Generalised hyperbolicity in singular space-times

C J S Clarke
Faculty of Mathematical Studies
University of Southampton
Southampton, SO17 1BJ, UK

Abstract A new concept analogous to global hyperbolicity is introduced, based on test fields. It is shown that the space-times termed here “curve integrable” are globally hyperbolic in this new sense, and a plausibility argument is given suggesting that the result applies to shell crossing singularities. If the assumptions behind this last argument are valid, this provides an alternative route to the assertion that such singularities do not violate cosmic censorship.

1. Introduction

Penrose’s strong cosmic censorship hypothesis [7] postulates that, subject to genericness conditions, space-time will be globally hyperbolic: i.e. strong causality holds and $J^+(p) \cap J^-(q)$ is compact for all $p, q$. The context in which this is usually discussed [6] is that of space-times where the metric is $C^2$ (ensuring the existence of unique geodesics in the classical sense). As result, a Lorentz manifold with a metric that fails to be $C^2$ at one point $p$ has to be viewed as a space-time from which $p$ is to be deleted, which usually results in a failure of global hyperbolicity and thus a breakdown in cosmic censorship.

On the other hand, there are increasingly many examples emerging of space-times that violate cosmic censorship in this way, but where there is a well posed initial value problem for test fields. Global hyperbolicity is sufficient, but not necessary for this. The physically meaningful condition is not global hyperbolicity, but the well-posedness of the field equations. This suggests that we need to redefine the notion of hyperbolicity (or equivalently, of cosmic censorship) so as to make direct reference to test fields, in situations where a putative singularity $p$ is an internal point of a Lorentz manifold $(M, g)$ with $g$ not being $C^2$ at $p$. (A general discussion of this idea is given in [2].)

We shall suppose throughout that we are working in a region in which there exist local coordinates (we regard $M \subset \mathbb{R}^4$) in which $g_{ij}$ and $g^{ij}$ are bounded but not necessarily continuous. This is not essential for many of the points discussed below, but it is not my aim here to discuss global issues. The definitions that follow below are then made with respect to a choice of some particular type of physical field $\phi$ (e.g. massless scalar) satisfying $L \phi = 0$ for a second order differential operator $L = L(g)$.

There is an underlying conflict (which will not be resolved here) between the view of geometrical general relativity, in which one deals with differentiability in the four-dimensional manifold, and the $3 + 1$ setting appropriate to the analysis of hyperbolic equations, in which function-spaces are defined on three-dimensional hypersurfaces. The ideal would be an integrated approach in which the full coupled Einstein-matter equations were handled in a manner consistent with their hyperbolic nature. In this paper I am the inhabiting a halfway house, in which the metric is being is being handled in terms of 4-D differentiability, while the matter is being regarded as a test field used to probe the metric and is described in $3 + 1$ terms. Thus for the case being considered where $L$ is second order, we shall take a foliation of the region which establishes a particular diffeomorphism with $\mathbb{R}^4 \times \mathbb{R}$ and regard the field $\phi$ as a map $\Phi : t \mapsto (\phi(., t), \dot{\phi}(., t))$ taking values in an appropriate function space (defined by the norms in the next section) on $\mathbb{R}^3$, and $L$ will have the form $d\Phi/dt + A\Phi$ for a three-dimensional differential operator $A$. For simplicity of notation, however, I shall usually not distinguish explicitly between $\Phi$ and $\phi$.

The extension of the “usual” definition of an operator to the case of a non-smooth metric is not always unambiguous, or even possible. In the case, however, of the wave operator which will be treated here we can regard $\Phi(t)$ as lying in $H^1(\mathbb{R}^3) \times H^0(\mathbb{R}3)$ ($H^i$ being the Hilbert space of $i$ times differentiable functions. For smooth $g$ and for $\psi, \phi \in C^\infty_0(\mathbb{R}^4) \times C^\infty_0(\mathbb{R}^4)$ we have that

$$\int \psi(\Box\phi) d^4x = - \iint \left[ \dot{\psi} \left( g^{00} \phi + g^{0\alpha} \phi,_{\alpha} \right) + \psi,_{\beta} \left( g^{00} \phi + g^{0\alpha} \phi,_{\alpha} \right) \right] \sqrt{-g} dx dt.$$
The $\mathbb{R}^3$ integral on the right hand side defines, for fixed $\phi$, a linear operator on $\psi$ which is bounded on $H^1(\mathbb{R}^3) \times H^0(\mathbb{R}^3)$, and is well defined for a general bounded invertible $g$. Hence there is an element $A(\Phi(t))$ of this space such that
\[ \int \psi(\Box \phi) d^4 x = - \int \langle \Psi(t), A(\Phi(t)) \rangle dt \]
thus defining $A$ (an unbounded operator on a dense domain) for general $g$.

In this context, I shall call $\phi$ a “solution” to $L \phi = 0$ if there exists a foliation with respect to which $d\Phi/dt + A \Phi = 0$. Note that this therefore does not imply that $\phi$ is $C^2$. A solution in this sense is also a weak solution in the sense that it has a locally integrable weak derivative (a distributional derivative that is a function) $\phi_{,k}$ satisfying
\[ \int \sqrt{-g} g^{jk} \phi_{,k} \chi_{,i} dV = 0 \] (1)
for all test functions $\chi$.

I then make the following definitions.

$M$ is $L$-globally hyperbolic if there is a spacelike surface $S$ such that there is a 1-1 correspondence (defined by taking the foliation for $\Phi$ to include $S$) between Cauchy data on $S$ (satisfying only local constraints, if any) and global solutions to $L \phi = 0$.

A point $p$ in $M$ is $L$-nakedly singular if it has no $L$-globally hyperbolic neighbourhood. Otherwise it is called $L$-inessential.

For simplicity of the later exposition I shall take Cauchy data that is $C^2 \times C^1$. This restriction on the initial conditions is stronger than is really required: the aim is to illustrate principles, not to obtain the best possible result.

2. Curve integrable space-times

Perhaps the most interesting of the singularities where the differentiability falls below $C^2$ are the shell-crossing singularities, which are not too unrealistic physically and may be tractable analytically. With a view to showing that these are inessential, I shall prove the following:

**Theorem.** Suppose given $(M, g)$ and a point $p$ in $M$ such that

(a) $g_{ij}$ and $g^{ij}$ are continuous

(b) $g_{ij}$ is $C^1$ in $M \setminus J^+(p)$

(c) weak derivatives $g_{ij,k}$ exist and are square integrable on $M$

(d) the distributionally defined $R_{ijkl}$ is a function

(e) there is a non-empty open set $C \subset \mathbb{R}^4$, and positive functions $M, N : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that, if $\gamma$ is a curve with $d\gamma/ds \in C$ for all $s$ then

(i) $\gamma$ is future timelike

(ii) the integrals

\[ I_{\gamma}(a) := \int_0^a |\Gamma_{jk}^i(\gamma(s))|^2 ds \quad \text{and} \quad J_{\gamma}(a) := \int_0^a |R_{ijkl}^i(\gamma(s))| ds \]

are convergent, with

\[ I_{\gamma}(a) < M(a), \quad J_{\gamma}(a) < N(a) \] (2)

and $M(a) \to 0$, $N(a) \to 0$ as $a \to 0$.

Then $p$ is $\Box$-inessential (where $\Box$ is the wave operator).
The conditions (a) and (c), introduced by Geroch and Traschen [5], are the minimal conditions for $R^i_{jkl}$ to be definable as a distribution by the usual coordinate formula in terms of $g_{ij}$ and $g^{ij}$. The set $C$ defines a range of timelike directions which are transverse to any shocks or caustics that may be present (this is illustrated in the next section). We refer to (c) by saying that the space-time is curve-integrable.

I conjecture that the same result holds if the definition of $f$ is altered to involve the simple modulus of $\Gamma$, rather than the modulus squared. The proof here, however, requires the stronger condition above, which essentially asserts that the quadratic and the linear (in $\Gamma$) parts of the Riemann tensor are separately integrable along the curves considered here. This form of the condition implies, of course, the square integrability stated separately in (c).

\[\vspace{10mm}\]

**Proof**

We will be able to apply standard theorems on the existence and uniqueness of solutions to linear partial differential equations, provided that we can establish an energy inequality for the solution. This in turn will require us to find a vector field $t$ whose covariant derivative is known to be bounded on spacelike surfaces. We achieve this by taking for the field the tangent vector to a suitable congruence of geodesics. The necessary steps are, therefore, to show the existence of the congruence and then to compute the covariant derivative of its tangent vector.

2.1 There exists a congruence of timelike geodesics whose tangent vector has an essentially bounded covariant derivative

**Proof of 2.1**

Since the definition of ‘inessential’ is local, we can shrink $M$ to a smaller neighbourhood of $p$ if necessary. Since the metric is continuous we can take a rotation of coordinates so that the surfaces $S_t = \{x \mid x^0 = t\}$ are spacelike in $M$ (shrunk if necessary). Suppose moreover that $p$ is at $t = 0$. Choose a fixed vector $t_0 \in C$ and let $S$ be the spacelike surface $S_{t_1}$, where $t_1 < 0$ is to be determined later. We denote the coordinates $x^\alpha$ ($\alpha = 1, 2, 3$) on $S$ by $\bar{y}$, and let $t_{\bar{y}}$ denote the vector with components $t_0$ at the point on $S$ with coordinates $\bar{y}$.

Next, we need to establish that the conditions on the metric are sufficient to ensure the existence of geodesics with initial tangents $t_{\bar{y}}$: we shall take these to form the required congruence. Let $S : \mathbb{R}^3 \to \mathbb{R}$ be a $C^\infty$ smoothing function, i.e. $S(x) \geq 0$, $S(x) = S(-x)$, $\int S(x)dx = 1$, support($S$) compact. Define $S_n(x) = n^4S(nx)$ and let $\Gamma^{(n)i}_{jk} = S_n \ast \Gamma^i_{jk}$ (where $\ast$ denotes convolution). For each $\bar{y} \in \mathbb{R}^3$ let $t^{(n)}_{\bar{y}}$ be the $\Gamma^{(n)}$-geodesic with initial tangent vector $t_{\bar{y}}$ and let $t^{(n)}_{\bar{y}}(s)$ be its tangent vector at parameter $s$.

We note that, for small enough $s$, say $s < s_1$, these tangent vectors have components in $C$ (with $s_1$ being a uniform bound, independent of $n$). This follows from the geodesic equation

\[
\frac{dt^{(n)i}_{\bar{y}}}{ds} = -\Gamma^{(n)i}_{jk}t^{(n)j}_{\bar{y}}t^{(n)k}_{\bar{y}} \tag{3}
\]

together with

\[
\left| \int_0^s \Gamma^{(n)i}_{jk}(\lambda(s'))ds' \right| = \left| \int S_n(z) \left[ \int_0^s \Gamma^{i}_{jk}(\lambda(s') + z)ds' \right] dz \right| \leq M(s)^{1/2}s^{1/2} \tag{4}
\]

for any curve with $\dot{\lambda} \in C$, which establishes a uniform bound for the right hand side of (3). Indeed, if

\[r = \min_{\bar{y}} \text{dist}(t^{(n)}_{\bar{y}}(0), C')\]

(\text{where } C' \text{ is the complement of } C) \text{ and } q = \max_{\bar{y}} \left| t^{(n)}_{\bar{y}}(0) \right|, \text{ then it suffices to take } s_1 \text{ such that } M(s_1)^{1/2}s^{1/2} < r/(8(q + r)^2) \text{ which will ensure that } |t^{(n)}_{\bar{y}} - t_{\bar{y}}| < r. \text{ The time } t_1 \text{ can now be specified as sufficiently close to } 0 \text{ to ensure that } p \text{ is covered by the curves up to } s_1.\]
We now examine the connecting vector $Y$, the basic tool being the geodesic deviation equation for the variation with $\vec{y}$ of $\kappa^{(n)}_{\vec{y}}$. Let $(e^{(n)_{\vec{y}}}_{a})_{a=0...3}$ be a parallelly propagated co-frame on $\kappa^{(n)}_{\vec{y}}$ coinciding with the coordinate basis at $s = 0$, with $e^{(n)_{\vec{y}}}_{a}$ the corresponding frame, and define

$$Y^{(n)_{\vec{y}}}_{\alpha a}(s) := \frac{\partial \kappa^{(n)_{\vec{y}}}_{i}}{\partial y^{a}} e^{(n)_{\vec{y}}}_{i}.$$

The geodesic deviation equation is then

$$\frac{d^{2}Y^{(n)_{\vec{y}}}_{\alpha a}}{ds^{2}} = e^{(n)_{\vec{y}}}_{i} R^{(n)_{\vec{y}}}_{i j k} t^{(n)_{\vec{y}}}_{j} t^{(n)_{\vec{y}}}_{k} Y^{(n)_{\vec{y}}}_{\alpha b}$$

subject to

$$Y^{(n)_{\vec{y}}}_{\alpha a}(0) = \delta^{a}_{\alpha}$$

and

$$\frac{dY^{(n)_{\vec{y}}}_{\alpha a}(0)}{ds} = \nabla_{\alpha} t_{\vec{y}0} = \Gamma^{(n)_{\vec{y}}}_{\alpha a} (t_{1}, \vec{y}) t_{0}^{i}.$$

Condition (e) implies that $e^{(n)_{\vec{y}}}_{a}$ are uniformly bounded, as are $t^{(n)_{\vec{y}}}_{i}$, so that (5) gives

$$\left| \frac{d^{2}Y^{(n)_{\vec{y}}}_{\alpha a}}{ds^{2}} \right| \leq Q \sigma \| Y^{(n)_{\vec{y}}}_{\alpha a} \|,$$

for some constant $Q$, where

$$\sigma := \sup_{i,j,k,l} | R^{(n)_{\vec{y}}}_{ijkl} |.$$

It follows that $\| Y \|$ is bounded by the solution $z$ of the majorizing equation

$$\frac{d^{2}z}{ds^{2}} = K \sigma z$$

subject to $z(0) = 1$, $dz(0)/ds = \| R^{(n)_{\vec{y}}}_{\alpha a} (t_{1}, \vec{y}) \| =: V$ (defining $V$).

This equation will imply that $z$, and hence $\| Y \|$, can be bounded in terms of the integral of $R^{(n)}$. Now the significance of the conditions (e) is that this integral can be bounded in terms of the integral of $R$. Indeed, since the integrals of the linear and quadratic parts of $R$ are separately bounded, we have inequalities of the following form (with $a, b, a_{1}, a_{2}$ constants)

$$\int_{0}^{s} |\partial_{i} \Gamma^{(n)_{\vec{y}}}_{jk} | ds' \leq \int_{0}^{s} | S_{n} * \partial_{i} \Gamma^{(n)_{\vec{y}}}_{jk} | ds' \leq \int dz S_{n}(z) \int_{0}^{s} |\partial_{i} \Gamma^{(n)_{\vec{y}}}_{jk} | ds' \leq a(J(s) + I(s))$$

$$\int_{0}^{s} | \Gamma^{(n)_{\vec{y}}}_{jk} | ds' \leq \int dz S_{n}(z) \int dy S_{n}(y) \int_{0}^{s} ds' |\Gamma^{(n)_{\vec{y}}}_{jk} | ds' \leq b I(s)$$

and hence

$$\int_{0}^{s} | R^{(n)_{\vec{y}}}_{ijkl} | ds' \leq \int_{0}^{s} \sigma ds' \leq a_{1} M(s) + a_{2} N(s) =: M_{1}(s),$$

say. To estimate the solution to (6) we then note that if $s_{2}$ is the first value of $s$ at which $|z| = 2$ (possibly $s_{2} = \infty$) then before $s_{2}$

$$\frac{d^{2}z}{ds^{2}} \leq 2Q \sigma$$
leading, for \(0 \leq s \leq s_2\), to
\[
z \leq 1 + V s + 2Q \int M_1(s) ds.
\]
Thus if we choose \(s_0\) so small that
\[
V s_0 + 2K \int_{s_0}^{s_1} M_1(s) ds < 1
\]
then we will have \(z < 2\) up to \(s_0\), and hence \(\|Y^{(n)}_{g_0}\| < 2\) in this interval.

This bound implies that the function \(\kappa : (\vec{y}, s) \mapsto \kappa_{g_0}^{(n)}(s)\) is equicontinuous, and so there is by Arzela-Ascoli a subsequence that tends to a limit. Choosing this subsequence gives meaning to the idea of limiting geodesics.

Having established this, essential boundedness of the derivative of the tangent vector follows in a similar way. If \(X^i = X^\alpha Y^{(n)}_{\vec{y}_\alpha} + X^0 \kappa_{\vec{y}}^{(n)}i\) then
\[
X^j t^{(n)}_{g_\alpha} = X^\alpha Y^{(n)}_{\vec{y}_\alpha} \kappa_{\vec{y}}^{(n)}j + \int_0^s \gamma_{\vec{y}}^{(n)}a \kappa_{\vec{y}}^{(n)}j_{ij} \kappa_{\vec{y}}^{(n)}l ds.
\]

We can now take the limit of this in \(L^\infty\) to obtain essential boundedness.

2.2 Solutions of the wave equation satisfy an energy inequality (12)

Proof of 2.2

The technique closely follows the account of Hawking and Ellis [6], section 7.4

We are concerned with solutions (cf (1)) to the wave equation
\[
\Box \phi \equiv g^{ij} \phi_{,ij} = 0.
\]
(7)

Suppose initially that \(\phi\) is \(C^2\) and define
\[
S^{ij} := (g^{ik}g^{jl} - \frac{1}{2} g^{ij} g^{kl}) \phi_{,k} \phi_{,l} - \frac{1}{2} g^{ij} \phi^2.
\]

We let \(U\) be a compact set bounded to the past by \(S\) (i.e. \(I^-(U) \cap \partial U \subset S\)) and to the future by a spacelike surface \(H = I^+(U) \cap \partial U\).

Working locally, from the continuity of \(g^{ij}\) we can choose \(U\) to be foliated by \(C^\infty\) spacelike surfaces \(S^U_t\). (We take \(0 \leq \tau \leq \tau_1\).)

Set \(U_\tau = \bigcup_{\tau' < \tau} S^U_{\tau'}\) and define
\[
E(\tau) := \int_{S^U_{\tau}} S^{ij} t_i n_j \sqrt{-g} d^3 x
\]
where \(n\) is the future normal to \(S^U_{\tau}\). Our aim is to estimate the norm
\[
\|\phi\|_{1,\tau}^3 = \left[ \int_{S^U_{\tau}} (\sum_i (\phi_{,i})^2 + \phi^2) d^3 x \right]^{1/2}
\]
which is related to \(E\) by
\[
kE(\tau) \leq (\|\phi\|_{1,\tau}^3) \leq K E(\tau)
\]
for positive \(k, K\). We also introduce
\[
\|\phi\|_{1,\tau}^4 = \left[ \int_{U_\tau} (\sum_i (\phi_{,i})^2 + \phi^2) d^4 x \right]^{1/2}
\]
\[
\leq \left[ \int_{0}^{\tau} (\|\phi\|_{1,\tau'}^3) dt' \right]^{1/2}
\]
(9)
and

\[ \| \phi \|^2 = \left[ \int_{U_r} \phi^2 \right]^{1/2}. \]

Stokes’ theorem yields

\[ \int_{U_r} (S^{ij} t_i \sqrt{-g} d^4 x) = (- \int_{S} + \int_{H_r}) S^{ij} t_i n_j \sqrt{-g} d^3 x. \tag{10} \]

By direct calculation

\[ S^{ij} = (g^{ij} \phi_j)(\phi_k k_l - \phi) \]

and so the left hand side of (10) becomes (with \( g_{kl} \phi_{;kl} = \Box \phi \))

\[ \int_{U_r} [\Box \phi - \phi ] \phi_k t^k + S^{ij} t_i \phi_{;j} \sqrt{-g} d^4 x. \]

Estimating all the terms by the bounds available gives

\[ \frac{1}{K} (\| \phi \|^1_{S, r})^2 \leq E(\tau) \leq E(0) + c \| \phi \|^0_r \| L \phi \|_r + c' \| \phi \|^2 \]

(11)

for constants \( c, c' \). If \( \Box \phi = 0 \) weakly, this becomes

\[ \frac{1}{K} (\| \phi \|^1_{S, r})^2 \leq E(t) \leq E(0) + C'(\| \phi \|^1_r)^2. \tag{12} \]

while if \( E(0) = 0 \), (11) and (9) give

\[ E(\tau) \leq c_1(\| L \phi \|^0) \]

(13)

for some constant \( c_1 \).

2.3 There exist unique solutions to the wave equation for \( C^2 \times C^1 \) initial conditions

Proof of 2.3

We briefly recall the standard arguments (see, for example [4]) which allow us to deduce uniqueness and existence of solutions from an energy estimate, using the symmetry of the wave operator.

Uniqueness is immediate: if the difference between two solutions is zero on \( S \) then \( E(0) = 0 \) and (13) then implies that the solutions are (pp) identical.

Let \( V_1 \) be the subset of \( L^2(U) \) consisting of \( C^\infty \) functions that are zero with a zero normal derivative on \( H := S_{n_1} \) and let \( V_0 \) be the subset of \( L^2(U) \) consisting of \( C^\infty \) functions that are zero with a zero normal derivative on \( S := S_0 \). Then the same uniqueness argument holds for both these data conditions, and we have equation (13), which implies there exists a constant \( c_2 \) such that

\[ \| \phi \| \leq C_2 \| L \phi \| \]

(14)

for \( \phi \in V_1 \) (from now on all norms are in \( L^2 \)).

To prove existence subject to conditions \( \phi = \phi_0, \dot{\phi} = \phi_1 \) on \( S \), choose a \( C^2 \) function \( f \) satisfying these conditions and look for a function \( \psi = \phi - f \) satisfying zero boundary conditions and \( L \psi = -L f = : \chi \). The required function \( \psi \) will satisfy

\[ \int_{U} \psi L w d^4 x = \int_{U} \chi w d^4 x \]

for all \( w \in V_1 \). From (14)

\[ \left| \int_{U} \chi w d^4 x \right| \leq c_2 \| \chi \| \| L \phi \| \]
so that the map \( k : Lw \mapsto \int_U \chi wd^4x \) is a bounded linear functional on \( LV_1 \). But \( LV_1 \) is dense in \( L^2 \), because \( V_0 \) is dense and if \( \int_U \phi Lwd^4x = 0 \) for all \( Lw \in LV_1 \) and \( \phi \in V_0 \) then we must have \( \phi = 0 \). So \( k \) defines an element \( \psi \) of \( L^2 \) such that \( \int_U \psi Lwd^4x = k(Lw) = \int_U \chi wd^4x \); so that \( \phi = f + \psi \) is the required solution.
This concludes the proof.

3. Application to dust caustic (shell crossing) space-times

Though there is as yet no rigorous proof, there are very strong indications [3] that shell-crossing spherically symmetric dust configurations produce relativistic solutions in which the flow lines of matter produce a caustic, as in the gravity-free case (see figure 1).

In what follows I shall be assuming the existence of such a space-time, in which the general form of the matter density is the same as that in the gravity-free case. Since the matter density determines the Riemann tensor and the connection via simple integrals in this case, we can pass from the density to the Riemann tensor immediately.

From catastrophe theory, the generic gravity free caustic is diffeomorphic to the following canonical form. If \( (r, t) \) are the essential coordinates in a spherically symmetric situation, and the flow lines of matter are parametrised by \( t \), with \( v := dr/dt \) constant on each line, then the lines when lifted to curves \( t \mapsto (r, t, v) \) in \( \mathbb{R}^3 \) (thought of as a reduction of the tangent bundle) rule the surface \( \Sigma \) with equation \( r = f(v, t) := vt - av^3 \) for a constant \( a \). The projection in the tangent bundle corresponds to the projection \( p : (r, t, v) \mapsto (r, t) \), and the critical points of \( p|\Sigma \) constitute the curve \( t = 3av^2 \) in \( \Sigma \). The caustic is the image under \( p \) of this critical point set, namely \( t = 3a^{1/3}(r/2)^{2/3} \). Each point on the ruled surface makes a contribution \( \rho = \rho_0(v)(\partial f/\partial v)^{-1} = \rho_0/(t - 3av^2) \) to the total density at the corresponding point of space-time, for some function \( \rho_0(v) \) giving the density distribution in velocity-space.

Integrability of the Riemann tensor along curves will depend on its behaviour near the caustic. Consider, therefore, a coordinate straight line cutting the caustic at the image of a point on the critical point set in \( \Sigma \) with velocity \( v_0 \), i.e. at \( r_0 = 2av_0^3 \), \( t_0 = 3av_0^2 \), the line being \( x = x_0 + \lambda(t - t_0) \). Setting \( t = t_0 + \tau \), \( v = v_0 + \nu \) and working to lowest significant order in \( \tau \) and \( \nu \), we obtain

\[
\nu \approx \left( \frac{v_0 - \lambda}{3av_0} \right)^{1/2} \tau^{1/2}
\]

for \( v_0 \neq 0 \) (where the condition \( v_0 > \lambda \) is required for transversality to the caustic) and

\[
\nu \approx -\left( \frac{\lambda \tau}{a} \right)^{1/3}
\]
for $v_0 = 0$. The key point arising from this as a consequence is that $\rho$ is integrable along the curve, a result which is diffeomorphism invariant and so applies to the generic caustic.

Passing to the relativistic case, as previously noted we assume that this behaviour of the density still holds, specifically when the metric is presented in coordinates linearly related to double null coordinates. (As described in detail in [3], the choice of coordinates becomes significant in general relativity, as opposed to the Newtonian case, because coordinate transformations – for instance, from curvature coordinates to double null coordinates – are typically specified by geometrical conditions and so are not, in this case, $C^\infty$.) With this assumption, the Riemann tensor is curve-integrable and (see the treatment of these coordinates in [1]) the connection coefficients are bounded. It would then be the case that the caustic is not a $\Box$-essential singularity, so that cosmic censorship is not violated.

References

[1] Clarke C.J.S. *The Analysis of Space-Time Singularities*, Cambridge University Press, 1993
[1] Clarke, C J S “Singularities: boundaries or internal points”, *Proceedings of the International Conference on Gravitation and Cosmology* 1995
[3] Clarke C.J.S. and O’Donnell N. “Dynamical extension through a space-time singularity,” *Rendiconti del seminario matematico, Università e Politecnico Torino* 50, (1) 39–60, 1992
[4] Egorov, Yu V and Shubin, M A *Partial Differential Equations I*, Springer Verlag, 1992
[5] Geroch, R and Traschen, J. *Phys. Rev.* D36, 1017-31, 1987.
[6] Hawking, S W and Ellis, G F R *The large scale structure of space-time*, Cambridge University Press, 1973
[7] Penrose R. “Singularities and Time Asymmetry,” in *General Relativity. An Einstein Centenary Survey*, ed S.W. Hawking and W. Israel, Cambridge University Press, 1979