GLOBAL ATTRACTORS FOR A FULL VON KARMAN BEAM TRANSMISSION PROBLEM

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Abstract. A nonlinear transmission problem for an elastic full von Karman beam is considered here. We prove that the system possesses a compact global attractor.

1. Introduction. In this paper we consider a nonlinear transmission problem for an elastic beam with the full von Karman nonlinearity. We assume that the beam, then in equilibrium, occupies an interval $[0, L]$. Let the part of the beam $(0, L_0)$, where $0 < L_0 < L$, is subjected to a structural damping while its complementary part $(L_0, L)$ is not.

The system of differential equations for the transverse displacements $\phi(x, t)$, $u(x, t)$ and the longitudinal displacements $\omega(x, t)$, $v(x, t)$ of the full von Karman part of the beam is as follows

$$
\begin{align*}
\beta_1 \phi_{tt} - \mu_1 \phi_{txxx} - \kappa \phi_{txx} + \lambda_1 \phi_{xxxx} - \left( [\phi_x (\omega_x + 1/2 \phi_x^2)]_x \right) &= g_1(x, t), \\
\rho_1 \omega_{tt} + \gamma \omega_t - (\omega_x + 1/2 \phi_x^2) &= g_2(x, t), \quad t > 0, \quad x \in (0, L_0) \\
\beta_2 u_{tt} - \mu_2 u_{txxx} + \lambda_2 u_{txxx} - \left( [u_x (v_x + 1/2 u_x^2)]_x \right) &= g_3(x, t), \\
\rho_2 v_{tt} - (v_x + 1/2 u_x^2) &= g_4(x, t), \quad t > 0, \quad x \in (L_0, L)
\end{align*}
$$

Here $\rho_1, \beta_1, \lambda_1, \kappa, \gamma$ for $i = 1, 2$ are positive constants.

System (1.1)-(1.4) is supplemented with the transmission boundary conditions

$$
\begin{align*}
\phi(L_0, t) &= u(L_0, t), \quad \omega(L_0, t) = v(L_0, t), \\
\phi_x(L_0, t) &= u_x(L_0, t), \quad \lambda_1 \phi_{xx}(L_0, t) = \lambda_2 u_{xx}(L_0, t), \quad \omega_x(L_0, t) = v_x(L_0, t), \\
(\lambda_1 \phi_{xxx} - \mu_1 \phi_{txx} - \kappa \phi_{tx}) &= (\lambda_2 u_{xxx} - \mu_2 u_{txx})(L_0, t)
\end{align*}
$$

and boundary conditions on the ends of the beam

$$
\phi_x(0, t) = 0, \quad \phi(0, t) = 0, \quad \omega(0, t) = 0, \quad u(L, t) = 0, \quad u_{xx}(L, t) = 0, \quad v(L, t) = 0.
$$

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[12] is related to the linear transmission problem between elastic and thermoelastic Kirchhoff beams with the classical Fourier law of heat conduction. In [13] the linear Kirchhoff problem with localized thermal dissipation of hereditary type is considered. In both works the exponential decay rate of the energy is shown. In paper [14] a nonlinear transmission problem for elastic and thermoelastic plates is under consideration. In case of the Berger type nonlinearities, the existence of a compact global attractor is established. Paper [8] is devoted to a linear transmission problem for a Kirchhoff-Timoshenko beam with different types of heat conduction, the exponential stability of the system is established.

In the present paper we investigate the long-time dynamics of an elastic beam described by the full von Karman model whose part is subjected to a structural damping.

The main aim of the present paper is to investigate the asymptotic behavior of the solutions to the problem considered no matter how small the dissipative part of the beam is. The complex structure of the nonlinear terms does not allow to prove the existence of an absorbing ball directly. To overcome this difficulty and to show the existence of a compact global attractor we establish the gradient property by means of a unique continuation result.

The paper is organized as follows. In Section 2 we formulate standard results on the existence of global attractors, introduce notations and state a well-posedness theorem for the problem considered. Section 3 is devoted to the asymptotic compactness of the system. In Section 4 the main result on the existence of a global attractor is established.

2. Preliminaries, notations, and well-posedness.

2.1. Abstract results on attractors. For the readers' convenience we recall some basic definitions and results from the theory of attractors.

Definition 2.1 ([1, 4, 5, 7, 15]). A global attractor of a dynamical system \((S_t, H)\) with the evolution operator \(S_t\) on a complete metric space \(H\) is defined as a bounded closed set \(A \subset H\) which is invariant \((S_t A = A \text{ for all } t > 0)\) and uniformly attracts all other bounded sets:

\[
\lim_{t \to \infty} \sup \{ \text{dist}_H(S_t y, A) : y \in B \} = 0 \quad \text{for any bounded set } B \text{ in } H.
\]

To establish the existence of attractor we use the concept of gradient systems. The main feature of these systems is that in the proof of the existence of a global attractor we can avoid a dissipativity property (existence of an absorbing ball) in the explicit form ([4]).

Definition 2.2 ([4, 5, 6, 7]). Let \(Y \subseteq H\) be a forward invariant set of a dynamical system \((S_t, H)\). A continuous functional \(L(y)\) defined on \(Y\) is said to be a Lyapunov function on \(Y\) for the dynamical system \((S_t, H)\) if \(t \mapsto L(S_t y)\) is a nonincreasing function for any \(y \in Y\).

The Lyapunov function is said to be strict on \(Y\) if the equation \(L(S_t y) = L(y)\) for all \(t > 0\) and for some \(y \in Y\) implies that \(S_t y = y\) for all \(t > 0\); that is, \(y\) is a stationary point of \((S_t, H)\).

The dynamical system is said to be gradient if there exists a strict Lyapunov function on the whole phase space \(H\).

Definition 2.3 ([4, 5, 6, 7]). A dynamical system \((X, S_t)\) is said to be asymptotically smooth if for any closed bounded set \(B \subset X\) that is positively invariant
(S_tB \subseteq B) one can find a compact set K = K(B) which uniformly attracts B, i.e.
\sup \{ \text{dist}_X(S_ty, K) : y \in B \} \to 0 \text{ as } t \to \infty.

In order to prove the asymptotical smoothness of system (1.1)-(1.10) we rely on
the compactness criterion due to [10], which is recalled below in an abstract version
formulated in [7].

**Theorem 2.4.** [7] Let (S_t, H) be a dynamical system on a complete metric space
H endowed with a metric d. Assume that for any bounded positively invariant set
B in H and for any \( \varepsilon > 0 \) there exists \( T = T(\varepsilon, B) \) such that
\[ d(S_Ty_1, S_Ty_2) \leq \varepsilon + \Psi_{\varepsilon, B, T}(y_1, y_2), y_i \in B, \]
where \( \Psi_{\varepsilon, B, T}(y_1, y_2) \) is a function defined on \( B \times B \) such that
\[ \liminf \liminf_{n \to \infty} \Psi_{\varepsilon, B, T}(y_1, y_2) = 0 \]
for every sequence \( y_n \in B \). Then \( (S_t, H) \) is an asymptotically smooth dynamical
system.

The following statement collects criteria on existence and properties of attractors
to gradient systems.

**Theorem 2.5** ([4, 6, 7]). Assume that \( (S_t, H) \) is a gradient asymptotically smooth
dynamical system. Assume its Lyapunov function \( L(y) \) is bounded from above on
any bounded subset of \( H \) and the set \( W_R = \{ y : L(y) \leq R \} \) is bounded for every
\( R \). If the set \( \mathcal{N} \) of stationary points of \( (S_t, H) \) is bounded, then \( (S_t, H) \) possesses
a compact global attractor. Moreover, the global attractor \( \mathfrak{A} \) consists of full trajectories
\( \gamma = \{ U(t) : t \in \mathbb{R} \} \) such that
\[ \lim_{t \to -\infty} \text{dist}_H(U(t), \mathcal{N}) = 0 \text{ and } \lim_{t \to +\infty} \text{dist}_H(U(t), \mathcal{N}) = 0. \]  

and
\[ \lim_{t \to +\infty} \text{dist}_H(S_t x, \mathcal{N}) = 0 \text{ for any } x \in H; \]  

that is, any trajectory stabilizes to the set \( \mathcal{N} \) of stationary points.

**2.2. Notations.** Let \( D \) be a bounded interval in \( \mathbb{R} \) and \( s \in \mathbb{R} \). We denote by \( H^s(D) \)
the standard Sobolev space of order \( s \) on a set \( D \) which we define as restriction (in
the sense of distributions) of the space \( H^s(\mathbb{R}) \) (introduced via Fourier transform).
We denote by \( \| \cdot \|_s \) the norm in \( H^s(D) \) which we define by the relation \( \| f \|_s^2 = \inf \{ \| g \|_{2, \mathbb{R}}^2 : g \in H^s(\mathbb{R}), g = f \text{ on } D \} \). We also use the notation \( \| \cdot \| = \| \cdot \|_0 \) for
the corresponding \( L_2 \)-norm and, similarly, \( \langle \cdot, \cdot \rangle \) for the \( L_2 \) inner product.
We denote by \( H^0_0(D) \) the closure of \( C_0^\infty(D) \) in \( H^s(D) \) (with respect to \( \| \cdot \|_s \)) and by \( H^1_{\{M\}}(D) \)
the closure of \( \{ f \in C^\infty(D) : f(M) = 0 \} \), where \( M \) can denote 0, \( L \), or \( L_0 \).

For the component \( \xi = (\phi, u) \) we define the space
\[ X = \{ (\phi, u) \in H^2_{(0)}(0, L_0) \times (H^2 \cap H^1_{(L)})/(0, L) : \phi(L_0) = u(L_0), \phi_x(L_0) = u_x(L_0) \}. \]
We also define a space for the component \( \zeta = (\omega, v) \)
\[ Y = \{ (\omega, v) \in H^1_{(0)}(0, L_0) \times H^1_{(L)}(L_0, L) : \omega(L_0) = v(L_0) \}. \]
We equip the space for \( Z = (\xi, \zeta) \)
\[ V = X \times Y \]
with the inner product
\[ (Z_1, Z_2)_V = \int_0^{L_0} \omega_1 \omega_2 dx + \lambda_1 \int_0^{L_0} \phi_{1xx} \phi_{2xx} dx + \int_0^{L} v_1 v_2 dx + \lambda_2 \int_0^{L} u_{1xx} u_{2xx} dx, \]
where \( Z_i = (\xi_i, \zeta_i), \ i = 1, 2. \)

We also define the space
\[
\tilde{Y} = L_2(0, L_0) \times L_2(L_0, L),
\]
and
\[
W = Y \times \tilde{Y}
\]
endowed with the inner product
\[
(Z_1, Z_2)_W = \beta_1 \int_0^{L_0} \phi_1 \phi_2 dx + \mu_1 \int_0^{L_0} \phi_1 x \phi_2 dx + \rho_1 \int_{L_0}^{L} \omega_1 \omega_2 dx
+ \beta_2 \int_{L_0}^{L} u_1 u_2 dx + \mu_2 \int_{L_0}^{L} u_1 u_2 x dx + \rho_2 \int_{L_0}^{L} v_1 v_2 dx.
\]

As the phase space we use
\[
H = V \times W.
\]

Throughout the paper we will denote by \( C \) a generic positive constant.

2.3. Well-posedness. To show the well-posedness of problem (1.1)-(1.10) we will need the following auxiliary result, which follows straightforward from embedding theorems (see, e.g. [1]).

Lemma 2.6. There exists a positive constant \( C \) such that for any \( z = (u; v) \in H^2(0, L) \times H^1(0, L) \) we have that
\[
\|z\|_{H^1(0, L)}^2 \leq C \left( Q(\xi) + \|u\|_{H^2(0, L)}^4 \right).
\]

Here \( Q(\xi) = \int_{\Omega} (v_x + u_x^2)^2 dx. \)

We will also use the following lemma.

Lemma 2.7. There exists a positive constant \( C \) such that for any \( g \in H^1(0, L) \) we have that
\[
\max_{\Omega} |g| \leq C \|g\|_{H^{1/2+\delta}(0, L)}, \ \text{for any} \ \delta > 0.
\]

The ideas of the proof can be found e.g. in [3].

We define the spaces of test functions
\[
L_T = \{ \Psi = (\Psi_1, \Psi_2, \Psi_3, \Psi_4) : \Psi \in L_2(0, T; V), \Psi_t \in L_2(0, T; W) \}
\]
and \( L^0_T = \{ \Psi \in L_T : \Psi(T) = 0 \}. \)

We also define positive self-adjoint operators
\[
N_1(\phi, u) = (\lambda_1 \phi_{xxx}, \lambda_2 u_{xxxx}) : D(N_1) \subset \tilde{Y} \mapsto \tilde{Y}
\]
and
\[
N_2(\omega, v) = (-\omega_{xx}, -v_{xx}) : D(N_2) \subset \tilde{Y} \mapsto \tilde{Y}
\]
with the domains
\[
D(N_1) = \{ (\phi, u) \in X \cap (H^4(0, L_0) \times H^4(L_0, L)) : \lambda_1 \phi_{xx}(L_0, t) = \lambda_2 u_{xx}(L_0, t), \lambda_1 \phi_{xxx}(L_0, t) = \lambda_2 u_{xxxx}(L_0, t), u_{xx}(L, t) = 0 \}
\]
and
\[
D(N_2) = \{ (\omega, v) \in Y \cap (H^2(0, L_0) \times H^2(L_0, L)) : \omega_x(L_0, t) = v_x(L_0, t) \}.\]
We also introduce a bounded operator $G : Y \to Y'$ as follows

$$
(G(\phi_1, u_1), (\phi_2, u_2))_{L^2(0, t) \times L^2(0, t)}
$$

$$
= \beta_1 \int_0^L \phi_1 \phi_2 dx + \mu_1 \int_0^L \phi_1 \phi_2 dx + \beta_2 \int_0^L u_1 u_2 dx + \mu_2 \int_0^L u_1 u_2 dx.
$$

It is easy to see that $G$ is an isomorphism of $Y$ onto $Y'$. Let us consider the operator $G^{-1} N : \mathcal{D}(G^{-1} N) \subset Y \to Y$.

In order to make our statements precise we need to introduce the definition of weak solutions to problem (1.1)-(1.10).

**Definition 2.8.** A function $Z(t) = (\xi(t), \zeta(t))$, where $\xi(t) = (\phi(t), u(t))$ and $\zeta(t) = (\omega(t), v(t))$, is said to be a weak solution to problem (1.1)-(1.10) on a time interval $[0, T]$ if

- $Z \in L^\infty(0, T; V)$, $Z_t \in L^\infty(0, T; W)$;
- $Z(0) = Z_0 = (\phi_0, u_0, \omega_0, v_0)$;
- for every $\Psi = (\Psi_1, \Psi_2, \Psi_3, \Psi_4) \in \mathcal{L}^0$,

$$
- \int_0^T \int_0^L (\beta_1 \phi_1 \Psi_{1t} + \rho_1 \omega_1 \Psi_{3t}) dx dt - \int_0^T \int_0^L (\beta_2 u_1 \Psi_{2t} + \rho_2 v_1 \Psi_{4t}) dx dt
- \mu_1 \int_0^T \int_0^L \phi_{1x} \Psi_{1zx} dx dt - \mu_2 \int_0^T \int_0^L u_{1x} \Psi_{2zx} dx dt + \kappa \int_0^T \int_0^L \phi_{1z} \Psi_{1x} dx dt
+ \int_0^T \int_0^L K_1(Z, \Psi) dx dt + \int_0^T \int_0^L K_2(Z, \Psi) dx dt
+ \lambda_2 \int_0^T \int_0^L u_{1x} \Psi_{2zx} dx dt = \int_0^T (\beta_1 \phi_1 \Psi_{1}(0) + \rho_1 \omega_1 \Psi_{3}(0)) dx
+ \int_0^T (\beta_2 u_1 \Psi_{2}(0) + \rho_2 v_1 \Psi_{4}(0)) dx + \mu_1 \int_0^T \phi_{1z} \Psi_{1x} dx dt + \mu_2 \int_0^T u_{1x} \Psi_{2x} dx dt
+ \frac{1}{2} \sum_{i=1,2} \int_0^T \int_0^L g_i \Psi_i dx dt,
$$

where

$$
K_1(Z, \Psi) = \int_0^L (\omega_x + \frac{\psi_x^2}{2} + \psi_x \phi_x \Psi_{3x}) dx, K_2(Z, \Psi) = \int_0^L (\omega_x + \frac{\psi_x^2}{2} + \psi_x \phi_x \Psi_{4x}) dx.
$$

The well-posedness result is as follows.

**Theorem 2.9.** Assume that

$$
g(x, t) = (g_1, g_2, g_3, g_4) \in L_2(0, T; W'),
$$

$$
U_0 = (Z_0, Z_1) = (\phi_0, u_0, \omega_0, v_0, \phi_1, u_1, \omega_1, v_1) \in H.
$$

Then for any interval $[0, T]$ there exists a unique weak solution $Z(t)$ to (1.1)-(1.10) with the initial data $U_0$. This solution possesses the following properties:

(i) $U(t, U_0) = (Z(t); Z_t(t)) \subset C(0, T; H)$.
(ii) The solution depends continuously on initial data, i.e., if $U_n \to U_0$ in the norm of $H$, then $U(t; U_n) \to U(t; U_0)$ in $H$ for each $t > 0$.
(iii) The energy equality

$$
\mathcal{E}(U(t)) + \kappa \int_0^t \int_0^L \psi_{2x}^2 dx dt + \int_0^t \int_0^L \omega_x^2 dx dt
$$
weak solutions can be shown by using the Galerkin method and relying on Lemma
Proof.
holds for every \( t > 0 \), where the energy functional \( \mathcal{E} \) is defined by the relation
\begin{equation}
\mathcal{E}(U(t)) = \frac{1}{2} \left[ \int_0^L (\beta_1 \phi_1^2 + \rho_1 \omega_1^2) \, dx + \int_0^L (\beta_2 u_1^2 + \rho_2 v_1^2) \, dx + \mu_1 \int_0^L \phi_1^2 \, dx + \lambda_1 \int_0^L \phi_1^2 \, dx + \lambda_2 \int_0^L u_2^2 \, dx + Q_1(Z) + Q_2(Z) \right]
\end{equation}
with
\begin{equation}
Q_1(Z) = \int_0^L \left( \omega_x + \frac{\phi_1^2}{2} \right)^2 \, dx, \quad Q_2(Z) = \int_0^L \left( v_x + \frac{u_2^2}{2} \right)^2 \, dx.
\end{equation}
(vi) If \( g = 0 \) and \((Z_0, Z_1) \in D = (\mathcal{D}(G^{-1}N_1) \times \mathcal{D}(N_2)) \times V\), weak solutions are strong, i.e. \((Z(t)) \in L_\infty(0,T; \mathcal{D}(G^{-1}N_1) \times \mathcal{D}(N_2)), Z_i(t) \in L_\infty(0,T; V), Z_{it}(t) \in L_\infty(0,T; W)\).

Proof. The proof is quite standard, here we present the sketch of it. The existence of weak solutions can be shown by using the Galerkin method and relying on Lemma 1.1. We choose orthonormal bases \( \tilde{e}_i = (e_i^1, e_i^2) \) in \( X \) and \( \tilde{e}_i = (e_i^3, e_i^4) \) in \( Y \) consisting of eigenvectors of operators \( N_1 \) and \( N_2 \). We define an approximate solution \( Z_m = (\xi_m, \zeta_m) \), where \( \xi_m = (\phi_m, u_m) = \sum_{i=1}^m d_i(t) \tilde{e}_i \) and \( \zeta_m = (\omega_m, v_m) = \sum_{i=1}^m h_i(t) \tilde{e}_i \) satisfying for \( i = 1, m \)
\begin{equation}
\beta_1 \int_0^L \phi_m e_1^1 \, dx + \mu_1 \int_0^L \phi_m e_1^2 \, dx + \beta_2 \int_0^L u_m e_2^1 \, dx + \mu_2 \int_0^L u_m e_2^2 \, dx + \lambda_1 \int_0^L \phi_m e_1^4 \, dx + \lambda_2 \int_0^L u_m e_2^4 \, dx + \kappa \int_0^L \phi_m e_1^3 \, dx \\
+ \int_0^L \left[ \phi_m \omega_x + 1/2 \phi_m^2 \right] e_1^1 \, dx + \int_0^L \left[ u_m v_x + 1/2 u_m^2 \right] e_2^2 \, dx \\
= \int_0^L g_1 e_1^1 \, dx + \int_0^L g_2 e_2^2 \, dx
\end{equation}
and
\begin{equation}
\rho_1 \int_0^L \omega_m e_3^3 \, dx + \rho_2 \int_0^L \omega_m e_4^4 \, dx + \gamma \int_0^L \omega_m e_3^3 \, dx + \int_0^L \left( \omega_m + 1/2 \phi_m^2 \right) e_3^3 \, dx \\
+ \int_0^L (u_m + 1/2 u_m^2) e_4^4 \, dx = \int_0^L g_3 e_3^3 \, dx + \int_0^L g_4 e_4^4 \, dx
\end{equation}
with initial conditions
\begin{align*}
Z_m(0) = (\phi_m(0), u_m(0), \omega_m(0), v_m(0)) &= (\phi_0, u_0, \omega_0, v_0) = Z_{0m}, \\
Z_{1m}(0) = (\phi_{1m}(0), u_{1m}(0), \omega_{1m}(0), v_{1m}(0)) &= (\phi_{1}, u_{1}, \omega_{1}, v_{1}) = Z_{1m}.
\end{align*}
Multiplying (2.12) by \( \phi_i'(t) \), (2.13) by \( h_i'(t) \) and summing up with respect to \( i \) from 1 to \( m \) we get
\begin{equation}
\mathcal{E}(Z_m(t), Z_{mt}(t)) + \kappa \int_0^t \int_0^L \phi_{mx}^2 \, dx \, d\tau + \gamma \int_0^t \int_0^L \omega_{mx}^2 dx \, d\tau
\end{equation}
\[ \Psi = (\Psi(\parallel Z_{0})_{t}) + \sum_{i=1,2}^{t} \int_{0}^{L} g_{i}Z_{it} \tilde{x} \, dx \tilde{t} + \sum_{i=3,4}^{t} \int_{0}^{L} g_{i}Z_{it} \tilde{x} \, dx \tilde{t}. \] (2.14)

Using the Gronwall’s lemma and Lemma 1 one can easily infer the estimate
\[ \|(Z_{m}(t), Z_{mt}(t))\|_{H} \leq C(T, \|U_{0}\|_{H}, \|g\|_{W^{*}}) \] (2.15)

and the following convergences
\[ \phi_{m} \to \phi, \text{ weak-* in } \mathcal{L}_{\infty}(0, T; H^{2}_{[0]}) \] (2.16)
\[ \omega_{m} \to \omega, \text{ strongly in } \mathcal{L}_{2}(0, T; H^{1}_{[0]}) \] (2.17)
\[ u_{m} \to u, \text{ weak-* in } \mathcal{L}_{\infty}(0, T; H^{2} \cap H^{1}_{[L]}) \] (2.19)
\[ v_{m} \to v, \text{ strongly in } \mathcal{L}_{2}(0, T; H^{1}_{[L]}) \] (2.21)

and, consequently, for any \( \epsilon > 1 \)
\[ \mathcal{L}_{2}(0, T; H^{2-\epsilon}(0, L)) \] (2.22)
\[ \mathcal{L}_{2}(0, T; H^{1-\epsilon}(0, L)) \] (2.23)
\[ \mathcal{L}_{2}(0, T; H^{2-\epsilon}(0, L)) \] (2.24)
\[ \mathcal{L}_{2}(0, T; H^{1-\epsilon}(0, L)) \] (2.25)

To prove the existence of weak solutions, one can resort to the standard limit procedure in (2.12). We only specify it for von Karman nonlinear terms. For any \( \Psi = (\Psi_{1}, \Psi_{2}, \Psi_{3}, \Psi_{4}) \in \mathcal{L}_{0}^{0} \) it follows from (2.12), (2.13) that
\[ -\beta_{1} \int_{0}^{T} \int_{0}^{L} \phi_{mt} \Psi_{1t} \tilde{x} \, dx \tilde{t} - \mu_{1} \int_{0}^{T} \int_{0}^{L} \phi_{mtx} \Psi_{1tx} \tilde{x} \, dx \tilde{t} - \beta_{2} \int_{0}^{T} \int_{0}^{L} u_{mt} \Psi_{2t} \tilde{x} \, dx \tilde{t} \]
\[ -\mu_{2} \int_{0}^{T} \int_{0}^{L} u_{mtxx} \Psi_{2tx} \tilde{x} \, dx \tilde{t} + \lambda_{1} \int_{0}^{T} \int_{0}^{L} \phi_{mtx} \Psi_{1tx} \tilde{x} \, dx \tilde{t} \]
\[ + \lambda_{2} \int_{0}^{T} \int_{0}^{L} u_{xx} \Psi_{2xx} \tilde{x} \, dx \tilde{t} \]
\[ + \kappa \int_{0}^{T} \int_{0}^{L} \phi_{mtx} \Psi_{1tx} \tilde{x} \, dx \tilde{t} + \int_{0}^{T} \int_{0}^{L} \phi_{mx} \left( \omega_{mx} + 1/2 \phi_{mx}^{2} \right) \Psi_{1tx} \tilde{x} \, dx \tilde{t} \]
\[ + \int_{0}^{T} \int_{0}^{L} \left[ u_{mx} \left( v_{mx} + 1/2u_{mx}^{2} \right) \right] \Psi_{2tx} \tilde{x} \, dx \tilde{t} = \int_{0}^{T} \int_{0}^{L} g_{1} \Psi_{1t} \tilde{x} \, dx \tilde{t} \]
\[ + \int_{0}^{T} \int_{0}^{L} g_{2} \Psi_{2t} \tilde{x} \, dx \tilde{t} + \beta_{1} \int_{0}^{T} \int_{0}^{L} \phi_{mt}(0) \Psi_{1t}(0) \tilde{x} \, dx \tilde{t} - \mu_{1} \int_{0}^{T} \int_{0}^{L} \phi_{mt}(0) \Psi_{1t}(0) \, dx \tilde{t} \]
\[ - \beta_{2} \int_{0}^{L} u_{mt}(0) \Psi_{2t}(0) \, dx - \mu_{2} \int_{0}^{L} u_{mtx}(0) \Psi_{2t}(0) \, dx \]
(2.26)

and
\[ -\rho_{1} \int_{0}^{T} \int_{0}^{L} \omega_{mt} \Psi_{3t} \tilde{x} \, dx \tilde{t} - \rho_{2} \int_{0}^{T} \int_{0}^{L} v_{mt} \Psi_{4t} \tilde{x} \, dx \tilde{t} + \gamma \int_{0}^{T} \int_{0}^{L} \omega_{mt} \Psi_{3t} \tilde{x} \, dx \tilde{t} \]
\[ + \int_{0}^{T} \int_{0}^{L} \left( \omega_{mx} + 1/2 \phi_{mx}^{2} \right) \Psi_{3t} \tilde{x} \, dx \tilde{t} + \int_{0}^{T} \int_{0}^{L} \left( v_{mx} + 1/2u_{mx}^{2} \right) \Psi_{4t} \tilde{x} \, dx \tilde{t} \]
\[ \begin{align*}
&= \int_0^L g_3 \Psi_3 dt + \int_0^L g_4 \Psi_4 dt + \rho_1 \int_0^L \omega_m(0) \Psi_1(0) dx \\
&\quad + \rho_2 \int_0^L v_m(0) \Psi_4(0) dx,
\end{align*} \tag{2.27} \]

where \((\Psi_1, \Psi_2)\) and \((\Psi_3, \Psi_4)\) are orthoprojections of \((\Psi_1, \Psi_2)\) and \((\Psi_3, \Psi_4)\) on

the first basis vectors \(e^1\) and \(e^3\) respectively, \(l \leq m\). Integrating by parts we get

\[ \begin{align*}
\int_0^T \int_0^L \left( \phi_{mx} \left( \omega_{mx} + 1/2 \phi_{mx}^2 \right) - \phi_x \left( \omega_x + 1/2 \phi_x^2 \right) \right) \Psi_{1lx} dt dx \\
+ \int_0^T \int_0^L \left( u_{mx} \left( v_{mx} + 1/2 u_{mx}^2 \right) - u_x \left( v_x + 1/2 u_x^2 \right) \right) \Psi_{2lx} dt dx \\
= \int_0^T \int_0^L \left( \phi_{mx} - \phi_x \right) \omega_x \Psi_{1lx} dx dt \\
- \int_0^T \int_0^L \left( \phi_{mx} \Psi_{1lx} + \phi_{mx} \Psi_{1xx} \right) (\omega - \omega_m) dx dt \\
+ \int_0^T \int_0^L \left( u_{mx} - u_x \right) v_x \Psi_{2lx} dx dt \\
- \int_0^T \int_0^L \left( u_{mx} \Psi_{2lx} + u_{mx} \Psi_{2xx} \right) (v - v_m) dx dt
\end{align*} \]

For fixed \(l\) we have from (2.22)–(2.25)

\[ \begin{align*}
&\left| \int_0^T \int_0^L \left( \phi_{mx} \left( \omega_{mx} + 1/2 \phi_{mx}^2 \right) - \phi_x \left( \omega_x + 1/2 \phi_x^2 \right) \right) \Psi_{1lx} dt dx \\
&\quad + \int_0^T \int_0^L \left( u_{mx} \left( v_{mx} + 1/2 u_{mx}^2 \right) - u_x \left( v_x + 1/2 u_x^2 \right) \right) \Psi_{2lx} dt dx \right| \\
\leq C \left( \int_0^T \| \phi_{mx} - \phi_x \| \| \omega_x \| \| \Psi_{1lx} \| dt + \int_0^T \| \phi_{mx} \| \| \Psi_{1lx} \| \| \omega - \omega_m \| dt \\
&\quad + \int_0^T \| u_{mx} - u_x \| \| v_x \| \| \Psi_{2lx} \| dt + \int_0^T \| u_{mx} \| \| \Psi_{2lx} \| \| v - v_m \| dt \\
&\quad + \int_0^T \| \phi_{mx} - \phi_x \| (\| \phi_{mx} \|^2 + \| \phi_x \|^2) \| \Psi_{1lx} \| dt \\
&\quad + \int_0^T \| u_{mx} - u_x \| (\| u_{mx} \|^2 + \| u_x \|^2) \| \Psi_{2lx} \| dt \right) \rightarrow 0, \ m \rightarrow \infty.
\end{align*} \]

and

\[ \begin{align*}
&\left| \int_0^T \int_0^L \left( \phi_{mx} \left( \omega_{mx} + 1/2 \phi_{mx}^2 \right) - \phi_x \left( \omega_x + 1/2 \phi_x^2 \right) \right) \Psi_{3lx} dx dt \\
&\quad + \int_0^T \int_0^L \left( u_{mx} \left( v_{mx} + 1/2 u_{mx}^2 \right) - u_x \left( v_x + 1/2 u_x^2 \right) \right) \Psi_{4lx} dx dt \right| \\
\leq \int_0^T \| \phi_{mx} - \phi_x \| (\| \phi_{mx} \| + \| \phi_x \|) \| \Psi_{3lx} \| dt \\
&\quad + \int_0^T \| u_{mx} - u_x \| (\| u_{mx} \| + \| u_x \|) \| \Psi_{4lx} \| dt \rightarrow 0, \ m \rightarrow \infty.
\end{align*} \]
Here and \(\tilde{m}\) \(tm\),

Now we consider the case \((\int \omega_{C}^{2} + \int L \tilde{m} \omega_{0}^{x} + \int P - \int L \tilde{e}_{Lx}^{2} + \int L \tilde{m} \omega_{x}^{2} + \int \phi \omega_{m}^{x} + \phi \omega_{mx}^{x} + \frac{1}{2} \phi_{m}^{2} + \int \tilde{m} \omega_{mx}^{x} + \frac{1}{2} \tilde{m}_{x}^{2} + \int \tilde{m} \omega_{0}^{x} + \frac{1}{2} \tilde{m}_{x}^{2}) dx = 0 (2.28)\)

and

\[
\rho \int_{0}^{L} \tilde{m} \omega_{0}^{x} \dot{x}^{2} + \rho_2 \int_{0}^{L} \tilde{m} \omega_{0}^{x} \dot{x}^{2} + \gamma \int_{0}^{L} \tilde{m} \omega_{0}^{x} \dot{x}^{2} + \int_{0}^{L} \tilde{m} \omega_{0}^{x} \dot{x}^{2} + \int_{0}^{L} \tilde{m} \omega_{0}^{x} \dot{x}^{2} = 0 (2.29)\]

with initial conditions

\(\tilde{Z}_{m}(0) = (\tilde{\phi}_{m}(0), \tilde{u}_{m}(0), \tilde{\omega}_{m}(0), \tilde{v}_{m}(0)) = (\phi_{1m}, u_{1m}, \omega_{1m}, v_{1m}) = Z_{1m} \in V, \)

\(\tilde{Z}_{tm}(0) = (\tilde{\phi}_{tm}(0), \tilde{u}_{tm}(0), \tilde{\omega}_{tm}(0), \tilde{v}_{tm}(0)) = (\phi_{2m}, u_{2m}, \omega_{2m}, v_{2m}) \in W.\)

Here

\[
(\phi_{2m}, u_{2m}) = -\tilde{P}_{m} G^{-1}(N_{1}(\phi_{m}, u_{m}))
\]

\[
+ ((\phi_{0m} \omega_{0mx} + \frac{1}{2} \phi_{0mx}^{2})_{x}, [u_{0mx}(\omega_{0mx} + \frac{1}{2} u_{0mx}^{2})]_{x}) (2.30)\]

\[
(\omega_{2m}, u_{2m}) = -N_{2}(\omega_{m0}, u_{m0}) + \frac{1}{2} \tilde{P}_{m} ((\phi_{0mx}^{2})_{x}, [u_{0mx}^{2}]_{x}), (2.31)\]

where \(\tilde{P}_{m}, \hat{P}_{m}\) are projectors on the first \(m\) basis vectors \(\hat{e}_{i}\) and \(\hat{e}_{i}\) respectively. It is easy to see that

\[
\|G^{-1} N_{1}(\phi_{m0}, u_{m0}) - G^{-1} N_{1}(\phi_{0m}, u_{0m})\|^{2}_{1}
\]

\[
\leq C(N_{1}(\phi_{m0}^{2}, u_{m0}) - (\phi_{0m}, u_{0m}), G^{-1} N_{1}(\phi_{m0}^{2}, u_{m0}) - (\phi_{0m}, u_{0m}))
\]

\[
\leq C\|N_{1}(\phi_{m0}, u_{m0}) - (\phi_{0m}, u_{0m})\|^{2}_{1},
\]

\[
\leq C\|N_{1}^{3/4}(\phi_{m0}^{2}, u_{m0}) - (\phi_{0m}, u_{0m})\| \to 0, m \to \infty (2.32)\]

and

\[
\|G^{-1} ((\phi_{0mx}^{2}, \omega_{0mx} + \frac{1}{2} u_{0mx}^{2})_{x}, [u_{0mx}(\omega_{0mx} + \frac{1}{2} u_{0mx}^{2})]_{x})_{x}, [u_{0mx}(\omega_{0mx} + \frac{1}{2} u_{0mx}^{2})]_{x}) \| \nu
\]

\[
\leq C(\|\phi_{0mx} - \phi_{0mx}\| \|w_{0xx}\| + \|\phi_{0xx}\|^{2} + \|\phi_{0xx}\|^{2} + \|\phi_{0xx}\|^{2} \|\omega_{0mx} - \omega_{0xx}\| \|v_{0xx}\| \|v_{0xx} - v_{0xx}\| \to 0, m \to \infty. (2.33)\]
Analogously,
\[
\begin{align*}
\|N_2(\omega_{m0}, v_{m0}) - N_2(\omega_0, v_0) + 1/2([\omega_{0mxx}]_x, [v_{0mxx}]_x) - 1/2([\omega_{0xx}]_x, [v_{0xx}]_x)\| \\
\leq C \left(\|v_{0mxx} - v_{0xx}\| + \|\omega_{0mxx} - \omega_{0xx}\| + \|\phi_{0mxx} - \phi_{0xx}\| \right) \rightarrow 0, \quad m \to \infty.
\end{align*}
\] *(2.34)*

Making use of (2.15), (2.32)–(2.34) it is easy to infer from (2.28)–(2.29) that
\[
\| (\tilde{Z}_m(t), \tilde{Z}_{m,t}(t)) \|_H \leq C(T, \| U_0 \|_D).
\] *(2.35)*

Then, it follows from (2.12)–(2.13) that
\[
\| Z_m(t) \|_D \leq C(T, \| U_0 \|_D).
\] *(2.36)*

Arguing as above for weak solutions one can show the existence of strong solutions which are also approximate solutions by the energy argument. \(\Box\)

**Remark 2.10.** The variational relation in (4) can be extended on the class of test functions from \(L_T\) by the appropriate limit transition, therefore, it is easy to see that in case \(g_i = 0\) strong solutions satisfy
\[
\int_0^T \int_0^{L_0} \left( \beta_1 \phi_{tt} z_1 dx + \mu_1 \int_0^{L_0} \phi_{tx} z_1 dx + \beta_2 \int_0^L u_{tt} z_2 dx + \mu_2 \int_0^L u_{tx} z_2 dx \\
- \lambda_1 \phi_{xxx} z_1 dx - \lambda_2 \int_0^L u_{xxx} z_2 dx + \kappa \int_0^L \phi_{1x} z_1 dx \\
+ \int_0^L \left[ \phi_x \left( \omega_x + 1/2 \phi_x^2 \right) \right] z_1 dx \\
+ \int_0^L \left[ u_x \left( v_x + 1/2 u_x^2 \right) \right] z_2 dx + \rho_1 \int_0^{L_0} \omega_{tt} y_1 dx + \gamma \int_0^{L_0} \omega_t y_1 dx \\
+ \int_0^L \left( \omega_x + 1/2 \phi_x^2 \right) y_1 dx + \rho_2 \int_0^L v_{tt} y_2 dx + \int_0^L \left( v_x + 1/2 u_x^2 \right) y_2 dx dt = 0
\] *(2.37)*

for any \((z_1, z_2), (y_1, y_2) \in L_2(0, T; Y)\).

**Remark 2.11.** It follows from Theorem 3 that system (1.1)–(1.10) generates a dynamical system \((S_t, H)\) with the nonlinear operator \(S_t U_0 = U(t)\), where \(U(t)\) is a weak solution to (1.1)–(1.10), if \(g_i = g_i(x), i = 1, \ldots, 4\).

3. **Asymptotic smoothness.** To describe the long-time behaviour of solutions to system (1.1)–(1.10) we show the following result on asymptotic smoothness.

**Theorem 3.1.** Let assumptions (2.9) hold true and all right-hand sides in equations (1.1)–(1.4) be autonomous, i.e.
\[
g_i = g_i(x), i = 1, \ldots, 4.
\] *(3.1)*

Let, moreover,
\[
\beta_1 \geq \beta_2, \quad \rho_1 \geq \rho_2, \quad \mu_1 \geq \mu_2, \quad \lambda_1 \leq \lambda_2.
\] *(3.2)*

Then the dynamical system \((S_t, H)\) generated by problem (1.1)–(1.10) is asymptotically smooth.
Proof. Let \( \bar{Z}(t) = (\dot{\phi}(t), \ddot{u}(t), \dot{\omega}(t), \ddot{v}(t)) \) and \( \ddot{Z}(t) = (\dot{\phi}(t), \ddot{u}(t), \dot{\omega}(t), \ddot{v}(t)) \) be two weak solutions to problem (1.1)-(1.10) with initial data \( \bar{U}_0 = (\phi_0, u_0, \omega_0, v_0, \dot{\phi}_1, \ddot{u}_1, \dot{\omega}_1, \ddot{v}_1) \) and \( \ddot{U}_0 = (\dot{\phi}_0, u_0, \omega_0, v_0, \ddot{\phi}_1, \dot{u}_1, \dddot{\omega}_1, \ddot{v}_1) \) respectively. We assume that \( \bar{U}_0 \) and \( \ddot{U}_0 \) lie in a ball \( B_R \) of radius \( R > 0 \). Then, it is easy to see from energy equality (2.10) and Lemma 2.6 that

\[
\| \bar{Z}, \ddot{Z} \|_H + \| (\bar{Z}, \ddot{Z}) \|_H \leq e^{TC(R)}. \tag{3.3}
\]

We consider the difference \( Z(t) = \bar{Z}(t) - \ddot{Z}(t) = (\phi(t), u(t), \omega(t), v(t)) \) which satisfies the problem

\[
\begin{align*}
\beta_1 \phi_{tt} - \mu_1 \phi_{ttxx} - \kappa \phi_{txxx} + \lambda_1 \phi_{xxxx} & = 0, \\
\rho_1 \omega_{tt} - \gamma \omega_1 - \left( \omega_x + \frac{\phi_x}{2} \left( \ddot{\phi}_x + \dddot{\phi}_x \right) \right) & = 0, \\
\beta_2 u_{tt} - \mu_2 u_{ttxx} + \lambda_2 u_{xxxxxx} & = 0, \\
\phi_x(L, t) = 0, \ \phi(L, t) = 0, \ \omega(L, t) = 0, \ u_{xx}(L, t) = 0, \ u(L, t) = 0, \ v(L, t) = 0, \\
\phi(L_0, t) = u(L_0, t), \ \omega(L_0, t) = v(L_0, t), \ \phi_x(L_0, t) = u_x(L_0, t), \\
\lambda_1 \phi_{xxx}(L_0, t) = \lambda_2 u_{xx}(L_0, t), \\
\omega_x(L_0, t) = v_x(L_0, t), \ (\lambda_1 \phi_{xxx} - \mu_1 \phi_{ttxx} - \kappa \phi_{txx})(L_0, t) = (\lambda_2 u_{xxx} - \mu_2 u_{ttxx})(L_0, t)
\end{align*}
\]

with initial conditions \( \bar{U}_0 = \ddot{U}_0 - \dddot{U}_0 \) in a weak sense.

First, by energy argument and integration over the interval \([t; T]\) we establish the following energy type equality

\[
\Phi(U(T)) + \gamma \int_t^T \int_0^L \omega_x^2 dx d\tau + \kappa \int_t^T \int_0^L \phi_x^2 dx d\tau = \Phi(U(t)) + \int_t^T H(\bar{Z}, \ddot{Z}) d\tau, \tag{3.4}
\]

where

\[
\Phi(U(t)) = \frac{1}{2} \left[ \rho_1 \int_0^L \omega_x^2 dx + \rho_2 \int_0^L v_x^2 dx + \beta_1 \int_0^L \phi_x^2 dx + \beta_2 \int_0^L u_x^2 dx \\
+ \mu_1 \int_0^L \phi_{txx}^2 dx + \mu_2 \int_0^L u_{txx}^2 dx + \lambda_1 \int_0^L \phi_{xxx}^2 dx + \int_0^L \omega_x^2 dx \\
+ \lambda_2 \int_0^L u_{xxx}^2 dx + \int_0^L v_x^2 dx \right]
\]

and

\[
H(\bar{Z}, \ddot{Z}) = \frac{1}{2} \int_0^L (\omega_x + \phi_x) \phi_x + \omega_x (\ddot{\phi}_x + \dddot{\phi}_x) \phi_{txx} dx + \int_0^L (\ddot{\phi}_x + \dddot{\phi}_x) \phi_{xxx} \omega_{tx} dx \\
+ \frac{1}{2} \int_0^L (\ddot{\phi}_x + \dddot{\phi}_x \phi_x + \dddot{\phi}_x) \phi_{xxx} \phi_{txx} dx + \frac{1}{2} \int_0^L (\ddot{\phi}_x + \dddot{\phi}_x) \phi_{xxx} \omega_{tx} dx
\]
We also choose non-negative functions \( \eta \) exist such that
\[
T \Phi(U(T)) + \gamma \int_0^T \int_t^T \int_0^{L_0} \omega_x'^2 dx \, d\tau \, dt + \kappa \int_0^T \int_t^T \int_0^{L_0} \phi_{xx}'^2 dx \, d\tau \, dt \\
= \int_0^T \Phi(U(t)) \, dt + \int_0^T H(\tilde{Z}, \tilde{Z}) \, d\tau. 
\]
(3.6)

The following computations can be justified by performing them on strong solutions and the limit procedure. We consider a function \( \eta(x) \in C^\infty \) and assume that there exist \( 0 < \delta < L_0 \), \( \tilde{\eta}, \tilde{\eta} > 0 \) such that
\[
\eta(0) = 0, \eta(L) = 0, \quad \eta'(x) \leq 0, x \in (0, L_0 - \delta), \\
\eta'(x) > 0, x \in (L_0 - \delta, L), \eta'(x) \geq \tilde{\eta} > 0, x \in (L_0 - \frac{\delta}{2}, L). 
\]
(3.7)
(3.8)
(3.9)

We also choose non-negative functions \( \alpha(x), \sigma(x) \in C^\infty \) such that
\[
\alpha(L_0 - \delta) = 0, \quad \alpha'(L_0 - \delta) = 0, \\
\alpha(x) = \tilde{\eta}, x \in (L_0 - \frac{\delta}{2}, L), \\
\alpha(x) = 0, x \in (0, L_0 - \delta) 
\]
(3.10)
(3.11)
(3.12)

and
\[
\sigma(L_0) = 0, \quad \sigma'(L_0) = 0, \\
\sigma(x) = \tilde{\sigma} = 2 \max\{\tilde{\eta}, \tilde{\eta}\}, x \in (0, L_0 - \frac{\delta}{2}), \\
\sigma(x) = 0, x \in (L_0, L). 
\]
(3.13)
(3.14)
(3.15)

We choose \( (z_1, z_2) = (\eta(x) \phi_x, \eta(x) u_x) \in Y \) and \( (y_1, y_2) = (\eta(x) \omega_x, \eta(x) v_x) \in Y \) in (2.37), integrate over \([0, T]\), sum up the results and take the difference for solutions \( \tilde{Z}(t), \tilde{Z}(t) \) and arrive at
\[
\begin{align*}
\beta_1 \int_0^T \int_0^{L_0} \phi_{tt} \eta(x) \phi_x dx \, dt + \mu_1 \int_0^T \int_0^{L_0} \phi_{tt} \eta(x) \phi_y dx \, dt \\
+ \mu_1 \int_0^T \int_0^{L_0} \phi_{tx} \eta'(x) \phi_x dx \, dt - \lambda_1 \int_0^T \int_0^{L_0} \phi_{xxx} \eta(x) \phi_x dx \, dt \\
- \lambda_1 \int_0^T \int_0^{L_0} \phi_{xxx} \eta'(x) \phi_x dx \, dt + \rho_1 \int_0^T \int_0^{L_0} \omega_{tt} \eta(x) \omega_x dx \, dt \\
+ \frac{1}{2} \int_0^T \int_0^{L_0} \left[ (\tilde{\phi}_x + \tilde{\phi}_x) (\omega_x + (\tilde{\phi}_x + \tilde{\phi}_x) \frac{\phi_x}{2}) \right] \eta(x) \phi_x dx \, dt \\
+ \frac{1}{2} \int_0^T \int_0^{L_0} \left[ (\tilde{\phi}_x + \tilde{\phi}_x) (\omega_x + (\tilde{\phi}_x + \tilde{\phi}_x) \frac{\phi_x}{2}) \right] \eta'(x) \phi_x dx \, dt \\
+ \frac{1}{2} \int_0^T \int_0^{L_0} \left[ \phi_x (\tilde{\omega}_x + \tilde{\omega}_x + \tilde{\phi}_x^2 + \tilde{\phi}_x^2) \right] \eta(x) \phi_x dx \, dt
\end{align*}
\]
+ \frac{1}{2} \int_0^T \int_0^{L_0} \left[ \phi_x (\ddot{\omega}_x + \dot{\omega}_x + \frac{\dot{\phi}_x^2}{2} + \frac{\ddot{\phi}_x^2}{2} ) \right] \eta'(x) \phi_x \, dx \, dt \\
+ \int_0^T \int_0^{L_0} (\omega_x + (\ddot{\phi}_x + \dot{\phi}_x) \frac{\phi_x}{2}) \eta(x) \omega_{xx} \, dx \, dt \\
+ \int_0^T \int_0^{L_0} (\omega_x + (\ddot{\phi}_x + \dot{\phi}_x) \frac{\phi_x}{2}) \eta'(x) \omega_x \, dx \, dt \\
+ \int_0^T \int_0^{L_0} \phi_{tx} \eta(x) \phi_{xx} \, dx \, dt + \kappa \int_0^T \int_0^{L_0} \phi_{tx} \eta'(x) \phi_x \, dx \, dt \\
+ \gamma \int_0^T \int_0^{L_0} \omega_t \eta(x) \omega_x \, dx \, dt + \int_0^T \int_0^{L_0} u_{tx} \eta(x) u_x \, dx \, dt \\
+ \mu_2 \int_0^T \int_0^{L_0} u_{tx} \eta(x) u_{xx} \, dx \, dt + \mu_2 \int_0^T \int_0^{L_0} u_{tx} \eta'(x) u_x \, dx \, dt \\
- \lambda_2 \int_0^T \int_0^{L_0} u_{xxx} \eta(x) u_{xx} \, dx \, dt - \lambda_2 \int_0^T \int_0^{L_0} u_{xxx} \eta'(x) u_x \, dx \, dt \\
+ \rho_2 \int_0^T \int_0^{L_0} v_{tx} \eta(x) v_x \, dx \, dt + \frac{1}{2} \int_0^T \int_0^{L_0} \left[ (\ddot{u}_x + \ddot{\underline{u}}_x)(v_x + (\ddot{u}_x + \ddot{\underline{u}}_x) \frac{u_x}{2}) \right] \eta(x) u_{xx} \, dx \, dt \\
+ \frac{1}{2} \int_0^T \int_0^{L_0} \left[ u_x (\dot{\ddot{v}}_x + \ddot{v}_x + \frac{\ddot{u}_x^2}{2} + \frac{\ddot{\underline{u}}_x^2}{2}) \right] \eta(x) u_{xx} \, dx \, dt \\
+ \frac{1}{2} \int_0^T \int_0^{L_0} \left[ u_x (\dot{\ddot{v}}_x + \ddot{v}_x + \frac{\ddot{u}_x^2}{2} + \frac{\ddot{\underline{u}}_x^2}{2}) \right] \eta'(x) u_x \, dx \, dt \\
+ \int_0^T \int_0^{L_0} (v_x + (\ddot{\underline{u}}_x + \ddot{u}_x) \frac{u_x}{2}) \eta(x) v_{xx} \, dx \, dt \\
+ \int_0^T \int_0^{L_0} (v_x + (\ddot{\underline{u}}_x + \ddot{u}_x) \frac{u_x}{2}) \eta'(x) v_x \, dx \, dt = 0. \quad (3.16)

Integrating by parts we obtain

\begin{align*}
\beta_1 \int_0^T \int_0^{L_0} \phi_t \eta(x) \phi_x \, dx \, dt + \rho_1 \int_0^T \int_0^{L_0} \omega_t \eta(x) \omega_x \, dx \, dt \\
+ \beta_2 \int_0^T \int_0^{L_0} u_{tx} \eta(x) u_x \, dx \, dt + \rho_2 \int_0^T \int_0^{L_0} v_{tx} \eta(x) v_x \, dx \, dt \\
= -\beta_1 \int_0^T \int_0^{L_0} \phi_t \eta(x) \phi_x \, dx \, dt - \rho_1 \int_0^T \int_0^{L_0} \omega_t \eta(x) \omega_x \, dx \, dt \\
+ \beta_1 \int_0^T \int_0^{L_0} \phi_t(x) \eta(x) \phi_x(T) \, dx + \rho_1 \int_0^T \int_0^{L_0} \omega_t \eta(x) \omega_x(T) \, dx \\
- \beta_1 \int_0^L \phi_t(0) \eta(x) \phi_x(0) \, dx - \rho_1 \int_0^L \omega_t(0) \eta(x) \omega_x(0) \, dx \\
- \beta_2 \int_0^L u_{tx} \eta(x) u_x \, dx \, dt - \rho_2 \int_0^L v_{tx} \eta(x) v_x \, dx \, dt
\end{align*}
It is easy to see that

\[ \lambda_1 \int_{L_0}^{L} \phi_{x_{xx}} \eta(x) \phi_{x_{xx}} dxdt - \lambda_1 \int_{L_0}^{L} \phi_{x_{xx}} \eta'(x) \phi_x dxdt \]

\[ - \lambda_2 \int_{L_0}^{L} u_{x_{xx}} \eta(x) u_{x_{xx}} dxdt - \lambda_2 \int_{L_0}^{L} u_{x_{xx}} \eta'(x) u_x dxdt \]
\[
\frac{3\lambda_1}{2} \int_0^T \int_0^{L_0} \phi_{xx}^2 \psi'(x) dx dt + \frac{3\lambda_2}{2} \int_0^T \int_0^L u_{xx}^2 \psi'(x) dx dt \\
+ \frac{\lambda_2(\lambda_2 - \lambda_1)\|\eta(0)\|}{2\lambda_1} \int_0^T u_{xx}^2(0) dt \\
+ \lambda_1 \int_0^T \int_0^{L_0} \phi_{xx} \eta''(x) \phi_x dx + \lambda_2 \int_0^T \int_0^L u_{xx} \eta''(x) u_x dx dt
\]

(3.19)

and
\[
\frac{1}{2} \int_0^T \int_0^{L_0} \left[ (\ddot{\phi}_x + \dot{\phi}_x)(\omega_x + (\ddot{\phi}_x + \dot{\phi}_x)\frac{\phi_x}{2}) \right] \eta(x) \phi_{xx} dx dt \\
+ \int_0^T \int_0^{L_0} (\omega_x + (\ddot{\phi}_x + \dot{\phi}_x)\frac{\phi_x}{2}) \eta(x) \omega_{xx} dx dt \\
+ \frac{1}{2} \int_0^T \int_0^{L_0} (\ddot{u}_{xx} + \dot{u}_{xx})(v_x + (\ddot{u}_{xx} + \dot{u}_{xx})\frac{u_x}{2}) \eta(x) u_{xx} dx dt \\
+ \int_0^T \int_0^{L_0} (v_x + (\ddot{u}_{xx} + \dot{u}_{xx})\frac{u_x}{2}) \eta(x) v_{xx} dx dt
\]

\[
= -\frac{1}{2} \int_0^T \int_0^{L_0} (\ddot{\phi}_{xx} + \dot{\phi}_{xx})(\omega_x + (\ddot{\phi}_x + \dot{\phi}_x)\frac{\phi_x}{2}) \eta(x) \phi_{xx} dx dt \\
- \frac{1}{2} \int_0^T \int_0^{L_0} (\omega_x + (\ddot{\phi}_x + \dot{\phi}_x)\frac{\phi_x}{2})^2 \eta'(x) dx dt \\
- \frac{1}{2} \int_0^T \int_0^{L_0} (\ddot{u}_{xx} + \dot{u}_{xx})(v_x + (\ddot{u}_{xx} + \dot{u}_{xx})\frac{u_x}{2}) \eta(x) u_{xx} dx dt \\
- \frac{1}{2} \int_0^T \int_0^{L_0} (v_x + (\ddot{u}_{xx} + \dot{u}_{xx})\frac{u_x}{2})^2 \eta'(x) dx dt.
\]

(3.20)

Next we substitute \((z_1, z_2) = (\alpha(x) \phi, \alpha(x) u) \in Y, (y_1, y_2) = (0, 0)\) into (2.37), integrate over \([0, T]\), sum up the results and take the difference for solutions \(\hat{Z}(t), Z(t)\).

\[
\beta_1 \int_0^T \int_0^{L_0} \phi_{tt} \alpha(x) \phi dx dt + \mu_1 \int_0^T \int_0^{L_0} \phi_{tt} \alpha(x) \phi_x dx dt \\
+ \mu_1 \int_0^T \int_0^{L_0} \phi_{tt} \alpha'(x) \phi dx dt + \beta_2 \int_0^T \int_0^{L_0} u_{tt} \alpha(x) u dx dt \\
+ \mu_2 \int_0^T \int_0^{L_0} u_{tt} \alpha(x) u_x dx dt + \mu_2 \int_0^T \int_0^{L_0} u_{tt} \alpha(x) u_{xx} dx dt
\]
Integrating by parts we come to

\[- \lambda_1 \int_0^T \int_{L_0}^{L_0} \phi_{xxx}\alpha(x)\phi_x \, dx \, dt - \lambda_1 \int_0^T \int_{L_0}^{L_0} \phi_{xxx}'(x)\phi_x \, dx \, dt\]
\[- \lambda_2 \int_0^T \int_{L_0}^{L_0} u_{xxx}\alpha(x)u_x \, dx \, dt - \lambda_2 \int_0^T \int_{L_0}^{L_0} u_{xxx}'(x)u_x \, dx \, dt\]
\[+ \kappa \int_0^T \int_{L_0}^{L_0} \phi_{1x}\alpha(x)\phi_x \, dx \, dt + \kappa \int_0^T \int_{L_0}^{L_0} \phi_{1x}'(x)\phi_x \, dx \, dt\]
\[+ \frac{1}{2} \int_0^T \int_{L_0}^{L_0} \left[ (\tilde{\phi}_x + \hat{\phi}_x)(\omega_x + (\tilde{\phi}_x + \hat{\phi}_x)\frac{\phi_x}{2}) \right] \alpha(x)\phi_x \, dx \, dt\]
\[+ \frac{1}{2} \int_0^T \int_{L_0}^{L_0} \left[ (\tilde{\phi}_x + \hat{\phi}_x)(\omega_x + (\tilde{\phi}_x + \hat{\phi}_x)\frac{\phi_x}{2}) \right] \alpha'(x)\phi_x \, dx \, dt\]
\[+ \frac{1}{2} \int_0^T \int_{L_0}^{L_0} \left[ \phi_x(\tilde{\omega}_x + \hat{\omega}_x + \frac{\tilde{\phi}_x^2}{2} + \frac{\hat{\phi}_x^2}{2}) \right] \alpha(x)\phi_x \, dx \, dt\]
\[+ \frac{1}{2} \int_0^T \int_{L_0}^{L_0} \left[ \phi_x(\tilde{\omega}_x + \hat{\omega}_x + \frac{\tilde{\phi}_x^2}{2} + \frac{\hat{\phi}_x^2}{2}) \right] \alpha'(x)\phi_x \, dx \, dt\]
\[+ \frac{1}{2} \int_0^T \int_{L_0}^{L_0} \left[ (\tilde{u}_x + \hat{u}_x)(v_x + (\tilde{u}_x + \hat{u}_x)\frac{u_x}{2}) \right] \alpha(x)u_x \, dx \, dt\]
\[+ \frac{1}{2} \int_0^T \int_{L_0}^{L_0} \left[ (\tilde{u}_x + \hat{u}_x)(v_x + (\tilde{u}_x + \hat{u}_x)\frac{u_x}{2}) \right] \alpha'(x)u_x \, dx \, dt\]
\[+ \frac{1}{2} \int_0^T \int_{L_0}^{L_0} u_x(\tilde{v}_x + \hat{v}_x + \frac{\tilde{u}_x^2}{2} + \frac{\hat{u}_x^2}{2}) \alpha(x) \, dx \, dt\]
\[+ \frac{1}{2} \int_0^T \int_{L_0}^{L_0} u_x(\tilde{v}_x + \hat{v}_x + \frac{\tilde{u}_x^2}{2} + \frac{\hat{u}_x^2}{2}) \alpha'(x) \, dx \, dt = 0. \quad (3.21)\]
Next we substitute \((z_1, z_2) = (\sigma(x)\phi, 0) \in Y, (y_1, y_2) = (\sigma(x)\omega, 0) \in Y\) into (2.37), integrate over \([0, T]\), sum up the results, take the difference for solutions \(Z(t), \hat{Z}(t)\). After integration by parts we come to

\[
- \frac{1}{2} \int_0^T \int_0^{L_0} \left[ \phi_x (\ddot{\omega}_x + \dot{\omega}_x + \frac{\dot{\phi}_x^2}{2} + \frac{\ddot{\phi}_x}{2}) \right] \alpha(x) \phi_x dx dt \\
- \frac{1}{2} \int_0^T \int_0^{L_0} \left[ \phi_x (\ddot{\omega}_x + \dot{\omega}_x + \frac{\dot{\phi}_x^2}{2} + \frac{\ddot{\phi}_x}{2}) \right] \alpha'(x) \phi_x dx dt \\
- \frac{1}{2} \int_0^T \int_0^{L_0} \left[ (\ddot{u}_x + \dot{u}_x)(v_x + (\ddot{u}_x + \dot{u}_x) \frac{\mu}{2}) \right] \alpha(x) u_x dx dt \\
- \frac{1}{2} \int_0^T \int_0^{L_0} \left[ (\ddot{u}_x + \dot{u}_x)(v_x + (\ddot{u}_x + \dot{u}_x) \frac{\mu}{2}) \right] \alpha'(x) u_x dx dt \\
- \frac{1}{2} \int_0^T \int_0^{L_0} \left[ u_x (\dddot{v}_x + \ddot{v}_x + \frac{\ddot{u}_x^2}{2} + \frac{\dddot{u}_x}{2}) \right] \alpha(x) u_x dx dt \\
- \frac{1}{2} \int_0^T \int_0^{L_0} \left[ u_x (\dddot{v}_x + \ddot{v}_x + \frac{\ddot{u}_x^2}{2} + \frac{\dddot{u}_x}{2}) \right] \alpha'(x) u_x dx dt \\
= \beta_2 \int_0^L u_0(T)u(T)\alpha(x) dx + \beta_1 \int_0^{L_0} \phi_0(T) \phi(T) \alpha(x) dx \\
- \beta_2 \int_0^L u_t(0)u(0)\alpha(x) dx - \beta_1 \int_0^{L_0} u_t(0)u(0)\alpha(x) dx \\
- \mu_2 \int_0^L u_{tx}(T)u_x(T)\alpha(x) dx + \mu_2 \int_0^L u_{tx}(0)u_x(0)\alpha(x) dx \\
- \mu_1 \int_0^{L_0} \phi_{tx}(T)\phi_x(T)\alpha(x) dx + \mu_1 \int_0^{L_0} \phi_{tx}(0)\phi_x(0)\alpha(x) dx. \quad (3.22)
\]
Collecting (3.2), (3.16)–(3.23) and taking into account properties of functions η, α, σ defined in (3.7)–(3.15) and using Lemma 2 and (3.3) we come to

\[
\sum_{i=1}^{4} \int_{0}^{T} \Psi_i(t)dt \leq C(\Phi(0) + \Phi(T) + \int_{0}^{T} \int_{0}^{L} \phi_x^2 dxdt \\
+ \int_{0}^{T} \int_{0}^{L} \omega_x^2 dxdt) + C(R,T) \int_{0}^{T} \text{lot}(\phi, \omega, u, v)dt, \quad (3.24)
\]

where

\[
\text{lot}(\phi, \omega, u, v) = \|\phi\|^2_{H^{2-\epsilon}(0,L_0)} + \|u\|^2_{H^{2-\epsilon}(L_0,L)} \\
+ \|\omega\|^2_{H^{1-\epsilon}(0,L_0)} + \|v\|^2_{H^{1-\epsilon}(L_0,L)} + \|u_t\|^2_{H^{1-\epsilon}(L_0,L)} \quad (3.25)
\]

for \(0 < \epsilon < 1/2\) and

\[
\Psi_1(t) = \tilde{\eta} \left( \frac{3\beta_2}{2} \int_{0}^{T} \int_{0}^{L} u_t^2 dxdt + \frac{\mu_2}{2} \int_{0}^{T} \int_{0}^{L} u_{xx}^2 dxdt \\
+ \frac{\lambda_2}{4} \int_{0}^{T} \int_{0}^{L} u_{tx}^2 dxdt + \frac{1}{4} \int_{0}^{T} \int_{0}^{L} u_x^2 dxdt + \frac{\rho_2}{2} \int_{0}^{T} \int_{0}^{L} v_{xx}^2 dxdt \right),
\]

\[
\Psi_2(t) = \tilde{\eta} \left( \frac{3\beta_2}{2} \int_{0}^{T} \int_{0}^{L} \phi_t^2 dxdt + \frac{\mu_1}{2} \int_{0}^{T} \int_{0}^{L} \phi_{tx}^2 dxdt \\
+ \frac{\lambda_1}{4} \int_{0}^{T} \int_{0}^{L} \phi_{xx}^2 dxdt + \frac{1}{4} \int_{0}^{T} \int_{0}^{L} \phi_t^2 dxdt + \frac{\rho_1}{2} \int_{0}^{T} \int_{0}^{L} \omega_t^2 dxdt \right),
\]

\[
\Psi_3(t) = \tilde{\eta} \left( \frac{3\beta_2}{2} \int_{0}^{T} \int_{0}^{L} \phi_t^2 dxdt + \frac{\mu_1}{2} \int_{0}^{T} \int_{0}^{L} \phi_{tx}^2 dxdt \\
+ \frac{\lambda_1}{4} \int_{0}^{T} \int_{0}^{L} \phi_{xx}^2 dxdt + \frac{1}{4} \int_{0}^{T} \int_{0}^{L} \phi_t^2 dxdt + \frac{\rho_1}{2} \int_{0}^{T} \int_{0}^{L} \omega_t^2 dxdt \right),
\]

and

\[
\Psi_4(t) = \tilde{\eta} \left( \frac{3\beta_2}{2} \int_{0}^{T} \int_{0}^{L} \phi_t^2 dxdt + \frac{\mu_1}{2} \int_{0}^{T} \int_{0}^{L} \phi_{tx}^2 dxdt \\
+ \frac{\lambda_1}{4} \int_{0}^{T} \int_{0}^{L} \phi_{xx}^2 dxdt + \frac{1}{4} \int_{0}^{T} \int_{0}^{L} \phi_t^2 dxdt + \frac{\rho_1}{2} \int_{0}^{T} \int_{0}^{L} \omega_t^2 dxdt \right).
\]

Consequently, (3.24) yields

\[
\int_{0}^{T} \Phi(t)dt \leq C(\Phi(0) + \Phi(T) + \int_{0}^{T} \int_{0}^{L} \phi_{xx}^2 dxdt)
\]
\[ + \int_0^T \int_0^{L_0} \omega_t^2 \, dx \, dt + C(R, T) \int_0^T \log(\phi, \omega, u, v) \, dt, \quad (3.26) \]

where the last term is defined by (3.25).

Then it follows from (3.4) with \( t = 0 \) and (3.6) that

\[
\int_0^T \Phi(t) \, dt \leq C \left( \Phi(0) + \Phi(T) + \int_0^T \int_0^{L_0} \phi_t^2 \, dx \, dt + \int_0^T \int_0^{L_0} \omega_t^2 \, dx \, dt \right)
+ C(R, T) \int_0^T \log(\phi, \omega, \phi, u) \, dt
\]

\[
\leq C \left( \Phi(0) + \Phi(T) + \int_0^T H(\tilde{Z}, \tilde{Z}) \, d\tau \right) + C(R, T) \int_0^T \log(\phi, \omega, \phi, u) \, dt.
\]

Next we substitute (3.27) into (3.6) and relying on (3.4) with \( t = 0 \) come to the estimate

\[
T \Phi(U(T)) + \int_0^T \int_0^T \int_0^{L_0} \omega_t^2 \, dx \, d\tau \, dt + \int_0^T \int_0^T \int_0^{L_0} \phi_t^2 \, dx \, d\tau \, dt
\leq C \left( \Phi(0) + \int_0^T H(\tilde{Z}, \tilde{Z}) \, d\tau \right) + \int_0^T \int_0^T H(\tilde{Z}, \tilde{Z}) \, d\tau \, dt
+ C(R, T) \int_0^T \log(\phi, \omega, \phi, u) \, dt.
\]

(3.28)

Now our remaining task is to estimate the nonlinear terms in (2.6). We begin with the third term in the right-hand side. Integrating by parts with respect to \( t \) and using Lemma 2 we obtain for any \( 0 < \epsilon < 1/2 \)

\[
\left| \int_0^T \int_0^{L_0} ((\ddot{u}_x + \ddot{u}_x)v_xu_t + (\dot{u}_x + \dot{u}_x)u_xv_t) \, dx \, dt \right|
\leq \int_0^T \int_0^{L_0} (\ddot{u}_x + \ddot{u}_x)v_xu_t \, dx \, dt + \max_{[0,T]} \left| \int_0^{L_0} (\ddot{u}_x + \ddot{u}_x)v_xu_t \, dx \right|
\leq C(T) \max_{[0,T]} \||\ddot{u}_x|| + ||\ddot{u}_x||\|v_x\|\|u\|_{2-\epsilon} + (||\ddot{u}_x||)\|v_x\|\|u\|_{2-\epsilon}
\leq C(T, R) \max_{[0,T]} \|u\|_{H^2-(L_0,L)}.
\]

(3.29)

\[
\left| \int_0^T \int_0^{L_0} (\dot{v}_x + \dot{v}_x)u_xu_t \, dx \, dt \right|
\leq C \int_0^T (||\dot{v}_x|| + ||\dot{v}_x||)\|u_t\|\|u\|_{2-\epsilon} \, dt
\leq C(R, T) \max_{[0,T]} \|u\|_{H^2-(L_0,L)}.
\]

(3.30)

Analogously,

\[
\left| \int_0^T \int_0^{L_0} (\ddot{u}_x + \ddot{u}_x + \ddot{u}_x)u_xu_t \, dx \right|
\leq C(R, T) \max_{[0,T]} \|u\|_{H^2-(L_0,L)}.
\]

(3.31)

Collecting (3.29)–(3.31) and handling the first three terms in (2.6) in the same way and repeating the same computations for the first three terms we infer from (3.28)

\[
\Phi(U(T)) \leq \frac{C_1(R)}{T} + C_2(R, T)h(\phi, \omega, \phi, u),
\]

(3.32)
where
\[
T > h > \text{we obtain } \phi
\]

\[h(\phi, \omega, \phi, u) = \max(\|\phi\|_{H^2 \cap (0, L_0)}^2 + \|u\|_{H^2 \cap (L_0, L)}^2 + \|\omega\|_{H^1 \cap (0, L_0)}^2)
\]

\[+ \|u\|_{H^2 \cap (L_0, L)}^2 + \|u\|_{H^2 \cap (L_0, L)}^2 + \|\phi\|_{H^2 \cap (L_0, L)}^2)
\]

for \(0 < \varepsilon < 1/2\). For any \(\varepsilon > 0\) it is possible to choose \(T > 0\) large enough to get (2.1) from (3.33) and to infer the statement of the theorem.

4. Existence of attractors.

**Theorem 4.1.** Let assumptions (2.9), (3.1) hold true. Then, the dynamical system \((S_t, H) \) generated by (1.1)-(1.10) is gradient.

**Proof.** Step 1. Regularity. Now we show that system (1.1)-(1.10) is gradient. The strict Lyapunov function is

\[
\mathcal{L}(t) = \mathcal{E}(t) - \sum_{i=1,2} \int_{0}^{L_0} g_i U_{t} dx - \sum_{i=3,4} \int_{0}^{L_0} g_i U_{t} dx.
\]

(4.1)

In order to prove this, we assume that \(\mathcal{L}(S_t U) = \mathcal{L}(U)\), i.e. from the energy equality we obtain \(\phi_1 = 0, \omega_1 = 0\) and \(\phi_2(t) = \phi(t + h) - \phi(t) = 0, \omega_2(t) = \omega(t + h) - \omega(t) = 0\) for arbitrary \(T > h > 0\). Consequently, \(U_d(t) = (\phi_d(t), u_d(t), \omega_d(t), v_d(t)) = U(t + h) - U(t)\), satisfies

\[
- \int_{0}^{T} \int_{L_0}^{L} (\beta_2 u_{dt} \Psi_2 - \rho_2 v_{dt} \Psi_4) dx dt - \mu_2 \int_{0}^{T} \int_{L_0}^{L} u_{dx} \Psi_{2dx} dx dt
\]

\[+ \lambda_2 \int_{0}^{T} \int_{L_0}^{L} u_{dx} \Psi_{2dx} dx dt
\]

\[+ \int_{0}^{T} \int_{L_0}^{L} (v_{dx} + \frac{u_{dx}}{2} (u_d(t + h) + u_d(t))) \left( \Psi_{4x} + \frac{1}{2} (u_d(t + h) + u_d(t)) \Psi_{2x} \right) dx dt
\]

\[+ \frac{1}{2} \int_{0}^{T} \int_{L_0}^{L} u_d \left( v_d(t + h) + v_d(t) + \frac{u_d(t + h)}{2} + \frac{u_d(t)}{2} \right) \Psi_{2x} dx dt
\]

\[= \int_{L_0}^{L} (\beta_2 u_d(0) \Psi_2(0) + \rho_2 v_d(0) \Psi_4(0)) dx + \mu_2 \int_{L_0}^{L} u_{dx}(0) \Psi_{2dx}(0) dx
\]

\[- \int_{L_0}^{L} (\beta_2 u_{dt}(T) \Psi_2(T) + \rho_2 v_{dt}(T) \Psi_4(T)) dx + \mu_2 \int_{L_0}^{L} u_{dx}(T) \Psi_{2dx}(T) dx.
\]

(4.2)

Namely, \(u_d\) and \(v_d\) are weak solutions to an overdetermined problem

\[
\beta_2 u_{dt} - \mu_2 u_{dt dx} + \lambda_2 u_{dxx} - \frac{1}{2} \left[ u_{dx} (v_d(t) + 1/2 u_d^2(t) + v_d(t + h) + 1/2 u_d^2(t + h)) \right]_x
\]

\[- \frac{1}{2} \left[ u_d(t + h) + u_d(t) \right] (v_{dx} + 1/2 u_{dx} (u_d(t + h) + u_d(t))) \right)_x = 0, t > 0, x \in (L_0, L),
\]

\[\rho_2 v_{dt} - (v_{dx} + 1/2 u_{dx} (u_d(t + h) + u_d(t))) = 0
\]

(4.3)

with boundary conditions

\[
u_d(L_0, t) = 0, v_d(L_0, t) = 0, u_{dx}(L_0, t) = 0, u_{dx}(L_0, t) = 0,
\]

\[v_{dx}(L_0, t) = 0, u_{dx}(L_0, t) = 0 u_{dx}(L, t) = 0, u_d(L, t) = 0, v_d(L, t) = 0.
\]

(4.4)

(4.5)
Now we prove the additional regularity of the solution to the overdetermined problem (4.3)–(4.6). We define the multiplier

$$\varphi(x, t) = (L - x)T^2 - 5(L - L_0)\left(t - \frac{T}{2}\right)^2.$$  (4.7)

It is easy to see that

$$\varphi(x, 0) \leq -\frac{(L - L_0)T^2}{4} = -\sigma_0, \quad \varphi(x, T) \leq -\sigma_0.$$  (4.8)

Moreover, there exist $t_0$ and $t_1$, such that and $0 < t_0 < \frac{T}{2} < t_1 < T$

$$\min_{[t_0, t_1]} \varphi(x, t) \geq -\frac{\sigma_0}{2}.$$  (4.9)

The following formal computations can be justified by performing them on the strong solutions to problem (4.3), (4.4) with boundary conditions

$$u_d(L_0, t) = 0, \quad v_d(L_0, t) = 0, \quad u_{dx}(L_0, t) = 0,$$
$$u_{dx}(L, t) = 0, \quad u_d(L, t) = 0, \quad v_d(L, t) = 0.$$  (4.10)

and strong solutions to problem (1.1)–(1.10) as the functions $u(t)$ and $v(t)$. It is easy to show by the energy arguments that weak solutions to (4.3), (4.4) can be approximated in the energy norm by such strong solutions. For the simplification we present here the formal scheme and substitute $\Psi_2 = e^{\tau \varphi}(L - x)u_{dx}$ and $\Psi_4 = e^{\tau \varphi}(L - x)v_{dx}$ into (4.2)

$$-\int_0^T \int_{L_0}^L (\beta_2 u_{dt} u_{dx} + \rho_2 v_{dt} v_{dx}) e^{\tau \varphi}(L - x) dx dt$$
$$+ 10\tau(L - L_0) \int_0^T \int_{L_0}^L (\beta_2 u_{dt} u_{dx} + \rho_2 v_{dt} v_{dx})(t - \frac{T}{2}) e^{\tau \varphi}(L - x) dx dt$$
$$- \mu_2 \int_0^T \int_{L_0}^L u_{dx} e^{\tau \varphi}(L - x) u_{dx} dx dt$$
$$+ 10\mu_2 \tau(L - L_0) \int_0^T \int_{L_0}^L u_{dx} e^{\tau \varphi}(t - \frac{T}{2})(L - x) u_{dx} dx dt$$
$$- 10\mu_2 \tau(L - L_0) \int_0^T \int_{L_0}^L u_{dx} (t - \frac{T}{2}) e^{\tau \varphi}(\tau T^2(L - x) + 1) u_{dx} dx dt$$
$$+ \mu_2 \int_0^T \int_{L_0}^L u_{dx}^2 e^{\tau \varphi} [\tau T^2(L - x) + 1] dt + \lambda_2 \int_0^T \int_{L_0}^L u_{dx} u_{dxx} e^{\tau \varphi}(L - x) dx dt$$
$$- 2\lambda_2 \int_0^T \int_{L_0}^L u_{dx}^2 e^{\tau \varphi} [\tau T^2(L - x) + 1] dx dt$$
$$+ \lambda_2 \tau T^2 \int_0^T \int_{L_0}^L u_{dxx} u_{dx} e^{\tau \varphi}(\tau T^2(L - x) + 2) dx dt$$
$$+ \int_0^T \int_{L_0}^L \left( v_{dx} + \frac{u_{dx}}{2}(u_x(t + h) + u_x(t)) \right) e^{\tau \varphi}(L - x) dx dt$$
After integration by parts we get

\[ - \int_0^T \int_{L_0}^L \left( v_{dx} + \frac{u_{dx}}{2} (u_x(t+h) + u_x(t)) \right) e^{\tau \varphi(L-x)} dx dt \]

\[ + \frac{1}{2} \int_0^T \int_{L_0}^L u_{dx} \left( v_x(t+h) + v_x(t) + \frac{u_x^2}{2} + \frac{u_x^2}{2} (u_x(t+h)) \right) e^{\tau \varphi(L-x)} dx dt \]

\[ - \frac{1}{2} \int_0^T \int_{L_0}^L u_{dx}^2 \left( v_x(t+h) + v_x(t) + \frac{u_x^2}{2} + \frac{u_x^2}{2} (u_x(t+h)) \right) e^{\tau \varphi(L-x)} dx dt \]

\[ = \int_{L_0}^L (\beta_2 u_{dt}(0) u_{dx}(0) + \rho_2 v_{dt}(0) v_{dx}(0)) e^{\tau \varphi(0)(L-x)} dx \]

\[ + \mu_2 \int_{L_0}^L u_{dx}(0) u_{dx}(0) e^{\tau \varphi(0)(L-x)} dx \]

\[ - \mu_2 \int_{L_0}^L u_{dx}(0) u_{dx}(0) e^{\tau \varphi(0)(L-x)} dx \]

\[ - \int_{L_0}^L (\beta_2 u_{dt}(T) u_{dx}(T) + \rho_2 v_{dt}(T) v_{dx}(T)) e^{\tau \varphi(T)(L-x)} dx \]

\[ - \mu_2 \int_{L_0}^L u_{dx}(T) u_{dx}(T) e^{\tau \varphi(T)(L-x)} dx \]

\[ + \mu_2 \int_{L_0}^L u_{dx}(T) u_{dx}(T) e^{\tau \varphi(T)(L-x)} dx, \]  

\[ (4.11) \]

After integration by parts we get

\[ \int_0^T \int_{L_0}^L \left( v_{dx} + \frac{u_{dx}}{2} (u_x(t+h) + u_x(t)) \right) \left( v_{dx} + \frac{u_{dx}}{2} (u_x(t+h) + u_x(t)) \right) e^{\tau \varphi(L-x)} dx dt \]

\[ - \int_0^T \int_{L_0}^L (\beta_2 u_{dt} u_{dx} + \rho_2 v_{dt} v_{dx}) e^{\tau \varphi(L-x)} dx dt - \mu_2 \int_0^T \int_{L_0}^L u_{dx} e^{\tau \varphi(L-x)} dx dt \]

\[ + \lambda_2 \int_0^T \int_{L_0}^L u_{dx} u_{dx} e^{\tau \varphi(L-x)} dx dt \]

\[ = - \frac{1}{2} \int_0^T \int_{L_0}^L (\beta_2 u_{dt}^2 + \rho_2 v_{dt}^2) e^{\tau \varphi(L-x)} dx dt \]

\[ - \frac{\mu_2}{2} \int_0^T \int_{L_0}^L u_{dx}^2 e^{\tau \varphi(L-x)} dx dt + \lambda_2 \frac{1}{2} \int_0^T \int_{L_0}^L u_{dx}^2 e^{\tau \varphi(L-x)} dx dt \]

\[ + \frac{1}{2} \int_0^T \int_{L_0}^L \left( v_{dx} + \frac{u_{dx}}{2} (u_x(t+h) + u_x(t)) \right)^2 e^{\tau \varphi(L-x)} dx dt \]

\[ - \frac{\lambda_2}{2} (L - L_0) \int_0^T u_{dx}^2 (L_0) e^{\tau \varphi(L_0)} dt - \frac{1}{2} (L - L_0) \int_0^T u_{dx}^2 (L_0) e^{\tau \varphi(L_0)} dt \]  

\[ (4.12) \]

Now we substitute \((\Psi_2, \Psi_4) = (e^{\tau \varphi(L-x)} + 1)u_d, 0)\) into \((4.3)\)

\[ - \beta_2 \int_0^T \int_{L_0}^L u_{dt}^2 e^{\tau \varphi(L-x)} dx dt \]

\[ + 10 \beta_2 (L - L_0) \int_0^T u_{dt} u_{dt} e^{\tau \varphi(L-x)} dx dt \]

\[ + \frac{T}{2} (L - L_0) \int_0^T u_{dx} u_{dx} e^{\tau \varphi(L-x)} dx dt \]
- \mu_2 \int_0^T \int_{L_0}^L u_{dx}^2 e^\tau \varphi [\tau T^2(L - x) + 1] dx dt
+ \mu_2 \tau T^2 \int_0^T \int_{L_0}^L u_{dx} u_{dt} e^\tau \varphi [\tau T^2(L - x) + 2] dx dt
+ 10 \mu_2 \tau (L - L_0) \int_0^T \int_{L_0}^L u_{dx} u_{dx} e^\tau \varphi [\tau T^2(L - x) + 1] (t - \frac{T}{2}) dx dt
- 10 \mu_2 \tau T^2 (L - L_0) \int_0^T \int_{L_0}^L u_{dx} u_{dt} e^\tau \varphi [\tau T^2(L - x) + 2] (t - \frac{T}{2}) dx dt
+ \lambda_2 \int_0^T \int_{L_0}^L u_{dx}^2 e^\tau \varphi [\tau T^2(L - x) + 1] dx dt
- 2 \lambda_2 \tau T^2 \int_0^T \int_{L_0}^L u_{dx} u_{dx} e^\tau \varphi [\tau T^2(L - x) + 2] dx dt
+ \lambda_2 \tau T^4 \int_0^T \int_{L_0}^L u_{dx} u_{dx} e^\tau \varphi [\tau T^2(L - x) + 3] dx dt
+ \int_0^T \int_{L_0}^L \left( u_{dx} + \frac{u_{dx}}{2} (u_x(t + h) + u_x(t)) \right) e^\tau \varphi
\left( \frac{1}{2} (u_x(t + h) + u_x(t)) \right) (u_{dx}[\tau T^2(L - x) + 1] - u_{dx}[\tau T^2(L - x) + 2]) dx dt
+ \frac{1}{2} \int_0^T \int_{L_0}^L u_{dx}^2 \left( v_x(t + h) + v_x(t) + \frac{v_x^2(t)}{2} + \frac{v_x(t + h)}{2} \right) e^\tau \varphi [\tau T^2(L - x) + 1] dx dt
- \frac{1}{2} \int_0^T \int_{L_0}^L u_{dx} u_{dx} \left( v_x(t + h) + v_x(t) + \frac{v_x^2(t)}{2} + \frac{v_x(t + h)}{2} \right) e^\tau \varphi [\tau T^2(L - x) + 2] dx dt
= \beta_2 \int_{L_0}^L \left( u_{aT}^2(0)e^\tau \varphi(0) - u_{aT}(T)e^\tau \varphi(T) \right) [\tau T^2(L - x) + 1] dx
+ \mu_2 \int_{L_0}^L \left( u_{dx}(0)u_{dx}(0)e^\tau \varphi(0) - u_{dx}(T)u_{dx}(T)e^\tau \varphi(T) \right) [\tau T^2(L - x) + 1] dx
- \mu_2 \tau T^2 \int_{L_0}^L \left( u_{dx}(0)u_{dx}(0)e^\tau \varphi(0) - u_{dx}(T)u_{dx}(T)e^\tau \varphi(T) \right) [\tau T^2(L - x) + 2] dx.
(4.13)

Distracting (4.11) from (4.13) and taking into account (4.12) we arrive at

\[ L_d(T) + H_d(0) - H_d(T) = lot + M_d(T), \quad (4.14) \]

where

\[ L_d(T) = \frac{1}{2} \left( 3 \beta_2 \int_0^T \int_{L_0}^L e^\tau \varphi [\tau T^2(L - x) + 1] u_{aT}^2 dx dt
+ \mu_2 \int_0^T \int_{L_0}^L e^\tau \varphi [\tau T^2(L - x) + 1] u_{aT}^2 dx dt
+ \rho_2 \int_0^T \int_{L_0}^L e^\tau \varphi [\tau T^2(L - x) + 1] v_{aT}^2 dx dt \]
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\[ \frac{d}{dt} \int_{L_0}^{T} e^{\tau T^2} (L - x) + 1 \left( v_d + 1/2 u_{dx}(t + h) + u_x(t) \right)^2 dx dt \]

\[ M_d(T) = 10 \mu_0 \tau (L - L_0) \int_{T_0}^{T} \int_{L_0}^{L} (t - T/2) u_{dx} e^{\tau T^2} (L - x) u_{dx} dx dt \]

\[ + 10 \rho_0 \tau (L - L_0) \int_{T_0}^{T} \int_{L_0}^{L} \left( t - T/2 \right) u_{d} e^{\tau T^2} (L - x) v_{dx} dx dt \]

and

\[ l_{oT} = 10 \beta_2 \tau (L - L_0) \int_{T_0}^{T} \int_{L_0}^{L} u_{dt} u_{dx} (t - T/2) e^{\tau T^2} (L - x) dx dt \]

\[ + \mu_2 \tau T^2 \int_{T_0}^{T} \int_{L_0}^{L} u_{dx} u_{dx} e^{\tau T^2} (L - x) + 2 dx dt \]

\[ - 10 \mu_2 \tau (L - L_0) \int_{T_0}^{T} \int_{L_0}^{L} u_{dx} e^{\tau T^2} (L - x) + 1 \left( t - T/2 \right) u_{dx} dx dt \]

\[ + \lambda_2 \tau T^2 \int_{T_0}^{T} \int_{L_0}^{L} u_{dx} u_{dx} e^{\tau T^2} (L - x) dx dt \]

\[ - \int_{T_0}^{T} \int_{L_0}^{L} \left( v_d + \frac{u_{dx}}{2} (u_x(t + h) + u_x(t)) \right) \left( v_d + \frac{u_{dx}}{2} (u_x(t + h) + u_x(t)) \right) \left( \tau T^2 (L - x) + 1 \right) e^{\tau T^2} dx dt \]

\[ + 1/2 \int_{T_0}^{T} \int_{L_0}^{L} u_{dx} \left( v_x(t + h) + v_x(t) \right) + \left( \frac{u_{dx}^2}{2} + \frac{u_x^2(t + h)}{2} \right) \left( v_x(t + h) + v_x(t) \right) - \int_{T_0}^{T} \int_{L_0}^{L} u_{dx} e^{\tau T^2} (L - x) dx dt \]

\[ - \frac{1}{2} \int_{T_0}^{T} \int_{L_0}^{L} u_{dx} \left( v_x(t + h) + v_x(t) \right) + \left( \frac{u_{dx}^2}{2} + \frac{u_x^2(t + h)}{2} \right) \left( v_x(t + h) + v_x(t) \right) - 1 \left( t - T/2 \right) e^{\tau T^2} (L - x) + 1 dx dt \]

\[ + 10 \beta_2 (L - L_0) \int_{T_0}^{T} \int_{L_0}^{L} u_{dx} e^{\tau T^2} (L - x) + 1 \left( t - T/2 \right) dx dt \]

\[ + 10 \mu_2 \tau (L - L_0) \int_{T_0}^{T} \int_{L_0}^{L} u_{dx} u_{dx} e^{\tau T^2} (L - x) dx dt \]

\[ - 10 \mu_2 \tau T^2 (L - L_0) \int_{T_0}^{T} \int_{L_0}^{L} u_{dx} u_{dx} e^{\tau T^2} (L - x) + 2 \left( t - T/2 \right) dx dt \]

\[ - 2 \lambda_2 \tau T^2 \int_{T_0}^{T} \int_{L_0}^{L} u_{dx} u_{dx} e^{\tau T^2} (L - x) + 2 dx dt \]

\[ + \lambda_2 \tau T^4 \int_{T_0}^{T} \int_{L_0}^{L} u_{dx} u_{dx} e^{\tau T^2 (L - x)} + 3 dx dt \]

\[ + \int_{T_0}^{T} \int_{L_0}^{L} \left( v_d + \frac{u_{dx}}{2} (u_x(t + h) + u_x(t)) \right) e^{\tau T^2} \left( \frac{1}{2} (u_x(t + h) + u_x(t)) \right) \left( u_{dx} \tau T^2 (L - x) + 1 \right) u_{dx} \left( 1 - u_d [\tau T^2 (L - x) + 2] dx dt \right) \]

\[ + \frac{1}{2} \int_{T_0}^{T} \int_{L_0}^{L} u_{dx}^2 \left( v_x(t + h) + v_x(t) \right) + \left( \frac{u_{dx}^2(t)}{2} + \frac{u_x^2(t + h)}{2} \right) e^{\tau T^2 (L - x) + 1} dx dt \]
Consequently, one can infer from (4.21) the following estimate
\[
e^{-\tau \sigma_0} \int_{t_0}^{t_1} E_d(t) dt - C_3 (E_d(t) + E_d(0)) \tau^2 e^{-\tau \sigma_0} \leq C(\tau, T) \int_0^T \int_{L_0}^L (u_{tt}^2 + u_{tx}^2 + u_{xx}^2 + v_{xx}^2) dx dt.
\]
It is easy to infer from (4.2) by the energy argument, Lemma 2, and the Gronwall’s lemma that there exists $C(T;U_0) > 0$ such that for $0 \leq s \leq t \leq T$

$$E_d(t) \leq E_d(s)e^{C(T;U_0)(t-s)} \tag{4.24}$$

and

$$E_d(s) \leq E_d(t)e^{C(T;U_0)(t-s)}. \tag{4.25}$$

If we choose $t = T$, $s = t$ in (4.24) and $s = 0$ in (4.25), we obtain

$$E_d(T) + E_d(0) \leq CE_d(t)e^{C(T;U_0)} \tag{4.26}$$

for any $t \in [0, T]$. Using (4.26) we estimate the left-hand side of (4.23) as follows

$$e^{-\frac{\tau}{2T}} \int_{t_0}^{t_1} E_d(t) dt - C_3(E_d(T) + E_d(0))\tau^2 e^{-\tau_0} \leq \left(CTe^{-\frac{\tau}{2T}}e^{-C(T;U_0)} - C_3\tau^2 e^{-\tau_0}\right) (E_d(T) + E_d(0)). \tag{4.27}$$

Choosing $\tau$ large enough and taking into account (4.23) we get

$$E_d(T) + E_d(0) \leq C(\tau, T) \int_0^T \int_{L_0}^L (u_d^2 + u_d^2 + u_d^2 + v_d^2) dx dt. \tag{4.28}$$

We note here that (4.28) was obtained for strong solutions to (4.2) with boundary conditions (4.10). Since these solutions are approximate in the energy norm we estimate the left-hand side of (4.23) as follows

$$\int_0^T \int_{L_0}^L -p_{xx} = u_d, \tag{4.29}$$

$$p(L_0) = 0, \; p(L) = 0. \tag{4.30}$$

Substituting $\Psi_2 = p$, $\Psi_3 = 0$ into (4.2) we get

$$\int_0^T \int_{L_0}^L u_d^2 dx dt + \int_0^T \int_{L_0}^L p_d^2 dx dt \leq C \int_0^T \int_{L_0}^L u_{dx} u_d dx dt$$

$$+ \int_0^T \int_{L_0}^L \left( v_{dx} + \frac{u_{dx}}{2}(u_x(t+h) + u_x(t)) \right) (u_x(t+h) + u_x(t)) dx dt$$

$$+ \int_0^T \int_{L_0}^L \left( v_x(t+h) + v_x(t) + \frac{u_x^2(t)}{2} + \frac{u_x^2(t+h)}{2} \right) p_x dx dt + \int_{L_0}^L u_{dt}(0)p(0) dx$$

$$+ \int_{L_0}^L u_{dx}(0)p_x dx + \int_{L_0}^L u_{dx}(T)p(T) dx + \int_{L_0}^L u_{dx}(T)p_x(T) dx$$

$$\leq C(||U_0||_{H^1}, T) \int_0^T \int_{L_0}^L (u_d^2 + v_d^2) dx dt + \varepsilon(E_d(0) + E_d(T))$$

$$+ C(\varepsilon)(||u_d(0)||^2 + ||u_d(T)||^2). \tag{4.31}$$
Next after substituting (4.31) into (4.28), dividing by $h^2$ and passing to the limit $h \to 0$ we come to

$$E(U_0(T)) \leq C(\|U_0\|_{H,T} \left( \int_0^T \int_{L_0}^L (u_{1x}^2 + u_t^2 + v_t^2)dxdt + \|u_t(0)\|^2 + \|u_t(T)\|^2 \right)$$

$$\leq C(\|U_0\|_{H,T}).$$

Consequently, $(u_t, v_t, u_{tt}, v_{tt}) \in H^2(L_0, L) \times H^1(L_0, L) \times H^1(L_0, L) \times L^2(L_0, L)$. Then, relying on (4.2), one can infer $(u_d, v_d) \in H^2(L_0, L) \times H^2(L_0, L)$ and, consequently, $(u_d, v_d, u_{dt}, v_{dt}) \in H^3(L_0, L) \times H^2(L_0, L) \times H^2(L_0, L) \times H^1(L_0, L)$.

Now we notice that $(u_t, v_t) = (w, z)$ is a weak solution to the problem

$$\beta_2 w_{tt} - \mu_2 w_{ttxx} + \lambda_2 w_{xxxx} - \left[ w_x(v_x + 1/2u_t^2) \right]_x - [u_x(z_x + w_xw_t)] = 0,$$

$$\rho_2 z_{tt} - (z_x + w_xw_t)_x = 0$$

with the same boundary conditions as in (4.5)–(4.6). Consequently, $(w_d, z_d) = (w(t + h) - w(t), z(t + h) - z(t))$ is a weak solution to the same problem with equations

$$\beta_2 w_{tt} - \mu_2 w_{ttxx} + \lambda_2 w_{xxxx} - \left[ w_x(v_x(t + h) + 1/2u_t^2) \right]_x$$

$$- [u_x(t)(v_x + 1/2u_t + u_t) + u_t(t))]_x - [u_x(t)(z_x + w_{dx}w_t(t) + w_x(t + h)u_{dx})]_x$$

$$- [u_x(t)(z_x(t + h) + w_x(t + h)u_{dx}(t + h))] = 0,$$

$$\rho_2 z_{tt} - (z_x + w_{dx}u_t(t) + w_x(t + h)u_{dx}) = 0.$$

Arguing as above we come to

$$E_{dd}(T) + E_{dd}(0) \leq c(r, T) \int_0^T \int_{L_0}^L \left[ w_{dx}^2 + w_{dt}^2 + w_{dx}^2 + z_{dx}^2 + w_{dx}^2 + u_{dx}^2 \right] dx dt,$$

where

$$E_{dd}(t) = \frac{1}{2} \left( \beta_2 \int_{L_0}^L w_{dt}^2 dx + \mu_2 \int_{L_0}^L w_{dtx}dx + \lambda_2 \int_{L_0}^L w_{dx} dx + \right.$$

$$\left. + \int_{L_0}^L (z_{dx} + w_{dx}u_t(t))^2 dx + \rho_2 \int_{L_0}^L z_{dx}^2 dx \right).$$

Arguing as for (4.28) we get $(u_{tt}, v_{tt}, u_{ttt}, u_{tttt}) \in H^2(L_0, L) \times H^1(L_0, L) \times H^1(L_0, L) \times L^2(L_0, L)$ and, consequently, $(u_d, v_d, u_{dt}, v_{dt}) \in H^4(L_0, L) \times H^3(L_0, L) \times H^3(L_0, L) \times H^2(L_0, L)$.

**Step 2. Carleman estimates.** Now we derive Carleman estimates (cf. [9] for smoother solutions). Let us consider the operator $P = \partial_x^2 \partial_x - \rho \partial_x^2$, where $\rho$ is a positive constant. We choose $\tilde{L} > L$ and introduce functions

$$r(x, t) = (x - \tilde{L})^2 - m(t - T)^2, \quad \eta(x, t) = e^{ut(x, t)},$$

$$q(x, t) = \tau \eta(x, t), \quad \theta(x, t) = e^{\theta(x, t)},$$

where $0 \leq t \leq T, x \in (L_0, L)$. We select $T > 0$ and $m \in (0, 1)$ as follows. Let

$$T > \max\{\sqrt{4\tilde{L}^2 + \sqrt{\bar{r}(\tilde{L} - L)}}, \frac{4\tilde{L}^2}{\sqrt{r(L - L)}} + 1\}.$$
Then, there exists a constant $0 < \sigma_1 < \frac{\sqrt{p(L-L)}}{4}$ such that $T^2 > 4\tilde{L}^2 + 4\sigma_1$. Then we choose $m \in (0, 1)$ such that

$$\frac{4\tilde{L}^2 + 4\sigma_1}{T^2} < m < \frac{\sqrt{p(L-L)} + 4\sigma_1}{4T} < \frac{\sqrt{p(L-L)}}{2T}.$$  \tag{4.41}$$

This choice is possible due to (4.40). Therefore,

$$r(x,0) = r(x,T) = \tilde{L}^2 - m \frac{T^2}{4} \leq -\sigma_1$$  \tag{4.42}

for any $x \in (L_0, L)$. Moreover, there exist $t_0, t_1$ such that $0 < t_0 < \frac{T}{2} < t_1 < T$ (chosen symmetrically around $\frac{T}{2}$)

$$\min_{x \in [L_0, L], t \in [t_0, t_1]} r(x,t) \geq \sigma_2, \quad 0 < \sigma_2 < (\tilde{L} - L)^2.$$  \tag{4.43}$$

Now we set

$$\tilde{v} = \theta v,$$

where we choose $v$ such that

$$v \in H^3((L_0, L) \times (0,T)), \quad v(L_0) = v_x(L_0) = v_{xx}(L_0) = v(L) = v_{xx}(L) = 0.$$ 

Direct computations show that

$$\theta P v = \tilde{v}_{txx} - \rho \tilde{v}_{xxx} + A_1 \tilde{v}_{xx} + A_2 \tilde{v}_x + A_3 \tilde{v}_t + A_4 \tilde{v}_{tx} + A_5 \tilde{v}_t + A_6 \tilde{v},$$  \tag{4.44}$$

where

$$A_1 = 3\rho q_x, \quad A_2 = 3\rho q_{xx} - 3\rho(q_x)^2 - q_t + (q_t)^2, \quad A_3 = -q_x, \quad A_4 = -2q_t, \quad A_5 = -2q_t + 2q_x q_t,$$

$$A_6 = \rho q_{xxx} - 3\rho q_x q_{xx} - q_{txx} + 2q_x q_{tx} + q_x q_t - q_x(q_t)^2 + \rho(q_x)^3.$$ 

We denote now

$$I_1 = -\rho \tilde{v}_{xxx} + \tilde{v}_{txx} + B_2 \tilde{v}_x + B_5 \tilde{v}_t,$$  \tag{4.45}$$

$$I_2 = B_1 \tilde{v}_{xx} + B_3 \tilde{v}_{tx} + B_3 \tilde{v}_t + B_6 \tilde{v},$$  \tag{4.46}$$

$$S = S_2 \tilde{v}_x + S_5 \tilde{v}_t + S_6 \tilde{v},$$  \tag{4.47}$$

where

$$B_1 = 3\rho q_x, \quad B_2 = -3\rho(q_x)^2 + (q_t)^2, \quad B_3 = -q_x, \quad B_4 = -2q_t, \quad B_5 = 2q_x q_t, \quad B_6 = -q_x(q_t)^2 + \rho(q_x)^3.$$  \tag{4.48}$$

and

$$S_2 = 3\rho q_{xx} - q_{tx}, \quad S_5 = -2q_x$$

$$S_6 = \rho q_{xxx} - 3\rho q_x q_{xx} - q_{txx} + 2q_x q_{tx} + q_x q_t.$$ 

Our next step is to estimate from below the expression

$$\int_0^T \int_{L_0}^L I_1 I_2 dx dt$$

$$= -\rho \int_0^T \int_{L_0}^L \tilde{v}_{xxx} B_1 \tilde{v}_{xx} dx dt - \rho \int_0^T \int_{L_0}^L \tilde{v}_{xxx} B_4 \tilde{v}_{tx} dx dt$$

$$- \rho \int_0^T \int_{L_0}^L \tilde{v}_{xxx} B_3 \tilde{v}_t dx dt - \rho \int_0^T \int_{L_0}^L \tilde{v}_{xxx} B_5 \tilde{v}_t dx dt + \int_0^T \int_{L_0}^L \tilde{v}_{txx} B_1 \tilde{v}_{xx} dx dt$$
Now we integrate by parts in the fourth term of the right-hand side of (4.49). Integrating by parts in the first three terms in the right-hand side of (4.50) we obtain

\[- \rho \int_0^T \int_{L_0}^L \ddot{v}_{x} B_1 \ddot{v}_x dx dt \]
\[= -3 \rho^2 \int_0^T \int_{L_0}^L \dddot{v}_{x} \dddot{v}_x q_x dx dt = \frac{3}{2} \rho^2 \int_0^T \int_{L_0}^L \dddot{v}_x^2 q_x dx dt, \quad (4.51)\]

\[- \rho \int_0^T \int_{L_0}^L \ddot{v}_{x} B_4 \ddot{v}_x dx dt \]
\[= -2 \rho \int_0^T \int_{L_0}^L \ddot{v}_{x} \ddot{v}_t q_t dx dt - 2 \rho \int_0^T \int_{L_0}^L \ddot{v}_x \dddot{v}_x q_x dx dt \]
\[= \rho \int_0^T \int_{L_0}^L \dddot{v}_x^2 q_t dx dt - \rho \int_0^T \int_{L_0}^L \dddot{v}_x^2 q_x(T) q_x(0) dx \]
\[+ \rho \int_0^T \int_{L_0}^L \dddot{v}_x^2(0) q_t(0) dx - 2 \rho \int_0^T \int_{L_0}^L \dddot{v}_x \dddot{v}_t q_x dx dt, \quad (4.52)\]

and

\[- \rho \int_0^T \int_{L_0}^L \ddot{v}_{x} B_3 \ddot{v}_t dx dt \]
\[= \rho \int_0^T \int_{L_0}^L \ddot{v}_{x} \dddot{v}_t q_x dx dt \]
\[= - \rho \int_0^T \int_{L_0}^L \ddot{v}_{x} \ddot{v}_{tt} q_x dx dt - \rho \int_0^T \int_{L_0}^L \ddot{v}_{x} \ddot{v}_t q_x dx dt \]
\[= - \frac{1}{2} \rho \int_0^T \int_{L_0}^L \dddot{v}_x^2 q_x dx dt + \rho \int_0^T \int_{L_0}^L \dddot{v}_x(0) \dddot{v}_x(0) q_x(0) dx \]
\[- \rho \int_0^T \int_{L_0}^L \dddot{v}_x(T) \dddot{v}_x(T) q_x dx dt - \rho \int_0^T \int_{L_0}^L \dddot{v}_x(T) q_x(T) q_x(0) dx \]
\[- \rho \int_0^T \int_{L_0}^L \dddot{v}_x q_x(T) q_x(0) dx - \rho \int_0^T \int_{L_0}^L \dddot{v}_x \dddot{v}_t q_x dx dt \]
\[+ \rho \int_0^T \int_{L_0}^L \dddot{v}_x \dddot{v}_t q_x dx dt + \frac{1}{2} \rho \int_0^T \int_{L_0}^L \dddot{v}_x^2(L) q_x(L) dx dt. \quad (4.53)\]

Now we integrate by parts in the fourth term of the right-hand side of (4.50)

\[- \rho \int_0^T \int_{L_0}^L \ddot{v}_{x} B_6 \ddot{v}_x dx dt \]
\[= - \rho \int_0^T \int_{L_0}^L \ddot{v}_{x} (\rho q_x^3 - q_x q_t^2) dx dt \]
\[
\rho \int_0^T \int_{L_0}^L \ddot{v}_{xx} \ddot{v}_x (\rho q_x^2 - q_x q_t^2) \, dx \, dt + \rho \int_0^T \int_{L_0}^L \ddot{v}_{xx} \ddot{v}(3\rho q_x^2 q_x - q_x q_t^2 - 2q_x q_t q_{tx}) \, dx \, dt \\
= -\rho \int_0^T \int_{L_0}^L \ddot{v}_x^2 (\frac{3}{2} \rho q_x^2 q_x - \frac{3}{2} q_x q_t^2 - 3q_x q_{tx}) \, dx \, dt \\
+ \frac{\rho}{2} \int_0^T \ddot{v}_x^2 (L) q_x (L) (\rho q_x^2 (L) - q_t^2 (L)) \, dt \\
+ \frac{\rho}{2} \int_0^T \int_{L_0}^L \ddot{v}^2 (3\rho q_x^2 q_{xxx} + 18 \rho q_x q_x q_{xx} + 6 \rho q_x^3 - q_x q_t q_{xx} - 6q_x q_{xx} q_t - 6q_x q_{xx}^2) \\
- 4q_x q_t q_{tx} - 6q_x q_t q_{tx} - 2q_x q_t q_{xxx}) \, dx \, dt.
\]
\]
Integrating by parts in the fifth, sixth, and seventh term in the right-hand side of (4.50) we come to
\[
\rho \int_0^T \int_{L_0}^L \dddot{v}_{tx} B_1 \ddot{v}_{xx} \, dx \, dt \\
= 3\rho \int_0^T \int_{L_0}^L \dddot{v}_{tx} \ddot{v}_{xx} q_x \, dx \, dt \\
= -3\rho \int_0^T \int_{L_0}^L \dddot{v}_{tx} \ddot{v}_{tx} q_x \, dx \, dt + 3\rho \int_{L_0}^L \dddot{v}_{tx} (T) \ddot{v}_{xx} (T) q_x (T) \, dx \\
- 3\rho \int_{L_0}^L \dddot{v}_{tx} (0) \ddot{v}_{xx} (0) q_x (0) \, dx - 3\rho \int_0^T \int_{L_0}^L \dddot{v}_{tx} \ddot{v}_{xx} q_{tx} \, dx \, dt \\
= \frac{3}{2} \rho \int_0^T \int_{L_0}^L \dddot{v}_{tx} q_x \, dx \, dt - \frac{3}{2} \rho \int_{L_0}^L \dddot{v}_x^2 (L) q_x (L) \, dx \\
+ 3\rho \int_{L_0}^L \dddot{v}_{tx} (T) \ddot{v}_{xx} (T) q_x (T) \, dx - 3\rho \int_{L_0}^L \dddot{v}_{tx} (0) \ddot{v}_{xx} (0) q_x (0) \, dx \\
- 3\rho \int_0^T \int_{L_0}^L \dddot{v}_{tx} \ddot{v}_{xx} q_{tx} \, dx \, dt, \\
\int_0^T \int_{L_0}^L \dddot{v}_{tx} B_3 \ddot{v}_t \, dx \, dt \\
= -2 \int_0^T \int_{L_0}^L \dddot{v}_{tx} \ddot{v}_t q_t \, dx \, dt \\
= \int_{L_0}^L \int_{L_0}^L \dddot{v}_{tx} q_{tx} \, dx \, dt - \int_{L_0}^L \dddot{v}_x^2 (T) q_t (T) \, dx + \int_{L_0}^L \dddot{v}_x^2 (0) q_t (0) \, dx,
\]
and
\[
\int_{L_0}^L \int_{L_0}^L \dddot{v}_{tx} B_3 \ddot{v}_t \, dx \, dt \\
= - \int_0^T \int_{L_0}^L \dddot{v}_{tx} \ddot{v}_t q_x \, dx \, dt = \frac{1}{2} \int_0^T \int_{L_0}^L \dddot{v}_t q_{tx} \, dx \, dt.
\]
After integration by parts in the eighth term in the right-hand side of (4.50) we get
\[
\int_0^T \int_{L_0}^L \dddot{v}_{tx} B_6 \ddot{v} \, dx \, dt \\
= \int_0^T \int_{L_0}^L \dddot{v}_{tx} \ddot{v} (\rho q_x^3 - q_x q_t^2) \, dx \, dt
\]
Next we integrate by parts in the ninth, tenth, and eleventh terms in the right-hand side of (4.50)

$$
\int_0^T \int_{L_0}^L \tilde{v}_x B_3 B_1 \tilde{v}_{xx} dx dt
$$

$$
= \int_0^T \int_{L_0}^L \tilde{v}_x \tilde{v}_{xx} dx dt
$$

$$
= \int_0^T \int_{L_0}^L \tilde{v}_x \tilde{v}_{xx} (-9 \rho^2 q_x^2 + 3 \rho q_x q_{tt}) dx dt
$$

$$
= \int_0^T \int_{L_0}^L \tilde{v}_x^2 (\frac{27}{2} \rho^2 q_x^2 q_{xx} - \frac{3}{2} \rho q_x q_{tt}) dx dt
$$

$$
- \frac{3 \rho}{2} \int_0^T \tilde{v}_x^2 (L) q_x (L) (3 \rho q_x^2 (L) - q_x^2 (L)) dt,
$$

(4.59)

$$
\int_0^T \int_{L_0}^L \tilde{v}_x B_2 B_3 \tilde{v}_{xx} dx dt
$$

$$
= -2 \int_0^T \int_{L_0}^L \tilde{v}_x \tilde{v}_{xx} (\rho q_x q_{tt}) dx dt
$$

$$
= -\int_{L_0}^L \tilde{v}_x^2 (T) (q_1^3 (T) - 3 \rho q_t q_x^2 (T)) dx + \int_{L_0}^L \tilde{v}_x^2 (0) (q_1^3 (0) - 3 \rho q_t (0) q_x^2 (0)) dx
$$

$$
+ \int_0^T \int_{L_0}^L \tilde{v}_x^2 (3 \rho q_t q_x^2 - 3 \rho q_{tt} q_x^2 - 6 \rho q_t q_x q_{xx}) dx dt,
$$

(4.60)

and

$$
\int_0^T \int_{L_0}^L \tilde{v}_x B_2 B_3 \tilde{v}_{xt} dx dt = \int_0^T \int_{L_0}^L \tilde{v}_x \tilde{v}_{xt} (3 \rho q_x^2 - q_x^2) dx dt
$$
\[
\frac{1}{2} \int_0^T \int_{L_0}^L \ddot{v}_t^2 (9\rho q_x^2 q_{xx} - q_{xx}q_t^2 - 2q_xq_tq_{xx}) dx dt \\
- \int_0^T \int_{L_0}^L \ddot{v}_x \ddot{v}_t (9\rho q_x^2 q_{xt} - q_{xt}q_t^2 - 2q_xq_tq_{xt}) dx dt \\
+ \int_0^T \int_{L_0}^L \ddot{v}_x (T) \ddot{v}_t (T) q_x (T) (3\rho q_x^2 (T) - q_t^2 (T)) dx \\
- \int_{L_0}^L \ddot{v}_x (0) \ddot{v}_t (0) q_x (0) (3\rho q_x^2 (0) - q_t^2 (0)) dx. \quad (4.61)
\]

Next we integrate by parts in the twelfth term in the right-hand side of (4.50)

\[
\int_0^T \int_{L_0}^L \ddot{v}_t B_2 B_5 \ddot{v} dx dt \\
= \int_0^T \int_{L_0}^L \ddot{v}_x \ddot{v}_x (4\rho q_x^2 q_{xx}^2 - 3\rho^2 q_t^2 - q_{xx}q_t^2) dx dt \\
= - \int_0^T \int_{L_0}^L \ddot{v}_x^2 (6\rho q_x^2 q_{xx}^2 q_t + 4\rho q_x^2 q_t q_{xx} - \frac{15}{2} \rho^2 q_x^2 q_{xx} - \frac{1}{2} q_{xx}q_t^2 - 2q_xq_{xx}q_t) dx dt. \quad (4.62)
\]

Integrating by parts in the thirteenth, fourteenth, and fifteenth terms in the right-hand side of (4.50) we obtain

\[
\int_0^T \int_{L_0}^L \ddot{v}_t B_4 B_5 \ddot{v} dx dt = -2 \int_0^T \int_{L_0}^L \ddot{v}_t \ddot{v}_x q_x q_t^2 dx dt \\
= \int_0^T \int_{L_0}^L \ddot{v}_x^2 (q_{xx}q_t^2 + 2q_xq_tq_{xx}) dx dt, \quad (4.64)
\]

and

\[
\int_0^T \int_{L_0}^L \ddot{v}_t B_3 B_5 \ddot{v} dx dt = -2 \int_0^T \int_{L_0}^L \ddot{v}_t q_x^2 q_t \ddot{v}_t dx dt \\
= \int_0^T \int_{L_0}^L \ddot{v}_x^2 (q_{xx}q_t + 2q_xq_tq_{xx}) dx dt \\
- \int_0^T \int_{L_0}^L \ddot{v}_x (T) q_x^2 (T) q_t (T) dx + \int_{L_0}^L \ddot{v}_x (0) q_x^2 (0) q_t (0) dx. \quad (4.65)
\]
Moreover, we note that
\[
\int_0^T \int_{L_0}^L \hat{\nu}_t B_5 B_6 \hat{\nu} \, dx \, dt = 2 \int_0^T \int_{L_0}^L \hat{\nu}_t \hat{\nu} (\rho q_4^4 q_t - q_5^2 q_2^2) \, dx \, dt
\]
\[
= - \int_0^T \int_{L_0}^L \hat{\nu}_t^2 (\rho q_4^4 q_t + 4 \rho q_4^3 q_t q_{xt} - 2q_x q_3^3 q_{tx} - 3q_x^2 q_4^2 q_{xt}) \, dx \, dt
\]
\[
+ \int_0^L \hat{\nu}^2(T)(\rho q_4^4(T) q_t(T) - q_x^2(T) q_3^3(T)) \, dx
\]
\[
- \int_0^L \hat{\nu}^2(0)(\rho q_4^4(0) q_t(0) - q_x^2(0) q_3^3(0)) \, dx. \tag{4.66}
\]

Now we take into account that
\[
q_x = 2 \tau \mu (x - L) \eta, \tag{4.67}
\]
\[
q_{xx} = 2 \tau \mu \eta + 4 \tau \mu^2 (x - L)^2 \eta, \tag{4.68}
\]
\[
q_{xxx} = 12 \tau \mu^2 (x - L) \eta + 8 \tau \mu^3 (x - L)^3 \eta, \tag{4.69}
\]
\[
q_{xxxx} = 12 \tau \mu^2 \eta + 48 \tau \mu^3 (x - L)^3 \eta + 16 \tau \mu^4 (x - L)^4 \eta, \tag{4.70}
\]
\[
q_t = -2 \tau \mu (t - \frac{T}{2}) \eta \mu, \tag{4.71}
\]
\[
q_{tt} = 4 \tau \mu^2 (t - \frac{T}{2})^2 \eta^2 - 2 \tau \mu \eta, \tag{4.72}
\]
\[
q_{tx} = -4 \tau \mu^2 m (x - L)(t - \frac{T}{2}) \eta, \tag{4.73}
\]
\[
q_{ttx} = -4 \tau \mu^2 m (t - \frac{T}{2}) \eta - 8 \tau \mu^3 m (x - L)^2 (t - \frac{T}{2}) \eta, \tag{4.74}
\]
\[
q_{ttt} = 8 \tau \mu^3 (t - \frac{T}{2})^2 \eta^2 (x - L)^2 \eta - 4 \tau \mu^2 (x - L) \eta \mu, \tag{4.75}
\]
\[
q_{tt}^t = 8 \tau \mu^3 m^2 (t - \frac{T}{2}) \eta - 8 \tau \mu^3 m^3 (t - \frac{T}{2}) \eta + 4 \tau \mu^2 m^2 (t - \frac{T}{2}) \eta, \tag{4.76}
\]
\[
q_{tttx} = -4 \tau \mu^2 m^2 (t - \frac{T}{2})^2 \eta - 8 \tau \mu^3 m (x - L)^2 \eta
\]
\[
+ 2 \tau \mu^4 m^2 (x - L) \eta (t - \frac{T}{2})^2 \eta, \tag{4.77}
\]
\[
q_{tttx} = 16 \tau \mu^3 (t - \frac{T}{2})^3 m^2 (x - L) \eta - 16 \tau \mu^4 m^3 (t - \frac{T}{2})^3 \eta + 8 \tau \mu^3 m^2 (t - \frac{T}{2}) (x - L) \eta, \tag{4.78}
\]
\[
q_{tttx} = -24 \tau \mu^3 m (x - L)(t - \frac{T}{2}) \eta - 16 \tau \mu^4 m (x - L)^3 (t - \frac{T}{2}) \eta. \tag{4.80}
\]

and estimate the last two terms in (4.53) as follows
\[
-\rho \int_0^T \int_{L_0}^L \hat{\nu}_{xx} \hat{\nu}_u q_{xx} \, dx \, dt = -\rho \int_0^T \int_{L_0}^L \hat{\nu}_{xx} \hat{\nu}_u (2 \tau \mu \eta + 4 \tau \mu^2 (x - L)^2 \eta) \, dt \, dx
\]
\[
\geq -\rho^2 \int_0^T \int_{L_0}^L (2 \tau \mu \eta + 2 \tau \mu^2 (x - L)^2 \eta) \hat{\nu}_{xx}^2 \, dx \, dt
\]
\[
- \int_0^T \int_{L_0}^L \hat{\nu}_u^2 (\frac{1}{2} \tau \mu \eta + 2 \tau \mu^2 (x - L) \eta) \, dx \, dt \tag{4.81}
\]
Analogously, the sum of the first terms in the right-hand sides in (4.50) equals to

\[ -4 \rho \int_0^T \int_{L_o}^L \tilde{v}_{xx} \tilde{v}_{tx} q \tilde{x} dx dt \]

\[ = 16 \rho \int_0^T \int_{L_o}^L \tilde{v}_{xx} \tilde{v}_{tx} (\tau \mu^2 (x - \tilde{L})(t - \frac{T}{2})) \eta dx dt \]

\[ \geq - \rho^2 \int_0^T \int_{L_o}^L 4 \tau \mu^2 \tilde{v}_{xx}^2 \eta dx dt - 16 \int_0^T \int_{L_o}^L \tilde{v}_{tt}^2 \tau \mu^2 (t - \frac{T}{2})^2 \eta dx dt. \quad (4.82) \]

Collecting the second terms in the right-hand sides of (4.58), (4.61), and (4.63) and estimating them from below we arrive at

\[ 12 \rho \int_0^T \int_{L_o}^L \tilde{v}_x q_x q_x \tilde{x} dx dt \]

\[ = - \rho \int_0^T \int_{L_o}^L \tilde{v}_x \tilde{v}_t (4 \tau^3 \mu^3 m \mu^2 (x - \tilde{L})(t - \frac{T}{2})) \eta^3 + 96 \tau^3 \mu^4 (x - \tilde{L})^3 (t - \frac{T}{2}) \eta dx dt \]

\[ \geq - \int_0^T \int_{L_o}^L \tilde{v}_t^2 (2 \tau^3 \mu^3 (x - \tilde{L})^2 \eta^3 + 4 \tau^3 \mu^4 (x - \tilde{L})^4 \eta^3) dx dt \]

\[ - \int_0^T \int_{L_o}^L \tilde{v}_t^2 (24 \tau^3 \mu^3 m^2 (t - \frac{T}{2})^2 \eta^3 + 48 \tau^3 \mu^4 m^2 (x - \tilde{L})^2 (t - \frac{T}{2})^2 \eta^3) dx dt. \quad (4.83) \]

Next, using (4.42)–(4.75) we infer that the sum of the term in the right-hand side of (4.51) and the first term in the right-hand side of (4.52) equals to

\[ \int_0^T \int_{L_o}^L \tilde{v}_{xx}^2 (3 \rho^2 \tau \mu \eta + 6 \rho^2 \tau \mu^2 (x - \tilde{L})^2 \eta + 4 \rho \tau \mu^2 (t - T)^2 m^2 \eta - 2 \rho \tau \mu m \eta) dx dt. \quad (4.84) \]

Here the relation in brackets can be made positive by choosing \( \mu \) large enough. The term in the right-hand side of (4.57) equals to

\[ \int_0^T \int_{L_o}^L \tilde{v}_{tt}^2 (\tau \mu \eta + 2 \tau \mu^2 (x - \tilde{L})^2 \eta) dx dt. \quad (4.85) \]

Collecting the first terms in the right-hand side of (4.53), (4.55), and (4.56) we obtain

\[ \int_0^T \int_{L_o}^L \tilde{v}_{tx}^2 (2 \rho \tau \mu \eta + 4 \rho \tau \mu^2 (x - \tilde{L})^2 \eta + 4 \rho \tau \mu^2 (t - \frac{T}{2})^2 m^2 \eta - 2 \rho \tau \mu m \eta) dx dt, \quad (4.86) \]

where the relation in brackets can also be made positive by choosing \( \mu \) large enough. Analogously, the sum of the first terms in the right-hand sides in (4.54), (4.59), (4.60), (4.63) equals to

\[ \int_0^T \int_{L_o}^L \tilde{v}_x^2 (72 \rho^2 \tau^3 \mu^3 (x - \tilde{L})^2 \eta^3 + 144 \rho^2 \tau^3 \mu^4 (x - \tilde{L})^4 \eta^3 \]

\[ + 48 \tau^3 \mu^3 (t - \frac{T}{2})^4 m^4 \eta^3 - 24 \tau^3 \mu^4 m^3 (t - \frac{T}{2})^2 \eta^3) dx dt \quad (4.87) \]

and the sum of the first terms in (4.58), (4.61), (4.64), (4.65) equals to

\[ \int_0^T \int_{L_o}^L \tilde{v}_x^2 (48 \rho \tau^3 \mu^3 (x - \tilde{L})^2 \eta^3 + 96 \rho \tau^3 \mu^4 (x - \tilde{L})^4 \eta^3 \]
where the relations in brackets can also be made positive by choosing $\mu$ large enough. We note that

$$
\int_0^T \int_{L_0} v^2 \left( \frac{15}{2} \rho^2 q_{xx} q_{xx} - \rho q_x q_{tt} - 4\rho q_x q_{txx} \right) dx dt
\geq 160\rho^2 \int_0^T \int_{L_0} \tilde{v}^2 \tau^5 (x - \tilde{L})^6 \eta^5 dx dt.
$$

(4.89)

This term can be made large enough by choosing $\tau$ so that the sum of all terms containing $\tilde{v}^2$ is positive. Collecting (4.50)–(4.89) with $\mu$ and $\tau$ large enough we obtain that there exist $C_1, C_2 > 0$ such that

$$
\int_0^T \int_{I_1} I_2 dx dt 
\geq C_1 \left[ \tau \mu^2 \int_0^T \int_{L_0} \eta (\tilde{v}^2_{xx} + \tilde{v}^2_{tt} + v^2_{x1x}) dx dt + \tau^3 \mu^4 \int_0^T \int_{L_0} \eta^3 (\tilde{v}^2_x + \tilde{v}^2_2) dx dt 
+ \tau^5 \mu^6 \int_0^T \int_{L_0} \eta^5 \tilde{v}^2 dx dt \right] 
-C_2 [ \mu \tau \eta (T) E_{v1}(T) + \mu \tau \eta (0) E_{v1}(0) + \mu^3 \tau^3 \eta^3 (T) E_{v2}(T) 
+ \mu^3 \tau^3 \eta^3 (0) E_{v2}(0) + \mu^5 \tau^5 \eta^5 (T) E_{v3}(T) + \mu^5 \tau^5 \eta^5 (0) E_{v3}(0) ],
$$

(4.90)

where

$$
E_{v1}(t) = \int_{L_0}^L v_{x1x}^2(t) dx + \int_{L_0}^L v_{x2}^2(t) dx
$$

(4.91)

$$
E_{v2}(t) = \int_{L_0}^L v_{x1}^2(t) dx + \int_{L_0}^L v_{x2}^2(t) dx
$$

(4.92)

$$
E_{v3}(t) = \int_{L_0}^L v^2(t) dx.
$$

(4.93)

It follows from (4.44), (4.45)–(4.47), (4.42)–(4.75) that

$$
2 \int_0^T \int_{I_1} I_2 dx dt
\leq \int_0^T \int_{L_0} |I_1 + I_2|^2 dx dt 
\leq C \left( \int_0^T \int_{L_0} \theta^2 |Pv|^2 dx dt + \int_0^T \int_{L_0} |Sv|^2 dx dt \right)
\leq C \left( \int_0^T \int_{L_0} \theta^2 |Pv|^2 dx dt + \tau^4 \mu^4 \int_0^T \int_{L_0} (\tilde{v}_{xx}^2 + \tilde{v}_{tt}^2) \eta^2 dx dt + \tau^6 \mu^6 \int_0^T \int_{L_0} \tilde{v}^2 \eta^2 dx dt \right).
$$

(4.94)

Collecting (4.90)–(4.94), and choosing $\tau$ large enough we obtain the following Carleman estimate

$$
\int_0^T \int_{L_0} \theta^2 |Pv|^2 dx dt
\geq C_1 \left[ \tau \mu^2 \int_0^T \int_{L_0} \eta \theta^2 (v_{xx}^2 + \tilde{v}_{tt}^2 + v_{x2}^2) dx dt 
+ \tau^3 \mu^4 \int_0^T \int_{L_0} \eta^3 \theta^2 (v_x^2 + \tilde{v}_t^2) dx dt + \tau^5 \mu^6 \int_0^T \int_{L_0} \eta^5 \theta^2 \tilde{v}^2 dx dt \right]
$$
- \( C_2 \left[ \mu \tau \eta(T) E_{v_1}(T) + \mu \tau \eta(0) E_{v_1}(0) + \mu^3 \tau^3 \eta^3(T) E_{v_2}(T) + \mu^3 \tau^3 \eta^3(0) E_{v_2}(0) \\
+ \mu^5 \tau^5 \eta^5(T) E_{v_3}(T) + \mu^5 \tau^5 \eta^5(0) E_{v_3}(0) \right]. \) (4.95)

The Carleman estimate (see the ideas of the proof in e.g. [2] and references therein) for the operator \( Q = \partial_t^2 - \rho \partial_x^2 \) for a function

\[
\hat{u} \in H^2((L_0, L) \times (0, T)), \quad \hat{u}(L_0) = \hat{u}_x(L_0) = \hat{u}(L) = 0
\]

is as follows

\[
\int_0^T \int_{L_0}^L \theta^2 |Q \hat{u}|^2 dxdt \\
\geq C_3 \left[ \tau \mu \int_0^T \int_{L_0}^L \eta \theta^2 (\hat{u}_x^2 + \hat{u}_t^2) dxdt + \tau^3 \mu^4 \int_0^T \int_{L_0}^L \eta^3 \theta^2 \hat{u}_t^2 dxdt \right] (4.96)
\]

\[
- C_4 \left[ \mu \tau \eta(T) E_{u_2}(T) + \mu \tau \eta(0) E_{u_2}(0) + \mu^3 \tau^3 \eta^3(T) E_{u_3}(T) + \mu^3 \tau^3 \eta^3(0) E_{u_3}(0) \right],
\]

Consequently, for a function

\[
u \in H^2((L_0, L) \times (0, T)), \quad u_{xx}(L_0) = u_{xxx}(L_0) = u_{xx}(L) = 0
\]

we have from (4.96) choosing \( \hat{u} = u_{xx} \)

\[
\int_0^T \int_{L_0}^L \theta^2 |Q u_{xx}|^2 dxdt \\
\geq C_3 \left[ \tau \mu \int_0^T \int_{L_0}^L \eta \theta^2 (u_{xx}^2 + u_{txx}^2) dxdt + \tau^3 \mu^4 \int_0^T \int_{L_0}^L \eta^3 \theta^2 u_{xx}^2 dxdt \right] (4.97)
\]

\[
- C_4 \left[ \mu \tau \eta(T) E_{u_4}(T) + \mu \tau \eta(0) E_{u_4}(0) + \mu^3 \tau^3 \eta^3(T) E_{u_1}(T) + \mu^3 \tau^3 \eta^3(0) E_{u_1}(0) \right],
\]

where

\[
E_{u_4}(t) = \int_{L_0}^L v_{xxx}^2(t) dx + \int_{L_0}^L v_{xxxx}^2(t) dx.
\]

Step 3. Observability inequality. From estimates (4.96), (4.97) and equations (4.3) and

\[
\beta_2 v_{dtx} - (v_{dx} + 1/2 u_{dx}(u_x(t + h) + u_x(t)))_{xx} = 0
\]

we infer the following estimate

\[
\tau \mu \int_0^T \int_{L_0}^L \eta \theta^2 (u_{xx}^2 + u_{dxx}^2) dxdt + \tau^3 \mu^4 \int_0^T \int_{L_0}^L \eta^3 \theta^2 u_{dx}^2 dx \\
+ \tau^2 \mu^2 \int_0^T \int_{L_0}^L \eta \theta^2 (v_{dxx}^2 + v_{dtx}^2 + v_{dt}^2) dxdt + \tau^3 \mu^4 \int_0^T \int_{L_0}^L \eta^3 \theta^2 (v_{dx}^2 + v_{dtt}^2) dxdt \\
+ \tau^5 \mu^6 \int_0^T \int_{L_0}^L \eta^4 \theta^2 v_{dtx}^2 dxdt - C \left[ \mu \tau \eta(T) E_{v_1}(T) + \mu \tau \eta(0) E_{v_1}(0) \right] (4.100)
\]

\[
+ \mu^3 \tau^3 \eta^3(T) E_{v_2}(T) + \mu^3 \tau^3 \eta^3(0) E_{v_2}(0) + \mu^5 \tau^5 \eta^5(T) E_{v_3}(T) + \mu^5 \tau^5 \eta^5(0) E_{v_3}(0) \\
+ \mu^3 \tau^3 \eta^3(T) E_{u_1}(T) + \mu^3 \tau^3 \eta^3(0) E_{u_1}(0) + \mu \tau \eta(T) E_{u_4}(T) + \mu \tau \eta(0) E_{u_4}(0) \right] \leq 0
\]
choosing in (4.100) \( \tau \) and \( \mu \) large enough and taking into account that \( E_{u_d,3}(t) \leq C E_{u_d,2}(t) \) we arrive at

\[
\tau \mu \int_{t_0}^{t_1} \int_{L_0}^{L} E_{u_d,v_d}(t)dxdt - C \mu^5 \tau^5 e^{-\sigma_0 \mu} [E_{u_d,v_d}(0) + E_{u_d,v_d}(T)] \leq 0, \tag{4.101}
\]

where

\[
E_{u_d,v_d}(t) = E_{u_d1}(t) + E_{u_d2}(t) + E_{u_d3}(t) + E_{u_d4}(t).
\]

We multiply equation (4.3) by \(-u_{dxx}\) and equation (4.4) by \(-v_{dxx}\) and integrate by parts over the intervals \([L_0, L]\) and \([s, t] \subset [0, T]\) to get

\[
E_{u_d4}(t) + E_{u_d1}(t) + E_{v_d1}(t)
\]

\[
\leq E_{u_d4}(s) + E_{u_d1}(s) + E_{v_d1}(s) + C_T \int_{s}^{t} (E_{u_d4}(\xi) + E_{u_d1}(\xi) + E_{v_d1}(\xi))d\xi. \tag{4.102}
\]

Analogously, after multiplication of equation (4.3) by \( u_{dt} \) and equation (4.4) by \( v_{dt} \) and integration by parts over the intervals \([L_0, L]\) and \([s, t] \subset [0, T]\) we arrive at

\[
E_{u_d1}(t) + E_{u_d2}(t) + E_{v_d2}(t)
\]

\[
\leq E_{u_d1}(s) + E_{u_d2}(s) + E_{v_d2}(s) + C_T \int_{s}^{t} (E_{u_d1}(\xi) + E_{u_d2}(\xi) + E_{v_d2}(\xi))d\xi. \tag{4.103}
\]

It follows from (4.102), (4.103), and the Gronwall’s lemma that for \(0 \leq s \leq t \leq T\)

\[
E_{u_d,v_d}(t) \leq E_{u_d,v_d}(s)e^{C_T(t-s)}. \tag{4.104}
\]

Analogously,

\[
E_{u_d,v_d}(s) \leq E_{u_d,v_d}(t)e^{C_T(t-s)}. \tag{4.105}
\]

Choosing \(t = T, \ s = t\) in (4.104) and \(s = 0\) in (4.105) and summing up the results we obtain

\[
E_{u_d,v_d}(T) + E_{u_d,v_d}(0) \leq E_{u_d,v_d}(t)e^{C_T T}. \tag{4.106}
\]

Substituting (4.106) into (4.101) and taking into account (4.43) we arrive at

\[
(\tau \mu (t_1 - t_0)e^{\sigma_2 \mu - C_T T} - C \mu^5 \tau^5 e^{-\sigma_1 \mu}) [E_{u_d,v_d}(0) + E_{u_d,v_d}(T)] \leq 0, \tag{4.107}
\]

where \(\mu\) can be chosen large enough, so that the constant \(\tau \mu (t_1 - t_0)e^{\sigma_2 \mu - C_T T} - C \mu^5 \tau^5 e^{-\sigma_1 \mu}\) is positive. This immediately gives \(u_d = v_d = 0\) for any \(t \geq 0\), consequently, the system \((S_t, H)\) is gradient.

Now we state our main result.

**Theorem 4.2.** Let assumptions of Theorem 3, Theorem 4, and Theorem 5 hold true. Moreover, let

\[
g_2(x) = g_1(x) = 0. \tag{4.108}
\]

Then, the dynamical system \((S_t, H)\) generated by (1.1)-(1.10) possesses a compact global attractor possessing properties (2.2), (2.3).
Proof. In view of Theorem 2, Theorem 4, and Theorem 5 our remaining task is to show the boundedness of the set of stationary points and the set $W_R = \{ Z : \mathcal{L}(Z) \leq R \}$, where $\mathcal{L}$ is given by (4.1).

The second statement follows immediately from the structure of function $\mathcal{L}$ and Lemma 1.

The first statement can be easily shown by the substitution of $\Psi = (\phi, u, \omega, v)$ into and application of energy-like estimates and Lemma 1 for stationary solutions. \hfill \square

Remark 4.3. From the point of view of applications it is interesting to consider the system

$$\begin{align*}
\beta_1 \phi_{tt} - \mu_1 \phi_{ttxx} - \kappa \phi_{txx} + \lambda_1 \phi_{xxxx} - \delta_1 \left( \left[ \phi_x \left( \omega_x + 1/2 \phi_x^2 \right) \right]_x \right) &= g_1(x, t), \\
\rho_1 \omega_{tt} + \gamma \omega_t - \delta_1 \left( \omega_x + 1/2 \phi_x^2 \right) &= g_2(x, t), \\
\beta_2 u_{tt} - \mu_2 u_{ttxx} + \lambda_2 u_{xxxx} - \delta_2 \left( \left[ u_x \left( v_x + 1/2 u_x^2 \right) \right]_x \right) &= g_3(x, t), \\
\rho_2 v_{tt} - \delta_2 \left( v_x + 1/2 u_x^2 \right)_x &= g_4(x, t),
\end{align*}$$

with the transmission boundary condition

$$\delta_1 (\omega_x + 1/2 \phi_x^2)(L_0, t) = \delta_2 (v_x + 1/2 u_x^2)(L_0, t)$$

instead of (1.6). This means that all the physical properties of two parts of the beam are different. However, in this case one can prove only the existence of weak solutions and this is an open question how to show the higher order estimates without use of strong solutions in the arguments.

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