INTEGRABLE QUANTUM MAPPINGS AND
QUANTIZATION ASPECTS OF INTEGRABLE
DISCRETE-TIME SYSTEMS*

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Abstract

We study a quantum Yang-Baxter structure associated with non-ultralocal lattice models. We discuss the canonical structure of a class of integrable quantum mappings, i.e. canonical transformations preserving the basic commutation relations. As a particular class of solutions we present two examples of quantum mappings associated with the lattice analogues of the KdV and MKdV equations, together with their exact quantum invariants.

1 Introduction

Discrete integrable models, in which the spatial dimension is discretized, but the time is continuous, have traditionally played an important role in mathematics and physics, both in the classical as well as in the quantum regime. On the quantum level the algebraic structure of integrable systems is discussed in terms of quantum groups [1]-[3]. The discretized version of such models has played a particular role in this respect, e.g. in the quantum inverse scattering method [4]. The models, in which also the time-flow is discretized (i.e. integrable lattices or partial difference equations), have been considered on the classical level in a number of papers [5, 6]. Recently, they have become of interest in connection with the construction of integrable mappings, i.e. finite-dimensional reductions of these integrable lattice equations, [7, 8]. Their integrability is to be understood in the sense that the discrete time-flow is the iterate of a canonical transformation, preserving a suitable symplectic structure, leading to invariants which are in involution with respect to this symplectic form. A theorem à la Liouville then tells us in analogy with the continuous-time situation that one can linearize the discrete-time flow on a hypertorus which is the intersection of the level sets of the

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Integrable mappings have been considered from a slightly different perspective also in the recent literature, cf. e.g. [10]-[13].

Integrable two-dimensional lattices arise, both on the classical as well as on the quantum level, as the compatibility conditions of a discrete-time ZS (Zakharov-Shabat) system

\[ L_n(\lambda) \cdot M_n(\lambda) = M_{n+1}(\lambda) \cdot L_n(\lambda) \, , \tag{1.1} \]

in which \( \lambda \) is a spectral parameter, \( L_n \) is the lattice translation operator at site \( n \), and the prime denotes the discrete time-shift corresponding to a translation in the second lattice direction. As \( L \) and \( M \), in the quantum case, depend on operators, the question of operator ordering becomes important. Throughout this paper we impose in the quantum case as a normal order the order which is induced by the lattice enumeration, with \( n \) increasing from the left to the right. Finite-dimensional mappings are obtained from (1.1) imposing a periodicity condition

\[ L_n(\lambda) = L_{n+P}(\lambda) \, , \quad M_n(\lambda) = M_{n+P}(\lambda) \]  

for some \( P \in \mathbb{N} \).

In a recent paper [14], we introduced a novel quantum structure that is appropriate for obtaining an integrable quantization of mappings of the so-called KdV-type, i.e. mappings derived from a lattice version of the KdV equation, cf. [7]. In this paper we will review the construction of such an integrable quantization of mappings of the so-called KdV-type, i.e. mappings derived from a lattice version of the MKdV equation. As was indicated in [8], it turns out that these mappings and their underlying integrable lattices are –on the classical level– as in (1.3) denote the factors in a matricial tensor product, i.e. \( A_{i_1,i_2,\ldots,i_M} = A_{i_1,i_2,\ldots,i_M}(\lambda_1, \lambda_2, \ldots, \lambda_M) \) denotes a matrix acting nontrivially only on the factors labeled by \( i_1, i_2, \ldots, i_M \) of a tensor product \( \otimes_\alpha V_\alpha \), of vector spaces \( V_\alpha \) and trivially on the other factors, cf. also e.g. [14, 15]. For example, in eq. (1.3), the subscripts \( \alpha, \beta = 1, 2, \cdots \) for the operator matrices \( L_{n,\alpha} \) denote the corresponding factor on which this \( L_n \) acts (acting trivially on the other factors), i.e. \( L_{n,1} = L_n(\lambda_1) \otimes 1, L_{n,2} = 1 \otimes L_n(\lambda_2) \). We suppress the explicit dependence on the spectral parameter \( \lambda = \lambda_1 \) respectively \( \lambda = \lambda_2 \), assuming that each value accompanies its respective factor in the tensor product.

Eq. (1.3) defines a proper Poisson bracket provided that the following relations hold for \( s^\pm = s^\pm(\lambda_1, \lambda_2) \) and \( r^\pm = r^\pm(\lambda_1, \lambda_2) \):

\[ s_{12}^+(\lambda_1, \lambda_2) = s_{21}^+(\lambda_2, \lambda_1) \, , \quad r_{12}^+(\lambda_1, \lambda_2) = -r_{21}^+(\lambda_2, \lambda_1) \, , \tag{1.4} \]

to ensure the skew-symmetry, and

\[
\begin{bmatrix} r_{12}^\pm, r_{13}^\pm \end{bmatrix} + \begin{bmatrix} r_{12}^\pm, r_{23}^\pm \end{bmatrix} + \begin{bmatrix} r_{13}^\pm, r_{23}^\pm \end{bmatrix} = 0 \, , \tag{1.5a}
\]

\[
\begin{bmatrix} s_{12}^\pm, s_{13}^\pm \end{bmatrix} + \begin{bmatrix} s_{12}^\pm, s_{23}^\pm \end{bmatrix} + \begin{bmatrix} s_{13}^\pm, s_{23}^\pm \end{bmatrix} = 0 , \tag{1.5b}
\]
to ensure that the the Jacobi identities hold for the Poisson bracket (1.3). The relation (1.5a for $r^\pm$ is nothing but the usual classical Yang-Baxter equation (CYBE). As a consequence of the ZS system (1.1) we have on the classical level a complete family of invariants of the mapping, namely by introducing the associated monodromy matrix $T(\lambda)$, obtained by gluing the elementary translation matrices $L_j$ along a line connecting the sites 1 and $P + 1$ over one period $P$, namely

$$T(\lambda) \equiv \prod_{n=1}^{P} L_n(\lambda).$$

(1.6)

In order to be able to integrate (1.3) to obtain Poisson brackets for the monodromy matrix we need in addition to these relations the extra relation

$$r_{12}^+ - s_{12}^- = r_{12}^- - s_{12}^+.$$

(1.7)

In the classical case the traces of powers of the monodromy matrix are invariant under the mapping as a consequence of

$$T'(\lambda) = M_{P+1}(\lambda)T(\lambda)M_1^{-1}(\lambda)$$

(1.8)

and the periodicity condition $M_{P+1} = M_1$, thus leading to a sufficient number of invariants which are obtained by expanding the traces in powers of the spectral parameter $\lambda$. The involution property of the classical invariants follows from the Poisson bracket

$$\{\text{tr}T(\lambda), \text{tr}T(\lambda')\} = 0$$

(1.9)

which can be derived from (1.3).

For the quantum mappings we will use the structure of [14, 21] which is the quantum analogue of this non-ultralocal Poisson structure. In the continuous-time case such a novel quantization scheme was proposed in ref. [22], in connection with the quantum Toda theory. Similar structures with continuous time flow have been introduced also for the quantum Wess-Zumino-Novikov-Witten (WZNW) theory with discrete spatial variable, cf. [23, 24]. When considering discrete-time flows some interesting new features arise, as was indicated in [14, 21]. In fact, the conventional point of view, that the $M$-part of the Lax equations does not need to be considered explicitly in order to construct quantum invariants, is no longer true. Therefore, one needs to establish the complete quantum algebra, containing commutation relations between the $L$-operators as well as between the $L$- and the $M$-operators, and between the $M$-operators themselves. As a consequence we will find, in the quantum mappings under consideration, non-trivial quantum corrections in the quantum invariants of the mappings. From an algebraic point of view, the basic algebraic relations for the monodromy matrices, that are relevant in the context of non-ultralocal models, are the algebras of currents introduced in various papers, [25]-[27], in different contexts. Interestingly enough, the relations between the monodromy matrix and the time-part of the Lax representation are very similar to the relations associated with the description of the cotangent bundle of a quantum group $(T^*G)_q$ [28].

The outline of this paper is as follows. In section 2 we introduce the basic ingredients of the non-ultralocal quantum $R, S$-structure. In section 3 we investigate the canonical structure of quantum gauge- or similarity transformations, leading to (integrable) quantum mappings. This leads to a ‘full’ Yang-Baxter structure including the discrete-time part of the Lax representation. In section 4
we present two examples of this structure: quantum mappings associated with the lattice KdV and with the lattice MKdV equation. In order to establish the quantum integrability of these mappings, we then develop in section 5 the ‘full’ quantum structure for the monodromy matrix, and show how to construct commuting families of exact quantum invariants for these mappings.

2 Non-ultralocal Yang-Baxter structure

We now define a quantum Yang-Baxter structure that is adequate for the mappings in this paper, i.e. discrete-time systems arising (both on the classical as well as quantum level) from compatibility equations of the form of (1.1).

We introduce the quantum $L$-operator $L_n(\lambda)$ at each site $n$ of a one-dimensional lattice, which is a matrix whose entries are quantum operators (acting on some properly chosen Hilbert space). The operators $L_n(\lambda)$ are supposed to have only non-trivial commutation relations between themselves on the same and nearest-neighbour sites, namely as follows

\begin{align}
R_{12}^+ L_{n,1} \cdot L_{n,2} &= L_{n,2} \cdot L_{n,1} R_{12}^- , \\
L_{n+1,1} \cdot S_{12}^+ L_{n,2} &= L_{n,2} \cdot L_{n+1,1} , \\
L_{n,1} \cdot L_{m,2} &= L_{m,2} \cdot L_{n,1} , \quad |n-m| \geq 2 .
\end{align}

These relations are the quantum analogue of the non-ultralocal Poisson bracket (1.3). We will show in section 4 that the quantum mappings provide examples of such a non-ultralocal quantum $R$-matrix structure.

The compatibility relations of the equations (2.1a)-(2.1c) lead to the following consistency conditions on $R^\pm$ and $S$

\begin{align}
R_{12}^\pm R_{13}^\pm R_{23}^\pm &= R_{23}^\pm R_{13}^\pm R_{12}^\pm , \\
R_{23}^\pm S_{12}^\pm S_{13}^\pm &= S_{13}^\pm S_{12}^\pm R_{23}^\pm ,
\end{align}

where $S_{12}^\pm = S_{21}^\mp$. Eq. (2.2a) is the quantum Yang-Baxter equation (QYBE’s) for $R^\pm$ coupled with an additional equation (2.2b) for $S^\pm$. They were given for the first time explicitly in [14], but they are implicit in the previous literature [25]-[27], where the relations (2.8) to be given below were given in more special situations. For a derivation of eqs. (2.2) see appendix A.

In order to establish that the structure given by the commutation relations (2.1) allows for suitable commutation relations for the monodromy matrix, we need to impose in addition to (2.2) that

\begin{equation}
R_{12}^\pm S_{12}^\pm = S_{12}^\mp R_{12}^\pm .
\end{equation}

Using these relations it is easy to establish that each sign of eqs. (2.2a),(2.2b) can be combined into a single equation as follows

\begin{equation}
R_{12}^\pm \left( R_{13}^\pm S_{13}^\pm \right) S_{12}^\pm \left( R_{23}^\pm S_{23}^\pm \right) = \left( R_{23}^\pm S_{23}^\pm \right) S_{12}^\mp \left( R_{13}^\pm S_{13}^\pm \right) R_{12}^\mp .
\end{equation}

At this point it is useful to introduce the following decomposition of the monodromy matrix (1.6)

\begin{equation}
T = T_n^+ \cdot T_n^- ,
\end{equation}

4
in which

\[
T^+_n(\lambda) = \prod_{j=n+1}^P L_j(\lambda) , \quad T^-_n(\lambda) = \prod_{j=1}^n L_j(\lambda) ,
\]

(2.6)

First, one derives for the monodromy matrices \( T^+_n \) and \( T^-_n \) the following set of relations

\[
R_{12}^+ T^+_{n,1} \cdot T^+_{n,2} = T^+_{n,2} \cdot T^+_{n,1} R_{12}^- , \quad (2.7a)
\]

\[
T^+_{n,1} \cdot S^+_{12} T^-_{n,2} = T^-_{n,2} \cdot S^-_{12} T^+_{n,1} , \quad (2.7b)
\]

for \( 2 \leq n \leq P - 1 \).

Next, taking into account the periodic boundary conditions we obtain for the monodromy matrix the commutation relations

\[
R_{12}^+ T_1 \cdot S^+_{12} T_2 = T_2 \cdot S^-_{12} T_1 R_{12}^- . \quad (2.8)
\]

Some details of the derivation of eqs. (2.7), (2.8) are presented in appendix B.

Remarks:

a) From the theory of quantum groups, \cite{3}, we know that the relations (2.7) are the defining relations for a quasi-triangular Hopf algebra \( \mathcal{A}^\pm \), generated by the entries of \( T^\pm_n \) and the unit, with defining relations (2.8). The co-algebra structure on \( \mathcal{A} \) is defined with the following coproduct

\[
\Delta(T^+_n) = T^+_n \otimes T^+_n , \quad \Delta(T^-_n) = T^-_n \otimes T^-_n ,
\]

(2.9)

in which the dot denotes a matrix product. The algebra generated by the unit and \( T \) is no longer a Hopf algebra, but instead of that \( T \) and the unit 1 generate a Hopf-ideal \( \mathcal{A} \) \[27\]

\[
\Delta(\mathcal{A}) \subset \mathcal{A}^\pm \otimes \mathcal{A} . \quad (2.10)
\]

In fact, the coproduct on the generators \( T \) is defined as

\[
\Delta(T) = \left( T^+ \otimes 1 \right) \cdot \left( 1 \otimes T \right) \cdot \left( T^- \otimes 1 \right) ,
\]

(2.11)

as a consequence of (2.5) and \(2.3\).

b) The classical limit of the quantum structure (2.1)-(2.2) is easily obtained by considering the quasi-classical expansion

\[
S_{12}^\pm = 1 \otimes 1 - h s_{12}^\pm + \mathcal{O}(h^2) ,
\]

\[
R_{12}^\pm = 1 \otimes 1 + h r_{12}^\pm + \mathcal{O}(h^2) ,
\]

(2.12)

In this limit the quantum commutation relations (2.1) yield the non-ultralocal Poisson bracket structure given in eqs. (1.3)-(1.5).
3 Quantum Mappings

We are interested in the canonical structure of discrete-time integrable systems, i.e. systems for which the time evolution is given by an iteration of mappings. If the mapping contains quantum operators, the commutation relations with the monodromy or Lax matrices become nontrivial and it is not a priori clear in this case that the Yang-Baxter structure is preserved. Furthermore, the traces of powers of the monodromy matrix are no longer trivially invariant as the cyclic property of the traces is no longer true for operator-valued arguments.

Let us comment why in the discrete-time case the spatial part of the Yang-Baxter equations is not sufficient for quantum integrable systems. When dealing with Yang-Baxter structures for continuous-time systems, cf. e.g. [4, 15], it is not necessary to consider explicitly the commutation relations (Poisson brackets on the classical level) involving the $M$-operator, i.e. the time-dependent part of the Lax pair. For instance, knowing the commutation relations for the spatial part, i.e. the $L$-operator, allows one to construct all relevant objects, namely the (conserved) Hamiltonians of the system as well as the corresponding $M$-operators, cf. [15] and [29] for the quantum case. In doing so, one may use the fact that the time evolution is governed by a canonical transformation which leaves the commutation relations between operators invariant. In other words one can reconstruct the allowed integrable continuous-time flows. In the discrete-time case this is no longer true. Firstly, on the lattice there is not a preferred ‘spatial’ versus ‘time-like’ direction, as both components appear on the same footing, and hence, there is no natural impetus to introduce reconstruction formulae expressing one of the Lax operators in terms of the other one. Secondly, the process of integrating an integrable continuous-time flow to a finite-step discrete-time flow is a highly non-trivial procedure, in which – a priori – it is not at all clear that the integrated time flow will lead to nice ‘local’ partial difference equations described by explicit closed-form expressions. Furthermore, in the time-discrete case the symplectic property is not automatically guaranteed, it is not a priori clear that the commutation properties of operators are invariant under the mapping. On the other hand, the symplectic property is a necessary ingredient for mappings to be integrable [4].

However, the explicit investigations of mappings obtained by reduction from integrable partial difference equations reveal that this is actually the case. Probably there is a highly nonlinear self-consistent mechanism at work in these systems. In order to get a grasp on this mechanism we feel that it is necessary to take the $M$-part of the Lax or ZS system into account, and investigate the full quantum structure involved in these systems, consisting of commutation relations between the $L$-part as well as of the $M$-part of the Lax pair.

In [14] and [21] we have introduced such a full Yang-Baxter structure taking account of the spatial as well as the time part of the Lax pair. The structure is obtained by supplying in addition to eqs. (2.1) the following equations.

\[ M_{n+1,1} \cdot S_{12}^+ L_{n,2} = L_{n,2} \cdot M_{n+1,1} \quad (3.1a) \]
\[ L'_{n,2} \cdot S_{12}^- M_{n,1} = M_{n,1} \cdot L'_{n,2} \quad (3.1b) \]

and

\[ R_{12}^- M_{n,1} \cdot M_{n,2} = M_{n,2} \cdot M_{n,1} R_{12}^- \quad (3.2a) \]
\[ M'_{n,1} \cdot S_{12}^+ M_{n,2} = M_{n,2} \cdot M'_{n,1} \quad (3.2b) \]

The trivial commutation relations are the following

\[ M_{n,1} \cdot L_{m,2} = L_{m,2} \cdot M_{n,1} \quad |n - m| \geq 2 \quad (3.3a) \]
\[ M_{n+1,1} \cdot L'_{m,2} = L'_{m,2} \cdot M_{n+1,1}, \quad |n - m| \geq 2, \]  
(3.3b)
\[ M_{n,1} \cdot M_{m,2} = M_{m,2} \cdot M_{n,1}, \quad |n - m| \geq 2, \]  
(3.3c)

in combination with
\[ M''_{n,1} \cdot M_{n,2} = M_{n,2} \cdot M''_{n,1}, \]  
(3.4a)
\[ M'_{n+1,1} \cdot M_{n,2} = M_{n,2} \cdot M'_{n+1,1}, \]  
(3.4b)
\[ M''_{n+1,1} \cdot L_{n,2} = L_{n,2} \cdot M''_{n+1,1}, \]  
(3.4c)

for multiple applications of the mapping. We shall not specify other commutation relations, as they do not belong to the Yang-Baxter structure. More precisely, one may notice in the explicit examples of section 4 that the commutation relations
\[ [L_n \otimes M_n], \quad [L_{n+1} \otimes M_n], \quad [M_{n+1} \otimes L'_n], \quad [M_{n+1} \otimes L'_{n-1}], \quad [M_{n+1} \otimes M_n], \]
are nontrivial, and they depend on the details of the system satisfying the Yang-Baxter equations. However, in order for the Yang-Baxter structure to be preserved under the mapping, we do not need information on these latter commutation relations. Let us now make a more explicit statement on the invariance of the non-ultralocal Yang-Baxter system. The full Yang-Baxter structure consists of two sets of nontrivial commutation relations
i) Eqs. (2.1a), (2.1b) and (3.1a)
ii) Eqs. (3.1b), (3.2a) and (3.2b).

Imposing the first set of equations for all \( n \), and the second set of equations for a fixed value of \( n \) but for all iterates of the mapping and using also trivial commutation relations, it follows that the first set of equations, and in particular the commutation relations between the matrices \( L_n \), is invariant under the mapping
\[ L_n \rightarrow L'_n = M_{n+1} L_n M_n^{-1}, \]  
(3.5)
see appendix C for some details.

For the quantum mappings under consideration here the operator \( L_n \) has a composite structure, i.e.
\[ L_n = V_{2n} \cdot V_{2n-1} \]  
(3.6)
and the commutation relations of the Yang-Baxter structure involving the \( L_n \) can be inferred from the commutation relations among the \( V_n \) themselves, as well as the commutation relations between the \( V_n \) and \( M_m \). In fact, imposing the commutation relations
\[ V_{n+1,1} \cdot S_{12}^+(n) V_{n,2} = V_{n,2} \cdot V_{n+1,1}, \]  
(3.7a)
\[ R_{12}^+ V_{n,1} \cdot V_{m,2} = V_{m,2} \cdot V_{n,1} R_{12}^- , \]  
(3.7b)
\[ V_{n,1} \cdot V_{m,2} = V_{m,2} \cdot V_{n,1}, \quad |n - m| \geq 2, \]  
(3.7c)
we obtain the relations (2.1) as can be easily verified.

In eq. (3.7a) the \( S_{12}^+(2n) \) is independent of \( n \) and is equal to the \( S_{12}^+ \) occurring in eq. (2.1b). For odd values of \( n \), \( S_{12}^+(n) \) may be a different solution of eqs. (2.1)-(2.3). The proof of eq. (2.1a) from eq. (3.7b) is essentially the same as the proof in appendix B showing how eq. (2.1a) is obtained.
from eqs. (2.1).

Next we impose the commutation relations between the operators $V_n$ and $M_n$. The only non-vanishing commutation relations involving $M_n$ are taken to be the following ones

$$
\begin{align*}
V_{2n+1} & \leftrightarrow V'_{2n-1}, \\
V_{2n} & \leftrightarrow M_n \\
V_{2n-1} & \leftrightarrow V'_{2n-2}, \\
V_{2n-2} & \leftrightarrow V'_{2n-3}.
\end{align*}
$$

and in addition we impose simple commutation relations between $M_n$ and $V_{2n-2}$ and $V'_{2n-1}$, respectively

$$
\begin{align*}
M_{n+1,1} \cdot S_{12} V_{2n,2} & = V_{2n,2} \cdot M_{n+1,1}, \\
V'_{2n-1,2} S_{12} \cdot M_{n,1} & = M_{n,1} \cdot V'_{2n-1,2}.
\end{align*}
$$

With the use of eqs. (3.8), (3.9) and (3.6) it is straightforward to derive eqs. (3.1) and (3.3). Eq. (3.4c) can also be shown replacing $L_n$ by $M_{n+1}^{-1} L_n M_n$ and taking account of the invariance of commutation relations under the mapping. The relations (3.7)-(3.9) are satisfied by the quantum mappings which will be considered in section 4. In section 5 we construct commuting families of quantum invariants on the basis of the full Yang-Baxter structure given above.

4 The Quantum Lattice KdV and MKdV System

Here we consider two examples of integrable quantum mappings coming from the lattice analogues of the KdV and MKdV equations.

a) The first example of a concrete integrable family of quantum mappings that exhibit the structure outlined above, is the mapping of the KdV type (i.e. mappings arising from the periodic initial value problem of lattice versions of the KdV equation \[7\]). These are rational mappings

$$
R^{2P} \to R^{2P} : (\{ v_j \}) \mapsto (\{ v'_j \})
$$

of the form

$$
v'_{2j-1} = v_{2j}, \quad v'_{2j} = v_{2j+1} + \frac{\epsilon \delta}{v_{2j}} - \frac{\epsilon \delta}{v_{2j+2}} \quad (j = 1, \cdots, P),
$$

imposing the periodicity condition $v_{i+2P} = v_i$. The mapping (4.1) has the Casimirs

$$
\sum_{j=1}^{P} v_{2j} = \sum_{j=1}^{P} v_{2j-1} = c,
$$

where $c$ is chosen to be invariant under the mapping, in which case we obtain a $(2P-2)$-dimensional generalization of the McMillan mapping \[10\].

The mapping (4.1) is obtained by reduction from the partial difference equation

$$
u(n, m + 1) - u(n + 1, m) = q - p + \frac{p^2 - q^2}{p + q + u(n, m) - u(n + 1, m + 1)}
$$

\[4.3\]

for fields $u$ defined at the sites $(n, m)$ of a two-dimensional lattice, $n, m \in \mathbb{Z}$. Eq. (4.3) is completely integrable in the sense that solutions can be obtained via the direct linearization method, i.e. by solving a linear integral equation with arbitrary measure and contour [6].

To investigate eq. (4.3) one can choose initial data on a staircase consisting of alternating horizontal and vertical steps, i.e.

$$a_{2j} = u(j, j), \quad a_{2j+1} = u(j+1, j), \quad (4.4)$$

and the solution above and below the staircase can be calculated from (4.3) completing elementary squares. In the case of periodic initial data

$$a_j = a_{j+2P}, \quad P \in \mathbb{N}, \quad (4.5)$$

the complete solution of (4.3) satisfies the periodicity property

$$u(n, m) = u(n + P, m + P). \quad (4.6)$$

The periodic solutions of (4.3) can be obtained via a $2P$-dimensional mapping which is defined in terms of the vertical shift:

$$u(n, m) \rightarrow u'(n, m) = u(n, m + 1). \quad (4.7)$$

In terms of the reduced variables

$$v_j = p + q + a_j - a_{j+2} \quad (4.8)$$

the $2P$-dimensional mapping is given by eq. (4.1).

To obtain the Yang-Baxter structure it is worthwhile to note that eq. (4.1) arises as the compatibility condition of a ZS system (1.1) with

$$L_j = V_{2j} \cdot V_{2j-1}, \quad M_j = \begin{pmatrix} u_j & 1 \\ \lambda_{2j} & 0 \end{pmatrix}, \quad (4.9)$$

$$V_i = \begin{pmatrix} v_i & 1 \\ \lambda_i & 0 \end{pmatrix}$$

in which $\lambda_{2j} = k^2 - q^2$, $\lambda_{2j+1} = k^2 - p^2$ and $\epsilon \delta = p^2 - q^2$. In fact, from the ZS condition (1.1) one obtains

$$u_j = v_{2j-1} - \frac{\epsilon \delta}{v_{2j}} \quad (4.10)$$

as well as the mapping (4.1). The corresponding classical invariants, obtained by expanding the trace of the monodromy matrix (1.6) in powers of $k^2$, are in involution, cf. eq. (1.9), with respect to the Poisson structure [7]

$$\{v_j, v_{j'}\} = \delta_{j+1,j'} - \delta_{j,j'+1}, \quad (4.11)$$

which was obtained using a Legendre transformation on an appropriately chosen Lagrangian [7]. This ensures that the mapping (4.1) is symplectic, i.e. the same Poisson brackets hold also for the primes variables $v'_j$. This property can also be checked easily by direct computation. On the basis of this a canonical transformation to action-angle variables can be found [7], thereby showing complete integrability in the sense of Liouville [7, 8]. In the quantum case the variables $v_j$ become hermitean.
operators on which we impose the following Heisenberg type of commutation relations, as a natural quantization of the Poisson relations (4.11), cf. [14, 21],

\[ [v_j, v_{j'}] = \hbar (\delta_{j, j'+1} - \delta_{j+1, j'}) \quad (4.12) \]

(where \( \hbar = i\hbar \)). It is easy to check that the quantum mapping (4.1) is a canonical transformation with respect to these commutation relations.

The special solution of the quantum relations (2.2), (2.3), which constitutes the \( R, S \)-matrix structure for the quantum mapping (4.1), together with the commutation relation (4.12), is given by

\[
\begin{align*}
R_{12}^+ &= R_{12}^- - S_{12}^+ + S_{12}^- \\
R_{12}^- &= 1 \otimes 1 + \hbar \frac{P_{12}}{\mu_1 - \mu_2} \\
S_{12}^+ &= 1 \otimes 1 - \frac{h}{\mu_2} F \otimes E, \quad S_{12}^- = S_{21}^+ ,
\end{align*}
\]

in which \( \mu_\alpha = k_\alpha^2 - q^2, \alpha = 1, 2 \) and the permutation operator \( P_{12} \) and the matrices \( E \) and \( F \) are given by

\[
P_{12} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\[
E = \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}, \quad F = \begin{pmatrix}
0 & 0 \\
1 & 0
\end{pmatrix} .
\] (4.13)

We mention the useful identity

\[
R_{12}^+ = \Lambda_1 \Lambda_2 R_{12}^\lambda \Lambda_1^{-1} \Lambda_2^{-1} ,
\] (4.15)

where \( \Lambda_\alpha = \mu_\alpha F + E, (\alpha = 1, 2) \), from which it is evident that it is not strictly necessary to introduce two different \( R \)-matrices \( R^\pm \).

The complete Yang-Baxter structure can now be derived from the mapping (4.1), the relation (4.10) for \( u_j \) and the commutation relation (4.12). In fact, from (4.12) one immediately obtains eq. (3.7a) with

\[
S^+_{12}(n) = 1 \otimes 1 - \frac{h}{\lambda_{n,2}} F \otimes E
\] (4.16)

with \( \lambda_{2j,2} = k_2^2 - q^2, \lambda_{2j-1,2} = k_2^2 - p^2 \), and also eqs. (3.7b) and (3.7c). These relations are at the basis of the \( L \) part, i.e. eqs. (2.1), of the Yang-Baxter structure. To derive the commutation relations (3.8), (3.9) one first checks by explicit calculation that the only nonvanishing commutation relations between the matrix \( M_n \) and the matrices \( V_m, V'_m \) are indeed given by eq. (3.8). Furthermore one has the commutation relations

\[
[M_{n+1} - V_{2n+1} \otimes V_{2n}], \quad \left[ M_{n+1} - V_{2n}' \otimes V_{2n+1}' \right] = 0
\] (4.17)

which with eq. (3.7a) and its counterpart in terms of the primed operators immediately yield eqs. (3.9). Finally the nontrivial commutation relations (3.2) follow from

\[
[M_n \otimes M_n] = 0 , \quad \left[ M_n' - V_{2n-1}' \otimes M_n \right] = 0
\] (4.18)
together with (3.9). The trivial commutation relations can be checked in a similar way.

Thus, the mapping (1.1) and its ZS system (1.9) with the commutation relation (1.12) satisfy the complete Yang-Baxter structure treated in sections 2 and 3.

Remark: The KdV mappings considered here are the discrete-time analogue of the quantum Volterra system treated in ref. [30]. Such systems are of interest, in connection with discretizations of the Virasoro algebra [31]-[34].

b) We now consider the example of the MKdV mappings, which is associated with the following $R,S$-matrice. Introducing

$$R_{12}(x) = \begin{pmatrix} qx - 1 & 0 & 0 & 0 \\ 0 & x - 1 & q - 1 & 0 \\ 0 & x(q - 1) & q(x - 1) & 0 \\ 0 & 0 & 0 & qx - 1 \end{pmatrix}, \quad S_{12} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ q \end{pmatrix}, \quad (4.19)$$

it is straightforward to check that the matrices of (4.19) obey for spectral parameter $x = \lambda_1/\lambda_2$ the following relations

$$R_{12} \Lambda_1 S_{21} \Lambda_2 = \Lambda_2 S_{12} \Lambda_1 R_{12}, \quad R_{12} \Lambda_1^{-1} S_{12} \Lambda_2^{-1} = \Lambda_2^{-1} S_{21} \Lambda_1^{-1} R_{12}, \quad (4.20)$$

in which $\Lambda_1 = \Lambda(\lambda_1) \otimes 1$, $\Lambda_2 = 1 \otimes \Lambda(\lambda_2)$ and

$$\Lambda(\lambda) = \begin{pmatrix} a & b \\ \lambda & d \end{pmatrix}, \quad (4.21)$$

Eq. (4.20) then yields a solution of the Yang-Baxter relations (2.2), (2.3) with

$$R_{12} = R_{12}^{-1}(\lambda_1, \lambda_2) = R_{12}(\lambda_1/\lambda_2),$$

$$R_{12}^+ = \Lambda_1 \Lambda_2 R_{12}(\lambda_1/\lambda_2)(\Lambda_1 \Lambda_2)^{-1} = R_{12} - S_{12}^+ + S_{12}^-, \quad (p_{2n} + r) e^{\varphi_{2n}} + (p_{2n+1} + r) e^{\varphi_{2n+1}},$$

$$S_{12}^+ = \Lambda_2 S_{12} \Lambda_2^{-1},$$

$$S_{12}^- = S_{12}^+ - \Lambda_1 S_{12} \Lambda_1^{-1} \quad (4.22)$$

As an example of a quantum mapping associated with this solution of the Yang-Baxter equation we consider

$$\varphi_{2n-1} = \varphi_{2n},$$

$$e^{\varphi_{2n}} = \frac{(p_{2n} - r) + (p_{2n+1} + r) e^{\varphi_{2n+2}}}{(p_{2n+1} - r) + (p_{2n} + r) e^{\varphi_{2n+2}}} e^{\varphi_{2n+1}} \frac{(p_{2n} - r) + (p_{2n} + r) e^{\varphi_{2n}}}{(p_{2n} - r) + (p_{2n+1} + r) e^{\varphi_{2n}}} \quad (p_{2n} - r) + (p_{2n+1} + r) e^{\varphi_{2n}},$$

$$n = 1, \cdots 2P$$

(4.23)

On the classical level this mapping is obtained by reduction from the lattice version of the MKdV equation [4]:

$$(p - r) \frac{w(n, m + 1)}{w(n + 1, m + 1)} - (q - r) \frac{w(n + 1, m)}{w(n + 1, m + 1)} = (p + r) \frac{w(n + 1, m)}{w(n, m)} - (q + r) \frac{w(n, m + 1)}{w(n, m)} \quad (4.24)$$
Choosing periodic initial values on a staircase of alternating horizontal and vertical steps, i.e.
\[ b_{2j} = w(j, j), \ b_{2j+1} = w(j + 1, j), \ b_{j+2P} = b_j, \] (4.25)
and defining the mapping in terms of the vertical shift, cf. (4.7) with \( u \rightarrow w \), we obtain eq. (4.23) in terms of the reduced variables \( \varphi_j \) defined by
\[ e^{\varphi_j} = \frac{b_j}{b_{j-2}} \] (4.26)

The MKdV mapping (4.23) arises as the compatibility condition of a ZS system (1.1) with \( L_n = V_{2n} \cdot V_{2n-1} \), cf. (3.6), and
\[ V_n = \Lambda_n \cdot V_n, \quad V_n = \begin{pmatrix} 1 & 0 \\ 0 & e^{\varphi_n} \end{pmatrix}, \quad \Lambda_n = \begin{pmatrix} p_n - r & 1 \\ \lambda & p_n + r \end{pmatrix}. \] (4.27)
in which \( \lambda = k^2 - r^2, \ p_{2n-1} = p, \ p_{2n} = q \) and
\[ M_n = \Lambda_{2n} \cdot \begin{pmatrix} 1 & 0 \\ 0 & e^{\gamma_n} \end{pmatrix}, \] (4.28)

In fact, working out (1.1) with (3.6) and (4.27), (4.28) one finds the mapping (4.23) and
\[ e^{\gamma_n} = \frac{(p_{2n} - r) + (p_{2n-1} + r)e^{\varphi_{2n}}}{(p_{2n-1} - r) + (p_{2n} + r)e^{\varphi_{2n}}}(e^{\varphi_{2n-1}} \] (4.29)

At this stage it is worthwhile to note that the \( v_j \) in eq. (4.27) and in eq. (4.9) are related via a gauge transformation of the type
\[ \begin{pmatrix} p_j - r \\ k^2 - r^2 \end{pmatrix} \begin{pmatrix} e^{\varphi_j} \\ (p_j + r)e^{\varphi_j} \end{pmatrix} = \begin{pmatrix} (p_j - r)b_{j-1} & b_j \\ (k^2 - r^2)b_{j-1} & 0 \end{pmatrix} \begin{pmatrix} v_j \\ k^2 - p_j^2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} (p_j - r)b_{j-2} & b_{j-1} \\ (k^2 - r^2)b_{j-2} & 0 \end{pmatrix}^{-1} \] (4.30)

and the Miura transformation relating the KdV and MKdV mappings is given by
\[ v_j = (p_{j-1} - r)b_{j-2}b_{j-1} + (p_j + r)b_jb_{j-1} \] (4.31)

In the classical case the mapping is completely integrable with \( P \) integrals in involution with respect to the (invariant) Poisson bracket
\[ \{ \varphi_j, \varphi'_j \} = \delta_{j, j+1} - \delta_{j, j-1} \] (4.32)

cf. eq. (1.9) and the expansion of \( \text{tr}T(\lambda) \) in powers of \( k^2 \).

In the quantum case we have the commutation relation
\[ [\varphi_j, \varphi_{j'}] = \hbar (\delta_{j, j'+1} - \delta_{j, j'-1}) \] (4.33)
implying in particular that
\[ e^{\varphi_n}e^{\varphi_{n+1}} = q e^{\varphi_{n+1}} e^{\varphi_n} , \quad q = e^{-h} \]  
(4.34)

Starting from (4.33) we find the commutation relations (3.7), in which the \( R_{12}^\pm, S_{12}^\pm \) are given by (4.22) with
\[ q = e^{-h} , \quad x = \frac{k_2^2 - r^2}{k_2^2 - r^2} \]  
(4.35)

and \( \Lambda_n \) given by (4.27) with \( \lambda = k^2 - r^2 \).

From the commutation relation (4.33), together with the explicit expression (4.29) for \( e^m \) it is straightforward to derive the remaining relations of the Yang-Baxter structure, i.e. eqs. (3.8), (3.9) and (3.2), completely analogously to the case of the KdV-type of mappings. The trivial commutation relations can also be checked directly. Thus with the MKdV-type of mappings we have another example of the complete Yang-Baxter structure presented in sections 2 and 3, but here the \( R_{12}^\pm \) and \( S_{12}^\pm \) correspond to different (trigonometric) solutions of the Yang-Baxter equations.

5 Quantum Invariants

In the classical case the trace of the monodromy matrix yields a sufficient number of invariants which are in involution. In the quantum case the trace is no longer invariant and we have to consider more general families of commuting operators.

Following the treatment of ref. [25], a commuting parameter-family of operators is obtained by taking (for details, cf. appendix D)
\[ \tau(\lambda) = tr (T(\lambda)K(\lambda)) , \]  
(5.1)

for any family of numerical matrices \( K(\lambda) \) obeying the relations
\[ K_1^{t_1} ( (t_1 S_{12}^-)^{-1} ) K_2 R_{12}^+ = R_{12}^- K_2^{t_2} ( (t_2 S_{12}^+)^{-1} ) K_1 . \]  
(5.2)

(We assume throughout that \( S_{12}^\pm \) and \( R_{12}^\pm \) are invertible). The left superscripts \( t_1 \) and \( t_2 \) denote the matrix transpositions with respect to the corresponding factors 1 and 2 in the matricial tensor product. Expanding (5.1) in powers of the spectral parameter \( \lambda \), we obtain a set of commuting observables of the quantum system in terms of which we can find a common basis of eigenvectors in the associated Hilbert space. We note that a matrix \( K(\lambda) \) is commonly introduced in connection with quantum boundary conditions other than periodic ones [25], but in relation to the quantum mappings of the present paper it is essential in the periodic case as well.

Furthermore the Yang-Baxter equations of section 3 lead to the following commutation relations between \( M \equiv M_{n=1} \) and the monodromy matrix \( T \),
\[ T_1 \cdot M_1^{-1} \cdot S_{12}^+ M_2 = M_2 \cdot S_{12}^- T_1 \cdot M_1^{-1} . \]  
(5.3)

Here we use the notation \( M_1 = M \otimes 1, M_2 = 1 \otimes M \) as usual for the factors 1 and 2 in the matricial tensor product. The derivation of eq. (5.3) is based on the commutation relation
\[ M_{n,1} \cdot (L_{n+1} \cdot L_n \cdot M_n^{-1})_2 = (L_{n+1} \cdot L_n \cdot M_n^{-1})_2 \cdot S_{12}^- M_{n,1} \]  
(5.4)
which is easily checked noting that
\[ L_{n+1} \cdot L_n \cdot M_n^{-1} = M_{n+2}^{-1} \cdot L'_{n+1} \cdot L'_n \] (5.5)
and using the commutation relation (3.1b) and the trivial relations (3.3b), (3.3c). Then with the use of (3.1a) it is found that
\[ M_1 \cdot S_{12}^+ \left( L_P \cdot L_{P-1} \cdots L_1 \cdot M^{-1} \right)_2 = L_{P,2} \cdot M_1 \cdot \left( L_{P-1} \cdots L_1 \cdot M^{-1} \right)_2 \]
\[ = (L_P \cdots L_3)_2 \cdot \left( L_2 \cdot L_1 \cdot M^{-1} \right)_2 \cdot S_{12}^- \cdot M_1 \] (5.6)
which is just eq. (5.3).

The commutation relation (2.8) for the monodromy matrices is invariant. This can be shown noting that eqs. (2.1) are invariant under the mapping and by repeating the derivation of eq. (2.8), but now with the updated variables \( L'_j \). It follows also directly from eq. (5.7) and the commutation relation (5.3). In fact,
\[ R_{12}^+ T'_1 \cdot S_{12}^+ T_2 = R_{12}^+ M_1 \cdot T_1 \cdot M_1^{-1} \cdot S_{12}^+ M_2 \cdot T_2 \cdot M_2^{-1} \]
\[ = R_{12}^+ M_1 \cdot M_2 \cdot S_{12}^- T_1 \cdot M_1^{-1} \cdot T_2 \cdot M_2^{-1} \]
\[ = M_2 \cdot M_1 \cdot S_{12}^+ R_{12}^+ T_1 \cdot M_1^{-1} \cdot T_2 \cdot M_2^{-1} \]
\[ = M_2 \cdot M_1 \cdot S_{12}^+ R_{12}^+ T_1 \cdot S_{12}^+ T_2 \cdot M_2^{-1} \cdot M_1^{-1} S_{12}^- \]
\[ = M_2 \cdot M_1 \cdot S_{12}^- T_2 \cdot S_{12}^- T_1 \cdot R_{12}^- M_2^{-1} \cdot M_1^{-1} S_{12}^- \]
\[ = M_2 \cdot M_1 \cdot S_{21}^- T_2 \cdot S_{12}^- T_1 \cdot R_{12}^- M_2^{-1} \cdot M_1^{-1} S_{12}^- \]
\[ = T_2' \cdot S_{21}^+ M_1 \cdot M_2 \cdot S_{12}^- T_1 M_1^{-1} \cdot M_2^{-1} S_{12}^+ \cdot R_{12}^- \]
\[ = T_2' \cdot S_{12}^- T_1 R_{12}^- \] (5.7)

Our aim is now to describe the integrability of the quantum mappings of section 4 which obey the commutation relations (2.8) and (5.3). For this we need to show that one can find a sufficient family of commuting invariants of the mapping. Let us thus use eq. (5.3) to calculate commuting families of quantum invariants in the case of the KdV and MKdV mappings of section 4.

In fact, introducing a tensor
\[ K_{12} = P_{12} K_1 K_2 \] (5.8)
where \( P_{12} \) is the permutation operator satisfying e.g.
\[ P_{12} A_1 = A_2 P_{12}, \; P_{12} A_2 = A_1 P_{12}, \; P_{12} = P_{21}, \; tr_2 P_{12} = 1_1 \] (5.9)
for matrices \( A \) not depending on the spectral parameter, and choosing \( \lambda_1 = \lambda_2 \), we can take the trace over left- and right hand side of (5.3) contracting with \( K_{12} \). This leads to
\[ tr_{12} \left( K_{12} (TM^{-1})_1 \cdot S_{12}^+ M_2 \right) = tr_2 \left( K_{2} T_2 M_2^{-1} \cdot tr_1 \left( P_{12} K_2 S_{12}^+ \right) M_2 \right) = tr(KT) \] (5.10)
provided that
\[ tr_1 \left( P_{12} K_2 S_{12}^+ \right) = 1_2. \] (5.11)
In eq. (5.10) \( tr_{12} = tr_1 tr_2 \) denotes the trace over the factors 1 and 2 in the direct product space of matrices, whereas \( tr_1 \) and \( tr_2 \) are restricted to only one of these factors. Under the same condition (5.11) we have that
\[ tr_{12} \left( K_{12} M_2 \cdot S_{12}^+(TM^{-1})_1 \right) = tr_1 \left( K_1 M_1 tr_2 \left( P_{12} K_1 S_{12}^- \right) (TM^{-1})_1 \right) = tr(KT'). \] (5.12)
A common solution to eqs. (5.12) and (5.13) is easily found, namely by taking

\[ K_2 = tr_1 \left\{ P_{12} \, t_1 \left( \left( t_1 S_{12}^+ \right)^{-1} \right) \right\} \]  

(5.13)

It can be shown that (5.14) will solve eq. (5.2).

Applying (5.14) to the examples of the KdV and MKdV mappings, we find in the KdV case, using (4.13),

\[ K(\lambda) = 1 + \frac{h}{\lambda} S_- , \quad S_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \]  

(5.14)

and in the case of the MKdV mapping, with the use of the relation

\[ S_{12}^- = 1 \otimes 1 + (q - 1) \left( \Lambda_1 \cdot S_- \cdot \Lambda_1^{-1} \otimes S_- \right) \]  

(5.15)

cf. (4.19) and (4.20), we find the following solution from (4.22)

\[ K(\lambda) = 1 + (q - r - 1) S_- \cdot \Lambda \cdot S_- \cdot \Lambda^{-1} \]

\[ \Lambda = \begin{pmatrix} q - r & 1 \\ \lambda & q + r \end{pmatrix} , \quad \lambda = k^2 - r^2 \]  

(5.16)

and again the \( K(\lambda) \) in combination with the \( R_{12}^+ \) and \( S_{12}^+ \) of eqs. (4.19), (4.20) satisfies eq. (5.2).

Hence, in the case of the KdV and MKdV mappings we have obtained a commuting family of quantum invariants that can be evaluated expanding \( \tau(\lambda) = trK(\lambda)T(\lambda) \) in powers of \( k^2 \).

For instance, in the KdV mapping the explicit expression of the invariants can be inferred from

\[ \tau(\lambda) = : \left( \prod_{j=1}^{2P} v_j \right) \left[ 1 + \sum_{1 \leq J_1 < \ldots < J_N \leq 2P} ^{J_{N+1} \geq J_1} \prod_{\nu=1}^{N} \frac{\hat{\lambda}_{\nu}}{v_{J_{\nu}+1} v_{J_{\nu}}} \right] : \]  

(5.17)

leading to find a full family of commuting invariants. In (5.17) : : denotes the normal ordering of the operators \( v_j \) in accordance with their enumeration, and \( \hat{\lambda}_J = \lambda_J \) for \( J \neq 2P \), \( \hat{\lambda}_{2P} = \lambda_{2P} + h \). Thus the quantum effect is only visible in the boundary terms associated with the factor \( 1/(v_1 v_{2P}) \).

As a very simple example we give the quantum invariant of the original McMillan mapping \([10]\), i.e. (4.1) for \( P = 2 \), namely

\[ x' = y , \quad y' = -x - \frac{2\epsilon \delta y}{\epsilon^2 - y^2} , \]  

(5.18)

for \( x = v_1 - \epsilon, \ y = v_2 - \epsilon, \) (choosing \( c = 2\epsilon \)) and where \( [y, x] = h \), having the invariant

\[ \mathcal{I} = (\epsilon^2 - y^2)(\epsilon^2 - x^2) - (\epsilon \delta + h)(2yx - \epsilon \delta) \]  

(5.19)

The invariant \( \mathcal{I} \) can be viewed as a Hamiltonian generating a continuous-time interpolating flow by \( \dot{x} = \frac{1}{h}[\mathcal{I}, x], \ \dot{y} = \frac{1}{h}[\mathcal{I}, y] \), whose solutions can be considered to be parametrized in terms of what we could call a quantum version of the Jacobi elliptic functions. More general two-dimensional quantum mappings have been studied in ref. \([33]\).

**Remark:**
The construction of quantum mappings can be generalized to a larger class of models, namely those
associated with the lattice Gel’fand-Dikii hierarchy, \[36\], as was explicitly shown in \[21\]. In this case the \(R, S\)-matrix structure is provided by the following solutions of the equations (2.2a,2.2b) and (2.3)

\[
R^{-}_{12} = R_{12} - S^{-}_{12} = \Lambda_1 \Lambda_2 R_{12} (\Lambda_1 \Lambda_2)^{-1},
\]

\[
R^{-}_{12} = 1 + \hbar \frac{P_{12}}{\lambda_1 - \lambda_2}, \quad S^{+}_{12} = S^{-}_{21} = 1 - \frac{\hbar}{\lambda_2} \sum_{i=1}^{N-1} E_{Ni} \otimes E_{iN}
\] (5.20)

\[
P_{12} = \sum_{i,j=1}^{N} E_{i,j} \otimes E_{j,i}, \quad \Lambda(\lambda) = \lambda E_{N,1} + \sum_{i=1}^{N-1} E_{i,i+1}
\] (5.21)

and \(E_{i,j}\) being the basic generators of \(\text{GL}_N\), i.e. \((E_{i,j})_{kl} = \delta_{ik}\delta_{jl}\). The relations of the full Yang-Baxter structure of sections 2 and 3 are satisfied for matrices \(L_n = V_{2n} \cdot V_{2n-1}\) and \(M_n\) of the form

\[
V_n = \Lambda_n \left(1 + \sum_{i>j=1}^{N} v_{i,j}(n)E_{i,j} \right), \quad M_n = \Lambda_{2n} \left(1 + \sum_{i>j=1}^{N} m_{i,j}(n)E_{i,j} \right)
\] (5.22)

in which \(\Lambda_n = \Lambda(\lambda_n), \quad \lambda_{2n} = \lambda_{2n+2} = \mu\) and the \(v_{i,j}(n)\) satisfy the following Heisenberg type of commutation relations, \((\hbar = i\hbar)\),

\[
[v_{i,j}(n), v_{k,l}(m)] = \hbar \left(\delta_{n,m+1}\delta_{k,j+1}\delta_{i,N}\delta_{l,1} - \delta_{m,n+1}\delta_{i,l+1}\delta_{k,N}\delta_{j,1}\right).
\] (5.23)

A commuting family of quantum invariants is obtained from

\[
\tau(\lambda) = \text{tr} K(\lambda) T(\lambda) , \quad K(\lambda) = 1 + (N - 1) \frac{\hbar}{\lambda} E_{N,N}
\] (5.24)

but for \(N \geq 3\) the expansion in powers of \(k^2\) does not yield enough invariants to establish complete integrability in the quantum case. For \(N \geq 3\) one needs additional invariants corresponding to higher order commuting families of operators. The construction of these higher order invariants needs the application of the fusion procedure \[29, 37, 38\] and is left to a future publication \[39\].


Appendix A

In order to derive the compatibility relations for the Yang-Baxter matrices $R$ and $S$, i.e. eqs. (2.2), from the commutation relations (2.1) for the Lax matrices $L$, we encounter four different types of combinations of matrices $L$. Embedding the $L$ matrices in a tensorial product of three copies of the matrix algebra, i.e. $L_n, j = 1, 2, 3$ acting on vector spaces $V \otimes V \otimes V$, and denoting

$$L_{n,1} = L_n \otimes 1 \otimes 1, \quad L_{n,2} = 1 \otimes L_n \otimes 1, \quad L_{n,3} = 1 \otimes 1 \otimes L_n,$$

we can distinguish the following types of combinations of matrices $L$ involving only coinciding and/or neighbouring sites:

i) $L_1 \equiv L_{n+2,1}, \quad L_2 \equiv L_{n+1,2}, \quad L_3 \equiv L_{n,3}$

In this case, no conditions on the $R$- or $S$ matrices will appear, because

$$L_1 \cdot S_{12}^+ L_2 \cdot S_{23}^+ L_3 = L_3 \cdot L_2 \cdot L_1,$$  \hspace{1cm} (A.1)

independently of the order in which the relation (2.1a) is applied.

ii) $L_1 \equiv L_{n+1,1}, \quad L_2 \equiv L_{n+1,2}, \quad L_3 \equiv L_{n,3}$

In this case, we have on the one hand

$$R_{12}^+ L_1 \cdot L_2 \cdot S_{13}^+ S_{23}^+ L_3 = L_2 \cdot L_1 \cdot R_{12}^+ S_{13}^+ S_{23}^+ L_3,$$  \hspace{1cm} (A.2)

whereas on the other hand we find

$$R_{12}^+ L_1 \cdot S_{13}^+ L_2 \cdot S_{23}^+ L_3 = R_{12}^+ L_3 \cdot L_1 \cdot L_2$$

$$= L_3 \cdot L_2 \cdot L_1 R_{12}^- = L_2 \cdot S_{23}^+ L_1 \cdot S_{13}^+ L_3 R_{12}^-.$$  \hspace{1cm} (A.3)

Comparing relations (A.2) and (A.3), we have

$$R_{12}^- S_{13}^+ S_{23}^- = S_{23}^+ S_{13}^+ R_{12}^-,$$  \hspace{1cm} (A.4)

which after relabelling of the vector spaces becomes eq. (2.2b).

iii) $L_1 \equiv L_{n+1,1}, \quad L_2 \equiv L_{n,2}, \quad L_3 \equiv L_{n,3}$

Take for this case the combination

$$R_{23}^+ L_1 \cdot S_{12}^+ L_2 \cdot S_{13}^+ L_3 = L_1 \cdot R_{23}^+ S_{12}^+ S_{13}^+ \cdot L_2 \cdot L_3,$$  \hspace{1cm} (A.5)

and compare this with

$$R_{23}^+ L_1 \cdot S_{12}^+ L_2 \cdot S_{13}^+ L_3 = R_{23}^+ L_2 \cdot L_3 \cdot L_1$$

$$= L_3 \cdot L_2 \cdot L_1 R_{23}^- = L_3 \cdot L_1 S_{13}^+ S_{12}^+ L_2 \cdot L_3 R_{23}^-,$$  \hspace{1cm} (A.6)

yielding eq. (2.2b) with the + sign.
iv) \[ L_1 \equiv L_{n,1}, \quad L_2 \equiv L_{n,2}, \quad L_3 \equiv L_{n,3} \]

In this case we have the standard braiding type of argument to find as a sufficient condition the quantum \( R \)-matrix relations (2.2a) for \( R^+ \) and \( R^- \).

**Appendix B**

In this appendix we establish the commutation relations between the monodromy matrices \( T \) and \( T^+_n, T^-_n \) of eq. (2.6), using the fundamental commutation relations of the matrices \( L_n \).

i) Using eqs. (2.1a), (2.1b) we can establish

\[ R^+_1 L_{n+1,1} \cdot L_{n,1} \cdot L_{n+1,2} \cdot L_{n,2} = R^+_1 L_{n+1,1} \cdot L_{n+1,2} \cdot S^+_2 L_{n,1} \cdot L_{n,2} \]

\[ = L_{n+1,2} \cdot L_{n+1,1} \cdot R^-_{12} S^+_2 L_{n,1} \cdot L_{n,2} , \quad (B.1) \]

which, by imposing the relation (2.3), reduces to

\[ = L_{n+1,2} \cdot L_{n+1,1} \cdot S^+_1 L_{n,2} \cdot L_{n,1} R^-_{12} , \]

\[ = L_{n+1,2} \cdot L_{n+1,1} \cdot L_{n,1} R^-_{12} . \quad (B.2) \]

By repeated application of eqs. (B.1) and (B.2) together with eq. (2.3) one shows that

\[ R^+_1 L_{P,1} \cdot \ldots \cdot L_{n+1,1} \cdot L_{P,2} \cdot \ldots \cdot L_{n+1,2} \]

\[ = R^+_1 L_{P,1} \cdot L_{P-1,1} \cdot L_{P,2} \cdot L_{P-1,2} \cdot \ldots \cdot L_{n+1,1} \cdot L_{n+2,2} \cdot L_{n+1,2} \]

\[ = R^+_1 L_{P,1} \cdot L_{P,2} \cdot S^+_2 L_{P-1,1} \cdot L_{P-1,2} \cdot S^+_2 L_{P-2,1} \cdot L_{P-2,2} \cdot \ldots \cdot L_{n+2,2} \cdot S^+_2 L_{n+1,1} \cdot L_{n+1,2} \]

\[ = L_{P,2} \cdot L_{P,1} S^+_2 L_{P-1,2} \cdot L_{P-1,1} S^+_2 L_{P-2,2} \cdot \ldots \cdot L_{n+1,2} S^+_2 L_{n+1,1} \cdot L_{n,1} R^-_{12} \]

\[ = L_{P,2} \cdot L_{P-1,2} \cdot \ldots \cdot L_{n+1,2} \cdot L_{P,1} \cdot L_{P-1,1} \cdot \ldots \cdot L_{n+1,1} R^-_{12} ; \quad (B.3) \]

leading to eq. (2.7a) for \( T^+_n \). A similar argument can be applied for \( T^-_n \). Furthermore, eq. (2.7b) is derived from eqs. (2.1b), (2.1c) by noting that

\[ L_{P,1} \cdot \ldots \cdot L_{n+1,1} \cdot S^+_2 L_{n,2} \cdot \ldots \cdot L_{1,2} \]

\[ = L_{P,1} \cdot \ldots \cdot L_{n+2,1} \cdot L_{n,2} \cdot L_{n+1,1} \cdot L_{n-1,2} \cdot \ldots \cdot L_{1,2} \]

\[ = L_{n,2} \cdot \ldots \cdot L_{2,2} \cdot L_{P,1} \cdot L_{1,2} \cdot L_{P-1,1} \cdot \ldots \cdot L_{n+1,1} \]

\[ = L_{n,2} \cdot \ldots \cdot L_{2,2} \cdot L_{1,2} L_{2,1} \cdot L_{P,1} \cdot \ldots \cdot L_{n+1,1} , \quad (B.4) \]

where in the last step we have used the commutation relation

\[ L_{P,1} \cdot L_{1,2} = L_{1,2} \cdot S^+_2 L_{P,1} \quad (B.5) \]

taking into account the periodic boundary conditions.

Finally, from (2.7) we immediately obtain

\[ R^+_1 T_1 \cdot S^+_2 T_2 = R^+_1 T^+_n \cdot T^+_n \cdot S^+_2 T^-_n \cdot T^-_n \]

\[ = T^+_n \cdot T^+_n \cdot R^-_{12} S^+_2 T^-_n \cdot T^-_n \]

\[ = T^+_n \cdot T^+_n \cdot S^+_2 T^-_n \cdot T^-_n R^-_{12} \]

\[ = T_2 \cdot S^+_2 T_1 R^-_{12} , \quad (B.6) \]

which is eq. (2.8).
Appendix C

i) We prove that eqs. (3.1a),(3.1b), together with (3.2b) are sufficient to ensure that the basic commutation relations between the matrices $L_n$, eqs. (2.1a), are preserved under the mapping

$$L_n \mapsto L'_n = M_{n+1} \cdot L_n \cdot M_n^{-1}.$$ 

In fact,

$$L'_{n+1,1} \cdot S_{12}^+ L'_{n,2} = L'_{n+1,1} \cdot S_{12}^+ M_{n+1,2} \cdot L_{n,2} \cdot M_n^{-1}$$

$$= M_{n+1,2} \cdot L'_{n+1,1} \cdot L_{n,2} \cdot M_n^{-1}$$

$$= M_{n+1,2} \cdot M_{n+2,1} \cdot L_{n+1,1} \cdot M_{n+1,1}^{-1} \cdot L_{n,2} \cdot M_n^{-1}$$

$$= M_{n+1,2} \cdot M_{n+2,1} \cdot L_{n+1,1} S_{12}^+ L_{n,2} \cdot M_{n+1,1}^{-1} \cdot M_n^{-1}$$

$$= L'_{n,2} \cdot M_{n,2} \cdot L'_{n+1,1} \cdot M_n^{-1}$$

$$= L'_{n,2} \cdot L'_{n+1,1}, \quad (C.1)$$

and similarly

$$R_{12}^+ L'_{n,1} \cdot L'_{n,2} = R_{12}^+ M_{n+1,1} \cdot L_{n,1} \cdot M_n^{-1} \cdot L'_{n,2}$$

$$= R_{12}^+ M_{n+1,1} \cdot L_{n,1} \cdot L'_{n,2} \cdot M_{n+1}^{-1} (S_{12}^-)^{-1}$$

$$= R_{12}^+ M_{n+1,1} \cdot M_{n+2,1} \cdot S_{12}^- L_{n,1} \cdot L_{n,2} \cdot M_{n+1}^{-1} \cdot M_n^{-1} (S_{12}^-)^{-1}$$

$$= M_{n+1,2} \cdot M_{n+2,1} \cdot S_{12}^+ R_{12}^+ L_{n,1} \cdot L_{n,2} \cdot M_{n+1}^{-1} \cdot M_n^{-1} (S_{12}^-)^{-1}$$

$$= M_{n+1,2} \cdot M_{n+2,1} \cdot S_{12}^+ L_{n,2} \cdot L_{n,1} \cdot M_{n+1}^{-1} \cdot M_n^{-1} (S_{12}^-)^{-1} R_{12}$$

$$= M_{n+1,2} \cdot L_{n,2} \cdot M_{n+1,1} \cdot L_{n,1} \cdot M_{n+1}^{-1} \cdot M_n^{-1} (S_{12}^-)^{-1} R_{12}$$

$$= L'_{n,2} \cdot M_{n,2} \cdot L'_{n+1,1} \cdot M_n^{-1} (S_{12}^-)^{-1} R_{12}$$

$$= L'_{n,2} \cdot L'_{n+1,1} R_{12}. \quad (C.2)$$

Finally, we have that

$$M'_{n+1,1} \cdot S_{12}^+ L'_{n,2} = M'_{n+1,1} \cdot S_{12}^+ M_{n+1,2} \cdot L_{n,2} \cdot M_n^{-1}$$

$$= M_{n+1,2} \cdot M'_{n+1,1} \cdot L_{n,2} \cdot M_n^{-1}$$

$$= M_{n+1,2} \cdot M'_{n+1,1} \cdot M_n^{-1}$$

$$= L'_{n,2} \cdot M'_{n+1,1}. \quad (C.3)$$

It is straightforward to check along similar lines that all trivial commutation relations remain so after applying the mapping.

Appendix D

In order to show that eq. (5.2) provides a commuting family of operators, let us give an argument similar to the one given by Sklyanin in [24]. Denoting by $T_i$ and $K_i$, ($i = 1, 2$) the monodromy matrix resp. matrix $K$ for two different values of the spectral parameter $\lambda_1$ resp. $\lambda_2$ and acting
in two different factors of a matricial tensor product, and denoting by $\tau_1, \tau_2$ the invariants (5.1) evaluated at these respective values of the spectral parameter, we have on the one hand, assuming that $[K_i \otimes T_j] = 0$,

$$\tau_1 \tau_2 = tr_1 (T_1 K_1) \ tr_2 (T_2 K_2) = tr_{1,2} \left\{ T_1 K_1 \ t_2 T_2 \ t_2 K_2 \right\}$$

$$= tr_{1,2} \left\{ t_2 (T_1 S_{1,2}^+ T_2) \ t_1 (t_1 S_{1,2}^+)^{-1} t_2 K_2 \right\}$$

$$= tr_{1,2} \left\{ R_{1,2}^+ T_2 S_{1,2}^- T_1 R_{1,2}^- \ t_{12} t_1 K_1 \ t_1 (t_2 S_{1,2}^{-1}) t_2 K_2 \right\} \quad (D.1)$$

whereas on the other hand we have

$$\tau_2 \tau_1 = tr_2 (T_2 K_2) \ tr_1 (T_1 K_1)$$

$$= tr_{1,2} \left\{ t_1 (T_2 S_{1,2}^- T_1) \ t_2 \ t_2 K_2 \ t_2 \left( t_1 S_{1,2}^- \right)^{-1} t_1 K_1 \right\} \quad (D.2)$$

from which it is clear that (D.1) and (D.2) can be identified provided that we have the following condition on the matrices $K$

$$\left( t_{12} R_{12}^+ \right)^{-1} t_1 K_1 \ t_1 \left( t_2 S_{12}^+ \right)^{-1} t_2 K_2 = t_2 K_2 \ t_2 \left( t_1 S_{12}^- \right)^{-1} t_1 K_1 \ \left( t_{12} R_{12}^+ \right)^{-1} \quad (D.3)$$

Eq. (D.3) is a very general condition for operator valued matrices $K$ of which the entries commute with the entries of $T$, which is sufficient to ensure that the $T(\lambda)$ form a parameter family of commuting operators. For numerical matrices $K(\lambda)$ eq. (D.3) leads to the condition (5.2) given in the main text.

References

[1] V.G. Drinfel’d, Quantum Groups, Proc. ICM Berkeley 1986, ed. A.M. Gleason, (AMS, Providence, 1987), p. 798.

[2] M. Jimbo, Lett. Math. Phys. 10 (1985) 63, ibid. 11 (1986) 247, Commun. Math. Phys. 102 (1986) 537.

[3] L.D. Faddeev, N.Yu. Reshetikhin and L.A. Takhtadzhyan, Algebra i Analiz. 1 (1988) 178 [in Russian].

[4] L.D. Faddeev, in Développements Récents en Théorie des Champs et Mécanique Statistique, eds. J.-B. Zuber and R. Stora, (North-Holland Publ. Co., 1984), p.561.

[5] M.J. Ablowitz and F.J. Ladik, Stud. Appl. Math. 55 (1976) 213, 57 (1977) 1; R. Hirota, J. Phys. Soc. Japan 43 (1977) 1424, 2074, 2079; 50 (1981) 3785; E. Date, M. Jimbo and T. Miwa, J. Phys. Soc. Japan 51 (1982) 4116, 4125, 52 (1983) 388, 761, 766.

[6] F.W. Nijhoff, G.R.W. Quispel and H.W. Capel, Phys. lett. 97A (1983) 125; G.R.W. Quispel, F.W. Nijhoff, H.W. Capel and J. van der Linden, Physica 125A (1984) 344.
[7] V.G. Papageorgiou, F.W. Nijhoff and H.W. Capel, Phys. Lett. 147A (1990) 106; H.W. Capel, F.W. Nijhoff and V.G. Papageorgiou, Phys. Lett. 155A (1991) 377.

[8] F.W. Nijhoff, V.G. Papageorgiou and H.W. Capel, *Integrable Time-Discrete Systems: Lattices and Mappings*, in Proc. of the Intl. Workshop on Quantum Groups, The Euler Intl. Math. institute, Leningrad, ed. P.P. Kulish, Springer Lecture Notes Math. 1510 (1992) 312.

[9] M. Bruschi, O. Ragnisco, P.M. Santini and G.-Z. Tu, Physica 49D (1991) 273.

[10] E.M. McMillan, in *Topics in Physics*, eds. W.E. Brittin and H. Odabasi, (Colorado Associated Univ. Press, Boulder, 1971), p. 219.

[11] G.R.W Quispel, J.A.G. Roberts and C.J. Thompson, Phys. Lett. A126 (1988) 419, Physica D34 (1989) 183.

[12] A.P. Veselov, Funct. Anal. Appl. 22 (1988) 83; Theor. Math. Phys. 71 (1987) 446; P.A. Deift and L.C. Li, Commun. Pure Appl. Math. 42 (1989) 963. J. Moser and A.P. Veselov, Preprint ETH (Zürich), 1989.

[13] Yu.B. Suris, Phys. Lett. 145A (1990) 113; Algebra i Analiz 2 (1990) 141 [in Russian].

[14] F.W. Nijhoff, H.W. Capel and V.G. Papageorgiou, Phys. Rev. 46A (1992) 2155.

[15] L.D. Faddeev and L.A. Takhtadzhyan, *Hamiltonian Methods in the Theory of Solitons*, (Springer Verlag, Berlin, 1987).

[16] J.M. Maillet, Phys. Lett. 162B (1985) 137, Nucl. Phys. B269 (1986) 54.

[17] A.G. Reyman and M.A. Semenov-Tian-Shanskii, Phys. Lett. 130A (1988) 456.

[18] O. Babelon and C. Viallet, Phys. Lett. 237B (1990) 411.

[19] J. Avan and M. Talon, Nucl. Phys. B352 (1991) 215.

[20] L.-C. Li and S. Parmentier, C.R. Acad. Sci. Paris, t. 307 Série I (1988) 279; Commun. Math. Phys. 125 (1989) 545.

[21] F.W. Nijhoff and H.W. Capel, Phys. Lett. 163A (1992) 49.

[22] O. Babelon and L. Bonora, Phys. Lett. 253B (1991) 365; O. Babelon, Commun. Math. Phys. 139 (1991) 619.

[23] A. Alekseev, L.D. Faddeev and M.A. Semenov-Tian-Shanskii, Peprint CERN-TH-5981/91.

[24] A. Alekseev, L.D. Faddeev and M.A. Semenov-Tian-Shanskii, in Proc. of the Intl. Workshop on Quantum Groups, The Euler Intl. Math. institute, Leningrad, ed. P.P. Kulish, Springer Lecture Notes Math. 1510 (1992) 148.

[25] E.K. Sklyanin, J. Phys. A21 (1988) 2375.

[26] N. Yu. Reshetikhin and M.A. Semenov-Tian-Shanskii, Lett. Math. Phys. 19 (1990) 133.
[27] G.I. Olshanskii, *Twisted Yangians and infinite-dimensional classical Lie algebras*, Preprint CWI (Amsterdam), 1990.

[28] A. Alekseev and L.D. Faddeev, Commun. Math. Phys. 141 (1991) 413.

[29] A.G. Izergin and V.E. Korepin, Sov. Phys. Dokl. 26 (1981) 653.

[30] A.Yu. Volkov, Preprint HU-TFT-92-6.

[31] J.L. Gervais, Phys. Lett. 160B (1985) 277,279.

[32] L.D. Faddeev and L.A. Takhtadzhyan, Springer Lect. Notes Phys. 246 (1986) 166.

[33] A.Yu. Volkov, Theor. Math. Phys. 74 (1988) 96.

[34] O. Babelon, Phys. Lett. 215B (1988) 523, ibid. 238B (1990) 234.

[35] G.R.W. Quispel and F.W. Nijhoff, Phys. Lett. 161A (1991) 419.

[36] F.W. Nijhoff, H.W. Capel, G.R.W. Quispel and V.G. Papageorgiou, Inverse Probl. 8 (1992) 597.

[37] P.P. Kulish and E.K. Sklyanin, Springer Lect. Notes Phys. 151 (1982) 61.

[38] P.P. Kulish, N. Yu. Reshetikhin and E.K. Sklyanin, Lett. Math. Phys. 5 (1981) 393.

[39] F.W. Nijhoff and H.W. Capel, *Integrability and Fusion Algebra for Quantum Mappings*, in preparation.