On a principle of cosmological uncertainty

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Abstract

We show that cosmological observations are subject to an intrinsic uncertainty which can be expressed in the form of an uncertainty relation similar to the Heisenberg principle. This is a consequence of the fact that the four dimensional space-time metric information is projected into the one-dimensional observational red-shift space, implying a limit on the amount of information which can be extracted about the underlying geometry. Since multiple space-time configurations can lead to the same red-shift, there is an unavoidable uncertainty about the determination of the space-time geometry. This suggests the existence of a limit about of the amount of information that cosmological observations can reveal about our Universe that no experiment could ever overcame, conceptually similar to what happens in quantum mechanics.

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Cosmology has some distinctive features respect to other branches of physics: we can observe the Universe only from one point, and we are unable to probe directly the geometry of space-time on large scales. We can only detect particles or radiation which have reached us, and extract from these observations information about the space-time between us and the sources. Most of cosmological scale distances can in fact only be deduced from red-shift measurements, assuming an underlying space-time metric. The Hubble law for example is based on the assumption of large scale isotropy and homogeneity of the metric describing the Universe. We know nevertheless that the Universe had a certain level of inhomogeneity at very early times, as predicted by inflation, which is supposed to be the seed for the later process of structure formation due to gravity. There has been an long debate \[1–5\] on whether this perturbations of the homogeneous space-time metric could explain cosmological observations, in particular if they could induce effects equivalent to a cosmological constant or dark energy. In this letter we consider what are the statistical effects of these inhomogeneities on the determination of background cosmological parameters, focusing on their complementary role rather than looking at them as alternatives to each other. Let’s consider a Friedman universe with scalar perturbations. In longitudinal gauge the metric can be written as

\[
\begin{align*}
\text{ds}^2 &= - (1 + 2\Psi) d\eta^2 + (1 - 2\Psi) \delta_{ij} dx^i dx^j ,
\end{align*}
\]

where we have assumed for simplicity the background to be flat and the perturbations to be of perfect fluid type. Let’s suppose \[6–8\] a photon is emitted from a source located at a comoving coordinate \(r_S\) and time \(t_S\) with a wave length \(\lambda_S\), and is reaching an observer at the center \(r_O = 0\), with a wavelength \(\lambda_O\). Its total redshift can be expressed as the sum of two components. One is due to the homogeneous background expansion, \(z_H\), and another due to the perturbations of the gravitation potential, \(z_I\):

\[
\begin{align*}
\frac{\lambda_O}{\lambda_S} &= 1 + z = 1 + z_H + z_I = \frac{a(t_O)}{a(t_S)} (1 + [\psi + \mathbf{v} \cdot \mathbf{n}]^O_S - 2 \int_{\lambda_S}^{\lambda_O} d\lambda \dot{\psi}) , \\
z_H &= \frac{a(t_O)}{a(t_S)} - 1 , \\
z_I &= (1 + z_H)([\psi + \mathbf{v} \cdot \mathbf{n}]^O_S - 2 \int_{\lambda_S}^{\lambda_O} d\lambda \dot{\psi}) .
\end{align*}
\]

In the following we will neglect the contribution coming from the difference in the potential and the peculiar velocity between the observer and the source to get

\[
z_I = (1 + z_H)(-2 \int_{\lambda_S}^{\lambda_O} d\lambda \dot{\psi}) \approx -2(1 + z_H) \Delta r \sum_{i=1}^{N} \psi ,
\]
where in the last approximate equality we have chosen the comoving coordinate as the affine parameter along the null geodesic, we have divided the interval \((r_S, 0)\) into \(N\) subintervals of equal comoving length \(\Delta r\) and approximated the integral with a summation.

The time evolution of the Fourier transform of the gravitational potential is given by:

\[
\dot{\psi} = \psi_P(k)T(k)\frac{d}{d\eta}\left(\frac{D(\eta)}{a(\eta)}\right),
\]

where \(D(\eta)\) is the growth factor, \(T(k)\) is the transfer function, and \(\phi_P(k)\) is the primordial perturbation. For a matter dominated universe the time derivative of the gravitational potential is zero, while for a Universe with cosmological constant there can be a significant contribution, which in the case of the cosmic microwave background radiation for example leads to the so called integrated ISW effect. While there is a redshift dependence for \(\frac{d}{d\eta}\left(\frac{D(\eta)}{a(\eta)}\right)\), we will neglect it here, and assume an average constant value \(\alpha\) along the geodesic. This implies that we can pull this constant factor out of the summation, and the only quantity summed is proportional to the primordial perturbation of the gravitational potential. Assuming this to be gaussian, we can approximate it as a discrete random variable \(\psi_i\) with uniform probability of having a positive, zero or negative value, corresponding to the standard deviation of the gaussian field \(\sigma_\psi\). This correspond to approximating the gaussian distribution with a uniform distribution with the three discrete values \(\psi_i = \{\sigma_\psi, 0, -\sigma_\psi\}\).

Finally we get

\[
z_I = -2(1 + z_H)\alpha \Delta r \sum_{i=1}^{N} \psi_i,
\]

where \(\psi_i\) is a random value in the \(i\)th sub-interval determined by the time independent part of the \(\psi(k, \eta)\), i.e. by \(\psi_P(k)T(k)\), and \(k\) is the momentum corresponding to the scale \(\Delta r\). For large scales \(T(k)\) is approximately 1, so \(\psi_i\) is of the order of the primordial spectrum \(\psi_P(k)\). From the above assumptions we can derive the probability distribution of \(z_I\). In this discrete model every space-time configuration along the photon path corresponds to assigning a set of \(N\) values to the random variable \(\phi_i\).

Let’s call \(\{n_+, n_-\}\), the number of positive and negative values of the random variable \(\phi_i\). The corresponding \(z_I\) will be given by

\[
z_I = -2(1 + z_H)\alpha \Delta r \sigma_\psi (n_+ - n_-) = \Delta z_{\text{min}}S,
\]

\[
\Delta z_{\text{min}} = -2(1 + z_H)\alpha \Delta r \sigma_\psi,
\]

\[
S = (n_+ - n_-),
\]
where we have introduced the $\Delta z_{\min}$, the minimum possible inhomogeneous contribution to the total redshift. According to our discrete model the total inhomogeneous redshift is a finite multiple of this minimum contribution, so we get:

$$-N \leq \frac{z}{\Delta z_{\min}} \leq N,$$

(11)

where the lower and upper bounds correspond to configurations of all positive or all negative values of $\psi_i$, i.e. $n_+ = S$ or $n_- = S$. Since every interval can have three possible values of $\psi_i$, we have a total of $3^N$ possible different configurations. Because of the fact that what it matters is the sum $(n_+ - n_-)$, but not the order in which positive or negative perturbations appear, different space-time configurations can lead to the same value of $z_I$. This is the origin of the intrinsic uncertainty on determining the origin of the total redshift, and consequently to estimate other background cosmological parameters such as $\Omega_\Lambda$.

The number of space-time configurations corresponding to a given value of $z_I$ can be calculated analytically. Let’s start from the number of configurations corresponding to a given set $\{n_-, n_+\}$ of positive and negative values of $\psi_i$, which is is given by:

$$N_c(n_-, n_+) = \frac{N!}{n_-!n_+!(N-n_- - n_+)!}.$$  

(12)

The number of different configurations corresponding to a given value of $z_I$, i.e. to a given value of $S = (n_+ - n_-)$, can be computed in the case of even $N$ and $S > 0$ as

$$N_S = \sum_{n_+ = S}^{(N+S)/2} N_c(n_+ - S, n_+) = 2F_1\left[(1 + S - N)/2; (S - N)/2; S + 1; 4\right] \frac{N!}{S!(N-S)!},$$  

(13)

where $2F_1$ is the Gauss hypergeometric function. Since the distribution is symmetric in $S$ respect to its central value, a similar expression can be found for $S < 0$. The formula derived above is consistent with the total number of possible configurations, i.e. we have

$$\sum_{S=-N}^{N} N_S = 3^N.$$  

(14)

The figure 1 shows a plot of the expected probability $P(S)$ and $P(z_I)$ for different values of $N$. As it can be seen the dispersion around the central value (corresponding to $z_I = 0$), is increasing as $N$ increases, while its average value remain zero because of the symmetry of the distribution. Since this dispersion correspond to an uncertainty on the estimation of $z_H$ from the total redshift $z$ because $z_H = z - z_I$, we deduce that an increased accuracy in estimating the effects of the inhomogeneities, i.e. a larger $N$, correspond to a larger
FIG. 1: The probability $P(S)$ and $P(z_i/\Delta z_{\text{min}})$ are plotted for different values of $N$. The thick line is for $N = 30$, the dotted line is for $N = 20$, and the dot-dashed line is for $N = 10$. The difference between the two distribution is due to the fact that as $N$ increases $\Delta r$ decreases, lowering $\Delta z_{\text{min}}$. As it can be seen the dispersion increases with $N$.

uncertainty on $z_H$, and consequently on the background cosmological parameters on which $z_H$ depend. Schematically we can write that

$$\Delta z_I \propto \frac{1}{N},$$

$$\Delta z_H \propto N,$$

(15)

(16)

(17)

where the proportionality symbol has to be interpreted qualitatively, since we have not quantitatively estimated it, but we have inferred it heuristically from the dispersion of the expected probability distribution $P(z_I)$ and from the intuitive argument that a larger number of subintervals $N$ corresponds to a more accurate modeling of the perturbations between the source and the observer, i.e. a smaller error $\Delta z_I$. We can express this with an uncertainty relation of the type:

$$\Delta z_I \Delta z_H \geq f(r_S - r_O),$$

(18)

where $f(r)$ should be a monotonous increasing function of the comoving coordinate, since the uncertainty increases with the comoving distance of the source, due to a larger number
of possible space-time configurations. Alternatively we could for example write it as:

\[ \Delta I \Delta \Omega_\Lambda \geq f(r) \]  \hspace{1cm} (19)

where \( \Delta I \) stands for uncertainty on the determination of the perturbative part of the metric, and \( \Omega_\Lambda \) is supposed to encode all the information about the homogeneous background, i.e. we are neglecting the effects of the curvature. A more accurate treatment could involve the use of stochastic calculus or of a random variable which can take more values following different physically motivated probability distributions. Nevertheless we expect that such improved approaches would not affect the the existence of such a lower bound, but could possibly modify its quantitative estimation.

The dependence of the lower bound on the comoving distance of the source can be interpreted in terms of information theory, as the consequence of the fact that from a single piece of information, i.e. the ratio between the energy of the photons at the source and at the observer, we are trying to extract an increasingly larger amount of information, i.e. the space-time configuration between the source and the observer. Since the number of possible space-time configurations goes like \( 3^N \), if we keep \( \Delta r \) constant, we are dealing with an increasingly larger number of possible space-time configurations. Vice versa if we are keeping \( N \) constant and increase \( r_S \) we will be able to resolve space-time inhomogeneities only at larger comoving scale \( r_S/N \), so \( \Delta I \) will increase.

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