The conformally invariant Laplace-Beltrami operator and factor ordering

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Abstract

In quantum mechanics the kinetic energy term for a single particle is usually written in the form of the Laplace-Beltrami operator. This operator is a factor ordering of the classical kinetic energy. We investigate other relatively simple factor orderings and show that the only other solution for a conformally flat metric is the conformally invariant Laplace-Beltrami operator. For non-conformally-flat metrics this type of factor ordering fails, by just one term, to give the conformally invariant Laplace-Beltrami operator.

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Factor ordering is a complication that has bedeviled quantum mechanics since its inception. The modern literature on this subject, while considerable, seems, outside of elementary texts, to be mostly concentrated in the field of canonical quantum gravity, where many factor orderings of the momentum term of the Wheeler-DeWitt equation have been proposed. These proposals go back to DeWitt [1], and we will mention only a few articles that are relevant to the problem in ordinary quantum mechanics. Komar [2], basing his arguments on Pauli’s work on quantum mechanics [3] suggests that the factor ordering of momentum terms must be equivalent to using momentum vectors lying along the Killing vectors of the configuration space (this assumes that the space admits a Riemannian metric). Many factor orderings of the Wheeler-DeWitt momentum terms exist, several for quantum cosmology, two important proposals being those of Misner [4] and Hartle and Hawking [5]. The main point is that it is never entirely clear when one passes from some product of classical quantities to the same product of their quantum analogues, exactly which ordering of the product to take, and there is an even more difficult problem, whether to interpolate functions of coordinate operators between momentum operators in such a way that the classical limit is preserved. This last point can be a dangerous procedure. This type of factor ordering can be used to transform the Hamiltonian of a free particle into that of a harmonic oscillator [6], generating a discrete spectrum from a continuous one. That is, if one writes the quantum analogue of a one-dimensional free particle Hamiltonian, \[ H = \frac{p^2}{2m} \], one can choose either

\[ \hat{H}\psi = -\frac{1}{2m} \partial_x^2 \psi , \tag{1} \]

or

\[ \hat{H}\psi = -\frac{1}{2m} f(x) \frac{1}{h(x)} \partial_x \left[ f \partial_x (h \psi) \right] , \tag{2} \]

where \( f \) and \( h \) are arbitrary functions. If we now choose \( f = 1/h^2 \) and \( h = \exp(-\frac{m\omega^2}{2}x^2) \), we have

\[ \hat{H} = -\frac{\hbar^2}{2m} \partial_x^2 + \frac{1}{2} m\omega^2 x^2 + \frac{\omega}{2} , \tag{3} \]

a harmonic oscillator with a slightly shifted spectrum.

There are, of course, many possible factor orderings of the original free particle Hamiltonian, including, in general, any algebraic combination of the commutator \([\hat{x}, \hat{p}_x]\) (although one usually prefers to keep the Hamiltonian as a second order differential operator). The usual solution to this problem is to exploit experimental evidence to exclude all but one (or a very reduced set) of the possible factor orderings.
One of the most common factor ordering problems is not usually regarded as being related to factor ordering. In a space of dimension $n$ parametrized by non-Cartesian coordinates (which may or may not be flat), the usual classical expression for the kinetic energy term of a one-particle Hamiltonian for a phase space $(p_a, q^a)$ is

$$\frac{1}{2m}g^{ab}p_ap_b, \quad a, b = 1, \ldots, n,$$

(4)

g_{ab} = g_{ab}(q^c)$. This may be converted to an operator by taking $\hat{p}_a = -i\partial/\partial q^a$. The simplest factor ordering, $g^{ab}\hat{p}_a\hat{p}_b$ does not seem to give the correct Hamiltonian in most cases. For example, for spherical coordinates in a flat, three-dimensional space, this factor ordering gives

$$-\frac{1}{2m} \left[ \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right],$$

(5)

and this expression, used in the Hamiltonian of a hydrogen atom, gives an energy spectrum that, for the magnetic quantum number, $m$, equal to zero is

$$E_{nlm} = \frac{me^4}{2[n - l - 1/2 + \sqrt{l^2 + 1/4}]^2},$$

(6)

and for $|m| > 0$ is

$$E_{nlm} = \frac{me^4}{2[n - l - 1/2 + \sqrt{l + \frac{1}{2}(1 + \sqrt{1+4m^2})^2 + 1/4}]}$$

(7)

with no restriction that $|m|$ be less than $l$. These expressions are only in accord with experiment for $(m = 0)$ s-states. The wave functions are similar to the usual ones, but for $m = 0$ the Legendre polynomials are replaced by Chebyshev polynomials and the radial functions have the form $e^{-r/a_0}r^{(1+2\sqrt{l^2+1/4})/2}L_{n-l-1}^{(2\sqrt{l^2+1/4})}(r/a_0)$, $a_0$ the Bohr radius and $L_{n-l-1}^{(\alpha)}$ Laguerre polynomials. For $m \neq 0$ the angular functions are based on $e^{im\varphi}(\sin \theta)^{(1+\sqrt{1+4m^2})/2}G_{l+1/2}^{(1+\sqrt{1+4m^2})/2}(\cos \theta)$, $G_{\lambda}^{\sigma}$ Gegenbauer polynomials and radial functions that are proportional to

$$e^{-r/a_0}r^{(1+\sqrt{1+4m^2})/2}L_{n-l-1}^{(2\sqrt{l+\frac{1}{2}(1+\sqrt{1+4m^2})^2+1/4})}(r/a_0)$$

(8)

The usual solution to this problem is to write the operator form of (4) in terms of a Laplace-Beltrami operator,

$$\frac{1}{2m} \frac{1}{\sqrt{g}} \hat{p}_a (\sqrt{gg^{ab}}) \hat{p}_b,$$

(9)
which is clearly a factor ordering of (11). This factor ordering, which is invariant under changes of the configuration variables $q^c$, has been very successful in constructing Hamiltonians that are in accord with experiment in many cases.

For a space that is not flat there is another coordinate-invariant operator besides the Laplace-Beltrami operator that is sometimes used as the quantum version of $g^{ab}p_ap_b$, the conformally invariant Laplace-Beltrami operator,

$$
\frac{1}{\sqrt{g}}\hat{p}_a(\sqrt{g}g^{ab}\hat{p}_b) - \frac{n-2}{4(n-1)}R ,
$$

(10)

(for $n > 1$), where $R$ is the Ricci scalar or scalar curvature. This curvature, as a function of the metric $g_{ab}$ only, is

$$
R = -g^{ab}\frac{g_{ab}}{g} + \frac{3}{4}g^{ab}g_{ac}g_{bc} - g^{ab}\frac{g_{ab}}{g} - \frac{1}{2}g_{ab}g_{ra,rb}g^{lr} + \frac{1}{4}g_{ab}g_{ra,rb}g^{lr} .
$$

(11)

It is interesting to note that over the years there has been some interest in relating the operator (10) to time reparametrization invariance. In a series of articles going back to Misner (4), (see Refs. (7) and (8)), a possible action for a relativistic particle moving in a curved background (and the similar cosmological minisuperspace actions) have a Hamiltonian constraint multiplied by a “lapse function” $N$ that serves as a Lagrange multiplier. If one rescales $N$ by an arbitrary function of the coordinates and insists that the quantum theory generated by the action be invariant under this rescaling, the momentum part of the quantum Hamiltonian constraint is necessarily (11), where $n$ is either the dimension of the space in which the particle moves or the dimension of the minisuperspace.

Since the conformally invariant Laplace-Beltrami operator has a similar relationship to the ordinary Laplace-Beltrami operator (9) as that between (11) and (2), we can ask what operators can be constructed from (11) that are similar to that used to construct (2) and whether (10) is included in this set of operators, that is, investigating factor orderings of the type

$$
g^{ab}p_ap_b \rightarrow \frac{1}{H(g_{cd})}\partial_a[A(g_{cd})\partial_bB(g_{cd})] ,
$$

(12)

where $H(g_{cd}), A(g_{cd}), B(g_{cd})$ are arbitrary functions of the metric with the constraint $AB = g^{ab}H$. The simplest functions that we can use are related to powers of the determinant $g$ of $g_{ab}$. One possibility is

$$
g^{ab}p_ap_b \rightarrow -\frac{1}{\sqrt{g}g^{\tilde{a}+\tilde{b}}\partial_a[g^{\tilde{a}}\sqrt{g}g^{ab}\partial_bg^{\tilde{b}}]} .
$$

(13)
In the general case no choice of $\bar{\alpha}$ and $\bar{\beta}$ can give an expression that includes (10), but if we consider only conformally flat metrics, $g_{ab} = \phi^2 \delta_{ab}$, we find that

$$\det g_{ab} = \phi^{2n} = g,$$

so $g_{ab} = g^{1/n} \delta_{ab}$, $g^{ab} = g^{-1/n} \delta_{ab}$, and the scalar curvature is

$$R = -\left(1 - \frac{1}{n}\right) g^{-1/n} \left[ \frac{\partial^2 g}{g} - \left( \frac{1}{2n} - \frac{3}{4} \right) \frac{(\partial g)^2}{g^2} \right],$$

where $\partial^2 g/g \equiv (g_{a,b}/g) \delta_{ab}$, $(\partial g)^2/g^2 \equiv (g_a g_b/g^2) \delta_{ab}$.

Assuming we want this factor ordering of $g^{ab} p_a p_b$ to give

$$\frac{1}{\sqrt{g}} \partial_a [\sqrt{g} g^{ab} \partial_b] + CR,$$

$C = \text{constant}$. Then we must eliminate first-order derivative terms, which implies $\bar{\alpha} + 2 \bar{\beta} = 0$. We find that we have (16) with $C = -\bar{\beta}(1 - 1/n)$ and

$$-\bar{\beta} \left(1 - \frac{1}{n}\right)^2 \left(\frac{1}{2n} + \frac{3}{4}\right) = \bar{\beta} \left(\bar{\alpha} + \frac{1}{2}\right) + \bar{\beta}(\bar{\beta} - 1) - \frac{\bar{\beta}}{n},$$

or

$$\bar{\beta} = -\frac{1}{2n} + \frac{1}{4}, \quad \bar{\alpha} = \frac{1}{n} - \frac{1}{2},$$

and finally,

$$C = -\frac{1}{4n} - \frac{2}{n}.$$

There is only one other solution, $\bar{\alpha} = \bar{\beta} = C = 0$. This means that there are only two possible solutions leading to (16), the ordinary Laplace-Beltrami operator $(C = 0)$ or the conformally invariant operator.

For a general metric $g_{ab}$ there are many more possible factor orderings. We can attempt a factor ordering that makes use of powers of the metric components, where we can use matrix identities such as ($\delta_j$ the Kronecker delta and $\varepsilon^{ij...k}$ the totally antisymmetric Levi-Civita symbol)

$$\delta_a \delta_r \cdots \delta_n = g_{a q r} g_{c d} \cdots g_{f y} g^{q a} g^{d c} \cdots g^{y f} g^{l n} \cdots g^{v k},$$

$$\frac{1}{n!} \varepsilon^{ij...k} \varepsilon_{l_1 m_1 ... l_n} \varepsilon_{m_1 m_2} g_{l_1} \cdots g_{p_1 q_1} g_{w_1 v} \cdots g_{k_n} = 1.$$
The most general factor ordering using these expressions we have been able to construct is

\[ g_{ab}g_{lm} \cdots g_{vw} g_{r1} \cdots g_{f1} g_{q1} g_{lm} \cdots g_{vw} g_{f1} = n^{q_1+q_2}. \]  

(22)

Considerable algebra leads to two conditions on the nine constants, \( q_1, q_2, r_1, r_2, m_1, m_2, \alpha, \beta, k_1 \) \((k_2 = n - k_1)\), that force the terms linear in derivatives to vanish. There are six independent terms in \( R \) as given in (11), and our factor ordering gives a “potential” term (which we would like to be \( CR, C \) the constant in [16]) with seven independent terms, each multiplied by combinations of the seven independent constants that remain after the two conditions from putting the terms linear in derivatives to zero are satisfied. It is remarkable that six of these seven terms have exactly the form of the six terms in \( R \) and that there is only a single extra term that has the form \( g_{ab} g_{ac} g_{cl} \). If it were not for this seventh term, we could find a set of constants that would give \( CR \). Unfortunately, the seven equations that relate the seven constants to each other and to \( C \) are inconsistent except for \( C = 0 \), so the only possible solution is the ordinary Laplace-Beltrami operator.

If we explicitly reduce the metric to conformally flat form, we have the two solutions given above, so the factor ordering given in (23) is consistent with our previous result. Since a conformally flat metric is characterized in a coordinate invariant way by a zero Weyl tensor, investigating the relation between the conformally-flat calculation and the general calculation in terms of the Weyl tensor may give some insights into the reasons the factor ordering we chose does not give the conformally invariant Laplace-Beltrami operator.

Finally, we have chosen a specific factor ordering similar to that given in (2), where we have taken powers of the metric components replacing the \( f \) and \( h \) functions of Eq. (2). Of course, this is a very specific factor ordering, and there is no
reason to believe that there is no factor ordering that can give the conformally invariant Laplace-Beltrami operator. In fact, Moss and Shiiki [9] have used the possibility of terms proportional to the commutator $[\hat{p}_c, \hat{q}_d] = -i\delta_{cd}$ to write a factor ordering that gives the conformally invariant Laplace-Beltrami operator by adding the term $R^a_b[\hat{p}_a, \hat{q}_b]$ to the ordinary Laplace-Beltrami operator. A search for a more general factor ordering could be the subject of future research.

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References

[1] B. DeWitt, *Phys. Rev.* **160**, 1113 (1967).

[2] A. Komar, *Phys. Rev.* **170**, 1195 (1968).

[3] W. Pauli, in *Handbook der Physik* **5/1**, p.40, S. Flügge, Ed. (Springer Verlag, Berlin, 1958).

[4] C. Misner, in *Magic without Magic: John Archibald Wheeler*, J. Klauder, Ed. (Freeman, San Francisco, 1972).

[5] J. Hartle and S. Hawking, *Phys. Rev. D* **28**, 2960 (1983).

[6] See K. Kuchař, Lecture notes on quantum theory, University of Utah (unpublished).

[7] I. G. Moss, *Ann. Inst. Henri Poincaré* **49**, 341 (1988).

[8] J. J. Halliwell, *Phys. Rev. D* **38**, 2468 (1988).

[9] I.G. Moss, and N. Shiiki, *Nucl.Phys.* **B565**, 345 (2000).