MITTAG-LEFFLER MODULES AND DEFINABLE SUBCATEGORIES. II

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This note on countably generated relative Mittag-Leffler modules is a continuation of [R3], henceforth referred to as Part I. Its terminology, preliminaries, and results are freely used throughout. (In particular, modules are left $R$-modules unless stated otherwise.) All unspecified citations are to that paper. Theorem 7.1(1) states:

The countably generated $K$-Mittag-Leffler modules are precisely the uniform $L$-pure images of $\omega$-limits.

Recall from Def. 7.4, an $L$-$\omega$-limit is an $\omega$-limit of finitely presented modules that is $L$-pure on images. Here, and throughout, $K$ and $L$ are definably dual classes of right, resp., left $R$-modules in the sense of Def. 2.5, i.e., they generate mutually elementarily dual definable subcategories, cf. Conv. 2.7. By the main theorem of [R1] (or Part I), the $K$-Mittag-Leffler modules are precisely the $L$-atomic modules. One could therefore replace $K$-Mittag-Leffler by $L$-atomic everywhere, as was done in [P3].

In Theorem 7.1(2) ‘uniform $L$-pure images’ were incorrectly omitted. The resulting discrepancy in the classical case was detected and communicated to me by Jan Trlifaj, for which I am very grateful.

After taking the opportunity to correct the statement of Thm. 7.1(2), I continue the study of uniform purity of epimorphisms in order to derive the main result, Thm. 3.1(2), which states that—provided $R \in \langle K \rangle$ (equivalently, $R \subseteq \langle L \rangle$, the definable subcategory generated by $L$)—every countably generated $K$-Mittag-Leffler module in $\langle L \rangle$ is a direct summand of a $\langle L \rangle$-preenvelope of a union of an $L$-pure $\omega$-chain of finitely presented modules. In conclusion I present a number of examples that starts with and grew out of the study of $L$-purity (of monomorphisms in $\mathbb{Z}$-Mod) for $L = \text{Div}$, the definable subcategory of divisible abelian groups.

1. THE CORRECTED THEOREM 7.1(2)

If $R \in \langle K \rangle$, the countably generated $K$-Mittag-Leffler modules are precisely the uniform $L$-pure images of unions of $L$-pure $\omega$-chains of finitely presented modules.

Deleting the incorrect application of Lemma 5.11—which simply does not apply to ‘right pure’ maps of §5.4 (but only to ‘left pure’ maps of §§5.1+2)—the proof given for Thm. 7.1(2) (in Part I) yields just that.

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2. Uniform purity

This entire section goes back to [R1, Lemma 3.9], where it was proved model-theoretically that countably generated relative Mittag-Leffler modules are relatively pure projective. Here I reiterate this material in terms introduced in Part I, especially some splitting behavior of relative Mittag-Leffler modules—the terms were coined recently, the proofs are old.

In [R2, Prop. 2.5] it was proved that every pure epimorphism from a module in \( \mathcal{L} \) onto a countably generated \( \mathcal{L} \)-atomic module splits. As pointed out there, this result is just a variant of [R1, Lemma 3.9]. Here I present another variant. To make the arguments more transparent, I distinguish three kinds of right \( \mathcal{L} \)-purity, i.e., purity for epimorphisms, and I break up the proof accordingly.

**Definition 2.1.**

1. \([R4, \text{Def.1.1(2)}]\). The map \( g : B \to C \) is an \( \mathcal{L} \)-pure epimorphism if for every tuple \( \bar{v} \in C \) and every pp formula \( \varphi \) it satisfies, there is a \( g \)-preimage \( \bar{b} \in B \) which satisfies a pp formula \( \psi \leq_{\mathcal{L}} \varphi \).

2. \([R3, \text{Def. 5.12}]\). The map \( g \) is a uniform \( \mathcal{L} \)-pure epimorphism if every tuple in \( \bar{v} \in C \) has a \( g \)-preimage \( \bar{b} \) such that \( \text{pp}_B(\bar{b}) \sim_{\mathcal{L}} \text{pp}_C(\bar{v}) \).

3. The map \( g \) is a strict uniform \( \mathcal{L} \)-pure epimorphism if for all tuples \( \bar{v} \in B \) and \( \bar{c} \in C \) with \( g(\bar{v}) = \bar{c} \) and \( \text{pp}_B(\bar{v}) \sim_{\mathcal{L}} \text{pp}_C(\bar{c}) \), every element \( c \in C \) has a \( g \)-preimage \( \bar{b} \) such that \( \text{pp}_B(\bar{b}) \sim_{\mathcal{L}} \text{pp}_C(\bar{c},c) \).

4. The prefix \( \mathcal{L} \) is omitted when it is all of \( R - \text{Mod} \).

5. Note that always \( \text{pp}_B(\bar{b}) \subseteq \text{pp}_C(\bar{c}) \) and that the properties above are an invariant of the definable subcategory generated by \( \mathcal{L} \).

6. In particular, w.l.o.g., \( \mathcal{L} = \langle \mathcal{L} \rangle \) everywhere in these concepts.

Upon iterating the strict condition one easily sees that strict uniform \( \mathcal{L} \)-pure epimorphisms are uniform \( \mathcal{L} \)-pure epimorphisms. In turn, by [R3, Rem. 2.1], uniform \( \mathcal{L} \)-pure epimorphisms are \( \mathcal{L} \)-pure epimorphisms.

**Remark 2.2.**

1. If \( B \in \mathcal{L} \) in (1), then \( \bar{v} \) in \( B \) satisfies the formula \( \varphi \) itself. Hence every \( \mathcal{L} \)-pure epimorphism \( g : B \to C \) is a pure epimorphism (i.e., \( R - \text{Mod-pure epic} \)).

2. If \( B \in \mathcal{L} \) in (2), then \( \bar{v} \) in \( B \) realizes the entire type \( \text{pp}_C(\bar{v}) \), hence \( \text{pp}_B(\bar{v}) = \text{pp}_C(\bar{v}) \). Thus every uniform \( \mathcal{L} \)-pure epimorphism \( g : B \to C \) is a uniform pure epimorphism (i.e., uniform \( R - \text{Mod-pure epic} \)).

3. Similarly, if \( B \in \mathcal{L} \), then every strict uniform \( \mathcal{L} \)-pure epimorphism \( g : B \to C \) is a strict uniform pure epimorphism (i.e., strictly uniform \( R - \text{Mod-pure epic} \)).

Arguments like the following are standard in model theory. This particular one is a major ingredient in the proof of aforementioned [R1, Lemma 3.9]. Note, \( \mathcal{L} = R - \text{Mod} \) here.

**Lemma 2.3.** Strict uniform epimorphisms onto countably generated modules split.

**Proof.** Suppose \( C = \langle c_0, c_1, c_2, \ldots \rangle \) and let \( g : B \to C \) be a strict uniform epimorphism. Starting from the empty tuple, choose a preimage \( b_0 \in B \) of \( c_0 \) of same pp type, and continue and successively choose preimages \( b_1, b_2, \ldots, b_i, \ldots \) of \( c_1, c_2, \ldots, c_i, \ldots \), respectively, such that \( \text{pp}_B(b_0, b_1, \ldots, b_i) = \text{pp}_C(c_0, c_1, \ldots, c_i) \) for all \( i \). Then the assignment \( h(c_i) = b_i \) defines a map, since if \( \sum_{i \leq n} c_i = 0 \) in \( C \) then \( \sum_{i \leq n} c_i = 0 \in \text{pp}_C(c_1, c_2, \ldots, c_i) = \text{pp}_B(b_1, b_2, \ldots, b_i) \), hence \( \sum_{i \leq n} b_i = 0 \) in \( B \). Applying the same argument to formulas expressing linear dependence shows, \( h \) is a homomorphism. Consequently, \( h \) is a section of \( g \), this proving that \( g \) splits. \( \square \)
**Remark 2.8.**

(1) By Theorem 7.1(1) (of Part I), every countably generated purity for monomorphisms (as opposed to epimorphisms, as considered previously).

...some pp formula holds for every tuple |

...summand of |

**Remark 2.7.**

Over any ring, if $M \subseteq N$, then $M$ is $\mathcal{L}$-pure in $N$ iff the following holds for every tuple $\overline{\pi}$ in $M$: if $\overline{\pi}$ satisfies a pp formula $\varphi$ in $N$, then it also satisfies some pp formula $\psi \leq_{\mathcal{L}} \varphi$ in $M$, see (the proof of) [R3, Lemma 5.3].

**Remark 2.8.** (1) By Theorem 7.1(1) (of Part I), every countably generated $\mathcal{L}$-atomic module is a uniform $\mathcal{L}$-pure image of an $\mathcal{L}$-ω-limit. If $\mathcal{L}$ is large enough
as to contain every such $\mathcal{L}$-$\omega$-limit, then, by the corollary, every countably generated $\mathcal{L}$-atomic module is a direct summand of an $\mathcal{L}$-$\omega$-limit.

(2) Similarly, Theorem 7.1(2), see above, can be improved on as follows in case $\mathcal{L}$ contains all unions of $\mathcal{L}$-pure chains of finitely presented modules and all absolutely pure modules (the latter is equivalent to $R_R \in \langle \mathcal{K} \rangle$).

Then the countably generated $K$-Mittag-Leffler modules are precisely the direct summands of unions of $\mathcal{L}$-pure chains of finitely presented modules.

(3) This is the case if $\mathcal{L} = R$-Mod. As then the unions in question are direct sums of finitely presented modules, we get back the classical result from [RG, p. 74, 2.2.2] that countably generated Mittag-Leffler-modules are pure-projective. This suggests the question posed in Section 4. But before asking more questions I combine these splitting facts with what is known about preenvelopes in definable subcategories.

3. Enter preenvelopes

Next we prove that every countably generated $K$-Mittag-Leffler module $N$ in $\mathcal{L}$ is a direct summand of a $\langle \mathcal{L} \rangle$-preenvelope of some $\mathcal{L}$-$\omega$-limit. We know from Theorem 7.1(1) that there is an $\mathcal{L}$-$\omega$-limit $M$ and a uniform $\mathcal{L}$-pure epimorphism $h : M \twoheadrightarrow N$. Let $D = \langle \mathcal{L} \rangle$ be the definable subcategory generated by $\mathcal{L}$. By [RS, Corollary 3.5(c)], every module $M$ has a $\mathcal{D}$-preenvelope, i.e., a map $\varepsilon_M : M \rightarrow D_M$, with $D_M \in D$, through which every other map from $M$ to a member of $\mathcal{D}$ factors. Applying this to the the $\mathcal{L}$-$\omega$-limit $M$, we see that $h$ factors through $\varepsilon_M$, which gives us $h_D : D_M \rightarrow N$ such that $h = h_D \varepsilon_M$. The same simple argument that shows $h_D$ is surjective also proves that it is, as a matter of fact, a uniform $\mathcal{L}$-pure epimorphism (remember, morphisms preserve pp formulas (and types)!). Thus Cor. 2.6 implies that $h_D$ splits, this making $N$ a direct summand of $D_M$, which proves (1) below. (Note for the converse that definable subcategories are closed under direct summands).

**Theorem 3.1.** Let $\mathcal{D} = \langle \mathcal{L} \rangle$, the definable subcategory generated by $\mathcal{L}$. Suppose $N$ is a countably generated $K$-Mittag-Leffler (= $\mathcal{L}$-atomic) module.

There is a $\mathcal{D}$-preenvelope $D$ of an $\mathcal{L}$-$\omega$-limit, $M$, and a uniform $\mathcal{L}$-pure epimorphism $h_D : D \rightarrow N$. For any such $D$ and $h_D$ the following holds.

1. $N \in D$ if and only if ($h_D$ splits and) $N$ is a direct summand of $D$.
2. If $R_R \in \langle \mathcal{K} \rangle$ (equivalently, $R_R^* \subseteq D$), then every countably generated $K$-Mittag-Leffler module in $\mathcal{D}$ is a direct summand of a $\mathcal{D}$-preenvelope of a union of an $\mathcal{L}$-pure $\omega$-chain of finitely presented modules.
3. If $\mathcal{D}$ contains $R_R^*$ and an $\mathcal{L}$-pure $\omega$-chain of finitely presented modules whose union is $M$, then $M \in \mathcal{D}$ is its own $\mathcal{D}$-preenvelope and, if $N$ is in $\mathcal{D}$ as well, $N$ is a direct summand of $M$.
4. If $\mathcal{D} = R$-Mod, then $M$ is a direct sum of finitely presented modules, hence $N$ is pure-projective. (This is, once again, the aforementioned classical result from [RG].)

Proof. (1) was proved above. For (2), just put (1) together with Theorem 7.1(2) above. For (3), as definable subcategories are closed under limits, $M \in D$, and it remains to apply (1). Finally, (4) is a special case of (3). □

**Remark 3.2.** There is another interesting module in $\mathcal{D}$, namely, the direct limit $D_\infty$ of $\mathcal{D}$-preenvelopes of the corresponding finitely presented modules constituting
the $L$-chain of the $L$-$\omega$-limit $M$. (Note, by [P3], these preenvelopes can be taken strict $D$-atomic (= $K$-Mittag-Leffler), which does not (seem to) make $D_\infty$, however, $D$-atomic, at least not automatically.) By properties of limits, there is an epimorphism $h_\infty : M \to D$. As $D_\infty \in D$, $h_\infty$ factors through $\varepsilon_M$. But it’s not clear (to me) if $h : M \to N$ factors through $D_\infty$ as well.

4. When are $L$-$\omega$-limits in $L$?

I have no conclusive answer. All I have is a simple criterion for any direct limit to be in $\langle L \rangle$. We need it only for $L$-$\omega$-limits, so, for simplicity, it is formulated here only for direct systems of partial order type $\omega$, i.e., for chains.

**Remark 4.1.** (1) Let for simplicity (and w.l.o.g.) $L = \langle L \rangle$, axiomatized by a collection, $\Psi$, of pp implications—written as pp pairs $\varphi/\psi$, i.e., it is assumed that $\psi \leq \varphi$, while the inverse implication, $\varphi \leq \psi$, (stating the closing of the pair) is an axiom of $L$. Cf. [P2].

Consider an $\omega$-system $A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \ldots$. Given $i \leq j$, let $f^j_i : A_i \to A_j$ be the corresponding composition of maps. Suppose $A$ is the limit of this chain.

Then $A \in L$ if and only if the following holds for every $\varphi/\psi \in \Psi$ and $i < \omega$.

If $a_i \in \varphi(A_i)$, then $f^j_i(a_i) \in \psi(A_j)$, for some $j \geq i$.

(2) Mutatis mutandis, the same holds for directed systems of any shape.

When constructing the $L$-$\omega$-limit $M_\Phi$ (in [R3, Not. 7.9]) for a given $L$-atomic module $N$, what condition on the $\omega$-system associated with $M_\Phi$ ensures its membership in $\langle L \rangle$?

5. When all embeddings are $L$-pure

**Example 5.1** (Divisible abelian groups, i.e., $R = \mathbb{Z}$). Let $L = \text{Div}$, the class of divisible abelian groups, which constitutes the subcategory defined by the closing of the pp pairs of the form $(x \doteq x)/(r|x)$, where $r$ runs over all nonzero integers.

(1) Every embedding of abelian groups is Div-pure: $\mathbb{Z}$ being an RD ring, every pp formula $\varphi$ is equivalent to a finite conjunction of basic RD-formulas, $r|\overline{a}$, where $\overline{a}$ is a row vector of integers. If one of those conjuncts has $r = 0$, then that conjunct is quantifier-free and is thus true of $\overline{r}$ in $M$ iff it is true of it in $N$. So, w.l.o.g., all $r \neq 0$ in $\varphi$. In that case, those conjuncts define everything in any divisible abelian group. Consequently, $\varphi(D) = D^n$ (where $n$ is the number of free variables in $\varphi$) in every $D \in \text{Div}$. This means, $\overline{x \doteq x} \leq_{\text{Div}} \varphi$, and therefore the formula $\psi$ in Rem. 2.7 can be taken to be $\overline{x \doteq x}$, which is always true—this showing that every embedding is Div-pure, as desired.

(2) Thus every chain of finitely generated (= finitely presented) abelian groups is a Div-chain, hence every union of such is a Div-$\omega$-limit, and hence every countable (= countably generated) abelian group is a Div-$\omega$-limit.

(3) Consequently, every countable abelian group is Div-atomic (and itself a Div-$\omega$-limit—no uniform epis needed).

(4) The Prüfer groups are examples of members of Div that are Div-$\omega$-limits of finite groups (which are obviously not in Div).
Note, \( K \) (the elementary dual of \( \mathcal{L} = \text{Div} \)) is the class of torsionfree (= flat) abelian groups. So the Div-atomic modules are exactly the \( \mathbb{Z} \)-Mittag-Leffler, or simply \( \mathbb{Z} \)-Mittag-Leffler, abelian groups.

Consequently, every abelian group is \( \mathbb{Z} \)-Mittag-Leffler, which is a special case of Goodearl’s result that every left module over a left noetherian ring \( R \) is \( R \)-Mittag-Leffler (and conversely), see [Goo, p.1] (or [R3, Cor.4.3(2)]) and Cor. 5.8 below.

Example 5.2 (Divisible modules over RD domains, special case). If \( R \) is a countable and left noetherian RD domain, the situation is the exact same as over \( \mathbb{Z} \), since ‘finitely generated’ = ‘finitely presented’ and ‘countable’ = ‘countably generated’.

Recall, a submodule \( N \) of a module \( M \) is said to be RD-pure or relatively divisible if \( rM \cap N = rN \) for every \( r \in R \). (This is purity for the pp formulas of the form \( r|x \) with \( r \in R \).) An RD ring is a ring over which RD-purity (relative divisibility) is (full) purity, or, equivalently, every finitely presented module is a direct summand of a direct sum of cyclically presented modules. A ring is RD if every pp formula is equivalent to a finite conjunction of basic RD-formulas, i.e., formulas of the form \( r|bx \) with \( r \in R \) and \( b \) a row vector over \( R \) if every unary pp formula is equivalent to a finite conjunction of unary basic RD-formulas \( r|sx \) \((r,s \in R)\). This property is left-right symmetric. See [P2, §2.4.2] or [PPR] for all this.

The argument in (1) above can be employed to yield a much farther reaching statement.

Proposition 5.3. Suppose, all embeddings of (left) \( R \)-modules are \( \mathcal{L} \)-pure.

(1) Then all countably presented modules are \( \mathcal{L} \)-atomic (= \( K \)-Mittag-Leffler).

(2) If \( R \) is, in addition, left noetherian, all modules are \( \mathcal{L} \)-atomic (= \( K \)-Mittag-Leffler).

Proof. (1). By hypothesis, every chain of morphisms of (left \( R \)-) modules is pure on images. Hence, every \( \omega \)-limit of finitely presented modules, in particular, every countably presented module, is an \( \mathcal{L} \)-\( \omega \)-limit and therefore \( \mathcal{L} \)-atomic (= \( K \)-Mittag-Leffler). (See, for example, [GT, Lemma 2.11 (or §13.2.2)] for the fact that countably presented modules are \( \omega \)-limits of finitely presented modules.)

(2). Every countably generated module is the union of an ascending chain of finitely generated submodules. Any chain is, by the first hypothesis, an \( \mathcal{L} \)-pure chain. On the other hand, as \( R \) is noetherian, finitely generated is the same as finitely presented. Consequently, every countably generated module is an \( \mathcal{L} \)-\( \omega \)-limit (even with all maps monic) and thus \( \mathcal{L} \)-atomic. Then every module is a directed union of an \( \omega \)-closed (i.e., closed under unions of countable subchains) system of \( \mathcal{L} \)-atomic submodules. By a result of Herbera–Trlifaj, see [R3, Cor. 11], this suffices to make all modules \( \mathcal{L} \)-atomic.

One obtains a classical instance with \( \mathcal{L} = R-\text{Mod} \) (and \( K = \text{Mod}-R \)), for which all embeddings of (left) modules are \( \mathcal{L} \)-pure (= pure) if and only if \( R \) is von Neumann regular if and only all embeddings of right modules are \( K \)-pure (= pure). As a von Neumann regular ring is one-sided noetherian if and only if it is semisimple artinian, (2) below should come to no surprise. For (1), recall from [RG] (or Thm. 3.1(4) above) that countably generated Mittag-Leffler modules are pure projective, hence projective in case they are flat.
Corollary 5.4. Let $R$ be von Neumann regular.

1. All countably presented (left or right) $R$-modules are Mittag-Leffler, hence projective.
2. If $R$ is, in addition, noetherian, all (left or right) $R$-modules are Mittag-Leffler.

Next we apply the proposition in order to extend Example 5.2 to arbitrary RD domains. Recall first that over any RD ring, on either side, the divisible modules are exactly the absolutely pure modules and the torsion-free modules are exactly the flat modules ([PPR, Lemma 2.16] or [P2, Prop.2.4.16]), so $\mathcal{L} = R\text{Div} = R\sharp$ and $\mathcal{K} = T\text{f}_R = \flat_R$ (with $\text{f}_R$ the class of torsion-free modules). In particular, $\mathcal{K}$-Mittag-Leffler is the same as $R$-Mittag-Leffler. If $R$ is a domain (RD or not), on either side, Div is a definable subcategory for the same simple reason as over $\mathbb{Z}$, see above; similarly $\text{f}_R$ is one too.

Remember, (3) below is a special case of Goodearl’s result mentioned in Example 5.1(6).

Corollary 5.5. Let $R$ be an RD domain and $\mathcal{L} = R\text{Div} = R\sharp$.

1. Then all countably presented left $R$-modules are $\mathcal{L}$-atomic (= $R$-Mittag-Leffler).
2. Similarly, for right modules.
3. If $R$ is, in addition, left noetherian, all modules are $\mathcal{L}$-atomic (= $R$-Mittag-Leffler).

Proof. The proof in Example 5.1(1) above that all embeddings are $\mathcal{L}$-pure, did in fact use only the fact that the ring $R = \mathbb{Z}$ was an RD domain. □

Note that by [PPR, Rem.5.8], not every left noetherian RD domain is right noetherian, so in contrast to (1) and (2), statement (3) is not left-right symmetric.

We now turn to a broader reason for this behavior of RD domains.

Lemma 5.6. If $R$ is left coherent, all embeddings in $R\text{-Mod}$ are $R\sharp$-pure.

Proof. By [P1, Thm.15.41], every injective (left) $R$-module has complete elimination of quantifiers if and only if $R$ is left coherent. (Taking an injective model of the largest theory of injectives shows that that this is equivalent to complete quantifier elimination universally for all injectives.)

Let $M \subseteq N$ be any inclusion in $R\text{-Mod}$. To show it is $R\sharp$-pure, let $\overline{\pi}$, an arbitrary tuple in $M$, satisfy a certain pp formula $\varphi$ in $N$. We must find a formula $\alpha \leq R\sharp \varphi$ that $\overline{\pi}$ satisfies in $M$. Being existential, $\varphi$ holds of $\overline{\pi}$ also in an injective envelope $E(N)$ of $N$. But there it is $R\sharp$-equivalent to a quantifier-free formula $\alpha$. Then $\alpha$ holds of $\overline{\pi}$ in $E(N)$, hence, being quantifier-free, also in $M$. □

Remark 5.7. This phenomenon is a consequence of the classical result of Eklof and Sabbagh that over a left coherent ring the theory of all (left) modules has a model-completion (and conversely). One would expect it to take place, mutatis mutandis, in other classes of modules having a model-completion, cf. [P1, §15.3].

Corollary 5.8. (1) If $R$ is left coherent, all countably presented left $R$-modules are $\mathcal{L}$-atomic (= $R$-Mittag-Leffler).
(2) (Goodearl) If $R$ is left noetherian, all modules are $\mathcal{L}$-atomic (= $R$-Mittag-Leffler).

Question 5.9 (Converses). The converse of (2) is part of Goodearl’s classical result, [Goo, p.1] (see [R1, Cor.2.7] or [R3, Cor.4.3(2)] for a model-theoretic proof).
Is the converse of (1) also true? How about the converse of the lemma? Are all inclusions $\mathcal{L}$-pure if all (countably presented) modules are $\mathcal{L}$-atomic (cf. Prop. 5.3)? At least for $\mathcal{L} = R\sharp$?

For a final application of the proposition I first introduce some terminology.

**Definition 5.10.** (1) [R5]. A **high formula** is a unary pp formula $\gamma$ that defines the entire module in every injective (equivalently, in every absolutely pure) module, i.e., $\gamma(E) = E$ for every injective $E$. The collection of all high formulas is denoted by $\Gamma$, with a left or right subscript for the ring, if necessary.

(2) A **left high ring** is a ring over which every unary left pp formula is equivalent to a finite conjunction of high formulas and quantifier-free formulas.

**Remark 5.11.**

(1) Clearly, $\Gamma$ is closed under finite conjunction (and so is the set of qf formulas). Hence over a high ring, every unary pp formula is equivalent to the conjunction of a high formula and a quantifier-free formula.

(2) High rings figured (without a specific name) in [R5, Cor. 6.24], where they were shown to admit a decomposition theorem for pure injectives into an ‘Ulm length 0 module’ and a ‘reduced module,’ see there for terminology.

(3) [R5, Rem. 6.25]. RD-domains are two-sided high, as every basic RD-formula is either high or quantifier-free, see [R5, Cor. 2.12(1)].

**Question 5.12.**

(1) Is highness of rings left-right symmetric? Presumably, not.

(2) Find examples of high rings other than RD domains.

**Remark 5.13.** The known arguments for (1) (and Question 6.4(1) below) to have affirmative answers when high formulas are replaced by RD-formulas do not seem to work here, since highness of a formula $A|x$ is not given by the matrix $A$ alone, but by the condition $l(A) \subseteq l(b)$, [R5, Prop. 2.8(2)].

Cor. 5.5 can be generalized to high Warfield rings. Recall from [Pu] (or [PPR] and [P2]) that a ring is called **left Warfield** if every finitely presented left module is a direct summand of a direct sum of cyclic finitely presented modules. This is equivalent to saying that every left pp formula (in the variables $\mathbf{r}$) is equivalent to a finite conjunction of formulas of the form $a | b\mathbf{r}$, where $a$ and $b$ are row vectors. (This is a special case of the much more general [PPR, Thm. 2.5] or [P2, Cor. 2.4.3].) Clearly, RD rings are left and right Warfield (and, by a result of Puninski, conversely, every two-sided Warfield ring is RD, see the above sources).

**Corollary 5.14.** Let $R$ be a left high, left Warfield ring and $\mathcal{L} = R\Div = R\sharp$.

1. Then all countably presented left $R$-modules are $\mathcal{L}$-atomic (= $R$-Mittag-Leffler).

2. If $R$ is, in addition, left noetherian, all modules are $\mathcal{L}$-atomic (= $R$-Mittag-Leffler).

**Proof.** We are going to verify that every embedding is $\mathcal{L}$-pure. To this end, consider modules $M \subseteq N$ and a tuple $\mathbf{r}$ in $M$ satisfying a certain pp formula $\varphi$ in $N$. All we have to do is find a pp formula $\psi \leq_\mathcal{L} \varphi$ that $\mathbf{r}$ satisfies in $M$. As $R$ is Warfield, $\varphi$ is equivalent to a conjunction of formulas $\varphi_i (i < m)$ of the form $a_i | b_i\mathbf{r}$ with $a_i$ and $b_i$ row vectors. Let $\theta$ be the (unary) formula $a_i | x, i < m$. By highness, every $\theta$ is equivalent to a conjunction $\gamma_i \land \alpha_i$ with $\gamma_i \in \Gamma$ and $\alpha_i$ quantifier-free. Note, $\varphi \sim \bigwedge_{i=0}^{m-1} \theta_i(b_i\mathbf{r})$. Let $\psi$ be the conjunction of all the $\alpha_i$'s. Clearly $\varphi$ satisfies $\psi$ in $N$. As $\psi$ is quantifier-free, it does so also in $M$. It remains to verify $\psi \leq_\mathcal{L} \varphi$. To this end let $L \in \mathcal{L} = R\sharp$ and $\mathbf{r} \in \psi(L)$, i.e., $b_i\mathbf{r} \in \alpha_i(L)$ for all $i$. By highness of $\gamma_i$,
we have also $\mathfrak{b}_i\varpi \in L = \gamma_i(L)$, hence $\mathfrak{b}_i\varpi \in \theta_i(L)$ for all $i$. Consequently, $\varpi \in \varphi(L)$, as desired.

6. Concluding remarks: high purity

One may call submodule $M$ of a module $N$ high-pure if $\gamma(N) \cap M = \gamma(M)$ for every $\gamma \in \Gamma$.

**Lemma 6.1.** Over domains, high purity implies RD-purity.

Consequently, over RD-domains (full) purity, high purity and RD-purity are all the same.

**Proof.** By [R5, Cor. 2.11(5)], the formula $r \mid x$ is high iff $r$ is not a right zero divisor. Over a domain this is true iff $r \neq 0$. For $r = 0$, the condition $rM \cap N = rN$ is trivial. For $r \neq 0$, we infer it from high purity. □

**Question 6.2.** What can be said about RD-rings in general?

**Remark 6.3.** As quantifier-free formulas always pass down, over left high rings, high purity is purity.

**Question 6.4.** (1) Is a ring high if high purity is purity?
(2) For full purity, RD-purity, and high purity, respectively, it suffices to inspect unary pp formulas. What about $\sharp$-purity? For what $\mathcal{L}$ would the same be true?

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