Maximal irreducibility measure
for spatial birth-and-death processes

Viktor Bezborodov * and Luca Di Persio †

University of Verona - Department of Computer Science

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Abstract

We prove that a spatial birth-and-death process is both $\phi$-irreducible and $\psi$-irreducible
under rather general conditions on the birth and death rates. It is also shown that every
maximal irreducibility measure is equivalent to the Lebesgue-Poisson measure on the space
of finite configuration.

1 Introduction

The basic question of stochastic stability analysis for a Markov process is whether the chain
is irreducible. The notion of irreducibility for countable state space Markov processes is not
directly transferable to Markov processes with continuous state spaces. The most widely used
generalization is the so called $\phi$-irreducibility, see, e.g., Myen and Tweedie [MT93]. The aim of
this paper is to prove that under certain general conditions the Lebesgue-Poisson measure is a
maximal irreducible measure for continuous-space birth-and-death processes. Roughly speaking
it means that, whatever the initial condition is, a set will be hit by the process with positive
probability if and only if it is of positive Lebesgue-Poisson measure.

*Email: viktor.bezborodov@univr.it
†Email: luca.dipersio@univr.it
We describe and define spatial birth-and-death processes in Section 4. The pioneering works on spatial birth-and-death processes are Preston [Pre75] and Holley and Stroock [HS78]. More recent studies of various related aspects include for example [FM04, GK06, FKK12], while we refer the interested reader to [DP10], for questions related to the connections with the theory of random walks in random media, and to [BCDP16, BDP16], and references therein, to what concerns the links with more applicative problems as those arising in financial and neurobiological settings.

The paper is organized as follows: in Section 2 we recall the notions of \( \phi \)-irreducibility and maximal irreducibility for measures; in Section 3 we recall the definition of the Lebesgue-Poisson measure; in Section 4 we describe the birth-and-death processes we consider and give our main result, Theorem 4.5; the proofs are collected in Section 5.

## 2 Irreducible and maximal irreducible measures

In what follows we shall adopt the notation used in [MT93]. Let \( X \) be a Polish space and \( \mathcal{B}(X) \) be its Borel \( \sigma \)-algebra. We will consider a Markov chain with transition probability kernel \( P \) and initial distribution \( \mu \) defined on the canonical space \( \Omega = \prod_{i=0}^{\infty} X \), with \( \Phi_n \) being the coordinate mappings,

\[ \Phi_n((x_0, x_1, \ldots)) = x_n. \]

The corresponding measure will be denoted by \( P_\mu \), so that for any Borel sets \( A_0, \ldots, A_n \in \mathcal{B}(\Omega) \),

\[
P_\mu(\Phi_0 \in A_0, \Phi_1 \in A_1, \ldots, \Phi_n \in A_n) = \int_{y_0 \in A_0} \ldots \int_{y_n \in A_n} \mu(dy_0)P(y_0, dy_1)\ldots P(y_{n-1}, dy_n).
\]

Let \( P_x \) denote the distribution of \( \Phi \) in \( \Omega \) when the initial distribution is the Dirac measure at \( x \), \( P_x\{\Phi_0 = x\} = 1 \). For any set \( A \in \mathcal{B}(X) \), \( \tau_A = \min\{n \geq 1 : \Phi_n \in A\} \) is called the first return time. Define also the return probabilities

\[ L(x, A) := P_x\{\tau_A < \infty\} = P_x\{\Phi \text{ ever enters } A\}. \]
Definition 2.1. A finite non-trivial measure $\phi$ is called $\phi$-irreducible for the chain $\Phi$ if $\phi(A) > 0$ implies that

$$L(x, A) > 0, \quad x \in X.$$ 

A finite non-trivial measure $\psi$ is called $\psi$-maximal irreducible for the chain $\Phi$ if

$$(\forall x \in X : L(x, A) > 0) \iff \psi(A) > 0.$$ 

The measures $\phi$ and $\psi$ from the above definition are called an irreducibility measure and a maximal irreducibility measure for $\Phi$, respectively. The next proposition provides a sufficient condition for an irreducibility measure to be a maximal irreducibility measure.

Proposition 2.2. If $\Phi$ is $\phi$-irreducible and the measure $\phi$ is such that $\phi\{y : P(y, A) > 0\} = 0$ whenever $\phi(A) = 0$, then $\Phi$ is $\psi$-irreducible with $\psi = \phi$.

3 Lebesgue-Poisson measure

The state space of a continuous-time, continuous-space birth and death process is

$$\Gamma_0(\mathbb{R}^d) = \{\eta \subset \mathbb{R}^d : |\eta| < \infty\},$$

where $|\eta|$ is the number of points of $\eta$. $\Gamma_0(\mathbb{R}^d)$ is often called the space of finite configurations. The space of $n$-point configuration is $\Gamma_0^{(n)}(\mathbb{R}^d) := \{\eta \subset \mathbb{R}^d : |\eta| = n\} \subset \Gamma_0(\mathbb{R}^d)$. We will use $\Gamma_0$ and $\Gamma_0^{(n)}$ as shorthands for $\Gamma_0(\mathbb{R}^d)$ and $\Gamma_0^{(n)}(\mathbb{R}^d)$, respectively. For $\eta, \zeta \in \Gamma_0$, $|\eta| = |\zeta| > 0$, we define

$$\rho(\eta, \zeta) := \min_{\varsigma} \max_{x \in \eta} \{ |\varsigma(x) - x| \}, \quad (3)$$

where minimum is taken over the set of all bijections $\varsigma : \eta \rightarrow \zeta$. Note than in (3) the notation $|\cdot|$ is used for the Euclidean distance in $\mathbb{R}^d$ (as opposed to the number of points as in $|\eta|$), which hopefully should not lead to ambiguity. For $\eta \in \Gamma_0$ and $a > 0$, let

$$B_{\rho}(\eta, a) := \{ \zeta \in \Gamma_0^{(|\eta|)} : \rho(\eta, \zeta) \leq a \}.$$
The $\sigma$-algebra can be defined as
\[
\mathcal{B}(\Gamma_0) = \sigma\left(\{\emptyset\}, \mathcal{B}_\rho(\eta, a), \eta \in \Gamma_0, a > 0\right).
\]

Let
\[
(\overline{\mathbb{R}^d})^n := \{(x_1, \ldots, x_n) \in (\mathbb{R}^d)^n \mid x_j \in \mathbb{R}^d, j = 1, \ldots, n, x_i \neq x_j, i \neq j\},
\]
and let $\text{sym}$ be the mapping
\[
\bigcup_{n=0}^{\infty} (\overline{\mathbb{R}^d})^n \ni (x_1, \ldots, x_n) \mapsto \{x_1, \ldots, x_n\} \in \Gamma_0.
\]

We are now going to define the Lebesgue-Poisson measure on $(\Gamma_0, \mathcal{B}(\Gamma_0))$. For any $n \in \mathbb{N}$, let $i_d^{\otimes n}$ be the restriction of the Lebesgue measure to $(\overline{\mathbb{R}^d})^n$. We denote by $\lambda^{(n)}$ the projection of this measure on $\Gamma_0^{(n)}$,
\[
\lambda^{(n)}(A) = i_d^{\otimes n}(\text{sym}^{-1} A), A \in \mathcal{B}(\Gamma_0^{(n)}).
\]

On $\Gamma_0^{(0)}$ the measure $\lambda^{(0)}$ is given by $\lambda^{(0)}(\{\emptyset\}) = 1$. The Lebesgue-Poisson measure on $(\Gamma_0, \mathcal{B}(\Gamma_0))$ is defined as
\[
\lambda := \sum_{n=0}^{\infty} \frac{1}{n!}\lambda^{(n)}.
\]

Let us note that the measure $\lambda$ is infinite.

## 4 Birth-and-death processes and the main result

Denote by $\mathcal{B}(\mathbb{R}^d)$ the Borel $\sigma$-algebra on $\mathbb{R}^d$. The evolution of a spatial birth-and-death process admits the following description. Two functions characterize the development in time, the birth rate $b : \mathbb{R}^d \times \Gamma_0(\mathbb{R}^d) \to [0, \infty)$ and the death rate $d : \mathbb{R}^d \times \Gamma_0(\mathbb{R}^d) \to [0, \infty)$. If the system is in state $\eta \in \Gamma_0$ at time $t$, then the probability that a new particle is added ("birth" event) in a bounded set $B \in \mathcal{B}(\mathbb{R}^d)$ over the time interval $[t; t + \Delta t]$ is
\[
\Delta t \int_B b(x, \eta)dx + o(\Delta t),
\]
while the probability that a particle \( x \in \eta \) is removed from the configuration ( "death" event), over time interval \([t; t + \Delta t]\) is

\[
d(x, \eta) \Delta t + o(\Delta t),
\]

and simultaneous events cannot occur. In other words, the rate at which a birth occurs in \( B \) is \( \int_B b(x, \eta) dx \), and the rate at which a particle \( x \in \eta \) dies is \( d(x, \eta) \), and no two events happen at the same time. Various aspects of birth-and-death processes are considered in, e.g., [FM04, GK06, KS06]. Here we focus our attention on the embedded Markov chain of the birth-and-death process, namely the Markov chain on \( \Gamma_0 \) with transition probabilities

\[
Q(\eta, \{ \eta \setminus \{x\} \}) = \frac{d(x, \eta)}{(B + D)(\eta)}, \quad x \in \eta, \; \eta \in \Gamma_0,
\]

\[
Q(\eta, \{ \eta \cup \{x\}, x \in U \}) = \frac{\int_{x \in U} b(x, \eta) dx}{(B + D)(\eta)}, \quad U \in \mathcal{B}(\mathbb{R}^d), \; \eta \in \Gamma_0,
\]

where \((B + D)(\eta) = \int b(x, \eta) dx + \sum_{x \in \eta} d(x, \eta)\) is the jump rate at \( \eta \).

Denote by \( Q_\alpha \) the distribution of the Markov chain on \(((\Gamma_0)^\infty, \mathcal{B}((\Gamma_0)^\infty))\) with transition probabilities (6) and initial value \( \alpha \in \Gamma_0 \). Here \( \mathcal{B}((\Gamma_0)^\infty) \) is the \( \sigma \)-algebra generated by the coordinate mappings. Let \( (\xi_n)_{n \in \mathbb{Z}_+} \) be the coordinate mappings \(((\Gamma_0)^\infty, \mathcal{B}((\Gamma_0)^\infty))\), that is \( \xi_n(\eta) = \eta_n \) for \( \eta = (\eta_0, \eta_1, \ldots) \in (\Gamma_0)^\infty \). Under \( Q_\alpha \), \( (\xi_n)_{n \in \mathbb{Z}_+} \) is a Markov chain with transition probabilities (6).

Concerning the functions \( b \) and \( d \), we assume that they are continuous functions in both variables, satisfying the following conditions

**Condition 4.1** (Sublinear growth). There exist \( c_1, c_2 > 0 \) such that

\[
\int_{\mathbb{R}^d} b(x, \eta) dx \leq c_1 |\eta| + c_2.
\]

**Condition 4.2.** We require

\[
\forall m \in \mathbb{N} : \sup_{x \in \mathbb{R}^d, |\eta| \leq m} d(x, \eta) < \infty.
\]

**Condition 4.3** (Non-degeneracy of \( d \)). The infimum

\[
\inf_{\eta \in \Gamma_0(\mathbb{R}^d), x \in \eta} d(x, \eta) > 0,
\]
**Condition 4.4** (Non-degeneracy of \(b\)). For some constants \(r > 0\) and \(c_3 > 0\),

\[
b(x, \eta) > c_3, \text{if there exists } y \in \eta, |x - y| \leq r, \tag{10}
\]

and \(b(x, \emptyset) > c_3\) for \(x \in B_\emptyset, B_\emptyset\) is some open ball in \(R^d\).

The following theorem constitutes the main result of the present paper.

**Theorem 4.5.** *The Lebesgue-Poisson measure \(\lambda\) is a maximal irreducibility measure for \((\xi_n)_{n \in \mathbb{N}}\).*

In other words,

\[
(\forall \alpha : Q_{\alpha} \{(\xi_n)_{n \in \mathbb{Z}^+} \text{ ever enters } A\} > 0) \iff \lambda(A) > 0.
\]

**Remark 4.6.** The second part of (10) means that points may come “out of nowhere”. We need such kind of condition in order for \(\emptyset\) not to be an absorbing state of the Markov chain \((\xi_n)_{n \in \mathbb{N}}\). Also, each of conditions (9) and (10) implies that every state \(\eta \in \Gamma_0, \eta \neq \emptyset\), is non-absorbing.

### 5 Proofs

**Proof of Proposition 2.2.** Let \(\phi\) be a measure satisfying conditions of the proposition. We first prove that

\[
\phi\{y : L(y, A) > 0\} = 0 \text{ whenever } \phi(A) = 0. \tag{11}
\]

Note that

\[
\{y : L(y, A) > 0\} = \bigcup_{n \in \mathbb{N}} \{y : P^n(y, A) > 0\}. \tag{12}
\]

For \(A \in \mathcal{B}(X)\) and \(k \in \mathbb{N}\), denote \(A^{(-k)} := \{x \in X : P^k(x, A) > 0\}\). To prove (11), we will proceed by induction and show that \(\phi\{y : P^n(y, A) > 0\} = 0\) as long as \(\phi(A) = 0\), for all \(n \in \mathbb{N}\). Assume that \(\phi\{y : P^n(y, A) > 0\} = 0\) whenever \(\phi(A) = 0\). Then, if \(\phi(A) = 0\),

\[
\phi\{y : P^{n+1}(y, A) > 0\} = \phi\{y : \int_{x \in X} P(y, dx)P^n(x, A) > 0\} \leq
\]

6
\[ \phi \{ y : \int_{x \in X} P(y, dx) I_{A(-m)}(x) > 0 \} = \phi \{ y : P(y, A^{(-m)}) > 0 \} = 0. \]

The base case is given in the condition, therefore \((11)\) holds.

Assume now that the statement of the proposition does not hold, so that \(\phi\) is not a maximal irreducible measure for \(\Phi\). Proposition 4.2.2 from [MT93] implies the existence of a maximal irreducible measure \(\psi'\) for \(\Phi\). Then there exists a set \(C \in \mathcal{B}(X)\) such that \(\phi(C) = 0\) whereas \(\psi'(C) > 0\). By definition of irreducibility, \(L(x, C) > 0\) for all \(x \in X\). By \((11)\), \(\phi \{ y : L(y, C) > 0 \} = 0\), hence \(\phi(X) = 0\), which contradicts to the non-triviality of \(\phi\).

Define a path of configurations as a finite sequence of configurations \(\zeta_0, \zeta_1, \ldots, \zeta_n\) such that \(|\zeta_k \triangle \zeta_{k+1}| = 1\), \(k = 0, \ldots, n - 1\), and if \(\zeta_{k+1} = \zeta_k \cup \{ z \}\), then \(|z - y| \leq \frac{r}{4}\) for some \(y \in \zeta_k\); that is, \(\zeta_{k+1}\) is obtained from \(\zeta_k\) either by adding one point to \(\zeta_k\) or by removing one point from \(\zeta_k\); in the case of the adding, it is required that the “new” point appears not further than \(\frac{r}{4}\) from an “old” one. If \(\zeta_k = \emptyset\), then we require \(\zeta_{k+1} = \{ x_\emptyset \}\), where \(x_\emptyset\) is the center of \(B_\emptyset\). We say that such a path has length \(n\), and we call \(\zeta_0\) and \(\zeta_n\) the starting vertex and the final vertex, respectively. Also, we say that \(\zeta_0, \zeta_1, \ldots, \zeta_n\) is a path from \(\zeta_0\) to \(\zeta_n\).

**Lemma 5.1.** For all \(\eta \in \Gamma_0\) there exists a path from \(\emptyset\) to \(\eta\).

**Proof.** We will show that there exists a path from \(\emptyset\) to \(\eta\) of length less than

\[
2\left( \sum_{x \in \eta} |x - x_\emptyset| \frac{4}{r} + |\eta| \right),
\]

where \(x_\emptyset\) is the center of \(B_\emptyset\).

Starting from \(\emptyset\) and only adding points, we see that there exists a path of length

\[
\leq \left( \sum_{x \in \eta} |x - x_\emptyset| \frac{4}{r} + |\eta| \right),
\]

with the starting vertex \(\emptyset\) and with the final vertex being some configuration \(\eta' \supset \eta\). Indeed, for each \(x \in \eta\) there exists a sequence of points \(x_\emptyset = x_0, x_1, \ldots, x_n = x\) such that \(|x_i - x_{i+1}| \leq \frac{r}{4}\) and \(n \leq |x - x_\emptyset| \frac{4}{r}\). Having reached \(\eta' \supset \eta\), we only need to delete some points from \(\eta'\).

**Lemma 5.2.** Let \(\emptyset = \eta_0, \eta_1, \ldots, \eta_n\) be a path. Then for every \(a > 0\)

\[
Q^n(\eta_0, B_\rho(\eta_n, a)) > 0.
\]
**Proof.** Without loss of generality we can assume $a < \frac{r}{2}$. Denote $A_k = B_\rho(\eta_k, a)$. We will first show that

$$\inf_{\eta \in A_k} Q(\eta, A_{k+1}) \geq \bar{c}_n$$

(13)

for some positive constant $\bar{c}_n$ that depends on $n$ but does not depend on the path we consider.

We have either $\eta_k \subset \eta_{k+1}$ or $\eta_k \supset \eta_{k+1}$. Consider first the case $\eta_k \subset \eta_{k+1}$. We know that $\eta_{k+1} = \eta_k \cup \{z\}$, where $|z - y| \leq \frac{r}{2}$ for some $y \in \eta_k$.

Take arbitrary $\eta \in A_k$. There exists $y' \in \eta$ such that $|y - y'| \leq a$. For $x \in B_\rho(z)$ we have then $|x - y'| \leq |x - z| + |z - y| + |y - y'| \leq a + \frac{r}{2} + a < r$. Moreover, if $x \in B_\rho(z) \setminus \eta$, then $\eta \cup \{x\} \in A_{k+1}$.

From (10) we obtain

$$Q(\eta, A_{k+1}) \geq \int_{x \in B_\rho(z)} b(x, \eta) dx \geq \int_{x \in B_\rho(z)} c_3 dx$$

$$= \frac{c_3 a^d v_d}{(B + D)(\eta)},$$

where $v_d$ is the volume of a unit ball in $\mathbb{R}^d$. By (8), the denominator of the last fraction is bounded in $\eta, \eta \in \bigcup_{k=0}^n \Gamma_{0}^{(k)}(\mathbb{R}^d)$. Therefore, (13) holds.

Now we turn our attention to the case when $\eta_k \supset \eta_{k+1}$. We may write $\eta_{k+1} = \eta_k \setminus \{y\}$ for some $y \in \eta_k$, and (13) follows from (9).

The statement of the lemma follows from (13), since

$$Q^n(\emptyset, B_\rho(\eta, a)) = \int_{\zeta_1, \zeta_2, \ldots, \zeta_n} Q(\emptyset, d\zeta_1)Q(\zeta_1, d\zeta_2)Q(\zeta_2, d\zeta_3) \times \ldots \times Q(\zeta_{n-1}, d\zeta_n)I_{\{\zeta_n \in B_\rho(\eta, a)\}}$$

$$\geq \int_{\zeta_1, \zeta_2, \ldots, \zeta_n} Q(\emptyset, d\zeta_1)Q(\zeta_1, d\zeta_2)Q(\zeta_2, d\zeta_3) \times \ldots \times Q(\zeta_{n-1}, d\zeta_n)I_{\{\zeta_k \in B_\rho(\eta, a), k=1, \ldots, n\}} \geq (\bar{c}_n)^n.$$

**Lemma 5.3.** Let $A \in \mathcal{B}(\Gamma_0)$, $\beta' \in \Gamma_{0}^{(n)}$ and $\lambda(A \cap B_\rho(\beta', \frac{r}{4})) > 0$. Then

$$Q^{2n}(\beta, A) > 0$$
for any $\beta \in B_\rho(\beta', \varphi)$. 

The idea of the proof. Let $\beta = \{x_1, \ldots, x_n\}$. The event $R$ described in the next sentence has positive probability. Let $\xi_1 = \beta \cup \{y_1\}$ for some $y_1 \in B_\varphi(x_1)$, $\xi_2 = \xi_1 \setminus \{x_1\}$, $\xi_3 = \xi_2 \cup \{y_2\}$ for some $y_2 \in B_\varphi(x_2)$, $\xi_4 = \xi_3 \setminus \{x_2\}$, and so on, so that $\xi_{2n} = \xi_{2n-1} \setminus \{x_n\}$. We will see that $Q_\beta(\xi_{2n} \in A | R) > 0$.

**Proof.** Fix $\beta = \{x_1, \ldots, x_n\}$. Consider a measurable subset $\Xi$ of $(\Gamma_0)^{2n}$,

$$\Xi = \left\{(\zeta_1, \ldots, \zeta_{2n}) \mid \zeta_{2k-1} = \{y_1, \ldots, y_k, x_k, \ldots, x_n\}, \zeta_{2k} = \{y_1, \ldots, y_k, x_{k+1}, \ldots, x_n\}, k = 1, \ldots, n, \right.$$ 

for some distinct $y_1, \ldots, y_n \in \mathbb{R}^d$ satisfying $|y_k - x_k| \leq \frac{r}{4}$

Define $R = \{(\xi_1, \ldots, \xi_{2n}) \in \Xi\}$.

By the Markov property,

$$Q^{2n}(\beta, A) = \int_{\zeta_1, \zeta_2, \ldots, \zeta_{2n}} Q(\beta, d\zeta_1)Q(\zeta_1, d\zeta_2)Q(\zeta_2, d\zeta_3) \times \ldots \times Q(\zeta_{2n-1}, d\zeta_{2n})I_{\{\zeta_{2n} \in A\}}$$

$$\geq \int_{\zeta_1, \zeta_2, \ldots, \zeta_{2n}} Q(\beta, d\zeta_1)Q(\zeta_1, d\zeta_2)Q(\zeta_2, d\zeta_3) \times \ldots$$

$$\times Q(\zeta_{2n-1}, d\zeta_{2n})I_{\{\zeta_1, \ldots, \zeta_{2n} \in \Xi\}}I_{\{([\zeta_1 \setminus \beta] \cup \{y_1\}) \times \ldots \times (\zeta_{2n-1} \setminus \zeta_{2n-2}) \in \text{sym}^{-1} A\}}.$$  

Here for singletons $S_1 = \{s_1\}, S_2 = \{s_2\}, \ldots, S_n = \{s_n\}$ we define

$$S_1 \supseteq S_2 \supseteq \ldots \supseteq S_n = (s_1, s_2, \ldots, s_n).$$

Note that $\zeta_{2n} = (\zeta_1 \setminus \beta) \cup (\zeta_3 \setminus \zeta_2) \cup \ldots (\zeta_{2n-1} \setminus \zeta_{2n-2})$ if $(\zeta_1, \ldots, \zeta_{2n}) \in \Xi$.

From the definition of the Lebesgue Poisson measure we have

$$l_\lambda^n(\text{sym}^{-1} A) = n!\lambda(A),$$

where $l_\lambda^n$ is the Lebesgue measure on $(\mathbb{R}^d)^n$.

Define a measure $\sigma$ on $\left(\bigotimes_{k=1}^n B_\varphi(x_k), \mathcal{B}(\bigotimes_{k=1}^n B_\varphi(x_k))\right)$ by
\[ \sigma(D) = \int_{\zeta_1, \ldots, \zeta_{2n}} Q(\beta, d\zeta_1)Q(\zeta_1, d\zeta_2)Q(\zeta_2, d\zeta_3) \times \ldots \times Q(\zeta_{2n-1}, d\zeta_{2n}) \times I_{\{(\zeta_1, \ldots, \zeta_{2n}) \in \Xi\}}I_{\{(\zeta_1 \gamma \zeta_4 \ldots \gamma \zeta_{2n-1}) \in D\}}, \quad D \in B\left(\prod_{k=1}^{n} B^{2}(x_{k})\right). \]

We can rewrite (14) as

\[ Q^{2n}(\beta, A) \geq \sigma(\text{sym}^{-1}A). \tag{16} \]

We will show that

\[ \sigma(D) \geq \tilde{c} \inf_{\eta \in \Gamma_0(\mathbb{R}^d), x \in \eta} \frac{d(x, \eta)}{\sup\{(B + D)(\eta) \mid |\eta| \leq n + 1\}} \geq \inf_{\eta \in \Gamma_0(\mathbb{R}^d), x \in \eta} \frac{d(x, \eta)}{\sup\{(B + D)(\eta) \mid |\eta| \leq n + 1\}}, \tag{17} \]

for some constant \( \tilde{c} > 0 \).

The statement of the lemma is a consequence of (15), (16) and (17). To establish (17) we only need to consider sets of the form \( D_1 \times \ldots \times D_n, \) \( D_j \in B(B^{2}(x_{j})) \). Define

\[ \Xi(D_1, \ldots, D_n) = \left\{ (\zeta_1, \ldots, \zeta_{2n}) \mid \zeta_{2k-1} = \{y_1, \ldots, y_k, x_k, \ldots, x_n\}, \zeta_{2k} = \{y_1, \ldots, y_k, x_k+1, \ldots, x_n\}, k = 1, \ldots, n, \right. \]

for some distinct \( y_k \in D_k \} \]

We have

\[ \sigma(D_1 \times \ldots \times D_n) = \int_{\zeta_1, \ldots, \zeta_{2n}} Q(\beta, d\zeta_1)Q(\zeta_1, d\zeta_2)Q(\zeta_2, d\zeta_3) \times \ldots \times Q(\zeta_{2n-1}, d\zeta_{2n})I_{\{(\zeta_1, \ldots, \zeta_{2n}) \in \Xi(D_1, \ldots, D_n)\}}. \]

Fix \( z_j \in D_j \). Using our assumptions on \( b \) and \( d \), we see that

\[ Q\left(\{z_1, \ldots, z_k, x_k, \ldots, x_n\}, \{z_1, \ldots, z_k, x_k+1, \ldots, x_n\}\right) = \frac{d(x_k, \{z_1, \ldots, z_k, x_k, \ldots, x_n\})}{(B + D)\{z_1, \ldots, z_k, x_k, \ldots, x_n\}} \]

\[ \geq \frac{d(x_k, \{z_1, \ldots, z_k, x_k, \ldots, x_n\})}{\sup\{(B + D)(\eta) \mid |\eta| \leq n + 1\}} \geq \inf_{\eta \in \Gamma_0(\mathbb{R}^d), x \in \eta} \frac{d(x, \eta)}{\sup\{(B + D)(\eta) \mid |\eta| \leq n + 1\}}, \]
and

\[
Q \left( \{ z_1, \ldots, z_k, x_{k+1}, \ldots, x_n \}, \left\{ z_1, \ldots, z_k, y_{k+1}, x_{k+1}, \ldots, x_n \mid y_{k+1} \in D_{k+1} \right\} \right) = \int_{y \in D_{k+1}} b(y, \{ z_1, \ldots, z_k, x_{k+1}, \ldots, x_n \}) dy \geq \frac{c_3 l_d(D_{k+1})}{(B + D)(\{ z_1, \ldots, z_k, x_{k+1}, \ldots, x_n \})},
\]

where \( l_d \) is the Lebesgue measure on \( \mathbb{R}^d \). Hence

\[
\sigma(D_1 \times \ldots \times D_n) \geq \left( \frac{n}{\lambda^{(n)}(D_1 \times \ldots \times D_n)} \right)^n \prod_{j=1}^n \frac{c_3 l_d(D_j)}{\sup\{(B + D)(\eta) : |\eta| \leq n + 1\}}.
\]

It remains to note that \( \prod_{j=1}^n l_d(D_j) = l_d^n(D_1 \times \ldots \times D_n) \).

**Proof of Theorem 4.5.** We will first establish \( \phi \)-irreducibility. Starting from any configuration, the process may go extinct in finite time: for all \( \eta \in \Gamma_0(\mathbb{R}^d) \)

\[
Q_\eta(\xi_k = \emptyset \text{ for some } k > 0) > 0.
\]

Therefore, it is sufficient to show that

\[
L(\emptyset, A) > 0 \quad \text{whenever } \lambda(A) > 0, A \in \mathcal{B}(\Gamma_0).
\] (18)

Let us take \( A \in \mathcal{B}(\Gamma_0) \) with \( \lambda(A) > 0 \). There exists \( n \in \mathbb{N} \) and \( \beta' \in \Gamma_0^{(n)} \) such that

\[
\lambda(A \cap \mathcal{B}_\rho(\beta', \frac{r}{4})) > 0.
\] (19)

By Lemma 5.1 there exists a path from \( \emptyset \) to \( \beta' \). Denote by \( m \) the length of this path. Applying Lemma 5.2 and Lemma 5.3 we get

\[
Q^{m+2n} (\emptyset, A) \geq \int_{\beta \in \mathcal{B}_\rho(\beta', \frac{r}{4})} Q^m (\emptyset, d\beta) Q^{2n} (\beta, A) > 0,
\]

which proves (18).

Now let us prove that \( \lambda \) is a maximal irreducibility measure for \( (\xi_n)_{n \in \mathbb{N}} \). Taking into account Proposition 2.2, we see that it suffices to show that for all \( A \subset \Gamma_0(\mathbb{R}^d) \) with \( \lambda(A) = 0 \) we have
\[ \lambda \{ \eta : Q(\eta, A) > 0 \} = 0. \] (20)

With no loss of generality, we assume that \( A \subset \Gamma_0^{(n)}(\mathbb{R}^d), \ n \geq 2. \) We have \( sym^{-1}(A) \subset (\mathbb{R}^d)^n \) and \( l_d^n (sym^{-1}(A)) = 0. \) Now, \( \eta \in \Gamma_0^{(n+1)}(\mathbb{R}^d) \) and \( Q(\eta, A) > 0 \) if and only if \( \eta \) may be represented as \( \xi \cup \{ x \}, \) where \( \xi \in A, \ x \in \mathbb{R}^d \setminus \xi. \) Then we also have for any \( y = (y_1, ..., y_{n+1}) \in sym^{-1}(\eta) \)

\[ \tilde{\Pi}_j y \in sym^{-1}(A) \]
for some \( j \in \{1, 2, ..., n+1\}, \) where \( \tilde{\Pi}_j y = (y_1, ..., y_{i-1}, y_{i+1}, ..., y_{n+1}) \in (\mathbb{R}^d)^n. \)

Since \( l_d^n (sym^{-1}(A)) = 0, \) we also have \( l_{d+1}^n (\tilde{\Pi}(\cdot)^{-1}(sym^{-1}(A))) = 0, \) and consequently

\[ \lambda \{ \eta : \eta \in \Gamma_0^{(n+1)}, Q(\eta, A) > 0 \} = 0. \] (21)

Similarly, if \( \eta \in \Gamma_0^{(n-1)}(\mathbb{R}^d) \) and \( Q(\eta, A) > 0, \) then for \( y \in sym^{-1}(\eta) \)

\[ l_d \{ z \in \mathbb{R}^d : (z, y) \in sym^{-1}(A) \} > 0. \] (22)

because a “newly born” point has an absolutely continuous distribution with respect to the Lebesgue measure on \( \mathbb{R}^d, \) in the sense that \( Q(\eta, \{ \eta \cup z \mid z \in D \}) = 0 \) if \( l_d(D) = 0. \) However, the set of all \( y \) satisfying (22) has zero Lebesgue measure, otherwise we would have

\[ l_d^n (sym^{-1}(A)) = \int l_d^{n-1}(dy) l_d \{ z : (z, y) \in sym^{-1}(A) \} > 0. \]

Therefore,

\[ \lambda \{ \eta : \eta \in \Gamma_0^{(n-1)}, Q(\eta, A) > 0 \} = 0. \] (23)

Note that in cases \( n = 0, 1 \) some changes should be made in the proofs of (21), (23), because of the special structure of \( \Gamma_0^{(0)}(\mathbb{R}^d) = \{ \emptyset \}. \)

Since

\[ \{ \eta : \eta \in \Gamma_0^{(k)}, P(\eta, A) > 0 \} = \emptyset, \]

\( k \neq n - 1, n + 1, n \geq 0, \) (21) and (23) imply (20). \( \square \)
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