$O(\tilde{d}, \tilde{d})$ Transformations and 3D Black Hole

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Abstract

We generalize the results of a previous paper by one of the authors to show a relationship among a class of string solutions through $O(\tilde{d}, \tilde{d})$ transformations. The results are applied to a rotating black hole solution of three dimensional general relativity discussed recently. We extend the black hole solution to string theory and show its connection with the three dimensional black string with nonzero momentum through an $O(\tilde{d}, \tilde{d})$ transformation of the above type.

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The search for the classical solutions of string theory and general relativity in various space-time dimensions has been an active area of research. Recent developments in understanding the symmetries of the string effective actions has lead to considerable progress in this direction. It has been shown that the string effective action is invariant under an $O(\tilde{d}, \tilde{d})$ group of symmetry transformations $[1]$, when the background configuration is independent of $\tilde{d}$ coordinates $[2]$. These symmetries have been utilized for generating several new solutions of string effective action $[3]$.

In this connection, an $O(\tilde{d}, \tilde{d})$ transformation was given in ref.$[4]$ to show a relationship between the background configuration of the ungauged string actions to the gauged ones$[5]$. In particular, it was shown that such transformations can be used to go from the background fields of the ungauged actions to the ones which can be interpreted as black branes with zero linear momentum along a number of directions. In this paper, we show that the results of ref.$[4]$ can be generalized further so that the transformed solution corresponds to the branes with nonzero momentum along these directions.

Our work was motivated partly by the structure of the rotating black hole solution of three dimensional general relativity discussed recently in ref.$[6]$(see also $[7]$) which we extend to string theory by inclusion of antisymmetric tensor and dilaton backgrounds. As an application of the results of the previous paragraph, we show that the three dimensional rotating black hole transforms to the moving black string $[8]$ by an $O(2,2)$ transformation. We
also find that the momentum of the black string is proportional to the black hole angular momentum.

We start by reviewing some basic results. It was shown in refs.\[1\] that the string effective action and the equations of motion are invariant under an $O(\tilde{d}, \tilde{d})$ transformation which acts on the background fields as a linear transformation $M \rightarrow \Omega M \Omega^T, \Phi \rightarrow \Phi$ where

$$M = \begin{bmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{bmatrix},$$

and $\Phi = \phi - ln\sqrt{detG}$. $\Omega$ is an $O(\tilde{d}, \tilde{d})$ matrix such that $\Omega^T \eta \Omega = \eta$, where

$$\eta = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$ \hspace{1cm} (2)

Then in ref.\[4\], the $O(\tilde{d}, \tilde{d})$ transformation on a general background configuration in $(D+d)$-dimensional space-time, specified by the metric $(G^0_{D+d})$ and the antisymmetric tensor $(B^0_{D+d})$, of the form\[5\]:

$$G^0_{D+d} = \begin{pmatrix} \Gamma_s & \frac{1}{2} \Gamma_1^T & \frac{1}{2} \Gamma_2^T \\ \frac{1}{2} \Gamma_1 & I_d & \Sigma^T \\ \frac{1}{2} \Gamma_2 & \Sigma & I_d \end{pmatrix},$$

and

$$B^0_{D+d} = \begin{pmatrix} \Gamma_a & -\frac{1}{2} \Gamma_1^T & \frac{1}{2} \Gamma_2^T \\ \frac{1}{2} \Gamma_1 & 0 & \Sigma^T \\ -\frac{1}{2} \Gamma_2 & -\Sigma & 0 \end{pmatrix},$$

were studied. In eqns.(3) and (4), $\Gamma_{s,a} = \pm \frac{1}{2}(\Gamma \pm \Gamma^T)$, are (anti-)symmetric parts of a $(D - d) \times (D - d)$ matrix $\Gamma$. $\Gamma_{1,2}, \Sigma$ are $d \times (D - d)$ and $d \times d$
matrices respectively. The backgrounds in eqns.(3)-(4) depend only on the first \((D - d)\) coordinates \(x^i \{ i = 1, \cdots, (D - d) \}.\) It was shown that the above backgrounds transform by an \(O(\bar{d}, \bar{d})\) \((\bar{d} = 2d)\) transformation of the form,

\[
\Omega^0 = \begin{pmatrix}
I_{D-d} & 0 & 0 & 0 & 0 \\
0 & \frac{I_d}{2} & -\frac{I_d}{2} & 0 & -\frac{I_d}{2} \\
0 & \frac{I_d}{2} & \frac{I_d}{2} & 0 & \frac{I_d}{2} \\
0 & 0 & 0 & I_{D-d} & 0 \\
0 & -\frac{I_d}{2} & -\frac{I_d}{2} & 0 & \frac{I_d}{2} \\
\end{pmatrix},
\]

(5)

to the following metric and antisymmetric tensor fields,

\[
\tilde{G}^0_{D+d} = \begin{pmatrix} G^0_D & 0 \\ 0 & I_d \end{pmatrix},
\]

(6)

\[
\tilde{B}^0_{D+d} = \begin{pmatrix} B^0_D & 0 \\ 0 & 0 \end{pmatrix},
\]

(7)

where

\[
G^0_D = \begin{pmatrix}
\left[ \Gamma_s - \frac{1}{4} \Gamma_1^T (I_d + \Sigma)^{-1} \Gamma_2 \right] & \frac{1}{2} \left[ \Gamma_1^T (I_d + \Sigma)^{-1} \right] \\
-\frac{1}{4} \Gamma_2^T (I_d + \Sigma^T)^{-1} \Gamma_1 & -\Gamma_2^T (I_d + \Sigma^T)^{-1} \\
\frac{1}{2} \left[ (I_d + \Sigma^T)^{-1} \Gamma_1 \right] & \frac{1}{2} \left[ (I_d - \Sigma)(I_d + \Sigma)^{-1} \right] \\
-(I_d + \Sigma)^{-1} \Gamma_2 & +(I_d + \Sigma^T)^{-1} (I_d - \Sigma^T) \end{pmatrix},
\]

(8)

\[
B^0_D = \begin{pmatrix}
\left[ \Gamma_a + \frac{1}{4} \Gamma_1^T (I_d + \Sigma)^{-1} \Gamma_2 \right] & -\frac{1}{2} \left[ \Gamma_1^T (I_d + \Sigma)^{-1} \right] \\
-\frac{1}{4} \Gamma_2^T (I_d + \Sigma^T)^{-1} \Gamma_1 & +\Gamma_2^T (I_d + \Sigma^T)^{-1} \\
\frac{1}{2} \left[ (I_d + \Sigma^T)^{-1} \Gamma_1 \right] & \frac{1}{2} \left[ -(I_d - \Sigma)(I_d + \Sigma)^{-1} \right] \\
+(I_d + \Sigma)^{-1} \Gamma_2 & +(I_d + \Sigma^T)^{-1} (I_d - \Sigma^T) \end{pmatrix},
\]

(9)

Corresponding dilaton field transformation is \(\tilde{\phi}^0_{D+d} = \phi^0_{D+d} - \ln[\text{det}(I_d + \Sigma)].\)

The background fields in eqns.(3)-(4) correspond to the ungauged string action \([5]\) whereas the ones in eqn.(3)-(7) correspond to the gauged ones with
respect to the vector gauging [5]. When the backgrounds in eqn. (3)-(4) represent the SL(2,R) WZW model, then the backgrounds (8)-(9) describe the two dimensional black hole [9, 10]. Moreover the structure of the metric and antisymmetric tensor in eqns.(6) and (7) point out a complete decoupling of $d$ of the coordinates. Hence the transformed background is a p-brane solution with zero momentum along these directions. An interesting aspect of the above $O(\tilde{d}, \tilde{d})$ transformation is that it does not depend on the background configuration and is represented by a unique matrix in a given space-time dimension.

We now generalize these results and obtain the backgrounds with nonzero momentum of the p-brane. More precisely, we obtain the $O(\tilde{d}, \tilde{d})$ transformed background from the following metric and antisymmetric tensor fields:

$$G_{D+d} = \begin{pmatrix}
\Gamma_s & \frac{1}{2\sqrt{2}} \sqrt{M - \left(\frac{J_l}{T}\right)^T} \Gamma_1 & \frac{1}{2\sqrt{2}} \sqrt{M + \left(\frac{J_l}{T}\right)^T} \Gamma_2 \\
\frac{1}{2\sqrt{2}} \sqrt{M - \left(\frac{J_l}{T}\right)^T} \Gamma_1 & \frac{1}{2} (M - \left(\frac{J_l}{T}\right)) I_d & \frac{1}{2} (\frac{J_l}{T})^2 \Sigma_T \\
\frac{1}{2\sqrt{2}} \sqrt{M + \left(\frac{J_l}{T}\right)^T} \Gamma_2 & \frac{1}{2} (\frac{J_l}{T})^2 \Sigma & \frac{1}{2} (M + \left(\frac{J_l}{T}\right)) I_d
\end{pmatrix},$$ (10)

and

$$B_{D+d} = \begin{pmatrix}
\Gamma_a & \frac{1}{2\sqrt{2}} \sqrt{M - \left(\frac{J_l}{T}\right)^T} \Gamma_1 & \frac{1}{2\sqrt{2}} \sqrt{M + \left(\frac{J_l}{T}\right)^T} \Gamma_2 \\
\frac{1}{2\sqrt{2}} \sqrt{M - \left(\frac{J_l}{T}\right)^T} \Gamma_1 & 0 & \frac{1}{2} (\frac{J_l}{T})^2 \Sigma_T \\
-\frac{1}{2\sqrt{2}} \sqrt{M + \left(\frac{J_l}{T}\right)^T} \Gamma_2 & -\frac{1}{2} (\frac{J_l}{T})^2 \Sigma & 0
\end{pmatrix},$$ (11)

where $r_+^2 = Ml^2 \sqrt{1 - \left(\frac{J_l}{MT}\right)^2}$. The $O(\tilde{d}, \tilde{d})$ transformation is a generalization
of the one given in eqn. (3) and is given by Ω =
\[
\begin{pmatrix}
I_{D-d} & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{\sqrt{2}}(\frac{r_t}{r_0})\frac{I_d}{2} & -\frac{1}{\sqrt{2}}(\frac{r_t}{r_0})\frac{I_d}{2} & 0 & -\frac{1}{\sqrt{2}}(\frac{r_t}{r_0})I_d & -\frac{1}{\sqrt{2}}(\frac{r_t}{r_0})I_d \\
0 & -\frac{1}{\sqrt{2}}(\frac{r_t}{r_0})\frac{I_d}{2} & -\frac{1}{\sqrt{2}}(\frac{r_t}{r_0})\frac{I_d}{2} & 0 & -\frac{1}{\sqrt{2}}(\frac{r_t}{r_0})I_d & -\frac{1}{\sqrt{2}}(\frac{r_t}{r_0})I_d \\
0 & 0 & I_{D-d} & 0 & 0 & 0 \\
0 & -\frac{1}{\sqrt{2}}(\frac{r_t}{r_0})\frac{I_d}{2} & -\frac{1}{\sqrt{2}}(\frac{r_t}{r_0})\frac{I_d}{2} & 0 & -\frac{1}{\sqrt{2}}(\frac{r_t}{r_0})I_d & -\frac{1}{\sqrt{2}}(\frac{r_t}{r_0})I_d \\
0 & -\frac{1}{\sqrt{2}}(\frac{r_t}{r_0})\frac{I_d}{2} & -\frac{1}{\sqrt{2}}(\frac{r_t}{r_0})\frac{I_d}{2} & 0 & -\frac{1}{\sqrt{2}}(\frac{r_t}{r_0})I_d & -\frac{1}{\sqrt{2}}(\frac{r_t}{r_0})I_d
\end{pmatrix}.
\] (12)

To obtain the transformed background, we first notice that the $G_{D+0}$ and $B_{D+0}$ in eqns. (3) and (4) are related to $G_{D+0}$ and $B_{D+0}$ in eqns. (10) and (11) by a constant coordinate transformation corresponding to the $O(\tilde{d}, \tilde{d})$ transformation:
\[
A_c = \begin{pmatrix} A & 0 \\ 0 & A^{-1T} \end{pmatrix},
\] (13)
where
\[
A = \begin{pmatrix} I_{D-d} & 0 & 0 \\ 0 & \sqrt{\frac{2}{M-(\frac{4}{r_+})}}I_d & 0 \\ 0 & 0 & \sqrt{\frac{2}{M+(\frac{4}{r_+})}}I_d \end{pmatrix}.
\] (14)

Then, using $\Omega^0M_0\Omega^0T = \tilde{M}_0$ and demanding $\Omega M \Omega^T = \tilde{M}$, where $M_0$, $\tilde{M}_0$, $M$, and $\tilde{M}$ are defined analogous to eqn. (11), one can prove that $\tilde{M}_0$ is related to $\tilde{M}$ by an $O(\tilde{d}, \tilde{d})$ transformation:
\[
\tilde{A}_c = \begin{pmatrix} \tilde{A} & 0 \\ 0 & \tilde{A}^{-1T} \end{pmatrix},
\] (15)
where
\[
\tilde{A} = \begin{pmatrix} I_{D-d} & 0 & 0 \\ 0 & \sqrt{\frac{M^2+JL+\sqrt{M^2-JL}}{2r_+}}I_d & \sqrt{\frac{M^2+JL-\sqrt{M^2-JL}}{2r_+}}I_d \\ 0 & \sqrt{\frac{M^2+JL-\sqrt{M^2-JL}}{2r_+}}I_d & \sqrt{\frac{M^2+JL+\sqrt{M^2-JL}}{2r_+}}I_d \end{pmatrix}.
\] (16)
The transformed fields \( \tilde{G}_{D+d}, \tilde{B}_{D+d} \) can therefore be obtained by a constant coordinate transformation given in eq. (16). For example, for the case when \( \Gamma = I, \Gamma_1 = \Gamma_2 = 0 \) the new metric is given by

\[
\begin{pmatrix}
I & 0 & 0 \\
0 & \frac{1}{2} \left[ (M(\frac{l}{r_+^2})^2 I_d - \Sigma)(I_d + \Sigma)^{-1} \right. \\
0 & \left. \frac{1}{2} \left[ (M(\frac{l}{r_+^2})^2 I_d + \Sigma)(I_d + \Sigma)^{-1} \right) \right. \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & \frac{1}{2} \left[ (M(\frac{l}{r_+^2})^2 I_d - \Sigma)(I_d + \Sigma)^{-1} \right. \\
0 & \left. \frac{1}{2} \left[ (M(\frac{l}{r_+^2})^2 I_d + \Sigma)(I_d + \Sigma)^{-1} \right) \right. \\
\end{pmatrix}
\]

General expression can also be obtained starting from eqns. (6)-(7).

It is now observed that the transformed metric has off-diagonal terms in the d-directions which were earlier decoupled. As a result, the new solution can be interpreted to represent a p-brane with nonzero momentum along these directions. This interpretation is further strengthened by the structure of the singularity for the case when \( J \neq 0 \). Since the metric in eqns. (6) and (17) are related by a constant coordinate transformation (16), the scalar curvature does not depend on \( J \). The inclusion of \( J \), therefore, does not generate any new singularity apart from the usual one at \( det(I + \Sigma) = 0 \) for the case when \( \phi_{0}^{D+d} = \text{constant} \). However, it generates momentum along some of the isometry directions.

As stated earlier, the above generalization was motivated by the space-time structure of the three dimensional rotating black hole \( \textbf{[6]} \). This solution has the metric of the form in eqn. (10). To show this, we start by writing...
down the three dimensional rotating black hole solution of [6]. It is given by
the metric:

\[ ds^2 = -N^2 dt^2 + \rho^2 (N\phi dt + d\phi)^2 + \left( \frac{L}{\rho} \right)^2 N^{-2} dr^2 \]  \hspace{1cm} (18)

where

\[ N^2 = \left( \frac{r}{\rho} \right)^2 \left( \frac{r^2 - r_+^2}{l^2} \right); N\phi = -\frac{J}{2\rho^2}; \rho^2 = r^2 + \frac{1}{2}(Ml^2 - r_+^2). \]  \hspace{1cm} (19)

This metric satisfies the Einstein equations for the comological constant \( \lambda = -\frac{1}{l^2} \). The metric in eqn.(18) can be transformed by a coordinate trans-
formation, \( r = r_+ \cosh \omega \) and an orthogonal transformation to

\[ G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2}(M - (\frac{L}{T})) & \frac{1}{2}(\frac{r^2}{T} \cosh 2\omega) \\ 0 & \frac{1}{2}(\frac{r^2}{T} \cosh 2\omega) & \frac{1}{2}(M + (\frac{L}{T})) \end{pmatrix} \]  \hspace{1cm} (20)

which we observe to be of the same form as in eqn.(10), when \( \Gamma_1 = \Gamma_2 = 0, \Gamma = 1, \) and \( \Sigma = \cosh \omega \).

We now extend this solution to string theory and obtain the transformed solution which correspond to the general result for \( \tilde{G}_{D+d}, \tilde{B}_{D+d} \). We find that it is precisely the moving black string of ref.[8]. The extension of the
solution in eqn.(20) is done by first observing that the metric depends on a single space-time coordinate. The string effective action with backgrounds dependent on a single space-time coordinate were studied in refs.[1]. In such cases, the gauge symmetries of the string effective action can be used to write
the metric and the antisymmetric tensor in the form

\[ G = \begin{pmatrix} -1 & 0 \\ 0 & G \end{pmatrix} \]  \hspace{1cm} and \hspace{1cm} \[ B = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} \]  \hspace{1cm} (21)
respectively. Then the equations of motion, following from the string effective action are,

\begin{align}
(\dot{\Phi})^2 + \frac{1}{4} Tr[\partial_b G \partial_b G] + \frac{1}{4} Tr[G^{-1}(\partial_b B)G^{-1}(\partial_b B)] - \Lambda &= 0, \\
(\dot{\Phi})^2 - 2\ddot{\Phi} - \frac{1}{4} Tr[\partial_b G \partial_b G] - \frac{1}{4} Tr[G^{-1}(\partial_b B)G^{-1}(\partial_b B)] - \Lambda - \frac{\partial \Lambda}{\partial \Phi} &= 0,
\end{align}

\begin{align}
- \dot{\Phi} \dot{G} + G \partial_b (G^{-1} \dot{G}) - \dot{BG}^{-1} \dot{B} &= 0,
\phi \dot{B} - \dot{B} G^{-1} \dot{G} + \dot{G} G^{-1} \dot{B} &= 0.
\end{align}

For our case, solution of the string effective action is obtained by making an ansatz that metric is the same as the black hole metric of general relativity, i.e. eqn.(20) and the dilaton \( \phi = c_0 \) is a constant. The equations of motion (22)-(25) then reduce to a set of differential equations for the only independent component of the antisymmetric tensor field. The solution, upto a constant shift, is given by

\begin{equation}
B = \frac{1}{2} \begin{bmatrix}
0 & -\left(\frac{r}{r^+}\right)^2 \cosh 2 \omega \\
\left(\frac{r}{r^+}\right)^2 \cosh 2 \omega & 0
\end{bmatrix}.
\end{equation}

The corresponding value of the cosmological constant is \( \Lambda = 4 \). Therefore, starting with a three dimensional rotating black hole metric of general relativity, we have obtained a consistent background for the string effective action given by the eqns.(20) and (26) together with a constant dilaton. This solution corresponds to a three dimensional rotating axion black hole.

Now, the connection of the transformed backgrounds for this case to the black string with nonzero momentum is established by writing down the
transformed fields. For the present case we get,

\[ \tilde{G} = \begin{bmatrix} -1 & 0 \\ 0 & \tilde{G} \end{bmatrix} \]  

(27)

where

\[ \tilde{G} = \frac{1}{1 + \cosh 2\omega} \begin{bmatrix} M\left(\frac{l}{r_+}\right)^2 - \cosh 2\omega & -\left(\frac{l}{r_+}\right)^2 \left(\frac{d}{d}\right) \\ -\frac{l}{r_+} \left(\frac{d}{d}\right) & M\left(\frac{l}{r_+}\right)^2 + \cosh 2\omega \end{bmatrix} \]  

(28)

and \( \tilde{B} = 0 \). The transformed dilaton background is given by \( \tilde{\phi} = c_0 - \ln \cosh^2 \omega \).

To interpret the solution in eqn.(28) we make the coordinate transformation \( \tilde{d} = \cosh \omega \). Then the dilaton background is given by \( \phi = c_1 - \ln \tilde{d} \) and the invariant distance can be written as:

\[ ds^2 = \frac{d\tilde{d}^2}{4\omega^2(1 - \tilde{d}^2)} + \left[ 1 + m_0 \frac{\tilde{d}}{\tilde{d}} \right] dx^2 \\
- \left[ 1 - m_0 \frac{\tilde{d}}{\tilde{d}} \right] dy^2 - \frac{m_0(\tilde{d})^2}{\tilde{d}} dxdy, \]  

(29)

which matches precisely with the moving black string of ref.[8] for \( \cosh^2 \alpha = \frac{1}{2}\left(\frac{M^2}{r_+^2} + 1\right) \).

To conclude, we have presented a relationship among a class of string backgrounds through \( O(\tilde{d}, \tilde{d}) \) transformations and given an explicit example of three dimensional rotating black hole. It will be interesting to apply our results to various other examples. An interesting case may be the investigation of four dimensional 2-branes, discussed in ref.[11], and its relationship with the four dimensional rotating black hole. We hope to return to this example in future.
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