GROTHENDIECK LOCAL DUALITY AND COHOMOLOGICAL HASSE PRINCIPLE FOR 2-DIMENSIONAL COMPLETE LOCAL RING

BELGACEM DRAOUIL

Abstract. We prove a local duality for some schemes associated to a 2-dimensional complete local ring whose residue field is an \(n\)-dimensional local field in the sense of Kato-Parshin. Our results generalize the Saito works in the case \(n = 0\) and are applied to study the Bloch-Ogus complex for such rings in various cases.

1. Introduction

Let \(A\) be a 2-dimensional complete local ring with finite residue field. The Bloch-Ogus complex associated to \(A\) has been studied by Saito in \([12]\). In this prospect, he calculated the homologies of this complex and obtained (for any integer \(n \geq 1\)) the following exact sequence

\[
0 \to (\mathbb{Z}/n)^r \to H^3(K, \mathbb{Z}/n(2)) \to \bigoplus_{v \in P} H^2(k(v), \mathbb{Z}/n(1)) \to \mathbb{Z}/n \to 0,
\]

where \(P\) denotes the set of height one prime ideals of \(A\), \(K\) is the fractional field of \(A\), \(k(v)\) is the residue field at \(v \in P\), and \(r = r(A)\) is an integer depending on the degeneracy of \(\text{Spec}A\) This result is based upon the isomorphism (\([12\), lemma 5.4]):

\[
H^4(X, \mathbb{Z}/n(2)) \simeq \mathbb{Z}/n,
\]

where \(X = \text{Spec}A \setminus \{x\}\); and \(x\) is the unique maximal ideal of \(A.A\) decate later, Matsumi in \([8]\) generalised the result by Saito to 3-dimensional complete regular local ring of positive characteristic. Indeed, he proved the exactness of the complex

\[
0 \to H^4(K, \mathbb{Z}/\ell(3)) \to \bigoplus_{v \in (\text{Spec}A)_1} H^3(k(v), \mathbb{Z}/\ell(1)) \to \mathbb{Z}/\ell \to 0
\]

for all \(\ell\) prime to \(\text{char}(A)\), where \((\text{Spec}A)_i\) indicates the set of all points in \(\text{Spec}A\) of dimension \(i\). Besides, if the ring \(A\) is not regular, then the map

\[
H^4(K, \mathbb{Z}/\ell(3)) \xrightarrow{\Psi_K} \bigoplus_{v \in (\text{Spec}A)_1} H^3(k(v), \mathbb{Z}/\ell(2))
\]

is non-injective.

We proved in ([3], th3) that \(\text{Ker} \Psi_K\) contains a sub-group of type \((\mathbb{Z}/\ell)\hat{r}_1'(A)\), where \(r_1'(A)\) is calculated as the \(\mathbb{Z}\)-rank of the graph of the exceptional fiber of a resolution of \(\text{Spec}A\). The main tools used in this direction are the isomorphism

\[
H^6(X, \mathbb{Z}/\ell(3)) \simeq \mathbb{Z}/\ell
\]

Date: November 2006.

1991 Mathematics Subject Classification. 11G20, 11G45, 14H30, 14C35,19F05.

Key words and phrases. Hasse principle, Purity, local duality.
and the perfect pairing

\[
H^i(X, \mathbb{Z}/\ell) \times H^{6-i}(X, \mathbb{Z}/\ell(3)) \longrightarrow H^6(X, \mathbb{Z}/\ell(3)) \simeq \mathbb{Z}/\ell
\]

for all \( i \geq 1 \) ([3], Section 0,D2).

In order to generalize the previous results, we consider a 2-dimensional complete local ring \( A \) whose residue field \( k \) is \( n \)-dimensional local field in the sense of Kato-Parshin. To be more precise, let \( X = \text{Spec} A \setminus \{ x \} \), where \( x \) is the unique maximal ideal of \( A \). Then the main result of this paper is the following

**Theorem (Theorem 3.1)**

There exist an isomorphism

\[
H^{4+n}(X, \mathbb{Z}/\ell(2+n)) \simeq \mathbb{Z}/\ell
\]

and a perfect pairing

\[
H^1(X, \mathbb{Z}/\ell) \times H^{3+n}(X, \mathbb{Z}/\ell(2+n)) \longrightarrow H^{4+n}(X, \mathbb{Z}/\ell(2+n)) \simeq \mathbb{Z}/\ell
\]

for all \( \ell \) prime to \( \text{char}(A) \).

We apply this result to calculate the homologies of the Bloch-Ogus complex associated to \( A \). Indeed, let \( \pi_1^{c.s}(X) \) be the quotient group of \( \pi_1^{ab}(X) \) which classifies abelian c.s coverings of \( X \) (see definition 4.1 below). We then prove the following

**Theorem (Theorem 4.2)**

Let \( A \) be a 2-dimensional complete normal local ring of positive characteristic whose residue field is \( n \)-dimensional local field. Then the exact sequence

\[
0 \longrightarrow \pi_1^{c.s}(X)/\ell \longrightarrow H^{3+n}(K, \mathbb{Z}/\ell(2+n))
\]

\[
\longrightarrow \bigoplus_{v \in P} H^{2+n}(k(v), \mathbb{Z}/\ell(1+n)) \longrightarrow \mathbb{Z}/\ell \longrightarrow 0
\]

holds.

Furthermore, if \( A \) is assumed to be regular then, as in the case of \( n = 0 \) considered by Saito [12], we prove the vanishing of the group \( \pi_1^{c.s}(X) \) using the recent paper [11] by Panin.

To prove these results, we rely heavily on the Grothendieck duality theorem for strict local rings (section 3) as well as the purity theorem of Fujiwara-Gabber, which we recall next.

In ([4], sentences just below Corollary 7.1.7), Fujiwara confirmed that the absolute cohomological purity in equicharacteristic is true. In other words, we get the following.

**Theorem of Fujiwara-Gabber**

Let \( T \) be an equicharacteristic Noetherian excellent regular scheme and \( Z \) be a regular closed subscheme of codimension \( c \). Then for an arbitrary natural number \( \ell \) prime to \( \text{char}(T) \), the following canonical isomorphism

\[
H^j_Z(T, \mathbb{Z}/\ell(j)) \simeq H^{1-2c}(Z, \mathbb{Z}/\ell(j-c))
\]

holds.

Finally, we complete the partial duality (1.7) in the case \( n = 1 \). So, we obtain the following.

**Theorem (Theorem 5.1)**

Let \( A \) be a 2-dimensional normal complete local ring whose residue field is one-dimensional local field. Then, for very \( \ell \) prime to \( \text{char}(A) \), the isomorphism

\[
H^5(X, \mathbb{Z}/\ell(3)) \simeq \mathbb{Z}/\ell
\]
and the perfect pairing

\[
H^i(X, \mathbb{Z}/\ell) \times H^{5-i}(X, \mathbb{Z}/\ell(3)) \to H^5(X, \mathbb{Z}/\ell(3)) \simeq \mathbb{Z}/\ell
\]

hold for all \(i \in \{0, ..., 5\}\).

Our paper is organised as follows. Section 2 devoted to some notations. Section 3 contains the main theorem of this work concerning the duality of the scheme \(X = \text{Spec}A \setminus \{x\}\), where \(A\) is a 2-dimensional complete local ring whose residue field is \(n\)-dimensional local field and \(x\) is the unique maximal ideal of \(A\). In section 4, we study the Bloch-Ogus complex associated to \(A\). In section 5, we investigate the particular case \(n = 1\).

2. Notations

For an abelian group \(M\) and a positive integer \(n\), we denote by \(M/n\) the cokernel of the map \(M \to M\). For a scheme \(Z\), and a sheaf \(F\) over the étale site of \(Z\), \(H^i(Z, F)\) denotes the \(i\)-th étale cohomology group. For a positive integer \(\ell\) invertible on \(Z\), \(\mathbb{Z}/\ell(1)\) denotes the sheaf of \(\ell\)-th root of unity and for an integer \(i\), we denote \(\mathbb{Z}/\ell(i) = (\mathbb{Z}/\ell(1))^\otimes i\).

A local field \(k\) is said to be \(n\)-dimensional local if there exists the following sequence of fields \(k_i\) (1 \(\leq\) \(i\) \(\leq\) \(n\)) such that
(i) each \(k_i\) is a complete discrete valuation field having \(k_{i-1}\) as the residue field of the valuation ring \(O_{k_i}\) of \(k_i\), and
(ii) \(k_0\) is a finite field.

For such a field, and for \(\ell\) prime to \(\text{Char}(k)\), the well-known isomorphism

\[
H^{n+1}(k, \mathbb{Z}/\ell(n)) \simeq \mathbb{Z}/\ell
\]

and for each \(i \in \{0, ..., n+1\}\) a perfect duality

\[
H^i(k, \mathbb{Z}/\ell(j)) \times H^{n+1-i}(k, \mathbb{Z}/\ell(n-j)) \to H^{n+1}(k, \mathbb{Z}/\ell(n)) \simeq \mathbb{Z}/\ell
\]

hold.

For a field \(L\), \(K_i(L)\) is the \(i\)-th Milnor group. It coincides with the \(i\)-th Quillen group for \(i \leq 2\). For \(\ell\) prime to \(\text{char}(L)\), there is a Galois symbol

\[
h_{i,L} K_iL/\ell \to H^i(L, \mathbb{Z}/\ell(i))
\]

which is an isomorphism for \(i = 0, 1, 2\) (\(i = 2\) is Merkur’jev-Suslin). In this context, we recall the Kato conjecture ([7], Conjecture 1, Section 1):

**Kato Conjecture**

For any field \(L\) and any \(\ell\) prime to \(\text{char}(L)\), the map \(h_{i,L}\) is bijective.

3. Local duality

We start this section by a description of the Grothendieck local duality. Let \(B\) denote a \(d\)-dimensional normal complete local ring with maximal ideal \(x'\). By Cohen structure theorem ([10], 31.1), \(B\) is a quotient of a regular local ring. Hence \(\text{Spec}B\) admits a dualizing complex. Now, assume in the first step that the residue field of \(B\) is separably closed (\(B\) is strictly local). Then, for \(X' = \text{Spec}B \setminus \{x'\}\) and for any \(\ell\) prime to \(\text{char}(A)\), there is a Poincaré duality theory ([15], Exposé I, Remarque 4.7.17). Namely, there is a trace isomorphism

\[
H^{2d-1}(X', \mathbb{Z}/\ell(d)) \to \mathbb{Z}/\ell
\]

and a perfect pairing
for all \( i \in \{0, \ldots, 2d - 1 \} \).

Assume at this point that the residue field \( k \) of \( B \) is arbitrary. Let \( k_s \) be a separable closure of \( k \). The strict henselization \( B^{sh} \) of \( B \) (with respect to the separably closed extension \( k_s \) of \( k \)) at the unique maximal ideal \( x \) of \( B \) is a strictly local ring. It coincides with the integral closure of \( B \) in the maximal unramified extension \( L^{ur} \) of the fraction field \( L \) of \( B \). Let \( x' \) be the maximal ideal of \( B^{sh} \) and let \( X' = \text{Spec}B^{sh}\setminus\{x'\} \). So, the Galois group of \( X' \) over \( X \) is \( \text{Gal}(L^{ur}/L) \) which is isomorphic to \( \text{Gal}(k_s/k) \). Then for any integer \( j \geq 0 \), we get the Hochschild-Serre spectral sequence ([9], Remark 2.21)

\[
E_2^{pq} = H^p(k, H^q(X', \mathbb{Z}/\ell (j))) \implies H^{p+q}(X, \mathbb{Z}/\ell (j))
\]

Let \( A \) denote a 2-dimensional normal complete local ring whose residue field is an \( n \)-dimensional local field. Let \( x \) be the unique maximal ideal of \( A \). Then by normality \( A \) admits at most one singularity at \( x \) in such a way that the scheme \( X = \text{Spec}A \setminus \{x\} \) becomes a regular scheme.

In what follows, we put
\( K \) : the fractional field of \( A \),
\( k \) : the residue field of \( K \),
\( P \) : the set of height one prime ideals of \( A \).

For each \( v \in P \) we denote by \( K_v \) the completion of \( K \) at \( v \) and by \( k(v) \) the residue field of \( K_v \).

Let \( X = \text{Spec}A \setminus \{x\} \) as above. Generalizing (1.2), (1.4), and (1.5), we get the following.

**Theorem 3.1**

For all \( \ell \) prime to \( \text{char}(A) \), the isomorphism

\[
H^{1+n}(X, \mathbb{Z}/\ell (2 + n)) \simeq \mathbb{Z}/\ell
\]

and the perfect pairing

\[
H^1(X, \mathbb{Z}/\ell) \times H^{3+n}(X, \mathbb{Z}/\ell(2+n)) \longrightarrow H^{4+n}(X, \mathbb{Z}/\ell(2+n)) \simeq \mathbb{Z}/\ell
\]

occur. Furthermore, this duality is compatible with duality (2.2) in the sense that the commutative diagram

\[
\begin{array}{c}
H^1(X, \mathbb{Z}/\ell) \quad \times \\
\downarrow i^* \\
H^1(k(v), \mathbb{Z}/\ell) \times H^{n+1}(k(v), \mathbb{Z}/\ell(n+1)) \\
\uparrow i_* \\
H^{3+n}(X, \mathbb{Z}/\ell(2+n)) \longrightarrow H^{4+n}(X, \mathbb{Z}/\ell(2+n)) \xrightarrow{\sim} \mathbb{Z}/\ell
\end{array}
\]

holds, where \( i^* \) is the map on \( H^1 \) induced from the map \( v \to X \) and \( i_* \) is the Gysin map.

**Proof.** The proof is slightly different from the proof of Theorem 1 in [2]. Let \( k_s \) be a separable closure of \( k \). We consider the strict henselization \( A^{sh} \) of \( A \) (with respect to the separably closed extension \( k_s \) of \( k \)) at the unique maximal ideal \( x \) of \( A \). Then, we denote \( x' \) the unique maximal ideal of \( A^{sh} \), \( X' = \text{Spec}A^{sh}\setminus\{x'\} \) and we use the spectral sequence (3.3). As \( k \) is \( n \)-dimensional local field, we have \( H^{n+2}(k, M) = 0 \) for any torsion module \( M \) and as \( X' \) is of cohomological dimension \( 2d - 1 \) ([14], the last paragraph of Introduction), we obtain

\[
H^{4+n}(X, \mathbb{Z}/\ell(2+n)) \simeq H^{n+1}(k, H^3(X', \mathbb{Z}/\ell (2 + n))) \\
\simeq H^{n+1}(k, \mathbb{Z}/\ell (n)) \quad \text{by (3.1)} \\
\simeq \mathbb{Z}/\ell \quad \text{by (2.1)}
\]
We prove now the duality (3.5). The filtration of the group \( H^{3+n}(X, \mathbb{Z}/\ell(2+n)) \) is
\[
H^{3+n}(X, \mathbb{Z}/\ell(2+n)) = E_{n+1}^{3+n} \supseteq E_{n+1}^{3+n} \supseteq 0
\]
which leads to the exact sequence
\[
0 \to E_{\infty}^{n+1,2} \to H^{3+n}(X, \mathbb{Z}/\ell(2+n)) \to E_{\infty}^{n,3} \to 0
\]
Since \( E_2^{p,q} = 0 \) for all \( p \geq n + 2 \) or \( q \geq 4 \), we see that
\[
E_{\infty}^{n,3} = E_{3}^{n,3} = \ldots = E_{\infty}^{n,3}.
\]
The same argument yields
\[
E_{\infty}^{n+1,2} = E_{4}^{n+1,2} = \ldots = E_{\infty}^{n+1,2}
\]
and \( E_3^{n+1,2} = \text{Coker } d_2^{n-1,3} \) where \( d_2^{n-1,3} \) is the map
\[
H^{n-1}(k, H^3(X', \mathbb{Z}/\ell(2+n))) \to H^{n+1}(k, H^2(X', \mathbb{Z}/\ell(2+n))).
\]
Hence, we obtain the exact sequence
\[
(3.7) \quad 0 \to \text{Coker } d_2^{n-1,3} \to H^{3+n}(X, \mathbb{Z}/\ell(2+n)) \to H^n(k, H^3(X', \mathbb{Z}/\ell(2+n))) \to 0
\]
Combining duality (2.2) for \( k \) and duality (3.2), we deduce that the group \( H^0(k, H^1(X', \mathbb{Z}/\ell)) \) is dual to the group \( H^{n+1}(k, H^2(X', \mathbb{Z}/\ell(2+n))) \) and the group \( H^2(k, H^0(X', \mathbb{Z}/\ell)) \) is dual to the group \( H^{n-1}(k, H^3(X', \mathbb{Z}/\ell(2+n))) \). On the other hand, we have the commutative diagram
\[
(3.8)
\]
\[
\begin{array}{ccc}
H^{n-1}(k, H^3(X', \mathbb{Z}/\ell(2+n))) & \times & H^2(k, H^0(X', \mathbb{Z}/\ell)) \\
\downarrow & & \downarrow \\
H^{n+1}(k, H^2(X', \mathbb{Z}/\ell(2+n))) & \times & H^0(k, H^1(X', \mathbb{Z}/\ell))
\end{array}
\]
given by the cup products and the spectral sequence (3.3), using the same argument as \([1]\), diagram 46). We infer that \( \text{Coker } d_2^{n-1,3} \) is the dual of \( \text{Ker } d_2^{n-1,3} \) where \( d_2^{0,1} \) is the boundary map for the spectral sequence \((3.4), j=0)
\[
(3.9) \quad 'E_2^{p,q} = H^p(k, H^q(X', \mathbb{Z}/\ell)) \to H^{p+q}(X, \mathbb{Z}/\ell)
\]
Similarly, the group \( H^n(k, H^3(X', \mathbb{Z}/\ell(2+n))) \) is dual to the group \( H^1(k, H^0(X', \mathbb{Z}/\ell)) \). The required duality is deduced from the following commutative diagram
\[
\begin{array}{ccc}
0 & \to & \text{Coker } d_2^{n-1,3} \\
\downarrow & & \downarrow \\
0 & \to & (\text{Ker } d_2^{n-1,3})^\vee
\end{array}
\]
\[
\begin{array}{ccc}
H^{3+n}(X, \mathbb{Z}/\ell(2+n)) & \to & H^n(k, H^3(X', \mathbb{Z}/\ell(2+n))) \\
\downarrow & & \downarrow \\
(H^1(X, \mathbb{Z}/\ell))^\vee & \to & (H^1(k, H^0(X', \mathbb{Z}/\ell))^\vee)
\end{array}
\]
where the upper exact sequence is (3.7) and the bottom exact sequence is the dual of the well-known exact sequence
\[
0 \to 'E_2^{1,0} \to H^1(k, H^0(X', \mathbb{Z}/\ell)) \to \text{Ker } d_2^{0,1} \to 0
\]
deduced from the spectral sequence (3.9) and where \((M)^\vee\) denotes the dual \( \text{Hom}(M, \mathbb{Z}/\ell) \) for any \( \mathbb{Z}/\ell \)-module \( M \).

Finally, to obtain the last part of the theorem, we remark that the commutativity of the diagram (3.6) is obtained by via a same argument (projection formula \([9]\), VI 6.5) and compatibility of traces \([9]\), VI 11.1) as \([1]\) to establish the commutative diagram in the proof of assertion ii) at page 791.
Corollary 3.2

With the same notations as above, the following commutative diagram

\[ H^{n+1}(k(v), \mathbb{Z}/\ell(n+1)) \longrightarrow H^{3+n}(X, \mathbb{Z}/\ell(2+n)) \]

\[ (H^1(k(v), \mathbb{Z}/\ell))^\vee \longrightarrow (H^1(X, \mathbb{Z}/\ell))^\vee \]

holds.

Proof. This is a consequence of diagram (3.6).

\[ \Box \]

The duality (3.5) will be completed (section 5) to a general pairing by replacing \( H^1(X, \mathbb{Z}/\ell) \) by \( H^i(X, \mathbb{Z}/\ell) \); for \( 0 \leq i \leq 5 \) in the case \( n = 1 \).

4. The Bloch-Ogus Complex

In this section, we investigate the study of the Bloch-Ogus complex associated to the ring \( A \) considered previously. So, let \( A \) be a 2-dimensional normal complete local ring whose residue field is an \( n \)-dimensional local field. Next, we define a group which appears in the homologies of the associated Bloch-Ogus complex of \( A \).

Definition 4.1

Let \( Z \) be a Noetherian scheme. A finite etale covering \( f : W \rightarrow Z \) is called a c.s covering if for any closed point \( z \) of \( Z \), \( z \times \mathbb{Z} \) \( W \) is isomorphic to a finite scheme-theoretic sum of copies of \( z \). We denote \( \pi_{c.s}^1(Z) \) the quotient group of \( \pi^{ab}_{1}(Z) \) which classifies abelian c.s coverings of \( Z \).

As above, let \( X = \text{Spec} A \setminus \{ x \} \). The group \( \pi_{c.s}^1(X) / \ell \) is the dual of the kernel of the map

\[ H^1(X, \mathbb{Z}/\ell) \longrightarrow \prod_{v \in P} H^1(k(v), \mathbb{Z}/\ell) \]

([12], section 2, definition and sentence just below). Now, we are able to calculate the homologies of the Bloch-Ogus complex associated to the ring \( A \).

Theorem 4.2

For all \( \ell \) prime to the characteristic of \( A \), the following sequence is exact.

\[ 0 \longrightarrow \pi_{c.s}^1(X) / \ell \longrightarrow H^{n+3}(K, \mathbb{Z}/\ell(n+2)) \longrightarrow \bigoplus_{v \in P} H^{n+2}(k(v), \mathbb{Z}/\ell(n+1)) \longrightarrow \mathbb{Z}/\ell \longrightarrow 0 \]

Proof. Consider the localisation sequence on \( X = \text{Spec} A \setminus \{ x \} \)

... \( \longrightarrow H^i(X, \mathbb{Z}/\ell(n+2)) \longrightarrow H^i(K, \mathbb{Z}/\ell(n+2)) \longrightarrow \bigoplus_{v \in P} H^i_{c.s}(X, \mathbb{Z}/\ell(n+2)) \rightarrow ... \)

Firstly, for any \( v \in P \), we have the isomorphisms

\[ H^i_{c.s}(X, \mathbb{Z}/\ell(2+n)) \cong H^i_{c.s}(\text{Spec} A_v, \mathbb{Z}/\ell(2+n)) \]

by excision. Secondly, we can apply the purity theorem of Fujiwara-Gabber (Introduction) for \( Z = v, T = \text{Spec} A_v \) and we find the isomorphisms

\[ H^{3+n}_{c.s}(\text{Spec} A_v, \mathbb{Z}/\ell(2+n)) \cong H^{1+n}(k(v), \mathbb{Z}/\ell(1+n)) \]

and

\[ H^{4+n}_{c.s}(\text{Spec} A_v, \mathbb{Z}/\ell(2+n)) \cong H^{2+n}(k(v), \mathbb{Z}/\ell(1+n)) \]

which lead to the isomorphisms.
\[ H^3+n (X, \mathbb{Z}/\ell (2+n)) \simeq H^{1+n} (k(v), \mathbb{Z}/\ell (1+n)) \]

and
\[ H^{4+n} (X, \mathbb{Z}/\ell (2+n)) \simeq H^{2+n} (k(v), \mathbb{Z}/\ell (1+n)). \]

Hence we derive the exact sequence
\[
\bigoplus_{v \in P} H^{1+n} (k(v), \mathbb{Z}/\ell (1+n)) \rightarrow H^{3+n} (X, \mathbb{Z}/\ell (2+n)) \rightarrow H^{4+n} (K, \mathbb{Z}/\ell (2+n)) \\
\rightarrow \bigoplus_{v \in P} H^{2+n} (k(v), \mathbb{Z}/\ell (1+n)) \rightarrow H^{4+n} (X, \mathbb{Z}/\ell (2+n)) \rightarrow 0
\]

The last zero on the right is a consequence of the vanishing of the group \( H^{4+n} (K, \mathbb{Z}/\ell (2+n)) \). Indeed, \( A \) is finite over \( O_L[[T]] \) for some complete discrete valuation field \( L \) having the same residue field with \( A \) [10, §31]. By Serre [13, chapI, Prop 14], \( cd_L(A) \leq cd_L(O_L[[T]]) \) and by Gabber [5], \( cd_L(O_L[[T]]) = n + 3 \) using the fact that \( cd_L(k) = n + 1 \).

Now, by the right square of the diagram (3.6), the Gysin map
\[
\bigoplus_{v \in P} H^{2+n} (k(v), \mathbb{Z}/\ell (1+n)) \rightarrow H^{4+n} (X, \mathbb{Z}/\ell (2+n))
\]
can be replaced by the map \( \bigoplus_{v \in P} H^{2+n} (k(v), \mathbb{Z}/\ell (1+n)) \rightarrow \mathbb{Z}/\ell \) after composing with the trace isomorphism \( H^{4+n} (X, \mathbb{Z}/\ell (2+n)) \simeq \mathbb{Z}/\ell \) (3.4). So, we obtain the exact sequence
\[
0 \rightarrow \text{Coker} \ g \rightarrow H^{3+n} (K, \mathbb{Z}/\ell (2+n)) \rightarrow \bigoplus_{v \in P} H^{2+n} (k(v), \mathbb{Z}/\ell (1+n)) \rightarrow \mathbb{Z}/\ell \rightarrow 0
\]

Finally, in view of the commutative diagram (3.10), we deduce that \( \text{Coker} \ g \) equals to the group \( \pi_1^{c.s} (X) / \ell \) taking in account (4.1).

Next, we assume further that \( A \) is regular. We will prove that the group \( \pi_1^{c.s} (X) / \ell \) vanishes.

**Theorem 4.3**

Let \( A = F_p((t_1))((t_2))...((t_n))[[X,Y]] \) of fraction field \( K \) and assume Kato conjecture (section 2), then the following Hasse principle complex for \( K \)
\[ 0 \rightarrow H^{3+n} (K, \mathbb{Z}/\ell (2+n)) \rightarrow \bigoplus_{v \in P} H^{2+n} (k(v), \mathbb{Z}/\ell (1+n)) \rightarrow \mathbb{Z}/\ell \rightarrow 0 \]

is exact.

**Proof.** Keeping in mind (4.2), it remains to prove the injectivity of the map
\[ \Psi_K : H^{3+n} (K, \mathbb{Z}/\ell (2+n)) \rightarrow \bigoplus_{v \in P} H^{2+n} (k(v), \mathbb{Z}/\ell (1+n)). \]

Let \( q \) be an integer and consider the sheaf \( H^q (\mathbb{Z}/\ell (n+2)) \) on \( \text{Spec} A \), the Zariskien sheaf associated to the presheaf \( U \rightarrow H^q (U, \mathbb{Z}/\ell (n+2)) \). As a consequence of a recent work of Panin [11], we conclude that the cohomology of this sheaf is calculated as the homology of the Bloch-Ogus complex, that is:
\[ H^q (K, \mathbb{Z}/\ell (n+2)) \rightarrow \bigoplus_{v \in P} H^{q-1} (k(v), \mathbb{Z}/\ell (n+1)) \rightarrow H^{q-1} (k(x), \mathbb{Z}/\ell (n)) . \]

So the group \( \text{Ker} \Psi_K \) is identified with the group \( H^0 ((\text{Spec} A)_{\text{Zar}}, H^{n+3} (\mathbb{Z}/\ell (n+2))) \).

On the other hand, the Bloch-Ogus spectral sequence
\[ H^p ((\text{Spec} A)_{\text{Zar}}, H^q (\mathbb{Z}/\ell (n+2))) \Rightarrow H^{p+q} (\text{Spec} A, \mathbb{Z}/\ell (n+2)) \]
gives the exact sequence
\[ H^{n+3} (\text{Spec} A, \mathbb{Z}/\ell (n+2)) \rightarrow H^0 ((\text{Spec} A)_{\text{Zar}}, H^{n+3} (\mathbb{Z}/\ell (n+2))) \rightarrow H^2 ((\text{Spec} A)_{\text{Zar}}, H^{n+2} (\mathbb{Z}/\ell (n+2))) \rightarrow H^{n+1} (\text{Spec} A, \mathbb{Z}/\ell (n+2)) \]
Since the ring $A$ is henselian, we obtain the isomorphism
\[ H^i(\text{Spec}A, \mathbb{Z}/(n + 2)) \simeq H^i(\text{Spec}K, \mathbb{Z}/(n + 2)), \ i \geq 0. \]

But the groups $H^{n+3}(\text{Spec}K, \mathbb{Z}/(3))$ and $H^{n+4}(\text{Spec}K, \mathbb{Z}/(3))$ vanish because the cohomological dimension of $k$ is $n + 1$. Thus we get the isomorphism
\[ H^0((\text{Spec}A)_{\text{Zar}}, H^{n+3}(\mathbb{Z}/(n + 2))) \longrightarrow H^2((\text{Spec}A)_{\text{Zar}}, H^{n+2}(\mathbb{Z}/(n + 2))) \]
which means that $\text{Ker} \Psi_K$ is isomorphic to the Cokernel of the map
\[ \bigoplus_{v \in P} H^{n+1}(k(v), \mathbb{Z}/(n + 1)) \longrightarrow H^n(k(x), \mathbb{Z}/(n)) \]
So, we must prove the surjectivity of this last map. Indeed, the Gersten-Quillen complex ([11], Theorem A)
\[ K_{n+2}(A)/\ell \longrightarrow K_{n+2}(K)/\ell \longrightarrow \bigoplus_{v \in P} K_{n+1}k(v)/\ell \longrightarrow K_nk(x)/\ell \longrightarrow 0 \]
is exact. On the other hand, we have the following commutative diagram
\[ \begin{array}{ccc}
K_{n+2}K/\ell & \longrightarrow & \bigoplus_{v \in P} K_{n+1}k(v)/\ell \\
\downarrow & & \downarrow \\
H^{n+2}(K, \mathbb{Z}/(n + 2)) & \longrightarrow & \bigoplus_{v \in P} H^{n+1}(k(v), \mathbb{Z}/(n + 1)) \rightarrow H^n(k(x), \mathbb{Z}/(n))
\end{array} \]
where the right vertical isomorphism comes from Kato conjecture. This yields that the map
\[ \bigoplus_{v \in P} H^{n+1}(k(v), \mathbb{Z}/(n + 1)) \longrightarrow H^n(k(x), \mathbb{Z}/(n)) \]
is surjective and we are done. \[ \square \]

**Remark 4.4**

1) The case $n = 0$, implies the following exact sequence
\[ 0 \longrightarrow H^3(K, \mathbb{Z}/(2)) \longrightarrow \bigoplus_{v \in P} H^2(k(v), \mathbb{Z}/(1)) \longrightarrow \mathbb{Z}/\ell \longrightarrow 0 \]
already obtained by Saito [12].

2) The case $n = 1$ leads to the following exact sequence
\[ 0 \longrightarrow H^4(K, \mathbb{Z}/(3)) \longrightarrow \bigoplus_{v \in P} H^3(k(v), \mathbb{Z}/(2)) \longrightarrow \mathbb{Z}/\ell \longrightarrow 0 \]
which is considered in [1].

3) The case $n = 2$. Let $A = \mathbb{F}_p((t))((u))[X, Y]$ of fraction field $K$. Then the following Hasse principle complex for $K$
\[ 0 \longrightarrow H^5(K, \mathbb{Z}/(4)) \longrightarrow \bigoplus_{v \in P} H^4(k(v), \mathbb{Z}/(3)) \longrightarrow \mathbb{Z}/\ell \longrightarrow 0 \]
is exact.

4) The case $n = 0$ has been used by Saito to study the class field theory of curves over one dimensional local field. Recently, Yoshida [16] provided an alternative approach which includes the equal characteristic case. In a forthcoming paper I use the case $n = 1$ to investigate the study of class field theory of curves over 2-dimensional local field.
5. The case \( n=1 \)

Let \( A \) denote a 2-dimensional complete normal local ring of positive characteristic whose residue field is one-dimensional local field. The aim of this section is to complete the duality (3.5) for \( i \geq 1 \). We prove the following.

**Theorem 5.1**

For all \( \ell \) prime to \( \text{char}(A) \), the isomorphism

\[
(5.1) \quad H^5(X, \mathbb{Z}/\ell(3)) \simeq \mathbb{Z}/\ell
\]

and the perfect pairing

\[
(5.2) \quad H^i(X, \mathbb{Z}/\ell) \times H^{5-i}(X, \mathbb{Z}/\ell(3)) \longrightarrow H^5(X, \mathbb{Z}/\ell(3)) \simeq \mathbb{Z}/\ell
\]

hold for all \( i \in \{0, ..., 5\} \).

**Proof.** The first isomorphism is given by (3.4). Next, we proceed to the second part of the theorem. As in the proof of Theorem 3.1, we consider the strict henselization \( A^\text{sh} \) of \( A \) (with respect to the separably closed extension \( k_s \) of \( k \)) at \( x \). If \( x' \) is the maximal ideal of \( A^\text{sh} \), we recall that we denote \( X' = \text{Spec} A^\text{sh}\setminus\{x'\} \) and we consider the spectral sequence (3.3). The filtration of the group \( H^i(X, \mathbb{Z}/\ell(3)) \) is

\[
H^i(X, \mathbb{Z}/\ell(3)) = E_0^i \supseteq E_1^i \supseteq E_2^i \supseteq 0
\]

where the quotients are given by

\[
\frac{E_0^i}{E_1^i} \simeq E_{\infty}^{0,i} \simeq \ker d_2^{0,i}
\]

\[
\frac{E_1^i}{E_2^i} \simeq E_{\infty}^{1,i-1} \simeq E_2^{1,i-1}, \quad \text{and}
\]

\[
E_2^i \simeq E_{\infty}^{2,i-2} \simeq \text{Coker } d_2^{0,i-1}
\]

The same computation is true for the group \( H^{5-i}(X, \mathbb{Z}/\ell) \) and the filtration

\[
H^{5-i}(X, \mathbb{Z}/\ell) = E_0^{5-i} \supseteq E_1^{5-i} \supseteq E_2^{5-i} \supseteq 0
\]

by considering the spectral sequence (3.4), \( j=0 \).

Now, combining duality (2.2) and duality (3.2) we observe that the group \( E_0^{0,j} \) is dual to the group \( 'E_2^{2,3-j} \) and the group \( E_2^{1,j} \) is dual to the group \( 'E_2^{1,3-j} \) for all \( 0 \leq j \leq 3 \). On the other hand, we have the commutative diagram

\[
\begin{array}{c}
\begin{array}{ccc}
H^0(k, H^i(X', \mathbb{Z}/\ell(3))) & \times & H^2(k, H^{3-i}(X', \mathbb{Z}/\ell)) \rightarrow H^2(k, \mathbb{Z}/\ell(1)) \rightarrow \mathbb{Z}/\ell \\
\downarrow & & \downarrow \\
H^2(k, H^{i-1}(X', \mathbb{Z}/\ell(3))) & \times & H^0(k, H^{4-i}(X', \mathbb{Z}/\ell)) \rightarrow H^2(k, \mathbb{Z}/\ell(1)) \rightarrow \mathbb{Z}/\ell
\end{array}
\end{array}
\]

given by the cup products and the spectral sequence (3.4), using the same argument as ([1], diagram 46). We infer that \( \text{Coker } d_2^{0,i-1} \) is the dual of \( \text{Ker}'d_2^{0,5-i} \) and \( \text{Ker} d_2^{0,i} \) is the dual of \( \text{Coker}'d_2^{0,2d-i} \) where \( 'd_2^{p,q} \) is the boundary map for the spectral sequence ((3.4), \( j=0 \))

\[
'E_2^{p,q} = H^p(k, H^q(X', \mathbb{Z}/\ell)) \rightarrow H^{p+q}(X, \mathbb{Z}/\ell).
\]

This is illustrated by the following diagram:
\[
H^1 X, \mathbb{Z}/(\ell(3)) = E^3_0
\]

where each pair of groups which are combined by \(\leftrightarrow\) consists of a group and its dual group.

We begin by calculating the dual group of \(E^1_1\), using the following commutative diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & E^1_2 & \rightarrow & E^1_1 & \rightarrow & E^1_0/E^1_2 & \rightarrow & 0 \\
\big\downarrow \wr & & \big\downarrow \wr & & \big\downarrow \wr & & \big\downarrow \wr & & \\
0 & \rightarrow & \left(\frac{E^2_{2d+1}}{E^2_{2d+1}}\right)^\vee & \rightarrow & \left(\frac{E^2_{2d}}{E^2_{2d}}\right)^\vee & \rightarrow & \left(\frac{E^2_{2d-1}}{E^2_{2d}}\right)^\vee & \rightarrow & 0
\end{array}
\]

where \((M)^\vee\) denotes the dual \(\text{Hom}(M, \mathbb{Z}/\ell)\) for any \(\mathbb{Z}/\ell\)-module \(M\) and where the left and right vertical isomorphisms are explained by the previous diagram. This yields that

\[
E^i_1 \simeq \left(\frac{E^5_{5-i}}{E^5_{5-i}}\right)^\vee
\]

Finally, the required duality between \(H^1 X, \mathbb{Z}/(\ell(d+1)) = E^3_0\) and \(E^5_{5-i} = H^5(\mathbb{Z}/\ell)\) follows from the following commutative diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & E^1_1 & \rightarrow & E^1_0 & \rightarrow & E^1_0/E^1_1 & \rightarrow & 0 \\
\big\downarrow \wr & & \big\downarrow \wr & & \big\downarrow \wr & & \big\downarrow \wr & & \\
0 & \rightarrow & \left(\frac{E^5_{5-i}}{E^5_{5-i}}\right)^\vee & \rightarrow & \left(\frac{E^5_{5-i}}{E^5_{5-i}}\right)^\vee & \rightarrow & \left(\frac{E^5_{5-i}}{E^5_{5-i}}\right)^\vee & \rightarrow & 0
\end{array}
\]

where the right vertical isomorphism is given by (5.3) and the left vertical isomorphism is the isomorphism (5.4).

\section*{References}

[1] Colliot-Thélène, J.L., Sansuc, J.J., Soulé, C. Torsion dans le groupe de Chow de codimension deux. Duke Math. Journal vol. 50 No.3 pp763-801 (1983)

[2] Draouil, B. Cohomological Hasse principle for the ring \(\mathbb{F}_p((t))[X,Y]\), Bull. Belg. Math. Soc. Simon Stevin 11, no. 2 (2004), 181-190

[3] Draouil, B., Douai, J.C. Sur l’arithmétique des anneaux locaux de dimension 2 et 3, Journal of Algebra 213 (1999), 499-512.

[4] Fujiwara, K. Theory of Tubular Neighborhood in Etale Topology Duke Math.J.80 (1995), 15-57.

[5] Gabber, O. Lecture at IHES, on March 1981.

[6] Kato, K A Hasse principle for two-dimensional global fields, J.reine angew.Math.366 (1986), 143-183.

[7] Kato, K. A generalisation of local class field theory by using K-theory II, J.Fac.Sci.Univ. Tokyo, 27 (1980), 603-683.

[8] Matsumi, K. Thesis, Arithmetic of three-dimensional complete regular local rings of positive characteristics Tōhoku University, Japan 1999.

[9] Milne, J.S. Étale Cohomology, Princeton University Press, Princeton 1980.

[10] Nagata, M. Local rings,Tracts in Mathematics Number 13, Intersciences Publishers. New York 1962

[11] Panin, I. The equi-characteristic case of the Gersten conjecture, preprint (2000), available on the K-theory server.
[12] Saito, S. Class field Theory for two-dimensional local rings Galois groups and their representations, Kinokuniya-North Holland Amsterdam, vol 12 (1987), 343-373
[13] Serre,J.P. Cohomologie Galoissiene ,L.N.M 5 Berlin-Heidelberg-New York 1965.
[14] SGA 4 Théorie des Topos et Cohomologie étale des Schémas. Lecture Notes in Math. vol.305, Springer-Verlag Berlin,Heidelberg,New York.
[15] SGA 5 Cohomologie l-adique et Fonctions L . Lecture Notes in Math. vol.589, Springer-Verlag Berlin,Heidelberg,New York
[16] Yoshida. Finitness theorems in the class field theory of varieties over local fields, Journal of Number Theory 101 (2003) 138-150

Département de Mathématiques, Faculté des Sciences de Bizerte, 7021 Jarzouna

TUNISIA

E-mail address: Belgacem.Draouil@fsb.rnu.tn