Landaus singularity of the effective coupling constant in quantum electrodynamics has been known for more than half a century [1]. The original derivation of the singularity was based on the summation of one loop diagrams of vacuum polarization tensor for photons in the perturbation theory. In the early days the validity of such an expansion was looked upon sceptically by many people including the authors themselves. However, after the advent of $1/N_f$ expansion technique by t’ Hooft, it was soon realised that this singularity appears at the leading order in $1/N_f$ expansion where $N_f$ is the number of speciees of electrons, also called the number of flavours. This implies that in the infinite flavour limit this singularity is exact provided the perturbation series in $1/N_f$ expansion converges. This issue of convergence of the perturbation series will be one of the central themes latter in the paper. There have been attempts to interpret this singularity in many ways. Landau and Pomeranchuk tried to argue that this singularity reflects the fact that at short distances strong vacuum polarization effects screen the electric charge completely [1]. Others, including Shirkov, have called it Landau ghost reflecting the internal inconsistency of quantum electrodynamics [2].

It should be mentioned here that the stability of ground state and the possible existence of an ultra-violet fixed point has been studied extensively in the lattice formulation of QED in the wide range of value of the fermion flavour $N_f$ by Kogut et al [3,4]. Using lattice formulation, Kim et al has argued for the triviality of QED [5]. On the other hand in massless QED in the continuum, Miransky argues that there exists a chiral symmetry breaking phase [6]. However, in these studies Landau singularity plays no role. There has been some recent studies of Landau singularity using lattice formulation of QED by Gockeler et al ([7] and references there in). These studies seem to suggest that chiral symmetry breaking allows QED to escape Landau singularity. But then chiral symmetry breaking, as their study shows, seems to be intimately connected with trivility of QED. Landau singularity has also been considered from a different perspective by Gies and Jaeckel [8], and by Langfeld et al [9]. The details of these approaches can be found in the cited references and will not be discussed here. Our approach will be very different from the ones described in the publications above. We will rather be interested in finding the meaning of Landau singularity than finding a way to escape it.

We want to consider the issue of stability of "ground state/vacuum state" of quantum theory in the unified and broad perspective of quantum field theory and the many-body theory. Therefore, at first, we consider many-body ground state of two purely quantum mechanical systems: "Coulomb system with large number of flavours of fermions" and "System of weakly interacting electron gas in a condensed matter system". These are discussed in section-I and section-II. After this, we return to the main theme of the paper. In section-III, we reproduce Dyson’s argument for the divergence of perturbation series in QED. In section-IV, we show the connection between the Landau singularity and instability of vacuum state in QED. It is argued in the Dysonian framework, how the divergence of the series
removes both these problem. In section-V, we explain how a divergent asymptotic series can give rise to physically meaningful physical quantities. In the last and concluding section, we comment on the non-perturbative aspects of QED. Dyson’s original arguments, as well as, our studies of Landau singularity and vacuum state shows that the physical observables in non-perturbative QED are non-analytic in the coupling $\epsilon$, $e^2$, as well as the inverse flavour, $1/N_f$, and the perturbative power series in these parameters cannot capture this behaviour. In the absence of non-perturbative theory, we suggest that there should be attempts to experimentally search for non-analyticity in some physical observables in QED.

I. GROUND STATE OF COULOMB SYSTEM WITH LARGE NUMBER OF FLAVOURS OF FERMIONS

We will be investigating in this paper the question of stability of vacuum state in quantum electrodynamics. In this context, it is interesting to have some information regarding the stability of quantum mechanical ground state of a large system of charged particle. Many-body theory of Coulomb system of fermions have been studied extensively in several publications. In quantum theory a many-body system with ground state energy $E_0(N)$ is called thermodynamically stable or simply stable if $E_0(N)/N$ is bounded below when the number of particles, $N \to \infty$. In this context, the question of stability of matter consisting of negatively charged electrons and positively charged nuclei is very important. It was in 1967 when Dyson and Lenard proved that, in the framework of nonrelativistic quantum mechanics, matter consisting of $N$ electrons and $K$ static point nuclei of charge $Z$ $(=N/K)$ is stable [10]. Subsequently, Lieb and his collaborators have made a very detailed investigation of the stability of matter in nonrelativistic as well as relativistic case [11, 12] and references there in. These studies seem to suggest that at high enough energies quantum electrodynamics may not be a well behaved theory. Thermodynamic stability of a system of particles interacting via Coulomb interaction is associated with the control of the short distance behaviour of the interaction. In the nonrelativistic limit, zero point kinetic energy of the fermions controls this short distance behaviour. However, in the relativistic limit, there is a need for certain bounds on the value of the fine structure constant [11, 12, 13]. Landau singularity is also believed to be associated with the "high energy / short distance" behaviour of the electromagnetic interaction [1].

Let us consider the large flavour case in some detail. In 3-dimensional space, let us consider a volume of linear dimension $R$ where there are $N$ number of positively charged and $N$ number of negatively charged fermions. Fermions of both positive and negative charges come with $N_f$ flavours. Particles are assumed to be of the same mass. We assume that $N >> N_f$, however, both the number of particles and number of flavours are assumed to be large. Taking into account the the Pauli exclusion principle for the case involving multflavour fermions in 3-dimensional space, the kinetic energy $K$, apart from some numerical factors, can be written [14] as $K \approx \hbar^2 N_f^{5/3} / R$. The potential energy $U$ due to Coulomb interaction is given by [11],

$$E(R) = K + U = \frac{\hbar^2 N_f^{5/3}}{2m R^2 N_f^{3/3}} - a_0 e^2 N_f^{4/3} / R$$

The ground state is obtained by minimizing the total energy, $E(R)$, with respect to $R$.

$$E_0 \approx -\frac{ma_0^2}{\hbar^2} (e^2 N_f^{1/3})^2 N ; \quad \epsilon_0 = E_0/N \approx -\frac{ma_0^2}{\hbar^2} (e^2 N_f^{1/3})^2$$

$$R_0 \approx \frac{\hbar^2}{ma_0} (N_f / N_f)^{1/3} \frac{1}{e^2 N_f^{1/3}} ; \quad r_0 = \frac{R_0}{(N_f / N_f)^{1/3}} \approx \frac{\hbar^2}{ma_0} \frac{1}{e^2 N_f^{1/3}}$$

When the number of flavour $N_f$ increases, the size of the small box $r_0$ decreases and when it approaches Compton wave length $\hbar/mc$, we can no longer use the nonrelativistic quantum mechanics. Therefore, the above estimates are correct only when

$$N_f << \frac{1}{a_0^3} \frac{1}{(\hbar/c)^3}$$

In the relativistic quantum theory of many-body systems, it is almost customary now to take $c|p|$ [11], where $c$ is the velocity of light, as the kinetic energy of individual particles. Therefore, the kinetic energy $K$ for the system of
$N$ particles with $N_f$ flavour in a region with linear dimension $R$ is given by, $K = (\hbar c N^4/3)/(RN_f^{1/3})$. Thus, in the relativistic case,

$$E(R) = K + U \approx \frac{\hbar c N^4/3}{RN_f^{1/3}} - a_o\frac{e^2N^4/3}{R}$$

(4)

In this case stability of ground state requires $E(R) \geq 0$. This leads to the condition,

$$N_f \leq \frac{1}{a_o}\frac{1}{(\rho e^2/\hbar c)^3}$$

(5)

which in the unit $c = \hbar = 1$, becomes $N_f \leq \frac{1}{(\rho e^2/\hbar c)^3}$. At this point a few comments are necessary. We considered a theory with large $N$ and large $N_f$, and it is implicit in our discussion that we are considering, $N \to \infty$, but we have not said much about the flavour $N_f$. From the equations above, we find that it also possible to take the limit, $N_f \to \infty$ but slower than $N$. Assuming that in the limit $N_f \to \infty$ and $e \to 0$, $e^2N_f^{1/3}$ = constant, and that it satisfies equation Eq.(5), it is easy to conclude that, in this limit, matter is stable. This mathematical limit, has the physical implication that when the charge, $e$ is small and the flavour $N_f$ is large, the relevant parameter that sets the stability criteria is $e^2N_f^{1/3}$. It is the value of $e^2N_f^{1/3}$ that decides the stability of the system. When $e^2N_f^{1/3} > \frac{1}{a_o}$, in other words, when the number of flavours $N_f > \frac{1}{(\rho e^2/\hbar c)^3}$, the many-body system is thermodynamically unstable.

In subsequent section, we will find that in quantum electrodynamics the relevant parameter is not $e^2N_f^{1/3}$ but $e^2N_f$.

II. FELDMAN MODEL OF WEAKLY INTERACTING ELECTRON GAS

The main theme of this paper, as announced in the abstract, is to look for the meaning of Landau singularity of the effective coupling constant in QED. There are some other contexts in which the effective coupling constant develops singularity. We have in mind some condensed matter systems, where these singularities have well defined physical meaning. In this section, we refer to the renormalisation group analysis of weakly interacting Fermi system with short range potential at finite density and zero temperature by Feldman et al [15, 16, 17]. The iterative renormalisation group transformations show that if there is an attractive interaction among the electrons in any angular momentum channel, then there appears similar type of singularity in some suitably defined running coupling constant. We briefly describe here the Feldman model of weakly interacting electron gas. This section is essentially a short description of the results taken from the very detailed review paper by Froehlich, Chen and Seifert [16].

The model of weakly interacting electron gas studied by Feldman et al is a condensed matter Fermi system in thermal equilibrium at some temperature $T$ (for simplicity, assume $T = 0$) and chemical potential $\mu$. On microscopic scale($\approx 10^{-8}$ cm), it can be described approximately in terms of non-relativistic electrons with short range two body interactions. The thermodynamic quantities such as conductivity depend only on physical properties of the system at mesoscopic length scale ($\approx 10^{-4}$ cm), and therefore, are determined from processes involving momenta of the order of $k_F$ around the Fermi surface, where the parameter, $\lambda >> 1$, should be thought of as a ratio of meso-to-microscopic length scale. This is generally referred to as the scaling limit(large $\lambda$, low frequencies) of the system. The most important observation of Feldman et al. is that in the scaling limit, systems of non-relativistic (free) electrons in $d$ spatial dimensions behave like a system of multi-flavoured relativistic chiral Dirac fermions in $1 + 1$ dimensions. The number of flavours $N \approx const. \lambda^{d-1}$. It is possible then to set up a renormalization group improved perturbation theory in $\lambda^{d-1}$ around the non-interacting electron gas, where in the large number of flavours $N$, play an important role in actual calculations.

A. Free Electron gas and the Multiflavour Relativistic Fermions in $1 + 1$ Dimensions

Let us consider a system of non-relativistic free electrons in $d$ spatial dimensions with the Euclidean action,

$$S_0(\psi^*, \psi) = \sum_\sigma \int d^{d+1}x \psi^*_\sigma(x)(i\partial_0 - \frac{1}{2m}\Delta - \mu)\psi_\sigma(x)$$

(6)

The Euclidean free fermion Green’s function, $G_{\sigma\sigma'}^0(x-y)$, where $\sigma$ and $\sigma'$ are the spin indices, $x = (t, \vec{x})$ and $y = (s, \vec{y})$, $t$ and $s$ are imaginary times, $t > s$, is given by,
\[ G^0_{\sigma\sigma'}(x-y) = \langle \psi^*_{\sigma}(x)\psi_{\sigma}(y) \rangle_{\mu} = -\delta_{\sigma\sigma'} \int (dk) \frac{e^{-ik_0(t-s)+i\vec{k}(\vec{x}-\vec{y})}}{ik_0 - (\frac{k^2}{2m} - \mu)} \]  

(7)

Where we have used \((dk) = \frac{1}{(2\pi)^{d+1}}d^{d+1}k\). In the scaling limit, the leading contributions to \(G^0_{\sigma\sigma'}(x-y)\) come from modes whose momenta are contained in a shell \(S_F^{(\lambda)}\) of thickness \(\frac{d\lambda}{2}\) around the Fermi surface \(S_F\). In order to approximate the Green’s function, let us introduce the new variables \(\vec{\omega}\), \(p_\parallel\), \(p_0\) such that \(k_F\vec{\omega} \in S_F\), \(p_0 = k_0\) and \(\vec{k} = (k_F + p_\parallel)\vec{\omega}\). If \(\vec{k} \in S_F^{(\lambda)}\), then \(p_\parallel \ll k_F\), and we can approximate the integrand of Eq.(7), by dropping \(p_0^2\) term in the denominator. In other words,

\[ G^0_{\sigma\sigma'}(x-y) = \delta_{\sigma\sigma'} \int \frac{d\sigma(\vec{\omega})}{(2\pi)^d} k_F^{d-1} e^{ik_F\vec{\omega}(\vec{x}-\vec{y})} G_c(t-s, \vec{\omega}(\vec{x}-\vec{y})) \]

(8)

where \(d\sigma(\vec{\omega})\) is the uniform measure on unit sphere and

\[ G_c(t-s, \vec{\omega}(\vec{x}-\vec{y})) = -\int \frac{dp_0 \, dp_\parallel}{2\pi} \frac{e^{-ip_0(t-s)+i\vec{p}\vec{\omega}(\vec{x}-\vec{y})}}{ip_0 - v_Fp_\parallel} \]

(9)

is the Green’s function of chiral Dirac fermion in 1 + 1 dimension. \(v_F = k_F/m\) is the Fermi velocity. The \(\vec{\omega}\)-integration in Eq.(8) can be further approximated by replacing it with summation over discrete directions \(\vec{\omega}_j\) by dividing the shell \(S_F^{(\lambda)}\) into \(N\) small boxes \(B_{\vec{\omega}_j}, j = 1, ..., N\) of roughly cubical shape. The box, \(B_{\vec{\omega}_j}\), is centered at \(\vec{\omega}_j \in S_F\) and has an approximate side length \(\frac{d\lambda}{2}\). The number of boxes, \(N = \Omega_{\lambda-1}d\lambda^{-1}\), where \(\Omega_{\lambda-1}\) is the surface volume of unit sphere in \(d\) spatial dimensions. The Green’s function is, now, given by

\[ G^0_{\sigma\sigma'}(x-y) = -\delta_{\sigma\sigma'} \sum_{\vec{\omega}_j} \int \frac{dp_0 \, dp_\parallel}{2\pi} \frac{e^{-ip_0(t-s)+i\vec{p}\vec{\omega}(\vec{x}-\vec{y})}}{ip_0 - v_Fp_\parallel} \]

(10)

where \(\vec{p} = p_\parallel\vec{\omega} + \vec{p}_\perp\) is a vector in \(B_{\vec{\omega}_j} - k_F\vec{\omega}_j\) and \(p_0 \in \mathcal{R}\). Thus in the scaling limit, the behaviour of a \(d\)-dimensional non-relativistic free electron gas is described by \(N = \Omega_{\lambda-1}d\lambda^{-1}\) flavours of free chiral Dirac fermions in \(1 + 1\) dimensional space-time. The propagator \(G_c(t-s, \vec{\omega}(\vec{x}-\vec{y}))\) depends on the flavour index \(\vec{\omega}\). But the energy of an electron or hole with momenta \(\vec{k}\) depends only on \(p_\parallel\), where \(p_\parallel = \vec{k} \cdot \vec{\omega} - k_F\) and \(\vec{\omega} = \frac{\vec{k}}{|\vec{k}|}\). It is proportional to \(p_\parallel\), just as for relativistic fermions in \(1 + 1\) dimensions.

**B. Renormalization Group Flow and the BCS Instability**

Large scale behaviour of the weakly interacting system is described by an effective action. To see how the effective action is calculated, let us consider a system with Euclidean action of the form,

\[ S(\psi^*, \psi) = S_0(\psi^*, \psi) + S_I(\psi^*, \psi) \]

(11)

where \(S_0(\psi^*, \psi)\) is the quadratic part given in Eq.(3) and \(S_I(\psi^*, \psi)\) is the quartic interaction term. For a weakly interacting electron gas,

\[ S_I(\psi^*, \psi) = g_0 k_F^{1-d} \int \sum_{\sigma, \sigma'} \delta(x_0 - y_0) \psi^*_\sigma(x)\psi_{\sigma'}(y)\delta(x - y) \]

(12)

The factor \(k_F^{1-d}\) ensures that \(g_0\) is dimensionless. The two body potential \(v(x)\) is assumed to be smooth and short range, and therefore, its Fourier transform \(\hat{v}(\vec{k})\) is also smooth. In order to calculate the effective actions, we first split the field variable into slow and fast modes, \(\hat{\psi} = \hat{\psi}_< + \hat{\psi}_\rangle\), where \(\text{Supp } \hat{\psi}_\rangle \subset R \times (R^d \setminus S_F^{(\lambda)})\) (region \(\rangle\)) and \(\text{Supp } \hat{\psi}_< \subset R \times S_F^{(\lambda)}\) (region \(<\)). We then integrate out the fast modes using the functional integral for fermions,

\[ e^{-S_{eff}(\hat{\psi}_>, \hat{\psi}_<)} = \frac{1}{Z} \int D\hat{\psi}_\rangle D\hat{\psi}_< e^{-S(\hat{\psi}_>, \hat{\psi}_\rangle, \hat{\psi}_<, \hat{\psi}_<)} \]

(13)
Now using the linked cluster theorem, we obtain

\[ e^{-S_{eff}(\hat{\psi},\hat{\psi})} = \exp (-S_{0,\gamma} - S_{1} > G_{\alpha}^{0} + \frac{1}{2} < S_{1}; S_{1} > G_{\alpha}^{0} - \frac{1}{3} < S_{1}; S_{1} > G_{\alpha}^{0} - ... ) \]  

(14)

The abbreviations, \(< a; b > < a; b; c > \) etc., denote the connected correlators. The subscripts \( G_{\alpha}^{0} \) indicate that the expectations \(< (.) >_{G_{\alpha}^{0}} \) are calculated using infrared cut-off free propagators in accordance with the functional measure,

\[ dP(\hat{\psi},\hat{\psi}) = (1/\Xi)D\hat{\psi}D\hat{\psi}e^{-S_{0}(\hat{\psi},\hat{\psi})} \]

The connected correlators can be evaluated using the Feynman diagram technique. Therefore, the effective action can be calculated once we know the amplitudes of connected Feynman diagrams. It is clear that the \( S_{eff} \) contains far more interactions than the original quartic interaction \( S_{1} \). However, for weakly interacting systems the original coupling remains dominant. To carry out the iterative renormalization group scheme of Feldman et al, we choose some large scale \( \lambda_{0} < \frac{1}{\sqrt{\nu_{0}}} \) and calculate the effective action, \( S_{eff} \) perturbatively to leading order in \( \frac{1}{\lambda_{0}} \). The effective action depends on modes corresponding to wave vectors located in the shell, \( S_{F}^{(\lambda)} \), of width \( \frac{k_{F}}{\lambda_{0}} \) around the Fermi surface. Now, we divide the shell \( S_{F}^{(\lambda)} \) into \( N = const. \lambda^{d-1} \) cubical boxes, \( B_{\alpha} \), of approximate side length \( \frac{\lambda_{0}}{\lambda} \). Next, we rescale all the momenta so that, instead of belonging to the boxes \( B_{\alpha} \), they are contained in boxes \( B_{\alpha} \), of side length \( \approx k_{F} \). These two steps are generally known as decimation of degrees of freedom and rescaling. The renormalization group scheme consists of iteration of these two steps.

Assume that the degrees of freedom corresponding to momenta not lying in \( S_{F}^{(\lambda)} \) have been integrated out. Let \( \psi_{\sigma}(k), k \in R \times S_{F}^{(\lambda)} \), denote these modes. The sector fields are defined as

\[ \psi_{\lambda,\sigma}(x) = \int_{R \times B_{\lambda}} (dk)e^{i(k_{\lambda} - \vec{k} \cdot \vec{\omega})x}\hat{\psi}_{\sigma}(k) \]  

(15)

It is easy to see that \( \psi_{\sigma}(x) = \sum_{\lambda,\sigma} e^{i\vec{k} \cdot \vec{\omega}} \hat{\psi}_{\lambda,\sigma}(x) \). Inserting the Fourier transform of sector fields in Eq.(6) and Eq.(12), and carrying out some algebraic manipulations, we obtain

\[ S_{0} = -\sum_{\lambda,\sigma} \int_{R \times (B_{\lambda} - k_{F} \vec{\omega})} (dp)\hat{\psi}_{\lambda,\sigma}^{*}(p)(ip_{0} - v_{F} \vec{\omega} \cdot \vec{p}) + O(\frac{1}{\lambda^{2}}) \hat{\psi}_{\lambda,\sigma}(p) \]  

(16)

\[ S_{1} = \frac{g_{0}}{2} \frac{1}{k_{F}} \sum_{\lambda_{1},...,\lambda_{d},\sigma,\sigma'} \tilde{v}(k_{F}(\bar{\lambda}_{1} - \bar{\omega}_{1})) \int (dp_{1})...(dp_{d}) (2\pi)^{d+1}\delta(p_{1} + p_{2} - p_{3} - p_{4}) \hat{\psi}_{\lambda_{1},\sigma}^{*}(p_{1}) \hat{\psi}_{\lambda_{2},\sigma'}(p_{2}) \hat{\psi}_{\lambda_{3},\sigma'}(p_{3}) \hat{\psi}_{\lambda_{4},\sigma}(p_{4}) + \text{terms of higher order in } \frac{1}{\lambda} \]  

(17)

Using cluster expansions to integrate out the degrees of freedom corresponding to momenta outside the shell \( S_{F}^{(\lambda_{0})} \), one can show that, at scale \( \frac{\lambda}{\lambda_{0}} \), the effective action has the form given by Eq.(13) and Eq.(14) except that \( \tilde{v}(k_{F}(\bar{\lambda}_{1} - \bar{\omega}_{1})) \) is replaced by a coupling function \( g(\bar{\omega}_{1}, \bar{\omega}_{2}, \bar{\omega}_{3}, \bar{\omega}_{4}) \approx \tilde{v}(k_{F}(\bar{\lambda}_{1} - \bar{\omega}_{1})) \) with \( \bar{\omega}_{1} + \bar{\omega}_{2} = \bar{\omega}_{3} + \bar{\omega}_{4} \).

Next, one considers the rescaling of the fields and the action. The fields are rescaled in such a that the supports of the Fourier transformed "sector fields" are boxes, \( B_{\alpha} = \lambda(B_{\alpha} - k_{F} \vec{\omega}) \), of roughly cubical shape with sides of length \( k_{F} \), and such that the quadratic part of the action remains unchanged to leading order in \( \frac{1}{\lambda} \). The first condition implies that \( p \mapsto \bar{p} = p/\lambda \) and \( x \mapsto \xi = \frac{x}{\lambda} \). The rescaled sector fields and their Fourier transforms are given by,

\[ \tilde{\psi}_{\lambda,\sigma}(x) = \lambda^{\alpha} \tilde{\psi}_{\lambda,\sigma}(\lambda x) ; \quad \hat{\psi}_{\lambda,\sigma}(\bar{p}) = \lambda^{\alpha-d-1} \hat{\psi}_{\lambda,\sigma}(\frac{\bar{p}}{\lambda}) \]  

(18)

Inserting the scaled Fourier transformed fields into quadratic part of action \( S_{0} \), it is easy to see that \( S_{0} \) remains unchanged if the scaling dimension \( \alpha = \frac{4}{2} \). Now, inserting the rescaled fields in the quadratic part of the action, we find that the quartic part has scaling dimension \( (1 - d) \). The quartic terms of higher degree in momenta as well as terms of higher degree in fields appearing in the effective action have smaller scaling dimensions. Thus the effective action in terms of scaled sector fields is,

\[ S_{eff} = \sum_{\lambda,\sigma} \int (dp)\hat{\psi}_{\lambda,\sigma}^{*}(\bar{p})(ip_{0} - v_{F} \vec{\omega} \cdot \vec{p})\hat{\psi}_{\lambda,\sigma}(\bar{p}) + \frac{1}{2} \frac{1}{\lambda^{d-1}} \sum_{\lambda_{1} + \lambda_{2} = \lambda_{3} + \lambda_{4}} g(\bar{\omega}_{1}, \bar{\omega}_{2}, \bar{\omega}_{3}, \bar{\omega}_{4}) \]
\[
\int (d\tilde{p}_1) \ldots (d\tilde{p}_4)(2\pi)^{d+1} \delta(\tilde{p}_1 + \tilde{p}_2 - \tilde{p}_3 - \tilde{p}_4) \tilde{\psi}_{\lambda_1,\sigma}^* (\tilde{p}_1) \tilde{\psi}_{\lambda_2,\sigma}^* (\tilde{p}_2) \tilde{\psi}_{\lambda_3,\sigma}^* (\tilde{p}_3) \tilde{\psi}_{\lambda_4,\sigma} (\tilde{p}_4)
\]

+ terms of higher order in \( \frac{1}{\lambda} \) \hspace{1cm} (19)

We see that the inverse propagator for the sector field is diagonal in \( \omega \), and it depends only on \( p_0 \) and \( p_\parallel = \tilde{\omega} \tilde{p} \) but not on \( p_\perp = \tilde{p} - (\tilde{\omega}, \tilde{p}) \tilde{\omega} \).

We are interested in the renormalization group flow equations in the leading order in \( \frac{1}{\lambda} \). Therefore, for carrying out the decimation of degrees of freedom, we will be interested only in those diagrams that contribute to the amplitude in the leading order in \( \frac{1}{\lambda} \). Before we consider these diagrams, let us consider the possible inter sector scattering geometries. How many independent \( g^{(0)}(\omega_1, \omega_2, \omega_3, \omega_4) \) exists? For \( d = 3 \), suppose \( \omega_3 \neq -\omega_4 \). On the unit sphere, there are \( N^{(0)} = \text{Const.} \lambda_0^{-1} \) different \( \omega \)’s. But all choices of \( \omega_1 \) and \( \omega_2 \) with \( \omega_1 + \omega_2 = \omega_3 + \omega_4 \) lie on a cone containing \( \omega_3 \) and \( \omega_4 \) and \( \omega_3 + \omega_4 \) as the symmetry axis. Therefore, there are \( O(\lambda_0^{-2}) \) choices. Only when \( \omega_3 = -\omega_4 \) that there are \( N^{(0)} = \text{Const.} \lambda_0^{-d-1} \) choices. Couplings involving incoming states with \( \omega_3 \neq -\omega_4 \) will be represented by \( g^{(0)}(\omega_1, \omega_2, \omega_3, \omega_4) \). Couplings that involve sectors \( \omega_3 = -\omega_4 \) or equivalently \( \omega_1 = -\omega_2 \) will be denoted by \( g_{BCS}(\omega_1, \omega_4) \). Because of rotational invariance, \( g^{(0)}_{BCS}(\omega_1, \omega_4) \) is a function of only the angle between \( \omega_1 \) and \( \omega_4 \).

The chemical potential receives corrections from connected diagrams with two external electron lines (self energy correction of the electrons). The contribution of order zero in \( \frac{1}{\lambda} \) comes only from the tadpole and turtle diagrams. These diagrams contain one internal interaction squiggles of order \( \frac{1}{\lambda_0} \) and there are \( N^{(0)} = \text{Const.} \lambda_0^{-1} \) choices of the inner particle sector. Therefore, the contribution is of order zero in \( \frac{1}{\lambda} \). The amplitude corresponding to these tadpole and turtle diagrams turns out to be \( p \)-independent but of order \( O(g^{(0)}) \). Thus, \( \delta \mu_1 = O(g^{(0)}/\lambda_0) \) and the renormalized electron propagator is

\[
\lambda_0^{-d} \left[ \delta \mu_0, \tilde{\omega} \right] = -\left[ \delta \mu_0, \tilde{W} \right].
\]

To find the evolution of the coupling constant \( g(\omega_1, \omega_2, \omega_3, \omega_4) \), we have to calculate amplitude of diagrams with four external legs. It is found that when \( \omega_1 + \omega_2 = \omega_3 + \omega_4 \neq 0 \), the coupling functions do not flow in the leading order in \( \frac{1}{\lambda} \). But for sector indices \( \omega_1 + \omega_2 = \omega_3 + \omega_4 = 0 \), the coupling functions, \( g^{(0)}_{BCS}(\omega_1, \omega_4) \), flow. The diagrams that contribute to the flow equation are the ladder diagrams with self energy insertion for the internal electron lines but with no other two legged subdiagram. The amplitude of such a diagram with \( n \) interaction squiggles and with zero incoming and outgoing box momenta of the particles is given by:

\[
(\frac{1}{\lambda_0^{-1}})^{n+1} \sum_{\omega_1, \ldots, \omega_n} (-1)^n \beta^n g_{BCS}(\omega, \omega_n) g_{BCS}(\omega_n, \omega_{n-1}) \ldots g_{BCS}(\omega_1, \omega_f)
\]

In the equation above, \((\omega, -\omega')\) and \((\omega', -\omega)\) are sector indices of incoming and outgoing electron lines respectively. Other sector indices correspond to the internal electron lines. \( \beta \) is a strictly positive number coming from the fermion loop integration in the Feynman diagram and is given by \( \beta = \int dk_x dk_\parallel dk_0 [k_0^2 + (v_F k_\parallel - \lambda \delta \mu_1)^2]^{-1} \). We find that the renormalized value of \( g_{BCS} \) is

\[
g^{(j+1)}_{BCS}(\omega, \omega') = g^{(j)}_{BCS}(\omega, \omega') + \sum_{n=1}^{\infty} \left( \frac{1}{\lambda_0^{-d}} \right)^n \sum_{\omega_1, \ldots, \omega_n} (-1)^n \beta^n g^{(j)}_{BCS}(\omega, \omega_n) g^{(j)}_{BCS}(\omega_n, \omega_{n-1}) \ldots g^{(j)}_{BCS}(\omega_1, \omega_f) + O\left( \frac{g^{(j)}}{\lambda_j} \right)
\]

The explicit expression for flow equation can be obtained by expanding the coupling functions \( g_{BCS}(\omega, \omega') = g_{BCS}(\omega, \omega') \) into spherical harmonics, \( g_{BCS}(\omega, \omega') = \sum g_h l(\omega, \omega') \). Up to terms of order \( \frac{1}{\lambda} \),

\[
g^{(j+1)} = \frac{g^{(j)}}{1 + \beta j g^{(j)}} + O\left( \frac{1}{\lambda} \right)
\]

To obtain the differential equation for the R.G. flow, let us define \( g^{(j)}_t := g^{(j)}(e^{\lambda t}) \), and consider a scale \( \lambda = e^{\lambda_0} \). Next, define \( g_t(t) := g^{(j)}(e^{\lambda_0} \lambda t) \). The coefficient \( \beta = \beta(t, t') \) vanishes in the limit \( t' \downarrow t \), therefore, \( \beta(t', t) = (t' - t) g(t) + O((t' - t)^2) \). Writing difference equation for the couplings \( g^{(j+1)}_t \) and \( g^{(j)}_t \), and dividing both sides of the difference equation by \( (t' - t) \), we finally obtain

\[
\frac{d}{dt} g^{(j)}(t) = -\gamma g^{(j)}(t)^2 + O(e^{-t} g^{(j)}(t)^2)
\]
where $\gamma = \gamma(t) > 0$. It is independent of $l$ and approximately independent of $t$, and therefore, we set $\gamma = \gamma_0$. The positivity of $\gamma$ follows from slow monotone growth of $\beta(t' - t)$ in $t'$. Neglecting the error term, the solution can be written as,

$$g_t = \frac{g_t(0)}{1 + \gamma_0 g_t(0)t}$$

(23)

If the coupling constants are positive or rather non-negative, $g_t(0) \geq 0$, the effective running coupling constant goes to zero, and we have the Landau-Fermi liquid phase. This phase consists of free quasi-particles which are electrons and holes with renormalized mass and the chemical potential. On the other hand, if there is an angular momentum channel, $l$, with attractive interactions ($g_t(0) < 0$) the flow diverges at a finite value of the scaling parameter, $t = -\frac{1}{\gamma_0 g_t(0)^{-1}}$. This singularity reflects the instability of the ground state.

The ground state of the perturbation theory described above was taken to be the non-interacting Fermi gas with Fock space constructed from the elementary excitations, electrons and holes. The renormalization group analysis shows that this is not the true ground state in the presence of attractive interaction. The singularity in the running coupling constant reflects just this fact. To see whether the true ground state is a BCS state, we need to know the nature of the pole. It is easily found that the residue at the pole of the effective running coupling constant has negative sign. This signifies the presence of Cooper pairs \[2,18\]. The true ground state is thus, the BCS ground state of superconductivity. In the following section, we shall see that the Landau pole, which looks very similar to the pole of the effective coupling constant in condensed matter system, has positive residue at the pole. This sets it apart from the BCS type of instability \[2,18\].

### III. DYSON’S ARGUMENTS FOR THE DIVERGENCE OF PERTURBATION SERIES IN COUPLING CONSTANT

Dyson’s arguments for the divergence of perturbation theory in QED is elegant in its’ simplicity. We will simply reproduce here his arguments \[19\].

Let us suppose that

$$F(e^2) = a_0 + a_1 e^2 + a_2 e^4 + ...$$

is a physical quantity which is calculated as a formal power series in $e^2$ by integrating the equations of motion of the theory over a finite or infinite time. Suppose that the series converges for some positive small value of $e^2$; this implies that $F(e^2)$ is an analytic function of $e$ at $e = 0$. Then for sufficiently small value of $e$, $F(-e^2)$ will also be a well-behaved analytic function with a convergent power series expansion. However, for $F(-e^2)$ we can also make a physical interpretation. In the fictitious world, like charges attract each other. The potential between static charges, in the classical limit of large distances and large number of elementary charges, will be just the Coulomb potential with the sign reversed. But it is clear that in the fictitious world the vacuum state as ordinarily defined is not the state of lowest energy. By creating a large number $N$ of electron-positron pairs, bringing the electrons in one region of space and the positrons in another separate region, it is easy to construct a pathological state in which the negative potential energy of the Coulomb forces is much greater than the total rest energy and the kinetic energy of the particles. Suppose that in the fictitious world the state of the system is known at a certain time to be an ordinary physical state with only a few particles present. There is a high potential barrier separating the physical state from the pathological state of equal energy. However, because of the quantum mechanical tunneling effect, there will always be a finite probability that in any finite time-interval the system will find itself in a pathological state. Thus every physical state is unstable against the spontaneous creation of many particles. Further, a system once in a pathological state will not remain steady; there will be rapid creation of more and more particles, an explosive disintegration of the vacuum by spontaneous polarization. In these circumstances it is impossible that the integration of the equation of motion of the theory over any finite or infinite time interval, starting from a given state of the fictitious world, should lead to well-defined analytic functions. Therefore $F(-e^2)$ can not be analytic and the series can not be convergent.

The central idea in Dyson’s arguments for the divergence of perturbation theory in coupling constant, as is evident from the lengthy discussion above, is that the convergence of the perturbation theory in coupling constant would lead to the existence of pathological states to which the normal states of QED would decay. These pathological states correspond to states of a quantum field theory whose vacuum state is unstable. Therefore, if quantum electrodynamics is a meaningful theory, the perturbation series must diverge (for more discussions related to these arguments, see \[20,21\] and references there in).

It should be noted that Dyson’s proof appeared much before the advent of asymptotic freedom in quantum field theories. The main point in Dyson’s proof is that, if the perturbative series is convergent, then for small value of
\( e^2 \), we can analytically continue to \(-e^2\) and then this series will also be convergent. Let us consider the series for the vacuum polarization (two point Green’s function for photons). Both the perturbative as well as the analytically continued series are assumed to be convergent. We can carry out loop wise summation. We can write the formal sum for one loop diagrams (the coupling is assumed to be small) and extract from it the effective coupling constant.

In the analytically continued theory, we obtain

\[
eff^2 = \frac{e^2}{1 - (-e^2)\ln \frac{\Lambda^2}{\alpha^2}}
\]  

(24)

When \( \Lambda \to \infty \), then \( \eff^2 \to 0 \), and therefore the analytically continued theory is asymptotically free. It is also easy to infer that at low energies the effective coupling constant increases. On purely formal grounds, the effective coupling constant behaves in the same way as the effective coupling constant in QCD. For large flavour limit of quantum electrodynamics can be equivalently described by the Lagrangian, 

\[ \mathcal{L} = \sum_{j=1}^{N_f} \bar{\psi}_j \left( i \gamma^\mu \partial_\mu + m - e \gamma^\mu A_\mu \right) \psi_j + \frac{1}{4} F_{\mu\nu}^2 \]  

(25)

where \( \psi_j \) and \( \bar{\psi}_j \) are the Dirac field and its' conjugate, \( j \) is the flavour index, and \( A_\mu \) and \( F_{\mu\nu} \) are the electromagnetic potential and the field strength respectively. We will investigate this model when the number flavours, \( N_f \), has both the positive and negative sign. We, therefore, introduce the notation \( |N_f| = \text{sign}(N_f) \times N_f \). For the Lagrangian, written above, \( \frac{1}{N_f} \) expansion is introduced by assuming that, in the limit \( |N_f| \to \infty \), \( e^2 |N_f| = \text{constant} = \alpha^2 \) (say) . Large flavour limit of quantum electrodynamics can be equivalently described by the Lagrangian,

\[
\mathcal{L} = \sum_{j=1}^{N_f} \bar{\psi}_j \left( i \gamma^\mu \partial_\mu + m - \frac{e}{\sqrt{|N_f|}} \gamma^\mu A_\mu \right) \psi_j + \frac{1}{4} F_{\mu\nu}^2
\]  

(26)

With this form of the Lagrangian, it is easier to set up Feynman diagram technique. To each photon and fermion line corresponds their usual propagator. Each vertex contributes a factor of \( \frac{1}{\sqrt{|N_f|}} \), each fermion loop contributes a factor of \((-1)\) for anticommuting fermions and a factor of \( N_f \) because of summation over fermion flavours. Using these rules, it is easy to set up \( 1/N_f \) expansion series for any physical observable. Just as in the case of perturbation theory in the coupling constant, the expansion in \( \frac{1}{N_f} \) allows us to express an observable \( F \) in the form,

\[
F \left( \frac{1}{N_f} \right) = Q_0 + \frac{1}{N_f} Q_1 + \frac{1}{N_f^2} Q_2 + \ldots
\]  

(27)

\( Q_0, Q_1, Q_2, \ldots \) are some functions of the coupling contant. Now suppose that the series converges for some small value of \( \frac{1}{N_f} \) ( large value of \( N_f \) ), then the observable function \( F \left( \frac{1}{N_f} \right) \) is analytic for \( \frac{1}{N_f} = 0 \) ( \( N_f = \infty \) ).
Therefore, we can consider a small negative value of \( \frac{1}{N_f} \) (large negative value of \( N_f \)) for which the function is analytic and convergent. In other words, the function \( F(\frac{1}{N_f}) \) can be analytically continued to small negative value of \( 1/N_f \) and the series thus obtained will be convergent. Latter, in the text, we will discuss the meaning of negative flavour. Here we just mention that, in the context of lattice QCD, fermions with finite number of negative flavours have been considered before (see [23, 24, 25] and references there in).

Let us calculate the effective coupling constant from the formal \( 1/N_f \) expansion series of the two point Green’s function. The series is assumed to be convergent, and therefore, for sufficiently large \( N_f \), one can restrict to the leading order term. The leading order term is given by the one-loop diagrams which can easily be evaluated to obtain the polarization from which one can read off the effective coupling constant. It is given by,

\[
e_{eff}^2(\Lambda^2) = \frac{e^2}{1 - \frac{e^2 N_f}{3\pi n} \ln \frac{\Lambda^2}{m^2}} \tag{28}
\]

If \( N_f \) is positive,

\[
e_{eff}^2(\Lambda^2) = \frac{e^2}{1 + \frac{e^2 N_f}{3\pi n} \ln \frac{\Lambda^2}{m^2}} \tag{29}
\]

It is to see that the effective coupling constant has a pole at finite but very large value of \( \Lambda^2 = m^2 \exp(3\pi^2/e^2) \). This is known as the Landau singularity of the effective coupling constant in QED. This is the central theme of the paper, and therefore, we repeat that the convergence of the \( 1/N_f \) expansion series allows us to choose a sufficiently large \( N_f \) and restrict to the leading order terms in \( 1/N_f \).

Thus:

• The appearance of Landau singularity in the effective running coupling constant in QED is intimately linked to the assumption that the \( 1/N_f \) expansion series converges.

We will now argue that Landau singularity leads to the instability of vacuum state in QED. We mentioned before that, the convergence of the series for sufficiently large positive \( N_f \), allows us to analytically continue it to negative \( N_f \), and the series will again be convergent. Thus, for \( N_f \) negative, we obtain,

\[
e_{eff}^2(\Lambda^2) = \frac{e^2}{1 + \frac{e^2 N_f}{3\pi n} \ln \frac{\Lambda^2}{m^2}} \tag{30}
\]

From this equation, it is easy to see that in the limit \( \Lambda \rightarrow \infty \), \( e_{eff}^2 \rightarrow 0 \). Therefore, the formal theory that we obtain from the analytical continuation of \( 1/N_f \) (for large \( N_f \)) to the small negative value of \( 1/N_f \) is (at least formally) asymptotically free. This seems to suggest that the physical meaning of the negative sign of \( N_f \) could possibly be traced in the free theory without the interaction term. In following sections, we shall argue that the choice of negative \( N_f \) for anticommuting fermions amounts to considering commuting fermions with positive \( N_f \).

For the Lagrangian given by Eq. (25) (in four dimensional Euclidean space), the partition function is given by functional integral,

\[
Z_{ac} = \int DA(x)D\bar{\psi}(x)D\psi(x)e^{\int d^4x(L)} \tag{31}
\]

Functonal integration with respect to the anticommuting fermion fields (grassman variables) gives,

\[
Z_{ac} = \int DA(x) \det^{N_f}(i\gamma^\mu \partial_\mu + m - e\gamma^\mu A_\mu) \exp\left( -\frac{1}{4} \int d^4xF^2_{\mu\nu} \right) \tag{32}
\]

On the other hand, if we consider the fermion fields to be commuting variables, then the integration above amounts to functional integration over complex fields, and we obtain,

\[
Z_c = \int DA(x) \det^{-N_f}(i\gamma^\mu \partial_\mu + m - e\gamma^\mu A_\mu) \exp\left( -\frac{1}{4} \int d^4xF^2_{\mu\nu} \right) \tag{33}
\]
Note that this expression could be obtained from the previous expression, simply by assuming that $N_f$ is negative. Therefore, anticommuting fermions with negative $N_f$ has the same partition function as the commuting fermions with positive $N_f$. Since, physically interesting observables can be calculated from the partition function, our claim is that the negative flavour anticommuting fermions are equivalent to the positive flavour commuting fermions.

We can also arrive at this conclusion using the Feynman diagram technique of the formal perturbation theory. Consider the two point Green’s functions for the photons using Lagrangian given by Eq.(26). First, we consider just one loop diagram and show how the contribution due to flavours appears in the calculations. There are two vertices and a fermion loop, each vertex contributes a factor of $\frac{\sqrt{N_f}}{Z}$, the fermion loop contributes a multiplicative factor of $(-1)$ because the fermions anticommute and a multiplicative factor of $N_f$ because of summation over flavours of the internal fermion lines. Now, if $N_f$ happens to be negative, the factor $(-1)$ and the factor $N_f$, combines to give the factor $|N_f|$. This is also the contribution if the fermions commute and the flavour is positive (the factor $(-1)$ is absent for commuting fermions). The same procedure applies for the multiloop diagrams. This shows that the choice of negative $N_f$ for anticommuting fermions amounts to considering commuting fermions with positive $N_f$. We have shown earlier in text that quantum electrodynamics with anticommuting fermions and negative value of $N_f$ is asymptotically free. Therefore, our purely formal quantum electrodynamics with commuting fermions and positive $N_f$ is asymptotically free. It is well known that the quantum field theory of free commuting fermions does not have stable vacuum state. It then follows that an interacting asymptotically free quantum field theory built around such a vacuum state can not be stable: all states in this theory will be pathological. These results follow from the single assumption that the $1/N_f$-expansion theory in QED is convergent, and therefore analytic in $1/N_f$. As per Dyson’s argument, this would lead to the decay of normal states of QED with anticommuting fermions to the pathological states of QED with commuting fermions via the process of quantum mechanical tunnelling. Therefore, the vacuum state of QED with flavour of anticommuting fermions can not be stable.

We have already seen that the convergence of $1/N_f$ expansion theory invariably leads to the appearance of Landau singularity.

Thus:-

- Landau singularity signals the instability of vacuum state of quantum electrodynamics.

We have, no where, in text shown that the $1/N_f$ expansion series diverges. It was only an assumption. There exist large number of publications which, based on the behaviour of the large order terms in the series (in coupling constant), suggest that the perturbation series is, possibly, divergent. Similar arguments can be extended to our case unless there is some magical cancellations in large orders of the series. But such magical cancellations, if any, would plague the theory with the instability of vacuum state and render it meaningless.

Thus:-

- Quantum electrodynamics with large number of flavours of (anticommuting) fermions will be a meaningful theory, only if the series in $1/N_f$ expansion diverges.

In the abstract, we announced the extraordinary success of quantum electrodynamics. How do we understand this success in the light of results obtained in this paper? In the following section, we discuss how a divergent series could possibly lead to a meaningful theory of quantum electrodynamics.

V. A BRIEF MATHEMATICAL DIGRESSION:ASYMPTOTIC EXPANSION SERIES

There is an important class of series, known as asymptotic series, which frequently appear in physical problems. These series behave like a convergent series up to a certain number of terms but after that it behaves like a divergent series. This type of series is called asymptotic series and is generally defined through a power series representation of a function. The behaviour of an asymptotic series is very transparent in the following example (a version of Stirling’s formula) given by Bender and Orszag (page 218 in [30]) for the asymptotic series expansion of factorial, $N! = (Z - 1)!$, in powers of $1/Z$.

$$(Z - 1)! = 2\pi/Z)^{1/2} e^{-Z} Z^Z \left( 1 + \frac{1}{12} Z^{-1} + \frac{1}{288} Z^{-2} - \frac{139}{51840} Z^{-3} - \frac{571}{2488320} Z^{-4} + \right.$$

$$\left. \frac{163879}{20901880} Z^{-5} + \frac{5246819}{7524679680} Z^{-6} - \frac{534703531}{902961561600} Z^{-7} - \frac{4483131259}{86684309913600} Z^{-8} + ... \right)$$
For example, for \( 0! \), the terms get smaller for a while but the 15th term becomes larger than 0.01, and the 35th is bigger than 10. On the other hand, the 35th term for 9! is \( 10^{10} / 10^{35} \) which is quite small but the 175th term is bigger than one, and the 199th term is nearly 10. This series has zero radius of convergence. A non-zero radius of convergence in \( 1/Z \) would also include some negative values around zero, and therefore, if the series for \( (Z-1)! \) converges for some large positive integer \( Z = (N + 1) \), it will also converge for some large negative integer \( Z = -(N-2) \). But \( (Z-1)! = (Z-N-2)! \) is infinity. Therefore, the the 1/Z expansion series of \( (Z-1)! \) for \( Z = N+1 \) can not converge. The series is meaningful only as an asymptotic expansion.

Mathematically, a function \( f(x) \) is said to have an asymptotic power series representation if for all \( n \),

\[
\lim_{x \to 0} \frac{f(x) - \sum_{i=0}^{n} a_i x^i}{x^n} = 0
\]

In other words,

\[
f(x) = \sum_{i=0}^{n} a_i x^i + O(x^n)
\]

This means that the error in estimating the function is of the same order as the last term in the series. To explain, let us consider the following function,

\[
F(x) = \int_0^\infty \frac{e^{-t}}{1 + xt} dt
\]

for real positive \( x \) and \( x \to 0 \). Since,

\[
\frac{1}{1 + xt} = 1 - xt + x^2 t^2 + ... + \frac{(-x t)^k}{1 + xt}
\]

we have,

\[
F(x) = \sum_{k=0}^{N} (-1)^k x^k k! + R_{N+1}(x) \quad |R_N(x)| = N! x^N
\]

The ratio of the two successive terms is

\[
\frac{x^k k!}{x^{(k-1)} (k-1)!} = xk
\]

This shows that the terms first decrease (since by assumption \( 0 < x << 1 \)) and then increase (when \( k > \frac{1}{x} \)). From this it follows that for a given value of \( x \), there exists a best approximation. In other words, for a fixed value of \( x \), only a definite accuracy can be achieved. However, the function defined by integral is well behaved and is non-analytic in \( x \) at \( x = 0 \).

We have seen in previous sections that perturbation series of quantum electrodynamics in coupling constant as well as in \( 1/N_f \) is divergent. Studies of the large order terms in perturbation series suggest that these series are probably asymptotic in nature. The fine structure constant of QED is 1/137 which is quite small, and therefore in the asymptotic series, one can consider terms up to a very large order. This can possibly explain the spectacular success of QED, in spite of fact that the series is divergent and asymptotic.

In our view it does not make sense to look for any kind of singularity in coupling constant of a theory with asymptotic expansion and try to draw any big physical conclusion. There are attempts to give meaning to this kind of series through Borel summation but, again in our view, one does not achieve more than what one obtains through the summation of asymptotic series as explained above.

**VI. CONCLUSIONS**

In this paper, we have discussed several aspects of perturbation theory in quantum electrodynamics. Most importantly, we have shown that the Landau singularity appears in the leading order terms in the \( 1/N_f \) expansion series. The restriction to leading order terms makes sense only when the series converges. We used Dysonian argument to
show that the convergence of the series leads to the instability of vacuum state in QED. This demonstrates that Landau singularity reflects the instability of vacuum state. These problems can be avoided if the series diverges. Divergent series as asymptotic series can provide physically meaningful results within some unavoidable errors depending on the value of the expansion parameter. The fine structure constant, which is the expansion parameter in QED, is small, and therefore, there is no surprise why QED is such a successful theory.

Divergence of the perturbation series, suggested by Dysonian argument, is to save the vacuum state from the catastrophic disintegration. But that is not the sole point of Dysonian argument. It also suggests non-analyticity of observables as a function of the coupling constant, $e^2$ when $e^2 = 0$ (similarly in $1/N_f$ for $N_f = \infty$). Note that the non-analyticity in coupling constant seems to be in the infrared (IR) region. On the other hand, the non-analyticity of $1/N_f$ expansion series for $N_f = \infty$ is connected with Landau singularity which is in the ultraviolet (UV) region. It is not clear how the IR and UV regions are connected. However, we must remember that we are dealing with pathological situations where there are singularities and divergences. The main point of Dysonian argument, in our view, is that the non-perturbative QED is non-analytic and this behaviour cannot be captured by a perturbative power series. More than fifty years of theoretical research has not been able to find a non-perturbative formulation which can remove the pathological aspects of QED. May be it is time for experimentalists to step in and look for ways to find non-analytical dependence of some physical observables on the coupling constant or flavours.

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