New geometric and field theoretic aspects of a radiation dominated universe II. Fundamental Cosmological Observers (FCOs)

Sujoy K. Modak$^{1,2,3,*}$

$^1$Facultad de Ciencias - CUICBAS, Universidad de Colima, Colima, C.P. 28045, México
$^2$KEK Theory Center, High Energy Accelerator Research Organization (KEK), Tsukuba, Ibaraki 305-0801, Japan
$^3$Physics Department, California State University, Fresno, CA 93740-8031, USA

We further extend our work [4] to give more insights on the radiation dominated stage of the early universe within the context of the quantum field theory in curved spacetime. Specifically, we focus on the fundamental cosmological observers (FCOs) who have a special status in the new spacetime metric introduced in [4]. These observers are the only observers who are accelerating radially and do not see an event horizon due to a coordinate singularity. Closer analysis, using the new coordinates, leads us to the definition of a new vacuum state and a compulsory gravitational particle creation for the fundamental cosmological observers. We further calculate the renormalized energy momentum tensors, both in FCO’s and comoving observer’s frame, and discuss their importance. Finally, we make a side by side comparison of this particle creation phenomena with the well known Unruh effect.

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I. INTRODUCTION

In general relativity, although coordinate systems and observers, by themselves, do not have a role to dictate fundamental physical laws (which should be covariantly defined), the choice of one frame over the other often helps to gain important physical insights. We see this happening in black holes where one has options to chose coordinates so that the coordinate singularity at the event horizon may appear (e.g., Schwarzschild coordinates) or may not appear (e.g., Kruskal coordinates). Also, if we want to single out a physical observer (like the asymptotic observer in Schwarzschild spacetime) we need to select a coordinate system suitable to that particular observer (i.e., Schwarzschild coordinates) in order to discuss relevant physics. This, to some extent is also true even in a flat spacetime where an observer with constant four acceleration encounters a horizon in Minkowski spacetime (e.g. Rindler observer) just because Rindler coordinates only cover one-fourth of the Minkowski spacetime. Physics become even more interesting by including quantum fields (even non-interacting case) into account which then introduces particle creation due to gravitational field (e.g., Hawking effect [1, 2]), due to the observer’s own motion (e.g., Unruh effect [3]) etc.

In a recent work [4] we found new coordinates (we shall refer to them as $(T, R)$ coordinates just for the sake of clarity of presentation) to describe the radiation dominated stage of the early universe by making a conformal transformation of the cosmological FRW coordinates. The use of $(T, R)$ coordinates express the radiation dominated universe, following the inflationary stage, as a spherically symmetric, inhomogeneous spacetime which then offers a new, unitarily inequivalent field quantization for massless scalar fields. We also discussed several physical aspects related with the static and non-static observers in this spacetime. This discovery was followed by a systematic discussion of particle creation phenomena with respect to the static observers in this new spacetime who finds the fundamental cosmological vacuum state as containing particles. However, we completely overlooked another side of the picture where the fundamental cosmological observers (hereafter FCOs) would also find the vacuum state corresponding to the newly found static
observers (hence-after \( T\)-vacuum state) to be full of particles. This aspect is even more important, because, our understanding of the cosmos, in many aspects, is based on the observations with respect to the FCOs. We can get their view of the universe by subtracting all relative motions of various structures in the cosmos and therefore this phenomena should be of our interest. Therefore, in principle, this particle creation process is of considerable interest. This paper will be a first step towards understanding this particle creation process and related issues within the framework of quantum field theory in curved space and applicable to a two dimensional set up. Further studies are required in four dimensions to comment on the impact of this particle creation process (either direct or indirect) in observational cosmology.

Specifically, in this paper we discuss physics associated with the FCOs which naturally leads us to a new example of gravitational particle creation process with a potential observational relevance. We start by studying various observer’s trajectory and highlight subtleties regarding the appearance and non-appearance of horizons for various observers in this new spacetime. Our main focus is on the FCOs who fall in a sub-class of observers have acceleration with respect to the \((T, R)\) static frame and do not encounter any horizon. Which is of course expected from the physical point of view. We then show, by staying within a toy model of two dimensional set up, that FCOs are exposed to the effect of gravitational particle creation due to the existence of a newly defined vacuum state (referred here as the \(T\)-vacuum) in this study. The resulting behaviour of the particle excitation number density is shown graphically. We also compute the components of renormalized energy-momentum tensor (REMT) in FCOs frame and using the \(T\)-vacuum and show none of the components have an unwanted behaviour such as divergence at any point on the spacetime. In fact, due to the covariance property of the REMT we are able to compute the same in the frame of comoving observers\(^1\). Finally, we make a thorough comparison of our study with the accelerated motion in Minkowski spacetime and cooresponding particle creation phenomena (the Unruh effect).

The organization of the paper is as follows. In section II we give a short summary of mathematics that leads us to the \((T, R)\) form of the radiation dominated universe. In section III we discuss the observer dependent horizons and the special status enjoyed by the FCOs. We also build a platform to discuss quantum field theory in curved space by foliating the spacetime both with constant space-slices (both constant FRW space (\(r\)) and constant new space (\(R\)) space-slices). Section IV then discusses the particle creation phenomena and computes the particle number density. The next section V is devoted to calculate the REMT in FCOs frame using the \(T\)-vacuum. We make a detailed comparison of this phenomena with the Unruh effect in section VI. Finally in VII we conclude.

II. NEW COORDINATES DESCRIBING THE RADIATION DOMINATED UNIVERSE

Here, we provide a brief review of our earlier work [4] related with the installation of a new coordinate system in the radiation dominated universe. For somewhat broader motivation and a detailed calculation we refer the reader to [4].

Let us start from the spatially flat FRW metric in comoving frame

\[
ds^2 = dt^2 - a^2(t)[dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2)],
\]

which in cosmological coordinates \((\eta, r, \theta, \phi)\), where \(\eta = \int \frac{dt}{a(t)}\), is given by

\[
ds^2 = a^2(\eta)[d\eta^2 - dv^2 - v^2(d\theta^2 + \sin^2\theta\,d\phi^2)].
\]

The scale factor \(a(t)\) is an exponential function of time in the inflationary and dark energy dominated stages, whereas, it satisfies the power law equation \(a(t) = a_0t^n\) for other stages of expansion, specifically, \(n = 1/2\) for the radiation dominated stage and \(n = 2/3\) for the matter dominated stage. The constant \(a_0 = \sqrt{2\mathcal{H}(e)}\) (where \(\mathcal{H}\) is the inflationary Hubble constant) for a universe starting from inflation and transiting into the radiation stage [5]. In the light-cone gauge \(u = \eta - r, v = \eta + r\) and \(r = \frac{1-u}{2}\)

\[
ds^2 = a^2du dv - \frac{(v-u)^2}{4}a^2(d\theta^2 + \sin^2\theta\,d\phi^2).
\]

In [4] we showed that, if we make a conformal transformation of the cosmological null coordinates, for a general functional dependence of \(a(t)\), then it is only for the radiation dominated case where the resulting spacetime poses important symmetries that allow more than one (unitarily inequivalent) field decompositions. In fact, for \(a(t) \propto \frac{1}{\sqrt{t}}\), a conformal transformation of null coordinates of the following form [4]

\[
U = T - R = \pm \frac{\mathcal{H}}{2}\, u^2; \ V = T + R = \frac{\mathcal{H}}{2}\, v^2
\]

(4)

(4) takes the above metric into a spherically symmetric form. The full spacetime is now a direct sum of the following two spacetime metrics applicable to the sub-Hubble and super-Hubble regions [4]

\[
ds^2 = F_1(T, R)(dT^2 - dR^2) - R^2d\Omega^2, \ (\text{for } U \geq 0; T \geq R)
\]

(5)

\(^1\) This route could be very useful for a four dimensional case where these REMT can be used to set up semiclassical Einstein-Friedman equations whose solution will then give a back-reacted metric with quantum corrections.
with
\[ F_I(T, R) = \frac{(\sqrt{T + R} + \sqrt{T - R})^2}{4\sqrt{T^2 - R^2}}. \]  

(6)

\[ ds^2 = F_{II}(T, R)(dT^2 - dR^2) - T^2d\Omega^2, \quad \text{for } U \leq 0; \ T \leq R \]  

(7)

with
\[ F_{II}(T, R) = \frac{(\sqrt{R + T} - \sqrt{R - T})^2}{4\sqrt{R^2 - T^2}}. \]  

(8)

We shall denote the sub-Hubble and super-Hubble regions (shortly we shall clarify this nomenclature), described by the metrics (5) and (7), as regions I and II, respectively. In region I, the new time and space coordinates are related with the cosmological time and space coordinates as
\[ T = (V + U)/2 = \frac{\mathcal{H}e^{\eta r}}{2}(q^2 + r^2) \]  

(9)

\[ R = (V - U)/2 = \mathcal{H}e^{\eta r}. \]  

(10)

In region II, the relationships between these two sets of coordinates are reversed, so that
\[ T = (V + U)/2 = \mathcal{H}e^{\eta r} \]  

(11)

\[ R = (V - U)/2 = \mathcal{H}e^{\eta r}. \]  

(12)

By expressing the conformal factors \( F_{I/II}(T, R) \rightarrow F_{I/II}(H, R) \), (where \( H = (\frac{2}{a})_{RD} \) is the Hubble parameter for radiation stage) we get \[ F_I(H, R) = \frac{1}{\mathcal{H}^2H^2} \]  

and \[ F_{II}(H, R) = \frac{1}{\pi^2H^2}. \]  

Thus the light-cone boundary for the new observers at \( T = R \) is nothing but the comoving Hubble radius at \( R = 1/H \). That justifies the above nomenclature for the metrics as sub and super-Hubble.

It is important to note that \( T \) remains timelike for both regions, I and II. There is no change of signature in (5) and (7). However, in the sub-Hubble region, the radius of the two sphere in (5) (given by \( R \)) remains unchanged with time, for a static observer in \((T, R)\) frame. The universe appears to be static, whose radial size does not change, for this observer. But there is a conformal change in the \( R - T \) sector of the metric because of the conformal time dependent factor. The root of this radial staticity lies in the fact that the rate of expansion of the universe with respect to the cosmological time coincides with the motion of the observer which is static in the new spacetime (only in region I) but moving in cosmological frame. However, for the super-Hubble region (II) (7), a static observer (at constant \( R \)) in the new spacetime, does see the universe expanding with time \( T \) since now the radius of the two-sphere is time-dependent (in fact it is given by \( T \)).

The constant \( T \) and \( R \) slices in \((\eta, r)\) plane is plotted in Fig. 1. Constant \( R \) trajectories (i.e., the static observers in \((T, R)\) frame) are freely falling in cosmological frame (i.e., parallel to the \( \eta \) axis) in the asymptotic past and future. However, they have acceleration and deceleration in between. With respect to the cosmological time, they start accelerating in the super-Hubble region and reach luminal velocity to reach the Hubble radius in a finite timescale. Once they reach the Hubble radius they start decelerating, thus leaving the light trajectory to become sub-Hubble, and finally become indistinguishable from the freely falling observer in cosmological frame in the asymptotic future. Since this motion includes acceleration/deceleration there is a possibility of particle creation phenomena which was discussed in [4]. Now that we have a clear idea of the trajectory of a static observer in \((T, R)\) frame in the cosmological \((\eta, r)\) frame, we move to the next section where we study the inverse case, i.e., the trajectory of the FCOs, who are static in cosmological \((\eta, r)\) frame, in the \((T, R)\) frame.

### III. Physical Observers and a New Spacetime Foliation

Let us start by considering the sub-Hubble metric (5) and a worldline given by \( T = G(R) \). Along this worldline the \((R, T)\) sector of the metric (5) becomes \( ds^2 = \Xi dR^2 \), where the conformal factor
\[ \Xi = \frac{(\sqrt{G(R)} + \sqrt{G(R) - R})^2}{4\sqrt{G(R)^2 - R^2}} \times (G^2(R) - 1). \]  

(13)

If this factor diverges for some allowed value of \( R > 0 \) and for a given \( T = G(R) \) that will indicate the presence of horizon for the observer satisfying the aforementioned conditions.
trajectory. We are now free to choose any observer trajectory and test if there will be a horizon or not for the concerned observer.

A. Observer with constant radial velocity

First, we consider a linear trajectory $T = G(R) = a_0 R + \beta_0$ where $a_0 \geq 1$ (necessary for region I) and $\beta_0 > 0$ are dimensionless constants. This will mean that the radial velocity of the observer is constant $dR/dT = 1/a_0$ and thus they have no acceleration. In this case it is easy to check that $\Xi$ never diverges. Therefore, any observer with constant radial velocity in $(T, R)$ frame do not encounter a horizon. This is somewhat analogous to the result in Minkowski spacetime where inertial observers with constant velocity do not encounter horizon.

B. Observer with constant radial acceleration

Next, let us consider a trajectory in region-I given by

$$T = G(R) = \frac{1}{a_0}(R + \beta_0)^{1/2},$$  \hspace{1cm} (14)

where $a_0$ is a dimensionless constant and $\beta_0$ is a dimensional constant. This trajectory represents an observer with constant radial acceleration $d^2R/dT^2 = 2a_0^2$. Using (13) it is straightforward to see that this observer will encounter at most two event horizons at $R = \frac{1}{2a_0} \pm \sqrt{\frac{1}{4a_0^2} + \frac{\beta_0}{a_0}}$. The appearance of horizon for an accelerated motion is reminiscent of the case for accelerated observers (Rindler observers) in the Minkowski spacetime who finds event (Rindler) horizon in the Minkowski spacetime and are confined to one-fourth of the full spacetime. The crucial difference of that with the present situation being, while the accelerated observer in Minkowski spacetime can be easily imagined and attributed to a class of physical observers, here the observers with a trajectory (14) has a little physical motivation – it is not clear if they are of physical interest to us in the understanding cosmology.

This situation, however, will change dramatically in the next sub-section where we will attribute a trajectory to a physical observer in cosmology, known as the Fundamental Cosmological Observer (observers with proper time registered as the conformal time $\eta = \int \frac{dt}{a(t)}$).

C. Fundamental Cosmological Observers (FCOs)

Let us consider the worldline in the sub-Hubble region-I

$$T = G(R) = \alpha_1 R^2 + \beta_1,$$  \hspace{1cm} (15)

where $\alpha_1$ and $\beta_1$ are dimensionful positive definite constants. These observers have a position dependent radial velocity $dR/dT = \frac{1}{2\alpha_1 R}$ and a nonzero, position-dependent radial acceleration $d^2R/dT^2 = -\frac{1}{4\alpha_1^2 R^3}$, (16)

whose magnitude is increasing as the observer approaches the center $R = 0$. That is, they tend to come to a rest close to the centre and become freely falling observer at their causal future. As evident from (15) these observers follow a parabolic trajectory in $(T, R)$ plane. Substituting this in (13) we find

$$\Xi = \frac{(2\alpha_1 R + 1)(\sqrt{(\alpha_1 R + 1)R + \beta_1} + (\alpha_1 R + 1)R - \beta_1)\sqrt{4\alpha_1 R^2 + \beta_1 + R}}{\sqrt{4\alpha_1 R^2 + \beta_1 - R}}.$$  \hspace{1cm} (17)

The first factor is always finite, so it is only the second factor which determines if $\Xi$ diverges. Clearly there are two divergences for the root $R_0 = \frac{1 + \sqrt{(1-4\alpha_1 \beta_1)}}{2\alpha_1}$ if $\beta_1 < 1/4\alpha_1$ implying, therefore, existence of two horizons. If $\beta_1$ vanishes there is only one horizon at $R_0 = 1/\alpha_1$. Further, there is a very special case for $\beta_1 = 1/4\alpha_1$ for which the second factor in (17) is $\sqrt{4\alpha_1}$. Therefore, among all observers satisfying (15), it is this and only this observer satisfying $4\alpha_1 \beta_1 = 1$ does not encounter any horizon anywhere in the spacetime.

It is genuinely interesting to show that the above special observers, satisfying (15) (with $\beta_1 = 1/4\alpha_1$), are none other than the FCOs at any position $r$ in region-I. To demonstrate that we use the relations (9) and (10). We find that the constant $r = r_0$ trajectories in $T - R$ plane satisfy an identical relationship like $T = \alpha_1 R^2 + \beta_1$ with $\beta_1 = 1/4\alpha_1$ implying, therefore, existence of two horizons. If $\beta_1$ vanishes there is only one horizon at $R_0 = 1/\alpha_1$. Further, there is a very special case for $\beta_1 = 1/4\alpha_1$ for which the second factor in (17) is $\sqrt{4\alpha_1}$. Therefore, among all observers satisfying (15), it is this and only this observer satisfying $4\alpha_1 \beta_1 = 1$ does not encounter any horizon anywhere in the spacetime. This is of course expected for the FCOs as well as for the comoving observers who are after all free to communicate with all spacetime events respecting causality.

Now let us turn our attention to the super-Hubble region-II. Everything that we have showed above for the
These observers correspond to the trajectory of the FCOs at constant \( r = r_0 \) with the identification \( \beta_1 = 1/4\alpha_1 = H\eta_0^2/2 \) in the \( T, R \) frame. Also, notice that at \( T = R \) (15) and (18) are identical. These observers have a time dependent velocity \( \frac{dT}{d\eta} = 2\alpha_1 T \) and a constant acceleration \( \frac{d^2T}{d\eta^2} = 2\alpha_1 \) in the region-II.

Now looking at the whole picture, we see that at the asymptotic past (i.e., the beginning of the radiation dominated universe \( \eta = \eta_0 \)) a FCO at \( r = r_0 \) is freely falling in region-II and subsequently it has the following interesting trajectory in the \( (T, R) \) frame: a FCO starts accelerating at a constant rate \( 2\alpha_1 = H\eta_0^2/2 \) as the time \( \eta \) increases and during this period the radial velocity also increases at a rate \( 2\alpha_1 T \). This is continued until the cosmological time \( \eta = \eta_0 \) or at the light-cone boundary. At this point the radial velocity \( \frac{dR}{dT} = 2\alpha_1 T = 1 \), i.e., it reaches the luminal velocity with respect to a freely falling observer in \( (T, R) \) frame. After this the FCO enters in region-I where it starts decelerating with respect to a static \( R \) observer satisfying (16) and coincides with the freely falling observer in \( (T, R) \) frame.

D. New spacetime foliation considering FCOs

Since we are about to construct a consistent quantum field theory keeping FCOs in focus, it is customary to show that there exists such a construction, and the first step in doing that is to foliate the spacetime in terms of the spacelike and timelike hypersurfaces. Particularly, we want to see how the constant \( \eta \) (which defines a “timeslice” or a spacelike hypersurface) and constant \( r \) (which defines a “space-slice” or timelike hypersurface) surfaces will look like. Since the relationships between the two sets of coordinates, in (9) and (10), are symmetric under the exchange of \( \eta \) and \( r \) (so is true for (11) and (12)) we have to be rather cautious to identify the time-slices and the space-slices. Below we provide a valid construction.

First, let us plot the \( \eta = \eta_0 \) timeslices. This follow from the relationships (9) and (10). Substituting \( r \) from (10) into (9) expresses \( T = T(R, \eta) \) and then a constant \( \eta \) slice will be the parabola

\[
T = \frac{R^2}{2H\eta_0^2} + \frac{1}{2}H\eta_0^2
\]

(19)

defined for the sub-Hubble region-I. We only consider the portion of this parabola that is below the Semi-Latus-Rectum (SLR). The SLR, the parabolic curve (19) and the Hubble horizon meet at one point. From that point onwards we extend the timeslice to the super-Hubble region-II. In order to do that now we consult the relationships (11) and (12). We substitute \( r \) from (11) and express \( R = R(T, \eta) \) and then the timeslice will simply be a portion of the parabola

\[
R = \frac{T^2}{2H\eta_0^2} + \frac{1}{2}H\eta_0^2
\]

(20)

for the super-Hubble region-II. This portion is chosen as the the parabolic curve on the right of SLR of (20). The complete \( \eta = const. \) hypersurface is now given by the union of the lower portion (of the SLR) of the parabola (19) and the right portion (of the SLR) of the parabola (20). This arrangement is plotted mathematically in Fig. 2.

In a similar way we can construct the timelike hypersurfaces or “spaceslices” with \( r = const. = r_0 \). For that, in region-I we now substitute \( \eta \) from (10) into (9) expresses \( T = T(R, r) \)

\[
T = \frac{R^2}{2H\eta_0^2} + \frac{1}{2}H\eta_0^2
\]

(21)

and in region-II we substitute \( \eta \) from (11) and express \( R = R(T, r) \), given by

\[
R = \frac{T^2}{2H\eta_0^2} + \frac{1}{2}H\eta_0^2
\]

(22)

Notice that they are identical to the other set (19) and (20) under the exchange of \( r \) with \( \eta \). Now the \( r = const. \) spaceslices are constructed by joining the upper portion (of the SLR) of the parabola (21) with the left portion (of the SLR) of the parabola (22) which will cover the whole regions of spacetime (I and II) covering the sub
and super-Hubble scales. These slices are shown in Fig. 3.

A combined plot including the spacelike and timelike hypersurfaces are shown in Fig. 4. These slices foliate the complete spacetime, (basically the \((T,R)\) plane; each point in this plane is a two sphere) with constant \(\eta\) (timeslices) and constant \(r\) (spaceslices), in a consistent manner so that the Cauchy problem is well posed. Recall that this foliation is independent of the other foliation discussed in Fig. 1.

Notice also, that, once again, in Fig. 4 the FCOs at \(r = \text{const.}\) (vertical curves) are, in fact, freely falling in \((T, R)\) frame only in the asymptotic past and future, but they are accelerated (and decelerated) radially in super (and sub) Hubble regions, respectively. FCOs attain the luminal velocity at the Hubble scale and this is exactly analogous to the case discussed in the last section for static observers in cosmological frame [4] who attain this pattern to be reciprocal because, they are, exactly analogous to the case discussed in Fig. 1.

IV. PARTICLE CREATION

Now we proceed on to another important part of our discussion which is gravitational particle creation for FCOs. In our earlier work [4] we provided a detailed discussion of particle creation phenomena with respect to the static observer in \((T,R)\) spacetime who finds the cosmological vacuum as containing particles. As showed in Fig. 1, these observers have non-trivial trajectory in FRW coordinates. Here we want to calculate the particle content for FCOs who are at \(r = \text{const.}\) and following the trajectories showed in Fig. 4. These observers will be exposed to a radiation due to their motion. This will be a consequence of a newly found vacuum state that we are about to introduce in this section. In fact, this effect is a parallel to other well-known effects, such as the Unruh effect (for accelerated observers in Minkowski space who envision the Minkowski vacuum as particle excited state) and the Hawking effect (where asymptotic static observers in Schwarzshild frame envision the Hartle-Hawking (for an eternal black hole geometry) or Unruh vacuum (for an evaporating black hole geometry) as particle excited states). We shall restrict ourselves for a two dimensional set up which will keep our analysis simpler, yet, physically very intuitive. The four dimensional calculation using spherical polar coordinates is a bit more involved than a two dimensional analysis and for this case it needs to be handled with more caution, including numerics.

A. Number density in two dimensions

In two dimensions (ignoring \(\theta, \phi\) coordinates) the field equations for the massless scalar field read \(\partial_{\mu}\partial_{\nu}\Phi = 0\).
for (2) and $\partial_U\partial_V \Phi = 0$ for both (5) and (7). The field operator, expanded in two bases as

$$\hat{\Phi} = \int_0^\infty \frac{d\omega}{\sqrt{4\pi\omega}} (e^{-i\omega u} a_\omega + e^{i\omega u} a_\omega^\dagger \text{ right moving})$$

$$= \int_0^\infty \frac{d\omega}{\sqrt{4\pi\Omega}} (e^{-\Omega u} b_\Omega + e^{\Omega u} b_\Omega^\dagger \text{ right moving})$$

(23)

The Bogolyubov coefficients relating the annihilation operator

$$a_\omega = \int_0^\infty d\Omega (\alpha_{\omega\Omega} b_\Omega - \beta_{\omega\Omega} b_\Omega^\dagger)$$

in terms of the sum of creation and annihilation operators of the other basis can be easily calculated as

$$\alpha_{\omega\Omega} = \frac{1}{2\pi} \sqrt{\frac{\omega}{\Omega}} \int_{-\infty}^{\infty} du e^{-\Omega u+i\omega u},$$

$$\beta_{\omega\Omega} = -\frac{1}{2\pi} \sqrt{\frac{\omega}{\Omega}} \int_{-\infty}^{\infty} du e^{\Omega u+i\omega u}.\quad (25)$$

The average particle number density for a given frequency is then given by

$$\langle n_\omega \rangle = \int_0^\infty d\Omega |\beta_{\omega\Omega}|^2$$

(27)

where, $n_\omega = a_\omega^\dagger a_\omega$ is the number operator defined in the cosmological basis and the expectation value $\langle 0_T|n_\omega|0_T \rangle$ is calculated in the vacuum state in the new basis, as defined by $b_\Omega|0_T \rangle = 0$. We refer to this vacuum state $|0_T \rangle$ as $T$-vacuum.

To calculate the coefficient (26) we first divide the integral for $u \leq 0$ and $u \geq 0$ and use appropriate relationships relating two null coordinates as appear in (4). After performing the integration (26) we can derive

$$|\beta_{\omega\Omega}|^2 = \frac{\omega}{8e\hbar^2\Omega^2} \left(1 + \sin(\frac{\omega^2}{\Omega e\hbar})\right) \Gamma^2[\frac{1}{2}, \frac{\omega^2}{2\Omega e\hbar}]$$

(28)

where $\Gamma$ is an upper incomplete gamma function. Equation (27) then provides average particle number density. Unfortunately, it is difficult to get an exact analytical result for the particle number density (27) using (28). We therefore use numerics and plot the number density in Fig 5. We have noted an infrared divergence in $\langle n_\omega \rangle$ which is unphysical and appears in other situations such as particle creation by the moving mirror [6]. This divergence is usually avoided just by making an infrared cutoff in $\Omega$ which is also the case in our case (Fig 5). This is a new result and quite significant from the physical perspective since it establishes the fact that the cosmological observers (FCOs) are exposed to a radiation just because of the existence of the new vacuum state $|0_T \rangle$ which can only be defined once we know the new coordinates (5) and (7) and field theory based on these coordinates.

In this section we discuss more about the new vacuum state that we found in the last section while quantizing the field equation in the new reference frame and use it to calculate various components of renormalized energy-momentum tensor applicable to the two dimensional set up discussed here.

We have already seen equations (23) and (24) give rise two choices of vacuum states — the standard cosmological vacuum state defined by $a_\omega|0_C \rangle = 0$ as well as a newly defined vacuum state which we call here the $T$-vacuum defined as $b_\Omega|0_T \rangle = 0$. An analogous situation arises for the Rindler-Minkowski case where one has the standard Minkowski vacuum and also the Rindler vacuum. However, as it is well known [8], the Rindler vacuum state is unphysical since it needs an infinite energy to be produced — that is, the expectation value of the energy-momentum tensor, in accelerated observer’s frame, calculated using the Rindler vacuum is divergent at the Rindler horizon (non-renormalizable and non-Hadamard). However, no such divergence is found while using the standard Minkowski vacuum which is well-defined in any region of the spacetime even when looking from the accelerated frame. The main conclusion drawn from this is the fact that although Minkowski vacuum state can be well-defined in flat spacetime Rindler vacuum is unphysical. This makes sense, because, had it been physical, observers at rest in Minkowski spacetime would have received radiation just because of the existence of such well defined vacuum state, which of course we do not see. It is an indirect evidence of the fact that the Rindler vacuum is unphysical. On the contrary, of course, Rindler observers are physical observers and they do find the Minkowski vacuum as a thermal state and register particle excitations which is the main point of the
Unruh effect [3].

With a clear idea of the above, now we want to examine here if the newly defined $T$-vacuum is well-defined everywhere or it has similar problems like the Rindler vacuum. To do this we first take the expectation values of the operators $(\partial_u \hat{\Phi})^2$ and $(\partial_U \hat{\Phi})^2$ in the cosmological and $T$-vacuum states respectively. This can be calculated from the field expansions (23) and (24) which gives (simply because $|0_C\rangle$ is annihilated by $a_u$ and $|0_T\rangle$ by $\partial_{\Omega}$.)

$$
\langle 0_C | (\partial_u \hat{\Phi})^2 | 0_C \rangle = \langle 0_T | (\partial_U \hat{\Phi})^2 | 0_T \rangle.
$$

(29)

We can use the above equation and the transformation law for derivative operators to express the quantity

$$
\langle 0_T | \frac{1}{2} (\partial_u \hat{\Phi})^2 | 0_T \rangle = \left( \frac{\partial U}{\partial u} \right)^2 \langle 0_T | \frac{1}{2} (\partial_U \hat{\Phi})^2 | 0_T \rangle,
$$

(30)

$$
= (\mathcal{H} cu)^2 \langle 0_C | \frac{1}{2} (\partial_u \hat{\Phi})^2 | 0_C \rangle,
$$

(31)

where for the second line we have used the relationship between $U(u)$ in (4), as well as equation (29). The above equation simply gives the relationship between the $uu$ component of the energy-momentum tensor $(\{T_{uu}\})$ evaluated in the $T$-vacuum with that evaluated in cosmological vacuum in the following way

$$
\langle 0_T | T_{uu} | 0_T \rangle = \langle 0_C | T_{uu} | 0_C \rangle.
$$

(32)

Such a relationship between the Rindler and Minkowski coordinates also holds for two dimensions and can be found in [8], where the relationship between the REMTs in two vacuums are related by

$$
\langle 0_R | T_{uu} | 0_R \rangle = \frac{1}{2a^2 u^2} \langle 0_M | T_{uu} | 0_M \rangle,
$$

(33)

where $a$ is the constant acceleration, $|0_M\rangle$ is the Minkowski vacuum and $|0_R\rangle$ is the Rindler vacuum. Notice that the coordinate dependent prefactor on the right hand side of the above equation diverges at the future (Rindler) horizon at $u = 0$. The only way to counter this and make the left hand side finite is another divergence in the REMT in Minkowski vacuum, which of course does not happen because it is well defined. This makes the Rindler vacuum unphysical and Minkowski observers do not have any radiation coming from it. In the same spirit it is necessary for us to consider (32) and check its legitimacy.

In order to calculate the l.h.s of (31) we first need to know the REMT in the cosmological vacuum. Since we are limited to a two dimensional set up, we recall various studies by Birrel, Davies, Fulling, Unruh and Bunch in the middle to late seventies [9] which provide a covariant, divergence free expression for REMT in the cosmological vacuum. This is given by

$$
\langle T_{\mu \nu} \rangle_C = \langle 0_C | T_{\mu \nu} | 0_C \rangle = \Theta_{\mu \nu} + \frac{R}{48\pi} g_{\mu \nu}
$$

(34)

where, $\Theta_{\mu \nu}$ is the constant acceleration,

$$
\Theta_{uu} = \frac{1}{24\pi} (\partial_u D_1 - \frac{1}{2} D_1^2)
$$

(35)

$$
\Theta_{vv} = \frac{1}{24\pi} (\partial_v D_2 - \frac{1}{2} D_2^2)
$$

(36)

$$
D_1 = \frac{1}{a^2} \partial_u (a^2), \quad & D_2 = \frac{1}{a^2} \partial_v (a^2)
$$

(37)

where $a(u, v) = \frac{\mathcal{H}}{2\pi} (u + v)$. This is the scale factor. In our case, for the radiation dominated universe we have $R = 0$, i.e., the second term in (34) is inevitably zero. That gives, $\langle T_{uu} \rangle_C = 0$. As for the others we get, using all of the above set of equations

$$
\langle T_{uu} \rangle_C = \langle T_{vv} \rangle_C = -\frac{1}{6\pi (u + v)^2}.
$$

(38)

Now substituting this in (32) we get

$$
\langle 0_T | T_{uu} | 0_T \rangle = -\frac{\mathcal{H}^2 \epsilon^2}{6\pi (1 + u/v)^2}.
$$

(39)

Similarly, one can easily calculate

$$
\langle 0_T | T_{vv} | 0_T \rangle = -\frac{\mathcal{H}^2 \epsilon^2}{6\pi (1 + u/v)^2},
$$

(40)

whereas, the other components $\langle T_{uu} \rangle_T = 0 = \langle T_{vv} \rangle_T$. These are extremely important results which prove the well-defined-ness of the $T$-vacuum. Notice that, none of the above expressions have singularity at the Hubble radius $(u = 0$ or $T = R)$ where two spacetime patches are glued together. Precisely, we have $\lim_{u \to 0} \langle T_{uu} \rangle_T = 0$ and $\lim_{u \to 0} \langle T_{vv} \rangle_T = -\frac{\mathcal{H}^2 \epsilon^2}{6\pi}$. Also, for asymptotic values of $u \to \pm \infty$, we have $\lim_{u \to \pm \infty} \langle T_{uu} \rangle_T = -\frac{\mathcal{H}^2 \epsilon^2}{6\pi}$, and $\lim_{u \to \pm \infty} \langle T_{vv} \rangle_T = 0$. On the other hand if we take similar limits on the advanced null coordinate, we get $\lim_{u \to 0} \langle T_{uu} \rangle_T = -\frac{\mathcal{H}^2 \epsilon^2}{6\pi}$, $\lim_{u \to 0} \langle T_{vv} \rangle_T = 0$, $\lim_{u \to \pm \infty} \langle T_{uu} \rangle_T = 0$, and $\lim_{u \to \pm \infty} \langle T_{vv} \rangle_T = -\frac{\mathcal{H}^2 \epsilon^2}{6\pi}$. The above analysis proves that the $T$-vacuum state that we introduce in this paper is well-defined and has a physical importance. Using (39) and (40) we can also calculate

$$
\langle 0_T | T_{\eta \nu} | 0_T \rangle = -\frac{\mathcal{H}^2 \epsilon^2}{12\pi} \left( 1 + \frac{\nu^2}{\eta^2} \right),
$$

(41)

$$
\langle 0_T | T_{\eta \nu} | 0_T \rangle = -\frac{\mathcal{H}^2 \epsilon^2}{12\pi} \left( \frac{r}{\eta} \right).
$$

(42)

One of the important aspects of calculating covariant, renormalized energy-momentum tensor is that the result can be transformed in other coordinates which are suitable to the observers of interest. In cosmology the most important observer is identified as the “comoving observer”. Comoving observers are the nearest analogy to the inertial observers in flat space which are not acted upon by any external force. The only difference in the comoving observers and FCOs is the time coordinate — FRW metric in the comoving frame is given by (1),
whereas it is in the cosmological frame is given by (2). Transforming the above results in comoving frame we get

\[
0_T[T_{tt}|0_T] = -\frac{He}{24\pi} \left( \frac{1}{t} + \frac{Hcr^2}{2t^2} \right),
\]

(43)

\[
0_T[T_{rr}|0_T] = -\frac{He^2c^2}{12\pi} \left( 1 + \frac{Hcr^2}{2t^2} \right),
\]

(44)

\[
0_T[T_{tr}|0_T] = Tr_n = -\frac{He^2c^2}{24\pi} \left( \frac{r}{t} \right).
\]

(45)

The \(tt\)-component is the energy density that is turning out to be negative here, thus giving an effect of repulsive gravity, and in a full four dimensional set up this would mean an accelerated expansion (the radiation pressure, treated classically, is already driving the background spacetime) just because of this term. This is however a two dimensional toy model and generalising this to a four dimensional set up is yet to be done. Likewise, the other diagonal term (\(rr\)-component) is interpreted as pressure. Usually, in the radiation-dominated universe we assume the radiation as a pressure-less dust, however, as we see above accounting for quantum effects such as above there is indeed a non-zero pressure coming from the particle creation. The off-diagonal components represent momentum-densities which are equivalent to the components of linear momentum. Since it is two dimensional model there is no shear stress acting here.

We should note again that the above analysis is restricted so far for two dimensions and the interpretations can’t be trivially generalized to a four dimensional set up. We are hoping shortly extending this for a full four dimensional set up in a forthcoming paper [7]. Nevertheless, this study should establish the physical importance of the new coordinates that we define here and more so for the \(T\)-vacuum state.

VI. COMPARISON WITH THE RINDLER-MINKOWSKI CASE AND UNRUH EFFECT

Even in the absence of gravity, the relationship between the Minkowski and Rindler metrics representing the flat spacetime has been a cornerstone for our understanding of QFT in curved space. Often discussions about particle creation in other cases, such as in black hole spacetimes, are compared with the Unruh effect. We intend to do the same with our discovery of the particle creation reported in the last section. This will help us to realize the importance of this study.

The first step is to compare the relationships between the coordinates. A comparison for the two dimensional case (angular coordinates are unchanged) is presented in the following tabular form:

| Item | Rindler—Minkowski | FRW \((\eta, r)\)—New \((T, R)\) coord. |
|------|------------------|--------------------------------|
| i) Metrics | Rindler: \(ds^2 = c^2\xi^2[(d\xi_0)^2 - (d\xi_1)^2]\) | New coord.: \(ds^2 = a^2(dy^2 - dr^2)\) |
| ii) Relationships | \(t = c\xi_1/a\) sinh\((a\xi_0)\) | \(T = \frac{He}{2}(\eta^2 + r^2)\); \(R = He\eta\) (in Reg. I) |
| iii) Coordinate span | \(-\infty \leq t \leq \infty; 0 \leq r \leq \infty\) | \(0 \leq \eta \leq \infty; 0 \leq r \leq \infty\) |
| iv) Completeness | \(-\infty \leq \xi_0 \leq \infty; -\infty \leq \xi_1 \leq \infty\) | \(0 \leq T \leq \infty; 0 \leq R \leq \infty\) |
| v) Overlapping regions | \((t, r)\) complete but \((\xi_0, \xi_1)\) incomplete | Both are complete |
| vi) Relevant observers | Rindler coordinates \((\xi_0, \xi_1)\) overlap only a quarter of Minkowski \((t, r)\) | FRW \((\eta, r)\) and new one \((T, R)\) overlap in all region |
| vii) Observers’ trajectory | Rindler (accelerated) observers | FCOS (starts acceleration in region II) and gets decelerated in region I) |
| viii) Spacelike hypersurface | Rindler time \(\xi_0 = \text{const.}\) hypersurface \(\xi_0 = \frac{1}{a}\tanh^{-1}(t/r)\) | Space coord. \(r = \text{const.}\) two parabolas in \((T, R)\) plane |
| | \(x^2 - t^2 = \frac{2a^2}{x^2}, \text{defined only for } x \geq |t|\) | \(T = \frac{Hr^2}{2} + \frac{1}{2}He\eta^2\) (region-I) |
| | Cosmological time \(\eta = \text{const.}\) hypersurface |
| | Rindler time \(\xi_0 = \text{const.}\) hypersurface |
| | \(T = \frac{Hr^2}{2} + \frac{1}{2}He\eta^2\) (region-II) |
| | \(R = \frac{Hr^2}{2} + \frac{1}{2}He\eta^2\) (region-II) |

Table 1: Analogy between the relationships of Rindler—Minkowski and FRW—\((T, R)\) coordinates.
Now, we want to compare the studies based on the quantum field theory side which will give a side by side picture of the Unruh effect with the present study. This is also shown in the following tabular form:

| Item                                    | Rindler—Minkowski                                                                 | FRW ($\eta, r$)—New ($T, R$) coord. |
|-----------------------------------------|-----------------------------------------------------------------------------------|--------------------------------------|
| i) Vacuum states                        | i) Rindler vacuum: $|0_R\rangle$                              | i) Cosmological vacuum: $|0_C\rangle$ |
|                                         | Minkowski vacuum: $|0_M\rangle$                              | $T$-vacuum: $|0_T\rangle$            |
| ii) Physical observers                  | ii) Both Rindler and Minkowski                                                   | ii) FCOs                             |
| iii) Physical vacuum                    | iii) $|0_M\rangle$                                                                   | iii) Both $|0_C\rangle$ and $|0_T\rangle$ |
| iv) Bogolyubov coefficient              | iv) $|\beta_{\omega}\Omega\rangle = \frac{\Omega}{\omega T} \Gamma(-i\Omega/a) \Gamma(i\Omega/a) \Gamma(\frac{i\Omega}{R/a})$ | iv) $|\beta_{\omega}\Omega\rangle = \frac{\omega}{\sqrt{\pi} \sigma T} \left( 1 + \sin\left(\frac{\omega^2}{4\pi T R}\right)\right) \Gamma^2\left[\frac{1}{2}, \frac{\omega^2}{2\pi T R}\right]$ |
| v) Particle number density              | v) $\langle n_\omega \rangle = \frac{1}{e^{\omega T/a} - 1}$                      | v) Numerically plotted in Fig. 5      |
| vi) Field modes                         | vi) Complete in Minkowski frame                                                   | vi) Complete in both Cosmological    |
|                                         | but incomplete in Rindler frame                                                   | and ($T, R$) frames                  |
| vii) Components of REMT                  | vii) $\lim_{u \to 0} \langle 0_R | T_{uu} | 0_R \rangle \to \infty$               | vii) $\lim_{u \to 0} \langle 0_T | T_{uu} | 0_T \rangle = 0$                  |
|                                         | $\lim_{v \to 0} \langle 0_R | T_{vv} | 0_R \rangle \to \infty$               | $\lim_{v \to 0} \langle 0_T | T_{vv} | 0_T \rangle = 0$                  |

Table 2: Comparison between the particle creation in this paper with the Unruh effect. While in Unruh effect both Minkowski and Rindler observers are of physical interest, here only the cosmological observer has a physical interest. However, while in Unruh effect only Minkowski vacuum is physical (and Rindler is not), here both the cosmological vacuum and the $T$-vacuum are physical.

**VII. CONCLUSIONS**

To conclude, we have extended our study [4] to understand further aspects of quantum field theory in the radiation dominated stage of early universe. Particularly, we have studied the Fundamental Cosmological Observers’ (FCOs) view of the universe who have a peculiar motion in the frame of the new coordinates ($(T, R)$ coordinates) that was first installed in [4]. Our motivation for focusing on those observers is rooted in the fact that in cosmology we have a tendency to understand the universe with respect to comoving observers who are closely related with the FCOs just by a redefinition of the time coordinate. Also, due to conformal flatness in the FCOs frame it is relatively clear to study field theory in their frame. We, in the Earth, do not share the FCO’s frame, but it is widely accepted that by subtracting all relative motions, such as (a) the Earth’s rotation around the Sun, (b) the Sun’s motion relative to the Local Standard of Rest (LSR), (c) the motion of LSR orbit in the Milky Way, (d) The Milky Way’s motion relative to the Local Group (LG), (e) the LG’s infall in Virgo Cluster of galaxies and finally (f) the speeding of Virgo cluster towards “The Great Attractor”, we can get FCO’s view of the Universe. In this work, apart from their peculiar trajectories in ($T$, $R$) frame, these observers are shown to be exposed to a radiation due to a new gravitational particle creation process. This particle creation process is a new discovery in this paper and it is possible due to the existence of a new vacuum state (we refer to that as $T$-vacuum) that appears to be a particle excited state in FCO’s reference frame.

We also made an analysis to show that the newly found $T$-vacuum is a physical vacuum state since it does not attribute undesired divergences in any component of the renormalized energy momentum tensor (REMT). We calculated REMT in $T$-vacuum both for the FCOs and later for comoving observers. The result shows that in both frames (FCO’s and comoving observer’s frame) the energy density is negative which is effectively giving an effect of repulsive gravity and giving an extra push to the expansion rate. Also, this quantum effect give a non-zero pressure and momentum densities which are otherwise non-existing for the radiation (dust) that is driving the background spacetime. Finally, we compare this particle creation phenomena with the well known Unruh effect and explain the similarity and differences which are key aspects to clearly understand our results.

We feel that understanding this effect with such finer details is very important as they can give important clues to understanding the Cosmos. This, however, can only be done by generalizing our work for a realistic four dimensional set up. Only then we shall be able to talk about any observational consequence which may be important for us. In any case, this paper adds to the literature a new aspect of particle creation, a la the Hawking and Unruh effects, that is important because these particles or more generally the radiation due to created particles will be registered by both the comoving and cosmological observers who play a deep role in our understanding of the cosmos.
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