Sub-subleading Soft Graviton Theorem in Generic Theories of Quantum Gravity

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Abstract

We analyze scattering amplitudes with one soft external graviton and arbitrary number of other finite energy external states carrying arbitrary mass and spin to sub-subleading order in the momentum of the soft graviton. Our result can be expressed as the sum of a universal part that depends only on the amplitude without the soft graviton and not the other details of the theory and a non-universal part that depends on the amplitude without the soft graviton, and the two and three point functions of the theory. For tree amplitudes our results are valid in all space-time dimensions while for loop amplitudes, infrared divergences force us to restrict our analysis to space time dimensions five or more. With this restriction the results are valid to all orders in perturbation theory. Our results agree with known results in quantum field theories and string theory.
1 Introduction and summary

Soft graviton theorem has been studied extensively in recent years [1–29], primarily due to its connection to asymptotic symmetries [30–39] (see [40] for a recent review). It relates the scattering amplitude of a set of finite energy particles and a low momentum (soft) graviton to an amplitude without the low momentum graviton. Soft theorems are also known to hold in string theories [41–55]. Our goal in this paper will be to analyze sub-subleading soft graviton theorem – that gives the result for the above mentioned scattering amplitude to the sub-subleading order in the energy of the soft graviton.
Our main result is that in a generic theory the sub-subleading soft graviton amplitude is given by a sum of a standard set of terms that are universal, independent of the theory, and a non-universal term that depends on the theory. The standard terms, reproduced in all but the last line of eq. (2.44), can be found e.g. in [6] after appropriate generalizations to arbitrary dimensions. On the other hand for scattering of \( N \) finite energy particles carrying momenta \( p_1, \cdots, p_N \) and a soft graviton carrying momentum \( k \) and polarization \( \varepsilon \), the correction term takes the form

\[
\{ \varepsilon_{\mu\nu} k_\rho k_\sigma - \varepsilon_{\nu\rho} k_\sigma k_\mu + \varepsilon_{\rho\sigma} k_\mu k_\nu \} \sum_{i=1}^{N} \frac{1}{p_i \cdot k} \sum_{i'} B^{\mu\nu\rho\sigma}_{i,i'}(p_i) \Gamma_{i \to i'} \tag{1.1}
\]

where the sum over \( i' \) runs over all on-shell states carrying the same mass and momentum as the external state \( i \) and \( \Gamma_{i \to i'} \) denotes the original amplitude without the soft graviton, and the \( i \)th state replaced by \( i' \). The quantity \( B^{\mu\nu\rho\sigma}_{i,i'}(p_i) \) is a function of the momentum \( p_i \) carried by the \( i \)-th external particle and depends on the quadratic and cubic terms in the one particle irreducible (1PI) effective action. For a given action \( B \) can be computed explicitly (see eqs.(2.44), (2.45)).

As our analysis is based on general properties of the 1PI effective action, our results are valid for any general coordinate invariant theory of gravity coupled to other fields, including string theory. For tree amplitudes there is no restriction on the number of space-time dimensions. However for loop amplitudes, infrared divergences [56] force us to restrict our analysis to five or more space-time dimensions. A more detailed investigation of soft graviton theorem in generic theories of gravity in four dimensions is left for future investigation.

If we focus our attention on the theory of massless fields in four dimensions, possibly obtained by integrating out other massive fields, then Weinberg-Witten theorem excludes the presence of interacting particles of spin \( > 2 \). For tree level scattering of massless particles of spin \( \leq 2 \) we can list all possible three point couplings that can possibly contribute to the function \( B^{\mu\nu\rho\sigma}_{i,i'}(p_i) \) appearing in (1.1). These have been listed in eqs.(4.19) and (4.37). Their contribution to \( B^{\mu\nu\rho\sigma}_{i,i'}(p_i) \) can be evaluated easily. In the spinor helicity representation they reproduce the results of [57]. Of course our general result (1.1) is more general and holds in any space-time dimensions and also for massive higher spin fields. In particular it can also be used to reproduce various results on sub-subleading soft graviton amplitudes in string theory [50,52] involving scattering of massless as well as massive fields.

Our analysis is based on the idea used in [54,55] in which the coupling of a soft graviton to the rest of the fields is obtained by covariantizing the gauge fixed 1PI effective action of the
finite energy particles with respect to the soft graviton background. It is natural to ask if the same technique can be used to extend the analysis to next order in soft momentum. However at the end of section 2 we have argued that at least this technique is not extendable to the next order.

In fact, there may be a deeper reason as to why for generic configuration of external states, soft theorems do not appear to extend beyond subleading order in gauge theories and sub-subleading order in gravity. It is now becoming increasingly evident that soft theorems are statements about (asymptotic) symmetries of the underlying theory [40]. In the case of QED, it was argued in [58] that if the soft theorems in QED were to extend beyond subleading order, the associated asymptotic symmetries will be ill-defined in the sense that the corresponding charges will be divergent. One expects similar divergences to occur in gravity, if one were to extend the emergence of soft theorems from asymptotic symmetries beyond sub-subleading order [34–36].

For special configurations of external states as in the case of MHV amplitude, it was shown in [8] that factorization theorem holds to all orders in graviton energy. In view of the discussion above this seems accidental. A more detailed investigation of such results from the perspective presented in this paper is left for future investigation.

The rest of the paper is organized as follows. In section 2 we analyze amplitudes with one external soft graviton and arbitrary number of other external states in any theory of gravity coupled to matter field to sub-subleading order in the soft momentum. The final result is given in (2.44), (2.45). These are the main results of our paper. In section 3 we show that our result (2.44), (2.45) depends only on the on-shell data of an amplitude without the soft graviton, even though individual terms in these equations depend on the off-shell continuation. Sections 4 and 5 involve comparing our general result with known tree level results in quantum field theories and string theory, and we find perfect agreement.

The usual S-matrix in four space-time dimensions suffers from infrared divergence in the presence of massless particles. Therefore for loop amplitudes we need to restrict our analysis to five or more space-time dimensions \( D \). Even though infrared divergences do not affect the usual S-matrix elements for \( D \geq 5 \), they may still alter the behaviour of an amplitude in the soft limit by producing additional singularities that are not included in our analysis of section 2. In section 6 we analyze this possibility in detail and show that no such additional divergences arise. Therefore we can trust the result of section 2 for loop amplitudes in \( D \geq 5 \).
Figure 1: The leading contribution.

2 Sub-subleading soft graviton theorem

We consider a general theory of gravity coupled to other matter fields and focus on a scattering amplitude involving one soft graviton of momentum $k$ and polarization $\varepsilon$, satisfying the constraints

$$k^2 = 0, \quad \varepsilon_{\mu\nu} = \varepsilon_{\nu\mu}, \quad k^\mu \varepsilon_{\mu\nu} = 0, \quad \varepsilon^\mu_\mu = 0.$$ (2.1)

The amplitude is given by a sum of two types of diagrams, shown in Figs. 1 and 2. Fig. 1 represents sum of all diagrams where the soft graviton is attached to one of the external finite energy lines via 1PI three point vertex. Fig. 2 contains the sum of the rest of the diagrams. The leading contribution in the soft limit $k \to 0$ comes from Fig. 1 due to the pole associated with the propagator carrying momentum $p_i + k$. Fig. 2 does not have such poles and therefore begins contributing at the subleading order.

We shall now describe separately the evaluation of these two classes of diagrams. In doing this we shall follow the strategy of [54, 55], i.e. first choose a covariant gauge fixing of the 1PI effective action of finite energy fields (including gravitons), expanded in a power series in the fields around flat space-time background, and then determine the coupling of the soft graviton to the finite energy fields by replacing the background flat metric by soft graviton background metric and ordinary derivatives by covariant derivatives computed using the soft graviton background metric. As in [55], the finite energy fields will be assumed to carry flat tensor indices associated with the tangent space group so that their covariant derivatives involve the spin connection and not the Christoffel symbol.
2.1 Evaluation of Fig. 2

In this section we shall analyze Fig. 2 which begins contributing at the subleading order. Let us denote this by \( \tilde{\Gamma}(\varepsilon, k; \varepsilon_1, p_1, \ldots, \varepsilon_N, p_N) \), where \((\varepsilon, k)\) are the polarizations and momentum of the soft graviton and \((\varepsilon_i, p_i)\) are the polarizations and momentum of the \(i\)-th external state. All external propagators are amputated in the definition of \( \tilde{\Gamma} \). We shall also assume that all the external fields are normalized correctly so that we do not need to keep track of wave-function renormalization factors in relating the amplitudes to S-matrix elements. We shall include an explicit momentum conserving delta function in the expression for the amplitude and treat the \( p_i \)'s and \( k \) as independent variables while taking derivative of the amplitude with respect to these momenta. We shall not impose any on-shell condition on \((\varepsilon_i, p_i)\) till the end after all the derivatives with respect to momenta are taken, but the soft graviton will be taken to be on-shell from the beginning. Finally we allow the polarization tensor \( \varepsilon_i \) to depend on \( p_i \) but no other external momenta and the polarization \( \varepsilon \) of the soft graviton to depend on \( k \) but no other momenta.

Our goal will be to express \( \tilde{\Gamma} \) in terms of the amplitude without the soft graviton shown in Fig. 3. This has the form

\[
\varepsilon_{1,\alpha_1} \cdots \varepsilon_{N,\alpha_N} \Gamma^{\alpha_1 \cdots \alpha_N}(p_1, \ldots p_N)(2\pi)^D \delta^{(D)}(p_1 + \cdots + p_N)
\]  

(2.2)

where we shall take the index \( \alpha \) to run over all the fields \( \Phi_\alpha \) present in the theory. We shall assume that all fields carry tangent space indices so that the fields \( \{\Phi_\alpha\} \) belong to some large reducible representation of the local Lorentz group. There is an ambiguity in defining the function \( \Gamma^{\alpha_1 \cdots \alpha_N}(p_1, \ldots p_N) \) since we can add to it any term that vanishes when \( \sum_i p_i = 0 \).
We shall not impose any restriction on how we resolve this ambiguity except for the (anti-)symmetry of $\Gamma^{\alpha_1 \ldots \alpha_N}(p_1, \ldots p_N)$ under the exchange $(\alpha_i, p_i) \leftrightarrow (\alpha_j, p_j)$ for any pair $(i, j)$. We also introduce the shorthand notation

$$\Gamma^{\alpha_i}_{(i)}(p_i) = \left( \prod_{j=1, j \neq i}^{N} \epsilon_{j, \alpha_j} \right) \Gamma^{\alpha_1 \ldots \alpha_N}(p_1, \ldots p_N)(2\pi)^D \delta(D)(p_1 + \cdots + p_N) \quad \text{(2.3)}$$

where in the argument of $\Gamma_{(i)}$ we have suppressed the momenta $p_j$ and polarizations $\epsilon_j$ for $j \neq i$.

We shall now determine the amplitude shown in Fig. 2 from the one in Fig 3 by noting the following. We can determine the coupling of a soft graviton to the finite energy fields by replacing, in the expression for (2.2) written in position space, all derivatives $\partial_\mu$ by covariant derivatives $D_\mu$, and eventually converting them to flat space index by contracting them with the inverse vielbein $E_\mu^a$. This procedure can be regarded as the result of covariantization of the amputated Green’s function with respect to the general coordinate transformation of the background soft graviton field.

To first order in the soft graviton field the inverse vielbein is given by

$$E_\mu^a = \delta_\mu^a - S_\mu^a \quad \text{(2.4)}$$

where $S_\mu^a$ is the soft graviton

$$S_{\mu \nu} = \epsilon_{\mu \nu} e^{ik \cdot x}, \quad \text{(2.5)}$$

and all indices are raised and lowered by the flat metric $\eta$. For constructing the covariant derivative we also need the expression for the spin connection $\omega_\mu^{ab}$ and Christoffel symbol $\Gamma_\mu^{\rho}$. 

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To first order in the soft graviton field these are given by
\[ \omega_{\mu}^{ab} = \partial^b S_{\mu}^a - \partial^a S_{\mu}^b = i e^{i k \cdot x} \left( k^b \varepsilon_{\mu}^a - k^a \varepsilon_{\mu}^b \right), \quad (2.6) \]
and
\[ \Gamma_{\mu\nu}^\rho = \partial_\mu S_\nu^\rho + \partial_\nu S_\mu^\rho - \partial_\rho S_{\mu\nu} = i e^{i k \cdot x} \left\{ k_\mu \varepsilon_\nu^\rho + k_\nu \varepsilon_\mu^\rho - k_\rho \varepsilon_{\mu\nu} \right\} . \quad (2.7) \]

The covariant derivative has two kinds of terms. Acting on a field \( \Phi_\alpha \) transforming in some (not necessarily irreducible) representation \( R \) of the Lorentz group, it has a piece
\[ \frac{1}{2} \omega_{\mu}^{ab} (J_{ab})_{\alpha}^\beta \Phi_\beta \quad (2.8) \]
where \( J_{ab} \) is the generator of the Lorentz group in the representation \( R \) normalized so that acting on a covariant vector field
\[ (J_{ab})^d_c = \delta_c^{bd} - \delta_c^{db}. \quad (2.9) \]
The second kind of term arises from the fact that when \( D_\mu \) is preceded by a \( D_\nu \) operation, we get a factor of
\[ -\Gamma_{\mu\nu}^\rho D_\rho. \quad (2.10) \]
Since \( \Gamma_{\mu\nu}^\rho \) already contains a factor of soft graviton field and since we shall work to first order in the soft graviton field, we can replace \( D_\rho \) by \( \partial_\rho \) in (2.10). This leads to the simple rule that for every pair of derivatives we get a factor of \(-\Gamma_{\mu\nu}^\rho \partial_\rho\).

Since in momentum space a derivative is replaced by \( i p_\mu \), the above considerations give the following expression for the amplitude in Fig 2 in terms of the amplitude in Fig 3 to order \( k^\rho \):
\[ (2\pi)^D \delta^{(D)} (p_1 + \cdots + p_N + k) \prod_{j=1}^N \varepsilon_{j,\alpha_j} \sum_{i=1}^N \left[ -\delta_{\beta_i}^{\alpha_i} \varepsilon_\mu p_\mu \frac{\partial}{\partial p_\mu} + k^b \varepsilon_\mu (J_{ab})_{\beta_i}^{\alpha_i} \frac{\partial}{\partial p_\mu} - \frac{1}{2} \delta_{\beta_i}^{\alpha_i} \left\{ k_\mu \varepsilon_\nu^\rho + k_\nu \varepsilon_\mu^\rho - k_\rho \varepsilon_{\mu\nu} \right\} p_\mu \frac{\partial^2}{\partial p_\mu \partial p_\nu} \right] \Gamma^{\alpha_1 \cdots \alpha_{i-1} \beta_i \alpha_{i+1} \cdots \alpha_N} (p_1, \ldots, p_N). \quad (2.11) \]

In this the first term inside the square bracket is the effect of multiplication by inverse vielbein to convert the space-time indices carried by the momenta to tangent space indices. The second term represents the effect of the spin connection term in the covariant derivative and the third factor represents the effect of the Christoffel symbol term in the covariant derivative. The shift
by $k$ of the argument of the delta function represents the effect of the multiplicative factor of $e^{ik\cdot x}$ from the soft graviton field in the position space representation of the amplitude.

We shall now show that we can bring the momentum conserving delta function in (2.11) to the right of the derivatives so that the derivatives also act on the delta function. We begin with second and the third terms in the square bracket. Their contribution to (2.11) may be expressed as

$$\prod_{j=1}^{N} \epsilon_{j,\alpha_j} \sum_{i=1}^{N} \left[ k_{\alpha_i}^\beta \frac{\partial}{\partial p_{i\mu}} + \frac{1}{2} \delta_\alpha^\alpha \left\{ k_{\mu}^{\rho} \varepsilon_{\nu}^{\beta} + k_{\nu}^{\beta} \varepsilon_{\mu}^{\rho} - k_{\rho}^{\mu} \varepsilon_{\mu\nu} \right\} p_{i\rho} \frac{\partial^2}{\partial p_{i\mu} \partial p_{i\nu}} \right] \left\{ \Gamma^{\alpha_1,\ldots,\alpha_{i-1},\beta_i,\alpha_{i+1},\ldots,\alpha_N}(p_1, \ldots, p_N)(2\pi)^D \delta^{(D)}(p_1 + \cdots p_N + k) \right\} + J_1 + J_2 \label{2.12}$$

where

$$J_1 = (2\pi)^D \left\{ \prod_{j=1}^{N} \epsilon_{j,\alpha_j} \right\} \Gamma^{\alpha_1,\ldots,\alpha_N}(p_1, \ldots, p_N) \frac{1}{2} \left\{ k_{\mu}^{\rho} \varepsilon_{\nu}^{\beta} + k_{\nu}^{\beta} \varepsilon_{\mu}^{\rho} - k_{\rho}^{\mu} \varepsilon_{\mu\nu} \right\} \sum_{i=1}^{N} p_{i\rho} \frac{\partial^2}{\partial p_{i\mu} \partial p_{i\nu}} \delta^{(D)}(p_1 + \cdots p_N + k) \label{2.13}$$

and

$$J_2 = -(2\pi)^D \left\{ \prod_{j=1}^{N} \epsilon_{j,\alpha_j} \right\} \sum_{i=1}^{N} \frac{\partial}{\partial p_{i\mu}} \delta^{(D)}(p_1 + \cdots p_N + k) \left[ k_{\alpha_i}^\beta \frac{\partial}{\partial p_{i\mu}} - \delta_\alpha^\alpha \left\{ k_{\mu}^{\rho} \varepsilon_{\nu}^{\beta} + k_{\nu}^{\beta} \varepsilon_{\mu}^{\rho} - k_{\rho}^{\mu} \varepsilon_{\mu\nu} \right\} p_{i\rho} \frac{\partial}{\partial p_{i\mu}} \right] \Gamma^{\alpha_1,\ldots,\alpha_{i-1},\beta_i,\alpha_{i+1},\ldots,\alpha_N}(p_1, \ldots, p_N). \label{2.14}$$

$J_1$ cancels the term where both derivatives in the last term in the square bracket in (2.12) act on the delta function, whereas $J_2$ cancels the terms in (2.12) where one momentum derivative acts on the delta function.

We shall first analyze $J_1$. In (2.13) we can replace $\partial^2/\partial p_{i\mu} \partial p_{i\nu}$ by $\partial^2/\partial k_{i\mu} \partial k_{i\nu}$ using the fact that the argument of the delta function contains sum of all the $p_i$'s and $k$. We can now bring the $\sum_{i=1}^{N} p_{i\rho}$ factor inside the derivative and finally replace it by $-k_{i\rho}$ using the delta function. This gives

$$J_1 = (2\pi)^D \left\{ \prod_{j=1}^{N} \epsilon_{j,\alpha_j} \right\} \Gamma^{\alpha_1,\ldots,\alpha_N}(p_1, \ldots, p_N) \frac{1}{2} \left\{ k_{\mu}^{\rho} \varepsilon_{\nu}^{\beta} + k_{\nu}^{\beta} \varepsilon_{\mu}^{\rho} - k_{\rho}^{\mu} \varepsilon_{\mu\nu} \right\}$$
\[
\frac{\partial^2}{\partial k_\mu \partial k_\nu} \left[ -k_\rho \delta^{(D)}(p_1 + \cdots + p_N + k) \right] = (2\pi)^D \left\{ \prod_{j=1}^N \epsilon_{j,\alpha_j} \right\} \Gamma^{\alpha_1 \cdots \alpha_N}(p_1, \ldots, p_N) \frac{1}{2} \left\{ k_\mu \varepsilon_\nu^\rho + k_\nu \varepsilon_\mu^\rho - k^\rho \varepsilon_{\mu\nu} \right\} \\
\left[ -k_\rho \frac{\partial^2}{\partial k_\mu \partial k_\nu} - \delta_\rho^\mu \frac{\partial}{\partial k_\nu} - \delta_\nu^\rho \frac{\partial}{\partial k_\mu} \right] \delta^{(D)}(p_1 + \cdots + p_N + k) = 0
\]

(2.15)

where in the last step we have used (2.1).

On the other hand in the expression \ref{2.14} for \( J_2 \) we can replace each of the \( \partial/\partial p_i \) operator acting on the momentum conserving delta function by \( \partial/\partial k_\mu \) and express it as

\[
J_2 = -(2\pi)^D \left\{ \prod_{j=1}^N \epsilon_{j,\alpha_j} \right\} \frac{\partial}{\partial k_\mu} \delta^{(D)}(p_1 + \cdots + p_N + k) \\
\times \sum_{i=1}^N \left[ k^\rho \varepsilon_\mu^a \left( J_{ab} \right)_{\beta_i}^{\alpha_i} - \delta_{\beta_i}^{\alpha_i} \left\{ k_\mu \varepsilon_\nu^\rho + k_\nu \varepsilon_\mu^\rho - k^\rho \varepsilon_{\mu\nu} \right\} p_i^\rho \frac{\partial}{\partial p_i^\nu} \right] \Gamma^{\alpha_1 \cdots \alpha_{i-1} \beta_i \alpha_{i+1} \cdots \alpha_N}(p_1, \ldots, p_N).
\]

(2.16)

Now Lorentz covariance of \( \Gamma^{\alpha_1 \cdots \alpha_N}(p_1, \ldots, p_N) \) implies

\[
\sum_{i=1}^N \left[ \left( J_{ab} \right)_{\beta_i}^{\alpha_i} \Gamma^{\alpha_1 \cdots \alpha_{i-1} \beta_i \alpha_{i+1} \cdots \alpha_N}(p_1, \ldots, p_N) - \left\{ p_i^a \frac{\partial}{\partial p_i^b} - p_i^b \frac{\partial}{\partial p_i^a} \right\} \Gamma^{\alpha_1 \cdots \alpha_N}(p_1, \ldots, p_N) \right] = 0.
\]

(2.17)

Using this \ref{2.16} may be expressed as

\[
J_2 = -(2\pi)^D \left\{ \prod_{j=1}^N \epsilon_{j,\alpha_j} \right\} \frac{\partial}{\partial k_\mu} \delta^{(D)}(p_1 + \cdots + p_N + k) \\
\sum_{i=1}^N \left[ k^\rho \varepsilon_\mu^a \left\{ p_i^a \frac{\partial}{\partial p_i^b} - p_i^b \frac{\partial}{\partial p_i^a} \right\} - \left\{ k_\mu \varepsilon_\nu^\rho + k_\nu \varepsilon_\mu^\rho - k^\rho \varepsilon_{\mu\nu} \right\} \right] p_i^\rho \frac{\partial}{\partial p_i^\nu} \Gamma^{\alpha_1 \cdots \alpha_N}(p_1, \ldots, p_N)
\]

\[
= (2\pi)^D \left\{ \prod_{j=1}^N \epsilon_{j,\alpha_j} \right\} \frac{\partial}{\partial k_\mu} \delta^{(D)}(p_1 + \cdots + p_N + k) k_\mu \varepsilon_\nu^\rho \sum_{i=1}^N \left( \frac{\partial}{\partial p_i^\nu} \Gamma^{\alpha_1 \cdots \alpha_N}(p_1, \ldots, p_N) \right).
\]

(2.18)

We now turn to the contribution from the first term inside the square bracket in \ref{2.11}. By expanding the delta function in Taylor series expansion in \( k \) and keeping terms up to order
\( k^\mu \), we get

\[-(2\pi)^D \left\{ \prod_{j=1}^{N} \epsilon_{j,\alpha_j} \right\} \delta^{(D)}(p_1 + \cdots + p_N) \varepsilon^{\rho}_{\nu} \sum_{i=1}^{N} p_{i\rho} \frac{\partial}{\partial p_{i\nu}} \Gamma^{\alpha_1 \cdots \alpha_N}(p_1, \ldots, p_N) + J_3 \] (2.19)

where

\[ J_3 = -(2\pi)^D \left\{ \prod_{j=1}^{N} \epsilon_{j,\alpha_j} \right\} \frac{\partial}{\partial k^\mu} \delta^{(D)}(p_1 + \cdots + p_N + k) k^\mu \varepsilon^{\rho}_{\nu} \sum_{i=1}^{N} p_{i\rho} \frac{\partial}{\partial p_{i\nu}} \Gamma^{\alpha_1 \cdots \alpha_N}(p_1, \ldots, p_N). \] (2.20)

Note that in a Taylor series expansion we would normally set \( k \) in the argument of \( \delta^{(D)} \) to zero after taking the derivative. However, to this order in the expansion in powers of \( k \), it does not make any difference. Using the relation

\[ \varepsilon^{\rho}_{\nu} \sum_{i=1}^{N} p_{i\rho} \frac{\partial}{\partial p_{i\nu}} \delta^{(D)}(p_1 + \cdots + p_N) = \varepsilon^{\rho}_{\nu} \sum_{i=1}^{N} p_{i\rho} \frac{\partial}{\partial p_{i\nu}} \delta^{(D)}(p_1 + \cdots + p_N) \]

we can express (2.19) as

\[- \left\{ \prod_{j=1}^{N} \epsilon_{j,\alpha_j} \right\} \varepsilon^{\rho}_{\nu} \sum_{i=1}^{N} p_{i\rho} \frac{\partial}{\partial p_{i\nu}} \{ \Gamma^{\alpha_1 \cdots \alpha_N}(p_1, \ldots, p_N)(2\pi)^D \delta^{(D)}(p_1 + \cdots + p_N) \} + J_3. \] (2.22)

The total contribution from Fig. 2, given in (2.11), is given by the sum of (2.12) and (2.22). We have seen from (2.15) that \( J_1 \) vanishes. On the other hand (2.18) and (2.20) shows that \( J_2 + J_3 \) vanishes. Furthermore, since we need the terms up to order \( k \), we can set \( k = 0 \) in the argument of the delta function in (2.12). This gives the net contribution to Fig. 2 to order \( k \) as

\[ I_0 = \prod_{j=1}^{N} \sum_{i=1}^{N} \left[ -\varepsilon^{\rho}_{\nu} p_{i\rho} \frac{\partial}{\partial p_{i\nu}} \delta^{\alpha_i} + k^\rho \varepsilon^{\alpha_i}_{\mu}(J_{ab})^{\alpha_i}_{\beta_i} \frac{\partial}{\partial p_{i\mu}} \right. \]

\[ -\frac{1}{2} \delta^{\alpha_i}_{\beta_i} \left\{ k^\mu \varepsilon^{\rho}_{\nu} + k^\nu \varepsilon^{\rho}_{\mu} - k^\rho \varepsilon^{\mu}_{\nu} \right\} p_{i\rho} \frac{\partial^2}{\partial p_{i\mu} \partial p_{i\nu}} \left. \right\{ \Gamma^{\alpha_1 \cdots \alpha_N}(p_1, \ldots, p_N)(2\pi)^D \delta^{(D)}(p_1 + \cdots + p_N) \}. \] (2.23)

Using (2.3) and the fact that \( \epsilon_j \) is independent of \( p_i \) for \( j \neq i \), (2.23) may be rewritten as

\[ I_0 = \sum_{i=1}^{N} \varepsilon^{\rho}_{\nu} p_{i\rho} \frac{\partial}{\partial p_{i\nu}} \delta^{\alpha_i} + k^\rho \varepsilon^{\alpha_i}_{\mu}(J_{ab})^{\alpha_i}_{\beta_j} \frac{\partial}{\partial p_{i\mu}} \]

\[ -\frac{1}{2} \delta^{\alpha_i}_{\beta_i} \left\{ k^\mu \varepsilon^{\rho}_{\nu} + k^\nu \varepsilon^{\rho}_{\mu} - k^\rho \varepsilon^{\mu}_{\nu} \right\} p_{i\rho} \frac{\partial^2}{\partial p_{i\mu} \partial p_{i\nu}} \left. \right\{ \Gamma^{\alpha_1 \cdots \alpha_N}(p_1, \ldots, p_N)(2\pi)^D \delta^{(D)}(p_1 + \cdots + p_N) \}. \] (2.24)
2.2 Evaluation of Fig. 1

We now turn to the evaluation of the contribution from Fig. 1. To evaluate this we begin by writing the quadratic term in the 1PI effective action:

\[ S^{(2)} = \frac{1}{2} \int \frac{d^Dq_1}{(2\pi)^D} \frac{d^Dq_2}{(2\pi)^D} \Phi_\alpha(q_1) \mathcal{K}^{\alpha\beta}(q_2) \Phi_\beta(q_2) (2\pi)^{D} \delta^{(D)}(q_1 + q_2), \]  

(2.25)

where we take

\[ \mathcal{K}^{\alpha\beta}(q) = \mathcal{K}^{\beta\alpha}(-q). \]  

(2.26)

For grassmann odd fields there will be an additional minus sign on the right hand side of (2.26), but the final result is unaffected by this. For this reason we shall proceed by taking the fields to be grassmann even. The full propagator computed from this action has the form

\[ \Xi(q)(q^2 + M^2)^{-1}, \]  

(2.27)

where

\[ \Xi(q) = i(q^2 + M^2)\mathcal{K}^{-1}(q). \]  

(2.28)

At this stage \( M \) is taken to be an arbitrary mass parameter. Lorentz covariance of \( \mathcal{K} \) and \( \Xi \) implies the relations

\[ \mathcal{K}^{\alpha\gamma}(q)(J^{ab})_\gamma^\beta + \mathcal{K}^{\gamma\beta}(q)(J^{ab})_\gamma^\alpha = q^a \frac{\partial \mathcal{K}^{\alpha\beta}(q)}{\partial q_b} - q^b \frac{\partial \mathcal{K}^{\alpha\beta}(q)}{\partial q_a}, \]  

(2.29)

\[ -\Xi^{\alpha\gamma}(q)(J^{ab})^\gamma_\beta - \Xi^{\gamma\beta}(q)(J^{ab})^\gamma_\alpha = q^a \frac{\partial \Xi^{\alpha\beta}(q)}{\partial q_b} - q^b \frac{\partial \Xi^{\alpha\beta}(q)}{\partial q_a}. \]  

(2.30)

For computing the propagator carrying momentum \( p_i + k \) in Fig. 1 we shall take \( M \) to be the mass \( M_i \) of the \( i \)-th incoming particle and call the corresponding \( \Xi(q) \) as \( \Xi^i(q) \). In that case the polarization vector \( \epsilon_{i,\alpha} \) and the momenta \( p_i \) of the \( i \)-th external state will satisfy the on-shell condition

\[ \epsilon_{i,\alpha} \mathcal{K}^{\alpha\beta}(p_i) = 0, \quad p_i^2 + M_i^2 = 0. \]  

(2.31)

We shall now determine the coupling of the soft graviton to a pair of finite energy particles by covariantizing the action (2.25) with respect to the background soft graviton field. We shall assume that while covariantizing, we replace ordinary derivatives by covariant derivatives and symmetrize under arbitrary permutations of these derivatives. This may differ from the actual action by terms proportional to the Riemann tensor of the soft graviton. The effect of such additional couplings will be taken care of separately.

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We now list the effect of coupling the action $S^{(2.1)}$ to the soft graviton field carrying momentum $k$ and polarization $\varepsilon$, up to sub-subleading order in the soft momentum $k$:

1. Since the soft graviton carries momentum $k$, the $\delta^{(D)}(q_1+q_2)$ is replaced by $\delta^{(D)}(q_1+q_2+k)$.

2. For every derivative $\partial_\mu$ acting on $\Phi_\beta$ or its derivatives, we get a term $-\varepsilon_\mu^\nu\partial_\nu$ from having to convert the space-time index associated with $\partial_\mu$ to tangent space index by the replacement $\partial_\mu \to E_\mu^\nu \partial_\nu$. This is done at the very last step after all the other steps mentioned below have been performed. Once this replacement is made, the indices can be contracted using the flat metric $\eta$.

3. For every derivative $\partial_\mu$ acting on $\Phi_\beta$ we get a term $\frac{1}{2}\omega_\mu^{ab}(J_{ab})_\beta \Phi_\gamma$ from having to replace ordinary derivatives by covariant derivatives.

4. For every pair of derivatives $\partial_\mu$ and $\partial_\nu$ acting on $\Phi_\beta$, we get an additional term $-\Gamma^\rho_\mu_\nu \partial_\rho \Phi_\beta$ due to the fact that $D_\mu$ acting on $D_\nu$ generates a term $-\Gamma^\rho_\mu_\nu D_\rho$. This factor is independent of the relative order of $D_\mu$ and $D_\nu$.

5. For every pair of derivatives $\partial_\mu$ and $\partial_\nu$ acting on $\Phi_\beta$ we get an additional term $\frac{1}{2}\partial_\mu \omega_\mu^{ab}(J_{ab})_\beta \Phi_\gamma$ due to the left-most derivative acting on the spin connection.

6. For every triplet of derivatives $\partial_\mu$, $\partial_\nu$ and $\partial_\rho$ acting on $\Phi_\beta$, we get an additional term $-\partial_\mu \Gamma^\gamma_\mu_\rho \partial_\sigma \Phi_\beta$ due to the left-most derivative acting on the Christoffel symbol.

Of these the first four effects also appeared in our analysis of Fig. 2 in section 2.1. The last two effects generate two powers of soft momentum and do not affect the evaluation of Fig. 2 which begins to contribute only at the subleading order. Using (2.6), (2.7) and the fact that in momentum space $\partial_\mu \Phi_\alpha$ is represented by $iq_\mu \Phi_\alpha(q)$, we get the following action describing the coupling of the soft graviton to the $\Phi$ field

$$
S^{(3)} = \frac{1}{2} \int \frac{d^Dq_1}{(2\pi)^D} \frac{d^Dq_2}{(2\pi)^D} \delta^{(D)}(q_1+q_2+k) \times \Phi_\alpha(q_1) \left[ -\varepsilon_\mu^\nu q_2^\nu \frac{\partial}{\partial q_2^\mu} \kappa^{\alpha\beta}(q_2) + \frac{1}{2}(k_b \varepsilon_a \mu - k_a \varepsilon_b \mu) \frac{\partial}{\partial q_2^\mu} \kappa^{\alpha\gamma}(q_2) (J_{ab})_\beta^\gamma \\
- \frac{1}{2} \frac{\partial^2}{\partial q_2^\mu \partial q_2^\nu} q_2^\rho (k_\mu \varepsilon_\nu^\rho + k_\nu \varepsilon_\mu^\rho - k_\rho \varepsilon_\mu \nu) + \frac{1}{4} \frac{\partial^2}{\partial q_2^\mu \partial q_2^\nu} q_2^\rho (k_\mu \varepsilon_\nu^a - k_\rho \varepsilon_\nu^b) (J_{ab})_\gamma \\
- \frac{1}{6} \frac{\partial^3}{\partial q_2^\mu \partial q_2^\nu \partial q_2^\rho} q_2^\rho (k_\mu \varepsilon_\nu^\sigma + k_\nu \varepsilon_\mu^\sigma - k_\rho \varepsilon_\mu \nu) \right] \Phi_\beta(q_2) .
$$

(2.32)
To (2.32) we could add an additional coupling of the fields \( \Phi_\alpha \) to the Riemann tensor constructed from the soft graviton:

\[
\tilde{S}^{(3)} \equiv \frac{1}{2} \int \frac{d^Dq_1}{(2\pi)^D} \frac{d^Dq_2}{(2\pi)^D} (2\pi)^D \delta^{(D)}(q_1 + q_2 + k) \mathcal{R}^{\mu_\rho\sigma\nu}_{\mu\rho\sigma\nu} \Phi_\alpha(q_1) \mathcal{B}^{\alpha\beta;\mu\rho\sigma\nu}(q_2) \Phi_\beta(q_2) .
\]  

(2.33)

where

\[
\mathcal{R}^{\mu_\rho\sigma\nu}_{\mu\rho\sigma\nu} \equiv \varepsilon^{\mu_\nu k_\rho k_\sigma} - \varepsilon^{\mu\sigma} k^{\nu} k^{\rho} - \varepsilon^{\nu\rho} k^{\sigma} k^{\mu} + \varepsilon^{\rho\sigma} k^{\mu} k^{\nu}
\]

(2.34)

is the linearized Riemann tensor of the soft graviton written in the momentum space. For the Riemann tensor we are using the convention

\[
\mathcal{R}^{\mu_\rho\sigma\nu}_{\mu\rho\sigma\nu} = \partial_\rho \Gamma^{\mu}_{\sigma\rho} - \partial_\sigma \Gamma^{\mu}_{\mu\rho} + \Gamma^{\mu}_{\rho\sigma} \Gamma^{\rho}_{\mu\nu} .
\]

(2.35)

includes an extra minus sign from having to convert \( \partial_\rho \) to \( i k_\rho \) when we go from position space description to momentum space description. (2.33) represents a non-minimal coupling of the soft graviton to the fields \( \Phi_\alpha \) that is not obtained from covariantization of the kinetic term. We can choose \( \mathcal{B}^{\alpha\beta;\mu\rho\sigma\nu}(q_2) = \mathcal{B}^{\beta\alpha;\mu\rho\sigma\nu}(-q_2 - k) \).

We now turn to the evaluation of Fig. 1. The propagator gives a factor of \( \Xi_{\alpha\beta}(-p_i) \{(p_i + k)^2 + M_i^2\}^{-1} = (2p_i \cdot k)^{-1} \Xi_{\alpha\beta}(-p_i) \) where now

\[
\Xi^i(p) \equiv \Xi(p)|_{M=M_i} = i (p^2 + M_i^2) K^{-1}(p) .
\]

(2.36)

Therefore the contribution to Fig. 1 is given by

\[
(2p_i \cdot k)^{-1} \epsilon_{i,\alpha} \left\{ \Gamma^{(3)\alpha\beta}(\epsilon, k; p_i, -p_i - k) + \tilde{\Gamma}^{(3)\alpha\beta}(\epsilon, k; p_i, -p_i - k) \right\} \Xi^i_{\alpha\beta}(-p_i - k) \Gamma^\delta_{(i)}(p_i + k) ,
\]

(2.37)

where \( \Gamma^{(3)} \) and \( \tilde{\Gamma}^{(3)} \) are the contributions of \( S^{(3)} \) and \( \tilde{S}^{(3)} \) to the 1PI three point vertices of two finite energy external states carrying labels \( \alpha \) and \( \beta \) and momenta \( p_i \) and \( -p_i - k \) and one external soft graviton carrying momentum \( k \) and polarization \( \epsilon \). We have from (2.32), (2.33)
\[ -\frac{1}{2} \frac{\partial^2 K^{\alpha\beta}(-p - k)}{\partial p_\mu \partial p_\nu}(-p_\rho - k_\rho) \left( k_\mu \varepsilon_\nu^\rho + k_\nu \varepsilon_\mu^\rho - k^\rho \varepsilon_{\mu\nu} \right) \]

\[ -\frac{1}{2} \frac{\partial^2 K^{\alpha\beta}(p)}{\partial p_\mu \partial p_\nu} p_\rho \left( k_\mu \varepsilon_\nu^\rho + k_\nu \varepsilon_\mu^\rho - k^\rho \varepsilon_{\mu\nu} \right) \]

\[ -\frac{1}{6} \frac{\partial^3 K^{\alpha\beta}(-p - k)}{\partial p_\mu \partial p_\nu \partial p_\rho} (p_\sigma + k_\sigma) k_\rho \left( k_\mu \varepsilon_\nu^\sigma + k_\nu \varepsilon_\mu^\sigma - k^\sigma \varepsilon_{\mu\nu} \right) \]

\[ -\frac{1}{6} \frac{\partial^3 K^{\alpha\beta}(p)}{\partial p_\mu \partial p_\nu \partial p_\rho} p_\sigma k_\rho \left( k_\mu \varepsilon_\nu^\sigma + k_\nu \varepsilon_\mu^\sigma - k^\sigma \varepsilon_{\mu\nu} \right) \]

(2.38)

and

\[ \bar{\Gamma}^{(3)}_{\alpha\beta}(\varepsilon, k; p_i, -p - k) = i \mathcal{R}_{\mu\rho\sigma\nu} B^{\alpha\beta;\mu\rho\sigma\nu}(-p) , \]

(2.39)

to order \(k^2\) in Taylor series expansion in powers of the soft momentum \(k\).

The contribution of (2.39) to (2.37) is easy to evaluate. Since we already have two factors of soft momentum in the vertex, we can set \(k = 0\) in the argument of \(\Xi^i\) and \(\Gamma^i\). Therefore this contribution is given by

\[ I_1 = \frac{i}{2} \sum_{i=1}^{N} (p_i \cdot k)^{-1} \mathcal{R}_{\mu\rho\sigma\nu} \epsilon_{i,\alpha} B^{\alpha\beta;\mu\rho\sigma\nu}(-p_i) \Xi_i^i(-p_i) \Gamma^i(p_i) . \]

(2.40)

In order to evaluate the contribution from the \(\Gamma^{(3)}\) part of the vertex to (2.37) we follow the following strategy:

1. First we replace all factor of \(K^{rs}(p_i)\) by \(K^{sr}(-p_i)\) using (2.20).

2. In each product of \(K, \Xi^i\) and \(J^{ab}\) factors, we first use (2.29), (2.30) to move the \(J^{ab}\) factors to the extreme right so that its index is contracted with that of \(\Gamma^i\). For this we have to rewrite (2.29), (2.30) as

\[ K^{\gamma\beta}(q)(J^{ab})^\alpha_{\gamma} = -K^{\alpha\gamma}(q)(J^{ab})^\beta_{\gamma} + q^a \frac{\partial K^{\alpha\beta}(q)}{\partial q_b} - q^b \frac{\partial K^{\alpha\beta}(q)}{\partial q_a} , \]

\[ \Xi^i_{\gamma\beta}(q)(J^{ab})^\alpha_{\gamma} = -\Xi^i_{\alpha\gamma}(q)(J^{ab})^\beta_{\gamma} - q^a \frac{\partial \Xi^i_{\alpha\beta}(q)}{\partial q_b} + q^b \frac{\partial \Xi^i_{\alpha\beta}(q)}{\partial q_a} . \]

(2.41)

3. We now expand \(K^{\alpha\beta}(-p_i - k), \Xi^i_{\gamma\beta}(-p_i - k)\) and \(\Gamma^i(p_i + k)\) in Taylor series expansion in powers of soft momenta to appropriate order relevant for computing the sub-subleading contribution to the amplitude.
4. The expression that results after this has products of (derivatives of) $\mathcal{K}$, $\Xi^i$ and $\Gamma_{(i)}$. We now use the derivatives of the relation $\mathcal{K}(q)\Xi^i(q) = i(q^2 + M_i^2)$ to transfer the derivatives from $\mathcal{K}$ to $\Xi^i$ to the maximal possible extent. This requires for example using the relations

\[
\frac{\partial \mathcal{K}(-p)}{\partial p_{\mu}} \Xi^i(-p) = -\mathcal{K}(-p) \frac{\partial \Xi^i(-p)}{\partial p_{\mu}} + 2i p^\mu,
\]

\[
\frac{\partial^2 \mathcal{K}(-p)}{\partial p_{\mu} \partial p_{\nu}} \Xi^i(-p) = -\frac{\partial \mathcal{K}(-p)}{\partial p_{\mu}} \frac{\partial \Xi^i(-p)}{\partial p_{\nu}} - \frac{\partial \mathcal{K}(-p)}{\partial p_{\nu}} \frac{\partial \Xi^i(-p)}{\partial p_{\mu}} - \mathcal{K}(-p) \frac{\partial^2 \Xi^i(-p)}{\partial p_{\mu} \partial p_{\nu}} + 2i \eta^{\mu\nu},
\]

\[
\frac{\partial^3 \mathcal{K}(-p)}{\partial p_{\mu} \partial p_{\nu} \partial p_{\rho}} \Xi^i(-p) = -\frac{\partial^2 \mathcal{K}(-p)}{\partial p_{\mu} \partial p_{\nu}} \frac{\partial \Xi^i(-p)}{\partial p_{\rho}} - \frac{\partial \mathcal{K}(-p)}{\partial p_{\rho}} \frac{\partial \Xi^i(-p)}{\partial p_{\mu} \partial p_{\nu}} - \frac{\partial \mathcal{K}(-p)}{\partial p_{\mu}} \frac{\partial^2 \Xi^i(-p)}{\partial p_{\rho} \partial p_{\nu}} - \frac{\partial \mathcal{K}(-p)}{\partial p_{\nu}} \frac{\partial^2 \Xi^i(-p)}{\partial p_{\rho} \partial p_{\mu}} - \mathcal{K}(-p) \frac{\partial^3 \Xi^i(-p)}{\partial p_{\mu} \partial p_{\nu} \partial p_{\rho}}.
\]

(2.42)

5. In the final step we use (2.31) to set to zero terms involving $\mathcal{K}$ without derivatives since they are always contracted with $\epsilon_{i,\alpha}$.

The final result for the contribution of (2.38) to (2.37) is given by

\[
I_2 = \frac{1}{2} \sum_{i=1}^{N} (p_i \cdot k)^{-1} \epsilon_{i,\alpha} \left( k_{a} \varepsilon_{b\mu} - k_{b} \varepsilon_{a\mu} \right) \left[ p_{i}^{\mu} (J^{ab})_{\delta}^{\alpha} \Gamma_{(i)}^{\delta}(p_i) + p_{i}^{\mu} k_{\rho} (J^{ab})_{\delta}^{\alpha} \frac{\partial \Gamma_{(i)}^{\delta}(p_i)}{\partial p_{i\rho}} \right] \\
- \frac{i}{4} \sum_{i=1}^{N} (2p_i \cdot k)^{-1} \left( k_{\rho} k_{a} \varepsilon_{b\mu} - k_{b} k_{\rho} \varepsilon_{a\mu} - k_{\mu} k_{a} \varepsilon_{b\rho} + k_{\mu} k_{b} \varepsilon_{a\rho} \right) \epsilon_{i,\alpha}
\]

\[
\frac{\partial \mathcal{K}^{\alpha\gamma}(-p_i)}{\partial p_{i\mu}} \frac{\partial \Xi^i(-p_i)}{\partial p_{\rho}} \left( J^{ab} \right)_{\delta}^{\beta} \Gamma_{(i)}^{\delta}(p_i)
\]

\[
+ \sum_{i=1}^{N} (p_i \cdot k)^{-1} \epsilon_{i,\alpha} \varepsilon_{\mu\nu} p_{i\mu} p_{i\nu} \left\{ \Gamma_{(i)}^{\alpha}(p_i) + k_{\rho} \frac{\partial \Gamma_{(i)}^{\alpha}(p_i)}{\partial p_{i\rho}} + \frac{1}{2} k_{\rho} k_{\sigma} \frac{\partial^2 \Gamma_{(i)}^{\alpha}(p_i)}{\partial p_{i\rho} \partial p_{i\sigma}} \right\} \\
+ \frac{i}{4} \left\{ \varepsilon_{\mu\nu} k_{\rho} k_{a} - \varepsilon_{\mu\nu} k_{b} k_{\rho} - \varepsilon_{\mu\sigma} k_{\rho} k_{\nu} + \varepsilon_{\rho\sigma} k_{a} k_{b} \right\}
\]

\[
\sum_{i=1}^{N} (p_i \cdot k)^{-1} \epsilon_{i,\alpha} \left\{ \frac{2}{3} p_{i\nu} \frac{\partial K_{\alpha\beta}(-p_i)}{\partial p_{i\mu}} \frac{\partial^2 \Xi_{\beta\delta}(-p_i)}{\partial p_{i\rho} p_{i\sigma}} - \frac{1}{3} \frac{\partial^2 K_{\alpha\beta}(-p_i)}{\partial p_{i\nu} p_{i\mu}} p_{i\mu} \frac{\partial \Xi_{\beta\delta}(-p_i)}{\partial p_{i\sigma}} \right\} \Gamma_{(i)}^{\delta}(p_i).
\]

(2.43)
2.3 Final result

Using (2.24), (2.40) and (2.43) we now get the total amplitude to sub-subleading order

\[ I = I_0 + I_1 + I_2 \]

\[ I = \sum_{i=1}^{N} (p_i \cdot k)^{-1} \epsilon_{i,\alpha} \varepsilon_{\mu \nu} p_i^\mu \varepsilon_{\gamma \beta} \Gamma^{\alpha \beta}_{(i)}(p_i) \]

\[ + \sum_{i=1}^{N} (p_i \cdot k)^{-1} \epsilon_{i,\alpha} \varepsilon_{\mu \nu} p_i^\mu k_a \left\{ p_i^\nu \frac{\partial}{\partial p_{ia}} - p_i^\nu \frac{\partial}{\partial p_{ib}} \right\} \delta_{\beta}^\alpha + (J_{ab})_\beta^\alpha \right\} \Gamma^{\beta}_{(i)}(p_i) \]

\[ - \frac{1}{2} \sum_{i=1}^{N} (p_i \cdot k)^{-1} \epsilon_{i,\alpha} \varepsilon_{ac} k_b k_d \left\{ p_i^\mu \frac{\partial}{\partial p_{ic}} - p_i^\mu \frac{\partial}{\partial p_{id}} \right\} \delta_{\beta}^\alpha + (J_{ad})_\beta^\alpha \right\} \Gamma^{\gamma}_{(i)}(p_i) \]

\[ + \frac{1}{2} \sum_{i=1}^{N} (p_i \cdot k)^{-1} \epsilon_{i,\alpha} \Delta_{\beta}^\alpha (-p_i, k) \Gamma^{\beta}_{(i)}(p_i) , \tag{2.44} \]

where

\[ \Delta_{\delta}^\alpha (-p_i, k) = \left\{ \varepsilon_{\mu \nu} k_\rho k_\sigma - \varepsilon_{\rho \sigma} k_\mu k_\nu - \varepsilon_{\alpha \rho} k_\mu k_\nu + \varepsilon_{\rho \sigma} k_\mu k_\nu \right\} \]

\[ \times \left\{ \frac{1}{3} i p_i^\mu \frac{\partial K^{\alpha \beta}(-p_i)}{\partial p_{i\mu}} \frac{\partial^2 \Xi^{i\beta}_{\gamma\delta}(-p_i)}{\partial p_{i\nu} \partial p_{i\sigma}} - \frac{1}{6} i \frac{\partial^2 K^{\alpha \beta}(-p_i)}{\partial p_{i\mu} \partial p_{i\nu}} p_i^\rho \frac{\partial \Xi^{i\beta}_{\gamma\delta}(-p_i)}{\partial p_{i\rho}} \right. \]

\[ + \frac{i}{4} \frac{\partial K^{\alpha \gamma}(-p_i)}{\partial p_{i\mu}} \frac{\partial \Xi^{i\beta}_{\gamma\delta}(-p_i)}{\partial p_{i\rho}} (J^{\rho\sigma})_\beta^\gamma \frac{1}{4} (J^{\mu\rho})_\alpha^\beta (J^{\nu\sigma})_\delta^\beta \]

\[ + \frac{i}{4} B^{\alpha \beta ; \mu \nu \rho \sigma} (-p_i) \Xi^{i\beta}_{\gamma\delta}(-p_i) \right\} . \tag{2.45} \]

Eqs. (2.44), (2.45) are our main results.

We end by making a few comments:

1. If the indices \( \alpha \) and \( \delta \) in (2.45) label scalar fields, then the tensor inside the curly bracket must be constructed from the vector \( p_i \) and the invariant tensor \( \eta \). Contraction of \( \eta \) with \( R_{\mu \nu \rho \sigma} \) vanishes as a result of (2.1). Therefore the only possibility is the tensor \( p_i^\mu p_i^\nu p_i^\rho p_i^\sigma \). The contraction of this with \( R_{\mu \nu \rho \sigma} \) vanishes due to antisymmetry of \( R \) in the first two indices and last two indices. Therefore (2.45) shows that for scalars there are no corrections to the sub-subleading soft graviton theorem. This is in agreement with known results.
2. (2.45) represents correction to the universal part of the sub-subleading factor. The first three terms on the right hand side show that unlike the leading and subleading soft factors, sub-subleading soft factors are sensitive to the (infrared-finite) loop corrections to the propagator. Even at tree level the contribution from these terms may be non-zero for higher spin fields – we shall discuss the case of Rarita-Schwinger fields in section 4.2. The fourth term represents an additional contribution due to spin-angular momentum of the finite energy particles and may give non-vanishing contribution even at tree-level. We shall discuss its contribution for a graviton line in section 4.1. The fifth and the final term shows that the sub-subleading factor depends on corrections to the three point function involving a soft graviton and a pair of finite energy particles, as given in eq. (2.33).

3. The line of argument followed here cannot be used to extend the analysis to higher order in the soft momentum. This is due to the fact that the contribution from Fig. 2 can have terms in which the linearized Riemann tensor of the soft graviton given in (2.34) is contracted with an arbitrary function of the finite external momenta $p_i$ – bearing no relation to the amplitude without the soft graviton. As a result terms of this type do not have factorized form and prevent us from extending the soft graviton theorem.

3 Consistency check

The right hand side of (2.44) apparently depends on off-shell data through its dependence of $\Gamma^\delta_{(i)}$. This arises from the following sources. A scattering amplitude of $n$ finite energy particles is given by the amplitude $\Gamma^{\alpha_1 \cdots \alpha_n}(p_1, \cdots p_n)$ after setting the external momenta $p_i$ on-shell, i.e. satisfy $p_i^2 + M_i^2 = 0$, and then contracting them with physical external polarization $\epsilon_{\iota, \alpha}$ satisfying (2.31). Therefore if we add to $\Gamma^{\alpha_1 \cdots \alpha_n}(p_1, \cdots p_n)$ (or equivalently to $\Gamma^\alpha_{(i)}$) a term proportional to $p_i^2 + M_i^2$ then the scattering amplitude of the finite energy particles do not change. On the other hand individual terms on the right hand side of (2.44) do get modified due to the derivative operation with respect to $p_i^\mu$. Acting on a term proportional to $p_i^2 + M_i^2$ this gives a terms proportional to $p_i^\mu$, which do not vanish on-shell. Similarly if we add to $\Gamma^\alpha_{(i)}(p_i)$ a term proportional to $\kappa^{\alpha \beta}(-p_i)\mathcal{M}_{(i)\beta}$ for any $\mathcal{M}_{(i)\beta}$, then the amplitudes involving finite energy external states do not get affected due to the on-shell condition (2.31). However the individual terms on the right hand side of (2.44) change under this transformation. Our goal will be to show that when we add all the contributions, the right hand side of (2.44) actually remains invariant under these deformations of $\Gamma^\alpha_{(i)}$.
First let us consider the effect of adding a term proportional to \(p_i^2 + M_i^2\) to \(\Gamma^\alpha_{\langle i\rangle}\). Using the fact that
\[
\left\{ \frac{p_i^a}{\partial p_{ia}} - \frac{p_i^b}{\partial p_{ib}} \right\} (p_i^2 + M_i^2) = 0 \quad (3.1)
\]
it is easy to check that the change of the right hand side of (2.44) vanishes after setting \(p_i^2 + M_i^2 = 0\).

Next let us consider the effect of shifting \(\Gamma^\alpha_{\langle i\rangle}\) by a term of the form \(K^\alpha_{\beta \gamma} (\xi_{\gamma i})\). It is easy to see that the first term on the right hand side of (2.44) does not change under this deformation as long as \(\epsilon_{i,\alpha}\) satisfies (2.31). For the terms in the second and the third lines on the right hand side of (2.44), we can use (2.29) to bring \(K\) to the left so that it is contracted with \(\epsilon_{i,\alpha}\). The result then vanishes by (2.31). Therefore we need to focus on the contribution from the last term on the right hand side of (2.44) given by
\[
\frac{1}{2} \sum_{i=1}^{N} (p_i \cdot k)^{-1} \epsilon_{i,\alpha} \Delta^\alpha_{\beta}(\xi_{\alpha i}) K^\delta_{\gamma \delta} M(i)_{\alpha \gamma}(p_i) . \quad (3.2)
\]
\(\Delta^\alpha_{\beta}(\xi_{\alpha i})\) has been given in (2.45). The contribution from the last term in (2.45) is proportional to \(\Xi^i_{\alpha \beta}(\xi_{\alpha i}) K^\delta_{\gamma \delta} M(i)_{\alpha \gamma}(p_i)\) and vanishes using the on-shell condition. The contribution from the rest of the terms may be manipulated as follows.

1. First we move all the \(J\)'s to the right using (2.41) so that the index of \(J\) is contracted with that of \(M\).

2. The resulting expression has products of (derivatives of) \(\Xi^i\) and \(K\) contracted with each other. We now transfer the derivatives from the left-most \(K\) to the right to the extent possible using (2.42) and its analog with \(K\) and \(\Xi^i\) exchanged:
\[
\frac{\partial \Xi^i(-p)}{\partial p_{\mu}} K(-p) = -\Xi^i(-p) \frac{\partial K(-p)}{\partial p_{\mu}} + 2 i p^\mu , \quad (3.3)
\]
\[
\frac{\partial^2 \Xi^i(-p)}{\partial p_{\mu} \partial p_{\nu}} K(-p) = - \frac{\partial \Xi^i(-p)}{\partial p_{\mu}} \frac{\partial K(-p)}{\partial p_{\nu}} - \frac{\partial \Xi^i(-p)}{\partial p_{\nu}} \frac{\partial K(-p)}{\partial p_{\mu}} + \Xi^i(-p) \frac{\partial^2 K(-p)}{\partial p_{\mu} \partial p_{\nu}} + 2 i \eta^{\mu \nu} .
\]

3. In the final step we set the terms in which the left-most \(K\) has no derivatives to zero using (2.31).

The net result of this analysis yields
\[
- \frac{i}{12} \sum_{i=1}^{N} (p_i \cdot k)^{-1} p_{i\rho} \epsilon_{i,\alpha} R^\mu_{\nu \lambda} \left[ \frac{\partial K}{\partial p_{\mu}} \frac{\partial \Xi^i}{\partial p_{\nu}} + \frac{\partial K}{\partial p_{\nu}} \frac{\partial \Xi^i}{\partial p_{\mu}} + \frac{\partial K}{\partial p_{\lambda}} \frac{\partial \Xi^i}{\partial p_{\rho}} + \frac{\partial K}{\partial p_{\rho}} \frac{\partial \Xi^i}{\partial p_{\lambda}} \right] \alpha \gamma M(i)_{\alpha \gamma}(p_i) = 0 \quad (3.4)
\]
where in the last line we have used the algebraic Bianchi identity of the Riemann tensor.

This shows that (2.44) is insensitive to the off-shell information in $\Gamma_{(i)}$, leading to the form given in (1.1). We shall now show that $\Delta_\alpha^\delta$ appearing in (2.45) depends only on the on-shell three point function involving the external soft graviton. We shall do this using factorization property of the full amplitude – namely that if we adjust the direction of $k$ so that $p_i \cdot k \rightarrow 0$, the amplitude (2.44) must factorize into a product of the on-shell three point function involving external states with momenta $p_i$, $k$ and $-p_i - k$ and the on-shell $N$-point function involving external states carrying momenta $p_1, \cdots, p_{i-1}, p_i + k, p_{i+1}, \cdots, p_N$. It then follows from (2.44) that $\Delta_\alpha^\delta$ in the limit $p_i \cdot k \rightarrow 0$ is determined in terms of the on-shell three point amplitude.

Our goal will be to show that the knowledge of $\Delta_\alpha^\delta$ in the $p_i \cdot k \rightarrow 0$ limit is enough to determine $\Delta_\alpha^\delta$ for general direction of $k$.

To proceed, let us suppress the indices $\alpha, \delta$ from $\Delta_\alpha^\delta$, and express (2.45) as

$$\Delta = \left\{ \varepsilon_{\mu\nu} k_\rho k_\sigma - \varepsilon_{\rho\nu} k_\mu k_\sigma - \varepsilon_{\mu\sigma} k_\rho k_\nu + \varepsilon_{\rho\sigma} k_\mu k_\nu \right\} B^{\mu\nu\rho\sigma}. \quad (3.5)$$

It is understood that $B$ carries the indices $\alpha, \delta$. $B$ depends on $p_i$ but not on $\varepsilon$ or $k$ to this order in expansion in powers of $k$. Without loss of generality we can assume that $B^{\mu\nu\rho\sigma}$ has the symmetries of the Riemann tensor. In this case the question of whether $\Delta$ is determined from on-shell three point function reduces to whether it is possible to add some terms to $B^{\mu\nu\rho\sigma}$ so that the contribution from this term to (3.5) vanishes for $p_i \cdot k = 0$ but not in general. In order to make use of the $p_i \cdot k = 0$ constraint, the additional terms in $B^{\mu\nu\rho\sigma}$ must be proportional to $p_i$. Let us make the most general ansatz for this ambiguity consistent with the symmetries of $B^{\mu\nu\rho\sigma}$:

$$p_i^\mu A^{\rho\sigma} - p_i^\rho A^{\mu\nu\rho\sigma} + p_i^\nu A^{\sigma\mu\rho} - p_i^\sigma A^{\nu\mu\rho}, \quad (3.6)$$

where $A^{\nu\rho\sigma}$ is antisymmetric under $\nu \leftrightarrow \sigma$. Substituting this into (3.5) we see that under this shift $\Delta$ changes by

$$4(\varepsilon_{\mu\nu} p_i^\rho k_\mu k_\sigma - \varepsilon_{\nu\sigma} p_i^\rho k_\mu k_\nu) A^{\rho\nu\sigma} \quad (3.7)$$

up to terms proportional to $p_i \cdot k$. Since this does not vanish identically for $p_i \cdot k = 0$, we see that different values of $A$ are still distinguishable near the pole at $p_i \cdot k = 0$. This can be rectified by taking $A^{\rho\nu\sigma}$ to be either proportional to $p_i^\rho B^{\nu\sigma}$ for any anti-symmetric tensor $B$, or proportional to $(\eta^{\rho\sigma} C^\nu - \eta^{\rho\nu} C^\sigma)$ for any vector $C^\nu$, or by taking it to be totally anti-symmetric in $\nu, \rho, \sigma$. It is easy to see that in the first case (3.6) vanishes identically, while in the last two cases (3.6) does not generate any change in (3.5). Therefore we conclude that there is no
ambiguity in determining $\Delta$ from its value near the pole at $p_i \cdot k = 0$, and therefore in terms of on-shell three point function.

4 Comparison with tree level results for massless fields

In this section we shall compare our final result, given in (2.44), (2.45) with some known results in the theory of massless fields at tree level.

4.1 Einstein-Maxwell theory

For Einstein-Maxwell theory without any higher derivative terms, the sub-subleading soft graviton theorem is known to include only the contribution from the first three lines on the right hand side of (2.44) [6]. Therefore $\Delta^{\alpha\beta}_i$ given in (2.45) must vanish for these theories. We shall now verify this explicitly.

First let us consider the case where the $i$-th external finite energy state is a photon. We shall choose the Feynman gauge. In this case the indices $\alpha, \delta$ can be taken to be covariant vector indices $m, n$, and $K^{mn}(q)$ is simply $-q^2 \eta^{mn}$. Therefore we have $\Xi^i_{mn}(q) = -i \eta_{mn}$ and the first three terms on the right hand side of (2.45) involving derivatives of $\Xi^i$ must vanish.

To compute the fourth term we recall that in this case the components of $J^{a\bar{b}}$ are given by (2.9). This gives

\[(J^{\mu\nu})^m_n (J^{\nu\sigma})_p^p = \eta^{\mu\sigma} \eta^{mn} \delta^\nu_n - \eta^{\sigma\nu} \eta^{pm} \delta^\nu_n - \eta^{\mu\nu} \eta^{pm} \delta^\sigma_n + \eta^{\rho\nu} \eta^{pm} \delta^\sigma_n. \tag{4.1}\]

This has to be contracted with $R_{\mu\nu\rho\sigma}$ given in (2.34). Using (2.1) one can easily verify that all the terms vanish. This shows that the contribution to (2.44) from the fourth term on the right hand side of (2.45) also vanishes.

It remains to analyze the contribution from the last term in (2.45). To calculate $B$ in this case we need to start with the Einstein-Maxwell action in Feynman gauge and compare with (2.32). Now the part of the Einstein-Maxwell action involving the gauge field, together with the gauge fixing term, is given by

\[B = \frac{1}{2} \int d^Dx \sqrt{-\det g} \left[ \frac{1}{4} (D_\mu A_\nu - D_\nu A_\mu) (D^\mu A_\nu - D^\nu A_\mu) - \frac{1}{2} D_\mu A^\mu D_\nu A^\nu \right] = \frac{1}{2} \int d^Dx \sqrt{-\det g \eta^{mn}} A_m D^n D_\rho A_n, \tag{4.2}\]
where we have used the fact that $R^\mu_{\nu\rho\sigma}$ vanishes as a consequence of (2.1). The right hand side of (4.2) is the covariantization of the free Maxwell action in Feynman gauge for which $K_{mn} = -q^2 \eta_{mn}$ and therefore the terms linear in the soft graviton field computed from (4.2) coincides with (2.32). Therefore in this case $B^{\alpha \beta; \mu \rho \sigma}$ vanishes. This in turn shows that the entire contribution to (2.44) from the $\Delta^\alpha_\beta$ term vanishes.

Next we turn to the case where the $i$-th external state is a finite energy graviton. We shall use de Donder gauge. In this case each of the indices $\alpha$, $\delta$ can be taken to be a pair of covariant vector indices $(mn)$, and we have $K_{mn,pq}^{\delta} = -q^2 \eta_{mp} \eta_{nq}^{\delta}$. In this gauge we have $\Xi_{mn,pq}^i (q) = -i \eta_{mp} \eta_{nq}$ and again the first three terms on the right hand side of (2.45) vanishes. On the other hand we have

$$J^{\mu \rho}_{mn,pq} = \delta^\mu_m \eta^\rho_p \delta^n_q - \delta^\rho_m \eta^\mu_p \delta^n_q + \delta^\mu_n \eta^\rho_q \delta^\rho_p - \delta^\rho_n \eta^\mu_q \delta^\rho_p .$$

This gives

$$\epsilon_{i,pq} R_{\mu \rho \sigma} (J^{\mu \rho}_{mn})_{pq} (J^{\nu \sigma}_{rs})_{mn}^{\delta} = 8 \epsilon_{i,pq} R_{r \ s \ q}^{\delta} .$$

where we have again used the fact that $R_{\mu \nu \rho \sigma} = 0$. Therefore the contribution to (2.44) from the fourth term in (2.45) is given by

$$- \sum_{i=1}^{N} (p_i \cdot k)^{-1} \epsilon_{i,pq} R_{r \ s \ q}^{\delta} \Gamma_{r \ s \ i} (p_i) .$$

It remains to calculate the contribution from the last term in (2.45). For this we need to determine $B$. This can be calculated in two different ways. The first approach will be to begin with Einstein action in de Donder gauge and then expand it in powers of the fluctuations $h_{mn}$ to quadratic order around a soft graviton background. This is then brought to the form $(1/2) \int \sqrt{-\det g} h_{mn} D^p D_q h_{mn} + \cdots$ where the $\cdots$ term, proportional to the Riemann tensor of the soft graviton, determines the action $\bar{S}^{(3)}$ in (2.33) and therefore $B^{\alpha \beta; \mu \rho \sigma}$ (see e.g. eq.(7.5.23) of [59]). The other possibility is to expand the Einstein action in the de Donder gauge in powers of the fluctuation $H_{mn}$ around the flat background to cubic order [60], split $H_{mn}$ as the sum of a soft and a finite energy parts, and then determine the coupling between a single soft graviton and a pair of finite energy gravitons. Comparing this with the action (2.32) one can determine the missing part $\bar{S}^{(3)}$. Both approaches yield

$$\bar{S}^{(3)} = \int d^D x \sqrt{-\det g} R^{mpnq} h_{mn} h_{pq} .$$

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1We omit the symmetrization under $m \leftrightarrow n$ and $p \leftrightarrow q$, and removal of the trace part, since they are taken care of by the symmetry and tracelessness of $h_{mn}$. 22
Comparing this with (2.33) we get
\[ \mathcal{R}_{\mu\nu\rho\sigma} B^{mn,pq;\mu\nu\rho\sigma} = 2 \mathcal{R}^{mpnq} . \]  
(4.7)

Using the fact that \(\Xi_{pq,rs}^i = -i\eta_{pr}\eta_{qs}\), the contribution from the last term in (2.45) to (2.44) is seen to be
\[ \sum_{i=1}^N (p_i \cdot k)^{-1} \epsilon_{i,pq} \mathcal{R}_{rs}^p q \Gamma_{(i)}^r (p_i) . \]  
(4.8)

This cancels (4.5). Therefore we see that even for external finite energy gravitons the sub-subleading soft graviton theorem in the Einstein-Maxwell theory is given by the first four lines on the right hand side of (2.44).

4.2 Fermions with minimal coupling to gravity

We shall now generalize the analysis of section 4.1 to the case of fermion fields minimally coupled to gravity. We shall work with real fermions by taking the real and imaginary parts of a complex field as independent fields – this effectively doubles the dimension of the \(\gamma\) matrices but makes them purely imaginary. First let us consider the case of Dirac field. Denoting the spinor indices by \(r,s\), we have
\[ \mathcal{K}_{rs}^r(-p) = \{\gamma^0 (p_\mu \gamma^\mu - M)\}_{rs} , \quad \Xi_{rs}^r(-p) = -i \{ (p_\mu \gamma^\mu + M) \gamma^0 \}_{rs} , \]  
(4.9)

where the \(\gamma^\mu\)'s satisfy
\[ \{ \gamma^\mu , \gamma^\nu \} = -2 \eta^{\mu\nu} , \quad (\gamma^\mu)^* = -\gamma^\mu , \quad (\gamma^0)^T = -\gamma^0 , \quad (\gamma^i)^T = \gamma^i \text{ for } 1 \leq i \leq (D - 1) . \]  
(4.10)

In this case the terms in (2.45) involving two derivatives of \(\mathcal{K}\) or \(\Xi\) vanish. Also for minimal coupling to gravity, \(B^{\alpha\beta;\mu\nu\rho\sigma}\) vanishes. This leaves us with the terms in the second line of (2.45). Now for spin 1/2 fermions \((J_{\mu\nu}^s)^r_s\), where \(r,s\) represent spinor indices, is given by
\[ (J_{S}^{\mu\nu})^r_s = -\frac{1}{2} (\gamma^{\mu\nu})_{rs} , \quad \gamma^{\mu\nu} = \frac{1}{2} (\gamma^{\mu} \gamma^{\nu} - \gamma^{\nu} \gamma^{\mu}) . \]  
(4.11)

The sign and normalization of \(J_{S}^{\mu\nu}\) defined in (4.11) can be shown to be consistent with that used in (2.9) by comparing the algebra of the \(J^{\mu\nu}\)'s in the spinor and the vector representation. On the other hand (1.9) gives
\[ \frac{\partial \mathcal{K}_{rs}^r(-p)}{\partial p_\mu} = (\gamma^0 \gamma^\mu)_{rs} , \quad \frac{\partial \Xi_{rs}^r(-p)}{\partial p_\rho} = -i (\gamma^\rho \gamma^0)_{rs} . \]  
(4.12)
and therefore
\[ \frac{\partial K^{rt}(-p)}{\partial p_{\mu}} \frac{\partial \Xi_{vu}(-p)}{\partial p_{\rho}} = -i (\gamma^0 \gamma_\mu \gamma^0)_{ru} = i (\gamma^\mu)_{ur}, \quad (4.13) \]
where in the last step we have used (4.10). Using this we see that sum of the terms in the second line of (2.45) is given by
\[ \Delta^r_s = R_{\mu \rho \nu \sigma} \left\{ \frac{1}{8} \gamma^{\nu \sigma} \gamma^{\rho \mu} - \frac{1}{16} \gamma^{\nu \rho} \gamma^{\mu \sigma} \right\}_{sr}. \quad (4.14) \]
In arriving at (4.14) we have used the fact that in order to interpret the product of $J$’s given in (2.45) as matrix multiplication as in (4.14) we have to transpose the matrices costing a sign. This does not change the sign of the second term but gives an additional minus sign in the first term. We now use the identity
\[ \gamma^{\nu \sigma} \gamma^{\rho \mu} = \gamma^{\nu \sigma \rho \mu} - (\eta^{\mu \nu} \gamma^{\rho \sigma} - \eta^{\mu \rho} \gamma^{\nu \sigma} + \eta^{\mu \sigma} \gamma^{\nu \rho}), \quad (4.15) \]
where $\gamma^{\nu \sigma \rho \mu}$ is the totally anti-symmetrized version of $\gamma^{\nu \sigma} \gamma^{\rho \mu}$. Using (2.1) and the algebraic Bianchi identity of $R_{\mu \rho \nu \sigma}$, we can see that individual terms in (4.14) vanish. Therefore $\Delta^r_s$ vanishes and the sub-subleading soft graviton amplitude is given by the terms in the first four lines on the right hand side of (2.44).

For the massless Rarita-Schwinger field $\psi_{a,r}$, with $a, b, c, d$ denoting vector indices and $r, s, t, u$ labelling spinor indices, we can fix harmonic gauge so that $K$ and $\Xi$ take simple form
\[ (K)^{a,r; b,s} = p_\mu (\gamma^0 \gamma^\mu)_{rs} \eta^{ab}, \quad (\Xi)^{a,r; b,s} = -i p_\mu (\gamma^\mu \gamma^0)_{rs} \eta^{ab}. \quad (4.16) \]
Also we have
\[ (J^{\mu \rho})^{a,r; b,s} = (J^{\mu \rho}_V)_a^b \delta_r^s + (J^{\mu \rho}_S)_a^b \delta_r^s, \quad (4.17) \]
where $J_V$ and $J_S$ denote the representation of $J$ in vector and spinor representations, given respectively in (2.9) and (4.11). Using (4.16) we again see that the contribution from the first line on the right hand side of (2.45) vanishes. For minimal coupling to gravity, the contribution from the third line also vanishes. In the second line of (2.45), noting that the first term is proportional to $(J^{\mu \rho}_S) J^{\nu \sigma}$ due to (4.13), we see that there are three kind of contributions from the first term, proportional to $J_S J_S$, $J_S J_V$ and $J_V J_V$. The second term in the second line of (2.45) is proportional to $(J_S + J_V)(J_S + J_V)$. The terms proportional to $J_S J_S$ have the same structure as (4.14) and vanish using (4.15). The terms proportional to $J_V J_V$ have the same
structure as (4.1) and vanish after contraction with \( R_{\mu\rho\nu\sigma} \). Therefore we are left with the term proportional to \( J_V J_S \) and \( J_S J_V \). Their contribution is given by

\[
\Delta_{a,r}^{b,s} = R_{\mu\rho\nu\sigma} \left\{ -\frac{1}{4} (\gamma^{\mu\rho})_{sr} (J^\sigma_V)_b^a + \frac{1}{8} (\gamma^{\mu\rho})_{sr} (J^\nu_V)_b^a + \frac{1}{8} (\gamma^{\nu\sigma})_{sr} (J^\mu_V)_b^a \right\} = 0 ,
\]

where in the last step we have used the symmetry of \( R_{\mu\rho\nu\sigma} \) under \( \mu,\rho \leftrightarrow \nu,\sigma \). Therefore even for massless Rarita Schwinger field minimally coupled to gravity, the contribution to the sub-subleading soft graviton theorem is given by the terms in the first four lines on the right hand side of (2.44).

### 4.3 Four dimensional quantum field theories with higher derivative corrections

Ref. [57] discussed soft graviton theorem for massless fields in four dimensions in the presence of higher derivative corrections. In this section we shall compare our results with the results of [57]. The relevant bosonic fields here include massless scalar \( \phi \), massless gauge field \( A_\mu \) and massless graviton. In the fermionic sector we can have massless spin 3/2 and spin 1/2 fields.

First let us consider the case of massless bosonic fields only. We shall choose harmonic gauge so that \( K_{\alpha\beta}(q) \) is given by \(-q^2 \delta_{\alpha\beta}\) and \( \Xi_{i\alpha\beta} = -i \delta_{\alpha\beta} \). In this case the contributions from the derivatives of \( \Xi_i \) in (2.45) vanish. Furthermore as seen in section 4.1 the contribution from the \( J^{\mu\rho} J^{\nu\sigma} \) term vanishes for scalar and the gauge fields, while for gravity this term cancels a term arising out of expansion of the Einstein-Hilbert action around a soft background. Therefore the contribution to (2.45) comes only from the interaction terms involving non-minimal coupling of gravity to other fields. It is easy to classify the possible terms that could contribute. They are

\[
\int d^4x \sqrt{-\det g} \phi R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}, \quad \int d^4x \sqrt{-\det g} R_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma},
\]

\[
\int d^4x \sqrt{-\det g} \phi R_{\mu\nu\rho\sigma} \widetilde{R}^{\mu\nu\rho\sigma}, \quad \int d^4x \sqrt{-\det g} R_{\mu\nu\rho\sigma} F^{\mu\nu} \widetilde{F}^{\rho\sigma},
\]

where \( R_{\mu\nu\rho\sigma} \) is the Riemann tensor, \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the gauge field strength and \( \widetilde{R}, \widetilde{F} \) denote Hodge duals:

\[
\widetilde{R}_{\mu\nu\rho\sigma} = \left( \sqrt{-\det g} \right)^{-1} \epsilon_{\mu\nu\mu'\nu'} R^{\mu\nu\rho\sigma}, \quad \widetilde{F}_{\mu\nu} = \left( \sqrt{-\det g} \right)^{-1} \epsilon_{\mu\nu\mu'\nu'} F^{\mu'\nu'}. \]

\[2\]We shall not consider theories with superreormalizable couplings e.g. a three point coupling without derivative between the massless scalars.
One could also consider a term with three Riemann tensors appropriately contracted, but when we take one of the external states to be soft and another on-shell, the vertex contains more than two powers of soft momentum and therefore does not contribute to the amplitude at the sub-subleading order. In higher dimensions the term with two Riemann tensors with their indices contracted gives rise to a three graviton vertex but in four dimensions this is equivalent to the sum of Gauss-Bonnet term which is a total derivative and terms involving Ricci tensor that vanish on-shell. Therefore this does not contribute in the soft limit.

The three point vertices listed in (4.19) affect the sub-subleading contribution by modifying the three point vertex in Fig. 1. Two of the external states of this vertex, including the soft graviton, are on-shell while the third one, representing the internal line, is nearly on-shell. Since we are to evaluate the leading contribution from this vertex in the soft limit, we can regard the internal line also as on-shell by decomposing the numerator factor $\Xi^i$ from the internal propagators into a sum over physical and unphysical polarizations and using the fact that in the final amplitude the contribution from the unphysical polarizations will cancel. Therefore the computation reduces to the problem of computing the contribution of (4.19) to an on-shell three point amplitude.

A further simplification in four dimensions comes from the fact that in four dimensions by appropriate choice of gauge the polarization tensor of a massless graviton can be taken to be the square of that of a massless photon carrying the same momentum. By making this choice we write

$$\varepsilon_{\mu\nu} = \varepsilon_\mu \varepsilon_\nu, \quad e_{\mu\nu} = e_\mu e_\nu,$$  \hspace{1cm} (4.21)

for the polarizations of soft and hard gravitons respectively. Then in the momentum space, to linearized order the Riemann tensors associated with the soft and the finite energy graviton fields take the form

$$R^{(s)}_{\mu\nu\rho\sigma} = \{\varepsilon_{\mu\nu} k_\rho k_\sigma - \varepsilon_{\mu\sigma} k_\nu k_\rho - \varepsilon_{\nu\rho} k_\sigma k_\mu + \varepsilon_{\rho\sigma} k_\nu k_\mu\} = (\varepsilon_{\mu\nu} k_\rho - \varepsilon_{\rho\mu} k_\nu)(\varepsilon_{\nu\sigma} k_\rho - \varepsilon_{\rho\nu} k_\sigma)$$ \hspace{1cm} (4.22)

$$R^{(h)}_{\mu\nu\rho\sigma} = \{e_{\mu\nu} p_\rho p_\sigma - e_{\mu\sigma} p_\nu p_\rho - e_{\nu\rho} p_\sigma p_\mu + e_{\rho\sigma} p_\nu p_\mu\} = (e_{\mu\nu} p_\rho - e_{\rho\mu} p_\nu)(e_{\nu\sigma} p_\rho - e_{\rho\nu} p_\sigma)$$ \hspace{1cm} (4.23)

respectively. Here $p$ denotes the momentum carried by the finite energy graviton. Using this we see that the contribution to the three point vertex from the $\phi R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$ term is proportional to

$$\{(\varepsilon_{\mu\nu} k_\rho - \varepsilon_{\rho\nu} k_\mu)(e^\mu p^\rho - e^\rho p^\mu)\}^2.$$  \hspace{1cm} (4.24)
Now in flat space-time background, a polarization vector $\varepsilon$ carried by a massless particle of momentum $k$ is defined to have helicity $\pm$ if

$$\epsilon_{\mu\nu\rho\sigma} (k^\rho \varepsilon^\sigma - k^\sigma \varepsilon^\rho) = \pm 2i (k_\mu \varepsilon_\nu - k_\nu \varepsilon_\mu). \quad (4.25)$$

Using this it is easy to see that

$$ (\varepsilon_\mu k_\rho - \varepsilon_\rho k_\mu) (e^\mu p^\rho - e^\rho p^\mu) = 0, \quad (4.26)$$

unless $\varepsilon$ and $e$ carry same helicity. For example if $\varepsilon$ has positive helicity and $e$ has negative helicity then we can write

$$ (\varepsilon_\mu k_\rho - \varepsilon_\rho k_\mu) (e^\mu p^\rho - e^\rho p^\mu) = \frac{1}{2i} \epsilon_{\mu\rho\mu'\rho'} (\varepsilon^\mu' k^\rho' - \varepsilon^\rho' k^\mu') (e^\mu p^\rho - e^\rho p^\mu) = - (\varepsilon^\mu' k^\rho' - \varepsilon^\rho' k^\mu') (e^{\rho'} p^{\mu'} - e^{\mu'} p^{\rho'}). \quad (4.27)$$

Since the two sides of this equation are negatives of each other the result vanishes. Therefore we shall take $\varepsilon$ and $e$ to have the same helicity. This analysis also shows that once we have chosen the helicity of the soft graviton, the contribution from the $\phi R_{\mu\nu\rho\sigma} \tilde{R}^{\mu\nu\rho\sigma}$ term differs from the one given in (4.24) by a factor of $\pm 2i$. Therefore we shall not analyze its contribution separately.

For the $R_{\mu\nu\rho\sigma} F^{\mu\rho} F^{\nu\sigma}$ term, the three point vertex receives a contribution proportional to

$$ \{ (\varepsilon_\mu k_\rho - \varepsilon_\rho k_\mu) (e^\mu p^\rho - e^\rho p^\mu) \} \{ (\varepsilon_\nu k_\sigma - \varepsilon_\sigma k_\nu) (e^\nu p^\sigma - e^\sigma p^\nu) \}, \quad (4.28)$$

where $e$ and $\tilde{e}$ represent the polarizations of the external and the internal photons. The previous argument now shows that this vanishes unless the helicities of $e$ and $\tilde{e}$ agree with that of $\varepsilon$. Since for soft external graviton, the momenta of the two photons connected to the vertex are nearly equal and opposite, this shows that $\tilde{e}$ is equal to $e$ (up to gauge transformation). Therefore (4.28) reduces to (4.24).

In order to compare this with the result of [57] we need to convert (4.24) to the spinor helicity notation (see e.g. [61, 62] for a review). We label each of the null vectors $p$ and $k$ by a pair of two component spinors

$$ p \to (\mu_\alpha, \tilde{\mu}_\dot{\alpha}), \quad k \to (\lambda_\alpha, \tilde{\lambda}_\dot{\alpha}), \quad (4.29) $$

via the relation

$$ p_\mu (\gamma^\mu)_{\alpha\dot{\alpha}} = \mu_\alpha \tilde{\mu}_{\dot{\alpha}}, \quad k_\mu (\gamma^\mu)_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}, \quad (4.30) $$

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and introduce the notation
\[ [\tilde{a} \tilde{b}] = \epsilon^{\alpha \beta} \tilde{a}_\alpha \tilde{b}_\beta, \quad \langle a b \rangle = \epsilon^{\alpha \beta} a_\alpha b_\beta, \tag{4.31} \]
where \( \epsilon = i \sigma_2 \), \( \sigma_i \)'s being Pauli matrices. In this notation we have
\[ p \cdot k = -\frac{1}{2} [\tilde{\lambda} \tilde{\mu}] \langle \lambda \mu \rangle. \tag{4.32} \]

For describing polarization vectors \( \varepsilon^\mu \) and \( e^\mu \) we introduce an auxiliary pair of spinors \((x_\alpha, \tilde{x}_{\dot{\alpha}})\) for the soft particle and another pair of spinors \((y_\alpha, \tilde{y}_{\dot{\alpha}})\) for the finite energy particle. In terms of these spinors we can label the normalized positive helicity polarization vectors \( \varepsilon \) and \( e \) as
\[ \varepsilon \rightarrow \sqrt{2} (x_\alpha, \tilde{\lambda}_{\dot{\alpha}}) / \langle \lambda x \rangle, \quad e \rightarrow \sqrt{2} (y_\alpha, \tilde{\mu}_{\dot{\alpha}}) / \langle \mu y \rangle. \tag{4.33} \]

Now we can easily generalize (4.32) as
\[ \varepsilon \cdot p = -\frac{1}{\sqrt{2}} [\tilde{\mu} \tilde{\lambda}] \langle \mu x \rangle, \quad e \cdot k = -\frac{1}{\sqrt{2}} [\tilde{\lambda} \tilde{\mu}] \langle \lambda y \rangle, \quad \varepsilon \cdot e = -[\tilde{\lambda} \tilde{\mu}] \langle \lambda x \rangle / \langle \mu y \rangle. \tag{4.34} \]

We can simplify our analysis by making the gauge choice \( y = \lambda \). In that case \( e \cdot k \) vanishes and we have
\[ \{(\varepsilon_\mu k_\rho - \varepsilon_\rho k_\mu)(e^\mu p^\rho - e^\rho p^\mu)\} = [\tilde{\lambda} \tilde{\mu}]^2. \tag{4.35} \]

Therefore for the three point vertex induced from any of the terms listed in (4.19), the soft factor associated with the amplitude in Fig. \( \text{II} \) is proportional to
\[ \frac{1}{2p \cdot k} \{(\varepsilon_\mu k_\rho - \varepsilon_\rho k_\mu)(e^\mu p^\rho - e^\rho p^\mu)\}^2 = -[\tilde{\lambda} \tilde{\mu}]^3 / \langle \lambda \mu \rangle. \tag{4.36} \]

This agrees with the result of [57].

Finally we consider the inclusion of spin 3/2 and spin 1/2 Dirac spinors \( \psi_\rho \) and \( \chi \). The terms in the action that can lead to the coupling of a soft graviton to a pair of finite energy nearly on-shell fermions are of the form
\[
\int d^4x \sqrt{-\det g} R^{\mu \nu \sigma} \bar{\psi}_\mu \gamma_{\nu \sigma} \partial_\rho \chi, \quad \int d^4x \sqrt{-\det g} \bar{R}^{\mu \nu \sigma} \bar{\psi}_\mu \gamma_{\nu \sigma} \gamma^5 \partial_\rho \chi. \tag{4.37} \]

For given helicity of \( \chi \) the contribution from the two terms are proportional to each other; so let us focus on the first term. Using (4.22) this leads to the following coupling between the soft graviton of momentum \( k \) and the finite energy (nearly) on-shell fermions of momentum \( p \):
\[ (\varepsilon_\mu k_\rho - \varepsilon_\rho k_\mu)(\varepsilon_\nu k_\sigma - \varepsilon_\sigma k_\nu) p^\rho \bar{\psi}^\mu \gamma^\nu \gamma^5 \chi. \tag{4.38} \]

\(^3\)The spinors \( \tilde{x} \) and \( \tilde{y} \) are necessary for describing negative helicity polarization vectors.
Using (4.25) and the corresponding result for the spinors in flat space:

\[ \epsilon_{\mu\nu\rho\sigma} \gamma^{\rho\sigma} = 2i \gamma_{\mu\nu} \gamma^5 = 2i \gamma^5 \gamma_{\mu\nu}, \]  

(4.39)

it is easy to see that the amplitude (4.38) vanishes unless the \( \bar{\psi} \) and \( \chi \) fields carry the same helicity as the soft graviton. For positive helicity of the soft graviton this means that

\[ \bar{\psi}_\rho \gamma^5 = \bar{\psi}_\rho, \quad \gamma^5 \chi = \chi. \quad \epsilon^{\mu\nu\rho\sigma} (p_\rho \bar{\psi}_\mu - p_\mu \bar{\psi}_\rho) = 2i (p^\sigma \bar{\psi}^\nu - p^\nu \bar{\psi}^\sigma) \]  

(4.40)

Therefore \( \bar{\psi}_\rho \) can be taken to be proportional to the positive helicity polarization vector \( e_\rho \) and that in spinor space both \( \bar{\psi}_\rho \) and \( \chi \) carry dotted index and can be taken to be proportional to \( \tilde{\mu}_\alpha \) introduced in (4.29). Up to overall normalization, the soft factor is then given by

\[ \frac{1}{2 \mathbf{p} \cdot \mathbf{k}} (\epsilon_\mu k_\rho - \epsilon_\rho k_\mu)(\epsilon_\nu k_\sigma - \epsilon_\sigma k_\nu) p^\mu (\gamma^{\sigma\nu})^{\dot{\alpha}\dot{\beta}} \tilde{\mu}_\alpha \tilde{\mu}_\beta. \]  

(4.41)

Now we have, using (4.30), (4.33)

\[ \epsilon_\nu k_\sigma (\gamma^{\sigma\nu})^{\dot{\alpha}\dot{\beta}} \tilde{\mu}_\alpha \tilde{\mu}_\beta \propto \frac{\langle \lambda x \rangle [\tilde{\lambda} \tilde{\mu}]^2}{\langle \lambda x \rangle} = [\tilde{\lambda} \tilde{\mu}]^2. \]  

(4.42)

Using this, and (4.32), (4.35) we see that (4.41) reduces to

\[ \frac{[\tilde{\lambda} \tilde{\mu}]^3}{\langle \lambda \mu \rangle} \]  

(4.43)

up to normalization factor. This is identical to (1.36), in agreement with [57].

For specific helicity configurations, the nature of soft theorems can be completely governed by the non-universal terms. An example of this is as follows. Consider a tree level 4-graviton amplitude \( \mathcal{M}_4(p_1^+, \ldots, p_4^+) \) in which all the gravitons have positive helicity. As is well known [61], in pure gravity this amplitude vanishes. However suppose we compute this amplitude in the theory where gravity is non-minimally coupled to a massless scalar via \( \int \sqrt{-\det g} \phi R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \). In this case the amplitude \( \mathcal{M}_4(p_1^+, \ldots, p_4^+) \) will not be zero due to the additional vertices involving the scalar, leading to a scalar exchange diagram. We can also see that in the limit \( p_4 \to 0 \) we get

\[ \mathcal{M}_4(p_1, \ldots, p_4) = \left[ \tilde{S}_1^{(2)} \mathcal{M}_3(p_1, p_2^+, p_3^+) + \tilde{S}_2^{(2)} \mathcal{M}_3(p_1^+, p_2, p_3^+) + \tilde{S}_3^{(2)} \mathcal{M}_3(p_1^+, p_2^+, p_3) \right] + O(E_n^2) \]  

(4.44)

where \( \tilde{S}_i^{(2)} \) is the sub-subleading factor given in eq.(4.36) with \( (e, p) \) replaced by \( (e_i, p_i) \) and the \( i \)-th 3 point amplitude on the right hand side of eq.(4.44) is an amplitude involving 2 gravitons
and a scalar with momentum $p_i$. In the above equation, there is no universal soft factor due to the fact that universal soft factors (to sub-subleading order) are precisely governed by the pure gravity three point vertices. These factors will dress a 3 graviton amplitude which is computed via Einstein-Hilbert Lagrangian and such an amplitude vanishes as all the gravitons have the same helicity.

5 Comparison with results from tree level string theory

In this section we shall compare our results with the results of [50,52] which computed bosonic string tree amplitudes with external graviton and other states in the soft limit.

5.1 Two tachyon two graviton amplitude

Ref. [50] computed the scattering amplitude involving a pair of external gravitons and a pair of external tachyons in the limit when one of the graviton momentum becomes soft. At the sub-subleading order the result of [50] contained an extra term besides the ones given by the first four lines on the right hand side of (2.44). If we denote by $k$ and $\varepsilon$ the momentum and polarization of the soft graviton, by $p_1$ and $e$ the momentum and polarization of the finite energy graviton, and by $p_2$ and $p_3$ the momenta of the tachyons then, up to an overall normalization, the extra term obtained in [50] (after correcting a typographical error and the overall sign) can be written as

$$\frac{-\alpha'}{4} \{ -k \cdot p_- \frac{p_1}{p_1 \cdot k} \varepsilon_{\mu \nu} \varepsilon_{\rho \sigma} p_{-\rho} + k \cdot p_1 \frac{p_2}{p_2 \cdot k} \varepsilon_{\mu \nu} \varepsilon_{\rho \sigma} p_{-\rho} + \frac{1}{p_1 \cdot k} k \cdot p_- k_\mu \varepsilon_{\mu \nu} p_{-\nu} p_{1\rho} \varepsilon_{\rho \sigma} p_{1\sigma} $$

$$-k_\mu \varepsilon_{\mu \nu} p_{-\nu} p_{1\rho} \varepsilon_{\rho \sigma} p_{-\sigma} \} ,$$

(5.1)

where

$$p_- = p_2 - p_3 .$$

(5.2)

Since $p_2$ and $p_3$ satisfy the on-shell condition $p_2^2 = p_3^2 = -m_T^2$ where $m_T^2$ is the tachyon mass, we have, using momentum conservation,

$$p_1 \cdot p_- = -(p_2 + p_3 + k) \cdot (p_2 - p_3) = \mathcal{O}(k) .$$

(5.3)

Using this we can express (5.1) as (up to term suppressed by additional powers of soft momentum)

$$\frac{-\alpha'}{8} \frac{1}{p_1 \cdot k} P^{(s)}_{\mu \nu \rho \sigma} R^{(h)\mu \rho \sigma} p_- p_{-\tau}$$

(5.4)
Figure 4: Possible sources of correction to the sub-subleading soft graviton theorem for two graviton, two tachyon scattering in string theory. $S$ denotes the soft graviton, $T$ denotes external tachyon, $H$ denotes finite energy external or internal graviton and $\phi$ denotes a finite energy internal dilaton. These represent additional contribution to Fig. 1 besides the one arising from the three point vertex (2.38) representing minimal coupling.

where $R^{(s)}$ and $R^{(h)}$ are the linearized Riemann tensors for the soft and finite energy external gravitons respectively:

$$R^{(s)}_{\mu\rho\nu\sigma} = \{\varepsilon_{\mu k}k_{\rho}k_{\sigma} - \varepsilon_{\mu\sigma}k_{\nu}k_{\rho} - \varepsilon_{\nu\rho}k_{\sigma}k_{\mu} + \varepsilon_{\rho\sigma}k_{\mu}k_{\nu}\}, \quad (5.5)$$

and

$$R^{(h)}_{\mu\rho\nu\sigma} = \{e_{\mu p_1 p_1 \nu} - e_{\mu\sigma}p_1 p_1 \rho - e_{\nu\rho}p_1 p_1 \mu + e_{\rho\sigma}p_1 p_1 \nu\}. \quad (5.6)$$

(5.4) may be written in a more suggestive form by noting that the three point coupling between the finite energy graviton of momentum $p_1$ and polarization $e$ and a pair of tachyons of momenta $p_2$ and $p_3$ has the form $e_{\mu\nu}\Gamma^{\mu\nu}$ where [50]

$$\Gamma^{\mu\nu} = p^\mu p^\nu / 4. \quad (5.7)$$

The three point coupling between the two tachyons and a dilaton is given by the same formula if we choose $e_{\mu\nu} \propto \eta_{\mu\nu}$. Therefore we can express (5.4) as

$$-\frac{\alpha'}{2} \frac{1}{p_1 \cdot k} R^{(s)}_{\mu\rho\nu\sigma} R^{(h)\rho\sigma\tau\rho} \Gamma^{\nu\tau}. \quad (5.8)$$

Eq. (5.8), being proportional to $(p_1 \cdot k)^{-1} R^{(s)}_{\mu\rho\nu\sigma}$, clearly has the structure of the corrections given in the last term on the right hand side of (2.44). We shall now explore their origin in some more detail. In the Siegel gauge $K^{\alpha\beta}(q)$ is proportional to $q^2$ and therefore $\Xi^i_{\alpha\beta}$ is independent of $q$. Therefore the contribution from the terms involving derivatives of $\Xi^i$ in (2.45) vanish. Also the quadratic term in $J$ vanishes for the tachyon and for the graviton it cancels against a term from the Einstein-Hilbert action as in section 4.1. Therefore the correction term (5.8)
can only come from a higher derivative three point coupling involving one soft graviton and a pair of finite energy particles. If in (5.8) we decompose

\[
\Gamma^\nu_{\tau} = \frac{1}{D} \Gamma^\rho_{\rho} \delta^\nu_{\tau} + \{\Gamma^\nu_{\tau} - \frac{1}{D} \Gamma^\rho_{\rho} \delta^\nu_{\tau}\}
\]

then the contribution from the first term to (5.8) gives the dilaton mediated coupling in Fig. 4 where we choose the internal line to be the dilaton \(\phi\). This requires a three point coupling proportional to

\[
\int d^Dx \, \phi \, R^{(s)}_{\mu\nu\rho\sigma} \, R^{(h)\mu\nu\rho\sigma},
\]

which comes via the correction to the effective action of the form

\[
\int d^Dx \, \sqrt{-\det g} \, R_{\mu\nu\rho\sigma} \, R^{\mu\nu\rho\sigma}.
\]

This is known to be present in the bosonic and heterotic string theory. Contribution from the second term in (5.9) to (5.8) can be identified as the graviton mediated amplitude where we pick the intermediate state in Fig. 4 to be a finite energy graviton \(H\). This requires a higher derivative three point coupling involving one soft and two finite energy gravitons of the form

\[
\int d^Dx \, R^{(s)}_{\mu\nu\rho\sigma} \, R^{(h)\mu\nu\rho\sigma} \, h_{\tau}^\nu.
\]

This can come from the following term in the original action

\[
\int d^Dx \, \sqrt{-\det g} \, R_{\mu\nu\rho\sigma} \, R^{\mu\nu\rho\sigma}.
\]

It is easy to verify that in the soft limit, the coupling of a soft graviton to a pair of finite energy gravitons computed from (5.12) and (5.13) are the same (up to overall normalization). For \(D = 4\) (5.13) does not contribute to the three point function since it is equivalent to the Gauss-Bonnet action on-shell. However in higher dimensions the contribution from this term does not vanish.

### 5.2 Scattering of gravitons and dilatons

Ref. [52] computed the scattering amplitude in the bosonic string theory for massless external states, and found corrections to the soft graviton theorem at sub-subleading order. If the soft particle carries polarization \(\varepsilon\) and momentum \(k\), and the finite energy particles carry momenta
\( p_1, \cdots p_N \) and polarizations \( e_1^{\mu}, \cdots e_N^{\mu} \), then the correction to the sub-subleading soft graviton theorem was found to be given by:

\[
\frac{\alpha'}{2} \varepsilon^{\mu \nu} \sum_{i=1}^{N} \left \{ k_\sigma p_i \eta_{\mu i} + k_\rho p_i \eta_{\mu \sigma} - \eta_{\mu i} \eta_{\sigma \nu} p_i \cdot k - \frac{1}{p_i \cdot k} k_\rho k_\sigma p_i p_\nu \right \} \Pi_i^{(\rho, \sigma)} \Gamma ,
\]

where \( \Gamma \) is the amplitude without the soft graviton and the operation \( \Pi_i^{(\rho, \sigma)} \) is defined as follows. If we label the polarization \( e_\rho^{\sigma i} \) as \( e_\rho^{\sigma i} \bar{e}_\sigma^{\rho i} \) then

\[
\Pi_i^{(\rho, \sigma)} = \frac{1}{2} \left[ e_\rho^{\sigma i} \frac{\partial}{\partial e_\rho^{\sigma i}} + \bar{e}_\rho^{\sigma i} \frac{\partial}{\partial \bar{e}_\rho^{\sigma i}} + e_\rho^{\sigma i} \frac{\partial}{\partial e_\sigma^{\rho i}} + \bar{e}_\rho^{\sigma i} \frac{\partial}{\partial \bar{e}_\sigma^{\rho i}} \right].
\]

In string theory \( e_\rho^{\sigma i} \) may be symmetric or anti-symmetric under the exchange \( \rho \leftrightarrow \sigma \). If we restrict to the symmetric case, representing graviton or dilaton state, then

\[
\Pi_i^{(\rho, \sigma)} \Gamma = e_\rho^{\sigma i} \Gamma_{\rho \tau}^{(\sigma)}(i) + e_\sigma^{\rho i} \Gamma_{\sigma \tau}^{(\rho)}(i).
\]

Now using the gauge invariance of \( \Gamma \):

\[
p_i, \rho \Gamma_{\rho \tau}^{(\sigma)}(i) = 0, \quad p_i, \tau \Gamma_{\rho \tau}^{(\sigma)}(i) = 0,
\]

one can express \( (5.17) \) as

\[
-\frac{\alpha'}{2} R_{(s) \mu \rho \sigma}^{(s)} \sum_{i=1}^{N} \frac{1}{p_i \cdot k} R_{(i) \mu \rho \sigma}^{(i)} \Gamma_{\tau \nu}^{(i)} ,
\]

where \( R_{(s)}^{(s)} \) has been defined in \( (5.5) \), and \( R_{(i)}^{(i)} \) is given by

\[
R_{(i) \mu \rho \sigma}^{(i)} = \left \{ e_{i, \mu \nu} p_\rho p_i \sigma - e_{i, \mu \sigma} p_\nu p_\rho - e_{i, \nu \rho} p_\sigma p_i \mu + e_{i, \rho \sigma} p_\mu p_i \nu \right \}.
\]

Eq. \( (5.19) \) has a structure identical to the one obtained in \( (5.8) \). As in that case, decomposing \( \Gamma_{\tau \nu}^{(i)} \) as

\[
\Gamma_{\tau \nu}^{(i)} = \frac{1}{D} \delta_{\tau \nu}^{(i)} \Gamma_{\rho}^{(i)} + \left \{ \Gamma_{\tau \nu}^{(i)} - \frac{1}{D} \delta_{\tau \nu}^{(i)} \Gamma_{\rho}^{(i)} \right \}
\]

we can interpret the contribution to \( (5.19) \) from the first term in \( (5.21) \) as due to an intermediate finite energy dilaton and the contribution to \( (5.19) \) from the rest of the terms in \( (5.21) \) as due to an intermediate finite energy graviton. The relevant three point interactions arise from \( (5.11) \) and \( (5.13) \).
5.3 Amplitude for two tachyons, one graviton and one massive particle

We shall now consider the four point scattering in bosonic string theory of a pair of tachyons carrying momenta $p_1$ and $p_2$, a rank 4 symmetric tensor field at the first massive level carrying momentum $p_3$ and polarization $\epsilon_3$ and a soft graviton carrying momentum $k$ and polarization $\varepsilon$. The full amplitude can be read out from eq. (70) of [50] with the following replacement:

$$k_3 \to k, \quad a_{3\mu} \to \epsilon_{\mu\nu}, \quad p_4 \to p_3, \quad H_{\mu\nu} \tilde{H}_{\rho\sigma} \to \epsilon_{3,\mu\nu\rho\sigma}.$$

(5.22)

With this we find that the leading and subleading soft graviton amplitudes agree with the expected result given in the first two lines on the right hand side of (2.44) if we take the amplitude without the soft graviton to be

$$-\frac{1}{16} \epsilon_{3,\mu\nu\rho\sigma} p_\mu^\rho p_\nu^\sigma p_\rho^- p_\sigma^-,$$

(5.23)

Given (5.23), sub-subleading contribution from the third and fourth lines on the right hand side of (2.44) take the form

$$\begin{align*}
\frac{3}{4} \left\{ \varepsilon^{\mu\tau} (p_1 + p_2) + \varepsilon^{\mu\tau} p_3 \frac{k \cdot p_-}{k \cdot p_3} \right\} \epsilon_{3,\mu\nu\rho\sigma} k^\nu p_\rho^- p_\sigma^- \\
- \frac{3}{8} \left\{ k \cdot (p_1 + p_2) + \frac{(k \cdot p_-)^2}{k \cdot p_3} \right\} \epsilon_{3,\mu\nu\rho\sigma} \epsilon^{\mu\nu} p_\rho^- p_\sigma^- \\
- \frac{3}{8} \left\{ \frac{\varepsilon_{\mu\nu} p_1^\mu p_2^\nu}{k \cdot p_1} + \frac{\varepsilon_{\mu\nu} p_2^\mu p_1^\nu}{k \cdot p_2} + \frac{\varepsilon_{\mu\nu} p_3^\mu p_3^\nu}{k \cdot p_3} \right\} \epsilon_{3,\mu\nu\rho\sigma} k^\mu k^\nu p_\rho^- p_\sigma^-.
\end{align*}$$

(5.24)

However the actual amplitude computed from eq. (70) of [50] has some additional terms. These are given by

$$-\frac{1}{k \cdot p_3} \mathcal{R}_{\mu\nu\rho\sigma} \left[ \frac{1}{2} p_1^\rho p_2^\sigma \epsilon_3^{\mu\nuab} p_{-a} p_{-b} - \frac{1}{4} p_3^\mu p_3^\nu \epsilon_3^{\sigmaabc} p_{-a} p_{-b} p_{-c} \right],$$

(5.25)

where $\mathcal{R}_{\mu\nu\rho\sigma}$ is given in (2.34). This form of the correction terms is consistent with the general form of the corrections to the sub-subleading soft graviton theorem given in (2.45), and can be traced to a non-minimal three point coupling between a soft graviton, a massive rank four symmetric tensor field and another massive field at the same mass level. The relevant diagram has the same structure as Fig. 4 with the finite energy external graviton replaced by the massive symmetric rank four tensor field and the internal line representing either a massive rank four symmetric tensor or another field at the same mass level.
6 Infrared divergences

In four space-time dimensions the loop amplitudes suffer from infrared divergences that have to be removed by either summing over final states and averaging over initial states \[1,2,63–65\], or by changing the description of the scattering states \[66–68\]. Hence in four dimensions the structure of soft theorems for loop amplitudes is sensitive to the divergent infra-red effects \[10,56\]. For this reason for loop amplitudes we focus on space-time dimensions \(D \geq 5\) for which the S-matrix elements are finite – at least before taking the soft limit. Our goal in this section will be to explore if our analysis of soft theorem in section 2 based on 1PI effective action, that includes loop amplitudes as well, could be affected by infrared issues in \(D \geq 5\) even though there are no divergences before taking the soft limit. We shall first consider the possible effects of soft divergences and then briefly discuss the effect of collinear divergences that can arise when some of the finite energy external states are massless.

6.1 Soft divergences

Soft divergences refer to divergences that arise from regions of loop momentum integration in which all components of the loop momentum becomes small. The absence of soft divergences in \(D \geq 5\) for amplitudes without soft external lines has been illustrated in Fig. 5. Here \(\Gamma\)'s represent amputated Green’s functions and the thin internal line carrying momentum \(\ell\) represents a massless soft line, i.e. we consider the limit \(\ell_{\mu} \to 0\). In this limit, if we pick the internal states carrying momenta \(p_j - \ell\) and \(p_i + \ell\) to be of the same mass as the external states carrying momentum \(p_j\) and \(p_i\) respectively, then in the \(\ell_{\mu} \to 0\) limit the integrand of the Feynman diagram goes as

\[
\mathcal{I} = \{\ell^2 \left(-2p_j \cdot \ell + p_j^2 + M_j^2\right) \left(2p_i \cdot \ell + p_i^2 + M_i^2\right)\}^{-1} \times \text{finite} \tag{6.1}
\]

where the \(\ell^2\) factor in the denominator comes from the propagator carrying momentum \(\ell\) and the \((-2p_j \cdot \ell + p_j^2 + M_j^2\) and \((2p_i \cdot \ell + p_i^2 + M_i^2)\) factors arise from the propagators carrying momenta \(p_j - \ell\) and \(p_i + \ell\) respectively. The on-shell condition for the external states carrying momenta \(p_i\) and \(p_j\) sets \(p_i^2 + M_i^2\) and \(p_j^2 + M_j^2\) to zero. Even though the integrand \(\mathcal{I}\) has four powers of \(\ell_{\mu}\) in the denominator and therefore diverges in the \(\ell_{\mu} \to 0\) limit, the integral \(\int d^D\ell \mathcal{I}\) is convergent for \(D \geq 5\). Similar power counting \[69\] shows that there are no collinear divergences – divergences arising from regions of loop momenta when one or more internal momenta of a massless state becomes collinear to the external momentum of a massless state. This will be
discussed in section \ref{sec:6.2}. Furthermore, adding more loops containing soft or collinear momenta does not lead to any new divergence.

Now the right hand side of (2.44) contains not just the amplitudes without soft external legs, but their derivatives with respect to external momenta, and absence of infrared divergence in the original amplitude does not necessarily imply absence of infrared divergence in its derivatives. To see this let us take a derivative of (6.1) with respect to \( p_\mu \) and then use the on-shell condition \( p_i^2 + M_i^2 = 0 \), \( p_j^2 + M_j^2 = 0 \). This generates an expression of the form

\[
\frac{\partial I}{\partial p_\mu} = \{\ell^2 \left( -2p_j \cdot \ell \right) \left( 2p_i \cdot \ell \right)^2 \}^{-1} \times \text{finite} \times (-2p_i^\mu) + \text{less divergent terms}.
\]  

(6.2)

Now in the small \( \ell_\mu \) limit the integrand has 5 powers of \( \ell_\mu \) in the denominator and therefore the integral has a logarithmic divergence in five dimensions. Similarly if we take two derivatives of \( I \) and then use the on-shell condition, then the leading and subleading divergent pieces are given by

\[
\frac{\partial^2 I}{\partial p_\mu \partial p_\nu} = \{\ell^2 \left( -2p_j \cdot \ell \right) \left( 2p_i \cdot \ell \right)^2 \}^{-1} \times \text{finite} \times (8p_i^\mu p_\nu^\nu) + \{\ell^2 \left( -2p_j \cdot \ell \right) \left( 2p_i \cdot \ell \right)^2 \}^{-1} \times \text{finite} \times (-2 \eta_\mu\nu) + \{\ell^2 \left( -2p_j \cdot \ell \right) \left( 2p_i \cdot \ell \right)^2 \}^{-1} \times \text{finite} \times (-2p_i^\nu) + \{\ell^2 \left( -2p_j \cdot \ell \right) \left( 2p_i \cdot \ell \right)^2 \}^{-1} \times \text{finite} \times (-2p_i^\mu) + \text{less divergent terms}.
\]  

(6.3)

The first term on the right hand side has six powers of \( \ell_\mu \) in the denominator in the small \( \ell_\mu \) limit. Therefore the integral is logarithmically divergent in six dimensions and linearly divergent in five dimensions. The contribution to the integral from the second, third and
fourth terms are free from divergence in six dimensions and are logarithmically divergent in five dimensions. It follows from this analysis that for \( D = 5, 6 \) the divergent parts of the derivatives of \( \Gamma_\gamma^{(i)} \) are of the form:

\[
D = 5 : \quad \frac{\partial \Gamma_\gamma^{(i)}}{\partial p_{ia}} = p_i^a \hat{\Gamma}_\gamma^{(i)} + \text{finite},
\]

\[
\frac{\partial^2 \Gamma_\gamma^{(i)}}{\partial p_{ia} \partial p_{ib}} = \eta^{ab} \hat{\Gamma}_\gamma^{(i)} + p_i^a \hat{\Gamma}^{\alpha b}_\gamma + p_i^b \hat{\Gamma}^{\alpha a}_\gamma + \text{finite}
\]

\[
D = 6 : \quad \frac{\partial \Gamma_\gamma^{(i)}}{\partial p_{ia}} = \text{finite},
\]

\[
\frac{\partial^2 \Gamma_\gamma^{(i)}}{\partial p_{ia} \partial p_{ib}} = p_i^a p_i^b \hat{\Gamma}^{\alpha \alpha}_\gamma + \text{finite}
\]

for some functions \( \hat{\Gamma}_\gamma^{(i)}, \hat{\Gamma}^{\alpha b}_\gamma \) and \( \hat{\Gamma}^{\alpha \alpha}_\gamma \).

We shall now argue that these divergences do not make the right hand side of (2.44) diverge. Since the divergences are more severe in \( D = 5 \) let us consider the \( D = 5 \) case – this will automatically extend to the \( D = 6 \) case. The potential sources of divergence are the terms involving derivatives of \( \Gamma_\gamma^{(i)} \) in the second, third and fourth lines on the right hand side of (2.44). Now using the first equation in (6.4) we see that the potentially divergent term on the second line is proportional to

\[
\frac{\partial \Gamma_\gamma^{(i)}}{\partial p_{ia}} = \text{finite},
\]

\[
\frac{\partial^2 \Gamma_\gamma^{(i)}}{\partial p_{ia} \partial p_{ib}} = p_i^a p_i^b \hat{\Gamma}^{\alpha \alpha}_\gamma + \eta^{ad} \hat{\Gamma}_\gamma^{(i)} + \eta^{ad} p_i^a p_i^d \hat{\Gamma}^{\alpha \alpha}_\gamma + \text{finite}
\]

for some functions \( \hat{\Gamma}_\gamma^{(i)}, \hat{\Gamma}^{\alpha b}_\gamma \) and \( \hat{\Gamma}^{\alpha \alpha}_\gamma \).

Therefore we see that the right hand side of (2.44) does not have any infrared divergence from the terms involving derivatives of \( \Gamma_\gamma^{(i)} \) for \( D \geq 5 \). One might worry that since the individual terms are divergent, one needs a regularization before claiming that they cancel. This can be done by keeping the external momenta slightly off-shell while computing the right hand side of (2.44). This is in any case needed to define derivatives with respect to \( p_{\mu i} \) for which we need to treat all components of \( p_i \) as independent.

Another potential source of infrared divergence on the right hand side of (2.44), (2.45) is the derivative of the self energy contribution proportional to \( K \) (and its inverse proportional
Figure 6: Infrared divergences in self-energy graphs. As usual the thin line denotes a particle carrying soft momentum.

to $\Xi$). Consider for example the contribution to $K^{\alpha\beta}$ from a diagram of the form shown in Fig. 6 with the thin line denoting a massless particle carrying soft momentum $\ell$. When the momentum $\ell$ is small, the integrand is proportional to

$$I' = \ell^2 (-2p \cdot \ell + p^2 + M^2)^{-1} \times \text{finite}.$$  \hspace{1cm} (6.6)

In this case $\int d^D \ell \, I'$ has no divergence from the small $\ell_\mu$ region for $D \geq 4$. However since for $p^2 + M^2 = 0$,

$$\frac{\partial^2 I'}{\partial p_\mu \partial p_\nu} = \ell^2 (-2p \cdot \ell)^{-1} \times 8 p^\mu p^\nu \times \text{finite} + \text{less divergent terms},$$  \hspace{1cm} (6.7)

and

$$\frac{\partial^3 I'}{\partial p_\mu \partial p_\nu \partial p_\rho} = -\{\ell^2 (-2p \cdot \ell)^4\}^{-1} \times 48 p^\mu p^\nu p^\rho \times \text{finite} + \text{less divergent terms},$$  \hspace{1cm} (6.8)

$\partial^2 I' / \partial p_\mu \partial p_\nu$ diverges logarithmically for $D = 5$ and $\partial^3 I' / \partial p_\mu \partial p_\nu \partial p_\rho$ diverges linearly for $D = 5$ and logarithmically for $D = 6$. It follows from this that the the first derivative of $K^{\alpha\beta}$ has no divergence for $D \geq 5$, but for $D = 5, 6$ the second and third derivatives of $K^{\alpha\beta}$ can have divergent pieces of the form

$$D = 5 : \quad \frac{\partial^2 K^{\alpha\beta}(-p_i)}{\partial p_{i\mu} \partial p_{i\nu}} = p_i^\mu p_i^\nu \tilde{K}^{\alpha\beta}(-p_i) + \text{finite},$$

$$\frac{\partial^3 K^{\alpha\beta}(-p_i)}{\partial p_{i\mu} \partial p_{i\nu} \partial p_{i\rho}} = (\eta^\mu \rho p_i^\nu + \eta^\mu \nu p_i^\rho + \eta^\nu \rho p_i^\mu) \tilde{K}^{\alpha\beta}(-p_i) + p_i^\mu p_i^\nu \tilde{K}^{\alpha\beta \rho}(-p_i) + p_i^\mu p_i^\rho \tilde{K}^{\alpha\beta \nu}(-p_i) + p_i^\rho p_i^\nu \tilde{K}^{\alpha\beta \mu}(-p_i) + \text{finite},$$

$$D = 6 : \quad \frac{\partial^2 K^{\alpha\beta}(-p_i)}{\partial p_{i\mu} \partial p_{i\nu}} = \text{finite},$$

$$\frac{\partial^3 K^{\alpha\beta}(-p_i)}{\partial p_{i\mu} \partial p_{i\nu} \partial p_{i\rho}} = p_i^\mu p_i^\nu p_i^\rho \tilde{K}^{\alpha\beta}(-p_i) + \text{finite},$$  \hspace{1cm} (6.9)
Figure 7: An apparently infrared divergent contribution to Fig. 2.

for some functions $\hat{K}_{\alpha\beta}(-p_i)$, $\hat{K}^{\alpha\beta\mu}(-p_i)$ and $\hat{K}^\prime_{\alpha\beta}(-p_i)$. Similar result holds for the derivatives of $\Xi^i$.

It is easy to check that these divergences also do not affect the final expression for the sub-subleading soft theorem given in (2.44). Via (2.45) this contains second derivative of $\mathcal{K}$ and $\Xi^i$ with respect to momenta and therefore has logarithmic divergence in $D = 5$. However the divergent piece in $\partial^2 \mathcal{K}/\partial p_\mu \partial p_\nu$ (and $\partial^2 \Xi/\partial p_\mu \partial p_\nu$) is proportional to $p^\mu p^\nu$. Substituting this into (2.45) and using (2.34) one can easily verify that the corresponding contributions vanish and therefore the final expression for the sub-subleading soft theorem is free from infrared divergences.

To summarize, we have shown that the right hand side of the sub-subleading soft theorem given in (2.44) is free from infrared divergences. Nevertheless since at the intermediate stages of the analysis one encounters derivatives of $\Gamma^\alpha_{(i)}$, $\mathcal{K}^{\alpha\beta}$ and $\Xi^i_{\alpha\beta}$ that are infrared divergent, one could worry whether all the terms have been properly accounted for. To this end we note that the original amplitude involving the soft graviton is manifestly free from infrared divergences for any finite value of the soft momentum. Therefore any difference between the original amplitude and (2.44) must be finite for any finite value of $k$. We shall now analyze whether there can be such finite pieces that are left over in the difference between the actual amplitude and the one given in (2.44).

Before proceeding further, it will be useful to get some insight into the origin of the apparent infrared divergences arising in the soft limit. Let us consider for example the diagram shown in Fig. 7 representing a possible contribution to Fig. 2. As long as $k$ is finite, this represents an infrared finite contribution in $D \geq 5$ since in the limit when $\ell$ becomes small there are at most four powers of $\ell$ in the denominator – one each from the propagators carrying momentum $p_i + \ell$ and $p_j - \ell$, and two from the propagator carrying momentum $\ell$. However if we take
the \( k \to 0 \) limit then the propagator carrying momentum \( p_i + k + \ell \) supplies another factor of \( \ell \) in the denominator, causing the integral to diverge logarithmically in \( D = 5 \). In \( D \geq 6 \) this still represents a finite integral, but if we attempt to expand the integrand in Taylor series expansion in \( k \), as is needed for computing the sub-subleading contribution, the next term in the Taylor series expansion will diverge logarithmically in \( D = 6 \) and linearly in \( D = 5 \).

These divergences explain the origin of the infrared divergences appearing in the naive Taylor series expansion (2.24) in powers of the soft momentum. For example in \( D = 5 \), the contribution of Fig. 7 can diverge as \( \ln(p_i \cdot k) \) as \( k^\mu \to 0 \), and this shows up as logarithmic divergence in the \( k \)-independent terms in the naive Taylor series expansion (2.24). On the other hand in \( D = 6 \), the contribution from Fig. 7 is finite in the \( k \to 0 \) limit, but has a subleading contribution proportional to \( p_i \cdot k \ln(p_i \cdot k) \). This shows up as a logarithmic divergence in the coefficient of the order \( k^\mu \) terms in the naive Taylor series expansion (2.24). A similar analysis can be carried out for the diagrams contributing to Fig. 1.

We now try to determine the tensor structures of the singular terms by analyzing the divergences in individual terms arising during the analysis in section 2.4. We shall illustrate this with an example. In expression (2.24) for Fig. 2 the \( k^\mu \) independent contribution (which represents a contribution to the subleading soft graviton amplitude) involving a single derivative with respect to \( p_i^\mu \) is expected to be logarithmically divergent. According to (6.4) the divergent term in \( \partial \Gamma_\alpha^{(i)} / \partial p_i^\mu \) is expected to be proportional to \( p_i^\mu \). Substituting this into (2.24) we see that the divergent part of this term is proportional to \( \varepsilon_{\mu \nu} p_i^\mu p_i^\nu \). This can also be seen directly from Fig. 7, but analyzing the divergent part of (2.24) yields the result in simpler fashion. Therefore we conclude that the possible error in the analysis of the \( k \) independent term in (2.24) in \( D = 5 \) is proportional to \( \varepsilon_{\mu \nu} p_i^\mu p_i^\nu \).

With this insight we shall now try to determine the tensor structures of the terms that could possibly diverge in the \( k \to 0 \) limit. First let us consider the subleading soft graviton theorem. In this case intermediate steps of the analysis involve at most one derivative of \( \Gamma_\alpha^{(i)} \) and two derivatives of \( K \) and \( \Xi^i \) with respect to the external momenta. These are free from divergences for \( D \geq 6 \), so we have to analyze the possible logarithmically divergent contributions for \( D = 5 \).

---

\(^4\)The reason that we can do this is due to the fact that the general formulae (2.24), (2.38) which express the amplitudes with a soft external state to ones without it, are valid for off-shell momenta of the finite energy external lines. Since for these there are no infrared divergences, the presence of infrared divergences in the Taylor series expansion of the original amplitudes in powers of the soft momentum \( k \) can be inferred from possible infrared divergences that arise in the Taylor series expansion of (2.24), (2.38) about on-shell external momenta.
The potentially infrared divergent terms from derivative of $\Gamma^\delta_{(i)}$ are

$$
\epsilon_{i,\alpha} (p_i \cdot k)^{-1} \varepsilon_{\mu b} p_i^\mu k_a p_i^b \frac{\partial}{\partial p_{ia}} \Gamma^\alpha_{(i)}(p_i) \quad \text{and} \quad -\epsilon_{i,\alpha} \varepsilon_{\mu b} p_i^\mu \frac{\partial}{\partial p_{ib}} \Gamma^\alpha_{(i)}(p_i)
$$

(6.10)

coming from Figs. 1 and 2 respectively. The divergent piece of $\partial \Gamma^\alpha_{(i)}(p_i) / \partial p_{i\mu}$ is proportional to $p_i^\mu$. Therefore the divergent pieces in both terms in (6.10) are proportional to

$$
\varepsilon_{\mu\nu} p_i^\mu p_i^\nu,
$$

(6.11)

and come with opposite coefficients. However one may worry that after the cancellation of the divergent pieces one may be left with an extra finite piece proportional to (6.11). This will generate an extra term of the form

$$
\sum_i \epsilon_{i,\alpha} \varepsilon_{\mu\nu} p_i^\mu p_i^\nu \widehat{\Gamma}^\alpha_{(i)}
$$

(6.12)

for some amplitude $\widehat{\Gamma}^\alpha_{(i)}$.

Another potential source of logarithmic divergence in $D = 5$ are the terms in (2.38) involving second derivative of $\mathcal{K}$, obtained after Taylor series expansion in soft momentum $k$ up to subleading order. After using the fact that the divergent part of $\partial^2 \mathcal{K}(-p) / \partial p_{i\mu} \partial p_{i\nu}$ is proportional to $p_i^\mu p_i^\nu$, and carefully examining all the terms proportional to $\partial^2 \mathcal{K}(-p) / \partial p_{i\mu} \partial p_{i\nu}$ appearing in the intermediate steps, one can see that the possible correction takes the form

$$
\sum_i \epsilon_{i,\alpha} \varepsilon_{\mu\nu} p_i^\mu p_i^\nu \widehat{\Gamma}^{\alpha}_{(i)},
$$

(6.13)

for some amplitude $\Gamma^{\alpha}_{(i)}$.

Since the possible ambiguities from both sources are proportional to $\varepsilon_{\mu\nu} p_i^\mu p_i^\nu$, they can be clubbed together. Therefore the net ambiguity in the subleading soft theorem takes the form of an additive term of the form

$$
\sum_i \epsilon_{i,\alpha} \varepsilon_{\mu\nu} p_i^\mu p_i^\nu \widehat{\Gamma}^{\alpha}_{(i)}
$$

(6.14)

for some amplitude $\widehat{\Gamma}^{\alpha}_{(i)}$. $\widehat{\Gamma}^{\alpha}_{(i)}$ has at most logarithmic divergence in the $k^\mu \to 0$ limit.

We shall now argue that an additional term of the form (6.14) in the subleading soft graviton amplitude is inconsistent with gauge invariance\footnote{Note that due to the way we have described the coupling of soft gravitons – by covariantizing the vertices without the soft graviton – possible corrections to the soft graviton amplitude should be invariant under gauge transformation of the soft graviton without using on-shell condition for other external states.} and therefore must vanish. For this let us
consider shifting \( \varepsilon_{\mu\nu} \) by a pure gauge term

\[
(k_\mu \xi_\nu + k_\nu \xi_\mu)
\]

(6.15)

for any vector \( \xi \) satisfying \( k \cdot \xi = 0 \). Then \( (6.14) \) changes by

\[
2 \sum_i \varepsilon_{i,\alpha} \xi \cdot p_i k \cdot p_i \hat{\Gamma}^\alpha_{(i)}.
\]

(6.16)

We now see that this does not vanish for general \( \xi \) and \( k \) unless \( \sum_i \varepsilon_{i,\alpha} p_i \mu p_i \nu \hat{\Gamma}^\alpha_{(i)} \) is proportional to \( \eta_{\mu\nu} \). In the latter case \( (6.14) \) itself vanishes. Therefore \( (6.14) \) is not gauge invariant, and the amplitude cannot have an additional contribution of the form given in \( (6.14) \). This shows that the subleading soft theorem is unaffected by infrared divergences for \( D \geq 5 \).

Next we consider sub-subleading soft graviton theorem. In this case the intermediate stages of the analysis involve at most two derivatives of \( \Gamma^\alpha_{(i)} \) and at most three derivatives of \( \mathcal{K} \) with respect to the external momenta. These are free from divergences for \( D \geq 7 \), so we need to analyze the cases \( D = 5 \) and \( D = 6 \). Let us first consider the case \( D = 6 \). In this case terms with single derivatives of \( \Gamma^\alpha_{(i)} \) are free from infrared divergences; so we need to analyze the terms with two derivatives of \( \Gamma^\alpha_{(i)} \). Such terms arise from two sources. First there is a contribution from Fig. 2 given by the last term in \( (2.24) \). Since the divergent part of \( \partial^2 \Gamma^\alpha_{(i)}(p_i)/\partial p_i \mu \partial p_i \nu \) is proportional to \( p_i^{\mu} p_i^{\nu} \) in \( D = 6 \), the divergent contribution to the last term in \( (2.24) \) is proportional to

\[
\varepsilon_{\mu\nu} p_i^{\mu} p_i^{\nu} (p_i \cdot k).
\]

(6.17)

The other contribution involving two derivatives of \( \Gamma^\alpha_{(i)} \) comes from Fig. 1 and is given by the Taylor series expansion of the \( \Gamma^\delta_{(i)} \) factor in \( (2.38) \). This is proportional to

\[
(p_i \cdot k)^{-1} \varepsilon_{\mu\nu} p_i^{\mu} p_i^{\nu} (p_i \cdot k)^2 ,
\]

(6.18)

which is the same as \( (6.17) \). Therefore in \( D = 6 \), the amplitude may have a potentially ambiguous contribution proportional to

\[
\sum_i \varepsilon_{i,\alpha} \varepsilon_{\mu\nu} p_i^{\mu} p_i^{\nu} (p_i \cdot k) \hat{\Gamma}^\alpha_{(i)},
\]

(6.19)

for some amplitude \( \hat{\Gamma}^\alpha_{(i)} \).

The potentially divergent self energy contributions in \( D = 6 \) come from terms involving three derivatives of \( \mathcal{K} \) in \( (2.38) \). Using the fact that the divergent part of \( \partial^3 \mathcal{K}/\partial p_\mu \partial p_\nu \partial p_\rho \) is
proportional to \( p^\mu p^\nu p^\rho \) one can see that the possible correction is proportional to

\[
\sum_i \epsilon_{i,\alpha} \varepsilon_{\mu\nu \rho} p_i^\mu p_i^\nu (p_i \cdot k) \tilde{\Gamma}_{(i)}^{\alpha},
\]  

(6.20)

for some amplitude \( \tilde{\Gamma}_{(i)}^{\alpha} \). This has the same form as (6.19) and can be clubbed with it. Therefore the net ambiguous term in the sub-subleading soft graviton theorem in \( D = 6 \) is an additive term of the form

\[
\sum_i \epsilon_{i,\alpha} \varepsilon_{\mu\nu \rho} p_i^\mu p_i^\nu (p_i \cdot k) \Gamma_{(i)}^{\alpha},
\]

(6.21)

for some amplitude \( \Gamma_{(i)}^{\alpha} \). In the \( k^\mu \to 0 \) limit \( \Gamma_{(i)}^{\alpha} \) can have logarithmic divergence.

Now under a gauge transformation of \( \varepsilon \) given in (6.15), (6.21) changes by

\[
2 \sum_i \epsilon_{i,\alpha} \xi \cdot p_i (k \cdot p_i)^2 \tilde{\Gamma}_{(i)}^{\alpha}.
\]

(6.22)

This does not vanish for general \( \xi \) and \( k \) satisfying \( k^2 = 0 \), \( \xi \cdot k = 0 \) unless \( \sum_i \epsilon_{i,\alpha} p_i^\mu p_i^\nu p_i^\rho \tilde{\Gamma}_{(i)}^{\alpha} \) is proportional to

\[
\eta_{\mu\nu} A_\rho + \eta_{\mu\rho} A_\nu + \eta_{\nu\rho} A_\mu,
\]

(6.23)

for some function \( A_\mu \). In this case (6.21) itself vanishes. Therefore adding a term of the form (6.21) to the amplitude is inconsistent with gauge invariance. This in turn proves that sub-subleading soft graviton theorem is unaffected by infrared divergences for \( D = 6 \).

One can carry out a similar analysis for sub-subleading soft graviton theorem in \( D = 5 \). In this case there are many types of terms that can have infrared divergences during the intermediate stages of the analysis, and therefore the possible ambiguity is given by the sum of all such terms. One finds that all such possibly divergent terms can be clubbed into the form

\[
\varepsilon_{\mu\nu} k_\rho A^{\mu\nu\rho}
\]

(6.24)

for some amplitude \( A^{\mu\nu\rho} \) which has at most logarithmic divergence as \( k^\mu \to 0 \). In particular the \( (p_i \cdot k) \) terms in the denominator are always cancelled. Without loss of generality we can take \( A^{\mu\nu\rho} \) to be symmetric in the indices \( \mu, \nu \). The requirement of gauge invariance now imposes the constraint

\[
\xi_\mu k_\nu k_\rho A^{\mu\nu\rho} = 0.
\]

(6.25)

\footnote{The divergent term (6.21) in \( D = 6 \) is a special case of this where we choose \( A^{\mu\nu\rho} \) to be \( \sum_i p_i^\mu p_i^\nu p_i^\rho \epsilon_{i,\alpha} \tilde{\Gamma}_{(i)}^{\alpha} \).}
This can be satisfied for general $\xi$ and $k$ satisfying $\xi \cdot k = 0$, $k^2 = 0$ if in the $k \to 0$ limit

$$A^{\mu\nu\rho} = P^{\mu} \eta^{\nu\rho} + P^{\nu} \eta^{\mu\rho} + Q^{\rho} \eta^{\mu\nu} + B^{\mu\nu\rho},$$  \hspace{1cm} (6.26)

for some function $P^{\mu}, Q^{\mu}, B^{\mu\nu\rho}$ with $B^{\mu\nu\rho}$ symmetric under $\mu \leftrightarrow \nu$ and satisfying

$$B^{\mu\nu\rho} + B^{\nu\mu\rho} = 0,$$  \hspace{1cm} (6.27)

for all $\mu, \nu, \rho$. It is easy to see that this, together with the relation $B^{\mu\nu\rho} = B^{\nu\mu\rho}$, gives

$$B^{\mu\nu\rho} = 0.$$  \hspace{1cm} (6.28)

Therefore we are left with the contribution to (6.26) from the terms proportional to the vectors $P$ and $Q$. However using (2.1) one can check that their contribution to the amplitude (6.24) vanishes. Therefore even in five dimensions the sub-subleading soft graviton theorem does not have any correction from the infrared divergent terms.

### 6.2 Collinear divergences

When some of the finite energy external states are massless, we can also have collinear divergences. Again as mentioned in footnote 4, we can analyze their effect by examining the presence of these divergences in (2.24), (2.38) and their derivatives in the on-shell limit.

Let us for example consider Fig. 5 representing a possible contribution to the $\Gamma^{\alpha\beta}_{\langle i}^\mu$ factor appearing in (2.24). Potential collinear divergences arise when one of the external states $i$ or $j$ represent massless particle. Let the $i$-th particle be massless. Without loss of generality we can choose a frame in which this particle moves along $x^{D-1}$ so that the only nonzero component of momenta are $p^0_i$ and $p^{D-1}_i$. For any momentum $p$ we now define $p^\pm = p^0 \pm p^{D-1}$ and $\vec{p} = (p_1^1, \cdots p^{D-2})$ so that $p^2 = -p^+ p^- + \vec{p}^2$. In this language collinear region will correspond to region of loop momentum integration where

$$\ell^+ \sim 1, \quad \ell_\parallel \sim \lambda, \quad \ell^- \sim \lambda^2,$$  \hspace{1cm} (6.29)

for some small $\lambda$. Therefore the small denominator factors of the integrand in Fig. 5 take the form

$$I|_{\ell^i = 0} \sim (-\ell^+ \ell^- + \ell_\parallel^2 - i\epsilon)^{-1} \{-(p^+_i + \ell^+)\ell^- + \ell_\parallel^2 - i\epsilon\}^{-1}.$$  \hspace{1cm} (6.30)

The collinear region is $-p^+_i \leq \ell^+ \leq 0$ since using the $i\epsilon$ prescription one can easily verify that outside this region the $\ell^-$ integration contour can be deformed away from the singularities 69.
Note that we have not included the denominator factor of the line carrying momentum $p_j - \ell$ since this remains finite in the limit \(6.29\). In this limit expression \(6.30\) goes as $\lambda^{-4}$ for $p_i^2 = 0$, whereas the $d\ell d^{D-2}\ell_\perp$ goes as $\lambda^D$. Therefore for $D \geq 5$ there are no divergences.

However now consider taking derivatives with respect to $p_{\mu i}$ by first keeping $p_i$ off-shell and and setting $p_i^2 = 0$ after taking the derivative. We get

$$\frac{\partial I}{\partial p_{\mu i}} \bigg|_{p_i^2=0} \sim (-\ell^+ \ell^- + \ell_\perp^2 - i\epsilon)^{-1} \{-(p_i^+ + \ell^+)\ell^- + \ell_\perp^2 - i\epsilon\}^{-2}(-2)(p_i^\mu + \ell^\mu), \quad (6.31)$$

$$\frac{\partial^2 I}{\partial p_{\mu i} \partial p_{\nu j}} \bigg|_{p_i^2=0} \sim (-\ell^+ \ell^- + \ell_\perp^2 - i\epsilon)^{-1} \{-(p_i^+ + \ell^+)\ell^- + \ell_\perp^2 - i\epsilon\}^{-3}(8)(p_i^\mu + \ell^\mu)(p_j^\nu + \ell^\nu). \quad (6.32)$$

In order to analyze these let us define $\tilde{\ell}^\mu$ via

$$\ell^\mu = \frac{\ell^+}{p_i^\mu} p_i^\mu + \tilde{\ell}_i^\mu \quad (6.33)$$

$\tilde{\ell}^+$ vanishes, and we have $\tilde{\ell}^- = \ell^-$ and $\tilde{\ell}_\perp = \ell_\perp$. When we substitute \(6.33\) into \(6.31\), \(6.32\) the terms proportional to $p_i^\mu$ are divergent since we now have six factors of $\lambda$ in the denominator of \(6.31\) and eight factors of $\lambda$ in the denominator of \(6.32\). However our analysis of \(6.31\) shows that divergent terms proportional to $p_i^\mu$ or $p_i^\nu$ do not cause any problem. If we choose the terms proportional to $\tilde{\ell}_i^\mu$ and/or $\tilde{\ell}_j^\nu$ then for $\mu, \nu = -$ it is easy to see that the degrees of divergence of \(6.31\) and \(6.32\) remain the same as \(6.30\) and therefore there is no divergence. However there is a potential problem if we choose $\mu, \nu = \perp$ in \(6.32\) since now the integrand goes as $\lambda^{-6}$ and the integration measure goes as $\lambda^D$. Therefore the integral is divergent in $D = 5, 6$.

We must however remember that we also have to take into account possible numerator factors from the vertices. If the internal graviton with momentum $\ell$ had been a physical graviton then it would always carry polarization transverse to $\ell$. This would couple to momentum components of $p_i$ transverse to $\ell$, giving a result proportional to $\ell_\perp^2$ and killing the divergence for $D \geq 5$. This would be the case if we work in a physical gauge where only the transverse components of the graviton propagate.\footnote{There is no conflict between choosing a physical gauge for the internal graviton and a covariant gauge for the 1PI action. We can compute the 1PI action using physical gauge, then subtract the gauge fixing term to get the gauge invariant 1PI action and then gauge fix it using covariant gauge condition. We can follow the same procedure if the internal particle had been a massless vector particle instead of a graviton. In this case we would only get a single factor of $\ell_\perp$ from the vertex. Naive power counting then shows that \(6.32\) is logarithmically divergent for $D = 5$. However since the numerator will have three powers of $\ell_\perp$, the apparently divergent term would vanish by $\ell_\perp \to -\ell_\perp$ symmetry.}

Alternatively if we use de Donder gauge where lon-
Longitudinal modes of the graviton also propagate, then the divergent contributions will vanish after summing over different Feynman diagrams [70].

A similar analysis can be carried out for Fig. [5] to show that there is no collinear divergence in the derivatives of this up to the desired order.

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References

[1] S. Weinberg, “Photons and Gravitons in s Matrix Theory: Derivation of Charge Conservation and Equality of Gravitational and Inertial Mass,” Phys. Rev. 135, B1049 (1964). doi:10.1103/PhysRev.135.B1049

[2] S. Weinberg, “Infrared photons and gravitons,” Phys. Rev. 140, B516 (1965). doi:10.1103/PhysRev.140.B516

[3] D. J. Gross and R. Jackiw, “Low-Energy Theorem for Graviton Scattering,” Phys. Rev. 166, 1287 (1968). doi:10.1103/PhysRev.166.1287

[4] R. Jackiw, “Low-Energy Theorems for Massless Bosons: Photons and Gravitons,” Phys. Rev. 168, 1623 (1968). doi:10.1103/PhysRev.168.1623

[5] C. D. White, “Factorization Properties of Soft Graviton Amplitudes,” JHEP 1105, 060 (2011) doi:10.1007/JHEP05(2011)060 [arXiv:1103.2981 [hep-th]].

[6] F. Cachazo and A. Strominger, “Evidence for a New Soft Graviton Theorem,” arXiv:1404.4091 [hep-th].

[7] B. U. W. Schwab and A. Volovich, “Subleading Soft Theorem in Arbitrary Dimensions from Scattering Equations,” Phys. Rev. Lett. 113, no. 10, 101601 (2014) doi:10.1103/PhysRevLett.113.101601 [arXiv:1404.7749 [hep-th]].
[8] S. He, Y. t. Huang and C. Wen, “Loop Corrections to Soft Theorems in Gauge Theories and Gravity,” JHEP 1412, 115 (2014) doi:10.1007/JHEP12(2014)115 [arXiv:1405.1410 [hep-th]].

[9] A. J. Larkoski, “Conformal Invariance of the Subleading Soft Theorem in Gauge Theory,” Phys. Rev. D 90, no. 8, 087701 (2014) doi:10.1103/PhysRevD.90.087701 [arXiv:1405.2346 [hep-th]].

[10] F. Cachazo and E. Y. Yuan, “Are Soft Theorems Renormalized?,” [arXiv:1405.3413 [hep-th]].

[11] N. Afkhami-Jeddi, “Soft Graviton Theorem in Arbitrary Dimensions,” [arXiv:1405.3533 [hep-th]].

[12] J. Broedel, M. de Leeuw, J. Plefka and M. Rosso, “Constraining subleading soft gluon and graviton theorems,” Phys. Rev. D 90, no. 6, 065024 (2014) doi:10.1103/PhysRevD.90.065024 [arXiv:1406.6574 [hep-th]].

[13] Z. Bern, S. Davies, P. Di Vecchia and J. Nohle, “Low-Energy Behavior of Gluons and Gravitons from Gauge Invariance,” Phys. Rev. D 90, no. 8, 084035 (2014) doi:10.1103/PhysRevD.90.084035 [arXiv:1406.6987 [hep-th]].

[14] C. D. White, “Diagrammatic insights into next-to-soft corrections,” Phys. Lett. B 737, 216 (2014) doi:10.1016/j.physletb.2014.08.041 [arXiv:1406.7184 [hep-th]].

[15] M. Zlotnikov, “Sub-sub-leading soft-graviton theorem in arbitrary dimension,” JHEP 1410, 148 (2014) doi:10.1007/JHEP10(2014)148 [arXiv:1407.5936 [hep-th]].

[16] C. Kalousios and F. Rojas, “Next to subleading soft-graviton theorem in arbitrary dimensions,” JHEP 1501, 107 (2015) doi:10.1007/JHEP01(2015)107 [arXiv:1407.5982 [hep-th]].

[17] Y. J. Du, B. Feng, C. H. Fu and Y. Wang, “Note on Soft Graviton theorem by KLT Relation,” JHEP 1411, 090 (2014) doi:10.1007/JHEP11(2014)090 [arXiv:1408.4179 [hep-th]].

[18] D. Bonocore, E. Laenen, L. Magnea, L. Vernazza and C. D. White, “The method of regions and next-to-soft corrections in DrellYan production,” Phys. Lett. B 742, 375 (2015) doi:10.1016/j.physletb.2015.02.008 [arXiv:1410.6406 [hep-ph]].
[19] A. Sabio Vera and M. A. Vazquez-Mozo, “The Double Copy Structure of Soft Gravitons,” JHEP 1503, 070 (2015) doi:10.1007/JHEP03(2015)070 [arXiv:1412.3699 [hep-th]].

[20] F. Cachazo, S. He and E. Y. Yuan, “New Double Soft Emission Theorems,” Phys. Rev. D 92, no. 6, 065030 (2015) doi:10.1103/PhysRevD.92.065030 [arXiv:1503.04816 [hep-th]].

[21] A. E. Lipstein, “Soft Theorems from Conformal Field Theory,” JHEP 1506, 166 (2015) doi:10.1007/JHEP06(2015)166 [arXiv:1504.01364 [hep-th]].

[22] S. D. Alston, D. C. Dunbar and W. B. Perkins, “$n$-point amplitudes with a single negative-helicity graviton,” Phys. Rev. D 92, no. 6, 065024 (2015) doi:10.1103/PhysRevD.92.065024 [arXiv:1507.08882 [hep-th]].

[23] Y. t. Huang and C. Wen, “Soft theorems from anomalous symmetries,” JHEP 1512, 143 (2015) doi:10.1007/JHEP12(2015)143 [arXiv:1509.07840 [hep-th]].

[24] J. Rao and B. Feng, “Note on Identities Inspired by New Soft Theorems,” JHEP 1604, 173 (2016) doi:10.1007/JHEP04(2016)173 [arXiv:1604.00650 [hep-th]].

[25] F. Cachazo, P. Cha and S. Mizera, “Extensions of Theories from Soft Limits,” JHEP 1606, 170 (2016) doi:10.1007/JHEP06(2016)170 [arXiv:1604.03893 [hep-th]].

[26] A. P. Saha, “Double Soft Theorem for Perturbative Gravity,” JHEP 1609, 165 (2016) doi:10.1007/JHEP09(2016)165 [arXiv:1607.02700 [hep-th]].

[27] A. Luna, S. Melville, S. G. Naculich and C. D. White, “Next-to-soft corrections to high energy scattering in QCD and gravity,” JHEP 1701, 052 (2017) doi:10.1007/JHEP01(2017)052 [arXiv:1611.02172 [hep-th]].

[28] C. Cheung, K. Kampf, J. Novotny, C. H. Shen and J. Trnka, “A Periodic Table of Effective Field Theories,” [arXiv:1611.03137] [hep-th].

[29] A. P. Saha, “Double Soft Theorem for Perturbative Gravity II: Some Details on CHY Soft Limits,” [arXiv:1702.02350] [hep-th].

[30] A. Strominger, “On BMS Invariance of Gravitational Scattering,” JHEP 1407, 152 (2014) doi:10.1007/JHEP07(2014)152 [arXiv:1312.2229 [hep-th]].
[31] T. He, V. Lysov, P. Mitra and A. Strominger, “BMS supertranslations and Weinberg's soft graviton theorem,” JHEP 1505, 151 (2015) doi:10.1007/JHEP05(2015)151 [arXiv:1401.7026 [hep-th]].

[32] A. Strominger and A. Zhiboedov, “Gravitational Memory, BMS Supertranslations and Soft Theorems,” JHEP 1601, 086 (2016) doi:10.1007/JHEP01(2016)086 [arXiv:1411.5745 [hep-th]].

[33] S. G. Avery and B. U. W. Schwab, “Burg-Metzner-Sachs symmetry, string theory, and soft theorems,” Phys. Rev. D 93, 026003 (2016) doi:10.1103/PhysRevD.93.026003 [arXiv:1506.05789 [hep-th]].

[34] M. Campiglia and A. Laddha, “Asymptotic symmetries of gravity and soft theorems for massive particles,” JHEP 1512, 094 (2015) doi:10.1007/JHEP12(2015)094 [arXiv:1509.01406 [hep-th]].

[35] M. Campiglia and A. Laddha, “Sub-subleading soft gravitons: New symmetries of quantum gravity?” Phys. Lett. B 764, 218 (2017) doi:10.1016/j.physletb.2016.11.046 [arXiv:1605.09094 [gr-qc]].

[36] M. Campiglia and A. Laddha, “Sub-subleading soft gravitons and large diffeomorphisms,” JHEP 1701, 036 (2017) doi:10.1007/JHEP01(2017)036 [arXiv:1608.00685 [gr-qc]].

[37] E. Conde and P. Mao, “BMS Supertranslations and Not So Soft Gravitons,” [arXiv:1612.08294 [hep-th]].

[38] T. He, D. Kapec, A. M. Raclariu and A. Strominger, “Loop-Corrected Virasoro Symmetry of 4D Quantum Gravity,” [arXiv:1701.00496 [hep-th]].

[39] M. Asorey, A. P. Balachandran, F. Lizzi and G. Marmo, “Equations of Motion as Constraints: Superselection Rules, Ward Identities,” [arXiv:1612.05886 [hep-th]].

[40] A. Strominger, “Lectures on the Infrared Structure of Gravity and Gauge Theory,” [arXiv:1703.05448 [hep-th]].

[41] M. Ademollo, A. D’Adda, R. D’Auria, F. Gliozzi, E. Napolitano, S. Sciuto and P. Di Vecchia, “Soft Dilations and Scale Renormalization in Dual Theories,” Nucl. Phys. B 94, 221 (1975). doi:10.1016/0550-3213(75)90491-5
[42] J. A. Shapiro, “On the Renormalization of Dual Models,” Phys. Rev. D 11, 2937 (1975). doi:10.1103/PhysRevD.11.2937

[43] B. U. W. Schwab, “Subleading Soft Factor for String Disk Amplitudes,” JHEP 1408, 062 (2014) doi:10.1007/JHEP08(2014)062 [arXiv:1406.4172 [hep-th]].

[44] M. Bianchi, S. He, Y. t. Huang and C. Wen, “More on Soft Theorems: Trees, Loops and Strings,” Phys. Rev. D 92, no. 6, 065022 (2015) doi:10.1103/PhysRevD.92.065022 [arXiv:1406.5155 [hep-th]].

[45] B. U. W. Schwab, “A Note on Soft Factors for Closed String Scattering,” JHEP 1503, 140 (2015) doi:10.1007/JHEP03(2015)140 [arXiv:1411.6661 [hep-th]].

[46] P. Di Vecchia, R. Marotta and M. Mojaza, “Soft theorem for the graviton, dilaton and the Kalb-Ramond field in the bosonic string,” JHEP 1505, 137 (2015) doi:10.1007/JHEP05(2015)137 [arXiv:1502.05258 [hep-th]].

[47] M. Bianchi and A. L. Guerrieri, “On the soft limit of open string disk amplitudes with massive states,” JHEP 1509, 164 (2015) doi:10.1007/JHEP09(2015)164 [arXiv:1505.05854 [hep-th]].

[48] A. L. Guerrieri, “Soft behavior of string amplitudes with external massive states,” Nuovo Cim. C 39, no. 1, 221 (2016) doi:10.1393/ncc/i2016-16221-2 [arXiv:1507.08829 [hep-th]].

[49] P. Di Vecchia, R. Marotta and M. Mojaza, “Soft Theorems from String Theory,” Fortsch. Phys. 64, 389 (2016) doi:10.1002/prop.201500068 [arXiv:1511.04921 [hep-th]].

[50] M. Bianchi and A. L. Guerrieri, “On the soft limit of closed string amplitudes with massive states,” Nucl. Phys. B 905, 188 (2016) doi:10.1016/j.nuclphysb.2016.02.005 [arXiv:1512.00803 [hep-th]].

[51] M. Bianchi and A. L. Guerrieri, “On the soft limit of tree-level string amplitudes,” arXiv:1601.03457 [hep-th].

[52] P. Di Vecchia, R. Marotta and M. Mojaza, “Subsubleading soft theorems of gravitons and dilatons in the bosonic string,” JHEP 1606, 054 (2016) doi:10.1007/JHEP06(2016)054 [arXiv:1604.03355 [hep-th]].
[53] P. Di Vecchia, R. Marotta and M. Mojaza, “Soft behavior of a closed massless state in superstring and universality in the soft behavior of the dilaton,” JHEP 1612, 020 (2016) doi:10.1007/JHEP12(2016)020 [arXiv:1610.03481 [hep-th]].

[54] A. Sen, “Soft Theorems in Superstring Theory,” arXiv:1702.03934 [hep-th].

[55] A. Sen, “Subleading Soft Graviton Theorem for Loop Amplitudes,” arXiv:1703.00024 [hep-th].

[56] Z. Bern, S. Davies and J. Nohle, “On Loop Corrections to Subleading Soft Behavior of Gluons and Gravitons,” Phys. Rev. D 90, no. 8, 085015 (2014) doi:10.1103/PhysRevD.90.085015 [arXiv:1405.1015 [hep-th]].

[57] H. Elvang, C. R. T. Jones and S. G. Naculich, “Soft Photon and Graviton Theorems in Effective Field Theory,” arXiv:1611.07534 [hep-th].

[58] M. Campiglia and A. Laddha, “Subleading soft photons and large gauge transformations,” JHEP 1611, 012 (2016) doi:10.1007/JHEP11(2016)012 [arXiv:1605.09677 [hep-th]].

[59] R. M. Wald, “General Relativity,” Chicago, Usa: Univ. Pr. (1984) 491p doi:10.7208/chicago/9780226870373.001.0001

[60] S. Y. Choi, J. S. Shim and H. S. Song, “Factorization and polarization in linearized gravity,” Phys. Rev. D 51, 2751 (1995) doi:10.1103/PhysRevD.51.2751 [hep-ph/9411092].

[61] H. Elvang and Y. t. Huang, “Scattering Amplitudes,” arXiv:1308.1697 [hep-th].

[62] L. J. Dixon, “A brief introduction to modern amplitude methods,” doi:10.5170/CERN-2014-008.31 [arXiv:1310.5353 [hep-ph]].

[63] T. Kinoshita, “Mass singularities of Feynman amplitudes,” J. Math. Phys. 3, 650 (1962). doi:10.1063/1.1724268

[64] T. D. Lee and M. Nauenberg, “Degenerate Systems and Mass Singularities,” Phys. Rev. 133, B1549 (1964). doi:10.1103/PhysRev.133.B1549

[65] F. Bloch and A. Nordsieck, “Note on the Radiation Field of the electron,” Phys. Rev. 52, 54 (1937). doi:10.1103/PhysRev.52.54
[66] P. P. Kulish and L. D. Faddeev, “Asymptotic conditions and infrared divergences in quantum electrodynamics,” Theor. Math. Phys. 4, 745 (1970) [Teor. Mat. Fiz. 4, 153 (1970)]. doi:10.1007/BF01066485

[67] J. Ware, R. Saotome and R. Akhoury, “Construction of an asymptotic S matrix for perturbative quantum gravity,” JHEP 1310, 159 (2013) doi:10.1007/JHEP10(2013)159 arXiv:1308.6285 [hep-th]].

[68] D. Kapec, M. Perry, A. M. Raclariu and A. Strominger, “Infrared Divergences in QED, Revisited,” arXiv:1705.04311 [hep-th].

[69] G. F. Sterman, “An Introduction to quantum field theory,” Cambridge University Press (1993).

[70] R. Akhoury, R. Saotome and G. Sterman, “Collinear and Soft Divergences in Perturbative Quantum Gravity,” Phys. Rev. D 84, 104040 (2011) doi:10.1103/PhysRevD.84.104040 arXiv:1109.0270 [hep-th]].