REMARK ON CALDERÓN’S PROBLEM FOR THE SYSTEM OF ELLIPTIC EQUATIONS

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Abstract. We consider the Calderón problem in the case of partial Dirichlet-to-Neumann map for the system of elliptic equations in a bounded two dimensional domain. The main result of the manuscript is as follows: If two systems of elliptic operators generate the same partial Dirichlet-to-Neumann map the coefficients can be uniquely determined up to the gauge equivalence.

1. Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^2$ with smooth boundary, let $\tilde{\Gamma}$ be an open set on $\partial\Omega$ and $\Gamma_0 = Int(\partial\Omega \setminus \tilde{\Gamma})$. Consider the following boundary value problem:

$$L(x, D)u = \Delta u + 2A\partial_\nu u + 2B\partial_\mu u + Qu = 0 \text{ in } \Omega, \quad u|_{\Gamma_0} = 0, \quad u|_{\tilde{\Gamma}} = f.$$  

Here $u = (u_1, \ldots, u_N)$ is a unknown vector function and $A, B, Q$ be smooth $N \times N$ matrices. Consider the following partial Dirichlet-to-Neumann map:

$$\Lambda_{A,B,Q}f = \partial_\nu u,$$

where $L(x, D)u = 0$ in $\Omega$, $u|_{\Gamma_0} = 0$, $u|_{\tilde{\Gamma}} = f$, where $\nu$ is the outward unit normal to $\partial\Omega$. This inverse problem is the generalization of so called Calderón’s problem (see [1]), which itself is the mathematical realization of Electrical Impedance Tomography (EIT). The goal of this paper is to extend the result obtained in [2] for the above problem in three-dimensional convex domain, which states that the coefficients of two systems of elliptic equations which principal part is the Laplace operator and which produce the same Dirichlet-to-Neumann map can be determined up to the gauge equivalence.

We have

**Theorem 1.1.** Let $A_j, B_j \in C^{5+\alpha}(\bar{\Omega}), Q_j \in C^{4+\alpha}(\bar{\Omega})$ for $j = 1, 2$ and some $\alpha \in (0, 1)$ and for the operators $L_j(x, D)$ of the form (1.1) with coefficients $A_j, B_j, Q_j$ and adjoint of these operators zero is not an eigenvalue. Suppose that $\Lambda_{A_1,B_1,Q_1} = \Lambda_{A_2,B_2,Q_2}$. Then

$$A_1 = A_2 \quad \text{and} \quad B_1 = B_2 \quad \text{on } \tilde{\Gamma},$$

and there exists an invertible matrix $Q \in C^{5+\alpha}(\bar{\Omega})$ such that

$$Q|_{\tilde{\Gamma}} = I, \quad \partial_\nu Q|_{\tilde{\Gamma}} = 0,$$

$$A_2 = 2Q^{-1}\partial_\nu Q + Q^{-1}A_1Q \quad \text{in } \Omega,$$
\begin{equation}
B_2 = 2Q^{-1}\partial_z Q + Q^{-1}B_1 Q \quad \text{in } \Omega,
\end{equation}

\begin{equation}
Q_2 = Q^{-1}Q_1 Q + Q^{-1}\Delta Q + 2Q^{-1}A_1 \partial_z Q + 2Q^{-1}B_1 \partial_z Q \quad \text{in } \Omega.
\end{equation}

The paper organized as follows. In section 3 we construct the complex geometric optics solutions for the boundary value problem (1.1). In section 4 we prove some asymptotic for coefficients of two operators $L_j(x,D)$ of the form (1.2) which generate the same Dirichlet-to-Neumann map. In section 5, from the asymptotic relations obtained in the section 4, it is proved that there exists a gauge transformation $Q$ which preserves the Dirichlet-to-Neumann map and such that it transforms the coefficient $A_1 \rightarrow A_2$. Then for the coefficients operators $Q^{-1}L_1(x,D)Q$ and $L_2(x,D)$ we obtain some system of integral-differential equations. Finally in the section 6 we study this integral-differential equation and show that the operators $Q^{-1}L_1(x,D)Q$ and $L_2(x,D)$ are the same.

\textbf{Notations.} Let $i = \sqrt{-1}$ and $\overline{\nu}$ be the complex conjugate of $z \in \mathbb{C}$. We set $\partial_z = \frac{1}{2}(\partial_{x_1} - i\partial_{x_2})$, $\partial_{\overline{\nu}} = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2})$ and

\[ \partial_{\overline{\nu}}^{-1}g = -\frac{1}{\pi} \int_{\Omega} \frac{g(\xi_1,\xi_2)}{\zeta - z} d\xi_1 d\xi_2, \quad \partial_z^{-1}g = -\frac{1}{\pi} \int_{\Omega} \frac{g(\xi_1,\xi_2)}{\zeta - \overline{\zeta}} d\xi_1 d\xi_2. \]

For any holomorphic function $\Phi$ we set $\Phi' = \partial_z \Phi$ and $\overline{\Phi}' = \partial_{\overline{\nu}} \Phi$, $\Phi'' = \partial_z^2 \Phi, \Phi'' = \partial_{\overline{\nu}}^2 \Phi$. Let $\overline{\nu} = (\nu_2, -\nu_1)$ be tangential vector to $\partial \Omega$. Let $W_{2,1}(\Omega)$ be the Sobolev space $W_{2}^1(\Omega)$ with the norm $\|u\|_{W_{2,1}(\Omega)} = \|\nabla u\|_{L^2(\Omega)} + \| \tau \| u\|_{L^2(\Omega)}$. Moreover by $\lim_{\eta \to \infty} \frac{\|f(\eta)\|_X}{\eta} = 0$ and $\|f(\eta)\|_X \leq C\eta$ as $\eta \to \infty$ with some $C > 0$, we define $f(\eta) = o_X(\eta)$ and $f(\eta) = O_X(\eta)$ as $\eta \to \infty$ for a normed space $X$ with norm $\| \cdot \|_X$, respectively. $\beta = (\beta_1, \beta_2)$, $\beta_i \in \mathbb{N}_+$, $|\beta| = \beta_1 + \beta_2$, $I$ is the identity matrix.

\textbf{2. Construction of the operators $P_B$ and $T_B$.}

Let $A, B$ be an $N \times N$ matrix with elements from $C^{5+\alpha}(\overline{\Omega})$ with $\alpha \in (0, 1)$. Consider the boundary value problem:

\begin{equation}
K(x,D)(U_0, \tilde{U}_0) = (2\partial_x U_0 + AU_0, 2\partial_{\overline{\nu}} \tilde{U}_0 + B\tilde{U}_0) = 0 \quad \text{in } \Omega, \quad U_0 + \tilde{U}_0 = 0 \quad \text{on } \Gamma_0.
\end{equation}

Without loss of generality we assume that $\Gamma$ is an ark with endpoints $x_{\pm}$.

We have

\textbf{Proposition 2.1.} (see [7]) Let $e$ be a positive number, $A, B \in C^{5+\alpha}(\Omega)$ for some $\alpha \in (0, 1)$, $\Psi \in C^\infty(\partial \Omega)$, $\overline{r}_{i,k}, \ldots, \overline{r}_{2,k} \in \mathbb{C}^3$ be arbitrary vectors and $x_1, \ldots, x_k$ be mutually distinct arbitrary points from the domain $\Omega$. There exists a solution $(U_0, \tilde{U}_0) \in C^{6+\alpha}(\overline{\Omega})$ to problem (2.1) such that

\begin{equation}
\partial^j \partial_{x_{\ell}} U_0(x_{\ell}) = \overline{r}_{j,\ell} \quad \forall j \in \{0, \ldots, 5\}, \quad \text{and } \forall \ell \in \{1, \ldots, k\},
\end{equation}

\begin{equation}
\lim_{x \to x_{\pm}} \frac{|U_0(x)|}{|x - x_{\pm}|^{98}} = \lim_{x \to x_{\pm}} \frac{|\tilde{U}_0(x)|}{|x - x_{\pm}|^{98}} = 0
\end{equation}

and

\begin{equation}
\|U_0 - \Psi\|_{C^{5+\alpha}(\overline{\Omega})} \leq e.
\end{equation}
We construct the matrix $\mathcal{C}$ and the matrix $\mathcal{P}$ as follows
\begin{equation}
\mathcal{C} = (\tilde{U}_0(1), \ldots, \tilde{U}_0(N)), \quad \mathcal{P} = (U_0(1), \ldots, U_0(N)) \in C^{6+\alpha}(\Omega)
\end{equation}
and for any $j \in \{1, \ldots, N\}$
\begin{equation}
\mathcal{K}(x, D)(U_0(j), \tilde{U}_0(j)) = 0 \quad \text{in } \Omega, \quad U_0(j) + \tilde{U}_0(j) = 0 \quad \text{on } \Gamma_0.
\end{equation}
By Proposition 2.1 for the equation (2.6) we can construct solutions $(U_0(j), \tilde{U}_0(j))$ such that
\[U_0(j)(\hat{x}) = \tilde{e}_j, \quad \forall j \in \{1, \ldots, N\},\]
where $\tilde{e}_j$ is the standard basis in $\mathbb{R}^N$.

By $\mathcal{Z}$ we denote the set of zeros of the function $q$ on $\overline{\Omega}$: $\mathcal{Z} = \{z \in \Omega; q(z) = 0\}$. Obviously $\text{card } \mathcal{Z} < \infty$. By $\kappa$ we denote the highest order of zeros of the function $q$ on $\overline{\Omega}$.

Using Proposition 9 of [?] we construct solutions $U_0^{(j)}$ to problem (3.9) such that
\[U_0^{(j)}(x) = \tilde{e}_j \quad \forall j \in \{1, \ldots, N\} \quad \text{and} \quad \forall x \in \mathcal{Z}.
\]
Set $\tilde{\mathcal{P}}(x) = (U_0^{(1)}(x), \ldots, U_0^{(N)}(x)), \tilde{\mathcal{C}}(x) = (\tilde{U}_0^{(1)}(x), \ldots, \tilde{U}_0^{(N)}(x))$. Then there exists a holomorphic function $\tilde{q}$ such that $\det \tilde{\mathcal{P}} = \tilde{q}(z)e^{-\frac{1}{2}\kappa^1 \text{tr} \tilde{\mathcal{P}}}$ in $\Omega$. Let $\tilde{\mathcal{Z}} = \{z \in \Omega; \tilde{q}(z) = 0\}$ and $\tilde{\kappa}$ the highest order of zeros of the function $\tilde{q}$.

By $\tilde{U}_0^{(j)}(x) = \tilde{e}_j$ for $x \in \mathcal{Z}$, we see that $\tilde{\mathcal{Z}} \cap \mathcal{Z} = \emptyset$. Therefore there exists a holomorphic function $r(z)$ such that $r|_{\mathcal{Z}} = 0$ and $(1 - r)|_{\tilde{\mathcal{Z}}} = 0$ and the orders of zeros of the function $r$ on $\mathcal{Z}$ and the function $1 - r$ on $\tilde{\mathcal{Z}}$ are greater than or equal to the max{$\kappa, \tilde{\kappa}$}.

We set
\begin{equation}
P_A f = \frac{1}{2} \mathcal{P} \partial_{\tilde{z}}^{-1}(\mathcal{P}^{-1} r f) + \frac{1}{2} \tilde{\mathcal{P}} \partial_{\tilde{z}}^{-1}(\tilde{\mathcal{P}}^{-1}(1 - r) f).
\end{equation}
Then
\[P_A^* f = -\frac{1}{2} r(\mathcal{P}^{-1})^* \partial_{\tilde{z}}^{-1}(\mathcal{P}^* f) - \frac{1}{2} (1 - r)(\tilde{\mathcal{P}}^{-1})^* \partial_{\tilde{z}}^{-1}(\tilde{\mathcal{P}}^* f).
\]
For any matrix $A \in C^{5+\alpha}(\overline{\Omega}), \alpha \in (0, 1)$, the linear operators $P_A, P_A^* \in \mathcal{L}(L^2(\Omega), W_2^1(\Omega))$ solve the differential equations
\[(-2\partial_{\tilde{z}} + A^*) P_A^* g = g \quad \text{in } \Omega \quad (2\partial_{\tilde{z}} + A) P_A g = g \quad \text{in } \Omega.
\]

In a similar way, using matrices $\mathcal{C}, \tilde{\mathcal{C}}$ we construct the operators
\[T_B f = \frac{1}{2} \mathcal{C} \partial_{\tilde{z}}^{-1}(\mathcal{C}^{-1} \tilde{r} f) + \frac{1}{2} \tilde{\mathcal{C}} \partial_{\tilde{z}}^{-1}(\tilde{\mathcal{C}}^{-1}(1 - \tilde{r}) f)
\]
and
\begin{equation}
T_B^* f = \frac{1}{2} r(\mathcal{C}^{-1})^* \partial_{\tilde{z}}^{-1}(\mathcal{C}^* f) - \frac{1}{2} (1 - r(\mathcal{C}^{-1})) \tilde{\mathcal{C}}^{-1} \partial_{\tilde{z}}^{-1}(\tilde{\mathcal{C}}^* f).
\end{equation}

For any matrix $B \in C^{5+\alpha}(\overline{\Omega}), \alpha \in (0, 1)$, the linear operators $T_B$ and $T_B^*$ solve the differential equation
\[(-2\partial_{\tilde{z}} + B^*) T_B g = g \quad \text{in } \Omega \quad \text{and} \quad (2\partial_{\tilde{z}} + B) T_B^* g = g \quad \text{in } \Omega.
\]

Finally we introduce two operators
\[\mathcal{R}_{r,B} g = e^{r(\mathcal{C} - \Phi)} T_B(e^{r(\Phi - \mathcal{C})} g) \quad \text{and} \quad \mathcal{R}_{r,B} g = e^{r(\Phi - \mathcal{C})} T_B(e^{r(\mathcal{C} - \Phi)} g).
\]
3. Step 1: Construction of complex geometric optics solutions.

Let $L_1(x, D)$ and $L_2(x, D)$ be the operators of the form (1.1) with the coefficients $A_j, B_j, Q_j$. In this step, we will construct two complex geometric optics solutions $u_1$ and $v$ respectively for operators $L_1(x, D)$ and $L_2(x, D)$.

As the phase function for such a solution we consider a holomorphic function $\Phi(z)$ such that $\Phi(z) = \varphi(x_1, x_2) + i\psi(x_1, x_2)$ with real-valued functions $\varphi$ and $\psi$. For some $\alpha \in (0, 1)$ the function $\Phi$ belongs to $C^{6+\alpha}(\overline{\Omega})$. Moreover
\begin{equation}
\partial_x \Phi = 0 \quad \text{in } \Omega, \quad \text{Im } \Phi|_{\Gamma_0} = 0.
\end{equation}

Denote by $\mathcal{H}$ the set of all the critical points of the function $\Phi$: $\mathcal{H} = \{z \in \overline{\Omega}; \Phi'(z) = 0\}$. Assume that $\Phi$ has no critical points on $\overline{\Gamma}$, and that all critical points are nondegenerate:
\begin{equation}
\mathcal{H} \cap \partial \Omega = 0, \quad \Phi'(z) \neq 0, \quad \forall z \in \mathcal{H}, \quad \text{card } \mathcal{H} < \infty.
\end{equation}

Let $\partial \Omega = \bigcup_{j=1}^N \gamma_j$. The following proposition was proved in [5].

**Proposition 3.1.** Let $\bar{x}$ be an arbitrary point in domain $\Omega$. There exists a sequence of functions $\{\Phi_\epsilon\}_{\epsilon \in (0, 1)} \in C^6(\overline{\Omega})$ satisfying (3.1), (3.2) and there exists a sequence $\{\bar{x}_\epsilon\}_{\epsilon \in (0, 1)}$ such that $\bar{x}_\epsilon \in \mathcal{H}_\epsilon = \{z \in \overline{\Omega}; \Phi'_\epsilon(z) = 0\}$, $\bar{x}_\epsilon \to \bar{x}$ as $\epsilon \to +0$, and
\begin{equation}
\text{Im } \Phi_\epsilon(\bar{x}_\epsilon) \notin \{\text{Im } \Phi_\epsilon(x); x \in \mathcal{H}_\epsilon \setminus \{\bar{x}_\epsilon\}\} \quad \text{and } \text{Im } \Phi_\epsilon(\bar{x}_\epsilon) \neq 0.
\end{equation}

Let the function $\Phi$ satisfy (3.1), (3.2) and $\bar{x}$ be some point from $\mathcal{H}$. Without loss of generality, we may assume that $\Gamma$ is an arc with the endpoints $x_\pm$.

Denote $Q_1(1) = -2\partial_x A_1 - B_1 A_1 + Q_1$, $Q_2(1) = -2\partial_x B_1 - A_1 B_1 + Q_1$.

Let $(U_0, \tilde{U}_0) \in C^{6+\alpha}(\overline{\Omega})$ be a solution to the boundary value problem:
\begin{equation}
\mathcal{K}(x, D)(U_0, \tilde{U}_0) = (2\partial_x U_0 + A_1 U_0, 2\partial_x \tilde{U}_0 + B_1 \tilde{U}_0) = 0 \quad \text{in } \Omega, \quad U_0 + \tilde{U}_0 = 0 \quad \text{on } \Gamma_0.
\end{equation}

The complex geometric optics solutions are constructed in [7], [8]. We remind the main steps. Let the pair $(U_0, \tilde{U}_0)$ be defined in the following way
\begin{equation}
U_0 = P_1 a, \quad \tilde{U}_0 = C_1 \bar{a},
\end{equation}
where $a(z) = (a_1(z), \ldots, a_N(z)) \in C^{5+\alpha}(\overline{\Omega})$ is the holomorphic vector function such that $\text{Im } a|_{\Gamma_0} = 0$, or
\begin{equation}
U_0 = P_1 a, \quad \tilde{U}_0 = -C_1 \bar{a},
\end{equation}
where $a(z) = (a_1(z), \ldots, a_N(z)) \in C^{5+\alpha}(\overline{\Omega})$ is the holomorphic vector function such that $\text{Re } a|_{\Gamma_0} = 0$.
\begin{equation}
C_1 = (\tilde{U}_0(1), \ldots, \tilde{U}_0(N)), \quad P_1 = (U_0(1), \ldots, U_0(N)) \in C^{6+\alpha}(\overline{\Omega})
\end{equation}

and for any $k \in \{1, \ldots, N\}$
\begin{equation}
\mathcal{K}(x, D)(U_0(k), \tilde{U}_0(k)) = 0 \quad \text{in } \Omega, \quad U_0(k) + \tilde{U}_0(k) = 0 \quad \text{on } \Gamma_0.
\end{equation}
In order to make a choice of $C_1, P_1$ let us fix a small positive number $\epsilon$. By Proposition [2.1] there exist solutions $(U_0(k), \widetilde{U}_0(k))$ to problem (3.3) for $k \in \{1, \ldots, N\}$ such that

$$(3.10) \quad \|U_0(k) - \bar{e}_k\|_{C^{5+\alpha}(\Gamma_0)} \leq \epsilon \quad \forall k \in \{1, \ldots, N\}.$$ 

This inequality and the boundary conditions in (3.3) on $\Gamma_0$ imply

$$(3.11) \quad \|\widetilde{U}_0(k) - \bar{e}_k\|_{C^{5+\alpha}(\Gamma_0)} \leq \epsilon \quad \forall k \in \{1, \ldots, N\}.$$ 

Let $e_1, e_2$ be smooth functions such that

$$(3.12) \quad e_1 + e_2 = 1 \quad \text{on } \Omega,$$

and $e_1$ vanishes in a neighborhood of $\partial \Omega$ and $e_2$ vanishes in a neighborhood of the set $\mathcal{H}$.

For any positive $\epsilon$ denote $G_\epsilon = \{x \in \Omega; \text{dist} (\text{supp } e_1, x) > \epsilon\}$. The following proposition proved in [?]:

**Proposition 3.2.** Let $B, q \in C^{5+\alpha}(\overline{\Omega})$ for some positive $\alpha \in (0, 1)$, the function $\Phi$ satisfy (3.1), (3.2) and $\bar{q} \in W^1_p(\overline{\Omega})$ for some $p > 2$. Suppose that $q|_\mathcal{H} = \bar{q}|_\mathcal{H} = 0$. Then the asymptotic formulae hold true:

$$(3.13) \quad \tilde{R}_{\tau,B}(e_1(q + \bar{q}/\tau))|_{\overline{\mathcal{G}_\epsilon}} = e^{\tau(\Phi - \Phi)} \left(\frac{m_{+,\bar{x}}e^{2i\tau\psi(\bar{x})}}{\tau^2} + o_{C^2(\overline{\mathcal{G}_\epsilon})}(\frac{1}{\tau^2})\right) \quad \text{as } |\tau| \to +\infty,$$

$$(3.14) \quad R_{\tau,B}(e_1(q + \bar{q}/\tau))|_{\overline{\mathcal{G}_\epsilon}} = e^{\tau(\Phi - \Phi)} \left(\frac{m_{-\bar{x}}e^{-2i\tau\psi(\bar{x})}}{\tau^2} + o_{C^2(\overline{\mathcal{G}_\epsilon})}(\frac{1}{\tau^2})\right) \quad \text{as } |\tau| \to +\infty.$$ 

Denote $q_1 = P_{A_1}(Q_1(1)U_0) - M_1$, $q_2 = T_{B_1}(Q_2(1)\widetilde{U}_0) - M_2 \in C^{5+\alpha}(\overline{\Omega})$, where the functions $M_1 \in \text{Ker}(2\partial_x + A_1)$ and $M_2 \in \text{Ker}(2\partial_x + B_1)$ are taken such that

$$(3.15) \quad q_1(\bar{x}) = q_2(\bar{x}) = \partial^3_x q_1(x) = \partial^3_x q_2(x) = 0, \quad \forall x \in \mathcal{H} \setminus \{\bar{x}\} \quad \text{and } \forall |\beta| \leq 5.$$ 

Moreover by (2.3) we can assume that

$$(3.16) \quad \lim_{x \to x_\pm} \frac{|q_1(x)|}{|x - x_\pm|^{98}} = \lim_{x \to x_\pm} \frac{|q_2(x)|}{|x - x_\pm|^{98}} = 0.$$ 

Next we introduce the functions $(U_{-1}, \widetilde{U}_{-1}) \in C^{5+\alpha}(\overline{\Omega}) \times C^{5+\alpha}(\overline{\Omega})$ as a solutions to the following boundary value problem:

$$(3.17) \quad K(x, D)(U_{-1}, \widetilde{U}_{-1}) = 0 \quad \text{in } \Omega, \quad (U_{-1} + \widetilde{U}_{-1})|_{\Gamma_0} = \frac{q_1}{2\Phi'} + \frac{q_2}{2\Phi'}.$$ 

We set $p_1 = -Q_1(1)(\frac{e^{q_1}}{\Phi} - U_{-1})+ L_1(x, D)(\frac{e^{q_1}}{2\Phi})$, $p_2 = -Q_1(1)(\frac{e^{q_2}}{\Phi} - \widetilde{U}_{-1})+ L_1(x, D)(\frac{e^{q_2}}{2\Phi'})$, $q_2 = T_{B_2}p_2 - M_2$, $\tilde{q}_1 = P_{A_1}p_1 - \tilde{M}_1 \in C^{5+\alpha}(\Omega)$, where $M_1 \in \text{Ker}(2\partial_x + A_1)$ and $M_2 \in \text{Ker}(2\partial_x + B_1)$ are taken such that

$$(3.18) \quad \partial^3_x \tilde{q}_1(x) = \partial^3_x \tilde{q}_2(x) = 0, \quad \forall x \in \mathcal{H} \quad \text{and } \forall |\beta| \leq 5.$$ 

By Proposition [3.2] there exist functions $m_{+,\bar{x}} \in C^{2+\alpha}(\overline{\mathcal{G}_\epsilon})$ such that

$$(3.19) \quad \tilde{R}_{\tau,B_1}(e_1(q_1 + \bar{q}_1/\tau))|_{\overline{\mathcal{G}_\epsilon}} = e^{\tau(\Phi - \Phi)} \left(\frac{m_{+,\bar{x}}e^{2i\tau\psi(\bar{x})}}{\tau^2} + o_{C^2(\overline{\mathcal{G}_\epsilon})}(\frac{1}{\tau^2})\right) \quad \text{as } |\tau| \to +\infty.$$
and
\begin{equation}
\mathcal{R}_{\tau, A_1}(q_2 + \frac{\bar{q}_2}{\tau})|_{\Gamma_{\alpha}} = e^{\tau(\Phi - \overline{\Phi})}\left(\frac{m_{-\xi}e^{-2i\tau\psi(\bar{\xi})}}{\tau^2} + o_{C^2(\Gamma_{\alpha})}\left(\frac{1}{\tau^2}\right)\right) \text{ as } |\tau| \to +\infty.
\end{equation}

For any \( \bar{x} \in \mathcal{H} \) we introduce the functions \( a_{\pm, \bar{x}}, b_{\pm, \bar{x}} \in C^{2+\alpha}(\Omega) \) as solutions to the boundary value problem
\begin{equation}
\mathcal{K}(x, D)(a_{\pm, \bar{x}}, b_{\pm, \bar{x}}) = 0 \quad \text{in } \Omega, \quad (a_{\pm, \bar{x}} + b_{\pm, \bar{x}})|_{\Gamma_0} = m_{\pm, \bar{x}}.
\end{equation}
We choose the functions \( a_{\pm, \bar{x}}, b_{\pm, \bar{x}} \) in the form
\begin{equation}
(a_{\pm, \bar{x}}, b_{\pm, \bar{x}}) = (p_1(x), p_2(x), \zeta_{\pm, \bar{x}}(\xi)),
\end{equation}
where \( p_1, p_2 \) is some holomorphic function and \( \zeta_{\pm, \bar{x}}(\xi) \) is some antiholomorphic function. Let \((\tilde{U}_2, \tilde{U}_{-2}) \in C^{5+\alpha}(\Omega) \times C^{5+\alpha}(\Omega)\) be solution to the boundary value problem
\begin{equation}
\mathcal{K}(x, D)(U_{-2}, \tilde{U}_{-2}) = 0 \quad \text{in } \Omega, \quad (U_{-2} + \tilde{U}_{-2})|_{\Gamma_0} = \frac{\bar{q}_1}{2\Phi'} + \frac{\bar{q}_2}{2\Phi'}.
\end{equation}

We introduce the functions \( U_{0, \tau}, \tilde{U}_{0, \tau} \in C^{2+\alpha}(\Omega) \) by
\begin{equation}
U_{0, \tau} = U_0 + \frac{U_{-1} - e_2 q_1/2\Phi'}{\tau} + \frac{1}{\tau^2}((e^{2i\tau\psi(\bar{x})}a_{+\bar{x}} + e^{-2i\tau\psi(\bar{x})}a_{-\bar{x}}) + U_{-2} - \frac{\bar{q}_1 e_2}{2\Phi'})
\end{equation}
and
\begin{equation}
\tilde{U}_{0, \tau} = \tilde{U}_0 + \frac{\tilde{U}_{-1} - e_2 q_2/2\Phi'}{\tau} + \frac{1}{\tau^2}((e^{2i\tau\psi(\bar{x})}b_{+\bar{x}} + e^{-2i\tau\psi(\bar{x})}b_{-\bar{x}}) + \tilde{U}_{-2} - \frac{\bar{q}_2 e_2}{2\Phi'}).
\end{equation}
We set \( \mathcal{O}_\varepsilon = \{ x \in \Omega; \text{dist}(x, \partial \Omega) \leq \varepsilon \}. \)
In [?] it is shown that there exists a function \( u_{-1} \) in complex geometric optics solution satisfies the estimate
\begin{equation}
\sqrt{\tau}|u_{-1}|_{L^2(\Omega)} + \frac{1}{\sqrt{|\tau|}}\|\nabla u_{-1}\|_{L^2(\Omega)} + \|u_{-1}\|_{W_{-\alpha}(\mathcal{O}_\varepsilon)} = o\left(\frac{1}{\tau}\right) \quad \text{as } \tau \to +\infty
\end{equation}
and such that the function
\begin{equation}
u_1(x) = U_{0, \tau}e^{r\Phi} + \tilde{U}_{0, \tau}e^{r\overline{\Phi}} - e^{r\Phi}\mathcal{R}_{\tau, B_1}(q_1 + \bar{q}_1/\tau) - e^{r\overline{\Phi}}\mathcal{R}_{\tau, A_1}(q_2 + \bar{q}_2/\tau) + e^{r\Phi}u_{-1}
\end{equation}
solves the boundary value problem
\begin{equation}
L_1(x, D)u_1 = 0 \quad \text{in } \Omega, \quad u_1|_{\Gamma_0} = 0.
\end{equation}
Similarly, we construct the complex geometric optics solutions to the operator \( L_2(x, D)^* \). Let \((V_0, \tilde{V}_0) \in C^{6+\alpha}(\Omega) \times C^{6+\alpha}(\Omega)\) be solutions to the following boundary value problem:
\begin{equation}
\mathcal{M}(x, D)(V_0, \tilde{V}_0) = ((2\partial_{\bar{x}} - B_{-2}^*)V_0, (2\partial_{\bar{x}} - A_{-2}^*)\tilde{V}_0) = 0 \quad \text{in } \Omega, \quad (V_0 + \tilde{V}_0)|_{\Gamma_0} = 0,
\end{equation}
such that
\begin{equation}
\lim_{x \to x_{\pm}} \frac{|V_0(x)|}{|x - x_{\pm}|^{\frac{95}{8}}} = \lim_{x \to x_{\pm}} \frac{|\tilde{V}_0(x)|}{|x - x_{\pm}|^{\frac{95}{8}}} = 0.
\end{equation}
Such a pair \((V_0, \tilde{V}_0)\) exists due to Proposition [2.1]. More specifically let
\begin{equation}
V_0 = C_2 \bar{b}, \quad \tilde{V}_0 = p_2 b,
\end{equation}
where \( b(z) = (b_1(z), \ldots, b_N(z)) \in C^{5+\alpha}(\Omega) \) is the holomorphic vector function such that \( \text{Im} \, b|_{\Gamma_0} = 0 \), or

\[
V_0 = C_2 \overline{b}, \quad \tilde{V}_0 = -\mathcal{P}_2 b,
\]

where \( \mathcal{P}_2 \) is the projection onto the space of holomorphic vector functions. Moreover, by Proposition 2.1 there exist solutions \((V_0(k), \tilde{V}_0(k))\) to problem (3.28) for \( k \in \{1, \ldots, N\} \) such that

\[
\| \tilde{V}_0(k) - \bar{c}_k \|_{C^{5+\alpha}({\Gamma_0})} \leq \epsilon \quad \forall k \in \{1, \ldots, N\}.
\]

This inequality and the boundary conditions in (3.28) on \( \Gamma_0 \) imply

\[
\| V_0(k) - \bar{c}_k \|_{C^{5+\alpha}(\Gamma_0)} \leq \epsilon \quad \forall k \in \{1, \ldots, N\}.
\]

In order to fix the choice of the operators \( P_{-B_2^*}, T_{-A_2^*} \) we take \( \mathcal{C} = C_2, \mathcal{P} = \mathcal{P}_2 \) and \( \tilde{\mathcal{C}} = \tilde{C}_2, \tilde{\mathcal{P}} = \tilde{P}_2 \). We set \( q_3 = P_{-A_2^*}(Q_1(2) \tilde{V}_0) - M_3, \ q_4 = T_{-B_2^*}(Q_2(2)V_0) - M_4 \in C^{5+\alpha}(\Omega) \), where \( Q_1(2) = Q_2 - 2\bar{\partial}_z B_2 - B_2^* A_2^*, \ Q_2(2) = Q_2^* - 2\bar{\partial}_z A_2^* - A_2^* B_2^* \) and \( M_3, M_4 \in Ker(2\bar{\partial}_z - A_2^*) \) are chosen such that

\[
q_3(x) = q_4(x) = \partial_x^\beta q_3(x) = \partial_x^\beta q_4(x) = 0, \quad \forall x \in \mathcal{H} \setminus \{\tilde{x}\} \quad \text{and} \quad \forall |\beta| \leq 5;
\]

\[
\lim_{x \to x_{\pm}} \frac{|q_j(x)|}{|x - x_{\pm}|^{98}} = 0 \quad \forall j \in \{3, 4\}.
\]

By (3.2) the functions \( \frac{q_3}{2\Phi}, \frac{q_4}{2\Phi} \) belong to the space \( C^{5+\alpha}(\Gamma_0) \). Therefore we can introduce the functions \( \bar{V}_1, \tilde{V}_1 \in C^{5+\alpha}(\Omega) \) as solutions to the following boundary value problem:

\[
\mathcal{M}(x, D)(\bar{V}_1, \tilde{V}_1) = 0 \quad \text{in} \ \Omega, \quad (V_1 + \bar{V}_1)|_{\Gamma_0} = -(\frac{q_3}{2\Phi} + \frac{q_4}{2\Phi}).
\]

Let \( p_3 = Q_1(2)(\frac{\epsilon_3 q_3}{2\Phi} + \bar{V}_1) + L_2(x, D)(\frac{q_3 q_4}{2\Phi}) \), \( p_4 = Q_2(2)(\frac{q_4 q_3}{2\Phi} + V_1) + L_2(x, D)(\frac{q_4 q_3}{2\Phi}) \) and \( \bar{q}_3 = (T_{-B_2^*} p_3 - \bar{M}_3), \ \bar{q}_4 = (P_{-A_2^*} p_3 - \bar{M}_4) \in C^{5+\alpha}(\Omega) \), where \( \bar{M}_3, \bar{M}_4 \in Ker(2\bar{\partial}_z - A_2^*) \), \( \bar{q}_3, \bar{q}_4 \) are chosen such that

\[
\partial_x^\beta \bar{q}_3(x) = \partial_x^\beta \bar{q}_4(x) = 0, \quad \forall x \in \mathcal{H} \quad \text{and} \quad \forall |\beta| \leq 5.
\]

By Proposition 3.2 there exist smooth functions \( \bar{m}_{\pm, \tilde{x}} \in C^{2+\alpha}(G_e), \tilde{x} \in \mathcal{H} \), independent of \( \tau \) such that

\[
\bar{R}_{\tau, -B_2^*}(e_1(q_3 + \tilde{q}_3/\tau))|_{\mathcal{G}_e} = \frac{\bar{m}_{\pm, \tilde{x}} e^{2i\tau(\psi - \psi(\tilde{x}))}}{\tau^2} + e^{2i\tau} O_{C^2(G_e)}(\frac{1}{\tau^2}) \quad \text{as} \quad |\tau| \to +\infty
\]

and

\[
R_{\tau, -A_2^*}(e_1(q_4 + \tilde{q}_4/\tau))|_{\mathcal{G}_e} = \frac{\bar{m}_{\pm, \tilde{x}} e^{-2i\tau(\psi - \psi(\tilde{x}))}}{\tau^2} + e^{-2i\tau} O_{C^2(G_e)}(\frac{1}{\tau^2}) \quad \text{as} \quad |\tau| \to +\infty.
\]
Using the functions \( \tilde{m}_{\pm,\bar{z}} \) we introduce functions \( \tilde{a}_{\pm,\bar{z}}, \tilde{b}_{\pm,\bar{z}} \in C^{2+\alpha}(\bar{\Omega}) \) which solve the boundary value problem

\[
\mathcal{M}(x, D)(\tilde{a}_{\pm,\bar{z}}, \tilde{b}_{\pm,\bar{z}}) = 0 \quad \text{in } \Omega, \quad (\tilde{a}_{\pm,\bar{z}} + \tilde{b}_{\pm,\bar{z}})|_{\Gamma_0} = \tilde{m}_{\pm,\bar{z}}.
\]

We choose \( \tilde{a}_{\pm,\bar{z}}, \tilde{b}_{\pm,\bar{z}} \) in the form

\[
(\tilde{a}_{\pm,\bar{z}}, \tilde{b}_{\pm,\bar{z}}) = (C_2(x)\tilde{a}_{\pm,\bar{z}}(\bar{z}), P_2(x)\tilde{b}_{\pm,\bar{z}}(\bar{z})),
\]

where \( \tilde{a}_{\pm,\bar{z}}(\bar{z}) \) is some antiholomorphic function and \( \tilde{b}_{\pm,\bar{z}}(\bar{z}) \) is some holomorphic function. By (3.2) the functions \( \frac{\tilde{a}_{1}}{2\Phi}, \frac{\tilde{a}_{4}}{2\Phi} \) belong to the space \( C^{5+\alpha}(\bar{\Omega}) \). Therefore there exists a pair \( (V_{-2}, \bar{V}_{-2}) \in C^{5+\alpha}(\bar{\Omega}) \times C^{5+\alpha}(\bar{\Omega}) \) which solves the boundary value problem

\[
\mathcal{M}(x, D)(V_{-2}, \bar{V}_{-2}) = 0 \quad \text{in } \Omega, \quad (V_{-2} + \bar{V}_{-2})|_{\Gamma_0} = -\left(\frac{q_3}{2\Phi} + \frac{\bar{q}_4}{2\Phi}\right).
\]

We introduce functions \( V_{0,\tau}, \bar{V}_{0,\tau} \in C^{2+\alpha}(\bar{\Omega}) \) by formulas

\[
\bar{V}_{0,\tau} = \bar{V}_0 + \frac{V_{-1} + \frac{e_2q_3/2\Phi}{\tau}}{\bar{\tau}} + \frac{1}{\bar{\tau}^2}\left(e^{2i\tau\psi(\bar{z})}\bar{b}_{+,\bar{z}} + e^{-2i\tau\psi(\bar{z})}\bar{b}_{-,\bar{z}} + \bar{V}_{-2} + \frac{e_2\bar{q}_3}{2\Phi}\right)
\]

and

\[
V_{0,\tau} = V_0 + \frac{V_{-1} + \frac{e_2q_4/2\Phi}{\tau}}{\tau} + \frac{1}{\tau^2}\left(e^{2i\tau\psi(\bar{z})}\tilde{a}_{+,\bar{z}} + e^{-2i\tau\psi(\bar{z})}\tilde{a}_{-,\bar{z}} + V_{-2} + \frac{e_2q_4}{2\Phi}\right).
\]

The last term \( v_{-1} \) in complex geometric optics solution satisfies the estimate

\[
\sqrt{|\tau|}||v_{-1}||_{L^2(\Omega)} + \frac{1}{\sqrt{|\tau|}}||\nabla v_{-1}||_{L^2(\Omega)} + ||v_{-1}||_{W^{1,7}(\partial_\Omega)} = o\left(\frac{1}{\tau}\right) \quad \text{as } \tau \to +\infty
\]

and such that the function

\[
v = V_{0,\tau}e^{-\tau\Phi} + \bar{V}_{0,\tau}e^{-\bar{\tau}\Phi} - e^{-\tau\Phi}\bar{R}_{-\tau,-B_2}\left(\tau_1(q_3 + \frac{\bar{q}_3}{\tau})\right) - e^{\bar{\tau}\Phi}R_{-\tau,-A_2}\left(\tau_1(q_4 + \frac{\bar{q}_4}{\tau})\right) + v_{-1}e^{-\tau\phi}
\]

solves the boundary value problem

\[
L_2(x, D)^*v = 0 \quad \text{in } \Omega, \quad v|_{\Gamma_0} = 0.
\]

We close this section with one technical proposition similar to one proved in [6]:

**Proposition 3.3.** Suppose that the functions \( C_i, P_i \in C^{6+\alpha}(\bar{\Omega}) \) for all \( i, j \in \{1, 2\} \) some \( \alpha \in (0, 1) \) given by (3.3)-(3.10), (3.32)-(3.34) satisfy

\[
\int_{\partial_\Omega} \{(\nu_1 + i\nu_2)\bar{\Phi}'(P_1a, P_2b) + (\nu_1 - i\nu_2)\Phi'(C_1a, C_2b)\}d\sigma = 0,
\]

for all holomorphic vector functions \( a, b \) such that \( \text{Im}a|_{\Gamma_0} = \text{Im}b|_{\Gamma_0} = 0 \). Then there exist a holomorphic function \( \Theta \in W^{1,7}_2(\Omega) \) and an antiholomorphic function \( \tilde{\Theta} \in W^{1,7}_2(\Omega) \) such that

\[
\tilde{\Theta}|_{\Gamma} = C_2^*C_1, \quad \Theta|_{\Gamma} = P_2^*P_1
\]

and

\[
\Theta = \tilde{\Theta} \quad \text{on } \Gamma_0.
\]
Proof.} First we show that for all holomorphic vector functions \( a, b \) such that \( \text{Im} a |_{\Gamma_0} = \text{Im} b |_{\Gamma_0} = 0 \) there exists a holomorphic function \( \tilde{\Psi} \) and antiholomorphic function \( \Psi \) such that
\[
\tilde{\Phi}'(C_1a, C_2b) - \Psi = \tilde{\Phi}'(P_1a, P_2b) - \tilde{\Psi} = 0 \quad \text{on} \quad \tilde{\Gamma} \quad \text{and} \quad ((\nu_1 - i\nu_2)\Psi + (\nu_1 + i\nu_2)\tilde{\Psi})|_{\Gamma_0} = 0.
\]
Also we observe that the equality (3.49) implies
\[
\mathcal{I} = \int_{\partial \Omega} \{(\nu_1 + i\nu_2)\Phi'(P_1a, P_2b) + (\nu_1 - i\nu_2)\tilde{\Phi}'(\tilde{C}_1(-\bar{a}), \tilde{C}_2\bar{b})\}d\sigma = 0,
\]
for all holomorphic vector functions \( a, b \) such that \( \text{Re} a |_{\Gamma_0} = \text{Im} b |_{\Gamma_0} = 0 \). Indeed,
\[
\mathcal{I} = \frac{1}{i} \int_{\partial \Omega} \{(\nu_1 + i\nu_2)\Phi'(P_1a, P_2b) + (\nu_1 - i\nu_2)\tilde{\Phi}'(\tilde{C}_1(-\bar{a}), \tilde{C}_2\bar{b})\}d\sigma = \frac{1}{i} \int_{\partial \Omega} \{(\nu_1 + i\nu_2)\Phi'(P_1a, P_2b) + (\nu_1 - i\nu_2)\tilde{\Phi}'(\tilde{C}_1(\bar{a}), \tilde{C}_2\bar{b})\}d\sigma = 0.
\]
Here, in order to get the last equality we used (3.49). Consider the extremal problem:
\[
J(\Psi, \tilde{\Psi}) = \|\tilde{\Phi}'(C_1a, C_2b) - \Psi\|^2_{L^2(\tilde{\Gamma})} + \|\Phi'(P_1a, P_2b) - \tilde{\Psi}\|^2_{L^2(\tilde{\Gamma})} \to \inf,
\]
\[
\frac{\partial \Psi}{\partial z} = 0 \quad \text{in} \quad \Omega, \quad \frac{\partial \tilde{\Psi}}{\partial z} = 0 \quad \text{in} \quad \Omega, \quad ((\nu_1 - i\nu_2)\Psi + (\nu_1 + i\nu_2)\tilde{\Psi})|_{\Gamma_0} = 0.
\]
Denote the unique solution to this extremal problem (3.53), (3.54) by \((\tilde{\Psi}, \tilde{\Psi})\). Applying the Fermat theorem, we obtain
\[
\text{Re}(\Phi'(P_1a, P_2b) - \tilde{\Psi}, \delta)_{L^2(\tilde{\Gamma})} + \text{Re}(\tilde{\Phi}'(\tilde{C}_1a, \tilde{C}_2b) - \tilde{\Psi}, \tilde{\delta})_{L^2(\tilde{\Gamma})} = 0
\]
for any \( \delta, \tilde{\delta} \) from \( \mathcal{W}^{\frac{1}{2}}(\Omega) \) such that
\[
\frac{\partial \delta}{\partial z} = 0 \quad \text{in} \quad \Omega, \quad \frac{\partial \tilde{\delta}}{\partial z} = 0 \quad \text{in} \quad \Omega, \quad (\nu_1 + i\nu_2)\delta|_{\Gamma_0} = -(\nu_1 - i\nu_2)\tilde{\delta}|_{\Gamma_0}
\]
and there exist two functions \( P, \tilde{P} \in \mathcal{W}^{\frac{1}{2}}(\Omega) \) such that
\[
\frac{\partial P}{\partial z} = 0 \quad \text{in} \quad \Omega, \quad \frac{\partial \tilde{P}}{\partial z} = 0 \quad \text{in} \quad \Omega,
\]
\[
(\nu_1 + i\nu_2)P = \Phi'(P_1a, P_2b) - \tilde{\Psi} \quad \text{on} \quad \tilde{\Gamma}, \quad (\nu_1 - i\nu_2)\tilde{P} = \tilde{\Phi}'(\tilde{C}_1a, \tilde{C}_2b) - \tilde{\Psi} \quad \text{on} \quad \tilde{\Gamma}
\]
and
\[
(P - \tilde{P})|_{\Gamma_0} = 0.
\]
Denote \( \Psi_0(z) = \frac{1}{2i}(P(z) - \overline{\tilde{P}(\overline{z})}) \) and \( \Phi_0(z) = \frac{1}{2}(P(z) + \overline{\tilde{P}(\overline{z})}) \). Equality (3.59) yields
\[
\text{Im} \Psi_0|_{\Gamma_0} = \text{Im} \Phi_0|_{\Gamma_0} = 0.
\]
Hence
\[
P = (\Phi_0 + i\Psi_0), \quad \overline{\tilde{P}} = (\Phi_0 - i\Psi_0).
\]
From (3.55), taking $\delta = \hat{\Psi}$ and $\tilde{\delta} = \tilde{\Psi}$, we have

$$
(3.62) \quad \text{Re} \int_{\tilde{\Gamma}} (\Phi'(C_1a, C_2b) - \hat{\Psi}, \bar{\Phi'}) + (\Phi'(P_1a, P_2b) - \tilde{\Psi}, \bar{\Phi'}) d\sigma = 0.
$$

By (3.57), (3.58) and (3.61), we have

$$
(3.63) \quad \text{Re} \int_{\tilde{\Gamma}} ((\nu_1 + i\nu_2)P, \Phi'(P_1a, P_2b)) + ((\nu_1 - i\nu_2)\bar{P}, \Phi'(C_1a, C_2b)) d\sigma = \text{Re} \int_{\tilde{\Gamma}} 2((\nu_1 + i\nu_2)(\Phi_0 + i\Psi_0)\Phi'(P_1a, P_2b)) + 2((\nu_1 - i\nu_2)(\Phi_0 - i\Psi_0)\Phi'(C_1a, C_2b)) d\sigma.
$$

By (3.49) and (3.60) we have

$$
(3.64) \quad \int_{\tilde{\Gamma}} 2\text{Re}((\nu_1 - i\nu_2)\Phi_0\Phi'(P_1a, P_2b)) + 2\text{Re}((\nu_1 + i\nu_2)\bar{\Phi}_0\bar{\Phi}'(C_1a, C_2b)) d\sigma = 0.
$$

By (3.52) and (3.60) we obtain

$$
(3.65) \quad (P_1a, P_2b)(x) = (\hat{\Psi}/\Phi')(z) = \tilde{\Xi}(z), \quad (C_1a, C_2b)(x) = (\Psi/\Phi')(\bar{z}) = \Xi(\bar{z}) \quad \text{on } \Gamma.
$$

In general the function $\Phi$ may have a finite number of zeros in $\Omega$. At these zeros $\Xi, \tilde{\Xi}$ may have poles. On the other hand observe that $\Xi, \tilde{\Xi}$ are independent of a particular choice of the function $\Phi$. Making small perturbations of these functions, we can shift the position of the zeros of the function $\Phi'$. Hence there are no poles for $\Xi, \tilde{\Xi}$. By (3.54) $((\nu_1 - i\nu_2)\Psi + (\nu_1 + i\nu_2)\bar{\Phi})|_{\Gamma_0} = 0$. Moreover, by the direct computations, $((\nu_1 + i\nu_2)\Phi' + (\nu_1 - i\nu_2)\bar{\Phi}')|_{\Gamma_0} = 0$. Therefore

$$
(3.66) \quad \tilde{\Xi}(z) = \Xi(\bar{z}) \quad \text{on } \Gamma_0.
$$

Consider $N$ holomorphic vector functions $b_j = (b_{1,j}, \ldots, b_{1,N})$ such that $Im b_j|_{\Gamma_0} = 0$ and determinant of the square matrix constructed from these vector functions not equal to zero at least at one point of domain $\Omega$. Then equality (3.65) can be written as

$$
(P_2^*P_1a, b_j) = \tilde{\Xi}_j(z) \quad \text{and} \quad (C_2^*C_1a, \tilde{b}_j) = \Xi_j(\bar{z}) \quad \text{on } \tilde{\Gamma}.
$$

Then

$$
P_2^*P_1a = B^{-1}\tilde{\Xi} \quad \text{and} \quad C_2^*C_1a = \bar{B}^{-1}\Xi \quad \text{on } \tilde{\Gamma}.
$$

Here $B$ is the matrix such that the row number $j$ equal $b_j^T$ and $\Xi(z) = (\Xi_1(z), \ldots, \Xi_N(z)), \tilde{\Xi} = (\Xi_1(\bar{z}), \ldots, \Xi_N(\bar{z}))$. Consider $N$ holomorphic vector functions $a_i$ such that $Im a_i|_{\Gamma_0} = 0$. Then

$$
P_2^*P_1a_i = B^{-1}z_i^T \quad \text{and} \quad C_2^*C_1a_i = \bar{B}^{-1}\Xi_i \quad \text{on } \tilde{\Gamma}.
From this equality we have
\[ \mathcal{P}_2^* \mathcal{P}_1 = B^{-1} \Pi A^{-1} \quad \text{and} \quad C_2^* C_1 = B^{-1} \tilde{\Pi} A^{-1} \quad \text{on } \Gamma. \]

Here \( A, \Pi, \tilde{\Pi} \) are matrix such that the row number \( i \) equal \( a_i, \tilde{z}_i \) and \( \tilde{z}_i \). We set
\[ \Theta = B^{-1} \Pi A^{-1} \quad \text{and} \quad \tilde{\Theta} = B^{-1} \tilde{\Pi} A^{-1}. \]

These formulae defines the functions \( \Theta, \tilde{\Theta} \) correctly except the point where determinants of matrix \( A \) and \( B \) are equal to zero. On the other hand it is obvious that functions \( \Theta, \tilde{\Theta} \) are independent of the choice of matrices \( A, B \). So if we assume that there exist a point of singularity of, say, the function \( \Theta \) by Proposition 2.1 we can make a choice matrices \( A, B \) such that determinants of these matrices do not equal to zero at this point and arrive to the contradiction. The equality (3.51) follows from (3.66) and the fact that \( \text{Im } B|_{\Gamma_0} = \text{Im } A|_{\Gamma_0} = 0 \). Indeed on \( \Gamma_0 \)
\[ \mathcal{P}_2^* \mathcal{P}_1 = B^{-1} \Pi A^{-1} = B^{-1} \tilde{\Pi} A^{-1} = C_2^* C_1. \]

Proof of the proposition is complete. \( \blacksquare \)

Let \( u_1 \) be the complex geometric optics solution given by (3.26) constructed for the operator \( L_1(x, D) \). Since the Dirichlet-to-Neumann maps for the operators \( L_1(x, D) \) and \( L_2(x, D) \) are equal there exists a function \( u_2 \) be a solution to the following boundary value problem:
\[ L_2(x, D)u_2 = 0 \quad \text{in } \Omega, \quad (u_1 - u_2)|_{\partial \Omega} = 0, \quad \partial \nu(u_1 - u_2) = 0 \quad \text{on } \Gamma. \]

Setting \( u = u_1 - u_2 \) we have
\[ L_2(x, D)u + 2A \partial_z u_1 + 2B \partial_{\tau} u_1 + Qu_1 = 0 \quad \text{in } \Omega, \]
where \( A = A_1 - A_2, B = B_1 - B_2 \) and \( Q = Q_1 - Q_2 \) and
\[ u|_{\partial \Omega} = 0, \quad \partial \nu u|_{\Gamma} = 0. \]

Let \( v \) be a function given by (3.47). Taking the scalar product of (3.67) with \( v \) in \( L^2(\Omega) \) and using (3.48) and (3.68), we obtain
\[ 0 = \int_{\Omega} (2A \partial_z u_1 + 2B \partial_{\tau} u_1 + Qu_1, v) dx. \]

Denote
\[ V = V_{0, \tau} e^{-\tau \Phi} + \tilde{V}_{0, \tau} e^{-\tau \Phi} - e^{-\tau \Phi} \tilde{\mathcal{R}}_{\tau, -B_z}(e_1(q_3 + \frac{q_3}{\tau})) - e^{-\tau \Phi} \mathcal{R}_{\tau, -A_z}(e_1(q_3 + \frac{q_3}{\tau})) \]
and
\[ U = U_{0, \tau} e^{\tau \Phi} + \tilde{U}_{0, \tau} e^{\tau \Phi} - e^{\tau \Phi} \tilde{\mathcal{R}}_{\tau, B_1}(e_1(q_1 + \frac{q_1}{\tau})) - e^{\tau \Phi} \mathcal{R}_{\tau, A_1}(e_1(q_2 + \frac{q_2}{\tau})). \]

We have

**Proposition 3.4.** Let \( u_1 \) is given by (3.26) and \( v \) is given by (3.47). Then the following asymptotic holds true
\[ \int_{\Omega} (2A \partial_z u_1 + 2B \partial_{\tau} u_1 + Qu_1, v) dx = \int_{\Omega} (2A \partial_z U + 2B \partial_{\tau} U + QU, V) dx + o\left(\frac{1}{\tau}\right) \quad \text{as } \tau \to +\infty, \]
where functions $U, V$ are determined by (3.71) and (3.70).

Proof of Proposition 3.4 is exactly the same as the proof of Proposition 5.1 from [7].

4. Step 2: Asymptotic

We introduce the following functionals
\[
\tilde{J}_{\tau} u = \frac{\pi}{2|\det \psi''(\bar{x})|^{1/2}} \left( \frac{u(\bar{x})}{\tau} - \frac{\partial^2 u(\bar{x})}{2\Phi''(\bar{x})\tau^2} + \frac{\partial^2 u(\bar{x})}{2\Phi''(\bar{x})}\frac{\partial u(\bar{x})}{\Phi''(\bar{x})\tau^2} + \frac{\partial u(\bar{x})\Phi''(\bar{x})}{2(\Phi''(\bar{x}))^2}\frac{\partial u(\bar{x})}{\Phi''(\bar{x})\tau^2} \right)
\]
and
\[
J_{\tau} u = \int_{\partial \Omega} u(\nu_1 - i\nu_2) e^{\tau(\Phi - \overline{\Phi})} d\sigma - \int_{\partial \Omega} \frac{(\nu_1 - i\nu_2)}{\Phi'} \partial_z \left( \frac{u}{2\tau^2\Phi'} \right) e^{\tau(\Phi - \overline{\Phi})} d\sigma.
\]

Using these notations and the fact that $\Phi$ is the harmonic function we rewrite the classical result of theorem 7.7.5 of [4] as

**Proposition 4.1.** Let $\Phi(z)$ satisfies (3.71), (3.72) and $u \in C^{5+\alpha}(\Omega), \alpha \in (0,1)$ be some function. Then the following asymptotic formula is true:

\[
(4.1) \quad \int_{\Omega} u e^{\tau(\Phi - \overline{\Phi})} dx = \sum_{\tilde{y} \in \mathcal{H}} e^{2\tau \psi(\tilde{y})} \tilde{J}_{\tau, \tilde{y}} u + J_{\tau} u + o\left( \frac{1}{\tau} \right) \quad \text{as } \tau \to +\infty.
\]

Denote
\[
H(x, \partial_z, \partial_{\bar{x}}) = 2A\partial_z + 2B\partial_{\bar{x}} + Q \quad \text{and} \quad J_{\tau} = \int_{\Omega} (H(x, \partial_z, \partial_{\bar{x}}) U, V) dx.
\]
where $U$ and $V$ are given by (3.71) and (3.70) respectively. We have

**Proposition 4.2.** The following asymptotic holds true

\[
\begin{align*}
0 &= \sum_{k=-1}^{1} \tau^{k} J_k + \frac{1}{\tau} ((J_+ + I_+ \Phi + K_+) (\bar{x}) e^{2\tau \psi(\bar{x})}) + (J_- + I_- \Phi + K_-) (\bar{x}) e^{-2\tau \psi(\bar{x})}) \\
&\quad + \int_{\Gamma} ((\nu_1 - i\nu_2) (\mathcal{A}U_0 e^{\tau \Phi}, V_0 e^{-\tau \Phi}) + (\nu_1 + i\nu_2) (\mathcal{B}U_0 e^{\tau \Phi}, V_0 e^{-\tau \Phi})) d\sigma \\
&\quad + o\left( \frac{1}{\tau} \right) \quad \text{as } \tau \to +\infty,
\end{align*}
\]

where
\[
J_1 = \int_{\partial \Omega} ((\nu_1 - i\nu_2) \Phi' (\tilde{U}_0, V_0) + (\nu_1 + i\nu_2) \Phi' (U_0, \tilde{V}_0)) d\sigma,
\]

\[
J_+ (\bar{x}) = \frac{\pi}{2|\det \psi''(\bar{x})|^{1/2}} ((-2\partial_z \mathcal{A}U_0, V_0) - (\mathcal{A}U_0, A_2^0 V_0) - (\mathcal{B}A_1 U_0, V_0) + (\mathcal{Q}U_0, V_0))(\bar{x}),
\]

\[
J_- (\bar{x}) = \frac{\pi}{2|\det \psi''(\bar{x})|^{1/2}} ((-\mathcal{A}B_1 \tilde{U}_0, \tilde{V}_0) - (2\partial \mathcal{B}U_0, V_0) - (\mathcal{B}U_0, B_2^0 \tilde{V}_0) + (\mathcal{Q}U_0, \tilde{V}_0))(\bar{x}),
\]
\[ I_{\pm, \Phi}(\bar{x}) = -\int_{\Omega} \left\{ (\nu_1 - i\nu_2)((2b_{\pm, \bar{x}}\Phi', V_0) + (2\Phi U_0, \bar{a}_{\pm, \bar{x}})) \right. \\
\left. + (\nu_1 + i\nu_2)((2a_{\pm, \bar{x}}\Phi', \bar{V}_0) + (2\Phi' U_0, \bar{b}_{\pm, \bar{x}})) \right\} d\sigma, \]

(4.6)

\[ K_+ = \tau \bar{\mathcal{S}}_{\tau, \bar{x}}(q_1, T_{B_1}^*(B_1^* A^* V_0) - A^* V_0) + 2T_{B_1}^*(\partial_2 B^* V_0) + T_{B_1}^*(B^*(A_2^* V_0 - 2\tau \Phi' V_0))) \]

(4.7)

\[ K_- = \tau \bar{\mathcal{S}}_{-\tau, \bar{x}}(q_3, P_{A_1}^*(2\partial_2 (A^* \bar{V}_0) - \tau \Phi' 2A^* \bar{V}_0) - B^* \bar{V}_0 + P_{A_1}^*(A_1^* B^* \bar{V}_0)) \]

(4.8)

**Proof.** By Proposition 3.3

\[ J_\tau = \int_{\Omega} (H(x, \partial_2, \partial_2) U, V) dx = o\left(\frac{1}{\tau}\right) \quad \text{as } \tau \to +\infty \]

Denote

\[ U_1 = -\bar{\mathcal{R}}_{\tau, B_1}(e_1(q_1 + q_1/\tau)), \quad \bar{U}_1 = -\mathcal{R}_{\tau, A_1}(e_1(q_2 + q_2/\tau)), \]

(4.9)

\[ \bar{V}_1 = -\bar{\mathcal{R}}_{-\tau, -B_2^*}(e_1(q_3 + q_3/\tau)), \quad V_1 = -\mathcal{R}_{-\tau, -A_2^*}(e_1(q_4 + q_4/\tau)). \]

(4.10)

Integrating by parts and using Proposition 4.1 we obtain

\[ \mathcal{M}_1 = \int_{\Omega} (2A \partial_2 U_{0, \tau} e^{\tau \Phi}) + 2B \partial_2 U_{0, \tau} e^{-\tau \Phi}) d\sigma = \]

\[ \int_{\Omega} ((-2\partial_2 AU_{0, \tau} e^{\tau \Phi}, V_{0, \tau} e^{-\tau \Phi}) - (2AU_{0, \tau} e^{\tau \Phi}, \partial_2 V_{0, \tau} e^{-\tau \Phi}) + (2B \partial_2 U_{0, \tau} e^{\tau \Phi}, V_{0, \tau} e^{-\tau \Phi})) d\sigma = \]

\[ + \int_{\Omega} (\nu_1 - i\nu_2)(A U_{0, \tau} e^{\tau \Phi}, V_{0, \tau} e^{-\tau \Phi}) d\sigma = e^{2i\tau\psi(\bar{x})} \bar{\mathcal{S}}_{\tau, \bar{x}}(-2\partial_2 AU_{0, \tau}, V_{0, \tau} - (2AU_{0, \tau}, \partial_2 V_{0, \tau}) + (2B \partial_2 U_{0, \tau}, V_{0, \tau})) \]

\[ + J_\tau(-2\partial_2 AU_{0, \tau}, V_{0, \tau}) - (2AU_{0, \tau}, \partial_2 V_{0, \tau}) + (2B \partial_2 U_{0, \tau}, V_{0, \tau})) \]

(4.11)

where \( \kappa_{0,j} \) are some constants independent of \( \tau \).

Integrating by parts we obtain that there exist constants \( \kappa_{1,j} \) independent of \( \tau \) such that

\[ \int_{\Omega} (2A \partial_2 (U_{0, \tau} e^{\tau \Phi}) + 2B \partial_2 (U_{0, \tau} e^{\tau \Phi}), V_{0, \tau} e^{-\tau \Phi}) d\sigma = \]

\[ (2A \partial_2 U_{0, \tau}, V_{0, \tau}) L^2(\Omega) + (2B \partial_2 U_{0, \tau} + \tau \Phi' U_{0, \tau}), V_{0, \tau}) L^2(\Omega) = \]

\[ \tau_{\kappa_{1,1}} + \kappa_{1,0} + \frac{\kappa_{1,1}}{\tau} + \frac{1}{\tau} e^{2i\tau\psi(\bar{x})}(2BB_{\tau, \bar{x}} \Phi', V_{0, \tau}) L^2(\Omega) + e^{-2i\tau\psi(\bar{x})}(2BB_{\tau, \bar{x}} \Phi', V_{0, \tau}) L^2(\Omega) \]

\[ + \frac{1}{\tau} e^{2i\tau\psi(\bar{x})}(2B \Phi' U_{0, \bar{a}_{+}}, V_{0, \tau}) L^2(\Omega) + e^{-2i\tau\psi(\bar{x})}(2B \Phi' U_{0, \bar{a}_{-}}, V_{0, \tau}) L^2(\Omega) + o\left(\frac{1}{\tau}\right). \]

(4.12)

Since by (3.5), (3.21), (3.28), (3.41) for the functions \( \bar{a}_{\pm, \bar{x}}, b_{\pm, \bar{x}} \) we have

\( (2B \Phi' U_{0, \bar{a}_{\pm}}, b_{\pm, \bar{x}}) = -4\partial_2 (\Phi' U_{0, \bar{a}_{\pm}}) \), and \( (2B b_{\pm, \bar{x}} \Phi', V_{0}) = -4\partial_2 (b_{\pm, \bar{x}} \Phi', V_{0}) \) in \( \Omega \)
from (4.12) we have

\[
\mathcal{M}_2 = \int_{\Omega} (2A\partial_2(\tilde{U}_0,\tau e^{\tau\Phi}) + 2B\partial_2(\tilde{U}_0,\tau e^{\tau\Phi}), V_0 e^{-\tau\Phi}) dx = \\
\tau \kappa_{1,1} + \kappa_{1,0} + \frac{\kappa_{1,-1}}{\tau} + \int_{\partial\Omega} \frac{(\nu_1 - i\nu_2)}{\tau}(e^{2i\tau\psi(\bar{\xi})}(2Bb_{+}\bar{\Phi}', V_0) + e^{-2i\tau\psi(\bar{\xi})}(2Bb_{-}\bar{\Phi}', V_0)) d\sigma \\
+ \int (\nu_1 - i\nu_2)(e^{2i\tau\psi(\bar{\xi})}(2\bar{\Phi}'\tilde{U}_0, a_{+}, \bar{\xi}) + e^{-2i\tau\psi(\bar{\xi})}(2\bar{\Phi}'\tilde{U}_0, a_{-}, \bar{\xi})) d\sigma = o(\frac{1}{\tau}).
\]

Integrating by parts we obtain that there exist constants \(\kappa_{2,j}\) independent of \(\tau\) such that

\[
\int_{\Omega} (2A\partial_2(U_{0,\tau} e^{\tau\Phi}), \tilde{V}_0, e^{-\tau\Phi}) dx = \tau \kappa_{2,1} + \kappa_{2,0} + \frac{\kappa_{2,-1}}{\tau} + \int_{\partial\Omega} \frac{(\nu_1 + i\nu_2)}{\tau}(e^{2i\tau\psi(\bar{\xi})}(2Aa_{+}\bar{\Phi}', \tilde{V}_0) + e^{-2i\tau\psi(\bar{\xi})}(2Aa_{-}\bar{\Phi}', \tilde{V}_0)) d\sigma \\
+ \int (\nu_1 + i\nu_2)(e^{2i\tau\psi(\bar{\xi})}(2\bar{\Phi}'\tilde{U}_0, \tilde{b}^{+}, \bar{\xi}) + e^{-2i\tau\psi(\bar{\xi})}(2\bar{\Phi}'\tilde{U}_0, \tilde{b}^{-}, \bar{\xi})) d\sigma = o(\frac{1}{\tau}).
\]

Since by (3.5), (3.21), (3.28), (3.41) for the functions \(a_{+}, \tilde{b}_{+}, \tilde{a}_{+}\) we have

\[
(2Aa_{+}\bar{\Phi}', \tilde{V}_0) = -4\partial\xi(a_{+}\bar{\Phi}', \tilde{V}_0) \quad \text{and} \quad (2A\bar{\Phi}'\tilde{U}_0, \tilde{b}^{+}) = -4\partial\xi(\bar{\Phi}'\tilde{U}_0, \tilde{b}^{+}) \quad \text{in} \quad \Omega
\]

we obtain from (4.14)

\[
\mathcal{M}_3 = \int_{\Omega} (2A\partial_2(U_{0,\tau} e^{\tau\Phi}) + 2B\partial_2(U_{0,\tau} e^{\tau\Phi}), \tilde{V}_0, e^{-\tau\Phi}) dx = \\
\tau \kappa_{2,1} + \kappa_{2,0} + \frac{\kappa_{2,-1}}{\tau} + \int_{\partial\Omega} \frac{(\nu_1 + i\nu_2)}{\tau}(e^{2i\tau\psi(\bar{\xi})}(2a_{+}\bar{\Phi}', \tilde{V}_0) + e^{-2i\tau\psi(\bar{\xi})}(2a_{-}\bar{\Phi}', \tilde{V}_0)) d\sigma \\
+ \int (\nu_1 + i\nu_2)(e^{2i\tau\psi(\bar{\xi})}(2\bar{\Phi}'\tilde{U}_0, \tilde{b}^{+}, \bar{\xi}) + e^{-2i\tau\psi(\bar{\xi})}(2\bar{\Phi}'\tilde{U}_0, \tilde{b}^{-}, \bar{\xi})) d\sigma = o(\frac{1}{\tau}).
\]

Integrating by parts, using (3.5) and Proposition 4.1 we obtain that there exists some constants \(\kappa_{3,j}\) independent of \(\tau\) such that

\[
\mathcal{M}_4 = \int_{\Omega} (2A\partial_2(\tilde{U}_0,\tau e^{\tau\Phi}) + 2B\partial_2(\tilde{U}_0,\tau e^{\tau\Phi}), \tilde{V}_0, e^{-\tau\Phi}) dx = \\
\int_{\Omega} ((2A\partial_2\tilde{U}_0,\tau e^{\tau\Phi}, \tilde{V}_0, e^{-\tau\Phi}) - (2\partial_2\bar{B}\tilde{U}_0,\tau e^{\tau\Phi}, \tilde{V}_0, e^{-\tau\Phi}) - (2B\bar{U}_0,\tau e^{\tau\Phi}, \partial_2\tilde{V}_0, e^{-\tau\Phi})) dx \\
+ \int_{\partial\Omega} (\nu_1 + i\nu_2)(e^{2i\tau\psi(\bar{\xi})}\tilde{\Phi}'\tilde{U}_0, \tilde{V}_0, e^{-\tau\Phi}) d\sigma = \\
e^{-2i\tau\psi(\bar{\xi})}\tilde{\Phi}'\tilde{U}_0(\tilde{V}_0, e^{-\tau\Phi}) - (2A\partial_2\tilde{U}_0, \tilde{V}_0, e^{-\tau\Phi}) - (2B\bar{U}_0, \partial_2\tilde{V}_0, e^{-\tau\Phi}) \\
+ \kappa_{3,1} + \frac{\kappa_{3,-1}}{\tau} + o(\frac{1}{\tau}).
\]

Integrating by parts and using Proposition 4.1 we obtain
Using (4.9), (3.19), (3.20) and Proposition 8 of \( M(4.18) \)
Integrating by parts and using Proposition 4.1 we have
\[
\int_{\partial \Omega} (\nu_1 + i\nu_2)(BU_1, V_{0,\tau}) e^{\tau(\Phi - \bar{\Phi})} d\sigma = (2BU_1, \partial_{\bar{z}}(V_{0,\tau} e^{\tau(\Phi - \bar{\Phi})}))_{L^2(\Omega)} + o\left(\frac{1}{\tau}\right) = \\
(\mathcal{B}T_{B_1}(e^{\tau(\Phi - \bar{\Phi})}e_1 q_1), \partial_{\bar{z}}V_{0,\tau} - 2\tau \bar{\Phi}'V_{0,\tau})_{L^2(\Omega)} + \int_{\partial \Omega} (\nu_1 + i\nu_2)(BU_1, V_{0,\tau}) e^{\tau(\Phi - \bar{\Phi})} d\sigma + o\left(\frac{1}{\tau}\right) = \\
e^{2i\tau\psi(x)} \mathfrak{F}_{\tau,\bar{x}}(q_1, T^{*}_{B_1}(B_1^* A^* V_0) - A^* V_0 + 2T^*_{B_1}(\partial_{\bar{z}} B^* V_0) + T^*_{B_1}(B^*(A_2^* V_0 - 2\bar{\Phi}'V_0))) \\
+ \int_{\partial \Omega} (\nu_1 + i\nu_2)(BU_1, V_{0,\tau}) e^{\tau(\Phi - \bar{\Phi})} d\sigma + o\left(\frac{1}{\tau}\right) \quad \text{as} \quad \tau \to +\infty.
\]

After integration by parts we have
\[
\mathcal{M}_6 = \int_{\Omega} (2A\partial_{\bar{z}}(U_1 e^{\tau\Phi}) + 2B\partial_{\bar{z}}(U_1 e^{\tau\Phi}), V_{0,\tau} e^{-\tau\bar{\Phi}}) dx = \\
\int_{\Omega} (A(-B_1 U_1 - e_1 q_1) - e_1 q_1) e^{\tau\Phi} - 2\partial_{\bar{z}} B_1(U_1 e^{\tau\Phi}), V_{0,\tau} e^{-\tau\bar{\Phi}} dx + \\
(2BU_1, \partial_{\bar{z}}V_{0,\tau})_{L^2(\Omega)} + \int_{\partial \Omega} (\nu_1 + i\nu_2)(BU_1, V_{0,\tau}) e^{\tau(\Phi - \bar{\Phi})} d\sigma + o\left(\frac{1}{\tau}\right) + \\
(2BU_1, \partial_{\bar{z}}V_{0,\tau})_{L^2(\Omega)} + \int_{\partial \Omega} (\nu_1 + i\nu_2)(BU_1, V_{0,\tau}) d\sigma.
\]

Using (4.9), (3.19), (3.20) and Proposition 8 of [?] we obtain that
\[
\mathcal{M}_6 = -\int_{\Omega} (AQ_1, V_{0,\tau}) dx + o\left(\frac{1}{\tau}\right) \quad \text{as} \quad \tau \to +\infty.
\]

Integrating by parts and using Proposition [4.11] we have
\[
\mathcal{M}_7 = \int_{\Omega} (2A\partial_{\bar{z}}(U_{0,\tau} e^{\tau\Phi}) + 2B\partial_{\bar{z}}(U_{0,\tau} e^{\tau\Phi}), V_{1,\tau} e^{-\tau\bar{\Phi}}) dx = \\
2 \int_{\Omega} (A(\partial_{\bar{z}} U_{0,\tau} + \tau \bar{\Phi}' U_{0,\tau}) e^{\tau\Phi} + B\partial_{\bar{z}} U_{0,\tau} e^{\tau\Phi}, V_{1,\tau} e^{-\tau\bar{\Phi}}) dx = \\
-2 \int_{\Omega} (P_{-A_2^*}(A(\partial_{\bar{z}} U_0 + \tau \bar{\Phi}' U_0) + B\partial_{\bar{z}} U_0), e_1 q_1 e^{\tau(\Phi - \bar{\Phi})} dx + o\left(\frac{1}{\tau}\right) = \\
-2 e^{2i\tau\psi(x)} \mathfrak{F}_{\tau,\bar{x}}(P_{-A_2^*}(A(\partial_{\bar{z}} U_0 + \tau \bar{\Phi}' U_0) + B\partial_{\bar{z}} U_0), q_1) + o\left(\frac{1}{\tau}\right)
\]
\[
+ o\left(\frac{1}{\tau}\right) \quad \text{as} \quad \tau \to +\infty.
\]
Integrating by parts and using Proposition 8 of [?] we have

\[ \mathcal{M}_8 = \int_{\Omega} (2A\partial_z(U_0,\tau e^{\mp\Phi}) + 2B\partial_z(U_0,\tau e^{\mp\Phi}), \tilde{V}_1 e^{-\tau\Phi}) dx = \]

\[ \int_{\Omega} \left( (-2\partial_z A U_0 + B\partial_z U_0, \tilde{V}_1) - (A U_{0,\tau}, -B_2^* \tilde{V}_1 - e_1 q_3) \right) dx + o\left( \frac{1}{\tau} \right) \]

(4.20) \[ + \int_{\partial\Omega} (\nu_1 - i\nu_2) (A U_0, \tilde{V}_1) d\sigma = -\int_{\Omega} (A U_{0,\tau}, q_3) dx + o\left( \frac{1}{\tau} \right) \quad \text{as} \quad \tau \to +\infty \]

and

\[ \mathcal{M}_9 = \int_{\Omega} (2A\partial_z(\tilde{U}_1 e^{\mp\Phi}) + 2B\partial_z(\tilde{U}_1 e^{\mp\Phi}), V_{0,\tau} e^{-\tau\Phi}) dx = \]

\[ \int_{\Omega} \left( (-\partial_z(2A^* V_{0,\tau}) + (B(-A_1 \tilde{U}_1 - e_1 q_2), V_{0,\tau}) \right) dx + o\left( \frac{1}{\tau} \right) \]

(4.21) \[ + \int_{\partial\Omega} (\nu_1 - i\nu_2) (A \tilde{U}_1, V_0) d\sigma = -\int_{\Omega} (B q_2, V_{0,\tau}) dx + o\left( \frac{1}{\tau} \right) \quad \text{as} \quad \tau \to +\infty. \]

Integrating by parts and using Proposition 4.1 we obtain

(4.22)

\[ \mathcal{M}_{10} = \int_{\Omega} (2A\partial_z(\tilde{U}_1 e^{\mp\Phi}) + 2B\partial_z(\tilde{U}_1 e^{\mp\Phi}), \tilde{V}_{0,\tau} e^{-\tau\Phi}) dx = \]

\[ \int_{\Omega} \left( (\tilde{U}_1, -\partial_z(2A^* \tilde{V}_{0,\tau}) + \tau \Phi' 2A^* \tilde{V}_{0,\tau}) + (B(-A_1 \tilde{U}_1 - e_1 q_2), \tilde{V}_{0,\tau}) \right) e^{\tau(\Phi - \Phi')} dx + \]

\[ + \int_{\partial\Omega} (\nu_1 - i\nu_2) (A \tilde{U}_1, \tilde{V}_0) e^{\tau(\Phi - \Phi')} d\sigma + o\left( \frac{1}{\tau} \right) = \]

\[ \int_{\Omega} (e_1 q_2, P_{A_1}(2\partial_z(\mathcal{A}^* \tilde{V}_{0,\tau}) - 2\tau \Phi' \mathcal{A}^* \tilde{V}_{0,\tau}) - \mathcal{B}^* \tilde{V}_0 + P_{A_1}(A_1^* \mathcal{B}^* \tilde{V}_0)) e^{\tau(\Phi - \Phi')} dx \]

\[ + \int_{\partial\Omega} (\nu_1 - i\nu_2) (A \tilde{U}_1, \tilde{V}_0) e^{\tau(\Phi - \Phi')} d\sigma + o\left( \frac{1}{\tau} \right) = \]

\[ e^{-2i\tau\psi(\bar{x})} + o\left( \frac{1}{\tau} \right) \quad \text{as} \quad \tau \to +\infty. \]

By (3.15) and Proposition 4.1 we obtain

(4.23)

\[ \mathcal{M}_{11} = \int_{\Omega} (2A\partial_z(\tilde{U}_0,\tau e^{\mp\Phi}) + 2B\partial_z(\tilde{U}_0,\tau e^{\mp\Phi}), \tilde{V}_1 e^{-\tau\Phi}) dx = \]

\[ \int_{\Omega} (2A\partial_z \tilde{U}_{0,\tau} + 2B(\partial_z \tilde{U}_{0,\tau} + \tau \Phi' \tilde{U}_{0,\tau}), \tilde{V}_1) e^{\tau(\Phi - \Phi')} dx = \]

\[ - \int_{\Omega} (e_1 q_3, T_{-B_2}^*(2A\partial_z \tilde{U}_{0,\tau} + 2B(\partial_z \tilde{U}_{0,\tau} + \tau \Phi' \tilde{U}_{0,\tau}) \right) e^{\tau(\Phi - \Phi')} dx + o\left( \frac{1}{\tau} \right) = \]

\[ -e^{-2i\tau\psi(\bar{x})} s_{-\tau,\bar{x}}(q_3, T_{-B_2}^*(2A\partial_z \tilde{U}_0 + 2B(\partial_z \tilde{U}_0 + \tau \Phi' \tilde{U}_0))) \]

\[ + o\left( \frac{1}{\tau} \right) \quad \text{as} \quad \tau \to +\infty. \]

By Proposition 4.1 there exist constants \( \kappa_{4,j} \) independent of \( \tau \) such that
exists a holomorphic matrix $\Theta$

\begin{equation}
(4.26)
\end{equation}

\begin{equation}
(4.28)
\end{equation}

\begin{equation}
(4.27)
\end{equation}

Proposition 4.3. Let all conditions of the proposition 4.1 holds true and

\begin{equation}
\sum_{k=1}^{12} M_k
\end{equation}

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Proof. From Proposition 4.2 for any function $\Phi$ which satisfies (3.1), (3.2) we have

\begin{equation}
(4.28)
\end{equation}

Then if $a(z) = (a_1(z), \ldots, a_N(z)), b(z) = (b_1(z), \ldots, b_N(z))$ are the holomorphic functions such that $\text{Im} \ a|_{\Gamma_0} = \text{Im} \ b|_{\Gamma_0} = 0$ the pairs $(P_1 a, C_1 \bar{\pi})$ and $(P_2 b, C_2 \bar{\pi})$ solve the problems (3.5) and (3.28) respectively. Therefore we can rewrite (4.28) as

\begin{equation}
(4.29)
\end{equation}

Thanks to (4.29) all assumptions of the Proposition 3.3 holds true. By Proposition 3.3 there exist holomorphic matrix $\Theta(z)$ and antiholomorphic matrix $\bar{\Theta}(z)$ with domain $\bar{\Omega}$ such that

\begin{equation}
(4.30)
\end{equation}

and

\begin{equation}
(4.31)
\end{equation}

From (3.10) and (3.35) and the classical regularity theory of systems of elliptic equations (see e.g [3]) we obtain that $\Theta, \bar{\Theta} \in C^{6+\alpha} (\bar{\Omega})$. Without loss of generality we may assume that

\begin{equation}
(4.32)
\end{equation}

Moreover by (3.10), (3.31)

\begin{equation}
(4.33)
\end{equation}

Observe that by (4.30)

\begin{equation}
(4.34)
\end{equation}

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\begin{equation}
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\begin{equation}
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\begin{equation}
(4.32)
\end{equation}

Moreover by (3.10), (3.31)

\begin{equation}
(4.33)
\end{equation}

Observe that by (4.30)
Since by the construction of the matrices $\mathcal{P}_j$
\[ 2\partial_z \mathcal{P}_1 + A_1 \mathcal{P}_1 = 0 \quad \text{in } \Omega \quad \text{and} \quad 2\partial_z \mathcal{P}_2^* - \mathcal{P}_2^* A_2 = 0 \quad \text{in } \Omega \]
and matrix $\Theta$ is holomorphic we have
\[ 2\partial_z (\mathcal{P}_1 \Theta^{-1}) + A_1 (\mathcal{P}_1 \Theta^{-1}) = 0 \quad \text{in } \Omega \setminus \mathcal{X}. \]
We compute
\[ (4.34) \quad 2\partial_z (\mathcal{P}_1 \Theta^{-1} \mathcal{P}_2^*) + A_1 (\mathcal{P}_1 \Theta^{-1} \mathcal{P}_2^*) - (\mathcal{P}_1 \Theta^{-1} \mathcal{P}_2^*) A_2 = 0 \quad \text{in } \Omega \setminus \mathcal{X}. \]
Thus the first equation in (4.26) holds true. By (4.33) the second equation in (4.26).

By (4.25), (4.33) on $\tilde{\Gamma}$ we have
\[ (4.35) \quad -2\partial_z Q = A_1 \mathcal{P}_1 \Theta^{-1} \mathcal{P}_2^* - \mathcal{P}_1 \Theta^{-1} \mathcal{P}_2^* A_2 = A_1 I - I A_2 = A_1 - A_2 = 0. \]

In order to prove the third equation in (4.26) we observe that there exists a matrix $T(x)$ with real-valued entries, det $T(x) \neq 0$, such that $\nabla = T(x)(\partial_\nu, \partial_\tau)$. Therefore $\partial_z = \frac{1}{2}((T_{11} + iT_{21})\partial_\nu + (T_{12} + iT_{22})\partial_\tau)$. By (4.35) on $\tilde{\Gamma}$ the following equation holds
\[ \partial_z Q = \frac{1}{2}((T_{11} + iT_{21})\partial_\nu Q + (T_{12} + iT_{22})\partial_\tau Q) = \frac{1}{2}((T_{11} + iT_{21})\partial_\nu Q + (T_{12} + iT_{22})\partial_\tau I) = \frac{1}{2}(T_{11} + iT_{21})\partial_\nu Q = 0. \]

The fact that determinant of the matrix $T$ is not equal zero implies that $(T_{11} + iT_{21}) \neq 0$. So from the above equation we have $\partial_z Q = 0$.

If det $Q(x_0) = 0$ then det $\mathcal{P}_1(x_0)\det \mathcal{P}_2(x_0) = 0$. Let matrices $\tilde{\mathcal{P}}_j$ be constructed as $\mathcal{P}_j$ but with the different choice of the pairs $(U_0(k), U_0(k))$, $(V_0(k), \tilde{V}_0(k))$ which are solutions to problem (3.5) and problem (3.28) respectively and satisfy (3.10), (3.35). In such a way we obtain another matrices $\mathcal{P}_j, \Theta, Q$ which satisfies to (4.26) with possibly different set $\mathcal{X}$. We denote such a matrix $\mathcal{P}_j, \Theta, Q$ as $\tilde{\mathcal{P}}_j, \tilde{\Theta}, \tilde{Q}$. By uniqueness of the Cauchy problem for the $\partial_z$ operator
\[ Q = \tilde{Q} \quad \text{on } \Omega \setminus \mathcal{X} \cup \tilde{\mathcal{X}} \quad \text{where} \quad \tilde{\mathcal{X}} = \{ x \in \bar{\Omega} | \det \tilde{\Theta} = 0 \}. \]

So, $\tilde{Q}(x_0) = 0$. On the other hand one can choose the matrices $\tilde{\mathcal{P}}_j$ such that det $\tilde{\mathcal{P}}_j(x_0) \neq 0$. Therefore we arrived to the contradiction. Proof of the proposition is complete. $\blacksquare$

Our next goal is to show that the matrix $Q$ is regular on $\bar{\Omega}$.

Now we prove that if operators $L_j(x, D)$ generate the same Dirichlet-to-Neumann map then the operators $L_j(x, D)^*$ generate the same Dirichlet-to-Neumann map.

**Proposition 4.4.** Let $A_j, B_j, Q_j \subset C^{5+\alpha}(\bar{\Omega})$ for $j = 1, 2$. If $\Lambda_{A_1, B_1, Q_1} = \Lambda_{A_2, B_2, Q_2}$ then $\Lambda_{A_1, B_1, R_1} = \Lambda_{A_2, B_2, R_2}$, where $R_j = -\partial_z A_j^* - \partial_\nu B_j^* + Q_j^*$ for $j \in \{1, 2\}$.

**Proof.** Let function $v_j$ solves the boundary value problem
\[ L_j(x, D)^* v_j = 0 \quad \text{in } \Omega, \quad v_j|_{\Gamma_0} = 0, \quad v_j|_{\bar{\Gamma}} = g \]
and $\tilde{u}_j$ be solution to the problem
\[ L_j(x, D) \tilde{u}_j = 0 \quad \text{in } \Omega, \quad \tilde{u}_j|_{\Gamma_0} = 0, \quad \tilde{u}_j|_{\bar{\Gamma}} = f. \]
By our assumption and Fredholm’s theorem solution for both problems exists for any \( f, g \in C^\infty_0(\tilde{\Gamma}) \). By the Green’s formula
\[
(L_j(x,D)\nu_j, \tilde{u}_j)_{L^2(\Omega)} - (\nu_j, L_j(x,D)\tilde{u}_j)_{L^2(\Omega)} = (\partial_\nu \nu_j, \tilde{u}_j)_{L^2(\Gamma)} + (\partial_\nu \tilde{u}_j)_{L^2(\Gamma)} + (A_j(\nu_1 - i\nu_2)g,f)_{L^2(\Gamma)} + (B_j(\nu_1 + i\nu_2)g,f)_{L^2(\Gamma)}.
\]
Subtracting the above formulae for different \( j \), using (4.25) and taking into account that \( \Lambda_{A_1,B_1,Q_1} = \Lambda_{A_2,B_2,Q_2} \) we have
\[
(\partial_\nu \nu_1 - \partial_\nu \nu_2, f)_{L^2(\Gamma)} = 0.
\]
Since the function \( f \) can be chosen an arbitrary from \( C^\infty_0(\tilde{\Gamma}) \) the proof of the proposition is complete. \[ \square \]

By Proposition 2.3 there exists a holomorphic matrix \( \tilde{\Gamma} \) such that in \( \Omega \)
\[
(2\partial_x U_0(k) - A_1^* U_0(k), 2\partial_x \tilde{U}_0(k) - B_1^* \tilde{U}_0(k)) = 0 \quad \text{in} \quad \Omega, \quad U_0(k) + \tilde{U}_0(k) = 0 \quad \text{on} \quad \Gamma_0
\]
and solutions \( (V_0(k), \tilde{V}_0(k)) \)
\[
(2\partial_x V_0(k) + A_2 V_0(k), 2\partial_x \tilde{V}_0(k) + B_2 \tilde{V}_0(k)) = 0 \quad \text{in} \quad \Omega, \quad V_0(k) + \tilde{V}_0(k) = 0 \quad \text{on} \quad \Gamma_0
\]
for \( k \in \{1, \ldots, N\} \) such that
\[
\|U_0(k) - \tilde{\epsilon}_k\|_{C^{\nu_0}(\Gamma_0)} + \|\tilde{V}_0(k) - \tilde{\epsilon}_k\|_{C^{\nu_0}(\Gamma_0)} \leq \epsilon \quad \text{for} \quad k \in \{1, \ldots, N\}. \tag{4.38}
\]
This inequality and the boundary conditions in (4.36) and in (4.37) imply
\[
\|\tilde{U}_0(k) - \tilde{\epsilon}_k\|_{C^{\nu_0}(\Gamma_0)} + \|V_0(k) - \tilde{\epsilon}_k\|_{C^{\nu_0}(\Gamma_0)} \leq \epsilon \quad \text{for} \quad k \in \{1, \ldots, N\}. \tag{4.39}
\]
We define matrices \( \mathcal{M}_1, \mathcal{M}_2, \mathcal{R}_1, \mathcal{R}_2 \) as
\[
\mathcal{M}_1 = (\tilde{U}_0(1), \ldots, \tilde{U}_0(N)), \quad \mathcal{R}_1 = (U_0(1), \ldots, U_0(N)), \quad \mathcal{M}_2 = (V_0(1), \ldots, V_0(N)), \quad \mathcal{R}_2 = (\tilde{V}_0(1), \ldots, \tilde{V}_0(N)). \tag{4.40}
\]

By Proposition 2.3 there exists a holomorphic matrix \( \mathcal{Y} \) such that the matrix function \( G = \mathcal{M}_1 \mathcal{Y}^{-1} \mathcal{M}_2^* \) solves the partial differential equation
\[
2\partial_x G + GA_2^* - A_1^* G = 0 \quad \text{in} \quad \Omega \setminus \{x \in \tilde{\Omega}|\text{det} \mathcal{Y} = 0\}, \quad G|_{\tilde{\Gamma}} = I, \quad \partial_\nu G|_{\tilde{\Gamma}} = 0. \tag{4.41}
\]
Observe that the matrix \( Q^{*-1} \) solves the following partial differential equation
\[
2\partial_x Q^{*-1} + Q^{*-1} A_2^* - A_1^* Q^{*-1} = 0 \quad \text{in} \quad \Omega \setminus \{x \in \tilde{\Omega}|\text{det} \mathcal{P}_1(x)\text{det} \mathcal{P}_2(x) = 0\}, \tag{4.42}
\]
\[
Q^{*-1}|_{\tilde{\Gamma}} = I, \quad \partial_\nu Q^{*-1}|_{\tilde{\Gamma}} = 0. \tag{4.43}
\]
Let matrices \( \tilde{\mathcal{P}}_j \) be constructed as \( \mathcal{P}_j \) but with the different choice of the pairs \( (U_0(k), \tilde{U}_0(k)), (V_0(k), \tilde{V}_0(k)) \) which are solutions to problem (3.5) and problem (3.28) respectively and satisfy (3.10), (3.35). In such a way we obtain another matrix \( \tilde{Q} \) which satisfies to (1.26) with possibly different set \( \mathcal{X} \). We denote such a matrix \( Q \) as \( \hat{Q} \). By uniqueness of the Cauchy problem for the \( \partial_x \) operator
\[
Q = \hat{Q} \quad \text{on} \quad \Omega \setminus \{x \in \tilde{\Omega}|\text{det} (\mathcal{P}_1 \mathcal{P}_2 \mathcal{P}_1 \mathcal{P}_2)(x) = 0\}. \tag{4.44}
\]
Let \( x_* \in \tilde{\Omega} \) be a point such that \( \text{det} (\mathcal{P}_1 \mathcal{P}_2)(x_*) = 0 \). We choose the matrices \( \tilde{\mathcal{P}}_j \) such that the determinants of these matrices are not equal to zero in some neighborhood of the point
Then by (4.44) the matrix \( G^{-1} \) can be defined on \( \Omega \) as the function from \( C^{5+\alpha}(\bar{\Omega}) \). Therefore the matrix \( G \) belongs to the space \( C^{5+\alpha}(\bar{\Omega}) \) and solves the equation (4.16) on \( \Omega \). The operator \( \tilde{L}_1(x,D) = Q^{-1}L_1(x,D)Q \) has the form
\[
\tilde{L}_1(x,D) = \Delta + 2A_2\partial_z + 2B_1\partial_z + \tilde{Q}_1,
\]
where
\[
\tilde{B}_1 = Q^{-1}(B_1Q + 2\partial_z Q), \quad \tilde{Q}_1 = Q^{-1}(Q_1Q + \Delta Q + 2A_1\partial_z Q + 2B_1\partial_z Q).
\]

The Dirichlet-to-Neumann maps of the operators \( L_1(x,D) \) and \( \tilde{L}_1(x,D) \) are the same. Let \( \tilde{u}_1 \) be the complex geometric optics solution for the differential operator \( \tilde{L}_1(x,D) \) constructed in the same way as solution for the operator \( L_1(x,D) \). (In fact we can set \( \tilde{u}_1 = Qu_1 \) where \( u_1 \) be the complex geometric optics solution given by (3.47) constructed for the operator \( L_1(x,D) \).) For elements of the complex geometric solution \( \tilde{u}_1 \) such as \( U_0, \tilde{U}_0, U_\tau, \tilde{U}_\tau \) we use the same notations as in construction of the function \( u_1 \). Since the Dirichlet-to-Neumann maps for the operators \( \tilde{L}_1(x,D) \) and \( L_2(x,D) \) are equal there exists a function \( u_2 \) be a solution to the following boundary value problem:
\[
L_2(x,D)u_2 = 0 \quad \text{in} \; \Omega, \quad (\tilde{u}_1 - u_2)|_{\partial\Omega} = 0, \quad \partial_\nu(\tilde{u}_1 - u_2) = 0 \quad \text{on} \; \Gamma.
\]

Setting \( \tilde{u} = \tilde{u}_1 - u_2, \tilde{B} = \tilde{B}_1 - B_2, \tilde{Q} = \tilde{Q}_1 - Q_2 \) we have
\[
L_2(x,D)\tilde{u} + 2\tilde{B}\partial_{\bar{\tau}}\tilde{u}_1 + \tilde{Q}\tilde{u}_1 = 0 \quad \text{in} \; \Omega
\]
and
\[
\tilde{u}|_{\partial\Omega} = 0, \quad \partial_\nu\tilde{u}|_{\Gamma} = 0.
\]

Let \( v \) be a function given by (3.47). Taking the scalar product of (4.46) with \( v \) in \( L^2(\Omega) \) and using (3.48) and (4.47), we obtain
\[
\int_{\Omega} (2\tilde{B}\partial_{\bar{\tau}}\tilde{u}_1 + \tilde{Q}\tilde{u}_1, v)dx = \int_{\Omega} (2\tilde{B}\partial_{\bar{\tau}}U + \tilde{Q}U, V)dx + o\left(\frac{1}{\tau}\right) = 0,
\]
where the function \( V \) given by (3.70) and
\[
U = U_{0,\tau}e^{\tau\Phi} + \tilde{U}_{0,\tau}e^{\tau\Phi} - e^{\tau\Phi}\tilde{R}_{\tau,B_1}(e_1(q_1 + \bar{q}_1/\tau)) - e^{\tau\Phi}\tilde{R}_{\tau,A_2}(e_1(q_2 + \bar{q}_2/\tau)).
\]

We have
Proposition 4.5. The following equalities are true
\begin{equation}
T_B^* (\Phi^* V_0) = T_B^* (\Phi' \overline{\Phi}^* V_0) = \Phi' T_B^* (\Phi^* \overline{V}_0) = T_B^* (\Phi' \overline{\Phi}^* \overline{V}_0) = 0 \quad \text{on } \tilde{\Gamma}
\end{equation}
and
\begin{equation}
I_{\pm, \Phi}(\bar{x}) = 0.
\end{equation}

Proof. Since the matrix \( \mathcal{P}_1 \) satisfies \( 2\partial_z \mathcal{P}_1 + A_2 \mathcal{P}_1 = 0 \) the matrix \( \mathcal{P}_2^* \mathcal{P}_1 \) is holomorphic in the domain \( \Omega \). Indeed,
\begin{equation}
2\partial_z(\mathcal{P}_2^* \mathcal{P}_1) = 2(\partial_z \mathcal{P}_2^* \mathcal{P}_1 + \mathcal{P}_2^* \partial_z \mathcal{P}_1) = -\mathcal{P}_2^* A_2 \mathcal{P}_1 + \mathcal{P}_2^* A_2 \mathcal{P}_1 = 0.
\end{equation}
This implies
\begin{equation}
\int_{\partial \Omega} (\nu_1 + i\nu_2) \Phi'(\mathcal{P}_1 a, \mathcal{P}_2 b)d\sigma = 0,
\end{equation}
By Proposition 4.2 \( C_2 \mathcal{C}_1 = \tilde{\Theta}(\tilde{\mathcal{C}}) \) on \( \tilde{\Gamma} \) where the function \( \tilde{\Theta} \) is antiholomorphic on \( \Omega \). So
\begin{equation}
\int_{\tilde{\Gamma}} (\nu_1 - i\nu_2) \Phi'(C_2 \mathcal{C}_1, \bar{b})d\sigma = \int_{\tilde{\Gamma}} (\nu_1 - i\nu_2) \Phi'(\tilde{\Theta} a, \bar{b})d\sigma = -\int_{\partial \Omega} (\nu_1 - i\nu_2) \Phi'(\tilde{\Theta} a, \bar{b})d\sigma.
\end{equation}
We write (4.54) as
\begin{equation}
\int_{\partial \Omega} (\nu_1 - i\nu_2) \Phi'((C_2 \mathcal{C}_1 - \tilde{\Theta}) a, \bar{b})d\sigma = 0.
\end{equation}
So, by corollary 7.1 of [6], from (4.55) we obtain
\begin{equation}
C_2 \mathcal{C}_1 = \tilde{\Theta} \quad \text{on } \partial \Omega.
\end{equation}
We observe that for construction of \( U_0 \) instead of the matrix \( \mathcal{C}_1 \) we can use \( \mathcal{C}_2 \). In that case the equality (4.56) has the form:
\begin{equation}
C_2 \tilde{C}_1 = \tilde{\Theta} \quad \text{on } \partial \Omega.
\end{equation}
We define \( T_B^* (\Phi' \overline{\Phi}^* V_0) \) on \( \mathbb{R}^2 \setminus \bar{\Omega} \) by formula (2.8). Now let \( y = (y_1, y_2) \in \tilde{\Gamma} \) be an arbitrary point and \( z = y_1 + iy_2 \). Then, thanks to (4.25), for any sequence \( \{y_j\} \in \mathbb{R}^2 \setminus \bar{\Omega} \) such that \( y_j \to y \) we have
\begin{equation}
T_B^* (\Phi' \overline{\Phi}^* V_0)(y_j) \to T_B^* (\Phi' \overline{\Phi}^* V_0)(y) \quad \text{as } j \to +\infty.
\end{equation}
Denote \( z_j = y_{j,1} + iy_{j,2} \). Indeed, by (2.8) and (4.25) the exist a constant \( C \) such that
\begin{equation}
|T_B^* (\Phi' \overline{\Phi}^* V_0)(y_j) - T_B^* (\Phi' \overline{\Phi}^* V_0)(y)| \leq C \int_{\Omega} \|\overline{\Phi}^*(\xi)\| \left| \frac{1}{z_j - \zeta} - \frac{1}{z - \zeta} \right| d\xi.
\end{equation}
Since by (4.25) $\|\tilde{B}^*(\xi)\|_1 = 0$ the sequence \(\left\{\|\tilde{B}^*(\xi)\|_1 \frac{1}{z_j - \zeta} - \frac{1}{z_k - \zeta} \right\}\) is bounded in $L^\infty(\Omega)$. Moreover for any positive $\delta$ the above sequence converges to zero in $L^\infty(\Omega \setminus B(y, \delta))$. Thus, from these facts and (4.59) we have (4.58) immediately.

By (4.56) and (4.57) we have

$$
(4.60)
T_{B_1}^r (\Phi' \tilde{B}^*V_0)(y_j) = \frac{1}{2} (C_1^{-1}r_{0,1})(y_j) \partial_z^{-1}(C_1^* \Phi' \tilde{B}^*V_0)(y_j)
$$

$$
+ \frac{1}{2} \tilde{c}_1^{-1} (1 - r_{0,1})(y_j) \partial_z^{-1}(\tilde{c}_1^* \Phi' \tilde{B}^*V_0)(y_j) = \frac{1}{\pi} r_{0,1}(z_j)(C_1^{-1})^* \int_{\partial \Omega} \frac{\partial_z(\tilde{c}_1^* \tilde{B}^*C_2 \tilde{B})}{z_j - \zeta} d\xi
$$

$$
- (1 - r_{0,1}(z_j))(\tilde{c}_1^{-1})^* \int_{\partial \Omega} \frac{(\nu_1 - i\nu_2) \tilde{B}^* \bar{B}}{z_j - \zeta} d\sigma = 0.
$$

Here, in order to obtain the last equality we used the fact that $z_j \notin \Omega$ and therefore the functions $\frac{1}{z_j - \zeta}$ are antiholomorphic on $\Omega$. From (4.58) and (4.60) $T_{B_1}^r (\Phi' \tilde{B}^*V_0)|_1 = 0$. The proof of remaining equalities in (4.50) is the same. Next we show that $I_{\pm, \Phi}(\bar{x}) = 0$. By (3.22), (3.42) we have

$$
I_{\pm, \Phi}(\bar{x}) = \int_{\partial \Omega} \left\{(\nu_1 - i\nu_2)((2C_2^* C_1 b_{\pm, \bar{x}} \Phi', \bar{b}) + (2\Phi' C_2^* C_1 \bar{a}, \bar{a}_{\pm, \bar{x}})) + (\nu_1 + i\nu_2)((2P_2^* P_1 a_{\pm, \bar{x}} \Phi', \bar{b}) + (2\Phi' P_2^* P_1 a, \bar{a}_{\pm, \bar{x}})) \right\} d\sigma.
$$

Since by (4.56) the restriction of the function $C_2^* C_1$ on $\partial \Omega$ coincides with the restriction of some antiholomorphic in $\bar{\Omega}$ function and by (4.52) the restriction of the function $P_2^* P_1$ on $\partial \Omega$ coincides with the restriction of some holomorphic in $\bar{\Omega}$ the equality (4.61) implies (4.51).

The proof of the proposition is complete. ■

We use the above proposition to prove the following:

**Proposition 4.6.** The following is true:

$$
(4.62)
\tilde{\Phi}' T_{B_1}^r (\tilde{B}^*V_0) = T_{B_1}^r (\tilde{\Phi}' \tilde{B}^*V_0),
$$

$$
(4.63)
\tilde{\Phi}' T_{-B_2}^r (\tilde{B}U_0) = T_{-B_2}^r (\tilde{\Phi}' \tilde{B}U_0).
$$

**Proof.** Denote $r = \tilde{\Phi}' T_{B_1}^r (\tilde{B}^*V_0) - T_{B_1}^r (\tilde{\Phi}' \tilde{B}^*V_0)$. Then this function satisfies

$$
2\partial_z r - \tilde{B}_1^* r = 0 \text{ in } \Omega.
$$

By Proposition 4.3

$$
r|_1 = 0.
$$
Proposition 4.7. Under conditions of Proposition 4.2 we have

\[ - \langle \mathcal{B} \mathcal{A}_2 U_0, V_0 \rangle - (Q_1(1)U_0, T_{B_1}^*(\mathcal{B}^*V_0)) + (\mathcal{Q} U_0, V_0) = 0 \quad \text{on } \Omega, \]

and

\[ (2 \partial_z \mathcal{B} \hat{U}_0, \hat{V}_0) + (\mathcal{B} \hat{U}_0, B_2^* \hat{V}_0) - (\mathcal{Q} \hat{U}_0, \hat{V}_0) - (Q_1(2)\hat{V}_0, T_{B_2}^*(\mathcal{B} \hat{U}_0)) = 0 \quad \text{on } \Omega. \]

Proof. We remind that \( \Phi \) satisfies (3.1), (3.2) and

\[ \text{Im } \Phi(\bar{x}) \notin \{ \text{Im } \Phi(x); x \in H \setminus \{ \bar{x} \} \}. \]

By Proposition 4.2 equality (4.2) holds true. Thanks to (4.66), (4.25) and Proposition 4.6 we can write it as

\[ (J_\pm + K_\pm)(\bar{x}) + I_{\pm, \Phi}(\bar{x}) = 0. \]

This equality and Proposition 4.5 imply

\[ (J_\pm + K_\pm)(\bar{x}) = 0. \]

By Propositions 4.1 and 4.6 we obtain

\[
\tilde{S}_{\tau, \bar{x}}(q_1, T_{B_1}^*(\mathcal{B}^*V_0)) = \mathcal{A}^*V_0 + 2T_{B_1}^*(\partial_z \mathcal{B}^*V_0) + T_{B_1}^*(\mathcal{B}^*(A_2^*V_0 - 2\tau \mathcal{F}'V_0))) = \\
-2\tau \tilde{S}_{\tau, \bar{x}}(q_1, T_{B_1}^*(\mathcal{B}^*\Phi V_0)) + o\left(\frac{1}{\tau}\right) = -2\tau \tilde{S}_{\tau, \bar{x}}(q_1, \mathcal{F}'T_{B_1}^*(\mathcal{B}^*V_0)) + o\left(\frac{1}{\tau}\right) = \\
-\frac{\pi}{2|\det \psi'(\bar{x})|^\frac{1}{2}}(2\partial_z q_1, T_{B_1}^*(\mathcal{B}^*V_0))(\bar{x}) + o\left(\frac{1}{\tau}\right) = \\
-\frac{\pi}{2|\det \psi'(\bar{x})|^\frac{1}{2}}(Q_1(1)U_0, T_{B_1}^*(\mathcal{B}^*V_0))(\bar{x}) + o\left(\frac{1}{\tau}\right),
\]

and

\[
-2\tilde{S}_{\tau, \bar{x}}(P_{A_2}^*(\mathcal{A}(\partial_z U_0 + \tau \Phi'U_0)) + \mathcal{B} \hat{\partial}_z U_{0, \tau}, q_4) = \\
-2\tilde{S}_{\tau}(P_{A_2}^*(\mathcal{A} \tau \Phi'U_0)), q_4) + o\left(\frac{1}{\tau}\right) = o\left(\frac{1}{\tau}\right).
\]

By (4.68) and (4.69)

\[
K_+(\bar{x}) = -\frac{\pi}{2|\det \psi'(\bar{x})|^\frac{1}{2}}(Q_1(1)U_0, T_{B_1}^*(\mathcal{B}^*V_0))(\bar{x}) + o\left(\frac{1}{\tau}\right).
\]

In the similar way we compute \( K_-(\bar{x}) \):

\[
\tilde{S}_{-\tau, \bar{x}}(q_2, P_{A_2}^*(2\partial_z (\mathcal{A}^*V_0 - \tau \Phi'2\mathcal{A}^*V_0) - \mathcal{B}^* \hat{V}_0 + P_{A_2}^*(A_2^* \mathcal{B}^* \hat{V}_0)) = \\
-2\tau \tilde{S}_{-\tau, \bar{x}}(q_2, P_{A_2}^*(\mathcal{F}' \mathcal{A}^*V_0)) + o\left(\frac{1}{\tau}\right) = o\left(\frac{1}{\tau}\right).
\]
and

\( (4.72) \)

\[-2\tilde{\mathbf{f}}_{\tau,x}^-(q_3, T_{-B_2}^*(2\tilde{\mathbf{A}}_2 \partial_2 \tilde{U}_0 + 2\tilde{\mathbf{B}}(\partial_2 \tilde{U}_0 + \tau \Phi' \tilde{U}_0))) = \]

\[-2\tilde{\mathbf{f}}_{\tau,x}^-(q_3, T_{-B_2}^*(\tau \tilde{\mathbf{B}} \Phi' \tilde{U}_0)) + o(\frac{1}{\tau}) = \frac{\pi}{2|\det \psi'(\tilde{x})|^{\frac{1}{2}}} (Q_1(2) \tilde{V}_0, T_{-B_2}^*(\tilde{\mathbf{B}} \tilde{U}_0)) + o(\frac{1}{\tau}). \]

By (4.71) and (4.72)

\( (4.73) \)

\[K_-(\tilde{x}) = \frac{\pi}{2|\det \psi'(\tilde{x})|^{\frac{1}{2}}} (Q_1(2) \tilde{V}_0, T_{-B_2}^*(\tilde{\mathbf{B}} \tilde{U}_0)) + o(\frac{1}{\tau}). \]

Substituting into (4.67) the right hand side of formulae (4.70) and (4.73) we obtain (4.64) and (4.65) as

By (4.71) and (4.72)

\( (4.74) \)

\[-(\tilde{\mathbf{B}}A_1 U_0, V_0) - (\tilde{Q}_1(1) U_0, T_{\tilde{B}_1}^*(\tilde{\mathbf{B}}^* V_0)) + (\tilde{Q} U_0, V_0) = 0 \quad \text{in } \Omega \]

and

\[-(2\partial_2 \tilde{\mathbf{B}} \tilde{U}_0, \tilde{V}_0) - (\tilde{\mathbf{B}} \tilde{U}_0, \tilde{B}_2 \tilde{V}_0) + (\tilde{Q} \tilde{U}_0, \tilde{V}_0) + (Q_1(2) \tilde{V}_0, T_{-B_2}^*(\tilde{\mathbf{B}} \tilde{U}_0)) = 0 \quad \text{in } \Omega. \]

The proof of the proposition is complete. ■

5. Step 3: End of the proof.

Let \( \tilde{\gamma} \) be a curve, without self-intersections which pass through the point \( \tilde{x} \) and couple points \( x_1, x_2 \) from \( \tilde{x} \) in such a way that the set \( \tilde{\gamma} \cap \partial \Omega \setminus \{x_1, x_2\} \) is empty. Denote by \( \Omega_1 \) a domain bounded by \( \tilde{\gamma} \) and part of \( \partial \Omega \) located between points \( x_1 \) and \( x_2 \). Then we set \( \Omega_{1,\xi} = \{x \in \Omega : \text{dist}(\Omega_1, x) < \xi\} \). By Proposition 2.1 for each point \( \tilde{x} \) from \( \Omega_{1,\xi} \) one can construct functions \( \tilde{U}_0^{(k)}, \tilde{V}_0^{(\ell)} \) satisfying (3.5), (3.28) such that

\[ \tilde{U}_0^{(k)}(\tilde{x}) = \tilde{e}_k, \quad \tilde{V}_0^{(\ell)}(\tilde{x}) = \tilde{e}_\ell \quad \forall k, \ell \in \{1, \ldots, N\}. \]

Then for each \( \tilde{x} \) there exists positive \( \delta(\tilde{x}) \) such that the matrices \( \{\tilde{U}^{(j)}_{0,i}\} \) and \( \{\tilde{V}^{(j)}_{0,i}\} \) are invertible for any \( x \in \tilde{B}(\tilde{x}, \delta(\tilde{x})) \). From the covering of \( \Omega_{1,\xi} \) by such a balls we take the finite subcovering \( \tilde{\Omega}_{1,\xi} \subset \cup_{k=1}^N \tilde{B}(x_k, \delta(x_k)) \). Then from (4.74) we have the differential inequality

\( (5.1) \)

\[ |\partial_2 \tilde{\mathbf{B}}_{ij}| \leq C_\xi \left( \sum_{k=1}^N |T_{-B_2}^*(\tilde{\mathbf{B}}^* \tilde{U}_0^{(k)})| + |\tilde{\mathbf{B}}| + |\tilde{\mathbf{Q}}| \right) \quad \text{in } \Omega_{1,\xi}, \quad \forall i, j \in \{1, \ldots, N\}. \]

Let \( \phi_0 \in C^2(\tilde{\Omega}) \) be a function such that

\( (5.2) \)

\[ \nabla \phi_0(x) \neq 0 \quad \text{in } \Omega_1, \quad \partial_\nu \phi_0|_\gamma \leq \alpha' < 0, \quad \phi_0|_\gamma = 0, \]

where \( \nu \) is the outward normal vector to \( \Omega_{1,\xi} \) and \( \chi_\xi \) be a function such that

\[ \chi_\xi \in C^2(\overline{\Omega_{1,\xi}}), \quad \chi_\xi = 1 \quad \text{in } \Omega_1, \]
and $\chi_\epsilon \equiv 0$ in some neighborhood of the curve $\partial \Omega_{1,\epsilon} \setminus \tilde{\Gamma}$. From (5.1), (4.50) we have

\begin{equation}
|\partial_2 (\chi_\epsilon \tilde{B}_i)| \leq C_\epsilon \sum_{k=1}^N |\chi_\epsilon T_{-B_2}^* (\tilde{B}^* \tilde{U}_{0}^{(k)})| + |\chi_\epsilon \tilde{B}|
\end{equation}

\begin{equation}
+ \frac{1}{2} |\chi_\epsilon, \partial_2 [\tilde{B}_i] + |\chi_\epsilon \tilde{Q}| | \ in \ \Omega_{1,\epsilon}, \ \forall i, j \in \{1, \ldots, N\},
\end{equation}

\begin{equation}
\chi_\epsilon \tilde{B}|_{\partial \Omega_{1,\epsilon}} = \partial_2 (\chi_\epsilon \tilde{B})|_{\partial \Omega_{1,\epsilon}} = 0.
\end{equation}

Set $\psi_0 = e^{\lambda_0}$ with positive $\lambda$ sufficiently large. Applying the Carleman estimate to the above inequality we have

\begin{equation}
\int_{\Omega_{1,\epsilon}} e^{2\tau \psi_0} \left( \frac{1}{\tau} |\nabla \chi_\epsilon \tilde{B}|^2 + \tau |\chi_\epsilon \tilde{B}|^2 \right) dx \leq C \int_{\Omega_{1,\epsilon}} \left( \sum_{k=1}^N |\chi_\epsilon T_{-B_2}^* (\tilde{B}^* \tilde{U}_{0}^{(k)})|^2 \right.
\end{equation}

\begin{equation}
+ \chi_\epsilon^2 (|\tilde{B}|^2 + |\tilde{Q}|^2) + \left. \frac{1}{2} |\chi_\epsilon, \partial_2 [\tilde{B}_i] + |\chi_\epsilon \tilde{Q}| | \in \Omega_{1,\epsilon}, \ \forall \tau \geq \tau_0. \right)
\end{equation}

By the Carleman estimate for the operator $\partial_2$ and (4.50) there exist $C$ and $\tau_0$ independent of $\tau$ such that

\begin{equation}
\int_{\Omega_{1,\epsilon}} |\chi_\epsilon T_{-B_2}^* (\tilde{B}^* \tilde{U}_{0}^{(k)})|^2 e^{2\tau \psi_0} dx \leq C \int_{\Omega_{1,\epsilon}} \left( |\chi_\epsilon, \partial_2 [\tilde{B}^* \tilde{U}_{0}^{(k)}]|^2 + |\chi_\epsilon \tilde{B}^* \tilde{U}_{0}^{(k)}|^2 \right) e^{2\tau \psi_0} dx
\end{equation}

and

\begin{equation}
\int_{\Omega_{1,\epsilon}} |\chi_\epsilon T_{B_1}^* (\tilde{B}^* V_0^{(k)})|^2 e^{2\tau \psi_0} dx \leq C \left( |\chi_\epsilon, \partial_2 [\tilde{B}^* V_0^{(k)}]|^2 + |\chi_\epsilon \tilde{B}^* V_0^{(k)}|^2 \right) e^{2\tau \psi_0} dx
\end{equation}

for all $\tau \geq \tau_0$.

Combining estimates (5.5), (5.6) we obtain that there exist a constant $C$ independent of $\tau$ such that

\begin{equation}
\int_{\Omega_{1,\epsilon}} e^{2\tau \psi_0} \left( \frac{1}{\tau} |\nabla \chi_\epsilon \tilde{B}|^2 + \tau |\chi_\epsilon \tilde{B}|^2 \right) dx
\end{equation}

\begin{equation}
\leq C \int_{\Omega_{1,\epsilon}} \chi_\epsilon^2 (|\tilde{B}|^2 + |\tilde{Q}|^2) + \sum_{k=1}^N |\chi_\epsilon, \partial_2 [T_{-B_2}^* (\tilde{B}^* \tilde{U}_{0}^{(k)})]|^2 + |\chi_\epsilon \tilde{B}^* \tilde{U}_{0}^{(k)}|^2 | e^{2\tau \psi_0} dx \ \forall \tau \geq \tau_0.
\end{equation}

For all sufficiently large $\tau$ the term $\int_{\Omega_{1,\epsilon}} |\chi_\epsilon \tilde{B}|^2 e^{2\tau \psi_0} dx$ absorbed by the integral on the left hand side. Moreover, thanks to the choice of the function $\chi_\epsilon$, we have supports of coefficients for the operator $[\chi_\epsilon, \partial_2]$ are located in the domain $\Omega_{1,\epsilon} \setminus \Omega_{1,\tilde{\epsilon}}$. Hence one can write the estimate (5.8) as

\begin{equation}
\int_{\Omega_{1,\epsilon}} e^{2\tau \psi_0} \left( \frac{1}{\tau} |\nabla \chi_\epsilon \tilde{B}|^2 + \tau |\chi_\epsilon \tilde{B}|^2 \right) dx \leq C \int_{\Omega_{1,\epsilon}} \chi_\epsilon^2 |\tilde{Q}|^2 e^{2\tau \psi_0} dx
\end{equation}

\begin{equation}
+ C \int_{\Omega_{1,\epsilon} \setminus \Omega_{1,\tilde{\epsilon}}} \sum_{k=1}^N \left( |\chi_\epsilon, \partial_2 [T_{-B_2}^* (\tilde{B}^* \tilde{U}_{0}^{(k)})]|^2 + |\chi_\epsilon \tilde{B}^* \tilde{U}_{0}^{(k)}|^2 \right) e^{2\tau \psi_0} dx \ \forall \tau \geq \tau_1.
\end{equation}

By Proposition 2.7 for each point $\hat{x}$ from $\Omega$ one can construct such a function $U_0^{(k)}, V_0^{(\ell)}$ satisfying (3.5), (3.28) such that

$U_0^{(k)}(\hat{x}) = \tilde{e}_k, \ V_0^{(\ell)}(\hat{x}) = \tilde{e}_\ell \ \forall k, \ell \in \{1, \ldots, N\}$. 
Then for each $\hat{x} \in \Omega_{1,\epsilon}$ there exists positive $\delta(\hat{x})$ such that the matrices $\{U_{0,i}^{(j)}\}$ and $\{V_{0,i}^{(j)}\}$ are invertible for any $x \in B(\hat{x}, \delta(\hat{x}))$. From the covering of $\Omega_{1,\epsilon}$ by such a balls we take the finite subcovering $\Omega \subset \cup_{k=N}^{N+N^*} B(x_k, \delta(x_k))$. Then there exists $C_\epsilon$ such that

$$
|Q| \leq C_\epsilon (|\tilde{B}| + \sum_{k=N+1}^{N+N^*} |T_{B_k^*}^* (\tilde{B}^* V_0^{(k)}))| \quad \text{in } \Omega_{1,\epsilon}.
$$

(5.10)

Combining (5.7), (5.9) and (5.10) we obtain that there exists a constant $C_5$ independent of $\tau$

$$
\int_{\Omega_{1,\epsilon}} e^{2\tau \psi_0 (\frac{1}{\tau} |\nabla (\chi_\epsilon \tilde{B})|^2 + \tau |\chi_\epsilon \tilde{B}|^2) dx \leq C_5 \int_{\Omega_{1,\epsilon} \setminus \Omega_{1,\epsilon}^\epsilon} \left( \sum_{k=1}^{N} \| [\chi_\epsilon, \partial_\epsilon] T_{B_k^*}^* (\tilde{B}^* \tilde{U}_0^{(k)}) \|^2 + \sum_{k=N+1}^{N+N^*} \| [\chi_\epsilon, \partial_\epsilon] T_{B_k^*}^* (\tilde{B}^* V_0^{(k)}) \|^2 \right)
$$

(5.11)

By (5.2) for all sufficiently small positive $\epsilon$ there exists a positive constant $\alpha < 1$ such that

$$
\psi_0 (x) < \alpha \quad \text{on } \Omega_{1,\epsilon} \setminus \Omega_{1,\epsilon}^\epsilon.
$$

(5.12)

Since $\hat{x} \in \text{supp } \tilde{B} \cap \tilde{\gamma}$ and thanks to the fact $\partial_\epsilon \phi_0 |_{\tilde{\gamma}} \leq \alpha^\prime < 0$ there exists $\kappa > 0$ such that

$$
\kappa e^{\tau} \leq \int_{\Omega_{1,\epsilon}} e^{2\tau \psi_0 |\chi_\epsilon \tilde{B}|^2 e^{2\tau \psi_0} dx \quad \forall \tau \geq \tau_1.
$$

(5.13)

By (5.12) we can estimate the right hand side of the inequality (5.9) as

$$
C_5 \int_{\Omega_{1,\epsilon} \setminus \Omega_{1,\epsilon}^\epsilon} \left( \sum_{k=1}^{N} \| [\chi_\epsilon, \partial_\epsilon] T_{B_k^*}^* (\tilde{B}^* \tilde{U}_0^{(k)}) \|^2 + \sum_{k=N+1}^{N+N^*} \| [\chi_\epsilon, \partial_\epsilon] T_{B_k^*}^* (\tilde{B}^* V_0^{(k)}) \|^2 \right) + \| [\chi_\epsilon, \partial_\epsilon] \tilde{B} \|^2 e^{2\tau \psi_0} dx \leq C_6 e^{\alpha \tau} \quad \forall \tau \geq \tau_1,
$$

(5.14)

where $C_5, C_6$ are positive constants independent of $\tau$. Using (5.13) and (5.14) in (5.9) we obtain

$$
\kappa e^{\tau} \leq C_7 e^{\alpha \tau} \quad \forall \tau \geq \tau_1.
$$

Since $\alpha < 1$ we arrived to the contradiction. Hence

$$
\tilde{B} = \tilde{Q} = 0 \quad \text{on } \Omega \setminus \mathcal{X}_0.
$$

The proof of theorem is complete. ■

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