Chern-Simons Gauge Theory coupled with BF Theory

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Abstract

We couple three-dimensional Chern-Simons gauge theory with BF theory and study deformations of the theory by means of the antifield BRST formalism. We analyze all possible consistent interaction terms for the action under physical requirements and find a new topological field theory in three dimensions with new nontrivial terms and a nontrivial gauge symmetry. We analyze the gauge symmetry of the theory and point out the theory has the gauge symmetry based on the Courant algebroid.

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1 Introduction

The Chern-Simons gauge theory in three dimensions is a Schwarz type topological field theory \[1\]. In this paper, we analyze nontrivial deformations of the Chern-Simons gauge theory in three dimensions as a topological field theory by the deformation theory of gauge symmetry.

The Chern-Simons gauge theory appears in many scenes of mathematics and physics. One of the main applications of the Chern-Simons gauge theory is to the knot theory \[2\]. The connections of the Chern-Simons gauge theory to several knot and link invariants are reviewed in [3]. The Einstein-Hilbert action in three dimensional gravitational theory can be formulated as a Chern-Simons gauge theory \[4\]. In the cubic string field theory, the action has the integral of the Chern-Simons type three-form \[5, 6\]. [7] reviews the Chern-Simons gauge theory and its applications. The perturbation theory has been discussed in [8].

Gauge symmetry is one of the fundamental principles of the quantum field theory. A deformation theory of the gauge theory \[9, 10\] is a powerful method to construct a new gauge theory or to prove impossibility of the construction of new gauge theories under a certain condition. We can construct gauge theories with generalized gauge algebras by this method. ‘Generalized’ means that the gauge algebra of the theory are not based on usual Lie groups but based on an extended algebra as a constraint system. In general, ‘structure constants’ depend on fields and are structure functions. Moreover in this case the gauge algebra is usually the open algebra, therefore we need analyze the gauge theory by the Batalin-Vilkovisky (antifield BRST) formalism.

The Chern-Simons gauge theory is constructed from a 1-form gauge field \(A^a\). The action of the abelian Chern-Simons theory is as follows:

\[
S_{ACS} = \int_X \frac{k_{ab}}{2} A^a \wedge dA^b, \tag{1}
\]

where \(k_{ab}\) is a symmetric constant tensor and \(X\) is a three dimensional manifold. Of course, this theory has the abelian gauge symmetry, \(\delta_0 A^a = dc^a\), where \(c^a\) is a gauge parameter.

Barnich and Henneaux have proved that we can only deform this theory to the known non-abelian Chern-Simons gauge theory from the consistency of the gauge symmetry and locality of the action \[9\]. That is, the only consistent gauge theory which we can obtain as
deformations of the action (1) is the nonabelian Chern-Simons gauge theory:

\[ S_{CS} = \int_X \left( \frac{k_{ab}}{2} A^a \wedge dA^b + \frac{1}{6} f_{abc} A^a \wedge A^b \wedge A^c \right), \]  

(2)

where \( f_{abc} \) satisfies the relation of the structure constants of the Lie algebra.

In the string field theory, we can generalize the cubic string field theory to the nonpolynomial string field theory with extended gauge algebras, \( A_\infty \)– or \( L_\infty \)–algebra structures.

It seems to be natural if we can deform the Chern-Simons gauge theory to a field theory with extended gauge algebras. However it is only analogical motivation, and relation of our theory with string field theory is out of scope in this paper.

A generalization of the Chern-Simons gauge theory has also been discussed in [15] or [16]. We consider an other generalization in this paper. Now we can couple the Chern-Simons gauge theory with an another Schwarz-type topological field theory, BF theory. We call this theory as the Chern-Simons-BF theory. Then we systematically analyze all the BRST cohomologies and possible deformations. We find a nontrivial new deformation of the gauge symmetry and a new action. We can consider that this theory is a higher dimensional generalization of the nonlinear gauge theory.

This paper is organized as follows. In section 2, we construct the superfield antifield formalism of the abelian Chern-Simons BF theory. In section 3, we analyze deformations of the abelian Chern-Simons BF theory and obtain all possible deformations. In section 4, we calculate the explicit action of our theory. In section 5, we analyze the gauge symmetry of the theory and find that the gauge symmetry has the Courant algebroid structure. Section 6 is conclusion and discussion.

2 Superfield Formalism of the Abelian Chern-Simons-BF Theory

We begin with a three dimensional base manifold \( X \), a target manifold \( M \) in \( N \) dimensions and smooth maps \( \phi : X \to M \) with local coordinate expression \( \{ \phi^i \} \). We also have a vector bundle \( E \) over \( X \).

In three dimensions, abelian Chern-Simons Theory with the abelian BF theory has the
following action:
\[ S_{\lambda} = \int_{X} \left( \frac{k_{ab}}{2} A^a \wedge dA^b - B_i \wedge d\phi^i \right), \quad (3) \]

where \( \phi^i \) is a 0-form scalar field, \( A^a \) is a 1-form and \( B_i \) is a 2-form gauge field and \( k_{ab} \) is a symmetric constant tensor. We call the spacetime integration of a BF term as BF theories, where \( A \) is a \( p \)-form, \( F \) is a curvature of \( A \) and \( B \) is a \( n - p - 1 \)-form in \( n \) dimensions. In three dimensions, there are two actions, the \( p = 1 \) action and the \( p = 0 \) action. The \( p = 2 \) action is equivalent to the \( p = 0 \) action if the term is integrated by parts.

The sign factor \(-1\) before the second term is introduced for convenience. We assume that \( k_{ab} \) is nondegenerate and has an inverse. However it is not necessary that \( k_{ab} \) is positive definite.

We can take different target spaces for the first term and the second term in the action (3). Indices \( a, b, c \), represent indices on the fiber of \( E \) and and \( i, j, k \), represent indices on \( M \), the tangent and cotangent space of \( M \).

We can add the following usual BF term to the action as a topological field theory:
\[ \int C_a \wedge dA^a, \quad (4) \]

where \( C_a \) is an auxiliary 1-form field. The action still have the abelian gauge symmetry. However if we make the local field redefinition \( C'_a = C_a + \frac{1}{2}k_{ab}A^b \), the theory reduces to the pure abelian BF theory, which deformation is already discussed in the papers [20].

We can consider the more general terms \((k_{ab}(\phi)/2) A^a \wedge dA^b \) or \( m_{ij}(\phi)B_i \wedge d\phi^j \) in the action, where \( k_{ab}(\phi) \) and \( m_{ij}(\phi) \) are functions of \( \phi^i \). However these terms reduces to the action (3) by local field redefinitions. If two actions coincide by a local redefinition of fields, two theories are equivalent at least classically. We call the theory with the action (3) the abelian Chern-Simons-BF theory.

This action has the following abelian gauge symmetry:
\[ \begin{align*}
\delta_0 \phi^i &= 0, & \delta_0 A^a &= dc^a, & \delta_0 c^a &= 0, \\
\delta_0 B_i &= dt_i, & \delta_0 t_i &= dv_i, & \delta_0 v_i &= 0, 
\end{align*} \quad (5) \]

where \( c^a \) is a 0-form gauge parameter and \( t_i \) is a 1-form gauge parameter. Since \( B_i \) is 2-form, we need a 'ghost for ghost' 0-form \( v_i \).
In order to analyze the theory by the antifield BRST formalism, first we take $c^a$ and $t_i$ to be the Grassmann odd FP ghosts with ghost number one, and $v_i$ to be a the Grassmann even ghost with ghost number two. Next we introduce the antifields for all the fields. Let $\Phi^+$ denote the antifields for the field $\Phi$. Note that the relations $\deg(\Phi) + \deg(\Phi^+) = 3$ and $\gh(\Phi) + \gh(\Phi^+) = -1$ are required, where $\deg(\Phi)$ and $\deg(\Phi^+)$ are the form degrees of the fields $\Phi$ and $\Phi^+$ and $\gh(\Phi)$ and $\gh(\Phi^+)$ are the ghost numbers of them. For functions $F(\Phi, \Phi^+)$ and $G(\Phi, \Phi^+)$ of the fields and the antifields, we define the antibracket as follows:

\[ (F, G) \equiv \left( F \left( \frac{\delta}{\partial \Phi} \frac{\delta}{\partial \Phi^+} - \frac{\delta}{\partial \Phi^+} \frac{\delta}{\partial \Phi} \right) G \right), \]  

(6)

where $\frac{\delta}{\partial \varphi}$ and $\frac{\delta}{\partial \varphi}$ are the right differentiation and the left differentiation with respect to $\varphi$, respectively. If $S, T$ are two functionals, the antibracket is defined as follows:

\[ (S, T) \equiv \int_X \left( S \left( \frac{\delta}{\partial \Phi} \frac{\delta}{\partial \Phi^+} - \frac{\delta}{\partial \Phi^+} \frac{\delta}{\partial \Phi} \right) T \right). \]  

(7)

The Batalin-Vilkovisky action with the antifields is constructed as follows:

\[ S_0 = \int_X \left( \frac{k_{ab}}{2} A^a \wedge dA^b - B_i \wedge d\phi^i - A^+_a \wedge dc^a + B^{+i} \wedge dt_i + t^{+i} \wedge dv_i \right). \]  

(8)

The gauge transformation is defined as $\delta_0 F = (S_0, F)$ in the BV action. Then the action (8) has the gauge transformation (the BRST transformation) (9). The BRST transformation on all fields are calculated as follows:

\[ \begin{align*}
\delta_0 c^+_a &= dA^+_a, \\
\delta_0 A^a &= k_{ab} dA^b, \\
\delta_0 A^a &= -dc^a, \\
\delta_0 c^a &= 0, \\
\delta_0 v^{+i} &= -dt^{+i}, \\
\delta_0 t^{+i} &= dB^{+i}, \\
\delta_0 B^{+i} &= -d\phi^i, \\
\delta_0 \phi^i &= 0, \\
\delta_0 B_i &= dt_i, \\
\delta_0 t_i &= -dv_i, \\
\delta_0 v_i &= 0.
\end{align*} \]  

(9)

In order to simplify notations and calculations, we rewrite notations by the superfield formalism. We combine the field, its antifield and their gauge descendant fields as superfield components. For $\phi^i$, $A^a$ and $B_i$, we define corresponding superfields as follows:

\[ \phi^i = \phi^i + B^{+i} + t^{+i} + v^{+i}, \]
\[ A^a = c^a + A^a + k^{ab} A^a_b + k^{ab} c^a_b, \]
\[ B_i = v_i + t_i + B_i + \phi^i_v. \]  \hspace{1cm} (10)

Then we define the total degree \(|F| \equiv ghF + \text{deg } F\). The component fields in a superfield have the same total degree. The total degrees of \(\phi^i\), \(A^a\) and \(B_i\) are 0, 1 and 2, respectively.

We introduce a notation \(\cdot\) as the \textit{dot product} among superfields in order to simplify the sign factors. The definitions and properties of the \textit{dot product} are listed in the appendix B.

The antibracket (3) and (7) are rewritten to the \textit{dot antibracket} on superfields and \textit{dot product}. The \textit{dot antibracket} of the superfields \(F\) and \(G\) is defined as
\[ \langle (F, G) \rangle \equiv (-1)^{(ghF + 1)(\text{deg } G - 3)}(-1)^{gh\Phi(\text{deg } \Phi - 3) + 3}(F, G), \]  \hspace{1cm} (11)

The properties are listed in the appendix B. We can rewrite the BV antibracket on two superfields \(F\) and \(G\) from (3) and (19) as follows:
\[ \langle (F, G) \rangle \equiv F \cdot \frac{\partial}{\partial A^a} \cdot k^{ab} \frac{\partial}{\partial A^b} \cdot G + F \cdot \frac{\partial}{\partial \phi^i} \cdot \frac{\partial}{\partial B_i} \cdot G - F \cdot \frac{\partial}{\partial B_i} \cdot \frac{\partial}{\partial \phi^i} \cdot G. \]  \hspace{1cm} (12)

We rewrite the Batalin-Vilkovisky action (8) for the abelian Chern-Simons-BF theory by the superfields as follows:
\[ S_0 = \int_X \left( \frac{k^{ab} A^a \cdot dA^b - B_i \cdot d\phi^i}{2} \right), \]  \hspace{1cm} (13)

where we integrate only 3-form part of the integrand. Integration on \(X\) is always understood as the integration of the 3-form part of the integrand. The BRST transformation for a superfield \(F\) under the action above is obtained as
\[ \delta_0 F = \langle (S_0, F) \rangle = S_0 \cdot \frac{\partial}{\partial A^a} \cdot k^{ab} \frac{\partial}{\partial A^b} \cdot F + S_0 \cdot \frac{\partial}{\partial \phi^i} \cdot \frac{\partial}{\partial B_i} \cdot F - S_0 \cdot \frac{\partial}{\partial B_i} \cdot \frac{\partial}{\partial \phi^i} \cdot F. \]  \hspace{1cm} (14)

Hence we can summarize the BRST transformations on \(\phi^i\), \(A^a\) and \(B_i\) as follows:
\[ \delta_0 \phi^i = \langle (S_0, \phi^i) \rangle = d\phi^i, \]
\[ \delta_0 A^a = \langle (S_0, A^a) \rangle = dA^a, \]
\[ \delta_0 B_i = \langle (S_0, B_i) \rangle = dB_i, \]  \hspace{1cm} (15)

which coincide with (3) if we expand them to the component fields. Equations of motion are
\[ d\phi^i = 0, \quad dA^a = 0, \quad dB_i = 0. \]  \hspace{1cm} (16)
$S_0$ must be BRST invariant. In fact,

$$\delta_0 S_0 = \langle (S_0, S_0) \rangle = 2 \int_X d \left( \frac{k^{ab}}{2} A^a \cdot dA^b - B_i \cdot d\phi^i \right)$$

(17)

therefore if the base manifold $X$ has no boundary, simply $\delta_0 S_0 = 0$. If $X$ has a boundary we can take two kinds of boundary conditions (i) $A^a_{//\partial X} = 0$ and $B_i_{//\partial X} = 0$, or (ii) $A^a_{/\partial X} = 0$ and $\phi^i_{/\partial X} = 0$, where the notation $//\partial X$ mean the components along the direction tangent to the boundary $\partial X$. We can also take different boundary conditions on each field component so as to satisfy BRST invariant condition of the action. In the rest of this paper, we select appropriate boundary conditions so as to satisfy $\delta_0 S_0 = 0$ if we consider $X$ with boundaries.

3 Deformation of Chern-Simons-BF Theory

Let us consider a deformation of the action $S_0$ perturbatively,

$$S = S_0 + gS_1 + g^2S_2 + \cdots,$$

(18)

where $g$ is a deformation parameter, or a coupling constant of the theory.

In order for the deformed BRST transformation $\delta$ to be nilpotent and make the theory consistent, the total action $S$ has to satisfy the following classical master equation:

$$\langle (S, S) \rangle = 0.$$  

(19)

Substituting (18) to (19), we obtain the $g$ power expansion of the master equation:

$$\langle (S, S) \rangle = \langle (S_0, S_0) \rangle + 2g\langle (S_0, S_1) \rangle + g^2[\langle (S_1, S_1) \rangle + 2\langle (S_0, S_2) \rangle] + O(g^3) = 0.$$  

(20)

We solve this equation order by order. Here we make the physical requirements for the solutions. We require the Lorentz invariance (Lorentzian case), or $SO(3)$ invariance (Euclidean case) of the action. We assume that $S$ is local. This means that $S$ is given by the integration of a local Lagrangian, $S = \int_X \mathcal{L}$. Furthermore we exclude the solution which is the BRST transformation is not deformed, for example, $\delta = \delta_0$, as a trivial one. This condition is realized by the assumption that each term contains at least one antifield for $S_i$, where $i \geq 1$.  

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At the 0-th order, we obtain $\delta_0 S_0 = \langle (S_0, S_0) \rangle = 0$, which is already satisfied from (17).

At the first order of $g$ in the Eq. (20),

$$\delta_0 S_1 = \langle (S_0, S_1) \rangle = 0,$$  \hspace{1cm} (21)

is required. $S_1$ is given by the integration of a *local* Lagrangian from the assumption:

$$S_1 = \int_M L_1,$$  \hspace{1cm} (22)

where $L_1$ can be constructed from the superfields $\phi^i, A^a$ and $B_i$. If a monomial in $L_1$ includes a differentiation $d$, its term is proportional to the equations of motion (16). Therefore its term can be absorbed to the abelian action (13) by the local field redefinitions of $\phi^i, A^a$ or $B_i$, and these terms are BRST trivial at the BRST cohomology. Hence the nontrivial deformation terms must not include the differentiation $d$ and we can write the candidate $L_1$ as

$$L_1 = \sum_{k,l} F_{kl, a_1...a_k i_1...i_l} (\phi^a) \cdot A^{a_1} \cdots A^{a_k} \cdot B_{i_1} \cdots B_{i_l},$$  \hspace{1cm} (23)

where $F_{kl, a_1...a_k i_1...i_l}(\phi^a)$ is a function of $\phi^i$. In order to consider the general deformations, we do not require the total degree of $L_1$ is 3. If the total degree of $L_1$ is not 3, the action $S_1$ includes a nonzero ghost number term. Then (21) is calculated as follows:

$$\delta_0 S_1 = \sum_{k,l} \int_X \left[ dF_{kl, a_1...a_k i_1...i_l} (\phi^a) \cdot A^{a_1} \cdots A^{a_k} \cdot B_{i_1} \cdots B_{i_l} ight. $$

$$+ \sum_{r=1}^k (-1)^{r-1} F_{kl, a_1...a_k i_1...i_l} (\phi^a) \cdot A^{a_1} \cdots dA^a \cdots A^{a_k} \cdot B_{i_1} \cdots B_{i_l} $$

$$+ \sum_{s=1}^l (-1)^s F_{kl, a_1...a_k i_1...i_l} (\phi^a) \cdot A^{a_1} \cdots A^{a_k} \cdot B_{i_1} \cdots dB_i \cdots B_{i_l} 

\left. \right] $$

$$= \sum_{k,l} \int_X d[F_{kl, a_1...a_k i_1...i_l} (\phi^a) \cdot A^{a_1} \cdots A^{a_k} \cdot B_{i_1} \cdots B_{i_l}].$$  \hspace{1cm} (24)

If there is no boundary in $X$, there is no restriction for $S_1$ and we obtain $\delta_0 S_1 = 0$. If there are boundaries in $X$, $\delta_0 S_1 = 0$ if

$$(F_{kl, a_1...a_k i_1...i_l} (\phi^a) \cdot A^{a_1} \cdots A^{a_k} \cdot B_{i_1} \cdots B_{i_l})/\delta X = 0.$$  \hspace{1cm} (25)

$S_1$ must be constructed from the terms which satisfy the requirements above. If we take the boundary condition (i) then (25) is satisfied if the terms include at least one $A^a$ or one $B_i$.

If we take (ii) then (25) is satisfied if the terms include at least one $A^a$ or one $\phi^i$. 

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At the second order of $g$,

$$
\left\langle \left( S_1, S_1 \right) \right\rangle + 2\left\langle \left( S_0, S_2 \right) \right\rangle = 0,
$$

(26)
is required. We cannot construct nontrivial $S_2$ to satisfy (26) from the integration of a local Lagrangian, because $\delta_0$-BRST transforms of the local terms are always total derivative. Therefore if we assume locality of the action, $S_2$ is BRST trivial (the Poincaré lemma), because we consider the local deformations on the space of field theories. If we solve the higher order $g$ part of the equation (20) recursively, we can find that $S_\alpha$ is BRST trivial for $\alpha \geq 2$. Therefore we can set $S_\alpha = 0$ for $\alpha \geq 2$. Then the condition (26) reduces to

$$
\left\langle \left( S_1, S_1 \right) \right\rangle = 0.
$$

(27)

This equation imposes the identities on the structure functions $F_{k l, a_1 \cdots a_k} i_1 \cdots i_l (\phi)$ in (23). Now we have obtained the possible deformations of the Chern-Simons-BF theory in three dimensions from (13) and (23) as follows:

$$
S = S_0 + gS_1 = \int_X \left( \frac{k_{ab}}{2} A^a \cdot dA^b - B_i \cdot d\phi^i \\
+ g \sum_{k, l} F_{k l, a_1 \cdots a_k} i_1 \cdots i_l (\phi) \cdot A^{a_1} \cdots A^{a_k} \cdot B_{i_1} \cdots B_{i_l} \right),
$$

(28)

with the condition (27) on the structure functions $F_{k l, a_1 \cdots a_k} i_1 \cdots i_l (\phi)$. The master equation (19) reduces to $\delta_0 S_1 + g/2 \left\langle \left( S_1, S_1 \right) \right\rangle = 0$. This is nothing but the Maurer-Cartan equation under the differential $\delta_0$.

## 4 Chern-Simons Sigma Model

As a nontrivial example, let us solve the condition (27) explicitly in case that the ghost number of the total action is zero. This assumption enables us to restrict the action to the following form:

$$
S = \int_X \left( \frac{k_{ab}}{2} A^a \cdot dA^b - B_i \cdot d\phi^i + f_{1a}^i (\phi) \cdot A^a \cdot B_i + \frac{1}{6} f_{2abc} (\phi) \cdot A^a \cdot A^b \cdot A^c \right),
$$

(29)

where we rewrite two structure functions $f_{1a}^i = gF_{11,a}^i$ and $\frac{1}{6} f_{2abc} = gF_{30,abc}$ for clarity.
If we substitute (28) to the condition (27), we obtain the identities on the structure functions $f_{1a}^i$ and $f_{2abc}$ as follows:

$$k^{ab}f_{1a}^i \cdot f_{1b}^j = 0,$$

$$\frac{\partial f_{1b}^j}{\partial \phi^i} \cdot f_{1c}^j = \frac{\partial f_{1c}^i}{\partial \phi^i} \cdot f_{1b}^j + k^{ej}f_{1e}^i \cdot f_{2be} = 0,$$

$$\left( f_{1d}^j \cdot \frac{\partial f_{2abc}}{\partial \phi^j} - f_{1e}^j \cdot \frac{\partial f_{2ab}}{\partial \phi^i} + f_{1b}^j \cdot \frac{\partial f_{2cda}}{\partial \phi^j} - f_{1a}^j \cdot \frac{\partial f_{2bd}}{\partial \phi^j} \right)$$

$$+ k^{ej}(f_{2eab} \cdot f_{2edf} + f_{2ace} \cdot f_{2dfe} + f_{2ead} \cdot f_{2bej}) = 0.$$

(30)

The BRST transformation of each field is calculated from the definition of the BRST transformation $\delta F = \langle (S, F) \rangle$:

$$\delta A^a = dA^a + k^{ab}f_{1b}^j \cdot B_j + \frac{1}{2}k^{ab}f_{2bcd} \cdot A^c \cdot A^d,$$

$$\delta B_i = dB_i + \frac{\partial f_{1b}^j}{\partial \phi^i} \cdot A^j \cdot B_j + \frac{1}{6}\frac{\partial f_{2bcd}}{\partial \phi^i} \cdot A^b \cdot A^c \cdot A^d,$$

$$\delta \phi^i = d\phi^i - f_{1b}^i \cdot A^b.$$

(31)

If we set all the antifields zero, we obtain the usual action without antifields as follows:

$$S = \int_X \left( \frac{k^{ab}}{2} A^a \wedge dA^b - B_i \wedge d\phi^i + f_{1a}^i(\phi)A^aB_i + \frac{1}{6}f_{2abc}(\phi)A^aA^bA^c \right),$$

(32)

with the gauge symmetry:

$$\delta A^a = dc^a + k^{ab}f_{1b}^i t_j + k^{ab}f_{2bcd}A^c \cdot A^d,$$

$$\delta B_i = dt_i + \frac{\partial f_{1b}^j}{\partial \phi^i} (A^j t_j - c^b B_j) + \frac{1}{2}\frac{\partial f_{2bcd}}{\partial \phi^i} A^b A^c A^d,$$

$$\delta \phi^i = -f_{1b}^i \cdot A^b.$$

(33)

The identities on the structure functions is obtained as:

$$k^{ab}f_{1a}^i(\phi)f_{1b}^j(\phi) = 0,$$

$$\frac{\partial f_{1b}^i(\phi)}{\partial \phi^j} f_{1c}^j(\phi) - \frac{\partial f_{1c}^i(\phi)}{\partial \phi^j} f_{1b}^j(\phi) + k^{ej}f_{1e}^i(\phi)f_{2be}(\phi) = 0,$$

$$\left( f_{1d}^j(\phi) \frac{\partial f_{2abc}}{\partial \phi^j} - f_{1c}^j(\phi) \frac{\partial f_{2ab}}{\partial \phi^i} + f_{1b}^j(\phi) \frac{\partial f_{2cda}}{\partial \phi^j} - f_{1a}^j(\phi) \frac{\partial f_{2bd}}{\partial \phi^j} \right)$$

$$+ k^{ej}(f_{2eab}(\phi)f_{2edf}(\phi) + f_{2ace}(\phi)f_{2dfe}(\phi) + f_{2ead}(\phi)f_{2bej}(\phi)) = 0.$$

(34)

If $f_{1a}^j = 0$ and $f_{2abc}$ is a constant, (34) reduces to the usual Jacobi identity of the Lie algebra structure constants and we have the nonabelian gauge symmetry. However in general $f_{2abc}(\phi)$ depends on the fields, and the theory has a generalization of the nonabelian gauge symmetry.
5 Courant Algebroid

Let us analyze the identities (30) on the structure functions \( f_1 \) and \( f_2 \), which is equivalent to (34). The gauge algebra under this theory is the Courant algebroid.

A Courant algebroid is introduced by Courant in order to analyze the Dirac structure as a generalization of the Lie algebra of the vector fields on the vector bundle. A Courant algebroid is a vector bundle \( E \to M \) and has a nondegenerate symmetric bilinear form \( \langle \cdot, \cdot \rangle \) on the bundle, a bilinear operation \( \circ \) on \( \Gamma(E) \)(the space of sections on \( E \)), an a bundle map \( \rho : E \to T \mathcal{M} \) satisfying the following properties:

1. \( e_1 \circ (e_2 \circ e_3) = (e_1 \circ e_2) \circ e_3 + e_2 \circ (e_1 \circ e_3) \),
2. \( \rho(e_1 \circ e_2) = [\rho(e_1), \rho(e_2)] \),
3. \( e_1 \circ F e_2 = F(e_1 \circ e_2) + (\rho(e_1)F)e_2 \),
4. \( e_1 \circ e_2 = \frac{1}{2} \mathcal{D}\langle e_1, e_2 \rangle \),
5. \( \rho(e_1)\langle e_2, e_3 \rangle = \langle e_1 \circ e_2, e_3 \rangle + \langle e_2, e_1 \circ e_3 \rangle \),

where \( e_1, e_2 \) and \( e_3 \) are sections of \( E \), \( F \) is a function on \( \mathcal{M} \). \( \mathcal{D} \) is a map from functions on \( \mathcal{M} \) to \( \Gamma(E) \) and is defined as \( \langle \mathcal{D}F, e \rangle = \rho(e)F \). Let \( e^a \) be basis of \( \Gamma(E) \) with respect to the fiber. Then (35) is written as

1. \( e^a \circ (e^b \circ e^c) = (e^a \circ e^b) \circ e^c + e^b \circ (e^a \circ e^c) \),
2. \( \rho(e^a \circ e^b) = [\rho(e^a), \rho(e^b)] \),
3. \( e^a \circ F e^b = F(e^a \circ e^b) + (\rho(e^a)F)e^b \),
4. \( e^a \circ e^b = \frac{1}{2} \mathcal{D}\langle e^a, e^b \rangle \),
5. \( \rho(e^a)\langle e^b, e^c \rangle = \langle e^a \circ e^b, e^c \rangle + \langle e^b, e^a \circ e^c \rangle \).

Let us consider the supermanifold \( \tilde{X} \) which bosonic part is a three dimensional manifold \( X \). In our topological field theory, a base space \( \mathcal{M} \) is the space of a (smooth) map from \( \tilde{X} \) to a target space \( \mathcal{M} \). Basis on \( \Gamma(E) \) is \( e^a = A^a \). We define a symmetric bilinear form \( \langle \cdot, \cdot \rangle \), a bilinear operation \( \circ \) and an a bundle map \( \rho \) as follows:

\[
\begin{align*}
e^a \circ e^b &\equiv \left( \langle S, e^a \rangle , e^b \right) , \\
\langle e^a , e^b \rangle &\equiv \left( \langle e^a , e^b \rangle \right) ,
\end{align*}
\]
\[ \rho(e^a) F(\phi) \equiv \left( (e^a, (S, F(\phi))) \right), \]
\[ \mathcal{D}(\ast) \equiv \left( (S, \ast) \right). \quad (37) \]

Then we can easily confirm that the gauge algebra satisfies the conditions 1 to 5 of the Courant algebroid by the identities \((30)\).

Conversely, first we take the basis on \(A^a\) on the fiber of the vector bundle \(\mathcal{E}\). We define the graded Poisson structure \((12)\) on the bundle \(\mathcal{E} \oplus T^* M\), where the grading on the fiber direction is shifted by 2. We define the operations \(\langle \cdot, \cdot \rangle, \circ\) and \(\rho\) as
\[ A^a \circ A^b = -k^{ac} k^{bd} f_{2 c de}(\phi) A^e, \]
\[ \langle A^a, A^b \rangle = k^{ab}, \]
\[ \rho(A^a) \phi^i = -f_{1 c} i(\phi) k^{ac}. \quad (38) \]

We can take a Darboux coordinate such that \(\langle A^a, A^b \rangle = k^{ab}\). Then the conditions 1 to 5 of the Courant algebroid are equivalent to the identities \((30)\) on \(f_1\) and \(f_2\). The action \(S\) is the BRST charge of the Courant algebroid. Since the master equation \((19)\) is equivalent to \((30)\), the relations 1 to 5 is represented by the master equation of the action \(S\).

6 Conclusion and Discussion

We have considered the Chern-Simons gauge theory in three dimensions, coupled with the BF theory which is another Schwarz-type topological field theory. We have analyzed all possible deformations of this theory by the antifield BRST formalism. Then it led us to a deformed new action with a new gauge symmetry. This 'nonlinear' gauge symmetry in our theory is an extension of the usual Lie algebra and the quantities corresponding to the structure constants are not constants and functions of the fields.

The 'nonlinear' Lie algebras in the nonlinear gauge theory are recently analyzed in the context of \(L_\infty\)-algebra \(\boxplus \boxplus \boxplus\), or the Lie algebroid \(\boxplus \boxplus\). These mathematical notions will be applicable to our theory. Here we have found that the gauge symmetry of the deformed topological field theory constructed in this paper has the gauge symmetry based on the Courant algebroid. Our theory is a first example of field theories with the Courant algebroid structure.
Since the deformed gauge theory is still a topological field theory, observables in this theory will define cohomological quantities. These are regarded as deformations of mathematical invariants obtained from the Chern-Simons gauge theory. In the Chern-Simons gauge theory, the coupling constant is quantized to the integer variable. However we have not treated such global aspects in this paper. The mathematical and physical aspects of this deformation should be studied.

We do not analyze the quantum theory in this paper. Since the gauge algebra in our theory is generally the open algebra, we have to use the BV formalism in order to make the gauge fixing and quantize the theory. We should analyze the correlation functions of the observables in detail.

Acknowledgments

The author would like to thank J. Stasheff, T. Strobl and P. Xu for valuable comments and discussions.

Appendix A, Antibracket

In three dimensions, we define the antibracket for functions $F(\Phi, \Phi^+)$ and $G(\Phi, \Phi^+)$ of the fields and the antifields as follows:

\[
(F, G) \equiv \frac{\overleftarrow{\partial} F}{\partial \Phi} \frac{\overrightarrow{\partial} G}{\partial \Phi^+} - \frac{\overleftarrow{\partial} G}{\partial \Phi^+} \frac{\overrightarrow{\partial} F}{\partial \Phi},
\]

(39)

where $\overleftarrow{\partial} / \partial \varphi$ and $\overrightarrow{\partial} / \partial \varphi$ are the right differentiation and the left differentiation with respect to $\varphi$, respectively. The following identity about left and right derivative is useful:

\[
\overrightarrow{\partial} F = (-1)^{(\text{deg } F - \text{deg } \varphi) \text{deg } \varphi + (\text{deg } F - \text{deg } \varphi) \text{deg } \varphi} \overleftarrow{\partial} F.
\]

(40)

If $S, T$ are two functionals, the antibracket is defined as follows:

\[
(S, T) \equiv \int_X \left( \frac{\overleftarrow{\partial} S \overrightarrow{\partial} T}{\partial \Phi} \frac{\overrightarrow{\partial} S \overleftarrow{\partial} T}{\partial \Phi^+} \right)
\]

(41)
The antibracket satisfies the following identities:

\[
(F, G) = -(-1)^{(\deg F - 3)(\deg G - 3) + (gh F + 1)(gh G + 1)} (G, F),
\]

\[
(F, GH) = (F, G)H + (-1)^{\deg F - 3} \deg G + (gh F + 1)gh G (F, H),
\]

\[
(FG, H) = F(G, H) + (-1)^{\deg G(\deg H - 3) + gh G(gh H + 1)} (F, H)G,
\]

\[
(-1)^{(\deg F - 3)(\deg H - 3) + (gh F + 1)(gh H + 1)} (F, (G, H)) + \text{cyclic permutations} = 0,
\]

where \( F, G \) and \( H \) are functions on fields and antifields.

**Appendix B, Dot Product**

It is convenient to combine fields to superfield to analyze BV actions. In order to simplify cumbersome sign factors, we introduce the dot product, dot Lie bracket, dot antibracket and dot differential.

For a superfield \( F(\Phi, \Phi^+) \) and \( G(\Phi, \Phi^+) \), The following identities are satisfied:

\[
FG = (-1)^{gh F gh G + \deg F \deg G} GF,
\]

\[
d(FG) = dFG + (-1)^{\deg F}FdG,
\]

at the usual products. The graded commutator of two superfields satisfies the following identities:

\[
[F, G] = -(-1)^{gh F gh G + \deg F \deg G} [G, F],
\]

\[
[F, [G, H]] = [[F, G], H] + (-1)^{gh F gh G + \deg F \deg G} [G, [F, H]].
\]

We introduce the total degree of a superfield \( F \) as \( |F| = gh F + \deg F \). We define the dot product on superfields as

\[
F \cdot G \equiv (-1)^{gh F \deg G} FG,
\]

and the dot Lie bracket

\[
[F, G] \equiv (-1)^{gh F \deg G} [F, G].
\]
We obtain the following identities of the *dot product* and the *dot Lie bracket* from (43), (44), (45) and (46):

\[
F \cdot G = (-1)^{|F||G|} G \cdot F,
\]
\[
[F, G] = -(-1)^{|F||G|} [G, F],
\]
\[
[F, [G, H]] = [[F, G], H] + (-1)^{|F||G|} [G, [F, H]],
\]

and

\[
d(F \cdot G) \equiv dF \cdot G + (-1)^{|F|} F \cdot dG.
\]

The *dot antibracket* of the superfields \(F\) and \(G\) is defined as

\[
\langle \langle F, G \rangle \rangle \equiv (-1)^{(ghF+1)(\deg G-3)}(-1)^{gh\Phi(\deg \Phi -3)+3} (F, G),
\]

Then the following identities are obtained from the equations (42) and (49):

\[
\langle \langle F, G \rangle \rangle = -(-1)^{|F||G|} \langle \langle G, F \rangle \rangle,
\]
\[
\langle \langle F, GH \rangle \rangle = \langle \langle F, G \rangle \rangle \cdot H + (-1)^{|F||G|} G \cdot \langle \langle F, H \rangle \rangle,
\]
\[
\langle \langle FG, H \rangle \rangle = F \cdot \langle \langle G, H \rangle \rangle + (-1)^{|G||H|} \langle \langle F, H \rangle \rangle \cdot G,
\]
\[
(-1)^{|F||H|} \langle \langle F, \langle \langle G, H \rangle \rangle \rangle \rangle + \text{cyclic permutations} = 0.
\]

We define the *dot differential* as

\[
\frac{\overrightarrow{\partial}}{\partial \varphi} \cdot F \equiv (-1)^{gh\varphi \deg F} \frac{\overrightarrow{\partial} F}{\partial \varphi},
\]
\[
F \cdot \frac{\overleftarrow{\partial}}{\partial \varphi} \equiv (-1)^{ghF \deg \varphi} \frac{\overleftarrow{\partial} F}{\partial \varphi}.
\]

Then, from the equation (50), we can obtain the formula

\[
\frac{\overrightarrow{\partial}}{\partial \varphi} \cdot F = (-1)^{|F|-|\varphi|} |\varphi| F \cdot \frac{\overleftarrow{\partial}}{\partial \varphi}.
\]

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