Connections on a non-symmetric (generalized) Riemannian manifold and gravity

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Abstract

Connections with (skew-symmetric) torsion on a non-symmetric Riemannian manifold satisfying the Einstein metricity condition (non-symmetric gravitational theory (NGT) with torsion) are considered. It is shown that an almost Hermitian manifold is NGT with torsion if and only if it is a nearly Kähler manifold. In the case of an almost contact metric manifold the NGT with torsion spaces are characterized and a possibly new class of almost contact metric manifolds is extracted. Similar considerations lead to a definition of a particular class of almost para-Hermitian and almost paracontact metric manifolds. Conditions are given in terms of the corresponding Nijenhuis tensors and the exterior derivative of the skew-symmetric part of the non-symmetric Riemannian metric.

Keywords: non-symmetric connection, Einstein metricity condition, Nijenhuis tensor

1. Introduction

In this paper we consider connections on a non-symmetric Riemannian manifold $(M, G = g + F)$ adopting the general assumption from the non-symmetric gravitational theory that the symmetric part $g$ of $G$ is non-degenerate. There are many examples of
generalized Riemannian manifolds, some of which deserve to be mentioned here: almost Hermitian, almost contact, almost para-Hermitian, almost paracontact, etc.

We show that the torsion and the covariant derivative of the non-symmetric metric determine a linear connection. The connection is uniquely and completely determined by the torsion and the Levi-Civita connection of the symmetric part of the non-symmetric metric. The Levi-Civita covariant derivative of the skew-symmetric part of the non-symmetric metric and the torsion supply a necessary condition for the existence of the connection.

1.1. Motivation from general relativity

General relativity (GR) was developed by Albert Einstein in 1916 [9] and contributed to by many others after 1916. In GR the equation $\text{d}s^2 = g_{ij}\text{d}x^i\text{d}x^j$, $(g_{ij} = g_{ji})$ is valid, where $g_{ij}$ are functions of a point. In GR, which is a four dimensional space-time continuum, metric properties depend on mass distribution. The magnitudes $g_{ij}$ are known as gravitational potential. Christoffel symbols, commonly expressed by $\Gamma^k_{ij}$, play the role of magnitudes which determine the gravitational force field. General relativity explains gravity as a curvature of space-time.

In GR the metric tensor is related by the Einstein equations $R_{ij} - \frac{1}{2}Rg_{ij} = T_{ij}$, where $R_{ij}$ is the Ricci tensor of the metric of space-time, $R$ is the scalar curvature of the metric, and $T_{ij}$ is the energy–momentum tensor of matter. In 1922, Friedmann [15] found a solution in which the universe may expand or contract, and later Lemaître [32] derived a solution for an expanding universe. However, Einstein believed that the universe was apparently static, and since static cosmology was not supported by the general relativistic field equations, he added the cosmological constant $\Lambda$ to the field equations, which became $R_{ij} - \frac{1}{2}Rg_{ij} + \Lambda g_{ij} = T_{ij}$.

From 1923 to the end of his life Einstein worked on various variants of unified field theory (non-symmetric gravitational theory (NGT)) [10]. This theory had the aim of unifying gravitation theory, which GR is related to, and the theory of electromagnetism. Introducing different variants of his NGT, Einstein used a complex basic tensor, with a symmetric real part and a skew-symmetric imaginary part. Starting from 1950, Einstein used the real non-symmetric basic tensor $G$, sometimes called the generalized Riemannian metric manifold.

Notice that in NGT the symmetric part $g_{ij}$ of the basic tensor $G_{ij} (G_{ij} = g_{ij} + F_{ij})$ is related to gravitation, and the skew-symmetric one $F_{ij}$ to electromagnetism. The same is valid for the symmetric part of the connection and the torsion tensor, respectively.

More recently the idea of a non-symmetric metric tensor appeared in Moffat’s non-symmetric gravitational theory [35]. In Moffat’s theory the skew-symmetric part of the metric tensor represents a Proca field (massive Maxwell field) which is a part of the gravitational interaction, contributing to the rotation of galaxies.

While on a Riemannian space the connection coefficients are expressed by virtue of the metric, $g_{ij}$, in Einstein’s work on NGT the connection between these magnitudes is determined by the so-called Einstein metricity condition, i.e. the non-symmetric metric tensor $G$ and the connection components $\Gamma^k_{ij}$ are connected with the equations

$$\frac{\partial G_{ij}}{\partial x^m} = \Gamma^p_{im}G_{pj} - \Gamma^p_{jm}G_{ip} = 0. \quad (1.1)$$

A generalized Riemannian manifold satisfying the Einstein metricity condition (1.1) is also called an NGT space [10, 34, 35].

The choice of a connection in NGT is not uniquely determined. In particular, in NGT there exist two kinds of the covariant derivative. For example, for tensor $a^i_j$:
We investigate connections on a generalized Riemannian manifold \((M, G = g + F)\) adopting the general assumption from NGT that the symmetric part \(g\) of \(G\) is non-degenerate. There are many examples of generalized Riemannian manifolds such as almost Hermitian, almost contact, almost para-Hermitian, almost paracontact etc. In these cases, the skew-symmetric part of \(G\) is played by the fundamental 2-form \(F\).

We show that the torsion and the covariant derivative of \(G\) determine a unique linear connection \(\nabla\) with given torsion \(T\) and covariant derivative \(\nabla G\) providing the covariant derivative of the skew-symmetric part \(F\) of \(G\), \(\nabla F\), satisfies a certain compatibility condition which we expressed in terms of the torsion and the covariant derivative of the symmetric part \(g\) of \(G\), \(\nabla g\) (theorem 2.2). The connection is unique and completely determined by the symmetric part \(g\) of \(G\).

We look for linear connections with torsion preserving the non-symmetric tensor \(G\). Surprisingly, we find that a linear connection \(\nabla\) preserves the non-symmetric metric, \(\nabla G = 0\), if and only if it preserves its symmetric and skew-symmetric parts, \(\nabla g = \nabla F = 0\).

We paid special attention to the case when the torsion of the connection preserving the Nijenhuis tensor and the exterior derivative of the skew-symmetric part \(F\) of \(G\) expressed in terms of the torsion. We show that there exists a unique connection preserving \(G\) provided its torsion is given and the above-mentioned PDE is satisfied (cf theorem 2.3).

We restricted our considerations to NGT, i.e. a generalized Riemannian manifold satisfying the Einstein metricity condition (1.1). We present a condition in terms of the Nijenhuis tensor and the exterior derivative of the skew-symmetric part \(F\) of \(G\) and show that this condition is necessary and sufficient for the existence of the connection with skew-symmetric torsion preserving the generalized Riemannian metric \(G\) (cf theorem 2.5). We note that such a condition was previously known for almost Hermitian, almost contact, almost para-Hermitian, almost paracontact manifolds [14, 17, 43]. We derive the results as a consequence of our considerations.

In section 3 we restrict our considerations to NGT, i.e. a generalized Riemannian manifold satisfying the Einstein metricity condition (1.1). We present a condition in terms of the Nijenhuis tensor and the exterior derivative of \(F\) which guarantees the existence of a unique linear connection with skew-symmetric torsion preserving the NGT structure and show that the torsion is equal to minus one third of the exterior derivative \(dF\) of \(F\) (theorem 3.1).

One of the main contributions is the application of this result to almost Hermitian and almost contact metric manifolds.

A careful analysis of our general condition for the existence of a connection with skew-symmetric torsion satisfying the Einstein metricity condition in the case of an almost Hermitian manifold allows us to conclude that an almost Hermitian manifold satisfies the Einstein metricity condition with respect to a connection with skew-symmetric torsion if and only if it is a nearly Kähler manifold (theorem 3.3). In other words, an almost Hermitian manifold is NGT with skew-symmetric torsion exactly when it is a nearly Kähler manifold. In this case the connection coincides with the Gray connection [22–24], which is the unique connection with skew-symmetric torsion preserving the nearly Kähler structure [14]. Nearly Kähler manifolds (called almost Tachibana spaces in [42]) were developed by Gray [22–24] and have been intensively studied since then in [8, 12, 30, 36–38]. Nearly Kähler manifolds also appear
in supersymmetric string theories (see e.g. [25, 31, 39, 40]. The first complete inhomogeneous examples of six-dimensional nearly Kähler manifolds were presented recently in [13].

Applying the general condition for the existence of a connection with skew-symmetric torsion satisfying the Einstein metricity condition to an almost contact metric manifold, we arrived at a possibly new class of almost contact metric manifolds, which we call here almost-nearly cosymplectic. We show that this is the precise class of almost contact metric manifolds which are NGT with skew-symmetric torsion, i.e. admitting a connection with skew-symmetric torsion satisfying the Einstein metricity condition. The simplest example of an almost-nearly cosymplectic manifold is a trivial circle bundle over a (compact) nearly Kähler manifold.

The cases of almost para-Hermitian and almost paracontact metric manifolds are also treated with respect to the NGT space with skew-symmetric torsion. We show that an almost para-Hermitian manifold admits an NGT connection with totally skew-symmetric torsion if and only if it admits a linear connection preserving the almost para-Hermitian structure with totally skew-symmetric torsion. We extract a possibly new class of almost paracontact metric structures which are NGT with skew-symmetric torsion. We characterize this class by an explicit formula for the covariant derivative with respect to the Levi-Civita connection of the fundamental 2-form.

2. The geometric model

The fundamental (0,2) tensor $G$ in a non-symmetric (generalized) Riemannian manifold $(M, G)$ is generally non-symmetric. It is decomposed into two parts, the symmetric part $g$ and the skew-symmetric part $F$, $G(X, Y) = g(X, Y) + F(X, Y)$, where

$$g(X, Y) = \frac{1}{2}(G(X, Y) + G(Y, X)), \quad F(X, Y) = \frac{1}{2}(G(X, Y) - G(Y, X)). \quad (2.1)$$

We assume that the symmetric part is non-degenerate and of arbitrary signature. Therefore, we obtain a well-defined $(1,1)$ tensor $A$ determined by the condition

$$F(X, Y) = g(AX, Y). \quad (2.2)$$

We look for a natural linear connection $\nabla$ preserving the generalized Riemannian metric $G$, $\nabla G = 0$ with torsion $T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$.

**Convection 2.1.** In the whole paper we shall use the capital Latin letters $X, Y, Z, \ldots$ to denote smooth vector fields on a smooth manifold $M$ which commute, $[X, Y] = 0$. Hence, $T(X, Y) = \nabla_X Y - \nabla_Y X$. In other words, if we have a local coordinate frame $\left\{ \frac{\partial}{\partial x^i}, s = 1, \ldots, \dim(M) \right\}$ we assume that the vector fields $X = \frac{\partial}{\partial x^i}$, $Y = \frac{\partial}{\partial x^j}$, $Z = \frac{\partial}{\partial x^k}, \ldots$. We also express some formulae in local coordinates. In particular, (2.2) can be written as $F_{ij} = A^j_i g_{ij}$.

We denote with $\nabla^*_g$ the Levi-Civita connection corresponding to the symmetric non-degenerate $(0,2)$ tensor $g$. The Koszul formula reads

$$g(\nabla^*_g X, Z) = \frac{1}{2}[Xg(Y, Z) + Yg(X, Z) - Zg(Y, X)]. \quad (2.3)$$

We denote the $(0, 3)$ torsion tensor with respect to $g$ with the same letter, $T(X, Y, Z) := g(T(X, Y), Z)$. 

2.1. Linear connections on generalized Riemannian manifolds

In this section we show that a linear connection $\nabla$ with a torsion tensor $T$ on a generalized Riemannian manifold is completely determined by the torsion and the covariant derivative $\nabla g$ of the symmetric part $g$ of $G$. More precisely, we have

**Theorem 2.2.** Let $(M, G = g + F)$ be a generalized Riemannian manifold and $\nabla^g$ be the Levi-Civita connection of $g$. Let $\nabla$ be a linear connection with torsion $T$ and denote the covariant derivative of the symmetric part $g$ of $G$ with $\nabla g$. Then $\nabla$ is uniquely determined by the following formula

$$g(\nabla_X Y, Z) = g(\nabla^g_X Y, Z) + \frac{1}{2} [T(X, Y, Z) + T(Z, X, Y) - T(Y, Z, X)]$$
$$- \frac{1}{2} [(\nabla_X g)(Y, Z) + (\nabla_Y g)(Z, X) - (\nabla_Z g)(Y, X)].$$

(2.4)

The covariant derivative $\nabla F$ of the skew-symmetric part $F$ of $G$ is given by

$$(\nabla_X F)(Y, Z) = (\nabla^g_X F)(Y, Z) + \frac{1}{2} [T(X, Y, AZ) + T(Z, X, AY)]$$
$$+ \frac{1}{2} [T(AZ, X, Y) + T(AZ, Y, X) + T(X, AY, Z) + T(Z, AY, X)]$$
$$+ \frac{1}{2} [(\nabla_X g)(AY, Z) - (\nabla_X g)(Y, AZ) - (\nabla_Y g)(AZ, X)]$$
$$+ \frac{1}{2} [(\nabla_X g)(AY, X) + (\nabla_Z g)(Y, X) - (\nabla_Z g)(Z, X)].$$

(2.5)

In particular, the exterior derivative $dF$ of $F$ satisfies

$$dF(X, Y, Z) = -T(X, Y, AZ) - T(Y, Z, AX) - T(Z, X, AY)$$
$$+ (\nabla_X F)(Y, Z) + (\nabla_Y F)(Z, X) + (\nabla_Z F)(X, Y).$$

(2.6)

Conversely, any three tensors $T, \nabla g, \nabla F$ satisfying (2.5) determine a unique linear connection via (2.4).

**Proof.** It follows from (2.2) that the (1,1) tensor $A$ is skew-symmetric with respect to the pseudo-Riemannian metric $g$, $g(AX, Y) = -g(X, AY)$. A simple calculation using (2.2) yields

$$(\nabla_X F)(Y, Z) = Xg(AY, Z) - g(A\nabla_X Y, Z) - g(AY, \nabla_X Z)$$
$$= (\nabla_X g)(AY, Z) + g(\nabla_X A) Y, Z).$$

(2.7)

From the definition of the covariant derivative of a $(0, 2)$ tensor, we have

$$(\nabla_X G)(Y, Z) = XG(Y, Z) - G(\nabla_X Y, Z) - G(Y, \nabla_X Z);$$
$$(\nabla_Y G)(Z, X) = YG(Z, X) - G(\nabla_Y Z, X) - G(Z, \nabla_Y X);$$
$$(\nabla_Z G)(Y, X) = ZG(Y, X) - G(\nabla_Z Y, X) - G(Y, \nabla_Z X).$$

(2.8)

Sum the first two equalities, subtract the third equality, and then use (2.1) and the definition of the torsion to get
Using the Koszul formula (2.3), we obtain from (2.9) that

\[
2 g (\nabla_X Y, Z) = 2 g (\nabla_X^g Y, Z) + dF(X, Y, Z) + G(Z, T(X, Y))
- G(Y, T(X, Z)) - G(T(Y, Z), X)
- (\nabla_Y G)(Y, Z) - (\nabla_Y G)(Z, X) + (\nabla_Z G)(Y, X).
\] (2.10)

The skew-symmetric part with respect to \( X, Y \) of (2.10) gives precisely the equation (2.6). Substitute (2.6) into (2.10) and use (2.1) to get (2.4). Insert the already proved (2.4) into

\[
(\nabla_X F)(Y, Z) = X F(Y, Z) - F(\nabla_X Y, Z) - F(Y, \nabla_X Z).
\]

Use (2.2) and (2.7) to obtain (2.5).

To complete the proof, observe that (2.5) implies (2.6). Indeed, taking the cyclic sum of (2.5) we get (2.6).

The converse follows from (2.4) by straightforward computation.

\[\square\]

2.2. Metric connections on a generalized Riemannian manifold

Here we investigate the existence of metric connections on a generalized Riemannian manifold. We have

**Theorem 2.3.** Let \((M, G = g + F)\) be a generalized Riemannian manifold and \(\nabla^g\) be the Levi-Civita connection of \(g\).

1. A linear connection \(\nabla\) preserves the generalized Riemannian metric \(G\) if and only if it preserves its symmetric part \(g\) and its skew-symmetric part \(F\), \(\nabla G = 0 \iff \nabla g = \nabla F = 0 \iff \nabla g = \nabla A = 0\).

2. If there exists a linear connection \(\nabla\) preserving the generalized Riemannian metric \(G\), \(\nabla G = 0\) with torsion \(T\) then the following condition holds

\[
(\nabla_X F)(Y, Z) = -\frac{1}{2} [T(X, Y, AZ) + T(Z, X, AY)]
- \frac{1}{2} [T(AZ, X, Y) + T(AZ, Y, X) + T(X, AY, Z) + T(Z, AY, X)].
\] (2.11)

In particular, the exterior derivative of \(F\) satisfies the following equality

\[
dF(X, Y, Z) = F(T(X, Y), Z) + F(T(Y, Z), X) + F(T(Z, X), Y), \quad \text{equivalently}
\]

\[
dF(X, Y, Z) = -T(X, Y, AZ) - T(Y, Z, AX) - T(Z, X, AY).
\] (2.12)

Conversely, if the condition (2.11) is valid then there exists a unique linear connection \(\nabla\) with torsion \(T\) preserving the generalized Riemannian metric \(G\) determined by the torsion \(T\) with the formula
Proof. Suppose we have \( 0 = \nabla G = \nabla g + \nabla F \). The cyclic sum of this equality yields
\[
(\nabla_X g)(Y, Z) + (\nabla_Y g)(Z, X) + (\nabla_Z g)(X, Y) = - (\nabla_X F)(Y, Z) - (\nabla_Y F)(Z, X) - (\nabla_Z F)(X, Y). 
\]
Observe that the left hand side is totally symmetric while the right hand side is totally skew-symmetric, which leads to the vanishing of each side
\[
(\nabla_X g)(Y, Z) + (\nabla_Y g)(Z, X) + (\nabla_Z g)(X, Y) = 0; 
(\nabla_X F)(Y, Z) + (\nabla_Y F)(Z, X) + (\nabla_Z F)(X, Y) = 0. 
\] (2.14)
On the other hand, the symmetric part of \( \nabla G \) with respect to the first two arguments gives
\[
(\nabla_X g)(Y, Z) + (\nabla_Y g)(X, Z) + (\nabla_X F)(Y, Z) + (\nabla_Y F)(X, Z) = 0
\]
which combined with the first identity in (2.14) yields
\[
(\nabla_X g)(Y, Z) + (\nabla_Y g)(X, Z) = \frac{1}{2} \left[ T(X, Y, Z) + T(Z, X, Y) - T(Y, Z, X) \right]. 
\] (2.13)

Remark 2.4. In local coordinates, (2.11) reads
\[
\nabla^i F_{jk} = -\frac{1}{2} \left[ T_{ij} A_k^i + T_{ik} A_j^i + T_{jk} A_i^i + T_{ij} A_k^i + T_{ik} A_j^i + T_{jk} A_i^i \right] 
\]
while (2.12) takes the form \( dF_{ijk} = -T_{ij} A_k^i - T_{ik} A_j^i - T_{jk} A_i^i \).

2.3. The torsion tensor, skew-symmetric torsion
It follows from theorem 2.3 that any linear connection preserving the generalized Riemannian metric is completely determined by the torsion tensor \( T \). In this section we investigate the torsion tensor. To this end we recall the definition of the Nijenhuis tensor \( N \) of the (1,1) tensor \( A \) (see e.g. [33]),
\[
N(X, Y) = [AX, AY] + A^2 [X, Y] - A[A X, Y] - A[X, A Y]. 
\] (2.16)
The Nijenhuis tensor is skew-symmetric by definition and it plays a fundamental role in almost complex (resp. almost para-complex) geometry. If \( A^2 = -1 \) (resp. \( A^2 = 1 \)) then the celebrated Newlander–Nirenberg theorem (see e.g. [33]) shows that an almost complex structure is integrable if and only if the Nijenhuis tensor vanishes.

Let \( \nabla \) be a linear connection preserving the generalized Riemannian metric \( G, \nabla G = 0 \). The \( \nabla \) preserves \( g, F \) and \( A, \nabla g = \nabla F = \nabla A = 0 \). Using the definition of the torsion and
the covariant derivative $\nabla A$ we can express the Nijenhuis tensor in terms of torsion $T$ and $\nabla A$ as follows

$$N(X, Y) = (\nabla_X Y) - (\nabla_Y X) - [X, Y] + A(\nabla_X Y) + A(\nabla_Y X) - T(AX, AY) - TA(AX, Y) + AT(X, AY).$$

We denote the Nijenhuis tensor of type $(0,3)$ with respect to $g$ with the same letter, $N_{XYZ} g = N_{XZA} Y F_{XY} A Z$. Set $\nabla A = 0$ into (2.17) and use (2.2) to get

$$N(X, Y, Z) = T(AX, AY, Z) - T(X, Y, A^2 Z) - T(AX, Y, AZ) - T(X, AY, AZ).$$

(2.18)

### 2.3.1. The skew-symmetric torsion

Linear connections preserving a (pseudo) Riemannian metric and having totally skew-symmetric torsion, i.e. the torsion satisfies the condition $T(X, Y, Z) = -T(X, Z, Y)$, in local coordinates, $T_{ijk} = -T_{ikj}$, (2.19) have become very attractive in the last 20 years mainly due to the relations with supersymmetric string theories (see e.g. [5, 6, 19, 21, 41] and references therein, for a mathematical treatment consult the nice overview [1]). The main point is that the number of preserved supersymmetries is equal to the number of parallel spinors with respect to such a connection. This property reduces the holonomy group of the connection to a subgroup of a group which is a stabilizer of a non-trivial spinor such as $SU(n)$, $Sp(n)$, $G_2$, $Spin(7)$. In these cases, such a connection is uniquely determined entirely by the structure induced by the parallel spinor [14, 17, 28, 41].

We investigate the case when a generalized Riemannian manifold admits a metric connection with skew-symmetric torsion. We have

**Theorem 2.5.** Let $(M, G = g + F)$ be a generalized Riemannian manifold and $\nabla^G$ be the Levi-Civita connection of $g$. If there exists a linear connection $\nabla$ preserving the generalized Riemannian metric $G$, $\nabla G = 0$ with totally skew-symmetric torsion $T$ then the following condition holds

$$N(X, Y, AZ) + N(X, Z, AY) = dF(X, Y, A^2 Z) + dF(X, Z, A^2 Y).$$

(2.20)

The torsion tensor satisfies the equality

$$T(AX, AY, Z) = -N(X, Y, Z) + dF(X, Y, AZ);$$

$$T(AX, Y, Z) = 2(\nabla_X Y) F(Y, Z) - dF(X, Y, Z).$$

(2.21)

The torsion connection $\nabla$ is determined by the formula

$$g(\nabla_X Y, Z) = g(\nabla_X Y, Z) + \frac{1}{2} T(X, Y, Z).$$

(2.22)

If, in addition, the skew-symmetric part $F$ of the generalized Riemannian metric $G$ is closed, $dF = 0$, then (2.20) and (2.21) hold by inserting $dF = 0$.

**Proof.** The equalities (2.21) follow by straightforward calculation from (2.19), (2.18), (2.12) and (2.11). Now, the equality (2.20) is an easy consequence of the first equality in (2.21). The formula (2.22) follows from (2.13) and (2.19).
In local coordinates (2.21) reads $T^{\alpha}_{\beta\gamma} A^\beta_{\gamma} = -N_{\beta\gamma} + dF_{\beta\gamma} A^\beta_{\gamma}$, $T^{\alpha}_{\beta\gamma} A^\beta_{\gamma} = 2\nabla^\beta F_{\beta\gamma} - dF_{\beta\gamma}$.

2.4. Eisenhart condition

Eisenhart [11] was one of the first who proposed that a connection with skew-symmetric torsion should be applied in general relativity. In the sense of Eisenhart’s definition (see [11]), a generalized Riemannian manifold is a differentiable manifold with a non-symmetric basic tensor $G(X, Y) = g(X, Y) + F(X, Y)$ with a connection explicitly defined by the equation

$$g(\nabla_X Y, Z) = \frac{1}{2}[XG(Y, Z) + YG(Z, X) - ZG(Y, X)].$$

(2.23)

It is easy to see that (2.23) can be written in the form

$$g(\nabla_X Y, Z) = \frac{1}{2}[XG(Y, Z) + YG(Z, X) - ZG(Y, X)] = g(\nabla_X^S Y, Z) + \frac{1}{2} dF(X, Y, Z),$$

which shows that the symmetric part $g$ of $G$ is covariantly constant, $\nabla g = 0$ and the torsion $T$ is totally skew-symmetric and determined by the equation $T(X, Y, Z) = dF(X, Y, Z)$. Using relations (2.7) and (2.17), we calculate that the Nijenhuis tensor satisfies

$$N(X, Y, Z) = (\nabla_X^S F)(Y, Z) - (\nabla_Y^S F)(X, Z) + (\nabla_Y^F)(Y, AZ) - (\nabla_X^F)(X, AZ).$$

(2.24)

If, in addition, the Eisenhart connection preserves the generalized Riemannian metric $G$, $\nabla g = \nabla F = 0$, then equation (2.24) reduces to (2.18) with $T = dF$.

2.5. Skew-symmetric torsion. Examples

A generalized Riemannian metric $G$ is equivalent to the choice of a pseudo-Riemannian metric $g$ and the 2-form $F$ (an (1, 1) tensor $A$ satisfying (2.2)), such that $G = g + F$. A generalized metric connection, i.e. a linear connection preserving $G$, is a Riemannian connection preserving the 2-form $F$ or, equivalently, a Riemannian connection preserving the (1, 1)-tensor $A$. This supplies a number of examples.

2.5.1. Almost Hermitian manifolds, $A^2 = -1$. Let us consider an almost Hermitian manifold $(M, g, A)$, i.e. a Riemannian manifold $(M, g)$ of dimension $n (= 2m \geq 4)$ endowed with an almost complex structure, i.e. endomorphism $A$, which satisfies $A^2 = -I$, $F(X, Y) = g(AZ, Y)$, $g(AZ, AY) = g(X, Y)$. The 2-form $F$ is the Kähler form (note the sign difference in the Kähler form in [14]).

In this case the Nijenhuis tensor (2.16) has the properties

$$N(AZ, Y, Z) = N(AZ, AY, Z) = N(AZ, AZ).$$

(2.25)

Then theorem 2.5 gives the following well-known result

**Corollary 2.6.** [14, 16, 41]. On an almost Hermitian manifold $(M, g, A)$ there exists a unique linear connection $\nabla$ preserving the generalized Riemannian metric $G = g + F$ with totally skew-symmetric torsion $T$ if and only if the Nijenhuis tensor is totally skew-symmetric, $N(X, Y, Z) = -N(X, Z, Y)$. The torsion $T$ is determined by $T(X, Y, Z) = N(X, Y, Z) + dF(AZ, AY, AZ)$. In particular, an almost Kähler manifold, $dF = 0$, admits such a connection if and only if it is a Kähler manifold, $N = 0$. 
2.5.2. Almost para-Hermitian manifolds, $A^2 = 1$. An almost para-Hermitian manifold $(M, g, A)$ is a Riemannian manifold $(M, g)$ endowed with endomorphism $A$ satisfying $A^2 = I$. $F(X, Y) = g(AX, Y)$, $g(AX, AY) = -g(X, Y)$. Such a manifold is of even dimension $2n$, the eigen-subbundles of the para-complex structure $A$ are of equal dimension $n$, and the metric $g$ is of neutral signature $(n, n)$. In this case the Nijenhuis tensor (2.16) has the properties (2.25) and theorem 2.5 gives the following well-known result (note the sign difference in the 2-form $F$ in [29]).

**Corollary 2.7.** [29] On an almost para-Hermitian manifold $(M, g, A)$ there exists a unique linear connection $\nabla$ preserving the generalized Riemannian metric $G = g + F$ with totally skew-symmetric torsion $T$ if and only if the Nijenhuis tensor is totally skew-symmetric, $N(X, Y, Z) = -N(X, Z, Y)$. The torsion $T$ is determined by $T(X, Y, Z) = -N(X, Y, Z) + dF(AX, AY, AZ)$. In particular, an almost para-Kähler manifold, $dF = 0$, admits such a connection if and only if it is a para-Kähler manifold, $N = 0$.

2.5.3. Almost contact metric structures. We consider an almost contact metric manifold $(M^{2n+1}, g, A, \eta, \xi)$, i.e. a $(2n+1)$-dimensional Riemannian manifold equipped with a 1-form $\eta$, a $(1, 1)$-tensor $A$ and a vector field $\xi$ dual to $\eta$ with respect to the metric $g$, $\eta(\xi) = 1$, $\eta(X) = g(X, \xi)$ such that the following compatibility conditions are satisfied (see e.g. [7])

$$A^2 = -\text{id} + \eta \otimes \xi,$$

$$F(X, Y) = g(AX, Y) = g(X, Y) - \eta(X)\eta(Y),$$

$$F(X, Y) = g(AX, Y), \quad A\xi = 0. \quad (2.26)$$

In this case the skew-symmetric part $F$ of $G = g + F$ is degenerate, $F(\xi, X) = 0$ and has rank $2n$. Then theorem 2.3 together with (2.26) implies $0 = g((\nabla_X A)Y, \xi) - (\nabla_X \eta)Y = g(\nabla_X \xi, Y)$, i.e. $\nabla\eta = \nabla\xi = 0$. Now (2.22) yields $0 = g(\nabla_X \xi, Y) + \frac{1}{2}T(X, \xi, Y)$, which shows that $\xi$ is a Killing vector field because $T$ is skew-symmetric and

$$d\eta = \xi . d\eta, \quad \xi . d\eta = 0, \quad (2.27)$$

where $\cdot$ denotes interior multiplication.

We obtain from (2.18) applying (2.26) and (2.27) that

$$N(X, Y, Z) + \eta(Z)d\eta(X, Y) = -T(AX, AY, Z) + T(X, Y, Z) - T(AX, AY, AZ) \quad (2.28)$$

The right hand side of equation (2.28) is totally skew-symmetric, which yields that the tensor

$$N^{ac} = N + d\eta \otimes \eta \quad (2.29)$$

is totally skew-symmetric, $N^{ac}(X, Y, Z) = -N^{ac}(X, Z, Y)$. In fact, the tensor $N^{ac}$ is usually called the Nijenhuis tensor in almost contact geometry (see e.g. [7]).

We get from (2.28) applying (2.26) and (2.27) that

$$N(\xi, Y, Z) = d\eta(Y, Z) - d\eta(AY, AZ) \quad (2.30)$$

We calculate from (2.16), taking into account (2.26) and (2.27), that

$$N(AX, AY, AZ) = -N(X, Y, Z) - \eta(Z)[d\eta(AX, AY) + d\eta(X, Y)] + \eta(X)N(\xi, Y, Z) - \eta(Y)N(\xi, X, Z) \quad (2.31)$$
The first formula in (2.21) together with (2.27) and (2.31) yields

\[ T(X, Y, Z) = -N(AX, AY, AZ) + dF(AX, AY, AZ) + \eta(X)d\eta(Y, Z) + \eta(Y)d\eta(Z, X) \]

\[ = N(X, Y, Z) + dF(AX, AY, AZ) + \eta(Z)d\eta(AX, AY) + d\eta(X, Y) \]

\[ - \eta(X)N(\xi, Y, Z) + \eta(Y)N(\xi, X, Z) + \eta(X)d\eta(Y, Z) + \eta(Y)d\eta(Z, X). \]

From this equality, applying (2.30), we obtain the following formula for the skew-symmetric torsion

\[ T(X, Y, Z) = N(X, Y, Z) + \eta(Z)d\eta(X, Y) + dF(AX, AY, AZ) \]

\[ + \eta(Z)d\eta(AX, AY) + \eta(Y)d\eta(AZ, AX) + \eta(X)d\eta(AY, AZ). \]  

(2.32)

Notice that using (2.16), (2.26) and (2.29), \( d\eta(AX, AY) = -N(X, Y, \xi) = -N^{ac}(X, Y, \xi) + d\eta(X, Y) \). Substitute the last equality into (2.32) and use the skew-symmetricity of \( N^{ac} \) to get the equality obtained in [14]

\[ T(X, Y, Z) = \eta \wedge d\eta + N^{ac} - d^4F - \eta(Z) \wedge (\xi \wedge N^{ac}), \]

\[ d^4F(X, Y, Z) = -dF(AX, AY, AZ). \]  

(2.33)

Theorem 2.5 gives the next well-known result (note the sign difference in the 2-form \( F \) in [14])

**Corollary 2.8.** [14]. On an almost contact metric manifold \( (M, g, A, F, \eta, \xi) \) there exists a unique linear connection \( \nabla \) preserving the generalized Riemannian metric \( G = g + F \) with totally skew-symmetric torsion \( T \) if and only if the almost contact Nijenhuis tensor \( N^{ac} \) is totally skew-symmetric, \( N^{ac}(X, Y, Z) = -N^{ac}(X, Z, Y) \) and \( \xi \) is a Killing vector field. The torsion \( T \) is determined by (2.33) or, equivalently, by (2.32).

### 2.5.4. Almost paracontact metric structures.

We consider an almost paracontact metric manifold \( \mathcal{M} = (\mathcal{M}^{2n+1}, g, A, \eta, \xi) \), i.e. a \((2n + 1)\)-dimensional pseudo-Riemannian manifold of signature \((n + 1, n)\) equipped with a 1-form \( \eta \) a \((1, 1)\)-tensor \( A \) and a vector field \( \xi \) dual to \( \eta \) with respect to the metric \( g \). \( \eta(\xi) = 1, \eta(X) = g(X, \xi) \) such that the following compatibility conditions hold (see e.g. [43])

\[ A^2 = \text{id} - \eta \otimes \xi, \quad g(AX, AY) = -g(X, Y) + \eta(X)\eta(Y), \]

\[ F(X, Y) = g(AX, Y), \quad A\xi = 0. \]  

(2.34)

In this case the skew-symmetric part \( F \) of \( G = g + F \) is degenerate, \( F(\xi, X) = 0 \) and has rank \( 2n \). Theorem 2.3 together with (2.34) implies \( 0 = g((\nabla_X A)AY, \xi) = -g(\nabla_X \xi, Y) \) for every \( X \in \mathcal{X}(\mathcal{M}) \). Now (2.22) yields \( 0 = g(\nabla_X \xi, Y) + \frac{1}{2} T(X, \xi, Y) \), which shows that \( \xi \) is a Killing vector field because \( T \) is skew-symmetric and \( d\eta = \xi \wedge T, \quad \xi.d\eta = 0 \). We obtain from (2.18) applying (2.34) and the last equality

\[ N(X, Y, Z) = \eta(Z)d\eta(X, Y) = -T(AX, AY, AZ) - T(X, AX, AZ) \]

(2.35)

The right hand side of equation (2.35) is totally skew-symmetric, which yields that the tensor \( N^{apc} = N - d\eta \otimes \eta \) is totally skew-symmetric, \( N^{apc}(X, Y, Z) = -N^{apc}(X, Z, Y) \). In fact, the tensor \( N^{apc} \) is usually called the *Nijenhuis tensor in almost paracontact geometry* (see e.g. [43]).

As in the previous case, theorem 2.5 gives the following well-known result (note the sign difference in the 2-form \( F \) in [43]).
Corollary 2.9. [43]. On an almost paracontact metric manifold \((M, g, A, F, \eta, \xi)\) there exists a unique linear connection \(\nabla\) preserving the generalized Riemannian metric \(G = g + F\) with totally skew-symmetric torsion \(T\) if and only if the almost paracontact Nijenhuis tensor \(N^\text{apc}\) is totally skew-symmetric, \(N^\text{apc}(X, Y, Z) = -N^\text{apc}(X, Z, Y)\) and \(\xi\) is a Killing vector field. The torsion \(T\) is determined by

\[
T(X, Y, Z) = \eta \wedge d\eta - N^\text{apc} - d^A F + \eta(Z) \wedge (\xi \wedge N^\text{apc}),
\]

\[
d^A F(X, Y, Z) = -dF(AX, AY, AZ).
\]

3. Einstein metricity condition (NGT)

In his attempt to construct a unified field theory (non-symmetric gravitational theory, briefly NGT) Einstein [10] considered a generalized Riemannian manifold and used the so-called metricity condition (1.1), which can be written as follows \(X G(Y, Z) - G(\nabla_X Y, Z) - G(Y, \nabla_X Z) = 0\), see also [34, 35] for a slightly more general right hand side of (1.1).

In view of (2.8), the definition of the torsion, (2.1) and (2.2), the metricity condition (1.1) can be written in the form

\[
(\nabla_X G)(Y, Z) = -G(T(X, Y), Z) \Leftrightarrow (\nabla_X (g + F))(Y, Z)
\]

\[
= -T(X, Y, Z) + T(X, Y, AZ).
\]

A general solution for the connection \(\nabla\) satisfying (1.1) is given in terms of \(g, F, T\) [27] (see also [35]). Here we show that the solution can be expressed in terms of the exterior derivative \(dF\) of \(F\) and find formulas for the covariant derivatives of \(\nabla g\) and \(\nabla F\).

Taking the cyclic sum in (3.1) and applying (2.6), we obtain

\[
(\nabla_X G)(Y, Z) + (\nabla_Y g)(Z, X) + (\nabla_Z g)(X, Y)
\]

\[
= -dF(X, Y, Z) - T(X, Y, Z) - T(Y, Z, X) - T(Z, X, Y).
\]

(3.2)

It is easy to observe that the left hand side of (3.2) is symmetric while the right hand side is skew-symmetric. Hence, we get

\[
(\nabla_X G)(Y, Z) + (\nabla_Y g)(Z, X) + (\nabla_Z g)(X, Y) = 0;
\]

\[
dF(X, Y, Z) = -T(X, Y, Z) - T(Y, Z, X) - T(Z, X, Y).
\]

(3.3)

In local coordinates, the formulas (3.3) have the form

\[
\nabla g_{jk} + \nabla g_{kj} + \nabla g_{kj} = 0; \quad dF_{ij} = -T_{ijk} - T_{jki} - T_{kji}.
\]

The symmetric part of (3.1) with respect to \(X, Y\) gives

\[
(\nabla_X g)(Y, Z) + (\nabla_Y g)(X, Z) + (\nabla_X F)(Y, Z) + (\nabla_Y F)(X, Z) = 0,
\]

which, combined with the first equality in (3.3), yields

\[
(\nabla_X g)(X, Y) = (\nabla_X F)(Y, Z) + (\nabla_Y F)(X, Z).
\]

(3.4)

Substitute (3.4) into (3.1) and use (2.6) to get

\[
(\nabla_X f)(X, Y) = \frac{1}{2} dF(X, Y, Z) + T(X, Y, Z) - T(Z, Y, AX) + T(Z, X,AY).
\]

(3.5)
We obtain inserting (3.5) into (3.4) that
\[(\nabla_X g)(Y, Z) = -\frac{1}{2}[T(X, Y, Z) - T(X, Y, A \mathcal{Z}) + T(X, Z, Y) - T(X, Z, A Y)]. \tag{3.6}\]

Note that in local coordinates we can write (3.5) and (3.6) in the form
\[\nabla_i F_{ij} = \frac{1}{2}[dF_{jk} + T_{ij} - T_{ij}A^k_i - T_{ik}A^j_j],\]
\[\nabla_i g_{jk} = -\frac{1}{2}[T_{ij} - T_{ij}A^k_k + T_{ik} - T_{ik}A^j_j].\]

Applying (3.6) and the second equation in (3.3) we obtain from (2.4) that
\[g(\nabla_X Y, Z) = g(\nabla_X Z, Y) - \frac{1}{2}[T(X, Y, Z) - T(X, Z, A \mathcal{Z}) - T(Y, Z, A X)]\]
\[= g(\nabla_X Z, Y) - \frac{1}{2}[dF(X, Y, Z) + T(Z, X, Y) + T(Y, Z, X)]\]
\[+ \frac{1}{2}[T(Z, X, A Y) + T(Z, Y, A X)]. \tag{3.7}\]

3.1. NGT involving the Nijenhuis tensor

We are going to involve the Nijenhuis tensor in our attempt to determine the torsion $T$. We start with an expression of the covariant derivative $\nabla A$. By substituting (3.5) and (3.6) into (2.7), we find
\[g((\nabla_X A)Y, Z) = g(\nabla_X F)(Y, Z) - (\nabla_X g)(AY, Z)\]
\[= \frac{1}{2}[dF(X, Y, Z) + T(Y, Z, X) + T(X, Y, AZ)\]
\[+ T(X, AY, Z) - T(X, Z, AY) - T(X, Z, AZ)]. \tag{3.8}\]

Substitute (3.8) into (2.17) to get after some calculations the following equality
\[N(X, Y, Z) = dF(X, Y, AZ) + \frac{1}{2}[dF(AX, Y, Z) + dF(X, AY, Z)]\]
\[+ \frac{1}{2}[T(Y, Z, AX) - T(X, Z, AY) + T(Y, AZ, X)\]
\[- T(X, AZ, Y) - T(X, AY, A^2Z) - T(AX, Y, A^2Z)]\]
\[- \frac{1}{2}[T(Z, AY, A^2X) - T(Z, AX, A^2X)\]
\[+ T(Z, AY, A^2X) - T(Z, AX, A^2Y)] - T(AX, AY, AZ). \tag{3.9}\]

3.2. NGT and skew-symmetric torsion

Now we consider the case of totally skew-symmetric torsion, $T(X, Y, Z) = -T(X, Z, Y)$. From the considerations above, we have

**Theorem 3.1.** A generalized Riemannian manifold $(M, G = g + \mathcal{F})$ admits a linear connection satisfying the Einstein metricity condition (1.1) with totally skew-symmetric torsion $T$ if and only if the Nijenhuis tensor $N$ and the exterior derivative of $\mathcal{F}$ satisfy the
relation

\[ N(X, Y, Z) = \frac{2}{3} dF(X, Y, AZ) + \frac{1}{3} dF(AX, Y, Z) \]
\[ + \frac{1}{3} dF(X, AY, Z) + \frac{1}{3} dF(AX, AY, AZ) \]
\[ - \frac{1}{6} [dF(A^2 X, Y, AZ) \]
\[ + dF(A^2 X, AY, Z) + dF(X, A^2 Y, AZ) - dF(X, AY, A^2 Z)] \]
\[ - \frac{1}{6} [dF(AX, A^2 Y, Z) - dF(AX, Y, A^2 Z)]. \] (3.10)

In this case the skew-symmetric torsion is completely determined by the exterior derivative of the skew-symmetric part of the generalized Riemannian metric,

\[ T(X, Y, Z) = \frac{1}{3} dF(X, Y, Z), \] in local coordinates, \( T_{ij\ell} = -\frac{1}{3} dF_{ij\ell}, \) (3.11)

the Einstein metricity condition has the form

\[ (\nabla_X g)(Y, Z) = \frac{1}{3} [dF(X, Y, Z) - dF(X, Y, AZ)] \] and it is equivalent to the following two conditions

\[ (\nabla_X g)(Y, Z) = -\frac{1}{6} [dF(X, Y, AZ) - dF(X, AY, Z)]; \]
\[ (\nabla_X F)(Y, Z) = \frac{1}{6} [2dF(X, Y, Z) - dF(X, Y, AZ) - dF(X, AY, Z)]. \] (3.12)

The connection is uniquely determined by the formula

\[ g(\nabla_X Y, Z) = g(\nabla_X^g Y, Z) - \frac{1}{6} dF(X, Y, Z) - \frac{1}{6} dF(X, AY, Z) + \frac{1}{6} dF(AX, Y, Z). \] (3.13)

The covariant derivative of \( F \) and \( A \) with respect to the Levi-Civita connection \( \nabla^g \) are given by

\[ (\nabla_X^g F)(Y, Z) = g((\nabla_X^g A) Y, Z) \]
\[ = \frac{1}{3} dF(X, Y, Z) + \frac{1}{3} dF(X, AY, AZ) - \frac{1}{6} dF(AX, Y, AZ) - \frac{1}{6} dF(AX, AY, Z). \] (3.14)

Proof. The second equation of (3.3) together with the skew-symmetric property of the torsion implies (3.11). Using the already obtained (3.11) reduces (3.9) to (3.10). Now, (3.12) and (3.13) follow from (3.5), (3.6), (3.7) and (3.11). Applying (3.11) and (3.12) to (2.5), we obtain
\[(\nabla_X F)(Y, Z) = (\nabla_Y F)(Y, Z) = \frac{1}{6}[dF(X, Y, AZ) + dF(Z, X, AZ)] - \frac{1}{6}[dF(X, AY, AZ) + dF(Z, AY, AX) + dF(Y, AX, AZ)]. \quad (3.15)\]

Substituting the second formula in (3.12) into (3.15) gives (3.14).

For the converse, checking that the connection determined by (3.13) satisfies the required properties provided (3.10) holds is straightforward. This completes the proof. \[\square\]

**Remark 3.2.** In local coordinates, (3.12) reads
\[\nabla_i g_{jk} = -\frac{1}{6}[dF_{js} A_i^s - dF_{sk} A_i^j], \quad \nabla_i F_{jk} = \frac{1}{6}[2dF_{jk} - dF_{js} A_i^s - dF_{sk} A_i^j]\]
while (3.14) takes the form
\[\nabla_i^g F_{jk} = \nabla_i^g A_j^s g_{sk} = \frac{1}{3}dF_{jk} + \frac{1}{3}dF_{is} A_i^j A_i^s - \frac{1}{6}dF_{js} A_i^j A_i^s - \frac{1}{6}dF_{sk} A_i^j A_i^s.\]

### 3.3. Almost Hermitian manifolds, \(A^2 = -1\)

The condition (3.10) in the almost Hermitian case takes the form
\[N(X, Y, Z) = dF(X, Y, AZ) + \frac{1}{3}dF(AX, Y, Z) + \frac{1}{3}dF(AX, AY, AZ). \quad (3.16)\]

Using the properties of the Nijenhuis tensor of an almost Hermitian manifold, (2.25), we get from (3.16) that the exterior derivative \(dF\) of the Kähler form satisfies
\[dF(X, Y, AZ) = dF(X, AY, Z) = dF(AX, Y, Z) = -dF(AX, AY, AZ), \quad (3.17)\]
i.e. \(dF\) is of type \((3, 0) + (0, 3)\) with respect to the almost complex structure \(A\). Substitute (3.17) into (3.16) to get \(N(X, Y, Z) = \frac{4}{3}dF(X, Y, AZ)\).

At this point we recall that an almost Hermitian manifold is said to be nearly Kähler if the covariant derivative of the almost complex structure \(A\) with respect to the Levi-Civita connection \(\nabla^g\) of the metric \(g\) is skew-symmetric,
\[(\nabla_X^g A)X = 0 \iff (\nabla_X^g F)(X, Y) = 0. \quad (3.18)\]

Nearly Kähler manifolds (called almost Tachibana spaces in [42]) were developed by Gray [22–24] and have been intensively studied since then in [8, 12, 30, 36–38]. Nearly Kähler manifolds in dimension 6 are Einstein manifolds with a positive scalar curvature. The Nijenhuis tensor \(N\) is a 3-form and is parallel with respect to the Gray characteristic connection (see [30]). This connection was defined by Gray [22–24] and it turns out to be the unique linear connection preserving the nearly Kähler structure and having totally skew-symmetric torsion (see [14]). Nearly Kähler manifolds also appear in supersymmetric string theories (see e.g. [25, 31, 39, 40] etc).

We obtain from theorem 3.1 that

**Theorem 3.3.** Let \((M, A, g, F)\) be an almost Hermitian manifold with a Kähler 2-form \(F\) considered as a generalized Riemannian manifold \((M, G = g + F)\). Then \((M, G)\) satisfies the
Einstein metricity condition (1.1) with totally skew-symmetric torsion $T$ if and only if it is a nearly Kähler manifold.

The skew-symmetric torsion is determined by the condition

$$T(X, Y, Z) = -\frac{1}{3} dF(X, Y, Z) = \frac{1}{4} N(X, Y, AZ).$$

(3.19)

The connection is unique given by the formula

$$g(\nabla_X Y, Z) = g(\nabla_X^g Y, Z) - \frac{1}{6} dF(X, Y, Z) = g(\nabla_X^g Y, Z) + \frac{1}{8} N(X, Y, AZ).$$

The Einstein metricity condition has the form $(\nabla_X G)(Y, Z) = \frac{1}{3} dF(X, Y, Z) - dF(X, Y, AZ)$. The covariant derivative of $g$ and the Kähler form $F$ are

$$(\nabla_X g)(Y, Z) = 0; \quad (\nabla_X F)(Y, Z) = \frac{1}{3} dF(X, Y, Z) - dF(X, Y, AZ).$$

Proof. Applying (3.19) and (3.17) to (2.5) we conclude that the nearly Kähler condition (3.18) holds. The rest of the theorem follows from theorem 3.1.

3.4. Almost para-Hermitian manifolds, $A^2 = 1$

The condition (3.10) in the almost para-Hermitian case takes the form

$$N(X, Y, Z) = \frac{1}{3} dF(X, Y, AZ) + \frac{1}{3} dF(AX, Y, Z)$$

$$+ \frac{1}{3} dF(X, AY, Z) + \frac{1}{3} dF(AX, AY, AZ),$$

which implies that the Nijenhuis tensor is totally skew-symmetric.

We obtain from theorem 3.1 and proposition 3.1 of [29].

Theorem 3.4. Let $(M, A, g, F)$ be an almost para-Hermitian manifold with a Kähler 2-form $F$ considered as a generalized Riemannian manifold $(M, G = g + F)$. Then $(M, G)$ satisfies the Einstein metricity condition (1.1) with totally skew-symmetric torsion $T$ if and only if the Nijenhuis tensor is totally skew-symmetric.

The skew-symmetric torsion is determined by $T(X, Y, Z) = -\frac{1}{3} dF(X, Y, Z)$, the Einstein metricity condition has the form $(\nabla_X G)(Y, Z) = \frac{1}{3} dF(X, Y, Z) - dF(X, Y, AZ)]$ and the equalities (3.12), (3.13) and (3.14) hold with $A^2 = 1$.

Remark 3.5. In particular, nearly para-Kähler manifolds defined by $(\nabla_X A)X = 0$ have a totally skew-symmetric Nijenhuis tensor, $dF$ share the properties (2.25) of the Nijenhuis tensor and $N(X, Y, Z) = \frac{1}{3} dF(X, Y, AZ)$ (proposition 5.1 of [29]). In this case the NGT connection with skew-symmetric torsion preserves the symmetric part of $G$, $\nabla g = 0$.

Remark 3.6. Combining theorem 3.4 and corollary 2.7 we conclude that an almost para-Hermitian manifold admits an NGT connection with totally skew-symmetric torsion if and
only if it admits a linear connection preserving the almost para-Hermitian structure with totally skew-symmetric torsion.

3.5. Almost contact metric structures

Let us consider the case of an almost contact metric structure which is defined in section 4.3. The fact that $g(\xi, \xi) = 1$ and (3.14) implies

$$(\nabla^g_X \eta)Z = g(\nabla^g_X \xi, Z) = -g((\nabla^g_X A)\xi, AZ) = \frac{1}{3}dF(X, AZ, \xi) + \frac{1}{6}dF(AZ, Z, \xi).$$

(3.20)

Consequently, we obtain

$$d\eta(X, Z) = \frac{1}{2}dF(X, AZ, \xi) + \frac{1}{2}dF(AZ, X, \xi),$$

$$d\eta(X, \xi) = 0, \quad d\eta(AZ, Z) = d\eta(X, AZ).$$

(3.21)

Applying (2.26) and (3.21), we simplify (3.10) to get

$$N(X, Y, Z) = dF(X, Y, AZ) + \frac{1}{3}dF(AZ, X, Y) + \frac{1}{3}dF(X, Y, Z) + \frac{1}{3}dF(AZ, Y, AZ) - \frac{1}{3}[d\eta(Y, Z)\eta(X) + d\eta(Z, X)\eta(Y) - d\eta(X, Y)\eta(Z)].$$

(3.22)

On the other hand, lemma 6.1 of [7], in our notation, reads

$$2g((\nabla^g_X A)Y, Z) = dF(X, Y, Z) - dF(X, AZ, Y) + N^{\eta}(Y, Z, AZ) + [d\eta(AZ, Y)\eta(X) - d\eta(X, AZ)\eta(Y)].$$

(3.23)

Substitute (3.14) into (3.23), use (3.21) and observe $N^{\eta}(Y, Z, AX) = N(Y, Z, AX)$ to get

$$N(Y, Z, AX) = -2d\eta(AZ, Y)\eta(X) + d\eta(X, AZ)\eta(Y) - d\eta(X, AZ)\eta(Y) - \frac{1}{3}dF(X, Y, AZ) + \frac{5}{3}dF(AZ, Y, AZ) - \frac{1}{3}dF(AZ, Y, AZ).$$

(3.24)

Setting $X = \xi$ into (3.24) gives $0 = -\frac{1}{3}dF(\xi, Y, Z) + \frac{5}{3}dF(\xi, AZ, Y) - 2d\eta(AY, Z)$, which, in view of (3.21), yields

$$dF(Y, AZ, \xi) = d\eta(Y, Z) = dF(AY, Z, \xi).$$

(3.25)

Applying (3.25) to (3.20) leads to

$$(\nabla^g_X \eta)Y = g(\nabla^g_X \xi, Y) = \frac{1}{2}d\eta(X, Y).$$

(3.26)

which shows that the vector field $\xi$ is a Killing vector field.

The equalities (3.22) and (3.25) imply

$$N(X, Y, \xi) = N(\xi, X, Y) = d\eta(X, Y).$$

(3.27)
Applying (3.27), we obtain from (3.24)
\[
N(X, Y, Z) = d\eta(Y, Z)\eta(X) + \frac{5}{3}d\eta(X, Y)\eta(Z) + d\eta(Z, X)\eta(Y) + \frac{1}{3}dF(X, Y, AZ) - \frac{5}{3}dF(AX, AY, AZ) - \frac{1}{3}dF(AX, Y, Z) - \frac{1}{3}dF(X, AY, Z).
\] (3.28)

Compare (3.22) with (3.28) to derive
\[
3dF(AX, AY, AZ) + dF(X, Y, AZ) + dF(AX, Y, Z) + dF(X, AY, Z) = 2(d\eta \wedge \eta)(X, Y, Z).
\] (3.29)

Using (2.26) and (3.25), we derive from (3.29) the following
\[
dF(AX, AY, AZ) + dF(X, Y, AZ) = \eta(X)d\eta(Y, Z) + \eta(Y)d\eta(Z, X).
\] (3.30)

Substitute (3.29) and (3.30) into (3.28) to get
\[
N(X, Y, Z) = \frac{4}{3}dF(AX, AY, AZ) + (d\eta \wedge \eta)(X, Y, Z),
\] (3.31)

which, in particular, shows that the Nijenhuis tensor \( N \) is totally skew-symmetric.

Applying (3.30) to (3.14) we obtain
\[
g(\nabla^\xi_X Y, Z) = \frac{1}{3}dF(X, Y, Z) - \frac{1}{6}\eta(Y)d\eta(Z, AX) + \frac{1}{3}\eta(X)d\eta(Y, AZ) + \frac{1}{6}\eta(Z)d\eta(Y, AX) - \frac{1}{3}dF(AX, AY, Z) + \frac{1}{6}\eta(Z)d\eta(Y, AX) - \frac{1}{2}\eta(Y)d\eta(AZ, X).
\] (3.32)

We derive from (3.32) that\[ g(\nabla^\xi_X Y, Z) + g(\nabla^\xi_A X, Z) = -\frac{1}{2}\eta(Y)d\eta(AZ, X) - \frac{1}{2}\eta(X)d\eta(AZ, Y), \]
i.e. the structure is nearly-cosymplectic, \( (\nabla^\xi_X A)X = 0 \) if and only if the 1-form \( \eta \) is closed, \( d\eta = 0 \).

It seems reasonable to consider the following

**Definition 3.7.** An almost contact metric manifold \( (M^{2n+1}, A, g, F, \eta, \xi) \) is said to be \textit{almost-nearly cosymplectic} if the Levi-Civita covariant derivative of the fundamental 2-form satisfies the following condition
\[
g((\nabla^\xi_X A)Y, Z) = -\frac{1}{3}dF(AX, AY, Z) + \frac{1}{6}\eta(Z)d\eta(Y, AX) - \frac{1}{2}\eta(Y)d\eta(AZ, X).
\] (3.33)

In particular, the vector field \( \xi \) is a Killing and (3.31) holds.

**Theorem 3.8.** Let \( (M, A, g, F, \eta, \xi) \) be an almost contact metric manifold with a fundamental 2-form \( F \) considered as a generalized Riemannian manifold \((M, G)\) with a generalized Riemannian metric \( G = g + F \). Then \((M, G)\) satisfies the Einstein metricity condition (1.1) with totally skew-symmetric torsion \( T \) if and only if it is an almost-nearly cosymplectic, i.e. (3.33) holds.
The skew-symmetric torsion is determined by the condition
\[
T(X, Y, Z) = -\frac{1}{3}dF(X, Y, Z) = -\frac{1}{4}N(AX, AY, AZ) + \frac{1}{3}[\eta(X)d\eta(Y, AZ) + \eta(Y)d\eta(Z, AX) + \eta(Z)d\eta(X, AY)].
\]
The connection is uniquely determined by the formula
\[
g(\nabla_X Y, Z) = g(\nabla^\xi_X Y, Z) - \frac{1}{6}dF(X, Y, Z) + \frac{1}{6}[\eta(X)d\eta(Y, Z) + \eta(Y)d\eta(X, Z)].
\]
The Einstein metricity condition has the form
\[
\nabla_X g(Y, Z) = \frac{1}{6}[dF(X, Y, Z) - dF(X, Y, AZ)].
\]
The covariant derivative of \(g\) and \(F\) are
\[
(\nabla_X g)(Y, Z) = \frac{1}{3}[dF(X, Y, Z) - dF(X, Y, AZ)] - \frac{1}{6}[\eta(Y)d\eta(Z, X) + \eta(Z)d\eta(Y, X)].
\]

3.6. Almost-nearly cosymplectic structures
We list some elementary properties of almost-nearly cosymplectic structures defined by (3.33).

First, due to (3.26) we have

**Corollary 3.9.** On an almost-nearly cosymplectic manifold the vector field \(\xi\) is a Killing.

**Corollary 3.10.** If an almost-nearly cosymplectic structure is normal then it is cosymplectic (co-Kähler), i.e. \(d\eta = dF = 0\).

**Proof.** An almost contact structure is normal if the almost contact Nijenhuis tensor vanishes, \(N^{ac} = 0\) which, in view of (2.29), gives \(N(X, Y, Z) = -\eta(Z)d\eta(X, Y)\). Now, (3.31) yields \(d\eta = 0\) and \(dF(AX, AY, AZ) = 0\). The last equality together with (3.25) implies \(dF = 0\). □

**Remark 3.11.** An almost-nearly cosymplectic structure is never contact because of (3.21). Indeed, if \(d\eta = F\) then \(d\eta\) is of type (1, 1) with respect to \(A\) which contradicts (3.21).

Suppose an almost-nearly cosymplectic structure has a closed 1-form \(\eta\), \(d\eta = 0\). Then (3.21) yields that \(\eta\) is \(\nabla^\xi\)-parallel, \(\nabla^\xi\eta = 0\). Then the distribution \(H = Ker(\eta)\) is involutive and therefore it is Frobenius integrable. The integral submanifold \(N^{2n}\) is a nearly Kähler manifold due to (3.33). Hence, an almost-nearly cosymplectic manifold with a closed 1-form \(\eta\) is locally a product of a nearly Kähler manifold with the real line.

More generally, we have \(L_\xi\eta = d\eta(\xi, \cdot) = 0\) due to (3.21). Since \(\xi\) is a Killing vector field, we calculate using (3.26), (3.25) and (3.32) that
\[
(L_\xi F)(X, Y) = (\nabla^\xi_F)(X, Y) - g(\nabla^\xi_X \xi, AY) + g(\nabla^\xi_X \xi, AX) = d\eta(Y, AX).
\]
Suppose $M^{2n+1}$ is compact and the almost contact metric structure is regular, i.e. the $2n$-dimensional quotient space $M^{2n+1}/\xi$ is a smooth manifold. If in addition $d\eta = 0$ then (3.34) and (3.32) implies that $M^{2n+1}$ is a trivial circle bundle over the nearly Kähler manifold $M^{2n+1}/\xi$. Conversely, starting with a nearly Kähler manifold $N^{2n}$ we obtain on $M^{2n+1} \times \mathbb{R}^+$ an almost-nearly cosymplectic structure taking $\eta = dt$.

### 3.7. Almost paracontact metric structures

An almost paracontact metric structure is defined in section 2.5.4. Equation (3.14) together with equation (2.34) implies the equalities

$$
(\nabla^h_X \eta) Y = g(\nabla^h_X \xi, Y) = -g((\nabla^h_X A)Y, \xi) = -\frac{1}{3}dF(X, AY, \xi) + \frac{1}{6}dF(AX, Y, \xi);
$$

$$
d\eta(X, Y) = -\frac{1}{6}dF(AX, Y, \xi) - \frac{1}{6}dF(X, AY, \xi),
$$

$$
d\eta(X, \xi) = 0, \quad d\eta(AX, Y) = d\eta(X, AY).
$$

With the help of (3.35) we write the condition (3.10) in the form

$$
N(X, Y, Z) = \frac{1}{3}dF(X, Y, AZ) + \frac{1}{3}dF(AX, Y, Z)
$$

$$
+ \frac{1}{3}dF(X, AY, Z) + \frac{1}{3}dF(AX, AY, AZ)
$$

$$
- \eta(X)d\eta(Y, Z) - \eta(Y)d\eta(Z, X) + \eta(Z)d\eta(X, Y).
$$

Now theorem 3.1 leads to the following

**Theorem 3.12.** Let $(M, A, g, F, \eta, \xi)$ be an almost paracontact metric manifold with a fundamental 2-form $F$ considered as a generalized Riemannian manifold $(M, G = g + F)$. Then $(M, G)$ satisfies the Einstein metricity condition (1.1) with totally skew-symmetric torsion $T$ if and only if the condition (3.14) holds.

The skew-symmetric torsion is determined by (3.11), the connection is uniquely determined by the formula (3.13). The covariant derivatives of $g$ and $F$ are given by (3.12).

**Proof.** We show that (3.14) implies (3.36) and then apply theorem 3.1.

The general formula from proposition 2.4 of [43], in our notation, reads

$$
2g((\nabla^h_X A)Y, Z) = dF(X, Y, Z) + dF(X, AY, AZ) - N^{\mathrm{sc}}(Y, Z, AX)
$$

$$
+ [d\eta(AY, Z) + d\eta(X, AZ) - d\eta(X, Z)]\eta(Y)
$$

$$
+ d\eta(AY, X)\eta(Z) - d\eta(AZ, X)\eta(Y).
$$

(3.37)
Substitute (3.14) into (3.37) and notice that \( N^{\text{fix}}(Y, Z, AX) = N(Y, Z, AX) \), so we get
\[
N(Y, Z, AX) = \frac{1}{3}dF(X, Y, Z) + \frac{1}{3}dF(X, AY, AZ) \\
+ \frac{1}{3}dF(AX, Y, AZ) + \frac{1}{3}dF(AX, AY, Z) \\
+ 2d\eta(AY, Z)\eta(X) + d\eta(AY, X)\eta(Z) - d\eta(AZ, X)\eta(Y). \quad (3.38)
\]

The defining equation (2.16) of the Nijenhuis tensor together with the properties listed in (2.34) and (3.35) implies \( N(X, Y, \xi) = -d\eta(AX, AY) = -d\eta(X, Y) \). Now, we easily get the equivalences of (3.38) and (3.36). The application of theorem 3.1 completes the proof. \( \square \)

In view of theorem 3.12 it seems reasonable to give the following

**Definition 3.13.** An almost paracontact metric manifold \((M, g, \eta, \xi, A)\) is called an almost paracontact NGT manifold with torsion if the following condition holds
\[
g((\nabla^A_X)Y, Z) = \frac{1}{3}dF(X, Y, Z) + \frac{1}{3}dF(X, AY, AZ) \\
- \frac{1}{6}dF(AX, Y, AZ) - \frac{1}{6}dF(AX, AY, Z).
\]

On an almost paracontact NGT manifold the equalities (3.35) hold, the Nijenhuis tensor is given by (3.36) and the Nijenhuis tensor \( N(AX, AY, AZ) \) is totally skew-symmetric and determined by
\[
N(AX, AY, AZ) = \frac{1}{3}dF(AX, AY, Z) + \frac{1}{3}dF(X, AY, AZ) \\
+ \frac{1}{3}dF(AX, Y, AZ) + \frac{1}{3}dF(X, Y, Z) \\
+ 2\eta(Z)d\eta(AX, Y) + 2\eta(Y)d\eta(AZ, X) + 2\eta(X)d\eta(AY, Z).
\]

An example can be obtained as follows. Starting with a nearly para-Kähler manifold \( N^{2n} \) one obtains on \( M^{2n+1} \times \mathbb{R}^+ \) an almost paracontact NGT structure taking \( \eta = dr \).

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