Regularity results and parametrices of semi-linear boundary problems of product type.

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Dedicated to Prof. Hans Triebel on the occasion of his 65th birthday

1 Introduction

This study focuses on semi-linear problems of the form
\[ Au + N(u) = f \quad \text{in} \quad \Omega \]
\[ Tu = \varphi \quad \text{on} \quad \Gamma := \partial \Omega. \]

Here \((f, \varphi)\) are the given data, and \(u\) the unknown. Problem (1) should be elliptic in some bounded, \(C^\infty\)-smooth region \(\Omega \subset \mathbb{R}^n\): that is \(A\) should be a linear differential operator in \(\Omega\) while \(T\) should be a trace operator such that the system \(\{A, T\}\) is elliptic in \(\Omega\). More generally, \(A\) could be suitably “pseudo-differential” as long as \(\{A, T\}\) is injectively elliptic in the Boutet de Monvel calculus of boundary problems. \(N(u)\) stands for a non-linearity which combines \(u(x)\) and its derivatives \(D^\alpha u\) in a polynomial way, roughly speaking.

The main point is the following frequently asked question: given a solution \(u\), does the presence of \(N(u)\) influence the regularity of \(u\)?

This problem can of course be phrased in various frameworks: to measure regularity, the Besov and Triebel–Lizorkin spaces \(B^s_{p,q}\) and \(F^s_{p,q}\) could be adopted for \(s \in \mathbb{R}\) and \(p, q \in [0, \infty]\) (though with \(p < \infty\) for \(F^s_{p,q}\)). But to simplify matters—and indeed to fix ideas—this survey deals with the Sobolev, or Bessel potential, spaces \(H^s_p(\Omega)\), where \(s \in \mathbb{R}\) and \(1 < p < \infty\). Now the solution may be known to exist in some a priori space, denoted \(H^s_{p_0}\) throughout, while data are given in other spaces having some integral exponent \(r \in [1, \infty]\). The case with \(r \neq p_0\) requires extra efforts, and the present paper deals with a flexible way of handling this.

The word “semi-linear” is often taken as an indication that solutions of problems like (1) will have practically the same regularity as in the linear case, ie as when \(N \equiv 0\). However, when \(r \neq p_0\) is allowed, it is more demanding to describe for which a priori spaces and data spaces this property of semi-linearity holds.

A classical way to obtain such conclusions is to improve the knowledge of \(u\) in finitely many steps (ie a boot-strap method). But one faces rather paining-taking difficulties when this method is applied to cases in which the a priori space for \(u\) is not “close enough” to the solution space associated to the data \(f\) and \(\varphi\) in the linear theory. (Such phenomena have been described in [Joh93, Joh95b] and in a joint work with T. Runst [JR97]).

However, in a recent article [Joh01] a different technique was worked out—it requires rather weaker assumptions than boot-strap methods do, it has cleaner proofs and in particular it also avoids the technicalities mentioned above. In short this approach is a much more flexible tool. It was exemplified for elliptic problems in full generality in [Joh01], where the crucial point was a specific parametrix formula...
for the non-linear problem (1); this formula is useful because one can read off a
given solution’s regularity directly.

The purpose of the present paper is to give a concise account of the resulting
technique and to present how the parametrices straightforwardly give regularity
improvements.

To give a very brief account of the outcome of the study (with examples to follow
further below), it is useful to introduce three parameter domains:

\[ D(A), \quad D(N), \quad D(L_u). \]  

(2)

Here \( D(A) \) consists of all the (pairs of) parameters \((s, p)\) for which the matrix-
formed operator \( A := (\frac{A^T}{2}) \) is defined on the space \( H^s \). This takes into account the
class of \( T \) (and of \( A \) in the pseudo-differential case).

Similarly \( D(N) \) contains all the \((s, p)\) for which \( N \) is defined on \( H^s \) and has
order less than that of \( A \) on this space. Finally, and most importantly, for any
\((s_0, p_0) \in D(N) \) and any given \( u \) in \( H^{s_0} \), there should exist some linear but possibly
\( u \)-dependent operator \( L_u \) such that

\[ N(u) = -L_u(u). \]  

(3)

When the operator \( L_u \) is studied in its own right on the scale \( H^s \), then \( D(L_u) \) contains the \((s, p)\) for which \( L_u \) is defined and has lower order than \( A \). In addition it is necessary to require of \( N \) and \( L_u \) that
\( D(N) \subset D(L_u) \).

In practice \( D(L_u) \) is much larger than \( D(N) \), and the more regular \( u \) is known
a priori to be, the larger \( D(L_u) \) will be. (While \( D(N) \) is the same independently of
any given solution \( u \).) This leads to a main feature:

On the one hand, using a boot-strap method it turns out that one can work inside
the domain \( D(A) \cap D(N) \); which is logical because \( N \) would lose “more derivatives”
on spaces outside \( D(N) \). On the other hand, the present parametrix methods work
well on the larger set

\[ D_u := D(A) \cap D(L_u). \]  

(4)

For this reason a given solution may be treated under much weaker initial assump-
tions on the data \((f, \varphi)\). Indeed, the regularity of \( u \) is read off from the following
parametrix formula (derived in [Joh01])

\[ u = P_u(N)(Rf + K\varphi) + Ru + (RL_u)^N u. \]  

(5)

Here \( (R, K) \) denotes a left-parametrix of \((\frac{A^T}{2})\), with associated smoothing operator
\( \mathcal{R} \); that is \( RA + KT = I - \mathcal{R} \). The parametrix \( P_u(N) \) is a finite Neumann series
with the linear operator \( RL_u \) as ‘quotient’. Consequently, with known mapping
properties of \( R, K \) and \( L_u \), the above formula (5) shows directly how the regularity
of \( u \) is determined by the data together with the a priori regularity of \( u \) itself (the
latter enters the term \((RL_u)^N) \).

Below this is explained in detail by means of an example.

2 The Framework

As a another simplification we may consider the following model problem, which is
rich enough to illustrate the points. Here and below \( \gamma_0 \) denotes the standard trace
(restriction) on \( \Gamma \):

\[ -\Delta u + u \partial_{x_1} u = f \quad \text{in} \quad \Omega \]
\[ \gamma_0 u = \varphi \quad \text{on} \quad \Gamma. \]  

(6)
The discussion of (6) will be carried out under the hypothesis that a solution \( u \) is given for some specific data \((f, \varphi)\) fulfilling

\[
\begin{align*}
  u &\in H^s_{p_0}(\Omega), \\
  f &\in H^{s-2}_t(\Omega), \\
  \varphi &\in B^{s-1/r}_{t,r}(\Gamma).
\end{align*}
\]

In general, the space \( H^s_{p}(\Omega) \) is defined by restriction to \( \Omega \) and \( B^{s,p}_{t,r}(\Gamma) \) is defined via local coordinates on \( \Gamma \).

With this set-up, the theme is whether \( u \) belongs to the space \( H^t_r(\Omega) \) too. (There are of course necessary conditions for this to be true, e.g. \( s_0 > 1/p_0 \) must hold for the boundary condition to make sense. It is tempting to require in an analogy that \( t > 1/r \), but it is a point that weaker assumptions will suffice; hence this discussion is postponed a little.) Using boot-strap arguments in treating this, difficulties occur as mentioned in the introduction. Indeed, for cases with, say \( p_0 \) and \( r \) close to 1 and \( \infty \) respectively, and small values of \( s_0 > t \), boot-strapping is possible, but careful arguments based on special estimates of \( u \partial_1 u \) are needed to avoid auxiliary spaces on which \( \gamma_0 u \) is undefined; cf [Joh95b].

With a more direct approach, the aim is to “invert” (6) by means of the formula

\[
  u = P^N_u(RDf + KD\varphi) + Ru + (RDLu)^N u. \tag{9}
\]

To explain the various quantities in (9), it is first noted that the linear problem corresponding to (6) is considered as an equation for the elliptic Green operator (when \( s > 1/p, 1 < p < \infty \)),

\[
  \mathcal{A} = \begin{pmatrix} -\Delta \\ \gamma_0 \end{pmatrix} : H^s_p(\Omega) \rightarrow H^{s-2}_p(\Omega) \oplus B^{s-1/p}_{t,r}(\Gamma). \tag{10}
\]

Then \((RD\;KD)\) is taken as a parametrix (belonging to the Boutet de Monvel calculus), i.e., it is continuous in the opposite direction in (10) and

\[
  (RD\;KD) \begin{pmatrix} -\Delta \\ \gamma_0 \end{pmatrix} = I - \mathcal{R}; \tag{11}
\]

here the range \( \mathcal{R}(H^s_p) \subset C^\infty(\Omega) \) for all \( s > 1/p \). (In fact \( \mathcal{R} = 0 \) is possible for this Dirichlet problem, but it is retained here to make it clear that also in general its presence is harmless.)

The second ingredient in (9) is a decomposition of the non-linear term as

\[
  u \partial_1 u = -LU(u), \quad LU \text{ linear}. \tag{12}
\]

More precisely, it is necessary to ensure that \( LU \) has certain mapping properties, hence it is defined by means of a universal extension operator from \( \Omega \) to \( \mathbb{R}^n \), say \( \ell_\Omega \), to be

\[
  -LU(v) = \pi_1(\ell_\Omega u, \ell_\Omega \partial_1 v) + \pi_2(\ell_\Omega u, \ell_\Omega \partial_1 v) + \pi_3(\ell_\Omega v, \ell_\Omega \partial_1 u). \tag{13}
\]

Here the \( \pi_j(\cdot, \cdot) \) are para-multiplication operators defined on \( \mathbb{R}^n \) (so that restriction to \( \Omega \) of each term on the right-hand side of (13) is understood). These are introduced using a Littlewood–Paley partition \( 1 = \sum_{j=0}^\infty \Phi_j(\xi) \) with smooth functions \( \Phi_j \) supported at \( \{ 2^{j-1} \leq |\xi| \leq 2^{j+1} \} \) for \( j > 0 \); then

\[
  \pi_1(g, h) = \sum_{j=2}^\infty (\Phi_0(D) + \cdots + \Phi_{j-2}(D))g \cdot \Phi_j(D)h, \tag{14}
\]
and \( \pi_3(g, h) := \pi_1(h, g) \) whilst \( \pi_2(g, h) \) gives the remainder in the formal decomposition of \( g \cdot h \).

Using this, the parametrix \( P_{u}^{(N)} \) of the non-linear problem (6) are now finally introduced as

\[
P_{u}^{(N)} = I + R_{D}L_{u} + \cdots + (R_{D}L_{u})^{(N-1)}, \quad N \in \mathbb{N}.
\]

They clearly depend on the given solution \( u \), and since both \( R_{D} \) and \( L_{u} \) are linear, so are the \( P_{u}^{(N)} \) on every \( H_{p}^{s}(\Omega) \) with \( (s, p) \in \mathbb{D}(A) \cap \mathbb{D}(L) \); cf (10) ff.

For the above model problem, the parameter domains from the introduction are, when \( N(u) = u \partial_{1}u \) and \( t_{+} = \max(0, t) \) denotes the positive part,

\[
\mathbb{D}(A) = \{ (s, p) \mid s > 1/p \} \quad \text{(16)}
\]
\[
\mathbb{D}(N) = \{ (s, p) \mid s > \frac{1}{2} + \left( \frac{2}{p} - \frac{n}{2} \right)_{+}, \quad s > \frac{2}{p} - 1 \} \quad \text{(17)}
\]
\[
\mathbb{D}(L_{u}) = \{ (s, p) \mid s + s_{0} > 1 + \left( \frac{n}{p_{0}} + \frac{n}{p} - n \right)_{+} \} \quad \text{(18)}
\]

The two first restrictions make the trace \( \gamma_{0}u \) and the product \( u \cdot \partial_{1}u \) well defined on \( H_{p}^{s}(\Omega) \), whilst the second condition in (17) implies that for some \( s' > s - 2 \) it holds that \( u \partial_{1}u \in H_{p}^{s'}(\Omega) \) for every \( u \in H_{p}^{s}(\Omega) \).

More noteworthy is it that the requirement in (18) is effectively weaker the better the a priori knowledge of \( u \) is: for higher values of \( s_{0} \) or larger values of \( p_{0} \), more pairs \( (s, p) \) fulfil the inequality.

A closer analysis shows that \( L_{u} \) has order \( \omega \) in the sense that

\[
L_{u}: H_{p}^{s}(\Omega) \to H_{p}^{s-\omega}(\Omega) \quad \text{for all } (s, p) \in \mathbb{D}(L_{u}),
\]

\[
\omega = 1 + \left( \frac{n}{p_{0}} - s_{0} \right)_{+} + \varepsilon, \quad \varepsilon \geq 0.
\]

Here \( \varepsilon > 0 \) is only necessary for \( s_{0} = \frac{n}{p_{0}} \). If one removes \( \ell_{\Omega} \) and the restriction to \( \Omega \) from \( L_{u} \), then the resulting operator is in the Hörmander class \( \text{OPS}_{1,1}^{\varepsilon} (\mathbb{R}^{n} \times \mathbb{R}^{n}) \), leading to an analogous continuity property.

It is important to observe that with \( u \in H_{p_{0}}^{s_{0}}(\Omega) \) for some \( (s_{0}, p_{0}) \) in \( \mathbb{D}(N) \), it follows from (17) that the order of \( L_{u} \) satisfies \( \omega < 2 \) in (19). In other words, \( L_{u} \) loses fewer derivatives than \( \Delta \). It is a peculiar fact that \( L_{u} \), once \( u \) is chosen, actually has constant order on all spaces regardless of whether they have parameter inside or outside \( \mathbb{D}(N) \) (by comparison, the boundary of \( \mathbb{D}(N) \) contains points where \( N \) attains the order 2).

Using the above continuity results for \( R_{D}, K_{D} \) and \( L_{u} \), one can now show that the parametrix formula holds and derive the regularity results.

**Remark 2.1** The operator \( L_{u} \) in (13) differs from the linearisations in J. M. Bony’s work [Bon81] because the \( \pi_{2}- \) term is a part of the operator instead of being treated as a negligible error term. In the present context this has to be so, for it does occur that this term has a non-negligible regularity and in addition it would not be natural to violate the identity \( L_{u}u = u \partial_{1}u \). For this reason it is suggested that one could call \( L_{u} \) the full paralinearisation of \( u \partial_{1}u \).

Moreover, the perhaps more natural linearisations \( v \mapsto u \partial_{1}v \) and the differential \( u \partial_{1}v \) do not work in this context, because they are not moderate in the terminology of [Joh01]. Indeed, on \( H_{p}^{s} \) they have order equal to \( s - 1 - s_{0} \) for large \( s \), and this has no upper limits for \( s \to \infty \); unlike \( L_{u} \) that has constant order with respect to \( s \) as observed above.

**Remark 2.2** About the above results it should be mentioned that the properties of linear elliptic problems were deduced for the \( H_{p}^{s} \)-scale in full generality by G. Grubb [Gru90], who extended the Boutet de Monvel calculus to these spaces (and to the
classical Besov spaces). In particular, this implies (10), and the statements following it. (For the $B_{p,q}^s$ and $F_{p,q}^s$ scales there is a similar extension of the calculus in [Joh96], which applies to the present problems in the same way.) For introductions to the calculus the reader may consult [Gru97, Gru91].

In the definition of $L_u$, the universal extension operator was constructed by V. Rychkov [Ryc99b, Ryc99a]. He showed that $\ell_\Omega$ can be taken such that for all $s$ and $p$ it is continuous $\ell_\Omega: H^s_p(\Omega) \to H^s_p(\mathbb{R}^n)$ (21) and that $r_\Omega \ell_\Omega = I$ holds on $H^s_p(\Omega)$ (in fact it was carried out for the Besov and Triebel–Lizorkin scales).

The para-multiplication operators $\pi_j(\cdot, \cdot)$ in (13) follow M. Yamazaki [Yam86] in the notation and the definition. To prove (19)–(20) it suffices to combine (21) with standard estimates of the $\pi_j(\cdot, \cdot)$; these are essentially found in [Yam86], but for a proof the reader may consult [Joh01], which presents a general study of non-linear operators of product type (encompassing sums of terms $D^\alpha u \cdot D^\beta u$ and more general expressions). For a full set of estimates of para-multiplication operators proved directly (without the somewhat heavier paradifferential techniques in [Yam86]), the reader may e.g. consult [Joh95a, Th 5.1].

3 Results involving parametrices

We shall now proceed to state the results for the model problem, that one obtains from the parametrices. Recall that $A = (\gamma_0 - \Delta)$ and $N(u) = u \partial_\Omega u$. Furthermore $\mathbb{D}(A)$, $\mathbb{D}(N)$ and $\mathbb{D}(L_u)$ are given as in (10) if, so that they fulfil the conditions described after (2).

The first point is to establish that the parametrix formula (9) really holds, and to give basic properties of the entering operators.

**Theorem 3.1** Assume that (6) and (7)–(8) hold for parameters fulfilling

\[(s_0, p_0) \in \mathbb{D}(A) \cap \mathbb{D}(N)\]
\[(t, r) \in \mathbb{D}(A) \cap \mathbb{D}(L_u) =: \mathbb{D}_u.\]  

(22)

Then (9) holds, and for $P_u^{(N)}$ as in (15),

\[\forall N \in \mathbb{N}_0, \forall (s, p) \in \mathbb{D}_u: P_u^{(N)}: H^s_p(\Omega) \to H^s_p(\Omega),\]  

(23)

\[\exists N \in \mathbb{N}_0: (R_D L_u)^N: H^{s_0}_{p_0}(\Omega) \to H^t_r(\Omega).\]  

(24)

In these lines the arrows stand for continuous, linear maps.

Actually (24) holds for all sufficiently large values of $N$, but usually it is enough to have a single such $N$.

The above Theorem 3.1 is a special case of an abstract result proved in [Joh01, Th 2.2]. The proof is not difficult in itself; it is formulated for a general situation specified by some lengthy, but essentially rather mild conditions labelled (I)–(V) in [Joh01, Sect. 2], and that these are fulfilled for the model problem considered in this paper is a consequence of the above Section 2.

Now one immediately gets

**Corollary 3.2** If (6), (11)–(8) and (22) all hold, then $u$ is also an element of $H^t_r(\Omega)$.

It is a major point of the paper that, to prove this, one may take $N$ as in (24), then the properties in (24), (23), (19) together with formula (9) show that $u \in H^t_r(\Omega)$. 

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It deserves to be emphasised that $(t, r)$ is assumed to lie in $\mathbb{D}(L_w)$ but not necessarily in the smaller parameter domain for the non-linear term, $\mathbb{D}(N)$. For this reason it is possible to conclude that given solutions may belong to spaces that are beyond the reach of the boot-strap method.

4 Final remarks

This paper focuses on semi-linear elliptic boundary problems, and even specialises to a simple model problem in order not to burden the exposition; the possible extensions are many, but the reader should get a good impression of the possibilities from the above. Within the framework of elliptic problems some of the generalisations are indicated after [1], but it is also possible to include semi-linear elliptic systems like the stationary Navier–Stokes equation or von Karman’s equations. This requires an extended notion of product type non-linearities, defined on sections of vector bundles. The reader may consult [Joh01] for this.

The general study in [Joh01, Sect. 2] also allow some applications to parabolic initial-boundary problems with non-linearities of product type. However, when such problems are non-homogeneous the compatibility conditions on the data set severe restrictions to how much the regularity can be improved; but even so it should be possible to work out some results in this area.

Concerning the tools, it is on the one hand clear that it is a fine theory of linear elliptic problems that enter, namely that of the Boutet de Monvel calculus. On the other hand, the treatment of the non-linear terms is based on para-multiplication operators on $\mathbb{R}^n$. This technique was essentially introduced (independently) by J. Peetre and H. Triebel around 1976-77 [Pee76, Tri77, Tri78] in order to analyse the pointwise product. One way to sum up the present paper could be to say that para-multiplication also may enter in a crucial way in treatments of certain non-linear perturbations of elliptic boundary problems.

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