DERIVATION OF COMPUTATIONAL FORMULAS FOR CERTAIN CLASS OF FINITE SUMS: APPROACH TO GENERATING FUNCTIONS ARISING FROM $p$-ADIC INTEGRALS AND SPECIAL FUNCTIONS

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Abstract. The aim of this paper is to construct generating functions for some families of special finite sums with the aid of the Newton-Mercator series, hypergeometric series, and $p$-adic integral (the Volkenborn integral). By using these generating functions, their functional equations, and their partial derivative equations, many novel computational formulas involving the special finite sums of (inverse) binomial coefficients, the Bernoulli type polynomials and numbers, Euler polynomials and numbers, the Stirling numbers, the (alternating) harmonic numbers, the Leibnitz polynomials and others. Among these formulas, by considering a computational formula which computes the aforementioned certain class of finite sums with the aid of the Bernoulli numbers and the Stirling numbers of the first kind, we present a computation algorithm and we provide some of their special values. Moreover, using the aforementioned special finite sums and combinatorial numbers, we give relations among multiple alternating zeta functions, the Bernoulli polynomials of higher order and the Euler polynomials of higher order. We also give decomposition of the multiple Hurwitz zeta functions with the aid of finite sums. Relationships and comparisons between the main results given in the article and previously known results have been criticized. With the help of the results of this paper, the solution of the problem that Charalambides [8, Exercise 30, p. 273] gave in his book was found and with the help of this solution, we also find very new formulas. In addition, the solutions of some of the problems we have raised in [48] are also given.

Keywords: Generating function, Finite sums, Special functions, Special numbers numbers and polynomials, multiple alternating zeta functions, $p$-adic integral, Computational algorithm

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1. Introduction, Definitions and Motivation

It is known that there are a large number of researchers have studied to find computational formulas for finite sums and infinite sums. Because it is often not easy to find computational formulas for any finite sum, involving special functions, special numbers, special polynomials, sums of higher powers of binomial coefficients. In order to find any computational formula for finite sums, still many new methods and techniques have been developed, investigated in mathematics, and also in other applied sciences. We know that finite sums and their computational formulas are special important mathematical tools

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most used by mathematicians, physicists, engineers and other scientists. Applications of the generating functions for special numbers and polynomials, and finite sums with their computational formulas have also been given by many different methods (cf. [1]-[58]). With this motivation, by an approach to generating functions arising from $p$-adic integrals and special functions, our purpose and motivations in this paper are to develop a computational methodology by deriving computational formulas for certain class of finite sums. The provided computational methodology provides the researchers a variety of methods that they can use in different fields and many situations.

In this study, we will construct generating functions that include special numbers and polynomials, and special finite sums. With the help of these generating functions and their functional equation, some new computational formulas will be given for these special finite sums. On the other hand, the main motivation of this paper is to construct and investigate generating functions, given by Theorem 1.2 and Theorem 1.3, with a great deal distinct applications for the following numbers $y(n, \lambda)$, represented with certain finite sum:

\begin{equation}
(1.1)
y(n, \lambda) = \sum_{j=0}^{n} \frac{(-1)^n}{(j+1)\lambda^{j+1}(\lambda - 1)^{n+1-j}}
\end{equation}

(cf. [48]).

The numbers $y(n, \lambda)$ reveal from the following zeta type function $Z_1(s; a, \lambda)$:

\begin{equation}
(1.2)
Z_1(s; a, \lambda) = \frac{\ln \lambda}{(\lambda - 1)(\ln a)^s} Li_s \left( \frac{1}{\lambda - 1} \right) + \frac{1}{(\ln a)^s} \sum_{n=0}^{\infty} \frac{y(n, \lambda)}{(n+1)^s},
\end{equation}

where $\lambda \in \mathbb{C} \setminus \{0, 1\}$ ($\frac{1}{|\lambda - 1|} < 1$; Re$(s) > 1$), $a \geq 1$, and $s \in \mathbb{C}$, (cf. [48], [49], [50]). The function $Li_s(\lambda)$ given on the right-hand side of the equation (1.2) is denoted the polylogarithm function:

\begin{equation}
Li_s(z) = \sum_{n=1}^{\infty} \frac{\lambda^n}{n^s},
\end{equation}

(cf. [53], [55]).

Our first goal is to construct the following generating functions for the numbers $y(n, \lambda)$ by aid the Newton-Mercator series which is the Taylor series for the logarithm function.

**Definition 1.1.** Let $\lambda \in \mathbb{R}$ with $\lambda \neq 0, 1$. Let $z \in \mathbb{C}$. The numbers $y(n, \lambda)$ are defined by the following generating function:

\begin{equation}
(1.3)
G(z, \lambda) = \sum_{n=0}^{\infty} (1 - \lambda)^{n+2} y(n, \lambda) z^n.
\end{equation}

This paper provide many new formulas that include not only the numbers $y(n, \lambda)$ and special numbers and polynomials and special finite sums, but also their generating functions. Among others, we list the following some of theorems involving these novel formulas. The proofs of the Theorems are given in detail in the following sections.
Theorem 1.2. Let \( \lambda \in \mathbb{R} \) with \( \lambda \neq 0, 1 \). Let \( z \in \mathbb{C} \) with \( \left| \frac{\lambda - 1}{\lambda} z \right| < 1 \). Then we have

\[
G(z, \lambda) = \ln \left( \frac{1 - \frac{\lambda - 1}{\lambda} z}{z(z - 1)} \right).
\]

To show the power of the function \( G(z, \lambda) \) and its various applications, it will be examined in detail in the next sections.

Subsequent, we come up with an interesting theorem asserting that the function \( G(z, \lambda) \) can be expressed as a hypergeometric function \( \text{2F1} \)

\[
G(z, \lambda) = \text{2F1} \left[ \frac{1, 1}{2}; \frac{1 - \lambda z}{\lambda} \right].
\]

Theorem 1.3. Let \( \lambda \in \mathbb{R} \) with \( \lambda \neq 0, 1 \). Let \( z \in \mathbb{C} \) with \( \left| \frac{\lambda - 1}{\lambda} z \right| < 1 \). Then we have

\[
G(z, \lambda) = \frac{(1 - \lambda) z}{\lambda(z - 1)} \text{2F1} \left[ \frac{1, 1}{2}; \frac{1 - \lambda z}{\lambda} \right].
\]

Applying Theorem 1.2 and Theorem 1.3, we derive many novel computational formulas and relations involving finite sums, special numbers, and special polynomials.

In addition, we present several eliciting and considerable remarks on these formulas and relations.

By using generating functions and their functional equations, we give some properties of the numbers \( y(n, \lambda) \). We show that the numbers \( y(n, \lambda) \) are closely associated with the Bernoulli numbers, the Euler numbers, the harmonic numbers, the alternating Harmonic numbers, the Apostol-Bernoulli numbers, the Stirling numbers, the Leibnitz numbers, and special finite sums.

Another important purpose of this paper is to provide solutions to some of the open problems that have been raised by the author \cite{48} and the Exercise 30 given by Charalambides \cite[Exercise 30, p. 273]{8} with the aid of the numbers \( y(n, \lambda) \) and their generating functions.

Some of our main contributions, derived from Theorem 1.2, Theorem 1.3, and hypergeometric functions, are listed by the following Theorems, among other results.

Theorem 1.4. Let \( m \in \mathbb{N}_0 \). Then, we have

\[
B_m(\lambda) = \sum_{n=0}^{m} (n + 1)! \lambda^{n+1} y(n, \lambda) S_2(m, n + 1),
\]

where \( B_m(\lambda) \) and \( S_2(m, n) \) denote the Apostol-Bernoulli numbers and the Stirling numbers of the second kind, respectively.

By applying to the equation (1.4), we derive the following a new relation involving the harmonic numbers \( H_n \) and the numbers \( y(n, \lambda) \).

Theorem 1.5. Let \( n \in \mathbb{N} \). Then we have

\[
H_{2n+2} - H_{n+1} = -\frac{1}{2(n + 1)} + (n + 1) \sum_{k=0}^{2n-1} \frac{\lambda^{k+2}}{2n-k} y(n, \lambda)
\]
\[ + (n + 1) (\lambda - 1) \sum_{k=0}^{2n} \frac{\lambda^{k+1} y(n, \lambda)}{2n + 1 - k}. \]

**Theorem 1.6.** Let \( n \in \mathbb{N} \). Then we have

\[ \sum_{j=0}^{n} \zeta(n+1-j)(-m, n+2) = \sum_{j=0}^{n} \zeta^{(d)}(s, x) = \frac{2^d}{x+v}, \]

where \( \zeta^{(d)}(s, x) \) denotes the multiple alternating Hurwitz zeta function (multiple Hurwitz-Euler eta function), which is given as follows (1.9):

\[ \zeta^{(d)}(s, x) = 2 \sum_{v=0}^{\infty} (-1)^v \left( \frac{1}{x+v} \right). \]

Note that proof of Theorem 1.6 will be presented in Section 6.

**Theorem 1.7.** Let \( m \in \mathbb{N}_0 \). Then, we have

\[ y(m, \lambda) = \sum_{v=0}^{m} \sum_{n=0}^{v} (-1)^{v-m} \frac{(\lambda - 1)^{v-m-1} B_n S_1(v, n)}{\lambda^{v+1} v!}. \]

We give two different proof of Theorem 1.7. The first proof related to generating functions and functional equation method. The second proof is associated with the \( p \)-adic integral method.

**Theorem 1.8.** Let \( f(\lambda) \) be an entire function and \( |\lambda| < 1 \). Then we have

\[ \sum_{v=0}^{\infty} f(v) \lambda^v + \sum_{m=1}^{\infty} \frac{f^{(m-1)}(0)}{m!} \sum_{n=0}^{m} (n+1)! \lambda^{n+1} y(n, \lambda) S_2(m, n+1) = 0. \]

### 1.1. Preliminaries

Throughout the paper, we use the following notation and definitions.

Let \( \mathbb{N}, \mathbb{Z}, \mathbb{R}, \) and \( \mathbb{C} \) denote the set of natural numbers, the set of integer numbers, the set of real numbers and the set of complex numbers, respectively. \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \). For \( z \in \mathbb{C} \) with \( z = x + iy \) \((x, y \in \mathbb{R})\); \( \text{Re}(z) = x \) and \( \text{Im}(z) = y \) and also \( \ln z \) denotes the principal branch of the many-valued function \( \Im(\ln z) \) with the imaginary part of \( \ln z \) constrained by

\[ -\pi < \text{Im}(\ln z) \leq \pi. \]

\[ 0^n = \begin{cases} 1, & n = 0 \\ 0, & n \in \mathbb{N}. \end{cases} \]

The well-known generalized hypergeometric function is given by

\[ \sum_{m=0}^{\infty} \frac{\prod_{j=1}^{p} (\alpha_j)^m}{\prod_{j=1}^{q} (\beta_j)^m} \frac{z^m}{m!}, \]
where the above series converges for all \( z \) if \( p < q + 1 \), and for \( |z| < 1 \) if \( p = q + 1 \). Assuming that all parameters have general values, real or complex, except for the \( \beta_j, j = 1, 2, ..., q \) none of which is equal to zero or a negative integer and also

\[
(\lambda)^v = \prod_{j=0}^{v-1} (\lambda + j),
\]

and \( (\lambda)^0 = 1 \) for \( \lambda \neq 1 \), where \( v \in \mathbb{N}, \lambda \in \mathbb{C} \). For the generalized hypergeometric function and their applications, it is also recommended to refer to the following resource (cf. [23, 45, 54], [30]; and references therein).

\[
(\lambda)^v = \prod_{j=0}^{v-1} (\lambda - j),
\]

and \( (\lambda)^0 = 1 \).

The Bernoulli polynomials of higher order, \( B_n^{(l)}(y) \), are defined by

\[
F_B(u, y; l) = \left( \frac{u}{e^u - 1} \right)^l e^{yu} = \sum_{n=0}^{\infty} B_n^{(l)}(y) \frac{u^n}{n!},
\]

(cf. [2, 9, 13, 19, 23, 24, 28, 32, 33, 34, 43, 44, 45, 47, 48, 52, 53]; and references therein).

When \( y = 0 \) in (1.12), we have the Bernoulli numbers of order \( k \)

\[
B_n^{(k)} = B_n^{(k)}(0),
\]

and when \( k = 1 \), we have the Bernoulli numbers

\[
B_n = B_n^{(1)},
\]

(cf. [9, 13, 19, 23, 24, 28, 32, 33, 34, 43, 44, 45, 47, 48, 52, 53]; and references therein).

The Euler polynomials of higher order, \( E_m^{(l)}(y) \), are defined by

\[
F_E(u, y; l) = \left( \frac{2}{e^u + 1} \right)^l e^{yu} = \sum_{m=0}^{\infty} E_m^{(l)}(y) \frac{u^m}{m!},
\]

The harmonic numbers \( H_n \) are defined by

\[
F_1(u) = \frac{\ln(1 - u)}{u - 1} = \sum_{n=1}^{\infty} H_n u^n,
\]

where \( H_0 = 0 \) and \( |u| < 1 \) (cf. [13, 32, 52, 53, 11]).

A relation between the numbers \( y \left( n, \frac{1}{2} \right) \) and \( H_n \) is given as follows:

\[
y \left( n, \frac{1}{2} \right) = 2^{n+2} \left( H_{\left[ \frac{n}{2} \right]} - H_n + \frac{(-1)^{n+1}}{n+1} \right),
\]

(cf. [49]).
The alternating Harmonic numbers $H_n$ are defined by

$$F_2(u) = \frac{\ln(1 + u)}{u - 1} = \sum_{n=1}^{\infty} H_n u^n,$$

where $|u| < 1$ (cf. [13], [15], [49], [52]).

In [49, Eq. (20)], we showed that the following formula

$$y\left(n, \frac{1}{2}\right) = 2^{n+2} \sum_{j=0}^{n} \frac{(-1)^{j+1}}{j+1}$$

is related to the following well-known alternating harmonic numbers

$$H_n = \sum_{j=1}^{n} \frac{(-1)^j}{j} = H_{\lfloor \frac{n}{2} \rfloor} - H_n,$$

(cf. [13], [15], [52, Eq. (1.5)], [49, Eq. (20)]).

The Stirling numbers of the first kind $S_1(v, d)$ are defined by

$$F_{s1}(u, d) = \frac{(\ln(1 + u))^d}{d!} = \sum_{v=0}^{\infty} S_1(v, d) \frac{u^v}{v!}$$

and

$$\langle u \rangle^d = \sum_{j=0}^{d} S_1(d, j) u^j$$

(cf. [9, 13, 19, 23, 24, 28, 32, 33, 34, 43, 44, 45, 46, 47, 48, 52, 53]; and references therein).

Combining (1.18) with the Lagrange inversion formula, a computation formula of the Stirling numbers of the first kind is given by

$$S_1(n, k) = \sum_{c=0}^{n-k} \sum_{j=0}^{e} (-1)^j \binom{c}{j} \binom{n+c-1}{k-1} \binom{2n-k}{n-k-c} \frac{j^{n-k+c}}{c!},$$

where $k = 0, 1, 2, \ldots, n$ and $n \in \mathbb{N}_0$ (cf. [8, Eq. (8.21), p. 291]).

The Stirling numbers of the second kind $S_2(v, d)$ are defined by

$$F_{s2}(u, k) = \frac{(e^u - 1)^d}{d!} = \sum_{v=0}^{\infty} S_2(v, d) \frac{u^v}{v!}$$

(cf. [9, 13, 19, 23, 24, 28, 32, 33, 34, 43, 44, 45, 46, 47, 48, 52, 53]; and references therein).

Using (1.21), a computation formula of the Stirling numbers of the second kind is given by

$$S_2(n, k) = \frac{1}{k!} \sum_{c=0}^{k} (-1)^{k-c} \binom{k}{c} c^n$$

where $k = 0, 1, 2, \ldots, n$ and $n \in \mathbb{N}_0$ (cf. [8, Eq. (8.19), p. 289]).
THE APOSTOL-BERNOULLI NUMBERS $B_v(\theta)$ ARE DEFINED BY

$$F_A(u, \theta) = \frac{u}{\theta e^u - 1} = \sum_{v=0}^{\infty} B_v(\theta) \frac{u^v}{v!}$$

(cf. [1]).

Combining (1.23) with (1.21), a computation formula of the Apostol-Bernoulli numbers $B_v(\theta)$ is given by

$$B_v(\theta) = \frac{n \theta}{(\theta - 1)^n} \sum_{c=0}^{n-1} (-1)^c c! \theta^{c-1} (\theta - 1)^{n-1-c} S_2(n - 1, c)$$

(cf. [1]).

The Leibnitz polynomials $L_m(x)$ are defined by the following generating function

$$G_l(x, u) = \ln (1 - u) + \ln (1 - xu) - \frac{1}{1 - u} (1 - xu) = \sum_{m=0}^{\infty} L_m(x) u^m,$$

where $|u| < 1$ (cf. [8, Exercise 16, p. 127]).

Using (1.25), the polynomials $L_m(x)$, whose degree is $m$, are given by

$$L_m(x) := \sum_{l=0}^{m} \ell(m, l) x^l,$$

where $\ell(m, l)$ denotes the Leibnitz numbers, defined by

$$\ell(m, l) = \frac{1}{(m + 1) \binom{m}{l}}$$

or

$$\ell(m, l) = \sum_{d=0}^{l} (-1)^{l-d} \frac{1}{m-d+1} \binom{l}{d},$$

where $l = 0, 1, 2, \ldots, m$ and $m \in \mathbb{N}_0$ (cf. [8, Exercise 16, p. 127], [51]).

The Bernstein basis functions are defined by

$$B_l^m(x) = \binom{m}{l} x^l (1 - x)^{m-l}$$

Integrate the following equation (1.28) with respect to $x$ from 0 to 1, we have

$$\frac{1}{(m/l)} \int_0^1 B_l^m(x) dx = \sum_{d=0}^{m-l} (-1)^{m-l-d} \binom{m-l}{d} \int_0^1 x^{m-d} dx.$$

Combining the above equation with (1.26), we have

$$\ell(m, l) = \sum_{d=0}^{m-l} (-1)^{m-l-d} \binom{m-l}{d} \frac{1}{m-d+1}. $$
The numbers \( Y_v(\lambda) \) are defined by
\[
(1.29) \quad g(u; \lambda) = \frac{2}{\theta^2 u + \theta - 1} = \sum_{v=0}^{\infty} Y_v(\theta) \frac{u^v}{v!},
\]
where
\[
Y_v(\lambda) = -\frac{2(v!) \theta^{2v}}{(1 - \theta)^{v+1}}.
\]
(cf. [42, Eq. (2.13)]).

The Bernoulli numbers of the second kind (the Cauchy numbers) \( b_v(0) \) are defined by
\[
(1.30) \quad F_{b_2}(u) = \frac{u}{\ln(1+u)} = \sum_{n=0}^{\infty} b_v(0) \frac{u^v}{v!},
\]
(cf. [13], [33, p. 116], [47], [53]; see also the references cited in each of these earlier works).

The numbers \( D_n \), which are so-called the Daehee numbers, are defined by
\[
(1.31) \quad F_{D_3}(u) = \ln(1+u) = \sum_{n=0}^{\infty} D_n \frac{u^n}{n!}, \quad (u \neq 0, |u| < 1)
\]
(cf. [19]).

By combining the Newton-Mercator series with (1.31), one has the following formula:
\[
(1.32) \quad D_n = (-1)^n \frac{n!}{n+1},
\]
(cf. [9, p. 117], [19], [32, p. 45, Exercise 19 (b)]).

The derangement numbers \( d_m \) are defined by the following generating function:
\[
(1.33) \quad F_d(u) = \frac{e^{-u}}{1 - u} = \sum_{m=0}^{\infty} d_m \frac{u^m}{m!}, \quad (|u| < 1)
\]
where
\[
d_m = \sum_{j=0}^{m} (-1)^j (m-j)! \binom{m}{j}.
\]
(cf. [7], [25]).

The Fibonacci-type polynomials in two variables are defined by the following generating function:
\[
(1.34) \quad H(t; x, y; k, m, l) = \sum_{n=0}^{\infty} \mathcal{G}_n(x, y; k, m, l) t^n = \frac{1}{1 - x^k t - y^m t^{m+l}},
\]
where \( k, m, l \in \mathbb{N}_0 \) (cf. [27]).

Using (1.34), we have the following explicit formula for the polynomials \( \mathcal{G}_n(x, y; k, m, l) \):
\[
\mathcal{G}_n(x, y; k, m, l) = \sum_{c=0}^{\left\lfloor \frac{n}{m+l+1} \right\rfloor} \binom{n - c (m + l - 1)}{c} y^{mc} x^{nk-mck-1ck},
\]
where \([a]\) is the largest integer \(\leq a\) (cf. \([27], [28]\)).

Let’s briefly summarize the next sections of the article.

In Section 2, we give the solution of the open problem 1, which has been proposed by the author \([48, p.57, Open problem 1]\) about the generating functions of the numbers \(y(n, \lambda)\). We give many properties of this function.

In Section 3, with the help of generating functions and their functional equations, we give many identities involving the numbers \(y(n, \lambda)\), the Bernoulli numbers of the second kind, the harmonic numbers, alternating Harmonic numbers, the Apostol-Bernoulli numbers, the Stirling numbers, the Leibnitz numbers, the Bernoulli numbers, and sums involving higher powers of inverses of binomial coefficients.

In Section 4, we give computation algorithm for the numbers \(y(n, \lambda)\). We also give some values of the numbers \(y(n, \lambda)\).

In Section 5, we give differential equations of the generating functions and their applications. We give some applications of these equation.

In Section 6, with the help of the numbers \(y(n, \lambda)\), we give decomposition of the multiple Hurwitz zeta functions involving the Bernoulli polynomials of higher order.

In Section 7, we give infinite series representations of the numbers \(y(n, \lambda)\) on entire functions.

In Section 8, we also construct the generating function for the numbers \(y(n, \lambda)\) with the help of Volkenborn integral on \(p\)-adic integers. We give some applications of the \(p\)-adic integral.

In Section 9, we conclusion about the results of this paper.

2. Generating functions for the numbers \(y(n, \lambda)\)

In this section, we construct generating functions for the numbers \(y(n, \lambda)\). We give some properties of these functions. By using these functions and their functional equations, we give many new computational formulas and relations for the numbers \(y(n, \lambda)\) and special finite sums.

We also give the solution of the following open problem, which has been proposed in \([48, p.57, Open problem 1]\):

What is generating function for the numbers \(y(n, 2)\) and the numbers \(y(n, \lambda)\)?

Its answer is given by Theorem 1.2.

Proof of Theorem 1.2. Substituting (1.1) into (1.3), after some calculations, we obtain

\[
G(z, \lambda) = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{1}{j + 1} \left(\frac{\lambda - 1}{\lambda}\right)^{j+1} z^n.
\]

The following result is obtained by decomposing the above series for which the Cauchy product of two infinite series has been applied.

\[
G(z, \lambda) = \frac{1}{z} \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n + 1} \left(\frac{1 - \lambda}{\lambda z}\right)^{n+1} \sum_{n=0}^{\infty} z^n.
\]
After combining the above equation with the following well-known Newton-Mercator series
\[
\ln(1 + z) = \sum_{j=0}^{\infty} \frac{(-1)^j z^{j+1}}{j+1}, \quad (|z| < 1)
\]
and geometric series
\[
\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \quad (|z| < 1)
\]
yields the assertion of Theorem 1.2. □

**Proof of Theorem 1.3.** Substituting \( p = 2, q = 1, \alpha_1 = \alpha_2 = 1, \beta_1 = 1 \) and \( u = \frac{1-\lambda}{\lambda} z \) into (1.11), we obtain
\[
\begin{align*}
2F_1 \left[ \frac{1}{2}, \frac{1-\lambda}{\lambda} z \right] & = \sum_{m=0}^{\infty} \frac{(1)^m (1)^m}{(2)^m m!} \left( \frac{1-\lambda}{\lambda} z \right)^m .
\end{align*}
\]
Multiplying both sides of the above equation by \( \frac{1-\lambda}{\lambda(z-1)} \), we get
\[
2F_1 \left[ \frac{1}{2}, \frac{1-\lambda}{\lambda} z \right] \frac{1}{z-1} = \frac{1}{z-1} \sum_{m=0}^{\infty} \frac{1}{m+1} \left( \frac{1-\lambda}{\lambda} z \right)^{m+1}.
\]
Combining the above equation with (1.4), we obtain
\[
\frac{(1-\lambda) z}{\lambda(z-1)} 2F_1 \left[ \frac{1}{2}, \frac{1-\lambda}{\lambda} z \right] = G(z, \lambda).
\]
Thus, proof of Theorem 1.3 is completed. □

Some special cases of the generating function, given in (1.4), are given as follows:

Substituting \( \lambda = -1 \) into (1.4), we have the following generating function for the numbers \( y(n, -1) \):
\[
g_1(z) = G(z, -1) = \frac{\ln(1 - 2z)}{z^2 - z} = \sum_{n=0}^{\infty} 2^{n+2} y(n, -1) z^n,
\]
where \( z \neq 0, z \neq 1 \) and \( |2z| < 1 \).

The function \( g_1(z) \) is associated with generating function for the finite sums of powers of inverse binomial coefficients. These relationships will be investigated in detail in the following Sections.

Substituting \( \lambda = 2 \) into (1.3), we have the following generating function for the numbers \( y(n, 2) \):
\[
g_2(z) = G(z, 2) = \frac{\ln(1 - \frac{z}{2})}{z^2 - z} = \sum_{n=0}^{\infty} (-1)^n y(n, 2) z^n,
\]
where \( z \neq 0, z \neq 1 \) and \( |z| < 1 \).
By using the function $g_2(z)$, relationships among the numbers $y(n, 2)$, the Bernoulli numbers, the Stirling numbers, and some special finite sums will be investigated in detail in the following sections.

Substituting $\lambda = \frac{1}{2}$ into (1.4), we have the following generating function for the numbers $y(n, \frac{1}{2})$:

$$g_3(z) = G(z, \frac{1}{2}) = \ln(1 + z) - z = \sum_{n=0}^{\infty} \frac{1}{2n+2} y(n, \frac{1}{2}) z^n,$$

where $z \neq 0$, $z \neq 1$ and $|z| < 1$.

By using the function $g_3(z)$, relationships among the numbers $y(n, \frac{1}{2})$, the Bernoulli numbers, the Stirling numbers, the Harmonic numbers, and some special finite sums will be investigated in detail in the following sections.

3. IDENTITIES DERIVED FROM GENERATING FUNCTION

In this section, using generating functions and their functional equations, we give very interesting and novel formulas and identities involving the numbers $y(n, \lambda)$, the Bernoulli numbers of the second kind, the harmonic numbers, alternating Harmonic numbers, the Apostol-Bernoulli numbers, the Stirling numbers, the Leibnitz numbers, the Bernoulli numbers, and sums involving higher powers of inverses of binomial coefficients.

We begin this section by giving proofs of some theorems given in the Section 1.

**Proof. of Theorem 1.5.** Multiply both sides of the equation (1.3) by the function $\ln (1 + \frac{1}{\lambda-1} z)$ and, with the help of the Newton-Mercator series, we obtain

$$\ln \left(1 - \frac{\lambda-1}{\lambda} z \right) \ln \left(1 + \frac{\lambda-1}{\lambda} z \right) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^n \frac{(1-\lambda)^n y(k, \lambda)}{(n+1-k) \lambda^{n-k+1}} z^{n+1}. \quad (3.1)$$

By using Abel’s summation formula, and using multiplication two of the Newton-Mercator series, Furduiu [14] gave the following formula:

$$\ln (1 - y) \ln (1 + y) = \sum_{v=1}^{\infty} \left( H_v - H_{2v} - \frac{1}{2v} \right) \frac{y^{2v}}{v}, \quad y^2 < 1. \quad (3.2)$$

Combining (3.1) with (3.2), we get

$$\sum_{m=1}^{\infty} \left( H_m - H_m - \frac{1}{2m} \right) \left( \frac{\lambda-1}{\lambda} \right)^{2m} \frac{2m}{m}$$

$$= \sum_{n=3}^{\infty} \sum_{k=0}^{n-3} (-1)^{n+1} \frac{(1-\lambda)^n y(k, \lambda)}{(n-k-2) \lambda^{n-k-2}} z^n$$

$$+ \sum_{n=2}^{\infty} \sum_{k=0}^{n-2} (-1)^{n+1} \frac{(1-\lambda)^{n+1} y(k, \lambda)}{(n-k-1) \lambda^{n-k-1}} z^n.$$
Therefore

\[ (3.3) \quad \sum_{n=1}^{\infty} \left( H_n - H_n - \frac{1}{2n} \right) \left( \frac{\lambda - 1}{\lambda} \right)^n \sum_{k=0}^{2n-2} \frac{(1 - \lambda)^{2n+1} y(k, \lambda)}{(2n-k-1) \lambda^{2n-k-1}} - \sum_{k=0}^{2n-1} \frac{(1 - \lambda)^{2n+1} y(k, \lambda)}{(2n-k) \lambda^{2n-k}} z^{2n+1}. \]

After making some necessary algebraic calculations in the previous equation, the coefficients of \( z^{2n} \) are equalized and the above equation we arrive at the desired result.

By using (3.3), we also get the following result:

**Corollary 3.1.** Let \( n \in \mathbb{N} \). Then we have

\[ (3.5) \quad (x + 1) L_n(x) - x L_{n-1}(x) = (-1)^n \lambda^{n+1} \left( x^{n+1} + 1 \right) y(n, \lambda) z^n. \]

Combining (3.4) with (1.3) and (1.25) yields

\[ -(x + 1) \sum_{n=0}^{\infty} L_n(x) \left( \frac{\lambda - 1}{\lambda} \right)^n z^n + x \sum_{n=1}^{\infty} L_{n-1}(x) \left( \frac{\lambda - 1}{\lambda} \right)^n z^n \]

\[ = - \sum_{n=0}^{\infty} (-1)^n (\lambda - 1)^{n+2} (x^{n+1} + 1) y(n, \lambda) z^n \]

\[ - \sum_{n=0}^{\infty} (-1)^n (\lambda - 1)^{n+1} (x^{n+1} + 1) y(n - 1, \lambda) z^n. \]

Comparing the coefficients of \( z^n \) on both sides of the above equation, we have following theorem:

**Theorem 3.2.** Let \( n \in \mathbb{N} \). Then we have

\[ (3.5) \quad (x + 1) L_n(x) - x L_{n-1}(x) = (-1)^n \lambda^{n+1} \left( x^{n+1} + 1 \right) \left( \lambda - 1 \right) y(n, \lambda) + y(n - 1, \lambda). \]

We set the following functional equation:

\[ g_3(z) F_{b2}(z) = \frac{1}{z - 1}. \]
where $|z| < 1$. Combining the above functional equation with (2.3) and (1.30), we get
\[ \sum_{n=0}^{\infty} \frac{1}{2^{n+2}} y \left( n, \frac{1}{2} \right) z^n \sum_{n=0}^{\infty} b_n(0) \frac{z^n}{n!} = - \sum_{n=0}^{\infty} z^n. \]

Therefore
\[ \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{1}{2^{j+2}} y \left( j, \frac{1}{2} \right) b_{n-j}(0) \frac{z^n}{(n-j)!} = - \sum_{n=0}^{\infty} z^n. \]

Comparing the coefficients of $z^n$ on both sides of the above equation, we have the following theorem:

**Theorem 3.3.** Let $n \in \mathbb{N}_0$. Then, we have
\[ \sum_{j=0}^{n} \frac{y \left( j, \frac{1}{2} \right) b_{n-j}(0)}{2^j (n-j)!} = -4. \]

We give a decomposition of the generating function $g_3(z)$ as follows:
\[ g_3(z) = F_2(z) - F_3(z). \]

Combining the above function with (1.31) and (1.16), we obtain
\[ \sum_{n=0}^{\infty} \frac{1}{2^{n+2}} y \left( n, \frac{1}{2} \right) z^n = \sum_{n=1}^{\infty} \mathcal{H}_n z^n - \sum_{n=0}^{\infty} D_n \frac{z^n}{n!}. \]

Comparing the coefficients of $z^n$ on both sides of the above equation, we have the following theorem:

**Theorem 3.4.** Let $n \in \mathbb{N}_0$. Then, we have
\[ y \left( n, \frac{1}{2} \right) = \frac{2^{n+2}}{n!} \left( n! \mathcal{H}_n - D_n \right). \]

**Corollary 3.5.** Let $n \in \mathbb{N}_0$. Then, we have
\[ y \left( n, \frac{1}{2} \right) = \frac{2^{n+2}}{n!} \left( n! \mathcal{H}_n - \sum_{j=0}^{n} B_j S_1(n, j) \right). \]

**Corollary 3.6.** Let $n \in \mathbb{N}_0$. Then, we have
\[ y \left( n, \frac{1}{2} \right) = 2^{n+2} \left( \mathcal{H}_n + \frac{(-1)^{n+1}}{n+1} \right). \]

**Corollary 3.7.** Let $n \in \mathbb{N}_0$. Then, we have
\[ y \left( n, \frac{1}{2} \right) = 2^{n+1} \mathcal{H}_{n+1}. \]
We set 
\[ F_2(z) = zF(z). \]

By using the above equation, we have
\[ \sum_{n=1}^{\infty} \mathcal{H}_n z^n = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} B_n \left( n, \frac{1}{2} \right) z^{n+1}. \]

After some elementary calculations, we arrive at the allowing result:

**Corollary 3.8.** Let \( n \in \mathbb{N} \). Then, we have
\[ H_n = \frac{1}{2^n+1} B_n \left( n-1, \frac{1}{2} \right). \]

Noting that with the aid of (3.8), we give a series representation of the function \( F_2(z) - g_3(z) \) as follows:
\[ F_2(z) - g_3(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} z^n. \]

By combining (2.3) with (3.11), we also arrive at (3.10).

It is time to give the first proof of Theorem 1.7. The following proof is related to generating functions and functional equation method.

**The first proof of Theorem 1.7.** Substituting \( l = 1 \), \( u = \ln \left( 1 + \frac{1}{\lambda} \right) \), \( y = 0 \), and \( z \neq 0 \) into (1.12), after some elementary calculations, we obtain
\[ \ln \left( 1 + \frac{1}{\lambda} \right) \frac{1}{z} = \sum_{n=0}^{\infty} B_n \frac{\ln \left( 1 + \frac{1}{\lambda} \right)^n}{n!}. \]

Combining the above equation with (1.3) and (1.18), we get
\[ \sum_{m=0}^{\infty} (1 - \lambda)^{m+2} y(m, \lambda) z^m = \frac{1}{z-1} \sum_{m=0}^{\infty} \sum_{n=0}^{m} B_n S_1(m, n) \left( \frac{1-\lambda}{\lambda} \right)^{m+1} z^m \]

since \( S_1(m, n) = 0 \) if \( n > m \). Assuming that \( |z| < 1 \), we obtain
\[ \sum_{m=0}^{\infty} (1 - \lambda)^{m+2} y(m, \lambda) z^m = - \sum_{v=0}^{\infty} z^v \sum_{m=0}^{\infty} \sum_{n=0}^{m} B_n S_1(m, n) \left( \frac{1-\lambda}{\lambda} \right)^{m+1} \frac{z^m}{m!}. \]

After some elementary calculations, the above equation yields
\[ \sum_{m=0}^{\infty} (1 - \lambda)^{m+2} y(m, \lambda) z^m = - \sum_{m=0}^{\infty} \sum_{v=0}^{m} \sum_{n=0}^{v} \frac{(1-\lambda)^{v+1}}{\lambda^v+1} B_n S_1(v, n) \frac{z^m}{v!}. \]

Now equating the coefficients of \( z^m \) on both sides of the above equation, we arrive at the desired result.

Substituting \( \lambda = \frac{1}{2} \) and \( \lambda = 2 \) into (1.10), we arrive at the following corollaries, respectively:
Corollary 3.9. Let $m \in \mathbb{N}_0$. Then, we have
\begin{equation}
y(m, \frac{1}{2}) = -2^{m+2} \sum_{v=0}^{m} \sum_{n=0}^{v} \frac{B_n S_1(v, n)}{v!}.
\end{equation}

Corollary 3.10. Let $m \in \mathbb{N}_0$. Then, we have
\begin{equation}
y(m, 2) = \sum_{v=0}^{m} \sum_{n=0}^{v} \frac{(-1)^{v-m} B_n S_1(v, n)}{2^{v+1}v!}.
\end{equation}

Remark 3.11. By combining the following well-known identity:
\begin{equation}
\sum_{n=0}^{m} B_n S_1(m, n) = \frac{(-1)^m m!}{m+1}
\end{equation}
(cf. [9, p. 117], [19], [32, p. 45, Exercise 19 (b)], [49, Eq. (20)]), with (1.10), we arrive at (1.1). Noting that there are many other proofs of (3.14). For example, Kim [19] gave proof of (3.14) by using the $p$-adic invariant integral on the set of $p$-adic integers. Kim represented the equation (3.14) by the notation $D_n$, which are so-called the Daehee numbers. Riordan [32, p. 45, Exercise 19 (b)] represented the equation (3.14) by the notation $(b)_n$

With the aid of the equations (3.14), (1.16), (1.31) and (1.32), we get some interesting formulas involving the Bernoulli numbers, the Stirling numbers of the first kind, the Daehee numbers, and the alternating Harmonic numbers.

Combining (1.16) with (1.31), we get
\[ F_3(u) = \frac{u - 1}{u} F_2(u) \]
Using the above equation, we get
\[ \sum_{n=0}^{\infty} D_n \frac{u^{n+1}}{n!} = \sum_{n=1}^{\infty} \mathcal{H}_n u^n - \sum_{n=1}^{\infty} \mathcal{H}_n u^n. \]
Therefore
\[ \sum_{n=1}^{\infty} D_n-1 \frac{u^n}{(n-1)!} = \sum_{n=2}^{\infty} \mathcal{H}_{n-1} u^n - \sum_{n=1}^{\infty} \mathcal{H}_n u^n. \]
Comparing the coefficients of $u^n$ on both sides of the above equation, we arrive at the following relation:
\[ D_{n-1} = (n - 1)! (\mathcal{H}_{n-1} - \mathcal{H}_n). \]
By the above equation and (1.32), we see that
\[ \mathcal{H}_{n-1} - \mathcal{H}_n = \frac{(-1)^{n-1}}{n}. \]
Combining (1.16) with (1.31), we also have
\[ F_2(u) = -\frac{u}{1-u} F_3(u) \]
Using the above equation, we obtain
\[ \sum_{n=1}^{\infty} H_n u^n = -u \sum_{n=0}^{\infty} u^n \sum_{n=0}^{\infty} D_n \frac{u^n}{n!}. \]

Therefore
\[ \sum_{n=1}^{\infty} \mathcal{H}_n u^n = -u \sum_{n=0}^{\infty} \sum_{j=0}^{n} D_j \frac{u^n}{j!}. \]

Comparing the coefficients of \( u^n \) on both sides of the above equation, we arrive at the following relation:

\[ (3.15) \quad \mathcal{H}_n = -\sum_{j=0}^{n-1} \frac{D_j}{j!}. \]

Substituting (1.32) into (3.15), and using (3.14), we arrive at the following theorem:

**Theorem 3.12.** Let \( n \in \mathbb{N} \). Then, we have
\[ \mathcal{H}_n = -\sum_{j=0}^{n-1} \frac{D_j}{j!}. \]

It is time to give the first proof of Theorem 1.4. The following proof is related to generating functions and functional equation method.

**Proof of Theorem 1.4.** Putting \( z = \frac{1}{\lambda} (e^w - 1) \) in (1.3) and combining with (1.4), (1.23) and (1.21), we obtain
\[ F_A(w, \lambda) = \sum_{n=0}^{\infty} (n+1)! \lambda^{n+1} y(n, \lambda) F_{s2}(w, n+1). \]

By using the above equation, we get
\[ \sum_{m=0}^{\infty} \mathcal{B}_m(\lambda) \frac{w^m}{m!} = \sum_{n=0}^{\infty} (n+1)! \lambda^{n+1} y(n, \lambda) \sum_{m=0}^{\infty} S_2(m, n+1) \frac{w^m}{m!}. \]

such that we here use the fact that \( S_2(m, n) = 0 \) if \( n > m \). Equating the coefficients of \( \frac{w^m}{m!} \) on both sides of the above equation, we get the desired result.

Combining (1.6) with (1.1), we arrive at the following corollaries:

**Corollary 3.13.** Let \( m \in \mathbb{N}_0 \). Then, we have
\[ \mathcal{B}_m(\lambda) = \frac{1}{\lambda-1} \sum_{n=0}^{m} \sum_{j=0}^{n} \frac{(-1)^n (n+1)!}{j+1} \left( \frac{\lambda}{\lambda-1} \right)^{n-j} S_2(m, n+1). \]
Corollary 3.14. Let $m \in \mathbb{N}_0$. Then, we have
\[
\mathcal{B}_m(\lambda) = \frac{1}{1 - \lambda} \sum_{n=0}^{m} \sum_{j=0}^{n} \frac{n + 2}{j + 1} \left( \frac{\lambda}{\lambda - 1} \right)^{n-j} D_{n+1} S_2(m, n + 1).
\]

Corollary 3.15. Let $m \in \mathbb{N}_0$. Then, we have
\[
\mathcal{B}_m(\lambda) = \frac{1}{1 - \lambda} \sum_{n=0}^{m} \sum_{j=0}^{n} \frac{n + 2}{j + 1} \left( \frac{\lambda}{\lambda - 1} \right)^{n-j} B_k S_1(n + 1, k) S_2(m, n + 1).
\]

Substituting $z = e^t - 1$ into (2.3), we get
\[
\frac{t}{(e^t - 2) (e^t - 1)} = \sum_{n=0}^{\infty} \frac{1}{2n+2} y \left( n, \frac{1}{2} \right) \left( e^t - 1 \right)^n.
\]

Combining the above equation with (1.23) and (1.21), we get
\[
\frac{1}{2t} \sum_{m=0}^{\infty} \mathcal{B}_m \left( \frac{1}{2} \right) \frac{t^m}{m!} \sum_{m=0}^{\infty} \mathcal{B}_m \frac{t^m}{m!} = \sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{n!}{2n+2} y \left( n, \frac{1}{2} \right) S_2(m, n + 1) \frac{t^m}{m!}
\]
or
\[
\frac{1}{2} \sum_{m=0}^{\infty} \mathcal{B}_m \left( \frac{1}{2} \right) \frac{t^m}{m!} = 2 \sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{(n + 1)!}{2n+1} y \left( n, \frac{1}{2} \right) S_2(m, n + 1) \frac{t^m}{m!}.
\]

After making some necessary algebraic calculations in the previous equations, the coefficients of $\frac{t^m}{m!}$ are equalized, we arrive at the following theorems:

Theorem 3.16. Let $m \in \mathbb{N}_0$. Then, we have
\[
\mathcal{B}_m \left( \frac{1}{2} \right) = \sum_{n=0}^{m} \frac{(n+1)!}{2n+1} y \left( n, \frac{1}{2} \right) S_2(m, n + 1).
\]

Theorem 3.17. Let $m \in \mathbb{N}_0$. Then, we have
\[
\sum_{n=0}^{m} \binom{m}{n} \mathcal{B}_n \left( \frac{1}{2} \right) \mathcal{B}_{m-n} = m \sum_{n=0}^{m-1} \frac{(n+1)!}{2n+1} y \left( n, \frac{1}{2} \right) S_2(m - 1, n + 1).
\]

Substituting $u = e^t - 1$ into (1.16), we get
\[
\frac{1}{2} \sum_{m=0}^{\infty} \mathcal{B}_m \left( \frac{1}{2} \right) \frac{t^m}{m!} = \sum_{m=0}^{\infty} \sum_{n=0}^{m} \frac{n!}{m!} H_n S_2(m, n) t^m.
\]

Combining the above equation with (3.16), we arrive at the following theorem:

Theorem 3.18. Let $m \in \mathbb{N}_0$. Then, we have
\[
\sum_{n=0}^{m} n! H_n S_2(m, n) = \sum_{n=0}^{m} \frac{(n+1)!}{2n} y \left( n, \frac{1}{2} \right) S_2(m, n + 1).
\]
Combining (1.4), (1.31) and (1.33), we get the following functional equation:

\[ \frac{\lambda - 1}{\lambda} F_3 \left( \frac{1 - \lambda}{\lambda} u \right) F_d(u) = e^{-u} G(u, \lambda). \]

By using the above equation, we obtain

\[ \frac{\lambda - 1}{\lambda} \sum_{m=0}^{\infty} d_m \frac{u^m}{m!} \sum_{n=0}^{\infty} \left( \frac{1 - \lambda}{\lambda} \right)^n D_n \frac{u^n}{n!} = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{(-1)^{n-j}}{(n-j)!} (1-\lambda)^{j+2} y(j, \lambda) u^n. \]

Therefore

\[ -\sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} \frac{1 - \lambda}{\lambda}^{n+1} D_{n-m} \frac{u^n}{n!} = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \frac{(-1)^{n-j}}{(n-j)!} (1-\lambda)^{j+2} y(j, \lambda) u^n. \]

Comparing the coefficients of \( u^n \) on both sides of the above equation, we have following theorem:

**Theorem 3.19.** Let \( n \in \mathbb{N}_0 \). Then, we have

\[ -\sum_{m=0}^{n} \binom{n}{m} \frac{(1 - \lambda)^{n+1}}{\lambda} D_{n-m} d_m = \sum_{j=0}^{n} \frac{(-1)^{n-j}}{(n-j)!} (1-\lambda)^{j+2} y(j, \lambda). \]

Using (1.32) and (1.1) in the above equation, we have

\[ \sum_{j=0}^{n} \frac{(-1)^{n-j}}{(n-j)!} (1-\lambda)^{j+2} y(j, \lambda) = \sum_{m=0}^{n} \sum_{j=0}^{m} (-1)^{n-m+j+1} \binom{n}{m} \binom{m}{j} \frac{1 - \lambda}{\lambda}^{n-m+1} \frac{(n-m)! (m-j)!}{n-m+1}. \]

After some elementary calculations, we also arrive at the following corollary:

**Corollary 3.20.** Let \( n \in \mathbb{N}_0 \). Then, we have

\[ \sum_{j=0}^{n} \frac{(-1)^{n-j}}{(n-j)!} (1-\lambda)^{j+2} y(j, \lambda) = \sum_{m=0}^{n} \sum_{j=0}^{m} (-1)^{n-m+j+1} \frac{1 - \lambda}{\lambda}^{n-m+1} \frac{n!}{(n-m+1) j!}. \]

4. **Computation algorithm for the numbers \( y(n, \lambda) \)

In this section, with the aid of (1.10), (1.20), and the definition of the Bernoulli numbers, we present a computation algorithm (Algorithm 1 with a procedure called `COMPUTE_Y_NUMBER`) for the numbers \( y(m, \lambda) \).
Algorithm 1 Let $m$ be nonnegative integer and $\lambda \in \mathbb{C}$. This algorithm includes a procedure called $\text{COMPUTE}_y\_\text{NUMBER}$ which returns the numbers $y(m, \lambda)$.

**procedure** $\text{COMPUTE}_y\_\text{NUMBER}(m: \text{nonnegative integer, } \lambda)$

**Local variables:** $v, n, y$

$v, n, y \leftarrow 0$

for all $v$ in $\{0, 1, 2, \ldots, m\}$ do

for all $n$ in $\{0, 1, 2, \ldots, v\}$ do

\[ y \leftarrow y + \left( \left( \text{Power}(-1, v - m) \ast \text{Power}(\lambda - 1, v - m - 1) \ast \text{BERNOULLI\_NUM}(n) \right) \ast \text{STIRLING\_FIRST\_NUM}(v, n) \right) / \left( \text{Power}(\lambda, v + 1) \ast \text{Factorial}(v) \right) \]

end for

end for

return $y$

end procedure

**Remark 4.1.** In Algorithm 1, the procedure $\text{BERNOULLI\_NUM}(n)$ corresponds to the procedure which gives the $n$-th Bernoulli number. In addition, the procedure $\text{STIRLING\_FIRST\_NUM}$ is corresponding to the procedure which computes the Stirling numbers of the first kind using the formula given in (1.20). For details about the procedure $\text{STIRLING\_FIRST\_NUM}$, the interested readers may refer to the paper [24].

By using the computation algorithm (Algorithm 1), we give some values of the numbers $y(m, \lambda)$ as follows:

\[
y(0, \lambda) = \frac{1}{\lambda(\lambda - 1)}, \\
y(1, \lambda) = \frac{-3\lambda + 1}{2\lambda^2(\lambda - 1)^2}, \\
y(2, \lambda) = \frac{11\lambda^2 - 7\lambda + 2}{6\lambda^3(\lambda - 1)^3}, \\
y(3, \lambda) = \frac{-25\lambda^3 + 23\lambda^2 - 13\lambda + 3}{12\lambda^4(\lambda - 1)^4}, \\
y(4, \lambda) = \frac{137\lambda^4 - 163\lambda^3 + 137\lambda^2 - 63\lambda + 12}{60\lambda^5(\lambda - 1)^5},
\]

and so on. Observe that $y(n, \lambda)$ is a rational function of the variable $\lambda$. The sequence of the leading coefficients of the polynomial in the numerator of the numbers $y(n, \lambda)$ is given as follows:

\[1, -3, 11, -25, 137, -147, 1089, -2283, 7129, -7381, 83711, \ldots\]
and so on. Taking absolute value of the each term above sequence, we get the following well-known sequence:

\((a(n))^{\infty}_{n=1} = \{1, 3, 11, 25, 137, 147, 1089, 2283, 7129, 7381, 83711, \ldots\}\)

and so on. The sequence \(a(n)\) is given by OEIS: A025529 with the following explicit formula (cf. [56]):

\[a(n) = \text{lcm}(1, 2, 3, \ldots, n)H_n,\]

where \(n \in \mathbb{N}\), \(H_n\) denotes the harmonic numbers.

5. Differential equations of the generating functions and their applications

In this section, we give partial derivative equations of the generating functions. By applying these equations, we give many identities and many novel recurrence relations involving the numbers \(y(n, \lambda)\), \(I(n, 0)\), and the special finite sums of (inverse) binomial coefficients.

Differentiating equation (1.4) with respect to \(z\), we obtain the following partial derivative equation:

\[(5.1) \quad (z^2 - z) \frac{\partial}{\partial z} \{G(z, \lambda)\} + (2z - 1) G(z, \lambda) = \frac{1 - \lambda}{\lambda + (1 - \lambda)z}.\]

Differentiating equations (2.1), (2.2), and (2.3) with respect to \(z\), we obtain the following partial derivative equations, respectively:

\[(5.2) \quad (z^2 - z) \frac{d}{dz} \{g_1(z)\} + (2z - 1) g_1(z) = \frac{2}{2z - 1},\]

\[(5.3) \quad (z^2 - z) \frac{d}{dz} \{g_2(z)\} + (2z - 1) g_2(z) = \frac{1}{z - 2},\]

and

\[(5.4) \quad (z^2 - z) \frac{d}{dz} \{g_3(z)\} + (2z - 1) g_3(z) = \frac{1}{z + 1}.\]

5.1. Recurrence relations derived from PDEs for the generating functions. Here, using equations (5.1)-(5.4), we give recurrence relations and identities involving the numbers \(y(n, \lambda)\), \(I(n, 0)\), and the special finite sums of (inverse) binomial coefficients.

Theorem 5.1. The numbers \(y(n, \lambda)\) satisfy the following derivative equations:

\[(5.5) \quad (\lambda - 1) \frac{d}{d\lambda} \{y(n, \lambda)\} + (n + 2)y(n, \lambda) = (-1)^{n+1} \sum_{k=0}^{n} \frac{(\lambda - 1)^{k-n-1}}{\lambda^{k+2}}.\]

and

\[\frac{d}{d\lambda} \{y(n, \lambda)\} + \frac{n + 2}{\lambda - 1} y(n, \lambda) = \frac{(-1)^n}{\lambda} \left(1 - \left(\frac{\lambda}{\lambda - 1}\right)^{n+1}\right).\]
Proof. Differentiating equation (1.4) with respect to \(\lambda\), we obtain
\[
\sum_{n=0}^{\infty} (1-\lambda)^{n+1} \left( (1-\lambda)^{n+1} \frac{d}{d\lambda} \{y(n, \lambda)\} - (n + 2)y(n, \lambda) \right) z^n = \frac{\partial}{\partial \lambda} \left\{ \ln \left( \frac{1-\lambda}{z(z-1)} \right) \right\}.
\]
Therefore
\[
\lambda^2 \sum_{n=0}^{\infty} (1-\lambda)^{n+1} \left( (1-\lambda)^{n+1} \frac{d}{d\lambda} \{y(n, \lambda)\} - (n + 2)y(n, \lambda) \right) z^n = \frac{1}{(1-\lambda/z)(1-z)}.
\]
Assuming that \(|(1-\lambda/z)| < 1\) and \(|z| < 1\), then we obtain
\[
\lambda^2 \sum_{n=0}^{\infty} (1-\lambda)^{n+1} \left( (1-\lambda)^{n+1} \frac{d}{d\lambda} \{y(n, \lambda)\} - (n + 2)y(n, \lambda) \right) z^n = \sum_{n=0}^{\infty} \frac{\lambda^n}{\lambda^k} z^n.
\]
Comparing the coefficients of \(z^n\) on both sides of the above equation, we arrive at the desired result.

Combining (1.3) with (5.1), we get
\[
(z^2 - z) \sum_{n=1}^{\infty} n (1-\lambda)^{n+2} y(n, \lambda) z^{n-1} + (2z - 1) \sum_{n=0}^{\infty} (1-\lambda)^{n+2} y(n, \lambda) z^n
\]
\[
= \sum_{n=0}^{\infty} (-1)^n \left( \frac{1-\lambda}{\lambda} \right)^{n+1} z^n.
\]
After some calculations in the above equation, after that equating the coefficients of \(z^n\) on both sides of the final equation, we get
\[
y(n-1, \lambda) + (\lambda - 1)y(n, \lambda) = \frac{(-1)^n}{(n+1)\lambda^{n+1}}.
\]
Combining the above equation with (3.14), we arrive at a recurrence relation for the numbers \(y(n, \lambda)\) as in the following theorem:

**Theorem 5.2.** Let \(n \in \mathbb{N}\). Then we have
\[
y(n-1, \lambda) + (\lambda - 1)y(n, \lambda) = \frac{1}{\lambda^{n+1} n!} \sum_{j=0}^{n} B_j S_1(n, j).
\]

**Remark 5.3.** Substituting \(x = 0\) into (3.5) and using the following well-known identity
\[
L_n(0) = l(n, 0) = \frac{1}{n+1},
\]
we also arrive at the equation (5.6).

**Remark 5.4.** By using (1.4) and (1.3), we get
\[
\sum_{n=0}^{\infty} (1-\lambda)^{n+2} y(n, \lambda) z^{n+1} - \sum_{n=0}^{\infty} (1-\lambda)^{n+2} y(n, \lambda) z^n = \sum_{n=0}^{\infty} \left( \frac{1-\lambda}{\lambda} \right)^{n+1} \frac{z^n}{n+1}.
\]
Equating the coefficients of $z^n$ on both sides of the above equation, we also arrive at the equation (5.6).

Substituting $\lambda = 2$ into (5.6), we have

$$y (n - 1, 2) + y (n, 2) = \frac{(-1)^n}{(n + 1) 2^n}.$$  

(5.7)

Combining the above equation with the following well-known identity

$$\frac{(-1)^n}{2n} n! = \sum_{j=0}^{n} E_j S_1 (n, j)$$

(cf. [16]), we arrive at the following result:

**Corollary 5.5.** Let $n \in \mathbb{N}$. Then we have

$$y (n - 1, 2) + y (n, 2) = \frac{1}{2} \sum_{j=0}^{n} E_j S_1 (n, j).$$

Combining (5.4) with (2.3), and assuming that $|z| < 1$, we get

$$(z^2 - z) \sum_{n=1}^{\infty} \frac{1}{2n+2} y \left( n, \frac{1}{2} \right) z^{n-1} = \sum_{n=0}^{\infty} (-1)^n z^n + (1 - 2z)$$

$$\times \sum_{n=0}^{\infty} \frac{1}{2n+2} y \left( n, \frac{1}{2} \right) z^n.$$

After some elementary calculations in the above equation, we obtain

$$\sum_{n=1}^{\infty} \frac{1}{2n+2} y \left( n, \frac{1}{2} \right) z^{n+1} - \sum_{n=1}^{\infty} \frac{1}{2n+2} y \left( n, \frac{1}{2} \right) z^n$$

$$= \sum_{n=0}^{\infty} (-1)^n z^n + \sum_{n=0}^{\infty} \frac{1}{2n+2} y \left( n, \frac{1}{2} \right) z^n - 2 \sum_{n=0}^{\infty} \frac{1}{2n+2} y \left( n, \frac{1}{2} \right) z^{n+1}.$$

Now equating the coefficients of $z^n$ on both sides of the above equation, we arrive at the following corollary:

**Corollary 5.6.** Let $n \in \mathbb{N}$. Then we have

$$2y \left( n - 1, \frac{1}{2} \right) - y \left( n, \frac{1}{2} \right) = (-1)^n \frac{2^{n+2}}{n + 1}.$$  

(5.8)

**Remark 5.7.** Substituting $\lambda = \frac{1}{2}$ into (5.6), we also arrive at the equation (5.4) and (5.8).

Substituting $\lambda = 2$ into (5.6), we also arrive the (5.7).

Substituting $\lambda = -1$ into (5.6), we get the following corollary:

**Corollary 5.8.** Let $n \in \mathbb{N}$. Then we have

$$y (n - 1, -1) - 2y (n, -1) = -\frac{1}{n + 1}.$$  

(5.9)
Combining (5.9) with the following well-known formula

\[ y(n, -1) = \frac{1}{2(n + 1)} \sum_{j=0}^{n} \frac{1}{\binom{n}{j}} \]

(cf. [48, Eq. (6.5)]), we get the following combinatorial sum:

**Corollary 5.9.** Let \( n \in \mathbb{N} \). Then, we have

\[ \sum_{j=0}^{n-1} \frac{1}{\binom{n-1}{j}} = \frac{2n}{n + 1} \sum_{j=0}^{n-1} \frac{1}{\binom{n}{j}}. \] (5.10)

Multiplying both sides of the equation (5.9) by \( 2(n+1) \), we arrive at the following result:

**Corollary 5.10.** Let \( n \in \mathbb{N} \). Then we have

\[ 2(n+1)y(n-1, -1) - 4(n+1)y(n, -1) = -2. \]

6. **Decomposition of the multiple Hurwitz zeta functions with the help of the numbers** \( y(n, \lambda) \)

In [49], by the aid of the numbers \( y(n, \lambda) \), we gave decomposition of the multiple Hurwitz zeta functions in terms of the Bernoulli polynomials of higher order. In this section, by using the same method in [49], we give decomposition of the multiple alternating Hurwitz zeta functions in terms of the Bernoulli polynomials of higher order, the Euler numbers and polynomials of higher order, and the Stirling numbers of the first kind. By combining these decomposition relations, we derive some formulas involving these numbers and polynomials.

It is time to give the proof of Theorem 1.6 as follows:

**Proof of Theorem 1.6.** Substituting \( \lambda = -e^{-t} \) into (1.1), we get

\[ y \left( n, -\frac{1}{e^t} \right) = \sum_{j=0}^{n} \frac{e^{t(n+2)}}{(j+1)(e^t + 1)^{n+1-j}}. \] (6.1)

Combining (6.1) with (1.13), we get

\[ y \left( n, -\frac{1}{e^t} \right) = \sum_{m=0}^{\infty} \sum_{j=0}^{n} \frac{E_{m}^{(n+1-j)}(n+2)}{(j+1)2^{n+1-j}m!} \frac{t^m}{m!}. \] (6.2)

Using (6.1), we also get

\[ y \left( n, -\frac{1}{e^t} \right) = \sum_{j=0}^{n} \frac{1}{(j+1)} \sum_{v=0}^{\infty} \frac{(-1)^v \binom{n-j}{v}}{v!(v+n-j)} e^{t(v+n+2)}, \] (6.3)

where \( |e^t| < 1 \). Using Taylor series of \( e^{tx} \) in (6.3) yields

\[ y \left( n, -\frac{1}{e^t} \right) = \sum_{j=0}^{n} \frac{1}{j+1} \sum_{v=0}^{\infty} \sum_{m=0}^{\infty} (-1)^v \frac{\binom{n+j}{v}}{v!(v+n-j)} (v+n+2)^m \frac{t^m}{m!}. \] (6.4)
After making the required calculations in (6.2) and (6.4), equating the coefficients of $\frac{n}{m}$ on both sides of the above equation, we obtain

\[(6.5) \sum_{j=0}^{n} \frac{1}{j+1} \left( \sum_{v=0}^{\infty} (-1)^v \binom{v+n-j}{v} \right) (v+n+2)^m - \frac{E_{m}^{(n+1-j)}(n+2)}{2^{n+1-j}} = 0.\]

Therefore, proof is completed. \(\Box\)

**Remark 6.1.** The well-known multiple Hurwitz-Euler eta function (or the multiple alternating Hurwitz function), which is given by the equation (1.9), can also represent as follows:

\[\zeta_E^{(d)}(s, x) = 2^d \sum_{v_1, v_2, \ldots, v_d=0}^{\infty} \frac{(-1)^{v_1+v_2+\cdots+v_d}}{(x+v_1+v_2+\cdots+v_d)^s},\]

where $\text{Re}(s) > 0$, $d \in \mathbb{N}$ and $x > 0$ (cf. [6], [12], [29], [40], [53]).

Combining the equation (6.5) with the equation (1.9), we also arrive at the following theorem.

**Theorem 6.2.** Let $m, n \in \mathbb{N}_0$. Then we have

\[(6.6) \sum_{j=0}^{n} \frac{1}{(j+1)2^{n+1-j}} \zeta_E^{(n+1-j)}(-m, n+2) = \sum_{j=0}^{n} \sum_{l=0}^{m} \sum_{l_1, l_2, \ldots, l_{n+1-j}=0}^{l} \binom{m}{l} \frac{(n+2)^{m-l}}{(l_1!)l_2!\cdots l_{n+1-j}!} (j+1)^{2^{n+1-j}}.\]

Putting $n = 0$ in (1.8), we get

\[E_{m+1}(2) = \zeta_E(-m, 2),\]

where

\[\zeta_E(s, x) = \zeta_E^{(1)}(s, x) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+x)^s}\]

(cf. [6], [12], [29], [40], [53]).

Substituting $n = 1$ into (1.8), we obtain

\[\zeta_E^{(2)}(-m, 3) - \zeta_E(-m, 3) = E_m^{(2)}(3) - E_m(3).\]

Substituting $n = 2$ into (1.8), we also obtain

\[3 \zeta_E^{(3)}(-m, 4) + 3 \zeta_E^{(2)}(-m, 4) + 8 \zeta_E(-m, 3) = 3E_m^{(3)}(4) + 3E_m^{(2)}(4) + 8E_m(4).\]

Substituting $n = 3$ into (6.10), we also obtain

\[15 \zeta_E^{(5)}(-m, 5) + 15 \zeta_E^{(4)}(-m, 5) + 9 \zeta_E^{(3)}(-m, 5) + 10 \zeta_E^{(2)}(-m, 5) + 24 \zeta_E(-m, 5) = 15E_m^{(5)}(5) + 15E_m^{(4)}(5) + 10E_m^{(3)}(5) + 10E_m^{(2)}(5) + 24E_m(5).\]
Since
\[
\binom{v+n-j}{v} = \binom{v+n-j}{n-j},
\]
by combining the following well-known identity
\[
\binom{v+n-j}{n-j} = \frac{1}{(n-j)!} \sum_{c=0}^{n-j} |S_1(n-j, c+1)| v^c,
\]
(cf. [10], [13], [53]) with the equation (6.4), we get
\[
y \left( n, -\frac{1}{e^t} \right) = \sum_{j=0}^{n} \sum_{c=0}^{n-j} \sum_{v=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^v (v+n+2) v^c}{(j+1) (n-j)!} |S_1(n-j, c+1)| \frac{t^m}{m!}.
\]

In [49], we gave the following results involving the multiple Hurwitz zeta functions and the Bernoulli polynomials of higher order:
\[
y \left( n, \frac{1}{e^t} \right) = \sum_{j=0}^{n} \frac{(-1)^n}{(j+1)^{n+1-j}} \zeta_{n+1-j}(-m, n+2) \frac{t^m}{m!},
\]
where \( \zeta_d(s, x) \) denotes the Hurwitz zeta functions, for \( d \in \mathbb{N} \), which defined by
\[
\zeta_d(s, x) = \sum_{v=0}^{\infty} \binom{v+d-1}{v} \frac{1}{(x+v)^s} = \sum_{v_1=0}^{\infty} \sum_{v_2=0}^{\infty} \cdots \sum_{v_d=0}^{\infty} \frac{1}{(x+v_1+v_2+\cdots+v_d)^s},
\]
where \( \Re(s) > d \), when \( d = 1 \), we have the Hurwitz zeta function
\[
\zeta(s, x) = \zeta_1(s, x) = \sum_{v=0}^{\infty} \frac{1}{(x+v)^s},
\]
(cf. [10], [17], [36]-[39], [53]).

It is clear that
\[
\zeta_d(-m, x) = \frac{(-1)^d m! B_{m+d}(x)}{(d+m)!}
\]
and
\[
\zeta(-m, x) = \zeta_1(-m, x) = -\frac{B_{m+1}(x)}{m+1}
\]
where \( m \in \mathbb{N}_0 \) (cf. [10], [17], [36]-[39], [53]).

For \( m, n \in \mathbb{N} \), we [49] also defined
\[
\sum_{j=0}^{n} \frac{1}{j+1} \left( \binom{n+1}{j} \zeta_{n+1-j}(-m, n+2) + \frac{(-1)^j B_{m+n+1-j}^{(n+1-j)} (n+2)}{(m+n+1-j)(n+1-j)!} \right) = 0.
\]
Substituting $\lambda = -e^{-2t}$ into (1.1), we get

$$y \left( n, -\frac{1}{e^{2t}} \right) = \sum_{j=0}^{n} \frac{(-1)^{j-1} e^{2t(n+2)}}{(j+1)(e^{2t} - 1)^{n+1-j}}.$$  

(6.11)

Combining (6.11) with (1.12) and (1.13), we get

$$y \left( n, -\frac{1}{e^{2t}} \right) = \sum_{j=0}^{n} \frac{(-1)^{j-1}}{(j+1)2^{n+1-j}} \sum_{m=0}^{\infty} (-1)^{j} \frac{B_{m+n+1-j}^{(n+1-j)}}{(m+n+1-j)(n+1-j)!} \frac{t^{m}}{m!}.$$  

(6.12)

Combining (6.11) with (1.12), we also get

$$y \left( n, -\frac{1}{e^{2t}} \right) = \sum_{j=0}^{n} \sum_{m=0}^{\infty} \sum_{c=0}^{m} (-1)^{j-1} \frac{B_{c+n+1-j}^{(n+1-j)}}{(c+n+1-j)(n+1-j)!} \frac{E_{m-c}^{(n+1-j)}(2n+4)}{(j+1)2^{n+1-j}} \frac{t^{m}}{m!}.$$  

(6.13)

Combining (6.12) with (6.13), we arrive at the following theorem:

**Theorem 6.3.** Let $n \in \mathbb{N}_0$. Then, we have

$$\sum_{j=0}^{n} \frac{(-1)^{j-1}}{(n+1-j)!} \frac{B_{m+n+1-j}^{(n+1-j)}}{(m+n+1-j)} (n+2) = \sum_{j=0}^{n} \frac{(-1)^{j-1}}{(n+1-j)!} \frac{B_{c+n+1-j}^{(n+1-j)}}{(c+n+1-j)} \frac{E_{m-c}^{(n+1-j)}(2n+4)}{(j+1)2^{n+1-j}} \frac{t^{m}}{m!}.$$  

Remark 6.4. Many other decompositions are obtained by continuing as above. It is known that the decomposition of the multiple Hurwitz zeta function is given by different techniques and methods in the literature. In this paper, we do not focus on the other kinds of decompositions.

7. **INFINITE SERIES REPRESENTATIONS OF THE NUMBERS $y(n, \lambda)$ ON ENTIRE FUNCTIONS**

In this section, we give some formulas containing the numbers $y(n, \lambda)$ with the help of power series of entire functions. In order to give these formulas, we need the following infinite series representation, which was given by Boyadzhiev [4]-[5]:

$$\sum_{m=0}^{\infty} \frac{f^{(m)}(0)}{m!} h(m) g^{m} = \sum_{m=0}^{\infty} \frac{h^{(m)}(0)}{m!} \sum_{j=0}^{m} S_{2}(m, j) y^{j} f^{(j)}(y),$$

where $f$ and $h$ are appropriate functions.

For a large class of entire functions and $|\lambda| < 1$, Boyadzhiev [5] gave the following novel formula:

$$\sum_{m=0}^{\infty} h(m) \lambda^{m} + \sum_{m=1}^{\infty} \frac{h^{(m-1)}(0)}{m!} B_{m}(\lambda) = 0.$$  

(7.1)

Combining (1.6) with (7.1), we arrive at the following corollary:
Corollary 7.1. Let $h(\lambda)$ be an entire function and $|\lambda| < 1$. Then we have
\begin{align}
\sum_{v=0}^{\infty} h(v)\lambda^v &= - \sum_{m=1}^{\infty} \frac{h^{(m-1)}(0)}{m!} \sum_{n=0}^{m} (n+1)! \lambda^{n+1} y(n, \lambda) S_2(m, n+1). 
\end{align}

Substituting $h(\lambda) = \cos \lambda$ into (7.2), with the aid of the Euler formula, for $|\lambda e^{\pm i}| < 1$, we obtain
\begin{align}
\sum_{v=0}^{\infty} \lambda^v \cos(v) &= \frac{1}{2} \sum_{v=0}^{\infty} \left( (\lambda e^i)^v + (\lambda e^{-i})^v \right) = \frac{1 - \lambda \cos 1}{1 - 2\lambda \cos 1 + \lambda^2}.
\end{align}

Combining (7.3) with (7.2), we have
\begin{align}
\sum_{m=1}^{\infty} \sum_{n=0}^{2m-1} \frac{(-1)^{m+1} (n+1)! S_2(2m-1, n+1)}{2(2m-1)!} \lambda^{n+1} y(n, \lambda)
&= \frac{1 - \lambda \cos 1}{1 - 2\lambda \cos 1 + \lambda^2},
\end{align}
where $|\lambda| < 1$.

Combining (7.4) with (1.34), we arrive at the following theorem:

Theorem 7.2. 
\begin{align}
\sum_{m=1}^{\infty} \sum_{n=0}^{2m-1} \frac{(-1)^{m+1} (n+1)! S_2(2m-1, n+1)}{2(2m-1)!} \lambda^{n+1} y(n, \lambda)
&= 1 + \sum_{n=1}^{\infty} \left( G_n(2 \cos 1, -1; 1, 1, 1) - G_{n-1}(2 \cos 1, -1; 1, 1, 1) \cos 1 \right) \lambda^n.
\end{align}

Noting that a considerable attribute of this formula is depended on the function $f$. For instance, if $f$ is a polynomial, then infinite series on the right-hand side of the equation (7.2) reduces to the finite sum. On account of this, one can easily compute the value of the infinite series on the left-hand side of this equation.

The Hurwitz-Lerch zeta function $\Phi(\lambda, z, b)$ is defined by
\begin{align}
\Phi(\lambda, z, b) &= \sum_{j=0}^{\infty} \frac{\lambda^j}{(j+b)^z},
\end{align}
where $b \in \mathbb{C} \setminus \mathbb{Z}_0^-$; $\lambda, z \in \mathbb{C}$ when $|\lambda| < 1$; $x > 1$ when $|\lambda| = 1$ (cf. [1], [53]).

The function $\Phi(\lambda, z, b)$ interpolates the Apostol-Bernoulli polynomials at negative integers, that is
\begin{align}
\Phi(\lambda, 1-n, b) &= -\frac{1}{n} B_n(b; \lambda),
\end{align}
where $n \in \mathbb{N}$, $|\lambda| < 1$ (cf. [1], [4], [5], [53]).

Since
\begin{align}
B_0(0; \lambda) &= 0
\end{align}
and
\[ \lambda B_1(1; \lambda) = 1 + B_1(\lambda) \]
and for \( n \geq 2, \)
\[ \lambda B_n(1; \lambda) = B_n(\lambda), \]
(cf. [1]), for \( b = 1, \) the equation (7.5) reduces to the following well-known formula:
\[ \Phi(\lambda, 1 - n, 1) = -\frac{1}{n} B_n(1; \lambda) \]
\[ = -\frac{1}{n\lambda} B_n(\lambda), \]
where \( n \in \mathbb{N} \) with \( n \geq 2. \)

Substituting \( f(z) = z^m \) (\( m \in \mathbb{N} \)) into (7.2), and using (1.6), we arrive at the another proof of Theorem 1.4.

8. CONSTRUCTION OF THE GENERATING FUNCTION \( G(t, \lambda) \) WITH THE HELP OF VOLKENBORN INTEGRAL ON \( p \)-ADIC INTEGERS

In this section, we present another construction of the generating function \( G(t, \lambda) \) with the help of Volkenborn integral on \( p \)-adic integers. We give \( p \)-adic integral representation of the function \( G(t, \lambda) \). We also give some applications of this integral representation.

Let \( \mathbb{Z}_p \) denote the set of \( p \)-adic integers. Also, let \( C^1 \) denote the set of continuous differentiable functions from \( \mathbb{Z}_p \) to a field with a complete valuation. With the aid of the indefinite sum of a continuous function \( f \) on \( \mathbb{Z}_p \), the Volkenborn integral is given by
\[
\int_{\mathbb{Z}_p} f(x) \, d\mu_1(x) = \lim_{N \to \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x),
\]
where \( f \in C^1 \) on \( \mathbb{Z}_p \) and \( \mu_1(x) \) denotes the Haar distribution, which is given by
\[
\mu_1(x + p^N \mathbb{Z}_p) = \mu_1(x) = \frac{1}{p^N}
\]
(cf. [20], [21], [35], Definition 55.1, p.167], [47], [57]); see also the references cited in each of these earlier works).

**Theorem 8.1.** Let \( \lambda \in \mathbb{Z}_p. \) Then, we have
\[
G(t, \lambda) = \frac{1 - \lambda}{\lambda(t - 1)} \int_{\mathbb{Z}_p} \left( 1 + \frac{1 - \lambda}{\lambda} t \right)^x d\mu_1(x).
\]

**Proof.** Substituting \( f(x; t, \lambda) = (1 + \frac{1 - \lambda}{\lambda} t)^x \) into the following integral equation, which is given in [35, p.169]:
\[
\int_{\mathbb{Z}_p} f(x + 1; t, \lambda) \, d\mu_1(x) = \int_{\mathbb{Z}_p} f(x; t, \lambda) \, d\mu_1(x) + \frac{d}{dx} f(x)|_{x=0},
\]
and after some elementary computations, we obtain
\[ G(t, \lambda) = \frac{1 - \lambda}{\lambda(t - 1)} \sum_{n=0}^{\infty} (-1)^n \left( \frac{1 - \lambda}{\lambda} t \right)^n \frac{1}{n + 1}. \]

Combining the above equation with the Newton-Mercator series, we arrive at the desired result. □

It is time to give the second proof of Theorem 1.7. The following proof is associated with the p-adic integral method.

The second proof of Theorem 1.7. Using (8.2), we get
\[ G(t, \lambda) = \frac{1 - \lambda}{\lambda(t - 1)} \sum_{n=0}^{\infty} (-1)^n \left( \frac{\lambda - 1}{\lambda} t \right)^n \frac{1}{n!} \int_{\mathbb{Z}_p} (x)^\mu_1 (x). \]

Combining the above equation with (1.19), we obtain
\[ G(t, \lambda) = \frac{1 - \lambda}{\lambda(t - 1)} \sum_{n=0}^{\infty} (-1)^n \left( \frac{\lambda - 1}{\lambda} t \right)^n \frac{1}{n!} \sum_{j=0}^{n} S_1(n, j) \int_{\mathbb{Z}_p} x^j \mu_1 (x). \]

Combining (8.4) with the following well-known formula
\[ B_j = \int_{\mathbb{Z}_p} x^j \mu_1 (x) \]
(c.f. [35, p.171]), and (1.3), we have
\[ \sum_{n=0}^{\infty} (1 - \lambda)^{n+2} y(n, \lambda) t^n = \frac{1 - \lambda}{\lambda(t - 1)} \sum_{n=0}^{\infty} \sum_{j=0}^{n} S_1(n, j) B_j \left( \frac{1 - \lambda}{\lambda} \right)^n t^n. \]

Therefore
\[ \sum_{n=0}^{\infty} (1 - \lambda)^{n+2} y(n, \lambda) t^n = \sum_{n=0}^{\infty} \sum_{d=0}^{n} \sum_{j=0}^{d} (-1)^d S_1(d, j) B_j \left( \frac{\lambda - 1}{\lambda} \right)^{d+1} t^n. \]

Comparing the coefficients of \( t^n \) on both sides of the above equation, we arrive at the equation (1.10). □

By using (8.3), we obtain
\[ \sum_{n=0}^{\infty} (1 - \lambda)^{n+2} y(n, \lambda) t^{n+1} = \sum_{n=0}^{\infty} (1 - \lambda)^{n+2} y(n, \lambda) t^n \]
\[ = \sum_{n=0}^{\infty} (-1)^{n+1} \left( \frac{\lambda - 1}{\lambda} \right)^{n+1} \frac{1}{n!} \int_{\mathbb{Z}_p} (x)^\mu_1 (x) t^n. \]
Combining the above equation with the Volkenborn integral in terms of the Mahler coefficients
\[ \int_{\mathbb{Z}_p} \left( \frac{x}{n} \right) \mu_1(x) = \frac{(-1)^n}{n+1} \]
(cf. [35, Proposition 55.3, p.168]), we get
\[ \sum_{n=0}^{\infty} (1 - \lambda)^{n+2} y(n, \lambda) t^{n+1} - \sum_{n=0}^{\infty} (1 - \lambda)^{n+2} y(n, \lambda) t^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \frac{(1 - \lambda)^{n+1}}{\lambda} t^n. \]
Comparing the coefficients of \( t^n \) on both sides of the above equation, we arrive at the equation (5.6).
Combining the following well-known identity which proved by Kim et al. [22]:
\[ D_m = \int_{\mathbb{Z}_p} (x)^m \mu_1(x) \]
where \( m \in \mathbb{N}_0 \), with (8.5), we get the following result:

**Theorem 8.2.** Let \( n \in \mathbb{N} \). Then, we have
\[ y(n - 1, \lambda) + (\lambda - 1)y(n, \lambda) = \frac{D_n}{\lambda^{n+1} n!}. \]

**9. Conclusion**

In this paper, we have given the solution of the Open problem 1, which has been proposed by the author [48, p.57, Open problem 1] about the generating functions of the numbers \( y(n, \lambda) \). We have also given many properties of this function. With the help of generating functions and their functional equations, we have derived many formulas associated with the numbers \( y(n, \lambda) \), the Bernoulli numbers of the second kind, the harmonic numbers, alternating Harmonic numbers, the Apostol-Bernoulli numbers, the Stirling numbers, the Leibnitz numbers, the Bernoulli numbers, and sums involving higher powers of inverses of binomial coefficients. Furthermore, we have provided an algorithm to compute the numbers \( y(n, \lambda) \). By using this algorithm, we have also computed some values of the numbers \( y(n, \lambda) \). In addition, we have presented differential equations of the generating functions with their applications. With the aid of the numbers \( y(n, \lambda) \), we have given decomposition of the multiple Hurwitz zeta functions involving the Bernoulli polynomials of higher order. We have also given infinite series representations of the numbers \( y(n, \lambda) \) on entire functions. By the aid of the Volkenborn integral on \( p \)-adic integers, we have constructed the generating function for the numbers \( y(n, \lambda) \) and given their some applications.

As a result, the results produced in this article are in a wide range and have the potential to attract the attention of many researchers. In the future, the examination of the properties of the numbers \( y(n, \lambda) \) will continue and it will be investigated which other numbers and polynomials these numbers are related to.
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