SILTING MODULES OVER COMMUTATIVE RINGS

LIDIA ANGELERI HÜGEL AND MICHAL HRBEK

Abstract. Tilting modules over commutative rings were recently classified in [12]; they correspond bijectively to faithful Gabriel topologies of finite type. In this note we extend this classification by dropping faithfulness. The counterpart of an arbitrary Gabriel topology of finite type is obtained by replacing tilting with the more general notion of a silting module.

1. Introduction

Silting modules were introduced in [3] as a common generalisation of tilting modules and of the support $\tau$-tilting modules from [1]. They are in bijection with 2-term silting complexes and with certain $t$-structures and co-$t$-structures in the derived module category. For certain rings, they are also known to parametrize universal localisations and wide subcategories of finitely presented modules [13, Theorem 4.5], [14, Corollary 5.15].

In this note, we give a classification of silting modules over commutative rings, establishing a bijective correspondence with Gabriel filters of finite type. This extends the results in [12] from the tilting to the silting case, and it is a further piece of evidence for the close relationship between silting modules and localisation theory.

Our result is achieved by investigating the dual notion of a cosilting module recently introduced in [8] as a generalisation of cotilting modules. Indeed, the dual of a silting module $T$ is a cosilting module $T^+$, and there is a duality between the modules in the silting class $\text{Gen} T$ and the cosilting class $\text{Cogen} T^+$. When $R$ is commutative, $\text{Cogen} T^+$ turns out to be the torsionfree class of a hereditary torsion pair of finite type. We can thus interpret the modules in $\text{Cogen} T^+$ as the $G$-torsionfree modules with respect to a Gabriel filter of finite type $G$. The silting class $\text{Gen} T$ is then the class of $G$-divisible modules. This defines a map assigning a Gabriel filter $G$ to every silting class $\text{Gen} T$. We show that this assignment is a bijection by constructing explicitly, for any $G$, a silting module $T$ which generates the $G$-divisible modules (Construction 4.5). We also provide a construction for a cosilting module cogenerated the $G$-torsionfree modules (Construction 5.2).

In general, not all cosilting modules arise as duals of silting modules. This is a phenomenon that already occurs for cotilting modules [5], see Example 5.4 for a cosilting example. If $R$ is a commutative noetherian ring, however, our classification yields bijections between silting classes, cosilting classes, Gabriel filters, and subsets of $\text{Spec}(R)$ closed under specialisation (Theorem 5.1). This generalises the classification of tilting and cotilting modules in [5, Theorem 2.11].

In fact, silting and cosilting classes are in bijection also over non-commutative noetherian rings. As a consequence, every definable torsion class of right modules over a left noetherian ring is generated by a silting module (Corollary 5.7). Finally,
extending a result from [7], we show that the only silting torsion pair of finite type over a commutative ring is the trivial one (Proposition 4.3).

The note is organized as follows. In Section 2 we investigate a finiteness condition which is shown to hold for silting classes, recovering a recent result from [13]. Section 3 is devoted to the duality between silting and cosilting classes. In Sections 4 and 5 we turn to commutative rings and prove our classification results. In [5,3] we further exhibit an example showing that the inclusion of silting modules in the class of finendo quasitilting modules proved in [3, Proposition 3.10] is proper.

2. Definability and finite type

Let $R$ be a ring, and let $\text{Mod-}R$ (respectively, $\text{R-Mod}$) denote the category of all right (respectively, left) $R$-modules. Denote by $\text{Proj-}R$ and $\text{proj-}R$ the full subcategory of $\text{Mod-}R$ consisting of all projective and all finitely generated projective modules, respectively. Given a subcategory $C$ of $\text{Mod-}R$, write $\text{Mor}(C)$ for the class of all morphisms in $\text{Mod-}R$ between objects in $C$, and denote

$$C^\perp = \{ M \in \text{Mod-}R | \text{Ext}^1_R(C, M) = 0 \}.$$

Given a map $\sigma$ in $\text{Mor}(\text{Proj-}R)$, we are going to investigate the class

$$D_\sigma := \{ X \in \text{Mod-}R \mid \text{Hom}_R(\sigma, X) \text{ is surjective} \}. $$

We say that $D_\sigma$ is of finite type if it is determined by a set of morphisms between finitely generated projective modules, i.e. there are $\sigma_i \in \text{Mor}(\text{proj-}R), i \in I$, such that $D_\sigma = \bigcap_{i \in I} D_{\sigma_i}$. As a shorthand, we say that $\sigma \in \text{Mor}(\text{Proj-}R)$ is of finite type if the class $D_\sigma$ is of finite type.

Recall that a class is said to be definable if it is closed under direct limits, direct products and pure submodules. We are going to see that $D_\sigma$ is definable if and only if it is of finite type.

**Lemma 2.1.** Let $\sigma \in \text{Mor}(\text{Proj-}R)$. Then

$$D_\sigma = (\text{Coker} \sigma)^\perp \cap D_{\sigma'},$$

where $\sigma' : P_{-1} \to \text{Im} \sigma$ is given by restricting the codomain of $\sigma$ to its image.

**Proof.** It is clear that $D_\sigma \subseteq D_{\sigma'}$. Then for any $M \in D_{\sigma'}$, a standard long exact sequence argument shows that $M \in D_\sigma$ if and only if $\text{Ext}^1_R(\text{Coker} \sigma, M) = 0$, finishing the proof. $\square$

**Lemma 2.2.** Let $\sigma \in \text{Mor}(\text{Proj-}R)$ be a map between projective modules. Then $D_\sigma = D_{\varphi}$, where $\varphi$ is a map between free modules.

**Proof.** Suppose that $\sigma : P_{-1} \to P_0$. Let $P'$ be a projective module such that $P_{-1} \oplus P'$ is free, and let $P''$ be a projective module such that $P_0 \oplus P' \oplus P''$ is free.

Then let $\varphi$ be the direct sum of the maps $\sigma : P_{-1} \to P_0, P' \xrightarrow{\sigma'} P', \text{and } 0 \xrightarrow{\sigma''} P''$. It is a routine check that $D_\sigma = D_{\varphi}$. $\square$

**Theorem 2.3.** Let $\sigma \in \text{Mor}(\text{Proj-}R)$. Then the following are equivalent:

(i) $D_\sigma$ is of finite type,
(ii) $D_\sigma$ is definable.

**Proof.** In the whole proof, let $\sigma : P_{-1} \to P_0$, and $\text{Coker} \sigma$.

(i) $\Rightarrow$ (ii): As an intersection of definable classes is a definable class, it is enough to show that $D_\sigma$ is definable if $\sigma \in \text{Mor}(\text{proj-}R)$. By [3, Lemma 3.9], $D_\sigma$ is closed under direct products and epimorphic images, it is thus enough to show that it is closed under direct sums and pure submodules. By Lemma 2.1 we have that $D_\sigma = D_{\sigma'} \cap C^\perp$, where $\sigma' : P_{-1} \to \text{Im} \sigma$ is $\sigma$ with codomain restricted to its image.
As $C$ is finitely presented, the class $C^+$ is definable by [10] Theorem 13.26. We finish the proof by showing that $D_{\sigma'}$ is closed under direct sums and submodules.

Let $(M_i \mid i \in I)$ be a family of modules from $D_{\sigma'}$, and $f : P_{-1} \to \bigoplus_{i \in I} M_i$ a map. As $P_{-1}$ is finitely generated, there is a finite subset $J \subseteq I$ such that $f$ factors through the direct summand $\bigoplus_{j \in J} M_i \simeq \prod_{j \in J} M_i$. Since $D_{\sigma'}$ is clearly closed under products, $f$ factorizes through $\sigma'$.

Let $M \in D_{\sigma'}$ and $\iota : N \subseteq M$ be an inclusion. Applying $\text{Hom}_R(-, \iota)$ on the exact sequence $0 \to K \to P_{-1} \xrightarrow{\varphi} \text{Im} \sigma \to 0$ yields

$$
0 \longrightarrow \text{Hom}_R(\text{Im} \sigma, M) \xrightarrow{\text{Hom}_R(\sigma^\prime, M)} \text{Hom}_R(P_{-1}, M) \xrightarrow{\varphi} \text{Hom}_R(K, M)
$$

By the assumption, the map $\varphi = 0$, and thus $\theta \psi = 0$. By left-exactness, all the vertical maps are injective, and therefore $\psi = 0$, showing that $\text{Hom}_R(\sigma^\prime, N)$ is surjective. Therefore, $N \in D_{\sigma'}$.

(ii) $\to$ (i): Using Lemma 2.2 we can without loss of generality assume that $P_{-1}$ and $P_0$ are free modules. Fix a free basis $X$ of $P_{-1}$, and write the set $X$ as a direct union $X = \bigcup_{i \in I} X_i$ of its finite subsets, inducing a presentation of $P_{-1}$ as a direct union of direct summands $F_i = R^{(X_i)}$. Denote $G_i = \sigma(F_i)$. Fix a free basis $Y$ of $P_0$. As $G_i$ is finitely generated for each $i$, there is a finite subset $Y_i \subseteq Y$ spanning $G_i$. Moreover, there are finite subsets $Y_i \subseteq Y$ such that $Y_i \cap Y_j = \emptyset$ for each $i \neq j$. The directed union $\bigcup_{i \in I} F_i' = \bigoplus_{i \in I} F_i'$ is a free direct summand of $P_0$, and the projection of the image of $\sigma$ onto the complement $R^{(Y \setminus \bigcup_{i \in I} Y_i)}$ is necessarily zero. Therefore, we can without loss of generality assume that $P_0 = \bigcup_{i \in I} F_i'$. Let $\sigma_i : F_i \to F_i'$ be the restriction of $\sigma$ onto $F_i'$, with codomain restricted to $F_i'$. We claim that $D_{\sigma} \subseteq D_{\sigma_i}$, for each $i \in I$. To prove this, let $M \in D_{\sigma'}$ and fix a map $f_i : F_i \to M$. As $F_i$ is a direct summand of $P_{-1}$, we can extend $f_i \to$ a map $f : P_{-1} \to M$. As $M \in D_{\sigma'}$, there is a map $g : P_0 \to M$ such that $\sigma = g \circ g$. Let $g_i$ be the restriction of $g$ to $F_i'$. Then $f_i = g_i \sigma_i$, proving that $M \in D_{\sigma_i}$. Denoting $D = \bigcap_{i \in I} D_{\sigma_i}$, we have $D_{\sigma} \subseteq D$.

Finally, we show that $D_{\sigma}$ is of finite type by proving $D \subseteq D_{\sigma}$. The class $D_{\sigma}$ is definable by the assumption, and the definability of the class $D$ is proved by implication (i) $\to$ (ii) of this Theorem. By [10] Lemma 6.9, it is enough to show that $M \in D$ implies $M \in D_{\sigma}$ for $M$ pure-injective.

Let $M \in D$ be pure-injective. Denote $C_i = \text{Coker} \sigma_i$ for all $i \in I$. By Lemma 2.1 we have that $\text{Ext}^1_R(C_i, M) = 0$ and $M \in D_{\sigma_i}$, where $\sigma_i$ is given by restricting the codomain of $\sigma_i$ to $C_i$. Since $M$ is pure-injective, we have by [10] Lemma 6.28 the following isomorphism:

$$
\text{Ext}^1_R(C, M) \simeq \text{Ext}^1_R(\lim_{i \in I} C_i, M) \simeq \lim_{i \in I} \text{Ext}^1_R(C_i, M).
$$

This shows that $M \in C^+$, applying $\text{Hom}_R(-, M)$ to the exact sequence $F_i \xrightarrow{\sigma_i} G_i \to 0$ we obtain that $\text{Hom}_R(\sigma_i, M)$ is an isomorphism for all $i \in I$. As inverse limit of a directed system of isomorphisms is an isomorphism, we obtain that

$$
\lim_{i \in I} \text{Hom}_R(\sigma_i, M) \simeq \text{Hom}_R(\lim_{i \in I} \sigma_i, M) \simeq \text{Hom}_R(\sigma', M).
$$
Lemma 2.1 yields $M$ one implication in a recent result due to Marks and Šťovíček.

$\sigma$ is an isomorphism, where again $\sigma'$ : $P_{-1} \to G$ is given by restricting the codomain of $\sigma$ to its image $G = \bigcup_{i \in I} G_i$. In other words, $M \in \mathcal{D}_\sigma$. As $M \in C^+ \cap \mathcal{D}_\sigma$, Lemma 2.1 yields $M \in \mathcal{D}_\sigma$ as desired. \hfill $\square$

3. Silting and cosilting modules

According to [5], an $R$-module $T$ is said to be silting if it admits a projective presentation $P_{-1} \stackrel{\sigma}{\to} P_0 \to T \to 0$ such that the class $\text{Gen} T$ of $T$-generated modules coincides with the class $\mathcal{D}_\sigma$. The class $\text{Gen} T$ is then called a silting class.

It is shown in [5] 3.5 and 3.10 that silting classes are definable torsion classes. From Theorem 2.3 we obtain that every silting class is of finite type. This reproves one implication in a recent result due to Marks and Šťovíček.

Theorem 3.1. [4] A map $\sigma$ in $\text{Mor}(\text{Proj}-R)$ is of finite type if and only if the class $\mathcal{D}_\sigma$ is a silting class.

Let us now turn to the dual notion. Following [5], an $R$-module $C$ is said to be cosilting if it admits an injective copresentation $0 \to C \to E_0 \stackrel{\sigma}{\to} E_1$ such that the class $\text{Cogen} C$ of $C$-cogenerated modules coincides with the class $\mathcal{C}_\sigma := \{ X \in \text{Mod-}R \mid \text{Hom}_R(X, \sigma) \text{ is surjective} \}$.

The class $\text{Cogen} C$ is then called a cosilting class.

It is shown in [5] that every cosilting module is pure-injective and that cosilting classes are definable torsionfree classes. In fact, there is a duality between the silting classes in $\text{Mod-}R$ and certain cosilting classes in $\text{R-Mod}$ (see also [5] 3.7 and 3.9). These cosilting classes will be characterized by the property below.

Definition 3.2. For any map $\sigma \in \text{Mor}(\text{Proj}-R)$, denote

\[ \mathcal{T}_\sigma = \{ X \in \text{R-Mod} \mid \sigma \otimes_R X \text{ is injective} \} \]

Given a map $\lambda$ between injective left $R$-modules, we say that the class $\mathcal{C}_\lambda$ (or, the map $\lambda$) is of cofinite type, if there is a set of maps $\sigma_i \in \text{Mor}(\text{proj}-R), i \in I$, such that $\mathcal{C}_\lambda = \bigcap_{i \in I} \mathcal{T}_{\sigma_i}$.

Let us investigate the duality. Assume that $R$ is a $k$-algebra over some commutative ring $k$. Given an $R$-module $M$, we denote by $M^+$ its dual with respect to an injective cogenerator of $\text{Mod-}k$, for example we can take $k = \mathbb{Z}$ and $M^+$ the character dual of $M$. To every definable category $\mathcal{C}$ of right (left) $R$-modules we can now associate a dual definable category of left (right) $R$-modules $\mathcal{C}^\vee$ which is determined by the property that a module $M$ belongs to $\mathcal{C}$ if and only if its dual module $M^+ \in \mathcal{C}^\vee$. This assignment defines a bijection between definable subcategories of $\text{Mod-}R$ and $\text{R-Mod}$, which restricts to a bijection between definable torsion classes and definable torsionfree classes and maps tilting classes to cotilting classes of cofinite type, see [5] Propositions 5.4 and 5.7 and Theorem 7.1. We are now going to prove the analogous result for silting and cosilting classes.

Lemma 3.3. (1) Let $\sigma \in \text{Mor}(\text{Proj}-R)$. Then $\mathcal{T}_\sigma = \mathcal{C}_{\sigma^+}$, and a left $R$-module $X$ belongs to $\mathcal{C}_{\sigma^+}$ if and only if $X^+ \in \mathcal{D}_\sigma$.

(2) If $\sigma \in \text{Mor}(\text{Proj}-R)$ has finite type, then $\mathcal{D}_\sigma$ and $\mathcal{C}_{\sigma^+}$ are dual definable categories, and a right $R$-module $Y$ belongs to $\mathcal{D}_\sigma$ if and only if $Y^+ \in \mathcal{C}_{\sigma^+}$.

(3) A map $\lambda$ between injective left $R$-modules has cofinite type if and only if there is a map $\sigma \in \text{Mor}(\text{Proj}-R)$ of finite type such that $\mathcal{C}_\lambda = \mathcal{C}_{\sigma^+}$.

Proof. (1),(2) By Hom-⊗-adjunction, for any left $R$-module $X$ there is a commutative diagram linking the maps $\text{Hom}_R(X, \sigma^+), (\sigma \otimes_R X)^+$ and $\text{Hom}_R(\sigma, X^+)$. This shows that $X \in \mathcal{C}_{\sigma^+}$ if and only if $X^+ \in \mathcal{D}_\sigma$, which in turn means that $(\sigma \otimes_R X)^+$ is surjective, or equivalently, $\sigma \otimes_R X$ is injective.
Furthermore, if \( \sigma \) is of finite type, the definable class \( D_\sigma \) contains a right \( R \)-module \( Y \) if and only if it contains its double dual \( Y^{++} \), see e.g. [15, 3.4.21]. This implies that \( Y \in D_\sigma \) if and only if \( Y^{++} \in C_\sigma^{-} \).

(3) Let \( \sigma_i \in \text{Mor}(\text{proj}-R), i \in I \), be a set of maps such that \( C_\lambda = \bigcap_{i \in I} T_{\sigma_i} \), and let \( \sigma = \bigoplus_{i \in I} \sigma_i \). Then \( C_\lambda = \bigcap_{i \in I} T_{\sigma_i} = T_\sigma = C_\sigma^{-} \) by (1), and \( D_\sigma = \bigcap_{i \in I} D_{\sigma_i} \), so the map \( \sigma \) is of finite type. Conversely, if \( C_\lambda = C_\sigma^{-} \) for a map \( \sigma \) of finite type, there are maps \( \sigma_i \in \text{Mor}(\text{proj}-R), i \in I \), such that \( D_\sigma = \bigcap_{i \in I} D_{\sigma_i} \), and \( C_\sigma^{-} = \bigcap_{i \in I} C_{\sigma_i^{-}} = \bigcap_{i \in I} T_{\sigma_i} \). □

**Proposition 3.4.** Let \( \sigma \in \text{Mor}(\text{Proj}-R) \), and let \( T = \text{Coker} \sigma \) be a silting module with respect to \( \sigma \). Then \( T^{++} \) is a cosilting left \( R \)-module with respect to the injective copresentation \( \sigma^{+} \). Moreover, \( \text{Gen} T \) and \( \text{Cogen} T^{++} \) are dual definable classes, and \( \text{Cogen} T^{++} \) is a cosilting class of cofinite type.

**Proof.** We have to verify \( \text{Cogen} T^{++} = C_\sigma^{+} \). The class \( C_\sigma^{+} \) is closed under submodules by [8, 3.5], so for the inclusion \( \subset \) it is enough to show that \( C_\sigma^{+} \) contains the direct product \( (T^{(\alpha)})^{+} \) for any cardinal \( \alpha \). Notice that the definable class \( D_\sigma \) contains \( T^{(\alpha)} \). The claim then follows from Lemma 3.3 as \( T^{(\alpha)} + \cong (T^{+})^{\alpha} \). For the inclusion \( \supset \), take \( X \in C_\sigma^{+} \). Then \( X^{+} \in D_\sigma = \text{Gen} T \), so there is an epimorphism \( T^{(\alpha)} \to X^{+} \) for some cardinal \( \alpha \). This yields a monomorphism \( X \hookrightarrow X^{++} \to (T^{+})^{\alpha} \), showing that \( X \in \text{Cogen} T^{++} \). □

From Theorem 3.1 and Lemma 3.3 we obtain

**Corollary 3.5.** The assignment \( \text{Gen} T \mapsto \text{Cogen} T^{++} \) defines a 1-1-correspondence between silting classes in \( \text{Mod}-R \) and cosilting classes of cofinite type in \( R-\text{Mod} \).

We now give a criterion for a torsionfree definable class to be of cofinite type.

**Lemma 3.6.** Let \( \mathcal{U} \) be a set of finitely presented left \( R \)-modules, and let \( (\mathcal{T}, \mathcal{F}) \) be the torsion pair in \( R-\text{Mod} \) generated by \( \mathcal{U} \), that is, \( \mathcal{F} = \{ M \in R-\text{Mod} \mid \text{Hom}_R(U, M) = 0 \text{ for all } U \in \mathcal{U} \} \). Then \( \mathcal{F} \) is a cosilting class of cofinite type.

**Proof.** For every \( U \in \mathcal{U} \) we choose a projective presentation \( \alpha_U \in \text{Mor}(\text{R-proj}) \), and we denote \( \sigma_U = \alpha_U^{-} \) and \( \sigma = \bigoplus_{U \in \mathcal{U}} \sigma_U \). Then, using that for any \( P \in \text{R-proj} \) and any \( X \in R-\text{Mod} \) there is a natural isomorphism \( P^{+} \otimes_R X \cong \text{Hom}_R(P, X) \), we see that \( \mathcal{F} = \bigcap_{U \in \mathcal{U}} T_{\sigma_U} = C_{\sigma^+} \) is a cosilting class of cofinite type. □

**Corollary 3.7.** If \( R \) is a left noetherian ring, the definable torsionfree classes in \( R-\text{Mod} \) coincide with the cosilting classes of cofinite type, and the assignment \( \text{Gen} T \mapsto \text{Cogen} T^{++} \) defines a 1-1-correspondence between silting classes in \( \text{Mod}-R \) and cosilting classes in \( R-\text{Mod} \). Moreover, the definable torsion classes in \( \text{Mod}-R \) coincide with the silting classes.

**Proof.** Let \( (\mathcal{T}, \mathcal{F}) \) be a torsion pair in \( R-\text{Mod} \) with \( \mathcal{F} \) being definable. By [11, Lemma 4.5.2], there is a torsion pair \( (\mathcal{U}, \mathcal{V}) \) in \( R-\text{mod} \) such that \( \mathcal{T} \) and \( \mathcal{F} \) consist of the direct limits of modules in \( \mathcal{U} \) and \( \mathcal{V} \), respectively, and \( \mathcal{F} = \{ M \in R-\text{Mod} \mid \text{Hom}_R(U, M) = 0 \text{ for all } U \in \mathcal{U} \} \). Then \( \mathcal{F} \) is a cosilting class of cofinite type by Lemma 3.4. In particular, every cosilting class is of cofinite type, and Corollary 3.5 yields the second statement.

For the last statement, recall from [6, Proposition 5.7] that the bijection in Corollary 3.5 extends to a bijection between definable torsion classes and definable torsionfree classes. By the discussion above, if \( \mathcal{T} \) is a definable torsion class, its dual definable class \( \mathcal{T}^{++} \) coincides with the dual definable class of a silting class. Now use that the assignment is injective. □
In general, a definable torsion class need not be silting, cf. Example 5.4. As for the dual result, it was recently shown in [18] that the definable torsionfree classes over an arbitrary ring are precisely the cosilting classes. But in general these classes are not of cofinite type, see again Example 5.4.

4. Silting classes over commutative rings

In this section, we classify silting classes over commutative rings, proving that they coincide precisely with the classes of divisibility by sets of finitely generated ideals.

The key to our classification are the following results relating cosilting modules of cofinite type with hereditary torsion pairs. Recall that a torsion pair \((\mathcal{T}, \mathcal{F})\) is hereditary if the torsion class \(\mathcal{T}\) is closed under submodules, or equivalently, the torsionfree class \(\mathcal{F}\) is closed under injective envelopes. Moreover, \((\mathcal{T}, \mathcal{F})\) has finite type if \(\mathcal{F}\) is closed under direct limits.

First of all, combining Lemma 3.6 with [12, Lemma 2.4], we obtain

**Corollary 4.1.** Let \(R\) be a ring. If \((\mathcal{T}, \mathcal{F})\) is a hereditary torsion pair of finite type in \(R\text{-Mod}\), then \(\mathcal{F}\) is a cosilting class of cofinite type.

For a commutative ring, also the converse holds true.

**Lemma 4.2.** Let \(R\) be a commutative ring. Let \(\lambda\) be a map between injective \(R\)-modules. If \(C_{\lambda}\) is a cosilting class of cofinite type, then \(C_{\lambda}\) is a torsionfree class in a hereditary torsion pair of finite type.

In particular, if \(R\) is a commutative noetherian ring, a torsion pair has finite type if and only if it is hereditary.

**Proof.** By assumption \(C_{\lambda} = \bigcap_{i \in I} \mathcal{T}_{\sigma_i}\) for a set of maps \(\sigma_i \in \text{Mor}(\text{proj}\text{-}R), i \in I\). It is then enough to prove the claim for each \(\mathcal{T}_{\sigma_i}\), or in other words, we can assume w.l.o.g. that \(C_{\lambda} = \mathcal{T}_{\sigma}\) for some \(\sigma \in \text{Mor}(\text{proj}\text{-}R)\). By Lemma 3.3, \(\mathcal{T}_{\sigma} = C_{\sigma^{+}}\) is a definable category, which is closed under submodules and extensions by [5, Lemma 2.3], so it is a torsion-free class closed under direct limits. It remains to show that is also closed under injective envelopes.

Let \(M \in \mathcal{T}_{\sigma}\), and consider the exact sequence induced by an injective envelope \(0 \to M \xrightarrow{\lambda} E(M) \to C \to 0\). Tensoring this sequence with \(\sigma\) yields a commutative diagram

\[
\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
0 & \xrightarrow{P_{-1} \otimes_R M} & P_{0} \otimes_R M \\
\downarrow & & \downarrow \\
P_{-1} \otimes_R E(M) & \xrightarrow{\sigma \otimes_R E(M)} & P_{0} \otimes_R E(M).
\end{array}
\]

The exactness of the columns follows from the projectivity of \(P_{-1}, P_{0}\), while the exactness of the first row follows by definition of \(\mathcal{T}_{\sigma}\). Since \(R\) is commutative, this is a commutative diagram in \(\text{Mod}\text{-}R\) (this is where we need the commutativity of \(R\)).

First, we claim that the inclusion \(P_{-1} \otimes_R \iota\) is an injective envelope of \(P_{-1} \otimes_R M\). Indeed, let \(P\) be a finitely generated projective such that \(P_{-1} \otimes_R P \simeq R^u\) for some \(u\). Then \((P_{-1} \otimes_R P) \otimes_R t = R^u \otimes_R t\) is essential by [2, Proposition 6.17(2)], and since \(E(M)^u \simeq R^u \otimes_R E(M)\) is injective, it is an injective envelope of \(M^u \simeq R^u \otimes_R M\). As \(R^u \otimes_R t = (P_{-1} \otimes_R t) \oplus (P \otimes_R t)\), we conclude that \(P_{-1} \otimes_R \iota\) is an injective envelope of \(P_{-1} \otimes_R M\).
If \( P_{-1} \otimes_R M \) is zero, then its injective envelope \( P_{-1} \otimes_R E(M) \) is also zero, and thus \( \sigma \otimes_R E(M) \) is injective. Towards a contradiction, suppose that \( P_{-1} \otimes_R M \) is non-zero, and the kernel of \( \sigma \otimes_R E(M) \) is non-zero. By the essentiality of \( P_{-1} \otimes_R i \), there is a non-zero element \( x \in P_{-1} \otimes_R M \) such that \( (\sigma \otimes_R E(M))(P_{-1} \otimes_R i)(x) = 0 \), which by a simple diagram chasing yields \( (\sigma \otimes_R M)(x) = 0 \), a contradiction to \( \sigma \otimes_R M \) being injective. Therefore, the kernel of \( \sigma \otimes_R E(M) \) is zero. In both cases, we showed that \( E(M) \in \mathcal{T}_\sigma \).

The last statement follows from Corollary 4.3.

It is well known that hereditary torsion pairs correspond bijectively to Gabriel filters. This will allow to establish a correspondence between silting classes and Gabriel filters. We first review the relevant notions.

**Reminder 4.3.** A filter \( \mathcal{G} \) of right ideals of \( R \) is a (right) Gabriel filter, if the following conditions hold true:

(i) if \( I \in \mathcal{G} \), then for any \( x \in R \) the ideal \( (I : x) = \{ r \in R \mid xr \in I \} \) belongs to \( \mathcal{G} \).

(ii) if \( J \) is a right ideal such that there is \( I \in \mathcal{G} \) with \( (J : x) \in \mathcal{G} \) for all \( x \in I \), then \( J \in \mathcal{G} \).

Further, \( \mathcal{G} \) is of finite type if it has a filter basis consisting of finitely generated ideals. We remark that a filter of ideals of a commutative ring with a filter basis of finitely generated ideals is a Gabriel filter (of finite type) if and only if it is closed under ideal multiplication, see e.g. [12, Lemma 2.3].

Every Gabriel filter \( \mathcal{G} \) induces a hereditary torsion pair \( (\mathcal{T}_\mathcal{G}, \mathcal{F}_\mathcal{G}) \) where

\[
\mathcal{F}_\mathcal{G} = \bigcap_{I \in \mathcal{G}} \text{Ker} \, \text{Hom}_R(R/I, -)
\]

is the class of \( \mathcal{G} \)-torsionfree modules. The assignment \( \mathcal{G} \mapsto (\mathcal{T}_\mathcal{G}, \mathcal{F}_\mathcal{G}) \) defines a bijection between Gabriel filters (of finite type) and hereditary torsion pairs (of finite type), see [17, Chapter VI, Theorem 5.1, and Chapter XIII, Proposition 1.2].

Given a Gabriel filter \( \mathcal{G} \), we say that a module \( M \in \text{Mod-}R \) is \( \mathcal{G} \)-divisible if \( MI = M \) for all \( I \in \mathcal{G} \). If \( \text{Div-}\mathcal{G} \) denotes the class of \( \mathcal{G} \)-divisible modules, then

\[
\text{Div-}\mathcal{G} = \bigcap_{I \in \mathcal{G}} \text{Ker} (- \otimes_R R/I).
\]

By \( \text{Hom}-\otimes \) adjunction, a module \( M \) is \( \mathcal{G} \)-divisible if and only if its dual \( M^+ \) is \( \mathcal{G} \)-torsionfree (cf. [17] Chapter VI, Proposition 9.2]). So, if the Gabriel filter \( \mathcal{G} \) is of finite type, \( \text{Div-}\mathcal{G} \) and \( \mathcal{F}_\mathcal{G} \) are dual definable classes, and it follows from Corollary 4.3 that \( \text{Div-}\mathcal{G} \) is a silting class.

Again, in the commutative case, we also have the converse.

**Proposition 4.4.** Let \( R \) be a commutative ring, and let \( \sigma \in \text{Mor}(\text{Proj-}R) \) be of finite type. Then there is a Gabriel filter of finite type \( \mathcal{G} \) such that \( \mathcal{D}_\sigma = \text{Div-}\mathcal{G} \).

**Proof.** By assumption \( \mathcal{D}_\sigma = \bigcap_{i \in I} \mathcal{D}_{\sigma_i} \) for a set of maps \( \sigma_i \in \text{Mor}(\text{proj-}R), i \in I \). If each \( \mathcal{D}_{\sigma_i} = \text{Div-}\mathcal{G}_i \), for some Gabriel filter of finite type \( \mathcal{G}_i \), then \( \mathcal{D}_\sigma = \bigcap_{i \in I} \text{Div-}\mathcal{G}_i = \text{Div-}\mathcal{G} \), where \( \mathcal{G} = \{ J \subseteq R \mid I_1 I_2 \cdots I_n \subseteq J \text{ whenever } I_1, I_2, \ldots, I_n \in \bigcup_{i \in I} \mathcal{G}_i \} \) is the smallest Gabriel filter of finite type containing \( \mathcal{G}_i \) for all \( i \in I \). So we can again assume w.l.o.g. that \( \sigma \in \text{Mor}(\text{proj-}R) \).

In Lemma 4.2 we showed that \( \mathcal{C}_{\sigma^+} \) is a hereditary torsionfree class of finite type. So there is a Gabriel filter \( \mathcal{G} \) of finite type such that \( \mathcal{C}_{\sigma^+} = \mathcal{F}_\mathcal{G} \), which amounts to \( \mathcal{D}_\sigma = \text{Div-}\mathcal{G} \). This completes the proof. \( \square \)
Combining the results above, we obtain the desired classification of silting classes over commutative rings. Here we give a direct proof by providing an explicit construction for a silting module corresponding to a Gabriel filter of finite type. It generalises the construction of a Fuchs-Salce tilting module in [12].

**Construction 4.5.** Let $R$ be a commutative ring and $G$ a Gabriel filter of finite type. Let $\mathcal{I}$ be the collection of all finitely generated ideals from $G$. For each $I \in G$, we fix a finite set of generators $\{x_0^I, x_1^I, \ldots, x_{n_I}^I\}$. The projective presentation

$$R^{n_I} \longrightarrow R \longrightarrow R/I \longrightarrow 0,$$

induces a projective presentation

$$R \xrightarrow{\sigma_I} R^{n_I} \longrightarrow \text{Tr}(R/I) \longrightarrow 0,$$

where $\sigma_I : R \to R^{n_I}$ is given by $\sigma_I(1) = (x_0^I, x_1^I, \ldots, x_{n_I}^I)$ and $\text{Tr}$ denotes the Auslander-Bridger transpose of $R/I$ (which is uniquely determined only up to stable equivalence). It is easy to check that $\mathcal{D}_{\sigma_I} = \{M \in \text{Mod-}R \mid M = IM\}$, and thus $\text{Div-}G = \bigcap_{I \in \mathcal{I}} \mathcal{D}_{\sigma_I}$.

Let now $\Lambda$ be the set of all finite sequences of pairs $(I, k)$, with $I \in \mathcal{I}$ and $0 \leq k < n_I$. The set includes the empty sequence denoted by $\emptyset$, and it is equipped with the operation of concatenation of sequences, for which we use the symbol $\sqcup$. Let $F$ be the free module on basis $\Lambda$, $F'$ the free module on basis $\Lambda \setminus \{\emptyset\}$, and $K$ the free module on basis $\Lambda \times \mathcal{I}$.

We define a map $\varphi_G : K \to F$ by its values on the designated basis elements: for any $(\lambda, I) \in \Lambda \times \mathcal{I}$ we set

$$\varphi_G((\lambda, I)) = \lambda - \sum_{k < n_I} x_k^I (\lambda \sqcup (I, k)).$$

We also define a map $\varphi'_G : K \to F'$ by the commutative diagram

$$\begin{array}{ccc}
K & \xrightarrow{\varphi} & F \\
| & \downarrow\sigma & | \\
| & p \downarrow & | \\
K & \xrightarrow{\varphi'_G} & F' \\
\end{array}$$

where $p$ denotes the canonical projection $F \to F'$ killing the coordinate $w$.

Let now $P_{-1} = K \oplus K$ and $P_0 = F \oplus F'$ and consider

$$P_{-1} \xrightarrow{\varphi_G} P_0 \longrightarrow T_G \longrightarrow 0,$$

where $\varphi_G$ is the direct sum of the maps $\varphi_G$ and $\varphi'_G$, and $T_G = C_G \oplus C'_G$.

**Proposition 4.6.** The module $T_G$ is a silting module with respect to the map $\varphi_G$, and $\text{Gen}(T_G) = \text{Div-}G$.

**Proof.** We divide the proof into several steps. Let us first fix some notation. Let $A = \sum_{I \in \mathcal{I}} \text{Ann}(I)$. Further, for every ideal $I \in \mathcal{I}$, we define $S_I = \text{Tr}(R/I) \otimes_R R/A$, and we set $\mathcal{S} = \{S_I \mid I \in \mathcal{I}\}$.

**Step I:** Every $I \in \mathcal{I}$ gives rise to a faithful ideal $(I + A)/A$ in the ring $R/A$. In other words, every $r \in R$ satisfying $r I \subseteq A$ must belong to $A$.

To see this, use that $I$ is finitely generated to find ideals $I_1, I_2, \ldots, I_l \in \mathcal{I}$ such that $r I \subseteq \sum_{j=1}^l \text{Ann}(I_j)$. Then $r(I_1 I_2 \cdots I_l) = 0$, and $r \in \text{Ann}(I_1 I_2 \cdots I_l) \subseteq A$.

**Step II:** An $R/A$-module $M$ satisfies $\text{Ext}^1_{R/A}(S_I, M) = 0$ if and only if $M = IM$. 


Indeed, the map $\sigma_I : R \rightarrow R^{n_I}$, $r \mapsto (r \cdot x_0^I, r \cdot x_1^I, \ldots, r \cdot x_{n_I-1}^I)$ induces a commutative diagram with exact rows

$$
\begin{array}{cccc}
R & \xrightarrow{\sigma_I} & R^{n_I} & \xrightarrow{\text{Tr}(R/I)} \xrightarrow{\text{Ext}_{1}} 0 \\
\downarrow & & \downarrow & \\
0 & \rightarrow & R/A & \xrightarrow{\mathbf{m}} (R/A)^{n_I} \rightarrow S_I & \rightarrow 0 \\
\end{array}
$$

because the kernel of $\mathbf{m}$, consisting of the elements $r \in R/A$ with $rI \subseteq A$, is trivial by Step I. It is now an easy observation that $\text{Ext}_1^{R/A}(S_I, M) = 0$ if and only if $M = (I + A)/A \cdot M = IM$.

**Step III:** Filtration of $C_G$ and $C'_G$.

For each $n < \omega$ denote by $\Lambda_n$ the set of all sequences from $\Lambda$ of length at most $n$.

Let $F_n$ be the span of $\Lambda_n$ in $F$, and let $G_n$ be the $\varphi_G$-image of the span of $\Lambda_{n-1} \times I$ in $K$. For $n = 0$ we have $F_0 = R\omega$, and we set $G_0 = \emptyset$. Let $C_n$ be the span of the image of $\Lambda_n$ in $C$, that is, $C_n = F_n/(F_n \cap G)$, where $G = \text{Im} \varphi_G$.

We claim that $F_n \cap G = AF_n + G_n$. For any $\lambda \in \Lambda_n$ and any $I \in \mathcal{I}$, the element $\varphi_G((\lambda, I)) = \lambda - \sum_{k<n} x_k^I (\lambda \cup (I, k))$ lies in $G$. Therefore, by multiplying by any $r \in \text{Ann}(I)$, we obtain $r\lambda \in G$. As clearly $G_n \subseteq F_n \cap G$, we have $AF_n + G_n \subseteq F_n \cap G$.

For the reverse inclusion, let $x \in F_n \cap G$. As $x \in G$, it is of the form

$$
x = \sum_{j=1}^m r_j \varphi_G((\lambda_j, I_j)) = \sum_{j=1}^m r_j (\lambda_j - \sum_{k<n_{j}} x_k^I (\lambda_j \cup (I_j, k)))
$$

for some $r_j \in R$, and $(\lambda_j, I_j)$ pairwise distinct elements of $\Lambda \times I$. We claim that if the length of some $\lambda_j$ exceeds $n - 1$, then $r_j \in A$. We prove this claim by backward induction on the length of $\lambda_j$. If $j$ is such that the length of $\lambda_j$ is maximal and exceeding $n$, it is clear from $x \in F_n$ that $r_j \in \text{Ann}(\{x_0^I, x_1^I, \ldots, x_{n_{j}-1}^I\}) = \text{Ann}(I) \subseteq A$. Suppose now that $\lambda_j$ is of length $k > n - 1$, and that all coefficients $r_i$ such that $\lambda_i$ has length $> k$ are in $A$. Then, since $x \in F_n$, the induction premise yields $r_j x_k^I \subseteq A$ for each $k = 0, 1, \ldots, n_{j}-1$. In other words, $r_j I \subseteq A$. By Step I, this implies that $r_j \in A$ as claimed. We proved that the coefficient $r_j$ is in $A$ for any $j$ such that the length of $\lambda_j$ exceeds $n - 1$, and thus $x \in AF + G_n$. But since $x \in F_n$, and $AF \cap F_n = AF_n$, we get $x \in AF_n + G_n$ as desired.

It follows that $C_n = F_n/(AF_n + G_n)$. Then $C_0 \simeq R/A$, and $C_{n+1}/C_n \simeq F_{n+1}/(F_n + AF_n + G_{n+1})$ for any $n \in \omega$. Therefore, $C_{n+1}/C_n \simeq F_{n+1}/(F_n + G_{n+1}) \otimes R/A$. The elements $\varphi_G((\lambda, I))$, where $\lambda$ is of length $n$, and $I \in \mathcal{I}$ generate $G_{n+1}$ modulo $F_n \cap G_{n+1}$. We obtain that $C_{n+1}/C_n$ is isomorphic to:

$$
\bigoplus_{\lambda \in \Lambda_n \setminus \Lambda_{n-1}} \bigoplus_{I \in \mathcal{I}} (R^{(\lambda \cup (I, k))}) \cap (\sum_{k<n} x_k^I (\lambda \cup (I, k))) R \otimes R/A \simeq \bigoplus_{\lambda \in \Lambda_n \setminus \Lambda_{n-1}} \bigoplus_{I \in \mathcal{I}} S_I.
$$

In particular, $C_G = \{R/A\} \cup S$-filtered, and the quotient $C_G/C_0$, which is clearly isomorphic to $C'_G$, is $S$-filtered.

**Step IV:** We claim that $\text{Gen}(T_G) = \text{Div}-G$.

Since $C'_G = C_G/C_0$, it is enough to show $\text{Gen}(C_G) = \text{Div}-G$. In $C_G$, the image of any basis element $\lambda$ is identified with the linear combination $\sum_{k<n} x_k^I (\lambda \cup (I, k))$ with coefficients from $I$, so $C_G \in \text{Div}-G$. Note that this implies that $C_G$ is an $R/A$-module. For the other inclusion, let $M \in \text{Div}-G$. It is clear that $AM = 0$, and therefore there is a surjection $\pi : (C_0)^{I(\bar{\omega})} \simeq (R/A)^{I(\bar{\omega})} \rightarrow M$. Since $\text{Ext}_1^{R/A}(S_I, M) = 0$ for every $I \in \mathcal{I}$ by Step II, we have by Step III and by the Eklof Lemma that $\text{Ext}_1^{R/A}(C_G', M) = 0$, and thus the $R/A$-homomorphism $\pi$ can be extended to a (surjective) map $C_G^{I(\bar{\omega})} \rightarrow M$, proving the claim.
Step V: The map \( \varphi_G \) induces a commutative diagram with exact rows

\[
\begin{array}{cccc}
K & \overset{\varphi_G}{\longrightarrow} & F & \longrightarrow C_G & \longrightarrow 0 \\
& & & & \\
0 & \longrightarrow & K/\text{Ann}(F) & \overset{\varphi_G}{\longrightarrow} & F/\text{Ann}(F) & \longrightarrow C_G & \longrightarrow 0
\end{array}
\]

and the analogous result holds for \( \varphi'_G \).

Indeed, the \( R/A \)-module \( C_G \) is the cokernel of \( \varphi_G \). Further, since \( K/\text{Ann}(F) \) is a free \( R/A \)-module with basis \( \Lambda \times I \), injectivity of \( \varphi_G \) amounts to showing that the elements \( \varphi_G((\lambda, I)) \) with \( (\lambda, I) \in \Lambda \times I \) form an \( R/A \)-linearly independent subset in \( F/\text{Ann}(F) \). To this end, we prove in next paragraph that for each \( n \in \omega \) the elements \( \varphi_G((\lambda, I)) \) where \( \lambda \) has length \( n \) form a linearly independent subset in the free \( R/A \)-module \( F_{n+1}/F_n \otimes_R R/A \) with basis \( \Lambda_{n+1} \setminus \Lambda_n \). Then indeed, as \( \varphi_G(\Lambda_{n-1} \times I) \subseteq F_n \) for each \( n > 0 \), a simple induction argument shows the linear independence of the \( \varphi_G \)-image of \( \Lambda_n \times I \) for each \( n \in \omega \), and thus of the \( \varphi_G \)-image of the whole basis \( \Lambda \times I \).

For any sequence \( \lambda \in \Lambda \) of length \( n \) and any \( I \in \mathcal{I} \), the image of \( (\lambda, I) \) in \( F_{n+1}/F_n \otimes_R R/A \) is equal to \( \sum_{k \leq n} (x_k + A)(\lambda \sqcup (I, k)) \). As these elements are linear combinations of pairwise disjoint subsets of \( \Lambda \), it is clear that their spans are independent in the free \( R/A \)-module \( F \otimes_R R/A \) with basis \( \Lambda \). To prove \( R/A \)-linearity independency, it remains to show that these elements have zero annihilator over \( R/A \). But that follows from Step I, as \( \text{Ann}(\sum_{k \leq n} (x_k + A)(\lambda \sqcup (I, k))) = \text{Ann}(R/A)((I + A)/A) = 0 \).

So \( \varphi_G \) is injective, and the proof of injectivity of \( \varphi'_G \) is completely analogous.

Step VI: \( \mathcal{D}_{\sigma_G} = \text{Div-} \mathcal{G} \).

Let \( M \in \mathcal{D}_{\sigma_G} \). We first show that \( AM = 0 \). For any \( m \in M \), define map \( f : K \to M \) by setting \( f((\lambda, I)) = m \) for each \( (\lambda, I) \in \Lambda \times I \). As \( \mathcal{D}_{\sigma_G} \subseteq \mathcal{D}_{\varphi_G} \), there is a map \( g : F' \to M \) such that \( f = g \varphi_G \). But \( \varphi'_G((w, I)) = \sum_{k \leq n} x_k(I, k) \) annihilates \( \text{Ann}(I) \). It follows that \( \text{Ann}(I)M = 0 \) for all \( I \in \mathcal{I} \), and thus \( AM = 0 \). Now, since \( M \in \mathcal{D}_{\sigma_G} \) also implies \( \text{Ext}^1_R(G, M) = 0 \) by Lemma 2.1, we can conclude as in Step IV that \( M \in \text{Gen } C_G = \text{Div-} \mathcal{G} \).

Conversely, let \( M \in \text{Div-} \mathcal{G} \), and let \( f : P_{-1} \to M \) be a map. By Step V we have a commutative diagram with exact rows

\[
\begin{array}{cccc}
P_{-1} & \overset{\sigma_G}{\longrightarrow} & P_0 & \longrightarrow T_G & \longrightarrow 0 \\
\pi \downarrow & & \psi \downarrow & & \\
0 & \longrightarrow & P_{-1}/\text{Ann}(P_{-1}) & \overset{\varphi_G}{\longrightarrow} & P_0/\text{Ann}(P_0) & \longrightarrow T_G & \longrightarrow 0
\end{array}
\]

where the vertical maps \( \pi \) and \( \psi \) are the canonical projections. As \( M \) is an \( R/A \)-module, the map \( f \) can be factorized through \( \pi \), say \( f = f'\pi \). Now \( T_G \) is \( \{ R/A \} \cup S \)-filtered by Step III, so the Eklof Lemma and Step II imply \( \text{Ext}^1_R(T_G, M) = 0 \). Therefore, there is a map \( h : P_0/\text{Ann}(P_0) \to M \) such that \( f' = h \varphi_G \). Then \( f = h\varphi_G \), proving that \( M \in \mathcal{D}_{\sigma_G} \).

Theorem 4.7. Let \( R \) be a commutative ring. There is a 1-1 correspondence between

(i) silting classes \( \mathcal{D} \) in \( \text{Mod-} R \),

(ii) Gabriel filters of finite type \( \mathcal{G} \) over \( R \).

The correspondence is given as follows:

\[
\Theta : \mathcal{G} \mapsto \text{Div-} \mathcal{G},
\]

\[
\Xi : \mathcal{D} \mapsto \{ I \subseteq R \mid M = IM \text{ for all } M \in \mathcal{D} \}.\]
Proof. By Proposition 4.3 and Proposition 4.6, both maps of the correspondence are well defined. By Proposition 4.3 it is clear that \( \Theta(\Xi(D)) = D \) for any silting class \( D \). That \( \Xi(\Theta(\mathcal{G})) = \mathcal{G} \) for any Gabriel topology of finite type follows from [17, Chapter VI, Theorem 5.1], and by character duality. \( \square \)

In [16], it was asked whether any tilting torsion pair \((\mathcal{T}, \mathcal{F})\) of finite type is classical (that is, \( \mathcal{T} \) is generated by a finitely presented tilting module). The answer turned out to be negative for general rings, but positive for commutative rings ([7]). We remark that for commutative rings, this means that \( \mathcal{F} \) is closed under direct limits if and only if \( \mathcal{F} = \{0\} \). We conclude this section with a generalization of this phenomenon for silting classes.

**Proposition 4.8.** Let \( R \) be a commutative ring, \( T \) a silting \( R \)-module, and \((D, \mathcal{F}) = (\text{Gen}(T), \text{Ker} \text{Hom}_R(T, -))\) the associated torsion pair. The following are equivalent:

(i) \( T \) is projective,

(ii) there is a finitely presented silting \( R \)-module generating \( D \),

(iii) \( \mathcal{F} \) is closed under direct limits,

(iv) \( D = \text{Gen}(Re) \) for a (central) idempotent \( e \in R \).

Proof. Denote \( A = \text{Ann}(T) = \text{Ann}(D) \). By [3, Proposition 3.13 and Lemma 3.4], \( T \) is a tilting \( R/A \)-module. Moreover, it is easy to check that \( R/A \in D \) if and only if \( D = \text{Mod--}R/A \), or equivalently, \( D = \text{Gen } R/A \). In this case, \( A \) is an idempotent ideal with \( D = \text{Ker Hom}_R(A, -) \subseteq (R/A)^{-1} \), cf. [3, Proposition 2.5].

(i) \( \rightarrow \) (iv): As \( T \) is also projective as an \( R/A \)-module, \( D = \text{Ker Ext}^1_{R/A}(T, -) = \text{Mod--}R/A \). Then \( R/A \in \text{Add}(T) \) is a projective \( R \)-module. Hence, \( R/A = Re \) for an idempotent \( e \in R \), and (iv) follows.

(ii) \( \rightarrow \) (iv): Let \( T' \) be a finitely presented silting module such that \( \text{Gen}(T') = D \). Then \( T' \) is a finitely presented tilting \( R/A \)-module, which is projective by [10, Proposition 13.2]). Hence, \( D = \text{Mod--}R/A \), and \( A \) is an idempotent ideal. Also, \( R/A \in \text{Add}(T') \) is finitely presented, and thus \( A \) is finitely generated. It follows from [9, Proposition 1.10(i)] that \( A = Rf \) for some idempotent \( f \in R \), and thus \( R/A = R(1 - f) \), proving (iv).

(iii) \( \rightarrow \) (iv): Consider the (tilting) torsion pair \((D, \mathcal{F}')\) in \( \text{Mod--}R/A \), where \( \mathcal{F}' = \text{Ker Hom}_{R/A}(T, -) \). Then \( \mathcal{F}' \) is closed under direct limits, and thus \( D = \text{Mod--}R/A \) by [11] or [12, Theorem 4.6]. In particular, \( A \) is an idempotent ideal.

We claim that \( A \) is finitely generated. Let us write \( A \) as a direct union of its finitely generated subideals, \( A = \varinjlim_{j \in J} I_j \). Denote by \( K_j \) the ideal such that \( R/K_j \) is the torsion-free quotient of \( R/I_j \) with respect to the torsion pair \((D, \mathcal{F})\). Then \( K_i \subseteq K_j \) whenever \( i \leq j \in J \). Since \( \mathcal{F} \) is closed under direct limits, we have that \( \varinjlim_{j \in J} R/K_j = R/\bigcup_{j \in J} K_j \) is in \( \mathcal{F} \), and thus zero, because \( R/A \in D \), and \( A \subseteq \bigcup_{j \in J} K_j \). It follows that there is \( j \in J \) such that \( R = K_j \), and therefore \( R/I_j \in D \). But \( D = \text{Mod--}R/A \), and \( I_j \subseteq A \), which forces \( I_j = A \).

We now conclude this implication as in (ii) \( \rightarrow \) (iv).

(iv) \( \rightarrow \) (i), (ii), (iii): As \( \mathcal{F} = \text{Gen}(R(1 - e)) \), condition (iii) is clear. Consider the map \( \sigma : R \rightarrow Re \oplus Re \) given by the canonical projection of \( R \) onto the first direct summand \( Re \). Then \( \text{Ker}(\sigma) = R(1 - e) \), and clearly \( D_\sigma = \text{Ker Hom}_R(R(1 - e), -) = \text{Gen}(Re) \). Hence, \( \text{Coker}(\sigma) = Re \) is a silting module generating \( \text{Gen}(Re) \). This proves (ii). Finally, \( T \in \text{Add} Re \), and thus \( T \) is projective. \( \square \)

The following example shows that, in contrast with tilting modules over commutative rings, we cannot replace “finitely presented” by “finitely generated” in Proposition 4.8(ii).
Example 4.9. Let \( k \) be a field, \( \kappa \) an infinite cardinal, and \( R = k^\kappa \). Consider the Gabriel filter \( \mathcal{G} \) over \( R \) with basis consisting of all principal ideals generated by elements of \( k^\kappa \), such that their support is cofinite in \( \kappa \). Let \( D = \text{Div-} \mathcal{G} \) be the associated silting class. Then \( A = \text{Ann}(D) = \sum_{I \in \mathcal{G}} \text{Ann}(I) \) is equal to \( k^{(|\kappa|)} \subseteq R \). Because \( A + I = R \) for any \( I \in \mathcal{G} \), we have that \( R/A \in D \), and therefore \( D = \text{Gen } R/A = \ker \text{Hom}_R(A,-) \subseteq (R/A)^\perp \) (cf. the proof of Proposition 4.8). We claim that \( R/A \) is a silting module.

For each \( a \in \kappa \), consider the idempotent \( e_a \in R \) with \( a \)-th component equal to 1, and all other components equal to zero. Taking the direct sum of the split exact sequences \( 0 \rightarrow e_a R \rightarrow R \rightarrow (1 - e_a)R \rightarrow 0 \), we obtain a split exact sequence \( 0 \rightarrow A \rightarrow R \rightarrow \bigoplus_{a \in \kappa} (1 - e_a)R \rightarrow 0 \), where \( D_\pi = \ker \text{Hom}_R(A,-) = D \subseteq (R/A)^\perp \). The map \( \sigma = \iota \oplus \pi \in \text{Mor(Proj-R)} \) then satisfies \( \text{Coker}(\sigma) = \text{Coker}(\iota) = R/A \), and as \( \iota \) is monic, \( D_\pi = (R/A)^\perp \cap D_\pi = D_\pi = \text{Gen } R/A \). We proved that \( R/A \) is silting.

Finally, note that \( R/A \) is not finitely presented, and thus not projective.

## 5. Cosilting modules over commutative rings

If \( R \) is a commutative noetherian ring, then all Gabriel filters and all hereditary torsion pairs are of finite type, and they correspond bijectively to subsets of \( \text{Spec}(R) \) closed under specialization. Recall that a subset \( P \subseteq \text{Spec}(R) \) is closed under specialization if \( p \in P \) implies that all prime ideals \( q \supseteq p \) belong to \( P \). Such \( P \) gives rise to a hereditary torsion pair \((\mathcal{T}(P), \mathcal{F}(P))\) where \( \mathcal{F}(P) = \{ M \in \text{Mod-R} \mid \text{Hom}_R(R/p, M) = 0 \text{ for all } p \in P \} \), and the assignment \( P \mapsto (\mathcal{T}(P), \mathcal{F}(P)) \) defines the stated bijection. For details we refer to [17] Chapter VI, §6.6.

### Theorem 5.1. If \( R \) is a commutative noetherian ring, there are bijections between

i. silting classes \( \mathcal{D} \) in \( \text{Mod-R} \),

ii. subsets \( P \subseteq \text{Spec}(R) \) closed under specialization,

iii. Gabriel filters \( \mathcal{G} \) over \( R \),

iv. cosilting classes \( \mathcal{C} \) in \( \text{Mod-R} \).

In particular, every cosilting class is of cofinite type.

**Proof.** Apply Corollary 3.7 and Theorem 4.7. \( \square \)

Next, we provide a construction for a cosilting module cogenerating the \( \mathcal{G} \)-torsion-free modules for a given Gabriel filter \( \mathcal{G} \). It is inspired by the construction of cotilting modules over commutative noetherian rings in [11].

### Construction 5.2. Let \( R \) be commutative, and \( \mathcal{G} \) be a Gabriel filter of finite type. Let \( (\mathcal{T}_G, \mathcal{F}_G) \) be the associated hereditary torsion pair from [4,3] that is, \( \mathcal{F}_G = \bigcap_{I \in \mathcal{G}} \ker \text{Hom}_R(R/I,-) \), and \( \mathcal{T}_G \) consists of the modules \( M \) for which every element \( m \in M \) is annihilated by some \( I \in \mathcal{G} \). Let us construct a cosilting module \( C_G \) such that \( \text{Cogen } C_G = \mathcal{F}_G \).

First, if \( \mathcal{F}_G \) is a hereditary torsion-free class, there is an injective module \( E \) with \( \text{Cogen } E = \mathcal{F}_G \). Indeed, we can put \( E = \prod \{ E(R/J) \mid R/J \in \mathcal{F}_G \} \). Then \( E \) is injective, \( E \in \mathcal{F}_G \), and any module from \( \mathcal{F}_G \) is easily seen to be cogenerated by \( E \).

Next, we let \( E_1 = \prod \{ E(R/I) \mid I \in \mathcal{G} \} \). Since \( \mathcal{G} \) is of finite type, \( \mathcal{F}_G \) is definable, and thus a precovering class. Let \( f : F \rightarrow E_1 \) be a \( \mathcal{F}_G \)-precover of \( E_1 \). Since \( E_1 \) is injective, we can extend \( f \) to a map \( \overline{f} : E(F) \rightarrow E_1 \). As \( E(F) \in \mathcal{F}_G \), the map \( \overline{f} \) is also an \( \mathcal{F}_G \)-precover of \( E_1 \). Then \( E_0 = E \oplus E(F) \) is an injective module in \( \mathcal{F}_G \). Denote by \( k : K \rightarrow E(F) \) the kernel of \( f \), and consider the following exact
sequence:

\[
0 \to E \oplus K \xrightarrow{\begin{pmatrix} 1_E & 0 \\ 0 & k \end{pmatrix}} E_0 \xrightarrow{(0 \ f)} E_1.
\]

We claim that \( C_\mathcal{G} = E \oplus K \) is a cosilting module with respect to the map \( \lambda = (0 \ f) \), and \( \text{Cogen} \left( C_\mathcal{G} \right) = \mathcal{F}_\mathcal{G} \).

Since \( \text{Cogen} (E) = \mathcal{F}_\mathcal{G} \), and \( K \) is isomorphic to a submodule of \( E(F) \in \mathcal{F}_\mathcal{G} \), we have \( \text{Cogen} (C_\mathcal{G}) = \mathcal{F}_\mathcal{G} \). Further, if \( M \in \mathcal{F}_\mathcal{G} \), then any map \( g : M \to E_1 \) factors through the \( \mathcal{F}_\mathcal{G} \)-precover \( f \) of \( E_1 \), so there is \( h : M \to E(F) \) such that \( g = fh = \lambda \left( \begin{pmatrix} 0 \\ h \end{pmatrix} \right) \), and thus \( M \in C_\lambda \). This proves that \( \mathcal{F}_\mathcal{G} \subseteq C_\lambda \).

Let now \( M \) be an \( R \)-module such that the \( T_\mathcal{G} \)-torsion part \( M' \) of \( M \) is non-zero. Choose any non-zero cyclic submodule \( R/I \) of \( M' \). As necessarily \( I \in \mathcal{G} \), there is a non-zero map \( g : R/I \to E_1 \), which extends to \( \bar{g} : M \to E_1 \). Suppose that there is \( h : M \to E_0 \) such that \( \bar{g} = \lambda h \). Then \( h_{1M} \) is a non-zero map \( M' \to E_0 \) with \( E_0 \in \mathcal{F}_\mathcal{G} \), a contradiction. Therefore, \( M \notin C_\lambda \). We have \( \mathcal{F}_\mathcal{G} = C_\lambda \) as desired.

**Corollary 5.3.** Let \( R \) be a commutative ring. With the notation of Constructions \[118\] and \[122\] \( \{ T_\mathcal{G} \mid \mathcal{G} \text{ a Gabriel filter of finite type} \} \) is a set of representatives, up to equivalence, of all silting \( R \)-modules, and \( \{ C_\mathcal{G} \mid \mathcal{G} \text{ a Gabriel filter of finite type} \} \) is a set of representatives, up to equivalence, of all cosilting \( R \)-modules of cofinite type.

We close this note with an example of a cosilting module which is not of cofinite type. The same module is also an example for a finendo quasitilting \[3\] Proposition 3.10).

**Example 5.4.** Let \( R \) be a commutative local ring with a non-zero idempotent maximal ideal \( m \) (e.g. any valuation domain with non-zero idempotent radical, such as the ring of Puiseux series over a field). We consider the module \( R/m \).

Since \( m \) is idempotent, the class \( \mathcal{C} = \text{Gen} (R/m) = \text{Add}(R/m) \) is a torsion class contained in \( (R/m)^\perp \). The natural projection \( R \to R/m \) is clearly seen to be a \( \mathcal{C} \)-preenvelope. The cokernel of this map is zero, and \[3\] Proposition 3.2 shows that \( R/m \) is a finendo quasitilting module. On the other hand, \( \mathcal{C} \) is not silting by Theorem \[4,7\]. Indeed, the only ideal \( R/m \) divisible by is \( R \). But \( \mathcal{C} \neq \text{Mod-}R \), because \( m \notin \mathcal{C} \), as \( m^2 = m \neq 0 \).

The same class \( \mathcal{C} \) is a cosilting class not of cofinite type. Indeed, \( \mathcal{C} \) is closed for direct products, and thus it coincides with \( \text{Cogen} (R/m) \). We prove that \( R/m \) is a cosilting module. Let \( 0 \to R/m \to E_0 \xrightarrow{\varphi} E_1 \) be the begining of the minimal injective coresolution of \( R/m \). Define an injective module \( E = \prod \{ E(R/J) \mid J \subseteq R \text{ such that } \text{Soc} R/J = 0 \} \). Let \( \sigma : E_0 \to E_1 \oplus E \) be the direct sum of \( \varphi \) and the zero map \( 0 \to E \). We prove that \( C_\sigma = \mathcal{C} \).

Note that the image of any map \( f : R/m \to E_1 \oplus E \) is contained in \( E_1 \) by the definition of \( E \). By the essentiality of the image of \( \varphi \) in \( E_1 \), \( f \) is actually a map \( R/m \to \text{Im} \varphi \). Since \( \text{Ext}_R^1(R/m, R/m) = 0 \) by the idempotency of \( m \), we have that \( R/m \in C_\sigma \), and thus \( \mathcal{C} \subseteq C_\sigma \).

Let now \( M \in \text{Mod-}R \) be such that \( mM \neq 0 \). Then \( M \) contains a cyclic submodule \( R/I \) with \( m \notin I \). Using injectivity, it is easy to show that \( R/I \notin \mathcal{C}_\sigma \). If \( \text{Soc} R/I = 0 \), then \( R/I \) injects into \( E \), and this injection clearly cannot be factorized through \( \sigma \). If \( \text{Soc} R/I = 0 \), let \( J \) be an ideal such that \( (R/I)/\text{Soc} R/I \cong R/J \). Then \( J \neq R \), because in such case \( \text{Soc} R/I = R/I \), implying that \( \text{Ann}(R/I) = m \), and thus \( R/I = R/m \), which is not the case. If \( \text{Soc} R/J \neq 0 \), the full preimage of this socle in \( R/I \) would be a non-trivial extension of two semisimple modules, which
does not exist by idempotency of $m$. Hence, $\text{Soc } R/J = 0$, and the composition of the projection $R/I \to R/J$ with inclusion $R/J \to E$ is a non-zero map $R/I \to E$. Again, this map cannot be factorized through $\sigma$. Hence, $R/I \notin C_\sigma$, and $C_\sigma = \text{Cogen } (R/m)$.

Finally, the class $C$ is not of cofinite type. Indeed, the only injective the class $C$ contains is zero, and thus it is not of cofinite type by Lemma 4.2.

References

[1] T. Adachi, O. Iyama, I. Reiten, $\tau$-tilting theory, Compos. Math. 150 (2014), 415–452.
[2] F.W. Anderson, K.R. Fuller, Rings and Categories of Modules, Second Edition, Springer-Verlag, 1992.
[3] L. Angeleri Hügel, F. Marks, J. Vitória, Silting modules, International Mathematics Research Notes 2015, doi:10.1093/imrn/rnv191.
[4] L. Angeleri Hügel, F. Marks, J. Vitória, Silting modules and ring epimorphisms, preprint, arXiv:1504.07169.
[5] L. Angeleri Hügel, D. Pospšíšil, J. Šťovíček, J. Trlifaj, Tilting, cotilting, and spectra of commutative noetherian rings. Transactions Amer. Math. Soc. 366 (2014), 3487–3517.
[6] S. Bazzoni, Cotilting and tilting modules over Prüfer domains, Forum Math. 19 (2007), 1005–1027.
[7] S. Bazzoni, I. Herzog, P. Průhoda, J. Šaroch, J. Trlifaj, in preparation.
[8] S. Breaz, F. Pop, Cosilting modules, preprint, arXiv:1501.05988.
[9] L. Fuchs, L. Salce, Modules over Non-Noetherian Domains, AMS, 2001.
[10] R. Göbel, J. Trlifaj, Approximations and Endomorphism Algebras of Modules, GEM 41, W. de Gruyter, Berlin 2006.
[11] D. Herbera, J. J. Šťovíček, J. Trlifaj, Cotilting modules over commutative noetherian rings. Journal of pure and applied algebra 218 (2014), 1696-1711.
[12] M. Hrbek, One-tilting classes and modules over commutative rings, preprint, arXiv:1507.02811.
[13] F. Marks, J. Šťovíček, Torsion classes, wide subcategories and localisations, preprint, arXiv:1505.03639.
[14] F. Marks, J. Šťovíček, in preparation.
[15] M. Prest, Purity, spectra and localisation, Cambridge University Press 2009.
[16] C. E. Parra, M. Saorín, Direct limits in the heart of a $t$-structure: the case of a torsion pair. Journal of pure and applied algebra 219 (2015), 4117-4143.
[17] B. Stenström. Rings of quotients. Springer-Verlag (1975).
[18] J. Wei, P. Zhang Cosilting complexes and AIR-cotilting modules, preprint, arXiv 1601.01385.

E-mail address: lidia.angeleri@univr.it
E-mail address: hrbmich@gmail.com