THE DEFOCUSING SCHRÖDINGER EQUATION WITH A NONZERO BACKGROUND: PAINLEVÉ ASYMPTOTICS IN TWO TRANSITION REGIONS

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Abstract. In this paper, we address the Painlevé asymptotics in the transition region \( |\xi| := |\frac{x}{t^2/3}| \approx 1 \) to the Cauchy problem of the defocusing Schrödinger (NLS) equation with a nonzero background. With the \( \bar{\partial} \)-generation of the nonlinear steepest descent approach and double scaling limit to compute the long-time asymptotics of the solution in two transition regions defined as

\[ P_{\pm 1}(x,t) := \{(x,t) \in \mathbb{R} \times \mathbb{R}^+ , \ 0 < |\xi - (\pm 1)|t^{2/3} \leq C \}, \]

we find that the long-time asymptotics of the NLS equation in both transition regions \( P_{\pm 1}(x,t) \) can be expressed in terms of the Painlevé II equation. We are also able to express the leading term explicitly in terms of the Airy function.

Keywords: defocusing NLS equation, Riemann-Hilbert problems, steepest descent method, Painlevé transcendent, long-time asymptotics.

MSC: 35Q55; 35P25; 35Q15; 35C20; 35G25.

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1. INTRODUCTION

The present paper is concerned with the Painlevé asymptotics in a transition region $|\xi| := \left|\frac{x}{t}\right| \approx 1$ for the Cauchy problem of the defocusing Schrödinger (NLS) equation with a nonzero background

$$i q_t + q_{xx} - 2(|q|^2 - 1)q = 0,$$

$$q(x, 0) = q_0(x) \sim \pm 1, \quad x \to \pm \infty.$$  

It is an important model that has been discussed extensively in classical texts like the book by Faddeev and Takhtajan [1]. Zakharov and Shabat first derived the Lax pair of the NLS equation [2]. The well-posedness of the NLS equation with the initial data in Sobolev spaces was proved [3, 4]. Zakharov and Shabat developed the inverse scattering transform (IST) for the focusing NLS equation with zero boundary conditions [5]. We mention the following works on the long-time asymptotics of the defocusing NLS equation. For the initial data in the Schwartz space $S(\mathbb{R})$ and using the IST method, Zakharov and Manakov obtained the long-time asymptotics of the NLS equation [6]. In 1981, Its presented a stationary phase method to analyze the long-time asymptotic behavior for the NLS equation [7]. In 1993, Deift and Zhou developed a nonlinear steepest descent method to rigorously obtain the long-time asymptotic behavior for the mKdV equation [8]. Later this method was extended to get the leading and high-order asymptotic behavior for the solution of the NLS equation (1.1) with the initial data $q_0 \in S(\mathbb{R})$ [9, 10]. Vartanian obtained the leading and first correlation terms in the asymptotic behavior of the NLS equation with finite density initial data [11–13]. Under much weaker weighted Sobolev initial data $q_0 \in H^{1,1}(\mathbb{R})$, Deift and Zhou gave the leading asymptotics, where the error is $O(t^{-1/2-\kappa})$, $0 < \kappa < 1/4$ [14]. In 2008, for the same space $q_0 \in H^{1,1}(\mathbb{R})$, Dieng and McLaughlin applied the $\bar{\partial}$-steepest descent method to obtain a sharp estimate, where the error is $O(t^{-3/4})$ [15]. Jenkins investigated the long-time/zero-dispersion limit of the solutions to the defocusing NLS equation associated with the step-like initial data [16]. Fromm, Lenells, and Quirchmayr studied the long-time asymptotics for the defocusing NLS equation with the step-like boundary condition [17].
In the study of Painlevé equations, one of the important topics is the asymptotic behavior of their solutions [18–22]. Especially some Painlevé equations, for example, the homogeneous Painlevé II equation, can be solved via a certain Riemann-Hilbert (RH) problem [23]. Although the study of the Painlevé equations originates from pure mathematics, they have found plenty of applications in many fields of mathematical physics, such as random matrix theory, statistical physics, and integrable systems.

The transition asymptotic regions for the Korteweg-de Vries equation were first described in Painlevé transcendents by Segur and Ablowitz in [24]. The connection between different regions was first understood in the case of the modified Korteweg-de Vries equation by Deift and Zhou [25]. Boutet de Monvel et al. discussed the Painlevé-type asymptotics for the Camassa-Holm equation by the nonlinear steepest descent method [26]. The connection between the tau-function of the Sine-Gordon reduction and the Painlevé III equation was given by the RH approach [27]. Charlier and Lenells carefully considered the Airy and higher order Painlevé asymptotics for the mKdV equation [28]. Recently, Huang and Zhang obtained Painlevé asymptotics for the whole mKdV hierarchy [29].

In the study of the fundamental rogue wave solutions in the limit of a large order of the focusing NLS equation with nonzero background

\[
\begin{align*}
iq_t + \frac{1}{2}q_{xx} + 2q(|q|^2 - 1) &= 0, \\
q(x, t) &\sim 1, \quad x \to \pm\infty,
\end{align*}
\]

Bilman, Ling, and Miller found that the limiting profile of the rogue wave of infinite order satisfies ordinary differential equations concerning space and time. The spatial differential equations were identified with certain members of the Painlevé-III hierarchy [30]. Recently, considering the focusing NLS equation with step-like oscillating background, Boutet de Monvel, Lenells, and Shepelsky found the asymptotics in a transition zone between two genus-3 sectors. Especially a local parametrix was constructed by solving an RH model problem associated with the Painlevé IV equation [31]. In this paper, we are concerned with the Painlevé transcendent to the defocusing NLS equation, which seems still unknown to the best of our knowledge.

For this purpose, we consider the Cauchy problem (1.1)-(1.2) with finite density initial data \( q_0 \in \tanh x + H^{4,4}(\mathbb{R}) \) that was studied by Cuccagna and Jenkins [32]. They derived the leading order approximation to the Cauchy problem (1.1)-(1.2) in the solitonic space-time region I: \( |\xi| < 1 \) (See Figure 1) by using the \( \partial \)-steepest descent method [32]

\[
q(x, t) = T(\infty)^{-2}q_{sol, N}(x, t) + O(t^{-1}).
\]

This result also proves the asymptotic stability of \( N \)-dark soliton solutions. A series of important works on various stability of solitons of the NLS equation can be found in [33–35]. In recent years, the \( \partial \)-steepest descent method [36, 37] has been successfully used to obtain the long-time asymptotics and
the soliton resolution for some integrable systems [38–42]. For the solitonless region II: \(|\xi| > 1\), we further obtain the large-time asymptotic behavior to the problem (1.1)-(1.2) in the form [43]

\[ q(x, t) = e^{-i\alpha(\infty)} \left( 1 + t^{-1/2} h(x, t) \right) + O(t^{-3/4}). \]

However, how to describe the transition region III: \(|\xi| \approx 1\), proposed by Cuccagna and Jenkins (see page 923 in [32]), seems to remain unknown to the best of our knowledge. In order to solve this problem, in this paper, we apply \(\bar{\partial}\)-techniques and double scaling limit method to find the asymptotics of solution \(q(x, t)\) in two transition regions defined as

\[ \mathcal{P}_{\pm 1}(x, t) := \{(x, t) \in \mathbb{R} \times \mathbb{R}^+, \quad 0 < |\xi - (\pm 1)|t^{2/3} \leq C\}. \]

1.1. Main results.

To state the theorem precisely introduce the normed spaces: \(L^{p,s}(\mathbb{R})\) defined with

\[ \|q\|_{L^{p,s}(\mathbb{R})} := \|\langle x \rangle^s q\|_{L^p(\mathbb{R})}, \]

where \(\langle x \rangle = \sqrt{1 + x^2}\) and \(H^{k,s}(\mathbb{R}) := H^k(\mathbb{R}) \cap L^{2,s}(\mathbb{R})\). The following theorem is our main results of this paper.

**Theorem 1.1.** Let \(\{r(z), R(z)\}\) and \(\{z_j\}_{j=0}^{N-1}\) be, respectively, the reflection coefficient and the discrete spectrum associated with the weighted Sobolev initial data \(q_0 \in \tanh(x) + H^{4,4}(\mathbb{R})\). Then the long-time asymptotics of the solution to the Cauchy problem (1.1)-(1.2) for the defocusing NLS equation in two transition regions are given by the following formulas.

Case I. For \((x, t) \in \mathcal{P}_{-1}(x, t)\),

\[ q(x, t) = e^{i\alpha(\infty)} \left( 1 + \tau^{-1/3} \beta(-1) \right) + O(t^{-1/2}), \]
where

\begin{equation}
\alpha(\infty) = -2i \sum_{j=0}^{N-1} \log \bar{z}_j + \int_0^\infty \frac{\nu(\zeta)}{\zeta} \, d\zeta, \quad \nu(\zeta) = -\frac{1}{2\pi} \log(1 - |r(\zeta)|^2),
\end{equation}

\begin{equation}
\beta(-1) = -i \left( u(s) e^{i\varphi_0} + \int_s^\infty u^2(\zeta) \, d\zeta \right), \quad s = \frac{8}{3}(\xi + 1)\tau^{3/2}, \quad \tau = \frac{3}{4}t.
\end{equation}

Here \( \varphi_0 = \arg(r(-1)) \) and \( u(s) \) is a solution of the Painlevé II equation

\begin{equation}
u''(s) = 2u^3(s) + su(s)
\end{equation}

with the asymptotics

\begin{equation}
u(s) \sim -|r(-1)|\text{Ai}(s) \sim -|r(-1)|^{1/2} \sqrt{\pi} s^{-1/4} e^{-2^{3/2} s^{3/2}}, \quad s \to +\infty,
\end{equation}

where \( \text{Ai}(s) \) is the classical Airy function.

Case II. For \((x,t) \in P_+\)

\begin{equation}q(x,t) = e^{i\alpha(\infty)} \left( 1 + \tau^{-1/3} \beta(1) \right) + \mathcal{O}\left( t^{-1/2} \right),
\end{equation}

where

\begin{equation}\alpha(\infty) = \int_0^\infty \frac{\nu(\zeta)}{\zeta} \, d\zeta, \quad \beta(1) = \frac{i}{2} \left( u(s) e^{i\varphi_0} + \int_s^\infty u^2(\zeta) \, d\zeta \right),
\end{equation}

\begin{equation}\varphi_0 = \arg(R(1)), \quad s = -\frac{8}{3}(\xi - 1)\tau^{3/2}, \quad \tau = \frac{3}{4}t.
\end{equation}

Here, the function \( R(z) \) is defined by (4.28) and \( u(s) \) is a solution of the Painlevé equation (1.6) with the asymptotics

\begin{equation}u(s) \sim |R(1)|^{1/2} \sqrt{\pi} s^{-1/4} e^{-2^{3/2} s^{3/2}}, \quad s \to +\infty.
\end{equation}

Remark 1.2. Comparing Case I and Case II in Theorem 1.1, we find that the formulae (1.4) and (1.8) have a similar asymptotic form. Their difference is that the discrete spectrum contributes to the phase \( \alpha(\infty) \) in (1.5), while the reflection coefficient contributes to the phase \( \alpha(\infty) \) in (1.9).

Remark 1.3. In the case of the KdV and Camassa-Holm equation [26, 44, 45], a given RH problem can be reduced into a local RH problem at a phase point \( z_0 \) with the nonlinear steepest method. For the non-generic case \( |r(0)| \neq 1 \), the local RH problem can then be analyzed and controlled by the norm \( (1 - |r(z_0)|^2)^{-1} \). While for the generic case \( |r(0)| = 1 \), it turns out that the norm \( (1 - |r(z_0)|^2)^{-1} \) blow up as \( z_0 \to 0 \). This indicates the presence of a new collisionless shock region for \( |r(0)| = 1 \).

Remark 1.4. By contrast to the results in Remark 1.3, in the case of the focusing NLS equation, Vartanian in [11–13] obtained the long-time asymptotics for (1.1)-(1.2) under the hypothesis \( |r|_{L^\infty(\mathbb{R})} < 1 \) including \( |r(\pm 1)| < 1 \), which is the non-generic case. However, for the generic case \( r(\pm 1) = 1 \), Cuccagna and Jerkins proposed a new way in [32] to remove the non-generic
condition \( \|r\|_{L^\infty(\mathbb{R})} < 1 \) by specially handling the singularity caused from \( |r(\pm 1)| = 1 \). This method allows to proceed as in the similar region, whose asymptotics matches the Painlevé asymptotics in self-similar region and thus no new shock wave asymptotic forms appear. Because of this, the Painlevé asymptotics given in Theorem 1.1 are still effective for the case \( |r(\pm 1)| = 1 \).

1.2. Plan of the proof.

We prove Theorem 1.1 by applying \( \bar{\partial} \)-steepest descent approach and double scaling limit to the defocusing NLS equation (1.1)-(1.2). The organization of our paper is as follows.

In Section 2, we quickly review some basic results, especially the construction of a basic RH formalism \( M(z) \) related to the Cauchy problem (1.1)-(1.2) for the defocusing NLS equation. For more details, see [32]. Following a brief review of Section 2, two main results of Theorem 1.1 are presented in Section 3 and Section 4, respectively.

In Section 3, we focus on the long-time asymptotic analysis for the defocusing NLS equation in the Painlevé sector \( P_{-1}(x,t) \) with the following steps. We first obtain a standard RH problem \( M^{(3)}(z) \) by removing the poles and singularity of the RH problem \( M(z) \) in Subsection 3.1. Then in Subsection 3.2, after a continuous extension of the jump matrix with the \( \bar{\partial} \)-steepest descent method, the RH problem \( M^{(5)}(z) \) is deformed into a hybrid \( \bar{\partial} \)-RH problem \( M^{(4)}(z) \), which can be solved by decomposing it into a pure RH problem \( M^{rhp}(z) \) and a pure \( \bar{\partial} \)-problem \( M^{(5)}(z) \). The RH problem \( M^{rhp}(z) \) can be approximated by a solvable Painlevé model via the local paramatrix near the critical point \( z = -1 \) in Subsection 3.3. The residual error comes from a small RH problem \( E(z) \) and the pure \( \bar{\partial} \)-problem \( M^{(5)}(z) \) in Subsection 3.4. Finally, summing up the estimates above yields the asymptotic behavior of the solutions of the defocusing NLS equation in terms of the real-valued solutions of the Painlevé II equation, and the proof of Theorem 1.1—Case I is in Subsection 3.5.

In Section 4, we investigate the asymptotics of the solution in the Painlevé sector \( P_{+1}(x,t) \) using a similar way as Section 3, and the proof of Theorem 1.1—Case II is in Subsection 4.5.

To clarify a series of deformations and approximations to the Painlevé model described above, we make the following chains as

\[
M(z) \xrightarrow{\text{Conjugating}} M^{(1)}(z) \xrightarrow{\text{Removing Poles}} M^{(2)}(z) \xrightarrow{\text{Removing Singularity}} M^{(3)}(z)
\]

\[
M^{(4)}(z) \xrightarrow{\text{Opening Contour}} \begin{cases} 
M^{rhp}(z) \xrightarrow{\text{Scaling}} M^{\infty}(k) \\
E(z) \xrightarrow{\text{Painlevé}} \\
M^{(5)}(z) := M^{(4)}(z)(M^{rhp}(z))^{-1}.
\end{cases}
\]

The main approximation in matching the local model with the Painlevé RH model \( M^{\infty}(k) \) is shown in Proposition 3.10. Two error functions \( E(z) \) and \( M^{(5)}(z) \) are given by Proposition 3.13 and Proposition 3.80, respectively.
2. Inverse scattering transform

In this section, we recall briefly main results about inverse scattering transform for the defocusing NLS equation that will be used in this paper. The details can be found in [32].

2.1. Jost functions.

The NLS equation (1.1) admits a Lax pair
\[
\begin{align*}
\psi_x &= L\psi, \quad L = L(z; x, t) = i\sigma_3 (Q - \lambda(z)), \\
\psi_t &= T\psi, \quad T = T(z; x, t) = -2\lambda(z)L + i(Q^2 - I)\sigma_3 + Q_x,
\end{align*}
\]
where
\[Q = Q(x, t) = \begin{pmatrix} 0 & \bar{q}(x, t) \\ q(x, t) & 0 \end{pmatrix}, \quad \lambda(z) = \frac{1}{2} (z + z^{-1}).\]

We define the Jost solutions of Lax pairs (2.1)-(2.2) with asymptotics
\[\psi^\pm(z) \sim Y^\pm e^{-it\theta(z)\sigma_3}, \quad x \to \pm \infty,
\]
where we denote
\[\psi^\pm(z) := \psi^\pm(z; x, t) \quad \text{and} \quad Y^\pm = I \mp \sigma_1 z^{-1}, \quad \det Y^\pm = 1 - z^{-2},\]

\[\theta(z) = \zeta(z)(xt^{-1} - 2\lambda(z)), \quad \zeta(z) = \frac{1}{2} (z - z^{-1}).\]

Making a transformation
\[m^\pm(z) = \psi^\pm(z)e^{it\theta(z)\sigma_3},\]
then \(m^\pm(z)\) satisfy the Volterra integral equations
\[m^\pm(z) = Y^\pm + \int_{\pm \infty}^x \left( Y^\pm e^{-i\zeta(z)(x-y)\sigma_3}Y^{-1} \right) \left( \Delta L^\pm m^\pm(y) \right) dy, \quad z \neq \pm 1,
\]

\[m^\pm(z) = Y^\pm + \int_{\pm \infty}^x \left( I + (x - y)L^\pm \right) \Delta L^\pm(z; y)m^\pm(z) dy, \quad z = \pm 1,
\]
where \(\Delta L^\pm(z; y) = i\sigma_3 (Q \mp \sigma_1)\).

The existence, analyticity and differentiability of \(m^\pm(z)\) can be proven directly. Here we list their properties [32].

**Lemma 2.1.** Denote \(m^\pm(z) = (m^\pm_1(z), m^\pm_2(z))\) and let \(q_0 \in \tanh(x) + L^{1,2}(\mathbb{R})\) and \(q'_0 \in W^{1,1}(\mathbb{R})\), then we have

- **Analyticity:** \(m^\pm_1(z)\) and \(m^\pm_2(z)\) can be analytically extended to \(z \in \mathbb{C}^-\), while \(m^\pm_1(z)\) and \(m^\pm_2(z)\) can be analytically extended to \(z \in \mathbb{C}^+\).
- **Symmetry:** \(m^\pm_1(z)\) and \(m^\pm_2(z)\) admit the symmetries
\[\psi^\pm(z) = \sigma_1 \overline{\psi^\pm(z)} \sigma_1 = \pm z^{-1} \psi^\pm(z^{-1}) \sigma_1.
\]
Lemma 2.2. Following properties \cite{32}. It is shown that the scattering data and the reflection coefficient have the following properties.

- Asymptotics: $m_1^\pm(z)$ and $m_2^\pm(z)$ have the asymptotic properties
  \[
  m_1^\pm(z) = e_1 + O\left(z^{-1}\right); \quad m_2^\pm(z) = e_2 + O\left(z^{-2}\right), \quad z \to \infty,
  \]
  \[
  m_1^\pm(z) = \pm \frac{1}{z} e_2 + O(1); \quad m_2^\pm(z) = \pm \frac{1}{z} e_1 + O(1), \quad z \to 0.
  \]

- For any $\delta > 0$ sufficiently small, the maps
  \[
  q \to \det[\psi_1^-, \psi_2^+], \quad q \to \det[\psi_1^+, \psi_1^-],
  \]
  are locally Lipschitz maps
  \[
  \{ q : q \in \tanh(x) + L^{1,2}, q' \in W^{1,\infty} \} \to W^{1,\infty}(\mathbb{R} \setminus (-\delta, \delta)).
  \]

2.2. Scattering data.

The Jost functions $\psi_{\pm}(z)$ admit the scattering relation
\[
\psi_-(z) = \psi_+(z)S(z),
\]
where $S(z)$ is the spectral matrix given by
\[
S(z) = \begin{pmatrix}
  s_{11}(z) & \overline{s_{21}(\bar{z})} \\
  s_{21}(z) & \overline{s_{22}(\bar{z})}
\end{pmatrix},
\]
and $s_{11}(z)$ and $s_{21}(z)$ are the scattering data, by which we define a reflection coefficient
\[
r(z) := \frac{s_{21}(z)}{s_{11}(z)}.
\]

It is shown that the scattering data and the reflection coefficient have the following properties \cite{32}.

Lemma 2.2. Let $q_0 \in \tanh x + H^{2,2}(\mathbb{R})$, then

- $s_{11}(z)$ can be analytically extended to $z \in \mathbb{C}^+$ while $s_{21}(z)$ and $r(z)$ are defined only for $z \in \mathbb{R} \setminus \{0, \pm 1\}$. Zeros of $s_{11}(z)$ in $\mathbb{C}^+$ are simple, finite and distribute on the unitary circle.

- The scattering data can be described by the Jost functions
  \[
  s_{11}(z) = \frac{\det[\psi_1^+(z), \psi_2^+(z)]}{1 - z^{-2}}, \quad s_{21}(z) = \frac{\det[\psi_1^+(z), \psi_1^-(z)]}{1 - z^{-2}}.
  \]

- For $z \in \mathbb{R} \setminus \{0, \pm 1\}$, we have
  \[
  |s_{11}(z)|^2 = 1 + |s_{21}(z)|^2 \iff |r(z)|^2 = 1 - |s_{11}(z)|^{-2} < 1.
  \]

- $r(z) \in H^1(\mathbb{R})$ and $\| \log(1 - |r(z)|^2) \|_{L^p(\mathbb{R})} < \infty$, $p \geq 1$.

- Symmetry: $S(z) = \sigma_1 S(\bar{z})\sigma_1 = \pm z^{-1} S(z^{-1})\sigma_1$.

- Asymptotics:
  \[
  |s_{21}(z)| = O(|z|^{-2}), \quad |z| \to \infty, \quad |s_{21}(z)| = O(|z|^2), \quad |z| \to 0.
  \]
  \[
  r(z) \sim z^{-2}, \quad z \to \infty, \quad r(z) \sim 0, \quad z \to 0.
  \]
RH problem 2.1. It can be verified that
\[ s_{11}(z) = \frac{s_+}{z \mp 1} + O(1), \quad s_{21}(z) = \mp \frac{s_-}{z \mp 1} + O(1), \]
where \( s_{\pm} = \det \left[ \psi_1^-(\pm 1), \psi_2^+(\pm 1) \right] \) and
\[
(2.11) \lim_{z \to \mp 1} r(z) = \mp 1.
\]

In the non-generic case, \( s_{11}(z) \) and \( s_{21}(z) \) are continuous at \( z = \pm 1 \) and \(|r(\pm 1)| < 1\).

2.3. A basic Riemann-Hilbert problem.
Denote \( H = \{0, 1, \cdots, N - 1\} \) and
\[
Z^+ = \{z_j | s_{11}(z_j) = 0, \quad z_j \in \mathbb{C}, \quad |z_j| = 1, \quad j \in H\},
\]
\[
Z^- = \{z_j | s_{22}(z_j) = 0, \quad z_j \in \mathbb{C}, \quad |z_j| = 1, \quad j \in H\}.
\]
Moreover, \( s_{11}(z) \) satisfies the trace formula
\[
(2.12) \quad s_{11}(z) = \prod_{j=1}^{N} \frac{z - z_j}{z - \bar{z}_j} \exp \left( -i \int_{\mathbb{R}} \frac{\nu(\zeta)}{\zeta - z} \, d\zeta \right),
\]
where \( z_j \in Z^+ \) and
\[
(2.13) \quad \nu(\zeta) = -\frac{1}{2\pi} \log \left(1 - |r(\zeta)|^2\right).
\]

Based on the properties of the scattering data \( s_{11}(z) \), we define
\[
M(z) = M(z; x, t) = \begin{cases} \left( \frac{m^-_1(z)}{s_{11}(z)}, \frac{m^+_2(z)}{s_{11}(z)} \right), & z \in \mathbb{C}_+, \\
\left( \frac{m^+_1(z)}{s_{11}(z)}, \frac{m^-_2(z)}{s_{11}(z)} \right), & z \in \mathbb{C}_- \end{cases}
\]
It can be verified that \( M(z) \) satisfies the following RH problem.

RH problem 2.1. Find a matrix-valued function \( M(z) \) which satisfies
- **Analyticity:** \( M(z) \) is meromorphic in \( \mathbb{C} \setminus \mathbb{R} \).
- **Symmetry:** \( M(z) = \sigma_1 \overline{M(\bar{z})} \sigma_1 = z^{-1} M(z^{-1}) \sigma_1 \).
- **Singularity:** \( M(z) \) has the singularity point \( z = 0 \) with \( zM(z) \to \sigma_1, \quad z \to 0 \).
- **Asymptotic behavior:** \( M(z) \to I, \quad z \to \infty \).
- **Jump condition:** \( M(z) \) satisfies the jump condition
\[
M_+(z) = M_-(z)V(z), \quad z \in \mathbb{R},
\]
where
\[
(2.14) \quad V(z) = \begin{pmatrix} 1 - |r(z)|^2 & -\overline{r(z)} e^{-2i\theta(z)} \\ r(z) e^{2i\theta(z)} & 1 \end{pmatrix},
\]
with
\[
\theta(z) = \frac{\zeta(z)}{t} - 2\zeta(z) \lambda(z) = \frac{x}{2t} (z - z^{-1}) - \frac{1}{2} (z^2 - z^{-2}).
\]
Figure 2. The poles \( z_j \in \mathbb{Z}^+, \bar{z}_j \in \mathbb{Z}^- \) and jump contour \( \mathbb{R} \) for the RH problem \( M(z) \).

\( \bullet \) Residue conditions: \( M(z) \) has simple poles at each points \( z_j \) in \( \mathbb{Z}^+ \cup \mathbb{Z}^- \) with the following residue conditions
\begin{align}
\text{Res}_{z = z_j} M(z) &= \lim_{z \to z_j} M(z) \begin{pmatrix} 0 & 0 \\ c_j e^{2i \theta(z_j)} & 0 \end{pmatrix}, \\
\text{Res}_{z = \bar{z}_j} M(z) &= \lim_{z \to \bar{z}_j} M(z) \begin{pmatrix} 0 & -\bar{c}_j e^{-2i \theta(\bar{z}_j)} \\ 0 & 0 \end{pmatrix},
\end{align}

where
\begin{align}
c_j &= \frac{s_{21}(z_j)}{s_{11}^*(z_j)} = \frac{4iz_j}{\int_{\mathbb{R}} |\psi_2^*(z_j, x)|^2 dx} = i z_j |c_k|.
\end{align}

This is an RH problem with jumps on the real axis and poles distributed on the unit circle. See Figure 2. The solution of the NLS equation can be given by the reconstruction formula
\begin{align}
q(x, t) = \lim_{z \to \infty} (zM(z))_{21}.
\end{align}

2.4. Classification of asymptotic regions.

The jump matrix \( V(z) \) admits the following factorizations
\begin{align}
V(z) &= \left\{ \begin{pmatrix} 1 & -\bar{r} e^{-2i \theta(z)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ r e^{2i \theta(z)} & 1 \end{pmatrix},
\begin{pmatrix} 1 & 0 \\ 1 - |r|^2 e^{2i \theta(z)} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{1-|r|^2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1-|r|^2 e^{-2i \theta(z)} & 1 \end{pmatrix} \right\}.
\end{align}

The long-time asymptotics of RH problem 2.1 is affected by the growth and decay of the oscillatory terms \( e^{\pm 2i \theta(z)} \) in the jump matrix \( V(z) \). Direct calculations show that
\begin{align}
\text{Re}(2i \theta(z)) = 2 \text{Re} z \text{Im} z \left( 1 + \frac{1}{(\text{Re}^2 z + \text{Im}^2 z)^2} \right) - 2 \xi \text{Im} z \left( 1 + \frac{1}{\text{Re}^2 z + \text{Im}^2 z} \right),
\end{align}

where \( \xi := \frac{\bar{r}}{\pi} \). The signature table of \( \text{Re}(2i \theta(z)) \) and distribution of phase points are shown in Figure 3. The signal of \( \text{Re}(2i \theta(z)) \) determines the decay.
regions of the oscillating factor $e^{\pm 2i\theta(z)}$, which inspires us to open the jump contour $\mathbb{R}$ with different factorizations of the jump matrix $V(z)$.

The stationary phase points are determined by the equation

$$\theta'(z) = -z^{-1}(\eta^2 - \xi \eta - 2) = 0,$$

where $\eta = z + z^{-1}$. The equation (2.21) admits two solutions

$$\eta_1 = \frac{1}{2}(\xi - \sqrt{\xi^2 + 8}), \quad \xi < -1,$$

$$\eta_2 = \frac{1}{2}(\xi + \sqrt{\xi^2 + 8}), \quad \xi > 1.$$

![Figure 3](image.png)

**Figure 3.** The signature table of $\text{Re}(2i\theta(z))$ and distribution of phase points. In the green region, we have $\text{Re}(2i\theta(z)) > 0$, which implies that $e^{-2i\theta(z)} \to 0$ as $t \to +\infty$; In the white region, $\text{Re}(2i\theta(z)) < 0$, which implies that $e^{2i\theta(z)} \to 0$ as $t \to +\infty$.

For the case $\xi < -1$, the two stationary phase points satisfy the equation

$$z^2 + 1 = \eta_1 z,$$

which has two solutions

$$\xi_j = -\frac{1}{2}\eta_1 + (-1)^{j+1}\sqrt{\eta_1^2 - 4}, \quad j = 1, 2, \text{ for } \xi < -1,$$

with $\xi_2 < -1 < \xi_1 < 0$. 


For the case $\xi > 1$, the two stationary phase points satisfy the equation
\begin{equation}
(2.26) \\
z^2 + 1 = \eta_2 z,
\end{equation}
which has two solutions
\begin{equation}
(2.27) \\
\xi_j = \frac{1}{2} \left| \eta_2 + (-1)^j \sqrt{\eta_2^2 - 4} \right|, \quad j = 1, 2, \text{ for } \xi > 1,
\end{equation}
with $0 < \xi_1 < 1 < \xi_2$.

The number of phase points located on jump contour $\mathbb{R}$ allows us to divide the half-plane $(x, t)$ into three asymptotic regions.

- For $|\xi| < 1$, there is no phase point on $\mathbb{R}$. See Figure 3(c) and 3(d).
  This case is a solitonic region studied by Cuccagana and Jenkins [32].
- For $|\xi| > 1$, there are two phase points on $\mathbb{R}$. See Figure 3(a) and 3(f). This case is a solitonless region studied by us [43].
- For $|\xi| \approx 1$, there is one phase point on $\mathbb{R}$. See Figure 3(b) and 3(e).
  This case is a transition region, which is an open question proposed by Cuccagana and Jenkins [32], and we will solve it in our present paper.

To describe the asymptotics in Figure 3(b) and 3(e), we aim to find the asymptotics of $q(x, t)$ in the transition region $\mathcal{P}^\pm_1(x, t)$ defined by (1.3) in the next Section 3 and Section 4, respectively.

3. Painlevé asymptotics in transition region $\mathcal{P}^{-1}_1(x, t)$

In this section, we study the Painlevé asymptotics in the region $(x, t) \in \mathcal{P}^{-1}_1(x, t)$. Here we consider the region $-C < (\xi + 1)t^{2/3} < 0$ which corresponds to Figure 3(a). For brevity, we denote
\begin{equation}
\mathcal{P}^{-1}_1(x, t) = \{(x, t) : -C < (\xi + 1)t^{2/3} < 0\}.
\end{equation}

In this case, the two stationary points $\xi_1, \xi_2$ defined by (2.25) are real and close to $z = -1$ at least the speed of $t^{-1/3}$ as $t \to +\infty$.

We make some modifications to the basic RH problem 2.1 to get a standard RH problem without poles and singularities by performing two essential operations.

3.1. Modifications to the basic RH problem.

Since the poles $z_j \in \mathbb{Z}^+$ and $\bar{z}_j \in \mathbb{Z}^-$ are finite, distributed on the unitary circle and far away from the jump contour $\mathbb{R}$ and critical line $\text{Im} \theta(z) = 0$, they exponentially decay when we change their residues into jumps on small circles. This allows us to first modify the basic RH problem 2.1 by removing these poles.
3.1.1. Removing poles.

To remove poles \( z_j \in \mathbb{Z}^+ \) and \( \bar{z}_j \in \mathbb{Z}^- \) and open the contour \((0, \infty)\) by the second matrix decomposition in (2.19), we define the function

\[
T(z) = \prod_{j=0}^{N-1} \left( \frac{z - z_j}{z - \bar{z}_j - 1} \right) \exp \left( -i \int_0^\infty \nu(\zeta) \left( \frac{1}{\zeta - z} - \frac{1}{2\zeta} \right) d\zeta \right),
\]

where \( \nu(\zeta) \) is given by (2.13). Then the following lemma holds.

**Lemma 3.1.** \cite{32} The function \( T(z) \) has the following properties

- **Analyticity:** \( T(z) \) is meromorphic in \( \mathbb{C} \setminus [0, \infty) \) with simple zeros at the points \( z_j \) and simple poles at the points \( \bar{z}_j \).
- **Symmetry:** \( T(\bar{z}) = T(z)^{-1} = T(z^{-1}) \).
- **Jump condition:**
  \[
  T_+(z) = T_-(z)(1 - |r(z)|^2), \quad z \in (0, \infty).
  \]
- **Asymptotic behavior:** Let
  \[
  T(\infty) := \lim_{z \to \infty} T(z) = \left( \prod_{j=0}^{N-1} \bar{z}_j \right) \exp \left( i \int_0^\infty \frac{\nu(\zeta)}{2\zeta} d\zeta \right).
  \]

Then the asymptotic expansion at infinity is

\[
T(z) = T(\infty) \left[ 1 - \frac{i}{z} \left( 2 \sum_{j=0}^{N-1} \text{Im} z_j - \int_0^\infty \nu(\zeta) d\zeta \right) + O(z^{-2}) \right].
\]

- **Boundedness:** The ratio \( \frac{\text{s}_{11}(z)}{T(z)} \) is holomorphic in \( \mathbb{C}^+ \) and \( \frac{\text{s}_{11}(z)}{T(z)} \) is bounded for \( z \in \mathbb{C}^+ \). Additionally, \( \frac{\text{s}_{11}(z)}{T(z)} \) extends as a continuous function on \( \mathbb{R}^+ \) with \( \left| \frac{\text{s}_{11}(z)}{T(z)} \right| = 1 \) for \( z \in (0, \infty) \).

For \( z_j \in \mathbb{Z}^+ \) on the circle \( |z| = 1 \), define

\[
\rho < \frac{1}{2} \text{min}\{ \min_{z_j, z_l \in \mathbb{Z}^+} |z_j - z_l|, \min_{z_j \in \mathbb{Z}^+} |\text{Im} z_j|, \min_{z_j, z_l \in \mathbb{Z}^+, \text{Im} \theta(z) = 0} |z_j - z_l| \},
\]

and make small circles at the center \( z_j \) and \( \bar{z}_j \) with the radius \( \rho \), respectively. The direction of each small circle in \( \mathbb{C}^+ \) is counterclockwise, and that of each small circle in \( \mathbb{C}^- \) is clockwise. See Figure 4.

Notice that the exponential factors in the residue conditions (2.15) and (2.16) increase with \( t \). To arrive at an RH problem with decreasing off-diagonal terms in the jump matrices, we further construct the interpolation.
Figure 4. The jump contour $\Sigma^{(1)}$ of $M^{(1)}(z)$.

function

$G(z) = \begin{cases} 
  \begin{pmatrix} 1 & z - z_j \\ -c_j e^{2i\theta(z_j)} & 1 \end{pmatrix}, & |z - z_j| < \rho, \\
  \begin{pmatrix} 1 & 0 \\ z - \bar{z}_j & 1 \end{pmatrix}, & |z - \bar{z}_j| < \rho, \\
  \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \text{otherwise,}
\end{cases}$

where $z_j \in \mathbb{Z}^+$ and $\bar{z}_j \in \mathbb{Z}^-$. Define a directed path

$\Sigma^{(1)} = \mathbb{R} \cup \bigcup_{j=0}^{N-1} \{z \in \mathbb{C} : |z - z_j| = \rho, \text{or } |z - \bar{z}_j| = \rho\}$

where the direction on $\mathbb{R}$ goes from left to right, depicted in Figure 4.

Denote two factorizations of jump matrix by

$M^{(1)} = T(\infty)^{-\sigma_3} M(z) T(z)^{\sigma_3}$,

then $M^{(1)}(z)$ satisfies the RH problem as follows.

**RH problem 3.1.** Find $M^{(1)}(z) = M^{(1)}(z; x, t)$ with properties

- $M^{(1)}(z)$ is analytical in $\mathbb{C} \setminus \Sigma^{(1)}$,
- $M^{(1)}(z) = \sigma_1 M^{(1)}(\bar{z}) \sigma_1 = z^{-1} M^{(1)}(z^{-1}) \sigma_1$. 
\* \( M^{(1)}(z) \) satisfies the jump condition
\[ M^{(1)}_+(z) = M^{(1)}_-(z)V^{(1)}(z), \]
where
\[
V^{(1)}(z) = \begin{cases} 
B_-^{-1}B_+, & z \in (0, \infty), \\
b_-^{-1}b_+, & z \in (-\infty, \xi_2) \cup (\xi_1, 0), \\
T(z)^{-\sigma_3}V(z)T(z)^{\sigma_3}, & z \in (\xi_2, \xi_1), 
\end{cases}
\]
\[
= \begin{cases} 
1 - \frac{z-z_j}{c_j}T^{-2}(z)e^{-2it\theta(z_j)}, & |z - z_j| = \rho, \\
0 & |z - \bar{z}_j| = \rho, 
\end{cases}
\]
\* \( M^{(1)}(z) \) admits the asymptotic behaviors
\[ M^{(1)}(z) = I + O(z^{-1}), \quad z \to \infty, \]
\[ zM^{(1)}(z) = \sigma_1 + O(z), \quad z \to 0. \]

Since the jump matrices on the circles \(|z - z_j| = \rho\) or \(|z - \bar{z}_j| = \rho\) exponentially decay to the identity matrix as \( t \to \infty \), it can be shown that the RH problem 3.1 is asymptotically equivalent to the following RH problem.

**RH problem 3.2.** Find \( M^{(2)}(z) = M^{(2)}(z; x, t) \) with properties
\* \( M^{(2)}(z) \) is analytical in \( \mathbb{C}\setminus\mathbb{R} \).
\* \( M^{(2)}(z) = \sigma_1 M^{(2)}(\bar{z})\sigma_1 = z^{-1}M^{(2)}(z^{-1})\sigma_1. \)
\* \( M^{(2)}(z) \) satisfies the jump condition
\[ M^{(2)}_+(z) = M^{(2)}_-(z)V^{(2)}(z), \]
where
\[
V^{(2)}(z) = \begin{cases} 
B_-^{-1}B_+, & z \in (0, \infty), \\
b_-^{-1}b_+, & z \in (-\infty, \xi_2) \cup (\xi_1, 0), \\
T(z)^{-\sigma_3}V(z)T(z)^{\sigma_3}, & z \in (\xi_2, \xi_1). 
\end{cases}
\]
\* \( M^{(2)}(z) \) admits the asymptotic behaviors
\[ M^{(2)}(z) = I + O(z^{-1}), \quad z \to \infty, \]
\[ zM^{(2)}(z) = \sigma_1 + O(z), \quad z \to 0. \]

**Proposition 3.2.** The solution of RH problem 3.1 can be approximated by the solution of RH problem 3.2
\[ M^{(1)}(z) = M^{(2)}(z) \left( I + O \left( e^{-ct} \right) \right), \]
where \( c \) is a constant.
Proof. The result is derived from the theorem of Beals-Coifman and the corresponding norm estimates.

Next we remove the spectral singularity $z = 0$ by an appropriate transformation.

3.1.2. Removing singularity.

In order to remove the singularity $z = 0$, we make a transformation

$$M^{(2)}(z) = \left( I + \frac{1}{z} \sigma_1 M^{(3)}(0)^{-1} \right) M^{(3)}(z), \quad (3.11)$$

then $M^{(3)}(z)$ satisfies the RH problem without spectral singularity.

**RH problem 3.3.** Find $M^{(3)}(z) = M^{(3)}(z; x, t)$ with properties

- $M^{(3)}(z)$ is analytical in $\mathbb{C} \setminus \mathbb{R}$.
- $M^{(3)}(z) = \sigma_1 M^{(3)}(\bar{z}) \sigma_1 = \sigma_1 M^{(3)}(0)^{-1} M^{(3)}(z^{-1}) \sigma_1$.
- $M^{(3)}(z)$ satisfies the jump condition

$$M^{(3)}_+(z) = M^{(3)}_-(z) V^{(2)}(z),$$

where $V^{(2)}(z)$ is given by (3.9).
- $M^{(3)}(z)$ admits the asymptotics

$$M^{(3)}(z) = I + O(1), \quad z \to \infty.$$

Proof. We show that if $M^{(3)}(z)$ satisfies the RH problem 3.3, then $M^{(2)}(z)$ satisfies the RH problem 3.2. Firstly, we verify the jump condition

$$M^{(2)}_+(z) = \left( I + \frac{1}{z} \sigma_1 M^{(3)}(0)^{-1} \right) M^{(3)}_+(z) = M^{(2)}_-(z) V^{(2)}(z). \quad (3.12)$$

To show the singularity of $z = 0$, substituting the expansion

$$M^{(3)}(z) = M^{(3)}(0) + z \tilde{M}^{(3)}(z),$$

into (3.12) yields

$$M^{(2)}(z) = \frac{1}{z} \sigma_1 + M^{(3)}(0) + z \tilde{M}^{(3)}(z) + \sigma_1 M^{(3)}(0)^{-1} \tilde{M}^{(3)}(z)$$

$$= \frac{1}{z} \sigma_1 + O(1), \quad z \to 0.$$

\]

3.2. Transformation to a hybrid $\bar{\partial}$-RH problem.

In this section, we open the jump contour $\mathbb{R} \setminus (\xi_2, \xi_1)$ by the $\bar{\partial}$ extension. Denote $\xi_0 = 0$, $\gamma = (\xi_0 + \xi_1)/2$, and

$$l_1 \in (0, |\gamma| \sec \phi), \quad l_2 \in (0, |\gamma| \tan \phi),$$

where $\phi = \phi(\xi)$. We define the following rays passing through $\xi_0, \xi_1$ and $\xi_2$

$$\Sigma_0 = e^{i(\pi - \phi)} l_1, \quad \Sigma_1 = \xi_1 + e^{i\phi} l_1, \quad \Sigma_2 = \xi_2 + e^{i(\pi - \phi)} \mathbb{R}^+, \quad \Sigma_3 = e^{i\phi} \mathbb{R}^+, \quad L = \gamma + e^{i\pi/2} l_2,$$
Figure 5. Open the jump contour $\mathbb{R} \setminus (\xi_2, \xi_1)$ along red rays and blue rays. The green regions are continuous extension sectors with $\text{Re} (2i\theta(z)) > 0$, while the white regions are continuous extension sectors with $\text{Re} (2i\theta(z)) < 0$.

$\Sigma_j$ and $\overline{L}$ denote their conjugate rays. These rays are opened at a sufficiently small angle $0 < \phi < \pi/4$ such that they all fall into their decaying regions, which correspond to the signature table of $\text{Re} (2i\theta(z))$. The opened sectors with the above jump lines are denoted by $\Omega_j$ and $\overline{\Omega}_j$. See Figure 5, which corresponds to Figure 3(a).

To determine the decaying properties of the oscillating factors $e^{\pm 2i\theta(z)}$, we especially estimate $\text{Re}(2i\theta(z))$ in regions $\Omega_j$, $j = 0, 1, 2, 3$.

Proposition 3.3. Let $\xi < -1$ and $(x, t) \in \mathcal{P}_{< -1}(x, t)$. Then the following estimates hold.

Case I. (corresponding to $z = 0$)

\begin{align*}
\text{Re}(2i\theta(z)) &\geq |\sin 2\phi||v|, \quad z \in \overline{\Omega}_0 \cup \Omega_3, \\
\text{Re}(2i\theta(z)) &\leq -|\sin 2\phi||v|, \quad z \in \Omega_0 \cup \overline{\Omega}_3,
\end{align*}

where $z = u + iv$ and $\phi = \text{arg } z$.

Case II. (corresponding to $z = \xi_1$)

\begin{align*}
\text{Re}(2i\theta(z)) &\leq -\frac{4}{|\xi_1|} u^2|v|, \quad z \in \Omega_1, \\
\text{Re}(2i\theta(z)) &\geq \frac{4}{|\xi_1|} u^2|v|, \quad z \in \overline{\Omega}_1,
\end{align*}

where $z = u + iv$.

Case III. (corresponding to $z = \xi_2$)

\begin{align*}
\text{Re}(2i\theta(z)) &\leq \begin{cases} 
- \frac{1}{8|\xi_2|} u^2|v|, & z \in \Omega_2 \cap \{|z| \leq 2\}, \\
-2\sqrt{2}|v|, & z \in \Omega_2 \cap \{|z| > 2\},
\end{cases} \\
\text{Re}(2i\theta(z)) &\geq \begin{cases} 
\frac{1}{8|\xi_2|} u^2|v|, & z \in \overline{\Omega}_2 \cap \{|z| \leq 2\}, \\
2\sqrt{2}|v|, & z \in \overline{\Omega}_2 \cap \{|z| > 2\},
\end{cases}
\end{align*}

where $z = \xi_2 + u + iv$. 
Proof. For the case I, we take $\Omega_3$ as an example to prove the estimate (3.13). Other cases can be proven similarly.

For $z \in \Omega_3$, denote the ray $z = |z|e^{i\varphi} = u + iv$ where $0 < \varphi < \phi$ and $u > v > 0$, and the function $F(s) = s + s^{-1}$ with $s > 0$. Then (2.20) becomes

\[(3.19) \quad \text{Re} (2i\theta(z)) = 2\varphi \left( F(|z|^2) - \xi \sec \varphi F(|z|) - 2 \right). \]

Since $\xi < -1$, we have $-\xi \sec \varphi F(|z|) > 0$, and so

\[F(|z|^2) - \xi \sec \varphi F(|z|) - 2 \geq F(|z|^2) - F(|z|) \geq v. \]

Substituting the above estimate into (3.19) gives

\[\text{Re} (2i\theta(z)) \geq v \sin 2\varphi, \]

which yields (3.13) for $z \in \Omega_3$.

For cases II and III, we take $\Omega_2$ as an example. In this case, for $z = \xi_2 + u + iv = \xi_2 + |z - \xi_2|e^{i\varphi} \in \Omega_2$, we have

\[u < 0, \ v = -u \tan \varphi > 0, \ |z|^2 = (u + \xi_2)^2 + u^2 \tan^2 \varphi, \]

with $0 < \varphi < \phi$. We prove the estimate (3.17) to the cases $|z| \leq 2$ and $|z| > 2$ respectively.

For the region $|z| \leq 2$, with (2.22) and (2.25), we have

\[\xi = \frac{\xi_2^4 + 1}{\xi_2^2 + \xi_2}. \]

Substituting the above formula into (2.20) gives

\[\text{Re} (2i\theta(z)) = \frac{2v}{|z|^4 (\xi_2^3 + \xi_2)} \left\{ \left[ (\xi_2^3 + \xi_2) u + \xi_2^2 - 1 \right] \right. \]

\[\left. \left[ (u + \xi_2)^2 + u^2 \tan^2 \varphi \right]^2 - (\xi_2^4 + 1) (1 + \tan^2 \varphi) u^2 \right. \]

\[+ \left. \left( -2\xi_2^5 + \xi_2^3 - \xi_2 \right) u + \xi_2^4 - \xi_2^6 \right\}. \]

With the properties

\[\xi_2 < -1, \ u < 0, \ (\xi_2^3 + \xi_2) u + \xi_2^2 - 1 \geq 0, \ (\xi_2^4 + \xi_2) u^3 \geq 0, \]

after removing the terms in order $u^4$ and $u^3$, we find that

\[(3.20) \quad \text{Re} (2i\theta(z)) \leq \frac{2v}{|z|^4 (\xi_2^3 + \xi_2)} (f(\xi_2)u^2 + g(\xi_2)u), \]

where

\[f(\xi_2) = 4\xi_2^6 + (9 + \tan^2 \varphi)\xi_2^4 - 2(3 + \tan^2 \varphi)\xi_2^2 - (1 + \tan^2 \varphi), \]

\[g(\xi_2) = \xi_2^2 + 3\xi_2^3 - 3\xi_2^5 - \xi_2. \]

Direct calculation shows that the function $f(\xi_2)$ is strictly decreasing in $\xi_2 \in (-\infty, -1)$, and hence

\[f(\xi_2) > f(-1) = 6 - 2\tan^2 \varphi. \]

For $\varphi \in (0, \frac{\pi}{2})$, we have $\tan^2 \varphi \in (0, 1)$, $f(-1) \in (4, 6)$, and so $f(\xi_2) > 4$. 


Similarly, \( g(\xi_2) \) is strictly increasing in \( \xi_2 \in (-\infty, -1) \) and \( g(-1) = 0 \), then \( g(\xi_2)u > 0 \). Therefore, (3.20) becomes

\[
\text{Re} \left( 2i\theta(z) \right) \leq \frac{8uv^2}{|z|^4 (|\xi_2|^2 + \xi_2)}.
\]

together with \( |z| \leq 2 \), gives us the estimate (3.17).

Next we prove (3.17) in the region \( |z| > 2 \). In this case, we notice that \( |z| = |u + \xi_2| \sec w > 2 \),

with \( w = \arg z \) and \( 0 < w < \varphi \), which implies that \( u + \xi_2 < -2 \cos w \).

Further with (2.20), we have

\[
(3.21) \quad \text{Re} \left( 2i\theta(z) \right) = 2v \left( 1 + \frac{1}{|z|^4} \right) \left( u + \xi_2 - \xi \frac{|z|^4 + |z|^2}{|z|^4 + 1} \right) \leq 2vh(\xi),
\]

where

\[
h(\xi) = -2 \cos w - (\xi + 1) \left( 1 + \frac{1}{4 \cos^2 w} \right) - 1 + \frac{1}{4 \cos^2 w}.
\]

Let \( t \to +\infty \), we have \( \xi \to -1^- \) and

\[
h(\xi) \to h(-1) = -2 \cos w - 1 + \frac{1}{4 \cos^2 w}.
\]

Since \( 0 < \varphi < \frac{\pi}{2} \), we have \( \frac{\sqrt{2}}{2} < \cos w < 1 \) and then \( h(-1) \in (-\frac{11}{4}, -\frac{2\sqrt{2}+1}{2}) \).

Therefore, there exists a large \( T \) such that when \( t > T \),

\[
(3.22) \quad h(\xi) \leq h(-1) + \frac{1}{2} < -\sqrt{2}.
\]

Substituting (3.22) into (3.21) gives the second estimate when \( |z| > 2 \) of (3.17). We finish the proof.

Next we open the contour \( \mathbb{R} \setminus [\xi_2, \xi_1] \) via continuous extensions of the jump matrix \( V^{(2)}(z) \) by defining appropriate functions.
Proposition 3.4. Let $q_0 \in \tanh(x) + H^4_{-4}(\mathbb{R})$ and define functions $R_j(z)(j = 0, 1, 2, 3)$ with boundary values

\begin{align}
R_j(z) &= \begin{cases} 
    r(z)T(z)^2, & z \in (-\infty, \xi_2) \cup (\xi_1, 0), \\
    r(\xi_j)T(\xi_j)^2, & z \in \Sigma_j, \ j = 0, 1, 2,
\end{cases} \\
R_j(z) &= \begin{cases} 
    \overline{r(z)}T(z)^{-2}, & z \in (-\infty, \xi_2) \cup (\xi_1, 0), \\
    \overline{r(\xi_j)}T(\xi_j)^{-2}, & z \in \Sigma_j, \ j = 0, 1, 2,
\end{cases} \\
R_3(z) &= \begin{cases} 
    \frac{r(z)}{1-|r(z)|^2}T_-(z)^2, & z \in (0, \infty), \\
    0, & z \in \Sigma_3,
\end{cases} \\
R_3(z) &= \begin{cases} 
    \frac{\overline{r(z)}}{1-|\overline{r(z)}|^2}T_-(z)^2, & z \in (0, \infty), \\
    0, & z \in \Sigma_3,
\end{cases}
\end{align}

where $r(\xi_0) = r(0) = 0$. Then there exists a constant $c = c(q_0)$ which only depends on $q_0$ such that

For $j = 1, 2,$

\begin{equation}
|\partial R_j| \leq c \left( |\varphi(\text{Re}(z))| + |r'(\text{Re}(z))| + |z - \xi_j|^{-1/2} \right), \ z \in \Omega_j \cup \overline{\Omega}_j.
\end{equation}

For $j = 0, 3,$

\begin{equation}
|\overline{\partial R_j}| \leq \begin{cases} 
    c \left( |\varphi(\text{Re}(z))| + |r'(\text{Re}(z))| + |z|^{-1/2} \right), & z \in \Omega_j \cup \overline{\Omega}_j, \\
    c|z - 1|, & \text{near } z = 1,
\end{cases}
\end{equation}

where $\varphi \in C_0^\infty(\mathbb{R}, [0, 1])$ with small support near $z = 1$.

Proof. The extension functions $R_j(z), j = 0, 1, 2$ and their $\overline{\partial}$ estimates are easy obtained by defining

\begin{equation}
R_j(z) = \cos(k \arg z)r(|z|)T(z)^2 + \left[1 - \cos(k \arg z)\right]r(\xi_j)T(\xi_j)^2,
\end{equation}

where $z \in \Omega_j, j = 0, 1, 2$ and $k = \frac{\pi}{25}$.

As for $R_3(z)$, in the non-generic case $|r(z)| < 1, z \in \mathbb{R}$, we define

\begin{equation}
R_3(z) = \frac{r(|z|)}{1-|r(z)|^2}T(z)^{-2}\cos(k \arg z), \ z \in \Omega_3.
\end{equation}

In the generic case $|r(1)| = 1$, as observed in (3.25)-(3.26), $R_3(z)$ is singular at $z = 1$, however, the singularity can be balanced by the factor $T(z)^{-2}$ following Cuccagna and Jenkins’s method [32].

It follows from Lemmas ?? that

\begin{equation}
\frac{r(z)}{1-|r(z)|^2}T_+(z) = \frac{s_{21}(z)}{s_{11}(z)} \left( \frac{s_{11}(z)}{T_+(z)} \right)^2 = \frac{S_{21}(z)}{S_{11}(z)} \left( \frac{s_{11}(z)}{T_+(z)} \right)^2,
\end{equation}

where

\begin{equation}
S_{21}(z) = \det(\psi_+^+(z), \psi_+^-(z)), \ S_{11}(z) = \det(\psi_-^-(z), \psi_+^+(z)).
\end{equation}

Then the denominator of each factor in the r.h.s. of (3.31) is non-zero and analytical in $\Omega_3$, with well defined nonzero limit on $\partial \Omega_3$. 
Introduce a cutoff function \( \chi_0(z), \chi_1(z) \in C_0^\infty(\mathbb{R}, [0, 1]) \) with small support near \( z = 0 \) and \( z = 1 \) respectively. Define the extension function \( R_3(z) = R_{11}(z) + R_{12}(z) \) with

\[
R_{11}(z) = (1 - \chi(z))\frac{r(|z|)}{1 - |r(|z|)|^2} T(z)^{-2} \cos(k \arg z),
\]

\[
R_{12}(z) = f(|z|)g(z) \cos(k \arg z) + \frac{i|z|}{k} \chi_0(\frac{\arg z}{\delta_0}) f'(|z|)g(z) \sin(k \arg z).
\]

where \( \delta_0 \) is a fixed small constant, \( f'(s) \) is the derivative of \( f(s) \), and

\[
g(s) = \frac{s_{11}(s)^2}{T_+(s)^2}, \quad f(s) = \chi_1(s) \frac{S_{21}(s)}{S_{11}(s)}.
\]

In this way, the effect of the singularity at \( z = 1 \) can be neutralized. The details of the proof can be found in [32]. We omit it here.

Further, with these functions \( R_j \), we define a matrix function

\[
R^{(3)}(z) = \begin{cases}
\begin{pmatrix} 1 & 0 \\ -R_j e^{2it\theta(z)} & 1 \end{pmatrix}, & z \in \Omega_j, \ j = 0, 1, 2, \\
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & z \in \bar{\Omega}_j, \ j = 0, 1, 2,
\end{cases}
\]

\[
(3.34)
\]

and a contour

\[
\Sigma^{(4)} = \Gamma \cup \Gamma \cup [\xi_2, \xi_1], \quad \Gamma = \bigcup_{j=0}^{3} \Sigma_j \cup L.
\]

Then the new matrix function

\[
M^{(4)}(z) = M^{(3)}(z)R^{(3)}(z)
\]

satisfies the following hybrid \( \bar{\partial} \)-RH problem.

**\( \bar{\partial} \)-RH problem 3.1.** Find \( M^{(4)}(z) = M^{(4)}(z; x, t) \) with properties

- \( M^{(4)}(z) \) is continuous in \( \mathbb{C} \setminus \Sigma^{(4)} \). See Figure 6.
- \( M^{(4)}(z) \) satisfies the jump condition

\[
M^+(z) = M^-(z)V^{(4)}(z),
\]
where

\[
V^{(4)}(z) = \begin{cases}
1 & z \in \Sigma_j, j = 1, 2; \\
1 - r(\xi_j)T(\xi_j)^{-2}e^{-2it\theta(z)} & z \in \Sigma_j, j = 1, 2; \\
1 - r(\xi_1)T(\xi_1)^{-2}e^{-2it\theta(z)} & z \in L; \\
1 - r(\xi_1)T(\xi_1)^{-2}e^{-2it\theta(z)} & z \in L; \\
T(z)^{-\sigma_3}V(z)T(z)^{\sigma_3}, & z \in (\xi_2, \xi_1).
\end{cases}
\]

- \(M^{(4)}(z) = I + O(z^{-1}), \ z \to \infty.\)
- For \(z \in \mathbb{C} \setminus \Sigma^{(4)},\) we have
  \[
  \partial M^{(4)}(z) = M^{(4)}(z)\partial R^{(3)}(z),
  \]
  where

\[
\partial R^{(3)}(z) = \begin{cases}
1 - \partial R_j e^{2it\theta(z)} & z \in \Omega_j, j = 0, 1, 2, \\
1 - \partial R_j e^{-2it\theta(z)} & z \in \Omega_j, j = 0, 1, 2, \\
1 & z \in \Omega_3; \\
0 & z \in \Omega_3.
\end{cases}
\]

We decompose \(M^{(4)}(z)\) into a pure RH problem \(M^{rh}(z)\) with \(\partial R^{(3)}(z) = 0\) and a pure \(\bar{\partial}\)-problem \(M^{(5)}(z)\) with \(\partial R^{(3)}(z) \neq 0\) in the form

\[
M^{(4)}(z) = M^{(5)}(z)M^{rh}(z).
\]

### 3.3. Long-time analysis on a pure RH problem.

In this subsection, we find a local solution \(M^{loc}(z)\), which approximates to \(M^{rh}(z)\) near \(z = -1.\) The pure RH problem is given as follows.

**RH problem 3.4.** Find \(M^{rh}(z) = M^{rh}(z; x, t)\) which satisfies

- \(M^{rh}(z)\) is analytical in \(\mathbb{C} \setminus \Sigma^{(4)}\). See Figure 6.
- \(M^{rh}(z)\) satisfies the jump condition

\[
M^{rh}_+(z) = M^{rh}_-(z)V^{(4)}(z),
\]

where \(V^{(4)}(z)\) is given by (3.36).
3.3.1. **Local paramatrix near** $z = -1$.

In the region $-C < (\xi + 1)^{2/3} < 0$, we first notice that $\xi \to -1^-$ as $t \to \infty$, further from (2.25), it is found that two phase points $\xi_1$ and $\xi_2$ also will merge to $z = -1$. Making the asymptotic expansion of the phase function $t\theta(z)$ near $z = -1$, we find that

$$t\theta(z) = t \left( (z + 1)^3 + \frac{1}{2} \sum_{n=4}^{\infty} (n - 1)(z + 1)^n \right) + (x + 2t) \left( (z + 1)^3 + \frac{1}{2} \sum_{n=2}^{\infty} (z + 1)^n \right)$$

$$:= \frac{4}{3} k^3 + sk + S(t; k),$$

where the scaled parameters are given by

$$k = \tau^{1/3}(z + 1), \quad s = \frac{8}{3}(\xi + 1)\tau^{2/3},$$

with $\tau = \frac{3}{4}t$. The first two terms $\frac{4}{3} k^3 + sk$ play a key role in matching the Painlevé model in the local region, and the remainder term is given by

$$S(t; k) = \frac{2}{3} \sum_{n=4}^{\infty} (n - 1)\tau^{\frac{3-n}{3}} k^n + \frac{4}{3}(\xi + 1) \sum_{n=2}^{\infty} \tau^{\frac{3-n}{3}} k^n.$$

Next we show that two scaled phase points $k_j = \tau^{1/3}(\xi_j + 1)$, $j = 1, 2$ are always in a fixed interval in the $k$-plane.

**Proposition 3.5.** *In the transition region* $\mathcal{P}_{<-1}(x, t)$ *and under scaling transformation (3.40), we have*

$$k_j \in (-3/4)^{1/3}\sqrt{2C}, (3/4)^{1/3}\sqrt{2C}), \quad j = 1, 2.$$

**Proof.** From (2.24), the phase point $\xi_1$ satisfies the equation

$$\xi_1^2 + 1 = \eta_1 \xi_1 \implies (\xi_1 + 1)^2 = (\eta_1 + 2)\xi_1.$$
Noting that \(\eta_1 + 2 < 0, \xi_1 > -1\), and using (2.22), we have
\[
(\eta_1 + 2)\xi_1 < -(\eta_1 + 2) = -\frac{1}{2}\xi + \frac{1}{2}\sqrt{\xi^2 + 8} - 2 \leq -2(\xi + 1).
\]
Substituting (3.44) into (3.43) gives
\[
(\xi_1 + 1)^2 < -2(\xi + 1) < 2Ct^{-2/3},
\]
which implies that
\[
k_1 = \tau^{1/3}(\xi_1 + 1) < (3/4)^{1/3}\sqrt{2C}.
\]
Further, by the symmetry \(\xi_1\xi_2 = 1\), we obtain
\[
\xi_2 + 1 = \frac{1}{\xi_1} + 1 = \frac{\xi_1 + 1}{\xi_1}.
\]
Therefore, for \(-1 < \xi_1 < 0\), \(k_2\) can be controlled by a constant
\[
k_2 = \tau^{1/3}(\xi_2 + 1) = \tau^{1/3}\frac{\xi_1 + 1}{\xi_1} > -(3/4)^{1/3}\sqrt{2C}.
\]

Let \(t\) be large enough so that \(\sqrt{2C}\tau^{-1/3} < \rho\) where \(\rho\) has been defined in (3.4). For a fix constant \(\varepsilon \leq \sqrt{2C}\), define two open disks
\[
\mathcal{U}_z(-1) = \{z \in \mathbb{C} : |z + 1| < \varepsilon\tau^{-1/3}\},
\]
\[
\mathcal{U}_k(0) = \{k \in \mathbb{C} : |k| < \varepsilon\},
\]
whose boundaries are oriented counterclockwise. The transformation defined by (3.40) defines a map \(z \mapsto k\) maps \(\mathcal{U}_z(-1)\) onto the disk \(\mathcal{U}_k(0)\) of the radius \(\varepsilon\) in the \(k\)-plane. Proposition 3.5 implies that for large \(t\), we have \(\xi_1, \xi_2 \in \mathcal{U}_z(-1)\), and also \(k_1, k_2 \in \mathcal{U}_k(0)\). See Figure 7.

We show that when \(t\) is sufficiently large, \(\xi\) is close to \(z = -1\) and \(z\) is close to \(z = -1\), the phase function \(t\theta(z)\) can be approximated by \(\frac{4}{3}k^3 + sk\). For this purpose, we first prove that the series \(S(t; k)\) converges uniformly in \(\mathcal{U}_k(0)\) and decays with respect to \(t\).
Proposition 3.6. Let $(x, t) \in \mathcal{P}_{<-1}(x, t)$, then for $k \in \mathcal{U}_k(0)$, we have the uniform convergence estimate

$$|S(t; k)| \leq ct^{-1/3}, \quad t \to +\infty,$$

where $c$ is only dependent on the radius of $\mathcal{U}_k(0)$.

Proof. We decompose the series $S(t; k)$ into

$$S(t; k) = S_1 + S_2,$$

where

$$S_1 = \frac{2}{3} \tau^{-1/3} k^4 \sum_{n=4}^{\infty} (n - 1) \left( \tau^{-1/3} k \right)^{n-4}, \quad S_2 = \frac{4}{3} (\xi + 1) \tau^{1/3} k^2 \sum_{n=2}^{\infty} \left( \tau^{-1/3} k \right)^{n-2}.$$

For the first series $S_1$, by $\lim_{n \to \infty} \frac{n-1}{n} = 1$ and $|k| < \varepsilon$, we have

$$|S_1| \leq \frac{2}{3} \tau^{-1/3} \varepsilon^4 \sum_{n=4}^{\infty} (n - 1) (\tau^{-1/3} \varepsilon)^{n-4} \leq ct^{-1/3},$$

which implies that $S_1$ converges uniformly and decays with respect to $t$. Similarly, by $|(|\xi + 1|^t)^{2/3}| < C$ and $|k| < \varepsilon$, we have

$$|S_2| \leq \left( \frac{3}{4} \right)^{-\frac{1}{3}} C \tau^{-1/3} \varepsilon^2 \sum_{n=2}^{\infty} (\tau^{-1/3} \varepsilon)^{n-2} \leq ct^{-1/3}.$$

To show $e^{i(t^k + s)}$ is bounded in the disk $\mathcal{U}_k(0)$ (see Figure 7 (b)), we give the estimates.

Proposition 3.7. Let $(x, t) \in \mathcal{P}_{<-1}(x, t)$, then for large $t$, we have

\begin{align*}
\text{(3.45)} & \quad \text{Re} \left[ i \left( \frac{4}{3} k^3 + sk \right) \right] \leq -\frac{8}{3} u^2 v, \quad k \in \Omega_j' , \\
\text{(3.46)} & \quad \text{Re} \left[ i \left( \frac{4}{3} k^3 + sk \right) \right] \geq \frac{8}{3} u^2 v, \quad k \in \overline{\Omega}_j',
\end{align*}

where $k = k_j + u + iv$ is the scaled variable.

Proof. We only show (3.45) and the proof of (3.46) is similar. The estimate (3.45) is equivalent to the following estimate

\begin{equation}
\text{(3.47)} \quad \text{Re} \left[ 2i \left( t(z + 1)^3 + (x + 2t)(z + 1) \right) \right] \leq -4tu^2 v, \quad z \in \Omega_j,
\end{equation}

where $z = \xi_j + u + iv$, $j = 1, 2$. For $j = 1$, $u > 0$ and $v > 0$, while for $j = 2$, $u < 0$ and $v > 0$. Since $\arg(z - \xi_1) < \frac{\pi}{4}$ and $\pi - \pi/4 < \arg(z - \xi_2) < \pi$, we have $v < |u|$ and direct calculations show that

$$\text{Re} \left[ 2i \left( t(z + 1)^3 + (x + 2t)(z + 1) \right) \right] \leq -4tu^2 v + 2tvf(\xi),$$

where

\begin{equation}
\text{(3.48)} \quad f(\xi) = -3(\xi_j + 1)^2 - 2(\xi + 1) - 6(1 + (-1)^{j+1})(\xi_j + 1)u.
\end{equation}
From (2.21) and (2.24), we obtain

\[(3.49) \quad \eta_1^2 - \xi \eta_1 - 2 = 0 \Rightarrow \eta_1 + 2 = \frac{(\xi + 1) \eta_1}{\eta_1 - 1},\]

\[(3.50) \quad \xi_j^2 + 1 = \eta_1 \xi_j \Rightarrow (\xi_j + 1)^2 = (\eta_1 + 2) \xi_j, \quad j = 1, 2.\]

Substituting (3.49) and (3.50) into (3.48) yields

\[(3.51) \quad f(\xi) = \frac{\xi + 1}{1 - \eta_1} (3 \eta_1 \xi_j + 2 \eta_1 - 2) - 6(1 + (-1)^{j+1})(\xi_j + 1)u,\]

in which

\[(3.52) \quad 3 \eta_1 \xi_j + 2 \eta_1 - 2 = \frac{1}{\xi_j} (\xi_j + 1) [(\xi_j - 1/2)^2 + 4/7].\]

Further substituting (3.52) into (3.51) yields

\[f(\xi) = (\xi_j + 1)g(\xi),\]

where

\[(3.53) \quad g(\xi) = \frac{\xi + 1}{(1 - \eta_1) \xi_j} [(\xi_j - 1/2)^2 + 4/7] - 6(1 + (-1)^{j+1})u.\]

For \(j = 1\), as \(t \to +\infty\), \(g(\xi) \to -6u < 0\). Thus there exists a sufficiently large time \(T\) such that for \(t > T\), \(g(\xi) < 0\), which together with \(\xi_1 + 1 > 0\), we get \(f(\xi) < 0\).

For \(j = 2\), \(\xi + 1 < 0\), \(\xi_2 < 0\), \(1 - \eta_1 > 0\), and then (3.53) yields

\[g(\xi) = \frac{\xi + 1}{(1 - \eta_1) \xi_2} [(\xi_2 - 1/2)^2 + 4/7] > 0,\]

which together with \(\xi_2 + 1 < 0\) gives \(f(\xi) < 0\). Finally, we obtain the result (3.47), which yields (3.45) by scaling \(u \to u \tau^{-1/3}, v \to v \tau^{-1/3}\).

The jump matrix decays to the identity matrix outside \(U_z(-1)\) exponentially and uniformly fast as \(t \to +\infty\), which enlightens us to construct the solution of \(M_{rhp}(z)\) as follows:

\[(3.54) \quad M_{rhp}(z) = \begin{cases} E(z), & z \in \mathbb{C} \setminus U_z(-1), \\ E(z)M_{loc}(z), & z \in U_z(-1), \end{cases}\]

where \(E(z)\) is an error function which will be determined later, and \(M_{loc}(z)\) is a solution to the following RH problem.

**RH problem 3.5.** Find \(M_{loc}(z) = M_{loc}(z; x, t)\) with properties

- \(M_{loc}(z)\) is analytical in \(U_z(-1) \setminus \Sigma_{loc}\), where \(\Sigma_{loc} = \Sigma^{(4)} \cap U_z(-1)\).

  See Figure 7 (a).
- \(M_{loc}(z)\) satisfies the jump condition

  \[M_{+}^{loc}(z) = M_{-}^{loc}(z)V^{loc}(z), \quad z \in \Sigma_{loc},\]

  \[\Box\]
where

\[
V^{\text{loc}}(z) = \begin{cases} 
  e^{-it\theta(z)}\bar{\theta}_3 \begin{pmatrix} 1 & 0 \\ r(\xi_j)T(\xi_j)^2 & 1 \end{pmatrix}, & z \in \Sigma_j, \ j = 1, 2, \\
  e^{-it\theta(z)}\bar{\theta}_3 \begin{pmatrix} 1 & -r(\xi_j)T(\xi_j)^{-2} \\ 0 & 1 \end{pmatrix}, & z \in \overline{\Sigma}_j, \ j = 1, 2, \\
  T(z)^{-\sigma_3}V(z)T(z)^{\sigma_3}, & z \in (\xi_2, \xi_1).
\end{cases}
\]

- \(M^{\text{loc}}(z)(M^\infty((z + 1)^{1/3}))^{-1} \to I, \ t \to +\infty, \ \text{uniformly for} \ z \in \partial U_2(-1).\)

Denote

\[R(z) := r(z)T^2(z),\]

then \(R(-1) = r(-1)T^2(-1).\) We show that in the \(U_k(0), \ M^{\text{loc}}(z)\) can be approximated by the solution \(M^\infty(k)\) of the model RH problem A.1 with \(r_0 = R(-1)\) based on the following estimates.

**Proposition 3.8.** Let \(r \in H^1(\mathbb{R}), \ (x, t) \in \mathcal{P}_{<-1}(x, t),\) then

\[
(3.55) \quad |R(z) e^{2it\theta(z)} - r_0 e^{is_{k^3/3 + 2ikk}}| \lesssim t^{-1/6}, \quad k \in (k_2, k_1),
\]

\[
(3.56) \quad |R(\xi_j) e^{2it\theta(z)} - r_0 e^{is_{k^3/3 + 2ikk}}| \lesssim t^{-1/6}, \quad k \in \Sigma_j' \cup \Sigma_j, \ j = 1, 2.
\]

**Proof.** For \(k \in (k_2, k_1),\) then \(z \in (\xi_2, \xi_1)\) and \(z\) is real,

\[
|e^{2it\theta(z)}| = 1, \quad |e^{i(\frac{2}{3}k^3 + 2sk)}| = 1.
\]

so we have

\[
(3.57) \quad |R(z) e^{2it\theta(z)} - r_0 e^{i(\frac{2}{3}k^3 + 2sk)}| \leq |R(z) - R(-1)| + |R(-1)| \left| e^{iS(t;k)} - 1 \right|.
\]

Since there is not discrete spectral in the disk \(U_2(-1),\) \(T(z)\) defined by (3.1) is analytical in the disk \(U_2(-1),\) it can be shown that

\[
(3.58) \quad \|R(z)\|_{H^1(\xi_2, \xi_1)} = \|r(z)\|_{H^1(\xi_2, \xi_1)}.
\]

Noticing that \(|k| < \sqrt{2C},\) with the Hölder inequality and (3.58),

\[
(3.59) \quad |R(z) - R(-1)| = \left| \int_{-1}^{z} R'(s)ds \right| \leq \|R'\|_{L^2(\xi_2, \xi_1)}|z + 1|^{1/2}
\]

\[
\quad \quad \leq \|r\|_{H^1(\xi_2, \xi_1)}|k|^{1/2} t^{-1/6} \leq ct^{-1/6}.
\]

By Proposition 3.6,

\[
(3.60) \quad \left| e^{iS(t;k)} - 1 \right| \leq |e^{S(t;k)}| - 1 \leq ct^{-1/3}.
\]

Substituting (3.59) and (3.60) into (3.57) gives the estimate (3.55).
For $k \in \Sigma_1$, denote $k = k_1 + u + iv$. By (3.45), $\left| e^{\left(\frac{r}{3}k^3 + 2sk\right)} \right|$ is bounded. Similarly to the case on the real axis, we can obtain the estimate (3.56). The estimate on the other jump contours can be given in the same way.

**Corollary 3.9.** Let $(x, t) \in \mathcal{P}_{<1}(x, t)$, then for large $t$, we have

$$V_{loc}(z) = V_{\infty}(k) + O(t^{-1/6}), \quad k \in \mathcal{U}_k(0),$$

From the symmetry of $T(z)$ in Proposition 3.1, we have $T(-1) = 1$, and $\arg(r(-1)T^2(-1)) = \arg(r(-1))$. Further based on above Corollary 3.9, we can show the following result.

**Proposition 3.10.** Let $(x, t) \in \mathcal{P}_{<1}(x, t)$, then for large $t$, we have

$$(3.61) \quad M_{loc}(z) = M_{\infty}(k) + O(t^{-1/6}), \quad k \in \mathcal{U}_k(0),$$

where $M_{\infty}(k)$ is given by (A.9) with the argument

$$(3.62) \quad \varphi_0 = \arg(r(-1)).$$

**Remark 3.11.** In the generic case, we have $r(-1) = 1$ [32] and then $\varphi_0 = 0$.

### 3.3.2. Small norm RH problem

We now consider the error function $E(z)$ defined by (3.54) and have the following RH problem.

**RH problem 3.6.** Find $E(z)$ with the properties

- $E(z)$ is analytical in $\mathbb{C} \setminus \Sigma^E$, where $\Sigma^E = \partial U_z(-1) \cup (\Sigma^{(4)} \setminus U_z(-1))$.
- $E(z)$ satisfies the jump condition

$$(3.63) \quad V^E(z) = \begin{cases} V^{(4)}(z), & z \in \Sigma^{(4)} \setminus U_z(-1), \\ M_{loc}(z), & z \in \partial U_z(-1). \end{cases}$$

See Figure 8.

- $E(z) = I + O(z^{-1}), \quad z \to \infty$.

We can estimate the jump matrix $V^E(z) - I$. 

![Figure 8. The jump contour $V^E(z)$ for $E(z)$](image-url)
Proposition 3.12. Let \( r \in H^1(\mathbb{R}) \). Then

\[
|V^E(z) - I| = \begin{cases} 
O(e^{-ct}), & z \in \Sigma^E \setminus U_z(-1), \\
O(t^{-1/3}), & z \in \partial U_z(-1).
\end{cases}
\]

Proof. For \( z \in \Sigma^E \setminus U_z(-1) \), by (3.63) and Proposition 3.3,

\[
|V^E(z) - I| = |V^4(z) - I| \lesssim O(e^{-ct}).
\]

For \( z \in \partial U_z(-1) \), by (3.63),

\[
|V^E(z) - I| = |M^\text{loc}(z) - I| \lesssim O(t^{-1/3}).
\]

Define the Cauchy integral operator

\[
C_{w^E} f = C_- \left( f \left( V^E(z) - I \right) \right),
\]

where \( w^E = V^E(z) - I \) and \( C_- \) is the Cauchy projection operator on \( \Sigma^E \).

By (3.64), a simple calculation shows that

\[
\|C_{w^E}\|_{L^2(\Sigma^E)} \lesssim \|C_-\|_{L^2(\Sigma^E)}\|V^E(z) - I\|_{L^\infty(\Sigma^E)} \lesssim O(t^{-1/3}).
\]

According to the theorem of Beals-Coifman, the solution of the RH problem 3.6 can be expressed by

\[
E(z) = I + \frac{1}{2\pi i} \int_{\Sigma^E} \frac{\mu_E(\zeta)(V^E(\zeta) - I)}{\zeta - z} \, d\zeta,
\]

where \( \mu_E \in L^2(\Sigma^E) \) satisfies \( (I - C_{w^E}) \mu_E = I \). Further, from (3.64), we have the estimates

\[
|V^E(z) - I|_{L^2} = O(t^{-1/3}), \quad \|\mu_E - I\|_{L^2} = O(t^{-1/3}),
\]

which imply that the RH problem 3.6 exists an unique solution. We make the expansion of \( E(z) \) at \( z = \infty \)

\[
E(z) = I + \frac{E_1}{z} + O\left( z^{-1} \right),
\]

where

\[
E_1 = \frac{1}{2\pi i} \int_{\Sigma^E} \mu_E(\zeta) (V^E(\zeta) - I) \, d\zeta.
\]

Proposition 3.13. \( E_1 \) and \( E(0) \) can be estimated as follows

\[
E_1 = -\tau^{-1/3} M^\infty_1(s) + O(t^{-2/3}),
\]

\[
E(0) = I - \tau^{-1/3} M^\infty_1(s) + O(t^{-2/3}),
\]

where \( M^\infty_1(s) \) is given by (A.10) with the argument (3.62).
Proof. By (3.63) and (3.67), we obtain that
\[
E_1 = -\frac{1}{2\pi i} \oint_{\partial U_{(-1)}} (V^E(\zeta) - I) \, d\zeta \quad \text{and} \quad \frac{1}{2\pi i} \int_{\Sigma E \setminus \partial U_{(-1)}} (V^E(\zeta) - I) \, d\zeta
\]
\[
= -\frac{1}{2\pi i} \int_{\partial U_{(-1)}} (V^E(\zeta) - I) \, d\zeta + O(t^{-2/3})
\]
\[
= -\tau^{-1/3} M^{1\text{loc}}(s) + O(t^{-2/3}),
\]
which gives (3.69) by the estimate (3.61).

In a similar way, we have
\[
E(0) = I + \frac{1}{2\pi i} \oint_{\partial U_{(-1)}} \frac{V^E(\zeta) - I}{\zeta} \, d\zeta + O(t^{-2/3})
\]
\[
= I - \tau^{-1/3} M^{1\text{loc}}(s) + O(t^{-2/3}),
\]
which yields (3.70) by the estimate (3.61). \(\square\)

3.4. Long-time analysis on a pure \(\bar{\partial}\)-problem.

Here we consider the long-time asymptotic behavior for the pure \(\bar{\partial}\)-problem \(M^{(5)}(z)\). Define the function
\[
M^{(5)}(z) = M^{(4)}(z) \left( M^{rh_{p}}(z) \right)^{-1},
\]
which satisfies the following \(\bar{\partial}\)-problem.

\(\bar{\partial}\)-problem 3.1. Find \(M^{(5)}(z)\) which satisfies

- \(M^{(5)}(z)\) is continuous and has sectionally continuous first partial derivatives in \(C \setminus (\mathbb{R} \cup \Sigma^{(4)})\).
- \(M^{(5)}(z) = I + O(z^{-1}), \quad z \to \infty.\)
- For \(z \in C\), \(M^{(5)}(z)\) satisfies the \(\bar{\partial}\)-equation
\[
\bar{\partial}M^{(5)}(z) = M^{(5)}(z) W^{(5)}(z),
\]
where
\[
W^{(5)}(z) := M^{rh_{p}}(z) \bar{\partial} R^{(3)}(z) \left( M^{rh_{p}}(z) \right)^{-1},
\]
and \(\bar{\partial} R^{(3)}(z)\) has been given in (3.34).

The solution of \(\bar{\partial}\)-RH problem 3.1 can be given by
\[
M^{(5)}(z) = I - \frac{1}{\pi} \iint_{C} \frac{M^{(5)}(\zeta) W^{(5)}(\zeta)}{\zeta - z} \, d\zeta \wedge d\bar{\zeta},
\]
which can be written as an operator equation
\[
(I - S)M^{(5)}(z) = I,
\]
where

\[
Sf(z) = \frac{1}{\pi} \iint_{C} \frac{f(\zeta)W^{(5)}(\zeta)}{\zeta - z} \, d\zeta \wedge d\bar{\zeta},
\]

Proposition 3.14. Let \( q_0 \in \tanh(x) + H^{4,4}(\mathbb{R}) \). Then the operator \( S \) admits the estimate

\[
\| S \|_{L^\infty \to L^\infty} \lesssim t^{-1/6},
\]

which implies the existence of \((I - S)^{-1}\) for large \( t \).

Proof. The estimate of the operator \( S \) on \( \Omega_3 \) and \( \Omega_3^c \) used (3.28), which was given by Cuccagna and Jenkins [32]. We estimate the operator \( S \) on \( \Omega_2 \) and other cases are similar. In fact, by (3.27), (3.34), (3.72) and (3.75), we have

\[
\| Sf \|_{L^\infty \to L^\infty} \leq c(I_1 + I_2 + I_3 + I_4),
\]

where

\[
\begin{align*}
I_1 &= \iint_{\Omega_2 \cap \{|z| \leq 2\}} F(\zeta, z) \, d\zeta \wedge d\bar{\zeta}, \\
I_2 &= \iint_{\Omega_2 \cap \{|z| > 2\}} F(\zeta, z) \, d\zeta \wedge d\bar{\zeta}, \\
I_3 &= \iint_{\Omega_2 \cap \{|z| \leq 2\}} G(\zeta, z) \, d\zeta \wedge d\bar{\zeta}, \\
I_4 &= \iint_{\Omega_2 \cap \{|z| > 2\}} G(\zeta, z) \, d\zeta \wedge d\bar{\zeta},
\end{align*}
\]

with

\[
F(\zeta, z) = \frac{1}{|\zeta - z|} |r'(\text{Re}(\zeta))| e^{\text{Re}(2it\theta)},
\]

\[
G(\zeta, z) = \frac{1}{|\zeta - \xi_2|} |\zeta - \xi_2|^{-1/2} e^{\text{Re}(2it\theta)}.
\]

Let \( z = x + iy \) and \( \zeta = \xi_2 + u + iv = |\zeta|e^{iw} \). Using Proposition 3.3 and the Cauchy-Schwartz’s inequality, we have

\[
I_1 = \int_0^{2 \sin w} \int_{-\xi_2 - 2 \cos w}^{\xi_2 - 2 \cos w} F(\zeta, z) \, d\zeta \, dv \\
\lesssim \int_0^{2 \sin w} |r'|L^2|v - y|^{-1/2} e^{-\frac{1}{8|\xi_2|^2} tv^4} \, dv \lesssim t^{-1/6},
\]

\[
I_2 = \int_{2 \sin w}^{\infty} \int_{-\xi_2 - 2 \cos w}^{\xi_2 - 2 \cos w} F(\zeta, z) \, d\zeta \, dv \\
\lesssim \int_{2 \sin w}^{\infty} |r'|L^2|v - y|^{-1/2} e^{-2\sqrt{2}tv} \, dv \lesssim t^{-1/2}.
\]

In a similar way, using Proposition 3.3 and the Hölder’s inequality with \( p > 2 \) and \( 1/p + 1/q = 1 \), we obtain

\[
I_3 \lesssim \int_0^{2 \sin w} v^{1/p - 1/2} |v - y|^{1/q - 1} e^{-\frac{1}{8|\xi_2|^2} tv^3} \, dv \lesssim t^{-1/6},
\]

\[
I_4 \lesssim \int_{2 \sin w}^{\infty} v^{1/p - 1/2} |v - y|^{1/q - 1} e^{-2\sqrt{2}tv} \, dv \lesssim t^{-1/2}.
\]

\( \square \)
This Proposition 3.14 implies that the operator equation (3.74) exists an unique solution, which can be expanded in the form

\[(3.77) \quad M^{(5)}(z) = I + \frac{M^{(5)}_1(x,t)}{z} + O(z^{-2}), \quad z \to \infty,\]

where

\[(3.78) \quad M^{(5)}_1(x,t) = \frac{1}{\pi} \int \int C M^{(5)}(z) W^{(5)}(z) d\zeta \wedge d\bar{\zeta}.\]

Take \(z = 0\) in (3.73), then

\[(3.79) \quad M^{(5)}(0) = I - \frac{1}{\pi} \int \int_{\mathbb{C}} M^{(5)}(z) W^{(5)}(z) \zeta \wedge d\bar{\zeta}.\]

**Proposition 3.15.** Let \(q_0 \in \tanh(x) + H^{4,4}(\mathbb{R})\). We have the following estimates

\[(3.80) \quad |M^{(5)}_1(x,t)| \lesssim t^{-1/2}, \quad |M^{(5)}(0) - I| \lesssim t^{-1/2}.\]

**Proof.** Similarly to the proof of Proposition 3.14, we take \(z \in \Omega_2\) as an example and divide the integration (3.78) on \(\Omega_2\) into four parts. Firstly, we consider the estimate of \(M^{(5)}_1(x,t)\). By (3.71) and the boundedness of \(M^{(4)}(z)\) and \(M^{rhp}(z)\) on \(\Omega_2\), we have

\[(3.81) \quad |M^{(5)}_1(x,t)| \lesssim I_1 + I_2 + I_3 + I_4,\]

where

\[I_1 = \int \int_{\Omega_2 \cap \{|z| \leq 2\}} F(\zeta, z) d\zeta \wedge d\bar{\zeta}, \quad I_2 = \int \int_{\Omega_2 \cap \{|z| > 2\}} F(\zeta, z) d\zeta \wedge d\bar{\zeta},\]

\[I_3 = \int \int_{\Omega_2 \cap \{|z| \leq 2\}} G(\zeta, z) d\zeta \wedge d\bar{\zeta}, \quad I_4 = \int \int_{\Omega_2 \cap \{|z| > 2\}} G(\zeta, z) d\zeta \wedge d\bar{\zeta},\]

with

\[F(\zeta, z) = |r'(\text{Re}(\zeta))| e^{\text{Re}(2it\theta)}, \quad G(\zeta, z) = |\zeta - \xi||1/2 e^{\text{Re}(2it\theta)}.\]

Let \(\zeta = \xi_2 + u + iv = |\zeta| e^{itw}\). By Cauchy-Schwartz’s inequality and Proposition 3.3, we have

\[I_1 \lesssim \int_0^{2\sin w} \int_{-\xi_2 - 2 \cos w}^{2\sin w} |r'(\text{Re}(\zeta))| e^{-1/2} \pi u^3 dudv \lesssim t^{-1/2},\]

\[I_2 \lesssim \int_{2\sin w}^{\infty} \int_{-\xi_2 - 2 \cos w}^{\infty} |r'(\text{Re}(\zeta))| e^{-2\sqrt{2}tv} dudv \lesssim t^{-3/2}.\]

By Hölder’s inequality with \(p > 2\) and \(1/p + 1/q = 1\) and Proposition 3.3, we have

\[I_3 \lesssim \int_0^{2\sin w} \int_{-\xi_2 - 2 \cos w}^{2\sin w} |u + iv|^{-1/2} e^{-1/2} \pi u^3 dudv \lesssim t^{-1/2},\]

\[I_4 \lesssim \int_{2\sin w}^{\infty} \int_{-\xi_2 - 2 \cos w}^{\infty} |u + iv|^{-1/2} e^{-2\sqrt{2}tv} dudv \lesssim t^{-3/2}.\]
Then we estimate $M^{(5)}(0) - I$. Notice that if $z \in \Omega_2$, then $|z|^{-1} \leq |\xi_2|^{-1}$. By (3.79) and (3.81), we get $|M^{(5)}(0) - I| \lesssim t^{-1/2}$.

To recover the potential from the reconstruction formula (2.18), we need the following estimate.

**Proposition 3.16.** $M^{(3)}(0)$ satisfies the estimate

$$M^{(3)}(0) = E(0) + \mathcal{O}(t^{-1/2}),$$

where $E(0)$ is given by (3.70).

**Proof.** Reviewing the series of transformations (3.35), (3.38), and (3.54), for $z$ large and satisfying $R^{(3)}(z) = I$, the solution of $M^{(3)}(z)$ is given by

$$M^{(3)}(z) = M^{(5)}(z)E(z).$$

By (3.77) and (3.80), we further obtain

$$M^{(3)}(z) = E(z) + \mathcal{O}(t^{-1/2}),$$

which yields (3.82) by taking $z = 0$.

3.5. **Proof of Theorem 1.1—Case I.**

**Proof.** Inverting the sequence of transformations (3.8), (3.10), (3.11), (3.35), (3.38), (3.54), and especially taking $z \to \infty$ vertically such that $R^{(3)}(z) = G(z) = I$, then the solution of RH problem 2.1 is given by

$$M(z) = T(\infty)^{\sigma_3}(I + z^{-1}\sigma_1M^{(3)}(0)^{-1})M^{(5)}(z)E(z)T(z)^{-\sigma_3} + \mathcal{O}(e^{-ct}).$$

Noticing that (3.2) and (3.82), further substituting asymptotic expansions (3.3), (3.68), (3.77) into the above formula, the reconstruction formula (2.18) yields

$$q(x,t) = T(\infty)^2 \left(1 - \frac{1}{2}\bar{\tau}^{-1/2} \left(u(s)e^{i\phi_0} + \int_s^\infty u^2(\zeta)d\zeta\right)\right) + \mathcal{O}(t^{-1/2}),$$

which leads to the result (1.4) in Theorem 1.1.

4. **Painlevé asymptotics in transition region $\mathcal{P}_{+1}(x,t)$**

In a way similar to Section 3, we study the Painlevé asymptotics in the region $(x,t) \in \mathcal{P}_{+1}(x,t)$ and consider the region $0 < (\xi - 1)t^{2/3} < C$, which corresponds to Figure 3(f). For brevity, we denote

$$\mathcal{P}_{>1}(x,t) = \{(x,t) : 0 < (\xi - 1)t^{2/3} < C\}.$$

In this case, the two stationary points $\xi_1, \xi_2$ defined by (2.25) are real and close to $z = 1$ at least the speed of $t^{-1/3}$ as $t \to +\infty$. 
4.1. Modifications to the basic RH problem.

Opening the contour \((0, \infty)\) needs the second matrix decomposition in (2.19), so we introduce the function

\[(4.1) \quad T(z) = \exp \left( -i \int_{0}^{\infty} \nu(\zeta) \left( \frac{1}{\zeta - z} - \frac{1}{2\zeta} \right) d\zeta \right),\]

and obtain the following proposition.

Lemma 4.1. [43] The function \(T(z)\) has the following properties

- \(T(z)\) is analytical in \(\mathbb{C} \setminus [0, \infty)\).
- \(T(z)\) has the symmetry \(\overline{T(\bar{z})} = T(z)^{-1} = T(z^{-1})\).
- \(T(z)\) satisfies the jump condition
  \[T_+(z) = T_-(z)(1 - |r(z)|^2), \quad z \in (0, \infty).\]
- Let
  \[(4.2) \quad T(\infty) := \lim_{z \to \infty} T(z) = \exp \left( i \int_{0}^{\infty} \frac{\nu(\zeta)}{2\zeta} d\zeta \right).\]

Then, \(|T(\infty)| = 1\) and the asymptotic expansion at infinity is

\[(4.3) \quad T(z) = T(\infty) \left[ 1 + \frac{i}{z} \left( \int_{0}^{\infty} \nu(\zeta) d\zeta \right) + O(z^{-2}) \right].\]

- The ratio \(\frac{s_{11}(z)}{T(z)}\) here has the same boundedness with that ratio in Lemma 3.1.

To remove poles on the unitary circle \(|z| = 1\) by converting their residues into jumps, we fix a constant \(\rho\) defined by (3.4) and introduce the interpolation function

\[
G(z) = \begin{cases} 
1 & |z - z_j| < \rho, \\
-c_j e^{2it\theta(z_j)} & |z - \bar{z}_j| < \rho, \\
& |z - z_j| = \rho, or |z - \bar{z}_j| = \rho, \\
& |z - z_j| = \rho, \\
1 & |z - \bar{z}_j| = \rho,
\end{cases}
\]

where \(z_j \in \mathbb{Z}^+\) and \(\bar{z}_j \in \mathbb{Z}^-\).

Define a contour

\[
\Sigma^{(1)} = \mathbb{R} \cup \bigcup_{j=0}^{N-1} \left\{ z \in \mathbb{C} : |z - z_j| = \rho, or |z - \bar{z}_j| = \rho \right\},
\]

and make the following transformation

\[(4.4) \quad M^{(1)}(z) = T(\infty)^{-\sigma_3} M(z) G(z) T(z)^{\sigma_3},\]
then $M^{(1)}(z)$ satisfies the RH problem as follows.

**RH problem 4.1.** Find $M^{(1)}(z) = M^{(1)}(z; x, t)$ satisfying

- $M^{(1)}(z)$ is analytical in $\mathbb{C} \setminus \Sigma^{(1)}$.
- $M^{(1)}(z) = \sigma_1 M^{(1)}(\bar{z}) \sigma_1 = z^{-1} M^{(1)}(z^{-1}) \sigma_1$.
- $M^{(1)}(z)$ satisfies the jump condition
  
  \[
  M_+^{(1)}(z) = M_-^{(1)}(z)V^{(1)}(z),
  \]

  where

  \[
  V^{(1)}(z) = \begin{cases}
  B_+^{-1}B_+, & z \in (0, \xi_1) \cup (\xi_2, \infty), \\
  T(z)^{-\sigma_3} V(z) T(z)^{\sigma_3}, & z \in (\xi_1, \xi_2), \\
  b_+^{-1}b_+, & z \in (-\infty, 0),
  \end{cases}
  \]

  with properties

  \[
  M^{(1)}(z) = I + O(z^{-1}), \quad z \to \infty,
  \]

  \[
  z M^{(1)}(z) = \sigma_1 + O(z), \quad z \to 0.
  \]

  Noticing that $V^{(1)}(z) \to I, t \to +\infty$ on the circles $|z - z_j| = \rho$ and $|z - \bar{z}_j| = \rho$, so we get rid of the exponential infinitesimal term in the jump matrices. The RH problem 4.1 is asymptotically equivalent to the RH problem 4.2 with the exponential error $O(e^{-ct})$.

  \[
  M^{(1)}(z) = M^{(2)}(z) (I + O(e^{-ct})),
  \]

  where $M^{(2)}(z)$ is the solution of the following RH problem.

**RH problem 4.2.** Find $M^{(2)}(z) = M^{(2)}(z; x, t)$ with properties

- $M^{(2)}(z)$ is analytical in $\mathbb{C} \setminus \mathbb{R}$.
- $M^{(2)}(z) = \sigma_1 M^{(2)}(\bar{z}) \sigma_1 = z^{-1} M^{(2)}(z^{-1}) \sigma_1$.
- $M^{(2)}(z)$ satisfies the jump condition
  
  \[
  M_+^{(2)}(z) = M_-^{(2)}(z)V^{(2)}(z),
  \]

  where

  \[
  V^{(2)}(z) = \begin{cases}
  B_+^{-1}B_+, & z \in (0, \xi_1) \cup (\xi_2, \infty), \\
  T(z)^{-\sigma_3} V(z) T(z)^{\sigma_3}, & z \in (\xi_1, \xi_2), \\
  b_+^{-1}b_+, & z \in (-\infty, 0).
  \end{cases}
  \]
• $M^{(2)}(z)$ admits the asymptotic behaviors

\[ M^{(2)}(z) = I + \mathcal{O}(z^{-1}), \quad z \to \infty, \]
\[ zM^{(2)}(z) = \sigma_1 + \mathcal{O}(z), \quad z \to 0. \]

Similarly to the case in Section 3.1.2, we remove the singularity by the following transformation

\[ M^{(2)}(z) = \left( I + \frac{1}{z} \sigma_1 M^{(3)}(0)^{-1} \right) M^{(3)}(z), \quad (4.7) \]

where $M^{(3)}(z)$ satisfies the following RH problem.

**RH problem 4.3.** Find $M^{(3)}(z) = M^{(3)}(z; x, t)$ which satisfies

• $M^{(3)}(z)$ is analytical in $\mathbb{C} \setminus \mathbb{R}$.
• $M^{(3)}(z) = \sigma_1 M^{(3)}(\bar{z}) \sigma_1 = \sigma_1 M^{(3)}(0)^{-1} M^{(3)}(z^{-1}) \sigma_1$.
• $M^{(3)}(z)$ satisfies the jump condition

\[ M_{+}^{(3)}(z) = M_{-}^{(3)}(z) V^{(2)}(z), \]

where $V^{(2)}(z)$ has been given in (4.6).
• $M^{(3)}(z) = I + \mathcal{O}(z^{-1}), \quad z \to \infty$.

### 4.2. Transformation to a hybrid $\bar{\partial}$-RH problem.

We define the contours and sectors. See Figure 9. We especially point out the properties of the real part of the phase function.

**Proposition 4.2.** Let $\xi > 1$ and $(x, t) \in \mathcal{P}_{>1}(x, t)$. The real part of the phase function $\text{Re} (2i\theta(z))$ has the following estimations.

**Case I.** (corresponding to $z = 0$)

\[ \text{Re} (2i\theta(z)) \geq |\sin 2\varphi| |v|, \quad z \in \Omega_0 \cup \overline{\Omega}_3, \quad (4.8) \]
\[ \text{Re} (2i\theta(z)) \leq -|\sin 2\varphi| |v|, \quad z \in \overline{\Omega}_0 \cup \Omega_3, \quad (4.9) \]

where $z = u + iv$ and $\varphi = \arg z$.

**Case II.** (corresponding to $z = \xi_1$)

\[ \text{Re} (2i\theta(z)) \geq \frac{4}{|\xi_1|} u^2 |v|, \quad z \in \Omega_1, \quad (4.10) \]
\[ \text{Re} (2i\theta(z)) \leq -\frac{4}{|\xi_1|} u^2 |v|, \quad z \in \overline{\Omega}_1, \quad (4.11) \]

where $z = \xi_1 + u + iv$.

**Case III.** (corresponding to $z = \xi_2$)

\[ \text{Re} (2i\theta(z)) \geq \begin{cases} \frac{1}{8|\xi_2|} u^2 |v|, & z \in \Omega_2 \cap \{|z| \leq 2\}, \\ 2\sqrt{2} |v|, & z \in \Omega_2 \cap \{|z| > 2\}, \end{cases} \quad (4.12) \]
\[ \text{Re} (2i\theta(z)) \leq \begin{cases} -\frac{1}{8|\xi_2|} u^2 |v|, & z \in \overline{\Omega}_2 \cap \{|z| \leq 2\}, \\ -2\sqrt{2} |v|, & z \in \overline{\Omega}_2 \cap \{|z| > 2\}, \end{cases} \quad (4.13) \]

where $z = \xi_2 + u + iv$. 
Figure 9. Open the jump contour $\mathcal{R} \setminus (\xi_1, \xi_2)$ along red rays and blue rays. The green regions are continuous extension sectors with $\text{Re}(2i\theta(z)) > 0$, while the white regions are continuous extension sectors with $\text{Re}(2i\theta(z)) < 0$.

Proof. The proof here is similar to the proof on the region $\mathcal{P}_{< -1}(x, t)$, see more details in Proposition 3.3.

To open the dashed part of the real axis $\mathbb{R}$ in Figure 9, we introduce the following extension functions $R_j(z)$, $j = 0, 1, 2, 3$.

**Proposition 4.3.** Let $q_0 \in \tanh(x) + H^{4,4}(\mathbb{R})$. Define functions $R_j(z)(j = 0, 1, 2, 3)$ with the boundary values

$$R_j(z) = \begin{cases} \frac{r(z)}{1 - |r(z)|^2} T_+^{-2}(z), & z \in (0, \xi_1) \cup (\xi_2, \infty), \\ R(\xi_j), & z \in \Sigma_j, \ j = 0, 1, 2, \end{cases}$$

$$\overline{R}_j(z) = \begin{cases} \frac{r(z)}{1 - |r(z)|^2} T_+^{-2}(z), & z \in (0, \xi_1) \cup (\xi_2, \infty), \\ \overline{R}(\xi_j), & z \in \Sigma_j, \ j = 0, 1, 2, \end{cases}$$

$$R_3(z) = \begin{cases} r(z)T^2(z), & z \in (-\infty, 0), \\ 0, & z \in \Sigma_3, \end{cases}$$

$$\overline{R}_3(z) = \begin{cases} \frac{r(z)}{T^2(z)}, & z \in (-\infty, 0), \\ 0, & z \in \Sigma_3, \end{cases}$$

where

$$R(\xi_j) = \frac{S_{21}(\xi_j)}{S_{11}(\xi_j)} \left( \frac{s_{11}(\xi_j)}{T_+^{2}(\xi_j)} \right)^2,$$

(4.14)

where $S_{21}(z)$ and $S_{11}(z)$ are given by (3.32). Then for $j = 0, 1, 2, 3$, there exists a constant $c = c(q_0)$, which only depends on $q_0$, such that

$$|\partial R_j| \leq c \left( |\varphi(\text{Re}(z))| + |r'(\text{Re}(z))| + |z - \xi_j|^{-1/2} \right),$$

for $z \in \Omega_j \cup \overline{\Omega}_j$, where $\varphi \in C_0^\infty(\mathbb{R}, [0, 1])$ with small support near $z = 1$.

Proof. The extension functions $R_j(z), j = 0, 1, 2$ are defined in the same way in (3.30). The function $R_3(z)$ can be treated by (3.29). The proof is similar to that in Proposition 3.4.
Define the function

\[
R(z) = \begin{cases} 
1 & R_je^{-2it\theta(z)}, \quad z \in \Omega_j, \quad j = 0, 1, 2, \\
0 & 1 \\
\frac{1}{R_je^{2it\theta(z)}} & 0 \\
-\frac{1}{R_3e^{2it\theta(z)}} & 1 \\
1 & -\frac{1}{R_3e^{-2it\theta(z)}} \\
0 & 1 \\
I, & \text{other regions,}
\end{cases}
\]

and the contour

\[
\Sigma(4) = \Gamma \cup \Gamma \cup [\xi_1, \xi_2], \quad \Gamma = \cup_{j=0}^{3} \Sigma_j \cup L.
\]

Then we obtain the following function

\[
M(4)(z) = M(3)(z)R(3)(z),
\]

satisfies the following mixed $\bar{\partial}$-RH problem.

\textbf{$\bar{\partial}$-RH problem 4.1.} Find $M(4)(z) = M(4)(z;x,t)$ satisfying

\begin{itemize}
  \item $M(4)(z)$ is continuous in $\mathbb{C} \setminus \Sigma(4)$. See Figure 9.
  \item $M(4)(z)$ satisfies the jump condition
    \[
    M_+(z) = M_-(z)V(4)(z),
    \]
    where
    \[
    V(4)(z) = \begin{cases} 
1 & -R(\xi_j)e^{-2it\theta(z)}, \quad z \in \Sigma_j, \\
0 & 1 \\
\frac{1}{R(\xi_j)e^{2it\theta(z)}} & 0 \\
1 & -\frac{1}{R(\xi_j)e^{-2it\theta(z)}} \\
0 & 1 \\
\frac{1}{R(\xi_j)e^{2it\theta(z)}} & 0 \\
T(z)^{-\sigma_3}V(z)T(z)^{\sigma_3}, & z \in (\xi_1, \xi_2).
\end{cases}
\]
  \item $M(4)(z)$ admits the asymptotic behavior
    \[
    M(4)(z) = I + O(z^{-1}), \quad z \to \infty.
    \]
\end{itemize}
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For $z \in \mathbb{C} \setminus \Sigma^{(4)}$, $\bar{\partial} M^{(4)}(z) = M^{(4)}(z) \bar{\partial} R^{(3)}(z)$, where

$$
\bar{\partial} R^{(3)}(z) = \begin{cases}
1 & \frac{\partial R_j e^{-2it\theta(z)}}{0} \\
0 & 1 \\
\frac{1}{\partial R_j e^{2it\theta(z)}} & 1 \\
0 & 1 \\
-\frac{\partial R_3 e^{2it\theta(z)}}{1} & 0 \\
1 & -\frac{\partial R_3 e^{-2it\theta(z)}}{0} \\
0 & 1
\end{cases}, \quad z \in \Omega_j, \quad j = 0, 1, 2,
$$

$$
\begin{cases}
\frac{1}{\partial R_j e^{2it\theta(z)}} & 1 \\
0 & 1 \\
\frac{1}{-\partial R_3 e^{2it\theta(z)}} & 0
\end{cases}, \quad z \in \Omega_3,
$$

Then we continue to decompose $M^{(4)}(z)$ into two parts:

$$
M^{(4)}(z) = \begin{cases}
M^{rhp}(z), & \bar{\partial} R^{(3)}(z) = 0 \\
M^{(5)}(z), & \bar{\partial} R^{(3)}(z) \neq 0
\end{cases}
$$

4.3. Asymptotic analysis on a pure RH problem.

The pure RH problem is described as follows.

**RH problem 4.4.** Find $M^{rhp}(z) = M^{rhp}(z; x, t)$ which satisfies

- $M^{rhp}(z)$ is analytical in $\mathbb{C} \setminus \Sigma^{(4)}$.
- $M^{rhp}(z)$ satisfies the jump condition
  $$
  M^{rhp}_+(z) = M^{rhp}_-(z) V^{(4)}(z),
  $$
  where $V^{(4)}(z)$ is given by (4.17).
- $M^{rhp}(z)$ has the same asymptotic behavior with $M^{(4)}(z)$.

4.3.1. Local paramatrix near $z = 1$.

In the region $\mathcal{P}_{>1}(x, t)$, the phase points $\xi_1$ and $\xi_2$ merge to 1 as $t \to +\infty$, and we have for small $kt^{-1/3}$

$$
t\theta(z) = -t \left( (z - 1)^3 + \frac{1}{2} \sum_{n=4}^{\infty} (-1)^{n+1} (n - 1) (z - 1)^n \right) + (x - 2t) \left( (z - 1) + \frac{1}{2} \sum_{n=2}^{\infty} (-1)^{n+1} (z - 1)^n \right)
$$

$$
:= \frac{4}{3} k^3 + sk + S(t; k),
$$

where

$$
k = -\tau^\frac{1}{3}(z - 1), \quad s = -\frac{8}{3} (\xi - 1) \tau^\frac{4}{3},
$$

$$
S(t; k) = \frac{2}{3} \sum_{n=4}^{\infty} (n - 1) \tau^{\frac{n}{3}} k^n - \frac{4}{3} (\xi - 1) \sum_{n=2}^{\infty} \tau^{\frac{n}{3}} k^n,
$$

with $\tau = \frac{3}{4} t$. The scaled phase points have the following properties.
Proposition 4.4. In the transition region $P_{>+1}(x,t)$, two phase points $k_j = -\tau^{1/3}(\xi_j - 1), \ j = 1, 2$ always lie in a fixed interval
\begin{equation}
(4.23) \quad k_j \in [-\frac{3}{4}^{1/3}\sqrt{2C}, \frac{3}{4}^{1/3}\sqrt{2C}] .
\end{equation}

For a fixed constant $\varepsilon \leq \sqrt{2C}$, define a neighbourhood
\begin{equation}
U_z(1) = \{ z \in \mathbb{C} : |z - 1| < \varepsilon \tau^{-1/3} \},
\end{equation}
\begin{equation}
\text{such that } \xi_1, \xi_2 \in U_z(1). \text{ The map } z \mapsto k \text{ maps } U_z(1) \text{ onto the disk } U_k(0) = \{ k \in \mathbb{C} : |k| < \varepsilon \} \text{ in the } k\text{-plane, where } k_1, k_2 \in U_k(0).\n\end{equation}

Proposition 4.5. In the region $P_{>+1}(x,t)$, the series $S(t;k)$ defined by (4.22) converges uniformly in $U_z(1)$ and decay concerning $t$.

From Proposition 4.5, the first two terms in (4.20) are crucial to obtain the Painlevé asymptotics. Additionally, we have the following proposition in the $k$-plane.

Proposition 4.6. In the transition region $P_{>+1}(x,t)$, when $t$ is large enough,
\begin{equation}
\text{Re} \left[ i \left( \frac{4}{3} k^3 + sk \right) \right] \geq \frac{8}{3} u^2 v, \text{ for } k \in \Omega'_j ,
\end{equation}
\begin{equation}
\text{Re} \left[ i \left( \frac{4}{3} k^3 + sk \right) \right] \leq -\frac{8}{3} u^2 v, \text{ for } k \in \overline{\Omega}'_j ,
\end{equation}
where $\Omega'_j$ is the scaled contour of $\Omega_j$ and $k = k_j + u + iv, j = 1, 2$.

Proof. The proofs for Proposition 4.4, 4.5, and 4.6 are similar to the proofs on the region $P_{<-1}(x,t)$. See more details in Proposition 3.5, 3.6, and 3.7. \hfill \Box

As $t \to +\infty$, the jump matrix decays to the unit matrix exponentially fast and uniformly outside a small neighborhood of $z = 1$. In this case, we construct the solution of $M^{rhp}(z)$ as follows:
\begin{equation}
M^{rhp}(z) = \begin{cases} E(z), & z \in \mathbb{C} \setminus U_z(1), \\ E(z)M^{loc}(z) , & z \in U_z(1). \end{cases}
\end{equation}

where $M^{loc}(z)$ satisfies the following RH problem.

RH problem 4.5. Find $M^{loc}(z) = M^{loc}(z;x,t)$ which satisfies
• Analytical in $U_z(1) \setminus \Sigma^{loc}$, where $\Sigma^{loc} = \Sigma^{(4)} \cap U_z(1)$. See Figure 9.
• Jump condition:
\begin{equation}
M^{loc}_+(z) = M^{loc}_-(z)V^{loc}(z) , \quad z \in \Sigma^{loc},
\end{equation}
where
\begin{equation}
V^{loc}(z) = \begin{cases} \begin{pmatrix} 1 & -R(\xi_j)e^{-2it\theta(z)} \\ 0 & 1 \end{pmatrix} , & z \in \Sigma_j, \ j = 1, 2, \\ \begin{pmatrix} 1 & 0 \\ \frac{1}{R(\xi_j)}e^{2it\theta(z)} & 1 \end{pmatrix} , & z \in \Sigma_j, \ j = 1, 2, \\ T(z)^{-\sigma_3}V(z)T(z)^{\sigma_3} , & z \in (\xi_1, \xi_2). \end{cases}
\end{equation}
Using Lemmas 2.1 and 4.1, each factor in the r.h.s. of (3.33) is analytical in 
\[ M^\nu(z)(M\infty((z - 1)\tau^{1/3}))^{-1} \to I, \quad t \to +\infty \text{ uniformly for } z \in \partial \mathcal{U}(1). \]

Since \( \xi_1, \xi_2 \to 1 \) as \( t \to +\infty \), it follows from (2.11) that \( r(\xi_j) \to -1 \) as \( t \to \infty \) for the generic case, which further causes singularity of the \( \frac{r(\xi_j)}{1 - |r(\xi_j)|^2} \) as \( \xi_j \to 1 \). However \( R(\xi_j) \) has no singularity as \( \xi_j \to 1 \) since this singularity can be neutralized by the factor \( T_-(\xi_j)^{-2} \). We define a cutoff function \( \chi(z) \in C_0^\infty(\mathbb{R}, [0, 1]) \) satisfying \( \chi(z) = 1, \quad z \in \mathbb{R} \cap \mathcal{U}(1) \), and write the function \( R(z) \) as \( \chi(z) = 1, \quad z \in \mathbb{R} \cap \mathcal{U}(1) \),

(4.27) \[ R(z) = (1 - \chi(z)) \frac{r(z)}{1 - |r(z)|^2} T(z)^{-2} + \chi(z) F(z) G(z)^2, \]

where \( F(z) := \frac{S_{21}(z)}{S_{11}(z)}, \quad G(z) := \frac{s_{11}(z)}{T_+(z)}. \)

Using Lemmas 2.1 and 4.1, each factor in the r.h.s. of (3.33) is analytical in \( \Omega_1 \cup \Omega_2 \), with well defined nonzero limit on \( \partial \Omega_1 \cup \partial \Omega_2 \).

**Lemma 4.7.** Let \( q_0 \in \tanh(x) + L^{1,2}(\mathbb{R}), q_0' \in W^{1,\infty}(\mathbb{R}), \) then \( F(z) \in H^1(\mathbb{R}). \)

**Proof.** In a similar to the proof of \( r(z) \in H^1(\mathbb{R}) \) in Lemma 2.2, fix a small \( \delta > 0 \), such that \( |z| < \delta \) and \( |z| < \delta \) have empty intersection. By using Lemma 2.1, we have

(4.29) \[ F(z) \in W^{1,\infty}(I_\delta) \cap H^1(I_\delta), \]

(4.30) \[ |\partial_j \xi| \leq \delta^{-1}, \quad j = 0, 1, \]

where \( I_\delta = \mathbb{R} \setminus ((-\delta, \delta) \cup (1 - \delta, 1 + \delta) \cup (-1 - \delta, -1 + \delta)). \)

Next to show

(4.31) \[ F(z) \in H^1(\mathbb{R} \setminus I_\delta). \]

For \( |z - 1| < 1 \), we write \( f(z) \) as

(4.32) \[ F(z) = \frac{-\pi_+ + \int_1^2 \partial_y S_{21}(y)dy}{s_+ + \int_1^2 \partial_y S_{11}(y)dy}, \]

by which, it follows that \( F'(z) \) is defined and bounded around \( z = 1 \). The same discussion holds at \( z = -1 \). \( F'(z) \) is defined and bounded around \( z = 0 \) by using symmetry \( F(z^{-1}) = \tilde{F}(z) \) and (4.30).

Finally, from (4.29) and (4.31), we conclude that \( F(z) \in H^1(\mathbb{R}). \)

Then we show the following proposition.
Proposition 4.8. Let \( q_0 \in \tanh(x) + L^{1,2}(\mathbb{R}), q_0' \in W^{1,\infty}(\mathbb{R}) \). Then for \( k \in (k_2, k_1) \),
\[
|R(z)e^{2it\theta(z)} - R(1)e^{8ik^3/3 + 2isk}| \lesssim t^{-1/6}.
\] (4.33)
And for \( k \in \Sigma_j', j = 1, 2 \),
\[
|R(\xi_j)e^{2it\theta(z)} - R(1)e^{8ik^3/3 + 2isk}| \lesssim t^{-1/6},
\] (4.34)
where \( \Sigma_j' \) is the scaled contour of \( \Sigma_j \).

Proof. For \( k \in (k_2, k_1) \), then \( z \in (\xi_2, \xi_1) \) and \( z \) is real,
\[
|e^{2it\theta(z)}| = 1, \quad |e^{i(2/3 k^3 + 2sk)}| = 1.
\]
so we have
\[
|R(z)e^{2it\theta(z)} - R(1)e^{i(2/3 k^3 + 2sk)}| \leq |G(z)|^2 |F(z) - F(1)|
\] (4.35)
\[
+ |F(1)||G(z) + G(1)||G(z) - G(1)| + |R(1)||e^{iS(t;k)} - 1|.
\]
By Lemma 4.7 and Hölder inequality,
\[
|F(z) - F(1)| = \left| \int_{-1}^{z} F'(s)ds \right| \leq \|F\|_{H^1} |z - 1|^{1/2} \leq ct^{-1/6}.
\] (4.36)
From (2.12) and (4.1), we have
\[
G(z) = \prod_{j=1}^{N} \frac{z - z_j}{z - \bar{z}_j} \exp \left( -i \int_{0}^{0} \frac{\nu(\zeta)}{\zeta - z} \frac{d\zeta}{2\zeta} \right) \exp \left( -i \int_{0}^{\infty} \frac{\nu(\zeta)}{2\zeta} d\zeta \right)
\]
with which,
\[
G(z) - G(1) = e^{-i \int_{-\infty}^{0} \frac{\nu(\zeta)}{\zeta - z} d\zeta} e^{-i \int_{0}^{\infty} \frac{\nu(\zeta)}{2\zeta} d\zeta} \left( \prod_{j=1}^{N} \frac{z - z_j}{z - \bar{z}_j} - \prod_{j=1}^{N} \frac{1 - z_j}{1 - \bar{z}_j} \right)
\]
\[
+ \prod_{j=1}^{N} \frac{1 - z_j}{1 - \bar{z}_j} e^{-i \int_{0}^{\infty} \frac{\nu(\zeta)}{2\zeta} d\zeta} \left( e^{-i \int_{-\infty}^{0} \frac{\nu(\zeta)}{\zeta - z} d\zeta} - e^{-i \int_{0}^{\infty} \frac{\nu(\zeta)}{\zeta - z} d\zeta} \right).
\]
In the r.h.s. all factor before two brackets have the absolute value \( \leq 1 \) for \( z \in \mathbb{R} \), while the terms in two brackets are controlled by \( |z - 1| \), therefore,
\[
|G(z) - G(1)| \leq c|z - 1|.
\] (4.37)
Substituting (4.36) and (4.38) into (4.35) gives
\[
|R(z)e^{2it\theta(z)} - R(1)e^{i(2/3 k^3 + 2sk)}| \leq ct^{-1/6}.
\] (4.38)
For \( k \in \Sigma_1 \), denote \( k = k_1 + u + iv \). By (4.24), \( e^{i(2/3 k^3 + 2sk)} \) is bounded. Similarly to the case on the real axis, we can obtain the estimate (4.34). The estimate on the other jump contours can be given in the same way. \( \square \)
Therefore, as $t \to +\infty$, the solution of the RH problem $M^{\text{loc}}(z)$ can be approximated by the solution of a limit model

$$M^{\text{loc}}(z) = M^\infty(k) + \mathcal{O}(t^{-1/6}),$$

where $M^\infty(k)$ is given by (A.9) with the argument

$$\varphi_0 = \arg \frac{S_{21}(1)}{S_{11}(1)} + 2 \sum_{j=0}^{N-1} \arg(1-z_j) - \int_0^\infty \frac{\nu(\zeta)}{2\zeta} d\zeta - \int_{-\infty}^0 \frac{\nu(\zeta)}{\zeta-1} d\zeta,$$

where the functions $S_{21}(z)$ and $S_{11}(z)$ are defined by (3.32), and $\nu(z)$ is given by (2.13).

4.3.2. Small norm RH problem.

We consider the error function $E(z)$ defined by (4.26). Denote $\Sigma^E = \partial U_z(1) \cup (\Sigma^{(4)} \setminus U_z(1))$.

**RH problem 4.6.** Find $E(z)$ with the properties as follows

- $E(z)$ is analytical in $\mathbb{C} \setminus \Sigma^E$.
- $E(z)$ satisfies the jump condition
  $$E_+(z) = E_-(z)V^E(z), \quad z \in \Sigma^E,$$
  where the jump matrix is given by
  $$V^E(z) = \begin{cases} V^{(4)}(z), & z \in \Sigma^{(4)} \setminus U_z(1), \\ M^{\text{loc}}(z), & z \in \partial U_z(1). \end{cases}$$
- As $z \to \infty$, $E(z) = I + \mathcal{O}(z^{-1})$.

Similarly to the analysis in the previous sector, we obtain

$$|V^E(z) - I| = \begin{cases} \mathcal{O}(e^{-cz}), & z \in \Sigma^E \setminus U_z(1), \\ \mathcal{O}(t^{-1/3}), & z \in \partial U_z(1), \end{cases}$$

and the RH problem 4.6 exists an unique solution

$$E(z) = I + \frac{1}{2\pi i} \int_{\Sigma^E} \frac{\mu_E(\zeta) (V^E(\zeta) - I)}{\zeta - z} d\zeta,$$

where $\mu_E \in L^2(\Sigma^E)$ satisfies $(I - C_{\psi_E}) \mu_E = I$. As $z \to \infty$,

$$E(z) = I + \frac{E_1}{z} + \mathcal{O}(z^{-1}),$$

where $E_1 = -\frac{1}{2\pi i} \int_{\Sigma^E} \mu_E(\zeta) (V^E(\zeta) - I) d\zeta$. In particular, $E_1$ and $E(0)$ have the following estimations.

**Proposition 4.9.**

$$E_1 = \tau^{-1/3} M_1^\infty(s) + \mathcal{O}(t^{-2/3}),$$

$$E(0) = I + \tau^{-1/3} M_1^\infty(s) + \mathcal{O}(t^{-2/3}).$$
4.4. Asymptotic analysis on a pure $\bar{\partial}$-problem.

In the subsection, we consider the pure $\bar{\partial}$-problem. By (4.19), we have

\begin{equation}
M^{(5)}(z) = M^{(4)}(z) \left( M^{rhp}(z) \right)^{-1},
\end{equation}

which satisfies the following $\bar{\partial}$-problem.

$\bar{\partial}$-problem 4.1. Find a matrix-valued function $M^{(5)}(z)$ which satisfies

- $M^{(5)}(z)$ is continuous and has sectionally continuous first partial derivatives in $\mathbb{C} \setminus (\mathbb{R} \cup \Sigma^{(4)})$.
- As $z \to \infty$, $M^{(5)}(z) = I + O(z^{-1})$.
- For $z \in \mathbb{C}$, we have
  \[ \bar{\partial}M^{(5)}(z) = M^{(5)}(z)W^{(5)}(z), \]
  where $W^{(5)}(z) := M^{rhp}(z)\bar{\partial}R^{(3)}(z) (M^{rhp}(z))^{-1}$ with $\bar{\partial}R^{(3)}(z)$ in (4.18) and $M^{rhp}(z)$ in (4.26).

We can prove that the solution of the $\bar{\partial}$-RH problem 4.1 exists and that the asymptotic unfolding at infinity satisfies

\begin{equation}
M^{(5)}(z) = I + \frac{M^{(5)}_1(x,t)}{z} + O(z^{-2}), \quad z \to \infty,
\end{equation}

where

\[ M^{(5)}_1(x,t) = \frac{1}{\pi} \iint_{\mathbb{C}} M^{(5)}(\zeta)W^{(5)}(\zeta) \, d\zeta \wedge d\bar{\zeta}. \]

with

\begin{equation}
|M^{(5)}_1(x,t)| \lesssim t^{-1/2}, \quad |M^{(5)}(0) - I| \lesssim t^{-1/2}.
\end{equation}

Proposition 4.10. $M^{(3)}(0)$ admits the estimate

\begin{equation}
M^{(3)}(0) = E(0) + O(t^{-1/2}),
\end{equation}

where $E(0)$ is given by (4.43).

4.5. Proof of Theorem 1.1—Case II.

Proof. Inverting the sequence of transformations (4.4), (4.5), (4.7), (4.16), (4.19), (4.26), and especially taking $z \to \infty$ vertically such that $R^{(3)}(z) = G(z) = I$, the solution of RH problem 2.1 is given by

\[ M(z) = T(\infty)^{\sigma_3} (I + z^{-1} \sigma_1 M^{(3)}(0)^{-1}) M^{(5)}(z) E(z) T(z)^{-\sigma_3} + O(e^{-ct}). \]

Noticing that (4.2) and (4.47), further substituting asymptotic expansions (4.3), (4.41) and (4.45) into the above formula, then the reconstruction formula (2.18) yields

\[ q(x,t) = T(\infty)^2 \left( 1 + \frac{1}{2} i \tau^{-\frac{1}{2}} \left( u(s) e^{i\varphi_0} + \int_s^\infty u^2(\zeta) d\zeta \right) \right) + O\left( t^{-\frac{1}{2}} \right), \]

which gives the result (1.8) in Theorem 1.1.

\[ \square \]
Appendix A. A model for $P_{-1}(x,t)$ and $P_{+1}(x,t)$

The long-time asymptotics in the region $P_{\mp 1}(x,t)$ are related to the solution $M^\infty(k)$ of the following RH problem.

Let $k \in \mathbb{C}$ be a complex variable, $k_1, k_2 \in \mathbb{R}$ and $r_0 \in \mathbb{C}$ be fixed constants, $s$ be a parameter, and $\Sigma_j$, $j = 1, \ldots, 6$, be rays passing through points $k_1, k_2$ at a fixed angle $\varphi$ with the real axis, see Figure 10. Denote $\Sigma^\infty = \cup_{j=1}^6 \Sigma_j$. We consider the following model RH problem.

**RH problem A.1.** Find $M^\infty(k) = M^\infty(k; s)$ with properties

- $M^\infty(k)$ is analytical in $\mathbb{C} \setminus \Sigma^\infty$.
- $M^\infty(k)$ satisfies the jump condition
  $$M^\infty_+(k) = M^\infty_-(k) V^\infty(k),$$

  where
  $$V^\infty(k) = \begin{cases}
  \begin{pmatrix}
  1 & 0 \\
  r_0 e^{2i\left(\frac{4}{3}k^3 + sk\right)} & 1 
  \end{pmatrix} := B_+, & k \in \Sigma_1 \cup \Sigma_2, \\
  \begin{pmatrix}
  1 & -r_0 e^{-2i\left(\frac{4}{3}k^3 + sk\right)} \\
  0 & 1 
  \end{pmatrix} := B^{-1}_-, & k \in \Sigma_3 \cup \Sigma_4, \\
  B^{-1}_- B_+, & k \in \Sigma_5 \cup \Sigma_6.
  \end{cases}$$

- $M^\infty(k) = I + O(k^{-1})$, $k \to \infty$.

The above RH problem can be transformed into a standard Painlevé II model via an appropriate equivalent deformation. For this purpose, we add four new auxiliary paths $L_j$, $j = 1, 2, 3, 4$, passing through the point $k = 0$ at the angle $\pi/3$ with real axis $\Sigma^\infty$ divide the complex plane into eight regions $\Omega_j$, $j = 1, \ldots, 8$. See Figure 11.

We further define

$$P(k) = \begin{cases}
  B^1_+, & k \in \Omega_2 \cup \Omega_4, \\
  B^{-1}_-, & k \in \Omega_6 \cup \Omega_8, \\
  I, & k \in \Omega_1 \cup \Omega_3 \cup \Omega_5 \cup \Omega_7,
  \end{cases}$$

and make a transformation

$$\tilde{M}^P(k) = M^\infty(k) P(k),$$

then we obtain a Painlevé model.
RH problem A.2. Find $\hat{M}^P(k) = \hat{M}^P(k; s)$ with properties

- $\hat{M}^P_+(k)$ is analytical in $\mathbb{C} \setminus \hat{\Sigma}^P$, where $\hat{\Sigma}^P = \bigcup_{j=1}^4 \{ L_j = e^{i(j-1)\pi/3} \mathbb{R}_+ \}$.
  See Figure 11.
- $\hat{M}^P_+(k)$ satisfies the jump condition
  
  $\hat{M}^P_+(k) = \hat{M}^P_-(k) \hat{V}^P(k), \quad k \in \hat{\Sigma}^P,$

  where

  $\hat{V}^P(k) = \begin{cases} 
  \begin{pmatrix} 1 & 0 \\ r_0 e^{2i(\frac{4}{3}k^3 + sk)} & 1 \end{pmatrix}, & k \in L_1 \cup L_2, \\
  \begin{pmatrix} 1 & -r_0 e^{2i(\frac{4}{3}k^3 + sk)} \\ 0 & 1 \end{pmatrix}, & k \in L_3 \cup L_4.
  \end{cases}$

- $\hat{M}^P(k) = I + O(k^{-1}), \quad k \to \infty.$

If the parameter $r_0$ is not real-valued, the solution to the RH problem A.2 is related to the Painlevé XXXIV equation. However, looking closely at the jump matrix, we find that the RH problem A.2 can be reduced into that usually associated with the Painlevé II equation by a gauge transformation. Let $\varphi_0 = \text{arg} r_0$, so $r_0 = |r_0| e^{i\varphi_0}$. Following the idea [23], we make the following transformation

$M^P(k) = e^{i(\frac{2n}{3} - \frac{2}{3}) \hat{\sigma}_3} \hat{M}^P(k),$

then $M^P(k)$ satisfies the RH problem.

RH problem A.3. Find $M^P(k) = M^P(k; s)$ with properties

- $M^P(k)$ is analytical in $\mathbb{C} \setminus \Sigma^P$, where $\Sigma^P = \bigcup_{j=1}^4 L_j$, see Figure 12.
- $M^P(k)$ satisfies the jump condition

  $M^P_+(k) = M^P_-(k) V^P(k), \quad k \in \Sigma^P,$
\[
\begin{pmatrix}
1 & 0 \\
-i|r_0| & 1 \\
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
i|r_0| & 1 \\
\end{pmatrix}
\]

\[L_2 \quad L_1\]

\[L_3 \quad L_4\]

\[
\begin{pmatrix}
1 & i|r_0| \\
0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
1 & -i|r_0| \\
0 & 1 \\
\end{pmatrix}
\]

**Figure 12.** The jump contours of \(M^P(k)\).

where

\[
V^P(k) = \begin{cases}
 e^{-i\left(\frac{4}{3}k^3 + sk\right)}\hat{\sigma}_3 \begin{pmatrix} 1 & 0 \\ i|r_0| & 1 \end{pmatrix}, & k \in L_1, \\
e^{-i\left(\frac{4}{3}k^3 + sk\right)}\hat{\sigma}_3 \begin{pmatrix} 1 & 0 \\ -i|\bar{r_0}| & 1 \end{pmatrix}, & k \in L_2, \\
e^{-i\left(\frac{4}{3}k^3 + sk\right)}\hat{\sigma}_3 \begin{pmatrix} 1 & i|\bar{r_0}| \\ 0 & 1 \end{pmatrix}, & k \in L_3, \\
e^{-i\left(\frac{4}{3}k^3 + sk\right)}\hat{\sigma}_3 \begin{pmatrix} 1 & -i|\bar{r_0}| \\ 0 & 1 \end{pmatrix}, & k \in L_4.
\end{cases}
\]

- \(M^P(k) = I + O(k^{-1}), \quad k \to \infty.\)

This RH problem model \(A.3\) is exactly a special case of the Painlevé II model. Therefore the solution of the RH problem model \(A.3\) is given by

\[
M^P(k) = I + \frac{M_1^P(s)}{k} + O\left(k^{-2}\right),
\]

where \(M_1^P(s)\) is given by

\[
M_1^P(s) = \frac{1}{2} \left( -i \int_s^\infty u(\zeta)^2d\zeta \int_s^\infty u(s) \zeta \right) + \frac{1}{2} \left( i \int_s^\infty u(\zeta)^2d\zeta \int_s^\infty u(s) \zeta \right),
\]

and for each \(C_1 > 0,\)

\[
\sup_{z \in \mathbb{C} \setminus \Sigma^P} \sup_{s \geq -C_1} |M^P(k, s)| < \infty.
\]

And \(u(s)\) is a solution of the Painlevé II equation

\[
u_{ss} = 2u^3 + su, \quad s \in \mathbb{R},
\]
With transformations (A.1) and (A.3), expanding $M^\infty(k)$ along the region $\Omega_3$ or $\Omega_7$ yields

$$M^\infty(k) = I + \frac{M^\infty_1(s)}{k} + O(k^{-2}),$$

(A.9)

where

$${M^\infty_1(s) = -\frac{i}{2} \begin{pmatrix} \int_s^\infty u(\zeta)^2 d\zeta & -e^{-i\varphi_0} u(s) \\ e^{i\varphi_0} u(s) & -\int_s^\infty u(\zeta)^2 d\zeta \end{pmatrix}}.$$ 

(A.10)

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Data Availability Statements
The data which supports the findings of this study is available within the article.

Conflict of Interest
The authors have no conflicts to disclose.

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