The Complexity of Fixed-Height Patterned Tile Self-Assembly

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Abstract

We characterize the complexity of the PATS problem for patterns of fixed height and color count in variants of the model where seed glues are either chosen or fixed and identical (so-called non-uniform and uniform variants). We prove that both variants are NP-complete for patterns of height 2 or more and admit $O(n)$-time algorithms for patterns of height 1. We also prove that if the height and number of colors in the pattern is fixed, the non-uniform variant admits a $O(n)$-time algorithm while the uniform variant remains NP-complete. The NP-completeness results use a new reduction from a constrained version of a problem on finite state transducers.

1 Introduction

Winfree [14] introduced the abstract tile assembly model (aTAM) to capture nanoscale systems of DNA-based particles aggregating to form intricate crystals, leading to an entire field devoted to understanding the theoretical limits of such systems (see surveys by Doty [3] and Patitz [11]). Ma and Lombardi [10] introduced the patterned self-assembly tile set synthesis (PATS) problem, of designing a tile set of minimum size that assembles into a given $n \times h$ colored pattern by attaching to an L-shaped seed.

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Czeizler and Popa [2] were the first to provide a proof that the PATS problem is NP-hard, thus establishing the problem as NP-complete. Subsequent work studied the hardness of the constrained version where the patterns have at most $c$ colors, called the $c$-PATS problem. This line of work proved the 60-PATS [12], 29-PATS [7], 11-PATS [8], and finally the 2-PATS [9] problems NP-complete.

Here we study the complexity of parameterized height-$h$ PATS and $c$-PATS problems where patterns have a specified fixed height $h$ and increasing width $n$. We consider both uniform and non-uniform model variants, where the glues along the seed are fixed and identical or chosen in tandem with the tile set, respectively. We characterize the computational complexity of these problems via the following results:

- The height-2 PATS problem is NP-complete in both models (Sec. 4).
- The uniform height-2 3-PATS problem is NP-complete (Sec. 5).
- The non-uniform height-$h$ $c$-PATS problem and uniform height-1 PATS problems admit $c \cdot O(h) n$-time and $O(n)$-time algorithms, respectively (Sec. 6).

The NP-completeness results also apply to patterns of height greater than 2. Thus the complexity of the PATS problem for all combinations of height, color, and uniformity are characterized, except uniform height-2 2-PATS.

The NP-hardness reductions are based on a reduction for a new variant of the minimum-state finite state transducer problem, originally proved NP-hard by Angluin [1] and by Vazirani and Vazirani [13]. In this variant, any solution transducer is also promised to satisfy additional constraints on its transitions. The reduction is also substantially simpler than the reduction given in [1] and uses input and output strings of just two symbols, rather than the three of [13].

2 Preliminaries

Patterns, tiles, assemblies, and seeds. Define $\mathbb{N}_k = \{1, 2, \ldots, k\}$. A pattern is a partial function $P : \mathbb{N}^2 \rightarrow C$, i.e. a function that maps a rectangular region of lattice points to a set of colors. If $\text{dom}(P) = \mathbb{N}_w \times \mathbb{N}_h$, then $P$ is a width-$w$ height-$h$ pattern. The codomain of $P$, i.e. the colors
seen in the pattern, is denoted color(P). A pattern P is c-color provided |color(P)| ≤ c.

A tile type t is a colored unit square with each edge labeled; these labels are called glues. A tile type’s color is denoted color(t). For a direction d ∈ {N, W, S, E}, t[d] denotes the glue assigned to side d of t. A tile type is non-rotatable, and thus is uniquely identified by its color and four glues. Instances of tile types, called tiles, are placed with their centers in \( \mathbb{N}^2 \).

An assembly is an arrangement of tiles from a set of tile types \( T \); formally a partial function \( A : \mathbb{N}^2 \rightarrow T \cup \{\emptyset\} \). A seed is an “L-shaped” assembly with domain \( \{(0,0)\} \cup \{(x,0) : x \in \mathbb{N}_w\} \cup \{(0,y) : y \in \mathbb{N}_h\} \) for some \( w, h \in \mathbb{N} \). The pattern of an assembly A is defined as \( P_A((x,y)) = color(A((x,y))) \) for \( (x,y) \in \text{dom}(A) \cap \mathbb{N}^2 \), i.e. the color pattern of A, excluding the seed.

RTASs. A rectilinear tile assembly system (RTAS) is a pair \( \mathcal{T} = (T, \sigma) \), where T is a set of tile types and \( \sigma \) is a seed. An assembly A yields an assembly \( A' \) with \( \text{dom}(A') = \text{dom}(A) \cup \{(x,y)\} \) provided \( (x-1,y), (x,y-1) \in \text{dom}(A) \) and \( A((x-1,y))[E] = A'((x,y))[E], A((x,y-1))[N] = A'((x,y))[S] \). The set of producible assemblies of an RTAS are those that can be yielded, starting with the seed assembly \( \sigma \). That is:

**RTAS Tiling Rule:** A tile of type t can be added to an assembly A at location \( (x,y) \) provided \( (x-1,y), (x,y-1) \in \text{dom}(A) \) and the east and north glues of the tiles at \( (x-1,y) \) and \( (x,y-1) \) are the same as the west and south glues of t, respectively.

As a result, tiling proceeds from southwest to northeast, i.e., a tile is first placed at \( (1,1) \), then at either \( (1,2) \) or \( (2,1) \), etc. The terminal assemblies of a RTAS are the producible assemblies that do not yield other (larger) assemblies. If every terminal assembly of the system has pattern P, the system is said to uniquely self-assemble P. An RTAS \( (T, \sigma) \) is directed, i.e. deterministic, provided for any distinct tile types \( t_1, t_2 \in T \), either \( t_1[W] \neq t_2[W] \) or \( t_1[S] \neq t_2[S] \).

Uniform RTASs. We also define a practical variant of a RTAS called a uniform RTAS. An RTAS \( (T, \sigma) \) is uniform provided there exist two glues \( \ell_E, \ell_N \) such that \( \sigma((x,0))[E] = \ell_E \) for all \( x \in \mathbb{N}_w \) and \( \sigma((0,y))[N] = \ell_N \) for all \( y \in \mathbb{N}_h \). In other words, the seed glues cannot be programmed and are generic.

The PATS problem. The pattern self-assembly tile set synthesis problem (PATS) \([10]\) asks for the minimum-size RTAS that uniquely self-assembles
a given rectangular color pattern, where the size of an RTAS \((T, \sigma)\) is \(|T|\), the number of tile types. Bounding the number of colors or height of the input pattern yields the following practically motivated special cases of PATS:

**Problem 2.1** (*c*-colored PATS or *c*-PATS). *Given a *c*-colored pattern* \(P\) *and integer* \(t\), *does there exist an RTAS of size \(\leq t\) that uniquely self-assembles \(P\)?*

**Problem 2.2** (Height-\(h\) PATS). *Given a height-\(h\) pattern* \(P\) *and integer* \(t\), *does there exist an RTAS of size \(\leq t\) that uniquely self-assembles \(P\)?*

Restricting the system to be uniform gives rise to uniform variants as well, contrasting with the conventional non-uniform variants.

![Pattern](image)

Figure 1: A height-2 2-color pattern and minimum-size RTASs uniquely assembling the pattern in the uniform and non-uniform models.

## 3 Minimum-State Finite State Transducer is NP-hard

We start with a reduction from a well-known NP-complete problem on integers, called 3-PARTITION, to a problem on finite state transducers or FSTs: deterministic finite automata where each transition is augmented with an output symbol and thus *transduces* an input string into an output string of the same length.

**Problem 3.1** (3-PARTITION). *Given a set of integers* \(A = \{a_1, a_2, \ldots, a_{3n}\}\) *with* \(\sum_{a_i \in A} a_i/n = p\) *and* \(p/4 < a_i < p/2\), *does there exist a partition of* \(A\) *into* \(n\) *sets, each with sum* \(p\)?

**Theorem 3.2** ([5]). 3-PARTITION is NP-hard.

\(^1\)3-PARTITION is strongly NP-hard, meaning that the problem is NP-hard when the elements of \(A\) are given in unary.
Formally, a FST is a 4-tuple $T = (\Sigma, Q, s_0, \delta)$, where $\Sigma$ is the alphabet, $Q$ is a finite set of states of $T$, $s_0 \in Q$ is the start state of $T$, and $\delta : Q \times \Sigma \to Q \times \Sigma$ is the transition function of $T$. An input-output quadruple $\delta(s_i, b) = (s_j, b')$ is a transition, specifically a $(b, b')$-transition or a $b$-transition. The size of $T$ is equal to $|Q|$. 

**Problem 3.3 (Encoding by FST).** Given two strings $S, S'$ and integer $K$, does there exist a FST with at most $K$ states that transduces $S$ to $S'$?

**Lemma 3.4.** Encoding by FST is NP-hard.

*Proof.* We borrow from [13] the approach of constructing $S$ and $S'$ by concatenating segments: pairs of input and output substrings of equal length that enforce specific structure in a solution FST. An input string $A$ and output string $B$ paired as a segment is denoted $A \rightarrow B$. The integer output by the reduction is $K = 3pn + n + 1$, where $n$ is the number of parts in the partition and $p$ the size of each part. The first segment is $0^{K-1}10^{K-1} \rightarrow 0^{K-1}10^{K-1}$. This segment enforces that a solution FST must have $K$ states; label them $s_1, s_2, \ldots, s_K$. Then for all $i < n$, $\delta(s_i, 0) = (s_{i+1}, 0)$ and $\delta(s_K, 0) = (s_1, 1)$.

![Figure 2](image-url) A solution FST for a toy reduction from 3-PARTITION to Encoding by FST with integers 1, 2, 1, 2 (invalid but used for illustrative purposes). The left-to-right states are $s_1$ to $s_K$, colored by their half-fixed interval or gray (for fixed singletons). Transitions above the states are $(1, 1)$-transitions. All others are $(0, 0)$-transitions except the lowermost, a $(0, 1)$-transition.

The problem of partitioning integers of $A$ into groups of size $p$ is implemented in the collection of 1-transitions that leave each state. Each state
has a 1-transition that either points to itself (a fixed singleton) or is one edge in a 3-cycle formed by two consecutive specified states and an unspecified third state (a half-fixed triple). Half-fixed triples are further organized into half-fixed intervals, each consisting of a group of $2a_i$ consecutive specified states and a group of $a_i$ consecutive unspecified states for some distinct $a_i$. The states are partitioned into three groups:

- States $s_1$ through $s_{2pn}$ are the specified halves of the half-fixed intervals.
- $n + 1$ equally-spaced fixed singletons in states $s_{2pn+1}, \ldots, s_K$.
- The remaining $pn$ states in states $s_{2pn+1}, \ldots, s_K$ partitioned into $n$ sets of $p$ consecutive states.

See Figure 2 for a toy example of the reduction.

The unspecified halves of the half-fixed intervals can be assigned to the third group of states if and only if the input 3-PARTITION instance has a solution. All that remains is to describe the segments that force the construction of a fixed singleton, half-fixed triple, and half-fixed interval.

**Fixed singleton.** The fixed singleton segment ensures that a given state $s_i$ has $\delta(s_i, 1) = (s_i, 1)$. This is done by moving the current state to $s_i$, transducing a 1 to a 1, and checking whether the current state is still $s_i$ (see Figure 3).

![Figure 3: The fixed single segment and corresponding FST structure enforced.](image)

**Half-fixed triple.** The half-fixed triple segment forces two specified fixed states $s_i$, $s_{i+1}$ and an unspecified free third state $s_j$ to have $\delta(s_i, 1) = (s_{i+1}, 1)$, $\delta(s_{i+1}, 1) = (s_j, 1)$, and $\delta(s_j, 1) = (s_i, 1)$ (see Figure 4).

The segment consists of two subsegments that each ensures a portion of the structure. The first, $0^{i-1}10^{K-i-10} \rightarrow 0^{i-1}10^{K-i-11}$, and ensures that $\delta(s_i, 1) = (s_{i+1}, 1)$. The second, $0^i110^{K-i-10} \rightarrow 0^i110^{K-i-11}$, ensures that
The half-fixed triple segment and corresponding FST structure enforced.

\[ \delta(s_{i+1}, 1) = (s_j, 1) \text{ and } \delta(s_j, 1) = (s_i, 1). \] The state \( s_j \) cannot be in a fixed state of another half-fixed triple segment with fixed states \( s'_i \), \( s'_{i+1} \) and free state \( s'_j \), as then either:

- \( s_j = s'_i \) and thus \( \delta(s_j, 1) = (s'_{i+1}, 1) \neq (s_i, 1) \) (and thus the segment \( 0^{i-1}10^{K-i-1}0 \to 0^{i-1}10^{K-i-1}1 \) is not transduced).
- \( s_j = s'_{i+1} \) and \( (s'_j, 1) = \delta(s'_{i+1}, 1) = (s_i, 1) \), so \( \delta(s'_j, 1) = (s_{i+1}, 1) \neq (s'_i, 1) \) (and thus the segment \( 0^{i'-1}10^{K-i'-1}0 \to 0^{i'-1}10^{K-i'-1}1 \) is not transduced).

\[ \delta(s_i, 1) = (s_{i+1}, 1) \text{ and } \delta(s_{i+1}, 1) = (s_i, 1). \] The state \( s_i \) cannot be in a fixed state of another half-fixed triple segment with fixed states \( s'_{i+1} \), \( s'_{i+2} \) and free state \( s'_i \), as then either:

- \( s_i = s'_{i+1} \) and thus \( \delta(s_i, 1) = (s_{i+2}, 1) \neq (s_i, 1) \) (and thus the segment \( 0^{i-1}10^{K-i-1}0 \to 0^{i-1}10^{K-i-1}1 \) is not transduced).
- \( s_i = s'_{i+2} \) and \( (s'_i, 1) = \delta(s'_{i+2}, 1) = (s_i, 1) \), so \( \delta(s'_i, 1) = (s_{i+1}, 1) \neq (s'_i, 1) \) (and thus the segment \( 0^{i'-1}10^{K-i'-1}0 \to 0^{i'-1}10^{K-i'-1}1 \) is not transduced).

**Half-fixed interval.** The half-fixed interval forces a collection of half-fixed triples with consecutive fixed states to also have consecutive free states. It does so by a simple traversal of the free states, checking that each has the expected pair of consecutive fixed states (see Figure 5).  

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Figure 4: The half-fixed triple segment and corresponding FST structure enforced.

Figure 5: The half-fixed interval segment for three consecutive free states and corresponding FST structure enforced.
Any solution transduction for the previous reduction uses an FST where each state has at most one incoming 1-transition and 0-transition, since every transition lies on a cycle (of length 1, 3, or $K$). Also, any solution transduction by an FST with $K$ states traverses $2K$ distinct transitions (with $K - 1$ (0,0)-transitions, 1 (0,1)-transition, and $K$ (1,1)-transitions). Any other solution FST must have at least $2K$ states and traverse at least $2K + 1$ distinct transitions: $2K$ 0-transitions and at least one 1-transition. Thus the following problem is also NP-hard by the prior reduction:

**Problem 3.5 (Promise encoding by FST).** Given two strings $S$, $S'$ and an integer $K$ with the following promises about any FST $T$ with at most $K$ states transducing $S$ to $S'$, does such a $T$ exist?

- Each state of $T$ has at most one incoming 0-transition.
- Each state of $T$ has at most one incoming 1-transition.
- When transducing $S$ to $S'$:
  - $K - 1$ distinct (0,0)-transitions are used.
  - $K$ distinct (1,1)-transitions are used.
  - 1 distinct (0,1)-transition is used.
  - The transitions are traversed in a unique specified order given as part of the input.

**Corollary 3.6.** The **Promise encoding by FST** problem is NP-hard.

### 4 Height-2 PATS is NP-complete

Göös and Orponen [6] establish that all the variations of the PATS problem considered here are in NP. So we need only consider their NP-hardness.

**Theorem 4.1.** The non-uniform height-2 PATS problem is NP-hard.

**Proof.** The pattern output by the reduction consists of a bottom row encoding $S$ and a top row encoding the sequence of transitions traversed when transducing $S$ to $S'$ (provided as part of the Promise encoding by FST instance). The bottom row encoding uses two colors, pink and red, corresponding to the two symbols in $S$. The top row encoding uses $2K$ colors,
one for each transition used in the transduction of $S$ to $S'$. The number of tile types permitted is $T = 2K + 2$: one type per color.

The north glues of the bottom row either encode $S$ (distinct north glues for the pink and red tile types) or $0^{|S|}$ (same glue). The latter is impossible, since then the leftmost $|S|$ locations of the top row are filled by many repetitions of the same $K$ transitions. So the north glues of the bottom row encode $S$.

A set of $2K$ tile types that assemble the top row is equivalent to a set of $2K$ transitions transducing $S$ to $S'$, with source and destination states corresponding to west and east glues. So the top row can be assembled using $2K$ tile types exactly when $S$ can be transduced to $S'$ using $2K$ transitions of the specified types traversed in the specified order. Thus the pattern can be assembled using a tile set of at most $2K$ types exactly when the corresponding instance of Promise encoding by FST has a solution transducer. \hfill \qed

Theorem 4.2. The uniform height-2 PATS problem is NP-hard.

Proof. The pattern output by the reduction is the following, and consists of a left input half and right transduction half:

\[
\begin{array}{c}
S \\
\square \cdots \square \\
\vdots \\
\end{array} \quad \square \text{(transduction)} \quad \begin{array}{c}
\text{top row} \\
\text{of input half} \\
\text{top row} \\
\text{of transduction half}
\end{array}
\]

The color patterns in the top rows of the input and transduction halves are identical to the bottom and top rows of the pattern used in the proof of Theorem 4.1 respectively. That is, they consist of two colors encoding $S$, and $2K$ colors encoding the sequence of transitions.

The bottom row of the input half consists of $|S|$ orange tiles; these must be of $|S|$ distinct types, otherwise the entire bottom row consists of orange tiles. The number of tile types output by the reduction is $T = |S| + 2K + 4$, thus any solution set of tile types has exactly the following tile types:

- $|S|$ orange tile types.
- 2 types used in the top row of the input half.
- $2K$ types used in the top row of the transduction half.
- White and black types used for the two tiles between the halves.
Since only one tile type per color is used in the top row of the input half, the north glues of the length-$|S|$ orange tile sequence must encode $S$. Also, since the bottom row of the transduction half consists of a length-$|S|$ sequence of orange tiles and these tiles are all contained in the bottom row of the input half, they must be the same sequence of tiles. So the north glues of the bottom row of the transduction half encode $S$, and thus any set of $2K$ tile types used in the top row of the right half corresponds exactly to a set of transitions in a solution FST for the Promise encoding by FST instance.

The addition of more rows with a new common color and increasing $T$ by 1 suffices to prove both the uniform and non-uniform variants NP-hard for greater heights.

5 Uniform Height-2 3-PATS is NP-complete

Problem 5.1 (Modified promise encoding by FST). Given two strings $S$, $S'$ and an integer $K \not\equiv 0 \pmod{3}$ with the following promises about any FST $T$ with at most $K$ states transducing $S$ to $S'$, does such a $T$ exist?

- The first and last symbols of $S'$ are 2.
- Each state of $T$ has at most one incoming 0-transition.
- Each state of $T$ has at most one incoming 1-transition.
- Every $(1,1)$-transition lies on a 1-cycle or 3-cycle of $(1,1)$-transitions.
- When transducing $S$ to $S'$:
  - $K - 1$ distinct $(0,0)$-transitions are used.
  - $K - 1$ distinct $(1,1)$-transitions are used.
  - 1 distinct $(0,1)$-transition is used.
  - 1 distinct $(1,2)$-transition is used.
  - The transitions are traversed in a unique specified order given as part of the input.

Lemma 5.2. The Modified promise encoding by FST problem is NP-hard.
Proof. The proof uses a modified version of the reduction from 3-PARTITION to min-state FST used in the proof of Lemma 3.4. A new state $s_{\text{new}}$ is added to the solution transducers with the following transitions as seen in Figure 6:

- A (0, 0)-transition from $s_{\text{new}}$ to $s_1$.
- (1, 2)-transition from $s_{\text{new}}$ to itself.
- A (0, 1)-transition from $s_K$ to $s_{\text{new}}$, replacing the (0, 1)-transition from $s_K$ to $s_1$.

![Figure 6: The modified min-state FST instances used in the reduction of Theorem 5.3.](image)

If the number of states in this transducer is a multiple of 3, add another fixed singleton state to the transducer. Reassign $K$ to be the number of states in the resulting transducer and label the states $s_1$ to $s_K$ in the order they are traversed by (0, 0)-transitions, starting with the state with a (1, 2)-transition to itself (formerly called $s_{\text{new}}$).

To enforce such solution transducers, replace a segment 0 → 1 in the strings $S$ and $S'$ output by the reduction with 01 → 12. Also add the segment 1 → 2 to the beginning and end of the strings to satisfy the last constraints of the FST instances in the problem formulation. The resulting modified strings $S$, $S'$ and integer $K$ yield an instance of MODIFIED PROMISE ENCODING BY FST that is “Yes” if and only if the 3-PARTITION instance was also “Yes”. □
Theorem 5.3. The uniform height-2, 3-PATS problem is NP-hard.

Proof. Let $P$ be the following width-$(1+|S'|+K^2)$, height-2 pattern over 3 colors \{■, □, ▼\}:

\[
\begin{array}{cccccccc}
\text{■} & S' & \cdots & \text{□} \cdots & \text{□} & w_1 & \text{□} \cdots & w_2 & \cdots & \text{□} w_{K-1} & \text{□} \\
|S'| & K-3 & & K-3 & & & K-3 & & K-3 & & K-3 \\
\end{array}
\]

where, for $1 \leq i \leq K-1$,

\[
w_i = \begin{cases} 
\text{■}(3i \text{ mod } K) - 1 & \text{if } 3i \leq K-3 \pmod{K} \\
\text{□} - 3i \text{ mod } K & \text{otherwise}
\end{cases}
\]

Notice that, for any $1 \leq i < j \leq K$, $w_i$ and $w_j$ differ in the position of 1. Split the pattern $P$ into the leftmost $|S'|+1$ columns and the remainder, called the transduction and FST-constructor gadgets, respectively. The FST-constructor gadget is further partitioned into $K$ rectangular subpatterns of width $K$.

Next, consider the constraints on RTASs with at most $|S'| + 2K + 2$ tile types that uniquely self-assemble $P$. Lemma 1 of Göös and Orponen [6] states that any smallest RTAS that uniquely self-assembles a pattern is directed. As we will prove, directed RTASs uniquely self-assembling $P$ have size at least $|S'| + 2K + 2$ tile types; thus we need only consider directed systems.

Let the north and east glues of the seed be 0. The leftmost $|S'|+1$ locations in the bottom row of $P$ are orange, with a cyan location following. So these positions must be tiled with orange tiles of pairwise-distinct type; the need for $|S'| + 1$ distinct orange tile types thus arises. Similarly, the leftmost $K-1$ cyan locations in the bottom row must use $K-1$ distinct cyan tile types. These tile types share the south glue 0, and since the system is directed, their west glues are pairwise distinct. Label these $K-1$ cyan tile types left-to-right $t_{00,2}, t_{00,3}, \ldots, t_{00,K}$ and the gray tile type immediately right $t_{01}$, as seen below. The cyan tile in the northwest corner of $P$ cannot have the same type as any of these $K-1$ types, since otherwise this tile can also appear in the southwest corner of $P$. Call this type $t_0$. There are $K$ tile types to be colored yet (illustrated as a dotted square).

\[\text{\footnote{In these later labels, the first subscript indicates the kind of transition of the FST that the tile type will be shown to simulate, e.g., $t_{00,i}$ is a (0,0)-transition, $t_{01}$ and (0,1)-transition, etc.}}\]
These $K$ tile types will turn out to be necessary, implying $(|S'|+1)+(K-1)+2+K-1+1 = S'+2K+2$ types total with $K-1$ colored gray and one colored orange. For this, we claim that the bottom row of all blocks but the first assemble identically by establishing that the gray tiles attaching to the southeast corner of the first two blocks are identical. Suppose not. Then the bottom row of the second block cannot reuse cyan tile types used in the bottom row of the first block. So the uncolored $K$ tile types must be one gray and $K-1$ cyan types with south glue 0. Thus the complete tile set includes only two gray tile types with the south glue 0.

Consider the gray tile attaching at the northeast corner of the first block. Its south glue is 0 and its west glue is equal to the east glue of the gray tile attaching to its immediate left. This contradicts the directedness of the system, since a cyan tile is provided with the same pair of west and south glues. Indeed, both gray tile types appear at the southeast corner of a block and to their east are cyan tiles attaching.

The verified claim brings following properties for all but the first block:

Property 1: For any $1 \leq i \leq K-1$, tiles attaching at the $i$-th top-row position of any two blocks but the first one have the same south glue; tiles attaching at the $K$-th top-row position (northeast corner) of any two blocks including the first one have the same south glue.

Property 2: Any such pair of tiles have pairwise-distinct east glues (and types).

Property 3: The assembly of the bottom row is provided with at least two different kinds of north glues.

Property 2 holds since a orange tile is placed in the northeast corner of only the last block. Thus without Property 3, $o(K^2)$ tile types would be necessary to place the orange tile. Observe that for each $1 \leq i \leq K-3$, the $i$-th position of exactly one block is gray and the counterpart of all other blocks are cyan; for each $K-2 \leq i \leq K$, the $i$-th position of only the last block is orange and the counterpart of all others are gray. Thus, Properties 1 and 2 imply that the tile type set must contain one orange and $K-1$ gray tile types whose south glue is equal to the north glue of $t_{01}$ and one gray and $K-2$ cyan tile types.
types with a common south glue.

We claim these requirements enforce that the north glue of $t_{01}$ is not 0. Suppose otherwise. Then the former requirement implies $K-2$ extra gray tile types with south glue 0. So at most 3 tile types, including $t_0$, have the non-0 south glue, and Property 3 cannot be satisfied. Thus, the north glue of $t_{01}$ is not 0; call it 1. Tiles attaching at the northeast corner of the blocks must all have distinct types due to Property 2, and now also their south glues must be 1. The $K$ uncolored tile types thus have south glue 1, and one is colored orange and all the others are colored gray.

Note that a $t_0$ tile cannot attach anywhere in the blocks. Indeed, it causes glue mismatch with the seed being placed on the bottom row, and in order for it to attach on the top row, it must share its south glue with $K-3$ cyan tile types due to Properties 1 and 2. In summary, any minimum tile set uniquely assembling $P$ consists of $K$ cyan tile types, $K$ gray ones, and $|S'| + 2$ orange ones.

Now we prove constraints on the glues of these types. With only $K-1$ cyan tile types with south glue 0, even the first block must assemble its bottom row as other blocks do. That is, the bottom row of all blocks assemble as $t_{00,2} t_{00,3} \cdots t_{00,K-1} t_{00}$. Thus, the east glue of $t_0$ is equal to the west glue of $t_{00,2}$, that is, $s_1$. Since $t_0$ does not appear in any block, Property 2 implies that the north glues of $t_{00,2}, t_{00,3}, \ldots, t_{00,K-2}$ are 0 and that the north glues of $t_{00,K-1}$ and $t_{00}$ are 1.

The top row of the last block is $w_K \begin{array}{cccc} \relax & \relax & \relax & \relax \\ 1 & 1 & 1 & 1 \\ \end{array} = \begin{array}{cccc} \relax & \relax & \relax & \relax \\ 1 & 1 & 1 & 1 \\ \end{array}$ and its last four positions are assembled as $t_{01} t_2 t_2 t_2$. This imposes that both the east and west glues of $t_2$ must be equal to the east glue of $t_0$, that is, $s_1$. Since $S'$ begins with 2, the east glue of $t_0$ is $s_1$. 

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Since \( S' \) ends with 2, no tile (necessarily of type \( t_{11,2}, t_{11,3}, \ldots, t_{11,K} \) by Properties 1 and 2) appearing at the northeast corner of a block has east glue \( s_1 \). Moreover, tiles attaching to their east are of type \( t_{00,2}, t_{00,3}, \ldots, t_{00,K} \) or \( t_{01} \), thus their east glues are in \( \{s_2, s_3, \ldots, s_K\} \). Without loss of generality, assign them as follows:

So the east glues of all tile types in the FST-construction gadget are in \( \{s_1, \ldots, s_K\} \). The east glues of \( t_{11,2}, \ldots, t_{11,K} \) are distinct and selected from \( \{s_1, \ldots, s_K\} \). Since \( t_{11,2}, \ldots, t_{11,K} \) share the south glue 1 with \( t_{2} \), the west glue of \( t_{2} \) is \( s_1 \), and the system is directed, the west glues are distinct and from \( \{s_2, s_3, \ldots, s_K\} \).

The only remaining flexibility in the design of the tile set is assigning west glues to \( t_{11,2}, \ldots, t_{11,K} \), corresponding to the assignment of \((1,1)\)-transition sources in Modified promise encoding by FST. All that remains is to prove this correspondence indeed holds.

The glue 0 is not in \( \{s_1, \ldots, s_K\} \), as otherwise a cyan or gray tile could appear in the southwest corner of \( P \). So none of the cyan, gray, or \( t_2 \) tile types has east glue 0 and thus a \( t_0 \) tile cannot attach anywhere but the northwest corner of \( P \). Also, observe that the tile set constraints imply that the north glue of a tile in the bottom row can be discerned by examining the color of the tile north and (possibly) northeast (a cyan tile north implies a 0 glue, a gray tile north implies a 0 or 1 glue if the color of the northeast tile is cyan or orange, respectively).

Finally, consider the tile types, excluding \( t_0 \) and all orange tile types except \( t_2 \), as a set of transitions of an FST, with \( t_{00,i} \) types as \((0,0)\)-transitions, \( t_{11,i} \) types as \((1,1)\)-transitions, \( t_{01} \) a \((0,1)\)-transition, and \( t_2 \) a \((1,2)\)-transition.
The constraints induced on the resulting transitions (e.g. that there are $K-1$ $(1,1)$-transitions because there are $K-1$ color-1 tile types with south glue 1) is found as a constraint on the transitions in the statement of Lemma 5.2. In particular, the choice of $w_i$'s in consecutive blocks requires that every $(1,1)$-transition lies on a 1-cycle or 3-cycle of $(1,1)$-transitions (the last constraint of Lemma 5.2). Thus there exists a solution FST to the Modified Promise Encoding by FST instance if and only if there exists a solution tile set. □

6 Efficiently Solvable PATS Problems

The non-uniform height-1 PATS problem is trivially solvable using one tile type for each color. This idea can be generalized for all patterns of fixed height:

**Theorem 6.1.** The non-uniform height-$h$ c-PATS problem admits a $c^{O(h)} n$-time algorithm.

**Proof.** Observe that the minimum-size tile set that assembles any given pattern has size at most $T = hc^h$, since there are $c^h$ possible columns and a distinct set of $h$ tiles for each column (whose appearance is programmed by the seed) can be used to assemble the pattern. The algorithm is a brute-force search for a smallest solution tile set (of size at most $T$), using dynamic programming to check each tile set in $O(hT^{h+2}n)$ time. Since a tile can be specified by a binary strength of length $4 \log T \log c$, there are at most $2^{4T \log cT} = (cT)^{4T}$ such tile sets. Thus the algorithm runs in $O((cT)^{4T}hT^{h+2}n) = T^{O(T)}n = c^{O(h)} n$ time.

All that remains is to describe the dynamic programming algorithm. Lemma 1 of Göös and Orponen [6] states that any smallest tile set is directed. The subproblems solved have the form: “Does the tile set deterministically assemble the first $i$ columns of the pattern with top-to-bottom sequence of tile types $t_1,t_2,\ldots,t_h$ in column $i$?”

All subproblems for column $i$ can be solved by checking each of at most $T \cdot T^h = T^{h+1}$ combinations of north seed glue in column $i$ and “Yes” subproblems for column $i-1$, and recording “Yes” for the top-to-bottom tile sequences resulting from deterministic assembly of column $i$ of the pattern and “No” for all other sequences. For each column, computing the “Yes” sequences takes $O(T^{h+1} \cdot hT)$ time and recording the solutions to all sub-
problems takes $O(T^h)$. So across all columns the algorithm takes $O(hT^h+2n)$ time.

As established in Section 5, a similar algorithm for the uniform model is impossible unless $P = NP$. Nevertheless, the uniform height-1 PATS problem can be solved in linear time using a pigeonhole argument and a DFS-based search for the longest repetitive suffix of a given height-1 pattern:

**Theorem 6.2.** The uniform height-1 PATS problem can be solved in $O(n)$ time.

**Proof.** Consider an input width-$n$ height-1 pattern $P$ for the uniform height-1 PATS problem as a word of length $n$ over an alphabet of colors $C$. Let $x, y$ be distinct suffixes of $P$ with $y$ a prefix of $x$ and $y$ as long as possible. Let $x = zy$ for some nonempty word $z$. Since $y$ is a prefix of $x$, $z$ is a period of $x$ and thus $x = z^izp$ for some $i \geq 1$ and prefix $zp$ of $z$. Moreover, $z$ is primitive (not a power of another word) since $y$ is as long as possible.

Let $m = n-|y|$. We prove that $m$ tile types are necessary and sufficient for a uniform RTAS to uniquely self-assemble $P$. For sufficiency, use a tile set that hardcodes the prefix of $P$ preceding $x$ with $m-|x|$ tile types and uses a repeating set of $|x|-|y|$ tile types to assemble the repetitions of $z$. For necessity, suppose $P$ can be uniquely self-assembled using strictly less than $m$ tile types. By pigeonhole, there must exist $1 \leq i < j \leq m$ such that the tiles at $(i,1)$ and $(j,1)$ have the same type. Then the suffixes $x'$ and $y'$ of $P$ starting at positions $(i,1)$ and $(j,1)$ are distinct suffixes of $P$ with $y' > y$, a contradiction with the previous choice of $x$ and $y$. So $m$ is the minimum number of tile types to uniquely assemble $P$.

All that remains is to prove that $y$ (and $x$) can be computed in $O(n)$ time. This can be done by DFS in a suffix tree, searching for the longest suffix ending at a non-leaf node. The suffix tree can be constructed from $P$ in $O(n)$ time [4], and the DFS done in the same running time. 

### 7 Conclusion

Our work here extends the extensive prior work on the parameterized $c$-PATS problem to also incorporate pattern height and uniformity, and finds a more delicate complexity landscape: limited height and colors do not make the PATS problem tractable, except when combined in the non-uniform
model, or in degenerate cases (height-1 or 1-PATS). A single combination of parameters and model remains unresolved; we conjecture the following:

**Conjecture 7.1.** The uniform height-2 2-PATS problem NP-hard.

We encourage further parameterized analysis of problems in tile self-assembly in support of recent efforts in developing a more complete understanding of the structural complexity of tile self-assembly (see [15]).

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