Analytical model of low-mass strange stars in $2 + 1$ space–time

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Abstract. It is quite fascinating to study the low-mass compact stars because of their enigmatic behaviour. In this paper, we have modelled this kind of low-mass strange stars based on the Heintzmann ansatz in $(2 + 1)$ dimensions. Attractive anisotropic force plays a significant role in restricting the upper mass limit (which is comparatively low) of the strange star. We have applied our model to some low-mass strange stars. Our model is useful to predict important parameters of the low-mass strange stars.

Keywords. Low mass; strange star; $(2 + 1)$ dimension.

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1. Introduction

The peculiar behaviour of some compact stars opens the door of the possibility of the low-mass neutron stars and the low-mass strange stars. One can explain the high braking index of the PSR J1640-4631 [1] and the smaller polar cap area of the PSR B0943+10 [2] by considering that these are low-mass strange stars. Though theoretically, the low-mass compact stars could be a product of a core-collapse supernova, in reality it is very unlikely that low-mass compact stars are created from the supernova. These low-mass compact stars are formed as massive white dwarf collapses due to accretion [3,4]. These compact stars could be either a neutron star or a strange star. There are a few ways to differentiate a neutron star and a strange star. The mass–radius relation of the star is one of the ways to differentiate a neutron star and a strange star. Mass of the neutron star, $M_{ns} \propto R^{-3}$, whereas the mass of the strange star, $M_{ss} \propto R^3$ [3,5–7]. A neutron star having $\sim 0.2M_{\odot}$ mass has a radius of $>15$ km, whereas a strange star with $\sim 0.2M_{\odot}$ mass has a radius of only $<5$ km. So, we see that low-mass neutron and strange stars with the same mass show a significant difference in radius whereas a neutron star and a strange star with $\sim 1M_{\odot}$ mass and above have almost the same radii [3].

Recently, lower-dimensional gravity has become valuable due to its simplicity in describing the geometry of the space–time. It has illuminated the haziness surrounding the four-dimensional gravity. Banados, Teitelboim and Zanelli (BTZ) described the $(2 + 1)$-dimensional space–time geometry with a negative cosmological constant which admits a black hole solution [8]. This work was revolutionary back then. It is easier to deal with a set of not-so-complicated equations since the system imitates the four-dimensional analysis. It is fascinating that BTZ black hole is a solution of low-energy string theory with a non-vanishing antisymmetric tensor and it resembles the exterior of a $(2 + 1)$-dimensional perfect fluid star. Keeping this in mind, Cruz and Zanelli [9] obtained the interior solution putting the upper limit for the mass regarding the generic equation of state $P = P(\rho)$. By a simple dimensional reduction, it is possible to get a $(2 + 1)$-dimensional perfect fluid solution with constant energy density which can be obtained from the Schwarzschild interior metric by comparing $(2 + 1)$ and $(3 + 1)$ gravities – it is an important interpretation given by García and Campuzano [10]. Mann and Ross [11] showed that it is possible for a $(2 + 1)$-dimensional star which is filled with dust ($\rho = 0$) to collapse to a black hole under certain conditions. There is an exact solution in $(2 + 1)$-dimensional gravity with a negative cosmological constant, for the critical collapse of a scalar field in the closed form given by Garfinkle [12].
Sá [13] also gave another solution assuming a polytropic equation of state of the form \( P = K \rho^{1+\frac{n}{2}} \), where \( K \) is the polytropic constant and \( n \) is the polytropic index. Sharma et al [14] have also taken a particular form of the mass function to study the interior of an isotropic star in (2 + 1)-dimensional gravity. On the other hand, Rahaman et al [15] and Bhar et al [16] have separately studied non-singular model for anisotropic stars based on the Krori and Barua (KB) ansatz in (2 + 1) dimensions. Some researchers [17,18] have presented a class of interior solutions corresponding to the BTZ exterior by using Finch and Skea ansatz.

The purpose of the present work is to construct a low-mass strange star model based on the Heintzmann ansatz [19] in (2 + 1) dimensions. The motivation for doing so is the curiosity of the role of anisotropy to bound the upper mass limit (which is comparatively low) of the strange star. The plan of this paper is as follows. In §2, we discuss the interior space–time of the low-mass strange stars. In §3, we look at some physical properties of the strange star. In the subsections, we discuss the matching condition with exterior BTZ solution, behaviour of energy density and anisotropic pressure, compactness, surface red-shift, energy condition and validity of the generalised TOV equation. In §4, we discuss several conditions imposed on the metric parameters and in §5, stability is explained. In §6, we apply our model on the three different compact objects. We discuss our results in §7.

2. Interior space–time

The line element which describes the interior space–time of a static spherically symmetric compact object in (2 + 1) dimensions is written as

\[
d s^2 = -e^{2\nu} dt^2 + e^{2\mu} dr^2 + r^2 d\Omega^2. \tag{1}
\]

The energy–momentum tensor for the matter distribution in the interior of the anisotropic star has the following standard form:

\[
T_{ij} = \text{diag}(\rho, p_r, p_t), \tag{2}
\]

where \( \rho, p_r \) and \( p_t \) represent the energy density, normal radial pressure and transverse pressure, respectively.

The Einstein’s equations for the metric with negative cosmological constant (\( \Lambda < 0 \)) in geometric units \((G = c = 1)\) can be written as

\[
2\pi \rho + \Lambda = e^{-2\mu} \frac{\partial \mu}{\partial r}, \quad \tag{3}
\]

\[
2\pi p_r - \Lambda = e^{-2\mu} \frac{\partial \nu}{\partial r}, \quad \tag{4}
\]

From eq. (3) we get the radial-dependent mass function (taking integration const. as unity [14]) as

\[
m(r) = \int_0^r 2\pi \rho \, d\tilde{r} = \frac{1}{2} (1 - e^{-2\mu} - \Lambda r^2). \tag{6}
\]

In 1916, Schwarzschild [20] first solved the exact solution of Einstein’s field equations; and in 1939, Oppenheimer, Volkoff and Tolman [21,22] successfully derived the balancing equations of the relativistic stellar structures from Einstein’s field equations. Since then, several scientists are trying to get a new exact solution of Einstein’s field equations for the interior region of the stars and unfolding several new aspects of nature. Some recent works on new exact solution are [23–37]. In this paper, we use Heintzmann’s exact solution in (2 + 1) dimensions to explore some of the new features of compact stars. According to Heintzmann [19]

\[
e^{2\nu} = A^2 (1 + ar^2)^3, \tag{7}
\]

\[
e^{-2\mu} = 1 - \frac{3ar^2}{2} \left[ 1 + C (1 + 4ar^2)^{-\frac{1}{2}} \right], \tag{8}
\]

where \( A \) and \( C \) are dimensionless constants and \( a \) is a constant with dimension of length\(^{-2}\) in geometric units.

Therefore, the mass function comes out as

\[
m(r) = \frac{3ar^2 [1 + C (1 + 4ar^2)^{-\frac{1}{2}}]}{4(1 + ar^2)} - \frac{\Lambda r^2}{2}. \tag{9}
\]

which is regular at the centre, i.e. \( m(r) = 0 \) at \( r = 0 \).

Solving eqs (3)–(8) we get

\[
\rho = \frac{3a[2aCr^2(1 - ar^2) + (4ar^2 + 1)^{3/2} + C]}{4\pi (ar^2 + 1)^{3/2} (4ar^2 + 1)^{3/2}} - \frac{\Lambda}{2\pi}, \tag{10}
\]

\[
p_r = \frac{3a[-ar^2[3C(4ar^2 + 1)^{-\frac{1}{2}} + 1] + 2]}{4\pi (ar^2 + 1)^{3/2}} \tag{11}
\]

\[
p_t = \frac{3a}{2\pi (ar^2 + 1)^2 (4ar^2 + 1)} \times \left[ 1 - ar^2 \left( \frac{3C(3ar^2 + 1)}{(4ar^2 + 1)^2} + (4ar^2 - 3) \right) \right] + \frac{\Lambda}{2\pi}. \tag{12}
\]

Then the central density and pressure are

\[
\rho_0 = \frac{3a(C + 1) - 2\Lambda}{4\pi}. \tag{13}
\]
The exterior space–time of the static spherically symmetric compact object is assumed to be described by the BTZ metric as follows:

\[
d s^2 = -(M_0 - \Lambda r^2) dr^2 + \frac{dr^2}{M_0 - \Lambda r^2} + r^2 d\Omega^2.
\]

The parameter \(M_0\) is the conserved charge associated with asymptotic invariance under time displacements. The continuity of \(g_{tt}\) and \(g_{rr}\) at the surface \((r = R)\) and vanishing of the normal pressure at the surface yield \(e^{2\nu(R)} = -M_0 - \Lambda R^2\), \(e^{-2\mu(R)} = -M_0 - \Lambda R^2\), \(0 = \frac{\Lambda}{2\pi} + \frac{\nu'(R)e^{-2\mu(R)}}{2\pi R}\). Solving eqs (16)–(18), we get

\[
A = \frac{1}{\sqrt{3}} \sqrt{\left(3M_0 + 2\Lambda R^2\right)^3 \left(M_0 + \Lambda R^2\right)^2},
\]

\[
a = \frac{\Lambda}{3M_0 + 2\Lambda R^2},
\]

\[
C = \frac{\sqrt{3}}{\Lambda R^2} \sqrt{\frac{M_0 + 2\Lambda R^2}{3M_0 + 2\Lambda R^2}} \times [M_0(4\Lambda R^2 + 2) + 2M_0^2 + \Lambda R^2(2\Lambda R^2 + 1)].
\]

The total mass of the compact object of radius \(R\) is given by

\[
M(R) = \frac{1}{2}(1 - e^{-2\mu(R)} - \Lambda R^2) = \frac{1}{2}(1 + M_0).
\]

### 3. Some physical properties

#### 3.1 Matching condition

For a physically acceptable model, both energy density and radial pressure should be monotonically decreasing functions in \(r\) and should be maximum at the centre. For our model, we have

\[
\frac{dp}{dr} = -\frac{3a^2r}{\pi (ar^2 + 1)^3 (4ar^2 + 1)^{3/2}}
\]

\[
dp_r = -\frac{3a^2r}{2\pi (ar^2 + 1)^3 (4ar^2 + 1)^{3/2}} \times [C(-6a^3r^6 + 12a^2r^4 + 12ar^2 + 3) + (4ar^2 + 1)^{5/2}] < 0,
\]

\[
\frac{dp_r}{dr} = -\frac{3a^2r}{2\pi (ar^2 + 1)^3 (4ar^2 + 1)^{3/2}} \times [3C(6a^2r^4 - ar^2 - 1) + \sqrt{4ar^2 + 1 (4a^2r^4 - 19ar^2 - 5)}] < 0.
\]

At the centre \((r = 0)\),

\[
\frac{dp}{dr} \bigg|_{r=0} = \frac{dp_r}{dr} \bigg|_{r=0} = 0
\]

and

\[
\frac{d^2p}{dr^2} \bigg|_{r=0} = -\frac{3a^2(3C + 5)}{2\pi} < 0.
\]

Equations (25)–(27) imply that central energy density and central pressure are maximum at the centre for any positive value of \(a\) and \(C\).

The measure of anisotropy for our model is given by the expression

\[
\Delta = p_t - p_r = \frac{3a^2r^2[C(6ar^2 + 3) + (4ar^2 + 1)^{3/2}]}{4\pi (ar^2 + 1)^2 (4ar^2 + 1)^{3/2}}.
\]

Based on the sign of the anisotropy parameter, the anisotropic force can be categorised into two: (i) the repulsive anisotropic force when \(p_t > p_r\), i.e., \(\Delta > 0\) and (ii) the attractive anisotropic force when \(p_t < p_r\), i.e., \(\Delta < 0\) [38]. This repulsiveness in anisotropic force enhances the stability of the star resulting in the star to be more compact than the isotropic one [39–41].

#### 3.2 Behaviour of energy density and pressure

The energy conditions such as null energy condition (NEC), weak energy condition (WEC), strong energy condition (SEC), dominant energy condition (DEC) and trace energy condition (TEC) [42,43] should be satisfied at every point in the interior of the compact star simultaneously. These energy conditions are as follows:

(i) NEC: \(\rho + p_t \geq 0\);
(ii) WEC: \(\rho + p_t \geq 0, \rho \geq 0\);
(iii) SEC: \(\rho + p_t \geq 0, \rho + p_r + 2p_t \geq 0\);
(iv) DEC: \(\rho > |p_t|\);
(v) TEC: \(\rho - p_r - 2p_t \geq 0\).
Figure 1. Graphical presentation of the accessible mass–radius region of our model.

Figure 2. The variation of the energy density with radial coordinate for the pulsar PSR B0943 + 10 (1st row for Λ = -0.001 km⁻², 2nd row for Λ = -0.005 km⁻² and 3rd row for Λ = -0.01 km⁻²).
3.4 Compactness and surface red-shift

The compactness of the star is given by

\[ u = \frac{m(r)}{r} = \frac{3ar[1 + C(1 + 4ar^2)^{-1}]}{4(1 + ar^2)} - \frac{r\Lambda}{2}. \]  

The corresponding red-shift is given by the expression

\[ Z_s = \frac{1}{\sqrt{1 - 2u}} - 1 = \frac{1}{\sqrt{1 + r\Lambda - \frac{3a[1 + C(1 + 4ar^2)^{-1}]}{2(1 + ar^2)}}} - 1. \]  

According to Buchdahl [44], the maximum value of \( u(r) \), i.e. \( \frac{m(r)}{r} \) max is \( \frac{4}{3} \) in \((3 + 1)\) dimensions. The maximum allowed value of the surface red-shift [45] is \( Z_s \leq 0.85 \) in \((3 + 1)\) dimensions. Cruz and Zanelli [46] have shown that Buchdahl’s theorem [44] also holds in \((2 + 1)\) dimensions. Therefore, we will use the same upper limit of the ratio of the mass to radius, i.e. \( \frac{m(r)}{r} \leq \frac{1}{3} \) and the surface red-shift \( Z_s \leq 0.85 \) in \((2 + 1)\) dimensions as we used in the case of \((3 + 1)\) dimensions.

3.5 Generalised TOV equation

The generalised TOV equation for the anisotropic system is written as

\[ \frac{d}{dr} \left( \frac{p_r - \Lambda}{2\pi} + \nu' (\rho + p_r) + \frac{1}{r} (p_r - p_t) \right) = 0. \]  

This equation represents the equilibrium condition of the system under gravitational force \( (F_g) \), hydrostatic force \( (F_h) \) and anisotropic force \( (F_a) \) as

\[ F_h + F_g + F_a = 0. \]
Figure 5. The variation of the compactness with radial coordinate for the pulsar PSR B0943 + 10 (1st row for $\Lambda = -0.001 \text{ km}^{-2}$, 2nd row for $\Lambda = -0.005 \text{ km}^{-2}$ and 3rd row for $\Lambda = -0.01 \text{ km}^{-2}$).

Figure 6. The variation of mass function for the pulsar PSR B0943 + 10.

where

$$F_g = -\nu' (\rho + p_r), \quad F_a = \frac{1}{r}(p_t - p_r)$$

and

$$F_h = -\frac{d}{dr} \left( p_r - \frac{\Lambda}{2\pi} \right).$$

The forms of the gravitational force ($F_g$), hydrostatic force ($F_h$) and anisotropic force ($F_a$) for the Heintzmann line element are given as follows:

$$F_g = \frac{9a^2r}{4\pi (ar^2 + 1)^3 (4ar^2 + 1)^{3/2}} \times \left[ C (14a^2r^4 + ar^2 - 1) + (4ar^2 + 1)^{1/2} (4a^2r^4 - 11ar^2 - 3) \right], \quad (33)$$

$$F_h = \frac{3a^2r}{2\pi (ar^2 + 1)^3 (4ar^2 + 1)^{3/2}} \times \left[ C(-18a^2r^4 + 3ar^2 + 3) + (4ar^2 + 1)^{1/2} (-4a^2r^4 + 19ar^2 + 5) \right], \quad (34)$$

$$F_a = -\frac{3a^2r [C(6ar^2 + 3) + (4ar^2 + 1)^{3/2}]}{4\pi (ar^2 + 1)^2 (4ar^2 + 1)^{3/2}}. \quad (35)$$
Figure 7. The variation of the red-shift with radial coordinate for the pulsar PSR B0943 + 10 (1st row for $\Lambda = -0.001 \text{ km}^{-2}$, 2nd row for $\Lambda = -0.005 \text{ km}^{-2}$ and 3rd row for $\Lambda = -0.01 \text{ km}^{-2}$).

Figure 8. Energy conditions at the interior of the star for the pulsar PSR B0943 + 10 (1st row for $\Lambda = -0.001 \text{ km}^{-2}$, 2nd row for $\Lambda = -0.005 \text{ km}^{-2}$ and 3rd row for $\Lambda = -0.01 \text{ km}^{-2}$).
4. Conditions on the metric parameters

Equations (25)–(27) indicate that both energy density and pressure are monotonically decreasing functions and get maximised at the centre for any values of \( a \) and \( C \) in a positive domain. The central pressure and \( \Lambda \) will be positive and real only when satisfying the conditions

\[
-3 < 3M_0 + 2\Lambda R^2 < 0. \tag{36}
\]

The positivity of \( C \) implies that

\[
M_0(4\Lambda R^2 + 2) + 2M_0^2 + \Lambda R^2(2\Lambda R^2 + 1) < 0. \tag{37}
\]

The dominating energy condition at the centre yields

\[
3\sqrt{3} \left( \frac{M_0 + 2\Lambda R^2}{3M_0 + 2\Lambda R^2} \right) \times [M_0(4\Lambda R^2 + 2) + 2M_0^2 + \Lambda R^2(2\Lambda R^2 + 1)] - 3\Lambda R^2 - 4\Lambda R^2(3M_0 + 2\Lambda R^2)\Lambda > 0. \tag{38}
\]

The strong energy condition at the surface implies that

\[
- \frac{\Lambda M_0 R^2 - M_0^2 - M_0 + 4\Lambda^2 R^4 + \Lambda R^2}{2\pi M_0 R^2 + 4\pi \Lambda R^4} \geq 0. \tag{39}
\]

Tables 1 and 2 present the analytical and numerical forms of solutions of eqs (36)–(39) along with Buchdahl condition [44] \( M < \frac{4}{9} R \), respectively.

5. Stability

In order to keep their model stable, one needs to keep the speed of sound in the interior region within the range

\[
0 \geq v^2 = \frac{dp_r}{d\rho} \leq 1. \tag{40}
\]

The adiabatic index for classical isotropic matter distribution (non-relativistic), should be \( \gamma > \frac{4}{3} \). But, for the relativistic fluid sphere, the minimum value of the adiabatic index gets some correction terms in addition with \( \frac{4}{3} \) which is written as

\[
\gamma = \frac{p_r + \rho \frac{dp_r}{d\rho}}{\frac{3}{p_r} \rho \frac{dp_r}{d\rho}} > \frac{4}{3} + \left( \frac{\kappa \rho p_r}{3 |p_r'| r} + \frac{3}{4} |p_r'| r \right)_{\text{max}}, \tag{41}
\]

where the second term on the right-hand side has its maximal value. This additional relativistic correction term could make the inside of a star unstable [47,48]. To avoid this situation, Moustakidis [49] proposed adiabatic index with a more strict condition. They proposed that the adiabatic index of a stable star must be greater than some existing critical value \( \gamma_{\text{crit}} \). The critical value
of the adiabatic index is given by

$$\gamma_{\text{crit}} = \frac{4}{3} + \frac{19}{21} \mu. \quad (42)$$

When we apply the stability conditions (eqs (40) and (42), we find that all the possible mass–radius region (see graph-1) is not stable. We show the stable mass–radius region in figure 10. Figure 11 shows that stability shortens the possible radius of the star for a given mass and cosmological constant. Though the causality breaks down for higher radius, within the star the adiabatic index (see figure 12) is always greater than $\gamma_{\text{crit}}$ [50,51].

6. Example

6.1 PSR B0943 + 10

Yue et al [2] showed that the small polar cap of PSR B0943 + 10 could be explained using Ruderman–Sutherland-type vacuum gap model if the pulsar has the mass and radii of about $0.02M_\odot$ and 2.6 km, respectively. We use our model for this pulsar and find out the useful parameters. If we use the mass of this pulsar as the only input parameter, then for $\Lambda = -0.001$, $\Lambda = -0.005$ and $\Lambda = -0.01$, our model predicts that the stable radii of this pulsar is in the range $0.066447 < R \leq 5.36149$, $0.066447 < R \leq 2.39773$ and $0.066447 < R \leq 1.69545$, respectively. We enlist some parameters of this pulsar calculated using our model in table 3. One can observe (figures 2 and 3) that matter–energy density, radial pressure and transverse pressure are maximum at the centre and decrease monotonically towards the boundary. Also, one can see that the radial pressure drops to zero at the boundary, while density does not. And although the energy density monotonically decreases, its value remains very high throughout the stellar system. Therefore, it may be justified to take these compact stars as strange stars where the surface density remains finite rather than the neutron stars for which the surface density vanishes at the boundary [5,52–56]. Figure 4 suggests that the anisotropic force is attractive for our model. The attractiveness of the anisotropic force disallows the formation of massive compact star [38]. In other words, it allows the formation of the low-mass compact star (strange star for our
Figure 11. The behaviour of sound velocity ($v_r$) within the star (1st row for $\Lambda = -0.001 \text{ km}^{-2}$, 2nd row for $\Lambda = -0.005 \text{ km}^{-2}$ and 3rd row for $\Lambda = -0.01 \text{ km}^{-2}$).

Figure 12. The behaviour of adiabatic index ($\gamma$) within the star (1st row for $\Lambda = -0.001 \text{ km}^{-2}$, 2nd row for $\Lambda = -0.005 \text{ km}^{-2}$ and 3rd row for $\Lambda = -0.01 \text{ km}^{-2}$).
case). Figure 8 ensures that all the energy conditions are satisfied in our model. It is apparent from figure 5 that the compactness, $u(r) = \frac{m(r)}{r}$ is an increasing function of the radial parameter and has maximum value below $\frac{4}{9}$ for all the six cases. According to Buchdahl [44], the maximum value of $u(r)$, i.e. $(m(r)/r)_{\text{max}}$ is $\frac{4}{9}$ in $(3+1)$ dimensions. Figure 7 shows that the value of surface gravitational red-shift has much less value than the maximum allowed value ($Z_{\text{r}} \leq 0.85$) [45] in $(3+1)$ dimensions. Figure 9 shows that the gravitational force ($F_g$), the hydrostatic force ($F_h$) and the anisotropic force ($F_a$) are in equilibrium in the interior region of the star.

### 6.2 PSR J1640-4631

The X-ray Pulsar PSR J1640-4631 high breaking index was explained by Chen [1] by considering the pulsar as a low-mass neutron star having mass about $0.1 M_\odot$. Chen [1] showed that the radius of this pulsar is about 29 km using the formula $R \propto M^{-1/3}$. Using our model, we obtained possible stable radius ranges for this pulsar: $0.332235 < R \leq 1.12541$, $0.332235 < R \leq 5.03209$ and $0.332235 < R \leq 3.55886$ for $\Lambda = -0.001$, $\Lambda = -0.005$ and $\Lambda = -0.01$, respectively. In this paper, we

### Table 1. List of the possible range of the radius and mass of the star and cosmological constant where $M_2$, $M_3$, $R_2$ and $R_6$ are the real positive roots of $M_2(x)$, $M_3(x)$, $R_2(x)$ and $R_6(x)$, respectively ($M_2(x)$, $M_3(x)$, $R_2(x)$, $R_6(x)$, $R_1$, $R_3$, $R_4$, $R_5$ and $R_7$ are given in Appendix A).

| Cosmological constant ($\Lambda$) | Mass ($M$) (km) | Radius ($R$) (km) |
|----------------------------------|----------------|------------------|
| $\Lambda \leq -0.395062$       | $0 < M < M_1$ | $2.25 M < R < R_1$ |
| $-0.395062 < \Lambda \leq -0.332447$ | $0 < M < 0.5$ | $2.25 M < R < R_1$ |
| $-0.332447 < \Lambda \leq -0.221879$ | $0.5 \leq M < M_1$ | $2.25 M < R < R_1$ |
| $-0.221879 < \Lambda < -0.197531$ | $0 < M < 0.5$ | $2.25 M < R < R_1$ |
| $\Lambda = -0.197531$ | $M = 0.5$ | $1.125 < R < 1.59099$ |
| $0.5 \leq M < 0.530149$ | $R_4 < R < R_5$ |
| $M = 0.530149$ | $1.26422 < R < 1.57526$ |
| $0.530149 < M < 0.536438$ | $R_6 < R < R_5$ |
| $\Lambda < -0.197531$ | $0 < M < M_3$ | $2.25 M < R < R_1$ |
| $M_2 < M < 0.5$ | $R_3 < R < R_1$ |
| $M = 0.530149$ | $0.561874 < R < 0.700116$ | $R_2 < R < R_1$ |
| $0.530149 < M < 0.536438$ | $\frac{1}{\sqrt{|\Lambda|}} R \leq R < \frac{1}{2 \sqrt{|\Lambda|}}$ | $R_2 < R < R_1$ |
| $\Lambda < -0.197531$ | $0 < M \leq 0.470588$ | $2.25 M < R < R_1$ |
| $0.470588 < M < M_3$ | $2.25 M < R \leq R_1$ and $R_3 \leq R < R_1$ |
| $M_3 < M \leq 0.5$ | $R_3 < R < R_1$ |
| $M = 0.5$ | $1.26422 < R < 1.57526$ |
| $0.5 < M < 0.530149$ | $R_6 < R < R_5$ |
| $M = 0.530149$ | $0.561874 < R < 0.700116$ | $R_2 < R < R_1$ |
| $0.530149 < M < 0.536438$ | $\frac{1}{\sqrt{|\Lambda|}} R \leq R < \frac{1}{2 \sqrt{|\Lambda|}}$ | $R_2 < R < R_1$ |
estimate some of the parameters of this pulsar using our model presented in table 4. All the graphs related to this pulsar are similar to the pulsar PSR B0943+10 (we do not present graphs of this pulsar in the present paper).

### 6.3 1E 1207.4-5209

Xu [3] showed that the radio-quiet object 1E 1207.4-5209 could be a low-mass bare strange star with mass and radii of $10^{-3} M_\odot$ and 1 km, respectively. The stable radius ranges predicted by our model are $0.00332235 < R < 1.21436$, $0.00332235 < R < 0.543077$, and $0.00332235 < R < 0.384013$ for $\Lambda = -0.001$, $\Lambda = -0.005$ and $\Lambda = -0.01$, respectively. Table 5 presents some parameters of this pulsar calculated using our model. Graphs related to this pulsar are similar to the pulsar PSR J1640-4631 (we do not show graphs of this pulsar in this paper).

### 7. Discussion and conclusions

In this paper, we have presented a new model of the anisotropic low-mass strange stars based on the Heintzmann ansatz in $(2 + 1)$ dimensions. Heintzmann’s solution is an exact and complete solution of Einstein’s equation which can be used to investigate the whole space-time. This solution has a boundary at a finite radius. Thus, one can match the Heintzmann solution onto the exterior solution at the boundary. Delgaty and Lake [57] showed that this Heintzmann metric ticks all conditions of a physically acceptable solution. Many researchers used this solution for star modelling in $(3+1)$ dimensions. Through this article, we have tried to investigate strange star behaviour using the Heintzmann metric in $(2 + 1)$ dimensions.

The attractive anisotropic force plays a significant role to bound the upper mass limit (which is comparatively

| $\Lambda$ | $M$ (km) | Possible radius ($R$) (km) | Stable radius ($R$) (km) |
| --- | --- | --- | --- |
| $-0.001$ | 0.001 | $0.00225 < R < 1.15457$ | $0.00225 < R < 0.999555$ |
| | 0.005 | $0.01125 < R < 2.58055$ | $0.01125 < R < 2.23108$ |
| | 0.01 | $0.0225 < R < 3.64739$ | $0.0225 < R < 3.14812$ |
| | 0.05 | $0.1125 < R < 8.11733$ | $0.1125 < R < 6.90814$ |
| | 0.1 | $0.225 < R < 11.405$ | $0.225 < R < 9.52047$ |
| | 0.5 | $15.8114 < R < 22.3607$ | Unstable |
| | 0.53 | $17.7598 < R < 22.1422$ | Unstable |
| $-0.005$ | 0.001 | $0.00225 < R < 0.51634$ | $0.00225 < R < 0.447015$ |
| | 0.005 | $0.01125 < R < 1.15406$ | $0.01125 < R < 0.99777$ |
| | 0.01 | $0.0225 < R < 1.63116$ | $0.0225 < R < 1.40788$ |
| | 0.05 | $0.1125 < R < 3.63018$ | $0.1125 < R < 3.08941$ |
| | 0.1 | $0.225 < R < 5.10046$ | $0.225 < R < 4.25769$ |
| | 0.5 | $7.07107 < R < 10.0$ | Unstable |
| | 0.53 | $7.94242 < R < 9.9023$ | Unstable |
| $-0.01$ | 0.001 | $0.00225 < R < 0.365108$ | $0.00225 < R < 0.316087$ |
| | 0.005 | $0.01125 < R < 0.816041$ | $0.01125 < R < 0.70553$ |
| | 0.01 | $0.0225 < R < 1.15341$ | $0.0225 < R < 0.995524$ |
| | 0.05 | $0.1125 < R < 2.56693$ | $0.1125 < R < 2.18455$ |
| | 0.1 | $0.225 < R < 3.60657$ | $0.225 < R < 3.01064$ |
| | 0.5 | $5.0 < R < 7.07107$ | Unstable |
| | 0.53 | $5.61614 < R < 7.00198$ | Unstable |
| $-0.1$ | 0.001 | $0.00225 < R < 0.115457$ | $0.00225 < R < 0.0999555$ |
| | 0.005 | $0.01125 < R < 0.258055$ | $0.01125 < R < 0.223108$ |
| | 0.01 | $0.0225 < R < 0.364739$ | $0.0225 < R < 0.314812$ |
| | 0.05 | $0.1125 < R < 0.811733$ | $0.1125 < R < 0.690814$ |
| | 0.1 | $0.225 < R < 1.1405$ | $0.225 < R < 0.952047$ |
| | 0.5 | $1.58114 < R < 2.23607$ | Unstable |
| | 0.53 | $1.77598 < R < 2.21422$ | Unstable |
Table 3. Value of some parameters of the pulsar PSR B0943 +10 estimated using our model.

| Cosmological constant (\(\Lambda\)) | Radius (\(R\)) (km) | Central energy density (\(\rho_0\)) (\(10^{14}\) g/cc) | Surface energy density (\(\rho R\)) (\(10^{14}\) g/cc) | Central pressure (\(p_0\)) (\(10^{3}\) dyne/cm\(^2\)) | Compactness (\(u\)) | Surface red-shift (\(Z_s\)) | A | a | c |
|-------------------------------------|----------------------|------------------------------------------------|------------------------------------------------|---------------------------------|-----------------|-----------------|---|---|---|
| -0.001                             | 1                    | 126.757                                           | 126.495                                         | 11.9498                         | 0.029532        | 0.0309081       | 0.970019       | 0.00035401       | 108.462       |
|                                     | 2.6                  | 18.8449                                           | 18.6192                                         | 11.1187                         | 0.011358        | 0.0115557       | 0.970027       | 0.00035257       | 13.7302       |
|                                     | 5.36                 | 4.45319                                           | 4.36293                                         | 8.00968                         | 0.0055097       | 0.00555566      | 0.970164       | 0.00034719       | 1.06836       |
| -0.005                             | 0.7                  | 259.054                                           | 257.788                                         | 58.6996                         | 0.042189        | 0.0450611       | 0.970020       | 0.00176822       | 42.6728       |
|                                     | 1.5                  | 56.7741                                           | 55.7879                                         | 52.3773                         | 0.019688        | 0.0202892       | 0.970041       | 0.00175728       | 7.14971       |
|                                     | 2.39                 | 22.399                                            | 21.9426                                         | 40.1662                         | 0.0123565       | 0.0125903       | 0.970163       | 0.00173615       | 1.09194       |
| -0.01                              | 0.5                  | 507.770                                           | 505.241                                         | 117.327                         | 0.059064        | 0.0648715       | 0.970020       | 0.00353631       | 41.7653       |
|                                     | 1                    | 127.657                                           | 125.606                                         | 106.541                         | 0.029532        | 0.0309081       | 0.970037       | 0.00351765       | 8.38975       |
|                                     | 1.69                 | 44.7972                                           | 43.8845                                         | 80.3318                         | 0.0174746       | 0.0179464       | 0.970163       | 0.00347231       | 1.09187       |

Table 4. Value of some parameters of the pulsar PSR J1640-4631 estimated using our model.

| Cosmological constant (\(\Lambda\)) | Radius (\(R\)) (km) | Central energy density (\(\rho_0\)) (\(10^{14}\) g/cc) | Surface energy density (\(\rho R\)) (\(10^{14}\) g/cc) | Central pressure (\(p_0\)) (\(10^{3}\) dyne/cm\(^2\)) | Compactness (\(u\)) | Surface red-shift (\(Z_s\)) | A | a | c |
|-------------------------------------|----------------------|------------------------------------------------|------------------------------------------------|---------------------------------|-----------------|-----------------|---|---|---|
| -0.001                             | 1                    | 634.023                                           | 632.238                                         | 80.4903                         | 0.147660        | 0.1912530       | 0.839452       | 0.00047258       | 414.783       |
|                                     | 7                    | 13.5744                                           | 12.3067                                         | 68.6351                         | 0.021094        | 0.0217861       | 0.840800       | 0.00045207       | 6.86262       |
|                                     | 11.25                | 5.32354                                           | 4.7192                                          | 51.5106                         | 0.013125        | 0.0133895       | 0.843205       | 0.00042244       | 1.34055       |
| -0.005                             | 1                    | 637.489                                           | 628.800                                         | 397.307                         | 0.147660        | 0.1912530       | 0.839459       | 0.00023540       | 81.7961       |
|                                     | 3                    | 73.7089                                           | 67.1735                                         | 347.918                         | 0.049220        | 0.0531802       | 0.839985       | 0.00226856       | 7.63430       |
|                                     | 5                    | 26.961                                            | 23.8806                                         | 259.167                         | 0.029532        | 0.0309081       | 0.843121       | 0.00211502       | 1.38794       |
| -0.01                              | 0.5                  | 2541.37                                           | 2523.70                                         | 801.036                         | 0.295320        | 0.5629580       | 0.839454       | 0.00471912       | 165.049       |
|                                     | 2                    | 165.243                                           | 151.671                                         | 707.789                         | 0.073830        | 0.0831624       | 0.839876       | 0.00455780       | 8.81130       |
|                                     | 3.55                 | 53.4698                                           | 47.3861                                         | 516.216                         | 0.0415944       | 0.0443836       | 0.843176       | 0.00422638       | 1.35674       |
Table 5. Value of some parameters of the pulsar 1E 1207.45209 estimated using our model.

| Parameter               | Value |
|-------------------------|-------|
| Cosmological constant   | 0.01  |
| Surface red-shift       | 0.386 |
| Compactness             | 0.538 |
| Surface energy density  | 0.384 |
| Energy density          | 0.384 |
| Pressure                | 0.384 |
| LMGR                    | 0.384 |

In this paper, we find the EoS having the form $p_r = \alpha \rho + \beta$, where $\alpha$ is a dimensionless constant and $\beta$ having a dimension of km$^{-2}$ is also a constant. Figure 13
Figure 13. The relation between radial pressure ($p_r$) and energy density ($\rho$) (EoS) in the stellar interior region (taking $M = 0.02M_\odot$, $R = 6.25$ km and $\Lambda = -0.001$ km$^{-2}$).

shows the relation between stellar interior radial pressure and energy density. Figure 13 indicates that EoS found in our model is on the softer side.

It is to be mentioned here that we are comparing our results with the data from a (3 + 1)-dimensional object, though for an observer in the $\theta = \pi/2$ or const. plane, the measured data will be approximately the same for both dimensions.

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Appendix A

\[ M_1 = -\frac{64}{243\Lambda} \]
\[ + \frac{4(729\Lambda+128)}{243\left(54\sqrt{-98304\Lambda^7-4374\Lambda^8(243\Lambda+128)+4096\Lambda^3}\right)^{\frac{1}{3}}} \]
\[ + \frac{2\left(54\sqrt{-98304\Lambda^7-4374\Lambda^8(243\Lambda+128)+4096\Lambda^3}\right)^{\frac{1}{3}}}{243\Lambda^2}, \]
\[ R_1 = \frac{1}{2}\sqrt{\frac{3}\Lambda + \frac{\sqrt{\Lambda^2(9-16M)}}{\Lambda^2}} - \frac{8M}{\Lambda}, \]  \hspace{1cm} (A2)
\[ R_3 = \frac{1}{2}\sqrt{\frac{M}{\Lambda} - \frac{\sqrt{M(17M-8)}}{\Lambda}}, \]  \hspace{1cm} (A3)
\[ R_4 = \frac{9}{8}\sqrt{M + \sqrt{M(17M-8)}}, \]  \hspace{1cm} (A4)
\[ R_5 = \frac{9}{8}\sqrt{8M + \sqrt{9-16M} - 3}, \]  \hspace{1cm} (A5)
\[ R_7 = \frac{1}{2}\sqrt{\frac{\sqrt{M(17M-8)} - M}{\Lambda}}, \]  \hspace{1cm} (A6)
\[ M_3(x) = 128 - 256x + 648\Lambda x^2 + 6561\Lambda^2 x^3 \]  \hspace{1cm} (A7)
\[ R_2(x) = 22\Lambda^5 x^{10} + (198\Lambda^4 M - 69\Lambda^4) x^8 \]
\[ + (54\Lambda^3 + 864\Lambda^3 M^2 - 522\Lambda^3 M^3) x^6 \]
\[ + (2160\Lambda^2 M^3 - 1944\Lambda^2 M^2 + 432\Lambda^2 M) x^4 \]
\[ + (2592\Lambda^4 M^4 - 3240\Lambda^3 M^3 + 1296\Lambda M^2 - 162\Lambda M) x^2 \]
\[ + 864M^5 - 1296M^4 + 648M^3 - 108M^2, \]  \hspace{1cm} (A8)
\[ M_2(x) = 1420541793\Lambda^5 x^8 + 2525407632\Lambda^4 x^7 \]
\[ + (217678236\Lambda^3 - 880066296\Lambda^4) x^6 \]
\[ + (107495420\Lambda^2 - 131513928\Lambda^3) x^5 \]
\[ + (136048896\Lambda^3 - 967458816\Lambda^2 + 254803968\Lambda) x^4 \]
\[ + (214990848\Lambda^2 - 318504960\Lambda + 16777216) x^3 \]
\[ + (127401984\Lambda - 25165824) x^2 \]
\[ + (12582912 - 15925248\Lambda) x - 2097152, \]  \hspace{1cm} (A9)
\[ R_6(x) = 94143178827M^2 - 564859072962M^3 \]
\[ + 112971845924M^4 - 753145430616M^5 \]
\[ + (-27894275208M + 223154201664M^2) \]
\[ - 55788504160M^3 + 446308403328M^4) x^2 \]
\[ - x^4(14693280768M - 66119763456M^2 \]
\[ + 73466403840M^3) + (362797056 - 3507038208M \]
\[ + 5804752896M^2) x^6 + (91570176 - 262766592M) x^8 \]
\[ + 5767168x^{10}. \]  \hspace{1cm} (A10)

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