The Riemann hypothesis and tachyonic off-shell string scattering amplitudes

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Received: 3 February 2022 / Accepted: 15 May 2022 / Published online: 22 May 2022
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Abstract The study of the 4-tachyon off-shell string scattering amplitude $A_4(s, t, u)$, based on Witten’s open string field theory, reveals the existence of poles in the $s$-channel and associated to a continuum of complex “spins” $J$. The latter $J$ belong to the Regge trajectories in the $t, u$ channels which are defined by $-J(t) = -1 - \frac{1}{2} t = \beta(t) = \frac{1}{2} + i \lambda$; $-J(u) = -1 - \frac{1}{4} u = \gamma(u) = \frac{1}{2} - i \lambda$, with $\lambda = \text{real}$. These values of $\beta(t), \gamma(u)$ given by $\frac{1}{2} \pm i \lambda$, respectively, coincide precisely with the location of the critical line of nontrivial Riemann zeta zeros $\zeta(z_n) = \frac{1}{2} \pm i \lambda_n = 0$. It is argued that despite assigning angular momentum (spin) values $J$ to the off-shell mass values of the external off-shell tachyons along their Regge trajectories is not physically meaningful, their net zero-spin value $J(k_1) + J(k_2) = J(k_3) + J(k_4) = 0$ is physically meaningful because the on-shell tachyon exchanged in the $s$-channel has a physically well defined zero-spin. We proceed to prove that if there were nontrivial zeta zeros (violating the Riemann Hypothesis) outside the critical line $\text{Real } z = 1/2$ (but inside the critical strip) these putative zeros don’t correspond to any poles of the 4-tachyon off-shell string scattering amplitude $A_4(s, t, u)$. We finalize with some concluding remarks on the zeros of $\sinh(z)$ given by $z = 0 + i \pi n$, continuous spins, non-commutative geometry and other relevant topics.

1 Tachyonic off-shell string scattering amplitudes

The Riemann’s hypothesis (RH) \cite{1–5} states that the non-trivial zeros of the Riemann zeta-function are of the form $z_n = 1/2 \pm i \lambda_n$. Trivial zeta zeros exist at $z_n = -2n$, for $n = \text{integer}$. The on-shell four-point dual string amplitude obtained by Veneziano is \cite{6,7} was

$$A_4 = A(s, t) + A(t, s) + A(u, s) + A(u, s)$$

where the Regge trajectories in the respective $s, t, u$ channels are:

$$-\alpha(s) = 1 + \frac{1}{2}s, \quad -\beta(t) = 1 + \frac{1}{2}t, \quad -\gamma(u) = 1 + \frac{1}{2}u.$$

In our notation we define the different channels as:

$$s = (k_1 + k_2)^2 = (k_3 + k_4)^2, \quad t = (k_2 - k_3)^2 = (k_4 - k_1)^2, \quad u = (k_1 - k_3)^2 = (k_4 - k_2)^2.$$

Our prior work \cite{8,9} was based on the study of the on-shell scattering amplitudes. A closer and more rigorous look reveals that this was not general enough because we overlooked to include the key study of the off-shell tachyon scattering amplitudes, which are crucial in arriving correctly at the desired conclusions. The incoming tachyons were on-shell but the external tachyons were off-shell with $k_3, k_4 = k_1^\ast$ (a complex-conjugate pair). We will show below that the analysis of the off-shell tachyon scattering amplitudes leads to the same conclusions as in \cite{8,9} due to a numerical “fluke”.

The 4-tachyon off-shell amplitude in Witten’s cubic string field theory \cite{10} is instrumental in describing the dynamics of the open bosonic string tachyon. Both the unstable vacuum and the true vacuum where the tachyon has condensed have been shown to be well-defined states in Witten’s cubic string field theory \cite{11,12}. Since tachyon condensation is an off-shell process, string field theory is the required setting for its analysis. We recall that the Higgs field in the Standard Model of particle physics has a tachyonic-like term in the potential...
and its shift to the true vacuum of the theory gives masses to most of the particles in the Standard Model including the part of the Higgs fields which acquires a positive mass [13].

The 4-tachyon off-shell $s - t$ amplitude found by [11, 12] is

$$A_4(s, t) = \frac{1}{4} \int_0^1 dx \ |x|^{s-2} |1-x|^{-t-2} \times \left( \frac{C(\frac{1}{2} + |x - \frac{1}{2}|)}{2\sqrt{\frac{1}{2} + |x - \frac{1}{2}|}} \right)^{M^2-4}$$

(5)

and it was obtained following the conformal mapping techniques that [14] used to derive the on-shell Veneziano amplitude from Witten’s cubic string field theory vertex. $C(x)$ is a very complicated expression defined by elliptic integrals, $M^2 = \sum_{i=1}^4 P_i^2$, and $x$ is the Koba–Nielsen cross-ratio. The full 4-tachyon off-shell amplitude can be obtained from Eq. (5) after performing the cyclic permutations $P_1 \rightarrow P_2 \rightarrow P_3 \rightarrow P_4 \rightarrow P_1$.

The notation and signature used by [11, 12] is

$$s = -(P_1 + P_2)^2, \quad t = -(P_2 + P_3)^2,$$

$$u = -(P_2 + P_4)^2$$

(6)

$$P^2 = P_\mu P^\mu = -E^2 + \vec{P} \cdot \vec{P}$$

(7)

with $P_1 + P_2 + P_3 + P_4 = 0$ And because it is different than ours, one must establish the following dictionary between their variables and ours

$$P_1 = \frac{i}{\sqrt{2}} k_1, \quad P_2 = \frac{i}{\sqrt{2}} k_2, \quad P_3 = -\frac{i}{\sqrt{2}} k_3,$$

$$P_4 = -\frac{i}{\sqrt{2}} k_4,$$

(8)

$$P^2 = -\frac{1}{2} k^2, \quad k^2 = k_\mu k^\mu \equiv E^2 - \vec{k} \cdot \vec{k},$$

(9)

$$\frac{1}{2} (k_1 + k_2)^2 = -(P_1 + P_2)^2, \quad \frac{1}{2} (k_2 - k_3)^2 = -(P_2 + P_3)^2, \ldots$$

(10)

and such that

$$M^2 = \sum_{i=1}^4 P_i^2 = -\frac{1}{2} (k_1^2 + k_2^2 + k_3^2 + k_4^2)$$

(11)

we will show that when the special condition holds

$$k_1^2 + k_2^2 + k_3^2 + k_4^2 = -8 \Rightarrow M^2 = \sum_{i=1}^4 P_i^2$$

$$= -\frac{1}{2} (k_1^2 + k_2^2 + k_3^2 + k_4^2) = 4$$

(13)

then the correction factor in the scattering amplitude ($\ldots)M^2 - 4$ of the 4 off-shell tachyons becomes precisely 1 (due to $M^2 - 4 = 0$) and one is still able to recover the same functional expression as the on-shell Veneziano string amplitude for those very special values of $k_1, k_2, k_3, k_4$. And these latter values are precisely those which allows us to establish a one-to-one correspondence between the poles of the string scattering amplitude and the critical line where the nontrivial zeros are located. Therefore, one could assert that this numerical “fluke” when the functional expressions for the on-shell and off-shell tachyon scattering amplitudes coincide is a reflection of a “coexistence” of the classical and quantum world.

The special condition $k_1^2 + k_2^2 + k_3^2 + k_4^2 = -8$, combined with the conservation of energy–momentum $k_1 + k_2 = k_3 + k_4$, and a judicious use of the definitions in Eq. (4) allows to prove that the sum

$$s + t + u = (k_1^2 + k_2^2 + k_3^2 + k_4^2) + 2k_2^2 + 2k_1 \cdot k_2 - 2k_2 \cdot k_3 - 2k_2 \cdot k_4$$

$$= -8 + 2k_2 \cdot (k_1 + k_2) - 2k_2 \cdot (k_3 + k_4)$$

$$= -8 + 2k_2 \cdot (k_1 + k_2) - 2k_2 \cdot (k_1 + k_2) = -8.$$  

(14)

This relationship $s + t + u = -8$ will be crucial in order to show below that the string amplitude can be rewritten in terms of products of zeta functions.

Hence, from the defining Regge trajectories (2) and Eq. (14) we obtain the following constraint

$$\alpha(s) + \beta(t) + \gamma(u) = 1.$$  

(15)

The last relationship can also be understood geometrically as the sums of the angles, in units of $\pi$, of an Euclidean triangle found in [15] where new relations among analyticity, Regge trajectories, the Veneziano string amplitudes and Moebius transformations were studied. Note that the author [15] uses a different convention for $\alpha, \beta, \gamma$ than ours.

There exists a well known relation [6, 16] among the $\Gamma$ functions in terms of $\zeta$ functions appearing in the expression for $A(s, t, u)$ when $\alpha, \beta$ fall inside the critical strip. In this case the integration region in the real line that defines the on-shell amplitude $A(s, t, u)$ in Eq. (1) can be divided into three parts and leads to the very important identity

$$A(s, t, u) = B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)} + \frac{\Gamma(\alpha)\Gamma(\gamma)}{\Gamma(\alpha + \gamma)}$$
involving the Euler beta function and the Riemann zeta function when \( \alpha + \beta + \gamma = 1 \), and \( \alpha, \beta \) are confined to the interior of the critical strip. Because the functional form of the on-shell amplitude coincides with the off-shell amplitude for very special values of \( k_1, k_2, k_3, k_4 \), as we shall prove, we may also use the expression of Eq. (16) for the off-shell amplitude.

The derivation behind Eq. (16) relies on the above constraint (15) \( \alpha + \beta + \gamma = 1 \) and the identities

\[
\sin \pi(\alpha + \beta) + \sin \pi(\alpha + \gamma) + \sin \pi(\beta + \gamma) = 4 \cos \frac{\pi \alpha}{2} \cos \frac{\pi \beta}{2} \cos \frac{\pi \gamma}{2},
\]

\[
\Gamma(\gamma) = \Gamma(1 - \alpha - \beta) \frac{\pi}{\Gamma(\alpha + \beta) \sin \pi(\alpha + \beta)}.
\]

plus the remaining cyclic permutations from which one can infer

\[
\Gamma(\alpha)\Gamma(\beta) = \Gamma(\alpha + \beta) \frac{\sin \pi(\alpha + \beta)}{\pi},
\]

\[
\Gamma(\alpha + \gamma) = \Gamma(\alpha)\Gamma(\beta) \frac{\sin \pi(\alpha + \gamma)}{\pi},
\]

\[
\Gamma(\beta + \gamma) = \Gamma(\alpha)\Gamma(\beta) \frac{\sin \pi(\beta + \gamma)}{\pi}.
\]

Therefore, Eq. (17) allow us to recast the l.h.s of (16) as

\[
A(s, t, u) = B(\alpha, \beta) = \frac{4}{\pi} \cos \frac{\pi \alpha}{2} \cos \frac{\pi \beta}{2} \cos \frac{\pi \gamma}{2} \Gamma(\alpha)\Gamma(\beta) \Gamma(\gamma)
\]

And, finally, the known functional relation

\[
(2\pi)^z \zeta(1 - z) = 2 \cos \frac{\pi z}{2} \Gamma(z) \zeta(z).
\]

in conjunction with the condition \( \alpha + \beta + \gamma = 1 \) such that \( (2\pi)^{\alpha+\beta+\gamma} = 2\pi \) is what establishes the important identity (16) expressing explicitly the string amplitude \( A(s, t, u) \) either in terms of zeta functions or in terms of \( \Gamma \) functions.

Having found the expression for \( A(s, t, u) \) (16) in terms of products of zeta functions it follows from the relation \( \alpha + \beta + \gamma = 1 \) that the location of the Riemann critical line of zeta zeros given by the complex numbers \( \beta = 1/2 + i\lambda, \gamma = \beta^* = 1/2 - i\lambda \) \( \Rightarrow \alpha = 0, \beta + \gamma = 1 \), corresponds to real-valued poles of the scattering amplitude \( A(s, t, u) = \frac{\xi(1-\alpha)}{\xi(\omega)} = \frac{\xi(1)}{\xi(0)} = -\infty \). Due to \( 1 - \beta = \beta^* = \gamma \) and \( 1 - \gamma = \gamma^* = \beta \) there is a pairwise exact cancellation of the numerator and the denominator in

\[
\frac{\xi(1-\beta)}{\xi(\beta)} \frac{\xi(1-\gamma)}{\xi(\gamma)} = 1
\]

(18c)

and \( A(s, t, u) \) reduces to \( \frac{\xi(1)}{\xi(0)} = -\infty \). Therefore, there is a pole of \( A(s, t, u) \) whenever \( \beta = 1/2 + i\lambda, \gamma = \beta^* = 1/2 - i\lambda \). Namely when \( \alpha, \beta \) lie in the critical line.

In the center of mass frame one has the following kinematical relations [17]

\[
s = (k_1 + k_2)^2 = 4E^2 = 4(\vec{p}^2 + M^2),
\]

\[
t = (k_4 - k_1)^2 = -2(\vec{p}^2)(1 + \cos \theta),
\]

\[
u = (k_3 - k_1)^2 = -2(\vec{p}^2)(1 - \cos \theta)
\]

(19)

where \( \theta \) is the scattering angle. From Eq. (19) one infers that \( s + t + u = 4M^2 \Rightarrow s + t + u = -8 \iff \alpha + \beta + \gamma = 1, \) when \( M^2 = -2 \) is the on-shell tachyon mass-squared.

For our purposes, in the upcoming discussions, the relevant value of \( \theta \) is chosen to be \( \theta = \frac{\pi}{2} \). In this case, the 4-tachyon on-shell amplitude has an on-shell tachyonic pole in the \( s \)-channel \( s = -2 \), and the on-shell values for \( t = u = -3 \) obeying \( s + t + u = -8 \), leading to \( \alpha = 0 \) and \( \beta = \gamma = \frac{1}{2} \). Because there is no zero at \( \frac{1}{2} \) there is no correspondence between the pole of this 4-tachyon on-shell amplitude and a zeta zero in this case.

We shall show below (in Eqs. 27–31) that these numerical values differ from the ones obtained in the 4-tachyon off-shell amplitude case as a result of performing an analytical continuation of the scattering angle \( \theta \) to complex values \( \theta = \frac{\pi}{2} - i\sigma \) leading then to the value of \( s = -2 \) associated to an on-shell tachyonic pole in the \( s \)-channel, and the following off-shell values for \( t = -3 - 2i\lambda; u = -3 + 2i\lambda \), which correspond respectively, to \( \alpha = 0 \) and the complex-conjugate pair \( \beta = \frac{1}{2} + i\lambda; \gamma = \frac{1}{2} - i\lambda \), which coincide with the location of the nontrivial zeta zeros.

In other words, the values of \( t = -3 - 2i\lambda; u = -3 + 2i\lambda \) corresponding to the scattering of 4 external off-shell tachyons leading to the on-shell tachyonic pole \( s = -2 \) in the \( s \)-channel, yield the values \( \beta = \gamma = \frac{1}{2} \), along the imaginary directions associated with the critical line : \( \frac{1}{2} \rightarrow \frac{1}{2} \pm i\lambda \), where the nontrivial zeta zeros reside. The key functional relation between the imaginary values of the points along the critical line and the imaginary parts of the scattering angles is given by \( 2\lambda = 3\sinh(\alpha) \).

By cyclic symmetry one may have poles in the \( t \) or the \( u \) channel as well given by \( t = -2, u = -2 \), respectively. This follows from \( \alpha + \beta + \gamma = 1 \) by setting \( \{\beta = 0; \alpha = \gamma^* = 1/2 + i\lambda\} \) (leading to a pole in the \( t \)-channel) or \( \{\gamma = 0; \alpha = \beta^* = 1/2 + i\lambda\} \) (leading to a pole in the \( u \)-channel).

Two questions arise: (i) Are there poles of \( A(s, t, u) \) that do not correspond to zeta zeros? (ii) Are there zeta zeros that
do not correspond to poles?. We found earlier that there is pole of the 4-tachyon on-shell amplitude when \( \alpha = 1 \) but there is no zeta zero at \( \beta = \gamma = \frac{1}{2} \), therefore one learns that the answer to the first question (i) is yes. Next we shall answer (ii).

The functional relation of the completed zeta function

\[
Z(z) = \pi^{-\frac{z}{2}} \Gamma\left(\frac{z}{2}\right) \zeta(z) = Z(1-z) = \pi^{-\frac{1-z}{2}} \Gamma\left(\frac{1-z}{2}\right) \zeta(1-z)
\]

is instrumental in showing why if there are nontrivial zeta zeros outside the critical Riemann line these zeros don’t correspond to poles of \( A(s, t, u) \).

Let us identify the sets of quartets of hypothetical nontrivial zeta zeros lying inside the critical strip\(^1\) \((0 < \text{Re} \, z < 1)\) at the locations described by

\[
a_n, \quad \beta_n = \alpha_n^*, \quad 1 - \alpha_n, \quad 1 - \beta_n = 1 - \alpha_n^*, \quad n = 1, 2, 3, \ldots
\]

respectively, such that

\[
\zeta(\alpha_n) = \zeta(\beta_n) = \zeta(1 - \alpha_n) = \zeta(1 - \beta_n) = 0
\]

and

\[
0 < \alpha_n + \beta_n < 1; \quad 0 < \gamma_n < 1.
\]

so that \( \alpha_n, \beta_n, \gamma_n \) are all confined inside the critical strip and whose values must be consistent with the condition \( \alpha_n + \beta_n + \gamma_n = 1 \), which is a required condition stemming from Eqs. (13, 14, 15), in order for the off-shell and on-shell amplitudes to coincide. Note that \( \gamma_n \) is real. The amplitude (16) in this case is

\[
A(s, t, u) = \frac{\zeta(1 - \alpha_n)}{\zeta(\alpha_n)} \frac{\zeta(1 - \beta_n)}{\zeta(\beta_n)} \frac{\zeta(1 - \gamma_n)}{\zeta(\gamma_n)} = \frac{\zeta(1 - \alpha_n)}{\zeta(\alpha_n)} \frac{\zeta(1 - \beta_n)}{\zeta(\beta_n)} \frac{\zeta(1 - \gamma_n)}{\zeta(\gamma_n)} = C_n \frac{\zeta(1 - \gamma_n)}{\zeta(\gamma_n)} = \text{real and finite.}
\]

This result (24) is a consequence of the above functional Eq. (20) since the ratio \( \frac{\zeta(1 - \alpha_n)}{\zeta(\alpha_n)} \) can be rewritten in terms of the Gamma functions as

\[
\frac{\zeta(1 - \alpha_n)}{\zeta(\alpha_n)} = \frac{\Gamma\left(\frac{\alpha_n}{2}\right)}{\Gamma\left(1 - \frac{\alpha_n}{2}\right)} = \pi^{\frac{1}{2} - \alpha_n} \frac{\pi^{\frac{\alpha_n}{2}}}{\pi^{\frac{1}{2}}}, \quad 0 < \text{Re}(\alpha_n) < 1
\]

Hence, from Eq. (25) one infers that the real constants \( C_n = |\frac{\zeta(1 - \alpha_n)}{\zeta(\alpha_n)}| = 0/0 \) are finite and nonzero. Consequently, the amplitude \( A(s, t, u) \) (24) is finite and devoid of poles because \( \frac{\zeta(1 - \gamma_n)}{\zeta(\gamma_n)} \) is finite when \( \gamma_n \) is real and constrained to obey \( 0 < \gamma_n < 1 \). The zeta function \( \zeta(z) \) has a simple pole at \( z = 1 \). Similar findings occur when \( \alpha_n, \beta_n \) lie to the right of the critical line such that

\[
2 > \alpha_n + \beta_n > 1; \quad -1 < \gamma_n < 0.
\]

Therefore, there are no poles in the r.h.s of Eq. (24) when the parameters \( \alpha_n, \beta_n, \gamma_n \) are restricted to obey the conditions described above. The case when the quartet of zeros \( \{\alpha_n, \alpha_n^*, 1 - \alpha_n, 1 - \alpha_n^*\} \) are not related to \( \beta_n \) via the relations displayed in Eq. (21), but still obeying \( \alpha_n + \beta_n + \gamma_n = 1 \) lead to the same conclusions. And one finally concludes that if there were nontrivial zeta zeros outside the critical Riemann line these putative zeros don’t correspond to poles of \( A(s, t, u) \). However, this fact alone does not necessarily mean that these zeros do not exist but only that if they existed they do not have a physical interpretation in terms of the poles of \( A(s, t, u) \).

We are going to prove next that one can actually satisfy our goals even if the incoming tachyons are off-shell; i.e. if \( k_1^2 \neq -2 \) and \( k_2^2 \neq -2 \), with the provision that the \( s \)-channel still obeys the on-shell condition \( (k_1 + k_2)^2 = -2 \) and the key algebraic condition \( s + t + u = -8 \) is still satisfied. In this case, all of the results in [8, 9] still hold and one can find exact solutions to all of the relevant equations. As stated earlier, the external tachyons were already off-shell with \( k_3, k_4 \) being a complex-conjugate pair.

Let us not impose now the on-shell conditions for the incoming tachyons (so that \( k_1^2 \neq -2 \) and \( k_2^2 \neq -2 \)) and search for solutions to the following system of 8 nonlinear equations

\[
s = (k_1 + k_2)^2 = (k_3 + k_4)^2 = -2 \Rightarrow \alpha(s)
\]

\[
t = (k_2 - k_3)^2 = (k_4 - k_1)^2 = -3 - 2i\lambda \Rightarrow \beta(t)
\]

\[
u = (k_2 - k_3)^2 = (k_4 - k_1)^2 = -3 + 2i\lambda \Rightarrow \gamma(u)
\]

\[
k_3^2 = -2 + 2i\xi \Rightarrow J(k_3^2)
\]

\[
k_4^2 = -2 - 2i\xi \Rightarrow J(k_4^2)
\]

\[
\beta \text{ and } \gamma \text{ are complex conjugates and lie in the critical line and the conditions } \alpha + \beta + \gamma = 1 \leftrightarrow s + t + u = -8 \text{ are satisfied. Conservation of angular momentum demands the sum of the spins in Eqs. (30,31) equals the zero-spin value in Eq. (27).}
\]

\(^1\) According to the Valle-de la Poussin theorem there are no zeros on the boundary of the critical strip.
From Eqs. (30, 31) one infers that \( k_3, k_4 \) are complex-valued and complex-conjugates \( k_3 = k_4^* \). And, in turn, from Eqs. (28, 29) one can then infer that \( k_1, k_2 \) must be real-valued. Let us choose an ansatz where the non-vanishing components in 26-dim for \( k_1, k_2, k_3, k_4 \) are of the form

\[
k_1 \equiv (E_1, p_1), \quad k_2 \equiv (E_2, p_2),
\]

\[
k_3 \equiv (E_3 + iE_3, p_3 + i\pi_3), \quad k_4 = k_3^* \equiv (E_3 - iE_3, p_3 - i\pi_3)
\]

(32)

and where we set to zero the remaining 24 transverse components to the bosonic string world-sheet. In this case, the total number of variables comprising \( k_1, k_2, k_3, k_4 \) is then given by \( 2 + 2 + 4 = 8 \) and which matches the number of 8 Eqs. (27–31).

Since one has a system of 8 nonlinear equations with 8 unknowns, there might not be solutions; or there might be one or many solutions. A closer inspection of the 8 nonlinear equations reveals that they are not independent. From the conservation of the energy-momentum \( k_1 + k_2 = k_3 + k_4 \) one can see that the second set of equations in the doublets of Eqs. (27–29) are not independent from the first set of equations. Thus there is a redundancy and in actuality there are 5 nonlinear equations plus one linear equation \( k_1 + k_2 = k_3 + k_4 \). Setting aside this subtlety, after some straightforward algebra one finds the following solutions in the lab frame

\[
k_1 = (0, p_1 = \frac{1 - \sqrt{3}}{\sqrt{2}}),
\]

\[
k_2 = (0, p_2 = \frac{1 + \sqrt{3}}{\sqrt{2}}),
\]

(33)

\[
k_3 = (0 + iE_3 = i\sqrt{2\xi^2 + \frac{3}{2}}),
\]

\[
p_3 + i\pi_3 = \frac{1}{\sqrt{2}} - i\sqrt{2\xi}
\]

(34)

\[
k_4 = k_3^* = (0 - iE_3 = -i\sqrt{2\xi^2 + \frac{3}{2}}),
\]

\[
p_3 - i\pi_3 = \frac{1}{\sqrt{2}} + i\sqrt{2\xi}
\]

(35)

and where the key relationship (obtained from the solutions) between \( \xi \) and \( \lambda \) turns out to be

\[
\sqrt{3} \xi = \lambda.
\]

(36)

From Eq. (33) one learns that \( k_1^2 \) and \( k_2^2 \) are Galois conjugates

\[
k_1^2 = -\left(\frac{1 + \sqrt{3}}{\sqrt{2}}\right)^2 = -2 + \sqrt{3} < 0, \quad \Rightarrow k_1^2 \neq -2.
\]

(37)

\[
k_2^2 = -\left(\frac{1 - \sqrt{3}}{\sqrt{2}}\right)^2 = -2 - \sqrt{3} < 0, \quad \Rightarrow k_2^2 \neq -2.
\]

(38)

From Eqs. (34, 35) one verifies also that \( k_3 \) and \( k_4 \) are complex conjugates. Equations (33–36) represent a considerable improvement of our previous findings in the appendix of [8,9]. In particular, Eq. (36) is far simpler than Eq. (A.18) in [8,9]. This is due to the fact that we are no longer imposing the on-shell conditions for the incoming tachyons \( k_1^2 = k_2^2 = -2 \). All the 4 tachyons are now off-shell.

To sum up, one can explicitly verify that

\[
(k_1 + k_2)^2 = (k_3 + k_4)^2 = -\left(\frac{2}{\sqrt{\xi}}\right)^2 = -2
\]

(39)

\[
k_3^2 = -(2\xi^2 + \frac{3}{2}) - \left(\frac{1}{\sqrt{2}} - i\sqrt{2\xi}\right)^2
\]

\[
= -2 + 2i\xi
\]

(40)

\[
k_4^2 = -(2\xi^2 + \frac{3}{2}) - \left(\frac{1}{\sqrt{2}} + i\sqrt{2\xi}\right)^2
\]

\[
= -2 - 2i\xi
\]

(41)

\[
(k_2 - k_3)^2 = (k_4 - k_1)^2
\]

\[
= -(2\xi^2 + \frac{3}{2}) - \left(\sqrt{\frac{3}{2}} + i\sqrt{2\xi}\right)^2
\]

\[
= -3 - 2i\sqrt{3}\xi = -3 - 2i\lambda
\]

(42)

the complex conjugate of Eq. (42) gives

\[
(k_2 - k_4)^2 = (k_3 - k_1)^2
\]

\[
= -3 + 2i\sqrt{3}\xi = -3 + 2i\lambda.
\]

(43)

And, finally, one can check the conservation of energy-momentum \( k_1 + k_2 = k_3 + k_4 = (0, \sqrt{\xi}) \) and that the key condition \( s + t + u = -2 - 3 - 2i\lambda - 3 + 2i\lambda = -8 \Rightarrow \alpha(s) + \beta(t) + \gamma(u) = 1 \) is obeyed.

To conclude, due to the fact that \( k_1^2 \) and \( k_2^2 \) are Galois conjugates, and \( k_3^2 \) and \( k_4^2 \) are complex conjugates, from the results in Eqs. (37, 38, 40, 41) one finally arrives at the sought-after condition displayed by Eq. (13)

\[
M^2 = \sum_{i=1}^{4} p_i^2 = -\frac{1}{2}(k_1^2 + k_2^2 + k_3^2 + k_4^2) = 4
\]

(44)

such that the correction factor in the 4-tachyon off-shell scattering amplitude (5) becomes unity and the functional expressions for the on-shell and off-shell amplitudes coincide in this very special case. And as a result, we were able to show that if there were nontrivial zeros violating the Riemann hypothesis, these zeros do not correspond to poles of the off-shell string scattering amplitude.

One must notice that when \( s + t + u = 4M^2 \neq -8 \leftrightarrow \alpha + \beta + \gamma \neq 1 \), then Eqs. (13, 14, 15) are no longer obeyed so the off-shell and on-shell amplitudes no longer coincide.

---

2 By writing \((k_1 + k_2)^2 = (k_3 + k_4)^2 = -2\) it is understood that it means \((k_1 + k_2)^2 = -2\) and \((k_3 + k_4)^2 = -2\), etc.
The special case \( s + t + u = 4M^2 = -8 \Rightarrow M^2 = -2 \) corresponds to the on-shell tachyonic pole \( s = -2 \) in the \( s \)-channel. However, when \( M^2 \neq -2 \) the 4-tachyon off-shell amplitudes given by Eq. (5) will differ considerably from the on-shell expression given in terms of the Euler beta functions (16), and one cannot rule out the possibility that the sets of quartets of putative zeta zeros off-the-critical line may now have an actual correspondence with new poles of the far more complicated off-shell amplitude (5). The study of this possibility is well beyond the scope of this work and requires extensive computer analysis beyond our capabilities.

Let’s assume for the moment that one can assign angular momentum (spin) values \( J \) to the off-shell mass values of these external off-shell tachyons despite the fact that it is only meaningful to assign \( J \) values to on-shell particles. Having found solutions for \( k_1, k_2, k_3, k_4 \) one can obtain the values of the angular momentum (spin) \( J \) carried by the tachyonic particles directly from their defining Regge trajectory. One then arrives at

\[
J(k_1) = 1 + \frac{1}{2} k_1^2 = \frac{\sqrt{3}}{2}, \quad J(k_2)
\]

\[
= 1 + \frac{1}{2} k_2^2 = -\frac{\sqrt{3}}{2} \Rightarrow J(k_1) + J(k_2) = 0
\]

\[
J(k_3) = 1 + \frac{1}{2} k_3^2 = i\xi = i\frac{\lambda}{\sqrt{3}}, \quad J(k_4)
\]

\[
= 1 + \frac{1}{2} k_4^2 = -i\xi
\]

\[
= -i\frac{\lambda}{\sqrt{3}} \Rightarrow J(k_3) + J(k_4) = 0
\]

Equations (45–46) are consistent with the fact that the well-defined spin of the on-shell tachyon exchanged in the \( s \)-channel \( (k_1 + k_2)^2 = -2 \) is given by

\[
J(s = (k_1 + k_2)^2) = -2
\]

\[
= 1 + \frac{1}{2} s = 1 + \frac{1}{2} (-2) = 0
\]

so that the net zero-spin value carried by the external off-shell tachyons is conserved. Likewise, the net value of the energy-momentum is also conserved \( k_1 + k_2 = k_3 + k_4 = (0, \sqrt{2}) \). Therefore, despite that assigning angular momentum (spin) values \( J \) to the off-shell mass values of the external off-shell tachyons along their Regge trajectories is not physically meaningful, their net zero-spin value is physically meaningful because the on-shell tachyon exchanged in the \( s \)-channel has a physically well defined zero-spin.

Because the angular momentum is the canonical conjugate to the angle in Quantum Mechanics, the putative complex values of \( J(t) = 1 + \frac{1}{2} t = -\beta(t) = -\left(\frac{3}{2} + i\lambda\right) \) can be interpreted as the canonically conjugates of the complex scattering angles \( \theta = \frac{\pi}{2} - i\sigma \), with \( 2\lambda = 3sinh(\sigma) \). Such analytical continuation to complex angles is tantamount to the complexification of world lines like it occurs in complexified Minkowski spacetime, typical of twistor theory. For a very recent extensive study of the complexification of world lines and many other topics pertaining the complexification of the \( S \)-matrix in a way consistent with causality we refer to [18].

In the conclusion we shall add some comments on Celestial conformal field theories and the representations of the \( D \)-dim conformal group \( SO(D, 2) \) leading to complex spins. Such representations are characterized by the complex conformal weight \( \Delta = \frac{D}{2} + iR \), a complex spin \( J = -\frac{D-2}{2} + iR \) and by a real \( A \) belonging to an irreducible representation of \( SO(D-2) \) [19].

In the particular case when \( D = 2 \), the group \( SO(2, 2) \) is associated to a space with a split signature in four-dimensions, a Klein space, and yields purely imaginary spins [19]. Since the bosonic string lives in a Lorentzian spacetime background one cannot recur to the complex spins belonging to the \( SO(D, 2) \) representations in order to describe the putative complex angular momentum values \( J \) assigned to the off-shell mass values of the off-shell tachyons. Hence the only reasonable physical interpretation one can assign to the above complex values \( J(t) = -\left(\frac{3}{2} + i\lambda\right) \) is that they are the canonically conjugates of the above-mentioned complex scattering angles \( \theta = \frac{\pi}{2} - i\sigma \), with \( 2\lambda = 3sinh(\sigma) \). Since the complex values of \( J(u) \) are not independent from \( J(t) \) because \( J(u) = -\left(\frac{1}{2} - i\lambda\right) = J(t)^* \) are the complex conjugates of \( J(t) \), strictly speaking, one has that the complex-conjugate pairs \( J(t), J(u) = J(t)^* \) can be interpreted as the canonically conjugates of the complex scattering angles \( \theta = \frac{\pi}{2} - i\sigma \).

Complex-valued energy-momenta and angular-momenta have physical significance. It is well known to the experts that the imaginary parts of the energies in scattering theory corresponds to the inverse lifetime of particle-resonances. The resonance-width is the inverse of the lifetime. However it is not clear how finding such poles in the off-shell string amplitudes is connected to actual particle resonances. Thus the presence of complex-valued energy-momenta and angular-momenta must be seen as a consequence of the analytical continuation from real-valued scattering angles to complex ones, rather than to the existence of particle resonances.

Related to the issue of complex energies and resonances, it is instructive to mention that the authors [20] found that the Riemann nontrivial zeta zeros defined the position and widths of the resonances of a quantized Artin dynamical system. The Artin dynamical system was defined on the fundamental region of the modular group \( SL(2, Z) \) on the Lobachevsky plane of negative curvature. In the quantum mechanical regime the system can be associated with a narrow infinitely-long waveguide stretched out to infinity along

3 We thank the referee for pointing this out.
the vertical axis, and a cavity resonator attached to it at the bottom. A physical interpretation of the Maass automorphic wave function was provided in the form of an incoming plane wave of a given energy entering the resonator, bouncing inside the resonator and scattering to infinity. As the energy of the incoming wave comes close to the eigenmodes of the cavity a pronounced resonance behaviour showed up in the scattering amplitude. The crux of the results of [20] were based on the existence of a hyperbolic triangle with one vertex lying at infinity along the $y$-axis corresponding to a cusp. Related to this geometrical configuration we should add that in [8,9] we explained that the solutions $\beta = \gamma^* = 1/2 + i \lambda$ have also a clear definite geometrical interpretation when a Euclidean triangle with 3 vertices degenerates into a vertical strip in the upper complex plane comprised of one vertex located at infinity (with the $\alpha = 0$ zero angle at the cusp) and the other two vertices (with angle $\pi/2$) located on the real axis and separated by a distance [15]

$$d = \frac{\Gamma(\beta)}{\Gamma(1-\beta)} \frac{\pi}{\sin(\pi \beta)} = \frac{\pi}{\sin(\pi/2 + i \pi \lambda)} = \frac{\pi}{\cosh \pi \lambda}. \quad (48)$$

Once again we must remind the reader that our notation for $\alpha, \beta, \gamma$ differs from [15].

Despite the fact that $\beta = \gamma^*$ are complex-valued their sum $\beta + \gamma = 1 = \text{real}$, thus the sum of the three angles of the triangle is still $\pi (\alpha + \beta + \gamma) = \pi$. Therefore, the discrete number of the imaginary parts of the nontrivial zeta zeros $\lambda_n$ are associated with a discrete number of possible distances between the two variable vertices of the triangles situated in the real axis of the complex plane and given by

$$d_n = \pi/cosh (\pi \lambda_n).$$

Physical systems with this type of hyperbolic spectrum of scales $d_n$ have been recently been investigated by [21] in connection to the Riemann hypothesis. In particular, these authors applied the infinite-component Majorana equation in a Rindler spacetime and focused on the $S$-matrix approach describing the bosonic open string for tachyonic states.

The author [22] studied the Riemann zeros as energy levels of a Dirac fermion in a potential built from the prime numbers in Rindler spacetime. The Hamiltonian was derived from the action of a massless Dirac fermion living in a domain of Rindler spacetime, in $1 + 1$ dimensions, that has a boundary given by the world line of a uniformly accelerated observer. The Riemann zeros appear as discrete eigenvalues immersed in the continuum.

We found that the poles of the off-shell (on-shell) scattering amplitude are associated with a continuum of values of $\beta = \frac{1}{2} + i \lambda, \gamma = \frac{1}{2} - i \lambda, \lambda$ real. The imaginary parts of the discrete zeta zeros are $\lambda_n = \lambda_1, \lambda_2, \ldots$ and correspond to the discrete values $\beta_n = \frac{1}{2} + i \lambda_n, \gamma_n = \frac{1}{2} - i \lambda_n$ that are also embedded in the continuum of values for $\beta, \gamma$ when $\lambda$ is real. In this case, the discrete values of $k_3, k_4$ are provided by Eqs. (34, 35) simply by setting

$$k_{3,(n)} = \left( i \sqrt{2\pi^2 + \frac{3}{2} - \frac{1}{\sqrt{2}} - i \sqrt{2\pi^2} } \right)$$

$$k_{4,(n)} = k_{3,(n)}^* = \left( - i \sqrt{2\pi^2 + \frac{3}{2} + \frac{1}{\sqrt{2}} + i \sqrt{2\pi^2} } \right),$$

where $\sqrt{3\pi^2} = \lambda_n$. Therefore, the discrete values of $k_{3,(n)}, k_{4,(n)}$ are given explicitly in terms of the imaginary parts of the zeta zeros $\lambda_n$. The values of $k_1, k_2$ remain the same as in Eq. (33) and have no dependence on $\lambda_n$.

More recently, physical properties of scattering amplitudes for point particles were mapped to the Riemann zeta function by [23]. Specifically, a closed-form amplitude in terms of the logarithmic derivative of the Landau–Riemann capital xi-function was constructed, and describing the on-shell four-point scattering of massless particles involving the tree-level exchange of a tower of heavy states in the $s$ and $u$ channels, with a spectrum of masses $m_n = \lambda_n$ such that $\zeta(\frac{1}{2} \pm i \lambda_n) = 0$. The Riemann xi-function has also appeared in the construction of quantum mechanical models (related to the Riemann hypothesis) and the Veneziano string amplitude which support the validity of the Riemann hypothesis, see [24,25].

2 Concluding remarks: continuous spins, non-commutative geometry, chaos, fractal strings and all that

It has been found that the study of the 4-tachyon off-shell string scattering amplitude $A_k(s, t, u)$, based on Witten’s open string field theory, reveals the existence of on-shell tachyonic poles $s = -2$ in the $s$-channel and which is associated to a continuum of complex “spins” $J$. The latter spins $J$ belong to the Regge trajectories in the $t, u$ channels which are defined by $- J(t) = - 1 - t \beta(t) = \frac{1}{2} + i \lambda; - J(u) = - 1 - \frac{1}{2} u = \gamma(u) = \frac{1}{2} - i \lambda$, with $\lambda = \text{real}$. These values of $\beta(t), \gamma(u)$ given by $\frac{1}{2} = \pm i \lambda$, respectively, coincide precisely with the location of the critical line of nontrivial Riemann zeta zeros $\zeta(z_n = \frac{1}{2} \pm i \lambda_n) = 0$.

We argued in Eqs. (45–47) that despite assigning angular momentum (spin) values $J$ to the off-shell mass values of the external off-shell tachyons along their Regge trajectories is not physically meaningful, their net zero-spin value is physically meaningful because the on-shell tachyon exchanged in the $s$-channel has a physically well defined zero-spin.
The complex-conjugate pairs of complex spins $J(t)$, $J(u)$ = $J(t)^*$ were interpreted as the canonically conjugates of the complex scattering angles $\theta = \frac{\pi}{2} - i\sigma$ resulting from an analytical continuation of the real scattering angle $\frac{\pi}{2}$ to complex values. The key functional relation between the imaginary values of the points along the critical line and the imaginary parts of the scattering angles was found to be given by $2\lambda = 3\sinh(\sigma)$. The discrete number of nontrivial zeta zeros are embedded in a continuum of values of $\frac{1}{2} \pm i\lambda$.

The introduction of complex Mandelstam variables like $t = -(3 - 2i\lambda)$ and $u = -(3 + 2i\lambda)$ results from performing an analytical continuation of the real scattering angle $\frac{\pi}{2}$ (in the center of mass frame) to complex values $\frac{\pi}{2} - i\lambda$. This is a reflection of the analytical continuation of the Riemann zeta function $\zeta(z) = \sum n^{-z}$ from the region of convergence $\text{Re}(z) > 1$ to the interior region of the critical strip $0 < \text{Re}(z) < 1$.

We then proceeded to prove that if there were nontrivial zeta zeros (violating the Riemann Hypothesis) outside the critical line $\text{Real } z = 1/2$ (but inside the critical strip) these putative zeros don’t correspond to any poles of the 4-tachyon off-shell string scattering amplitude $A_4(s, t, u)$. However, when $s + t + u = 4M^2 \neq -8 \leftrightarrow \alpha + \beta + \gamma \neq 1$, the off-shell and on-shell amplitudes no longer coincide and one cannot rule out the possibility that the sets of quartets of putative zeta zeros off-the-critical line may now have an actual correspondence with new poles of the far more complicated off-shell amplitude given by Eq. (5).

There are other functions than the Riemann zeta function whose zeros lie in a vertical line in the complex plane. For example, the zeros of $\sinh(z) = 0$ are $z = 0 + i\pi n$ with $n = 0, \pm 1, \pm 2, \pm 3, \cdots$. This tower of equally spaced zeros along the $y$-axis resemble the distribution of spins of the continuous spin representations. Particles with continuous spin have a long history since Wigner’s construction of continuous spin representations of the Poincare group for massless particles [26]. Photons and tachyons with continuous spin were studied a while back by [27].

There are two classes of unitary infinite dimensional representations, one being massless and named continuous (or infinite) spin particles and the other constituted by tachyonic particles. For a long time no field theory was known for these infinite dimensional representations preventing the study of its properties even at the free level [13]. Only recently a field theory for continuous spins particles was proposed [28] triggering a new wave of interest on the subject. For a recent review and earlier references see [29, 30].

The irreducible unitary representations of the Poincare group in $D = 4$ can be labelled by the quadratic Casimir operator $C_2 = P^2 = P_\mu P^\mu$ associated to the mass-shell condition, and the quartic Casimir operator $C_4 = -\frac{1}{4} P^\mu P_\rho J^{\mu \nu} J_{\mu \rho} P^\nu$, the square of the Pauli-Lubanski vector $W^\mu = \epsilon^{\mu \nu \rho \tau} J_{\nu \rho} P_\tau$. The irreducible unitary representations of the Poincare group in other dimensions than $D = 4$ can be found in [31].

A nice description of the continuous spin representations can be found in [13]. In $D = 4$, the scalar spin-0 tachyon belongs to a one-dim representation. The spin-$s$ tachyon $(s \neq 0)$, with $s$ integer or half-integer, has for quadratic Casimir $C_2 = -m^2 < 0$, a quartic Casimir $C_4 = -m^2(s + 1)s$, and belongs to an infinite-dim representation of the Poincare group with an infinite tower of states labeled by $l = \pm(s + 1), \pm(s + 2), \pm(s + 3), \ldots, \infty$.

The “continuous” spin tachyon, has for quadratic Casimir $C_2 = -m^2 < 0$, a quartic Casimir $C_4 = -\rho^2$, where $\rho$ is a real number (the value of the “continuous spin”). The “bosonic” continuous spin tachyon belongs to an infinite-dim representation of the Poincare group with an infinite tower of states labeled by $l = 0, \pm 1, \pm 2, \ldots, \infty$. Whereas, the “fermionic” continuous spin tachyon belongs to an infinite-dim representation of the Poincare group with an infinite tower of states labeled by $l = \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{5}{2}, \ldots, \infty$. The massless “bosonic/fermionic” continuous spin representations have zero for their quadratic Casimir $C_2 = m^2 = 0$, a value of $C_4 = -\rho^2$ for $\rho$ real (the value of the “continuous spin), and a similar tower of states as above.

Continuous and complex spins are also present in Celestial Conformal Field Theories which are based in introducing conformal correlation functions living on the celestial torus (in contrast to a celestial sphere) resulting from the representations of $SO(2, 2)$ associated with a Klein space of split signature $(2, 2)$ in four-dimensions [32,33]. The four-point function contains non-trivial information of the spectrum, and in some cases can be related to the three-point structure constants [34]. In particular, the decomposition of [32,33] has found that not only conformal primaries (fields) with continuous boost weight are exchanged in the four-point function, but also their so-called light-ray transforms [19] with continuous and complex spin [34–36]. A review of celestial conformal field theories (CCFTs) can be found in [37,38].

Witten [7] was motivated by ideas from non-commutative geometry and introduced the non-commutative star product of three string fields $\Psi_1 \star \Psi_2 \star \Psi_3$ to construct the cubic vertex. Connes’ approach to the Riemann hypothesis relied on non-commutative geometry and Adelic products [39]. Since our results are based on the study of 4-tachyon off-shell scattering amplitudes which required Witten’s open string field theory, it is warranted to investigate the role of non-commutative geometry even further.

Tachyons were essential in the recent study of chaotic scattering of highly excited strings [40]. Motivated by the desire to understand chaos in the S-matrix of string theory, the authors studied tree level scattering amplitudes involving highly excited strings. The excited string is formed by repeatedly scattering photons off an initial tachyon (the DDF formalism) and they computed the scattering amplitude of
one arbitrary excited string and any number of tachyons in bosonic string theory. Pertaining to chaos, Dyson pointed out that the statistics of random matrix theory revealed important connections to the level statistics of the (two-point) pair correlation functions of the imaginary parts of the zeta zeros as conjectured by Montgomery [41]. It is warranted to explore deeper the role of tachyons in chaotic scattering of highly excited strings and the zeta zeros.

We found poles in the $4$-tachyon off-shell string scattering amplitude $A_4(s, t, u)$ and associated with the values of $\beta(t)$, $\gamma(u)$ given by $\frac{1}{2} \pm i\kappa$, respectively, with $\kappa$ real, which coincide precisely with the location of the critical line of non-trivial Riemann zeta zeros. This “pole-continuum” encoded by the values of $\gamma = \gamma^* = \frac{1}{2} + i\kappa$ spanning the critical line resembles the location of black hole singularities. The Schwarzschild black hole singularity at $r = 0$ is a spacelike singularity and is represented by a line in the Penrose diagram. All matter that crosses the horizon falls towards the singularity.

Roughly speaking, this pole-continuum (the critical line) behaves like an attractor where the quartets of putative zeros outside the critical line flow into. This picture of zeta zeros flowing towards the critical line was advocated by Lapidus [42] in his study of fractal strings. His main conjecture is that under the action of the modular flow, the spacetime geometries become increasingly symmetric and crystal-like, hence, arithmetic. Correspondingly, the zeros of the associated zeta functions eventually condense onto the critical line, towards which they are attracted, thereby explaining why the Riemann hypothesis must be true. This picture deserves further investigation.

The Riemann hypothesis is a manifestation of the level statistics of the (two-point) pair correlation functions of the imaginary parts of the zeta zeros as conjectured by Montgomery [41]. It is warranted to explore deeper the role of tachyons in chaotic scattering of highly excited strings and the zeta zeros.

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