The Least-core and Nucleolus of Path Cooperative Games *

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Abstract. Cooperative games provide an appropriate framework for fair and stable profit distribution in multiagent systems. In this paper, we study the algorithmic issues on path cooperative games that arise from the situations where some commodity flows through a network. In these games, a coalition of edges or vertices is successful if it enables a path from the source to the sink in the network, and lose otherwise. Based on dual theory of linear programming and the relationship with flow games, we provide the characterizations on the CS-core, least-core and nucleolus of path cooperative games. Furthermore, we show that the least-core and nucleolus are polynomially solvable for path cooperative games defined on both directed and undirected network.

1 Introduction

One of the important problems in cooperative game is how to distribute the total profit generated by a group of agents to individual participants. The prerequisite here is to make all the agents work together, i.e., form a grand coalition. To achieve this goal, the collective profit should be distributed properly so as to minimize the incentive of subgroups of agents to deviate and form coalitions of their own. This intuition is formally captured by several solution concepts, such as the core, the least-core, and the nucleolus, which will be the focus of this paper.

In this paper, we consider a kind of cooperative game models, path cooperative games (PC-games), arising from the situations where some commodity (traffic, liquid or information) flows through a network. In these games, each player controls an edge or a vertex of the network (called edge path cooperative games or vertex edge path cooperative games, respectively), a coalition of players wins if it enables a path from the source to the sink, and lose otherwise. We will focus on the algorithmic problems on game solutions of path cooperative games, especially core related solutions.

Path cooperative games have a natural correspondence with flow games. Flow games were first introduced by Kalai and Zemel [13] and studied extensively by many researchers. When there are public arcs in the network, the core of the

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flow game is nonempty if and only if there is a minimum \((s,t)\)-cut containing no public arcs. And in this case, the core can be characterized by the minimum \((s,t)\)-cuts \([13, 18]\), and the nucleolus can also be computed efficiently \([5, 17]\).

Recently, Aziz et al. \([1]\) introduced the threshold versions of monotone games, including PC-games as a special case. Yoram \([3]\) showed that computing \(\epsilon\)-core for threshold network flow games is polynomial time solvable for unit capacity networks, and NP-hard for networks with general capacities. For PC-games defined on series-parallel graphs, Aziz et al. \([1]\) showed that the nucleolus can be computed in polynomial time. However, the complexity of computing the nucleolus for general PC-games remains open; from the algorithmic point of view, the solution concepts of general PC-games have not been systematically discussed.

The algorithmic problems in cooperative games are especially interesting, since except for the fairness and rationality requirements in the solution definitions, computational complexity is suggested be taken into consideration as another measure of rationality for evaluating and comparing different solution concepts (Deng and Papadimitriou \([6]\)). Until now, various interesting complexity and algorithmic results have been investigated. On one hand, efficient algorithms have been proposed for computing the core, the least-core and the nucleolus for, such as, assignment games \([20]\), cardinality matching games \([14]\), unit flow games \([5]\) and weighted voting games \([8]\). On the other hand, negative results are also given. For example, the problems of computing the nucleolus and testing whether a given distribution belongs to the core or the nucleolus are proved to be NP-hard for minimum spanning tree games \([9, 10]\), flow games and linear production games \([5, 11]\).

The main contribution of this work is the efficient characterizations of the CS-core, least-core and the nucleolus of PC-games, based on linear programming technique and the relationship with flow games. These characterizations yield directly to efficient algorithms for the related solutions. The organization of the paper is as follows.

In section 2, the relevant definitions in cooperative game are introduced. In section 3, we first define PC-games (edge path cooperative game and vertex path cooperative game), and then give the the characterizations of the core and CS-core. Section 4 is dedicated to the efficient description of the least-core for PC-games. In section 5, we prove that the nucleolus is polynomially solvable for both edge and vertex path cooperative games.

## 2 Preliminaries

A cooperative game \(\Gamma = (N, \gamma)\) consists of a player set \(N = \{1, 2, \cdots, n\}\) and a characteristic function \(\gamma : 2^N \rightarrow R\) with \(\gamma(\emptyset) = 0\). For each coalition \(S \subseteq N\), \(\gamma(S)\) represents the profit obtained by \(S\) without help of other players. The set \(N\) is called the grand coalition. In what follows, we assume that \(\gamma(S) \geq 0\) for all \(S \subseteq N\), and \(\gamma(\emptyset) = 0\).

An imputation of \(\Gamma\) is a payoff vector \(x = (x_1, \ldots, x_n)\) such that \(\sum_{i \in N} x_i = \gamma(N)\) and \(x_i \geq \gamma(\{i\})\), \(\forall i \in N\). The set of imputations is denoted by \(I(\Gamma)\). Throughout this paper, we use the shorthand notation \(x(S) = \sum_{i \in S} x_i\). Given a payoff vector \(x \in I(\Gamma)\), the excess of coalition \(S \subseteq N\) with respect to \(x\) is defined...
as: \( e(x, S) = x(S) - \gamma(S) \). This value measures the degree of \( S \)'s satisfaction with the payoff \( x \).

**Core.** The core of a game \( \Gamma \), denoted by \( C(\Gamma) \), is the set of payoff vectors satisfying that, \( x \in C(\Gamma) \) if and only if \( e(x, S) \geq 0 \) for all \( S \subseteq N \). These constraints, called group rationality, ensure that no coalition would have an incentive to split from the grand coalition \( N \), and do better on its own.

**Least-core.** When \( C(\Gamma) \) is empty, it is meaningful to relax the group rationality constraints by \( e(x, S) \geq \varepsilon \) for all \( S \subseteq N \). We shall find the maximum value \( \varepsilon^* \) such that the set \( \{ x \in I(\Gamma) : e(x, S) \geq \varepsilon^*, \forall S \subseteq N \} \) is nonempty. This set of imputations is called the least-core, denoted by \( LC(\Gamma) \), and \( \varepsilon^* \) is called the value of \( LC(\Gamma) \) or \( LC \)-value.

**Nucleolus.** Now we turn to the concept of the nucleolus. A payoff vector \( x \) generates a \( 2^n \)-dimensional excess vector \( \theta(x) = (e(x, S_1), \cdots, e(x, S_2^n)) \), whose components are arranged in a non-decreasing order. That is, \( e(x, S_i) \leq e(x, S_j) \) for \( 1 \leq i < j \leq 2^n \). The nucleolus, denoted by \( \eta(\Gamma) \), is defined to be a payoff vector that lexicographically maximizes the excess vector \( \theta(x) \) over the set of imputations \( I(\Gamma) \). It was proved by Schmeidler [19] that the nucleolus of a game with the nonempty imputation set contains exactly one element.

**Monotone games and simple games.** A game \( \Gamma = (N, \gamma) \) is monotone if \( \gamma(S') \leq \gamma(S) \) whenever \( S' \subseteq S \). A game is called a simple game if it is a monotonic game with \( \gamma : 2^N \to \{0, 1\} \) such that \( \gamma(\emptyset) = 0 \) and \( \gamma(N) = 1 \). Simple games can be usually used to model situations where there is a task to be completed, a coalition is labeled as winning if and only if it can complete the task. Formally, coalition \( S \subseteq N \) is winning if \( \gamma(S) = 1 \), and losing if \( \gamma(S) = 0 \). A player \( i \) is called a veto player if he or she belongs to all winning coalitions. It is easy to see that, in a simple game, \( i \) is a veto player if and only if \( \gamma(N) = 1 \) but \( \gamma(N \setminus \{i\}) = 0 \).

For simple games, Osborne [16] and Elkind et al. [7] gave the following result on the core and the nucleolus.

**Lemma 1** A simple game \( \Gamma = (N, \gamma) \) has a nonempty core if and only if there exists a veto player. Moreover,

1. \( x \in C(\Gamma) \) if and only if \( x_i = 0 \) for each \( i \in N \) who is not a veto player;
2. when \( C(\Gamma) \neq \emptyset \), the nucleolus of \( \Gamma \) is given by \( x_i = \frac{1}{k} \) if \( i \) is a veto player and \( x_i = 0 \) otherwise, where \( k \) is the number of veto players.

**CS-core.** Taking coalition structure into consideration, we can arrive at another solution concept, CS-core. Given a cooperative game \( \Gamma = (N, \gamma) \), a coalition structure over \( N \) is a partition of \( N \), i.e., a collection of subsets \( CS = \{ C^1, \cdots, C^k \} \) with \( \bigcup_{j=1}^k C^j = N \) and \( C^i \cap C^j = \emptyset \) for \( i \neq j \) and \( i, j \in \{1, \cdots, k\} \). A vector \( x = (x_1, \cdots, x_n) \) is a payoff vector for a coalition structure \( CS = \{ C^1, \cdots, C^k \} \) if \( x_i \geq 0 \) for all \( i \in N \), and \( x(C^j) = \gamma(C^j) \) for each \( j \in \{1, \cdots, k\} \).

In general, an outcome of the game \( \Gamma \) is a pair \((CS, x)\), where \( CS \) is a coalition structure and \( x \) is a corresponding payoff vector. The CS-core of the game
\( \Gamma = (N, \gamma) \), denoted by \( C_{cs}(\Gamma) \), is the set of outcomes \((CS, x)\) satisfying the constraints of “group rationality”. That is,

\[
C_{cs}(\Gamma) = \{ (CS, x) : \forall C \in CS, x(C) = \gamma(C) \text{ and } \forall S \subseteq N, x(S) \geq \gamma(S) \}.
\]

A stronger property that is also enjoyed by many practically useful games is superadditivity. The game \( \Gamma = (N, \gamma) \) is superadditive if it satisfies \( \gamma(S_1 \cup S_2) \geq \gamma(S_1) + \gamma(S_2) \) for every pair of disjoint coalitions \( S_1, S_2 \subseteq N \). This implies that the agents can earn at least as much profit by working together within the grand coalition. Therefore, for superadditive games, it is always assumed that the agents form the grand coalition. For a (non-superadditive) game \( \Gamma = (N, \gamma) \), we can define a new game \( \Gamma^* = (N, \gamma^*) \) by setting

\[
\gamma^*(S) = \max_{CS \in CS_S} \gamma(CS), \forall S \subseteq N
\]

where \( CS_S \) denotes the space of all coalition structures over \( S \) and \( \gamma(CS) = \sum_{C \in CS} \gamma(C) \). It is easy to verify that the game \( \Gamma^* \) is superadditive, and it is called the superadditive cover of \( \Gamma \). The relationship between the \( CS \)-core of \( \Gamma \) and the core of its superadditive cover \( \Gamma^* \) is presented in the following lemma [4,12].

**Lemma 2** A cooperative game \( \Gamma = (N, \gamma) \) has nonempty \( CS \)-core if and only if its superadditive cover \( \Gamma^* = (N, \gamma^*) \) has a non-empty core. Moreover, if \( C(\Gamma^*) \neq \emptyset \), then \( C_{cs}(\Gamma) = C(\Gamma^*) \).

### 3 Path Cooperative Game and Its Core

Let \( D = (V, E; s, t) \) be a connected flow network with unit arc capacity (called unit flow network), where \( V \) is the vertex set, \( E \) is the arc set, \( s, t \in V \) are the source and the sink of the network respectively. In this paper, an \((s, t)\)-path is referred to a directed path from \( s \) to \( t \) that visits each vertex in \( V \) at most once.

Let \( U, W \subseteq V \) be a partition of the vertex set \( V \) such that \( s \in U \) and \( t \in W \), then the set of arcs with heads in \( U \) and tails in \( W \) is called an \((s, t)\)-edge-cut, denoted by \( \bar{E} \subseteq E \). An \((s, t)\)-vertex-cut is a vertex subset \( V \subseteq V \setminus \{s, t\} \) such that \( D \setminus V \) is disconnected. An \((s, t)\)-edge(vertex)-cut is minimum if its cardinality is minimum. In the remainder of the paper, \((s, t)\)-edge(vertex)-cuts will be abbreviated as edge(vertex)-cut \( S \) for short. Given an edge-cut \( \bar{E} \), we denote its indicator vector by \( H_{\bar{E}} \in \{0,1\}^{|E|} \), where \( H_{\bar{E}}(e) = 1 \) if \( e \in \bar{E} \), and 0 otherwise. The indicator vector of a vertex-cut is defined analogously.

Now we introduce two kinds of path cooperative games (PC-games), edge path cooperative games and vertex path cooperative games.

**Definition 1 (Path cooperative game, PC-game)** Let \( D = (V, E; s, t) \) be a unit flow network.

1. The associated edge path cooperative game (EPC-game) \( \Gamma_E = (E, \gamma_E) \) is:
   - The player set is \( E \);
2. The associated vertex path cooperative game (VPC-game) $\Gamma_V = (V, \gamma_V)$ is:
   - The player set is $V \setminus \{s, t\}$;
   - $\forall T \subseteq V, \begin{cases} 
   \gamma_V(T) = 1 & \text{if induced subgraph } D[T] \text{ admits an } (s, t)\text{-path;} \\
   \gamma_V(T) = 0 & \text{otherwise.}
   \end{cases}$

Clearly, PC-games fall into the class of simple games. Therefore, we can get the necessary and sufficient condition of the non-emptiness of the core directly from Lemma 1.

**Proposition 1** Given an EPC-game $\Gamma_E$ and a VPC-game $\Gamma_V$ associated with network $D = (V, E; s, t)$, then

1. $C(\Gamma_E) \neq \emptyset$ if and only if the size of the minimum edge-cut of $D$ is 1;
2. $C(\Gamma_V) \neq \emptyset$ if and only if the size of the minimum vertex-cut of $D$ is 1.

Moreover, when the core of a PC-game is nonempty, the only edge (vertex) in the edge (vertex)-cut is a veto player, both the core and the nucleolus can be given directly. In the following two sections, we only consider PC-games with empty core.

We note that PC-games also have a natural correspondence with flow games and in what follows, we will reveal the close relationship between flow games and PC-games. Let $D = (V, E; s, t)$ be a unit flow network. Given $N \subseteq E$, each edge $e \in N$ is controlled by one player, i.e., we can identify the set of edges $N$ with the set of players. Edges not under control of any players, in $E \setminus N$, are called public arcs; they can be used freely by any coalition. Thus, a unit flow network with player set $N$ is denoted as $D(N) = (V, E; s, t)$

**Definition 2 (Simple flow game)** The simple flow game $\Gamma_f(N) = (N, \gamma)$ associated with the unit network $D(N)$ is defined as:

1. The player set is $N$;
2. $\forall S \subseteq N, \gamma(S)$ is the value of the max-flow from $s$ to $t$ in $D[S \cup (E \setminus N)]$ (using only the edges in $S$ and public edges).

Flow game is a classical combinatorial optimization game, which has been extensively studied. The core of the flow game $\Gamma_f(N)$ is nonempty if and only if there is a minimum edge-cut without public edges [18]. In this case, the core is exactly the convex hull of the indicator vectors of minimum edge-cuts without public edges in $D$ [13,18], and the nucleolus can also be computed in polynomial time [5,17].

Now we turn to discuss the $CS$-core of PC-games. It is easy to see that for the network $D$ without public edges, the associated flow game is the superadditive cover of the corresponding EPC-game. Thus, the nonemptiness of $CS$-core of EPC-game is followed directly from Lemma 2.
Proposition 2. Given an EPC-game $\Gamma_E$ associated with network $D = (V, E; s, t)$, then the CS-core of $\Gamma_E$ is nonempty and it is exactly the convex hull of the indicator vectors of minimum edge-cuts of $D$.

For a VPC-game, we can also establish some relationship with a flow game. Given a network $D = (V, E; s, t)$, we transform it into a new network $D_V$ in the following way.

1. For each $v \in V \setminus \{s, t\}$, split it into two distinct vertices $v'$ and $v''$;
2. Connect $v'$ and $v''$ by a new directed edge $e_v = (v', v'')$. The set of all such edges is denoted by $E_V$;
3. For original edge $e = (u, v) \in E$, transform it into a new edge $e = (u'', v')$ in $D_V$ ($s = s'' = s'$ and $t = t' = t''$).

Proposition 3. Given an VPC-game $\Gamma_V$ associated with network $D = (V, E; s, t)$, then the CS-core of $\Gamma_V$ is nonempty and it is exactly the convex hull of the indicator vectors of minimum vertex-cuts of $D$.

4 Least-core of PC-Games

In this section, we first discuss the least-core of EPC-games. Throughout this section, $\Gamma_E$ is an EPC-game associated with the network $D = (V, E; s, t)$ with $|E| = n$. Denote by $P$ the set of all $(s, t)$-paths in $D$, and $|P| = m$. According to the definitions of EPC-game and the least-core, it is shown that $\text{LC}(\Gamma_E)$ can be formulated as the following linear program:

$$\begin{align*}
\max & \quad \varepsilon \\
\text{s.t.} & \quad x(E) = 1 \\
& \quad x(P) \geq 1 + \varepsilon \quad \forall P \in P \\
& \quad x_i \geq 0 \quad \forall i \in E
\end{align*}$$

In spite that the number of the constrains in (1) may be exponential in $|E|$, the $\mathcal{LC}$-value and a least-core imputation can be found efficiently by ellipsoid algorithm with a polynomial-time separation oracle. Let $(x, \varepsilon)$ be a candidate solution for LP($\mathcal{LC}_E$). We first check whether constraints $x(E) = 1$ and $x(e) \geq 0$
(∀e ∈ E) are satisfied. Then, checking whether x(P) ≥ 1 + ε (∀P ∈ P) are satisfied is transformed to solving the shortest (s, t)-path in D with respect to the edge length x(e) (∀e ∈ E), and this can also be done in polynomial time.

In what follows, we aim at giving a succinct characterization of the least-core for EPC-games. First give the linear program model of the max-flow problem on D and its dual:

\[
\text{LP(flow)}: \quad \begin{array}{ll}
\text{max} & \sum_{j=1}^{m} y_j \\
\text{s.t.} & \sum_{P_{ij} \in P_j} y_j \leq 1 & i = 1, 2, ..., n \\
& y_j \geq 0 & j = 1, 2, ..., m
\end{array}
\]

(2)

\[
\text{DLP(flow)}: \quad \begin{array}{ll}
\text{min} & \sum_{i=1}^{n} x_i \\
\text{s.t.} & \sum_{e_i \in P_j} x_i \geq 1 & j = 1, 2, ..., m \\
& x_i \geq 0 & i = 1, ..., n
\end{array}
\]

(3)

Due to max-flow and min-cut theorem, the optimum value of (2) and (3) are equal, and the set of optimal solutions of (3) is exactly the convex hull of the indicator vectors of the minimum edge-cut of D, which is denoted by \( C_E \). On the other hand, it is known that the core of the flow game \( \Gamma_f \) defined on \( D(E) \) is also the convex hull of the indicator vectors of the minimum edge-cut of D. Hence, we have

**Theorem 1** Let \( \Gamma_E \) and \( \Gamma_f \) be an EPC-game and a flow game defined on \( D = (V, E; s, t) \), respectively, \( f^* \) be the value of the max-flow of D. Then,

\[ x \in \mathcal{LC}(\Gamma_E) \text{ if and only if } x = z/f^* \text{ for some } z \in C_E. \]

Proof. Let \( x = (1 + \epsilon)z \) be a transformation, then (1) can be rewritten as

\[
\text{max} \ \epsilon \\
\text{s.t.} \quad \begin{array}{ll}
z(P) = 1/(1 + \epsilon) & \forall P \in P \\
z(e) \geq 1 & \forall e \in E \\
z_i \geq 0 & \forall e_i \in E
\end{array}
\]

(4)

Combining the first constraint \( z(E) = 1/(1 + \epsilon) \) and the objective function \( \min\{1 + \epsilon\} \), it is easy to see that linear program (4) is the same as \( \text{DLP(flow)} \) (3). Since the optimal value of (3) is also \( f^* \), Theorem 1 thus follows. \( \square \)

Based on the relationship between a VPC-game and the corresponding flow game discussed in Section 3, we can obtain a similar result on the least-core for VPC-games (The proof is omitted).

**Theorem 2** Let \( \Gamma_V = (E, \gamma_V) \) be a VPC-game defined on \( D = (V, E; s, t) \), \( f^* \) be the value of the max-flow of D, then

\[ x \in \mathcal{LC}(\Gamma_V) \text{ if and only if } x = z/f^* \text{ for some } z \in C_V. \]

Here \( C_V \) is the convex hull of the indicator vectors of minimum vertex-cuts in D.
Theorem 1 and 2 show that for the unit flow network, the least-core of the PC-game is equivalent to the core of the corresponding flow game in the sense of scaling down by $1/f^*$. Hence, all the following problems for PC-games can be solved efficiently:

- Computing the $\mathcal{L}C$-value;
- Finding an imputation in $\mathcal{L}C(\Gamma_E)$ and $\mathcal{L}C(\Gamma_V)$;
- Checking whether a given imputation is in $\mathcal{L}C(\Gamma_E)$ or $\mathcal{L}C(\Gamma_V)$.

**Remark.** Path cooperative games have close relationship with a non-cooperative two-person zero-sum game, called path intercept game [21]. In this model, an “evader” attempts to select a path $P$ from the source to the sink through a given network. At the same time, an “interdictor” attempts to select an edge $e$ in this network to detect the evader. If the evader traverses through arc $e$, he is detected; otherwise, he goes undetected. The interdictor aims to find a probabilistic “edge-inspection” strategy to maximize the average probability of detecting the evader. While for the evader, he wants to find a “path-selection strategy” to minimize the interdiction probability. Aziz et al. [2] observed that the mixed Nash Equilibrium of path intercept games is the same as the least-core of EPC-games. With max-min theorem in matrix game theory, the same result can be obtained based on the similar analysis as in the proof of Theorem 1.

5 **Nucleolus of PC-games**

In this section, we aim at showing that the nucleolus of PC-games can be computed in polynomial time. Given a game $\Gamma = (N, \gamma)$, Kopelowitz [15] showed that the nucleolus $\eta(\Gamma)$ can be obtained by recursively solving the following standard sequence of linear programs $SLP(\eta(\Gamma))$:

$$\max_{\varepsilon} \varepsilon$$

$$LP_k \quad (k = 1, 2, \ldots) \quad \text{s.t.}$$

$$x(S) = \gamma(S) + \varepsilon_r, \forall S \in J_r, \quad r = 0, 1, \ldots, k-1$$

$$x(S) \geq \gamma(S) + \varepsilon_r, \forall \emptyset \neq S \subset N \setminus \bigcup_{r=0}^{k-1} J_r$$

$$x \in \mathcal{I}(\Gamma).$$

Initially, set $J_0 = \{\emptyset, N\}$ and $\varepsilon_0 = 0$. The number $\varepsilon_r$ is the optimal value of the $r$-th program $LP_r$, and $J_r = \{S \subseteq N : x(S) = \gamma(S) + \varepsilon_r, \forall x \in X_r\}$, where $X_r = \{x \in R^n : (x, \varepsilon_r) \text{ is an optimal solution of } LP_r\}$.

As in the last section, we first discuss the nucleolus of EPC-games. Let $\Gamma_E$ be the EPC-game associated with network $D = (V, E; s, t)$ with $|E| = n$, $\mathcal{P}$ be the set of all $(s, t)$-paths and $f^*$ be the value of the max-flow of $D$. Denote $\mathcal{E}_r$ be the set of coalitions consisting of one-edge coalitions and path coalitions, i.e.,

$$\mathcal{E}_r = \{\{e\} : e \in E\} \cup \{P \subseteq E : P \in \mathcal{P} \text{ is an } (s, t)\text{-path}\}.$$
We show that the sequential linear programs \( SLP(\eta(\Gamma_E)) \) of EPC-game \( \Gamma_E \) can be simplified as follows.

\[
\begin{align*}
&\text{max} && \varepsilon \\
&\text{s.t.} && \begin{cases} 
  x(e) = \varepsilon_r, & \forall e \in E_r, r = 0, 1, ..., k - 1 \\
  x(e) \geq \varepsilon, & \forall e \in E \setminus \bigcup_{r=0}^{k-1} E_r \\
  x(P) = 1/f^* + \varepsilon_r, & \forall P \in \mathcal{P}_r, r = 0, 1, ..., k - 1 \\
  x(P) \geq 1/f^* + \varepsilon, & \forall P \in \mathcal{P} \setminus \bigcup_{r=0}^{k-1} \mathcal{P}_r \\
  x(e) \geq 0, & \forall e \in E \\
  x(E) = 1.
\end{cases}
\end{align*}
\]

where \( \varepsilon_r \) is the optimum value of \( LP_r \), \( X_r = \{ x \in \mathbb{R}^n : (x, \varepsilon_r) \text{ is an optimal solution of } LP_r \} \), \( \mathcal{P}_r = \{ P \in \mathcal{P} : x(P) = 1 + \varepsilon_r, \forall x \in X_r \} \) and \( E_r = \{ e \in E : x(e) = \varepsilon_r, \forall x \in X_r \} \). Initially, \( \varepsilon_0 = 0 \), \( \mathcal{P}_0 = \emptyset \) and \( E_0 = \emptyset \).

**Proposition 4** The nucleolus \( \eta(\Gamma_E) \) of EPC-game \( \Gamma_E \) defined on the network \( D = (V, E; s,t) \) can be obtained by computing the linear programs \( LP'_k \) in (5).

**Proof.** Firstly, we show that in sequential linear programs \( SLP(\eta(\Gamma)) \), only the constrains corresponding to the the coalitions in \( \mathcal{E}_F \) (i.e., the one-edge coalitions and path coalitions) are necessary in determining the nucleolus \( \eta(\Gamma_E) \).

In fact, for any winning coalition \( S \subseteq N \) (not a path), \( S \) can be decomposed into a path \( P \) and some edges \( E' = S \setminus E(P) \). Then,

\[
x(S) - \gamma(S) = x(P) - 1 + \sum_{e \in E'} x(e) \geq x(P) - 1.
\]

Since \( x(e) \geq 0 \) for all \( e \in E' \), \( S \) cannot be fixed before \( P \) or any \( e \in E' \). After \( P \) and all \( e \in E' \) are fixed, \( S \) is also fixed, i.e., \( S \) is redundant. If \( S \) is a losing coalition, then \( S \) is a set of edges with \( \gamma(S) = 0 \) and \( x(S) - \gamma(S) = \sum_{x \in S} x(e) \geq x(e), \forall e \in S \). That is to say, \( S \) cannot be fixed before any \( e \in S \). When all edges in \( S \) are fixed, \( S \) is fixed accordingly, i.e. \( S \) is also redundant in this case. Therefore, deleting all the constrains corresponding to the coalitions not in \( \mathcal{E}_F \) will not change the result of \( SLP(\eta(\Gamma)) \).

The key point in remainder of the proof is the correctness of the third and the forth constrains in (5), where we replace the original constrains \( x(P) = 1 + \varepsilon_r \) and \( x(P) \geq 1 + \varepsilon \) in \( SLP(\eta(\Gamma)) \) with new constrains \( x(P) = 1/f^* + \varepsilon_r \) and \( x(P) \geq 1/f^* + \varepsilon \), respectively.

In the process of solving the sequential linear programs, the optimal values increase with \( k \). Since \( \mathcal{C}(\Gamma_E) = \emptyset \), we know \( \varepsilon_1 < 0 \). Note that we can always find an optimal solution such that \( \varepsilon_1 > -1 \) (for example \( x(e) = \frac{1}{n}, \forall e \in E \) is a feasible solution of the linear programming of \( \mathcal{L}(\Gamma_E) \)).

We can divide the process into two stages. The first stage is the programs with \( -1 < \varepsilon_r < 0 \). In this case, the constraints \( x(e) \geq \varepsilon, \forall e \in E \) cannot effect the optimal solutions of the current programs, because \( x(e) \geq 0 \). Ignoring the invalid constrains we can get (5) directly.

The second stage is the programs with \( \varepsilon_r \geq 0 \). When the programs arrive at this stage, we can claim that all paths have been fixed. Otherwise, if there is
a path satisfying \( x(p) = 1 + \varepsilon_r \geq 1 \), then we have \( x(p) = 1 \) (note \( x(E) = 1 \)), contradicting with the precondition the value of maximum flow \( f^* \geq 2 \). Then we can omit the path constraints in this stage and then this implies (5).

This completes the proof of Proposition 4. \( \square \)

In the following, by making use the known results on the nucleolus of flow games, we shall show that the nucleolus of PC-games can be solved in polynomial time. Let \( \Gamma_f = (E, \gamma) \) be the flow game defined on the unit flow network \( D = (V, E; s, t) \). It is easy to show that the sequential linear programs \( LP(\eta(\Gamma_f)) \) can be simplified as \( \bar{LP}_k, (k = 1, 2, \ldots) \):

\[
\begin{align*}
\max \quad & \varepsilon \\
\text{s.t.} \quad & x(e) = \varepsilon_r, \quad \forall e \in E_r, r = 0, 1, \ldots, k - 1 \\
& x(P) = 1 + \varepsilon_r, \quad \forall P \in \mathcal{P}_r, r = 0, 1, \ldots, k - 1 \\
& x(e) \geq \varepsilon_r, \quad \forall e \in E \cup \bigcup_{r=0}^{k-1} E_r \\
& x(P) \geq 1 + \varepsilon, \quad \forall P \in \mathcal{P} \cup \bigcup_{r=0}^{k-1} \mathcal{P}_r \\
& x(E) = f^*, \quad \forall \mathcal{P} \in \mathcal{P}_k \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ Quad
then we have $x^*(e) > \varepsilon^*$. Thus, $E_1 = \tilde{E}_1$, $P_1 = \tilde{P}_1$ can be shown analogously. The other direction of the result can be shown similarly. That is, the conclusion holds for $k = 1$.

For the rest iteration $k = 2, 3, \cdots$, the proof can be carried out in a similar way. Here we omit the detail of the proof. Since the nucleolus of flow game can be found in polynomial time, it follows that the nucleolus of EPC-game is also efficiently solvable. □

As for the nucleolus of VPC-games, we also show that it is polynomially solvable based on the relationship between a VPC-game and the corresponding flow game demonstrated in Section 3. Due to the space limitation, the proof of the following theorem is omitted.

**Theorem 4** The nucleolus of VPC-games can be solved in polynomial time.

**PC-games on undirected networks.** Given an undirected network $D = (V, E; s, t)$, we construct a directed network $\overrightarrow{D} = (V, \overrightarrow{E}; s, t)$ derived from $D$ as follows (see the following figure):

1. For edge $e \in E$ with end vertices $v_1$ and $v_2$, transform it into two directed edges $\overrightarrow{e}_{v_1} = (v_{11}, v_{12})$ and $\overrightarrow{e}_{v_2} = (v_{21}, v_{22})$;
2. Connect the two directed edges into a directed cycle via two supplemental directed edges $\overrightarrow{e}_1$ and $\overrightarrow{e}_2$.

Thus, the EPC-game defined on undirected network $D = (V, E; s, t)$ is transformed to an EPC-game defined on the constructed directed network $\overrightarrow{D} = (V, \overrightarrow{E}; s, t)$. Furthermore, it is easy to check that there exists one-to-one correspondence for the game solution (such as, the core, the least-core and the nucleolus) between the two games. As for a VPC-game defined on an undirected network, we first transform it into EPC-game on an undirected network as demonstrated in Section 3, and then transform it to EPC-game on a directed network in the same way as above. Henceforth, the algorithmic results for PC-games can be generalized from directed networks to undirected networks.

**Theorem 5** Computing the least-core and the nucleolus can be done in polynomial time for both EPC-games and VPC-games defined on undirected networks.

**References**

1. Haris Aziz, Felix Brandt, and Paul Harrenstein. Monotone cooperative games and their threshold versions. In *Proceedings of the 9th International Conference on Autonomous Agents and Multiagent Systems*, volume 1, pages 1107–1114, 2010.
2. Haris Aziz and Troels Bjerre Sørensen. Path coalitional games. *arXiv preprint arXiv:1103.3310*, 2011.
3. Yoram Bachrach. The least-core of threshold network flow games. In *Mathematical Foundations of Computer Science 2011*, pages 36–47. Springer, 2011.
4. Georgios Chalkiadakis, Edith Elkind, and Michael Wooldridge. Computational aspects of cooperative game theory. *Synthesis Lectures on Artificial Intelligence and Machine Learning*, 5(6):1–168, 2011.
5. Xiaotie Deng, Qizhi Fang, and Xiaoxun Sun. Finding nucleolus of flow game. *Journal of combinatorial optimization*, 18(1):64–86, 2009.
6. Xiaotie Deng and Christos H Papadimitriou. On the complexity of cooperative solution concepts. *Mathematics of Operations Research*, 19(2):257–266, 1994.
7. Edith Elkind, Leslie Ann Goldberg, Paul W Goldberg, and Michael Wooldridge. Computational complexity of weighted threshold games. In *Proceedings of the National Conference on Artificial Intelligence*, volume 22, page 718, 2007.
8. Edith Elkind and Dmitrii Pasechnik. Computing the nucleolus of weighted voting games. In *Proceedings of the 12th Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 327–335, 2009.
9. Ulrich Faigle, Walter Kern, Sándor P Fekete, and Winfried Hochstättler. On the complexity of testing membership in the core of min-cost spanning tree games. *International Journal of Game Theory*, 26(3):361–366, 1997.
10. Ulrich Faigle, Walter Kern, and Jeroen Kuipers. Note computing the nucleolus of min-cost spanning tree games is np-hard. *International Journal of Game Theory*, 27(3):443–450, 1998.
11. Qizhi Fang, Shanfeng Zhu, Maocheng Cai, and Xiaotie Deng. On computational complexity of membership test in flow games and linear production games. *International Journal of Game Theory*, 31(1):39–45, 2002.
12. Gianluigi Greco, Enrico Malizia, Luigi Palopoli, and Francesco Scarcello. On the complexity of the core over coalition structures. In *IJCAI*, volume 11, pages 216–221. Citeseer, 2011.
13. Ehud Kalai and Eitan Zemel. Generalized network problems yielding totally balanced games. *Operations Research*, 30(5):998–1008, 1982.
14. Walter Kern and Daniel Paulusma. Matching games: the least-core and the nucleolus. *Mathematics of Operations Research*, 28(2):294–308, 2003.
15. Alexander Kopelowitz. Computation of the kernels of simple games and the nucleolus of n-person games. Technical report, DTIC Document, 1967.
16. Martin J Osborne and Ariel Rubinstein. A course in game theory. *Cambridge, Massachusetts*, 1994.
17. Jos Potters, Hans Reijnierse, and Amit Biswas. The nucleolus of balanced simple flow networks. *Games and Economic Behavior*, 54(1):205–225, 2006.
18. Hans Reijnierse, Michael Maschler, Jos Potters, and Stef Tijs. Simple flow games. *Games and Economic Behavior*, 16(2):238–260, 1996.
19. David Schmeidler. The nucleolus of a characteristic function game. *SIAM Journal on applied mathematics*, 17(6):1163–1170, 1969.
20. Tamás Solymosi and Tirukkanamangai ES Raghavan. An algorithm for finding the nucleolus of assignment games. *International Journal of Game Theory*, 23(2):119–143, 1994.
21. Alan Washburn and Kevin Wood. Two-person zero-sum games for network interdiction. *Operations Research*, 43(2):243–251, 1995.