Abstract. In this paper we enumerate the number of \((k, r)\)-Fuss-Schröder paths of type \(\lambda\). Y. Park and S. Kim studied small Schröder paths with type \(\lambda\). Generalizing the results to small \((k, r)\)-Fuss-Schröder paths with type \(\lambda\), we give a combinatorial interpretation for the number of small \((k, r)\)-Fuss-Schröder paths of type \(\lambda\) by using Chung-Feller style. We also give two sets of sparse noncrossing partitions of \([2(k + 1)n + 1]\) and \([2(k + 1)n + 2]\) which are in bijection with the set of all small and large, respectively, \((k, r)\)-Fuss-Schröder paths of type \(\lambda\).

1. Introduction

A Dyck path of length \(n\) is a lattice path from \((0, 0)\) to \((n, n)\) using east steps \(E = (1, 0)\) and north steps \(N = (0, 1)\) such that it stays weakly above the diagonal line \(y = x\). It is well-known that the number of all Dyck paths of length \(n\) is given by the famous Catalan numbers

\[
\frac{1}{n+1} \binom{2n}{n}.
\]

A large Schröder path of length \(n\) is a lattice path from \((0, 0)\) to \((n, n)\) using east steps \(E\), north steps \(N\), and diagonal steps \(D = (1, 1)\) staying weakly above the diagonal line \(y = x\). The number of all large Schröder paths of length \(n\) is

\[
\frac{1}{n} \sum_{k=1}^{n} \binom{n}{k-1} \binom{n}{k} 2^k.
\]

A small Schröder path of length \(n\) is a large Schröder path with no diagonal steps on the diagonal line. The number of all small Schröder paths of length \(n\) is the half of the number of large Schröder paths of length \(n\). Note that a Dyck path is a large Schröder path without using diagonal steps.

For a large Schröder path (and hence for a Dyck path), its type is the integer partition formed by the length of the maximal adjacent east steps. For example, the large Schröder path \(NENNNEEDNEEDE\) has type \(\lambda = (2, 2, 1, 1)\). The enumeration of Dyck paths by type is done since Kreweras \[4\] introduced noncrossing partitions while the enumeration of large Schröder paths.

\textit{Key words and phrases.} Fuss-Schröder paths, type, sparse noncrossing partitions.
paths by type is recently done by An, Eu, and Kim [1]. The number of small Schröder paths of given type is not the half of the number of large Schröder paths and it is enumerated by Park and Kim [6].

Now we introduce Fuss analogue of Dyck and Schröder paths. Given a positive number $k$, a $k$-Fuss-Catalan path of length $n$ is a path from $(0,0)$ to $(n,kn)$ using east steps $E$ and north steps $N$ such that it stays weakly above the line $y = kx$. The number of all $k$-Fuss-Catalan paths of length $n$ is given by the Fuss-Catalan numbers

$$\frac{1}{kn+1} \binom{(k+1)n}{n}$$

and Armstrong [2] enumerates the number of $k$-Fuss-Catalan paths of given type.

For $k, r \ (1 \leq r \leq k)$, a large $(k, r)$-Fuss-Schröder path of length $n$ is a path $\pi$ from $(0,0)$ to $(n,kn)$ using east steps, north steps, and diagonal steps that satisfies the following two conditions:

(C1) the path $\pi$ never passes below the line $y = kx$, and

(C2) the diagonal steps of $\pi$ are only allowed to go from the line $y = kj + r - 1$ to the line $y = kj + r$, for some $j$.

A small Fuss-Schröder path is a large Fuss-Schröder path with no diagonal steps touching the line $y = kx$. The number of small $(k, r)$-Fuss-Schröder paths with fixed length and number of diagonal steps is independent of $r$ and it is given by Eu and Fu [3].

We will provide the number of small $(k, r)$-Fuss-Schröder paths of given length and type. We also give two conjectures about Fuss-Schröder paths and sparse noncrossing partitions which might be useful for the formula for the number of large $(k, r)$-Fuss-Schröder paths of given type and length.

2. D Y C K A N D S C H R Ö D E R P A T H S B Y T Y P E

In this section, we introduce previous results about the numbers of Dyck and Schröder paths with given length and type.

Given an integer partition $\lambda$, we set $m_\lambda := m_1(\lambda)!m_2(\lambda)!m_3(\lambda)!\cdots$, where $m_i(\lambda)$ is the number of parts of $\lambda$ equal to $i$. Note that $m_\lambda$ here is not the monomial symmetric function. We use $|\lambda|$ for the sum of the parts of $\lambda$.

First, we begin with the number of Dyck paths of given type. Kreweras [4] shows the following theorem using recursions and Liaw et al. [5] give a bijective proof.

**Theorem 1.** The number of

1) Dyck paths of length $n$ with type $\lambda = (\lambda_1, \ldots, \lambda_\ell)$

2) noncrossing partitions of $[n]$ with type $\lambda$
is
\[ \frac{n(n-1) \cdots (n-\ell(\lambda)+2)}{m_\lambda}. \]

If diagonal steps are allowed, Theorem 1 is generalized to Schröder path cases. An, Eu, and Kim [1] enumerate the number of large Schröder paths of given length and type.

**Theorem 2.** The number of

1. large Schröder paths of length \( n \) with type \( \lambda = (\lambda_1, \ldots, \lambda_\ell) \)
2. sparse noncrossing set partitions of \([n+|\lambda|+1]\) with arc type \( \lambda \)

is
\[ \frac{1}{|\lambda|+1} \binom{n}{|\lambda|} \binom{n+1}{\ell} \ell! \]

It is well-known that the number of small Schröder paths of length \( n \) is the half of the number of large Schröder paths of length \( n \). This is not the case when we count the number of small Schröder paths of fixed type. Park and Kim [6] provide the number of small Schröder paths of given length and type.

**Theorem 3.** The number of

1. small Schröder paths of length \( n \) with type \( \lambda = (\lambda_1, \ldots, \lambda_\ell) \)
2. large Schröder paths of length \( n \) with type \( \lambda \) with no diagonal steps after the last up step
3. connected sparse noncrossing set partitions of \([n+|\lambda|+1]\) with arc type \( \lambda \)

is given by
\[ \frac{1}{n+1} \binom{n-1}{|\lambda|-1} \binom{n+1}{\ell} \ell! \]

3. **Fuss-Schröder paths**

In this section, we consider Fuss analogue of Dyck and Schröder paths of given length and type, i.e. Fuss-Catalan paths and Fuss-Schröder paths of fixed length \( n \) with type \( \lambda \). Our goal is to enumerate the number of small Fuss-Schröder paths of length \( n \) with type \( \lambda \).

First, we begin with the number of Fuss-Catalan paths of given type.

**Theorem 4 ([2]).** The number of \( k \)-Fuss-Catalan paths of type \( \lambda \) is
\[ \frac{(kn)!}{m_\lambda \cdot (kn+1-\ell(\lambda))!}. \]

The following shows that the number of large \((k, r)\)-Fuss-Schröder paths of given length and type is independent of \( r \).
Lemma 5. Let $A_{n,\lambda}^{(k,i)}$ be the set of all large $(k,r)$-Fuss-Schröder paths of length $n$ with type $\lambda$. Then there is a bijection between $A_{n,\lambda}^{(k,i)}$ and $A_{n,\lambda}^{(k,j)}$ for $1 \leq i < j \leq k$.

Proof. Let $\pi$ be a path in $A_{n,\lambda}^{(k,i)}$. If $\pi$ has a diagonal step $D$ from the level $kt+i-1$ to the level $kt+i$, then it can be decomposed into $\mu_1 D \mu_2 \omega N \mu_3$, where $\omega$ is the section of east steps on level $kt+j-1$, $N$ is the north step from the level $kt+j-1$ to $kt+j$, $\mu_1$ goes from $(0,0)$ to the level $kt+i-1$, $\mu_2$ goes from the level $kt+i$ to the level $kt+j-1$, and $\mu_3$ goes from the level $kt+j$ to $(n, kn)$. Let $\tau$ be the path $\mu_1 N \mu_2 \omega D \mu_3$. Then $\tau$ has a diagonal step from the level $kt+j-1$ to the level $kt+j$ and its type is the same as the type of $\pi$. By applying the similar operations to all diagonal steps in $\pi$, we get a path in $A_{n,\lambda}^{(k,j)}$. \hfill $\Box$

Similarly, one can show that the number of small $(k,r)$-Fuss-Schröder paths of given length and type is independent of $r$. The following theorem provides the number of small Fuss-Schröder paths of fixed length and type.

Theorem 6. The number of small $(k,r)$-Fuss-Schröder paths of type $\lambda$ ($1 \leq r \leq k$) is

$$\binom{n-1}{|\lambda|-1} \frac{nk}{\ell-1} \frac{\ell!}{m_{\lambda}} = \frac{1}{nk+1} \binom{n-1}{\ell} \binom{nk+1}{\ell} \frac{\ell!}{m_{\lambda}}.$$ 

When $k = 1$, this becomes

$$\binom{n-1}{|\lambda|-1} \frac{n}{\ell-1} \frac{\ell!}{m_{\lambda}} = \frac{1}{n+1} \binom{n-1}{\ell} \binom{n+1}{\ell} \frac{\ell!}{m_{\lambda}}.$$ 

When $|\lambda| = n$, this becomes

$$\binom{nk}{\ell-1} \frac{\ell!}{m_{\lambda}} = \frac{1}{nk+1} \binom{nk+1}{\ell} \frac{\ell!}{m_{\lambda}}.$$ 

Note that the number of small $(k,r)$-Fuss-Schröder paths of the case $k = 1$ is the number of small Schröder paths since paths staying above the line $y = x$ are considered. In the case of $|\lambda| = n$, the number of small $(k,r)$-Fuss-Schröder paths is the number of $k$-Fuss-Catalan paths since we count paths without using diagonal steps only.

Proof of Theorem 6. The number of small $(k,r)$-Fuss-Schröder paths of type $\lambda$ ($1 \leq r \leq k$) is independent of $r$ (by Lemma 5), and $r = k$ is assumed in this proof.

To show the number of small $(k,k)$-Fuss-Schröder paths of type $\lambda = \lambda_1 \lambda_2 \ldots \lambda_\ell$ is $\frac{1}{nk+1} \binom{\lambda-1}{\ell} \binom{nk+1}{\ell} \frac{\ell!}{m_{\lambda}}$, we first consider all the paths from $(0,0)$ to $(n, kn)$ of
type \( \lambda \) using east steps, north steps, and diagonal steps such that the diagonal steps are only allowed to go in the \( ik \)th rows for \( 2 \leq i \leq n \). Being such a path, \( n - |\lambda| \) rows are chosen from \( 2k, 3k, \ldots, nk \)th rows for \( n - |\lambda| \) diagonal steps, and \( \ell \) lines are needed from \( nk + 1 \) horizontal lines for \( \ell \) east runs (i.e. maximal consecutive east steps). Since the \( \ell \) east runs are ordered in \( \frac{n!}{m \lambda} \) different ways, there are total \( \left( \frac{n-1}{|\lambda|-1} \right) (nk+1) \frac{n!}{m \lambda} \) paths satisfying the above conditions.

We say a diagonal step of the paths has a flaw if it is located below the line \( y = kx + k \), and a north step has a flaw if it is below the line \( y = kx \). North steps in \((ik-1)\)th, \((ik-2)\)th, \ldots, \((ik-k+1)\)th rows between \( y = kx \) and \( y = kx + k \) are also considered as flawed steps if a flawed diagonal step is contained in the \( ik \)th row. Note that each path has \( j \) flaws for some \( j \in [1, kn] \) if and only if it is not a small \((k, k)\)-Fuss-Schröder path. The following rules identify \( nk \) flawed paths corresponding to a small \((k, k)\)-Fuss-Schröder path by increasing the number of flaws from the small \((k, k)\)-Fuss-Schröder path one by one, and vice versa.

1. Increasing one flaw:
   1a) See a given path as a sequence on \( \{E, N, D\}^{(k+1)n} \), and take the leftmost east run which is located right before D, NE, any flawed step, or nothing (i.e. the east run is the last steps). Move the east run to the left on the sequence until the path get exactly one more flaw on condition that the east run does not pass D, NE, or any flawed step without an increase on the number of flaws. Example 7 is given for this basic case.
   1b) If the leftmost east run passes D, NE, or any flawed step without an increase on the number of flaws as seen in Example 8, stop the moving right after passing the step, and do (1a) for the following east runs again.
   1c) In the case that the leftmost east run in (1a) contains the first east step of the sequence, apply \((-1, -k)\)-circular shifts (i.e. shifting each step of the path along a vector \((-1 \mod n, -k \mod kn)\) to get another path) repeatedly until obtaining a new path from \((0, 0)\) to \((n, kn)\) such that the first step is not an east step and the \( k \)th row doesn’t have a diagonal step. If there is no such path, the original path already has \( kn \) flaws and we may stop. See Example 9. To keep type \( \lambda \) for the new path, we consider two east runs of the original path are separately even if they are connected after circular shifts.
   1d) If a new path is obtained in (1c), take the leftmost one (say the \( s \)th east run) and the the rightmost one (say the \( (s+t) \)th east run) among east runs located right before D, NE, any flawed step, or on the same line with another east run. Move the \( (s+t-u) \)th east
run to the position of the \((s + t - u - 1)\)th east run \((0 \leq u \leq t - 1)\) and, after that, apply (1a) to the \(s\)th east run. There are two examples for the circular shift in Example [10].

(2) Decreasing one flaw:

(2a) Note that there is always an east run right before the leftmost flawed step on a sequence. Move the east run to the right until it meets D, E, NE, or the second leftmost flawed step if those steps exist. Otherwise, the east run becomes the last steps of the sequence. Look at the second part in Example [7].

(2b) If another east run is located right before flawless D or flawless NE which is prior the east run used in (2a) as Figure [2(b)] in Example [8], move the another one to the right until meeting with next D, NE or a flawed step.

(2c) If two east runs share the same horizontal line (i.e. east lines are back to back on a sequence) while any east run rearrangement is done, move the right east run to the right until it meets a next east run or a horizontal line \(y = (w - 1)k\) where \(wk\)th row is the highest row containing no diagonal step among \(ik\)th rows \((2 \leq i \leq n)\) as Example [10].

(2d) If (2c) has ever been applied like both examples in Example [10], then we need a \((-w + 1) \mod n, (-w + 1)k \mod kn\)-circular shift. Otherwise, we may stop.

Therefore, there are \(\frac{1}{nk + 1}\left(\binom{n - 1}{\lambda - 1}\frac{n}{\lambda}\right)\frac{\ell}{m\lambda}\) small \((k, k)\)-Fuss-Schröder paths of type \(\lambda\). \(\square\)

In the next 4 examples, we consider paths from \((0, 0)\) to \((n, kn)\) of type \(\lambda\) using east steps, north steps, and diagonal steps such that the diagonal steps are only allowed to go in the \(ik\)th rows for \(2 \leq i \leq n\) where \(n = 4\), \(k = 2\), and \(\lambda = (2, 1)\).

**Example 7.** The 2-east run in Figure [11(a)] is the leftmost one among east runs located right before D, NE, any flaw step, or nothing. Move the 2-east run to the left one unit on the sequence so that the path get one more flaw, a diagonal step below the line \(y = 2x + 2\), as in Figure [11(b)].

In the reverse, the leftmost (and unique) flawed step is a diagonal step, and there is a 2-east run right before the flawed step in Figure [11(b)]. Since there is no D, E, NE, or the second leftmost flawed step after the 2-east run, move the 2-east run to the right until it becomes the last steps of the sequence as in Figure [11(a)].

**Example 8.** The 1-east run in Figure [2(a)] is the leftmost one among east runs right before D, NE, any flaw step, or nothing. Moving the 1-east run to the left, it passes D without increase on the number of flaws. Hence, stop the
moving right after passing the diagonal step, and move the second leftmost one, the 2-east run, to the left one unit on the sequence so that the path get one more flaw, a north step below the line $y = 2x$, as in Figure 2(b).

Inversely, the leftmost flawed step is the highest north step in Figure 2(b), and move a 2-east run right before the flawed step until it becomes the last steps of the sequence. After that, move the 1-east run, right before flawless D prior the 2-east run, to the right 2 units until meeting with NE as in Figure 2(a).

**Example 9.** If $(-1, -2)$-circular shift is applied to the path in Figure 3 only once, we get a path containing a diagonal step on the 2nd row. If $(-1, -2)$-circular shift is applied twice, a path having 2-east run as first steps is obtained.
Three \((-1, -2)\)-circular shifts generate a disconnected path not from \((0, 0)\) to \((4, 8)\). Hence, one more flaw cannot be added, and it is natural since the path in Figure 3 is already containing 8 flaws fully.

Figure 3. Rules for adding and subtracting flaws in Theorem 6(1c).

Example 10.

1. If \((-1, -2)\)-circular shift is applied to the path in Figure 4(a) twice, we obtain a new path from \((0, 0)\) to \((4, 8)\) such that the first step is not an east step and the 2nd row doesn’t have a diagonal step as in Figure 4(b). To keep type \(\lambda\) for the new path, 3 east steps on the same horizontal line after circular shifts are considered as two east runs, 2-east run and 1-east run in order, separately. Now in Figure 4(b), take the leftmost one (2-east run) and the rightmost one (1-east run) among east runs right before D, NE, any flawed step, or on the same line with another east run. The 1-east run is already in the (next) position of the 2-east run on the sequence, and the 2-east run is moved to the left one unit so that a path gets one more flawed north step as shown in Figure 4(c).

To the opposite direction, the 2-east run right before the leftmost flawed step in Figure 4(c) is moved to the right one unit on the sequence and meets another east run as in Figure 4(b). Since two east runs share the same horizontal line, and 6th \((w = 3)\) row is the highest row containing no diagonal step among \(ik\)th rows \((2 \leq i \leq n)\), and the right east run (1-east run) is already on the horizontal line \(y = 4\), the only necessary thing to get a path in Figure 4(a) is a \((2, 4)\)-circular shift.

2. Similar to the previous case, \((-1, -2)\)-circular shift is applied to the path in Figure 5(a) twice. Take the leftmost one (2-east run) and the rightmost one (1-east run) of the newly obtained path in Figure 5(b),
(a) 4 flaws   (b) circular shifting   (c) 5 flaws

Figure 4. Rules for adding and subtracting flaws in Theorem 6(1c) and 6(2c).

and move the 1-east run to the position of the 2-east run on the sequence. As the last step, the 2-east run is moved to the left one unit to get one more flawed north step as shown in Figure 5(c).

Conversely, the 2-east run right before the leftmost flawed step in Figure 5(c) is moved to the right one unit on the sequence and meets another east run. Then, move the right east run (1-east run) between two east runs sharing the same horizontal line to the right until it meets a horizontal line $y = 4$ as in Figure 5(b). Lastly, a (2, 4)-circular shift is applied to get a path in Figure 5(a).

(a) 5 flaws   (b) circular shifting   (c) 6 flaws

Figure 5. Rules for adding and subtracting flaws in Theorem 6(1d) and 6(2d).
4. Fuss-Schröder paths and noncrossing partitions

In this section, to extend the results of small \((k, r)\)-Fuss-Schröder paths with type \(\lambda\) to large \((k, r)\)-Fuss-Schröder paths with type \(\lambda\), we introduce sparse noncrossing partitions which are in bijection with the set of \((k, r)\)-Fuss-Schröder paths of type \(\lambda\).

A noncrossing partition of \([n] := \{1, 2, \ldots, n\}\) is a pairwise disjoint subsets \(B_1, B_2, \ldots, B_l\) of \([n]\) whose union is \([n]\) in which, if \(a\) and \(b\) belong to one block \(B_i\) and \(x\) and \(y\) to another block \(B_j\), they are not arranged in the order \(axby\). Note that, if the elements 1, 2, \ldots, \(n\) are equally-spaced dots on a horizontal line, and all the successive elements of the same block are connected by arc above the line, then no arches cross each other for a noncrossing partition of \([n]\). A noncrossing partition is called sparse if no two consecutive integers are in the same block. We give an order to the blocks by the order of the smallest element of each block. Connected components of a noncrossing partition of \([n]\) are \(\{1, 2, \ldots, i_1\}, \{i_1 + 1, i_1 + 2, \ldots, i_2\}, \ldots, \{i_t + 1, i_t + 2, \ldots, n\}\) where \(i_1\) is the greatest element of the block containing 1, \(i_2\) is the greatest element of the block containing \(i_1 + 1\), and so on. The arc type of a noncrossing partition is the integer partition obtained from the numbers of connected arcs.

In Figure 6, a noncrossing partition of \([10]\) is written by 4 ordered blocks. Two connected components are \(\{1, 2, \ldots, 8\}\) and \(\{9, 10\}\), and the arc type of the given noncrossing partition is \((3, 2, 1)\). Since 2 and 3 are in the same block, this partition is not a sparse noncrossing partition.

![Figure 6](image_url)

**Figure 6.** Noncrossing partition \((\{1, 5, 6, 8\}, \{2, 3, 4\}, \{7\}, \{9, 10\})\) of \([10]\).

Before introducing a special noncrossing partition, we need more notations and labellings concerning \((k, k)\)-Fuss-Schröder paths. On a skew shape from \((0, 0)\) to \((n, kn)\), the \((n - i)k\)th row is labeled by \(i(k + 1) + 2\) for \(i \geq 0\), and the horizontal line \(y = (n - i)k + j\) is labeled by \(i(k + 1) + 1 - j\) if \(0 \leq j < k\). See Figure 7 for example.

We can represent a \((k, k)\)-Fuss-Schröder path of length \(n\) as a sequence \(s_1 \leq s_2 \leq \cdots \leq s_n\) such that \(s_i\)’s are the labels in which \(E\) and \(D\) steps located. In Figure 7, the sequence corresponding to the path is 1124. Then, we trace sequence representations of \((k, k)\)-Fuss-Schröder path as follows:

1. Start with two numbers 1 and 2 on a horizontal line, and read a sequence from \(s_1\) to \(s_n\).
(2) For \( m \) consecutive east steps labeled by \( s_j = s_{j+1} = \cdots = s_{j+m-1} \),
   (a) replace each number \( i > s_j \) with \( i + m(k+1) \), and
   (b) replace \( s_j \) with \( s_j, s_j + 1, s_j, s_j + 2, s_j, \ldots, s_j, s_j + m(k+1), s_j \).

(3) For a diagonal step labeled by \( s_j \),
   (a) replace each number \( i > s_j \) with \( i + k + 1 \), and
   (b) replace \( s_j \) with \( s_j, s_j + 1, s_j, s_j + 2, s_j, \ldots, s_j, s_j + (k+1), s_j \).

(4) For a north step, do nothing.

Then the resulting sequence gives a sparse noncrossing partition of \([2(k+1)n + 2]\) in which the elements at the number \( i \) positions are the elements of \( i \)th block. See Example 11.

**Example 11.** The sparse noncrossing partition corresponding to a \((k, k)\)-Fuss-Schröder path in Figure 7 is given in Figure 8.

![Figure 8](image-url)

See the following conjecture about characteristics of sparse noncrossing partitions from \((k, k)\)-Fuss-Schröder paths.

**Conjecture 12.** The set of large \((k, k)\)-Fuss-Schröder paths of type \( \lambda = \lambda_1\lambda_2 \cdots \lambda_\ell \) of length \( n \) is in bijection with the set of sparse noncrossing partitions of \([2(k+1)n + 2]\) with 2 connected components such that
(1) the arc type is 
\((k+1)\lambda_1, (k+1)\lambda_2, \ldots, (k+1)\lambda_\ell, (k+1)^{n-|\lambda|}\),
(2) the set of \((i(k+1)+2)\)th blocks consists of \(n-|\lambda|\) blocks of arc type 
\((k+1)\) and \(|\lambda|\) singleton blocks where \(0 \leq i \leq n-1\), and
(3) the set of last \(t(k+1)\) blocks has at least \(t(k-1)+1\) singleton blocks 
for \(t \geq 1\).

Note that, if the second connected component is a singleton block, the 
corresponding \((k,k)\)-Fuss-Schröder path is a small \((k,k)\)-Fuss-Schröder path. 
Hence, we have a next conjecture.

**Conjecture 13.** The set of small \((k,k)\)-Fuss-Schröder paths of type \(\lambda = \lambda_1\lambda_2\ldots\lambda_\ell\) of length \(n\) is in bijection with the set of connected sparse non-
crossing partitions of \([2(k+1)n+1]\) such that 
(1) the arc type is 
\((k+1)\lambda_1, (k+1)\lambda_2, \ldots, (k+1)\lambda_\ell, (k+1)^{n-|\lambda|}\),
(2) the set of \((i(k+1)+2)\)th blocks consists of \(n-|\lambda|\) blocks of arc type 
\((k+1)\) and \(|\lambda|\) singleton blocks where \(0 \leq i \leq n-1\), and
(3) the set of last \(t(k+1)\) blocks has at least \(t(k-1)+1\) singleton blocks 
for \(t \geq 1\).

However, it has a different correspondence for the case that the first con-
nected component is a singleton block. In this case, all the partitions counts 
\((k,k)\)-Fuss-Schröder paths starting with a diagonal step.

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