Velocity Memory Effect for polarized gravitational waves

P.-M. Zhang, a C. Duval, b G.W. Gibbons c and P.A. Horvathy a,d

a Institute of Modern Physics, Chinese Academy of Sciences, Lanzhou, China
b Aix Marseille Univ, Université de Toulon, CNRS, CPT, Marseille, France
c D.A.M.T.P., Cambridge University, U.K.
d Laboratoire de Mathématiques et de Physique Théorique, Université de Tours, France
E-mail: zhpm@impcas.ac.cn, duval@cpt.univ-mrs.fr, G.W.Gibbons@damtp.cam.ac.uk, horvathy@lmpt.univ-tours.fr

Received March 6, 2018
Accepted April 26, 2018
Published May 8, 2018

Abstract. Circularly polarized gravitational sandwich waves exhibit, as do their linearly polarized counterparts, the Velocity Memory Effect: freely falling test particles in the flat after-zone fly apart along straight lines with constant velocity. In the inside zone their trajectories combine oscillatory and rotational motions in a complicated way. For circularly polarized periodic gravitational waves some trajectories remain bounded, while others spiral outward. These waves admit an additional “screw” isometry beyond the usual five. The consequences of this extra symmetry are explored.

Keywords: gravitational waves / theory, quantum field theory on curved space

ArXiv ePrint: 1802.09061
1 Introduction

In the Velocity Memory Effect test particles initially at rest fly apart with nonvanishing constant velocity after a burst of a gravitational wave (GW) has passed [1–10]. This statement, which contradicts that of Zel’dovich and Polnarev [16] (to whom the Memory Effect is usually attributed who claimed simple displacement with zero relative velocity), appears, in a clear and precise form in a seminal paper of Ehlers and Kund [1] published 11 years before [2], 12 years before [16] and 23 years before [17]. In the authors’ own words: “After a pulse has swept over the particles they have constant velocities...”. The effect, although very small, could theoretically be observed through the Doppler effect [3, 10]. For further details and references see, e.g., [11–15].

Our previous investigations [7–9] focused at linearly polarized sandwich waves.1 However those recent observations by LIGO, and VIRGO [18, 19] which have revolutionized the field indicate that these gravitational waves are not linearly polarised, and one may wonder if our findings pertain to linear polarisation.

In this paper we extend our investigations to polarized gravitational waves with the conclusion that they behave, essentially as linearly polarised, albeit with considerable additional complications.

We start by studying circularly polarized (approximate) sandwich waves. Circular polarization is indeed expected to arise for a binary (either black holes or neutron stars) and looking along the orbital rotation axis, this is circularly polarized. For core-collapse supernovæ if the star was rotating beforehand, then circular polarization will be induced by this component. The other aspect is that for rotating systems the rotation axis is the one along which the GW wave emission is generally greatest, on average one will preferentially see circularly polarized emission. Circularly polarized sandwich waves might arise, e.g., for coalescing black holes [18] or neutron star merger [19]. Our results may therefore apply to the suggestion of [20] that astrometric data from GAIA may allow detection of long wavelength gravitational radiation emitted by binary systems. See also [21].

In what follows we begin by studying what happens for a simplified model, namely for an oscillating profile combined with a Gaussian envelope, see our eqs. (2.1)–(3.1) depicted

---

1The spacetime of a sandwich wave is flat outside an interval \([U_i, U_f]\) of a “non-relativistic time” coordinate, \(U\).
in figure 1 below, which could be thought of as (a naive) idealization of those “chirps” in figure 1 of [18].

Although we did not succeed finding analytic solutions let alone for our simplified model, our numerical calculations indicate that, while the motion in the inside-zone \([U_i, U_f]\) is highly complicated, in the (approximately) flat before- and after-zones of a(n approximate) sandwich wave, the motion is still along straight lines with constant velocity, consistently with . . . Newton’s First law.

Then, dropping the Gaussian envelope, we turn to purely periodic (non-sandwich) gravitational wave trains, which are suspected to arise (but unsuccessfully searched for so far) for rapidly rotating neutron stars [22], and/or for primordial waves in the inflationary universe [23, 24].

Periodic waves, modelled by the uniformly rotating profile in (4.1) have interesting theoretical properties. In the linearly polarized case analytic solutions can be found in terms of Mathieu functions. Another remarkable property is that they carry, in addition to the general 5-parameter isometry [2, 25–29] a curious “screw symmetry” [26, 27, 29].

2 Motions in a plane gravitational wave

In Brinkmann (B) coordinates \((X, U, V)\) the profile of a plane GW is given by the symmetric and traceless \(2 \times 2\) matrix \(H(U) = H_{ij}(U)\) [25, 30],

\[
\begin{align*}
\mathrm{d}s^2 &= \delta_{ij} dX^i dX^j + 2 dU dV + H_{ij}(U) X^i X^j dU^2, \\
H_{ij}(U) X^i X^j &= \frac{1}{2} A_+ (U) ((X^1)^2 - (X^2)^2) + A_\times (U) X^1 X^2,
\end{align*}
\]

where \(A_+\) and \(A_\times\) are the amplitudes of the + and \(\times\) polarization states. Let us record for further reference that our 4D GW can be viewed as the “Bargmann manifold” of a \(2 + 1\) dimensional non-relativistic system, namely of a time-dependent anisotropic oscillator [31–33]. The motion of a spinless test particle in the (2.1) background is described by geodesics, \(X^\mu(\sigma)\), whose equations of motion can be derived by varying the natural geodesic Lagrangian \(\frac{1}{2} g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu\), i.e.,

\[
\mathcal{L} = \frac{1}{2} (\dot{X}_i \dot{X}^i + 2 \ddot{U} V + H_{ij}(U) X^i X^j (\dot{U})^2)
\]

where \(\dot{X}^\mu = dX^\mu/\mathrm{d}\sigma\). Variation with respect to \(V\) yields that \(\ddot{U} = 0\) allowing us to choose \(\sigma = U\) and providing us finally with

\[
\begin{align*}
\frac{d^2 X}{dU^2} - H(U) X &= 0, \\
H(U) &= \frac{1}{2} \left( A_+ A_\times - A_\times A_+ \right), \\
\frac{d^2 V}{dU^2} + \frac{1}{4} \frac{dA_+}{dU} ((X^1)^2 - (X^2)^2) + A_+ \left( X^1 \frac{dX^1}{dU} - X^2 \frac{dX^2}{dU} \right) \\
&\qquad + \frac{1}{2} \frac{dA_\times}{dU} X^1 X^2 + A_\times \left( X^2 \frac{dX^1}{dU} + X^1 \frac{dX^2}{dU} \right) &= 0.
\end{align*}
\]

In what follows, we limit our investigations (as before [7–9]) to the motion of particles which are at rest in the before-zone, \(X(U) = X_0\) for \(U \leq U_i\).

\footnote{Our naive plots actually fit better neutron star merger, cf. figure 2 of [19].}
Figure 1. Profile of circularly polarized sandwich waves for (i) large and (ii) small thickness parameter \( \lambda \). For \( \lambda \to \infty \) the Gaussian profile approximates a polarized impulsive wave, cf. (3.5), and for \( \lambda \to 0 \) it becomes weak but wide. The colors cyan and magenta refer to the orthogonal components \( A_+ \) and \( A_\times \) of the spin-2 field \( H(U) \). Note the different scales of the figures.

### 3 Circularly polarized waves

If \( A_+(U) = 0 \) or \( A_\times(U) = 0 \), the wave is linearly polarized; the motions were studied in our previous papers [7–9]. Turning to polarized waves and approximating the sandwich by a Gaussian, we consider now

\[
A_+(U) = A_0 \frac{\lambda}{\sqrt{\pi}} e^{-\lambda^2 U^2} \cos(\omega U), \quad (3.1a)
\]

\[
A_\times(U) = B_0 \frac{\lambda}{\sqrt{\pi}} e^{-\lambda^2 U^2} \cos(\omega U - \phi). \quad (3.1b)
\]

For simplicity we chose, henceforth, \( A_0 = B_0 = C^2 = \text{const.} \) and \( \phi = \pi/2 \). Then the polarization is circular with an \( U \)-dependent amplitude,

\[
A_+ = D(U) \cos(\omega U), \quad A_\times = D(U) \sin(\omega U). \quad (3.3)
\]

Our main interest lies in determining \( X(U) \); displacements of the \( V \) coordinate are of the second order [8]. Eq. (2.3a) describes, in “Bargmann” language, an anisotropic harmonic oscillator in transverse space with time-dependent (rotating) force. The latter is non-vanishing only in the (approximate) inside zone.\(^5\)

The profiles for different width-parameters \( \lambda \) are depicted in figure 1. The numerically calculated trajectories and velocities, shown in figures 2 and 3, hint at the following behavior:

1. In the large-\( \lambda \) regime (i) the Gaussian is thin and high and the motion is roughly along straight lines with constant velocity, except in a short (\( \sim O(1/\lambda) \) transitory “inside”

\(^3\)Changing the relative phaseshift \( \phi \) has a strong influence on geodesics. For \( \phi = 0 \) we have \( A_+ \propto A_\times \) and the wave is once again linearly polarized.

\(^4\)More sophisticated formulæ were considered in [34]:

\[
A_+ = A(\omega(U))^{2/3} \cos \varphi(U), \quad \varphi(U) = \int_{U_0}^U \omega(U')dU', \quad \omega(U) = (k(U - U_0))^{3/2}. \quad (3.2)
\]

\(^5\)The “before”, \( U < U_i \), “inside” \( U_i \leq U \leq U_f \) and “after” \( U_f < U \) zones are defined only approximately by the requirement that the Gaussian factor be small.
Figure 2. (a) Trajectories (b) velocities after the passage of a circularly polarized Gaussian sandwich wave (3.1) in the impulsive limit $\lambda \to \infty$. The blue and red colors refer to the transverse components $X^1$ and $X^2$. The trajectory is bent in the (narrow) inside zone and then escapes with non-vanishing constant velocity in the flat after-zone.

Figure 3. (a) Trajectories (b) velocities for small $\lambda$, describing test particle motion in a wide but weak circularly polarized Gaussian sandwich GW (3.1). The blue and red colors refer to the transverse components $X^1$ and $X^2$. A particle initially at rest has a complicated motion in the (approximate) inside zone however it escapes with non-zero constant velocity in the flat after-zone.

region, where the trajectory is sharply bent and the velocity changes rapidly from zero to some non-zero value figure 2 — reminiscent of motion in an impulsive wave [9, 35–37].

2. In the small-$\lambda$ regime (ii) the profile is wide and low, with many oscillations in the inside zone. Apart of some fine “denting”, the motion is “reasonably regular” in the inside zone, see figure 3. The effect of “denting” becomes important, though, when the velocity is plotted. The trajectories suffer a rather weak rotation.

In the outside zones, $U < U_i$ and $U > U_f$, everything smoothes out for both regimes: the trajectory appears, just like in the linearly polarized case, to follow straight lines with constant velocity. The difference between the large and small-$\lambda$ regimes lies in the inside zone: motion in the after zone is always simple.

Although we have not been able to find analytic solutions, we have arguments to confirm our observations made above.
Firstly, the behavior in the outside zones is obvious from the Bargmann point of view, which says that the non-relativistic motions in $d+1$ dimensions are the projections of lightlike geodesics in a $(d+1,1)$ dimensional Lorentz manifold along a covariantly constant null direction [31–33]. Our metric (2.1) describes, in particular, non-relativistic motions in 2+1 dimensions in the potential $\frac{1}{2}H_{ij}X^iX^j$. Where $H \approx 0$, we have therefore a free non-relativistic particle. Our observation made above is therefore Newton’s First Law.

Next, integrating (2.3a) from $U_i$ to $U_f$ yields the accumulated change of velocity as

$$\Delta \dot{X} = \int_{U_i}^{U_f} H(U) \dot{X}(U) dU = \left( \frac{1}{2} \int_{U_i}^{U_f} (A_+(U)X^1(U) + A_\times(u)X^2(U)) dU \left) \right. \right.$$

which generalizes eq. (VI.1) of [9] from linear to circular polarization. The integral can be evaluated numerically and the result is consistent with what is seen on the figures, both in the large- and small-$\lambda$ regimes, figures 2 and 3, respectively. (We checked this for $\lambda = 20$ by evaluating numerically the integral.

When $\lambda \to \infty$ the profile (2.1)–(3.1) goes over to the impulsive form [9, 35–37]

$$A_+(U) = C \delta(U) \cos(\omega U), \quad A_\times(U) = C \delta(U) \sin(\omega U). \quad (3.5)$$

The remarkable result is that evaluating the velocity jump (3.4) the off-diagonal component $A_\times$, and in fact the circular frequency $\omega$ drop out, leaving us with the same expression as in the linearly polarized case [9], namely with

$$\dot{X}(0^+) = c_0X_0 \quad \text{where} \quad c_0 = \frac{1}{2} C \text{diag}(1, -1). \quad (3.7)$$

At the impulsive limit the polarization has no effect, consistently with the findings of Podolský and Veselý, [38], who argued that the resulting particle motions are identical regardless of the sandwich we started with before shrinking. An intuitive explanation is that when $\lambda \to \infty$ the inside zone shrinks to a single instant $U=0$ and the extra structure has no time to act.

The “Tissot” diagram (figure 4), which shows the evolution of a small circle of particles initially at rest, exhibits the characteristic “breathing”, seen before for linear polarization [7–9]. Polarized waves exhibit also a new effect: unlike in the linearly polarized case, the “breathing” is combined with a rather complicated precession in transverse space, shown in figure 5.

### 4 Periodic waves

Waves with periodic profile

$$A_+(U) = A_0 \cos(\omega U), \quad A_\times(U) = B_0 \cos(\omega U - \phi), \quad (4.1)$$

---

For arbitrary $A_0, B_0$ and $\phi$ (3.7) is generalized to

$$\dot{X}(0^+) = c_0X_0 \quad \text{where} \quad c_0 = \frac{1}{2} \begin{pmatrix} A_0 & B_0 \cos \phi \\ B_0 \cos \phi & -A_0 \end{pmatrix}. \quad (3.6)$$
Figure 4. “Tissot” diagram: the trajectories in the inside zone of a circularly polarized gravitational wave with wide ($\lambda = .01$) Gaussian profile “breathe”, as they do for linear polarisation.

Figure 5. In the inside zone of a circularly polarized GW the transverse trajectory precesses in a rather complicated way.

(although not yet observed) are believed to be generated by, e.g., rapidly rotating neutron stars [22], or in the early inflationary universe [23, 24]. They differ from our previously considered sandwich waves by the absence of damping. Such waves behave oppositely to impulsive waves: intuitively, they have only an inside zone.

We first note that in the linearly polarized case $B_0 = 0$ the matrix $H(U)$ in (2.3a) is diagonal, and the equations of motion (2.3a) can be solved analytically: the separate, leaving us with two uncoupled Mathieu equations,

$$\ddot{X} - \frac{1}{2} \cos(\omega U) \begin{pmatrix} A_0 & 0 \\ 0 & -A_0 \end{pmatrix} X = 0,$$

(4.2)
In a linearly polarized periodic wave with no relative phase shift, $\phi = 0$, the transverse coordinate $X(U)$ oscillates along a “dented” straight line. The initial conditions are $\dot{X}^1(U=0) = \dot{X}^2(U=0) = 0$ (at rest for $U=0$) at initial position $X^1(U=0) = -1, X^2(U=0) = 1/2$.

whose general solutions are combinations of even and odd (Mathieu cosine resp. Mathieu sine) functions $C(a,q,x)$ resp. $S(a,q,x)$. Choosing the initial conditions $\dot{X}(U=0) = 0$ (particle at rest at $U=0$), the solutions are pure Mathieu cosine functions,

$$
X^1(U) = c_1 \cos\left(\frac{A_0}{\omega^2} \left[-\left(\frac{\omega}{2}\right)U\right]\right),
$$

$$
X^2(U) = c_3 \cos\left(\frac{-A_0}{\omega^2} \left[-\left(\frac{\omega}{2}\right)U\right]\right),
$$

where the coefficients $c_1$ and $c_3$ are determined by the initial conditions $X(0)$. Plotting either the Mathieu cosine or alternatively, solving the equations numerically, yields figure 6.

In the skew-diagonal case $A_0 = 0$ the polarization is again linear, and the problem is equivalent the previous one, since it can be brought to diagonal form by introducing new coordinates, $Y^1 = (X^1 + X^2), Y^2 = (X^1 - X^2)$, yielding uncoupled Mathieu equations for the latter. If the phase lag $\phi = 0$, we obtain again (4.3).

The general case is a combination of the previous ones, however we could find numerical solutions only. Limiting ourselves again to circular polarization, $A_0 = B_0 = C^2$, we found that the trajectory is a sort of “ellipse, dented with an epicycle” cf. figure 7. The choice of $\phi$ appears to have little effect. Choosing $\phi = \pi/2$, for example, yields a “dented elliptic” trajectory along which the particle goes around: the trajectory is similar to the one in figure 7.

Unfolding the trajectory in figure 7 as shown in figure 8 shows that the particle moves on a “dented” spiral with expanding radius.

### 5 Symmetries of circularly polarised periodic waves

General plane waves have a 5-parameter isometry group [25–28], recently identified as the Carroll group with broken rotations [29]: for time-dependent anisotropic profile $U$-translational and transverse-space rotational symmetry are both broken. However for a periodic wave, the two broken transformations can be combined to yield an additional, 6th isometry [26, 27, 29]. Let us consider indeed the periodic (Brinkmann) profile in the particular case $A_0 = B_0 = C^2, C > 0$ and $\phi = \pi/2$ in (4.1), i.e.,

$$
H_{ij}(U)X^i X^j = C^2 \left(\frac{1}{2} \cos(\omega U) \left[ (X^1)^2 - (X^2)^2 \right] + \sin(\omega U) X^1 X^2 \right)
$$

Rotating motion was found also in a different background [39].
Figure 7. The (transverse-space) trajectories in the periodic GW (4.1) are reminiscent of “ellipses dented with epicycles”.

Figure 8. For appropriate initial conditions the highly “dented” trajectories in the periodic GW background (4.1) spirals outward with a growing radius.

where \( \omega = \text{const.} \). Using complex coordinates, \( Z = X^1 + iX^2 \), (5.1) is

\[
H_{ij}X^iX^j = \frac{C^2}{2} \text{Re}(e^{-i\omega U} Z^2),
\]

which is plainly invariant under the transformation

\[
U \rightarrow U + e, \quad Z \rightarrow e^{i\omega e} Z, \quad V \rightarrow V,
\]

or in real form,

\[
U \rightarrow U + e, \quad X \rightarrow R_{\frac{1}{2}\omega e} X, \quad V \rightarrow V,
\]

where \( R(\cdot) \) is the rotation matrix in the plane. These “screw transformations”, which combine \( U \)-translations with transverse rotations, are therefore isometries for all \( e \in \mathbb{R} \) [26, 27, 29];
they belong to the Bargmann group (itself a subgroup of the Poincaré group), but not to its Carroll subgroup (for which $U$ is fixed).

A nice insight is due to P. Kosinski [40] to whom we are grateful for allowing us to include the following. From the Bargmann point of view [31–33] the profile (5.1) describes an anisotropic oscillator in the transverse plane with time-dependent frequency. Viewed as a classical mechanical system, its motion can be described by the Lagrangian

$$L = \frac{1}{2} \dot{X}^2 + \frac{C^2}{4} \left( \cos \omega t \left( (X^1)^2 - (X^2)^2 \right) + 2 \sin \omega t \ X^1 X^2 \right),$$

for which both rotations and time translations are manifestly broken due to anisotropy and to time-dependence, respectively. We record for further use the associated equations of motion,

$$\ddot{X}^1 = \frac{C^2}{2} \left( X^1 \cos \omega t + X^2 \sin \omega t \right),$$

$$\ddot{X}^2 = \frac{C^2}{2} \left( X^1 \sin \omega t - X^2 \cos \omega t \right),$$

which are coupled, driven by a time-dependent rotating linear force. However, switching to a rotating frame by setting

$$\begin{pmatrix} X^1 \\ X^2 \end{pmatrix} = \begin{pmatrix} \cos \frac{1}{2} \omega t & -\sin \frac{1}{2} \omega t \\ \sin \frac{1}{2} \omega t & \cos \frac{1}{2} \omega t \end{pmatrix} \begin{pmatrix} Y^+ \\ Y^- \end{pmatrix},$$

the Lagrangian (5.5) can be written as

$$L = \frac{1}{2} \dot{Y}^2 + \frac{\omega}{2} \left( \dot{Y}^- Y^+ - \dot{Y}^+ Y^- \right) + \frac{1}{2} \Omega_+^2 (Y^+)^2 + \frac{1}{2} \Omega_-^2 (Y^-)^2,$$

$$\Omega_+^2 = \frac{\omega^2 + 2C^2}{4}, \quad \Omega_-^2 = \frac{\omega^2 - 2C^2}{4},$$

which describes a time independent (but still anisotropic) oscillator put into a constant magnetic field. It has therefore natural time-translational symmetry, $U \rightarrow U + e$ (whereas rotations are still broken); transforming back to the Brinkmann coordinates provides us precisely with the “screw symmetry” (5.4) for the time-dependent system (5.5).

To get further insight, we note that the effective force in the $Y^-$ direction can be attractive, vanish or repulsive depending on the sign of $\Omega_-^2$; the $Y^+$ direction is always repulsive, because $\Omega_+^2 > 0$. The equations of motion which correspond to (5.8) are

$$\ddot{Y}^+ - \omega \dot{Y}^- - \Omega_+^2 Y^+ = 0,$$

$$\ddot{Y}^- + \omega \dot{Y}^+ - \Omega_-^2 Y^- = 0.$$

Particularly simple motions arise when

$$\omega^2 = 2C^2 \quad \text{when} \quad \Omega_- = 0,$$

when

$$\omega = 2\sqrt{C^2}.$$
When amplitude and frequency are correlated as in (5.10), a particle at rest at $U = 0$ can, for appropriate initial conditions spiral outward. Here we took $X^1(0) = 0, X^2(0) = 1, \dot{X}^1(0) = \dot{X}^2(0) = 0$. The increase of the radius is manifest.

so that the oscillator becomes maximally anisotropic. The attractive term drops out from (5.8), leaving us with a one-sided “inverted” (repulsive) oscillator in the $Y^+$ direction (plus an effective magnetic field). Simple solutions can be found readily in this case [43, 44]

\begin{align}
Y^+ &= \frac{\sqrt{2}}{C} c_1 + c_2 \cos Ct + c_3 \sin Ct, \\
Y^- &= -c_1 t - \sqrt{2} c_2 \sin Ct + \sqrt{2} c_3 \cos Ct + c_4, 
\end{align}

(5.11a) (5.11b)

where $c_i, i = 1, \ldots, 4$ are constants. Then the inverse of (5.7) provides us with trajectories in Brinkmann coordinates.

Choosing appropriate initial conditions we can arrange to get both outwards-spiraling (figure 9) or periodic (figure 10) motion.

In the spiraling case the velocity diagram essentially the same as for the trajectories and is therefore not reproduced here.

Periodic motions, (figure 10), are obtained when the initial conditions satisfy $\dot{X}^1(0) + (\omega/2)X^2(0) = 0$. The velocity diagram (called the *hodograph*) shows a “flower-like” pattern, see figure 11.

In what follows, we shall explore in more depth the symmetry enhancement of polarised gravitational waves. We shall follow the treatment [27] (see also [1, 26, 41, 42]). Row 10 of table II of [27] confirms that the metric (2.1) has, in the general case, 5 Killing vectors, namely $\partial_{V}$, and

\[ F(U) \partial_1 + E(U) \partial_2 - (X^1 F'(U) + X^2 E'(U)) \partial_V, \]

(5.12)

where

\[
\begin{cases}
F A_+ + E A_\times = F'' \\
F A_\times - E A_+ = E''
\end{cases}
\]

(5.13)
Figure 10. When amplitude and frequency are correlated as in (5.10), a particle at rest at $U = 0$ can move periodically when appropriate initial conditions are chosen. Here we took $X^1(0) = 0$, $X^2(0) = 1$, $\dot{X}^1(0) = -1/\sqrt{2}$, $\dot{X}^2(0) = 0$.

Figure 11. Velocity diagram (hodograph) in the periodic case of figure 10.

Equivalently, the symmetries are thus determined by 4 first-order linear ODEs and have therefore 4 linearly independent solutions. Inserting the values read off from (4.1) yields,

\[
\begin{align*}
(C^2/2) \left( - F \cos(\omega U) + E \sin(\omega U) \right) &= F'' \\
(C^2/2) \left( F \sin(\omega U) + E \cos(\omega U) \right) &= E''.
\end{align*}
\]

(5.14)

It is worth noting that these same equations could have been obtained by asking, as in [45]: "which time-dependent translations,

\[
X \rightarrow X + a(U), \quad a(t) = \begin{pmatrix} E(U) \\ F(U) \end{pmatrix}
\]

(5.15)
preserve the equations of motion (5.6)?” Comparing (5.14) with (5.6) leads to the remarkable conclusion that the symmetry equations, (5.14), are, after rescaling, identical to the equations of motion satisfied by the coordinates. Having solved the latter we had also determined the isometries: they form a central extension of the Newton-Hooke group with broken rotations [43, 44].

Noether’s theorem applied to the geodesic Lagrangian (2.2) associates a constant of the motion along \( X^\mu (\sigma) \),
\[
E[\mathcal{K}\nu] = K^\mu \frac{dX^\nu}{d\sigma},
\]
(5.16)
to any Killing vector field \( K^\mu \). For those 5 standard generators this has been done, e.g., in [43, 44]; for \( \partial_V \) in particular we get \( \dot{U} = \text{const} \) which confirms that \( U \) is indeed an affine parameter. For the extra “screw symmetry” (5.4) we find in turn an extra Killing vector,
\[
K^\mu \partial_\mu = \partial_U + \frac{\omega}{2}(X^1 \partial_2 - X^2 \partial_1),
\]
(5.17)
to which Noether associates
\[
E = H_{ij}X^i X^j + \dot{V} + \frac{\omega}{2}(X^1 \dot{X}^2 - X^2 \dot{X}^1)
\]
(5.18)
as we could check numerically for the non-trivial periodic solution depicted in figure 10.

The Lie algebra of Killing vectors can also be determined. If \( F, E = (F_1, E_1), (F_2, E_2) \) respectively then the Lie bracket of two vector fields is of the form
\[
- (F_1 F'_2 - F_2 F'_1 + E_1 E'_2 - E_2 E'_1) \partial_V.
\]
(5.19)

Then using (5.13) we deduce that the Lie bracket of every 6 pairs of such vector fields is a constant multiple of the central element \( \partial_V \). Thus the five dimensional symmetry group is a central extension of a four dimensional Abelian group.

We now turn to the structure of the enhanced algebra. We compute the bracket
\[
\left[ \partial_U + (\omega X^1 \partial_2 - X^2 \partial_1), F \partial_1 + E \partial_2 + (X^1 F' + X^2 E') \partial_V \right] = F' \partial_1 + E' \partial_2 + (X^1 F'' + X^2 E'') \partial_V + \omega (X^1 F' - X^2 E') \partial_V.
\]
(5.20)

The subalgebra could be 2,3, or 4 dimensional. If three dimensional, it could be one of the Bianchi algebras. This case has been investigated in [41]. Inspection reveals that it is not our case. Somewhat earlier [1] the issue of extra symmetries had also been considered. For a six dimensional group two cases are listed, ours and the case considered in [41, 42]. It is claimed that there is a four-dimensional simply transitive subgroup but no details are given.

6 Conclusion

Our investigations confirm that under the influence of a polarized sandwich wave (imitated here by an oscillating Gaussian profile) test particles originally at rest move, in the (approximately flat) before and after zones, along straight lines with constant velocity — just as they do in the linearly polarized case and as expected theoretically. However motion in the inside zone is a rather complicated combination of oscillations and precessions.
Shrinking the Gaussian to a Dirac delta by letting the thickness parameter go to infinity, much of the complications (including the precessions) disappear cf. [38] and the behavior in an impulsive wave [9, 37, 38] is recovered.

Widening the Gaussian by letting the thickness parameter go to zero yields instead, after a suitable rescaling, periodic GWs, which “only have an inside zone”. The precession as well as the “breathing” of the “dented” trajectories become manifest. The model exhibits a curious, additional “screw symmetry” noticed before [26, 27, 29].

Note added. Bondi [46], in his note establishing the physical reality of gravitational waves and putting an end to the ongoing controversy writes: “Consider a set of test particles at rest […] before the arrival of the wave. Then, after the passage of the wave, the particle that was at rest will have [a non-trivial] velocity 4-vector…”

Finally, the recent papers [47] and [48] contribute to our understanding of the memory effect.

Acknowledgments

We thank L. Blanchet, M. Faber, B. Kocsis, P. Kosinski, A. Lasenby, S.T.C. Siklos for their interest and advice and correspondence. PH is indebted the Institute of Modern Physics of the Chinese Academy of Sciences in Lanzhou for hospitality. Support by the National Natural Science Foundation of China (Grant No. 11575254) is acknowledged.

References

[1] J. Ehlers and W. Kundt, *Exact solutions of the gravitational field equations*, in *Gravitation: An Introduction to Current Research*, L. Witten ed., Wiley (1962).

[2] J-M. Souriau, *Ondes et radiations gravitationnelles*, Colloques Internationaux du CNRS No 220, 243, Paris (1973).

[3] V.B. Braginsky and K.S. Thorne, *Gravitational-wave bursts with memory experiments and experimental prospects*, *Nature* 327 (1987) 123.

[4] H. Bondi and F.A.E. Pirani, *Energy conversion by gravitational waves*, *Nature* 332 (1988) 212.

[5] H. Bondi and F.A.E. Pirani, *Gravitational Waves in General Relativity. 13: Caustic Property of Plane Waves*, *Proc. Roy. Soc. Lond.* A 421 (1989) 395 [SPIRE].

[6] L.P. Grishchuk and A.G. Polnarev, *Gravitational wave pulses with ‘velocity coded memory’*, *Sov. Phys. JETP* 69 (1989) 653 [SPIRE].

[7] P.M. Zhang, C. Duval, G.W. Gibbons and P.A. Horvathy, *The Memory Effect for Plane Gravitational Waves*, *Phys. Lett.* B 772 (2017) 743 [arXiv:1704.05997] [SPIRE].

[8] P.M. Zhang, C. Duval, G.W. Gibbons and P.A. Horvathy, *Soft gravitons and the memory effect for plane gravitational waves*, *Phys. Rev.* D 96 (2017) 064013 [arXiv:1705.01378] [SPIRE].

[9] P.M. Zhang and P.A. Horvathy, *Memory Effect for Impulsive Gravitational Waves*, *Class. Quant. Grav.* 35 (2018) 065011 [arXiv:1709.02299] [SPIRE].

[10] A. Lasenby, *Black holes and gravitational waves*, talks given at the *Royal Society Workshop on ‘Black Holes’*, Chichley Hall, U.K. (2017) and KIAA, Beijing, China (2017).

[11] M. Favata, *The gravitational-wave memory effect*, *Class. Quant. Grav.* 27 (2010) 084036 [arXiv:1003.3486] [SPIRE].
A.I. Harte, Strong lensing, plane gravitational waves and transient flashes, *Class. Quant. Grav.* 30 (2013) 075011 [arXiv:1210.1449] [SPIRE].

J. Winicour, Global aspects of radiation memory, *Class. Quant. Grav.* 31 (2014) 205003 [arXiv:1407.0259] [SPIRE].

T. Mäöder and J. Winicour, The sky pattern of the linearized gravitational memory effect, *Class. Quant. Grav.* 33 (2016) 175006 [arXiv:1605.01273] [SPIRE].

P.D. Lasky, E. Thrane, Y. Levin, J. Blackman and Y. Chen, Detecting gravitational-wave memory with LIGO: implications of GW150914, *Phys. Rev. Lett.* 117 (2016) 061102 [arXiv:1605.01415] [SPIRE].

Ya. B. Zel’dovich and A.G. Polnarev, Radiation of gravitational waves by a cluster of superdense stars, *Astron. Zh.* 51 (1974) 30 [Sov. Astron. 18 (1974) 17].

V.B. Braginsky and L.P. Grishchuk, Kinematic Resonance and Memory Effect in Free Mass Gravitational Antennas, *Sov. Phys. JETP* 62 (1985) 427 [SPIRE].

Virgo and LIGO Scientific collaborations, B.P. Abbott et al., GW170814: A Three-Detector Observation of Gravitational Waves from a Binary Black Hole Coalescence, *Phys. Rev. Lett.* 119 (2017) 141101 [arXiv:1709.09660] [SPIRE].

Virgo and LIGO Scientific collaborations, B. Abbott et al., GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral, *Phys. Rev. Lett.* 119 (2017) 161101 [arXiv:1710.05832] [SPIRE].

C.J. Moore, D. Mihaylov, A. Lasenby and G. Gilmore, Astrometric Search Method for Individually Resolvable Gravitational Wave Sources with Gaia, *Phys. Rev. Lett.* 119 (2017) 261102 [arXiv:1707.06239] [SPIRE].

S.A. Klioner, Gaia-like astrometry and gravitational waves, *Class. Quant. Grav.* 35 (2018) 045005 [arXiv:1710.11474] [SPIRE].

Virgo and LIGO Scientific collaborations, B.P. Abbott et al., All-sky Search for Periodic Gravitational Waves in the O1 LIGO Data, *Phys. Rev. D* 96 (2017) 062002 [arXiv:1707.02667] [SPIRE].

M. Kamionkowski, A. Kosowsky and A. Stebbins, Statistics of cosmic microwave background polarization, *Phys. Rev. D* 55 (1997) 7368 [astro-ph/9611125] [SPIRE].

M. Kamionkowski and E.D. Kovetz, The Quest for B Modes from Inflationary Gravitational Waves, *Ann. Rev. Astron. Astrophys.* 54 (2016) 227 [arXiv:1510.06042] [SPIRE].

H. Bondi, F.A.E. Pirani and I. Robinson, Gravitational waves in general relativity. 3. Exact plane waves, *Proc. Roy. Soc. Lond. A* 251 (1959) 519.

R. Sippel and H. Gonner, Symmetry Classes of PP Waves, *Gen. Rel. Grav.* 18 (1986) 1229 [SPIRE].

C.G. Torre, Gravitational waves: Just plane symmetry, *Gen. Rel. Grav.* 38 (2006) 653 [gr-qc/9907089] [SPIRE].

C. Duval, G.W. Gibbons, P.A. Horvathy and P.M. Zhang, Carroll symmetry of plane gravitational waves, *Class. Quant. Grav.* 34 (2017) 175003 [arXiv:1702.08284] [SPIRE].

H.W. Brinkmann, Einstein spaces which are mapped conformally on each other, *Math. Ann.* 94 (1925) 119 [SPIRE].

L.P. Eisenhart, Dynamical trajectories and geodesics, *Annals Math.* 30 (1928) 591.
[32] C. Duval, G. Burdet, H.P. Künzle and M. Perrin, Bargmann Structures and Newton-cartan Theory, Phys. Rev. D 31 (1985) 1841 [inSPIRE].

[33] C. Duval, G.W. Gibbons and P. Horvathy, Celestial mechanics, conformal structures and gravitational waves, Phys. Rev. D 43 (1991) 3907 [hep-th/0512188] [inSPIRE].

[34] L. Blanchet, Gravitational Radiation from Post-Newtonian Sources and Inspiralling Compact Binaries, Living Rev. Rel. 17 (2014) 2 [arXiv:1310.1528] [inSPIRE].

[35] R. Penrose, The geometry of impulsive gravitational waves, in General Relativity, Papers in Honour of J.L. Synge, L. O’Raifeartaigh ed., Clarendon Press, Oxford, U.K., (1972), pp. 101–115.

[36] C. Barrabes and P.A. Hogan, Singular Null Hypersurfaces in General Relativity, World Scientific, (2003).

[37] J. Podolský, C. Sämann, R. Steinbauer and R. Švarc, The global existence, uniqueness and C^1-regularity of geodesics in nonexpanding impulsive gravitational waves, Class. Quant. Grav. 32 (2015) 025003 [arXiv:1409.1782] [inSPIRE].

[38] J. Podolský and K. Veselý, New examples of sandwich gravitational waves and their impulsive limit, Czech. J. Phys. 48 (1998) 871 [gr-qc/9801054] [inSPIRE].

[39] W. Kulczycki and E. Malec, Axial gravitational waves in FLRW cosmology and memory effects, Phys. Rev. D 96 (2017) 063523 [arXiv:1706.09620] [inSPIRE].

[40] P. Kosiński, private communication (2018).

[41] S.T.C. Siklos, Einstein’s Equations and Some Cosmological Solutions, in Relativistic Astrophysics and Cosmology: Proceedings of the XIVth GIFT International Seminar on Theoretical Physics X. Fustero and E. Vedaguer eds., World Scientific, (1984), pp. 201–248.

[42] S.T.C. Siklos, Some Einstein spaces and their global properties, J. Phys. A 14 (1981) 395.

[43] P.M. Zhang, G.W. Gibbons and P.A. Horvathy, Kohn’s theorem and Newton-Hooke symmetry for Hill’s equations, Phys. Rev. D 85 (2012) 045031 [arXiv:1112.4793] [inSPIRE].

[44] P.M. Zhang, P.A. Horvathy, K. Andrzejewski, J. Gonera and P. Kosiński, Newton-Hooke type symmetry of anisotropic oscillators, Annals Phys. 333 (2013) 335 [arXiv:1207.2875] [inSPIRE].

[45] G.W. Gibbons and C.N. Pope, Kohn’s Theorem, Larmor’s Equivalence Principle and the Newton-Hooke Group, Annals Phys. 326 (2011) 1760 [arXiv:1010.2455] [inSPIRE].

[46] H. Bondi, Plane gravitational waves in general relativity, Nature 179 (1957) 1072 [inSPIRE].

[47] M. Faber and M. Suda, Influence of gravitational waves on circular moving particles, J. Mod. Phys. 9 (2018) 651 [arXiv:1704.07668] [inSPIRE].

[48] A. Ilderton, Screw-symmetric gravitational waves: a double copy of the vortex, arXiv:1804.07290 [inSPIRE].