The Baer Invariant of Semidirect and Verbal Wreath Products of Groups *

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Abstract

W. Haebich (1977, Journal of Algebra 44, 420-433) presented some formulas for the Schur multiplier of a semidirect product and also a verbal wreath product of two groups. The author (1997, Indag. Math., (N.S.), 8(4), 529-535) generalized a theorem of W. Haebich to the Baer invariant of a semidirect product of two groups with respect to the variety of nilpotent groups of class at most $c \geq 1$, $N_c$. In this paper, first, it is shown that $\mathcal{V}M(B)$ and $\mathcal{V}M(A)$ are direct factors of $\mathcal{V}M(G)$, where $G = B \bowtie A$ is the semidirect product of a normal subgroup $A$ and a subgroup $B$ and $\mathcal{V}$ is an arbitrary variety. Second,

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it is proved that $\mathcal{N}_c M(B\lhd A)$ has some homomorphic images of Haebich’s type. Also some formulas of Haebich’s type is given for $\mathcal{N}_c M(B\lhd A)$, when $B$ and $A$ are cyclic groups. Third, we will present a formula for the Baer invariant of a $\mathcal{V}$-verbal wreath product of two groups with respect to the variety of nilpotent groups of class at most $c \geq 1$, where $\mathcal{V}$ is an arbitrary variety. Moreover, it is tried to improve this formula, when $G = AWr_\mathcal{V} B$ and $B$ is cyclic. Finally, a structure for the Baer invariant of a free wreath product with respect to $\mathcal{N}_c$ will be presented, specially for the free wreath product $AWr_\mathcal{V} B$ where $B$ is a cyclic group.

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1. INTRODUCTION AND MOTIVATION

As stated in my previous paper (2001, Journal of Algebra 235, 15-26), there are some results on the Schur multiplier and the Baer invariant of the direct product of groups by I. Schur [16], J. Wiegold [18], M.R.R. Moghaddam [13], G.Ellis [4], and the author [12].

It is known that a semidirect product of two groups is a generalization of a direct product of groups. Also, we know that a verbal wreath product is a kind of semidirect product. Therefore, it is interesting to find some formulas for the Schur multiplier or the Baer invariant of a semidirect product and a verbal wreath product.

In 1972, K.I. Tahara [17, (Theorem 2.2.5)], using cohomological methods, gave a structure for a semidirect product of two groups. W. Haebich [6]
in 1977, presented some formulas for the Schur multiplier of a semidirect product and a $\mathcal{V}$-verbal wreath product of two groups, where $\mathcal{V}$ is an arbitrary variety. His method was based on presentations of groups.

In 1972, N. Blackburn [2] gave an explicit formula for the Schur multiplier of a standard wreath product of two groups. Also E.W. Read [15] in 1976, found an explicit formula for the Schur multiplier of a wreath product (we know that a standard wreath product is a special kind of a wreath product). It should be mentioned that their formulas were more explicit than Haebich’s one. But Haebich’s results were more general than Blackburn and Read’s formulas.

In 1997, the author [10] generalized a result of W. Haebich [6, (Theorem 1.7)] to the Baer invariant of a semidirect product with respect to the variety $\mathcal{N}_c$.

Now, in this paper, we concentrate on the Haebich’s method in order to find some structures for the Baer invariant of a semidirect product and a verbal wreath product of two groups with respect to the variety of nilpotent groups of class at most $c$, $\mathcal{N}_c$.

More precisely, first, using the functorial property of the Baer invariant, we show that $\mathcal{V}M(B)$ and $\mathcal{V}M(A)$ are direct factors of $\mathcal{V}M(G)$, where $G = B \triangleright A$ is the semidirect product of $A$ by $B$ and $\mathcal{V}$ is an arbitrary variety of groups (Theorem 3.2). Also, in Section 3, we find some homomorphic images and in finite case subgroups with structures similar to Haebich’s type [6,(Theorems 2.2 and 2.3)] for the Baer invariant of a semidirect product $B \triangleright A$ with respect to the variety $\mathcal{N}_c$ (Theorem 3.6). Moreover, if $B$ and $A$ are cyclic groups, then we show that the above structures are isomorphic to
\(\mathcal{N}_c M(B \times A)\) (Theorem 3.7 and Corollary 3.11).

In Section 4, we concentrate on a verbal wreath product and first, we find a structure for \(\mathcal{N}_c M(A \text{Wr}_V B)\) similar to Haebich’s type (Theorem 4.2). Second, we improve this structure for a homomorphic image and in finite case for a subgroup of \(\mathcal{N}_c M(A \text{Wr}_V B)\) (Theorem 4.5). Moreover, if \(B\) is cyclic, then we prove that the last structure is isomorphic to \(\mathcal{N}_c M(A \text{Wr}_V B)\) (Theorem 4.6).

Finally, in Section 5, using the previous results, we try to find a structure for the Baer invariant of a free wreath product with respect to the variety \(\mathcal{N}_c\). Specially, we present a formula for \(\mathcal{N}_c M(A \text{Wr}_n B)\), when \(B\) is cyclic and \(\mathcal{N}_c M(A \text{Wr}_n B)\) is finite (Theorem 5.5), and also a formula when \(A\) and \(B\) are both cyclic groups (Theorem 5.6).

2. NOTATION AND PRELIMINARIES

We assume that the reader is familiar with the notions of variety of groups, verbal and marginal subgroups (see [14]). Let \(G\) be a group with a free presentation \(G \cong F/R\). Then the Baer invariant of \(G\) with respect to the variety \(V\), denoted by \(V M(G)\) is defined to be

\[V M(G) = \frac{R \cap V(F)}{[RV^*F]} ,\]

where \(V(F)\) is the verbal subgroup of \(F\) and

\([RV^*G] = \langle v(g_1, \ldots, g_{i-1}, g_i a, g_{i+1}, \ldots, g_n) v(g_1, \ldots, g_i, \ldots g_n)^{-1} \mid a \in R, 1 \leq i \leq n, v \in V, g_i \in G, n \in \mathbb{N} \rangle ,\]

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which is independent of the choice of the presentation of $G$ and it is always an abelian group (see [1,5,9] for further properties of the Baer invariant).

In special case, if $V$ is the variety of abelian groups, $A$, then the Baer invariant of a group $G$ will be

$$\frac{R \cap F'}{[R,F]}$$

the Schur multiplier of $G$, where, in finite case, by Hopf’s formula [7] is isomorphic to the second cohomology group of $G$.

Also, if $V$ is the variety of nilpotent groups of class at most $c \geq 1$, $N_c$, then the Baer invariant of a group $G$ will be

$$N_cM(G) = \frac{R \cap \gamma_c(F)}{[R,\ cF]}$$

where $[R,\ cF] = [R,\ F,\ F,\ \ldots,\ F]$. The above notion is also called the $c$-nilpotent multiplier of $G$ (see [3]).

In order to deal with the Baer invariant of a semidirect product we need a presentation for the semidirect product which is given as follows.

**Lemma 2.1** [6] Suppose $G$ is a semidirect product (or a splitting extension) of $A$ by $B$ under $\theta : B \rightarrow Aut(A)$ and

$$1 \rightarrow R_1 \rightarrow F_1 \overset{\nu_1}{\rightarrow} A \rightarrow 1 \ , \ 1 \rightarrow R_2 \rightarrow F_2 \overset{\nu_2}{\rightarrow} B \rightarrow 1$$

are free presentations for $A$ and $B$, respectively. Then

$$1 \rightarrow R \rightarrow F \rightarrow G \rightarrow 1$$

is a free presentation for $G$, where
(i) \( F = F_1 \ast F_2 \), the free product of \( F_1 \) and \( F_2 \);  
(ii) \( R = R_1^F R_2^F S \);  
(iii) \( S = \langle f_1^{-1} f_2 f_1 | f_1, f_2 \in F_1; f_2, f_2 \in F_2; \nu_1(f_1) = \theta(\nu_2(f_2))(\nu_1 f_1) \rangle^F \).

W. Haebich [6] in 1977, using the above lemma prove the following theorem about the Schur multiplier of a semidirect product.

**Theorem 2.2** [6] Let \( G \) be a semidirect product of \( A \) by \( B \) under \( \theta \) and \( F_1/R_1 \) and \( F_2/R_2 \) be presentations for \( A \) and \( B \), respectively. Then, by the above notation, the following isomorphism holds:

\[
M(G) \cong M(B) \oplus \frac{S \cap F'}{[R_2, F_1][S, F]} .
\]

A generalization of the above theorem was presented by the author in 1997 as follows.

**Theorem 2.3** [10] Let \( G \) be a semidirect product of \( A \) by \( B \) under \( \theta : B \rightarrow Aut(A) \) and \( N_c \) be the variety of nilpotent groups of class at most \( c (c \geq 1) \). Then

\[
N_c M(G) \cong N_c M(B) \oplus \frac{S \cap \gamma_{c+1}(F)}{\prod [R_2, F_1, F_2]_c [S, cF]} ,
\]

where

\[
\prod [R_2, F_1, F_2]_c = \langle [r_2, f_1, \ldots, f_c] | f_i \in F_1 \cup F_2, 1 \leq i \leq c, \exists k f_k \in F_1 \rangle^F .
\]

In particular, \( N_c M(B) \) can be regarded as a direct factor of \( N_c M(G) \).

We also assume that the reader is familiar with notions of a verbal product and the cartesian subgroup of a free product (see [11,14]). We need the following properties of the cartesian subgroup.
Lemma 2.4 [14] Let \( \{A_i | i \in I \} \) be a family of groups and \( \prod_{i \in I}^* A_i \) be the free product of \( A_i \)'s, and let \( [A_i]^* \) be the cartesian subgroup of the above free product. Then

(i) The cartesian subgroup avoids the constituents i.e \( [A_i]^* \cap A_j = 1 \), for all \( j \in I \).

(ii) If \( a \in \prod_{i \in I}^* A_i \), then \( a = a_{i_1}a_{i_2}\ldots a_{i_m}c \), where \( a_{i_j} \neq 1 \) for all \( j \), \( i_1 < i_2 < \ldots < i_m \) and \( c \in [A_i]^* \). The elements \( a_{i_j} \) and \( c \) are uniquely determined by \( a \) and chosen order of \( I \).

Lemma 2.5 [14] Let \( \mathcal{V} \) be a variety of groups defined by the set of laws \( V \). Then

(i) If \( V(A) \) is the verbal subgroup of \( A = \prod_{i \in I}^* A_i \), then \( V(A) \cap A_i = V(A_i) \) for all \( i \in I \).

(ii) If \( A = \prod_{i \in I}^* A_i \), then

\[
V(A) = (\prod_i^* V(A_i))(V(A) \cap [A_i]^*) .
\]

Now, in the following, you can find the definition of a verbal wreath product.

Definition 2.6 Given arbitrary groups \( A \) and \( B \), let \( A_b \) be an isomorphic copy of \( A \) for each \( b \in B \) and denote by \( a_b \) the element of \( A_b \) mapped to \( a \in A \).

Consider the \( \mathcal{V} \)-verbal product of \( A_b \)'s, \( G_V = \prod_{b \in B}^V A_b \) corresponding to the variety \( \mathcal{V} \) with a set of words \( V \), which is isomorphic to the quotient \( C/C_V \), where \( C = \prod_{b \in B}^* A_b \) and \( C_V = V(C) \cap [A_b]^* \). The map \( a_b \mapsto a_{bb'} \) for all \( a \in A \), \( b \in B \) and fixed \( b' \in B \) induces automorphisms \( \theta_s b' \) and \( \theta_V b' \) of \( C \) and \( G_V \), respectively, i.e.
\[ \theta_* : B \rightarrow Aut(C) \quad , \quad \theta_V : B \rightarrow Aut(G_V) \]

\[ b' \mapsto \theta_* b' : C \rightarrow C \quad b' \mapsto \theta_V b' : G_V \rightarrow G_V \]

\[ a_b \mapsto a_{bb'} \quad \bar{a}_b \mapsto \bar{a}_{bb'} \]

It can be proved that \( \theta_* : B \rightarrow Aut(C) \) and \( \theta_V : B \rightarrow Aut(G_V) \) are monomorphisms.

Now, the \( \mathcal{V}\)-verbal wreath product of \( A \) by \( B \) is the semidirect product of \( G_V \) by \( B \) under \( \theta_V \), denoted by \( AWr_V B \). The two special cases of the \( \mathcal{V}\)-verbal wreath product are important, the free wreath product and the standard wreath product.

(i) If \( \mathcal{V} \) is the variety of all groups, then the \( \mathcal{V}\)-verbal wreath product is the free wreath product which is the semidirect product of \( C \) by \( B \) under \( \theta_* \), denoted by \( AWr_* B \).

(ii) If \( \mathcal{V} = A \) is the variety of abelian groups then the \( \mathcal{V}\)-verbal wreath product is the standard wreath product which is the semidirect product of the direct product \( \prod_{b \in B} A_b \) by \( B \) under

\[ \theta : B \rightarrow Aut(\prod_{b \in B} A_b) \]

\[ b' \mapsto \theta b' : \prod_{b} A_b \rightarrow \prod_{b} A_b \]

\[ \{a_b\}_b \mapsto \{a_{bb'}\}_b \]

denoted by \( AWr B \) or \( A \wr B \).

Now, a presentation for a \( \mathcal{V}\)-verbal wreath product \( AWr_V B \) is given based on the presentations of \( A \) and \( B \) as follows.
**Lemma 2.7** [6] Let $A$ and $B$ be two groups with the following presentations:

\[ 1 \rightarrow R_1 \rightarrow F_1 \xrightarrow{\nu_1} A \rightarrow 1 \ , \ 1 \rightarrow R_2 \rightarrow F_2 \xrightarrow{\nu_2} B \rightarrow 1 \ . \]

Then

\[ 1 \rightarrow R \rightarrow F \rightarrow AWR_V B \rightarrow 1 \]

is a free presentation for the $V$-verbal wreath product $AWR_V B$, where

(i) $F = F_1 * F_2$, the free product of $F_1$ and $F_2$;

(ii) $R = R_2 S_V$;

(iii) $S_V = [R_2, F_1]^F R_1^F R_V$;

(iv) $R_V = V(H) \cap [(F_1^F)^H]$ with $H = F_1^{F_2}$ (that is, $F_1^F$) and

\[ [(F_1^F)^H] = < [x,y] | x \in (F_1^u)^H, y \in (F_1^v)^H, u,v \in F_2 > . \]

Now, the following theorem gives a structure of the Schur multiplier of a $V$-verbal wreath product.

**Theorem 2.8** [6] In accordance with the above notation, the following isomorphism holds.

\[ M(AWR_V B) \cong M(A) \oplus M(B) \oplus \frac{C_V}{[C_V, AWR_* B]} . \]

**Corollary 2.9**

\[ M(AWR_* B) \cong M(A) \oplus M(B) . \]

**Proof.** We know that $AW_* B = AWR_V B$, where $V = \{1\}$. So $V(C) = 1$ and then $C_V = 1$. Now by Theorem 2.8 the result holds. □
Corollary 2.10

\[ M(A \wr B) \cong M(A) \oplus M(B) \oplus \frac{[A_b]^*}{[[A_b]^*, AW_{r_V}B]}, \]

where \([A_b]^*\) is the cartesian subgroup of the free product \(\prod_{b \in B} A_b\).

**Proof.** As mentioned before \(A \wr B = AW_{r_V}B\), where \(V\) is the variety of abelian groups. Therefore \(V(C) = C'\) and \(C_V = V(C) \cap [A_b]^* = [A_b]^*\) since \([A_b]^* \subseteq C'\). Now, the result holds by Theorem 2.8. \(\square\)

We also need the following lemmas in our investigation.

**Lemma 2.11** [16] Let \(A\) be a subgroup of a group \(G\), and \(N\) be a normal subgroup of \(G\) and let \(\{M_i| i \in I\}\) be a family of normal subgroups of \(G\). Then

\[ [A \prod_i M_i, N] = [A, N] \prod_i [M_i, N]. \]

**Lemma 2.12**

If \(H\) is a subgroup of a finite abelian group \(G\), then \(G\) has a subgroup isomorphic to \(G/H\). Consequently, if \(A\) is a homomorphic image of a finite abelian group \(G\), then \(G\) has a subgroup isomorphic to \(A\).

### 3. SOME RESULTS ON THE BAER INVARIANT OF A SEMIDIRECT PRODUCT

First of all, by the functorial property of the Baer invariant, we are going to generalize somehow Theorems 2.2 and 2.3 to an arbitrary variety.

**Theorem 3.1** Using the notion of the Baer invariant, we can consider the following functor from the category of all groups to the category of all
abelian groups:

$$\mathcal{V}M(-) : \text{Groups} \to \text{Ab},$$

where \( \mathcal{V} \) is an arbitrary variety of groups.

**Proof.** See [5,9].

**Theorem 3.2** Let \( G = B \bowtie_A \theta \) be the semidirect product of \( A \) by \( B \) under \( \theta \). Then \( \mathcal{V}M(B) \) and \( \mathcal{V}M(A) \) are direct factors of \( \mathcal{V}M(G) \), where \( \mathcal{V} \) is an arbitrary variety.

**Proof.** By the property of a semidirect product, we have the following split exact sequence:

$$1 \to A \xrightarrow{h} G \xrightarrow{f} B \to 1.$$

So there exists a homomorphism \( g : B \to G \) such that \( f \circ g = I_B \). Now, applying the functorial property of \( \mathcal{V}M(-) \) (Theorem 3.1), the following exact sequence splits:

$$1 \to \ker(\mathcal{V}M(f)) \hookrightarrow \mathcal{V}M(G) \xrightarrow{\mathcal{V}M(f)} \mathcal{V}M(B) \to 1,$$

and \( \mathcal{V}M(f) \circ \mathcal{V}M(g) = I_{\mathcal{V}M(B)}. \) Hence, we have

$$\mathcal{V}M(G) \cong \mathcal{V}M(B) \oplus \ker(\mathcal{V}M(f)).$$

i.e. \( \mathcal{V}M(B) \) is a direct factor of \( \mathcal{V}M(G) \).

To prove that \( \mathcal{V}M(A) \) is a direct factor of \( \mathcal{V}M(G) \), by a similar argument as above, we should consider the following split exact sequence:

$$1 \to \mathcal{V}M(A) \xrightarrow{\mathcal{V}M(h)} \mathcal{V}M(G) \xrightarrow{\text{nat}} \mathcal{V}M(G) / \text{Im}(\mathcal{V}M(h)) \to 1. \square$$
Now, in the following, we state a theorem of W. Haebich which is improvements somehow the structure of the complementary factor of $M(B)$ in $M(G)$ in Theorem 2.2.

**Theorem 3.3** [6] By the assumptions and notations of Theorem 2.2, we have

$$M(G) \cong M(B) \oplus \frac{T \cap K'}{[T, K]} ,$$

where $K = F_1 \ast B$ and

$$T = \langle f_1^{-1} f_1[b, f_1] \mid f_1, f_1 \in F_1 ; b \in B ; \nu_1 f_1 = (\theta \nu_2 b)(\nu_1(f_1)) >^K \rangle .$$

Now, we are going to find a structure similar to the above for the complementary factor, $N_c M(B)$ of $N_c M(G)$ in Theorem 2.3.

First we need the following lemmas.

**Lemma 3.4** [10] Considering the assumptions and notations of Lemma 2.1, the following statements hold:

(i) $R_1$ and $[R_2, F_1]$ are subgroups of $S$;

(ii) $R = R_2 S$;

(iii) $R \cap \gamma_{c+1}(F) = (R_2 \cap \gamma_{c+1}(F_2))(S \cap \gamma_{c+1}(F))$, for all $c \geq 1$;

(iv) $[R, c F] = [R_2, c F_2] \prod [R_2, F_1, F_2] c[S, c F]$, for all $c \geq 1$.

**Lemma 3.5** [6] Let $C$ and $\overline{C}$ be two normal subgroups of $A$ and let $A$ be a subgroup of a group $B$. Let $\phi$ and $\overline{\phi}$ be two homomorphisms from $B$ to any groups such that

$$(A \cap Ker \phi)C = (A \cap Ker \overline{\phi})\overline{C} .$$

Then the map $\psi : \phi(A)/\phi(C) \longrightarrow \overline{\phi}(A)/\overline{\phi}(\overline{C})$ given by

$$(\phi a) \phi(C) \longmapsto (\overline{\phi} a) \overline{\phi}(\overline{C})$$
is an isomorphism.

Now we are in a position to state and prove the main results of this section.

**Theorem 3.6** Let $G$ be a semidirect product of $A$ by $B$ under $\theta : B \to Aut(A)$, and $\mathcal{N}_c$ be the variety of nilpotent groups of class at most $c$. Then, using the notation of Theorem 3.3, there exists an epimorphism as follows:

$$\mathcal{N}_c M(G) \longrightarrow \mathcal{N}_c M(B) \oplus \frac{T \cap \gamma_{c+1}(K)}{[T, \gamma_c K]}.$$  

So, in finite case, $\mathcal{N}_c M(G)$ has a subgroup isomorphic to

$$\mathcal{N}_c M(B) \oplus \frac{T \cap \gamma_{c+1}(K)}{[T, \gamma_c K]}.$$  

**Proof.** By the universal property of a free product, let $\delta$ be the natural homomorphism from $F = F_1 * F_2$ onto $K = F_1 * B$ induced by $\nu_2 : F_2 \to B$ and the identity on $F_1$. If $b = \nu_2(f_2)$ and $(\theta b)(\nu_1(f_1)) = \nu_1(f_1)$, then

$$\delta(f_1^{-1} \mathcal{T}_1[f_2, f_1]) = \delta(f_1)^{-1}\delta(\mathcal{T}_1)[\delta(f_2), \delta(f_1)] = f_1^{-1}\mathcal{T}_1[b, f_1].$$

Thus $\delta(S) = T$ and hence $\delta([S, \gamma F]) = [\delta(S), \gamma \delta(F)] = [T, \gamma K]$.

Also, we have

$$\delta(S \cap \gamma_{c+1}(F)) \subseteq \delta(S) \cap \delta(\gamma_{c+1}(F)) = T \cap \gamma_{c+1}(K).$$

To show the reverse containment, let $u \in T \cap \gamma_{c+1}(K)$. Then there exists $x \in S$ and $y \in \gamma_{c+1}(F)$ such that $\delta(x) = u = \delta(y)$, and so $yx^{-1} \in Ker \delta$. It is easy to see that $Ker \delta = R_2^F$, so $yx^{-1} \in R_2^F$ and hence $y \in R_2^F S$.

By Lemma 3.4 parts (i) and (ii) we have $R = R_2^F S$. Therefore

$$y \in R \cap \gamma_{c+1}(F).$$
Also, by Lemma 3.4 part \((iii)\) \(R \cap \gamma_{c+1}(F) = (R_2 \cap \gamma_{c+1}(F_2))(S \cap \gamma_{c+1}(F)).\) 
Since \(R_2 \cap \gamma_{c+1}(F_2) \subseteq \text{Ker}\delta\), \(u = \delta(y) \in \delta(R \cap \gamma_{c+1}(F)) = \delta(S \cap \gamma_{c+1}(F))\), thus
\[
\delta(S \cap \gamma_{c+1}(F)) = T \cap \gamma_{c+1}(K).
\]
Hence
\[
\frac{T \cap \gamma_{c+1}(K)}{T, cK} = \frac{\delta(S \cap \gamma_{c+1}(F))}{\delta([S, cF])}.
\]
Now, putting \(\phi = \delta : F \to K, \overline{\phi} = \text{identity} : F \to F, A = S \cap \gamma_{c+1}(F)\) and 
\[
C = \prod [R_2, F_1, F_2]_c[S, cF];
\]
we have
\[
(A \cap \text{Ker}\phi)C = (S \cap \gamma_{c+1}(F) \cap R_2^F) \prod [R_2, F_1, F_2]_c[S, cF]
\]
\[
= (S \cap \gamma_{c+1}(F) \cap R_2^F)[S, cF] \quad \text{(since } \prod [R_2, F_1, F_2]_c \subseteq R_2^F \cap \gamma_{c+1}(F) \text{ and } S)\]
\[
= (S \cap \gamma_{c+1}(F) \cap R_2^F[R_2, F_1]^F][S, cF]
\]
\[
= (S \gamma_{c+1}(F) \cap [R_2, F_1]^F][S, cF] \quad \text{(since } F = F_2 \bowtie F_1[F_1, F_2] \text{ and } S \leq F_1[F_1, F_2])
\]
\[
= (\gamma_{c+1}(F) \cap [R_2, F_1]^F][S, cF] \quad \text{(By Lemma 3.4 (i))}
\]
\[
(A \cap \text{Ker}\overline{\phi})C.
\]
So, by Lemma 3.5, we have the following isomorphism:
\[
\frac{T \cap \gamma_{c+1}(K)}{T, cK} \cong \frac{S \cap \gamma_{c+1}(F)}{\delta([S, cF])} \cong \frac{S \cap \gamma_{c+1}(F)}{\gamma_{c+1}(F) \cap [R_2, F_1]^F}[S, cF]
\]
(Note that \(\prod [R_2, F_1, F_2]_c \subseteq \text{Ker}\delta\) and \(\prod[R_2, F_1, F_2]_c[S, cF] \subseteq (\gamma_{c+1}(F) \cap [R_2, F_1]^F][S, cF]\).
Therefore there exists a natural epimorphism

\[
\frac{S \cap \gamma_{c+1}(F)}{\prod[R_2, F_1, F_2]_c[S, cF]} \to \frac{S \cap \gamma_{c+1}(F)}{(\gamma_{c+1}(F) \cap [R_2, F_1])_c[S, cF]}
\]

Now, by Theorem 2.3 there exists the following epimorphism

\[
\mathcal{N}_cM(G) \longrightarrow \mathcal{N}_cM(B) \oplus \frac{T \cap \gamma_{c+1}(K)}{[T, cK]}
\]

Moreover, if \( G \) is finite, then \( \mathcal{N}_cM(G) \) is also finite and hence by Lemma 2.12

\[
\mathcal{N}_cM(B) \oplus \frac{T \cap \gamma_{c+1}(K)}{[T, cK]}
\]

is isomorphic to a subgroup of \( \mathcal{N}_cM(G) \). \( \Box \)

Now, using the above result, we can present an improved formula with respect to that of Theorem 2.3 for the Baer invariant of a semidirect product of an arbitrary group by a cyclic group with respect to the variety \( \mathcal{N}_c \).

**Theorem 3.7** Let \( G \) be a semidirect product of an arbitrary group \( A \) by a cyclic group \( B \) under \( \theta \) and \( \mathcal{N}_c \) be the variety of nilpotent groups of class at most \( c \). Then, by the previous notation, there exists the following isomorphism

\[
\mathcal{N}_cM(G) \cong \mathcal{N}_cM(B) \oplus \frac{T \cap \gamma_{c+1}(K)}{[T, cK]}
\]

**Proof.** Applying the previous proof, it is enough to show that

\[
\prod[R_2, F_1, F_2]_c = \gamma_{c+1}(F) \cap [R_2, F_1]^F
\]

Since \( B \) is cyclic, we can consider the following presentation for \( B \)

\[
1 \longrightarrow R_2 = \langle y^n \rangle \longrightarrow F_2 = \langle y \rangle \overset{\nu_2}{\twoheadrightarrow} B \longrightarrow 1
\]
So $R_2$ has no commutators and hence by commutator calculus, specially on basic commutators, we can conclude the required equality (see also [11] section 4).

Thus

$$\frac{T \cap \gamma_{c+1}(K)}{[T, cK]} \cong \frac{S \cap \gamma_{c+1}(F)}{\prod[R_2, F_1, F_2]_c[S, cF]} ,$$

and so, by Theorem 2.3, the result holds. □

**Note 3.8** In order to find a relation between $\mathcal{N}_c M(A)$ and the quotient group $T \cap \gamma_{c+1}(K)/[T, cK]$, it is enough to consider the following natural epimorphism:

$$\mathcal{N}_c M(A) = \frac{R_1 \cap \gamma_{c+1}(F_1)}{[r_1, cF_1]} \longrightarrow \frac{(R_1 \cap \gamma_{c+1}(F_1))[T, cK]}{[T, cK]} \leq \frac{T \cap \gamma_{c+1}(K)}{[T, cK]} .$$

Note that $R_1 \leq T$ and $[R_1, cF_1] \leq [T, cK]$ and in finite case $T \cap \gamma_{c+1}(K)/[T, cK]$ has a subgroup which is isomorphic to a subgroup of $\mathcal{N}_c M(A)$.

In the following theorem, we try to provide a considerable simplification of the quotient group $T \cap \gamma_{c+1}(K)/[T, cK]$ and relate its structure more closely to that of $A$.

**Theorem 3.9** With the previous notation and assumption the following isomorphism holds:

$$\frac{U \cap \gamma_{c+1}(D)}{[U, cD]} \cong \frac{T \cap \gamma_{c+1}(K)/[T, cK]}{(R_1 \cap \gamma_{c+1}(F_1))(\gamma_{c+1}(K) \cap [R_1, B]^K)[T, cK]/[T, cK]} ,$$

where $D = A \ast B$ and $U = \langle a^{-1}(\theta(b)(a))[b, a] | a \in A, b \in B >^D$.

**Proof.** let $\eta : K = F_1 \ast B \to D = A \ast B$ be the natural homomorphism induced by $\nu_1 : F_1 \to A$ and the identity on $B$. Clearly $\text{Ker}\eta = R_1^K$. 

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\( \eta(T) = U \) and hence \( \eta([T, cK]) = [U, cD] \).

Also, similar to the proof of Theorem 3.6 we can show that

\[
\eta(T \cap \gamma_{c+1}(K)) = U \cap \gamma_{c+1}(D).
\]

Now, using Lemma 3.5 and a similar method of the proof of Theorem 3.6, if we put \( \phi = \eta : K \rightarrow D, \bar{\phi} = \text{identity} : K \rightarrow K, A = T \cap \gamma_{c+1}(K) \) and

\[
C = [T, cK];
\]

\[
C = (R_1 \cap \gamma_{c+1}(F_1))(\gamma_{c+1}(K) \cap [R_1, B]^K)[T, cK],
\]

then we have

\[
\frac{U \cap \gamma_{c+1}(D)}{[U, cD]} = \frac{\eta(T \cap \gamma_{c+1}(K))}{\eta([T, cK])} \cong \frac{T \cap \gamma_{c+1}(K)}{(R_1 \cap \gamma_{c+1}(F_1))(\gamma_{c+1}(K) \cap [R_1, B]^K)[T, cK]}. \tag{17}
\]

**Corollary 3.10** If \( A \) is a cyclic group, then

\[
\frac{U \cap \gamma_{c+1}(D)}{[U, cD]} \cong \frac{T \cap \gamma_{c+1}(K)}{[T, cK]}.
\]

**Proof.** Since \( A \) is cyclic, we can consider \( A = F_1/R_1 \), where \( F_1 = \langle x \rangle \) and \( R_1 = \langle x^n \rangle \). So \( R_1 \) has no commutators and hence by commutator calculus we have

\[
\gamma_{c+1}(K) \cap [R_1, B]^K \subseteq [T, cK].
\]

Also, since \( A \) is cyclic, so \( N_cM(A) = 1 \) and hence

\[
R_1 \cap \gamma_{c+1}(F_1) = [R_1, cF_1] \leq [T, cK].
\]

Thus, by Theorem 3.9 the result holds. \( \square \)
**Corollary 3.11** If $A$ and $B$ are cyclic groups, then

$$N_c M(B \triangleright A) \cong \frac{U \cap \gamma_{c+1}(D)}{[U, \ cD]}.$$  

**Proof.** By Theorem 3.7 and Corollary 3.10 the result holds. □

### 4. THE BAER INvariant OF A VERBAL WREATH PRODUCT

In this section, first, using a presentation for a verbal wreath product given in [6], (Lemma 2.7), we are going to present a formula similar to that of Theorem 2.3 for the Baer invariant of a $V$-verbal wreath product with respect to the variety of nilpotent groups, $N_c$, where $V$ is an arbitrary variety of groups.

The following theorem is vital in the proof of the results of this section.

**Theorem 4.1** By the notation of Lemma 2.7, the following equalities hold.

(i) $R \cap \gamma_{c+1}(F) = (R_2 \cap \gamma_{c+1}(F_2))(S_V \cap \gamma_{c+1}(F))$, for all $c \geq 1$;

(ii) $[R, \ cF] = [R_2, \ cF_2] \prod[R_2, F_1, F_2]_{c}[S_V, \ cF]$, for all $c \geq 1$, 

where $\prod[R_2, F_1, F_2]_{c}$ was defined in Theorem 2.3.

**Proof.** First, by definition of $S_V$ and $R_V$ in Lemma 2.7, it is easy to see that

$$R_1 \leq S_V, \ [R_2, F_1] \leq S_V, \ R_V \leq V(H) \leq H = F_1^{F_2} = F_1[F_1, F_2],$$

$$S_V \leq F_1[F_1, F_2], \ F = F_2 \triangleright F_1[F_1, F_2] \text{ and } S_V \leq F.$$ 

(i) By Lemma 2.5

$$\gamma_{c+1}(F) = \gamma_{c+1}(F_1)\gamma_{c+1}(F_2) \prod[F_1, F_2]_{c+1},$$
where
\[
\prod[F_1, F_2]_{c+1} = < [F_1, F_2, F_{i_1}, \ldots, F_{i_{c-1}}] \mid i_j \in \{1, 2\}, 1 \leq j \leq c - 1 >
\]
and \(\prod[F_1, F_2]_{c+1} \subseteq F\) (see also [13]). Now, by Lemma 2.4, we have
\[
R \cap \gamma_{c+1}(F) = R \cap \gamma_{c+1}(F_1) \prod[F_1, F_2]_{c+1}
\]
\[
= (R \cap \gamma_{c+1}(F_2))(S \cap \gamma_{c+1}(F_1) \prod[F_1, F_2]_{c+1})
\]
\[
= (R \cap \gamma_{c+1}(F_2))(S \cap \gamma_{c+1}(F)) .
\]

(ii) We use induction on \(c\). If \(c = 1\), then
\[
[R, F] = [R_2 S, F]
\]
\[
\subseteq [R_2, F][R_2 S, F] , \text{ (since } S \subseteq F)\]
\[
= [R_2, F][R_2, F_1] F[S, F] , \text{ (since } F = < F_1, F_2 >)\]
\[
= [R_2, F][R_2, F_1][S, F] , \text{ (since } [R_2, F_1] \leq S) .
\]
The reverse containment is clear. Hence \([R, F] = [R_2, F_2][R_2, F_1][S, F]\).

Now, suppose \([R, kF] = [R_2, kF_2] \prod[R_2, F_1, F_2]_{k}[S, kF]\). Then we have
\[
[R, k+1F] = [[R, kF], F]
\]
\[
= [[R_2, kF_2] \prod[R_2, F_1, F_2]_{k}[S, kF], F] \text{ (by induction hypothesis)}
\]
\[
= [[R_2, kF_2], F][\prod[R_2, F_1, F_2]_{k}, [S, kF], F]
\]
\[
(\text{since } [S, kF], \prod[R_2, F_1, F_2]_{k} \subseteq F)
\]
\[
\subseteq [[R_2, kF_2], F_2] \prod[R_2, F_1, F_2]_{k+1}[S, k+1F] .
\]
Therefore, by induction we have

\[ [R, \ cF] = [R_2, \ cF_2] \prod [R_2, \ F_1, \ F_2]_c [S_V, \ cF] \quad \text{for all } c \geq 1. \]

Now, we are in a position to prove one of the main results of this section.

**Theorem 4.2** Let \( G \) be the \( \mathcal{V} \)-verbal wreath product of \( A \) by \( B \), where \( \mathcal{V} \) is an arbitrary variety of groups. Then, using the previous notation, the following isomorphism exists.

\[
\mathcal{N}_c M(G) = \mathcal{N}_c M(A \text{Wr}_\mathcal{V} B) \cong \mathcal{N}_c M(B) \oplus \frac{S_V \cap \gamma_{c+1}(F)}{\prod [R_2, \ F_1, \ F_2]_c [S_V, \ cF]}.
\]

**Proof.** By the previous assumptions and notations the following natural homomorphisms exist

\[ F \xrightarrow{\varphi} F \xrightarrow{\eta} F \xrightarrow{\varphi} F \xrightarrow{\varphi} F \]

By Theorem 4.1 the following isomorphisms hold:

\[
\frac{R \cap \gamma_{c+1}(F)}{[R, \ cF]} \cong (\eta \varphi)(R \cap \gamma_{c+1}(F))
\]

\[
\cong (\eta \varphi)(R_2 \cap \gamma_{c+1}(F_2)) (\eta \varphi)(S_V \cap \gamma_{c+1}(F)). \quad (*)
\]

Consider the following two natural homomorphisms:

\[
\frac{F_1 * F_2}{[R_2, \ cF_2]^F} \xrightarrow{h} F_1 * \frac{F_2}{[R_2, \ cF_2]^F} \xrightarrow{g} \frac{F_1 * F_2}{[R_2, \ cF_2]^F},
\]

given by

\[
\overline{f_1} \mapsto f_1 \quad f_1 \mapsto \overline{f_1}
\]

\[
\overline{f_2} \mapsto \overline{f_2} \quad \overline{f_2} \mapsto \overline{f_2}.
\]

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It is easy to see that $h \circ g = 1$ and $g \circ h = 1$ i.e $h$ and $g$ are isomorphisms.

Thus, we have

$$
\frac{F_1 * F_2}{[R_2, \, cF_2]^F} = \varphi(F) \cong F_1 * \frac{F_2}{[R_2, \, cF_2]}.
$$

Also

$$
\varphi(F_2) = \frac{F_2[R_2, \, cF_2]^F}{[R_2, \, cF_2]^F} = \frac{F_2[R_2, \, cF_2][R_2, \, cF_2, F_1]^F}{[R_2, \, cF_2][R_2, \, cF_2, F_1]^F} \cong \frac{F_2}{[R_2, \, cF_2]} \quad \text{(By the second isomorphism Theorem)}
$$

and

$$
\varphi(F_1[F_1, F_2]) \cong \varphi(F_1)[\varphi(F_1), \varphi(F_2)] \cong F_1[F_1, \frac{F_2}{[R_2, \, cF_2]}].
$$

(Note that

$$
\varphi(F_1) = \frac{F_1[R_2, \, cF_2]^F}{[R_2, \, cF_2]^F} \cong \frac{F_1}{F_1 \cap [R_2, \, cF_2]^F} = \frac{F_1}{1} \cong F_1.
$$

Therefore

$$
\varphi(F) \cong F_1 * \frac{F_2}{[R_2, \, cF_2]} \cong \frac{F_2}{[R_2, \, cF_2]} \cong F_1[F_1, \frac{F_2}{[R_2, \, cF_2]}] \cong \varphi(F_2)[\varphi(F_1), \varphi(F_2)].
$$

Clearly $\text{Ker}(\eta) = \varphi(\prod[R_2, F_1, F_2]_{c[S_V, \, cF]})$

$$
\leq \varphi(F_1)[\varphi(F_1), \varphi(F_2)]. \quad (**)
$$

So by [6, (Lemma 1.3)] we have

$$
(\eta \varphi)(F) \cong \frac{\varphi(F)}{\text{Ker}(\eta)} = \frac{\varphi(F)}{\varphi(\prod[R_2, F_1, F_2]_{c[S_V, \, cF]})}
$$
\[ \varphi(F_2) \cong \frac{\varphi(F_1)[\varphi(F_1), \varphi(F_2)]}{\text{Ker}(\eta)} \, . \]

By (\ast\ast), it is easy to see that \((\eta \varphi)(F_2) \cong \varphi(F_2)\) and \((\eta \varphi)(F_1) \cong \varphi(F_1)/\text{Ker}(\eta)\).

Thus we have

\[
(\eta \varphi)(F) \cong \varphi(F_2) \triangleleft \frac{\varphi(F_1)[\varphi(F_1), \varphi(F_2)]}{\text{Ker}(\eta)}
\]

\[ \cong (\eta \varphi)(F_2) \triangleleft (\eta \varphi)(F_1)[(\eta \varphi)(F_1), (\eta \varphi)(F_2)] \, . \]

So, we can conclude that

\[
(\eta \varphi)(R_2 \cap \gamma_{c+1}(F_2)) \cap (\eta \varphi)(S_V \cap \gamma_{c+1}(F))
\]

\[ \subseteq (\eta \varphi)(F_2) \cap (\eta \varphi)(F_1)[(\eta \varphi)(F_1), (\eta \varphi)(F_2)] = 1 \]

Hence, by considering (*), we have

\[
\frac{R \cap \gamma_{c+1}(F)}{[R, \varphi(F_2)]} \cong (\eta \varphi)(R_2 \cap \gamma_{c+1}(F_2)) \oplus (\eta \varphi)(S_V \cap \gamma_{c+1}(F)) \, .
\]

On the other hand, we have the following isomorphisms.

\[
(\eta \varphi)(R_2 \cap \gamma_{c+1}(F_2)) = \frac{(R_2 \cap \gamma_{c+1}(F_2))\text{Ker}(\eta \varphi)}{\text{Ker}(\eta \varphi)}
\]

\[ \cong \frac{R_2 \cap \gamma_{c+1}(F_2)}{(R_2 \cap \gamma_{c+1}(F_2)) \cap \text{Ker}(\eta \varphi)} \quad \text{(by the second isomorphism Theorem)}
\]

\[ \cong \frac{R_2 \cap \gamma_{c+1}(F_2)}{[R_2, \varphi(F_2)]} \quad \text{(by Lemma 2.5)}
\]

\[ \cong \mathcal{N}_c M(B) \, . \]

Also

\[
(\eta \varphi)(S_V \cap \gamma_{c+1}(F)) = \frac{(S_V \cap \gamma_{c+1}(F))\text{Ker}(\eta \varphi)}{\text{Ker}(\eta \varphi)}
\]

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\[\cong \frac{S_V \cap \gamma_{c+1}(F)}{(S_V \cap \gamma_{c+1}(F)) \cap \text{Ker}(\eta_{\phi})} \quad (\text{by the second isomorphism Theorem})\]
\[= \frac{S_V \cap \gamma_{c+1}(F)}{\prod[R_2, F_1, F_2][S_V, cF]} \quad (\text{by Lemma 2.5}) .\]

Therefore
\[N_c M(\text{BWr}_V A) \cong \frac{R \cap \gamma_{c+1}(F)}{[R, cF]}\]
\[\cong N_c M(B) \oplus \frac{S_V \cap \gamma_{c+1}(F)}{\prod[R_2, F_1, F_2][S_V, cF]} .\]

Now, we can obtain the following corollary if we put \(c = 1\).

**Corollary 4.3 [6]** Suppose \(G = A \text{Wr}_V B\) is the \(\mathcal{V}\)-verbal wreath product of \(A\) by \(B\), where \(\mathcal{V}\) is a variety of groups. Then
\[M(\text{AWr}_V B) \cong M(B) \oplus \frac{S_V \cap F'}{[R_2, F_1][S_V, F]} .\]

Now, in what follows, we are going to improve the structure of the complementary factor of \(N_c M(B)\) in \(N_c M(\text{AWr}_V B)\).

**Notation 4.4** By considering the previous notation, let \(\delta : F_1 \ast F_2 \rightarrow F_1 \ast B\) be the natural epimorphism, where \(A \cong F_1/R_1\) and \(B \cong F_2/R_2\) are presentations for \(A\) and \(B\). Put \(T_V = \delta(S_V)\) and \(K = F_1 \ast B\). Using Lemmas 3.4 and 2.7, Theorem 4.1 and similar properties of \(S_V\) to that of \(S\), we can rewrite the proof of Theorem 3.6 to get the following Theorem about the Baer invariant of a verbal wreath product.

**Theorem 4.5** By the notation above, there exists the following epimorphism:
\[N_c M(\text{AWr}_V B) \rightarrow N_c M(B) \oplus \frac{T_V \cap \gamma_{c+1}(K)}{[T_V, cK]} .\]
Also, in finite case, $N_c M(A \text{Wr}_V B)$ has a subgroup isomorphic to

$$N_c M(B) \oplus \frac{TV \cap \gamma_{c+1}(K)}{[TV_{c}, K]}.$$ 

Now, by putting a condition on $B$, and a similar proof to that of Theorem 3.7, we can present a formula for a verbal wreath product which is somehow better than that of Theorem 4.2.

**Theorem 4.6** By the above notation, let $G$ be a $V$-verbal wreath product of an arbitrary group $A$ by a cyclic group $B$, where $V$ is any variety of groups. Then

$$N_c M(A \text{Wr}_V B) \cong N_c M(B) \oplus \frac{TV \cap \gamma_{c+1}(K)}{[TV_{c}, K]}. \quad \frac{TV \cap \gamma_{c+1}(K)}{[TV_{c}, K]} = \frac{TV \cap \gamma_{c+1}(K)}{[TV_{c}, K]}. \quad \frac{TV \cap \gamma_{c+1}(K)}{[TV_{c}, K]}.$$

**Note 4.7** Note that W. Haebich in [6] prove that

$$M(A \text{Wr}_V B) \cong M(B) \oplus \frac{TV \cap K'}{[TV, K]}.$$

where $A$ and $B$ are two arbitrary groups.

5. **The Baer Invariant of a Free Wreath Product**

In this final section we try to find a structure for the Baer invariant of a free wreath product with respect to the variety of nilpotent groups. We recall that a free wreath product is in fact a $V$-verbal wreath product where $V$ is the variety of all groups, i.e. $V = \{1\}$. Then, by the previous notation, Lemma 2.7, we have $V(H) = 1$ and so $R_V = 1$ and $S_V = [R_2, F_1]^\Sigma R_1^\Sigma$. Now, by the definition of the natural epimorphism $\delta : F_1 \ast F_2 \rightarrow F_1 \ast B, \delta(R_2) = 1$ and hence $TV = \delta(S_V) = R_1^{K} = R_1[R_1, K]^{K}$. 

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**Lemma 5.1** By the above notation, we have

(i) \([TV, cK] = [R_1, cF_1]D_c\),

where \(D_c = \prod_{G_i \in \{F_1, B\}} [R_1, G_1, \ldots, G_c]^K\);

(ii) \(TV \cap \gamma_{c+1}(K) = (R_1 \cap \gamma_{c+1}(F_1))E_c\),

where \(E_c = [R_1, K]^K \cap \gamma_{c+1}(K) \cap [F_1, B]\).

**Proof.**

(i) \([TV, cK] = [R_1, cK]^K = [R_1, K]^K = [R_1, cF_1]D_c\) (by commutator calculus and Lemma 2.11).

(ii) \(TV \cap \gamma_{c+1}(K) = R_1^K \cap \gamma_{c+1}(K) = R_1[R_1, K]^K \cap \gamma_{c+1}(K) \subseteq (R_1 \cap \gamma_{c+1}(F_1))(R_1, K)^K \cap \gamma_{c+1}(K)\) (by Lemmas 2.4 and 2.5)

\[= (R_1 \cap \gamma_{c+1}(F_1))E_c.\]

The reverse inclusion can be seen easily. \(\Box\)

**Lemma 5.2** By the above notation, we have the following isomorphism:

\[
\frac{TV \cap \gamma_{c+1}(K)}{[TV, cK]} \cong N_cM(A) \oplus \frac{E_c}{D_c}.
\]

**Proof.** Put

\[
\varphi : K \xrightarrow{\text{nat}} \frac{K}{[TV, cK]} \quad \text{and} \quad K = F_1 * B \xrightarrow{\xi = \text{nat}} \frac{F_1}{[R_1, cF_1]} * B \xrightarrow{\rho} \frac{\xi(K)}{\xi(D_c)}.
\]

Clearly \(ker(\rho \xi) = [R_1, cF_1]D_c = [TV, cK] = ker \varphi\), so \(\varphi = \rho \xi\). Now similar to the proof of Theorem 3.9 in [6] we can show that

\[
\varphi(TV \cap \gamma_{c+1}(K)) = \varphi(R_1 \cap \gamma_{c+1}(F_1)) \oplus \varphi(E_c)
\]

and

\[
\varphi(R_1 \cap \gamma_{c+1}(F_1)) \cong \xi(R_1 \cap \gamma_{c+1}(F_1)) \cong N_cM(A).
\]
Also

\[ \varphi(E_c) = \frac{E_c([R_1, cF_1]D_c)}{[R_1, cF_1]D_c \cap E_c} \approx \frac{E_c}{([R_1, cF_1] \cap E_c)D_c} \cdot \]

Since \([R_1, cF_1] \cap E_c = 1\), so \(\varphi(E_c) \cong E_c/D_c\). Hence the result holds. \(\square\)

Now, we are in a position to find a structure for the \(c\)-nilpotent multiplier of a free wreath product.

**Theorem 5.3** Let \(G\) be the free wreath product of \(A\) by \(B\) i.e. \(G = A \mathcal{W} r_B\). Then there exists an epimorphism as follows:

\[ \mathcal{N}_cM(A \mathcal{W} r_B) \twoheadrightarrow \mathcal{N}_cM(A) \oplus \mathcal{N}_cM(B) \oplus \frac{E_c}{D_c} \cdot \]

Also, in finite case, the above structure is isomorphic to a subgroup of \(\mathcal{N}_cM(A \mathcal{W} r_B)\).

**Proof.** By Theorem 4.5 and Lemma 5.2 the result holds. \(\square\)

**Theorem 5.4** Suppose \(A\) and \(B\) are two groups such that \(B\) is cyclic. Then

\[ \mathcal{N}_cM(A \mathcal{W} r_B) \cong \mathcal{N}_cM(A) \oplus \mathcal{N}_cM(B) \oplus \frac{E_c}{D_c} \cong \mathcal{N}_cM(A) \oplus \frac{E_c}{D_c} \cdot \]

**Proof.** By Theorem 4.6, Lemma 5.3 the result holds. \(\square\)

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