The average dual surface of a cohomology class and minimal simplicial decompositions of infinitely many lens spaces

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Abstract

Discrete normal surfaces are normal surfaces whose intersection with each tetrahedron of a triangulation has at most one component. They are natural Poincaré duals to 1-cocycles with $\mathbb{Z}/2\mathbb{Z}$-coefficients. We show that for a simplicial poset and a fixed cohomology class the average Euler characteristic of the associated discrete normal surfaces only depends on the $f$-vector of the triangulation. As an application we determine the minimum simplicial poset representations, also known as crystallizations, of lens spaces $L(2k, q)$, where $2k = qr + 1$.

Analyzing compact three-manifolds by cutting them into pieces, in particular tetrahedra, has a long and successful history. Depending on the author, a “triangulated three-manifold” can have several different meanings. At one extreme are abstract simplicial complexes where a face is completely determined by its vertices and a given three-manifold $M$ is triangulated by an abstract simplicial complex $\Delta$ if the geometric representations of $\Delta$ are homeomorphic to $M$. At the other extreme are Delta-complexes which are the structures most commonly used in modern algorithmic low-dimensional topology. In a Delta-complex the interiors of the cells are open simplices. Giving the closed cells more flexibility may allow one to present $M$ in a very succinct manner. See, for instance [11]. In between these two are simplicial posets. Here the closed cells are simplices, but more than one face can have the same set of vertices. So in this setting two vertices and two edges are sufficient to triangulate a circle. A basic result is that any $d$-dimensional closed PL-manifold can be given a simplicial poset triangulation with $d + 1$ vertices, the minimum possible [13], [7].

In all three cases one of the fundamental problems is to determine the smallest possible triangulations of a given three-manifold. In [8] Jaco, Rubenstein and Tillman produced the first infinite family of irreducible three-manifolds whose minimal presentation as Delta-complexes could be proven. We use their ideas in Theorem 2.6 to accomplish the same for simplicial posets which are lens spaces of the form $L(2k, q)$ with $2k = qr + 1$. Along the

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way we will find that for a simplicial poset $\Delta$ the Euler characteristic of the average discrete normal surface dual to a fixed cohomology class $\phi$ in $H^1(\Delta; \mathbb{Z}/2\mathbb{Z})$ is independent of $M$ and $\phi$. It only depends on the $f$-vector of $\Delta$!

After setting notation in Section 1, the precise meaning of the previous sentence is explained in Section 2. In the last section we show how to use this to prove that minimal simplicial poset representations of $L(2k, q)$ with $2k = qr + 1$ have $4(q + r)$ tetrahedra.

1 Notation

Regular CW-complexes in which all closed cells are combinatorially simplices have appeared under a variety of names. These include semi-simplicial complexes [6], Boolean cell complexes [4], and simplicial posets [15]. The last is the most frequent in the combinatorics literature, and since we will be concerned with questions of an enumerative nature we will use it throughout. In any case, the reader will be well-served with the idea that simplicial posets are analogous to abstract simplicial complexes where a set of vertices may determine more than one face.

Throughout $\Delta$ is a simplicial poset. As usual $f_0, f_1, f_2$ and $f_3$ will refer to the number of vertices, edges, triangles and tetrahedra of the complex. We use $L(p, q)$ to stand for the lens space given by $S^3/(\mathbb{Z}/p\mathbb{Z})$ where $S^3 = \{ (z_1, z_2) \in \mathbb{C}^2 : |z_1| = |z_2| = 1 \}$ with $\mathbb{Z}/p\mathbb{Z}$ as the $p^{th}$-roots of unity acting by $e^{\frac{2\pi i}{p}} \cdot (z_1, z_2) = (e^{\frac{2\pi i}{p}} z_1, e^{\frac{2\pi i}{p}} z_2)$.

A crystallization of a three-manifold $M$ without boundary is a simplicial poset with exactly four vertices which is homeomorphic as a topological space to $M$. A well-known crystallization of $L(p, q)$ is formed by taking the join of two circles each of which consists of $2p$ vertices and edges. Then quotient out by the $\mathbb{Z}/p\mathbb{Z}$ action. We will call this the standard crystallization of $L(p, q)$. For future reference we observe that the standard crystallization of $L(p, q)$ has $4p$ tetrahedra.

Except where otherwise noted, all chains, cochains and their corresponding homology and cohomology groups will be with $\mathbb{Z}/2\mathbb{Z}$ coefficients.

A surface $S$ contained in $\Delta$ is normal if for every tetrahedron $T$ of $\Delta$ each component of $S \cap T$ is combinatorially equivalent to one the three possibilities in Figure 1 (taken directly from [8]). Suppose that $S$ is a normal surface such that for each $T$ the intersection $S \cap T$ has at most one component. In [14] these types of normal surfaces were called discrete normal surfaces. Given a discrete normal surface $S$ every triangle of $\Delta$ intersects $S$ in either 0 or 2 edges. Hence, if we define a function $\psi : C^1(\Delta) \to \mathbb{Z}/2\mathbb{Z}$ by $\psi(e)$ is one if $e$ intersects $S$, and zero otherwise, then $\psi$ is a $\mathbb{Z}/2\mathbb{Z}$ 1-cocycle of $\Delta$. Conversely, if $\psi$ is a $\mathbb{Z}/2\mathbb{Z}$ 1-cocycle of $\Delta$, then we can easily produce $S_\psi$ so that $S_\psi$ is a discrete normal surface and $\psi$ is the 1-cocycle given by the previous construction applied to $S_\psi$. Evidently $S_\psi$ as defined above is unique up to combinatorial equivalence. We note that $S_\psi$ is a $\mathbb{Z}/2\mathbb{Z}$-Poincaré dual of $\psi$. We also note that if $\psi$ is the zero 1-cocycle, then $S_\psi$ is the empty set which we consider to be a discrete normal surface of Euler characteristic zero.

Given a discrete normal surface $S$ and a tetrahedron $T$ we call $T$ of type $A, B,$ or $C$ according to Figure 1. If $\psi$ is a 1-cocycle in $C^1(\Delta)$, then the associated discrete normal
surface is \( S_\psi \) and we denote by \( A_\psi, B_\psi \) and \( C_\psi \) the number of tetrahedrons of type \( A, B \) and \( C \) respectively in \( \Delta \). Edges \( e \) with \( \psi(e) = 0 \) are called \( \psi \)-even edges. Similarly, edges \( e \) such that \( \psi(e) = 1 \) are called \( \psi \)-odd edges.

\[ \text{Proof.} \]

Let \( \Delta \) be a simplicial poset. For any cohomology class \([\phi] \in H^1(\Delta; \mathbb{Z}/2\mathbb{Z})\) the average of \( 2e_\phi - (B_\psi + 2C_\psi) \) for all cocycles \( \psi \) in \([\phi]\) is \( f_1 - \frac{3}{2} f_3 \).

\[ \text{Proof.} \]

Let \( n \) be the number of vertices of \( \Delta \) and \( c \) the number of components of \( \Delta \). Choose a representative 1-cocycle \( \sigma \in [\phi] \). Consider

\[ Z = \sum_{u \in C^0(\Delta)} 2e_{\sigma+\delta(u)} - (B_{\sigma+\delta(u)} + 2C_{\sigma+\delta(u)}). \]

The coboundary map \( \delta_0 : C^0 \rightarrow C^1 \) has a \( c \)-dimensional kernel, so \( Z \) counts each cocycle in \([\phi] \) \( 2^c \)-times. However, this has no effect on the fact that the average value of \( 2e_\phi - (B_\psi + 2C_\psi) \) is \( Z/2^n \).

Let \( T \) be a tetrahedron in \( \Delta \) with distinct vertices \( i, j, k, l \). If \( T \) is of type \( C \) for \( \sigma \), then \( T \) occurs in the sum doubled (i.e., the \( 2C \) term) \( 2^{n_4} + 2^{n_4} \) times corresponding to \( u(i) = u(j) = u(k) = u(l) = 0 \) and \( u(i) = u(j) = u(k) = u(l) = 1 \). Furthermore, \( T \) occurs singly in \( 4 \cdot 2^{n_4} + 4 \cdot 2^{n_4} \) terms corresponding to when exactly one or three of \( u(i), u(j), u(k), u(l) \) equals one. Thus \( T \) contributes \(-12 \cdot 2^{n_4} \) to \( Z \). The same type of analysis shows that tetrahedron of type \( A \) or \( B \) also contribute \(-12 \cdot 2^{n_4} \) to \( Z \). If \( e = \{i, j\} \) is a \( \sigma \)-even edge, then \( e \) contributes \( 2 \cdot 2 \cdot 2^{n_4} \) to \( Z \) corresponding to \( u(i) = u(j) = 0 \) or \( 1 \). Similarly, \( \sigma \)-odd edges also add \( 2 \cdot 2 \cdot 2^{n_4} \) to \( Z \). Hence \( Z = 2^n f_1 - 12 \cdot 2^{n_4} \).

\[ \text{Formulas (1) and (2) in the following corollary are from [8].} \]

\[ \text{Corollary 2.2.} \]

Let \( \Delta \) be a simplicial poset whose geometric realization is a closed three-manifold. If \([\phi] \in H^1(\Delta; \mathbb{Z}/2\mathbb{Z})\), then the average of \( \chi(S_\psi) \) over all cocycles \( \psi \) in \([\phi]\) is

\[ \frac{4f_0 - f_3}{8} = \frac{5f_0 - f_1}{8}. \]
Proof. Let $\psi$ be a cocyle in $[\phi]$ and let $N_1$ be the complement of a small regular neighborhood $N_0$ of $S_\psi$ in $\Delta$. Then $N_1$ is homotopy equivalent to $\Delta_\psi$, the subcomplex of $\Delta$ spanned by the $\psi$-even edges of $\Delta$. Hence,

\[
2\chi(S_\psi) = 2\chi(N_0) = \chi(\partial N_0) = \chi(N_1) = 2\chi(N_1) = 2\chi(\Delta_\psi).
\]

The Euler characteristic of $\Delta_\psi$ (and hence $S_\psi$) can be computed directly as

\[
\chi(S_\psi) = f_0 - e_\psi + \frac{B_\psi + 4C_\psi}{2} - C_\psi.
\]

Now apply the previous lemma and the fact that in any simplicial poset whose geometric realization is a three-manifold without boundary, $f_3 = f_1 - f_0$. \qed

Remark 2.3. An $f$-vector formula for the average Euler characteristic of $S_\psi$ when $\Delta$ is a three-dimensional normal pseudomanifold can be obtained by using the fact that $\chi(\partial N_1) = 2\chi(N_1) - 2\chi(\Delta)$.

As an example of possible applications of Corollary 2.2 we consider a simple example.

Example 2.4. Let $\Delta$ be the boundary of an 11-vertex two-neighborly 4-polytope. In this case $5f_0 - f_1 = (55 - 55)/8 = 0$. In addition, each vertex link occurs twice as a discrete normal surface and has Euler characteristic two. Hence $\Delta$ must contain a discrete normal surface $S$ with negative Euler characteristic. As nonorientable closed surfaces do not embed in the three-sphere, $S$ must be orientable with genus at least two.

Our other application of Corollary 2.2 is to determine the size of a minimal crystallization of $L(2k, q)$ whenever $2k = rq + 1$. In other words, $2k - 1 = qr$, with $q$ and $r$ odd positive integers. In preparation, we recall Bredon and Wood’s main result concerning which nonorientable surfaces embed in $L(2k, q)$. The following theorem is implied by [3, Theorem 6.1].

Theorem 2.5. [3] Assume that $2k = qr + 1$, $q$ and $r$ odd positive integers. Then a closed nonorientable surface $S$ embeds in $L(2k, q)$ if and only if its Euler characteristic is $\frac{4q - r - 2i}{2}$, with $i$ a nonnegative integer.

Theorem 2.6. Suppose $2k = qr + 1$, with $q$ and $r$ positive odd integers. Any minimal crystallization of $L(2k, q)$ has $4(q + r)$ tetrahedra.

Proof. The theorem is well-known for $k = 1$, so we assume $k \geq 2$. Let $\Delta$ be a minimal crystallization of $L(2k, q)$. The recent preprints by Casali and Cristofori [5] and (independently) Basak and Datta [2] produce crystallizations of $L(2k, q)$ with $4(q + r)$ facets. Hence, $f_3(\Delta) \leq 4(q + r)$. So it remains to prove the reverse inequality. Let $\phi$ be the nontrivial element of $H^1(\Delta, \mathbb{Z}/2\mathbb{Z})$. Applying Corollary 2.2 we can find a cocycle $\psi$ in $[\phi]$ such that

\[
4f_0 - f_3 \leq 8\chi(S_\psi).
\]
Since $H^2(L(2k, q); \mathbb{Z})$ is zero $S_\psi$ is not orientable. By Theorem 2.5 the Euler characteristic of the nonorientable components of $S_\psi$ sum to at most $(4 - q - r)/2$. What about orientable components? Except for spheres, these components do not increase the Euler characteristic of $S_\psi$. We claim that $S_\psi$ has no sphere components. Suppose $S_\psi$ has a sphere component. Then $\Delta - S_\psi$ has at least two components, so $\Delta_\psi$ also has at least two components. That implies that there exist vertices $v_1, v_2$ so that $\psi(e) = 1$ for all edges $e$ between $v_1$ and $v_2$. Let $X$ be the subcomplex of $\Delta$ spanned by $v_1$ and $v_2$. Then the natural inclusion map from $H_1(X; \mathbb{Z}/2\mathbb{Z}) \rightarrow H_1(\Delta; \mathbb{Z}/2\mathbb{Z})$ is surjective. See, for instance, [9]. Of course, this is impossible since $\psi$ is nontrivial in cohomology but evaluates to zero on all generators of $H_1(X; \mathbb{Z}/2\mathbb{Z})$.

Since $\chi(S_\psi) \leq (4 - q - r)/2$ equation (3) implies that $16 - f_3 \leq 4(4 - q - r)$ so $f_3 \geq 4(q + r)$ as required.

When $q = 1$ and $r = 2k - 1$ the standard crystallization of $L(2k, 1)$ has the required number of tetrahedra. In other cases more sophistication is needed. See [2] and [5].

**Remark 2.7.** Using 1-dipole moves and connected sum with a balanced sphere whose $h$-vector is $(1, 0, 2, 0, 1)$ it is possible to show that determining the minimal crystallization of a closed 3-manifold $M$ is equivalent to determining all possible $f$-vectors of balanced simplicial posets homeomorphic to $M$. See [7] for an explanation of dipole moves, [15] for a balanced sphere with $h$-vector $(1, 0, 2, 0, 1)$, and [9] for a recent definition of balanced simplicial poset.

### 3 Problems

- Are there other families of 3-manifolds for which the idea of the proof of Theorem 2.6 works?
- Are there higher dimensional analogs of Corollary 2.2?
- Extensions to manifolds with boundary?
- An abstract simplicial complex is a flag complex if all minimal nonfaces are one-dimensional. It is known that if $\Delta$ is a three-dimensional flag homology sphere, then $f_1 - 5f_0 + 16 \geq 0$ [12]. By Corollary 2.2 this is equivalent to showing that the average discrete normal surface has Euler characteristic less than or equal to two. Could this fact be proven directly? Could this be extended to arbitrary closed three-manifolds? For evidence that this might be true see [1].
- Can the extra information available when $\Delta$ is a balanced simplicial poset be used to get a better understanding of the discrete normal surfaces in $\Delta$?
- Are there examples of normal pseudomanifolds where the analog of Corollary 2.2 can be used effectively?
• While Corollary 2.2 holds for abstract simplicial complexes, it appears that a deeper understanding of the distribution of the $\chi(S_\psi)$ may be needed to make it a useful method capable of producing sharp simplicial $f$-vector results.

• Can the face ring see discrete normal surfaces?

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