Wilson loops in large N QCD on a sphere

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Abstract.

Wilson loop averages of pure gauge QCD at large N on a sphere are calculated by means of Makeenko-Migdal loop equation.
The exact solvability of QCD$_2$ was noticed long time ago by A.A.Migdal\cite{Migdal}. To realize it, one can integrate over $U(N)$ link variables using only the orthogonality of group characters.

Following this, an exact solution of QCD$_2$ on an arbitrary 2-manifold was obtained in ref. \cite{ref2}.

In particular, the partition function for a closed surface of a genus $g$ and of area $A$ (coupling constant absorbed into area) has the form:

$$Z_g(A) = \sum_R d_R^{2(1-g)} \exp\left(-\frac{A}{2N}C_R\right)$$  \hspace{1cm} (1)

where $d_R$ is the dimension of an $R$-th irreducible representation,

$$d_R = \prod_{1 \leq i < j \leq N} \left(1 + \frac{n_i - n_j}{j - i}\right)$$  \hspace{1cm} (2)

$C_R$ is the value of the second Casimir operator,

$$C_R = \sum_{i=1}^{N} n_i(n_i + N + 1 - 2i)$$  \hspace{1cm} (3)

where $n_i$ are highest weight components of $R \equiv \{n_1 \geq n_2 \geq \ldots \geq n_N\}$.

The Wilson loop average has the form \cite{ref2}:

$$W_g(C) = \sum_{R_1,\ldots,R_M} \Phi_{R_1\ldots R_M} \prod_{k=1}^{M} d_R^{2(1-g_k)} \exp\left(-\frac{A_k}{2N}C_{R_k}\right)$$  \hspace{1cm} (4)

where $M$ is the number of windows, $A_k$’s are their areas, $g_k$ is the “genus per window” and the coefficient $\Phi_{R_1\ldots R_M}$ is the $U(N)$ group factor related to a form of the given contour \cite{ref2} (see ref. \cite{ref2} for details).

Recently, the partition function (11) has been reexpanded over $1/N$ and at large $N$ it has been interpreted as a sum over branched coverings \cite{ref3}.

In order to develop a large $N$ technique appropriate for calculations, say, of loop averages, it is tempting to rewrite the partition function (11) at large $N$.

\footnote{In particular, for a contour without self-intersections, this factor is the product, taken over all pairs of neighboring windows $<ij>$, of Wigner coefficients $\Phi_{R_i,R_j}$ \cite{ref12} below.}

\footnote{The similar interpretation has been known for the Wilson loop operators as well \cite{ref2}.}
as a path integral over continuous Young tables [3]. In the paper of Douglas
and Kazakov [4], in the framework of this approach, a third order phase
transition on a sphere was found. Recently, Boulatov has calculated the
Wilson loop average for the simplest contour [7]. The goal of present paper
is the calculation of an arbitrary Wilson loop average at large N on a sphere.
First, we repeat briefly the steps of [3]-[7]. In the spherical case, \( g = 0 \), the
sum becomes divergent at small areas. Hence, in the large \( N \) limit, a
non-trivial saddle-point should exist. Let us introduce the continuous
function [5],

\[
h(x) = \lim_{N \to \infty} \frac{1}{N} \left( i - \frac{N}{2} - n_i \right); \quad x = \frac{i}{N} - \frac{1}{2} \tag{5}
\]

Then the saddle-point equation takes the form

\[
\frac{A}{2} h = \int \frac{dy \rho(y)}{h - y} \tag{6}
\]

Here,

\[
\rho(h) = \frac{dx}{dh} \tag{7}
\]

obeys the inequality

\[
\rho(h) \leq 1 \tag{8}
\]

If eq.(8) is ignored, then the solution of eq.(8) is the semi-circle distribution,

\[
\rho(h) = \frac{1}{\pi} \sqrt{A - A^2 h^2} \tag{9}
\]

which is valid for the areas \( A < \pi^2 \). For the areas \( A > \pi^2 \), the inequality (8)
is crucial and the solution of eq.(8) is [4]:

\[
\rho(h) = \begin{cases} 
-\frac{2}{\pi a h} \sqrt{(a^2 - h^2)(h^2 - b^2)} \Pi_1 \left( -\frac{b^2}{h^2}, \frac{b}{a} \right) & \text{for } -a < h < -b \\
1 & \text{for } -b < h < b \\
\frac{2}{\pi a h} \sqrt{(a^2 - h^2)(h^2 - b^2)} \Pi_1 \left( -\frac{b^2}{h^2}, \frac{b}{a} \right) & \text{for } b < h < a
\end{cases} \tag{10}
\]
where $\Pi_1(x, k)$ is the complete elliptic integral of the third kind with the modulus $k = \frac{b}{a}$ and parameters are to be determined from the equations

$$a(2E - k'^2K) = 1 \quad \quad aA = 4K \quad \quad (11)$$

At the critical value, $A_c = \pi^2$, the third order phase transition takes place.

Since the saddle-point solution is known in both phases, one can calculate the Wilson loop averages (4) at large $N$ on a sphere.

The simplest Wilson loop on a sphere corresponds to $M = 2$ and $g_1 = g_2 = \frac{1}{2}$ (see (4)),

$$W(A_1, A_2) = \frac{1}{NZ_0(A)} \sum_{R,S} d_R d_S \Phi_{RS} \exp\left(-\frac{A_1}{2N} C_R - \frac{A_2}{2N} C_S\right) \quad \quad (12)$$

where

$$\Phi_{RS} = \int dU \chi_R(U) \chi_S(U^\dagger) \text{tr} (U) \quad \quad (13)$$

is the multiplicity of an irreducible representation $S$ in the tensor product of $R$ with the fundamental representation $f$; $A = A_1 + A_2$ is the total area.

The expression for this quantity valid for both phases is [4]:

$$W(A_1, A_2) = \oint_C \frac{dh}{2\pi i} e^{A_1 h - f(h)} = \oint_C \frac{dh}{2\pi i} e^{-A_2 h + f(h)} \quad \quad (14)$$

where

$$f(h) = \int \frac{dy}{h - y} \quad \quad (15)$$

and the contour $C$ encircles the cut of $f(h)$ (contour $\bar{C}$ encircles the cut in the opposite direction). Obviously, $W(A_1, 0) = W(0, A_2) = 1$.

In the weak coupling phase, $f(h) = \frac{A}{2} h - \sqrt{\frac{A^2 h^2}{4} - A}$, and in terms of the new variable $z = \sqrt{\frac{A}{i f}}$ the integral (14) takes the form

$$W_{wc}(A_1, A_2) = \frac{1}{i \sqrt{A}} \oint_C \frac{dz}{2\pi i} \left(1 + \frac{1}{z^2}\right) e^{i \alpha z + i \frac{A}{z}} \quad \quad (16)$$
where $\alpha = \frac{A_1}{\sqrt{A}}$ and $\beta = \frac{A_2}{\sqrt{A}}$. Expanding the exponential and taking the residue at zero, one find

$$W_{wc}(A_1, A_2) = \frac{1}{i\sqrt{A}} \oint_C dz \left( 1 + \frac{1}{z^2} \right) \sum_{n=0}^{\infty} \frac{(2n+1)!}{n!(n+1)!} \left( \alpha^{n+1} \beta^n z + \alpha^n \beta^{n+1} \frac{1}{z} \right)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)!} \left( \alpha \beta \right)^n = \sqrt{\frac{A_1 + A_2}{A_1 A_2}} J_1 \left( \sqrt{\frac{A_1 A_2}{A_1 + A_2}} \right) .$$

(17)

In what follows we will also need $A \to \infty$ limit of (14). As it has been shown in ref. \[7\], the function (15) in this case takes the form $f(x) = \log \frac{2x+1}{2x-1}$. For example, in the case $A_1 = \infty$, we have

$$W_{sc}(\infty, A_2) = \frac{1}{2} \int_0^{\infty} dx \frac{x - 1}{x} e^{-\frac{A_2}{2x-1}} = e^{-\frac{A_2}{2}}$$

(18)

i.e., standard QCD$_2$ area law.

In principle, all set of (4) can be calculated in the same fashion. However, since an expression for simple loop (14) is known, all others Wilson loops can be derived easily from the loop equation.

The equation for Wilson loops derived by Yu.Makeenko and A.Migdal a long time ago \[8\], has especially simple form in two dimensions \[1\]:

$$\hat{R}_i W_n(C) = W_q(C^1_i) W_{n-q-1}(C^2_i) \quad i = 1, \ldots, n$$

(19)

where $n$ is a number of self-intersections of contour $C$. Each equation of system (19) corresponds to given $i$-th point of self-intersection of $C$ ($C^1_i$ and $C^2_i$ are results of disconnection of contour $C$ in this point)\[3\]. Here, differential operator $\hat{R}_i$ acting in $i$-th self-intersection point is the linear combination of derivatives with respect to areas of windows $i_1, ..., i_4$ neighboring at this point (in what follows, we use the notation $\partial_k \equiv \frac{\partial}{\partial A_k}$),

$$\hat{R}_i = \partial_{i_1} - \partial_{i_2} + \partial_{i_3} - \partial_{i_4}$$

(20)

where different signs correspond to windows having common boundary \[4\].

\[3\] Actually, on a sphere (as well as on any compact surface), the number of independent equations of the system (19) can be less than $n$. Nevertheless, it turns out that in the spherical case the system (19) is sufficient to derive a complete set of loop averages.

\[4\] We do not fix the total sign here and below since it is always can be restored from the condition $W(0, ..., 0) = 1$. 

5
If all Wilson averages for loops with number of self-intersection less than $n$, are known, then we know an explicit form of RHS of (19). Solving recurrently the system (19) we get all set of loop averages.

Starting point of (19) is equation for 8-like contour drawn on Fig.2. In this case, the RHS of (19) has the form of product of two simple loops, $W_0$ (eq.(14)), which could not be derived from the Makeenko-Migdal equation. The equation, corresponding to Fig.2, has the form

$$
\hat{R} W_1(A_1, A_2, A_3) = \oint \frac{dx}{2\pi i} \oint \frac{dy}{2\pi i} e^{A_1 x + A_2 y - f(x) - f(y)}
$$

(21)

where

$$
\hat{R} = \partial_1 + \partial_2 - 2\partial_3
$$

(22)

The natural anzatz is:

$$
W_1(A_1, A_2, A_3) = \oint \frac{dx}{2\pi i} \oint \frac{dy}{2\pi i} e^{A_1 x + A_2 y - f(x) - f(y)} F
$$

(23)

Then, function $F = F(x, y; A_1, A_2, A_3)$ should satisfy the equation:

$$
(\hat{R} + x + y) F = 1
$$

(24)

The solution of (24) is:

$$
F = \frac{1 + e^{-(x+y)(eA_1 + A_3)}}{x + y} \xi(A_1, A_2, A_3)
$$

(25)

where function $\xi(A_1, A_2, A_3)$ should satisfy the equation:

$$
\hat{R} \xi = 0
$$

(26)
and $c$ is arbitrary constant. An explicit form of function $\xi$ and value of constant $c$ should be defined from obvious conditions:

\[ W_1(0, A_2, A_3) = W_0(A_2, A_3) \quad , \quad (27) \]

\[ W_1(A_1, 0, A_3) = W_0(A_1, A_3) \quad , \quad (28) \]

which fix unambiguously the function $F$. The result is:

\[ W_1(A_1, A_2, A_3) = \oint \frac{dx}{2\pi i} \oint \frac{dy}{2\pi i} e^{A_1x + A_2y - f(x) - f(y)} \quad x + y \quad (29) \]

Let us notice that this expression can be equally rewritten as

\[ W_1(A_1, A_2, A_3) = P \oint \frac{dx}{2\pi i} \oint \frac{dy}{2\pi i} e^{A_1x + (A_1 + A_3)y - f(x) - f(y)} \quad x - y \quad (30) \]

where $P$ means that integration over $x$ should be performed after integration over $y$. The equivalence between (29) and (30) corresponds to the fact that the contour drawn on Fig.2 coincides with one drawn on Fig.3.

The known answers for a plane [9]-[11] are follow from (29) and (30) as well:

\[ W_1(A_1, A_2, \infty) = e^{-\frac{A_1 + A_2}{2}} \quad (31) \]

\[ W_1(A_1, \infty, A_3) = e^{-A_1 - \frac{A_3}{2}} (1 - A_1) \quad (32) \]

\[ W_1(\infty, A_2, A_3) = e^{-A_2 - \frac{A_3}{2}} (1 - A_2) \quad . \quad (33) \]

In the case of two self-intersections there are two different types of contours (Fig.4).

Since $W_0$ and $W_1$ are known, we are able to write down explicitly an equation for $W_2$’s. For example, for contour Fig.4 (a), we have

\[ \hat{R}_1 W_2(A_1, A_2, A_3, A_4) = W_0(A_1, A_2 + A_3 + A_4) W_1(A_2, A_3, A_1 + A_4) \quad (34) \]

\[ \hat{R}_2 W_2(A_1, A_2, A_3, A_4) = W_0(A_3, A_1 + A_2 + A_4) W_1(A_1, A_2, A_3 + A_4) \quad (35) \]

where

\[ \hat{R}_1 = \partial_1 + \partial_2 - 2\partial_4 \quad , \quad (36) \]
\[ \hat{R}_2 = \partial_2 + \partial_3 - 2\partial_4. \]  

(37)

Repeating the steps of previous case, we derive

\[ W_2(A_1, ..., A_4) = \oint dx_1 e^{-f(x_1)} \ldots \oint dx_3 e^{-f(x_3)} \frac{e^{A_1 x_1 + A_2 x_2 + A_3 x_3}}{(x_1 + x_2)(x_2 + x_3)}. \]  

(38)

while for contour Fig.4 (b), the answer is

\[ W_2(A_1, ..., A_4) = \oint dx_1 e^{-f(x_1)} \ldots \oint dx_3 e^{-f(x_3)} \frac{e^{A_1 x_1 + (A_2 + A_3) x_2 + A_4 x_3}}{(x_1 + x_2)(x_3 - x_2)}. \]  

(39)

It is also instructive to consider one more example, the contours where pair (or pairs) of windows are separated by more than one self-intersection point. The first example of such a contour we match while consider the case of three self-intersections. The contour drawn on the Fig.5 has two windows, 1 and 4, separated by three self-intersection points. In this case, the solution of loop equation is

\[ W_3(A_1, ..., A_5) = \oint dx_1 \oint dy_2 e^{-f(x)} e^{A_1 x + A_4 y} (e^{A_2 (x+y)} + e^{A_3 (x+y)} - e^{(A_2 + A_3) (x+y)}). \]  

(40)

Let us notice that the structure of expression under integral in (40) is in one to one correspondence to structure of the plane solution which we easily reproduce from (41):

\[ W_3(A_1, ..., A_4, \infty) = e^{-\frac{A_1 + A_4}{2}} (e^{-A_2} + e^{-A_3} - e^{-A_2 - A_3}). \]  

(41)

Now, following previous consideration, we are able to formulate general solution for arbitrary \( W_n(C) \):

\[ W_n(C) = P \oint dx_1 \ldots \oint dx_n \prod_{k=1}^m e^{A_k x_k - f(x_k)} \prod_{\langle ij \rangle} \frac{1}{x_i \pm x_j} \sum_{q=1}^{n_{ij}>1} (-1)^q e^{A_{ij}^q (x_\pm x_j)} \]  

(42)

where: \( m \) is number of windows of the disk topology appearing as a result of disconnection of \( C \) at all self-intersection points (we associate variables
$x_k$ with all such a windows) $\mathcal{A}_k$ is an area of such a window the second product goes over all pairs $<ij>$ of windows from this set which are separated by (at least one) self-intersection point; minus sign of “propagator” $\frac{1}{x_i+x_j}$ corresponds to the case of self-overlapping, when $i$-th window is internal with respect to $j$-th one; $P$-symbol means that, in the latter case, integration over $x_j$ should be performed before integration over $x_i$; if $i$-th and $j$-th windows are separated by $n_{ij} > 1$ self-intersection points then the sum goes over all disconnections at such points and $\mathcal{A}_{ij}^q$ is the area of the overlapping region appeared as result of $q$-th disconnection (see eq.5).

Eq. (12) is the answer for arbitrary Wilson loop average in large $N$ pure QCD on a sphere. Let us emphasize that it is valid in both weak and strong coupling phases. A complete information about phase transition is contained in the function $\rho(x)$ ($f(x)$ as well). This function is given by eq.(9) in the weak coupling phase and by eq.(10) in the strong coupling phase.

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5 Though choice of $x_k$ is not unambiguous (we can attach it to “internal” or to “external” regions of the window), it doesn’t lead to ambiguity in the answer.
6 It can be equal to sum of areas $A_k$ of several original windows as we have shown in (31) and (39).
7 Actually, $n_{ij}$ is the odd number.
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