Gauge Invariance and Factorisation 
in Exclusive Meson Production

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Abstract

The structure of the nonperturbative vector meson vertex function complicates the proof of the factorisation theorem for the reaction $\gamma^* p \rightarrow V p$. It leads to additional contributions but, in a simple model for the vertex function, gauge invariance ensures that they cancel and factorisation is preserved.
1. Introduction

With the measurement of exclusive vector meson electroproduction at high photon energy and virtuality at HERA [1], a new possibility for investigating the interplay of hard and soft physics is emerging. The original crude calculations of the process [2], based on the diagrams of figure 1, have since been refined by several authors [3]. These diagrams contain a hard part, calculated with perturbation theory, but there is also a nonperturbative part. Recently, the mechanism of factorisation of the calculable hard amplitude and the nonperturbative component, given by the non-diagonal parton densities and the meson wave function, has been discussed in a rather general framework [4].

It is the purpose of the present paper to discuss the problems associated with the structure of the vertex function at the top right-hand corner of the diagrams. This couples the vector meson to the internal quark loop. It is nonperturbative and so our knowledge of it is far from complete, though its analytic structure is known [5]. If we neglect spin, it is a function $V(u, v)$ of the squared 4-momenta on the two quark legs, $u = k^2$ and $v = (q' - k)^2$. In the case of a nonrelativistic system, such as the $\Upsilon$, it is a good approximation to take $V$ to be strongly peaked at values of $u$ and $v$ such that the quarks are very close to their mass shells. But it is doubtful [6] that this is a good approximation for the $J/\psi$, and for the $\rho$ surely it is not. In this case, $V(u, v)$ has branch points in each of its variables, $u$ and $v$, associated with “normal thresholds” and possibly also “anomalous thresholds” [5]. We show that even in leading power this results in a breakdown of the desired factorisation of the sum of the diagrams of figure 1.

However, this does not necessarily imply that the factorisation theorem is invalid. The set of diagrams of figure 1 is not QCD gauge invariant. To achieve gauge invariance, they must be supplemented by other graphs in which either or both of the gluons couples directly into the nonperturbative vector meson vertex function. The question we discuss is whether these additional diagrams restore the factorisation theorem.

We are not able to give a definitive answer to this question, because of the need to introduce two further nonperturbative vertex functions, and we know as little about these as we do about $V$. However, we report a calculation based on an explicit simple model in which we find that the factorisation theorem is indeed restored. This encourages the belief that its validity may be general.

Our model is a simple one in which all three vertex functions have the analytic structure that is expected on general grounds [5] and in which they are related in such a way that the complete leading-power amplitude for the exclusive meson production is gauge
invariant. We then find that the amplitude contains leading contributions from diagrams that cannot be separated into a hard production process and soft wave function corrections. However, gauge invariance leads to a cancellation of the unwanted contributions and leaves the factorisation theorem intact. The final formula can be obtained from the diagrams of figure [1] by a redefinition of the rules for calculating them: basically, this amounts to ignoring the branch points in the vertex function $V$.

In section 2, we review some aspects of the factorisation theorem in the case where the structure of the nonperturbative vertex function $V$ may be neglected. In section 3 we explain the complications that arise when its structure is taken into account, and introduce a simple model for this structure. Its consequences are analysed in section 4.

![Fig.1](image-url) The leading amplitude for a structureless meson vertex.

2. Structureless vertex function

According to usual ideas, unless [6] the vector meson is at least as heavy as the $\Upsilon$, its complete vertex function $V(u,v)$ is nonperturbative for small values of $u$ and $v$. Its high-$u$ or high-$v$ tail may be calculated from perturbative QCD, but we do not consider this tail because it contributes a correction to the amplitude that is a nonleading power of the perturbative coupling $\alpha_s(Q^2)$. To begin with, we take the nonperturbative vertex to be structureless. For simplicity, we take the meson to be spinless, and also pretend that the photon and the quarks have no spin. The coupling of the gluons to the scalar quarks is given by $-ig(r_\mu + r'_\mu)$, where $r$ and $r'$ are the momenta of the directed quark lines, and the coupling of the photon is $ie$, where $e$ has the dimensions of mass. We use light-cone coordinates and work in a frame where $q, q', P, P'$ have zero transverse components and large ‘$+$’ and ‘$-$’ components respectively. We concentrate on the forward production, so that the transverse component of the momentum transfer $\Delta = q' - q = \ell - \ell' = P - P'$ vanishes, $\Delta_\perp = 0$. 

3
For scalar quarks, there is a seagull two-gluon vertex. In order to obtain a gauge-invariant set of diagrams, we must supplement those of figure 1 with diagrams that involve the seagull. However, we shall calculate in Feynman gauge and suppose that \( W^2 \gg Q^2 \), where \( W \) is the \( \gamma^*p \) energy, \( W^2 = (q + P)^2 \). In this case the diagrams that involve the seagull contribute to the amplitude a nonleading power of \( W \): the diagrams of figure 1 are the only ones that contribute in leading power when \( V \) is constant. We study the case of large \( Q^2 \), \( Q^2 \gg m_V^2 \), so that we neglect all masses. Then for forward production \( \Delta = x_{\text{Bj}} P \).

The lower bubble in the diagrams of figure 1 in principle has a complicated structure. However, we shall assume that the main contribution comes from values of its subenergy \( \sigma \) that are not too large, \( \sigma = (P - \ell)^2 \ll W^2 \). Then \( \ell_+ \ll q_+ \) and the dependence of the lower bubble on \( \sigma \) may be approximated by \( \delta(\sigma) \). Further, when the upper parts of the diagrams are added together the main contribution arises from small values of \( \ell_- \), \( \ell_- \ll P_- \), so that also \( \ell'_- \ll P_- \) and \( \ell^2 \sim \ell'^2 \sim \ell_+^2 \). This may be seen most simply by calculating the imaginary part of the amplitude, where the left-most of the two quarks to which \( \ell' \) is attached is on shell. Hence the important part of the lower bubble effectively has the structure

\[
F^{\mu\nu}(\ell, \ell', P) \approx \delta(P_+ - \ell_+) F(\ell_+^2) P^\mu P^\nu, \tag{1}
\]

which is defined to include both gluon propagators and all colour factors. A similar expression was found by Cheng and Wu [7] in a tree model for the lower bubble, though we do not need to restrict ourselves to such a simple model. We assume that \( F \) restricts the gluon momentum to be soft, \( \ell_+^2 \ll Q^2 \). In the high energy limit it suffices to calculate

\[
M = \int \frac{d^4\ell}{(2\pi)^4} T^{\mu\nu} F_{\mu\nu} \approx \int \frac{d^4\ell}{4(2\pi)^4} T_{++} F_{--}, \tag{2}
\]

where

\[
T^{\mu\nu} = T^{\mu\nu}(\ell, \ell', q) = T_{a}^{\mu\nu} + T_{b}^{\mu\nu} + T_{c}^{\mu\nu} \tag{3}
\]

is the sum of the upper parts of the diagrams in figure 1.

The lower amplitude \( F_{\mu\nu} \) in the diagrams of figure 1 is symmetric with respect to the two gluon lines. This symmetry of the lower amplitude allows us to replace the properly-symmetrised upper amplitude \( T_{\text{sym}}^{\mu\nu} \) with the unsymmetrised amplitude \( T^{\mu\nu} \) corresponding to the sum of the diagrams in figure 1. The symmetrised amplitude is

\[
T_{\text{sym}}^{\mu\nu}(\ell, \ell', q) = \frac{1}{2}[T^{\mu\nu}(\ell, \ell', q) + T^{\nu\mu}(-\ell', -\ell, q)]. \tag{4}
\]
The two exchanged gluons together must form a colour singlet and so, at least for our calculation that takes account only of the lowest order in $\alpha_s(Q^2)$, the symmetrised amplitude $T_{\text{sym}}^{\mu\nu}(\ell, \ell', q)\ell_\mu\ell'_\nu = 0$. Writing this equation in light-cone components and setting $\ell_\perp = \ell'_\perp$, we see that for the small values of $\ell_-, \ell'_-, \ell_+$ and $\ell'_+$ that we need,

$$T_{\text{sym},++} \sim \ell_\perp^2$$

for $\ell_\perp^2 \to 0$. Here we have used the fact that the tensor $T_{\text{sym}}^{\mu\nu}$, which is built from $\ell'$, $\ell$ and $q$, has no large ‘−’ components. The $\ell_-$ integration makes this equation hold also for the original, unsymmetrised amplitude:

$$\int d\ell_- T_{++} \sim \ell_\perp^2.$$  

This is the crucial feature of the two-gluon amplitude that will simplify the calculation and lead to the factorising result of the next sections.

It is convenient to begin with the contribution from diagram a) of figure 1 to the $\ell_-$ integral of $T_{++}$, which is required in (2):

$$\int d\ell_- T_{a,++} = -4eg^2q_+ \int \frac{d^4k}{(2\pi)^3} \frac{z(1-z)}{N^2 + (k_\perp \ell_\perp)^2} \frac{V(k^2, (q' - k)^2)}{k^2(q' - k)^2}. $$

Here $N^2 = z(1-z)Q^2$, $z = k_+/q_+$ and the condition $\ell_+ = 0$, enforced by the $\delta$-function in (4), has been anticipated.

Now $\int d\ell_- T_{b,++}$ and $\int d\ell_- T_{c,++}$ each carry no $\ell_\perp$ dependence. So to ensure the validity of (6) the sum of the three diagrams must be

$$\int d\ell_- T_{++} = 4eg^2q_+ \int \frac{d^4k}{(2\pi)^3} z(1-z) N \frac{V(k^2, (q' - k)^2)}{k^2(q' - k)^2},$$

where

$$N = \left[ \frac{1}{(N^2 + k_\perp^2)^2} - \frac{1}{N^2 + (k_\perp + \ell_\perp)^2} \right] \sim \frac{\ell_\perp^2}{N^4}. $$

We have used the softness of the wave function, which results in the dominant contribution to the integral arising from values of $k_\perp^2 \ll Q^2$, and the rotational symmetry, which makes $k_\perp \cdot \ell_\perp$ integrate to 0. Note the $1/Q^4$ behaviour obtained after a cancellation of $1/Q^2$ contributions from the individual diagrams. This cancellation, which is closely related to the well-known effect of colour transparency [8], has been discussed in [2] in the framework of vector meson electroproduction.
Introduce the light-cone wave function of the meson
\[ \psi(z, k_{\perp}^2) = -i \frac{q'_+}{2} \int dk_- dk_+ \frac{V(k^2, (q' - k)^2)}{(2\pi)^4 k^2 (q' - k)^2} \delta(k_+ - zq'_+). \] (11)

The final result following from (2) and (9) is a convolution of the production amplitude of two on-shell quarks and the light-cone wave function:
\[ M = i e g^2 W^2 \left( \int \frac{d^2 \ell_{\perp}}{2(2\pi)^3} \ell_{\perp}^2 F(\ell_{\perp}^2) \right) \int dz \int d^2 k_{\perp} \frac{z(1 - z)}{N^4} \psi(z, k_{\perp}^2). \] (12)

This corresponds to the \(O(\ell_{\perp}^4)\) term in the Taylor expansion of the contribution (3) from figure (1a).

In a more general analysis the above \(\ell_{\perp}\) integral of \(F\) has to be replaced by an expression proportional to the non-diagonal gluon density which introduces an \(W^2\) dependence and an additional \(Q^2\) dependence.

3. Modelling the meson wave function

In the previous section we have supposed that the vertex function \(V\) has no structure. We used the gauge invariance of the amplitude to argue that there is a cancellation among the three diagrams of figure (1a). Of course, we may instead obtain the same result by explicit calculation of each of the three diagrams. This may be done most simply by calculating their imaginary parts and using the known fact that the complete amplitude must be pure-imaginary when its energy dependence is \((W^2)^{1/0}\). Alternatively, the contributions to the amplitude itself may be calculated by doing the \(k_-\) integration, completing the integration contour with an infinite semicircle and taking appropriate pole residues.

Consider figure (1b) for example. In either method of calculation, if \(V\) has no structure the upper quark line gets put on shell. However, if we now take account of the known structure of \(V\), there is an additional contribution to the imaginary part corresponding to cutting the graph through the vertex function — the \(k_-\) integration would need to take account also of branch points of \(V\), not just the poles of the propagators. The gauge-invariance argument breaks down because the set of diagrams by itself is no longer gauge invariant: one must add to it diagrams where either or both of the gluons couples directly into the vertex function.

In order to study this, we use the simplest model for the quark-quark-meson vertex function \(V(u, v)\) that incorporates at least part of its known branch-point structure. It is the purely nonperturbative vertex function that we are modelling: it goes to zero
suitably rapidly when either of the squared 4-momenta $u$ or $v$ of the quarks becomes large. We do not consider its perturbative tail, which would be obtained by exchanging a perturbative gluon between the quarks. It is a familiar notion \[\text{[5]}\] that the correct \textit{analytic} properties of nonperturbative amplitudes are those corresponding to Feynman graphs, even though the \textit{numerical} values of such graphs have no physical significance. In order to model the vertex function $V$, therefore, we use the simple Feynman graph of figure 2a), where the line that joins the quarks is a scalar (like the quarks themselves in our simple calculations) which couples to them with strength $\lambda'$ and where the right-hand internal vertex is taken to be a constant $\lambda$. This model has branch points in each of the variables $u$ and $v$, and it has the appropriate softness.

The diagram of figure 3 by itself gives no consistent description of meson production since it lacks gauge invariance. This problem is not cured by just adding the two diagrams 1b) and c) with the blob replaced by the vertex of figure 2a). It is necessary also to include diagrams where a gluon is coupled into the vertex (see, e.g., figure 2b)). Furthermore, diagrams where both gluons couple into the vertex have to be included.

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Fig. 2a) Model for the nonperturbative vertex function $V$. b) Diagram with a gluon coupling into the vertex.

Fig. 3 Diagram for meson production with the vertex modelled by scalar particle exchange.

When the vertex function of figure 2a) is used in figure 1a), we obtain figure 3. The expression for the upper part of the diagram is (8) with

$$\int \frac{d^4k'}{(2\pi)^4} \frac{i\lambda\lambda'^2}{k'^2(q' - k)^2(k - k')^2}. \quad (13)$$

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As will be demonstrated in the next section, the complete result can nevertheless be extracted directly from (8) and (13).

4. Gauge invariance and factorisation

To obtain gauge invariance within the model of the vertex illustrated in figure 2a), diagrams with the two gluons attached to the quark loop in all possible ways have to be added to that of figure 3. In addition to figure 3 there are nine such diagrams. They are shown in figure 4.

The remaining diagrams contributing to meson production within the above simple model for the meson wave function.

The same gauge invariance arguments that lead to (7) apply to the sum of all the diagrams in figures 3 and 4. Therefore, the complete result for $T_{++}$, which is now defined by the sum of the upper parts of all these diagrams, can be obtained by extracting the $\ell_{\perp}^2$ term at leading order in $W^2$ and $Q^2$. Such a term, with a power behaviour $\sim \ell_{\perp}^2 W^2/Q^4$, is obtained from the diagram in figure 3 (see (8) and (13)) by expanding around $\ell_{\perp} = 0$. We now argue that none of the other diagrams gives rise to such a leading-order $\ell_{\perp}^2$ contribution.
It is simpler to evaluate the contributions to the amplitude itself, rather than its imaginary part. We continue the evaluation of the diagram in figure 3 by performing the \( k_- \) and \( k'_- \) integrations in (8) and (13). Consider the region where \( k_+ > k'_+ \) and pick up the poles in such a way that the propagators with momenta \( k' \) and \( q' - k \) go on-shell. The resulting expression reads

\[
\int d\ell \cdot T_{\text{figure 3, +}} = \frac{-ieg^2\lambda\lambda' q_+}{(2\pi)^5} \int dz d^2k_+ \frac{z(1-z)}{[N^2 + (k_+ + \ell_+)^2]k^2_+} \times
\]

\[
\int dz' d^2k'_+ \left[ \frac{1}{(z-z')(k'_+^2/z' + k_+^2/(1-z)) + (k_- - k'_-)k_+^2} \right],
\]

where \( z' = k'_+ / q_+ \) and the integration is restricted to \( z > z' \). A corresponding expression can be obtained for the region \( z < z' \).

The above manipulations can also be applied to all the diagrams of figure 4. One arrives at the following results. The upper parts of diagrams a) – d) have no \( \ell_\perp \) dependence at all since the \( \ell_- \) integration puts the quark propagator connecting the two gluon vertices on-shell. Diagrams h) and i) vanish after the \( \ell_- \) integration because the integrand has two poles, both of which lie on the same side of the real axis. To analyse the remaining diagrams e) – g) it is convenient to route the momentum \( \ell \) as far as possible to the left. After all the ‘\(-\)’ integrations are performed it becomes obvious that each diagram contributes only at order \( W_2^2 / Q^4 \). To obtain an \( \ell_\perp^2 \) term one of the off-shell propagators left of the gluon vertices has to be expanded around \( \ell_\perp = 0 \). This brings the power down to \( W_2^2 / Q^6 \). Therefore, no leading-order \( \ell_\perp^2 \)-contribution is obtained.

The above discussion shows that the complete answer is given by the \( \ell_\perp^2 \) term from the Taylor expansion of (8). The amplitude \( M \) is precisely the one of (12) and (11), with \( V(k^2, (q' - k)^2) \) given by (13).

We have also checked the correctness of this simple factorising result by explicitly reducing all diagrams of figure 4 to a form similar to that given in (14) for the diagram of figure 3. The required cancellations occur on the level of the integrands, before the \( k_\perp, k'_\perp, z \) and \( z' \) integrations.

5. Conclusions

The mechanism of factorisation in exclusive meson production has been analysed in the framework of a simple scalar model. In this model the meson is formed by two scalar quarks interacting via the exchange of a scalar boson. From a calculation of all
contributing diagrams within the restriction of two-gluon exchange the following picture emerges.

The complete result contains leading contributions from diagrams that cannot be factorised into quark-pair production and meson formation. Nevertheless, in Feynman gauge the answer to the calculation can be anticipated by looking only at one particular factorising diagram. The reason for this simplification is gauge invariance. In the dominant region where the transverse momentum $\ell_\perp$ of the two $t$-channel gluons is small, gauge invariance requires the complete quark part of the amplitude to be proportional to $\ell_\perp^2$. The leading $\ell_\perp^2$ dependence comes exclusively from one diagram. Thus, the complete answer can be obtained from this particular diagram, which has the property to factorise explicitly if the two quark lines are cut. The resulting amplitude can be written in a factorised form.

It was our intention to demonstrate the above mechanism of gauge invariance induced cancellations in as simple and as explicit a way as possible. Our analysis supplements the otherwise much more general and complete discussion of [4] by making it more explicit and by handling the known structure of the nonperturbative meson vertex function.

Our calculation has many important limitations. It is restricted to the very simplest model of the meson vertex function, with only the simplest features of the known branch-point structure. However, a generalisation to a more complicated wave function structure seems straightforward. We have assumed that the main contribution to the diagrams in figures 1, 3 and 4 comes from values of the subenergy of the lower bubble that are not too large. Nevertheless, the integration over this subenergy will provide the energy growth of our cross section, corresponding to the rapid rise at small $x$ of the gluon distribution observed at HERA. We have not discussed the subtleties involved in factorising the off-diagonal gluon density, nor the behaviour in the region where the longitudinal momentum fraction of one of the quarks becomes small.

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