Richardson-Gaudin description of pairing in atomic nuclei

Stijn De Baerdemacker
Ghent University, Department of Physics and Astronomy, Proeftuinstraat 86, 9000 Gent, Belgium
Department of Physics, University of Toronto, Toronto, Ontario M5S 1A7, Canada
Department of Physics, University of Notre Dame, Notre Dame, IN 46556-5670, USA
E-mail: stijn.debaerdemacker@ugent.be

Abstract. The present contribution discusses a connection between the exact Bethe Ansatz eigenstates of the reduced Bardeen-Cooper-Schrieffer (BCS) Hamiltonian and the multi-phonon states of the Tamm-Dancoff Approximation (TDA). The connection is made on the algebraic level, by means of a deformed quasi-spin algebra with a bosonic Heisenberg-Weyl algebra in the contraction limit of the deformation parameter. Each exact Bethe Ansatz eigenstate is mapped on a unique TDA multi-phonon state, shedding light on the physics behind the Bethe Ansatz structure of the exact wave function. The procedure is illustrated with a model describing neutron pairing in $^{56}$Fe.

1. Introduction
Pairing correlations comprise an important part of the low-energy structure of atomic nuclei [1, 2]. Due to the short-range nucleon-nucleon interaction, two individual nucleons have a tendency to couple to a zero total angular momentum pair [3]. In analogy with the Bardeen-Cooper-Schrieffer (BCS) theory of superconductivity in metals [4], it was suggested that the ground state of an even-even atomic nucleus is a condensate of zero total angular momentum recoupled Cooper pairs [5]. This mechanism is capable of explaining the systematic $J^\pi = 0^+$ spin assignment of the ground state in even-even atomic nuclei, as well as the energy required to excite these nuclei from the ground state to the first excited state, the so-called pairing gap. In the nuclear shell model [6, 7], the pairing interaction is effectively modelled by means of the BCS Hamiltonian. The standard approach to solve the corresponding Schrödinger equation is to employ the BCS approximation, which assumes that the ground state is a coherent state of Cooper pairs [4]. While this assumption is valid in the thermodynamic limit, certain issues arise for finite-size systems. First, the total number of particles is not a conserved quantity in the BCS wavefunction, so pair-number fluctuations cause significant uncertainties in the ground-state energy. Second, BCS theory predicts a sharp phase transition to the superconducting phase at a critical interaction strength, in contrast with the smooth transitions observed in finite-size systems [8]. For sufficiently small finite-size systems, the basis of Slater determinants is small enough to extract exact spectroscopic information in the full basis [9]. However, with the recent developments in superconducting nanograins [10] and exotic nuclei far from the valley of stability, there is a need for exact canonical techniques to diagonalise the BCS Hamiltonian.
Richardson & Sherman have shown that the reduced, or level-independent, BCS Hamiltonian is exactly solvable with a Bethe Ansatz wavefunction, provided the parameters in the wavefunction are the solution of a set of non-linear algebraic equations, the so-called Richardson-Gaudin equations [11, 12]. The significance of the RG approach is that the diagonalisation of a Hamiltonian living in an exponentially scaling Hilbert space, is reduced to solving a set of algebraic equations scaling linearly with the system size. However the RG equations have proven to be very difficult to solve numberically [13], due to the presence of singular values in the interaction strengths, where the RG variables become degenerate. Recent innovative approaches [14, 15, 16, 17] have removed this obstacle, however the physical interpretation of the RG variables, including their singular behaviour, remained enigmatic.

The present contribution will discuss how the RG variables of the reduced BCS Hamiltonian are related to the elementary excitation modes of the Tamm-Dancoff Approximation (TDA), and how this can shed light on the physical nature of the RG variables.

2. Richardson’s solution of the reduced BCS Hamiltonian
Consider one family of fermions, living in a single-particle Hilbert space composed of \( n \) single-particle levels. We assume rotational symmetry, so the good quantum numbers of every level are given by the set \( (n_i, l_i, s_i, j_i, m_i) \), and we abbreviate the set \( (n_i, j_i, l_i, s_i) \) with the index \( i \) \((i = 1 \ldots n)\) [6]. The single-particle energies and degeneracies of the levels are given by respectively \( \varepsilon_i \) and \( \Omega_i = 2j_i + 1 \). The reduced BCS Hamiltonian reads

\[
\hat{H} = \sum_{i=1}^{n} \varepsilon_i \hat{n}_i + g \sum_{i,k=1}^{n} \hat{S}^\dagger_i \hat{S}_k,
\]

(1)

with \( \hat{S}^\dagger \) and \( \hat{S} \) respectively the pair creation/annihilation operator, and \( \hat{n}_i \) the number operator

\[
\hat{n}_i = \sum_{m_i=\mp j_i}^{j_i} \hat{a}^\dagger_{i,m_i} \hat{a}_{i,m_i}, \quad \hat{S}^\dagger_i = \sum_{m_i=\mp j_i}^{j_i} (-)^{j_i+m_i} \hat{a}^\dagger_{i,m_i} \hat{a}^\dagger_{i,-m_i}, \quad \hat{S}_i = (\hat{S}_i^\dagger)^\dagger.
\]

(2)

The set of operators \( \{ \hat{S}_i^\dagger, \hat{S}_i, \hat{S}_i^0 \} \) (with \( \hat{S}_i^0 = \frac{1}{2} \hat{n}_i - \frac{1}{4} \Omega_i \)) spans an \( su(2) \) quasi-spin algebra for the level \( i \). The allowed \( su(2)_i \) representations are \( |d_i, \mu_i⟩ \) with \( d_i = \frac{1}{4} \Omega_i - \frac{1}{2} v_i \) and \( \mu_i = \frac{1}{2} n_i - \frac{1}{4} \Omega_i \), where \( v_i \) denotes the seniority of the level \( i \), \( i.e. \) the number of particles not pairwise coupled to zero [18].

The Hamiltonian (1) can be diagonalised by means of the Bethe Ansatz product wavefunction

\[
|\psi⟩ = \prod_{\alpha=1}^{N} \left( \sum_{i=1}^{n} \frac{S_\alpha^\dagger}{2\varepsilon_i - E_\alpha} \right) |\theta⟩,
\]

(3)

with \( N \) the number of pairs and \( |\theta⟩ \) the particle vaccuum, provided the RG variables \( E_\alpha \) \((\alpha = 1 \ldots N)\) are a solution of the RG equations

\[
1 + 2g \sum_{i=1}^{n} \frac{d_i}{2\varepsilon_i - E_\alpha} - 2g \sum_{\beta \neq \alpha}^{N} \frac{1}{E_\beta - E_\alpha} = 0 \quad (\forall \alpha = 1 \ldots N).
\]

(4)

The associated energy of the eigenstate (3) is then given by \( E = \sum_{\alpha=1}^{N} E_\alpha \).
3. The RG equations and the TDA

The TDA assumes that the eigenstates of a quantum many-body Hamiltonian can be approximated at first order by means of harmonic multi-phonon states of an elementary excitation modes [6, 19]. The elementary TDA modes of the BCS Hamiltonian are generalised pair creation operators

\[ \hat{b}_k^\dagger = \sum_{i=1}^{n} Y_{ki} \hat{\phi}_i^\dagger. \]

Note that there are a total of \( n \) different TDA modes, with the lowest one \( (k = 1) \) referred to as the collective mode. Thanks to the \( su(2) \) symmetry of the BCS Hamiltonian, the elementary TDA modes \( (5) \) coincide with the exact wavefunctions for one single pair (eq. (3) with \( N = 1 \)), and consequently, the energy of the elementary TDA mode can be interpret as a RG variable for a system with 1 pair. Therefore, the multi-phonon states of the TDA seem as a good approximations for the Bethe Ansatz wave functions of the reduced BCS Hamiltonian (3). This is confirmed by calculating the overlaps of the multi-phonon states with the exact wavefunctions [20]. Depending on the interaction strength \( g \), one finds multi-phonon states overlapping maximally with the exact ground state or first excited state of the system. For weak interaction strengths \( g \), the overlap is largest with multi-phonon states reminiscent of the single-particle fermionic structure of the non-interacting system, whereas for strong \( g \), the largest overlap is found for fully collective TDA states. The physical underpinning of these results is provided by the Pauli principle. By assuming that the wavefunction can be approximated by means of a multi-phonon state, we have transformed a fermion pair into a genuine boson, hence violating the Pauli principle. For a better understanding of this process, it would be interesting to adiabatically switch off the Pauli principle in the exact eigenstate (3), and investigate which TDA multi-phonon state we will recover. For this purpose, we can employ a deformation of the \( su(2) \) quasi-spin [17], such that we recover the standard quasi-spin \( su(2) \) in one limit of the deformation parameter and the bosonic Heisenberg-Weyl algebra \( hw(1) \) in the other. This algebra is given by

\[ [\hat{S}_0^0, \hat{S}_1^\dagger] = \hat{S}_1^\dagger, \quad [\hat{S}_0^0, \hat{S}_i] = -\hat{S}_i, \quad [\hat{S}_1^\dagger, \hat{S}_i] = \xi \hat{S}_i + (\xi - 1) \frac{1}{2} \Omega, \]

with \( \xi \) acting as the deformation parameter. It can be verified that \( hw(1) \) is obtained for \( \xi = 0 \), whereas we retain \( su(2) \) for \( \xi = 1 \). Remarkably, if we employ the deformed algebra \( (6) \) instead of the standard \( su(2) \), the reduced BCS Hamiltonian \( (1) \) remains exactly solvable along the whole path of deformation \( (\xi \in [0, 1]) \). The deformed RG equations are given by

\[ 1 + g \sum_{i=1}^{n} \frac{\frac{1}{2} \Omega - \xi \nu_i}{2 \varepsilon_i - E_\alpha} - 2g \xi \sum_{\beta \neq \alpha} \frac{1}{E_\beta - E_\alpha} = 0, \quad (\forall \alpha = 1 \ldots N). \]

Again, for \( \xi = 1 \), we obtain the standard RG equations \( (4) \), whereas for \( \xi = 0 \), we obtain \( N \) copies of the secular equation of TDA. So, we can use the deformed RG equations to adiabatically switch off the Pauli principle and observe to which TDA multi-phonon we converge. This has been done for the model describing neutron superfluidity of \(^{56}\)Fe, as discussed in [14]. In Figure 1, the evolution of the ground state RG variables for an interaction strength of \( g = -0.500\text{MeV} \) are plotted as a function of the deformation parameter \( \xi \). The \( \xi = 1 \) RG variables are given by open circles, the \( \xi = 0 \) TDA solutions by open squares and the reference single-particle energies by filled circles. The followed paths of every RG variable are plotted as full lines starting at the RG variables and ending at the TDA solutions. From this figure, one can deduce that the RG variables evolve towards a \([3, 3, 0, 5, 0, \ldots]\) TDA multi-phonon structure, where the notation \([\nu_1, \nu_2, \ldots, \nu_n]\) refers to the TDA multi-phonon decomposition, i.e. the number \( \nu_k \) of elementary
Figure 1. The RG variables (open circles) of the neutron superfluidity model of $^{56}$Fe with $g = -0.5$MeV [14]. The corresponding TDA modes (open squares) are connected with the RG variables by means of full lines. The single-particle energies (full circles) are given as reference.

TDA modes $k$ (see eq. (5)) in the TDA multi-phonon state [17]. We can repeat this procedure for every interaction strength $g$ of the model. The results are given in Figure 2 and Table 1. Figure 2 plots the RG variables (solid lines) of the ground state as a function of the interaction strength $|g|$, as well as the 5 lowest TDA elementary modes (dashed lines). The singular values are accentuated by vertical dotted lines. The main result from the calculations is that every time the interaction strength crosses a critical point, the TDA multi-phonon decomposition changes. For instance, we can read from the Table that at $g = -0.923$MeV, the TDA decomposition changes from $[3,3,0,5,0,\ldots]$ (see Figure 1) to $[11,0,0,0,0,\ldots]$. As the latter decomposition is solely built from the collective $k = 1$ TDA mode, we can label this regime as the collective regime for the ground state.

| $|g_c|$ [MeV] | 1   | 2   | 3   | 4   | 5   | 6   | $\ldots$ | 10    |
|----------------|-----|-----|-----|-----|-----|-----|-----------|-------|
| 0.000          | 3   | 2   | 1   | 4   | 1   | 0   | $\ldots$ | 0     |
| 0.245          | 3   | 3   | 0   | 4   | 1   | 0   | $\ldots$ | 0     |
| 0.272          | 3   | 3   | 0   | 5   | 0   | 0   | $\ldots$ | 0     |
| 0.923          | 11  | 0   | 0   | 0   | 0   | 0   | $\ldots$ | 0     |

Table 1. The TDA decomposition $[\nu_1, \nu_2, \ldots, \nu_{10}]$ of the ground state as a function of the interaction strength. The singular interaction strengths $|g_c|$ are given in the first column, and the rows give the TDA decomposition for interaction strengths larger than $|g_c|$.

4. Conclusions
In conclusion, a clear-cut connection is made between the exact Bethe Ansatz wavefunctions of the reduced BCS Hamiltonian and the multi-phonon predictions of the TDA. This is done by means of a deformation of the quasi-spin algebra, allowing us to switch off the Pauli principle in an adiabatic way. The procedure is illustrated with a model describing neutron pairing in $^{56}$Fe. The singular interaction strengths mark the points where the TDA decomposition of the Bethe Ansatz wavefunction changes.
Figure 2. The Real and Imaginary part of the RG variables (solid lines) of the ground state of the neutron super fluidity model of $^{56}$Fe [14] as a function of the interaction strength $|g|$. The solutions of the 5 lowest elementary TDA modes (dashed lines) are also plotted.

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