GRAVITATIONAL FIELD AND EQUATIONS OF MOTION OF NONLINEAR COSMIC STRING

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Abstract

For the case of tension tensor containing nonlinear terms in $l^a$, we give generalization of Vilenkin metrics and equations of motion of cosmic string. Dynamics of nonlinear string in (1+1)-dimensional universe is discussed.

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1 Introduction

Consider a stringful medium meaning a variety of thin, infinite non-intersecting strings which fill up all given space-time $V_4$ in a continuous way. Energy-momentum tensor of the stringlike medium is of the form \[ T^{\alpha\beta} = \mu(u^\alpha u^\beta - l^\alpha l^\beta), \] where $u^\alpha = \frac{dx^\alpha}{d\tau}$ is time-like ($u^\alpha u_\alpha < 0$), and $l^\alpha = \frac{dx^\alpha}{d\rho}$ is space-like ($l^\alpha l_\alpha > 0$) vectors. Vector $u^\alpha$ describes dynamics of a separate string of the medium while $l^\alpha$ describes its space orientation.

We rewrite the tensor (1) in the following form:

\[ T^{\alpha\beta} = \mu u^\alpha u^\beta - t^{\alpha\beta}, \] (2)

where

\[ t^{\alpha\beta} = \mu l^\alpha l^\beta, \] (3)

and consider (3) as a tension tensor of the stringful medium arising as the result of oscillation of separate string. Indeed, from conservation of the energy-momentum $\nabla_\beta T^{\alpha\beta} = 0$, taking into account continuousness condition of the medium and each string separately, we have equations of motion of a separate string

\[ \frac{D^2 x^\alpha}{d\tau^2} - \frac{D^2 x^\alpha}{d\rho^2} = 0. \] (4)

Solutions of these equations, with initial or initial and boundary conditions, are, as it is well known, travelling and standing waves, respectively. Therefore, these are such waves that create abovementioned tensions in the medium. In the case, considered oscillations are not only small but also linear. Thus, expression (3) is a linear tension tensor of the stringful medium.

In paper [3], using energy-momentum tensor (2) with a linear tension tensor, the metrics of gravitational field created by straight cosmic string has been obtained. In subsequent papers, metrics of $V_4$ created by, for example, rotating cosmic strings [4], cosmic strings with propagating kinks along it [5], oscillating strings in the form of standing waves [6] were considered. In these papers behavior of light rays and test particles was investigated [7-10]. In addition, equations of motion of cosmic strings (4) in some external [11, 12]
and selfconsistent [13, 14] gravitational fields had been studied. However, all these results, being obtained on the basis of equations (1)-(4), have a linear character too.

We remind, however, that cosmic strings are supposed to appear at early stages of the evolution of universe from the scalar fields arisen as the result of phase transitions in vacuum [15]. Simplest lagrangian describing such fields is the nonlinear Higgs lagrangian for complex scalar field \( \chi(x) \),

\[
\mathcal{L} = \partial_\alpha \chi^* \partial^\alpha \chi + m_0^2 \chi^* \chi - \lambda (\chi^* \chi)^2. \tag{5}
\]

So, nonlinear character of the field (\( \lambda \) is a self-coupling constant) should lead to nonlinearity of the cosmic string generated by this field. Particularly, oscillations of such a string should be essentially nonlinear. It follows that the tension tensor of the stringful medium will have nonlinear contributions. Let’s construct nonlinear tension tensor of the stringful matter.

## 2 Nonlinear tension tensor

Consider a single oscillating string in the medium at some fixed time, i.e. a position of the string on hypersurface \( \tau = \text{const} \). This hypersurface is parametrised by the variable \( \rho \) (see Fig.1). In Fig.1, all coordinate axises are shown, for simplicity, as a single line.

Let some point \( A \) on the string, in an equilibrium state, is given by coordinates \( X_\alpha^o \). Under oscillations, this point takes position \( X^\alpha \). Denote the deflection of point \( A \) from the origin by \( x^\alpha_1 \), i.e.

\[
x^\alpha_1 = X^\alpha - X_\alpha^o. \tag{6}
\]

Supposing that oscillations are small, we have

\[
\frac{x^\alpha_1}{X^\alpha} \simeq \frac{x_1}{X_\alpha^o} = \varepsilon \ll 1. \tag{7}
\]

Therefore,

\[
x^\alpha_1 = \varepsilon X^\alpha, \tag{8}
\]

where \( \varepsilon \) is a small parameter. It can be introduced, for example, as a ratio between oscillation amplitude \( a \) of string and half-length of wave \( L/2 \), i.e.\( \varepsilon = \)
2a/L. Using definition of \( l^\alpha \), one can show that deflection of oscillating string from the equilibrium state, due to (8), is described by the vector

\[
l_1^\alpha = \frac{dx_1^\alpha}{d\rho} = \varepsilon \frac{dX^\alpha}{d\rho} = \varepsilon l^\alpha, \tag{9}
\]

where \( l^\alpha \) is the vector defining orientation of the string. Since the string is one-dimensional object, its oscillations produce tension along the string. So, the tension vector \( S^\alpha \) is proportional to \( l^\alpha \), i.e.

\[
S^\alpha = \mu l^\alpha, \tag{10}
\]

where \( \mu \) is some coefficient. Oscillations along each of coordinate axes are due to projections of the tension vector \( S^\alpha \) on the corresponding axis, i.e. due to its normal component \( S^\alpha (n) \). Its components on coordinate axes are components of tension tensor \( t^{\alpha\beta} \). From Fig.1 and (10) it can be seen that

\[
t^{\alpha\beta} = S^\alpha \sin \psi^\beta = \mu l^\alpha \sin \psi^\beta, \tag{11}
\]

where angle \( \psi^\beta = (x^\beta; l^\alpha) \). Again, from Fig.1 it follows that

\[
\tan \psi^\beta = \frac{dx_1^\beta}{d\rho} = l_1^\beta. \tag{12}
\]

For small angles the following approximation is valid (up to third power of the angle):

\[
\sin \psi^\beta = \frac{\tan \psi^\beta}{\sqrt{1 + \tan^2 \psi^\gamma}} \simeq \tan \psi^\beta (1 - \frac{1}{2} \tan^2 \psi^\gamma),
\]

which, in accordance to (12), can be written as

\[
\sin \psi^\beta \simeq l_1^\beta (1 - \frac{1}{2} l^\gamma l_1^\gamma). \tag{13}
\]

Inserting this equation to (11), using the earlier found relation (9), and restricting consideration by the abovementioned approximation we obtain the nonlinear tension tensor:

\[
t^{\alpha\beta} = \mu l^\alpha l^\beta (1 - \frac{\varepsilon^2}{2} l^\gamma l_1^\gamma), \tag{14}
\]

where \( \mu = \varepsilon \mu \) is linear mass density of the strings.
3 Gravitational field of nonlinear cosmic string

From eqs. (2) and (14) one can derive nonlinear energy-momentum tensor of the stringful medium, namely,

\[ T^{\alpha\beta} = \mu [u^\alpha u^\beta - l^\alpha l^\beta (1 - \frac{\epsilon^2}{2} l^\gamma l_\gamma)] \]  

(15)

and use it to find metrics of gravitational field created by alone nonlinear cosmic string.

All further calculations will be made with supposing that gravitational field is weak, i.e., as usually, taking \( g_{\alpha\beta} = \delta_{\alpha\beta} + h_{\alpha\beta} \), where \( h_{\alpha\beta} \) are small contributions to pseudo-Euclidean metrics \( \delta_{\alpha\beta} \). Then, Einstein equations can be linearised, and in harmonic coordinates take the well-known form

\[ \Box h_{\alpha\beta} = -16\pi\gamma T_{\alpha\beta}, \]  

(16)

where \( T_{\alpha\beta} = T_{\alpha\beta} - \frac{1}{2} \delta_{\alpha\beta} T \). To derive \( T_{\alpha\beta} \) one have to take into account that vectors \( u^\alpha \) and \( l^\alpha \) obey the following orthonormal gauge conditions:

\[ u^\alpha u^\alpha + l^\alpha l^\alpha = 0, \quad u^\alpha l^\alpha = 0. \]  

(17)

Thus, taking into account (17) we have finally

\[ T^{\alpha\beta} = \mu [u^\alpha u^\beta - l^\alpha l^\beta (1 - \frac{\epsilon^2}{2} l^\gamma l_\gamma)] \delta_3 (x - x_0). \]  

(18)

Now, we put additional restrictions on the form of space-time interval under study. Namely, we will find it in the orthogonal form. For this purpose, we take \( l^0 = 0 \). Besides, we will suppose that cosmic string is immobile as whole, i.e. \( u^k = 0 \). Under these conditions, equations (16) in the component form are:

\[ \Box h_{00} = 4\pi\gamma \epsilon^2 \delta_{00} \delta_3 (x - x_0), \]  

(19)

\[ \Box h_{kl} = 16\pi\gamma \mu [l_k l_l (1 + \frac{\epsilon^2}{2} l^m l_m) + \delta_{kl} l^m l_n (1 + \frac{\epsilon^2}{4} l^m l_m)] \delta_3 (x - x_0). \]  

(20)

Equations (19)-(20) have the form of D’Alambert equations, so its solutions are retarded potentials. Expanding the potentials in power series of
the parameter $|x - x'|/x^0$ and retaining only leading terms we find timelike component,

$$h_{00} = -\gamma \mu \varepsilon^2 \delta_0 \int \frac{\delta_3(x - x')}{|x - x'|} dV' = -2\gamma \mu \varepsilon^2 \ln \left( \frac{r}{r_0} \right).$$

(21)

To find spacelike components we need to fix orientation of the string. We suppose that the string is placed along the $Oz$ axis. Then, $l^1 = l^2 = 0, l^3 = 1$ and so

$$h_{kl} = -4\gamma \mu \int_V \left[ l_k' l'_l (1 + \frac{\varepsilon^2}{2} l^m l'_m) + \delta_{kl} l^m l'_n (1 + \frac{\varepsilon^2}{4} l^m l'_m) \right] dx^0 |x - x'|^{-1} \delta_3(x - x_0) dV' = -8\gamma \mu (1 + \frac{\varepsilon^2}{4}) \ln \left( \frac{r}{r_0} \right) \delta_{kl}.$$  

(22)

Subsequently, the interval of gravitational field created by nonlinear cosmic string has the form:

$$dS^2 = (1 - 2\gamma \mu \varepsilon^2 \ln \left( \frac{r}{r_0} \right)) dx^0 - (1 + 8\gamma \mu (1 + \frac{\varepsilon^2}{4}) \ln \left( \frac{r}{r_0} \right)) \delta_{kl} dx^k dx^l.$$  

(23)

At the limit $\varepsilon \to 0$, the eq.(23) reproduce the Vilenkin’s interval [3].

4 Particle in the field of nonlinear cosmic string

In accord with the paper [16] consider motion of nonrelativistic particle in the gravitational field (23). Corresponding equations of motion

$$\frac{d^2 x^k}{dt^2} = -\frac{1}{2} h_{00,k} = \gamma \mu \varepsilon^2 \frac{\eta^k}{r}$$

(24)

can be easily solved in $\{x, y\}$ plain. Introducing cylindrical coordinates we have, in the particular case of rotational motion ($r = R$), the following equation for the angular velocity of the particle:

$$\Omega = \varepsilon \sqrt{\frac{\gamma \mu}{R}}.$$  

(25)
Knowing this velocity one can calculate the power of the corresponding gravitational radiation. Denoting the mass of particle by $m$ and using results of paper [17] we obtain the power of radiation:

$$N = \frac{8}{5} c^{-5} m^2 l^4 \Omega^6 = \frac{8}{5} c^{-5} R^{-2} \gamma^4 m^2 \mu^3 \varepsilon^6. \quad (26)$$

Let the mass be equal to mass of proton, $m = m_p \sim 1, 710^{-27} kg$. Thickness of the string, for example, in theories of great unification, is of the order of the corresponding Compton wavelength, i.e. $R \sim 10^{-30} m$ [18] while linear mass density $\mu \sim 10^{21} kg/m$. Therefore, power of gravitational radiation per proton is of the order of $\sim \varepsilon^6 10^{-14} W$atts.

As to frequency of gravitational radiation, it is two times of the frequency of orbital motion of the particle, due to [17]. In our case, we thus have

$$f = 2\Omega = 2\varepsilon \sqrt{\frac{\gamma \mu}{R}}. \quad (27)$$

Using the abovementioned numerical values it is easy to find that the frequency is of the order of $\sim \varepsilon 10^{27} sec^{-1}$. We see from this that despite smallness of the parameter $\varepsilon$ the frequency of gravitational radiation can be very large (see Conclusions).

5 Equations of motion of nonlinear cosmic string

Let us find equations of motion of cosmic string taking into account nonlinear terms as well. To this end, we insert (15) into energy-momentum conservation law and use the medium continuousness condition $\nabla_\beta (\mu u^\beta) = 0$ in it, and continuousness condition of each string $\nabla_\beta (\mu l^\beta) = 0$. Then we obtain the following equation:

$$\frac{Du^\alpha}{d\tau} - \frac{Dl^\alpha}{d\rho} (1 - \frac{\varepsilon^2}{2} l^\gamma l_\gamma) + \varepsilon^2 \frac{Dl^\gamma}{d\rho} l^\alpha l_\gamma = 0. \quad (28)$$
Since spacelike vector $l^\alpha$ in isometric frame of coordinates $\{\tau, \rho\}$ obey the condition $\frac{Dl^\gamma}{d\rho}l_\gamma = 0$[19] the equations of motion can be written in the form

$$\frac{Du^\alpha}{d\tau} - \frac{Dl^\alpha}{d\rho}(1 - \frac{\varepsilon^2}{2} l^\gamma l_\gamma) = 0.$$

(29)

In pseudo-Euclidean space-time it takes simple form

$$\frac{d^2 x^\alpha}{d\tau^2} - \frac{d^2 x^\alpha}{d\rho^2}(1 - \frac{\varepsilon^2}{2} \frac{dx^\gamma}{d\rho} \frac{dx_\gamma}{d\rho}) = 0.$$

(30)

In Newtonian limit eq.(29) takes the form

$$\frac{d^2 x^k}{dt^2} - \frac{d^2 x^k}{d\rho^2}(1 + \frac{\varepsilon^2}{2} \frac{dx^m}{d\rho} \frac{dx_m}{d\rho}) = 0,$$

(31)

that completely coincides with wellknown equation describing nonlinear oscillations of string in classical mechanics (see, for example ref.[20]). One can easily prove this taking solution of eq. (31) in the form $x^k = a^k(t) \sin \frac{n \pi \rho}{L}$.

6 Motion of nonlinear cosmic string in (1+1) - dimensional universe

Let us solve equations of motion (29) in the given gravitational field. As an example of simple external $V_4$ we take (1+1)-dimensional Freedman-type space-time, namely,

$$ds^2 = dx^0^2 - R(x^0)^2 dx^1^2 = d\tau^2 - R(\tau)^2 dx^1^2.$$

(32)

Due to smallness of the coefficient at the nonlinear term, solution of the eqs.(29) can be presented in series,

$$x^1 = x^1_0 + \xi^1,$$

(33)

and the contribution $\xi^1$ have the order of $\varepsilon^2$. Then, the original equation of motion can be presented as two equations,

$$\frac{D^2 x^1_0}{d\tau^2} - \frac{D^2 x^1_0}{d\rho^2} = 0$$

(34)
\[
\frac{d^2 \xi^1}{d\tau^2} - \frac{d^2 \xi^1}{d\rho^2} + \frac{\varepsilon^2}{2} \frac{d^2 x^1_0}{d\rho^2} \frac{d^2 x^1_0}{d\rho}.
\]

(35)

General solution of the eq.(34) in the metrics (32) has been obtained in ref.[21]. Here, we present only the following anzatz:

\[
x^1_0 = \Phi(\rho + \tau) - \int \frac{d\tau}{R(\tau)} + C,
\]

(36)

where \( \Phi \) is an arbitrary function, \( C \) is a constant. Take this function as propagating sine wave, \( \Phi(\rho + \tau) = a \sin \Omega(\rho + \tau) \). Inserting the explicit form of propagating wave into eq. (36) we cast equation of perturbed motion into standard form,

\[
\frac{d^2 \xi^1}{d\tau^2} - \frac{d^2 \xi^1}{d\rho^2} = \varepsilon^2 \Psi(\rho + \tau)
\]

(37)

with the r.h.s.

\[
\Psi(\rho + \tau) = \left(\frac{A}{2}\right)^3 \Omega^4 \left[\sin 3\Omega(\rho + \tau) + \sin \Omega(\rho + \tau)\right].
\]

(38)

Particular solution of this equation can be written

\[
\xi^1(\rho, \tau) = \frac{\varepsilon^2}{2} \int_0^\tau \left(\int_{\rho - \tau + \Theta}^{\rho + \tau - \Theta} \Psi(\chi + \Theta) d\chi\right) d\Theta.
\]

(39)

Calculation of this integral shows that perturbation function contains non-periodical terms. In the other words, nonlinearity of the equations of motion of cosmic string yields changes in the oscillations amplitude of wave propagating along it, namely, the amplitude increases in time proportionally to \( x^0 \).

On the other hand, if we insert the term arising from multiplication of \( \varepsilon^2 \) to Riemann-Cristoffel symbols into (35), that corresponds to the metrics (32), the amplitude increases in time even more, namely, proportionally to \( (x^0)^2 \).
7 Conclusions

Deriving nonlinear tension tensor of stringful matter (14) we obtained space-time interval (23) generated by nonlinear cosmic string. We derived also nonlinear equations of motion of cosmic string (29) containing term of second order in $l^\alpha$. On the basis of this equations we briefly analyzed dynamics of nonlinear cosmic string in (1+1)-dimensional Freedman model of the universe.

As to the value of the coefficient $\varepsilon^2$ entering the expressions that we found, it seems to be less than $\lambda$ in the lagrangian (5). Since in the framework of inflationary models $\lambda \simeq 10^{-14}[15]$ then $\varepsilon \gtrsim 10^{-7}$.

As a consequence, frequency of gravitational radiation of particles rotating around the cosmic string has a minimal order of $\sim 10^{20} sec^{-1}$, and corresponding field quanta have a minimal energy $\sim 10^{-4} Gev$. This value is much bigger than the energy of gravitons radiated, for example, by oscillations of loops of linear cosmic string [22], and also of synchrotron radiation arising from kink-like deformations of linear cosmic string [23].

The considered above nonlinear string model can also be applied as in [19] to the compound hadron conception, for the meson describing, in particular.

And at last, it is necessary to point out that the discussing string model is described by such nonlinear hyperbolic equations that drastically differ from the wellknown Liuwille [19] and Korteweg - de Vries equations.

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Fig. 1.

Available on request from authors via e-mail.
References

[1] P.Letelier, Phys. Rev. D 20, 1294, (1979).

[2] D.Mitchell, N.Turok. Phys. Rev. Lett. 58, 1577, (1987).

[3] A.Vilenkin, Phys. Rev. D 24, 2082, (1981).

[4] S.Deser, R.Jackiw, and G.t’Hooft, Ann.Phys.(N.Y.) 152, 220, (1984).

[5] D.Garfinkle, Phys. Rev. D 37, 257, (1988).

[6] T.B.Omarov and L.M.Chechin, Reports NAS Kazakhstan, N2, 43, (1998); Abstracts of Plenary Lectures and Contributed Papers, 15th Int. Conf. on Gen. Relat. and Grav., Puna, India, 186, (1997).

[7] D.N.Vollick and W.G.Unruch, Phys. Rev. D 42, 2621, (1990).

[8] A.Banerjee and N.Banerjee, Phys. Lett. A 160, 119, (1991).

[9] Y.Gamboa, A.Segui-Santonia, Class.& Quant. Grav. 9, L111, (1992).

[10] T.Vachaspati, Phys.Rev. D 35, 1767, (1987).

[11] A.Larsen, N.Sanchez, Phys. Rev. D 50, 7493, (1994).

[12] H.J.de Vega and I.L.Eququiza, Phys. Rev. D 53, 3296, (1996).

[13] J.Quashnock and D.Spregel, Phys. Rev. D 42, 2505, (1990).

[14] T.B.Omarov and L.M.Chechin, Reports NAS Kazakhstan, N1, 4, (1995).

[15] A.D.Linde, Particle Physics and Inflationary Cosmology. (Harwood, Switzerland, 1990).

[16] D.Garfinkle and T.Vachaspati, Phys. Rev. D 37, 257, (1988).

[17] I.Bichak and V.Rudenko, Gravitational waves in general relativity and problem of its detecting. (Moscow, Moscow State Univ., 1987) (in Russian).

[18] D.Garfinkle, Phys.Rev. D 41, 1112, (1990).
[19] B.M. Barbashov and V.V. Nesterenko, *Relativistic string model in hadron physics*. (Moscow, Energoatomizdat, 1987) (in Russian).

[20] L.D. Akulenko and S.V. Nesterov, Mechanics of Solid, N1, 17, (1996) (in Russian).

[21] H.J. de Vega and N. Sanchez, Phys. Rev. D 50, 7202, (1994).

[22] T. Vachaspati and A. Vilenkin, Phys. Rev. D 31, 3052, (1985).

[23] A. Cresswell and R. Zimmerman, Phys. Rev. D 42, 2527, (1990).