Further Explorations of the Parameter Space of 3D Lattice Gauge Theories

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Abstract

State sum models can be used to construct the partition functions of 3D lattice gauge theories based on involutory Hopf algebras, A, of which the group algebras, CG, are a particular case. Transfer matrices can be obtained from such partition functions by carrying out the state sum construction on a manifold with boundary. The parameter space of these transfer contains various Hamiltonians of physical interest. The 2D quantum double Hamiltonians of Kitaev can be obtained from such transfer matrices for specific values of these parameters. An initial study of such models has been carried out in [12]. In this paper we study other regions of this parameter space to obtain some new and known models. The new model comprise of Hamiltonians which “partially” confine the excitations of the quantum double Hamiltonians which are usually deconfined. The state sum construction is not invariant by translations and thus it is natural to expect to obtain disordered Hamiltonians from the transfer matrices resulting from these constructions. Thus one set of known models consist of the disordered quantum double Hamiltonians. Finally we obtain quantum double Hamiltonians perturbed by magnetic fields which have been considered earlier in the literature to study the stability of topological order to perturbations.

1 Introduction

Topologically ordered systems have gained wide attention in recent years due to some of its consequences in topological quantum computation and emergence of new phases of matter among many others [1]. Among the different types of systems exhibiting topological order the ones with long-ranged entangled (LRE) ground states are the ones which are thought to be most useful for quantum computation. The earliest proposals of such systems are the quantum double Hamiltonians of Kitaev [13, 14]. The toric code is the simplest example of a 2D lattice systems which contains anyons as low energy excitations and have degenerate LRE states as ground states. These were further generalized by the Levin-Wen models [3] or the string-net models which described more general anyonic excitations by directly taking a unitary fusion category as inputs. These models are also quantum double models based on weak Hopf algebras as noted in [4] and can thus be constructed via the algorithm of Kitaev. Several other models inspired by the usefulness of the toric code as a stabilizer code have been constructed of which the topological color codes [5] are an example which have also been experimentally implemented [6].

State sum models have been used to construct the Levin-Wen models [3] using Turaev-Viro invariants [7] and chain-mail link invariants [8, 9]. Kitaev’s toric code has also related to Turaev-Viro codes [10]. Both the Levin-Wen model and Toric code models can be thought of as Hamiltonian realizations of TQFTs based on state sums of Turaev-Viro. The Levin-Wen model corresponds to the Barrett-Westbury invariant [11] and the toric code corresponds to a special case of the Kuperberg invariant [12]. This has been especially noted in [15]. We showed this explicitly in [12] where we embedded the 2D quantum double models based on an involutory Hopf algebra, A, in an enlarged parameter space, that of the 3D generalized lattice gauge theories based on

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these algebras $A$. The group algebras $\mathbb{C}G$ are a special case of these algebras $A$ resulting in the lattice gauge theories familiar to physicists. The toric code occurs when we choose $G = \mathbb{Z}_2$.

The transfer matrices were shown to be made up of four kinds of parameters, $U(A, z_T, \xi_T, \xi_S)$ \cite{12}. This transfer matrix is then written down as a product of local operators acting on vertices, plaquettes and links. One of the consequences of deriving the quantum double Hamiltonians of Kitaev from the transfer matrices of the corresponding lattice gauge theories in three dimensions is that we can find other phases which are different from that described by Kitaev's quantum double. Some examples include quasi-topological phases which result from a condensation of the excitations \cite{17} of the quantum double phase of Kitaev. This leads to increased ground state degeneracy for the condensed phases. Examples of these phases were studied in \cite{12}. These phases including the quantum double phases of Kitaev were obtained when we considered the parameters $z_S$ and $\xi_T$ in the transfer matrix. In \cite{18} we showed that we could obtain the quantum double phases of Kitaev by writing down models which included $z_T$ or $\xi_S$.

In this paper we go further in this direction to see what other new phases we can find by considering more parameters in the transfer matrix which were not included in \cite{12, 18}. We consider three types of models here. Two of them do not include $z_T$ and $\xi_S$ while the third includes them. The first two sets of models comprise of the disordered quantum double Hamiltonians of Kitaev and a new Hamiltonian which lead to "partial" confinement of the excitations of the quantum double phase of Kitaev. We use the term partial to emphasize on the fact that the models are such that the excitations can be moved a few steps with an energy cost after which they become deconfined. According to the terms added we can confine the excitations for any number of steps that we wish to. We will also call these models $n$-step confined models in the text to follow. The models which include the other two parameters, $z_T$ and $\xi_S$ are the quantum double Hamiltonians perturbed by magnetic fields. These can be thought of as local perturbations to the exactly solvable Hamiltonians of Kitaev. Due to the usefulness of these models to realize fault tolerant quantum computation, it is necessary to study the stability of the topological order to local perturbations \cite{2}. These models have already been considered in the literature and we write them down here just for the sake of completion and to drive home the point that they are well within the parameter space of the three dimensional lattice gauge theories.

The contents of the paper are organized as follows. Section 2 gives a brief review of the construction of the transfer matrix of the generalized lattice gauge theories. The section also includes an introduction to the mathematical preliminaries that go into the construction of the partition function and the transfer matrices. The algebra of operators, which include the quantum double relations between the vertex and plaquette operators, are written down. The models obtained from this transfer matrix are described in section 3. An outlook is presented in section 4.

2 Partition Function and Transfer Matrix of Generalized Lattice Gauge Theories

The partition function of lattice gauge theory is a well known example of a classical partition function built out of local weights associated to plaquettes of an oriented 3D lattice, where the gauge degrees of freedom are elements of a gauge group $G$ living on the edges of the lattice. A typical action for these models is the so called Ising-gauge action defined by $S = \frac{1}{2} \sum_p (\text{tr}(U_p) + \text{tr}(U_p^{-1}))$, where the sum runs over the plaquettes of the lattice and $U_p$ is the holonomy of a plaquette $p$. The partition function which describes the model is given by\footnote{The precise definition of $Z$ will be given in section 4.2}

$$Z_{sg} = \sum_{\text{conf.}} e^{-\beta S} = \prod_p M_{sg}(U_p) \text{, with } M_{sg}(U_p) = \exp\{-\beta/2 (\text{tr}(U_p) + \text{tr}(U_p^{-1}))\}.$$  

In above equation $M_{sg}(U_p)$ is the local weight for the spin-gauge model. Due to the invariance of the local action under cyclic permutation, the local weight is also invariant under this cyclic permutation, which makes it a class function, $M: G \rightarrow \mathbb{C}$ (equivalently, $M_{sg}(g) = M_{sg}(gh^{-1})$). This construction can be generalized by choosing $M(g)$ as being any class function $M: G \rightarrow \mathbb{C}$. More over, we can associate local weights, $\Delta(l)$, to the edges $l$ of the lattice, such that the partition function is now given by

$$Z = \prod_p M(p) \prod_l \Delta(l). \quad (1)$$
We can reproduce the usual lattice gauge theories (with any action choice) by making appropriate choices for $M(p)$ and $\Delta(l)$. The partition function in Eq. (1) is called the partition function of generalized lattice gauge theory. Starting from this partition function we can obtain a transfer matrix whose logarithm gives us Hamiltonian operators, and thus dynamical quantum models. These quantum models are parametrized by functions of the parameters of the generalized lattice gauge theories. In [12] it was shown that the quantum double Hamiltonians of Kitaev, of which the toric code is a special case, can be obtained from this approach. In other words it was shown how to embed such models into the parameter space of these generalized lattice gauge theories.

In [12] it was shown that this partition function can be built out of the structure constants of an involutory Hopf algebra $A$ and a 3-manifold of the form $\Sigma \times S^1$, where $\Sigma$ is some compact 2-manifold and $S^1$ is the 1-dimensional sphere. We did not consider all possible deformations of the generalized lattice gauge theory partition function in [12], working only with a specific kind of deformation (one parameter deformation) of the gauge theory partition function. Now we will allow other deformations by letting the parameters be any element of the center of the algebra $A$ and its dual algebra, $A^*$. We only work with group algebras $CG$ of a discrete group $G$ here. Nevertheless the methods presented here hold for any involutory Hopf algebra. We now recall the definition of the group algebra $CG$.

### 2.1 The Group Algebra $CG$

The group algebra $CG$ has its multiplication defined by the multiplication of a group $G$, which we consider to be discrete and finite. Let $\{\phi_g : g \in G\}$ be a basis of the vector space indexed by the group elements and let $\{\Psi^g : g \in G\}$ be the dual basis such that $\Psi^g(\phi_h) = \delta(g,h)$. In this basis the multiplication and co-multiplication are defined by

\[
\phi_a \cdot \phi_b := \phi_{ab} \quad \Rightarrow \quad m_{ab}^c = \delta(ab,c)
\]

\[
\psi^a \cdot \psi^b := \delta(a,b)\psi^a \quad \Rightarrow \quad \Delta_{ab}^c = \delta(a,c)\delta(b,c)
\]

and we can easily see that the unit and the co-unit of the algebra are

\[
e = \phi_{g_0} \quad \Rightarrow \quad e_a = \delta(a,g_0)
\]

\[
\epsilon = \sum_{g \in G} \psi^g \quad \Rightarrow \quad \epsilon^a = 1
\]

where $g_0$ is the identity element of the group.

Finally the antipode map is defined by

\[
S(\phi_a) = \phi_{a^{-1}} \quad \Rightarrow \quad S_a^b = \delta(ab, g_0).
\]

It is not difficult to see that the group algebra structure constants satisfy all the axioms of Hopf algebras [19, 12]. An important thing about the group algebra is the fact that it is an involutory Hopf algebra, which means that $S^2 \equiv 1$.

### 2.2 Constructing a Partition Function with a Hopf Algebra

The construction begins by associating a tensor to each face and link of the lattice associated to a closed manifold. These tensors play the role of local weights. These tensors are then contracted resulting in a scalar which is the partition function.

For a plaquette with edges labelled by the elements $a, b, c$ and $d$ we associate a tensor $M_{abcd}$, as shown in figure 1(a). For a link with four plaquettes glued to it’s edges, labelled by $x, y, z$ and $t$, we associate a tensor $\Delta_{xyzt}$ as shown in figure 1(b). These tensors are defined in terms of the algebra and co-algebra structure constants of the group algebra $A = CG$ and parametrized by the elements of the center of $A$ and $A^*$, $z$ and $\xi$

\[
M_{abcd}(z) = \text{tr} (z \phi_a \phi_b \phi_c \phi_d),
\]

3
(a) The tensor $M_{abcd}$ associated to a plaquette of the lattice.

(b) The tensor $\Delta^{xyzt}$ associated to a link of the lattice.

Figure 1: Local weights associated to the plaquettes and links of the lattice.

\[ M_{abcd} \]

\[ \Delta^{xyzt} \]

and

\[ \Delta^{xyzt}(z^*) = \text{co-tr} \left( \xi \Psi^x \Psi^y \Psi^z \Psi^{t'} \right). \]

where $\text{tr}(\phi_g) = |G| \delta(g, g_0)$ is the trace in the regular representation and $\text{co-tr}(\Psi^g) = 1, \forall g$. The partition function is obtained by contracting the indices of the tensors associated to the plaquettes and links. However we need to take care of the orientation of the lattice while performing this contraction. If the plaquette and link orientation matches the contraction is made directly, otherwise it is done through the antipode tensor $S^g_y$.

At the end the partition function will be of the form

\[ Z(z, \xi) = \prod_p M_{abcd}(z) \prod_l \Delta^{xyzt}(\xi) \prod_{l'} S^{z_{l'}} {x'_l}, \]

where $l'$ runs over the links with mismatching orientations.

We can now extract a transfer matrix out of this partition function after making a distinction between timelike and spacelike directions of the 3D lattice, as shown in figure 2 for a small piece of the lattice. Since there is now a distinction between the timelike and spacelike parts there is nothing forcing the parameters of the timelike and spacelike plaquette weights to be the same. So for a more general description we have all spacelike plaquette weights parametrized by the element $z_S^2$ while the timelike ones are parametrized by $z_T$. In a similar manner the spacelike and timelike link weights are parametrized by $\xi_S$ and $\xi_T$ respectively. Therefore the partition function is now a function of the form $Z(z_S, z_T, \xi_S, \xi_T)$.

2.3 The Transfer Matrix

From the partition function we have just defined we can get a transfer matrix $U$ such that its trace is equal to the partition function. This transfer matrix may depend on the same parameters as the partition function, namely $U = U(z_S, z_T, \xi_S, \xi_T)$. The operator $U$ acts on the links of the 2-dimensional lattice where the quantum states lives. A local Hilbert space $\mathcal{V}_l$ associated to each link $l$ (with basis $\{ |g|_l : g \in G \}$). The Hilbert space of

\[ \text{The parameters } z_S \text{ need not be the same for all the spacelike plaquettes. This shows that this construction need not obey translational invariance. This fact will be exploited to obtain disordered Hamiltonians in the next section. The same fact holds for the other parameters as well.} \]

\[ \text{The partition function also depends on the group } G \text{ and the lattice } \mathcal{L}, \text{ so it is actually a function of the form } Z(G, L, z_S, z_T, \xi_S, \xi_T). \text{ For brevity we denote it just as } Z(z_S, z_T, \xi_S, \xi_T). \]
the full system is given by $H = V_1 \otimes V_2 \otimes \cdots \otimes V_n$ and a vector of this space is a linear combination of vectors of the form

$$|g_1\rangle_1 \otimes |g_2\rangle_2 \otimes \cdots \otimes |g_n\rangle_n.$$ 

The procedure to get such a transfer matrix was shown in [12] using a diagrammatic notation to manage the tensors that builds the partition function and also the transfer matrix. We are now considering a more general parametrization than the one considered in [12]. The way to get the transfer matrix is similar to the procedure shown in [12], resulting in the same operators as in [12] but now in a bigger parameter space. We just write down the results in what follows. The transfer matrix is

$$U(z_S, z_T, \xi_S, \xi_T) = \prod_p B_p(z_S) \prod_l C_l(z_T, \xi_S) \prod_v A_v(\xi_T),$$

where $B_p(z_S)$ is an operator which acts on the links at edge of the plaquette $p$, $A_v(\xi_T)$ an operator which acts on the links sharing the vertex $v$ and $C_l(z_T, \xi_S)$ is an operator which acts on a single link. The operators $B_p(z_S)$ and $A_v(\xi_T)$ are called the plaquette and vertex operators, as before. All the parameters in $U(z_S, z_T, \xi_S, \xi_T)$ are central elements of $A$ and $A^\ast$. It can be seen in [20] that the central elements of $CG$ are written in terms of the conjugacy classes $[C]$ of $G$ as

$$z = \sum_C \beta_C z_C, \quad \text{with} \quad z_C = \sum_{g \in [C]} \phi_g,$$

while the central elements of $A^\ast$ we write in terms of the irreducible representations $R$ of $G$ as

$$\xi = \sum_R \alpha_R \xi_R, \quad \text{with} \quad \xi_R = \sum_{g \in G} \chi_R(g) \Psi_g,$$

where $\chi_R(g)$ is the trace of $g$ in the representation $R$. Hence the parameters of the transfer matrix can be written as

$$z_S = \sum_C \beta_C z_C \quad \text{and} \quad z_T = \sum_C \beta_C z_C \quad (2)$$

$$\xi_S = \sum_R \alpha_R \xi_R \quad \text{and} \quad \xi_T = \sum_R \alpha_R \xi_R \quad (3)$$

The plaquette and vertex operators are linear functions in its parameters. That is

$$B_p(z_S) = \sum_C \beta_C B_p(z_C) = \sum_C \beta_C B_p^C, \quad (4)$$

$$A_v(\xi_T) = \sum_R \alpha_R A_v(\xi_T) = \sum_R \alpha_R A_v^R. \quad (5)$$

where the operators $B_p^C$ and $A_v^R$ act on the links as shown in the picture in figure 3 and are given by

$$\text{Figure 3: A plaquette and a vertex of the lattice.}$$
Thus the final expression for the transfer matrix is

\[
B_p^C = |G| \sum_{\{a_i\}_{i=1}^n} \delta(a_1^{-1}a_2a_3^{-1}a_4, C)T_1^{a_1} \otimes T_2^{a_2} \otimes T_3^{a_3} \otimes T_4^{a_4},
\]

\[
A_v^R = \sum_{g \in G} \chi_R(g)R_3(g^{-1}) \otimes R_4(g^{-1}) \otimes L_5(g) \otimes L_6(g)
\]

where \(\delta(g, C) = 1\) if \(g \in [C]\) and \(\delta(g, C) = 0\) if \(g \notin [C]\) and the operators \(L_i(\phi_g), R_i(\phi_g)\) and \(T_i(\Psi^g)\) are operators which act on a single link defined as

\[
L_i(\phi_g)|h\rangle_l = |gh\rangle_l,
\]

\[
R_i(\phi_g)|h\rangle_l = |hg\rangle_l,
\]

\[
T_i(\Psi_g)|h\rangle_l = \delta(g, h)|h\rangle_l.
\]

These operators are linear on its parameters, in other words, \(L(z) = \sum g^z L(\phi_g)\) (this property also holds for \(R(z)\) and \(T(z^*)\)). Sometimes we use the short notation \(L^g = L(\phi_g), R^g = R(\phi_g)\) and \(T^g = T(\Psi^g)\). The link operator \(C_l(z_T, \xi_S)\) can also be written in terms of the \(L_1\) and \(T_1\) operator as we shall see next.

The main difference between this model in the one we have considered in [12] is the link operator \(C_l(z_T, \xi_S)\). Unlike in [12] this operator is no longer proportional to identity, now it takes the form

\[
C_l(z_T, \xi_S) = |G|T_1(\xi_S)L_1(z_T) = |G| \left( \sum_R a_R \frac{T_1(\xi_R)}{T_1^R} \right) \left( \sum_C b_C \frac{L_1(z_C)}{L_1^C} \right),
\]

where

\[
L_1^C = R_1^C = \sum_{g \in [G]} L_1^g = \sum_{g \in [G]} R_1^g
\]

\[
T_1^R = \sum_{g \in G} \chi_R(g)T_1^g
\]

Thus the final expression for the transfer matrix is

\[
U(z_S, z_T, \xi_S; \xi_T) = |G|^n \prod_{p} \left( \sum_C \beta_C B_p(z_C) \right) \prod_{l} \left[ \left( \sum_R a_R T_1^R \right) \left( \sum_C b_C L_1^C \right) \right] \prod_{v} \left( \sum_R a_R A_v^R \right),
\]

where the parameters on the left hand side are related to the coefficients on the right hand side by the equations [2] and [3].

### 2.4 Algebra of the operators

In order to find the algebra that the operators \(B_p, A_v\) and \(C_l\) satisfy one should first look at the algebra of the operators \(L_i, R_i\) and \(T_i\). It is not difficult to see that the following relations holds

\[
L_i^gL_i^h = L_i^{gh} \quad R_i^gR_i^h = R_i^{gh} \quad T_i^gT_i^h = \delta(g, h)T_i^g
\]

Now using Eq. [12] and the orthogonality relations on the characters \(\chi_R(g)\) we can show the algebra of the operators which build the transfer matrix, namely \(B_p^C, A_v^R, L_i^R\) and \(T_1^R\).

The set of operators \(\{B_p^C\}\) is a complete basis of orthogonal projectors which generate the plaquette operators, which means

\[
B_p^CB_p^{C'} = \delta(C, C')B_p^C, \quad \text{and} \quad |G|^{-1} \sum_C B_p^C = 1.
\]

Same way the set of \(\{A_v^R\}\) is a complete basis of orthogonal projectors which generate the vertex operator, in other words

\[
A_v^RA_v^{R'} = \delta(R, R')A_v^R, \quad \text{and} \quad \sum_R A_v^R = 1.
\]
The plaquette and vertex operator still commute for any choice of the parameters $z_S$ and $\xi_T$, so we can write
\[ [B^C_p, A^R_v] = 0 \quad \Rightarrow \quad [B_p(z_S), A_v(\xi_T)] = 0, \quad \forall z_S, \xi_T. \]

The set of operators \( \{L^C_i\} \) and \( \{T^R_i\} \) are also complete sets of orthogonal projectors (\( L^C_i L^C_j = \delta(C, C')L^C_i \) and \( T^R_i T^R_j = \delta(R, R')T^R_i \)), however the operator \( T^R_i \) does not commute with the link operator but it does commute with the plaquette operator, in the same way as the operator \( L^C_i \) does not commute with the plaquette operator but it does commute with the vertex operator. Thus we can write
\[ [T^R_i, B^C_p] = 0, \quad [T^R_i, A^R_v] \neq 0, \quad [L^C_i, B^C_p] \neq 0, \quad [L^C_i, A^R_v] = 0. \] (15)

Therefore, the link operator \( C_i(z_T, \xi_S) \) commutes with the plaquette and vertex operators only for some choices of the parameters \( z_T \) and \( \xi_S \), which means the quantum model obtained from the transfer matrix containing this link operator is in general not solvable.

3 Examples of Models from the Transfer Matrix

The fully parametrized transfer matrix obtained in the previous section helps us construct a number of other interesting models. We have seen that the quantum double Hamiltonians are only one special class of models in this parameter space. Here we will look at what other possibilities exist in the extended parameter space. The first set of examples consist of disordered quantum double Hamiltonians. These are quantum double Hamiltonians which do not have translational invariance. This is due to the appearance of coefficients, for the vertex and plaquette terms, which are not constant but depend on the vertex \( v \) and plaquette \( p \). These models continue to be in the quantum double phase. The trace of the transfer matrix for these Hamiltonians also help us obtain their partition functions.

We can produce solvable models even in the presence of the parameters \( z_T \) and \( \xi_S \). We discussed a class of such models in [18] where we exhibited exactly solvable models which continued to remain in the quantum double phase described by modified vertex operators or plaquette operators. Here we show another class of models that can be obtained from the transfer matrices of lattice gauge theories that continue to remain exactly solvable but are in a “partially” confined phase with respect to the original quantum double phase. By this we mean that the deconfined quantum double excitations are now confined up to a few steps due to the addition of an extra term in the Hamiltonian. The number of steps for which they are confined can be controlled by adding the appropriate term in the Hamiltonian. We will call these models \( n \)-step confined models. These models comprise our second set of examples. They are examples where the self-duality of the quantum double Hamiltonians is broken. We also discuss the ground states, it’s degeneracy apart from the excited states of the model.

Finally we write down models that are obtained by using the remaining two parameters \( z_T \) and \( \xi_S \) along with \( z_S \) and \( \xi_T \). The transfer matrix is now significantly modified as new single qudit operators, acting on individual links, appear. They do not commute with the vertex and plaquette operators in general. However for certain special values of parameters they commute with products of vertex and plaquette operators as we shall see when we consider these models later in this section. The models obtained at these values are quantum double Hamiltonians perturbed by generalized “magnetic” fields. They have the interpretation of magnetic fields in the case the input algebra \( \mathcal{A} \) is \( \mathbb{C}(Z_2) \). For the remaining values of the parameters we obtain more complicated terms in the Hamiltonian which we will briefly touch upon. These models are not exactly solvable and are outside the phase described by the quantum double Hamiltonians, at least for sufficiently large values of the parameters.

3.1 Disordered Quantum Double Hamiltonians (QDH)

The transfer matrices used to obtain these models only use \( z_S \) and \( \xi_T \). The other two parameters are set to \( z_T = \eta \) and \( \xi_S = \epsilon \) for all the timelike plaquettes and spacelike links. To obtain the disordered QDH models we associate a different central element of the algebra and it’s dual to every spacelike plaquette and timelike link respectively. This leads to the following transfer matrix
\[ U(\mathcal{A}, z_{S,p}, \xi_{T,v}) = \prod_p B_p(z_{S,p}) \prod_v A_v(\xi_{T,v}) \] (16)
where \( z_{S,p} \) and \( \xi_{T,v} \) are the parameters for the plaquette \( p \) and vertex \( v \) respectively. The plaquette and vertex operators commute with each other in this transfer matrix and each of the operators is a sum of projectors. Thus it is easy to take the logarithm of these matrices to obtain the disordered QDHs.

Let us look at these Hamiltonians in the case when \( A = C(Z_2) \). Denote the basis elements of \( C(Z_2) \) by \( \{ \phi_1, \phi_{-1} \} \) with \( \phi_{-1}^2 = \phi_1 \), and the basis elements of the dual \( C(Z_2)^* \) by \( \{ \psi^1, \psi^{-1} \} \) with the product \( \psi^i \psi^j = \delta(i,j) \psi^i \). The central elements of \( C(Z_2) \) and \( C(Z_2)^* \) can be written as \( z = \alpha_1 \phi_1 + \alpha_{-1} \phi_{-1} \) and \( \xi = \beta_1 \psi^1 + \beta_{-1} \psi^{-1} \) respectively. By assigning each spacelike plaquette and timelike link with \( z_{S,p} \) and \( \xi_{T,v} \) respectively we obtain the transfer matrix as

\[
U(C(Z_2), z_{S,p}, \xi_{T,v}) = \prod_p B_p(z_{S,p}) \prod_v A_v(\xi_{T,v})
\]

where the plaquette operators are given by

\[
B_p(z_{S,p}) = \alpha_{1,p} B_p^1 + \alpha_{-1,p} B_p^{-1}
\]

with

\[
B_p^{\pm 1} = \frac{1 \pm \sigma^x_i \otimes \sigma^x_j \otimes \sigma^z_i \otimes \sigma^z_j}{2}
\]

and the vertex operators are given by

\[
A_v(\xi_{T,v}) = \left( \frac{\beta_{1,v} + \beta_{-1,v}}{2} \right) A_v^1 + \left( \frac{\beta_{1,v} - \beta_{-1,v}}{2} \right) A_v^{-1}
\]

with

\[
A_v^{\pm 1} = \frac{1 \pm \sigma^x_i \otimes \sigma^x_j \otimes \sigma^y_i \otimes \sigma^y_j}{2}
\]

The action is shown in figure 3.

The disordered QDH can now be written as

\[
H = \sum_v \left( \ln \left( \frac{\beta_{1,v} + \beta_{-1,v}}{2} \right) A_v^1 + \ln \left( \frac{\beta_{1,v} - \beta_{-1,v}}{2} \right) A_v^{-1} \right) + \sum_p \left( \ln \alpha_{1,p} B_p^1 + \ln \alpha_{-1,p} B_p^{-1} \right).
\]

As it can be seen this Hamiltonian breaks translational invariance and is made up of a sum of commuting projectors. When translational invariance is restored we recover the familiar toric code Hamiltonian.

The ground state of this Hamiltonian is easily obtained by projecting on to the smallest of \( \beta_{1,v} \) and \( \beta_{-1,v} \) for every vertex \( v \) and \( \alpha_{1,p} \) and \( \alpha_{-1,p} \) for every plaquette \( p \). The ground state degeneracy is the same as the usual toric code and the winding operators,

\[
X_{C_1^*;C_2^*}(C_1^*;C_2^*) = \prod_{j \in C_1^*;C_2^* (C_1^*;C_2^*)} \sigma^x_j
\]

\[
Z_{C_1^*;C_2^*}(C_1^*;C_2^*) = \prod_{k \in C_1^*;C_2^* (C_1^*;C_2^*)} \sigma^z_k
\]

where the non-contractible loops \( C_1, C_2, C_1^*; C_2^* \) are defined on the direct and dual lattice respectively as shown in the figure 3 commute with the Hamiltonian.

The excitations correspond to the other value of the coefficients for each vertex and plaquette. The string operators (ribbon operators in the case of the non-Abelian groups [17]) creating the excitations are the same as in the QDH case. For the specific case of \( C(Z_2) \) we have the string operators, creating charge or vertex excitations at the end points of the string \( \gamma \) along the direct lattice, as

\[
V_\gamma = \prod_{j \in \gamma} \sigma^y_j
\]

and those of the fluxes or plaquette excitations at the end points of the string \( \gamma^* \) along the dual lattice as

\[
P_{\gamma^*} = \prod_{k \in \gamma^*} \sigma^x_k
\]

These are shown in figure 3. As is well known these excitations are deconfined by which we mean that there is no cost in energy for moving them around by stretching the string creating them. Moreover the fusion rules and braiding statistics are the same as in the translationally invariant toric code. Thus we conclude that the disordered QDH given in Eq. (22) continues to remain in the toric code phase.
3.2 Quantum Double Hamiltonian with \( n \)-Step Confined Excitations

We first consider the case with \( n = 2 \). To obtain this we write down an example of exactly solvable Hamiltonian made up of the QDH vertex and plaquette operators along with new terms made of these operators which have 2-step confined low energy excitations unlike the QDH case where all the excitations are completely deconfined.

The transfer matrix used to obtain these models contain only the \( z_S \) and \( \xi_T \) parameters.

The transfer matrix can be written as

\[
U(A, z_S, \xi_T) = \prod_p B_p(z_S) \prod_v A_v(\xi_T).
\]  

In the case of \( C(Z_2) \) we can write down the Hamiltonian with the 2-step confined charges and fluxes as follows

\[
H = \sum_v (\alpha_1 A^1_v + \alpha_{-1} A^{-1}_v) + \sum_p (\beta_1 B^1_p + \beta_{-1} B^{-1}_p)
+ \sum_{<ij>} \left( \alpha'_1 A^1_{v_i} A^1_{v_j} + \alpha'_2 A^1_{v_i} A^{-1}_{v_j} + \alpha'_3 A^{-1}_{v_i} A^1_{v_j} + \alpha'_4 A^{-1}_{v_i} A^{-1}_{v_j} \right)
+ \sum_{<ij^*>} \left( \beta'_1 B^1_{v_i} B^1_{v_j} + \beta'_2 B^1_{v_i} B^{-1}_{v_j} + \beta'_3 B^{-1}_{v_i} B^1_{v_j} + \beta'_4 B^{-1}_{v_i} B^{-1}_{v_j} \right)
\]  

where \( <ij> \) and \( <ij^*> \) are nearest neighbor vertices in the direct and dual lattices respectively. All the terms in this Hamiltonian commute with each other and are sums of projectors. The ground states are given by the usual toric code Hamiltonian. The degeneracy does not change as the winding operators in the toric code case, given by Eq. 23 and Eq. 24, continue to commute with this Hamiltonian and thus help create the new states from a given ground state. In particular on a torus the degeneracy is four.

The interesting feature of this model occurs when we look at the excitations. As in the toric code case the string operators creating charge and flux excitations are given by Eq. 25 and Eq. 26 respectively. However in this model when we create a charge or flux excitation we also excite the direct or the dual link given by the \( A^1_{v_i} A^1_{v_j} \) or the \( B^1_{v_i} B^1_{v_j} \) term respectively. This creates link excitations along the string where the operator

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4 We assume that the coefficients of these two terms in the Hamiltonian are the smallest and hence violating these will lead to excitations. There is no loss of generality in making this assumption.
given by Eq. (25) or Eq. (26) acts. However the creation of these link excitations occurs only for two steps after which they are deconfined as in the toric code case. Thus we say that these excitations as 2-step confined excitations. These partially confined excitations are shown in figure (6) and (7).

(a) The pair of charges with energy $E$.

(b) Moving by one step increases the energy by $\Delta E$.

(c) Moving one more step outside the shaded region does not cost any energy.

(d) Once outside the shaded region the charges are deconfined.

Figure 6: The 2-step confined charge excitations.

(a) The pair of fluxes with energy $E$.

(b) Moving the flux by one step increases the energy by $\Delta E$.

(c) Once outside the shaded region they are deconfined.

Figure 7: The 2-step confined flux excitations.

It is easy to see that the dyonic excitations are also confined in a similar manner. Thus we have a model based on lattice gauge theory which is exactly solvable, has ground state degeneracy and has excitations which are confined up to two steps or can be thought of as being partially confined.

There is a natural way to increase the number of steps for which these particles are confined. This is achieved by coupling more number of vertex and plaquette operators. For example to obtain three-step confinement we add terms of the form $A^{1}_{v_{i}} A^{1}_{v_{j}} A^{1}_{v_{k}}$ where $i,j,k$ are nearest neighbors on the direct lattice. The corresponding plaquette terms are $B^{1}_{p^{*}_{i}} B^{1}_{p^{*}_{j}} B^{1}_{p^{*}_{k}}$, where $i^{*},j^{*},k^{*}$ are nearest neighbor vertices on the dual lattice. The energy of these excitations are more when compared to the original QDH as we violate more terms in the Hamiltonian to obtain them. This argument can easily be extended to any number of steps. The corresponding figures of the $n$-step confined excitations will have a larger shaded region where they are confined when compared to the $n=2$ case.

This model can be constructed on any triangulation of a two dimensional lattice keeping the confinement properties unchanged. This is easy to see as the additional terms of pairing neighboring vertex and plaquette operators is independent of triangulation. The model can also be easily extended to all group algebras and more generally to all involutory Hopf algebras without any obstacle.

In the case of $\mathbb{C}(\mathbb{Z}_{n})$ we can also add the parameters $z_T$ and $\xi_S$, to include the single qudit terms in the transfer matrix and keep the above properties of confinement. We will illustrate why this is so for $\mathbb{C}(\mathbb{Z}_{2})$. The argument extends easily for other $n$. We have the following relation

$$A^{1}_{v_{i}} A^{1}_{v_{j}} T_{l} A^{1}_{v_{i}} A^{1}_{v_{j}} = A^{1}_{v_{i}} A^{1}_{v_{j}}$$ (29)

where $T_{l} = \frac{1+\sigma_{l}}{2}$ and $v_1, v_2$ are the end points of the link $l$. This reduces the model to the previous Hamiltonian showing confinement. From this identity we see that we can include the parameters $\xi_S$ in the transfer matrix without any effect to the properties of the Hamiltonian considered earlier.
3.3 Perturbed QDHs

These are constructed out of transfer matrices which include the parameters $z_T$ or $\xi_S$. Their inclusion introduces single qudit operators $L_l(z_T)$ and $T_l(\xi_S)$ on the link $l$ respectively. These terms do not commute in general with the vertex and plaquette operators thereby making the process of taking their logarithms and hence obtaining the Hamiltonian difficult. However for certain parameters we can still take the logarithm to obtain a Hamiltonian. We will illustrate this in the case of $A = C(Z_2)$.

Let us include the parameter $z_T$ into the transfer matrix. This brings in the operator

$$L_l(z_T) = \left(\frac{x_1 + x_{-1}}{2}\right) (1 + \sigma^T_l) + \left(\frac{x_1 - x_{-1}}{2}\right) (1 - \sigma^T_l)$$

(30)

when $z_T = x_1\phi_1 + x_{-1}\phi_{-1}$. When $L_l(z_T) = \sigma^T_l$ we can have a product of two plaquette operators adjacent to the link $l$ commute with $L_l(z_T)$. We can then take the logarithms to obtain the following Hamiltonian

$$H = \sum_v \left(\ln(\alpha_1)A^1_v + \ln(\alpha_{-1})A^{-1}_v\right) + \sum_p \left(\ln(\beta_1)B^1_p + \ln(\beta_{-1})B^{-1}_p\right) + i\pi \left(\frac{1 - \sigma^T_l}{2}\right).$$

(31)

This model resembles adding magnetic perturbations to the QDH for the simplest case of $C(Z_2)$. The dual version of this involves using the parameter $\xi_S$ in the transfer matrix to obtain the magnetic field operator $\sigma^z_l$ on the links.

4 Outlook

A systematic procedure was used to obtain the models with interesting properties which included studying all possible ways of taking the logarithms of the transfer matrices of generalized lattice gauge theories. In [12] this led to quasi-topological phases with increased ground state degeneracy occurring due to condensed excitations of the QDH. In this paper another way of taking the logarithm led to partially confined excitations. We went further to include other parameters in the transfer matrix which took us away from the topologically ordered phases.

The state sum procedure used in [12] and further explored here can be thought of as a method to construct the quantum double of a given input algebra. Using this principle we can use other inputs to obtain the quantum doubles of more general objects leading to more interesting models. One such input are groupoid algebras which are examples of quantum groupoids. Such considerations lead to confined excitations in pure lattice gauge theories [21]. In [4] the quantum doubles of weak Hopf algebras were constructed. These reproduced the Levin-Wen models. Our considerations will embed the Levin-Wen models in the parameter space of lattice gauge theories based on these weak Hopf algebras. The weak Hopf algebras used in [4] were the ones constructed from a unitary fusion category by Kitaev and Kong in [15]. In particular by using weak Hopf algebras of [15] as inputs in the construction of [12] we will be able to obtain the operators creating excitations in the Levin-Wen models and finally we can confine and condense these quasi particles by using the methods in this paper.

The state sum construction can be used to construct the transfer matrices of lattice theories with gauge and matter fields by adding matter degrees of freedom, acted upon by the gauge fields, on the vertices of the triangulated lattice. This construction was shown to produce exactly solvable quantum models in one and two dimensions in [22]. The methods of this paper when applied to the transfer matrix constructed in [22] is bound to give many new interesting models.

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