Uniqueness in inverse acoustic scattering with phaseless near-field measurements

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Abstract

This paper is devoted to the uniqueness of inverse acoustic scattering problems with the modulus of near-field data. By utilizing the superpositions of point sources as the incident waves, we rigorously prove that the phaseless near-fields collected on an admissible surface can uniquely determine the location and shape of the obstacle as well as its boundary condition and the refractive index of a medium inclusion, respectively. We also establish the uniqueness in determining a locally rough surface from the phaseless near-field data due to superpositions of point sources. These are novel uniqueness results in inverse scattering with phaseless near-field data.

Keywords: uniqueness, phaseless, inverse scattering, near field, point source, acoustic wave

1 Introduction

Inverse scattering theory is concerned with the determination of underlying target scatterer from the incident wave and the measured near-field or far-field data. In particular, the inverse scattering theory of time-harmonic waves is of great importance in various applications such as radar detection, sonar inspection, nondestructive testing and modern medical diagnostics. The time-harmonic inverse scattering problems are typically based on complex-valued data. Hence, in terms of the accessibility to the corresponding phase information, the measured data in inverse scattering problems can be classified into two types: phased/full data and phaseless or intensity-only/modulus-only data. Over the past several decades, the inverse scattering problems with full measured data (both phase and intensity) have been mathematically and numerically studied intensively in the literature (see, e.g. [12, 22] and the references therein). Recently, a great deal of effort has been devoted to phaseless inverse scattering problems [11, 17, 24, 26, 25]. The motivation for investigating phaseless inverse problems is mainly due to the fact that such phase information is extremely difficult
to be measured accurately or even completely unavailable in a rich variety of realistic scenarios. As a result, only the phaseless data can be practically obtained in these cases.

The inverse scattering problem with one incident plane wave and phaseless far-field data is challenging due to the *translation invariance property*, namely, the modulus of the far field pattern is invariant under translations \([27, 33]\). Specifically speaking, the location of the scatterer cannot be uniquely determined by the phaseless far-field data. Nevertheless, shape recovery from phaseless data is still possible. Actually, quite a number of reversion schemes have been proposed to reconstruct the shape of the scatterer from the modulus-only far-field data with a single incident plane wave, see \([16, 17, 18, 28, 29, 30]\). We also refer to \([10, 15, 51]\) for the relevant numerical studies. It was established in \([34]\) that the radius of a small sound-soft ball can be uniquely determined from a single intensity-only far-field datum.

It is often desirable to develop corresponding techniques to tackle the difficulty of translation invariance. An effective attempt in this direction is the superposition of distinct incident plane waves proposed in \([49]\). This idea leads to the multi-frequency Newton iteration algorithm \([49, 50]\) and the fast imaging algorithm at a fixed frequency \([51]\). Further, by the superposition of two incident plane waves, uniqueness results were established in \([46]\) under some a priori assumptions.

Recently, the reference ball technique was introduced in \([52]\) to break the translation invariance in phaseless inverse acoustic scattering problem. By incorporating an suitably chosen ball into the scattering system as well as the superposition of incident plane wave and point sources, the authors in \([52]\) rigorously prove that the location and shape of the obstacle as well as its boundary condition or the refractive index can be uniquely determined by the modulus of far-field patterns. We would like to point out that the idea of adding a reference ball to the scattering system was first proposed in \([31]\) to numerically enhance the resolution of the linear sampling method. The reference ball technique was used in \([47]\) to alleviate the requirement of the a priori assumptions in \([46]\). Similar strategies of adding reference objects or sources to the scattering system have also been extensively applied to the theoretical analysis and numerical approaches for different models of phaseless inverse scattering problems \([13, 14, 19, 20, 21, 53]\). In the absence of any additional reference object, the uniqueness can be established by the superposition of incident point sources and phaseless far-field data, see \([43]\).

In this paper, we will deal with the uniqueness issue concerning the inverse acoustic scattering problems with incident point sources and phaseless near-field data. In the areas of optics and engineering sciences, the phaseless inverse scattering with near-field data is also known as phase retrieval problem \([35, 36]\). The inverse scattering problems with phaseless near-field data have been studied numerically (see, e.g. \([7, 8, 9, 11, 14, 11, 14]\)), and few studies have been made on the theoretical aspects of uniqueness for the inverse scattering problems. A recent result on uniqueness in \([23]\) was related to the reconstruction of a potential with the phaseless near-field data for point sources on a spherical surface and an interval of wavenumbers, which was extended in \([24]\) to determine the wave speed in generalized 3-D Helmholtz equation. The uniqueness of a coefficient inverse scattering problem with phaseless near-field data has been established in \([26]\). We also refer to \([25, 39, 40]\) for some recovery algorithms.
for the inverse medium scattering problems with phaseless near-field data. The
stability analysis for linearized near-field phase retrieval in X-ray phase contrast
imaging can be found in [37].

In this work, we establish the uniqueness via superposition of incident point
sources, which does not rely on any additional reference/interfering scatterer.
By introducing the concept of an admissible surface or curve, together with
the superposition of point sources, we rigorously proved that the bounded scatterer
(impenetrable obstacle or medium inclusion) and the locally perturbed half-plane
(a.k.a locally rough surface) could be uniquely determined from the phaseless near-field measurements. For the uniqueness of inverse scattering by
locally rough surfaces with phaseless far-field data, we refer to [47]. A key feature
of this study is that we make use of the limited-aperture phaseless near-field
data co-produced by the scatterer and point sources, thus the configuration is
practically more feasible than the cases of using the phaseless scattered data.

The rest of this paper is arranged as follows. Section 2 is devoted to the
inverse scattering problem of uniquely determining a bounded scatterer. Then
in section 3, we study the uniqueness results on phaseless inverse scattering by
locally perturbed half-planes.

2 Uniqueness for inverse scattering by bounded
scatterers

2.1 Problem setting

We begin this section with the acoustic scattering problems for an incident
plane wave. Assume \( D \subset \mathbb{R}^3 \) is an open and simply-connected domain with \( C^2 \)
boundary \( \partial D \). Denote by \( \nu \) the unit outward normal to \( \partial D \) and by \( S^2 := \{ x \in \mathbb{R}^3 : |x| = 1 \} \) the unit sphere in \( \mathbb{R}^3 \). Let \( u'(x, d) = e^{ikx \cdot d} \) be a given incident
plane wave, where \( d \in S^2 \) and \( k > 0 \) are the incident direction and wavenumber,
respectively. Then, the obstacle scattering problem can be formulated as: to
find the total field \( u = u' + u^s \) which satisfies the following boundary value
problem (see [13]):

\[
\begin{align*}
\Delta u + k^2 u &= 0 \quad \text{in } \mathbb{R}^3 \setminus D, \\
\mathcal{B}u &= 0 \quad \text{on } \partial D, \\
\lim_{r = |x| \to \infty} r \left( \frac{\partial u^s}{\partial r} - iku^s \right) &= 0,
\end{align*}
\]

where \( u^s \) denotes the scattered field and (3) is the Sommerfeld radiation condition. Here \( \mathcal{B} \) in (2) is the boundary operator defined by

\[
\begin{cases}
\mathcal{B}u = u & \text{for a sound-soft obstacle}, \\
\mathcal{B}u = \frac{\partial u}{\partial \nu} + iku & \text{for an impedance obstacle},
\end{cases}
\]

where \( \lambda \) is a real parameter. This boundary condition (4) covers the Dirichlet/sound-soft boundary condition, the Neumann/sound-hard boundary condition \((\lambda = 0)\),
and the impedance boundary condition \((\lambda \neq 0)\).
The medium scattering problem is to find the total field \( u = u^i + u^s \) that fulfills
\[
\Delta u + k^2 n(x) u = 0 \quad \text{in} \quad \mathbb{R}^3, \quad (5)
\]
\[
\lim_{r = |x| \to \infty} r \left( \frac{\partial u^s}{\partial r} - i k u^s \right) = 0, \quad (6)
\]
where the refractive index \( n(x) \) of the inhomogeneous medium is piecewise continuous such that \( \text{Re}(n) > 0, \text{Im}(n) \geq 0 \) and \( 1 - n(x) \) is supported in \( D \).

The direct scattering problems (1)–(3) and (5)–(6) admit a unique solution (see, e.g., \([6, 12, 38]\)), respectively, and the scattered wave \( u^s \) has the following asymptotic behavior
\[
u^s(x, d) = \frac{e^{i k |x|}}{|x|} \left\{ u^\infty(\hat{x}, d) + O \left( \frac{1}{|x|} \right) \right\}, \quad |x| \to \infty
\]
uniformly in all observation directions \( \hat{x} = x/|x| \in S^2 \). The analytic function \( u^\infty(\hat{x}, d) \) defined on the unit sphere \( S^2 \) is called the far field pattern or scattering amplitude (see \([12]\)).

Now, we turn to introducing the inverse acoustic scattering problem for incident point sources with limited-aperture phaseless near-field data. To this end, we first introduce the following definition of admissible surfaces.

**Definition 2.1** (Admissible surface). An open surface \( \Gamma \) is called an admissible surface with respect to domain \( \Omega \) if

(i) \( \Omega \subset \mathbb{R}^3 \setminus \overline{D} \) is bounded and simply-connected;
(ii) \( \partial \Omega \) is analytic homeomorphic to \( S^2 \);
(iii) \( k^2 \) is not a Dirichlet eigenvalue of \( -\Delta \) in \( \Omega \);
(iv) \( \Gamma \subset \partial \Omega \) is a two-dimensional analytic manifold with nonvanishing measure.

**Remark 2.1.** We would like to point out that this requirement for the admissibility of \( \Gamma \) is quite mild and thus can be easily fulfilled. For instance, \( \Omega \) can be chosen as a ball whose radius is less than \( \pi/k \) and \( \Gamma \) is chosen as an arbitrary corresponding semisphere.

For a generic point \( z \in \mathbb{R}^3 \setminus \overline{D} \), the incident field due to the point source located at \( z \) is given by
\[
\Phi(x, z) := \frac{e^{i k |x - z|}}{4 \pi |x - z|}, \quad x \in \mathbb{R}^3 \setminus (\overline{D} \cup \{z\}),
\]
which is also known as the fundamental solution to the Helmholtz equation. Denote by \( v^p_i(x, z) \) and \( v^p_\infty(\hat{x}, z) \) the near-field and far-field pattern generated by \( D \) corresponding to the incident field \( \Phi(x, z) \). Define
\[
v(x, z) := v^p_i(x, z) + \Phi(x, z), \quad x \in \mathbb{R}^3 \setminus (\overline{D} \cup \{z\})
\]
and
\[
v^\infty(\hat{x}, z) := v^p_\infty(\hat{x}, z) + \Phi^\infty(\hat{x}, z), \quad \hat{x} \in S^2,
\]
where \( \Phi^\infty(\hat{x}, z) := e^{-i k \hat{x} \cdot z}/(4 \pi) \) is the the far-field pattern of \( \Phi(x, z) \).

For two generic and distinct source points \( z_1, z_2 \in \mathbb{R}^3 \setminus \overline{D} \), we denote by
\[
v^i(x; z_1, z_2) := \Phi(x, z_1) + \Phi(x, z_2), \quad x \in \mathbb{R}^3 \setminus (\overline{D} \cup \{z_1\} \cup \{z_2\}), \quad (7)
\]
the superposition of these point sources. Then, by the linearity of direct scattering problem, the near-field co-produced by $D$ and the incident wave $v^i(x; z_1, z_2)$ is given by

$$v(x; z_1, z_2) := v(x, z_1) + v(x, z_2), \quad x \in \mathbb{R}^3 \backslash (\overline{D} \cup \{z_1\} \cup \{z_2\}).$$

With these preparations, we formulate the phaseless inverse scattering problems as the following.

**Problem 2.1** (Phaseless inverse scattering by a bounded scatterer). Assume that $\Gamma$ and $\Sigma$ are admissible surfaces with respect to $\Omega$ and $G$, respectively, such that $\Omega \cap \overline{G} = \emptyset$. Let $D$ be the impenetrable obstacle with boundary condition $\mathcal{B}$ or the inhomogeneous medium with refractive index $n$. Given the phaseless near-field data

$$\{v(x, z_0) : x \in \Sigma\}, \quad \{v(x, z) : x \in \Sigma, \ z \in \Gamma\}, \quad \{v(x, z_0) + v(x, z) : x \in \Sigma, \ z \in \Gamma\}$$

for a fixed wavenumber $k > 0$ and a fixed $z_0 \in \mathbb{R}^3 \backslash (\overline{D} \cup \Gamma \cup \Sigma)$, determine the location and shape $\partial D$ as well as the boundary condition $\mathcal{B}$ for the obstacle or the refractive index $n$ for the medium inclusion.

We refer to Figure 1 for an illustration of the geometry setting of Problem 2.1. The uniqueness of this problem will be analyzed in the next subsection.

### 2.2 Uniqueness result

Now we present the uniqueness results on phaseless inverse scattering. The following theorem shows that Problem 2.1 admits a unique solution, namely, the geometrical and physical information of the scatterer boundary or the refractive index for the medium can be simultaneously and uniquely determined from the modulus of near-fields.
Theorem 2.1. Assume that $\Gamma$ and $\Sigma$ are admissible surfaces with respect to $\Omega$ and $G$, respectively, such that $\overline{\Gamma} \cap \Omega = \emptyset$. For two scatterers $D_1$ and $D_2$, suppose that the corresponding near-fields satisfy that

$$|v_1(x, z_0)| = |v_2(x, z_0)|, \quad \forall x \in \Sigma, \quad (8)$$

and

$$|v_1(x, z)| = |v_2(x, z)|, \quad \forall (x, z) \in \Sigma \times \Gamma \quad (9)$$

and

$$|v_1(x, z_0) + v_1(x, z)| = |v_2(x, z_0) + v_2(x, z)|, \quad \forall (x, z) \in \Sigma \times \Gamma \quad (10)$$

for an arbitrarily fixed $z_0 \in \mathbb{R}^3 \setminus (D \cup \Gamma \cup \Sigma)$. Then we have

(i) If $D_1$ and $D_2$ are two impenetrable obstacles with boundary conditions $\mathcal{B}_1$ and $\mathcal{B}_2$ respectively, then $D_1 = D_2$ and $\mathcal{B}_1 = \mathcal{B}_2$.

(ii) If $D_1$ and $D_2$ are two medium inclusions with refractive indices $n_1$ and $n_2$ respectively, then $n_1 = n_2$.

Proof. From (8), (9) and (10), we have for all $x \in \Sigma, z \in \Gamma$

$$\text{Re}\left\{ v_1(x, z_0)\overline{v_1(x, z)} \right\} = \text{Re}\left\{ v_2(x, z_0)\overline{v_2(x, z)} \right\}, \quad (11)$$

where the overline denotes the complex conjugate. According to (8) and (9), we denote

$$v_j(x, z_0) = r(x, z_0)e^{i\alpha_j(x, z_0)}, \quad v_j(x, z) = s(x, z)e^{i\beta_j(x, z)}, \quad j = 1, 2,$$

where $r(x, z_0) = |v_j(x, z_0)|$, $s(x, z) = |v_j(x, z)|$, $\alpha_j(x, z_0)$ and $\beta_j(x, z)$, are real-valued functions, $j = 1, 2$.

Since $\Sigma$ is an admissible surface of $G$, by Definition 2.1 and the analyticity of $v_j(x, z)$ with respect to $x$, we have $s(x, z) \neq 0$ for $x \in \Sigma, z \in \Gamma$. Further, the continuity yields that there exists open sets $\Sigma \subset \Sigma$ and $\Gamma_0 \subset \Gamma$ such that $s(x, z) \neq 0, \forall (x, z) \in \Sigma \times \Gamma_0$. Similarly, we have $r(x, z_0) \neq 0$ on $\Sigma$. Again, the continuity leads to $r(x, z_0) \neq 0$ on an open set $\Sigma_0 \subset \Sigma$. Therefore, we have $r(x, z_0) \neq 0, s(x, z) \neq 0, \forall (x, z) \in \Sigma_0 \times \Gamma_0$. In addition, taking (11) into account, we derive that

$$\cos[\alpha_1(x, z_0) - \beta_1(x, z)] = \cos[\alpha_2(x, z_0) - \beta_2(x, z)], \quad \forall (x, z) \in \Sigma_0 \times \Gamma_0.$$

Hence, either

$$\alpha_1(x, z_0) - \alpha_2(x, z_0) = \beta_1(x, z) - \beta_2(x, z) + 2m\pi, \quad \forall (x, z) \in \Sigma_0 \times \Gamma_0 \quad (12)$$

or

$$\alpha_1(x, z_0) + \alpha_2(x, z_0) = \beta_1(x, z) + \beta_2(x, z) + 2m\pi, \quad \forall (x, z) \in \Sigma_0 \times \Gamma_0 \quad (13)$$

holds with some $m \in \mathbb{Z}$.

First, we shall consider the case (12). Since $z_0$ is fixed, let us define $\gamma(x) := \alpha_1(x, z_0) - \alpha_2(x, z_0) - 2m\pi$ for $x \in \Sigma_0$, and then, we deduce for all $(x, z) \in \Sigma_0 \times \Gamma_0$

$$v_1(x, z) = s(x, z)e^{i\beta_1(x, z)} = s(x, z)e^{i\beta_2(x, z) + i\gamma(x)} = v_2(x, z)e^{i\gamma(x)}.$$

From the reciprocity relation [2] Theorem 3 for point sources, we have

$$v_1(z, x) = e^{i\gamma(x)}v_2(z, x), \quad \forall (x, z) \in \Sigma_0 \times \Gamma_0.$$
Then, for every \( x \in \Sigma_0 \), by using the analyticity of \( v_j(z, x)(j = 1, 2) \) with respect to \( z \), we have \( v_1(z, x) = e^{i\gamma(x)}v_2(z, x) \), \( \forall z \in \partial \Omega \). Let \( w(z, x) = v_1(z, x) - e^{i\gamma(x)}v_2(z, x) \), then
\[
\begin{aligned}
\Delta w + k^2 w &= 0 \quad \text{in } \Omega, \\
w &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

By the assumption of \( \Omega \) that \( k^2 \) is not a Dirichlet eigenvalue of \( -\Delta \) in \( \Omega \), we find \( w = 0 \) in \( \Omega \). Now, the analyticity of \( v_j(z, x)(j = 1, 2) \) with respect to \( z \) yields
\[
v_1(z, x) = e^{i\gamma(x)}v_2(z, x), \quad \forall z \in \mathbb{R}^3 \setminus (\overline{D_1} \cup \overline{D_2} \cup \{x\}).
\]
i.e., for all \( z \in \mathbb{R}^3 \setminus (\overline{D_1} \cup \overline{D_2} \cup \{x\}) \),
\[
v_{D_1}^s(z, x) + \Phi(z, x) = e^{i\gamma(x)}[v_{D_2}^s(z, x) + \Phi(z, x)]. \tag{14}
\]

By the Green’s formula \cite[Theorem 2.5]{[12]} one can readily deduce that \( v_{D_1}^s(z, x)(j = 1, 2) \) are bounded for \( z \in \mathbb{R}^3 \setminus (\overline{D_1} \cup \overline{D_2}) \), which, together with \( \Phi(z, x) \) is bounded for \( z \in \mathbb{R}^3 \setminus (\overline{D_1} \cup \overline{D_2} \cup \{x\}) \). Hence, by letting \( z \to x \), we obtain \( e^{i\gamma(x)} = 1 \), and
\[
v_{D_1}^s(z, x) = v_{D_2}^s(z, x), \quad \forall (x, z) \in \Sigma_0 \times \mathbb{R}^3 \setminus (\overline{D_1} \cup \overline{D_2} \cup \{x\}).
\]

And thus, the far-field patterns coincide, i.e.
\[
v_{D_1}^s(\hat{x}, z) = v_{D_2}^s(\hat{x}, z), \quad \forall (x, \hat{x}) \in \Sigma_0 \times S^2.
\]

Now, from the mixed reciprocity relation \cite[Theorem 3.16]{[12]} for the obstacle or \cite[Theorem 2.2.4]{[?]} for the inhomogeneous medium, we have
\[
u_1^s(x, -\hat{z}) = u_2^s(x, -\hat{z}), \quad \forall (x, \hat{z}) \in \Sigma_0 \times S^2.
\]

Further, the analyticity of \( u_j^s(x, d)(j = 1, 2) \) with respect to \( x \) yields \( u_1^s(x, d) = u_2^s(x, d), \quad \forall (x, d) \in \partial G \times S^2 \). By the similar discussion of \( \Phi(z, x) \) for \( u_1^s(x, d) - u_2^s(x, d) \) on \( G \), we have
\[
u_1^s(x, d) = u_2^s(x, d), \quad \forall (x, d) \in \mathbb{R}^3 \setminus (\overline{D_1} \cup \overline{D_2}) \times S^2.
\]

Therefore, we obtain
\[
u_1^s(\hat{x}, d) = u_2^s(\hat{x}, d), \quad \forall \hat{x}, d \in S^2. \tag{15}
\]

Next, we are going to show that the case \( \Phi(z, x) \) does not hold. Suppose that \( \Phi(z, x) \) is true, then following a similar argument, we see that there exists \( \eta(x) \) such that \( v_1(z, x) = e^{i\eta(x)}v_2(z, x) \) for \( x \in \Sigma_0, z \in \mathbb{R}^3 \setminus (\overline{D_1} \cup \overline{D_2} \cup \{x\}) \), i.e.
\[
v_{D_1}^s(z, x) + \Phi(z, x) = e^{i\eta(x)}[v_{D_2}^s(z, x) + \Phi(z, x)].
\]

Then, from the boundedness of \( v_{D_1}^s(z, x) \), it can be seen that \( \Phi(z, x) - e^{i\eta(x)}\Phi(z, x) \) is bounded for all \( z \in \mathbb{R}^3 \setminus (\overline{D_1} \cup \overline{D_2} \cup \{x\}) \). Since
\[
\Phi(z, x) - e^{i\eta(x)}\Phi(z, x) = [1 - e^{i\eta(x)}] \frac{\cos(k |z - x|)}{4\pi |z - x|} + i[1 + e^{i\eta(x)}] \frac{\sin(k |z - x|)}{4\pi |z - x|},
\]

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Further, noticing $v\in z$, we deduce $e^{i \eta (x)} = 1$, and thus, $v_1(z, x) = \overline{v_2(z, x)}$ for $z \in \mathbb{R}^3 \setminus (D_1 \cup D_2 \cup \{x\})$. We claim that $v_1^\infty(z, x) \neq 0$ for $\hat{z} \in S^2$. Otherwise, if $v_1^\infty(z, x) \equiv 0$ for $\hat{z} \in S^2$, then from Rellich Lemma [12, Theorem 2.14], we have $v_1(z, x) = 0$ for all $z \in \mathbb{R}^3 \setminus (D_1 \cup \{x\})$. Further, from $v_1(z, x) = v_1^0(z, x) + \Phi(z, x)$ and the boundedness of $v_1^0(z, x)$, we deduce $\Phi(z, x)$ is bounded for all $z \in \mathbb{R}^3 \setminus (D_1 \cup \{x\})$, which is a contradiction. Then, the continuity leads to $v_1^\infty(z, x) \neq 0 \forall \hat{z} \in S$, where $S \subset S^2$ is an open set. By taking $\bar{z} \in S$, $z = \rho \bar{z}$, and using the definition of far-field pattern (see [12, Theorem 2.6]), we obtain

$$\lim_{\rho \to \infty} e^{-ik\rho} v_1(\rho \bar{z}, x) = v_1^\infty(\bar{z}, x)$$

and

$$\lim_{\rho \to \infty} e^{ik\rho} \overline{v_2(\rho \bar{z}, x)} = v_2^\infty(\bar{z}, x).$$

Further, noticing $v_1(\rho \bar{z}, x) = \overline{v_2(\rho \bar{z}, x)}$ and $v_1^\infty(\bar{z}, x) \neq 0$, we have

$$\lim_{\rho \to \infty} e^{2ik\rho} = \frac{v_2^\infty(\bar{z}, x)}{v_1^\infty(\bar{z}, x)},$$

which is a contradiction. Hence, the case (13) does not hold.

Having verified (15), we shall complete our proof as the consequences of two existing uniqueness results. For the inverse obstacle scattering, by Theorem 5.6 in [12], we have $D_1 = D_2$ and $B_1 = B_2$, and for inverse medium scattering, Theorem 10.5 in [12] leads to $n_1 = n_2$.

**Remark 2.2.** We would like to point out that an analogous uniqueness result in two dimensions remains valid after appropriate modifications of the fundamental solution, the radiation condition and the admissible surface. So we omit the 2D details.

**Remark 2.3.** We would like to remark that a similar result on uniqueness can be also obtained by using the superposition of a fixed plane wave and some point sources as the incident fields.

## 3 Uniqueness for inverse scattering by locally perturbed half-planes

### 3.1 Problem statement

We begin this section with the precise formulations of the model scattering problem. Assume that the real-valued function $f \in C^2(\mathbb{R})$ has a compact support. Let $\Gamma = \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_2 = f(x_1), x_1 \in \mathbb{R}\}$ be the locally perturbed curve and $D = \{x \in \mathbb{R}^2 \mid x_2 > f(x_1), x_1 \in \mathbb{R}\}$ be the locally perturbed half-plane above curve $\Gamma$. Denote by $\Gamma_p = \Gamma \setminus \Gamma_c$ the local perturbation. For a generic point $z \in D$, the incident field $u^i$ due to the point source located at $z$ is given by

$$u^i(x, z) := \frac{i}{4} H_0^{(1)}(k|x - z|) = \frac{i}{4} J_0(k|x - z|) + \frac{1}{4} Y_0(k|x - z|), \, x \in D \setminus \{z\},$$

which is also known as the fundamental solution to the Helmholtz equation with wavenumber $k > 0$, where $J_0$ and $Y_0$ are the Bessel functions of the first kind.
and the second kind of order 0, respectively. Then, the scattering problem can be formulated as: find the scattered field \( u^s \), such that

\[
\Delta u^s + k^2 u^s = 0 \quad \text{in } D,
\]

\[
B_c u = 0 \quad \text{on } \Gamma_c,
\]

\[
B_p u = 0 \quad \text{on } \Gamma_p,
\]

\[
\lim_{r=|x| \to \infty} \sqrt{r} \left( \frac{\partial u^s}{\partial r} - ik u^s \right) = 0,
\]

where \( u = u^i + u^s \) denotes the total field. Here \( B_c \) and \( B_p \) in (17)-(18) are the boundary operators defined by

\[
B_c u = \begin{cases} u, & \text{for a sound-soft perturbed half-plane}, \\ \frac{\partial u}{\partial \nu}, & \text{for a sound-hard perturbed half-plane} \end{cases}
\]

and

\[
B_p u = \begin{cases} u, & \text{on } \Gamma_p, \\ \frac{\partial u}{\partial \nu} + \lambda u, & \text{on } \Gamma_p, \end{cases}
\]

where \( \nu \) is the unit normal on \( \Gamma \) directed into \( D \), \( \Gamma_p \cap \Gamma_p = \emptyset \), \( \lambda \in C(\Gamma_p, I) \) and \( \text{Im}\lambda \geq 0 \). The mixed boundary condition (21) is rather general in the sense that it covers the usual Dirichlet/sound-soft boundary condition (\( \Gamma_p = \emptyset \)), the Neumann/sound-hard boundary condition (\( \Gamma_p = \emptyset \) and \( \lambda = 0 \)), and the impedance boundary condition (\( \Gamma_p = \emptyset \) and \( \lambda \neq 0 \)).

The existence of a unique solution to the scattering problem (16)-(19) by a sound-soft perturbed half-plane with \( \Gamma_p = \emptyset \) was established in [54, 3] by a variational method or in [48] by an integral equation method, and the well-posedness of the problem (16)-(19) by a sound-hard perturbed half-plane with \( \Gamma_p = \emptyset \) and \( \lambda = 0 \) was studied in [42] by the integral equation method. In three-dimension case, we refer to [45] for the integral equation method.

The well-posedness of the problem (16)-(19) by a sound-soft perturbed half-plane with \( \Gamma_p \neq \emptyset \) can be obtained similarly by the variational method as shown in [54, 3], while the existence and uniqueness of solutions to the problem (16)-(19) by a sound-hard perturbed half-plane can be established by the variational method with an even expansion and extension of the solution (an odd expansion and extension of the solution in [54, 3]) and a simple modification of the Dirichlet-to-Neumann operator in [54, 3] which does not affect the properties.

In the following we are going to consider the inverse scattering problem by the locally perturbed half-plane for incident point sources with limited-aperture phaseless near-field data. Similar to Definition 2.1, we first introduce the description of admissible curves.

**Definition 3.1** (Admissible curve). An open curve \( \Lambda \) is called an admissible curve with respect to domain \( \Omega \) if

(i) \( \overline{\Omega} \subset D \) is bounded and simply-connected;

(ii) \( \partial \Omega \) is analytic homeomorphic to \( \mathbb{S} \);

(iii) \( k^2 \) is not a Dirichlet eigenvalue of \( -\Delta \) in \( \Omega \);

(iv) \( \Lambda \subset \partial \Omega \) is a one-dimensional analytic manifold with nonvanishing measure.
Remark 3.1. It can be readily seen that Definition 3.1 is the one-dimensional version of Definition 2.1. Thus it is easy to find an admissible pair of $(\Omega, \Lambda)$. For example, $\Omega$ can be chosen as a disk whose radius is less than $2.4048/k$ and $\Lambda$ is chosen as an arbitrary corresponding semicircle.

Analogous to the arguments in the previous section, for two generic and distinct source points $z_1, z_2 \in D$, we denote by
\[
u^i(x; z_1, z_2) := u^i(x, z_1) + u^i(x, z_2), \quad x \in D \setminus \{z_1 \cup \{z_2\}\},
\] the superposition of these point sources. Then, by the linearity of direct scattering problem, the total near-field is given by
\[
u(x; z_1, z_2) := u(x, z_1) + u(x, z_2), \quad x \in D \setminus \{z_1 \cup \{z_2\}\}.
\]

We are now in the position to formulate the phaseless inverse scattering problems under consideration.

**Problem 3.1** (Phaseless inverse scattering by locally perturbed half-planes).
Let $\Gamma$ be the locally perturbed curve with boundary condition $B_c$ and $B_p$. Assume that $\Lambda$ and $\Sigma$ are admissible curves with respect to $\Omega$ and $G$, respectively. Given the phaseless near-field data
\[
\{|u(x, z_0)| : x \in \Sigma\},
\{|u(x, z)| : x \in \Sigma, \ z \in \Lambda\},
\{|u(x, z_0) + u(x, z)| : x \in \Sigma, \ z \in \Lambda\},
\] for a fixed wavenumber $k > 0$ and a fixed $z_0 \in D \setminus (\Lambda \cup \Sigma)$, determine the locally perturbed curve $\Gamma$ as well as the boundary condition $B_c$ and $B_p$.

For an illustration of the above problem, we refer to Figure 2. The next subsection will be devoted to the uniqueness issue of this problem.

### 3.2 Uniqueness results

Before we present the uniqueness result on phaseless inverse scattering, the following reciprocity relation for the total fields is needed.

\[
\text{Figure 2: An illustration of the phaseless inverse scattering by a locally perturbed half-plane.}
\]
Lemma 3.1. Let \( u^s(x, z) \) be the scattered field satisfying (16)–(19). Then we have
\[
  u(x, z) = u(z, x), \quad \forall x, z \in D, \; x \neq z.
\]  
(23)

Proof. The proof of the reciprocity relation is similar to that of Theorem 3.1.4 in [32], so the details are omitted.

Let \( \Gamma_j = \{ x \in \mathbb{R}^2 \mid x_2 = f_j(x_1), x_1 \in \mathbb{R} \} \) be the locally perturbed curve with the real-valued function \( f_j \in C^2(\mathbb{R}) \) having a compact support, \( j = 1, 2 \). Denote by \( D_j = \{ x \in \mathbb{R}^2 \mid x_2 > f_j(x_1), x_1 \in \mathbb{R} \} \) the domain above \( \Gamma_j \), \( j = 1, 2 \), and by \( D_0 = D_1 \cap D_2 \).

Denote by \( u^1 \) and \( u^2 \) the scattered field and the total field generated by \( \Gamma_j \), respectively, corresponding to the incident field \( u^i(x, z), \; j = 1, 2 \). Now, the following theorem shows that Problem 3.1 admits a unique solution, namely, the geometrical and physical information of the locally perturbed plane can be simultaneously and uniquely determined from the modulus of total near-fields.

Theorem 3.1. Let \( \Gamma_1 \) and \( \Gamma_2 \) be two locally perturbed curves with boundary conditions \( \mathcal{B}_{c,1}, \mathcal{B}_{p,1} \) and \( \mathcal{B}_{c,2}, \mathcal{B}_{p,2} \), respectively. Assume that \( \Lambda \) and \( \Sigma \) are admissible curves with respect to \( \Omega \) and \( G \), respectively, such that \( \mathcal{G} \subset D_0 \), \( \mathcal{G} \subset \subset D_0 \) and \( \Omega \cap \mathcal{G} = \emptyset \). Suppose that the corresponding total near-fields satisfy that
\[
|u_1(x, z_0)| = |u_2(x, z_0)|, \quad \forall x \in \Sigma,
\]
\[
|u_1(x, z)| = |u_2(x, z)|, \quad \forall (x, z) \in \Sigma \times \Lambda
\]
(24) and
(25)
and
\[
|u_1(x, z_0) + u_1(x, z)| = |u_2(x, z_0) + u_2(x, z)|, \quad \forall (x, z) \in \Sigma \times \Lambda
\]
(26)
for an arbitrarily fixed \( z_0 \in D_0 \setminus (\Lambda \cup \Sigma) \). Then we have \( \Gamma_1 = \Gamma_2, \mathcal{B}_{c,1} = \mathcal{B}_{c,2} \) and \( \mathcal{B}_{p,1} = \mathcal{B}_{p,2} \).

Proof. In terms of (24), (25) and (26), we have for all \( x \in \Sigma, \; z \in \Lambda \)
\[
\text{Re} \left\{ u_1(x, z_0)\overline{u_1(x, z)} \right\} = \text{Re} \left\{ u_2(x, z_0)\overline{u_2(x, z)} \right\}.
\]
(27)
Using (24) and (25), we denote
\[
u_j(x, z_0) = r_j(x, z_0)e^{i\alpha_j(x, z_0)}, \quad u_j(x, z) = s_j(x, z)e^{i\beta_j(x, z)}, \quad j = 1, 2,
\]
where \( r_j(x, z_0) = |u_j(x, z_0)|, \; s_j(x, z) = |u_j(x, z)|, \; \alpha_j(x, z_0) \) and \( \beta_j(x, z) \), are real-valued functions, \( j = 1, 2 \).

Due to the fact that \( \Sigma \) is an admissible curve of \( G \), Definition 3.1 and the analyticity of \( u_j(x, z) \) with respect to \( x \) imply that \( s(x, z) \neq 0 \) for \( x \in \Sigma, \; z \in \Lambda \). Moreover, by the continuity we deduce that there exists open sets \( \Sigma \subset \Lambda \) and \( \Lambda_0 \subset \Lambda \) such that \( s(x, z) \neq 0 \), \( \forall (x, z) \in \Sigma \times \Lambda_0 \). Analogously, we obtain \( r(x, z_0) \neq 0 \) on \( \Sigma \). The continuity also leads to \( r(x, z_0) \neq 0 \) on an open set \( \Sigma_0 \subset \Sigma \). Hence, we have \( r(x, z_0) \neq 0, \; s(x, z) \neq 0, \; \forall (x, z) \in \Sigma_0 \times \Lambda_0 \). Furthermore, we derive from (27) that
\[
\cos[\alpha_1(x, z_0) - \beta_1(x, z)] = \cos[\alpha_2(x, z_0) - \beta_2(x, z)], \quad \forall (x, z) \in \Sigma_0 \times \Lambda_0.
\]
Therefore, either
\[
\alpha_1(x, z_0) - \alpha_2(x, z_0) = \beta_1(x, z) - \beta_2(x, z) + 2m\pi, \ \forall (x, z) \in \Sigma_0 \times \Lambda_0
\]  
(28)
or
\[
\alpha_1(x, z_0) + \alpha_2(x, z_0) = \beta_1(x, z) + \beta_2(x, z) + 2m\pi, \ \forall (x, z) \in \Sigma_0 \times \Lambda_0
\]  
(29)
holds with some \(m \in \mathbb{Z}\).

We first deal with the case (28). Note that \(z_0\) is fixed, we can define \(\gamma(x) := \alpha_1(x, z_0) - \alpha_2(x, z_0) - 2m\pi\) for \(x \in \Sigma_0\), thus we deduce for all \((x, z) \in \Sigma_0 \times \Lambda_0\)
\[
u_1(x, z) = s(x, z)e^{i\beta_1(x, z)} = s(x, z)e^{i\beta_2(x, z) + i\gamma(x)} = u_2(x, z)e^{i\gamma(x)}.
\]
By the reciprocity relation (23), we arrive at
\[
u_1(x, z) = e^{i\gamma(x)}u_2(x, z), \ \forall (x, z) \in \Sigma_0 \times \Lambda_0.
\]
Now, for every \(x \in \Sigma_0\), the analyticity of \(u_j(x, z)\) with respect to \(z\) leads to
\[
u_1(x, z) = e^{i\gamma(x)}u_2(x, z), \ \forall z \in \partial \Omega.\]
Define \(w(z, x) = \nu_1(x, z) - e^{i\gamma(x)}u_2(x, z)\), then
\[
\begin{cases}
\Delta w + k^2 w = 0 & \text{in } \Omega, \\
w = 0 & \text{on } \partial \Omega.
\end{cases}
\]  
(30)
Since \(k^2\) is not a Dirichlet eigenvalue of \(-\Delta\) in \(\Omega\), (30) implies that \(w = 0\) in \(\Omega\).

So, the analyticity of \(u_j(x, z)\) with respect to \(z\) yields
\[
u_1(x, z) = e^{i\gamma(x)}u_2(x, z), \ \forall z \in D_0 \setminus \{x\}.
\]
namely, for all \(z \in D_0 \setminus \{x\}\),
\[
u_1^*(x, z) + u'(z, x) = e^{i\gamma(x)}[u_2^*(z, x) + u'(z, x)].
\]  
(31)
From the boundedness of \(u_j^*(x, z)\) for \(z \in D_j \setminus \{x\}\) and (14), we see \((1 - e^{i\gamma(x)})u'(z, x)\) is bounded for \(z \in D_0 \setminus \{x\}\). Therefore, by letting \(z \to x\), we find \(e^{i\gamma(x)} = 1\). Furthermore, the continuity of the scattered fields and the reciprocity relation (29) lead to
\[
u_1^*(x, z) = u_2^*(x, z), \ \forall z \in D_0.
\]
By a similar argument of (30) for \(w(x, z) = \nu_1^*(x, z) - u_2^*(x, z)\) on \(G\), we have
\[
u_1^*(x, z) = u_2^*(x, z), \ \forall x, z \in D_0.
\]  
(32)
Next we shall show that the case (29) does not hold. Suppose that (29) is true, then following a similar argument, we find that for every \(x \in \Sigma_0\), there exists \(\eta(x)\) such that \(u_1(x, z) = e^{i\eta(x)}u_2(x, z)\) for all \(z \in D_0 \setminus \{x\}\), that means
\[
u_1^*(x, z) + u'(z, x) = e^{i\eta(x)}[u_2^*(z, x) + u'(z, x)].
\]
Then, it can be seen from the boundedness of \(u_j^*(x, z)\) that \(u'(z, x) - e^{i\eta(x)}u'(z, x)\) is bounded for all \(z \in D_0 \setminus \{x\}\). By letting \(z \to x\) in
\[
u'(z, x) - e^{i\eta(x)}u'(z, x) = \frac{e^{i\eta(x)} - 1}{4} Y_0(k|x - z|) + \frac{e^{i\eta(x)} + 1}{4} J_0(k|x - z|),
\]

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we find that \( e^{i\eta(x)} = 1 \), which implies \( u_1(z, x) = \overline{u_2(z, x)} \) for \( z \in \partial D_0 \setminus \{x\} \). Now, following the same ideas in Theorem 2.1, we deduce that the case does not hold.

Once the case is verified, we would conclude that \( \Gamma_1 = \Gamma_2 \). Otherwise, assume that \( \Gamma_1 \neq \Gamma_2 \). Then, without loss of generality, there exists \( x^* \in \partial D_0 \) such that \( x^* \in \Gamma_1 \) and \( x^* \in D_2 \). Define

\[
z_n := x^* - \frac{1}{n} \nu(x^*), \quad n = 1, 2, ...
\]

such that \( z_n \in D_0 \) for sufficiently large \( n \). Then, from the reciprocity relation and the smoothness of \( u_2^s(x^*, z) \) in \( D_2 \), we have

\[
\lim_{n \to \infty} B_{p,1} u_2^s(x^*, z_n) = \lim_{n \to \infty} B_{p,1} u_2^s(z_n, x^*) = B_{p,1} u_2^s(x^*, x^*).
\]

On the other hand, the boundary conditions \( 15 \) and \( 32 \) imply that

\[
\lim_{n \to \infty} B_{p,1} u_1^s(x^*, z_n) = \lim_{n \to \infty} B_{p,1} u_1^s(z_n, x^*) = -\lim_{n \to \infty} B_{p,1} u^t(x^*, z_n) = \infty,
\]

which is a contradiction. Therefore \( \Gamma_1 = \Gamma_2 \). Further, similar to the proof of Theorem 5.6 in [17], we obtain \( \mathcal{B}_{c,1} = \mathcal{B}_{c,2} \) and \( \mathcal{B}_{p,1} = \mathcal{B}_{p,2} \).

**Remark 3.2.** We want to point out that an analogous uniqueness result in three dimensions remains valid subject to some modifications of the fundamental solution and the admissible curve.

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