Re-entrant phase transitions in non-commutative quantum mechanics

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Abstract. We discuss the (2+1)-dimensional Dirac oscillator in a magnetic field in noncommutative quantum mechanics. The system is known to be characterised by a left right-chiral phase transition in ordinary Quantum mechanics. We show that the momentum noncommutativity shifts the known phase transition while the space noncommutativity introduces a new right-left chiral quantum phase transition giving rise to the intriguing phenomenon of re-entrant phase transition observed in condensed matter as well as in black hole physics.

1. Introduction

On the basis that both string theory and quantum gravity indicate that space-time can be non-commutative [1–4] several quantum mechanical systems have been studied by a number of authors to determine the role of noncommutativity parameters on a variety of physical observables [5–14].

The Dirac oscillator on the other hand is one of the very few relativistic systems which is exactly solvable [15–22]. In mathematical physics the Dirac oscillator has become a paradigm for the realization of covariant quantum models and it has found applications both in nuclear [23–25] and subnuclear [26, 27] physics as well as in quantum optics [28–30]. Very recently a one dimensional version of the Dirac oscillator has been realised experimentally [31] for the first time with realistic prospects to realise in the near future the two dimensional version of the Dirac oscillator which may be feasible using networks of microwave coaxial cables [32–34]. It is interesting to note that a further interaction in the form of a homogeneous magnetic field can still be incorporated in the Dirac oscillator keeping the system still exactly solvable [28, 35–38] and this combined system has quite interesting properties. In some recent papers it has been shown that for this combined system there is a chirality phase transition if the magnitude of the magnetic field either exceeds or is less than a critical value $B_{cr}$ (which also depends on the oscillator strength) [39, 40]. A consequence of this chirality phase transition is that the spectrum is different for $B > B_{cr}$ and $B < B_{cr}$, $B$ being the magnetic field strength.

Our objective here is to analyse the same system, a 2D Dirac oscillator within a constant magnetic field, but in the framework of non-commutative space and momentum coordinates. It will be shown that in this case the critical value of the magnetic field depends not only on the oscillator strength but on the non-commutativity parameters as well. The interesting feature which we would like to emphasise, is that beside the two left- and right-chiral phases present also in the commutative case, in the non-commutative scenario there appear also a new third
left phase and so a second quantum phase transition (right-left) which both disappear as the space non-commutativity parameter vanishes \[41\]. This is the well known phenomenon of re-entrant phase transitions which has been pointed out in condensed matter as well as in black hole physics \[42, 43\]. While the details of the full calculation can be found in \[41\] we illustrate here the main results. We shall discuss the spectrum and the degeneracy of the various energy levels in all three phases. Another point which we address is how to characterise the phase transition(s). It will be shown that such quantum phase transitions can be described in terms of the magnetisation of the system.

2. Formulation of the problem

To begin with we note that the Hamiltonian for the \((2 + 1)\) dimensional Dirac oscillator in the noncommutative plane in the presence of a homogeneous magnetic field is given by

\[ H = c \sigma \left( \hat{p} - i m \omega \beta \hat{x} + \frac{e}{c} \hat{A} \right) + \beta mc^2, \]  

(1)

where \(c\) is the velocity of light and \(\sigma, \beta = \sigma_z\) denote Pauli matrices. We now choose the vector potential as

\[ \hat{A} = (\frac{-B}{2} \hat{y}, \frac{B}{2} \hat{x}, 0). \]  

(2)

The commutation relation between coordinates and momenta are given by

\[ [\hat{x}, \hat{y}] = i \theta, \quad [\hat{p}_x, \hat{p}_y] = i \eta, \quad [\hat{x}_i, \hat{p}_j] = i \hbar (1 + \frac{\theta \eta}{4 \hbar^2}) \delta_{ij}, \quad \theta, \eta \in \mathbb{R}. \]  

(3)

Written in full the above Hamiltonian reads

\[ H = c \left( mc \hat{\Pi} - \hat{\Pi} + \frac{\beta}{mc} \right), \]  

(4)

where \(\hat{\Pi}_\pm\) are given by

\[ \hat{\Pi}_- = \hat{p}_x - i \hat{p}_y - i \left( \frac{eB}{2c} - m\omega \right) \hat{x} - \left( \frac{eB}{2c} - m\omega \right) \hat{y}; \]  

\[ \hat{\Pi}_+ = \hat{p}_x + i \hat{p}_y + i \left( \frac{eB}{2c} - m\omega \right) \hat{x} - \left( \frac{eB}{2c} - m\omega \right) \hat{y}. \]  

(5)

It is now necessary to express the noncommuting coordinates and momenta in terms of commuting ones. This can be achieved using the Seiberg-Witten map and are given by

\[ \hat{x} = x - \frac{\theta}{2 \hbar} \hat{p}_y, \quad \hat{p}_x = p_x + \frac{\eta m}{2 \hbar} \hat{y}, \]  

\[ \hat{y} = y + \frac{\theta m}{2 \hbar} \hat{p}_x, \quad \hat{p}_y = p_y - \frac{\eta m}{2 \hbar} \hat{x}, \]  

(6)

where \((x, y)\) and \((p_x, p_y)\) denote commuting coordinates and momenta. Now using the relation (6) the Hamiltonian (4) can be written as

\[ H = c \left( mc \hat{\Pi} - \hat{\Pi} + \frac{\beta}{mc} \right), \]  

(7)

where \(\hat{\Pi}_\pm\) are more conveniently written after introducing the following frequencies out of the parameters of the problem:

\[ \tilde{\omega} = \frac{eB}{2mc}, \quad \omega_\theta = \frac{2 \hbar}{m \theta}, \quad \omega_\eta = \frac{\eta m}{2 \hbar m}. \]  

(8)
Then we find:

\[ \hat{\Pi}_\pm = \left(1 - \frac{\tilde{\omega} - \omega}{\omega_y}\right)(p_x \pm i p_y) \pm i m(\tilde{\omega} - \omega - \omega_\eta)(x \pm i y). \]  

(9)

From the above relations one finds that there is a critical value of the magnetic field \(B_{\text{cr}}\)

\[ B_{\text{cr}} = \frac{2mc}{e}(\omega + \omega_\eta) = \frac{2c}{e}(m\omega + \frac{2\eta}{\hbar}), \]

(10)
such that for \(B = B_{\text{cr}}\) (or \(\tilde{\omega} = \omega + \omega_\eta\)) there are no interactions in the model and the Hamiltonian represents a free particle (only kinetic energy). Note that the critical magnetic field \(B = B_{\text{cr}}\) actually depends on the oscillator frequency as well as the momentum non-commutativity parameter \((\eta\) or \(\omega_\eta\)). We see therefore that the momentum non-commutativity parameter \(\eta\) shifts the value of the critical filed relative to the value in the commutative case, where the system is well known to undergo a quantum phase transition.

More importantly we note that space non commutativity \((\theta \neq 0)\) introduces an additional critical value for the magnetic field \(B^*_{\text{cr}}\) which corresponds to the the condition \(\tilde{\omega} = \omega + \omega_\theta\):

\[ B^*_{\text{cr}} = \frac{2mc}{e}(\omega + \omega_\theta) = \frac{2c}{e}(m\omega + \frac{\hbar}{\theta}). \]

(11)

At this value of the magnetic field the kinetic part of the Hamiltonian will vanish. Physically we can visualise the system at this critical value as being made up by only potential energy. This second critical point signals a new quantum phase transition which is absent in the commutative limit, whereas the quantum phase transition at \(B = B_{\text{cr}}\) is only shifted by the momentum non commutativity parameter \(\eta\) (and this shift goes smoothly to zero when \(\eta \to 0\)).

In the following section we shall analyse the spectrum as a function of the magnetic field separately in each of the three regions: when \(B < B_{\text{cr}}\), \(B_{\text{cr}} < B < B^*_{\text{cr}}\) and \(B > B^*_{\text{cr}}\).

We shall discuss the problem according to the magnitude of the magnetic field (or the cyclotron frequency \(\tilde{\omega}\)) relative to the other parameters (Dirac oscillator frequency \(\omega\) and NC frequencies \(\omega_i, \omega_\eta\)). In this regard we note that as the non commutative (NC) parameters \(\eta, \theta \to 0\) (the commutative limit) the frequency \(\omega_\eta\) vanishes, while \(\omega_\theta \to \infty\). Therefore without any loss of generality we may assume the following relation between \(\omega_\eta\), and \(\omega_\theta\):

\[ \omega_\eta < \omega_\theta. \]

(12)

3. Weak Magnetic Field: \(B < B_{\text{cr}}\) \((\tilde{\omega} < \omega + \omega_\eta)\)

In this case we find from Eq. (9):

\[ \hat{\Pi}_\pm = \lambda(p_x \pm i p_y) \mp i\nu(x \pm y), \]

(13)

\[ \lambda = 1 - \frac{\tilde{\omega} - \omega}{\omega_y} > 0, \quad \nu = m(\omega + \omega_\eta - \tilde{\omega}) > 0. \]

(14)

To this end let us define the following set of creation and annihilation operators:

\[ a_x = \frac{i}{\sqrt{2\nu\lambda\hbar}}(\lambda p_x - i\nu x), \quad a_x^\dagger = \frac{-i}{\sqrt{2\nu\lambda\hbar}}(\lambda p_x + i\nu x), \]

\[ a_y = \frac{i}{\sqrt{2\nu\lambda\hbar}}(\lambda p_y - i\nu y), \quad a_y^\dagger = \frac{-i}{\sqrt{2\nu\lambda\hbar}}(\lambda p_y + i\nu y), \]

(15)
and then the so called circular (or chiral) annihilation and creation operators:

\[ a_L = \frac{1}{\sqrt{2}} (a_x + i a_y), \quad a_L^\dagger = \frac{1}{\sqrt{2}} (a_x - i a_y), \]
\[ a_R = \frac{1}{\sqrt{2}} (a_x - i a_y), \quad a_R^\dagger = \frac{1}{\sqrt{2}} (a_x + i a_y). \]  \hfill (16)

It can easily be verified that the operators in (16) satisfy the following commutation relations

\[ [a_R, a_L^\dagger] = [a_L, a_R^\dagger] = 1, \quad [a_R, a_L] = [a_R^\dagger, a_L^\dagger] = [a_R^\dagger, a_L] = [a_R, a_L^\dagger] = 0. \]  \hfill (17)

Thus \((a_R^\dagger, a_R, a_L, a_L^\dagger)\) represent creation and annihilation operators for a pair of independent harmonic oscillators. We shall now analyse the spectrum using the creation and annihilation operators defined in (Eq. 16).

As a consequence we obtain:

\[ \hat{\Pi}_+ = -2 i \sqrt{\nu \lambda \hbar} a_L, \quad \hat{\Pi}_- = +2 i \sqrt{\nu \lambda \hbar} a_L^\dagger. \]  \hfill (18)

The eigenvalue equation can be written as

\[ +2 i \sqrt{\nu \lambda \hbar} a_L^\dagger \psi_L^{(2)} = \epsilon_\pm \psi_L^{(1)}, \quad -2 i \sqrt{\nu \lambda \hbar} a_L \psi_L^{(1)} = \epsilon_\pm \psi_L^{(2)}, \quad \epsilon_\pm = \frac{E \pm mc^2}{\hbar} c. \]  \hfill (19)

Let us now examine whether there are solutions with energies \(E = \pm mc^2\) i.e., \(\epsilon_\pm = 0\). It turns out to be convenient to work in standard polar coordinates:

\[ r = \sqrt{x^2 + y^2}, \quad \varphi = \arctan \left( \frac{y}{x} \right), \quad \rightarrow \quad (x, y) = (r \cos \varphi, r \sin \varphi), \]

and then we can easily find the form of the annihilation and creation operators:

\[ a_L = \frac{e^{i \varphi}}{2 \sqrt{\nu \lambda \hbar}} \left\{ \lambda \hbar \left( \partial_r + \frac{i}{r} \partial_\varphi \right) + \nu r \right\}, \]
\[ a_L^\dagger = \frac{e^{-i \varphi}}{2 \sqrt{\nu \lambda \hbar}} \left\{ -\lambda \hbar \left( \partial_r - \frac{i}{r} \partial_\varphi \right) + \nu r \right\}, \]
\[ a_L \psi_L^{(1)} = 0, \quad \rightarrow \quad \left\{ \partial_r + \frac{i}{r} \partial_\varphi + \frac{\nu}{\lambda \hbar} r \right\} \psi_L^{(1)}(r, \varphi) = 0, \]
\[ a_L^\dagger \psi_L^{(2)} = 0, \quad \rightarrow \quad \left\{ \partial_r - \frac{i}{r} \partial_\varphi - \frac{\nu}{\lambda \hbar} r \right\} \psi_L^{(2)}(r, \varphi) = 0. \]  \hfill (20)

Let us first consider the first of Eqs. (20). This is a first order partial differential equation which can be easily solved by the method of separation of the variables. For the wave function we make the ansatz \(\psi_L^{(2)} = u(r)Y(\varphi)\) and find with straightforward calculations that \(u(r) = r^M e^{-\frac{\nu}{\lambda \hbar} r^2} + c_1\) and \(Y(\varphi) = e^{iM\varphi} + c_2\) with \(M, c_1\) and \(c_2\) constants. Physical boundary conditions are imposed on the wave function and single-valuedness implies that \(M\) can only take integer values \(M = 0, \pm 1, \pm 2, \cdots\), but normalisability further restrict \(M\) to non-negative values only \((M \geq 0)\). Therefore the normalised solutions of the equation \(a_L \psi_L^{(1)} = 0\) are:

\[ \psi_L^{(1)} = u_{0,M} = C r^M e^{-\frac{\nu}{\lambda \hbar} r^2} e^{iM\varphi}, \quad M = 0, 1, 2, \cdots. \]  \hfill (21)

with \(C = (\nu / (\lambda \hbar))^{\frac{M+1}{2} / (\sqrt{\pi} \Gamma(M + 1))}\).
A spinor solution of the Hamiltonian equation with the upper component \( \psi^{(1)}_{L,0} \) given by Eq. (21) can be associated with a null lower component \( \psi^{(2)}_{L,0} = 0 \) to build a (singlet like) spinor state solution of the full Hamiltonian eigenvalue equation \( H \psi = E \psi \) with energy \( E = +mc^2 \) for which again both intertwining solutions are satisfied:

\[
(\epsilon_- = 0) \quad E = +mc^2, \quad \psi_{L,0} = \begin{pmatrix} u_{0,M} \\ 0 \end{pmatrix} \quad M = 0, 1, 2, \ldots \quad \text{(solution of Eqs. (19))}.
\]

This is an acceptable (normalized) solution of our eigenvalue problem. Note that this zero mode has an infinite degeneracy with respect to the positive values of \( M \), – also note that here, and thereafter, by zero modes we mean states that are annihilated by the corresponding annihilation operators entering the problem, and singlet (doublet) are referred to spinors with only one (both) component(s) non-vanishing–.

The second of Eqs. (20) can be solved in a similar way and we do not give the details here.

It is easily found that the solution \( \psi^{(2)}_L = CR^{-M} e^{x\varphi} e^{iM\varphi} \) can be made regular in the origin by choosing negative values of \( M \) but it is always diverging at large radial distances \( (r \rightarrow \infty) \). While one would still be able to construct a spinor (singlet) solution of the hamiltonian eigenvalue equations, c.f. Eqs. (20), corresponding to \( E = -mc^2 \) or \( (\epsilon_+ = 0) \) this would be un-normalizable for any value of \( M \) and hence unphysical and is therefore discarded.

We now turn to a detailed discussion of the excited states and their energy eigen-values which are best discussed by considering the second order eigenvalue equation \( H^2 \psi_L = E^2 \psi_L \) which allows to disentangle the two components \( \psi^{(1)}_L \) and \( \psi^{(2)}_L \). Then form Eq. (7) we obtain the decoupled equations:

\[
\hat{\Pi}_- \hat{\Pi}_+ \psi^{(1)}_L = \epsilon_+ \epsilon_- \psi^{(1)}_L, \quad (23a)
\]

\[
\hat{\Pi}_+ \hat{\Pi}_- \psi^{(2)}_L = \epsilon_+ \epsilon_- \psi^{(2)}_L, \quad (23b)
\]

where \( \epsilon_+ \epsilon_- = (E^2 - m^2 c^4)/c^2 \). By using the explicit form of the creation and annihilation operators given in Eqs. (15), (16) we find:

\[
a_{L}^\dagger a_L = \frac{\hat{\Pi}_- \hat{\Pi}_+}{4\nu \lambda \hbar} = \frac{1}{2\hbar} \left[ H_{0}^{2D} - L_z - \hbar \right], \quad (24a)
\]

\[
a_L a_{L}^\dagger = \frac{\hat{\Pi}_+ \hat{\Pi}_-}{4\nu \lambda \hbar} = \frac{1}{2\hbar} \left[ H_{0}^{2D} - L_z + \hbar \right], \quad (24b)
\]

where we have indicated the angular momentum in the two dimensional commuting plane \((x,y)\) by the operator \( L_z = xp_y - yp_x \) and:

\[
H_{0}^{2D} = \frac{\lambda}{2\nu} \left( p_x^2 + p_y^2 \right) + \frac{\nu}{2\lambda} (x^2 + y^2). \quad (25)
\]

We see therefore that the second order Hamiltonians in Eqs. (23)\(^1\) can be related to \( H_{0}^{2D} \), the Hamiltonian of a well known \textit{and exactly solvable} non-relativistic system –that of a two dimensional isotropic (or circular) harmonic oscillator (of unit frequency and mass \( \nu/\lambda \))-\(\). The eigen-functions and eigenvalues of this system are well known \cite{52} and can be readily used to solve the second order Hamiltonians of Eqs. (23) since the angular momentum operator \( L_z \) commutes with \( H_{0}^{2D} \). The complete set of eigenfunctions of \( H_{0}^{2D} \) and the corresponding

\(^1\) Eqs. (24) prove indirectly the relation \([a_L, a_{L}^\dagger] = 1\).
eigenvalues are conveniently derived using the polar coordinates of the \((x, y)\) plane and they are identified by a radial quantum number \(n_r = 0, 1, 2, \cdots\) and the angular momentum quantum number \(M = 0, \pm 1, \pm 2, \cdots\) \([52]\):

\[
H_{\text{2D}}^{\text{M}} u_{n_r, M}(r, \varphi) = \varepsilon u_{n_r, M}(r, \varphi),
\]

(26)

whose eigenstates and eigenvalues are given by:

\[
\varepsilon = \hbar \left( |M| + 1 + 2n_r \right),
\]

(27a)

\[
u_{n_r, M}(r, \varphi) = C_{n_r, M} r^{|M|} \, e^{-\frac{\pi}{\hbar} r^2} \, \frac{1}{\sqrt{\Gamma(|M| + 1 + 2n_r)}} \, _1 F_1 (-n_r, |M| + 1; \frac{\nu}{\hbar} r^2) \, e^{i M \varphi},
\]

(27b)

where \(C_{n_r, M}\) are normalisation constants that can be easily computed as:

\[
C_{n_r, M} = \frac{1}{\sqrt{\pi}} \left( \frac{\nu}{\hbar} \right)^{|M|+1} \frac{\sqrt{\Gamma(|M| + 1 + 2n_r)}}{\Gamma(|M| + 1) \sqrt{\Gamma(n_r + 1)}}.
\]

(28)

The relativistic eigenvalues of the second order equations are then easily found. The solution of Eq. (23a) is obtained with the help of Eq. (24a) and is found to be:

\[
\psi^{(1)}_L(r, \varphi) = C_1 \, \nu_{n_r, M}(r, \varphi),
\]

(29)

\[
\epsilon_+, \epsilon_- = 4 \nu \lambda \hbar \left( \frac{1}{2 \hbar} \left[ \hbar \left( |M| + 1 + 2n_r \right) - \hbar M - \hbar \right] \right),
\]

(30)

where \(C_1\) is a spinor normalisation constant. From the above equation the excited states eigenvalues \((n_r \geq 1)\) of the original Dirac equation can be extracted:

\[
E^\pm_N = \pm mc^2 \sqrt{1 + \frac{\lambda \nu \hbar}{m^2 c^4}} \, N = \pm mc^2 \sqrt{1 + \zeta_L} \, N, \quad N = n_r + \frac{|M| - M}{2}, \quad N = 1, 2, \ldots,
\]

(31)

and the quantity \(\zeta_L\) is defined as:

\[
\zeta_L = \zeta_L(\vec{\omega} - \vec{\omega}_r; \omega, \omega_\theta) = \frac{4 \lambda \nu \hbar}{m^2 c^2} \left( 1 - \frac{\vec{\omega} - \vec{\omega}}{\omega_\theta} \right) (\vec{\omega} - \omega - \omega_\theta).
\]

(32)

We see that every energy level \(E_N\) is highly degenerate. In particular every level an infinite degeneracy with respect to the non-negative values of \(M, (M \geq 0)\), while there is a finite degeneracy \(D = N + 1\) with respect to the negative values of \(M\).

The lower component \(\psi^{(2)}_L\) of the eigen-solution of the Dirac Hamiltonian is found by using the intertwining relation in Eq. (19):

\[
\psi^{(2)}_L(r, \varphi) = -2i \sqrt{\frac{\lambda \nu \hbar}{\epsilon_+}} a_L C_1 \, \nu_{n_r, M}(r, \varphi),
\]

(33)

\[
= -i \frac{C_1}{\epsilon_+} e^{i \varphi} \left\{ \lambda \hbar \left( \partial_r + \frac{i}{r} \partial_\varphi \right) + \nu \right\} \, \nu_{n_r, M}(r, \varphi).
\]

We find that the sign of the angular quantum number \((M)\) identifies two classes of eigen-states which we discuss separately.

\(M \geq 0\) (infinite degeneracy):

In this case using the explicit expression of the wave functions \(\nu_{n_r, M}\) and with the aid of the recurrence relation \([53]\):

\[
\frac{d}{dz} \, _1 F_1 (a, b; z) = \frac{a}{b} \, _1 F_1 (a + 1, b + 1; z),
\]

(34)
and after taking into account that for $M \geq 0$ the normalisation constants in Eq. (28) satisfy $C_{n_r,M} = C_{n_r-1,M+1} \sqrt{\lambda h/\nu} (M + 1)/\sqrt{\nu}$ we find explicitly:

$$\psi_L^{(2)} = +i C_1 \sqrt{4\nu \lambda m_n \epsilon_+} u_{n_r-1,M+1} = +i C_1 \sqrt{\epsilon_- \epsilon_+} u_{n_r-1,M+1}. \quad (35)$$

Finally the normalised spinor solution for positive values of $M$ can be put in the form:

$$D = \infty, \quad \psi_L^{(\pm,n_r,M)} = \frac{1}{\sqrt{2}} \left( \begin{pmatrix} \sqrt{1 + \frac{mc^2}{E_N}} u_{n_r,M} \\ \pm i \sqrt{1 - \frac{mc^2}{E_N}} u_{n_r-1,M+1} \end{pmatrix} \right), \quad N = n_r = 1, 2, \ldots, \quad (36)$$

where the upper (lower) sign corresponds respectively to the positive (negative) branch of the spectrum, and the normalisation condition is, here and thereafter, the usual one for two component spinors, i.e. $\langle \psi | \psi \rangle = \langle \psi_1 | \psi_1 \rangle + \langle \psi_2 | \psi_2 \rangle = 1$.

**$M < 0$ (finite degeneracy):**

In this case when solving for the lower limit of the spinor solution in Eq. (33) one has to use a different recurrence relation satisfied by the confluent hypergeometric function [53]:

$$z \frac{d}{dz} \phantom{\psi} F_1(a, b; z) = (b - 1) \left[ \phantom{\psi} F_1(a, b - 1; z) - \phantom{\psi} F_1(a, b; z) \right], \quad (37)$$

and after taking into account that for $M < 0$ the normalisation constants in Eq. (28) satisfy $C_{n_r,M} = -C_{n_r,M+1} \sqrt{\nu (n_r - M)}/\sqrt{\lambda h}/M$ we find explicitly:

$$\psi_L^{(2)} = -i C_1 \sqrt{4\nu \lambda h (n_r - M)} \epsilon_+ u_{n_r,M+1} = -i C_1 \sqrt{\epsilon_- \epsilon_+} u_{n_r,M+1}, \quad (38)$$

and the final expression of the spinor solution for the $N$-th energy level in the case of negative values of $M$ is:

$$D = N + 1, \quad \psi_L^{(\pm,n_r,M)} = \frac{1}{\sqrt{2}} \left( \begin{pmatrix} \sqrt{1 + \frac{mc^2}{E_N}} u_{n_r,M} \\ \mp i \sqrt{1 - \frac{mc^2}{E_N}} u_{n_r,M+1} \end{pmatrix} \right), \quad N = n_r - M = 1, 2, \ldots, \quad (39)$$

where again the upper (lower) sign correspond to the positive (negative) branch of the spectrum.

Also we note the energy gap is dependent on the commutative parameters $\theta, \eta$.

4. **Intermediate Magnetic Field: $B_{cr} < B < B^*_{cr}$ ($\omega_N < \tilde{\omega} - \omega < \omega_\theta$)**

In this case we find from Eq. (9):

$$\Pi_\pm = \lambda (p_x \pm i p_y) \pm i \mu (x \pm i y), \quad (40)$$

$$\lambda = 1 - \frac{\tilde{\omega} - \omega}{\omega_\theta} > 0, \quad \mu = m (\tilde{\omega} - \omega - \omega_\eta) > 0. \quad (41)$$

The Dirac Hamiltonian is in this case expressed solely in terms of the right handed creation and annihilation operators so in this region of parameters the system is said to be in the right chiral phase.

As regards the zero modes in the right phase we find that the normalisable solution is given in terms of the eigenfunction $\tilde{u}_0,M = C r^{-M} e^{-\frac{\pi r}{\sqrt{\nu}}} + C r^{-M} e^{\frac{\pi r}{\sqrt{\nu}}}$ and now it is infinitely degenerate with
respect to the non-positive values of \( M \) (\( M \leq 0 \)), and is a spin down singlet with \( E = -mc^2 \) (negative branch of the spectrum):

\[
\epsilon_+ = 0 \quad E = -mc^2, \quad \psi_{R,0} = \begin{pmatrix} 0 \\ \tilde{u}_{0,M} \end{pmatrix}, \quad M = 0, -1, -2 \ldots , \quad \text{(solution of Eqs. (??))}
\]

which is normalized to unity if the constant \( C \) is chosen as:

\[
C = \frac{(\mu/\lambda \hbar)^{|M|+1}}{\sqrt{\pi} \Gamma(|M| + 1)}.
\]

The excited levels (\( n_r \geq 1 \)) are then derived from the second order equations c.f. Eq (23) similarly to what has been done in the left chiral phase. We find it convenient to solve the lower component \( \psi_R^{(2)} \) from Eq. (23b) and then compute the upper component \( \psi_R^{(1)} \) with the first of the intertwining relations. We find:

\[
E_\pm^N = \pm mc^2 \sqrt{1 + \frac{\lambda \mu \hbar}{m^2 c^2} N} = \pm mc^2 \sqrt{1 + \zeta_R N} \quad N = n_r + \frac{M + |M|}{2} \quad N = 1, 2 \ldots
\]

and the quantity \( \zeta_R \) is defined as:

\[
\zeta_R = \zeta_R(\bar{\omega} - \omega; \omega_\eta, \omega_\theta) = 4 \frac{\mu \lambda \hbar}{m^2 c^2} = 4 \frac{\hbar}{mc^2} \left( 1 - \frac{\bar{\omega} - \omega}{\omega_\theta} \right) (\bar{\omega} - \omega - \omega_\eta).
\]

We see again that every energy level \( E_N \) is highly degenerate and, as expected, there is an infinite degeneracy with respect to the non-positive values of \( M \) (\( M \leq 0 \)), while the degeneracy is finite, \( D = N + 1 \), with respect to the positive values of \( M \). The corresponding eigen-solutions are:

\( M > 0 \) (finite degeneracy):

In this case using the explicit expression of the wave functions \( \tilde{u}_{nr,M} \) we find the normalised spinor solution for positive values of \( M \) can be put in the form:

\[
D = N + 1, \quad \psi_R^{(\pm,n_r,M)} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mp i \sqrt{1 + \frac{mc^2}{E_N}} \tilde{u}_{nr,M-1} \\ \sqrt{1 - \frac{mc^2}{E_N}} \tilde{u}_{nr,M} \end{pmatrix}, \quad N = n_r + M = 1, 2, \ldots
\]

where the upper (lower) sign corresponds respectively to the positive (negative) branch of the spectrum.

\( M \leq 0 \) (infinite degeneracy):

\[
D = \infty, \quad \psi_R^{(\pm,n_r,M)} = \frac{1}{\sqrt{2}} \begin{pmatrix} \pm i \sqrt{1 + \frac{mc^2}{E_N}} \tilde{u}_{n_r-1,M-1} \\ \sqrt{1 - \frac{mc^2}{E_N}} \tilde{u}_{n_r,M} \end{pmatrix}, \quad N = n_r = 1, 2, \ldots
\]

Thus for \( B_{cr} < B < B_{cr}^* \), the \( E = -mc^2 \) state is a singlet while for the other energy values the positive and negative energy states are paired in doublets. It may also be noticed that the energy gap is not \( 2mc^2 \) but depends on the magnetic field intensity \( B \) as well as on the non-commutativity parameters \( \eta \) and \( \theta \).
5. Strong Magnetic Field: $B > B_{ex}^*$ ($\tilde{\omega} - \omega > \omega_\theta$).

We see that relative to case 2 (see Eq. (40)) now the quantity $\lambda = 1 - (\tilde{\omega} - \omega)/\omega_\theta$ becomes negative and therefore the operators $\hat{\Pi}_\pm$ are given by

$$
\hat{\Pi}_\pm = -[\hat{p}_x \pm i \hat{p}_y] \mp i \mu(x \pm i y) ,
$$

where:

$$
\delta = \frac{\tilde{\omega} - \omega}{\omega_\theta} - 1 > 0 ,
$$

and $\mu$ is given as in Eq. 41 and we see that $\Pi_\pm$ in Eq. (47) differ by an overall negative sign with the operators in Eq. (13), of case 1 (the region with a weak magnetic field), the so called left chiral phase– with the substitutions: $\lambda \rightarrow \delta, \nu \rightarrow \mu$. We expect therefore this region of very strong magnetic field ($\tilde{\omega} - \omega > \omega_\theta$) to be described as left chiral phase.

We examine first whether there are solutions with energies $E = \pm mc^2$ i.e., $\epsilon_\pm = 0$, and find now that the normalisable solution is given in terms of the eigenfunction $u'_{0,M} = C^r r^M e^{-\pi \xi r^2}$, infinitely degenerate with respect to the non-negative values of $M$, $(M \geq 0)$, and is a spin-up singlet with $E = \pm mc^2$ (positive branch of the spectrum):

$$
(\epsilon_- = 0) \quad E = \pm mc^2, \quad \psi_{R, 0} = \left( \begin{array}{c} u'_{0, M} \\ 0 \end{array} \right) , \quad M = 0, 1, 2, \cdots , \quad \text{(solution of Eqs. (??))}
$$

which is normalized to unity if the constant $C^r$ is chosen as:

$$
C^r = (\mu/(\delta \hbar)) \frac{\xi_{\theta/2}}{(\pi \Gamma(M + 1))}.
$$

The excited levels ($n_r \geq 1$) are then derived from the second order equations c.f. Eq (23) in exactly the same way as done in the left chiral phase (case 1).

The spectrum and the eigenfunctions are then:

$$
E^\pm_N = \pm mc^2 \sqrt{1 + \frac{4 \delta \mu \hbar}{m^2 c^2} N} = \pm mc^2 \sqrt{1 + \zeta'_L N} \quad N = n_r + \left| \frac{M}{2} \right| N = 1, 2, \cdots
$$

and the quantity $\zeta'_L$ is defined by:

$$
\zeta'_L = \zeta'_L(\tilde{\omega} - \omega; \omega_\eta, \omega_\theta) = \frac{4 \delta \mu \hbar}{m^2 c^2} = \frac{\hbar}{mc^2} \left( \frac{\tilde{\omega} - \omega}{\omega_\theta} - 1 \right) (\tilde{\omega} - \omega - \omega_\eta).
$$

$M \geq 0$ (infinite degeneracy):

The normalised spinor solution for positive values of $M$ can be put in the form:

$$
D = \infty, \quad \psi_L^{(+, n_r, M)} = \frac{1}{\sqrt{2}} \left( \begin{array}{c} \sqrt{1 + \frac{mc^2}{E_N}} u_{n_r, M} \\ \pm i \sqrt{1 - \frac{mc^2}{E_N}} u_{n_r - 1, M + 1} \end{array} \right) , \quad N = n_r = 1, 2, \cdots ,
$$

where the upper (lower) sign corresponds respectively to the positive (negative) branch of the spectrum.

$M < 0$ (finite degeneracy):

The final expression of the spinor solution for the $N$-th energy level in the case of negative values of $M$ is:

$$
D = N + 1, \quad \psi_L^{(\pm, n_r, M)} = \frac{1}{\sqrt{2}} \left( \begin{array}{c} \sqrt{1 + \frac{mc^2}{E_N}} u_{n_r, M} \\ \pm i \sqrt{1 - \frac{mc^2}{E_N}} u_{n_r, M + 1} \end{array} \right) , \quad N = n_r - M = 1, 2, \cdots ,
$$
where again the upper (lower) sign correspond to the positive (negative) branch of the spectrum.

Note that the spinor structure of the eigenstates of the excited levels in this new left chiral phase is identical to that of the initial left phase except for the relative sign between the lower and upper components (in addition of course to the replacement of the parameters).

6. Discussion

In Fig. 1 we show the positive branch of the spectrum for the first few energy levels as function of the dimensionless quantity $\xi = \hbar(\tilde{\omega} - \omega)/(mc^2)$ which is directly related to the intensity of the external magnetic field through $\tilde{\omega} = eB/(2mc)$. Note that while in the two left phases the positive branch of the spectrum starts with the singlet zero mode at $E = +mc^2$ in the right phase the lowest positive energy state (the ground state) is a doublet with $E = +mc^2\sqrt{1 + \zeta_R}$. In Fig. 1 one clearly makes out the two points of non-analyticity, $\xi = \hbar\omega_\eta/(mc^2)$ and $\xi = \hbar\omega_\theta/(mc^2)$, which signal the two quantum phase transitions. The same behaviour is of course shown by all the excited states. In the commutative limit ($\eta, \theta \to 0$) $\omega_\eta \to 0$ and $\omega_\theta \to \infty$ and therefore we obtain that while the first phase transition (left-right) is shifted towards the commutative critical value $\xi = 0$ ($\tilde{\omega} = \omega$), the second phase transition (right-left) disappears since it is moved to $\xi = \infty$ ($\tilde{\omega} = \infty$).

We now would like to discuss a physical observable, closely related to the spectrum, the magnetisation, which has the potential of characterising the quantum phase transitions of the system. Indeed for every energy level $E_N^{(i)}$, $i = L, R, L'$, we can define the magnetisation $M_N^{(i)}$ by:

$$M_N^{(i)} = -\frac{\partial E_N^{(i)}}{\partial B}.$$  

In each of the quantum phases that we have discussed previously the energy eigenvalues can be written as $E_N^{(i)} = mc^2\sqrt{1 + \zeta(i)}N$ with the functions $\zeta(i)$ as in Eqs. (31,44,51) and the

\begin{figure}[h]
\centering
\includegraphics[width=0.7\textwidth]{figure1.png}
\caption{Energy eigenvalues of the positive branch $E_N^{(i)}$, in units of $mc^2$, as a function of the dimensionless ratio $\xi = \hbar(\tilde{\omega} - \omega)/(mc^2)$. Shown are the ground state, solid line (blue) and of the first few excited states, dashed (red), dotted (light green) and dot-dashed (green). There are two values of the magnetic field (or $\tilde{\omega}$) for which there is a quantum phase transition (a point of non analyticity in the energy eigenvalues). In the commutative limit ($\eta, \theta \to 0$) $\omega_\eta \to 0$ and $\omega_\theta \to \infty$ and the spectrum goes smoothly into the one discussed in [39] with only one phase transition at $\xi = 0$.}
\end{figure}
The magnetisation of the energy levels of the positive branch of the spectrum \( E_N^+ \) in units of the Bohr magneton \( \mu_B = e \hbar / 2m_e c \) (and assuming \( m_e/m = 1 \)), as a function of the dimensionless ratio \( \xi = \hbar (\tilde{\omega} - \omega) / (mc^2) \). Shown are the ground state, solid line (blue) and first few excited energy levels, dashed (red), dotted (light green) and dot-dashed (green). At the two critical points (\( \xi = \hbar \omega / mc^2 \) and \( \xi = \hbar \omega / mc^2 \)) the magnetisation presents a discontinuity for each energy level.

The magnetisation is then easily found to be given by:

\[
M_N^{(i)} = - \frac{mc^2}{2\sqrt{1 + N\xi(i)}} N \left( \frac{\partial \xi(i)}{\partial B} \right) = -2 \mu_B \frac{m_e}{m} \frac{N}{\sqrt{1 + N\xi(i)}} \left( 1 + \frac{\omega_\eta}{\omega_\theta} - 2 \frac{\tilde{\omega} - \omega}{\omega_\theta} \right). \tag{55}
\]

where \( m_e \) is the electron mass and \( \mu_B = e \hbar / 2m_e c \) is the Bohr magneton.

In Fig. 2 we show the magnetisation of our system assuming \( m_e = m \). We find that for every level the magnetisation has a finite discontinuity at the two critical points. We point out a very interesting feature which is closely tied to the presence of non-commutativity. We clearly see that in the right chiral phase there is a fixed point of vanishing magnetisation (\( M_N^{(R)} = 0 \)) at \( \tilde{\omega} - \omega = (\omega_\eta + \omega_\theta) / 2 \), exactly the midpoint of the two critical points. Clearly this will remain so even when computing a thermal average of the magnetisation at a given temperature. Since \( M_N^{(R)} = 0 \) for every level its average value, at any given temperature, will also vanish. So this can be taken as a prediction of our model for the system under consideration: in the presence of non-commutativity there is a fixed point with vanishing magnetisation in the right chiral phase.

7. Conclusions

Within a non-commutativity scenario both in the space and momentum coordinates we have studied the (2+1) dimensional Dirac oscillator in a constant homogenous magnetic field. The one dimensional version of the Dirac oscillator has been recently experimentally realised and observed in the laboratory [31] with realistic prospects of realising soon a 2-dimensional version. Here we have solved exactly the eigenvalue equations according to the strength of the magnetic field. The system in the absence of non-commutativity is known to have a left-right chiral phase transition for \( B = B_{ct} = \frac{2mc}{e} \omega \). We find that the non-commutativity of the momentum coordinates (\( \eta \)–parameter, see Eq. (3)) simply shifts the left-right chiral phase transition. The presence of non-commutativity in the space coordinates (\( \theta \)–parameter, see Eq. (3)) changes the picture quite dramatically, because it introduces a new right-left phase transition at \( B = B_{cr}^* = \frac{2mc}{e} (\omega + \omega_\theta) \) which is absent in the commutative limit (as \( \theta \to 0, \omega_\theta \to \infty \)). Therefore the system is shown to exhibit the interesting phenomenon of reentrant phase transitions.

We have also discussed the magnetisation of the energy levels of the system and we have found that this physical observable has the potential of being able to characterise the two quantum
phase transitions since it offers a finite discontinuity (for every energy level) at the two critical points. A rather peculiar prediction for this observable is the existence of a fixed point with vanishing magnetisation, for every energy level, exactly located at the midpoint of the two critical points. Work is in progress to present a full thermodynamic treatment of the system based on the solution presented here.

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