EQUIVALENCE AFTER EXTENSION FOR COMPACT OPERATORS ON BANACH SPACES

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Abstract. In recent years the coincidence of the operator relations equivalence after extension and Schur coupling was settled for the Hilbert space case, by showing that equivalence after extension implies equivalence after one-sided extension. In the present paper we investigate consequences of equivalence after extension for compact Banach space operators. We show that generating the same operator ideal is necessary but not sufficient for two compact operators to be equivalent after extension. In analogy with the necessary and sufficient conditions for compact Hilbert space operators to be equivalent after extension, in terms of their singular values, we prove, under certain additional conditions, the necessity of a similar relationship between the $s$-numbers of two compact Banach space operators that are equivalent after extension, for arbitrary $s$-functions.

We investigate equivalence after extension for operators on $\ell^p$-spaces. We show that two operators that act on different $\ell^p$-spaces cannot be equivalent after one-sided extension. Such operators can still be equivalent after extension, for instance all invertible operators are equivalent after extension, however, if one of the two operators is compact, then they cannot be equivalent after extension. This contrasts the Hilbert space case where equivalence after one-sided extension and equivalence after extension are, in fact, identical relations.

Finally, for general Banach spaces $X$ and $Y$, we investigate consequences of an operator on $X$ being equivalent after extension to a compact operator on $Y$. We show that, in this case, a closed finite codimensional subspace of $Y$ must embed into $X$, and that certain general Banach space properties must transfer from $X$ to $Y$. We also show that no operator on $X$ can be equivalent after extension to an operator on $Y$, if $X$ and $Y$ are essentially incomparable Banach spaces.

1. Introduction

Equivalence after extension (EAE) is an equivalence relation on bounded Banach space operators that first appeared in the study of integral equations [2]; see Definition 1.1 below for its formal definition, as well as the definitions of the other operator relations discussed in this paragraph. Part of the advances made after the introduction of this notion came from the observation that it coincided with another equivalence relation referred to as matricial coupling (MC); in [2] only the implication (MC) $\Rightarrow$ (EAE) is used, and proved, while the reverse implication (EAE) $\Rightarrow$ (MC) was settled in [3]. A few years later the operator relations again appeared, when in [4] it was shown that a third operator relation, named Schur coupling (SC), implies equivalence after extension and matricial coupling, and the question was posed whether these three operator relations coincide, i.e., if (EAE) =

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(MC) ⇒ (SC). Apart from an affirmative answer in the case of Fredholm operators of index 0 (and without the index constraint in the case of Hilbert space operators), little progress was made until recently. In [16] the three operator relations were shown to coincide for the classes of Hilbert space operators with closed range and Banach space operators that can be approximated in norm by an invertible operator, leading to an affirmative answer in the case of Hilbert space operators on separable Hilbert spaces. The general Hilbert space case was settled by Timotin in [17] by showing that equivalence after extension implies another operator relation, namely equivalence after one-sided extension (EAOE), which was shown to imply Schur coupling in [3]. Specifically in the case of compact Hilbert space operators, a characterization for two compact operators to be equivalent after extension is presented by Timotin in [17] in terms of their singular values (cf. Theorem 2.8 below).

In the current paper we focus on the notions of equivalence after extension and equivalence after one-sided extension for compact Banach space operators.

In the sequel the term operator will be short for bounded linear operator and invertibility of an operator will imply that the inverse is a (bounded) operator as well. All Banach spaces are assumed to be over \( \mathbb{C} \) and for given Banach spaces \( X \) and \( Y \) we write \( B(X, Y) \) for the space of bounded linear operators from \( X \) to \( Y \), abbreviated to \( B(X) \) in case \( X = Y \). By \( X \oplus Y \) we denote the \( \ell^2 \)-direct sum of Banach spaces \( X \) and \( Y \). The identity operator on a Banach space \( X \) will be denoted by \( \text{id}_X \). With these definitions out of the way, we are ready to formulate the operator relations discussed in the first paragraph.

**Definition 1.1.** Let \( T \in B(X) \) and \( S \in B(Y) \) be Banach space operators.

1. We will say that \( T \) and \( S \) are **equivalent after extension** if there exist Banach spaces \( X' \) and \( Y' \) and invertible operators invertible operators \( E \in B(Y \oplus Y', X \oplus X') \) and \( F \in B(X \oplus X', Y \oplus Y') \) such that

\[
\begin{bmatrix}
T & 0 \\
0 & \text{id}_{X'}
\end{bmatrix} = E
\begin{bmatrix}
S & 0 \\
0 & \text{id}_{Y'}
\end{bmatrix}
F.
\]

N.B. if \( T \) and \( S \) are equivalent after extension then this can always be achieved with \( X' = Y \) and \( Y' = X \) (cf. [16, Lemma 4.1]). Throughout the rest of this paper this will be tacitly assumed.

2. We will say that \( T \) and \( S \) are **equivalent after one-sided extension** if \( T \) and \( S \) are equivalent after extension and one of the Banach spaces \( X' \) or \( Y' \) can be chosen as the trivial Banach space \( \{0\} \).

3. We will say that \( T \) and \( S \) are **matricially coupled** if there exists an invertible operator \( U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \in B(X \oplus Y, X \oplus Y) \) with inverse \( V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \in B(X \oplus Y, X \oplus Y) \), so that \( T = U_{11} \) and \( S = V_{22} \).

4. We will say that \( T \) and \( S \) are **Schur coupled** if there exists an operator \( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in B(X \oplus Y, X \oplus Y) \) such that \( A \in B(X) \) and \( D \in B(Y) \) are invertible and \( T = A - BD^{-1}C \) and \( S = D - CA^{-1}B \).

Timotin’s solution in [17], that equivalence after extension of Hilbert space operators implies their equivalence after one-sided extension, relies first of all on the spectral theorem (after reducing the general situation to that of positive operators without loss of generality). This technique is, of course, not available for Banach space operators without restricting to smaller classes of operators. Secondly, Timotin’s argument relies in an essential way on the identical geometry that all Hilbert
spaces share, in particular, that any Hilbert space can be embedded into any other of greater dimension. This is not possible for Banach spaces in general. As we shall see, the geometries of the underlying spaces play a crucial role in the possibility of two operators to be equivalent after (one-sided) extension. In fact, Corollary 4.3 will show that Timotin’s result, that equivalence after extension implies equivalence after one-sided extension, does not generalize to general Banach spaces.

Despite the lack of consistent geometrical structure across different Banach spaces, some positive results are achievable. Compact Banach space operators that are equivalent after extension are shown in Theorem 2.6 to necessarily generate the same ideal. This property is however not sufficient to imply equivalence after extension as illustrated by Example 2.9.

Using the general theory of \( s \)-numbers for Banach space operators as replacement for Hilbert space operators’ singular values, one is still able to regain the necessity of certain relationships between the \( s \)-numbers of compact Banach space operators that are equivalent after extension or generate the same ideal, much akin to similar results by Timotin and Schatten, cf. Theorem 2.8 and Proposition 3.5 and Theorem 2.7 and Proposition 3.2 below.

Banach space operators being equivalent after extension, with one of the operators compact, has far-reaching consequences for the geometry of the underlying Banach spaces. An elementary application of the Pitt-Rosenthal Theorem shows that the geometries of different \( \ell^p \)-spaces are such that no compact operator on an \( \ell^p \)-space can be equivalent after extension to any operator on a different \( \ell^p \)-space (cf. Proposition 4.5). We can go even further, by showing that no compact operator on a Banach space \( Y \) can be equivalent after extension to any operator on a Banach space \( X \), if \( X \) and \( Y \) are essentially incomparable Banach spaces (cf. Theorem 5.2).

If a compact operator on a Banach space \( Y \) is equivalent after extension to any operator on a Banach space \( X \), then a finite codimensional subspace of \( Y \) must embed into \( X \) (cf. Theorem 5.3). The salient point of this result is that, for a Banach space operator to be equivalent after extension to a compact Banach space operator, the underlying Banach spaces’ geometries must be “compatible enough” to allow for such an embedding. In fact, any Banach space property that \( X \) may have, that is also transferred to its closed subspaces, and preserved under taking direct sums with finite dimensional spaces, transfers from \( X \) to \( Y \) (cf. Proposition 5.6).

We briefly describe the structure of the paper.

In Section 2, we will prove one of our main results, Theorem 2.6. That compact Banach space operators that are equivalent after extension, necessarily generate the same (operator) ideal. The proof relies on Proposition 2.2 which establishes what may be termed a “finite rank perturbed conjugation relationship” that exists between all compact operators on Banach spaces that are equivalent after extension. In providing Example 2.9, we show that generating the same (operator) ideal is not sufficient for compact Banach space operators to be equivalent after extension.

In Section 3, we investigate \( s \)-number relationships for compact Banach space operators. In Proposition 3.2, we prove one direction of Schatten’s characterization for Hilbert space compact operators generating the same ideal in terms of their singular values. Assuming, for a given \( s \)-function \( s \), that the “finite rank perturbed conjugation relationship” from Proposition 2.2 behaves well with respect to \( s \), we can establish the necessity of an \( s \)-number relationship for compact Banach space
operators that are equivalent after extension which is directly analogous to Timotin’s for Hilbert space compact operators, Theorem 2.8.

We investigate equivalence after extension for operators on $\ell^p$-spaces in Section 4. The Pitt-Rosenthal Theorem (Theorem 4.2) plays a crucial role in our results. Employing this theorem, we show in Proposition 4.3 that no operators on different $\ell^p$-spaces can ever be equivalent after one-sided extension. This immediately establishes the existence of very simple operators on $\ell^p$ that are equivalent after extension, but are not equivalent after one-sided extension, cf. Corollary 4.4. This shows that Timotin’s result, Theorem 4.1, does not generalize to Banach spaces. We conclude the section by showing in Proposition 4.5 that no operator on an $\ell^p$-space can be equivalent after extension to a compact operator on a different $\ell^p$-space.

Finally, in Section 5, we investigate some of the consequences of a Banach space operator being equivalent after extension to a compact Banach space operator. For Banach spaces $X$ and $Y$, we prove in Theorem 5.2 that if $X$ and $Y$ are essentially incomparable (cf. Definition 5.1), then no operator on $X$ can be equivalent after extension to a compact operator on $Y$. On the other hand, if an operator on $X$ is equivalent after extension to a compact operator on $Y$, Theorem 5.3 shows that a closed finite codimensional subspace of $Y$ must embed into $X$. Also, Proposition 5.6 shows that any Banach space property that is transferred to closed subspaces and preserved under the taking of direct sums with finite dimensional spaces, transfers from $X$ to $Y$. Finally, Corollary 5.7 gives some specific examples of such properties.

2. Operator ideals generated by compact operators and equivalence after extension

This section will establish that two compact operators on Banach spaces that are equivalent after extension must necessarily generate the same (operator) ideal. The converse implication is not true in general, not even in the Hilbert space case, as illustrated in Example 2.9 below.

**Definition 2.1.** Let $T \in B(X, Y)$ be a Banach space operator. For any Banach spaces $Z_1$ and $Z_2$, we define

$$I_T(Z_1, Z_2) := \bigcup_{n \in \mathbb{N}} \left\{ \sum_{j=1}^{n} R_j T R_j' \mid R_j \in B(Y, Z_2), \ R_j' \in B(Z_1, X) \right\}.$$

By $I_T$ we will denote the (proper) class $\bigcup_{Z_1, Z_2} I_T(Z_1, Z_2)$, and refer to $I_T$ as the operator ideal generated by $T$.

It is easy to see that $I_T$ is, in fact, an operator ideal in the sense of Pietsch [12, Chapter I], provided $T \neq 0$. In this case we note that $I_T$ also contains all finite rank operators.

The following proposition will be a crucial ingredient in the current and following section. It establishes what may be termed “a finite rank perturbed conjugation relationship” that exists between compact operators on Banach spaces that are equivalent after extension.

We note that the symmetry of equivalence after extension allows us to exchange the roles of $T$ and $S$ in the following result without any loss of generality.
Proposition 2.2. Let $T \in B(X)$ and $S \in B(Y)$ be compact Banach space operators that are equivalent after extension. Then there exist operators $G \in B(Y, X)$, $H \in B(X, Y)$ and a finite rank operator $R \in B(X)$ such that $T = GSH + R$.

Proof. Since $T$ and $S$ are equivalent after extension, there exist invertible operators $E \in B(Y \oplus X, X \oplus Y)$ and $F \in B(X \oplus Y, Y \oplus X)$ satisfying

\[
\begin{bmatrix}
T & 0 \\
0 & \text{id}_Y
\end{bmatrix} = E \begin{bmatrix}
S & 0 \\
0 & \text{id}_X
\end{bmatrix} F.
\]

Furthermore, by \cite[Theorem 2.1]{16}, we may choose operators $G_{11}, G_{21}, G_{22}, H_{11}, H_{21}$ and $H_{22}$ in such a way that

\[
E = \begin{bmatrix}
G_{11} & T \\
G_{21} & G_{22}
\end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix}
H_{11} & \text{id}_Y \\
H_{21}T & H_{22}
\end{bmatrix}.
\]

From

\[
\begin{bmatrix}
T & 0 \\
0 & \text{id}_Y
\end{bmatrix} = \begin{bmatrix}
G_{11} & T \\
G_{21} & G_{22}
\end{bmatrix} \begin{bmatrix}
S & 0 \\
0 & \text{id}_X
\end{bmatrix} + \begin{bmatrix}
H_{11} & \text{id}_Y \\
H_{21}T & H_{22}
\end{bmatrix}
\]

we notice that $T = G_{11}SH_{11} + TH_{21}T$, which we may rearrange to

\[
T(\text{id}_X - H_{21}T) = G_{11}SH_{11}.
\]

The operator $\text{id}_X - H_{21}T$ is Fredholm, and of index zero, cf. \cite[Corollary XI.4.3]{7}. Hence there exists an invertible operator $L \in B(X)$ and finite rank operator $K \in B(X)$ satisfying $\text{id}_X - H_{21}T = L + K$, cf. \cite[Theorem XI.5.3]{7}. Define $G := G_{11}$, $H := H_{11}L^{-1}$ and $R := -TKL^{-1}$. Then

\[
T = G_{11}SH_{11}L^{-1} - TKL^{-1} = GSH + R
\]

and $R$ has finite rank. \hfill \Box

Before proving Theorem \cite[2.6]{2.4}, our main result in this section, we give a number of consequences of the previous result that we will need in later sections.

The following corollary is an immediate consequence of Proposition \cite[2.2]{2.4}.

Corollary 2.3. Let $T \in B(X)$ and $S \in B(Y)$ be compact Banach space operators that are equivalent after extension. The operator $T$ has finite rank if and only if $S$ has finite rank.

We remark that in the previous proposition and corollary compactness of both operators is required. Equivalence after extension of a finite rank operator with a second operator does not imply that the second operator has finite rank. In fact, in many of the original examples (cf. \cite{2}) it is shown that an integral operator is equivalent after extension (or rather, matricially coupled) to an operator on a finite dimensional space from which, amongst others, it can be concluded that the integral operator is Fredholm.

We briefly give an alternative proof of the previous corollary using Lemma \cite[2.3]{2.4} which may be of independent interest.

Lemma 2.4. Let $T \in B(X)$ and $S \in B(Y)$ be Banach space operators that are equivalent after extension. The operator $T$ has closed range if and only if $S$ has closed range.
Proof. Let $T$ and $S$ be equivalent after extension. Then there exist invertible operators $E : Y \oplus X \to X \oplus Y$ and $F : X \oplus Y \to Y \oplus X$, so that $F[T_0 0] = E[0 0 id_Y]F$. Elementary arguments will establish that $T$ has closed range if and only if $[T_0 0 id_Y]$ has closed range, and also that $S$ has closed range if and only if $[S_0 0 id_X]$ has closed range. Since $E$ and $F$ are invertible, $[T_0 0 id_Y]$ has closed range if and only if $[T_0 0 id_Y]$ has closed range. □

Alternative proof of Corollary 2.3. Let $T \in B(X)$ and $S \in B(Y)$ be compact Banach space operators that are equivalent after extension. Assume $T$ has finite rank. Then $T$ has closed range, and hence, by Lemma 2.4, $S$ also has closed range. Since $S$ is compact and has closed range it must have finite rank. The converse follows similarly. □

We also note that, if the Banach spaces $X$ and $Y$ are non-trivial, we may choose the operators $G$ and $H$ from Proposition 2.2 so as to be non-zero. This will be required later for establishing Lemma 3.3 and Proposition 3.5.

Corollary 2.5. Let $T \in B(X)$ and $S \in B(Y)$ be compact Banach space operators that are equivalent after extension. Let $G \in B(Y,X)$, $H \in B(X,Y)$ and the finite rank operator $R \in B(X)$ be such that $T = GSH + R$ (cf. Proposition 2.2). If $X \neq \{0\}$ and $Y \neq \{0\}$, then $G$ and $H$ can both be chosen to be non-zero.

Proof. If $S$ is of finite rank, then, by Corollary 2.3, so is $T$. We can hence choose $G$ and $H$ as any non-zero operators and set $R := T - GSH$.

If $S$ has infinite rank, then both $G$ and $H$ must have infinite rank, hence must be non-zero. □

Using Proposition 2.2, our main result in this section becomes a matter of routine:

Theorem 2.6. Let $T \in B(X)$ and $S \in B(Y)$ be non-zero compact Banach space operators. If $T$ and $S$ are equivalent after extension, then $IT = IS$.

Proof. By Proposition 2.2, there exist operators $G, H, G', H'$ and finite rank operators $R$ and $R'$ such that $T = GSH + R$ and $S = G'TH' + R'$. We note that $GSH \in IS$, and also, since $R$ is of finite rank, that $R \in IS$. We conclude that $T = GSH + R \in IS$, and hence $IT \subseteq IS$. Similarly, $IS \subseteq IT$, and hence, $IT = IS$. □

The converse of Theorem 2.6 is false. We will briefly elaborate on this claim.

In [10] Schatten characterized ideals of compact operators on Hilbert spaces in terms of the properties of their singular values. For ideals generated by single compact operators [15, Theorem 12] specializes to the following:

Theorem 2.7. Let $T$ and $S$ be compact operators on a Hilbert space $H$ and let $\{t_n\}$ and $\{s_n\}$ denote their respective sequences of singular values. The following are equivalent:

1. The operators $T$ and $S$ generate the same ideal in $B(H)$.
2. There exist constants $M > 0$ and $m \in \mathbb{N}$ such that both $t_{m(n-1)+j} \leq Mt_n$ and $s_{m(n-1)+j} \leq Ms_n$ hold for all $n \in \mathbb{N}$, and $j \in \{1, \ldots, m-1\}$.
With the previous result, one can easily find examples of compact operators on ℓ² that are not equivalent after extension by finding compact operators that do not generate the same ideal. E.g., the compact diagonal operators  
\[ a_n \]  
and  
\[ 2^{-n} \]  
on ℓ² are not equivalent after extension, where, for any bounded sequence  
\( \{a_n\} \subseteq \mathbb{C} \), by  
\( [a_n] \in B(\ell^2) \)  
we denote the diagonal operator  
\[ (x_1, x_2, \ldots) \mapsto (a_1 x_1, a_2 x_2, \ldots), \]  
with  
\( (x_1, x_2, \ldots) \in \ell^2 \).

In [17] Timotin established the following characterization connecting the equivalence after extension of compact operators on Hilbert spaces to their singular values satisfying a specific relationship [17, Theorem 6.3].

**Theorem 2.8.** Let  
\( T \)  
and  
\( S \)  
be compact operators on Hilbert spaces and let  
\( \{t_n\} \)  
and  
\( \{s_n\} \)  
denote their respective sequences of singular values. The following are equivalent:

1. The operators  
\( T \)  
and  
\( S \)  
are equivalent after extension.
2. There exist constants  
\( \delta \in (0, 1) \)  
and  
\( m \in \mathbb{N} \), such that either

\[ \delta \leq \frac{s_n}{t_{n+m}} \leq \delta^{-1} \]  
or

\[ \delta \leq \frac{t_n}{s_{n+m}} \leq \delta^{-1} \]

holds for all  
\( n \in \mathbb{N} \).

Using this result, we can now show that the converse of Theorem 2.6 is false:

**Example 2.9.** Consider the two compact diagonal operators  
\[ 2^{-n} \]  
and  
\[ 2^{-2n} \]  
on ℓ². We note that, for any  
\( m, n, j \in \mathbb{N} \), since  
\( 2^{-2mn} \leq 2^{-n} \)  
and  
\( 2^{-2j} \leq 1 \), that

\[ 2^{-2(m(n-1)+j)} = 2^{m2^{-2mn}2^{-2j}} \leq 2^{m2^{-n}}. \]

Similarly, if, in addition  
\( m \geq 2 \), then  
\( 2^{-mn} \leq 2^{-2n} \)  
and  
\( 2^m \leq 2^{2m} \), so that

\[ 2^{-(m(n-1)+j)} = 2^{m2^{-mn}2^{-j}} \leq 2^{m2^{-2n}} \leq 2^{2mn2^{-2n}}. \]

Therefore, by Theorem 2.7 (therein taking  
\( m := 2 \)  
and  
\( M := 2^4 = 16 \) ),  
\([2^{-n}] \)  
and  
\([2^{-2n}] \)  
genenerate the same ideal on ℓ². Hence we conclude that  
\( I_{[2^{-n}]} = I_{[2^{-2n}]} \).

On the other hand, for any fixed  
\( m \in \mathbb{N} \),

\[ \frac{2^{-n}}{2^{-2(n+m)}} = 2^{n+2m} \to \infty \]

and

\[ \frac{2^{-2n}}{2^{-(n+m)}} = \frac{1}{2^{n-m}} \to 0, \]

as  
\( n \to \infty \). Hence, by Theorem 2.8 the operators  
\([2^{-n}] \)  
and  
\([2^{-2n}] \)  
on ℓ² are not equivalent after extension.

### 3. General s-number relationships of compact operators that are equivalent after extension

In this section we investigate the possibilities of extending the Hilbert space case results of Theorems 2.7 and 2.8 to the Banach space setting. For both results we only prove the implication (1) ⇒ (2), where the role of the singular values are now played by s-numbers. In Proposition 3.5 we require an additional condition on the
s-numbers, relating to the “finite rank perturbed conjugation relationship” from Proposition 2.2 for our extension of the implication (1) ⇒ (2) from Theorem 2.8.

The proof of the reverse implication (2) ⇒ (1) in Theorem 2.8 given in [17] relies heavily on the fact that the operator relations equivalence after extension and equivalence after one-sided extension coincide, which is not the case in the general Banach space setting, as will be shown in the next section. Hence it is not clear whether it is even reasonable to think that the reverse implication may extend to the Banach space case.

We begin by defining s-functions and s-numbers which play the same role as singular values of operators on Hilbert spaces. For a more complete treatment of these objects we refer the reader to [12, 13].

**Definition 3.1.** By an s-function we will mean a rule for assigning to any operator $T \in B(X, Y)$ for any Banach spaces $X$ and $Y$, a sequence of numbers $\{s_n(T)\}$, the sequence of s-numbers of $T$, satisfying the following conditions:

1. For every $T \in B(X, Y)$, $\|T\| = s_1(T) \geq s_2(T) \geq \ldots \geq 0$.
2. For every $m, n \in \mathbb{N}$ and $S, T \in B(X, Y)$, $s_{n+m-1}(S+T) \leq s_n(S) + s_m(T)$.
3. For $T \in B(X, Y)$, $S \in B(Y, Z_2)$ and $R \in B(Z_1, X)$, with $Z_1$ and $Z_2$ arbitrary Banach spaces, and for every $n \in \mathbb{N}$, $s_n(STR) \leq \|S\| \|R\| s_n(T)$.
4. If $T \in B(X, Y)$ and $\text{rank} T < n$, then $s_n(T) = 0$.
5. For all $n \in \mathbb{N}$, $s_n(\text{id}_X) = 1$.

Since in the case of Hilbert space operators all s-functions coincide (with the singular values [12, Theorem 11.3.4]), the following result generalizes the necessity of (2) in Theorem 2.7.

**Proposition 3.2.** Let $T \in B(X)$ and $S \in B(Y)$ be Banach spaces operators. If $I_T = I_S$, then there exist constants $M > 0$ and $m \in \mathbb{N}$ such that, for any s-function $s$,

$$s_{m(n-1)+j}(T) \leq Ms_n(S) \quad \text{and} \quad s_{m(n-1)+j}(S) \leq Ms_n(T) \quad (n \in \mathbb{N})$$

for all $j \in \{0, \ldots, m-1\}$.

**Proof.** If $I_T = I_S$, then there exists a constant $m$ and operators $R_j, R_j'' \in B(X, Y)$ and $R_j', R_j'' \in B(Y, X)$ (by choosing some to be zero, if need be) for $j \in \{1, \ldots, m\}$, such that $S = \sum_{j=1}^{m} R_j T R_j'$ and $T = \sum_{j=1}^{m} R_j'' S R_j''$. For all $n \in \mathbb{N}$, by the properties of s-functions, we have

$$s_{m(n-1)}(S) = s_{mn-m}(S) = s_{mn-m} \left( \sum_{j=1}^{m} R_j T R_j' \right)$$

$$\leq s_n \left( R_1 T R_1' \right) + s_{(m-1)n-(m-1)} \left( \sum_{j=2}^{m} R_j T R_j' \right)$$

$$\leq \ldots$$

---

1Pietsch’s axioms for s-functions across [13, 12] are different. What we call an s-function, Pietsch calls an additive s-function in [12], and an s-scale in [13].
exchanging the roles of $T$ and $H$.

Proposition 3.5. The definition and subsequent proposition. from Theorem 2.8 to the Banach space setting. This is made precise in the following

Definition 3.4. Let $s$ be an s-function. Let $T \in B(X)$ and $F \in B(X)$ be Banach space operators with $F$ of finite rank. For every $n \in \mathbb{N}$, let $M_{n,F} > 1$ be such that

$$s_n(T + F) \leq M_{n,F} s_{n+\text{rank}_F}(T),$$

with $M_{n,F} = \infty$ in case $s_{n+\text{rank}_F}(T) = 0$. We will say that $T$ is $s$-shift-compatible with respect to $F$ if the sequence $\{M_{n,F}\}_{n \in \mathbb{N}}$ can be chosen so as to be bounded, i.e., that $\sup_{n \in \mathbb{N}} \{M_{n,F}\} < \infty$.

Proposition 3.5. Let $T \in B(X)$ and $S \in B(Y)$ be compact Banach space operators that are equivalent after extension. Let $G', H \in B(Y, X)$ and $G, H' \in B(X, Y)$ be non-zero operators and $R' \in B(X)$ and $R \in B(Y)$ of finite rank such that $S = GTH + R$ and $T = G'SH' + R'$ (cf. Proposition 2.2 and Corollary 2.5). By possibly exchanging the roles of $T$ and $S$, we may assume $\text{rank} R' = \max\{\text{rank} R', \text{rank} R\}$. We note that
If \( G'SH' \) is \( s \)-shift-compatible with respect to \( R' \), then there exists a constant \( \delta \in (0,1) \) such that
\[
\delta < \frac{s_n(T)}{s_{n+\text{rank } R'}(S)} < \delta^{-1} \quad (n \in \mathbb{N}).
\]

**Proof.** Since \( G'SH' \) is \( s \)-shift-compatible with respect to \( R' \), there exists a constant \( M > 1 \) such that, for every \( n \in \mathbb{N} \),
\[
s_n(T) \leq s_n(G'SH' + R') \\
\leq M s_{n+\text{rank } R'}(G'SH') \\
\leq M \|G'\| \|H'\| s_{n+\text{rank } R'}(S).
\]
On the other hand, since \( \text{rank } R' \geq \text{rank } R \), Lemma [3.3] and the properties of \( s \)-functions yield
\[
s_{n+\text{rank } R'}(S) \leq s_{n+\text{rank } R'}(GTH + R) \leq \|G\| \|H\| s_n(T).
\]
Choosing \( 0 < \delta < \max\{\|G\| \|H\|, M \|G'\| \|H'\|, 1\}^{-1} \) we conclude that
\[
\delta < \frac{1}{\|G\| \|H\|} \leq \frac{s_n(T)}{s_{n+\text{rank } R'}(S)} \leq M \|G'\| \|H'\| < \delta^{-1}.
\]
\( \square \)

### 4. Equivalence after extension for operators on \( \ell^p \)-spaces

In this section we consider operators on different \( \ell^p \)-spaces. Here, for \( 1 \leq p \leq \infty \), by \( \ell^p \) we will denote the sequence space \( \ell^p(\mathbb{N}) \). The results obtained here illustrate that an “incompatibility” in the geometry of the underlying Banach spaces on which operators act has consequences for whether certain classes of operators can be equivalent after (one-sided) extension.

The fact that all Hilbert spaces have “the same geometry” allows for the establishment of the following result, due to Timotin [17, Theorem 5.4]:

**Theorem 4.1.** Let \( T \in B(H_1) \) and \( S \in B(H_2) \) be Hilbert space operators. Then the following are equivalent:

1. The operators \( T \) and \( S \) are equivalent after extension.
2. The operators \( T \) and \( S \) are equivalent after one-sided extension.

This result does not carry over to the case of Banach space operators, as we will see from the results and examples presented below, cf. Corollary [3.4].

All results in this section hinge on The Pitt-Rosenthal Theorem [10, Theorem 5.14]:

**Theorem 4.2 (The Pitt-Rosenthal Theorem).** Any operator in \( B(\ell^p, \ell^q) \), where \( 1 \leq q < p < \infty \), is compact.

Obviously, every Hilbert space can be isometrically embedded into any other Hilbert space of higher dimension, i.e., all Hilbert spaces essentially have the same geometry. This is not true in the general Banach space case: The Pitt-Rosenthal Theorem even implies that no infinite dimensional subspace of \( \ell^p \) is topologically isomorphic to a subspace of \( \ell^q \) (and vice versa) when \( 1 \leq p \neq q < \infty \), [10, Corollary 5.10].

**Proposition 4.3.** No operators \( T \in B(\ell^p) \) and \( S \in B(\ell^q) \) are ever equivalent after one-sided extension whenever \( 1 \leq p \neq q < \infty \).
Proof. Suppose \( T \in B(\ell^p) \) and \( S \in B(\ell^q) \) are equivalent after one-sided extension, where \( 1 \leq p \neq q < \infty \). Then (by perhaps exchanging the roles of \( T \) and \( S \) if necessary) there exists a Banach space \( X \) and operators \( A \in B(\ell^q, \ell^p), B \in B(X, \ell^p), C \in B(\ell^q, \ell^q) \) and \( D \in B(\ell^q, X) \) such that
\[
\begin{bmatrix}
A & B
\end{bmatrix}: \ell^q \oplus X \to \ell^p \quad \text{and} \quad \begin{bmatrix}
C & D
\end{bmatrix}: \ell^q \to \ell^q \oplus X
\]
are invertible and
\[
T = \begin{bmatrix}
A & B
\end{bmatrix} \begin{bmatrix}
S & 0
0 & \text{id}_X
\end{bmatrix} \begin{bmatrix}
C
D
\end{bmatrix}.
\]

Let \( G \in B(\ell^p, \ell^q) \) and \( H \in B(\ell^q, X) \) be such that \( \begin{bmatrix}
g_{ij}
\end{bmatrix} \) is the inverse of \( \begin{bmatrix}
a_{ij}
\end{bmatrix} \), i.e.,
\[
\begin{bmatrix}
id_{\ell^q}
0
0
\end{bmatrix} = \begin{bmatrix}
g
H
\end{bmatrix} \begin{bmatrix}
a
b
\end{bmatrix} = \begin{bmatrix}
G & GA
H & HB
\end{bmatrix}.
\]

By the Pitt-Rosenthal Theorem, either \( A \in B(\ell^q, \ell^p) \) or \( G \in B(\ell^p, \ell^q) \) is compact, so that \( \text{id}_{\ell^q} = GA \) is also compact, which is absurd. We conclude that \( T \) and \( S \) cannot be equivalent after one-sided extension.

Corollary 4.4. Let \( T \in B(\ell^p) \) and \( S \in B(\ell^q) \) be invertible, where \( 1 \leq q \neq p < \infty \). Then \( T \) and \( S \) are equivalent after extension, but are not equivalent after one-sided extension.

Proof. All invertible operators are equivalent after extension. The previous result shows that \( T \) and \( S \) cannot be equivalent after one-sided extension.

Although operators \( T \in B(\ell^p) \) and \( S \in B(\ell^q) \) can still be equivalent after extension whenever \( 1 \leq q \neq p < \infty \), by another application of the Pitt-Rosenthal Theorem we will now show that this cannot occur if one of the operators is compact.

Proposition 4.5. Let \( T \in B(\ell^p) \) and \( S \in B(\ell^q) \) with \( 1 \leq q \neq p < \infty \). If either \( T \) or \( S \) is compact, then \( T \) and \( S \) cannot be equivalent after extension.

Proof. By perhaps exchanging the roles of \( S \) and \( T \), we may assume that \( S \) is compact. The equivalence after extension of \( T \) and \( S \) implies that there exist invertible operators \( F = \begin{bmatrix}
F_{11} & F_{12}
F_{21} & F_{22}
\end{bmatrix} \in B(\ell^q \oplus \ell^q, \ell^p \oplus \ell^p) \) and \( E = \begin{bmatrix}
E_{11} & E_{12}
E_{21} & E_{22}
\end{bmatrix} \in B(\ell^q \oplus \ell^p, \ell^p \oplus \ell^q) \) such that
\[
\begin{bmatrix}
T & 0
0 & \text{id}_{\ell^p}
\end{bmatrix} = E \begin{bmatrix}
S & 0
0 & \text{id}_{\ell^p}
\end{bmatrix} F = \begin{bmatrix}
\cdots
\cdots
\end{bmatrix} E_{21}SF_{12} + E_{22}F_{22}.
\]

By the Pitt-Rosenthal Theorem, either \( E_{21} \in B(\ell^p, \ell^q) \) or \( F_{22} \in B(\ell^q, \ell^p) \) is compact. Therefore, since \( S \) is compact, \( E_{21}SF_{12} + E_{22}F_{22} = \text{id}_{\ell^q} \) is compact, which is absurd. We conclude that \( T \) and \( S \) cannot be equivalent after extension whenever \( T \) or \( S \) is compact.

A curious consequence of the previous corollary is that compact operators with identical definitions, but acting on different \( \ell^p \)-spaces, cannot be equivalent after extension. This illustrates the importance that the geometry of the underlying spaces play in the possibility of certain classes of operators on them being equivalent after extension. As before, for any bounded sequence \( \{a_n\}_{n \in \mathbb{N}} \in \mathbb{C} \), by \( \{a_n\} \in B(\ell^p) \) we will denote the diagonal operator \( [a_n] : (x_1, x_2, \ldots) \mapsto (a_1x_1, a_2x_2, \ldots) \) with \( (x_1, x_2, \ldots) \in \ell^p \).
Example 4.6. Let $T$ and $S$ both denote the compact diagonal operator $[n^{-1}]$ acting on $\ell^p$ and $\ell^q$ respectively, with $1 \leq p \neq q < \infty$. Then $T$ and $S$ are not equivalent after extension.

5. Equivalence after extension for compact operators on general Banach spaces

In the previous section we have seen that no compact operator on an $\ell^p$-space can ever be equivalent after extension to an operator on a different $\ell^p$-space. In this section we will prove similar results for compact operators on more general Banach spaces.

Definition 5.1. Let $X$ and $Y$ be Banach spaces.

1. An operator $S \in B(X,Y)$ is called inessential if, for all operators $T \in B(Y,X)$, the operator $i_{1X} - TS$ is Fredholm. The set of inessential operators in $B(X,Y)$ is denoted by $\mathcal{J}(X,Y)$.
2. The Banach spaces $X$ and $Y$ are said to be essentially incomparable if $B(X,Y) = \mathcal{J}(X,Y)$.
3. The Banach spaces $X$ and $Y$ are said to be totally incomparable if no infinite dimensional subspace of $X$ is topologically isomorphic to a subspace of $Y$, and vice versa.

Total incomparability was introduced by Rosenthal in [14]. The notion of an inessential operator originated in [11] and essentially incomparability was introduced in [8] (see [8] Theorem 2) for a characterization of pairs of spaces that are essentially incomparable. We note that essential incomparability of Banach spaces is symmetric, i.e., $B(X,Y) = \mathcal{J}(X,Y)$ if and only if $B(Y,X) = \mathcal{J}(Y,X)$, [8] Proposition 1. Furthermore, total incomparability implies essential incomparability, but the converse is false (cf. [8]).

The Pitt-Rosenthal Theorem implies that different $\ell^p$-spaces are totally incomparable [10] Corollary 5.10, and hence essentially incomparable. The following result is therefore a generalization of Proposition 4.5 and, together with [8] Theorem 1, yields many more examples of pairs of Banach spaces on which no operators on the one space can be equivalent after extension to a compact operator on the other. A more exotic example of a pair of essentially incomparable spaces is any $C(K)$-space (which has the Dunford-Pettis property [11] Theorem 5.4.5) and the Tsielerson space (which is reflexive [11] Theorem 10.3.2), which are then essentially incomparable by [8] Theorem 1.

Theorem 5.2. Let $X$ and $Y$ be infinite dimensional Banach spaces that are essentially incomparable. Then no compact operator $S \in B(Y)$ is equivalent after extension to any operator $T \in B(X)$.

Proof. Suppose $S \in B(Y)$ is compact and equivalent after extension to $T \in B(X)$. Then there exist invertible operators $E = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} \in B(Y \oplus X, X \oplus Y)$ and $F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \in B(X \oplus Y), Y \oplus X)$, so that

$$\begin{bmatrix} T & 0 \\ 0 & \text{id}_Y \end{bmatrix} = E \begin{bmatrix} S & 0 \\ 0 & \text{id}_X \end{bmatrix} F = \begin{bmatrix} \cdots & \cdots \\ \cdots & \cdots \\ E_{21}SF_{12} + E_{22}F_{22} \end{bmatrix}. $$

Therefore $\text{id}_Y = E_{21}SF_{12} + E_{22}F_{22}$. Since $X$ and $Y$ are essentially incomparable, it follows that $\text{id}_Y - E_{22}F_{22}$ is Fredholm. On the other hand, $\text{id}_Y - E_{22}F_{22} = E_{21}SF_{12}$.
is compact. That \( \text{id}_Y - E_{22}F_{22} \) is both Fredholm and compact, implies that \( Y \) is finite dimensional, contrary to our assumption. We conclude that \( T \) and \( S \) cannot be equivalent after extension.

In our next result, we show that a consequence of an operator on some Banach space \( X \) being equivalent after extension to a compact operator on another Banach space \( Y \), is that a complemented subspace of finite codimension in \( Y \) must necessarily embed into \( X \). In other words, the geometry of \( X \) must be “compatible enough” with that of \( Y \) to allow for such an embedding.

**Theorem 5.3.** Let \( X \) and \( Y \) be Banach spaces and \( T \in B(X) \) and \( S \in B(Y) \) operators which are equivalent after extension. If \( S \) is compact, then there exists a closed subspace of \( Y \) of finite codimension that is topologically isomorphic to a closed subspace of \( X \).

**Proof.** Let \( T \in B(X) \) and \( S \in B(Y) \), with \( S \) compact, be equivalent after extension. Then there exist invertible operators \( E = [E_{11}, E_{12}] \in B(Y \oplus X, X \oplus Y) \) and \( F = [F_{11}, F_{12}] \in B(X \oplus Y, Y \oplus X) \) such that
\[
\begin{bmatrix}
T & 0 \\
0 & \text{id}_Y
\end{bmatrix} = E \begin{bmatrix}
S & 0 \\
0 & \text{id}_X
\end{bmatrix} F = \begin{bmatrix}
\cdots & \cdots \\
E_{11}SF_{12} + E_{22}F_{22}
\end{bmatrix}.
\]
Since \( E_{22}F_{22} = \text{id}_Y - E_{21}SF_{12} \) and \( S \) is compact, \( E_{22}F_{22} \) is Fredholm, and therefore has finite dimensional kernel. In particular, \( F_{22} \in B(Y, X) \) has finite dimensional kernel. Since all finite dimensional spaces are complemented, there exists a complement, denoted \( Y_1 \), of \( \ker F_{22} \) in \( Y \), i.e., \( Y \) is topologically isomorphic to \( \ker F_{22} \oplus Y_1 \).

We claim that \( \inf \{ \| F_{22}y \| \mid y \in Y_1, \| y \| = 1 \} > 0 \). Suppose to the contrary that there exists a sequence \( \{y_n\} \subseteq Y_1 \), with \( \| y_n \| = 1 \) for all \( n \in \mathbb{N} \), such that \( F_{22}y_n \rightarrow 0 \). Then \( y_n - E_{21}SF_{12}y_n = E_{22}F_{22}y_n \rightarrow 0 \) as \( n \rightarrow \infty \). Since \( S \) is compact and \( \| y_n \| = 1 \) for all \( n \in \mathbb{N} \), there exists a subsequence \( \{y_{n_k}\} \) of \( \{y_n\} \) such that \( E_{21}SF_{12}y_{n_k} \) converges, with limit denoted \( y \). But then \( y_{n_k} = E_{22}F_{22}y_{n_k} + E_{21}SF_{12}y_{n_k} \rightarrow 0 + y \) as \( k \rightarrow \infty \). Since \( \{y_{n_k}\} \subseteq Y_1 \), with \( \| y_{n_k} \| = 1 \) for all \( n \in \mathbb{N} \), we obtain \( y \in Y_1 \) and \( \| y \| = 1 \). Since \( y \in Y_1 \) and \( Y_1 \) is a complement of \( \ker F_{22} \), we obtain \( 0 \neq F_{22}y = \lim_{k \rightarrow \infty} F_{22}y_{n_k} = 0 \), which is absurd. We conclude that \( \inf \{ \| F_{22}y \| \mid y \in Y_1, \| y \| = 1 \} > 0 \).

Now defining \( X_1 := \overline{\text{ran}(F_{22}|_{Y_1})} \), the operator \( F_{22}|_{Y_1} : Y_1 \rightarrow X_1 \) is bijective with bounded inverse. Therefore \( X_1 \) is complete, and hence closed. The operator \( F_{22}|_{Y_1} \) is then the sought topological isomorphism.

We recall that a Banach space \( X \) is called prime [1] Definition 2.2.5], if every infinite dimensional complemented subspace of \( X \) is topologically isomorphic to \( X \). Standard examples of prime spaces are the space of convergent sequences \( c_0 \) (both endowed with the uniform norm), and \( \ell^p \) with \( 1 \leq p \leq \infty \) (cf. [I]). The following corollary follows immediately from the previous result:

**Corollary 5.4.** Let \( X \) and \( Y \) be Banach spaces with \( Y \) prime. If \( S \in B(Y) \) is compact and equivalent after extension to some \( T \in B(X) \), then \( X \) contains a copy of \( Y \).

Theorem 5.3 shows that any Banach space property that \( X \) may have, that is also transferred to its closed subspaces also and preserved under the taking of direct
sums with finite dimensional spaces, must transfer to \( Y \). We make this precise by stating the following definition and subsequent result.

**Definition 5.5.** Let \( X, Y \) and \( Z \) be a Banach spaces and let \( (P) \) be a Banach space property.

1. We will say \( X \) **transfers property** \( (P) \) **to closed subspaces**, if every closed subspace of \( X \) has property \( (P) \).
2. We will say \( (P) \) is **preserved under direct sums with finite dimensional spaces**, if \( Y \oplus Z \) has property \( (P) \), whenever \( Y \) has property \( (P) \) and \( Z \) is finite dimensional.

**Proposition 5.6.** Let \( (P) \) be a Banach space property, and let \( X \) and \( Y \) be Banach spaces with \( T \in B(X) \) and \( S \in B(Y) \) equivalent after extension. If \( S \) is compact, \( X \) transfers property \( (P) \) to closed subspaces, and \( (P) \) is preserved under direct sums with finite dimensional spaces, then \( Y \) has property \( (P) \).

**Proof.** By Theorem 5.3, there exists a complemented subspace \( Y_1 \) of \( Y \) with finite codimension that is topologically isomorphic to a closed subspace of \( X \). Since \( X \) transfers property \( (P) \) to closed subspaces, \( Y_1 \) has property \( (P) \). Also, \( (P) \) is preserved under direct sums with finite dimensional spaces, so \( Y \) has property \( (P) \), because \( Y \) is topologically isomorphic to a direct sum of \( Y_1 \) and a finite dimensional complement of \( Y_1 \). \[\Box\]

We briefly demonstrate some applications of the previous proposition in the next corollary.

We refer the reader to [6] for definitions of the Radon-Nikodym property and the Dunford-Pettis property. A Banach space is said to have the **hereditary Dunford-Pettis property** if each of its closed subspaces has the Dunford-Pettis property.

**Corollary 5.7.** Let \( X \) and \( Y \) be Banach spaces with \( T \in B(X) \) and \( S \in B(Y) \) equivalent after extension. If \( S \) is compact, then:

1. If \( X \) is isomorphic a Hilbert space, then so is \( Y \).
2. If \( X \) is separable, then so is \( Y \).
3. If \( X \) is reflexive, then so is \( Y \).
4. If \( X \) has the Radon-Nikodym property, then so does \( Y \).
5. If \( X \) has the hereditary Dunford-Pettis property, then so does \( Y \).

**Proof.** The results all follow from Proposition 5.6.

If \( X \) is respectively isomorphic to a Hilbert space, separable or has the hereditary Dunford-Pettis property, then straightforward arguments will show that each of these properties is respectively transferred to closed subspaces and preserved under taking direct sums with finite dimensional spaces. This establishes (1), (2) and (5).

Every closed subspace of a reflexive space is reflexive [9, Theorem 1.11.16], and an elementary argument will establish that direct sums of reflexive spaces with finite dimensional spaces are reflexive, establishing (3).

Every Banach space with the Radon-Nikodym property transfers the Radon-Nikodym property to its closed subspaces [6, Theorem III.3.2], and an elementary argument will establish that the direct sum of a Banach space with the Radon-Nikodym property and a finite dimensional space has the Radon-Nikodym property, establishing (4). \[\Box\]
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