Pattern selection of three components Gray-Scott model

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Abstract. The reaction-diffusion system demonstrates a variety of dynamical behaviours, and has become a standard model for explaining complex Turing patterns. In this work we have performed the analytical analysis of the three components Gray-Scott reaction-diffusion system. The analytical conditions for Turing instability about the homogeneous steady state has been derived. The linear stability is theoretically discussed. To determine the nature of pattern amplitude equation is derived by using weakly nonlinear analysis, which enumerates about the rich dynamical behaviour of this model, e.g. spot-, strip- and hexagon-patterns.

1. Introduction

Turing reaction-diffusion (RD) system [1] has been immensely used in the study of self-organized phenomena [2, 3] and has played a key role in explaining pattern formation scenario in different fields, such as physics, chemistry, biology, neurosciences, geology and ecology [4]. The affluent dynamical behavior of the RD model leads to various spatio-temporal patterns, like soliton, strips, hexagon, spots, peaks, travelling waves, etc. [5].

The general form of Turing RD equation is

$$\frac{\partial U}{\partial t} = D U \nabla^2 U + F(U, \mu),$$ (1)

where the vector $U$ is the morphogen concentrations of multi-reactants, $F$ accounts for the reaction kinetics, $D$ is the diagonal matrix of positive constant diffusion coefficients whereas $\mu$ represents the bifurcation parameter. The Turing instability is also known as diffusion driven instability, emerges numerous spatial patterns which are primarily stationary in nature, termed as labyrinthine, hexagonal array, stripes, mixture of spots, cold spot, hot spot, white eye and black eye patterns etc. [6-9].

Typically, stationary Turing patterns generated with binary reactants have two general types of form, i.e. the spot type which forms a hexagonal array pattern and strip type which forms a labyrinthine pattern [10], and blending of the patterns can occur in nature [11]. There are also some non-stationary patterns which can be detected with spatio-temporal models, named as, periodic traveling wave, traveling wave, target, spiral, interacting spiral chaos, spatio-temporal chaos, etc. [7,12-15].

Moreover, some complex dynamic patterns have also been observed. It was predicted that the interaction of two modes can produce superlattice pattern [16] which was then explored with the bistable system [17]. It was also found the twinkling eye pattern in which each spot has $2\pi/3$ phase shift with its nearest neighboring spots [18]. Another type is breathing mode which was studied
numerically and analytically with the help of activator-inhibitor model [19]. In three components RD model the moving hexagon pattern has been found [20]. Majority of the complex patterns are scrutinized with the help of two-layer linearly coupled models.

Although the phenomena like pulse propagation or formation of spatially inhomogeneous structures are described by the two components RD system, however, certain scenarios like instability of homogenous steady state (SS) by a travelling wave are dealt with the three components RD system. The three components RD system was firstly introduced in [20], which became a prototype model for the analysis of the rich variety of structures [20-24]. These structures can sustain the motion like annihilation, scattering, repulsion, collision, self-replication, attraction, spontaneous generation and breathing. As far as the applications of RD equation are concerned, the multi-reactants RD system can be used to analyze species invasion phenomena, e.g. cancer invasion as the spatio-temporal expansion of tumor tissue [25,26]. A computational study of oxygen transport in the body of a living organism was performed, to explore the consequences of numerous hypothesis [27]. The interaction between cell density and immune feedback was explained theoretically with the help of RD equation [28]. In ecology RD equations are broadly used in certain phenomenon such as spatial patterns formation in the distribution of population in homogeneous environments [4,15]. Using Cole-Hopf transformation the exact and numerical solutions of nonlinear RD equation was obtained which is the generalization of the Fisher-Burgers equations [29].

Nevertheless, the Gray-Scott (GS) model is still the prototypical example of spatio-temporal Turing pattern formation in RD systems. From the numerical simulation with the 2D binary GS RD equation some knotty spatio-temporal patterns were reported: spot multiplication, self-replication, travelling spot, spot annihilation, growing strips, labyrinthine, wave splitting and spatio-temporal chaos [30-32]. In addition, some analytical results for the GS model have been presented, e.g. the stability properties of 1D model, the pulse splitting in 2D and 3D [33,34] and a framework for different regimes of GS model [35,36].

In theoretical analysis the amplitude equation, i.e. the parameter values near the Turing bifurcation boundary in a RD system, is a useful tool [12]. A small inhomogeneous perturbation caused by the Turing instability destabilizes the homogenous SS in the system [13]. The linear stability analysis of an amplitude equation describes the upheaval and stability of different forms of Turing structures [37]. Weakly nonlinear analysis plays a major role in the derivation of amplitude equation, and to analyze the emergence of different patterns. Subsequently the method was accomplished for interacting population model like, competition models [37], predator prey model [38] etc.

However, the analysis of three components RD system with the help of amplitude equation has not been done yet. Because of the complexity of amplitude equation and weakly nonlinear analysis, only a few systems have been chosen for the weakly nonlinear analysis including Brusselator model [39] and other models [40]. The derivation of amplitude equation for the Swift-Hohenberg model has been extensively used for the convective structures emerging from Benard-Marangoni instability or non-Boussinesq Benard convection. In this article, we are interested in the theoretical analysis for the three components GS model. We have derived the analytical conditions for the Turing instability about the homogeneous SS in the three components GS model, which interpret the stability of various forms of Turing patterns as well as the structural transition between them [41,42]. To know more about the nature and dynamics of the pattern, we obtain the amplitude equation by using weakly nonlinear analysis and deduce parameter values for the stability condition about the Turing bifurcation boundary.

2. Turing instability

GS model, which describes an autocatalytic system, is well-known for having a diversity of different spatio-temporal pattern for different parameter values. e.g. spike, self-replication of pulse, travelling waves and spatio-temporal chaos [43-45]. With an additive reactant, the extended GS model equation for three components is written as,
\[
\frac{\partial u}{\partial t} = D_u \nabla^2 u - uv^2 - uw^2 + F(1-u);
\]
\[
\frac{\partial v}{\partial t} = D_v \nabla^2 v + uv^2 - (F + k_v)v;
\]
\[
\frac{\partial w}{\partial t} = D_w \nabla^2 w + uw^2 - (F + k_w)w,
\]

where \(u\) is the density distribution of activator, \(v\) and \(w\) are density distributions of two inhibitors. \(D_u\), \(D_v\) and \(D_w\) are the corresponding diffusion coefficients of the morphogens. \(F\) is the feed rate of \(u\); \((F + k_v)\) and \((F + k_w)\) are the removal rate of \(v\) and \(w\), respectively. In this expression, there is no direct interaction term between \(v\) and \(w\), but they interact indirectly with each other through intake of activator \(u\). For \(v\) and \(w\) are similar chemical species, they ought to form patterns belonging to the same kind of binary RD model, depending on diffusion coefficients and conversion rates.

The equilibrium solutions can be obtained from the nonnegative solutions of the equations,
\[
\begin{align*}
&f(u,v,w) = -uv^2 - uw^2 + F(1-u) = 0; \\
g(u,v) = uv^2 - (F + k_v)v = 0; \\
h(u,w) = uw^2 - (F + k_w)w = 0.
\end{align*}
\]

The equilibrium points are: 1) \((1,0,0)\), which implies extinction of both inhibitors; 2) \((u^*,v^*,w^*)\), corresponding to the interior equilibrium points:
\[
\begin{align*}
&u^* = \frac{F - \sqrt{F^2 - 4F(A^2 + B^2)}}{2F}, \\
v^* = \frac{2AF}{F - \sqrt{F^2 - 4F(A^2 + B^2)}}, \\
w^* = \frac{2BF}{F - \sqrt{F^2 - 4F(A^2 + B^2)}},
\end{align*}
\]

where \(A = F + k_v\) and \(B = F + k_w\). Assuming \((u^*,v^*,w^*)\) as the SS solution of system, Equation (3), the Jacobian of the system at \((u^*,v^*,w^*)\) is given by,
\[
J = \begin{pmatrix}
\frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} & \frac{\partial f}{\partial w} \\
\frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} & \frac{\partial g}{\partial w} \\
\frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} & \frac{\partial h}{\partial w}
\end{pmatrix}_{(u^*,v^*,w^*)} = \begin{pmatrix}
 f^*_{11} & f^*_{12} & f^*_{13} \\
 f^*_{21} & f^*_{22} & 0 \\
 f^*_{31} & 0 & f^*_{33}
\end{pmatrix},
\]

where \(f^*_{11} = -v^2 - w^2 - F\), \(f^*_{12} = -2uv\), \(f^*_{13} = -2uw\), \(f^*_{21} = v^2\), \(f^*_{22} = 2uv - A\), \(f^*_{31} = w^2\) and \(f^*_{33} = 2uv - B\). The characteristic equation of Jacobian is
\[
\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3 = 0,
\]

where
To satisfy the Routh-Hurwitz criteria [13], the real parts of all eigenvalues, i.e., \( \text{Re}(\lambda) \), are less than zero, which means that when these conditions are satisfied then the interior equilibrium point \((u^*, v^*, w^*)\) is asymptotically stable:

\[
\begin{align*}
    c_1^* &> 0, \quad c_3^* > 0, \quad c_1^* c_2^* - c_3^* > 0. \\
\end{align*}
\]

Now by introducing diffusion to the system, the GS model with diffusion term becomes Equation (2). Diffusion driven instability or Turing instability takes place, when a small amplitude nonhomogeneous perturbation near the homogeneous SS destabilizes the temporally stable homogeneous SS [13]. The Turing instability conditions are obtained from the linear stability analysis of the Equation (2) around the homogeneous SS \((u^*, v^*, w^*)\) [13]. We perturb the homogenous SS \((u^*, v^*, w^*)\) of the three components GS model with a small amplitude heterogeneous perturbation. By adding perturbation terms into the system and linearizing the homogenous SS, the Jacobian of the linearized system is given by,

\[
J_1 = \begin{pmatrix}
    -v^2 - w^2 - F - D_u k^2 & -2 uv & -2 uv \\
    v^2 & 2 uv - A - D_v k^2 & 0 \\
    w^2 & 0 & 2 uv - B - D_w k^2
\end{pmatrix} = \begin{pmatrix}
    j_{11} & j_{12} & j_{13} \\
    j_{21} & j_{22} & 0 \\
    j_{31} & 0 & j_{33}
\end{pmatrix}. \tag{9}
\]

\(\text{Det}(J_1) = 0\) gives

\[
\lambda^3 + c_1 \lambda^2 + c_2 \lambda + c_3 = 0, \tag{10}
\]

where

\[
\begin{align*}
    c_1 &= D_u + D_v + D_w - j_{11} - j_{22} - j_{33}; \\
    c_2 &= (D_u j_{11} + D_v j_{22} + D_w j_{33}) k^4 - (D_u j_{12} + D_v j_{13} + D_w j_{23} + D_u j_{11} + D_v j_{12} + D_w j_{22}) k^2 \\
    &+ j_{11} j_{22} - j_{12} j_{21} + j_{13} j_{33} - j_{13} j_{31} + j_{22} j_{33}; \\
    c_3 &= (D_u^2 D_v + D_u D_v^2 + D_v D_u^2) k^6 - (D_u D_v j_{11} + D_v D_u j_{22} + D_u D_v j_{33}) k^4 \\
    &+ (D_u j_{12} j_{33} + D_v j_{13} j_{31} - D_v j_{13} j_{31} + D_u j_{11} j_{22} - D_u j_{12} j_{21}) k^2 + \\
    &j_{12} j_{21} j_{33} - j_{13} j_{23} j_{31} + j_{13} j_{23} j_{31}. \tag{11}
\end{align*}
\]

The Routh-Hurwitz criteria is satisfied only when

\[
\begin{align*}
    c_1^* &> 0, \quad c_3^* > 0, \quad c_1^* c_2^* - c_3^* > 0. \\
\end{align*}
\]

When Equation (12) is satisfied, the entire eigenvalues have negative real parts and the homogenous SS is considered to be stable. When one of the real parts of the eigenvalue crosses zero whilst the other two eigenvalues still have negative real parts, the Turing instability occurs [1], [46, 47]. Let \( \lambda_1, \lambda_2 \) and \( \lambda_3 \) be the roots of Equation (10), from the properties of roots of a cubic equation we then have,

\[
\begin{align*}
    \lambda_1 + \lambda_2 + \lambda_3 &= -c_1; \\
    \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1 &= c_2; \\
    \lambda_1 \lambda_2 \lambda_3 &= -c_3; \\
    - (\lambda_1 + \lambda_2)(\lambda_2 + \lambda_3)(\lambda_3 + \lambda_1) &= c_1 c_2 - c_3. \tag{13}
\end{align*}
\]
At the edge of Turing bifurcation, \( k = k_c \), one of the roots is zero. Let \( \lambda_i \left( k^2 = k_c^2 \right) = 0 \), \( \text{Re} \left( \lambda_i \right) < 0 \). Taking \( c_3 \left( k_c^2 \right) = 0 \), then from Equation (12) we have \( c_1 \left( k_c^2 \right) > 0 \), \( c_2 \left( k_c^2 \right) > 0 \) and \( c_3 \left( k_c^2 \right) c_2 \left( k_c^2 \right) - c_3 \left( k_c^2 \right) > 0 \). The system is stable as long as \( c_3 \left( k_c^2 \right) > 0 \) for all values of \( k \). But it is unstable when \( c_3 \left( k_c^2 \right) < 0 \) for at least one value of \( k \). The expression for \( c_3 \left( k_c^2 \right) \) is

\[
c_3 = I_a k^6 + I_b k^4 + I_c k^2 + I_d,
\]

where \( I_a = D_u D_w D_w \), \( I_b = -(D_v D_w j_{11} + D_u D_w j_{22} + D_u D_v j_{33}) \), \( I_c = D_u j_{22} j_{33} + D_v j_{11} j_{33} - D_v j_{13} j_{31} + D_u j_{11} j_{22} - D_u j_{12} j_{21} \) and \( I_d = j_{12} j_{21} j_{33} - j_{11} j_{22} j_{33} + j_{13} j_{22} j_{31} \).

Since the diffusion coefficients are always positive, i.e. \( I_a > 0 \), and from the 2nd condition of Equation (12) we get \( I_d > 0 \). At \( k = k_c \) the minima for \( c_3 \) occurs, where

\[
k_c^2 = \left( -I_b + \sqrt{I_b^2 - 3I_a I_c} \right) / 3I_a,
\]

is positive if \( I_c < 0 \) or \( I_b < 0 \), and \( I_b^2 > 3I_a I_c \). Then the Turing bifurcation boundary can be written as,

\[
2I_b^3 - 9I_a I_b I_c - 2\sqrt{(I_b^2 - 3I_a I_c)^3 + 27I_a^2 I_d} = 0.
\]

The dispersion relation curve is always influential for better understanding of spatial pattern formation. To explore the response of diffusion, we plot the dispersion relation curve in Figure 1 for different values of \( D_v \). By changing the value of \( D_v \), the Turing mode changes from stable to unstable. It is obvious that \( \text{Re} \left( \lambda_i \right) \) increases by decreasing \( D_v \); for \( D_v = 152.61 \), the curve reaches the point of \( \text{Re} \left( \lambda \right) = 0 \) and Turing instability takes place. At this Turing bifurcation boundary, the homogeneous SS changes from stable to unstable. With further decreasing \( D_v \) we can get a range of \( k \)-values for which the Turing mode is unstable. The condition for Turing instability is thus derived. For the weakly nonlinear analysis, we will take \( D_v \) as our bifurcation parameter.

![Dispersion Relation Curve](image)

**Figure 1.** Typical dispersion relation curve for different values of \( D_v \): \( D_{v1} = 100.443 \), \( D_{v2} = 152.61 \) and \( D_{v3} = 174 \). The parameter set is \( \left[ F, k_c, k_w, D_u, D_v \right] = [0.025, 0.0043, 0.046, 290, 7.3] \).
3. Weakly nonlinear analysis

To find more about the nature and dynamics of the pattern the amplitude Equation [48,49] is to be obtained. When parameter $D_v$ is close to its critical value, i.e. $D_v = D_v^c$, then the eigenvalues associated with Turing critical mode are near to zero and it mutates slowly. On the other side the modes except Turing critical mode attenuate rapidly. So only perturbation with $k$ about $k_c$ is taken into account. There are two ways to get the coefficient of amplitude equation: (a) multiple scale analysis; (b) symmetrical analysis. Multiple scale analysis is a more versatile approach for derivation of amplitude equation [49, 50] and we use it in this work to derive the amplitude equation for the study of pattern stability, where the amplitude equation is about Turing bifurcation point with respect to the perturbation of wave number near to the critical mode of the Equation (2). A system of three pairs of active mode $(k_l, -k_l)$ for $(l = 1, 2, 3)$ describes the solutions where each mode makes an angle of $2\pi/3$.

The linearized form of Equation (2) around homogeneous SS is given in matrix form by

$$
\frac{\partial}{\partial t} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = L \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \frac{1}{2} \begin{pmatrix} f_{uu}u^2 + f_{uv}v^2 + f_{uw}w^2 + 2f_{uv}uv + 2f_{uw}uw + 2f_{vw}vw \\
2g_{uv}u^2 + 2g_{uw}uv + g_{vw}v^2 \\
h_{uw}u^2 + 2h_{uv}uv + h_{vw}v^2 
\end{pmatrix}
$$

\begin{equation}
+ \frac{1}{6} \begin{pmatrix} f_{uu}u^3 + 3f_{uv}u^2v + f_{uw}u^2w + 3f_{vuw}uv^2 + 3f_{vw}vw^2 \\
3f_{uw}uv^2 + f_{vw}vw^2 + 6f_{uvw}uvw \\
g_{uu}u^3 + 3g_{uv}u^2v + 3g_{uw}uw^2 + g_{vw}v^3 \\
h_{uu}u^3 + 3h_{uv}u^2v + 3h_{uw}uw^2 + h_{vw}v^3 
\end{pmatrix},
\end{equation}

where

$$
L = \begin{pmatrix}
-v^2 - w^2 - F + D_u \nabla^2 & -2uv & -2uw \\
v^2 & 2uv - A + D_v \nabla^2 & 0 \\
w^2 & 0 & 2uv - B + D_w \nabla^2 
\end{pmatrix},
$$

$$
f_{uu} = 0, f_{uv} = -2u, f_{uw} = -2u, f_{vw} = -2v, f_{uw} = -2w, f_{vw} = 0, g_{uv} = 0, g_{uw} = 0, g_{vw} = 0, h_{uw} = 0, h_{vw} = 0, h_{uw} = 2w, h_{vw} = 2w, f_{uuw} = 0, f_{uvw} = 0, f_{uvw} = 0, f_{wu} = 0, f_{vw} = 0, f_{uw} = 0, f_{uvw} = 0, f_{uvw} = 0, f_{uwv} = 0, g_{uu} = 0, g_{uv} = 0, g_{uw} = 0, g_{vw} = 0, h_{uu} = 0, h_{uv} = 0, h_{uw} = 0, h_{vw} = 0, h_{uw} = 0.
$$

Taking $U^*_r$ as the homogenous SS and $U^*_0$ as the eigen vectors of the linearized operator $L$, which represents the directions of the eigenmodes, Similarly, $E_i$ and $\overline{E}_i$ as the amplitude associated with the modes $k_i$ and $-k_i$ respectively. At the Turing bifurcation point, $D_v = D_v^c$, the general solution of Equation (2) is given by

$$
U^* = U^*_r + \sum_{l=1}^3 U^*_0 \left[ E_l \exp(ik_l \cdot r) + \overline{E}_l \exp(-ik_l \cdot r) \right].
$$

We perturb $(u,v,w)$ and $t$ along with the bifurcation parameter $D_v$ in term of the perturbation variable $\epsilon$ as follows:
Substituting Equation (20) into Equation (18) we get
\[
L = L^e + \varepsilon \begin{pmatrix}
0 & 0 & 0 \\
0 & \nabla^2 & 0 \\
0 & 0 & 0
\end{pmatrix} + \varepsilon^2 \begin{pmatrix}
0 & 0 & 0 \\
0 & \nabla^2 & 0 \\
0 & 0 & 0
\end{pmatrix} + \varepsilon^3 \ldots,
\]
where
\[
L^e = \begin{pmatrix}
-v^2 - w^2 - F + D_u\nabla^2 & -2uv & -2uw \\
v^2 & 2uv - A + D_v\nabla^2 & 0 \\
w^2 & 0 & 2uv - B + D_w\nabla^2
\end{pmatrix}.
\]

The time derivative of \( E_l \) in term of perturbation variable is given by,
\[
\frac{\partial E_l}{\partial t} = \varepsilon \frac{\partial E_l}{\partial t_1} + \varepsilon^2 \frac{\partial E_l}{\partial t_2} + O(\varepsilon^3).
\]
Substituting Equation (20) into Equation (17) and getting the coefficients of \( \varepsilon^0, \varepsilon^1, \varepsilon^2 \) and \( \varepsilon^3 \), we can derive the amplitude equation as below.

By collecting the coefficient of \( O(\varepsilon) \) terms we get the following linear system,
\[
L^e \begin{pmatrix} u_1 \\ v_1 \\ w_1 \end{pmatrix} = 0,
\]
where \( L^e \) is called the linear operator of the system at \( D_v = D_v^e \). The solution at the leading order is of the form,
\[
\begin{pmatrix} u_1 \\ v_1 \\ w_1 \end{pmatrix} = \begin{pmatrix} f_a \\ f_b \\ 1 \end{pmatrix} M_0(x, y, t_1, t_2),
\]
where \( f_a = (D_v k_v^2 - 2uw + B)/w^2 \) and \( f_b = (2uw - B - D_u k_u^2)v^2/(2uv - A - D_v k_v^2)w^2 \). \( (f_a, f_b, 1) \) is the eigenvector associated with zero eigenvalue at which the original system becomes unstable, \( M_0 \) represents the planform of the solution which satisfies \( \Delta M_0 = -k_v^2 M_0 \). We take \( M_0 \) as the linear combination of the three modes
\[
M_0(x, y, t_1, t_2) = \sum_{l=1}^{3} W_l(t_1, t_2) \exp(i k_l \cdot r) + c.c.,
\]
which allows the planforms for the strip pattern or spot pattern, whereas \( W_l \) represents the amplitude of the mode \( \exp(i k_l \cdot r) \) for \( l = 1, 2, 3 \) and \( c.c. \) stands for the complex conjugate.

Similarly, we obtain the coefficients of \( O(\varepsilon^2) \):
To get the nontrivial solution of nonhomogeneous Equation (27) the right-hand side must be orthogonal to the kernel of \(L^c\) which is the adjugate matrix of \(cL\). This is called as Fredholm's solvability condition which ascertains the unknown coefficients of perturbation terms in Equation (20).

For finding the amplitude of the leading order solution through its temporal expansion equation, known as amplitude equation, we take homogeneous adjoint problem solution in the form of

\[
\hat{M}_0(x, y, t, t_2) = \begin{cases} 
\hat{M}_0(x, y, t, t_2) = \begin{cases} 
\hat{F}_x &= 2w^2 (D_x k_x^2 - 2uv + A) \\
\hat{F}_y &= 2w^2 (D_y k_y^2 - 2uw + B) \\
\hat{F}_{(x)} &= \hat{F}_x + \hat{F}_y \\
\hat{F}_{(y)} &= \hat{F}_x - \hat{F}_y \\
\end{cases}
\end{cases}
\]

From the orthogonality condition, \((1, g_a, g_b) F_x, F_y, F_{(x)} = 0\), where \(F_x, F_y\) and \(F_{(x)}\) are the coefficients of the \(\exp(ik_\cdot r)\) terms in \(F_x, F_y\) and \(F_{(x)}\), respectively. For instance, substituting Equation (25) in Equation (27) and getting the coefficients of \(\exp(ik_\cdot r)\), we get

\[
\begin{pmatrix}
F_x \\
F_y \\
F_{(x)}
\end{pmatrix} = \begin{pmatrix}
f_a \\
-g_b \\
0
\end{pmatrix} W_1 + \begin{pmatrix}
f_b \\
-g_a \\
0
\end{pmatrix} W_2 - \begin{pmatrix}
F_x \\
G_1 \\
H_1
\end{pmatrix} \hat{W}_2 \hat{W}_3,
\]

where \(F_x = f_{ww} + 2f_{vy}f_b + 2f_{yy}f_b + f_{uu}f_a + f_{uy}f_a\), \(G_1 = 2g_{wv}f_a g_b + g_{uv}f_b^2 + g_{uu}f_a^2\) and \(H_1 = h_{ww}f_a + h_{uu}f_a^2\).}

Applying the solvability condition, we get

\[
(f_a + f_b g_a + g_b) (\partial W_2/\partial t_1) = -D_2 f_a g_b k_x^2 W_3 + (F_x + g_a G_1 + g_b H_1) \hat{W}_2 \hat{W}_3.
\]

Similarly, for the coefficients of \(\exp(i k_2 \cdot r)\) and \(\exp(i k_3 \cdot r)\) we get the following results:

\[
(f_a + f_b g_a + g_b) (\partial W_2/\partial t_1) = -D_2 f_a g_b k_y^2 W_3 + (F_y + g_a G_1 + g_b H_1) \hat{W}_3 \hat{W}_1;
\]

\[
(f_a + f_b g_a + g_b) (\partial W_2/\partial t_1) = -D_2 f_a g_b k_x^2 W_3 + (F_x + g_a G_1 + g_b H_1) \hat{W}_2 \hat{W}_1.
\]

By solving Equation (27) we get

\[
\begin{pmatrix}
u_2 \\
\end{pmatrix} = \begin{pmatrix}
\begin{pmatrix}
X_0 \\
Y_0 \\
Z_0
\end{pmatrix} + \sum_{i=1}^{3} \begin{pmatrix}
X_i \\
Y_i \\
Z_i
\end{pmatrix} \exp(i k_i \cdot r) + \sum_{i=1}^{3} \begin{pmatrix}
X_{il} \\
Y_{il} \\
Z_{il}
\end{pmatrix} \exp(i 2k_i \cdot r) + \begin{pmatrix}
X_{12} \\
Y_{12} \\
Z_{12}
\end{pmatrix} \exp(i (k_i - k_2) \cdot r)
\end{pmatrix} + \begin{pmatrix}
X_{23} \\
Y_{23} \\
Z_{23}
\end{pmatrix} \exp(i (k_2 - k_3) \cdot r) + \begin{pmatrix}
X_{31} \\
Y_{31} \\
Z_{31}
\end{pmatrix} \exp(i (k_3 - k_1) \cdot r) + \text{c.c.}
\]

Substituting Equation (32) into Equation (27) and collecting the coefficients of \(\exp(0)\), \(\exp(i k_\cdot r)\), \(\exp(2ik_\cdot r)\) and \(\exp(i (k_i - k_2) \cdot r)\) we find:
\[
\begin{align*}
\begin{pmatrix}
X_0 \\
Y_0 \\
Z_0
\end{pmatrix} &= \left( \begin{pmatrix}
-v^2 - w^2 - F & -2uv & -2uw \\
v^2 & 2uv - A & 0 \\
w^2 & 0 & 2uv - B
\end{pmatrix} \right)^{-1}
\begin{pmatrix}
F_1 \\
G_1 \\
H_1
\end{pmatrix}
\left( |W_1|^2 + |W_2|^2 |W_3|^2 \right) \\
&= \begin{pmatrix}
Z_{x_0} \\
Z_{y_0} \\
Z_{z_0}
\end{pmatrix} \left( |W_1|^2 + |W_2|^2 |W_3|^2 \right); \\
\begin{pmatrix}
X_{11} \\
Y_{11} \\
Z_{11}
\end{pmatrix} &= -\frac{1}{2} \begin{pmatrix}
-v^2 - w^2 - F - 4k_v^2 D_u & -2uv & -2uv \\
v^2 & 2uv - A - 4k_v^2 D_v & 0 \\
w^2 & 0 & 2uv - B - 4k_v^2 D_w
\end{pmatrix}^{-1}
\begin{pmatrix}
F_1 \\
G_1 \\
H_1
\end{pmatrix} W_1^2 \\
&= \begin{pmatrix}
Z_{x_1} \\
Z_{y_1} \\
Z_{z_1}
\end{pmatrix} W_1^2; \\
\begin{pmatrix}
X_{12} \\
Y_{12} \\
Z_{12}
\end{pmatrix} &= \begin{pmatrix}
-v^2 - w^2 - F - 3k_v^2 D_u & -2uv & -2uv \\
v^2 & 2uv - A - 3k_v^2 D_v & 0 \\
w^2 & 0 & 2uv - B - 3k_v^2 D_w
\end{pmatrix}^{-1}
\begin{pmatrix}
F_1 \\
G_1 \\
H_1
\end{pmatrix} W_1 |\vec{W}_2| \\
&= \begin{pmatrix}
Z_{x_2} \\
Z_{y_2} \\
Z_{z_2}
\end{pmatrix} W_1 |\vec{W}_2|.
\end{align*}
\]

We can get the coefficients of other terms of Equation (32) by changing the suffixes. Taking coefficient of \(O(e^3)\), we get
\[
L' \begin{pmatrix}
u_3 \\
v_3 \\
w_3
\end{pmatrix} = \begin{pmatrix}G_X \\
G_y \\
G_z
\end{pmatrix},
\] (36)

where
Collecting the coefficients of \( \exp(ik \cdot r) \) in Equation (37) we get

\[
\begin{align*}
G_x^{(1)} &= f_a \left( \frac{\partial W}{\partial t} + \frac{\partial Y_1}{\partial t} \right) + \begin{pmatrix}
0 & 0 & 0 \\
0 & D_v^{(2)} & 0 \\
0 & 0 & 0
\end{pmatrix} W_1 + \begin{pmatrix}
f_a \\
f_b \\
1
\end{pmatrix} y_1, \\
G_y^{(1)} &= \begin{pmatrix}
0 & 0 & 0 \\
0 & D_v^{(1)}k_c & 0 \\
0 & 0 & 0
\end{pmatrix} W_1 + \begin{pmatrix}
f_a \\
f_b \\
1
\end{pmatrix} y_1, \\
G_z^{(1)} &= \begin{pmatrix}
0 & 0 & 0 \\
0 & D_v^{(1)}k_c & 0 \\
0 & 0 & 0
\end{pmatrix} W_1 + \begin{pmatrix}
f_a \\
f_b \\
1
\end{pmatrix} y_1,
\end{align*}
\]

where

\[
F_2 = f_a f_{uv} Z_{z_1} + f_a f_{uv} Z_{v_1} + f_b f_{uv} Z_{z_1} + f_b f_{uv} Z_{v_1} + f_{uw} f_{uv} Z_{z_1} + f_{uw} f_{uv} Z_{v_1} + f_{uw} f_{uv} Z_{x_1} + f_{uw} f_{uv} Z_{y_1} + f_{uv} f_{uv} Z_{z_1} + f_{uv} f_{uv} Z_{v_1} + f_{uv} f_{uv} Z_{x_1} + f_{uv} f_{uv} Z_{y_1} + f_{uv} f_{uv} Z_{x_1} + f_{uv} f_{uv} Z_{y_1} + f_{uv} f_{uv} Z_{x_1} + f_{uv} f_{uv} Z_{y_1}.
\]

\[
G_2 = f_{aw} g_{uw} Z_{z_1} + f_{aw} g_{uw} Z_{v_1} + f_{aw} g_{uw} Z_{x_1} + f_{aw} g_{uw} Z_{y_1} + f_{aw} g_{uw} Z_{x_1} + f_{aw} g_{uw} Z_{y_1} + f_{aw} g_{uw} Z_{x_1} + f_{aw} g_{uw} Z_{y_1} + f_{aw} g_{uw} Z_{x_1} + f_{aw} g_{uw} Z_{y_1} + f_{aw} g_{uw} Z_{x_1} + f_{aw} g_{uw} Z_{y_1} + f_{aw} g_{uw} Z_{x_1} + f_{aw} g_{uw} Z_{y_1}.
\]

\[
H_2 = f_{ah} h_{uw} Z_{z_1} + f_{ah} h_{uw} Z_{v_1} + f_{ah} h_{uw} Z_{x_1} + f_{ah} h_{uw} Z_{y_1} + f_{ah} h_{uw} Z_{x_1} + f_{ah} h_{uw} Z_{y_1} + f_{ah} h_{uw} Z_{x_1} + f_{ah} h_{uw} Z_{y_1} + f_{ah} h_{uw} Z_{x_1} + f_{ah} h_{uw} Z_{y_1} + f_{ah} h_{uw} Z_{x_1} + f_{ah} h_{uw} Z_{y_1}.
\]
Using Fredholm solvability condition as in $O(\varepsilon^2)$ case, we find \( (g_a, g_b)(G_x^r, G_y^r, G_z^r) = 0 \).

Thus, \[
(f_a + f_b g_a + g_b) \left((\partial W_1/\partial t_2) + (\partial Y_1/\partial t_1)\right) = -k_f f_a g_a \left(D_{v}^{(2)} W_1 + D_{v}^{(1)} Y_1\right) + (F_1 + g_a G_1 + g_b H_1)
\]
\[
\left(\overline{W}_2 F_3 + \overline{F}_2 W_3\right) + (F_3 + g_a G_2 + g_b H_2) |W_1|^2 W_1 + (F_3 + g_a G_3 + g_b H_3) |W_2|^2 + |W_3|^2 W_1.
\]

The amplitude $E_i$ can be expressed as,
\[
E_i = \varepsilon W_i + \varepsilon^2 Y_i + O(\varepsilon^3).
\]

The amplitude equation corresponding to $E_i$ can be obtained from Equations. (40) and (23),
\[
\tau_0 \left(\partial E_i/\partial t\right) = \mu_0 E_i + \delta H^2 E_3 - \left(q_1 |E_1|^2 + q_2 \left(|E_2|^2 + |E_3|^2\right)\right) E_i,
\]
where \( \mu_0 = \left(D_{v}^{(3)} - D_{v}^{(1)}\right)/D_{v}^{(3)}, \tau_0 = (f_a + f_b g_a + g_b)/D_{v}^{(3)} k_f g_a, q_1 = -(F_3 + g_a G_3 + g_b H_3)/D_{v}^{(3)} k_f g_a, q_2 = -(F_2 + g_a G_2 + g_b H_2)/D_{v}^{(3)} k_f g_a \) and \( h = (F_1 + g_a G_1 + g_b H_1)/D_{v}^{(3)} k_f g_a \). Similarly, by changing the subscript of $E$, the other two equations are given here:
\[
\tau_0 \left(\partial E_2/\partial t\right) = \mu_0 E_2 + \delta H^2 E_1 - \left(q_1 |E_1|^2 + q_2 \left(|E_2|^2 + |E_3|^2\right)\right) E_2;
\]
\[
\tau_0 \left(\partial E_3/\partial t\right) = \mu_0 E_3 + \delta H^2 E_2 - \left(q_1 |E_1|^2 + q_2 \left(|E_2|^2 + |E_3|^2\right)\right) E_3.
\]

4. Stability analysis of amplitude equation

The emergence and stability of various Turing pattern can be obtained with the help of stability analysis of amplitude Equation

Any respective amplitude in Equations. (41)-(43) can be decomposed into mode $\gamma_i = E_i$ and a corresponding phase angle $\phi_i$, where \( i = 1, 2, 3 \). Replacing $E_i = \gamma_i \exp(\imath \phi_i)$ into Equations. (41)-(43) and separating the real and imaginary parts, we obtain the following differential equations with the real variable as,
\[
\tau_0 \left(\partial \phi/\partial t\right) = -h^2 \gamma_1^2 \gamma_2^2 + \gamma_2^2 \gamma_3^2 + \gamma_3^2 \gamma_1^2 \sin \phi;
\]
\[
\tau_0 \left(\partial \gamma_1/\partial t\right) = \mu_0 \gamma_1 + h \gamma_2 \gamma_3 \cos \phi - b_3 \gamma_1^2 - b_2 \left(\gamma_2^2 + \gamma_3^2\right) \gamma_1;
\]
\[
\tau_0 \left(\partial \gamma_2/\partial t\right) = \mu_0 \gamma_2 + h \gamma_3 \gamma_1 \cos \phi - b_3 \gamma_2^2 - b_2 \left(\gamma_2^2 + \gamma_3^2\right) \gamma_2;
\]
\[
\tau_0 \left(\partial \gamma_3/\partial t\right) = \mu_0 \gamma_3 + h \gamma_1 \gamma_2 \cos \phi - b_3 \gamma_3^2 - b_2 \left(\gamma_2^2 + \gamma_3^2\right) \gamma_3,
\]
where $\gamma = \gamma_1 + \gamma_2 + \gamma_3$. Equation (44) characterizes a four-dimensional dynamical system.

Based on the SS of the system, there exists four kinds of solutions:

1. Stationary state, $\gamma_1 = \gamma_2 = \gamma_3 = 0$, which is stable for $\mu_0 < \mu_c = 0$ and unstable for $\mu_0 > \mu_c = 0$;

2. Stripe pattern, $\gamma_1 = \sqrt{\mu_0/q_1} = 0$ and $\gamma_2 = \gamma_3 = 0$, which is stable for $\mu_0 > \mu_b = h^2 q_1/(q_2 - q_1)^2$ and unstable for $\mu_0 < \mu_b$;

3. Hexagonal pattern, $\gamma_1 = \gamma_2 = \gamma_3 = 0$ (multiplied by $h$), which exists when $\mu_0 > \mu_c = h^2 (2q_1 + q_2)/(q_2 - q_1)^2$ or is expressed as $\mu_0 < \mu_b = h^2 (2q_1 + q_2)/(q_2 - q_1)^2$ or else unstable;

4. The mixed states are given by $\gamma_1 = |h|(q_2 - q_1)$ and $\gamma_2 = \gamma_3 = \sqrt{(\mu_0 - q_1 \gamma_1^2)/(q_1 + q_2)}$ for $\mu_0 = q_1 \gamma_1^2$, which are always unstable.

Hence the amplitude equation gives a scheme about different patterns emerging from the numerous thresholds of $\mu_0$.

5. Conclusions
In this article we have analyzed three components GS RD system. With the help of linear stability analysis, we have deduced conditions for Turing instability which guarantee the pattern formation. The amplitude equation gives a scheme about the nature and dynamics of the pattern near the Turing bifurcation boundary. Multiple scale analysis is the key step toward weakly nonlinear analysis. The stability analysis of the amplitude equation deduces different SS solutions. There are four kinds of SS solutions for the derived amplitude equations and each of the SS combines with stability, equate to different kinds of Turing pattern, i.e. the spot pattern, strip pattern and mix of the two.

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