REGULARIZATION OF RELATIVE HOLONOMIC $\mathcal{D}$-MODULES

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Abstract. Let $X$ and $S$ be complex analytic manifolds where $S$ plays the role of a parameter space. Using the sheaf $\mathcal{D}_{X \times S/S}^\infty$ of relative differential operators of infinite order, we construct functorially the regular holonomic $\mathcal{D}_{X \times S/S}$-module $\mathcal{M}_{\text{reg}}$ associated to a relative holonomic $\mathcal{D}_{X \times S/S}$-module $\mathcal{M}$, extending to the relative case classical theorems by Kashiwara-Kawai: denoting by $\mathcal{M}^\sim$ the tensor product of $\mathcal{M}$ by $\mathcal{D}_{X \times S/S}^\infty$ we explicit $\mathcal{M}^\sim$ in terms of the sheaf of holomorphic solutions of $\mathcal{M}$. As a consequence of the relative Riemann-Hilbert correspondence we conclude that $\mathcal{M}^\sim$ and $\mathcal{M}_{\text{reg}}^\sim$ are isomorphic.

Contents

1. Introduction 1
2. A short reminder on the relative Riemann-Hilbert correspondence 4
3. Technical Lemmas 7
4. Main result 9
References 13

1. Introduction

The relative framework we deal with is associated to a projection

$$p : X \times S \rightarrow S$$

where $X$ and $S$ are complex manifolds. Throughout this work we identify the relative cotangent bundle $T^*(X \times S/S)$ to $T^*X \times S$ and $d_X$ and $d_S$ will denote respectively the complex dimension of $X$ and of $S$. Let $\mathcal{D}_{X \times S/S}$ be the subsheaf of $\mathcal{D}_{X \times S}$ of operators commuting with $p^{-1}O_S$ and let Mod$_{\text{coh}}(\mathcal{D}_{X \times S/S})$ be the abelian category of coherent $\mathcal{D}_{X \times S/S}$-modules. A $\mathcal{D}_{X \times S/S}$-holonomic module is a coherent $\mathcal{D}_{X \times S/S}$-module whose characteristic variety is contained in a product $\Lambda \times S$ where $\Lambda$ is $\mathbb{C}^\ast$-conic analytic lagrangian in $T^*X$ (cf. [20], [23], [19]). The datum of a strict (i.e, a $p^{-1}O_S$-flat) holonomic $\mathcal{D}_{X \times S/S}$-module is equivalent to the datum of a flat family of holonomic $\mathcal{D}_X$-modules with characteristic variety contained in $\Lambda$. 

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Let $\mathcal{D}_X^{\infty}$ denote the subsheaf of $\mathcal{D}_X^{\infty}$ of operators commuting with $p^{-1}\mathcal{O}_S$. As pointed out in [25, Rem. 2, p. 406], the sheaf of rings $\mathcal{D}_X^{\infty}$ is faithfully flat over $\mathcal{D}_X^{\infty} \times S/S$. Indeed the method of the proof of [25, Th. 3.4.1] which concerns the relative microdifferential case out of the zero section of $T^*(X \times S/S)$ adapts to the sheaves $\mathcal{D}_X^{\infty}$ and $\mathcal{D}_X^{\infty} \times S/S$.

The relative setting means here that $\mathcal{D}_X$ and $\mathcal{D}_X^{\infty}$ are replaced respectively by $\mathcal{D}_X \times S/S$ and $\mathcal{D}_X^{\infty} \times S/S$ and that we consider relative holonomic modules. Our main main result is Theorem 8 which proves a relative version of the following Kashiwara-Kawai’s Theorem ([12, Th. 1.4.9]): Let $\mathcal{D}_X^{\infty}$ denote the sheaf of linear differential operators on $X$ with possibly infinite order. To any holonomic $\mathcal{D}_X$-module one associates $M^{\infty} := \mathcal{D}_X^{\infty} \otimes_{\mathcal{D}_X} M$ and, if $F = \text{Sol}M$, then $M^{\infty} \simeq R\text{Hom}(F, \mathcal{O}_X)$.

The same authors introduce in loc.cit a regular holonomic $\mathcal{D}_X$-module $M_{\text{reg}}$ contained in $M^{\infty}$ and prove in [12, Theorem 5.2.1] a $\mathcal{D}_X^{\infty}$-isomorphism:

$$M^{\infty} \simeq M_{\text{reg}}^{\infty}$$

In (b) of Theorem [12] we extend this result to the relative setting. The proof is based on the relative Riemann-Hilbert correspondence obtained in [3] & [6] since one previous step is to prove that $\langle \cdot \rangle_{\text{reg}} \simeq \text{RH}S(\text{Sol} \cdot)[d_X]$. The latter isomorphism is a contribution to the understanding of the functor $\text{RH}^S$.

The task is not trivial although we dispose of a good notion of regularity recalled below, as well as of the inspiration provided by the techniques in [12]. Let us explain why:

One big difference from the absolute to the relative case is that the triangulated category of $\mathcal{D}_X \times S/S$-complexes having bounded holonomic cohomologies $(\mathcal{D}_b^{b, \text{hol}}(\mathcal{D}_X \times S/S))$ is not stable under the inverse image functor by morphisms $f \times \text{Id} : X' \times S \rightarrow X \times S$. Such constraint entails a loss of several functorial properties (for instance localization, algebraic supports cohomology).

The notions of $S$-$\mathbb{R}$- and $S$-$\mathbb{C}$-constructibility were introduced in [19] for objects in $\mathcal{D}_b^{b}(p^{-1}\mathcal{O}_S)$ as well as a natural duality and a middle perversity $t$-structure on the triangulated category $\mathcal{D}_b^{b, c}(p^{-1}\mathcal{O}_S)$ whose objects have $S$-$\mathbb{C}$-constructible cohomologies. A perverse object with perverse dual is then equivalent to the datum of a flat family of perverse sheaves on $X$.

The lack of functorialities in $\mathcal{D}_b^{b, \text{hol}}(\mathcal{D}_X \times S/S)$ prevents from stating an irregular relative Riemann-Hilbert correspondence by simply adapting the strategies used in the absolute case as treated by D’Agnolo-Kashiwara (cf. [3]). For a satisfactory functorial behaviour, regularity is necessary as proved in [3], [6].

Recall that a regular holonomic $\mathcal{D}_X \times S/S$-module is a holonomic $\mathcal{D}_X \times S/S$-module satisfying the following condition: the (derived) holomorphic restriction to each fiber of $p$ is a regular holonomic complex on $X$. We also consider
the associated triangulated category \((\mathcal{D}_\text{hol}^b(\mathcal{D}_{X/S/S}))\) of complexes having bounded regular holonomic cohomologies.

It is then natural to ask what kind of "regularity" can be associated to any holonomic \(\mathcal{D}_{X/S/S}\)-module.

Recall that the relative Riemann-Hilbert equivalence was first proved in [9] assuming that \(d_S = 1:\)

The functor \(p\text{Sol} : \mathcal{M} \mapsto R\text{Hom}_{\mathcal{D}_{X/S/S}}(\mathcal{M}, \mathcal{O}_{X/S})[d_X]\) from \(\mathcal{D}_\text{hol}^b(\mathcal{D}_{X/S/S})\) to \(\mathcal{D}_{\mathcal{C}^\text{reg}}^b(p^{-1}\mathcal{O}_S)\) admits a right and left adjoint denoted by \(\text{RH}^S\) and thus \(p\text{Sol}\) is an equivalence of categories.

In [6], the same authors proved that this equivalence holds true for arbitrary \(d_S\).

In the absolute case (meaning that \(S = \text{pt}\)) we recover Kashiwara’s regular Riemann-Hilbert correspondence, and, if \(X = \text{pt}\), we get the natural duality on the bounded derived category of complexes with \(\mathcal{O}_S\)-coherent cohomologies.

We now precise our results:

If \(\mathcal{M}\) is a holonomic \(\mathcal{D}_{X/S/S}\)-module, we define

\[\mathcal{M}^\infty := \mathcal{D}_{X/S/S}^\infty \otimes_{\mathcal{D}_{X/S/S}} \mathcal{M}\]

and we generalize this definition by flatness to \(\mathcal{D}_{\text{hol}}^b(\mathcal{D}_{X/S/S})\).

In our main result (Theorem 5) we prove that if \(\mathcal{M}\) is an object of \(\mathcal{D}_{\text{hol}}^b(\mathcal{D}_{X/S/S})\) and \(F = p\text{Sol}\mathcal{M}\) then \(\mathcal{M}^\infty \simeq R\text{Hom}^{p-1}_{\mathcal{O}_S}(F, \mathcal{O}_{X/S})\) (to compare with [12] Th. 1.49). As a consequence one concludes in Theorem 12 that if \(\mathcal{M}\) is a holonomic \(\mathcal{D}_{X/S/S}\)-module then \(\mathcal{M}_{\text{reg}} \simeq \text{RH}^S(p\text{Sol}\mathcal{M})\) and so (1) holds true in this setting.

The simplest example is the following: for a submanifold \(Z\) of \(X\), one has

\[\text{RH}^S(C_{Z/S} \otimes p^{-1}\mathcal{O}_S)^\infty[-d_X]\]

\[\simeq T\text{Hom}(C_{Z/S}, \mathcal{O}_{X/S})^\infty \simeq B^\infty_{Z/S/X/S}[-d]\]

\[\simeq R\text{Hom}(C_{Z/S}, \mathcal{O}_{X/S})\]

where \(d\) is the codimension of \(Z\).

Another example is provided by [12] page 814, replacing \(a \in \mathbb{C}\) by a holomorphic function \(a(s)\) without zeros on some open \(S := \Omega \subset \mathbb{C}\). For \(X = \mathbb{C}\), we consider the \(\mathcal{D}_{X/S/S}\)-module (holonomic, non regular) defined by

\[(x^2\partial_x - a(s))u(x, s) = 0\]

We then obtain (cf. page 815 of [12]) an equivalent system substituting the generator \(u\) by \(u_0 = u\) and introducing \(u_1 = -x\partial_x u\),

\[
\begin{align*}
  x\partial_x u_0 + u_1 &= 0 \\
  -a(s)u_0 - xu_1 &= 0
\end{align*}
\]

After multiplication by matrices in \(\mathcal{D}^\infty_{X/S/S}\) (the matrices provided by [12] which now depend on the parameter \(s\)), one concludes a \(\mathcal{D}^\infty_{X/S/S}\)-isomorphism from the \(\mathcal{D}^\infty_{X/S/S}\)-module extension of (2) to the \(\mathcal{D}^\infty_{X/S/S}\)-module (with generators \(w_0, w_1\)) extension of the regular holonomic \(\mathcal{D}_{X/S/S}\)-module.
\[ \begin{aligned} x w_0 - a(s) w_1 &= 0 \\ x \partial_x w_1 &= 0 \end{aligned} \]

We remark that [12] uses microlocal technics for the proof of the regularity of \( M_{reg} \). With the more recent notion of microsupport ([13]) and the results on [26], the necessary tools in the relative framework (see Section Technical Lemmas) are easier to prove. Together with the relative Riemann-Hilbert correspondence, our task is much simplified, in particular we no longer need to microlocalize.

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2. A short reminder on the relative Riemann-Hilbert correspondence

Below we summarize the background from [19], [20], [5], [6] we shall need in the sequel.

2.a. Holonomic and regular holonomic \( \mathcal{D}_{X \times S/S} \)-modules.

(a) We say that a \( p^{-1} \mathcal{O}_S \)-module is strict if it is flat over \( p^{-1} \mathcal{O}_S \).

(b) We recall that \( M \in \text{Mod}_{\text{coh}}(\mathcal{D}_{X \times S/S}) \) is holonomic if the characteristic variety \( \text{Char}(M) \) is contained in \( \Lambda \times S \), where \( \Lambda \) is analytic \( \mathbb{C}^* \)-conic lagrangian subset of \( T^* X \); we denote by \( \mathcal{D}_b^{\text{hol}}(\mathcal{D}_{X \times S/S}) \) the associated triangulated category whose objects are the bounded complexes with holonomic cohomologies.

(c) There is a well defined duality functor
\[
\mathcal{D} : \mathcal{D}_b^{\text{hol}}(\mathcal{D}_{X \times S/S}) \to \mathcal{D}_b^{\text{hol}}(\mathcal{D}_{X \times S/S})^{\text{op}}
\]
given by
\[
\mathcal{D} M := R\mathcal{H}om_{\mathcal{D}_{X \times S/S}}(M, \mathcal{D}_{X \times S/S} \otimes_{\mathcal{O}_{X \times S}} \Omega_{X \times S/S}^{-1})[d_X]
\]
where \( \Omega_{X \times S/S} \) denotes the sheaf of relative differential forms of maximal degree.

(d) \( \mathcal{D} \) is an involution, i.e. \( \mathcal{D} \mathcal{D} = \text{Id} \).

(e) We recall a tool introduced in [19], the holomorphic restriction to each fiber of \( p \):
\[
\forall s \in S, L_i^s(\cdot) := \cdot \otimes_{p^{-1} \mathcal{O}_S} p^{-1}(\mathcal{O}_S/\mathcal{J}_s)
\]
where \( \mathcal{J}_s \) is the maximal ideal of functions vanishing in \( s \).

(f) A Nakayama’s Lemma variation: Let \( M \in \mathcal{D}_b^{\text{hol}}(\mathcal{D}_{X \times S/S}) \) and assume that \( L_i^s M = 0 \) for each \( s_o \in S \). Then \( M = 0 \).

(g) Let \( M \) be an object of \( \mathcal{D}_b^{\text{hol}}(\mathcal{D}_{X \times S/S}) \). Then \( \mathcal{D} M \) is concentrated in degree zero and \( \mathcal{H}^0 \mathcal{D} M \) is strict if and only if \( M \) is itself concentrated in degree zero and \( \mathcal{H}^0 M \) is a strict \( \mathcal{D}_{X \times S/S} \)-module.
Remark 1. Note that $\rem$ does not preserve perversity.

(i) $\mathcal{D}_{\text{hol}}^b(D_{X \times S/S})$ is stable by duality.

(j) $\text{Mod}_{\text{hol}}(D_{X \times S/S})$ and $\text{Mod}_{\text{hol}}(D_{S \times S})$ are closed under taking extensions in $\text{Mod}(D_{X \times S/S})$ and subquotients in $\text{Mod}_{\text{coh}}(D_{X \times S/S})$.

2.b. $S$-constructibility. We say that a sheaf $L$ of $p^{-1}\mathcal{O}_S$-modules is $S$-locally constant coherent if, locally on $X \times S$, $L$ is isomorphic to $p^{-1}G$ where $G$ is an $\mathcal{O}_S$-coherent module. Such an $L$ is also called $S$-local system. We recall the following full triangulated subcategories of $\mathcal{D}^b(p^{-1}\mathcal{O}_S)$.

- An object $F \in \mathcal{D}^b(p^{-1}\mathcal{O}_S)$ is an object of $\mathcal{D}_{\text{C}}^b(p^{-1}\mathcal{O}_S)$ if there exists a $C$-analytic stratification $(X_\alpha)_{\alpha \in A}$ of $X$, such that $\forall j \in \mathbb{Z}, \forall \alpha \in A, \mathcal{H}^j(F)|_{X_\alpha \times S}$ is $S$-locally constant coherent. We say for short that $F$ is $S$-C-constructible.
- Replacing $C$-analyticity by subanalyticity with respect to the real analytic manifold $X_\mathcal{R}$ underlying $X$, we obtain the notion of $S - \mathcal{R}$-constructibility and the corresponding triangulated category $\mathcal{D}_{\mathcal{R}}^b(p^{-1}\mathcal{O}_S)$. $\mathcal{D}_{\mathcal{C}}^b(p^{-1}\mathcal{O}_S)$ is a full subcategory of $\mathcal{D}_{\mathcal{R}}^b(p^{-1}\mathcal{O}_S)$.
- If $F \in \mathcal{D}_{\mathcal{R}}^b(p^{-1}\mathcal{O}_S)$ then for each $x \in X$, $F|_{\{x\} \times S}$ belongs to $\mathcal{D}_{\text{coh}}^b(\mathcal{O}_S)$.
- There is a natural duality functor $D : \mathcal{D}_{\mathcal{R}}^b(p^{-1}\mathcal{O}_S) \to \mathcal{D}_{\mathcal{R}}^b(p^{-1}\mathcal{O}_S)^{\text{op}}$ which is an involution given by

$$DF = R\mathcal{H}om_{p^{-1}\mathcal{O}_S}(F, p^{-1}\mathcal{O}_S)[2d_X]$$

- $\mathcal{D}_{\mathcal{C}}^b(p^{-1}\mathcal{O}_S)$ is stable by duality.

2.c. A middle perversity $t$-structure on $\mathcal{D}_{\mathcal{C}}^b(p^{-1}\mathcal{O}_S)$. We consider the two subcategories $\mathcal{D}_{\mathcal{C}}^{\leq 0}(p^{-1}\mathcal{O}_S)$ and $\mathcal{D}_{\mathcal{C}}^{0}\mathcal{O}_S(p^{-1}\mathcal{O}_S)$ of $\mathcal{D}_{\mathcal{C}}^b(p^{-1}\mathcal{O}_S)$ defined as follows:

$F \in \mathcal{D}_{\mathcal{C}}^{\leq 0}(p^{-1}\mathcal{O}_S)$ (resp. $F \in \mathcal{D}_{\mathcal{C}}^{0}\mathcal{O}_S(p^{-1}\mathcal{O}_S)$) if for an adapted $\mu$-stratification $(X_\alpha)_{\alpha \in A}$, noting $i_\alpha : X_\alpha \hookrightarrow X$

$$\forall \alpha \text{ and } \forall j > - \dim(X_\alpha), \quad \mathcal{H}^j(i_\alpha^{-1}F) = 0$$

(resp.) $\forall \alpha \text{ and } \forall j < - \dim(X_\alpha), \quad \mathcal{H}^j(i_\alpha^1F) = 0$.

We say that $F$ of $\mathcal{D}_{\mathcal{C}}^b(p^{-1}\mathcal{O}_S)$ is perverse if $F \in \mathcal{D}_{\mathcal{C}}^{\leq 0}(p^{-1}\mathcal{O}_S)$ and $F \in \mathcal{D}_{\mathcal{C}}^{0}\mathcal{O}_S(p^{-1}\mathcal{O}_S)$, that is $F$ belongs to the heart of the $t$-structure defined above.

Remark 1. Note that $D$ is not $t$-exact for this $t$-structure, in particular it does not preserve perversity.

Theorem 2 ([20]). For a given object $F \in \mathcal{D}_{\mathcal{C}}^b(p^{-1}\mathcal{O}_S)$, $F$ and $DF$ are perverse if and only if $\forall s_0 \in S, Li_{s_0}^*(F)$ is perverse in $\mathcal{D}_{\mathcal{C}}^b(C_X)$. 


2.d. **Link with holonomicity.** We have the following link with holonomic $D_{X \times S/S}$-modules. Let us note $p\text{Sol}\,\mathcal{M} = R\mathcal{H}\text{om}_{D_{X \times S/S}}(\mathcal{M}, \mathcal{O}_{X \times S})[d_X]$ and $p\text{DR}\,\mathcal{M} = R\mathcal{H}\text{om}_{D_{X \times S/S}}(\mathcal{O}_{X \times S}, \mathcal{M})[d_X]$.

Then (cf. [4], [19], [20]):

- $\text{Sol, DR} : D_{\text{hol}}(D_{X \times S/S})$ take values in $D_{b,c}^b(p^{-1}\mathcal{O}_S)$ and $D^p\text{Sol} = p\text{DR} = p\text{Sol}\,D$.
- If $\mathcal{M} \in \text{Mod}_{\text{hol}}(D_{X \times S/S})$ then $p\text{DR}\,\mathcal{M}$ is perverse (cf. [3] Theorem 4.1).
- If $F$ is such that $DF$ is perverse then $R\mathcal{H}^S(F)$ is concentrated in degree zero (cf. [3] Th. 4.1). In particular, for any holonomic $D_{X \times S/S}$-module, $R\mathcal{H}^S(p\text{Sol}\,\mathcal{M})$ is concentrated in degree zero.
- Given $\mathcal{M} \in D_{b,c}^b(D_{X \times S/S})$, $\mathcal{M}$ and $D\mathcal{M}$ are strict $D_{X \times S/S}$-modules if and only if $p\text{Sol}\,\mathcal{M}$ and $p\text{DR}\,\mathcal{M} = D^p\text{Sol}\,\mathcal{M}$ are perverse.

2.e. **The functor $R\mathcal{H}^S$.** With the subanalytic tools developed in [17], [18], the functor $R\mathcal{H}^S$ was first introduced in [20], followed by [5] (case $d_S = 1$) and by [3] (general case). Kashivara’s functor $\mathcal{H}$ (cf. [3]) is recovered with $d_S = 0$. Below we give a short reminder of its construction and main results:

Let $\rho_S : X \times S \rightarrow X_{sa} \times S$ be the natural morphism of sites introduced in [18]. The functor $\rho_S^{-1}$ admits a left adjoint $\rho_S^\circ$ which is exact. We note $\mathcal{O}_{X_{sa} \times S}$ the relative subanalytic sheaf on $X_{sa} \times S$ associated in [18] to the subanalytic sheaf $\mathcal{O}_{X_{sa} \times S}^\circ$ on $(X \times S)_{sa}$ (introduced in [15], see also [22]).

The functor $R\mathcal{H}^S$ on $D^b(p^{-1}\mathcal{O}_S)_{\text{op}}$ is given by

$$R\mathcal{H}^S(*) := \rho_S^{-1} R\mathcal{H}\text{om}_{p^{-1}\mathcal{O}_S}(\rho_S^\circ(*), \mathcal{O}_{X_{sa} \times S}^\circ)[d_X]$$

**Theorem 3** ([20], [3], [6]). (a) $R\mathcal{H}^S$ induces an equivalence of categories: $D_{b,c}^b(p^{-1}\mathcal{O}_S)_{\text{op}} \rightarrow D_{\text{hol}}(D_{X \times S/S})$ compatible with duality.
(b) $F$ is perverse with a perverse dual if and only if $R\mathcal{H}^S(F)$ is strict and concentrated in degree zero.
(c) For $F \in D_{b,c}^b(p^{-1}\mathcal{O}_S)$ and $\mathcal{M} \in D_{\text{hol}}(D_{X \times S/S})$, we have a natural isomorphism in $D_{b,c}^b(p^{-1}\mathcal{O}_S)$

$$R\mathcal{H}\text{om}_{D_{X \times S/S}}(\mathcal{M}, R\mathcal{H}^S(F)[-d_X]) \sim R\mathcal{H}\text{om}_{D_{X \times S/S}}(\mathcal{M}, R\mathcal{H}\text{om}_{p^{-1}\mathcal{O}_S}(F, \mathcal{O}_{X \times S}))$$

2.f. **Topological aspects of $\mathcal{O}_S$.** $\mathcal{O}_S$ is a sheaf of complete bornological algebras (multiplicatively convex sheaf of Fréchet algebras over $S$). In the category of sheaves of complete bornological modules over $\mathcal{O}_S$ (denoted by $\text{Born}(\mathcal{O}_S)$), C. Houzel (cf. [7]) introduced a notion of tensor product $\widehat{\otimes} \mathcal{O}_S \cdot$.

To the latter one associates a family of functors $\widehat{\otimes} \mathcal{M}$ on the category of bornological vector spaces, depending functorially on $\mathcal{M} \in \text{Born}(\mathcal{O}_S)$ (cf. [26] Section 3.4). We have

\[
\mathcal{O}_{X \times S}|_x \times S \simeq \mathcal{O}_{X_x \times S} \widehat{\otimes} \mathcal{O}_S
\]

Then [4] shows that $\mathcal{O}_{X \times S}|_x \times S$ is a so-called FN-free as well as a DFN-free $\mathcal{O}_S$-module (cf. [26] page 25 for the definition and also [23]).
In particular, given another complex manifold \( Y \), we have
\[
\mathcal{O}_{X \times Y \times S}|_{\{x,y\} \times S} \simeq (\mathcal{O}_{X \times S}|_{\{x\} \times S}) \otimes_{\mathcal{O}_S} (\mathcal{O}_{Y \times S}|_{\{y\} \times S})
\]

3. Technical Lemmas

3.a. Complements on \( S\mathbb{R}\)-constructible sheaves. We refer to [13, Chapter VIII] for the background on constructibility.

Notation 4. For short we shall keep the notations \( p \) as well as \( p^{-1}\mathcal{O}_S \) without referring to the manifold \( X \) whenever there is no risk of ambiguity.

Let \( X \) and \( Y \) be complex manifolds. Let \( q_1 : X \times Y \times S \rightarrow X \times S \) be the first projection and \( q_2 : X \times Y \times S \rightarrow Y \times S \) be the second projection, which is illustrated by the following commutative diagram below.

\[
\begin{array}{ccc}
X \times Y \times S & \xrightarrow{q_1} & X \times S \\
q_2 \downarrow & & \downarrow p \\
Y \times S & \xrightarrow{p} & S
\end{array}
\]

Lemma 5. For any \( F \in \text{D}^b_{\mathbb{R}-c}(p^{-1}\mathcal{O}_S) \) on \( X \times S \) and any object \( \mathcal{G} \) of \( \text{D}^b(p^{-1}\mathcal{O}_S) \) on \( Y \times S \) the functorial morphism
\[
T(F) := q_1^{-1} R\mathcal{H}\text{om}_{p^{-1}\mathcal{O}_S}(F, p^{-1}\mathcal{O}_S) \otimes_{p^{-1}\mathcal{O}_S} q_2^{-1}\mathcal{G}
\]
\[
\xrightarrow{T' \mathcal{G}} R\mathcal{H}\text{om}_{p^{-1}\mathcal{O}_S}(q_1^{-1}F, q_2^{-1}\mathcal{G})
\]
is an isomorphism.

Proof. The proof is now simpler than that of Lemma B.3 of [12] since we dispose of the notion of microsupport and of its properties (cf. [13, Chap.V]). It is sufficient to check the isomorphism locally. Furthermore, arguing by induction on the length of \( F \), we may assume that \( F \) is in degree zero, that is, \( F \) is an \( S\mathbb{R} \)-constructible sheaf.

We recall the following result (cf. Lemma A.9 in the complete version of [10], https://arxiv.org/pdf/2203.05444.pdf):

Lemma 6. Let \( F \) be an \( S\mathbb{R} \)-constructible sheaf on \( X \times S \). Then there exist
- a locally finite covering \((U(\sigma))_{\sigma \in \Delta} \) of \( X \) by open subanalytic relatively compact subsets of \( X \),
- for each \( \sigma \in \Delta \) a coherent \( \mathcal{O}_S \)-module \( G_\sigma(F) \) on \( S \),
- and an epimorphism \( \bigoplus_{\sigma \in \Delta} \mathcal{O}_U(\sigma) \boxtimes G_\sigma(F) \rightarrow F \).

Let us assume for a moment that \( F = \mathbb{C}_U \boxtimes G \) for some open relatively compact subanalytic subset \( U \) of \( X \) and for some coherent \( \mathcal{O}_S \)-module \( G \). In that case, the proof of Lemma 6 is as follows. Regarding the left hand term of (7) we have a chain of isomorphisms:
\[
q_1^{-1} R\mathcal{H}\text{om}_{p^{-1}\mathcal{O}_S}(\mathbb{C}_U \boxtimes G, p^{-1}\mathcal{O}_S) \otimes_{p^{-1}\mathcal{O}_S} q_2^{-1}\mathcal{G}
\]
\[
\simeq q_1^{-1} R\mathcal{H}\text{om}(\mathbb{C}_{U \times S}, R\mathcal{H}\text{om}(p^{-1}G, p^{-1}\mathcal{O}_S)) \otimes_{p^{-1}\mathcal{O}_S} q_2^{-1}\mathcal{G}
\]
Lemma 7. Let $\mathcal{M} \in \mathcal{D}_{\text{hol}}^b(\mathcal{D}_{X \times S/S})$ Then we have a natural isomorphism

$$q_1^{-1} R\mathcal{H}\mathcal{O}\mathcal{M}(\mathcal{M}, \mathcal{O}_{X \times S}) \otimes q_1^{-1} R\mathcal{H}\mathcal{O}\mathcal{M}_{p^{-1}\mathcal{O}_S}(p^{-1}G, p^{-1}\mathcal{O}_S) \otimes^L q_1^{-1}\mathcal{O}_{Y \times S} \simeq q_1^{-1} R\mathcal{H}\mathcal{O}\mathcal{M}(\mathcal{M}, \mathcal{O}_{X \times S}) \otimes R\mathcal{H}\mathcal{O}\mathcal{M}_{p^{-1}\mathcal{O}_S}(p^{-1}G, q_2^{-1}\mathcal{O}_S) \simeq q_1^{-1} R\mathcal{H}\mathcal{O}\mathcal{M}(\mathcal{M}, \mathcal{O}_{X \times Y \times S})$$

Isomorphism (a) follows by [13, Prop. 5.4.14 (ii)] and isomorphism (b) follows by the coherence of $G$.

Similarly, the right hand term of (7) becomes isomorphic to

$$R\mathcal{H}\mathcal{O}\mathcal{M}(q_1^{-1}\mathcal{C}_{X \times S}, R\mathcal{H}\mathcal{O}\mathcal{M}_{p^{-1}\mathcal{O}_S}(p^{-1}G, q_2^{-1}\mathcal{O}_S))$$

We have

$$q_1^{-1} R\mathcal{H}\mathcal{O}\mathcal{M}(\mathcal{C}_{X \times S}, \mathcal{C}_{X \times S}) \simeq q_1^{-1} R\mathcal{H}\mathcal{O}\mathcal{M}(\mathcal{C}_{X \times Y \times S}, \mathcal{C}_{X \times Y \times S})$$

Thus, for $F = \mathbb{C}_U \boxtimes G$, Lemma 5 follows by [13, Prop. 5.4.14 (ii)].

As a consequence, Lemma 6 holds true for sheaves of the form

$$(*) \bigoplus_{T \in \Delta} \mathbb{C}_{U(T)} \boxtimes G_{a}(F).$$

We shall now prove the general case ($F \in \text{Mod}_{\mathbb{R}_{-c}}(p^{-1}\mathcal{O}_S)$) by a standard argument. The epimorphism of Lemma 6 induces the following exact sequence

$$0 \rightarrow F' \rightarrow K \rightarrow F \rightarrow 0$$

where $K$ has the form $(*)$, thus $K$ and $F'$ belong to $\text{Mod}_{\mathbb{R}_{-c}}(p^{-1}\mathcal{O}_S)$. We consider the associated distinguished triangles

$$T(F) \rightarrow T(K) \rightarrow T(F') \rightarrow +1$$

Thus (7) reads $T(K) \simeq T'(K)$ in $\mathbb{D}_{\mathbb{R}_{-c}}^b(p^{-1}\mathcal{O}_S)$. There exist integers $N < M$ only depending on $T, T'$ and $\mathcal{J}$ such that the $j$-cohomology groups of $T(\bullet), T'(\bullet)$, with $\bullet$ replaced by $F, F', K$ (see (7)), vanish for $j \notin [N, M]$. We have $\mathcal{H}^j T(K) \simeq \mathcal{H}^j T'(K)$ thus $\mathcal{H}^N T(F) \rightarrow \mathcal{H}^N T'(F)$ is injective (since $\mathcal{H}^{N-1} T(F') = 0 = \mathcal{H}^{N-1} T'(F')$). As $F$ is arbitrary, the same holds true for $F$ replaced by $F'$. By the Five Lemma it follows that $\mathcal{H}^N T(F) \simeq \mathcal{H}^N T'(F')$ and so $\mathcal{H}^N T(F') \simeq \mathcal{H}^N T'(F')$ again because $F$ is arbitrary. We then pursue recursively this argument which ends after a finite number of steps.

q.e.d.

3.b. A complement on relative holonomic modules. Let $X$ and $Y$ be complex manifolds and let us consider diagram (9).

Lemma 7. Let $\mathcal{M} \in \mathcal{D}_{\text{hol}}^b(\mathcal{D}_{X \times S/S})$ Then we have a natural isomorphism

$$q_1^{-1} R\mathcal{H}\mathcal{O}\mathcal{M}_{\mathcal{D}_{X \times S/S}}(\mathcal{M}, \mathcal{O}_{X \times S}) \otimes_{p^{-1}\mathcal{O}_S} q_2^{-1}\mathcal{O}_{Y \times S} \simeq R\mathcal{H}\mathcal{O}\mathcal{M}_{q_1^{-1}\mathcal{D}_{X \times S/S}}(q_1^{-1}\mathcal{M}, \mathcal{O}_{X \times Y \times S})$$
Proof. We adapt Proposition 1.4.3 of [12]. Since the morphism is well-defined, it is enough to prove that, for any \( x \in X, y \in Y \), it induces an isomorphism

\[
R\mathcal{H}om_{D_{X \times S/S}}(M, \mathcal{O}_{X \times S})|_{\{x\} \times S} \otimes \mathcal{O}_S \mathcal{O}_Y|_{\{y\} \times S} \\
\simeq R\mathcal{H}om_{q_1^{-1}D_{X \times S/S}}(q_1^{-1}M, \mathcal{O}_{X \times Y \times S})|_{\{(x,y)\} \times S}
\]

For any \( s \in S \), in a neighbourhood of \((x, s)\), we now replace \( M \) by a bounded locally free \( D_{X \times S/S} \)-resolution \((D_{X \times S/S}^k, p_k)_{k \in \mathbb{Z}} \to Q_{fS}M\). Then we may assume that \( R\mathcal{H}om_{D_{X \times S/S}}(M, \mathcal{O}_{X \times S})|_{\{x\} \times S} \) is quasi-isomorphic to the complex \((\mathcal{O}_{X \times S}^k|_{\{x\} \times S}, p_k^T)\) and that \( R\mathcal{H}om_{q_1^{-1}D_{X \times S/S}}(q_1^{-1}M, \mathcal{O}_{X \times Y \times S})|_{\{(x,y)\} \times S} \) is quasi-isomorphic to the complex \((\mathcal{O}_{X \times Y \times S}^k|_{\{(x,y)\} \times S}, p_k^T)\).

We have \( \mathcal{O}_{X \times S}^k|_{\{x\} \times S} \simeq \mathcal{O}_{X,x}^k \hat{\otimes} \mathcal{O}_S \) and \( \mathcal{O}_{X \times Y \times S}^k|_{\{(x,y)\} \times S} \simeq \mathcal{O}_{X \times Y,\{x,y\}} \hat{\otimes} \mathcal{O}_S \) and so \( \mathcal{O}_{X \times S}^k|_{\{x\} \times S} \) as well as \( \mathcal{O}_{X \times Y \times S}^k|_{\{(x,y)\} \times S} \) are FN-free \( \mathcal{O}_S \)-modules in the sense of [23].

Since \( R\mathcal{H}om_{D_{X \times S/S}}(M, \mathcal{O}_{X \times S})|_{\{x\} \times S} \) has \( \mathcal{O}_S \)-coherent cohomologies we are in conditions of applying Proposition 3.13 of [20], and, in view of [13], to conclude quasi isomorphisms

\[
(\mathcal{O}_{X \times Y \times S}^k|_{\{(x,y)\} \times S}, p_k^T)|_{\{(x,y)\} \times S} \simeq (\mathcal{O}_{X \times S}^k, p_k^T)|_{\{x\} \times S} \hat{\otimes} \mathcal{O}_S \mathcal{O}_Y|_{\{y\} \times S} \\
\simeq R\mathcal{H}om_{D_{X \times S/S}}(M, \mathcal{O}_{X \times S})|_{\{x\} \times S} \otimes \mathcal{O}_S \mathcal{O}_Y|_{\{y\} \times S} \quad \text{q.e.d.}
\]

4. Main Result

4.a. Statement and proof of the main result. Let \( \Delta \) denote the diagonal of \( X \times X \).

The canonical section of \( i_{\Delta \times S}^1 D_{\Delta \times S}^\infty (\mathcal{O}_{X \times X \times S}) \otimes \mathcal{O}_{X \times S} \mathcal{O}_X|_{\Delta \times S} \) corresponding to the global section 1 of \( D_{X \times S/S}^\infty \) allows to define an isomorphism of sheaves of rings

\[
D_{X \times S/S}^\infty \simeq i_{\Delta \times S}^{-1} B_{\Delta \times S}^\infty \mathcal{O}_{X \times X \times S} \otimes \mathcal{O}_{X \times S} \mathcal{O}_X|_{\Delta \times S} \\
\simeq i_{\Delta \times S}^{-1} R\Gamma_{\Delta \times S}(\mathcal{O}_{X \times X \times S}) \otimes \mathcal{O}_{X \times S} \mathcal{O}_X|_{\Delta \times S}[dx]
\]

Theorem 8. Let \( M \in D_{b, \text{fl}}(D_{X \times S/S}) \).

Let \( F = \text{Sol} M = R\mathcal{H}om_{D_{X \times S/S}}(M, \mathcal{O}_{X \times S}) \). Then we have a natural isomorphism in \( D_{b}(D_{X \times S/S}) \)

\[
M^\infty \simeq R\mathcal{H}om_{p^{-1}D_{X \times S/S}}(F, \mathcal{O}_{X \times S})
\]

Proof. In view of [20], we have isomorphisms

\[
D_{X \times S/S} \otimes_{D_{X \times S/S}} M \\
\simeq i_{\Delta \times S}^{-1} R\Gamma_{\Delta \times S}(\mathcal{O}_{X \times X \times S}) \otimes \mathcal{O}_{X \times S} \mathcal{O}_X|_{\Delta \times S}[dx] \otimes D_{X \times S/S} \mathcal{D} \mathcal{D} \mathcal{M} \\
\simeq i_{\Delta \times S}^{-1} R\Gamma_{\Delta \times S}(R\mathcal{H}om_{q_1^{-1}D_{X \times S/S}}(q_1^{-1} \mathcal{D} \mathcal{M}, \mathcal{O}_{X \times X \times S}))[2dx]
\]

According to Lemma 7, we have

\[
R\mathcal{H}om_{q_1^{-1}D_{X \times S/S}}(q_1^{-1} \mathcal{D} \mathcal{M}, \mathcal{O}_{X \times X \times S})
\]
According to Lemma 5 with $X$ a natural isomorphism $\text{Id}$ is an immediate consequence of Theorem 8 since $\text{Id}$ follows by the sequence of isomorphisms defined in Definition 10. If $X \otimes$ of the functors $\mathcal{M}$ there exists a coherent ideal $J$ which is regular holonomic.

Lemma 11. $\mathcal{M}$ is a holonomic $\mathcal{D}_{X \times S/S}$-module, we denote by $\mathcal{M}_{\text{reg}}$ the subsheaf of $\mathcal{M} = \mathcal{O}_{X \times S}$ of local sections $u$ satisfying the following condition: there exists a coherent ideal $\mathcal{J}$ in $\mathcal{D}_{X \times S/S}$ such that $\mathcal{J}u = 0$ and $\mathcal{D}_{X \times S/S}/\mathcal{J}$ is regular holonomic.

Lemma 11. $\mathcal{M}_{\text{reg}}$ is a $\mathcal{D}_{X \times S/S}$-module.
Proof. The proof is similar to that in Prop.1.1.20 in [12]. If \( u \) is a local section of \( M_{\text{reg}} \), let \( \mathcal{J} \) be a left ideal of \( \mathcal{D}_{X \times S/S} \) as in Definition [10] and let \( P \in \mathcal{D}_{X \times S/S} \); then the left ideal \( \mathcal{J}' \) of \( \mathcal{D}_{X \times S/S} \) such that \( QP \in \mathcal{J}' \) is coherent and \( \mathcal{D}_{X \times S/S}/\mathcal{J}' \) is isomorphic to a coherent \( \mathcal{D}_{X \times S/S} \)-submodule of \( \mathcal{D}_{X \times S/S}/\mathcal{J} \) hence, in view of (j) of 2.a, it is regular holonomic so that conditions on Definition [10] are satisfied by \( Pu \).

q.e.d.

Clearly the correspondence

\[
M \in \text{Mod}_{\text{hol}}(\mathcal{D}_{X \times S/S}) \mapsto M_{\text{reg}} \in \text{Mod}_{\text{hol}}(\mathcal{D}_{X \times S/S})
\]

defines a left exact functor.

**Theorem 12.** (a) Let \( N \in \text{Mod}_{\text{hol}}(\mathcal{D}_{X \times S/S}) \). Then \( N = N_{\text{reg}} \).

(b) Let \( N \in \text{Mod}_{\text{hol}}(\mathcal{D}_{X \times S/S}) \). Then \( N_{\text{reg}} \) is a regular holonomic \( \mathcal{D}_{X \times S/S} \)-module isomorphic to \( \text{RH}^S(\mathcal{P}\text{Sol}N) \). In particular \( N^\infty \simeq N_{\text{reg}}^\infty \).

**Proof.** (a) By the assumption of regularity, we derive a natural inclusion \( N \subset N_{\text{reg}} \). Let us now prove the inclusion \( N_{\text{reg}} \subset N \). Let \( u \) be a local section of \( N_{\text{reg}} \) and let \( \mathcal{J} \) be a left coherent ideal of \( \mathcal{D}_{X \times S/S} \) such that \( \mathcal{J}u = 0 \) and such that \( \mathcal{D}_{X \times S/S}/\mathcal{J} \) is regular holonomic. We thus deduce a natural morphism \( \phi : \mathcal{D}_{X \times S/S}/\mathcal{J} \to N^\infty \) as the composition of \( \mathcal{D}_{X \times S/S}/\mathcal{J} \to \mathcal{D}_{X \times S/S}u \hookrightarrow N^\infty \). Applying Corollary [9] b) to the cohomologies of degree zero with \( M = \mathcal{D}_{X \times S/S}/\mathcal{J} \), \( \phi \) factors through \( N \) thus \( \mathcal{D}_{X \times S/S}u \subset N \).

(b) According to Theorem [8] we have a \( \mathcal{D}_{X \times S/S} \)-linear isomorphism

\[
\Phi : N^\infty \simeq \text{RH}^S(\mathcal{P}\text{Sol}N)^\infty
\]

In view of (a) we conclude by a similar argument that \( \Phi(N_{\text{reg}}) \) is contained in \( \text{RH}^S(\mathcal{P}\text{Sol}N) \). Similarly, using \( \Phi^{-1} \), we conclude that \( N_{\text{reg}} \) contains \( \Phi^{-1}(\text{RH}^S(\mathcal{P}\text{Sol}N)) \). Thus \( \Phi \) provides the desired \( \mathcal{D}_{X \times S/S} \)-isomorphism.

q.e.d.

4.b. **Example.** We will assume that \( d_S = 1 \). Our goal is to explicit \( (\bullet)^\infty \) in the case of the relative hermitian duality(cf. [21]) by proving the relative variant of Remark 2.1 in [10].

We denote by \( \mathcal{D}_{bX \times S} \) the sheaf of distributions on the really analytic manifold \( X_\mathbb{R} \times S_\mathbb{R} \) underlying \( X \times S \) and by \( \mathcal{D}_{bX \times S/S} \) the subsheaf of \( \mathcal{D}_{bX \times S} \) of germs of distributions holomorphic along \( S \). We call \( \mathcal{D}_{bX \times S/S} \) the sheaf of relative distributions. We denote by \( \overline{X} \) the complex conjugate manifold of the manifold \( X \). We recall the main results (Theorem 2) in [21].

(a) The relative Hermitian duality functor

\[
C^S_{X,\overline{X}}(\bullet) := \text{RH}_0m_{X \times S/S}(\bullet, \mathcal{D}_{bX \times S/S})
\]

induces an equivalence

\[
C^S_{X,\overline{X}} : \mathcal{D}^b_{\text{hol}}(\mathcal{D}_{X \times S/S}) \xrightarrow{\sim} \mathcal{D}^b_{\text{hol}}(\mathcal{D}_{X \times S/S})^{\text{op}}
\]

(b)

\[
C^S_{X,\overline{X}} \circ C^S_{X,\overline{X}} \simeq \text{Id}.
\]
We note that, by definition of $\mathcal{O}_{\overline{X}, X}$, we regard $X \times S$ as usual we regard $X \times \overline{X}$ as a complexification of $X$. Hence

$$R\mathcal{H}\text{om}_{\mathcal{D}^{\infty}_{X \times S/S}}(M, \mathcal{O}_{X \times \overline{X} \times S})[2d]$$

Let $q_1$ denote the projection $X \times \overline{X} \times S \rightarrow X \times S$ and let $q_2$ denote the projection $X \times \overline{X} \times S \rightarrow \overline{X} \times S$. We have

$$R\mathcal{H}\text{om}_{\mathcal{D}^{\infty}_{X \times S/S}}(M, \mathcal{O}_{X \times \overline{X} \times S})[2d]$$

where $(a')$ follows by Lemma 7, $(b')$ follows by Lemma 1, $(c')$ follows by a similar argument as in $(b)$, $(d')$ follows by $(c)$ and $(e')$ follows by Theorem 8.
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