ON A GENERALIZED MAXIMUM PRINCIPLE FOR A TRANSPORT-DIFFUSION MODEL WITH log-MODULATED FRACTIONAL DISSIPATION

HONGJIE DONG AND DONG LI

Abstract. We consider a transport-diffusion equation of the form \( \partial_t \theta + v \cdot \nabla \theta + \nu A \theta = 0 \), where \( v \) is a given time-dependent vector field on \( \mathbb{R}^d \). The operator \( A \) represents log-modulated fractional dissipation: \( A = \frac{|\nabla|}{\log^\gamma (\lambda + |\xi|)} \) and the parameters \( \nu \geq 0, \beta \geq 0, 0 \leq \gamma \leq 2, \lambda > 1 \). We introduce a novel nonlocal decomposition of the operator \( A \) in terms of a weighted integral of the usual fractional operators \( |\nabla|^s \), \( 0 \leq s \leq \gamma \) plus a smooth remainder term which corresponds to an \( L^1 \) kernel. For a general vector field \( v \) (possibly non-divergence-free) we prove a generalized \( L^\infty \) maximum principle of the form \( \| \theta(t) \|_{L^\infty} \leq e^{Ct} \| \theta_0 \|_{L^\infty} \) where the constant \( C = C(\nu, \beta, \gamma) > 0 \). In the case \( \text{div}(v) = 0 \) the same inequality holds for \( \| \theta(t) \|_{L^p} \) with \( 1 \leq p \leq \infty \). At the cost of an exponential factor, this extends a recent result of Hmidi [7] to the full regime \( d \geq 1, 0 \leq \gamma \leq 2 \) and removes the incompressibility assumption in the \( L^\infty \) case.

1. Introduction

We consider the transport equation with log-modulated fractional dissipation of the form
\[
\begin{cases}
\partial_t \theta + v \cdot \nabla \theta + \nu A \theta = 0, & (t, x) \in (0, \infty) \times \mathbb{R}^d, \\
A \theta := \frac{|\nabla|}{\log^\gamma (\lambda + |\xi|)} \theta, \\
\theta(0, x) = \theta_0,
\end{cases}
\]
where \( \nu \geq 0 \), and \( v = v(t, x) : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d \) is a given vector field, possibly non-divergence-free. The basic unknown is the scalar function \( \theta = \theta(t, x) \) which is usually termed “active scalar”. The operator \( A \) is defined via the Fourier transform
\[
\hat{A}_f(\xi) = \frac{|\xi|\gamma}{\log^\beta (\lambda + |\xi|)} \hat{f}(\xi), \quad \xi \in \mathbb{R}^d,
\]
where the parameters \( 0 \leq \gamma \leq 2, \beta \geq 0, \lambda > 1 \). It is termed log-modulated fractional dissipation since it is the usual fractional Laplacian \( |\nabla|^\gamma \) divided by a logarithm symbol. Along the way we will also consider a variant of the operator \( A \) which is denoted as \( A_1 = |\nabla|^\gamma \log^{-\beta} (\lambda - \Delta) \), i.e.
\[
\hat{A}_1 f(\xi) = \frac{|\xi|\gamma}{\log^{\beta} (\lambda + |\xi|^2)} \hat{f}(\xi), \quad \xi \in \mathbb{R}^d.
\]
The main objective of this paper is to prove some maximum principles for the operator \( A \) and \( A_1 \) in Lebesgue spaces.

The transport-diffusion model \([\square]\) is a natural generalization of several linear and nonlinear fluid equations such as the two-dimensional surface quasi-geostrophic
equations, fractional Burgers equations, vortex patch models, and Boussinesq systems. See, for instance, the recent work \cite{9, 4, 3, 10, 6, 5} and references therein. In these problems, the velocity \( v \) is typically related to the active scalar \( \theta \) by a constitutive relation \( v = \mathcal{T}(\theta) \) where \( \mathcal{T} \) could be some singular integral operator or more generally a nonlocal operator. To obtain local and global wellposedness results for the nonlinear problems, an important first step is to get a priori \( L^p \) \( 1 \leq p \leq \infty \) estimates of solutions. Specific to the linear problem (1.1), one needs to prove \( L^p \) bounds on the active scalar \( \theta \) independent of the size of \( v \). We refer to these types of results as \( L^p \) maximum principle estimates. In this respect, the two-dimensional dissipative surface quasi-geostrophic equations can be regarded as a (nonlinear) version of (1.1) and they correspond to the case \( \beta = 0 \) in the operator \( \mathcal{A} \). A classical result is due to A. Córdoba and D. Córdoba \cite{2} who proved the following

\[
\| \theta(t) \|_p \leq \| \theta(0) \|_p, \quad \forall \ t \geq 0, \ 1 \leq p \leq \infty.
\]

In a recent article, Hmidi \cite{7} initiated the study of (1.1) and obtained the following important maximum principle:

**Theorem 1.1** (Hmidi \cite{7}). Let the dimension \( d = 2, 3 \) and let \( \nu \geq 0, 0 \leq \gamma \leq 1, \beta \geq 0, \lambda \geq e^{3+2\alpha} \). Assume the velocity \( v \) is divergence-free, i.e. \( \nabla \cdot v = 0 \). Then any smooth solution of (1.1) satisfies

\[
\| \theta(t) \|_p \leq \| \theta(0) \|_p, \quad \forall \ t \geq 0, \ 1 \leq p \leq \infty.
\]

To prove Theorem 1.1, Hmidi used the theory of \( C_0 \)-semigroup of contractions on \( L^p \) \( (1 < p < \infty) \) for the family of convolution kernels \((K_t)_{t \geq 0}\) defined by

\[
\hat{K}_t(\xi) = e^{-\frac{|\xi|^2}{\log(\lambda + 1)}},
\]

The key step is to get the positivity of the kernel \( K_t \). For this purpose, Hmidi used the Askey’s criterion for characteristic functions \cite{1}. The restrictions on the dimension \( d \) and the parameters \( (\gamma, \beta, \lambda) \) are mainly due to the use of this criterion. Hmidi conjectured that the maximum principle should hold for all dimensions \( d \geq 1 \) and the full range \( 0 \leq \gamma \leq 2 \) and \( \beta \geq 0 \). The purpose of this paper is to give an affirmative answer to this question at the cost of a harmless exponential factor.

**Theorem 1.2** (Generalized maximum principle, \( L^\infty \) case). Let \( \nu \geq 0, \ d \geq 1, 0 \leq \gamma \leq 2 \) and \( \beta \geq 0, \lambda > 1 \). Assume \( \theta = \theta(t, x) \) is a smooth solution of (1.1) which decays at spatial infinity, i.e., for any fixed \( t \geq 0 \),

\[
\lim_{|x| \to \infty} \theta(t, x) = 0,
\]

Then we have

\[
\| \theta(t) \|_\infty \leq e^{Ct} \| \theta_0 \|_\infty, \quad \forall \ t \geq 0,
\]

where \( C > 0 \) is a constant depending only on \( (\nu, d, \gamma, \beta, \lambda) \).

**Remark 1.3.** The same result holds if we replace the dissipation operator \( \mathcal{A} \) by \( \mathcal{A}_1 \) in (1.1). As we shall see in Section 2, the proof for \( \mathcal{A}_1 \) case is actually simpler. The decay condition (1.2) is fairly weak as most smooth solutions to these type of fluid equations typically belong to the Sobolev space \( C_0^q \mathcal{H}_x^s \) which can easily imply (1.2). We should also stress that we do not assume any divergence-free condition on \( v \) in Theorem 1.2. This can have applications for compressible fluid equations.
To prove Theorem 1.2, we shall use a completely new idea which avoids the use of Askey’s criterion. Namely we introduce a novel nonlocal decomposition of the operator $A$ (see Section 2 for more details) in terms of a weighted integral of the usual fractional operators $|\nabla|^s$, $0 \leq s \leq \gamma$ plus a smooth remainder term which corresponds to an $L^1$ kernel. Thanks to this new decomposition, we shall only need to appeal to the classic maximum principle for the fractional Laplacian operators. In a similar vein, one can even consider a weighted integral of a parameterized family of nonlocal operators each of which obeys a maximum principle. However we shall not pursue this generality here.

As was already mentioned, Theorem 1.2 deals with the $L^\infty$ norm and no special assumption is needed on the velocity field $v$. On the other hand for more general $L^p$-norms with $1 \leq p < \infty$, the divergence-free condition on the vector field $v$ has to be assumed, as one needs to calculate the time derivative of the $L^p$ norm and perform integration by parts.

**Theorem 1.4** (Generalized maximum principle, $L^p$ case). Let $\nu \geq 0$, $d \geq 1$, $0 \leq \gamma \leq 2$ and $\beta \geq 0$, $\lambda > 1$. Assume the vector field $v = v(t, x)$ is divergence free, i.e. $\nabla \cdot v = 0$. If $\theta = \theta(t, x)$ is a smooth solution of (1.1), then for any $1 \leq p \leq \infty$, we have

$$\|\theta(t)\|_p \leq e^{Ct}\|\theta_0\|_p, \quad \forall t > 0,$$

where $C > 0$ is a constant depending only on $(\nu, d, \gamma, \beta, \lambda)$.

**Remark 1.5.** Both Theorem 1.2 and Theorem 1.4 hold in the periodic boundary condition case. In the periodic setting, the decay condition (1.3) is no longer needed as the periodic domain is compact.

It is an interesting question whether one can prove the sharp constant is $C = 0$ in both Theorem 1.2 and Theorem 1.4. We conjecture this is indeed the case at least for a generic set of parameters.

The rest of this article is organized as follows. In Section 2 we introduce the nonlocal decomposition for both the operator $A$ and the operator $A_1$. In Section 3 we give the proof for Theorem 1.2 and Theorem 1.4 for the operator $A_1$. The case $0 \leq \gamma \leq 1$ of $A$ is also covered there. In Section 4 we complete the proof of the main theorems for the operator $A$ in the regime $1 < \gamma \leq 2$.

We conclude the introduction by setting up some

**Notations.**

- For any two quantities $X$ and $Y$, we denote $X \lesssim Y$ if $X \leq CY$ for some constant $C > 0$. Similarly $X \gtrsim Y$ if $X \geq CY$ for some $C > 0$. We denote $X \sim Y$ if $X \lesssim Y$ and $Y \lesssim X$. We shall write $X \lesssim z_1, z_2, \ldots, z_k$ if $X \leq CY$ and the constant $C$ depends on the quantities $(Z_1, \cdots, Z_k)$. Similarly we define $\gtrsim z_1, \cdots, z_k$ and $\sim z_1, \cdots, z_k$.

- For any $f$ on $\mathbb{R}^d$, we denote the Fourier transform of $f$ has

$$\mathcal{F} f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-i\xi \cdot x} \, dx.$$

The inverse Fourier transform of any $g$ is given by

$$\mathcal{F}^{-1} g(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} g(\xi) e^{i\xi \cdot x} \, d\xi.$$

---

1We thank Edriss Titi for this suggestion.
• For any real number \( x \), the sign function \( \text{sgn}(x) \) is defined as follows:

\[
\text{sgn}(x) = \begin{cases} 
-1, & \text{if } x < 0, \\
0, & \text{if } x = 0, \\
1, & \text{if } x > 0.
\end{cases}
\]

For any complex \( z \) with \( \Re(z) > 0 \), the Gamma function \( \Gamma(z) \) is given by the expression

\[
\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt.
\]

• We will also occasionally need to use the Littlewood-Paley frequency projection operators. Let \( \varphi(\xi) \) be a smooth bump function supported in the ball \( |\xi| \leq 2 \) and equal to one on the ball \( |\xi| \leq 1 \). For each dyadic number \( N \in 2^\mathbb{Z} \) we define the Littlewood-Paley operators

\[
\widehat{P_{\leq N}}f(\xi) := \varphi(\xi/N)\hat{f}(\xi),
\]

\[
\widehat{P_{> N}}f(\xi) := [1 - \varphi(\xi/N)]\hat{f}(\xi),
\]

\[
\widehat{P_{N}}f(\xi) := [\varphi(\xi/N) - \varphi(2\xi/N)]\hat{f}(\xi).
\]

Similarly we can define \( P_{< N}, P_{\geq N}, \) and \( P_{M< N} = P_{\leq N} - P_{\leq M} \), whenever \( M \) and \( N \) are dyadic numbers.

Acknowledgements. H. Dong was partially supported by the NSF under agreements DMS-0800129 and DMS-1056737.

2. The nonlocal decomposition

We start with the following lemma which establishes the nonlocal decomposition of the log-modulated fractional dissipation operator \( |\nabla|^{\gamma} \log^\beta (\lambda + |\nabla|) \) in the regime \( 0 \leq \gamma \leq 1 \). One should notice the subtle difference between this operator and the operator \( A_1 \) in the logarithmic term. By a simple change of variable \( |\xi| \to |\xi|^2 \), the decomposition of the operator \( A_1 \) is addressed in the next corollary. After that we establish the decomposition for the operator \( A \) in the regime \( 1 < \gamma \leq 2 \). The proof will be more involved due to certain first order negative corrections.

Lemma 2.1 (Nonlocal decomposition, case \( 0 \leq \gamma \leq 1 \)). Let \( d \geq 1, 0 \leq \gamma \leq 1 \) and \( \beta > 0, \lambda > 1 \). Then we have the decomposition:

\[
\frac{|\nabla|^{\gamma}}{\log^{\beta}(\lambda + |\nabla|)} = C_\beta \int_0^{\tau} \tau^{\beta-1} |\nabla|^{\gamma-\tau} d\tau + P,
\]

where \( P \) is a smooth Fourier multiplier which maps \( L^p \) to \( L^p \) for all \( 1 \leq p \leq +\infty \).

More precisely, for any function \( f \)

\[
(Pf)(x) = (K*f)(x) = \int K(x-y)f(y)dy,
\]

and \( \|K\|_{L^1} \leq \text{Const} \).

Proof of Lemma 2.1. On the Fourier side the identity (2.1) is equivalent to the following

\[
\frac{|\xi|^{\gamma}}{\log^{\beta}(\lambda + |\xi|)} = C_\beta \int_0^{\tau} \tau^{\beta-1} |\xi|^{\gamma-\tau} d\tau + P(\xi).
\]
To show (2.2) we start with the simple identity
\[
\frac{1}{\log^\beta(\lambda + |\xi|)} = \frac{1}{\Gamma(\beta)} \int_0^\infty \tau^{\beta-1}(\lambda + |\xi|)^{\gamma - \tau} d\tau. \tag{2.3}
\]
Hence
\[
\frac{(\lambda + |\xi|)^\gamma}{\log^\beta(\lambda + |\xi|)} = \frac{1}{\Gamma(\beta)} \int_0^\gamma \tau^{\beta-1}(\lambda + |\xi|)^{\gamma - \tau} d\tau \\
+ \frac{1}{\Gamma(\beta)} \int_\gamma^\infty \tau^{\beta-1}(\lambda + |\xi|)^{\gamma - \tau} d\tau.
\]

We then set \( C_\beta = \frac{1}{\Gamma(\beta)} \) and obtain (2.2) with
\[
P(\xi) = \int_0^\gamma \tau^{\beta-1}(\lambda + |\xi|)^{\gamma - \tau} d\tau \tag{2.4}
\]
\[
+ \int_\gamma^\infty \tau^{\beta-1}(\lambda + |\xi|)^{\gamma - \tau} d\tau \tag{2.5}
\]
\[
+ \frac{|\xi|^\gamma - (\lambda + |\xi|)^\gamma}{\log^2(\lambda + |\xi|)}. \tag{2.6}
\]

It remains for us to show the \( L^1 \) boundedness of \( \mathcal{F}^{-1}(P) \). We first deal with the piece (2.4). By the Fundamental Theorem of Calculus, we have for any \( 0 \leq t \leq 1, \)
\[
(\lambda + |\xi|)^t - |\xi|^t = \int_0^\lambda t(s + |\xi|)^{t-1} ds. \tag{2.7}
\]
If \( 0 < t < 1, \) then
\[
(s + |\xi|)^{t-1} = \frac{1}{\Gamma(1-t)} \int_0^\infty y^{-t} e^{-y(s+|\xi|)} dy.
\]
Since for \( y > 0 \) the Poisson kernel \( \mathcal{F}^{-1}(e^{-y|\xi|}) \) is positive, it follows easily that \( \mathcal{F}^{-1}((s + |\xi|)^{t-1}) \) is a non-negative function and furthermore,
\[
\|\mathcal{F}^{-1}((s + |\xi|)^{t-1})\|_{L^1_x} = s^{t-1}.
\]
By (2.7), we get
\[
\|\mathcal{F}^{-1}((\lambda + |\xi|)^t - |\xi|^t)\|_{L^1_x} = \lambda^t. \tag{2.8}
\]
Plugging the above estimate into (2.4), we get
\[
\|\mathcal{F}^{-1}(2.4)\|_{L^1_x} \leq C_\beta \int_0^\gamma \tau^{\beta-1} \lambda^{\gamma - \tau} d\tau < \infty,
\]
which is clearly good for us.

For (2.5), we just note that for \( \tau > \gamma, \)
\[
\|\mathcal{F}^{-1}((\lambda + |\xi|)^{\gamma - \tau})\|_{L^1_x} = \lambda^{\gamma - \tau},
\]
and hence
\[
\|\mathcal{F}^{-1}(2.5)\|_{L^1_x} \leq C_\beta \int_\gamma^\infty \tau^{\beta-1} \lambda^{\gamma - \tau} d\tau < +\infty,
\]
where we used the fact that \( \lambda > 1. \)
Finally we deal with the contribution of (2.6). By (2.3), it is obvious that
\[
\left\| \mathcal{F}^{-1}\left( \frac{1}{\log^\beta(\lambda + |\xi|)} \right) \right\|_{L^1_x} = \frac{1}{\log^\beta \lambda} < \infty,
\]
since \(\lambda > 1\). Now in (2.6) we may assume \(0 < \gamma < 1\) (the cases \(\gamma = 0\) and \(\gamma = 1\) are trivial). By (2.8) and Young’s inequality, we get
\[
\|\mathcal{F}^{-1}(2.6)\|_{L^1_x} \leq \|\mathcal{F}^{-1}\left( \frac{1}{\log^\beta(\lambda + |\xi|)} \right)\|_{L^1_x} \cdot \|\mathcal{F}^{-1}((\lambda + |\xi|)\gamma - |\xi|\gamma)\|_{L^1_x} \leq \frac{1}{\log^\beta \lambda} \cdot \lambda^\gamma < \infty.
\]
\(\square\)

By a simple substitution \(|\xi| \to |\xi|^2\), we can deduce the nonlocal decomposition of the operator \(A_1\) from Lemma 2.1. Of course, one still needs to check the \(L^1\) boundedness of the error term under such nonlinear substitution.

**Corollary 2.2** (Nonlocal decomposition for the operator \(A_1\)). Let \(d \geq 1, 0 \leq \gamma \leq 2\) and \(\beta > 0, \lambda > 1\). Then we have the decomposition:
\[
\frac{\nabla|\gamma|}{\log^\beta(\lambda - \Delta)} = C_\beta \int_0^\gamma \tau^{\beta - 1}|\nabla|^{\gamma - \tau} d\tau + P,
\]
where \(P\) is a smooth Fourier multiplier which maps \(L^p\) to \(L^p\) for all \(1 \leq p \leq +\infty\). More precisely, for any function \(f\)
\[
(Pf)(x) = (K * f)(x) = \int_{\mathbb{R}^d} K(x - y)f(y) dy,
\]
and \(\|K\|_{L^1_x} \leq \text{Const.}\)

**Proof of Corollary 2.2** On the Fourier side, the identity (2.9) is equivalent to the following (the value of the constant \(C_\beta\) can be adjusted slightly)
\[
\frac{|\xi|^\gamma}{\log^\beta(\lambda + |\xi|^2)} = C_\beta \int_0^\gamma \tau^{\beta - 1}(|\xi|^2)^{\frac{\gamma - \tau}{2}} d\tau + P(\xi),
\]
(2.10)

By using a similar derivation as in the beginning part of the proof of Lemma 2.1 (see in particular (2.4)–(2.6), and replace \(|\xi|\) by \(|\xi|^2\), \(\gamma\) by \(\gamma/2\), we obtain (2.10) with
\[
P(\xi) = C_\beta \int_0^\gamma \tau^{\beta - 1}\left((\lambda + |\xi|^2)^{\frac{\gamma - \tau}{2}} - (|\xi|^2)^{\frac{\gamma - \tau}{2}}\right) d\tau \\
+ C_\beta \int_\gamma^\infty \tau^{\beta - 1}(\lambda + |\xi|^2)^{\frac{\gamma - \tau}{2}} d\tau \\
+ \frac{(|\xi|^2)^{\frac{\gamma - 1}{2}} - (\lambda + |\xi|^2)^{\frac{\gamma - 1}{2}}}{\log^\beta(\lambda + |\xi|^2)}.
\]
Note that \(0 \leq \gamma/2 \leq 1\) and the fact \(\|\mathcal{F}^{-1}(a + |\xi|^2)^{-s}\|_{L^1} = a^{-s}\) for any \(s > 0\) and \(a > 0\). By using a similar analysis as in the proof of Lemma 2.1, it is then not difficult to check that \(\mathcal{F}^{-1}(P)\) is an \(L^1\) bounded kernel. \(\square\)

We now consider the more involved \(1 < \gamma \leq 2\) case for the operator \(A\). One should compare the decomposition (2.11) with (2.1).
Lemma 2.3 (Nonlocal decomposition, case 1 < γ ≤ 2). Let d ≥ 1, 1 < γ ≤ 2 and β > 0, λ > 1. Then we have the decomposition:

\[ \frac{|\nabla|^\gamma}{\log^2(\lambda + |\nabla|)} \]

\[ = C_\beta \int_{\gamma-1}^{\gamma} \tau^{\beta-1}|\nabla|^\gamma \tau^\gamma d\tau + C_\beta \int_{\gamma-1}^{\gamma-1} \tau^{\beta-1}(|\nabla|^\gamma - \lambda \tau |\nabla|^\gamma - \tau^\gamma) d\tau + P, \]

(2.11)

where P is a smooth Fourier multiplier which maps \( L^p \) to \( L^p \) for all \( 1 \leq p \leq +\infty \). More precisely, for any function f

\[ (Pf)(x) = (K * f)(x) = \int K(x - y) f(y) dy, \]

and \( \|K\|_{L^1} \leq \text{Const.} \)

Proof of Lemma 2.3. Throughout this proof we shall use the letter P to denote the symbol of an \( L^1 \rightarrow L^1 \) bounded operator. For the convenience of notation, we allow the value of P to vary from line to line. We begin with two elementary estimates. Let \( \phi \in C_0^\infty(\mathbb{R}^d) \) be a radial smooth cut-off function such that \( \phi(x) = 1 \) for \( |x| \leq 2 \) and \( \phi(x) = 0 \) for \( |x| \geq 3 \). For any constant \( C > 0 \) define \( \phi_{< C}(x) := \phi(x/C) \) and \( \phi_{> C}(x) = 1 - \phi_{< C}(x) \). Then for any \( \gamma > 0, C > 0, s \geq 0 \), we have the following

\[ \|F^{-1}(\frac{|\xi|}{\lambda + |\xi|})\|_{L^1_L^1} < \infty, \]

(2.12)

\[ \|F^{-1}(\frac{|\xi|^{-s} \phi_{> C}(\xi)}{1 + |\xi|})\|_{L^1^1} \leq C_1 (1 + s)^d + C^s, \]

(2.13)

where the constant \( C_1 \) is independent of s. To prove (2.12), one can use a scaling argument to reduce to the \( \lambda = 1 \) case. The result then follows easily from the binomial expansion of \( (|\xi|/(1 + |\xi|))^{-\gamma} = (1 - \frac{1}{1 + |\xi|})^{-\gamma} = \sum_{n \geq 0} C_{\gamma,n}(1 + |\xi|)^{-n} \), the \( L^1 \)-boundedness of the operators \( (1 + |\xi|)^{-n} \) (namely \( \|F^{-1}((1 + |\xi|)^{-n})\|_{L^1_L^1} \leq 1 \) for any \( n \geq 0 \)), and the fact that \( \sum_{n \geq 0} |C_{\gamma,n}| < +\infty \) (note that \( C_{\gamma,n} \) has a definite sign for \( n \) sufficiently large). To prove (2.13) one can again use scaling to reduce to the case \( C = 1 \). For \( s \geq 2 \) the result is obvious by using integration by parts. For \( 0 < s \leq 2 \), we note that

\[ \|F^{-1}(1 + |\xi|)^{-s} \phi_{> C}(\xi)\|_{L^1^1} \lesssim 1, \]

and on the support of \( \phi_{> C}(\xi) \),

\[ |\xi|^{-s} - (1 + |\xi|)^{-s} \sim |\xi|^{-s-1}. \]

One can then use the Littlewood-Paley operators to bound (since we are summing over \( N \geq N_0 \) the convergence in \( L^1 \) is of no problem):

\[ \|F^{-1}(|\xi|^{-s} \phi_{> 1})\|_{L^1_L^1} \lesssim 1 + \|P_{\geq 1}((1 + |\nabla|)^{-s} - (1 + |\nabla|)^{-s})\delta_0\|_{L^1_L^1} \]

\[ \lesssim 1 + \sum_{N \text{ dyadic: } N \geq 1} N^{-s-1}\|P_N \delta_0\|_{L^1_L^1} \]

\[ \leq \text{Const.} \]

Observe that the implied constants are uniform in s since \( 0 < s \leq 2 \). This settles (2.13).
By (2.3), we write
\[ \Gamma(\beta) \frac{|\xi|^\gamma}{\log^\gamma (\lambda + |\xi|)} = \int_0^{\gamma-1} \tau^{\beta-1} |\xi|^\gamma (\lambda + |\xi|)^{-\tau} d\tau \] (2.14)
\[ + \int_{\gamma-1}^{\gamma} \tau^{\beta-1} |\xi|^\gamma (\lambda + |\xi|)^{-\tau} d\tau \] (2.15)
\[ + \int_\gamma^{\infty} \tau^{\beta-1} |\xi|^\gamma (\lambda + |\xi|)^{-\tau} d\tau. \] (2.16)

We first deal with (2.16). Rewrite
\[ |\xi|^\gamma (\lambda + |\xi|)^{-\tau} = (\frac{|\xi|}{\lambda + |\xi|})^\gamma \cdot (\lambda + |\xi|)^{\gamma-\tau}. \]

By using (2.12) we obtain
\[ \|F^{-1}(\Phi_{10\lambda}(\xi))\|_{L^1_x} \lesssim \int_\gamma^{\infty} \tau^{\beta-1} \lambda^{\gamma-\tau} d\tau < \infty. \]

Next we turn to (2.15). By inserting a smooth cut-off function \( \phi_{10\lambda}(\xi) \), we have
\[ (2.15) = \int_{\gamma-1}^{\gamma} \tau^{\beta-1} |\xi|^\gamma (1 + \frac{\lambda}{|\xi|})^{-\tau} d\tau \phi_{10\lambda}(\xi) + P \]
\[ = \int_{\gamma-1}^{\gamma} \tau^{\beta-1} |\xi|^\gamma d\tau \phi_{10\lambda}(\xi) + P \]
\[ + \int_{\gamma-1}^{\gamma} \tau^{\beta-1} |\xi|^\gamma (1 + \frac{\lambda}{|\xi|})^{-\tau} - 1) d\tau \phi_{10\lambda}(\xi). \] (2.17)

On the other hand, by using the binomial expansion of the function \((1 + t)^{-s} = \sum_{n \geq 0} C_{n,s} t^n\) and the estimate (2.13), it is not difficult to check that
\[ \|F^{-1}(\Phi_{10\lambda}(\xi))\|_{L^1_x} \lesssim \int_{\gamma-1}^{\gamma} \tau^{\beta-1} (\sum_{n \geq 1} |C_{n,\tau}| \cdot \lambda^n \cdot (10\lambda)^{-(n+\tau-\gamma)} \cdot (1 + n + \tau - \gamma)^{d+1}) d\tau \]
\[ < +\infty. \]

Hence we have proved
\[ (2.15) = \int_{\gamma-1}^{\gamma} \tau^{\beta-1} |\xi|^\gamma d\tau + P. \]

We turn now to the final piece (2.14). The main idea is similar to that of (2.15). We again insert a smooth cut-off \( \phi_{10\lambda}(\xi) \) and observe that when \(|\xi| \geq 10\lambda\), we have
\[ (1 + \frac{\lambda}{|\xi|})^{-\tau} = 1 - \tau \frac{\lambda}{|\xi|} + \sum_{n \geq 2} C_{n,\tau} (\frac{\lambda}{|\xi|})^n. \]

Note that \(0 \leq \tau \leq 1\) and we have to keep terms up to the linear term. Then clearly
\[ (2.14) = \int_0^{\gamma-1} \tau^{\beta-1} (|\xi|^\gamma - \tau \lambda |\xi|^\gamma) d\tau + P. \]

The desired decomposition (2.11) now follows. \(\square\)
3. Proof of Theorem 1.2 and Theorem 1.4 for the operator $A_1$

In this section we give the proofs of Theorems 1.2 and 1.4 for the operator $A_1$. The proofs for the operator $A$ is slightly more involved and will be given in the next section.

Proof of Theorem 1.2 for the operator $A_1$. Assume first $0 < \gamma \leq 2$ and $\beta > 0$. By (2.9) we can write

$$A_1 = L + P,$$

(3.1)

where $L = C_\beta \int_0^\gamma \tau^{\beta-1} |\nabla^{\gamma-\tau} d\tau$ and $\|L\|_{L^1_x} < +\infty$. Now take $\lambda_1 > \nu \|P\|_{L^1_x}$ and define $f(t, x) = e^{-\lambda_1 t} \theta(t, x)$. Fix $T > 0$ and consider

$$\sup_{0 \leq t \leq T, x \in \mathbb{R}^d} \|f(t, x)\| = M > 0.$$

Without loss of generality we can assume

$$\sup_{0 \leq t \leq T, x \in \mathbb{R}^d} f(t, x) = M > 0.$$

By using the decay condition (1.2) and a simple compactness argument in $t$, we conclude that there exists $(t_0, x_0)$ such that

$$f(t_0, x_0) = M.$$

We now show that $t_0 = 0$. Indeed if $0 < t_0 \leq T$, we compute

$$\begin{align*}
(\partial_t f)(t_0, x_0) &= -\lambda_1 f(t_0, x_0) - \nu(A_1 f)(t_0, x_0) \\
&= -\lambda_1 M - \nu(Lf)(t_0, x_0) - \nu(Pf)(t_0, x_0).
\end{align*}$$

(3.2)

Now by Corollary 2.2 we have

$$\|Pf(t_0)\|_{L^\infty_x} \leq \|P\|_{L^1_x} \|f(t_0)\|_{L^\infty_x} \leq \|P\|_{L^1_x} \cdot M.$$

(3.3)

For any $0 < s < 2$, by using the fractional representation

$$\begin{align*}
(|\nabla|^s g)(x) &= C_s \lim_{\epsilon \to 0} \int_{|y-x| > \epsilon} \frac{g(x) - g(y)}{|x-y|^{d+s}} dy,
\end{align*}$$

it is easy to see $|\nabla|^s f(t_0, x_0) \geq 0$ and hence

$$Lf(t_0, x_0) \geq 0.$$

(3.4)

Plugging (3.3) and (3.4) into (3.2), we reach a contradiction:

$$\begin{align*}
(\partial_t f)(t_0, x_0) < -(\lambda_1 - \nu \|P\|_{L^1_x}) M < 0.
\end{align*}$$

Therefore we conclude that $t_0 = 0$ and clearly the estimate (1.3) follows.

It remains to prove the case $\gamma = 0$ and $\beta > 0$. But in this case the operator $\log^{-\beta}(\lambda - \Delta)$ corresponds to an $L^1$-bounded convolution kernel. Hence we just need to repeat the previous argument with $A_1 = P$ and $\lambda_1 > \nu \|P\|_{L^1_x}$. We omit the repetitive details.

Finally we complete the
Proof of Theorem 1.4 for the operator \( A_1 \). Without loss of generality we assume \( \nu > 0, 0 < \gamma \leq 2 \) and \( \beta > 0 \). Let \( 1 \leq p < \infty \). Multiplying both sides of (1.1) by \( |\theta|^{p-1} \text{sgn}(\theta) \), integrating by parts and using the fact that \( v \) is divergence free, we obtain

\[
\frac{1}{p} \frac{d}{dt} (\|\theta(t)\|_p^p) = -\nu \int_{\mathbb{R}^d} (A_1 \theta)|\theta|^{p-1} \text{sgn}(\theta) \, dx
\]

\[
= -\nu \int_{\mathbb{R}^d} (L \theta)|\theta|^{p-1} \text{sgn}(\theta) \, dx - \nu \|(P \theta)|\theta|^{p-1}\|_{L^1_x},
\]

where in the last equality we have used the decomposition (3.1).

Since for any \( 0 \leq s < 2, 1 \leq p < \infty \), we have

\[
\int_{\mathbb{R}^d} (|\nabla|^s \theta)|\theta|^{p-1} \text{sgn}(\theta) \, dx \geq 0,
\]

it follows easily that

\[
\int_{\mathbb{R}^d} (L \theta)|\theta|^{p-1} \text{sgn}(\theta) \, dx \geq 0.
\]

By Hölder, we have

\[
\|(P \theta)|\theta|^{p-1}\|_{L^1_x} \leq \|P\|_{L^1_x} \|\theta\|_p^p.
\]

Plugging the above estimates into (3.5) and integrating in time, we get for any \( 1 \leq p < \infty \),

\[
\|\theta(t)\|_{L^p_x} \leq e^{\nu \|P\|_{L^1_x} t} \|\theta_0\|_{L^p_x}.
\]

The case \( p = \infty \) follows by a limiting argument \( p \to \infty \). Clearly (1.4) holds by setting \( C = \nu \|P\|_{L^1_x} \).

\[
\square
\]

4. Proof of the main theorems for the operator \( A \)

In this section we describe the proofs of the main theorems for the operator \( A \). We shall only need to consider the case \( 1 < \gamma \leq 2 \). Thanks to Lemma 2.1 the case \( 0 \leq \gamma \leq 1 \) is already covered in the previous section. In the case \( 1 < \gamma \leq 2 \) we have to use the decomposition (2.11) in Lemma 2.3. The extra complication is due to the negative term \(-\lambda \tau |\nabla|^{\gamma - \tau - 1}\) which in principle can cause the maximum principle to fail. The way out of this difficulty is to note that the main term \( |\nabla|^{\gamma - \tau}\) is stronger than this negative term by an order of \( |\nabla|^{-1}\). The following lemma quantifies this observation. In some sense it gives the maximum principle for "mixed" operators.

Lemma 4.1. Let \( 0 < s_1 < s_2 < 2 \) and \( C_1 > 0 \). Consider the operator

\[
L = |\nabla|^{s_2} - C_1 |\nabla|^{s_1}.
\]

Then for any smooth function \( g \) which attains its maximum at some point \( x_0 \), we have

\[
(Lg)(x_0) \geq -C_1 C_d \|g\|_\infty \cdot (2 - s_1) \left(1 + \left(s_2(2 - s_2)\right)^{-\frac{s_2}{2s_2 - s_1}}\right).
\]

where \( C_d > 0 \) is some constant depending only on the dimension \( d \). In particular if \( s_1 = s_2 - 1 \) and \( 1 < s_2 < 2 \), then we have the estimate

\[
(Lg)(x_0) \geq -C_1 C'_d \|g\|_\infty (1 + (2 - s_2)^{1-s_2}),
\]

where \( C'_d > 0 \) is another constant depending only on the dimension \( d \).
Proof of Lemma 4.1. To begin we need to derive the explicit constant appearing in the integral representation of the fractional operators $|\nabla|^s$ with $0 < s < 2$. Recall that for any $0 < \alpha < d$, the Riesz potential $|\nabla|^{-\alpha}$ has the following explicit representation (cf. pp 117 of Stein [8])

$$\left(|\nabla|^{-\alpha} f\right)(x) = c_{\alpha,d} \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d+\alpha}} dy,$$

with

$$c_{\alpha,d} = \frac{\Gamma\left(\frac{d-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right) 2^{\alpha} \pi^{\frac{d}{2}} \alpha}.$$

For $0 < s < 2$ by writing $|\nabla|^s = -\Delta |\nabla|^{(2-s)}$ and integrating by parts, we get

$$\left(|\nabla|^s f\right)(x) = C_{s,d} \lim_{\epsilon \to 0} \int_{|y-x| > \epsilon} \frac{f(x) - f(y)}{|x-y|^{d+s}} dy,$$

where

$$C_{s,d} = \frac{s}{2^{2-s} \pi^{\frac{d}{2}} \Gamma\left(\frac{2-s}{2}\right)}.$$

By using the asymptotics $\Gamma(z) \sim z^{-1}$ for $z \to 0$, it is easy to see that

$$C_{s,d} \sim \frac{s}{\Gamma\left(\frac{2-s}{2}\right)} \sim d(2-s).$$

and in particular for all $0 < s < 2$,

$$C_{s,d} \lesssim_d 1.$$  \hspace{1cm} (4.3)

We now write

$$(\left(|\nabla|^{s_2} - C_1 |\nabla|^{s_1}\right) g)(x_0) = C_{s_2,d} \lim_{\epsilon \to 0} \int_{|y-x_0| > \epsilon} \frac{g(x_0) - g(y)}{|x_0-y|^{d+s_2}} dy$$

$$- C_1 C_{s_1,d} \lim_{\epsilon \to 0} \int_{|y-x_0| > \epsilon} \frac{g(x_0) - g(y)}{|x_0-y|^{d+s_1}} dy.  \hspace{1cm} (4.5)$$

Observe that $g(x_0) - g(y) \geq 0$ for all $y$. We now separate the $y$-integral into two regimes. The first regime is $\{ y : |y-x_0| \leq \min\{1, C_2\} \}$, where $C_2 > 0$ is a constant such that (here we use (4.3) to bound $C_{s_1,d}$)

$$(C_2)^{s_1-s_2} C_{s_2,d} \gtrsim_d C_1.$$

By using (4.3), we have (the notation $\sim_{C_1,d}$ means up a constant depending on $C_1$ and $d$)

$$C_2 \sim_{C_1,d} \left( C_2(2-s_2) \right)^\frac{1}{s_2-s_1}.$$  \hspace{1cm} (4.4)

In the first regime, it is easy to check that the first integral bounds the second integral in (4.5). The second regime is just the complement $\{ y : |y-x_0| > \min\{1, C_2\} \}$. In this case we simply discard the first integral and bound the second
integral by $\|g\|_{L^\infty}$ which produces a term of the form (below $C_d$ denotes a constant depending only on the dimension $d$):

$$- C_1 C_d \frac{s_1}{\Gamma\left(\frac{\gamma - s_1}{2}\right)} \|g\|_\infty \int_{r > \min\{1, C_2\}}^{r^{\gamma - s_1}} dr \geq C_{1, d} - \|g\|_\infty \cdot (2 - s_1) \left(1 + (s_2(2 - s_2))^{-\frac{1}{2 - s_2}}\right).$$

This settles (4.1).

Finally (4.2) is a simple consequence of (4.1). □

The following corollary will be used in the proof of Theorem 1.2.

**Corollary 4.2.** Let $d \geq 1$, $1 < \gamma \leq 2$ and $\beta > 0$, $\lambda > 1$. Then for any smooth function $g$ which attains its maximum at some point $x_0$, we have

$$\left(\frac{|\nabla|^{\gamma}}{\log^\beta(\lambda + |\nabla|)} g\right)(x_0) \geq -(C_{d, \beta, \gamma}^1 \lambda + \|P\|_{L^1_x})\|g\|_\infty,$$

where $C_{d, \beta, \gamma}^1$ is some constant depending only on $(d, \alpha, \beta, \gamma)$, and $P$ is the operator defined in (2.11).

**Proof of Corollary 4.2.** By Lemma 2.3, Lemma 4.1, and the fact that $(|\nabla|^{\gamma - \tau} g)(x_0) \geq 0$, we have

$$\left(\frac{|\nabla|^{\gamma}}{\log^\beta(\lambda + |\nabla|)} g\right)(x_0) \geq C_{\beta} \int_0^{\gamma - 1} \tau^{\beta - 1} \left(|\nabla|^{\gamma - \tau} - \lambda \tau|\nabla|^{\gamma - \tau - 1}\right) g(x_0) d\tau - \|P\|_{L^1_x} \|g\|_\infty \geq \left(1 + (2 - \gamma + \tau)^{1-\gamma + \tau})d\tau - \|P\|_{L^1_x} \|g\|_\infty \geq - (C_{d, \beta, \gamma}^1 \lambda + \|P\|_{L^1_x}) \|g\|_\infty.$$

We are now ready to complete the proof of Theorem 1.2 for the operator $A$, case $1 < \gamma \leq 2$. With the help of Corollary 4.2, the proof is similar to the proof for the operator $A_1$ in section 3, one only needs to consider $f(t, x) = e^{-\lambda t} \theta(t, x)$ with $\lambda_1 > \nu(C_{d, \beta, \gamma}^1 \lambda + \|P\|_{L^1_x})$, where the constant $C_{d, \beta, \gamma}^1$ is the same as in Corollary 4.2. The rest of the proof is now the same as in Section 3. We omit the details. □

We next turn to the proof of Theorem 1.4. The following lemma establishes a form of maximum principle for the mixed operator $L = |\nabla|^s - C_1 |\nabla|^{s - 1}$ in the $L^p$, $1 \leq p < \infty$ setting.

**Lemma 4.3.** Let $1 < \gamma \leq 2$, $1 < s < \gamma$, $C_1 > 0$ and consider the operator

$$L = |\nabla|^s - C_1 (\gamma - s)|\nabla|^{s - 1}.$$

Then for any $1 \leq p < \infty$ and any smooth $g \in L^p$, we have the bound

$$\int_{\mathbb{R}^d} (Lg) |g|^{p-1} \text{sgn}(g) \, dx \geq -C_d \|g\|_p^p,$$

where $C_d > 0$ is some constant depending only on $(d, \gamma)$. 

\[12\ H.\ DONG\ AND\ D.\ LI\]
Proof of Lemma 4.3. The proof is analogous to that of Corollary 4.2 with suitable modifications. Without loss of generality we assume $C_1 = 1$. We begin with a simple consequence of the Young’s inequality

$$
(s - 1) \int_{\mathbb{R}^d} \int_{|y - x| > A} \frac{|f(x) - f(y)|}{|x - y|^{d + s}}(|f(x)|^{p-1} + |f(y)|^{p-1}) \, dx \, dy
\leq C_d \|f\|_p^p \cdot A^{1-s}, \tag{4.7}
$$

where $C_d > 0$ is a constant depending only on the dimension $d$. The estimate (4.7) is a simple consequence of the Young’s inequality

$$
|f(y)| \cdot |f(x)|^{p-1} \leq \frac{p-1}{p} |f(x)|^p + \frac{1}{p} |f(y)|^p,
$$

and Fubini.

Next we need to transform the LHS of (4.6) suitably. By a symmetrization in the variable $x$ and $y$, we have

$$
\int_{\mathbb{R}^d} (|\nabla| y |g|^{p-1} \text{sgn}(g)) \, dx
= C_{s,d} \int_{\mathbb{R}^d} \int_{|x - y|^{d+s}} \frac{g(x) - g(y)}{|x - y|} \, dy |g(x)|^{p-1} \text{sgn}(g(x)) \, dx
= \frac{C_{s,d}}{2} \int_{\mathbb{R}^d} \frac{h(x, y)}{|x - y|^{d+s}} \, dx \, dy, \tag{4.8}
$$

where

$$
h(x, y) = (g(x) - g(y)) \left( |g(x)|^{p-1} \text{sgn}(g(x)) - |g(y)|^{p-1} \text{sgn}(g(y)) \right).
$$

Note that for any real numbers $a, b$, we have

$$
(a - b) \left( |a|^{p-1} \text{sgn}(a) - |b|^{p-1} \text{sgn}(b) \right) \geq 0.
$$

Therefore $h(x, y) \geq 0$ for all $x, y$. The advantage of the expression (4.8) is that the integrand is always non-negative. Similar expression holds for the operator $|\nabla|^{s-1}$.

Let $A > 0$ be a constant whose value will be specified later. By using (4.8) and (4.3), we have

$$
\text{LHS of (4.6)} \geq C_d s(2 - s) \int_{\mathbb{R}^d} \int_{|y - x| < A} \frac{h(x, y)}{|x - y|^{d+s}} \, dx \, dy \tag{4.9}
- C_d' (s - 1)(\gamma - s) \int_{\mathbb{R}^d} \int_{|y - x| < A} \frac{h(x, y)}{|x - y|^{d+s-1}} \, dx \, dy \tag{4.10}
- C_d' (s - 1)(\gamma - s) \int_{\mathbb{R}^d} \int_{|y - x| > A} \frac{h(x, y)}{|x - y|^{d+s-1}} \, dx \, dy, \tag{4.11}
$$

where $C_d$ and $C_d'$ are constants depending only on the dimension $d$.

By (4.7), we have

$$
(4.11) \geq -C_d'R_p\|g\|_P^p A^{1-s} (\gamma - s), \tag{4.12}
$$

On the other hand, by choosing $A = C_d/C_d'$, we have

$$
C_d \cdot s(2 - s) \cdot \frac{1}{A} \geq C_d'(s - 1)(\gamma - s),
$$
and
\[ (4.9) + (4.10) \geq 0. \]
Substituting the value of $A$ into (4.12), we obtain (4.6). □

Finally we are ready to complete the

Proof of Theorem 1.4 for the operator $A$ in the regime $1 < \gamma \leq 2$. Thanks to Lemma 2.3 and Lemma 4.3, we essentially only have to repeat the proof for the operator $A_1$ in Section 3. In place of (3.5), we have
\[
\frac{1}{p} \frac{d}{dt}(\|\theta(t)\|_p^p) \leq \nu(\|P\|_{L^1_x} + \lambda C_{d,\gamma,\beta}) \|\theta(t)\|_p^p.
\]

The estimate (1.4) follows immediately. □

References

[1] Askey, R. Radial Characteristic Functions, University of Wisconsin-Madison, Mathematics Research Center, 1262, 1973.
[2] Córdoba, A.; Córdoba, D. A maximum principle applied to quasi-geostrophic equations, Comm. Math. Phys. 249 (2004), no. 3, 511–528.
[3] Chan, C.H.; Czubak, M.; Silvestre, L. Eventual regularization of the slightly supercritical fractional Burgers equation, Discrete Contin. Dyn. Syst. 27 (2010), no. 2, 847–861.
[4] Dong, H.; Du, D.; Li, D. Finite time singularities and global well-posedness for fractional Burgers equations, Indiana Univ. Math. J. 58 (2009), no. 2, 807–821.
[5] Dabkowski, M.; Kiselev, A.; Silvestre, L.; Vicol, V. Global well-posedness of slightly supercritical active scalar equations, preprint (2012).
[6] Dabkowski, M.; Kiselev, A.; Vicol, V. Global well-posedness for a slightly supercritical quasigeostrophic equation, Nonlinearity 25 (2012), no. 5, 1525–1535.
[7] Hmidi, T. On a maximum principle and its application to the logarithmically critical Boussinesq system, Anal. PDE, 4 (2011), No. 2, 247–284.
[8] Stein, Elias M. Singular integrals and differentiability properties of functions. Princeton Mathematical Series, No. 30 Princeton University Press, Princeton, N.J. 1970.
[9] Tao, T. Global regularity for a logarithmically supercritical hyperdissipative Navier-Stokes equation, Anal. PDE 2 (2009), no. 3, 361–366.
[10] Wu, J. Global regularity for a class of generalized magnetohydrodynamic equations, J. Math. Fluid Mech. 13 (2011), no. 2, 295–305.

(H. Dong) Division of Applied Mathematics, Brown University, 182 George Street, Providence, RI 02912, USA
E-mail address: Hongjie_Dong@brown.edu

(D. Li) Department of Mathematics, University of British Columbia, Vancouver BC Canada V6T 1Z2
E-mail address: mpdongli@gmail.com