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New Solution for the Earth’s Free Wobble and Its Geophysical Implications

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Abstract  In this paper, the theory of the free wobble of the triaxial Earth is developed and new conclusions are drawn: the Euler period should actually be expressed by the first kind of complete elliptic integral; the trace of the free polar motion is elliptic and the orientations of its semi-minor and major axes are approximately parallel to the Earth’s principal axes $A$ and $B$, respectively. In addition, the present theory shows that there is a mechanism of frequency-amplitude modulation in the Chandler wobble, which might be a candidate for explaining the correlation between the amplitude and period of the Chandler wobble.

Keywords triaxial Earth; elliptic free polar motion; variation of Chandler period; frequency-amplitude modulation

CLC number P223

Introduction

The Earth rotates in a way that depends on its shape, its internal structure, and its initial rotating state (neglecting the external torques). Therefore, the dynamic shape of the Earth (biaxial or triaxial) might play a significant role in understanding and modeling the rotation of the Earth.

Conventionally, the Earth’s equatorial principal moments of inertia $A$ and $B$ are assumed to be equivalent to simplify the Euler’s dynamic equations, otherwise no analytical solutions to the Euler’s dynamic equations would be found[1-3]. However, the recent measurements have reported that, all the Earth’s principal moments of inertia, $A$, $B$ and $C$, are different from each other[4-10]. The hypothesis, $A = B$, although simplifying the solutions to the traditional analytical ones[1-3], makes the actual rotation state of the Earth theoretically unknown (but we can obtain it at a certain accuracy by various kinds of observations).

On the basis of the temporal gravity models, Shen et al. (2008)[8], Shen et al. (2008)[9] and Chen et al. (2009)[10] deduced the temporal inertia moment tensor of the Earth as well as the secular and high frequency
variations of length of day. However, these studies had adopted linearization when solving the Euler-Livioulle equations. Shen et al. (2007)[11] did an elementary study on the free Euler motion of a triaxial rigid Earth without any linear approximation, and found that the triaxial nature could give rise to a small fluctuation in the length of day (LOD), which had not been predicted before. This study aims to develop the theory of Shen et al. (2007) to obtain new solutions to the Euler’s dynamic equations as well as a new expression for the Euler period. The trace of free polar motion is found to be an ellipse with the following features: (1) The length of its semi-major axis is around 0.67 mas larger than that of the semi-minor axis. (2) The orientations of its semi-major and minor axes are related to the orientations of the Earth’s principal axes A and B. (3) There is a mechanism of frequency-amplitude modulation in the Chandler wobble when the Earth’s triaxiality is concerned.

1 Solutions to Euler’s dynamic equations for the triaxial rigid Earth

Adopting the Earth’s principal axial coordinate $o_i-x_i,y_i,z_i$ whose $o x_i, o y_i, o z_i$ axes are aligned with the (rigid) Earth’s principal axes $A, B$ and $C$, respectively, one gets the well-known Euler’s dynamic equations\[1-3\]

\[
\begin{align*}
A\dot{\omega}_1 - (B-C)\omega_2\omega_3 &= L_1 \\
B\dot{\omega}_2 - (C-A)\omega_3\omega_1 &= L_2 \\
C\dot{\omega}_3 - (A-B)\omega_1\omega_2 &= L_3
\end{align*}
\]

where $\omega$ and $L$ are the angular velocity vector and external torque, with $\omega_i$ and $L_i$ ($i = 1, 2, 3$) denoting their three components respectively. Since only the free rotation of the rigid Earth is concerned, the external torque $L=0$. Moreover, setting $\alpha = (C-B)/A$, $\beta = (C-A)/B$, $\gamma = (B-A)/C$, Eq.(1) can be equivalently expressed as

\[
\begin{align*}
\dot{\omega}_1 + \alpha \omega_3\omega_2 &= 0 \\
\dot{\omega}_2 - \beta \omega_1\omega_3 &= 0 \\
\dot{\omega}_3 + \gamma \omega_2\omega_1 &= 0
\end{align*}
\]

From Eqs.(2) to (4), one gets the following relations:

\[
\begin{align*}
\beta \omega_3^2 + \alpha \omega_2^2 &= C_{12} \\
\gamma \omega_2^2 + \beta \omega_1^2 &= C_{23} \\
\alpha \omega_1^2 - \gamma \omega_3^2 &= C_{31}
\end{align*}
\]

where $C_i$ ($i, j = 1, 2, 3; i \neq j$) are constants to be determined. According to Eq.(5), the polar motion should not be circular but elliptic, say, $m_i^2/a^2 + m_j^2/b^2 = 1$ ($a/b = \sqrt{\alpha/\beta} = 0.996663$, and $\sqrt{ab} = m_{\delta}$, where $m_{\delta} = 200$ mas is the mean amplitude of the free polar motion. Thus, one could obtain that $a = (\alpha/\beta)^{4/3} m_{\delta}$, $b = (\beta/\alpha)^{4/3} m_{\delta}$. Here $m = m_1 + im_2 = (\omega_1 + i\omega_3)/\Omega$ is the complex coordinate of the rotational pole, where $\Omega$ is the mean rotation rate of the Earth, and $|m|$ is the amplitude of the free polar motion ($a \leq |m| \leq b$).

Without loss of generality, here we assume at the initial time, $\omega$ locates in the $o-xz$ plane, and $\omega_i(0) = \omega_i$, then $|m(0)| = a = (\alpha/\beta)^{4/3} m_{\delta}$, $\omega_i(0) = 0$, $\omega_i(0) = a\alpha i\omega_3(0)$.

On the basis of Eqs.(5) to (7), one gets

\[
\begin{align*}
C_{12} &= \beta \omega_3^2(0) = \beta \Omega^2a^2 \\
C_{23} &= \beta \omega_3^2(0) = \beta \Omega^2 \\
C_{31} &= \alpha \omega_1^2(0) - \gamma \omega_3^2(0) = \alpha \omega_1^2 - \gamma \Omega^2a^2
\end{align*}
\]

From Eqs.(5) and (6), expressing $\omega_1$ and $\omega_3$ as the functions of $\omega_2$, and substituting them into Eq.(3), one gets

\[
\frac{d\omega_i}{dt} - \sqrt{(C_{23} - \gamma \omega_2^2)(C_{12} - \alpha \omega_2^2)} = 0
\]

From the above differential equation, one gets

\[
\int_{\omega_i(0)}^{\omega_i(t)} \frac{d\omega_i}{\sqrt{(C_{23} - \gamma \omega_2^2)(C_{12} - \alpha \omega_2^2)}} = t
\]

Setting

\[
m = \frac{\gamma C_{12}}{\alpha C_{23}} \in (0,1), \quad \varphi = \arcsin \left( \frac{\alpha}{\sqrt{C_{12}}} \omega_2 \right)
\]

then one gets

\[
\frac{1}{\sqrt{\alpha C_{23}}} \int_{\varphi(0)}^{\varphi(t)} \frac{d\varphi}{\sqrt{1 - m \sin^2 \varphi}} = t, \quad 0 < m < 1
\]

where $\varphi(0) = 0$, due to our choice that $\omega_2(0) = 0$.

Eqs.(1) to (12) is a brief review of the theory of Shen et al. (2007)[11] (correction: Eq.(19) in the Reference [11] should be replaced by Eq.(12) of this study). Eq.(12) can be re-expressed as

\[
F = \int_{0}^{\varphi} \frac{d\varphi}{\sqrt{1 - m \sin^2 \varphi}} = t\sqrt{\alpha C_{23}}
\]

where $F$ is the non-complete elliptic integral of the first kind[12]. Given $m$ and $F$, we can obtain the ellip-
tic-function solutions \( sn( t\sqrt{\alpha C_{23}} ) \) and \( cn( t\sqrt{\alpha C_{23}} ) \), both of which have the same period \( 4K(m) \). Here,
\[
K(m) = \int_0^\pi \frac{d\varphi}{\sqrt{1 - m \sin^2 \varphi}}
\]
(14)
is the complete elliptic integral of the first kind\([12]\). Then, \( sn( t\alpha \) and \( cn( t\alpha \), both of which have the same period \( 4K(m) \).

Here,
\[
\int_0^\pi \frac{d\varphi}{\sqrt{1 - m \sin^2 \varphi}} = \frac{\gamma}{\alpha a^2} = \frac{\gamma}{\sqrt{\alpha \beta}} m_0^2
\]
(16)

Substituting Eq.(15) into Eqs.(5) and (6), one has
\[
\omega_1 = \sqrt{\frac{C_{12}}{\beta}} \cos \varphi = \sqrt{\frac{C_{12}}{\beta}} \sin(t) = \Omega a \sin(t)
\]
(17)
\[
\omega_2 = \sqrt{\frac{C_{23} - \rho \alpha^2}{\beta}} = \Omega \sqrt{1 - m \sin^2(t)} = \Omega \sin(t)
\]
(18)

Eqs.(15) to (18) are just the solutions we research for the rotation of the triaxial rigid Earth.

2 Euler wobble of the triaxial rigid Earth

The main results obtained in the last section can be summarized as follows.

First, in the Earth’s principal axial coordinate system, the Earth’s angular velocity could be written as
\[
\begin{align*}
\omega_1 &= \Omega a \sin(u) \\
\omega_2 &= \Omega b \sin(u) \\
\omega_3 &= \Omega \sin(u) - 1
\end{align*}
\]
(19)
where \( a/b = \sqrt{\alpha/\beta} \), \( \omega_1 = \Omega(1 + m_1) \), \( cn \), \( sn \) and \( dn \) are the Jacobian elliptic functions, and
\[
u = \frac{1}{\Omega \sqrt{\alpha \beta}} \int_0^\varphi \frac{d\varphi}{\sqrt{1 - m \sin^2 \varphi}} = \frac{\gamma}{\sqrt{\alpha \beta}} m_0^2,
\]
(20)

Second, \( \omega_1 \) and \( \omega_2 \) have a common period, namely the Euler period
\[
T_E = \frac{4}{\Omega \sqrt{\alpha \beta}} \int_0^\pi \frac{d\varphi}{\sqrt{1 - m \sin^2 \varphi}}, \quad m = \frac{\gamma}{\sqrt{\alpha \beta}} m_0^2
\]
(21)

Taking into account that \( (cn, sn) \) tend to \( (cos, sin) \) when \( m \) tends to zero (Abramowitz and Stegun, 1964) and noting that \( m = 3.941901 \times 10^{-14} \) in the present case, \( (cn, sn) \) and \( (cos, sin) \) can be regarded as equivalent respectively in practical cases, and then Eq.(19) will reduce to
\[
\begin{align*}
m_1 &= a \cos \sigma \xi t \\
m_2 &= b \sin \sigma \xi t \\
m_3 &= \sqrt{1 - m \sin^2 \sigma \xi t} - 1 = \sqrt{1 + m/2 (\cos 2 \sigma \xi t - 1)} - 1
\end{align*}
\]
(22)

where \( \sigma \xi = 2\pi/T_E \) is the Euler frequency (one must keep in mind that \( T_E \) is newly defined by Eq.(21)).

Folgueira and Souchay (2005)\([13]\) discussed the free polar motion of the triaxial and elastic Earth in Hamiltonian formalism, and found both the longitude and latitude of the pole oscillate with the semi-Chandler period. Eqs.(5), (19) and (22) strongly suggest that, for the real Earth with \( A < B < C \), the trace of free polar motion is no longer a circle but an ellipse, of which the length of the semi-minor axis is \( \sqrt{\alpha/\beta} = 0.996663 \) times of that of the semi-major axis. The elliptic polar motion obviously leads to a fluctuation of the latitude of the pole with the semi-Euler period while the circular one only leads to an Euler-period fluctuation. This result coincides well with Reference [13], if we note that the Euler period for the rigid Earth changes to the Chandler period for the real Earth.

Last, \( \omega_3 \) fluctuates with the semi-Euler period \( T_E/2 \), just as explained by the last equation of Eq.(22). That is to say, the triaxiality would give rise to a semi-Euler period fluctuation of the length of day (LOD).

Table 1 Values of the relevant parameters

| Parameters | Values |
|------------|--------|
| \( \alpha \) | \( \pm 6 \times 10^{-8} \) |
| \( \beta \) | \( \pm 6 \times 10^{-8} \) |
| \( \gamma \) | \( \pm 6 \times 10^{-8} \) |
| \( \Omega \) | 7.922115 \times 10^{-7} \text{ rad/s} |
| \( m_0 \) | 200 mas |
Van Hoolst and Dehant (2002)\cite{14} pointed out that the triaxiality of the Earth (i.e., $A \neq B \neq C$) could reduce the values of the Chandler and FCN (Free Core Nutation) frequencies. By using Eqs.(19) to (21) and the parameters listed in Table 1, we can obtain the Euler period $T_E = 304.461118$ sidereal days, as well as $m_i (i = 1, 2, \omega)$ and the amplitude of free wobble, $\theta$ (see Fig.1). One could note that the triaxiality could lengthen the Euler period by about 0.0017 sidereal day, or 146.606144 seconds (see the following text for the Euler period of the biaxial Earth), which is in agreement with Reference [14]. The amplitude of $m_2$ is about 0.67 mas larger than that of $m_1$ and this difference could be detected by the very long baseline interferometry (VLBI), which can reach an accuracy better than 0.1 mas at present\cite{15}. However, $m_3$ is on the order of $10^{-15}$ which equals to a variation of $10^{-10}$ s in the LOD, and it is 4 to 5 orders of magnitude smaller than present measuring accuracy\cite{15}. In other words, the variation of $\omega_3$ is so small that we can assume it is time independent for safety.

Fig.1 Parameters of the Euler wobble in the Earth’s principal axial coordinate system

Conventionally, there are two categories of solutions to Euler’s dynamic equations: one assumes $A = B$ and is most frequently adopted by geophysical scientists; the other, assuming $A \neq B$, is usually studied by scientists of theoretical physics.

If $A = B$ is set, then $a = b$ and Eq.(22) will reduce to the solutions to the conventional case. Obviously, $A = B$ leads to a round free wobble of the Earth. In this case, the Euler frequency and period are respectively defined by $\sigma_E^0 = \frac{C - A}{A} \Omega = \frac{2 \alpha + \gamma - \alpha' \gamma}{2 - \gamma + \alpha' \gamma} \Omega$ (here $\bar{A} = A + B$) and $T_E^0 = \frac{2 \pi}{\sigma_E^0}$. According to these definitions, one can obtain that $T_E^0 = 304.459417$ sidereal days, which is a little shorter than the period corresponding to the triaxial case.

Some famous physical scientists have developed the elliptic function-solutions for the Euler's dynamic equations. They considered the conservations of energy $E$ and angular momentum $M$ (with $M_i (i = 1, 2, 3)$ as its three components), and put forward that\cite{2,3}

$$\begin{align*}
A \omega_1^2 + B \omega_2^2 + C \omega_3^2 &= 2E \\
A' \omega_1^2 + B' \omega_2^2 + C' \omega_3^2 &= M^2
\end{align*}$$

(23)

Then from Eqs.(1) and (23), they obtained

$$\begin{align*}
\omega_1 &= \frac{2EC - M^2}{\sqrt{A(C - A)}} \text{cn}(\tau) \\
\omega_2 &= \frac{2EC - M^2}{\sqrt{B(C - B)}} \text{sn}(\tau) \\
\omega_3 &= \frac{M^2 - 2EA}{\sqrt{C(C - A)}} \text{dn}(\tau)
\end{align*}$$

(24)

where

$$\begin{align*}
\tau &= \int_0^s \frac{ds}{\sqrt{(1-s^2)(1-ms^2)}} \\
\text{sn}(\tau) &= s \\
\text{cn}(\tau) &= \sqrt{1 - \text{sn}^2(\tau)} \\
\text{dn}(\tau) &= \sqrt{1 - m \text{sn}^2(\tau)}
\end{align*}$$

(25)

Although Eq.(24) is somewhat similar to Eq.(19),
and also leads to the conclusion that $\omega_1/\omega_2 = \sqrt{\alpha/\beta}$ as we have obtained in this study, two critical differences between them should be addressed.

First, we have not assumed the conservations of energy $E$ and angular momentum $M$, which hamper us to extend the solutions to the nonrigid Earth. Especially, the conservation of energy would not hold due to the coupling dissipation of the internal Earth even when the external torques are ignored. Thus, our solutions might be applied to the real Earth (e.g., Section 3) but further investigations are needed.

Second, we have chosen polar motion data but not $E$ or $M$ as the parameter of the elliptic integral. It is obvious that polar motion data could be more easily and precisely measured.

Thus, our method seems to be more adoptable relying on the above two advantages. Here we address that, the new Euler period should be expressed by Eq.(21) and the free wobble should be described by Eq.(19), to which Eq.(22) is only an approximation.

### 3 Frequency-amplitude modulation of Chandler wobble

For the real Earth, the Euler frequency $\sigma_c$ should be replaced by the Chandler one $\sigma_C$, and Eq.(22) should be changed into

$$\begin{align*}
    m_1 &= a \cos \sigma_c t, \\
    m_2 &= b \sin \sigma_c t,
\end{align*}$$

Noting $m_0 = \sqrt{ab}$, thus the Chandler period is$^{[16]}$

$$T_C = \frac{2\pi}{\sigma_C} = \frac{T_E}{0.7} = \frac{40}{7\Omega \sqrt{ab}} \int_0^{\pi/2} \frac{\pi}{\sqrt{1 - m \sin^2 \phi}} d\phi,$$

$$m = \frac{r}{\sqrt{ab}} m_0^2$$

Eq.(27) shows that $T_C = T_C(m_0)$, or inversely, $m_0 = m_0(\sigma_C)$.

Long term observations show that the Chandler period, $T_C$, and the amplitude of Chandler wobble, $\theta$, are time dependent and might be positively correlated with each other. Chandler (1891)$^{[17]}$ had first suggested the possible existence of this phenomenon. Iijima (1965)$^{[18]}$ analyzed the ILS data for the period 1900.0-1963.2 with a 0.1-year sampling, and found that the Chandler period varies from about 1.1 to 1.2 years and that the smaller period happens when the Chandler component has a smaller amplitude and vice versa. Carter (1981)$^{[19]}$ obtained the Chandler period with a variation of 10 days within three years based on the analysis of the polar motion data. Gao (1997)$^{[20]}$ concluded that $T_C$ might have a 10-day fluctuation in correlation with $\theta$ during the last several decades. Höffner (2003)$^{[21]}$ found that the Chandler wobble owns a period variation between 422 and 438 days with an estimated standard deviation of only 0.48 days, while its amplitude varies from 0.15 to 0.20 mas with a time dependence similar to the period.

From the traditional theory, Jochmann (2003)$^{[22]}$ deduced that the mass redistribution does contribute to the variation of the Chandler period, but he found its effect is too small to excite the period variations observed. Thus, the mechanism of the correlation between the amplitude and period of the Chandler wobble remains unexplained.

The present study finds that the triaxial nature of the Earth can partly explain the positive correlation between $T_C$ and $m_0$. By examining Eqs.(26) and (27) (or Eq.(21) and (22)), one could find a frequency-amplitude modulation mechanism in the Chandler wobble if we regard $a$ and $b$ as the instantaneous semi-minor and major axes, respectively, and notes that $m_0(t) = \sqrt{a(t)b(t)}$ will give rise to the variation of $T_C$ or $\sigma_C$ (variations of $a$ and $b$ correspond to the amplitude modulation while the variation of $\sigma_C$ corresponds to the frequency modulation). Since the range of variation of $m_0$ is around 100 to 300 mas, $m$ varies in an interval around $(1 \sim 9) \times 10^{-14}$, which ensures Eqs.(21) and (27) being the increasing functions of $m$. Thus, a theoretical model for the positive correlation between $T_C$ and $m_0$ is established. Chen et al. (2009)$^{[16]}$ had adopted the data EOP (IERS) C04, which contains Earth orientation parameters (EOP) ranging from Jan. 1962 to Aug. 2007, to check the theory proposed by the present study. They found that the predicted Chandler wobble coincides well with the observations and $m_0$ (denoted by $\theta$ ) and $T_C$ fluctuate synchronously (see Fig.2 cited from [16]), which is the key clue of the mechanism of frequency-amplitude modulation. Thus, the theory of the present study is rather reliable$^{[16]}$. 

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**References**

- Carter (1981)
- Gao (1997)
- Höffner (2003)
- Jochmann (2003)
- Chen et al. (2009)
4 Conclusion

The present study shows that the conventional treatment, i.e., setting $A = B$, will inevitably bring a misunderstanding (i.e., the trace of the pole is round) and a discrepancy of about 0.67 mas in the free polar motion. Further, the hypothesis $A = B$ ensnrods the mechanism of the frequency-amplitude modulation of the Chandler wobble, which is the nature of a rotating triaxial body and has been long observed. Here we develop a new form of elliptic integral as the solution to Euler’s dynamic equations, and give new expressions of the Euler and Chandler periods as well as the free motional trace of the pole, and obtain a theoretical model for the frequency-amplitude modulation of the free wobble which might be a candidate to explain the correlation between the amplitude and period of the Chandler wobble. Obviously, further investigations, taking into account the viscoelasticity of the mantle, the role of the ocean and the Earth’s core[23], are needed.

The triaxial nature of the Earth prolongs the Euler period by about 0.0017 sidereal day, forces the trace of the free polar motion into an ellipse with an ellipticity of about $1/298.67 = (b-a)/b$, leads to tiny fluctuations in $\omega_3$ (with an amplitude about $10^{-10}$ rad/s in one Chandler period) and LOD (with an amplitude about $10^{-10}$ s in one Chandler period), and gives rise to the frequency-amplitude modulation of the Chandler wobble. Thus, presently, we should address the Earth’s triaxial nature in the field of polar motion (including the Chandler wobble), but neglect its impact on $\omega_3$ or LOD. One should note that this conclusion might be only valid to the case of free wobble, and the variation of $\omega_3$ of the triaxial Earth might be substantially important if the external torques are taken into account. On the basis of the above reasons[11,13,14,16], it can be concluded that the hypothesis $A = B$ ensnrods many important aspects in the Earth’s rotation and we should be careful in treating the Earth as a biaxial body considering the rapid development of our measurement technologies.

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