UNIFORM CENTRAL LIMIT THEOREM FOR SELF NORMALIZED SUMS IN HIGH DIMENSIONS

DEBRAJ DAS

ABSTRACT. In this article, we are interested in high dimensional central limit theorem for self normalized sums. Specifically, we are interested in the normal approximation of \( \left( \frac{\sum_{i=1}^{n} X_{i,1}}{\sqrt{\sum_{i=1}^{n} X_{i,1}^2}} , \ldots , \frac{\sum_{i=1}^{n} X_{i,p}}{\sqrt{\sum_{i=1}^{n} X_{i,p}^2}} \right) \) in \( \mathbb{R}^p \) uniformly over the class of hyper-rectangles \( A^p = \{ \prod_{j=1}^{p} [a_j, b_j] \cap \mathbb{R} : -\infty \leq a_j \leq b_j \leq \infty , j = 1, \ldots , p \} \), where \( X_1, \ldots , X_n \) are non-degenerate independent \( p \)-dimensional random vectors with each having independent and identically distributed (iid) components. We investigate the optimal cut-off rate of \( \log p \) below which the uniform central limit theorem (UCLT) holds, under variety of moment conditions. When \( X_{ij} \)'s have \( (2+\delta) \)th absolute moment for some \( 0 < \delta \leq 1 \), the optimal rate of \( \log p \) is \( o(n^{\delta/(2+\delta)}) \). If \( \log p = cn^{\delta/(2+\delta)} \) for some sufficiently small \( 0 < c < 1 \), then UCLT is valid with an error \( \epsilon^{(1+\delta)/3} \). When \( X_{ij} \)'s are independent and identically distributed (iid) across \( (i, j) \), even \( (2+\delta) \)th absolute moment of \( X_{11} \) is not needed. Only under the condition that \( X_{11} \) is in the domain of attraction of the normal distribution, the growth rate of \( \log p \) can be made to be \( o(\eta_n) \) for some \( \eta_n \rightarrow 0 \) as \( n \rightarrow \infty \).

The natural question is how far we can push the growth rate of \( \log p \) for the UCLT to hold. We establish that the rate of \( \log p \) can be pushed to \( \log p = o(n^{1/2}) \) if we assume the existence of fourth moment of \( X_{ij} \)'s. Moreover, if \( \log p = cn^{1/2} \) for sufficiently small \( \epsilon > 0 \), the UCLT holds with an error \( \epsilon^{1/3} \). By an example, it is shown however that the rate of growth of \( \log p \) can not further be improved from \( n^{1/2} \) as a power of \( n \), even if \( X_{ij} \)'s are iid across \( (i, j) \) and \( X_{11} \) is stochastically bounded. As an application, we found respective versions of the high dimensional UCLT for component-wise Student’s t-statistic, which is useful in high dimensional inference. A very important aspect of the novel UCLT of self normalized sums or of Student’s t-statistic is that it does not require the existence of some exponential moments even when dimension \( p \) grows exponentially with some power of \( n \), as opposed to the UCLT of normalized sums (cf. Chernozhukov et al. (2017), Kuchibhotla et al. (2021), Das and Lahiri (2020)). Only the existence of some absolute moment of order \( \in [2, 4] \) is sufficient.

1. INTRODUCTION

Let \( X_1, \ldots , X_n \) be independent random vectors in \( \mathbb{R}^p \) each with mean 0. Suppose that \( X_i = (X_{i,1}, \ldots , X_{i,p}) \), \( n \geq 1 \), and for any \( j \in \{1, \ldots , p\} \),

\[
S_{nj} = X_{1j} + \cdots + X_{nj}, \quad B^2_{nj} = \sum_{i=1}^{n} EX_{ij}^2, \quad V^2_{nj} = \sum_{i=1}^{n} X_{ij}^2.
\]
Then the component-wise normalized and the component-wise self-normalized sum of the random vectors $X_1, \ldots, X_n$ are respectively defined as

$$U_n = \left( \frac{S_{n1}}{B_{n1}}, \ldots, \frac{S_{np}}{B_{np}} \right)' \quad \text{and} \quad T_n = \left( \frac{S_{n1}}{V_{n1}}, \ldots, \frac{S_{np}}{V_{np}} \right)'.$$  

Define the Gaussian random vector $Z_n$ on $\mathbb{R}^p$ with mean 0 and covariance matrix $\text{Corr}(S_n)$, the correlation matrix of $S_n = (S_{n1}, \ldots, S_{np})'$. Distributional approximation of $U_n$ using the Gaussian random vector $Z_n$ is extensively studied in fixed $p$ setting as well as when $p$ increases with $n$. In this paper we study the Gaussian approximation of $T_n$ when $p$ can grow exponentially with $n$. More precisely, we investigate the rate of growth of $p$ required to ensure the uniform central limit [hereafter referred to as UCLT] given by

$$(1.1) \quad \tau_{n, \mathcal{A}} \equiv \sup_{A \in \mathcal{A}} \left| \mathbf{P}(T_n \in A) - \mathbf{P}(Z_n \in A) \right| \to 0 \quad \text{as} \quad n \to \infty,$$

where $\mathcal{A}$ is a suitable collection of convex subsets of $\mathbb{R}^p$. Typical choices of $\mathcal{A}$ include

(i) $\mathcal{A}_{\text{dist}} = \{ (-\infty, a_1] \times \ldots \times (-\infty, a_p] : a_1, \ldots, a_p \in \mathbb{R} \}$, the collection of all left-infinite rectangles, leading to the Kolmogorov distance between the distributions of $T_n$ and $Z_n$,

(ii) $\mathcal{A}_{\text{max}} = \{ (-\infty, t] \times \ldots \times (-\infty, t] : t \in \mathbb{R} \} = \{ \{ \max_{1 \leq j \leq p} T_{nj} \leq t \} : t \in \mathbb{R} \}$, the collection of left infinite hyper-cubes, leading to the Kolmogorov distance between the distributions of $\max_{1 \leq j \leq p} T_{nj}$ and $\max_{1 \leq j \leq p} Z_{nj}$, and

(iii) $\mathcal{A}_{\text{re}} = \{ \prod_{j=1}^p [a_j, b_j] \cap \mathcal{R} : -\infty \leq a_j \leq b_j \leq \infty \text{ for } j = 1, \ldots, p \}$, the collection of all hyper rectangles, among others. Since $\mathcal{A}_{\text{max}} \subset \mathcal{A}_{\text{dist}} \subset \mathcal{A}_{\text{re}}$, we only study the growth rate of $p$ in $\tau_{n, \mathcal{A}_{\text{re}}} \to 0$ as $n \to \infty$. As an application of the UCLT (1.1) for the self-normalized random vector $T_n$, we also investigate the UCLT of the component-wise studentized random vector $W_n = (W_{n1}, \ldots, W_{np})$ over the class of sets $\mathcal{A}_{\text{re}}$, where $W_{nj}$ is defined as

$$W_{nj} = \frac{\sqrt{n} \bar{X}_{nj}}{\sqrt{(n-1)^{-1} \sum_{i=1}^n (X_{ij} - \bar{X}_{nj})^2}},$$

with $\bar{X}_{nj} = n^{-1} \sum_{i=1}^n X_{ij}, j \in \{1, \ldots, p\}$. We show that all the results that hold for $T_n$ are also true for $W_n$.

Similar to (1.1), we say that UCLT of the normalized random vector $U_n$ holds over a class of sets $\mathcal{A}$ provided

$$(1.2) \quad \rho_{n, \mathcal{A}} \equiv \sup_{A \in \mathcal{A}} \left| \mathbf{P}(U_n \in A) - \mathbf{P}(Z_n \in A) \right| \to 0, \quad \text{as} \quad n \to \infty.$$  

When $p$ is fixed, under some conditions on the truncated second moment of $\|X_1\|, \ldots, \|X_n\|$ ($\| \cdot \|$ denotes the Euclidean norm), convergence of $\rho_{n, \mathcal{A}_{\text{re}}}$ to 0 as $n \to \infty$ follows from the classic Lindeberg’s Central Limit Theorem (CLT) (cf. Theorem 11.1.6 of Athreya and Lahiri (2006)). When $p$ grows with $n$, then there is a series of interesting results available in the literature. In a seminal paper, Chernozhukov et al. (2013) showed that when the random variables $X_{ij}$’s are sub-exponential, $\rho_{n, \mathcal{A}_{\text{re}}} \to 0$ as $n \to \infty$ provided $\log p = o(n^{1/7})$. Chernozhukov et al. (2017) improved their results and established the same growth rate of $p$ when $\mathcal{A} = \mathcal{A}_{\text{re}}$ in
Under the same assumption of sub-exponentiality of $X_{ij}$’s when $A = A^\alpha$, the rate of $p$ is improved to $\log p = o(n^{1/3})$ by Chernozhukov et al. (2019) and Koike (2019). Kuchibhotla et al. (2021) considered $A = A^{max}$ in (1.2) and established that $\log p$ can grow like $o(n^{1/4})$ under the assumption of sub-Exponentiality of $|X_{ij}|^{\alpha}$’s for some $0 < \alpha \leq 2$. Under similar setting and with the additional assumption that the random vectors $X_i$’s are iid having log-concave density, Fang and Koike (2020) improved the growth rate of $p$ to $\log p = o(n^{1/3}(\log n)^{-2/3})$ in (1.2) with $A = A^{\infty}$. Recently, Das and Lahiri (2020) further improved the rate of $p$ to $\log p = o(n^{1/2})$ in (1.2) with $A = A^{\infty}$ when the random vectors $X_i$’s are independent and they have independent & identically distributed (iid) sub-Gaussian components symmetric around 0. Some negative results are also available in the literature. For example, Fang and Koike (2020) showed that when $X_{ij}$’s are iid across $(i,j)$ with $E(X_{11}^4) \neq 0$ and some other conditions are true, the rate of $p$ in (1.2) with $A = A^{max}$ can not be improved further from $\log p = o(n^{1/3})$. On the other hand, Das and Lahiri (2020) showed using an example that the rate $\log p = o(n^{1/2})$ in (1.2) with $A = A^{max}$ can not be improved as a power of $n$ when the underlying setup is symmetric around 0. Some results in the direction of matching the $n^{-1/2}$ scaling of classical Berry-Esseen theorem in high dimensional setup are also established; see for example Lopes (2020) and Kuchibhotla and Rinaldo (2020).

In this paper we study the UCLT of $T_n$ and $W_n$ over the class $A^{\infty}$ where $X_1, \ldots, X_n$ are non-degenerate independent $p$–dimensional random vectors with each having iid components. The aim for considering $T_n$ and $W_n$ as an alternative to $U_n$ is to reduce the underlying moment assumptions from existence of some exponential moments to the existence of some absolute polynomial moments of order $\leq 4$. The motivation is due to the well-established fact that the one dimensional limit theorems for self-normalized sums or the studentized sums hold under much weaker moment conditions than that for the normalized sums. Indeed we show that when $E|X_{ij}|^{2+\delta} < \infty$ for all $(i,j)$ for some $0 < \delta \leq 1$, then the rate $\log p = o(n^{\delta/(2+\delta)})$ can be achieved in the UCLT (1.1) of $T_n$ with $A = A^{\infty}$.

Moreover if $\log p = o(n^{1/3})$, then $\tau_{n,A^{\infty}} < \epsilon^{1+\delta)/3}$ for sufficiently small $\epsilon > 0$. Therefore the rate $\log p = o(n^{1/3})$ can be achieved in (1.1) under the existence of third absolute moments of underlying random variables, whereas the underlying random variables are assumed to be sub-exponential to achieve the similar rate of $\log p$ in (1.2) [cf. Corollary 1.1 and Proposition 1.1 in Fang and Koike (2020)]. Moreover, the rate of $\log p$ in (1.1) can be improved to $o(n^{1/2})$ under the existence of fourth moment and $\tau_{n,A^{\infty}} < \epsilon$ if $\log p = o(n^{1/2})$ for sufficiently small $\epsilon \in (0, 1)$. However similar to the normalized case (cf. Das and Lahiri (2020)), the rate of $\log p$ can not be significantly improved from $o(n^{1/2})$ even when $X_{ij}$’s are iid across $(i,j)$ and $X_{11}$ is stochastically bounded. When $X_{ij}$’s are all iid, then even no moment condition is required to achieve sub-exponential growth rate of $p$ in (1.1), only the assumption of $X_{11}$ belonging to the domain of attraction of the normal distribution is sufficient. Moreover, all of these results still hold if we define $\tau_{n,A}$ with $T_n$ replaced by $W_n$. The backbone behind establishing the UCLT of $W_n$ from that of
$T_n$ is the well-known identity between $T_{nj}$ and $W_{nj}$, given by

$$P(W_{nj} \geq x) = P(T_{nj} \geq x \left(\frac{n}{n + x^2 - 1}\right)^{1/2}), \text{ for any } x \geq 0,$$

for all $1 \leq j \leq p$ [cf. Efron (1969)]. Therefore in the UCLT, similar growth rates of $p$ can be achieved under much weaker moment conditions if we perform component-wise studentization instead of component-wise standardization. This fact is very interesting from the perspective of statistical inference as well. In most of practical problems, the underlying population variance is not known and hence studentization becomes essential.

The proofs of the main results here crucially depend on a set of non-uniform Berry-Esseen type bounds in the one dimensional CLT for self-normalized sum of independent random variables which, in turn, heavily depend on the Cramér type large deviation for one dimensional self normalized sums. Another property that is also very useful for our proofs is the sub-Gaussian tail behaviour of the self normalized sum of independent random variables which is essentially ensured by the assumption of symmetry of the distribution of the underlying random variables.

The assumption of symmetry may be replaced by the assumption of the uniform stochastically boundedness of the components of $T_n$ (see also Giné et al. (1997)). However, we do not consider such generality here and restrict our attention to the collection of symmetric random variables $X_{ij}$, $1 \leq i \leq n$, $1 \leq j \leq p$. The literature on one dimensional self-normalized sums is well developed. Preliminary results in self-normalized sums are due to Darling (1952), Efron (1969) and Logan et al. (1973). Efron (1969) pointed out the crucial relation (1.3) which essentially implies that the limit distribution of $T_n$ and $W_n$ coincide when $p$ is fixed. Under the assumption that $p = 1$ and $X_i$’s are iid, Logan et al. (1973) and later Chistyakov and Götzte (2004b) found the asymptotic distribution of $T_n$ when $X_i$ is in the domain of attraction of some stable law. Uniform Berry-Esseen bounds for $T_n$ and $W_n$ in one dimension were obtained by Slavov (1985) and Hall (1988) and later refined by Bentkus and Götzte (1996) and Bentkus et al. (1996). Wang and Jing (1999) essentially improved these uniform results and established a non-uniform Berry-Esseen bound for $T_n$ and $W_n$. Shao (1997, 1999) established a Cramér type large deviation result for $T_n$ and $W_n$ when $p = 1$, which was later improved and generalized by Jing et al. (2003). Jing et al. (2003), among other interesting results, also established sharp non-uniform Berry-Esseen bound for $T_n$ under $(2 + \delta)$th absolute moment condition when $p = 1$. Robinson and Wang (2005) considered iid setup in one dimension and established Cramér type large deviation result for $T_n$ without any moment assumption. Recently, Sang and ge (2017) improved existing Cramér type large deviation results for $T_n$ in one dimension, among other results, under the assumption of the existence of fourth moment. For an elaborate and systematic presentation of the results for self-normalized sums, one can look into Peña et al. (2009) and Wang and Shao (2013).

The rest of the paper is organized as follows. We state the results on growth rate of $\log p$ for self-normalized sums in Section 2. Respective results corresponding to the component-wise student’s t-statistic are presented in Section 3. Proofs of all the results are presented in Section 4.
2. Main Results

In this section we are going to present our results. For the rest of the paper, we assume \( \{X_n\}_{n \geq 1} \) to be a sequence of independent random vectors in \( \mathbb{R}^p \). For each \( i \in \{1, \ldots, n\} \), \( X_i = (X_{i1}, \ldots, X_{ip})' \). For each \( i \), \( \{X_{ij} : j \in \{1, \ldots, p\}\} \), are iid with distribution symmetric around 0. We are going to explore the rate of growth of the dimension \( p \) for the UCLT (1.1) over \( \mathcal{A}^{re} \) to hold, under different moment conditions. Note that in our setup \( Z_n = Z \) for all \( n \geq 1 \), where \( Z \) is the standard Gaussian random vector on \( \mathbb{R}^p \) with mean 0 and covariance matrix \( I_p \), the identity matrix of order \( p \). We divide this section in two sub-sections. In the first sub-section, we start with exploring the growth rate of \( p \) in the general setting when \( \mathbb{E}|X_{11}|^{(2+\delta)} \) are finite for some \( 0 < \delta \leq 1 \). Then we assume that \( X_{ij} \)'s are iid across \( (i, j) \) and drop the assumption of finiteness of \((2+\delta)\)th absolute moment of \( X_{ij} \)'s. We only assume that \( X_{11} \) belongs to the domain of attraction of the normal distribution. In the second sub-section we explore the rate of \( p \) when \( \max_{1 \leq i \leq n} \mathbb{E}|X_{11}|^{4} = O(1) \), as \( n \to \infty \). There we show by an example that this rate can not be improved even when \( X_{ij} \)'s are iid across \( (i, j) \) and \( X_{11} \) is stochastically bounded. Before moving to the sub-sections, we are going to define few notations.

For a sequence of random variables \( \{Y_i\}_{i=1}^n \), define \( \sigma_n^2(Y) = n^{-1} \sum_{i=1}^n \text{Var}(Y_i) \). Also for any \( k > 0 \), define \( \beta_{n,k}(Y) = n^{-1} \sum_{i=1}^n \mathbb{E}|Y_i|^{2+k} \) and the Lyapunov’s ratio

\[
\delta_{n,x}(Z) = \frac{n \mathbf{P}\{|Z| > \kappa_{n,x}(Z)\} + n \left[ \kappa_{n,x}(Z) \right]^{-1} \mathbb{E}\{Z I(|Z| \leq \kappa_{n,x}(Z))\}}{n \beta_{n,k}(Y)},
\]

where \( \kappa_{n,x}(Z) = \sup \left\{ s : n s^{-2} \mathbb{E}\{Z^2 I(|Z| \leq s)\} \geq 1 + x^2 \right\} \), for some random variable \( Z \). Whenever \( X_{ij} \)'s are the underlying random variables, then we simply write \( \sigma_n^2; \beta_{n,k}, d_{n,k} \) and \( \kappa_n \).

2.1. Rate of growth of \( p \) under \((2+\delta)\)th absolute moment. In this sub-section, we state three theorems on the growth rate of \( \log p \) in the UCLT over \( \mathcal{A}^{re} \). The first two are under the assumption of existence of \((2+\delta)\)th absolute moments of \( X_{ij} \)'s. On the other hand, Theorem 3 shows that sub-exponential growth rate of \( p \) is possible in iid case even when we only assume that \( X_{11} \) is in the domain of attraction of the normal distribution.

**Theorem 1.** Let \( \mathbb{E}|X_{11}|^{2+\delta} < \infty \) for all \( n \geq 1 \) and for some \( 0 < \delta \leq 1 \), \( d_{n,\delta} \to \infty \) as \( n \to \infty \). Then we have

\[
\tau_{n,\mathcal{A}^{re}} \to 0 \quad \text{as} \quad n \to \infty,
\]

provided \( \log p = o(d_{n,\delta}^2) \).

Theorem 1 shows that the UCLT (1.1) holds with \( \mathcal{A} = \mathcal{A}^{re} \) for \( p \) growing at the rate \( \exp\left(o\left(d_{n,\delta}^2\right)\right) \) with the sample size \( n \). In particular, for some \( 0 < \delta \leq 1 \) if \( m \leq \mathbb{E}|X_{11}|^{2}, \mathbb{E}|X_{11}|^{2+\delta} \leq M \) for all \( n \geq 1 \) for some constants \( m, M > 0 \), then \( \log p \) can grow like \( o\left(n^{\delta/(2+\delta)}\right) \) in the UCLT over \( \mathcal{A}^{re} \). Now a natural question is:
Does there exist an asymptotic upper bound on $\tau_{n,A^c}$ other than the trivial bound 1, even when $\log p$ is exactly of order $d_{n,\delta}^2$? Next theorem gives an answer to this question.

**Theorem 2.** Consider the setup of Theorem 1. Then there exists a positive constant $c \leq 1/8$ such that whenever $\log p \leq c d_{n,\delta}^2$ with $0 < \epsilon < c$,

$$\limsup_{n \to \infty} \tau_{n,A^c} < \epsilon^{(1+\delta)/3}.$$  

Theorem 2 is a refinement of Theorem 1 and shows that the uniform error of UCLT over $A^c$, namely, $\tau_{n,A^c}$, decreases in the power of $(1 + \delta)/3$ of the multiplier $\epsilon$ in the rate bound $p \leq \exp(c d_{n,\delta}^2)$ for a nontrivial set of $\epsilon$ bounded above by $1/8$. In particular, if $m \leq \mathbb{E}|X_{n1}|^2, \mathbb{E}|X_{n1}|^3 \leq M$ for all $n \geq 1$ for some constants $m, M > 0$, then $\tau_{n,A^c}$ is bounded by $\epsilon^{2/3}$ whenever $\log p \leq cn^{1/3}$.

Now there are two directions of improving Theorem 1 and Theorem 2. One direction is to establish sub-exponential rate of $p$ even when $(2 + \delta)$th absolute moment does not exist, whereas the other one is to improve the growth rate of $p$ under higher moment conditions. The later one is relegated to the next subsection. The first direction is considered in the next theorem when $X_{ij}$'s are iid and no moment condition on $X_{11}$ is assumed. The only assumption is that $X_{11}$ is in the domain of attraction of the normal distribution.

**Theorem 3.** Let $X_{ij}$'s are iid copies of $X_{11}$ for all $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, p\}$. Let $X_{11}$ be in the domain of attraction of the normal distribution with the distribution of $X_{11}$ being non-degenerate and symmetric around 0. Define $\omega_n = [\delta_{n,0}]^{-1/6}$. Then we have

$$\tau_{n,A^c} \to 0 \text{ as } n \to \infty,$$

provided $\log p = o(\omega_n^2)$. Moreover there exists a positive constant $c \leq 1/8$ such that whenever $\log p \leq c \omega_n^2$ with $0 < \epsilon < c$,

$$\limsup_{n \to \infty} \tau_{n,A^c} < \epsilon^{5/3}.$$  

Note that $X_{11}$ is in the domain of attraction of normal distribution if and only if

$$\lim_{r \to \infty} \frac{\mathbb{E} \mathbb{P}(|X_{11}| > r)}{\mathbb{E} \mathbb{P}(|X_{11}| \leq r)} = 0 \text{ (cf. Theorem 4 at page 323 of Chow and Teicher 1997).}$$

Since due to Lemma 1.3 of Bentkus and Götze (1996), $\kappa_{n,0} = \mathbb{E}\{X_{11}^2 I(|X_{11}| \leq \kappa_{n,0})\}$, the necessary and sufficient condition in turn implies that $\delta_{n,0} \to 0$ as $n \to \infty$ due to Lemma 1 of Görgö et al. (2003). However, $\mathbb{E}|X_{11}|^{2+\delta}$ may not exist for any $\delta > 0$ and hence we can not use Theorem 2. In particular, if we only know that $\mathbb{E}X_{11}^2 \log(1 + |X_{11}|)^\alpha < \infty$ for some $\alpha > 0$, then we can not apply Theorem 2. However here $\delta_{n,0} \leq (\log(1 + \sqrt{n}))^{-\alpha}$ and hence using theorem 3 UCLT over $A^c$ holds if $\log p = o((\log n)^{\alpha/3})$. Therefore, $\log p$ can still grow sub-exponentially with $n$ even without existence of absolute polynomial moment $\geq 2$.

**Remark 2.1.** Our results of this section should be compared with the UCLT results available in the literature in case of the normalized sums. The rate $\log p = o(n^{1/4})$ in the UCLT $\rho_{n,A^n_{\text{max}}} \to 0$ of the normalized sum $U_n$ is established in Kuchibhotla et al. (2021) when $\mathbb{E}\exp(\ell|X_{ij}|^a) < \infty$ for some $0 < \alpha \leq 2$ and $t > 0$ (i.e. $X_{ij}$'s are sub-Weibull), for all $(i,j)$. On the other hand, here we show that to
achieve \( \log p = o(n^{1/4}) \) in the UCLT \( \tau_{n,A^{re}} \to 0 \) of the self-normalized sum \( T_n \) we need \( \mathbb{E}|X_{ij}|^{8/3} < \infty \), for all \((i,j)\). Again the rate \( \log p = o(n^{1/3}(\log n)^{-2/3}) \) in \( \rho_{n,A^{re}} \to 0 \) is obtained by Fang and Koike (2020) under the assumption of the random vectors \( X_i \)'s having log-concave densities (i.e. when \( X_{ij} \) are sub-exponential for all \((i,j)\)). Whereas for the rate \( \log p = o(n^{1/3}) \) in \( \tau_{n,A^{re}} \to 0 \), we need only the existence of third absolute moments of \( X_{ij} \) for all \((i,j)\).

Remark 2.2. The rate of \( \log p \) obtained in Theorem 3 should be optimal with respect to the underlying moment conditions. This is because the proof of Theorem 3 essentially relies on a non-uniform Berry-Esseen result for the self-normalized sums, obtained by Jing et al. (2003). This non-uniform Berry-Esseen result is sharp and hence can not be improved, as shown by Chistyakov and Götze (2004a). However, it is not clear at this time whether the rate of \( \log p \) in Theorem 3 is optimal or not. If \( \mathbb{E}|X_{11}|^{2+6\delta} < \infty \) for some \( 0 < \delta \leq 1 \), then \( \delta_{n,0} \leq n^{-\delta/2} \) and hence the rate of \( \log p \) as obtained by Theorem 3 is \( o(n^{5/6}) \). This rate is clearly not as good as the rate \( o(n^{3(2+\delta)}) \) which can be achieved by applying Theorem 4.

### 2.2. Rate of growth of \( p \) under fourth moment

In this sub-section we try to improve the results obtained in the previous sub-section on the rate of growth of \( p \) under higher moment conditions. In particular, we show that \( \log p \) can grow like \( o(n^{1/2}) \) under the existence of fourth moment. Additionally, the UCLT over \( A^{re} \) holds with an error \( \epsilon \) whenever \( \log p = cn^{1/2} \) for a non trivial set of \( \epsilon > 0 \). Now the natural question is if the growth rate of \( \log p \) can be improved further from \( n^{1/2} \) as a power of \( n \) under stronger moment conditions. The answer is no even when \( X_{ij} \)'s are iid across \((i,j)\) and \( X_{11} \) is stochastically bounded. The rate of growth of \( \log p \) in the UCLT of \( T_n \) over \( A^{re} \) essentially stabilizes at \( n^{1/2} \) whenever fourth or higher absolute moments of \( X_{ij} \) exists. We are now ready to state our first theorem of this sub-section.

**Theorem 4.** Let \( \max_{i=1,\ldots,n} \mathbb{E}|X_{i1}|^4 \leq O(1) \) and \( \lim \inf_{n \to \infty} \sigma_n^2 > 0 \). Then we have

\[
\tau_{n,A^{re}} \to 0 \quad \text{as} \quad n \to \infty,
\]

provided \( \log p = o(n^{1/2}) \). Moreover, there exists a positive constant \( c \leq 1/8 \) such that whenever \( \log p = cn^{1/2} \) with \( 0 < \epsilon < c \),

\[
\limsup_{n \to \infty} \tau_{n,A^{re}} < \epsilon.
\]

The next result shows that for a UCLT even over the smaller class of sets \( A^{max} \), the \( o(n^{1/2}) \) upper bound on \( \log p \) can not be significantly improved upon.

**Theorem 5.** Let \( X_{ij} \)'s be iid Rademacher variables, i.e. \( X_{ij} = 1 \) or \(-1\) each with probability \( 1/2 \) and be independent across \( i \in \{1,\ldots,n\} \) and \( j \in \{1,\ldots,p\} \). If \( \lim\sup_{n \to \infty} n^{-(\kappa+1/2)} \log p > 0 \) for some \( \kappa > 0 \), then

\[
(2.1) \quad \tau_{n,A^{max}} \to 0 \quad \text{as} \quad n \to \infty.
\]

From Theorem 4 and Theorem 5 it is clear that the best possible growth rate of \( \log p \) in the UCLT (1.1) of the self normalized random vector \( T_n \) over the class of sets \( A^{re} \) or \( A^{max} \) or \( A^{dist} \) is \( o(n^{1/2}) \) as a power of \( n \).

**Remark 2.3.** The proof of Theorem 5 follows essentially through the same line of the proof of Theorem 3 of Das and Lahiri (2020). Moreover, under the setup
of Theorem 5 a little stronger version of (2.1) holds. It can be shown that if 
\( \limsup_{n \to \infty} \left[ (n \log n)^{-1/2} \log p \right] > 0 \), then also (2.1) is true. The proof depends essentially on the choice of a set \( A \in {\mathcal{A}}^{\text{max}} \). This choice again depends on the choice of a function \( f(n) \). Instead of considering the lower bound of \( f(n) \) to be \( n^{\kappa/4} \) as considered in Das and Lahiri (2020), here we need to consider \( 2(\log n)^{1/4} \) as the lower bound on \( f(n) \). Then choosing \( \sqrt{n} f(n) \) when \( 2\sqrt{n} \log n < \log p < \left[ \frac{3 \log n}{2(1+\eta)} + \frac{n}{2(1+\eta)^2} \right] \), \( f(n) = n^{1/4} \) when \( \left[ \frac{3 \log n}{2(1+\eta)} + \frac{n}{2(1+\eta)^2} \right] \leq \log p \leq \log \left[ \frac{\sqrt{2}}{\sqrt{n} + 4 + \sqrt{n} \eta^{n/2}} \right] \) and \( f(n) = n^{1/4} \) when \( p > \frac{\sqrt{2}}{\sqrt{n} + 4 + \sqrt{n} \eta^{n/2}} \) with the choice of \( \eta > 0 \) such that \( \left[ \frac{1}{(1+\eta)^2} + \frac{1}{\eta(1+\eta)^2} - 1 \right] \geq 0 \), all the steps of the proof go through and the stronger version follows.

3. An application to Student’s t statistic

In this section we apply the results obtained in the previous section for finding UCLT for high dimensional component-wise Student’s t statistic. Recall that based on the mean zero random vectors \( X_1, \ldots, X_n \), the corresponding student’s t statistic is \( W_n = (W_n^1, \ldots, W_n^p)' \), where

\[
W_{nj} = \frac{\sqrt{n} X_{nj}}{\sqrt{(n-1)^{-1} \sum_{i=1}^n (X_{ij} - \bar{X}_{nj})^2}},
\]

where \( \bar{X}_{nj} = n^{-1} \sum_{i=1}^n X_{ij} \). The statistic \( W_n \) can be used for drawing high dimensional inference based on the sample \( X_1, \ldots, X_n \). In fact in most of the practical applications, component-wise studentization is more natural to perform than component-wise standardization since component-wise population variances are not in general available. In statistical terms, \( W_n \) is always a statistic irrespective of whether underlying variance structure is known or unknown. This is not true for the standardized sum \( U_n \). To that end, we are interested in high dimensional UCLT for \( W_n \) by investigating the quantity given by

\[
(3.1) \quad \gamma_{n,A} \equiv \sup_{A \in {\mathcal{A}}} \left| P(W_n \in A) - P(Z \in A) \right|
\]

for some class \( A \) of convex subsets of \( \mathbb{R}^p \). We show that results similar to Section 2 continue to hold for \( \gamma_{n,A} \) whenever \( A = {\mathcal{A}}^{\text{re}} \) or \( {\mathcal{A}}^{\text{max}} \) or \( {\mathcal{A}}^{\text{dist}} \). As mentioned in the Section 1 the identity between \( T_{nj} \) and \( W_{nj} \) that is crucial in establishing a high dimensional UCLT for \( W_n \) from that for \( T_n \) is given by

\[
(3.2) \quad P(W_{nj} \geq x) = P(T_{nj} \geq x \left( \frac{n}{n + x^2 - 1} \right)^{1/2}),
\]

for any \( x \geq 0 \), and all \( j \in \{1, \ldots, p\} \). As a result the asymptotic distribution of \( T_{nj} \) and \( W_{nj} \) are same for all \( j \in \{1, \ldots, p\} \). To that end, we are now ready to state the high dimensional UCLT result for \( W_n \).

**Theorem 6.** Statements of the Theorems 1, 2, 3, 4 are true if \( \tau_{n,A^{\text{re}}} \) is replaced by \( \gamma_{n,A^{\text{re}}} \).

Theorem 6 shows that all the high dimensional UCLT results that we developed for self-normalized sums, also hold for component-wise Student’s t-statistic. Therefore the same rate of \( \log p \) can be achieved in the high dimensional UCLT
over \( \mathcal{A}^{\tau} \) under much weaker moment conditions if we use studentization in place of standardization. For log \( p \) to grow like \( o(n^{1/4}) \) in the UCLT for \( W_n \) over \( \mathcal{A}^{\tau} \) we need \( \mathbb{E}|X_{ij}|^{8/3} < \infty \), for all \((i, j)\), to be true, whereas the same rate of \( p \) in the UCLT of \( U_n \) is established for the class \( \mathcal{A}^{max} \) under the condition that \( \mathbb{E}\exp \left( t|X_{ij}|^3 \right) < \infty \) for some \( 0 < \alpha \leq 2 \) and \( t > 0 \) (i.e. \( X_{ij} \)'s are sub-Weibull), for all \((i, j)\) , in Kuchibhotla et al. (2021). Fang and Koike (2020) obtained the rate \( \log p = o(n^{1/3}(\log n)^{-2/3}) \) in \( \rho_{n, A^{\tau}} \to 0 \) under the assumption of the random vectors \( X_i \)'s having log-concave densities (i.e. when \( X_{ij} \) are sub-exponential for all \((i, j)\)). Whereas for the rate \( \log p = o(n^{1/3}) \) in \( \tau_{n, A^{\tau}} \to 0 \), we need only the existence of third absolute moments of \( X_{ij} \) for all \((i, j)\). Again to achieve \( \log p = o(n^{1/2}) \) in \( \gamma_{n, A^{\tau}} \to 0 \) we need \( \mathbb{E}|X_{ij}|^4 < \infty \), for all \((i, j)\). Whereas \( X_{ij} \)'s are assumed to be sub-Gaussian to achieve \( \log p = o(n^{1/2}) \) in \( \rho_{n, A^{\tau}} \to 0 \), as established in Das and Lahiri (2020). Additionally under the more specialized iid structure of \( X_i \)'s, sub-exponential growth rate of \( p \) can be achieved in (3.1) with \( \mathcal{A} = \mathcal{A}^{\tau} \) without any moment conditions, only the assumption of \( X_{11} \) being in the domain of attraction of normal distribution is sufficient.

4. PROOFS OF THE RESULTS

Suppose, \( \Phi(\cdot) \) and \( \phi(\cdot) \) respectively denote the cdf and pdf of the standard normal random variable in any dimension. Define \( N_i = (N_{i1}, \ldots, N_{ip})' \), \( i \in \{1, \ldots, n\} \) where \( N_{ij} \)'s are iid \( N(0, 1) \) random variables for all \( i \in \{1, \ldots, n\} \) and \( j \in \{1, \ldots, p\} \). For any vector \( t = (t_1, \ldots, t_p) \in \mathcal{R} \), let \( t_{(j)} \) and \( t^{(j)} \) respectively denote the \( j \)th element after sorting the components of \( t \) in increasing order and in decreasing order. (We use boldface font only for \( t \) to avoid some notational conflict later on. All other vectors are denoted using regular font). For any random variable \( H \), \( P(H \leq x) \) is assumed to be 1 if \( x = \infty \). Any absolute constant is denoted by \( A \). \( C, C_1, C_2, \ldots \) denote generic constants which depend only on the underlying distribution of the random vectors \( X_1, \ldots, X_n \).

Recall that for a sequence of random variables \( \{Y_i\}_{i \geq 1}, \sigma_n^2(Y) = n^{-1} \sum_{i=1}^n Var(Y_i) \), for any \( k > 0 \), \( \beta_{n,k}(Y) = n^{-1} \sum_{i=1}^n E|Y_i|^{2+k} \) and the Lyapunov’s ratio is

\[
ed_{n,k}(Y) = \left( \frac{\sqrt{n} \sigma_n(Y)}{n \beta_{n,k}(Y)} \right)^{1/(2+k)} \]

For some random variable \( Z \) recall that

\[
d_{n,z}(Z) = nP\left(|Z| > \kappa_{n,z}(Z) \right) + n[\kappa_{n,z}(Z)]^{-1}E\{ZI(|Z| \leq \kappa_{n,z}(Z))\} + n[\kappa_{n,z}(Z)]^{-3}E\{|Z|^3 I(|Z| \leq \kappa_{n,z}(Z))\}, \]

where \( \kappa_{n,z}(Z) = \sup \left\{ s : ns^{-2}E\{Z^2 I(|Z| \leq s)\} \geq 1 + x^2 \right\} \). Whenever \( X_{ij} \)'s are the underlying random variables, then we simply write \( \sigma^2_{ij}, \beta_{n,k}, d_{n,k} \) and \( \kappa_{n,x} \).

We will need some lemmas which are stated next. Proofs of the theorems are given in Section 4.2 below.

4.1. Auxiliary Lemmas.

**Lemma 1.** For any \( t > 0 \), \( \frac{1}{t} \geq \frac{1 - \Phi(t)}{\phi(t)} \geq \frac{2}{\sqrt{t^2 + 4} + t} \).


Proof of Lemma 1: This inequality is proved in Birnbaum (1942).

Lemma 2. Let $Z_1, \ldots, Z_n$ be independent random variables which are symmetric around 0. Then for any $x \in \mathcal{R}$ and $x \neq 0$, we have
\[
\left| P\left( \frac{\sum_{i=1}^{n} Z_i}{\sqrt{\sum_{i=1}^{n} Z_i^2}} \leq x \right) - \Phi(x) \right| \leq \left( 1 + \frac{1}{\sqrt{2\pi |x|}} \right) e^{-x^2/2}.
\]
Proof of Lemma 2: Due to symmetry of $Z_i$’s around 0, it is enough to establish the statement of the lemma for $x > 0$. Now, by Lemma 4.3 of Wang and Jing (1999), we have
\[
P\left( \frac{\sum_{i=1}^{n} Z_i}{\sqrt{\sum_{i=1}^{n} Z_i^2}} > x \right) \leq e^{-x^2/2}.
\]
Now using Lemma 1, we have $(1 - \phi(x)) \leq x^{-1} \phi(x)$. Combining these two facts we are done.

Lemma 3. Let $Z_1, \ldots, Z_n$ be independent random variables with $E|Z_i|^{2+\delta} < \infty$ for all $i \in \{1, \ldots, n\}$ for some $0 < \delta \leq 1$. Then for $0 \leq |x| \leq d_{n,\delta}(Z)$, we have
\[
P\left( \frac{\sum_{i=1}^{n} Z_i}{\sqrt{\sum_{i=1}^{n} Z_i^2}} \leq x \right) - \Phi(x) \leq A(1 + |x|)^{1+\delta} e^{-x^2/2} \left[ d_{n,\delta}(Z) \right]^{-(2+\delta)},
\]
where $A > 0$ is an absolute constant.

Proof of Lemma 3: This lemma follows from (2.8) and (2.9) of Theorem 2.3 of Jing et al. (2003).

Lemma 4. Suppose that $Z, Z_1, \ldots, Z_n$ are iid non-degenerate random variables. $Z$ is in the domain of attraction of the normal law. Recall the definition of $\delta_{n,0}(Z)$ from Section 4. Define, $\omega_n(Z) = \left[ \delta_{n,0}(Z) \right]^{-1/6}$. Then there exists a constant $C_3 > 0$, independent of $n, x$, such that for $n \geq C_3$,
\[
|P\left( \frac{\sum_{i=1}^{n} Z_i}{\sqrt{\sum_{i=1}^{n} Z_i^2}} \leq x \right) - \Phi(x) | \leq C_3 \delta_{n,0}(Z) \max\{1, x^5\} e^{-x^2/2},
\]
whenever $2|x| \leq \omega_n(Z)$.

Proof of Lemma 4: When $2|x| \leq 2$, then (4.1) follows directly from Theorem 1.4 of Bentkus and Götze (1996). Now let us look into when $2 < 2|x| \leq \omega_n(Z)$. Recall the definitions of $\kappa_{n,x}(Z)$ and $\delta_{n,x}(Z)$ from Section 2. Then from Remark 3 of Robinson and Wang (2005), we have
\[
P\left( \frac{\sum_{i=1}^{n} Z_i}{\sqrt{\sum_{i=1}^{n} Z_i^2}} \leq x \right) - \Phi(x) \leq A \delta_{n,|x|}(Z) (1 + |x|)^{-1} e^{-x^2/2},
\]
for some absolute constant $A > 0$, whenever $\delta_{n,|x|}(Z) \leq 1$. Now the aim is to replace $\delta_{n,|x|}(Z) (1 + |x|)^{-1}$ in RHS of (4.2) with $\delta_{n,0}(Z) |x|^\delta$ when $2 \leq 2|x| \leq \omega_n(Z)$. To do that we need both upper and lower bound on $\kappa_{n,|x|}(Z)$ in terms of $\kappa_{n,0}(Z)$. It is easy to see that $\kappa_{n,|x|}(Z) \leq \kappa_{n,0}(Z)$ for any $x$. Regarding the lower bound, we claim that $\kappa_{n,0}(Z) / (1 + x^2) \leq \kappa_{n,|x|}(Z)$ for large enough $n$ uniformly for $2 \leq 2|x| \leq \omega_n(Z)$. Note that using Lemma 1.3 of Bentkus and Götze (1996) we have
\[
\frac{\kappa_{n,0}(Z)}{1 + x^2} = \sup \left\{ s : ns^{-2} E Z^2 I(|Z| \leq s(1 + x^2)) = 1 + x^2 \right\}
\]
and \( k_{n,0}(Z)/\sqrt{n} \) to be positive for sufficiently large \( n \). Hence, \( k_{n,0}(Z)/(1+x^2) \to \infty \) as \( n \to \infty \), uniformly for \(|x| \leq \omega_n(Z)\) since \( \omega_n(Z) = o(n^{1/4}) \). To show \( k_{n,0}(Z)/(1+x^2) \leq \kappa_{n,|x|}(Z) \) uniformly in \( 2 \leq 2|x| \leq \omega_n(Z) \) for sufficiently large \( n \), enough to show \( \mathbb{E}Z^2I(|Z| \leq k_{n,0}(Z)) \leq (1+x^2)\mathbb{E}Z^2I(|Z| \leq k_{n,0}(Z))/(1+x^2) \), uniformly in \( 2 \leq 2|x| \leq \omega_n(Z) \), for sufficiently large \( n \). This is indeed true since \( \mathbb{E}Z^2I(|Z| \leq s) \) is slowly varying as a function of \( s \) at infinity [cf. Lemma 1(a) of Cörgő et al. (2003)] and the fact that \( k_{n,0}(Z)/(1+x^2) \to \infty \) as \( n \to \infty \) uniformly for \(|x| \leq \omega_n(Z)\). Therefore we have for sufficiently large \( n \),

\[
\delta_{n,|x|}(Z) = n\mathbb{P}\left(|Z| > \kappa_{n,|x|}(Z)\right) + n\left[\kappa_{n,|x|}(Z)\right]^{3}\mathbb{E}\left\{ [Z]^3I(|Z| \leq \kappa_{n,|x|}(Z))\right\} \\
\leq n\mathbb{P}\left(|Z| > \kappa_{n,0}(Z)/(1+x^2)\right) + n\left(1+x^2\right)^3\left[\kappa_{n,0}(Z)\right]^{3}\mathbb{E}\left\{ [Z]^3I(|Z| \leq \kappa_{n,0}(Z))\right\} \\
= 3\left(1+x^2\right)^3\delta_{n,0}(Z),
\]

uniformly in \( 2 \leq 2|x| \leq \omega_n(Z) \). From (4.2) and (4.4), for sufficiently large \( n \) we have

\[
\mathbb{P}\left(\sum_{i=1}^{n} Z_i/\sqrt{\sum_{i=1}^{n} Z_i^2} \leq x\right) - \Phi(x) \leq 3A(1+x^2)^3\delta_{n,0}(Z)(1+|x|)^{-1}e^{-x^2/2} \\
\leq 96A\delta_{n,0}(Z)|x|^5e^{-x^2/2},
\]

whenever \( x \in \left\{\left[3(1+y^2)^3\delta_{n,0}(Z)\right] \leq 1\right\} \cap \left\{2 \leq 2|y| \leq \omega_n(Z)\right\} \). Again apply Lemma 1.3 of Bentkus and Götze (1996) and Lemma 1 of Csörgő et al. (2003) to claim that \( \delta_{n,0}(Z) \to 0 \) as \( n \to \infty \). Therefore, (4.5) follows from (4.3).

**Lemma 5.** Let \( Z_1, \ldots, Z_n \) be independent random variables which are symmetric around 0, \( \max_{1 \leq i \leq n} \mathbb{E}|Z_i|^4 = O(1) \) and \( \liminf_{n \to \infty} \sigma_n^2(Z) > 0 \). Then whenever \(|x| \leq n^{1/4}\), for sufficiently large \( n \) we have

\[
\mathbb{P}\left(\sum_{i=1}^{n} Z_i/\sqrt{\sum_{i=1}^{n} Z_i^2} \leq x\right) - \Phi(x) \leq C_2\left(\frac{x^2}{n^{3/2}} + \frac{|x|^3}{n} + \frac{1}{\sqrt{n}}\right)e^{-x^2/2},
\]

for some constant \( C_2 > 0 \), independent of \( n, x \).

**Proof of Lemma 5** When \( 0 \leq x \leq n^{1/4} \), the statement is a direct consequence of Corollary 2.2 of Sang and Ge (2017). The statement for \( 0 \geq x \geq -n^{1/4} \) follows due to symmetry of the distribution of \( Z_i \)'s around 0.

**Lemma 6.** For any positive integer \( m \),

\[
\sqrt{2\pi} \ m^{m+1/2}e^{-m} \leq m! \leq m^{m+1/2}e^{-m+1}.
\]

This is the well-known Stirling’s formula. See for example Robbins (1955).

**Lemma 7.** Let \( Z_1, \ldots, Z_n \) be a sequence of mean zero independent random vectors in \( \mathbb{R}^p \) with \( Z_i = (Z_{i1}, \ldots, Z_{ip}) \), \( i \in \{1, \ldots, n\} \) and let \( \{Z_{i1}, \ldots, Z_{ip}\} \) be iid for each \( i \in \{1, \ldots, n\} \) with \( \sigma_{n,j}^2 = \sum_{i=1}^{n} Z_{ij}^2 \). Define, \( l_1(x) = \max\left\{\mathbb{P}\left(d_{n,1}^{-1}\sum_{i=1}^{n} (-Z_{i1}) \leq x\right), \Phi(x)\right\}, \quad d_1(x) = \mathbb{P}\left(d_{n,1}^{-1}\sum_{i=1}^{n} (-Z_{i1}) \leq x\right) - \Phi(x), \quad l_2(x) = \max\left\{\mathbb{P}\left(d_{n,1}^{-1}\sum_{i=1}^{n} (-Z_{i1}) \leq x\right)\right\}
\]


\[\sum_{i=1}^{n} Z_{i} \leq x\), \Phi(x)\} and \(d_{2}(x) = \left| P \left( d_{1}^{-1} \sum_{i=1}^{n} Z_{i} \leq x \right) - \Phi(x) \right|.\] Also define, \(D_{n} = \text{diag}(d_{n1}, \ldots, d_{np})\). Then we have
\[
\left| P \left( D_{n}^{-1} \sum_{i=1}^{n} Z_{i} \in \bigcap_{j=1}^{p} \{ [a_{j}, b_{j}] \cap \mathcal{R} \} \right) - P \left( n^{-1/2} \sum_{i=1}^{n} N_{i} \in \bigcap_{j=1}^{p} \{ [a_{j}, b_{j}] \cap \mathcal{R} \} \right) \right| \\
\leq L_{1}(\mathbf{a}) + L_{2}(\mathbf{b}),
\]
where \(\mathbf{a} = (a_{1}, \ldots, a_{p})', \mathbf{b} = (b_{1}, \ldots, b_{p})'\),
\[
L_{1}(\mathbf{a}) = \left[ \sum_{k=1}^{p} \left( \prod_{j \neq k} l_{1}(-a^{(j)}) \right) d_{1}(-a^{(k)}) \right], \quad L_{2}(\mathbf{b}) = \left[ \sum_{k=1}^{p} \left( \prod_{j \neq k} l_{2}(b^{(j)}) \right) d_{2}(b^{(k)}) \right].
\]

**Proof of Lemma 7.** Note that \(D_{n}^{-1} \sum_{i=1}^{n} Y_{i} = (W_{1}, \ldots, W_{p})'\) where \(W_{j} = d_{nj}^{-1} \sum_{i=1}^{n} Y_{ij}, j \in \{1, \ldots, p\}\). Then, using the nature of \(Y_{ij}\)’s, it is easy to see that \(\{W_{1}, \ldots, W_{p}\}\) are iid. Similarly, since \(N_{ij} \sim N(0, 1)\) are iid, the \(p\)-variables \(\left( n^{-1/2} \sum_{i=1}^{n} N_{ij} \right), \ldots, \left( n^{-1/2} \sum_{i=1}^{n} N_{ip} \right)\) are also iid. Hence we have
\[
\left| P \left( D_{n}^{-1} \sum_{i=1}^{n} Z_{i} \in \bigcap_{j=1}^{p} \{ [a_{j}, b_{j}] \cap \mathcal{R} \} \right) - P \left( n^{-1/2} \sum_{i=1}^{n} N_{i} \in \bigcap_{j=1}^{p} \{ [a_{j}, b_{j}] \cap \mathcal{R} \} \right) \right| \\
= \left| \prod_{j=1}^{p} P \left( W_{j} \in [a_{j}, b_{j}] \cap \mathcal{R} \right) - \prod_{j=1}^{p} P \left( n^{-1/2} \sum_{i=1}^{n} N_{ij} \in [a_{j}, b_{j}] \cap \mathcal{R} \right) \right| \\
\leq \left[ \sum_{k=1}^{p} \left( \prod_{j \neq k} \min \{ l_{1}(-a_{j}), l_{2}(b_{j}) \} \right) \right] \left[ d_{1}(-a_{k}) + d_{2}(b_{k}) \right] \\
\leq \sum_{k=1}^{p} \left( \prod_{j \neq k} l_{1}(-a_{j}) \right) \left[ d_{1}(-a_{k}) \right] + \sum_{k=1}^{p} \left( \prod_{j \neq k} l_{2}(b_{j}) \right) \left[ d_{2}(b_{k}) \right] \\
= \sum_{k=1}^{p} \left( \prod_{j \neq k} l_{1}(-a^{(j)}) \right) \left[ d_{1}(-a^{(k)}) \right] + \sum_{k=1}^{p} \left( \prod_{j \neq k} l_{2}(b^{(j)}) \right) \left[ d_{2}(b^{(k)}) \right]
\]
The last equality is due to the following fact:
If \((G_{1}, H_{1}), \ldots, (G_{p}, H_{p})\) are iid random vectors in \(\mathcal{R}^{2}\), then for any \(t_{1}, \ldots, t_{p} \in \mathcal{R},\)
\[
\sum_{k=1}^{p} \left[ \left( \prod_{j \neq k} \max \{ P \left( G_{j} \leq t_{j} \right), P \left( H_{j} \leq t_{j} \right) \} \right) \right] \left| P \left( G_{k} \leq t_{k} \right) - P \left( H_{k} \leq t_{k} \right) \right| \\
= \sum_{k=1}^{p} \left[ \left( \prod_{j \neq k} \max \{ P \left( G_{1} \leq t_{(j)} \right), P \left( H_{1} \leq t_{(j)} \right) \} \right) \right] \left| P \left( G_{1} \leq t_{(k)} \right) - P \left( H_{1} \leq t_{(k)} \right) \right],
\]
where \(\{t_{(1)}, t_{(2)}, \ldots, t_{(p)}\}\) are obtained after sorting \(\{t_{1}, \ldots, t_{p}\}\) in increasing order. Therefore we are done.

**4.2. Proofs of the main results. Proof of Theorem 1** Suppose that \(T_{n} = \left( \frac{\sum_{i=1}^{n} X_{i}}{\sqrt{\sum_{i=1}^{n} X_{i}^{2}}} \right) \) and \(S_{n} = n^{-1/2} \sum_{i=1}^{n} N_{i} \). Let \(T = (T_{n1}, \ldots, T_{np})'\) and \(S = (S_{n1}, \ldots, S_{np})'\). Clearly \(T_{nj}\)'s are iid and \(S_{nj}\)'s are iid for \(j \in \{1, \ldots, p\}\).
We can use Lemma 7 with $Z_i = X_i$ for $i \in \{1, \ldots, n\}$, to obtain
\[
\left| P\left( T_n \in \prod_{j=1}^{p} \{[a_j, b_j] \cap R\} \right) \right| - P\left( S_n \in \prod_{j=1}^{p} \{[a_j, b_j] \cap R\} \right) \leq L_1(a) + L_2(b),
\]
where $L_1(a)$ and $L_2(b)$ are as defined in Lemma 7. Since all the assumptions are also satisfied if we replace $\{X_1, \ldots, X_n\}$ by $\{-X_1, \ldots, -X_n\}$, it is enough to show
\[
(4.6) \quad \sup_{t_1 \leq t_2 \leq \cdots \leq t_p} L((t_1, \ldots, t_p)) = \sup_{t_1 \leq t_2 \leq \cdots \leq t_p} \left[ \sum_{j=1}^{p} \left( \prod_{j \neq k} l(t_j) \right) d(t_k) \right] = o(1), \quad n \to \infty.
\]
Here, $l(x) = \max \left\{ P\left( T_{n1} \leq x \right), P\left( S_{n1} \leq x \right) \right\}$ and $d(x) = P\left( T_{n1} \leq x \right) - P\left( S_{n1} \leq x \right)$. Note that we are done if we can show $L(t) = L((t_1, \ldots, t_p)) \leq A_n$ for sufficiently large $n$, where $A_n$ does not depend on $t$, and $A_n = o(1)$ as $n \to \infty$.

Since $\log p = o(d_{n,\delta}^2)$ and $d_{n,\delta} \to \infty$ as $n \to \infty$, there exists a sequence of positive numbers $a_n$ increasing to $\infty$ such that $\log p = O\left( a_n^{-3} d_{n,\delta}^2 \right)$. Without loss of generality assume $a_n^3 = o(d_{n,\delta}^2)$. Now fix $t = (t_1, \ldots, t_p')$ in $\mathbb{R}^p$ such that $t_1 \leq t_2 \leq \cdots \leq t_p$. Then there exist integers $l_1, l_2, l_3$, depending on $n$, such that $0 \leq l_1, l_2, l_3 \leq p$ and
\[
(4.7) \quad t_1 \leq t_2 \leq \cdots \leq t_{l_1} < -a_n^{-1} d_{n,\delta},
\]
\[
-t_{l_2} < t_{l_2} \leq \cdots \leq t_{l_3} < a_n^{-1} d_{n,\delta},
\]
\[
-t_{l_3} < t_{l_3} \leq \cdots \leq t_{l_p} < 1.
\]
Since $a_n^3 = o(d_{n,\delta})$ and $a_n \to \infty$ as $n \to \infty$, hence due to Lemma 1 and Lemma 3, we have for sufficiently large $n$,
\[
(4.8) \quad l(x) \leq I\left( x > a_n^{-1} d_{n,\delta} \right) + \left[ 1 - \frac{2\phi(1)}{\sqrt{\delta n}} + A 2^{1+\delta} e^{-x^2/2 d_{n,\delta}^{-1}} \right] I\left( x < 1 \right) + \left[ 1 - \frac{2\phi(x)}{\sqrt{x^2 + 4 + x}} + A (1 + x)^{1+\delta} e^{-x^2/2 d_{n,\delta}^{-1}} \right] I\left( x \in \left[ 1 - a_n^{-1} d_{n,\delta}, 1 \right) \right)
\]
\[
\leq I\left( x > a_n^{-1} d_{n,\delta} \right) + \left[ 1 - \frac{\phi(1)}{\sqrt{\delta n}} \right] I\left( x < 1 \right) + \left[ 1 - \frac{2\phi(x)}{\sqrt{x^2 + 4 + x}} + A (1 + x)^{1+\delta} e^{-x^2/2 d_{n,\delta}^{-1}} \right] I\left( x \in \left[ 1 - a_n^{-1} d_{n,\delta}, 1 \right) \right).
\]

for any $x \in \mathbb{R}$. $I(\cdot)$ is the indicator function. Again due to Lemma 1, Lemma 2 and Lemma 3 for sufficiently large $n$ we have for any $x \in \mathbb{R}$,
\[
(4.9) \quad d(x) \leq \left[ \left( 1 + \frac{1}{\sqrt{2\pi} |x|} \right) e^{-x^2/2} I\left( |x| > a_n^{-1} d_{n,\delta} \right) \right] + \left[ A (1 + |x|)^{1+\delta} e^{-x^2/2 d_{n,\delta}^{-1}} \right] I\left( |x| \leq a_n^{-1} d_{n,\delta} \right)
\]
\[
\leq 2 e^{-x^2/2} I\left( |x| > a_n^{-1} d_{n,\delta} \right) + A 2^{1+\delta} a_n^{-1} e^{-x^2/2} I\left( |x| \leq a_n^{-1} d_{n,\delta} \right).
\]

Therefore from equations (4.6) - (4.9), we have
\[
L(t) \leq I_1(t) + I_2(t) + I_3(t) + I_4(t),
\]
where

\[ I_1(t) = \left( 1 - \frac{\phi(1)}{\sqrt{5} + 1} \right)^{l_2 - 1} \times \left( \prod_{j=l_2+1}^{l_3} \left[ 1 - (4\pi)^{-1/2} a_n d_{n,\delta}^{-1} e^{-t_2^2/2} \right] \right) \]

\[ * \left( \sum_{k=1}^{l_1} 2e^{-t_k^2/2} \right) * I(l_1 \geq 1), \]

\[ I_2(t) = \left( 1 - \frac{\phi(1)}{\sqrt{5} + 1} \right)^{l_2 - 1} \times \left( \prod_{j=l_2+1}^{l_3} \left[ 1 - (4\pi)^{-1/2} a_n d_{n,\delta}^{-1} e^{-t_j^2/2} \right] \right) \]

\[ * \left( \sum_{k=l_2+1}^{l_2} A^{2+\delta} a_n^{-1} d_{n,\delta}^{-1} e^{-t_j^2/2} \right) * I(l_2 - l_1 \geq 1), \]

\[ I_3(t) = \left( 1 - \frac{\phi(1)}{\sqrt{5} + 1} \right)^{l_2} * I(l_3 - l_2 \geq 1) \]

\[ \times \left( \sum_{k=l_2+1}^{l_3} A^{2+\delta} a_n^{-1} d_{n,\delta}^{-1} e^{-t_j^2/2} \left( \prod_{j=l_2+1}^{l_3} \left[ 1 - (4\pi)^{-1/2} a_n d_{n,\delta}^{-1} e^{-t_j^2/2} \right] \right) \right). \]

\[ I_4(t) = \left( 1 - \frac{\phi(1)}{\sqrt{5} + 1} \right)^{l_2} \times \left( \prod_{j=l_2+1}^{l_3} \left[ 1 - (4\pi)^{-1/2} a_n d_{n,\delta}^{-1} e^{-t_j^2/2} \right] \right) \]

\[ * \left( \sum_{k=l_2+1}^{p} 2e^{-t_k^2/2} \right) * I(p - l_3 \geq 1). \]

Bound on \( I_1(t) + I_4(t) \): Note that since \( \log p = O(a_n^{-3} d_{n,\delta}^2) \), by looking into (4.17) we have for some \( 0 < M < \infty \),

\[ I_1(t) + I_4(t) \leq \left( \sum_{k=1}^{l_1} 2e^{-t_k^2/2} \right) I(l_1 \geq 1) + \left( \sum_{k=l_3+1}^{p} 2e^{-t_k^2/2} \right) I(p - l_3 \geq 1) \]

\[ \leq (l_1 + p - l_3) \left( 2e^{-a_n^{-2} d_{n,\delta}^2/2} \right) \]

\[ \leq p \left( 2e^{-a_n^{-2} d_{n,\delta}^2/2} \right) \]

\[ \leq 2e^{a_n^{-3} d_{n,\delta}^2(M - a_n^2/2)} \]

(4.11) \[ = A_{1n} \quad \text{(say)} \]

Bound on \( I_2(t) \): Let \( d^{-1} = \left[ 1 - \frac{\phi(1)}{\sqrt{5} + 1} \right] \). Then

\[ I_2(t) \leq \left( \left[ 1 - \frac{\phi(1)}{\sqrt{5} + 1} \right]^{l_2 - 1} \right) \left( \sum_{k=l_2+1}^{l_2} A^{2+\delta} a_n^{-1} d_{n,\delta}^{-1} e^{-t_j^2/2} \right) I(l_2 - l_1 \geq 1) \]

\[ \leq A^{2+\delta} d^{-1} \left( l_2 - 1 \right)^{-1} a_n^{-1} d_{n,\delta}^{-1} \]

\[ \leq A^{2+\delta} d a_n^{-1} d_{n,\delta}^{-1} \left[ \sup_{x>0} (xd^{-x}) \right]. \]
Now, \( \sup_{x>0} (xe^{-x}) = (\log c)^{-1} c^{-(\log c)^{-1}} \) for any \( c > 1 \). Therefore we have

\[
(4.12) \quad I_2(t) \leq \left( A2^{1+\delta}d(\log d)^{-1}d^{-1} \right) a_n^{-1}(1+\delta) = A_{2n} \quad \text{(say)}
\]

Bound on \( I_3(t) \): Note that if \( (l_3 - l_2) = 0 \) then \( I_3(t) = 0 \) and there is nothing more to do. Hence assume \( (l_3 - l_2) \geq 1 \). Then we have

\[
(4.13) \quad I_3(t) \leq \left( \sum_{k=l_2+1}^{l_3} A2^{1+\delta}a_n^{-1}(1+\delta) \prod_{j=l_2+1}^{l_3} \left[ 1 - (4\pi)^{-1/2}a_n\delta d_n^{-1}e^{-t_j^2/2} \right] \right) = I_{31}(t) \quad \text{(say)}.
\]

We are going to check the monotonicity of \( I_{31}(t) \) with respect to \( t_{l_2+1}, \ldots, t_{l_3} \). Note that

\[
\frac{\partial I_{31}(t)}{\partial t_l} = \left[ A2^{1+\delta}a_n^{-1}(1+\delta) \prod_{j=l_2+1}^{l_3} \left[ 1 - (4\pi)^{-1/2}a_n\delta d_n^{-1}e^{-t_j^2/2} \right] \right] \times \\
\left[ \sum_{k=l_2+1}^{l_3} \left[ 1 - (4\pi)^{-1/2}a_n\delta d_n^{-1}e^{-t_k^2/2} \right]^{-1} \left( 4\pi a_n\delta d_n^{-1}e^{-t_k^2/2} \right) - 1 \right]
\]

Hence for any \( l = l_2 + 1, \ldots, l_3 \), \( \frac{\partial I_{31}(t)}{\partial t_l} \leq 0 \) if and only if

\[
(4.14) \quad \sum_{j=l_2+1}^{l_3} \frac{z_j}{1 - z_j} \geq 1,
\]

where \( z_j = (4\pi)^{-1/2}a_n\delta d_n^{-1}e^{-t_j^2/2} \) for \( j \in \{l_2+1, \ldots, l_3\} \). Note that since \( 1 \leq t_{l_2+1} \leq \cdots \leq t_{l_3} > z_{l_2+1} \geq \cdots \geq z_{l_3} > 0 \) for sufficiently large \( n \). Hence for sufficiently large \( n \),

\[
\frac{z_{l_2+1}}{1 - z_{l_2+1}} \geq \cdots \geq \frac{z_{l_3}}{1 - z_{l_3}},
\]

due to the fact that \( z/(1-z) \) is increasing for \( z \in (0, 1) \). Therefore from \( 4.14 \) we can say that \( I_{31}(t) \) is non-increasing in \( \{t_{l_2+1}, \ldots, t_{l_3}\} \) and non-decreasing in \( \{t_{m+1}, \ldots, t_l\} \) where \( (m - l_2) \) is a non-negative integer not more than \( (l_3 - l_2) \). Again note that \( 1 \leq t_{l_2+1} \leq \cdots \leq t_m \leq t_{m+1} \leq \cdots t_{l_3} \leq a_n^{-1}d_n\delta \). Hence from \( 4.13 \) we have

\[
I_3(t) \leq I_{31}((t^{(1)}), (t^{(2)})^\prime)
\]

where \( t^{(1)} \) is an \((m - l_2) \times 1\) vector with each component being 1 and \( t^{(2)} \) is an \((l_3 - m) \times 1\) vector with each component being \( a_n^{-1}d_n\delta \). Therefore using the fact that \( \log p = O(a_n^{-3}d_n\delta) \) we can say that there exists \( M \in (0, \infty) \) such that for
sufficiently large $n$,

$$I_3(t) \leq (m - l_2) \left[ 1 - (4\pi)^{-1/2} a_n d_n \epsilon^{-1/2} \right]^{m-l_2-1} (A^{2^{1+\delta}} a_n^{1+\delta} d_n^{1+\delta} \epsilon^{-1/2})$$

\begin{align*}
&+ (l_3 - m)(A^{2^{1+\delta}} a_n^{-(1+\delta)} d_n^{-1} \epsilon^{-(2^{-1} a_n^{-2} d_n^{2})}) \\
&\leq 2 \left[ \exp \left( \log(m - l_2) - (m - l_2) ((4\pi)^{-1/2} a_n d_n \epsilon^{-1/2}) \right) \right] (A^{2^{1+\delta}} a_n^{-(1+\delta)} d_n^{-1} \epsilon^{-1/2}) \\
&+ p(A^{2^{1+\delta}} a_n^{-(1+\delta)} d_n^{-1} \epsilon^{-(2^{-1} a_n^{-2} d_n^{2})}) \\
&\leq 2 \left[ \exp \left( \sup_{x>0} \left( \log x - x ((4\pi)^{-1/2} a_n d_n \epsilon^{-1/2}) \right) \right) \right] (A^{2^{1+\delta}} a_n^{-(1+\delta)} d_n^{-1} \epsilon^{-1/2}) \\
&+ \left( A^{2^{1+\delta}} a_n^{-(1+\delta)} d_n^{-1} \epsilon^{-(2^{-1} a_n^{-2} d_n^{2})} \right) \\
&\leq ((4\pi)^{1/2} a_n^{-1} d_n \epsilon^{-1/2}) (A^{2^{1+\delta}} a_n^{-(1+\delta)} d_n^{-1} \epsilon^{-1/2}) \\
&+ \left( A^{2^{1+\delta}} a_n^{-(1+\delta)} d_n^{-1} \epsilon^{-(2^{-1} a_n^{-2} d_n^{2})} \right) \\
(4.15) &= A_{3n} \quad \text{(say)}
\end{align*}

Combining (4.11), (4.12) and (4.15), we have for sufficiently large $n$,

$$I(t) \leq A_{1n} + A_{2n} + A_{3n} = A_n \quad \text{(say)}.$$ 

Note that $A_n$ does not depend on the choice of $t$ and also $A_n \to 0$ as $n \to \infty$. Therefore, the proof of Theorem 1 is now complete.

**Proof of Theorem 2** We are going to follow the same steps as in the proof of Theorem 1. Note that we are done if in (4.10) we can show $L_n(t) \leq \epsilon^{(1+\delta)/3}/2$ for sufficiently large $n$, irrespective of the choice of $t$.

Now take $8c = \min \left\{ \left| A^{2^{1+\delta}} \sqrt{2\pi (\sqrt{2}+1)} \right|^{-3}, 1 \right\}$. Fix $t = (t_1, \ldots, t_p)$ in $\mathbb{R}^p$ such that $t_1 \leq t_2 \leq \cdots \leq t_p$. Then there exist integers $l_4, l_5, l_6$, depending on $n$, such that $0 \leq l_4, l_5, l_6 \leq p$ and

\begin{align*}
&t_1 \leq t_2 \leq \cdots \leq t_4 < -\epsilon^{1/3} d_n \delta \\
&-\epsilon^{1/3} d_n \delta \leq t_{l_4+1} \leq t_{l_4+1} \leq \cdots \leq t_{l_5} < 1 \\
&1 \leq t_{l_5+1} \leq t_{l_5+1} \leq \cdots \leq t_{l_6} \leq \epsilon^{1/3} d_n \delta \\
&\epsilon^{1/3} d_n \delta < t_{l_6+1} \leq t_{l_6+1} \leq \cdots \leq t_p \\
(4.16)
\end{align*}

Now use the same definitions of $l(x)$ and $d(x)$, as in the proof of Theorem 1. Then due to Lemma 12 and 3 and the fact that $\epsilon \leq c < 1$, we have for sufficiently
large $n$,

$$
I(x) \leq I(x > \epsilon^{1/3}d_{n,\delta}) + \left[ 1 - \frac{2\phi(1)}{\sqrt{5} + 1} + A2^{1+\delta}e^{-1/2}d_{n,\delta}^{-2(2+\delta)} \right] I(x < 1)
\leq I(x > \epsilon^{1/3}d_{n,\delta}) + \left[ 1 - \frac{\phi(1)}{\sqrt{5} + 1} \right] I(x < 1)
$$

(4.17) 

and

(4.18)

$$
d(x) \leq \left[ 2e^{-x^2/2} \right] I(|x| > \epsilon^{1/3}d_{n,\delta}) + \left[ A2^{1+\delta}\epsilon^{1+\delta/3}d_{n,\delta}^{-1}e^{-x^2/2} \right] I(|x| \leq \epsilon^{1/3}d_{n,\delta})
$$

for any $x \in \mathbb{R}$, for sufficiently large $n$. Therefore from equations (4.16)-(4.19) we have for sufficiently large $n$,

(4.19) 

$$
L(t) \leq J_1(t) + J_2(t) + J_3(t) + J_4(t),
$$

where

$$
J_1(t) = \left[ 1 - \frac{\phi(1)}{\sqrt{5} + 1} \right]^{l_5-1} \ast \left( \prod_{j=l_5+1}^{l_6} \left[ 1 - \frac{d_{n,\delta}^{-1}e^{-x^2/2}}{\sqrt{2\pi}\left( \sqrt{\epsilon^{2/3} + 4d_{n,\delta}^{-2} + \epsilon^{1/3}} \right)} \right] \right)
$$

* \left( \sum_{k=1}^{l_4} 2e^{-l_4^2/2} \right) \ast I(l_4 \geq 1),

$$
J_2(t) = \left[ 1 - \frac{\phi(1)}{\sqrt{5} + 1} \right]^{l_5-1} \ast \left( \prod_{j=l_5+1}^{l_6} \left[ 1 - \frac{d_{n,\delta}^{-1}e^{-x^2/2}}{\sqrt{2\pi}\left( \sqrt{\epsilon^{2/3} + 4d_{n,\delta}^{-2} + \epsilon^{1/3}} \right)} \right] \right)
$$

* \left( \sum_{k=l_5+1}^{l_6} A2^{1+\delta}\epsilon^{1+\delta/3}d_{n,\delta}^{-1}e^{-l_2^2/2} \right) \ast I(l_2 - l_5 \geq 1),

$$
J_3(t) = \left[ 1 - \frac{\phi(1)}{\sqrt{5} + 1} \right]^{l_5} \ast I((l_6 - l_5) \geq 1)
$$

* \left( \sum_{k=l_5+1}^{l_6} A2^{1+\delta}\epsilon^{1+\delta/3}d_{n,\delta}^{-1}e^{-l_2^2/2} \right) \ast \left( \prod_{j=l_5+1}^{l_6} \left[ 1 - \frac{d_{n,\delta}^{-1}e^{-x^2/2}}{\sqrt{2\pi}\left( \sqrt{\epsilon^{2/3} + 4d_{n,\delta}^{-2} + \epsilon^{1/3}} \right)} \right] \right),

$$
J_4(t) = \left[ 1 - \frac{\phi(1)}{\sqrt{5} + 1} \right]^{l_6} \ast \left( \prod_{j=l_6+1}^{l_6} \left[ 1 - \frac{d_{n,\delta}^{-1}e^{-x^2/2}}{\sqrt{2\pi}\left( \sqrt{\epsilon^{2/3} + 4d_{n,\delta}^{-2} + \epsilon^{1/3}} \right)} \right] \right)
$$

* \left( \sum_{k=l_6+1}^{p} 2e^{-l_6^2/2} \right) \ast I((p - l_6) \geq 1).
Bound on $J_1(t) + J_4(t)$: Since $\log p = \epsilon d_{n,\delta}^2$, from (4.19) we have

$$ J_1(t) + J_4(t) \leq \left( \sum_{k=1}^{l_4} 2e^{-t_4^2/2} \right) I(l_4 \geq 1) + \left( \sum_{k=l_4+1}^p 2e^{-t_4^2/2} \right) I((p-l_6) \geq 1) $$

$$ \leq p \left( 2e^{-\epsilon^2/3}d_{n,\delta}^2 \right) $$

$$ = 2 \exp(\epsilon d_{n,\delta}^2 - \epsilon^{2/3}d_{n,\delta}/2) $$

$$ \leq 2 \exp(-\epsilon^{2/3}d_{n,\delta}^2(1/2 - \epsilon^{1/3}) $$

$$ < \epsilon^{(1+\delta)/3}/12, \tag{4.20} $$

for large enough $n$, since $\epsilon < c \leq 1/8$.

Bound on $J_2(t)$: Noting that $d^{-1} = \left[ 1 - \frac{\phi(1)}{\sqrt{5} + 1} \right]$ and $\epsilon < 1$, we have for sufficiently large $n$,

$$ J_2(t) \leq I_2(t) \leq \left( \left[ 1 - \frac{\phi(1)}{\sqrt{5} + 1} \right]^{l_2-1} \left( \sum_{k=1}^{l_2} A2^{1+\delta}e^{(1+\delta)/3}d_{n,\delta}^{-1}e^{-t_2^2/2} \right) I((l_2 - l_1) \geq 1) \right) $$

$$ \leq \left( d^{-1}(\log d)^{-1}d^{-1}(\log d)^{-1} \right) A2^{1+\delta}e^{(1+\delta)/3}d_{n,\delta}^{-1} < \epsilon^{(1+\delta)/3}/12. \tag{4.21} $$

Bound on $J_3(t)$: Write $z_n = d_{n,\delta}^{-1}e^{-1/2} \left[ \sqrt{2\pi} \left( \sqrt{\epsilon^{2/3} + 4d_{n,\delta}^{-2} + \epsilon^{1/3}} \right) \right]^{-1}$. Through the same line of arguments as in bounding $I_3(t)$, we have for some non-negative integer $q \in [l_5, l_6]$ and for sufficiently large $n$,

$$ J_3(t) \leq (q - l_5) \left[ 1 - z_n \right]^{q-l_5-1} \left( A2^{1+\delta}e^{(1+\delta)/3}d_{n,\delta}^{-1}e^{-1/2} \right) $$

$$ + (l_6 - q) \left( A2^{1+\delta}e^{(1+\delta)/3}d_{n,\delta}^{-1}e^{-2^{-1/2}/d_{n,\delta}^2} \right) $$

$$ \leq 2 \left[ \exp \left( \log(q - l_5) - (q - l_5)z_n \right) \right] \left( A2^{1+\delta}e^{(1+\delta)/3}d_{n,\delta}^{-1}e^{-1/2} \right) $$

$$ + p \left( A2^{1+\delta}e^{(1+\delta)/3}d_{n,\delta}^{-1}e^{-2^{-1/2}/d_{n,\delta}^2} \right) $$

$$ \leq 2 \left[ \exp \left( \sup_{x>0} \left[ \log x - xz_n \right] \right) \right] \left( A2^{1+\delta}e^{(1+\delta)/3}d_{n,\delta}^{-1}e^{-1/2} \right) $$

$$ + \left( A2^{1+\delta}e^{(1+\delta)/3}d_{n,\delta}^{-1}e^{-2^{-1/2}/d_{n,\delta}^2} \right) $$

$$ \leq 2z_n^{-1} \left( A2^{1+\delta}e^{(1+\delta)/3}d_{n,\delta}^{-1}e^{-1/2} \right) + \left( A2^{1+\delta}e^{(1+\delta)/3}d_{n,\delta}^{-1}e^{-1/2} \right) $$

$$ \leq \left[ \sqrt{2\pi} \left( \sqrt{\epsilon^{2/3} + 4d_{n,\delta}^{-2} + \epsilon^{1/3}} \right) \right] A2^{1+\delta}e^{(1+\delta)/3} $$

$$ + \left( A2^{1+\delta}e^{(1+\delta)/3}d_{n,\delta}^{-1}e^{-1/2} \right) $$

$$ \leq \epsilon^{(1+\delta)/3}/4 + \epsilon^{(1+\delta)/3}/12, \tag{4.22} $$

since $\epsilon^{1/3} < \epsilon^{1/3} \leq [A2^{1+\delta}\sqrt{2\pi}(\sqrt{2} + 1)]^{-1}$. Now combining (4.16)-(4.22), the proof of Theorem 2 is complete.
**Proof of Theorem 3**: We are going to prove part (b) only. Part (a) follows from part (b) by taking $\epsilon \to 0$. We are going to follow the same steps as in the proof of Theorem 2 with different estimates of $l(x)$ and $d(x)$.

Now take $c = \min \left\{ \left\{ 4C_3 \sqrt{2\pi (\sqrt{2} + 1)} \right\}^{-3}, 1/8 \right\}$ where $C_3$ is the constant defined in Lemma 4 with $Z$ replaced by $X_{11}$. Recall that $\omega_n = \delta_n^{-1/6}$ where $\delta_n = \delta_n,0 (X_{11})$. Fix $t = (t_1, \ldots, t_p)'$ in $\mathcal{R}^p$ such that $t_1 \leq t_2 \leq \cdots \leq t_p$. Then there exist integers $l_7, l_8, l_9$, depending on $n$, such that $0 \leq l_7, l_8, l_9 \leq p$ and

\[
\begin{align*}
1 \leq & \ t_9 + 1 \leq t_{9 + 1} \leq \cdots \leq t_8 \leq \epsilon^{1/3} \omega_n \\
\epsilon^{1/3} \omega_n & < t_{8 + 1} \leq t_{9 + 1} \leq \cdots \leq t_p.
\end{align*}
\]

(4.23)

Use the same definitions of $l(x)$ and $d(x)$, as in the proof of Theorem 1. Note that using Lemma 1 of Cörgő et al. (2003), we have $\delta_n \to 0$ as $n \to \infty$. Then due to Lemma 4 and the fact that $\epsilon \leq c < 1/8$, we have for sufficiently large $n$,

\[
\begin{align*}
l(x) \leq & \ I(x > \epsilon^{1/3} \omega_n) + \left[ 1 - \frac{2 \phi(1)}{\sqrt{5} + 1} + C_3 \delta_n e^{-1/2} \right] I(x < 1) \\
& \quad + \left[ 1 - \frac{2 \phi(x)}{\sqrt{x^2 + 4 + x}} + C_3 \delta_n e^{-1/2} \right] I(x \in \left[ 1, \epsilon^{1/3} \omega_n \right]) \\
& \quad + \left[ 1 - \frac{\phi(1)}{\sqrt{5} + 1} \right] I(x < 1) \\
& \quad + \left[ 1 - \frac{\omega_n^{-1} e^{-x^2/2}}{\sqrt{2\pi (\sqrt{e^2/3 + 4\omega_n^2} + \epsilon^{1/3})}} \right] I(x \in \left[ 1, \epsilon^{1/3} \omega_n \right]),
\end{align*}
\]

and

\[
\begin{align*}
d(x) \leq & \ 2 e^{-x^2/2} I(|x| \leq \epsilon^{1/3} \omega_n) + \left[ C_3 \delta_n e^{-x^2/2} \right] I(|x| \leq 2) \\
& \quad + \left[ C_3 \delta_n e^{-x^2/2} \right] I(1 < |x| \leq \epsilon^{1/3} \omega_n) \\
& \quad + \left[ C_3 \delta_n e^{-x^2/2} \right] I(|x| \leq \epsilon^{1/3} \omega_n) \\
& \quad + \left[ C_3 \delta_n e^{-x^2/2} \right] I(|x| \leq \epsilon^{1/3} \omega_n) \\
& \quad + \left[ C_3 \delta_n e^{-x^2/2} \right] I(|x| \leq \epsilon^{1/3} \omega_n) \\
& \quad + \left[ C_3 \delta_n e^{-x^2/2} \right] I(|x| \leq \epsilon^{1/3} \omega_n)
\end{align*}
\]

(4.25)

for any $x \in \mathcal{R}$, for sufficiently large $n$. Therefore from equations (4.30)-(4.25) we have for sufficiently large $n$,

\[
L(t) \leq J_5(t) + J_6(t) + J_7(t) + J_8(t),
\]

(4.26)

where similar to the proof of Theorem 2 we have

\[
J_5(t) + J_8(t) \leq p \left( 2 e^{-2^{-1/2^3} \omega_n^2} \right) \leq 2 \exp \left( - \epsilon^{2/3} \omega_n^2 (1/2 - \epsilon^{1/3}) \right) < \epsilon^{5/3}/12,
\]

(4.27)

for large enough $n$, since $\epsilon < 1/8$. Again similar to $J_2(t)$, it can be shown that

\[
J_6(t) \leq \left( d^{-1} (\log d)^{-1} d^{-1} (\log d)^{-1} \right) C_3 \epsilon^{5/3} \omega_n^{-1} < \epsilon^{5/3}/12,
\]

(4.28)
for sufficiently large \( n \) where \( d^{-1} = \left[ 1 - \frac{\phi(1)}{\sqrt{5} + 1} \right] \). The only thing that remains to bound is \( J_7(t) \) which is

\[
J_7(t) = \left( 1 - \frac{\phi(1)}{\sqrt{5} + 1} \right)^{l_8} \ast I \big( (l_9 - l_8) \geq 1 \big) \ast \left( \prod_{j \neq k} C_3 \epsilon^{5/3} \omega_n^{-1} e^{-t_j^2/2} \left( 1 - \frac{\omega_n^{-1} e^{-t_j^2/2}}{\sqrt{2\pi} (\sqrt{\epsilon^{2/3} + 4 \omega_n^{-2} + \epsilon^{1/3})} \right) \right). \]

Note that \( J_7(t) \) can be obtained from \( J_3(t) \) by replacing \( A^{1+\delta} \) by \( C_3 \), \( \epsilon^{(1+\delta)/3} \) by \( \epsilon^{5/3} \) and \( d_n, \delta \) by \( \omega_n \). Therefore writing \( z_{n1} = \omega_n^{-1} e^{-1/2} \left[ \sqrt{2\pi} (\sqrt{\epsilon^{2/3} + 4 \omega_n^{-2} + \epsilon^{1/3})} \right]^{-1} \), similar to the treatment to \( J_3(t) \) we have

\[
J_7(t) \leq 2 z_{n1}^{-1} \left( C_3 \epsilon^{5/3} \omega_n^{-1} e^{-1/2} \right) + p \left( C_3 \epsilon^{5/3} \omega_n^{-1} \exp \left( - \epsilon^{2/3} \omega_n^2 / 2 \right) \right) \leq \left[ \sqrt{2\pi} (\sqrt{\epsilon^{2/3} + 4 \omega_n^{-2} + \epsilon^{1/3})} \right] C_3 \epsilon^{5/3} \left( C_3 \epsilon^{5/3} d_n^{-1} \exp \left( - \epsilon^{2/3} \omega_n^2 (1/2 - \epsilon^{1/3}) \right) \right) \]

\[
(4.29) < \epsilon^{5/3} / 4 + \epsilon^{5/3}/12.
\]

Therefore combining \( 4.26 \) and \( 4.29 \), the proof is complete.

**Proof of Theorem 4** Here also we will follow the same route as in the proof of Theorem 3. Now take \( c = \min \left\{ 12 C_2 \sqrt{2\pi} (\sqrt{2} + 1) \right\}^{-3} \), where \( C_2 \) is the constant defined in Lemma 5 with \( Z \) replaced by \( X \). Fix \( t = (t_1, \ldots, t_p) \) in \( \mathbb{R}^p \) such that \( t_1 \leq t_2 \leq \cdots \leq t_p \). Then there exist integers \( l_{10}, l_{11}, l_{12}, \) depending on \( n \), such that \( 0 \leq l_{10}, l_{11}, l_{12} \leq p \) and

\[
t_1 \leq t_2 \leq \cdots \leq t_{l_{10}} < -\epsilon^{1/3} n^{1/4} \]

\[
-\epsilon^{1/3} n^{1/4} - l_{10+1} \leq t_{l_{10}+1} \leq \cdots \leq t_{l_{11}} < 1 \]

\[
1 \leq t_{l_{11}+1} \leq t_{l_{11}+2} \leq \cdots \leq t_{l_{12}} \leq \epsilon^{1/3} n^{1/4} \]

\[
(4.30) \epsilon^{1/3} n^{1/4} \leq t_{l_{12}+1} \leq t_{l_{12}+2} \leq \cdots \leq t_p.
\]

Now Lemma 13 and the fact that \( \epsilon \leq c < 1/8 \), we have for sufficiently large \( n \),

\[
l(x) \leq I \left( x > \epsilon^{1/3} n^{1/4} \right) + \left[ 1 - \frac{2 \phi(1)}{\sqrt{5} + 1} + 3 C_2 n^{-1/2} e^{-1/2} \right] I \left( x < 1 \right) \]

\[
+ \left[ 1 - \frac{2 \phi(x)}{\sqrt{2^3 + 4 + x}} + C_2 (2^{2/3} n^{-1} + c n^{-1/4} + n^{-1/2}) e^{-x^2/2} \right] I \left( x \in [1, \epsilon^{1/3} n^{1/4}] \right) \]

\[
\leq I \left( x > \epsilon^{1/3} n^{1/4} \right) + \left[ 1 - \frac{\phi(1)}{\sqrt{5} + 1} \right] I \left( x < 1 \right) \]

\[
(4.31) + \left[ 1 - \frac{n^{-1/2} e^{-x^2/2}}{2\pi (\sqrt{2^{2/3} + 4 n^{-1/2} + \epsilon^{1/3}})} \right] I \left( x \in [1, \epsilon^{1/3} n^{1/4}] \right),
\]
and

\[(4.32) \quad d(x) \leq 2e^{-x^2/2} \mathbb{I}(|x| > e^{1/3} n^{1/4}) + \mathbb{I}\left(1/2 < |x| \leq e^{1/3} n^{1/4}\right) + 3C_2 e n^{-1/4} e^{-x^2/2} \mathbb{I}\left(1/2 < |x| \leq e^{1/3} n^{1/4}\right). \]

Clearly (4.31) and (4.32) are respectively same as (4.24) and (4.25), but after replacing \(\omega_n\) by \(n^{1/4}\) and \(C_3 e^{5/3}\) by \(3C_2 e\). Therefore, the rest of the proof follows exactly following the arguments of the proof of Theorem 3.

**Proof of Theorem 5.** Note that \(\sum_{i=1}^n X_{i1}^2 = \cdots = \sum_{i=1}^n X_{ip}^2 = n\) and hence the setup of Theorem 5 becomes exactly the same setup of Theorem 3 of Das and Lahiri (2020). Therefore the proof follows from the proof of Theorem 3 of Das and Lahiri (2020).

**Proof of Theorem 6.** We are going to prove the version of Theorem 3 for \(W_n\). Theorem 4 can be proved by taking \(\epsilon \to 0\). Proof of the versions of Theorem 6 for \(W_n\) are analogous. Now for proving the version of Theorem 6 for \(T_n\) we need to show

\[(4.33) \quad \lim_{n \to \infty} \sup_{B \in \mathcal{A}_{\mathbb{R}}} \mathbb{P}\left(W_n \in B \setminus \Phi(B)\right) < e^{(1+\delta)/3}, \]

whenever \(\log p = \epsilon * d_{n, \delta}^2\) for some \(\epsilon < c\) with some \(0 < c \leq 1/8\). Now note that the conclusion of Lemma 7 is still true if we replace \(D_n\) by \(\tilde{D}_n = \text{diag}(\tilde{d}_{n1}, \ldots, \tilde{d}_{np})\) in its statement where \(\tilde{d}_{nj} = \frac{n}{n-1} \sum_{i=\ell}^n (Z_{ij} - \bar{Z}_{nj})^2\) and \(\bar{Z}_{nj} = n^{-1} \sum_{i=\ell}^n Z_{ij}\), \(j \in \{1, \ldots, p\}\). Therefore using Lemma 7 to prove (4.33) it is enough to show that

\[(4.34) \quad \sup_{t_1 \leq t_2 \leq \cdots \leq t_p} \bar{L}((t_1, \ldots, t_p))' = \sup_{t_1 \leq t_2 \leq \cdots \leq t_p} \left[ \sum_{j=1}^p \left( \prod_{j \neq k} \hat{F}(t_j) \right) \hat{d}(t_k) \right] < e^{(1+\delta)/3}/2, \]

for large enough \(n\). Here, \(\hat{F}(x) = \max\left\{ \mathbb{P}\left(W_{n1} \leq x\right), \mathbb{P}\left(S_{n1} \leq x\right)\right\}\) and \(d(x) = \left| \mathbb{P}\left(W_{n1} \leq x\right) - \mathbb{P}\left(S_{n1} \leq x\right)\right|\), with \(S_n = (S_{n1}, \ldots, S_{np})' = n^{-1/2} \sum_{i=1}^n N_i\). Now fix \(t = (t_1, \ldots, t_p)\). Next step is to find bounds on \(\hat{F}(x)\) and \(d(x)\). Now from (1.3) and using that the distribution of \(T_{n1}\) and \(W_{n1}\) are both symmetric around 0, we have

\[\mathbb{P}\left(W_{n1} \leq x\right) = \mathbb{P}\left(T_{n1} \leq x\left(\frac{n}{n + x^2 - 1}\right)^{1/2}\right),\]
for any $x \in \mathcal{R}$. Therefore using Lemma 3 for sufficiently large $n$ we have
\[
\mathbb{P}(W_{n1} \leq x) - \Phi(x) \\
\leq \mathbb{P}(T_{n1} \leq x) - \Phi(x) \\
\leq A(1 + |x|) \left( \frac{n}{n + x^2 - 1} \right)^{1/2} \exp \left( - \frac{1}{2} x^2 \left( \frac{n}{n + x^2 - 1} \right) \right) d_{n, \delta}^{-2(2 + \delta)} \\
+ 3 \phi \left( \frac{n}{n + x^2 - 1} \right)^{1/2} |x| \left( \frac{n}{n + x^2 - 1} \right)^{1/2} - x \\
\leq A(1 + 2|x|)^{1+\delta} \exp \left( - \frac{1}{2} x^2 \left( \frac{n}{n + x^2 - 1} \right) \right) d_{n, \delta}^{-2(2 + \delta)} \\
+ 3(\sqrt{2\pi})^{-1} \left( |x|^3 + |x| \right) \exp \left( - \frac{1}{2} x^2 \left( \frac{n}{n + x^2 - 1} \right) \right) [n(n - 1)]^{-1/2}
\]
(4.35)
\[
\leq 3A(1 + 2|x|)^{1+\delta} \exp \left( - \frac{1}{2} x^2 \left( \frac{n}{n + x^2 - 1} \right) \right) d_{n, \delta}^{-2(2 + \delta)},
\]
whenever $|x| \leq d_{n, \delta}$. Again whenever $|x| \geq 1$, then using Lemma 1 and Lemma 4.3 of Wang and Jing (1999) we have
\[
\mathbb{P}(W_{n1} \leq x) - \Phi(x) \\
\leq \mathbb{P}(T_{n1} \leq x) - \Phi(x) \\
\leq \exp \left( - \frac{1}{2} x^2 \left( \frac{n}{n + x^2 - 1} \right) \right) + \Phi(x) \\
\leq \left( 1 + \frac{1}{\sqrt{2\pi}|x|} \right) \exp \left( - \frac{1}{2} x^2 \left( \frac{n}{n + x^2 - 1} \right) \right).
\]
(4.36)

Now define $8c = \min \{ [A3^{2+\delta} \sqrt{2\pi}((\sqrt{2} + 1)]^{-3}, 1 \}$ and consider the partition of $\{t_1, \ldots, t_p\}$ as in the proof of Theorem 2. Write $x_1 = x \left( \frac{n}{n + x^2 - 1} \right)^{1/2}$. Then due to (4.35) we have for sufficiently large $n$,
\[
I(x) \leq I \left( x > c^{1/3} d_{n, \delta} \right) + \left[ 1 - \frac{2\phi(1)}{\sqrt{5} + 1} + A3^{2+\delta} d_{n, \delta}^{-2(2 + \delta)} \right] I \left( x < \frac{n}{n + x^2 - 1} \right) \\
+ \left[ 1 - \frac{2\phi(x)}{\sqrt{5} + 1} + A3^{2+\delta} \epsilon^{(1+\delta)/3} d_{n, \delta}^{-1} e^{-x^2/2} \right] I \left( x \in \left[ \frac{n}{n + x^2 - 1}, 1 \right] \right) \\
\leq I \left( x > c^{1/3} n^{1/4} \right) + \left[ 1 - \frac{\phi(1)}{\sqrt{5} + 1} \right] I \left( x < 1 \right) \\
+ \left[ 1 - \frac{d_{n, \delta}^{-1} e^{-x^2/2}}{\sqrt{2\pi} \left( \sqrt{e^{2/3} + 4d_{n, \delta}^{-2} + e^{1/3}} \right)} \right] I \left( x \in \left[ 1, c^{1/3} d_{n, \delta} \right] \right).
\]
(4.37)

The third part of the second inequality follows due to the facts that $A3^{2+\delta} \sqrt{2\pi}((\sqrt{2} + 1)\epsilon^{(2+\delta)/3} < 1/8$ & $x_1 \leq x$ for $x \geq 1$ and by noting that $g(y) = y \left[ 1 - n(n+y-1)^{-1} \right]$ is non-decreasing when $y \geq 1$. Again due to (4.35) and (4.36) we have
\[
d(x) \leq \left[ 2e^{-x^2/2} \right] I \left( |x| > c^{1/3} d_{n, \delta} \right) + \left[ A3^{2+\delta} \epsilon^{(1+\delta)/3} d_{n, \delta}^{-1} e^{-x^2/2} \right] I \left( |x| \leq c^{1/3} d_{n, \delta} \right)
\]
(4.38)
for any $x \in \mathcal{R}$, for sufficiently large $n$. Therefore from equations (4.37) and (4.38) we have for sufficiently large $n$,

\begin{equation}
\tilde{J}(t) \leq \tilde{J}_1(t) + \tilde{J}_2(t) + \tilde{J}_3(t) + \tilde{J}_4(t),
\end{equation}

where with $t_{k_1} = t_k \left( \frac{n}{n + t_l^2} \right)^{1/2}$ for any $k \in \{1, \ldots, p\}$,

\begin{align*}
\tilde{J}_1(t) &= \left( \left[ 1 - \frac{\phi(1)}{\sqrt{5} + 1} \right] t_5 - 1 \right) * \left( \prod_{j=l_5+1}^{l_6} \left[ 1 - \frac{d_{n,\delta}^{-1} e^{-t_j^2/2}}{\sqrt{2\pi} \left( \sqrt{e^{2/3} + 4d_{n,\delta}^{-2} + e^{1/3}} \right)} \right] \right)\\
&\quad * \left( \sum_{k=1}^{l_4} 2e^{-t_k^2/2} \right) I \left( l_4 \geq 1 \right),
\end{align*}

\begin{align*}
\tilde{J}_2(t) &= \left( \left[ 1 - \frac{\phi(1)}{\sqrt{5} + 1} \right] t_5 - 1 \right) * \left( \prod_{j=l_5+1}^{l_6} \left[ 1 - \frac{d_{n,\delta}^{-1} e^{-t_j^2/2}}{\sqrt{2\pi} \left( \sqrt{e^{2/3} + 4d_{n,\delta}^{-2} + e^{1/3}} \right)} \right] \right)\\
&\quad * \left( \sum_{k=l_5+1}^{l_4} A^{2+\delta} e^{(1+\delta)/3} d_{n,\delta}^{-1} e^{-t_k^2/2} \right) I \left( (l_2 - l_1) \geq 1 \right),
\end{align*}

\begin{align*}
\tilde{J}_3(t) &= \left( \left[ 1 - \frac{\phi(1)}{\sqrt{5} + 1} \right] t_5 - 1 \right) * I \left( (l_5 - l_5) \geq 1 \right)\\
&\quad * \left( \sum_{k=l_5+1}^{l_6} A^{2+\delta} e^{(1+\delta)/3} d_{n,\delta}^{-1} e^{-t_k^2/2} \right) \left( \prod_{j=l_5+1}^{l_6} \left[ 1 - \frac{d_{n,\delta}^{-1} e^{-t_j^2/2}}{\sqrt{2\pi} \left( \sqrt{e^{2/3} + 4d_{n,\delta}^{-2} + e^{1/3}} \right)} \right] \right),
\end{align*}

\begin{align*}
\tilde{J}_4(t) &= \left( \left[ 1 - \frac{\phi(1)}{\sqrt{5} + 1} \right] t_5 - 1 \right) * \left( \prod_{j=l_5+1}^{l_6} \left[ 1 - \frac{d_{n,\delta}^{-1} e^{-t_j^2/2}}{\sqrt{2\pi} \left( \sqrt{e^{2/3} + 4d_{n,\delta}^{-2} + e^{1/3}} \right)} \right] \right)\\
&\quad * \left( \sum_{k=l_6+1}^{p} 2e^{-t_k^2/2} \right) I \left( (p - l_6) \geq 1 \right).
\end{align*}

Clearly $\{\tilde{J}_i\}_{i=1}^4$ can be obtained from $\{J_i\}_{i=1}^4$ in the proof of Theorem 2 by replacing $t_k$’s by $t_{k_1}$’s for all $k \in \{1, \ldots, p\}$. Therefore $\tilde{J}_1(t)$, $\tilde{J}_2(t)$ and $\tilde{J}_3(t)$ can be dealt with exactly similarly as for $J_1(t)$, $J_2(t)$ and $J_3(t)$ in the the proof of Theorem 2 and noting that $n/(n + x^2 - 1)$ converges to 1 as $n \rightarrow \infty$ whenever $|x| = o(n)$. Hence enough to show $\tilde{J}_3(t) \leq \epsilon^{(1+\delta)/3}/4$ for sufficiently large $n$. Note that

\begin{equation}
\tilde{J}_3(t) \leq \sum_{k=l_5+1}^{l_6} A^{2+\delta} e^{(1+\delta)/3} d_{n,\delta}^{-1} e^{-t_k^2/2} \left( \prod_{j=l_5+1}^{l_6} \left[ 1 - \frac{d_{n,\delta}^{-1} e^{-t_j^2/2}}{\sqrt{2\pi} \left( \sqrt{e^{2/3} + 4d_{n,\delta}^{-2} + e^{1/3}} \right)} \right] \right),
\end{equation}

(4.40)

$= \tilde{J}_{31}(t)$ (say).

and

\begin{equation}
\frac{\partial \tilde{J}_{31}(t)}{\partial t_l} = A^{2+\delta} e^{(1+\delta)/3} d_{n,\delta}^{-1} \left[ \frac{n(n-1)t_l}{n + t_l^2 - 1} \right] e^{-t_l^2/2} \left( \prod_{j=l_2+1}^{l_3} \sum_{k=l_2+1}^{l_3} \left( \frac{z_k}{1 - z_k} - 1 \right) \right),
\end{equation}
where $\tilde{z}_k = d_{n, \delta}^{-1} e^{-t_k^2/2} \left[ \sqrt{2\pi} \left( \sqrt{\epsilon^{2/3} + 4d_{n, \delta}^{-2} + \epsilon^{1/3}} \right) \right]^{-1}$, $k = 1, \ldots, p$. Hence for any $l = l_2 + 1, \ldots, l_3$, \[ \frac{\partial \tilde{J}_{31}(t)}{\partial t_l} \geq 0 \text{ if and only if} \]

\[ \sum_{j=l_2+1}^{l_3} \frac{\tilde{z}_j}{1 - \tilde{z}_j} \geq 1, \]

Now using the fact that $f(y) = ny/(n + y - 1)$ is non-decreasing for $y > 0$, we can claim that $1 > \tilde{z}_{l_2+1} \geq \cdots \geq \tilde{z}_{l_3} > 0$ for sufficiently large $n$, since $1 \leq t_{l_2+1} \leq \cdots \leq t_{l_3}$. Hence for sufficiently large $n$,

\[ \frac{\tilde{z}_{l_2+1}}{1 - \tilde{z}_{l_2+1}} \geq \cdots \geq \frac{\tilde{z}_{l_3}}{1 - \tilde{z}_{l_3}}, \]

due to the fact that $z/(1 - z)$ is increasing for $z \in (0, 1)$. Therefore from (4.41) we can say that $\tilde{J}_{31}(t)$ is non-increasing in $\{t_{l_2+1}, \ldots, t_m\}$ and non-decreasing in $\{t_{m+1}, \ldots, t_{l_3}\}$ where $(m - l_2)$ is a non-negative integer not more than $(l_3 - l_2)$. Hence we can follow the steps which leads to (4.22) in the proof of Theorem 2 and complete the proof of (4.33).

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**Department of Mathematics and Statistics, Indian Institute of Technology, Kanpur, Uttar Pradesh 208016, India**

*Email address: rajdas@iitk.ac.in*