Absolute continuity of the spectrum of the periodic
Schrödinger operator in a layer and in a smooth cylinder

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Abstract

We consider the Schrödinger operator \( H = -\Delta + V \) in a layer or in a \( d \)-dimensional cylinder. The potential \( V \) is assumed to be periodic with respect to some lattice. We establish the absolute continuity of \( H \), assuming \( V \in L^p_{\text{loc}} \), where \( p \) is a real number greater than \( d/2 \) in the case of a layer, and \( p > \max(d/2, d-2) \) for the cylinder.  

1 Introduction

Let \( M \) be a smooth \( k \)-dimensional compact Riemannian manifold, let also

\[
\Xi = M \times \mathbb{R}^m, \quad d := \dim \Xi = k + m.
\]

We are interested in the type of the spectrum of the Schrödinger operator \( H = -\Delta + V \) in a cylinder \( \Xi \). The function \( V \) is supposed to be periodic. If \( M \) is a manifold with boundary, we study the operator \( H \) with various boundary conditions at \( \partial \Xi = \partial M \times \mathbb{R}^m \). We are going to prove that, under some assumptions on \( V \), the spectrum of \( H \) is absolutely continuous (see Theorems 2.1 and 2.2 below).

The points of \( \Xi \) are denoted by \((x, y)\), \( x \in M, y \in \mathbb{R}^m \). Let \( \Gamma \) be a lattice in \( \mathbb{R}^m \),

\[
\Gamma = \left\{ l = \sum_{j=1}^{m} l_j b_j, \quad l_j \in \mathbb{Z} \right\},
\]

where \( \{b_j\}_{j=1}^m \) is a basis of \( \mathbb{R}^m \). Assume that \( V \) is periodic over the "longitudinal" variables:

\[
V(x, y + l) = V(x, y), \quad x \in M, \ y \in \mathbb{R}^m, \ l \in \Gamma.
\]

Thanks to \( V \) being periodic, it is enough to know \( V \) on \( M \times \Omega \), where

\[
\Omega = \left\{ y = \sum_{j=1}^{m} y_j b_j, \quad y_j \in [0, 1) \right\}
\]

is an elementary cell of \( \Gamma \).

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Let us introduce the reader to the main results regarding absolute continuity of $H$. Usually, in the sufficient conditions it is assumed that the potential $V$ belongs to $L_p(M \times \Omega)$ or to a Lorentz space $L^0_{p,\infty}(M \times \Omega)$. We recall that if $N$ is a set of finite measure, then $L_p(N) \subset L_p(N) \subset L^0_{p,\infty}(N)$ for all $\varepsilon > 0$.

The two-dimensional case, $d = 2$ ($\Xi$ is a whole plane or a strip), has been studied in much detail. In [1, 10, 7], the absolute continuity of $H$ is proved for $V \in L_p$, $p > 1$. From now on, we consider only $d \geq 3$.

The case of $k = 0$, corresponding to the operator in the whole space, is also well studied. In [8], the absolute continuity is established in the "critical" case $V \in L^0_{d/2,\infty}(\Omega)$ for all $d \geq 3$ (see also [3]). In [13], the case $k = 1$ ($M$ is a line segment, $\Xi$ is a plane-parallel layer) is studied, and for $V \in L^0_{p,\infty}(M \times \Omega)$, where $p = \max(d/2, d - 2)$, the absolute continuity of $H$ is obtained. The author also considers the third type boundary condition. Finally, the case $k \geq 2$ is studied in [4], and it is established that $H$ is absolutely continuous if $V \in L_{d-1}(M \times \Omega)$.

In the present paper, we prove (see Theorem 2.1 below) the absolute continuity of $H$ with $V \in L_p(M \times \Omega)$ for all $p > d/2$ in the following cases: 1) $\partial M = \emptyset$; 2) $M$ is a line segment, $k = 1$; 3) $d = 3$ or 4. If $M$ is a manifold with boundary, $k > 1$, and $d > 4$, we obtain only $V \in L_{d-2}(M \times \Omega)$ as a sufficient condition. In the case $k = 1$ we also consider the third type boundary condition (see Theorem 2.2).

All mentioned results are obtained using the Thomas scheme [14], its key point is to study the operator family

$$H(\xi) = -\Delta_x + (-i\nabla_y + \xi)^*(-i\nabla_y + \xi) + V(x, y),$$

where $\xi$ is called quasimomentum. To obtain the resolvent estimates for the free operator $H_0(\xi)$, corresponding to $V = 0$, we use the spectral cluster estimates from [12] (the idea of using these estimates arose in [8]).

## 2 Formulation of the result

Let $M$ be a compact smooth Riemannian manifold with or without boundary, $\dim M = k$. Consider a $d$-dimensional cylinder

$$\Xi = M \times \mathbb{R}^m, \quad d = k + m \geq 3.$$ 

Let $\Gamma$ be a lattice (1.1), let $\Omega$ be a cell (1.3), and let $V(x, y)$ be a real-valued function, satisfying (1.2). Assume that

$$V \in L_{d/2}(M \times \Omega). \quad (2.1)$$

Consider the following quadratic form in $L_2(\Xi)$:

$$h[u, u] = \int_{\Xi} \left( |\nabla u(x, y)|^2 + V(x, y) |u(x, y)|^2 \right) \, dx \, dy, \quad \text{Dom} \, h = H^1(\Xi). \quad (2.2)$$

If $M$ has a boundary, $\partial M \neq \emptyset$, then we denote (2.2) by $h_N$. In this case we are also going to study a form $h_D = h_N \mid_{H^1_0(\Xi)}$. 

2
It is well known that, assuming (2.1), the form \( h \) (resp. \( h_N, h_D \)) is closed and semi-bounded from below. In \( L_2(\Xi) \), it corresponds to a semi-bounded operator \( H \) (resp. \( H_N, H_D \)), which is called the Schrödinger operator in \( \Xi \) (resp. the Schrödinger operator with Dirichlet or Neumann boundary conditions).

**Theorem 2.1.** Let \( M \) be a compact smooth Riemannian manifold with or without boundary, \( \dim M = k, \Xi = M \times \mathbb{R}^m, d = k + m \geq 3 \). Let \( \Gamma \) be a lattice (1.1), let \( V \) be a real-valued \( \Gamma \)-periodic function in \( \Xi \). Assume that \( V \in L_p(M \times \Omega) \), where

- \( p > d/2, \) if \( \partial M = \emptyset; \)
- \( p > d/2, \) if \( \partial M \neq \emptyset \) and \( k = 1 \) (\( M \) is a line segment);
- \( p > d/2, \) if \( \partial M \neq \emptyset \) and \( d = 3 \) or \( d = 4; \)
- \( p > d - 2, \) if \( \partial M \neq \emptyset \) and \( d \geq 5. \)

Then the spectra of \( H (\partial M = \emptyset), H_N \) and \( H_D (\partial M \neq \emptyset) \) are absolutely continuous.

In the case of a layer (\( M \) is a line segment), Suslina’s result [13] (see Theorem 4.5 below) allows us to consider the case of the third type boundary condition. Let \( k = 1, \Xi = [0, a] \times \mathbb{R}^m \), let also \( \sigma \) be a real \( \Gamma \)-periodic function on \( \partial \Xi = \{0; a\} \times \mathbb{R}^m \). Consider a quadratic form

\[
\begin{align*}
\sigma_{\gamma}[u, u] &= \int_{\Xi} \left( |\nabla u(x, y)|^2 + V(x, y)|u(x, y)|^2 \right) \, dx \, dy \\
&\quad + \int_{\mathbb{R}^m} \left( \sigma(a, y)|u(a, y)|^2 - \sigma(0, y)|u(0, y)|^2 \right) \, dy, \quad \text{Dom} \, \sigma_{\gamma} = H^1(\Xi). \tag{2.3}
\end{align*}
\]

If \( \sigma \in L_m(\{0, a\} \times \Omega) \), then the form (2.3) is closed and semi-bounded from below (see [9]). In the case \( \sigma = 0 \) the form \( \sigma_{\gamma} \) coincides with \( h_N \).

**Theorem 2.2.** Let \( \Xi = [0, a] \times \mathbb{R}^m \), \( d = m + 1 \geq 3 \), let \( \Gamma \) be a lattice (1.1). Let \( V \) be a \( \Gamma \)-periodic function on \( \Xi \), \( V \in L_p([0, a] \times \Omega) \) with \( p > d/2 \). Let \( \sigma \) be a \( \Gamma \)-periodic function on \( \partial \Xi \), satisfying

\[
\sigma \in L_q(\{0, a\} \times \Omega), \quad \text{where} \quad q = 2 \, \text{for} \, d = 3, \quad q = 2d - 2 \, \text{for} \, d \geq 4. \tag{2.4}
\]

Then the spectrum of the Schrödinger operator \( H_\sigma \), corresponding to the form (2.3), is absolutely continuous.

**Remark 2.3.** Theorem 2.1 can be reformulated in the matrix case. Let \( V \) be an \((n \times n)\)-matrix-valued function on \( \Xi \) such that \( V(x, y)^* = V(x, y) \), (1.2) holds, and \( V \in L_p(M \times \Omega), p > d/2 \). The quadratic form

\[
\begin{align*}
h[u, u] &= \int_{\Xi} \left( |\nabla u(x, y)|^2 + \langle V(x, y)u(x, y), u(x, y) \rangle \right) \, dx \, dy
\end{align*}
\]

is closed and semi-bounded on the domains \( H^1(\Xi, \mathbb{C}^n) \) and \( H^1_0(\Xi, \mathbb{C}^n) \). These forms correspond to the self-adjoint operators \( H, H_N, H_D \) in \( L_2(\Xi, \mathbb{C}^n) \). In the cases of a manifold without
boundary, a layer, and 3- and 4-dimensional cylinders, the spectra of such operators are absolutely continuous. In the case of a \( d \)-dimensional cylinder, \( d > 4 \), the spectra of \( H_N \) and \( H_D \) are absolutely continuous whenever \( V \in L_p(M \times \Omega) \), \( p > d - 2 \). The proof of Theorem 2.1 is valid for the matrix case without changes. A matrix analog of Theorem 2.2 can also be obtained.

It is convenient for us to interpret \( \Omega \) as an \( m \)-dimensional torus \( \mathbb{T} = \mathbb{R}^m / \Gamma \). Let us introduce an additional parameter \( \xi \in \mathbb{C}^m \), and consider the following quadratic forms. In the case of a manifold without boundary let

\[
\begin{align*}
  h(\xi)[v, v] &= \int_{M \times \Omega} \left(|\nabla_x v|^2 + \langle (\nabla_y + i\xi)v, (\nabla_y + i\xi)v \rangle + V|v|^2\right) dx dy, \\
  \text{Dom } h(\xi) &= H^1(M \times \mathbb{T}).
\end{align*}
\]

If \( \partial M \neq \emptyset \), then the form (2.5) will be denoted by \( h_N(\xi) \), and let also

\[
\begin{align*}
  h_D(\xi) &= h_N(\xi) |_{H^1_0(M \times \mathbb{T})}.
\end{align*}
\]

In the case of a layer, \( \Xi = [0, a] \times \mathbb{R}^m \), consider also a form

\[
\begin{align*}
  h_\sigma(\xi)[v, v] &= h_N(\xi)[v, v] + \int_{\Omega} (\sigma(a, y)|v(a, y)|^2 - \sigma(0, y)|v(0, y)|^2) dy, \\
  \text{Dom } h_\sigma(\xi) &= H^1([0, a] \times \mathbb{T}).
\end{align*}
\]

These forms are sectorial (the definition and main properties of sectoriality can be found in [5, Ch. VI, VII]), and they correspond to analytic operator families \( H(\xi), H_N(\xi), H_D(\xi), \) and \( H_\sigma(\xi) \) respectively. For real \( \xi \), these operators are self-adjoint.

Let \( b_1 \) be the first vector in the basis of \( \Gamma \). The conditions on the potential are dilatation-invariant, so we can assume \( |b_1| = 1 \).

**Theorem 2.4.** Suppose the conditions of Theorem 2.1 or Theorem 2.2 are satisfied. Then, for every \( \lambda \in \mathbb{C} \) and \( \xi \in \mathbb{R}^m \), \( \xi \perp b_1 \), there exists \( \tau_0 \) such that for \( |\tau| > \tau_0 \) the operator \( (H((\pi + i\tau)b_1 + \xi) - \lambda I) \) is invertible and

\[
\begin{align*}
  \| (H((\pi + i\tau)b_1 + \xi) - \lambda I)^{-1} \| &\leq C|\tau|^{-1}.
\end{align*}
\]

We prove this Theorem in §4. In a standard way (see, for example, [2] or [6]) Theorem 2.4 implies Theorems 2.1 and 2.2.

## 3 Spectral cluster estimates

For a self-adjoint operator \( P \), we denote by \( E_k(P) = E_P(([k - 1]^2; k^2]) \) its spectral projector onto a subspace, corresponding to an interval \(([k - 1]^2; k^2]) \). The following Theorem is proved in [12].

**Theorem 3.1.** Let \( N \) be a compact \( C^\infty \)-smooth \( d \)-dimensional Riemannian manifold without boundary, let \( P \) be an elliptic second-order differential operator on \( N \) with positive-definite symbol. Then

\[
\begin{align*}
  \| E_k(P)f \|_{L_2(N)} &\leq C k^{d(1/p-1/2)-1/2} \| f \|_{L_p(N)}, \quad f \in L_p(N), \quad 1 \leq p \leq \frac{2(d + 1)}{d + 3}.
\end{align*}
\]
By duality, this yields

**Corollary 3.2.** Under the assumptions of Theorem 3.1, the following inequality holds:

\[
\| E_k f \|_{L_q(N)} \leq C k^{d(1/2 - 1/q) - 1/2} \| f \|_{L_2(N)}, \quad f \in L_2(N), \quad \frac{2(d + 1)}{d - 1} \leq q \leq +\infty. \tag{3.1}
\]

**Theorem 3.3.** Let \( N_0 \) be a compact smooth Riemannian manifold without boundary, \( \dim N_0 = d - 1 \). Let \( P_0 \) be a second-order elliptic differential operator on \( N_0 \) with positive-definite symbol. Consider an elliptic operator \( P = 1 \otimes P_0 - \frac{d^2}{dx^2} \otimes 1 \) on a manifold \( N = [0, a] \times N_0 \) (\( x \) denotes a local coordinate on \([0, a]\)). Then, for \( P \) on \( N \) with either Dirichlet or Neumann boundary conditions, the estimate (3.1) holds.

**Proof.** We shall give proof for the Dirichlet problem, the Neumann case is analogous. The statement of Theorem is invariant with respect to dilatations over \( x \), so we can assume \( a = \pi \). In this case, the spectral projector \( E_k \) of \( P \) is an integral operator with kernel

\[
K(x, x', y, y') = \sum_{j^2 + \lambda_n \leq (k-1)^2 \ k^2} \frac{2}{\pi} \sin(jx) \sin(jx') \varphi_n(y) \overline{\varphi_n(y')}, \tag{3.2}
\]

where \( \{\lambda_n\}, \{\varphi_n\} \) are eigenvalues and eigenfunctions of \( P_0 \). We introduce three operators: an operator \( \tilde{E}_k \), acting on functions from \( L_2([0, 2\pi] \times N_0) \) as an integral operator with the same kernel (3.2), an operator of zero extension \( T: L_2(N) \to L_2([0, 2\pi] \times N_0) \), and a restriction operator \( S: L_2([0, 2\pi] \times N_0) \to L_2(N) \). Obviously, \( E_k = S \tilde{E}_k T \). Furthermore, \( \tilde{E}_k = \frac{1}{2\pi} (\tilde{E}_k^{(1)} - \tilde{E}_k^{(2)}) \), where \( \tilde{E}_k^{(1)} \) and \( \tilde{E}_k^{(2)} \) are integral operators with kernels

\[
K^{(1)}(x, x', y, y') = \sum_{j^2 + \lambda_n \leq (k-1)^2 \ k^2} (e^{ij(x-x')} + e^{-ij(x-x')}) \varphi_n(y) \overline{\varphi_n(y')},
\]

\[
K^{(2)}(x, x', y, y') = \sum_{j^2 + \lambda_n \leq (k-1)^2 \ k^2} (e^{ij(x+x')} + e^{-ij(x+x')}) \varphi_n(y) \overline{\varphi_n(y')}.
\]

The operator \( \tilde{E}_k^{(1)} \) is a spectral projector of \( -\frac{d^2}{dx^2} \otimes 1 + 1 \otimes P_0 \) on \([0, 2\pi] \times N_0\) with periodic boundary conditions over \( x \). The last operator is an elliptic operator on a manifold \( S^1 \times N_0 \) without boundary, and it satisfies (3.1). Similarly, (3.1) holds for \( \tilde{E}_k^{(2)} \), and so for \( \tilde{E}_k \) and \( E_k \).

The proof for the Neumann case can be obtained by replacing \( \sin(jx) \) with \( \cos(jx) \), in this case \( \tilde{E}_k = \frac{1}{2\pi} (\tilde{E}_k^{(1)} + \tilde{E}_k^{(2)}) \).

In [11], the following result is proved.

**Theorem 3.4.** Let \( N \) be a compact smooth Riemannian manifold with boundary, \( \dim N = d \geq 3 \). Let \( P \) be an elliptic second-order differential operator on \( N \) with positive-definite symbol and with Dirichlet or Neumann boundary conditions. Then, for \( 5 \leq q \leq 4, \) \( 4 \leq q \leq \infty, \) if \( d = 3 \), \( d = 4 \), \( d \geq 4 \), the estimate (3.1) holds. For \( 2 \leq q \leq 4, \) \( d \geq 4 \), the estimate is replaced with

\[
\| E_k f \|_{L_q(N)} \leq C k^{d(1/2 - 1/q) + 2/q - 1} \| f \|_{L_2(N)} , \tag{3.4}
\]

5
4 Proof of Theorem 2.4

For simplicity, denote $H((\pi + i\tau)b_1 + \xi)$ by $H(\tau)$. Let

$$H_0(\tau) = H(\tau)|_{V = 0, \sigma = 0}, \quad H_0 = H_0(0).$$

The operator $H_0$ is a self-adjoint second-order elliptic differential operator on a manifold $M \times \mathbb{T}$. Let $E_k$ denote its spectral projector onto $[(k - 1)^2; k^2]$. For a manifold $M$, we introduce

**Condition A(q).** $M$ satisfies the property that for every $\xi \in \mathbb{R}^m$, $\langle \xi, b_1 \rangle = 0$, there exist $\varepsilon > 0$ and $C > 0$ such that

$$\|E_k f\|_{L^q(M \times \mathbb{T})} \leq C k^{1/2 - \varepsilon} \|f\|_{L^2(M \times \mathbb{T})}, \quad \forall f \in L^2(M \times \mathbb{T}).$$

It is easy to see that $A(q)$ implies $A(\tilde{q})$ if $\tilde{q} < q$.

Let $\{\mu_j\}$ and $\{\varphi_j(x)\}$ be eigenvalues and eigenfunctions of the Laplace operator $-\Delta_x$ on $M$ with the corresponding (Dirichlet or Neumann) boundary conditions. Then the eigenvalues of $H_0(\tau)$ are of the form

$$h_{j,n}(\tau) = |n + \pi b_1 + \xi|^2 + \mu_j - \tau^2 + 2i\tau \langle n + \pi b_1, b_1 \rangle,$$

and the normalized eigenfunctions are

$$\varphi_{j,n}(x, y) = |\Omega|^{-1/2} \varphi_j(x) e^{i(n, y)}, \quad j \in \mathbb{N}, \ n \in \tilde{\Gamma},$$

where $\tilde{\Gamma}$ is the dual lattice,

$$\tilde{\Gamma} = \left\{ n = \sum_{j=1}^m n_j \tilde{b}_j, \ n_j \in \mathbb{Z} \right\}, \quad \langle b_k, \tilde{b}_j \rangle = 2\pi \delta_{kj}.$$ 

Notice that $\langle n, b_1 \rangle \in 2\pi \mathbb{Z}$. This gives

$$|h_{j,n}(\tau)| \geq |\text{Im} h_{j,n}(\tau)| = 2|\langle n + \pi b_1, b_1 \rangle||\tau| \geq 2\pi |\tau|.$$ 

Then, for $|\tau| > 0$, the operator $H_0(\tau)$ is invertible and

$$\|H_0(\tau)^{-1}\| \leq (2\pi |\tau|)^{-1}, \quad \tau \neq 0. \quad (4.1)$$

Consider also an operator $|H_0(\tau)|^{-1/2}$ such that

$$|H_0(\tau)|^{-1/2} \varphi_{j,n} = |h_{j,n}(\tau)|^{-1/2} \varphi_{j,n}.$$ 

The following Lemma is elementary.

**Lemma 4.1.** Let $0 < \varepsilon < 1/2$. Then the sums

$$\sum_{k=1}^{\infty} \frac{k^{1-2\varepsilon}}{|k^2 - \tau^2 + |\tau|}, \quad \sum_{k=1}^{\infty} \frac{k^{1-2\varepsilon}}{|(k - 1)^2 - \tau^2 + |\tau|}$$ 

are finite and uniformly bounded with respect to $\tau$ for $|\tau| > 1$. 

6
Proof. For certainty, consider the first sum. Without loss of generality, we can assume \( \tau > 0 \). If \( k^2 \geq 2 \tau^2 \), then the denominator can be replaced with \( \frac{1}{2} k^2 \), and this implies that the "tail" of the sum converges uniformly. Therefore, we may consider only \( k^2 < 2 \tau^2 \). In this case,

\[
\sum_{k < 2 \tau} \frac{k^{1-2\varepsilon}}{|k^2 - \tau^2| + |\tau|} \leq 2 |\tau|^{1-2\varepsilon} \sum_{k < 2 \tau} \frac{1}{|k^2 - \tau^2| + |\tau|} \leq 2 |\tau|^{2-2\varepsilon} \sum_{k < 2 \tau} \frac{1}{|k - \tau| + 1}.
\]

The last sum is bounded, because

\[
|\tau|^{2-2\varepsilon} \int_0^{2 \tau} \frac{dk}{|k - \tau| + 1} = 2 |\tau|^{2-2\varepsilon} \int_\tau^{2 \tau} \frac{dk}{k - \tau + 1} = 2 |\tau|^{2-2\varepsilon} \ln(\tau + 1). \]

**Theorem 4.2.** Assume that Condition A\((q)\) holds. Then, for some \( \tau_0 > 0 \),

\[
\| |H_0(\tau)|^{-1/2} f \|_{L^q(\Omega \times \mathbb{T})} \leq C \| f \|_{L^2(\Omega \times \mathbb{T})}, \quad \forall |\tau| > \tau_0, f \in L^2(\Omega \times \mathbb{T}). \tag{4.3}
\]

Proof. Let \( E_k \) be a spectral projector of \( H_0 \) onto \((k-1)^2, k^2\). Then

\[
\| |H_0(\tau)|^{-1/2} f \|_{L^q(\Omega \times \mathbb{T})} \leq \sum_{k=1}^{\infty} \| E_k |H_0(\tau)|^{-1/2} f \|_{L^q(\Omega \times \mathbb{T})}
\]

\[
\leq C \sum_{k=1}^{\infty} k^{1/2-\varepsilon} \| E_k |H_0(\tau)|^{-1/2} f \|_{L^2(\Omega \times \mathbb{T})} \leq C \sum_{k=1}^{\infty} k^{1/2-\varepsilon} \| E_k |H_0(\tau)|^{-1/2} f \|_{L^2(\Omega \times \mathbb{T})}.
\]

from which, using Cauchy-Bunyakovsky-Schwarz inequality, we obtain

\[
\| |H_0(\tau)|^{-1/2} f \|_{L^q(\Omega \times \mathbb{T})}^2 \leq C \| f \|_{L^2(\Omega \times \mathbb{T})}^2 \sum_{k=1}^{\infty} k^{1-2\varepsilon} \| E_k |H_0(\tau)|^{-1/2} f \|_{L^2(\Omega \times \mathbb{T})}^2.
\]

The eigenvalues of \( H_0 \) are \(|n + \pi b_1 + \xi|^2 + \mu_j, n \in \mathbb{N} \), \( j \in \mathbb{N} \). The range of \( E_k \) corresponds to the pairs \((j, n)\) such that \((k-1)^2 \leq |n + \pi b_1 + \xi|^2 + \mu_j < k^2\). So,

\[
\| E_k |H_0(\tau)|^{-1/2} f \|_{L^2(\Omega \times \mathbb{T})}^2 = \max_{|n + \pi b_1 + \xi|^2 + \mu_j \in [(k-1)^2, k^2]} \frac{1}{|h_{j,n}(\tau)|} \leq \max_{|n + \pi b_1 + \xi|^2 + \mu_j \in [(k-1)^2, k^2]} \sqrt{\frac{\sqrt{2}}{|n + \pi b_1 + \xi|^2 + \mu_j - \tau^2| + |\tau|}}.
\]

Finally, we need to show that the sum

\[
\sum_{k=1}^{\infty} \max_{|n + \pi b_1 + \xi|^2 + \mu_j \in [(k-1)^2, k^2]} \frac{k^{1-2\varepsilon}}{|n + \pi b_1 + \xi|^2 + \mu_j - \tau^2| + |\tau|} \tag{4.4}
\]

is finite and uniformly bounded for \(|\tau| > \tau_0\).

To do this, we notice that in all the terms (maybe, all but one) we can replace \(|n + \pi b_1 + \xi|^2 + \mu_j \) with \((k-1)^2\) or \(k^2\), and the term will not decrease, because, if \(|\tau| \notin [k-1; k] \), then, after one of these substitutions, the denominator may only decrease. The term, for which \(|\tau| \in [k-1; k] \), can be estimated by \(Ck^{-2\varepsilon}\) and does not affect the convergence. So, it is enough to consider two sums \((4.4)\): we replace \(|n + \pi b_1 + \xi|^2 + \mu_j \) with \((k-1)^2\) in the first one, and with \(k^2\) in the second one. Their boundness follows from Lemma 4.1.
We need the following fact to prove Theorem 2.4:

**Lemma 4.3.** Let \((M, \mu)\) be a measurable space with \(\sigma\)-finite measure, let \(V \in L^p(M), 1 \leq p < \infty\). Then for every \(\delta > 0\) there exists \(c(\delta)\) such that

\[
\int_M |Vf|d\mu \leq \delta \|f\|_{L^{2p'}(M)}\|g\|_{L^{2p'}(M)} + c(\delta)\|f\|_{L_2(M)}\|g\|_{L_2(M)}, \quad f, g \in L^{2p'}(M),
\]

where \(p'\) is the conjugate index to \(p\).

**Proof.** The function \(V\) can be expressed in the form

\[
V = V_1 + V_2, \quad \text{where} \quad \|V_1\|_{L^p(M)} \leq \delta, \quad V_2 \in L_\infty(M).
\]

By Hölder inequality,

\[
\int_M |Vfg|d\mu \leq \delta \|f\|_{L^{2p'}(M)}\|g\|_{L^{2p'}(M)} + \|V_2\|_{L_\infty(M)}\|f\|_{L_2(M)}\|g\|_{L_2(M)}.
\]

**Theorem 4.4.** Let \(M\) satisfy \(A(q)\) for some \(q \in (2, 2d/(d - 2))\). Let \(V \in L_p(M \times \mathbb{T})\), where \(p = q/(q-2)\). Then the operator \((H(\tau) - \lambda I)\) is invertible for \(|\tau| > \tau_0\), and \(|(H(\tau) - \lambda I)^{-1}| \leq C|\tau|^{-1}\).

**Proof.** The condition on \(V\) is invariant with respect to adding a constant. So, without loss of generality, we can assume \(\lambda = 0\). It is enough to prove the following statement: for any \(u \in \text{Dom}(H(\tau)), \|u\|_{L_2(M \times \mathbb{T})} = 1\), there exists \(v \in \text{Dom}(H(\tau)), \|v\|_{L_2(M \times \mathbb{T})} = 1\), such that

\[
|(H(\tau)u, v)| \geq C|\tau|, \quad |\tau| > \tau_0.
\]

Let \(H_0(\tau) = \Phi_0(\tau)|H_0(\tau)|\) be the polar decomposition of \(H_0(\tau)\). We set

\[
v = \Phi_0(\tau)u.
\]

Then,

\[
(H_0(\tau)u, v) = (|H_0(\tau)|u, u) \geq 2\pi|\tau|
\]

by (4.1), and

\[
(H_0(\tau)u, v) = ||H_0(\tau)|^{1/2}u||^2_{L_2(M \times \mathbb{T})} = ||H_0(\tau)|^{1/2}v||^2_{L_2(M \times \mathbb{T})}.
\]

Let us estimate the term \((Vu, v)\) using Lemma 4.3 and Theorem 4.2:

\[
|(Vu, v)| \leq \delta \|u\|_{L_q(M \times \mathbb{T})}\|v\|_{L_q(M \times \mathbb{T})} + c(\delta) \leq C\delta ||H_0(\tau)|^{1/2}v||_{L_2(M \times \mathbb{T})}\|H_0(\tau)|^{1/2}u||_{L_2(M \times \mathbb{T})} + c(\delta) = C\delta (H_0(\tau)u, v) + c(\delta).
\]

This implies

\[
|(H(\tau)u, v)| \geq (1 - C\delta)(H_0(\tau)u, v) - c(\delta) \geq 2\pi(1 - C\delta)|\tau| - c(\delta) \geq C_1|\tau| \quad \text{for} \quad |\tau| > \tau_0, \delta < 1/C.
\]
Proof of Theorem 2.4, the case of a manifold without boundary.

If \( \partial M = \emptyset \), then Corollary 3.2 implies \( A(q) \) for all \( q < 2d/(d-2) \). From Theorem 4.4, we get (2.6) for any \( p > d/2 \). \( \blacksquare \)

Proof of Theorem 2.4, the case of Dirichlet or Neumann boundary conditions.

If \( k = 1 \) (\( M \) is a line segment), then Theorem 3.3 again yields \( A(q) \) for all \( q < 2d/(d-2) \). And all \( p > d/2 \) are suitable.

If \( d = 3 \), then Theorem 3.4 gives \( A(q) \) only if \( q < 6 \), so we need \( p > 3/2 \).

If \( d > 4 \), then, again by Theorem 3.4, Condition \( A(q) \) holds for \( q < (2d-4)/(d-3) \), and the corresponding condition on \( V \) is \( V \in L_p(M \times \Omega) \), where \( p > d-2 \). \( \blacksquare \)

To study the third type boundary condition, we use the following result from [13].

**Theorem 4.5.** Let \( k = 1 \), \( M = [0, a] \), and assume that \( \sigma \) satisfies (2.4). Then

\[
\int_{\Omega} |\sigma(0, y)| \left| \left( |H_0(\tau)|^{-1/2} u \right)(0, y) \right|^2 \, dy \\
+ \int_{\Omega} |\sigma(a, y)| \left| \left( |H_0(\tau)|^{-1/2} u \right)(a, y) \right|^2 \, dy \leq \tilde{c}(\tau) \| u \|^2_{L_2([0, a] \times \Omega)},
\]

where \( \lim_{|\tau| \to \infty} \tilde{c}(\tau) = 0 \) uniformly over \( \xi' \) and \( u \in L_2([0, a] \times \Omega) \).

Proof of Theorem 2.4, the case of the third type boundary condition.

Let \( p > d/2 \), \( q = 2p' < 2d/(d-2) \). Theorem 3.3 guaranties \( A(q) \). Let \( V \in L_p([0, a] \times \Omega) \).

For an arbitrary \( u \in \text{Dom}(H_\sigma) \), \( \| u \|_{L_2([0, a] \times \Omega)} = 1 \), let \( v \) be defined by (4.5). Then

\[
(H_\sigma(\tau)u, v) = (H_0(\tau)u, v) + (Vu, v) + \int_{\Omega} \sigma(a, y)u(a, y)v(a, y) \, dy - \int_{\Omega} \sigma(0, y)u(0, y)v(0, y) \, dy.
\]

The first two terms are estimated in (4.7) and (4.6). Let us estimate the last one (the same can be done for the remaining term). Theorem 4.5 gives

\[
\left| \int_{\Omega} \sigma(0, y)u(0, y)v(0, y) \, dy \right| \leq \frac{1}{2} \int_{\Omega} |\sigma(0, y)| \left( |u(0, y)|^2 + |v(0, y)|^2 \right) \, dy \\
\leq \frac{\tilde{c}(\tau)}{2} \left( \| H_0(\tau) \|_{L_2([0, a] \times \Omega)}^{1/2} \| u \|^2_{L_2([0, a] \times \Omega)} + \| H_0(\tau) \|_{L_2([0, a] \times \Omega)}^{1/2} \| v \|^2_{L_2([0, a] \times \Omega)} \right) = \tilde{c}(\tau) (H_0(\tau)u, v).
\]

Hence,

\[
|(H_\sigma(\tau)u, v)| \geq (H_0(\tau)u, v) \left( 1 - C\delta - 2\tilde{c}(\tau) \right) - c(\delta) \geq 2\pi (1 - C\delta - 2\tilde{c}(\tau)) |\tau| - c(\delta), \quad |\tau| > \tau_0,
\]

where \( \delta \) and \( \tau_0 \) are chosen in such a way that \( C\delta + 2\tilde{c}(\tau) < 1, \ |\tau| > \tau_0 \). The last estimate implies (2.6). \( \blacksquare \)
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