Velocity operator approach to a Fermion system

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Abstract

In this paper, we formulate a velocity operator approach to a three-dimensional (3D) Fermion system. Following Sunakawa, introducing density and velocity operators, we treat 3D quantum fluid dynamics in the system. We get a collective Hamiltonian in terms of collective variables. The lowest order collective Hamiltonian is diagonalized. This diagonalization leads us to a Bogoliubov transformation for Boson-like operators.

Keywords: Collective motion of a three-dimensional Fermion system; velocity operator; vortex motion; Grassmann numbers

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1 Introduction

To approach elementary excitations in a Fermi system, Tomonaga and Emery gave basic ideas in their collective motion theories [1, 2]. On the other hand, Sunakawa’s discrete integral equation method for a Fermi system [3] may be expected to also work well for a collective motion problem. In the preceding papers [4, 5], introducing density operators \( \rho_k \) and associated variables \( \pi_k \) and defining exact momenta \( \Pi_k \) (collective variables), we could get an exact canonically conjugate momenta approach to one- and three-dimensional (1D and 3D) Fermion systems. In the present paper, we formulate a velocity operator approach to a 3D Fermion system. Following Sunakawa, after introducing momentum density operators \( g_k \), we define velocity operators \( v_k \) which denote classical fluid velocities. We derive a collective Hamiltonian in terms of the collective variables \( v_k \) and \( \rho_k \) for the elementary excitations including vortex motions. The lowest order collective Hamiltonian is diagonalized. This diagonalization leads to a Bogoliubov transformation for Boson-like operators [6].

In §2 first we introduce collective variables \( \rho_k \) and associated momentum density variables \( g_k \) and give the commutation relations between them. The velocity operator \( v_k \) is defined by a discrete integral equation and commutation relations between the velocity operators are also given. In §3 the dependence of the original Hamiltonian on \( v_k \) and \( \rho_k \) is determined. This section is also devoted to a calculation of a constant term in the collective Hamiltonian. Its lowest order is diagonalized and leads to a Bogoliubov transformation for Boson-like operators. Finally in §4 some discussions and further outlook are given.

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2 Collective variable and velocity operator

Let us define the Fourier component of the density operator \( \rho(x) = \psi^\dagger(x) \psi(x) \) as
\[ \rho_k \equiv \frac{1}{\sqrt{N}} \sum p a_p^\dagger p - k a_p + k - p, \quad \rho_0 = \frac{1}{\sqrt{N}} \sum p a_p^\dagger a_p = \sqrt{N}, \quad (N: \text{total number of Fermion}). \quad (2.1) \]
We here consider a spin-less Fermion and have used the anti-commutation relations (CR)s among \( a_p^\dagger \)'s and \( a_p \)'s.

The Fourier component of the interaction \( V \) is given by
\[ V_k = \frac{1}{2m} \sum q \sum p \sum p^\dagger \rho_k p - k, \quad V_0 = \frac{1}{\sqrt{N}} \sum p a_p^\dagger a_p = 0. \quad (2.2) \]

The CRs (2.6) and (2.7) are quite the same as the Sunakawa’s [7].

Along the same way as the one in I and II and the Sunakawa’s [3], we prove two important CRs
\[ [T, \rho_k] = \hbar k \delta \cdot \cdot \cdot, \quad [T, \rho_k] = \hbar k \delta \cdot \cdot \cdot, \quad [T, \rho_k] = \hbar k \delta \cdot \cdot \cdot. \quad (2.2) \]

The symbol \( \delta \cdot \cdot \cdot \) means that the term \( p = k \) and \( q = k' \) at the same time should be omitted. The CRs (2.6) and (2.7) are quite the same as the Sunakawa’s [7].

As pointed out by him, Fourier transforms of the operators \( \rho_k \) and \( v_k \) and the CRs among them are identical with those found by Landau [8] for the fluid dynamical density operator and the velocity operator.

Then, it turns out that the quantum mechanical operator \( v(x) \), which satisfies the famous CR
\[ [v^{(i)}(x), v^{(j)}(x')] = \frac{i \hbar}{m} \delta(x - x') \rho(x)^{-1} \text{det} \psi(x)^{(k)} (i, j, k) \text{ cyclic}, \quad (2.8) \]
corresponds to the fluid dynamical velocity. We also have \[ [v(x), \rho(x')] = - \frac{i \hbar}{m} \nabla_x \delta(x - x'). \]

3 \( v_k, \rho_k \)-dependence of the Hamiltonian

We derive here a collective Hamiltonian in terms of the \( v_k \) and \( \rho_k \). Following Sunakawa, we expand the kinetic operator \( T \) in a power series of the velocity operator \( v_k \) as follows:
\[ T = T_0 + \sum_{\rho \neq \rho_0} T_1(\rho; p, p) v_p + \sum_{\rho \neq \rho_0} T_2(\rho; p, q) v_p v_q + \cdots, \quad T_2(\rho; p, q) = T_2(\rho; q, p), \quad (3.1) \]
in which \( T_n(n \neq 0) \) are unknown expansion coefficients. In order to determine their explicit expressions, we take the CRs between \( T \) and \( \rho_k \) as follows:
\[ [T, \rho_k] = \hbar T_1(\rho; p, k) \rho_k + 2 \hbar k \sum_{\rho \neq \rho_0} T_2(\rho; p, k) v_p k + \cdots, \quad (3.2) \]
From (2.3) and (2.5), we can calculate the CRs between \( T \) and \( \rho_k \) as follows:
\[ [T, \rho_k] = \frac{\hbar}{m} k \cdot v_k \rho_k = \frac{\hbar}{m} k \cdot v_k \rho_k + \frac{\hbar}{m \sqrt{N}} \sum_{\rho \neq \rho_0} k \rho_k p k \cdot v_p, \quad (3.3) \]

\[ [[T, \rho_k], \rho_k] = \frac{\hbar^2}{m} \delta_{k, k'} k' \rho_k k' \rho_k k' k' + \cdots, \quad [[T, \rho_k], \rho_k] = 0, \cdots. \]
Comparing the above results with the CRs (3.2), we can determine the coefficients $T_n(n \neq 0)$. Then we can express the kinetic part $T$ in terms of the $\rho_k$ and $v_k$ as follows:

$$T = T_0(\rho) + \frac{1}{2m} \sum_{\mathbf{k} \neq 0} v_k \cdot v_{-k} + \frac{1}{2m N} \sum_{\mathbf{p}+\mathbf{q} \neq 0} \rho_{\mathbf{p}+\mathbf{q}} v_{\mathbf{p}} \cdot v_{\mathbf{q}}, \quad (v_0 = 0). \quad (3.4)$$

Up to the present stage, all the expressions have been derived without any approximation. To determine $T_0(\rho)$, we expand it in a power series of $\rho_k$ and take the CRs $[T_0(\rho), v^{(i)}_k]$ as follows:

$$\begin{align*}
[v^{(i)}_k, T_0(\rho)] &= \hbar k_i C_1(k) + 2\hbar k_i \sum_{\mathbf{p} \neq 0} C_2(\mathbf{p}; k) \rho_{\mathbf{p}} + \cdots, \\
[C_2(\mathbf{p}; k), T_0(\rho)] &= 2\hbar^2 k_i^2 C_2(\mathbf{k}'; k) + 6\hbar^2 k_i^4 \sum_{\mathbf{p} \neq 0} C_3(\mathbf{p}; \mathbf{k}'; k) \rho_{\mathbf{p}} + \cdots.
\end{align*} \quad (3.5)$$

From (2.5), we have a discrete integral equation

$$[v^{(i)}_k, T_0(\rho)] = [g^{(i)}_k, T_0(\rho)] - \frac{1}{2\sqrt{mN}} \sum_{\mathbf{p} \neq 0} \rho_{\mathbf{p}+\mathbf{k}} [v^{(i)}_k, T_0(\rho)]. \quad (Denote [g^{(i)}_k, T_0(\rho)] as f^{(i)}(\rho; \mathbf{k})]. \quad (3.6)$$

With the aid of (3.4) and using two CRs of (2.4), $f^{(i)}(\rho; \mathbf{k})$ is calculated approximately as

$$f^{(i)}(\rho; \mathbf{k}) \approx [g^{(i)}_k, T] - \frac{\hbar}{mN} \sum_{\mathbf{p} \neq 0} \rho_{\mathbf{p}+\mathbf{k}} [g^{(i)}_k, \rho_{\mathbf{p}+\mathbf{k}}] v_{\mathbf{p}} \cdot v_{\mathbf{q}} - \frac{1}{2m} \sum_{\mathbf{p} \neq 0} [g^{(i)}_k, v_{\mathbf{p}} \cdot v_{-\mathbf{p}}]
$$

$$= [g^{(i)}_k, T] - \frac{\hbar}{mN} \sum_{\mathbf{p} \neq 0} \rho_{\mathbf{p}+\mathbf{k}} [v^{(i)}_k, T_0(\rho)]\left\{ (k \cdot v_{\mathbf{p}}) v_{\mathbf{q}} + k_i v_{\mathbf{p}} \cdot v_{\mathbf{q}} \right\}
$$

$$- \frac{\hbar}{2m \sqrt{N}} \sum_{\mathbf{p} \neq 0} \left\{ (k \cdot v_{\mathbf{p}}) v_{\mathbf{k}+\mathbf{p}} + v_{\mathbf{k}+\mathbf{p}} (k \cdot v_{\mathbf{p}}) - p_i v_{\mathbf{p}} \cdot v_{\mathbf{k}+\mathbf{p}} + p_i v_{\mathbf{p}} \cdot v_{\mathbf{k}+\mathbf{p}} \right\}
$$

$$= [g^{(i)}_k, T] - \Lambda^{(i)}_k \equiv \Lambda^{(i)}_k = \frac{\hbar}{mN} \sum_{\mathbf{p} \neq 0} \rho_{\mathbf{p}+\mathbf{k}} (k \cdot v_{\mathbf{p}}) v_{\mathbf{q}}, \quad (3.7)$$

where

$$[g^{(i)}_k, T] = \frac{\hbar^3}{2m \sqrt{N}} \sum_{\mathbf{p} \neq 0} \left\{ \left( p_i + \frac{k_i}{2} \right) a^\dagger_{\mathbf{p}} a_{\mathbf{p}+\mathbf{k}} + \frac{\hbar^3}{2mN} \sum_{\mathbf{p} \neq 0} \rho_{\mathbf{p}+\mathbf{k}} a^\dagger_{\mathbf{p}} \cdot k_i a_{\mathbf{p}+\mathbf{k}}, \quad (3.8)$$

$$\Lambda^{(i)}_k \equiv \frac{\hbar}{2m \sqrt{N}} \sum_{\mathbf{p} \neq 0} \rho_{\mathbf{p}+\mathbf{k}} (k \cdot v_{\mathbf{p}}) v_{\mathbf{k}+\mathbf{p}} + v_{\mathbf{k}+\mathbf{p}} (k \cdot v_{\mathbf{p}}) - p_i v_{\mathbf{p}} \cdot v_{\mathbf{k}+\mathbf{p}} + p_i v_{\mathbf{p}} \cdot v_{\mathbf{k}+\mathbf{p}} \right\}. \quad (3.10)$$

From now on, using $\rho_0 = \sqrt{N}$, we make approximations for $v_{\mathbf{k}}, \rho_{\mathbf{k}}, a_0$ and $a^\dagger_0$ as

$$v_{\mathbf{k}} \approx \frac{\hbar k}{2} (\theta a_{\mathbf{k}} - a^\dagger_{-\mathbf{k}}), \quad \rho_{\mathbf{k}} \approx \theta a_{\mathbf{k}} - a^\dagger_{-\mathbf{k}}, \quad a_0 \approx \sqrt{N} \theta, \quad a^\dagger_0 \approx \sqrt{N} \theta, \quad (3.11)$$

where $\theta$ and $\bar{\theta}$ are the Grassmann numbers [9]. Using $\rho_{\mathbf{p}+\mathbf{k}} \approx \sqrt{N} \delta_{\mathbf{q}, \mathbf{k}+\mathbf{p}}$, $\Lambda^{(i)}_k$ in (3.7) is computed as

$$\Lambda^{(i)}_k = \frac{\hbar^3}{4m \sqrt{N}} \sum_{\mathbf{p} \neq 0} k \cdot p (k_i - p_i) \left( \theta a_{\mathbf{k}} - a^\dagger_{-\mathbf{k}} \right) \left( \theta a_{\mathbf{p}} - a^\dagger_{-\mathbf{p}} \right)
$$

$$= \frac{\hbar^3}{4m \sqrt{N}} \sum_{\mathbf{p} \neq 0} k \cdot p (k_i - p_i) \rho_{\mathbf{p}+\mathbf{k}} + \theta \theta \frac{\hbar^3}{m} \left( k_i \cdot k_i - k^2 \right) \rho_{\mathbf{k}}, \quad (3.12)$$
where the last term in the last line is obtained by extracting the terms with $p = \frac{k}{2}$ or $p = -\frac{k}{2}$ in the $\theta \theta$ term of the third line. As is clear from its structure, it evidently vanishes. The remaining terms in the $\theta \theta$ term are small and can be neglected. The terms $\Lambda^{2(i)}_{k}$ and $\Lambda^{3(i)}_{k}$ in (3.12) and (3.11), which have never been seen in the previous paper [5], are also calculated similarly as the above, respectively and are given as

$$\Lambda^{2(i)}_{k} = \frac{h^{3}k_{i}}{4m\sqrt{N}} \sum p \cdot (k-p) \cdot p \left( \theta a_{k-p} - a_{(k-p)}^{\dagger} \right) \left( \theta a_{p} - a_{p}^{\dagger} \right)$$

(3.13)

$$\Lambda^{3(i)}_{k} = -\frac{h^{3}}{4m\sqrt{N}} \sum p \cdot (k-p) \rho \cdot p \cdot k - \theta \theta \frac{3h^{3}k_{i}^{2}}{16m} \rho \cdot k,$$

(3.14)

here in (3.13) we use the relation $\theta \theta = 1$.

Substituting $[g^{(i)}_{k}, T] = \frac{h^{3}k_{i}^{2}}{4m} \rho \cdot k$ and $\Lambda^{(i)}_{k}$'s, we get an approximate formula for $f^{(i)}(\rho; k)$ as

$$f^{(i)}(\rho; k) = \frac{7h^{3}k_{i}^{2}}{16m} \rho \cdot k - \frac{h^{3}}{4m\sqrt{N}} \sum p \cdot k \left( (k_{i}+p_{i}) \cdot k \cdot p - p_{i} (k+p) \cdot p \right) \rho \cdot p \rho \cdot p \cdot k$$

(3.15)

$$= \frac{7h^{3}k_{i}^{2}}{16m} \rho \cdot k - \frac{h^{3}}{4m\sqrt{N}} \sum p \cdot k \left( (k_{i}+p_{i}) \cdot k \cdot p - p_{i} (k+p) \cdot p \right) \rho \cdot p \rho \cdot p \cdot k.$$  

In the last line, the numerical factor $\frac{7}{4}$ and the term $-p^{2}(k_{i}-p_{i})$, which have also never been seen in the previous paper [5], arise due to the consideration of the terms $\Lambda^{2(i)}_{k}$ and $\Lambda^{3(i)}_{k}$. Further substituting (3.15) into (3.6) and picking up the next leading term, we have

$$\left[ v^{(i)}_{k}, T_{0}(\rho) \right] \approx \frac{7h^{3}k_{i}^{2}}{16m} \rho \cdot k - \frac{h^{3}}{4m\sqrt{N}} \sum p \cdot k \left( (k_{i}+p_{i}) \cdot k \cdot p - p_{i} (k+p) \cdot p \right) \rho \cdot p \rho \cdot p \cdot k.$$  

(3.16)

To keep a favored form $\sum p \cdot k \left( (k_{i}+p_{i}) \cdot k \cdot p - p_{i} (k+p) \cdot p \right)$ occurred in [5], in the curly brackets, we had better to treat $k_{i}-p_{i}$ and $\frac{7}{4}p_{i}$ equally and to lead $p \cdot (p+k)$. So, we take $p_{i} = \frac{8}{15}k_{i}$ and then $k_{i}-p_{i} = \frac{7}{15}k_{i}$. Thus, we have a final expression for $\left[ v^{(i)}_{k}, T_{0}(\rho) \right]$ up to the order of $\frac{1}{\sqrt{N}}$ as

$$\left[ v^{(i)}_{k}, T_{0}(\rho) \right] = \frac{7h^{3}k_{i}^{2}}{4m} \rho \cdot k - \frac{14h^{3}k_{i}}{15} \frac{8m\sqrt{N}}{N} \sum p \cdot k \left( k_{i}^{2} \cdot p \cdot (p+k) \rho \cdot p \rho \cdot p \cdot k. \right.$$  

(3.17)

From (3.16) and (2.6), we get the following CRs between $v^{(i)}_{k}$ and $T_{0}(\rho)$:

$$\left[ v^{(i)}_{k}, v^{(i)}_{k}, T_{0}(\rho) \right] = -\frac{7h^{4}k_{i}k_{j}'}{4m} \delta_{k'} \cdot k - \frac{14h^{4}k_{i}k_{j}'}{15m\sqrt{N}} \left( k^{2}+k \cdot k'+k'^{2} \right) \rho \cdot k \cdot k'$$

(3.18)

$$\left[ v^{(i)}_{k}, v^{(i)}_{k}, v^{(i)}_{k}, T_{0}(\rho) \right] = \frac{14h^{5}k_{i}k_{j}'}{15} \frac{8m\sqrt{N}}{N} \left( k^{2}+k \cdot k'+k'^{2} \right),$$

$$\left[ v^{(i)}_{k}, v^{(i)}_{k}, T_{0}(\rho) \right] = 0.$$
By the procedure similar to the previous one, we can determine the coefficients $C_2$ and $C_3$ in (3.5) as

$C_2(k; k) = \frac{7}{152m} \delta_{k', -k}$ and $C_3(k'; k; k) = -\frac{7}{152m} \sqrt{N} (k^2 + k^2 + k') \delta_{k, -k'}$.

which are the same forms as the Sunakawa’s $C_2$ and $C_3$ except having different numerical factors. This is also due to the consideration of the terms $\Lambda^{(2)}_k$ and $\Lambda^{(3)}_k$. Then get an approximate form of $T_0(\rho)$ in terms of variables $\rho_k$ as

$T_0(\rho) = C_0 + \frac{7}{152m} \sum_{k \neq 0} k^2 \rho_k \rho_{-k} - \frac{14}{24mN} \sum_{p \neq 0, q \neq 0} (p^2 + q^2) \rho_p \rho_q \rho_{-p} \rho_{-q}. \quad (3.19)$

Substituting (3.19) into (3.24) and using the underlying identity,

$\sum_{p \neq 0, q \neq 0, p + q \neq 0} (p^2 + q^2) \rho_p \rho_q \rho_{-p} \rho_{-q} = \sum_{p \neq 0, q \neq 0} p \rho_p \rho_q \rho_{-p} \rho_{-q} = \sum_{p \neq 0, q \neq 0} p \cdot q \rho_p \rho_q \rho_{-p} \rho_{-q}, \quad (3.20)$

the constant term $C_0$ is computed as

$C_0 = T - \frac{7}{152m} \sum_{k \neq 0} k^2 \rho_k \rho_{-k} - \frac{1}{2m} \sum_{k} v_k \cdot v_{-k} - \frac{1}{2mN} \sum_{p + q \neq 0} \rho_p \rho_q \rho_{-p} \rho_{-q}. \quad (3.21)$

Using (3.11), we can calculate the third term in (3.21) and then reach to a result such as

$\frac{1}{2m} \sum_{k} v_k \cdot v_{-k} = \frac{\hbar^2}{2m} \sum_{k} k^2 (\bar{\rho}_a k - a^\dagger k \theta) \bar{\rho}_a k - a^\dagger k \theta )$ \quad (3.22)

As for the forth and last terms in (3.21), they become vanishing due to $\theta \theta = 0$ and $\bar{\theta} \bar{\theta} = 0$. Substituting these into (3.21) and replacing $\rho_k \rho_{-k}$ by $<\rho_k \rho_{-k}>_{Ave}$, we have

$C_0 = (1 - \bar{\theta} \theta) T + \bar{\theta} \theta \frac{\hbar^2}{2m} \sum_{k} k^2 - \frac{3\hbar^2}{32m} \sum_{k} k^2 \rho_k \rho_{-k} \equiv \sum_{k} \frac{h^2 k^2}{2m} - \frac{3\hbar^2}{32m} \sum_{k} k^2 <\rho_k \rho_{-k}>_{Ave}. \quad (3.23)$

where the quantity $<\rho_k \rho_{-k}>_{Ave}$ stands for the average value of $\rho_k \rho_{-k}$ and we have used the relation $\bar{\theta} \theta = 1$. It is surprising to see that the $C_0$ coincides with the constant term in the ground-state energy given by Tomonaga [11] except the last term. Using (3.24), (3.19) and (3.23), we can express the Hamiltonian $H(= H^I + H^II)$ in terms of $\rho_k$ and $v_k$ as follows:

$H^I = \frac{N(N-1)}{2\Omega} \nu(0) + \sum_{k \neq 0} \left[ -\sqrt{\frac{\hbar^2 k^2}{2m}} - \sqrt{\frac{\hbar^2}{4m} \nu(k)} + \frac{1}{2m} \sum_{k} k^2 v_k \cdot v_{-k} + \left( \frac{7\hbar^2 k^2}{32m} + \frac{N}{2\Omega} \nu(k) \right) \rho_k \rho_{-k} \right] \quad (3.24)$

$H^II = \frac{1}{2mN} \sum_{p + q \neq 0} \rho_p \rho_q \rho_{-p} \rho_{-q} + \frac{14}{152mN} \sum_{p \neq 0, q \neq 0} \rho_p \rho_q \rho_{-p} \rho_{-q} \quad (3.24)$

which is similar to the Sunakawa’s Hamiltonian [3] except the term $<\rho_k \rho_{-k}>_{Ave}$ in $H^I$.

Now, let us introduce the Boson annihilation and creation operators defined as

$\alpha_k = \sqrt{\frac{m E_k}{2m k^2 E_k}} \rho_k \rho_{-k} + \frac{1}{\sqrt{2m k^2 E_k}} k \cdot v_k; \quad \alpha_k^\dagger = \sqrt{\frac{m E_k}{2m k^2 E_k}} \rho_k \rho_{-k} + \frac{1}{\sqrt{2m k^2 E_k}} k \cdot v_{-k}, \quad (k \neq 0). \quad (3.25)$
Using (3.25), \( \lambda_k = \frac{\hbar^2 k^2}{2mE_k} \) and (3.11), the collective variables \( \rho_k \) and \( \nu_k (k \neq 0) \) are expressed as

\[
\rho_k = \frac{N}{2 \Omega} \frac{\hbar k}{2\sqrt{\lambda_k}} \left( \alpha_{-k} + \alpha_{-k}^\dagger \right), \quad
\nu_k = \frac{\hbar k}{2\sqrt{\lambda_k}} \left( \alpha_{k} - \alpha_k^\dagger \right) \approx \frac{\hbar k}{2} \left( \bar{\alpha}_{k} - \alpha_{-k}^\dagger \right).
\]

The \( H^I \) in (3.24) is considered as the lowest order Hamiltonian within the scope of the present approximation. With the use of an inverse transformation of (3.25), the lowest order Hamiltonian \( H^I \) denoted as \( H_0 \) is diagonalized as

\[
H_0 = E_0^G + \sum_{k \neq 0} E_k \alpha_k^\dagger \alpha_k^\dagger,
\]

\[
E_0^G = \frac{N (N - 1)}{2 \Omega} \nu(0) + \frac{1}{2} \sum_{k \neq 0} \left( E_k - \frac{\sqrt{7} \hbar^2 k^2}{4m} - \frac{2 \sqrt{7} N}{7 \Omega} \nu(k) - \frac{3 \hbar^2 k^2}{16m} < \rho_k \rho_{-k}>_{Ave} \right),
\]

where \( E_k \) is the quasi-particle energy but with the modified single-particle energy \( \frac{\sqrt{7}}{2} \varepsilon_k \) in the lowest approximation and we have used \( [\nu_k, \rho_k ] = \hbar k \). In this sense, the zero point energy of the collective mode is included in the above diagonalization. The quantity \( E_0^G \) in (3.27) corresponds to the lowest order ground-state energy \([10, 6]\) but with the average value \( < \rho_k \rho_{-k}>_{Ave} \). The part of the Hamiltonian, \( H^II \), is also represented in terms of the Boson operators \( \alpha_k \) and \( \alpha_k^\dagger \) though we omit here its explicit expression. Further we have a Bogoliubov transformation for Boson-like operators \( \bar{\alpha}_k \) and \( \alpha_k^\dagger \) as

\[
\alpha_k = \frac{(E_k + \varepsilon_k) \bar{\alpha}_k + (E_k - \varepsilon_k) \alpha_k^\dagger}{2 \sqrt{\varepsilon_k E_k}}, \quad \alpha_{-k}^\dagger = \frac{(E_k - \varepsilon_k) \bar{\alpha}_k + (E_k + \varepsilon_k) \alpha_k^\dagger}{2 \sqrt{\varepsilon_k E_k}},
\]

which already appeared in the preceding papers \([4, 5]\) and is of the same form as the Bogoliubov transformation for the usual Bosons \([6]\) except the introduction of the Grassmann numbers \( \theta \) and \( \bar{\theta} \). The above kind of the diagonalization (3.27) has also been given by Sunakawa in the Boson system \([11]\).

4 Discussions and further outlook

In this paper, we have proposed a velocity operator approach to a 3D Fermion system. After introducing collective variables, the velocity operator approach to the 3D Fermion system could be provided. Particularly, an interesting problem of describing elementary excitations occurring in a strongly correlated system may be possibly treated as an elementary exercise. For such problems, for example, see textbooks \([12, 13]\). By applying the velocity operator approach to such a problem, an excellent description of the elementary excitations in the 3D Fermion system will be expected to reproduce various possible behaviors including excited energies. Because the present theory is constructed to take into account important many-body correlations which have not been investigated sufficiently for a long time in the historical ways for such a problem. Here the velocity operator \( v_k^{(i)} \) is defined through the discrete integral equation which is necessarily accompanied by the inhomogeneous term \( f^{(i)}(\rho; k) \). It is a very important task how to evaluate this term which brings an essential effect to the construction of the theory for the velocity operator approach. More precisely, in the last line
of (3.7), we have taken into account the contributions from all of the second term $\Lambda^{(2)}_{k}$, the third one $\Lambda^{(3)}_{k}$ and the forth one $\Lambda^{(4)}_{k}$. Such an evaluation is quite different from the manner adopted in the preceding papers \[4, 5\] in which only the term $\Lambda^{(2)}_{k}$ had been taken into account. As a result, the present evaluation leads to another approximate formula for $f^{(i)}(\rho; k)$ and another final form of $[v_{k}^{(i)}(\theta), T_{0}(\rho)]$ which are of course different from those given in those papers. Under the use of the CRs (3.18) between $v_{k}^{(i)}$ and $T_{0}(\rho)$, thus, the correct expression for $T_{0}(\rho)$ could be determined in terms of $v_{k}$ and the constant $C_{0}$ appearing in the expansion of a modified $T_{0}(\rho)$ turns out to involve the term $<\rho_{k}\rho_{-k}>_{Ave}$ which is not determined yet.

However, now, making use of the inverse transformation of (3.25), we determine a notable term as $<\rho_{k}\rho_{-k}>_{Ave} = \frac{\hbar^{2}k^{2}}{2mE_{k}} <(\alpha_{-k}^{\dagger}\alpha_{k}^{\dagger})(\alpha_{k}\alpha_{-k}^{\dagger})>_{Ave} = \frac{\hbar^{2}k^{2}}{2mE_{k}} = \frac{\varepsilon_{k}}{E_{k}}$. Using the last two energies $E_{k}$ and $\varepsilon_{k}$ in (3.27), here we approximate $E_{k}$ and the term $-\frac{3\hbar^{2}k^{2}}{16m}<\rho_{k}\rho_{-k}>_{Ave}$ in $E_{0}^{G}$ of (3.27) as $\frac{\sqrt{7}}{2}\varepsilon_{k}$ and $-\frac{3\sqrt{7}}{28}\varepsilon_{k}$, respectively. Then the lowest order ground-state energy $E_{0}^{G}$ is rewritten as $E_{0}^{G} = \frac{N(N-1)}{2\Omega} \nu(0) + \frac{1}{2} \sum_{k \neq 0} \left( E_{k} - \frac{17\sqrt{7}}{28} \varepsilon_{k} - \frac{2\sqrt{7}}{7} \frac{m}{\Omega} \nu(k) \right)$. Further we derive the Bogoliubov transformation for Boson-like operators $\overline{\theta}a_{k}$ and $a_{k}^{\dagger}\theta$ which already appeared in the preceding papers \[4, 5\] and is the same form as the famous Bogoliubov transformation for the usual Bosons \[7\] but which brings the modified single-particle energy $\sqrt{\frac{7}{2}}\varepsilon_{k}$ in the quasi-particle excitation energy $E_{k}$ in the lowest order approximation. Thus, we could provide the modified and then the correct theory for the velocity operator approach. Based on such approach, it is expected that a new field of exploration of elementary excitation in a 3D Fermi system may open up with aid of the velocity operator approach whose new development may appear elsewhere.

The connection of the present theory with the fluid dynamics was been mentioned briefly by Sunakawa. Restrict the Hilbert space to a subspace in which the vortex operator satisfies $\text{rot} v(x) = 0, (k \times v_{k}) = 0$ leading effectively to $[v_{k}^{(i)}, v_{k}^{(j)}] = 0$. He transformed the quantum-fluid Hamiltonian (3.24) to one in configuration space and obtained the classical-fluid Hamiltonian for the case of irrotational flow \[7, 11\]. From this view point, very recently, we have developed a quantum-fluid approach to vortex motion in nuclei \[14\]. In this work we have introduced Clebsch parameterization making it possible to describe canonically quantum-fluid dynamics and have done Clebsch-parameterized gauge potential possessing Chern-Simons number (quantized helicity) \[15, 16, 17\]. A study of Chern-Simons form shed new light on the aspect of the behavior of the quantum fluid.

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References

[1] S. Tomonaga, *Prog. Theor. Phys.* **5**, 544 (1950).

[2] V.J. Emery, Theory of the One-Dimensional Electron Gas in *Highly Conducting One-Dimensional Solids*, Editors, J.T. Devreese, R.P. Evrard and V.E. van Dore, Physics of Solids and Liquids, Springer US, 1979, pp 247-303.

[3] S. Sunakawa, Y. Yoko-o and H. Nakatani, *Prog. Theor. Phys.* **27**, 589 (1962).

[4] S. Nishiyama and J. da Providência, *Int. J. Mod. Phys.* E**24**, 1550045 (2015).

[5] S. Nishiyama and J. da Providência, *Int. J. Mod. Phys.* E**25**, 1650057 (2016).

[6] N.N. Bogoliubov, *J. Phys.* **11**, 23 (1947).

[7] S. Sunakawa, Y. Yoko-o and H. Nakatani, *Prog. Theor. Phys.* **27**, 600 (1962).

[8] L. Landau, *J. Phys.* **5**, 71 (1941).

[9] F.A. Berezin, *The Method of Second Quantization*, Academic Press, New York and London, 1966.

[10] N.N. Bogoliubov and D.N. Zubarev, *Soviet Phys. JETP* **1**, 83 (1955).

[11] S. Sunakawa, Y. Yoko-o and H. Nakatani, *Prog. Theor. Phys.* **28**, 127 (1962).

[12] P. Ring and P. Schuck, *The nuclear many body problem*, Texts and monographs in Physics (Springer-Verlag, Berlin, Heidelberg and New York 1980).

[13] J.W. Negele and H. Orland, *Quantum Many-Particle Systems*, Frontiers in Physics, Volume 68, Addison-Wesley Publishing Company, 1988.

[14] S. Nishiyama and J. da Providência, *Int. J. Mod. Phys.* E**26**, 1750020 (2017).

[15] A. Jackiw and S.-Y. Pi, *Phys. Rev.* D**61**, 105015 (2000).

[16] A. Jackiw, V.P. Nair and S.-Y. Pi, *Phys. Rev.* D**62**, 085018 (2000).

[17] A. Jackiw, CRM Series in Mathematical Physics, *Lectures on fluid dynamics*, A Particle Theorist’s View of Supersymmetric, Non-Abelian, Noncommutative Fluid Mechanics and d-Branes, 2002 Springer-Verlag, New York, Inc.