Homological Mirror Symmetry for higher dimensional pairs-of-pants

Sasha Polishchuk

January 28, 2020
This is joint work with Yanki Lekili

The goal is to prove the equivalence of the wrapped Fukaya category of $n$ dimensional pairs-of-pants with the derived category of coherent sheaves on $x_1x_2\ldots x_{n+1} = 0$.

Inspired by Auroux’s calculation of the partially wrapped Fukaya category of the symmetric powers of punctured surfaces.

Main idea: introduce stops to simplify the endomorphism algebra of the set of generators. Identify corresponding nc resolution of $x_1x_2\ldots x_{n+1} = 0$ on the B-side.

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Pair-of-pants

Let $\Sigma$ be the 3-punctured sphere with the set of two stops $\Lambda$.

The partially wrapped Fukaya category $\mathcal{W}(\Sigma, \Lambda)$ is generated by the Lagrangians $L_0, L_1, L_2$. 

**FIGURE 1. Pair-of-pants**
There exists a unique grading structure (given by the line field on $\Sigma$) such that the endomorphism algebra is concentrated in degree 0.

**FIGURE 2.** Endomorphism algebra of a generating set
Endomorphism algebra

There exists a unique grading structure (given by the line field on $\Sigma$) such that the endomorphism algebra is concentrated in degree 0.
Auslander order

On the B-side, we consider the algebra of the node

\[ R = k[x_1, x_2]/(x_1 x_2). \]

The Auslander order is given by

\[ A = \text{End}_R(R/(x_1) \oplus R/(x_2) \oplus R). \]

It is easy to see that \( A \) is isomorphic to the algebra associated with the above quiver with relations, so we get an equivalence

\[ \text{Perf}(A) \simeq \mathcal{W}(\Sigma, \Lambda). \]

In general, the Auslander order of a nodal curve \( C \) is \( \mathcal{E}nd(I \oplus \mathcal{O}_C) \), where \( I \) is the ideal sheaf of the nodes. The above equivalence generalizes to Auslander orders over nodal chains and rings.
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Localization

On the A-side, removing the stops corresponds to taking the quotient by the subcategory generated by the objects $T_1, T_2$ supported near the stops.

We can express them in terms of $L_0, L_1, L_2$ as follows:

$$T_1 \sim \{ L_0 \xrightarrow{u_1} L_1 \xrightarrow{u_2} L_2 \}$$

$$T_2 \sim \{ L_2 \xrightarrow{v_2} L_1 \xrightarrow{v_1} L_0 \}$$

Can identify corresponding objects on the B-side: we get simple modules at vertices $L_0$ and $L_2$ of the quiver. As a corollary, get an equivalence

$$\mathcal{W}(\Sigma) \sim D^b(R).$$
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$$\mathcal{V}(\Sigma) \cong D^b(R).$$
Consider $\Pi_n$, the complement to $n + 2$ generic hyperplanes in $\mathbb{P}^n$, as an exact symplectic manifold. Since $\mathbb{P}^n = \text{Sym}^n(\mathbb{P}^1)$, have an identification

$$\Pi_n = \text{Sym}^n(\mathbb{P}^1 \setminus \{p_0, p_1, \ldots, p_{n+1}\}).$$

More generally, we consider

$$M_{n,k} = \text{Sym}^n(\Sigma_k), \text{ where } \Sigma_k = \mathbb{P}^1 \setminus \{p_0, p_1, \ldots, p_k\}$$

(for $k \geq n$). Away from a small neighborhood of the diagonal, the symplectic form can be arranged to be induced by one on the surface.

We fix two points $q_1, q_2$ on one of the boundary components of the punctured sphere, and consider the induced hypersurfaces $\Lambda_i = q_i \times \text{Sym}^{n-1}(\Sigma_k)$. We will use either $\Lambda_1$ or $\Lambda = \Lambda_1 \cup \Lambda_2$ as stops.
Symmetric products

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Generating Lagrangians

We start with the same collection of Lagrangians on $\Sigma_k$ as before:

By Auroux’s theorem, the products $L_S := L_{i_1} \times \cdots \times L_{i_n}$, for $S = \{i_1 < \ldots < i_n\} \subset [0, k]$, generate $\mathcal{W}(M_{n,k}, \Lambda)$. 
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Computation on the A-side. I

We can compute (cohomology of) morphism spaces between generating objects in \( \mathcal{W}(M_{n,k}, \Lambda) \).

For every proper subinterval \([i, j] \subset [0, k]\), set

\[
\mathcal{A}_{[i,j]} = \begin{cases} 
\mathbf{k}[x_i, \ldots, x_{j+1}]/(x_i \ldots x_{j+1}) & \text{if } i > 0, j < k, \\
\mathbf{k}[x_1, \ldots, x_{j+1}] & \text{if } i = 0, j < k, \\
\mathbf{k}[x_i, \ldots, x_k] & \text{if } i > 0, j = k,
\end{cases}
\]

**Proposition.** For \( S = [i_1, j_1] \sqcup [i_2, j_2] \sqcup \ldots \sqcup [i_r, j_r] \) with \( j_s + 1 < i_{s+1} \), one has

\[
\text{End}(L_S) \simeq \mathcal{A}(S, S) := \mathcal{A}_{[i_1, j_1]} \otimes \mathcal{A}_{[i_2, j_2]} \otimes \ldots \otimes \mathcal{A}_{[i_r, j_r]},
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Computation on the A-side. II

The subsets $S, S' \subset [0, k]$ are called close if there exists a bijection $g : S \rightarrow S'$ with $g(i) \in \{i - 1, i, i + 1\}$. In this case there exists a decomposition

$$S = S_0 \sqcup \bigsqcup_a I_a \sqcup \bigsqcup_b J_b,$$

where $I_a$ and $J_b$ are subintervals, such that

$$S' = S_0 \sqcup \bigsqcup_a (I_a + 1) \sqcup \bigsqcup_b (J_b - 1).$$

**Proposition.** One has

$$\text{Hom}(L_S, L_{S'}) \simeq \begin{cases} 0, & S, S' \text{ not close}, \\ \mathcal{A}(S_0, S_0) \otimes \bigotimes_a \mathcal{A}'_{I_a} \otimes \bigotimes_b \mathcal{A}'_{J_b}, & S, S' \text{ close}, \end{cases}$$

where $\mathcal{A}'_{[i, j]} = k[x_{i+1}, \ldots, x_j]$. 
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where $\mathcal{A}'_{[i,j]} = k[x_{i+1}, \ldots, x_j]$. 
Can compute compositions as well.
Example: \( n = 2, \ k = 3 \) (Sym\(^2\) of 4-punctured sphere).
Get quiver with relations over \( R = \mathbb{k}[x_1, x_2, x_3]/(x_1 x_2 x_3) \)

Relations:
\[
\begin{align*}
    u_i v_i &= x_i = v_i u_i, \\
    u_3 u_2 &= v_2 v_3 = u_2 u_1 = v_1 v_2 = 0 \\
    u_3 u_1 &= u_1 u_3, \\
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Relations:

$$u_i v_i = x_i = v_i u_i, \quad u_3 u_2 = v_2 v_3 = u_2 u_1 = v_1 v_2 = 0$$

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Our algebra $A$ of endomorphisms turns out to be the same as the algebra $B(k, n)$ defined combinatorially in [Ozsváth, Szabó], Kauffman states, bordered algebras and a bigraded knot invariant

They use bimodules over such algebras to define a categorification of the Alexander polynomial of a knot.

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Since $c_1(M_{n,k}) = 0$, the symplectic manifold $M_{n,k}$ can be equipped with a $\mathbb{Z}$-grading structure. The grading structures naturally form a torsor over $H^1(M_{n,k}, \mathbb{Z}) \cong \mathbb{Z}^k$.

All our Lagrangians $L_S$ are contractible, so they can be graded (uniquely up to a shift by $\mathbb{Z}$).

**Proposition.** For any assignment of degrees, $\deg(x_i) = d_i \in \mathbb{Z}$, $i = 1, \ldots, k$, there is a unique $\mathbb{Z}$-grading on the algebra

$$\mathcal{A} = \bigoplus_{S,S'} \text{Hom}(L_S, L_{S'})$$

coming from some choices of $\deg(f_{S,S'}) = d_{S,S'} \in \mathbb{Z}$, for $S, S'$ close, up to a transformation of the form $d_{S,S'} \mapsto d_{S,S'} + d_{S'} - d_S$. 
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\]
Let $R = R_{[1,k]} = k[x_1, \ldots, x_k]/(x_1 \ldots x_k)$. We construct an nc-resolution of $R$.

$$B = B_{[1,k]} := \text{End}_R \left( \bigoplus_{I \subset [1,k], I \neq \emptyset} R/(x_I) \right),$$

where the summation is over all nonempty subintervals of $[1, k]$, $x_I = \prod_{i \in I} x_i$.

E.g, for $k = 2$, this is precisely the Auslander order.

For each subinterval $I \subset [1, k]$, denote by $P_I$ the corresponding projective module over $B$. Note that $\text{End}_B(P_I) = R/(x_I)$. So we have a fully faithful embedding

$$i_B^R : \text{Perf}(R) \rightarrow \text{Perf}(B) : R \mapsto P_{[1,k]}$$
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Localization on the $B$-side

We also have the right adjoint functor to $i^B_R$,

$$r^B_R : D^b(B) \to D^b(R) : M \mapsto \text{Hom}_B(P[1,k], M).$$

For a pair of nonempty disjoint subintervals $I, J \subset [1, k]$, such that $I \sqcup J$ is also a subinterval, can define a $B$-module $M\{I, J\}$, so that we have an exact sequence

$$0 \to P_I \to P_{I \sqcup J} \to P_J \to M\{I, J\} \to 0.$$

**Proposition.** Assume $k$ is regular. Then $r^B_R$ induces an equivalence

$$D^b(B)/ \ker(r^B_R) \simeq D^b(R),$$

and $\ker(r^B_R)$ is generated by the modules $(M\{[i], [i + 1, j]\}, M\{[j], [i, j - 1]\})_{i < j}$. 
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Matching the $A$-side with the $B$-side

For $n = k - 1$ the Lagrangians $L_S$ are numbered by subsets $S \subset [0, k]$ with $|S| = k - 1$. Now we define the correspondence between such $L_S$ and subintervals $I \subset [1, k]$ by

$$L_{[0,k]\{i,j\}} \leftrightarrow [i + 1, j],$$

where $0 \leq i < j \leq k$.

**Theorem.** This extends to an isomorphism of algebras $A \simeq B$, so that we get an equivalence of categories

$$\mathcal{W}(\Pi_{k-1}, \Lambda) \simeq \text{Perf}(B_k).$$

The $\mathbb{Z}$-grading on the left is the unique one with $\deg(x_i) = 0$. Furthermore, the subcategory corresponding to stops matches with $\ker(r^B_R)$, so for $k$ regular, we deduce

$$\mathcal{W}(\Pi_{k-1}) \simeq D^b(k[x_1, \ldots, x_k]/(x_1 \ldots x_k)).$$
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The $\mathbb{Z}$-grading on the left is the unique one with $\deg(x_i) = 0$.

Furthermore, the subcategory corresponding to stops matches with $\ker(r^B_R)$, so for $k$ regular, we deduce

$$\mathcal{W}(\Pi_{k-1}) \simeq D^b(\mathbb{k}[x_1, \ldots, x_k]/(x_1 \ldots x_k)).$$
Addional features

1. Can similarly identify the nc resolution of $R = \mathbf{k}[x_1, \ldots, x_k]/(x_1 \ldots x_k)$ corresponding to $\mathcal{W}(\Pi_{k-1}, \Lambda_1)$ (only one stop). It is given by

$$\mathcal{B}^\circ := \text{End}_R(R/(x_1) \oplus R/(x_{[1,2]}) \oplus \ldots \oplus R/(x_{[1,k-1]}) \oplus R).$$

2. There is a semiorthogonal decomposition

$$\text{Perf}(\mathcal{B}^\circ) = \langle \text{Perf}(R/(x_k)), \ldots, \text{Perf}(R/(x_2)), \text{Perf}(R/(x_1)) \rangle$$

and a semiorthogonal decomposition

$$\text{Perf}(\mathcal{B}) = \langle C_1, \ldots, C_N, \text{Perf}(\mathcal{B}^\circ) \rangle$$

where each $C_i$ is of the form $\text{Perf}(R/(x_1, x_j))$ for some $j \geq 2$. 

Additional features

1. Can similarly identify the nc resolution of 
   \( R = k[x_1, \ldots, x_k]/(x_1 \ldots x_k) \) corresponding to \( \mathcal{W}(\Pi_{k-1}, \Lambda_1) \) (only one stop). It is given by

   \[ \mathcal{B}^\circ := \text{End}_R(R/(x_1) \oplus R/(x_{[1,2]}) \oplus \ldots \oplus R/(x_{[1,k-1]}) \oplus R). \]

2. There is a semiorthogonal decomposition

   \[ \text{Perf}(\mathcal{B}^\circ) = \langle \text{Perf}(R/(x_k)), \ldots, \text{Perf}(R/(x_2)), \text{Perf}(R/(x_1)) \rangle \]

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   where each \( C_i \) is of the form \( \text{Perf}(R/(x_1, x_j)) \) for some \( j \geq 2 \).
Abelian covers

For $k > 2$ we have a natural isomorphism $\pi_1(\Pi_{k-1}) \cong \mathbb{Z}_k$. Fix a homomorphism

$$\phi : \pi_1(\Pi_{k-1}) \cong \mathbb{Z}_k \to \Gamma,$$

where $\Gamma$ is a finite abelian group, and let

$$\pi : M \to \Pi_{k-1}$$

be the corresponding finite covering.

Let $G = \text{Hom}(\Gamma, \mathbb{G}_m)$ denote the dual abelian group scheme to $\Gamma$, and let

$$G \to \mathbb{G}_m^k$$

be the homomorphism corresponding to $\phi$.

**Theorem.** For $k$ regular, we have an equivalence

$$\mathcal{W}(M) \cong D^b_G(k[x_1, \ldots, x_k]/(x_1 \ldots x_k))$$
Abelian covers

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**Theorem.** For $k$ regular, we have an equivalence

$$\mathcal{W}(M) \cong D_G^b(k[x_1, \ldots, x_k] / (x_1 \ldots x_k))$$
Let \( w = \sum_{i=1}^{k} \prod_{j=1}^{k} x_j^{a_{ij}} \) be an invertible polynomial described by the matrix of exponents \((a_{ij})\).

Let

\[
M_w := \{ (x_1, \ldots, x_k) \in (\mathbb{C}^*)^k \mid w(x_1, \ldots, x_k) = 1 \}
\]

be the punctured Milnor fiber.

We have a covering map

\[
\pi : M_w \to \Pi_{k-1}
\]

given by \((x_1, x_2, \ldots, x_k) \to (\prod_{j=1}^{k} x_j^{a_{1j}}, \prod_{j=1}^{k} x_j^{a_{2j}}, \ldots, \prod_{j=1}^{k} x_j^{a_{kj}})\)

where we view \(\Pi_{k-1}\) as a hypersurface in \((\mathbb{C}^*)^k\) via the identification

\[
\Pi_{k-1} = \{ (x_1, \ldots, x_k) \in (\mathbb{C}^*)^k : x_1 + x_2 + \ldots + x_k = 1 \}.
\]
Punctured Milnor fibers for invertible polynomial

Let \( w = \sum_{i=1}^{k} \prod_{j=1}^{k} x_j^{a_{ij}} \) be an invertible polynomial described by the matrix of exponents \((a_{ij})\).

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\]
The group of deck transformations of this covering map is

\[ \Gamma = \{(t_1, t_2, \ldots, t_k) \in G_m^k : \forall i, t_1^{a_{i1}} t_2^{a_{i2}} \ldots t_k^{a_{ik}} = 1 \}, \]

which is exactly the group of diagonal symmetries of \( w \).

Let \( G = \text{Hom}(\Gamma, G_m) \) be the dual abelian group.

**Corollary.** For \( k \) regular we have an equivalence

\[ \mathcal{W}(M_w) \simeq D^b_G(k[x_1, \ldots, x_k]/(x_1 \ldots x_k)). \]
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**Corollary.** For $k$ regular we have an equivalence

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