ANGULAR INFORMATION RESOLUTION FROM CO-PRIME ARRAYS IN RADAR

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ABSTRACT
Angular resolution can be improved by using co-prime arrays instead of uniform linear arrays (ULA) with the same number of elements. We investigate how the possible co-prime angle resolution is related to the angle resolution from a full ULA of the size equal to the virtual same number of elements. We take into account not only the resulting beam width but also the fact that fewer measurements are acquired by co-prime arrays. This fact is especially relevant in compressive acquisition typical for compressive sensing. This angular resolution is called angular information resolution as it is computed from the intrinsic geometrical structure of data models that is characterized by the Fisher information. Based on this information-geometry approach, we compare angular information resolution from co-prime arrays and from the two ULAs. This novel resolution analysis is applied in a one-dimensional azimuth case. Numerical results demonstrate the suitability in radar-resolution analysis.

Index Terms— resolution, information geometry, co-prime arrays, compressive sensing, radar

1. INTRODUCTION
Resolution is defined by the minimum distance between two objects that still can be resolved. Besides the sensing bandwidth, the signal-to-noise ratio (SNR) is also crucial in the ability to resolve close objects (e.g. [1]). The sensing bandwidth of angle processing is given by the wavelength and the antenna aperture size: either actual as in a full uniform array or virtual as in the case of co-prime arrays. Our resolution analysis includes SNR and processing gain (PG) as being critical with the spatial acquisition of fewer samples what is typical for compressive sensing (CS).

When seeking the resolution limits, we keep exploring a practical combination of information geometry (IG) and CS in radar ([2]). IG is stochastic signal processing that treats the stochastic inferences as structures in differential geometry (e.g. [3-5]). The intrinsic geometrical structure of measurement models is conveniently characterized in terms of the Fisher information metric. Accordingly, potential resolution of sensors is based on information distances on such statistical manifolds.

When focusing on the system level, we also check how the resolution analysis suits sparse-signal processing (SSP), and provides the limits in high resolution. SSP can be seen as a refinement of existing processing (e.g. [7-8]).

Both IG and CS have a potential to improve radar measurements (needed for angles) are combined with temporal measurements (needed for range and doppler). Both IG and CS are aimed at sensing incoherence (e.g. [6]). In radar, SSP can be seen as a refinement of existing processing (e.g. [7-8]).

Information resolution has been studied (e.g. [2], [5] and [10]) but not related to co-prime arrays or compressive acquisition as typical for CS.

Sparse sensing by co-prime sampling has been studied separately in time and in space (e.g. [9]), also combined (e.g. [11]) and fully combined in radar ([12], summarized in 2.2 in this paper).

In this paper, we focus on effects to angular resolution from fewer measurements acquired by active co-prime linear arrays (LAs). Moreover, since we keep exploring CS at the system level, we also investigate how measurements from the co-prime LAs suit SSP in the back-end.

1.1. Related Work

In Section 2, co-prime LAs are presented in active radar with CS (as also indicated in [9]). In addition, SSP with spatial measurements from co-prime LAs is also given (as introduced in [12], with main contributions of suitability of co-prime LAs to SSP with optimal PG).

In Section 3, potential angular resolution is derived based on information distances, with main contributions of including the effects of fewer measurements from sparse sensing. In Section 4, numerical results with SSP and the resolution analysis are shown. In the end, conclusions are drawn and future work indicated.

2. CO-PRIME ARRAYS IN ACTIVE CS RADAR

In standard CS, compressive acquisition applies in the analog domain as analog-to-information conversion (AIC, e.g. [6]) after reception with a full number of antenna
elements. In radar, such AIC causes drawbacks such as: many analog delicate computations, SNR loss, changed stochastic behavior of radar data, etc. Since all the AIC difficulties shall better be avoided, we investigate compressive acquisition before reception, i.e. sparse sensing, and moreover, try exploring the existing means in a radar system: waveforms and antenna arrays (AA) for temporal and spatial acquisition, respectively (e.g. [12]).

In this work, we keep exploring AAs for spatial sparse sensing with co-prime LAs, and focus on the angular resolution. Moreover, while focusing on the system level, we also demonstrate SSP in the back-end.

2.1. Co-prime linear arrays

Co-prime arrays are defined by a pair of uniform LAs (ULAs) formed by 𝑀 and 𝐿 elements and with an inter-element spacing of 𝐿𝑑 and 𝑀𝑑, respectively, where 𝑀 and 𝐿 are co-prime integers and 𝑑 is a parameter usually equal to the half-wavelength ([9]). In the case of 𝑀 = 6 and 𝐿 = 5, the element positions of the pair of co-prime arrays are shown in Fig. 1. Due to the co-prime spatial sampling, the elements of the array coincide only at the positions that are a multiple integer of 𝑀𝐿𝑑, e.g. 0 and 30𝑑 in Fig. 1.

Fig. 1. Positioning of of co-prime arrays with 𝑀=6 and 𝐿=5.

A major advantage of co-prime arrays is a potential to achieve high angle resolution using a reduced number of sensors. Namely, 𝑀𝐿 beams with resolution of the order 1/𝑀𝐿 can be achieved by two co-prime arrays of order 𝑀 and 𝐿 connected to 𝑀-point and 𝐿-point DFT filter banks generating 𝑀 and 𝐿 outputs, respectively (Fig.2 from [9]). Each output, defined as: 𝐻(𝑒𝑖𝜔) = 𝐻(𝑒𝑖(2𝜋𝑛𝑀𝐿/𝑀𝐿)) and 𝐺(𝑒𝑖𝜔) = 𝐺(𝑒𝑖(2𝜋𝑛𝐿𝑀/𝑀𝐿)), for 0 ≤ 𝑚 ≤ 𝑀 − 1, 0 ≤ 𝑙 ≤ 𝐿 − 1, and 𝜔 = 𝜋sinθ, corresponds to shifted versions, in increments multiple of 2𝜋/𝑀𝐿, of the responses 𝐻(𝑒𝑖𝜔) and 𝐺(𝑒𝑖𝜔), obtained from low-pass responses with cut-off spatial frequencies π/𝑀 and π/𝐿 and decimated by a factor 𝑀 and 𝐿, respectively. The product of responses at the 𝑡th and 𝑙th outputs:

\[F_{lm}(e^{j\omega}) = G(e^{j(2\pi n/ML)})H(e^{j(2\pi m/ML)})\] (1)

for 0 ≤ 𝑛 ≤ 𝑀𝐿 − 1, is characterized by a unique pass-band centered at 2𝜋𝑛/ML with width 2𝜋/ML. In other words, there is only one overlapping beam among the 𝐻 beams of 𝐺(𝑒𝑖𝜔) and the 𝐿 beams of 𝐻(𝑒𝑖𝜔), as indicated in Fig. 2. Moreover, from the ML combinations of the two responses, different ML overlapping beams are obtained, exactly as in the case of an ML DFT filter bank for an ULA with ML elements.

Measurements from co-prime receive arrays have been applied to DOA, i.e. angle (e.g. [9], [13] and [14]), or angle-frequency processing (e.g. [11]). These processing techniques are based on covariance (2nd order statistics) what is not very convenient for radar processing.

2.2. SSP with co-prime AA measurements

Raw radar measurements 𝑦 (e.g. [15]) can be modeled as:

\[\mathbf{y} = \mathbf{Ax} + \mathbf{z},\] (2)

by a sensing matrix 𝐴, a sparse radar profile 𝑥, signals 𝐴𝑥 and a (complex Gaussian) receiver-noise vector 𝑧 with zero mean and equal variances 𝜎, 𝜎(𝑧) = exp(−|𝑧|²/2). When 𝑥 is sparse, the usual SSP, e.g. LASSO, applies as:

\[\mathbf{x}_{\text{opt}} = \arg \min \{ ||\mathbf{y}-\mathbf{Ax}||^2 + \eta ||\mathbf{x}||_1\},\] (3)

with the 𝑙1-norm ||𝑥||1 promoting the sparsity, the 𝑙2-norm ||𝑦−𝐴𝑥||2 minimizing the noise, and a regularization parameter 𝜂 that balances between the two tasks. In radar, the parameter 𝜂 is closely related to the detection threshold (e.g. [7-8])). An underdetermined system can be solved i.e. 𝑀 measurements in 𝑦 can be enough for 𝑁 outputs in 𝑥, because of the sparsity, i.e. only 𝐾 nonzeros in 𝑥, 𝑀 < 𝑁, 𝐾 < 𝑀, and incoherence of 𝐴 (e.g. [6]).

The basic SSP from (3) uses complex-valued measurements directly what is preferred in radar because of higher processing gain (PG, e.g. [15]). The covariance-based processing works with co-prime receive arrays leading to power-based SSP. Moreover, the covariance estimation needs training data or snapshots that are hardly available from a radar system. Finally, power-based SSP can hardly work for all radar parameters (i.e. range, doppler and angles) at once as desired for optimal PG.

Therefore, we prefer exploring transmit-receive co-prime arrays as more appropriate for active radar ([12]). As indicated in [9], with co-prime integer numbers 𝑀 and 𝐿 of receive and transmit elements, respectively, an outcome 𝐶𝑚(𝑡) of an 𝑚th receive filter (whose pattern is known for all ML angles, as in [9], and shown in Fig. 2), at time 𝑡 and angle 𝜃𝑛, 0 ≤ 𝑛 ≤ 𝑀𝐿 − 1, can be modeled as:

\[\mathbf{C}_{m,n}(t) = \sum_{l=1}^{L} r_{ml}(t) \left( H_{m,n,l}G_{l,n,m} \right) + Z_{m}(t) \] (4)

where 𝑅(𝑡) is an echo at 𝑡 from 𝜃𝑛, a pair (𝐦, 𝐿) is unique for 𝑎𝑛(𝑚, 𝐿) = 𝜋sin𝜃𝑛, e.g. (𝐦, 𝐿) = (1, 1) for 𝜃𝑛 = 𝑁, in Fig. 2), 𝐻 and 𝐺 are responses of the 𝑚th receive, and an 𝑙th transmit filter (interpreted over all ML angles, as in Fig. 2), respectively, and 𝑍(𝑡) is the DFT of the noise.

The received data 𝐶(𝑡) contain already co-prime products 𝐻𝐺, and moreover, the temporal part 𝑅(𝑡) remains unchanged. Finally, we create an 𝑁×1 data vector 𝐶(𝑡) with \[\mathbf{c}(t) = \sum_{m=1}^{M} \mathbf{C}_{m}(t),\] being its 𝑛th element.
Now we can build a model suitable for SSP, with a vector \( \mathbf{y}(t) \) of radar measurements at ML virtual elements coming from inverse DFT of the \( M \) received data sorted in \( c(t) \), as:

\[
\mathbf{y}(t) = \mathbf{F} c(t) = \mathbf{F} x(t) + \mathbf{z}(t),
\]

where \( c(t) \) is the \( N_1 \) co-prime data vector, \( \mathbf{F} \) is an \( M \times N \) steering matrix whose \( n^{th} \) column: \( \text{vec}[\mathbf{g}_n^T \mathbf{h}_n^T] \) at \( \omega_n \), has \( ML \) distinct (virtual) positions \( i, i \in [0 \text{ to } 2ML-(M+1)] \). The steering values: receive \( h_{m,n} \) and transmit \( g_{i,n} \) write as: \( h_{m,n} = e^{j m \omega_n} \) and \( g_{i,n} = e^{j i M \omega_n} \), respectively. Such a steering matrix \( \mathbf{F} \) applies also to an LA of size \( N \).

Hence, only \( M \) received data is acquired by the AA for an \( N_1 \) angle profile \( \mathbf{x}(t) \), \( M < N, N = M \), directly with less AA elements and without AIC.

The spatial data \( \mathbf{y}(t) \) from (5) can be extended to doppler and range by modeling \( \mathbf{x}(t) \) over a coherent processing time \( t \). Thus, the echo \( x_n(t) \) of a target at angle \( \theta_n \), delay \( \tau_n \) and doppler \( f_n \) is modeled as a replica of a (single) transmitted signal \( s(t) \) shifted in time by \( \tau_n \) and in frequency by \( f_n \), and with amplitude \( \alpha(\theta_n, \tau_n, f_n) \), as:

\[
x_n(t) = s(t - \tau_n) \exp(j2\pi f_n t) \alpha(\theta_n, \tau_n, f_n) \tag{6}
\]

For the simplicity, we elaborate a range-only case in pulse radar. (The extension to doppler is straightforward.) Temporal sampling is not compressive yet but kept Nyquist in an \( N \times N \) matrix \( \mathbf{Y} \) of spatial measurements \( \mathbf{y}(t) \) taken over \( N_t \) time samples, for \( N_t \) estimates of delays via an \( N \times N \) model matrix \( \mathbf{S} \) aiming for an \( N \times N \), angle-range profile matrix \( \mathbf{X} \). A data model writes as \( \mathbf{Y} = \mathbf{FX} + \mathbf{Z} \), whose vector form is suitable for SSP from (3). The model matrices \( \mathbf{S} \) and \( \mathbf{F} \) in the combined model are mutually incoherent by their physical nature. Namely, shifts in time and shifts in phase are correctly isolated by the physics. This also holds for shifts in frequency in the doppler matrix.

Recall that receive-receive, i.e. passive, co-prime arrays provide \( M-L \) measurements. ML products of two sums are involved in the covariance estimate, each written as: \( \sum_n r_n(t)^2 h_{m,n} + \sum_m r_m(t) g_{i,n} \). The only outcome: \( \sum_n r_n(t)^2 h_{m,n} \), matters while many cross-products of measurements from an LA given the unknown parameter \( \omega \). The second derivative used in (8), writes as:

\[
\frac{d^2 \ln \mathbf{p}(\mathbf{y}|\omega)}{d \omega^2} = -2 \mu_1 (\alpha/\gamma) Re(a_1^T y),
\]

where the matched-filtering (MF) value: \( a_1^T y \) also appears. The expected value is \(-2\mu_1^2 \alpha^2/\gamma \) as used in (8). Note \( G(\omega) \) that PG from an AA configuration comes from the sums \( \mu_i^2 \), and that the edge elements contribute most.

In the accuracy analysis, the metric \( G(\omega) \) is typically applied to the Cramer-Rao bound (CRB) of the mean squared error (MSE) of an unbiased estimator \( \hat{\omega} \) of \( \omega \), i.e. MSE(\( \hat{\omega} \)) \( \geq \text{CRB}(\omega) = 1/G(\omega) \) (e.g. [16]).

In the resolution analysis, information distances on this 1D statistical manifold are simply computed (e.g. [2]), as:

\[
d(\omega, \omega + \delta \omega) = \int \omega + \delta \omega \sqrt{G(\omega)} \ d\omega = 2 \delta \omega \sqrt{2 \text{SNR} \sum \mu_i^2}. \tag{9}
\]

Information resolution is higher if the information distance is larger. With the same separation \( \delta \omega \) between two angles, the information distance differs only because of \( G(\omega) \). Thus, the same information distance at \( \delta \omega \) can be achieved by different LA configurations but only with appropriate input SNR. Since our goal is to assess changes in information resolution of different LAs, we compare information distances from (9). (Finding the information resolution at \( \delta \omega_{\text{min}} \) from \( d(\omega, \omega + \delta \omega_{\text{min}}) \) at which two angles can be resolved is another goal, e.g. [2].)
4. NUMERICAL RESULTS

Numerical results on angular information resolution with co-prime LAs are given here. Moreover, while focusing on the system level, we also demonstrate SSP in the back-end.

4.1. Co-prime arrays and SSP

Angular resolution of co-prime LAs (being connected to a DFT filter bank, \(M=6\) and \(L=5\) as in Fig. 1) is indicated by a single beam of all the \(ML\) beam responses, in Fig. 3a. The co-prime response is comparable with the response of an ULA with \(ML\) elements (Fig. 3b). The advantages of the co-prime array solution are also clear when compared to an ULA with \(M\) receive elements, as shown in Fig. 3c.

Numerical tests with the co-prime LAs measurements from (4) demonstrate angle processing from (5), and also angle-range processing indicated in (6) in Fig. 4 and Fig. 5, respectively. SSP from (3) is performed by yall1 ([17]). In both cases, 12 nonzeros are randomly located over the estimation grid. The true amplitude \(\alpha\) of a nonzero in \(x\) is kept constant (so-called SW0) and given by its SNR, \(\text{SNR} = |\alpha|^2/\gamma\), where \(\gamma\) is fixed: \(\gamma=1\). A realistic case of range processing in pulse radar is merged with the angle processing. The \(N_xN_t\) sensing matrix \(S\) contains delayed replicas of a transmitted pulse that is a linearly frequency modulated (LFM) waveform, with the bandwidth equal to the sampling frequency.

This angle-range processing demonstrates the potential of the co-prime LAs in CS radar whose optimal PG is feasible because spatial and temporal data are merged in \(y\), and used in SSP of a radar profile \(x\). The extension also enlarges the sparsity of such a radar profile because the same targets are looked in a larger parameter space.

4.2. Angular information distances from co-prime LAs

The advantages of the co-prime array solution are evident from beam widths when compared to ULAs with \(M\) and \(ML\) elements, as in Fig. 3. Besides the beams from 4.1, we also explore the whole potential angular resolution based on information distances in an azimuth-only case whose measurements are acquired from LAs, as explained in Section 3. In particular, since fewer measurements are acquired by co-prime arrays, we investigate differences in the possible angle resolution with co-prime arrays in comparison with the full ULA of size \(M\) and of size \(ML\). For fair comparison of the PG effects, we let the transmit array as well as the target echo \(\alpha\) be equal in all the three receive LA cases. Moreover, \(\alpha\) is kept constant and equal to one, \(\alpha=1\), so that the target (input) SNR can be ignored, \(\text{SNR} = |\alpha|^2/\gamma = 1\). The data model is common as given by (7) while LA configurations differ per case as:

- \(\text{ULA } M\): \(\{\mu_i\} = \{0 \ 1 \ ... \ M-1\}\);
- \(\text{ULA } ML\): \(\{\mu_i\} = \{0 \ 1 \ ... \ ML-1\}\); and
- co-prime \((M,L)\) receive: \(\{\mu_i\} = \{0 \ L \ 2L \ ... \ (M-1)L\}\).
co-prime LAs perform fairly close to the full ULA with results from the three LAs, information resolution with 25 times weaker SNR. Thus, receive elements, the co-prime LAs reaches the same

However, although having $M$ receive elements only, the co-prime LAs perform fairly close to the full ULA with $ML$ elements. The information distances (i.e. the potential high resolution) from the co-prime LAs are significant at small $\delta \omega$, as shown in Fig. 7. E.g. at the specific $\delta \omega/\pi$ of 2/ML (that is the DFT bin size in Fig. 3a-b) the information distance is 3.4960 as compared to 8.7203 of the full ULA.

As given in (9), for the same information resolution, this is to be compensated by 6.22 times stronger SNR. This also holds for the accuracy given by CRB($\omega$). Furthermore, co-prime LAs perform also much better than the full ULA with $M$ receive elements. At the typical $\delta \omega/\pi$ of 2/ML, the information distance is 3.4960 as compared to 0.6992 of the full ULA. This means that with the same number of receive elements, the co-prime LAs reaches the same information resolution with 25 times weaker SNR. Thus, regarding the angular information resolution, the co-prime LAs are much closer to the full ULA with $ML$ elements than to the full ULA with $M$ elements.

5. CONCLUSIONS

Potential angular resolution is crucial when using co-prime AAs that can be convenient for spatial sparse sensing in the front-end of a sensor with CS. In the back-end, the resolution potential is also relevant in SSP as it poses the limits to the SSP high-resolution performance.

Accordingly, we investigate not only the resulting beam width that depends on the AA configuration size, but also the effects of fewer measurements that are acquired by co-prime AA. These PG effects can be seen in angular information resolution because it is computed from the intrinsic geometrical structure of data models that is characterized by the Fisher information.

Based on this information-geometry approach to angular resolution, we can conclude that active co-prime LAs with ($M+L$) elements perform much more closely to the full ULA with $ML$ elements than to the full ULA with the same number $M$ of receive elements. Moreover, we can also conclude that the concept of information resolution is appropriate for the resolution analysis in radar because of the completeness of the crucial effects it can take into account: the AA configuration and the input SNR.

5.1. Future work

In further work on information resolution, we will extend the analysis to all radar parameters: range, doppler and angle(s) in order to be able to resolve close targets in the whole parameter space. Moreover, while focusing on the system level, we will also explore the ease of achieving high resolution per parameter. Finally, we will keep designing the SSP estimation grid based on the information resolution what also involves proper sensing incoherence.

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