On the Theory of Capacities on Locally Compact Spaces and its Interaction with the Theory of Balayage

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Abstract
The paper deals with the theory of inner (outer) capacities on locally compact spaces with respect to general function kernels, the main emphasis being placed on the establishment of alternative characterizations of inner (outer) capacities and inner (outer) capacitary measures for arbitrary sets. The analysis is substantially based on the close interaction between the theory of capacities and that of balayage. As a by-product, we provide a rigorous justification of Fuglede’s theories of inner and outer capacitary measures and capacitability (Acta Math., 1960). The results obtained are largely new even for the logarithmic, Newtonian, Green, $\alpha$-Riesz, and $\alpha$-Green kernels.

Keywords
Inner and outer capacities and capacitary measures · Inner and outer balayage · Capacitability · Perfect kernels · Maximum principles · Principle of positivity of mass

Mathematics Subject Classification (2010) 31C15

1 Introduction
The classical theory of capacities of compact sets on $\mathbb{R}^n$, $n \geq 2$, was initiated by Wiener [31], and further developed by Frostman [14], Riesz [24], and de la Vallée-Poussin [30]. The main tool is here the vague topology, for the energy is vaguely lower semicontinuous on positive measures, whereas the set of admissible measures in the principal minimum energy problem is vaguely compact.

The modern theory of inner and outer capacities of arbitrary sets was originated in the pioneering papers by Cartan [6, 7]. His approach is mainly based on the discovery that the cone of all positive measures on $\mathbb{R}^n$, $n \geq 3$, of finite Newtonian energy is complete in the strong topology, determined by the energy norm.
The systematic use of both the strong and the vague topologies enabled Fuglede [16] to develop a theory of inner (outer) capacities and inner (outer) capacitary measures on a locally compact space $X$, generalizing Cartan’s theory to quite a large class of positive definite kernels $\kappa$ on $X$ satisfying the principle of consistency.

Similar ideas were further explored by Ohtsuka [26] for vector-valued Radon measures $((\mu_i))_{i \in I}$ on $X$, influenced by external fields; see also relevant papers by the present author [33–36], where the measures $((\mu_i))_{i \in I}$ were even allowed to be infinite dimensional.

In this work we continue the study of inner (outer) capacities and inner (outer) capacitary measures of sets $A \subset X$ for suitable kernels $\kappa$ on $X$, the main emphasis being placed on the establishment of alternative characterizations of these concepts for arbitrary $A$. (In the existing literature on this subject, mainly pertaining to the classical kernels on $\mathbb{R}^n$, one usually imposed upon $A$ some topological restrictions, see e.g. [1, p. 138], [10, p. 241–243], [22, p. 138–139], [29, 31].)

The alternative characterizations of inner capacities and inner capacitary measures thereby obtained are given in particular by Theorems 4.1–4.3, 7.2, 9.1–9.3 and Corollary 9.1. The analysis performed is substantially based on the close interaction between the theory of capacities and that of balayage (sweeping). Concerning the theory of balayage on a locally compact space $X$ with respect to a suitable kernel $\kappa$, see the author’s recent paper [40]; see also Sections 3, 11 below for further relevant results, in particular Theorems 3.1, 11.1 for a number of alternative characterizations of inner and outer swept measures.

We also investigate the convergence of inner capacities, inner capacitary measures, and their potentials for monotone families of sets (Theorems 8.1–8.3).

Based on the theory of inner capacities and inner capacitary measures thereby developed, we further carry out a similar analysis for outer capacities and outer capacitary measures. See in particular alternative characterizations of outer capacities and outer capacitary measures described in Theorems 10.4–10.7, as well as convergence results for monotone families of sets presented in Theorems 10.2 and 10.3. 1

To provide an example of the results obtained, 2 consider a perfect kernel $\kappa$ satisfying the first and second maximum principles (for definitions see Section 2), and let $\mathcal{E}^+$ stand for the cone of all positive (scalar) Radon measures $\mu$ on $X$ of finite energy:

$$\kappa(\mu, \mu) := \int \kappa(x, y) d(\mu \otimes \mu)(x, y) = \int \kappa \mu d\mu < \infty,$$

$\kappa \mu(\cdot) := \int \kappa(\cdot, y) d\mu(y)$ being the potential of $\mu$. Given arbitrary $A \subset X$, we denote by $\mathcal{E}^+_A$ the cone of all $\mu \in \mathcal{E}^+$ concentrated on $A$, by $\mathcal{E}^+_A$ the closure of $\mathcal{E}^+_A$ in the strong topology on $\mathcal{E}^+$, determined by the energy norm $\|\mu\| := \sqrt{\kappa(\mu, \mu)}$, and by $\mathcal{E}^+_A$ the class of all $\mu \in \mathcal{E}^+_A$ with the property $\kappa \mu \leq 1$ $\mu$-a.e. on $X$. If moreover the set $A$ is of finite inner capacity $c(A)$, there exists the unique inner equilibrium measure $\gamma_A$, introduced as the (unique) solution to the minimum energy problem over the class

$$\Gamma^+_A := \{ \mu \in \mathcal{E}^+ : \kappa \mu \geq 1 \text{ n.e. on } A \},$$

where ‘n.e. on $A$’ means that the inequality holds everywhere on $A$ except for $E \subset A$ with $c(E) = 0$ (see Section 6).

1In the investigation of these outer concepts we must however impose upon $X$ and $A$ certain topological restrictions, e.g. that the space $X$ be second-countable while the sets $A \subset X$ be Borel.

2In this work we are only concerned with the characterizations of capacities expressed in potential theoretical terms, leaving aside those given in terms of transfinite diameters, Chebyshev constants, extremal lengths, or Dirichlet integrals of suitable functions (cf. e.g. [8, 13, 15, 25, 27, 28, 32]).
Theorem 1.1 (see Section 9) The inner equilibrium measure $\gamma_A$ can alternatively be characterized by means of any one of the following (equivalent) assertions (a)–(f):

(a) $\gamma_A$ is the unique measure of minimum potential in the class $\Gamma^+_A$, that is, $\gamma_A \in \Gamma^+_A$ and
$$\kappa \gamma_A = \min_{\mu \in \Gamma^+_A} \kappa \mu \text{ on } X.$$

(b) $\gamma_A$ is the unique measure maximizing $G(\mu) := 2 \mu(X) - \|\mu\|^2$ over $E'_A$, that is, $\gamma_A \in E'_A$ and
$$G(\gamma_A) = \max_{\mu \in E'_A} G(\mu) \quad (= c_*(A)).$$

(c) $\gamma_A$ is the unique measure maximizing $\mu(X)$ over $\hat{E}'_A$, that is, $\gamma_A \in \hat{E}'_A$ and
$$\gamma_A(X) = \max_{\mu \in \hat{E}'_A} \mu(X) \quad (= c_*(A)).$$

(d) $\gamma_A$ is the unique measure in $E'_A$ satisfying the two inequalities
$$\gamma_A(X) \geq \|\gamma_A\|^2 \geq c_*(A).$$

(e) $\gamma_A$ is the unique measure in $E'_A$ having the property
$$\kappa \gamma_A = 1 \text{ n.e. on } A.$$

(f) $\gamma_A$ is the unique measure in $E^+_A$ satisfying any one of the three limit relations
$$\gamma_K \to \gamma_A \text{ strongly in } E^+_A, \quad \gamma_K \to \gamma_A \text{ vaguely in } E^+_A, \quad \kappa \gamma_K \uparrow \kappa \gamma_A \text{ pointwise on } X,$$
where $K$ ranges over the upward directed set $\mathcal{C}_A$ of all compact subsets of $A$, whereas $\gamma_K$ denotes the only measure in $E'^+_K$ with $\kappa \gamma_K = 1 \text{ n.e. on } K$.

Remark 1.1 Being valid for arbitrary $A \subset X$, Theorem 1.1 seems to be largely new even for the Green kernels associated with the Laplacian on Greenian sets in $\mathbb{R}^n$, $n \geq 2$ (in particular, for the Newtonian kernel on $\mathbb{R}^n$, $n \geq 3$), as well as for the $\alpha$-Riesz kernels $|x - y|^{\alpha - n}$ and the associated $\alpha$-Green kernels on $\mathbb{R}^n$, where $\alpha < 2 \leq n$.

Remark 1.2 In the particular case where $E'^+_A$ is strongly closed (which occurs e.g. if the set $A$ is closed or just quasiclosed, cf. Lemma 3.3), (b)–(e) in Theorem 1.1 remain valid with $E'_A$ replaced throughout by $E'^+_A$. Moreover, such refined characterizations of $\gamma_A$ actually hold true even for any perfect kernel $\kappa$, except for (e) where $\kappa$ is additionally required to satisfy Frostman’s maximum principle (cf. Theorem 7.2).

Remark 1.3 If moreover $X$ is second-countable (or, more generally, $\sigma$-compact and perfectly normal) while $A$ is Borel, then Theorem 1.1 still holds with the inner concepts replaced throughout by the outer concepts (for details see Section 10).

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3It follows that there is in general no $\nu \in \hat{E}'_A$ with $\kappa \nu = 1 \text{ n.e. on } A$ (unless of course $E'^+_A$ is strongly closed, cf. Remark 1.2). Indeed, let $A := B_r := \{|x| < r\} \subset \mathbb{R}^n$, where $n \geq 3$ and $r \in (0, \infty)$, and let $\kappa(x, y)$ be the Newtonian kernel $|x - y|^{2-n}$. If there were $\nu \in \hat{E}'_{B_r}$, $\nu \in E'^+_B$ with $\kappa \nu = 1 \text{ n.e. on } B_r$, then, according to (e), $\nu$ would necessarily serve as $\gamma_{B_r}$. But $\gamma_{B_r}$ is the positive measure of total mass $r^{n-2}$, uniformly distributed over the sphere $\{|x| = r\}$; therefore, $\gamma_{B_r} \notin E'^+_B$. Contradiction.
2 On the Theory of Potentials on Locally Compact Spaces

In this section we review some basic facts of the theory of potentials on a locally compact (Hausdorff) space $X$, to be used throughout the paper. We denote by $\mathcal{M}$ the linear space of all (real-valued scalar Radon) measures $\mu$ on $X$ equipped with the vague topology of pointwise convergence on the class $C_0(X)$ of all continuous functions $f : X \to \mathbb{R}$ of compact support, and by $\mathcal{M}^+$ the cone of all positive $\mu \in \mathcal{M}$.

Lemma 2.1 (see [4, Section IV.1, Proposition 4]) For any lower semicontinuous (l.s.c.) function $\psi : X \to [0, \infty]$, the mapping $\nu \mapsto \int \psi \, d\nu$ is vaguely l.s.c. on $\mathcal{M}^+$, the integral here being understood as an upper integral.

For any set $A \subset X$, denote by $\mathcal{M}^+_A$ the cone of all $\mu \in \mathcal{M}^+$ concentrated on $A$, which means that $A^c := X \setminus A$ is locally $\mu$-negligible, or equivalently that $A$ is $\mu$-measurable and $\mu = \mu|_A$, $\mu|_A := 1_A \cdot \mu$ being the trace of $\mu$ to $A$ [12, Section IV.14.7]. (Note that for $\mu \in \mathcal{M}^+_A$, the indicator function $1_A$ of $A$ is locally $\mu$-integrable.) The total mass of $\mu \in \mathcal{M}^+_A$ is $\mu(X) = \mu_+(A)$, $\mu_+(A)$ and $\mu^*(A)$ denoting the inner and the outer measure of $A$, respectively. If moreover $A$ is closed, or if $A^c$ is contained in a countable union of sets $Q_j$ with $\mu^*(Q_j) < \infty$, then for any $\mu \in \mathcal{M}^+_A$, $A^c$ is $\mu$-negligible, that is, $\mu^*(A^c) = 0$. In particular, if $A$ is closed, $\mathcal{M}^+_A$ consists of all $\mu \in \mathcal{M}^+$ with support $S(\mu) \subset A$, cf. [4, Section III.2.2].

Given $A \subset X$, denote by $\mathcal{C}_A$ the upward directed set of all compact subsets $K$ of $A$, where $K_1 \subseteq K_2$ if and only if $K_1 \subset K_2$. If a net $(x_K)_{K \in \mathcal{C}_A} \subset Y$ converges to $x_0 \in Y$, $Y$ being a topological space, then we shall indicate this fact by writing $x_K \to x_0$ in $Y$ as $K \uparrow A$.

Lemma 2.2 (cf. [16, Lemma 1.2.2]) For any l.s.c. function $\psi : X \to [0, \infty]$, any measure $\mu \in \mathcal{M}^+$, and any $\mu$-measurable set $A \subset X$,

$$\int \psi \, d\mu|_A = \lim_{K \uparrow A} \int \psi \, d\mu|_K.$$

A kernel $\kappa$ on $X$ is meant to be a symmetric, l.s.c. function $\kappa : X \times X \to (-\infty, \infty]$ that is positive ($\geq 0$) unless the space $X$ is compact. Given (signed) measures $\mu, \nu \in \mathcal{M}$, define the potential and the mutual energy by

$$\kappa(\mu, \nu) := \int \kappa(x, y) \, d\mu(y), \quad \kappa(\mu, \nu) := \int \kappa(x, y) \, d(\mu \otimes \nu)(x, y),$$

respectively, provided the right-hand side is well defined as a finite number or $\pm \infty$ (for more details see [16, Section 2.1]). For $\mu = \nu$, $\kappa(\mu, \nu)$ defines the energy $\kappa(\mu, \mu)$ of $\mu$. In particular, if the measures are positive, then $\kappa(\mu, \mu)$, resp. $\kappa(\mu, \nu)$, is always well defined and represents a l.s.c. function of $(x, \mu) \in X \times \mathcal{M}^+$, resp. of $(\mu, \nu) \in \mathcal{M}^+ \times \mathcal{M}^+$ (the principle of descent [16, Lemma 2.2.1], cf. Lemma 2.1 and footnote 5).

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4If the latter holds, $A^c$ is said to be $\mu$-$\sigma$-finite [12, Section IV.7.3]. This in particular occurs if the measure $\mu$ is bounded (that is, with $\mu(X) < \infty$), or if the locally compact space $X$ is $\sigma$-compact (that is, representable as a countable union of compact sets [2, Section I.9, Definition 5]).

5The latter case can often be reduced to the former by introducing a kernel $\kappa' := \kappa + q \geq 0$, where $q \in \mathbb{R}$, which is always possible, for a l.s.c. function on a compact space is lower bounded.
Lemma 2.3  For any \( A \subset X \) and any \( \mu \in \mathcal{M}_A^+ \),
\[
\kappa(\mu, \mu) = \lim_{K \uparrow A} \kappa(\mu|_K, \mu|_K).
\]

Proof  If \( \kappa \geq 0 \), this follows from Lemma 2.2 noting that \((\mu \otimes \mu)|_{K \times K} = \mu|_K \otimes \mu|_K\), whereas the remaining case of compact \( X \) is treated as described in footnote 5.

From now on we shall always assume that a kernel \( \kappa \) is pseudo-positive \([16, \text{p.} 150]\), which means that \( \kappa(\mu, \mu) \geq 0 \) for all \( \mu \in \mathcal{M}^+ \), or equivalently, for all \( \mu \in \mathcal{M}^+ \) of compact support, cf. Lemma 2.3. A (pseudo-positive) kernel \( \kappa \) is said to be strictly pseudo-positive if \( \kappa(\mu, \mu) > 0 \) for all \( \mu \in \mathcal{M}^+ \), \( \mu \neq 0 \), of compact support.

Denote by \( \mathcal{E} \) the set of all (signed) \( \mu \in \mathcal{M} \) of finite energy \( \kappa(\mu, \mu) \), and by \( \mathcal{E}^+ \) its subset consisting of all positive \( \mu \in \mathcal{E} \). Given \( A \subset X \), also denote
\[
\mathcal{E}_A^+ := \mathcal{E} \cap \mathcal{M}_A^+, \quad \mathcal{E}_A^{++} := \{ \mu \in \mathcal{E}_A^+ : \mu(X) = 1 \}.
\]

and\(^6\)
\[
\omega(A) := \inf_{\mu \in \mathcal{E}_A^{+++}} \kappa(\mu, \mu) \in [0, \infty].
\]

The value
\[
c_*(A) := 1/\omega(A) \in [0, \infty]
\]
is said to be the (Wiener) inner capacity of the set \( A \) (with respect to the kernel \( \kappa \)).

It is seen from Lemma 2.3 that \( c_*(A) \) would be the same if the admissible measures in definition (2.1) were required to be of compact support. This in turn implies that\(^7\)
\[
c_*(A) = \lim_{K \uparrow A} c(K).
\]

Lemma 2.4 (cf. [16, Lemma 2.3.1])  For any \( A \subset X \),
\[
c_*(A) = 0 \iff \mathcal{E}_A^{+++} = \{0\} \iff \mathcal{E}_K^{+++} = \{0\} \text{ for all } K \in \mathcal{C}_A.
\]

Lemma 2.5  Given \( \mu \in \mathcal{E}^+ \), let \( A \subset X \) be a \( \mu \)-measurable and \( \mu \)-\( \sigma \)-finite set with \( c_*(A) = 0 \). Then \( A \) is \( \mu \)-negligible.

Proof  As \( A \) is \( \mu \)-measurable and \( \mu \)-\( \sigma \)-finite, \( \mu^*(A) = 0 \) will follow once we show that \( \mu(K) = 0 \) for every compact \( K \subset A \), but this is obvious in view of Lemma 2.4.

To avoid trivialities, assume throughout that \( \mathcal{E}^+ \neq \{0\} \), or equivalently (cf. Lemma 2.4)
\[
c(X) > 0;
\]
then the kernel \( \kappa \) is necessarily not identically infinite: \( \kappa \neq +\infty \).

A proposition \( \mathcal{P}(x) \) involving a variable point \( x \in X \) is said to hold nearly everywhere (n.e.) on \( A \subset X \) if \( c_*(E) = 0 \), \( E \) being the set of all \( x \in A \) where \( \mathcal{P}(x) \) fails. Replacing here \( c_*(E) \) by \( c^*(E) \) we arrive at the concept of quasi-everywhere (q.e.) on \( A \subset X \).

\(^6\)As usual, the infimum over the empty set is taken to be \(+\infty\). We also agree that \( 1/(+\infty) = 0 \) and \( 1/0 = +\infty \).

\(^7\)We write \( c(A) \) in place of \( c_*(A) \) if \( A \) is capacitable, that is, if \( c_*(A) = c^*(A) \), where \( c^*(A) \) is the outer capacity of \( A \), defined as \( \inf c_*(D) \), \( D \) ranging over all open sets containing \( A \). This occurs, for instance, if \( A \) is compact [16, Lemma 2.3.4] or open.
Assume for a moment that \( A = K \subset X \) is a compact set with \( w(K) < \infty \); then the infimum in Eq. 2.1 is an actual minimum, the class \( \mathcal{E}^+ \) being vaguely compact. A measure \( \lambda_K \in \mathcal{E}^+ \) of minimal energy \( \kappa(\lambda_K, \lambda_K) = w(K) \) is said to be a capacitary distribution of unit mass on \( K \). If moreover \( w(K) > 0 \) (which in particular occurs if the kernel is strictly pseudo-positive), one can introduce the (normalized) measure \( \gamma_K := \lambda_K/w(K) \in E^+_K \), called a capacitory measure on the (compact) set \( K \). A capacitory measure \( \gamma_K \) is in general non-unique, and it has the following properties [16, Theorem 2.4]:

\[
\gamma_K(X) = \kappa(\gamma_K, \gamma_K) = c(K),
\]

\[
\kappa \gamma_K \geq 1 \text{ n.e. on } K,
\]

\[
\kappa \gamma_K \leq 1 \text{ on } S(\gamma_K),
\]

\[
\kappa \gamma_K = 1 \text{ } \gamma_K\text{-a.e. on } X.
\]

But if \( A \subset X \) is noncompact, the class \( \mathcal{E}^+ \) is no longer vaguely compact, and the infimum in Eq. 2.1 is in general not attained among the admissible measures. In other words, for noncompact \( A \), minimum energy problem (2.1) is in general unsolvable.

### 2.1 Potential-Theoretic Principles

A kernel \( \kappa \) is said to be positive definite if the energy \( \kappa(\mu, \mu) \) is \( \geq 0 \) for all \( \mu \in \mathfrak{M} \) (whenever it is well defined); then \( \mathcal{E} \) is a pre-Hilbert space with the inner product \( \langle \mu, \nu \rangle := \kappa(\mu, \nu) \) and the energy seminorm \( \|\mu\| := \sqrt{\kappa(\mu, \mu)} \) [16, Section 3.1]. The topology on \( \mathcal{E} \) determined by this seminorm is said to be strong. A positive definite kernel \( \kappa \) is said to satisfy the energy principle (or to be strictly positive definite) if the seminorm \( \|\cdot\| \) is a norm. (Concerning various potential-theoretic principles, see e.g. the comprehensive work by Ohtsuka [26].)

If a kernel \( \kappa \) is positive definite, the potential of any \( \mu \in \mathcal{E} \) is (well defined and) finite quasi-everywhere on \( X \) [16, p. 165, Corollary]; furthermore, the potentials of any two given \( \mu, \nu \in \mathcal{E} \) are equal nearly everywhere on \( X \) if and only if \( \|\mu - \nu\| = 0 \) [16, Lemma 3.2.1]. Therefore, for a strictly positive definite kernel \( \kappa \),

\[
\kappa \mu = \kappa \nu \text{ n.e. on } X \iff \mu = \nu.
\]

A positive definite kernel \( \kappa \) is said to be consistent if every strong Cauchy net \( (\mu_s) \subset \mathcal{E}^+ \) converges strongly to any of its vague limit points \( \mu \) [16, Section 3.3].10 (Such \( \mu \) do exist e.g. if the kernel \( \kappa \) is strictly positive definite [16, Lemma 2.5.1].)

A positive definite kernel \( \kappa \) is said to be perfect if the energy and consistency principles are both fulfilled [16, Section 3.3]. It follows that for a perfect kernel \( \kappa \), the cone \( \mathcal{E}^+ \) is

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8If Frostman’s maximum principle holds (see Section 2.1), the capacitory measures \( \gamma_K \) are also called equilibrium measures on \( K \), for their potentials take the constant value 1 for nearly all \( x \in K \). (In fact, it follows from Eq. 2.6 that \( \kappa \gamma_K \leq 1 \) on \( X \), which together with Eq. 2.5 gives \( \kappa \gamma_K = 1 \) n.e. on \( K \).)

9Any positive definite kernel is of course pseudo-positive, but not the other way around. Except for Sections 2.2 and 4, in this study we shall mainly focus on positive definite kernels, thereby enjoying the advantages of a pre-Hilbert structure on the space \( \mathcal{E} \).

10Since the space \( \mathfrak{M} \) equipped with the vague topology does not necessarily satisfy the first axiom of countability, the vague convergence cannot in general be described in terms of sequences. We follow Moore and Smith’s theory of convergence [23], based on the concept of nets. However, if a locally compact space \( X \) is second-countable, then the space \( \mathfrak{M} \) is first-countable [39, Lemma 4.5], and the use of nets in \( \mathfrak{M} \) may often be avoided.
complete in the induced strong topology, and moreover every strong Cauchy net \((\mu_s)\) in \(\mathcal{E}^+\) converges to the same unique limit \(\mu_0 \in \mathcal{E}^+\) both strongly and vaguely. In fact, all the vague limit points of such \((\mu_s)\) must coincide, being equal to the (unique) strong limit \(\mu_0\); since the vague topology on \(\mathcal{M}\) is Hausdorff, the same \(\mu_0\) must also serve as the vague limit of \((\mu_s)\) (see [2, Section I.9.1, Corollary]).

The following two maximum principles will often be used in the sequel. A kernel \(\kappa\) is said to satisfy Frostman’s maximum principle (= the first maximum principle) if for any \(\mu \in \mathcal{M}^+\) with \(\kappa \mu \leq 1\) on \(S(\mu)\), the same inequality holds on all of \(X\);\(^{11}\) and it is said to satisfy the domination principle (= the second maximum principle) if for any \(\mu, \nu \in \mathcal{E}^+\) with \(\kappa \mu \leq \kappa \nu\) \(-a.e.\), the same inequality is fulfilled everywhere on \(X\).

### 2.2 Principle of Positivity of Mass

The principle of positivity of mass goes back to Deny (see e.g. [9]). In the generality stated in Theorem 2.1, it seems to be new.

**Theorem 2.1** Assume a locally compact space \(X\) is \(\sigma\)-compact, a kernel \(\kappa\) is strictly pseudo-positive, and the first and second maximum principles are fulfilled. For any \(\mu, \nu \in \mathcal{E}^+\) with \(\kappa \mu \leq \kappa \nu\) \(-a.e.\) on \(X\), then

\[
\mu(X) \leq \nu(X). \tag{2.9}
\]

**Proof** Assume first that \(\kappa \geq 0\). The space \(X\) being \(\sigma\)-compact, there is an increasing sequence \((K_j)\) of compact sets with the union \(X\). The kernel being strictly pseudo-positive, for each \(j\) large enough, we have \(0 < c(K_j) < \infty\) (cf. (2.2) and (2.3)), and there is therefore an equilibrium measure \(\gamma_{K_j}\) on the (compact) set \(K_j\) (cf. footnote 8). By the countable subadditivity of inner capacity on universally measurable sets [16, Lemma 2.3.5],

\[
\kappa \gamma_{K_j} = 1 = \kappa \gamma_{K_{j+1}} \text{ n.e. on } K_j,
\]

hence \(\gamma_{K_j}\)-a.e. (Lemma 2.5), and so \(\kappa \gamma_{K_j} \leq \kappa \gamma_{K_{j+1}}\) on all of \(X\), by the domination principle. Thus the sequence \((\kappa \gamma_{K_j})\) increases pointwise on \(X\) to some function \(g \geq 0\), and moreover

\[
g = 1 \text{ n.e. on } X, \tag{2.10}
\]

again by the countable subadditivity of inner capacity on universally measurable sets.

Given \(\mu, \nu \in \mathcal{E}^+\) with \(\kappa \mu \leq \kappa \nu\) n.e. on \(X\), the same inequality is fulfilled \(\gamma_{K_j}\)-a.e. (Lemma 2.5); therefore, by Fubini’s theorem,

\[
\int \kappa \gamma_{K_j} \, d\mu = \int \kappa \mu \, d\gamma_{K_j} \leq \int \kappa \nu \, d\gamma_{K_j} = \int \kappa \gamma_{K_j} \, d\nu.
\]

By the monotone convergence theorem [4, Section IV.1, Theorem 3] applied to the increasing sequence \((\kappa \gamma_{K_j})\) of positive functions with the upper envelope \(g\) (cf. Eq. 2.10), we obtain

\[
\int g \, d\mu \leq \int g \, d\nu,
\]

whence Eq. 2.9. For, \(g = 1\) \(-\mu\)-a.e. as well as \(-\nu\)-a.e. (Lemma 2.5), \(X\) being \(\sigma\)-compact.

It remains to consider the case of compact \(X\); then \(0 < c(X) < \infty\), and there is therefore an equilibrium measure \(\gamma_X\) on the whole of \(X\). Moreover, \(\kappa \gamma_X = 1\) holds true n.e. on \(X\),

\(^{11}\)For the purposes of the present paper, it is enough to require this only for \(\mu\) of finite energy.
hence $\mu$-a.e. as well as $\nu$-a.e. (Lemma 2.5). Likewise, $\kappa \mu \leq \kappa \nu \gamma_X$-a.e., the measure $\gamma_X$ being of finite energy. This altogether gives
\[
\mu(X) = \int \kappa \gamma_X \, d\mu = \int \kappa \mu \, d\gamma_X \leq \int \kappa \nu \, d\gamma_X = \int \kappa \gamma_X \, d\nu = \nu(X)
\]
as desired. \hfill \square

**Remark 2.1** It is seen from the above proof that in the case of compact $X$, the domination principle is unnecessary for the validity of Theorem 2.1.

### 2.3 Examples

The $\alpha$-Riesz kernels $|x - y|^{\alpha - n}$ of order $\alpha \in (0, 2]$, $\alpha < n$, on $\mathbb{R}^n$, $n \geq 2$ (thus in particular the Newtonian kernel $|x - y|^{2-n}$ on $\mathbb{R}^n$, $n \geq 3$), satisfy the energy and consistency principles as well as the first and second maximum principles [22, Theorems 1.10, 1.15, 1.18, 1.27, 1.29]. So do the associated $\alpha$-Green kernels on an arbitrary open subset of $\mathbb{R}^n$, $n \geq 2$ [19, Theorems 4.6, 4.9, 4.11]. The $(2)$-Green kernel on a planar Greenian set is likewise strictly positive definite [10, Section I.XIII.7] and consistent [11], and it fulfills the first and second maximum principles (see [1, Theorem 5.1.11], [10, Section I.V.10]). Furthermore, all these kernels also satisfy the principle of positivity of mass (Theorem 2.1).

Finally, the restriction of the logarithmic kernel $-\log |x - y|$ to a closed disc in $\mathbb{R}^2$ of radius $< 1$ is strictly positive definite and fulfills Frostman’s maximum principle [22, Theorems 1.6, 1.16]; hence, it is perfect [16, Theorem 3.4.2], and satisfies the principle of positivity of mass (cf. Theorem 2.1 and Remark 2.1). However, the domination principle then fails in general; it does hold only in a weaker sense where the measures $\mu, \nu$ involved in the above-quoted definition (see Section 2.1) meet the additional requirement that $\nu(\mathbb{R}^2) \leq \mu(\mathbb{R}^2)$, cf. [25, Theorem 3.2].

## 3 On the Theory of Inner Balayage on Locally Compact Spaces

In what follows, we shall use the notions and notations introduced in Section 2.

The theory of inner balayage of $\mu \in E^+$ to arbitrary $A \subset X$, generalizing Cartan’s theory [7] of inner Newtonian balayage to a suitable kernel $\kappa$ on an arbitrary locally compact space $X$, was developed in the author’s recent paper [40, Sections 4–8]. To review some basic facts of this theory, to be used in the sequel, throughout this section the energy, consistency, and domination principles are required to be fulfilled.

Assume for a moment that $A = K \subset X$ is compact. Based on the above-mentioned principles, one can prove by generalizing the classical Gauss variational method (see [5, 7], cf. also [22, Section IV.5.23]) that for any given $\mu \in E^+$, there exists $\mu^K \in E^+_K$ uniquely determined within $E^+_K$ by the equality $\kappa \mu^K = \kappa \mu$ n.e. on $K$.\footnote{Concerning the uniqueness, see Eq. 2.8 applied to the space $X := K$ and the kernel $\kappa' := \kappa|_{K \times K}$.} This $\mu^K$ is said to be the balayage of $\mu \in E^+$ onto (compact) $K$.

If now $A \subset X$ is arbitrary, there is in general no $\mu^A \in E^+_A$ having the property $\kappa \mu^A = \kappa \mu$ n.e. on $A$.\footnote{Compare with Theorem 3.1(c).} Nevertheless, a substantial theory of inner balayage of $\mu \in E^+$ to arbitrary $A$ was developed [40, Sections 4–8], and this was performed by means of several alternative approaches described in Theorem 3.1 below.
To make an exposition of the theory of inner balayage similar to that of inner capacitary measures, cf. [16, Section 4.1], we have however chosen here to start with another approach, different from those utilized in [40].

Given \( \mu \in \mathcal{E}^+ \) and \( A \subset X \), denote
\[
\Gamma_{A,\mu}^+ := \{ \nu \in \mathcal{E}^+ : \kappa \nu \geq \kappa \mu \text{ n.e. on } A \}. \tag{3.1}
\]

**Definition 3.1** The inner balayage \( \mu^A \) of \( \mu \in \mathcal{E}^+ \) to \( A \subset X \) is defined as the measure of minimum energy in the class \( \Gamma_{A,\mu}^+ \), that is, \( \mu^A \in \Gamma_{A,\mu}^+ \) and
\[
\| \mu^A \|^2 = \min_{\nu \in \Gamma_{A,\mu}^+} \| \nu \|^2. \tag{3.2}
\]

**Lemma 3.1** The inner balayage \( \mu^A \) is unique (if it exists).

**Proof** It is clear from a strengthened version of countable subadditivity for inner capacity, presented by Lemma 3.2, that the class \( \Gamma_{A,\mu}^+ \) is convex. Therefore, the lemma follows by standard arguments based on a pre-Hilbert structure on the space \( \mathcal{E} \). \( \square \)

**Lemma 3.2** For arbitrary \( A \subset X \) and universally measurable \( U_j \subset X, j \in \mathbb{N}, \)
\[
c^*(\bigcup_{j \in \mathbb{N}} A \cap U_j) \leq \sum_{j \in \mathbb{N}} c^*(A \cap U_j).
\]

**Proof** See [16, p. 158, Remark], compare with [16, Lemma 2.3.5]. \( \square \)

### 3.1 On the Existence of \( \mu^A \) and its Alternative Characterizations

As before, we denote by \( \mathcal{E}_A^+ \) the convex cone of all \( \nu \in \mathcal{E}^+ \) concentrated on \( A \subset X \), and by \( \mathcal{E}_A' \) the closure of \( \mathcal{E}_A^+ \) in the strong topology on \( \mathcal{E}^+ \).

**Theorem 3.1** For any \( \mu \in \mathcal{E}^+ \) and any \( A \subset X \), the (unique) inner balayage \( \mu^A \), introduced by Definition 3.1, does exist, and it satisfies the three relations
\[
\kappa \mu^A = \kappa \mu \text{ n.e. on } A, \tag{3.3}
\]
\[
\kappa \mu^A = \kappa \mu \text{ } \mu^A \text{-a.e.,} \tag{3.4}
\]
\[
\kappa \mu^A \leq \kappa \mu \text{ on } X. \tag{3.5}
\]

Furthermore, \( \mu^A \) can alternatively be characterized by means of any one of the following (equivalent) assertions (a)–(d):

---

14 Minimum energy problem (3.2) makes sense, for \( \Gamma_{A,\mu}^+ \neq \emptyset \) (because obviously \( \mu \in \Gamma_{A,\mu}^+ \)).

15 In fact, if a set \( \Gamma \subset \mathcal{E} \) is convex, and if \( \gamma, \gamma' \in \Gamma \) are two measures of minimum energy, then an application of the parallelogram identity in the pre-Hilbert space \( \mathcal{E} \) gives
\[
0 \leq \| \gamma - \gamma' \|^2 = 2\| \gamma \|^2 + 2\| \gamma' \|^2 - \| \gamma + \gamma' \|^2 \leq 0,
\]
for \( (\gamma + \gamma')/2 \in \Gamma \). Hence \( \gamma = \gamma' \), by the energy principle.

16 For the Newtonian kernel \( |x-y|^{-n} \) on \( \mathbb{R}^n, n \geq 3 \), this goes back to Cartan [7, p. 253]. Also note that Lemma 3.2 actually holds true for any pseudo-positive kernel on a locally compact space.
(a) \( \mu^A \) is the unique measure of minimum potential in the class \( \Gamma^+_{A,\mu} \), that is, \( \mu^A \in \Gamma^+_{A,\mu} \)

\[
\kappa \mu^A = \min_{\nu \in \Gamma^+_{A,\mu}} \kappa \nu \text{ on } X.
\]

(b) \( \mu^A \) is the unique orthogonal projection of \( \mu \) in the pre-Hilbert space \( \mathcal{E} \) onto the (convex, strongly complete) cone \( \mathcal{E}'_A \), that is, \( \mu^A \in \mathcal{E}'_A \) and

\[
\|\mu - \mu^A\| = \min_{\nu \in \mathcal{E}'_A} \|\mu - \nu\|.
\]

(c) \( \mu^A \) is the unique measure in \( \mathcal{E}'_A \) having the property

\[
\kappa \mu^A = \kappa \mu \text{ n.e. on } A.
\]

(d) \( \mu^A \) is the unique measure in \( \mathcal{E}'_A \) satisfying any one of the three limit relations

\[
\begin{align*}
\mu^K &\to \mu^A \text{ strongly in } \mathcal{E}'_A \text{ as } K \uparrow A, \\
\mu^K &\to \mu^A \text{ vaguely in } \mathcal{E}'_A \text{ as } K \uparrow A, \\
\kappa \mu^K &\uparrow \kappa \mu^A \text{ pointwise on } X \text{ as } K \uparrow A,
\end{align*}
\]

where \( \mu^K \) denotes the only measure in \( \mathcal{E}'_K \) with \( \kappa \mu^K = \kappa \mu \text{ n.e. on } K \).

Proof It was shown in [40, Section 4] that there exists \( \mu' \in \mathcal{E}'_+ \) uniquely determined by any one of assertions (a)–(d) (with \( \mu^A \) replaced by \( \mu' \)), and that it satisfies (3.3)–(3.5) (again with \( \mu^A \) replaced by \( \mu' \)).

It thus remains to show that the same \( \mu' \) meets Definition 3.1, and so it serves as the (unique) inner balayage \( \mu^A \). To this end, one can assume the two conditions

\[
c_*(A) > 0 \text{ and } \kappa \mu \neq 0 \text{ n.e. on } A
\]

(3.6)

to be fulfilled, for if not, zero measure belongs to \( \Gamma^+_{A,\mu} \), and therefore

\[
\mu^A = 0 = \mu'.
\]

the former equality being obvious from Definition 3.1, while the latter being clear from (a). Applying (c), we now conclude from Eq. 3.6 that then necessarily

\[
\mu' \neq 0.
\]

As \( \mu' \in \Gamma^+_{A,\mu} \) by Eq. 3.3, the theorem will follow once we show that

\[
\|\mu'\| \leq \|\nu\|,
\]

(3.8)

\( \nu \in \Gamma^+_{A,\mu} \) being arbitrarily given. But, by virtue of (a), \( \kappa \mu' \leq \kappa \nu \) everywhere on \( X \), which yields, by the Cauchy–Schwarz (Bunyakovski) inequality,

\[
\|\mu'\|^2 = \int \kappa \mu' \, d\mu' \leq \int \kappa \nu \, d\mu' = \langle \nu, \mu' \rangle \leq \|\nu\| \cdot \|\mu'\|,
\]

and dividing by \( \|\mu'\| \neq 0 \) (cf. Eq. 3.7) results in Eq. 3.8.

\( \square \)

17 Concerning the concept of orthogonal projection, see e.g. [12, Theorem 1.12.3]. (Observe that, being a strongly closed subset of the strongly complete cone \( \mathcal{E}'_+ \), \( \mathcal{E}'_A \) is likewise strongly complete.)

18 A kernel \( \kappa \) was defined in [40] to be positive; however, this restriction on \( \kappa \) is unnecessary for the validity of the above-quoted results from [40].
Corollary 3.1 Assume that $X$ is $\sigma$-compact while $\kappa$ satisfies Frostman’s maximum principle. For any $\mu \in \mathcal{E}^+$ and any $A \subset X$, the inner balayage $\mu^A$ is of minimum total mass in the class $\Gamma^+_{A,\mu}$, that is,

$$\mu^A(X) = \min_{\nu \in \Gamma^+_{A,\mu}} \nu(X).$$

(3.9)

Proof Since $\mu^A \in \Gamma^+_{A,\mu}$, we only need to show that $\mu^A(X) \leq \nu(X)$ for all $\nu \in \Gamma^+_{A,\mu}$, but this follows immediately from Theorem 3.1(a) by making use of the principle of positivity of mass (Theorem 2.1).

Remark 3.1 However, extremal property (3.9) cannot serve as an alternative characterization of inner balayage, for it does not determine $\mu^A$ uniquely within $\Gamma^+_{A,\mu}$. Indeed, consider the $\alpha$-Riesz kernel $|x - y|^{\alpha-n}$ of order $0 < \alpha \leq 2$, $\alpha < n$, on $\mathbb{R}^n$, $n \geq 2$, and a proper, closed subset $A$ of $\mathbb{R}^n$ that is not $\alpha$-thin at infinity [21, Definition 3.1] (take, for instance, $A := \{|x| \geq 1\}$). Then for any $\mu \in \mathcal{E}^+_A$,

$$\mu^A \neq \mu \quad \text{and} \quad \mu^A(\mathbb{R}^n) = \mu(\mathbb{R}^n),$$

(3.10)

the former being obvious e.g. from Theorem 3.1(b), whereas the latter holds true by [38, Corollary 5.3]. Noting that $\mu, \mu^A \in \Gamma^+_{A,\mu}$, we conclude by combining Eqs. 3.9 and 3.10 that there are actually infinitely many measures of minimum total mass in $\Gamma^+_{A,\mu}$, for so is every measure of the form $a\mu + b\mu^A$, where $a, b \in [0, 1]$ and $a + b = 1$.

3.2 The Case of the Strongly Closed $\mathcal{E}^+_A$

In the particular case where the class $\mathcal{E}^+_A$ is strongly closed, assertions (b) and (c) in Theorem 3.1 remain valid with $\mathcal{E}^+_A$ in place of $\mathcal{E}^+_A \setminus A$, and these refined characterizations of inner balayage do hold, for instance, if the set $A$ is closed or just quasiclosed (see Definition 3.2 and Lemma 3.3).

Definition 3.2 (Fuglede [17]) A set $A \subset X$ is said to be quasiclosed if it can be approximated in outer capacity by closed sets, that is, if

$$\inf \{ c^*(A \triangle F) : F \text{ closed, } F \subset X \} = 0,$$

$\triangle$ being the symmetric difference.

Lemma 3.3 If $A \subset X$ is quasiclosed, then the cone $\mathcal{E}^+_A$ is strongly closed; hence

$$\mathcal{E}^+_A \setminus A = \mathcal{E}^+_A.$$

Proof The kernel $\kappa$ being perfect, the strong convergence of any net $(\nu_k) \subset \mathcal{E}^+_A$ to $\nu_0 \in \mathcal{E}^+$ implies the vague convergence to the same limit. Since for quasiclosed $A$, $\mathcal{E}^+_A$ is vaguely closed in consequence of [18, Corollary 6.2], the lemma follows.
4 Characterizations of Inner Capacity for Pseudo-Positive Kernels

In this section, \( \kappa \) is pseudo-positive (see Section 2). For any \( \mu \in \mathcal{E}^+ \) and \( A \subset X \), denote

\[
G(\mu) := 2\mu(X) - \kappa(\mu, \mu) \in (-\infty, \infty],
\]

(4.1)

\[
\mathcal{E}_A^+ := \{ \nu \in \mathcal{E}_A^+ : \kappa \nu \leq 1 \text{ \( \nu \)-a.e. on } X \}.
\]

(4.2)

**Theorem 4.1** For any set \( A \subset X \) and any pseudo-positive kernel \( \kappa \),

\[
c_*(A) = \sup_{v \in \mathcal{E}_A^+} G(v).
\]

(4.3)

**Remark 4.1** Formula (4.3) was declared by Fuglede [16, p. 162, Eq. (1)]; however, the proof given in [16, p. 162, footnote 1] was incomplete, being actually only applicable to a relatively compact set \( A \) and a strictly pseudo-positive kernel \( \kappa \). This in turn has led to the incompleteness of some other proofs in [16]; thus the theories of inner and outer capacitary measures and capacitability [16, Sections 4.2–4.5] turned out to be unjustified. In the present work we have restored Fuglede’s theories in full generality by showing that formula (4.3) does indeed hold as stated (see Section 4.1 for a proof).

The following Theorems 4.2 and 4.3 will be proved in Sections 4.2 and 4.3 below.

**Theorem 4.2** Given \( A \subset X \) and a pseudo-positive kernel \( \kappa \), assume that either \( \kappa \geq 0 \) or \( c(K) < \infty \) for all compact \( K \subset A \). Then

\[
c_*(A) = \sup_{v \in \mathcal{E}_A^+} v(X).
\]

(4.4)

**Theorem 4.3** For any pseudo-positive kernel \( \kappa \) and any compact set \( K \subset X \) with \( c(K) \in (0, \infty) \), extremal problems (4.3) and (4.4) are both solvable with precisely the same maximizing measures, forming the class of capacitary measures \( \gamma_K \) on \( K \).

**Remark 4.2** Regarding Theorem 4.3, assume for a moment that \( A \subset X \) is noncompact, and that \( c_*(A) < \infty \). We shall show below that if moreover the kernel \( \kappa \) is perfect, whereas the class \( \mathcal{E}_A^+ \) is strongly closed (or in particular if \( A \) is quasiclosed), then each of extremal problems (4.3) and (4.4) is uniquely solvable with precisely the same maximizing measure, serving simultaneously as the unique inner capacitary measure \( \gamma_A \) (see Theorem 7.2). This result admits a further generalization to arbitrary \( A \), which however additionally requires upon the perfect kernel \( \kappa \) the first and the second maximum principles (see Theorem 9.1).

### 4.1 Proof of Theorem 4.1

Assume \( c_*(A) > 0 \), since otherwise \( \mathcal{E}_A^+ = \{0\} \) by Lemma 2.4, and formula (4.3) is trivial. There is also no loss of generality in assuming that

\[
\kappa(\nu, \nu) > 0 \text{ for all bounded } \nu \in \mathcal{E}_A^+, \nu \neq 0,
\]

(4.5)

This occurs e.g. if either \( X \) is noncompact, or \( \kappa \) is strictly pseudo-positive, or \( c_*(A) < \infty \).
for if not, both sides in Eq. 4.3 are equal to $+\infty$. In fact, if there exists $\xi \in \mathcal{E}_A^+$, $\xi \neq 0$, with $\xi(X) < \infty$ and $\kappa(\xi, \xi) = 0$, then $c_*(A) = \infty$, for $\xi' := \xi/\xi(X) \in \mathcal{E}_A^+$, whence

$$0 \leq w(A) \leq \kappa(\xi', \xi') = 0.$$  

Noting also that $h\xi \in \mathcal{E}_A^+$ for all $h > 0$, whereas

$$G(h\xi) = 2h\xi(X) - h^2\kappa(\xi, \xi) = 2h\xi(X) > 0,$$

we arrive at Eq. 4.3 by letting $h \to \infty$.

Suppose first that the space $X$ is compact; then all the measures on $X$ must be bounded. Excluding $\nu = 0$, for every given $\nu \in \mathcal{E}_A^+$ write $\nu = q\mu$, where $\mu \in \mathcal{E}_A^+$ and $q := \nu(X) \in (0, \infty)$. Then $G(\nu)$ takes the form $2q - q^2\kappa(\mu, \mu)$, which attains its maximum in $q \in (0, \infty)$ at $q_0 := 1/\kappa(\mu, \mu) \in (0, \infty)$, cf. Eq. 4.5, and this maximum equals $q_0 = 1/\kappa(\mu, \mu) \in (0, \infty)$. Taking now the supremum over $\mu \in \mathcal{E}_A^+$ we obtain

$$\sup_{\nu \in \mathcal{E}_A^+} G(\nu) = \left[ \inf_{\mu \in \mathcal{E}_A^+} \kappa(\mu, \mu) \right]^{-1} = c_*(A).$$  

(4.6)

In particular,

$$\sup_{\nu \in \mathcal{E}_X^+} G(\nu) = c(X),$$  

(4.7)

the space $X$ being compact.

To complete the proof, it remains to consider noncompact $X$; then $\kappa \geq 0$ by convention. By Lemma 2.2 applied to $\psi := \kappa$, and subsequently to $\psi := 1$,

$$G(\nu) = \lim_{K \uparrow A} G(\nu|_K) \text{ for all } \nu \in \mathcal{E}_A^+.$$  

(4.8)

Therefore,

$$\sup_{\nu \in \mathcal{E}_A^+} G(\nu) = \lim_{K \uparrow A} \left( \sup_{\nu \in \mathcal{E}_K^+} G(\nu) \right) = \lim_{K \uparrow A} c(K) = c_*(A),$$

where the first equality follows easily from Eq. 4.8,\(^20\) while the second holds by Eq. 4.7, applied to the compact space $X := K$ and the kernel $\kappa|_{K \times K}$. Formula (4.3) is thus proved.

### 4.2 Proof of Theorem 4.2

Likewise as in the preceding proof, assume $c_*(A) > 0$, since otherwise $\mathcal{E}_A^+ = \mathcal{E}_M^+ = \{0\}$ (Lemma 2.4), and formula (4.4) is trivial.

In the case where $\kappa \geq 0$, one can also assume Eq. 4.5 to be fulfilled, for if not, both sides in Eq. 4.4 equal $+\infty$. In fact, if there exists a bounded measure $\xi \in \mathcal{E}_A^+$, $\xi \neq 0$, with $\kappa(\xi, \xi) = 0$, then $c_*(A) = \infty$ (see Section 4.1). On the other hand, since $\kappa\xi \geq 0$ on $X$, we infer from $\kappa(\xi, \xi) = \int_X \kappa \xi^2 d\xi = 0$ that $\kappa\xi = 0$ -a.e. This in turn implies that $h\xi \in \mathcal{E}_A^+$ for all $h > 0$, which leads to Eq. 4.4 by taking $h$ infinitely large.

\(^{20}\)Indeed, since $\mathcal{E}_K^+ \subset \mathcal{E}_K^+ \subset \mathcal{E}_A^+$ whenever $K \subset K' \subset A$,

$$\sup_{\nu \in \mathcal{E}_A^+} G(\nu) \geq \lim_{K \uparrow A} \left( \sup_{\nu \in \mathcal{E}_K^+} G(\nu) \right).$$

For the opposite inequality, observe from Eq. 4.8 that for every $\nu \in \mathcal{E}_A^+$,

$$G(\nu) = \lim_{K \uparrow A} G(\nu|_K) \leq \lim_{K \uparrow A} \left( \sup_{\nu \in \mathcal{E}_K^+} G(\nu) \right),$$

and the claim follows by letting $\nu$ range over $\mathcal{E}_A^+$.  

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We stress that under condition (4.5), the restriction of \( \kappa \) to \( K \times K, K \subset A \) being compact, represents a strictly pseudo-positive kernel on the (compact) space \( K \).

It follows that under either of the hypotheses of the theorem, for each \( K \in \mathcal{C}_A \) sufficiently large, we have \( 0 < c(K) < \infty \), and hence a capacitary measure \( \gamma_K \) on \( K \) does exist. Since \( \gamma_K(X) = c(K) \) while \( \gamma_K \in \mathcal{E}_K^+ \) (cf. (2.4) and (2.7)),

\[
c(K) = \gamma_K(X) \leq \sup_{\nu \in \mathcal{E}_K^+} v(X) \leq \sup_{\nu \in \mathcal{E}_A^+} v(X). \tag{4.9}
\]

On the other hand, it is clear from Eqs. 4.1 and 4.2 that for every \( \nu \in \mathcal{E}_A^+ \),

\[
G(\nu) \geq v(X), \tag{4.10}
\]

because \( \kappa (\nu, v) = \int \kappa \nu \, d\nu \leq v(X) \). On account of Eq. 4.3, we therefore get

\[
\sup_{\nu \in \mathcal{E}_A^+} v(X) \leq \sup_{\nu \in \mathcal{E}_A^+} G(\nu) \leq \sup_{\nu \in \mathcal{E}_A^+} G(\nu) = c_* (A). \tag{4.11}
\]

Combining this with Eq. 4.9, and then letting \( K \uparrow A \) results in Eq. 4.4.

4.3 Proof of Theorem 4.3

This is clear from Eqs. 4.6, 4.9, and 4.11 (applied to compact \( A := K \)).

5 An Alternative Proof of Theorem 4.1

Throughout this section, a kernel \( \kappa \) is assumed to be strictly positive definite. Then Theorem 4.1 admits an alternative proof, to be particularly useful in the sequel. It is based on the following crucial auxiliary result, being of independent interest.

Theorem 5.1 For any \( A \subset X \), the following two assertions are equivalent:

(i) \( c_*(A) < \infty \).

(ii) \( v(X) < \infty \) for all \( \nu \in \mathcal{E}_A^+ \).

Remark 5.1 Actually, (i) \( \Rightarrow \) (ii) holds true even for any pseudo-positive kernel. This follows in the same manner as in the proof below, the only difference being in replacing \( w(K) > 0 \) by \( w(K) \geq 0 \), see Eq. 5.2.

Remark 5.2 On the contrary, (ii) \( \Rightarrow \) (i) would not hold in general if the requirement of the energy principle were omitted. (Take \( X \) to be compact, \( \kappa \equiv 0, A := X \). Then \( c_*(A) = \infty \), though the measures on the compact space \( X \) must be bounded.)

5.1 Proof of Theorem 5.1

To prove that (i) implies (ii), let there on the contrary be \( \nu \in \mathcal{E}_A^+ \) with \( v(X) = \infty \). Then

\[
\lim_{K \uparrow A} v(K) = v_*(A) = v(X) = \infty, \tag{5.1}
\]
and hence one can choose \( K_0 \in \mathcal{C}_A \) so that \( v(K) > 0 \) for all \( K \geq K_0 \) \((K \in \mathcal{C}_A)\). Since for each of those \( K \), the measure \( v|_K/v(K) \) belongs to \( \mathcal{E}_K^+ \),

\[
\frac{\|v|_K\|^2}{v(K)^2} \geq w(K) > 0. \tag{5.2}
\]

Letting now \( K \uparrow A \) we obtain \( w(A) = 0 \) by virtue of Eqs. 2.2, 5.1 and Lemma 2.3. Thus \( c_*(A) = \infty \), which contradicts (i).

Assume now that \( c_*(A) = \infty \); we need to construct \( \omega \in \mathcal{E}_A^+ \) with \( \omega(X) = \infty \).

Choose an increasing sequence of compact sets \( K_j \subset A \), \( j \in \mathbb{N} \), such that

\[
\lim_{j \to \infty} c(K_j) = c_*(A) = \infty. \tag{5.3}
\]

Since \( c(K_j) < \infty \) by the energy principle, an application of Lemma 3.2 gives

\[
c_*(A \setminus K_j) = \infty \quad \text{for all} \quad j.
\]

Therefore, there exists a compact set \( L_j \subset A \setminus K_j \) with \( c(L_j) > j^4 \), or equivalently

\[
w(L_j) < j^{-4}.
\]

We can certainly assume that \( L_j \subset K_{j+1} \), for if not, we replace \( K_j+1 \) by \( K_{j+1} \cup L_j \), which does not affect limit relation (5.3). Define

\[
\omega(f) := \sum_{j \in \mathbb{N}} \lambda_{L_j}(f) \quad \text{for all} \quad f \in C_0(X),
\]

where \( \lambda_{L_j} \) is the (unique) solution to extremal problem (2.1) for the set \( L_j \). Then \( \omega \) represents a positive Radon measure on \( X \), because any compact subset of \( X \) has points in common with at most finitely many \( L_j \). Furthermore, the measure \( \omega \) is concentrated on the union of \( L_j \), \( j \in \mathbb{N} \), and hence on the (larger) set \( A \). Besides,

\[
\omega(X) = \sum_{j \in \mathbb{N}} \lambda_{L_j}(X) = \sum_{j \in \mathbb{N}} 1 = +\infty,
\]

the sets \( L_j \) being mutually disjoint. It thus remains to verify that

\[
\kappa(\omega, \omega) < \infty. \tag{5.4}
\]

To this end, observe that \( \omega \) is the vague limit of the partial sums

\[
\lambda_N := \sum_{j=1}^{N} \lambda_{L_j} \quad \text{(as} \quad N \to \infty),
\]

because for every \( f \in C_0(X) \), \( \omega(f) = \lambda_N(f) \) for all \( N \) sufficiently large. Noting that, by the triangle inequality,

\[
\|\lambda_N\| \leq \sum_{j=1}^{N} \|\lambda_{L_j}\| < \sum_{j=1}^{N} j^{-2} < \sum_{j \in \mathbb{N}} j^{-2} < \infty \quad \text{for all} \quad N,
\]

we therefore conclude by the principle of descent that

\[
\kappa(\omega, \omega) \leq \liminf_{N \to \infty} \kappa(\lambda_N, \lambda_N) < \infty,
\]

whence Eq. 5.4. This establishes (ii)⇒(i), thereby completing the whole proof.
5.2 An Alternative Proof of Theorem 4.1

If \( c_*(A) = 0 \), Eq. 4.3 is trivial, for then \( \delta_A^+ = \{0\} \) by Lemma 2.4. If now \( c_*(A) = \infty \), there exists \( v_0 \in \delta_A^+ \) with \( v_0(X) = \infty \) (Theorem 5.1), and Eq. 4.3 holds in the sense that its both sides are equal to \( +\infty \).

It is left to verify Eq. 4.3 in the remaining case \( 0 < c_*(A) < \infty \); then, again by Theorem 5.1, all \( \nu \in \delta_A^+ \) must be bounded. Excluding the measure \( \nu = 0 \), write \( \nu = q\mu \), where \( \mu \in \bar{\mathcal{E}}_A^+ \) and \( q := \nu(X) \in (0, \infty) \). Then \( G(\nu) \) takes the form \( 2q - q^2\|\mu\|^2 \), which attains its maximum in \( q \in (0, \infty) \) at \( q_0 = 1/\|\mu\|^2 \in (0, \infty) \) (the kernel being strictly positive definite), and this maximum equals \( q_0 = 1/\|\mu\|^2 \in (0, \infty) \). Taking now the supremum over \( \mu \in \bar{\mathcal{E}}_A^+ \) we obtain

\[
\sup_{\nu \in \delta_A^+} G(\nu) = \left[ \inf_{\mu \in \delta_A^+} \|\mu\|^2 \right]^{-1} = c_*(A),
\]

and the proof is complete.

6 Basic Facts on Inner Capacities and Inner Capacitary Measures

In the rest of the paper we shall assume a kernel \( \kappa \) to be perfect (see Section 2).

6.1 Inner Capacitary Measures

Suppose for a moment that a set \( A = K \subset X \) is compact, and \( w(K) < \infty \), cf. Eq. 2.1. Then there exists the unique capacitary distribution \( \lambda_K \) of unit mass on \( K \), minimizing the energy over the class \( \bar{\mathcal{E}}_K^+ \): \( \lambda_K \in \bar{\mathcal{E}}_K^+ \) and \( \|\lambda_K\|^2 = w(K) \).

In fact, the existence of this \( \lambda_K \) is due to the vague compactness of the class \( \bar{\mathcal{E}}_K^+ \), whereas the uniqueness follows easily by means of standard arguments based on the convexity of the class \( \bar{\mathcal{E}}_K^+ \) and a pre-Hilbert structure on the space \( \mathcal{E} \) (cf. footnote 15). Noting that then necessarily \( w(K) > 0 \) by the energy principle, one can also consider the (normalized) capacitary measure \( \gamma_K \) on the (compact) set \( K \), introduced by

\[
\gamma_K := \lambda_K / w(K) = c(K)\lambda_K \in \bar{\mathcal{E}}_K^+
\]

and having properties (2.4)–(2.7).

However, if \( A \subset X \) is noncompact, \( \bar{\mathcal{E}}_A^+ \) is no longer vaguely compact, and the infimum in Eq. 2.1 is in general not attained among the admissible measures — not even for a perfect kernel. To overcome this inconvenience, Fuglede has formulated the (dual) minimum energy problem over the class \( \Gamma_A^+ \subset \mathcal{E} \) (see below), which leads to the same concept of inner capacity, but it is already solvable (see [16, Theorem 4.1]).

Given \( A \subset X \), let \( \Gamma_A^- \) denote the class of all (signed) \( v \in \mathcal{E} \) having the property \( \kappa v \geq 1 \) n.e. on \( A \), and \( \Gamma_A^+ \) its subclass consisting of all positive measures:

\[
\Gamma_A^+ := \{ v \in \delta^+ : \kappa v \geq 1 \ \text{n.e. on } A \}.
\]

Due to the energy principle accepted in the present section, [16, Theorem 4.1] can be specified as follows.
Theorem 6.1 For any \( A \subset X \),
\[
\inf_{\nu \in \Gamma_A} \| \nu \|_2^2 = \inf_{\nu \in \Gamma_A^+} \| \nu \|_2^2 = c_+(A).
\]

If now \( c_+(A) < \infty \), then each of these two infima is an actual minimum with the same unique minimizing measure \( \gamma_A \in \Gamma_A^+ \subset \Gamma_A \), called the inner capacitary measure of \( A \) and having the properties\(^{21}\)
\[
\gamma_A(X) = \| \gamma_A \|_2^2 = c_+(A),
\]
\[
\kappa \gamma_A \geq 1 \text{ n.e. on } A,
\]
\[
\kappa \gamma_A \leq 1 \text{ on } S(\gamma_A).
\]
\[
\kappa \gamma_A = 1 \text{ } \gamma_A\text{-a.e. on } X.
\]

If moreover Frostman’s maximum principle is fulfilled, then also\(^{22}\)
\[
\kappa \gamma_A = 1 \text{ n.e. on } A.
\]

Proof By \([16, Lemma 3.2.2]\) with \( t = 1 \), \( \|
u\|_2^2 \geq c_+(A) \) for all \( \nu \in \Gamma_A \), and hence
\[
\inf_{\nu \in \Gamma_A^+} \| \nu \|_2^2 \geq \inf_{\nu \in \Gamma_A} \| \nu \|_2^2 \geq c_+(A).
\]

It is thus enough to consider the case where \( c_+(A) < \infty \). The uniqueness of the solution to the minimum energy problem over \( \Gamma_A \) (resp. \( \Gamma_A^+ \)) follows from the convexity of \( \Gamma_A \) (resp. \( \Gamma_A^+ \)), which in turn is obvious from Lemma 3.2.

The (unique) measure \( \gamma \in \Gamma_A \) of minimal energy in the class \( \Gamma_A \) (resp. \( \Gamma_A^+ \)) does exist, and moreover it satisfies (6.3)–(6.6) (see [16, Theorem 4.1]). It thus remains to prove that the same \( \gamma \) solves the minimum energy problem over \( \Gamma_A^+ \). In view of Eq. 6.8, this will follow once we show that this \( \gamma \) is positive. Assume \( c_+(A) > 0 \), for if not, \( \gamma = \gamma_A = 0 \).

Analyzing the proof of Theorem 4.1 in [16], we deduce the following.

Lemma 6.1 The above measure \( \gamma \) is the strong limit of \( (\gamma_{K_j}) \), where \( (K_j) \) is an increasing sequence of compact sets \( K_j \subset A \) with \( c(K_j) > 0 \) and such that\(^{23}\)
\[
\lim_{j \to \infty} c(K_j) = c_+(A),
\]
while \( \gamma_{K_j} = \lambda_{K_j}/w(K_j) \in \delta_{K_j}^+ \) is the (unique) capacitary measure on \( K_j \).

Since the kernel is perfect, \( \gamma_{K_j} \to \gamma \) also vaguely. Hence \( \gamma \) is indeed positive, the cone \( \mathfrak{M}^+ \) being vaguely closed [4, Section III.1, Proposition 14].

Remark 6.1 In general, \( \gamma_A \) is not concentrated on the set \( A \) itself, but on its closure in \( X \). For instance, if \( A := B_r \) is an open ball \( \{|x| < r\} \), \( r > 0 \), in \( \mathbb{R}^n \), \( n \geq 3 \), and \( \kappa(x,y) \) is the Newtonian kernel \( |x - y|^{2-n} \), then \( \gamma_{B_r} \) is the positive measure of total mass \( r^{n-2} \), uniformly distributed over the sphere \( \{|x| = r\} \). Thus \( S(\gamma_{B_r}) \cap B_r = \emptyset \).

\(^{21}\) It follows that \( c_+(A) < \infty \iff \Gamma_A^+ \neq \emptyset \iff \Gamma_A \neq \emptyset \).
\(^{22}\) Then \( \gamma_A \) is also said to be the inner equilibrium measure of \( A \) (cf. footnote 8).
\(^{23}\) Such a sequence \( (K_j) \) exists by virtue of Eq. 2.2.
We shall now show that the problem of the determination of the inner capacitary measure for arbitrary \( A \) can always be reduced to that for a (Borel) \( \mathcal{K}_\sigma \)-set \( A' \subset A \).\(^\text{24}\) This observation will be particularly useful in the proof of Theorem 8.2 (see Section 8.2), dealing with the convergence of inner capacitary measures and their potentials.

**Theorem 6.2** For arbitrary \( A \subset X \), there exists a \( \mathcal{K}_\sigma \)-set \( A' \subset A \) such that, for every \( H \) with the property \( A' \subset H \subset A \),

\[
c_*(A') = c_*(H) = c_*(A).
\]

If now \( c_*(A) < \infty \), then also

\[
\gamma_{A'} = \gamma_H = \gamma_A.
\]

**Proof** Given \( A \subset X \), choose an increasing sequence \( (K_j) \subset A \) of compact sets such that \( c(K_j) \to c_*(A) \), and let \( A' \) denote the union of all those \( K_j \). By Eq. 2.2 applied to \( A' \) (or alternatively by [16, Lemma 2.3.3], the sets \( K_j \) being universally measurable),

\[
c_*(A') = \lim_{j \to \infty} c(K_j),
\]

and hence \( c_*(A') = c_*(A) \). For any \( H \) such that \( A' \subset H \subset A \), Eq. 6.10 now follows from the monotonicity of the inner capacity:

\[
c_*(A') \leq c_*(H) \leq c_*(A) = c_*(A').
\]

Assume moreover that \( c_*(A) < \infty \). Noting that \( \overline{\Gamma}_+^A \subset \overline{\Gamma}_H^+ \subset \overline{\Gamma}_A^+ \), we derive Eq. 6.11 from Eq. 6.10 by use of the uniqueness of the inner capacitary measure. \( \square \)

### 6.2 Relations Between the Minimum Energy Problems Over \( \overline{\Gamma}_A^+ \) and \( \overline{\epsilon}_A^+ \)

Despite the fact that the minimum energy problem over the class \( \overline{\epsilon}_A^+ \) is in general unsolvable, it is often useful to consider it in parallel with the (dual) minimum energy problem over the class \( \overline{\Gamma}_A^+ \). To analyze relations between these two problems, assume \( w(A) < \infty \), for then (and only then) \( \overline{\epsilon}_A^+ \) is nonempty, and problem (2.1) makes sense. To avoid trivialities, assume also that \( w(A) > 0 \), for if not, problem (2.1) admits no solution by the energy principle. Thus, in the rest of this section,

\[
0 < c_*(A) < \infty,
\]

and hence there exists the (unique) inner capacitary measure \( \gamma_A \neq 0 \) (Theorem 6.1).

**Definition 6.1** A net \( (\mu_s)_{s \in S} \subset \overline{\epsilon}_A^+ \) is said to be **minimizing** in problem (2.1) if

\[
\lim_{s \in S} \|\mu_s\|^2 = w(A).
\]

As \( \overline{\epsilon}_A^+ \) is nonempty, so is the set \( \mathbb{M}(A) \) consisting of all minimizing nets \( (\mu_s)_{s \in S} \).

**Lemma 6.2** For every \( (\mu_s)_{s \in S} \in \mathbb{M}(A) \),

\[
\mu_s \to \xi_A \text{ strongly and vaguely},
\]

\(^{24}\)A set \( A \subset X \) is said to be of class \( K_\sigma \) (in short, a \( K_\sigma \)-set) if it is representable as a countable union of compact subsets.
where

$$\xi_A := \gamma_A / \varsigma_+(A) \in \mathcal{E}^+. \quad (6.15)$$

\textbf{Proof} We shall first show that for any \((\mu_s)_{s \in S}\) and \((\nu_t)_{t \in T}\) from \(\mathbb{M}(A)\),

$$\lim_{(s, t) \in S \times T} \| \mu_s - \nu_t \| = 0, \quad (6.16)$$

\(S \times T\) being the directed product of the directed sets \(S\) and \(T\) (see e.g. [20, p. 68]). In fact, due to the convexity of the class \(\mathcal{E}_A^+\), for any \((s, t) \in S \times T\) we have

$$\| \mu_s + \nu_t \|^2 \geq 4w(A).$$

Applying the parallelogram identity in \(\mathcal{E}\) therefore gives

$$0 \leq \| \mu_s - \nu_t \|^2 \leq -4w(A) + 2\| \mu_s \|^2 + 2\| \nu_t \|^2,$$

which yields Eq. 6.16 when combined with Eq. 6.13.

Taking the two nets in Eq. 6.16 to be equal, we see that every \((\nu_t)_{t \in T} \in \mathbb{M}(A)\) is strong Cauchy. But \((\nu_t)_{t \in T}\) is vaguely bounded, for \(\nu_t(X) = 1\) for all \(t \in T\), and hence vaguely relatively compact [4, Section III.1, Proposition 15]. If \(\nu_0\) denotes a vague limit point of \((\nu_t)_{t \in T}\), then \(\nu_t \to \nu_0\) strongly, the kernel \(\kappa\) being perfect. The same unique \(\nu_0\) also serves as the strong limit of any other net \((\mu_s)_{s \in S} \in \mathbb{M}(A)\), which is obvious from Eq. 6.16. Since the strong convergence implies the vague convergence,

$$\mu_s \to \nu_0 \text{ strongly and vaguely.}$$

To prove that this \(\nu_0\) actually equals \(\xi_A\), \(\xi_A\) being introduced by Eq. 6.15, it is thus enough to construct a minimizing net converging to \(\xi_A\) strongly. Let a sequence \((K_j)\) of compact subsets of \(A\) be as in Lemma 6.1; it is clear from Eq. 6.9 that

$$\lambda_{K_j} \in \mathbb{M}(A),$$

\(\lambda_{K_j} \in \mathcal{E}^+_A \subset \mathcal{E}_A^+\) solving problem (2.1) for the set \(K_j\). But by Lemma 6.1, \(c(K_j)\lambda_{K_j} \to \gamma_A\) strongly, which combined with Eqs. 6.9 and 6.15 shows that, indeed,

$$\lambda_{K_j} \to \xi_A \text{ strongly,}$$

thereby completing the proof of the lemma.

\textbf{Remark 6.2} In view of Eq. 6.14, it is natural to call the measure \(\xi_A \in \mathcal{E}^+\) extremal in problem (2.1). It is obvious from Eqs. 6.3 and 6.15 that

$$\xi_A(X) = 1, \quad \| \xi_A \|^2 = w(A). \quad (6.17)$$

Nevertheless, the extremal measure \(\xi_A\) may not be the solution to problem (2.1), being in general not concentrated on \(A\) (cf. Remark 6.1).

\textbf{Theorem 6.3} Under assumption (6.12), the three assertions are equivalent:

(i) \textbf{The extremal measure} \(\xi_A\) \textbf{is concentrated on} \(A\).

(ii) \textbf{Problem} (2.1) \textbf{has the (unique) minimizing measure} \(\lambda_A \in \mathcal{E}^+_A\):

$$\| \lambda_A \|^2 = \min_{\nu \in \mathcal{E}_A^+} \| \nu \|^2,$$

called the capacitary distribution of unit mass on \(A\).
The inner capacitory measure $\gamma_A$ serves simultaneously as the (unique) solution to the minimum energy problem over the class $\{v \in \delta_A^+ : v(X) = c_\ast(A)\}$.

If any one of these (i)–(iii) holds true, then actually

$$\lambda_A = \xi_A = \gamma_A/c_\ast(A). \quad (6.18)$$

**Proof** The fact that (i) implies (ii) (with $\lambda_A := \xi_A$) follows directly from Eq. 6.17.

Let now (ii) hold. Since the trivial sequence $(\lambda_A)$ is minimizing:

$$(\lambda_A) \in \mathbb{M}(A),$$

we conclude from Lemma 6.2 that then necessarily $\lambda_A = \xi_A$ (the strong topology on $\delta$ being Hausdorff), whence (i). Also, combining $\lambda_A = \xi_A$ with Eq. 6.15 gives Eq. 6.18.

Finally, the equivalence of (ii) and (iii) is obvious for reasons of homogeneity. □

**Corollary 6.1** Assume assumption (6.12) is fulfilled. For (i)–(iii) in Theorem 6.3 to hold, it is sufficient that $\delta_A^+$ be strongly closed (or in particular that $A$ be quasiclosed).

**Proof** If $\delta_A^+$ is strongly closed (or in particular if $A$ is quasiclosed, cf. Lemma 3.3), then the extremal measure, being the strong limit of any minimizing net, must be concentrated on $A$. Thus (i) holds true, and hence so do both (ii) and (iii). □

## 7 Further Alternative Characterizations of $\gamma_A$

As in Section 6, in the present section a kernel $\kappa$ is assumed to be perfect.

We first establish necessary and sufficient conditions for (extremal) problems (4.3) and (4.4) to be uniquely solvable (Theorem 7.1), which will further be utilized to characterize the inner capacitary measure $\gamma_A$ for $A$ with the strongly closed $\delta_A^+$ (Theorem 7.2).

**Theorem 7.1** For any $A \subset X$ with $0 < c_\ast(A) < \infty$,\(^{25}\) each of assertions (i)–(iii) in Theorem 6.3 is equivalent to either of the following (iv) and (v):

(iii) \quad There exists a solution $\theta_A$ to problem (4.3), that is, $\theta_A \in \delta_A^+$ and

$$G(\theta_A) = \max_{v \in \delta_A^+} G(v) \quad ( = c_\ast(A)).$$

(iv) \quad There exists a solution $\theta_A$ to problem (4.3), that is, $\theta_A \in \delta_A^+$ and

$$G(\theta_A) = \max_{v \in \delta_A^+} G(v) \quad ( = c_\ast(A)).$$

(v) \quad There exists a solution $\sigma_A$ to problem (4.4), that is, $\sigma_A \in \hat{\delta}_A^+$ and

$$\sigma_A(X) = \max_{v \in \delta_A^+} v(X) \quad ( = c_\ast(A)).$$

If any one of these (i)–(v) is fulfilled, then $\theta_A$ and $\sigma_A$ are unique, and moreover

$$\theta_A = \sigma_A = \gamma_A = c_\ast(A)\lambda_A.$$

**Proof** It follows from the proof given in Section 5.2 (see in particular formula (5.5)) that the solution $\theta_A$ to problem (4.3) exists if and only if so does the solution $\lambda_A$ to problem

\(^{25}\)If $c_\ast(A) = 0$, then $\delta_A^+ = \hat{\delta}_A^+ = \{0\}$ (Lemma 2.4), and problems (4.3) and (4.4) are obviously solvable with the same unique solution $\theta_A = \sigma_A = 0 \quad ( = \gamma_A)$.\)
(2.1), and in the affirmative case they are related to one another as follows:

$$\theta_A = c_*(A)\lambda_A = \gamma_A,$$

the latter equality being valid by Eq. 6.18. This proves the equivalence of (ii) and (iv), as well as the uniqueness of $\theta_A$, the solution $\lambda_A$ to problem (2.1) being unique.

Furthermore, (iv) implies (v) with $\sigma_A := \gamma_A = \theta_A$. Indeed, $\gamma_A \in \hat{\mathcal{E}}_A^+$ by Eq. 6.6, which together with Eqs. 4.4 and 6.3 gives

$$\gamma_A(X) \leq \sup_{\nu \in \hat{\mathcal{E}}_A^+} \nu(X) = c_*(A) = \gamma_A(X),$$

and hence the same $\sigma_A$ must be the (unique) solution to problem (4.3).

**Theorem 7.2** Given a set $A \subset X$ with $c_*(A) < \infty$, assume that the class $\mathcal{E}_A^+$ is strongly closed (or in particular that $A$ is quasiclosed). Then the inner capacitary measure $\gamma_A$ can alternatively be characterized as the unique solution to either of extremal problems (4.3) or (4.4); that is, $\gamma_A \in \hat{\mathcal{E}}_A^+ \subset \mathcal{E}_A^+$ and

$$\max_{\nu \in \hat{\mathcal{E}}_A^+} G(\nu) = G(\gamma_A) \quad (= c_*(A)), \quad (7.1)$$

$$\max_{\nu \in \hat{\mathcal{E}}_A^+} \nu(X) = \gamma_A(X) \quad (= c_*(A)). \quad (7.2)$$

Furthermore, $\gamma_A$ is uniquely determined within $\mathcal{E}_A^+$ by each of (a) or (b), where

(a) $\gamma_A(X) \geq \|\gamma_A\|^2 \geq c_*(A)$.

(b) $\kappa \gamma_A \geq 1$ n.e. on $A$, and $\kappa \gamma_A \leq 1$ $\gamma_A$-a.e.

If moreover Frostman’s maximum principle is fulfilled, then $\gamma_A$ is the only measure in $\mathcal{E}_A^+$ having the property

(c) $\kappa \gamma_A = 1$ n.e. on $A$.

**Proof** Let the hypotheses of the theorem be fulfilled. Also assume that $c_*(A) > 0$, for if not, $\mathcal{E}_A^+ = \hat{\mathcal{E}}_A^+ = \{0\}$ (Lemma 2.4), and the theorem holds trivially with $\gamma_A = 0$. Taking Corollary 6.1 into account, we then conclude from Theorem 7.1 that the inner capacitary measure $\gamma_A$ can alternatively be characterized as the unique maximizing measure in either of problems (4.3) and (4.4), which establishes Eqs. 7.1 and 7.2.

Besides, it is obvious from Eqs. 6.3, 6.4, and 6.6 that $\gamma_A$ has properties (a) and (b). If now $\mu$ is another measure in $\mathcal{E}_A^+$ satisfying (a), then, in view of Eq. 4.3,

$$c_*(A) = \sup_{\nu \in \hat{\mathcal{E}}_A^+} G(\nu) \geq G(\mu) = 2\mu(X) - \|\mu\|^2 \geq \|\mu\|^2 \geq c_*(A).$$

Being therefore the maximizing measure in problem (4.3), $\mu$ necessarily equals $\gamma_A$.

Assume now that some $\chi \in \hat{\mathcal{E}}_A^+$ meets (b). By the former relation in (b), we have $\chi \in \Gamma_A^+$, hence $\|\chi\|^2 \geq c_*(A)$, whereas the latter relation in (b) yields

$$\|\chi\|^2 = \int \kappa \chi \, d\chi \leq \chi(X).$$
Thus (b) is actually reduced to (a), whence $\chi = \gamma_A$ (cf. the preceding paragraph).

Assume finally that Frostman’s maximum principle holds; then $\gamma_A$ satisfies (c), see Theorem 6.1. Given $\tau \in \mathcal{E}_A^+$ meeting (c), we need to prove that then necessarily $\tau = \gamma_A$, which would follow immediately if $\tau$ were shown to fulfill (b), or equivalently
\[
\kappa \tau \leq 1 \text{ } \tau\text{-a.e.} \tag{7.3}
\]

We may verify this only for positive $\kappa$, because the remaining case of compact $X$ can be reduced to the former case in the manner described in footnote 5.

For any compact $K \subset A$, then $\kappa \tau|_K \leq \kappa \tau$ on $X$; therefore, $\kappa \tau|_K \leq 1$ holds n.e. on $K$, hence everywhere on $S(\tau|_K) \subset K$, the potential of a positive measure being l.s.c. on $X$, and an application of Frostman’s maximum principle gives
\[
\kappa \tau|_K \leq 1 \text{ on } X.
\]

Since $\tau|_K \to \tau|_A = \tau$ vaguely as $K \uparrow A$ (cf. Lemma 2.2 with $\psi := f \in C_0^+(X)$),\footnote{As usual, $C_0^+(X)$ denotes the subclass of $C_0(X)$ consisting of all $f \geq 0$.}
\[
\kappa \tau \leq \liminf_{K \uparrow A} \kappa \tau|_K \leq 1 \text{ on } X
\]
(the principle of descent), whence Eq. 7.3.

\[\square\]

Remark 7.1 The requirement of the strong closedness of the class $\mathcal{E}_A^+$ is essential for the validity of Theorem 7.2. Indeed, if $\mathcal{E}_A^+$ is strongly unclosed, it may happen that $\gamma_A \notin \mathcal{E}_A^+$ (see Remark 6.1), and then both Eqs. 7.1 and 7.2 do fail to hold. Furthermore, even if Frostman’s maximum principle is fulfilled, there may exist no $\nu \in \mathcal{E}_A^+$ with $\kappa \nu = 1$ n.e. on $A$ (see footnote 3). In view of these facts, Theorem 9.1 (see Section 9), generalizing Theorem 7.2 to arbitrary $A$, is of particular interest to the present study.

8 Convergence of Inner Capacitary Measures and their Potentials

As in Sections 6 and 7, in this section a kernel $\kappa$ is assumed to be perfect.

Theorem 8.1 For arbitrary $A \subset X$ with $c_*(A) < \infty$,
\[
\gamma_K \to \gamma_A \text{ strongly and vaguely in } \mathcal{E}_A^+ \text{ as } K \uparrow A. \tag{8.1}
\]
If moreover the first and the second maximum principles both hold, then also
\[
\kappa \gamma_K \uparrow \kappa \gamma_A \text{ pointwise on } X \text{ as } K \uparrow A,
\]
and hence
\[
\kappa \gamma_A = \sup_{K \in \mathcal{E}_A} \kappa \gamma_K \text{ on } X. \tag{8.2}
\]

Corollary 8.1 For any $A, H \subset X$ such that $H \subset A$ and $c_*(A) < \infty$,
\[
\|\gamma_A - \gamma_H\|^2 \leq \|\gamma_A\|^2 - \|\gamma_H\|^2. \tag{8.3}
\]
If moreover Frostman’s maximum principle is fulfilled, equality prevails here:
\[
\|\gamma_A - \gamma_H\|^2 = \|\gamma_A\|^2 - \|\gamma_H\|^2. \tag{8.4}
\]
Proof Since $\gamma_A \in \Gamma_H^+$, while $\gamma_H$ minimizes $\|v\|^2$ over $v \in \Gamma_H^+$, we derive Eq. 8.3 from [16, Lemma 4.1.1]. Assume now that Frostman’s maximum principle is satisfied. Noting that then $\kappa \gamma_A = 1$ holds n.e. on each $K \in \mathfrak{C}_H$, hence $\gamma_K$-a.e. (Lemma 2.5), whereas

$$\gamma_K \rightarrow \gamma_H \text{ strongly in } \mathfrak{C}^+ \text{ as } K \uparrow H,$$

we obtain

$$\kappa(\gamma_A, \gamma_H) = \lim_{K \uparrow H} \kappa(\gamma_A, \gamma_K) = \lim_{K \uparrow H} \int \kappa \gamma_A \, d\gamma_K = \lim_{K \uparrow H} \gamma_K(X) = \lim_{K \uparrow H} \|\gamma_K\|^2 = \|\gamma_H\|^2,$$

whence Eq. 8.4. (The forth equality here is valid by Eq. 6.3 with $A := K$, while the first follows from Eq. 8.5 with the aid of the Cauchy–Schwarz inequality.)

Corollary 8.2 (Monotonicity property) If $c_\star(A) < \infty$, and if the first and the second maximum principles both hold, then for any $H \subset A$,

$$\kappa \gamma_H \leq \kappa \gamma_A \text{ on } X.$$

Proof Since obviously $\mathfrak{C}_H \subset \mathfrak{C}_A$, Eq. 8.6 is implied by Eq. 8.2.

Theorem 8.2 Given arbitrary $A \subset X$, assume there exists an increasing sequence $(U_j)$ of universally measurable sets $U_j \subset X$ such that

$$A = \bigcup_{j \in \mathbb{N}} A_j, \text{ where } A_j := A \cap U_j.$$

Then

$$c_\star(A) = \lim_{j \to \infty} c_\star(A_j). \quad (8.7)$$

If now $c_\star(A) < \infty$, then also

$$\gamma_{A_j} \rightarrow \gamma_A \text{ strongly and vaguely}, \quad (8.8)$$

$$\kappa \gamma_A = \lim_{j \to \infty} \kappa \gamma_{A_j} \text{ n.e. on } X. \quad (8.9)$$

If moreover the first and the second maximum principles both hold, then Eq. 8.9 can be refined as follows:

$$\kappa \gamma_{A_j} \uparrow \kappa \gamma_A \text{ on } X. \quad (8.10)$$

Remark 8.1 Limit relations (8.7) and (8.8) generalize Fuglede’s results [16] (Lemma 2.3.3 and Theorem 4.2), established in the case where $A$ is the union of an increasing sequence of universally measurable subsets (and hence $A$ is likewise universally measurable).\(^{27}\) The generalization obtained (see Section 8.2 for a proof) is substantially based on Theorem 6.2, which makes it possible to reduce the analysis to that for Borel sets.

Theorem 8.3 Let $A$ be the intersection of a decreasing net (resp. a decreasing sequence) $(A_t)_{t \in T}$ of closed (resp. quasiclosed) sets $A_t \subset X$ such that

$$c_\star(A_t) < \infty \text{ for some } t_0 \in T. \quad (8.11)$$

\(^{27}\)Such $(U_j)$ necessarily exists if the space $X$ is second-countable or, more generally, $\sigma$-compact.

\(^{28}\)See also [22, Section II.2.9, Remark] pertaining to the Riesz kernels on $\mathbb{R}^n$. 

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Then

\[ \gamma_{A_t} \to \gamma_A \text{ strongly and vaguely,} \quad (8.12) \]

and hence

\[ \lim_{t \in T} c_s(A_t) = c_s(A). \quad (8.13) \]

If moreover the first and the second maximum principles both hold, then also

\[ \kappa \gamma_{A_t} \downarrow \kappa \gamma_A \text{ pointwise n.e. on } X. \quad (8.14) \]

**Remark 8.2** Assumption (8.11) is essential for the validity of Theorem 8.3, and this is the case even for the Newtonian kernel \(|x - y|^{2-n}\) on \(\mathbb{R}^n, n \geq 3\). Indeed, let \(K \subset \mathbb{R}^n\) be compact; the closed sets \(A_j := K \cup \{|x| \geq j\}, j \in \mathbb{N}\), form a decreasing sequence with the intersection \(K\). However, \(c(K) < \infty\), whereas \(c(A_j) = \infty\) for all \(j\), and so Eq. 8.13 does fail to hold.

### 8.1 Proof of Theorem 8.1

Lemma 6.2 is crucial to the present proof.

Assume \(c_s(A) > 0\), for if not, the theorem holds trivially with \(\gamma_K = \gamma_A = 0\). Then for all \(K \in \mathcal{C}_A\) sufficiently large \((K \geq K_0)\), we have \(c(K) > 0\), and hence there exists the (unique) \(\lambda_K \in \tilde{\mathcal{E}}^K \subset \tilde{\mathcal{E}}^A\) with

\[ \|\lambda_K\|^2 = w(K) = 1/c(K). \]

It is clear from Eq. 2.2 that these \(\lambda_K\) form a minimizing net, which according to Lemma 6.2 must converge to the extremal measure \(\xi_A\) strongly and vaguely. Combined with Eqs. 2.2, 6.1, and 6.15, this establishes Eq. 8.1.

Assume now additionally that the first and the second maximum principles are both fulfilled. For any \(K, K' \in \mathcal{C}_A\) such that \(K_0 \leq K \leq K'\), then

\[ \kappa \gamma_K = \kappa \gamma_{K'} = \kappa \gamma_A = 1 \text{ n.e. on } K, \]

which is derived from Eq. 6.7 by use of the countable subadditivity of inner capacity on universally measurable sets. Since each of these equalities holds \(\gamma_K\)-a.e. on \(X\) (Lemma 2.5), the second maximum principle gives

\[ \kappa \gamma_K \leq \kappa \gamma_{K'} \leq \kappa \gamma_A \text{ on } X. \]

Thus \(\lim_{K \uparrow A} \kappa \gamma_K\) exists everywhere on \(X\), and moreover it does not exceed \(\kappa \gamma_A\). To complete the proof, it remains to show that

\[ \kappa \gamma_A \leq \lim_{K \uparrow A} \kappa \gamma_K \text{ on } X, \]

which however follows immediately from the vague convergence of the net \((\gamma_K)_{K \in \mathcal{C}_A}\) to \(\gamma_A\) by the principle of descent.

---

29If \(A\) is the intersection of a decreasing net of closed \(A_t\), Eq. 8.12 was declared in [16, Lemma 4.2.1]; however, its proof was incomplete, being based on [16, Section 2.5, Eq. (1)] (cf. Remark 4.1 above).
8.2 Proof of Theorem 8.2

Applying Theorem 6.2 we see that there exist $K_\sigma$-sets $A'_j \subset A_j$, $j \in \mathbb{N}$, and $A' \subset A$ such that $A'_j \subset A'_{j+1}$ and

$$c_*(A'_j) = c_*(A_j), \quad c_*(A') = c_*(A).$$

Then

$$\widehat{A}_j := A'_j \cup (A' \cap U_j) \subset A_j, \quad j \in \mathbb{N},$$

form an increasing sequence of $K_\sigma$-sets, whose union

$$\widehat{A} := \bigcup_{j \in \mathbb{N}} \widehat{A}_j \supset A'$$

is a $K_\sigma$-subset of $A$. The $K_\sigma$-sets being universally measurable,

$$c_*(A) = c_*(\widehat{A}) = \lim_{j \to \infty} c_*(\widehat{A}_j) = \lim_{j \to \infty} c_*(A_j),$$

whence Eq. 8.7. The second equality here holds by [16, Lemma 2.3.3], whereas the first (resp. the third) holds by Eq. 6.10 with $H := \widehat{A}$ (resp. with $A := A_j$ and $H := \widehat{A}_j$).

If now $c_*(A)$ is finite, then so is $c_*(\widehat{A})$, and [16, Theorem 4.2] (applied to the universally measurable sets $\widehat{A}_j$, $j \in \mathbb{N}$, and $\widehat{A}$) gives

$$\gamma_{\widehat{A}_j} \to \gamma_{\widehat{A}}$$

strongly and vaguely,

the kernel $\kappa$ being perfect. This proves Eq. 8.8, for, by Theorem 6.2,

$$\gamma_{\widehat{A}_j} = \gamma A_j, \quad \gamma_{\widehat{A}} = \gamma A.$$

In view of the strong convergence of $(\gamma A_j)$ to $\gamma A$, [16, Lemma 3.2.4] yields

$$\kappa \gamma A \geq \liminf_{j \to \infty} \kappa \gamma A_j \quad \text{n.e. on } X.$$

The opposite being valid everywhere on $X$ by the vague convergence of $(\gamma A_j)$ to $\gamma A$ (the principle of descent), Eq. 8.9 follows.

If the first and the second maximum principles are both fulfilled, the monotonicity property (Corollary 8.2) implies that $(\kappa \gamma A_j)$ increases pointwise on $X$, and moreover

$$\lim_{j \to \infty} \kappa \gamma A_j \leq \kappa \gamma A \quad \text{on } X.$$

The opposite being obvious by the principle of descent (cf. above), this gives Eq. 8.10.

8.3 Proof of Theorem 8.3

As the net $(\|\gamma_{A_t}\|)$ is decreasing, we see from Eq. 8.3 that $(\gamma A_t)$ is strong Cauchy in $\mathcal{E}^+$, and there is therefore the unique $\gamma \in \mathcal{E}^+$ such that

$$\gamma A_t \to \gamma$$

strongly and vaguely, (8.15)

whence

$$\|\gamma\|^2 = \lim_{t \in T} \|\gamma_{A_t}\|^2 = \lim_{t \in T} c_*(A_t) \geq c_*(A).$$

(8.16)

Since $\kappa \gamma A_t \leq 1$ on $S(\gamma A_t)$ for every $t$, cf. Eq. 6.5, we infer from the vague convergence of $(\gamma A_t)_{t \in T}$ to $\gamma$ that $\kappa \gamma \leq 1$ on $S(\gamma)$, hence $\|\gamma\|^2 = \int \kappa \gamma d\gamma \leq \gamma (X)$, and consequently

$$G(\gamma) = 2\gamma (X) - \|\gamma\|^2 \geq \|\gamma\|^2.$$
Suppose first that \( A \) is the intersection of a decreasing net \((A_t)_{t \in T}\) of closed sets. Then \( \gamma \) belongs to the class \( \mathcal{E}_A^+ \), for every \( t \in T \), \( \mathcal{E}_A^+ \) being strongly closed. Since for a closed set \( F \subset X \), \( \mathcal{E}_F^+ \) consists of all \( v \in \mathcal{E}^+ \) supported by \( F \), \( \gamma \) is supported by every \( A_t \), and hence by the intersection of \( A_t \) over all \( t \). Thus

\[
\gamma \in \mathcal{E}_A^+, \tag{8.18}
\]

which in view of Eq. 4.3 yields

\[
G(\gamma) \leq \sup_{v \in \mathcal{E}_A^+} G(v) = c_*(A).
\]

Combining this with Eqs. 8.16 and 8.17, we obtain

\[
c_*(A) \geq G(\gamma) \geq \|\gamma\|^2 \geq c_*(A),
\]

whence \( \gamma = \gamma_A \), by the uniqueness of the solution to extremal problem (4.3) (see Theorem 7.2). Substituting \( \gamma = \gamma_A \) into Eq. 8.15 gives Eq. 8.12.

Let in addition the first and the second maximum principles be both fulfilled. By Corollary 8.2, then the net \((\kappa \gamma A_t)_{t \in T}\) decreases pointwise on \( X \), and moreover

\[
\kappa \gamma A(x) \leq \lim_{t \in T} \kappa \gamma A_t(x) \quad \text{for all} \quad x \in X. \tag{8.19}
\]

The strong topology on \( \mathcal{E} \) having a countable base of neighborhoods, it follows from Eq. 8.12 that there exists a subsequence \((\gamma_{A_t})_{j \in \mathbb{N}}\) of the net \((\gamma_{A_t})_{t \in T}\) that converges strongly to \( \gamma_A \). Applying [16, Lemma 3.2.4] we therefore conclude that equality in fact prevails in inequality (8.19) for nearly all \( x \in X \), which establishes Eq. 8.14.

Let \( A \) now be the intersection of a decreasing sequence \((A_t)_{t \in T}\) of quasiclosed sets. In view of the fact that a countable intersection of quasiclosed sets is likewise quasiclosed [17, Lemma 2.3], the proof of Eqs. 8.12 and 8.14 is essentially the same as above, the only difference being in that of Eq. 8.18. Noting that for each \( t \), the cone \( \mathcal{E}_A^+ \) is strongly closed (Lemma 3.3), we obtain \( \gamma \in \mathcal{E}_A^+ \) (cf. above), which means that \( (A_t)^c \) is locally \( \gamma \)-negligible. Being thus a countable union of locally \( \gamma \)-negligible sets, \( A^c \) is likewise locally \( \gamma \)-negligible [4, Section IV.5.2], which proves Eq. 8.18.

### 9 Characterizations of \( c_*(A) \) and \( \gamma_A \) for Arbitrary \( A \)

As in Sections 6–8, a kernel \( \kappa \) is perfect. The purpose of this section is to establish alternative characterizations of the inner capacity \( c_*(A) \) and the inner capacitary measure \( \gamma_A \) for arbitrary sets \( A \subset X \). To achieve such a level of generality, assume in addition that the first and the second maximum principles are both fulfilled. Denote

\[
\widehat{\mathcal{E}}_A^+ := \{ v \in \mathcal{E}_A^+ : \kappa v \leq 1 \text{ on } X \},
\]

\( \mathcal{E}_A^+ \) being the closure of \( \mathcal{E}_A^+ \) in the strong topology on \( \mathcal{E}^+ \).

\( \widehat{\mathcal{E}}_A^+ \) would be the same if \( \kappa v \leq 1 \) on \( X \) were replaced by the apparently weaker inequality \( \kappa v \leq 1 \text{-a.e.} \), which is obvious from the lower semicontinuity of the potential \( \kappa v \) on \( X \), \( v \) being positive, and Frostman’s maximum principle.
Theorem 9.1 \((31)\) For arbitrary \(A \subset X\),
\[
c_*(A) = \sup_{v \in \mathcal{E}_A'} G(v), \quad (9.1)
\]
\[
c_*(A) = \sup_{v \in \mathcal{E}_A'} v(X). \quad (9.2)
\]
If now \(c_*(A) < \infty\), the inner equilibrium measure \(\gamma_A\) can alternatively be characterized as the unique solution to either of extremal problems \((9.1)\) or \((9.2)\), that is, \(\gamma_A \in \mathcal{E}_A' \subset \mathcal{E}_A'\) and
\[
\max_{v \in \mathcal{E}_A'} G(v) = G(\gamma_A) \quad (\text{or} \ c_*(A)), \quad (9.3)
\]
\[
\max_{v \in \mathcal{E}_A'} v(X) = \gamma_A(X) \quad (\text{or} \ c_*(A)). \quad (9.4)
\]
Furthermore, \(\gamma_A\) is uniquely characterized within \(\mathcal{E}_A'\) by each of \(\text{a})\) or \(\text{c})\), where

\(\text{a})\) \(\gamma_A(X) \geq \|\gamma_A\|^2 \geq c_*(A)\).

\(\text{c})\) \(\kappa \gamma_A = 1 \text{ n.e. on } A\).

Remark 9.1 As noted in Remark 7.1, in the case where \(\mathcal{E}_A^+\) is strongly unclosed, problems \((4.3)\) and \((4.4)\) of maximizing \(G(v)\), resp. \(v(X)\), over \(\mathcal{E}_A^+\), resp. \(\hat{\mathcal{E}}_A^+\), are in general unsolvable. To overcome this inconvenience, we have now enlarged the classes of admissible measures so that the extremal values remain the same, but the extremal problems be already solvable — even for arbitrary \(A\). See \((9.1)\)–\((9.4)\).

Remark 9.2 Being valid for arbitrary \(A \subset X\), Theorem 9.1 seems to be new even for the Green kernels associated with the Laplacian on Greenian sets in \(\mathbb{R}^n, n \geq 2\) (in particular, for the Newtonian kernel on \(\mathbb{R}^n, n \geq 3\), as well as for the \(\alpha\)-Riesz kernels \(|x−y|^{α−n}\) and the associated \(\alpha\)-Green kernels on \(\mathbb{R}^n\), where \(α < 2 \leq n\).

Theorem 9.2 For arbitrary \(A \subset X\) with \(c_*(A) < \infty\), the inner equilibrium measure \(\gamma_A\) can alternatively be characterized as the unique measure of minimum potential in the class \(\Gamma_A^+\), that is, \(\gamma_A \in \Gamma_A^+\) and
\[
\kappa \gamma_A = \min_{v \in \Gamma_A^+} \kappa v \text{ on } X. \quad (9.5)
\]
If moreover \(X\) is \(σ\)-compact, then \(\gamma_A\) is also of minimum total mass in \(\Gamma_A^+\), that is,
\[
\gamma_A(X) = \min_{v \in \Gamma_A^+} v(X). \quad (9.6)
\]
Corollary 9.1 If the space \(X\) is \(σ\)-compact, then for any \(A \subset X\),
\[
c_*(A) = \inf_{v \in \Gamma_A^+} v(X). \quad (9.7)
\]
Proof If \(c_*(A) = \infty\), then \(\Gamma_A^+ = \emptyset\) (see footnote 21), and the right-hand side in Eq. 9.7 equals \(+\infty\) by convention. In the remaining case \(c_*(A) < \infty\), Eq. 9.7 is implied by Eq. 9.6.

\(\text{\(31\)Compare with Theorems 4.1–4.3 and 7.2.}\)

\(\text{\(32\)Recall that \(\gamma_A\) is the unique measure of minimum energy in } \Gamma_A^+ \text{ (Theorem 6.1), and this holds true even for an arbitrary perfect kernel.}\)
**Theorem 9.3** For arbitrary \( A \subset X \) with \( c_*(A) < \infty \), \( \gamma_A \) can alternatively be characterized as the unique measure in \( E^+ \) satisfying any one of the three limit relations

\[
\begin{align*}
\gamma_K \to \gamma_A & \text{ strongly in } E^+ \text{ as } K \uparrow A, \\
\gamma_K \to \gamma_A & \text{ vaguely in } E^+ \text{ as } K \uparrow A, \\
\kappa \gamma_K \uparrow \kappa \gamma_A & \text{ pointwise on } X \text{ as } K \uparrow A,
\end{align*}
\]

where \( \gamma_K \) denotes the only measure in \( E^+_K \) having the property \( \kappa \gamma_K = 1 \) n.e. on \( K \).

**Remark 9.3** Consider the \( \alpha \)-Riesz kernel \(|x - y|^{\alpha - n}\) of order \( \alpha \in (0, 2], \alpha < n \), on \( \mathbb{R}^n \), \( n \geq 2 \). Then the concept of inner equilibrium measure as well as its alternative characterizations, provided by formulae (9.5), (9.8), and (9.9), can be generalized to arbitrary \( A \subset \mathbb{R}^n \) that is inner \( \alpha \)-thin at infinity — even if the inner \( \alpha \)-Riesz capacity of \( A \) is infinite. (See the author’s recent results established in [37, Section 5] and [38, Sections 1.3, 2.1, 2.3].) We emphasize that in these generalizations, the inner \( \alpha \)-Riesz equilibrium measure \( \gamma_A \) as well as the admissible measure in extremal problem (9.5) must be allowed to have infinite energy.

### 9.1 Interaction Between the Concepts of Equilibrium and Swept Measures

The proofs of Theorems 9.1–9.3 (see Sections 9.2–9.4) are based on the close interaction between the concepts of inner equilibrium and inner swept measures, discovered in Lemma 9.1, as well as on the results on inner balayage established in Section 3.

**Lemma 9.1** For any \( A, Q \subset X \) such that \( A \subset Q \) and \( c_*(Q) < \infty \),

\[
(\gamma Q)^A = \gamma_A.
\]

**Proof** By Definition 3.1 and Lemma 3.1 with \( \mu := \gamma Q \), \((\gamma Q)^A\) is the unique measure minimizing the energy over \( \Gamma^+_A, \gamma Q \), whereas \( \gamma_A \) is the unique solution to the minimum energy problem over \( \Gamma^+_A \) (see Theorem 6.1). It is thus enough to show that

\[
\Gamma^+_A, \gamma Q = \Gamma^+_A.
\]

This however is obvious in view of the fact that for any given \( \nu \in \Gamma^+_A, \gamma Q \cup \Gamma^+_A \),

\[
\kappa \nu \geq 1 = \kappa \gamma Q \text{ n.e. on } A,
\]

which in turn is derived from Eqs. 3.1, 6.7, and 6.2 (the first two being applied to \( \gamma Q \)) by making use of Lemma 3.2.

### 9.2 Proof of Theorem 9.1

Assume \( c_*(A) < \infty \), for if not, Eq. 9.1, resp. Eq. 9.2, holds in the sense that its both sides equal \( +\infty \), which is obvious from Eq. 4.3, resp. Eq. 4.4, in view of the inclusion \( E^+_A \subset E'_A \), resp. \( E^+_A \subset E'_A \). Also assume that \( c_*(A) > 0 \), for if not, \( E'_A = E^+_A = \{0\} \) by Lemma 2.4, and the theorem holds trivially with \( \gamma_A = 0 \).

For each \( \nu_0 \in E'_A \), there is a sequence \((\nu_j) \subset E^+_A\) converging to \( \nu_0 \) both strongly and vaguely, the kernel \( \kappa \) being perfect. Noting that \( \nu(X) \) is vaguely lower semicontinuous on \( M^+ \), whereas \( \|\nu\|^2 \) is strongly continuous on \( E^+ \), we obtain

\[
G(\nu_0) \leq \liminf_{j \to \infty} G(\nu_j) \leq \sup_{v \in E^+_A} G(v),
\]
and letting now \( \nu \) range over \( \mathcal{E}_A' \) gives
\[
\sup_{\nu \in \mathcal{E}_A'} G(\nu) \leq \sup_{\nu \in \mathcal{E}_A^+} G(\nu).
\]
The opposite being obvious, combining this with Eq. 4.3 establishes Eq. 9.1.

The class \( \mathcal{E}_A' \) being strongly closed, \( \gamma_A \in \mathcal{E}_A' \) (cf. Lemma 6.1 or Theorem 8.1). Furthermore, \( \gamma_A \in \mathcal{E}_A' \) for \( \kappa \gamma_A \leq 1 \) on \( X \) by Frostman’s maximum principle.

Since according to Lemma 9.1 with \( Q := A \),
\[
\gamma_A = (\gamma_A)^A,
\]
we infer from Theorem 3.1(b) that
\[
\min_{\nu \in \mathcal{E}_A'} \| \gamma_A - \nu \|^2 = 0,
\]
the minimum being attained at the unique \( \gamma_A \in \mathcal{E}_A' \). Consequently,
\[
c_*(A) = \| \gamma_A \|^2 = \max_{\nu \in \mathcal{E}_A'} \left( 2 \int \kappa \gamma_A \, d\nu - \| \nu \|^2 \right),
\]
the maximum being likewise attained at the same (unique) \( \gamma_A \in \mathcal{E}_A' \).

On account of the fact that \( \kappa \gamma_A \leq 1 \) on \( X \), we therefore obtain
\[
c_*(A) = \gamma_A(X) \leq \sup_{\nu \in \mathcal{E}_A'} \nu(X) \leq \sup_{\nu \in \mathcal{E}_A'} G(\nu) = \max_{\nu \in \mathcal{E}_A'} G(\nu) = G(\gamma_A) = c_*(A),
\]
the first inequality being valid by \( \gamma_A \in \mathcal{E}_A' \), and the last two equalities by Eq. 9.3. This proves both Eqs. 9.2 and 9.4. Furthermore, the solution \( \gamma_A \) to problem (9.2) is unique, which is seen from Eq. 9.14 in view of the uniqueness of the solution to problem (9.1).

According to Eqs. 6.3 and 6.7, \( \gamma_A \) has properties (a) and (c). It thus remains to show that each of these (a) and (c) determines \( \gamma_A \) uniquely within \( \mathcal{E}_A' \).

If \( \mu \) is another measure in \( \mathcal{E}_A' \) satisfying (a), then, by Eq. 9.3,
\[
c_*(A) = \max_{\nu \in \mathcal{E}_A'} G(\nu) \geq G(\mu) = 2 \mu(X) - \| \mu \|^2 \geq \| \mu \|^2 \geq c_*(A).
\]
Thus \( \mu \) must be the (unique) solution to problem (9.1), and so indeed \( \mu = \gamma_A \).

Assume finally that \( \tau \) is another measure in \( \mathcal{E}_A' \) with \( \kappa \tau = 1 \) n.e. on \( A \); then
\[
\kappa \tau = \kappa \gamma_A \text{ n.e. on } A,
\]
which is clear from Eq. 6.7 by use of Lemma 3.2. But according to Theorem 3.1(c) with \( \mu := \gamma_A \), \( (\gamma_A)^A \) is the only measure in \( \mathcal{E}_A' \) whose potential equals \( \kappa \gamma_A \) nearly everywhere on \( A \). Hence, \( \tau = (\gamma_A)^A \), which combined with Eq. 9.12 gives \( \tau = \gamma_A \).

---

33 The inner equilibrium measure \( \gamma_A \) serves here as the (unique) orthogonal projection of \( \gamma_A \) in the pre-Hilbert space \( \mathcal{E} \) onto the convex, strongly complete cone \( \mathcal{E}_A' \).
9.3 Proof of Theorem 9.2

On account of Eq. 9.12, we infer from Theorem 3.1(a) with \( \mu := \gamma_A \) that \( \gamma_A \) is the only measure of minimum potential in the class \( \Gamma_A, \gamma_A \), i.e.

\[
\kappa \gamma_A = \min_{\nu \in \Gamma_A, \gamma_A} \kappa \nu \text{ on } X,
\]

which combined with Eq. 9.11 with \( Q := A \) results in Eq. 9.5. Similarly, Eq. 9.6 follows by substituting Eqs. 9.11 and 9.12 (the former with \( Q := A \)) into Eq. 3.9 with \( \mu := \gamma_A \).

9.4 Proof of Theorem 9.3

In view of Eq. 9.12, we derive this from Theorem 3.1(d) with \( \mu := \gamma_A \) noting that the unique \( \nu \in E^+_K \) with \( \kappa \nu = \kappa \gamma_A \) n.e. on \( K \) obviously coincides with the unique measure in \( E^+_K \) whose potential equals 1 n.e. on \( K \).

10 On Convergence and Alternative Characterizations of Outer Capacities and Outer Capacitary Measures

Assuming as usual that a kernel \( \kappa \) is perfect, we shall now make the following two additional hypotheses concerning a locally compact space \( X \):

(\( H_1 \)) \( X \) is perfectly normal.\(^{34}\)

(\( H_2 \)) \( X \) is \( \sigma \)-compact.

Remark 10.1 Given a locally compact space \( X \), both (\( H_1 \)) and (\( H_2 \)) are fulfilled if \( X \) is second-countable; but not the other way around. Indeed, according to [3, Section IX.2, Corollary to Proposition 16], \( X \) is second-countable if and only if it is metrizable and \( \sigma \)-compact. Being therefore metrizable, a second-countable, locally compact space \( X \) must be perfectly normal [3, Section IX.1, Proposition 2]; whereas the converse fails in general by virtue of [3, Section IX.2, Exercise 13(b)].

Theorem 10.1 Any Borel subset of a locally compact space \( X \) satisfying hypotheses (\( H_1 \)) and (\( H_2 \)) and endowed with a perfect kernel \( \kappa \), is capacitable.

Proof This follows directly from [16, Theorem 4.5]. \( \square \)

In the remainder of this section, assume \( A \subset X \) to be Borel, hence capacitable:

\[
c_\ast(A) = c^\ast(A) \ (=: c(A)).
\]

If moreover \( c(A) < \infty \), then the (unique) inner capacitary measure \( \gamma_A \) (see Theorem 6.1) serves simultaneously as the outer capacitary measure \( \gamma^\ast_A \), defined as the (unique) measure

---

\(^{34}\)By Urysohn’s theorem [3, Section IX.1, Theorem 1], a Hausdorff space \( Y \) is said to be normal if for any disjoint closed \( F_1, F_2 \subset Y \), there exist disjoint open \( D_1, D_2 \) such that \( F_i \subset D_i \) (\( i = 1, 2 \)). A normal space \( Y \) is said to be perfectly normal [3, Section IX.4, Exercise 7] if each closed subset of \( Y \) is a countable intersection of open sets.
of minimum energy in the class $\Gamma_A^*$, where
\[ \Gamma_A^* := \{ \nu \in \mathcal{E}^+ : \kappa \nu \geq 1 \text{ q.e. on } A \}. \]
Indeed, the equality $\gamma_A = \gamma_A^*$ can be concluded from [16, Theorem 4.3] noting that
\[ \Gamma_A^+ = \Gamma_A^*, \]
which in turn follows from the fact that $A \cap \{ \kappa \nu < 1 \}$ is Borel, hence capacitable.

This makes it possible to utilize the results on inner capacities and inner capacitary measures, obtained in Sections 6–9 above, in order to perform a similar analysis for outer capacities and outer capacitary measures. We are thus led to the following Theorems 10.2–10.7 (compare with Theorems 4.1–4.3, 7.2, 8.1, 8.3, 9.1–9.3).

**Theorem 10.2** For any Borel $A \subset X$ with $c^*(A) < \infty$,
\[ \gamma_K \rightarrow \gamma_A^* \text{ strongly and vaguely in } \mathcal{E}^+ \text{ as } K \uparrow A. \]
If moreover the first and the second maximum principles both hold, then also
\[ \kappa \gamma_K \uparrow \kappa \gamma_A^* \text{ pointwise on } X \text{ as } K \uparrow A. \]

**Theorem 10.3** Let $A$ be the intersection of a decreasing sequence $(A_j)$ of Borel, quasi-closed sets $A_j \subset X$ such that $c^*(A_j) < \infty$ for some $j_0$. Then
\[ \gamma_{A_j}^* \rightarrow \gamma_A^* \text{ strongly and vaguely as } j \rightarrow \infty. \]
If moreover the first and the second maximum principles both hold, then also
\[ \kappa \gamma_{A_j}^* \downarrow \kappa \gamma_A^* \text{ pointwise q.e. on } X. \]

**Theorem 10.4** For any Borel $A \subset X$,
\[ c^*(A) = \sup_{\nu \in \mathcal{E}^+_A} G(\nu) = \sup_{\nu \in \mathcal{E}^-_A} \nu(X). \quad (10.1) \]
If moreover the class $\mathcal{E}^+_A$ is strongly closed (or in particular if $A$ is quasiclosed), and if $c^*(A) < \infty$, then each of the two extremal problems appearing in formula (10.1) has the same unique solution, and it exactly equals the outer capacitary measure $\gamma_A^*$. This $\gamma_A^*$ can also be uniquely characterized within $\mathcal{E}^+_A$ by either of (a') or (b'), where
\[ \begin{align*}
(a') & \quad \gamma_A^*(X) \geq \|\gamma_A^*\|^2 \geq c^*(A). \\
(b') & \quad \kappa \gamma_A^* \geq 1 \text{ q.e. on } A, \text{ and } \kappa \gamma_A^* \leq 1 \gamma_A^* \text{-a.e.}
\end{align*} \]
If in addition Frostman’s maximum principle is fulfilled, then $\gamma_A^*$ is the only measure in $\mathcal{E}^+_A$ having the property
\[ (c') \quad \kappa \gamma_A^* = 1 \text{ q.e. on } A. \]

In the following three theorems assume additionally that the (perfect) kernel $\kappa$ satisfies both the first and the second maximum principles.

**Theorem 10.5** For any Borel $A \subset X$,
\[ c^*(A) = \sup_{\nu \in \mathcal{E}^+_A} G(\nu) = \sup_{\nu \in \mathcal{E}^-_A} \nu(X). \quad (10.2) \]
If now $c^*(A) < \infty$, then each of the two extremal problems appearing in formula (10.2) has the same unique solution, and it exactly equals the outer equilibrium measure $\gamma_A^*$. This $\gamma_A^*$ can also be uniquely characterized within $E_A^\prime$ by either of (a') or (c'), where

(a') $\gamma_A^*(X) \geq \|\gamma_A^*\|^2 \geq c^*(A)$.
(c') $\kappa \gamma_A^* = 1$ q.e. on $A$.

**Theorem 10.6** For any Borel $A \subset X$ with $c^*(A) < \infty$, the outer equilibrium measure $\gamma_A^*$ can alternatively be characterized as the unique measure of minimum potential in the class $\Gamma_A^*$, that is,

$$\kappa \gamma_A^* = \min_{\nu \in \Gamma_A^*} \kappa \nu \text{ on } X.$$  

Furthermore, $\gamma_A^*$ is also of minimum total mass in the class $\Gamma_A^*$, that is,

$$\gamma_A^*(X) = \min_{\nu \in \Gamma_A^*} \nu(X).$$

**Theorem 10.7** For any Borel $A \subset X$ with $c^*(A) < \infty$, $\gamma_A^*$ can alternatively be characterized as the unique measure in $E_A^+$ satisfying any one of the three limit relations

$$\gamma_K^* \rightarrow \gamma_A^* \text{ strongly in } E_A^+ \text{ as } K \uparrow A,$$

$$\gamma_K^* \rightarrow \gamma_A^* \text{ vaguely in } E_A^+ \text{ as } K \uparrow A,$$

$$\kappa \gamma_K^* \uparrow \kappa \gamma_A^* \text{ pointwise on } X \text{ as } K \uparrow A,$$

where $\gamma_K^*$ denotes the only measure in $E_K^+$ having the property $\kappa \gamma_K^* = 1$ q.e. on $K$.

### 11 Alternative Characterizations of Outer Swept Measures

In this section we assume that a locally compact space $X$ is $\sigma$-compact and perfectly normal, and that a kernel $\kappa$ is perfect and satisfies the domination principle.

For $\mu \in E_A^+$ and $A \subset X$, denote

$$\Gamma_{A,\mu}^* := \{\nu \in E_A^+ : \kappa \nu \geq \kappa \mu \text{ q.e. on } A\}. \quad (11.1)$$

**Definition 11.1** The outer balayage $\mu^{*A}$ of $\mu \in E_A^+$ to $A \subset X$ is defined as the measure of minimum energy in the class $\Gamma_{A,\mu}^*$, that is, $\mu^{*A} \in \Gamma_{A,\mu}^*$ and

$$\|\mu^{*A}\|^2 = \min_{\nu \in \Gamma_{A,\mu}^*} \|\nu\|^2.$$  

Observe that this definition differs from that of inner balayage (Definition 3.1) only by replacing an exceptional set in Eq. 3.1 by that of outer capacity zero, cf. Eq. 11.1.

**Theorem 11.1** For any $\mu \in E_A^+$ and any Borel $A \subset X$, there exists the unique outer balayage $\mu^{*A}$, introduced by Definition 11.1, and it satisfies the three relations

$$\kappa \mu^{*A} = \kappa \mu \text{ q.e. on } A,$$

$$\kappa \mu^{*A} = \kappa \mu \text{ } \mu^{*A}-\text{a.e.},$$

$$\kappa \mu^{*A} \leq \kappa \mu \text{ on } X.$$
The outer balayage $\mu^* A$ actually coincides with the inner balayage $\mu^ A$, and it can alternatively be characterized by means of any one of the following (equivalent) assertions (a')–(d'):

(a') $\mu^* A$ is the unique measure of minimum potential in $\Gamma^*_{A, \mu}$, that is, $\mu^* A \in \Gamma^*_{A, \mu}$ and

$$\kappa \mu^* A = \min_{\nu \in \Gamma^*_{A, \mu}} \kappa \nu \text{ on } X.$$  

(b') $\mu^* A$ is the unique orthogonal projection of $\mu$ in the pre-Hilbert space $\mathcal{E}$ onto the (convex, strongly complete) cone $\mathcal{E}'_A$, that is, $\mu^* A \in \mathcal{E}'_A$ and

$$\|\mu - \mu^* A\| = \min_{\nu \in \mathcal{E}'_A} \|\mu - \nu\|.$$  

(c') $\mu^* A$ is the unique measure in $\mathcal{E}''_A$ having the property

$$\kappa \mu^* A = \kappa \mu \text{ q.e. on } A.$$  

(d') $\mu^* A$ is the unique measure in $\mathcal{E}''_A$ satisfying any one of the three limit relations

$$\mu^* K \to \mu^* A \text{ strongly in } \mathcal{E}''_A \text{ as } K \uparrow A,$$

$$\mu^* K \to \mu^* A \text{ vaguely in } \mathcal{E}''_A \text{ as } K \uparrow A,$$

$$\kappa \mu^* K \uparrow \kappa \mu^* A \text{ pointwise on } X \text{ as } K \uparrow A,$$

where $\mu^* K$ denotes the only measure in $\mathcal{E}''_K$ with $\kappa \mu^* K = \kappa \mu \text{ q.e. on } K$.

If moreover the class $\mathcal{E}''_A$ is strongly closed (or in particular if the set $A$ is quasiclosed), then (b') and (c') remain valid with $\mathcal{E}''_A$ in place of $\mathcal{E}'_A$.

**Proof** We first observe that the classes $\Gamma^+_{A, \mu}$ and $\Gamma^*_{A, \mu}$, introduced by Eqs. 3.1 and 11.1, respectively, coincide:

$$\Gamma^+_{A, \mu} = \Gamma^*_{A, \mu}, \quad (11.2)$$

which follows from the fact that for any $\mu, \nu \in \mathcal{E}^+$, the set $A \cap \{\kappa \mu \neq \kappa \nu\}$ is Borel, hence capacitable (Theorem 10.1). This in turn implies that the inner balayage $\mu^ A$ serves simultaneously as the outer balayage $\mu^* A$, i.e.

$$\mu^* A = \mu^ A, \quad (11.3)$$

and all we need to prove can therefore be immediately derived from Theorem 3.1 and Lemma 3.3. \qed

In the following two corollaries assume in addition that *Frostman's maximum principle* is fulfilled.

**Corollary 11.1** For any $\mu \in \mathcal{E}^+$ any any Borel $A \subset X$, the outer balayage $\mu^* A$ is of minimum total mass in the class $\Gamma^*_{A, \mu}$, that is,

$$\mu^* A (X) = \min_{\nu \in \Gamma^*_{A, \mu}} \nu (X).$$

**Proof** This follows by substituting Eqs. 11.2 and 11.3 into Eq. 3.9. \qed
Corollary 11.2 For any Borel sets $A, Q \subset X$ such that $A \subset Q$ and $c^* (Q) < \infty$,

$$(\gamma^*_Q)^* A = \gamma^*_A.$$  

Proof As observed in Section 10, $\gamma_A = \gamma^*_A$ and $\gamma_Q = \gamma^*_Q$. Substituting this and Eq. 11.3 with $\mu := \gamma^*_Q$ into Eq. 9.10 establishes the corollary.

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