Friedmann Cosmology and Almost Isotropy

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Abstract: In the Friedmann Model of the universe, cosmologists assume that spacelike slices of the universe are Riemannian manifolds of constant sectional curvature. This assumption is justified via Schur’s Theorem by stating that the spacelike universe is locally isotropic. Here we define a Riemannian manifold as almost locally isotropic in a sense which allows both weak gravitational lensing in all directions and strong gravitational lensing in localized angular regions at most points. We then prove that such a manifold is Gromov-Hausdorff close to a length space \( Y \) which is a collection of space forms joined at discrete points. Within the paper we define a concept we call an “exponential length space” and prove that if such a space is locally isotropic then it is a space form.

1 Introduction

The Friedmann Model of the universe is a Lorentzian manifold satisfying Einstein’s equations which is assumed to be a warped product of a a space form with the real line. \( \text{Fra} \) \( \text{Peeb} \) \( \text{CW} \). Recall that a space form is a complete Riemannian manifold with constant sectional curvature. This assumption is “justified” in the references above by stating that the spacelike universe is locally isotropic:

Definition 1.1 A Riemannian manifold \( M \) is \( R \) locally isotropic if for all \( p \in M \) and for every element \( g \in SO(n, \mathbb{R}) \) there is an isometry between balls, \( f_g : B_p(R) \rightarrow B_p(R) \), such that \( f_g(p) = p \) and \( df_g = g : T_pM \rightarrow T_pM \).

Clearly if \( M \) is locally isotropic then its sectional curvature \( K_p(\sigma) \) depends only on \( p \) not the 2 plane \( \sigma \subset T_pM \), and, by Schur’s lemma, it has constant sectional curvature (c.f. \( \text{Fra} \) p.142).

Now the universe is not exactly locally isotropic and is only an approximately so. To deal with this, cosmologists test perturbations of the Friedmann model and look for measurable effects on light rays. The most popular perturbation is the Swiss Cheese model in which holes are cut out of the standard model and replaced with Schwarzschild solutions \( \text{Kan} \) \( \text{DyRd} \). The effects of these clumps of mass have been tested using random distribution \( \text{HoWa} \) and fractal distribution \( \text{GabLab} \) of the massive regions. However all these studies of possible cosmologies are making the assumption that the Friedmann model is stable in some sense.

It should be noted that Schur’s Lemma is not stable. Noncompact examples by Gribkov and compact examples by Currier show that Riemannian manifolds whose sectional curvature satisfies

\[
|K_p(\sigma) - K_p| < \epsilon \quad \forall \text{ 2 planes } \sigma \subset T_pM
\]

(1.1)
can still have

\[
\max_{p \in M} |K_p| - \min_{q \in M} |K_q| = 1,
\]

(1.2)

and thus do not have almost constant sectional curvature \( \text{Grib} \) \( \text{Cur} \). The only stability theorem for Schur’s Lemma has been proven by Nikolaev, and it makes an integral approximation on the pointwise sectional curvature variation \( \text{Nik} \). Furthermore before one could even apply Nikolaev’s

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Stability Theorem in our situation, one would need to investigate whether a space which is almost isotropic in some sense has almost constant sectional curvature at each point.

In this paper we show that the implication that a locally isotropic Riemannian manifold is a space form is stable with respect to the Gromov Hausdorff topology on Riemannian manifolds. This stability uses a definition of almost isotropy which the author has constructed to allow observed inhomogeneities in the universe including strong and weak gravitational lensing as long as the weak lensing is very weak and the strong lensing is localized in an angular sense. [Theorem 1.2] and Theorem 1.1. Furthermore the Swiss Cheese models of the universe are almost isotropic in the sense that will be used in this paper [Kan] [DyRo].

We will begin by providing a more angular rephrasing of the definition of local isotropy which is equivalent to Definition 1.1.

Definition 1.2 A Riemannian manifold $M$ is $R$ locally isotropic if for all $p \in M$ there is a radius $R_p > 0$ less than the injectivity radius at $p$ and a function $F_p : [0, \pi] \times [0, R) \times [0, R)$ such that for all unit vectors $v, w \in S^{n-1} \subset TM_p$ and for all $s, t \in (0, R)$ we have

$$d_M(\exp_p(sv), \exp_p(tw)) = F_p(d_S(v, w), s, t).$$

(1.3)

where $d_S(v, w)$ is the angle between $v$ and $w$. We will call $F_p$ the isotropy function about $p$ and $R_p$ the radius of isotropy at $p$. Furthermore $M$ is uniformly isotropic on a region $U$, if $F_p$ is constant for $p \in U$.

Note that as described above, a locally isotropic manifold is a space form. Thus $F_p$ must increase in its first variable and $F_p(\pi, t, t) = 2t$ for $t$ sufficiently small. While $F_p$ is not assumed to be constant here, by Schur’s Lemma, it must be constant and in fact

$$F_p(\theta, s, t) = F_K(\theta, s, t)$$

(1.4)

where $F_K(\theta, s, t)$ is the length of the third side of a triangle with angle $\theta$ between sides of lengths $s$ and $t$ in $\mathbb{H}^n$, $S^n$ or $\mathbb{E}^n$ of constant sectional curvature $K$. This is a well known function, e.g.

$$F_0(\theta, s, t) = s^2 + t^2 - 2st \cos \theta.$$ 

(1.5)

We have now defined local isotropy as a property of geodesics emanating from a point, a property that can be measured astronomically if one assumes that light travels along spacial geodesics. This is true if we have given the spacelike slice of the universe the Fermat metric (which incidentally is proportional to the restricted metric in the Friedmann model) c.f. pages 90-92 and 141-143 in Fra.

In the following definition we approximate local stability in a sense which will allow weak gravitational lensing of some geodesics and strong gravitational lensing of those that enter a region $W \subset M$. Since geodesics entering $W$ behave unpredictably, we will restrict our relatively good behavior to geodesics emanating from $p$ outside a tubular neighborhood $T_\epsilon(W)$ of $W$.

Definition 1.3 Given $\epsilon < 1$, $R > 1$, a Riemannian manifold $M^n$ and a subset $W \subset M^n$, we say that $M^n$ is locally $(\epsilon, R)$-almost isotropic off of $W$ if for all $p \in M \setminus T_\epsilon(W)$, we have a set of tangent vectors

$$T_p = T_{p,W} = \{v \in B_0(R) \subset TM_p : \exp_p([0, 1])v \cap W = \emptyset\}$$

(1.6)

and a function $F_p : [0, \pi] \times [0, R) \times [0, R) \to [0, 2R)$ satisfying

(a) $F_p(\theta_1, a, b) < F_p(\theta_2, a, b) \ \forall \theta_1 < \theta_2$

(b) $F_p(\pi, t, t) > t$

(e) $F_p(0, 0, R) = R$

(1.7) 

(1.8) 

(1.9)
such that

$$d_M(\exp_p(v), \exp_p(w)) - F_p(d_S(v/|v|, w/|w|), |v|, |w|) < \epsilon \quad \forall v, w \in T_p.$$  \hspace{1cm} (1.10)

We will call $F_p$ the almost isotropy function about $p$ and $R$ the isotropy radius. We will say that $M^n$ is uniformly almost isotropic on a region $U$ if $F_p$ can be taken to be constant for $p \in U$. See Figure 1.

![Figure 1: We've filled $W \subset M^2$ in black and outlined $\exp_p(T_p)$.

This definition captures the concept that the universe looks almost the same in many directions as an angular view, $T_p$, but allows for some directions to be poorly behaved after they pass through a region, $W$, with strong gravitational lensing effects. Small gravitational lensing is absorbed in the flexibility of (1.10). Note that assumptions (a) (b) and (c) on $F$ all hold on isotropy functions. Condition (b) guarantees that there is a geodesic $\exp_p(tv)$ whose end points are at least a distance $R$ apart. This condition will replace the standard injectivity radius condition often imposed on Riemannian manifolds when studying their limits. By only requiring that $d_M(\exp_p(-Rv), \exp_p(Rv)) > R$ and not $= 2R$, we are not demanding that $p$ be a midpoint of any long minimizing geodesic as is the case with an injectivity radius bound. This is not a strong assumption to make for certainly it would seem that there should be two opposing directions in the sky that are far apart from each other.

We now wish to impose some restrictions on the size of the set $W$ where $M$ fails to be almost isotropic using an angular measurement. The idea we are trying to capture is that very few directions in the sky exhibit strong gravitational lensing.

**Definition 1.4** A subset $W$ of $M$ is $(\epsilon, R)$-almost unseen if for all $p \in M \setminus T_\epsilon(W)$, the set of directions from $p$ passing through $W \cap B_p(R)$,

$$S_p = \{v/|v| : v \in B_0(R) \setminus T_{p,W} \subset TM_p\},$$ \hspace{1cm} (1.11)

where $T_{p,W}$ was defined in (1.6), is contained in a disjoint set of balls,

$$S_p \subset \bigcup_{j=1}^N B_{w_j}(\epsilon_j) \subset S^{n-1} \subset TM_p$$ \hspace{1cm} (1.12)

where $B_{w_j}(3\epsilon_j)$ are disjoint and $\epsilon_j < \epsilon$. See Figure 2.
In our theorems we will also assume that \( W \subset \bigcup_k B_{q_k}(\epsilon) \) where \( d(q_k, q_j) > 2R \). In some sense this means we are making the assumption that the “black holes” are small (which they appear to be from a cosmic perspective) and are far between. Ones which are closer together can be fit in a common \( \epsilon \) ball. Spaces with thin wormholes that are long do not satisfy this condition. To allow for such spaces we can cut off the worm holes and smooth them out. In which case we are really only concerned with the universe on “our side” of the wormholes, our connected region.

We may now state the main theorem and then its cosmological implications. Here \( d_{GH}(X_1, X_2) \) denotes the Gromov-Hausdorff distance between \( X_1 \) and \( X_2 \). Section 2 contains a description of this distance between spaces. We will also use the notation \( \text{Ricci}(M^n) \geq (n-1)H \) to denote that the standard assumption that Ricci curvature is bounded below in the sense that

\[
\text{Ricci}_p(v,v) \geq (n-1)H g(v,v) \quad \forall p \in M^n, \forall v \in T M_p. \tag{1.13}
\]

**Theorem 1.1** Given \( H > 0, n \in \mathbb{N}, \bar{R}, R > 0, D > 0 \) and \( \delta > 0 \) there exists

\[
\epsilon = \epsilon(H, n, \bar{R}, R, D, \delta) > 0
\]

such that if \( \bar{B}_p(D) \subset M^n \) is a closed ball in a complete Riemannian manifold with the Ricci\((M^n) \geq (n-1)H \) such that \( M^n \) is locally \((\epsilon,R)\)-almost isotropic off of \( W \) where \( W \) is an \((\epsilon,R)\)-almost unseen set contained in uniformly disjoint balls,

\[
W \subset \bigcup_j B_{q_j}(\epsilon) \text{ where } B_{q_j}(\bar{R}) \text{ are disjoint}, \tag{1.15}
\]

then

\[
d_{GH}(B_p(D) \subset M^n, B_y(D) \subset Y) < \delta \tag{1.16}
\]

where \( Y \) is an \( n \) dimensional Riemannian manifold with constant sectional curvature \( K \geq H \) and injectivity radius greater than \( R \). Furthermore \( M^n \) is uniformly \((\epsilon+\delta)\)-almost isotropic on \( B_p(D) \),

\[
|F_q(\theta, s, t) - F_K(\theta, s, t)| < \delta \quad \forall q \in B_p(D),
\]

where \( F_K \) is the isotropy function of a simply connected space form of sectional curvature \( K \).
Figure 3: Here we see Weak Gravitational Lensing all over the bumpy almost isotropic $M_i$. There is Strong Gravitational Lensing at the necks of the $M_i$ which is restricted to the the subsets, $W_{M_i}$, marked in black. These $M_i$ converge in the Gromov Hausdorff sense to $Y$ with a single point in $W_Y$.

Note that if $M$ is compact then we can take $D$ to be the diameter and we need not deal with the closed balls. In this case $Y$ will be a compact space form. This does not require that $Y$ has positive curvature since it may be a torus or a compact quotient of hyperbolic space. If one further adds the condition that $M$ is a simply connected compact manifold, then $Y$ must be a sphere (see Remark 2.1).

If $M$ is noncompact, $F_q$ may change slowly from point to point in $M$ such that the $K$ in (1.17) depends on the ball. See Example 2.1. Nevertheless one can use (1.17) to control the growth of the change in $K$ because the same $\delta$ holds for all balls.

Cosmologically, Theorem 1.1 says that if a space has sufficiently small weak gravitational lensing and sufficiently localized strong gravitational lensing as viewed from most points in space and if the strong gravitational lensing is caused by regions which are contained in sufficiently small balls then in fact one can estimate the distances between stars whose light has not passed through regions of strong gravitational lensing using the standard formulas involving only the angle between them as viewed from earth and the distance to the two stars (1.17). Of course one must estimate $K$ as usual, but this can be done using astronomical data measured from earth, and then the same $K$ can be used in all directions and from any basepoint (not just earth).

The first equation (1.16) is a bit more complicated to describe quickly other than to say in some rough, not smooth sense the space is close to a space of constant curvature. A discussion of the Gromov-Hausdorff distance, denoted $d_{GH}$ can be found in Section 2 and a good reference is [BBI].

The following immediate corollary of Anderson’s Smooth Convergence Theorem clarifies this closeness if one adds an additional upper Ricci curvature bound [And].

**Corollary 1.1** Given $H > 0$, $n, \in \mathbb{N}$, $R > 0$, $D > 0$ $\delta > 0$ there exists $\epsilon = \epsilon(H, n, R, D, \delta) > 0$ such that if $B_p(D) \subset M^n$ is closed ball in a complete Einstein Riemannian manifold with $|\text{Ricci}| \leq H$ $\text{injrad} \geq \iota_0$ $\text{diam}(M^n) \leq D$ and $M^n$ is locally $(\epsilon, R)$-almost isotropic, then $B_p(D)$ is $C^\infty$ close to a ball in a compact space form.

This $C^\infty$ closeness allows one to study the properties of the universe using a smooth variation of the standard Friedmann model. That is, spaces which are $C^\infty$ close to a space form can be studied just by smoothly varying the metric on a space form and do not have possible additional topology as might occur in the case when it is only Gromov-Hausdorff close to a space form.

In some sense these two results are not as natural as one would hope because one would like to permit a sequence of Schwarzschild universes that do not approach a Riemannian manifold but
such that if where $Y > R$ dimension space form with injectivity radius $W$ been smoothly attached with bounded Ricci curvature. These black holes would then be included in of actual space in which all black holes have been cut out and replaced with Euclidean balls that have been almost isotropic and then Theorem and Corollary can be applied.

In the next theorem, we allow for an almost isotropic space which includes black holes and any other sort of region that is badly behaved without having to cut and paste the manifold. This has the advantage that one need not make any assumption on the local curvature of the space in these bad regions allowing for undiscovered phenomenon like short wormholes or networks of wormholes or any other distortion of space that is restricted to a collection of small balls. It also matches the conditions of the Swiss Cheese Models of the universe studied in [Kan] [DyRo].

We replace the Ricci bound in Theorem by a significantly more general ball packing assumption and obtain a slightly weaker result that neatly matches the idea of black holes joining pairs of universes.

**Definition 1.5** Given a Riemannian manifold and a map $f : (0, R) \times (0, \infty) \to \mathbb{N}$ we say that $M$ has the $f$ ball packing property if for any $s \in (0, R)$ and $t \in (0, \infty)$ the maximum number of disjoint balls of radius $s$ contained in a ball of radius $t$ is bounded by $f(s, t)$.

Note that Gromov’s compactness theorem says that a sequence of Riemannian manifolds has a converging subsequence iff there exists a function $f$ such that all manifolds satisfy the same $f$ ball packing property. When a Riemannian manifold has $\text{Ricci} \geq -(n-1)H$ then by the Bishop-Gromov Volume Comparison Theorem, $f$ is an explicit function of $H$ involving $\sinh \frac{1}{H}$. The limits of sequences of manifolds which satisfy the same $f$ ball packing property are not necessarily manifolds but are complete length spaces.

**Theorem 1.2** Given $n \in \mathbb{N}$, $\bar{R}, R > 0$, $D > 0$ $\delta > 0$ and a map $f : (0, R) \times (0, \infty) \to \mathbb{N}$ there exists $\epsilon = \epsilon(n, R, \bar{R}, D, \delta, f) > 0$ such that if $B_p(D) \subseteq M^n$ is a closed ball in a complete Riemannian manifold with the $f$ ball packing property such that $M^n$ is locally $(\epsilon, R)$-almost isotropic off an $(\epsilon, R)$-almost unseen set of the form described in [BiCr] then

$$d_{GH}(B_p(D) \subseteq M^n, B_{\bar{p}}(D) \subseteq Y) < \delta$$

(1.18)

where $Y$ is a complete length space with a subset $W_Y = \{y_1, y_2, \ldots\}$, such that $d_Y(y_j, y_k) \geq 2\bar{R}$ and such that if $Y'$ is a connected component of $Y \setminus W_Y$ then the closure $\text{Cl}(Y')$ is isometric to an $n$ dimensional space form with injectivity radius $R$.

Furthermore a connected region of $M \setminus W_M$, $M'$, is uniformly $(\epsilon+\delta)$-almost isotropic everywhere,

$$d_{GH}(B_p(D) \cap \text{Cl}(M'), B_{\bar{p}}(D) \cap \text{Cl}(Y')) < \delta.$$  

(1.19)

and

$$|F_q(\theta, s, t) - F_K(\theta, s, t)| < \delta \quad \forall q \in B_p(D) \cap M',$$

(1.20)

where $F_K$ is the isotropy function of the space form $\text{Cl}(Y')$ and $K$ depends on $Y'$. 

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The basic idea here is that the space \( Y \) is a space created by joining together space forms at single points. It is not a smooth manifold because at the points where the space forms are joined there is no local chart. Interestingly the space forms that are used to create \( Y \) do not need to have the same curvature: \( Y \) could be a sphere joined to a plane at a point. It is possible that more than two space forms are joined at a single point and that the space forms can be joined at multiple points. However only finitely many may meet at any given, as can be seen since the limit space must be locally compact. It is also possible that \( Y \) is a hyperbolic space joined to countably many spheres at countably many points. It is possible that \( Y \) could be a single space form with two points that are set equal to one another like a gateway.

Cosmologically, one can think of the points as black holes or gates or some unknown phenomenon and the different space forms as being the universe as seen on various sides of these points.

An example of a sequence of \( M_i \) converging to \( Y \) with a single bad point \( W_Y \) is a sequence of rescalings of the Schwarzschild metric which converges to a pair of planes joined at a point. It is easy to see how to extend this example using cutting and paste techniques to give examples where \( Y \) is any space which is of the form described in Theorem 1.2. See Lemma 3.2, Corollary 3.3 and Example 3.1.

Note that the uniform almost isotropy achieved in (1.17) and (1.20) is not a consequence of the Gromov-Hausdorff closeness (1.16) and (1.18). These equations provide significant angular information about the Riemannian manifold while Gromov Hausdorff closeness can only be used to estimate distances. It is quite possible that two manifolds be very close in the Gromov Hausdorff sense and yet have very different formulas for the length of the third side of a triangle. Consider the surface of a smooth ball versus the surface of a golf ball and the wild behavior of geodesics on the latter. The proof (1.17) and (1.20) involves an extension of Grove Petersen’s Arzela Ascoli theorem \[GrPet\] and makes strong use of the almost isotropy condition.

We now present a quick survey of the contents of this paper pointing out key results which may be useful to mathematicians who study length spaces and non-Euclidean geometry. We also provide the definition of a space we call an **Exponential Length space** and relate it to the above project.

Section 2 has a review of Gromov Hausdorff theory. In it we make the usual conversion of Theorem 1.2 into a theorem regarding limits \( Y \) of sequences of Riemannian manifolds \( M_i \) which are locally \((\epsilon_i, R)\)-almost isotropic off \((\epsilon_i, R)\)-almost unseen sets \( W_{M_i} \) where \( \epsilon_i \to 0 \) [Theorems 2.1]. Ordinarily, even with an assumption of \( \text{Ricci} \geq (n-1)H \), such a limit space \( Y \) is a complete length space with no well defined exponential map. Gromov proved that between every pair of points in the limit space there is at least one length minimizing curve which achieves the distance between the points \[Gr\]. However, it is quite possible to have two length minimizing curves which overlap for some time and then diverge. This makes it very difficult to control what happens to the \((\epsilon_i, R)\)-almost isotropy in the limit process. To prepare for this difficulty, we extend the Grove-Petersen Arzela-Ascoli Theorem \[GrPet\] to a theorem concerning the limits of almost equicontinuous functions on converging spaces [Definition 2.5 and Theorem 2.3].

Section 3 has examples of Riemannian manifolds which are almost isotropic off almost unseen sets and contains a couple of technical lemmas regarding such Riemannian manifolds which are used in subsequent sections.

Section 4 studies the limiting behavior of the exponential maps of the sequences \( M_i \) proving Theorem 4.1. These limit exponential maps are not defined on a set \( W_Y \subset Y \) [Definition 4.1] but are homeomorphisms onto their images and describe length minimizing curves. Note that in this section no assumption is made that the almost unseen sets \( W_{M_i} \) need to be contained in unions of uniformly disjoint balls as in (1.15).

In Section 5 we add this last condition (e.g. (1.15) with \( \epsilon = \epsilon_i \)) on the \( W_{M_i} \) and use it to prove that \( W_Y \) is discrete [Lemma 5.1]. We then show that the exponential maps constructed on \( Y \) are locally surjective onto balls in \( Y \) as long as they avoid \( W_Y \).
At this point in the paper, enough properties of the limit space \( Y \) will have been proven to proceed and we will no longer need to refer to the sequence of \( M_i \). Thus the remaining sections will be written about complete length spaces that share these properties. We make the following definition.

**Definition 1.6** A complete length space \( Y \) is called an **exponential length space** off a set of points \( W_Y \), if there exists a continuous function \( R_y > 0 \) and an exponential dimension \( n \in \mathbb{N} \) such that for all \( y \in Y \setminus W_Y \), there is a continuous 1:1 function, \( \exp_y : B_0(r_y) \to B_y(r_y) \), and

\[
\exists r_y = d_Y(y, W_Y) \in (0, R_y) \text{ such that } \exp_y : B_0(r_y) \to B_y(r_y) \text{ is onto.} \tag{1.21}
\]

Furthermore for fixed \( v \in S^{n-1} \), \( \exp_y(tv) \) is a length minimizing curve for \( t \in [0, R_y) \).

If \( \inf_y (R_y) = R > 0 \) exists we call \( R \) the **exponential radius**.

It should be noted that there is no assumption that the exponential functions \( \exp_x \) vary continuously in the variable \( x \) and that this allows us to avoid the issues involved in defining a tangent bundle. Note that the exponential radius plays a role similar to the injectivity radius of a Riemannian manifold. Zhongmin Shen has informed the author that complete Finsler spaces are also exponential length spaces and in fact the exponential map is a \( C^1 \) diffeomorphism in that case. Example \[6.6\] is an exponential length space off a single point.

In Section \[6\] we prove a few lemmas concerning exponential length spaces \( Y \) off discrete subsets \( W_Y \) with positive exponential radius. In Lemma \[6.1\] we prove that if \( Y' \) is a connected component of \( Y \setminus W_Y \), then for any \( y \in Y' \) the exponential map, \( \exp_y \), is a homeomorphism from \( B_0(R) \) onto \( B_y(R) \cap Cl(Y') \). The proof of this theorem involves the Invariance of Domain Theorem, a strong topological result stating that a subset of \( \mathbb{R}^n \) which is homeomorphic to a ball in \( \mathbb{R}^n \) is an open subset of \( \mathbb{R}^n \) (c.f. \cite{EilSt}).

In Section \[7\] we add in the condition of local isotropy which was proven to hold on the limit spaces \( Y \) of the \( M_i \) in Theorem \[4.1\]. Once again we make a definition to describe such spaces.

**Definition 1.7** An exponential length space \( Y \) is **locally isotropic** off \( W_Y \) if it has exponential radius greater than \( R > 0 \) and for all \( x \in Y \setminus W_Y \) there is a function \( F_x : [0, \pi] \times [0, R] \times [0, R] \) which is continuous and satisfies

\[
a) \quad F_x(\theta_1, a, b) \leq F_x(\theta_2, a, b) \quad \forall \theta_1 < \theta_2 \tag{1.22}
b) \quad F_x(\pi, t, t) \geq t \tag{1.23}
c) \quad F_x(0, 0, R) = R \tag{1.24}
\]

such that

\[
d_Y(\exp_x(tv), \exp_x(sw)) = F_x(d_S(v, w), t, s) \quad \forall v, w \in S^{n-1}, \forall s, t \in [0, R). \tag{1.25}
\]

We will call \( R \) the **isotropy radius** and \( F_x \) the isotropy function about \( x \).

Note that \[(1.22) - (1.24)\] and \[(1.25)\] are the natural limits of \[(1.7) - (1.9)\] and \[(1.10)\] in the definition \[Def \[1.3\]\] of a locally \((\epsilon, R)\)-almost isotropic Riemannian manifold as \( \epsilon \) is taken to 0.

In Sections \[8\] through \[12\] we prove the following theorem and its corollary.

**Theorem 1.3** If \( Y \) is a locally isotropic exponential length space off of a discrete set \( W_Y \) then the closure of any connected component \( Y' \) of \( Y \setminus W_Y \) is a space form and its exponential structure matches that of the space form. In particular \( F_y(\theta, s, t) = F_K(\theta, s, t) \) for all \( y \in Y' \) where \( K \) is the sectional curvature of that space form.
**Corollary 1.2** If $Y$ is an exponential length space which is locally isotropic then it is a complete Riemannian manifold with constant sectional curvature.

Theorem 1.3 implies Theorem 1.2 when combined with the work in the earlier sections. The proof of Theorem 1.3 is broken into sections which focus on different properties of exponential length spaces. We hope that they will prove interesting to mathematicians who study length spaces and non-Euclidean geometry.

In Section 7 we first prove that a locally isotropic exponential length space is uniformly locally minimizing [Definition 7.2, Lemma 7.1]. That is there is a uniform distance such that pairs of points which are less than that distance apart have unique length minimizing curves running between them. We then construct local isometries between balls centered in $Y'$ [Lemma 7.4 and Corollary 7.7] and then show that $Cl(Y')$ is isometric to a locally isotropic exponential length space everywhere whose isotropy functions do not depend upon the base point [Lemma 7.8]. In some weak sense we now have that $Cl(Y')$ has constant sectional curvature but we do not yet have smoothness.

In Section 8 we prove that uniformly locally minimizing exponential length spaces are extended exponential length spaces [Definition 8.1]. That is, the exponential maps on these spaces are not just defined on balls but can be extended to maps on all of $\mathbb{R}^n$ with good properties. There is no assumption of local isotropy made in this section. To extend the exponential map continuously to all of $\mathbb{R}^n$ we introduce the concept of exponential curves [Definition 8.1, Lemma 8.3 and Lemma 8.4]. We show that in such spaces, all length minimizing curves are exponential curves [Lemma 8.2] and all exponential curves are locally length minimizing.

In Section 9 we study the concept of a conjugate point on an extended exponential length space [Definition 9.1 and Definition 9.2]. Through a series of lemmas, we show that, in such a space, if a ball has no conjugate points then it is mapped by the exponential map as a local homeomorphism [Theorem 9.1]. This extends the traditional theorem in Riemannian geometry which obtains a local diffeomorphism using the Inverse Function Theorem in regions without conjugate points. We apply the Invariance of Domain Theorem here to replace the Inverse function theorem because we do not yet have smoothness.

In Section 10 we reintroduce the hypothesis of local isotropy. We prove that we can extend isometries between subsets to isometries of balls [Lemma 10.2] and use them to prove that the distances between points on closely located pairs of exponential curves depends only on the angle between them [Lemma 10.4]. We then show that the distance to conjugate points is a constant on $Y$ [Lemma 10.6].

In Section 11 we add the condition that $Y$ is simply connected. We then prove $Y$ is homeomorphic to $\mathbb{R}^n$, $S^n$ or $H^n$ via the exponential map [Theorem 11.1]. In Lemma 11.8 we extend Lemma 7.4 to triangles of all sizes. Then we construct global isometries in Lemmas 11.1 and 11.2.

In Section 12 we complete the proof of Theorem 1.9. The key ingredient is Birkhoff’s Theorem which states that if a space $X$ has locally unique length minimizing curves and any isometry on subsets of $X$ extends to a global isometry then the space must be $S^n$, $H^n$ or $\mathbb{E}^n$ [Bir]. We then prove Lemma 12.2 matching the exponential structure of the length space to that of the space form and complete the proof of Theorem 2.1.

Finally in Section 13 we prove Theorem 1.1 by demonstrating that when a lower bound on Ricci curvature is assumed one cannot have a limit space which contains pairs of space forms joined at a point. The proof consists of a careful measurement of the volumes of balls using the Bishop-Gromov Volume Comparison Theorem. Recall that such a comparison holds on the limit space $Y$ by Colding’s Volume Convergence Theorem [Co].

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2 Limits

In this section we review the definition of Gromov-Hausdorff convergence, Gromov’s compactness theorems [Gr] and Grove-Petersen’s Arzela-Ascoli Theorem [GrPe] and some other key concepts. We extend this Arzela-Ascoli Theorem to functions which are only almost equicontinuous [Definition 2.5 and Theorem 2.3]. Finally we reduce Theorem 1.2 to the following theorem.

**Theorem 2.1** Given $n \in \mathbb{N}$, $\bar{R}, R > 0$, $\delta > 0$ and a map $f : (0, R) \times (0, \infty) \rightarrow \mathbb{N}$, $\epsilon_i \rightarrow 0$, if $M^\iota_i$ are complete Riemannian manifolds with the $f$ ball packing property such that $M^\iota_i$ is locally $(\epsilon_i, R)$-almost isotropic off an $(\epsilon_i, R)$-almost unseen set $W_{M_i}$ of the form described in (2.4) with $\epsilon = \epsilon_i$ respectively then a subsequence of the $M_i$ converges to a complete length space $Y$ with a countable collection of points $W_Y = \{y_1, y_2, \ldots\}$, such that $d_Y(y_i, y_k) \geq 2R$ and such that if $Y'$ is a connected component of $Y \backslash W_Y$ then $Cl(Y')$ is isometric to an $n$ dimensional space form.

Furthermore if $M'_i$ are connected regions of $M_i \backslash W_{M_i}$ containing points $q_i$ converging to $x \in Y$, then $GH \lim_{i \rightarrow \infty}(Cl(M'_i), q_i) = (Cl(Y'), x)$ For fixed $s, t \in [0, R)$, $\theta \in [0, \pi]$, the almost isotropy functions converge,

$$\lim_{i \rightarrow \infty} F_{q_i}(\theta, s, t) = F_K(\theta, s, t)$$

(2.1)

where $K$ is the sectional curvature of $Cl(Y')$.

**Remark 2.1** Note that in [SoWei], it is proven that if the $M_i$ are compact and simply connected, then the limit space $Y$ is its own universal cover. So in that case $Y'$ is not just a space form but it must be $S^2$, as the sphere is the only compact simply connected space form. Thus compact simply connected $M$ in Theorems 1.2 and 1.1 are close to spaces where $Y'$ and $Y$ are spheres respectively.

Those who are experts in this theory will immediately see why this theorem implies Theorem 1.1 and can skip ahead to the next section. The discussion of equicontinuity [Defn 2.5 and Thm 2.3] can be referred to as needed later on.

We now provide the necessary background for following the remainder of this paper. We recommend [BB1] as a reference for non-experts.

**Definition 2.1** A metric space is a set of points, $X$, and a distance function $d : X \times X \rightarrow [0, \infty)$ such that $d(x, y) = 0$ iff $x = y$, $d(x, y) = d(y, x)$ and $d(x, y) + d(y, z) = d(x, z)$ for all $x, y, z \in X$.

**Definition 2.2** A complete length space is a metric space such that between every pair of points there is a length minimizing (rectifiable) curve joining them whose length is the distance between the points.

Gromov’s definition of the Gromov-Hausdorff distance between two spaces involves infima of the Hausdorff distances between all possible embeddings of these spaces. We will not be using this definition but rather a very useful property of the Gromov-Hausdorff distance which relates the concept to maps between the two spaces.

**Definition 2.3** A function $\phi : X \rightarrow Y$ is an $\epsilon$-almost isometry if it is

$$\epsilon$$-almost distance preserving: $|d_Y(\phi(x_1), \phi(x_2)) - d_X(x_1, x_2)| < \epsilon$

(2.2)
and

\[ \epsilon\text{-almost onto } T_\epsilon(\phi(X)) \supset Y. \]  

(2.3)

Note that \( \phi \) need not be continuous.

Recall that two spaces are isometric when there is a map, called an isometry, between them which is 1:1, onto and distance preserving. Actually the fact any map that is distance preserving is 1:1 allows the two spaces to be shaped quite differently. For example \( X \) could be a circle and \( Y \) could be a thin torus and we would get an almost isometry from \( X \) to \( Y \) by embedding \( X \) in \( Y \) and an almost isometry from \( Y \) to \( X \) by mapping rings to single points. Notice that both the topology and the dimensions of \( X \) and \( Y \) are quite different.

The following lemma [c.f. [BBI]], will be used in place of a definition both when proving and when applying Theorem 1.2.

Lemma 2.2 Suppose \( X \) and \( Y \) are metric spaces then

\[ \text{if } d_{GH}(X, Y) < \epsilon \text{ then there is a } (2\epsilon)\text{-almost isometry from } X \text{ to } Y, \]  

(2.4)

and

\[ \text{if there is an } \epsilon\text{-almost isometry from } X \text{ to } Y \text{ then } d_{GH}(X, Y) < 2\epsilon. \]  

(2.5)

If it were not for the annoyance of the change of the \( \epsilon \) to \( 2\epsilon \), the existence of an almost isometry between 2 spaces would make a wonderful definition of Gromov-Hausdorff distance.

Cosmologically, Theorem 1.1 then says that we have an almost isometry between an almost isotropic manifold, \( M \) and a space form \( Y \). This means that distances between points can be estimated using an almost isometry to the space form. The actual almost isometry is not produced in this paper which makes this difficult to apply cosmologically. Nevertheless, there are implications. Space forms have lots of isometries. In fact all balls whose radius is less than the injectivity radius are isometric to each other. Using the almost isometry between \( M \) and \( Y \) we get the fact that all balls of this size are almost isometric in \( M \), that space looks pretty much the same from point to point. This is a stronger fact than (1.17) because some balls may contain components of \( W \) where there is strong gravitational lensing and now one can estimate the distances between stars which cannot see each other without passing through \( W \). In fact, one can use the region of space near earth as a sample ball (which does not contain any strong gravitational lensing) and then know that distant regions (even those containing strong gravitational lensing) are almost isometric. The actual black hole would have to be on the scale of the error, \( \epsilon \), in the almost isometry but on the cosmological scale things could be well understood. It is often assumed that regions around black holes look just like Euclidean space, here we have proven that they must be close to a space form but not necessarily in a smooth way. Theorem 1.2 essentially has the same result where we compare \( M' \) the connected component of \( M \) (the part of space which can be reached without passing through \( W \)) to a space form \( Y' \).

Once one has an understanding of Gromov-Hausdorff distance, one can define the convergence of metric spaces. That is metric spaces \( X_i \) converge to a metric space \( Y \) iff \( d_{GH}(X_i, Y) \) converges to 0. This definition is too restricted for applications with unbounded limit spaces so Gromov defined the following pointed Gromov Hausdorff convergence.

Definition 2.4 [Gromov] If each \( x_i \) is in a complete metric space \( X_i \), we say \((X_i, x_i)\) converges to \((X_0, x_0)\) in the pointed Gromov Hausdorff sense if for all \( D > 0 \) the closed balls \( B(x_i, R) \subset X_i \) converge in the Gromov Hausdorff sense to \( B(x_0, D) \subset X_0 \).

He then proved the Gromov Compactness Theorem:
Theorem 2.2 [Gromov] If $X_i$ are complete length spaces that satisfy a uniform $f$ ball packing condition, then for any $x_i \in X_i$ a subsequence of $(X_i, x_i)$ converges to a complete length space $(Y, y)$ in the pointed Gromov-Hausdorff sense. Conversely, if $(X_i, x_i)$ converge to a complete length space $(Y, y)$ then they satisfy a uniform $f$ ball packing condition.

As a consequence any sequence of complete Riemannian manifolds with a uniform lower bound on Ricci curvature converges to a complete length space. Note, however, that in general the limit space will not be a manifold. For example a sequence of hyperboloids can converge to a cone and a sequence of paraboloids to a half line.

Note also that the closeness in Gromov’s compactness theorem is on compact regions, not on the whole manifold at once. This is why there are balls $B_p(D)$ mentioned in the Theorem’s [12] and [11]. This condition is necessary in the statement of these theorems as can be seen in the following example.

Example 2.1 Suppose $M_2^2$ are warped product manifolds with the metrics,

$$g = dt^2 + \sinh^2(K_s(t)t)g_0,$$

where $K_s(t)$ is increasing and smooth such that $K_s(t) = 1$ on $[0, 1]$ and $K_s(t) = 1+(\ln(t))^{1/3}(1/s)$ on $[2, e^{27s}]$ and $K_s(t) = 4+1/(se^{27s}+1)$ on $[e^{27s}+1, \infty)$. Note that this can be done smoothly keeping $K'(t)$ and $K''(t)$ both less than $5/s$ on $[0, 2]$ and both less than $5/(se^{27s}+1)$ on $[e^{27s}, e^{27s}+1]$.

On a 2 dimensional warped product, the sectional curvature for $q \in \partial B_p(t)$ is

$$\text{Sect}_q = -\frac{\partial^2}{\partial t^2}(\sinh(K_s(t)t)) - \frac{\partial}{\partial t}(\cosh(K_s(t)t)(K'_s(t)t + K_s(t)) - \frac{(\sinh(K_s(t)t))K'_s(t)t + K_s(t)^2}{\sinh(K_s(t)t)} - \frac{(\cosh(K_s(t)t)(K''_s(t)t + K'_s(t) + K'_s(t))}{\sinh(K_s(t)t)}) = -(K_s(t) - K'_s(t)t)^2 - (K''_s(t)t + K'_s(t) + K'_s(t))(\coth(K_s(t)t)) - (K_s(t))^2 + 2K_s(t)K'_s(t)t - (K'_s(t)t)^2 - (K''_s(t)t + 2K'_s(t))(\coth(K_s(t)t))].$$

Suppose we fix a number $R > 0$. For any $p \in M_2^2$, let $r = d(p, p_0)$ then $q \in B_p(R) \subset Ann_{p_0}(r - R, r + R)$, so $t = d(q, p_0) \in (r - R, r + R)$. Then the sectional curvature at $q$ is close to a constant $-\langle K_s\rangle^2$ for sufficiently large $s$ as follows:

$$|\text{Sect}_q + K_s(r)| \leq |K_s(t)|^2 - K_s(r)|^2| + |2K_s(t)K'_s(t)t| + |(K'_s(t)t)^2| + |K''_s(t)t| + |2K'_s(t)|$$

$$\leq 2 \max_{a \in [r - R, r + R]} (2K_s(a)K'_s(a)R) + |2K_s(t)K'_s(t)t| + |(K'_s(t)t)^2| + |K''_s(t)t| + |2K'_s(t)|,$$

which can be shown to be small by examining the following sets of cases:

First, we have

$$2K_s(a)K'_s(a)R = 0 \text{ for } a \in [0, 1],$$

$$2K_s(a)K'_s(a)R \leq 2(1 + (\ln(2))^{1/3}(1/s))(5/s)R \text{ for } a \in [1, 2],$$

$$2K_s(a)K'_s(a)R \leq |2(1 + (\ln(a))^{1/3}/s)((1/3)(\ln(a))^{-2/3}/(as))R|.$$
Then we bound

\[ 2K_s(t)K'_s(t)t + |(K'_s(t)t)^2 + |K''_s(t)t| + |2K'_s(t)| \leq 0 \text{ for } t \in [0,1], \]

\[ 2K_s(t)K'_s(t)t + |(K'_s(t)t)^2 + |K''_s(t)t| + |2K'_s(t)| \leq 2(1 + (Ln(2))^{1/3}(1/s))(5/s)2 + 100/s^2 + 10/s + 10/s \text{ for } t \in [1,2], \]

\[ 2K_s(t)K'_s(t)t + |(K'_s(t)t)^2 + |K''_s(t)t| + |2K'_s(t)| \leq 2(1 + (Ln(2))^{1/3}(1/s))(5/s)2 + 100/s^2 + 10/s + 10/s \text{ for } t \in [1,2], \]

\[ 2K_s(t)K'_s(t)t + |(K'_s(t)t)^2 + |K''_s(t)t| + |2K'_s(t)| \leq 2(1 + (Ln(2))^{1/3}(1/s))(5/s)2 + 100/s^2 + 10/s + 10/s \text{ for } t \in [1,2], \]

\[ 2K_s(t)K'_s(t)t + |(K'_s(t)t)^2 + |K''_s(t)t| + |2K'_s(t)| \leq 2(1 + (Ln(2))^{1/3}(1/s))(5/s)2 + 100/s^2 + 10/s + 10/s \text{ for } t \in [1,2], \]

\[ 2K_s(t)K'_s(t)t + |(K'_s(t)t)^2 + |K''_s(t)t| + |2K'_s(t)| \leq 0 \text{ for } t \in [e^{27s^3} + 1, \infty]. \]

That is, for all $R > 0$ and $\epsilon > 0$, there exists $s$ sufficiently large that $|sect_q - K_s(d(p,p_0))^2| < \epsilon$ for all $q,p \in M^n_s$ such that $d(p,q) < R$. Since the distance between geodesics emanating from $p$ can be estimated from above and below by integrating the curvature, this implies that there exists $s$ sufficiently large depending only on $R$ and $\epsilon$ such that

\[ |d_{M_s}(\exp_p(tv), \exp_p(sw)) - F_{K_s}(d(p,p_0))^2(d_S(v,w), t, s)| < \epsilon' \quad (2.7) \]

for all $t, s < R$, for all $p \in M^n_s$.

Since the $M_s$ also have curvature uniformly bounded below by $-25$, they satisfy the conditions of Theorem $2.1$ and so balls of a fixed radius $D$ approach a space form. However different balls in $M_s$ will approach different space forms. In particular, the ball near the center of $M_s$ is a space of constant curvature 1 while a ball far away from the center will have constant curvature greater than 16.

Now it is common to refer to the concept of points $p_i$ in the $X_i$ converging to a point $z$ in the limit $Y$. This is made rigorous if one uses the almost isometries $\phi_i : X_i \to Y$ from Lemma $2.2$. We first choose the isometries $\phi_i$ to fix a particular convergence onto the limit space. For example when a hyperboloid converges to a cone the $\phi_i$ can be rotated many different ways. We need to fix the $\phi_i$ to discuss particular points. Then we say $p_i$ converge to $z$ if $\phi_i(p_i)$ converge to $z$ as points in $Z$. Note that given any sequence of $p_i \in B(x_i, D)$ we know a subsequence converges because $B(y, D)$ is compact.

We can now prove that Theorem $2.1$ implies Theorem $1.2$.

**Proof:** Suppose on the contrary that Theorem $2.1$ is false for some $n \in \mathbb{N}$, $\tilde{R}, R > 0$, $D > 0, \delta > 0$ and a map $f : (0, R) \times (0, \infty) \to \mathbb{N}$. So there is a sequence of $\epsilon_i$ converging to 0 and a sequence
of manifolds $M_i$ which satisfy the $f$ ball packing property and are locally $(\epsilon_i, R)$-almost isotropic off $(\epsilon_i, R)$-almost unseen sets, $W_i$ of the form described in (1.15) with $\epsilon = \epsilon_i$ such that for all $i$, (1.18) and (1.20) don’t hold for any complete length space $Y$ of the form described in the theorem. However Theorem 2.1 states that they must converge in the pointed Gromov-Hausdorff sense to exactly such a space $Y$, which means that for $i$ sufficiently large, depending on $\delta$, we do in fact have a length space $Y$ satisfying (1.18) and (1.20) which contradicts the above. 

Example 2.2 The ball packing condition is necessary in Theorem 2.1 because one can take a sequence of hyperbolic manifolds with constant curvature $K$ and take $K$ to negative infinity. Each space is actually isotropic but the sequence does not have a converging subsequence approaching a length space $Y$.

It is an open question as to whether the ball packing condition is necessary in Theorem 1.2. To try to prove this theorem without the ball packing condition would involve adapting the Gromov Compactness Theorem to say something about sequences of manifolds which don’t have converging subsequences, a daunting task.

Clearly the first step towards proving Theorem 2.1 will be to apply Gromov’s Compactness Theorem to obtain a limit space $Y$. However, to study the isotropy on $Y$ we will need an exponential map. We will construct such an exponential map by taking the limit of a subsequence of the exponential maps defined on $M_i$.

There is already an extension of Arzela Ascoli Theorem to Gromov Hausdorff situations by Grove and Petersen [GrPet], which states that if a sequence of continuous functions $f_i : X_i \to Y_i$ are equicontinuous and $(X_i, x_i) \to (X, x)$ and $(Y_i, f(x_i)) \to (Y, y)$ in the pointed Gromov Hausdorff sense then a subsequence of the $f_i$ converge to a limit function $f : X \to Y$. This implies that curves which are parametrized by arclength converge and that length minimizing curves converge to length minimizing curves but it does not control the angular behavior of the exponential maps.

In general the exponential maps are not well controlled under Gromov-Hausdorff convergence. For example, take a length space $Y$ consisting of 3 line segments meeting at a point. Suppose we have a sequence of Riemannian surfaces $M_i$ which converge to a $Y$ shaped $Y$. Note that the exponential maps must converge to functions which are no longer injective and that there are minimizing curves which diverge from one another after initially overlapping. See Figure 4.

Figure 4: Pairs of geodesics running from $p_i$ to $x_i$ and from $p_i$ to $y_i$ converge to a pair of minimizing curves in $Y$ running from $p_\infty$ to $x_\infty$ and from $p_\infty$ to $y_\infty$. This limit pair start as an identical curve and then diverge.
In this paper we can use almost isotropy to control the exponential maps to some extent and this will allow us to create better exponential maps on our limit spaces. To do so we need the following more general theorem which will allow the maps $f_i$ not to be continuous.

**Definition 2.5** A sequence of functions between compact metric spaces, $f_i : X_i \to Y_i$, is said to be uniformly almost equicontinuous if there exists $\epsilon_i$ decreasing to 0 such that for all $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that

$$d_Y(f_i(x_1), f_i(x_2)) < \epsilon + \epsilon_i \text{ whenever } d_{X_i}(x_1, x_2) < \delta_\epsilon. \quad (2.8)$$

**Theorem 2.3** If $f_i : X_i \to Y_i$, is uniformly almost equicontinuous between complete length spaces $(X_i, x_i) \to (X, x)$ and $(Y_i, f_i(x_i)) \to (Y, y)$ converge in the Gromov Hausdorff sense where $X$ and $Y$ are compact, then a subsequence of the $f_i$ converge to a continuous limit function $f : X \to Y$.

**Proof:** As in Petersen, choose countable dense subsets $A_i = \{a_1^i, a_2^i, \ldots\} \subset X_i$, such that $a_j^i \to a_j \in X$ where $A = \{a_1, a_2, \ldots\}$ is dense in $X$. Then subsequences of $f_i(a_j^i)$ converge using the pointed convergence of the $Y_i$ and the precompactness of balls in the $Y_i$. So we can thus apply the standard diagonalization argument to get a subsequence of the $f_i$ which converges on these countable dense sets to some function $f : A \to Y$.

We need only show $f$ is continuous on $A$ and then we can extend it to a continuous function on $X$. For all $\epsilon > 0$ take $N$ sufficiently large that $\epsilon_N < \epsilon/2$ and $\delta < \delta_{\epsilon/2}$ so

$$d_Y(f_i(x_1), f_i(x_2)) < \epsilon \text{ whenever } d_{X_i}(x_1, x_2) < \delta, \quad i \geq N. \quad (2.9)$$

So now given $a_j, a_k \in A$ such that $d_X(a_j, a_k) < \delta/2$, taking $i \geq N$ sufficiently large that $d_{X_i}(a_j^i, a_k^i) < \delta$, so

$$d_Y(f_i(a_j^i), f_i(a_k^i)) < \epsilon. \quad (2.10)$$

Then taking $i \to \infty$ we get

$$d_Y(f(a_k), f(a_j)) < \epsilon. \quad (2.11)$$

\[ \square \]

3 Almost Isotropy off Almost Unseen Sets

In this section we provide some examples of Riemannian manifolds which are almost isotropic off almost unseen sets [Definition 1.4]. We also prove two technical lemmas regarding such Riemannian manifolds which will be needed later.

**Lemma 3.1** A ball of radius $r$ in a space form $N$ of constant sectional curvature $K$ will be $(\epsilon, R)$-almost unseen if

$$r < F_K(\epsilon, \epsilon, \epsilon)$$

where $F_K$ is the almost isotropy function of $N$ and $R < \text{ the injectivity radius of } N$.

**Proof:** If $p \in M_r \setminus T_r(B_q(r))$ then by our choice of $R$ and the symmetry of space forms $S_p$ is exactly one ball. Now let $v$ be such that $exp_p(d(p, q)v) = q$ and let $w \notin S_p$. Then

$$F_K(d_S(v, w), d(p, q), d(p, q)) \geq r, \quad (3.2)$$

and the radius, $\theta$, of $S_p$ must satisfy

$$F_K(\theta, d(p, q), d(p, q)) \geq r, \quad (3.3)$$

as well. So by (3.2),

$$F_K(\theta, \epsilon, \epsilon) < F_K(\theta, d(p, p_1), d(p, p_1)) = r < F_K(\epsilon, \epsilon, \epsilon), \quad (3.4)$$

which implies that $\theta < \epsilon$ by the monotonicity of $F$. \[ \square \]
We can now apply this lemma to construct examples of spaces with almost unseen Schwarzschild necks and interesting limit spaces $Y$.

**Lemma 3.2** For any fixed pair of space forms $N_1^3$ and $N_2^3$ and points $p_i \in N_i$ and given any $r$ sufficiently small, we can construct a Riemannian manifold $M_r$ which is isometric to $N_i \setminus B_{p_i}(r)$ on two regions and has a Schwarzschild neck of diameter less than $16(r + 4r^2)$ so that taking $r$ to zero we get a sequence of Riemannian manifolds satisfying the conditions of Theorem [4] which converge to a length space $Y = N_1 \cup N_2 / p_1 \sim p_2$.

**Proof:** Recall that on a ball of sufficiently small radius $r$ the metric in a space form $N_i$ may be described as a warped product metric $dt^2 + f_K(t)^2g_0$ where $t \in [0, r)$, $g_0$ is the standard metric on the sphere and $f_K$ is either $\sinh(\sqrt{-K}t)$, $t$ or $\sin(\sqrt{K}t)$.

Recall that the Schwarzschild metric on $\mathbb{R}^3 \setminus \{0\}$ can also be described with a warped product metric as

$$g_{sch} = (1 + \frac{m}{2t})^4 g_{Euch} = (1 + \frac{m}{2t})^4(dt^2 + t^2 g_0).$$

(c.f. [SchYau], [Br]) Setting $s = m^2/(4t)$, we see that we get an isometric inversion so that the Schwarzschild solution is asymptotically flat in both directions. In fact we can describe how to glue our Schwarzschild neck for $t \geq m/2$ into $N_1$ and then repeat the process to glue in $s \geq m/2$ into $N_2$.

Given $r > 0$ as above and such that $f_K(r) < 2r$, let $m_r < r$ such that $\sinh(\sqrt{-K}r) = 16r^2 < 16$. By Corollary 3.3, we can verify that $\sinh(\sqrt{-K}r)$ passes into $N_1 \setminus B_{p_i}(r)$ on two regions and has a Schwarzschild neck of diameter less than $16(r + 4r^2)$ so that taking $r$ to zero we get a sequence of Riemannian manifolds satisfying the conditions of Theorem 2.1 which converge to a length space $Y = N_1 \cup N_2 / p_1 \sim p_2$.

**Proof:** Recall that on a ball of sufficiently small radius $r$ the metric in a space form $N_i$ may be described as a warped product metric $dt^2 + f_K(t)^2g_0$ where $t \in [0, r)$, $g_0$ is the standard metric on the sphere and $f_K$ is either $\sinh(\sqrt{-K}t)$, $t$ or $\sin(\sqrt{K}t)$.

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Given $r > 0$ as above and such that $f_K(r) < 2r$, let $m_r < r$ such that $\sinh(\sqrt{-K}r) = 16r^2 < 16$. By Corollary 3.3, we can verify that $\sinh(\sqrt{-K}r)$ passes into $N_1 \setminus B_{p_i}(r)$ on two regions and has a Schwarzschild neck of diameter less than $16(r + 4r^2)$ so that taking $r$ to zero we get a sequence of Riemannian manifolds satisfying the conditions of Theorem 2.1 which converge to a length space $Y = N_1 \cup N_2 / p_1 \sim p_2$.

**Proof:** Recall that on a ball of sufficiently small radius $r$ the metric in a space form $N_i$ may be described as a warped product metric $dt^2 + f_K(t)^2g_0$ where $t \in [0, r)$, $g_0$ is the standard metric on the sphere and $f_K$ is either $\sinh(\sqrt{-K}t)$, $t$ or $\sin(\sqrt{K}t)$.

Recall that the Schwarzschild metric on $\mathbb{R}^3 \setminus \{0\}$ can also be described with a warped product metric as

$$g_{sch} = (1 + \frac{m}{2t})^4 g_{Euch} = (1 + \frac{m}{2t})^4(dt^2 + t^2 g_0).$$

(c.f. [SchYau], [Br]) Setting $s = m^2/(4t)$, we see that we get an isometric inversion so that the Schwarzschild solution is asymptotically flat in both directions. In fact we can describe how to glue our Schwarzschild neck for $t \geq m/2$ into $N_1$ and then repeat the process to glue in $s \geq m/2$ into $N_2$.

Given $r > 0$ as above and such that $f_K(r) < 2r$, let $m_r < r$ such that $\sinh(\sqrt{-K}r) = 16r^2 < 16$. By Corollary 3.3, we can verify that $\sinh(\sqrt{-K}r)$ passes into $N_1 \setminus B_{p_i}(r)$ on two regions and has a Schwarzschild neck of diameter less than $16(r + 4r^2)$ so that taking $r$ to zero we get a sequence of Riemannian manifolds satisfying the conditions of Theorem 2.1 which converge to a length space $Y = N_1 \cup N_2 / p_1 \sim p_2$.

**Proof:** Recall that on a ball of sufficiently small radius $r$ the metric in a space form $N_i$ may be described as a warped product metric $dt^2 + f_K(t)^2g_0$ where $t \in [0, r)$, $g_0$ is the standard metric on the sphere and $f_K$ is either $\sinh(\sqrt{-K}t)$, $t$ or $\sin(\sqrt{K}t)$.

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Given $r > 0$ as above and such that $f_K(r) < 2r$, let $m_r < r$ such that $\sinh(\sqrt{-K}r) = 16r^2 < 16$. By Corollary 3.3, we can verify that $\sinh(\sqrt{-K}r)$ passes into $N_1 \setminus B_{p_i}(r)$ on two regions and has a Schwarzschild neck of diameter less than $16(r + 4r^2)$ so that taking $r$ to zero we get a sequence of Riemannian manifolds satisfying the conditions of Theorem 2.1 which converge to a length space $Y = N_1 \cup N_2 / p_1 \sim p_2$.
construct a sequence of smooth Riemannian manifolds $M_r$ (locally as above) satisfying the conditions of Theorem 2.1 which converge to the space

$$Y = \bigcup N_i / \pi_{i,k} P(p_{i,k})$$

(3.10)

This includes the possibility of a single $N_i$ with an even number of points.

**Example 3.1** In order to construct limit spaces $Y$ with many space forms meeting at a single point we don’t use the Schwarzschild metric as a model. Instead we can take any manifold with an arbitrary number of asymptotically flat ends and cut it off and rescale it down appropriately to glue it into a collection of small balls.

We now prove some technical lemmas regarding almost unseen sets.

We begin by noting that most directions in terms of the volume of $S^{n-1}$ behave in an almost isotropic manner when a space is almost isotropic off of an almost unseen set (recall Definition 1.4).

**Lemma 3.4** Given $n \in \mathbb{N}, R > 0$, and a map $f : (0, R) \times (0, \infty) \to \mathbb{N}$. For all $h > 0$, and for all

$$\epsilon \in (0, \min\{\frac{h}{2}, \frac{\pi}{4f(h/2, R + h/2)}\})$$

(3.11)

there exists

$$\theta(n, f, R, h) := \frac{\pi}{2f(h/2, R + h/2)}$$

(3.12)

such that if $M_n$ is $\epsilon, R$ almost isotropic off an $(\epsilon, R)$-almost unseen set and satisfies the $f$ ball packing property then for all $t \in [0, R)$,

$$F_q(\theta, t, t) < h \quad \forall t \in [0, R), \forall \theta < \theta(n, f, R, h).$$

(3.13)

Note that one cannot expect to control $F$ better than $\epsilon$ because its behavior is defined by $\epsilon$. Note also that the bound on $\theta$ does not depend on $\epsilon$.

**Proof:** Suppose on the contrary that

$$F_q(\theta, t, t) \geq h.$$  

(3.14)

Since $F$ is nondecreasing in $\theta$ and $M$ is almost isotropic this means that

$$d_M(\exp_q(tv), \exp_q(tw)) > h - \epsilon > h/2,$$

(3.15)

whenever $v, w \in S_q$ such that $d_S(v, w) \geq \theta$. Recall that

$$S_q \subset \bigcup_{j=1}^N B_{w_j}(\epsilon_j) \subset S^{n-1} \subset TM_q$$

(3.16)

in Definition 1.4.

Now in $S^{n-1}$ with the standard metric $d_S$, there are at least

$$N_\theta = \frac{\pi}{(\theta + 2\epsilon)}$$

(3.17)

disjoint balls of radius $\theta/2 + \epsilon$. If any of these balls is centered in $S_q$, then it is centered in a ball $B_{w_j}(\epsilon_j)$, and so it contains a ball of radius $\theta/2 < \theta/2 + \epsilon - \epsilon_j$ centered in

$$B_{w_j}(3\epsilon_j) \setminus B_{w_j}(\epsilon_j) \subset S^{n-1} \setminus S_q.$$  

(3.18)

Let $v_1, v_2, \ldots v_{N_\theta}$ be the centers of these balls. Then by (3.15), $B_{\exp_q(tv_i)}(h/2)$ are disjoint as well and contained in $B_q(t + h/2) \subset B_q(R + h/2)$. But by the $f$ ball packing property there are at most $f(h/2, R + h/2)$ disjoint balls of radius $h/2$ in a ball of radius $R + h/2$. Thus $f(h/2, R + h/2) \geq N_\theta \geq \pi/(\theta + 2\epsilon)$, and

$$\theta \geq \frac{\pi}{f(h/2, R + h/2)} - 2\epsilon \geq \frac{\pi}{2f(h/2, R + h/2)}.$$  

(3.19)
4 Almost Isotropy and Exponential Maps

We begin with a definition.

**Definition 4.1** Let

\[ W_Y := \{ y \in Y : \text{there exist } x_i \in W_{M_i} \text{ converging to } y \}, \quad (4.1) \]

and let

\[ W_\infty := \{ y \in Y : \text{there does not exist } x_i \in M_i \setminus W_{M_i} \text{ converging to } y \} \subset W_Y. \quad (4.2) \]

Note that \( W_\infty \) are the points that cannot be examined using the almost isotropy properties of the \( M_i \). To prove Theorem 1.2 we will show \( W_Y \) is discrete using the fact that the \( W_{M_i} \) consist of uniformly disjoint balls \((1.15)\), however, in this section we will make no assumption on the \( W_{M_i} \) other than the fact that they are “almost unseen”. We prove the following theorem.

**Theorem 4.1** Let \( M_i \) be a sequence of Riemannian manifolds possibly with boundary that are \((\epsilon_i,R)\)-almost isotropic off an \((\epsilon_i,R)\)-almost unseen set \( W_{M_i} \) which includes the boundary of \( M_i \) if it exists. Suppose further that \( M_i, p_i \) converge to a complete length space \( Y, y \) in the pointed Gromov-Hausdorff sense.

For all \( x \in Y \setminus W_\infty \), there is a continuous map \( \exp_x : B_0(R) \subset \mathbb{R}^n \to B_y(R) \subset Y \) which is a homeomorphism onto its image \( \exp_x(B_0(R)) \). Furthermore, the curves \( \exp_x(tv) \) for \( t \in [0,R] \) are length minimizing. Furthermore there is isotropy in the sense that there exist functions \( F_x : [0,\pi] \times [0,R) \to [0,2R) \) satisfying \((1.22)-1.24\) and \((1.24)\).

Note that \((1.22)-1.24\) are just the natural limits of \((1.7)-1.9\) of Definition 1.3.

In general the exponential map won’t be surjective, as can be seen in the case where \( Y \) is a sphere and a plane joined at a point. In that case \( \exp_x \) will only map onto the intersection of \( B_x(R) \) with the plane containing \( x \).

We prove this theorem through a series of lemmas.

One of the special properties of Gromov-Hausdorff Convergence is that if we have a sequence of curves, \( C_i : [0,L_i] \to M_i \), parametrized by arclength with \( L_i \leq L \), then a subsequence has a limit \( C_\infty : [0,L_\infty] \to Y \) which is a curve parametrized by arclength (although \( L_\infty = \lim_i L_i \) might be 0). This follows from the generalized Arzela Ascoli Theorem [GrPet]. Furthermore if the \( C_i \) are length minimizing, so is \( C_\infty \).

This allows us to make the following definition.

**Definition 4.2** Let \( v_i \in S^{n-1} \setminus S_q \), such that the curves \( \exp_{q_i}(tv_i) \) for \( t \in [0,R] \) converge to a limit curve, then \( C_{\{v_i\}} : [0,R] \to Y \) is their limit curve.

There is no natural relationship between the \( v_i \) from the different tangent cones \( TM_{q_i} \). For this reason we fix an identification between all the \( TM_{q_i} \). Each identification is determined only up to \( SO(n-1) \) but we need to make a choice. Thus all the \( S^{n-1} \setminus S_q \) can be thought of as subsets of the same \( S^{n-1} \). By Definition 1.3 it is easy to see that the \( S^{n-1} \setminus S_q \) converge to this \( S^{n-1} \).

**Lemma 4.1** Suppose \( v \in S^{n-1} \) and \( v_i, w_i \in S^{n-1} \setminus S_{q_i} \) are both sequences converging to \( v \), such that the curves \( \exp_{q_i}(tv_i) \) converge to a limit curve \( C_{\{v_i\}}(t) \), then \( \exp_{q_i}(tv_i) \) also converges to the same limit curve without having to take a subsequence. In particular

\[
\limsup_{i \to \infty} F_{q_i}(\theta_i,t,t) = 0 \text{ if } \theta_i \to 0. \quad (4.3)
\]
Proof: By the \((\varepsilon_i, R)\)-almost isotropy
\[
d_M(\exp_{q_i}(tv_i), \exp_{q_i}(tw_i)) < F_{q_i}(d_S(v_i, w_i), t, t) + \varepsilon_i \quad \forall t \in [0, R].
\] (4.4)

Now we know a subsequence of \(\exp_{q_{i_j}}(tw_{i_j})\) must converge to a limit curve \(C_{(w_{i_j})}\). We need only show that \(C_{(w_{i_j})} = C_{(v_i)}\).

Using these exponential maps.

It is known to be surjective. It is also possible that this exponential map depends on the choice of \(q\) \(\in \exp\map in the sense that \(q\) \(\in \) are uniformly almost equicontinuous for all \(i\).

We can now apply Theorem 2.3 to our exponential functions.

Lemma 4.2 If we assume that \(M_i \to Y\) are locally \((\varepsilon_i, R)\)-almost isotropic off sets \(W_{M_i}\) which are \((\varepsilon_i, R)\)-almost unseen and \(\epsilon_i\) converges to 0, then we can show that the maps,
\[
\exp_{q_i} : [0, R] \times (S^{n-1} \setminus S_{q_i}) \to B_{q_i}(R),
\] (4.6)
are uniformly almost equicontinuous for all \(q_i \in M_i \setminus W_{M_i}\). Thus for any \(x \in Y \setminus W_\infty\), there is a subsequence of the \(i\) with a continuous limit map \(\exp_x : [0, R] \times S^{n-1} \to B_y(R)\) such that \(\exp_x(0v) = x\) for all \(v \in S^{n-1}\) and,
\[
d_Y(\exp_x(av), \exp_x(bv)) \leq |b - a|.
\] (4.7)

It should be noted that at this stage the limit exponential map is not necessarily an exponential map in the sense that \(\exp_x(tv)\) is a minimizing curve parametrized proportional to arclength. Nor is it known to be surjective. It is also possible that this exponential map depends on the choice of the sequence of \(q_i \in M_i\) converging to \(x \in Y\). Nevertheless we can set up a local isotropy of sorts using these exponential maps.

Proof: Let \(f_i : [0, R] \times (S^{n-1} \setminus S_{p_i}) \to M_i\) be defined \(f_i(s, v) = \exp_{p_i}(sv)\). See Figure 5.

![Figure 5](image)

Figure 5: Here we see \(f_i([0, R] \times (S^{n-1} \setminus S_{p_i})) \subset M_i\), each looking like a disk with a wedge removed to avoid \(W_{M_i}\), converging to a limit \(\exp_x(B_0(R)) \subset Y\), which is a disk in a plane but not a ball in \(Y\).

By the \((\varepsilon_i, R)\)-almost unseen property, \(([0, R] \times (S^{n-1} \setminus S_{q_i}), (0, v_0))\) converges to \(([0, R] \times S^{n-1}, (0, v_0))\) in the pointed Gromov Hausdorff sense. Furthermore \(f_i(0, v_0) = q_i\) and we are already given \((M_i, p_i) \to (Y, y), q_i \to x\) so \((M_i, q_i)\) converges to \((Y, x)\). So we need only verify that \(f_i\) are uniformly equicontinuous and can use the functions \(F_i\) of the almost isotropy to do so.
Given any $h > 0$, let $\theta < \theta(n, f, R, h/2)$ of Lemma 4.3, so for $i$ sufficiently large that
\[ \epsilon_i < \min \left\{ \frac{h}{4}, \frac{\pi}{4(f(R + h/4))} \right\} \]  
we have
\[ F_q(\theta, t, t) < h/2 \quad \forall t \in [0, R). \]  
So
\[ d_M(f_i(s_1, v_1), f_i(s_2, v_2)) \leq d_M(f_i(s_1, v_1), f_i(s_1, v_2)) + d_M(f_i(s_1, v_2), f_i(s_2, v_2)) \]  
(4.10)
\[ < F_i(d_S(v_1, v_2), s_1, s_1) + (\epsilon_i + |s_1 - s_2|) \]  
(4.11)
\[ < h/2 + \epsilon_i + h/2, \]  
(4.12)  
as long as $d_S(v_1, v_2) < \theta$ and $|s_1 - s_2| < h/2$.

Thus we are uniformly almost equicontinuous and the rest follows from Lemma 4.2.

**Lemma 4.3** For $Y$ as in Lemma 4.2, there are continuous functions $F_x : [0, R] \times S^{n-1} \to R$ defined at each point $x \in Y \setminus W_\infty$, such that $F$ satisfies conditions 1.22-1.24 and
\[ F_x(d_S(v, w), t, s) = d_Y(exp_x(tv), exp_x(sw)) = \lim_{i \to \infty} F_{q_i}(d_S(v_i, w_i), t, s) \]  
(4.13)
for any $v, w$ in $S^{n-1}$. Here $v_i, w_i \in S^{n-1} \times S_q$ converge to $v$ and $w$ respectively and $exp_x$ and $F_x$ are defined using the same sequence of $q_i$.

**Proof:** First let $q_i \in M_i \setminus W_{M_i}$ converge to $x$ and let $v_i \to v$ and $w_i \to w$. Then by Lemma 4.2, $exp_{q_i}(tv_i) \to exp_x(tv)$ and so
\[ d_Y(exp_x(tv), exp_x(sw)) = \lim_{i \to \infty} d_M(exp_{q_i}(tv_i), exp_{q_i}(sw_i)) = \lim_{i \to \infty} F_{q_i}(d_S(v_i, w_i), t, s). \]  
(4.14)  
In particular the limit on the right hand side exists. However this limit clearly depends only on the angle and the lengths, so
\[ d_Y(exp_x(tv), exp_x(sw)) = F_x(d_S(v, w), t, s). \]  
(4.15)
Furthermore, since $F_{q_i}$ satisfy (4.11)-(4.13) of the definition of almost isotropy, $F_x$ satisfies 1.22-1.24.

**Lemma 4.4** For $Y$ and $exp_y$ as in Lemma 4.2, $d(exp_y(tv), y) = t$ for all $t \in [0, R)$ and so $exp_y(tv)$ is a minimizing curve parametrized proportional to arclength.

Note the geodesics in the $M_i$ were not assumed to be length minimizing but that the almost isotropy implies that they are almost length minimizing.

**Proof:** Fix $v \in S^n$ and $t \leq R$ and let $z = exp_y(tv)$. Then there exists $v_i \to v$, such that $z_i = exp_{q_i}(tv_i) \in B_{q_i}(R)$ converge to $z$ by Lemma 4.2. First note that,
\[ d_Y(y, z) = \lim_{i \to \infty} d_M(q_i, z_i) \leq t. \]  
(4.16)  
On the other hand by the triangle inequality and (c) in the definition of almost isotropy,
\[ d_Y(y, z) = \lim_{i \to \infty} d_M(q_i, z_i) \geq \lim_{i \to \infty} d_M(exp_{q_i}(Rtv_i), q_i) - (R - t) \]  
(4.17)
\[ \geq \lim_{i \to \infty} F_{q_i}(0, 0, R) - (R - t) = R - R + t = t. \]  
(4.18)
We also get some interesting properties from the triangle inequality.

**Lemma 4.5** \( F_x(\theta_1, t_1, s) + F_x(\theta_2, s, t_2) \geq F_x(\theta_1 + \theta_2, t_1, t_2) \)

**Proof:** Just apply the triangle inequality to \( \exp_y(t_1v_1) \), \( \exp_y(t_2v_2) \) and \( \exp_y(sw) \) where \( d_S(v_1, w) = \theta_i \). □

**Corollary 4.6** \( F_x(\pi/k, t, t) > t/k \).

**Proof:** Apply Lemma 4.5 repeatedly, so that \( kF_x(\pi/k, t, t) \geq F_x(\pi, t, t) \) and then apply property (b) of Lemma 4.3. □

Although we have defined \( \exp_x \) as a function of two variables, a length and a unit vector, we know \( \exp_x(0v) = \exp_x(0w) \) for all \( v \) and \( w \), so we can consider it as a function of \( \mathbb{R}^n \).

**Lemma 4.7** For fixed \( x \in Y \), \( \exp_x : B_0(R) \to B_x(R) \) is a one to one map.

**Proof:** Suppose not, then there exists \( v, w \in S^{n-1} \) and \( t, s \in (0, R] \) such that \( \exp_x(tv) = \exp_x(sw) \). By Lemma 4.3, \( t = s \). So by Lemma 4.3 we have \( F_x(\theta, t, t) = 0 \) for some \( \theta > 0 \). Since \( F_x \) is nondecreasing in its first variable and there exists \( k \) sufficiently large that \( \theta > \pi/k \) we have \( F_x(\pi/k, t, t) = 0 \). This contradicts Corollary 4.6. □

Putting these lemmas together we have Theorem 4.1.

We need only show that \( W_\infty \) is discrete to prove that \( Y \) is an exponential length space. To prove this we need additional conditions on the sequence \( M_i \). This can be seen because the \( M_i \) could be a pair of planes which are connected by Schwarzschild solutions at an increasingly dense set of points and still satisfy all the conditions used in this section. In the next two sections we show how additional conditions can be found and satisfied.

## 5 Local Surjectivity

In this section we use the condition that the bad sets \( W_{M_i} \), which are avoided in the definition of the almost isotropy are each contained in a union of balls of decreasing radii that are a uniform distance apart (1.16).

Recall the definitions of \( W_\infty \subset W_Y \) in Definition 4.1 and the exponential map defined in Theorem 4.1.

**Lemma 5.1** If \( M_i \) converge to a space \( Y \) in the Gromov Hausdorff sense and subsets \( W_{M_i} \) satisfy (1.12), then \( W_Y \) is a countable collection of points \( \{y_j\} \) such that \( D_Y(y_j, y_k) \geq 2R \) and so \( W_\infty \) is an empty set.

**Proof:** Given \( y_1, y_2 \in W_\infty \). By the definition of \( W_\infty \), we know there exists \( x_i \to y_1 \) and \( z_i \to y_2 \) where \( x_i, z_i \in W_{M_i} \). Since the radius of the balls in \( W_{M_i} \) decreases to 0, but \( d_{M_i}(x_i, z_i) \to d_Y(y_1, y_2) > 0 \), eventually \( x_i \) and \( z_i \) will be in distinct balls, and thus \( d_{M_i}(x_i, z_i) \geq 2R \) and the lemma follows. □

**Lemma 5.2** Suppose \( M_i \to Y \) satisfy all the conditions of Theorem 4.1, then for all \( x \in Y \setminus W_Y \), there is an \( r_x > 0 \) such that \( \exp_x : B_0(r_x) \to B_x(r_x) \) is onto.

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Recall that in general it does not map onto $B_x(R)$ as seen in Figure 6. Note also that $\exp_x$ is not shown to map onto any balls about $x$ if $x$ is in $W_Y$.

**Proof:** First by Lemma 4.2, we know that for all $x \in Y \setminus W_Y \subset Y \setminus W_\infty$ we have an exponential function $\exp_x$ which is the limit of some selected subsequence $\exp_n$, restricted to almost isotropic directions where $q_i \in M_i$ converge to $x$.

To prove this lemma we first take $r_x = d_Y(x, W_\infty)/2 > 0$ since $W_\infty$ is discrete (by Lemma 5.1). Taking $q_i \in M_i$ as above converging to $x$ eventually $d_{M_i}(q_i, W_{M_i}) > r_x$ as well, so $tv, sw \in T_{q_i}$ of Definition 3.2. Then $\exp_n(tv)$ converges to a continuous function $\exp_n(tv)$, these limits must agree. So in fact for any $v_i$ converging to $v \in S^{n-1}$ and $t_i \to t$ in $[0, r_x)$, we have,

$$\exp_x(tv) = \lim_{i \to \infty} \exp_n(t_i v_i). \quad (5.2)$$

Now for any $z \in B_x(r_x)$ there exist

$$z_i \in B_{q_i}(r_x) \subset M_i \setminus W_{M_i} \quad (5.3)$$

converging to $z$ and there exist

$$v_i \in B_0(r_x) \cap T_{q_i} \text{ such that } \exp_{q_i}(v_i) = z_i, \quad (5.4)$$

so a subsequence of the $v_i$ converge to some $v \in B_0(r_x)$ such that $\exp_x(v) = z$.

**Example 5.1** Let $M_i$ be a pair of planes each with $i$ balls removed, $B_{(0, 1/k)}(1/(ik^2))$ where $k = 1, \ldots$. Replace the pairs of corresponding balls with smoothly attached Schwarzschild solutions as in Lemma 5.2. Then $M_i$ are locally $(\epsilon_i, 1)$-almost isotropic off $W_{M_i}$ equal to the collection of edited balls. These $W_{M_i}$ are $(\epsilon_i, 1)$-almost unseen but do not satisfy (1.13). The limit space $Y$ is a pair of planes joined at the points $(0, 1/k)$ where $k = 1, 2, \ldots$. This set of points is $W_\infty$. The limit exponential map based at $(0, 0)$ on one plane does not map onto any points in the other plane except for those in $W_\infty$, and so $\exp(0, 0)$ does not map onto any balls no matter how small.
Note that Lemmas 5.2 and Theorem 4.1 imply that the limit space $Y$ of $M_i$ satisfying the combined hypotheses is a locally isotropic exponential length space off of $W_Y$ where the exponential maps are chosen depending on the $p_i \in M_i$ converging to $x \in Y$ and the subsequences. Later we will show that in fact no subsequences or choices were necessary because the limit space will be a space form regardless of the choices made and the limit space is unique.

I conjecture that one could replace (1.15) with a lower bound on Ricci curvature. Thus far all arguments possibly leading to such a statement are lengthy, and so this question has not been pursued in this paper.

6 Exponential Length Spaces

This section focuses on exponential length spaces, $Y$, off a set of discrete points $W_Y$. Recall definitions 1.6 and 1.7 from the Introduction. To simplify notation, $Y’$ will be a connected component of $Y \setminus W_Y$ and its closure will be denoted $\text{Cl}(Y’)$. Ultimately we will prove Theorem 1.3 that $\text{Cl}(Y’)$ is a space form.

**Example 6.1** A sphere $S^2$ and a plane $\mathbb{E}^2$ joined at a point $p$ is an exponential length space off that point. The exponential function can be taken to agree with the exponential function of each space form and is not defined at the point $p$. Note that the image of $\exp_q$ for $q \in S^2$ is contained in $S^2$, so that in this sense the two components of the space with $p$ removed don’t “see” each other.

We now prove some lemmas about exponential length spaces that are not necessarily locally isotropic.

**Lemma 6.1** If $Y$ is an exponential length space off of a set $W_Y$ then for all $y \in Y \setminus W_Y$,

$$d_Y(\exp_y(sv), \exp_y(tw)) \geq |s - t| \quad \forall s, t \in [0, R], v, w \in S^{n-1}. \quad (6.1)$$

**Proof:** If not, there exists $s > t \in [0, R]$ and $v, w \in S^{n-1}$ such that

$$d_Y(\exp_y(sv), \exp_y(tw)) < s - t. \quad (6.2)$$

But then by the triangle inequality,

$$d_Y(\exp_y(sv), y) < s - t + d_Y(y, \exp_y(tw)) = s - t + t = s \quad (6.3)$$

which contradicts the length minimizing property of the exponential map. □

**Lemma 6.2** If $Y$ is an exponential length space off of a set $W_Y$ then for all $x \in Y \setminus W_Y$, $\exp_x$ is a homeomorphism from $B_0(R_x)$ to its image $\exp_x(B_0(R_x))$. In particular, if $q_i, q \in \exp_x(B_0(R_x))$ and $q_i$ converge to $q$ then

$$\lim_{i \to \infty} \exp_x^{-1}(q_i) = \exp_x^{-1}(q). \quad (6.4)$$

**Proof:** Since $\exp_x : B_0(R_x) \to \exp_x(B_0(R_x))$ is continuous and 1:1 and onto, we need only show that the inverse map is continuous in the sense described in 6.4. This convergence is the same whether it is defined in the relative topology of $\exp_x(B_0(R_x))$ or on $Y$ itself. Note that there exists an $\epsilon > 0$ such that $d_Y(q, x) < R_x - \epsilon$ so eventually the $d(q_i, x) < R_x - \epsilon/2$. Then $v_i = \exp_x^{-1}(q_i) \in B_0(R_x - \epsilon/2)$ have a converging subsequence to some $v_\infty \in B_0(R_x)$. By the continuity of $\exp_x$,

$$\exp_x(v_\infty) = \lim_{i \to \infty} \exp_x(v_i) = \lim_{i \to \infty} q_i = q. \quad (6.5)$$

So $v_\infty = \exp_x^{-1}(q)$ and this is true for any subsequence of the $v_i$. Thus the limit of the $v_i$ exists and we get (6.4). □
Note that the images under $\exp_x$ of open sets are relatively open in $\exp_x(B_0(R_x))$ but are not necessarily open. In Example [1.1] if $x \in \mathbb{S}^2$ is taken to be within distance $\pi/2$ of the plane, then $\exp_x(R_x)$ is contained completely in $\mathbb{S}^2$ and yet it contains the point in $W_Y$ which is not in its interior.

By the definition of an exponential length space, we know that $\exp_x$ maps onto $B_x(r_x)$ thus we have the following.

**Corollary 6.3** If $Y$ is an exponential length space off a set $W_Y$ then for all $x \in Y \setminus W_Y$, $\exp_x$ is a homeomorphism from $B_0(r_x)$ to $B_x(r_x)$.

The following Lemma now describes the whole image of the exponential map proving that the images under $\exp_x$ of open sets are indeed relatively open in $Y'$.

**Lemma 6.4** If $Y$ is an exponential length space off a discrete set $W_Y$, let $Y'$ be a connected comp of $Y \setminus W_Y$. Then for all $y \in Y'$, $\exp_y : B_0(R_y) \to B_y(R_y) \cap (\text{Cl}(Y'))$ is onto and is therefore a homeomorphism.

**Proof:** Let $z \in B_y(R) \cap Y'$. By the discreteness of $W_Y$ and connectedness of $Y'$, there is a continuous curve $c : [0, 1] \to B_y(R) \subset Y'$ which is not necessarily minimizing from $c(0) = y$ to $c(1) = z$. Let

$$T := c^{-1}(\exp_y(B_0(R_y))).$$

(6.6)

We need only show that $1 \in T$.

Let $t_0 = \sup T$. So there exists $t_i \in T$ such that $c(t_i) = \exp_y(v_i)$ where $|v_i| = d_Y(c(t_i), y)$ converges to $d_Y(c(t_0), y) < R_y$ by the continuity and location of $c$. Thus a subsequence of the $v_i$ converges to $v \in B_0(R_y)$ and, since $\exp_y$ is continuous, $\exp_y(v) = c(t_0)$ and $t_0 \in T$.

Let $x = c(t_0)$ and let $\delta > 0$ such that $\delta < \min\{r_x, R - d(x, y)\}$, so $\exp_x^{-1}$ is defined on $B_x(\delta)$. Please consult Figure 7 while reading this proof.

![Figure 7: Applying the Invariance of Domain Theorem.](image)

Since $\exp_y$ is continuous, $\exp_y^{-1}(B_x(\delta))$ is an open set in $B_0(R)$ containing $v$, so there is $\delta' > 0$ such that $B_v(\delta') \subset \exp_y^{-1}(B_x(\delta))$.

Let $U = \exp_x^{-1}(\exp_y(B_v(\delta'))) \subset \mathbb{R}^n$. We claim that $U$ is homeomorphic to $B_v(\delta') \subset \mathbb{R}^n$. This would then imply that $U$ is an open set by the Invariance of Domain Theorem (c.f. [EiSt]) which
states that if \( U_i \) are homeomorphically homeomorphic subsets of \( \mathbb{R}^n \) then \( U_1 \) open implies that \( U_2 \) is open as well. Then \( \exp_x(U) \) is open as well by Corollary 6.3. Since \( c \) is continuous, \( c^{-1}(\exp_x(U)) \) is open, but

\[
t_0 \in c^{-1}(\exp_x(U)) \subset c^{-1}(\exp_y(B_0(R_y))) = T,
\]

so \( t_0 = \sup T = 1 \).

Thus it suffices to prove our claim. To do so, we need only show that

\[
\exp_x^{-1} \circ \exp_y : B_c(\delta') \to U
\]

is continuous and so is its inverse (since we already know it is 1:1 and onto). Now, \( \exp_y \) is continuous by definition and \( \exp_x^{-1} \) is continuous by Corollary 6.3 and the fact that \( \exp_y(B_c(\delta')) \subset B_x(r_x) \). On the other hand, if \( u_i \in U \) converge to \( u \in U \) then \( \exp_x(u_i) \) converges to \( \exp_x(u) \) by the continuity of \( \exp_x \) and since \( \exp_y^{-1} \) is continuous on \( \exp_y(B_0(R_y)) \) as in (6.4) we get \( \exp_y^{-1}(\exp_x(u_i)) \) converges to \( \exp_y^{-1}(\exp_x(u)) \) and we are done. \( \square \)

The author would like to thank Prof. Vasquez of CUNY for drawing her attention to the Invariance of Domain Theorem.

**Lemma 6.5** If \( Y \) is an exponential length space off a discrete set \( W_y \), then \( Cl(Y') \) has a simply connected universal cover. Furthermore if \( Y \) is locally isotropic then so is the universal cover of \( Cl(Y') \).

**Proof:** For all \( x \in Cl(Y') \) there exists \( y \in Y' \) such that \( x \in B_y(R) \). Since all \( B_y(R) \cap Cl(Y') \) are simply connected, this means that \( Cl(Y') \) is locally simply connected. We can then lift the exponential length structure on these balls isometrically up to the universal cover. \( \square \)

Note that the connectedness of \( Y' \) is a necessary condition in the above Lemma as it is possible that \( Y' \) could be a bouquet of length spaces attached at a point in \( W_y \). Recall from Definition 1.6 that all \( x \in Y \setminus W_y \) have an \( r_x > 0 \) such that \( \exp_x : B_0(r_x) \to B_x(r_x) \) is a homeomorphism without having to restrict to \( Cl(Y') \).

Finally we close with a useful little lemma to deal with the fact that noncompact exponential length spaces may not be bounded below.

**Lemma 6.6** If \( Y \) is an exponential length space off \( W_y \) and then there exists a positive function \( R'_y < R_y/4 \) such that if \( x_i \in B_y(R'_y) \) then \( R_{x_i} > R'_y \) and \( d(x_i, x_j) < R'_{x_i} \).

**Proof:** Just take \( R'_y \) small enough that for all \( x_i \in B_y(R'_y), R_{x_i} > R_y/2, \) so \( d(x_1, x_2) < R_y/4 + R_y/4 < R_{x_i} \). \( \square \)

## 7 Locally Isotropic Exponential Length Spaces

Recall the definition of a locally isotropic exponential length space in Definition 1.7.

We begin with two definitions: the first is classical and the second is useful for our purposes.

**Definition 7.1** A length space is *locally minimizing* if locally there exist unique minimizing curves between pairs of points.

**Definition 7.2** A length space is *uniformly locally minimizing* if there exists \( R > 0 \) such that if \( d(x, y) < R \) then there exist unique minimizing curves between \( x \) and \( y \).

Recall that the existence of a minimizing geodesic is a global property of all complete length spaces by definition. Here we will show uniqueness. Note that it is necessary to assume that \( Y \) is locally isotropic to state this lemma. The standard cone over a circle with an opening angle \( < \pi \) is an exponential length space and it is not locally minimizing.
Lemma 7.1 Let \( Y \) denote an everywhere locally isotropic exponential length space. Then for any length minimizing curve \( c(s) \) from \( y \) to \( x \in B_y(R) \), there exists a unique \( v \in S^{n-1} \) such that \( c(s) = \exp_y(sv) \).

This has an immediate corollary which follows from Lemma 7.6.

Corollary 7.2 Let \( Y \) denote an everywhere locally isotropic exponential length space. Then it is locally minimizing and if \( Y \) has a positive exponential radius \( R \) then it is uniformly locally minimizing.

Before we prove this lemma we need a technical lemma.

Lemma 7.3 If \( Y \) is a locally isotropic exponential length space off a set \( W_Y \), then for all \( y \in Y \setminus W_Y \), \( a, b \in (0, R) \), \( \theta \in (0, \pi) \), \( F_y(\theta, a, b) > F_y(0, a, b) \).

Proof: Recall that by by (1.23) \( F_y(\pi, a, a) > 0 \). Given any \( \theta > 0 \), there exists a natural number \( k \) such that \( \theta > \pi/(2k) \) so by (1.22), \( F_y(\theta, a, b) > F_y(\pi/(2k), a, b) \). However by the triangle inequality applied to a polygon of \( 2k \) points alternatively distances \( a \) and \( b \) from \( y \), we know

\[
k(F_y(\pi/(2k), a, b) + F_y(\pi/(2k), b, a)) > F_y(\pi, a, a).
\]

Thus by the symmetry of \( F_y \), we have \( F_y(\pi/(2k), a, b) = F_y(\pi/(2k), b, a) > 0 \).

We can now prove our Lemma 7.1.

Proof of Lemma 7.1 Let \( C : [0, L] \rightarrow Y \) be a length minimizing curve from \( y \) to \( x \). Then \( L = d_Y(x, y) < R \) so \( C([0, L]) \subset B_y(R) \). Since \( W_Y = \emptyset \) in our hypotheses, we have \( r_y = R_y = R \) and Corollary 7.6 implies that \( \exp_y^{-1} : B_y(R) \rightarrow B_0(R) \) is continuous. Thus we can define a continuous map \( v(s) = \exp_y^{-1}(C(s)) \) and we need only show that \( v(s)/|v(s)| \) is constant for \( s > 0 \).

If not then there exists \( s \in (0, L) \) such that \( v(s)/|v(s)| \neq v(1)/|v(1)| \) and they have some angle \( \theta \) between them. Thus using \( C \) is parametrized by arclength and Lemma 7.3 we have,

\[
L = d_Y(y, x) = d_Y(x, C(s)) + d_Y(C(s), y) = d_Y(\exp_y(Lv(1)), \exp_y(v(s))) + s = F_y(\theta, L, s) + F_y(\theta, s, 0) > F_y(0, L, s) + s = (L - s) + s = L
\]

which is a contradiction.

Thus we can set \( v = v(1)/|v(1)| \) and since we know \( \exp_y \) is \( 1:1 \) on \( B_0(R) \), \( v \) must be unique.

Note \( Y \) is locally minimizing because on any \( B_x(R)/2 \), any pair of points is at most \( R \) apart.

We can now prove a significantly stronger technical lemma.

Lemma 7.4 If \( Y \) is a locally isotropic exponential length space off a set \( W_Y \), then for all \( y \in Y \setminus W_Y \), \( a, b \in (0, R) \), we can solve \( F_y(\theta, a, b) = d \) uniquely for \( \theta \) if a solution exists unless \( a = b = 0 \).

Proof: Suppose \( F_y(\theta_1, a, b) = F_y(\theta_2, a, b) = F_0 \) with \( \theta_2 > \theta_1 \). Without loss of generality because \( F_y \) is symmetric in its last two variables we may assume \( b \geq a \).

Then, by (1.22) \( F_y(\theta_1, a, b) = F_y(\theta_2, a, b) = F_0 \) for all \( \theta \in [\theta_1, \theta_2] \). So

\[
d_Y(\exp_y(aw), \exp_y(bv)) = F_0 \forall v, w \text{ s.t. } d_S(v, w) \in [\theta_1, \theta_2].
\]

Fix \( v_0, w_0 \in S^{n-1} \) such that \( d_S(v_0, w_0) = \theta_2 \) and look at the triangle between the points \( y, \exp_y(aw_0) \) and \( \exp_y(bv_0) \). Join the latter two points by a length minimizing curve \( c(t) \) parametrized by arclength such that \( c(0) = \exp_y(bv_0) \). Since \( b < R \) we know \( c(t) \in B_y(R) \) for \( t < R - b \). Thus since \( \exp_y^{-1} : B_y(R) \rightarrow B_0(R) \) is a continuous map, there exists continuous curves \( v(t) \in S^{n-1} \) and \( b(t) < R \)
Thus we are done.

Inverse is well defined. We need only verify that $|b - b| \geq t$ for small $t > 0$.

However, by Lemma 7.5, $\exp_y(bv_0)$ is the unique minimizing curve joining $y$ to $\exp_y(bv_0)$ so

$$b = d_Y(\exp_y(bv_0), y) < d_Y(\exp_y(bv_0), \exp_y(b(t)v(t))) + d_Y(\exp_y(b(t)v(t)), y) = t + b(t).$$

Furthermore $\exp_y(sv(t))$ is the unique minimizing curve joining $y$ to $\exp_y(b(t)v(t))$ so

$$b(t) = d_Y(\exp_y(b(t)v(t)), y) < d_Y(\exp_y(bv_0), \exp_y(b(t)v(t))) + d_Y(\exp_y(bv_0), y) = t + b.$$  

Thus $|b(t) - b| < t$ and we have a contradiction.

**Lemma 7.5** Suppose $Y$ is a locally isotropic exponential length space off a discrete set $W_Y$. If $y \in Y'$ and $g \in S0(n)$ then there is an isometry $f_g : B_y(R_y) \cap Cl(Y') \to B_y(R_y) \cap Cl(Y')$ such that $f_g(x) = \exp_y(g(\exp_y^{-1}(x)))$.

**Proof:** By Lemma 7.4 we know $\exp_y : B_0(R_y) \to B_y(R_y) \cap Cl(Y')$ is a homeomorphism so the inverse is well defined. We need only verify that $f_g$ is an isometry.

For any $x_1, x_2 \in B_y(R_y)$, we know there exists $s_i \in [0, R_y)$ and $v_i \in S^{n-1}$ such that $\exp_y(s_i v_i) = x_i$, and since $g$ is an isometry on $S^{n-1}$, we have

$$d_Y(f_g(x_1), f_g(x_2)) = F_y(d_S(g(v_1), g(v_2)), s_1, s_2) = F_y(d_S(v_1, v_2), s_1, s_2) = d_Y(x_1, x_2).$$

Thus we are done.

**Lemma 7.6** Suppose $Y$ is a locally isotropic exponential length space off a discrete set $W_Y$. Given any $y \in Y$, there exists $R = R'_y > 0$ such that if $y_1, y_2 \in B_y(R) \cap Y'$ have a length minimizing curve $\gamma$ running from $y_1$ to $y_2$ of length $d$ such that $\gamma(d/2) \notin W_Y$, then there is an isometry $f : B_{\gamma(d/2)}(R) \cap Cl(Y') \to B_{\gamma(d/2)}(R) \cap Cl(Y')$ which fixes $\gamma(d/2)$ and maps $y_1$ to $y_2$ and $y_2$ to $y_1$.

**Proof:** First we set $R'_y > 0$ as defined in Lemma 6.3. So $R_{y_1}, R_{y_2} > R$.

By Lemma 6.3, we know $\exp_{\gamma(d/2)} : B_0(R) \to B_y(R) \cap Cl(Y')$ is a homeomorphism. Furthermore $\exp_{\gamma(d/2)}^{-1}(\gamma(0))$ and $\exp_{\gamma(d/2)}^{-1}(\gamma(d))$ are both vectors of length $d/2 < R$. Thus there is an isometry of $\mathbb{R}^n$ fixing 0 that interchanges these two vectors. Call it $g$.

So we can define the isometry

$$f : B_{\gamma(d/2)}(R) \cap Cl(Y') \to B_{\gamma(d/2)}(R) \cap Cl(Y')$$

as in Lemma 6.3

$$f(x) = f_g(x) = \exp_{\gamma(d/2)}(g(\exp_{\gamma(d/2)}^{-1}(x))).$$

By the definition of $g$, $f_g(y_1) = y_2$ and visa versa.
Corollary 7.7 Suppose $Y$ is a locally isotropic exponential length space off a discrete set $W_Y$ and $x, y \in \text{Cl}(Y')$, then there is an isometry
\[
f : B_x(R/2) \cap \text{Cl}(Y') \to B_y(R/2) \cap \text{Cl}(Y')
\]
where $R = \min_{z \in B_y(2d(x,y))} R_z$.

Note that $x, y$ must be in the closure of the same connected component $Y'$ or this is not true as can be seen when $Y$ is a sphere joined to a plane at a point.

Proof: First we assume $x, y \in Y'$. Let $C : [0, L] \to Y' \cap B_y(2d(x,y))$ be a piecewise length minimizing curve running from $x$ to $y$. We only need to show that for all $t$ there is an isometry $f_t : B_x(R) \cap \text{Cl}(Y') \to B_{C(t)}(R) \cap \text{Cl}(Y')$ for all $t \in [0, L]$.

We know $f_0$ exists trivially. Now if $f_s$ exists then for $s$ near $t$, $f_t$ exists as well using the isometry from Lemma 7.9 to get from $B_{C(s)}(R) \cap \text{Cl}(Y')$ to $B_{C(t)}(R) \cap \text{Cl}(Y')$. Furthermore by the standard Arzela Ascoli Theorem, if $f_t$ exist and $t_i \to t$ then a subsequence converges to a limit isometry with the same domain and range as the required $f_t$. So $f_t$ is defined on open and closed intervals and we are done.

Now suppose $x, y \in \text{Cl}(Y')$. Then there exists $x_i \in Y'$ and $y_i \in Y'$ converging to $x$ and $y$ respectively. By the above, we have isometries $f_i : B_{x_i}(R) \cap \text{Cl}(Y') \to B_{y_i}(R) \cap \text{Cl}(Y')$. For all $r < R$ we can restrict these isometries to the closed ball $B_x(r)$ and we can apply Arzela Ascoli to say that a subsequence converges to a map
\[
f_r : B_x(r) \cap \text{Cl}(Y') \to B_y(R) \cap \text{Cl}(Y')
\]
which preserves distances and is 1:1. In particular $f_{R/2}$ is an isometry from $B_x(R/2) \cap Y'$ to $B_y(R/2) \cap Y'$.

Lemma 7.8 If $Y$ is a locally isotropic exponential length space off a discrete set $W_Y$ then we can define a locally isotropic exponential length structure everywhere on $Y'$ which is isometric to the original metric restricted to $Y'$. This new exponential structure has $\exp_x : B_0(R_x) \to B_x(R_x) \cap \text{Cl}(Y')$.

We do not claim this extension is unique and clearly the extension will depend on which connected component of $Y \setminus W_Y$ we are completing. Note any radius $< R$ will do just as in Lemma 7.7.

Later in Lemma 12.2 we will show to what extent isometries preserve length structures. See also Example 10.1 below. It should also be noted that we have not claimed that $\inf_{y \in Y} R_y > 0$. In fact $Y$ could be a Riemannian manifold with constant sectional curvature $-1$ and a cusp end so that $\inf_{y \in Y} R_y = 0$.

Proof: For all $x \in \text{Cl}(Y') \setminus Y'$, let $R_x = R_{x'}$ as in Lemma 7.4 and let $y \in B_x(R_x')$. Then just define $\exp_x : B_0(R_x/2) \to B_x(R_x/2)$ by taking the isometry $f : B_y(R_x/2) \cap \text{Cl}(Y') \to B_x(R_x/2) \cap \text{Cl}(Y')$ defined in Lemma 7.9 and let $\exp_x(v) = f(\exp_y(v))$ and we are done.

Lemma 7.9 If $Y$ is a locally isotropic exponential length space then we can define a locally isotropic exponential length structure everywhere on $Y$ which is isometric to the original metric on $Y$ but has $F_x = F_y$ for all $x, y \in Y$ and $s, t < \min\{R_x, R_y\}$.

Proof: Fix any $y \in Y$ and for any $x \in Y$ define $R$ as in Lemma 11.4 and then define $\exp_x : B_0(R/2) \to B_x(R/2)$ by taking the isometry $f : B_y(R/2) \to B_x(R/2)$ defined in that lemma and let $\exp_x(v) = f(\exp_y(v))$.

Later in Lemma 12.2 we will show this new exponential length structure agrees with the old exponential length structure in some sense. However, this is not necessary at this time. We will only apply this lemma occasionally and will not in general assume that $F_y$ is constant.
8 Exponential Curves and Locally Minimizing Spaces

In this section we will assume that $Y$ is an exponential length space everywhere which is locally minimizing. So $W_Y = \emptyset$ and $r_x = R_x$ in Definition 1.6. For simplicity, we will take $R_x$ to be small enough both to satisfy the properties of $R_x$ of the definition of exponential length space and the $R_x$ of the local minimizing property [Definition 7.1].

The following definition should be thought of intuitively as the standard differential equation for a geodesic in a Riemannian manifold adjusted to make sense in an exponential length space. We do not yet claim that such curves exist and are unique.

**Definition 8.1** An exponential curve is a curve $C : [a, b] \to Y$ such that for all $t \in [a, b]$, there exists $v(t) \in S^{n-1}$ and satisfying

$$C(s) = \exp_{C(t)}((s-t)v(t)) \quad \forall s \in [t, t+R_t/4] \cap [a, b].$$

(8.1)

Here $R_t = R_{c,t} = R'_{c(t)}$ where $R'_c$ is defined in Lemma 6.6.

The following lemma is immediately seen from the definition.

**Lemma 8.1** If $C$ is an exponential curve on $[a, b]$ and on $[b, c]$ and on $[t_1, t_2]$ where $t_1 \in (a, b)$ and $t_2 \in (b, c)$, then $C$ is exponential on $[a, c]$.

**Lemma 8.2** If $Y$ is a locally minimizing exponential length space everywhere then all length minimizing curves in $Y$ are exponential curves.

Note that if $Y$ has a nonempty $W_Y$ then length minimizing curves joining points in distinct connected components of $Y \setminus W_Y$ are not exponential as can be seen in the example with the sphere attached to the plane at one point [Example 6.1].

**Proof:** Let $c : [a, b] \to C(Y')$ be a length minimizing curve. For any $t \in [a, b]$ let $t' = \min\{t + R_c, b\}$. Since $t' - t \leq R_b < R'_c(t)$ and $Y$ is locally minimizing, $c([t, t'])$ is a unique length minimizing curve running from $c(t)$ to $c(t')$. Let $v_t \in S^{n-1}$ be defined as $\exp_{c(t)}^{-1}(c(t'))$. Then by the definition of an exponential length space, we know $\exp_c((s-t)v(t))$ with $s \in [t, t'] \subset [t, t+R_c]$ is also a length minimizing curve from $c(t)$ to $c(t')$. Thus these curves agree and we are done.

**Lemma 8.3** If $Y$ is a locally minimizing exponential length space everywhere, the function $\exp_y$ can be extended uniquely so that $\exp_y : \mathbb{R}^n \to Y$, such that $\exp_y(sv)$ is an exponential curve for all $s \in \mathbb{R}$.

**Proof:** Fix $y \in Y$ and $v \in S^{n-1}$. We treat (8.1) like a differential equation. We can call the possible solution $C(t)$.

We know $C(t) = \exp_y(tv)$ is defined for $t \in [0, R_y]$, so now we must extend it. Clearly if $C(t)$ is defined on an open set it can be defined on a closed set by extending it continuously.

It suffices to show that if $C(s)$ satisfies (8.1) for $t \in [0, a]$ then it does for $t \in [0, a+R_a/8]$ as well. Although $R_a$ may decrease, we will have proven that $C$ is defined on a right open set, and since it is defined on a closed set, it is defined on all of $[0, \infty)$.

Assume $C(s)$ is defined on $[0, a]$. So for all $t \in [0, a]$ $C(s) = \exp_{C(t)}((s-t)v_t) \quad \forall s \in [t, t+R_a/4] \cap [0, a]$ (8.2)

So it is minimizing on $[a - R_a/2, a]$ by the definition of an exponential length space the fact that $R_a \geq R_{c(a)}$. Let $s' = a - R_a/4$. Since $C : [a - R_a/2, a] \to B_{C(s')}(R/2)$ is minimizing and $R_a \geq R_{c(a-R_a/4)}$ by Lemma 6.9, we know that there exists some $w \in S^{n-1}$ such that $C(s) = \exp_{C(s')((s-s')w)} \quad \forall s \in [s', s' + R_{C(s')}] \subset [s', a]$. (8.3)
Extend the definition of \( C \) to \([s', a + R_a/4] \subset [s', s' + R_a]\) using this \( w \).

\[
C(s) := \exp_{C(s')}((s - s')w) \quad \forall s \in [s', a + R_a/4].
\]  

(8.4)

This extension is a length minimizing curve on \([a - R_a/4, a + R_a/4]\) thus for all \( t \in [a, a + R_a/4] \) we know

\[
C(s) = \exp_{C(t')}((s - t)v) \quad \forall s \in [t, a + R_a/4].
\]  

(8.5)

Using (8.2) for \( t \in [0, a - R_a/4] \) and (8.3) for \( t \in [a - R_a/4, a + R_a/4] \) we see that \( C \) is an exponential curve on \([0, a + R_a/4]\). In fact it is slightly better than exponential since \( R_a \geq R_{a + R_a/4} \).

Since this was true for all \( a > 0 \), \( C \) is exponential on \([0, \infty)\).

Note further that \( \exp_y \) is not invertible. The Implicit Function Theorem is used to show that a lack of conjugate points on \([0, a + R_a/4]\) occurs at \( \exp_y(t_0v) \) for \( t \rightarrow 0 \), since at any point \( a \) where they might split, we are forced to have both satisfy (8.5) with the same \( v \) at \( t = a - R/4 \). So \( \exp_y(tv) \) has been extended uniquely to all \( t \in [0, \infty) \) using this solution \( C(t) \). 

We now show this extended exponential map is continuous. It is not 1:1 as can be seen when \( Y = S^n \) or \( Y = T^n \).

**Lemma 8.4** If \( Y \) is a locally minimizing exponential length space everywhere the extended exponential map based at any fixed point is continuous.

**Proof:** Suppose \( t_i \rightarrow t \) and \( v_i \rightarrow v \in S^{n-1} \), we need to show \( \exp_y(t_i v_i) \) converges to \( \exp_y(tv) \). Clearly we need only show \( \exp_y(tv_i) \) converges to \( \exp_y(tv) \) since \( |t_i - t| \rightarrow 0 \), \( \exp_y(tv_i) \) is parametrized proportional to arclength and the triangle inequality holds.

By Arzela Ascoli Theorem a subsequence of \( c_i(t) = \exp_y(tv_i) \) converges to a curve \( C(t) \) which is parametrized by arclength. Well we know \( R_t = R_{c_i,t} = \min_{s \in [a,t]} R'_{c_i(s)}/2 \) where \( R'_{c_i} \) is defined in Lemma 6.6 so that \( R'_{c_i} < R_z \) for all \( z \) near \( x \). In particular

\[
R_{c_i,t} > R_{\min,t} = \min_{x \in B_y(t + R_{\max})} R_x
\]  

(8.6)

where \( R_{\max} = \max_{x \in B_y(t)} R'_{x} \). So each \( c_i \) is a minimizing curve on intervals of length \( R_{\min,t} \) in \([0, t]\). Since this is uniform in \( i \), the same holds for \( C \).

Since \( C \) is minimizing on intervals then it must be exponential on those intervals by Lemma 6.3. Thus it must be an exponential curve by Lemma 6.1. Since it must agree with \( \exp_y(tv) \) for small \( t \), it must be its unique extension by Lemma 6.3. Thus the \( \exp_y(tv_i) \) must have converged without taking a subsequence and we are done.

To show that the exponential map is open in Riemannian manifolds it is necessary to avoid conjugate points. So we need to make a similar argument in this case. We can study conjugate points in any space with an extended exponential map so we will do so in the following separate section.

### 9 Extended Exponential Length Spaces

In this section we generalize the properties of the exponential map and its relationship with conjugate points. Recall that in Riemannian manifolds a conjugate point \( y \) occurs at \( \exp_y(t_0v) \) iff \( d(\exp_y(t_0v)) \) is not invertible. The Implicit Function Theorem is used to show that a lack of conjugate points on a ball implies that \( \exp_y \) is a local diffeomorphism on that ball. Here we have no differentiability, but we can use the definition of a conjugate point which refers only to length minimizing curves and we can obtain a local homeomorphism using the Invariance of Domain Theorem.

We begin with a definition.
Definition 9.1 A complete length space \( Y \) is an extended exponential length space if there exists \( n \in \mathbb{N} \) for all \( y \in Y \) there exists a map \( \exp_y : \mathbb{R}^n \to Y \) which is continuous and there exists a continuous function \( R_y > 0 \) such that \( \exp_y : B_0(R_y) \to B_y(R_y) \) is a homeomorphism.

We also assume that for all \( t > s > 0 \), for all \( y \in Y \) and \( v \in S^{n-1} \) there exists \( w \in S^{n-1} \) satisfying
\[
\exp_y(tv) = \exp_{\exp_y(sv)}((t-s)w)
\] (9.1)
and all length minimizing curves are of the form \( \exp_y(tv) \) with \( t \geq 0 \).

We have already proven that uniformly locally length minimizing exponential length spaces are extended exponential length spaces in Lemmas \( 9.2 \) and \( 9.3 \). Conversely, since the exponential maps in an extended exponential length space are assumed to be invertible up to some radius \( R_y > 0 \) and since here we assume length minimizing curves are exponential, we know that extended exponential length spaces are locally minimizing.

Example 9.1 Note that the definition of an extended exponential length space works only in the positive direction. We want \( \exp_y(tv) \) to be an exponential map when \( t \) runs from \(-1 \) to \( 1 \) but this is not necessarily the case. For example, one could define an extended exponential length structure on \( \mathbb{E}^2 \) where
\[
\exp_0(t(\cos(\theta), \sin(\theta))) = (t\cos(\theta^2/\pi), t\sin(\theta^2/\pi))
\] (9.2)
for \( \theta \in [0, \pi] \) and
\[
\exp_0(t(\cos(\theta), \sin(\theta))) = (t\cos(\theta), t\sin(\theta))
\] (9.3)
for \( \theta \in [-\pi, 0] \) and \( \exp_0(tv) \) would have a corner at 0.

Lemma 9.1 If \( Y \) is an extended exponential length space then for all \( y \in Y \) \( \exp_y : \mathbb{R}^n \to Y \) is surjective.

Proof: For all \( x \in Y \), there is a length minimizing curve from \( y \) to \( x \) by the definition of a complete length space. By Definition 9.1 that curve must be exponential and have the form \( \exp_y(tv) \).

Recall that a cut point of \( y \) has two distinct length minimizing curves joining it to \( y \).

Lemma 9.2 In an exponential length space \( Y \). If \( \exp_y(tv) \) is length minimizing on \( [0, L] \) then it has no cut points before \( L \).

Proof: Let \( x = \exp_y(Lv) \). Suppose that \( \exp_y(tv) \) has a cut point at \( t_0 \in (0, L) \). Then there exists two distinct length minimizing curves from \( x \) to \( y \) which both agree with \( \exp_y((L-s)v) \) for \( s \in [0, L-t_0] \) and then diverge. Since length minimizing curves are exponential curves, they cannot diverge, so they agree everywhere and there is no cut point.

Now we make a definition of conjugate point for extended exponential spaces which does not agree exactly with the definition in Riemannian geometry but is the appropriate extension for our purposes.

Definition 9.2 In an exponential length space, an exponential curve \( \exp_y(tv) \) has a conjugate point \( \exp_y(t_0v) \) at \( t_0 > 0 \) if there exists \( v_i \neq w_i \) both converging to \( v \) and \( t_i, s_i \to t_0 \) such that \( \exp_y(t_iv_i) = \exp_y(s_iw_i) \).

Recall that in a Riemannian manifold, not all conjugate points take this form, but any point which has this property is a conjugate point (c.f. \( \text{degC} \)).

As in Riemannian manifolds, some conjugate points are also cut points. It is easy to see that the first conjugate point along an exponential curve must have \( t_0 > R_y \) because \( \exp_y : B_0(R_y) \to B_y(R_y) \) is one to one.

We now follow with lemmas extending standard theory of conjugate points from Riemannian manifolds to this setting.
Lemma 9.3 In an extended exponential length space Y: if \( \exp_y(tv) \) is an exponential curve with no conjugate or cut points before \( L \), then it is length minimizing on \([0, L]\).

Proof: Suppose \( d_y(\exp_y(Lv), y) < L \), then by continuity, for \( t \) near \( L \) \( d_y(\exp_y(tv), y) < t \). Let \( t_0 = \inf\{t : d(\exp_y(tv), y) < t\} \geq R_y \). Let \( s_i \) decrease to \( t_0 \), then there are length minimizing curves \( \exp_y(t_i v_i) \) running from \( y \) to \( \exp_y(s_i v) \) of length \( t_i < s_i \). If a subsequence of \( v_i \) converges to \( v \) then we have a conjugate point.

Otherwise a subsequence must converge to some \( w \) giving a length minimizing curve \( \exp_y(tw) \) from \( y \) to \( \exp_y(t_0v) \). The latter must then be a cut point. \( \Box \)

Lemma 9.4 Suppose \( Y \) is an extended exponential length space. If \( y \in Y \) has no conjugate points before \( t_0 > 0 \) then \( \exp_y : B_0(t_0) \to B_y(t_0) \) is locally one-to-one.

Proof: We need only show that for all \( s_0v \in B_0(t_0) \) there exists a neighborhood \( U \subset B_0(t_0) \) of \( s_0v \) such that \( \exp_y : U \to \exp_y(U) \) is 1:1. If not, \( \exists(t_i v_i) \neq (s_i w_i) \) both converging to \( s_0v \) s.t. \( \exp_y(t_i v_i) = \exp_y(s_i w_i) \). Then \( v_i, w_i \to v \) so there is a conjugate point at \( s_0 < t_0 \) unless \( v_i = w_i \).

However if \( v_i = w_i \) then there are increasingly small exponential loops converging on the point \( \exp_y(t_0v) \):
\[
\exp_y(t_i v_i) = \exp_y(s_i v_i) \text{ with } s_i \neq t_i. \quad (9.4)
\]

However if we take \( r = R_{\exp_y(t_i v_i)}/2 \) then for \( i \) sufficiently large we have \( R_{\exp_y(t_i v_i)} > r \) and so it cannot have a loop shorter than \( r \). Taking \( i \) possibly larger we get \( |s_i - t_i| < r \) and a contradiction. \( \Box \)

Lemma 9.5 Suppose \( Y \) is an extended exponential length space. If \( y \in Y \) has no conjugate points before \( t_0 > 0 \) then \( \exp_y : B_0(t_0) \to B_y(t_0) \) is open.

Proof: Since there are no conjugate points we can apply Lemma 9.4 so \( \forall v \in B_0(R) \exists \epsilon > 0 \) s.t. \( \exp_y : B_v(\epsilon) \to \text{im}(B_v(\epsilon)) \) is 1:1 and continuous.

I claim it is also open. So I first show \( \exp_{y}^{-1} : \text{im}(B_v(\epsilon/2)) \to B_v(\epsilon/2) \) is continuous. Let \( x_i \in \text{im}(B_v(\epsilon/2)) \) and \( x_i \to x \in \text{im}(B_v(\epsilon/2)) \). Then \( \exists v_i \in \text{im}(\exp_y(v_i) = x_i \text{, and a subsequence of } v_i \text{ converging to some } v_\infty \in B_v(\epsilon) \) by continuity of \( \exp_y \), \( x_i \to \exp_y(v_\infty) \). Thus \( \exp_y(v_\infty) = x \), but \( \exp_{y}^{-1} \) is unique so all subsequences of the \( v_i \) must converge to the same \( v_\infty \), so in fact \( v_i \) themselves must converge to \( v_\infty \) and \( \exp_{y}^{-1} \) is continuous.

Now we prove that \( \exp_y \) is open. Let \( U \in B_0(R) \) be any open set. We must show \( \exp_y(U) \) is open. That is for any \( v \in U \), we need to find an \( r_v \) such that \( B_{\exp_y(v)}(r_v) \subset \exp_y(U) \). Please consult Figure, while reading this proof.

Now take any \( \delta > 0 \) sufficiently small that \( B_v(\delta) \subset U \) and \( \delta < \epsilon \). Thus \( \exp_y(B_v(\delta)) \) is homeomorphic to \( B_v(\delta) \) and is relatively open as a subset of \( U \). For \( \delta \) sufficiently small
\[
\exp_y(B_v(\delta)) \subset B_{\exp_y(v)}(r_{\exp_y(v)}). \quad (9.5)
\]

Now
\[
\exp_{\exp_y(v)}^{-1}(\exp_y(B_v(\delta))) \subset B_0(r_{\exp_y(v)}) \subset \mathbb{R}^n \quad (9.6)
\]
is homeomorphic to the open set \( B_v(\delta) \subset \mathbb{R}^n \), so it must be open by the Invariance of Domain Theorem.

So using \( \exp_{\exp_y(v)} \) as a homeomorphism, we map the sets in Figure back to the corresponding sets in Figure and conclude that \( \exp_y(B_v(\delta)) \) is a relatively open set in \( B_{\exp_y(v)}(r_{\exp_y(v)}) \), so it is open. Thus there exists \( r_v > 0 \) such that \( B_{\exp_y(v)}(r_v) \subset \exp_y(B_v(\delta) \subset \exp_y(U) \) and we are done. \( \Box \)
Theorem 9.1 Suppose $Y$ is an extended exponential length space with no conjugate points about a given point $y$ before $t = t_0$, then the exponential map $\exp_y$ is a local homeomorphism from $B_0(t_0)$ onto $B_y(t_0)$. So if $Y$ is simply connected it is a homeomorphism.

Proof: The exponential map is continuous, open, locally 1:1 and onto by Lemmas 9.5, 9.4 and 9.1, so it is a local homeomorphism, thus when applied to a simply connected space it is a homeomorphism.

10 Local Isotropy and Conjugate Points

We now return to locally isotropic exponential length spaces $Y$ with $W_Y$ empty. Our goal is to show that the universal covers of such spaces must be $S^n$, $\mathbb{H}^n$ or $E^n$. Recall that when proving that complete simply connected Riemannian manifolds with constant sectional curvature are limited to these three cases, one first shows that either there are no conjugate points or there is exactly one conjugate point and thus the exponential map can be inverted to get either a map from the Riemannian manifold to $\mathbb{R}^n$ or to $S^n$.

Here we complete this first step [Lemma 10.6]. To show that all exponential curves have conjugate points at the same locations, we first construct isometries of balls along the exponential curves [Lemma 10.2] and then study the behavior of exponential curves which are located near each other [Lemma 10.4]. (c.f. [26(c)].)

Recall that we’ve proven that all locally isotropic exponential length spaces have extended exponential length structures. Recall also that the definition of an exponential curve is only in a positive direction so that in general $\exp_y(tv)$ is not an exponential curve through $t = 0$ [Example 9.1].

Lemma 10.1 For all $y$ in a locally isotropic exponential length space $Y$, the curve $c(t) = \exp_y(tv)$ is exponential for all $t \in \mathbb{R}$.

Proof: We need only show there exists $\epsilon > 0$ such that $d_Y(\exp_y(-\epsilon v), \exp_y(\epsilon v)) = 2\epsilon$ since all length minimizing curves are exponential and then we’d be able to apply Lemma 8.1.

Now

$$d_Y(\exp_y(\epsilon v), \exp_y(-\epsilon v)) = F_y(\pi, \epsilon, \epsilon) > F_y(\theta, \epsilon, \epsilon)$$  (10.1)
for all $\theta < \pi$ by Lemma 7.4 combined with 10.22 of the definition of locally isotropic. Thus we have a one point set:
\[ \partial B_{y}(\epsilon) \cap \partial B_{\exp_{y}(\epsilon v)}(2\epsilon) = \{ \exp_{y}(-\epsilon v) \} \] (10.2)

Take $\epsilon < R_{y}/3$ sufficiently small that if $x \in B_{y}(\epsilon)$ we know $R_{x} > 2R_{y}/3$. Set $x = \exp_{y}(\epsilon v)$. Then there exists $w \in S^{n-1}$ such that $\exp_{z}(tw) = \exp_{y}((\epsilon - t)v)$ because $\exp_{y}((\epsilon - t)v)$ is minimizing on $(0, \epsilon) \subseteq [0, R_{y}]$. On the other hand $\exp_{z}(tw)$ is minimizing on $[0, 2\epsilon] \subseteq [0, R_{z}]$ so $d_{Y}(\exp_{z}(2\epsilon v), x) = 2\epsilon$ and by 10.2 $\exp_{z}(2\epsilon v) = \exp_{y}(-\epsilon v)$ and we are done.

Lemma 10.2 Let $Y$ denote an everywhere locally isotropic exponential length space with isotropy radius $R$, where $R$ may be infinity.

Suppose $f : A \rightarrow B$ is an isometry and $A \subseteq B_{x}(R/2)$ and $B \subseteq B_{f(x)}(R/2)$. Then we can extend the map $f$ so that $f : B_{x}(R/2) \rightarrow B_{f(x)}(R/2)$ is an isometry.

We do not claim this extension is unique.

Proof: First of all $Y$ can be temporarily given a locally isotropic exponential length structure such that $F_{x} = F_{y}$ for all $x, y \in Y$ by Lemma 7.9

Let $y = f(x)$, $A' = \exp_{x}^{-1}(A) \subseteq B_{0}(R)$ and $B' = \exp_{y}^{-1}(B) \subseteq B_{0}(R)$. Let $d_{f_{x}} : A' \rightarrow B'$ be defined as
\[ d_{f_{x}}(v) = \exp_{y}^{-1}(f(\exp_{x}(v))). \] (10.3)

We claim $d_{f_{x}}$ is in $SO(n)$. Clearly $d_{f_{x}}(0) = 0$ and $|d_{f_{x}}(v)| = |v|$ for all $v \in A'$ by the choice of $x$ and $y$. Thus for all $v, w \in A'$ we have
\[ |d_{f_{x}}(v) - d_{f_{x}}(w)|^{2} = |v|^{2} + |w|^{2} - 2|v||w|\cos(\theta), \] (10.4)

where $\theta = d_{S}(d_{f_{x}}(v), d_{f_{x}}(w))$. Now using $F_{x} = F_{y}$ we have
\[ F_{x}(\theta, |v|, |w|) = d_{Y}(\exp_{x}(d_{f_{x}}(v)), \exp_{x}(d_{f_{x}}(w))) = d_{Y}(\exp_{y}(v), \exp_{y}(w)) \] (10.5)
\[ = F_{y}(d_{S}(v, w), |v|, |w|) = F_{x}(d_{S}(v, w), |v|, |w|). \] (10.6)

This implies $\theta = d_{S}(v, w)$ by Lemma 7.3 so
\[ |d_{f_{x}}(v) - d_{f_{x}}(w)|^{2} = |v|^{2} + |w|^{2} - 2|v||w|\cos(\theta) = |v - w| \] (10.7)

and we have our claim.

Now since $d_{f_{x}}$ is in $SO(n)$, although it may not completely be determined depending on the size of $A'$, we can extend $d_{f_{x}}$ to an element of $SO(n)$ mapping $B_{0}(R) \rightarrow B_{0}(R)$. Note that $df \cdot g$ is also a possible extension of $df$ as long as $g$ is in the subgroup of $SO(n)$ preserving $A'$.

We can extend the definition of $f$ as $f(z) = \exp_{y}(d_{f_{x}}(\exp_{z}^{-1}(z)))$ which agrees with $f$ on $A$.

Then given $x_{i} \in B_{x}(R)$ let $s_{i}v_{i} = \exp_{x_{i}}^{-1}(x_{i})$ with $|v_{i}| = 1$ and we have
\[ d_{Y}(f(x_{1}), f(x_{2})) = F_{y}(d_{S}(d_{f_{x}}(v_{1}), d_{f_{x}}(v_{2})), s_{1}, s_{2}) \] (10.8)
\[ = F_{y}(d_{S}(v_{1}, v_{2}), s_{1}, s_{2}) \] (10.9)
\[ = F_{x}(d_{S}(v_{1}, v_{2}), s_{1}, s_{2}) \] (10.10)
\[ = d_{Y}(x_{1}, x_{2}). \] (10.11)

so we have an isometry.
By Lemma 10.2 we can extend this isometry to an isometry determined by the fact that the exponential curves are length minimizing, we can define an isometry $f$ by our choice of $tv_i, v_i \in S^{n-1}$ such that $d_S(v, w) < \theta_{L, x, v}$.

**Proof:** If not then there exist $w_i, v_i \in S^{n-1}$, $d_S(v_i, w_i) \to 0$ and $t_i \in [0, L]$ such that

$$d_Y(\exp_x(t_iw_i), \exp_x(t_i)) \geq r.$$  \hspace{1cm} (10.13)

Since we know there exist converging subsequences of $t_iw_i$ and of $t_i v_i$, and they must converge to the same $tv_i$ and the exponential map is continuous, we get a contradiction. \qed

**Corollary 10.3** Suppose $Y$ is a locally isotropic exponential length space everywhere. Given any $L > 0$ and $r \in (0, R/4)$ there exists a sector defined by some $\theta_{L, x, r} > 0$ such that

$$d_Y(\exp_x(tw), \exp_x(tv)) < r \quad \forall t \in [0, L]$$  \hspace{1cm} (10.12)

for all $v, w \in S^{n-1}$ such that $d_S(v, w) < \theta_{L, x, v}$.

**Proof:** If $d_S(v, w) \to 0$ then $\theta_{L, x, v} \to 0$. Since we know there exist converging subsequences of $tv_i, v_i$ and $t_i v_i$, and they must converge to the same $tv_i$ and the exponential map is continuous, we get a contradiction. \qed

**Lemma 10.4** Suppose $Y$ is a locally isotropic length space everywhere with isotropy radius $R$ and $N \in \mathbb{N}$. Then we can extend the isotropic behavior in thin sectors: given

$$d_S(v_1, w_1) = d_S(v_2, w_2) = \theta < \min\{\theta_{N+1}R/4, \theta_{N+1}R/4, \theta_{N+1}R/4, R/4\},$$  \hspace{1cm} (10.14)

and

$$d_S(v_1, w_1) = d_S(v_2, w_2) = \bar{\theta} < \min\{\theta_{N+1}R/4, \theta_{N+1}R/4, \theta_{N+1}R/4, R/4\},$$  \hspace{1cm} (10.15)

and $d_S(w_1, w_2) = d_S(v_2, w_2)$, then

$$d_Y(\exp_x(tv_1), \exp_x(sw_1)) = d_Y(\exp_x(tw_1), \exp_x(sw_2))$$  \hspace{1cm} (10.16)

and

$$d_Y(\exp_x(tv_1), \exp_x(s\bar{w}_1)) = d_Y(\exp_x(tw_2), \exp_x(s\bar{w}_2))$$  \hspace{1cm} (10.17)

and

$$d_Y(\exp_x(s\bar{w}_1), \exp_x(sw_1)) = d_Y(\exp_x(s\bar{w}_2), \exp_x(sw_2))$$  \hspace{1cm} (10.18)

whenever $t, s, \bar{s} \in [0, NR/4]$ with $|t - s| < R/4$ and $|t - \bar{s}| < R/4$.

**Proof:** By our choice of $\theta$, for all $k = 0, \ldots, N$ we know that $\exp_x([0, kR/4], w_i) \subset T_{\exp_x([0, kR/4], R/4)}$ and $\exp_x([0, kR/4], \bar{w}_i) \subset T_{\exp_x([0, kR/4], R/4)}$. We will prove 10.16 - 10.18 inductively with $t, s, \bar{s} \in [0, kR/4]$ and $k$ increasing to $N$.

We start with $k = 0$. Choose $g \in S0(n)$ with $v_1$ to $v_2$, $w_1$ to $w_2$ and $w_1$ to $\bar{w}_2$, then by Lemma 10.6 there is an isometry $g : B_{x_1}(R) \to B_{x_2}(R)$ which maps $exp_x([0, R/4], v_1)$ to $exp_x([0, R/2], v_1)$, $exp_x([0, R/4], w_1)$ to $exp_x([0, R/2], w_1)$ and $exp_x([0, R/4], \bar{w}_1)$ to $exp_x([0, R/2], \bar{w}_1)$ and we have 10.16 - 10.18 for $t, s, \bar{s} \in [0, R/2]$.

Suppose we have 10.16 - 10.18 for $t, s, \bar{s} \in [0, kR/4]$ Set $y_1 = exp_x((kR/4), v_1)$ and $y_2 = exp_x((kR/4), v_2)$ and let

$$A = B_{y_1}(R/2) \cap (exp_x([0, kR/4], v_1) \cup (exp_x([0, kR/4], w_1) \cup (exp_x([0, kR/4], \bar{w}_1)).$$  \hspace{1cm} (10.19)

Let

$$B = B_{y_2}(R/2) \cap (exp_x([0, kR/4], v_2) \cup (exp_x([0, kR/4], w_2) \cup (exp_x([0, kR/4], \bar{w}_2)).$$  \hspace{1cm} (10.20)

By the fact that 10.16 - 10.18 holds for $t, s, \bar{s} \in [0, kR/4]$ and all other distances in $A$ and $B$ are determined by the fact that the exponential curves are length minimizing, we can define an isometry $f : A \to B$ such that $f(exp_x((tv_1))) = exp_x((tw_1)), f(exp_x((sw_1)) = exp_x((sw_2)) and

$$f(exp_x(s\bar{w}_1)) = exp_x(s\bar{w}_2).$$  \hspace{1cm} (10.21)

By Lemma 10.2 we can extend this isometry to an isometry $f : B_{y_1}(R/2) \to B_{y_2}(R/2)$.

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Since by the choice of $\theta$,
\[
\exp_x([kR/4 - R/4, kR/4 + R/4]w_i) \subset B_{y_i}(R/2)
\] (10.22)
and, by the choice of $\hat{\theta}$,
\[
\exp_x([kR/4 - R/4, kR/4 + R/4]\hat{w}_i) \subset B_{y_i}(R/2)
\] (10.23)
these exponential curves restricted to $[kR/4 - R/4, kR/4]$ are in $A$. Since isometries map exponential curves to exponential curves we know $f$ must map these entire segments to the corresponding exponential curves extended as well. That is $f(\exp_x(tv_1)) = \exp_x(tv_2)$, $f(\exp_x(sw_1)) = \exp_x(sw_2)$ and $f(\exp_x(s\hat{w}_1)) = \exp_x(s\hat{w}_2)$ for $t, s \in ((k - 1)R/4, (k + 1)R/4)$ and we have (10.16)-(10.18) for $t, s \in ((k - 1)R/4, (k + 1)R/4)$.

Thus by induction we have (10.16)-(10.18) for $t, s, \bar{s} \in ((k - 1)R/4, (k + 1)R/4)$ where $k = 0..N$. Given any $t, s \in [0, NR/4]$ such that $|t - s| < R/4$ we set $k = \lfloor 4 \min_{t,s} R/4 \rfloor$ and similarly for $\bar{s}$.

**Lemma 10.5** In a locally isotropic exponential length space $Y$, if $\exp_y(tv)$ is length minimizing on $[0, L]$ then it has no conjugate points before $L$.

**Proof:** Suppose that $\exp_y(tv)$ has a conjugate point in $(0, L)$. So there exists $v_i \neq w_i$ both converging to $v$ and $t_i, s_i \to t_0$ such that $\exp_y(t_i v_i) = \exp_y(s_i w_i)$. For $i$ sufficiently large,
\[
d_S(v_i, w_i) < \min\{\theta_{2L,y,R/4}\} \text{ and } |s_i - t_i| < \min\{R/4, L - t_0\}.
\] (10.24)
Let $\bar{v}_i$ be chosen such that $d_S(\bar{v}_i, v) = d_S(v_i, w_i)$, so by Lemma 10.3 we get
\[
d_Y(\exp_y(t_i v_i), \exp_y(s_i \bar{v}_i)) = d_Y(\exp_y(t_i v_i), \exp_y(s_i w_i)) = 0,
\] (10.25)
and\[
d_Y(\exp_y(s_i \bar{v}_i), \exp_y(t_i \bar{v}_i)) = d_Y(\exp_y(t_i v_i), \exp_y(s_i w_i)) = 0.
\] (10.26)
Since $\exp_y(tv)$ is length minimizing curve up to $L$, and $t_i, s_i < L$, (10.26) implies that
\[
s_i \leq d_Y(y, \exp_y(t_i v_i)) = t_i
\] (10.27)
while (10.26) implies that $t_i \leq d_Y(y, \exp_y(s_i v_i)) = s_i$. Thus $s_i = t_i$ and both $\exp_y(t_i \bar{v}_i)$ and $\exp_y(t_i v)$ are distinct length minimizing curves which gives us a cut point before $L$. This is impossible on a length minimizing curve by Lemma 9.2.

**Lemma 10.6** Let $Y$ denote an everywhere locally isotropic exponential length space. If there is an exponential curve with a conjugate point at $t_0$ then every exponential curve from a point in $Y$ has a conjugate point at $t_0$. In fact so do all exponential curves in any covering space of $Y$.

Note that this is not true for cut points as can be seen when $Y$ is a cylinder.

**Proof:** Fix $x, y \in Y$ and $v, \bar{v} \in S^{n-1}$.

If an exponential curve $\exp_x(tv)$ has a conjugate point at $t_0$ then there exist distinct exponential curves $\exp_x(t_0 v_i), \exp_x(t_0 w_i)$ running from $x$ to $\exp_x(t_0 v_i) = \exp_x(s_i w_i)$ converging to it with $t_i, s_i \to t_0$ and $v_i, w_i \to v$. Eventually
\[
\max\{d_S(v_i, v), d_S(w_i, v), d_S(v_i, w_i)\} < \min\{\theta_{(N+1)R/4,x,v,R/4}, \theta_{(N+1)R/4,y,w,R/4}\}
\] (10.28)
where we take $N > 8t_0/R$.  

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Choose \( \bar{v}_i \) and \( \bar{w}_i \) in \( S^{n-1} \) such that
\[
d_S(\bar{v}, \bar{v}_i) = d_S(v, v_i) \quad d_S(\bar{v}, \bar{w}_i) = d_S(v, w_i) \quad d_S(\bar{w}_i, \bar{v}_i) = d_S(w_i, v_i) \quad (10.29)
\]

Then by Lemma 10.4, we have
\[
d_Y(\exp_x(tv_i), \exp_x(sv)) = d_Y(\exp_y(t\bar{v}_i), \exp_y(s\bar{v})) \quad (10.30)
\]
\[
d_Y(\exp_x(tw_i), \exp_x(sv)) = d_Y(\exp_y(t\bar{w}_i), \exp_y(s\bar{v})) \quad (10.31)
\]
\[
d_Y(\exp_x(tv_i), \exp_x(sw_i)) = d_Y(\exp_y(t\bar{v}_i), \exp_y(s\bar{w}_i)) \quad (10.32)
\]

whenever \( t, s \in [0, NR/4] \) with \( |t - s| < R/4 \).

Since \( t_i, s_i \rightarrow t_0 \), eventually they are close and \( \in [0, NR/4] \) thus
\[
0 = d_Y(\exp_x(t_iv_i), \exp_x(s_iw_i)) = d_Y(\exp_y(t_i\bar{v}_i), \exp_y(s_i\bar{w}_i)). \quad (10.33)
\]

\[\square\]

11 Simply Connected Locally Isotropic Spaces

In this section, we study simply connected locally isotropic exponential length spaces proving that they are homeomorphic to either \( \mathbb{R}^n \) or \( S^n \) [Theorem 11.1] and constructing global isometries [Lemmas 11.1 and 11.2]. We also prove a nice result about triangles [Lemma 11.3].

**Theorem 11.1** A simply connected locally isotropic exponential length space \( Y \) is homeomorphic to \( \mathbb{R}^n \) or \( S^n \) where \( n \) is the exponential dimension of \( Y \). The homeomorphism is \( \exp_Y : U \rightarrow Y \) where \( U = \mathbb{R}^n \) if \( Y \) is unbounded and, if \( Y \) is bounded, \( U = B_0(D) \) ~ where \( v \sim w \) if \( |v| = |w| = D \) where \( \text{diam}(Y) = D \).

**Proof:** Clear if there are no conjugate points then we are done by Lemma 9.4. Otherwise there is a first conjugate point at some \( t_0 > R \) and by Lemma 10.4 every exponential curve has a first conjugate point at the same \( t_0 > R \). Choose one point \( p \in Y \). Then sup\(_{q\in Y} d_Y(p, q) \leq t_0 \) by Lemmas 10.5 and 9.2.

Since this is true for all \( p \in Y \), we have \( t_0 = \text{diam}(Y) \). Furthermore \( \exp_{p_0} : B_0(t_0) \rightarrow B_0(t_0) \) is a homeomorphism by Lemma 9.1. If \( \exp_p \) maps \( \partial B_0(t_0) \) to a single point then we are done.

Fix \( v \in S^{n-1} \). We know there exists \( v_i \neq \bar{v}_i \) both converging to \( v \) and \( s_i, t_i \rightarrow to_0 \) such that \( \exp_p(t_i v_i) = \exp_p(s_i \bar{v}_i) \). Note that either \( t_i \geq t_0 \) or \( s_i \geq t_0 \) since \( \exp_{p_0} \) is 1:1 on \( B_0(t_0) \).

Let \( \theta_i = d_S(v_i, v) \) and \( \text{theta}_i = d_S(\bar{v}_i, v) \) so \( \theta_i \rightarrow 0 \) and \( \bar{\theta}_i \rightarrow 0 \). Eventually
\[
\max\{\theta_i, \bar{\theta}_i\} < \theta_{2t_0,p,R/4}/2. \quad (11.1)
\]

So applying Lemma 10.1 taking \( p \) to \( p, v_i \) to itself and \( \bar{v}_i \) to some vector \( w_i \), we get \( \exp_p(t_i w_i) = \exp_p(s_i \bar{v}_i) \). So in fact
\[
\exp_p(t_i w_i) = \exp_p(t_i v_i). \quad (11.2)
\]

Since \( d_S(v_i, w_i) = \bar{\theta}_i \leq \theta_i + \bar{\theta}_i \rightarrow 0 \). we know eventually it is < \( \theta_{2t_0,p,R/4}/2 \). Given any \( \bar{v}, \bar{w} \in S^{n-1} \) such that \( d_S(\bar{v}, \bar{w}) = \theta_i \) we can apply Lemma 10.2 again mapping \( p \) to \( p, v_i \) to \( \bar{v} \) and \( w_i \) to \( \bar{w} \) to get
\[
\exp_p(t_i \bar{v}) = \exp_p(t_i \bar{w}) \quad (11.3)
\]

Now for any \( w \in S^{n-1} \), \( d_S(w, v_0) = k_i \theta_i + \phi_i \) where \( \phi_i < \theta_i \). So there exists \( \bar{w}_i \in S^{n-1} \) such that \( d_S(w, \bar{w}_i) < \theta_i \). Applying \( 10.3 \) repeatedly along an exponential curve from \( w_i \) to \( v_0 \) at intervals of length \( \theta_i \), we get \( \exp_p(t_i v) = \exp_p(t_i \bar{w}_i) \). Taking \( i \) to infinity and using the continuity of \( \exp_p \), \( t_i \rightarrow t_0, w_i \rightarrow w \) we get \( \exp_y(t_0 v_0) = \exp_y(t_0 w) \). \[\square\]
Lemma 11.1 Suppose $Y$ is a simply connected locally isotropic exponential length space. If $y_1, y_2 \in Y$ and $g \in S0(u)$ then there is an isometry $f_g : Y \to Y$ such that $f_g(x) = \exp_{y_2} (g(\exp_{y_1}^{-1}(x)))$ for all $x \in B_g(D)$ where $D = \text{diam}(Y) \in (0, \infty]$.

Proof: By Lemma 11.1 we know that $\exp_{y_1}^{-1}$ is well defined on $B_g(t_0)$ where $t_0$ is the first conjugate point and the diameter. Here $t_0$ may be infinity. We need only verify that $f_g$ is an isometry from $B_g(D) \to B_{y_2}(D)$ since then it is forced to be a global isometry by continuity since $\partial B_g(D)$ is a single point for all $y \in Y$.

Since $Y$ is simply connected we need only verify that $f_g$ is a local isometry. That is, for all $x \in Y$, there exists $r > 0$ such that $f_g$ restricted to $B_x(r)$ maps isometrically onto $B_{f(x)}(r)$. We will choose $r < D - d_Y(x, y_1)$ to avoid trouble in the $S^n$ case.

Now fix $x_1 \in B_{y_1}(D)$ and let $s_1v_1 = \exp_{y_1}^{-1}(x_1)$. Let

$$\theta_1 = \min\{\theta_{(D, y_1, R/4)}, \theta_{D, y_2, R/4}\}, \quad (11.4)$$

Since $\exp_{y_1}$ is a homeomorphism, we can take $r$ to be sufficiently small that for all $x \in B_{x_1}(r)$, we have $d_S(\exp_{y_1}^{-1}(x)/|\exp_{y_1}^{-1}(x)|, v_1) < \theta_1$.

For any $z_1, z_2 \in B_{x_1}(r)$, let $s_iw_i = \exp_{y_1}^{-1}(z_i)$. Since $d_S(w_1, v_1) < \theta$, we can apply Lemma 11.1 to see that

$$d_Y(z_1, z_2) = d_Y(\exp_{y_1}(s_1w_1), \exp_{y_1}(s_2w_2)) = d_Y(\exp_{y_1}(s_1w_1), \exp_{y_1}(s_2w_2)) = d_Y(f_g(z_1), f_g(z_2)). \quad (11.5)$$

Note that the above Lemma implies that if we have a triangle formed by two length minimizing curves of lengths $a < D$ and $b < D$ and any angle $\theta$ between them then the length of the third side is determined depending only on $\theta$, $a$ and $b$.

Corollary 11.2 Suppose $Y$ is a simply connected locally isotropic exponential length space then its isotropy radius is the diameter $D$ or infinity in the unbounded case.

We now prove that given a triangle with sides of length $a$, $b$ and $c$ we can determine the angle opposite $c$, however we must restrict our lengths to avoid the pole which causes indeterminacy even in $S^n$.

Lemma 11.3 If $Y$ is a simply connected locally isotropic exponential length space with diameter $D \in [R, \infty]$ then for all $y \in Y$, $a, b, c \in (0, D)$, if $d_Y(\exp_y(bv), \exp_y(av)) = c$ then $d_S(v, w) = \theta(a, b, c)$. In particular, if $a + b + c = 2D$ then $\theta(a, b, c) = \pi$.

Proof: Corollary 11.2 and Lemma 11.1 imply the existence of $\theta(a, b, c)$.

Look at the triangle between the points $x_2$, $\exp_{x_2}(aw_2)$ and $\exp_{x_2}(bw_2)$. Join the later two points by a length minimizing curve $C(t)$ parametrized by arclength such that $C(0) = \exp_{x_2}(bv_2)$. Note that since $d_Y(x_2, C(t)) \leq D$ then $a + t \leq D$ and $b + (c - t) \leq D$ so $a + b + c \leq 2D$.

If $a + b + c = 2D$ then $C(t)$ hits the point $\bar{x}_2 = \partial B_{x_2}(D)$. Since $C(t)$ runs minimally to this point and $d_{\exp_{x_2}(tv_2)}(\bar{x}_2) = D - t$ including $t = a$, we know $C(t) = \exp_{x_2}(t + a)w_2$ for $t \in [0, D - a]$. Similarly $C(c - t) = \exp_{x_2}(t + b)v_2$. So we have an exponential curve running from $x_2$ through $\bar{x}_2$ and back to $x_2$ and $a + b + c = 2D$. Since this curve must be minimizing on segments of length $D = t_0$ by the by Corollary 11.3 we have $d_Y(\exp_{x_2}(2D)w_2), \exp_{x_2}((D/2)w_2)) = D$. However, we can join these points by a curve through $x_2$ of length $D$ so that curve must be an exponential curve and so $d_S(v_2, w_2) = \pi$ by Lemma 11.1.


We can now use Lemma 10.2 to extend isometries between subdomains of $Y$ to all of $Y$.

**Theorem 11.2** If $Y$ is a locally isotropic exponential length space which is simply connected. If $f : A \rightarrow B$ is an isometry between subsets of $Y$, then there is an extension of $f$ to an isometry from $Y$ to $Y$.

**Proof:** If $Y$ is unbounded then by Corollary 11.2, $R = \infty$ and this is a consequence of Lemma 10.2.

Suppose $Y$ is bounded. If $A = Y$ then there is nothing to do. If there exists $x \in Y \setminus A$ then $A \subset B_y(D)$ where $d_y(y, x) = D$. So we can apply Corollary 11.2 and Lemma 10.2 to extend $f$ to $B_y(D)$. Let $f(x)$ be the only point in $Y \setminus B_x(D)$. Then $f$ is continuous so it must be an isometry.

As a consequence of this theorem we know that there are global isometries mapping any point to any other point, that there are global isometries mapping any triangle to any other congruent triangle. In the unbounded case, combined with the fact that there the space is globally minimizing, we can use Busemann’s Theorem to state that $Y$ is either Euclidean or Hyperbolic space $[Bu]$.

12 Birkhoff’s Theorem and Local Isotropy

In this section we apply Birkhoff’s Theorem to complete proofs of Theorems 1.3 and 2.1 $[Birk]$. This theorem dates to 1941 and there are similar earlier theorems by Busemann which characterize $\mathbb{H}^n$ and $\mathbb{E}^n$ $[Birk]$. The difficulty with Busemann’s theorems is that they assume extendibility of the geodesics as minimizing curves and this is not true in the sphere. Birkhoff’s Theorem is stated in the following proof.

**Theorem 12.1** If $Y$ is a locally isotropic exponential length space which is simply connected then $Y$ is isometric to $\mathbb{S}^n$, $\mathbb{H}^n$ or $\mathbb{R}^n$ where $n$ is the exponential dimension of $Y$.

**Proof:** By Birkhoff’s Theorem $[Birk]$ any length space such that the following hold is such a simply connected space form:

a) $\bar{Y}$ has locally unique minimal geodesics.

b) any isometry on subsets of $\bar{Y}$ extends to an isometry of the whole space.

Now (a) follows from Lemma 7.1 and (b) follows from Theorem 11.2. Then we apply Theorem 11.1 to show that $n$ matches the dimension of the space form.

**Remark 12.1** We could also prove this theorem without quoting Birkhoff, but rather using a more recent theorem which states that all simply compact two point homogeneous manifolds are Riemannian manifolds. Our space is simply compact by the definition of exponential length space and it can be seen to be two point homogeneous by taking $A$ to contain exactly two points in Theorem 11.2. Once $Y$ is isometric to a Riemannian manifold, we know the exponential curves are geodesics because they are locally minimizing. We also know it is an $n$ dimensional manifold by Theorem 11.1 where $n$ is the exponential dimension of $Y$. Then we can apply Theorem 11.2 again with $A$ and $B$ sharing a geodesic and forcing the existence of isometries rotating around that geodesic to get constant sectional curvature. All simply connected space forms must be either $\mathbb{S}^n$, $\mathbb{H}^n$ or $\mathbb{E}^n$ (c.f. $[doC]$).

This would appear to be sufficient to complete the paper but we must relate the exponential structure of the space form to that of the exponential length space.

**Lemma 12.2** If $X$ and $Y$ are locally isotropic exponential length spaces off $W_X$ and $W_Y$ that are isometric to each other then they have the same exponential length structure. That is $f : Y \rightarrow X$. 

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an isometry implies that for all \( y \in Y \setminus W_Y \) mapped to \( f(y) \in X \setminus W_X \) there exists \( g_y \in S0(n) \) such that
\[
f(exp_y(v)) = exp_{f(z)}(g_y v).
\] (12.1)
and if \( R \) is the minimum of the isotropy radii of \( X \) and \( Y \), we have
\[
F_y(\theta, s, t) = F_{f(y)}(\theta, s, t) \quad \forall s, t < R.
\] (12.2)

Note that without assuming local isotropy on \( X \) this is false since \( \mathbb{R}^n \) can be given two distinct exponential length structures that are both isometric to Euclidean space. See Example 12.1. Note also that \( g_x \) need not be continuous in \( x \).

**Proof:** First since we are only making a statement about \( y \in Y \setminus W_Y \) and \( f(y) \in X \setminus W_X \) we can first choose \( Y' \) and \( X' \) to be their respective connected components and then extend the exponential structures to \( Cl(Y') \) and \( Cl(X') \) respectively using Lemma 13.8.

Thus, without loss of generality, we may assume \( W_Y \) and \( W_X \) are empty sets and can then apply all our lemmas concerning such spaces, using the fact that they are extended exponential length spaces.

Since \( \gamma(t) = exp_p(tv) \) is a length minimizing curve for \( t \in (0, R) \), \( f(\gamma(t)) \) must be as well. So \( f(\gamma(t)) \) is an exponential curve starting at \( f(p) \). Thus there is a map \( g_p : \mathbb{R}^n \to \mathbb{R}^n \) such that \( f(exp_p(v)) = exp_{f(p)}(g_p(v)) \). By Lemma 10.1 we have \( g_p(tv) = t g_p(v) \) even for negative \( t \). Using the fact that exponential curves are parameterized proportional to arclength we see that \( |g_p(v)| = L(f(\gamma([0, L]))) = L(\gamma([0, L])) = |v| \). So we need only verify that \( g_p \) is an isometry from \( S^{n-1} \) to \( S^{n-1} \).

By Lemma 10.1 we know that for \( \epsilon > 0 \) sufficiently small
\[
F_y(\pi, \epsilon, \epsilon) = d_Y(exp_y(\epsilon v), exp_y(-\epsilon v)) = 2 \epsilon
\] (12.3)
and
\[
F_{f(y)}(\pi, \epsilon, \epsilon) = d_Y(exp_{f(y)}(\epsilon v), exp_{f(y)}(-\epsilon v)) = 2 \epsilon.
\] (12.4)

We will prove that for any isometry \( f \) and \( g_f \) as above,
\[
F_y(\pi/2^k, \epsilon, \epsilon) = F_{f(y)}(\pi/2^k, \epsilon, \epsilon)
\] (12.5)
and
\[
d_S(v_1, v_2) = \pi/2^k \text{ implies that } d_S(g_f(v_1), g_f(v_2)) = \pi/2^k
\] (12.6)
by induction on \( k \).

When \( k = 0 \), (12.5) holds by Lemma 10.1 as described above. On the other hand \( d_S(v_1, v_2) = \pi \) implies \( v_1 = -v_2 \) so \( g_f(v_1) = -g_f(v_2) \) and we are done.

Assuming it is true for \( k = j \), let \( v_1, v_2 \in S^{n-1} \) be chosen such that \( d_S(v_1, v_2) = \pi/2^j \). Let \( w = (v_1 + v_2)/|v_1 + v_2| \). So
\[
d_Y(exp_y(\epsilon v_1), exp_y(\epsilon w)) = F_y(\pi/2^{j+1})d_Y(exp_y(\epsilon v_2), exp_y(\epsilon w))
\] (12.7)
Thus by the isometry,
\[
dx(exp_{f(y)}(\epsilon g_f(v_1)), exp_{f(y)}(\epsilon g_f(w))) = F_y(\pi/2^{j+1}) = d_X(exp_{f(y)}(\epsilon g_f(v_2)), exp_{f(y)}(\epsilon g_f(w))).
\] (12.8)
Thus by Lemma 13.4 \( d_S(g_f(v_1), g_f(w)) = d_S(g_f(v_2), g_f(w)) \). Since the triangle inequality gives, \( d_S(g_f v_1, g_f w) + d_S(g_f v_2, g_f w) \geq d_S(g_f v_1, g_f v_2) \), we know \( d_S(g_f v_1, g_f w) = d_S(g_f v_2, g_f w) \geq \pi/2^{j+1} \). By the properties of \( F_{f(y)} \) this implies that
\[
F_{f(y)}(\pi/2^{j+1}, \epsilon, \epsilon) \leq F_{f(y)}(d_S(g_f v_1, g_f w), \epsilon, \epsilon)
\] (12.9)
\[
= d_X(exp_{f(y)}(\epsilon g_f(v_1)), exp_{f(y)}(\epsilon g_f(w))) = F_y(\pi/2^{j+1}).
\] (12.10)
However, the same holds for the isometry $f^{-1}$ so we get the opposite inequality. Thus we get \((12.5)\) for \(k = j + 1\) and then Lemma \(6.4\) gives us \((12.6)\).

Thus applying \((12.6)\) and the properties of \(g_i\) shown at the top, we get,

\[
F_y(\pi/2^k, s, t) = d_Y(exp_y(tv_1), exp_y(sv_2)) = d_X(exp_f(y)(tg_f v_1), exp_f(y)(sg_f v_2)) \quad (12.11)
\]

\[
= F_f(y)(\pi/2^k, s, t) \quad \forall s, t < \min\{R_y, R_f(y)\}. \quad (12.12)
\]

Now choose any \(j \in \mathbb{N}\) such that \(j\pi/2^k < \pi\) and any \(v_1, v_2\) with \(d_S(v_1, v_2) = j\pi/2^k < \pi\). Let \(c\) be a length minimizing curve from \(c(0) = exp_y(tv_1)\) to \(c(L) = exp_y(sv_2)\) where \(L = F_f(y)(j\pi/2^k, s, t)\). Note that \(L < R\), so \(c\) cannot leave \(B_y(R)\). Then \(f(c(t))\) is length minimizing from \(exp_f(y)(tg_f v_1)\) to \(exp_f(y)(sg_f v_2)\) so \(L = F_f(y)(d_S(g_f v_1, g_f v_2), s, t)\).

We claim that \(v_i = exp_y^{-1}(c(t))/|exp_y^{-1}(c(t))|\) is in the minimizing geodesic segment between \(v_1\) and \(v_2\) for all \(t \in [0, L]\). If not at some point \(t\), there is an element \(g \in S0(n)\) which maps \(v_i\) to \(v_i\) but moves \(v_t\). Then the isometry \(f_g\) of Lemma \(11.1\) maps \(c(t)\) to another length minimizing curve with the same end points. But our space is locally minimizing, so there cannot be a second such curve and we have a contradiction.

Thus we can choose \(t_0 = 0 < t_1 < t_2 < \ldots < t_j = L\) so that \(c(t_h) = exp_y(s_h w_h)\) where \(d_S(w_h, w_{h+1}) = \pi/2^k\). This can be done using the intermediate value theorem and continuity of \(exp_y^{-1}\) on \(B_y(R)\). By \((12.6)\) \(d_S(g_f w_h, g_f w_{h+1}) = \pi/2^k\), so in fact

\[
\rightarrow d_S(g_f v_1, g_f v_2) \leq \sum_{i=0}^{j-1} d_S(g_f w_h, g_f w_{h+1}) = j\pi/2^k. \quad (12.13)
\]

Since this whole argument works for \(f^{-1}\) as well we get

\[
d_S(v_1, v_2) = j\pi/2^k \text{ implies that } d_S(g_r f(v_1), g_f(v_2)) = j\pi/2^k. \quad (12.14)
\]

Using this fact and the fact that \(L\) was preserved under the isometry \(f\), we get

\[
F_y(j\pi/2^k, s, t) = L = F_f(y)(j\pi/2^k, s, t) \quad \forall s, t < R = \min\{R_y, R_f(y)\}. \quad (12.15)
\]

Now given any \(\theta \in [0, \pi]\) and any \(v, w\) such that \(d_S(v, w) = \theta\) there exists \(v_i, w_i\) such that \(d_S(v_i, w_i) = j_i\pi/2^k\) so that substituting these \(v_i\) and \(w_i\) in \((12.5)\) and \((12.6)\) and taking \(i \rightarrow \infty\) we get

\[
d_S(v, w) = d_S(g_r f(v), g_f(w)) \quad (12.16)
\]

and

\[
F_y(\theta, s, t) = F_f(y)(\theta, s, t) \quad \forall s, t < R = \min\{R_y, R_f(y)\}. \quad (12.17)
\]

We can now prove that a locally isotropic exponential length space is a collection of manifolds with constant sectional curvature joined at discrete points.

**Proof of Theorem 1.3**

For the first part we need only show that if \(\bar{Y}\) is the universal cover of \((\text{Cl}(Y'))\) then it is either \(S^n\), \(H^n\) or \(E^n\) with their standard metrics. By Lemma \(7.3\) we know the \(\text{Cl}(Y')\) is isometric to a locally isometric exponential length space and so by Lemma \(6.3\) its universal cover \(\bar{Y}\) exists and is a simply connected locally isometric exponential length space. So by Theorem \(12.2\) \(\bar{Y}\) is isometric to \(S^n, H^n\) or \(E^n\). Thus \(\text{Cl}(Y')\) is isometric to a space form of some constant curvature \(K\). Lemma \(12.2\) then says that the exponential structures match and \(F_y(\theta, s, t) = F_K(\theta, s, t)\) for all \(y \in Y'\).

Once this is known we use the fact that \(W_Y\) is discrete, to piece together the various connected components of \(Y' \cap W_Y\) in a countable way.

\[\square\]
Finally we can prove Theorem 2.1 which implies Theorem 1.2.

**Proof of Theorem 2.1** By Gromov’s Compactness Theorem and the f ball packing property, we know that a subsequence of these $M_i$ (also called $M_i$) converges to some complete length space $Y$.

By Theorem 1.1 Lemma 5.1 and Lemma 6.4 we know that $Y$ is a locally isotropic exponential length space off a discrete set $W_x$. We can thus apply Theorem 1.3 to obtain the required properties of $Y$.

Since the exponential length structure originally given to $Y$ as a limit of the $M_i$ was defined to satisfy

$$\lim_{i \to \infty} F_{q_i}(\theta, s, t) = F_y(\theta, s, t)$$

(12.18)

and since Theorem 1.3 says that the exponential length structure on any $Cl(Y')$ must match that of a space form with constant sectional curvature $K$, we see that $F_y(\theta, s, t) = F_K(\theta, s, t)$. This implies 2.1.

13 Ricci Curvature

In this section we will apply Theorem 2.1 combined with the lower Ricci curvature bound to prove Theorem 1.2.

**Proof of 1.2** First note that if $M$ has a fixed lower bound on Ricci curvature then by Bishop-Gromov, it satisfies the conditions of Theorem 1.2. Furthermore $Y$ can be described as a limit of $M_i$ satisfying this uniform Ricci curvature bound just as in Theorem 2.1 except that now we have additional measure properties on $Y$ proven by Colding. We will show $Y$ is a space form by showing $Y = Cl(Y')$.

If $Y \neq Cl(Y')$ then by Theorem 1.2 we know that $Y$ contains at least two space forms $Cl(Y')$ and $Cl(Y'')$ joined at a common point $y_0$. Let $p_i \to y$. We know $vol(B_{y_i}(r))$ converges to $vol(B_y(r))$ by Colding’s Volume Convergence Theorem [Co]. By the properties of $Cl(Y')$ and $Cl(Y'')$, we know

$$vol(B_y(r)) \geq V(n, K', r) + V(n, K'', r) \forall r > 0.$$ (13.1)

On the other hand, since we don’t have dense bad points then there exists $y_1$ near $y$ in $Y'$, such that $vol(B_{y_1}(r)) = V(n, K', r)$ for all $r$ sufficiently small. Let $d(y_1, y_0) = r_1$. Take an annulus about $y_1$ which includes $y_0$ but no other bad points. By volume comparison with $H = \min \{K', K''\}$:

$$Vol(Ann_{y_1}(r_1 - r, r_1 + r)) \leq \frac{V(n, H, r_1 + r) - V(n, H, r_1 - r)}{V(n, H, r_1 - r)} Vol(B_x(r_1 - r)).$$ (13.2)

By the space forms:

$$Vol(Ann_{y_1}(r_1 - r, r_1 + r)) \geq \frac{Vol(Ann \cap Y') + Vol(Ann \cap Y'')}{Vol(B_x(r_1 - r))} = \frac{V(n, K', r_1 + r) - V(n, K', r_1 - r) + V(n, K'', r)}{V(n, K', r_1 - r)}.$$ (13.3)

Putting this together and using $r_1 - r < r_x$, $Vol(B_x(r_1 - r)) = V(n, K', r_1 - r)$ we have

$$\frac{V(n, H, r_1 + r) - V(n, H, r_1 - r)}{V(n, H, r_1 - r)} \geq \frac{V(n, K', r_1 + r) - V(n, K', r_1 - r) + V(n, K'', r)}{V(n, K', r_1 - r)}.$$ (13.5)

Now we can take any $y_1$ close to $y_0$ and set $r_1 = 2r$.

$$\frac{V(n, H, 3r) - V(n, H, r)}{V(n, H, r)} \geq \frac{V(n, K', 3r) - V(n, K', r)}{V(n, K', r)}.$$ (13.6)

Taking $r \to 0$ and using the fact $\lim_{r \to 0} V(n, H, r)/r^n = \omega_n$, we get the impossible limit:

$$\frac{3^n - 1^n}{1^n} \geq \frac{3^n - 1^n + 1^n}{1^n}. $$ (13.7)

\[\square\]
References

[And] M. T. Anderson. *Convergence and rigidity of Manifolds under Ricci curvature bounds*, Invent. Math. 102 (1990), no. 2, 429–445.

[Bic] J. Bicak, *Selected solutions of Einstein’s field equations: their role in general relativity and astrophysics*, Lect. Notes Phys. 540 (2000) 1-126.

[Birk] G. Birkhoff. *Metric foundations of geometry*. Proc. Nat. Acad. Sci. U. S. A. 27, (1941). 402–406.

[BiCr] R. Bishop, R. Crittenden. *Geometry of manifolds*. Reprint of the 1964 original. AMS Chelsea Publishing, Providence, RI, 2001.

[Br] H. L. Bray. *Black holes and the Penrose inequality in general relativity*, Proceedings of the ICM, Beijing 2002, vol. 2, 257–272.

[BBI] D. Burago, Y. Burago, S. Ivanov, *A course in Metric Geometry*. Graduate Studies in Mathematics Vol. 33, AMS, 2001.

[Bu] H. Busemann. *On Leibnitz’s Definition of planes*, Amer. J. Math. 63, 101–111 (1941);

[Co] T.H. Colding. *Ricci curvature and volume convergence*. Ann. of Math. (2) 145 (1997), no. 3, 477–501.

[CW] N. J. Cornish, Neil J. and J.R. Weeks. *Measuring the shape of the universe*. Notices Amer. Math. Soc. 45 (1998), no. 11, 1463–1471.

[Cur] R. J. Currier. *Spheres with locally pinched metrics*. Proc. Amer. Math. Soc. 106 (1989), no. 3, 803–805.

[doC] M. DoCarmo. *Riemannian Geometry*. Translated by Francis Flaherty. Mathematics: Theory & Applications. Birkhauser Boston, Inc., Boston, MA, 1992.

[DyRo] C. C. Dyer and R. C. Roeder, *Astrophys. J. 180 L31* (1973)

[EilSt] S. Eilenberg and N. Steenrod. *Foundations of algebraic topology*. Princeton University Press, Princeton, New Jersey, 1952.

[Fra] T. Frankel. *Gravitational Curvature: An Introduction to Einstein’s Theory*. W.H. Freeman and Company, 1979.

[GabLab] A. Gabrielli and F. S. Labini, *Fluctuations in galaxy counts: a new test for homogeneity versus fractality* Europhys. Lett. vol. 54, issue 3, 1 May 2001

[Grib] I. V. Gribkov. *A multidimensional problem on the correctness of Schur’s theorem*. (Russian) Mat. Sb. (N.S.) 120(162) (1983), no. 3, 426–440.

[Gr] M. Gromov. *Metric structures for Riemannian and non-Riemannian spaces*. PM 152, Birkhauser, 1999.

[GrPet] K. Grove and P. Petersen. *Manifolds near the boundary of existence*. Journal of Differential Geometry, Vol 33 (1991) 379-394.

[HoWa] D. E. Holz and R. M. Wald. *New Method for Determining the cumulative gravitational lensing effects in inhomogeneous universes*, Physical Review D. Vol 58, 063501 (1998)
[Kan] R. Kantowski. *Corrections in the Luminosity-Redshift Relations of the Homogeneous Friedmann Models*, The Astrophysics Journal, Vol 155, January 1969.

[Kras] A. Krasinski. *Physics in an inhomogeneous universe*, Cambridge University Press (1996)

[KSHM] D. Kramer, E. Stephani, E. Hertl and M.A.H. MacCallum, *Exact solutions of Einstein field equations* Cambridge University Press (1990)

[Nik] I. G. Nikolaev. *Stability problems in a theorem of F. Schur*. (English) Comment. Math. Helv. 70 (1995), no. 2, 210–234.

[Peeb] P.J.E. Peebles *Principles of Physical Cosmology*. Princeton University Press, 1993.

[SchYau] R. Schoen and S.-T. Yau. *On the Proof of the Positive Mass Conjecture in General Relativity*. Comm. Math. Phys. 65 (1979) 45-76.

[SoWei] C. Sormani and G. Wei, *Hausdorff Convergence and Universal Covers*, Transactions of the American Mathematical Society 353 (2001) 3585-3602.