ T_{i_{k-1}})(f_{i_k}),$$

where, for $i \in I$, $T_i$ is the Lusztig’s braid automorphism associated with $i$. Finally, if $a = (a_1, \ldots, a_r) \in \mathbb{N}^r$, then we put

$$F_i(a) := F_{i_1}^{(a_1)} \cdot \ldots \cdot F_{i_r}^{(a_r)},$$

where

$$x^{(a)} := \frac{x^a}{[a]!}$$

for an element $x$ of $U_q^- (\mathfrak{g})$ and $a \in \mathbb{N}$,

$$[a]! = [1] \cdot \ldots \cdot [a]$$

for $a \in \mathbb{N}$, and

$$[a] = \frac{q^a - q^{-a}}{q - q^{-1}}$$

for $a \in \mathbb{N}$. Lusztig has proved that for each $a \in \mathbb{N}^r$ there exists unique element $b$ of $\mathcal{B}$ such that the elements $F_i(a)$ and $b$ are congruent modulo $q \cdot \mathcal{L}$. We denote the map induced in this way by $\Phi_i$.

Let $\Lambda$ be the preprojective algebra associated with $Q$. For a dimension vector $d$ we denote by $\Lambda_d$ the variety of the nilpotent $\Lambda$-modules with dimension vector $d$. By $\text{Irr} \, \Lambda$ we denote the set of the irreducible components of the varieties $\Lambda_d$, $d \in \mathbb{N}^I$. Kashiwara and Saito have

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Yong Jiang proved that there is a natural bijection $\Psi$ between $B$ and $\text{Irr} \Lambda$. Consequently, we have the function $\Psi_1 := \Psi \circ \Phi_1 : \mathbb{N}^r \to \text{Irr} \Lambda$. We describe this map explicitly.

For $i \in I$ we denote by $I_i$ the injective envelope of the simple $\Lambda$-module at $i$. For $k \in [1, r]$ we put

$$V_{i,k} := S_{i_1, \ldots, i_k} I_k.$$ 

Here, for $i \in I$ and a $\Lambda$-module $V$, we denote by $S_i V$ the maximal submodule of $V$ whose composition factors are isomorphic to $S_i$. Moreover, if $j_1, \ldots, j_t \in I$ and $t > 1$, then $S_{j_1, \ldots, j_t} V := S_{j_1} (S_{j_2, \ldots, j_t} V)$. Next, for $k \in [1, r]$ we denote by $k^-$ the maximal $s \in [1, k - 1]$ such that $i_s = i_k$ (we put $k^- := 0$ if there is no such $s$). There is a natural embedding $V_{i,k^-} \hookrightarrow V_{i,k}$ and we put $M_{i,k} := V_{i,k}/V_{i,k^-}$. If $a \in \mathbb{N}^r$, then we denote by $Z_i(a)$ the closure of the set consisting of the $\Lambda$-modules $X$ such that there exists a filtration

$$0 = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_r = X$$

such that $X_k/X_{k-1} \simeq M_{i,k}^{a_k}$ for each $k \in [1, r]$. Then $Z_i(a) \in \text{Irr} \Lambda$ and $\Psi_1(a) = Z_i(a)$ for each $a \in \mathbb{N}^r$. 
