Trees of metric compacta and trees of manifolds

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Introduction

We present a construction, called the *limit of a tree system of spaces* (or, less formally, a *tree of spaces*). The construction is designed to produce compact metric spaces that resemble fractals, out of more regular spaces, such as closed manifolds, compact polyhedra, compact Menger manifolds, etc. Such spaces are potential candidates to be homeomorphic to ideal boundaries of infinite groups.

A very special case of this construction, *trees of manifolds* (known also as *Jakobsche spaces*), has been studied in the literature (see [AS], [J1], [J2], [St]). We present here a different approach, much more general, and, as we believe, much more convenient for establishing various basic properties of the resulting spaces, in a more general setting. Already in the case of trees of manifolds, using this approach we clarify, correct and extend so far known results and properties.

*A wider context for the results of the present paper.*

Our motivation for dealing with the general construction as presented in this paper comes from an attempt to understand which topological spaces are boundaries of hyperbolic groups. This problem, stated e.g. in [KK] as Question A, or as Problem 2.1 in "Problems on boundaries of groups and Kleinian groups" by M. Kapovich (see the web page http://aimath.org/pggt/Boundaries), remains widely open. An overview of the limited knowledge concerning this problem can be found in Section 17 of [KB] or in the introduction in [PS].

The present paper initiates a larger research project, investigated by the author, concerning the above problem. We briefly outline the aims and expected lines of further development in this project.

- In a paper under preparation we describe a vast class of topological spaces called *trees of polyhedra*. These spaces are compact, metrizable, have finite topological dimension, and typically they are "wild" (e.g. they are usually not locally contractible, and hence not ANR). They are obtained as limits of some tree systems, and depend uniquely up to homeomorphism on certain finite data, part of which is a finite collection of compact polyhedra. Thus, the spaces are given not by a universal characterization in terms of a list of properties, but rather by a sort of "presentation" (similar in spirit to a presentation of a group in combinatorial group theory). Typically, the same space has many distinct "presentations", and clarification of the relationship between such "presentations" will be one of the challenges.

- Next part of the project consists of identifying boundaries of various classes of groups as explicit trees of polyhedra. We have formulated several conjectures in this direction.

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based on some results from the literature (see Remark 2.C.8), and on our new partial results. One of these conjectures deals with Gromov boundaries of all groups obtained by any procedure of strict hyperbolization (e.g. the one described in [CD]). Another conjecture concerns CAT(0) boundaries of a large class of right angled Coxeter groups, namely the ones that enjoy some partial hyperbolicity property, weaker than word hyperbolicity. We have already confirmed these conjectures in some cases when the boundaries are trees of manifolds (in arbitrary dimension), trees of graphs, or trees of various 2-dimensional polyhedra which can be recognized as the Menger curve or the Sierpiński curve. It seems that boundaries of numerous (maybe even most of) other groups so far studied in the literature are trees of polyhedra.

- A question arises, for which hyperbolic groups their Gromov boundaries are not trees of polyhedra. The following examples seem to belong to this class:
  1. groups whose Gromov boundaries are Menger compacta of dimension $\geq 2$ (see [DO] for examples of such groups, with boundaries of dimension 2 and 3); we suspect that Menger compacta satisfy stronger disjoint disk properties than any trees of polyhedra of the same topological dimension;
  2. 7-systolic groups, as defined in [JS], with boundaries of dimension $\geq 3$; by [Sw1], Gromov boundaries of 7-systolic groups contain no copy of the 2-disk, which cannot happen for a tree of polyhedra in dimensions $\geq 3$;
  3. topologically rigid hyperbolic groups, examples of which have been constructed in [KK]; we expect that trees of polyhedra always admit homeomorphisms whose dynamics is different from that occurring for the induced action of a group on its boundary.

It seems to be known that boundaries of all hyperbolic groups fall in the class of spaces called Markov compacta (as defined e.g. in [Dr]). We work on showing that trees of polyhedra coincide with a subclass of Markov compacta of some rather simple form. This suggests a possibility to introduce certain notion of degree of complexity for Markov compacta, with the lowest degree corresponding to trees of polyhedra. A big challenge is to explore more fully the territory of hyperbolic groups with higher degree boundaries (i.e. boundaries which are not trees of polyhedra). For example, one can ask for new strict hyperbolization procedures, resulting with groups of higher degree boundaries. One can also ask for explicit description of such boundaries, perhaps starting with degree just above the lowest one.

The content of the paper.

In Part 1 we introduce the notions of a tree system of metric compacta (Section 1.B) and its limit (Section 1.C). We then show that such limit is always a compact metrizable space (Section 1.D). In Section 1.E we introduce the notion of isomorphism of tree systems. We also present a class of natural examples - dense tree systems of closed (topological) manifolds, and a class of spaces obtained as their limits - the trees of manifolds $M$. As we show in Part 2 of the paper, trees of manifolds $M$ coincide with the spaces studied earlier by W. Jakobsche in [J2] (for $M$ oriented) and by P. Stallings in [St] (for non-orientable $M$). Our exposition of the case of non-orientable manifolds $M$, in Subsection 1.E.3, concerns all topological manifolds (and not only PL ones, as in [St]), and it frees the description from certain inconvenient and unnecessary condition present in Stallings’ approach (see
Remark 1.E.3.3).

In Part 2 of the paper we show that limits of tree systems can be expressed as inverse limits of some inverse systems of spaces. In Section 2.A we describe inverse systems naturally associated to tree systems, and in Section 2.B we show that their inverse limits canonically coincide with limits of the corresponding tree systems. Section 2.C deals with a subclass of tree systems called peripherally ANR, and describes inverse systems of particularly nice form associated to such tree systems. This allows to relate our construction of limit of a tree system, in the case of dense tree systems of manifolds, to some earlier constructions from the literature, notably the construction of Jakobsche [J2]. In Section 2.D we apply associated inverse systems to provide estimates from above for the topological dimension of limits of tree systems. We indicate some general cases in which these estimates are sharp.

In Part 3 of the paper we introduce some natural and useful operations on tree systems. In Section 3.A we describe an operation of consolidation, by which the spaces appearing in the initial system are merged together into bigger spaces, constituting naturally a new tree system. We show that this operation does not affect the limit. As an application, we derive equalities (up to homeomorphism) between the limits of various different tree systems of manifolds. We also include a correction to the main result of H. Fischer in [Fi], in which he identifies boundaries of certain right angled Coxeter groups as some explicit trees of manifolds $M$ (see Theorem 3.A.3 in the text, and a comment after its statement). In Section 3.B we show how to decompose a compact metric space $X$ into pieces which form a tree system, so that $X$ naturally coincides with the limit of this tree system. We also show how to use such decompositions to determine limits of certain tree systems. In Section 3.C we introduce subdivision of a tree system, an operation opposite to consolidation, which generalizes the operation of decomposition from Section 3.B. In Section 3.D we apply operations of subdivision and consolidation to study orbits under homeomorphisms in trees of manifolds. The presented method is potentially applicable to more general tree systems. Finally, in Section 3.E we apply operations of consolidation and subdivision to provide a much more flexible description of certain trees of manifolds than those so far present in the literature (see Theorem 3.E.2 and Corollary 3.E.4 in the text). In the companion paper [Sw2], we use this description to identify ideal boundaries of many groups as trees of manifolds. In particular, we show in [Sw2] that trees of manifolds in arbitrary dimension appear as Gromov boundaries of certain hyperbolic groups.

1. Tree systems and their limits.

1.A Some terminology and notation concerning trees.

Trees under our consideration will be usually countable infinite, and locally infinite. We denote by $V_T$ the set of all vertices of a tree $T$, and by $O_T$ the set of its all oriented edges. For any $e \in O_T$, we denote by $\alpha(e), \omega(e)$ the initial and the terminal vertex of $e$, respectively. We also denote by $\bar{e}$ the same geometric edge as $e$, but oppositely oriented. For any $t \in V_T$, we denote by $N_t = \{e \in O_T : \alpha(e) = t\}$ the set of all oriented edges of $T$ with initial vertex $t$.

We denote the combinatorial (embedded) paths in $T$ as sequences of consecutive vertices $[t_0, t_1, \ldots, t_m]$, or as sequences $[e_1, \ldots, e_m]$ of consecutive oriented edges, or shortly
by $[t, s]$ if $t$ and $s$ are the ends of the path. An infinite combinatorial path $[t_0, t_1, \ldots]$ in $T$ is called a ray, and denoted usually by $\rho$. We denote by $\rho(0)$ the initial vertex $t_0$, and by $e_1(\rho)$ the initial oriented edge $(t_0, t_1)$ of a ray $\rho$.

We denote by $E_T$ the set of ends of $T$, i.e. the set of equivalence classes of rays in $T$ with respect to the relation of coincidence except possibly at some finite initial parts. We denote the end determined by a ray $\rho$ as $[\rho]$.

Let $S$ be a subtree of $T$. We then distinguish the set

$$N_S = \{e \in O_T : \alpha(e) \in V_S \text{ and } \omega(e) \notin V_S\}.$$ 

Note that, in case when $S$ is reduced to a single vertex $t$, this notation agrees with the earlier introduced notation for the set $N_t$. A finite subtree of $T$ will be usually denoted by $F$, and we shall consider the poset $(\mathcal{F}_T, \subset)$ of all finite subtrees of $T$.

1.B Tree systems of spaces.

Recall that a family of subsets of a compact metric space is null if for each $\epsilon > 0$ all but finitely many of these subsets have diameters less than $\epsilon$.

1.B.1 Definition. A tree system of metric compacta is a tuple $\Theta = (T, \{K_t\}, \{\Sigma_e\}, \{\phi_e\})$ such that:

- (TS1) $T$ is a countable tree;
- (TS2) to each $t \in V_T$ there is associated a compact metric space $K_t$;
- (TS3) to each $e \in O_T$, there is associated a nonempty compact subset $\Sigma_e \subset K_{\alpha(e)}$, and a homeomorphism $\phi_e : \Sigma_e \to \Sigma_{\tilde{e}}$ such that $\phi_{\tilde{e}} = \phi_e^{-1}$;
- (TS4) for each $t \in V_T$ the family $\Sigma_e : e \in N_t$ is null and consists of pairwise disjoint sets.

We call $T$ the underlying tree, $K_t : t \in V_T$ the constituent spaces, $\Sigma_e : e \in O_T$ the peripheral subspaces, and $\phi_e : e \in O_T$ the connecting maps of the tree system $\Theta$.

Remark. In future applications of tree systems of spaces we will often additionally require that for any $t \in V$ the family $\Sigma_e : e \in N_t$ is dense in the space $K_t$ (which means that the union of this family is a dense subset). However, to establish many basic properties of tree systems of metric compacta we do not need this requirement.

1.B.2 Example: tree system of manifolds. Let $T$ be a countable tree. Let $M_t : t \in V_T$ be a family of closed manifolds of the same dimension $n$. For each $t \in V_T$ and each $e \in N_t$ let $\Delta_e$ be a collared $n$-dimensional disk embedded in $M_t$, and suppose that each $\mathcal{D}_t = \{\Delta_e : e \in N_t\}$ is a null family of pairwise disjoint subsets of $M_t$.

For each $t \in V_T$ put

$$K_t = M_t \setminus \bigcup \{\text{int}(\Delta_e) : e \in N_t\}.$$

This defines a family $\{K_t\}$ in a tree system of manifolds.

For each $e \in O_T$, put $\Sigma_e = \partial \Delta_e$, and note that $\Sigma_e \subset K_{\alpha(e)}$. For each $e \in O_T$ consider also a homeomorphism $\phi_e : \Sigma_{\alpha(e)} \to \Sigma_{\omega(e)}$ between the corresponding $(n - 1)$-spheres, so
that \( \phi_e = \phi_e^{-1} \) for each \( e \). A tree system of manifolds is a tuple \( M = (T, \{K_t\}, \{\Sigma_e\}, \{\phi_e\}) \) as described above.

Intuitively, such a system \( M \) may be viewed as a pattern for a connected sum operation applied at the same time to a countable (in general infinite) family of closed manifolds.

1.C Limit of a tree system of spaces.

We now describe the limit of a tree system \( \Theta = (T, \{K_t\}, \{\Sigma_e\}, \{\phi_e\}) \), denoted \( \lim \Theta \), starting with its description as a set. Denote by \( \#\Theta \) the quotient

\[
(\bigcup_{t \in V_T} K_t)/\sim
\]

of the disjoint union of the sets \( K_t \) by the equivalence relation \( \sim \) induced by the equivalences \( x \sim \phi_e(x) \) for any \( e \in O_T \) and any \( x \in \Sigma_e \). This set may be viewed as obtained from the family \( K_t : t \in V \) as a result of gluings provided by the maps \( \phi_e \). Observe that any set \( K_t \) canonically injects in \( \#\Theta \). Define \( \lim \Theta \) to be the disjoint union \( \#\Theta \cup E_T \), where \( E_T \) is the set of ends of \( T \).

To put appropriate topology on the set \( \lim \Theta \) we need some terminology. Given a family \( A \) of subsets in a set \( X \), we say that \( U \subset X \) is saturated with respect to \( A \) (shortly, \( A \)-saturated) if for any \( A \in A \) we have either \( A \subset U \) or \( A \cap U = \emptyset \).

For any finite subtree \( F \) of \( T \), denote by \( V_F, O_F \) the sets of vertices and of oriented edges in \( F \), respectively. Denote also by

\[
\Theta_F = (F, \{K_t : t \in V_F\}, \{\Sigma_e : e \in O_F\}, \{\phi_e : e \in O_F\})
\]

the tree system of spaces \( \Theta \) restricted to \( F \). The set \( K_F := \#\Theta_F \), equipped with the natural quotient topology (with which it is clearly a metrizable compact space), will be called the partial union of the system \( \Theta \) related to \( F \). Observe that any partial union \( K_F \) is canonically a subset in \( \#\Theta \), and thus also in \( \lim \Theta \).

For any finite subtree \( F \) put \( A_F = \{\Sigma_e : e \in N_F\} \), and view the elements of this family as subsets in the partial union \( K_F \). Observe that the family \( A_F \) consists of pairwise disjoint compact sets, and that it is null. Let \( U \subset K_F \) be a subset which is saturated with respect to \( A_F \). Put \( N_U := \{e \in N_F : \Sigma_e \subset U\} \) and

\[
D_U := \{t \in V_T : [\omega(e), t] \cap V_F = \emptyset \text{ for some } e \in N_U\}
\]

(i.e. \( D_U \) is the set of these vertices \( t \in V_T \setminus V_F \) for which the shortest path in \( T \) connecting \( F \) to \( t \) starts with an edge \( e \in N_U \)). Here we mean in particular that the vertices \( \omega(e) : e \in N_U \) all belong to \( D_U \).

Again, for any finite subtree \( F \) of \( T \) let \( R_F \) be the set of all rays \( \rho \) in \( T \) with \( \rho(0) \in V_F \), and with all other vertices outside \( V_F \). Note that the map \( \rho \mapsto [\rho] \) is then a bijection from \( R_F \) to the set \( E_T \) of all ends of \( T \). Given a subset \( U \subset K_F \) as above (i.e. saturated with respect to \( A_F \)), put \( R_U := \{\rho \in R_F : e_1(\rho) \in N_U\} \) and \( E_U := \{[\rho] \in E_T : \rho \in R_U\} \).

Finally, for any subset \( U \subset K_F \) as above, put

\[
G(U) := U \cup (\bigcup_{t \in D_U} K_t) \cup E_U,
\]
where all summands in the above union are viewed as subsets of $\lim \Theta$; clearly, $G(U)$ is then also a subset in $\lim \Theta$. We consider the topology in the set $\lim \Theta$ given by the basis $\mathcal{B}$ consisting of all sets $G(U)$, for all finite subtrees $F$ in $T$, and all open subsets $U \subset K_F$ saturated with respect to $\mathcal{A}_F$. We leave it as an easy and instructive exercise to check that the family $\mathcal{B}$ is closed under finite intersections, and hence it satisfies the axioms for a basis of topology.

1.C.1 Proposition. For any tree system $\Theta$ of metric compacta the limit $\lim \Theta$, with topology given by the above described basis $\mathcal{B}$, is a metrizable compact space. Moreover, for each finite subtree $F$ of $T$ the canonical inclusion (as a set) of the partial union $\#\Theta_F$ in $\lim \Theta$ is a topological embedding. In particular, for any $t \in V_T$ the canonical inclusion of $K_t$ in $\lim \Theta$ is a topological embedding. Finally, both families of subsets $\{K_t : t \in V_T\}$ and $\{\Sigma_e : e \in O_T\}$ in $\lim \Theta$ are null (with respect to any metric compatible with the topology).

The proof of Proposition 1.C.1 is rather long and uses certain separately constructed auxiliary metric space, and its properties. This construction and the proof of the proposition occupy Section 1.D.

1.C.2 Remark. If we skip the assumption that each of the families of subsets $\{\Sigma_e : e \in N_t\}$ in $K_t$, for any $t \in V_T$, is null, then the above description still defines a compact topological space $\lim \Theta$, but then the limit is in general not Hausdorff. Moreover, canonical inclusions $K_t \rightarrow \lim \Theta$ are then in general not embeddings (though they are always continuous).

1.D Limit $\lim \Theta$ as perfect quotient of a compact metric space - proof of Proposition 1.C.1.

In this section we prove Proposition 1.C.1 by giving an alternative description of the limit $\lim \Theta$ of a tree system $\Theta$ of metric compacta. In this description, we first construct an auxiliary compact metric space $T(\Theta^*)$, then show that $\lim \Theta$ is a topological quotient of $T(\Theta^*)$, and finally we note that the quotient map is perfect, thus preserving the property of metrisability.

We start with describing the metric space $T(\Theta^*)$. Let $\Theta = (T, \{K_t\}, \{\Sigma_e\}, \{\phi_e\})$ be any tree system of metric compacta. For each $e \in O_T$ we choose a base point $b_e \in \Sigma_e$ in such a way that $\phi_e(b_e) = b_{\bar{e}}$, and we denote by $\Theta^* = (T, \{K_t\}, \{\Sigma_e\}, \{\phi_e\}, \{b_e\})$ the extended tuple of data. We define first $T(\Theta^*)$ as a set, by putting

$$T(\Theta^*) = E_T \sqcup (\bigsqcup_{t \in V_T} K_t)/\sim$$

where $\sqcup$ is the disjoint union, $E_T$ is the set of ends of the tree $T$, and $\sim$ is the equivalence relation induced by the equivalences $b_e \sim b_{\bar{e}}$ for all $e \in O_T$.

We now describe a metric in $T(\Theta^*)$. To do this, consider a family of metrics $d_t$ in the spaces $K_t$ such that

$$\sum_{t \in V_T} \text{diam}(K_t, d_t) < \infty.$$
Identifying each $K_t$ naturally with an appropriate subset of $T(\Theta^*)$, we define a metric $d_T$ in $T(\Theta^*)$ as follows. If $x, y \in K_t$, put $d_T(x, y) := d_t(x, y)$. If $x \in K_t$ and $y \in K_s$ for some $t \neq s$, let $\gamma = [e_1, \ldots, e_m]$ be the path in $T$ connecting $t$ to $s$; we then put

$$d_T(x, y) := d_t(x, b_{e_1}) + \sum_{i=1}^{m-1} d_{\omega(e_i)}(b_{e_i}, b_{e_{i+1}}) + d_s(b_{e_m}, y).$$

If $x \in K_t$ and $y \in E_T$, let $\gamma = [e_1, e_2, \ldots]$ be the ray in $T$ connecting $t$ to $y$; put

$$d_T(x, y) := d_t(x, b_{e_1}) + \sum_{i=1}^{\infty} d_{\omega(e_i)}(b_{e_i}, b_{e_{i+1}}).$$

Finally, if $x, y \in E_T$, let $\gamma = [\ldots, e_0, e_1, \ldots]$ be the bi-infinite path in $T$ connecting $x$ to $y$; put

$$d_T(x, y) := \sum_{i=-\infty}^{\infty} d_{\omega(e_i)}(b_{e_i}, b_{e_{i+1}}).$$

We skip a straightforward verification that $d_T$ is well defined and is a metric in $T(\Theta^*)$.

1.D.1 Lemma. The metric space $(T(\Theta^*), d_T)$ is compact.

Proof: Let $(x_i)$ be any sequence of elements in $T(\Theta^*)$. We will show that it contains a convergent subsequence. The proof splits into several cases.

Case 1. Suppose that there is a subsequence $(x_{i_k})$ consisting of elements from a single subset $K_t \subset T(\Theta^*)$. Then, by compactness of $(K_t, d_t)$, there is a subsequence convergent to a point $x_0 \in K_t$ with respect to the metric $d_t$. Since on $K_t$ the metrics $d_T$ and $d_t$ coincide, the proof in this case is complete.

Case 2. Suppose that there is $t \in V_T$ and a subsequence $(x_{i_n})$ such that:

- (1) for each $n$ we have $x_{i_n} \notin K_t$;
- (2) if $x \in K_t$ is any point, and if we denote by $\gamma_n$ the corresponding path in $T$ started at $t$ as in the definition of the distance $d_T(x, x_{i_n})$, then the first oriented edges $e_{1_n}$ in the paths $\gamma_n$ are pairwise distinct.

By compactness of $K_t$, up to passing to a further subsequence, we may assume that the sequence $b_{e_{1_n}}$ converges to a point $x_0 \in K_t$. Viewing $x_0$ as a point in $T(\Theta^*)$, we claim that the subsequence $(x_{i_n})$ converges to $x_0$. To see this, for each $n$ let $D_n$ be the set of final vertices of all paths $\gamma$ in $T$ starting with $e_{1_n}$. Put also $\sigma_n = \sum_{s \in D_n} \text{diam}(K_s, d_s)$ and note that, since the sets $D_n$ are pairwise disjoint, we have $\lim_{n \to \infty} \sigma_n = 0$. Since we also have an obvious estimate $d_T(b_{e_{1_n}}, x_{i_n}) \leq \sigma_n$, the claim above follows.

Case 3: the remaining case. If there are no subsequences as in Cases 1 and 2, it is not hard to show that, after fixing any vertex $t \in V_T$ there is a ray $\rho = (e_1, e_2, \ldots)$ in $T$ started at $t$ and a subsequence $(x_{i_n})$ such that:

- (1) for each $n$ we have $x_{i_n} \notin K_t$;
- (2) if $x \in K_t$, and if we denote by $\gamma_n$ the corresponding path in $T$ starting at $t$ as in the definition of the distance $d_T(x, x_{i_n})$, then the length of the common initial part of $\rho$ and $\gamma_n$ grows indefinitely as $n \to \infty$. 

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Put $x_0 = [\rho] \in E_T \subset T(\Theta^*)$ to be the end of $T$ determined by $\rho$. We claim that the subsequence $(x_{i_n})$ converges to $x_0$.

To prove the claim, for each edge $e_k$ of $\rho$ denote by $D_k$ the set of final vertices of all paths $\eta$ in $T$ starting with $e_k$. Put also $\sigma_k = \sum_{s \in D_k} \text{diam}(K_s, d_s)$ and note that, since we have strict inclusions $D_{k+1} \subset D_k$, it holds $\lim_{k \to \infty} \sigma_k = 0$. On the other hand, for each $n$ let $e_{k(n)}$ be the final edge in the common initial part of $\rho$ and $\gamma_n$. We then have the estimates $d_T(b_{e_{k(n)}}, x_0) \leq \sigma_{k(n)}$ and $d_T(b_{e_{k(n)}}, x_{i_n}) \leq \sigma_{k(n)}$, which together imply that $d_T(x_0, x_{i_n}) \leq 2\sigma_{k(n)}$. Since by assumption (2) above we have $k(n) \to \infty$ as $n \to \infty$, this implies that $\lim_{n \to \infty} x_{i_n} = x_0$, as claimed. This completes the proof.

Consider the equivalence relation $\approx$ in the set $T(\Theta^*)$ induced by the equivalences $x \approx \phi_e(x)$ for all $e \in O_T$ and all $x \in \Sigma_e \setminus \{b_e\} \subset K_{\alpha(e)} \subset T(\Theta^*)$. The equivalence classes of this relation are either singletons or the two element sets $\{x, \phi_e(x)\}$ related to the equivalences above. Moreover, we have canonical identification, as sets, of $\lim \Theta$ and the quotient set $T(\Theta^*)/\approx$. A much less obvious fact is the following.

**1.D.2 Lemma.** Two topologies in $\lim \Theta$, the one given by the basis $\mathcal{B}$ described in Section 1.C, and the quotient topology in $T(\Theta^*)/\approx$ induced by the metric topology in $(T(\Theta^*), d_T)$, do coincide.

In the proof of Lemma 1.D.2, as well as in some later arguments in this paper, we will need the following result, which is a direct consequence of [Dav, Proposition 3 on page 14].

**1.D.3 Lemma.** Let $K$ be a compact metric space and let $\mathcal{A} = \{A_n : n \in N\}$ be a family of its pairwise disjoint closed subsets such that $\lim_{n \to \infty} \text{diam}(A_n) = 0$.

1. For any $n$ and any open neighbourhood of $A_n$ in $K$ there is a smaller open neighbourhood $V$ of $A_n$ which is saturated with respect to $\mathcal{A}$ (i.e. for each $j$ either $A_j \subset V$ or $A_j \cap V = \emptyset$).

2. For any point $p \in K \setminus \bigcup_{n=1}^{\infty} A_n$ and any open neighbourhood of $p$ in $K$ there is a smaller open neighbourhood of $p$ which is saturated with respect to $\mathcal{A}$.

**Proof of Lemma 1.D.2:** Denote by $q : T(\Theta^*) \to \lim \Theta = T(\Theta^*)/\approx$ the quotient map of the relation $\approx$. Recall that point preimages $q^{-1}(x)$ under this map are sets of cardinality either 1 or 2. In order to prove the lemma, it is sufficient to show the following two claims.

**Claim 1.** Let $G(U)$ be any subset of $\lim \Theta$ from the basis $\mathcal{B}$. Then the preimage $q^{-1}(G(U))$ is open in $(T(\Theta^*), d_T)$.

**Claim 2.** Let $x$ be any point in $\lim \Theta$ and let $r > 0$ be a real number. Then there is a set $G(U) \in \mathcal{B}$ such that $x \in G(U)$ and $q^{-1}(G(U)) \subset B_r(q^{-1}(x))$, where $B_r(q^{-1}(x)) = \{y \in T(\Theta^*) : d_T(\Theta^*)(y, q^{-1}(x)) \leq r\}$.

We sketch the proof of Claim 2, skipping the very similar argument for proving Claim 1. We need to consider three cases.

**Case 1.** Suppose that for some $t \in V_T$ we have $x \in K_t \setminus \bigcup \{\Sigma_e : e \in N_t\}$. For each $e \in N_t$ let $D_e$ be the set of endpoint of all finite paths in $T$ starting with the edge $e$ (in particular, $\omega(e) \in D_e$), and put $\sigma_e = \sum_{s \in D_e} \text{diam}(K_s)$. Since the sets $D_e : e \in N_t$ are pairwise disjoint, there are only finitely many $e \in N_t$ with $\sigma_e \geq r^2$. Denote by $N_0 \subset N_t$ the subset consisting of all such edges $e$. Let $U \subset K_t$ be an open subset which is saturated with
forward corollary. nullity, which follows from existence of Lebesgue numbers for open coverings of compact metric space \( T \in K \).

Proof of Proposition 1.C.1: We first show that \( \lim \Theta \) is a metrizable compact space. By Lemmas 1.D.1 and 1.D.2, it is homeomorphic to the quotient \( T(\Theta^*)/\approx \) of a compact metric space \( T(\Theta^*) \), hence being also compact. The quotient map \( q : T(\Theta^*) \to \lim \Theta \) is then clearly continuous and closed. Since the point preimages of \( q \) are either singletons or two element sets, they are all compact, and thus \( q \) is a perfect map. Since the image of a metric space through a perfect map is metrisable (see [Eng, Theorem 4.4.15]), the first assertion follows.

In view of Lemma 1.D.2, the fact that canonical inclusions \( K_t \subset \lim \Theta \) are all topological embeddings follows from the fact that the spaces \( (K_t, d_t) \) isometrically embed in \( (T(\Theta^*), d_T) \). The analogous assertion for all (finite) partial unions \( \#\Theta_F \) is then a straightforward corollary.

To prove the last assertion, we use the following purely topological characterization of nullity, which follows from existence of Lebesgue numbers for open coverings of compact metric spaces:

- A family \( \{Y_\lambda\} \) of subsets in a compact metric space \( X \) is null iff for each open covering \( U \) of \( X \) almost every subset \( Y_\lambda \) is contained in some \( U \in U \).
Since the families \(\{K_t : t \in V_T\}\) and \(\{\Sigma_e \cup \Sigma_{\bar{e}} : e \in O_T\}\) are clearly null in \(T(\Theta^*)\), the above characterization (together with Lemma 1.D.2) easily implies nullity of the families \(\{K_t\}\) and \(\{\Sigma_e\}\) in \(\lim \Theta\).

This completes the proof.

1.D.4 Remark: \(\lim \Theta\) is a perfect quotient of an inverse limit. The metric space \(T(\Theta^*)\) has the following alternative description as the limit of an inverse system of metric compacta. For each finite subtree \(F \subset T\) denote by \(F(\Theta^*)\) the quotient space \(\sqcup_{t \in V_F} K_t/\sim\), where \(\sim\) is induced by the equivalences \(b_e \sim b_{\bar{e}}\) for all \(e \in O_F\). Note that each \(F(\Theta^*)\) is obviously a compact metric space. Consider the family of spaces \(\{F(\Theta^*) : F \in \mathcal{F}_T\}\), and for any pair \(F_1 \subset F_2\) consider the map \(\pi_{F_2F_1} : F_2(\Theta^*) \to F_1(\Theta^*)\) described as follows.

If \(x \in K_t \subset F_2(\Theta^*)\) for some \(t \in V_{F_1}\), put \(\pi_{F_2F_1}(x) = x\). If \(x \in K_t \subset F_2(\Theta^*)\) for some \(t \notin V_{F_1}\), let \(\gamma\) be the shortest path in \(T\) from a vertex of \(F_1\) to \(t\), and let \(e_1\) be the first edge in this path. Put \(\pi_{F_2F_1}(x) = b_{e_1} \in K_{\alpha(e_1)} \subset F_1(\Theta^*)\).

We skip the easy verification of the facts that each \(\pi_{F_2F_1}\) is well defined, continuous, and that \(\pi_{F_2F_1} \circ \pi_{F_3F_2} = \pi_{F_3F_1}\) for any triple \(F_1 \subset F_2 \subset F_3\). We denote by \(I(\Theta^*)\) the inverse system \(\{(F(\Theta^*) : F \in \mathcal{F}_T), \{\pi_{F_2F_1} : F_1 \subset F_2\}\}\), where \(\mathcal{F}_T\) is the poset of all finite subtrees of \(T\). We leave it as an exercise to show that the inverse limit \(\lim \mathcal{F}_T\) is canonically homeomorphic with \(T(\Theta^*)\). This description of \(T(\Theta^*)\) shows that the limit \(\lim \Theta\) can be expressed as a perfect quotient of the inverse limit of compact metric spaces naturally associated to \(\Theta\).

1.E An isomorphism of tree systems of spaces.

Let \(\Theta = (T, \{K_t\}, \{\Sigma_e\}, \{\phi_e\})\) and \(\Theta' = (T', \{K'_t\}, \{\Sigma'_e\}, \{\phi'_e\})\) be two tree systems of spaces. An isomorphism \(F : \Theta \to \Theta'\) is a tuple \(F = (\lambda, \{f_t\})\) such that:

(I1) \(\lambda : T \to T'\) is an isomorphism of trees;
(I2) for each \(t \in V_T\) the map \(f_t : K_t \to K'_{\lambda(t)}\) is a homeomorphism;
(I3) for each \(e \in N_t\) we have \(f_t(\Sigma_e) = \Sigma'_{\lambda(e)}\);
(I4) for each \(e \in N_t\) the following commutation rule holds: \(\phi'_{\lambda(e)} \circ (f_{\alpha(e)}|\Sigma_{\alpha(e)}) = f_{\omega(e)} \circ \phi_e\).

An easy consequence of the definition of the limit (of a tree system of metric compacta) is the following.

1.E.1 Lemma. If \(\Theta, \Theta'\) are isomorphic tree systems of metric compacta then their limits \(\lim \Theta\) and \(\lim \Theta'\) are homeomorphic.

1.E.2 Example: Toruńczyk’s Lemma and dense tree systems of oriented manifolds.

The following result proved by Henryk Toruńczyk (see [J1]) has interesting consequences concerning existence of isomorphisms for certain natural classes of tree systems of manifolds. (Throughout this paper, by a manifold we mean a topological manifold.) Recall that a family of subsets of a topological space is dense if the union of this family is a dense subset.

1.E.2.1 Toruńczyk’s Lemma. Let \(M\) be a compact n-dimensional topological manifold with or without boundary, and let \(\mathcal{D}, \mathcal{D}'\) be two families of collared n-disks in \(\text{int}(M)\) such
that each family consists of pairwise disjoint sets and both families are null and dense. Then each homeomorphism \( h : \partial M \to \partial M \) which is extendable to a homeomorphism of \( M \) admits an extension to a homeomorphism \( H : M \to M \) which maps \( D \) to \( D' \). More precisely, this means that there is an associated bijective map \( \nu : D \to D' \) such that for each \( \Delta \in D \) the restriction \( H|_{\Delta} \) maps \( \Delta \) homeomorphically on the disk \( \nu(\Delta) \in D' \).

The above lemma provides motivation for the following.

**1.E.2.2 Definition.** Let \( \mathcal{M} \) be a tree system of closed manifolds, with families of manifolds \( \{M_t\} \) and disks \( \{\Delta_e\} \) as in Example 1.B.2. We say that this system is dense if for each \( t \in V_T \) the family \( D_t = \{\Delta_e : e \in N_t\} \) is dense in the manifold \( M_t \).

We denote the constituent spaces of a dense system of manifolds \( M_t \) by \( M^\circ_t \). The symbol \( M^\circ_t \) is meant to contain information both of the space itself and of the peripheral subspaces contained in this space. Given \( M_t \), it follows from Lemma 1.E.2.1 that the corresponding \( M^\circ_t \) is unique up to a homeomorphism preserving the peripheral subspaces. We will call any constituent space of form \( M^\circ_t \) (viewed again as equipped with its standard family of peripheral subspaces) a densely punctured manifold \( M_t \).

A recursive application of Toruńczyk’s Lemma, together with Lemma 1.E.1, immediately yield the following.

**1.E.2.3 Proposition.** Let \( M \) be a closed connected oriented topological manifold. Let \( \mathcal{M}, \mathcal{M}' \) be two dense tree systems of manifolds such that

1. all manifolds in the corresponding families \( \{M_t\} \) and \( \{M'_t\} \) are homeomorphic to \( M \),
2. all maps \( \phi_e : \Sigma_e = \partial \Delta_e \to \Sigma_{\bar{e}} = \partial \Delta_{\bar{e}} \) respect orientations, i.e. reverse the induced orientations on the corresponding spheres, and the same holds for all maps \( \phi'_{e'} \).

Then the tree systems \( \mathcal{M}, \mathcal{M}' \) are isomorphic, and consequently, their limits are homeomorphic.

Note that, due to the above proposition, each closed connected oriented manifold \( M \) determines uniquely up to isomorphism the dense tree system of manifolds satisfying conditions (1) and (2) as in the proposition. We denote this tree system by \( \mathcal{M}(M) \) and call it the dense tree system of manifolds \( M \) or the Jakobsche system for \( M \). The latter term is motivated by the fact that the tree systems \( \mathcal{M}(M) \) are intimately related to some inverse sequences described by W. Jakobsche in [J2]. We describe this relationship in Part 2 of the paper, especially in Example 2.C.7.

As a consequence of Lemma 1.E.1, \( M \) as above determines, uniquely up to homeomorphism, the compact metric space \( \lim \mathcal{M}(M) \), which we denote by \( \mathcal{X}(M) \) and call the tree of manifolds \( M \) or the Jakobsche space for \( M \). In Jakobsche’s paper [J2] this space is denoted by \( \mathcal{X}(M, \{M\}) \), and it is obtained as inverse limit of an inverse sequence mentioned in the previous paragraph.

**Remark.** In [J2] Jakobsche considered also a more general class of spaces obtained as inverse limits and determined uniquely up to homeomorphism by a finite or infinite family \( \mathcal{N} \) of closed connected oriented manifolds of the same dimension. We discuss the corresponding dense tree systems of manifolds, for finite families \( \mathcal{N} \), in Example 3.A.2.
1.E.3 Example: the tree of non-orientable manifolds $N$.

Recall the following rather well known fact.

1.E.3.1 Lemma. Let $N$ be a closed connected non-orientable topological manifold of dimension $n$, and let $D$ be any collared $n$-disk in $N$. Then there is a homeomorphism $h$ of $N$ such that $h(D) = D$ and $h|_D$ reverses the orientation of $D$.

A recursive application of Toruńczyk’s Lemma 1.E.2.1 and Lemma 1.E.3.1 yields the following.

1.E.3.2 Proposition. Let $N$ be a closed connected non-orientable topological manifold, and let $M$, $M'$ be any two dense tree systems of manifolds such that all manifolds in the corresponding families $\{M_t\}$ and $\{M'_t\}$ are homeomorphic to $N$. Then the tree systems $M$, $M'$ are isomorphic, and consequently, their limits are homeomorphic.

The tree system as above, uniquely determined by $N$, will be denoted $M(N)$ and called the dense tree system of manifolds $N$, or the Jakobsche system for $N$. Its limit, denoted $X(N)$, will be called the tree of manifolds $N$ or the Jakobsche space for $N$.

1.E.3.3 Remark. Proposition 1.E.3.2 clarifies the picture with trees of non-orientable manifolds, as documented so far in the literature. Namely, in the paper [St] Paul Stallings describes these spaces in a way which could be translated to our setting as follows: the tree of manifolds $N$ is the limit of a dense tree system as in Proposition 1.E.3.2 satisfying some additional technical condition for the connecting maps called dense orientation condition (we do not recall this condition). Proposition 1.E.3.2 shows that this additional condition plays in fact no role. Moreover, Proposition 1.E.3.2 applies to all topological manifolds, while the methods used by Stallings in [St] allowed him to deal only with PL manifolds.

2. Inverse systems associated to tree systems of spaces.

In this part of the paper we are interested in expressing some trees of spaces as inverse limits of appropriately associated inverse systems. There are numerous motivations for getting such expressions. First, they allow to relate the spaces obtained as limits of tree systems with certain previously studied classes of topological spaces defined in terms of inverse limits (e.g. Jakobsche trees of manifolds, Markov compacta as defined in [Dr], etc.). Even more importantly, ideal boundaries of spaces and groups can usually be expressed as inverse limits, so having trees of spaces expressed in a similar way may help to recognize some of these boundaries as specific trees of spaces (see Remark 2.C.8 for examples of such approach occurring in the literature).

It also turns out that an expression of the limit of a tree system as an inverse limit may be useful for detecting some of the topological properties of the limit. For example, in Section 2.D below, we use such expression to calculate or estimate topological dimension of the limit. It is possible that many other properties of limits of tree systems can be deduced from properties of appropriate associated inverse systems.
2.A Extended spaces and maps.

Let \( \Theta = (T, \{K_t\}, \{\Sigma_e\}, \{\phi_e\}) \) be a tree system of metric compacta. Suppose that for each \( e \in O_T \) we are given a compact metric space \( \Delta_e \), its compact subspace \( S_e \), and a homeomorphism \( \varphi_e : S_e \to \Sigma_e \).

For any \( t \in V_T \) let \( T(t) \) be a subtree of \( T \) spanned on the set \( \{t\} \cup \{\omega(e) : e \in N_t\} \) (i.e. \( t \) and all vertices adjacent to \( t \)). We define a tree system

\[
\Theta(t) = (T(t), \{K'_t\}, \{\Sigma'_e\}, \{\phi'_e\})
\]

as follows. Put \( K'_t = K_t \) and \( K'_{\omega(e)} = \Delta_e \) for each \( e \in N_t \). Put \( \Sigma'_e = \Sigma_e \) and \( \Sigma'_e = S_e \) for each \( e \in N_t \). Finally, put \( \phi'_e = \varphi_e^{-1} \) and \( \phi'_e = \varphi_e \) for each \( e \in N_t \). Denote by \( \hat{K}_t = \lim \Theta(t) \) the limit of the above tree system. Observe that \( K_t \) and the sets \( \Delta_e : e \in N_t \) are then canonically the subspaces of \( \hat{K}_t \) (in particular, they are compact subspaces). Moreover, the sets \( \Sigma_e \) and \( S_e \), viewed in the above way as subspaces of \( \hat{K}_{\alpha(e)} \), do coincide. We call any family \( \hat{K}_t : t \in V_t \) as above a family of extended spaces for \( \Theta \).

2.A.1 Examples.

1. For each \( e \in O_T \) put \( \Delta_e = S_e = \Sigma_e \) and \( \varphi_e = id_{\Sigma_e} \). This defines what we call the trivial family of extended spaces for \( \Theta \). Note that in this family we have \( \hat{K}_t = K_t \) for each \( t \in V_T \).

2. Let \( M \) be a tree system of manifolds as in Example 1.B.2. Let \( \Delta_e : e \in O_T \) be the family of \( n \)-disks as in this example, and put \( S_e = \partial \Delta_e \) and \( \varphi_e = id_{\Sigma_e} \) for each such \( e \). It is not hard to show, using Lemma 1.D.3, that then we have \( \hat{K}_t = M_t \) for each \( t \in V_T \). Thus, as a family of extended spaces for \( M \) we can take the initial family of manifolds \( M_t \). We will call the family \( \{\hat{K}_t = M_t\} \) as above the standard family of extended spaces for \( M \).

3. The previous example can be generalized as follows. Let \( \Theta = (T, \{K_t\}, \{\Sigma_e\}, \{\phi_e\}) \) be any tree system of metric compacta. For each \( e \in N_T \) put \( \Delta_e = cone(\Sigma_e) \) and \( S_e \) to be the base of this cone, and let \( \varphi_e \) be the identity on \( S_e = \Sigma_e \). We call the associated family \( \hat{K}_t : t \in V_T \) the family of conically extended spaces for \( \Theta \).

Given a family of extended spaces for \( \Theta \) determined, as above, by a family of pairs \( (\Delta_e, S_e) : e \in O_T \), consider the family \( \hat{K}_e : e \in O_T \) of subspaces defined by \( \hat{K}_e = \hat{K}_{\omega(e)} \setminus (\Delta_e \setminus S_e) \). Suppose that we are given a family of maps \( \delta_e : \hat{K}_e \to \Delta_e \) such that \( \delta_e|\Sigma_e = \phi_e \). Any tuple \( E = (\{\hat{K}_t : t \in V_T\}, \{\delta_e : e \in O_T\}) \) as above will be called a family of extended spaces and maps for \( \Theta \).

To any family \( E \) of spaces and maps as above we associate an inverse system of metric compacta, denoted \( S_E \), as follows. Let \( (\mathcal{F}_T, \subset) \) be the poset of all finite subtrees of \( T \). For any \( F \in \mathcal{F}_T \) let \( \hat{K}_F \) be the quotient of the disjoint union

\[
\hat{K}_F = \bigcup_{t \in V_F} [\hat{K}_t \setminus \bigcup_{e \in N_t \cap O_F} (\Delta_e \setminus S_e)] / \sim
\]

where \( \sim \) is the equivalence relation induced by the equivalences of form \( x \sim \phi_e(x) \) for all \( e \in O_F \) and all \( x \in \Sigma_e \) (where we view \( \Sigma_e \) and \( \Sigma_{\bar{e}} \) canonically as subsets in \( \hat{K}_{\alpha(e)} \) and
\(\hat{K}_{\omega(e)}\), respectively). This gives us the family \(\hat{K}_F : F \in \mathcal{F}_T\) of spaces in the inverse system \(S_\mathcal{E}\).

We now turn to defining the maps \(h_{F_1F_2} : \hat{K}_{F_2} \to \hat{K}_{F_1}\), for all pairs \(F_1 \subseteq F_2\) of finite subtrees of \(T\). We do this in two steps, as follows. First, suppose that \(V_{F_2} = V_{F_1} \cup \{t\}\) for some \(t \notin V_{F_1}\). Let \(e\) be the unique oriented edge in \(F_2\) with \(\omega(e) = t\). Viewing canonically \(\Delta_e\) as a subset of \(\hat{K}_{F_1}\) and both \(\hat{K}_{F_1}, \hat{K}_e\) as subsets of \(\hat{K}_{F_2}\), we put

\[
h_{F_1F_2}(x) = \begin{cases} 
    x & \text{for } x \in \hat{K}_{F_1} \setminus (\Delta_e \setminus S_e) \subseteq \hat{K}_{F_2} \\
    \delta_e(x) & \text{for } x \in \hat{K}_e \subseteq \hat{K}_{F_2}
\end{cases}
\]

In the second step, for any pair \(F' \subseteq F\) of finite subtrees of \(T\) we consider a sequence \(F_1 \subseteq F_2 \subseteq \ldots \subseteq F_m\) such that \(F_1 = F', F_m = F\) and the pairs \(F_i, F_{i+1}\) are of the form as in the first step. We then put

\[
h_{F,F'} = h_{F_1F_2} \circ h_{F_2F_3} \circ \ldots \circ h_{F_{m-1}F_m}
\]

and we observe that the resulting map does not depend on the choice of the above sequence \(F_1, \ldots, F_m\) (which is in general not unique). A related observation is that \(h_{F_1F_3} = h_{F_1F_2} \circ h_{F_2F_3}\) whenever \(F_1 \subseteq F_2 \subseteq F_3\).

**2.A.2 Definition.** Let \(\Theta\) be a tree system of metric compacta, and let \(\mathcal{E}\) be an associated family of extended spaces and maps for \(\Theta\). The **associated inverse system for \(\Theta\) induced by \(\mathcal{E}\)** is the tuple

\[
S_\mathcal{E} = (\{\hat{K}_F : F \in \mathcal{F}_T\}, \{h_{F,F'} : F \subseteq F'\}).
\]

In the next section we show (see Theorem 2.B.4) that in some cases one can use limit of an associated inverse system \(S_\mathcal{E}\) as an alternative description of the limit of a tree system.

**2.A.3 Remark.** Any associated inverse system \(S_\mathcal{E}\) as above has various cofinal subsequences. Since any cofinal subsystem is viewed as equivalent to the original one (for example, its inverse limit is canonically the same), this allows simpler (in certain sense) descriptions of the system \(S_\mathcal{E}\), which might be more convenient for some purposes. More precisely, let \(F_1 \subseteq F_2 \subseteq \ldots\) be an infinite increasing sequence of finite subtrees of \(T\). Clearly, this sequence is cofinal with \(\mathcal{F}_T\) iff \(\bigcup_{i=1}^\infty V_{F_i} = V_T\). For any such cofinal sequence \((F_i)\) the tuple

\[
S_{\mathcal{E},(F_i)} = (\{\hat{K}_{F_i} : i \in \mathbb{N}\}, \{h_{F_jF_i} : i < j\})
\]

is an inverse sequence equivalent to the inverse system \(S_\mathcal{E}\).

**2.B. Limit of a tree system as inverse limit.**

In order to relate the limit \(\lim \Theta\) with the inverse limit of some associated inverse system \(S_\mathcal{E}\) for \(\Theta\), we need one more condition on the associated family \(\mathcal{E}\) of extended spaces and maps. We use the notation as in the previous section.

Let \(\gamma = (t_0, t_1, \ldots, t_m)\) be any finite combinatorial path in \(T\) of length \(m \geq 2\). For \(i = 1, \ldots, m\) let \(e_i = (t_{i-1}, t_i)\) be the consecutive oriented edges in \(\gamma\). We then have

\[
\Delta_{e_m} \subseteq \hat{K}_{e_{m-1}} \delta_{e_{m-1}} \Delta_{e_{m-1}} \subseteq \cdots \delta_{e_2} \Delta_{e_2} \subseteq \hat{K}_{e_1} \delta_{e_1} \Delta_{e_1}
\]
and we denote by \( \delta_\gamma : \Delta_{e_m} \to \Delta_{e_1} \) the composition map \( \delta_\gamma = \delta_{e_1} \circ \cdots \circ \delta_{e_{m-2}} \circ \delta_{e_{m-1}}|_{\Delta_{e_m}} \).

2.B.1 Definition. We say that an associated family \( \mathcal{E} \) of extended spaces and maps is fine if for each \( e \in \mathcal{O}_T \) the family of images \( \delta_\gamma(\Delta_{e_m}) \) (where \( e_m \) is the terminal edge in \( \gamma \)), for all combinatorial paths \( \gamma \) in \( T \) of length \( \geq 2 \) and starting with \( e_1 = e \), is a null family of subsets in \( \Delta_e \).

Remark. The condition of fineness does not follow automatically from the nullity conditions in \( \Theta \) (for families \( \{ \Sigma_e : e \in \mathcal{N}_t \} \)).

Next definition describes a subclass of fine families \( \mathcal{E} \) of extended spaces and maps which occur in practical situations where we express limits of tree systems as inverse limits.

2.B.2 Definition. Let \( \mathcal{E} = \{ (\hat{K}_t), \{ \delta_e \} \} \) be a family of extended spaces and maps related to a family of pairs \( \{ (\Delta_e, S_e) \} \). We say that \( \mathcal{E} \) is contracting if there are metrics \( \hat{d}_t \) on the extended spaces \( \hat{K}_t \) and a constant \( 0 < \epsilon < 1 \) such that

1. for each \( e, e' \in \mathcal{O}_T \) such that \( e' \neq \hat{e} \) and \( \alpha(e') = \omega(e) \) the restricted map \( \delta_e|_{\Delta_{e'}} \) is a \( \epsilon \)-contraction with respect to the metrics \( \hat{d}_{\omega(e)} \) in \( \Delta_{e'} \subset \hat{K}_{\omega(e)} \) and \( \hat{d}_{\alpha(e)} \) in \( \Delta_e \subset \hat{K}_{\alpha(e)} \); more precisely, for any \( x, y \in \Delta_{e'} \) we have
   \[
   \hat{d}_{\alpha(e)}(\delta_e(x), \delta_e(y)) \leq \epsilon \cdot \hat{d}_{\omega(e)}(x, y);
   \]

2. if \( \epsilon > 0 \) then there is another constant \( C > 0 \) such that for each \( t \in V_T \) we have
   \[
   \text{diam}(\hat{K}_t, \hat{d}_t) \leq C.
   \]

We omit the proof of the following apparent fact.

2.B.3 Fact. Any contracting family \( \mathcal{E} \) of extended spaces and maps is fine.

2.B.4 Theorem. Let \( \Theta \) be a tree system of metric compacta, let \( \mathcal{E} \) be an associated family of extended spaces and maps for \( \Theta \), and suppose that \( \mathcal{E} \) is fine. Let \( S_{\mathcal{E}} \) be the associated inverse system of compact topological spaces. Then the limit \( \lim_{\mathcal{T}} \Theta \) is canonically homeomorphic to the inverse limit \( \lim_{\mathcal{T}} S_{\mathcal{E}} \).

Proof: The proof consists of three parts. In the first part we describe explicitly strings \( (x_F)_{F \in \mathcal{F}_T} \) which represent points of \( \lim_{\mathcal{T}} S_{\mathcal{E}} \). In the second part, we use this description to define a natural map \( \beta : \lim_{\mathcal{T}} S_{\mathcal{E}} \to \lim \Theta \), which is a bijection. Finally, in the third part we prove that \( \beta \) is a homeomorphism.

Recall that, by definition of inverse limit, each element \( x \in \lim_{\mathcal{T}} S_{\mathcal{E}} \) is a tuple \( (x_F)_{F \in \mathcal{F}_T} \), with \( x_F \in \hat{K}_F \), such that \( h_{F'F}(x_F) = x_{F'} \) whenever \( F' \subset F \). We will show that there are two kinds of such tuples in \( \lim_{\mathcal{T}} S_{\mathcal{E}} \). At first, we consider tuples \( (x_F) \) satisfying the following property:

\((*)\) for any cofinal subsequence \( F_1 \subset F_2 \subset \ldots \) in the poset \( \mathcal{F}_T \) the sequence \( x_{F_n} \) eventually stabilizes, i.e. there is \( N \) such that for all \( n \geq N \) we have \( x_{F_n} \in K_{F_n} \subset \hat{K}_{F_n} \) and, under the natural inclusions \( K_{F_n} \subset K_{F_{n+1}} \), we have the equalities \( x_{F_{n+1}} = x_{F_n} \).

A large class of such tuples can be described as follows. Let \( y \in \hat{K}_t \) for some \( t \in V_T \). This \( y \) determines the tuple \( (x^y_F) \) as follows. If \( F \) contains \( t \), we have the canonical inclusion
$K_t \subset \hat{K}_F$, and we put $x^y_F = x$. If $F$ does not contain $t$, let $F'$ be the smallest subtree of $T$ containing both $F$ and $t$. Viewing $K_t$ as a subset of $\hat{K}_{F'}$, we put $x^y_F := h_{F',F}(x)$. It is easy to check that $x^y = (x^y_F)$ is then a string, i.e. $x^y \in \lim_\prec S_\mathcal{E}$, and it obviously satisfies condition $(\star)$. Moreover, we have the following.

**Claim 1.** Each string $(x_F) \in \lim_\prec S_\mathcal{E}$ which satisfies property $(\star)$ has form $(x^y_F)$, as described above.

The claim follows by observing that if $y$ is the element to which some sequence $x_{F_n}$ stabilizes, then any other such sequence stabilizes to the same element, and consequently the string necessarily has the asserted form $(x^y_F)$.

We now turn to strings $(x_F)$ not satisfying condition $(\star)$). A large class of such strings is induced by ends $z \in E_T$. Given any $z \in E_T$, we define the tuple $(x^z_F)$ as follows. For any $F \in \mathcal{F}_T$ let $\rho_F$ be the minimal (with respect to inclusion) ray starting at a vertex of $F$ and such that $[\rho_F] = z$. Let $\gamma_n = (e'_1, \ldots, e'_n)$ be the initial path of length $n$ in $\rho_F$, and let $\delta_{\gamma_n} : \Delta e' \to \Delta e'_n$ be the map described at the beginning of this section. Note that the sequence $(\hat{Q}_n)$ of compact subsets in $\Delta e'_n$ given by $Q_n = \delta_{\gamma_n}(\Delta e'_n)$ is then decreasing and, due to fineness of $\mathcal{E}$, we have $\lim_{n \to \infty} \operatorname{diam}(Q_n) = 0$. Consequently, the intersection $\cap Q_n$ is a single point in $\Delta e'_n \subset \hat{K}_F$, and we take it as $x^z_F$. We make the following observations (omitting their straightforward proofs):

1. $(x^z_F)$ is a string, i.e. it belongs to $\lim_\prec S_\mathcal{E}$,
2. $(x^z_F)$ does not satisfy condition $(\star)$,
3. if $z \neq z'$ then $(x^z_F) \neq (x^{z'}_F)$.

**Claim 2.** Each element $(x_F) \in \lim_\prec S_\mathcal{E}$ which does not satisfy condition $(\star)$ has a form $(x^z_F)$ as above, for some ray $z \in E_T$. Moreover, $z \to (x^z_F)$ is a bijective correspondence between the set of ends of $T$ and the set of elements of $\lim_\prec S_\mathcal{E}$ not satisfying $(\star)$.

To prove the claim, note that for each string $(x_F)$ not satisfying $(\star)$ there is a cofinal sequence $F_1 \subset F_2 \subset \ldots$ for which elements $x_{F_n}$ change infinitely often. By passing to a subsequence, we may assume that $x_{F_{n+1}} \neq x_{F_n}$ (i.e. it is not true that $x_{F_{n+1}} = x_{F_n} \in K_{F_n} \subset K_{F_{n+1}}$) for all $n$. Observe that in this situation we have $x_{F_n} \in \Delta e_n$ for some unique $e_n \in N_{F_n}$, and that the edges $e_n$ induce uniquely a ray $\rho$ in $T$ of form $\rho = (e_1, \ldots, e_2, \ldots, e_3, \ldots)$. It is not hard to realize that then $(x_F) = (x^{[\rho]}_F)$, which yields the first assertion of the claim. The second assertion follows then from observation (3) stated just before Claim 2.

We are now ready to describe a natural map $\beta : \lim_\prec S_\mathcal{E} \to \lim \Theta$, which will be our candidate for a homeomorphism. If a string $x = (x_F)$ satisfies condition $(\star)$, there is a point $y$ to which $(x_F)$ stabilizes, and this $y$ belongs to some $K_t$. Viewing $K_t$ as a subset in $\lim \Theta$, we put $\beta(x) = y$. If $x$ does not satisfy $(\star)$, it corresponds uniquely to some end $z$ of $T$.Viewing ends of $T$ as elements of $\lim \Theta$, we put $\beta(x) = z$. We skip a direct verification of the fact that so described $\beta$ is well defined, and that it is a bijection.

Since both spaces $\lim_\prec S_\mathcal{E}$ and $\lim \Theta$ are compact, to prove that $\beta$ is a homeomorphism, it is sufficient to show that it is continuous. To do this, we will show that for any open set $G(U) \in \mathcal{B}$ its preimage $\beta^{-1}(G(U))$ is open in $\lim_\prec S_\mathcal{E}$. More precisely, viewing $\lim_\prec S_\mathcal{E}$ as subspace of the product $\prod_F \hat{K}_F$, we will show that $\beta^{-1}(G(U)) = W \cap \lim_\prec S_\mathcal{E}$.
for some open set $W$ in $\prod_F \hat{K}_F$. Suppose that $U$ is an open subset of $K_{F_0}$ saturated with respect to the family $\mathcal{A}_{F_0}$. Put

$$\hat{U} = U \cup \bigcup \{\Delta_e : e \in N_{F_0} \text{ and } \Sigma_e \subset U\}$$

and note that $\hat{U}$ is an open subset of $\hat{K}_{F_0}$. Take $W = \prod_F W_F$, where $W_{F_0} = \hat{U}$ and $W_F = \hat{K}_F$ for $F \neq F_0$. Clearly, $W$ is open in $\prod_F \hat{K}_F$. It is also not hard to deduce from Claims 1 and 2, and from the descriptions of strings, that $\beta^{-1}(G(U)) = W \cap \lim_{\mathcal{E}} \mathcal{S}_\mathcal{E}$. This completes the proof.

2.B.5 Example: tree of spheres is a sphere.

For arbitrary $n \geq 1$, consider the dense tree system $\mathcal{M}(S^n)$ of $n$-spheres, as described in Example 1.E.2. Next result is an application of Theorem 2.B.4.

2.B.5.1 Proposition. The limit $\lim \mathcal{M}(S^n)$, i.e. the tree of spheres $\mathcal{X}(S^n)$, is homeomorphic to $S^n$.

Before proving Proposition 2.B.5.1, we make a comment concerning the tree system $\mathcal{M}(S^n)$.

2.B.5.2 An alternative description of the tree system $\mathcal{M}(S^n)$.

Recall that the unique topological space $(S^n)^\circ$ obtained from $S^n$ by deleting interiors of $n$-disks $D$ from any null and dense family $\mathcal{D}$ consisting of pairwise disjoint collared disks is called the $(n - 1)$-dimensional Sierpiński compactum. (This follows from [Ca] if $n \neq 4$, and in the remaining case the argument in [Ca] also holds true in view of the later proofs of the Annulus Theorem and the Approximation Theorem for $n = 4$, due to F. Quinn [Qu].) The space $(S^n)^\circ$ contains a family of distinguished subsets, called peripheral spheres, which coincides with the family of boundaries $\partial D$ of the disks $D \in \mathcal{D}$. The tree system $\mathcal{M}(S^n)$ can be described as follows. It is the unique tree system of metric compacta in which all constituent spaces are $(n - 1)$-dimensional Sierpiński compacta, and families of peripheral subsets in all of these spaces coincide with the families of peripheral spheres.

Proof of Proposition 2.B.5.1:

Consider a family $\mathcal{E} = (\{\hat{K}_t\}, \{\delta_e\})$ of extended spaces and maps for $\mathcal{M}(S^n)$ described as follows. As the family $\{\hat{K}_t\}$ take the standard family of extended spaces, as in Example 2.A.1(2). More precisely, identifying each $K_t$ with $(S^n)^\circ = S^n \setminus \bigcup\{\text{int}(D) : D \in \mathcal{D}\}$, take $\hat{K}_t = S^n$. The corresponding pairs $(\Delta_e, S_e) : e \in N_t$ are then canonically identified with the pairs $(D, \partial D) : D \in \mathcal{D}$, and the maps $\varphi_e$ with the identities $\text{id}_{\partial D}$. Moreover, the corresponding subspaces $\hat{K}_e$ are then homeomorphic to $n$-disks. The family of extended maps $\{\delta_e\}$ in $\mathcal{E}$, with $\delta_e : \hat{K}_e \to \Delta_e$, is described as follows. First, denote by $\hat{K}_e^*$ the quotient of $\hat{K}_e$ obtained by collapsing all disks $\Delta_e : e' \in O_{\omega(e)} \setminus \{\overline{e}\}$ to points. Thus, $\hat{K}_e^*$ is homeomorphic to the space obtained from the Sierpiński compactum $(S^n)^\circ$ by collapsing all but one peripheral spheres. Since the latter space is homeomorphic to the $n$-disk, with boundary corresponding to the noncollapsed peripheral sphere in $(S^n)^\circ$ (see e.g. Lemma 3.D.3 below), we get that the pair $(\hat{K}_e^*, \partial \hat{K}_e)$ is homeomorphic to the pair $(D^n, \partial D^n)$. Let $q_e : \hat{K}_e \to \hat{K}_e^*$ be the associated quotient map. Take also any homeomorphism $j_e : \hat{K}_e^* \to \Delta_e$ such that $j_e|_{\Sigma_e} : \Sigma_{\overline{e}} \to \Sigma_e = S_e \subset \Delta_e$ coincides with $\varphi_{\overline{e}}$. Put $\delta_e = j_e \circ q_e$. 

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Note that, by definition of $\delta_e$, each restriction $\delta_e|_{\Delta_{e'}} : e' \in O_{\omega(e)} \setminus \{\bar{e}\}$ is a 0-contraction, so that $\mathcal{E}$ is a contracting family of extended spaces and maps, and in particular it is fine (see Fact 2.B.3). It follows then from Theorem 2.B.4 that $\lim_{\leftarrow} \mathcal{M}(S^n) = \lim_{\leftarrow} S_{\mathcal{E}}$.

Note that the inverse system $S_{\mathcal{E}}$ is a system of $n$-spheres with all bonding maps cell like and cellular. Indeed, the point preimages of these bonding maps are either points or collared $n$-disks in the corresponding $n$-spheres, which obviously implies their cell likeness and cellularity. Consequently, these maps are near homeomorphisms, i.e. they can be approximated by homeomorphisms (compare e.g. Theorem (3c.1) in [DJ], which collects in one statement results proved by many authors). Since an inverse limit of near homeomorphisms is a near homeomorphism ([Br]), it follows that $\lim_{\leftarrow} S_{\mathcal{E}}$, and hence also $\lim_{\leftarrow} \mathcal{M}(S^n)$, is homeomorphic to the $n$-sphere. This completes the proof.

A different argument for proving Proposition 2.B.5.1 is sketched in Remark 3.B.12.2.

### 2.C Tree systems with ANR peripheral subspaces.

This section is devoted to showing that any tree system $\Theta$ in which all peripheral subspaces $\Sigma_e$ are ANRs admits a fine family $\mathcal{E} = (\{\hat{K}_t\}, \{\delta_e\})$ of extended spaces and maps in which $\{\hat{K}_t\}$ is the family of conically extended spaces (see Example 2.A.1(3)) and $\{\delta_e\}$ is an associated family of 0-contracting maps. See Proposition 2.C.2 for precise statement of this main result of the section.

Recall that a compact metric space $\Sigma$ is an ANR (absolute neighbourhood retract) if for any embedding of $\Sigma$ in another compact metric space $K$ there is a neighbourhood of $\Sigma$ in $K$ which retracts on $\Sigma$. Recall also that every compact polyhedron is an ANR. Lemma 2.C.3 below presents a different characterization of ANR spaces, more convenient for our purposes.

#### 2.C.1 Definition. Let $\mathcal{E} = (\{\hat{K}_t\}, \{\delta_e\})$ be a family of extended spaces and maps for a tree system $\Theta$.

1. We say that $\Theta$ is **peripherally ANR** if all peripheral subspaces $\Sigma_e : e \in OT$ of $\Theta$ are ANRs.
2. We say that $\mathcal{E}$ is **conical** if the corresponding family $\{\hat{K}_t\}$ is the family of conically extended spaces for $\Theta$.
3. We say that $\mathcal{E}$ is **0-contracting** if for each $e, e' \in OT$ such that $e' \neq \bar{e}$ and $\alpha(e') = \omega(e)$ the restricted map $\delta_e|_{\Delta_{e'}}$ is a 0-contraction; equivalently, for all $e, e'$ as above $\delta_e$ maps the subspace $\Delta_{e'} \subset \hat{K}_e$ to a point.

Note that a 0-contracting family $\mathcal{E}$ is automatically fine (compare Fact 2.B.3) and thus, due to Theorem 2.B.4, can be used to express the limit $\lim_{\leftarrow} \Theta$ as an inverse limit. The main result of this section is the following.

#### 2.C.2 Proposition. Each peripherally ANR tree system has a conical and 0-contracting family of extended spaces and maps.

The proof of Proposition 2.C.2 requires various preparations. We start with a lemma which characterizes ANR spaces $\Sigma$ in terms of maps to the cones over $\Sigma$. Since this characterization is an obvious reformulation of the definition of an ANR space, we omit its proof.
2.C.3 Lemma. Let $\Sigma$ be a compact metric space, and let $cone(\Sigma)$ be the cone over $\Sigma$, with $\Sigma$ canonically identified as the cone base. Then the following two conditions are equivalent:

1. for any metric compactum $K$ containing $\Sigma$ as a subspace there is a continuous map $f : K \to cone(\Sigma)$ such that $f|_\Sigma = id_\Sigma$;
2. $\Sigma$ is an ANR.

We now turn to discussing decompositions of the constituent spaces $K_t$ induced by families of their peripheral subsets. For any $e \in O_T$, put

$$A_e = \{\Sigma_{e'} : e' \neq \bar{e} \text{ and } \alpha(e') = \omega(e)\}.$$ 

Clearly, $A_e$ is a null family of pairwise disjoint compact subsets of the space $K_{\omega(e)}$. We will view $A_e$ as a decomposition of $K_{\omega(e)}$ into closed subsets by considering all singletons $\{x\}$ with $x \notin \cup A_e$, together with the sets from $A_e$, as elements of this decomposition. Equivalently, we identify the family $A_e$ with the decomposition of $K_{\omega(e)}$ in which $A_e$ is the set of nondegenerate elements. By the fact that the family $A_e$ is null we immediately get the following (compare [Dav], Proposition 3 on p. 14).

2.C.4 Fact. For each $e \in O_T$ the decomposition $A_e$ of the space $K_{\omega(e)}$ is upper semicontinuous.

Denote by $K_{\omega(e)}/A_e$ the quotient space of the decomposition $A_e$. Since any quotient of an upper semicontinuous decomposition of a metric space is metrizable (see [Dav], Proposition 2 on p. 13), we get

2.C.5 Fact. For each $e \in O_T$ the quotient $K_{\omega(e)}/A_e$ is a compact metrizable space, and the subspace $\Sigma_{\bar{e}}$ canonically topologically embeds in this quotient.

Fact 2.C.5 together with Lemma 2.C.3 yield the following.

2.C.6 Corollary. Suppose that the peripheral subset $\Sigma_{\bar{e}}$ is an ANR. Then there is a continuous map $f_e : K_{\omega(e)}/A_e \to cone(\Sigma_{\bar{e}})$ which is identical on $\Sigma_{\bar{e}}$. Consequently, there is an induced continuous map $f'_e : K_{\omega(e)} \to cone(\Sigma_{\bar{e}})$ which is identical on $\Sigma_{\bar{e}}$ and which maps each subset $\Sigma_{e'} \in A_e$ to a point.

Proof of Proposition 2.C.2: Let $\{K_t : t \in V_T\}$ be the associated family of conically extended spaces for $\Theta$, as described in Example 2.A.1(3), and let $\{K_e : e \in O_T\}$ be the corresponding family of subspsces. For each $e \in O_T$ define a map $\delta'_e : \hat{K}_e \to cone(\Sigma_e)$ by

$$\delta'_e(x) = \begin{cases} f'_e(x) & \text{if } x \in K_{\omega(e)} \subset \hat{K}_e \\ f'_e(\Sigma_{e'}) & \text{if } x \in \Delta_{e'} \subset \hat{K}_e, \text{ for some } e' \in N_{\omega(e)} \setminus \{\bar{e}\}, \end{cases}$$

where $f'_e$ is a map as in Corollary 2.C.6.

After identifying $cone(\Sigma_{\bar{e}})$ with $cone(\Sigma_e)$ via cone of the map $\phi_{\bar{e}} : \Sigma_{\bar{e}} \to \Sigma_e$, and then identifying $cone(\Sigma_{\bar{e}})$ with $\Delta_{\bar{e}}$, the above described map $\delta'_e$ gives the map $\delta_e : \hat{K}_e \to \Delta_{\bar{e}}$ such that:

1. $\delta_e$ maps each subset $\Delta_{e'} \subset \hat{K}_e$, for $e' \in N_{\omega(e)} \setminus \{\bar{e}\}$, to a point;
2. the restriction of $\delta_e$ to $\Sigma_{\bar{e}}$ coincides with $\phi_{\bar{e}}$. 

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Thus, the tuple $E = (\{\hat{K}_e\}, \{\delta_e\})$ is a conical and 0-contracting family of extended spaces and maps for $\Theta$, as required. This finishes the proof.

2.C.7 Example: Jakobsche’s inverse sequences.

As we have already mentioned in Section 1.E, Włodzimierz Jakobsche has introduced and studied (in his papers [J1, J2]) the spaces which appear in the present paper as limits of dense tree systems of manifolds, in particular the tree systems of manifolds $M$, as described in Proposition 1.E.2.3 (the Jakobsche spaces $\mathcal{X}(M)$). Jakobsche has introduced those spaces as inverse limits of certain inverse sequences of manifolds, which we can describe in our terms as follows.

Given a closed oriented manifold $M$, let $\mathcal{M} = \mathcal{M}(M)$ be the dense tree system of manifolds $M$ (as defined at the end of Section 1.E). Let $E = (\{\hat{K}_t\}, \{\delta_e\})$ be any fine conical family of extended spaces and maps for $\mathcal{M}$ (existence of which is justified by Proposition 2.C.2). Let $S_E = (\{\hat{K}_F\}, \{h_{F,F'}\})$ be the associated inverse system for $\mathcal{M}$, as in Definition 2.A.2. Note that the spaces $\hat{K}_F$ in this system are homeomorphic to iterated connected (oriented) sum of copies of $M$.

Jakobsche has considered cofinal inverse subsequences $S_{E,(F_i)}$ of the system $S_E$, as described in Remark 2.A.3, in which the sequences $F_1 \subset F_2 \subset \ldots$ of finite subtrees of $T$ satisfy the following conditions:

1. $F_1$ is a subtree of $T$ reduced to a single vertex;
2. for each $i \geq 1$ the vertex set $V_{F_{i+1}} \setminus V_{F_i}$ consists of vertices adjacent to $F_i$ (i.e. of form $\omega(e)$ for some $e \in N_{F_i}$).

A choice of a sequence $(F_i)$ in the above way yields an inverse sequence with the following properties:

1. each manifold $\hat{K}_{F_{i+1}}$ is obtained from $\hat{K}_{F_i}$ by means of operations of connected sum, with copies of the manifold $M$, performed at a finite set of pairwise disjoint connecting disks in the manifold $\hat{K}_{F_i}$;
2. each map $h_{F_{i+1},F_i}$ maps each new copy of $M$ in $\hat{K}_{F_{i+1}}$, attached to $\hat{K}_{F_i}$ by means of connected sum at appropriate connecting disk, to this disk, and it is the identity in the complement of the union of the connecting disks.

2.C.8 Remark. There are at least two examples in the literature where ideal boundaries of groups have been identified as certain trees of manifolds, and this has been achieved by referring to the description of these spaces as limits of Jakobsche’s inverse sequences, as described in Example 2.C.7.

First, there is a paper [Fi] by Hanspeter Fischer, in which the author identifies the CAT(0) boundaries of right angled Coxeter groups having manifold nerves as trees of the corresponding manifolds. A minor mistake in the statement of the main result of [Fi] is corrected in Theorem 3.A.3 below. The work of Fisher is complemented by the paper [PS], by Piotr Przytycki and the author, where a vast class of trees of manifolds, in dimensions $\leq 3$, is realized as Gromov boundaries of Coxeter groups which are hyperbolic.

A different class of spaces, and associated hyperbolic groups, is studied by Paweł Zawisła in [Z1]. He shows, among others, that Gromov boundary of a 7-systolic 3-dimensional orientable simplicial pseudomanifold is the tree of 2-tori (known also as the Pontriagin sphere). A related class of spaces is exhibited, for which the Gromov boundary is the tree of projective planes.
2.D Dimension of the limit of a tree system.

In this section we use associated inverse systems to estimate or calculate the topological dimension of limits of the corresponding tree systems.

An obvious estimate, implied by the fact that each constituent space $K_t$ embeds in $\lim \Theta$, is

$$\dim(\lim \Theta) \geq \sup \{ \dim(K_t) : t \in V_T \}.$$  \hspace{1cm} (2.D.1)

Our goal is to provide some upper bounds for $\dim(\lim \Theta)$, which in some cases, together with (2.D.1), yield equalities. Clearly, if $\sup \{ \dim(K_t) : t \in V_T \} = \infty$ then $\dim(\lim \Theta) = \infty$ as well. Thus, we restrict our attention to tree systems which have a universal finite upper bound for the dimensions of the constituent spaces $K_t$.

2.D.1 Proposition. Let $\Theta = (T, \{K_t\}, \{\Sigma_e\}, \{\phi_e\})$ be a tree system of metric compacta such that

$$\sup \{ \dim(K_t) : t \in V_T \} = n < \infty.$$

Suppose that for some constant $0 \leq c < 1$ for each $e \in O_T$ there is a retraction $r_e : K_{\alpha(e)} \to \Sigma_e$ such that for each $e' \in N_{\alpha(e)} \setminus \{e\}$ the restriction $r_e|_{\Sigma_{e'}}$ is a $c$-contraction. Suppose also that if the constant $c$ in the previous assumption is positive, there exists another constant $C > 0$ such that $\text{diam}(K_t) \leq C$ for each $t \in V_T$ Then $\dim(\lim \Theta) = n$.

Note that any family of maps $\{r_e : e \in O_T\}$ as in Proposition 2.D.1, together with the trivial family of extended spaces for $\Theta$ (see Example 2.A.1(1)), form a contracting system $E = (\{\hat{K}_F\}, \{h_{F,F'}\})$ of extended spaces and maps for $\Theta$. Moreover, by Lemma 2.B.3, $E$ is then fine. Thus, Proposition 2.D.1 is a special case of the following slightly more general result.

2.D.2 Proposition. Let $\Theta = (T, \{K_t\}, \{\Sigma_e\}, \{\phi_e\})$ be a tree system of metric compacta such that

$$\sup \{ \dim(K_t) : t \in V_T \} = n < \infty.$$

Suppose that $\Theta$ admits a fine family $E$ of extended spaces and maps, in which the corresponding family of extended spaces is trivial. Then $\dim(\lim \Theta) = n$.

Proof of Proposition 2.D.2: Since we have an easy estimate

$$\dim(\lim \Theta) \geq \sup \{ \dim(K_t) : t \in V_T \} = n,$$

it is sufficient to show the converse inequality $\dim(\lim \Theta) \leq n$.

Let $S_E = (\{\hat{K}_F\}, \{h_{F,F'}\})$ be the inverse system associated to $E$. Since the family $E$ is fine, Theorem 2.B.2 implies that $\lim \Theta \cong \lim \leftarrow S_E$. Since the family of extended spaces in $E$ is trivial, for each finite subtree $F \subset T$ we have $\hat{K}_F = \#\{K_t : t \in V_F\}$ (finite partial union), and hence $\dim(\hat{K}_F) = \max \{\dim(K_t) : t \in V_F\} \leq n$. Moreover, by the properties of inverse limits we have $\dim(\lim \leftarrow S_E) \leq \sup_{F \in \mathcal{F}_T} \dim(\hat{K}_F)$. This gives the required converse inequality $\dim(\lim \Theta) \leq n$, thus proving the proposition.
2.D.3 Example: tree of internally punctured manifolds with boundary.

We show how to apply Proposition 2.D.1 to calculate dimension of the limit for the following class of tree systems $\Theta = (T, \{K_t\}, \{\Sigma_e\}, \{\phi_e\})$. Fix any $n \geq 1$, and suppose that for each $t \in V_T$

$$K_t = M_t \setminus \bigcup \{\text{int}(D) : D \in \mathcal{D}\} \quad \text{and} \quad \{\Sigma_e : e \in N_t\} = \{\partial D : D \in \mathcal{D}\},$$

where $M_t$ is a compact $n$-dimensional topological manifold with nonempty boundary, and $\mathcal{D}$ is a null and dense in $M_t$ family of pairwise disjoint collared $n$-disks $D \subset \text{int}(M_t)$. We call any $\Theta$ as above a dense tree system of internally punctured manifolds with boundary. We claim that for any such $\Theta$ the topological dimension $\dim(\lim \Theta) = n - 1$.

To prove the claim, fix any $t \in V_T$ and any $e \in N_t$, and suppose that $\Sigma_e = \partial D_0$, where $D_0 \in \mathcal{D}$. Observe that by collapsing all peripheral subsets $\Sigma_{e'} : e' \in N_t \setminus \{e\}$ to points one gets the quotient space $K_t/\sim_e$ homeomorphic to $M_t/\text{int}(D_0)$, via a homeomorphism which sends $\partial M_t \cup \partial D_0 \subset K_t/\sim_e$ identically to $\partial M_t \cup \partial D_0 \subset M_t \setminus \text{int}(D_0)$ (see Lemma 3.D.3 below). We denote a homeomorphism $K_t/\sim_e \to M_t/\text{int}(D_0)$ as above by $h_e$. Observe also that, since $\partial M_t \neq \emptyset$, there exists a retraction $g_e : M_t \setminus \text{int}(D_0) \to \partial D_0 = \Sigma_e$. (Existence of such a retraction is pretty obvious when $M_t$ is either smooth or PL, and it requires some effort in topological category; the latter case is carefully dealt with in [Z2]!) Thus, putting $r_e := g_e \circ h_e$, we obtain a family $\{r_e\}$ of retractions as in the assumption of Proposition 2.D.1, with constant $c = 0$. Since we clearly have $\dim(K_t) = n - 1$ for each $t$, Proposition 2.D.1 directly implies that $\dim(\lim \Theta) = n - 1$.

2.D.3.1 Remark. Let $\mathcal{M}$ be a dense tree system of internally punctured manifolds with boundary, as defined above. Let $\mathcal{M}$ be a connected topological manifold with nonempty boundary, oriented or non-orientable, and suppose that all manifolds $M_t$ used to describe $\mathcal{M}$ are homeomorphic to $M$. Suppose also that, in case when $\mathcal{M}$ is oriented, all connecting maps $\phi_e$ of $\mathcal{M}$ respect orientations, i.e. they reverse the induced orientations on the corresponding spheres. By the arguments as before (see Examples 1.E.2 and 1.E.3) one easily shows that the system $\mathcal{M}$ depends uniquely up to isomorphism on $\mathcal{M}$ only. We will call such system $\mathcal{M}$ the dense tree system of internally punctured manifolds $\mathcal{M}$, and its limit $\lim \mathcal{M}$ the tree of internally punctured manifolds $\mathcal{M}$, denoted $\mathcal{X}_{\text{int}}(\mathcal{M})$.

2.D.4 Remark. The following example shows that the equality as in the assertion of Proposition 2.D.1 or 2.D.2 does not hold universally. Let $\mathcal{M}$ be a dense tree system of closed oriented $(n + 1)$-dimensional manifolds $\{M_t\}$. Then the corresponding constituent spaces $K_t$ (obtained from the manifolds $M_t$ as in Example 1.B.2) can be easily shown to have the topological dimension $\dim(K_t) = n$. In particular, we have $\sup \{\dim(K_t) : t \in V_T\} = n$. On the other hand, it is known that $\dim(\lim \mathcal{M}) = n + 1$, see Proposition (2.2) in [J2]. In the special case when all $M_t$ are $(n + 1)$-spheres, this follows also from Proposition 2.B.5.1.

The next estimate concerns peripherally ANR tree systems $\Theta$ (as defined in Section 2.C). This estimate implies in particular that if $\dim(\lim \Theta) \neq \sup \{\dim(K_t) : t \in V_T\}$ then $\dim(\lim \Theta) = \sup \{\dim(K_t) : t \in V_T\} + 1$. Note that the examples from Remark 2.D.4 fall into the category of tree systems satisfying this last equality.
2. D. 5 Proposition. For any peripherally ANR tree system of metric compacta \( \Theta = (T, \{K_t\}, \{\Sigma_e\}, \{\phi_e\}) \) we have

\[
\dim(\lim \Theta) \leq \max \left( \sup_{t \in V_t} \dim(K_t), \sup_{e \in O_T} \dim(\Sigma_e) + 1 \right).
\]

In particular, if \( \Theta \) is peripherally ANR and if \( \sup_t \dim(K_t) = n < \infty \) then \( \dim(\lim \Theta) \in \{n, n + 1\} \).

Proof: Consider any conical and 0-contracting family \( \mathcal{E} \) of extended spaces and maps for \( \Theta \), the existence of which is justified by Proposition 2.C.2, and recall that it is fine. Let \( S_\mathcal{E} \) be the associated inverse system for \( \Theta \) induced by \( \mathcal{E} \). For each \( t \in V_T \) the family \( \{K_t\} \cup \{\Delta_e : e \in N_t\} \) is a countable closed covering of \( \hat{K}_t \) and hence, by the countable union theorem (see [Eng, Theorem 7.2.1]), we have \( \dim(\hat{K}_t) = \max \left( \dim(K_t), \sup_{e \in N_t} \dim(\Delta_e) \right) \).

Since the family \( \mathcal{E} \) is conical, and since we have the equality \( \dim(\text{cone}(X)) = \dim(X) + 1 \) for any compact metric space, it follows that

\[
\dim(\hat{K}_t) = \max \left( \dim(K_t), \sup_{e \in N_t} \dim(\Sigma_e) + 1 \right).
\]

Using this, and applying once again the countable union theorem (this time to a finite union), we get the following estimate for each finite subtree \( F \) of \( T \):

\[
\dim(\hat{K}_F) \leq \max \left( \sup_{t \in V_t} \dim(K_t), \sup_{e \in O_T} \dim(\Sigma_e) + 1 \right).
\]

Finally, since by Theorem 2.B.4 we have \( \lim \Theta \cong \lim_{\leftarrow} S_\mathcal{E} \), and since by the properties of inverse limits we have \( \dim(\lim_{\leftarrow} S_\mathcal{E}) \leq \sup_{F \in F_T} \dim(\hat{K}_F) \), we get the required estimate for \( \dim(\lim \Theta) \), as in the first assertion of the proposition.

The second assertion follows immediately from the first one, and from the inequality \( \sup_{e \in O_T} \dim(\Sigma_e) \leq \sup_{t \in V_t} \dim(K_t) \) implied by the inclusions \( \Sigma_e \subset K_{\alpha(e)} \).

3. Modifications of tree systems.

In this part of the paper we describe some natural operations on tree systems which do not affect their limits. We also present several applications of these operations for justifying various properties of trees of manifolds. We are convinced that these operations provide a powerful tool for the future study of more general classes of trees of spaces.

3.A Consolidation of a tree system.

We describe an operation which turns one tree system of spaces into another by merging the constituent spaces of the initial system, and forming a new system out of bigger pieces (corresponding to a family of pairwise disjoint subtrees in the underlying tree of the initial system). As we show below (see Theorem 3.A.1), this operation does not affect the limit of a system.
Let $\Theta = (T, \{K_t\}, \{\Sigma_e\}, \{\phi_e\})$ be a tree system of metric compacta. Let $\Pi$ be a partition of a tree $T$ into subtrees, i.e., a family of subtrees $S \subset T$ such that the vertex sets $V_S : S \in \Pi$ are pairwise disjoint and cover all of $V_T$. We allow that some of the subtrees $S \in \Pi$ are just single vertices of $T$.

We define a consolidation of $\Theta$ with respect to $\Pi$, denoted $\Theta_\Pi$, to be the following tree system $(T_\Pi, \{K_S\}, \{\Sigma_e\}, \{\phi_e\})$. As a vertex set $V_\Pi$ of the tree $T_\Pi$ we take the family $\Pi$, and as the edge set $O_\Pi$ the set $\{e \in O_T : e \notin \bigcup_{S \in \Pi} O_S\}$. Clearly, for each oriented edge $e \in O_\Pi$ the initial vertex $\alpha_\Pi(e)$ is this subtree $S \in \Pi$ for which $\alpha(e) \in V_S$. Similarly, $\omega_\Pi(e)$ is this subtree $S \in \Pi$ for which $\omega(e) \in V_S$.

For any subtree $S \in \Pi$ denote by $\Theta_S$ the restriction of $\Theta$ to $S$, and put $K_S := \lim \Theta_S$.

Note that for any $e \in O_\Pi$ we have the canonical inclusion $\Sigma_e \subset K_{\alpha_\Pi(e)}$, and if we put $N_S = \{e \in O_\Pi : \alpha_\Pi(e) = S\}$, the family $\Sigma_e : e \in N_S$ of subsets of $K_S$ consists of pairwise disjoint sets. Moreover, by Proposition 1.C.1, the subsets $\Sigma_e \subset K_{\alpha_\Pi(e)}$ are all compact and each family $\Sigma_e : e \in N_S$ is null. This justifies that the just described tuple

$$\Theta_\Pi := (T_\Pi, \{K_S : S \in \Pi\}, \{\Sigma_e : e \in O_\Pi\}, \{\phi_e : e \in O_\Pi\})$$

is a tree system of metric compacta.

3.A.1 Theorem. For any tree system $\Theta$ of metric compacta, and its any consolidation $\Theta_\Pi$, the limits $\lim \Theta$ and $\lim \Theta_\Pi$ are canonically homeomorphic.

Proof: We first describe the natural map $i_\Pi : \lim \Theta_\Pi \to \lim \Theta$, and then show it is a homeomorphism.

To define $i_\Pi(x)$ for all $x \in \lim \Theta_\Pi$, we consider three possible positions of $x$ in $\lim \Theta_\Pi$. First, suppose that $x \in K_t \subset K_S \subset \lim \Theta_\Pi$, for some $t \in V_S$ and some $S \in \Pi$. View $x$ as an element of $\lim \Theta$ via the canonical inclusion $K_t \subset \lim \Theta$, and put $i_\Pi(x) = x$. Second, suppose that $x \in E_S \subset \lim \Theta_S = K_S \subset \lim \Theta_\Pi$ is an end of some tree $S \in \Pi$. Since we have the canonical inclusion $E_S \subset E_T$, $x$ is also an end of $T$, and hence an element of $\lim \Theta$. We put again $i_\Pi(x) = x$. Finally, in the remaining case $x$ is an end of the tree $T_\Pi$. Suppose that $x$ is represented by a ray $\rho = (S_0, S_1, \ldots)$ in the tree $T_\Pi$. Then $\rho$ determines the ray $\rho_T$ in $T$ by

$$\rho_T = (e_1, [\omega(e_1), \alpha(e_2)], e_2, [\omega(e_2), \alpha(e_3)], e_3, \ldots),$$

where $e_i$ is the edge in $T$ connecting $S_{i-1}$ to $S_i$, and $[\omega(e_i), \alpha(e_{i+1})]$ are the paths in $T$ (sometimes perhaps empty) connecting the corresponding vertices. Put $i_\Pi(x) = [\rho_T] \in E_T \subset \lim \Theta$. We skip the straightforward verification that $i_\Pi$ is well defined, and that it is a bijection.

Since any continuous bijection between compact metric spaces is a homeomorphism, to prove that $i_\Pi$ is a homeomorphism, it is sufficient to show that it is continuous. To do this, we will show that for any open set $V \subset \lim \Theta$ from the basis $B$ the preimage $i_\Pi^{-1}(V)$ is open in $\lim \Theta_\Pi$. For this we need two claims.

Claim 1. For any $S \in \Pi$ the restriction $i_\Pi|_{K_S}$ is a homeomorphism on its image (a topological embedding).
Since $K_S$ is compact and $i_\Pi$ is injective, to prove Claim 1 it is sufficient to show that $i_\Pi|_{K_S}$ is continuous. Let $G(U) \in \mathcal{B}$, where $U$ is an open subset in $K_F$ for some finite $F \subset T$, and $U$ is $\mathcal{A}_F$-saturated. We need to show that $K_S \cap i_\Pi^{-1}(G(U))$ is open in $K_S$. Suppose first that $F \cap S = \emptyset$. It is not hard to realize that in this case $K_S \cap i_\Pi^{-1}(G(U))$ is either empty or coincides with $K_S$, and thus the assertion follows. Suppose then that $F \cap S \neq \emptyset$. Viewing $K_{F \cap S}$ as subset in $K_F$, put $U_S := U \cap K_{F \cap S}$, and note that $U_S$ is open in $K_{F \cap S}$ and saturated with respect to the family $\mathcal{A}^\Theta_{F_S}$ of peripheral subsets in $K_{F \cap S}$ viewed as a partial union of the system $\Theta_S$. Moreover, it is not hard to observe that $K_S \cap i_\Pi^{-1}(G(U)) = G(U_S) \in \mathcal{B}_S$, where $\mathcal{B}_S$ is the standard basis in the limit $K_S$ of the system $\Theta_S$ (as described in Section 1.C). Thus Claim 1 follows.

**Claim 2.** For any finite subtree $F_0 \subset T_\Pi$ the restriction $i_\Pi|_{K_{F_0}}$ is a homeomorphism on its image.

Since $K_{F_0}$ is a finite union of its compact subsets of form $K_S$, it is compact itself. Moreover, in view of Claim 1, the same fact implies that $i_\Pi|_{K_{F_0}}$ is continuous. Since this map is also injective, the assertion of Claim 2 follows.

Coming back to the proof that $i_\Pi$ is continuous, let $V = G(U)$ for some open and $\mathcal{A}_F$-saturated subset $U \subset K_F$, where $F \subset T$ is some finite subtree. Let $F_\Pi$ be the subtree of $T_\Pi$ spanned by the vertices represented by those subtrees $S \in \Pi$ which intersect $F$; clearly, $F_\Pi$ is finite. Note that, by Claim 2, the set $U_\Pi := K_{F_\Pi} \cap i^{-1}_\Pi(V)$ is open in $K_{F_\Pi}$ (because $i_\Pi(K_{F_\Pi}) \cap V$ is open in $i_\Pi(K_{F_\Pi})$). Moreover, since $U$ is $\mathcal{A}_F$-saturated, it is not hard to realize that $U_\Pi$ is $\mathcal{A}_{F_\Pi}$-saturated, where $\mathcal{A}_{F_\Pi}$ is the appropriate family of peripheral subsets of the system $\Theta_\Pi$ in its partial union $K_{F_\Pi}$. Finally, observe that $i^{-1}_\Pi(V) = G(U_\Pi) \in \mathcal{B}_\Pi$ ($\mathcal{B}_\Pi$ denotes the standard basis for the topology in $\lim \Theta_\Pi$), and hence this preimage is open in $\lim \Theta_\Pi$. This finishes the proof.

The next example illustrates how one can apply the procedure of consolidation to justify that limits of various classes of tree systems are homeomorphic.

### 3.A.2 Example: dense trees of finite families of manifolds.

Let $\mathcal{N} = \{M_1, \ldots, M_k\}$ be a finite family of closed connected oriented topological manifolds of the same dimension. Let $\mathcal{M} = (T, \{K_t\}, \{\Sigma_e\}, \{\phi_e\})$ be a dense tree system of manifolds from $\mathcal{N}$. For each $t \in V_T$ let $i_t \in \{1, \ldots, k\}$ be this index for which the space $K_t$ has a form $M^{\circ}_{i_t}$, as described in Section 1.E, just after Definition 1.E.2.2. We say that $\mathcal{M}$ is 2-saturated if for each $t \in V_T$ and every $j \in \{1, \ldots, k\}$ there are at least two distinct edges $e \in N_t$ such that $i_{\omega(e)} = j$.

If $\mathcal{M}$ is 2-saturated, it is not hard to construct a partition $\Pi$ of $T$ such that for each $S \in \Pi$ the vertex set $V_S$ contains exactly one vertex $s$ with $i_s = j$, for each $j \in \{1, \ldots, k\}$ (in particular, the cardinality of each $V_S$ is $k$). Consider the consolidated tree system $\mathcal{M}_\Pi$ for the partition $\Pi$. It is easy to note that the constituent spaces of $\mathcal{M}_\Pi$ are all of form $(M_1 \# \ldots \# M_k)^\circ$, and that $\mathcal{M}_\Pi$ is (isomorphic to) the dense tree system of manifolds $M_1 \# \ldots \# M_k$. In view of Theorem 3.A.1 and Proposition 1.E.2.3 we get the following.

### 3.A.2 Proposition.

For any 2-saturated dense tree system $\mathcal{M}$ of manifolds from $\mathcal{N}$ the limit $\lim \mathcal{M}$ is homeomorphic to the Jakobsche space $X(M_1 \# \ldots \# M_k)$. 

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The above observation can be generalized as follows. Let $\mathcal{N}$ be a family of manifolds as above, and let $\mu = (m_1, \ldots, m_k)$ be a tuple of arbitrary positive integers. If $\mathcal{M}$ is a tree system as above, one can easily construct a partition $\Pi_{\mu}$ such that for each $S \in \Pi_{\mu}$ and each $j \in \{1, \ldots, k\}$ the vertex set $V_S$ contains exactly $m_j$ elements $s$ with $i_s = j$. By the argument as above, we get that $\lim \mathcal{M} = \lim \mathcal{M}_{\Pi_{\mu}}$ is homeomorphic to the Jakobsche space $\mathcal{X}(m_1 M_1 \# \ldots \# m_k M_k)$, where each $m_i M_i$ is the connected sum of $m_i$ copies of $M_i$. As a consequence, we get the following.

3.A.2.2 Corollary. Let $M_1, \ldots, M_k$ be any closed connected oriented topological manifolds of the same dimension, and let $m_1, \ldots, m_k$ be arbitrary positive integers. Then the Jakobsche spaces $\mathcal{X}(M_1 \# \ldots \# M_k)$ and $\mathcal{X}(m_1 M_1 \# \ldots \# m_k M_k)$ are homeomorphic.

Now, let $\mathcal{K} = \{N_1, \ldots, N_k\}$ be a family of closed connected topological manifolds of the same dimension, at least one of which is non-orientable. We also assume that the orientable manifolds in $\mathcal{K}$ have no distinguished orientation. By a tree system of manifolds from $\mathcal{K}$ we mean a tree system of manifolds in which every constituent space has a form $N^\circ$ for some $N \in \mathcal{K}$, and in which the connecting maps are arbitrary homeomorphisms between the corresponding spherical peripheral subspaces. Note that a connected sum $N_1 \# \ldots \# N_k$ (again, with arbitrary connecting homeomorphisms) is unique up to homeomorphism (this is a consequence of Lemma 1.E.3.1).

The arguments as above, enriched by application of Lemma 1.E.3.1, yield the following variation on Proposition 3.A.2.1 and Corollary 3.A.2.2.

3.A.2.3 Proposition. Let $\mathcal{K} = \{N_1, \ldots, N_k\}$ be a family of closed connected topological manifolds of the same dimension, at least one of which is non-orientable. Let $\mathcal{M}$ be any 2-saturated dense tree system of manifolds from $\mathcal{K}$. Then the limit $\lim \mathcal{M}$ is homeomorphic to the Jakobsche space $\mathcal{X}(N_1 \# \ldots \# N_k)$ (as defined in Example 1.E.3). Moreover, for any positive integers $m_1, \ldots, m_k$ the space $\mathcal{X}(m_1 N_1 \# \ldots \# m_k N_k)$ is also homeomorphic to $\mathcal{X}(N_1 \# \ldots \# N_k)$.

Another application of the results of this section is the following correction to the main result from the paper [Fi] by Hanspeter Fischer.

3.A.3 Theorem. Suppose $W$ is a right angled Coxeter group whose nerve is a flag PL triangulation of a closed oriented manifold $M$, and let $\overline{M}$ be the same manifold with reversed orientation. Then the CAT(0) boundary of $W$ (i.e. the boundary of the Coxeter-Davis complex for $W$) is the Jakobsche space $\mathcal{X}(M \# \overline{M})$.

In the original (wrong) statement of this result in [Fi] instead of the space $\mathcal{X}(M \# \overline{M})$, there appears the space $\mathcal{X}(M)$, which is in general different (for example, it is not hard to show that the spaces $\mathcal{X}(CP^2)$ and $\mathcal{X}(CP^2 \# \overline{CP^2})$ are not homeomorphic, by referring to the properties of their Čech cohomology rings).

Proof of Theorem 3.A.3: We indicate a necessary minor modification of the argument provided in [Fi]. The author argues by showing that $\partial W$ is homeomorphic to the inverse limit of some inverse sequence of manifolds of the form as in Example 2.C.7. A part of his argument which requires correction is this (see the beginning of the proof of Theorem 3.7 in [Fi]). Let $X_W$ be the Coxeter-Davis complex for $W$, obtained as the union $X_W =$
∪\{gQ : g ∈ W\}, where Q is the Davis’ cell for W (topologically equal to the cone over the manifold M). Let |g| be the word norm for elements g ∈ W with respect to the standard generating set. Let X_k = ∪\{gQ : |g| ≤ k\} and let M_k = ∂X_k. The author claims that M_k is the connected sum of the appropriate number of copies of M, but it is clear from the way X_k is formed out of copies of Q that in fact M_k is the connected sum of copies of both M and \overline{M}. Moreover, since any two adjacent copies of Q in X_W are obtained from one another by reflection, M_{k+1} is obtained from M_k by connected sum with copies of M if k is odd, and with copies of \overline{M} if k is even (in particular, M_0 = ∂Q = M). From this, using the remaining arguments of Fischer, one deduces that ∂W is homeomorphic to the limit of a 2-saturated dense tree system of manifolds M and \overline{M}. By Proposition 3.A.2.1, this yields the assertion.

3.B Tree decomposition of a compact metric space.

In Sections 3.B and 3.C we describe an operation on a tree system that is inverse to that of consolidation (as described in Section 3.A). In this section, we start with an elementary case when the initial tree system is trivial, i.e. the underlying tree T is reduced to a single vertex. The operation is then called tree decomposition of a compact metric space.

We start with introducing terminology related to the concept of tree decomposition. The main result of the section, which relates this concept to that of a tree system and its limit, is Theorem 3.B.10 below.

At the end of the section we show how to use tree decompositions to prove that limits of some tree systems are homeomorphic to some explicit spaces (Example 3.B.12).

3.B.1 Definition. An elementary splitting of a compact metric space K is a triple (A, \{Y, Z\}) of compact subspaces of K such that Y ∪ Z = K, Y ∩ Z = A, and A is a nonempty proper subset both in Y and in Z. The set A is called the separator of the splitting, and the sets Y, Z are the halfspaces. Moreover, the sets Y \ A and Z \ A will be called the open halfspaces of the splitting. If H is any halfspace of the splitting above (i.e. H = Y or H = Z), we denote by \hat{H} the corresponding open halfspace, and by H^c the complementary (or opposite) halfspace (equal to K \ \hat{H}).

Note that for any splitting as above we have the following:
(1) the set K \ A is disconnected and the open halfspaces are the unions of connected components in this set;
(2) K is canonically homeomorphic to the space Y ∪_A Z obtained from the disjoint union of Y and Z by gluing through the identity on A; equivalently, K is the limit of the tree system whose underlying tree T is a single edge, the constituent spaces K_i are Y and Z, the peripheral subspaces Σ_e coincide with A, and the connecting maps φ_e are the identities on A.

3.B.2 Definition. Given two elementary splittings (A_i, \{Y_i, Z_i\}), i = 1, 2, of a compact metric space K, we say they do not cross if for at least one pair of halfspaces H_1, H_2 selected from those splittings we have H_1 ∩ H_2 = ∅.

3.B.3 Remark. Note that the noncrossing condition H_1 ∩ H_2 = ∅ has the following consequences: (1) A_1 ∩ A_2 = ∅, (2) H_1 ⊂ \hat{H}_2^c and H_2 ⊂ \hat{H}_1^c.
3.B.4 Definition. Given three pairwise noncrossing splittings \((A_i, \{Y_i, Z_i\}), i = 1, 2, 3\), of a compact metric space \(K\), we say that \(A_2\) separates \(A_1\) from \(A_3\) if for appropriately chosen halfspace \(H\) for \(A_2\) we have \(A_1 \subset H\) and \(A_3 \subset H^c\).

We will be interested in countable (usually infinite) families of pairwise noncrossing splittings satisfying some additional properties, which we now describe.

3.B.5 Definition. Let \(\mathcal{C} = (A_\lambda, \{Y_\lambda, Z_\lambda\})_{\lambda \in \Lambda}\) be a family of pairwise noncrossing splittings of a compact metric space \(K\). We say that \(\mathcal{C}\) is discrete if for any two separators \(A, A'\) from \(\mathcal{C}\) there is only finitely many separators in \(\mathcal{C}\) that separate \(A\) from \(A'\).

Any discrete family \(\mathcal{C}\) of pairwise noncrossing splittings of \(K\) determines the associated family of domains obtained by splitting, and the dual tree \(T_\mathcal{C}\). We start with describing the domains.

Consider any separator \(A\) from \(\mathcal{C}\) and any halfspace \(H\) related to \(A\). The pair \((A, H)\) determines a domain \(\Omega_{A,H}\) described as follows. For any separator \(A_\lambda\) in \(\mathcal{C}\) consider this halfspace \(H_\lambda\) which contains \(A\). We have to consider the following two cases. First, suppose that for each \(\lambda \in \Lambda\) we have \(H \subset H_\lambda\). We then put \(\Omega_{A,H} := H\) and note that this domain is disjoint with all separators from \(\mathcal{C}\) other than \(A\). In the second case, there is some \(\lambda \in \Lambda\) with \(H \not\subset H_\lambda\). By discreteness, there is also such \(\lambda\) for which no separator from \(\mathcal{C}\) separates \(A\) from \(A_\lambda\). Denote by \(\Lambda_{A,H}\) the set of all \(\lambda \in \Lambda\) for which \(H \not\subset H_\lambda\) and no separator from \(\mathcal{C}\) separates \(A\) from \(A_\lambda\). By what was said above, this set is nonempty. Put

\[
\Omega_{A,H} := H \cap \bigcap_{\lambda \in \Lambda_{A,H}} H_\lambda
\]

and note that this set satisfies the following properties:

1. \(\Omega_{A,H}\) is compact, contains the sets \(A\) and \(A_\lambda : \lambda \in \Lambda_{A,H}\), and it is disjoint with all other separators from \(\mathcal{C}\);
2. for each \(\lambda \in \Lambda_{A,H}\) we have \(\Omega_{A_\lambda,H_\lambda} = \Omega_{A,H}\).

A domain in \(K\) induced by \(\mathcal{C}\) is any subset \(\Omega = \Omega_{A,H}\) as above. A domain \(\Omega\) is called adjacent to a separator \(A\) of \(\mathcal{C}\) if it contains \(A\) or equivalently, if \(\Omega = \Omega_{A,H}\) for some halfspace \(H\) related to \(A\). For each separator \(A\) of \(\mathcal{C}\) there are exactly two domains induced by \(\mathcal{C}\) and adjacent to \(A\).

We are now ready to describe the dual tree \(T_\mathcal{C}\) of a discrete family of pairwise noncrossing splittings \(\mathcal{C}\). As a vertex set \(V_\mathcal{C}\) we take the set of all domains for \(\mathcal{C}\), as described above. As the set \(E_\mathcal{C}\) of orieted edges we take the set of all pairs \((A, H)\) as above, and we put \(\omega_\mathcal{C}(A, H) = \Omega_{A,H}\) and \((A, H) = (A, H^c)\). It is an easy exercise to show that the graph obtained in the above way is connected, and contains no loops, thus being a tree. We denote this tree by \(T_\mathcal{C}\) and call the dual tree of \(\mathcal{C}\).

Remark. Note that for any discrete family \(\mathcal{C}\) of pairwise noncrossing splittings of a compact metric space \(K\) the corresponding dual tree \(T_\mathcal{C}\) is locally countable (and hence countable). More precisely, given a vertex \(\Omega\) of \(T_\mathcal{C}\) (i.e. a domain for \(\mathcal{C}\)), consider the set \(N_\Omega\) of all edges \((A, H)\) in \(T_\mathcal{C}\) satisfying \(\alpha_\mathcal{C}(A, H) = \Omega\). Note that the corresponding family of open halfspaces in \(K\), \(\{\hat{H} : (A, H) \in N_\Omega\}\) consists of pairwise disjoint nonempty open
subsets of $K$. Since any compact metric space is separable, it follows that the set $N_{\Omega}$ is countable, thus justifying the remark.

The above construction of the dual tree motivates the first part of the following.

3.B.6 Definition.  
(1) A discrete family $C$ of pairwise noncrossing splittings of a compact metric space $K$ will be called a tree decomposition of $K$.

(2) A tree decomposition $C = \{(A_\lambda, \{Y_\lambda, Z_\lambda\})\}_{\lambda \in \Lambda}$ of $K$ is fine if for each $\epsilon > 0$ the set $\{\lambda \in \Lambda : \min[\text{diam}(Y_\lambda), \text{diam}(Z_\lambda)] > \epsilon\}$ is finite.

3.B.7 Example. Let $K = \lim \Theta$ be the limit of a tree system $\Theta$ of metric compacta, and suppose that it is essential, in the sense that for any $e \in O_T$ the set $\Sigma_e$ is a proper subset in $K_{\alpha(e)}$. For any edge $e \in O_T$ denote by $T_e$ this subtree obtained by removing (the interior of) $e$ from $T$ which contains the terminal vertex $\omega(e)$. Put $H_e$ to be the limit of the restricted system $\Theta_{T_e}$. Note that for each $e \in O_T$ the triple $(\Sigma_e, \{H_e, H_{\bar{e}}\})$, viewed as consisting of subsets of $K$, is an elementary splitting of $K$. Note also that the family of all splittings of $K$ having this form yields a tree decomposition of $K$. We denote this decomposition by $\mathcal{C}(\Theta)$. Moreover, this decomposition is fine, which can be deduced by the same methods as in the proof of the last assertions in Proposition 1.C.1 (as given in Section 1.D).

One of the consequences of fineness of a tree decomposition is the following.

3.B.8 Fact. Let $C$ be a fine tree decomposition of a compact metric space $K$. Then for any domain $\Omega$ for $C$ the family $A_\Omega$ of all separators from $C$ adjacent to $\Omega$ is null.

Proof: Note that $A_\Omega$ coincides with the family of those separators $A$ from $C$ for which there is a halfspace $H$ (related to $A$) such that $\alpha_C(A, H) = \Omega$. Since any opposite halfspace $H^c$ to a halfspace $H$ as in the previous sentence contains $\Omega$, and since then $\text{diam}(H) \geq \text{diam}(\Omega) > 0$, it follows from fineness of $C$ that the family of halfspaces $\{H : \alpha_C(A, H) = \Omega\}$ is null. Consequently, since we have inclusions $A \subset H$, the family $A_\Omega$ is also null.

3.B.9 Definition. Given a fine tree decomposition $C$ of a compact metric space $K$, the tree system $\Theta_C$ associated to $C$ is described as follows. The underlying tree for $\Theta_C$ is the dual tree $T_C$. For each vertex $t \in V_C$ represented by some domain $\Omega$ we put $K_t = \Omega$. For each oriented edge $e = (A, H) \in O_C$ we put $\Sigma_e = A$ and $\phi_e = \text{id}_A$. In view of Fact 3.B.8, this well defines a tree system of metric compacta (which is moreover essential).

Remark. It is not hard to realize that if $\mathcal{C}(\Theta)$ is the tree decomposition of $\lim \Theta$ described in Example 3.B.7 then the associated tree system $\Theta_{\mathcal{C}(\Theta)}$ is canonically isomorphic with $\Theta$.

The main result of this section is the following.

3.B.10 Theorem. For any fine tree decomposition $C$ of a compact metric space $K$ the limit $\lim \Theta_C$ of the associated tree system is canonically homeomorphic to $K$.

To prove Theorem 3.B.10 we need the following result which exhibits consequences of fineness.
3.B.11 Fact. Let \( C \) be a fine tree decomposition of a compact metric space \( K \).

1. For each domain \( \Omega \) for \( C \) the family of halfspaces from \( C \) given by \( \mathcal{H}_\Omega := \{ H : \alpha_C(A, H) = \Omega \} \) is null.

2. For each ray \( \rho = (e_1, e_2, \ldots) \) in the dual tree \( T_C \), with \( e_i = (A_i, H_i) \), the corresponding family \( \{ H_i \} \) of halfspaces is null in \( K \).

Proof: The proof of part (1) has already been given in the proof of Fact 3.B.8. To prove part (2), put \( \Omega = \alpha_C(e_1) \) and note that for each \( i \) we have \( \Omega \subset H_i^\circ \). Consequently, we have \( \text{diam}(H_i^\circ) \geq \text{diam}(\Omega) > 0 \). It follows then from fineness of \( C \) that \( \lim_{i \to \infty} \text{diam}(H_i) = 0 \), which completes the proof.

Proof of Theorem 3.B.10: We first describe some natural bijection \( \beta : \lim \Theta_C \to K \) and then show that \( \beta \) is a homeomorphism.

To define \( \beta(x) \) for any \( x \in \lim \Theta_C \), consider first the case when \( x \in \#\Theta_C \). It means that \( x \) is contained in some constituent space of the system \( \Theta_C \), i.e., in some domain \( \Omega \) for \( C \). Viewing \( \Omega \) as subset of \( K \), we put \( \beta(x) = x \). In the second case, suppose that \( x \in E_{T_C} \subset \lim \Theta_C \) is an end of the dual tree \( T_C \). Suppose that \( x = [\rho] \), where \( \rho = (e_1, e_2, \ldots) \) is an appropriate ray in \( T_C \). Denoting \( e_i = (A_i, H_i) \), we get the sequence of halfspaces \( H_1 \supset H_2 \supset \ldots \) with \( \lim_{i \to \infty} \text{diam}(H_i) = 0 \) (see Fact 3.B.11(2)). Thus the intersection \( \bigcap_{i=1}^\infty H_i \) is a single point of \( K \). We denote this point by \( y \) and put \( \beta(x) = y \). We leave it as an exercise to show that \( y \) depends only on \( x \) (and not on a representative ray \( \rho \)). We also skip an easy observation that \( \beta \) is a bijection.

Since both spaces \( K \) and \( \lim \Theta_C \) are compact, to prove that \( \beta \) is a homeomorphism, it is sufficient to show that \( \beta \) is continuous. To do this, for arbitrary open set \( W \subset K \) and any \( x \in \beta^{-1}(W) \) we will indicate some open set \( G(U) \) from the basis \( B_C \) for \( \lim \Theta_C \) such that \( x \in G(U) \subset \beta^{-1}(W) \).

Suppose first that \( x \in E_{T_C} \). Let \( \{ H_i \} \) be a sequence of halfspaces as in the definition of \( \beta(x) \) above. Since the family \( \{ H_i \} \) is null, and since \( \beta(x) \in H_i \) for each \( i \), there is \( m \) such that \( H_m \subset W \). Put \( \Omega := \Omega_{A_m, H_m} \) and \( U := \Omega \setminus A_m \). Clearly, \( U \) is an open subset in \( \Omega \) which is saturated with respect to the family \( A_\Omega \) (consisting of all separators from \( C \) contained in \( \Omega \)). Moreover, it is not hard to see that \( \beta(G(U)) = H_m \setminus A_m \subset W \), and hence \( G(U) \subset \beta^{-1}(W) \). Furthermore, since \( \beta(x) \in H_{m+1} \subset H_m \setminus A_m = \beta(G(U)) \), we get that \( x \in G(U) \), which completes the proof in the first case.

Now, suppose that \( x \in \#\Theta_C \). i.e. \( x \in \Omega \subset \#\Theta_C \subset \lim \Theta_C \) for some constituent space \( \Omega \) of \( \Theta_C \). We need to consider two subcases, even though the argument in these two subcases is almost the same. In the first subcase, suppose that \( x \) is not contained in any separator \( A \) from \( C \). Besides the family \( A_\Omega \) as in the previous paragraph (and as in Fact 3.B.8) consider also the related family of halfspaces \( \mathcal{H}_\Omega \) as described in Fact 3.B.11(1). Consider also the bijective function \( h : A_\Omega \to \mathcal{H}_\Omega \) given by \( h(A) = H \) for any pair \( (A, H) \) as in the descriptions of \( A_\Omega \) and \( \mathcal{H}_\Omega \). Let \( r > 0 \) be a real number such that the ball \( B_r(x) \) in \( K \) (centered at \( x = \beta(x) \in K \)) is contained in \( W \). Since the family \( A_\Omega \) is null, it follows from Lemma 1.D.3(2) that there is an open neighbourhood \( U \) of \( x \) in \( \Omega \) which is contained in the smaller ball \( B_{\frac{r}{2}}(x) \) in \( \Omega \) (for the metric restricted from \( K \)) and which is \( A_\Omega \)-saturated. Since the family \( \mathcal{H}_\Omega \) is null, we may assume without loss of generality that whenever some separator \( A \in A_\Omega \) is contained in \( U \) then \( \text{diam}(h(A)) < \frac{r}{2} \). Consequently,
the set
\[ U' := U \cup \bigcup \{ H \in \mathcal{H}_\Omega : H = h(A) \text{ and } A \subset U \} \]
is contained in the ball \( B_r(x) \) in \( K \), and hence in \( W \). On the other hand, \( U \) defines the set \( G(U) \) for the basis \( B_C \) in \( \lim \Theta_C \), and it is not hard to realize that \( \beta^{-1}(U') = G(U) \). Since we also have \( x \in U \subset G(U) \), the proof is completed in this subcase.

In the remaining subcase, we consider \( x \in A \subset \Omega \) for some domain \( \Omega \) for \( C \) and some separator \( A \) from \( C \). Let \( \Omega' \) be the second (where the first one is \( \Omega \)) domain for \( C \) containing \( A \). We repeat the argument as in the previous subcase, for \( \Omega \) replaced with \( \Omega \cup \Omega' \), \( \mathcal{A}_\Omega \) replaced with \( \mathcal{A}_\Omega \cup \mathcal{A}_{\Omega'} \setminus \{ A \} \), and \( \mathcal{H}_\Omega \) replaced with \( \mathcal{H}_\Omega \cup \mathcal{H}_{\Omega'} \setminus \{ H, H^c \} \), where \( H \) and \( H^c \) are the halfspaces corresponding to the separator \( A \). We skip further details, thus finishing the proof.

3.B.12 Example: boundary tree of disks is a disk.

For any \( n \geq 2 \), let \( D^n \) be the \( n \)-disk, and let \( D \) be a dense and null family of pairwise disjoint collared \((n - 1)\)-disks in the boundary sphere \( \partial D^n \). Note that, due to Toruńczyk’s Lemma 1.E.2.1 (followed by Alexander’s trick), the tuple \((D^n, D)\) is unique up to homeomorphism. Consider the unique tree system in which all constituent spaces are homeomorphic to \( D^n \), and all families of peripheral subspaces coincide, up to ambient homeomorphism, with \( D \). We denote this tree system by \( \mathcal{M}_\partial(D^n) \) and call it the dense boundary tree system of \( n \)-disks.

3.B.12.1 Lemma. The limit \( \lim \mathcal{M}_\partial(D^n) \) is the \( n \)-disk.

Proof: We will describe a fine tree decomposition \( \mathcal{C} \) of the \( n \)-disk \( D^n \) such that the associated tree system \( \Theta_\mathcal{C} \) is isomorphic to \( \mathcal{M}_\partial(D^n) \). By applying Theorem 3.B.10, and in view of Lemma 1.E.1, this gives the assertion.

To construct \( \mathcal{C} \), view \( D^n \) as the standard \( n \)-disk in \( R^n \), and consider the group \( M\text{ö}b(D^n) \) of all Möbius transformations of \( D^n \), i.e. those Möbius transformations of \( R^n \) which preserve \( D^n \). Viewing \( \text{int}(D^n) \) as the Poincare disk model for the hyperbolic \( n \)-space, we will think of \( M\text{ö}b(D^n) \) as the group of all hyperbolic isometries in its action on the completion of the hyperbolic \( n \)-space by its ideal boundary.

Let \( D \) be any null and dense family of pairwise disjoint round \((n - 1)\)-disks in \( \partial D^n \). For each \( D \in D \), let \( H_D \) be the hyperbolic halfspace in \( D^n \) such that \( H_D \cap \partial D^n = D \), and let \( H_D^c \) be the opposite halfspace. Denote by \( A_D \) the hyperplane in \( D^n \) bounding \( H_D \), and by \( s_D \) the hyperbolic reflection with respect to \( A_D \), which clearly belongs to \( M\text{ö}b(D^n) \).

Let \( \Gamma \) be the subgroup of \( M\text{ö}b(D^n) \) generated by all elements \( s_D : D \in D \). Obviously, \( \Gamma \) is then an infinitely generated free reflection group with the fundamental domain

\[ \Omega_0 := \bigcap_{D \in D} H_D^c. \]

Algebraically, \( \Gamma \) is the free product of its order 2 subgroups \( \langle s_D \rangle : D \in D \).

Let \( \mathcal{A} \) be the family of reflection hyperplanes in \( D^n \) for all reflections from \( \Gamma \). In other words, \( \mathcal{A} \) is the family of all images through elements of \( \Gamma \) of the hyperplanes \( A_D : D \in D \). Each \( A \in \mathcal{A} \) splits \( D^n \) into two components. Denote the closures of these components in \( D^n \) by \( Y \) and \( Z \), and observe that \( (A, \{Y, Z\}) \) is an elementary splitting of \( D^n \). Denote by
C the set of elementary splittings \((A, \{Y, Z\})\) as above, for all \(A \in A\). It is fairly clear that \(C\) is then a discrete family of pairwise noncrossing splittings of \(D^n\), i.e. a tree decomposition of \(D^n\), and that it is fine.

It remains to show that the tree system \(\Theta_C\) associated to \(C\) is isomorphic to \(M_\partial(D^n)\). To see this, note that for each \(D \in D\) the domain \(\Omega_{A_D, H_D}\) coincides with the fundamental domain \(\Omega_0\) for \(\Gamma\). It is not hard to see that this domain is homeomorphic to the \(n\)-disk. The separators of \(C\) contained in \(\Omega_0\) are exactly \(A_D : D \in D\), and they clearly form a null and dense family of collared and pairwise disjoint \((n-1)\)-disks in the boundary \(\partial \Omega_0\). Consequently, the constituent space \(\Omega_0\) of \(\Theta_C\), together with the family \(A_D : D \in D\) of its all peripheral subspaces, is homeomorphic to \((D^n, \mathcal{D})\). To see that the same is true for all other constituent spaces of \(\Theta_C\), note that each such space has a form \(\gamma \Omega_0\) for some \(\gamma \in \Gamma\), and the corresponding family of peripheral subspaces has a form \(\gamma A_D : D \in D\). This shows that \(\Theta_C\) is isomorphic to \(M_\partial(D^n)\), thus completing the proof.

3.B.12.2 Remark. Let \(C\) be a family of elementary splittings of \(D^n\), as constructed in the proof of Lemma 3.B.12.1. Restricting \(C\) to the boundary \(\partial D^n\), we obviously get a fine tree decomposition of the \((n-1)\)-sphere, and we denote it by \(C|_{\partial D^n}\). Moreover, the domains of this new decomposition are the intersections of the domains of \(C\) with \(\partial D^n\), and it is not hard to verify that they are the \((n-2)\)-dimensional Sierpiński compacta \((S^{n-1})^o\). Thus, these Sierpiński compacta are the constituent spaces of the associated tree system \(\Theta_{C|_{\partial D^n}}\), and the peripheral subspaces correspond exactly to the peripheral spheres in these Sierpiński compacta. It follows that the associated tree system \(\Theta_{C|_{\partial D^n}}\) is isomorphic to the tree system \(M(S^{n-1})\). Since, by Theorem 3.B.10, we have

\[
\lim \Theta_{C|_{\partial D^n}} = \partial D^n = S^{n-1},
\]

it follows that \(\lim M(S^{n-1}) = S^{n-1}\). Thus, we get an alternative (and more elementary) proof of Proposition 2.B.5.1.

3.B.13 Remark/Example/Exercise. Using a similar argument as in the proof of Lemma 3.B.12.1 one can identify, up to homeomorphism, limits of various other tree systems. For example, one can show that the limit of any dense tree system of internally punctured \(n\)-disks (see Example 2.D.3) is homeomorphic to the \((n-1)\)-dimensional Sierpiński compactum \((S^n)^o\). Once this is known, one can use a consolidation procedure from Section 3.A to show that the limit of any dense tree system of internally punctured connected planar surfaces is homeomorphic to the Sierpiński curve.

3.C Subdivision of a tree system.

Generalizing the concepts from the previous section, we describe in this section the operation of subdivision of a tree system, opposite to the operation of consolidation described in Section 3.A.

Let \(\Theta = (T, \{K_t\}, \{\Sigma_e\}, \{\phi_e\})\) be a tree system, let \(t \in V_T\) be any vertex, and suppose that \(C\) is a tree decomposition of the space \(K_t\). We say that \(C\) does not cross \(\Theta\) if for each separator \(A\) of \(C\) and each \(e \in N_t\) there is a halfspace \(H\) for \(A\) such that \(\Sigma_e \subset \tilde{H}\). This means in particular that \(A \cap \Sigma_e = \emptyset\).
Furthermore, given a tree decomposition $\mathcal{C}$ of $K_t$ not crossing $\Theta$, and any edge $e \in N_t$, we say that $\mathcal{C}$ is discrete at $e$ if for some (and hence any) separator $A$ of $\mathcal{C}$ there are only finitely many separators in $\mathcal{C}$ that separate $A$ from $\Sigma_e$ (i.e. finitely many separators $A'$ from $\mathcal{C}$ such that for some halfspace $H$ related to $A'$ we have $A \subset H$ and $\Sigma_e \subset H^c$). It is not hard to realize that $\mathcal{C}$ is discrete at $e$ iff $\Sigma_e \subset \Omega$ for some domain $\Omega \subset K_t$ for $\mathcal{C}$. We also have the following sufficient condition for discreteness at $e$.

3.C.1 Lemma. Let $\Theta = (T, \{K_t\}, \{\Sigma_e\}, \{\phi_e\})$ be a tree system, let $\mathcal{C}$ be a tree decomposition not crossing $\Theta$ of some constituent space $K_t$, and let $e \in N_t$. If $\mathcal{C}$ is fine and if $\text{diam}(\Sigma_e) > 0$, then $\mathcal{C}$ is discrete at $e$.

Proof: Let $A$ be any separator from $\mathcal{C}$, and let $H_A$ be this halfspace for $A$ which does not contain $\Sigma_e$. Let $A'$ be any separator from $\mathcal{C}$ that separates $A$ from $\Sigma_e$, with $A \subset \hat{H}$ and $\Sigma_e \subset \hat{H}^c$ for the corresponding halfspaces $H, H^c$ for $A'$. Note that $H_A \subset \hat{H}$, and consequently

$$\text{diam}(H) \geq \text{diam}(H_A) > 0 \quad \text{and} \quad \text{diam}(H^c) \geq \text{diam}(\Sigma_e) > 0.$$ 

In view of fineness of $\mathcal{C}$, this implies that there are only finitely many separators $A'$ as above, which completes the proof.

3.C.2 Definition. A tree decomposition $\mathcal{C}$ of a space $K_t$ from a tree system $\Theta$ is compatible with $\Theta$ if it does not cross $\Theta$ and if it is discrete at $e$ for each $e \in N_t$.

3.C.3 Example. Given a tree system $\Theta = (T, \{K_t\}, \{\Sigma_e\}, \{\phi_e\})$ and a vertex $t \in V_T$, we exhibit a class of of families $\mathcal{C}_t$ of elementary splittings of the space $K_t$ which are tree decompositions of of $K_t$ compatible with $\Theta$.

pairwise noncrossing and which do not cross $\Theta$. We present a useful criterion for $\mathcal{C}_t$ to be discrete and discrete at all $e \in N_t$ (and hence to be compatible with $\Theta$). We will use this criterion in Section 3.D.

Let $x_0 \in K_t \setminus \bigcup \{\Sigma_e : e \in N_t\}$. Suppose that a family $\mathcal{C}_t$ of elementary splittings of $K_t$ has a form

$$\mathcal{C}_t = \{(A_k, \{H_k, H_k^c\}) : k \in \mathbb{N}\},$$

where the following three conditions hold:

1. splittings from $\mathcal{C}_t$ do not cross $\Theta$, i.e. each separator $A_k$ is disjoint with each of the sets from the family $\mathcal{A}_t = \{\Sigma_e : e \in N_t\}$ and each halfspace $H_k$ is $\mathcal{A}_t$-saturated,
2. $H_{k+1} \subset \hat{H}_k$ for each natural $k$,
3. $\bigcap_{k=1}^{\infty} H_k = \{x_0\}$ (equivalently, $x_0 \in \bigcap_{k=1}^{\infty} H_k$ and $\lim_{k \to \infty} \text{diam}(H_k) = 0$).

We then get the following.

3.C.3.1 Lemma. If a family $\mathcal{C}_t$ satisfies conditions (1)-(3) above then $\mathcal{C}_t$ is fine, discrete and discrete at all $e \in N_t$. In particular, it is a tree decomposition of $K_t$, and it is compatible with $\Theta$.

Proof: By condition (2), splittings in $\mathcal{C}_t$ are pairwise noncrossing, and $\mathcal{C}_t$ is discrete. By condition (3), $\mathcal{C}_t$ is fine. Thus, in view of condition (1), $\mathcal{C}_t$ is a tree decomposition of $K_t$. Moreover, by condition (3), for each $e \in N_t$ there is $k$ such that $\Sigma_e \cap H_k = \emptyset$. It follows that for each $e \in N_t$ the family $\mathcal{C}_t$ is discrete at $e$. This completes the proof.
3.C.4 Definition. A tree decomposition of a tree system $\Theta$ is a family $\mathcal{C} = \{C_t : t \in V_T\}$ of tree decompositions $C_t$ of the spaces $K_t$ which are all compatible with $\Theta$.

We now describe some elementary splittings of the limit space $\lim \Theta$ induced by any elementary splittings from any tree decomposition $C = \{C_t\}$ of $\Theta$. Let $t$ be an arbitrary vertex in the underlying tree $T$ of $\Theta$ and let $(A, \{H, H^c\})$ be an elementary splitting of the space $K_t$ belonging to $C_t$. Note that both subsets $H$ and $H^c$ of $K_t$ are saturated with respect to the family $A_t = \{\Sigma_e : e \in N_t\}$. In particular, it makes sense to speak of the subsets $G(H)$ and $G(H^c)$ in $\lim \Theta$ of the form described in Section 1.C (just before Proposition 1.C.1). Moreover, when we view $A$ as the subset of $\lim \Theta$, we clearly have $G(H) \cap G(H^c) = A$, where all three subsets in this expression are compact subspaces of $\lim \Theta$. Thus, the triple $(A, \{G(H), G(H^c)\})$ is an elementary splitting of the limit space $\lim \Theta$. For each $t \in V_T$ we put $C^\lim_t = \{(A, \{G(H), G(H^c)\}) : A$ is a separator in $C_t\}$

3.C.5 Lemma. Let $\Theta$ be a tree system of metric compacta, and let $C(\Theta)$ be the associated tree decomposition of the limit $\lim \Theta$. Then for any tree decomposition $C$ of $\Theta$ the family $C^\lim = (\bigcup_{t \in V_T} C^\lim_t) \cup C(\Theta)$ is a tree decomposition of $\lim \Theta$.

Proof: The fact that the elementary splittings from $C^\lim$ are pairwise noncrossing follows directly from the fact that the families $C_t$ do not cross $\Theta$. The discreteness of $C^\lim$ follows fairly directly from discreteness of each $C_t$ at each $e \in N_t$. We omit further details.

3.C.6 Proposition. Under notation of Lemma 3.C.5, the tree decomposition $C^\lim$ is fine iff each tree decomposition $C_t$ from $C$ is fine.

Proof: One implication, namely that fineness of $C^\lim$ implies fineness of every $C_t$, is immediate just by observing that halfspaces for $C_t$ are simply restrictions to $K_t$ of the appropriate halfspaces for $C^\lim$. The converse implication requires much more effort and some preparatory claims. We start with a claim which gives a useful characterization of fineness of a tree decomposition.

Claim 1. A tree decomposition $C$ of a compact $K$ is fine iff for any $\epsilon > 0$ there is a finite collection $\Omega_1, \ldots, \Omega_m$ of domains for $C$ such that if $H$ is any halfspace from $C$ not containing any of the above domains then $\text{diam}(H) < \epsilon$.

To prove Claim 1, suppose first that $C$ is fine. Given $\epsilon > 0$, let $(A_i, \{Y_i, Z_i\}) : i = 1, \ldots, q$ be all elementary splittings in $C$ for which $\text{diam}(Y_i) \geq \epsilon$ and $\text{diam}(Z_i) \geq \epsilon$. Let $\Omega_1, \ldots, \Omega_m$ be the set of all domains for $C$ adjacent to some of the separators $A_1, \ldots, A_q$. Since each separator has exactly two adjacent domains, the above set of domains is finite. Let $H$ be a halfspace from $C$ not containing any of the above domains. It is not hard to realize that then for any $1 \leq i \leq q$ either $Y_i$ or $Z_i$ is contained in $H^c$, and hence
diam($H^c$) ≥ $\epsilon$. On the other hand, the separator $A$ corresponding to $H$ is clearly distinct from each of the separators $A_1, \ldots, A_q$, and hence diam($H$) < $\epsilon$, as required.

To prove the converse implication in Claim 1, fix $\epsilon > 0$ and let $\Omega_1, \ldots, \Omega_m$ be some domains associated to $\epsilon$, as in the assumption of the implication. Note that, by the assumption, for any splitting $(A, \{Y, Z\}) \in C$ with at least one halfspace not containing any of the above domains we have min(diam($Y$), diam($Z$)) < $\epsilon$. Thus, it is sufficient to show that the number of splittings with both halfspaces containing some of the domains $\Omega_1, \ldots, \Omega_m$ is finite. Translating this to the language of the dual tree, we need to know that for any finite set $V_0$ of vertices in $T_C$ the set of edges in $T_C$ which separate some two of the vertices from $V_0$ is finite. Since this set of edges clearly coincides with the set of edges in the subtree of $T_C$ spanned by $V_0$, this completes the proof of Claim 1.

We come back to the tree decomposition $C_{\lim}$. For each $t \in V_T$ and any $e \in N_t$ let $H_e$ be this halfspace from $C(\Theta)$ associated to the separator $\Sigma_e$ which does not contain $K_t$. For each halfspace $H$ from $C_t$ put $\mathcal{H}_H := \{H_e : e \in N_t$ and $\Sigma_e \subset H\}$. The next preparatory claim provides some estimate for the diameter of a halfspace from $C_{\lim}$ in terms of diameters of appropriate halfspaces from $C$ and $C(\Theta)$.

**Claim 2.** Let $t$ be any vertex of $T$ and let $H$ be any halfspace from $C_t$. Then the induced halfspace $G(H)$ from $C_{\lim}$ has form $G(H) = H \cup (\bigcup \mathcal{H}_H)$ and its diameter is estimated by

$$diam(G(H)) \leq diam(H) + 2 \cdot \max\{diam(H') : H' \in \mathcal{H}_H\}.$$  

The first assertion of Claim 2, i.e. that $G(H) = H \cup (\bigcup \mathcal{H}_H)$, follows directly from the definition of $G(H)$. To prove the second assertion, we estimate the distance of any two points of $G(H)$. Suppose that $x, y$ are some points of $G(H)$ not contained in $H$. Then, by the first assertion, there exist $e, e' \in N_t$ such that $x \in H_e$ and $y \in H_{e'}$. Let $x' \in \Sigma_e$ and $y' \in \Sigma_{e'}$ be arbitrary points. Since $\Sigma_e = H \cap H_e$ and $\Sigma_{e'} = H \cap H_{e'}$, we get the estimate

$$d(x, y) \leq d(x, x') + d(x', y') + d(y', y) \leq diam(H_e) + diam(H) + diam(H_{e'})$$

which can be further estimated from above by the right hand side term in the desired inequality. In case when one or both point $x, y$ belong to $H$ a similar (even simpler) estimate can be obtained in the same way. This completes the proof of Claim 2.

We come back to the proof of the following implication: if each tree decomposition $C_t$ from $C$ is fine then $C_{\lim}$ is fine. We use the characterization of fineness given in Claim 1. Fix any $\epsilon > 0$. Since $C(\Theta)$ is fine (see the last comment in Example 3.B.7), we choose a finite subset $V_0 \subset V_T$ such that any halfspace $H'$ from $C(\Theta)$ which does not contain any of the spaces $K_t : t \in V_0$ satisfies diam($H'$) < $\frac{\epsilon}{7}$. Note that this implies also that for any $s \in V_T \setminus V_0$ we have diam($K_s$) < $\frac{\epsilon}{7}$. For each $t \in V_0$, using the fact that $C_t$ is fine, choose some domains $\Omega_1^t, \ldots, \Omega_m^t$, with $m_t \geq 1$, such that any halfspace $H$ from $C_t$ which does not contain any of these domains satisfies diam($H$) < $\frac{\epsilon}{7}$. Put $\mathcal{Z} := \bigcup_{t \in V_0} \{\Omega_1^t, \ldots, \Omega_m^t\}$ and note that $\mathcal{Z}$ is a finite family of domains for the tree decomposition $C_{\lim}$. We claim that any halfspace $H_{\lim}$ from $C_{\lim}$ which does not contain any of the domains from $\mathcal{Z}$ satisfies diam($H_{\lim}$) < $\epsilon$.  

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To prove the above claim, suppose first that $H_{\lim} = H$ for some halfspace $H$ from $C(\Theta)$. Since $H = H_{\lim}$ does not contain any of the domains from $Z$, it also does not contain any of the spaces $K_t : t \in V_0$. By the choice of $V_0$, this implies that $\text{diam}(H_{\lim}) < \varepsilon < \frac{\varepsilon}{3}$, as required.

In the remaining case we have $H_{\lim} = G(H)$, where $H$ is a halfspace from $C_t$ in the space $K_t$, for some $t \in V_T$. We will estimate the diameter of $H_{\lim} = G(H)$ using Claim 2. To do this, we claim that $\text{diam}(H) < \frac{\varepsilon}{3}$. Indeed, if $H \subset K_t$ for some $t \in V_0$, this estimate follows from our choice of the domains $\Omega_1^t, \ldots, \Omega_m^t$, in view of the fact that $H$ does not contain any of them. If $H \subset K_s$ for some $s \in V_T \setminus V_0$, we get $\text{diam}(H) \leq \text{diam}(K_s) < \frac{\varepsilon}{3}$, where the last inequality follows from the choice of $V_0$.

We now estimate diameters of the halfspaces $H' \in H_H$. Since any such $H'$ (being a halfspace from $C(\Theta)$) does not contain any of the spaces $K_t : t \in V_0$ (because it does not contain any of the domains from $Z$), it follows from the choice of $V_0$ that $\text{diam}(H') < \frac{\varepsilon}{3}$.

In view of Claim 2, the estimates from the two previous paragraphs yield the inequality $\text{diam}(G(H)) < \varepsilon$, as required. By Claim 1, the tree decomposition $C_{\lim}$ is then fine, which completes the proof.

3.C.7 Definition. Let $\Theta$ be a tree system of metric compacta. A subdivision of $\Theta$ is any tree system of form $\Theta_{C_{\lim}}$, for any fine tree decomposition $C_{\lim}$ as in Lemma 3.C.5.

3.C.8 Proposition. Let $\Xi$ be any subdivision of $\Theta$. Then $\Theta$ can be canonically obtained from $\Xi$ by means of a consolidation. Moreover, the limits $\lim \Xi$ and $\lim \Theta$ are canonically homeomorphic.

Proof: Let $\Xi = \Theta_{C_{\lim}}$. Note that the constituent spaces of the tree system $\Xi$ are precisely the constituent spaces of the systems $\Theta_{C_t}$ for all $t \in V_T$. Since each $C_t$ is a fine tree decomposition of $K_t$, it follows from Theorem 3.B.10 that $\lim \Theta_{C_t} = K_t$. This shows that $\Theta$ is a consolidation of $\Xi$, for the canonical partition of the dual tree $T_{C_{\lim}}$ into subtrees $T_{C_t}$. The second assertion follows either from Theorem 3.B.10 and Lemma 3.C.5 or, in view of the first assertion, from Theorem 3.A.1.

3.D Homogeneity of trees of manifolds.

In this section we prove the following.

3.D.1 Proposition. Let $M$ be a closed connected topological manifold (either oriented or non-orientable). Then the tree of manifolds $M$, i.e. the Jakobsche space $\mathcal{X}(M)$ (as defined in Section 1.E), is homogeneous.

This result has been proved before in [J2] (for oriented $M$) and in [St] (for non-orientable $M$ which are PL). The proof we present here uses the technique of subdivisions and consolidations of tree systems.

In the proof of this result in [J2], Jakobsche uses his approach to trees of manifolds in terms of limits of inverse sequences (as we have explained in Example 2.C.7). He describes some modifications of the involved inverse sequences which are very similar to the modifications of tree systems that we use below. However, in his paper he never proves independence of the inverse limits on the modifications performed. Our results from Part
3 of this paper (notably, Theorem 3.A.1 and Proposition 3.C.8) fill this gap, and allow to present a rigorous proof of Proposition 3.D.1.

The argument presented below, and the technique used in it, has a potential for extensions. It can be applied to various other classes of tree systems (e.g. to tree systems of polyhedra mentioned in the introduction), to study orbits of the group of homeomorphisms of the corresponding limit space. One easy instance of such extension is presented below, as Proposition 3.D.7.1.

We start with few technical preparatory results.

3.D.2 Lemma. Let \( M \) be a closed connected topological manifold, oriented or non-orientable, and let \( \mathcal{M} \) be the dense tree system of manifolds \( M \), with the underlying tree \( T \). Then the points of the limit \( \mathcal{X}(M) = \lim \mathcal{M} \) corresponding to the set \( E_T \) of the ends of \( T \) are all in the same orbit of the group of homeomorphisms of \( \mathcal{X}(M) \).

Proof: Let \( z_1, z_2 \in E_T \subset \lim \mathcal{M} \). Using Toruńczyk’s Lemma 1.E.2.1 (together with Lemma 1.E.3.1 in case when \( M \) is non-orientable) it is not hard to get an automorphism of the tree system \( \mathcal{M} \) for which the corresponding automorphism of the underlying tree \( T \) maps \( z_1 \) to \( z_2 \). Clearly, this automorphism of \( \mathcal{M} \) induces a homeomorphism of \( \lim \mathcal{M} \) which maps \( z_1 \) to \( z_2 \). This finishes the proof.

Next result is an extension of Lemma 5 from [J1], and it appears implicitly inside the proof of Lemma 7.1 in [J2].

3.D.3 Lemma. Let \( M \) be an \( n \)-dimensional compact topological manifold with boundary, and let \( \mathcal{D} \) be a null and dense family of pairwise disjoint collared \( n \)-disks contained in the interior of \( M \). Let \( M/\mathcal{D} \) be the quotient space obtained by collapsing all disks \( D \in \mathcal{D} \) to points, i.e. the quotient space of the decomposition of \( M \) induced by \( \mathcal{D} \). Then \( M/\mathcal{D} \) is homeomorphic to \( M \), via a homeomorphism which is identical on \( \partial M \).

Proof: It follows from a theorem of Bing (Theorem 7.2 in [Fr]) that the decomposition of \( M \) induced by \( \mathcal{D} \) is shrinkable (see Section II.5 of [Dav] for the definition of shrinkability). By Theorem 5.3 in [Dav], this implies that the quotient map \( M \rightarrow M/\mathcal{D} \) can be approximated by homeomorphisms, which clearly implies our assertion.

We will need the following corollary to Lemma 3.D.3.

3.D.4 Corollary. Let \( M \) and \( \mathcal{D} \) be as in Lemma 3.D.3, and let \( x_0 \) be an interior point of \( M \) not contained in any \( D \in \mathcal{D} \). Then there is a sequence of collared \( n \)-disks \( Q_k \) contained in \( \text{int}(M) \), such that

1. for each \( k \) the boundary \( \partial Q_k \) is disjoint with the union of \( \mathcal{D} \),
2. for each \( k \) we have \( Q_{k+1} \subset \text{int}(Q_k) \),
3. \( \{x_0\} = \cap_k Q_k \).

Proof: By Lemma 3.D.3, the quotient \( M/\mathcal{D} \) is homeomorphic to \( M \). Let \( A_\mathcal{D} \) be the set of points in the quotient \( M/\mathcal{D} \) corresponding to the collapsed elements of \( \mathcal{D} \). Clearly, \( A_\mathcal{D} \) is then countable infinite, and \( x_0 \in (M/\mathcal{D}) \setminus A_\mathcal{D} \). Obviously, there exists a sequence of collared \( n \)-disks \( P_k \) in \( M/\mathcal{D} \) such that

1. for each \( k \) the boundary \( \partial P_k \) is disjoint with \( A_\mathcal{D} \),
2. for each \( k \) we have \( P_{k+1} \subset \text{int}(P_k) \),

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splits into the constituent spaces

For each \( k \) put \( Q_k = q^{-1}(P_k) \), where \( q : M \to M/D \) is the quotient map. To see that the sequence \( Q_k \) is as required, it obviously suffices to show that each \( Q_k \) is a collared \( n \)-disk in \( M \). To do this, consider the decomposition of \( Q_k \) induced by the family \( D_k = \{ D \in D : D \subset Q_k \} \). Then we clearly have \( P_k = Q_k/D_k \), and we denote the quotient map by \( q_k \). Arguing as in the proof of Lemma 3.D.3, we get that the above decomposition is shrinkable, and hence \( q_k \) can be approximated by homeomorphisms. This shows \( Q_k \) is an \( n \)-disk. By applying the same argument to the complement \( M \setminus \text{int}(Q_k) \) (instead of \( Q_k \)), we get that \( Q_k \) is collared, which finishes the proof.

A crucial ingredient in the proof of Proposition 3.D.1, is the following result, in the proof of which we use the technique of modifications of tree systems, developed in Sections 3.A-3.C above.

**3.D.5 Lemma.** Let \( M \) be a closed connected manifold, oriented or non-orientable, and let \( \mathcal{M} = (T, \{ K_t \}, \{ \Sigma_e \}, \{ \phi_e \}) \) be the dense tree system of manifolds \( M \). Let

\[
x \in K_{t_*} \setminus \{ e \in N_{t_*} \} \subset \lim \mathcal{M}
\]

for some \( t_* \in V_T \), and let \( y \in E_T \subset \lim \mathcal{M} \). Then there is a homeomorphism of \( \lim \mathcal{M} \) which maps \( x \) to \( y \).

**Proof:** Denoting \( n = \dim M \), recall that for all \( t \in V_T \) we have

\[
K_t = M_t^o = M_t \setminus \text{int}(D) : D \in D,
\]

where \( M_t \) is a homeomorphic copy of \( M \), and where \( D \) is a null and dense family of pairwise disjoint collared \( n \)-disks in \( M_t \). View \( x \) as a point of \( M_{t_*} \) and consider a sequence \( Q_k \) of \( n \)-disks as in Corollary 3.D.4, for \( x_0 = x \). For each \( k \) consider the elementary splitting \( (A_k, \{ H_k, H_k^c \}) \) of \( K_{t_*} \) given by \( A_k := \partial Q_k \) and \( H_k := Q_k \cap K_{t_*} \), and denote by \( C_{t_*} \) the set of these elementary splittings for all natural \( k \). By Lemma 3.C.3.1, the family \( C_{t_*} \) is a tree decomposition of \( K_{t_*} \) compatible with \( \mathcal{M} \).

Let \( C^x = \{ C_t : t \in V_T \} \) be a tree decomposition of \( \mathcal{M} \) such that \( C_{t_*} \) is as above and \( C_t = \emptyset \) for \( t \neq t_* \). Let \( \mathcal{M}^x = (T^x, \{ K_t^x \}, \{ \Sigma_e^x \}, \{ \phi_e^x \}) \) be the subdivision of \( \mathcal{M} \) induced by \( C^x \). The underlying tree \( T^x \) may be viewed as obtained from \( T \) by expanding the vertex \( t_* \) into an infinite polygonal ray \( \rho = (t_0, t_1, \ldots) \). We may identify all other vertices of \( T^x \) bijectively with the vertices in \( V_T \setminus \{ t_* \} \). The set of the edges of \( T \) adjacent to \( t_* \) canonically splits into subsets which may be identified with the sets of edges of \( T^x \) adjacent to the vertices \( t_i \) (for \( i = 0, 1, \ldots \)). The edges of \( T^x \) disjoint with the ray \( \rho \) are in the natural bijective correspondence with the edges of \( T \) not adjacent to \( t_* \). Accordingly, the constituent spaces \( K_t : t \neq t_* \) remain to be the constituent spaces in \( \mathcal{M}^x \) at the corresponding vertices, with the families of peripheral subspaces unchanged. In particular, they remain to be the densely punctured manifolds \( M \). The constituent space \( K_{t_*} \) of \( \mathcal{M} \) splits into the constituent spaces \( K_{t_i}^x : i \geq 0 \) of \( \mathcal{M}^x \), which are described as follows. We have \( K_{t_0}^x := H_t^c \cap K_{t_*} = K_{t_*} \setminus \text{int}(Q_1) \), and for \( i \geq 1 \) we have \( K_{t_i}^x := (H_{i+1}^c \setminus H_i) \cap K_{t_*} = K_{t_*} \cap (Q_i \setminus \text{int}(Q_{i+1})) \). It is not hard to note that \( K_{t_0}^x \) has still the form of the densely punctured manifold \( M \), while each of \( K_{t_i}^x : i \geq 1 \) is the densely punctured sphere \( S^n \).
By Proposition 3.C.8, we have the canonical identification of the limits \( \lim M \) and \( \lim M^x \). The point \( x \), viewed as an element of \( \lim M^x \), clearly corresponds to the end of the tree \( T^x \) induced by the ray \( \rho \), i.e. \( x = [\rho] \in E_{T^x} \). On the other hand, the point \( y \) viewed as an element of \( \lim M^x \) still corresponds to an end of the underlying tree, i.e. \( y \in E_{T^x} \subset \lim M^x \).

We now apply to the system \( M^x \) an operation of consolidation, as described in Section 3.A. More precisely, for each \( i \geq 1 \) choose any vertex \( s_i \) in \( T^x \) adjacent to \( t_i \) and distinct from both \( t_{i-1} \) and \( t_{i+1} \), and denote by \( \Pi \) the partition of \( T^x \) consisting of the subtrees \( S_i : i \geq 1 \) spanned on the pairs \( t_i, s_i \) (these subtrees are just edges), and of subtrees reduced to vertices for all remaining vertices of \( T^x \). Let \( \mathcal{M}^x_{\Pi} \) be the tree system obtained from \( M^x \) by consolidation with respect to \( \Pi \). By Proposition 3.C.8, the limit \( \lim \mathcal{M}^x_{\Pi} \) is canonically homeomorphic to \( \lim M^x \), and hence also to \( \lim M \).

From what was said above about the system \( M^x \) it is not hard to deduce that \( \mathcal{M}^x_{\Pi} \) is a dense tree system of manifolds \( M \). Moreover, the point \( x \) (viewed now as a point of \( \lim \mathcal{M}^x_{\Pi} \)) clearly corresponds to the end of the tree \( T^x_{\Pi} \) induced by the ray \( \rho_1 = (S_1, S_2, \ldots) \). On the other hand, the point \( y \) still corresponds to an end of of the underlying tree, i.e. \( y \in E_{T^x_{\Pi}} \subset \lim \mathcal{M}^x_{\Pi} \). Thus, by Lemma 3.D.2, there is a homeomorphism of \( \lim \mathcal{M}^x_{\Pi} \) which maps \( x \) to \( y \), which finishes the proof.

The next result is a consequence of Lemma 3.D.5.

**3.D.6 Corollary.** Let \( M \) and \( \mathcal{M} \) be as in Lemma 3.D.5, let \( x \in \Sigma_{e^*} \subset \lim M \) for some \( e^* \in O_T \), and let \( y \in E_T \subset \lim \mathcal{M} \). Then there is a homeomorphism of \( \lim \mathcal{M} \) which maps \( x \) to \( y \).

**Proof:** Denoting \( n = \dim M \), view the constituent space \( K_{\alpha(e^*)} \) of \( M \) as obtained from \( M_{\alpha(e^*)} = M \) by deleting interiors of disks \( D \) from a null and dense family \( \mathcal{D} \) of pairwise disjoint collared \( n \)-disks. Let \( D^* \in \mathcal{D} \) be this disk for which \( \partial D^* = \Sigma_{e^*} \). Let \( Q \) be a collared \( n \)-disk in \( M_{\alpha(e^*)} \) with \( D^* \subset Q \) and with \( \partial Q \) disjoint with the union of \( \mathcal{D} \) (existence of such \( Q \) follows by the arguments as in the proof of Lemma 3.D.4).

Consider the elementary splitting \( (A, \{H, H^c\}) \) of \( K_{\alpha(e^*)} \) given by \( A = \partial Q \) and \( H = Q \cap K_{\alpha(e^*)} \). Let \( \mathcal{M}^Q \) be the subdivision of \( M \) induced by this single splitting. Let \( t_H \) be the vertex of the underlying tree \( T^Q \) of \( \mathcal{M}^Q \) corresponding to the constituent space \( H \). Viewing \( \omega(e^*) \) naturally as a vertex in \( T^Q \), note that \( t_H \) and \( \omega(e^*) \) are adjacent, and denote by \( S_Q \) the subtree of \( T^Q \) consisting of these two vertices and the edge which connects them. Consider the consolidation \( \mathcal{M}^Q_{\Pi} \) of \( \mathcal{M}^Q \) for the partition \( \Pi \) consisting of the subtree \( S_Q \) and the singleton subtrees for all other vertices. Clearly, the limits \( \lim \mathcal{M}^Q_{\Pi} \) and \( \lim M \) are canonically homeomorphic.

Note that \( \mathcal{M}^Q_{\Pi} \) is again a dense tree of manifolds \( M \), and that \( x \), naturally viewed as element of \( \lim \mathcal{M}^Q_{\Pi} \), belongs to the constituent space \( K_{S_Q} \subset \lim \mathcal{M}^Q_{\Pi} \), and lies outside its all peripheral subspaces. Since the point \( y \), viewed as element of \( \lim \mathcal{M}^Q_{\Pi} \), still corresponds to an end of the underlying tree, the corollary follows by applying Lemma 3.D.5.

**Proof of Proposition 3.D.1:** The proposition is a direct consequence of Lemmas 3.D.2, 3.D.5 and Corollary 3.D.6.
3.D.7 Remarks.

The argument as above in this section yields also the following.

3.D.7.1 Proposition. Let $M$ be a connected compact topological manifold with boundary, either oriented or non-orientable, and let $\mathcal{M}$ be the dense tree system of internally punctured manifolds $M$, as defined in Remark 2.D.3.1. Then, viewing the boundaries $\partial M_t$ as subsets of the constituent spaces $K_t = M_t^\circ$ of $M$, all points of $K_t \setminus \partial M_t \subset \lim M$ (for all $t$) and all points of $E_T \subset \lim M$ are in the same orbit of the group of homeomorphisms of $\lim M = \mathcal{X}_{int}(M)$.

A more or less straightforward extension of the arguments of this section allows to prove rigorously the following result of Jakobsche (Theorem (8.1) in [J2]), in the extended form when non-orientable manifolds are allowed. We omit the details. Recall that, given any natural number $m$, a topological space $X$ is $m$-homogeneous if the group of its homeomorphisms acts transitively on the set of all $m$-tuples of pairwise distinct points of $X$.

3.D.7.2 Proposition. For any closed connected topological manifold $N$ (oriented or non-orientable), and for any natural $m$, the tree of manifolds $N$, i.e. the Jakobsche space $\mathcal{X}(N)$, is $m$-homogeneous.

3.E Weakly saturated tree systems of manifolds.

In this section we present another application of the operations of consolidation and subdivision. It concerns dense trees of finite families of manifolds, as in Example 3.A.2, and provides a significant strengthening of Propositions 3.A.2.1 and 3.A.2.3. More precisely, we will show that under a much weaker assumption than 2-saturation, the limit of a dense tree of a finite family of manifolds $\{M_1, \ldots, M_k\}$ is still homeomorphic to the space $\mathcal{X}(M_1 \# \ldots \# M_k)$. The weaker condition will be called weak saturation. The results of this section are used by the author (in another paper [Sw2]) to show that various trees of manifolds, in arbitrary dimension, appear as Gromov boundaries of some hyperbolic groups.

Let $\mathcal{N} = \{M_1, \ldots, M_k\}$ be a finite family of closed connected topological manifolds of the same dimension, either all oriented, or at least one of which is non-orientable. Let $\mathcal{M} = (T, \{K_t\}, \{\Sigma_e\}, \{\phi_e\})$ be a dense tree system of manifolds from $\mathcal{N}$. As in Example 3.A.2, for each $t \in V_T$ let $i_t \in \{1, \ldots, k\}$ be this index for which $K_t$ has a form $M_{i_t}^\circ$. Recall also that a half-tree in a tree is its any subtree obtained by deleting the interior of any edge.

3.E.1 Definition We say that a dense tree system $\mathcal{M}$ of manifolds from $\mathcal{N}$ is weakly saturated if for each $j \in \{1, \ldots, k\}$ any half-tree in the underlying tree $T$ contains a vertex $t$ with $i_t = j$ (equivalently, for each $j \in \{1, \ldots, k\}$ the set $V^j_t = \{t \in V_T : i_t = j\}$ spans $T$).

The main result of this section is the following.

3.E.2 Theorem. Let $\mathcal{M}$ be any weakly saturated dense tree system of manifolds from a finite family $\mathcal{N} = \{M_1, \ldots, M_k\}$ (where $M_i$ are closed connected topological manifolds of the same dimension, either all oriented, or at least one of which is non-orientable). Then the limit $\lim \mathcal{M}$ is homeomorphic to the Jakobsche space $\mathcal{X}(M_1 \# \ldots \# M_k)$. 40
Proof: The proof consists of showing that, by applying certain consolidation followed by some subdivision, the tree system $\mathcal{M}$ can be transformed into a 2-saturated dense tree system of manifolds from $\mathcal{N}$. Due to invariance of the limit under operations as above, and in view of Propositions 3.A.2.1 and 3.A.2.3, this will give the assertion.

We use the following notation. Given an oriented edge $e \in O_T$ (where $T$ is the underlying tree of the system $\mathcal{M}$), denote by $T_e$ this half-tree obtained by deleting from $T$ the interior of $e$, which contains the terminal vertex $\omega(e)$.

We describe appropriate consolidation and subdivision simultaneously, accordingly with the following scheme. We successively choose finite subtrees $S$ of a partition $\Pi$ of the tree $T$ (which will induce a desired consolidation $\mathcal{M}_{\Pi}$ of $\mathcal{M}$), and finite tree decompositions $\mathcal{C}_S$ of the corresponding constituent spaces $K_S$ in $\mathcal{M}_{\Pi}$ (which will give the tree decomposition $C = \{\mathcal{C}_S : S \in \Pi\}$ inducing a desired subdivision of $\mathcal{M}_{\Pi}$). The choices of subtrees $S$ and decompositions $\mathcal{C}_S$ are made inductively, using an auxilliary ordering of the vertices of $T$ into a sequence $(u_n)_{n \geq 1}$, as follows.

**Step 1.** Put $S_1 = \{u_1\}$ (i.e. the subtree consisting of a single vertex $u_1$), and put $\Pi_1 = \{S_1\}$. Furthermore, let $\mathcal{C}_{S_1}$ be the empty tree decomposition of the corresponding space $K_{S_1} = K_{u_1}$.

**Step 2.** Having constructed a finite family $\Pi_n$ of finite subtrees, and a corresponding family of decompositions $\mathcal{C}_S : S \in \Pi_n$, we keep as a part of inductive assumption the following properties (which clearly hold true for $n = 1$):

(i0) the subtrees in $\Pi_n$ are pairwise disjoint,

(i1) $u_n \in \bigcup\{V_S : S \in \Pi_n\}$,

(i2) the union $\bigcup\{V_S : S \in \Pi_n\}$ is the vertex set of a subtree of $T$, which we denote $T_n$,

(i3) for each $S \in \Pi_n$, for each separator $A$ of the decomposition $\mathcal{C}_S$, and for any edge $e \in N_S$ there is a halfspace $H$ for $A$ such that $\Sigma_e \subset H$.

For each subtree $S \in \Pi_n \setminus \Pi_{n-1}$ and for each domain $\Omega \subset K_S$ for $\mathcal{C}_S$, choose arbitrary pairwise distinct oriented edges $e_1, \ldots, e_k, e_1', \ldots, e_k'$ from $N_S$, not belonging to $T_n$, and such that for $1 \leq j \leq k$ the corresponding peripheral subsets $\Sigma_{e_j}$ and $\Sigma_{e_j'}$ are contained in $\Omega$. This is possible since, due to finiteness of $\mathcal{C}_S$, $\Omega$ has nonempty interior in $K_S$, and by denseness of $\mathcal{M}$, it thus contains infinitely many peripheral subsets $\Sigma_e$ with $e \in N_S$ and not belonging to $T_n$. For each $1 \leq j \leq k$ choose a vertex $t_j$ in $T_{e_j}$ and $t_j'$ in $T_{e_j'}$ such that $i_{t_j} = i_{t_j'} = j$. This is possible since $\mathcal{M}$ is weakly saturated. Further, for each $j$ choose a finite subtree $S_j(\Omega)$ in $T_{e_j}$ containing the vertices $\omega(e_j)$ and $t_j$, and a finite subtree $S_j'(\Omega)$ in $T_{e_j'}$ containing the vertices $\omega(e_j')$ and $t_j'$. We also require that, putting

$$\Pi_{n+1} = \Pi_n \cup \bigcup_{\Omega} \{S_1(\Omega), \ldots, S_k(\Omega), S_1'(\Omega), \ldots, S_k'(\Omega)\}$$

(where $\Omega$ runs through all domains in all spaces $K_S : S \in \Pi_n \setminus \Pi_{n-1}$), we have $u_{n+1} \in \bigcup\{V_S : S \in \Pi_{n+1}\}$. This clearly holds true if $u_{n+1} \in \bigcup\{V_S : S \in \Pi_n\}$; otherwise, this can be assured as follows. Let $s$ be the vertex in $T_n$ which is closest to $u_{n+1}$ in $T$, and let $e$ be the first oriented edge on the path from $s$ to $u_{n+1}$. Let $S \in \Pi_n$ be this subtree for which $s \in V_S$, and let $\Omega \subset K_S$ be this domain for $\mathcal{C}_S$ for which $\Sigma_e \subset \Omega$. We then choose $e_1$ as above so that additionally we have $u_{n+1} \in T_{e_1}$ (i.e. we put $e_1 = e$), and then choose
\( S_1(\Omega) \) so that it contains \( u_{n+1} \). As a consequence of all our choices above, the family \( \Pi_{n+1} \) satisfies conditions (i0)-(i2).

Now, for each \( S = S_j(\Omega) \) or \( S = S'_j(\Omega) \) as above, we choose an appropriate tree decomposition \( \mathcal{C}_S \) of the space \( K_S \). To describe it, note that \( K_S \) (together with its peripheral subspaces of the system \( \mathcal{M}_{\Pi} \)) is homeomorphic to the densely punctured manifold, denoted \( M_S \), which is a connected sum of the manifolds \( M_t : t \in V_S \). We denote by \( \Delta_e \) the disks in \( M_S \) corresponding to the peripheral subspaces \( \Sigma_e \) of \( K_S \). We also denote by \( \Delta_j \) and \( K_j \) the spaces \( \Delta_{e_j} \) and \( K_{t_j} \), if \( S = S_j(\Omega) \), and the spaces \( \Delta_{e'_j} \) and \( K_{t'_j} \), if \( S = S'_j(\Omega) \).

Choose any \( \Delta_{e_0} \subset M_S \) such that \( \Sigma_{e_0} \subset K_j \) and note that, by applying Toruńczyk’s Lemma 1.E.2.1 to the manifolds \( M_S \setminus \text{int}(\Delta_{e_0}) \) and \( M_S \setminus \text{int}(\Delta_j) \), we get a homeomorphism \( h : M_S \to M_S \) (preserving the orientation if all manifolds in \( \mathcal{N} \) are oriented) which maps \( \Delta_{e_0} \) onto \( \Delta_j \), and which preserves the family of all disks \( \Delta_e \) in \( M_S \). We denote by \( h^0 : K_S \to K_S \) the restricted homeomorphism of the densely punctured manifold. Now, we consider the finite tree decomposition \( \mathcal{C}_S \) of \( K_S \) which is induced by pushing through \( h^0 \) the original tree decomposition of \( K_S \) into constituent spaces \( K_i : t \in V_S \) (of the system \( \mathcal{M} \) restricted to \( S \)). Obviously, \( \mathcal{C}_S \) satisfies property (i3) above, and it also has the following property:

\( (*) \) all domains \( \Omega \subset K_S \) for \( \mathcal{C}_S \) are densely punctured manifolds from \( \mathcal{N} \), and this \( \Omega \) which contains the peripheral subspace \( \Sigma_j = \partial \Delta_j \) is homeomorphic to \( K_j \), i.e. to the densely punctured manifold \( M_j \).

We now put \( \Pi = \bigcup_{i=1}^{\infty} \Pi_n \) and note that, by conditions (i0) and (i1), \( \Pi \) describes a decomposition of the tree \( T \) into finite subtrees. We thus consider the induced consolidation \( \mathcal{M}_{\Pi} \). By condition (i3), and by finiteness of the decompositions \( \mathcal{C}_S \), the family \( \mathcal{C} = \{ \mathcal{C}_S : S \in \Pi \} \) is a tree decomposition of the system \( \mathcal{M}_{\Pi} \). Denoting by \( \mathcal{M}' \) the tree system obtained from \( \mathcal{M}_{\Pi} \) by the subdivision induced by \( \mathcal{C} \) (i.e. putting \( \mathcal{M}' = (\mathcal{M}_{\Pi})_{\text{clim}} \)), we get from the construction, and in particular from the property \( (*) \) above, that \( \mathcal{M}' \) is a 2-saturated dense tree system of manifolds from \( \mathcal{N} \). By the comment in the first paragraph of the proof, Theorem 3.E.2 follows.

As an application of Theorem 3.E.2, we now describe a class of inverse sequences of manifolds whose limits are the Jakobsche spaces \( \mathcal{X}(M_1 \# \ldots \# M_k) \). This class of inverse sequences is much more flexible, and much more convenient to deal with, than the corresponding class considered by Jakobsche in [J2]. For this reason, it can be used to identify boundaries of some spaces and groups as appropriate trees of manifolds, see [Sw2].

3.E.3 Definition. Let \( \mathcal{N} \) be a finite family of closed connected \( n \)-dimensional topological manifolds, either all oriented, or at least one of which is non-orientable. Let

\[ \mathcal{J} = (\{ X_i : i \geq 1 \}, \{ \pi_i : i \geq 1 \}) \]

be an inverse sequence consisting of closed connected topological \( n \)-manifolds \( X_i \) and maps \( \pi_i : X_{i+1} \to X_i \). Assume furthermore that if the manifolds in \( \mathcal{N} \) are oriented then all \( X_i \) are also oriented. We say that \( \mathcal{J} \) is a weak Jakobsche inverse sequence for \( \mathcal{N} \) if for all \( i \geq 1 \) and all \( M \in \mathcal{N} \) one can choose finite families \( \mathcal{D}^M_i \) of collared \( n \)-disks in \( X_i \) such that:

1. for each \( i \geq 1 \) the families \( \mathcal{D}^M_i : M \in \mathcal{N} \) pairwise do not intersect, and the disks in the union \( \mathcal{D}_i := \bigcup_{M \in \mathcal{N}} \mathcal{D}^M_i \) are pairwise disjoint;
(2) for each $i \geq 1$ the map $\pi_i$ maps the preimage $\pi_i^{-1}(X_i \setminus \{\text{int}(D) : D \in D_i\})$ homeomorphically onto $X_i \setminus \{\text{int}(D) : D \in D_i\}$;

(3a) $X_1$ is homeomorphic to one of the manifolds from $\mathcal{N}$, and if the manifolds in $\mathcal{N}$ are oriented, we require that this homeomorphism respects orientations;

(3b) for each $i \geq 1$, for each $M \in \mathcal{N}$, and for any $D \in D_i^M$ the preimage $\pi_i^{-1}(D)$ is homeomorphic to $M \setminus \text{int}(\Delta)$, where $\Delta$ is some collared $n$-disk in $M$; furthermore, if the manifolds in $\mathcal{N}$ are oriented, we require that the above homeomorphism respects the orientations induced from $X_{i+1}$ and from $M$;

(4) if $i < j$, $D \in D_i$, $D' \in D_j$, then $\pi_{i,j}(D') \cap \partial D = \emptyset$, where $\pi_{i,j} := \pi_i \circ \pi_{i+1} \circ \ldots \circ \pi_{j-1}$;

(5) for each $i \geq 1$ the family $\{\pi_{i,j}(D) : j \geq i, D \in D_j\}$ of subsets of $X_i$ is null, i.e. the diameters of these subsets converge to 0; here $\pi_{i,i}$ denotes the identity map on $X_i$;

(6) for any $i \geq 1$ and each $M \in \mathcal{N}$ the set $\bigcup_{j=i}^{\infty} \pi_{i,j}(\bigcup D_j^M)$ is dense in $X_i$.

Remarks.

(1) It follows from conditions (1), (2), (3a) and (3b) that each $X_i$ is the connected sum of a family of manifolds each homeomorphic to one of the manifolds in $\mathcal{N}$; moreover, if the manifolds in $\mathcal{N}$ are oriented, the above mentioned homeomorphisms and the connected sum respect the orientations.

(2) In the case when the manifolds in $\mathcal{N}$ are oriented, conditions (1)-(5) in Definition 3.E.3 coincide with conditions (1)-(6) in [J], Section 2, p. 82.

(3) Condition (6) in Definition 3.E.3 implies condition (7) in [J], but it is essentially weaker than the conjunction of conditions (7) and (8) of [J] (except the case when the family $\mathcal{M}$ consists of a single manifold $M$, in which (6) is equivalent to the conjunction of (7) and (8), as it was observed and exploited in [Fi] and [Z1]).

3.E.4 Corollary. Given $\mathcal{N}$ as in Definition 3.E.3, and any weak Jakobsche inverse sequence $\mathcal{J}$ for $\mathcal{N}$, its limit $\lim_{\rightarrow} \mathcal{J}$ is homeomorphic to the space $\mathcal{X}(M_1 \# \ldots \# M_k)$.

Proof: First, observe that by conditions (1)-(5) of Definition 3.E.3, there is a tree system $\mathcal{M}$ of manifolds from $\mathcal{N}$ such that $\mathcal{J}$ has the form of an inverse sequence associated to $\mathcal{M}$, as in Example 2.C.7, for an appropriate choice of a conical family of extended spaces and maps. More precisely, the constituent spaces of $\mathcal{M}$ coincide with the spaces $Y$ of the following two kinds:

(1) for any $i \geq 1$ put

$$D'_i = \{D \in D_i \text{ such that there is no } j < i \text{ with } \pi_{i,j}(D) \subset D' \text{ for some } D' \in D_j\},$$

and set $Y = X_1 \setminus \bigcup_{i=1}^{\infty} \bigcup_{D \in D_i} \pi_{1,i}(\text{int}(D))$;

(2) for any $m \geq 1$, any $\Delta \in D_m$, and any $i \geq m+1$ put $D_{\Delta,i}$ to be the family of all $D \in D_i$ such that $\pi_{m,i}(D) \subset \Delta$ and there is no $m+1 \leq j < i$ with $\pi_{i,j}(D) \subset D'$ for some $D' \in D_j$; set $Y = \pi_m^{-1}(\Delta) \setminus \bigcup_{i=m+1}^{\infty} \bigcup_{D \in D_{\Delta,i}} \pi_{m+1,i}(\text{int}(D))$.

We skip further explanations and justifications concerning this first observation, and we note that, due to Theorem 2.B.4, we have $\lim_{\rightarrow} \mathcal{J} = \lim \mathcal{M}$.

Next, it follows fairly directly from condition (6) of Definition 3.E.3 that the tree system $\mathcal{M}$ of manifolds from $\mathcal{N}$, as above, is dense and weakly saturated. The assertion follows then directly from Theorem 3.E.2.
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