Abstract

We study the late time evolution of flat and negatively curved FRW models with a perfect fluid matter source and a scalar field having an arbitrary non-negative potential function $V(\phi)$. We prove using a dynamical systems approach four general results for a large class of non-negative potentials which show that almost always the universe ever expands. In particular, for potentials having a local zero minimum, flat and negatively curved FRW models are ever expanding and the energy density asymptotically approaches zero. We investigate the conditions under which the scalar field asymptotically approaches the minimum of the potential. We discuss the question of whether a closed FRW with ordinary matter can avoid recollapse due to the presence of a scalar field with a non-negative potential.

1 Introduction

Scalar fields currently play a prominent role in the construction of cosmological scenarios aiming to describe the structure and evolution of the early universe. The standard inflationary idea requires that there be a period of slow-roll evolution of a scalar field (the inflaton) during which its potential energy drives the universe in a quasi-exponential expansion [1]. Scalar fields also arise naturally in alternative theories of gravity which aim to extend general relativity, e.g., higher order gravity theories, scalar-tensor and
string theories. In higher derivative gravity, due to its conformal relation with general relativity [2], scalar fields appear in the Einstein frame with a self-interaction, nonlinear potential term which mimics the higher order curvature properties of the original (Jordan) frame. Generalizations of the original Brans-Dicke theory lead to scalar-tensor theories with scalar field self-interactions and dynamical couplings to matter. Further generalizations can be achieved by considering multiple scalar fields [3]. Similar results can be proved for simple bosonic string theories [4].

Most of the studies of scalar-field cosmologies with the dynamical systems methods are restricted to Friedmann-Robertson-Walker (FRW) models (see for example [5, 6] and references therein). On the other hand, there are also important investigations in spatially homogeneous Bianchi cosmologies with an exponential potential [7], as well as in models containing both a perfect fluid of ordinary matter and a scalar field with an exponential potential, the so-called “scaling” cosmologies [8].

It is important to investigate the general properties shared by all FRW models with a scalar field irrespectively of the particular choice of the potential. In this paper we study the time evolution of flat and negatively curved FRW models with a perfect fluid and a scalar field with an arbitrary non-negative potential function $V(\phi)$ and prove some general results for a large class of such functions. More precisely, we first show that $\phi$ almost always diverges in the past. Furthermore, for potentials having a local zero minimum, flat and negatively curved FRW models are ever expanding and the energy density asymptotically approaches zero. If in addition the potential has a unique global zero minimum, the scalar field almost always asymptotically approaches zero. Similarly for exponential potentials it is shown that in an expanding universe the energy density $\rho$ of ordinary matter and the Hubble function $H$ asymptotically approach zero, while the scalar field $\phi$ diverges to $+\infty$.

The plan of the paper is as follows. In the next Section we write down the field equations assuming an arbitrary non-negative potential, as a constrained four-dimensional dynamical system. In Section 3 and prove the above mentioned four propositions under mild conditions on the potential. In Section 4 we discuss the closed universe recollapse conjecture for a $k = 1$ FRW universe, containing a perfect fluid and a scalar field with a non-negative potential.
2 Scalar-field cosmologies

In General Relativity the evolution of FRW models with a scalar field (ordinary matter is described by a perfect fluid with energy density $\rho$ and pressure $p$) are governed by the Friedmann equation,\(^1\)

$$
\left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} = \frac{1}{3} \left( \rho + \frac{1}{2} \dot{\phi}^2 + V(\phi) \right),
$$

(1)

the Raychaudhuri equation,

$$
\frac{\ddot{a}}{a} = -\frac{1}{6} \left( \rho + 3p + 2\dot{\phi}^2 - 2V \right),
$$

(2)

the equation of motion of the scalar field,

$$
\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + V'(\phi) = 0,
$$

(3)

and the conservation equation,

$$
\dot{\rho} + 3(\rho + p)\frac{\dot{a}}{a} = 0.
$$

(4)

Here $V(\phi)$ is the potential energy of the scalar field and $V' = dV/d\phi$. We assume an equation of state of the form $p = (\gamma - 1)\rho$, with $0 \leq \gamma \leq 2$.

Formally the FRW model with a scalar field is obtained if we add to the matter content of the classical Friedmann universe a perfect fluid with energy density $\varepsilon := \frac{1}{2}\dot{\phi}^2 + V$ and pressure $q := \frac{1}{2}\dot{\phi}^2 - V$. However, this fluid violates the strong energy condition, i.e., $\varepsilon + 3q = 2\dot{\phi}^2 - 2V$ may be negative. It is precisely this violation that leads to inflation in the very early universe [1].

From Eqs. (1)-(4) we see that the state $(a, \dot{a}, \rho, \phi, \dot{\phi}) \in \mathbb{R}^5$ of the system lies on the hypersurface defined by the constraint (1) and the remaining evolution equations can be written as a five-dimensional dynamical system. In the case of a flat, $k = 0$ model, the dimension of the dynamical system reduces to four.

\(^1\)We adopt the metric and curvature conventions of [9]. Here, $a(t)$ is the scale factor, an overdot denotes differentiation with respect to time $t$, and units have been chosen so that $c = 1 = 8\pi G$. 

3
In the remaining of the paper we assume that the potential of the scalar field is an arbitrary non-negative (at least $C^2$) function. Gradually we introduce further conditions on the potential that allow us to analyze the general properties of the resulted dynamical system. Setting $\dot{\phi} = y$, $\dot{a}/a = H$ the evolution equations (2)-(4) for the flat model become

$$
\begin{align*}
\dot{\phi} &= y, \\
\dot{y} &= -3Hy - V'(\phi), \\
\dot{\rho} &= -3\gamma \rho H, \\
\dot{H} &= -\frac{1}{2}y^2 - \frac{\gamma}{2} \rho,
\end{align*}
$$
\begin{equation}
\text{(5)}
\end{equation}

subject to the constraint

$$
3H^2 = \rho + \frac{1}{2}y^2 + V(\phi).
$$
\begin{equation}
\text{(6)}
\end{equation}

Therefore, the phase space of the dynamical system (5) is the set

$$
\Sigma := \left\{ (\phi, y, \rho, H) \in \mathbb{R}^4 : 3H^2 = \rho + \frac{1}{2}y^2 + V(\phi) \right\}.
$$

3 Flat and negatively curved FRW with an arbitrary non-negative potential

A theorem due to Foster [10] states that for expanding flat FRW models in vacuum with a scalar field having a non-negative potential, $\phi$ almost always diverges in the past. The following Proposition is a generalization of this result in the case we include a barotropic fluid.

**Proposition 1** Let $x \in \Sigma$ and let $O^- (x)$ be the past orbit of $x$ under (5). Then $\phi$ is almost always unbounded on $O^- (x)$.

**Proof.** Let $x \in \Sigma$ be such that $\phi$ is bounded on $O^- (x)$. Using the same type of argument as in Theorem 1 of [10], one can show that the $\alpha$–limit set of $x$ lies on the plane $\rho = 0$, $y = 0$. From (5) we see that the only invariant sets lying on the plane $\rho = 0$, $y = 0$, are equilibrium points of the form $(\phi, y, \rho, H) : \rho = 0, y = 0$ and $\phi_1 : V'(\phi_1) = 0$. Taking into account the
constraint (6), we see that the past limit set \( \alpha (x) \) consists of equilibrium points of the form \((\phi_1, 0, 0, H_1)\), where \( H_1^2 = V(\phi_1)/3 \).

We will show that no such equilibrium is past asymptote to an open neighborhood of \( x \). To reduce the dimension of the dynamical system (5) we use the constraint (6) to eliminate \( \rho \). The evolution equations become

\[
\begin{align*}
\dot{\phi} &= y, \\
\dot{y} &= -3Hy - V'(\phi), \\
\dot{H} &= -\frac{3\gamma}{2}H^2 - \frac{2 - \gamma}{4}y^2 + \frac{\gamma}{2}V(\phi).
\end{align*}
\]  

(7)

For an expanding universe, any equilibrium point of (7) can be written as \( q = (\phi_1, 0, \sqrt{V_0/3}) \), where \( \phi_1 : V'(\phi_1) = 0 \) and \( V_0 := V(\phi_1) > 0 \). It is straightforward to see that the eigenvalues of the Jacobian matrix of (7) at \( q \) are

\[-\gamma \sqrt{3V_0}, \quad \frac{1}{2} \left(-\sqrt{3V_0} \pm \sqrt{(3V_0 - 4V''(\phi_1))}\right).\]  

(8)

At least two of them have negative real parts and, therefore the center manifold theorem implies that there exists a 2 or 3-dimensional stable manifold through \( q \) (see for example [11]). That means that all trajectories asymptotically approaching \( q \) as \( t \to \infty \), lie on a 2 or 3-dimensional invariant manifold. We conclude that all solution curves asymptotically approaching \( q \) as \( t \to -\infty \), lie on a 0 or 1-dimensional invariant manifold.

If all of the eigenvalues have negative real parts the assertion follows immediately from the linearization theorem. In particular, if \( V''(\phi_1) > 0 \), i.e., \( \phi_1 \) is a local minimum of the potential, then the equilibrium point \((\phi_1, 0, \sqrt{V_0/3})\) is future asymptotically stable. This equilibrium solution corresponds to a vacuum de Sitter space with a cosmological constant equal to \( \sqrt{V_0} \).

Unfortunately the above Proposition cannot be applied to potentials having a zero local minimum \( V(\phi_1) \), because for \( V_0 = 0 \), the eigenvalues (8) have zero real parts. However, a large class of potentials used in scalar-field cosmological models have a zero local minimum. Examples of potentials belonging to this class are polynomial potentials of the form \( V(\phi) = \lambda \phi^{2n} \), generalized and logarithmic potentials studied by Barrow and Parsons [12], or potentials in conformally related theories of gravity, for example

\[ V(\phi) = V_\infty \left(1 - e^{-\sqrt{2/3}\phi}\right)^2 \]  

(9)
which arises in the conformal frame of the $R + \alpha R^2$ theory [2, 13]. To study the late time evolution of these models, we adopt a different approach.

In the following discussion we assume that the potential function of the scalar field is non-negative with a local minimum $V(0) = 0$, but otherwise arbitrary. For simplicity, we present only the flat case, $k = 0$, but the results can be extended to the case of negatively curved FRW models (cf. Remark after Proposition 3).

The equilibrium points of (5) are $(\phi, y, \rho, H)$ : $\rho = 0, y = 0$ and $\phi : V'(\phi) = 0$. Using the constraint (6) and the assumed $V_{\text{min}} = 0$ we see that an equilibrium point is the $(0,0,0,0)$. Physically, the equilibrium point $(0,0,0,0)$, corresponds to an empty universe. Linearization of (5) near the equilibrium point $(0,0,0,0)$ shows that the eigenvalues of the Jacobian matrix at that point have zero real parts (the assumptions on the potential imply that $V'(\phi) = 0$ at $\phi = 0$ and also that $V''(0) \geq 0$), consequently the Hartman-Grobman Theorem does not apply [11]. Therefore, we cannot conclude about the stability of the equilibrium. However, looking at the system (5), one arrives at the conclusion that for an ever-expanding universe, $H(t)$ and $\rho(t)$ are decreasing functions and $\phi(t)$ oscillates around the minimum of the potential with a decreasing amplitude due to the damping factor $3H\dot{\phi}$. These intuitive arguments are made more precise in Propositions 2 and 3 below.

We first show that an initially expanding flat universe remains ever-expanding. In fact, the set $\{(\phi, y, \rho, H) \in \mathbb{R}^4 : H = 0\}$ is invariant under the flow of (5) with the constraint (6). Therefore, the sign of $H$ is invariant. (If the sign of $H$ could change, a solution curve starting with say, a positive $H$, would pass through the point $(0,0,0,0)$ which is an equilibrium solution of (5), thus violating the fundamental existence and uniqueness theorem of differential equations). We conclude that if $H(t_0) > 0$, then $H(t) > 0$ for all $t > t_0$.

In the following Proposition the potential function has a local minimum, $V(0) = 0$, but we do not exclude the existence of other local minima with a higher value of $V$.

**Proposition 2** Suppose that $V \geq 0$ and $V(\phi) = 0 \iff \phi = 0$. Suppose also that if $A \subseteq \mathbb{R}$ is such that $V$ is bounded on $A$, then $V'$ is bounded on $A$. Then $\lim_{t \to +\infty} \rho = 0 = \lim_{t \to +\infty} y$.

**Proof.** Consider the trajectory passing through an arbitrary point $x_0 = (\phi, y, \rho, H) \in \Sigma$ with $H > 0$ at $t = t_0$. Since $H(t)$ is decreasing and positive, it
follows that \( \lim_{t \to \infty} H(t) \) exists and is a non-negative number \( \eta \); also, \( H(t) \leq H(t_0) \), for all \( t \geq t_0 \). We then deduce from (6) that each of the terms \( \rho, \frac{1}{2}y^2 \) and \( V \) is bounded by \( 3H(t_0)^2 \). Let \( A = \{ \phi : V(\phi) \leq 3H(t_0)^2 \} \). Then the trajectory is such that \( \phi \) stays inside \( A \).

From (5) we get
\[
- \int_{t_0}^{t} \left( \frac{1}{2}y^2 + \frac{\gamma}{2} \rho \right) dt = H(t) - H(t_0),
\]
and taking the limit as \( t \to \infty \), we obtain
\[
\frac{1}{2} \int_{t_0}^{+\infty} (y^2 + \gamma \rho) dt = H(t_0) - \eta.
\]
Therefore,
\[
\int_{t_0}^{\infty} (y^2(t) + \gamma \rho(t)) dt < \infty.
\] (10)

In general, if \( f \) is a non-negative function, the convergence of \( \int_{t_0}^{\infty} f(t) dt \) does not imply that \( \lim_{t \to \infty} f(t) = 0 \), unless the derivative of \( f \) is bounded. It can be shown that \( \lim_{t \to \infty} f(t) = 0 \), provided that \( f' \) is bounded from above.\(^2\)

In our case,
\[
\frac{d}{dt} (y^2 + \gamma \rho) = 2y \dot{y} + \gamma \dot{\rho} = -6Hy^2 - 2yV'(\phi) - 3\gamma^2 \rho H \leq -2yV'(\phi).
\]

As we already remarked, \( y \) is bounded; also, by our assumption on \( V \), \( V'(\phi) \) is bounded. We conclude that the derivative of the function \( y^2 + \gamma \rho \)
\[^2\]If \( f > 0 \), \( \dot{f} < k \), where \( k \) is a positive constant and \( \int_{t_0}^{\infty} f dt < +\infty \) then, \( \lim_{t \to +\infty} f(t) = 0 \). Here is the proof: Suppose to the contrary that there exists \( \varepsilon > 0 \) such that one can find arbitrarily large \( t \) such that \( f(t) > \varepsilon \). Choose such a \( t \) and then define \( s_0 < t \) so that \( t - s_0 = \frac{\varepsilon}{k} \). One has
\[
\forall s < t, \int_{s}^{t} \dot{f}(s) ds < (t - s) k \Rightarrow f(s) > f(t) - (t - s) k \Rightarrow
\]
\[
\int_{s_0}^{t} f(s) ds > (t - s_0) f(t) - \frac{1}{2} k (t - s_0)^2 > \frac{\varepsilon^2}{2k}.
\]
The assumption \( \int_{t_0}^{+\infty} f dt < +\infty \) implies that \( \lim_{s \to +\infty} \int_{s}^{+\infty} f ds = 0 \). This contradicts the fact that one can find inside any interval \( (s, +\infty) \) a subinterval \( (s_0, t) \) such that \( \int_{s_0}^{t} f(s) ds > \frac{\varepsilon^2}{2k} \).
is bounded from above and therefore, (10) implies that \( \lim_{t \to \infty} y^2(t) = 0 \) and \( \lim_{t \to \infty} \rho(t) = 0. \)

The conditions in Proposition 2 are satisfied by a large class of potential functions, for example by the generalized and logarithmic potentials studied in [12]. In particular, this class includes potentials which do not approach zero as \( \phi \to \infty \) (cf. equation (9)).

**Remark.** It is easy to conclude that \( \lim_{t \to +\infty} \rho = 0 \), without any assumption of boundedness of \( V' \). We only have to consider the case \( \eta > 0 \) because if \( \eta = 0 \) then (6) entails that \( \lim_{t \to \infty} \rho = 0 \). By (5b), \( \rho \) is decreasing, thus \( \exists \lim_{t \to +\infty} \rho := \rho_0 \). We shall show that \( \rho_0 = 0 \).

Suppose that \( \rho_0 > 0 \). Then from (5b) again we get \\( \rho \leq -3\eta\rho_0\gamma \). Thus

\[
\rho(t) - \rho(t_0) = \int_{t_0}^t \dot{\rho}(s) \, ds < -3\eta\rho_0\gamma (t - t_0),
\]

which implies that \( \rho \) becomes negative for large \( t \), a contradiction.

If we assume in addition that \( V'(\phi) > 0 \) for \( \phi > 0 \) and \( V'(\phi) < 0 \) for \( \phi < 0 \) (thus, 0 is the only local minimum of \( V \)), then we can conclude that, if the initial value of \( H \) is less than \( \sqrt{V(\infty)}/3 \), then \( \lim_{t \to +\infty} \phi \) is equal to 0. More precisely, we have the following result.

**Proposition 3** Suppose that \( V'(\phi) > 0 \) for \( \phi > 0 \) and \( V'(\phi) < 0 \) for \( \phi < 0 \). Then, under the assumptions of Proposition (2), \( \lim_{t \to +\infty} \phi \) exists and is equal to +\( \infty \), or -\( \infty \) or 0.

**Proof.** We know that \( \exists \lim_{t \to \infty} H(t) = \eta. \) If \( \eta = 0 \), then from (6) we obtain \( \lim_{t \to \infty} V(\phi) = 0 \). The assumption on the potential implies that \( \lim_{t \to \infty} \phi = 0 \). So suppose that \( \eta > 0 \). From (6) we get \( \lim_{t \to +\infty} V(\phi(t)) = 3\eta^2 \). Thus there exists \( t' \) such that \( V(\phi) > 3\eta^2/2 \) for all \( t > t' \). It follows that \( \phi \) cannot become zero for some \( t > t' \) because \( V(0) = 0 \). Thus, \( \phi \) has a constant sign for all \( t > t' \).

Suppose that \( \phi > 0 \) for all \( t > t' \). Since \( V \) is increasing as a function of \( \phi \) on \( (0, +\infty) \), we have \( \lim_{t \to +\infty} V(\phi(t)) = 3\eta^2 \leq \lim_{\phi \to +\infty} V(\phi) \). We consider two cases:

(i) If \( \lim_{t \to +\infty} V(\phi(t)) = \lim_{\phi \to +\infty} V(\phi) \) then obviously \( \lim_{t \to +\infty} \phi = +\infty \).

(ii) If \( \lim_{t \to +\infty} V(\phi(t)) < \lim_{\phi \to +\infty} V(\phi) \) then there exists \( \phi \geq 0 \) such that \( \lim_{t \to +\infty} V(\phi(t)) = V(\phi) \). Since \( V \) is continuous and strictly increasing, it follows that \( \lim_{t \to +\infty} \phi = \phi \). From (5b), taking into account that
$\lim_{t \to +\infty} y = 0$ and $H$ is bounded, we deduce that, $\lim_{t \to +\infty} \dot{y} = -V'(\phi) < 0$. Hence, there exists $t'' > t'$ such that for all $t > t''$, $\dot{y} < -V'(\phi)/2$. But this entails immediately that

$$y(t) - y(t'') = \int_{t''}^{t} \dot{y} dt < \frac{-V'(\phi)}{2} (t - t''),$$

i.e., $y(t)$ takes arbitrarily large negative values as $t$ increases, which is not possible because $\lim_{t \to +\infty} y = 0$.

Thus, if $\phi > 0$ for all $t > t'$ then $\lim_{t \to +\infty} \phi = +\infty$. Similarly, the case $\phi < 0$ for all $t > t'$ leads to $\lim_{t \to +\infty} \phi = -\infty$. 

It follows that if initially, $3H^2(t_0) < \min\{\lim_{\phi \to +\infty} V(\phi), \lim_{\phi \to -\infty} V(\phi)\}$, then $\lim_{t \to +\infty} H(t) = 0$. Indeed, we know that $\lim_{t \to +\infty} \phi$ is equal to 0 or $+\infty$ or $-\infty$. If, say, $\lim_{t \to +\infty} \phi = +\infty$, then by (6), $3\eta^2 = \lim_{t \to +\infty} V(\phi(t)) = \lim_{\phi \to +\infty} V(\phi) > 3H^2(t_0)$. This is impossible, since $H(t)$ is decreasing and thus $H(t_0) \geq \eta$. Likewise, $\lim_{t \to +\infty} \phi = -\infty$ leads to a contradiction. Hence, $\lim_{t \to +\infty} \phi = 0$ and this implies that $\lim_{t \to +\infty} V(\phi(t)) = 0$ and, again by (6), $\lim_{t \to +\infty} H(t) = 0$.

We use as an example the potential (9) to visualize the above result. This potential has a long and flat plateau. For large values of $\phi$, the potential $V$ is almost constant, $V_\infty = \lim_{\phi \to +\infty} V(\phi)$, thus $V$ has the general properties for inflation to commence. According to Proposition 3, if the initial value of $H$ is less than $V_\infty$, then $\phi, H \to 0$ as $t \to \infty$. For initial values of $H$ greater than $V_\infty$, $\phi$ may diverge to $+\infty$, i.e., the scalar field may reach the flat plateau of the potential.

**Remark.** The above results can be extended to the case of negatively curved FRW models, $k = -1$. Setting $x = 1/a$ the evolution equations (2)-(4) become

$$\dot{x} = -Hx,$$
$$\dot{\phi} = y,$$
$$\dot{y} = -3Hy - V'(\phi),$$
$$\dot{\rho} = -3\gamma \rho H,$$
$$\dot{H} = -\frac{1}{2}y^2 - \frac{\gamma}{2} \rho + kx^2,$$

subject to the constraint

$$3H^2 + 3kx^2 = \rho + \frac{1}{2}y^2 + V(\phi).$$
With slight modifications we can repeat the same type of arguments to show that Propositions 2 and 3 still hold and, in addition, \( \lim_{t \to +\infty} x = 0 \).

Up to now, we have assumed that the potential is non-negative and has a minimum. Therefore, the above three Propositions do not apply directly to the important case of an exponential potential, \( \text{vis. } V(\phi) = V_0 e^{-\lambda \phi} \), with \( V_0 \) and \( \lambda \) positive constants. It is already known that the state corresponding to \( \rho = 0 \), \( y = 0 \) and \( \phi \to \infty \) is an equilibrium of the projection of the system (5) on the \( \phi - y \) plane. This result has been established using phase plane analysis (cf. [14]) on the reduced two-dimensional dynamical system. Using similar arguments as in Propositions 2 and 3, we complete our analysis by showing the global result that, for expanding flat models with an exponential potential, \( \phi \to \infty \) as \( t \to \infty \). More precisely, we have the following result.

**Proposition 4** Let \( V \) be a potential function with the following properties.\(^3\)

1. \( V \) is non-negative and \( \lim_{\phi \to -\infty} V(\phi) = +\infty \).
2. \( V' \) is continuous and \( V'(\phi) < 0 \).
3. If \( A \subseteq \mathbb{R} \) is such that \( V \) is bounded on \( A \), then \( V' \) is bounded on \( A \).

Then \( \lim_{t \to +\infty} y = 0 = \lim_{t \to +\infty} \rho \), and \( \lim_{t \to +\infty} \phi = +\infty \).

**Proof.** It is easy to see that the set \( \rho > 0 \) is invariant under the flow of (5), therefore \( \rho \) is nonzero if initially \( \rho(t_0) \) is nonzero. Now from (4) it follows that \( H \) is never zero, thus it cannot change sign. Hence, \( H \) is always non-negative if \( H(t_0) > 0 \). Further, \( H \) is decreasing in view of (5d), thus \( \exists \lim_{t \to +\infty} H = \eta \geq 0 \).

We know from (5d) that

\[
\frac{1}{2} \int_{t_0}^{+\infty} (y^2 + \gamma \rho) \, dt = H(t_0) - \eta < +\infty. \tag{13}
\]

As in Proposition 2, \( \frac{d}{dt} (y^2 + \gamma \rho) \leq -2yV'(\phi) \). From this and (13) we deduce that \( \lim_{t \to +\infty} y = \lim_{t \to +\infty} \rho = 0 \), as in Proposition 2.

We now show that \( \lim_{t \to +\infty} \phi = +\infty \) as in Proposition 3. It follows from (6) that \( \lim_{t \to +\infty} V(\phi) = 3\eta^2 \). Since \( V \) is strictly decreasing as a function of \( \phi \), we have \( V(\phi) > \lim_{\phi \to +\infty} V(\phi) \) for all \( \phi \), thus \( \lim_{t \to +\infty} V(\phi(t)) \geq \lim_{\phi \to +\infty} V(\phi) \). We consider two cases:

(i) If \( \lim_{t \to +\infty} V(\phi(t)) = \lim_{\phi \to +\infty} V(\phi) \) then obviously \( \lim_{t \to +\infty} \phi = +\infty \).

\(^3\)Note that \( V(\phi) = V_0 e^{-\lambda \phi} \) has all these properties. We do not exclude potential functions with \( \lim_{\phi \to +\infty} V(\phi) > 0 \).
(ii) If \( \lim_{t \to +\infty} V(\phi(t)) > \lim_{\phi \to +\infty} V(\phi) \), then there exists a unique \( \bar{\phi} \) such that \( \lim_{t \to +\infty} V(\phi(t)) = V(\bar{\phi}) \). It then follows that \( \lim_{t \to +\infty} \phi(t) = \bar{\phi} \).

From (5b) we deduce that \( \lim_{t \to +\infty} y = -V'(\bar{\phi}) > 0 \); thus, there exists \( t' \) such that \( \dot{y} \geq -V'(\bar{\phi})/2 \) for all \( t > t' \). From this we get that \( y(t) - y(t') > -V'(\bar{\phi}) (t - t') / 2 \), which is not possible because \( \lim_{t \to +\infty} y = 0 \). We conclude that \( \lim_{t \to +\infty} \phi = +\infty \).

If in addition the potential is such that \( \lim_{\phi \to +\infty} V(\phi) = 0 \), then we conclude that \( H \to 0 \) as \( t \to \infty \). This is obviously true for an exponential potential.

### 4 Discussion and the \( k = +1 \) FRW model

In this paper we have investigated the qualitative properties of flat and negatively curved FRW models containing a barotropic fluid and a scalar field with an arbitrary non-negative potential. For potentials having a unique minimum \( V(0) = 0 \), we have shown that in an expanding universe, the energy density \( \rho \) of ordinary matter, the Hubble function \( H \) and the scalar field \( \phi \) asymptotically approach zero. This result was rigorously proved without referring to the exact details of the potential.

However, the situation is more delicate for positively curved FRW models with a scalar field. In the present state of the universe the scalar field oscillates around the minimum of the potential and it is unobservable, in the sense that the energy density \( \rho \) of ordinary matter dominates over the energy density \( \varepsilon \) of the scalar field, hence we expect an almost classical Friedmanian evolution. Using (3) it is easy to see that the energy density of the field satisfies

\[
\dot{\varepsilon} = -3H \dot{\phi}^2
\]

i.e., in an expanding universe \( \varepsilon \) is a decreasing function of time. Since the energy density of ordinary matter also decreases, it may happen that in a future time, \( \varepsilon \) be comparable to \( \rho \). In particular, for closed \( (k = 1) \) models, once the scale factor reaches its maximum value and recollapse commences i.e., \( H < 0 \), the term \( 3H \dot{\phi} \) in (3) is no longer a damping factor, but acts as a driving force which forces the field \( \phi \) to oscillate with larger and larger amplitude. If this be the case, the repulsive effect of the cosmological term may drastically change the evolution of a classical FRW model.
These intuitive ideas show that it is of interest to study the late time evolution of the dynamical system (11) with \( k = +1 \). The main problem we are faced with is the following: Can a closed universe filled with ordinary matter and a scalar field avoid recollapse? In the following we give a partial answer to this question for an arbitrary non-negative potential having a unique minimum \( V(0) = 0 \).

A closed Friedmann universe is considered almost synonymous to a recollapsing universe. This is mainly due to our experience with the dust and radiation filled Friedmann models usually treated in textbooks. That this picture is misleading follows clearly from an example found by Barrow et al [15] according to which an expanding Friedmann model with spatial topology \( S^3 \) satisfying the weak, the strong, the dominant energy conditions and the generic condition may not recollapse. Thus the problem of recollapse of a closed universe to a second singularity is not trivial already in the Friedmann case.

Among the known global results concerning the closed universe recollapse conjecture, let us recall that a closed universe recollapses provided that the strong energy condition (SEC) is satisfied and there exists a maximal space-like hypersurface \( \Sigma \), i.e., the expansion is zero on \( \Sigma \) (cf. [16]). Therefore, in FRW models it suffices to check if the scale factor has a maximum.

In the following we assume an equation of state of the form \( p = (\gamma - 1)\rho \), \( \gamma \in \left(\frac{2}{3}, 2\right) \).\(^4\) Equation (4) can be integrated to give

\[
\rho = \text{const} \times a^{-3\gamma}.
\]

Assume that at time \( t_0 \) (now) the values of \( \phi \) and \( \dot{\phi} \) are very small in the sense that

\[
\varepsilon_0 := \frac{1}{2} \dot{\phi}_0^2 + V_0 \ll \rho_0 \quad \text{and} \quad 2\dot{\phi}_0^2 - 2V_0 \ll \rho_0 + 3p_0
\]

so that the total stress-energy tensor satisfies the SEC. This is a plausible assumption since the scalar field is unobservable in the present universe. We write (3) (in first-order approximation) as

\[
\ddot{\phi} + 3H\dot{\phi} + m^2 \phi = 0
\]

where \( m^2 := V''(0) \). This is the equation of motion of an harmonic oscillator with a variable damping factor \( 3H \). For a slowly varying function \( H \)

\(^4\)The range of \( \gamma \) is chosen in accordance to the conditions for recollapse of Barrow et al [15].
this equation can be solved using the Kryloff-Bogoliuboff [17] approximation. We find that the amplitude of the scalar field varies as \( \sim a^{-3/2} \). Since the amplitude of \( \dot{\phi} \) has the same time dependence and in our approximation \( \varepsilon = \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} m^2 \phi^2 \), it follows that

\[
\varepsilon \sim a^{-3}.
\]

Comparing this with the time dependence of the density \( \rho \sim a^{-3\gamma} \) we arrive at the following picture for the evolution of these universes. If \( \gamma \leq 1 \) the initial conditions (15) imply that

\[
\varepsilon = \frac{1}{2} \dot{\phi}^2 + V \ll \rho \quad \text{and} \quad 2 \dot{\phi}^2 - 2V \ll \rho + 3p,
\]

for all \( t \geq t_0 \), that is, the SEC on the total stress-energy tensor is satisfied for all \( t \geq t_0 \). Hence the universe follows the classical Friedmannian evolution and has a time of maximum expansion. Therefore, with initial conditions (15) the approximation (16) suggests that if \( \gamma \leq 1 \), the universe recollapses. The case \( \gamma = 1 \) is particularly interesting since it corresponds to a dust-filled universe which perfectly approximates our Universe. For \( \gamma > 1 \), since \( \rho \) decreases faster than the energy density \( \varepsilon \) of the scalar field, \( \varepsilon \) eventually dominates over \( \rho \). Hence this model may, or may not have a time of maximum expansion.

The above analysis may serve as an indication of the richness of possible scenarios for expanding closed FRW models with a scalar field. However, spurious conclusions may be drawn from an approximate analysis, especially if it is referred to the asymptotic behavior of the solutions. Similarly, the determination of the stability of equilibrium solutions does not reveal the complete global behavior of a dynamical system of dimension greater than two. A rigorous proof of the closed universe recollapse conjecture may come from the investigation of the global structure of the solutions of (11) with \( k = +1 \).

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