Optimal large-time estimates and singular limits for the thermoelastic plate equations with Fourier’s law

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Abstract

In this paper, we study asymptotic behaviors for the classical thermoelastic plate equations equipping Fourier’s law of heat conduction in the whole space $\mathbb{R}^n$, where we introduce a reduction methodology basing on third-order (in time) differential equations and WKB analysis. We derive optimal growth estimates when $n \leq 3$, bounded estimates when $n = 4$, and decay estimates when $n \geq 5$ for the vertical displacement in the $L^2$ norm. Particularly, the new critical dimension $n = 4$ for the decisive role between the plate model and Fourier’s law of heat conduction is discovered. Moreover, concerning the small thermal parameter in Fourier’s law, we study the singular limit problem. We not only show global (in time) convergence of the vertical displacements between the thermoelastic plates and the structurally damped plates, but also rigorously demonstrate a new second-order profile of solution. We believe our methodology can settle several closely related problems in thermoelasticity.

Keywords: thermoelastic plate equations, Cauchy problem, optimal estimates, asymptotic profiles, singular limits, WKB analysis.

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1 Introduction

It is widely known that thin plates theory arises in numerous applications of engineering, for example, raft foundation, road pavement, airport runway, etc. In order to describe thin plate motions under different practical circumstances, there are various mathematical models in the plate’s field, including Mindlin-Timoshenko model, von Kármán model, thermoelastic plate equations and viscoelastic plates. In the paper, we will study some asymptotic behaviors (in terms of large-time or small thermal parameter) for the classical thermoelastic plate equations in the whole space $\mathbb{R}^n$. Before introducing our aims, we will sketch out some historical background of the thermoelastic plates.

1.1 Background of the thermoelastic plate equations

Let us consider a homogeneous, elastic and thermally isotropic plate subjecting to a temperature distribution. Associating with the second law of thermodynamics for irreversible process (relates
the entropy to the elastic strains), the well-known thermoelastic plate equations with Fourier’s law of heat conduction, namely,

\[
\begin{aligned}
\begin{cases}
\frac{\partial^2 u}{\partial t^2} + \Delta^2 u + \Delta \theta = 0, \\
\epsilon \theta_t - \Delta \theta - \Delta u_t = 0,
\end{cases}
\end{aligned}
\]

\tag{1}

has been established, where the scalar unknowns \( u = u(t, x) \) and \( \theta = \theta(t, x) \) denote, respectively, the vertical displacement and the temperature (relative to some reference temperature). The constant \( \epsilon > 0 \), which is the so-called thermal parameter, contains some parameters from the specific heat of the body, thermal conductivity as well as the mass density per unit volume. Note that we omitted other physical constants for simplicity. Concerning the heuristic derivations of the model (1), we refer interested readers to [20, 19].

In recent years, the thermoelastic plate equations (1) in the bounded domains or unbounded domains (even the whole space \( \mathbb{R}^n \)) have caught a lot of attentions (see [18, 42, 30, 25, 29, 24, 21, 28, 23, 33, 7, 31, 41, 22, 36, 37, 8, 9] and references therein). We stress that topics of the thermoelastic plates are energetic not only in the community of PDEs but also of controllability and dynamical systems perspectives. According to the theme of this work, we just briefly introduce the progressive progress in the corresponding Cauchy problem of (1) with \( \epsilon = 1 \), namely,

\[
\begin{aligned}
\begin{cases}
\frac{\partial^2 u}{\partial t^2} + \Delta^2 u + \Delta \theta = 0, \\
\theta_t - \Delta \theta - \Delta u_t = 0, \\
u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad \theta(0, x) = \theta_0(x),
\end{cases}
\end{aligned}
\]

\tag{2}

By constructing a natural energy of the coupled system (2) as follows:

\[
U^N(t, x) := \left( u_t(t, x), \Delta u(t, x), \theta(t, x) \right),
\]

the author of [41] proved some decay estimates of \( U^N(t, \cdot) \) in the \( L^2 \) or \( L^\infty \) norms with a wide range of weighted \( L^1 \) datum, whose proof is based on energy method as well as Fourier splitting techniques. Later, associated with energy estimates in the Fourier space, the authors of [36] derived sharp decay estimates for \( U^N(t, \cdot) \) in the \( \hat{H}^k \) norm carrying \( k \geq 0 \), where initial datum are assumed in energy spaces with additionally \( L^1 \) regularity. In order to verify the optimality of the next decay estimate (it also was shown in [30, Theorem 3.6] and [41, Theorem 2.1 with \( \gamma = 0 \)]):

\[
\|U^N(t, \cdot)\|_{\hat{H}^k} \lesssim (1 + t)^{-\frac{n}{4} - \frac{d}{2}} \|U^N(0, \cdot)\|_{L^1} + e^{-ct} \|U^N(0, \cdot)\|_{\hat{H}^k},
\]

the characteristic roots of (2) in the Fourier space have been calculated explicitly that is the sharp decay rate for an energy term. Additionally, from their works, there exists a unique solution to (2) satisfying

\[
u \in \mathcal{C}([0, \infty), \hat{H}^2), \quad u_t \in \mathcal{C}([0, \infty), L^2), \quad \theta \in \mathcal{C}([0, \infty), L^2),
\]

for \( u_0 \in \hat{H}^2 \) and \( u_1, \theta_0 \in L^2 \) originated from the vector unknown (3). Subsequently, a quasilinear Cauchy problem was investigated by [37]. Under this situation, a natural question emerges as follows:

\textit{How does Fourier’s law of heat conduction influence on the plate model for large-time?}
In the language of energy term, i.e. (3), a series of works [30, 41, 36, 37] said that Fourier’s law of heat conduction plays a decisive role. Nevertheless, if one consider a corresponding nonlinear problem of (2) equipping the nonlinearity \( N[u] \) of the solution itself \( u = u(t, x) \), an important step is to understand the above-mentioned question with respect to \( u \) rather than \( U^N \). Concerning large-time of the solution itself, the complete classification is still unknown, but we will state the new classification in the \( L^2 \) framework by the critical dimension \( n = 4 \) in this paper. In particular, the large-time behaviors of \( u(t, \cdot) \) to the coupled system (2) fully determined by the plate model in physical dimensions \( n = 1, 2, 3 \) as a new phenomenon.

To the best of authors’ knowledge, asymptotic behaviors (e.g. estimates in the \( L^2 \) norm and asymptotic profiles) of the solution itself, i.e. the vertical displacement \( u(t, \cdot) \), so far did not be studied deeply. Indeed, these studies are not generalizations of the previous works [30, 41, 36] due to some hidden influences of oscillations from the plate model. Let us explain this challenge briefly by an example. If one uses the sharp quantitative estimate in [36, Equation (2.3)] that is

\[
|\xi|^2 |\hat{u}(t, \xi)| \leq \frac{13}{3} e^{\frac{|\xi|^2 t}{32}} |\hat{U}^N(0, \xi)|,
\]

by simply dividing \( |\xi|^2 \) in the previous line, the next unbounded estimates appears:

\[
\|u(t, \cdot)\|_{L^2} \leq \left( \int_0^1 r^{n-5} e^{-2\alpha r^2 t} dr \right)^{1/2} \left( \|U^N(0, \cdot)\|_{L^1} + e^{-ct} \|\hat{U}^N(0, \cdot)\|_{H^{-2}(|\xi| \geq 1)} \right)
\]

due to some singularities for \( n = 1, \ldots, 4 \) as \( r \downarrow 0 \). Furthermore, another difficulty is the rigorous justification of optimal estimates in the sense of same upper bound and lower bound estimates for large-time. Especially, due to some oscillating effects in the thermoelastic plate equations (2), one may expect growth estimates rather than decay estimates for lower dimensions as \( t \gg 1 \).

Finally, we underline that the difficulties mentioned in preceding part of the paragraph not only occurs in thermoelastic plates, but also appear in other coupled systems, e.g. thermoelasticity (two temperatures, types II & III, or second sound), porous elasticity and Bresse systems because of some applications of the energy method. Over the last couple of decades, another effective approach to treat linear coupled systems, which is the so-called diagonalization procedure, had been deeply developed (see, for example, [40, 43, 16, 26, 39, 27]). Some asymptotic representations of a micro-energy can be formulated by diagonalization procedure, but sharp asymptotic behaviors of the solution itself, i.e. each element in the corresponding model, are still open to the best of authors’ knowledge. To overcome these difficulties, with a deep root from the monograph [17], we will propose a reduction methodology, that is the reduction to higher-order (in time) partial differential equations associated with refined time-frequency analysis, to deal with the thermoelastic plate equations (2). This approach can recover some oscillations to compensate strong singularities in lower dimensions (see Remark 3.2) and enlighten the construction of asymptotic profiles. More detail explanation will be addressed afterwards. We expect our methodology can be widely applied in other hyperbolic-parabolic coupled systems and hyperbolic-hyperbolic coupled systems.
1.2 Main purposes of this paper

In the present manuscript, we consider the following Cauchy problem for the classical thermoelastic plate equations with Fourier's law of heat conduction:

\[
\begin{align*}
\begin{cases}
u_{tt} + \Delta^2 u + \Delta \theta & = 0, \\
\epsilon \theta_t - \Delta \theta - \Delta u_t & = 0, \\
u(0, x) = u_0(x), \quad u_1(0, x) & = u_1(x), \quad \theta(0, x) = \theta_0(x),
\end{cases}
\end{align*}
\tag{4}
\]

with a positive constant $\epsilon$ as we introduced in Subsection 1.1. Our first purpose of the work is to study large-time asymptotic behaviors of solutions to (4) carrying $\epsilon = 1$ without loss of generality.

In Section 3, by applying the reduction procedure for the coupled system, the main consideration immediately becomes the third-order (in time) evolution equation (12). Then, with the aid of refined WKB analysis and Fourier analysis, we derive optimal estimates of solutions in the $L^2$ norm. Particularly among them, the next optimal estimates hold:

\[
\|u(t, \cdot)\|_{L^2} \simeq t^{1-n/4} \quad \text{for any } n \geq 1 \text{ and any } t \gg 1,
\]

under a new nontrivial assumption $B_{u_1, \theta_0} \neq 0$ in (7) for initial datum. It implies infinite time sharp $L^2$-blowup of the vertical displacement if and only if for the physical dimensions $n = 1, 2, 3$ with polynomial growth (the decisive part is the plate model); neither growth nor decay estimates for $n = 4$ (exquisite interplay between plates and Fourier's law); sharp decay polynomially with the factors from plates and Fourier's law. One may see Remark 2.2 for the in-depth explanation. As a by-product, we will determine asymptotic profiles of solution by diffusive-plates and heat model in the $L^2$ framework.

Let us formally take into account the limit case with the vanishing thermal parameter, i.e. $\epsilon = 0$, in the thermoelastic plate equations (4), which turns out to be the structurally damped plate equation (see [34, 5, 6] and references therein) whose initial datum are inherited from (4) as follows:

\[
\begin{align*}
\begin{cases}
u_{tt}^0 + \Delta^2 u_0^0 - \Delta u_0^0 & = 0, \\
u^0(0, x) = u_0(x), \quad u_1^0(0, x) & = u_1(x), \quad \theta(0, x) = \theta_0(x),
\end{cases}
\end{align*}
\tag{5}
\]

Formally, (5) seems to be the limit model for the coupled system (4) with the vanishing thermal parameter $\epsilon = 0$ whereas the last data $\theta(0, x) = \theta_0(x)$ will lose. For this reason, our second purpose is to investigate the small-parameter asymptotic behaviors of solutions to (4). In Section 4, we rigorously justify singular limits (limiting procedures) from the thermoelastic plate equations (4) to the structurally damped plate equation (5) as the thermal parameter tends to zero, for instance,

\[
u \to u^0 \text{ in } L^\infty([0, \infty), L^2(\mathbb{R}^n)) \text{ as } \epsilon \downarrow 0 \text{ for } n \geq 3 \text{ with the rate } \epsilon.
\]

Furthermore, by applying multi-scale analysis and energy method in the Fourier space, we derive a formal expansion of solutions with respect to $\epsilon$, and rigorously justify the correction for the second-order asymptotic expansion, for example,

\[
u \to u^0 + \epsilon u^{I,1} \text{ in } L^\infty([0, \infty), L^2(\mathbb{R}^n)) \text{ as } \epsilon \downarrow 0 \text{ for } n \geq 3 \text{ with the improved rate } \epsilon^2,
\]

where $u^{I,1} = u^{I,1}(t, x)$ that is the solution to inhomogeneous structural damped plates equipping the source term $\Delta u_t^0 - \Delta^2 u^0$ (see precisely in (11) later).
This paper is organized as follows: Section 2 will show some main results of this work, whose proofs will be given in Section 3 and Section 4, respectively. Eventually, some concluding remarks in Section 5 about the application of our methodology complete the manuscript.

1.3 Notations

Let us introduce some notations which will be used all over this work. We define the following zones of the Fourier space:

\[ \mathcal{Z}_{\text{int}}(\varepsilon_0) := \{ |\xi| \leq \varepsilon_0 \ll 1 \}, \quad \mathcal{Z}_{\text{bdd}}(\varepsilon_0, N_0) := \{ \varepsilon_0 \leq |\xi| \leq N_0 \}, \quad \mathcal{Z}_{\text{ext}}(N_0) := \{ |\xi| \geq N_0 \gg 1 \}, \]

in which, the cut-off functions \( \chi_{\text{int}}(\xi), \chi_{\text{bdd}}(\xi), \chi_{\text{ext}}(\xi) \in \mathcal{C}^\infty \) having their supports in the corresponding zones \( \mathcal{Z}_{\text{int}}(\varepsilon_0), \mathcal{Z}_{\text{bdd}}(\varepsilon_0/2, 2N_0) \) and \( \mathcal{Z}_{\text{ext}}(N_0) \), respectively, such that

\[ \chi_{\text{bdd}}(\xi) = 1 - \chi_{\text{int}}(\xi) - \chi_{\text{ext}}(\xi) \quad \text{for all} \quad \xi \in \mathbb{R}^n. \]

The symbols of pseudo-differential operators \(|D|^s\) and \(\langle D \rangle^s\) are denoted by \(|\xi|^s\) and \(\langle \xi \rangle^s\) with \(s \in \mathbb{R}\), respectively, where \(\langle \xi \rangle := \sqrt{1 + |\xi|^2}\) is the Japanese bracket.

The symbol \(f \lesssim g\) means: there exists a positive constant \(C\) fulfilling \(f \leq Cg\), which may be varied in different lines, analogously, for \(f \gtrsim g\). Additionally, the relation \(f \simeq g\) holds if and only if \(f \lesssim g\) and \(f \gtrsim g\) are retained meanwhile.

To end the introduction, let us recall the weighted \(L^1\) space

\[ L^{1,1} := \left\{ f \in L^1 \mid \|f\|_{L^{1,1}} := \int_{\mathbb{R}^n} (1 + |x|) |f(x)| \, dx < \infty \right\} \]

so that \(\|f\|_{L^1} \leq \|f\|_{L^{1,1}}\) notably. The mean of a summable function \(f\) is denoted by

\[ P_f := \int_{\mathbb{R}^n} f(x) \, dx. \]

The Sobolev spaces of negative order are defined by (see, for example, [35])

\[ \dot{H}^{-s} := \left\{ f \in \mathcal{F}' \mid \|f\|_{\dot{H}^{-s}} := \|(-\Delta)^{-\frac{s}{2}} f\|_{L^2} < \infty \right\}, \]

where \(\mathcal{F}'\) is the factor space \(S'\backslash \mathcal{P}\) with \(\mathcal{P}\) denoting the space of all polynomials. We mark the \(L^2\) space with the localization \(\chi_{\text{int}}(D)\) or \(\chi_{\text{int}}(\xi)\) by \(L^2_\chi\).

2 Main results

2.1 Asymptotic behaviors of solution for large-time

Let us introduce two positive parameters

\[ \alpha_\pm := \sqrt[3]{\frac{1}{2}} \left( 3\sqrt{69} + 11 \right) \pm \sqrt[3]{\frac{1}{2}} \left( 3\sqrt{69} - 11 \right). \]

We formulate the first result (with the detail explanation in Remark 2.2) on optimal estimates of solution \((u, \theta)\) to the linear thermoelastic plate equations (4) with \(\varepsilon = 1\).
Theorem 2.1. Let us consider the Cauchy problem (4) with \( \epsilon = 1 \) and initial datum \( u_0 \in H^2 \cap L^1 \) and \( u_1, \theta_0 \in L^2 \cap L^1 \) for any \( n \geq 1 \). Then, the solution \((u, \theta)\) satisfies the following estimates for \( t \gg 1 \):

\[
\|u(t, \cdot)\|_{L^2} \lesssim t^{-\frac{n}{2}} \|u_0\|_{L^2 \cap L^1} + t^{1 - \frac{n}{2}} \|(u_1, \theta_0)\|_{(L^2 \cap L^1)^2},
\]

\[
\|\theta(t, \cdot)\|_{L^2} \lesssim t^{-\frac{n}{2}} \|u_0\|_{H^2 \cap L^1} + t^{-\frac{n}{2}} \|(u_1, \theta_0)\|_{(L^2 \cap L^1)^2}.
\]

Assuming \( u_1, \theta_0 \in L^{1,1} \) additionally associated with the positive constant

\[
B_{u_1, \theta_0} := \left| \int_{\mathbb{R}^n} \left( \frac{2 - \alpha_-}{3} u_1(x) + \theta_0(x) \right) \, dx \right| + \left| \int_{\mathbb{R}^n} \left( \frac{\alpha_+^2 + \alpha_-^2 + 4\alpha_-}{6\alpha_-} u_1(x) + \theta_0(x) \right) \, dx \right| \neq 0,
\]

then the solution \((u, \theta)\) satisfies the following optimal estimates for \( t \gg 1 \):

\[
t^{1 - \frac{n}{2}} B_{u_1, \theta_0} \lesssim \|u(t, \cdot)\|_{L^2} \lesssim t^{1 - \frac{n}{2}} \|(u_0, u_1, \theta_0)\|_{(L^2 \cap L^1) \times (L^2 \cap L^1)^2},
\]

\[
t^{-\frac{n}{2}} B_{u_1, \theta_0} \lesssim \|\theta(t, \cdot)\|_{L^2} \lesssim t^{-\frac{n}{2}} \|(u_0, u_1, \theta_0)\|_{(H^2 \cap L^1) \times (L^2 \cap L^1)^2}.
\]

Remark 2.1. Because of the study in coupled systems, the initial datum \( u_1, \theta_0 \) always share the same time-dependent coefficient (no matter decay rate or growth rate) and regularity. Thus, the combined effect in the quantity \( B_{u_1, \theta_0} \) from two datum is spontaneous, which is quite different from the researches of scalar single equations (e.g. damped wave/plate models).

Remark 2.2. Our first innovation is the new classification (with respect to dimension \( n \)) of interplay between the plate model and Fourier’s law of heat conduction. Recalling the large-time behavior of the pure plate model

\[
\begin{cases}
v_{tt} + \Delta^2 v = 0, & x \in \mathbb{R}^n, t > 0, \\
v(0, x) = v_0(x), & v_t(0, x) = v_1(x), & x \in \mathbb{R}^n,
\end{cases}
\]

under some \( L^2 \) assumptions with additionally \( L^1 \) regularity (even \( L^{1,1} \) regularity) on initial datum and \( |P_{u_1}| \neq 0 \), the recent work [13] got the optimal growth estimates

\[
\|v(t, \cdot)\|_{L^2}^2 \simeq \begin{cases} t^{2 - \frac{n}{2}} & \text{if } n \leq 3, \\ \log t & \text{if } n = 4,
\end{cases}
\]

for \( t \gg 1 \). Reviewing Theorem 2.1, we actually obtained the optimal estimates of the thermoelastic plate equations (4) with \( \epsilon = 1 \) by the following one:

\[
\|u(t, \cdot)\|_{L^2}^2 \simeq \begin{cases} t^{2 - \frac{n}{2}} & \text{if } n \leq 3, \\ 1 & \text{if } n = 4, \\ t^{-\frac{n+4}{2}} & \text{if } n \geq 5,
\end{cases}
\]

as \( t \gg 1 \). Then, we may realize that the growth rates for the pure plate model (8) and the thermoelastic plate equations (2) if \( n \leq 3 \) are the same; the growth rate \( \log t \) in the pure plate equation (8) if \( n = 4 \) can be improved by the bounded one due to Fourier’s law of heat conduction from the thermoelastic plates (2); the decay rate \( t^{2 - \frac{n}{2}} \) for the thermoelastic plate model if \( n \geq 5 \) is contributed by two portions: the factor \( t^2 \) comes from the bi-Laplacian operator in the plate model (8) and
another factor $t^{-\frac{n}{2}}$ originates from the heat conduction since the Gaussian kernel. Surely, the heat conduction generates decay properties. In conclusion, by the summary in Table 1, we may answer the question proposed in the introduction.

It is worth noting that all large-time properties in Table 1 are optimal in the sense of same behaviors for upper bound and lower bound of the vertical displacement in the $L^2$ norm. In particular, concerning physical dimensions $n = 1, 2, 3$, we amazedly found that the large-time properties of the thermoelastic plate equations (4) are not influenced by the heat conduction.

Remark 2.3. The second contribution in Theorem 2.1 is to arrive at optimal estimates under a new nontrivial assumption $B_{u_1, \theta_0} \neq 0$ in (7) for initial datum. This condition (7) is also valid for either $u_1 \neq 0$ or $\theta_0 \neq 0$.

Remark 2.4. The optimal estimates for $(u, \theta)$ not only imply some underlying phenomena of the linear thermoelastic plates, but also prepare some crucial estimates to study global (in time) existence of solutions to nonlinear models. For examples, [1, 2] considered (4) with $N[u]$ on the right-hand side. Motivated by Theorem 2.1, by constructing a time-weighted Sobolev space

$$\mathcal{Y}_s(T) := \sup_{t \in [0, T]} \left( \left(1 + t\right)^{-1+\frac{n}{2}} ||u(t, \cdot)||_{L^2} + \left(1 + t\right)^{-1+\frac{n}{2}} ||\theta(t, \cdot)||_{L^2} \right)$$

It is optimal estimates of the solution.

$$+ \left(1 + t\right)^{-1+\frac{n}{2}} \left(||u(t, \cdot)||_{H^{2+s}} + ||\theta(t, \cdot)||_{H^s}\right),$$

we may prove global (in time) existence results for some nonlinear problem of thermoelastic plates equipping some nonlinear functions $N[u]$ (e.g. the power-type nonlinearities $|u|^p$ or $||D|^a u|^p$ with $0 < a < 2$) even $N[u, \theta]$, whose philosophy is standard and similar to the one in [32].

As a by-product, we also can describe asymptotic profiles for large-time. Before stating the corollary, let us introduce two functions

$$J_0(t, x) := \frac{1}{(a_0 - a_1)^2 + a_2^2} \mathcal{F}^{-1}_{\xi \to x} \left( \frac{1}{|\xi|^2} \left( e^{-a_0|\xi|^2 t} - \cos(a_2|\xi|^2 t) e^{-a_1|\xi|^2 t} \right) \right), \quad (9)$$

$$J_1(t, x) := \frac{1}{(a_0 - a_1)^2 + a_2^2} \mathcal{F}^{-1}_{\xi \to x} \left( \frac{\sin(a_2|\xi|^2 t)}{a_2 |\xi|^2} e^{-a_1|\xi|^2 t} \right), \quad (10)$$

and the auxiliary functions with $j = 0, 1$ serving for the temperature variable as follows:

$$J_{2+j}(t, x) := \mathcal{F}^{-1}_{\xi \to x} \left( (|\xi|^{-2} \partial^2_t + |\xi|^2) \hat{J}_j(t, |\xi|) \right),$$

* The terminology “property” specializes the time-dependent coefficient in the $L^2$ estimates of the solution.

| Dimensions         | $n \leq 3$ (Lower-dimensions) | $n = 4$ (Critical-dimension) | $n \geq 5$ (Higher-dimensions) |
|--------------------|--------------------------------|----------------------------|-------------------------------|
| Pure plate’s property | $t^{2-\frac{n}{2}}$            | $\log t$                   | $-$                           |
| Heat’s property (Fourier’s law) | $t^{-\frac{n}{2}}$ | $t^{-2}$ | $t^{-\frac{n}{2}}$ |
| Thermoelastic plate’s property | $t^{2-\frac{n}{2}}$ | $1$ | $t^{-\frac{n-4}{2}}$ |
| Crucial influence | Pure plates                  | Pure plates + Fourier’s law | Fourier’s law                  |

Table 1: Influence from the plate model and Fourier’s law of heat conduction

The terminology “property” specializes the time-dependent coefficient in the $L^2$ estimates of the solution.
where \( a_0 := (1 + \alpha_-)/3, a_1 := (2 - \alpha_-)/6 \) and \( a_2 = \sqrt{3\alpha_+}/6 \) are positive constants.

**Corollary 2.1.** Under the same assumption on initial datum as those in Theorem 2.1, the following refined estimates hold for \( t \gg 1 \):

\[
\|u(t, \cdot) - J_0(t, \cdot)P\psi_0 - J_1(t, \cdot)P\psi_1\|_{L_t^\infty L_x^2} \lesssim t^{\frac{1}{2} - \frac{4}{n}} \|(u_0, u_1, \theta_0)\|_{L^1 \times (L^1)^2},
\]

\[
\|\theta(t, \cdot) - J_2(t, \cdot)P\psi_0 - J_3(t, \cdot)P\psi_1\|_{L_t^\infty L_x^2} \lesssim t^{-\frac{1}{2} - \frac{n}{2}} \|(u_0, u_1, \theta_0)\|_{L^1 \times (L^1)^2},
\]

with the combined datum

\[
\Psi_0(x) := 2a_1u_1(x) + \theta_0(x) \quad \text{and} \quad \Psi_1(x) := (a_0^2 + a_1^2 - a_1^2)u_1(x) + (a_0 - a_1)\theta_0(x).
\]

**Remark 2.5.** The refined estimates in Corollary 2.1 tell us that the solution \((u, \theta)^T\) behaves like

\[
\left(\begin{array}{c} J_0(t, x) \\ \left(|D|^{-2}\partial_t^2 + |D|^2\right)J_0(t, x) \end{array}\right) \text{ and } \left(\begin{array}{c} J_1(t, x) \\ \left(|D|^{-2}\partial_t^2 + |D|^2\right)J_1(t, x) \end{array}\right) \text{ as } t \to \infty \text{ in the } L^2 \text{ framework, which gives the first-order profile.}
\]

These dominant functions \(J_0(t, x)\) and \(J_1(t, x)\) can be understood by a linear combination of the diffusion-plates

\[
\mathcal{F}_{\xi \to x}^{-1}\left(\cos(a_2|\xi|^2t)\right) e^{-a_1|\xi|^2t}, \quad \mathcal{F}_{\xi \to x}^{-1}\left(\frac{\sin(a_2|\xi|^2t)}{a_2|\xi|^2}\right) e^{-a_1|\xi|^2t}
\]

and the Gaussian kernel

\[
\mathcal{F}_{\xi \to x}^{-1}\left(e^{-a_0|\xi|^2t}\right) = \frac{1}{(4\pi a_0 t)^{n/2}} e^{-\frac{|\xi|^2}{4a_0 t}}
\]

with the singularity \(|\xi|^{-2}\) near \(|\xi| = 0\).

**Remark 2.6.** We may ensure an improvement \(t^{-\frac{1}{2}}\) as \(t \gg 1\) by subtracting the corresponding profiles of \((u(t, \cdot), \theta(t, \cdot))\) in the \(L^2\) norm for all dimensions. The effect is similar to generalized diffusion phenomena (see, for instance, [38]).

### 2.2 Asymptotic behaviors of solution for the small thermal parameter

In the first place, we state the global (in time) convergence results between \(u^\epsilon = u^\epsilon(t, x)\) of the thermoelastic plate equations (4) and \(u^0 = u^0(t, x)\) of the structurally damped plate equation (5) in the \(L^2\) framework.

**Theorem 2.2.** Let us consider the thermoelastic plate equations (4) with the small thermal parameter \(0 < \epsilon \ll 1\), and the structurally damped plate equation (5).

- Let \(u_0 \in H^2 \cap L^1\), \(u_1 \in L^2 \cap L^1\) and \(u_1 + \theta_0 \in L^2\). Then, the convergence of energy terms holds:

\[
\sup_{t > 0} \left(\|u^\epsilon(t, \cdot) - u^0(t, \cdot)\|_{L^2} + \|\Delta u^\epsilon(t, \cdot) - \Delta u^0(t, \cdot)\|_{L^2}\right) \leq C \epsilon \left(\|u_1 + \theta_0\|_{L^2} + \|u_0\|_{H^2 \cap L^1} + \|u_1\|_{L^2 \cap L^1}\right)
\]

for any \(n \geq 1\), where \(C > 0\) is independent of \(t\) and \(\epsilon\).
• Let \( u_0 \in L^2 \cap L^1, u_1 \in L^2 \cap L^{1,1} \) with \(|P_{u_1}| = 0\) and \( u_1 + \theta_0 \in \dot{H}^{-2} \). Then, the convergence of the vertical displacement holds:

\[
\sup_{t > 0} \left\| u^\varepsilon(t, \cdot) - u^0(t, \cdot) \right\|_{L^2} \leq C \varepsilon \left( \|u_1 + \theta_0\|_{\dot{H}^{-2}} + \|u_0\|_{L^2 \cap L^1} + \|u_1\|_{L^2 \cap L^{1,1}} \right)
\]

for any \( n \geq 3 \), where \( C > 0 \) is independent of \( t \) and \( \varepsilon \).

**Remark 2.7.** The restriction \( u_1 \in L^{1,1} \) with \(|P_{u_1}| = 0\) is to guarantee global (in time) convergence result for \( n \geq 3 \). Instead of weighted \( L^1 \) hypothesis in our theorem, by assuming \( u_1 \in L^1 \), we are able to get the convergence result for \( n \geq 5 \) only since

\[
\int_0^t \| \text{Int}(\xi)|\xi|^{-1} e^{-c|\xi|^2\tau} \|_{L^2}^2 \, d\tau \lesssim \int_0^\infty (1 + \tau)^{1-\frac{9}{2}} d\tau < \infty,
\]

when \( 1 - n/2 < -1 \) that is \( n > 4 \). Obviously, if one considers a stronger assumption \( u_1 \in \dot{H}^{-2} \), we can arrive at a global (in time) convergence for any \( n \geq 1 \).

**Remark 2.8.** In the second convergence result of Theorem 2.2, we require \( u_1 + \theta_0 \in \dot{H}^{-2} \) which is trivial providing that \( u_1 + \theta_0 \equiv 0 \). That is to say that the thermoelastic plate equations (4) with special data (for the temperature variable) \( \theta_0(x) = -u_1(x) \) may simplify our consideration of singular limits.

**Remark 2.9.** The last theorem shows that

\[
u^\varepsilon \to u^0 \quad \text{in} \quad L^\infty([0, \infty), H^2(\mathbb{R}^n)) \quad \text{for} \quad n \geq 3,
\]

\[
\Delta u^\varepsilon \to \Delta u^0 \quad \text{in} \quad L^\infty([0, \infty), L^2(\mathbb{R}^n)) \quad \text{for} \quad n \geq 1,
\]

as \( \varepsilon \downarrow 0 \) with the rate of convergence \( \varepsilon \). It implies a new relation for the thin plates equipping different dissipative mechanisms: thermal damping versus structural damping. By using the similar method to our proof additionally with the Hausdorff-Young inequality in the phase space, we expect some convergence results still hold in \( L^\infty([0, \infty), H^{2,p}(\mathbb{R}^n)) \) with \( 2 \leq p < \infty \), even in \( L^\infty([0, \infty) \times \mathbb{R}^n) \).

Before showing the next result to get a faster convergence rate, let us introduce a new function

\[
u^{I,1}(t,x) = -\frac{2}{\sqrt{3}} \int_0^t \sin \left( \frac{\sqrt{3}}{2} |D|^2 (t-\tau) \right) e^{-\frac{1}{2} |D|^2 (t-\tau)} \left( u^0_1(\tau, x) - \Delta u^0(\tau, x) \right) \, d\tau \quad (11)
\]

as a second-order profile for small \( \varepsilon > 0 \). Actually, \( \nu^{I,1}(t,x) \) solves the structurally damped plates with the source term \( \Delta u^0_1(t,x) - \Delta^2 u^0(t,x) \) on the right-hand side and vanishing initial conditions.

**Theorem 2.3.** Let us consider the thermoelastic plate equation (4) with \( \theta_0(x) \equiv -u_1(x) \) as well as the small thermal parameter \( 0 < \varepsilon \ll 1 \), and the structurally damped plate equation (5).

• Let \( u_0 \in H^2 \cap L^1 \) and \( u_1 \in L^2 \cap L^1 \). Then, the further convergence of energy terms holds:

\[
\sup_{t > 0} \left( \left\| u^\varepsilon(t, \cdot) - \nu^{I,1}_0(t, \cdot) - \varepsilon \nu^{I,1}(t, \cdot) \right\|_{L^2} + \left\| \Delta u^\varepsilon(t, \cdot) - \Delta u^0(t, \cdot) - \varepsilon \Delta u^{I,1}(t, \cdot) \right\|_{L^2} \right) \\
\leq C \varepsilon^2 \left( \|u_0\|_{H^2 \cap L^1} + \|u_1\|_{L^2 \cap L^1} \right)
\]

for any \( n \geq 1 \), where \( C > 0 \) is independent of \( t \) and \( \varepsilon \).
• Let \( u_0 \in L^2 \cap L^1 \) and \( u_1 \in L^2 \cap L^{1,1} \) with \(|P_{u_1}| = 0\). Then, the further convergence of the vertical displacement holds:

\[
\sup_{t>0} \left\| u'(t, \cdot) - u^0(t, \cdot) - \epsilon u^{I,1}(t, \cdot) \right\|_{L^2} \leq C \epsilon^2 (\|u_0\|_{L^2 \cap L^1} + \|u_1\|_{L^2 \cap L^{1,1}})
\]

for any \( n \geq 3 \), where \( C > 0 \) is independent of \( t \) and \( \epsilon \).

**Remark 2.10.** The last theorem shows that

\[
u^\epsilon \to u^0 + \epsilon u^{I,1} \quad \text{in} \quad L^\infty \left([0, \infty), H^2(\mathbb{R}^n)\right) \quad \text{for} \quad n \geq 3,
\]

\[
u^\epsilon_t \to u^0_t + \epsilon u^{I,1}_t, \quad \Delta \nu^\epsilon \to \Delta u^0 + \epsilon \Delta u^{I,1} \quad \text{in} \quad L^\infty \left([0, \infty), L^2(\mathbb{R}^n)\right) \quad \text{for} \quad n \geq 1,
\]
as \( \epsilon \downarrow 0 \) with the rate of convergence \( \epsilon^2 \). Moreover, comparing Theorem 2.2 with Theorem 2.3, by subtracting the additional term \( \epsilon u^{I,1}(t, \cdot) \) in the \( L^2 \) norm (or its derivatives), the rate of convergence has been improved by a factor \( \epsilon \). We conclude the accurate profile \( \epsilon u^{I,1} \) since it is the \( \epsilon \)-order term.

**Remark 2.11.** Concerning the convergence results of \( u'(t, \cdot) \) for lower dimensions \( n = 1, 2 \), we actually can demonstrate local (in time) convergence with the corresponding rates \( \epsilon \) or \( \epsilon^2 \) (when we subtract the second-order profiles), respectively. However, the global (in time) results are still open.

## 3 Large-time asymptotic behaviors of solutions

This section will contribute to the study of large-time asymptotic profiles for the linear thermoelastic plate equations (4), namely, the proofs of Theorem 2.1 and Corollary 2.1. To begin with, let us act the heat operator \( \partial_t - \Delta \) on (4)_1 and plug (4)_2 with \( \epsilon = 1 \) into the resultant, which lead to

\[
0 = (\partial_t - \Delta)(u_{tt} + \Delta^2 u) + \Delta(\partial_t - \Delta)\theta
\]

\[
= u_{ttt} - \Delta u_{tt} + 2\Delta^2 u_t - \Delta^3 u,
\]
carrying initial data \( u_{tt}(0, x) = -\Delta^2 u_0(x) - \Delta \theta_0(x) \). For this reason, we just need to investigate the following Cauchy problem for the third-order (in time) evolution equation:

\[
\begin{cases}
u_{ttt} - \Delta u_{tt} + 2\Delta^2 u_t - \Delta^3 u = 0, & x \in \mathbb{R}^n, \ t > 0, \\
u(0, x) = u_0(x), \ u_t(0, x) = u_1(x), \ u_{tt}(0, x) = -\Delta^2 u_0(x) - \Delta \theta_0(x), & x \in \mathbb{R}^n.
\end{cases}
\]

**Remark 3.1.** We analyze the classification of (12) from the scalar evolution equation’s viewpoint. Indeed, the partial differential operator appearing on (12), namely,

\[
\mathcal{L} := \partial_t^3 - \Delta \partial_t^2 + 2\Delta^2 \partial_t - \Delta^3,
\]
is not of Kovalevskian type (one may check [10, Section 3.1]). Its principal part in the sense of Petrovsky is given by \( \mathcal{L} \) itself carrying the symbol

\[
-i\tau^3 - |\zeta|^2 \tau^2 + 2i|\zeta|^4 \tau + |\zeta|^6 = -i\tau^2(\tau^2 - 2|\zeta|^4) - |\zeta|^2(\tau^2 - |\zeta|^4).
\]

If the linear partial differential operator \( \mathcal{L} \) is a 2-evolution operator (see [10, Chapter 3, Definition 3.2]), then the corresponding equation of (13) has only real and distinct roots for all \( \zeta \in \mathbb{R}^n \setminus \{0\} \). That is to say \( \tau^2 = 2|\zeta|^4 \) due to its imaginary coefficient whereas it yields a contradiction. For this reason, \( \mathcal{L} \) is not a 2-evolution operator, and we may not apply the general theory for such class of operators to derive a well-posedness result in \( L^2 \), particularly, the solution itself in lower dimensions. Therewith, we are going to show some relevant results requiring additional \( L^1 \) regularity.
After getting some results of the vertical displacement $u = u(t, x)$, according to

$$\theta = (-\Delta)^{-1}(u_{tt} + \Delta^2 u) = (-\Delta)^{-1}u_{tt} - \Delta u,$$  

(14)

we also can derive qualitative properties of the temperature $\theta = \theta(t, x)$ immediately. Again, the symbol of $(-\Delta)^{-1}$ is denoted by $|\xi|^2$.

3.1 Pointwise estimates of solutions in the Fourier space

As preparations for deriving $L^2$ estimates of solutions to the linear Cauchy problem (12), let us initially focus on the model in the Fourier space, namely,

$$\begin{cases}
\hat{u}_{ttt} + |\xi|^2\hat{u}_{tt} + 2|\xi|^4\hat{u}_t + |\xi|^6\hat{u} = 0, \\
\hat{u}(0, \xi) = \hat{u}_0(\xi), \quad \hat{u}_t(0, \xi) = \hat{u}_1(\xi), \quad \hat{u}_{tt}(0, \xi) = -|\xi|^4\hat{u}_0(\xi) + |\xi|^2\hat{\theta}_0(\xi),
\end{cases} \quad \xi \in \mathbb{R}^n, \ t > 0,$$

(15)

where the partial Fourier transform with respect to spatial variables $x$ was employed. Due to its corresponding characteristic equation

$$\lambda^3 + |\xi|^2\lambda^2 + 2|\xi|^4\lambda + |\xi|^6 = 0$$

has one real root and two complex (non-real) conjugate roots as follows:

$$\lambda_1 = a_0|\xi|^2 := -\frac{1}{2} + \frac{\alpha_-}{3}|\xi|^2,$$

$$\lambda_{2,3} = -a_1|\xi|^2 + ia_2|\xi|^2 := -\frac{2}{3} - \frac{\alpha_-}{6}|\xi|^2 \pm i\frac{\sqrt{3}\alpha_+}{6}|\xi|^2,$$

where two positive numbers $\alpha_{\pm}$ are defined in (6). For the readers’ convenience, we state $a_0 \approx 0.57$, $a_1 \approx 0.22$ and $a_2 \approx 1.31$. It is remarkable that $\lambda_1 < 0$ and $\Re\lambda_{2,3} < 0$ for any $\xi \in \mathbb{R}^n \setminus \{0\}$. Hence, for the sake of pairwise distinct characteristic roots, the solution to (15) is uniquely given by

$$\hat{u}(t, \xi) = \hat{K}_0(t, |\xi|)\hat{u}_0(\xi) + \hat{K}_1(t, |\xi|)\hat{u}_1(\xi) + \hat{K}_2(t, |\xi|)\left(-|\xi|^4\hat{u}_0(\xi) + |\xi|^2\hat{\theta}_0(\xi)\right)$$

$$= \left(\hat{K}_0(t, |\xi|) - |\xi|^4\hat{K}_2(t, |\xi|)\right)\hat{u}_0(\xi) + \hat{K}_1(t, |\xi|)\hat{u}_1(\xi) + |\xi|^2\hat{K}_2(t, |\xi|)\hat{\theta}_0(\xi),$$

(16)

in which the kernels in the Fourier space have the representations

$$\hat{K}_0(t, |\xi|) := \sum_{j=1,2,3} \exp(\lambda_j t) \prod_{k=1,2,3, k \neq j} \frac{\lambda_k}{\lambda_j - \lambda_k},$$

$$\hat{K}_1(t, |\xi|) := -\sum_{j=1,2,3} \exp(\lambda_j t) \prod_{k=1,2,3, k \neq j} \frac{\lambda_k}{\lambda_j - \lambda_k},$$

$$\hat{K}_2(t, |\xi|) := \sum_{j=1,2,3} \exp(\lambda_j t) \prod_{k=1,2,3, k \neq j} \frac{1}{\lambda_j - \lambda_k}.$$

From the Fourier transform of (14), the solution $\hat{\theta} = \hat{\theta}(t, \xi)$ is represented by

$$\hat{\theta}(t, \xi) = (|\xi|^{-2}\partial_t^2 + |\xi|^2) \left(\hat{K}_0(t, |\xi|) - |\xi|^4\hat{K}_2(t, |\xi|)\right)\hat{u}_0(\xi)$$

$$+ (|\xi|^{-2}\partial_t^2 + |\xi|^2)\hat{K}_1(t, |\xi|)\hat{u}_1(\xi) + (\partial_t^2 + |\xi|^4)\hat{K}_2(t, |\xi|)\hat{\theta}_0(\xi).$$
By lengthy but straightforward computations, one may observe
\[
\hat{K}_0(t, |\xi|) = \frac{a_1^2 + a_2^2}{(a_0 - a_1)^2 + a_2^2} e^{-a_0 |\xi|^2 t} + \frac{a_0^2 - 2a_0a_1}{(a_0 - a_1)^2 + a_2^2} \cos(a_2|\xi|^2 t)e^{-a_1 |\xi|^2 t} \\
+ \frac{a_0(a_0a_1 - a_1^2 + a_2^2)}{a_2[(a_0 - a_1)^2 + a_2^2]} \sin(a_2|\xi|^2 t)e^{-a_1 |\xi|^2 t},
\]
and with the aid of \(1 - \cos y = 2\sin^2(y/2)\),
\[
\hat{K}_1(t, |\xi|) = \frac{2a_1}{(a_0 - a_1)^2 + a_2^2 |\xi|^2} \left( e^{-a_0 |\xi|^2 t} - \cos(a_2|\xi|^2 t)e^{-a_1 |\xi|^2 t} \right) + \frac{a_2^2 - a_1^2}{(a_0 - a_1)^2 + a_2^2} \frac{\sin(a_2|\xi|^2 t)}{a_2 |\xi|^2} e^{-a_1 |\xi|^2 t} \\
= \frac{2a_1}{(a_0 - a_1)^2 + a_2^2 |\xi|^2} \left( (1 - \cos(a_2|\xi|^2 t)) e^{-a_1 |\xi|^2 t} + e^{-a_0 |\xi|^2 t} \right) \\
+ \frac{a_2^2 - a_1^2}{(a_0 - a_1)^2 + a_2^2} \frac{\sin(a_2|\xi|^2 t)}{a_2 |\xi|^2} e^{-a_1 |\xi|^2 t},
\]
moreover, by the similar approach to the last chain, we claim
\[
\hat{K}_2(t, |\xi|) = \frac{1}{(a_0 - a_1)^2 + a_2^2 |\xi|^4} \left( e^{-a_0 |\xi|^2 t} - \cos(a_2|\xi|^2 t)e^{-a_1 |\xi|^2 t} \right) + \frac{a_0 - a_1}{(a_0 - a_1)^2 + a_2^2} \frac{\sin(a_2|\xi|^2 t)}{a_2 |\xi|^4} e^{-a_1 |\xi|^2 t} \\
= \frac{2}{(a_0 - a_1)^2 + a_2^2 |\xi|^4} \left( (1 - \cos(a_2|\xi|^2 t)) e^{-a_1 |\xi|^2 t} + e^{-a_0 |\xi|^2 t} \right) \\
+ \frac{a_0 - a_1}{(a_0 - a_1)^2 + a_2^2} \frac{\sin(a_2|\xi|^2 t)}{a_2 |\xi|^4} e^{-a_1 |\xi|^2 t}.
\]
It provides a path for us to compensate the singularity $|\xi|^{-2}$ as $|\xi| \to 0$ by the recovering oscillating component, whose justification will be shown in the next subsection. This is exactly one of the advantages for using the reduction methodology. We have to point out that our approach also can be used even if we do not know the explicit form of representations. The alternative idea is to employ asymptotic expansions (see, for example, [3, 4] in third-order PDEs of acoustic waves).

Summarizing the last discussions and using

$$e^{-a_0|\xi|^2 t} \int_0^1 e^{(a_0-a_1)|\xi|^2 t} d\tau \leq e^{-a_1|\xi|^2 t}$$

since $a_0 - a_1 = \alpha_- / 2 > 0$, we are able to conclude some pointwise estimates.

**Proposition 3.1.** The following pointwise estimates for the solutions hold in the Fourier space:

$$\chi_{\text{int}}(\xi)|\hat{u}(t, \xi)| \lesssim \chi_{\text{int}}(\xi)e^{-c|\xi|^2 t}|\hat{u}_0(\xi)| + \chi_{\text{int}}(\xi) \left( t + \frac{|\sin(a_2|\xi|^2 t)|}{|\xi|^2} \right) e^{-c|\xi|^2 t} \left( |\hat{u}_1(\xi)| + |\hat{\theta}_0(\xi)| + \frac{1}{|\xi|^2} (|\hat{u}_1(\xi)| + |\hat{\theta}_0(\xi)|) \right),$$

$$\left(1 - \chi_{\text{int}}(\xi)\right)|\hat{u}(t, \xi)| \lesssim \left(1 - \chi_{\text{int}}(\xi)\right)e^{-ct} \left( |\hat{u}_0(\xi)| + \frac{1}{|\xi|^2} (|\hat{u}_1(\xi)| + |\hat{\theta}_0(\xi)|) \right),$$

and

$$\chi_{\text{int}}(\xi)|\hat{\theta}(t, \xi)| \lesssim \chi_{\text{int}}(\xi)|\xi|^2 e^{-c|\xi|^2 t}|\hat{u}_0(\xi)| + \chi_{\text{int}}(\xi)e^{-c|\xi|^2 t} \left( |\hat{u}_1(\xi)| + |\hat{\theta}_0(\xi)| \right),$$

$$\left(1 - \chi_{\text{int}}(\xi)\right)|\hat{\theta}(t, \xi)| \lesssim \left(1 - \chi_{\text{int}}(\xi)\right)e^{-ct} \left( |\xi|^2 |\hat{u}_0(\xi)| + |\hat{u}_1(\xi)| + |\hat{\theta}_0(\xi)| \right),$$

with a suitable constant $c > 0$.

**Proof.** From the explicit formulas of kernels, one derives

$$\chi_{\text{int}}(\xi) \left| \hat{K}_0(t, |\xi|) - |\xi|^4 \hat{K}_2(t, |\xi|) \right| \lesssim \chi_{\text{int}}(\xi) e^{-c|\xi|^2 t},$$

$$\chi_{\text{int}}(\xi) \left| \hat{K}_1(t, |\xi|) + \chi_{\text{int}}(\xi)|\xi|^2|\hat{K}_2(t, |\xi|) \right| \lesssim \chi_{\text{int}}(\xi) \left( t + \frac{|\sin(a_2|\xi|^2 t)|}{|\xi|^2} \right) e^{-c|\xi|^2 t},$$

as well as

$$\left(1 - \chi_{\text{int}}(\xi)\right) \left| \hat{K}_0(t, |\xi|) - |\xi|^4 \hat{K}_2(t, |\xi|) \right| \lesssim \left(1 - \chi_{\text{int}}(\xi)\right) e^{-ct},$$

$$\left(1 - \chi_{\text{int}}(\xi)\right) \left| \hat{K}_1(t, |\xi|) + \left(1 - \chi_{\text{int}}(\xi)\right)|\xi|^2|\hat{K}_2(t, |\xi|) \right| \lesssim \left(1 - \chi_{\text{int}}(\xi)\right) |\xi|^{-2} e^{-ct},$$

with a suitable constant $c > 0$. Then, we may arrive at the desired estimate of $\hat{u}(t, \xi)$ by plugging into the representation (16). For another, due to some straightforward computations that

$$|\partial^2_t \hat{K}_j(t, |\xi|)| \lesssim |\xi|^{4-2j} e^{-c|\xi|^2 t}$$

for $j = 0, 1, 2$, we can employ

$$|\hat{\theta}(t, \xi)| \leq \frac{1}{|\xi|^2} |\hat{u}_t(t, \xi)| + |\xi|^2 |\hat{u}(t, \xi)|$$

to derive the aim estimate of $\hat{\theta}(t, \xi)$. The proof is complete eventually. \qed
Next, let us introduce two multipliers in the Fourier space (they are the Fourier transform of (9) and (10), respectively) as follows:

\[
\hat{J}_0(t, |\xi|) := \frac{1}{(a_0 - a_1)^2 + a_2^2 |\xi|^2} \left( e^{-a_0|\xi|^2 t} - \cos(a_2|\xi|^2 t) e^{-a_1|\xi|^2 t} \right),
\]

\[
\hat{J}_1(t, |\xi|) := \frac{1}{(a_0 - a_1)^2 + a_2^2} \frac{\sin(a_2|\xi|^2 t)}{a_2 |\xi|^2} e^{-a_1|\xi|^2 t},
\]

and those generalized from the operator $|D|^{-2} \partial_t^2 + |D|^2$ for $j = 0, 1$ as follows:

\[
\hat{J}_{2+j}(t, |\xi|) := (|\xi|^{-2} \partial_t^2 + |\xi|^2) \hat{J}_j(t, |\xi|).
\] (18)

Then, we can arrive at the next result easily.

**Proposition 3.2.** The following refined estimates for the solutions hold in the Fourier space:

\[
\begin{align*}
\chi_{\text{int}}(\xi) \left| \hat{u}(t, \xi) - \hat{J}_0(t, |\xi|) \hat{\Psi}_0(\xi) - \hat{J}_1(t, |\xi|) \hat{\Psi}_1(\xi) \right| &\lesssim \chi_{\text{int}}(\xi) e^{-c|\xi|^2 t} |\hat{u}_0(\xi)|, \\
\chi_{\text{int}}(\xi) \left| \hat{\theta}(t, \xi) - \hat{J}_2(t, |\xi|) \hat{\Psi}_0(\xi) - \hat{J}_3(t, |\xi|) \hat{\Psi}_1(\xi) \right| &\lesssim \chi_{\text{int}}(\xi) |\xi|^2 e^{-c|\xi|^2 t} |\hat{u}_0(\xi)|,
\end{align*}
\]

with a suitable constant $c > 0$, where the combined datum in the Fourier space are defined by

\[
\hat{\Psi}_0(\xi) := 2a_1 \hat{u}_1(\xi) + \hat{\theta}_0(\xi) \quad \text{and} \quad \hat{\Psi}_1(\xi) := (a_0^2 + a_2^2 - a_1^2) \hat{u}_1(\xi) + (a_0 - a_1) \hat{\theta}_0(\xi).
\]

**Remark 3.3.** We realize the leading terms of $\hat{\theta}(t, \xi)$ being constituted by the linear combination of

\[
e^{-a_0|\xi|^2 t}, \quad \cos(a_2|\xi|^2 t) e^{-a_1|\xi|^2 t}, \quad \sin(a_2|\xi|^2 t) e^{-a_1|\xi|^2 t}.
\]

Moreover, the leading terms of $\hat{\theta}(t, \xi)$ are the pseudo-differential operator (symbol’s sense) $|\xi|^{-2} \partial_t^2 + |\xi|^2$ acting on those for $\hat{u}(t, \xi)$, which is strongly motivated by the first equation of (4).

### 3.2 Sharp estimates for the Fourier multipliers

In the last part, we found the vital Fourier multipliers $\hat{J}_0(t, |\xi|)$ and $\hat{J}_1(t, |\xi|)$ to describe asymptotic profiles of the vertical displacement in the phase space. So, our first lemma is to analyze them finely in the sense of optimal estimates as large-time.

**Proposition 3.3.** The following optimal estimates for the multipliers hold:

\[
\begin{align*}
|| \chi_{\text{int}}(D) J_0(t, |D|) ||_{L^2} &\simeq t^{1-\frac{4}{n}}, \\
|| \chi_{\text{int}}(D) J_1(t, |D|) ||_{L^2} &\simeq t^{1-\frac{4}{n}},
\end{align*}
\]

as $t \gg 1$ for any $n \geqslant 1$.

**Proof.** Let us recall

\[
\sup_{z \neq 0} |z^{-1} \sin z| =: L_0 > 0.
\]
By the same approach as the one in (17), with some positive constants $\tilde{c}$, we notice

$$
\|\chi_{\text{int}}(D)J_0(t, |D|)\|_{L^2}^2 + \|\chi_{\text{int}}(D)J_1(t, |D|)\|_{L^2}^2
\lesssim \left\| \chi_{\text{int}}(\xi) \frac{\sin(c|\xi|^2t)}{|\xi|^2} e^{-c|\xi|^2t} \right\|^2_{L^2} + t^2 \left\| \chi_{\text{int}}(\xi)e^{-c|\xi|^2t} \right\|^2_{L^2}
\lesssim \int_0^{\varepsilon_0} \left| \sin(\tilde{c}r^2t) \right|^2 r^{-n-5} e^{-2cr^2t} \, dr + t^2 \int_0^{\varepsilon_0} r^{-n-1} e^{-2cr^2t} \, dr
\lesssim t^2 \int_0^{\varepsilon_0} \left( \frac{\sin(\tilde{c}r^2t)}{|\tilde{c}r^2t|^2} + 1 \right) \left( r^2 t \right)^{\frac{1}{2}} e^{-2cr^2t} d(r^2 t)^{\frac{1}{2}}
\lesssim t^2 \int_0^{\varepsilon_0 \sqrt{T}} \left( \frac{\sin(\tilde{c}w^2)}{|\tilde{c}w^2|^2} + 1 \right) w^{-n-1} e^{-2cw^2} \, dw
\lesssim t^2 \int_0^{\varepsilon_0 \sqrt{T}} w^{-n-1} e^{-2cw^2} \, dw \lesssim t^2 - \frac{\pi}{2}
$$

(19)

for any $n \geq 1$ and any $t \gg 1$, where we applied polar coordinates and the change of variable $w = r\sqrt{t}$ in the last chains.

The lower bound for $J_1(t, |D|)$ is estimated easily by

$$
\|\chi_{\text{int}}(\xi)\tilde{J}_1(t, |\xi|)\|_{L^2}^2 \gtrsim t^2 \int_0^{\varepsilon_0} \left| \frac{\sin(a_2r^2t)}{|a_2r^2t|^2} r^{-n-1} e^{-2a_2r^2t} \right| \, dr
\gtrsim t^2 \int_0^{\varepsilon_0 \sqrt{T}} \left| \frac{\sin(a_2w^2)}{|a_2w^2|} \right|^2 w^{-n-1} e^{-2a_2w^2} \, dw \gtrsim t^2 - \frac{\pi}{2},
$$

since the fact that

$$
\lim_{t \to \infty} \int_0^{\varepsilon_0 \sqrt{T}} \left| \frac{\sin(a_2w^2)}{|a_2w^2|} \right|^2 w^{-n-1} e^{-2a_2w^2} \, dw < \infty.
$$

Finally, let us turn to the lower bound estimates for $J_0(t, |D|)$. The proof is harder than those in [4] with the reason of $a_0 > a_1$. There exist two positive constants $0 < k_0 < k_1$ such that

$$
\frac{\pi}{2} < a_2k_0^2 \leq a_2r^2t \leq a_2k_1^2 < \pi \text{ for any } r \in [k_0/\sqrt{T}, k_1/\sqrt{T}],
$$

(20)

and one of the conditions hold: $\Pi_0 > 0$ or $\Pi_0 < 0$, where

$$
\Pi_0 := \frac{1}{2} \left| \cos(a_2k_0^2) \right|^2 e^{-2a_2k_0^2} - e^{-2a_0k_0^2}.
$$

Focusing on Figure 1, the domain with red color stands for $\Pi_0 < 0$, and the blue one indicates $\Pi_0 > 0$ under the rule (20). Next, we will choose $(k_0, k_1)$ in the range of blue color if $n \geq 4$ and the red one if $n \leq 3$.

As a consequence, by using a suitable triangle inequality $2|f - g|^2 \geq |f|^2 - 2|g|^2$ and monotonic
increasing property for the cosine function since \( a_2 r^2 t \in (\pi/2, \pi) \), we may derive

\[
\left\| \chi_{\text{int}}(\xi) \hat{J}_0(t, |\xi|) \right\|_{L^2}^2 \geq \left\| \frac{1}{|\xi|^2} \cos(a_2 |\xi|^2 t) e^{-a_1 |\xi|^2 t} - \frac{1}{|\xi|^2} e^{-a_0 |\xi|^2 t} \right\|_{L^2(k_0/\sqrt{7} \leq |\xi| \leq k_1/\sqrt{7})}^2
\]

\[
\geq \omega_n \int_{k_0/\sqrt{7}}^{k_1/\sqrt{7}} \cos(a_2 r^2 t)|2 r^{-n-5} e^{-2a_1 r^2 t} \, dr - \omega_n \int_{k_0/\sqrt{7}}^{k_1/\sqrt{7}} r^{-n-5} e^{-2a_0 r^2 t} \, dr
\]

\[
\geq \omega_n \int_{k_0/\sqrt{7}}^{k_1/\sqrt{7}} r^{-5} \left( \frac{1}{2} \cos(a_2 k_0^2)^2 e^{-2a_1 k_0^2} - e^{-2a_0 k_0^2} \right)
\]

\[
\geq \begin{cases} 
\omega_n (k_1^{-4} - k_0^{-4}) \Pi_0 r^{-\frac{a-4}{2}} & \text{if } n \neq 4,
\omega_4 \log(k_1/k_0) \Pi_0 & \text{if } n = 4,
\end{cases}
\]

\[
\geq t^{2-\frac{4}{n}}
\]

for any \( n \geq 1 \) because \( \Pi_0/(n-4) > 0 \) if \( n \neq 4 \) and \( \Pi_0 > 0 \) if \( n = 4 \) always hold, where \( \omega_n \) denoted the \((n-1)\) dimensional measure of the unite sphere. Summarizing all derived estimates, our proof is totally finished.

Moreover, we state another upper bound estimate. Its proof is the same as those of Proposition 3.3 that is

\[
\int_0^{\xi_0} |\sin(r^2 t)|^2 r^{-n-3} e^{-2r^2 t} \, dr \lesssim t^{1-\frac{4}{n}} \int_0^{\xi_0 \sqrt{7}} |\sin(w^2)|^2 w^{n+1} e^{-2w^2} w^4 \, dw \lesssim t^{1-\frac{4}{n}},
\]

so we omit its derivation in detail.

**Corollary 3.1.** The following upper bound estimates for the multiplier hold:

\[
\left\| \chi_{\text{int}}(D) \left| \sin(|D|^2 t) \right| e^{-c|D|^2 t} \right\|_{L^2} \lesssim t^{\frac{1}{2} - \frac{4}{n}},
\]

as \( t \gg 1 \) for any \( n \gg 1 \).

Recalling the structure of multipliers in Remark 3.3, different from \( \hat{J}_{1,2}(t, |\xi|) \) in Proposition 3.3, there is no any singularity as \( |\xi| \to 0 \). Then, by applying the derived results in [12, 14], we can obtain the next two optimal estimates without any difficulty.
Proposition 3.4. The following optimal estimates for the multipliers hold:

\[ \| \chi_{\text{int}}(D) J_3(t, |D|) \|_{L^2} \approx t^{-\frac{n}{4}}, \]
\[ \| \chi_{\text{int}}(D) J_4(t, |D|) \|_{L^2} \approx t^{-\frac{n}{4}}, \]

as \( t \gg 1 \) for any \( n \geq 1 \).

3.3 Optimal estimates and asymptotic profiles: Proof of Theorem 2.1

Motivated by the previous estimates (19), some applications of the Plancherel theorem and the Hausdorff-Young inequality in Proposition 3.1 yield immediately that

\[ \| u(t, \cdot) \|_{L^2} \lesssim \| \chi_{\text{int}}(\xi) \hat{u}(t, \xi) \|_{L^2} + \| (1 - \chi_{\text{int}}(\xi)) \hat{u}(t, \xi) \|_{L^2} \]
\[ \leq \| \chi_{\text{int}}(\xi) e^{-c|\xi|^2t} \|_{L^2} \| u_0 \|_{L^1} + \| \chi_{\text{int}}(\xi) \left( t + \frac{|\sin(a_2|\xi|^2t)|}{|\xi|^2} \right) e^{-c|\xi|^2t} \|_{L^2} \| u_1 \|_{L^1} + \| \theta_0 \|_{L^1} \]
\[ + e^{-ct} (\| u_0 \|_{L^2} + \| u_1 \|_{L^2} + \| \theta_0 \|_{L^2}) \]
\[ \lesssim t^{-\frac{n}{4}} \| u_0 \|_{L^2 \cap L^1} + t^{1-\frac{n}{4}} (\| u_1 \|_{L^2 \cap L^1} + \| \theta_0 \|_{L^2 \cap L^1}), \]

and

\[ \| \theta(t, \cdot) \|_{L^2} \lesssim \| \chi_{\text{int}}(\xi) |\xi|^2 e^{-c|\xi|^2t} \|_{L^2} \| u_0 \|_{L^1} + \| \chi_{\text{int}}(\xi) e^{-c|\xi|^2t} \|_{L^2} (\| u_1 \|_{L^1} + \| \theta_0 \|_{L^1}) \]
\[ + e^{-ct} (\| u_0 \|_{H^2} + \| u_1 \|_{L^2} + \| \theta_0 \|_{L^2}) \]
\[ \lesssim t^{-1-\frac{n}{4}} \| u_0 \|_{H^2 \cap L^1} + t^{-\frac{n}{4}} (\| u_1 \|_{L^2 \cap L^1} + \| \theta_0 \|_{L^2 \cap L^1}), \]

as \( t \gg 1 \), where we applied Proposition 3.3 from the upper side. By an analogous way as well as Proposition 3.2, we obtain

\[ \| u(t, \cdot) - J_0(t, |D|) \Psi_0 - J_1(t, |D|) \Psi_1 \|_{L^2} = \| \left( \hat{K}_0(t, |\xi|) - |\xi|^2 \hat{K}_2(t, |\xi|) \right) \hat{u}_0(\xi) \|_{L^2} \]
\[ \lesssim t^{-\frac{n}{2}} \| u_0 \|_{L^1}, \quad (21) \]

and

\[ \| \theta(t, \cdot) - J_2(t, |D|) \Psi_0 - J_3(t, |D|) \Psi_1 \|_{L^2} = \| \left| \xi \right|^2 \hat{\theta}_0(\xi) + |\xi|^2 \left( \hat{K}_0(t, |\xi|) - |\xi|^2 \hat{K}_2(t, |\xi|) \right) \hat{u}_0(\xi) \|_{L^2} \]
\[ \lesssim t^{-1-\frac{n}{2}} \| u_0 \|_{L^1}, \]

as \( t \gg 1 \). Employing the triangle inequality, one has

\[ \| u(t, \cdot) - J_0(t, \cdot) P \Psi_0 - J_1(t, \cdot) P \Psi_1 \|_{L^2} \leq \| \hat{u}(t, \xi) - \hat{J}_0(t, |\xi|) \hat{\Psi}_0(\xi) - \hat{J}_1(t, |\xi|) \hat{\Psi}_1(\xi) \|_{L^2} \]
\[ + \left\| \hat{J}_0(t, |\xi|) \left( \hat{\Psi}_0(\xi) - P \Psi_0 \right) \right\|_{L^2} + \left\| \hat{J}_1(t, |\xi|) \left( \hat{\Psi}_1(\xi) - P \Psi_1 \right) \right\|_{L^2} \]
\[ =: I_2(t) + I_3(t) + I_4(t). \]

Indeed, the derived estimate (21) implies

\[ I_2(t) \lesssim t^{-\frac{n}{2}} \| u_0 \|_{L^1}. \]
Because of the decomposition of combined datum \( \hat{\Psi}_j(\xi) - P_{\Psi_j} = A_j(\xi) - iB_j(\xi) \) for \( j = 0, 1 \) with
\[
A_j(\xi) := \int_{\mathbb{R}^n} \Psi_j(x) \left( \cos(x \cdot \xi) - 1 \right) dx \quad \text{and} \quad B_j(\xi) := \int_{\mathbb{R}^n} \Psi_j(x) \sin(x \cdot \xi) dx.
\]

By using [12, Lemma 2.2], the auxiliary functions can be estimated by
\[
|A_j(\xi)| + |B_j(\xi)| \lesssim |\xi| \|\Psi_j\|_{L^{1,1}}
\]
for \( j = 0, 1 \). Therefore, similarly to the second line of (19), it gives
\[
I_3(t) + I_4(t) \lesssim \sum_{j=0,1} \|\chi_{\text{int}}(\xi)|\xi|\tilde{J}_j(t, |\xi|)\|L^\infty\|\Psi_j\|_{L^{1,1}}
\lesssim \left( \|\chi_{\text{int}}(\xi)\frac{\sin(c|\xi|^2 t)}{|\xi|}e^{-c|\xi|^2 t}\|_{L^2} + t \|\chi_{\text{int}}(\xi)\frac{e^{-c|\xi|^2 t}}{|\xi|}\|_{L^2} \right) (\|u_1\|_{L^{1,1}} + \|\theta_0\|_{L^{1,1}})
\lesssim t^{\frac{3}{2} - \frac{n}{4}} (\|u_1\|_{L^{1,1}} + \|\theta_0\|_{L^{1,1}})
\]
as \( t \gg 1 \), in which we considered Corollary 3.1. In other words,
\[
\|u(t, \cdot) - J_0(t, \cdot)P_{\Psi_0} - J_1(t, \cdot)P_{\Psi_1}\|_{L^\infty} \lesssim t^{-\frac{n}{4}} \|u_0\|_{L^1} + t^{\frac{1}{2} - \frac{n}{4}} (\|u_1\|_{L^{1,1}} + \|\theta_0\|_{L^{1,1}}) \tag{22}
\]
for large-time. Similarly, one may also obtain
\[
\|\theta(t, \cdot) - J_2(t, \cdot)P_{\Psi_0} - J_3(t, \cdot)P_{\Psi_1}\|_{L^\infty} \lesssim t^{-\frac{n}{4}} \|u_0\|_{L^1} + t^{\frac{1}{2} - \frac{n}{4}} (\|u_1\|_{L^{1,1}} + \|\theta_0\|_{L^{1,1}}) \tag{23}
\]
for large-time. The last two estimates demonstrate Corollary 2.1. Lastly, since (22), (23) as well as Propositions 3.3, 3.4 holding for large-time, Minkowski’s inequality results
\[
\|u(t, \cdot)\|_{L^2} \geq \|J_0(t, \cdot)\|_{L^\infty} \|P_{\Psi_0} + \|J_1(t, \cdot)\|_{L^\infty} \|P_{\Psi_1}\|_2 - \|u(t, \cdot) - J_0(t, \cdot)P_{\Psi_0} - J_1(t, \cdot)P_{\Psi_1}\|_{L^\infty}
\gtrsim t^{\frac{3}{2} - \frac{n}{4}} (\|P_{\Psi_0}\| + \|P_{\Psi_1}\|) - t^{-\frac{n}{4}} \|u_0\|_{L^1} - t^{\frac{1}{2} - \frac{n}{4}} (\|u_1\|_{L^{1,1}} + \|\theta_0\|_{L^{1,1}})
\gtrsim t^{\frac{3}{2} - \frac{n}{4}} (\|P_{\Psi_0}\| + \|P_{\Psi_1}\|)
\]
for \( t \gg 1 \) due to \( |P_{\Psi_0}| + |P_{\Psi_1}| \neq 0 \) and \( u_0 \in L^1 \) as well as \( u_1, \theta_0 \in L^{1,1} \), moreover,
\[
\|\theta(t, \cdot)\|_{L^2} \gtrsim t^{-\frac{n}{4}} (\|P_{\Psi_0}\| + \|P_{\Psi_1}\|) - t^{\frac{1}{2} - \frac{n}{4}} \|u_0\|_{L^1} - t^{\frac{3}{2} - \frac{n}{4}} (\|u_1\|_{L^{1,1}} + \|\theta_0\|_{L^{1,1}})
\gtrsim t^{-\frac{n}{4}} (\|P_{\Psi_0}\| + \|P_{\Psi_1}\|).
\]

In conclusion, rewriting the condition in an elegant form, our proof of Theorem 2.1 is complete.

## 4 Singular limits problem for the vanishing thermal parameter

Let us turn to the second purpose of this paper. For the sake of readability and exactitude, we restate the classical thermoelastic plate equations (4) by the next way:

\[
\begin{cases}
\dddot{u} + \Delta^2 u + \Delta \theta = 0, & x \in \mathbb{R}^n, \ t > 0, \\
\epsilon \ddot{\theta} - \Delta \theta - \Delta u_t = 0, & x \in \mathbb{R}^n, \ t > 0, \\
u(0, x) = u_0(x), \ u_t(0, x) = u_1(x), \ \theta(0, x) = \theta_0(x), & x \in \mathbb{R}^n,
\end{cases}
\tag{24}
\]
where $\epsilon > 0$ denotes the thermal parameter. We recall that the formal limit model with $\epsilon = 0$ was introduced in (5). Thus, our aim is to demonstrate $u^\epsilon \to u^0$ in some norms, and find a correct second-order profile for small $\epsilon$. Before rigorously demonstrating the desired convergence results, it is important to understand the influence of the small thermal parameter $\epsilon$, which will provide some profiles in a formal way. Throughout this section, we set $\epsilon > 0$ to be a small number, and $C > 0$ to be a suitable constant independent of $t$ as well as $\epsilon$.

4.1 Formal analysis of the thermoelastic plate equations

In order to analyze the influence of small parameter and asymptotic profiles of solution to (24) with respect to $0 < \epsilon \ll 1$, motivated by the boundary layer theory and the multi-scale analysis (see, for example, [11]), the solution $(u^\epsilon, \theta^\epsilon)$ of the thermoelastic plate equations (24) owns the following expansions:

\[
\begin{align*}
  u^\epsilon(t, x) &= \sum_{j \geq 0} \epsilon^j \left( u^{I,j}(t, x) + u^{L,j}(\epsilon^{-1}t, x) \right), \\
  \theta^\epsilon(t, x) &= \sum_{j \geq 0} \epsilon^j \left( \theta^{I,j}(t, x) + \theta^{L,j}(\epsilon^{-1}t, x) \right),
\end{align*}
\]

where all terms in the last representations are assumed to be sufficiently smooth. Here, $u^{I,j} = u^{I,j}(t, x)$, $\theta^{I,j} = \theta^{I,j}(t, x)$ stand for the dominant profiles for each order expansion, and $u^{L,j} = u^{L,j}(z, x)$, $\theta^{L,j} = \theta^{L,j}(z, x)$ with $z := \epsilon^{-1}t$ denote the profiles decaying to zero as $z \to \infty$. These components will be fixed later.

By employing the perturbation theory, we plug the new ansatz (25) into two equation of (24). We immediately arrive at

\[
\begin{align*}
  &\sum_{j \geq 0} \epsilon^j (u^{I,j}_{tt} + \epsilon^{-2} u^{L,j}_{zz}) + \sum_{j \geq 0} \epsilon^j (\Delta^2 u^{I,j} + \Delta^2 u^{L,j}) + \sum_{j \geq 0} \epsilon^j (\Delta \theta^{I,j} + \Delta \theta^{L,j}) = 0, \\
  &\sum_{j \geq 0} \epsilon^j (\epsilon \theta^{I,j}_{t} + \theta^{L,j}_{z}) - \sum_{j \geq 0} \epsilon^j (\Delta \theta^{I,j} + \Delta \theta^{L,j}) - \sum_{j \geq 0} \epsilon^j (\Delta u^{I,j}_{t} + \epsilon^{-1} \Delta u^{L,j}_{z}) = 0.
\end{align*}
\]

Matching the order of $\epsilon^{j+1}$ (here, we consider $j \in \mathbb{Z}$ formally), we claim

\[
\begin{align*}
  &u^{I,j+1}_{tt} + \Delta^2 u^{I,j+1} + \Delta \theta^{I,j+1} = -u^{L,j+3}_{zz} - \Delta^2 u^{L,j+1} - \Delta \theta^{L,j+1}, \\
  &\theta^{I,j} - \Delta \theta^{I,j+1} - \Delta u^{I,j+1}_{t} = -\theta^{L,j+1} + \Delta \theta^{L,j+1} + \Delta u^{L,j+2}_{z}.
\end{align*}
\]

Letting $\epsilon \downarrow 0$, i.e. $z \to \infty$, the right-hand sides of (26) tend to zero so that

\[
\begin{align*}
  &u^{I,j+1}_{tt} + \Delta^2 u^{I,j+1} + \Delta \theta^{I,j+1} = 0, \\
  &\theta^{I,j} - \Delta \theta^{I,j+1} - \Delta u^{I,j+1}_{t} = 0,
\end{align*}
\]

(27)

carrying $u^{I,j} \equiv 0 \equiv \theta^{I,j}$ if $j$ becomes negative, as well as

\[
\begin{align*}
  &u^{L,j+3}_{zz} + \Delta^2 u^{L,j+1} + \Delta \theta^{L,j+1} = 0, \\
  &\theta^{L,j+1} - \Delta \theta^{L,j+1} - \Delta u^{L,j+2}_{z} = 0,
\end{align*}
\]

(28)

carrying $u^{L,j} \equiv 0 \equiv \theta^{L,j}$ if $j$ becomes negative. Later we will derive the profiles by carefully determining their initial conditions.
From the initial conditions in the original problem (24) with the ansatz (25), one notices

\[
\begin{cases}
  u_0(x) = u^{I,0}(0, x) + u^{L,0}(0, x) + \sum_{j \geq 1} \epsilon^j (u^{I,j}(0, x) + u^{L,j}(0, x)), \\
u_1(x) = \epsilon^{-1} u^{L,0}_{z}(0, x) + u^{I,0}_{t}(0, x) + u^{L,1}_{z}(0, x) + \sum_{j \geq 1} \epsilon^j (u^{I,j}_{t}(0, x) + u^{L,j+1}_{z}(0, x)), \\
\theta_0(x) = \theta^{I,0}(0, x) + \theta^{L,0}(0, x) + \sum_{j \geq 1} \epsilon^j (\theta^{I,j}(0, x) + \theta^{L,j}(0, x)).
\end{cases}
\]

(29)

Under our consideration that three initial datum in (24) are independent of \(\epsilon\) in general, we may naturally take vanishing value of \(u^{I,j}(0, x), u^{L,j}(0, x), u^{I,j}_{t}(0, x), u^{L,j+1}_{z}(0, x)\) as well as \(\theta^{I,j}(0, x), \theta^{L,j}(0, x)\) for any \(j \geq 1\). The condition (29) can be reset as

\[
\begin{cases}
  u^{I,0}_{t}(0, x) + u^{L,0}_{t}(0, x) = u_0(x), \\
u^{I,0}_{t}(0, x) + u^{L,1}_{z}(0, x) = u_1(x), \\
\theta^{I,0}_{t}(0, x) + \theta^{L,0}_{t}(0, x) = \theta_0(x).
\end{cases}
\]

Indeed, we cannot fix all initial data in the present step. In the next subsection, we will follow the methodology of the derivation for the Prandtl equation in the boundary layer theory to get the explicit value of initial conditions.

### 4.2 Formal derivation of asymptotic profiles for the small thermal parameter

To begin with our deductions, let us take \(j = -1\) in (27), and \(j = -3, -2\) in (28), respectively, to find easily

\[
\begin{cases}
  u^{I,0}_{tt} + \Delta^2 u^{I,0} + \Delta \theta^{I,0} = 0, & x \in \mathbb{R}^n, \ t > 0, \\
-\Delta \theta^{I,0} - \Delta u^{I,0}_{t} = 0, & x \in \mathbb{R}^n, \ t > 0, \\
u^{I,0}(0, x) = u^{I,0}_0(x), \ u^{I,0}_{t}(0, x) = u^{I,0}_{1}(x), & x \in \mathbb{R}^n,
\end{cases}
\]

(30)

in which \((30)_1+(30)_2\) yields the single plate equation with structural damping \(u^{I,0}_{tt} + \Delta^2 u^{I,0} - \Delta u^{I,0}_{t} = 0\), moreover,

\[
\begin{cases}
  u^{L,0}_{zz} = 0, \ u^{L,1}_{zz} = 0, \ \Delta u^{L,0}_{z} = 0, \ & x \in \mathbb{R}^n, \ z > 0, \\
u^{L,0}_{z}(0, x) = u^{L,0}_{0}(x), \ u^{L,0}_{z}(0, x) = u^{L,1}_{z}(0, x) = 0, \ u^{L,1}_{z}(0, x) = u^{L,1}_{1}(x), & x \in \mathbb{R}^n,
\end{cases}
\]

(31)

equipping \(u^{I,0}(0, x) + u^{L,0}(0, x) = u_0(x)\) and \(u^{I,0}_{t}(0, x) + u^{L,1}_{z}(0, x) = u_1(x)\). By constructing two new functions \(u^{P,j} = u^{P,j}_{t}(t, x)\) with \(j = 0, 1\) such that

\[
u^{P,0}(t, x) := u^{L,0}(\epsilon^{-1} t, x) + u^{I,0}_{0}(x), \ u^{P,1}(t, x) := \epsilon u^{L,1}(\epsilon^{-1} t, x) + t u^{I,0}_{1}(x),
\]

(32)

they satisfy the following equations (we used (31), actually):

\[
\begin{cases}
  \partial_t^2 u^{P,0} = 0, \ \partial_t^2 u^{P,1} = 0, & x \in \mathbb{R}^n, \ t > 0, \\
u^{P,0}_{t}(0, x) = u_0(x), \ u^{P,0}_{t}(0, x) = u^{P,1}_{t}(0, x) = 0, \ u^{P,1}_{t}(0, x) = u_1(x), & x \in \mathbb{R}.
\end{cases}
\]

The solutions of the previous models can be easily obtained by \(u^{P,0}(t, x) = u_0(x)\) and \(u^{P,1}(t, x) = tu_1(x)\). Again with \(\epsilon \downarrow 0\) in (32), it gives \(u^{I,0}_{0}(x) = u_0(x)\) as well as \(u^{I,0}_{1}(x) = u_1(x)\), respectively.
Coming back to (31), due to all vanishing initial datum, we arrive at \( u^{L,0}(z, x) \equiv 0 \equiv u^{L,1}(z, x) \) for any \( z \geq 0 \) and \( x \in \mathbb{R}^n \). Furthermore, we claim \( u^{I,0}(t, x) \equiv u^0(t, x) \) which is the solution to the structurally damped plate equation (5). The formal expansion (25) so far can be updated by

\[
\begin{align*}
\left\{
\begin{array}{l}
\epsilon \frac{\partial^2 u}{\partial t^2} = \Delta u + \theta \epsilon^{-1} u + \sum_{j \geq 1} \epsilon^j u^{I,j}(t, x), \\
\theta(t, x) = \theta^I, \quad x \in \mathbb{R}^n, \quad t > 0,
\end{array}
\right.
\end{align*}
\]

(33)

The second relation in (30) may say \( \theta^I(t, x) = -u^I(t, x) \), therefore, \( \theta^I(0, x) =: \theta^0_0(x) = -u_1(x) \).

In addition, taking \( j = 1 \) in (28), we are able to find

\[
\begin{align*}
\left\{
\begin{array}{l}
u_{tzz}^I = -\Delta \theta^L_0, \\
\theta^L_0 = -\Delta \theta^L_0 = 0, \\
u_{tzz}^I(0, x) = u_{tzz}^I(0, x) = 0, \quad \theta^L_0(0, x) = u_1(x) + \theta_0(x), \quad x \in \mathbb{R}^n,
\end{array}
\right.
\end{align*}
\]

where we employed \( \theta^0_0(0, x) = \theta_0(x) - \theta^0_0(x) = u_1(x) + \theta_0(x) \). The second equation is the standard heat model which can be uniquely solved by

\[
\theta^L_0(\epsilon^{-1} t, x) = e^{-|D|^2 t \epsilon^2} (u_1(x) + \theta_0(x)) = \frac{e^{n/2}}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|y-a|^2}{4t}} (u_1(y) + \theta_0(y)) dy.
\]

(34)

Combining the last representation and the derived differential equation, we obtain

\[
u_{tzz}^L(\epsilon^{-1} t, x) = \left( \frac{1}{|D|^2} (e^{-|D|^2 t \epsilon^2} - 1) + \frac{1}{\epsilon} \right) (u_1(x) + \theta_0(x)).
\]

Summarizing the last statements, we are able to conclude the next result for formal higher-order profiles of solution (24).

**Proposition 4.1.** The solution \( (u^\epsilon, \theta^\epsilon) \) to the Cauchy problem for thermoelastic plate equations (24) with the small \( \epsilon > 0 \) formally have the following asymptotic expansions:

\[
\begin{align*}
\left\{
\begin{array}{l}
u^\epsilon(t, x) = u^0(t, x) + \epsilon u^{I,1}(t, x) + \sum_{j \geq 1} \epsilon^j \left( u^{I,j}(t, x) + u^{L,j}(\epsilon^{-1} t, x) \right), \\
\theta^\epsilon(t, x) = -u^0_t(t, x) + e^{-|D|^2 |z|^2 \epsilon^2} (u_1(x) + \theta_0(x)) + \sum_{j \geq 1} \epsilon^j \left( \theta^{I,j}(t, x) + \theta^{L,j}(\epsilon^{-1} t, x) \right),
\end{array}
\right.
\end{align*}
\]

where

- \( u^0 = u^0(t, x) \) is the solution to the structurally damped plate equation (5);
- \( u^{I,1} = u^{I,1}(t, x) \) is the solution to the inhomogeneous structurally damped plate equation

\[
\begin{align*}
\left\{
\begin{array}{l}
u_{tt}^{I,1} + \Delta^2 u^{I,1} - \Delta u^{I,1} = -\Delta^2 u^0 + \Delta u^0_t, \quad x \in \mathbb{R}^n, \quad t > 0, \\
u^{I,1}(0, x) = u^{I,1}_t(0, x) = 0, \quad x \in \mathbb{R}^n,
\end{array}
\right.
\end{align*}
\]
• the pair of \( u^{l,j+1} = u^{l,j+1}(t,x) \) and \( \theta^{l,j} = \theta^{l,j}(t,x) \) for \( j \geq 1 \) is the solution to (27) with vanishing datum;
• the pair of \( u^{L,j+1} = u^{L,j+1}(z,x) \) and \( \theta^{L,j} = \theta^{L,j}(z,x) \) with \( z = \epsilon^{-1}t \) for \( j \geq 1 \) is the solution to (28) with vanishing datum.

Remark 4.1. For the reason of Fourier’s law of heat conduction, according to Proposition 4.1, we did not observe any initial layer (i.e. a rapid change with respect to the small thermal parameter \( \epsilon \)). The profile (34) originates from the heat equation only. In other words, we expect some changes polynomially in terms of \( 1/\epsilon \) rather than exponential change \( e^{-1/\epsilon} \) (see, for example, between the damped waves and the heat equation [15]).

4.3 Rigorous justification of singular limits: Proof of Theorem 2.2

Let us introduce two quantities to describe the first-order errors as follows:

\[
\begin{align*}
U(t,x) &:= u'(t,x) - u^0(t,x), \\
\Theta(t,x) &:= \theta'(t,x) - \theta^0(t,x) = \theta'(t,x) + u^0_t(t,x),
\end{align*}
\]

which fulfill the inhomogeneous thermoelastic plate equations

\[
\begin{align*}
U_{tt} + \Delta^2 U + \Delta \Theta &= 0, & x \in \mathbb{R}^n, & t > 0, \\
\epsilon \Theta_t - \Delta \Theta - \Delta U_t &= \epsilon F, & x \in \mathbb{R}^n, & t > 0, \\
U(0,x) &= U_1(0,x) = 0, & \Theta(0,x) &= u_1(x) + \theta_0(x), & x \in \mathbb{R}^n,
\end{align*}
\]

where the source term \( F = F(t,x) \) can be expressed by

\[
F(t,x) = -\theta^0_t(t,x) = u^0_{tt}(t,x) = \Delta u^0_t(t,x) - \Delta^2 u^0(t,x)
\]

that is independent of \( \epsilon \).

To prove our desired result, we follow the idea in Section 3 to get the first-order (in time) coupled system in the Fourier space

\[
\begin{align*}
\hat{U}_t - \hat{V} &= 0, & \xi \in \mathbb{R}^n, & t > 0, \\
\hat{V}_t - \hat{W} &= 0, & \xi \in \mathbb{R}^n, & t > 0, \\
\epsilon \hat{W}_t + |\xi|^2 \hat{W} + (1+\epsilon)|\xi|^4 \hat{V} + |\xi|^6 \hat{U} &= \epsilon |\xi|^2 \hat{F}, & \xi \in \mathbb{R}^n, & t > 0, \\
\hat{U}(0,\xi) = \hat{V}(0,\xi) = 0, & \hat{W}(0,\xi) &= |\xi|^2 \left( \hat{u}_1(\xi) + \hat{\theta}_0(\xi) \right), & \xi \in \mathbb{R}^n,
\end{align*}
\]

where we used \( \hat{V} := \hat{U}_t, \hat{W} := \hat{V}_t \) and the equation

\[
\epsilon \hat{U}_{tt} + |\xi|^2 \hat{U}_t + (1+\epsilon)|\xi|^4 \hat{U} + |\xi|^6 \hat{U} = \epsilon |\xi|^2 \hat{F}.
\]

Then, we construct a suitable energy

\[
\hat{E} := \frac{1}{2} \left( \frac{1}{2} |\xi|^2 \hat{V} + \epsilon \hat{W} |^2 + \frac{\epsilon}{1+\epsilon} |\xi|^4 \hat{U} |^2 + (1+\epsilon) \hat{V} |^2 + \frac{1}{4} |\xi|^4 |\hat{V}|^2 + \frac{1-\epsilon}{2(1+\epsilon)} |\xi|^8 |\hat{U}|^2 \right).
\]
Taking time-derivative of the energy, we can derive
\[
\frac{d}{dt} \tilde{E} = \Re \left[ \left( \frac{1}{2} |\xi|^2 \tilde{V}_t + \varepsilon \tilde{W}_t \right) \left( \frac{1}{2} |\xi|^2 \tilde{V} + \varepsilon \tilde{W} \right) \right] + \frac{\varepsilon}{1 + \varepsilon} |\xi|^4 \Re \left[ \left( |\xi|^2 \tilde{V} + (1 + \varepsilon) \tilde{W} \right) \left( |\xi|^2 \tilde{U} + (1 + \varepsilon) \tilde{W} \right) \right] + \frac{1 - \varepsilon}{4(1 + \varepsilon)} |\xi|^4 \Re \left( \tilde{V} \tilde{W} \right) \\
= -\frac{1 - \varepsilon}{2} |\xi|^6 \tilde{V}^2 - \frac{\varepsilon}{2} |\xi|^2 \tilde{W}^2 + \varepsilon |\xi|^2 \Re \left[ \hat{F} \left( \frac{1}{2} |\xi|^2 \tilde{V} + \varepsilon \tilde{W} \right) \right] \\
\leq 1 + \frac{4\varepsilon(1 - \varepsilon)}{8(1 - \varepsilon)} e^{2|\xi|^2 |\hat{F}|^2} \leq C e^{2|\xi|^2 |\hat{F}|^2}.
\]

Recalling that \( \hat{F} = -|\xi|^2 \hat{u}_t^0 - |\xi|^4 \hat{u}^0 \), we need to get some estimates of \( \hat{u}_t^0 \) as well as \( \hat{u}^0 \) in the Fourier space. We apply the partial Fourier transform to (5), namely,
\[
\begin{cases}
\hat{u}_t^0 + |\xi|^4 \hat{u}_t^0 + |\xi|^2 \hat{u}_t^0 = 0, & \xi \in \mathbb{R}^n, \ t > 0, \\
\hat{u}_t^0(0, \xi) = \hat{u}_0(\xi), \quad \hat{u}_t^0(0, \xi) = \hat{u}_1(\xi), & \xi \in \mathbb{R}^n,
\end{cases}
\]
whose solution and its time-derivative are given by
\[
\hat{u}_t^0(t, \xi) = \left[ \frac{1}{\sqrt{3}} \sin \left( \frac{\sqrt{3}}{2} |\xi|^2 t \right) + \cos \left( \frac{\sqrt{3}}{2} |\xi|^2 t \right) \right] e^{-\frac{1}{2} |\xi|^2 t} \hat{u}_0(\xi) + \frac{2}{\sqrt{3}} \sin \left( \frac{\sqrt{3}}{2} |\xi|^2 t \right) e^{-\frac{1}{2} |\xi|^2 t} \hat{u}_1(\xi), \\
\hat{u}_t^0(t, \xi) = -\frac{2}{\sqrt{3}} |\xi|^2 \sin \left( \frac{\sqrt{3}}{2} |\xi|^2 t \right) e^{-\frac{1}{2} |\xi|^2 t} \hat{u}_0(\xi) + \left[ \cos \left( \frac{\sqrt{3}}{2} |\xi|^2 t \right) - \frac{1}{\sqrt{3}} \sin \left( \frac{\sqrt{3}}{2} |\xi|^2 t \right) \right] e^{-\frac{1}{2} |\xi|^2 t} \hat{u}_1(\xi).
\]

According to
\[
|\tilde{U}_t(t, \xi)|^2 + |\xi|^4 |\tilde{U}(t, \xi)|^2 \leq \frac{C}{|\xi|^4} \tilde{E}(t, \xi) \leq C e^2 |\hat{u}_1(\xi) + \hat{\theta}_0(\xi)|^2 + C e^2 |\xi|^{-2} \int_0^t |\hat{F}(\tau, \xi)|^2 d\tau \\
\leq C e^2 |\hat{u}_1(\xi) + \hat{\theta}_0(\xi)|^2 + C e^2 \int_0^t e^{-c|\xi|^2 \tau} d\tau \left( |\xi|^4 |\hat{u}_0(\xi)|^2 + |\xi|^2 |\hat{u}_1(\xi)|^2 \right),
\]
we integrate it over \( \mathbb{R}^n \) to deduce
\[
||U_1(t, \cdot)||^2_{L^2} + ||\Delta U(t, \cdot)||^2_{L^2} \leq C e^2 ||u_1 + \theta_0||^2_{L^2} + C e^2 \int_0^t ||\chi_{int}(\xi)|| |\xi|^3 e^{-c|\xi|^2 \tau} |\hat{u}_0(\xi)||^2_{L^2} d\tau \\
+ C e^2 \int_0^t ||\chi_{int}(\xi)|| |\xi|^3 e^{-c|\xi|^2 \tau} |\hat{u}_1(\xi)||^2_{L^2} d\tau + C e^2 \left( ||u_0||^2_{H^2} + ||u_1||^2_{L^2} \right) \\
\leq C e^2 ||u_1 + \theta_0||^2_{L^2} + C e^2 \left( ||u_0||^2_{H^2} + ||u_1||^2_{L^2} \right) \\
+ C e^2 \left( \int_0^t (1 + \tau)^{-\frac{3}{2}} d\tau ||u_0||^2_{L^1} + \int_0^t (1 + \tau)^{-\frac{1}{2}} d\tau ||u_1||^2_{L^1} \right) \\
\leq C e^2 \left( ||u_1 + \theta_0||^2_{L^2} + ||u_0||^2_{H^2 \cap L^1} + ||u_1||^2_{L^2 \cap L^1} \right),
\]
where the next estimate was used:
\[
\int_0^t (1 - \chi_{int}(\xi)) e^{-c|\xi|^2 \tau} d\tau = \frac{1 - \chi_{int}(\xi)}{c|\xi|^2} \left( 1 - e^{-c|\xi|^2 t} \right) \lesssim \frac{1 - \chi_{int}(\xi)}{|\xi|^2}.
\]

Then, we complete the first desired consequence in the sense of derivatives.
For another, we already know
\[ |\hat{U}(t, \xi)|^2 \leq C \epsilon^2 \frac{|\hat{u}_1(\xi) + \hat{\theta}_0(\xi)|^2}{|\xi|^4} + C \epsilon^2 \int_0^t e^{-c|\xi|^2 \tau} \left( |\xi|^2 |\hat{u}_0(\xi)|^2 + \frac{|\hat{u}_1(\xi)|^2}{|\xi|^2} \right) d\tau. \]
In order to achieve global (in time) convergence result for \( n \geq 3 \) we apply our assumption \( |P_{u_1}| = 0 \) so that
\[ |\hat{u}_1(\xi)|^2 \leq C \left( |P_{u_1}|^2 + |\xi|^2 \|u_1\|_{L^2}^2 \right) = C |\xi|^2 \|u_1\|_{L^2}^2. \]
Thus, one gets
\[ \|U(t, \cdot, \xi)\|_{L^2}^2 \leq C \epsilon^2 \|u_1 + \theta_0\|_{H^{-2}}^2 + C \epsilon^2 \left( \|u_0\|_{L^2}^2 + \|u_1\|_{L^2}^2 \right) \]
\[ + C \epsilon^2 \left( \int_0^t (1 + \tau)^{-\frac{3}{2}} d\tau \|u_0\|_{L^2}^2 + \int_0^t (1 + \tau)^{-\frac{3}{2}} d\tau \|u_1\|_{L^2}^2 \right) \]
\[ \leq C \epsilon^2 \left( \|u_1 + \theta_0\|_{H^{-2}}^2 + \|u_0\|_{L^2 \cap L^1}^2 + \|u_1\|_{L^2 \cap L^1}^2 \right) \]
for \( n \geq 3 \) since \( (1 + \tau)^{-\frac{3}{2}} \in L^1 \) if \( n \geq 3 \). It finishes our proof.

### 4.4 Rigorous justification of second-order profiles: Proof of Theorem 2.3

Strongly motivated by Proposition 4.1, we may construct the second-order error terms
\[ \begin{cases} 
U^s(t, x) = u^s(t, x) - u^0(t, x) - \epsilon u^{I,1}(t, x), \\
\Theta^s(t, x) = \theta^s(t, x) - \theta^0(t, x) - \epsilon \theta^{I,1}(t, x),
\end{cases} \tag{35} \]
where the profile \( u^{I,1} = u^{I,1}(t, x) \) fulfills (33) and the other profile \( \theta^{I,1} = \theta^{I,1}(t, x) \) actually satisfies
\[ \theta^{I,1} = (-\Delta)^{-1} \left( u^{I,1}_{tt} + \Delta^2 u^{I,1} \right). \]
Since \((u^{I,1}, \theta^{I,1})\) is the solution to
\[ \begin{cases} 
u^{I,1}_{tt} + \Delta^2 u^{I,1} + \Delta \theta^{I,1} = 0, & x \in \mathbb{R}^n, t > 0, \\
\theta^0 - \Delta \theta^{I,1} - \Delta u^{I,1}_{t} = 0, & x \in \mathbb{R}^n, t > 0, \\
u^{I,1}(0, x) = u^{I,1}_t(0, x) = \theta^{I,1}(0, x) = 0, & x \in \mathbb{R}^n,
\end{cases} \]
the difference \((U^s, \Theta^s)\) solves the inhomogeneous thermoelastic plate equations as follows:
\[ \begin{cases} 
u^s_{tt} + \Delta^2 U^s + \Delta \Theta^s = 0, & x \in \mathbb{R}^n, t > 0, \\
\epsilon \Theta^s_t - \Delta \Theta^s - \Delta U^s_t = \epsilon^2 F^s, & x \in \mathbb{R}^n, t > 0, \\
u^s(0, x) = U^s_t(0, x) = \Theta^s(0, x) = 0, & x \in \mathbb{R}^n,
\end{cases} \]
where the assumption \( u_1(x) + \theta_0(x) \equiv 0 \) was proposed, and the source term on the second equation in the above is
\[ F^s(t, x) = -\theta^{I,1}_t(t, x) = \Delta^{-1} \partial_t \left( u^{I,1}_{tt}(t, x) + \Delta^2 u^{I,1}(t, x) \right) \]
\[ = u^{I,1}_{tt}(t, x) - \Delta u^{I,1}_t(t, x) + u^{I,1}_t(t, x) \]
\[ = -\Delta^2 u^{I,1}(t, x) + \Delta u^{I,1}_t(t, x) - \Delta^2 u^0(t, x) + \Delta u^0_t(t, x). \]
With the aid of the same procedure as those in last subsection, we may estimate
\[
|\hat{U}^s_t(t, \xi)|^2 + |\xi|^4|\hat{U}^s(t, \xi)|^2 \leq C\epsilon |\xi|^{-2} \int_0^t |\hat{F}^s(\tau, \xi)|^2 \, d\tau
\]
\[
\leq C\epsilon |\xi|^6 \int_0^t \left( |\hat{u}^{1,1}_t(\tau, \xi)|^2 + |\hat{u}^0(\tau, \xi)|^2 \right) d\tau + C\epsilon |\xi|^2 \int_0^t \left( |\hat{u}^{1,1}_t(\tau, \xi)|^2 + |\hat{u}^0_t(\tau, \xi)|^2 \right) d\tau.
\]
Moreover, the Fourier transform of \((33)\) is given by
\[
\left\{ \begin{array}{l}
\hat{u}^{1,1}_t + |\xi|^2 \hat{u}^{1,1}_t + |\xi|^4 \hat{u}^{1,1} = -|\xi|^4 \hat{u}^0 - |\xi|^2 \hat{u}^0, \quad \xi \in \mathbb{R}^n, \ t > 0, \\
\hat{u}^{1,1}(0, \xi) = \hat{u}^{1,1}_t(0, \xi) = 0, \quad \xi \in \mathbb{R}^n.
\end{array} \right.
\]
Its solution can be determined by the use of Duhamel's principle, i.e.
\[
\hat{u}^{1,1}(t, \xi) = -\frac{2}{\sqrt{3}} \int_0^t \sin \left( \frac{\sqrt{3}}{2} |\xi|^2 (t - \tau) \right) e^{-\frac{1}{2} |\xi|^2 (t - \tau)} \left( |\xi|^2 \hat{u}^0(\tau, \xi) + \hat{u}^0_t(\tau, \xi) \right) \, d\tau.
\]
Some direct estimates imply
\[
|\hat{u}^{1,1}(t, \xi)| \leq C(t) e^{-\frac{1}{2} |\xi|^2 t} \left( |\xi|^2 |\hat{u}^0(\xi)| + |\hat{u}^1(\xi)| \right),
\]
\[
|\hat{u}^{1,1}_t(t, \xi)| \leq C(t) |\xi|^2 e^{-\frac{1}{2} |\xi|^2 t} \left( |\xi|^2 |\hat{u}^0(\xi)| + |\hat{u}^1(\xi)| \right).
\]
Summarizing the last estimates, we claim
\[
|\hat{U}^s_t(t, \xi)|^2 + |\xi|^4|\hat{U}^s(t, \xi)|^2 \leq C\epsilon \int_0^t e^{-\frac{1}{2} |\xi|^2 \tau} \left( |\xi|^6 |\hat{u}^0(\xi)|^2 + |\xi|^2 |\hat{u}^1(\xi)|^2 \right) \, d\tau,
\]
\[
|\hat{U}^s(t, \xi)|^2 \leq C\epsilon \int_0^t e^{-\frac{1}{2} |\xi|^2 \tau} \left( |\xi|^2 |\hat{u}^0(\xi)|^2 + \frac{|\hat{u}^1(\xi)|^2}{|\xi|^2} \right) \, d\tau.
\]
By using the same approach as those for Theorem 2.2, the next estimates can be done:
\[
\|U^s(t, \cdot)\|^2_{L^2} + \|\Delta U^s(t, \cdot)\|^2_{L^2} \leq C\epsilon^4 \left( \|u_0\|^2_{H^2 \cap L^1} + \|u_1\|^2_{L^2 \cap L^1} \right)
\]
for \(n \geq 1\), and
\[
\|U^s(t, \cdot)\|^2_{L^2} \leq C\epsilon^4 \left( \|u_0\|^2_{L^2 \cap L^1} + \|u_1\|^2_{L^2 \cap L^{1.1}} \right)
\]
for \(n \geq 3\). Then, our proof is completed.

5 Concluding remarks

In the final section, we will give some remarks concerning wide applications of the reduction methodology proposed in this work. The typical example of applications is the system of thermoelasticity, which can describe the elasticity and thermal behavior of elastic, heat conducting media. Let us concentrate on the following Cauchy problem for the thermoelastic system in 1D:
\[
\left\{ \begin{array}{l}
u_{tt} - \alpha u_{xx} + \gamma_1 \theta_x = 0, \quad x \in \mathbb{R}, \ t > 0, \\
\theta_t - \kappa \theta_{xx} + \gamma_2 u_{xx} = 0, \quad x \in \mathbb{R}, \ t > 0, \\
u(0, x) = u_0(x), \ u_t(0, x) = u_1(x), \ \theta(0, x) = \theta_0(x), \ x \in \mathbb{R},
\end{array} \right. \quad (36)
\]
where \( u = u(t, x) \) and \( \theta = \theta(t, x) \) denote the elastic displacement and the temperature difference to the equilibrium, individually. In the model (36), physical constants \( \alpha > 0 \) is the elasticity modules, \( \kappa > 0 \) is the thermal conductivity and the thermoelastic coupling coefficients \( \gamma_1, \gamma_2 \) fulfill \( \gamma_1 \gamma_2 > 0 \).

Following the same approach in this work, we may reduce (36) to the third-order (in time) PDEs

\[
\begin{cases}
    u_{ttt} - \kappa u_{txx} - (\gamma_1 \gamma_2 + \alpha) u_{txx} + \kappa \alpha u_{xxxx} = 0, & x \in \mathbb{R}, \ t > 0, \\
    u(0, x) = u_0(x), \ u_t(0, x) = u_1(x), \ u_{tt}(0, x) = \alpha u''_0(x) - \gamma_1 \theta'_0(x), & x \in \mathbb{R}.
\end{cases}
\]

We conjecture that by using Fourier analysis, asymptotic analysis as well as WKB method, some qualitative properties of solutions to (36), particularly, the solution itself \( u = u(t, x) \), can be obtained.

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