DEFORMATIONS OF FUCHSIAN SYSTEMS
OF LINEAR DIFFERENTIAL EQUATIONS
AND THE SCHLESINGER SYSTEM.

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To the centenary of the Schlesinger system

Abstract. We consider holomorphic deformations of Fuchsian systems parameterized by the pole loci. It is well known that, in the case when the residue matrices are non-resonant, such a deformation is isomonodromic if and only if the residue matrices satisfy the Schlesinger system with respect to the parameter. Without the non-resonance condition this result fails: there exist non-Schlesinger isomonodromic deformations. In the present article we introduce the class of the so-called isoprincipal deformations of Fuchsian systems. Every isoprincipal deformation is also an isomonodromic one. In general, the class of the isomonodromic deformations is much richer than the class of the isoprincipal deformations, but in the non-resonant case these classes coincide. We prove that a deformation is isoprincipal if and only if the residue matrices satisfy the Schlesinger system. This theorem holds in the general case, without any assumptions on the spectra of the residue matrices of the deformation. An explicit example illustrating isomonodromic deformations, which are neither isoprincipal nor meromorphic with respect to the parameter, is also given.

NOTATION

• $\mathbb{C}$ stands for the complex plane.
• $\mathbb{CP}^1$ stands for the extended complex plane (= the Riemann sphere):
  $\mathbb{CP}^1 = \mathbb{C} \cup \infty$.
• $\mathbb{C}^n$ stands for the $n$-dimensional complex space.
• In the coordinate notation, a point $t \in \mathbb{C}^n$ will be written as $t = (t_1, \ldots, t_n)$.
• $\mathbb{C}^n_*$ is the set of points $t \in \mathbb{C}^n$, whose coordinates $t_1, \ldots, t_n$ are pairwise different:
  $\mathbb{C}^n_* = \mathbb{C}^n \setminus \bigcup_{1 \leq i, j \leq n} \{ t : t_i = t_j \}$.
• $\mathbb{M}_k$ stands for the set of all $k \times k$ matrices with complex entries.
• $[,]$ denotes the commutator: for $A, B \in \mathbb{M}_k$, $[A, B] = AB - BA$.
• $I$ stands for the identity matrix of an appropriate dimension.
0. Introduction

The systematic study of linear differential equations in the complex plane with coefficients dependent on parameters has been started by Lazarus Fuchs in the late eighties of the nineteenth century [FuL1], [FuL2], [FuL3], [FuL4]. In particular, L. Fuchs investigated the equations whose monodromy does not depend on such parameters. These investigations were continued in the beginning of the twentieth century by L. Schlesinger [Sch2], [Sch3], [Sch4], R. Fuchs [FuR1], [FuR2], and R. Garnier [Gar].

L. Schlesinger’s research was closely related to the Hilbert 21st problem (a.k.a. the Riemann-Hilbert monodromy problem), which requires to construct a Fuchsian system with prescribed monodromy (for the explanation of terminology see Section 1 of the present article). In the paper [Sch2], which appeared exactly one hundred years ago – in 1905, L. Schlesinger proposed the idea that it would be very fruitful to study the deformations of Fuchsian systems

\[ \frac{dY}{dx} = \sum_{1 \leq j \leq n} \frac{Q_j(t)}{x - t_j} \cdot Y, \]

where the residues \( Q_j \) depend holomorphically on the pole loci \( t = (t_1, \ldots, t_n) \), and investigate the dependence of the solution \( Y \) on \( t \), as well as on \( x \).

Emphasizing this idea, L. Schlesinger explained that he was guided by the analogy with the theory of algebraic functions, where he had studied algebraic functions as functions of both the “main variable” and the loci of ramification points considered as parameters (see [Sch1, pp. 287-288]).

Also in the paper [Sch2], the system of PDEs

\[
\begin{align*}
\frac{\partial Q_j}{\partial t_k} &= \frac{[Q_j, Q_k]}{t_j - t_k}, & 1 \leq j, k \leq n, k \neq j, \\
\frac{\partial Q_j}{\partial t_j} &= -\sum_{1 \leq k \leq n, k \neq j} \frac{[Q_j, Q_k]}{t_j - t_k}, & 1 \leq j \leq n,
\end{align*}
\]

which is now known as the Schlesinger system, was introduced and the statement that the holomorphic deformation (0.1) is isomonodromic if and only if its coefficients \( Q_j(t) \) satisfy the system (0.2) was formulated. (See page 294 of [Sch2], four bottom lines of this page.) This formulation was repeated in the book [Sch3], pp. 328-329, and later in the paper [Sch4], p. 106. (References to the earlier paper [Sch2] are relatively rare. Usually, one refers to the more recent paper [Sch4].)

Over the years the Schlesinger system and the isomonodromic deformations of Fuchsian systems were extensively studied; we would like to mention in particular the papers of T. Miwa [Miwa] and of B. Malgrange [Mal], where it was proved that the Schlesinger system enjoys the Painlevé property (its solutions are meromorphic functions in the universal covering space over \( \mathbb{C}_n^* \)), and the book by A.R. Its and V.Yu. Novokshenov [ItNo], where the connections between the isomonodromic deformations and the transcendents of Painlevé were revealed.

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1L. Fuchs died in 1902.
2A student of L. Fuchs.
3The son of L. Fuchs.
However, in the 1990s the famous negative solution to the Riemann-Hilbert monodromy problem due to A.A. Bolibrukh (see \cite{Boli1, Boli2}) gave a strong motivation for the revision of the classical results, concerning the isomonodromic deformations of the Fuchsian systems.

For example, it should be noted that in the original works \cite{Sch2}–\cite{Sch4} of L. Schlesinger no assumptions concerning the non-resonance of the matrices $Q_j$ are made. In such generality the above-cited statement of L. Schlesinger fails. If a holomorphic deformation \eqref{0.1} of Fuchsian systems is such that the residues $Q_j$ satisfy the Schlesinger system \eqref{0.2}, then this deformation is isomonodromic, but the converse statement is not true, in general.

It was also A.A. Bolibrukh who constructed the first explicit example of the non-Schlesinger isomonodromic deformation. In this example the monodromy is non-trivial and the residues $Q_j(t)$ are rational functions of $t$ (see \cite{Boli3} and \cite{Boli4} – in both papers the same example appears as Example 2; see also \cite{Boli5} Section 3 of the review paper \cite{Boli5}, where this example appears as Example 3).

At the same time it was shown independently in \cite{Kats1} that almost every isomonodromic deformation of Fuchsian systems with generic rational solutions is non-Schlesinger (for more details see Remark 5.1 in Section 5 of this article).

Thus the isomonodromic property of the deformation \eqref{0.1} implies the Schlesinger system for the residues $Q_j(t)$ under the non-resonance condition, but not in general.

Unfortunately, careless treatment of the non-resonance condition is very common in the history of problems related to the monodromy of Fuchsian systems. It can also be found in some works of V. Volterra and of G.D. Birkhoff (see \cite{Gant} Chapter XV, \S 9 for details). This tradition continues to certain extent in the above-mentioned paper \cite{Miwa} of T. Miwa on the Painlevé property of isomonodromic deformations: in this paper the non-resonance condition appears as the equation (2.22), but is omitted both in the introduction and in the formulation of the main result.

Without the assumption of non-resonance the main result of \cite{Miwa} does not hold: there exist isomonodromic deformations of the form \eqref{0.1}, where the residues $Q_j(t)$ are not meromorphic in the universal covering space over $\mathbb{C}^*_\ast$. (The appropriate example is presented in Section 5 of this article. We note that this phenomenon does not occur in the above-mentioned example of the non-Schlesinger isomonodromic deformation due to A.A. Bolibrukh: in that example the residues $Q_j(t)$ are rational functions of $t$.)

The main goal of the present work is to answer the following question: how to describe the class of holomorphic deformations \eqref{0.1} with the property that the residues $Q_j(t)$ satisfy the Schlesinger system \eqref{0.2}, when one omits the non-resonance assumption?

The presentation of our results is organized as follows.

In the first section after this introduction we recall the basic notions concerning the Fuchsian system and introduce a certain canonical multiplicative decomposition of the fundamental solution in a neighborhood of its singular point $t_j$. This is the so-called regular-principal factorization: the fundamental solution is represented as the product of a regular factor, holomorphic and invertible at the point $t_j$, and a principal factor, holomorphic (multi-valued) and invertible everywhere except at $t_j$. This principal factor is the multiplicative analogue of the principal part in
the Laurent decomposition: it contains the information about the nature of the singularity.

In Section 2 we introduce the main notion of the present article (Definition 2.9): the so-called isoprincipal\(^6\) families of Fuchsian systems. These are the holomorphic families (0.1) with the property that all the principal factors of a suitably normalized fundamental solution \(Y(x,t)\) are, in a certain sense, preserved. We show that every isoprincipal family is also isomonodromic and that the converse is true, when the non-resonance condition is in force.

In Section 3 we formulate and prove our main result (Theorem 3.1) that the family (0.1) is isoprincipal if and only if the residues \(Q_j(t)\) satisfy the Schlesinger system (0.2). This result holds in the general case, without the assumption of non-resonance.

In the next section we discuss the isoprincipal deformation of a given Fuchsian system. Using our Theorem 3.1 we also outline how to establish the Painlevé property of the Schlesinger system and indicate possible generalizations.

Finally, in Section 5 we illustrate the general theory with explicit examples of the isoprincipal families of Fuchsian systems with generic rational solutions, developed in [Kats1], [Kats2] and [KaVo2].

1. Fuchsian differential systems

1.1. Fuchsian differential systems. A Fuchsian differential system is a linear system of ordinary differential equations of the form

\[
\frac{dY}{dx} = \left( \sum_{1 \leq j \leq n} \frac{Q_j}{x-t_j} \right) Y,
\]

where \(Q_j, 1 \leq j \leq n\), are square matrices of the same dimension, say \(Q_j \in \mathbb{M}_k\), and \(t_1, \ldots, t_n\) are pairwise distinct points of the complex plane \(\mathbb{C}\). The variable \(x\) "lives" in the punctured Riemann sphere \(\mathbb{C}P^1 \setminus \{t_1, \ldots, t_n\}\), the "unknown" \(Y\) is an \(\mathbb{M}_k\)-valued matrix function of \(x\). Under the condition

\[
\sum_{1 \leq j \leq n} Q_j = 0
\]

the point \(x_0 = \infty\) is a regular point for the system (1.1). If this condition is satisfied (which we always assume in the sequel), then in a neighborhood of the point \(x_0 = \infty\) there exists a fundamental solution \(Y = Y(x)\) of (1.1) satisfying the initial condition

\[
Y(x) \big|_{x=\infty} = I.
\]

This solution \(Y\) can be analytically continued into the multi-connected domain \(\mathbb{C}P^1 \setminus \{t_1, \ldots, t_n\}\). However, for \(x \in \mathbb{C}P^1 \setminus \{t_1, \ldots, t_n\}\) the value of \(Y\) at the point \(x\) depends, in general, on the path \(\alpha\) from \(x_0 = \infty\) to \(x\), along which the analytic continuation is performed:

\[
Y = Y(x, \alpha).
\]

\(^6\)Iso- (from \(\text{iσος}\) - equal - in Old Greek) is a combining form.
More precisely, $Y$ depends not on the path $\alpha$ itself, but on its homotopy class in $\mathbb{C}P^1 \setminus \{t_1, \ldots, t_n\}$. Thus $Y$ is a multi-valued holomorphic function in the punctured Riemann sphere $\mathbb{C}P^1 \setminus \{t_1, \ldots, t_n\}$ or, better to say, $Y$ is a singled-valued holomorphic function on the universal covering surface of the punctured Riemann sphere $\mathbb{C}P^1 \setminus \{t_1, \ldots, t_n\}$ with the distinguished point $\infty$.

1.2. Universal covering spaces. Recall (see [5], Chapter 1, Sections 3 - 5) if need be) that the universal covering space $\text{cov}(\mathcal{X}; x_0)$ of an arcwise connected topological space $\mathcal{X}$ with the distinguished point $x_0 \in \mathcal{X}$ is the set of pairs $(x, \alpha)$, where $x$ is a point in $\mathcal{X}$ and $\alpha$ is a homotopy class of continuous mappings

$$
\alpha : \{s \in \mathbb{R} : 0 \leq s \leq 1\} \to \mathcal{X}, \quad \alpha(0) = x_0, \alpha(1) = x.
$$

Such a mapping $\alpha$ is called a path in $\mathcal{X}$ from $x_0$ to $x$. A path in $\mathcal{X}$ from $x_0$ to $x_0$ is called a loop with the distinguished point $x_0$.

The product $\beta \cdot \alpha$ of two paths $\alpha, \beta$ in $\mathcal{X}$, where $\alpha$ is a path from $a$ to $b$ and $\beta$ is a path from $b$ to $c$, is defined as the path from $a$ to $c$, obtained by going first along $\alpha$ from $a$ to $b$ and then along $\beta$ from $b$ to $c$:

$$(\beta \cdot \alpha)(s) = \begin{cases} 
\alpha(2s), & 0 \leq s \leq \frac{1}{2}, \\
\beta(2s - 1), & \frac{1}{2} \leq s \leq 1.
\end{cases}
$$

With respect to this product, the homotopy classes of loops in $\mathcal{X}$ with the distinguished point $x_0$ form the so-called fundamental group $\pi(\mathcal{X}; x_0)$ of the space $\mathcal{X}$ with the distinguished point $x_0$. The fundamental group $\pi(\mathcal{X}; x_0)$ acts on the universal covering space $\text{cov}(\mathcal{X}; x_0)$ (on the right) as the group of deck transformations

$$(x, \alpha) \mapsto (x, \alpha\gamma), \quad (x, \alpha) \in \text{cov}(\mathcal{X}; x_0), \gamma \in \pi(\mathcal{X}; x_0).$$

1.3. Monodromy. Let $Y = Y(x, \alpha)$ be the solution of (1.1) - (1.3) defined on the universal covering surface $\text{cov}(\mathbb{C}P^1 \setminus \{t_1, \ldots, t_n\}; \infty)$. For each loop $\gamma \in \pi(\mathbb{C}P^1 \setminus \{t_1, \ldots, t_n\}; \infty)$ let us consider the function $Y_\gamma$ defined on $\text{cov}(\mathbb{C}P^1 \setminus \{t_1, \ldots, t_n\}; \infty)$ by

$$(1.5) \quad Y_\gamma(x, \alpha) \overset{\text{def}}{=} Y(x, \alpha\gamma).$$

The expression $\overset{\text{def}}{=}$ means that the value of $Y_\gamma$ at the point $(x, \alpha)$ of the universal covering surface $\text{cov}(\mathbb{C}P^1 \setminus \{t_1, \ldots, t_n\}; \infty)$ is obtained by the analytic continuation of the solution $Y$ of (1.1) - (1.3): first along the loop $\gamma$ from the distinguished point $x_0 = \infty$ to itself, then along the path $\alpha$ from $x_0$ to $x$.

Thus $Y_\gamma$ is also a fundamental solution of the linear system (1.1) and, therefore, there exists a unique invertible constant matrix $M_\gamma \in \mathfrak{M}_k$ such that

$$(1.6) \quad Y_\gamma(x, \alpha) = Y(x, \alpha) \cdot M_\gamma, \quad (x, \alpha) \in \text{cov}(\mathbb{C}P^1 \setminus \{t_1, \ldots, t_n\}; \infty).$$

**Definition 1.1.** Let $Y = Y(x, \alpha)$ be the solution of (1.1) - (1.3) on the universal covering surface $\text{cov}(\mathbb{C}P^1 \setminus \{t_1, \ldots, t_n\}; \infty)$ and let $\gamma \in \pi(\mathbb{C}P^1 \setminus \{t_1, \ldots, t_n\}; \infty)$.

The constant (with respect to $x$) matrix $M_\gamma \in \mathfrak{M}_k$, which appears in the identity (1.6), is said to be the monodromy matrix of the solution $Y$, corresponding to the loop $\gamma$. 
Note that for a pair of loops \( \gamma_1, \gamma_2 \in \pi(\mathbb{CP}^1 \setminus \{t_1, \ldots, t_n\}) \) and the corresponding monodromy matrices \( M_{\gamma_1}, M_{\gamma_2} \) of the solution \( Y \) it holds that

\[
Y(x, \alpha \gamma_1 \gamma_2) = (Y \cdot M_{\gamma_2})(x, \alpha \gamma_1) = Y(x, \alpha \gamma_1) \cdot M_{\gamma_2} = Y(x, \alpha) \cdot M_{\gamma_1} \cdot M_{\gamma_2}.
\]

Therefore, the monodromy matrices of \( Y \) satisfy the following multiplicative identity:

\[
(1.7) \quad M_{\gamma_1 \gamma_2} = M_{\gamma_1} M_{\gamma_2} \quad \forall \gamma_1, \gamma_2 \in \pi(\mathbb{CP}^1 \setminus \{t_1, \ldots, t_n\}).
\]

This means that the mapping \( \gamma \mapsto M_\gamma \) is a linear representation of the fundamental group \( \pi(\mathbb{CP}^1 \setminus \{t_1, \ldots, t_n\}) : \infty \).

**Definition 1.2.** Let \( Y \) be the solution of (1.1) – (1.3) on the universal covering surface \( \text{cov}(\mathbb{CP}^1 \setminus \{t_1, \ldots, t_n\}); \infty) \). The linear representation of the fundamental group \( \pi(\mathbb{CP}^1 \setminus \{t_1, \ldots, t_n\}) : \infty) \)

\[
(1.8) \quad \gamma \mapsto M_\gamma, \quad \gamma \in \pi(\mathbb{CP}^1 \setminus \{t_1, \ldots, t_n\}) : \infty),
\]

where \( M_\gamma \) denotes the monodromy matrix of the solution \( Y \), corresponding to the loop \( \gamma \), is called the monodromy representation of the solution \( Y \).

### 1.4. The regular-principal factorization for a fundamental solution of a Fuchsian system: single-valued case.

Each of the points \( t_j, 1 \leq j \leq n \), is a singularity of the solution \( Y \). This means that at least one of the two functions \( Y \) and \( Y^{-1} \) is not holomorphic at \( t_j \). More information about the nature of the singularity at \( t_j \) can be obtained from a certain multiplicative decomposition of the solution \( Y \) near the point \( t_j \), which is called the regular-principal factorization.

In order to explain the idea of the regular-principal factorization, let us assume for the moment that the solution \( Y = Y(x) \) is single-valued in the domain \( \mathbb{CP}^1 \setminus \{t_1, \ldots, t_n\} \) – that is, the monodromy representation of \( Y \) is trivial:

\[
M_\gamma = I \quad \forall \gamma \in \pi(\mathbb{CP}^1 \setminus \{t_1, \ldots, t_n\}); \infty).
\]

For \( 1 \leq j \leq n \) let \( \mathcal{V}_j \) be an open simply connected neighborhood of \( t_j \) in \( \mathbb{C} \), such that \( t_k \not\in \mathcal{V}_j \) for \( k \neq j \). Then it follows, for instance, from G.D. Birkhoff’s results on factorization of matrix functions holomorphic in the annulus (see [Birkhoff §7]) that in the punctured neighborhood \( \mathcal{V}_j \setminus \{t_j\} \) the solution \( Y(x) \) admits a factorization of the form

\[
Y(x) = H_j(x) \cdot P_j(x), \quad x \in \mathcal{V}_j \setminus \{t_j\},
\]

where the function \( H_j(x) \) is holomorphic and invertible in the entire (non-punctured) neighborhood \( \mathcal{V}_j \) and the function \( P_j(x) \) is holomorphic, single-valued and invertible in the punctured plane \( \mathbb{C} \setminus \{t_j\} \). The factors \( H_j(x) \) and \( P_j(x) \) are said to be, respectively, the regular factor and the principal factor of the solution \( Y \) at its singular point \( t_j \).

In the general case, when the monodromy representation of \( Y \) may be non-trivial, the regular-principal factorization of \( Y \) is more involved.

Indeed, on the one hand the solution \( Y \) is normalized at the distinguished point \( x_0 = \infty \). In order to consider \( Y \) in a neighborhood of \( t_j \), we have to choose a homotopy class of paths, connecting the distinguished point \( x_0 = \infty \) with this neighborhood of \( t_j \), and such a choice is not unique.

On the other hand, even in the single-valued case \( x_0 = \infty \) is, in general, a singular point of the principal factor \( P_j \). In the general case \( P_j \) will have to be considered as a function on a universal covering surface of the punctured plane.
Let \( Y \) be the solution of (1.13) on the universal covering surface \( \operatorname{cov}(\mathbb{C}P^1 \setminus \{t_1, \ldots, t_n\}; \infty) \).

For \( 1 \leq j \leq n \) assume that:

(i) \( V_j \) is a domain in \( \mathbb{C}P^1 \setminus \{t_1, \ldots, t_n\} \);

(ii) \( p_j \) is a point in the domain \( V_j \);

(iii) \( \alpha_j \) is a path in \( \mathbb{C}P^1 \setminus \{t_1, \ldots, t_n\} \) from the distinguished point \( x_0 = \infty \) to the point \( p_j \).

Define the function \( Y_j \) on the universal covering surface \( \operatorname{cov}(V; p_j) \) by the analytic continuation of the solution \( Y \) first along the path \( \alpha \), then inside the domain \( V \):

\[
Y_j(x, \beta) \overset{\text{def}}{=} Y(x, \beta \cdot \alpha_j),
\]

where \( x \in V \) and \( \beta \) is a path in \( V \) from \( p \) to \( x \).

Then the function \( Y_j \), holomorphic in \( \operatorname{cov}(V; p_j) \), is said to be the branch of the solution \( Y \) in the domain \( V \), corresponding to the path \( \alpha \).

1.5. **Branches of the solution of a Fuchsian system in a neighborhood of the singular point.** We propose the following terminology:

**Definition 1.3.** Let \( Y \) be the solution of (1.13) on the universal covering surface \( \operatorname{cov}(\mathbb{C}P^1 \setminus \{t_1, \ldots, t_n\}; \infty) \).

For \( 1 \leq j \leq n \) assume that:

(i) \( V_j \) is a domain in \( \mathbb{C}P^1 \setminus \{t_1, \ldots, t_n\} \);

(ii) \( p_j \) is a point in the domain \( V_j \);

(iii) \( \alpha_j \) is a path in \( \mathbb{C}P^1 \setminus \{t_1, \ldots, t_n\} \) from the distinguished point \( x_0 = \infty \) to the point \( p_j \).

Let \( \beta_j \) be the loop in the punctured neighborhood \( V_j \setminus \{t_j\} \) with the distinguished point \( p_j \) which makes one positive circuit of \( t_j \), and let \( \gamma_j \) be the loop in the punctured sphere \( \mathbb{C}P^1 \setminus \{t_1, \ldots, t_n\} \) with the distinguished point \( x_0 = \infty \), defined by

\[
\gamma_j \overset{\text{def}}{=} \alpha_j^{-1} \cdot \beta_j \cdot \alpha_j
\]

(1.10)

Then:

- the loop \( \beta_j \) is said to be the small loop around \( t_j \) in the punctured neighborhood \( V_j \setminus \{t_j\} \);

- the loop \( \gamma_j \) is said to be the big loop around \( t_j \), corresponding to the path \( \alpha_j \).
Remark 1.5. Note that for a suitable choice of the paths \( \alpha_1, \ldots, \alpha_n \) the corresponding big loops \( \gamma_1, \ldots, \gamma_n \) generate the fundamental group \( \pi(\mathbb{C}P^1 \setminus \{t_1, \ldots, t_n\}; \infty) \). These generators are not free: choosing \( \alpha_1, \ldots, \alpha_n \) carefully we can ensure, for example, that \( \gamma_1 \cdots \gamma_n = 1 \).

We observe that the surface \( \text{cov}(\mathbb{C} \setminus \{p_j\}; \{p_j\}) \) in Definition 1.3 is naturally embedded into the universal covering surface \( \text{cov}(\mathbb{C} \setminus \{t_j\}; \{p_j\}) \), which is isomorphic to the Riemann surface of the logarithm \( \ln \zeta \).

Although the basic properties of the function \( \ln \zeta \) are very well-known, we shall discuss them in some detail, because they are important for our future considerations.

1.6. The Riemann surface of \( \ln \zeta \). For each fixed \( \zeta \in \mathbb{C} \setminus \{0\} \) the equation

\[
e^\lambda = \zeta
\]

has a countable set of solutions \( \lambda = \lambda(\zeta) \overset{\text{def}}{=} \ln \zeta \). These solutions can be parameterized as

\[
(1.11) \quad \ln \zeta = \ln |\zeta| + i \arg \zeta, \quad \text{where } \ln |\zeta| \in \mathbb{R}
\]

and \( \arg \zeta \) is an equivalence class of real numbers (the values of \( \arg \zeta \)) modulo addition by \( 2\pi \).

Now we explain how the function \( \ln \zeta \) can be defined as a single-valued holomorphic function on the universal covering surface \( \text{cov}(\mathbb{C} \setminus \{0\}; 1) \) of the punctured plane \( \mathbb{C} \setminus \{0\} \) with the distinguished point \( \zeta_0 = 1 \).

Let us choose a point \( \zeta \in \mathbb{C} \setminus \{0\} \) and a path \( \vartheta \in \mathbb{C} \setminus \{0\} \) from \( \zeta_0 = 1 \) to \( \zeta \). Then there exists a unique \( \theta \in \mathbb{R} \) such that, up to homotopy in \( \mathbb{C} \setminus \{0\} \), the path \( \vartheta \) can be parameterized as follows:

\[
(1.12) \quad \vartheta(s) = e^{(\ln |\zeta| + i\theta)s}, \quad 0 \leq s \leq 1
\]

(here \( \ln |\zeta| \) is the real-valued logarithm). The real number \( \theta \) is said to be the value of \( \arg \zeta \) corresponding to the path \( \vartheta \). In this manner we establish a 1-to-1 correspondence between the values of \( \arg \zeta \) and the homotopy classes of paths in \( \mathbb{C} \setminus \{0\} \) from \( \zeta_0 = 1 \) to \( \zeta \).

Thus the function \( \arg \zeta \) is defined as a single-valued continuous function on \( \text{cov}(\mathbb{C} \setminus \{0\}; 1) \). Accordingly, the function \( \ln \zeta \) is defined by \( \overset{\text{1.11}}{=\ln \zeta} \) as a single-valued holomorphic function on the universal covering surface \( \text{cov}(\mathbb{C} \setminus \{0\}; 1) \). In the sequel we shall often refer to the surface \( \text{cov}(\mathbb{C} \setminus \{0\}; 1) \) as the Riemann surface of \( \ln \zeta \).

The fundamental group \( \pi(\mathbb{C} \setminus \{0\}; 1) \) is cyclic, generated by the loop with the distinguished point \( \zeta_0 = 1 \) which makes one turn counterclockwise around the origin. The corresponding deck transformation of \( \text{cov}(\mathbb{C} \setminus \{0\}; 1) \) is denoted by

\[
(1.13) \quad \zeta \mapsto \zeta \cdot e^{2\pi i},
\]

so that the following monodromy relations hold:

\[
(1.14) \quad \arg(\zeta e^{2\pi i}) = \arg \zeta + 2\pi, \quad \ln(\zeta e^{2\pi i}) = \ln \zeta + 2\pi i.
\]

1.7. Transplants of functions defined on the Riemann surface of \( \ln \zeta \). Let us consider the universal covering surface \( \text{cov}(\mathbb{C} \setminus \{t_j\}; \{p_j\}) \), where \( t_j \) is some point in the complex plane \( \mathbb{C} \) and \( p_j \) is a distinguished point in the punctured plane \( \mathbb{C} \setminus \{t_j\} \).

Let us choose some value \( \theta_j \) of \( \arg(p_j - t_j) \) and let \( \vartheta_j \) denote the corresponding path (see (1.12)) in \( \mathbb{C} \setminus \{0\} \) from \( \zeta_0 = 1 \) to \( \zeta = p_j - t_j \). Let us define a mapping
from the universal covering surface \( \text{cov}(\mathbb{C} \setminus \{t_j\}; \{p_j\}) \) into the Riemann surface of \( \ln \zeta \) as follows.

To each point \((x, \alpha) \in \text{cov}(\mathbb{C} \setminus \{t_j\}; \{p_j\})\), where \( \alpha \) is a path in \( \mathbb{C} \setminus \{t_j\} \) from \( p_j \) to \( x \), we associate the point \((x - t_j, \alpha_{t_j} \cdot \theta_j) \in \text{cov}(\mathbb{C} \setminus \{0\}; 1)\), where the path \( \alpha_{t_j} \), which leads in \( \mathbb{C} \setminus \{0\} \) from \( p_j - t_j \) to \( x - t_j \), is obtained by the parallel translation of the path \( \alpha \):

\[
\alpha_{t_j}(s) = \alpha(s) - t_j, \quad 0 \leq s \leq 1.
\]

This mapping is an isomorphism between \( \text{cov}(\mathbb{C} \setminus \{t_j\}; \{p_j\}) \) and the Riemann surface of \( \ln \zeta \). It will be denoted by

\[
(1.15) \quad x \mapsto x - t_j, \quad x \in \text{cov}(\mathbb{C} \setminus \{t_j\}; \{p_j\}), \quad x - t_j \in \text{cov}(\mathbb{C} \setminus \{0\}; 1),
\]

and the inverse mapping will be denoted by

\[
(1.16) \quad x \mapsto x + t_j, \quad x \in \text{cov}(\mathbb{C} \setminus \{0\}; 1), \quad x + t_j \in \text{cov}(\mathbb{C} \setminus \{t_j\}; \{p_j\}).
\]

Accordingly, we shall denote the deck transformation of \( \text{cov}(\mathbb{C} \setminus \{t_j\}; \{p_j\}) \), corresponding to the loop with the distinguished point \( p_j \) which makes one positive circuit around \( t_j \) in \( \mathbb{C} \setminus \{t_j\} \), by

\[
(1.17) \quad x \mapsto t_j + (x - t_j) \cdot e^{2\pi i}, \quad x \in \text{cov}(\mathbb{C} \setminus \{t_j\}; \{p_j\}).
\]

With this notation we can consider the function \( \ln(x - t_j) \) as a function of \( x \), holomorphic on the universal covering surface \( \text{cov}(\mathbb{C} \setminus \{t_j\}; \{p_j\}) \) and such that (see (1.14))

\[
(1.18) \quad \ln((x - t_j) \cdot e^{2\pi i}) = \ln(x - t_j) + 2\pi i.
\]

More generally, we propose the following

**Definition 1.6.** Let a function \( E(\zeta) \) be defined on the Riemann surface of \( \ln \zeta \). Let \( t_j \in \mathbb{C} \), let \( p_j \in \mathbb{C} \setminus \{t_j\} \) and let us choose a value \( \theta_j \) of \( \arg(p_j - t_j) \). For each \( x \in \text{cov}(\mathbb{C} \setminus \{t_j\}; \{p_j\}) \) let \( x - t_j \) denote the image of \( x \) in the Riemann surface of \( \ln \zeta \) under the isomorphism (1.15).

The function \( E(x - t_j) \), defined as a function of \( x \) on the universal covering surface \( \text{cov}(\mathbb{C} \setminus \{t_j\}; \{p_j\}) \), is said to be the transplant of the function \( E(\zeta) \) into \( \text{cov}(\mathbb{C} \setminus \{t_j\}; \{p_j\}) \), corresponding to the value \( \theta_j \) of \( \arg(p_j - t_j) \).

1.8. The regular-principal factorization for a fundamental solution of a Fuchsian system: general case.

**Theorem 1.7.** Let \( Y \) be the solution of the Fuchsian system (1.11)–(1.12), satisfying the initial condition (1.13). For \( j = 1, \ldots, n \) assume that:

(i) \( V_j \subset \mathbb{C} \) is an open simply connected neighborhood of the singular point \( t_j \), such that \( t_k \notin V_j \) for \( k \neq j \);

(ii) \( p_j \) is a point in the punctured neighborhood \( V_j \setminus \{t_j\} \);

(iii) \( \alpha_j \) is a path in \( \mathbb{CP}^1 \setminus \{t_1, \ldots, t_n\} \) from the distinguished point \( x_0 = \infty \) to the point \( p_j \).

Then for each \( j, 1 \leq j \leq n \), the branch \( Y_{\alpha_j} \) of the solution \( Y \) in the punctured neighborhood \( V_j \setminus \{t_j\} \), corresponding to the path \( \alpha_j \), admits the factorization

\[
Y_{\alpha_j}(x) = H_{\alpha_j}(x) \cdot P_{\alpha_j}(x), \quad x \in \text{cov}(V_j \setminus \{t_j\}; \{p_j\}),
\]

where the factors possess the following properties:
(R) the matrix function $H_{\alpha_j}(x)$ is holomorphic and invertible in the entire \(^7\) (non-punctured) neighborhood $V_j$; 
(P) the matrix function $P_{\alpha_j}(x)$ is holomorphic and invertible on the universal covering surface $\text{cov}(\mathbb{C} \setminus \{t_j\}; p_j)$.

**Definition 1.8.** The factorization (1.19), where the factors $H_{\alpha_j}$ and $P_{\alpha_j}$ possess the properties (R) and (P), is said to be the regular-principal factorization of the branch $Y_{\alpha_j}$ of the solution $Y$ in a punctured neighborhood of the singular point $t_j$. The factors $H_{\alpha_j}$ and $P_{\alpha_j}$ are said to be, respectively, the regular factor and the principal factor of the branch $Y_{\alpha_j}$.

**Proof of Theorem 1.7.** For $j = 1, \ldots, n$ let $\gamma_j$ be the big loop around $t_j$, corresponding to the path $\alpha_j$ (see Definition 1.4), and let $M_{\gamma_j}$ be the corresponding monodromy matrix of $Y$.

Then, according to Definitions 1.1 and 1.3, the monodromy matrix $M_{\gamma_j}$ is given by

$$M_{\gamma_j} = Y_{\alpha_j}(t_j + (x - t_j) \cdot e^{2\pi i}), \quad \forall x \in \text{cov}(V_j \setminus \{t_j\}; p_j),$$

where $Y_{\alpha_j}$ is the branch of $Y$ in $V_j \setminus \{t_j\}$, corresponding to the path $\alpha_j$.

Since the matrix $M_{\gamma_j}$ is invertible, there exists a matrix denoted by $\ln M_{\gamma_j}$, such that

$$e^{\ln M_{\gamma_j}} = M_{\gamma_j}.$$

Let us choose a transplant $\ln(x - t_j)$ of the function $\ln \zeta$ into $\text{cov}(\mathbb{C} \setminus \{t_j\}; p_j)$. Then, according to (1.18), the matrix function

$$\frac{1}{(x - t_j)^{2\pi i} \ln M_{\gamma_j}} \overset{\text{def}}{=} e^{\ln(x - t_j)} \cdot \frac{1}{2\pi i} \ln M_{\gamma_j}, \quad x \in \text{cov}(\mathbb{C} \setminus \{t_j\}; p_j),$$

which is holomorphic and invertible on $\text{cov}(\mathbb{C} \setminus \{t_j\}; p_j)$, satisfies the relation

$$((x - t_j) \cdot e^{2\pi i})^{\ln M_{\gamma_j}} = (x - t_j)^{\ln M_{\gamma_j}} \cdot M_{\gamma_j}, \quad x \in \text{cov}(\mathbb{C} \setminus \{t_j\}; p_j).$$

Hence, in view of (1.20), the branch $Y_{\alpha_j}$ of the solution $Y$ in $V_j \setminus \{t_j\}$ has the form

$$Y_{\alpha_j}(x) = \Phi(x) \cdot (x - t_j)^{\ln M_{\gamma_j}}, \quad x \in \text{cov}(\mathbb{C} \setminus \{t_j\}; p_j),$$

where $\Phi(x)$ is a matrix function, holomorphic, invertible and single-valued in the punctured neighborhood $V_j \setminus \{t_j\}$.

Now, according to [Birk], §7, we can factorize the function $\Phi(x)$ as

$$\Phi(x) = \Phi_+(x) \cdot \Phi_-(x), \quad x \in V_j \setminus \{t_j\}$$

\(^7\)In particular, the function $H_{\alpha_j}(x)$ is single-valued in $V_j$.

\(^8\)Here we refer to [Gant, Chapter VIII, Section 8]. Such a matrix $\ln M_{\gamma_j}$ is not unique, of course, but for our purposes any choice of $\ln M_{\gamma_j}$ will do.
where $\Phi_+(x)$ and $\Phi_-(x)$ are matrix functions, single-valued, holomorphic and invertible in, respectively, the entire (non-punctured) neighborhood $V_j$ and the punctured plane $\mathbb{C} \setminus \{t_j\}$. We set

$$H_{\alpha_j}(x) \overset{\text{def}}{=} \Phi_+(x),$$

$$P_{\alpha_j}(x) \overset{\text{def}}{=} \Phi_-(x) \cdot (x - t_j)^{\frac{1}{2\pi i} \ln M_{\alpha_j}}$$

and obtain the desired factorization (1.19) with the properties (R) and (P). This completes the proof. □

**Remark 1.9.** Of course, the principal and regular factors of the branch $Y_{\alpha_j}$ of the solution $Y$ in a punctured neighborhood of the singularity $t_j$ are determined only up to the transformation

$$(1.21) \quad P_{\alpha_j}(x) \rightarrow T(x) \cdot P_{\alpha_j}(x), \quad H_{\alpha_j}(x) \rightarrow T(x)^{-1} \cdot H_{\alpha_j}(x),$$

where $T(x)$ is an invertible entire matrix function. However, once the choice of, say, the regular factor $H_{\alpha_j}$ is fixed, the principal factor $P_{\alpha_j}$ is uniquely determined.

Moreover, if we choose a different path from $x_0$ to $p_j$, say $\alpha'_j$, then, in view of Definitions 1.3 and 1.3, the branches $Y_{\alpha_j}$ and $Y_{\alpha'_j}$ of the solution $Y$ are related by

$$Y_{\alpha'_j}(x) = Y_{\alpha_j}(x) \cdot M_{\alpha_j^{-1} \alpha'_j}, \quad x \in \text{cov}(\mathbb{C} \setminus \{t_j\}; p_j),$$

where $M_{\alpha_j^{-1} \alpha'_j}$ is the monodromy matrix of $Y$, corresponding to the loop $\alpha_j^{-1} \alpha'_j \in \pi(\mathbb{CP}^1 \setminus \{t_1, \ldots, t_n\}; \infty)$. Hence the branch $Y_{\alpha'_j}$ admits the regular-principal factorization

$$Y_{\alpha'_j}(x) = H_{\alpha'_j}(x) \cdot P_{\alpha'_j}(x), \quad x \in \text{cov}(\mathbb{C} \setminus \{t_j\}; p_j),$$

where

$$H_{\alpha_j}(x) = H_{\alpha'_j}(x), \quad x \in V_j,$$

$$P_{\alpha_j}(x) = P_{\alpha'_j}(x) \cdot M_{\alpha_j^{-1} \alpha'_j}, \quad x \in \text{cov}(\mathbb{C} \setminus \{t_j\}; p_j).$$

Thus the regular factor at the singular point $t_j$ can be chosen independently of the choice of the path $\alpha_j$ and will be denoted simply by $H_j(x)$.

**Remark 1.10.** Up to now, we have made no use of the fact that the system (1.1) is Fuchsian (that is, each singularity is a simple pole for the coefficients of the system). In particular, the regular - principal factorization (1.19) of the fundamental solution of a linear differential system also takes place in a neighborhood of the isolated singularity, where the coefficients of the system have a higher order pole or even an essential singular point.

However, in the special case when the system is Fuchsian, the general form of the fundamental solution in a neighborhood of its singular point is quite well known (see, for instance, [Gant, Chapter XV, §10]) and thus much more precise statements concerning the principal factors of the solution of the system (1.1) can be made.

If the matrix $Q_j$ is non-resonant\(^9\), then the principal factor $P_{\alpha_j}$ can be chosen in the form

$$(1.22) \quad P_{\alpha_j}(x) = (x - t_j)^{A_{\alpha_j}},$$

\(^9\) A square matrix $Q$ is said to be non-resonant if distinct eigenvalues of $Q$ do not differ by integers or, in other words, if the spectra of the matrices $Q + nI$ and $Q$ are disjoint for every $n \in \mathbb{Z} \setminus 0$. 


where $A_{\alpha_j}$ is a matrix, similar to the matrix $Q_j$:

$$A_{\alpha_j} = C_{\alpha_j}^{-1} Q_j C_{\alpha_j},$$

where $C_{\alpha_j}$ is an invertible matrix. In the general case (without the assumption that the matrix $Q_j$ is non-resonant) the principal factor $P_{\alpha_j}$ can be chosen in the form

$$P_{\alpha_j}(x) = (x - t_j)^{Z_{\alpha_j}} \cdot (x - t_j)^{A_{\alpha_j}},$$

where $Z_{\alpha_j}$ is a diagonalizable matrix with integer eigenvalues $l_1, \ldots, l_k$ and $A_{\alpha_j}$ is a non-resonant matrix, whose eigenvalues $\lambda_p$ are related to the eigenvalues $\hat{\lambda}_p$ of the matrix $Q_j$ by the equations

$$\hat{\lambda}_p = \lambda_p - l_p, \quad 1 \leq p \leq k.$$

The matrices $Z_{\alpha_j}, A_{\alpha_j}$ also possess certain additional properties, but we shall not go into further details, because in the sequel our considerations will be mostly based on the existence of the regular - principal factorization (1.19) described in Theorem 1.7 rather than on the specific form of the factors.

2. Holomorphic Families of Fuchsian Differential Systems

2.1. Families of Fuchsian systems, parameterized by the pole loci. In the present paper we consider a family of linear differential systems of the form

$$dY = \left( \sum_{1 \leq j \leq n} Q_j(t) \cdot \frac{t_j}{x - t_j} \right) Y.$$

The variable $x$ "lives" in the punctured Riemann sphere $\mathbb{C}P^1 \setminus \{t_1, \ldots, t_n\}$, where $t_1, \ldots, t_n$ are pairwise distinct points of the complex plane $\mathbb{C}$. However, now $t_1, \ldots, t_n$ are not fixed but serve as the parameters of the family. The string $t = (t_1, \ldots, t_n)$ is considered as a point of $\mathbb{C}^n$ and the residue matrices $Q_j$ are assumed to depend on the parameter $t$. The "unknown" $Y$ is a square matrix function depending both on the "main variable" $x$ and on the parameter $t$.

**Definition 2.1.** Assume that the matrix functions $Q_j(t), 1 \leq j \leq n$, are defined and holomorphic for $t \in \mathcal{D}$, where $\mathcal{D}$ is a domain in $\mathbb{C}^n$. Then the family (2.1) is said to be a holomorphic family of Fuchsian systems, parameterized by the pole loci.

In what follows we assume that the condition

$$\sum_{1 \leq j \leq n} Q_j(t) = 0, \quad t \in \mathcal{D},$$

holds. Thus for every fixed $t \in \mathcal{D}$ the point $x = \infty$ is a regular point for the system (2.1), considered as a differential system with respect to $x$.

For each fixed $t \in \mathcal{D}$, the fundamental solution $Y = Y(x, t)$ of the differential system (2.1) with the initial condition

$$Y(x, t) \big|_{x=\infty} = I$$

is defined and holomorphic as a function of $x$ on the universal covering surface $\text{cov}(\mathbb{C}P^1 \setminus \{t_1, \ldots, t_n\}; \infty)$.

---

10. In view of (2.2), for each fixed $t \in \mathcal{D}$ the point $x = \infty$ is regular for the system (2.1) and hence the initial condition (2.3) can be posed.
In the present section our goal is to compare the properties of \( Y(x,t) \), such as the monodromy representation or the principal factors, for different \( t \). More precisely, we would like to understand what does it mean that "the monodromy representation or the principal factors of \( Y(x,t) \) are the same for different \( t \)? In the previous section these notions were defined in terms of homotopy classes of paths on the punctured Riemann sphere \( \mathbb{CP}^1 \setminus \{t_1, \ldots, t_n\} \). However, for different \( t = (t_1, \ldots, t_n) \) the domains \( \mathbb{CP}^1 \setminus \{t_1, \ldots, t_n\} \) are different. Thus one ought to explain how to consider "the same paths for different \( t \). This can be done because we can confine ourselves to local considerations.

2.2. Cylindrical neighborhoods of points in \( \mathbb{C}^n \).

**Definition 2.2.** Let \( W_j, 1 \leq j \leq n \), be subsets of the complex plane \( \mathbb{C} \).

The Cartesian product
\[
W = W_1 \times \cdots \times W_n \subset \mathbb{C}^n
\]
is said to be the cylindrical set with the bases \( W_j, 1 \leq j \leq n \).

**Definition 2.3.** Let \( t = (t_1, \ldots, t_n) \) be a point in \( \mathbb{C}^n \). For \( j = 1, \ldots, n \) let \( W_j \) be an open neighborhood of \( t_j \) in \( \mathbb{C} \), such that:

(i) the set \( W_j \) is simply connected;
(ii) the set \( \mathbb{C} \setminus \overline{W_j} \) is connected.

Denote by \( W \) the cylindrical set with the bases \( W_j, 1 \leq j \leq n \).

The set \( W \) is said to be an open cylindrical neighborhood of the point \( t \) in \( \mathbb{C}^n \).

**Definition 2.4.** Let subsets \( S, O \) of \( \mathbb{C}^n \) be such that:

(i) the set \( O \) is open;
(ii) the closure \( \overline{S} \) is compact;
(iii) \( \overline{S} \subset O \).

Then we say that the set \( S \) is compactly included in \( O \) and denote this relation by
\[
S \subset O.
\]

2.3. Isomonodromic families of Fuchsian systems. Let \( t^0 \) be a point in the domain\(^{11} \) \( D \) and let \( W \) be a cylindrical neighborhood of \( t^0 \), such that \( W \subset D \). Then
\[
(2.4) \quad \overline{W}_p \cap \overline{W}_q = \emptyset, \quad 1 \leq p, q \leq n, p \neq q,
\]
hence\(^{12} \) the set \( \mathbb{C} \setminus \bigcup_k \overline{W}_k \) is connected.

For a fixed \( t \in W \) each homotopy class of loops in the punctured sphere \( \mathbb{CP}^1 \setminus \{t_1, \ldots, t_n\} \) with the distinguished point \( x_0 = \infty \) has a representative which is a loop in the perforated sphere \( \mathbb{CP}^1 \setminus \bigcup_k \overline{W}_k \).

\(^{11}\)Recall that \( D \) is the domain where the residue matrices \( Q_j(t) \) from \( \text{[21]} \) are defined and holomorphic. Since \( D \subset \mathbb{C}^n \), the coordinates \( t^0_1, t^0_2, \ldots, t^0_n \) of every point \( t^0 \in D \) are pairwise distinct.

\(^{12}\)Here we refer to a relatively delicate result from general topology: if \( K_1, K_2 \) are two disjoint compact subsets of \( \mathbb{R}^m \) and each of two sets \( \mathbb{R}^m \setminus K_1, \mathbb{R}^m \setminus K_2 \) is connected, then the set \( \mathbb{R}^m \setminus \{K_1 \cup K_2\} \) is connected, as well. (See for example \cite{HW}, Corollary of Theorem VI.10.) Actually we do not need this result in full generality. For our goal it is enough to consider only the case of \( m = 2 \) and some very special compact sets \( K \subset \mathbb{R}^2 \), such as finite unions of disks, etc. In this particular case, the above stated result is elementary.
On the other hand, if $\gamma$ is a loop in the perforated sphere $\mathbb{CP}^1 \setminus \bigcup_k \overline{W_k}$ with the distinguished point $x_0 = \infty$, then for each fixed $t \in \mathcal{W}$ the loop $\gamma$ serves as a path of the analytic continuation with respect to $x$ for the solution $Y(x, t)$ of (2.1) – (2.3) and one can consider the corresponding monodromy matrix $^{13}M_\gamma(t)$.

Although the loop $\gamma$ does not depend on $t$, the corresponding monodromy matrix $M_\gamma(t)$ does, in general. We distinguish the following special case:

**Definition 2.5.** Let (2.1) be a holomorphic family of Fuchsian systems, where the residue matrices $Q_j(t)$ are holomorphic in a domain $\mathcal{D} \subseteq \mathbb{C}_+$ and satisfy (2.2). For each $t \in \mathcal{D}$, let $Y(x, t)$ be the solution of (2.1) – (2.3).

The family (2.1) is said to be an isomonodromic family of Fuchsian systems with the distinguished point $x_0 = \infty$ if for every $t^0 \in \mathcal{D}$ there exists a cylindrical open neighborhood $\mathcal{W} \Subset \mathcal{D}$ of $t^0$, such that the following holds.

For every loop $\gamma$ in the perforated sphere $\mathbb{CP}^1 \setminus \bigcup_k \overline{W_k}$ with the distinguished point $x_0 = \infty$ and every pair of points $t', t'' \in \mathcal{W}$ the monodromy matrices $M_\gamma(t')$, $M_\gamma(t'')$ of the solutions $Y(x, t')$, $Y(x, t'')$, which correspond to this loop $\gamma$, are equal:

$$M_\gamma(t') = M_\gamma(t'')$$  \hspace{1cm} (2.5)

**Remark 2.6.** Note that if the family (2.1) is isomonodromic with the distinguished point $x_0 = \infty$ and $t^0 \in \mathcal{D}$, then the monodromy matrices of $Y$ are constant with respect to $t$ in every cylindrical open neighborhood $\mathcal{W}$ of $t^0$, such that $\mathcal{W} \Subset \mathcal{D}$.

### 2.4. Isoprincipal families: the informal definition

Now we introduce the notion of the *isoprincipal* family of Fuchsian systems, which is the central notion in the present article.

Again, we assume that (2.1) is a holomorphic family of Fuchsian systems, where the residue matrices $Q_j(t)$ are holomorphic in a domain $\mathcal{D} \subseteq \mathbb{C}_+$ and satisfy (2.2), and consider the solution $Y(x, t)$ of (2.1) – (2.3).

According to Proposition 1.7, for each fixed $t \in \mathcal{D}$ a branch $Y_{\alpha_j}$ of the solution $Y$ in a neighborhood of $t_j$ admits the regular-principal factorization, but now both the regular factor $H_j$ and the principal factor $P_{\alpha_j}$ may depend on $t$:

$$Y_{\alpha_j}(x, t) = H_j(x, t) \cdot P_{\alpha_j}(x, t).$$ \hspace{1cm} (2.6)

For example, if the principal factor is chosen in the form (2.1), then $Z_{\alpha_j} = Z_{\alpha_j}(t)$, $A_{\alpha_j} = A_{\alpha_j}(t)$ and

$$P_{\alpha_j}(x, t) = (x - t_j)^{Z_{\alpha_j}(t)} \cdot (x - t_j)^{A_{\alpha_j}(t)}.$$ \hspace{1cm} (2.7)

Roughly speaking, the family (2.1) is isoprincipal if for every $j$, $1 \leq j \leq n$, the matrices $Z_{\alpha_j}$ and $A_{\alpha_j}$ in (2.7) do not depend on $t$: $Z_{\alpha_j}(t) \equiv Z_{\alpha_j}$, $A_{\alpha_j}(t) \equiv A_{\alpha_j}$, and

$$P_{\alpha_j}(x, t) = (x - t_j)^{Z_{\alpha_j}} \cdot (x - t_j)^{A_{\alpha_j}}.$$ \hspace{1cm} (2.8)

If the matrices $Z_{\alpha_j}$ and $A_{\alpha_j}$ in (2.8) do not depend on $t$, then the principal factor $P_{\alpha_j}(x, t)$ of the form (2.8) possesses the following property: *it depends only on the difference $x - t_j$.*

For our goals, the specific form (2.8) of the principal factors is of no importance. We just need each principal factor $P_{\alpha_j}(x, t)$ to depend only on the difference $x - t_j$.

\[\text{See Definition}\ (\text{13})]}
This means that there exist functions $E_{\alpha_j}$, such that
\begin{equation}
(2.9) \quad P_{\alpha_j}(x, t) = E_{\alpha_j}(x - t_j),
\end{equation}
or, in the language of differential equations,
\begin{equation}
(2.10a) \quad \frac{\partial P_{\alpha_j}}{\partial t_\ell} = 0, \quad \ell \neq j,
\end{equation}
\begin{equation}
(2.10b) \quad \frac{\partial P_{\alpha_j}}{\partial x} = -\frac{\partial P_{\alpha_j}}{\partial t_j} = \frac{dE_{\alpha_j}}{d\zeta} \bigg|_{\zeta = x - t_j}.
\end{equation}

The formal definition of the isoprincipal family of Fuchsian systems (see Definition 2.9 below) is more involved, since for each $t$ the branch $Y_{\alpha_j}(x, t)$ and the principal factor $P_{\alpha_j}(x, t)$ depend on the choice of a path $\alpha_j$ in the punctured sphere $\mathbb{C}P^1 \setminus \{t_1, \ldots, t_n\}$, which connects the distinguished point $x_0 = \infty$ with a neighborhood of $x = t_j$.

Moreover, for each $t$ the principal factor $P_{\alpha_j}(x, t)$ should be a transplant of a function $E_{\alpha_j}(\zeta)$, holomorphic on the Riemann surface of $\ln \zeta$, into a universal covering surface over $\mathbb{C} \setminus \{t_j\}$ and these transplants should be defined coherently with respect to $t$.

2.5. Isoprincipal families: the formal definition.

**Definition 2.7.** Let a function $E(\zeta)$ be defined on the Riemann surface of $\ln \zeta$. Let $W_j$ be a simply connected domain, compactly included in $\mathbb{C}$. Let $p_j \in \mathbb{C} \setminus W_j$ and let us choose a branch of $\arg(p_j - t_j)$, continuous with respect to $t_j$ in $W_j$ (since $W_j$ is simply connected and $W_j \subseteq \mathbb{C} \setminus \{p_j\}$, such a choice can be made). For every $t_j \in W_j$ let us specify the value of $\arg(p_j - t_j)$ in this manner and consider the corresponding transplant $E(x - t_j)$ of the function $E(\zeta)$ into $\text{cov}(\mathbb{C} \setminus \{t_j\}; p_j)$.

Then the family of transplants $\{E(x - t_j)\}_{t_j \in W_j}$ is said to be coherent with respect to $t_j$ in $W_j$.

**Definition 2.8.** Let $t^0$ be a point in a domain $D \subseteq \mathbb{C}^n$ and let $W, V$ be a pair of open cylindrical neighborhoods of the point $t^0$, such that

$$W \subseteq \mathbb{C}^n \quad \text{and} \quad V \subseteq \{\{V \cap D\}.}$$

Then the pair $W, V$ is said to be a nested pair of open cylindrical neighborhoods of the point $t^0$ in $D$.

**Definition 2.9** (Formal definition of the isoprincipal family). Let $\mathcal{D}$ be a holomorphic family of Fuchsian systems, where the residue matrices $Q_j(t)$ are holomorphic in a domain $D \subseteq \mathbb{C}^n$ and satisfy $2.2$. For $t \in D$ let $Y(x, t)$ be the solution of $2.1 - 2.3$.

The family $\mathcal{D}$ is said to be an isoprincipal family of Fuchsian systems with the distinguished point $x_0 = \infty$ if for every $t^0 \in D$ there exists a nested pair of open cylindrical neighborhoods of $t^0$: $\mathcal{V} = V_1 \times \cdots \times V_n$, $\mathcal{W} = W_1 \times \cdots \times W_n$, $\mathcal{V} \subseteq \mathcal{V}$, such that the following holds.

\footnote{See Definition 1.6.}
For every path $\alpha_j$ in the perforated sphere $\mathbb{CP}^1 \setminus \bigcup_k \mathcal{W}_k$ from the distinguished point $x_0 = \infty$ to a point $p_j \in \mathcal{V}_j \setminus \mathcal{W}_j$, $1 \leq j \leq n$, there exists a coherent family of transplants $\{E_{\alpha_j}(x - t_j)\}_{t_j \in \mathcal{V}_j}$ of a function $E_{\alpha_j}(\zeta)$, holomorphic and invertible on the Riemann surface of $\ln \zeta$, such that for each $t \in W$ the branch $Y_{\alpha_j}(x, t)$ of the solution $Y(x, t)$ in the punctured domain $\mathcal{V}_j \setminus \{t_j\}$ admits the representation

\begin{equation}
Y_{\alpha_j}(x, t) = H_j(x, t) \cdot E_{\alpha_j}(x - t_j), \quad x \in \text{cov}(\mathcal{V}_j \setminus \{t_j\}; p_j),
\end{equation}

where $H_j(x, t)$ is a function, holomorphic (with respect to $x$) and invertible in the entire domain $\mathcal{V}_j$.

**Remark 2.10.** In view of Definition 1.8 Definition 2.9 means that the family (2.1) is isoprincipal with the distinguished point $x_0 = \infty$ if every branch of the solution $Y(x, t)$ in a neighborhood of each singular point $x = t_j$ admits the regular-principal factorization (2.11), where the principal factor is the appropriately shifted copy of a function $E_{\alpha_j}(\zeta)$, which is holomorphic and invertible on the Riemann surface of $\ln \zeta$ and does not depend on $t$. The meaning of the words ”appropriately shifted copy” is made precise in Definition 2.9: this is what we call a coherent family of transplants of $E_{\alpha_j}(\zeta)$.

Thus Definition 2.9 is a formal interpretation of the informal definition in Section 2.4.

We would also like to note that it suffices to consider only the branches corresponding to a certain choice of the paths $\alpha_1, \ldots, \alpha_n$ – the same choice as the one mentioned in Remark 2.13 below.

2.6. Every isoprincipal family is an isomonodromic one.

**Theorem 2.11.** Let (2.1) be a holomorphic family of Fuchsian systems, where the residue matrices $Q_j(t)$ are holomorphic in a domain $\mathcal{D} \subseteq \mathbb{C}^*_+$ and satisfy (2.2).

Assume that the family (2.1) is isoprincipal with the distinguished point $x_0 = \infty$.

Then this family is isomonodromic with the distinguished point $x_0 = \infty$.

Before we turn to the proof of Theorem 2.11 let us introduce the following "$t$-dependent" counterpart of Definition 1.4.

**Definition 2.12.** Let $t^0$ be a point in the domain $\mathcal{D}$ and let $\mathcal{W} \subseteq \mathcal{V}$ be a nested pair of open cylindrical neighborhoods of $t^0$ in $\mathcal{D}$. For $1 \leq j \leq n$ let $\alpha_j$ be a path in the perforated sphere $\mathbb{CP}^1 \setminus \bigcup_k \mathcal{W}_k$ from the distinguished point $x_0 = \infty$ to a point $p_j \in \mathcal{V}_j \setminus \mathcal{W}_j$.

Furthermore, assume that $\beta_j$ is the loop in the annulus $\mathcal{V}_j \setminus \mathcal{W}_j$ with the distinguished point $p_j$ which makes one positive circuit of the set $\mathcal{W}_j$, and let $\gamma_j$ be the loop in the perforated sphere $\mathbb{CP}^1 \setminus \bigcup_k \mathcal{W}_k$ with the distinguished point $x_0 = \infty$, defined by

\begin{equation}
\gamma_j \overset{\text{def}}{=} \alpha_j^{-1} \cdot \beta_j \cdot \alpha_j.
\end{equation}

Then:

- the loop $\beta_j$ is said to be the small loop around the set $\mathcal{W}_j$ in the annulus $\mathcal{V}_j \setminus \mathcal{W}_j$;
- the loop $\gamma_j$ is said to be the big loop around the set $\mathcal{W}_j$, corresponding to the path $\alpha_j$. 
Remark 2.13. Similarly to the case of a fixed \( t \) (see Remark 1.9), for a suitable choice of the paths \( \alpha_1, \ldots, \alpha_n \) the corresponding big loops \( \gamma_1, \ldots, \gamma_n \) generate the fundamental group \( \pi(\mathbb{C}^1 \setminus \bigcup_k \mathbb{W}_k; \infty) \).

Proof of Theorem 2.11. Let \( t^0 \) be a point in \( \mathcal{D} \) and let \( \mathcal{W} \subseteq \mathcal{V} \) be a nested pair of open cylindrical neighborhoods of \( t^0 \) as in Definition 2.9.

In view of Remark 2.13, it suffices to prove that if \( \alpha_j \) is a path in \( \mathbb{C}^1 \setminus \bigcup_k \mathbb{W}_k \) from the distinguished point \( x_0 = \infty \) to a point \( p_j \in \mathcal{V}_j \setminus \mathbb{W}_j \) and \( \gamma_j \) is the corresponding big loop around \( \mathbb{W}_j \), then the monodromy matrix \( M_{\gamma_j}(t) \) of \( Y(x, t) \) does not depend on \( t \):

\[
M_{\gamma_j}(t) = \text{const}, \quad t \in \mathcal{W}.
\]

In view of (2.10), for each fixed \( t \in \mathcal{W} \) the monodromy matrix \( M_{\gamma_j}(t) \) is given by

\[
M_{\gamma_j}(t) = Y_{\alpha_j}^{-1}(x, t) \cdot Y_{\alpha_j}(t_j + (x - t_j) e^{2\pi i}, t), \quad \forall x \in \text{cov}(\mathcal{V}_j \setminus \{t_j\}; p_j),
\]

where \( Y_{\alpha_j}(x, t) \) is the branch of \( Y(x, t) \) in \( \mathcal{V}_j \setminus \{t_j\} \), corresponding to the path \( \alpha_j \).

Substituting the expression (2.4) for \( Y_{\alpha_j} \) into the above identity and taking into account that the factor \( H_j(x, t) \) is a single-valued function of \( x \), we obtain

\[
M_{\gamma_j}(t) = E_{\alpha_j}^{-1}(x - t_j) \cdot E_{\alpha_j}((x - t_j) e^{2\pi i})
\]

\[
= (E_{\alpha_j}^{-1}(\zeta) \cdot E_{\alpha_j}(\zeta e^{2\pi i})) \bigg|_{\zeta=x-t_j}, \quad \forall x \in \text{cov}(\mathcal{V}_j \setminus \{t_j\}; p_j).
\]

Thus the function \( E_{\alpha_j}^{-1}(\zeta) \cdot E_{\alpha_j}(\zeta e^{2\pi i}) \), holomorphic on the Riemann surface of \( \ln \zeta \), is constant with respect to \( \zeta \) on a certain non-empty open subset of this surface. Therefore, this function is identically constant on the Riemann surface of \( \ln \zeta \) and we write

\[
(2.14) \quad M_{\gamma_j}(t) = E_{\alpha_j}^{-1}(\zeta) \cdot E_{\alpha_j}(\zeta e^{2\pi i}) \quad \forall t \in \mathcal{W}, \quad \zeta \in \text{cov}(\mathbb{C} \setminus \{0; 1\}).
\]

But the right-hand side of the last identity does not depend on \( t \), hence we obtain (2.13). \( \square \)

2.7. Every non-resonant isomonodromic family is an isoprincipal one.

The converse to Theorem 2.11 is only conditionally true: it holds under the assumption that all the matrices \( Q_j \) are non-resonant (see footnote 9). In general, however, an isomonodromic family can be non-isoprincipal. The appropriate counterexample will be presented in Section 3 of this paper.

Lemma 2.14. Let \( \{2.4\} \) be a holomorphic family of Fuchsian systems, where the residue matrices \( Q_j(t) \) are holomorphic in a domain \( \mathcal{D} \subseteq \mathbb{C}_n^* \) and satisfy (2.2).

Assume that this family is isomonodromic with the distinguished point \( x_0 = \infty \).

Then the family \( \{2.4\} \) is isospectral in the following sense: for every pair of points \( t', t'' \in \mathcal{D} \) and each \( j, \ 1 \leq j \leq n \), the spectra \( \text{spec} Q_j(t') \) and \( \text{spec} Q_j(t'') \) are equal: \( ^{15} \)

\[
(2.15) \quad \text{spec} Q_j(t') = \text{spec} Q_j(t''), \quad \forall t', t'' \in \mathcal{D}, \ 1 \leq j \leq n.
\]

Theorem 2.15. Let \( \{2.4\} \) be a holomorphic family of Fuchsian systems, where the residue matrices \( Q_j(t) \) are holomorphic in a domain \( \mathcal{D} \subseteq \mathbb{C}_n^* \) and satisfy (2.2).

Assume that this family satisfies the following conditions:

\(^{15}\)As usual, the spectra are considered "with multiplicities".
Therefore, the modromy matrix hence since the family \((2.1)\) is isomonodromic, we have the regular-principal factorization \(\gamma\) Let where \(\text{spec} \left( e^{2\pi i A_{\alpha_j}(t)} \right) = \text{spec} \left( e^{2\pi i Q_j(t)} \right). \) Let \(\gamma_j\) be the big loop around \(W_j\), corresponding to the path \(\alpha_j\). Then the monodromy matrix \(M_{\gamma_j}(t)\) of \(Y(x, t)\), corresponding to the loop \(\gamma_j\) is given by \[(2.16)\quad M_{\gamma_j}(t) = Y_{\gamma_j}^{-1}(x, t) \cdot Y_{\alpha_j}(t_j + (x - t_j)e^{2\pi i}, t) = e^{2\pi i A_{\alpha_j}(t)}.\]

Therefore, \[
\text{spec}(M_{\gamma_j}(t)) = \text{spec} \left( e^{2\pi i Q_j(t)} \right), \quad t \in \mathcal{W}.
\]

Since the family \((2.1)\) is isomonodromic, we have \(M_{\gamma_j}(t) = M_{\gamma_j}(t^0), \quad t \in \mathcal{W},\) hence \[
\text{spec} \left( e^{2\pi i Q_j(t)} \right) = \text{spec} \left( e^{2\pi i Q_j(t^0)} \right), \quad t \in \mathcal{W}.
\]

This means that the spectra \(\text{spec} Q_j(t)\) and \(\text{spec} Q_j(t^0)\) coincide modulo integers. But the function \(Q_j(t)\) is continuous with respect to \(t\) in \(\mathcal{W}\), hence \[
\text{spec} Q_j(t) = \text{spec} Q_j(t^0), \quad t \in \mathcal{W}.
\]

Since the above identity holds for all \(t\) in a neighborhood of every point \(t^0 \in \mathcal{D}\), we obtain the identity \((2.15)\). \(\square\)

Proof of Theorem 2.16 Let \(t^0\) be a point in \(\mathcal{D}\) and let \(\mathcal{W} \subseteq \mathcal{V}\) be a nested pair of open cylindrical neighborhoods of \(t^0\) in \(\mathcal{D}\) as in Definition 2.3.

As in the proof of Lemma 2.14 for \(j = 1, \ldots, n\) let us choose a path \(\alpha_j\) in the perforated sphere \(\mathbb{CP}^1 \setminus \bigcup_k W_k\) from the distinguished point \(x_0 = \infty\) to a point \(p_j \in V_j \setminus W_j\) and consider for each fixed \(t \in \mathcal{W}\) the branch \(Y_{\gamma_j}(x, t)\) of the solution \(Y(x, t)\) of \((2.1) - (2.3)\) in the punctured domain \(V_j \setminus \{t_j\}\), corresponding to this path \(\alpha_j\).
Since by Lemma 2.14 the family (2.1) is isospectral, the matrix $Q_j(t)$ is non-resonant for every $t \in \mathcal{D}$. Hence, in view of Remark 1.10 (see the expressions (1.22), (1.23)), the branch $Y_{\alpha_j}$ admits the regular - principal factorization

$$Y(x, t) = H_j(x, t) \cdot (x - t_j)^{A_{\alpha_j}(t)}, \quad x \in \text{cov}(\mathcal{V}_j \setminus \{t_j\}; p_j),$$

where the matrix $A_{\alpha_j}(t)$ is similar to the matrix $Q_j(t)$. Therefore,

$$\text{spec}(A_{\alpha_j}(t)) = \text{spec}(A_{\alpha_j}(t^0)), \quad t \in \mathcal{W}. \tag{2.17}$$

In view of Definition 2.9 it remains to prove that the matrix $A_{\alpha_j}(t)$ does not actually depend on $t$.

Let $\gamma_j$ be the big loop around $\mathcal{W}_j$, corresponding to the path $\alpha_j$. Then the monodromy matrix $M_{\gamma_j}(t)$ of $Y(x, t)$, corresponding to the loop $\gamma_j$, is given by (see (2.16))

$$M_{\gamma_j}(t) = e^{2\pi i A_{\alpha_j}(t)}, \quad t \in \mathcal{W}.$$ 

Since the family (2.1) is isomonodromic, we have

$$M_{\gamma_j}(t) = M_{\gamma_j}(t^0), \quad t \in \mathcal{W},$$

hence

$$e^{2\pi i A_{\alpha_j}(t)} = e^{2\pi i A_{\alpha_j}(t^0)}, \quad t \in \mathcal{W}.$$ 

But, in view of (2.17), the last identity implies

$$A_{\alpha_j}(t) = A_{\alpha_j}(t^0) = \text{const}, \quad t \in \mathcal{W}. \tag*{□}$$

3. Isoprincipal Families of Fuchsian Systems and the Schlesinger System

3.1. The Schlesinger system. A natural question arises: how to express the property of a family of Fuchsian systems

$$\frac{dY}{dx} = \left( \sum_{1 \leq j \leq n} \frac{Q_j(t)}{x - t_j} \right) Y, \tag{3.1}$$

to be isoprincipal in terms of the residues matrix functions $Q_j(t)$? Here, as always, we assume that the residue matrices $Q_j(t)$ are holomorphic in a domain $\mathcal{D} \subseteq \mathbb{C}_n^*$ and satisfy

$$\sum_{1 \leq j \leq n} Q_j(t) \equiv 0, \quad t \in \mathcal{D}. \tag{3.2}$$

It turns out that an answer to this question is given by the so-called Schlesinger system of PDEs:

$$\begin{align*}
\frac{\partial Q_i}{\partial t_j} &= \frac{[Q_i, Q_j]}{t_i - t_j}, & 1 \leq i, j \leq n, \ i \neq j, \\
\frac{\partial Q_i}{\partial t_i} &= - \sum_{1 \leq j \leq n; j \neq i} \frac{[Q_i, Q_j]}{t_i - t_j}, & 1 \leq i \leq n. \tag{3.3}
\end{align*}$$

The following theorem is the main result of the present article.
Theorem 3.1 (The main result). Let (3.1) be a holomorphic family of Fuchsian systems, where the residue matrices $Q_j(t)$ are holomorphic in a domain $D \subseteq \mathbb{C}^n_*$ and satisfy (3.2).

Then the family (3.1) is isoprincipal with the distinguished point $x_0 = \infty$ if and only if the residue matrices $Q_j(t)$ satisfy with respect to $t$ the Schlesinger system (3.3) in the domain $D$.

Remark 3.2. Note that (3.3) implies

$$\frac{\partial}{\partial t_j} \sum_{1 \leq i \leq n} Q_i = 0, \quad 1 \leq j \leq n,$$

that is, $\sum_i Q_i(t)$ is a first integral of the Schlesinger system.

In particular, if functions $Q_j(t)$, $1 \leq j \leq n$, satisfy the Schlesinger system (3.3) in the domain $D$ and at some point $t^0 \in D$ it holds that

$$\sum_{1 \leq j \leq n} Q_j(t^0) = 0,$$

then these functions $Q_j(t)$ satisfy the relation (3.2).

Remark 3.3. We would like to stress that in Theorem 3.1 no assumptions on the spectra of the residue matrices $Q_j(t)$ are made. Thus Theorem 3.1 for the isoprincipal families of Fuchsian systems can be viewed as an amended version of L. Schlesinger’s statement, concerning the isomonodromic deformations (see [Sch2] and the introduction of the present article).

In the case of the isomonodromic families of Fuchsian systems Theorem 3.1 implies the following:

(i) If the residue matrices $Q_j(t)$ satisfy the Schlesinger system (3.3), then the family (3.1) is isoprincipal and hence by Theorem 2.11 also isomonodromic.

(ii) If the family (3.1) is isomonodromic and, in addition, all the residue matrices $Q_j(t)$ are non-resonant (at least at some point), then by Theorem 2.15 the family (3.1) is isoprincipal and hence the residue matrices $Q_j(t)$ satisfy the Schlesinger system (3.3).

We remark that in the statement (ii) the assumption of non-resonance for the residues $Q_j(t)$ cannot be omitted: in Section 5 we shall present an example of the isomonodromic family (3.1), where the residue matrices $Q_j(t)$ are resonant and do not satisfy the Schlesinger system (3.3) (thus contradicting the statement of L. Schlesinger).

Nevertheless, our proof of the "only if" part of the Theorem 3.1 largely follows the original proof of L. Schlesinger for the isomonodromic case (see also [IKSY] Section 3.5], where the modern adaptation of Schlesinger’s proof is presented). In particular, the overdetermined linear system (3.6) which appears in Proposition 3.6 below and is crucial in the derivation of the Schlesinger system, can be found in [Sch2] Section II].

The proof of Theorem 3.1 will be split into parts and presented as a series of propositions, culminating with Propositions 3.15 and 3.16.
3.2. The auxiliary system related to the isomonodromic family of Fuchsian systems. In order to prove Theorem 3.1, we have to study the partial derivatives of the solution $Y(x, t)$, satisfying the initial condition

$$Y(x, t) \big|_{x = \infty} = I,$$

with respect to the parameters $t_1, \ldots, t_n$.

First of all, let us choose and fix a point $t^0 \in \mathcal{D}$ and let $\mathcal{W} \Subset \mathcal{D}$ be a cylindrical open neighborhood of $t^0$. Since the coefficients of the system (3.1) and the initial condition (3.4) depend holomorphically on $t$, the solution $Y(x, t)$ is holomorphic jointly in $x$ and $t$ in the Cartesian product $\text{cov}(\mathbb{C}P^1 \setminus \bigcup_k \overline{W_k}; \infty) \times \mathcal{W}$.

In particular, the partial derivatives $\frac{\partial Y}{\partial t_j}(x, t^0)$ are defined and holomorphic with respect to $x$ in $\text{cov}(\mathbb{C}P^1 \setminus \bigcup_k \overline{W_k}; \infty)$. Since these definitions of $\frac{\partial Y}{\partial t_j}(x, t^0)$ agree for various choices of $\mathcal{W}$ as long as $\mathcal{W}$ is sufficiently small, we conclude that for each fixed $t^0 \in \mathcal{D}$ the partial derivatives $\frac{\partial Y}{\partial t_j}(x, t^0)$ are defined and holomorphic as functions of $x$ on the same surface as the function $Y(x, t^0)$ itself – the universal covering surface $\text{cov}(\mathbb{C}P^1 \setminus \{t_1^0, \ldots, t_n^0\}; \infty)$.

It turns out that in terms of these partial derivatives of $Y$ the property of the family (3.1) to be isomonodromic can be expressed as follows:

**Proposition 3.4.** Let (3.1) be a holomorphic family of Fuchsian systems, where the residue matrices $Q_j(t)$ are holomorphic in a domain $\mathcal{D} \subseteq \mathbb{C}_n^*$ and satisfy (3.2).

Then the family (3.1) is isomonodromic with the distinguished point $x_0 = \infty$ if and only if the solution $Y(x, t)$ of (3.1), (3.4) satisfies a linear system of the form

$$
\begin{cases}
\frac{\partial Y}{\partial x} = \sum_{1 \leq j \leq n} Q_j(t) \frac{x - t_j}{x - t_j} \cdot Y, \\
\frac{\partial Y}{\partial t_j} = T_j(x, t) \cdot Y, \quad 1 \leq j \leq n,
\end{cases}
$$

where for each $t \in \mathcal{D}$ the functions $T_j(x, t), \ 1 \leq j \leq n$, are single-valued holomorphic with respect to $x$ in $\mathbb{C}P^1 \setminus \{t_1, \ldots, t_n\}$.

**Definition 3.5.** Let the family (3.1) of Fuchsian systems be isomonodromic with the distinguished point $x_0 = \infty$.

The system (3.5) with the single-valued coefficients $T_j(x, t), \ 1 \leq j \leq n$, which appears in Theorem 3.4, is said to be the auxiliary linear system related to the isomonodromic family (3.1) of Fuchsian systems.

**Proof of Proposition 3.4.** The first equation of the system (3.5) is just the Fuchsian system (3.1) itself, hence we only need to prove that for each fixed $t \in \mathcal{D}$ the logarithmic derivatives

$$T_j(x, t) \overset{\text{def}}{=} \frac{\partial Y}{\partial t_j}(x, t) \cdot Y^{-1}(x, t),$$

which a priori are defined as holomorphic functions of $x$ on the universal covering surface $\text{cov}(\mathbb{C}P^1 \setminus \{t_1, \ldots, t_n\}; \infty)$, are single-valued in the punctured sphere $\mathbb{C}P^1 \setminus \{t_1, \ldots, t_n\}$. 
Let us choose a point $t^0 \in D$ and a cylindrical open neighborhood $\mathcal{W} \subseteq D$ of $t^0$. Let $\gamma \in \pi(\mathbb{C}P^1 \setminus \bigcup_k \overline{\mathcal{W}_k}; \infty)$ and let us denote by
\[
x \mapsto x \cdot \gamma, \quad x \in \text{cov}(\mathbb{C}P^1 \setminus \bigcup_k \overline{\mathcal{W}_k}; \infty)
\]
the deck transformation of the universal covering surface $\text{cov}(\mathbb{C}P^1 \setminus \bigcup_k \overline{\mathcal{W}_k}; \infty)$, corresponding to this loop $\gamma$.

According to Definition 1.1, the monodromy matrix $M_\gamma(t)$ of the solution $Y(x, t)$, which corresponds to the loop $\gamma$, is given by
\[
M_\gamma(t) = Y^{-1}(x, t) \cdot Y(x\gamma, t), \quad x \in \text{cov}(\mathbb{C}P^1 \setminus \bigcup_{1 \leq k \leq n} \overline{\mathcal{W}_k}; \infty), \quad t \in \mathcal{W}.
\]

The monodromy matrix $M_\gamma(t)$ does not depend on $x$ and hence is a holomorphic single-valued\(^{17}\) function of $t$ in $\mathcal{W}$. Differentiating the equality
\[
Y(x\gamma, t) = Y(x, t) \cdot M_\gamma(t)
\]
with respect to $t_j$, we obtain
\[
\frac{\partial Y}{\partial t_j}(x\gamma, t) = \frac{\partial Y}{\partial t_j}(x, t) \cdot M_\gamma(t) + Y(x, t) \cdot \frac{\partial M_\gamma}{\partial t_j}(t).
\]

Therefore, the logarithmic derivative
\[
T_j(x, t) = \frac{\partial Y}{\partial t_j}(x, t) \cdot Y^{-1}(x, t)
\]
satisfies the monodromy relation
\[
T_j(x\gamma, t) = T_j(x, t) + Y(x, t) \cdot \frac{\partial M_\gamma}{\partial t_j}(t) \cdot Y^{-1}(x\gamma, t).
\]

The last equality implies that the following two statements are equivalent\(^{18}\):

1. The monodromy matrix $M_\gamma(t)$ does not depend on $t_j$:
   \[
   \frac{\partial M_\gamma}{\partial t_j}(t) \equiv 0, \quad t \in \mathcal{W}.
   \]

2. It holds that
   \[
   T_j(x\gamma, t) \equiv T_j(x, t), \quad x \in \text{cov}(\mathbb{C}P^1 \setminus \bigcup_{1 \leq k \leq n} \overline{\mathcal{W}_k}; \infty), \quad t \in \mathcal{W}.
   \]

However, the statement 2) holds for every $\gamma \in \pi(\mathbb{C}P^1 \setminus \bigcup_k \overline{\mathcal{W}_k}; \infty)$ if and only if for each $t \in \mathcal{W}$ the function $T_j(x, t)$ is a single-valued function of $x$ in $\mathbb{C}P^1 \setminus \{t_1, \ldots, t_n\}$.

In view of Definition 2.5, this completes the proof. $\square$

---

\(^{16}\)See (1.4).

\(^{17}\)Recall (see Definition 2.2) that all the bases $\mathcal{W}_k$ of the cylindrical neighborhood $\mathcal{W}$ are simply connected.

\(^{18}\)This equivalence is stronger than the statement of Proposition 3.6 in the sense that it holds for each individual loop $\gamma$ and each individual index $j$. 
3.3. The auxiliary system related to the isoprincipal family of Fuchsian systems. Now we turn to the case, when the family \((3.1)\) is not only isomonodromic but, moreover, isoprincipal. We claim that in this special case the auxiliary linear system \((3.5)\) can be written explicitly in terms of the residues \(Q_j(t)\).

Proposition 3.6. Let \((3.1)\) be a holomorphic family of Fuchsian systems, where the residue matrices \(Q_j(t)\) are holomorphic in a domain \(D \subseteq \mathbb{C}^n\) and satisfy \((3.2)\).

Assume that the family \((3.1)\) is isoprincipal with the distinguished point \(x_0 = \infty\).

Then the solution \(Y(x, t)\) of \((3.1), (3.4)\) satisfies the following auxiliary system:

\[
\begin{align*}
\frac{\partial Y}{\partial x} &= \sum_{1 \leq j \leq n} \frac{Q_j(t)}{x - t_j} \cdot Y, \\
\frac{\partial Y}{\partial t_j} &= -\frac{Q_j(t)}{x - t_j} \cdot Y, \quad 1 \leq j \leq n.
\end{align*}
\]

In the proof of Proposition 3.6 we shall use the following

Lemma 3.7. Let \(U, V\) be simply-connected domains in the complex plane \(\mathbb{C}\), such that:

(i) \(U \subseteq V\);
(ii) the set \(V \setminus U\) is connected.

Let \(W\) be a domain in \(\mathbb{C}^n\) and let \(H(x, t)\) be a function of \(x \in V\) and \(t \in W\), possessing the following properties:

(a) the function \(H(x, t)\) is holomorphic (jointly in \(x\) and \(t\)) in \(\{V \setminus U\} \times W\);
(b) for each fixed \(t \in W\) the function \(H(x, t)\) is holomorphic with respect to \(x\) in the entire domain \(V\).

Then the function \(H(x, t)\) is holomorphic (jointly in \(x\) and \(t\)) in the domain \(V \times W\).

Remark 3.8. Lemma 3.7 is a special case of the well-known Hartogs lemma. However, in this simple case the conclusion follows immediately from the Cauchy integral formula.

Indeed, let \(\beta\) be a smooth loop in the annulus \(V \setminus \overline{W}\) which makes an \(n\) positive circuit of the set \(W\) and let \(\Delta \in W\) be a polydisk:

\[
\Delta = \Delta_1 \times \cdots \Delta_n.
\]

Then for each \(t \in \Delta\) and \(x \in \overline{W}\) it holds that

\[
H_j(x, t) = \frac{1}{2\pi i} \oint_{\beta_j} \frac{H_j(\zeta, t)}{\zeta - x} d\zeta
\]

\[
= \frac{1}{(2\pi i)^{n+1}} \oint_{\beta_j} \oint_{\partial \Delta_n} \cdots \oint_{\partial \Delta_1} \frac{H_j(\zeta, \tau_1, \ldots, \tau_n)}{(\tau_1 - t_1) \cdots (\tau_n - t_n)} d\tau_1 \cdots d\tau_n \oint_{\zeta - x} \frac{d\zeta}{\zeta - x},
\]

where \(\partial \Delta_k\) denotes the boundary of the disk \(\Delta_k\). Since the contours of integration lie in the domain \(\{V \setminus \overline{W}\} \times W\), where \(H_j(x, t)\) is jointly holomorphic in \(x\) and \(t\) (in particular, continuous), the integral represents a function, jointly holomorphic in \(x\) and \(t\) for \(x\) in a neighborhood of \(\overline{W}\) and \(t \in \Delta\).
Lemma 3.7 implies that it is jointly holomorphic in open neighborhoods of \( t \).

For \( \ell = 1, \ldots, n \) let us consider the logarithmic derivative

\[
T_\ell(x, t) = \frac{\partial Y}{\partial \ell}(x, t) \cdot Y^{-1}(x, t).
\]

Since by Theorem 2.11 the isoprincipal family (3.1) is also isomonodromic, Proposition 3.4 implies that for each fixed \( t \in \mathcal{W} \) the function \( T_\ell(x, t) \) is single-valued holomorphic with respect to \( x \) in the punctured sphere \( \mathbb{C}P^1 \setminus \{t_1, \ldots, t_n\} \). We have to prove that

\[
T_\ell(x, t) = -\frac{Q_\ell(t)}{x - t_\ell} \quad (3.8)
\]

To this end let us introduce the auxiliary function

\[
F_\ell(x, t) \overset{\text{def}}{=} T_\ell(x, t) + \frac{Q_\ell(t)}{x - t_\ell} \quad (3.9)
\]

which is single-valued holomorphic with respect to \( x \) in the punctured sphere \( \mathbb{C}P^1 \setminus \{t_1, \ldots, t_n\} \). We are going to show that for \( j = 1, \ldots, n \) the following statement holds:

\( \star \) The point \( x = t_j \) is a removable singularity of the function \( F_\ell(x, t) \).

Then, according to Liouville theorem, the function \( F_\ell(x, t) \) is constant with respect to \( x \). Since, as follows from (3.4), (3.9)

\[
T_\ell(x, t) \big|_{x=\infty} = F_\ell(x, t) \big|_{x=\infty} = 0, \quad (3.10)
\]

we can then conclude that

\[
F_\ell(x, t) \equiv 0,
\]

which leads to the desired result (3.8).

Now we turn to the proof of the statement (\( \star \)). To begin with, let us choose and fix a path \( \alpha_j \) in the perforated sphere \( \mathbb{C}P^1 \setminus \bigcup_k \mathcal{W}_k \) from the distinguished point \( x_0 = \infty \) to a point \( p_j \in \mathcal{V}_j \setminus \overline{\mathcal{W}_j} \). Then, according to Definition 2.9, for each \( t \in \mathcal{W} \) the branch \( Y_{\alpha_j}(x, t) \) of the solution \( Y(x, t) \) in \( \mathcal{V}_j \setminus \{t_j\} \) admits the regular-principal factorization

\[
Y_{\alpha_j}(x, t) = H_j(x, t) \cdot E_{\alpha_j}(x - t_j), \quad x \in \text{cov}(\mathcal{V}_j \setminus \{t_j\}; p_j), \quad (3.11)
\]

where the family \( \{E_{\alpha_j}(x - t_j)\}_{t_j \in \mathcal{W}_j} \) is a coherent family of transplants of a function \( E_{\alpha_j}(\zeta) \), holomorphic and invertible on the Riemann surface of \( \ln \zeta \), and the function \( H_j(x, t) \) is holomorphic with respect to \( x \) and invertible in the entire (non-punctured) domain \( \mathcal{V}_j \).

Then the principal factor \( E_{\alpha_j}(x - t_j) \) is jointly holomorphic in \( x \) and \( t_j \) in \( \text{cov}(\mathbb{C} \setminus \overline{\mathcal{W}_j}; p_j) \times \mathcal{W}_j \), and hence the regular factor

\[
H_j(x, t) = Y_{\alpha_j}^{-1}(x, t) \cdot E_{\alpha_j}(x - t_j)
\]

is jointly holomorphic (single-valued) in \( x \) and \( t \) in \( \{\mathcal{V}_j \setminus \overline{\mathcal{W}_j}\} \times \mathcal{W} \). Since the function \( H_j(x, t) \) is also holomorphic with respect to \( x \) in the entire domain \( \mathcal{V}_j \), Lemma 3.4 implies that it is jointly holomorphic in \( x \) and \( t \) in \( \mathcal{V}_j \times \mathcal{W} \).
Thus we can differentiate the equality \((3.11)\) with respect to \(t_\ell, 1 \leq \ell \leq n\). First, we consider the case \(\ell \neq j\). Then, since
\[
(3.12) \quad \frac{\partial E_{\alpha_j}(x - t_j)}{\partial t_\ell} = 0, \quad \ell \neq j,
\]
we obtain for \(x \in \mathcal{V}_j \setminus \{t_j\}\)
\[
T_\ell(x, t) = \frac{\partial Y_{\alpha_j}(x, t)}{\partial t_\ell} = \frac{\partial H_j(x, t)}{\partial t_\ell} \cdot E_{\alpha_j}(x - t_j) = \frac{\partial H_j(x, t)}{\partial t_\ell} \cdot H_j^{-1}(x, t) \cdot Y_{\alpha_j}(x, t).
\]
Hence
\[
F_\ell(x, t) = \frac{\partial H_j(x, t)}{\partial t_\ell} \cdot H_j^{-1}(x, t) + \frac{Q_\ell(t)}{x - t_\ell}, \quad x \in \mathcal{V}_j \setminus \{t_j\}, \quad \ell \neq j,
\]
which proves the statement \((\ast)\) in the case \(\ell \neq j\).

Next we differentiate the equality \((3.11)\) with respect to \(t_j\). Since
\[
(3.13) \quad \frac{\partial E_{\alpha_j}(x - t_j)}{\partial x} = - \frac{\partial E_{\alpha_j}(x - t_j)}{\partial t_j} = \frac{dE_{\alpha_j}(\zeta)}{d\zeta} \bigg|_{\zeta = x - t_j},
\]
we obtain
\[
T_j(x, t) = \frac{\partial H_j(x, t)}{\partial t_j} \cdot H_j^{-1}(x, t) - 
\]
\[
- H_j(x, t) \cdot \frac{\partial E_{\alpha_j}(x - t_j)}{\partial x} \cdot E_{\alpha_j}(x - t_j) \cdot H_j^{-1}(x, t), \quad \ell \in \mathcal{V}_j \setminus \{t_j\}; \quad p_j.
\]
On the other hand, differentiating the equality \((3.11)\) with respect to \(x\), we get
\[
\frac{\partial Y_{\alpha_j}(x, t)}{\partial x} \cdot Y_{\alpha_j}^{-1}(x, t) = \frac{\partial H_j(x, t)}{\partial x} \cdot H_j^{-1}(x, t) + 
\]
\[
+ H_j(x, t) \cdot \frac{\partial E_{\alpha_j}(x - t_j)}{\partial x} \cdot E_{\alpha_j}^{-1}(x - t_j) \cdot H_j^{-1}(x, t), \quad \ell \in \mathcal{V}_j \setminus \{t_j\}; \quad p_j,
\]
hence
\[
F_j(x, t) = \left( \frac{\partial H_j(x, t)}{\partial x} + \frac{\partial H_j(x, t)}{\partial t_j} \right) H_j^{-1}(x, t) + 
\]
\[
+ \frac{Q_j(t)}{x - t_j} - \frac{\partial Y_{\alpha_j}(x, t)}{\partial x} \cdot Y_{\alpha_j}^{-1}(x, t), \quad \ell \in \mathcal{V}_j \setminus \{t_j\}; \quad p_j.
\]
Taking into account that \(Y_{\alpha_j}(x, t)\) is a branch of the solution \(Y(x, t)\) of the Fuchsian system \((3.1)\), we obtain
\[
F_j(x, t) = \left( \frac{\partial H_j(x, t)}{\partial x} + \frac{\partial H_j(x, t)}{\partial t_j} \right) H_j^{-1}(x, t) - \sum_{1 \leq k \leq n} \frac{Q_k(t)}{x - t_k}, \quad x \in \mathcal{V}_j \setminus \{t_j\}.
\]
Thus the function \(F_j(x, t)\) has at \(x = t_j\) a removable singularity, which proves the statement \((\ast)\) in the case \(j = \ell\). \(\square\)

**Remark 3.9.** Note that the relations \((3.12), (3.13)\), which are instrumental in the proof of Proposition 3.6 are precisely the relations \((2.10)\) in the informal definition of the isoprincipal family of Fuchsian systems in Section 2.4.
Remark 3.10. We observe that the auxiliary linear system \( (3.6) \) leads to a linear system for the regular factor \( H_j(x, t) \) of the regular-principal factorization \( (3.11) \).

Indeed, in view of \( (3.12) \) and \( (3.13) \), we can differentiate the equality \( (3.11) \) to obtain

\[
\frac{\partial H_j}{\partial t}(x, t) \cdot H^{-1}_j(x, t) = \frac{\partial Y_{\alpha_j}}{\partial t}(x, t) \cdot Y^{-1}_{\alpha_j}(x, t), \quad j \neq \ell,
\]

and therefore

\[
(3.14) \begin{cases}
\frac{\partial H_j}{\partial t}(x, t) = -Q_{j \ell} \frac{x - t_j}{x - t_\ell} \cdot H_j, \quad \ell \neq j, \\
\frac{\partial H_j}{\partial t}(x, t) + \frac{\partial H_j}{\partial x}(x, t) = \sum_{1 \leq \ell \leq n} Q_{j \ell} \frac{x - t_j}{x - t_\ell} \cdot H_j.
\end{cases}
\]

Using the change of variables

\[
(3.15) \quad \zeta = x - t_j,
\]

\[
(3.16) \quad L_j(\zeta, t) \overset{\text{def}}{=} H_j(\zeta + t_j, t),
\]

one can rewrite the system \( (3.14) \) in the following form:

\[
(3.17) \begin{cases}
\frac{\partial L_j}{\partial t}(\zeta, t) = -Q_{j \ell} \frac{\zeta + t_j - t_\ell}{\zeta + t_j - t_\ell} \cdot L_j, \quad \ell \neq j, \\
\frac{\partial L_j}{\partial \zeta}(\zeta, t) = \sum_{1 \leq \ell \leq n} Q_{j \ell} \frac{\zeta + t_j - t_\ell}{\zeta + t_j - t_\ell} \cdot L_j.
\end{cases}
\]

The system \( (3.17) \) is nothing more than the system \( (3.6) \) with the constraint

\[ x - t_j = \zeta = \text{const}. \]

Note that, although \( x = t_j \) is a singularity of the Fuchsian system \( (3.1) \) and the auxiliary system \( (3.6) \), the right-hand side of the system \( (3.17) \) is holomorphic with respect to \( \zeta \) at \( \zeta = 0 \) (compare with Remark 1.2 in [Ma1]). This occurs because the function \( L_j(\zeta, t) \), defined in \( (3.16) \), is holomorphic with respect to \( \zeta \) and invertible at \( \zeta = 0 \).

3.4. The Frobenius theorem. The auxiliary system \( (3.6) \), related to the iso-principal family \( (3.1) \) is an overdetermined system of PDEs. The criterion for the existence of solution of such a system is known (see [Na, Section 2.11]) as the Frobenius theorem:

**Theorem 3.11** (The Frobenius theorem for Pfaffian systems). Let \( \Omega_p \) and \( \Omega_q \) be domains in, respectively, \( \mathbb{C}^p \) and \( \mathbb{C}^q \). Consider the following system of PDEs:

\[
(3.18) \quad \frac{\partial \lambda}{\partial t_j}(\lambda, \mu), \quad 1 \leq j \leq q,
\]

where \( \lambda(\mu) \) is an unknown \( \mathbb{C}^p \)-valued function of the variable \( \mu = (\mu_1, \ldots, \mu_q) \in \mathbb{C}^q \) and \( \phi_j(\lambda, \mu) = (\phi_{1j}(\lambda, \mu), \ldots, \phi_{pj}(\lambda, \mu)), \quad 1 \leq j \leq q, \) are given \( \mathbb{C}^p \)-valued functions, holomorphic with respect to \( \lambda, \mu \) in the domain \( \Omega_p \times \Omega_q \).

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19 A Pfaffian system is a system of the form \( (3.6) \).
Then the following statements (i) and (ii) are equivalent:

(i) For every pair of points $\lambda^0 \in \Omega_p$, $\mu^0 \in \Omega_q$ there exists a solution $\lambda(\mu)$ of the system (3.18), holomorphic in a neighborhood of $\mu^0$ and satisfying the initial condition

$$\lambda(\mu^0) = \lambda^0.$$ 

(ii) The $\mathbb{C}$-valued functions $\phi_{i,j}(\lambda, \mu)$ satisfy the equations

$$\frac{\partial \phi_{i,j}(\lambda, \mu)}{\partial \mu_k} + \sum_{1 \leq \ell \leq p} \frac{\partial \phi_{i,j}(\lambda, \mu)}{\partial \lambda_{\ell}} \cdot \phi_{\ell,k}(\lambda, \mu) = \frac{\partial \phi_{i,k}(\lambda, \mu)}{\partial \mu_j} + \sum_{1 \leq \ell \leq p} \frac{\partial \phi_{i,k}(\lambda, \mu)}{\partial \lambda_{\ell}} \cdot \phi_{\ell,j}(\lambda, \mu),$$

$$1 \leq i \leq p, \ 1 \leq j, k \leq q,$$ 

in the domain $\Omega_p \times \Omega_q$.

**Definition 3.12.** The condition (3.19), formulated in the Frobenius theorem 3.11, is said to be the compatibility condition for the overdetermined system of PDEs (3.18). The overdetermined system of PDEs (3.18) which satisfies the compatibility condition (3.19) is said to be compatible.

**Remark 3.13.** Note that the compatibility condition (3.19) can be obtained by substituting the equations (3.18) into the identity

$$\frac{\partial^2 \lambda_i}{\partial \mu_j \partial \mu_k} = \frac{\partial^2 \lambda_i}{\partial \mu_k \partial \mu_j}, \quad 1 \leq i \leq p, \ 1 \leq j, k \leq q.$$ 

Thus the statement (ii) of the Frobenius theorem (3.11) follows from the statement (i) immediately.

In what follows, we shall often deal with linear overdetermined systems of PDEs, depending on a parameter. Because of the global existence theorem for such systems, the following stronger version of the Frobenius theorem (3.11) holds in this case:

**Theorem 3.14 (The Frobenius theorem for linear systems with a parameter).** Let $\Omega_q$ and $\Omega_r$ be domains in, respectively, $\mathbb{C}^q$ and $\mathbb{C}^r$. Consider the following linear systems of PDEs:

$$\frac{\partial \lambda}{\partial \mu_j} = \Phi_j(\mu, \tau) \cdot \lambda, \quad 1 \leq j \leq q,$$

where $\mu \in \Omega_q$ is the "main" variable, $\tau \in \Omega_r$ is a parameter and $\Phi_j(\mu, \tau), 1 \leq j \leq q$, are $\mathbb{M}_p$-valued functions, holomorphic with respect to $\mu$ in the domain $\Omega_q$ for each fixed $\tau \in \Omega_r$.

Then:

(i) For each fixed value of the parameter $\tau$, say $\tau = \tau^0 \in \Omega_r$, the linear system

$$\frac{\partial \lambda}{\partial \mu_j} = \Phi_j(\mu, \tau^0) \cdot \lambda, \quad 1 \leq j \leq q,$$

Note that, according to the uniqueness theorem for ordinary differential equations, such a solution $\lambda(\mu)$ is necessarily unique.
has a fundamental solution\(^{21}\) \(\lambda(\mu, \tau^0)\) if and only if the functions \(\Phi_j(\mu, \tau^0)\) satisfy the equations

\[
\frac{\partial \Phi_j(\mu, \tau^0)}{\partial \mu} - \frac{\partial \Phi_k(\mu, \tau^0)}{\partial \mu_j} = [\Phi_k(\mu, \tau^0), \Phi_j(\mu, \tau^0)], \quad \mu \in \Omega_q, \quad 1 \leq j, k \leq q.
\]

(ii) If the functions \(\Phi_j(\mu, \tau)\) are jointly holomorphic with respect to \(\mu, \tau\) in the domain \(\Omega_q \times \Omega_r\) and satisfy the equations

\[
\frac{\partial \Phi_j(\mu, \tau)}{\partial \mu_k} - \frac{\partial \Phi_k(\mu, \tau)}{\partial \mu_j} = [\Phi_k(\mu, \tau), \Phi_j(\mu, \tau)], \quad \mu \in \Omega_q, \tau \in \Omega_r, \quad 1 \leq j, k \leq q,
\]

then for every point \(\mu^0 \in \Omega_q\) and every \(\mathcal{M}_p\)-valued function \(\lambda^0(\tau)\), holomorphic and invertible in \(\Omega_r\), there exists a unique fundamental solution \(\lambda(\mu, \tau)\) of the system \((3.20)\), jointly holomorphic with respect to \(\mu, \tau\) in the domain \(\Omega_q \times \Omega_r\) and satisfying the initial condition

\[
\lambda(\mu, \tau) \big|_{\mu=\mu^0} = \lambda^0(\tau).
\]

3.5. Proof of Theorem 3.1

Proposition 3.15. Let \(3.1\) be a holomorphic family of Fuchsian systems, where the residue matrices \(Q_j(\mu)\) are holomorphic in a domain \(\mathcal{D} \subseteq \mathbb{C}^*_p\) and satisfy \((3.22)\). Assume that the family \((3.1)\) is isoprincipal with the distinguished point \(x_0 = \infty\). Then the residues \(Q_1(\mu), \ldots, Q_n(\mu)\) satisfy the Schlesinger system \((3.20)\) in the domain \(\mathcal{D}\).

Proof. According to Proposition \(3.16\), the solution \(Y(x, t)\) of \((3.1), (3.4)\) satisfies the auxiliary linear system \((3.6)\). Therefore, \(Y(x, t)\) is the fundamental solution of the overdetermined linear system \((3.6)\) with the initial condition

\[
Y(\infty, t^0) = I,
\]

where \(t^0\) is an arbitrary fixed point in the domain \(\mathcal{D}\).

Hence, in view of Theorem \(3.14\), the linear system \((3.6)\) (which is a special case of the system \((3.20)\) with \(q = n + 1, r = 0, \lambda(\mu) = Y(x, t)\)) is compatible. The compatibility condition \((3.22)\) takes in this case the form of the following two equations:

\[
\frac{\partial}{\partial x} \left( \frac{Q_j}{x-t_j} \right) + \sum_{1 \leq i \leq n} \frac{\partial}{\partial t_i} \left( \frac{Q_i}{x-t_i} \right) = \sum_{1 \leq i \leq n} \frac{[Q_i, Q_j]}{(x-t_i)(x-t_j)}, \quad 1 \leq j \leq n,
\]

\[
\frac{\partial}{\partial t_i} \left( \frac{Q_i}{x-t_i} \right) - \frac{\partial}{\partial t_j} \left( \frac{Q_j}{x-t_j} \right) = \frac{[Q_i, Q_j]}{(x-t_i)(x-t_j)}, \quad 1 \leq i, j \leq n.
\]

Since the residues \(Q_j\) do not depend on \(x\), from the equation \((3.26)\) we obtain

\[
\sum_{1 \leq i \leq n} \frac{\partial Q_i}{\partial t_j} \cdot \frac{1}{x-t_i} = \sum_{1 \leq i \leq n} \frac{[Q_i, Q_j]}{(x-t_i)(x-t_j)}, \quad 1 \leq j \leq n.
\]

Both sides of the last equality are rational functions of \(x\), which are holomorphic in the punctured sphere \(\mathbb{C}^p \setminus \{t_1, \ldots, t_n\}\) and have simple poles at the points \(x = t_i\),

\(^{21}\)In other words, an \(\mathcal{M}_p\)-valued solution \(\lambda(\mu, \tau^0)\), such that \(\det \lambda(\mu, \tau^0) \neq 0\) for \(\mu \in \Omega_q\).
Let us choose Assume that the residue matrices $V(3.30)$ of $(3.1), (3.4)$. In view of Remark 3.10, we consider the overdetermined linear equations $(3.27)$.

We set $\zeta$ where the factors $H_j(x, t^0)$ and $P_{\alpha_j}(x, t^0)$ are holomorphic with respect to $x$ and invertible in, respectively, $V_j$ and $\text{cov}(\mathbb{C} \setminus \{t^0_j\}; p_j)$.

From here we proceed in four steps: 

**Step 1.** Firstly, we construct a function $E_{\alpha_j}(\zeta)$, holomorphic and invertible on the Riemann surface of $\ln \zeta$ and such that the principal factor $P_{\alpha_j}(x, t^0)$ is a transplant of $E_{\alpha_j}(\zeta)$ into $\text{cov}(\mathbb{C} \setminus \{t^0_j\}; p_j)$:

$$P_{\alpha_j}(x, t^0) = E_{\alpha_j}(x - t^0_j).$$

To this end we choose some value $\theta^0_j$ of $\text{arg}(p_j - t^0_j)$ and consider the corresponding isomorphism from the Riemann surface of $\ln \zeta$ onto $\text{cov}(\mathbb{C} \setminus \{t^0_j\}; p_j)$ (see (1.16)):

$$\zeta \xrightarrow{\text{arg}(p_j - t^0_j) = \theta^0_j} \zeta + t^0_j, \quad \zeta \in \text{cov}(\mathbb{C} \setminus \{0\}; 1), \quad \zeta + t^0_j \in \text{cov}(\mathbb{C} \setminus \{t^0_j\}; p_j).$$

We set

$$E_{\alpha_j}(\zeta) \overset{\text{def}}{=} P_{\alpha_j}(\zeta + t^0_j, t^0),$$

where $\zeta + t^0_j$ denotes the image in $\text{cov}(\mathbb{C} \setminus \{t^0_j\}; p_j)$ of the point $\zeta \in \text{cov}(\mathbb{C} \setminus \{0\}; 1)$ under the isomorphism $\theta^0_j$.

**Step 2.** Secondly, we construct the regular factor $H_j(x, t)$ for the solution $Y(x, t)$ of $(3.1), (3.4)$. In view of Remark 3.10, we consider the overdetermined linear system

$$
\begin{align*}
\frac{\partial L_j(\zeta, t)}{\partial t_i} &= -\frac{Q_j(t)}{\zeta + t_j - t_i} \cdot L_j(\zeta, t), \quad 1 \leq i \leq n, \ i \neq j, \\
\frac{\partial L_j(\zeta, t)}{\partial t_j} &= \sum_{1 \leq \ell \leq n} \frac{Q_j(t)}{\zeta + t_j - t_i} \cdot L_j(\zeta, t),
\end{align*}
$$

with the initial condition

$$L_j(\zeta, t) \big|_{t=t^0} = H_j(\zeta + t^0_j, t^0).$$
where $H_j(x, t^0)$ is the regular factor in the factorization (3.32). Here $\zeta \in \mathbb{C}$ is a small parameter:

\[(3.32) \quad |\zeta| < \epsilon, \quad \text{where } \epsilon > 0 \text{ is such that} \]
\[x_1 \in W_j \implies \{x : |x - x_1| < \epsilon \} \subset V_j, \quad 1 \leq j \leq n.\]

We claim that the overdetermined linear system (3.30), depending on the parameter $\zeta$, is compatible (see Calculation 1 below).

Therefore, the system (3.30) has a solution $L_j(\zeta, t)$, satisfying the initial condition (3.31). This solution $L_j(\zeta, t)$ is jointly holomorphic in $\zeta, t$ for $|\zeta| < \epsilon, t \in \mathcal{W}$ and invertible.

We define the function $H_j(x, t)$ by

\[(3.33) \quad H_j(x, t) \overset{\text{def}}{=} L_j(x - t_j, t), \quad t \in \mathcal{W}, \quad x \in V_j\]

**Step 3.** Thirdly, we consider the product

\[(3.34) \quad Z_{\alpha_j}(\zeta, t) \overset{\text{def}}{=} L_j(\zeta, t) \cdot E_{\alpha_j}(\zeta).\]

Note that, in view of (3.33), for $t = t^0$ we have

\[Z_{\alpha_j}(\zeta, t^0) = Y_{\alpha_j}(\zeta + t_j^0, t^0),\]

hence the function $Z_{\alpha_j}(\zeta, t^0)$ satisfies the linear system

\[(3.35) \quad \frac{dZ_{\alpha_j}(\zeta, t^0)}{d\zeta} = \sum_{1 \leq i \leq n} \frac{Q_i(t^0)}{\zeta + t_j^0 - t_i} \cdot Z_{\alpha_j}(\zeta, t^0).\]

Also, as follows from (3.30), we have

\[(3.36) \quad \frac{\partial Z_{\alpha_j}(\zeta, t)}{\partial t_i} = -\frac{Q_i(t)}{\zeta + t_j - t_i} \cdot Z_{\alpha_j}(\zeta, t), \quad 1 \leq i \leq n, \quad i \neq j,\]

\[(3.37) \quad \frac{\partial Z_{\alpha_j}(\zeta, t)}{\partial t_j} = \sum_{1 \leq i \leq n, \ i \neq j} \frac{Q_i(t)}{\zeta + t_j - t_i} \cdot Z_{\alpha_j}(\zeta, t).\]

Furthermore, the function $Z_{\alpha_j}(\zeta, t)$ can be shown (see Calculation 2 below) to satisfy with respect to $\zeta$ the linear system

\[(3.38) \quad \frac{\partial Z_{\alpha_j}(\zeta, t)}{\partial \zeta} = \sum_{1 \leq i \leq n} \frac{Q_i(t)}{\zeta + t_j - t_i} \cdot Z_{\alpha_j}(\zeta, t).\]

We note that for each fixed $t \in \mathcal{W}$ the linear differential system (3.38) with respect to $\zeta$ has no singularities in the punctured domain

\[V_{ij} \overset{\text{def}}{=} \{\zeta : \zeta + t_j \in V_j \setminus \{0\}\},\]

hence its fundamental solution $Z_{\alpha_j}(\zeta, t)$ is holomorphic with respect to $\zeta$ on a universal covering surface over the domain $V_{ij}$. Therefore, the function

\[L_j(\zeta, t) = Z_{\alpha_j}(\zeta, t) \cdot E_{\alpha_j}^{-1}(\zeta)\]

is holomorphic with respect to $\zeta$ and invertible on this universal covering surface. On the other hand, the function $L_j(\zeta, t)$ is also single-valued holomorphic with respect to $\zeta$ and invertible in the open disk $\{\zeta : |\zeta| < \epsilon\}$ (see (3.32)). Hence the function $L_j(\zeta, t)$ is single-valued holomorphic with respect to $\zeta$ and invertible in the non-punctured domain $V_{ij} \cup \{0\}$. 


It follows that the function \( H_j(x, t) \), defined in (3.33), is holomorphic (single-valued) with respect to \( x \) and invertible in the entire domain \( \mathcal{V}_j \).

**Step 4.** Finally, we consider the coherent family of transplants \( \{E_{\alpha_j}(x - t_j)\}_{t_j \in \mathcal{W}_j} \) (see Definition 2.7), corresponding to the unique branch of \( \arg(p_j - t_j) \) which is continuous with respect to \( t_j \) in \( \mathcal{W}_j \) and takes the value \( \theta^0 \), chosen at Step 1, at the point \( t_j = t_j^0 \).

For each \( t \in \mathcal{W} \) we define the function \( Y_{\alpha_j}(x, t) \) by

\[
Y_{\alpha_j}(x, t) \overset{\text{def}}{=} H_j(x, t) \cdot E_{\alpha_j}(x - t_j) = Z_{\alpha_j}(x - t_j, t), \quad x \in \text{cov}(\mathcal{V}_j \setminus \{t_j\}; p_j).
\]

Note that in view of (3.29), (3.31) this definition agrees for \( t = t^0 \) with (3.26) and the notation \( Y_{\alpha_j}(x, t^0) \) for the branch of the solution \( Y(x, t^0) \) in \( \mathcal{V}_j \setminus \{t_j^0\} \), corresponding to the path \( \alpha_j \).

Now we show that for every \( t \in \mathcal{W} \) the function \( Y_{\alpha_j}(x, t) \), defined in (3.39), is the branch of the solution \( Y(x, t) \) of (3.1), (3.4) in the punctured domain \( \mathcal{V}_j \setminus \{t_j\} \), corresponding to the path \( \alpha_j \).

First of all, in view of (3.38), the function \( Y_{\alpha_j}(x, t) \) satisfies with respect to \( x \) the system (3.41):

\[
\frac{\partial Y_{\alpha_j}}{\partial x}(x, t) = \sum_{1 \leq i \leq n} \frac{Q_i(t)}{x - t_i} \cdot Y_{\alpha_j}(x, t).
\]

Hence for each \( t \in \mathcal{W} \) the function \( Y_{\alpha_j}(x, t) \) can be analytically continued along the path \( \alpha_j \), in the opposite direction: from \( p_j \) to \( \infty \). The value of this continuation at \( x = \infty \) will be denoted by \( \hat{Y}(\infty, t) \); in particular it holds that

\[
\hat{Y}(\infty, t^0) = Y(\infty, t^0) = I.
\]

Furthermore, the value \( \hat{Y}(\infty, t) \) can be considered as the initial value at the distinguished point \( x_0 = \infty \) for a fundamental solution \( \hat{Y}(x, t) \) of the Fuchsian system (3.1), defined on the universal covering surface \( \text{cov}(\mathbb{C}P^1 \setminus \{t_1, \ldots, t_n\}; \infty) \).

The function \( Y_{\alpha_j}(x, t) \) is the branch of this solution \( \hat{Y}(x, t) \) in the punctured domain \( \mathcal{V}_j \setminus \{t_j\} \), corresponding to the path \( \alpha_j \).

Now we note that, in view of (3.30) – (3.37),

\[
\frac{\partial \hat{Y}}{\partial t_i}(x, t) = -\frac{Q_i(t)}{x - t_i} \cdot \hat{Y}(x, t), \quad 1 \leq i \leq n;
\]

in particular,

\[
\frac{\partial \hat{Y}}{\partial t_i}(x, t) \bigg|_{x=\infty} = 0, \quad 1 \leq i \leq n.
\]

Combining (3.41) with (3.40), we observe that the solution \( \hat{Y}(x, t) \) satisfies the initial condition

\[
\hat{Y}(x, t) \bigg|_{x=\infty} = I, \quad t \in \mathcal{W},
\]

and by the uniqueness theorem for linear differential systems coincides with the fundamental solution \( Y(x, t) \) of (3.1), (3.4).

Thus the function \( Y_{\alpha_j}(x, t) \), defined in (3.39), is the branch of the solution \( Y(x, t) \) of (3.1), (3.4) in the punctured domain \( \mathcal{V}_j \setminus \{t_j\} \), corresponding to the path \( \alpha_j \). The equality (3.39) itself can now be considered as the regular-principal factorization of the branch \( Y_{\alpha_j}(x, t) \). In view of Definition 2.7 we conclude that the family \( \{Y_{\alpha_j}\} \) is isoprincipal with the distinguished point \( x_0 = \infty \).
In order to complete the proof, it remains to present the calculations, omitted in the above reasonings.

**Calculation 1.** We show that the overdetermined linear system (3.30), depending on the parameter \( \zeta \), is compatible.

In this case the compatibility condition (3.23) of Theorem (3.14) is represented by the following two equalities:

\[
(3.42a) \quad \frac{\partial}{\partial t_j} \left( \frac{Q_i}{\zeta + t_j - t_i} \right) + \sum_{1 \leq k \leq n \atop k \neq j} \frac{\partial}{\partial t_i} \left( \frac{Q_k}{\zeta + t_j - t_k} \right) = \sum_{1 \leq k \leq n \atop k \neq j} \left[ \frac{Q_k, Q_i}{(\zeta + t_j - t_k)(\zeta + t_j - t_i)} \right], \quad 1 \leq i \leq n, \ i \neq j,
\]

and

\[
(3.42b) \quad \frac{\partial}{\partial t_k} \left( \frac{Q_i}{\zeta + t_j - t_i} \right) - \frac{\partial}{\partial t_i} \left( \frac{Q_k}{\zeta + t_j - t_k} \right) = \sum_{1 \leq k \leq n \atop k \neq j} \left[ \frac{Q_k, Q_i}{(\zeta + t_j - t_k)(\zeta + t_j - t_i)} \right], \quad 1 \leq i, k \leq n, \ i, k \neq j.
\]

The equality (3.42a) can be simplified as follows:

\[
\frac{\partial Q_i}{\partial t_j} \cdot \frac{1}{\zeta + t_j - t_i} + \sum_{1 \leq k \leq n \atop k \neq j} \frac{\partial Q_k}{\partial t_i} \cdot \frac{1}{\zeta + t_j - t_k} = \sum_{1 \leq k \leq n \atop k \neq i, j} \frac{Q_k, Q_i}{\zeta - t_i} \left( \frac{1}{\zeta + t_j - t_k} - \frac{1}{\zeta + t_j - t_i} \right).
\]

In view of (3.3), the right-hand side of the last equality can be rewritten as

\[
\sum_{1 \leq k \leq n \atop k \neq i, j} \frac{\partial Q_k}{\partial t_i} \left( \frac{1}{\zeta + t_j - t_k} - \frac{1}{\zeta + t_j - t_i} \right) = \frac{\partial Q_j}{\partial t_i} \cdot \frac{1}{\zeta + t_j - t_i} + \sum_{1 \leq k \leq n \atop k \neq j} \frac{\partial Q_k}{\partial t_i} \cdot \frac{1}{\zeta + t_j - t_k},
\]

where we have used the fact that, in view of Remark 3.2

\[
\sum_{1 \leq k \leq n} \frac{\partial Q_k}{\partial t_i} = 0, \quad 1 \leq i \leq n.
\]

Since, as follows from (3.3),

\[
\frac{\partial Q_i}{\partial t_j} = \frac{\partial Q_j}{\partial t_i}, \quad 1 \leq i, j \leq n, \ i \neq j,
\]

we conclude that the equality (3.42a), indeed, holds. The equality (3.42b) can be verified analogously.
Calculation 2. We show that the function $Z_{\alpha j}(\zeta, t)$, defined in (3.34), satisfies with respect to $\zeta$ the equation (3.38).

This can be done as follows. We consider the auxiliary function

$$X_{\alpha j}(\zeta, t) \equiv \frac{\partial Z_{\alpha j}(\zeta, t)}{\partial \zeta} - \sum_{1 \leq i \leq n} Q_i(t) \cdot Z_{\alpha j}(\zeta, t).$$

It will be shown below that $X_{\alpha j}(\zeta, t)$ satisfies the linear system

$$\begin{align*}
\frac{\partial X_{\alpha j}}{\partial t_i} &= -\frac{Q_i}{\zeta + t_j - t_i} \cdot X_{\alpha j}, \quad 1 \leq i \leq n, i \neq j, \\
\frac{\partial X_{\alpha j}}{\partial t_j} &= \sum_{1 \leq i \leq n, i \neq j} \frac{Q_i}{\zeta + t_j - t_i} \cdot X_{\alpha j}.
\end{align*}$$

(Notes that this system is the same as the system (3.30) for the function $L_j$.)

Since, in view of (3.35), the solution $X_{\alpha j}(\zeta, t)$ of the linear system (3.44) satisfies the initial condition

$$X_{\alpha j}(\zeta, t) \big|_{t=t^0} = 0,$$

the uniqueness theorem for linear differential systems implies

$$X(\zeta, t) \equiv 0.$$

Therefore, the function $Z_{\alpha j}(\zeta, t)$ satisfies the equation (3.38).

Now let us prove that $X_{\alpha j}(\zeta, t)$ satisfies, indeed, the linear system (3.44). In view of (3.37), we have

$$\frac{\partial X_{\alpha j}}{\partial t_j} = \sum_{1 \leq i \leq n, i \neq j} Q_i \cdot \frac{\partial}{\partial \zeta} \left( \frac{Z_{\alpha j}}{\zeta + t_j - t_i} \right) - \sum_{1 \leq i \leq n} \frac{\partial Q_i}{\partial t_j} \left( \frac{Q_i}{\zeta + t_j - t_i} \right) \cdot Z_{\alpha j} -$$

$$- \sum_{1 \leq i, k \leq n} Q_i Q_k \left( \frac{1}{\zeta + t_j - t_i} \right) \left( \frac{1}{\zeta + t_j - t_k} \right) \cdot Z_{\alpha j}$$

$$= \sum_{1 \leq i \leq n, i \neq j} \frac{Q_i}{\zeta + t_j - t_i} \cdot \frac{\partial Z_{\alpha j}}{\partial \zeta} - \sum_{1 \leq i \leq n} \frac{\partial Q_i}{\partial t_j} \cdot \frac{1}{\zeta + t_j - t_i} \cdot Z_{\alpha j} -$$

$$- \sum_{1 \leq i, k \leq n} \frac{Q_i Q_k}{(\zeta + t_j - t_i)(\zeta + t_j - t_k)} \cdot Z_{\alpha j}.$$
Now we substitute (3.3) to obtain
\[
\frac{\partial X_{\alpha_j}}{\partial t_j} = \sum_{i=1, i \neq j}^{n} \frac{Q_j}{\zeta + t_j - t_i} \frac{\partial Z_{\alpha_i}}{\partial \zeta} - \sum_{i=1, i \neq j}^{n} \frac{Q_i Q_j}{(\zeta + t_j - t_i)(\zeta + t_j - t_k)} \cdot Z_{\alpha_j} - \sum_{i,k=1, i \neq j}^{n} \frac{Q_i Q_k}{\zeta + t_j - t_i}(\zeta + t_j - t_k) \cdot Z_{\alpha_j}.
\]

\[
= \left( \sum_{i=1, i \neq j}^{n} \frac{Q_j}{\zeta + t_j - t_i} \right) \cdot \left( \frac{\partial Z_{\alpha_j}}{\partial \zeta} - \sum_{k=1, k \neq j}^{n} \frac{Q_k}{\zeta + t_j - t_k} \cdot Z_{\alpha_j} \right)
\]

\[
= \sum_{i=1, i \neq j}^{n} \frac{Q_j}{\zeta + t_j - t_i} \cdot X_{\alpha_j}.
\]

The first equation of the system (3.44) is obtained analogously. \[\Box\]

4. ISOMONODROMIC AND ISOPRINCIPAL DEFORMATIONS OF FUCHSIAN SYSTEMS

**Definition 4.1.** Let a Fuchsian system
\[
(4.1) \quad \frac{dY}{dx} = \left( \sum_{1 \leq j \leq n} \frac{Q_j^0}{x - t_j^0} \right) Y,
\]

where \( t^0 = (t_1^0, \ldots, t_n^0) \in \mathbb{C}_+^n \), \( Q_1^0, \ldots, Q_n^0 \in \mathcal{M}_k \) and
\[
(4.2) \quad \sum_{1 \leq j \leq n} Q_j^0 = 0,
\]

be given.

Let a holomorphic family of Fuchsian systems
\[
(4.3) \quad \frac{dY}{dx} = \left( \sum_{1 \leq j \leq n} \frac{Q_j(t)}{x - t_j} \right) Y,
\]

where the residue matrices \( Q_j(t) \) are holomorphic in a neighborhood \( \mathcal{D} \subset \mathbb{C}_+^n \) of \( t^0 \), be such that:
\[
(4.4) \quad \sum_{1 \leq j \leq n} Q_j(t) \equiv 0, \quad t \in \mathcal{D},
\]

\[
(4.5) \quad Q_j(t^0) = Q_j^0, \quad 1 \leq j \leq n.
\]

If the holomorphic family of Fuchsian systems (4.3) is isoprincipal (respectively, isomonodromic) with the distinguished point \( x_0 = \infty \), then it is said to be an isoprincipal (respectively, isomonodromic) deformation with the distinguished point \( x_0 = \infty \) of the Fuchsian system (4.1).

**Remark 4.2.** Note that, according to Theorem 2.11, an isoprincipal deformation (4.3) of a given Fuchsian system (4.1) is also an isomonodromic one. As Theorem 2.11 implies, the converse is true under the condition that all the matrices \( Q_1^0, \ldots, Q_n^0 \) are non-resonant.
Now we are going to address the question of the existence of an isoprincipal deformation of a given Fuchsian system. It follows from Theorem 3.1 and Remark 3.2 that the holomorphic family (4.3) is an isoprincipal deformation with the distinguished point $x_0 = \infty$ of the Fuchsian system (4.1) if and only if the residues $Q_j(t)$ satisfy the Schlesinger system

$$\begin{align*}
\frac{\partial Q_i}{\partial t_j} &= \frac{[Q_i, Q_j]}{t_i - t_j}, \quad 1 \leq i, j \leq n, \; i \neq j, \\
\frac{\partial Q_i}{\partial t_i} &= -\sum_{1 \leq j \leq n \atop j \neq i} \frac{[Q_i, Q_j]}{t_i - t_j}, \quad 1 \leq i \leq n.
\end{align*}$$

(4.6)

Thus the question is, whether the Cauchy problem for the Schlesinger system with the initial condition (4.5) is solvable. An answer to this question follows from the Frobenius theorem (Theorem 3.11):

**Proposition 4.3.** Let $t^0 = (t^0_1, \ldots, t^0_n) \in \mathbb{C}^n$, $Q^0_1, \ldots, Q^0_n \in M_k$ be given.

Then there exist a neighborhood $D \subset \mathbb{C}^n$ of $t^0$ and unique matrix functions $Q_1(t), \ldots, Q_n(t)$, holomorphic in $D$, which satisfy the Schlesinger system (4.6) and the initial condition (4.5).

**Proof.** According to Theorem 3.11 and Remark 3.13, we have to verify the identity

$$\frac{\partial^2 Q_i}{\partial t_j \partial t_k} = \frac{\partial^2 Q_i}{\partial t_k \partial t_j}, \quad 1 \leq i, j, k \leq n,$$

substituting the expressions (4.6) for the partial derivatives of $Q_i$.

In the case $i = k \neq j$ we have

$$\frac{\partial^2 Q_i}{\partial t_j \partial t_k} = \frac{\partial}{\partial t_j} \left( \sum_{1 \leq k \leq n \atop k \neq i} \frac{[Q_j, Q_i]}{t_i - t_j} \right) = \frac{[Q_i, [Q_j, Q_i]] - [Q_i, Q_j]}{(t_i - t_j)^2} + \sum_{1 \leq k \leq n \atop k \neq i} \frac{[Q_i, [Q_j, Q_k]] + [Q_k, [Q_i, Q_j]]}{(t_i - t_j)(t_i - t_k)} = \frac{[Q_i, [Q_j, Q_i]] - [Q_i, Q_j]}{(t_i - t_j)^2} - \sum_{1 \leq k \leq n \atop k \neq i} \frac{[Q_j, [Q_k, Q_i]]}{(t_i - t_j)(t_i - t_k)} = \frac{\partial}{\partial t_i} \left( \frac{[Q_i, Q_j]}{t_i - t_j} \right) = \frac{\partial^2 Q_i}{\partial t_j \partial t_i},$$

where we have used the Jacobi identity

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0, \quad \forall A, B, C \in M_k.$$

In the case $k \neq i \neq j$ the computations are analogous and will be omitted. \hfill $\square$

**Remark 4.4.** Proposition 4.3 was originally proved by L. Schlesinger in [Sch2]. Here we would also like to mention the paper [Boa] by P. Boalch, where some general considerations concerning the compatibility of systems from a class, containing the Schlesinger system, are presented.
As an immediate consequence of Theorem 3.11 and Proposition 4.3, we obtain

**Theorem 4.5.** Let a Fuchsian system \( \mathcal{W} ∈ \mathcal{V} \) be a nested pair of open cylindrical neighborhoods of \( t^0 \) in \( \mathbb{C}^n \) and for \( 1 ≤ j ≤ n \) let \( α_j \) be a path from \( x_0 = \infty \) to a point \( p_j \in \mathcal{V}_j \setminus \overline{\mathcal{W}_j} \). Let \( Y(x, t^0) \) be the solution of \( \mathcal{W} \) with the distinguished point \( x_0 = \infty \) and consider the solution \( Y(x, t) \) of \( \mathcal{W} \) with the initial condition

\[
Y(x, t^0) \bigg|_{x=∞} = I
\]

and let

\[
Y_{α_j}(x, t^0) = H_j(x, t^0) \cdot E_{α_j}(x - t^0_j), \quad x ∈ \text{cov}(\mathcal{V}_j \setminus \{t^0_j\}; p_j)
\]

be the regular-principal factorization of the appropriate branch \( Y_{α_j}(x, t^0) \) in the punctured domain \( \mathcal{V}_j \setminus \{t^0_j\} \) (here the function \( E_{α_j}(\zeta) \), holomorphic and invertible in the Riemann surface of \( \ln ζ \), is defined in the same way as in the proof of Proposition 3.10 – see (3.29)).

First we assume that the family \( \mathcal{W} \), where the residue matrices \( Q_j(t) \) are holomorphic in \( V \), is an isoprincipal deformation of the Fuchsian system \( \mathcal{W} \) with the distinguished point \( x_0 = \infty \) and consider the solution \( Y(x, t) \) of \( \mathcal{W} \) with the initial condition

\[
(4.7) \quad Y(x, t) \bigg|_{x=∞} = I.
\]

Since \( Y(x, t) \) is jointly holomorphic in \( x \) and \( t \) and invertible in the domain \( \text{cov}(\mathbb{C}P^1 \setminus \bigcup_k \mathcal{W}_k; \infty) × \mathcal{W} \), so is the "ratio"

\[
R(x, t) \overset{\text{def}}{=} Y(x, t) \cdot Y^{-1}(x, t^0).
\]

Moreover, taking into account the initial condition (4.7) and the fact that the family \( \mathcal{W} \) is by Theorem 2.11 isomonodromic, we reach the following conclusion:

**(R)** The function \( R(x, t) \) is holomorphic (single-valued) with respect to \( x, t \) and invertible in the domain \( \{\mathbb{C}P^1 \setminus \bigcup_k \mathcal{W}_k\} × \mathcal{W} \) and it holds that

\[
R(x, t) \bigg|_{x=∞} = I, \quad t ∈ \mathcal{W}.
\]

Similarly, considering the coherent family of transplants \( \{E_{α_j}(x - t_j)\}_{t_j ∈ \mathcal{W}_j} \), we observe that the "ratio"

\[
F_j(x, t) \overset{\text{def}}{=} E_{α_j}(x - t_j) \cdot E_{α_j}^{-1}(x - t_0_j), \quad x ∈ \text{cov}(\mathbb{C} \setminus \overline{\mathcal{W}_j}; p_j), \quad t ∈ \mathcal{W}
\]

is holomorphic (single-valued) with respect to \( x, t \) and invertible in \( \{\mathbb{C} \setminus \overline{\mathcal{W}_j}\} × \mathcal{W} \). Since the family \( \mathcal{W} \) is assumed to be isoprincipal, there exist functions \( H_j(x, t) \) (the regular factors of \( Y(x, t) \)), such that:

**(H)** Each function \( H_j(x, t) \), \( 1 ≤ j ≤ n \), is holomorphic (single-valued) with respect to \( x, t \) and invertible in \( \mathcal{V}_j \times \mathcal{W} \);

**(Pb)**

\[
F_j(x, t) \cdot H_j^{-1}(x, t^0) = H_j^{-1}(x, t) \cdot R(x, t), \quad t ∈ \mathcal{W}, \quad x ∈ \mathcal{V}_j \setminus \overline{\mathcal{W}_j}, \quad 1 ≤ j ≤ n.
\]
Note that the function $F_j(x, t) \cdot H_j^{-1}(x, t^0)$ on the left-hand side of the equality (Pb) is holomorphic (single-valued) with respect to $x, t$ and invertible in $\{ \mathbb{C} \setminus \overline{W_j} \} \times \mathcal{W}$. It is determined entirely in terms of the initial data $t^0, Q^0_1, \ldots, Q^0_n$. The equality (Pb) itself can be viewed as a factorization problem for this function, where one looks for the factors $H_j(x, t), R(x, t)$, possessing the analyticity properties (H) and (R). If such factors can be found, then, reversing the reasonings above, one arrives at the isoprincipal deformation of the Fuchsian system (4.1).

The uniqueness of the solution, possessing the properties (H) and (R), for the factorization problem (Pb) follows immediately from the Liouville theorem. The existence of this solution for $t$ in a sufficiently small neighborhood $\mathcal{W}$ of $t^0$ can be established by elementary means, since for $t = t^0$ the solution exists (it is trivial: $R = I$).

However, analyzing the factorization problem (Pb) more carefully and systematically (see, for instance, [14K §5]) and taking into account Theorem 3.1, one can reach the stronger conclusion that the functions $Q_j(t)$, which satisfy the Schlesinger system (4.6) with the initial condition (4.5), are meromorphic in the universal covering space $\text{cov}(\mathbb{C}_n^*, t^0)$. This result was obtained by B. Malgrange in [Ma1] and (in the non-resonant case) by T. Miwa in [Miwa]. Our proof, which involves the isoprincipal deformations and is outlined above, will be presented in more detail elsewhere.

**Remark 4.7.** As was already mentioned (see Remarks 1.10 and 2.16), most of the considerations of the present article need not be restricted to the case of linear differential systems with only Fuchsian singularities.

For instance, one can consider linear systems of the form

$$\frac{dY}{dx} = \left( \sum_{j=1}^{n} \sum_{k=0}^{p_j} \frac{Q_{j,k}}{(x-t_j)^{k+1}} \right) Y,$$

where

$$\sum_{j=1}^{n} Q_{j,0} = 0.$$

In this case the regular-principal factorization of the solution and the notion of the isoprincipal family

$$\frac{dY}{dx} = \left( \sum_{j=1}^{n} \sum_{k=0}^{p_j} \frac{Q_{j,k}(t)}{(x-t_j)^{k+1}} \right) Y$$

can be introduced as in Definitions 1.8, 2.9.

Furthermore, the auxiliary linear system related to the isoprincipal family takes the form (compare with (3.5))

$$\begin{cases}
\frac{\partial Y}{\partial x} = \left( \sum_{j=1}^{n} \sum_{k=0}^{p_j} \frac{Q_{j,k}(t)}{(x-t_j)^{k+1}} \right) Y, \\
\frac{\partial Y}{\partial t_j} = - \left( \sum_{k=0}^{p_j} \frac{Q_{j,k}(t)}{(x-t_j)^{k+1}} \right) Y, \quad 1 \leq j \leq n.
\end{cases}$$
The compatibility condition for this overdetermined system is given by the system
\[
\begin{align*}
\frac{\partial Q_{i,k}}{\partial t_j} &= \sum_{0 \leq \ell \leq p_i - k, 0 \leq q \leq p_j \atop j \neq i} (-1)^l \binom{l+q}{l} \frac{Q_{i,k+l}, Q_{j,q}}{(t_i - t_j)^{q+l+1}} \quad 0 \leq k \leq p_i, 1 \leq i, j \leq n, i \neq j, \\
\frac{\partial Q_{i,k}}{\partial t_i} &= - \sum_{1 \leq j \leq n \atop j \neq i} \sum_{0 \leq \ell \leq p_i - k, 0 \leq q \leq p_j \atop j \neq i} (-1)^l \binom{l+q}{l} \frac{Q_{i,k+l}, Q_{j,q}}{(t_i - t_j)^{q+l+1}} \quad 0 \leq k \leq p_i, 1 \leq i \leq n,
\end{align*}
\]
which itself is compatible and contains the Schlesinger system \(^\text{4.6}\) as a special case (corresponding to \(p_1 = \cdots = p_n = 0\)).

5. An example

In order to illustrate the distinction between the isoprincipal and the isomonodromic deformations of Fuchsian systems in the case when the non-resonance condition is violated, we present the following explicit example.

Let us consider the linear differential system
\[
\frac{dY}{dx} = \begin{pmatrix}
-1 & 0 \\
0 & -1 \\
\end{pmatrix}
\begin{pmatrix}
\frac{x(x-1)}{(x-2)(x-3)} \\
0 \\
\end{pmatrix} Y.
\]
Note that the system (5.1) is of the form
\[
\frac{dY}{dx} = \left( \frac{Q_0}{x} + \frac{Q_0}{x-1} + \frac{Q_0}{x-2} + \frac{Q_0}{x-3} \right) Y,
\]
where
\[
Q_0 = -Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Q_2 = -Q_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},
\]
and hence it is Fuchsian and resonant. Moreover, \(x = \infty\) is a regular point of the system (5.1), since
\[
Q_0 + Q_1 + Q_2 + Q_3 = 0.
\]
The solution \(Y(x)\) satisfying the initial condition \(Y|_{\infty} = I\) is a rational matrix functions (in particular, single-valued); it has the following form:
\[
Y(x) = \begin{pmatrix}
x \\
x - 1 \\
0 \\
\end{pmatrix}
\begin{pmatrix}
0 \\
x - 2 \\
x - 3 \\
\end{pmatrix}
def Y(x).
\]
The principal factors of \(Y(x)\) can be chosen\(^\text{22}\) in the form
\[
\begin{align*}
P_0(x) &= \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}, & P_1(x) &= \begin{pmatrix} 1 & 0 \\ x - 1 & 0 \end{pmatrix}, \\
P_2(x) &= \begin{pmatrix} 1 & 0 \\ 0 & x - 2 \end{pmatrix}, & P_3(x) &= \begin{pmatrix} 1 & 0 \\ 0 & x - 3 \end{pmatrix}.
\end{align*}
\]
Now we are going to construct the isoprincipal deformation of the system (5.1). For simplicity, we assume that the singular points \(x = 1, 2, 3\) are fixed while the
\footnote{\(\text{22}\)See Remark 1.9.}
position of one singularity \((x=0)\) varies: \(x = t\). Then we need to determine a holomorphic family of matrix functions \(Y(x,t)\) such that

\[
Y(x,t) \big|_{x=\infty} = I, \quad Y(x,t) \big|_{t=0} = Y^0(x),
\]

\(P_0(x-t)\) is the principal factor of \(Y(x,t)\) at \(x = t\) and for \(k = 1, 2, 3\) \(P_k(x)\) is the principal factor of \(Y(x,t)\) at \(x = k\). Such a family is unique and consists of rational matrix functions of the form:

\[
Y(x,t) = \begin{pmatrix}
\frac{x-t}{x-1} & 0 \\
0 & \frac{x-2}{x-3}
\end{pmatrix}.
\]

The function \(Y(x,t)\) satisfies with respect to \(x\) the Fuchsian system

\[
\frac{dY}{dx} = \begin{pmatrix}
t - 1 \\
(x-t)(x-1) & 0 \\
(x-2)(x-3)
\end{pmatrix}
\begin{pmatrix}
0 \\
-1
\end{pmatrix}
Y = \left(\frac{Q_0^0}{x-t} + \frac{Q_1^0}{x-1} + \frac{Q_2^0}{x-2} + \frac{Q_3^0}{x-3}\right)Y,
\]

and the constant functions

\(Q_j(t) \equiv Q_j^0, \quad j = 0, 1, 2, 3,\)

satisfy, of course, the Schlesinger system

\[
\begin{cases}
\frac{dQ_j}{dt} = \frac{[Q_0, Q_j]}{t-j}, & j = 1, 2, 3, \\
\frac{dQ_0}{dt} = \sum_{j=1}^{3} \frac{[Q_j, Q_0]}{t-j}
\end{cases}
\]

with the initial condition

\(Q_j(0) = Q_j^0, \quad j = 0, 1, 2, 3.\)

However, the deformation \(Y(x,t)\) is not the only possible isomonodromic deformation of the system \((5.1)\). Indeed, let us consider a family of rational functions

\[
Y(x,t) = \begin{pmatrix}
x-t \\
(x-1)(x-3)(t-3)
\end{pmatrix}.
\]

where \(h(t)\) is a function, holomorphic in a neighborhood of \(t = 0\) (outside this neighborhood \(h(t)\) may have arbitrary singularities). Such a family also satisfies
the normalizing conditions \[5.2\]; moreover, we have

\[
Y(x, t)^{-1} = \begin{pmatrix}
(x-2)(t-3) & 2th(t) \\
(x-2)(t-3) & x-3
\end{pmatrix},
\]

\[
\frac{\partial Y(x, t)}{\partial x} Y(x, t)^{-1} = \begin{pmatrix}
t-1 & 2t(x-t)h(t) \\
(x-t)(x-1) & (x-1)(x-2)(x-3)(t-3)
\end{pmatrix}.
\]

Thus we obtain the following deformation of the system \[5.1\]:

\[
(5.4) \quad \frac{dY}{dx} = \begin{pmatrix}
t-1 & 2t(x-t)h(t) \\
(x-t)(x-1) & (x-1)(x-2)(x-3)(t-3)
\end{pmatrix} Y = \begin{pmatrix}
Q_0(t) & Q_1(t) & Q_2(t) & Q_3(t)
\end{pmatrix} Y,
\]

where

\[
Q_0(t) = Q_0^0, \quad Q_1(t) = \begin{pmatrix}
-1 & -t(t-1)h(t) \\
0 & t-3
\end{pmatrix},
\]

\[
Q_2(t) = \begin{pmatrix}
0 & 2t(t-2)h(t) \\
0 & t-3
\end{pmatrix}, \quad Q_3(t) = \begin{pmatrix}
0 & -th(t) \\
0 & -1
\end{pmatrix}.
\]

Since the monodromy of \(Y(x, t)\) for every \(t\) is trivial, the deformation \(5.4\) is isomonodromic, but if \(h(t) \neq 0\), then it is not isopincipal and the coefficients \(Q_j(t)\) do not satisfy the Schlesinger system. Moreover, since we only require \(h(t)\) to be holomorphic in a neighborhood of \(t = 0\), the behavior of the functions \(Q_j(t)\) outside this neighborhood may be arbitrary. In particular, these functions \(Q_j(t)\) need not be meromorphic with respect to \(t\) in \(\mathbb{C} \setminus \{1, 2, 3\}\).

Remark 5.1. The example presented above is based on the theory of holomorphic families \[1.3\] of Fuchsian systems, whose solutions \(Y(x, t)\) are generic rational matrix functions of \(x\) for every fixed \(t\). This theory was developed in \[Kats1\], \[Kats2\] and \[KaVo2\] (see also the electronic version of the latter work \[KaVo2-e\]). For such Fuchsian systems the number of poles \(n\) is even, so we write \(2n\) instead of \(n\). All such Fuchsian systems with this property can be parameterized as follows. If \(k\) is the dimension of the square residue matrices \(Q_j\), then to each pole \(t_j\) a \(k - 1\)-vector is related as a ”free” parameter. Therefore, to each \(2n\)-tuple \(t = (t_1, \ldots, t_{2n})\) the total of \((k - 1) \times 2n\) complex parameters is related. To every choice of these parameters (satisfying a certain non-degeneracy condition) corresponds a different system of the form \[1.3\], whose solution \(Y(x, t)\) is a generic rational matrix function. Considering \(t\) as variable, we assign \((k - 1) \times 2n\) complex parameters to each \(t \in \mathbb{C}^{2n}\).

Thus the families \[1.3\], possessing the property that \(Y(x, t)\) is a generic rational matrix function of \(x\) for each fixed \(t\), can be parameterized by \((k - 1) \times 2n\).
complex valued functions of $t$, and these functional parameters are free. To obtain holomorphic families, we have to require that these complex valued functions are holomorphic. Of course, the monodromy of any rational matrix function of $x$ is trivial. Hence we can parameterize the class of the isomonodromic deformations of Fuchsian systems with generic rational solutions by $(k - 1) \times 2n$ functional parameters, and these parameters can be arbitrary (up to the mentioned non-degeneracy condition) holomorphic functions of $t$.

It turns out that the deformation corresponding to a given choice of the functional parameters is isoprincipal if and only if all these parameters are constant functions. In particular, the class of the isomonodromic deformations (considered without the non-resonance condition) is much richer than its subclass of the iso-principal deformations.

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