Improved Bounds for the Graham-Pollak Problem for Hypergraphs

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Abstract

For a fixed $r$, let $f_r(n)$ denote the minimum number of complete $r$-partite $r$-graphs needed to partition the complete $r$-graph on $n$ vertices. The Graham-Pollak theorem asserts that $f_2(n) = n - 1$. An easy construction shows that $f_r(n) \leq (1 + o(1))(\binom{n}{\lfloor \frac{r}{2} \rfloor})$, and we write $c_r$ for the least number such that $f_r(n) \leq c_r(1 + o(1))(\binom{n}{\lfloor \frac{r}{2} \rfloor})$.

It was known that $c_r < 1$ for each even $r \geq 4$, but this was not known for any odd value of $r$. In this short note, we prove that $c_{295} < 1$. Our method also shows that $c_r \to 0$, answering another open problem.

Keywords: Hypergraph, Decomposition, Graham-Pollak

1 Introduction

The edge set of $K_n$, the complete graph on $n$ vertices, can be partitioned into $n - 1$ complete bipartite subgraphs: this may be done in many ways, for example by taking $n - 1$ stars centred at different vertices. Graham and Pollak [4, 5] proved that the number $n - 1$ cannot be decreased. Several other proofs of this result have been found, by Tverberg [8], Peck [7], and Vishwanathan [9, 10], among others.

Generalising this to hypergraphs, for $n \geq r \geq 1$, let $f_r(n)$ be the minimum number of complete $r$-partite $r$-graphs needed to partition the edge set of $K_n^{(r)}$, the complete $r$-uniform hypergraph on $n$ vertices (i.e., the collection of all $r$-sets from an $n$-set). Thus the Graham-Pollak theorem asserts that $f_2(n) = n - 1$. For $r \geq 3$, an easy upper bound of $\binom{n-\lceil \frac{r}{2} \rceil}{\lceil \frac{r}{2} \rceil}$ may be obtained by generalising the star example above. Indeed, for $r$ even, having ordered the vertices, consider

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the collection of \(r\)-sets whose \(2nd, 4th, \ldots, rth\) vertices are fixed. This forms a complete \(r\)-partite \(r\)-graph, and the collection of all \(\binom{n-r/2}{r/2}\) such is a partition of \(K_n^{(r)}\). For \(r\) odd, we instead fix the \(2nd, 4th, \ldots, (r-1)th\) vertices, yielding a partition into \(\binom{n-(r+1)/2}{(r-1)/2}\) parts.

Alon [1] showed that \(f_3(n) = n - 2\). More generally, for each fixed \(r \geq 1\), he showed that

\[
\frac{2}{\binom{2r/2}{(r/2)}} \left(1 + o(1)\right) \left(\frac{n}{r/2}\right) \leq f_r(n) \leq (1 - o(1)) \left(\frac{n}{r/2}\right),
\]

where the upper bound follows from the construction above. Writing \(c_r\) for the least \(c\) such that \(f_r(n) \leq c(1 + o(1)) \left(\frac{n}{r/2}\right)\), the above results assert that \(c_2 = 1\), \(c_3 = 1\), and \(\frac{2}{\binom{2r}{(r/2)}} \leq c_r \leq 1\) for all \(r\). How do the \(c_r\) behave?

Cioabă, Kündgen and Verstraëte [2] gave an improvement (in a lower-order term) to Alon’s lower bound, and Cioabă and Tait [3] showed that the construction above is not sharp in general, but Alon’s asymptotic bounds (i.e., the above bounds on \(c_r\)) remained unchanged. Recently, Leader, Miličević and Tan [6] showed that \(c_r \leq \frac{14}{15}\) for each even \(r \geq 4\). However, they could not improve the bound of \(c_r \leq 1\) for any odd \(r\) – the point being that the construction above is better for \(r\) odd than for \(r\) even (the exponent of \(n\) is \((r-1)/2\) for \(r\) odd versus \(r/2\) for \(r\) even), and so is harder to improve.

In this note, we give a simple argument to show that \(c_{295} < 1\). Our method also shows that \(c_r \to 0\), answering another question from [6].

It would be interesting to know what happens for smaller odd values of \(r\): for example, is \(c_5 < 1\)? Determining the precise value of \(c_4\) (i.e., the asymptotic behaviour of \(f_4(n)\)) would also be of great interest, as would determining the decay rate of the \(c_r\). See [6] for several related questions and conjectures.

## 2 Main Result

The motivation for our proof is as follows. The key to the approach used in [6] in proving \(c_r < 1\) for each even \(r \geq 4\) was to investigate the minimum number of products of complete bipartite graphs, that is, sets of the form \(E(K_{a,b}) \times E(K_{c,d})\), needed to partition the set \(E(K_n) \times E(K_n)\). Writing \(g(n)\) for this minimum value, it is trivial that \(g(n) \leq (n-1)^2\), by taking the products of the complete bipartite graphs appearing in a decomposition of \(K_n\) into \(n-1\) complete bipartite graphs. It was shown in [6] that \(g(n) \leq \left(\frac{14}{15} + o(1)\right) n^2\). It turned out that this upper bound on \(g(n)\) was enough (via an iterative construction) to bound \(c_r\) below 1 for each even \(r \geq 4\).

Now, as remarked above, for \(r\) odd the construction in the Introduction is much better than for \(r\) even. In fact, while there are many iterative ways to redo the construction when \(r\) is even, passing from \(n/2\) to \(n\), these fail when \(r\) is odd: it turns out that an extra factor is introduced at each stage. However, rather unexpectedly, we will see that (at least if \(r\) is large) if we partition into many pieces, instead of just two pieces, then the gain we obtain from the 14/15 improvement
in $g(n)$ outweighs the loss arising from this extra factor – even though this extra factor grows as the number of pieces grows.

A minimal decomposition of a complete $r$-partite $r$-graph $K_n^{(r)}$ is a partition of the edge set into $f_r(n)$ complete $r$-partite $r$-graphs. A block is a product of the edge sets of two complete bipartite graphs. Similarly, a minimal decomposition of $E(K_n) \times E(K_n)$ is a partition of $E(K_n) \times E(K_n)$ into $g(n)$ blocks. Finally, for a set $V$, we may write $E(V)$ to denote the edge set of the complete graph on $V$, that is, the set of all 2-subsets of $V$.

**Theorem 1.** Let $r = 2d + 1$ be fixed. Then for each $k$ there exists $\epsilon_k$, with $\epsilon_k \to 0$ as $k \to \infty$, such that for all $n$ we have

$$f_r(kn) \leq \left\lfloor \frac{14}{15} + d \left( \frac{14}{15} \right) \right\rfloor + o(1) \left( \frac{kn}{d} \right).$$

(Here the $o(1)$ term is as $n \to \infty$, with $k$ and $d$ fixed.)

**Proof.** In order to decompose the edge set of $K_n^{(r)}$, we start by splitting the $kn$ vertices into $k$ equal parts, say $V \left( K_n^{(r)} \right) = V_1 \cup V_2 \cup \cdots \cup V_k$, where $|V_i| = n$ for each $i$. We consider the $r$-edges based on their intersection sizes with the $k$ vertex classes. For each partition of $r$ into positive integers $r_1 + r_2 + \cdots + r_l$ with $r_1 \leq r_2 \leq \cdots \leq r_l$ and for each collection of $l$ vertex classes $V_{i_1}, V_{i_2}, \ldots, V_{i_l}$, the set of $r$-edges $e$ with $|e \cap V_{i_j}| = r_j$ for all $j$ can be decomposed into $f_{r_1}(n)f_{r_2}(n)\cdots f_{r_l}(n)$ complete $r$-partite $r$-graphs: take a complete $r_j$-partite $r_j$-graph from a minimal decomposition of $K_n^{(r_j)}$ for each $j$, and form a complete $r$-partite $r$-graph by taking the product of them.

Note that if at least three values of the $r_j$ are odd, then $f_{r_1}(n)f_{r_2}(n)\cdots f_{r_l}(n) = O(n^{d-1})$, as $f_s(n) \leq \left\lfloor \frac{n}{2} \right\rfloor$ for any $s$. So the set of $r$-edges $e$ with $|e \cap V_i|$ is odd for at least three distinct $V_i$ can be decomposed into $Cn^{d-1}$ complete $r$-partite $r$-graphs, for some constant $C$ depending on $d$ and $k$.

Let $C'$ be the number of partitions of $r$ into at most $d - 1$ positive integers where exactly one of them is odd. Then we observe that the set of $r$-edges $e$ such that $e$ intersects with at most $d - 1$ vertex classes and $|e \cap V_i|$ is odd for exactly one $V_i$ can be decomposed into at most $C'k^{d-1}n^d$ complete $r$-partite $r$-graphs.

We are now only left with two partitions of $r$: $r = 1 + 2 + 2 + \cdots + 2$ and $r = 2 + 2 + \cdots + 2 + 3$. The first case corresponds to the set of $r$-edges with $r_1 = 1, r_2 = \cdots = r_{d+1} = 2$. For each of the $\binom{d}{2}$ collections of $d$ vertex classes $V_{i_1}, V_{i_2}, \ldots, V_{i_d}$, we claim that the set of $r$-edges $\{e : |e \cap V_{i_j}| = 2, j = 1, 2, \ldots, d\}$ can be decomposed into $g(n)^{d/2}$ or $ng(n)^{(d-1)/2}$ complete $r$-partite $r$-graphs, depending on whether $d$ is even or odd. This is done by pairing up the $V_{i_j}$s (or all but one of the $V_{i_j}$s if $d$ is odd), and forming complete $r$-partite $r$-graphs using products of blocks in a minimal decomposition of $E(K_n) \times E(K_n)$. [For example, for $d = 4$, we would take a decomposition of $E(V_{i_1}) \times E(V_{i_2})$ into blocks $E_x \times E_x, 1 \leq x \leq g(n)$, and similarly a decomposition of $E(V_{i_3}) \times E(V_{i_4})$ into blocks $G_x \times H_x, 1 \leq x \leq g(n)$, and now the set of all]
9-edges \( e \) with \( |e \cap V_i| = 2 \) for all \( 1 \leq j \leq 4 \) may be decomposed into \( g(n)^2 \) complete 9-graphs by taking the \( E_x \times F_x \times G_y \times H_y \times (V_{i1} \cup V_{i2} \cup V_{i3} \cup V_{i4})^c \) for \( 1 \leq x, y \leq g(n) \).}

Finally, the second case corresponds to the set of \( r \)-edges with \( r_1 = r_2 = \cdots = r_{d-1} = 2, r_d = 3 \). These can be decomposed in a similar fashion. Indeed, for each collection of \( d \) vertex classes \( V_{i1}, V_{i2}, \ldots, V_{id} \), the set of \( r \)-edges \( \{e : |e \cap V_i| = 3 \text{ and } |e \cap V_j| = 2, j = 1, 2, \ldots, d-1 \} \) can be decomposed into \( n^2 g(n)^{(d-2)/2} \) or \( n^2 (n-1)^{(d-1)/2} \) complete \( r \)-partite \( r \)-graphs, depending on whether \( d \) is even or odd. There are \( d(\binom{n}{d}) \) such sets of \( r \)-edges.

Combining the above and the bound on \( g(n) \), we have

\[
f_r(kn) \leq \begin{cases} 
\binom{k}{d} g(n)^{\frac{d}{2}} + d(\binom{k}{d}) n^2 g(n)^{\frac{d-1}{2}} + C'k^{d-1}n^d + Cn^{d-1} & \text{(if } d \text{ even)} \\
\binom{k}{d} g(n)^{\frac{d}{2}} + d(\binom{k}{d}) n^2 g(n)^{\frac{d-1}{2}} + C'k^{d-1}n^d + Cn^{d-1} & \text{(if } d \text{ odd)}
\end{cases}
\]

\[
\leq \binom{k}{d} \left( \frac{14}{15} \right)^{\left\lfloor \frac{d}{2} \right\rfloor} n^d + d(\binom{k}{d}) \left( \frac{14}{15} \right)^{\left\lfloor \frac{d-1}{2} \right\rfloor} n^d + C'k^{d-1}n^d + o(n^d)
\]

\[
\leq \left( \frac{14}{15} \right)^{\left\lfloor \frac{d}{2} \right\rfloor} + d \left( \frac{14}{15} \right)^{\left\lfloor \frac{d-1}{2} \right\rfloor} + \epsilon_k \left( 1 + o(1) \right) \binom{kn}{d}.
\]

Corollary 2. Let \( r \geq 295 \) be a fixed odd number. Then there exists \( c < 1 \) such that

\[
f_r(n) \leq c(1 + o(1)) \binom{n}{\left\lfloor \frac{r}{2} \right\rfloor}.
\]

Proof. As above, write \( r = 2d + 1 \). It is straightforward to check that for \( d \geq 147 \) we have \( \left( \frac{14}{15} \right)^{\left\lfloor \frac{d}{2} \right\rfloor} + d \left( \frac{14}{15} \right)^{\left\lfloor \frac{d-1}{2} \right\rfloor} < 1 \). Choosing \( c \) such that

\[
c = \left( \frac{14}{15} \right)^{\left\lfloor \frac{d}{2} \right\rfloor} + d \left( \frac{14}{15} \right)^{\left\lfloor \frac{d-1}{2} \right\rfloor} + \epsilon_k < 1,
\]

we have \( f_r(kn) \leq c(1 + o(1)) \binom{kn}{d} \) for all \( n \). However since the function \( f_r(n) \) is monotone in \( n \), and \( k \) is constant as \( n \) varies, it follows that \( f_r(n) \leq c(1 + o(1)) \binom{n}{d} \) for all \( n \). \( \square \)

From Theorem 1, we have

\[
c_{2d+1} \leq \left( \frac{14}{15} \right)^{\left\lfloor \frac{d}{2} \right\rfloor} + d \left( \frac{14}{15} \right)^{\left\lfloor \frac{d-1}{2} \right\rfloor}
\]

for every \( d \). Also, it is easy to see that \( c_{2d} \leq c_{2d+1} \). Indeed, by excluding a vertex in the complete \((2d + 1)\)-graph on \( n + 1 \) vertices, the complete \((2d)\)-partite \((2d)\)-graphs induced from the complete \((2d + 1)\)-partite \((2d + 1)\)-graphs in a minimal decomposition of \( K_n^{(2d+1)} \) form a decomposition of \( K_n^{(2d)} \), implying that \( f_{2d}(n) \leq f_{2d+1}(n + 1) \). Hence we have the following.
Corollary 3. The numbers $c_r$ satisfy
\[
c_r \leq \frac{r}{2} \left( \frac{14}{15} \right)^{r/4} + o(1).
\]

Corollary 3 implies that $c_r \to 0$ as $r \to \infty$, proving Conjecture 16 in [6].

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