HÖLDER ESTIMATES FOR HOMOTOPY OPERATORS ON STRICTLY PSEUDOCONVEX DOMAINS WITH $C^2$ BOUNDARY

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Abstract. We derive a new homotopy formula for a strictly pseudoconvex domain of $C^2$ boundary in $\mathbb{C}^n$ by using a method of Lieb and Range and obtain estimates in Lipschitz spaces for the homotopy operators. For $r > 1$ and $q > 0$, we obtain a $\Lambda_{r+1/2}$ solution $u$ to $\bar{\partial} u = f$ for a $\bar{\partial}$-closed $(0,q)$-form $f$ of class $\Lambda_r$ in the domain. We apply the estimates to obtain boundary regularities of $\mathcal{D}$-solutions for a domain in $\mathbb{C}^n \times \mathbb{R}^m$.

1. Introduction

The main purpose of this paper is to show the boundary regularity for $\bar{\partial}$ solutions in a strictly pseudoconvex domain $D$ in $\mathbb{C}^n$ under the minimal smoothness condition of the boundary $\partial D \in C^2$. We will also derive a homotopy formula for the domain $D$,

$$\varphi = \bar{\partial} H_q \varphi + H_{q+1} \partial \varphi, \quad q \geq 1$$

that admits a derivative estimate. Here $\varphi$ is a $(0,q)$-form in $\mathcal{D}$ and $\varphi, \bar{\partial} \varphi$ are in $C^1(\mathcal{D})$. We will prove the following $C^{r+1/2}$ estimate.

**Theorem 1.1.** Let $r \in [1, \infty)$ and $1 \leq q \leq n$. Let $D$ be a bounded strictly pseudoconvex domain of $C^2$ boundary in $\mathbb{C}^n$. If $r + 1/2$ is non integer, then

$$|H_q \varphi|_{C^{r+1/2}(\mathcal{D})} \leq C_r(D)|\varphi|_{C^r(\mathcal{D})},$$

where $C_r(D) < \infty$ depends only on $r$ and the domain $D$.

The study of regularities of $\bar{\partial}$ solutions via integral representations has a long history. The sup-norm estimate of $\bar{\partial}$ solutions was proved by Grauert-Lieb [25] and Henkin [28] for $(0,1)$-forms (the forms are thus $\bar{\partial}$-closed). Kerzman [31] obtained $L^p$ and $C^\beta$ estimates of $\bar{\partial}$ solutions for $(0,1)$-forms and all $\beta < 1/2$, and Øvrelid [46] obtained a homotopy formula with homotopy operators admitting $L^p$ estimates for all $(0,q)$-forms. Lieb [35] obtained the $L^\infty$ and the $C^\beta$ estimates of $\bar{\partial}$ solutions for $(0,0)$-forms. Finally, Henkin and Romanov [30] achieved the $C^{1/2}$ estimate of $\bar{\partial}$ solutions for continuous $(0,1)$-forms. The two $\bar{\partial}$-solution operators in [25,28] make essential uses of the Henkin-Ramírez functions constructed independently by Henkin [28] and Ramírez [50]. On the other hand, Stein showed that the $C^{1/2}$ estimate is optimal for $\bar{\partial}$-closed continuous $(0,1)$-forms in the unit ball of $\mathbb{C}^n$ for $n > 1$ (see [29,31]).

Note that Treves [58] studied the boundary regularity for the Leray-Koppelman homotopy operator. Noticeably the $C^{1/2}$ estimate, valid for all continuous $(0,q)$-forms that are not necessarily $\bar{\partial}$-closed, was first obtained by Range-Siu [53] for a homotopy operator $T_q$. However, to the author’s best knowledge it remains open if the $\bar{\partial}$ solution operators in the above-mentioned results have a boundary regularity beyond the $C^{1/2}$ estimate when they

2010 Mathematics Subject Classification. 32A06, 32T15, 32W05.

Key words and phrases. Strongly pseudoconvex domains, homotopy formula, Lipschitz estimates.
act on the continuous forms that are not $\overline{\partial}$-closed. There are, of course, important results under the conditions that $\varphi$ is $\partial$-closed and $r$ is a positive integer $k$: Siu [55] proved the $C^{k+1/2}$ estimate for $T_1$ and Alt [3] obtained analogous results for the two $\overline{\partial}$ solution operators of Kerzman [31] and Grauert-Lieb [25] for $(0, 1)$-forms. For $\partial$-closed $(0, q)$-forms $\varphi$ with $q \geq 1$, Lieb and Range [36] constructed a $\overline{\partial}$ solution operator $H_q$ and proved (1.2) when $\partial D \subset C^{k+2}$, and in [37, 38] they also showed that Kohn’s canonical solution $u$ to $\overline{\partial}u = \varphi$ is in $C^{k+1/2}(\overline{D})$ when $\varphi \in C^k(\overline{D})$ and $\partial D \subset C^\infty$. The above-mentioned results are for strictly pseudoconvex domains. Range [52] obtained a Hölder estimate for $\overline{\partial}$ solutions in finite type pseudoconvex domains of $C^2$. There are derivative estimates for $\overline{\partial}$ solutions in convex domains of D’Angelo finite type $M$. We mention that Alexandre [2] obtained the Hölder estimate for $\overline{\partial}$ solutions that are not $\overline{\partial}$-closed. Here are some related results. An interior estimate of gaining one derivative for $\varphi \in C^r$ with a non-integer $r$ was obtained by Webster [60]. We mention that Alexandre [2] obtained the $C^{1/m}$ estimate for a homotopy operator on convex finite-type domains. For the $\overline{\partial}$ operator in a domain in a strictly pseudoconvex hypersurface $M$ in $C^n$ with $n \geq 4$, the interior $C^k$ estimate was obtained by Webster [61] and Ma-Michel [39] proved a boundary regularity for homotopy operators. Gong-Webster [24] obtained an interior $C^{k+1/2}$ estimate for Henkin homotopy operators when the $M$ is in $C^{k+2}$. Range and Siu [53] proved the $C^\beta$ estimate for all $\beta < 1/2$ for $\overline{\partial}$ solutions of continuous $(0, q)$-forms on the transversal intersection of strictly pseudoconvex domains; see also Poljakov [49] for related results. It is open if the $C^{1/2}$ estimate holds for continuous forms in this situation. Higher order derivative estimates for $\overline{\partial}$ solutions were obtained by Brinkmann [4], Michel [40], and Michel-Perotti [42] for the intersection. Peters [47] constructed a new homotopy operator for the weakly transversal intersection of strictly pseudoconvex domains and obtained higher order derivative estimates with some loss of derivatives. Note that all these results require the boundary of domains to be sufficiently smooth. For weakly pseudoconvex domains with $C^\infty$ boundary, Michel [41] and Michel-Shaw [43] respectively constructed homotopy operators with the $C^\infty$ regularity for the domains and for their transversal intersection.

Theorem 1.1 does not require that $\varphi$ is $\partial$-closed. We will derive a homotopy operator for a strictly pseudoconvex domain $D$ with $C^2$ boundary, by perfecting the formulation of the Lieb-Range $\overline{\partial}$ solution operator. The homotopy operator has the form

\begin{equation}
H_q \varphi(z) = \int_{C^n} \Omega^0_{0,q-1}(z, \zeta) \wedge E\varphi(\zeta) + \int_{C^n \setminus D} \Omega^0_{0,q-1}(z, \zeta) \wedge [\overline{\partial}, E]\varphi(\zeta)
\end{equation}

for $z \in D$ and $q > 0$. Here $E : C(\overline{D}) \to C_0(\mathbb{C}^n)$ is a linear extension operator constructed by Calderón [6] and Stein [57], and it satisfies two important properties

$$|Ef|_{C^n, r} \leq C_r |f|_{\overline{\partial}, r}, \quad |Ef|_{\Lambda_r(\mathbb{C}^n)} \leq C_r |f|_{\Lambda_r(\overline{D})}.$$ 

Here $C^r(\overline{D})$ with norm $|\cdot|_{\overline{\partial}, r}$ is the Hölder space; the $\Lambda_r(\overline{D})$ with norm $|\cdot|_{\Lambda_r(\overline{D})}$ is the Lipschitz space (see Definition 3.9). We mention two main features in $H_q$: the first is a regularized Henkin-Ramírez map, introduced in this paper, for a strictly pseudoconvex domain with $C^2$ boundary, and the second is the commutator $[\overline{\partial}, E]$, defined by $[\overline{\partial}, E]\varphi = \overline{\partial}E\varphi - E\overline{\partial}\varphi$. The
commutator has an important property:
\[
[\bar{\partial}, E] f = 0, \quad \text{in } \overline{D}.
\]
Combining with the property \([\bar{\partial}, E]: \Lambda_r(\overline{D}) \to \Lambda_{r-1}(\mathbb{C}^n)\), the commutator is a smooth cut-off operator losing one derivative. We mention closely related previous work. Lieb-Range [36] first introduced the Seeley extension operator for their \(\bar{\partial}\) solution operators in strictly pseudoconvex domains and the extension has been a basic technique in other situations. Ma-Michel [39] used it for a suitable domain in a strictly pseudoconvex real hypersurface in \(\mathbb{C}^n\) for \(n \geq 4\) and Alexandre [2] used it for finite type convex domains. If \(\varphi\) is \(\partial\)-closed, we obtain \([\bar{\partial}, E] \varphi = \bar{\partial} E \varphi\) and the \(H_q\) is an analogue of the Lieb-Range \(\bar{\partial}\) solution operator. We should mention that the important commutator \([\bar{\partial}, E]\) was introduced by Peters [47] and it has been used by Michel [41], Michel-Shaw [43], and others.

A detailed version of Theorem 1.1 (Theorem 5.2) yields the following.

**Corollary 1.2.** Let \(r > 1\) and \(0 < q < n\). Let \(D\) be a bounded strictly pseudoconvex domain of \(C^2\) boundary in \(\mathbb{C}^n\). Let \(\varphi \in \Lambda_r(\overline{D})\) be a \(\partial\)-closed \((0,q)\)-form in \(D\). Then there is a solution \(u \in \Lambda_{r+1/2}(\overline{D})\) to \(\partial u = \varphi\) in \(D\).

Our \(C^{k+1/2}\) estimate improves the regularity results of Siu [55] (for \(q = 1\)) and Lieb-Range [36] (for all \(q\)) for the case when \(k\) is an integer bigger than 1 and \(\partial D \in C^{k+2}\). When \(\partial D \in C^\infty\) additionally, the improvement for all \(r > 0\) was obtained by Greiner-Stein [26, Thm. 16.7(c), p. 174] and Phong-Stein [48] (for \(q = 1\)), and by Chang [8, Thm. 4.10 (iii)] with \(U = D, q \geq 1\). The case \(q = n\), which is not included in the corollary, is simple: We need the domain to be Lipschitz, but not necessarily pseudoconvex, while solutions gain a full derivative; see Proposition 3.13 for details.

We will also obtain a Bochner-Martinelli-Leray-Koppelman formula: If \(f\) is a \(C^1\) function in \(\overline{D}\) with \(\overline{\partial} f \in C^1(\overline{D})\), then
\[
f = H_0 f + H_1 \overline{\partial} f,
\]
where \(D\) is strictly pseudoconvex with \(C^2\) boundary and
\[
H_0 f = \int_{\mathbb{C}^n \times D} \Omega^1_{0,0} \wedge [\overline{\partial}, E]\ f.
\]
Here \(\Omega^1_{0,0}\) is a Cauchy-Fantappiè form of the above-mentioned regularized Henkin-Ramirez function. In connection with previous work, \(H_0 f\) is a holomorphic projection analogous to \(\widetilde{H}_0 f = \int_{\partial D} \Omega^1_{0,0} f\). We will show in Theorem 5.2 that when \(r > 1\) the holomorphic projection \(H_0\) maps \(\Lambda_r(\overline{D})\) continuously into itself. For \(\widetilde{H}_0\), Elgueta [17] obtained a similar estimate with a minor loss of regularity and Ahern-Schneider [1] obtained a sharp estimate that actually holds for all \(r > 0\). See also Phong-Stein [48] for the regularity of Bergman and Szegő projections for strictly pseudoconvex domains with \(C^\infty\) boundary.

As mentioned earlier, one of our main results is a homotopy formula in (1.1) and (1.3), which admits Hölder estimates in \(\overline{D}\). Using \(H_q\), we will study the elliptic differential
\[
D := \overline{\partial} z + dt
\]
in \((z, t) \in \mathbb{C}^n \times \mathbb{R}^m\), introduced by Treves [58]. Let \(D \times S\) be a product domain in \(\mathbb{C}^n \times \mathbb{R}^m\).

A \(k\)-form \(\varphi\) in \(D \times S\) is said of mixed type \((0, k)\) if
\[
\varphi(z, t) = [\varphi]_0(z, t) + \cdots + [\varphi]_k(z, t)
\]
where \([\varphi]\) has type \((0,i)\) in \(z\) and degree \(k-i\) in \(t\). By a \(\mathcal{D}\)-closed form \(\varphi\), we mean \(\mathcal{D}\varphi = 0\). We have the following.

**Theorem 1.3.** Let \(1 \leq r \leq \infty\). Let \(D\) be a bounded strictly pseudoconvex domain with \(C^2\) boundary and let \(S\) be a bounded star-shaped domain in \(\mathbb{R}^m\). Let \(\varphi\) be a \(\mathcal{D}\)-closed form of mixed type \((0,k)\) in \(D \times S\) with \(k \geq 1\). If \(\varphi \in C^r(\overline{D} \times \overline{S})\), there is a solution \(u \in C^r(\overline{D} \times \overline{S})\) satisfying \(\mathcal{D}u = \varphi\).

Hanges and Jacobowitz [27] proved the interior \(C^\infty\) regularity of a \(\mathcal{D}\)-solution on a smooth domain \(\Omega\) in \(\mathbb{C}^n \times \mathbb{R}^m\) under a strictly Levi convex condition.

We further mention some important results concerning \(\overline{\partial}\) or \(\overline{\partial}_b\) solutions. The \(C^\infty\) regularity results of \(\overline{\partial}\) solutions were achieved by Kohn [33] for smoothly bounded strictly pseudoconvex domains, by Kohn [33] for \(n = 2\) and Catlin [7] for pseudoconvex domains of finite D’Angelo type [14], and by Kohn for smoothly bounded pseudoconvex domains [34]. McNeal [44] obtained exact subelliptic estimates for finite type convex domains. The regularity of \(\mathcal{D}\) and \(\partial_b\) solutions for \((0,1)\)-forms was obtained by Chang-Nagel-Stein [9], Fefferman-Kohn [18], and Christ [12] for finite type pseudoconvex domains in \(\mathbb{C}^2\), and by Fefferman-Kohn-Machedon [19] for finite type domains in \(\mathbb{C}^n\) with diagonalizable Levi-form. Note that Shaw [54] obtained the exact \(C^{1/m}\) estimate of \(\overline{\partial}_b\) solutions for \((0,1)\)-forms in the boundary of an ellipsoid of finite type. For \((0,q)\)-forms, the Hölder estimates for \(\partial_b\) solutions were finally achieved by Koenig [32] for finite type CR manifolds with comparable eigenvalues in the Levi form.

We now state two questions.

**Question 1.** Let \(0 < q < \infty\). Let \(D\) be a bounded strictly pseudoconvex domain with \(C^2\) boundary in \(\mathbb{C}^n\). Let \(\varphi\) be a \(\overline{\partial}\)-closed \((0,q)\)-form in \(D\). If \(\varphi \in \Lambda_r(\overline{D})\) and \(0 < r \leq 1\), does there exist \(u \in \Lambda_{r+1/2}(\overline{D})\) satisfying \(\overline{\partial}u = \varphi\) in \(D\)?

As mentioned early, when \(\partial D \in C^\infty\), positive results for the above question via Kohn’s solution are in [8,26,48] for all \(r > 0\). Corollary 1.2 gives a positive answer when \(r > 1\). The result of Kohn [34] and Corollary 1.2 give rise to the following question.

**Question 2.** Let \(0 < q < \infty\). Let \(D\) be a bounded pseudoconvex domain in \(\mathbb{C}^n\) with \(C^2\) boundary. Let \(\varphi\) be a \(\overline{\partial}\)-closed \((0,q)\)-form in \(D\). If \(\varphi \in C^\infty(\overline{D})\), does there exist \(u \in C^\infty(\overline{D})\) satisfying \(\overline{\partial}u = \varphi\) in \(D\)?

Finally, we should mention that Chaumat and Chollet [10] obtained a \(\overline{\partial}\) solution with a loss of \(n-q-1\) derivatives, when \(D\) is a convex domain of \(C^2\) boundary and \(r \in \mathbb{N}\). They also obtained other results. Michel-Shaw [43] also showed that when \(D\) is an annulus domain \(\Omega_1 \setminus \overline{\Omega_2}\), where \(\Omega_1\) is a bounded strictly pseudoconvex domain with \(C^\infty\) boundary, \(\Omega_2\) is a pseudoconvex domain which is relatively compact in \(\Omega_1\) and has \(C^2\) boundary, there exists a solution \(u \in C^\infty(\overline{D})\) to \(\overline{\partial}u = f\), if \(f\) is a \(\overline{\partial}\)-closed \((0,q)\)-form in \(C^\infty(\overline{D})\) and \(0 < q < n-1\).

The paper is organized as follows. In section 2 we derive the homotopy formula. In section 3 we recall the Whitney and Stein extension operators from [57] and use them to obtain regularized defining functions for domains with \(C^2\) boundary and describe equivalent norms of \(\Lambda_r(\overline{D})\). Section 4 contains the main estimation of this paper, assuming the existence of regularized Henkin-Ramírez functions. The latter are derived in section 5 for which we
follow the classical construction of Henkin-Ramírez functions. The final section contains two homotopy formulae for the $D$-complex and the proof of Theorem 1.3.

**Acknowledgments.** The author is grateful to Andreas Seeger for helpful discussions on the real interpolation theory.

2. The homotopy formula and the commutator

In this section we derive a homotopy formula, inspired by Lieb-Range [36], Peters [47], and Michel-Shaw [43]. We derive it by keeping the minimum smoothness conditions on the domain and the forms. We will apply it to prove our main results, after the regularized Henkin-Ramírez functions are constructed in section 5.

We first recall the Leray-Koppelman homotopy formula. Let $D$ be a bounded domain with $C^1$ boundary. Let $g^1 : D \times \partial D \to \mathbb{C}^n$ be a $C^1$ mapping satisfying
\[ g^1(z, \zeta) \cdot (\zeta - z) \neq 0, \quad \forall \zeta \in \partial D, z \in D. \]

Let $g^0(z, \zeta) = \overline{\zeta - z}$ and $w = \zeta - z$. Define
\[
\omega^i = \frac{1}{2\pi i} \frac{g^i \cdot dw}{g^i \cdot w}, \quad \Omega^i = \omega^i \wedge (\overline{\partial \omega^i})^{n-1},
\]
\[
\Omega^{01} = \omega^0 \wedge \omega^1 \wedge \sum_{\alpha + \beta = n-2} (\overline{\partial \omega^0})^\alpha \wedge (\overline{\partial \omega^1})^\beta.
\]

Here both differentials $d$ and $\overline{\partial}$ are in $z, \zeta$ variables. We have
\[
\omega^i \wedge (\overline{\partial \omega^i})^\alpha = \frac{g^i \cdot dw \wedge (\overline{\partial (g^i \cdot dw)})^\alpha}{(2\pi \sqrt{-1} g^i \cdot w)^{\alpha+1}}, \quad \alpha = 1, 2, \ldots
\]

We decompose $\Omega^i = \sum \Omega^i_{0,q}$ and $\Omega^{01} = \sum \Omega^{01}_{0,q}$, where $\Omega^i_{0,q}$ (resp. $\Omega^{01}_{0,q}$) has type $(0, q)$ in $\zeta$ and type $(n, n-1-q)$ (resp. $(n, n-2-q)$) in $z$. Set $\Omega^{1}_{0,-1} = 0$ and $\Omega^{01}_{0,-1} = 0$. By the Koppelman lemma [11, p. 263], we have
\[
\overline{\partial}_z \Omega^1_{0,q} + \overline{\partial}_z \Omega^1_{0,q-1} = 0, \quad q \geq 0,
\]
\[
\overline{\partial}_z \Omega^{01}_{0,q} + \overline{\partial}_z \Omega^{01}_{0,q-1} = \Omega^{01}_{0,q} - \Omega^{1}_{0,q}, \quad q \geq 0.
\]

We need to know how the sign changes, when the exterior differential interchanges with integration. Following notations in Chen-Shaw [11, p. 263], we define
\[
\int_{y \in M} u(x, y) dy^I \wedge dx^I = \left\{ \int_{y \in M} u(x, y) dy^I \right\} dx^I
\]
for a continuous function $u$ in a manifold $M$. If $d_x$ is the exterior differential in $x$-variables, we have
\[
d_x \int_M \phi(x, y) = (-1)^{\dim M} \int_M d_x \phi(x, y).
\]

The Leray-Koppelman homotopy formula [11, p. 273] for a $(0, q)$-form $\varphi$ is given by
\[
\varphi(z) = \overline{\partial}_z T_q \varphi + T_{q+1} \overline{\partial}_z \varphi, \quad z \in D, \quad 1 \leq q \leq n,
\]
\[
\varphi(z) = \int_{\partial D} \Omega^1_{0,q} \varphi + T_1 \overline{\partial} \varphi, \quad q = 0,
\]
with
\[ T_q \varphi = - \int_{\partial D} \Omega_{0,q-1}^0 \wedge \varphi + \int_D \Omega_{0,q-1}^0 \wedge \varphi, \quad q \geq 1. \]

**Proposition 2.1.** Let \( D \subset \mathbb{C}^n \) be a domain with \( C^1 \) boundary and let \( \mathcal{U} \) be a bounded neighborhood of \( \overline{D} \). Let \( g^0(z, \zeta) = \overline{\zeta} - \zeta \). Let \( g^1(z, \zeta) = W(z, \zeta) \) where \( W \in C^1(D \times (\mathcal{U} \setminus D)) \) is a Leray mapping, that is that \( W \) is holomorphic in \( z \in D \) and satisfies
\[ \Phi(z, \zeta) := W(z, \zeta) \cdot (\zeta - z) \neq 0, \quad z \in D, \quad \zeta \in \mathcal{U} \setminus D. \]
Let \( \varphi \) be a \((0,q)\)-form in \( \overline{D} \). Suppose that \( \varphi \) and \( \overline{\partial} \varphi \) are in \( C^1(\overline{D}) \). Then in \( D \)
\[ \begin{align*}
\varphi &= \partial H_q \varphi + H_{q+1} \overline{\partial} \varphi, \quad 1 \leq q \leq n, \\
\varphi &= H_0 \varphi + H_1 \overline{\partial} \varphi, \quad q = 0,
\end{align*} \]
where
\[ H_q \varphi := \int_\mathcal{U} \Omega_{0,q-1}^0 \wedge E \varphi + \int_{\mathcal{U} \setminus D} \Omega_{0,q-1}^0 \wedge [\overline{\partial}, E] \varphi, \quad q > 0, \]
\[ H_0 \varphi := \int_{\partial D} \Omega_{0,0}^1 \varphi - \int_{\mathcal{U} \setminus D} \Omega_{0,0}^1 \wedge E \overline{\partial} \varphi = \int_{\mathcal{U} \setminus D} \Omega_{0,0}^1 \wedge [\overline{\partial}, E] \varphi. \]

**Proof.** In the formulae, the extension \( E \) constructed in [57] will be recalled in Lemma 3.11 below. The \( E \) is defined for functions. We thus define \( E \varphi \) by applying \( E \) componentwise to its coefficients, which results in a form of the same type. We always assume that \( E \varphi \) has a compact support in \( \mathcal{U} \), by using a cut-off function.

Assume that \( q \geq 1 \). Let us modify the solution operator \( T_q \) given by (2.4)-(2.6), by applying the method of Lieb-Range [36] via the linear extension \( E \). The \( \Omega_{0,1}^0 \) has total degree \( 2n - 2 \). Applying Stokes’ formula and (2.2)-(2.3), we get
\[ - \int_{\zeta \in \partial D} \Omega_{0,q-1}^0 \wedge \varphi = \int_{\zeta \in \mathcal{U} \setminus D} \Omega_{0,q-1}^0 \wedge \overline{\partial} \zeta E \varphi + \int_{\zeta \in \mathcal{U} \setminus D} \overline{\partial} \zeta \Omega_{0,q-1}^0 \wedge E \varphi \]
\[ = \int_{\mathcal{U} \setminus D} \Omega_{0,q-1}^0 \wedge \overline{\partial} E \varphi \]
\[ - \int_{\mathcal{U} \setminus D} (\overline{\partial} z \Omega_{0,q-2}^0 \wedge E \varphi + \Omega_{0,q-1}^1 \wedge E \varphi - \Omega_{0,q-1}^0 \wedge E \varphi) \]
\[ = \int_{\mathcal{U} \setminus D} \Omega_{0,q-1}^0 \wedge \overline{\partial} E \varphi - \overline{\partial} z \int_{\mathcal{U} \setminus D} \Omega_{0,q-2}^0 \wedge E \varphi \]
\[ + \int_{\mathcal{U} \setminus D} (-\Omega_{0,q-1}^1 \wedge E \varphi + \Omega_{0,q-1}^0 \wedge E \varphi). \]

Let us apply \( \overline{\partial} \) to the last 4 terms. The second of the four terms becomes zero. The third also becomes zero since it is holomorphic for \( q = 1 \) and it is zero for \( q > 1 \). Thus we obtain for \( z \in D \)
\[ - \overline{\partial} \int_{\zeta \in \partial D} \Omega_{0,q-1}^0(z, \zeta) \wedge \varphi(\zeta) + \overline{\partial} \int_{\zeta \in \partial D} \Omega_{0,q-1}^0(z, \zeta) \wedge \varphi(\zeta) \]
\[ = \overline{\partial} \int_{\mathcal{U} \setminus D} \Omega_{0,q-1}^0(z, \zeta) \wedge \overline{\partial} E \varphi(\zeta) + \overline{\partial} \int_{\mathcal{U}} \Omega_{0,q-1}^0(z, \zeta) \wedge E \varphi(\zeta). \]
So far, we have used \( \varphi \in C^1(\overline{D}) \). Assume now that \( \overline{\partial}\varphi \in C^1(\overline{D}) \). Using the last 4 terms in (2.11) in which \( \varphi \) is replaced by \( \overline{\partial}\varphi \), we obtain

\[
(2.13) \quad - \int_{\partial D} \Omega_{0,q}^{01} \wedge \overline{\partial}\varphi + \int_D \Omega_{0,q}^{0} \wedge \overline{\partial}\varphi = \int_{U \setminus D} \Omega_{0,q}^{01} \wedge \overline{\partial}E\overline{\partial}\varphi + \int_{U \setminus D} \Omega_{0,q}^{01} \wedge \overline{\partial}\varphi - \int_{U \setminus D} \Omega_{0,q}^{0} \wedge E\overline{\partial}\varphi + \int_D \Omega_{0,q}^{0} \wedge \overline{\partial}\varphi.
\]

On the right-hand side, the first term can be written via the commutator as \( \overline{\partial}E\overline{\partial}\varphi = (\overline{\partial}E - E\overline{\partial})\overline{\partial}\varphi \). Since \( q \geq 1 \), the third is zero. The second, when combined with the first term on the right-hand side of (2.12), gives us the desired commutator for \( \varphi \). Adding (2.12)-(2.13) yields (2.7).

To derive (2.10), we recall that \( \Omega_{0,-1}^{01} = 0 \) and by (2.13) we get

\[
- \int_{\partial D} \Omega_{0,0}^{01} \wedge \overline{\partial}\varphi + \int_D \Omega_{0,0}^{0} \wedge \overline{\partial}\varphi = \int_{U \setminus D} \Omega_{0,0}^{01} \wedge \overline{\partial}E\overline{\partial}\varphi - \int_{U \setminus D} \Omega_{0,0}^{01} \wedge \overline{\partial}\varphi + \int_D \Omega_{0,0}^{0} \wedge E\overline{\partial}\varphi = H_0\overline{\partial}\varphi - \int_{U \setminus D} \Omega_{0,0}^{0} \wedge E\overline{\partial}\varphi.
\]

Thus we have verified (2.8) with \( H_0 \) being defined in (2.10), while the second expression of \( H_0 \) in (2.10) follows from Stokes’ formula and \( \overline{\partial}\varphi \Omega_{0,0}^{1} = 0 \) by (2.1).

Throughout the paper, \( | \cdot |_{D,r} \), or \( | \cdot |_r \) for abbreviation, denotes the Hölder \( C^r \) norm, \( r \in [0, \infty) \), for differential forms or functions on a domain \( D \). We finish the section with the following interior estimate of Webster [60].

**Proposition 2.2.** Let \( r \in [0, \infty) \). Let \( U \) be a bounded domain in \( C^n \) with \( \overline{D} \subset U \). Let \( L\psi = \int_U \Omega_{0,q}^{0} \wedge \psi \) with \( 0 \leq q \leq n \). Then

\[
(2.14) \quad |L\psi|_{D,r} \leq C^*_r|\psi|_{U,r},
\]

\[
(2.15) \quad |L\psi|_{D,r+1} \leq C^*_r|\psi|_{U,r}, \quad r \notin \mathbb{N},
\]

where \( C^*_r \leq C_r(U) \) dist\( (D, \partial U) \)^\(-\alpha r - c_1 \) and \( C_r(U) \) depends only on \( r \) and the diameter of \( U \).

### 3. Regularized defining functions and preliminaries for Lipschitz estimates.

In this section we define a **regularized** defining function for a domain \( D \) by Whitney’s extension so that the derivatives of the extension have optimal growth rates near the boundary of \( D \). The defining function will play an important role in our estimates. We recall an extension operator of Stein [57] and basic facts about the Lipschitz space \( \Lambda_r \) and its equivalent norms. The equivalent norms are used for the \( \Lambda_{r+1/2} \) estimate when \( r + 1/2 = 2, 3, \ldots \). We will also recall some basic results on the real interpolation theory. The interpolation will be used for \( C^{r+1/2} \) estimates when \( r = 2, 3, \ldots \), which as mentioned in the introduction, improves the regularity result of Lieb-Range. While the results of this sections might be known to the reader, we formulate them for the purpose of this paper. We will also specify the dependence of the various constants on the domains, which is used to address the stability of estimates of the homotopy operators in Theorem 5.2. We will conclude the section with a regularity result for the \( \overline{\partial} \) equation of top type.
Let us first introduce notations. For \( r \in \mathbb{R}, [r] \) denotes the largest integer \( k \leq r \). For two sets \( A, B \) in \( \mathbb{R}^n \), \( \text{dist}(A, B) \) denotes inf\( \{|a - b|: a \in A, b \in B\} \). For \( \alpha \in \mathbb{N}^n \), let

\[
\partial^\alpha_x f := \partial^\alpha_{i_1} \ldots \partial^\alpha_{i_n} f(x)
\]
denote the partial derivative function in \( x \) and \( \partial^\alpha_x f \) also denotes the set of all partial derivatives of order \( k \). Let \( D \) be a domain in \( \mathbb{R}^n \). Let \( C^r(D) \) denote the set of functions \( f \) in \( D \) such that \( \partial^\alpha_x f \) extend to functions \( f^{(\alpha)} \in C^{r-|\alpha|}(\overline{D}) \) for \( |\alpha| := \alpha_1 + \ldots + \alpha_n \leq r \). For a continuous function \( f \) in \( \overline{D} \), \( |f|_{L^\infty(D)} \) denotes \( |f|_{L^\infty(D)} \), while \( |f|_{\overline{D}, r} \) also denotes the Hölder norm for \( f \in C^r(\overline{D}) \) when \( 0 < r < 1 \). For \( f \in C^r(\overline{D}) \), define

\[
|f|_{\overline{D}, r} := \max_{|\alpha| \leq r} |f^{(\alpha)}|_{\overline{D}, r-|\alpha|}.
\]

3.1. Regularized defining functions. Let \( F \) be a closed set in \( \mathbb{R}^n \) and let \( r \in (0, \infty) \). We recall the following definition.

**Definition 3.1.** ([62, p. 64], [57, p. 194]) Let \( F \) be a non-empty closed subset of \( \mathbb{R}^n \). A function \( f \) in \( F \) is said in \( C^r_w(F) \) in terms of the functions \( f^{(\alpha)} \) in \( F \) for \( \alpha \in \mathbb{N}^n \) and \( |\alpha| \leq r \), if \( f^{(0)} = f \) and there is a finite constant \( A \) so that \( |f^{(\alpha)}(x)| \leq A \) for all \( x \in F \), while \( R_\alpha \), defined by

\[
(3.1) \quad P_\alpha(x, p) := \sum_{|\beta| + |\alpha| \leq r} \frac{f(\alpha+\beta)(p)}{\beta!}(x-p)^\beta, \quad R_\alpha(x, p) := f^{(\alpha)}(x) - P_\alpha(x, p)
\]

has the properties: (i) \( |R_\alpha(x, p)| \leq A|x-p|^{r-|\alpha|} \) for \( x, p \in F \) and \( |\alpha| \leq r \); (ii) when \( r \in \mathbb{N} \), for each \( p \in F \) and \( \epsilon > 0 \) there is \( \delta > 0 \) so that for \( x, x' \in F \) with \( |x-p| + |x'-p| < \delta \),

\[
(3.2) \quad |R_\alpha(x, x')| \leq \epsilon|x-x'|^{r-|\alpha|}.
\]

As observed by Whitney [62], condition (3.2) is essential and consequently all \( f^{(\alpha)} \) are continuous in \( F \) for \( |\alpha| \leq r \). Following Stein [57, p. 173], we define \( |f|^w_{F, r} \) to be the infimum of the constants \( A \) for all possible choices of \( f^{(\alpha)} \) for \( 0 \leq |\alpha| \leq r \).

**Proposition 3.2.** Fix \( r \in [0, \infty) \). Let \( F \) be a closed subset of \( \mathbb{R}^n \). There is an extension operator \( E_r : C^r_w(F) \rightarrow C^r(\mathbb{R}^n) \) so that \( |E_r f|_{\mathbb{R}^n} \leq C_r |f|^w_{F, r} \). Moreover, for \( x \in F^c := \mathbb{R}^n \setminus F \) with \( d(x) := \text{dist}(x, F) < 1 \),

\[
(3.3) \quad |\partial^\alpha_x E_r f - P_\alpha(x, x_*)| \leq C_r \sup_{|\beta| \leq \alpha, x' \in \partial{\overline{F}}: |x-x_*| < 4d(x)} |R_\beta(x', x')d(x)^{|\beta| - |\alpha|}|, \quad |\alpha| \leq r;
\]

\[
(3.4) \quad |\partial^\alpha_x E_r f| \leq C_r |f|^w_{\overline{D}, r}(1 + d(x))^{-k}, \quad x \in F^c, \quad k = 0, 1, 2, \ldots,
\]

where \( |x-x_*| = d(x) \) with \( x_* \in F \), and \( \partial^\alpha E_r f = 0 \) for \( |\beta| > r \).

**Proof.** Inequality (3.4) is proved in Stein [57, p. 178] and Glaeser [21] when \( |\alpha| \leq r + 1 \) and stated in [21, p. 31] for all \( \alpha \). We present here a proof for the reader’s convenience. Recall from [57, p. 169-170] the following properties: (i) There are \( \varphi_k \in C^\infty_0(F^c) \) so that \( \sum \varphi_k = 1 \) in \( F^c \), \( 0 \leq \varphi_k \leq 1 \), and

\[
(3.5) \quad |\partial^\alpha_x \varphi_k| \leq C_\alpha \text{dist}(x, F)^{-|\alpha|}, \quad \text{supp} \varphi_k \subset Q_k,
\]

where \( Q_k \) are cubes satisfying \( \frac{1}{2} \text{diam} Q_k \leq \text{dist}(Q_k, F) \leq 5 \text{diam} Q_k \), and \( F^c = \bigcup_k Q_k \). (ii) Each point in \( F^c \) is contained in at most \( N_0 \) of cubes \( Q_k \). Here \( N_0 \) and \( C_\alpha \) are independent of \( F \).
For each $Q_k$, fix $p_k \in F$ such that $\text{dist}(F, Q_k) = \text{dist}(p_k, Q_k)$. We choose \( \{f^{(a)} : |a| \leq r\} \) so that the constant $A$ in Definition 3.1 satisfies $A \leq 2|f|^w_r$. Let $P(x, p) = \sum_{|a| \leq r} \frac{1}{a!} f^{(a)}(p)(x - p)^a$. Define $E_r f = f$ in $F$ and

\[
E_r f(x) = \sum_i' P(x, p_i) \varphi_i(x), \quad x \in F^c.
\]

Here the sum with the prime is over the $i$ satisfying $\text{dist}(Q_i, F) < 1$. When $d(x) < 1$ and $x \in Q_i$, we have $\text{dist}(Q_i, F) < 1$. Thus we drop the prime, by assuming $d(x) < 1$. Recall from [57, Lemma, p. 177] that

\[
P_{\beta}(x, p) - P_{\beta}(x, q) = \sum_{|\gamma| \leq r-|\beta|} R_{\beta + \gamma}(p, q) \frac{(x - p)^\gamma}{\gamma!}, \quad p, q \in F;
\]

\[
\partial_x^\alpha P_{\beta}(x, p) = P_{\beta}(x, p), \quad p \in F.
\]

For $x \in F^c$, we fix $x_\ast \in F$ such that $|x - x_\ast| = \text{dist}(x, F)$. Suppose that $|\alpha| > 0$. Then

\[
\partial_x^\alpha E_r f = \sum_{\beta, \alpha-\beta \in \mathbb{N}^n} \sum_k \left( \frac{\alpha}{\beta} \right) \partial_x^\beta P(x, p_k) \partial_x^{\alpha-\beta} \varphi_k
\]

\[
= P_\alpha(x, x_\ast) + \sum_{\beta, \alpha-\beta \in \mathbb{N}^n} \sum_k \left( \frac{\alpha}{\beta} \right) (P_{\beta}(x, p_k) - P_{\beta}(x, x_\ast)) \partial_x^{\alpha-\beta} \varphi_k.
\]

We only need to consider the terms with $\partial_x^{\alpha-\beta} \varphi_k \neq 0$. Thus $x \in Q_k$ and there are at most $N_0$ of such $\varphi_k$’s. When $\partial_x^{\alpha-\beta} \varphi_k \neq 0$, we have $|x - p_k| \leq \text{diam}(Q_k) + \text{dist}(Q_k, F) \leq 3 \text{dist}(Q_k, F) \leq 3d(x)$. Then by (3.6)

\[
|P_{\beta}(x, x_\ast) - P_{\beta}(x, p_k)| |\partial_x^{\alpha-\beta} \varphi_k| \leq C \sum_{|\gamma| \leq r-|\beta|} |R_{\beta + \gamma}(x, x_\ast, p_k)| |d(x)|^{\beta + |\gamma| - |\alpha|}.
\]

Combining it with (3.8) yields

\[
|\partial_x^\alpha E f - P_\alpha(x, x_\ast)| \leq C \sum_{0 \leq |\beta| \leq r} |R_{\beta}(x, x_\ast, p_k)| |d(x)|^{\beta - |\alpha|}.
\]

We also have $|x_\ast - p_k| \leq |x_\ast - x| + |x - p_k| \leq 4d(x)$. Hence $|\partial_x^\alpha E f - P_\alpha(x, x_\ast)| \leq C'|f|^w_r d(x)^r - |\alpha|$. Also the continuity of $\partial_x^\alpha E f$ comes from

\[
|\partial_x^\alpha E f - P_\alpha(x, x_\ast)| \leq C' \epsilon_{x, x_\ast} d(x)^{|\alpha|}.
\]

with $\epsilon_{x, x_\ast} \to 0$ as $x$ tends to $x_0 \in \partial D$. We have proved (3.3), while (3.4) follows directly from (3.3). When $r \in \mathbb{N}$, (3.4) implies that $|E_{r, f}|_r \leq C|f|^w_r$. When $r > [r]$, (3.4) for $k = [r] + 1$ also implies that $|\partial_x^k E_{r, f}|_{[r], [r]-[r]} \leq C|f|^w_r$; see the proof in [57, Thm. 3, p. 173].

**Remark 3.3.** When $F = \overline{D}$ for a domain $D$ in $\mathbb{R}^n$ and $f \in C^w_r(\overline{D})$, $D$ is dense in $F$ and $f^{(\alpha)}$ are uniquely determined by the values of $f$ in $D$; in fact $f^{(\alpha)} = \partial^\alpha f$ in $D$. In this case, the above $E_r$ is a linear operator for a fixed sequence $p_k$ appeared in the above proof.

We first identify $C^r(\overline{D})$ with $C^w_r(\overline{D})$ under a mild condition on the domain $D$.

**Lemma 3.4.** Let $r \geq 1$ and $L \geq 1$. Let $D$ be a domain in $\mathbb{R}^n$. Assume that any two points $p, q$ in $\overline{D}$ can be connected by $\gamma$, a union of finitely many line segments in $\overline{D}$, so that $\gamma$ has length at most $L|p - q|$ and $\gamma \cap \partial D$ is a finite set. Then $C^r(\overline{D}) = C^w_r(\overline{D})$ and $|f|_{\overline{D}, r} \leq |f|^w_{\overline{D}, r} \leq C_r L^r |f|_{\overline{D}, r}$. 
Proof. When $|\alpha| = [r]$, we have $R_\alpha(x,y) = \partial_x^\alpha f - \partial_y^\alpha f$. By the continuity of $\partial_x^\alpha f$ and the definition of Hölder ratio, we get $|R_\alpha(x,y)| \leq |f|_{D^{\alpha}} |x-y|^{r-[r]}$ and get (3.2) by the continuity of $\partial^\alpha f$.

Assume that $|\alpha| < [r]$. Let $\gamma: [0,1] \to \overline{D}$ be a piecewise linear curve with $\gamma(0) = p, \gamma(1) = q$. Suppose that $\gamma(t) \in D$ and $|\gamma'(t)| \leq L|p-q|$ for $t \in (t_k, t_{k+1})$ with $t_0 = 0, \ldots, t_N = 1$. Choose an increasing $C^\infty$ function $\hat{s}$ such that $\hat{s}(0) = 0$, $\hat{s}(1) = 1$, and all derivatives of $\hat{s}$ vanish at 0, 1. Let $s(t) = t_k + (t_{k+1} - t_k)\hat{s}((t-t_k)/(t_{k+1} - t_k))$ for $t \in [t_k, t_{k+1}]$. Then $s(t_j) = t_j$, $s^{(i)}(t_j) = 0$ for all $\ell \geq 0$, and $0 \leq s'(t) \leq C$, where $C$ is independent of $t_i, N$. Then $t \mapsto \gamma(s(t))$ is a $C^\infty$ curve connecting $p, q$. Let $g(t)$ still denote $g(\gamma(s(t)))$. We have $|\gamma'(t)| \leq CL|p-q|$.

Let $g(t) = R_\alpha(\gamma(t), p)$. Then $g$ is in $C^1([0,1])$. We have

$$g(1) = \sum_i \int_0^1 \partial_{x_i} |x = \gamma(s_1)| R_\alpha(x, p) \gamma'_i(s_1) ds_1.$$ 

Also, $\partial_x^2 R_\alpha(x, p)$ vanishes at $x = p$ if $|\beta| + |\alpha| \leq r$. Therefore,

$$g(1) = \sum_{i_1, \ldots, i_l} \int_0^1 \cdots \int_0^{s_{i_1} - 1} \partial_{x_{i_1}} \cdots \partial_{x_{i_l}} R_\alpha(\gamma(s_1), p) \gamma'_{i_1}(s_1) \cdots \gamma'_{i_l}(s_1) ds_{i_1} \cdots ds_{i_l},$$

for summing over $k_1, \ldots, k_l$ with $k_1 + \cdots + k_l = i$ and $i = [r] - |\alpha|$. We obtain

$$|R_\alpha(q, p)| \leq C_1 (CL|p-q|)^{|r|-|\alpha|} \max_{t \in [0,1], |\beta| = [r]-|\alpha|} |\partial_x^\beta R_\alpha(\gamma(t), p)|.$$ 

Note that $\partial_x^\beta R_\alpha(x, p) = R_{\beta+\alpha}(x, p)$. The lemma is verified.

The proof also yields the following inequality.

Proposition 3.5. Let $D$ be as in Lemma 3.4. Let $P(x, p)$ be the Taylor polynomial of $f$ of degree $k$ about $p \in \overline{D}$. Then for $x \in \overline{D}$

$$|f(x) - P(x, p)| \leq C_k L^k |x - p|^k \sup_{x' : |\alpha| = k} |\partial^\alpha f(x') - \partial^\alpha f(p)|,$$

where $x' \in \overline{D}$ and $|x' - p| \leq L|x - p|$.

Definition 3.6 ([57], p. 189). Let $D$ be a domain in $\mathbb{R}^n$. We say that $\partial D$ is minimally smooth if the following conditions hold: There are positive numbers $\epsilon, N, M$, and a sequence of open subsets $U_1, U_2, \ldots$ of $\mathbb{R}^n$ so that the following hold:

(i) If $x \in \partial D$, then $B(x, \epsilon) \subset U_i$; $B(x, \epsilon)$ is the ball of center $x$ and radius $\epsilon$.

(ii) No point of $\mathbb{R}^n$ is contained in more than $N$ of the $U_i$’s.

(iii) For each $i$ there exists a domain $D_i$ in $\mathbb{R}^n$, defined by $x_{n_i} > \varphi_i(x_{n_i}')$ for $x = (x_1, \ldots, x_n)$, $x_{n_i}' = (x_1, \ldots, \hat{x_{n_i}}, \ldots, x_n)$ so that $U_i \cap D = U_i \cap D_i$ and

$$|\varphi_i(u) - \varphi_i(v)| \leq M|u - x|, \quad u, v \in \mathbb{R}^{n-1}.$$ 

We will denote by $C_r(D)$ a finite number depending on the above $M, N, \epsilon$, and $r$.

Note that a bounded domain in $\mathbb{R}^n$ has a (strong) Lipschitz boundary, i.e. its boundary is locally the graph of a Lipschitz function in some smooth coordinates, if and only if its boundary is minimally smooth.
Lemma 3.7. Let $D$ be a bounded domain in $\mathbb{R}^n$ with $C^2$ boundary. Let $\rho_0 \in C^2(\overline{D})$ with $\partial \rho_0 \neq 0$ in $\partial D$ and $\rho_0 \leq 0$ in $\overline{D}$. There exists a real function $\rho \in C^2(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n \setminus \overline{D})$ such that $\rho = \rho_0$ in $\overline{D}$, and for $0 < d(x) := \text{dist}(x, D) < 1$,
\begin{align}
|\partial_x \rho| &\leq C_1 L^2 |\rho_0|_{D,2} (1 + d(x))^{2-i}, \quad i = 0, 1, 2, \ldots, \\
|\partial \rho(x) - \partial \rho_0(x_*)| &\leq C L^2 |x - x_*| \max_{y \in D, |y - x_*| \leq 4L |x - x_*|} |\partial^2 \rho_0|, \\
|\partial^2 \rho(x) - \partial^2 \rho_0(x_*)| &\leq C L^2 \omega_2(|x - x_*|),
\end{align}
where $x_* \in \partial D$, $|x - x_*| = \text{dist}(x, D)$, and
\[
\omega_2(\delta) = \sup_{x' \in \overline{D}, x \in \partial D, |x' - x| \leq 4L \delta} |\partial^2 \rho_0(x') - \partial^2 \rho_0(x)|.
\]

If $0 < d(x) < \min_{y \in \partial D} \{1, |\partial_y \rho_0|/(C L^2 |\partial^2 \rho_0|_0)\}$, then
\[
|\partial_x \rho| \geq \frac{1}{2} \min_{y \in \partial D} |\partial_y \rho_0|, \quad \rho(x) \geq \frac{1}{2} \min_{y \in \partial D} |\partial_y \rho_0| d(x).
\]

Proof. Applying (3.3) and Proposition 3.2 to $\rho = E_2 \rho_0$ and $F = \overline{D}$, we obtain
\[
|\rho(x)| \leq C |(\partial \rho, \partial^2 \rho)(x_*)||x - x_*| + C \sup_{x', |\alpha| \leq 2} |R_\alpha(x', x_*)| |d(x)|^{|\alpha|},
\]
\[
|\partial \rho(x) - \partial \rho_0(x_*)| \leq C |\partial^2 \rho(x_*)||x - x_*| + C \sup_{x', |\alpha| \leq 2} |R_\alpha(x', x_*)| |d(x)|^{|\alpha| - 1},
\]
\[
|\partial^2 \rho(x) - \partial^2 \rho_0(x_*)| \leq C \sup_{x', |\alpha| \leq 2} |R_\alpha(x', x_*)| |d(x)|^{|\alpha| - 2},
\]
where $x' \in \partial D$ and $|x' - x_*| \leq 4|x - x_*|$. Here $R_\alpha(x', x_*)$ is the Taylor remainder of $\rho$ defined by (3.1) with $r = 2$. By Proposition 3.5, we have for $|\alpha| \leq 2$
\[
|R_\alpha(x', x_*)| \leq C(L|x' - x_*|)^{2-|\alpha|} \sup_{x'' \in \overline{D}, |x'' - x_*| \leq L|x' - x_*|} |\partial^2 \rho(x'') - \partial^2 \rho(x_*)|.
\]
This gives us (3.11)-(3.12). We get (3.10) from (3.4) and $|E_2 \rho_0|_2 \leq C L^2 |\rho_0|_{D,2}$ by Lemma 3.4. Estimate (3.13) follows directly from (3.11).

We call the above $\rho$ a regularized $C^2$ defining function of $D$. We will also need the following version of Stokes’ theorem.

Lemma 3.8. Let $m$ be a positive integer, and let $b \in \mathbb{R}$. Let $\mathcal{V}$ be a bounded domain in $\mathbb{R}^n$ with $C^3$ boundary. Assume that $B$ and $S$ are functions in $C^1(\mathcal{V})$ and for $x \in \mathcal{V}$ and $i = 0, 1,$
\[
|\partial^i_x S| < C \text{dist}(x, \partial \mathcal{V})^{m-i}, \quad |\partial^i_x B| \leq C(1 + \text{dist}^{b-i}(x, \partial \mathcal{V})), \quad C = C(B, S) < \infty.
\]
Assume further that $b + m > 0$. Then $\int_{\mathcal{V}} B(x) \partial_x S \, dx = - \int_{\mathcal{V}} S(x) \partial_x B \, dx$.

Proof. Let $d(x) = \text{dist}(x, \partial \mathcal{V})$. We know that $B \partial_x S$ is Lebesgue integrable in $\mathcal{V}$, since $|B(x) \partial_x S| \leq C(1 + \text{dist}^{m+b-1}(x, \partial \mathcal{V}))$. Analogously, $S(x) \partial_x B$ is integrable in $\mathcal{V}$.

Let $N_\delta$ be the set of $x \in \mathcal{V}$ satisfying $d(x) < \delta$. Take $\chi_\ell \in C^\infty(\mathcal{V} \setminus N_{1/\ell})$ so that $0 \leq \chi_\ell \leq 1$, $\chi_\ell = 1$ in $\mathcal{V} \setminus N_{2/\ell}$, and $|\partial_x \chi_\ell| \leq C d(x)^{-1}$. Then we have
\[
\left| \int_{\mathcal{V}} \partial_x ((1 - \chi_\ell) BS) \, dx \right| \leq C \int_{N_{2/\ell}} (1 + d(x))^{m-1+b} \, dx,
\]
which tends to 0 as $\ell \to \infty$ as $V$ is bounded and $\partial V \in C^1$. Also, \( \int_V((1-\chi_\ell)S(x)\partial_x B \, dx, \int_V((1-\chi_\ell)B)(x)\partial_x S \, dx, \) and \( \int_V(SB)(x)\partial_x \chi_\ell \, dx \) tend to 0 as $\ell \to \infty$. By Stokes’ theorem, we have
\[
\int_V(SB)(x)\partial_x \chi_\ell \, dx + \int_V(\chi_\ell B)(x)\partial_x S \, dx + \int_V(\chi_\ell S)(x)\partial_x B \, dx = 0.
\]

Letting $\ell \to \infty$, we get the identity. \qed

### 3.2. Equivalent norms

For $0 < r \leq 1$, the Lipschitz space $\Lambda_r(V^n)$ is the set of functions $f \in L^\infty(V^n)$ such that
\[
|f|_{\Lambda_r(V^n)} := |f|_{R^n} := |f|_{L^\infty(V^n)} + \sup_{y \in V^n} \frac{\Delta_y f}{|y|^r}, \quad 0 < r < 1,
\]
\[
|f|_{\Lambda_1(V^n)} := |f|_{L^\infty(V^n)} + \sup_{y \in V^n \setminus \{0\}} \frac{\Delta^2_y f}{|y|^r}.
\]
Here $\Delta_y f(x) := f(x + y) - f(x)$ and thus $\Delta^2_y f(x) = f(x + 2y) + f(x) - 2f(x + y)$. When $r > 1$, we define $\Lambda_r(V^n)$ to be the set of functions $f \in C^r[V^n]$ satisfying
\[
|f|_{\Lambda_r(V^n)} := |f|_{L^\infty(V^n)} + |\partial f|_{\Lambda_{r-1}(V^n)} < \infty.
\]

By [57, Prop. 8, p. 146], the $|f|_{\Lambda_r(V^n)}$ is equivalent with the H"older norm $| \cdot |_{R^n,r}$ by [57, Prop. 8, p. 146].

**Definition 3.9.** Let $F$ be a closed subset in $\mathbb{R}^n$. Let $r \in (0, \infty)$. We write $f \in \Lambda_r(F)$ if there exists $\tilde{f} \in \Lambda_r(V^n)$ such that $\tilde{f}|_F = f$. Define $|f|_{\Lambda_r(F)}$ to be the infimum of $|\tilde{f}|_{\Lambda_r(V^n)}$ for all such extensions $\tilde{f}$. Sometimes $|f|_{\Lambda_r}$ denotes $|f|_{\Lambda_r(V^n)}$ for abbreviation.

We now discuss equivalent norms of the spaces $\Lambda_r$. The following lemma is in McNeal-Stein [45]. We need a quantified version.

**Proposition 3.10.** Let $0 < r < \infty$. Then $f \in \Lambda_r(V^n)$ if and only if there is a decomposition $f = \sum_{k \geq 0} f_k$ so that $f_k \in C^\infty(V^n)$ and
\[
|\partial^i f_k|_{L^\infty(V^n)} \leq A 2^{k(i-r)}, \quad i = 0, \ldots, [r] + 1.
\]
Furthermore, the smallest constant $A_r(f)$ of $A$ is equivalent with $|f|_{\Lambda_r}$, i.e. $c_r A_r(f) \leq |f|_{\Lambda_r} \leq C_r A_r(f)$ for some positive numbers $c_\cdot, C_\cdot$ independent of $f$.

**Proof.** The lemma is proved by Greiner-Stein [26, p. 142] for $0 < r < 1$. For $0 < r \leq 1$, the existence of decomposition is proved in [26, p. 145]. The decomposition is also valid for $r > 1$ since $f_k(x) = \int \varphi_k(t)f(x - t) \, dt$, each $\varphi_k \in C^\infty(V^n)$ is rapidly decreasing, and hence $\partial_x f_k(x) = \int \varphi_k(t)\partial_x f(x - t) \, dt$.

Assume that $1 \leq r < 2$ and (3.14) holds. We have $|f|_{L^\infty} \leq \sum A 2^{-rk} \leq C_1 A$. We decompose
\[
\Delta_y f(x - y) = \sum_{k \leq N} \Delta_y^2 f_k(x - y) + \sum_{k > N} \Delta_y^2 f_k(x - y).
\]
Let \( p. 171 \), there is a regularized distance function \( \Delta \)
\[ (3.17) \]

Proof. We follow the proof in [57]. We first recall an extension for each \( \Lambda_i \). Assume now that \( f \) is an integer and \( f \in \Lambda_r(\mathbb{R}^n) \) so that \( f \Delta \leq 2f \Delta r \). Let us consider the case \( f \in \Lambda_r(\mathbb{R}^n) \) for all \( i = 0, \ldots, k \). Assume now that \( f \) is an integer and \( f \in \Lambda_r(\mathbb{R}^n) \). We now verify (3.16) for \( f \). We will use a linear extension operator from Stein [57] to prove the following.

**Proposition 3.11.** Let \( D \) be a domain in \( \mathbb{R}^n \) where \( \partial D \) is minimally smooth. There is a continuous linear extension operator \( E : C^0(\overline{D}) \to C^0(\mathbb{R}^n) \) so that \( Ef = f \) on \( D \)
\[ (3.15) \]
\[ (3.16) \]

Proof. We follow the proof in [57]. We first recall an extension for each \( D_i \). To simplify the notation, we drop the \( i \) and assume \( n_i = n \). Thus \( D \) is defined by \( x_n > \varphi(x') \). Set \( D^c = \mathbb{R}^n \setminus D \), \( \overline{D}^c = \mathbb{R}^n \setminus \overline{D} \), and let \( d(x) \) be the distance of \( x \) from \( D \). By [57, Thm. 2, p. 171], there is a regularized distance function \( \Delta \in C^\infty(\overline{D}^c) \) vanishing on \( \partial D \) so that
\[ (3.17) \]
Then we choose a rapidly decreasing function \( \psi \in C^\infty([1, \infty)) \) so that
\[ (3.18) \]
We have a linear extension operator
\[ (3.19) \]
where \( L^\infty_k(D) \) is the set of functions \( f \) in \( D \) such that the distributional derivative \( \partial^i f \in L^\infty(D) \) for all \( i = 0, \ldots, k \), and the constant \( C_k \) depends only on the upper bound of \( M \) and \( k, p \); see [57, Thm. 5', p. 181].

Assume now that \( f \in \Lambda_r(\overline{D}) \). We verify (3.15) by using an argument in Greiner-Stein [26, p.p. 146–147] in which \( D \) is a half-space. By the definition of \( \Lambda_r(\overline{D}) \), \( f \) has an extension \( \tilde{f} \in \Lambda_r(\mathbb{R}^n) \) so that \( |\tilde{f}|_{\Lambda_r} \leq 2|f|_{\Lambda_r} \). Take a decomposition \( \tilde{f} = \sum f_j \) satisfying (3.14). By (3.19), we have \( \mathcal{E}f = \sum \mathcal{E}(f_j|D) \) and the decomposition satisfies (3.14), i.e. \( |\mathcal{E}(f_j|D)|_{L^\infty_k(D)} \leq 2C_k |f_j|_{\Lambda_r(\overline{D})} \). This shows that \( |\mathcal{E}f|_{\Lambda_r(\mathbb{R}^n)} \leq C'' |f|_{\Lambda_r(\overline{D})} \). We have verified (3.15) for \( D = D_i \).

We now verify (3.16) for \( D_i \). When \( r \) is an integer and \( f \in C^r(\overline{D}) \), we need only to verify, by (3.19), that \( \partial^i \mathcal{E}f \) is continuous in \( \mathbb{R}^n \). And if \( \alpha = r - |r| > 0 \), we need to estimate the \( C^\alpha \) norm of \( g = \partial^i \mathcal{E}f \). Let us consider the case \( |r| = 0 \) first. The continuity of \( \mathcal{E}f \) follows from the continuity of \( f \), \( |f|_{L^\infty} < \infty \), the rapidly decreasing property of \( \psi \), and
\[ (3.20) \]
where we have used the rapid decrease of \( \psi \) and \( \varphi \).
where \( R_0(x, \lambda) := f(x', x_n + \lambda \delta^*(x)) - f(x', x_n + \delta^*(x)). \) To estimate the Hölder ratio at two points \( u, v \) in \( \mathbb{R}^n \), we may assume that \( u, v \) are in \( D^c \). Let \( d = |u - v| \). Since \( \delta^* \) vanishes on \( \partial D \) and \( \partial \delta^* \) is bounded in \( \overline{D}^c \), then by connecting \( u, v \) in the line segment we show that

\[
|\delta^*(u) - \delta^*(v)| \leq C|u - v|.
\]

Computing the Hölder ratio of each term in \( R_0 \), we obtain

\[
|R_0(u, \lambda) - R_0(v, \lambda)| \leq C_1 \lambda^\alpha |f|_\alpha d^\alpha.
\]

We have verified (3.16) for \( 0 \leq r < 1 \).

For \( k = [r] > 0 \), a \( k \)-th derivative \( \partial^k \mathcal{E} f \) is a sum of \( \mathcal{E} \partial^k f \) and terms of the form

\[
I(x) = \partial^1(\xi_1 \delta^*(x) + \cdots + \partial^1(\xi_j \delta^*(x)) \int_1^\infty \lambda^j \partial^j f(x', x_n + \lambda \delta^*(x)) \psi(\lambda) d\lambda,
\]

with \( j + \xi_1 + \cdots + \xi_j \leq k, j > 0 \) and \( i \leq j \). By the result for \([r] = 0\), we know that \( \mathcal{E} \partial^k f \) is continuous and has the desired estimate in \( |\,|_\alpha \) norm. With \( j > 0 \) and (3.18), we subtract the Taylor polynomial of \( \partial^j f(x', x_n + \lambda \delta^*(x)) \) in \( \lambda \) of degree \( k - j \) about \( \lambda = 1 \) from \( \partial^j f(x', x_n + \lambda \delta^*(x)) \) and apply a Taylor remainder formula to express \( I(x) \) as a linear combination of

\[
\tilde{I}(x) = \eta(x) \int_1^\infty \lambda^j R_k f(x, \lambda) \psi(\lambda) d\lambda,
\]

where \( \eta(x) := \delta^*(x)^{k-j} \partial^i(\xi_j \delta^*(x) + \cdots + \partial^i(\xi_j \delta^*(x)) \lambda^j R_k f(x, \lambda) \psi(\lambda) d\lambda, \)

\[
R_k f(x, \lambda) := \int_1^\lambda (\lambda - \theta)^{k-j} \{ \partial^j f(x', x_n + \theta \delta^*(x)) - \partial^j f(x', x_n + \lambda \delta^*(x)) \} d\theta.
\]

By (3.17), we have \( |\eta(x)| \leq C_0 \). It is now clear that \( \tilde{I} \) and hence \( I \) is continuous in \( \overline{D}^c \) and vanishes in \( \partial D \). Therefore \( \mathcal{E} f \in C^{[r]}(\mathbb{R}^n) \).

For the Hölder ratio, let \( \alpha = r - [r] \). Take two points \( x, x' \in \mathbb{R}^n \setminus D \). If \( x' \in \partial D \), we have \( R_k f(x', \lambda) = 0 \) and

\[
|\tilde{I}(x) - \tilde{I}(x')| \leq C_0^r |f|_{\overline{D}^c} \delta^*(x)^\alpha \leq C_0^r |f|_{\overline{D}^c} |x - x'|^\alpha.
\]

Let \( d = |x - x'| \) and let \( d_L \) be the distance from \( \partial D \) to the line segment \( L \) connecting \( x, x' \). We consider two cases: (i) \( d_L \leq d \); (ii) \( d_L > d \). In the first case, we take \( x'' \in \partial D \) with distance at most \( d \) from \( L \). Then \( |x - x''| \leq 2d \) and \( |x' - x''| \leq 2d \). We get

\[
|\tilde{I}(x) - \tilde{I}(x')| \leq |\tilde{I}(x) - \tilde{I}(x'')| + |\tilde{I}(x'') - \tilde{I}(x')| \leq C |f|_\alpha |x - x''|^{\alpha} + |x' - x''|^{\alpha} \leq C |f|_\alpha d^\alpha.
\]

In the second case, we have

\[
|\tilde{I}(x) - \tilde{I}(x')| \leq C \lambda^{k+1} |f|_{[r]} |x - x'|^{\alpha}.
\]

Thus \( \tilde{\eta}(x) := \int_1^\infty \lambda^j R_k f(x, \lambda) \psi(\lambda) d\lambda \) satisfies

\[
|\eta(x)(\tilde{\eta}(x) - \tilde{\eta}(x'))| \leq C |f|_{[r]} |x - x'|^{\alpha}.
\]

By (3.17), we have \( |\partial \eta(x)| \leq C_1 d_L^{-1} \). Thus,

\[
|\eta(x) - \eta(x')| \leq \sup_{\xi \in L} |\partial \xi \eta| |x - x'| \leq C d_L^{-1} |x - x'| \text{ and } |\tilde{\eta}(x')| \leq C |f|_{[r]} d^{\alpha} d_L^{-\alpha},
\]

and we obtain

\[
|\tilde{\eta}(x')(\eta(x) - \eta(x'))| \leq C |f|_{[r]} d^{\alpha} |x - x'|^{\alpha},
\]

Furthermore, \( d_L \geq d(x') - |x - x'| \geq d(x') - d_L \). We simplify (3.21) and combine it with (3.20) to conclude \( |\tilde{I}(x) - \tilde{I}(x')| \leq C |f|_{[r]} |x - x'|^{\alpha} \), for the second case.

Therefore, we have verified (3.16) for each \( D \). In the general case, we can verify that the linear extension operator \( \mathcal{E} \) defined in [57, p. 191, formula (31)] satisfies (3.15)-(3.16). We leave the details to the reader.
3.3. Real interpolation. We recall some basic results of the real interpolation via the $K$-method of Peetre from Butzer-Berens [5, Sect. 3.2, p. 165]. Let $X_0, X_1$ be two Banach spaces embedded continuously in a linear Hausdorff space $\mathcal{X}$. Define

$$|f|_{X_0 \cap X_1} = \max\{|f|_{X_0}, |f|_{X_1}\}, \quad |f|_{X_0 + X_1} = \inf_{f = f_0 + f_1}(|f_0|_{X_0} + |f_1|_{X_1}).$$

The $(X_0, X_1)$ is called an interpolation pair of Banach spaces in $\mathcal{X}$. Define

$$K(t, f; X_0, X_1) = \inf_{f = f_0 + f_1}(|f_0|_{X_0} + tf_1|_{X_1}), \quad t > 0, \quad f \in X_0 + X_1.$$

Let $\theta \in (0, 1)$. If $f \in X_{\theta, \infty; K}$, we mean that

$$|f|_{\theta; X_0, X_1} := \sup_{t > 0} \{t^{-\theta}K(t, f; X_0, X_1)\} < \infty.$$

Then $(X_0, X_1)_\theta := X_{\theta, \infty; K}$ with norm $\cdot_{\theta; X_0, X_1}$ is a Banach space, while $X_0 \cap X_1 \subset (X_0, X_1)_\theta \subset X_0 + X_1$ are continuous embeddings. Following Triebel [59, Sect. 2.7, p. 200-202], we let $C^r(\mathbb{R}^n)$ be the closure of the space of rapidly decreasing functions in $\mathbb{R}^n$ in $\Lambda_r(\mathbb{R}^n)$. Then by [59, Thm. 1, p. 201; Thm. (g), p. 50] we have

$$C^{r_0}(\mathbb{R}^n), C^{r_1}(\mathbb{R}^n) = C^{(1-\theta)r_0+\theta r_1}(\mathbb{R}^n), \quad 0 < \theta < 1, \quad 0 < r_0 < r_1 < \infty$$

in equivalent norms. Let $(Y_0, Y_1)$ be an interpolation couple of Banach spaces continuously embedded in a linear Hausdorff space $\mathcal{Y}$. If $T: \mathcal{X} \to \mathcal{Y}$ is linear, and if

$$\|Tf_i\|_{Y_i} \leq M_i\|f_i\|_{X_i}, \quad i = 0, 1$$

then $\|Tf\|_{(Y_0, Y_1)} \leq M_0^{1-\theta}M_1^\theta\|f\|_{(X_0, X_1)}$; see [5, Thm. 3.2.23, p. 180] or [59, p. 26].

In summary, we can apply the following:

**Proposition 3.12.** Let $C^r = C^r(\mathbb{R}^n)$. Let $a_i, b_i$ be positive real numbers satisfying $a_0 < a_1$ and $b_0 < b_1$. Let $T: C^{a_0} \to C^{b_0}$ be a linear operator such that $|T f|_{C^{a_i}} \leq M_i|f|_{C^{a_i}}$, for $i = 0, 1$. Then in equivalent norms, $|T f|_{C^{a_0}} \leq C_{r, b, \theta}M_0^{1-\theta}M_1^\theta|f|_{C^{a_0}}$ for $0 < \theta < 1$, $a_0 = (1-\theta)a_0 + \theta a_1$, and $b_0 = (1-\theta)b_0 + \theta b_1$.

3.4. $\overline{\partial}$ solutions for the top type. As an application of the extension and interpolation, we estimate a $\overline{\partial}$ solution for forms of type $(0,n)$. Let $C^r_{(0,q)}(\overline{D})$ (resp. $\Lambda^r_{(0,q)}(\overline{D})$) be the set of $(0,q)$-forms in $D$ of which the coefficients are in $C^r(\overline{D})$ (resp. $\Lambda_r(\overline{D})$). It seems that the following statement has not appeared in the literature.

**Proposition 3.13.** Let $D$ be a bounded domain in $\mathbb{C}^n$.

(i) Suppose that any two points $p, q$ in $D$ can be joined by a broken line segment $\gamma$ in $\overline{D}$ of length at most $L|p-q|$, while $\gamma \cap \partial D$ is a finite set. For each $r \in (0, \infty) \setminus \mathbb{N}$, there is a linear map $T_r: C^r_{0,n}(\overline{D}) \to C^{r+1}_{0,n-1}(\mathbb{C}^n)$, which depends on $r$, so that $\overline{\partial}T_r\varphi = \varphi$ in $D$ and $|T_r\varphi|_{C^{r+1}_{0,n-1}} \leq C_r(D)|\varphi|_{C^r_{0,n}}$.

(ii) Assume that $\partial D$ is Lipschitz. There is a linear operator $S: C_{0,n}(\overline{D}) \to C_{0,n-1}(\mathbb{C}^n)$ so that $\overline{\partial}S\varphi = \varphi$ and $|S\varphi|_{\Lambda^{r+1}_{(0,n-1)}} \leq C_r(D)|\varphi|_{\Lambda^r_{(0,n)}}$ with $C_r(D) < \infty$ for all $r \in (0, \infty)$.

**Proof.** (i) We apply the Whitney extension $E_r$ for $\overline{D}$ via Lemma 3.4 and Proposition 3.2. Fix an open ball $B$ containing $\overline{D}$. By the Leray-Koppleman solution operator $T_n$ for $B$ and estimate in [60], we get the conclusion.
(ii) Let $E: C^0(\overline{D}) \to C^0(C^n)$ be the bounded linear Stein extension. Thus $E: \Lambda_a(\overline{D}) \to \Lambda_a(C^n)$ is bounded for all $a \in (0, \infty)$. We first consider the case of a non-integer $r$. We have $\varphi = f dz_1 \wedge \cdots \wedge dz_n$. By (3.15), we may assume that $D$ is relatively compact in a ball $B_0$. Replacing $\varphi$ by $E\varphi$, we may assume that $\varphi \in \Lambda_r(C^n)$. Take a sequence $\varphi_j \in C^\infty(C^n)$ satisfying $\varphi_j \to \varphi$ in $L^\infty(D)$. Then we have $T_n \varphi_j \to T_n \varphi$ in $L^\infty(D)$. Since $\partial T_n \varphi_j = \varphi_j$, we get $\partial T_n \varphi = \varphi$ in the sense of distributions. By (2.15), we get $|T_n \varphi|_{\Omega_{r+1}} \leq C_r |\varphi|_{B_0;r}$. By (3.15) again, we conclude that

$$|ET_n \varphi|_{\Omega_{n,r+1}} \leq C_r |\varphi|_{B_0;r}, \quad r > 0.$$ 

When $r$ is a positive integer, the estimate follows from interpolation by Proposition 3.12 as follows. We consider a linear operator

$$\tilde{T}_n := \chi ET_n: C_{(0,n)}(C^n) \to C_{(0,n-1)}(C^n),$$

where $\chi \in C^\infty_0(C^n)$ and $\chi = 1$ in $B_r$. By (3.23), we have $|\tilde{T}_n \varphi|_{\Lambda_{r+1}} \leq M_r |\varphi|_{\Lambda_r}$ for $r \in (0, \infty) \setminus N$. By Proposition 3.12, we get the same estimate for all positive integer $r$. □

**Remark 3.14.** The constant $C_r(D)$ in Proposition 3.13 (i) depends on $L$ and the diameter of $D$ and the $C_r(D)$ in (ii) depends on the constants $\epsilon, M, N$ in Definition 3.6, as well as the diameter of $D$ by Proposition 2.2.

## 4. Estimates for the Homotopy Operators

In this section we first introduce a regularized Leray map to study strictly pseudoconvex domains with low regularity. The main estimates are derived under the assumption of the existence of a *regularized* Henkin-Ramírez function for the homotopy operators $H_q$.

**Definition 4.1.** Let $D$ be a bounded domain of class $C^2$ and define

$$D_\delta = \{z \in C^n: \text{dist}(z, \overline{D}) < \delta\}, \quad D_{-\delta} = \{z \in D: \text{dist}(z, \partial D) > \delta\}, \quad \delta > 0.$$ 

We say that $W$ is a *regularized* Leray mapping in $D_\delta \times (D_\delta \setminus D_{-\delta})$, if for some positive number $\delta$ the following hold

(i) $W: D_\delta \times (D_\delta \setminus D_{-\delta}) \to C^n$ is a $C^1$ mapping, and $W(z, \zeta)$ is holomorphic in $z \in D_\delta$.

(ii) $W(z, \zeta) \cdot (\zeta - z) \neq 0$ for $z \in D$ and $\zeta \in D_\delta \setminus D$.

(iii) For each $z \in D_\delta$, we have $W(z, \cdot) \in C^1(\overline{D} \setminus D_{-\delta})$ and

$$|\partial_i^i W(z, \zeta)| \leq C_{i+1} |W(z, \cdot)|_{\Omega_1} (1 + \text{dist}^{1-i}(\zeta, D)), \quad \zeta \in D_\delta \setminus \overline{D}, \quad 0 \leq i < \infty.$$ 

The first two properties are the standard requirements for the Leray maps. The third is new. The existence of a regularized $C^2$ defining function for a domain with $C^2$ boundary is proved in Lemma 3.7. The Whitney extension of a strictly convex function $\rho$ in $\overline{D}$ remains strictly convex in a neighborhood of $\overline{D}$. Therefore, we have the following.

**Example 4.2.** Let $D$ be defined by $\rho_0 < 0$ in $U$ with $\overline{D} \subset U$. Suppose that $\rho_0$ is a $C^2$ strictly convex function in $U$. Let $\rho$ be a Whitney extension of $\rho_0|_{\overline{D}}$ as in Lemma 3.7. Then $W(z, \zeta) = (\rho_{\zeta_1}, \ldots, \rho_{\zeta_n})$ is a regularized Leray mapping.

We now derive our main estimates. Recall the homotopy operator

$$H_q \varphi = \int_U \Omega_{0,q-1}^0 \wedge E\varphi + \int_{U \setminus D} \Omega_{0,q-1}^0 \wedge [\overline{D}, E]\varphi.$$
Lemma 4.3. Let \( (i) \) We divide \([0, δ] \) in \([0, δ] \) and its first-order derivatives. Then
\[
\int_{U \setminus D} \Omega_{0,q}^{01}(z, ζ) \wedge [Ω, E] = φ(ζ).
\]
From now on, we take \( g^0(z, ζ) = ζ - ζ \) and \( g^1(z, ζ) = W(z, ζ) \). We require that \( W \) is a regularized Laplace.
We will denote by \( \hat{δ}^k \) a derivative of order \( k \) in \((z, ζ)\), and by \( N_k(ζ - ζ) \) a monomial in \( ζ - z, ζ - ζ \) of degree \( k \). Let \( A(w) \) denote a polynomial in \( w, w \), where \( N_k \) and \( A \) may differ when they recur. We can write (4.1) as a linear combination of
\[
Kf(z) := \int_{U \setminus D} f(ζ) A(\hat{δ}^k W(z, ζ), z, ζ) N_1(ζ - ζ) dV(ζ), \quad 1 ≤ l < n,
\]
(4.2) \( \Phi(ζ, z) = W(z, ζ) \cdot (ζ - z) \),
where \( f \) is a coefficient of the form \([Ω, E] = φ(ζ) \). In particular \( f \) vanishes on \( D \). Here \( \hat{δ} W \) denotes \( W \) and its first-order derivatives.
To derive our main estimates, we start with the following lemma.

Lemma 4.3. Let \( β ≥ 0, α ≥ 0, \) and let \( 0 < δ < 1/2 \).

(i) If \( α < β + 1/2, \) then \( \int_0^1 \int_0^{s_0+1} \frac{s_1+1 dt ds}{(δ + s)^{2+β}} \leq Cδ^{α-β-1/2}. \)

(ii) \( \int_{s=δ}^{s=2δ} \int_0^{s_0+1} dt ds \leq Cδ^{α-β+3/2}. \)

Proof. (i). We divide \([0, 1] \times [0, 1] \) in the \((s, t)\)-plane in three regions

\[
P: δ + s ≥ t, \quad Q: δ + s ≤ t^2, \quad R: t ≥ δ + s ≥ t^2.
\]
The integral in \( P \) is bounded above by
\[
\int_{s=0}^{1} \int_{t=0}^{δ+s} \frac{s^{α+1} dt ds}{(δ + s)^{3+β}} ≤ \int_{0}^{1} \frac{s^{α+1} ds}{(δ + s)^{2+β}} ≤ \int_{0}^{1} (δ + s)^{α-1-β} ds,
\]
which is less than \( Cδ^{α-β-1/8} \). In \( Q \), it is bounded by
\[
\int_{s=0}^{1} \int_{t=0}^{τ+δ+s} \frac{s^{α+1} dt ds}{t^{6+2β}} ≤ \int_{s=0}^{1} \frac{s^{α+1} ds}{(δ + s)^{3+β+5/2}},
\]
which is less than \( Cδ^{α-β-1/2} \). In \( R \), it has a similar bound as
\[
\int_{s=0}^{1} \int_{t=0}^{τ+δ+s} \frac{s^{α+1} dt ds}{(δ + s)^{3+β}} ≤ \int_{s=0}^{1} \frac{s^{α+1} ds}{(δ + s)^{3+β+5/2}}.
\]
(ii). We divide the domain \([δ, 2δ] \times [0, 1] \) in three regions

\[
P: s ≥ t; \quad Q: s ≤ t^2; \quad R: t ≥ s ≥ t^2, \quad t ≤ δ^{1/2}.
\]
The integral in \( P \) is bounded above by \( \int_{δ}^{2δ} s^{α-β} \int_{0}^{s} dt ds ≤ Cδ^{α-β+2} \). In \( Q \), it is bounded by
\[
\int_{δ}^{2δ} s^{α+1} \int_{0}^{1} t^{-2(β+1)} dt ds ≤ Cδ^{α-β+3/2}. \quad \text{In } R, \text{ it is bounded by } \int_{δ}^{2δ} s^{α-β} \int_{0}^{1} t^{-2(β+1)} dt ds ≤ Cδ^{α-β+3/2}.
\]
Proposition 4.4. Let $1 \leq r < \infty$ and $\alpha = r - [r]$. Let $D$ be a strictly pseudoconvex domain defined by $\rho < 0$ with $\rho \in C^{2}(U)$ and $\overline{D} \subset U$. Suppose that $\partial \rho \neq 0$ in $\partial D$. Let $W$ be a regularized Leray mapping of $D$ in $D_{\delta} \times (D_{\delta} \setminus D_{-\delta})$ and let $\Phi, Kf$ be defined by (4.2)-(4.3). Assume that there is a finite open covering $\{\omega_{1}, \ldots, \omega_{m}\}$ of $\partial D$ and $C^{1}$ coordinate maps $\Psi_{i} : \zeta \rightarrow (s, t) = (s_{1}, s_{2}, t_{3}, \ldots, t_{2n})$ defined in $\omega_{i}$ such that $s_{1} = \rho(\zeta)$ and for $z \in \omega_{i} \cap D, \zeta \in \omega_{i} \setminus D$

\begin{equation}
(4.12) |\Phi(z, \zeta)| \geq c_{*}(d(z) + s_{1}(\zeta) + |s_{2}(z, \zeta)| + |t(z, \zeta)|^{2}),
\end{equation}

\begin{equation}
(4.14) |\Phi(z, \zeta)| \geq c_{*}(|z - \zeta|^{2}), \quad |z - \zeta| \geq c_{*}(s_{2}, t)(z, \zeta),
\end{equation}

where $c_{*} > 0$ is a constant. Suppose that $f$ vanishes in $D$. Then the following hold:

(i) Suppose that $f \in C^{0}_{\alpha - 1}(U)$. If $r \geq 1$, then for $z \in D$ and $d(z) = \text{dist}(z, \partial D)$

\begin{equation}
(4.15) |\partial_{z}^{[r]} Kf| \leq C_{r}d(z)^{\alpha - 1/2}\|f\|_{\text{tt}, r - 1}, \quad 0 \leq \alpha < 1/2,
\end{equation}

\begin{equation}
(4.17) |\partial_{z}^{[r] + 2} Kf| \leq C_{r}d(z)^{\alpha - 3/2}\|f\|_{\text{tt}, r - 1}, \quad 1/2 \leq \alpha < 1.
\end{equation}

In particular, \(\|Kf\|_{\text{tt}, r + 1/2} \leq C_{r}\|f\|_{\text{tt}, r - 1}\) for $\alpha \neq 1/2$.

(ii) Suppose that $f \in \Lambda^{0}_{\alpha - 1}(U)$ and $r > 1$. Then \(\|Kf\|_{\Lambda^{\alpha - 1/2}_{r + 1/2}(D)} \leq C_{r}\|f\|_{\Lambda^{\alpha - 1}(U)}\).

Here $C_{r}$ depends on $r$, $c_{*}$, sup\(z \in D_{\delta}|W(z, \cdot)|_{D_{\delta} \setminus D_{-\delta}; 1}\), and $C^{1}$ norms of $\Psi_{i}$ and the sup norms of $(\det \partial_{\zeta}^{j}\Psi_{i})^{-1}$.

Proof. By the assumption, we have $\partial D \in C^{1}$. We have $\Phi(z, \zeta) \neq 0$, for $z \in D$ and $\zeta \in U \setminus D$. The latter contains the support of $f$. We will first consider the case $f \in C^{\alpha - 1}_{r}$. We will distribute the first $[r] - 1$ derivatives of $Kf(z)$ directly to the integrand when $[r] > 1$. We will then apply the integration by parts in $\zeta$ variables to derive a new formula.

(i) By the assumption, we have $f \in C^{0}_{\alpha - 1}(U)$ and $f = 0$ in $D$. We may assume that $U = D_{\delta}$. We write $\partial_{z}^{[r] - 1}\{Kf(z)\}$ as a linear combination of $K_{1}f(z)$ with

\begin{equation}
(4.11) \hat{K}_{1}f(z) := \int_{U \setminus \overline{D}} f_{1}(z, \zeta) W_{1}(z, \zeta) \, dV(z),
\end{equation}

\begin{equation}
(4.13) f_{1}(z, \zeta) = f(\zeta)A_{1}(W_{1}(z, \zeta), z, \zeta), \quad W_{1}(z, \zeta) = (\partial_{\zeta}W(z, \zeta), \partial_{\zeta}^{k} \partial_{\zeta}^{j} W(z, \zeta)),
\end{equation}

\begin{equation}
(4.16) \mu_{0} + \mu_{1} + \mu_{2} \leq [r] - 1, \quad 1 - \mu_{0} + \mu_{2} \geq 0, \quad k_{0} \leq [r] - 1,
\end{equation}

where $A_{1}$ is a polynomial.

Let us explain how $\partial D \in C^{2}$ suffices the estimation. Since $\hat{\partial}_{\zeta}W(z, \zeta)$ is holomorphic in $z$, its $z$-derivatives in a suitable neighborhood of $D_{\delta/2}$ can be estimated by the sup-norm of $\hat{\partial}_{\zeta}W$ in $D_{\delta}$ by using the Cauchy formula. In $U \setminus \overline{D}$, the integrand in $K_{1}f$ is smooth in $z$. As $\zeta \in U \setminus \overline{D}$ approaches $\partial D$, the rate of growth of a $\zeta$-derivative of $W_{1}$ is bounded by a precise negative power of $d(\zeta)$. The latter can be dominated by the order of vanishing of $f(\zeta)$ along $\partial D$. Let us record the estimate

\begin{equation}
(4.18) |\partial_{\zeta}^{i} \partial_{\zeta}^{j} W_{1}(z, \zeta)| \leq C_{i+j}(|W_{1}|d(\zeta))^{-i},
\end{equation}

\begin{equation}
(4.20) C_{k}(|W_{1}|) := C_{k}(\sup_{z \in D_{\delta}} |W(z, \cdot)|_{D_{\delta}; 1}),
\end{equation}

for $\zeta \in U \setminus \overline{D}, z \in D_{\delta/2}$ and $i, j = 0, 1, \ldots$. Our argument relies essentially on that $f(\zeta)$ is independent of $z$.

We now provide the details of the proof. We will use integration by parts as in Elgueta [17], Ahern-Schneider [1], and Lieb-Range [36] to reduce the exponent of $\Phi$ to the original $n - j$. 


In our case, the integration by parts needs to be carried out by using Lemma 3.8, since $W_1(z, \zeta)$ is $C^\infty$ in $\zeta \in D_\delta \setminus \overline{D}$ and it is, however, merely $C^1$ in $\zeta \in D^c$. To this end, we write (4.8) as

$$K_1 f(z) := \int_{\U \setminus \overline{D}} \frac{h_1(z, \zeta)}{\Phi^{n-j+\mu_1}(z, \zeta)} dV(\zeta),$$

with

$$h_1(z, \zeta) = f_1(z, \zeta) \frac{N_{1-\mu_0+\mu_2}(\zeta - z)}{|\zeta - z|^{2j+2\mu_2}}.$$

Using a partition of unity in $\zeta$ space and replacing $f$ by $\chi f$ for a $C^\infty$ cut-off function, we may assume that

$$\text{supp } f \subset B_0 \setminus D, \quad u(z, \zeta) := \partial_{\zeta_i^*} \Phi(z, \zeta) \neq 0$$

for some $i^*$. Here $B_0$ is a small open set in $\U$ containing a given $\zeta_0 \in \partial D$. Recall that $\partial D \in C^1$. We have for $\zeta \in B_0 \setminus \overline{D}$

$$|\partial_{\zeta_i^*} W(z, \zeta)| \leq C_i(|W|_1)(1 + d(\zeta)^{1-i}), \quad i = 0, 1, \ldots,$$

$$|\partial_{\zeta_i^*} W_1(z, \zeta)| + \left|\partial_{\zeta_j} \frac{1}{u(z, \zeta)}\right| \leq C_i(|W|_1)d(\zeta)^{-i}, \quad i = 0, 1, \ldots$$

Up to a constant multiple, we rewrite $K_1 f$ as

$$K_1 f = \int_{C^n \setminus \overline{D}} u(z, \zeta)^{-1} h_1(z, \zeta) \partial_{\zeta_i^*} \Phi^{-(n-j+\mu_1-1)}(z, \zeta) dV(\zeta).$$

Since $f \in C^{r-1}$ vanishes identically in $\overline{D}$, then $\partial^i f$ vanishes in $\partial D$ for $i \leq [r] - 1$. Thus, by Taylor’s theorem,

(4.13) \quad $|\partial^i f(\zeta)| \leq C_i |f|_{D^{i, r-1}} d(\zeta)^{r-1-i}, \quad \zeta \in D_\delta, \quad 0 \leq i \leq [r] - 1.$

Suppose that $[r] > 1$. Fix $z \in D$. Thus $|z - \zeta|^{-2j-2\mu_2}$ is $C^\infty$ in $\zeta \in C^n \setminus D$. Recall that $f_1(z, \zeta) = f(\zeta) A_1(W_1(z, \zeta), \zeta, \zeta)$. Using (4.11) and (4.13), a straightforward computation shows that

(4.14) \quad $|\partial_{\zeta_i^*}((u^{-1} h_1)(z, \zeta))| \leq C_{i, j}(z) C_{i+j}(|W|_1) |f|_{D^{i, r-1}} d(\zeta)^{r-1-i},$

for $\zeta \in D_\delta \setminus \overline{D}$ and $j \in \mathbb{N}$. Here $C_{i, j}(z) < \infty$ because $z \in D$. In particular, this allows us to apply the integration by parts to transform $K_1 f$.

When $[r] > 1$, we apply Stokes’ theorem via Lemma 3.8 in which $S(\zeta) = u(z, \zeta)^{-1} h_1(z, \zeta)$, $B(\zeta) = \Phi(z, \zeta)^{-(n-j+\mu_1-1)}$, $m = [r] - 1$, and $b = 0$. (Recall that we fix $z \in D$. Thus $B$ is $C^1$ in $D^c$ and $C^\infty$ in its interior.) Up to a constant multiple, we have

$$K_1 f(z) = \int_{C^n \setminus \overline{D}} \frac{\partial_{\zeta_i^*} \{u(z, \zeta)^{-1} h_1(z, \zeta)\}}{\Phi^{n-j+\mu_1-1}(z, \zeta)} dV(\zeta), \quad \forall z \in D.$$

We also have, up to a constant multiple depending on $z, |W|_1$,

$$|\partial_{\zeta_i^*} (u(z, \zeta)^{-1} (\partial_{\zeta_i^*} \circ u(z, \zeta)^{-1})^\ell \{h_1(z, \zeta)\})| \leq |f|_{D^{i, r-1}} d(\zeta)^{r-1-\ell-i},$$

$$|\partial_{\zeta_i^*} \Phi^{-(n-j+\mu_1)+\ell}| \leq C_{i, \ell}(z) C_{i+\ell}(|W|_1)(1 + d(\zeta)^{1-i}).$$
If \( \mu_1 - \ell > 0 \), we have \( [r] - 1 - \ell > [r] - 1 - \mu_1 \geq 0 \) by (4.10). Applying integration by parts \( \mu_1 \) times via Lemma 3.8 till \( \mu_1 - \ell = 0 \), we obtain
\[
K_1 f(z) := \int_{\mathbb{C}^n \setminus \overline{D}} \frac{h_2(z, \zeta)}{\Phi^{n-j}(z, \zeta)} dV(\zeta), \quad \forall z \in D
\]
with
\[
h_2(z, \zeta) := (\partial_{\zeta^i} \circ u(z, \zeta)^{-1})^\mu_1 \{h_1(z, \zeta)\}.
\]
By the product and quotient rules, we can write \( h_2(z, \zeta) \) as a linear combination of
\[
(4.16)
\]
with
\[
\tilde{N}_\lambda(z - \zeta) = \frac{N_{1-\mu_0+\mu_2-\nu_1+\nu_2}(\zeta - z)}{\zeta - z^{2j+2\mu_2+2\nu_2}},
\]
\[
(4.17)
f_2(z, \zeta) = A_2(\tilde{W}_1(z, \zeta), z, \zeta) \partial_{\zeta^1}^\nu_1 \tilde{W}_1(z, \zeta) \cdots \partial_{\zeta^1}^\nu_1 \tilde{W}_1(z, \zeta) \partial_{\zeta^1}^\nu_1 f.
\]
Furthermore, each \( \tilde{W}_1 \) is one of \( \partial_\zeta W(z, \zeta), \partial_{\zeta^1}^k \partial_\zeta W(z, \zeta), u_1^{-1}(z, \zeta) \). And
\[
(4.18)
\nu_1 + \cdots + \nu_1 \leq \mu_1, \quad 1 - \mu_0 + \mu_2 - \nu_1 + \nu_2 \geq 0,
\]
\[
(4.19)
\lambda := (1 - \mu_0 + \mu_2 - \nu_1 + \nu_2) - (2j + 2\mu_2 + 2\nu_2).
\]
We have proved that \( \partial_{\zeta}^{-1} K f \) is a linear combination of \( K_1 f \), while \( K_1 f \) is a linear combination of
\[
(4.20)
K_2 f(z) = \int_{\mathbb{C}^n \setminus \overline{D}} f_2(z, \zeta) \tilde{N}_\lambda(z - \zeta) \frac{dV(\zeta)}{\Phi^{n-j}(z, \zeta)}.
\]
Since \( f(\zeta) = 0 \) in \( \overline{D} \), it is easy to see that \( K_2 f \in C^\infty(D) \).

We want to estimate \( K_2 f(z) \) in terms of distance \( d(z) \). To achieve the estimate that has the form (4.6), we need to count the numbers of derivatives in the expression of \( f_2 \). In (4.17), we have applied \( \nu_4 + \cdots + \nu_4 \) extra derivatives on \( \tilde{W}_1 \). Set
\[
\lambda' = \nu_4 + \cdots + \nu_4.
\]
Since \( |\zeta - z| \geq d(\zeta) \) and \( [r] - 1 - \nu_3 \geq 0 \) by (4.10) and (4.18), we obtain for \( \zeta \in \overline{D} \),
\[
|\partial_{\zeta}^\nu f| \leq C |f|^{r-1} d(\zeta)^{r-1-\nu_3}, \quad |\partial_{\zeta}^\nu \tilde{W}_1(z, \zeta) \cdots \partial_{\zeta}^\nu \tilde{W}_1(z, \zeta)| \leq C(|W|_1) d(\zeta)^{-\lambda'},
\]
where the last inequality follows from (4.11). Hence their \( z \)-derivatives can be estimated by \( |W|_{D_3,1} \). Then we have proved that for \( z \in D \) and \( \zeta \notin \overline{D} \)
\[
(4.21)
|f_2(z, \zeta)| \leq C(|W|_1) |f|^{r-1} d(\zeta)^{r-1-\lambda' - \nu_3}
\]
by using \( [r] - 1 - \nu_3 - \lambda' \geq 0 \) and \( d(\zeta) \leq |\zeta - z| \). By the definition of \( \tilde{N}_\lambda \), we have
\[
|\tilde{N}_\lambda(z - \zeta)| \leq |\zeta - z|^\lambda, \quad |h_2(z, \zeta)| \leq C(|W|_1) |f|^{r-1} d(\zeta)^{\alpha} |\zeta - z|^{[r]-1-\nu_3-\lambda' - \lambda.}
\]
We have just estimated $h_2$. Since $f(\zeta)$ does not depend on $z$, the $z$-derivatives of $f_2(z, \zeta)$, given by (4.17), satisfy
\[ |\partial_z f_2(z, \zeta)| \leq C_1 |f|_{r-1} d(\zeta)^\alpha |\zeta - z|^{[r]-1-\nu_3-\lambda'}, \quad [r] - 1 - \nu_3 - \lambda' \geq 0, \]
\[ |\partial_z^2 \tilde{N}_\lambda(z - \zeta)| \leq C_\ell |\zeta - z|^\lambda - \ell, \]
\[ |\partial_z^2 \Phi^{-(n-j)}(z, \zeta)| \leq C_\ell |W|_1 |\Phi^{-(n-j)-\ell}(z, \zeta)|. \]

We estimate $\tilde{N}_\lambda$ first. We have
\[ \lambda = (1 - \mu_0 + \mu_2 - \nu_1 + \nu_2) - (2j + 2\mu_2 + 2\nu_2) \]
\[ = 1 - 2j - \mu_0 - \mu_2 - \nu_1 - \nu_2 \geq 1 - 2j - \mu_0 - \mu_2 - \mu_1 + \nu_3 + \lambda' \]
\[ \geq 2 - 2j - [r] + \nu_3 + \lambda' \]
by the first inequalities in (4.18) and (4.10). Thus $[r] - 1 - \nu_3 + \lambda \geq 1 - 2j$. For $h_3$ given by (4.15), we have
\[ \partial_z^2 h_3(z, \zeta) \leq C(|W|_1) |f|_{r-1} d(\zeta)^\alpha |\zeta - z|^{1-2j-\ell}. \] (4.22)

We have expressed $\partial_\zeta^{[r]-1} K f$ as a linear combination of $K_2 f$ by exhausting all derivatives of $f$. Let $z \in D$. We want to show that
\[ \partial_z^2 K_2 f(z) \leq C(|W|_1) |f|_{r-1} d(z)^{-1+(\alpha+1/2)}, \quad \alpha + 1/2 < 1, \] (4.23)
\[ \partial_z^2 K_2 f(z) \leq C(|W|_1) |f|_{r-1} d(z)^{-1+(\alpha-1/2)}, \quad \alpha + 1/2 \geq 1. \] (4.24)

For $\ell = 2, 3$, we compute $\partial_z^\ell K_2 f$ by differentiating the integrand directly. The $\partial_z^2 K_2 f$ is a sum of three kinds of terms
\[ J_0 f(z) = \int_{C^n \setminus \overline{D}} \frac{f_2(z, \zeta)}{\Phi^{n-j}(z, \zeta)} \partial_z^2 \left\{ \tilde{N}_\lambda(z - \zeta) \right\} dV(\zeta), \]
\[ J_1 f(z) = \int_{C^n \setminus \overline{D}} \frac{f_2(z, \zeta)}{\Phi^{n-j+1}(z, \zeta)} \partial_z \left\{ \tilde{N}_\lambda(z - \zeta) \right\} dV(\zeta), \]
\[ J_2 f(z) = \int_{C^n \setminus \overline{D}} \frac{f_2(z, \zeta)}{\Phi^{n-j+2}(z, \zeta)} \left\{ \tilde{N}_\lambda(z - \zeta) \right\} dV(\zeta), \]

where $f_2$ still has the form (4.17) while $\nu_4, \ldots, \nu_i$ are unchanged. Therefore we obtain
\[ |J_0 f(z)| \leq C(|W|_1) |f|_{r-1} \int_{U \setminus \overline{D}} \frac{d(\zeta)^\alpha}{|\Phi(z, \zeta)||^{n-j}|\zeta - z|^{1+2j}} dV(\zeta), \]
\[ |J_1 f(z)| \leq C(|W|_1) |f|_{r-1} \int_{U \setminus \overline{D}} \frac{d(\zeta)^\alpha}{|\Phi(z, \zeta)||^{n-j+1}|\zeta - z|^{2j}} dV(\zeta), \]
\[ |J_2 f(z)| \leq C(|W|_1) |f|_{r-1} \int_{U \setminus \overline{D}} \frac{d(\zeta)^\alpha}{|\Phi(z, \zeta)||^{n-j+2}|\zeta - z|^{-1+2j}} dV(\zeta). \]

Recall that $1 \leq j < n$. For $z \in D$ and $\zeta \notin D$, we have $C' |\zeta - z| \geq |\Phi(z, \zeta)| \geq C |\zeta - z|^2$. Thus it suffices to estimate the last integral for $j = n - 1$. Set
\[ \tilde{J}_2(z) := \int_{U \setminus \overline{D}} \frac{d(\zeta)^\alpha}{|\Phi(z, \zeta)||^{3}|\zeta - z|^{2n-3}} dV(\zeta). \]
Fix \( \zeta_0 \in \partial D \) and a small neighborhood \( \omega_0 \) of \( \zeta_0 \). Let \( z \in \omega_0 \cap D \) and \( \zeta \in \omega_0 \setminus D \). Note that \( r(\zeta) \approx \text{dist}(\zeta, \partial D) = d(\zeta) \). We now use the assumption that
\[
|\Phi(z, \zeta)| \geq c_*(d(z) + s_1(\zeta) + |s_2(z, \zeta)| + |t(z, \zeta)|^2), \quad |\zeta - z| \geq c_*|s_2(t)(z, \zeta)|.
\]
We also have
\[
d(\zeta)/C \leq r(\zeta) = s_1(\zeta) \leq |(s_1(\zeta), s_2(z, \zeta))|, \quad \zeta \in D_\delta \setminus \overline{D}.
\]
Using polar coordinates for \( (s_1, s_2) \in \mathbb{R}^2 \) and \( (t_3, \ldots, t_{2n}) \in \mathbb{R}^{2n-2} \), we obtain for \( z \in \omega_0 \)
\[
\tilde{I}_2(z) \leq C \int_{s=1}^{1} \int_{t=0}^{1} s^{\alpha+1} ds dt \frac{dV(\zeta)}{(d(z) + s + t^2)^\beta},
\]
which is less than \( Cd(z)^{\alpha-1} \) by Lemma 4.3 in which \( \beta = 0 \) and \( 0 \leq \alpha < 1/2 \). We have verified (4.23).

Consider now the case \( 1/2 < \alpha < 1 \). This requires us to estimate \( \partial^2_2 K_2 f \), which is a sum of terms
\[
\tilde{J}_i f(z) := \int_{C_{\alpha,1} \overline{D}} \frac{f_2(z, \zeta)}{\Phi^{n-j+i}(z, \zeta)} \partial^2_{z-i} \left\{ \tilde{N}(\zeta - z) \right\} dV(\zeta)
\]
for \( i = 0, 1, 2, 3 \). The worst term is \( \tilde{J}_3 f(z) \) with \( j = n - 1 \) and \( i = 3 \). We have
\[
|\tilde{J}_3 f(z)| \leq C'(|W||f|_r) \int_{s=1}^{1} \int_{t=0}^{1} s^{\alpha+1} ds dt \frac{dV(\zeta)}{(d(z) + s + t^2)^4},
\]
which is less than \( C|f|_{r-1} d(z)^{\alpha-3/2} \) by Lemma 4.3 with \( \beta = 1 \) and \( 1/2 < \alpha < 3/2 \).

\( (ii) \). We now consider the estimate in the \( \Lambda_{r+1/2} \) space.

**Case 1**, \( \alpha \neq 0, 1/2 \). Recall that \( Kf \) is \( C^\infty \) in \( D \) and its derivatives on a compact subset of \( D \) can be estimated easily by the sup norm of \( f \). When \( \alpha \neq 0, 1/2 \), by the Hardy-Littlewood lemma for Hölder spaces we get the estimate in (ii) from (i) immediately. Note that the same argument by the Hardy-Littlewood also gives us \( |Kf|_{D,k+1/2} \leq C_k |f|_{\Lambda,k-1} \) when \( k \) is a positive integer, which is however a weaker version of (ii) for the \( \Lambda_{k+1/2} \) estimate.

**Case 2**, \( \alpha = 1/2 \). In this case (4.23) says that \( |\partial^2_2 K_2 f| \leq C|f|_{r-1} \text{dist}(z)^{-1} \). We remark that if we have \( \partial D \in C^\infty \), then by a version of Hardy-Littlewood lemma (see [45]), we could conclude that \( K_2 f \in \Lambda_1 \). Since \( \partial D \) is only \( C^2 \), We need another proof for the case \( \alpha = 1/2 \), by using the estimates in (i).

In fact we will provide an argument that actually works for \( 0 < \alpha < 1 \). Let us show that
\[
|K_2 f|_{\Lambda_{r+1/2}} \leq C_r( |W||f|_r ) |f|_{\Lambda_{r-1}}.
\]
We may assume that the \( f \) vanishes when \(|(s_1, s_2)| > 1 \) or \(|t| > 1 \). We consider a dyadic decomposition with \( A^+_k := \{(s_1, s_2): 2^{-k-1} < |(s_1, s_2)| < 2^{-k+1}, s_1 \geq 0 \} \) for \( k = 1, 2, \ldots \).

Take a partition of unity \( \{\chi_k\} \) such that \( \text{supp} \chi_k \) is contained in \( A^+_k \), \( \sum_k \chi_k = 1 \) in \( \cup_k A^+_k \), and \( |\partial^j \chi_k| \leq C_j 2^{jk} \) for \( j = 0, 1, \ldots \). Set \( K_2 f = \sum_{k \geq 1} g_k \) with
\[
g_k(z) = \int_{A^+_k} \chi_k(s_1(\zeta), s_2(z, \zeta)) f_2(z, \zeta) \frac{\tilde{N}(\zeta - z)}{\Phi^{n-j}(z, \zeta)} dV(\zeta), \quad z \in D.
\]
In connection with Question 1 in the introduction, one can approximate
Remark 4.5. Using (4.23), we have
$$I_{i,\ell}(z) := \int_{\mathbb{T}} \partial_{\bar{z}}^{i-\ell} \{ \chi_k(s_1(\zeta), s_2(z(\zeta))) \} \partial_z^i \left\{ f_2(z, \zeta, \bar{N}_\lambda(\zeta - z)) \right\} dV(\zeta), \quad z \in D$$
for \(\ell = 0, 1, \ldots, i\). Again the worst term occurs to \(j = n - 1\) and \(\ell = i\). Thus,
$$|I_{i,\ell}(z)| \leq \int_{A_k^1} \int_{t \in \mathbb{R}^{2n-2}, |t| < 1} C'_{|W|_1}|f|_{r-1} \cdot \frac{2^{(i-\ell)k} s_1^{2\Delta(t+|t|)}}{(s_1 + |s_2| + |t|)^{2\Delta} (s_1 + |s_2| + |t|)^{2n-3}} ds_1 ds_2 dt \leq C(|W|_1)|f|_{r-1} 2^{-k(\alpha-1)}$$,
where the last integral is estimated in two regions \(s \leq t^2\) and \(s \geq t^2\). Now assertion (ii) for \(0 < \alpha < 1\) follows from Proposition 3.11 and Lemma 3.10.

Case 3, \(r > 1\) an integer. We will achieve the \(\Lambda_{r+1/2}\) estimate by the real interpolation theory. Fix \(d\Omega^J\) with \(|I| = q > 0\) and fix \(d\Omega^J\) with \(|J| = q-1\). Let \(\{\psi\}_J\) denote the coefficients of \(d\Omega^J\) for a \((0, q-1)\)-form \(\psi\). Consider the linear mapping
$$L_J: f \mapsto \left\{ \int_{\mathbb{T}} \Omega_{0,q}^{01} \wedge [\partial, E](f d\Omega^J) \right\}_J.$$
Assume that \(r \geq 2\) be an integer. Let \(E\) be the linear extension operator for functions defined in \(\mathbb{T}\), given in Proposition 3.11. For the interpolation theory to be applicable, it is crucial that there is no other restriction to \(f\). Using (4.24), we have
$$|EL_J f|_{C^{\alpha, r-1}} \leq C_1 |L_J f|_{C^{\alpha, r-1} + \frac{1}{2}} \leq C_1 C_k(|W|_1)|f|_{C^{\alpha, r-1-\epsilon}}.$$  
Using (4.23), we have
$$|EL_J f|_{C^{\alpha, r+\epsilon + \frac{1}{2}}} \leq C_1 C_{k+1}(|W|_1)|f|_{C^{\alpha, r-1+\epsilon}}.$$  
The estimate follows from interpolation via Proposition 3.12. The assertion (ii) is proved.

**Remark 4.5.** In connection with Question 1 in the introduction, one can approximate \(\varphi \in \Lambda_1(C^0)\) by bounded \(C^1\) forms \(\varphi_j\) in \(C^0\), which converges in the sup norm to \(\varphi\). However, we do not have a useful limit for \(H_q \varphi_j\) as \(j \to \infty\), in order to conclude that \(\partial u = \varphi\).

We now turn to the estimate of holomorphic projection \(H_0\). The analogous estimate for the boundary operator in (2.5) is in Ahern-Schneider [1], where \(\partial D \in C^\infty\) is used. We need to restrict to \(r > 1\), requiring \(\partial D \in C^2\) only.

**Lemma 4.6.** Let \(0 \leq \alpha < 1\), \(0 < \beta < 1/2\), and \(n \geq 2\). Then
$$\int_0^1 \int_0^1 \frac{s^{\alpha+1} t^{2n-3}}{(\delta + s + t^2)^{n+2}} dt ds \leq \frac{C_n}{1 - \alpha} \delta^{\alpha-1}.$$

**Proof.** We estimate the integrals \(I\) of the integrand by a covering of \([0, 1] \times [0, 1]\):

(i) \(\delta \leq t^2 \leq s\).
$$I \leq \int_0^1 \int_{t=0}^{\sqrt{s}} \frac{s^{\alpha+1} t^{2n-3}}{s^{n+2}} dt ds \leq \int_0^1 s^{\alpha-2} ds \leq \frac{1}{1 - \alpha} \delta^{\alpha-1}.$$

(ii) \(\delta \leq s \leq t^2\).
$$I \leq \int_0^1 \int_{s=0}^{t^2} \frac{s^{\alpha+1} t^{2n-3}}{t^{2n+4}} ds dt \leq \int_0^1 t^{2n-3} dt \leq \frac{1}{1 - \alpha} \delta^{\alpha-1}. $$
(iii) \( t^2 \leq \delta \leq s \).

\[
I \leq \int_0^{\sqrt{\delta}} \int_{s=\delta}^{1} \frac{s^{n+1}t^{2n-3}}{s^{n+2}} \, ds \, dt = \delta^{n-1} = \delta^{n-1}.
\]

(iv) \( s \leq \delta \leq t^2 \).

\[
I \leq \int_0^{\delta} \int_{t=\sqrt{s}}^{1} \frac{s^{n+1}t^{2n-3}}{t^{2n+4}} \, ds \, dt \leq \delta^{n+2}\delta^{-3} = \delta^{n-1}.
\]

(v) \( t^2 \leq s \leq \delta \).

\[
I \leq \int_0^{\sqrt{\delta}} \int_{t=0}^{s} \frac{s^{n+1}t^{2n-3}}{\delta^{n+2}} \, ds \, dt \leq \delta^{-2} \int_0^{\delta} s^{n+2} \, ds \leq \delta^{n-1}.
\]

(vi) \( s \leq t^2 \leq \delta \).

\[
I \leq \int_0^{\sqrt{\delta}} \int_{s=0}^{t^2} \frac{s^{n+1}t^{2n-3}}{\delta^{n+2}} \, ds \, dt \leq \delta^{n-1}.
\]

\( \square \)

**Proposition 4.7.** Let \( r > 1 \). Let \( D, \Phi, g^1 \) be as in Proposition 4.4. Suppose that \( f \in C^1(C^n) \) is a function vanishing in \( D \). Then

\[
\|H_0f\|_{\Lambda_r(\overline{D})} \leq C_r(\|W_1\|f\|_{\Lambda_r(\partial D)}), \quad r > 1,
\]

\[
|\partial^2z H_0f(z)| \leq C_1(\|W_1\| \operatorname{dist}(z, \partial D))^{-1} |f|_{1, \partial D}, \quad z \in D.
\]

**Proof.** Let \( k = \lfloor r \rfloor \geq 1 \). We first consider the case \( f \in C^r(\overline{D}) \). The above proof for \( H_i f \) with \( i > 0 \) can be adapted easily. Let \( \partial^k H_0f \) be a \( (k + 1) \)-th order derivative of \( H_0f \). It is a linear combination of

\[
Kf(z) = \int_{\partial D} f(\zeta) \partial^k H_0f(z, \zeta) \, dV(\zeta).
\]

Let \( z_0 \in \partial D \). Using a partition of unity, we may assume that for a neighborhood \( B_0 \) of \( z_0 \) in \( C^n \) and for some \( j \), we have

\[
\operatorname{supp} f \subset B_0 \setminus D; \quad u(z, \zeta) := \partial_z \Phi(z, \zeta) \neq 0, \quad z, \zeta \in B_0.
\]

Applying integration by parts \( k - 1 \) times, we write \( Kf \) as a linear combination of \( K_1 f \) with

\[
K_1 f (x) := \int_{\partial D} \frac{f_1(z, \zeta)}{\Phi^{n+2}(z, \zeta)} \, dV(\zeta), \quad \forall z \in D
\]

with \( f_1(z, \zeta) = A(W_1(z, \zeta), z, \zeta) \partial^{\nu_0} W_1 \cdots \partial^{\nu_\ell} W_1 \partial^{\nu_k} f \) and

\[
\nu_0 + \cdots + \nu_\ell = k - 1.
\]

Since \( f(\zeta) = 0 \) in \( \overline{D} \), it is easy to see that \( K_1 f \in C^\infty(D) \). We have for \( \alpha > 0 \)

\[
|K_1 f(z)| \leq C(\|W_1\|f) r \int_0^1 \int_0^1 \frac{s^{n+1}t^{2n-3}}{(d(z) + s + t)^{n+2}} \, ds \, dt \leq C(\|W_1\|f) r |d(z)|^{n+1} - 1.
\]

This gives us the desired estimate when \( r \) is non integer. When \( r \) is a positive integer, the estimate follows from interpolation by Proposition 3.12. \( \square \)
5. Regularized Henkin-Ramírez functions

We now discuss our result for strictly pseudoconvex domains. We first strengthen the classical Henkin-Ramírez functions via the following result.

**Proposition 5.1.** Let $D$ be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with $C^2$ boundary. Suppose that $\rho_0 \in C^2(\mathcal{U})$, $\partial D = \{ z \in \mathcal{U} : \rho_0 = 0 \}$, and $\partial \rho_0 \neq 0$ in $\partial D$. Let $D_\delta = \{ z \in \mathbb{C}^n : \operatorname{dist}(z,D) < \delta \}$ and $D_{-\delta} = \{ z \in D : \operatorname{dist}(z,\partial D) > \delta \}$. Let $\rho = E_2(e^{\ell_0\rho_0} - 1)$ be a regularized $C^2$ defining function of $D$, where $L_0 > 0$ is sufficiently large so that $e^{\ell_0\rho_0} - 1$ is strictly plurisubharmonic in a neighborhood $\omega$ of $\partial D$. There exist $\delta > 0$ and functions $W$ satisfying the following.

(i) $W$ is defined in $D_\delta \times (D_\delta \setminus D_{-\delta})$, $\Phi(z,\zeta) = W(z,\zeta) \cdot (\zeta - z) \neq 0$ for $\rho(z) \leq \rho(\zeta)$ and $\zeta \neq z$, $W(\cdot,\zeta)$ is holomorphic in $D_\delta$ for $z \in D_\delta$, and $W \in C^1(D_\delta \times (D_\delta \setminus D_{-\delta}))$.

(ii) If $|\zeta - z| < \epsilon$ and $\zeta \in D_\delta \setminus D_{-\delta}$, then $\Phi(z,\zeta) = F(z,\zeta)M(z,\zeta)$, $M(z,\zeta) \neq 0$ and

\[
F(z,\zeta) = -\sum \frac{\partial \rho}{\partial \zeta_j}(z_j - \zeta_j) + \sum a_{jk}(\zeta)(z_j - \zeta_j)(z_k - \zeta_k),
\]

\[
\operatorname{Re} F(z,\zeta) \geq \rho(\zeta) - \rho(z) + |\zeta - z|^2/C, \quad \text{with } (M, F) \in C^1(D_\delta \times (D_\delta \setminus D_{-\delta})) \text{ and } a_{jk} \in C^\infty(\mathbb{C}^n).
\]

(iii) For $z \in D_\delta$ and $\zeta \in D_\delta \setminus D$, we have

\[
|\partial_\zeta \partial_\zeta W(z,\zeta)| \leq C_{i,j}(D, L_0, |\rho_0|_{\mathcal{U}_2}) \sum_{j_1 + j_2 = j} \delta^{-i-j1}\left\{ 1 + d(\zeta, \partial D)^{1-j_2} \right\}.
\]

The $(W_1, \ldots, W_n)$ is called a regularized Henkin-Ramírez map.

**Proof.** When $\rho$ is strictly plurisubharmonic, the proof for (i) and (ii) is in Øvrelid [46] and see also Henkin-Leiterer [29, Thm. 2.4.3, p. 78; Thm. 2.5.5, p. 81], and Range [51, Prop. 3.1, p. 284]. In [23], the Henkin-Ramírez functions for a family of strictly pseudoconvex domains are studied. Therefore, only (iii) is new.

Fix $\delta_0$ so that $\overline{D_{\delta_0}} \setminus D$ is contained in $\mathcal{U} \cap \omega$. We have

\[
\delta_1 = \min\{ \rho(\zeta) : \zeta \in \partial D_{\delta_0} \} > \delta_0/C.
\]

We have

\[
\sum_{j,k} \frac{\partial^2 \rho(\zeta)}{\partial \zeta_j \partial \zeta_k} \xi_j \xi_k \geq C_0 |t|^2, \quad \zeta \in \omega,
\]

with $C_0 > 0$. Define $D_c = \{ z \in D_{\delta_0} : \rho(z) < c \}$. We take

\[
F(z,\zeta) := -\sum \frac{\partial \rho}{\partial \zeta_j}(z_j - \zeta_j) - \sum a_{ij}(\zeta)(z_i - \zeta_i)(z_j - \zeta_j),
\]

where $a_{jk} \in C^\infty(\mathbb{C}^n)$ with $|a_{jk}(\zeta) - \frac{\partial^2}{\partial \zeta_j \partial \zeta_k} \rho| < 1/C$ for $\zeta \in \mathcal{U}$.

Fix $\epsilon$ sufficiently small so that for $|\zeta - z| < \epsilon$ and $\zeta, z \in D_{\delta_0} \setminus D_{-\delta_0}$,

\[
\operatorname{Re} F(z,\zeta) \geq r(\zeta) - r(z) + |z - \zeta|^2/C_0.
\]

Let $\chi$ be a $C^\infty$ function satisfying $\chi(\zeta) = 1$ for $|\zeta| < 3\epsilon/4$ and $\chi(\zeta) = 0$ for $|\zeta| > 7\epsilon/8$. Take $\delta_2 < \frac{1}{4C_0} (\frac{3\epsilon}{4})^2$ and $\delta_2 \in (0, \delta_1)$. For $z \in D_{\delta_2}^+, \zeta \in D_{\delta_1}^+ \setminus D_{-\delta_2}^+$, and $|\zeta - z| > 3\epsilon/4$, we have

\[
\operatorname{Re} F(z,\zeta) \geq r(\zeta) - r(z) + |z - \zeta|^2/C_0 \geq -2\delta_2 + \frac{1}{C_0} \left( \frac{3\epsilon}{4} \right)^2 > \frac{1}{2C_0} \left( \frac{3\epsilon}{4} \right)^2.
\]
Thus we can define
\begin{equation}
(5.3) \quad f(z, \zeta) = \begin{cases} 
\partial_z (\chi (\zeta - z) \log F(z, \zeta)) & \text{if } 3\epsilon/4 < |\zeta - z| < 7\epsilon/8, \\
0 & \text{otherwise},
\end{cases}
\end{equation}
for \( z \in D^*_d \) and \( \zeta \in D^*_d \setminus D^*_d \). Define
\begin{equation}
(5.4) \quad u(z, \zeta) = T_D^* f(\cdot, \zeta)(z), \quad \forall z \in D^*_d, \ \zeta \in D^*_d \setminus D^*_d.
\end{equation}

Here \( T_0 = T_D^* \) is a linear \( \overline{\partial} \) solution operator that admits an interior super-norm estimate on \( D^*_d \) (see [29, Thm. 2.3.5, p. 76]). Namely, for any \( \overline{\partial} \)-closed \((0,1)\)-form \( \varphi \) in \( D^*_d \), \( \overline{\partial} T_0 \varphi = \varphi \) and
\begin{equation}
(5.5) \quad |T_0 \varphi|_{C^0(D^*_d)} \leq C_0^*|\varphi|_{C^0(D^*_d)},
\end{equation}
for \( \delta \in (0, \delta) \). By the linearity of \( T_0 \) and Proposition 2.2, the \( u(z, \zeta) \) and \( \partial_z u(z, \zeta) \) are uniformly continuous in \( D^*_d \times (D^*_d \setminus D^*_d) \). We also have
\[ |f(z, \cdot)|_{D^*_d \setminus D^*_d, 1} \ \leq C(|\rho|_2), \ z \in D^*_d; \quad |u(z, \cdot)|_{D^*_d \setminus D^*_d, 1} \ \leq C^*_d C(|\rho|_2), \ z \in D^*_d. \]

Define for \( z \in D^*_d \) and \( \zeta \in D^*_d \setminus D^*_d \),
\begin{equation}
(5.6) \quad \Phi(z, \zeta) := \begin{cases} 
\frac{F(z, \zeta) e^{-u(z, \zeta)}}{\log F(z, \zeta) - u(z, \zeta)} & \text{if } |\zeta - z| \leq 3\epsilon/4, \\
0 & \text{otherwise}.
\end{cases}
\end{equation}

Then \( |\Phi(\cdot, \zeta)|_{D^*_d, 0} \leq C|\rho|_2 \) for \( \zeta \in D^*_d \setminus D^*_d \). Also
\[ |\Phi(\cdot, \cdot)|_{D^*_d \setminus D^*_d, 1} \ \leq C(C^*_d, |\rho|_2), \ z \in D^*_d. \]

Fix \( \delta^*_d \in (0, \delta^*_d) \). By Hefer’s decomposition theorem [29, p. 81], there are continuous linear mappings \( T_j : \mathcal{O}(D^*_d) \rightarrow \mathcal{O}((D^*_d)^2) \) so that \( h(\zeta) - h(z) = \sum_{j=1}^n T_j h(z, \zeta) (\zeta_j - z_j) \). Then we have
\begin{equation}
(5.7) \quad \Phi(z, \zeta) - \Phi(z, \zeta) = \sum T_j \Phi(\cdot, \zeta)(z, \zeta_j - z_j).
\end{equation}

Set \( W_j(z, \zeta) = T_j \Phi(\cdot, \zeta)(z, \zeta) \). We know that \( T_j \Phi(\cdot, \zeta)(z, \eta) \) is holomorphic in \( z, \eta \). We express the boundedness of \( T_j \) as
\begin{equation}
(5.8) \quad |T_j h|_{C^0((D^*_d)^2)} \leq C^*_j |h|_{C^0(D^*_d)}, \quad h \in \mathcal{O}(D^*_d), \quad j = 1, \ldots, n.
\end{equation}

The linearity and continuity of \( T_j \) imply that \( W_j \) and its first-order derivatives in \( \zeta \) are continuous. Since \( W_j \) is holomorphic in \( z \), the Cauchy formula implies that \( W \) is in \( C^1(D^*_d \times (D^*_d \setminus D^*_d)) \) by shrinking \( \delta \) slightly.

We now use the fact that \( \rho \) is a regularized \( C^2 \) defining function for the domain \( D \) to estimate the higher order derivatives of \( \Phi(z, \zeta) \) for \( \zeta \notin \partial D \). We restrict
\[ z \in D^*_d, \quad \zeta \in D^*_d \setminus \partial D. \]

Here we take \( \delta^*_d \in (0, \delta^*_d) \). This also allows us to use Cauchy inequality in the \( z \) variables.

By \( (5.3), (5.4) \) and the linear estimate \( (5.5) \), we first see that for each \( j \), \( \partial^2_\zeta u(z, \zeta) \) are continuous in \((z, \zeta) \in D^*_d \times (D^*_d \setminus \partial D) \). Moreover,
\[ |\partial^2_\zeta u(\cdot, \cdot)|_{D^*_d, 0} \leq C_j(C^*_d, |\rho|_2)(1 + d(\zeta)^{1-j}), \quad \zeta \in D^*_d \setminus \partial D. \]
Here we have use $|\partial_{\zeta}^j \rho(z)| \leq C_j (1 + d(z)^{1-j})$ as well as the product rule for

$$
\log F(z, \zeta) = \log \Re F(z, \zeta) + \log \left(1 + \frac{\Im F(z, \zeta)}{\Re F(z, \zeta)}\right), \quad 3\epsilon/4 < |\zeta - z| < 7\epsilon/8,
$$

where $z \in D_\delta^*, \zeta \in D_\delta^* \setminus \overline{D}$. By (5.6), we get $|\partial_{\zeta}^j \Phi(\cdot, \zeta)|_{D_\delta^* \setminus 0} \leq C_j (1 + d(\zeta)^{1-j})$. Here and in what follows, we let

$$
C_j := C_j(C_0^*, \ldots, C_n^*, |\rho|_2).
$$

By the linearity and continuity of $T_j$ and the holomorphicity of $T_j \Phi(\cdot, \zeta)(z, \eta)$ in $\eta$, we have

$$
\partial_{\zeta}^j W(z, \zeta) = \sum \left(\begin{array}{c} \alpha \\beta \end{array} \right) \partial_{\eta}^{\alpha - \beta} \left| \eta = \zeta \right| T_\zeta \partial_{\zeta}^\beta \Phi(\cdot, \zeta)(z, \eta)
$$

for $z \in D_\delta^*$ and $\zeta \in D_\delta^* \setminus \overline{D}$. By the linearity of estimate (5.8) for $T_\zeta$ and Cauchy inequalities applied to the last term, we get

$$
|\partial_{\zeta}^j W(\cdot, \zeta)|_{D_\delta^* \setminus 0} \leq C_j \sum_{j_1 + j_2 = j} \text{dist}(D_{\delta^*}, \partial D_{\delta^*})^{-j_1} (1 + d(\zeta)^{1-j_1})
$$

for $j = 1, 2, \ldots$. By Cauchy inequalities, we get

$$
|\partial_{\zeta}^j W(\cdot, \zeta)|_{D_\delta^* \setminus 0} \leq C_j \sum_{j_1 + j_2 = j} \text{dist}(D_{\delta^*}, \partial D_{\delta^*})^{-i - j_2} (1 + d(\zeta)^{1-j_2})
$$

for $\zeta \in D_{\delta^*} \setminus \overline{D}$. Finally, we fix $\delta \in (0, \delta_0)$. We have achieved (5.1).

**Theorem 5.2.** Let $D = \{z \in U: \rho_0 < 0\}$ be a strictly pseudoconvex domain with $C^2$ boundary that is relatively compact in $U$, where $\rho_0 \in C^2(U)$ and $d\rho_0 \neq 0$ in $\partial D$. Let $H_q$ be defined by (2.9) and (2.10), where $q^1 = W$ is the regularized Henkin-Ramirez function $D_\delta \times (D_\delta \setminus D_{\delta^*})$ as in Proposition 5.1 and $\Phi(z, \zeta) = W(z, \zeta) \cdot (\zeta - z)$. Let $\varphi$ be a $(0, q)$-form such that $\varphi, \overline{\partial} \varphi$ are in $C^1(\overline{D})$. Then in $D$

$$
\varphi = \overline{\partial} H_q \varphi + H_{q+1} \overline{\partial} \varphi, \quad 1 \leq q \leq n,
$$

$$
\varphi_0 = H_0 \varphi_0 + H_1 \overline{\partial} \varphi_0.
$$

Moreover, we have

$$
\|H_q \varphi\|_{\Lambda_{r+1/4}(\overline{D})} \leq C_r(D) \|\varphi\|_{\Lambda_r(\overline{D})}, \quad r > 1, q > 0,
$$

$$
\|H_q \varphi\|_{\Lambda_{q/2}} \leq C_1(D) \|\varphi\|_{\Lambda_{q/1}}, \quad q > 0,
$$

$$
\|H_0 \varphi\|_{\Lambda_r(\overline{D})} \leq C_r(D) \|\varphi\|_{\Lambda_r(\overline{D})}, \quad r > 1,
$$

$$
\|\partial_{\zeta}^2 H_0 \varphi(z)\| \leq C_1(D) \text{dist}(z, \partial D)^{-1} \|\varphi\|_{\Lambda_{q/1}, \overline{D}}, \quad z \in D.
$$

The constants $C_1(D), C_r(D)$ are stable under a small $C^2$ perturbation. They depend on the $\epsilon, N, M$ in Definition 3.6, the $L$ in Lemma 3.4, $|\partial \rho_0|_{D_\delta \setminus D_{\delta^*} \setminus 0}$, $|\overline{\partial} \rho_0|_{D_\delta \setminus D_{\delta^*} \setminus 0}$, $|\partial \rho_0|_{\overline{D}_{\delta}^2}$, as well as the constants $L_0, \delta, C_0^*, \ldots, C_n^*$ in the proof of Proposition 5.1. Therefore, $C_r(D) \leq \tilde{C}_r(D, \epsilon) < \infty$ for all $r \in (0, \infty)$ when $\tilde{D}$ has a defining function $\tilde{\rho}$ such that $|\rho_0 - \tilde{\rho}_0|_{\partial U \setminus \tilde{D}} < \epsilon$ for sufficiently small $\epsilon$.

**Proof.** Let $d(z) = \text{dist}(z, \partial D)$. Let $\rho = E_2(\epsilon^{L_0 \rho_0} - 1)$ as in Proposition 5.1. Let us first choose the local coordinates described in Proposition 4.4. As in [29, p. 73], we take

$$
s_1 = \rho(\zeta), \quad s_2 = \text{Im}(\rho \cdot (\zeta - z)), \quad t = (\text{Re}(\zeta'), \text{Im}(\zeta' - z')).$$
Let $F$ be as in Proposition 5.1. First, we have $|F(z, \zeta)| \geq \Re(F(z, \zeta)) \geq c_1|\zeta - z|^2$. Note that $ho(\zeta) \approx d(\zeta)$ and $-\rho(z) \approx d(z)$. Then we have $2|F(\zeta, z)| \geq |s_1| + \rho(\zeta) - \rho(z) + c_1|\zeta - z|^2 \geq c'(d(z) + s_1 + |s_2| + |t|^2)$. Since $z \in D$ and $\zeta \notin \overline{D}$, we also have $d(z) \leq |\zeta - z|$, $d(\zeta) \leq |\zeta - z|$, $ho(\zeta) \approx d(\zeta) \leq |\zeta - z|$. Hence $d(z) + d(\zeta) + s_1 + |s_2| + |t| \leq C|\zeta - z|$. Thus, we have verified (4.4)-(4.5) in which $\Phi = FM$. We obtain the desired estimates by Proposition 5.1, Proposition 2.2, Proposition 4.4 (ii), and Proposition 4.7.

\[ \square \]

6. Boundary regularities of the elliptic differential complex for the Levi-flat Euclidean space

We consider the complex for the exterior differential $\mathcal{D} := \partial_t + \overline{\partial}_z$, where $(z, t)$ are coordinates of $\mathbf{C}^n \times \mathbf{R}^M$. We will also write it as $\mathcal{D} = \partial^0 + \overline{\partial}$.

The Poincaré lemma for a $q$-form on a bounded star-shaped domain $S$ has the form

\[ \phi = d^0 R_q \phi + R_{q+1} d^0 \phi, \quad q > 0; \quad \phi = R_1 d^0 \phi + \phi(0), \quad q = 0; \]

\[ R_q \phi(t) := \int_{\theta \in [0, 1]} H^* \phi(t, \theta). \]

Here $H(t, \theta) = \theta t$ for $(t, \theta) \in S \times [0, 1]$; see [56, p. 224] and [58, p. 105]. If $\phi(t) = f(t) dt_1 \wedge \cdots \wedge dt_q$, we have

\[ R_q \phi(t) = \left\{ \int_0^1 f(\theta t) \theta^{q-1} d\theta \right\} \sum (-1)^{j-1} t_j dt_1 \cdots \hat{d}_j \cdots dt_q. \]

It is immediate that for $q > 0$

\[ |R_q \phi|_{S, r} \leq C_r |\phi|_{S, r}, \quad 0 \leq r < \infty. \quad (6.1) \]

Here $C_r$ depends only on the diameter of $S$. By the interpolation argument, we obtain

\[ |R_q \phi|_{\Lambda_r(\overline{S})} \leq C_r(D) \rho |\phi|_{\Lambda_r(\overline{S})}, \quad 0 < r < \infty, \quad (6.2) \]

provided $S$ is a bounded Lipschitz domain.

A differential form $\varphi$ is called of mixed type $(0, q)$ if $\varphi = \sum_{i=0}^q [\varphi]_i$, where

\[ [\varphi]_i = \sum_{|I| = i, |I| + |J| = q} a_{IJ} d\overline{z}_I \wedge dt_J. \]

Thus $[\varphi]_i = 0$, for $i > n$. The $\mathcal{D}$ acts on a function $f$ and a $(0, q)$-form as follows

\[ \mathcal{D} f = \sum \frac{\partial f}{\partial t_m} dt_m + \sum \frac{\partial f}{\partial z_\alpha} d\overline{z}_\alpha, \quad \mathcal{D} \sum a_{IJ} d\overline{z}_I \wedge dt_J = \sum \mathcal{D} a_{IJ} \wedge d\overline{z}_I \wedge dt_J. \]

We have $\mathcal{D}^2 = 0$ and $d^0 \overline{\partial} + \overline{\partial} d^0 = 0$. We also have

\[ [\mathcal{D} \varphi]_0 = d^0[\varphi]_0, \quad [\mathcal{D} \varphi]_i = d^0[\varphi]_{i+1} + \overline{\partial}[\varphi]_i, \quad 0 < i \leq n. \]

For $\varphi = \sum \varphi_{IJ} d\overline{z}_I \wedge dt_J = \sum \varphi_{IJ} dt_J \wedge d\overline{z}_I$ on $\overline{\mathcal{D}} \times \overline{S}$, define

\[ H_i \varphi = \sum_{|I| = i} H_i (\varphi_{IJ} d\overline{z}_I) \wedge dt_J, \quad R_i \varphi = \sum_{|I| = i} R_i (\varphi_{IJ} dt_J) \wedge d\overline{z}_I. \]

Thus $H_i \varphi = H_i [\varphi]_i$, while $R_{q-i} \varphi = R_{q-i} [\varphi]_i$ if $\varphi$ has the (mixed) type $(0, q)$. 

\[ \square \]
Definition 6.1. Let $0 < r \leq 1$ and $0 \leq \alpha < 1$. Let $\Lambda^{0}_r(\overline{D} \times \overline{S})$ be the set of continuous functions $f$ in $\overline{D} \times \overline{S}$ so that $t \to |f(\cdot, t)|_{\Lambda^r(\overline{D})}$ is bounded in $\overline{S}$. For $a > 1$ and $k \in \mathbb{N}$, let $\Lambda^{a,k}_r(\overline{D} \times \overline{S})$ be the set of functions $f$ so that $\partial^a_j \partial^j f$ are in $\Lambda^{a-i,0}_r(\overline{D} \times \overline{S})$ for $j \leq k$ and $i < a$. Define $C^{a,k}_r(\overline{D} \times \overline{S})$ analogously.

We now derive the following homotopy formulae.

Proposition 6.2. Let $1 \leq q \leq n + m$. Let $D$ be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with $\partial D \in C^2$ and let $S$ be a bounded domain in $\mathbb{R}^m$ so that $\partial S \subset S$ for $\theta \in [0, 1]$. Let $\varphi$ be a mixed $(0, q)$-form in $D \times S$.

(i) If $\varphi \in C^{1,1}_r(\overline{D} \times \overline{S})$ and $\overline{\partial} \varphi$ are in $C^{1,0}_r(\overline{D} \times \overline{S})$, then

\begin{align}
\varphi &= DT_q \varphi + T_{q+1} \partial \varphi, \\
T_q \varphi &= R_q H_0[\varphi]_0 + \sum_{i > 0} H_i[\varphi]_i.
\end{align}

(ii) If $\varphi \in C^1(\overline{D} \times \overline{S})$ and $\overline{\partial}[\varphi]_q(\cdot, 0) \in C^1(\overline{D})$, then

\begin{align}
\varphi &= D \overline{T}_q \varphi + \overline{T}_{q+1} \partial \varphi, \\
\overline{T}_q \varphi(z, t) &= H_q[\varphi]_q(z, 0) + \sum_{i < q} R_{q-i}[\varphi]_i(z, \cdot)(t).
\end{align}

Proof. For the related homotopy formulae when $H_i$'s are replaced by the Leray-Koppelman homotopy operators, see Treves [58, Sect. VI.7.12, Sect. VI.7.13, p. 294] for suitable $q$'s and [22] for arbitrary $q$.

(i) Recall that the homotopy operators $H_i$ are linear. To derive the homotopy formulae for $\mathcal{D}$, we will use the following estimates from (5.12):

\[ |H_i[\varphi]_i|_{\mathcal{D}, 0} \leq C|\varphi|_{\mathcal{D}, 1}, \quad i = 0, 1, \ldots n. \]

Thus if $\psi_j$ converges to $\psi$ in $C^1(\overline{D})$ norm, then

\[ \lim_{j \to \infty} H_i \psi_j = H_i \psi. \]

Also for $\psi \in C^{1,1}_r(\overline{D} \times \overline{S})$, we have

\[ \frac{\partial}{\partial t_j} H_i \psi(\cdot, t) = H_i \frac{\partial}{\partial t_j} \psi(\cdot, t). \]

Analogously, if $\psi_j$ converges to $\psi$ in $C^0(\overline{S})$, then $\lim_{j \to \infty} R_i \psi_j = R_i \psi$. For $\psi \in C^{1,0}_r(\overline{D} \times \overline{S})$, we have $\frac{\partial}{\partial t_j} R_i \psi = R_i \frac{\partial}{\partial t_j} \psi$. Note that $\overline{\partial}$ commutes with the pull-back $H^*$ of $H(t, \theta) = \theta t$. By (2.3), we have

\begin{align}
\partial^0 H_i \varphi &= -H_i \partial^0 \varphi, \quad \varphi \in C^{1,1}_r(\overline{D} \times \overline{S}), \\
\partial R_i \varphi &= -R_i \partial \varphi, \quad \varphi \in C^{1,0}_r(\overline{D} \times \overline{S}).
\end{align}

Let us start with the integral representation of $[\varphi]_0$ in $D$. Since $\varphi$ has total degree $q$, then $\deg_\partial [\varphi]_0 = q > 0$. We apply (5.10) for functions and the Poincaré formula for $\partial^0$ by (6.9). Thus for $[\varphi]_0 \in C^{1,1}_r$, we obtain in $D \times S$

\[ [\varphi]_0 = H_0[\varphi]_0 + H_1 \overline{\partial}[\varphi]_0 = (\partial R_q H_0[\varphi]_0 + R_{q+1} \partial^0 H_0[\varphi]_0) + H_1 \overline{\partial}[\varphi]_0. \]
Since \( \partial_z \Omega_{0,0}^1(z, \zeta) = 0 \), we have \( d^0 R_q H_0[\varphi]_0 = D R_q H_0[\varphi]_0 \). Combining it with \( d^0[\varphi]_0 = [D\varphi]_0 \), we express (6.11) as

\[
(6.12) \quad [\varphi]_0 = D R_q H_0[\varphi]_0 + R_{q+1} H_0[\varphi]_0 + H_1 \partial\varphi_0.
\]

Analogously, for \( [\varphi]_j \in C_+^{1,1} (\overline{D} \times S) \), we get

\[
(6.13) \quad [\varphi]_j = \overline{\partial} H_j [\varphi]_j + H_{j+1} \partial [\varphi]_j = D H_j [\varphi]_j - d^0 H_j [\varphi]_j + H_{j+1} \partial [\varphi]_j.
\]

By (6.9) and \( d^0[\varphi]_j = [D\varphi]_j - \overline{\partial}[\varphi]_{j-1} \), we obtain

\[
\sum_{j > 0} (-d^0 H_j [\varphi]_j + H_{j+1} \overline{\partial} [\varphi]_j) = \sum_{j > 0} (H_j [D\varphi]_j - H_j \overline{\partial} [\varphi]_{j-1} + H_{j+1} \partial [\varphi]_j) = -H_1 \overline{\partial} [\varphi]_0 + \sum_{j > 0} H_j [D\varphi]_j.
\]

Here we have used \( H_{n+1} = 0 \). Combining it with (6.12) and (6.13), we obtain

\[
\varphi = D R_q H_0[\varphi]_0 + R_{q+1} H_0[\varphi]_0 + \sum_{j > 0} D H_j [\varphi]_j + \sum_{j > 0} H_j [D\varphi]_j,
\]

which gives us (i).

(ii) By \( [\varphi]_j \in C^1(\overline{S} \times S) \) and the Poincaré lemma, we obtain

\[
\varphi = [\varphi]_q + \sum_{i < q} (d^0 R_{q-i}[\varphi]_i + R_{q+1-i} d^0[\varphi]_i) = [\varphi]_q + \sum_{i < q} D R_{q-i}[\varphi]_i + R_{q+1-i} [D\varphi]_i.
\]

Here we have used \( \overline{\partial} R_{q-i}[\varphi]_i = -R_{q-i} \overline{\partial}[\varphi]_i \) for \( i < q \) by (6.9) and \( R_{q+1-i} d^0[\varphi]_i = R_{q+1-i} [D\varphi]_i - R_{q-i} \overline{\partial}[\varphi]_i \). We express

\[
\sum_{i < q} R_{q+1-i} [D\varphi]_i = \sum_{i < q} R_{q+1-i} ([d^0\varphi]_i + [\overline{\partial}\varphi]_i + 1) = -R_1 [d^0\varphi]_q + \sum_{i < q} R_{q+1-i} [D\varphi]_i + 1,
\]

because \( [\overline{\partial}\varphi]_0 = 0 \). We have \( [\varphi]_q(z, t) - R_1 d^0[\varphi]_q(z, t) = [\varphi]_q(z, 0) \). We now apply the homotopy formula (2.7) to obtain

\[
[\varphi]_q(\cdot, 0) = \overline{\partial} H_q [\varphi]_q(\cdot, 0) + H_{q+1} \overline{\partial} [\varphi]_q(\cdot, 0) = D H_q [\varphi]_q(\cdot, 0) + H_{q+1} [D\varphi]_{q+1}(\cdot, 0).
\]

Combining the identities, we get (ii). \qed

**Theorem 6.3.** Let \( q > 0 \). Let \( D \) be a strictly pseudoconvex domain with \( C^2 \) boundary. Let \( S \) be a bounded star-shaped domain in \( \mathbb{R}^m \). Let \( \varphi \) be a \( D \)-closed \((0,q)\)-form in \( C^1(\overline{D} \times S) \). Then there exists a solution \( u \in C^1(\overline{D} \times S) \) to \( Du = \varphi \). Furthermore, the following properties hold.

(i) Suppose that \( [\varphi]_0 = 0 \) and \( \overline{\partial}\varphi \in C_+^{1,0} (\overline{D} \times S) \). If \( \varphi \in \Lambda_+^{r,k}(\overline{D} \times S) \) with \( k \in \{0,1, \ldots, \infty\} \) and \( r \in (1, \infty) \), the \( u \) is in \( \Lambda_+^{r+1/2,k}(\overline{D} \times S) \).

(ii) Let \( r \in [1, \infty] \). If \( \varphi \in C^r(\overline{D} \times S) \), the \( u \) is in \( C^r(\overline{D} \times S) \). If \( \varphi \in \Lambda_+^{r,k}(\overline{D} \times S) \) and \( S \) is a Lipschitz domain, the \( u \) is in \( \Lambda_+(\overline{D} \times S) \) for \( r > 1 \).

**Proof.** (i) follows from (6.4) and (5.11). (ii) follows from (6.6), (6.1), and (5.13). Indeed, we have \( \overline{\partial}[\varphi]_q(\cdot, 0) = 0 \) as \( D\varphi = 0 \). We first obtain the assertion when \( r \) is non-integer. The general case is obtained via interpolation for the Lipschitz domain \( D \times S \). \qed
REFERENCES

[1] P. Ahern and R. Schneider, Holomorphic Lipschitz functions in pseudoconvex domains, Amer. J. Math. 101 (1979), no. 3, 543–565, DOI 10.2307/2373797. MR533190

[2] W. Alexandre, C^k-estimates for the ∂-equation on convex domains of finite type, Math. Z. 252 (2006), no. 3, 473–496, DOI 10.1007/s00209-005-0812-y. MR2207755

[3] W. Alt, Hölderabschätzungen für Ableitungen von Lösungen der Gleichung ∂u = f bei streng pseudokonvexem Rand, Manuscripta Math. 13 (1974), 381–414 (German, with English summary). MR0352536

[4] Ch. Brinkmann, Lösungssoperatoren für den Cauchy-Riemann-Komplex auf Gebieten mit stückweise glattem, streng pseudokonvexen Rand in allgemeiner Lage mit C^k-Abschätzungen, Diplomarbeit, 1–157, Bonn, 1984.

[5] P. L. Butzer and H. Berens, Semi-groups of operators and approximation, Die Grundlehren der mathematischen Wissenschaften, Band 145, Springer-Verlag New York Inc., New York, 1967. MR0230022

[6] A.-P. Calderón, Lebesgue spaces of differentiable functions and distributions, Proc. Sympos. Pure Math., Vol. IV, American Mathematical Society, Providence, R.I., 1961, pp. 33–49. MR0143037

[7] D. Catlin, Subelliptic estimates for the ∂-Neumann problem on pseudoconvex domains, Ann. of Math. (2) 126 (1987), no. 1, 131–191, DOI 10.2307/1971347. MR898054

[8] D.-C. E. Chang, Optimal L^p and Hölder estimates for the ∂-equation on strongly pseudoconvex domains, Trans. Amer. Math. Soc. 315 (1989), no. 1, 273–304, DOI 10.2307/2001384. MR937241

[9] D.-C. Chang, A. Nagel, and E. M. Stein, Estimates for the ∂-Neumann problem in pseudoconvex domains of finite type in \mathbb{C}^2, Acta Math. 169 (1992), no. 3-4, 153–228, DOI 10.1007/BF02392760. MR1194003

[10] J. Chaumat and A.-M. Chollet, Estimations höldériennes pour les équations de Cauchy-Riemann dans les convexes compacts de \mathbb{C}^n, Math. Z. 207 (1991), no. 4, 501–534, DOI 10.1007/BF02571405 (French). MR1119954

[11] S.-C. Chen and M.-C. Shaw, Partial differential equations in several complex variables, AMS/IP Studies in Advanced Mathematics, vol. 19, American Mathematical Society, Providence, RI; International Press, Boston, MA, 2001. MR1800297 (2001m:32071)

[12] M. Christ, Regularity properties of the ∂b equation on weakly pseudoconvex CR manifolds of dimension 3, J. Amer. Math. Soc. 1 (1988), no. 3, 43–61, DOI 10.2307/1990950. MR928903

[13] A. Cumenge, Sharp estimates for ∂ on convex domains of finite type, Ark. Mat. 39 (2001), no. 1, 1–25, DOI 10.1007/BF02388789. MR1807801

[14] J. P. D’Angelo, Real hypersurfaces, orders of contact, and applications, Ann. of Math. (2) 115 (1982), no. 3, 615–637, DOI 10.2307/2007015. MR657241

[15] K. Diederich, B. Fischer, and J. E. Fornæss, Hölder estimates on convex domains of finite type, Math. Z. 232 (1999), no. 1, 43–61, DOI 10.1007/PL00004758. MR1714279

[16] K. Diederich, J. E. Fornæss, and J. Wiegerinck, Sharp Hölder estimates for ∂ on ellipsoids, Manuscripta Math. 56 (1986), no. 4, 399–417, DOI 10.1007/BF01168502. MR860730

[17] M. Elgueta, Extension to strictly pseudoconvex domains of functions holomorphic in a submanifold in general position and \mathbb{C}^\infty up to the boundary, Illinois J. Math. 24 (1980), no. 1, 1–17. MR550648

[18] C. L. Fefferman and J. J. Kohn, Hölder estimates on domains of complex dimension two and on three-dimensional CR manifolds, Adv. in Math. 69 (1988), no. 2, 223–303, DOI 10.1016/0001-8708(88)90002-3. MR946264

[19] C. L. Fefferman, J. J. Kohn, and M. Machedon, Hölder estimates on CR manifolds with a diagonalizable Levi form, Adv. Math. 84 (1990), no. 1, 1–90, DOI 10.1016/0001-8708(90)90036-M. MR1075233

[20] G. B. Folland and E. M. Stein, Estimates for the ∂b complex and analysis on the Heisenberg group, Comm. Pure Appl. Math. 27 (1974), 429–522. MR0367477

[21] G. Glaeser, Étude de quelques algèbres tayloriennes, J. Analyse Math. 6 (1958), 1–124; erratum, insert to 6 (1958), no. 2, DOI 10.1007/BF02790231 (French). MR0101294

[22] X. Gong, A Frobenius-Nirenberg theorem with parameter, J. reine angew. Math., available at https://doi.org/10.1515/crelle-2017-0051.

[23] X. Gong and K.-T. Kim, The \overline{\partial}-equation on variable strictly pseudoconvex domains, Math. Z. (2017), available at https://doi.org/10.1007/s00209-017-1211-z.
REFERENCES

[24] X. Gong and S. M. Webster, *Regularity for the CR vector bundle problem II*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **10** (2011), no. 1, 129–191. MR2829316

[25] H. Grauert and I. Lieb, *Das Ramirzscbe Integral und die Lösung der Gleichung \( \partial f = \alpha \) im Bereich der beschränkten Formen*, Rice Univ. Studies **56** (1970), no. 2, 29–50 (1971) (German). MR0273057

[26] P. C. Greiner and E. M. Stein, *Estimates for the \( \overline{\partial} \)-Neumann problem*, Princeton University Press, Princeton, N.J., 1977. Mathematical Notes, No. 19. MR0499319

[27] N. Hanges and H. Jacobowitz, *The Euclidean elliptic complex*, Indiana Univ. Math. J. **46** (1997), no. 3, 753–770, DOI 10.1512/iumj.1997.46.1397. MR1488336

[28] G. M. Henkin, *Integral representation of functions which are holomorphic in strictly pseudoconvex regions, and some applications*, Mat. Sb. (N.S.) **78 (120)** (1969), 611–632 (Russian). MR0249660

[29] G. M. Henkin and J. Leiterer, *Theory of functions on complex manifolds*, Monographs in Mathematics, vol. 79, Birkhäuser Verlag, Basel, 1984. MR774049

[30] G. M. Henkin and A. V. Romanov, *Exact Hölder estimates of the solutions of the \( \partial \)-equation, Izv. Akad. Nauk SSSR Ser. Mat. **35** (1971), 1171–1183 (Russian). MR0293121

[31] N. Kerzman, *Hölder and \( L^p \) estimates for solutions of \( \partial u = f \) in strongly pseudoconvex domains*, Comm. Pure Appl. Math. **24** (1971), 301–379. MR0281944

[32] K. D. Koenig, *On maximal Sobolev and Hölder estimates for the tangential Cauchy-Riemann operator and boundary Laplacian*, Amer. J. Math. **124** (2002), no. 1, 129–197. MR1879002

[33] J. J. Kohn, *Harmonic integrals on strongly pseudo-convex manifolds. II*, Ann. of Math. (2) **79** (1964), 450–472. MR0208200

[34] ______, *Global regularity for \( \partial \) on weakly pseudo-convex manifolds*, Trans. Amer. Math. Soc. **181** (1973), 273–292. MR0344703

[35] I. Lieb, *Die Cauchy-Riemannschen Differentialgleichungen auf streng pseudokonvexen Gebieten. Beschränkte Lösungen*, Math. Ann. **190** (1970/1971), 6–44 (German). MR0283235 (44 #468)

[36] I. Lieb and R. M. Range, *Lösungsopteroren für den Cauchy-Riemann-Komplex mit \( C^k \)-Abschätzungen*, Math. Ann. **253** (1980), no. 2, 145–164, DOI 10.1007/BF01578911 (German). MR597825

[37] ______, *Integral representations and estimates in the theory of the \( \overline{\partial} \)-Neumann problem*, Ann. of Math. (2) **123** (1986), no. 2, 265–301, DOI 10.2307/1971272. MR835763

[38] ______, *Estimates for a class of integral operators and applications to the \( \overline{\partial} \)-Neumann problem*, Invent. Math. **85** (1986), no. 2, 415–438, DOI 10.1007/BF01388997. MR846935

[39] L. Ma and J. Michel, *Local regularity for the tangential Cauchy-Riemann complex*, J. Reine Angew. Math. **442** (1993), 63–90, DOI 10.1515/crll.1993.442.63. MR1234836

[40] J. Michel, *Randregularität des \( \overline{\partial} \)-Problems für stückweise streng pseudokonvexe Gebiete in \( \mathbb{C}^n \)*, Math. Ann. **280** (1988), no. 1, 45–68, DOI 10.1007/BF01474180 (German). MR928297

[41] ______, *Integral representations on weakly pseudoconvex domains*, Math. Z. **208** (1991), no. 3, 437–462, DOI 10.1007/BF02571538. MR1134587

[42] J. Michel and A. Perotti, *\( C^k \)-regularity for the \( \overline{\partial} \)-equation on strictly pseudoconvex domains with piecewise smooth boundaries*, Math. Z. **203** (1990), no. 3, 415–427, DOI 10.1007/BF02570747. MR1038709

[43] J. Michel and M.-C. Shaw, *A decomposition problem on weakly pseudoconvex domains*, Math. Z. **230** (1999), no. 1, 1–19, DOI 10.1007/PL00004685. MR1671846

[44] J. D. McNeal, *Estimates on the Bergman kernels of convex domains*, Adv. Math. **109** (1994), no. 1, 108–139. MR1302759

[45] J. D. McNeal and E. M. Stein, *Mapping properties of the Bergman projection on convex domains of finite type*, Duke Math. J. **73** (1994), no. 1, 177–199, DOI 10.1215/S0012-7094-94-07307-9. MR1257282

[46] N. Øvrelid, *Integral representation formulas and \( L^p \)-estimates for the \( \overline{\partial} \)-equation*, Math. Scand. **29** (1971), 137–160. MR0324073

[47] K. Peters, *Solution operators for the \( \overline{\partial} \)-equation on nontransversal intersections of strictly pseudoconvex domains*, Math. Ann. **291** (1991), no. 4, 617–641, DOI 10.1007/BF01445231. MR1135535

[48] D. H. Phong and E. M. Stein, *Estimates for the Bergman and Szegő projections on strongly pseudoconvex domains*, Duke Math. J. **44** (1977), no. 3, 695–704. MR0450623

[49] P. L. Poljakov, *Banach cohomology on piecewise strictly pseudoconvex domains*, Mat. Sb. (N.S.) **88(130)** (1972), 238–255 (Russian). MR0301238

[50] E. Ramírez de Arellano, *Ein Divisionsproblem und Randintegraldarstellungen in der komplexen Analysis*, Math. Ann. **184** (1969/1970), 172–187 (German). MR0269874
[51] R. M. Range, *Holomorphic functions and integral representations in several complex variables*, Graduate Texts in Mathematics, vol. 108, Springer-Verlag, New York, 1986. MR847923

[52] _____, *Integral kernels and Hölder estimates for $\overline{\partial}$ on pseudoconvex domains of finite type in $\mathbb{C}^2$*, Math. Ann. 288 (1990), no. 1, 63–74, DOI 10.1007/BF01444521. MR1070924

[53] R. M. Range and Y.-T. Siu, *Uniform estimates for the $\overline{\partial}$-equation on domains with piecewise smooth strictly pseudoconvex boundaries*, Math. Ann. 206 (1973), 325–354, DOI 10.1007/BF01355986. MR0338450

[54] M.-C. Shaw, *Optimal Hölder and $L^p$ estimates for $\overline{\partial}_b$ on the boundaries of real ellipsoids in $\mathbb{C}^n$*, Trans. Amer. Math. Soc. 324 (1991), no. 1, 213–234, DOI 10.2307/2001504. MR1005084

[55] Y.-T. Siu, *The $\overline{\partial}$ problem with uniform bounds on derivatives*, Math. Ann. 207 (1974), 163–176. MR0330515

[56] M. Spivak, *A comprehensive introduction to differential geometry. Vol. I*, 3rd ed., Publish or Perish, Inc., Houston, TX, 1999.

[57] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N.J., 1970. MR0290095

[58] F. Treves, *Hypo-analytic structures*, Princeton Mathematical Series, vol. 40, Princeton University Press, Princeton, NJ, 1992. Local theory. MR1200459

[59] H. Triebel, *Interpolation theory, function spaces, differential operators*, 2nd ed., Johann Ambrosius Barth, Heidelberg, 1995. MR1328645

[60] S. M. Webster, *A new proof of the Newlander-Nirenberg theorem*, Math. Z. 201 (1989), no. 3, 303–316, DOI 10.1007/BF01214897. MR999729 (90f:32012)

[61] _____, *On the local solution of the tangential Cauchy-Riemann equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire 6 (1989), no. 3, 167–182 (English, with French summary). MR995503

[62] H. Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Trans. Amer. Math. Soc. 36 (1934), no. 1, 63–89, DOI 10.2307/1989708. MR1501735

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