Dissipative dynamics of vortex arrays in anisotropic traps

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We discuss the dissipative dynamics of vortex arrays in trapped Bose-condensed gases and analyze the lifetime of the vortices as a function of trap anisotropy and the temperature. In particular, we distinguish the two regimes of the dissipative dynamics, depending on the relative strength of the mutual friction between the vortices and the thermal component, and the friction of the thermal particles on the trap anisotropy. We study the effects of heating of the thermal cloud by the escaping vortices on the dynamics of the system.

I. INTRODUCTION

Recent experimental evidence [1,2] for vortices in weakly interacting Bose-condensed gases stimulated a number of publications on structure of a rotating state of superfluids (see [3] for a review). Latest state of the art experimental technology makes possible creation and study of sufficiently large vortex arrays [4], as well as non-destructive measurements of time evolution in a system with a single vortex [5]. Such possibilities allow, in principle, experimentally clarify the nature of interaction of vortices with thermal excitations. This issue plays an important role for understanding of vortex dynamics in numerous related systems, ranging from superconductors to superfluid cores of neutron stars (see [6] for a review), and still remains a controversial issue in the literature [7].

In our recent papers we investigated the dissipative dynamics of a single vortex [8] and a large vortex array [9] in a static (non-rotating) trap at finite temperatures. By this we assumed a very fast relaxation of the thermal component towards its static configuration due to, say, the anisotropy of the trap. Since actual experiments are performed in traps, which can be made nearly perfect, the rotation of the thermal component can persist for a very long time. Moreover, in the case of a large vortex array, the decay of vortices reaching the border of the condensate may be accompanied by a substantial heating of the thermal cloud, which, in turn, affects the strength of the friction forces acting on the remaining vortices. Thus, in order to understand an experiment in realistic conditions and use its results for a fine test of vortex dynamics theories, one has to have a solid description of dissipative dynamics including various temperature effects as well as the role of the trap anisotropy.

In this Letter we develop a simple, but yet a feature rich, model of dissipative dynamics of large vortex arrays in realistic anisotropic traps at finite temperatures. We analyze the relaxation kinetics of a system consisting of the rotating thermal component interacting with a condensate the vortices. We show that the trap anisotropy, or any other mechanism of angular momentum dissipation, is absolutely crucial for the onset of the dissipative dynamics. On the basis of our analysis we distinguish the two regimes of the evolution. The first one corresponds to the previously studied static case [8], i.e. to the situation when the trap is sufficiently anisotropic and the rotation of the thermal cloud quickly ceases. In this case the vortex array decays in a non-exponential fashion (a power law for the number of vortices at sufficiently long times). Remarkably, in this limit the lifetime of vortices does not depend on the parameters of anisotropy. In the opposite case, i.e. when the trap is nearly perfect, the friction of the vortices on the thermal cloud is stronger than the friction of the thermal cloud on the trap asymmetry. This makes the vortices and the cloud stick to each other: the condensate rotates with the same angular velocity as the thermal particles and the angular velocity of the both components decreases exponentially with a time constant explicitly depending on the parameters of the trap asymmetry.

We discuss the heating of the thermal component due to the vortex array decay. The developed model, together with certain simplifications, allows us to obtain a simple analytical description of dissipative dynamics and compare the results with the experimental data [9]. We believe that the presented model can be used for indirect study of vortex arrays dynamics based on the thermal component thermometry (proposed earlier in [10]).

II. THE DESCRIPTION OF THE MODEL

We consider an evolving vortex array in an anisotropic trap characterized by the radial and the longitudinal frequencies $\omega_\rho$ and $\omega_z$. The equilibrium vortex configuration in a rotating trap (in the experiment the this corresponds to the situation when the stirring beam is on) is considered in [8]. Further on we assume that the initial angular velocity of the trap rotation $\Omega_0 \gg \Omega_\rho$, where $\Omega_\rho$ is the critical velocity corresponding to the appearance of the first vortex. In this case the equilibrium number of vortices $N_\rho \gg 1$, and their spatial distribution is uniform.

As we will see, the relaxation of vortex arrays occurs on a time scale, exceeding the relaxation time in the thermal cloud. Hence, we may consider the evolution of the system in quasi-equilibrium approximation. This means, that at any given moment of time, the thermal cloud is characterized by its equilibrium density profile, cor-
responding to a certain temperature $T$, and its angular velocity of the uniform macroscopic rotation $\Omega$ (see Fig. 1). To derive the equations of motion for the whole system, consisting of the condensate with the vortices and the thermal cloud, we first write down the conservation laws. Assuming that the magnetic trap does not dissipate energy, we write down the conservation of the total energy

$$\frac{d}{dt} \left( \frac{I(T)\Omega^2}{2} + E(T) + E_v \right) = 0,$$

where $E_v$ is the energy of the condensate with the vortices (see below),

$$E(T) = 3T \frac{\zeta(4)}{\zeta(3)} N(T),$$

is the internal energy of the non-condensed particles,

$$I(T) = mR_T^2 N(T) \frac{\zeta(4)}{\zeta(3)},$$

is the momentum of inertia and $R_T = (2T/m\omega_p^2)^{1/2}$ is the radial size of the thermal cloud. Here

$$N(T) = \frac{\zeta(3)}{\tilde{\omega}} T^3 = \frac{T^3}{\bar{c}_v^2} N,$$

is the number of thermal particles, $\tilde{\omega} = (\omega_0\omega_p^2)^{1/3}$, $T_c = \bar{\omega}(N/\zeta(3))^{1/3}$ is the BEC transition temperature, and $N$ is the total number of particles in the trapped gas. Eqs. (2), (3) imply that $T \gg \mu$, where $\mu = n_0g$ is the chemical potential, $g = 4\pi\hbar^2a/m$, $a$ is the two-body scattering length, and $m$ is the mass of the gas particles. At lower temperatures, $T \lesssim \mu$, the size of the thermal cloud matches the size of the condensate and, hence, the temperature measurements become very difficult. Accordingly, in our quantitative discussions we confine ourself to the case $T \gg \mu$.

In the Thomas-Fermi limit ($\mu \gg \hbar\omega_{\rho}$) the condensate is characterized by the longitudinal $R_c = (2\mu/m\omega_p^2)^{1/2}$ and the transverse $R_c = (2\mu/m\omega_p^2)^{1/2}$ sizes, so that $R_c \gg l_c$, where $l_c = (\hbar^2/m\mu)^{1/2}$ is the correlation (healing) length in the condensate. The velocity of the superfluid flow, generated by a rotating vortex grid, imitates the flow of a normal liquid with the angular velocity $\Omega_v$: $v_s = [\Omega_v, r]$. The corresponding density of the vortex lines is given by

$$n_v = \frac{\Omega_v}{\pi R_c},$$

where $\kappa = \hbar/m$ is the vortex circulation (vortices with a larger circulation are unstable). The angular velocity $\Omega_v$ must not be very large. Indeed, the maximum velocity of the superflow, which coincides with the velocity of the superfluid on the border of the condensate $\sim \Omega_v R_c$ must not reach the velocity of Bogolyubov sound $c_s = \sqrt{\mu/m}$, i.e.

$$\Omega_v \ll \frac{c_s}{R_c} \sim \omega_p,$$

since otherwise the superflow becomes unstable with respect to the spontaneous creation of excitations already at $T = 0$ (see Landau arguments in [11]: the nature of the rotational friction in such a situation is an interesting question by itself and is not considered here, see [12] for a detailed discussion). Since the angular velocity $\Omega_v$ is directly related with the density of the vortex array (see Eq. (3)), the condition (6) limits the maximum number of vortices in a stable vortex configuration confined in a harmonic trap of a given frequency:

$$N_v \sim \pi R_c^2 \frac{\Omega_v}{\pi \kappa} \lesssim \frac{\mu}{\hbar\omega_{\rho}} \gg 1,$$

which is a large number in the Thomas-Fermi limit.

Since the density of the condensate energy is logarithmically singular in the vicinity of the vortex line, the energy of the vortex array, $E_v$, consists of two parts:

$$E_v = \int d^2r \rho_s \frac{v_s^2}{2} = E_0 + I_v \frac{\Omega_v^2}{2},$$

where $E_0$ is the part of the vortex energy independent on the velocities of the vortices, and amounts to $N_v$ times energy of a single vortex at rest, $E_0 = \pi mn_0 R_c \log(R_c/l_c)$, see [11]. The quantity $I_v = mn_0 R_c \pi R_c^2/2$ can be called as the momentum of inertia for the condensate with the vortices (moment of inertia of the vortex array). The ratio of the first to the second term in the r.h.s. of Eq. (8) is $\sim \log(R_c/l_c)/N_v \ll 1$. This shows that in a sufficiently large vortex array, $N_v \gg 1$, its rest energy is negligible and the whole internal energy of the condensate coincides with the energy of its macroscopic rotation.

The second equation comes from the angular momentum balance.

FIG. 1. The schematic view of a condensate with vortices in a rotating thermal cloud.
\[
\frac{d}{dt}(I(T)\Omega + M_v) = -\alpha \Omega,
\]
where \(M_v\) is the angular momentum of the vortex array and the phenomenological coefficient \(\alpha\) describes the interaction of the thermal particles scattered by the trap anisotropy. In fact, the form of r.h.s. in Eq. (9) corresponds to the dissipation of the angular momentum by a friction force proportional to the velocity of the gas cloud. This relation between the friction force and the gas velocity is a very general one and holds for a friction force acting on an obstacle in a moving media both in the collisionless and hydrodynamic limits \([13]\), at least for sufficiently small velocities. The choice of the friction coefficient \(\alpha\) depends on the details of the anisotropy model and requires a microscopic calculation (see below).

Since in the reference frame rotating with the superfluid the trap asymmetry moves with a subsonic velocity (see the discussion leading to Eq. (8)), the condensate particles are not scattered by the trap and, therefore, no term proportional to the condensate angular velocity can appear in the r.h.s. of Eq. (9).

Unlike in the calculation of the condensate energy \([3]\), the expression for the angular momentum of a vortex has no singularity and, hence, the angular momentum of the vortex array coincides with the angular momentum of the superfluid

\[
M_v = \int d^2 r \rho_s r v_s = I_v \Omega_v.
\]

Eqs. (10),(11) are still insufficient, since they comprise a system of two equations for the three unknown functions \(\Omega(t), \Omega_v(t)\) and \(T(t)\). The formulation of the model is completed by the addition of the equation of motion for the vortices. It follows from the Magnus law with the friction force originating from the interaction of the vortex lines with the thermal component. In fact it is the continuity equation for the vortex gas moving towards the border of the condensate (see Appendix A for the derivation, which closely follows the arguments given in \([14]\)):

\[
\frac{d\Omega_v}{dt} + \gamma \Omega_v (\Omega_v - \Omega) = 0,
\]

where \(\gamma = 2\kappa \rho_s D/(\kappa^2 \rho^2 + D^2)\), \(\rho_s = mn_0\) and \(\rho\) are the superfluid density and the total density of the gas, and \(D\) is the mutual friction coefficient, characterizing the interaction of the vortex line with the excitations.

Eqs. (10),(11) and (12) constitute the complete set of equations describing the dissipative dynamics of vortex arrays interacting with thermal particles. The initial condition at \(t = 0\), when the rotation of the trap stops (i.e. when the stirring beam is off), are

\[
\Omega(0) = \Omega_v(0) = \Omega_0,\ T(0) = T_0 > \mu.
\]

The formulation of the model should be completed by ascribing a certain value to the friction coefficient \(\alpha\). This can be done, using the results of the discussion of the rotational properties of trapped gas, presented in \([4]\).

Consider an ideal gas, splined up in an infinitely long trap, characterized by the two close transverse frequencies \(\omega_x\) and \(\omega_y\), so that \(\omega_{x,y} = \omega(1 \pm \epsilon)^{1/2}\), where \(\epsilon\) is the parameter of anisotropy. Then, after the rotation of the trapping stops, the gas continues rotation with the angular velocity, decreasing exponentially with the characteristic time constant given by the expression \([4]\):

\[
\lambda = \text{Re} \frac{1}{4\tau} \left( 1 - \sqrt{1 - \frac{\epsilon^2}{c_{\epsilon}^2}} \right),
\]

where \(c_{\epsilon} = 1/4\omega\tau\) and \(\tau\) is the characteristic time between the collisions in the thermal cloud. The relaxation time \(\tau\) can be estimated as \(\tau^{-1} \sim n\sigma \sqrt{T/m}\), where \(n\) is the gas density and \(\sigma\) is the collisional cross section. To generalize this result for the time \(\tau\), obtained for a gas sample at \(T > T_c\), for the case of a Bose condensed gas, we change the total density of the gas to the condensate density (see the argumentation in \([13]\)).

In our model, when the condensate is absent, or the interaction between the vortices and the thermal component is negligible (this corresponds to \(\gamma \sim D = 0\)), the vortices and the thermal excitations evolve independently. From Eq. (10) for the temperature change in the thermal cloud we have

\[
\frac{\Delta T}{T} \sim \frac{I \Omega^2}{E(T)} \sim \frac{\Omega^2}{\omega^2} \ll 1.
\]

The latter inequality follows from Eq. (11) and guarantees that the temperature of the system does not change. In this approximation Eq. (11) reads

\[
\frac{d\Omega}{dt} = -\frac{\alpha}{I} \Omega,
\]

and gives the following simple decay law for the angular velocity of the thermal cloud

\[
\Omega(t) = \Omega(0) \exp(-t/\tau_A),
\]

where \(\tau_A = I/\alpha\) is the relaxation time related to the friction of the moving gas and the trap anisotropy. Introducing the requirement, that the presented model recovers the result of Gneezy-Odelin \([14]\), we set \(\tau_A^{-1} = \lambda\) and find, that \(\alpha = I\lambda\).

The appearance of imaginary part in the relaxation rate \([14]\) for \(\omega_{\sigma} \gg \epsilon^{-1}\) implies an oscillatory behavior in the onset of equilibrium of a free gas cloud (see \([14]\) for the discussion). As we shall see, the evolution of the combined system of vortices and thermal excitations explicitly depends on the relaxation rate \([15]\) only in the so called "weak anisotropy" limit, when both the vortex array and the non-condensed gas rotate with the same angular velocity. Since, according to Eq. (10), the number of vortices in the vortex array is proportional to the angular velocity \(\Omega_v\), such oscillations would imply either
III. ANALYSIS OF THE DISSIPATIVE DYNAMICS.

As discussed in the previous section, Eqs. (9), (11) and (13) form a close set of equations describing the evolution of a large \( N_v \gg 1 \) vortex array emerged in a rotating thermal cloud. Although the equations look simple, their analytical solution requires certain simplifications.

First of all we qualitatively analyze the heating of the thermal cloud in the course of dissipation of rotational energy of the gas. According to Eq. (13) the rotational energy of the thermal cloud is always less than its internal energy and therefore makes no contribution to the internal energy of the thermal cloud. According to Eq. (14) the rotational energy of the gas. According to Eq. (14) the rotational energy of the gas.

The parameter \( \eta \) distinguishes two limiting regimes of relaxation. First we turn to the static trap limit, when \( \tau_R \ll \tau_v (\eta \ll 1) \). In this case at the initial stage the friction of the thermal cloud on the trap anisotropy is much stronger than the friction on the condensate. Therefore, at this stage the relaxation of the thermal cloud should proceed in the same way as in the absence of vortices, i.e. the angular velocity \( \Omega \) exponentially drops on the time scale \( \tau_A \ll \tau_R \ll \tau_v \), while the angular velocity of a condensate remains almost unchanged, \( \dot{\Omega}_v \approx 0 \). Neglecting the first term in the brackets of r.h.s. of Eq. (13) and using Eq. (3) we reduce Eq. (13) to Eq. (15). This qualitative picture remains valid until the thermal cloud nearly stops and its angular velocity reaches the very small value \( \lesssim \eta \Omega^2_v / \Omega_0 \ll \Omega_0 \). At this point the trap becomes effectively static and, setting \( \Omega_v \approx 0 \) in Eq. (11), we find the equation of motion for the condensate angular velocity

\[
\frac{d\Omega_v}{dt} = -\gamma \Omega_v^2.
\]

If the time dependence of the coefficient \( \gamma \) can be neglected, that either for \( A \ll 1 \), or for very small times when \( \Omega_v \approx 0 \), the solution of this equation is given by the expression

\[
\Omega_v(t) = \frac{\Omega_0}{1 + t/\tau_v},
\]

which recovers the result previously obtained in [10].

To analyze the solutions of Eq. (20) in the strong heating limit, \( A \gg 1 \), we note that \( \gamma \propto T \) and rewrite Eq. (20) with the help of Eq. (17) and the dimensionless units for the condensate angular velocity \( \tilde{\Omega}_v = \Omega_v / \Omega_0 \) and the time \( \tilde{t} = t / \tau_R (T_0) \):

\[
\frac{d\tilde{\Omega}_v}{d\tilde{t}} = -(1 + A(1 - \tilde{\Omega}_v^2))^{1/4} \tilde{\Omega}_v^2.
\]

Integrating it we find the exact implicit solution \( \Omega_v(\tilde{t}) \):

\[
\frac{(1 + A - A\tilde{\Omega}_v^2)^{3/4}}{(1 + A)\tilde{\Omega}_v} + \frac{A\tilde{\Omega}_v}{2(1 + A)^{5/4} F_{a,b,c}(x)} = \frac{A\tilde{\Omega}_v^2}{1 + A} = \tilde{t} + C,
\]

where \( F_{a,b,c}(x) \) is the hypergeometric function and the integration constant \( C \) is to be determined from the initial condition \( \tilde{t} = 0 \). In the considered limit of large \( A \) we find, that

\[
\eta = \frac{(I + I_v)\gamma \Omega_0}{\alpha} = \frac{\tau_R}{\tau_v}.
\]

characterizes the relative strength of the mutual interaction between the vortices and the thermal particles and the interaction of the thermal cloud and the trap asymmetry. Here \( \tau_v^{-1} = \gamma \Omega_0 \) and \( \tau_R^{-1} = \alpha / (I + I_v) \) are the characteristic relaxation times, determined respectively by the interaction of thermal component with the vortices and with the trap anisotropy.

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\]

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periodic oscillations of the vortex density (if the oscillations of \( \Omega \), are sufficiently small), or a reversible destruction and nucleation of vortices close to the border of the condensate. Well below \( T_c \), the nucleation of vortices is slow compared with kinetic times \( E / \hbar \omega_p^2 \), and, thus, the number of vortices, as well as the angular velocity of the condensate to the internal energy of the thermal cloud.

Indeed, the term containing the time derivative of the temperature dependence of the momentum of inertia of a thermal cloud.

\[
\rho v^2 = \frac{\rho \Omega^2_v}{\Omega_0^2}
\]

This allows us to neglect the temperature dependence of velocity of the condensate.

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\[
\frac{d\Omega_v}{dt} = -\gamma \Omega_v^2.
\]

If the time dependence of the coefficient \( \gamma \) can be neglected, that either for \( A \ll 1 \), or for very small times when \( \Omega_v \approx 0 \), the solution of this equation is given by the expression

\[
\Omega_v(t) = \frac{\Omega_0}{1 + t/\tau_v},
\]

which recovers the result previously obtained in [10].

To analyze the solutions of Eq. (20) in the strong heating limit, \( A \gg 1 \), we note that \( \gamma \propto T \) and rewrite Eq. (20) with the help of Eq. (17) and the dimensionless units for the condensate angular velocity \( \tilde{\Omega}_v = \Omega_v / \Omega_0 \) and the time \( \tilde{t} = t / \tau_R (T_0) \):

\[
\frac{d\tilde{\Omega}_v}{d\tilde{t}} = -(1 + A(1 - \tilde{\Omega}_v^2))^{1/4} \tilde{\Omega}_v^2.
\]

Integrating it we find the exact implicit solution \( \Omega_v(\tilde{t}) \):

\[
\frac{(1 + A - A\tilde{\Omega}_v^2)^{3/4}}{(1 + A)\tilde{\Omega}_v} + \frac{A\tilde{\Omega}_v}{2(1 + A)^{5/4} F_{a,b,c}(x)} = \frac{A\tilde{\Omega}_v^2}{1 + A} = \tilde{t} + C,
\]

where \( F_{a,b,c}(x) \) is the hypergeometric function and the integration constant \( C \) is to be determined from the initial condition \( \tilde{t} = 0 \). In the considered limit of large \( A \) we find, that
Then, for small \( \tilde{\Omega}_v \), which corresponds to large times, when the rotation of the condensate is almost stopped, the hypergeometric function can be expanded in series of its small argument and in the lowest order in \( 1/A \) we obtain the asymptotic expression for the condensate angular velocity

\[
\Omega_v = \frac{\Omega_0}{0.4 + t/\tau_v(T_f)},
\]

which is valid as soon as \( t \gg \tau_v(T_f) \). This means, that the quantity \( \tau_v(T_f) \) plays the role of the vortex array life time and is mainly determined by the friction coefficients at the final temperature. This fact is fairly intuitive, since the strength of the interaction of the vortices with the thermal cloud quickly increases with the rise of the temperature and thus most of the relaxation of the system occurs when its temperature is close to its final value. Remarkably, the dynamics of the vortex array in a static trap limit, that is when anisotropy is strong, turns out to be independent from the parameters of the anisotropy.

In the opposite, rotating trap limit, when the anisotropy is weak and \( \tau_R \gg \tau_v \) \((\eta \gg 1)\), the friction between the vortices and the thermal cloud is stronger than the friction of the thermal excitation on the trap asymmetry. As the result, the both components stick to each other due to their mutual interaction and dissipate the angular momentum together, as a single rigid body. Neglecting the last term in the brackets of r.h.s. of Eq. (25) for the initial conditions given by (12) we find the solution \( \Omega = \Omega_v \). Using this relation between the angular velocities in Eq. (1) we arrive at

\[
\frac{d\Omega_v}{dt} = -\frac{\alpha}{T + I_v} \Omega_v.
\]

Again, when the heating of the gas sample is small, the coefficient \( \alpha \) is time independent, and the solution of this equation is simple:

\[
\Omega_v(t) = \Omega_0 \exp(-t/\tau_R),
\]

i.e. in this, “rotating trap” approximation, the power law changes into the simple exponential decrease of the angular velocities, with the same time constant for both the vortex array and the thermal cloud.

To discuss the solutions of Eq. (23) in the strong heating limit \( A \gg 1 \) we have to make certain assumptions about the trap anisotropy. For example, according to Eq. (13), when the trap asymmetry is small, \( 4\omega_\text{r}T \ll 1 \), we have \( \alpha \propto T^2 \). Then Eq. (23) has the solution

\[
\Omega_v = \frac{2\exp(-t/\tau_R(T_f))}{1 + \exp(-2t/\tau_R(T_f))}.
\]

This means that again, the relaxation time at the finite temperature \( \tau_R(T_f) \) plays the role of the vortex array life time. At larger times, \( t \gg \tau_v(T_f) \), the angular velocity is small and the solution has an approximate asymptotic form

\[
\Omega_v = 2\Omega_0 \exp(-t/\tau_R(T_f)).
\]

Remarkably, in logarithmical approximation the solution holds also for a situation, when the trap asymmetry is strong, \( 4\omega_\text{r}T \gg 1 \), and \( \alpha \propto T^6 \) (see Eq. (13)).

Using the presented above analysis we now can draw a qualitative picture of dissipative dynamics. To do this, we introduce the dimensionless ratio of the momenta of inertia of the thermal cloud and the vortex array: \( \zeta = I/I_v \sim (T_0/\mu)^4(n_0 a^3)^{1/2} \). Since in a weakly interacting gas the gaseous parameter is small: \((n_0 a^3)^{1/2} \ll 1\), at \( T \sim \mu \) the condensate is heavier than the thermal cloud, and both the rotational energy and the angular momentum are dominated by the condensate contributions. Thus, in the rotating trap limit \((\eta \ll 1)\) the relaxation time \( \tau_R \) is mainly determined by \( I_v \) and is much larger than the spinning down time of a free thermal cloud \( \tau_A \). In the static trap limit the role of the parameter \( \zeta \) is more subtle. At sufficiently large times, \( t \gg \tau_A \), we can neglect the term \( I \Omega \) in Eq. (1), which thus transforms into \( I_v \Omega = -\alpha \Omega \). Comparing this equation with Eq. (20) we find that \( \Omega = \eta \Omega_0^2/\Omega_0 \), i.e. the thermal cloud angular velocity decays as a power law and closely follows the relaxation of the vortex array. In the same time the heating parameter \( \zeta \) is related to the parameter \( \zeta \): strong heating of the sample is only possible when \( \zeta \gg 1 \). As the temperature grows, the number of the condensed particles decreases and at some point the condensate may become so small, that \( \zeta \ll 1 \). One can see, that the crossover between the two regimes occurs at the temperature below \( T_c \), so the condensate does not disappear. Later on, the heating of the system stops and the system relaxes according to Eqs. (23) and (24) for \( \eta \ll 1 \) and \( \eta \gg 1 \) respectively.

The presented analysis shows that the anisotropy, or any other mechanism of angular momentum dissipation, is absolutely crucial for the onset of dissipative dynamics. It is seen already from the fact, that when \( \tau_R \to \infty \), Eq. (24) gives \( \Omega_v(t) = \Omega(t) = \Omega_0 \), i.e. the rotation of the vortex grid persists forever.

Although the presented analysis is sufficient for a qualitative understanding of the dissipative dynamics in the whole parameters range, the quantitative description requires a numerical solution of the initial Eqs. (1) and (11). For the sake of completeness we provide the results of such calculation based on the experimental parameters of (44). We use the following values for the model parameters: the condensate density \( n_0 = 2 \times 10^{14} \text{cm}^{-3} \), the trap frequencies \( \omega_\rho = 2\pi \times 109 \text{s}^{-1} \) and \( \omega_\zeta = 2\pi \times 11.7 \text{s}^{-1} \), and the temperature \( T_0 \approx \mu = 80 \text{nK} \). Then, the condensate size is \( R_c = 4 \mu\text{m} \) and, at the reported frequency of the stirring \( \Omega_0 = 2\pi \times 135 \text{s}^{-1} \), the vortex array consists of \( N_v \sim 10 \) vortices.
FIG. 2. Solid (dashed) line represents $\Omega_v (\Omega)$ obtained by numerical solution of Eqs. (1), (9), and (11) for the parameters of the ENS experiment. Dotted and dash-dotted lines represent approximate solution (25) with $\tau_R$ calculated for the strong ($\epsilon \gg \epsilon_c$) and the weak ($\epsilon \ll \epsilon_c$) anisotropy limits.

We provide the results of the numerical calculations for the trap anisotropy parameter $\epsilon = 0.06$ (see Figs. 2 and 3). Our model system turns out to well in the rotating trap limit ($\eta \sim 10$) and, in the full agreement with our qualitative analysis, $\Omega \approx \Omega_v$ and either of the angular velocities relaxes exponentially (see the solid and the dashed lines on Fig. 2). The solid line on Fig. 3 illustrates the heating of the system in the course of the dissipative dynamics.

FIG. 3. Solid line represents temperature of the thermal cloud obtained by numerical solution of Eqs. (1), (9), and (11) corresponding to Fig. 2. Dotted and dash-dotted lines represent temperature obtained from Eq. (17) using approximate solutions presented in Fig. 2.

The comparison of the exact numerical solution with our approximate analytical results is a bit difficult, since this value of the anisotropy parameter is close to $\epsilon_c$ (see Eq. (13)). The dashed and the dotted dashed lines on Figs. 2 and 3 represents the analytical results based on Eq. (25) with $\tau_R$ calculated in the strong $\epsilon \gg \epsilon_c$ and the weak $\epsilon \ll \epsilon_c$ anisotropy limits in Eq. (13) respectively. We note, that all of the presented results are in a good agreement with the experimentally found vortex array life time $\sim 1s$.

IV. CONCLUSION

In this Letter we formulate a realistic model, describing the evolution of a large vortex array in a rotating thermal cloud interacting with the trap anisotropy. The model allows both the simple analytical description of the most important limiting cases as well as a very straightforward numerical analysis.

First we show that the angular momentum dissipation mechanism, such as the trap anisotropy is absolutely crucial for the onset of dissipative dynamics of vortices in trapped Bose-condensed gases. We demonstrate, that if the total angular momentum of the system is conserved, the rotation of the vortex array persists forever and, consequently, the number of the vortices in the condensate remains constant.

Since the flow of the condensate is a superflow, the condensate itself can not interact with the trap anisotropy. On the contrary, the thermal excitations are scattered by the trap and can dissipate the angular momentum. Thus, it is just the mutual friction between the vortices and the gas particles that allows the condensate to transfer its angular momentum to the excitations and, subsequently, to the trap. Depending on the relative strength of interaction of the thermal gas with vortices and with the trap anisotropy we distinguish the two regimes of the dissipative dynamics.

First we identify the limit, when the trap anisotropy is very strong and the rotation of the thermal cloud quickly stops. Then the vortex array interacts with the thermal cloud at rest and subsequently looses the vortices due to the mutual friction between the vortices and the excitations (static trap limit). This case has been previously studied in [9] for a single vortex and in [10] for a large vortex array.

In the opposite limit, when the trap is nearly perfect, the thermal cloud stops slowly. In this approximation the mutual friction between the vortices and the thermal cloud makes the two components stick to each other, forming a single rotating object with the momentum of inertia equal to the sum of the momenta of inertia of the condensate and the thermal cloud. This composite body is usually much heavier than the thermal cloud and hence its relaxation proceeds on a time scale much larger than that in the Guery-Odelin scenario [14], where the calculation was done above $T_c$ and only the thermal cloud was taken into account.

Another feature of a large vortex array evolution is the heating of the thermal component caused by the energy
released by the vortices escaping the condensate. We show, that though the heating itself can be very substantial, it does not change the qualitative picture of the dissipative dynamics. Since the friction coefficients $\alpha$ and $\gamma$ both grow very quickly with the temperature increase, most of the relaxation processes occur when the temperature is already high, i.e. when the temperature is close to $T_f$. Hence, for estimation purposes, one can use a much simpler calculation with $T = \text{Const}$ with all the model parameters taken at the final temperature.

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**APPENDIX A: DYNAMICS OF VORTEX ARRAYS.**

To derive the equation of motion for the vortices inside a rotating thermal cloud, we closely follow the derivation presented in [10]. The configuration of the superfluid is described by the superfluid velocity $\mathbf{v}_s$ and by the two continuous functions: the average vortex density $n_v(t, \mathbf{r})$ and velocity $\mathbf{v}_v(t, \mathbf{r})$. Since the number of vortex lines is a locally conserved quantity, the introduced functions satisfy the continuity equation:

$$\frac{\partial n_v}{\partial t} + \frac{\partial (\mathbf{v}_sn_v)}{\partial \mathbf{r}} = 0. \tag{A1}$$

The velocity of the superfluid satisfies Maxwell-like equations

$$\text{rot} \mathbf{v}_s = 2\pi \kappa n_v \hat{z}, \quad \text{div} \mathbf{v}_s = 0, \tag{A2}$$

where $\hat{z}$ is a unit vector along the axis of the cylinder [1]. These equations are solved with the boundary condition that the normal component of $\mathbf{v}_s$ vanishes on border of the condensate. The vortex velocity is related to the superfluid velocity through the Magnus law [1]:

$$\rho_s [\mathbf{v}_v - \mathbf{v}_s, \kappa] = \mathbf{F}, \tag{A3}$$

where $\rho_s$ is the superfluid (mass) density and $\mathbf{F}$ is the friction force acting on the vortex line (per unit length).

At finite temperatures, the force $\mathbf{F}$ originates from the scattering of thermal excitations from the moving vortices and consists of two terms:

$$\mathbf{F} = -D(\mathbf{v}_v - \mathbf{v}_n) + D' [\kappa \hat{z}, (\mathbf{v}_v - \mathbf{v}_n)], \tag{A4}$$

where $\mathbf{v}_n$ is the velocity of the thermal gas at the position of the vortex line. Here $D$ and $D'$ are the longitudinal and transverse friction coefficients respectively. Both of them are temperature dependent and $D' = D = \rho_n$, where $\rho_n$ is the density of the normal component. The friction coefficient $D$ is also temperature dependent and ranges from $D \propto k_B T^{5/2} / \mu^{1/2} h^3$ for very small temperatures $T \ll \mu$ to $D \approx 0.1 k_B T^{5/2} \mu^{1/2} / h^3$ for $\mu \ll T \lesssim T_c$.

As discussed in [10], the vortex density is spatially uniform: $n_v(t, \mathbf{r}) = n_v$ and the superfluid velocity inside the vortex array can be found from Eq. (A2) by using the Stokes theorem. It has no radial component (i.e. it satisfies the necessary boundary condition) and is given by

$$\mathbf{v}_s(\mathbf{r}) = \pi [\kappa, \mathbf{r}] n_v. \tag{A5}$$

The solution of Eqs. (A3) and (A4), with $\mathbf{v}_s$ from Eq. (A3) and $\mathbf{v}_n = \{\Omega, \mathbf{r}\}$, can be represented in the form: $\mathbf{v}_v = v^{(\rho)} \hat{r} + v^{(\phi)} [\kappa, \mathbf{r}]$, where

$$v^{(\rho)} = (v_s - v_n) \frac{\kappa \rho_s D}{\kappa^2 \rho^2 + D''} \tag{A6}$$

and

$$v^{(\phi)} = \frac{D'^2 v_n + \kappa \rho_s (v_s \rho_s + D' v_n)}{\kappa^2 \rho^2 + D''} \tag{A7}$$

are the radial and tangential components of the velocity field, and $\rho = \rho_v + \rho_n$ is the total density. Eqs. (A6) and (A7) show, that if the condensate rotates together with the normal component, $v_s = v_n$, then $v^{(\rho)} = v_n$ and the radial vortex velocity is absent, i.e. the vortex and the thermal particles move together and are in the equilibrium in the reference frame defined by the rotating cloud.

Using Eqs. (A5) and (A6), we rewrite Eq. (A1) in the form

$$\frac{dn_v(t)}{dt} + \gamma n_v(t) (\pi \kappa n_v(t) - \Omega) = 0,$$

where $\gamma = 2 \kappa \rho_s D / (\kappa^2 \rho^2 + D'') \approx 2D / \kappa \rho$. Finally, expressing the vortex density through the angular velocity of the condensate rotation (see Eq. (3)), we obtain the desired equation of motion for the vortices (10).

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