Dynamical symmetry in a minimal dimeric complex

E Sadurní and Y Hernández-Espinosa

1 Instituto de Física, Benemérita Universidad Autónoma de Puebla, Apartado Postal J-48, 72570 Puebla, Mexico
2 Instituto de Física, Universidad Nacional Autónoma de México, Apartado Postal, 04510 Ciudad de México, Mexico

E-mail: sadurni@ifuap.buap.mx

Received 21 May 2018, revised 29 April 2019
Accepted for publication 30 May 2019
Published 27 June 2019

Abstract
The emergence of non-configurational symmetry is studied in a minimal example. The system under scrutiny consists of a dimeric hexagonal complex with configurational $C_3$ symmetry, formulated as a tight-binding model. An accidental three-fold degeneracy point in parameter space is found; it is shown that an internal $U(3)$ symmetry group operates on Hilbert space, but not on configuration space. The corresponding discrete Wigner functions for the irreducible representations of $C_6 \cong C_3 \times Z_2$ are utilized to show that a $6 \times 6$ phase space is sufficient to exhibit an invariant subset. The dynamical symmetry is thus identified with a discrete semi-plane. Some implications on other known hidden symmetries of continuous systems are qualitatively discussed.

Keywords: hidden symmetry, discrete Wigner function, geometric phase, polymer, three-fold degeneracy

1. Introduction

The concept of dynamical symmetry has been historically relevant for a better understanding of superintegrability in quantum mechanics. The Coulomb problem [1–4] and the isotropic oscillator [5–9] are the paradigms that nature has chosen in the form of atoms and springs, showing us clear realizations of Lie algebraic structures whose dimension exceeds that of physical space. The usefulness of dynamical symmetry, at least in the case of atoms, resides in the explanation of accidental degeneracy, as shown in many textbooks. It also provides a suitable scaffolding for the description of more complex atomic systems around integrability via useful quantum numbers and the transitions between them—this is the cornerstone of
spectroscopy. To the authors’ knowledge, there remains an open problem in mathematical physics regarding the reverse statement: that every degeneracy corresponds to a symmetry of a system, regardless of whether its realization is in configuration space or in phase space. Under reasonable assumptions and starting from a prescribed degeneracy of states, one may try to show that the only relevant feature is the existence of a symmetry Lie group, and in this respect the bound states of both the hydrogen atom and a constrained four-dimensional isotropic oscillator have the same $SO(4)$ structure [10–12]. Furthermore, the full (Cartan) classification of Lie groups exhausts all semi-simple possibilities. But this game is far more complicated when one also recognizes the many forms of degeneracy, either finite (compact group), infinite (non-compact group) or maybe of unknown multiplicity as a function of energy (Diophantine problem). Therefore, if a system possesses a set of parameters for which some levels ‘accidentally’ coalesce, it is difficult in general to show the existence of symmetry in the space of observables (especially if they are canonically conjugate), whereas the trivial answer of an internal symmetry in Hilbert space adds little to our knowledge, despite its correctness.

It is worthwhile to pose these questions in the case of finite systems, where the concept of phase space is attainable only if the appropriate Wigner function [13] is employed in the description of the corresponding states. Regardless of the existence of symmetry, such functions can always be put in terms of lattice sites and unimodular eigenvalues of a certain finite group. This entails the use of two indices that define a two-dimensional phase space where all hidden symmetries can be exposed. In this work we are interested in the spectrum of an electron hopping on a hexagonal polymer made of three dimers, which happens to possess an accidental three-fold degeneracy point. Due to finiteness, we shall be able to answer the question in the previous paragraph without the use of canonical operators $x$ and $p = -i\hbar \partial /\partial x$. Regarding the more general case, the simplicity of our approach will help to build more complex realizations until a continuous limit is reached.

In connection with the applications of the present work, it is important to mention that dimers are a special realization of q-bits, similar to two-level atoms, provided that each monomer consists of a single-level potential well. The existence of symmetry operators for certain values of the model’s parameters can be used to obtain a quantity that remains undisturbed by evolution or other unitary operations. Also, in the control of certain states, a geometric (Berry) phase [14] can be identified when the system is driven through a loop. For three-fold degeneracy points, this quantity has been carefully computed [15–17]. More applications related to hexagonal or dimeric structures can be found [18–22]; in particular, we have applied three-fold degeneracy to achieve level inversion and negative couplings in a tight-binding chain equipped with a quasi-spin polarizer [23].

Structure of the paper: in section 2 we describe tight-binding polymers with degeneracy, reaching the conclusion that an accidentally degenerate system of three dimers is indeed minimal. We proceed to the diagonalization of a model Hamiltonian and a full Hamiltonian with pairwise site-to-site coupling, and we describe the geometric configurations containing triplets. The dynamical algebra of the problem is explicitly written. In section 3 we define a $6 \times 6$ phase space and find the discrete Wigner function of a triplet for three cases: three-fold accidental degeneracy, two-fold degeneracy with unbroken $C_3$ and no degeneracy with broken time reversal invariance. For the first case, we give a description of the invariant locus in such a phase space. In section 4 we discuss the phase space for a collection of polymers and the emergence of continuous symmetries.

3 The unconstrained 3D oscillator, of course, has the larger $U(3)$ as symmetry group and $sp(6,R)$ as dynamical algebra.
2. On polymers and minimal systems

Our line of reasoning consists in finding the simplest polymeric complexes where degeneracy can be regarded as accidental, i.e. level crossing that is not inherent to polygonal $C_n$ or permutational $S_n$ groups. In addition to this requirement, our main assumption is that our system contains single-level sites and all couplings between sites are positive quantities; this is the case when localized wave functions are real and positive, for the integral overlaps that constitute tunneling amplitudes must have such a quality. A good example is a spherically symmetric ground state of an atom in the vicinity of a positively charged ion; another example is a carbon $\pi$ orbital orthogonal to the molecular plane in benzene or graphene [24]. Complex couplings are irrelevant in open configurations, while their presence in loops represent magnetic fluxes, but here we deal with systems in the absence of additional external fields, therefore only real positive hopping amplitudes will be considered in this section.

Our polymers are indeed minimal. The simplest tight binding systems with one, two and three identical dimers are modelled by $2 \times 2$, $4 \times 4$ and $6 \times 6$ matrices, respectively. Each dimer also enjoys the $S_2$ permutational symmetry. The $2 \times 2$ system is our building block. The $4 \times 4$ array has no accidental degeneracy in parameter space: if all couplings are equal, a singlet and a triplet are found, but this can be seen as a symmetry of the configurational (geometric) type. If the bonds are represented by lengths the array will correspond to a tetrahedron in three dimensions. The $6 \times 6$ system with broken $C_6$ but unbroken $C_3$ symmetry has a crossing point in the spectrum, containing one of the doublets and one of the singlets. The configuration lies somewhere between a star graph and a hexagon, with broken reflection symmetry, see figure 1. In the jargon of chemical physics this would be referred to as a broken $C_3 v$ symmetry. The accidental crossing of three levels was reported for the first time in [23] by one of the authors, without deeper analysis. The corresponding couplings are not identical, therefore no higher-dimensional construction can map the system into a regular polytope.

Other small non-dimeric systems can be discussed, but once more, their degeneracies are inferred from configuration space. An equilateral trimer has a doublet and a singlet. A polygon with $C_5$ symmetry has two doublets and a singlet, and the existence of a higher (quadruple) degeneracy point corresponds to equal couplings of all elements; the corresponding graph has $S_5$ symmetry and can be embedded in four-dimensional space as a polytope whose projections are regular tetrahedra, which constitute again a symmetry of the configurational type.

2.1. Our physical system

Microwave resonators [25], bent waveguides with corners [26] and, in general, arrays of potential wells such as molecular structures, motivate the introduction of tight-binding Hamiltonians. We focus on planar arrays with $C_3$ symmetry for simplicity, although an iso-spectral study with dimers in three dimensions can be found in [27].

Consider the hexagon and the star graph in figure 1. Each point or site is denoted by $(m,n)$ where $m=1,2,3$ is the dimer label and $n=1,2$ is the intra-dimer site. The regular hexagonal configuration with time reversal invariance is known to possess two doublets and two singlets. This remains true if the $C_6$ symmetry is broken (as in the figure), but $C_3$ is preserved: the real positive couplings

$$\Delta_{(m,n,l,k)} = \int d^2 x \psi_{m,n}(x) H \psi_{l,k}(x), \quad \psi_{m,n}(x) \equiv \langle x | m,n \rangle$$ (1)
make $H = H^*$ in this reduced Hilbert space of localized wave functions, so the two conjugate representations of $C_3$ must have eigenfunctions with the same energy (forming thus real wavefunctions by linear combinations). This is the doublet of the equilateral triangle; there is also a singlet. The spectrum is ordered as 2,1. With the additional dimeric structure, we obtain two copies of such levels, but their energies are repelled (or separated) by a strong intra-dimer coupling, giving rise to energies ordered as 1,2,2,1 (inverted copies) or 2,1,2,1 (non-inverted copies). See the spectra in figure 2(a). For example, the coupling $\Delta_s$ that favours the star configuration can be parameterized by the orientational angle $\theta$, such that $\Delta_s \sim 0$ when $\theta \sim 0$ due to a large distance between sites, as shown in the geometry of the array in figure 1(c). This corresponds to a hexagon, and it is found that the levels distributed as 1,2,2,1. Also, $\Delta_b \sim 0$ when $\theta \sim \pi/2$ and $\Delta_s$ reaches a maximum, corresponding to a star graph. Here the level distribution is found to be 2,1,2,1. The phenomenon of level inversion is explained by the existence of symmetric and antisymmetric states in each dimer, producing positive and negative effective couplings between eigenstates of neighbouring dimers. A sign inversion of the coupling produces the exchange $1,2 \mapsto 2,1$. A diagram with the coupling scheme of dimeric eigenstates is shown in figure 3, where the signs of overlaps are visible. Thus, the star graph and the hexagon must have 2,1,2,1 and 1,2,2,1 level distributions, respectively. The existence of a triple point is now unavoidable: one can move continuously from the hexagonal configuration to the star configuration by a rotation of dimers around their centres; from a geometric perspective, the continuity of the spectrum is obvious. With the evolution with respect to the angle $\theta$, one finally infers that there must be a critical value $\theta_c$ for which the lower singlet and doublet levels have crossed, as marked in figure 2(a) by a vertical dashed line. This critical angle does not correspond to a restoration of $C_3$, as can be seen by the first panel of figure 1, so the resulting three-fold degeneracy must be ‘accidental’. We must prove that this is the outcome of a hidden symmetry in phase space. These general considerations can be substantiated by using a concrete model Hamiltonian. Retaining only nearest-neighbour couplings between sites, one has the following operator:
5

\[ H = \begin{pmatrix}
E_0 & \Delta_d & 0 & 0 & 0 & \Delta_h \\
\Delta_d & E_0 & \Delta_h & 0 & 0 & \Delta_s \\
0 & \Delta_h & E_0 & \Delta_d & 0 & 0 \\
0 & \Delta_s & \Delta_d & E_0 & \Delta_h & 0 \\
0 & 0 & \Delta_h & E_0 & \Delta_s & \Delta_d \\
\Delta_h & \Delta_s & 0 & \Delta_s & \Delta_d & E_0 \\
\end{pmatrix}, \]

where the canonical basis \( e_1, e_2 \) corresponds to the first dimer, \( e_3, e_4 \) to the second dimer and \( e_5, e_6 \) to the third dimer. We introduced \( \Delta_d = \Delta_{(m,1,m,2)} \) as intra-dimer couplings, and

---

**Figure 2.** Evolution of energy levels with a geometrical parameter, numbers indicate level degeneracy. Panel (a) shows the level inversion and the crossing point for unbroken \( C_3 \). Panel (b) shows a level splitting due to explicit \( C_3 \) symmetry breaking where at least one of the dimer lengths is different. Panel (c) shows the level evolution at critical angle as a function of one of the couplings; this corresponds to a one-parameter family of Hamiltonians with triplets. (a) Inversion. (b) Repulsion. (c) Formula \( \Delta s = F(\Delta h) \).

|                  | Antisymmetric | Symmetric |
|------------------|---------------|-----------|
| Hexagon          | (a)           | (b)       |
|                  | [Diagram]     | [Diagram] |
| Star             | (c)           | (d)       |
|                  | [Diagram]     | [Diagram] |

**Figure 3.** Overlaps between dimeric eigenstates. Panels (a) and (b) show two hexagonal configurations. The antisymmetric dimer states in (a) necessarily produce negative overlaps according to sign multiplication (blue and yellow), whereas symmetric states in (b) give positive overlaps. In (c) and (d), both symmetric and antisymmetric states produce positive couplings, if the \( C_3 \) symmetric scheme is imposed.
with \( (m) \) the relation of congruence \( \text{mod} \ 3 \) (for \( m = 1, 2, 3 \)), are the couplings that favour the hexagonal configuration around \( \theta = 0 \). Similarly \( \Delta_e = \Delta_{(m,1,0,1)} \) are couplings at the centre of the array that favour the star configuration around \( \theta = \pi/2 \). In this notation, we may also have \( \Delta_e = \Delta_{(m,2,n,2)} \) at \( \theta = 3\pi/2 \), but since the spectrum is unchanged under a reflection of the full system, we restrict to values \( \theta \in [0, \pi/2] \), where our model Hamiltonian (2) is a continuous function of geometric parameters. From this Hamiltonian we can obtain the analytical condition for the existence of triple degeneracy as follows. Let us ignore the on-site energy \( E_0 \) without loss of generality. First, we diagonalize each dimeric block with a Hadamard matrix

\[
U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad H^{(1)} = (I_3 \otimes U)^\dagger H (I_3 \otimes U).
\]

Then we gather all lower dimer levels in a \( 3 \times 3 \) block by means of permutations; similarly for upper dimer levels. The permutation matrix \( P \) is such that

\[
H^{(2)} = \begin{pmatrix} \Delta_d & \Delta_+ & 0 & 0 \\ \Delta_+ & \Delta_d & \Delta_+ & 0 \\ 0 & -\Delta_+ & -\Delta_d & 0 \\ -\Delta_+ & 0 & -\Delta_- & -\Delta_+ \end{pmatrix},
\]

with \( \Delta_{\pm} \equiv (\Delta_d \pm \Delta_e)/2 \). Now we use the \( C_3 \) basis to diagonalize both \( 3 \times 3 \) blocks in the diagonal of \( H^{(2)} \):

\[
U_{C_3} = \frac{1}{\sqrt{3}} \begin{pmatrix} e^{i2\pi/3} & 1 & e^{i4\pi/3} \\ e^{i4\pi/3} & e^{i2\pi/3} & 1 \\ 1 & e^{-i2\pi/3} & e^{-i4\pi/3} \end{pmatrix}, \quad H^{(3)} = (I_2 \otimes U_{C_3})^\dagger H (I_2 \otimes U_{C_3})
\]

with the following result

\[
H^{(3)} = \begin{pmatrix} X_+ & Y \\ Y^\dagger & X_- \end{pmatrix}, \quad X_{\pm} = \text{diag} \left\{ \pm (\Delta_d - \Delta_{\pm}), \pm (\Delta_d - \Delta_{\pm}), \pm (\Delta_d + 2\Delta_{\pm}) \right\}
\]

\[
Y = \text{diag} \left\{ (\Delta_d + i\sqrt{3}\Delta_e)/2, (\Delta_d - i\sqrt{3}\Delta_e)/2, -\Delta_e \right\}.
\]

Finally, we observe that this operator contains uncoupled \( 2 \times 2 \) blocks, so we diagonalize them to obtain the spectrum:

\[
E_{\text{doublets}}^{\pm} = -\Delta_d \pm \sqrt{\left( \Delta_d - \Delta_{\pm} \right)^2 + \left( \Delta_d + 2\Delta_{\pm} \right)^2} + 3 \left( \Delta_d/2 \right)^2
\]

\[
E_{\text{singlets}}^{\pm} = \Delta_d \pm \sqrt{\left( \Delta_d + \Delta_e \right)^2 + \Delta_d^2}.
\]

Here we see that \( E_{\text{doublets}} = E_{\text{singlets}} \) is attainable. After some trivial algebraic steps, this condition results in

\[
\frac{\Delta_e}{\Delta_d} = \frac{(\Delta_e/\Delta_d)}{\sqrt{1 + (\Delta_e/\Delta_d)^2}} \equiv F(\Delta_e/\Delta_d),
\]

\[
\frac{\Delta_e}{\Delta_d} = \frac{(\Delta_e/\Delta_d)}{\sqrt{1 + (\Delta_e/\Delta_d)^2}} \equiv F(\Delta_e/\Delta_d).
\]

\[
\frac{\Delta_e}{\Delta_d} = \frac{(\Delta_e/\Delta_d)}{\sqrt{1 + (\Delta_e/\Delta_d)^2}} = F(\Delta_e/\Delta_d).
\]
which constrains the star and hexagonal couplings to a curve, with coordinates $\Delta_s/\Delta_d, \Delta_h/\Delta_d$ normalized with respect to intra-dimer $\Delta_d$. Therefore there is a one-parameter family of Hamiltonians with triple points, and none of its members contain equal couplings (no permutational symmetry) except for the trivial case $\Delta_h = \Delta_s = 0$. The value of $\theta_c$ must change as a function of the curve’s coordinates. According to (9), triple points are impossible for $\Delta_s > \Delta_d$, and $\theta_c$ does not exist. Very weak intra-dimer couplings are an example of this. Sometimes, the localized waves are modeled by exponential tails (e.g. in a cylindrical resonator the Bessel function $K$ can be approximated by decaying exponentials [28]), so the couplings (1) also decay exponentially as functions of the separation distance. With the help of (9) and trivial geometrical considerations, the critical angle $\theta_c$ can be obtained as a function of inter dimer separations. For dimers that are far apart, the triple degeneracy is removed. Only for tight systems do we see the phenomenon in question, but this does not preclude the existence of a Berry phase for more dilute arrays of dimers where the loop parameterized by $\theta$ does not encounter the triple point. See figure 4 for energy plots of the degenerate triplet as a function of the distance between sites. Panels (a) and (b) are a comparison between model Hamiltonian (2) and a tight-binding Hamiltonian of six sites with couplings between all pairs and modelled by an exponential law that decays with the separation distance.

2.2. Dynamical algebra

As in any finite system, the Hamiltonian can be put in terms of $SU(N)$ Hermitian generators. Here $N = 6$, but a close inspection of (2) makes us consider the direct product of irreducible representations $(1/2, 1)$ of $SU(2) \otimes SU(2)$. Moreover, if the $C_3$ symmetry is unbroken, the second $SU(2)$ factor can be replaced by the cyclic group generated by a $2\pi/3$ rotation of the polymer plane. Let us show this explicitly: we have vector couplings of the type

$$H = v_0 \otimes 1_2 + \sum_{j=1,2,3} v_j \otimes \sigma_j$$

(10)

with

$$v_0 = \frac{\Delta_s}{2} (J_+ + J_- + J_+^2 + J_-^2)$$

$$v_1 = \Delta_d + \Delta_h (J_+ + J_- + J_+^2 + J_-^2)$$

$$v_2 = i\Delta_h (J_+^2 + J_- - J_+^2 - J_-)$$

$$v_3 = -\frac{\Delta_s}{2} (J_+ + J_- + J_+^2 + J_-^2)$$

(11)

and the algebraic relations

$$[J_+, J_-] = 2J_3, \quad J_+^2 = J_-^2 = 0, \quad J_3 = \text{diag}\{1, 0, -1\},$$

$$[\sigma_+, \sigma_-] = 2\sigma_3, \quad \sigma_+^2 = \sigma_-^2 = 0, \quad \sigma_3 = \text{diag}\{1, -1\}.$$  

(12)

Because of the vanishing powers $J_+^0 = 0$, a $2\pi/3$ rotation operator has the form $T = J_+ + J_-^2$, and can be introduced in (11):

---

4 A more general Hamiltonian may require the product of fundamental representations of $SU(2) \otimes SU(3)$, but here we have a dot product of three-dimensional vectors, as shown in (10), instead of eight-dimensional vectors. Therefore, it is not necessary to employ the full set of Gell–Mann matrices.
\[ v_0 = -v_3 = \frac{\Delta_s}{2} (T + T^\dagger), \quad v_1 = \Delta_d + \Delta_h (T + T^\dagger), \quad v_2 = i\Delta_h (T^\dagger - T), \]

\[ T^3 = 1, \quad [T, T^\dagger] = 0, \]

which further reduces the algebra to \([v_\lambda, v_\mu] = 0, \forall \lambda, \mu\). The accidental degeneracy is produced by \([H, I] = 0\) if

\[ I = \alpha|E_{\text{doublet}}^-\rangle\langle E_{\text{singlet}}^-| + \beta|E_{\text{doublet}}^-\rangle\langle E_{\text{doublet}}^+| + \gamma|E_{\text{doublet}}^+\rangle\langle E_{\text{singlet}}^-| + \text{h.c.} \]

Figure 4. Energy levels for various configurations of distance and angles with triple degeneracy \(E_{\text{doublets}}^- = E_{\text{singlet}}^-\). (a) Eigenstates of a tight-binding Hamiltonian with all-site pairwise interactions and (b) eigenstates of the model Hamiltonian (2). \(\theta\) is given in radians, and the distance is given in units of dimeric length.

\[ T = \frac{\Delta_s}{2} (T + T^\dagger), \quad v_1 = \Delta_d + \Delta_h (T + T^\dagger), \quad v_2 = i\Delta_h (T^\dagger - T), \]

\[ T^3 = 1, \quad [T, T^\dagger] = 0, \]

which further reduces the algebra to \([v_\lambda, v_\mu] = 0, \forall \lambda, \mu\). The accidental degeneracy is produced by \([H, I] = 0\) if

\[ I = \alpha|E_{\text{doublet}}^-\rangle\langle E_{\text{singlet}}^-| + \beta|E_{\text{doublet}}^-\rangle\langle E_{\text{doublet}}^+| + \gamma|E_{\text{doublet}}^+\rangle\langle E_{\text{singlet}}^-| + \text{h.c.} \]
where h.c. stands for hermitean conjugate of the first three terms. This invariant is valid for non-zero coefficients and only when (9) holds.

The following remarks are at order: a $a_6 \times 6$ diagonal matrix with $3 + 2 + 1$ equal eigenvalues commutes with a $u(3) + u(2) + u(1)$ subalgebra of $u(6)$; however the triple degeneracy point ensures commutability with $u(3)$ acting exclusively on this sector. The other 3 eigenvalues are either non-degenerate or the degeneracy is explained by configurational plus time reversal symmetry. The $u(3)$ component of interest has 9 parameters. Some of the transformations of this $3 \times 3$ subspace are redundant functions of the Hamiltonian or the invariant $I$; therefore, we need a phase space to characterize the true symmetry of the system. It is important to note that unitary operators containing redundant functions of $H$ do not constitute new symmetries.

3. Phase space in a $6 \times 6$ grid

Important work on finite fields [29, 30] and rings [31–33] can be used as a guide in our constructions, but for even dimension we must start from scratch. In our system, the $C_6$ symmetry is never fulfilled, but our aim is to use the eigenstates of $C_6 \cong C_3 \times Z_2$ as momentum marks in the vertical axis, conjugate to the six positions $(m, n)$ in the horizontal axis. One can pass from one axis to the other by linear combinations analogous to Fourier transforms. For the $C_3$ generator we have $T|m, n\rangle = |(m + 1), n\rangle$ where $(m) \equiv m \mod 3$. The transformed states are denoted by a subindex 1:

$$|k, n\rangle_1 = \frac{1}{\sqrt{3}} \sum_{q=1,2,3} e^{2\pi i q k/3} |q, n\rangle, \quad T|k, n\rangle_1 = e^{-2\pi i k/3} |k, n\rangle_1, \quad k = 1, 2, 3. \tag{15}$$

For the $Z_2$ part, we have the obvious symmetric and antisymmetric combinations, denoted by the subindex 2:

$$|q, s\rangle_2 = \frac{1}{\sqrt{2}} \left((-1)^s |q, 1\rangle + |q, 2\rangle\right), \quad s = 1, 2. \tag{16}$$

The full transformation is defined then as

$$|k, s\rangle_{1,2} = \frac{1}{\sqrt{6}} \sum_{q=1,2,3,r=1,2} e^{2\pi i q k/3} (-1)^r |q, r\rangle. \tag{17}$$

Because of unitarity in (3) and (5), the inverse is well defined. The wavefunctions are denoted by $\psi_{|q, n\rangle} = \langle q, n | \psi \rangle$, $\psi_{|k, s\rangle} = |k, s \rangle \langle \psi |$.

In order to define a bona fide Wigner function, it is necessary to analyze each factor in $C_6$ separately, instead of using its eigenphases $1^{1/6}$ directly; this is related to the fact that 6 is

---

5 It what follows, we use 1, 2, 3 excluding 0.
not a prime. The factor $C_3$ allows a simple treatment in terms of $1^{1/3}$ and the corresponding 3 eigenstates of $T$; however $Z_2$ needs at least three summands (not two) in the Wigner function in order to recover the correct marginal distributions and the properties indicated in appendices A and B.

3.1. Wigner function for $Z_2$

There is a covariant definition provided in [34–36] for any spin, in particular $s = 1/2$. For two-level systems one may also resort to [37]. Here we are not interested in marginal distributions over the three possible axes $\sigma_x, \sigma_y, \sigma_z$, or in general values of angular momentum, but only in the two eigenphases $\pm 1$ of $Z_2$. Our phase space is two-dimensional. The most general function $W(\beta, l)$ of variables $\beta = 1, 2, l = 1, 2$ that yields the correct marginals is

$$W(1, 1) = A|\psi_1|^2 + B|\psi_2|^2 + 2cR \{ \psi_1 \psi_2^* \},$$
$$W(2, 1) = (1 - A)|\psi_1|^2 - B|\psi_2|^2 - 2cR \{ \psi_1 \psi_2^* \},$$
$$W(1, 2) = (1/2 - A)|\psi_1|^2 + (1/2 - B)|\psi_2|^2 + (1 - 2c)R \{ \psi_1 \psi_2^* \},$$
$$W(2, 2) = (A - 1/2)|\psi_1|^2 + (B + 1/2)|\psi_2|^2 + (2c - 1)R \{ \psi_1 \psi_2^* \},$$

(18)

where $A, B, C$ are arbitrary real numbers and $R \{ c \} \equiv (c + c^*)/2$. These relations satisfy $\sum_\beta W(\beta, l) = |\psi|^2$, $\sum_l W(\beta, l) = |\tilde{\psi}_\beta|^2$, where $\tilde{\psi}_\beta = \langle l|\psi \rangle$ for an arbitrary state $\psi$ and its finite Fourier transform is $\tilde{\psi}_\beta = ((-1)^\beta \psi_1 + \psi_2)/\sqrt{2}$ in accordance with (16). The relations (18) can also be written as

$$W(\beta, l) = \sum_{a=1,2,b=1,2} C_{a,b}^\beta \psi_a \psi_b^*. $$

(19)

In the continuous case, the covariance of the Wigner function under translations can be obtained straightforwardly from very general definitions given by Cohen [38]. The properties of our bona fide discrete Wigner function (19) under translations in space and under the exchange of $Z_2$ phases (i.e. discrete momenta) are in agreement with other definitions provided in [33, 39]; the reader may consult appendix A for a full derivation in the general case of six sites. For a very special choice $A = 1/2, B = 0, C = 1/4$, the non-zero coefficients $C_{a,b}^\beta$ can be put explicitly in terms of $I, \beta$, so we may use

$$C_{1,1}^\beta = \frac{1 - (-1)^I}{4}, \quad C_{1,2}^\beta = \frac{1 + (-1)^I}{4}, \quad C_{2,1}^\beta = C_{2,2}^\beta = \frac{(-1)^\beta + 1}{4}. $$

(20)

This particular choice is for computational purposes and for definiteness, and it is not detrimental to the generality of our discussion. To see this in detail, we refer the reader to appendix B, where the problem of degeneracy shall be treated in terms of the $C_3$ basis alone.

3.2. Wigner function for $C_3$

Here we proceed with the unimodular eigenvalues of $C_3$ operators in the construction of the Wigner function; the formula is then

$$W_C(q', k) = \frac{1}{3} \sum_{q=1,2,3} \psi(q) \psi(q'-q) \exp \left[ \frac{2\pi i k}{3} (2q - q') \right], \quad (q' - q) \equiv q' - q \mod 3. $$

(21)
It is confirmed that
\[
\sum_{q'=1,2,3} W_{C_3}(q', k) = \left| \frac{1}{\sqrt{3}} \sum_{a=1,2,3} \psi_a e^{i2\pi ak/3} \right|^2
\]  
and
\[
\sum_{k=1,2,3} W_{C_3}(q', k) = \left| \psi(-q') \right|^2
\]
or, equivalently
\[
\sum_{k=1,2,3} W_{C_3}(-q' \mod 3, k) = \left| \psi_{q'} \right|^2.
\]

3.3. Full Wigner function

The aim now is to give a full definition for \( C_6 \). Using our wavefunctions with two indices \((m, n)\), we recover the correct marginal distributions with the definition
\[
W(l_1, l_2; \beta_1, \beta_2) = \sum_{a=1,2,b=1,2} \sum_{q=1,2,3} \psi_{(q)}^a \psi_{(l_2-q)}^b \exp \left[ \frac{2\pi i \beta_2}{3} (2q - l_2) \right].
\]

The wavefunctions of interest are contained in the degenerate triplet of (2) under the condition (9). We study the behaviour of (25) for these three solutions in various regimes:

(i) Critical angle \( \theta = \theta_c \) with time reversal symmetry.
(ii) A slightly perturbed configuration away from the triple point, i.e.
\[
F(\Delta_s/\Delta_d) \rightarrow F(\Delta_s/\Delta_d) + \delta.
\]
(iii) Critical angle with a small magnetic field applied with the minimal substitution
\[
\Delta_h \rightarrow \Delta_h e^{i\Phi}.
\]

Let us denote by \( \mathcal{H} \) the Hilbert space of localized states labeled by site numbers; the elements of this basis can be regarded as positions, hence \( \mathcal{H} \) can be associated with configuration space \( X \). Similarly, let \( C \) be the space of phase eigenstates, which corresponds to \( \mathcal{H} \) under finite Fourier transforms; \( C \) can be associated with momentum space \( P \). We adopt the following order: our \( 6 \times 6 \) grid is made of rows labeled in top-to-bottom order as \((k, q) = (1, 1), (2, 1), (3, 1), (1, 2), (2, 2), (3, 2)\), corresponding to the \( C_1 \times Z_2 \) eigenphases. The columns are polymer sites in the order \((m, n) = (1, 1), (2, 1), (3, 1), (1, 2), (2, 2), (3, 2)\). In each row of figure 5 we show three phase space portraits of the triplet. The first row (a)–(c) corresponds to \( \theta = \theta_c \), \( \Delta_h/\Delta_d = 1.9 \) and arbitrary linear combinations of (orthogonal) degenerate states, displaying full occupation of the grid. The second row shows \( \theta \neq \theta_c \), \( \Delta_s = F(1.9) + 0.1 \) and a splitting into a doublet (d), (e) and a singlet (f), where the latter displays only two fringes of occupied pixels. The third row (g)–(i) is the triplet in the eigenphase basis, where the supports of the \( W \) functions do not intersect each other; the parameters of this last set of portraits are remarkably similar to the case \( \theta = \theta_c \), \( \Delta_h/\Delta_d = 1.9 \) but with a slight time reversal symmetry breaking \( \Phi = 10^{-3} \times \pi/2 \) for which the two conjugate representations of \( C_3 \) must have different energies, albeit very close to each other \( \Delta E \sim 10^{-11} \). The small splitting has a strong effect on phase space portraits, as can be verified by comparing the first and third rows of figure 5.
The emergence of hidden symmetry is now evident: by comparing the first and third rows of figure 5, we see that all linear combinations of the triplet will reveal full occupation of phase space in various configurations with the same energy. The reason we only consider the lower semi plane is that the upper semi plane is completely correlated to the former, as can be noted by combining (g) with various coefficients. It is worthwhile mentioning that the portraits (d), (e) can be obtained also by combinations of (g), (i). In this case, the resulting rows (3, 1), (3, 2) of (d), (e) are not empty due to interference terms (cross terms) in $W$ that can be regarded as discrete versions of the cat’s smile [40].

As a conclusion, the locus of symmetry is the lower discrete semi plane in phase space, instead of the nine-dimensional Lie manifold $U(3)$. 

**Figure 5.** Discrete phase space of triplets. First row: critical angle with time reversal symmetry, three-fold degeneracy. Arbitrary linear combinations of panels (a)–(c) reveal an invariant structure of at least three rows of pixels, with arbitrary occupation probability. Second row: a doublet (d) (e) and a singlet (f) obtained from an increase in the star coupling $\Delta_s$; panel (f) already shows zero probability fringes for a state of well defined $C_3$ phase, depicted as white cells (color code at the bottom). Third row: time reversal symmetry breaking with a complex coupling (magnetic flux) showing that only $C_3$ eigenfunctions are allowed. $\mathcal{H}$ is the Hilbert space of on-site functions and $\mathcal{C}$ its finite Fourier transform.
The emergence of continuous symmetries

The formalism developed so far can be used to infer the properties of phase space for continuous systems; the existence of a correct discrete Wigner function is indispensable to this end. We provide remarks on the necessary conditions for a continuous symmetry to emerge (more rigorous proofs may be concocted using appropriate measures and metrics). We shall formulate our result in terms of degeneracies. Let us consider a larger system built by a collection of hexamers on which our single particle can jump. The total Hilbert space of localized states is then \( \mathcal{H}_{\text{tot}} = \bigoplus_i \mathcal{H}_i \) —a sum, not a product—while its Fourier transform is the direct sum of spaces with momentum eigenstates \( \mathcal{C}_{\text{tot}} = \bigoplus_j \mathcal{C}_j \), supplemented by the eigenstates of an additional transformation that hops from polymer \( i \) to polymer \( i + 1 \). We identify the space of position and momentum states of the full system as \( \mathbf{X} \leftrightarrow \mathcal{H}_{\text{tot}}, \mathbf{P} \leftrightarrow \mathcal{C}_{\text{tot}} \). A specific value of the position, denoted by \( X \), is identified with a localized eigenstate in the space \( \mathbf{X} \), similarly for momentum \( P \) and space \( \mathbf{P} \), see figure 6. Using our previous results, the diagonal blocks \( \mathcal{H}_i \times \mathcal{C}_j \) possess invariant subsets (red boxes) that can be juxtaposed to form a continuous infinite set. This is possible when the number of copies \( N \to \infty \) and the number of polymers configured in critical angles is \( g \to \infty \). We fix their ratio \( g/N \to \mathcal{L} \) with \( \mathcal{L} \) a finite length. If the invariant set of the full system remains discrete, we shall have \( \mathcal{L} \to 0 \) and if the invariant set is unbounded, we shall have \( \mathcal{L} \to \infty \), which necessitates a non-compact group. We focus on compact groups, therefore \( 0 < \mathcal{L} < \infty \) as a minimal requirement. The total number of quanta for a degenerate energy is \( 3g \) and the action can be written as \( S = 3\hbar g = 3\hbar N \mathcal{L} \). With this, we venture the following statement for ensembles of discrete systems:

1. A hidden symmetry is continuous and one-dimensional if the associated degeneracy is proportional to the number of quanta, i.e. the action.
2. A system in \( d > 1 \) continuous variables built from an ensemble is integrable if \( g \sim N^{d-2} \) and superintegrable if \( g \sim N^p \) with \( p \geq d - 1 \).
The converse statement is evidently false, since integrable Hamiltonians can be written in terms of integrals of the motion in arbitrary combinations, with the possibility of removing degeneracy. In our case, it is remarkable that traditional examples of integrable and superintegrable systems follow a pattern consistent with 1 and 2:

4.1. The Coulomb problem

The bound states of the non-relativistic hydrogen atom obey $I_{\text{max}} = n - 1$, $g = n^2$, for the $n$th level, ignoring a trivial factor of 2 from spin. The system is superintegrable with an algebra $so(4) = so(3) + so(3)$.

4.2. The isotropic harmonic oscillator

The group in the previous example solves this problem in 3$d$ as well. Moshinsky showed a larger $U(3)$ using ladder operators [41] and $g \sim N^2$. In 2$d$, one has $U(2)$, the system is also superintegrable and $g \sim N$.

4.3. Two commensurate oscillators

The degeneracy grows linearly with $N$ for all solutions of $pn + qm = N$, with $n, m$ the oscillator quanta and $p, q \in \mathbb{N}$ relatively primes. The subtleties of the symmetry group $U(2)$ can be found in [42].

4.4. The Dirac–Coulomb problem

The system enjoys a supersymmetry that allows to find the bound spectrum exactly. However, the degeneracy is linear in the number of quanta: only the angular momentum projections and the $l, l + 1$ orbitals are degenerate (without Lamb shift). The Lippmann–Johnson operator [43, 44] is a symmetry generator, in addition to the $SO(3)$ group and the Dirac (spin–orbit) operator $K$, but its presence only helps to sustain an additional two-fold degeneracy; in fact, its square can be put in terms of $K^2$ and $H^2$ [45] equations (5) and (6). Any possible hidden symmetry must be then discrete, and should be accommodated in bi-spinorial degrees of freedom. The explicit invariant sets are not reported in the literature, but a Wigner function is proposed in [46].

4.5. The Landau electron

This problem has infinite degeneracy. The presence of only one chiral harmonic excitation number (and not two) in the Hamiltonian reveals commutability with the Heisenberg algebra, i.e. non-compact group, in addition to cylindrical symmetry.

4.6. The Dirac oscillator

This problem has infinite degeneracy [47], but the symmetry algebra can be decomposed into compact and non-compact subalgebras [48]. The compact part is $so(4)$ and the total degeneracy is quadratic, obtained from a linear factor coming from the sequence $(N \pm 1, j \mp 1), (N \pm 2, j \mp 2), ...$ and another linear factor from the $2j + 1$ projections. The system is superintegrable. For related spin-orbit systems, see [49].
4.7. Circular and square cavities, separability

Diophantine equations offer a challenge for the computation of degeneracies. Without delving into Hilbert’s tenth problem, it suffices to say that degeneracies \( g = 1, 2 \) occur irregularly for a square box, and are at least two-fold \( \pm m \) for a circular shape. Since \( d = 2 \), our statement 2 is not contradicted. Moreover, at high energies, the formula \( n^2 + m^2 = E \) is that of a circle, so the square box has approximate linear degeneracy for large \( E \). Similarly, the circular shape of radius \( R \) and energy \( E = \frac{k^2}{2} \) obeys \( J_m(kR) = 0 \); using the asymptotic form of the Bessel function, the energy equation \( \cos[kR - \frac{\pi}{4}(2m + 1)] = 0 \) is solved by \( kR = \frac{\pi(2m + 4n + 3)}{4} \) with \( m, n \geq 0 \), and the combination \( 2n + m \) also has linear degeneracy. Indeed, it is a simple exercise to put the classical energy in terms of two action variables for this problem, defining a family of constant energy curves in the plane of actions. Despite of this, the two classical problems are not recognized in the literature as superintegrable.

5. Conclusions

We have found a triple degeneracy point in a system consisting of three dimers. The resulting geometry does not correspond to a recognizable configurational symmetry. It was argued that polymers with a lesser number of sites do not show this phenomenon; in this sense, the system is regarded as minimal. In order to understand the nature of hidden symmetry, we defined a finite phase space and studied the behaviour of the corresponding Wigner function at critical angle and under two types of symmetry breaking: time reversal and polygonal. Since this could be done in the simplest possible case, we went further and built a collection of these objects with the purpose of explaining hidden continuous symmetries starting from degeneracies. Some paradigmatic systems follow our pattern.

As an outlook, we envisage a two-dimensional periodic construction made of critically configured polymers, with the aim of emulating the emergence of purely geometric magnetic fields, with applications to transport properties of electromagnetic waves and the artificial realization of the (non-anomalous) quantum Hall effect without charge carriers [50]. From the point of view of anomalous spectral statistics in \( C_3 \) geometries [51] and false T violation [52], this construction would be plausible and desirable.

Also, of potential interest, is the application of discrete groups to molecular vibrations, where normal modes correspond to symmetry bases in models made of springs. A convenient way to visualize discretely such oscillations suggests itself by means of our discrete Wigner function, as a continuation of [53–55].

Appendix A. Properties of the discrete Wigner function

A.1. Discrete translation property

In parallel to the continuous case, we shall prove some properties of our discrete Wigner function. A translation of our set can be achieved by applying an element of the cyclic group \( C_6 \), decomposable into a translation in \( C_3 \) and a translation in \( Z_2 \). Our kernel in (25) is also decomposable into products, where the \( C_3 \) part is a well-known exponential indicated below:

\[
C^{a,\beta}_{a,\beta} \times e^{2\pi i \beta (2q - l_2)/3},
\]

(A.1)
therefore (25) can be conveniently rewritten as:

\[ W(l_1, l_2; \beta_1, \beta_2) = \sum_{q=1,2,3} A_q(l_1, \beta_1) \exp \left[ \frac{2\pi \beta_2}{3} (2q - l_2) \right] \]

\[ = \sum_{a,b=1,2} B^{a,b}(l_2, \beta_2) C_{a,b}^{l_1,\beta_1}, \quad (A.2) \]

where

\[ A_q(l_1, \beta_1) = \sum_{a,b=1,2} \psi_{a,q} \psi_{b,l_1,q}^* C_{a,b}^{l_1,\beta_1} \quad (A.3) \]

and

\[ B^{a,b}(l_2, \beta_2) = \sum_{q=1,2,3} \exp \left[ \frac{2\pi \beta_2}{3} (2q - l_2) \right] \psi_{a,q} \psi_{b,l_2,q}^* \quad (A.4) \]

are Wigner-like functions for \( Z_2 \) and \( C_3 \) respectively. A translation in position under \( C_3 \) by \( k \) units has the following effect \( \psi_{a,q} \mapsto \psi_{a,q+k} \) and \( \psi_{b,l_2,q}^* \mapsto \psi_{b,(l_2-k),q}^* \); therefore, we proceed to substitute these relations in \( B^{a,b}(l_2, \beta_2) \) given by (A.4), which yields:

\[ \sum_{q'=1,2,3} \exp \left[ \frac{2\pi \beta_2}{3} (2q' - (l_2 + 2k)) \right] \psi_{a,q'} \psi_{b,(l_2+2k)-q'} \]

\[ = B^{a,b}(l_2 + 2k, \beta_2), \quad (A.5) \]

where only \( (l_2 + 2k) \mod 3 \) is considered. Now we verify the translation property in \( \beta_2 \) by \( k \) units:

\[ B^{a,b}(l_2, \beta_2 + k) = \sum_{q=1,2,3} \exp \left[ \frac{2\pi \beta_2}{3} (2q - l_2) \right] \psi_{a,q} \psi_{b,(l_2-k)}^* \]

\[ = \sum_{q=1,2,3} \exp \left[ \frac{2\pi \beta_2}{3} (2q - l_2) \right] \exp \left[ \frac{2\pi k}{3} (q - (l_2 - q)) \right] \psi_{a,q} \psi_{b,(l_2-k)}^* \]

\[ = \sum_{q=1,2,3} \exp \left[ \frac{2\pi \beta_2}{3} (2q - l_2) \right] \exp \left[ \frac{2\pi \beta_2}{3} (2q - l_2) \right] \psi_{a,q} \exp \left[ \frac{-2\pi (l_2 - q)k}{3} \right] \psi_{b,(l_2-k)}^*. \quad (A.6) \]

We recognize that this is equivalent to \( \psi_{a,q} \mapsto \exp \left[ \frac{2\pi \beta_2}{3} (2q - l_2) \right] \psi_{a,q} \) and \( \psi_{b,l_2-k,q}^* \mapsto \exp \left[ \frac{-2\pi (l_2 - q)k}{3} \right] \psi_{b,(l_2-k)}^* \) which are the correct phase translations in \( C_3 \). The same property holds for \( Z_2 \) as we show now. Using \( A_q(l_1, \beta_1) \) and the particular choice of \( C_{a,b}^{l_1,\beta_1} \) in the following expression

\[ A_q(1,1) = \frac{1}{2} \psi_{1,q} \psi_{1,(l_2-q)}^* + \frac{1}{4} (\psi_{1,q} \psi_{2,(l_2-q)}^* + \psi_{2,q} \psi_{1,(l_2-q)}^*), \]

\[ A_q(2,1) = \frac{1}{2} \psi_{1,q} \psi_{2,(l_2-q)}^* - \frac{1}{4} (\psi_{1,q} \psi_{2,(l_2-q)}^* + \psi_{2,q} \psi_{1,(l_2-q)}^*), \]

\[ A_q(1,2) = \frac{1}{2} \psi_{2,q} \psi_{1,(l_2-q)}^* + \frac{1}{4} (\psi_{1,q} \psi_{2,(l_2-q)}^* + \psi_{2,q} \psi_{1,(l_2-q)}^*), \]

\[ A_q(2,2) = \frac{1}{2} \psi_{2,q} \psi_{2,(l_2-q)}^* - \frac{1}{4} (\psi_{1,q} \psi_{2,(l_2-q)}^* + \psi_{2,q} \psi_{1,(l_2-q)}^*), \quad (A.7) \]
we find that upon a translation of wave components \(\psi_{1(j)} \mapsto \psi_{2(j)}\), \(\psi_{2(j)} \mapsto \psi_{1(j)}\) with \(j = qJ_2 - q\), the functions satisfy \(A_q(11) \mapsto A_q(12), A_q(21) \mapsto A_q(22), A_q(12) \mapsto A_q(11)\) and \(A_q(22) \mapsto A_q(21)\). Also, using the transformation \(\psi_{mJ} \mapsto e^{i\delta} \psi_{mJ}\) where only the following \(Z_2\) phase factors are allowed \(e^{i\delta} = \pm 1\), the wave functions are transformed as \(\psi_{1(j)} \mapsto +\psi_{1(j)}\) and \(\psi_{2(j)} \mapsto -\psi_{2(j)}\) with \(j = qJ_2 - q\). This finally leads to the desired map \(A_q(11) \mapsto A_q(21), A_q(21) \mapsto A_q(11), A_q(12) \mapsto A_q(21)\) and \(A_q(22) \mapsto A_q(12)\). Similarly for \(\psi_{1(j)} \mapsto -\psi_{1(j)}\) and \(\psi_{2(j)} \mapsto +\psi_{2(j)}\).

The overall effect of translations studied in (A.5)-(A.7) yields the correct transformation properties of the full \(C_6\) Wigner function, as previously announced.

**A.2. Composition property**

Another property of interest is the correct probability amplitude under composition, as presented in [56] by O’Connell and Wigner for the continuous case. Here we establish the discrete version as follows:

\[
|\langle \psi | \phi \rangle|^2 \propto \sum_{i,j,k} W^\psi(i, j; k, l) W^\phi(i, j, k, l).
\]  
(A.8)

We use again the Wigner-like functions (A.3) and (A.4) to treat the \(Z_2\) and \(C_3\) cases separately. For \(C_3\) we proceed as follows:

\[
|\langle \psi | \phi \rangle|^2 \propto \sum_{a, b=1, 2, i, j=1, 2, 3} B^a_b(i, j) B^b_a(i, j).
\]  
(A.9)

The sum over \(i\) and \(j\) can be written as shown below:

\[
\sum_i B^b_a(i, j) B^a_b(i, j) = \sum_i \sum_{q, q' = 1, 2, 3} \exp \left[ \frac{4\pi q}{3} (q - (i - q')) \right] \psi_{a, (q)} \psi_{b, (i - q')} \phi_{a, (q')} \phi_{b, (i - q')}.
\]  
(A.10)

Here, in order to obtain non vanishing terms we need \(q - (i - q') = 0\), simplifying the sum over \(q\) and \(q'\) to a sum over just one index. The sum over \(i\) and \(q\) can be rewritten if we make the appropriate substitution \((q) \mapsto m\) and \((i - q) \mapsto n\) where \(m, n = 1, 2, 3\), obtaining thus

\[
\sum_i B^b_a(i, j) B^a_b(i, j) = 3 \sum_{m, n=1, 2, 3} \psi_{a, m} \phi_{b, n} \phi_{a, n} \psi_{b, n}.
\]  
(A.11)

and if we sum over \(a\) and \(b\) we also find

\[
|\langle \psi | \phi \rangle|^2 = \frac{1}{3} \sum_{a, b=1, 2, i, j=1, 2, 3} B^b_a(i, j) B^a_b(i, j).
\]  
(A.12)

For the \(Z_2\) case we have the following choice of coefficients (note the phase change with respect to (A.7))

\[
A^\psi_{m,n}(1, 1) = \frac{1}{2} \psi_{1, (m)} \psi_{1, (n)}^* + \frac{1}{2\sqrt{2}} (e^{i\pi/4} \psi_{1, (m)} \psi_{2, (n)}^* + e^{-i\pi/4} \psi_{2, (m)} \psi_{1, (n)}^*)
\]

\[
A^\psi_{m,n}(2, 1) = \frac{1}{2} \psi_{1, (m)} \psi_{1, (n)}^* - \frac{1}{2\sqrt{2}} (e^{i\pi/4} \psi_{1, (m)} \psi_{2, (n)}^* + e^{-i\pi/4} \psi_{2, (m)} \psi_{1, (n)}^*)
\]

\[
A^\psi_{m,n}(1, 2) = \frac{1}{2} \psi_{2, (m)} \psi_{2, (n)}^* + \frac{1}{2\sqrt{2}} (e^{-i\pi/4} \psi_{1, (m)} \psi_{2, (n)}^* + e^{i\pi/4} \psi_{2, (m)} \psi_{1, (n)}^*)
\]

\[
A^\psi_{m,n}(2, 2) = \frac{1}{2} \psi_{2, (m)} \psi_{2, (n)}^* - \frac{1}{2\sqrt{2}} (e^{-i\pi/4} \psi_{1, (m)} \psi_{2, (n)}^* + e^{i\pi/4} \psi_{2, (m)} \psi_{1, (n)}^*)
\]  
(A.13)
and
\[
\sum_{ij} A^a_{ij}(i,j) A^a_{ij}(i,j) = \frac{1}{2} \left[ \psi_{1,m}^* \phi_{1,m}^\dagger \phi_{1,m} + \psi_{2,m}^2 \phi_{2,m}^\dagger \phi_{2,m} + \psi_{1,m}^* \phi_{2,m}^\dagger \phi_{2,m} + \psi_{2,m}^* \phi_{1,m}^\dagger \phi_{1,m} \right]
\]
\[
= \frac{1}{2} \sum_{i,j=1,2} \psi_{i,m} \phi_{j,m}^* \phi_{j,m} \phi_{j,m}^*.
\] (A.14)

Applying the sum on \(m, n\) leads to the desired result:
\[
|\langle \psi | \phi \rangle|^2 = 2 \sum_{m, n} \sum_{ij} A^a_{mn}(i,j) A^a_{mn}(i,j).
\] (A.15)

### Appendix B. Generality of coefficients for the \(Z_2\) Wigner function

The specific choice of coefficients in (20) is not detrimental to the generality of our description in terms of degeneracies. We prove this as follows: let \(\psi, \phi, \eta\) be three degenerate states of energy but with different eigenvalues of the \(C_3\) generator (\(2\pi/3\) rotation \(T\)), and take an arbitrary linear combination \(|\Psi\rangle = \alpha|\psi\rangle + \beta|\phi\rangle + \gamma|\eta\rangle\). Then the Wigner function corresponding to \(\Psi\) has three continuous parameters; if all but one of them vanish, then the support of the Wigner function will be given by two horizontal fringes. The fringes are disjoint for the three cases \((\alpha = 1, \beta = \gamma = 0), (\beta = 1; \alpha = \gamma = 0)\) and \((\gamma = 1, \alpha = \beta = 0)\) (as shown in figures 5(g)–(i)). Let us consider an arbitrary \(C_3\) eigenfunction in the components of \(|\Psi\rangle\)
\[
|\Psi_{n,(q)}\rangle = (1/\sqrt{3}) \exp[-i2\pi \kappa q/3] |\psi_n\rangle
\] (B.1)
for arbitrary \(\psi_n\). Then, the relevant sum in the Wigner function (25) over \(C_3\) sites has the form
\[
\sum_{q=1,2,3} \exp \left[ -\frac{2\pi}{3} (\kappa q + \kappa q - \kappa l_2) \right] \exp \left[ \frac{2\pi}{3} \beta_3 (2q - l_2) \right] = 3 \delta_{k, \beta_3},
\] (B.2)
regardless of the coefficients \(c_{n,k}^{a,b}\) in (25). Finally, the full variation of \(\alpha, \beta, \gamma\) will produce Wigner functions whose supports populate the whole grid. This is the invariant locus.

### ORCID iDs

E Sadurní Ⓟ https://orcid.org/0000-0003-1653-9409

### References

[1] Stevenson A F 1941 Phys. Rev. 59 842–3
[2] Bander M and Itzykson C 1966 Rev. Mod. Phys. 38 330–45
[3] Fock V 1935 Z. Phys. 98 145–54
[4] Pauli W 1926 Z. Phys. 36 336–63
[5] Bargmann V and Moshinsky M 1960 Nucl. Phys. 18 697
[6] Bargmann V 1936 Z. Phys. 99 576–82
[7] Bargmann V and Moshinsky M 1961 Nucl. Phys. 23 177–99
[8] Kramer P and Moshinsky M 1966 Nucl. Phys. 82 241–74
[9] Kramer P and Moshinsky M 1968 Nucl. Phys. A 107 481–522
[10] Kustanheimo P and Stiefel E 1965 J. Reine Angew. Math. 218 204

18
