Confinement and Superfluidity in one-dimensional Degenerate Fermionic Cold Atoms

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The physical properties of arbitrary half-integer spins \( F = N - 1/2 \) fermionic cold atoms trapped in a one-dimensional optical lattice are investigated by means of a low-energy approach. Two different superfluid phases are found for \( F \geq 3/2 \) depending on whether a discrete symmetry is spontaneously broken or not: an unconfined BCS pairing phase and a confined molecular superfluid instability made of 2N fermions. We propose an experimental distinction between these phases for a gas trapped in an annular geometry. The confined-unconfined transition is shown to belong to the \( Z_N \) generalized universality class. We discuss on the possible Mott phases at 1/2N filling.

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In the recent past, spectacular experimental progress has allowed to cool down alkali atoms below Fermi temperatures offering the promising perspective to study strongly correlated electronic effects, such as high-temperature superconductivity, in a new context. On top of strong correlations, ultracold atomic systems provide also an opportunity to investigate the effect of spin degeneracy since in general alkali fermionic atoms have (hyperfine) spins \( F > 1/2 \) in their lowest hyperfine manifold. Simplest examples are spin-3/2 atoms such as \(^9\)Be, \(^{132}\)Cs, and \(^{135}\)Ba atoms. In this Letter, we elucidate the generic physical features of half-integer spins \( F = N - 1/2 \) fermionic cold atoms in the particular case of a one-dimensional optical lattice. The low-energy physical properties of these systems are known to be described by a Hubbard-like Hamiltonian:

\[
\mathcal{H} = -t \sum_{i,\alpha} \left[ c^\dagger_{\alpha,i} c_{\alpha,i+1} + \text{H.c.} \right] + \sum_i U_i \sum_{j, \beta} \sum_{M = -J}^{J} P^\dagger_{JM,i} P_{JM,i}, \tag{1}
\]

where \( c^\dagger_{\alpha,i} \) (\( \alpha = 1, ..., 2N \)) is the fermion creation operator corresponding to the \( 2F + 1 = 2N \) atomic states. The pairing operators in Eq. (1) are defined through the Clebsch-Gordan coefficient for two indistinguishable particles: \( P^\dagger_{JM,i} = \sum_{\alpha \beta} \langle J M | F, F; \alpha, \beta \rangle c^\dagger_{\alpha,i} c^\dagger_{\beta,i} \). The interactions are SU(2) spin-conserving and depend on \( U_i \) parameters corresponding to the total spin of two spin-F particles which takes only even integers value due to Pauli’s principle: \( J = 0, 2, ..., 2N - 2 \). Even in this simple scheme the interaction pattern is still involved. It is thus highly desirable to focus on a much simpler paradigmatic model that incorporates the relevant physics of higher-spin degeneracy. In this respect, we consider a two coupling-constant version of model (1) with \( U_2 = ... = U_{2N-2} \neq U_0 \):

\[
\mathcal{H} = -t \sum_{i,\alpha} \left[ c^\dagger_{\alpha,i} c_{\alpha,i+1} + \text{H.c.} \right] + \frac{U}{2} \sum_i \rho_i^2 + V \sum_i \pi_i^2, \tag{2}
\]

\( \rho_i = \sum_{\alpha} c^\dagger_{\alpha,i} c_{\alpha,i} \) being the particle number operator and \( U = 2U_2, V = U_0 - U_2 \). In Eq. (2), the singlet BCS pairing operator for spin-F fermions is \( \pi_i^2 = \sqrt{2N} P_{00,i} = c^\dagger_{\alpha,i} J_{\alpha\beta} c^\dagger_{\beta,i} \), the matrix \( J \) being the natural generalization of the familiar antisymmetric tensor \( \epsilon = i \sigma_2 \) to spin \( F > 1/2 \). Such a singlet-pairing operator has been extensively studied in the context of two-dimensional frustrated quantum magnets. When \( V = 0 \), i.e. \( U_0 = U_2 \), model (2) corresponds to the SU(2N) Hubbard model. The Hamiltonian (2) for \( V \neq 0 \) still displays a large symmetry since it is invariant under the Sp(2N) group which consists of unitary matrices \( U \) that satisfy \( U^* J U \) = \( J \). In the spin \( F = 1/2 \) case, model (2) reduces to the SU(2) Hubbard chain since SU(2) \( \simeq \) Sp(2). In the spin-3/2 case, models (2) and (1) have an exact SO(5) \( \simeq \) Sp(4) symmetry. This Sp(2N) symmetry considerably simplifies the problem but may appear rather artificial. However, we expect that, for generic and small interactions, the original SU(2) spin-rotational invariance will be dynamically enlarged at sufficiently low energy. A second reason to consider the Sp(2N) symmetric model (2) stems from the fact that the Sp(2N) and SU(2N) groups share the same center, the Z_2 group. Moreover, the striking physical properties of the system rely on the existence of a \( Z_N \) symmetry which is also a symmetry of the SU(2) model (1). This \( Z_N \) symmetry, that simply amounts to a global redefinition of the fermion...
phase, is properly defined as the coset between the center $\mathbb{Z}_{2N}$ of the SU$(2N)$ group and the center $\mathbb{Z}_2$ of the Sp$(2N)$ or SU$(2)$ one: $\mathbb{Z}_N = \mathbb{Z}_{2N}/\mathbb{Z}_2$ with

$$Z_{2N} : c_{a,i} \rightarrow e^{i\pi n/N}c_{a,i}, \quad n = 0, \ldots, 2N - 1,$$

the $\mathbb{Z}_2$ symmetry being $c_{a,i} \rightarrow -c_{a,i}$. The $\mathbb{Z}_N$ symmetry, defined by Eq. (3) with $n = 0, \ldots, N - 1$, provides an important physical ingredient not present in the $F = 1/2$ case. The stabilization of a quasi-long-range BCS phase for $F > 1/2$ requires the spontaneous breaking of this $\mathbb{Z}_N$ symmetry since the singlet pairing $\pi^1_L$ is not invariant under this symmetry. Since $\mathbb{Z}_N$ may be also viewed as a discrete subgroup of the global U$(1)$ charge symmetry (see Eq. (3)), it is tempting to interpret, for $F > 1/2$, the breaking of $\mathbb{Z}_N$ as the reminiscence of the spontaneous global U$(1)$ charge breaking that characterizes the BCS phase in higher dimensions. In contrast, if $\mathbb{Z}_N$ is not broken, the BCS instability is suppressed and the leading superfluid instability, which has to be a singlet under the $\mathbb{Z}_N$ symmetry, is a molecular object made of $2N$ fermions. In the following, the delicate competition between these superfluid instabilities will be investigated by means of a low-energy approach.

-Phase diagram. The low-energy effective field theory associated with Eq. (2) is obtained, as usual, from the continuum description of the lattice electronic operators in terms of right and left moving Dirac fermions: $c_{a,i}/\sqrt{\alpha_0} \rightarrow R_{a}(x)e^{ik_F x} + L_{a}(x)e^{-ik_F x}, x = i a_0, a_0$ being the lattice spacing and $k_F$ the Fermi momentum [6]. Away from half-filling (i.e. $N$ atoms per site), it separates into two commuting density and spin pieces: $\mathcal{H} = \mathcal{H}_d + \mathcal{H}_s$ with $[\mathcal{H}_d, \mathcal{H}_s] = 0$. The U$(1)$ density sector is described by a bosonic field $\Phi$ and its dual $\Theta$ whose dynamics is governed by the free-boson Hamiltonian:

$$\mathcal{H}_d = \frac{v_F}{2} \left[ \frac{1}{K_d} \left( \partial_\Phi^2 \right)^2 + K_d \left( \partial_\Theta^2 \right)^2 \right],$$

where $v_F = v_F \left[ 1 + (2V + UN(2N - 1))/(N \pi v_F) \right]^{1/2}$ ($v_F = 2t_0 \sin(\kappa a_0)$ being the Fermi velocity) and $K_d = [1 + (2V + UN(2N - 1))/(N \pi v_F)]^{-1/2}$ are the Luttinger parameters. The conserved quantities in this U$(1)$ sector are the total particle number and left moving Dirac fermions: $\mathcal{N} = \int dx \ (R_R^a R_a + L_L^a L_a) = \sqrt{2N/\pi} \int dx \ \partial_\Phi \Phi$ and $\mathcal{J} = \int dx \ (-R_R^a R_a - L_L^a L_a) = \sqrt{2N/\pi} \int dx \ \partial_\Theta \Theta$ respectively. For incommensurate fillings, due to the density wave order parameters $\sigma_k$ and $\mu_k$, $k = 1, \ldots, N - 1$, which are dual to each other by means of the Kramers-Wannier (KW) duality symmetry. This duality transformation maps the $\mathbb{Z}_N$ symmetry, broken in the low-temperature phase ($\langle \sigma_k \rangle \neq 0$ and $\langle \mu_k \rangle = 0$), onto a $\mathbb{Z}_N$ symmetry which is broken in the high-temperature phase where $\langle \mu_k \rangle \neq 0$ and $\langle \sigma_k \rangle = 0$. At the critical point, the theory is self-dual with a $\mathbb{Z}_N \times \mathbb{Z}_N$ symmetry and its universal properties are captured by the $\mathbb{Z}_N$ parafermion CFT [6]. In the simplest $N = 2$ case, there is a simple free-field representation of the unperturbed SU$(2)$ CFT

![FIG. 1: Phase diagram of model (2). Dashed lines denote cross-over lines whereas the solid; this results in strong constraints on the possible phases line marks the phase transition in the $\mathbb{Z}_N$ universality class between phases II and III.](image-url)
terms of the real fermions which has been extensively used in the context of two-leg ladders \[7\]. Introducing real fermions \(\xi^{\alpha}_{R,L}\) and \(\tilde{\xi}^{\alpha}_{R,L}\), \(\alpha = 1,...,5\) to describe respectively the \(Z_2\) and \(SO(5)\) gauge theories, the interacting part of Eq. \(6\) becomes \(\hat{H}_{\text{int}}^{\alpha} = g_{\alpha} (\xi^{\alpha}_{R,L} \tilde{\xi}^{\alpha}_{R,L})^2 + g_{\alpha} \xi^{\alpha}_{R,L} \tilde{\xi}^{\alpha}_{R,L}\). The latter model has been studied recently to describe a \(SO(5)\) symmetric two-leg ladder \[10\]. For generic \(N\), the phase diagram of Eq. \(6\) can be elucidated by means of a two-loop renormalization group (RG) analysis. As depicted in Fig. 1 it consists of three regions \[11\]. Region I is a generalized spin-density-wave (SDW) phase which is obtained when \(U\) and \(V\) are positive. In that case, \(g_{\perp,\parallel} \to 0\) in the infrared (IR) limit and the interaction is marginal irrelevant. Up to a spin-velocity anisotropy, the low-energy properties of this phase are similar to that of the repulsive SU(2N) Hubbard chain with \(2N-1\) gapless spin excitations \[12, 13\]. In contrast, a spin gap opens along a special symmetric ray \(g_{\parallel} = g_{\perp} = g^* > 0\) where the interacting part of the Hamiltonian \(6\) can be rewritten in a SU(2N) invariant form:

\[
\hat{H}_{\text{int}}^{\alpha} = g^* \left( I_{R}^a I_{L}^a + I_{R}^b I_{L}^b \right) = g^* I_{R}^a I_{R}^a.
\]

The Hamiltonian \[8\], which governs the IR properties of phase II, takes the form of the SU(2N) Gross-Neveu (GN) model which is an integrable massive field theory \[14\]. It is instructive to have a simpler understanding of the spin-gap formation from the underlying \(Z_N\) symmetry. The \(Z_N\) and \(\tilde{Z}_N\) symmetries that define the low-T and high-T phases of this Ising model admit a representation in terms of the fermions \((R,L)\). Indeed, the \(Z_N\) group \[3\] can be generated in this chiral fermion basis with help of the unitary operator \(U = e^{i\pi/N}\): \(UR(L)\alpha U^\dagger = e^{i\pi/N} RL\alpha\). Similarly, the dual \(Z_N\) symmetry can be defined by \(U = e^{i\pi/2N}: UR(L)\alpha U^\dagger = e^{i\pi/2N} RL\alpha\). The ground state of model \(6\) displays long-range order associated with the order parameter \(\text{Tr}(g) = R_|\alpha L_\alpha e^{i\pi/2\sqrt{N}}\). In phase II, we find that the \(Z_{2N} = Z_2 \times Z_N\) symmetry remains unbroken while \(\tilde{Z}_N\) is spontaneously broken. The \(Z_N\) Ising model thus be- comes to its high-T phase and a spectral gap is formed. In the second spin-gapped phase (III) of Fig. 1, defined by \(V < 0\) and \(V < NU/2\), the RG flow is now attracted along the asymptote: \(g_{\parallel} = -g_{\perp} = g^* > 0\). In that case, the interacting part of the IR Hamiltonian becomes

\[
\hat{H}_{\text{int}}^{\alpha} = g^* \left( I_{R}^a I_{L}^a - I_{R}^b I_{L}^b \right),
\]

which can be recast as a SU(2N) GN model \[8\] by means of a duality transformation \(D\) on the fermions: \(DRL(D)^{-1} = \tilde{R}(L)\) with \(R_{\alpha} = J_{\alpha\beta} R_{\beta}\) and \(L_{\alpha} = L_{\alpha}\). This transformation acts on the currents as: \(I_{R}^a (R(L)) = I_{R}^a (R(L))\) and \(I_{R}^a (R(L)) = -I_{R}^a (R(L))\) so that \(D\) indeed maps onto \(\tilde{D}\). Besides the opening of a spectral gap, we thus find that phase III possesses a hidden symmetry at low energy i.e. a SU(2N) symmetry generated by the dual currents \((I_{R}^a (R(L)), \tilde{I}_{R}^a (R(L)))\). In fact, one has \(\tilde{D}D^{-1} = \hat{U}\) so that \(D\) identifies to the KW duality of the \(Z_N\) Ising model. In phase III, the latter model thus belongs to its low-T phase and the \(Z_N\) symmetry is spontaneously broken whereas \(\tilde{Z}_{2N} = Z_2 \times \tilde{Z}_N\) remains unbroken. In summary, the existence of these two distinct spin-gapped phases is a non-trivial consequence of higher-spin degeneracy and does not occur in the \(F = 1/2\) case. The emergence of the spin-gap stems from the spontaneous breakdown of the \(Z_N\) or \(\tilde{Z}_N\) discrete symmetries. As we shall see now, these symmetries are central to the striking physical properties displayed by these phases.

**Spin superfluidity.** The low-energy properties of the spin sector of phase II can be extracted from the integrability of the SU(2N) GN model \[9\]. Its spectrum consists into \(2N-1\) branches that transform in the SU(2N) representations \[14\]. These eigenstates are labelled by quantum numbers associated with the conserved quantities of the SU(2N) low-energy symmetry: \(Q_{R}^{a} = \int dx \; (I_{R}^a I_{L}^a - I_{R}^b I_{L}^b), \; a = (1,...,N)\) and \(Q_{\parallel} = \int dx \; (I_{R}^a - I_{R}^b), \; i = (1,...,N - 1)\). Due to the Sp(2N) symmetry of model \[6\], the \(Q_{R}^{a}\) numbers are conserved whereas the \(Q_{R}^{a}\) charges are only good quantum numbers at low energy. The spin spectrum in phase III can be obtained from the duality symmetry \(D\) and consists into \(2N-1\) branches which transform in the representations of the dual group \(SU(2N)\). The dual quantum numbers are now given by: \(\tilde{Q}_{R}^{a} = Q_{R}^{a}\) and \(\tilde{Q}_{\parallel} = \int dx \; (I_{R}^a I_{L}^a + I_{R}^b I_{L}^b) = \int dx \; (I_{R}^a - I_{R}^b) = J_{\parallel}\). We thus observe that the low-lying excitations in phase III carry quantized spin \(1/2\) currents in the \(\perp\) direction. In this sense, the phase III might be viewed as a partially spin-superfluid phase.

**Confinement.** We shall now determine the nature of the dominant electronic instabilities of the different phases of Fig. 1. To this end let us consider operators \(O_{n,j}\) which carry particle number \(n\) and current \(j\): \(\text{Tr}(F_{n,j} O_{n,j}) = n(j)O_{n,j}\). Using the \(Z_{2N}\) and \(\tilde{Z}_{2N}\) generators, \(U\) and \(\tilde{U}\), we find that \(O_{n,j}\) carry \(Z_{2N}\) and \(\tilde{Z}_{2N}\) charges \(n\) and \(j\) respectively. In phase II, the full \(Z_{2N} = Z_2 \times Z_N\) symmetry \[9\] is unbroken so that it costs a finite energy gap to excite states that either break the \(Z_2\) or \(\tilde{Z}_N\) symmetries. The dominant instabilities must thus be neutral under the \(Z_{2N}\) symmetry and the resulting order parameters \(O_{n,j}\) are characterized by \(n = 0 \text{ mod } 2N\) and \(j = 0 \text{ mod } 2\). In particular, there is no quasi-long-range BCS order in phase II since the lattice singlet pairing operator \(\tilde{\sigma}_{x}^i\) carries a charge \(n = 2\) under the \(Z_{2N}\) symmetry \[9\]. The \(Z_{2N}\) symmetry thus confines the electronic charge to multiple of \(2Ne\) i.e. the leading superfluid instability in phase II
is a composite object made of 2N fermions. In this respect, the dominant order parameters in phase II are: \( \rho_{2k}\xi = L^1_k R_\alpha \) and \( \Pi^{(N)}_{\alpha\beta} = \epsilon^{\alpha_1...\alpha_{2N}} L^\dagger_{\alpha_1}...L^\dagger_{\alpha_{2N}} \) which are, respectively, the 2k_\xi component of the atomic density \( \rho \) and the uniform component of the lattice SU(2N)-singlet superconducting instability made of 2N fermions: \( \Pi^{(N)} (i) = \epsilon^{\alpha_1...\alpha_{2N}} \epsilon^{\alpha_{2N+1}} \). These orders are power-law fluctuating: \( \langle \rho_{2k}(x) \rho_{2k}(0) \rangle \sim x^{-K_d/N} \). For \( K_d < N \), the leading instability is \( \rho_{2k} \), which gives rise to an atomic-density wave (ADW) phase whereas for \( K_d > N \) a SU(2N) molecular-superfluid (MS) phase is stabilized (see Fig. 1) with order parameter \( \Pi^{(N)}_{\alpha\beta} \). The properties of phase III are obtained from those of phase II with help of the duality symmetry \( D : (N \leftrightarrow J, Z_N \leftrightarrow \tilde{Z}_N, K_d \leftrightarrow 1/K_d) \). Low-energy excitations in phase III carry now \( n = 0 \) mod 2 and \( j = 0 \) mod 2N since the symmetry \( \tilde{Z}_N = Z_N \times \tilde{Z}_N \) remains unbroken. We find now the confinement of atomic currents and the emergence of a quasi-long-range BCS pairing phase. Under the duality \( D \) symmetry, the ADW phase is mapped onto a BCS phase for \( K_d > 1/N \) with order parameter \( \pi_0^\dagger = R^\dagger_{\alpha} J_{\alpha} L^\dagger_{\beta} \) whereas the MS phase is mapped onto a molecular density-wave (MDW) phase with order parameter \( \rho^N_{2k_\xi} = \epsilon^{\alpha_1...\alpha_{2N}} J_{\alpha_\gamma_1}...J_{\alpha_{2N}} R_{\gamma_1}...R_{\gamma_B} L^\dagger_{\beta_1}...L^\dagger_{\beta_N} \) which emerges when \( K_d < 1/N \). The spontaneous breaking of the \( Z_N \) symmetry [13] thus accounts for the emergence of the BCS superfluid phase and the spin-superfluidity phenomenon discussed above. The possible occurrence of two different superfluid phases II and III may be probed experimentally. Consider for example a gas trapped in an optical potential of length \( L \) with an annular geometry and moving with tangential velocity \( V \). This amounts to imposing a total particle current in the system \( J = 4NV/V_0 \) where \( V_0 = \hbar mL \). In the superfluid phase III since the low energy excitations carry currents \( j = 0 \) mod 2N we expect the total energy \( E(V) \) to display degenerate minima for quantized velocities: \( V_n = nV_0/2 \) irrespective of the value of the spin \( F \). In contrast in the phase II where currents are unconfined, we expect the degenerate minima of \( E(V) \) at \( V_n = nV_0/2N \).

-The \( Z_N \) phase transition. The nature of the quantum phase transition between the two spin-gapped phases II and III can be determined through the duality symmetry \( D \). On the self-dual line \( g_\perp = 0 \), i.e. 2V = NU, there is a separation of the Sp(2N) and \( Z_N \) degrees of freedom. Though the Sp(2N) sector remains gapful when \( U < 0 \), the effective \( Z_N \) Ising model is at its self-dual critical point and governs the phase transition. The \( Z_N \) quantum criticality for \( N = 2, 3 \) may be revealed by considering the ratio \( R_N(x) = \langle (\pi_0^\dagger (x) \pi_0 (0)) \rangle / \langle \Pi^{(N)}_{\alpha\beta} (x) \Pi^{(N)}_{\alpha\beta} (0) \rangle \) which displays a power-law decay with a universal exponent: \( R_N(x) \sim x^{-2N(N-1)/(N+2)} \). For larger \( N \) the phase transition is non-universal. For \( N \geq 4 \), a strongly relevant perturbation is indeed generated which takes the form of the second thermal operator \( \epsilon_2 \) of the \( Z_N \) CFT with scaling dimension 12/(\( N + 2 \)). The resulting model is integrable and the transition is either of first-order or in the U(1) universality class depending on \( N \) and the sign of the coupling constant of \( \epsilon_2 \). In the \( N = 4 \) case, i.e. a special case of the Ashkin-Teller model, the \( Z_4 \) criticality can emerge with the introduction of the interaction \( \lambda \langle R^0_{\alpha\beta} R^0_{\alpha\beta} \rangle^2 \) which can eliminate the operator \( \epsilon_2 \) by a fine-tuning of \( \lambda \).

-Mott phases. At the commensurate 1/2N filling, i.e. one atom per site, an umklapp term \( \sim \cos (\sqrt{8\pi N \Phi}) \) is generated in the density sector and becomes relevant when \( K_d < 1/N \) leading to the opening of a density gap. We further distinguish between three different Mott phases [10]. The first one lies in the SDW region I and is qualitatively similar to the one encountered in the pure SU(2N) Hubbard chain [13]. The two others have a spin gap and can be distinguished with respect to the confinement properties of atomic currents. In the region II we find for large enough \( V \) a 2k_\xi-ordered ADW and spin-Peierls ordering with a 2N ground-state degeneracy. In the region III the cross-over line between BCS and MDW phases identifies to the Mott transition line. At filling 1/2N, while the BCS phase remains, the MDW regime locks into a 2Nk_\perp MDW and displays a dimerized bond ordering for all \( N \). This Mott phase is unusual since there is no one-particle density long-range fluctuation due to the confinement of atomic currents. Remarkably enough we find that this MDW Mott phase is the only gapped phase directly connected to the BCS superfluid phase.

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