Local Popularity Based Collaborative Filters

Kishor Barman  
School of Technology and Computer Science  
Tata Institute of Fundamental Research  
Mumbai, India  
Email: kishor@tcs.tifr.res.in

Onkar Dabeer  
School of Technology and Computer Science  
Tata Institute of Fundamental Research  
Mumbai, India  
Email: onkar@tcs.tifr.res.in

Abstract—Motivated by applications such as recommendation systems, we consider the estimation of a binary random field $X$ obtained by unknown row and column permutations of a block random constant matrix. The estimation of $X$ is based on observations $Y$, which are obtained by passing entries of $X$ through a binary symmetric channel (BSC) (representing noisy user behavior) and an erasure channel (representing missing data). We analyze an estimation algorithm based on local popularity. We study the bit error rate (BER) in the limit as the matrix size approaches infinity and the erasure rate approaches unity at a specified rate. Our main result identifies three regimes characterized by the cluster size and erasure rate. In one regime, the algorithm asymptotically zero BER, in another regime the BER is bounded away from 0 and 1/2, while in the remaining regime, the algorithm fails and BER approaches 1/2. Numerical results for the Movielens dataset and comparison with earlier work is also given.

I. INTRODUCTION

Recommendation systems are commonly used in e-commerce to suggest relevant content to users. One approach considers the user-item rating matrix, predicts the missing entries, and recommends items based on the predicted values (for example, see [1]). Recently, a number of researchers have considered mathematical models for this problem and studied fundamental limits. One model assumes the rating matrix to be a low-rank random matrix ([2]–[4]), and then bounds on the number of samples needed to recover the complete matrix with high probability are obtained. In another model ([5], [6]), the rating matrix is assumed to be obtained from a block constant matrix by applying unknown row and column permutations, a noisy discrete memoryless channel representing noisy user behavior, and an erasure channel denoting missing entries. The goal for such a model is not matrix completion, but estimation of the underlying “noiseless” matrix. In [5], [6], the probability of error in recovering the entire matrix for fixed erasure rate is considered, and threshold results reminiscent of the channel coding theorem (but with different scaling) are established.

In this paper, we consider the model in [6], but we allow the erasure rate to approach unity, and focus on the BER - the probability of error that a specific recommendation fails. We analyze the BER for a specific algorithm, which makes recommendations based on “local popularity”. Such an analysis is of interest for two reasons:

- It gives an upper bound on achievable BER;
- The local popularity algorithm used is motivated by algorithms used in practice [7], and has lower complexity compared to those in the above mentioned references.
- The algorithm has competitive empirical performance on real datasets such as the Movielens data [8]. For example, we compare the algorithm with OptSpace [3] on Movielens data. While OptSpace uses ratings on the scale 1-5 given by Movielens, in our algorithm we quantize the ratings as follows: 4,5 are mapped to 1, while 1-3 are mapped to 0. (Similarly, the output of OptSpace is quantized to $\{0,1\}$.) We find that the local algorithm yields a BER of 0.091, while on the same test data, OptSpace gives a BER of 0.107. Thus the performance of both algorithms is similar. (More detailed simulation results will be presented in a future publication.)

In this paper, we seek to understand the reason for the competitive performance of the relatively simple local algorithm by analyzing its BER for the model proposed in [5]. Suppose that the matrix is of size $n \times n$ and the erasure probability $\epsilon = 1 - c/n^\alpha$. If $\alpha \in [0,1/2)$, then our main result says that if the cluster size is greater than $n^{\alpha - \gamma_n}$ where $\gamma_n \to 0$, then the BER approaches 0, but if the cluster size is less than $n^{\alpha - \gamma}$, $\gamma > 0$, the BER is bounded away from zero and a lower bound is obtained in terms of the observation noise and $\gamma$. For $\alpha > 1/2$, BER always approaches 1/2. Due to space constraints we only provide an outline of the proofs; the details with additional results will be reported in a journal submission.

The rest of the paper is organized as follows. In Section II we describe our model, the local popularity algorithm, and establish notation. The main results are stated and discussed in Section III. The proof of the main result is given in Section IV and some related lemmas are established in Section V. The conclusion of given in Section V.

II. BASIC SETUP

In Section II-A we describe our model, and discuss a local popularity based algorithm in Section II-B.

A. The Model

We consider an $n \times n$ rating matrix $X$ whose entries are binary. The rows of the matrix represent users and the columns represent items. Suppose $A = \{A_i\}_{i=1}^r$ and $B = \{B_i\}_{i=1}^r$ are row and column partitions respectively, representing sets of similar users and items. We assume that for all $i = 1, \ldots, r$ we have $|A_i| = |B_i| = k$. The sets $A_i \times B_j$ are the clusters.
Theorem 1. We use the following local algorithm (the erasure probability \(c\) stronger result for approaches zero. In the following theorem, we establish a more detailed motivation of this model, we refer to [5], [6].

We consider the case of binary entries and uniform cluster size is for simplicity, and like in [6], these can be relaxed. For more detailed motivation of this model, we refer to [5], [6].

B. A Local Popularity Algorithm

Without loss of generality suppose the first row belongs to \(A_1\). Upon observing \(Y\), we want to recommend an item (a column) to the user 1. In this paper we study a particular “local” algorithm, which only uses pairwise row correlations. Let the number of commonly sampled entries between two rows (similarity) \(s_{ij} := \sum_{k=1}^{n} 1(Y_{i,k} \neq \ast) \cdot 1(Y_{j,k} \neq \ast) \cdot 1(Y_{i,k} = Y_{j,k})\), where \(1(\cdot)\) denotes the indicator function. We use the following local algorithm (\(\text{local_algo}(T)\)) to recommend an item \(j_0\) to user 1.

\[
\text{local_algo}(T) : \\
1) \text{(Select the top } T \text{ nearest rows)} \text{ Compute } s_{ij}, \text{ for } i = 1, 2, \ldots, n. \text{ Select the top } T \text{ rows with the highest values of similarity, where } T \text{ is a parameter whose choice is discussed later.} \\
2) \text{(Pick the most popular column)} \text{ Among the columns } j \text{ such that } Y(1,j) = \ast, \text{ select the column having maximum number of } 1\text{'s among the top } T \text{ neighbors. Break ties randomly.}
\]

Suppose we represent each row by a vertex in a graph with an edge between vertex \(i\) and \(j\) iff \(s_{ij} > 0\). Then to recommend an item to user 1, the above algorithm depends only on the rows neighboring to user 1, and chooses the most popular item among the top few neighbors. Hence we use the adjective “local popularity”. We study the probability of error for this algorithm, denoted as \(P_e[\text{local_algo}(T)] := P_r[X(1,j_0) = 0]\).

III. MAIN RESULT

From the results in [6], it follows that for \(k > c_1 n^{\alpha} \log n\), \(\alpha \in [0, 1/2]\), with high probability we can recover the entire matrix \(X\) using a “local” algorithm, and hence the BER also approaches zero. In the following theorem, we establish a stronger result for \(\text{local_algo}(T)\).

Theorem 1. Suppose \(\alpha \in [0, 1/2]\) and \(c > 0\). Assume that the erasure probability \(\epsilon = 1 - \frac{1}{n^c}\), the BSC error probability \(p \in [0, 1/2]\), and \(r\) goes to infinity with \(n\).

- (Large cluster size) If there exists a sequence \(\gamma_n \geq 0\) such that \(\gamma_n \to 0\) and \(k \geq n^{\alpha - \gamma_n}\), then \(P_e[\text{local_algo}(k)] \to 0\) as \(n \to \infty\).
- (Small cluster size) If there is a constant \(\gamma > 0\) such that \(k \leq n^{\alpha - \gamma}\), then

\[
\lim \inf_{n \to \infty} P_e[\text{local_algo}(k)] \geq \frac{p^{1/2}}{p^{1/2} + (1 - p)^{1/2}}.
\]

In Theorem 1 we restrict ourselves to \(\alpha \in [0, 1/2]\). For \(\alpha < 1/2\), as we show in Section IV, all the rows picked by Step 1 of the algorithm are from \(A_1\) (“good”) with high probability. But, for \(\alpha > 1/2\), most of the rows picked are from outside \(A_1\) (“bad”), and hence the algorithm breaks down. Due to lack of space, the results for \(\alpha > 1/2\) will be presented in subsequent publications. In the rest of this paper, we present a proof of Theorem 1.

IV. PROOF OF THEOREM 1

In this section we present the proof of Theorem 1. To begin with, we introduce some notation.

Notation: By \(X \sim B(n, p)\) we mean that a random variable \(X\) is binomially distributed with parameters \(n\) and \(p\). For a real valued function \(f(n)\), by \(\Omega(f(n)), \Theta(f(n))\) and \(o(f(n))\) we represent the standard asymptotic order notation (see for example [9, p. 433]). We say that \(f(n) = \Theta(g(n))\) if \(\lim_{n \to \infty} \frac{f(n)}{g(n)} = 1\). For a matrix \(X\), \(X(:, j)\) denotes the \(j\)th column of \(X\). For a vector \(y \in \{0, 1\}^n\), \(|y|_0\), \(|y|_1\) and \(|\bar{y}|\) represent number of 0’s, number of 1’s and the total number of 0’s and 1’s respectively. For a sequence of events \(\{E_n\}\), if \(P[E_n] \to 1\) with \(n\), then we say that \(E_n\) occurs w.h.p.

Analysis of Step 1 of the algorithm: We show that w.h.p. the top \(k\) rows are all from \(A_1\). We observe that for \(i \in A_1 \setminus \{1\}\), \(s_{i1} \sim B(n, p)\) with \(p := (1 - \epsilon)^2(1 - p)^2 + p^2\). For \(i \notin A_1\) we observe that \(s_{i1}\) is a mixture of binomials with \(\mathbb{E}[s_{i1}] = np_0\) for \(p_0 := (1 - \epsilon)^2 < p \). We omit the proofs of the following two lemmas, which are consequences of the Chernoff bound [10, Theorem 1.1] together with a union bound.

**Lemma 1 (Overlap with “good” rows).** For \(\delta \in (0, 1)\), we have

\[
Pr\left[\min_{i \in A_1} s_{i1} \leq np_0(1 - \delta)\right] \leq ke^{-np_0 \delta^2/3} =: p_1.
\]

**Lemma 2 (Overlap with “bad” rows).** For \(\delta \in (0, 1)\), we have

\[
Pr\left[\max_{i \in A_1} s_{i1} \geq np_0(1 + \delta)^2\right] \leq e^{-n \frac{p_0}{3} - 2re^{-r/6}} =: p_2.
\]

Since \(p_0 > p\), we can choose a small enough constant \(\delta_0\) such that \(np_0(1 - \delta_0) > np_0(1 + \delta_0)^2\). Let \(E_1\) denote the event that there is an error in Step 1 of the algorithm, i.e., we choose some rows from outside \(A_1\) in the top \(k\) users. Using Lemma 1 and Lemma 2 we obtain

\[
Pr[E_1] \leq Pr\left[\min_{i \in A_1} s_{i1} \leq \max_{i \in A_1} s_{i1}\right] \leq p_1 + p_2 = o(1). \quad (1)
\]

Here (a) follows since \(np_0 = \Theta(np_0) = \Theta(n^{1-2\alpha})\), and \(r\) increases to infinity with \(n\). This implies that w.h.p. Step 1 of \(\text{local_algo}\) does not contribute to the error.
Analysis of Step 2 of the algorithm: We assume that Step 1 picks all the \( k \) “good” neighbors. (i.e., we condition on the event \( E_1 \).)

**Large cluster size:** Suppose \( k \geq n^{\alpha-n} \) for \( \gamma_n = o(1) \). Let \( j_{\text{max}} \) denote the most popular column chosen by \( \text{local_algo}(k) \), and suppose \( X_k \) and \( Y_k \) denotes the matrices \( X \) and \( Y \) respectively, restricted to the top \( k \) rows. Since we have conditioned on \( E_1 \), we observe that for a column \( j \) such that \( X(1, j) = 1 \), we have \( |Y_k(:, j)|_1 \sim B(k, (1 - \epsilon)(1 - p)) \). Define \( \mu_Y := \mathbb{E}[|Y_k(:, j)|_1] \) and \( \sigma_Y^2 := \text{Var}(|Y_k(:, j)|_1) \) to obtain the following two lemmas.

**Lemma 3 (Many 1’s in the most popular column).** For different values of \( k \), we have the following lower bounds on \( |Y_k(:, j_{\text{max}})|_1 \):

1. If \( k = n^{\alpha-n} \) such that \( \gamma_n \geq 0 \) and \( \gamma_n \to 0 \), then w.h.p. \( |Y_k(:, j_{\text{max}}})|_1 \geq \min\{\sqrt{\log n}, 1\} \) = \( t_1(n) \).
2. If \( k = n^{\alpha_n} \) for \( \gamma_n \geq 1 \), then w.h.p. \( |Y_k(:, j_{\text{max}})|_1 \geq \max\{\mu_Y, \min\{\sigma_Y^2, \sqrt{\log n}\}\} =: t_2(n) \).

**Proof:** The proof is given in Section V.A

**Lemma 4 (1’s form majority in the most popular column).** Let \( \text{local_algo} \) makes vanishingly small probability of error. In the following, by Lemma 3 and Lemma 4 we can prove that the local algorithm makes vanishingly small probability of error. Now we use Lemma 3 and Lemma 4 to prove that the local algorithm makes vanishingly small probability of error. We define \( t(k, n) := t_1(n) \) if \( k = n^{\alpha-n} \) for \( \gamma_n \to 0 \), and \( t(k, n) := t_2(n) \) if \( k = n^{\alpha-n} \) for \( \gamma_n \geq 1 \) (here \( t_1(n) \) and \( t_2(n) \) are as defined in Lemma 3). Suppose \( M := \{ \bar{y} \in \{0, 1, *\}^k : (|\bar{y}| - |\bar{y}|_0) \to \infty \}, \) and \( |\bar{y}| \geq t(k, n) \).

We also observe that for a column \( j \),

\[ X_k(:, j) \rightarrow Y_k(:, j) \rightarrow \{j_{\text{max}} = j\}, \]

i.e., the random variables \( \{X_k(:, j), Y_k(:, j), \{j_{\text{max}} = j\}\} \) form a Markov chain. We are interested in finding the overall probability of error. In the following, by \( p_{k,j}(\bar{y}) \) we mean \( Pr[Y_k(:, j) = \bar{y}]_{j_{\text{max}} = j, E_1^c} \). Then we have

\[ \sum_{\bar{y} \in \{0, 1, *\}^k} Pr[\text{local_algo}(k)] = \sum_{\bar{y} \in \{0, 1, *\}^k} Pr[X(1, j) = 0|j_{\text{max}} = \bar{y}]_{E_1^c} + o(1) \]

\[ \sum_{\bar{y} \in \{0, 1, *\}^k} Pr[X(1, j) = 0, Y_k(:, j) = \bar{y}]_{j_{\text{max}} = j, E_1^c} + o(1) \]

\[ \sum_{\bar{y} \in \{0, 1, *\}^k} Pr[X(1, j) = 0] \cdot p_{k,j}(\bar{y}) + o(1) \]

**V. PROOFS OF LEMMAS**

To prove Lemma 3 and Lemma 4, we need the following theorem. Suppose \( Q(t) \) denotes the upper tail of a standard normal distribution, i.e., \( Q(t) := \frac{1}{\sqrt{2\pi}} \int_t^\infty e^{-t^2/2} dt \).

**Theorem 2 (Moderate deviations for binomial distribution).** Suppose \( X_n \sim B(n, p_n) \). If \( t_n \to \infty \) in such a way that \( t_n^2 = o\left(VaR(X_n)\right) = o(np_n(1 - p_n)) \), then

\[ Pr[X_n > np_n + t_n \sqrt{np_n(1 - p_n)}] = Q(t_n). \]

The above theorem is an adaptation of a theorem about moderate deviations of binomials when \( p_n \) is a constant [11]. The proof is very similar to the one presented in [11] for the constant probability case, and is omitted here due to lack of space.
A. Proof of Lemma 3

1) Recall that we have conditioned on the event that all the rows in the top $k$ neighbors chosen by $\text{local\_algo}(k)$ are “good”. Suppose $k = n^\alpha g_n$ for $g_n \geq 1$. By following a very similar analysis as in the first part, we see that $|X(1,j)| = 1$. Thus $|S| \sim B(n, 1/2)$ and due to Chernoff bound we have w.h.p. $|S| \geq n/3$. For a column $j \in S$ we see that $|Y_k(:,j)|_1 \sim B(k, (1-\epsilon)(1-p))$, and they are independent for different values of $j$. Thus for $j \in S$,

$$Pr\left[|Y_k(:,j)|_1 \geq t \right] \geq Pr\left[|Y_k(:,j)|_1 = t \right]$$

(a) \[ \geq \binom{k}{t} \left( (1-\epsilon)(1-p) \right)^t c^{-k-t} \]

(b) \[ \geq \left( \frac{k}{t} \right)^t \left( \frac{c(1-p)}{n^\alpha} \right)^t e^{-2\ln(2)c/n^\gamma_n}, \text{ for large } n \]

(c) \[ \geq \left( \frac{c(1-p)}{t} \right)^t e^{-2\ln(2)c}. \]

where (a) is true since $1 - (1-\epsilon)(1-p) \geq \epsilon$, (b) follows since $\epsilon = 1 - c/n^\alpha$, $1 - x \geq e^{-2\ln(2)x}$ for $x \in [0, 1/2]$, and (c) is true because $\gamma_n \geq 0$. Since w.h.p. $|S| \geq n/3$, we now have

$$Pr\left[|Y_k(:,j)|_1 \geq t \right] \leq Pr\left[\max_{j \in S}|Y_k(:,j)|_1 \geq t \right]|S| \geq n/3] + o(1)$$

(a) \[ \leq \left( 1 - \left( \frac{c(1-p)}{n^\alpha} \right)^t \right)^{n/3} e^{-2\ln(2)c/n^\gamma_n} + o(1) \]

(b) \[ \leq e^{-\frac{\binom{k}{t}}{c(1-p)^t} e^{-2\ln(2)c}} + o(1) \]

Hence for $t = \sqrt{\log n}$, (5) has the following counterpart,

$$Pr[|Y_k(:,j)|_1 < t] \leq e^{-\frac{\binom{k}{t}}{c(1-p)^t} e^{-2\ln(2)c}} + o(1)$$

or

$$Pr[|Y_k(:,j)|_1 < t] \leq e^{-\frac{\binom{k}{t}}{c(1-p)^t} e^{-2\ln(2)c}} + o(1)$$

But in Lemma 4 we need better bounds for $g_n \to \infty$, and we consider this case now. Recall that for $j \in S$, $\mu_j = E[|Y_k(:,j)|] = c(1-p)g_n$ and $\sigma^2_j = Var(|Y_k(:,j)|) = g_n(1-p)(1-\epsilon)(1-p)$. We define $t_n := \min \left\{ \sigma_j^{1/4}, \sqrt{\log n} \right\}$. Since $\sigma_j \to \infty$, we have $t_n = o(\sigma_j^2)$, and then Theorem 2 implies that for a column $j \in S$,

$$Pr[|Y_k(:,j)|_1 > \mu_j + t_n\sigma_j] \geq \min \{ a, 1 - \Omega \left( \frac{1}{\sqrt{n\log n}} \right) \} \geq \frac{1}{2}$$

where $a$ is true because $Q(t) = \frac{1}{\sqrt{2\pi t}} e^{-t^2/2}$ (Lemma 1.2), and (b) is true since $t_n \leq \sqrt{\log n}$. Since w.h.p. $|S| \geq n/3$, we have

$$Pr[|Y_k(:,j)|_1 \leq \mu_j + t_n\sigma_j] \leq Pr\left[\max_{j \in S}|Y_k(:,j)|_1 \leq \mu_j + t_n\sigma_j \right]|S| \geq n/3] + o(1)$$

(a) \[ \leq \left( 1 - \Omega \left( \frac{1}{\sqrt{n\log n}} \right) \right)^{n/3} + o(1) = o(1). \]

Thus w.h.p. $|Y_k(:,j)|_1 \geq \mu_j + t_n\sigma_j$, if $g_n \to \infty$. We have already observed that w.h.p. $|Y_k(:,j)|_1 \geq \sqrt{\log n}$. Thus the lemma is implied.

B. Proof of Lemma 4

Lemma 3 gives us a lower bound for $|Y_k(:,j)|_1$ that holds w.h.p.. Next we find an upper bound for $|Y_k(:,j)|_0$ to prove Lemma 4.

First we condition on the event that $X(1,j) = 1$. We observe that

$$|Y_k(:,j)|_0 \to |Y_k(:,j)|_1 \to \{j_{max} = j\}.$$
We have considered estimation of a binary random field obtained by permuting rows and columns of a block constant matrix, by observing a sub-sampled and noisy version. It would be interesting to analyze the performance of “local” algorithms on a more general class of matrices obtained from realizations of a “smooth” stochastic process. Further, non-uniform sampling models are also of interest.

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