ON SOME GENERAL MULTIPLYING SOLUTIONS RESULTS OF
A ROBIN PROBLEM

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Abstract. By applying Ricceri’s variational principle, we demonstrate the
existence of solutions for the following Robin problem

\[
\begin{align*}
- \text{div} \left( \omega_1(x) |\nabla u|^{p(x)-2} \nabla u \right) &= \lambda \omega_2(x) f(x, u), \quad x \in \Omega \\
\omega_1(x) |\nabla u|^{p(x)-2} \frac{\partial u}{\partial \nu} + \beta(x) |u|^{p(x)-2} u &= 0, \quad x \in \partial \Omega,
\end{align*}
\]

in \( W^{1,p(x)}_{\omega_1,\omega_2}(\Omega) \) under some appropriate conditions.

1. Introduction

Let \( \Omega \subset \mathbb{R}^N \) (\( N \geq 2 \)) be a bounded smooth domain. Assume that \( \omega_1 \) and \( \omega_2 \) are weight functions. The aim of this study is to discuss the three solutions for the following Robin problem

\[
\begin{align*}
- \text{div} \left( \omega_1(x) |\nabla u|^{p(x)-2} \nabla u \right) &= \lambda \omega_2(x) f(x, u), \quad x \in \Omega \\
\omega_1(x) |\nabla u|^{p(x)-2} \frac{\partial u}{\partial \nu} + \beta(x) |u|^{p(x)-2} u &= 0, \quad x \in \partial \Omega,
\end{align*}
\]

where \( \frac{\partial u}{\partial \nu} \) is the outer unit normal derivative of \( u \) with respect to \( \partial \Omega \), \( \lambda > 0 \), \( p , q \in C(\Omega) \) with \( \inf_{x \in \Omega} p(x) > 1 \), and \( \beta \in L^\infty(\partial \Omega) \) such that \( \beta^- = \inf_{x \in \partial \Omega} \beta(x) > 0 \).

In recent years, the investigation of the existence of weak solutions of partial differential equations involving weighted \( p(x) \)-Laplacian in variable exponent (weighted or unweighted) Sobolev spaces has been very popular (see \([4, 6, 9, 11, 12, 17, 23]\)). Because some such type of equations can explain several physical problems such as electrorheological fluids, image processing, elastic mechanics, fluid dynamics and calculus of variations, see \([14, 18, 20, 24]\).

The Robin problem involving \( p(x) \)-Laplacian was studied by several authors, see \([1, 7, 10, 15, 21]\). In 2013, Tsouli et al. \([22]\) obtained some results about weak solutions of the following Robin problem

\[
\begin{align*}
- \text{div} \left( |\nabla u|^{p(x)-2} \nabla u \right) &= \lambda f(x, u), \quad x \in \Omega \\
|\nabla u|^{p(x)-2} \frac{\partial u}{\partial \nu} + \beta(x) |u|^{p(x)-2} u &= 0, \quad x \in \partial \Omega
\end{align*}
\]

using the variational methods under some suitable conditions for the function \( f \).

In addition, they showed that the problem \((1.2)\) has at least three solutions.

In the light of the articles mentioned above, we discuss the existence of multiplicity solutions of the problem \((1.1)\) in the variable exponent Sobolev spaces \( W^{1,p(x)}_{\omega_1,\omega_2}(\Omega) \) with respect to two different weight functions \( \omega_1 \) and \( \omega_2 \). Moreover, we introduce a more general norm in compared to the norm given by Deng \([10]\).
Finally, we find more general results than [22] using the technical approach, which is mainly based on Ricceri’s theorem.

2. Notation and preliminaries

Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^N \) with a smooth boundary \( \partial \Omega \). Then, the set is defined by

\[
C_+ (\overline{\Omega}) = \left\{ p \in C (\overline{\Omega}) : \inf_{x \in \Omega} p(x) > 1 \right\},
\]

where \( C (\overline{\Omega}) \) consists of all continuous functions on \( \overline{\Omega} \). For any \( p \in C_+ (\overline{\Omega}) \), we indicate

\[
p_- = \inf_{x \in \Omega} p(x) \quad \text{and} \quad p_+ = \sup_{x \in \Omega} p(x).
\]

Let \( p \in C_+ (\overline{\Omega}) \) and \( 1 < p^- \leq p(\cdot) \leq p^+ < \infty \). The space \( L^{p(\cdot)}(\Omega) \) is defined by

\[
L^{p(\cdot)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ is measurable and } \int_{\Omega} |u(x)|^{p(x)} \, dx < \infty \right\}
\]

with the (Luxemburg) norm

\[
\| u \|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \varrho_{p(\cdot)} \left( \frac{u}{\lambda} \right) \leq 1 \right\},
\]

where

\[
\varrho_{p(\cdot)}(u) = \int_{\Omega} |u(x)|^{p(x)} \, dx.
\]

, see [16]. If \( \| u \|_{p(\cdot), \omega} = \| u \omega^{\frac{1}{p(x)}} \|_{p(\cdot)} < \infty \), then \( u \in L^{p(\cdot)}_{\omega}(\Omega) \) where \( \omega \) is a weight function from \( \Omega \) to \((0, \infty)\). It is known that the space \( \left( L^{p(\cdot)}_{\omega}(\Omega), \| \cdot \|_{p(\cdot), \omega} \right) \) is a Banach space. The dual space of \( L^{p(\cdot)}_{\omega}(\Omega) \) is \( L^{r(\cdot)}_{\omega^*}(\Omega) \) where \( \frac{1}{p(\cdot)} + \frac{1}{r(\cdot)} = 1 \) and \( \omega^* = \omega^{1-r(\cdot)} = \omega^{-\frac{1}{r(\cdot)}} \). If \( \omega \in L^\infty (\Omega) \), then \( L^{p(\cdot)}_{\omega} = L^{p(\cdot)} \), see [3, 5].

**Proposition 1.** (see [3, 12]) For all \( u, v \in L^{p(\cdot)}_{\omega}(\Omega) \), we have

(i) \( \| u \|_{p(\cdot), \omega} < 1 \) (resp. \( = 1, > 1 \)) if and only if \( \varrho_{p(\cdot), \omega}(u) < 1 \) (resp. \( = 1, > 1 \)),

(ii) \( \| u \|_{p(\cdot), \omega}^p \leq \varrho_{p(\cdot), \omega}(u) \leq \| u \|_{p(\cdot), \omega}^{p^+} \) with \( \| u \|_{p(\cdot), \omega} > 1 \),

(iii) \( \| u \|_{p(\cdot), \omega} \leq \varrho_{p(\cdot), \omega}(u) \leq \| u \|_{p(\cdot), \omega}^{p^+} \) with \( \| u \|_{p(\cdot), \omega} < 1 \),

(iv) \( \min \left\{ \| u \|_{p(\cdot), \omega}^{p^-}, \| u \|_{p(\cdot), \omega}^{p^+} \right\} \leq \varrho_{p(\cdot), \omega}(u) \leq \max \left\{ \| u \|_{p(\cdot), \omega}^{p^-}, \| u \|_{p(\cdot), \omega}^{p^+} \right\} \),

(v) \( \min \left\{ \varrho_{p(\cdot), \omega}(u)^{\frac{1}{p^-}}, \varrho_{p(\cdot), \omega}(u)^{\frac{1}{p^+}} \right\} \leq \| u \|_{p(\cdot), \omega} \leq \max \left\{ \varrho_{p(\cdot), \omega}(u)^{\frac{1}{p^-}}, \varrho_{p(\cdot), \omega}(u)^{\frac{1}{p^+}} \right\} \),

(vi) \( \varrho_{p(\cdot), \omega}(u - v) \to 0 \) if and only if \( \| u - v \|_{p(\cdot), \omega} \to 0 \)

where \( \varrho_{p(\cdot), \omega}(u) \) is defined by the integral \( \int_{\Omega} |u(x)|^{p(x)} \omega(x) \, dx \).

**Definition 1.** Let \( \omega^{-\frac{1}{r(\cdot)}} \in L^1_{\text{loc}} (\Omega) \). The space \( W^{k,p(\cdot)}_{\omega}(\Omega) \) is defined by

\[
W^{k,p(\cdot)}_{\omega}(\Omega) = \left\{ u \in L^{p(\cdot)}_{\omega}(\Omega) : D^\alpha u \in L^{p(\cdot)}_{\omega}(\Omega), 0 \leq |\alpha| \leq k \right\}
\]
equipped with the norm
\[ \|u\|^{k,p(\cdot)} = \sum_{0 \leq |\alpha| \leq k} \|D^\alpha u\|_{p(\cdot),\omega} \]

where \( \alpha \in \mathbb{N}_0^N \) is a multi-index, \( |\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_N \) and \( D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \ldots \frac{\partial^{\alpha_N}}{\partial x_N^{\alpha_N}} \). In particular, the space \( W^{1,p(\cdot)}_{\omega}(\Omega) \) is defined by
\[ W^{1,p(\cdot)}_{\omega}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega) \right\} \]
equipped with the norm
\[ \|u\|^{1,p(\cdot)} = \|u\|_{p(\cdot),\omega} + \|\nabla u\|_{p(\cdot),\omega}. \]
The space \( W^{-1,r(\cdot)}_{\omega} (\Omega) \) is the topological dual for \( W^{1,p(\cdot)}_{\omega}(\Omega) \) where \( \frac{1}{p(\cdot)} + \frac{1}{r(\cdot)} = 1 \) and \( \omega^* = \omega^{-1,r(\cdot)} = \omega^{-p(\cdot)/r(\cdot)}. \) Moreover, the space \( W^{1,p(\cdot)}_{\omega}(\Omega) \) is a separable and reflexive Banach space, see [5].

Let \( \omega_1^{-1/p(\cdot)}, \omega_2^{-1/p(\cdot)-1} \in L_{loc}^1(\Omega) \). The space \( W^{1,p(\cdot)}_{\omega_1,\omega_2}(\Omega) \) is defined by
\[ W^{1,p(\cdot)}_{\omega_1,\omega_2}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega) \right\} \]
equipped with the norm
\[ \|u\|^{1,p(\cdot)}_{\omega_1,\omega_2} = \|\nabla u\|_{p(\cdot),\omega_1} + \|u\|_{p(\cdot),\omega_2}. \]

Let \( d\sigma \) be the measure on \( \partial\Omega \). We can define the space \( L^{p(\cdot)}_{\omega}(\partial\Omega) \) similarly by
\[ L^{p(\cdot)}_{\omega}(\partial\Omega) = \left\{ u \mid u : \partial\Omega \to \mathbb{R} \text{ measurable and } \int_{\partial\Omega} |u(x)|^{p(x)} \omega(x) d\sigma < +\infty \right\} \]
with the Luxemburg norm \( \|\cdot\|^{p(\cdot)}_{\omega,\partial\Omega} \). Then \( \left( L^{p(\cdot)}_{\omega}(\partial\Omega), \|\cdot\|^{p(\cdot)}_{\omega,\partial\Omega} \right) \) is a Banach space. If \( \omega \in L^\infty(\Omega) \), then \( L^{p(\cdot)}_{\omega} = L^{p(\cdot)} \).

**Theorem 1.** (see [5]) Let \( \omega_1^{-\alpha(\cdot)} \in L^1(\Omega) \) with \( \alpha(x) \in \left( \frac{N}{p(x)}, \infty \right) \cap \left[ \frac{1}{p(x)-1}, \infty \right] \).

If we define the variable exponent \( p_\ast(x) = \frac{\alpha(x) p(x)}{\alpha(x) + 1} \) with \( N < p_\ast \), then we have the embeddings \( W^{1,p(\cdot)}_{\omega_1,\omega_2}(\Omega) \hookrightarrow W^{1,p_\ast(\cdot)}(\Omega) \) and \( W^{1,p(\cdot)}_{\omega_1,\omega_2}(\Omega) \hookrightarrow C(\bar{\Omega}). \)

**Corollary 1.** Since \( W^{1,p(\cdot)}_{\omega_1,\omega_2}(\Omega) \hookrightarrow C(\bar{\Omega}) \), then there exists a constant \( c_1 > 0 \) such that
\[ \|u\|_{\infty} \leq c_1 \|u\|^{1,p(\cdot)}_{\omega_1,\omega_2} \]
for any \( u \in W^{1,p(\cdot)}_{\omega_1,\omega_2}(\Omega) \) where \( \|u\|_{\infty} = \sup_{x \in \Omega} u(x) \) for \( u \in C(\bar{\Omega}) \).

For a set \( A \subset \Omega \), denote by \( p^-(A) = \inf_{x \in A} p(x) \) and \( p^+(A) = \sup_{x \in A} p(x) \). We define
\[ p^\beta(x) = (p(x))^\beta = \begin{cases} \frac{(N-1)p(x)}{N-p(x)}, & \text{if } p(x) < N, \\ +\infty, & \text{if } p(x) \geq N, \end{cases} \]
and
\[ p^\beta_r(x) = \frac{r(x)-1}{r(x)} p^\beta(x) \]
for any \( x \in \partial\Omega \), where \( r \in C(\partial\Omega) \) with \( r^- = \inf_{x \in \partial\Omega} r(x) > 1 \).
Theorem 3. Assume that the boundary of \( \Omega \) possesses the cone property and \( p \in C(\overline{\Omega}) \) with \( p^- > 1 \). Suppose that \( \omega_1 \in L^{r(\cdot)}(\partial \Omega) \), \( r \in C(\partial \Omega) \) with \( r(x) > \frac{p^0(x)}{p^0(x) - 1} \) for all \( x \in \partial \Omega \). If \( q \in C(\partial \Omega) \) and \( 1 \leq q(x) < p^0(r(x)) \) for all \( x \in \partial \Omega \), then there is a compact embedding \( W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\partial \Omega) \). In particular, there is a compact embedding \( W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\partial \Omega) \), where \( 1 \leq q(x) < p^0(x) \) for all \( x \in \partial \Omega \).

Corollary 2. Assume that the boundary of \( \Omega \) possesses the cone property and \( p \in C(\overline{\Omega}) \) with \( p^- > 1 \). If \( q \in C(\partial \Omega) \) and \( 1 \leq q(x) < p^0(x) \) for all \( x \in \partial \Omega \), then there is a compact embedding \( W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\partial \Omega) \).

Corollary 3. If \( p(x) < p^0\star(x) < p^0(\cdot), \forall x \in \partial \Omega \), then we have the compact embeddings \( W^{1,p(\cdot)}(\Omega) \hookrightarrow W^{1,p^\ast(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\partial \Omega) \).

Theorem 4. Assume that the boundary of \( \Omega \) possesses the cone property and \( p \in C(\overline{\Omega}) \) with \( p^- > 1 \). If \( q \in C(\partial \Omega) \) and \( 1 \leq q(x) < p^\ast(x) \) for all \( x \in \partial \Omega \), then there is a compact embedding \( W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\partial \Omega) \), where

\[
p^\ast(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & \text{if } p(x) < N \\ +\infty, & \text{if } p(x) \geq N. \end{cases}
\]

Corollary 4. Let \( N < p^\ast \) and \( 1 \leq q(x) < (p^\ast)^\ast(x) \) for all \( x \in \partial \Omega \), then we have the compact embedding \( W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\partial \Omega) \).

Proof. By Theorem 3 and Theorem 4, we have the continuous embedding \( W^{1,p(\cdot)}(\Omega) \hookrightarrow W^{1,p^\ast(\cdot)}(\Omega) \) and the compact embedding \( W^{1,p^\ast(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\partial \Omega) \). Thus it is easy to see that the compact embedding \( W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\partial \Omega) \) is valid. \( \Box \)

If we apply the technique in [10, Theorem 2.1], then we prove the following theorem similarly. Moreover, due to this theorem we can find out the existence of weak solutions of the problem (11).

Theorem 4. Let \( \beta \in L^\infty(\partial \Omega) \) such that \( \beta^- = \inf_{x \in \partial \Omega} \beta(x) > 0 \). Then, the norm \( \| u \|_{\beta(x)} \) is defined by

\[
\| u \|_{\beta(x)} = \inf \left\{ \tau > 0 : \int_\Omega \omega_1(x) \left| \nabla u(x) \right|^p dx + \int_{\partial \Omega} \beta(x) \left| \frac{u(x)}{\tau} \right|^p \left| u(x) \right|^p dx d\sigma \leq 1 \right\}
\]

for any \( u \in W^{1,p(\cdot)}_\omega(\Omega) \). Moreover, \( \| \cdot \|_{\beta(x)} \) and \( \| \cdot \|_{1,p(\cdot)}^{\omega_1,\omega_2} \) are equivalent on \( W^{1,p(\cdot)}_\omega(\Omega) \).

Proposition 2. Let \( I_{\beta(x)}(u) = \int_\Omega \omega_1(x) \left| \nabla u(x) \right|^p dx + \int_{\partial \Omega} \beta(x) \left| \frac{u(x)}{\tau} \right|^p \left| u(x) \right|^p dx d\sigma \)

with \( \beta^- > 0 \). For any \( u, u_k \in W^{1,p(\cdot)}_{\omega_1,\omega_2}(\Omega) \) (\( k = 1, 2, \ldots, \)), we have

(i) \( \| u \|_{\beta(x)}^p \leq I_{\beta(x)}(u) \leq \| u \|_{\beta(x)}^{p^\ast} \) with \( \| u \|_{\beta(x)} \geq 1 \),

(ii) \( \| u \|_{\beta(x)}^p \leq I_{\beta(x)}(u) \leq \| u \|_{\beta(x)}^{p^\ast} \) with \( \| u \|_{\beta(x)} \leq 1 \),

(iii) \( \min \{ \| u \|_{\beta(x)}^p, \| u \|_{\beta(x)}^{p^\ast} \} \leq I_{\beta(x)}(u) \leq \max \{ \| u \|_{\beta(x)}^p, \| u \|_{\beta(x)}^{p^\ast} \} \).
(iv) \( \| u - u_k \|_{\beta(x)} \to 0 \) if and only if \( I_{\beta(x)}(u - u_k) \to 0 \) (as \( k \to \infty \)), 
(v) \( \| u_k \|_{\beta(x)} \to \infty \) if and only if \( I_{\beta(x)}(u_k) \to \infty \) (as \( k \to \infty \)).

The following Proposition can be proved by Proposition 2.2 in [13].

**Proposition 3.** Let us define the functional \( L_{\beta(x)} : W^{1,p(\cdot)}_{\omega_1,\omega_2}(\Omega) \to \mathbb{R} \) by
\[
L_{\beta(x)}(u) = \int_{\Omega} \frac{\omega_1(x)}{p(x)} |\nabla u|^{p(x)} \, dx + \int_{\partial \Omega} \frac{\beta(x)}{p(x)} |u(x)|^{p(x)} \, d\sigma
\]
for all \( u \in W^{1,p(\cdot)}_{\omega_1,\omega_2}(\Omega) \). Then we obtain \( L_{\beta(x)} \in C^1 \left( W^{1,p(\cdot)}_{\omega_1,\omega_2}(\Omega), \mathbb{R} \right) \) and
\[
L'_{\beta(x)}(u)(v) = \int_{\Omega} \frac{\omega_1(x)}{p(x)} |\nabla u|^{p(x)-2} \nabla u \nabla v \, dx + \int_{\partial \Omega} \frac{\beta(x)}{p(x)} |u(x)|^{p(x)-2} uv \, d\sigma
\]
for any \( u, v \in W^{1,p(\cdot)}_{\omega_1,\omega_2}(\Omega) \). In addition, we have the following properties

(i) \( L'_{\beta(x)} : W^{1,p(\cdot)}_{\omega_1,\omega_2}(\Omega) \to W^{-1,p'(\cdot)}_{\omega_1,\omega_2}(\Omega) \) is continuous, bounded and strictly monotone operator,

(ii) \( L'_{\beta(x)} : W^{1,p(\cdot)}_{\omega_1,\omega_2}(\Omega) \to W^{-1,p'(\cdot)}_{\omega_1,\omega_2}(\Omega) \) is a mapping of type \((S_+)\), i.e., if \( u_n \to u \) in \( W^{1,p(\cdot)}_{\omega_1,\omega_2}(\Omega) \) and \( \limsup_{n \to \infty} L'_{\beta(x)}(u_n)(u_n - u) \leq 0 \), then \( u_n \to u \) in \( W^{1,p(\cdot)}_{\omega_1,\omega_2}(\Omega) \)

(iii) \( L'_{\beta(x)} : W^{1,p(\cdot)}_{\omega_1,\omega_2}(\Omega) \to W^{-1,p'(\cdot)}_{\omega_1,\omega_2}(\Omega) \) is a homeomorphism.

**Theorem 5.** (see [19]) Let \( X \) be a separable and reflexive real Banach space; \( \Phi : X \to \mathbb{R} \) a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on \( X^* \); \( \Psi : X \to \mathbb{R} \) a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Moreover, assume that

(i) \( \lim_{\|u\| \to \infty} (\Phi(u) + \lambda \Psi(u)) = \infty \) for all \( \lambda > 0 \),

(ii) there are \( r \in \mathbb{R} \) and \( u_0, u_1 \in X \) such that \( \Phi(u_0) < r < \Phi(u_1) \),

(iii) \( \inf_{u \in \Phi^{-1}(-\infty,-r]} \Psi(u) > \frac{\Phi(u_1) - \Phi(u_0)}{\Phi(u_1) - \Phi(u_0)} \).

Then there exist an open interval \( \Lambda \subset (0,\infty) \) and a positive constant \( \rho > 0 \) such that for any \( \lambda \in \Lambda \) the equation \( \Phi'(u) + \lambda \Psi'(u) = 0 \) has at least three solutions in \( X \) whose norms are less than \( \rho \).

3. The Main Result

Throughout the paper we assume that the following conditions:

(I) \( |f(x,t)| \leq h(x) + c_2 |t|^{s(x)-1} \) for any \( (x,t) \in \Omega \times \mathbb{R} \), \( c_2 > 0 \), where the function \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) is a Carathéodory function, \( h(x) \in L^{\frac{p(x)}{p(x)-1}}(\Omega) \), \( h(x) \geq 0 \) and \( s(x) \in C_+(\Omega) \), \( 1 < s^- = \inf_{x \in \Omega} s(x) \leq s^+ = \sup_{x \in \Omega} s(x) < p^- \) with \( p^- \) for all \( x \in \Omega \).

(II) \( (i) \ f(x,t) < 0 \) for all \( (x,t) \in \Omega \times \mathbb{R} \), and \( |t| \in (0,1) \),

(ii) \( f(x,t) \geq k > 0 \), when \( |t| \in (t_0,\infty) \), \( t_0 > 1 \).
Let \( u \in W^{1,p(\cdot)}_{\omega_1, \omega^2_2}(\Omega) \). Then the functional \( \Phi_\lambda(u) \) is defined by
\[
\Phi_\lambda(u) = L_{\beta(\cdot)}(u) + \lambda \Psi(u),
\]
where \( \Psi(u) = -\int_\Omega F(x,u)\,dx \) and \( F(x,t) = \int_0^t f(x,y)\,dy \). Moreover, \( \Phi_\lambda(u) \) is called energy functional of the problem \((1.1)\).

It is obvious that \( \left(L'_{\beta(x)}\right)^{-1} : W^{-1,q(\cdot)}_{\omega^1_1, \omega^2_2}(\Omega) \rightarrow W^{1,p(\cdot)}_{\omega^1_1, \omega^2_2}(\Omega) \) exists and continuous, because \( L'_{\beta(x)} : W^{1,p(\cdot)}_{\omega^1_1, \omega^2_2}(\Omega) \rightarrow W^{-1,q(\cdot)}_{\omega^1_1, \omega^2_2}(\Omega) \) is a homeomorphism by Proposition \( \Box \)

Moreover, due to the assumption \((I)\) it is well known that \( \Psi \in C^1 \left(W^{1,p(\cdot)}_{\omega^1_1, \omega^2_2}(\Omega), \mathbb{R}\right) \)
with the derivatives given by \( \langle \Psi'(u), v \rangle = -\int_\Omega f(x,u)v\,dx \) for any \( u, v \in W^{1,p(\cdot)}_{\omega^1_1, \omega^2_2}(\Omega) \), and \( \Psi' : W^{1,p(\cdot)}_{\omega^1_1, \omega^2_2}(\Omega) \rightarrow W^{-1,q(\cdot)}_{\omega^1_1, \omega^2_2}(\Omega) \) is completely continuous by \([2\text{ Theorem 2.9}].

Therefore, \( \Psi' : W^{1,p(\cdot)}_{\omega^1_1, \omega^2_2}(\Omega) \rightarrow W^{-1,q(\cdot)}_{\omega^1_1, \omega^2_2}(\Omega) \) is compact.

**Definition 2.** We call that \( u \in W^{1,p(\cdot)}_{\omega^1_1, \omega^2_2}(\Omega) \) is a weak solution of the problem \((.1)\) if
\[
\int_\Omega \omega_1(x) |\nabla u|^{p(x)-2} \nabla u \nabla v\,dx + \int_\Omega \beta(x) |u(x)|^{p(x)-2} u v\omega_2\,dx = 0
\]
for all \( v \in W^{1,p(\cdot)}_{\omega^1_1, \omega^2_2}(\Omega) \). We point out that if \( \lambda \in \mathbb{R} \) is an eigenvalue of the problem \((1.1)\), then the corresponding \( u \in W^{1,p(\cdot)}_{\omega^1_1, \omega^2_2}(\Omega) - \{0\} \) is a weak solution of \((1.1)\).

**Theorem 6.** There exist an open interval \( \Lambda \subset (0, \infty) \) and a positive constant \( \rho > 0 \) such that for any \( \lambda \in \Lambda \), the problem \((1.1)\) has at least three solutions in \( W^{1,p(\cdot)}_{\omega^1_1, \omega^2_2}(\Omega) \) whose norms are less than \( \rho \).

**Proof.** We only need to prove the conditions \((i), (ii)\) and \((iii)\) in Theorem \([5\text{ Proposition }\Box] \)

Using Proposition \([5\text{ Proposition }\Box] \) we get
\[
L_{\beta(x)}(u) = \int_\Omega \frac{\omega_1(x)}{p(x)} |\nabla u|^{p(x)}\,dx + \int_{\partial \Omega} \frac{\beta(x)}{p(x)} |u(x)|^{p(x)}\,d\sigma
\]
\[
= \frac{1}{p} \|u\|_{p_{\beta(x)}}^p
\]
(3.1)

for any \( u \in W^{1,p(\cdot)}_{\omega^1_1, \omega^2_2}(\Omega) \) with \( \|u\|_{p_{\beta(x)}} > 1 \).

In addition, due to \((I)\) and Hölder inequality, we have
\[
-\Psi(u) = \int_\Omega F(x,u)\,dx = \int_\Omega \left( \int_0^t f(x,t)\,dt \right)\,dx
\]
\[
\leq \int_\Omega \left( h(x)|u(x)| + \frac{c_2}{s(x)} |u(x)|^{s(x)} \right)\,dx
\]
\[
\leq 2 \|h\|_{s(\cdot),\Omega} \|u\|_{s(\cdot),\Omega} + \frac{c_2}{s} \int_\Omega |u(x)|^{s(x)}\,dx.
\]
(3.2)
By Corollary 4, there exist the continuous embedding $W^{1,p(.)}_{\omega_1,\omega_2}(\Omega) \hookrightarrow L^{s(.)}(\Omega)$ and the inequality

\[
(3.3) \quad \int_{\Omega} |u(x)|^{s(x)} \, dx \leq \max \left\{ \|u\|_{s_1, \Omega}^{-}, \|u\|_{s_1, \Omega}^{+} \right\} \leq c_3 \|u\|_{\beta(x)}^{+}.
\]

If we use (3.2) and (3.3), then we get

\[
(3.4) \quad -\Psi(u) \leq 2C_7 \|h\|_{\beta(x)} \|u\|_{\beta(x)} + \frac{c_4}{s} \|u\|_{\beta(x)}^{+}.
\]

For any $\lambda > 0$ we can write

\[
L_{\beta(x)}(u) + \lambda \Psi(u) \geq \frac{1}{p^*} \|u\|_{\beta(x)}^{p^*} - 2\lambda C_7 \|h\|_{\beta(x)} \|u\|_{\beta(x)} - \frac{c_4}{s} \lambda \|u\|_{\beta(x)}^{+}
\]

by (3.1) and (3.4). Since $1 < s^+ < p^*$, \( \lim_{\|u\|_{\beta(x)} \to \infty} (L_{\beta(x)}(u) + \lambda \Psi(u)) = \infty \) for all $\lambda > 0$ and the proof of $(i)$ is completed.

Due to $\frac{\partial F(x,t)}{\partial t} = f(x,t)$ and $(II)$, it is easy to see that $F(x,t)$ is increasing and decreasing for $t \in (t_0, \infty)$ and $(0,1)$ with respect to $x \in \Omega$, respectively. Since $F(x,t) \geq kt$ uniformly for $x$, we have $F(x,t) \to \infty$ as $t \to \infty$. Then for a real number $\delta > t_0$, we can obtain

\[
(3.5) \quad F(x,t) \geq 0 = F(x,0) \geq F(x,\tau), \quad \text{for all} \quad x \in \Omega, \quad t > \delta, \quad \tau \in (0,1).
\]

Let $\beta, \gamma$ be two real numbers such that $0 < \beta < \min \{1, c_1\}$, where $c_1$ is given in Corollary 4 and $\gamma > \delta (\gamma > 1)$ satisfies $\gamma \|\beta\|_{1,\partial \Omega} > 1$. If we use relation (3.5), then we have $F(x,t) \leq F(x,0) = 0$ for $t \in [0, \beta]$, and

\[
(3.6) \quad \int_{\Omega} \sup_{0 \leq t \leq \beta} F(x,t) \, dx \leq \int_{\Omega} F(x,0) \, dx = 0.
\]

Using $\gamma > \delta$ and (3.5), we have $\int F(x,\delta) \, dx > 0$ and

\[
(3.7) \quad \frac{1}{p^*} \frac{\beta^+}{\gamma^{p^*}} \int_{\Omega} F(x,\delta) \, dx > 0.
\]

If we use the inequalities in (3.6) and (3.7), then we get

\[
\int_{\Omega} \sup_{0 \leq t \leq \beta} F(x,t) \, dx \leq 0 < \frac{1}{c_1^{p^*} \gamma^{p^*}} \int_{\Omega} F(x,\delta) \, dx.
\]

Define $u_0, u_1 \in W^{1,p(.)}_{\omega_1,\omega_2}(\Omega)$ with $u_0(x) = 0$ and $u_1(x) = \gamma$ for any $x \in \Omega$. If we take $r = \frac{1}{p^*} \left(\frac{\beta}{c_1}\right)^{p^*}$, then $r \in (0,1)$, $L_{\beta(x)}(u_0) = \Psi(u_0) = 0$ and

\[
L_{\beta(x)}(u_1) = \int_{\partial \Omega} \frac{\beta(x)}{p(x)} \gamma^{p(x)} \, d\sigma \geq \frac{\gamma^{p^*}}{p^*} \int_{\partial \Omega} \beta(x) \, d\sigma = \frac{1}{p^*} \gamma^{p^*} \|\beta\|_{1,\partial \Omega}
\]

\[
\geq \frac{1}{p^*} > r.
\]
Thus we have $L_{β(x)}(u_0) < r < L_{β(x)}(u_1)$ and

$$\Psi(u_1) = -\int_\Omega F(x, u_1) dx = -\int_\Omega F(x, \gamma) dx < 0.$$ 

Then the proof of (ii) is obtained.

On the other hand, we have

$$\frac{(L_{β(x)}(u_1) - r) \Psi(u_0) + (r - L_{β(x)}(u_0)) \Psi(u_1)}{L_{β(x)}(u_1) - L_{β(x)}(u_0)} = -r \frac{\Psi(u_1)}{L_{β(x)}(u_1)} = \frac{r}{\int_\Omega F(x, \gamma) dx} \geq 0.$$

Now, let $u \in W^{1, p, \gamma}_w(\Omega)$ with $L_{β(x)}(u) \leq r < 1$. Since $\frac{1}{p} L_{β(x)}(u) \leq L_{β(x)}(u) \leq r$, we obtain

$$I_{β(x)}(u) \leq p^+ r = \left(\frac{β}{c_1}\right)^{p^+} < 1.$$ 

Due to Proposition [2], we see that $\|u\|_{β(x)} < 1$ and

$$\frac{1}{p^+} \|u\|_{β(x)}^{p^+} \leq \frac{1}{p^+} I_{β(x)}(u) \leq L_{β(x)}(u) \leq r.$$

Then using Corollary [1], we can get

$$|u(x)| \leq c_1 \|u\|_{β(x)} \leq c_1 (p^+ r)^{\frac{1}{p^+}} = β$$

for all $u \in W^{1, p, \gamma}_w(\Omega)$ and $x \in Ω$ with $Φ(u) \leq r$.

The last inequality implies that

$$-\inf_{u \in Φ^{-1}((−∞, r])} Ψ(u) = \sup_{u \in Φ^{-1}((−∞, r])} Ψ(u) \leq \int_\Omega \sup_{0 ≤ t ≤ β} F(x, t) dx \leq 0.$$ 

Then we have

$$-\inf_{u \in Φ^{-1}((−∞, r])} Ψ(u) < r \frac{\int_\Omega F(x, \gamma) dx}{\int_\partial Ω \frac{β(x)}{p(x)} \gamma^{p(x)} dσ}$$

and

$$\inf_{u \in Φ^{-1}((−∞, r])} Ψ(u) > \frac{(Φ(u_1) - r) \Psi(u_0) + (r - Φ(u_0)) \Psi(u_1)}{Φ(u_1) - Φ(u_0)}.$$

This completes the proof.  

References

[1] M. Allaoui, Robin problems involving the $p(x)$-Laplacian. Appl. Math. and Comp. 332 (2018), 457-468.

[2] M. Allaoui, A. R. El Amrouss, A. Ourraoui, Existence and multiplicity of solutions for a Steklov problem involving the $p(x)$-Laplace operator. Electron. J. Differential Equations 2012(132) (2012), 1-12.

[3] I. Aydin, Weighted variable Sobolev spaces and capacity. J. Funct. Spaces Appl. 2012, (2012).

[4] I. Aydin, C. Unal, Weighted stochastic field exponent Sobolev spaces and nonlinear degenerate elliptic problem with nonstandard growth. Hacettepe J. of Math. and Stat. 49(4) (2020), 1386-1396.
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[5] I. Aydin, C. Unal, The Kolmogorov–Riesz theorem and some compactness criterions of bounded subsets in weighted variable exponent amalgam and Sobolev spaces. Collect. Math. 71 (2020), 349-367.

[6] I. Aydin, C. Unal, Three solutions to a Steklov problem involving the weighted p(x)-Laplacian. Rocky Mountain J. Math. (Accepted for publication) (2020), https://arxiv.org/pdf/2005.10344.pdf.

[7] N. T. Chung, Some remarks on a class of p(x)–Laplacian Robin eigenvalue problems. Mediterr. J. Math. 15(147) (2018), 1-14.

[8] S. G. Deng, Eigenvalues of the p(x)-Laplacian Steklov problem. J. Math. Anal. Appl. 339 (2008), 925-937.

[9] S. G. Deng, A local mountain pass theorem and applications to a double perturbed p(x)-Laplacian equations. Appl. Math. Comput. 211 (2009), 234-241.

[10] S. G. Deng, Positive solutions for Robin problem involving the p(x)-Laplacian. J. Math. Anal. Appl. 360 (2009), 548-560.

[11] X. L. Fan, Solutions for p(x)-Laplacian Dirichlet problems with singular coefficients. J. Math. Anal. Appl. 312 (2005), 464-477.

[12] X. L. Fan, Q. Zhang, Existence of solutions for p (x)-Laplacian Dirichlet problem. Nonlinear Anal. 52 (2003), 1843-1852.

[13] B. Ge, Q. M. Zhou, Multiple solutions for a Robin-type differential inclusion problem involving the p(x)-Laplacian. Math. Methods Appl. Sci. 40(18) (2017), 6229-6238.

[14] T. C. Halsey, Electrorheological fluids, Science 258(5083) (1992), 761-766.

[15] K. Kefi, On the Robin problem with indefinite weight in Sobolev spaces with variable exponents. Z. Anal. Anwend. 37 (2018), 25-38.

[16] O. Kováčik, J. Rákosník, On spaces Lp(x) and Wk,p(x). Czechoslovak Math. J. 41(116)(4) (1991), 592-618.

[17] M. Mihăilescu, Existence and multiplicity of solutions for a Neumann problem involving the p(x)-Laplace operator. Nonlinear Anal. 67 (2007), 1419-1425.

[18] M. Mihăilescu, V. Rădulescu, A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids. Proc. R. Soc. Lond. Ser. A 462 (2006), 2625-2641.

[19] B. Ricceri, On three critical points theorem. Arch. Math. (Basel) 75 (2000), 220-226.

[20] M. Růžička, Electrorheological Fluids: Modeling and Mathematical Theory. Lecture Notes in Mathematics 1748, Springer, 2000.

[21] N. Tsouli, O. Darhouche, Existence and multiplicity results for nonlinear problems involving the p(x)-Laplace operator. Opuscula Math. 34(3) (2014), 621-638.

[22] N. Tsouli, O. Chakrone, O. Darhouche, M. Rahmani, Existence and multiplicity of solutions for a Robin problem involving the p(x)-Laplace operator. Hindawi Publishing Corporation, Conference Papers in Mathematics Volume 2013, Article ID 231898, 7 pp.

[23] C. Unal, I. Aydin, Compact embeddings of weighted variable exponent Sobolev spaces and existence of solutions for weighted p(x)-Laplacian. Complex Var. Elliptic Equ. (Accepted for publication) (2020).

[24] V. V. Zhikov, Averaging of functionals of the calculus of variations and elasticity theory. Math. USSR Izv. 9 (1987), 33-66.

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