In a temperature range around ten Kelvins or below, the physical properties of amorphous materials differ significantly from the properties of crystals and show striking universal features (see [1] for a collection of reviews). Most of these features can be explained by assuming that in the amorphous solid exists a collection of dynamic defects, which are atoms or groups of atoms, oscillating in two well potentials. At low temperatures the thermal activation is suppressed and the oscillation happens by quantum tunnelling from one potential minimum to the other, forming in this way what is called a two-level system (TLS). In the two-dimensional Hilbert space spanned by the ground states of the two wells, the effective Hamiltonian of the TLS is

$$H_{TLS} = \frac{\Delta}{2} \sigma_z - \frac{\Lambda}{2} \sigma_x \equiv \frac{1}{2} \begin{pmatrix} \Delta & -\Lambda \\ -\Lambda & -\Delta \end{pmatrix}$$

(1)

where \(\sigma_x\) and \(\sigma_z\) are Pauli matrices, while \(\Delta\) and \(\Lambda\) are called the asymmetry of the potential and the tunnel splitting, respectively. The Hamiltonian (1) may be diagonalized by an orthogonal
transformation $O$, $H'_{\text{TLS}} = O^T H_{\text{TLS}} O = \frac{1}{2} \sigma_z$, to obtain the excitation energy of the TLS, $\varepsilon \equiv \sqrt{\Delta^2 + \Lambda^2}$—by the superscript $T$ we denote in general the transpose of a matrix. The parameters $\Delta$ and $\Lambda$ do not have the same values for all the TLSs, but are distributed with the density $V P(\Delta, \Lambda)$, where $V$ is the volume of the solid. According to the standard tunneling model (STM), this distribution is $P(\Delta, \Lambda) = P_0/\Delta$, with $P_0$ a constant. If expressed through the variables $\varepsilon$ and $u \equiv \Lambda/\varepsilon$, the distribution function becomes $P(\varepsilon, u) = P_0/(u \sqrt{1 - u^2})$.

A phonon, or any other strain in the solid body, perturbs $H_{\text{TLS}}$ by $H_1 \equiv (\delta/2) \sigma_z$, $[1, 2, 3, 4, 5]$: The perturbation $\delta$ is linear in the strain field, $S_{ij}$, $[5, 1]$ and in general may be written as $\delta \equiv 2\gamma_{ij} S_{ij}$—here, as everywhere in this paper, we assume $\delta \equiv 2\gamma_T S_T$—so the three components, $\delta_{12}, \delta_{13}, \delta_{23}$, which, we say it again, are scalars under coordinate transformations. Therefore, the tensor $\delta$ may be written as $\delta \equiv 2\gamma_T S_T + 2\gamma_T S_T$, and $\delta$ becomes the vector $\gamma \equiv [R]^T \cdot T$. Since $[S]$ is also transformed into $S \equiv (S_{xx}, S_{yy}, S_{zz}, 2S_{yx}, 2S_{zx}, 2S_{xy})^T$, the component of the interaction hamiltonian, $\delta$, is written simply as $\delta \equiv 2T^T \cdot [R] \cdot S$, which, we say it again, is a scalar under coordinate transformations.

Now notice the analogy between $\delta$ and the elastic energy density, $u$, that exists in a deformed body. In abbreviated subscript notations, $u = \frac{1}{2} S^T \cdot \epsilon \cdot S$, where $\epsilon$ is the $6 \times 6$ matrix of the elastic stiffness constants. Under a coordinate transformation, $S$ transforms into $S' = [N] \cdot S$, where $[N]$ is the $6 \times 6$ matrix of the transformation (see Ref. [7], Eq. 3.34), so

$$u = u' = \frac{1}{2} (S')^T \cdot \epsilon \cdot S' = \frac{1}{2} S^T \cdot [N]^T \cdot \epsilon \cdot [N] \cdot S.$$  

(2)

Since (2) should be valid for any $S$ and any transformation, then $[\epsilon]$ should remain unchanged—$[\epsilon] = [N]^T \cdot [\epsilon] \cdot [N]$—under the symmetry transformations of the crystalline lattice. From this argument follow all the properties of the matrix $[\epsilon]$ which are characteristic to the symmetries of the crystal under consideration [7].

The same is true for $\delta$. Here $T$ transforms in the same way as $S$ under coordinate transformations—$T' = [N] \cdot T$. Therefore $\delta = \delta' = T^T \cdot [N]^T \cdot [R] \cdot [N] \cdot S$ and, like for $[\epsilon]$,
we have \( [R] = [N]^T \cdot [R] \cdot [N] \) for any symmetry transformation \([N]\). In conclusion, \([R]\) has the same structure as the tensor of elastic stiffness constants (see also \([6]\)).

If a material is isotropic, \([c]\) has only two independent parameters—the Lamé constants, \(\lambda\) and \(\mu\): \(c_{1IJ} = 2\mu\delta_{IJ} + \lambda\) for \(I, J \leq 3\) and \(c_{1IJ} = \delta_{I1}\delta_{J1}\) for \(I\) or \(J\) bigger than 3. In normal subscripts \(c_{ijkl}\) can be written in the more compact form, \(c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\). Based on the arguments above, the same is true for \([R]\); let us denote the independent parameters of \([R]\) by \(\zeta\) and \(\xi\), and, in normal subscripts, \(R_{ijkl} = \zeta\delta_{ij}\delta_{kl} + \xi(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\). For a more straightforward reference to the STM, it might be even more convenient to denote \(\tilde{\zeta} = \xi + 2\zeta\) and introduce the reduced tensor \([\tilde{r}]\) by \(r_{ijkl} \equiv R_{ijkl}/\tilde{\gamma} = \zeta'\delta_{ij}\delta_{kl} + \xi'(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\), where \(\zeta' + 2\xi' = 1\).

Using such a form of the coupling constants tensor, we calculated in \([6]\) the scattering rate of phonons on the TLSs and showed that we recover the results of the STM after we average over the directions of the TLSs, assuming that they are isotropically oriented. The two constants of the STM, \(\gamma_l\) and \(\gamma_t\), are related to the parameters in this model by \(\gamma_l = \tilde{\gamma}C_l\) and \(\gamma_t = \tilde{\gamma}C_t\), with

\[
C_l = \frac{1}{15} (15 - 40\xi' + 32(\xi')^2) \quad \text{and} \quad C_t = \frac{4}{15}(\xi')^2. \quad (3)
\]

From Eqs. \((3)\) we see immediately that \(C_l > C_t \geq 0\) for any real \(\xi\), as it is observed experimentally in general \([6]\).

What is also interesting to note, is that by calculating \(\gamma_l\) and \(\gamma_t\) from the experimental data, we can calculate \(\xi\) and \(\zeta\) and determine completely the tensor of coupling constants, \([R]\). For example the two different sets of values for \(P_0\gamma_l\) and \(P_0\gamma_t\) in fused silica, reported by Golding et al \([8]\) \((P_0\gamma_l^2 = 1.4 \times 10^{-5} \text{ J/m}^3\) and \(P_0\gamma_t^2 = 0.63 \times 10^{-5} \text{ J/m}^3\)) and Hunklinger and Arnold \([9]\) \((P_0\gamma_l^2 = 2.0 \times 10^{-5} \text{ J/m}^3\) and \(P_0\gamma_t^2 = 0.89 \times 10^{-5} \text{ J/m}^3\)), cited also by Black \([10]\), give the same solutions for \(\xi'\): \(\xi'_1 = 0.55\) and \(\xi'_2 = 1.2\).

If we now go back to the interaction of a single TLS with a strain field and we assume that a transversal wave of strain field \(S = (0, 0, 0, S, 0, 0)^T\), is propagating through the solid, we find \(\delta = 4\tilde{\gamma}\xi' t_y t_z S\). This means that any TLS of \(t_y = 0\) or \(t_z = 0\) (i.e. any TLS which is contained in a plane perpendicular either to the propagation direction or to the polarization direction) will not be perturbed by this wave.

Observing that for both values of \(\xi'\) calculated above, the corresponding values of \(\xi'\) \((= 1 - 2\xi')\) are negative, we show an interesting polarization effect of the TLS ensemble, due to an external stress. Let us assume that we apply a longitudinal stress along the \(z\) direction, \(S = (0, 0, S, 0, 0, 0)^T\). This stress gives a perturbation \(\delta = 2\tilde{\gamma}t_z^2 St_z + 2\tilde{\gamma}\xi' S(t_y^2 + t_z^2)\) and we notice that the two terms in the expression of \(\delta - \delta_1 = 2\tilde{\gamma}t_z^2 St_z\) and \(\delta_2 = 22\tilde{\gamma}\xi'(t_y^2 + t_z^2)S\) have opposite signs. This means that, if e.g. \(\tilde{\gamma}t_z^2 S > 0\), the energy splitting, and therefore the excitation energy, of the TLSs oriented along the strain increase, while the energy splitting and the excitation energy of the TLSs oriented perpendicular to the strain direction decrease. In other words, the strain polarises the TLS ensemble.

In conclusion, we used a model for the interaction of two-level systems (TLS) with arbitrary strain fields, introduced in \([6]\), which assumes that to any TLS is associated a direction, \(\hat{t}\), and a tensor of coupling constants to the strain field, \([[R]]\), and we showed on general grounds that \([[R]]\) has the same structure with respect to the symmetry transformations of the solid as the tensor of stiffness constants, \([[c]]\), from the elasticity theory. Some immediate consequences of this formalism are that in isotropic solids, on average, the longitudinal phonons interact stronger with the TLSs than the transversal ones, \((\gamma_l > \gamma_t\), in the language of the standard tunnelling model), a transversal wave does not interact with the TLSs contained in one of the two planes that are perpendicular either to the wave propagation direction or to the polarization direction, and a strain applied to the body may polarize the TLS ensemble.
Acknowledgments
This work was partly supported by the U. S. Department of Energy Office of Science under the Contract No. DE-AC02-06CH11357 and by the NATO grant EAP.RIG 982080. DVA acknowledges the hospitality of the University of Jyväskylä, where part of this work has been done, and the financial support from the Academy of Finland.

References
[1] Esquinazi P 1998 Tunneling systems in amorphous and crystalline solids (Springer) ISBN 3-540-63960-8
[2] Philips W A 1972 J. Low Temp. Phys. 7 351
[3] Anderson P W, Halperin B I and Varma C M 1972 Phil. Mag. 25 1
[4] Jäckle J 1972 Z. Phys. 257 212
[5] Leggett A J, Chakravarty S, Dorsey A T, Fisher M P A, Garg A and Zwerger W 1987 Rev. Mod. Phys. 59 1–85
[6] Anghel D V, Kühn T, Galperin Y M and Manninen M 2007 Phys. Rev. B 75 064202
[7] Auld B A 1990 Acoustic Fields and Waves in Solids, 2nd Ed. (Robert E. Krieger Publishing Company) ISBN 0-89874-783
[8] Golding B, Graebner J E and Schutz R J 1976 Phys. Rev. B 14 1660
[9] Hunklinger S and Arnold W 1998 Tunneling systems in amorphous and crystalline solids in [1]
[10] Black J L 1978 Phys. Rev. B 17 2740