Unique solvability of an equilibrium problem for a Kirchhoff-Love plate with a crack along the boundary of a flat rigid inclusion

G E Semenova and N P Lazarev

1 North-Eastern Federal University, Yakutsk, Russia
E-mail: nyurgun_lazarev@mail.ru

Abstract. A new model of an equilibrium problem for a Kirchhoff-Love plate with a flat cylindrical rigid inclusion and an interfacial crack is considered. As in previous works, we consider a rigid inclusion defined with the help of a cylindrical surface, but unlike the known models relating to the crack theory, we suppose that traces of derivatives of vertical displacements (deflections) satisfy certain boundary conditions. These conditions determine constant angles of normal fibers along an entire flat cylindrical inclusion. The interfacial crack is located on the boundary of the rigid inclusion. A condition of mutual non-penetration of opposite crack faces is given as an inequality on the crack curve. We prove the existence and uniqueness of a solution for this variational problem.

1. Introduction

Justification and examination of the most accurate mathematical models describing the stress-strain state of composites and, in particular, elastic plates with inhomogeneities in the form of inclusions and cracks, is a promising direction of scientific research. It is well known that the difference between the coefficients of thermal expansion and moduli elasticity for composite materials often leads to initiation of cracks (delamination) and ruptures at the boundary interface of different materials. In this regard, it is important to analyze high-level mathematical models of elastic bodies with delaminated inclusions and to investigate dependence of solutions on the variation of physical parameters of inclusions. For inhomogeneous bodies with a crack along the boundary of a rigid or elastic inclusion, problems are further complicated by relations describing mechanical interaction of the inclusion and the supporting matrix.

It is well known that classical linear crack problems in solid mechanics are characterized by linear boundary conditions imposed at the crack faces. Such linear models allow the opposite crack faces to penetrate each other which leads to inconsistency with practical situations [1]. Since the beginning of 1990, the crack theory with non-penetration conditions of inequality type is under active study [1–7]. Using the variational methods, various problems for bodies with rigid inclusions or elastic inclusions have been successfully formulated and investigated [2,4,8–23]. An overview of methods for analysis of solids with inclusions and cracks was presented by Mura [24].

In the present work, we consider a new model of an equilibrium problem for a composite plate with a flat cylindrical rigid inclusion and an interfacial through crack. The plate matrix is assumed to be elastic. As in previous works, we suppose that rigid inclusion is defined...
by a cylindrical inclusion, but unlike the known models [16–18], we suppose that traces of
derivatives of vertical displacements satisfy certain boundary conditions. These conditions
determine constant angles of normal fibers along the entire flat cylindrical inclusion. We consider
a case for an inhomogeneous plate with a crack situated at the inclusion-matrix interface. We
prove the existence and uniqueness of solution for this equilibrium problem.

2. Formulation of a variational problem

Let us formulate an equilibrium problem for an elastic plate containing a flat cylindrical rigid
inclusion. We consider the case of the delaminated inclusion, when the crack passes through
the inclusion interface. Let $\Omega \subset \mathbb{R}^2$ be a bounded with a smooth boundary $\Gamma$. Suppose that a
smooth unclosed curve $\gamma$ lies strictly inside $\Omega$, i.e. $\bar{\gamma} \subset \Omega$. We require that the curve $\gamma$ can be
extended up to the outer boundary $\Gamma$ in such a way that $\Omega$ is divided into two subdomains $\Omega_1$, $\Omega_2$ with Lipschitz boundaries. The latter condition is sufficient to fulfil the Korn and Poincare
inequalities in the domain $\Omega_\gamma = \Omega \setminus \bar{\gamma}$ [1].

For simplicity, suppose the plate has a uniform thickness $2h = 2$. Let us assign a three-
dimensional Cartesian space $\{x_1, x_2, z\}$ with the set $\{\Omega_\gamma\} \times \{0\} \subset \mathbb{R}^3$ corresponding to the middle plane of the plate. We assume that a flat rigid inclusion is described by the cylindrical
surface $x = (x_1, x_2) \in \gamma, -1 \leq z \leq 1$, where $|z|$ is the distance to the middle plane. Elastic
part of the plate corresponds to the domain $\bar{\Omega\setminus\gamma}$. Depending on the direction of the normal $\nu = (\nu_1, \nu_2)$ to $\gamma$ we will speak about a positive face $\gamma^+$ or a negative face $\gamma^-$ of the curve $\gamma$. The jump $[q]$ of the function $q$ on the curve $\gamma$ is found by the formula $[q] = q^+ - q^−, q^+ = q|\gamma^+, q^- = q|\gamma^-$. We suppose that there is a crack which is located on the positive side $(\gamma^+)$ of the inclusion’s surface.

Denote by $\chi = \chi(x) = (W, w)$ the displacement vector of the mid-surface points ($x \in \Omega_\gamma$), by $W = (w_1, w_2)$ the displacements in the plane $\{x_1, x_2\}$, and by $w$ the displacements along the axis $z$.

The strain and integrated stress tensors are denoted by $\varepsilon_{ij} = \varepsilon_{ij}(W), \sigma_{ij} = \sigma_{ij}(W)$, respectively [1]:

$$
\varepsilon_{ij}(W) = \frac{1}{2} \left( \frac{\partial w_j}{\partial x_i} + \frac{\partial w_i}{\partial x_j} \right), \quad \sigma_{ij}(W) = a_{ijkl} \varepsilon_{kl}(W), \quad i, j = 1, 2,
$$

where $\{a_{ijkl}\}$ is the given elasticity tensor, assumed as usual to be symmetric and positive
definite:

$$
a_{ijkl} = a_{klji} = a_{jikl}, \quad i, j, k, l = 1, 2, \quad a_{ijkl} \in L^\infty(\Omega_\gamma),
$$

$$
a_{ijkl} \xi_i \xi_j \xi_k \xi_l \geq c_0 |\xi|^2, \quad \forall \xi, \quad \xi_i = \xi_{ji}, \quad i, j = 1, 2, \quad c_0 = \text{const} > 0.
$$

A summation convention over repeated indices is used in the sequel. Next we denote the bending
moments by formulas [1]

$$
m_{ij}(w) = d_{ijkl} w_{,kl}, \quad i, j = 1, 2, \quad (w_{,kl} = \frac{\partial^2 w}{\partial x_k \partial x_l}),
$$

where the tensor $\{d_{ijkl}\}$ has the same properties as the tensor $\{a_{ijkl}\}$. Let $B(\cdot, \cdot)$ be a bilinear form defined by the equality

$$
B(\chi, \bar{\chi}) = \int_{\Omega_\gamma} \{\sigma_{ij}(W) \varepsilon_{ij}(W) + m_{ij}(w) \bar{w}_{,ij}\} dx,
$$

2
where \( \chi = (W, w) \), \( \bar{\chi} = (\bar{W}, \bar{w}) \). The potential energy functional of the plate has the following representation [1]:

\[
\Pi(\chi) = \frac{1}{2} B(\chi, \chi) - \int_{\Omega} F \chi dx, \quad \chi = (W, w),
\]

where vector \( F = (f_1, f_2, f_3) \in L^2(\Omega)^3 \) describes the body forces [1]. Introduce the Sobolev spaces

\[
H^1,0(\Omega_\gamma) = \{ v \in H^1(\Omega_\gamma) \mid v = 0 \text{ on } \Gamma \},
\]

\[
H^2,0(\Omega_\gamma) = \{ v \in H^2(\Omega_\gamma) \mid v = \frac{\partial w}{\partial n} = 0 \text{ on } \Gamma \},
\]

\[
H(\Omega_\gamma) = H^1,0(\Omega_\gamma)^2 \times H^2,0(\Omega_\gamma).
\]

Note that the following inequality

\[
B(\chi, \chi) \geq c \| \chi \|^2 \quad \forall \chi \in H(\Omega_\gamma), \quad (\| \chi \| = \| \chi \|_{H(\Omega_\gamma)})
\]

(1)

with a constant \( c > 0 \) independent of \( \chi \), holds for the bilinear form \( B(\cdot, \cdot) \) [1].

**Remark 1.** The inequality (1) yields the equivalence of the standard norm and the semi-norm determined by the bilinear form \( B(\cdot, \cdot) \) in the space \( H(\Omega_\gamma) \).

We suppose that the following general nonpenetration condition for cracks in Kirchhoff–Love plates is imposed [1,2].

\[
[W]_\nu \geq \| \frac{\partial w}{\partial \nu} \| \quad \text{on } \gamma
\]

(2)

Due to presence of a flat rigid inclusion in the plate, restrictions of the functions describing displacements \( \chi \) to the curve \( \gamma \) satisfy a special kind of relations. We introduce the space which allows us to characterize the properties of the flat rigid inclusion. Let us first introduce some notation:

\[
R(\gamma) = \{ \zeta(x) = (\rho, l) \mid \rho(x) = b(x_2, -x_1) + (c_1, c_2); \quad l(x) = a_0 + a_1 x_1 + a_2 x_2, \quad x \in \gamma \},
\]

where \( b, c_1, c_2, a_0, a_1, a_2 \in \mathbb{R} \). In addition we should impose the rigid properties of the flat inclusion in plate by the following relations

\[
\frac{\partial w^-}{\partial x_1} = a_1, \quad \frac{\partial w^-}{\partial x_2} = a_2 \quad \text{on } \gamma.
\]

(3)

It is important to note that these conditions (3) determine constant angles of normal fibers along an entire flat inclusion. These additional conditions determine the difference between this problem’s statement and those formulations for problems related to plates with thin rigid inclusions that were considered, for example, in [16–18]. Therefore, in the case of flat inclusion, we introduce the following set of admissible functions

\[
K = \{ \chi = (W, w) \in H(\Omega_\gamma) \mid [W]_\nu \geq \| \frac{\partial w}{\partial \nu} \|, \quad \frac{\partial w^-}{\partial x_i} = a_i \text{ on } \gamma; \quad \chi|_{\gamma^-} \in R(\gamma) \},
\]

where \( w = a_0 + a_1 x_1 + a_2 x_2 \) on \( \gamma^- \).

Let us give a variational formulation of the new problem. It is required to find a solution \( \xi = (U, u) \in K \), such that

\[
\Pi(\xi) = \inf_{\chi \in K} \Pi(\chi).
\]

(4)
Theorem. There exists a unique solution $\xi = (U, u) \in K$ of problem (4).

Proof. Obviously, set $K$ is convex. Let us prove the closedness. Let us suppose that $\chi_n \to \chi$ strongly in $H(\Omega_i)$ as $n \to \infty$, $\chi_n \in K$. We can establish for the horizontal displacements $W = (w_1, w_2)$ that

$$W(x) = \rho(x) = b(x_2, -x_1) + (c_1, c_2) \text{ on } \gamma^-$$

for some $b, c_1, c_2 \in \mathbb{R}$, as it was provided in the [16]. Let us consider deflection’s functions $w$. In this case, the traces $w_n \to w$, $\frac{\partial w_n}{\partial x_i} \to \frac{\partial w}{\partial x_i}$ converge in $L_2(\gamma)$, $i = 1, 2$, and therefore there exists a subsequence still denoted by $n$, such that

$$w_n = a_0 + a_1 x_1 + a_2 x_2, \quad \frac{\partial w_n}{\partial x_i}$$

converge a. e. on $\gamma$ to functions $w$, $\frac{\partial w}{\partial x_i}$, $i = 1, 2$, respectively. Since $\frac{\partial w_n}{\partial x_i} = a_i$ a.e. on $\gamma$, $i = 1, 2$, we have that $\{a_i\}$, $i = 1, 2$, converge to some numbers $\hat{a}_i$, $i = 1, 2$. Therefore, $\frac{\partial w}{\partial x_i} = \hat{a}_i$, $i = 1, 2$. Next, in view of the following convergence

$$w_n = a_0 + a_1 x_1 + a_2 x_2 \to w \quad \text{a.e. on } \gamma,$$

we get

$$\lim_{n \to \infty} a_0 = w - \hat{a}_1 x_1 + \hat{a}_2 x_2 \quad \text{a.e. on } \gamma.$$ 

Finally, there exists a limit $\lim_{n \to \infty} a_0 = \hat{a}_0$ and $w = \hat{a}_0 + \hat{a}_1 x_1 + \hat{a}_2 x_2$ a.e. on $\gamma$. Using well-known approaches, it is possible to show that $\chi$ satisfies (2), see [1, 2]. The functional $\Pi(\chi)$ is coercive, G-differentiable and weakly lower semicontinuous in the reflexive space $H(\Omega_i)$.

As a result, these established properties of the functional $\Pi$ and the set $K$ lead to the existence of a solution $\xi$ by the Weierstrass theorem. Furthermore, it satisfies the variational inequality [1, 2]

$$\xi \in K, \quad B(\xi, \chi - \xi) \geq \int_{\Omega} F(\chi - \xi) \, dx \quad \forall \chi \in K. \quad (5)$$

Supposing that there are two distinct solutions, in view of (1) and Remark 1 we can easily get from (5) uniqueness of the solution $\xi$.

Acknowledgments

This work has been supported by the mega-grant of the Russian Federation Government (project no. 14.Y26.31.0013).

References

[1] Khludnev A M and Kovtunenko V A 2000 Analysis of Cracks in Solids (Southampton: WIT-Press)
[2] Khludnev A M 2010 Elasticity Problems in Nonsmooth Domains (Moscow: Fizmatlit)
[3] Lazarev N and Grigoryev M 2017 AIP Conf. Proc. 1907 030033
[4] Itou H and Khludnev A M 2016 Math. Method. Appl. Sci. 39 4980–93
[5] Rudy E 2016 ESAIM-Math. Model. Num. 50 995–1009
[6] Lazarev N and Neustroeva N 2018 Springer Proc. Math. Statist. 253 67–77
[7] Lazarev N P and Rudy E M 2014 Z. Angew. Math. Mech. 94 730–9
[8] Khludnev A M 2013 Arch. Appl. Mech. 83 1493–509
[9] Khludnev A and Negri M 2012 Z. Angew. Math. Mech. 92 341–54
[10] Khludnev A M and Shcherbakov V V 2017 Math. Mech. Solids 22 2180–95
[11] Shcherbakov V V 2016 Z. Angew. Math. Mech. 96 1306–17
[12] Shcherbakov V V 2016 Z. Angew. Math. Phys. 67 71
[13] Kazarinov N A, Rudy E M, Slesarenko V Y and Shcherbakov V V 2018 Comput. Math. Math. Phys. 58 761–74
[14] Faella L and Khludnev A 2016 *Math. Method. Appl. Sci.* **39** 3381–90
[15] Kovtunenko V A and Leugering G 2016 *SIAM J. Control Optim.* **54** 1329–51
[16] Khludnev A M 2011 *J. Appl. Indust. Math.* **5** 582–94
[17] Shcherbakov V V 2014 *J. Appl. Indust. Math.* **8** 97–105
[18] Lazarev N 2015 *Bound. Value Probl.* **2015** 180
[19] Khludnev A M, Faella L and Popova T S 2017 *Math. Mech. Solids.* **22** 1–14
[20] Khludnev A M and Popova T S *Q. Appl. Math.* **74** 705–18
[21] Lazarev N and Everstov V 2019 *Z. Angew. Math. Mech.* **99** e201800268
[22] Lazarev N and Semenova G 2019 *Bound. Value Probl.* **2019** 87
[23] Khludnev A M 2010 *Eur. J. Mech. A. Solids.* **29** 392–99
[24] Mura T 1987 *Micromechanics of Defects in Solids* (Dordrecht: Martinus Nijhoff Publishers)