The Chow Ring of the Moduli Space of Abelian Threefolds

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In this paper we determine the structure of the Chow ring of the Delaunay-Voronoi compactification $\tilde{A}_3$ of the moduli space of principally polarized abelian threefolds. This compactification was introduced by Namikawa and studied by Tsushima. We shall use equivariant classes on level coverings of $\tilde{A}_3$. We also compare this ring with the Chow ring of the moduli space of stable genus 3 curves as determined by Faber.

§0 Introduction

Let $A_g$ be the moduli stack of principally polarized abelian varieties of dimension $g$ and $X_g$ the universal family of abelian varieties over $A_g$. The stack $A_g$ is not complete; it can be completed to a space $A^*_g$, the so-called Satake compactification. In Faltings-Chai [F-C] smooth compactifications of $A_g$ are constructed. These compactifications are not unique as they depend on the choice of a polyhedral decomposition of the cone of positive definite symmetric bilinear forms in $g$ real variables. However, a specific choice of polyhedral decomposition (the Delaunay-Voronoi decomposition) gives for low values of $g$ a good (smooth) compactification, cf. [N]. For $g = 1$ it gives the unique compactification, while for $g = 2$ one gets the blow-up of the Satake compactification as considered by Igusa. For $g = 3$ one gets the compactification $\tilde{A}_3$ as constructed by Namikawa and studied by Tsushima in [N] and [T 1,2]. This is a smooth stack and it has a Chow ring. In the following $\tilde{A}_g$ will denote this Delaunay-Voronoi compactification.

The Chow ring of $\tilde{A}_2$ was determined by Mumford in [M3], since he determined the Chow ring of $\overline{M}_2$. In this paper we determine the structure of the Chow ring of $\tilde{A}_3$. It is a ring with four generators. We do this by bounding the ranks of the Chow groups and then use the intersection numbers of equivariant classes to get the generators. In [F] Faber determined the Chow ring of $\overline{M}_3$. For our choice of compactification the Torelli morphism extends and we can use this morphism $t : \overline{M}_3 \to \tilde{A}_3$ of degree 2 to compare both rings.

I like to thank Carel Faber and Valery Alexeev for some helpful remarks. I also thank Sam Grushevsky for pointing out an inaccuracy.

This version corrects two inaccuracies of the 1998 JAG paper
§1 Preliminaries

The Satake compactification $\mathcal{A}_g^*$ of the stack $\mathcal{A}_g$ over $\mathbb{Z}$ admits a stratification

$$\mathcal{A}_g^* = \bigcoprod_{j=0}^g \mathcal{A}_j.$$ 

Let $\mathcal{A}_g[\ell]$ denote the moduli stack over $\mathbb{Z}[1/\ell]$ of principally polarized abelian varieties of dimension $g$ with a full symplectic level $\ell$ structure. There is a morphism $p_\ell : A_g[\ell] \to \mathcal{A}_g$ which is equivariant for the natural action of $G_\ell := Sp(2g, \mathbb{Z}/\ell\mathbb{Z})$ on $\mathcal{A}_g[\ell]$. The degree of this morphism is

$$\gamma = \gamma_\ell = \ell^{g(2g+1)} \prod_{p|\ell} \prod_{1 \leq j \leq g} (1 - p^{-2j}).$$

The morphism $p_\ell$ extends to the Satake compactification $\mathcal{A}_g^*[\ell]$. For $\ell \geq 3$ we denote by

$$\mu_g = \mu_g(\ell) = \frac{1}{2} \ell^{2g} \prod_{p|\ell} \prod_{1 \leq j \leq g} (1 - p^{-2j})$$

the number of maximal dimensional cusps of $\mathcal{A}_g^*[\ell]$ (each of them is a copy of $\mathcal{A}_{g-1}^*[\ell]$).

The toroidal compactifications we need are described by Namikawa in [N] and by Faltings-Chai [F-C], see also [T1]. There one also finds the toroidal compactification of the universal abelian variety, see [F-C, p. 195]. The toroidal compactifications of Faltings-Chai carry a rank $g$ vector bundle, the Hodge bundle $E$. It is given over $\mathcal{A}_g$ by $s^*(\Omega_X/S)$ for each principally polarized abelian variety $X/S$ corresponding to $S \to \mathcal{A}_g$ with zero section $s$. The pull back $\pi^*(E)$ under $\pi : \tilde{X}_g \to \tilde{\mathcal{A}}_g$ of the Hodge bundle to the universal scheme $\tilde{X}_g$ over $\tilde{\mathcal{A}}_g$ is isomorphic to the sheaf of relative logarithmic differentials $\Omega_{\tilde{X}}(d\log\infty)/\pi^*(\Omega_{\tilde{\mathcal{A}}_g}(d\log\infty))$, cf. [F-C, p.195]. We denote the Chern classes of the Hodge bundle by $\lambda_i$. The first Chern class $\lambda_1$ is ample on $\mathcal{A}_g$ and defines a morphism $q : \tilde{\mathcal{A}}_g \to \mathcal{A}_g^*$ to the Satake compactification. The $\mathbb{Q}$-subalgebra generated by the $\lambda_i$ was determined in [vdG]. The proof there is given only for $CH^*(\mathcal{A}_g)$ and here we shall use no more than that.

The Chow groups we discuss in this paper are always taken with $\mathbb{Q}$-coefficients. We use equivariant classes on the level $\ell$-spaces, cf. [E-G]. The classes of loci are always taken in the sense of $Q$-classes, cf. [M3]. So if $Y \subset \tilde{\mathcal{A}}_g$ is a subvariety then the class of $Y$ in $CH^*(\tilde{\mathcal{A}}_g)$ is taken with coefficient $1/n$, where $n$ is the order of the automorphism group of the completed (semi-)abelian variety corresponding to the generic point of $Y$. The automorphisms must preserve the group structure of the semi-abelian part and extend to the completion. We shall frequently use the following standard fact: if $Y \subset X$ is a closed subvariety then we have an exact sequence

$$CH_i(Y) \to CH_i(X) \to CH_i(X - Y) \to 0.$$
We need several loci in the moduli space. We denote by $A_{g_1,\ldots,g_r}$ the locus of products of principally polarized abelian varieties of dimensions $g_1,\ldots,g_r$ with $\sum g_i = g$. There is a map

$$A_{g_1} \times \cdots \times A_{g_r} \longrightarrow A_{g_1,\ldots,g_r} \subseteq A_g$$

of finite degree. The closure of $A_{g_1,\ldots,g_r}$ in a toroidal compactification $\tilde{A}_g$ is denoted by $\tilde{A}_{g_1,\ldots,g_r}$. We shall use the notation

$$\beta_t = q^{-1}(\prod_{j \leq g-t} A_j)$$

for the locus of semi-abelian varieties with torus rank $\geq t$ on a toroidal compactification $\tilde{A}_g$ and

$$\tilde{A}_g^{(t)} = \tilde{A}_g - \beta_{t+1}$$

for the space of rank $\leq t$ degenerations. Sometimes we shall write $\tilde{A}_g^{(\geq t)}$ for $\beta_t$ and $\tilde{A}_g^{(\leq t)}$ for $\tilde{A}_g^{(t)}$.

An important remark here is that the space of rank 1-degenerations $\tilde{A}_g^{(1)}$ is canonical, i.e., does not depend on a choice of polyhedral decomposition, cf. [M2]. The boundary $\tilde{A}_g^{(1)} - A_g$ is a divisor $D$. It admits a map of generic degree 2

$$j: X_{g-1} \rightarrow D$$

which identifies $D$ with the universal family of Kummer varieties over $A_{g-1}$. We have the following lemma, cf. [M2]:

(1.1) **Lemma.** The pull back $j^*(D)$ of the $Q$-class of $D$ restricted to a generic fibre $X$ equals $-\Theta$ with $\Theta$ the theta divisor on $X$.

(1.2) **Lemma.** The cycle class $\lambda_1^{(g+1-t)+1}$ vanishes on $\beta_t$.

**Proof.** This follows from the fact that $\lambda_1$ is ample on $A_g^*$ and the dimension of $A_{g-t}$. Indeed, a multiple of $\lambda_1$ is represented by a hyperplane and the dimension of $\beta_t$ is $\binom{g+1-t}{2}$.

The stack $\tilde{A}_g$ is defined over $\mathbb{Z}$ and one can consider the Chow ring and the subring generated by the Chern classes $\lambda_i$ of the Hodge bundle $E$ and the ‘boundary’ classes $\sigma_j$ for all fibres $\tilde{A}_g \otimes \mathbb{F}_p$ and $\tilde{A}_g \otimes \mathbb{C}$. In this paper we restrict ourselves to the complex case because we use references to papers where this restriction is made. I see no serious obstacles for extending these results to all fibres.

§2 **Ring Structure for $g = 1$ and $g = 2$.**

In this section we describe the Chow rings for $g = 1$ and $g = 2$. The formulation that we give is intended to bring out similarities in ring structure. The structure of the Chow
ring of $\tilde{A}_1$ is well known. Recall that $\lambda_1$ denotes the first Chern class of the Hodge bundle; we denote the $\mathbb{Q}$-class of the boundary $\beta_1$ by $\sigma_1$.

**Theorem 1.** The Chow ring of $\tilde{A}_1$ is generated by $\lambda_1$ and $\sigma_1$ and is isomorphic to

$$\mathbb{Q}[\lambda_1, \sigma_1]/(\lambda_1 \sigma_1, \sigma_1 - 12 \lambda_1) \cong \mathbb{Q}[\lambda_1]/(\lambda_1^2).$$

Note that if we work over $\mathbb{Z}$ the relation $\sigma_1 \lambda_1$ should be replaced by $2 \sigma_1 \lambda_1 = 0$ because $2 \sigma_1$ represents the class of a ‘physical’ point. We thus expect over $\mathbb{Z}$ the ring $\mathbb{Z}[\lambda_1]/(24 \lambda_1^2)$, cf. the result of [E-G].

The structure of the Chow ring of the moduli stack $\mathcal{M}_2$ of stable curves of genus 2 was determined by Mumford in [M3]. We can identify the stack $\mathcal{M}_2$ with $\tilde{A}_2$ and obtain thus the structure of the Chow ring of $\tilde{A}_2$. Here $\tilde{A}_2$ is a canonical toroidal compactification and it was already considered by Igusa. Because we use this result for $g = 3$ we reprove this result by a method we also use for $g = 3$.

For every level $\ell$ there is a canonical compactification $\tilde{A}_2[\ell]$ of $A_2[\ell]$, the moduli space of principally polarized abelian surfaces with a full symplectic level $\ell$ structure. This compactification is described in [Y]. The compactification is obtained by adding a divisor $D[\ell]$ to $A_2[\ell]$. The finite group $G_\ell \cong Sp(2g, \mathbb{Z}/\ell \mathbb{Z})$ acts on $\tilde{A}_2[\ell]$ and there is an equivariant finite morphism $p_\ell: \tilde{A}_2[\ell] \rightarrow \tilde{A}_2$. Classes in the Chow group $CH^k(\tilde{A}_2[\ell])$ which are $G_\ell$-invariant give rise to classes in $CH^k(\tilde{A}_2)$.

We denote by $\sigma_i[\ell] \in CH^i(\tilde{A}_2[\ell])$ the $i$-th elementary symmetric function in the components of the divisor $D[\ell]$. Since it is $G_\ell$-invariant it defines a class $\sigma_i = \sigma_i[1]$ in $CH^i(\tilde{A}_2)$ and this class has the property

$$p_\ell^*(\sigma_i) = \ell^i \sigma_i[\ell].$$

Besides these elements we have the Chern classes $\lambda_i$ of the Hodge bundle $E$ on $\tilde{A}_2$. In [vdG] I showed that these Chern classes satisfy in the Chow ring $CH^*(A_g)$ the relation

$$(1 + \lambda_1 + \lambda_2)((1 - \lambda_1 + \lambda_2) = 1$$
on $A_2$. The proof of the following theorem will show that this relation still holds on $\tilde{A}_2$.

**Theorem 2.** The Chow ring of $\tilde{A}_2$ is generated by $\lambda_1, \lambda_2, \sigma_1$ and is isomorphic to

$$\mathbb{Q}[\lambda_1, \lambda_2, \sigma_1]/I_2$$

with $I_2$ the ideal generated by the relations

$$(1 + \lambda_1 + \lambda_2)(1 - \lambda_1 + \lambda_2) = 1,$$

$$\lambda_2 \sigma_1 = 0,$$

$$\sigma_1^2 = 22 \sigma_1 \lambda_1 - 120 \lambda_1^2.$$

The ranks of the Chow groups are: 1, 2, 2, 1.

We can write this ring also as:

$$\mathbb{Q}[\lambda_1, \sigma_1]/(\lambda_1^2 \sigma_1, (\sigma_1 - 10 \lambda_1)(\sigma_1 - 12 \lambda_1)).$$
The proof will consist of two steps: i) we bound the ranks of $\text{CH}^k(\tilde{A}_2)$ for $k = 1, 2, 3$ and ii) by using intersection numbers we show that the above generators are sufficient. The relations then follow from the intersection numbers.

We begin by bounding the ranks. There is a well-known finite morphism $A_3 \rightarrow A_2 - A_{1,1}$ which shows the triviality of the Chow groups of $A_2 - A_{1,1}$. This shows that $\text{CH}^1(\tilde{A}_2)$ is generated by the class of $\tilde{A}_{1,1}$ and by $\sigma_1$. Since $A_{1,1}$ has trivial Chow ring we see that $\text{CH}^2$ is generated by classes living (in codimension 1) on $D$. Now $D$ is the image of $X_1 \rightarrow D$ and has in codimension 1 two generators: the class of the zero section and the class of a fibre. Since $\tilde{A}_2$ is unirational we find that $\text{CH}^3$ has rank 1.

(2.1) Conclusion. The ranks $r_k = \text{rank}(\text{CH}^k(\tilde{A}_2))$ are bounded by $r_1 \leq 2, r_2 \leq 2$. Moreover $r_0 = 1, r_3 = 1$.

Now we calculate intersection numbers. We start with the intersection numbers for the classes $\sigma_i$. One calculates

$$\deg(\sigma_3[\ell]) = \#0\text{-dim strata} = \frac{1}{3} \ell \mu_1(\ell) \mu_2(\ell) = \frac{1}{12} \gamma / \ell^3.$$ 

leading to $\sigma_3 = 1/12$ on the stack. Indeed, each of the $\mu_2(\ell)$ 1-dimensional cusps carries $\mu_1(\ell)$ 0-dimensional cusps and each of these occurs on three transversally intersecting 1-dimensional cusps. A calculation (cf. [Y, T1]) shows that for two different components $F, G$ of $D[\ell]$ that intersect we have: $\deg(F^2G) = -2$. We thus find

$$\deg(\sigma_1[\ell] \sigma_2[\ell]) = 3 \deg(\sigma_3([\ell])) - 2\ell \mu_1[\ell] \mu_2[\ell] = -\frac{1}{4} \gamma / \ell^3.$$ 

Using proportionality (in cohomology) and the fact that $\lambda_1^2$ can be represented by a compactly supported cycle on $\tilde{A}_2$ (use the Satake compactification) we find $\deg(\lambda_1^2) = (1/2880) \gamma$. Furthermore, for $i : \tilde{X}_1 \rightarrow \tilde{A}_2$ we have

$$i_* (1_{\tilde{X}_1}) = 2\sigma_1$$

and $i^*(\sigma_1)$ on $\tilde{X}_1$ is in a fibre $F$ of $X_1 \rightarrow A_1$ equal to $-2\Theta$ with $\Theta$ the theta divisor there. On the other hand $i^*(\lambda_1) = [F]/24$, where $F$ is a general fibre of $\tilde{X}_1 \rightarrow A_1$. This gives $\deg(\lambda_1 \sigma_1[\ell]^2) = (-1/24) \gamma / \ell^2$. The relation $12\lambda_1 = \sigma_1$ for $g = 1$ gives

$$\sigma_2 = 6\lambda_1 \sigma_1$$

for $g = 2$. We summarize these intersection numbers in a table.

**Table 2a 2 × 1.**

|   | $\lambda_1$ | $\sigma_1$ |
|---|-------------|-------------|
| $\lambda_1^2$ | $1/2880$ | 0 |
| $\lambda_1 \sigma_1$ | 0 | $-1/24$ |

The table shows that $\lambda_1$ and $\sigma_1$ are linearly independent in codimension 1 as are $\lambda_1^2$ and $\lambda_1 \sigma_1$ in codimension 2. Thus we have the generators. The restriction of $\lambda_2$ to any boundary class is zero since the Hodge bundle possesses a line bundle quotient on the closure $\tilde{D}$ of $D$. 

Moreover, \(\deg(\lambda_1 \lambda_2) = 1/5760\) since \(\lambda_1^2 = 2\lambda_2\) on \(A_2\) and this now shows that \(\lambda_2^2 = 2\lambda_2\) on \(\tilde{A}_2\). We also know that \(\sigma_1^2\) is a linear combination of \(\lambda_1^2\) and \(\lambda_1 \sigma_1\) and can determine the coefficients if we know \(\deg(\sigma_1^2)\). We know that \(\sigma_1^2 = -120 \lambda_1^2 + a \lambda_1 \sigma_1\) for some \(a \in \mathbb{Q}\). In order to determine \(a\) we shall use the following lemma.

\[\begin{align*}
\textbf{(2.2) Lemma.} & \quad \text{The class of the products of elliptic curves satisfies} \\
& \quad [\tilde{A}_{1,1}] = 5\lambda_1 - \frac{1}{2} \sigma_1.
\end{align*}\]

This is well known, cf. [Ig]; compare also [Mu2]. One can use the modular form of weight 10 which vanishes along \(A_{1,1}\). Since \(12\lambda_1 = \sigma_1\) in \(g = 1\) we get \((12\lambda_1 - \sigma_1)(5\lambda_1 - (1/2)\sigma_1) = 0\). This gives the relation

\[\sigma_1^2 = 22\sigma_1 \lambda_1 - 120\lambda_1^2\]

that we want and also the value for \(\deg(\sigma_1^3)\).

\[\begin{array}{|c|c|c|}
\hline
\sigma_3 & \sigma_2 \sigma_1 & \sigma_1^3 \\
\hline
1/4 & -1/4 & -11/12 \\
\hline
\end{array}\]

\[\begin{align*}
\textbf{(2.3) Remark.} & \quad \text{The } Q\text{-classes } \sigma_i \text{ with } i = 1, \ldots \text{ satisfy } \sigma_1 = [\beta_1], \sigma_2 = [\beta_2] \text{ here for } g = 2, \text{ but in general } \sigma_i \text{ is not a multiple of } \beta_i.
\end{align*}\]

Consider now the class \(Z\) of the closure of the image cycle of \(\tilde{j}: \tilde{A}_1 \to \tilde{A}_2\) obtained from sending \([X]\) to \([X \times E]\), with \(E\) a fixed generic elliptic curve. The pull back of \(\lambda_1\) is \(\lambda_1\) on \(\tilde{A}_1\) and the pull back of \(\sigma_1\) is \(\sigma_1\). In other words, \(12\lambda_1 - \sigma_1\) vanishes under pull back. If we write \(Z = a \lambda_1^2 + b \lambda_1 \sigma_1\) this gives \((12/2880) a - (-1/24) b = 0\), i.e. \(a = -10 b\).

\[\begin{align*}
\textbf{(2.4) Proposition.} & \quad \text{The } Q\text{-class of the closure of the locus of trivial extensions of elliptic curves } [B_2] \text{ is} \\
& \quad [B_2] = 120 \lambda_2 - \sigma_2.
\end{align*}\]

Note also that one has (cf. Mumford [M2] p. 368)

\[\begin{align*}
[B_2] = N_0 \sigma_1 = (5\lambda_1 - \sigma_1/2)\sigma_1.
\end{align*}\]

(cf. Mumford [M2] p. 368), and the two formulas agree by the relations \(\lambda_1^2 = 2\lambda_2\), \(\sigma_1^2 = 22\sigma_1 \lambda_1 - 120\lambda_1^2\) and \(\sigma_2 = 6\lambda_1 \sigma_1\).

\[\begin{align*}
\textbf{§3 Ring Structure for } g = 3.
\end{align*}\]

We wish to describe the full Chow ring of the standard Delaunay-Voronoi compactification for \(g = 3\). For a precise description of this compactification see Tsushima [T1]. Tsushima constructs a smooth compactification \(\tilde{A}_3[\ell]\) for all \(\ell \geq 3\). These compactifications are equipped with an action of \(G_\ell = Sp(6, \mathbb{Z}/\ell\mathbb{Z})\) and with equivariant morphisms

\[p_\ell \colon \tilde{A}_3[\ell] \to \tilde{A}_3.\]
Let $\gamma = \# G_\ell$ here be the order of this group.

The Chow ring $CH_\mathbb{Q}(\tilde{A}_3)$ is defined to be the invariant part of the Chow ring of $\tilde{A}_3[\ell]$ in the sense of [E-G]. It contains the $G_\ell$-invariant classes on $\tilde{A}_3[\ell]$. As a subring it contains the ring generated by the $\lambda_i$.

The difference $\tilde{A}_3[\ell] - A_3[\ell]$ is a union of divisors. Let $\sigma_i[\ell] \in CH_i_\mathbb{Q}(\tilde{A}_3[\ell])$ be the $i$-th symmetric function of these divisors and set $\sigma_i = \sigma_i[1]$. From the ramification properties along these divisors we derive

$$ p_\ell^*(\sigma_i) = \ell^i \sigma_i[\ell]. $$

**Theorem 3.** The Chow ring of $\tilde{A}_3$ is generated by the $\lambda_i$, $i = 1, 2, 3$ and the $\sigma_i$, $i = 1, 2$ and is isomorphic to the graded ring (subscript is degree)

$$ \mathbb{Q}[\lambda_1, \lambda_2, \lambda_3, \sigma_1, \sigma_2]/I, $$

with $I$ the ideal generated by the relations:

$$ (1 + \lambda_1 + \lambda_2 + \lambda_3)(1 - \lambda_1 + \lambda_2 - \lambda_3) = 1, $$

$$ \lambda_3 \sigma_1 = \lambda_3 \sigma_2 = \lambda_1^2 \sigma_2 = 0, $$

$$ \sigma_1^3 = 2016 \lambda_3 - 4 \lambda_1^3 \sigma_1 - 24 \lambda_1 \lambda_2 \sigma_2 + \frac{11}{3} \sigma_2 \sigma_1, $$

$$ \sigma_2^2 = 360 \lambda_3^2 \sigma_1 - 45 \lambda_1^2 \sigma_1^2 + 15 \lambda_1 \lambda_2 \sigma_1, $$

$$ \sigma_1^2 \sigma_2 = 3 \sigma_2^2 - 30 \lambda_1^2 \sigma_1^2 + 2 \lambda_1 \lambda_2 \sigma_2. $$

The ranks of the Chow groups are: 1, 2, 4, 6, 4, 2, 1.

For the proof we proceed as before. We first bound the ranks of the Chow groups. Alternatively, this could be done by using the results of Faber [F] and the morphism $t: \overline{M}_3 \to \tilde{A}_3$, see §4. Analysis of the generators for $\overline{M}_3$ leads to bounds on the ranks of the Chow groups. But we proceed here more directly and use parametrizations (as in [F]) to bound the ranks.

**3.1 Proposition.** The tautological ring of $A_3$ generated by the Chern classes $\lambda_i$ of the Hodge bundle is isomorphic to $\mathbb{Q}[\lambda_1]/(\lambda_1^4)$.

**Proof.** In [vdG] I showed that on $A_3$ the relations

$$ (1 + \lambda_1 + \lambda_2 + \lambda_3)(1 - \lambda_1 + \lambda_2 - \lambda_3) = 1, $$

$$ \lambda_3 = 0 $$

hold. This implies the result. □

Recall that the locus of abelian varieties whose theta divisor is singular yields a divisor $N_0$ as defined by Mumford. Its class is given by $N_0 = 18 \lambda_1 - 2 \sigma_1$ on $\tilde{A}_3$, cf. [I], [M2].

Another divisor is obtained as the zero divisor of a modular form of weight 140. Let $\psi$ be the modular form of weight 140 which is the product of the 8-th powers of the 35 $\theta[\epsilon]$ for $\epsilon \neq 0$ and let $\Psi$ be its zero divisor on $\tilde{A}_3$. It is not difficult to see that the intersection of $\Psi$ and $N_0$ is $240 A_{2,1}$. Analysis of how they intersect on $D$ leads to the following value for the class of $\tilde{A}_{2,1}$.
(3.2) Proposition. The $Q$-class of $\tilde{A}_{2,1}$ on $\tilde{A}_3$ is given by
\[
[\tilde{A}_{2,1}] = \frac{21}{2} \lambda_1^2 - \frac{5}{2} \lambda_1 \sigma_1 + \frac{1}{8} \sigma_1^2 + \frac{1}{24} \sigma_2.
\]

Proof. Let $R$ be the $Q$-class of the closure of the locus of semi-abelian varieties of $t$-rank 1 whose abelian part is a product of elliptic curves. We have $R = 5 \lambda_1 \sigma_1 - \sigma_2$ which comes from genus 2, cf. Lemma (2.2). We have from [I] and [T1]:
\[
N_0 \Psi = 240 [\tilde{A}_{2,1}] + 10 R.
\]

We thus have
\[
(18 \lambda_1 - 2 \sigma_1)(140 \lambda_1 - 15 \sigma_1) = 240 [\tilde{A}_{2,1}] + 50 \lambda_1 \sigma_1 - 10 \sigma_2.
\]

(3.3) Proposition. The ring $CH^\ast (\tilde{A}_3)$ is generated by $\lambda_1$ and the images of classes on $D$.

Proof. We know $[A_{2,1}] = (21/2)\lambda_1^2$ on $A_3$. On the other hand we have the Torelli map of degree 2
\[
t : \mathcal{M}_3(\mathbb{C}) \longrightarrow (A_3 - A_{2,1})(\mathbb{C}) \quad C \mapsto \text{Jac}(C)
\]
and from the fact that $CH^\ast (\mathcal{M}_3) = \mathbb{Q}[\lambda_1]/(\lambda_1^2)$ (cf. [F]) we see that $CH^\ast (A_3)$ is generated by $\lambda_1$ and $CH^\ast (A_{2,1})$. The description of the Chow rings of $A_1$ and $A_2$ implies that $CH^\ast (A_{2,1})$ is generated by $p_2^\ast (\mu_1)$ (with $\mu_1$ the first Chern class of the Hodge bundle on $A_2$). But this class can be obtained from $\lambda_1[A_{2,1}]$. So $CH^\ast (A_3)$ is generated by $\lambda_1$. □

(3.4) Corollary. The ring $CH^\ast (A_3)$ is generated by $\lambda_1$. The group $CH^1(\tilde{A}_3)$ is generated by $\lambda_1$ and $\sigma_1$. These are linearly independent.

The linear independence follows e.g. from $\lambda_1^6 \neq 0$ and $\lambda_1^5 \sigma_1 = 0$, which follows immediately by using the Satake compactification.

We now stratify the moduli space according to torus rank:
\[
\tilde{A}_3 = A_3 \cup \tilde{A}_3^{t=1} \cup \tilde{A}_3^{t=2} \cup \tilde{A}_3^{t=3},
\]
with $\text{codim}(\tilde{A}_3^{t=j}) = j$.

In view of Proposition (3.3) we now first determine the Chow groups of $\tilde{A}_3^{t \geq 1} = \tilde{A}_3 - A_3$. From Section 1 we have a morphism $j : \tilde{X}_2 \rightarrow \tilde{A}_3^{t \geq 1}$ with $\tilde{X}_2$ the universal fibering over $\tilde{A}_2$ and this is of degree 2 on $\tilde{X}_2$. On $\tilde{X}_2$ we have the pull-backs $\tilde{\lambda}_i, \tilde{\sigma}_i$ of the classes $\lambda_i, \sigma_i$ on $\tilde{A}_2$ and the class $t$ coming from the theta-divisor on the universal abelian surface. We normalize $t$ such that $t|S$ is trivial, where $S$ is the zero-section. The class of $S$ is denoted $s$. 

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(3.5) Lemma. The Chow group \( CH^1(\tilde{A}_2) \) is generated by \( t, \tilde{\lambda}_1 \) and \( \tilde{\sigma}_1 \).

Proof. Consider the generic fibre \( X_\eta \) with dual variety \( \hat{X}_\eta \). For the codimension 1 Chow group we have \( CH^1(X_\eta) = \mathbb{Z} \oplus \hat{X}_\eta \). But the only section of \( \hat{X}_2 \) over \( A_2 \) is the zero section. If \( L \) is a line bundle on \( \hat{X}_2 \) then \( L \otimes t^m \) is trivial on \( X_\eta \) for some \( m \) and \( \pi_*(L) \) is a line bundle on \( \tilde{A}_2 \) with \( \pi^*(\pi_*(L)) \cong L \). So \( L \) can be written as \( L \cong t^{-m} \otimes \pi^*(\lambda_1^p \otimes \sigma_1^q) \).

\[ \square \]

Using the \( g = 2 \) results and Lemma 1.1 we obtain the following table:

**Table 3a.** 4 \times 1 (i.e. codim 4 \times codim1) on \( \tilde{A}_2 \).

| 4\1 | \( \lambda_1 \) | \( \tilde{\sigma}_1 \) | \( t \) |
|-----|------------------|------------------|-----|
| \( \lambda_1^3t \) | 0                | 0                | 1/1440 |
| \( \tilde{\lambda}_1^2s \) | 1/2880           | 0                | 0    |
| \( \tilde{\lambda}_1^2\tilde{\sigma}_1s \) | 0                | -1/24            | 0    |

It follows from this that \( CH^1(\tilde{A}_3^{t\geq 1}) \) is generated by three elements: \( \lambda_1\sigma_1, \sigma_1^2 \) and \( \sigma_2 \). There is a relation between the generators of \( CH^2(\tilde{A}_3) \) and of \( CH^1(\tilde{A}_3^{t\geq 1}) \). By Lemma 1.1 we have:

\[ \sigma_1^2 \equiv -t \mod (\lambda_1\sigma_1) \text{ on } \tilde{A}_3^{(t \leq 1)}. \]

(3.6) Proposition. The Chow group \( CH^2(\tilde{A}_3) \) is generated by 4 elements: \( \lambda_1^2, \lambda_1\sigma_1, \sigma_1^2 \) and \( \sigma_2 \).

Next we turn to \( CH^3(\tilde{A}_3) \). This group is generated by the generator \( \lambda_1^3 \) of \( CH^3(\tilde{A}_3) \) and the codimension 2 classes on \( \tilde{A}_3^{t\geq 1} \). To determine these we decompose the space \( \tilde{A}_3^{t\geq 1} \) in the following four pieces (of dimensions 5, 4, 4, 3):

\( U := \tilde{A}_3^{t=1} - \pi_2^{-1}(A_{1,1}) - S, \quad \Xi := \pi_2^{-1}(A_{1,1}), \quad \tilde{A}_3^{t\geq 2} \) and \( S^0 := S - (S \cap \pi^{-1}(A_{1,1})) \).

Note that \( \Xi \) is a quotient of \( \text{Sym}^2(A_1) \) and it is fibred over \( \text{Sym}^2(A_1) = \mathbb{A}^2 \). It is a \( \mathbb{P}^2 \)-bundle over \( \mathbb{A}^2 \), from which we deduce \( CH^1(\Xi) \cong \mathbb{Q}, CH^2(\Xi) \cong \mathbb{Q} \) and \( CH^k(\Xi) = 0 \) for \( k \geq 3 \). In accordance with Lemma (3.5) we have \( CH^1(U) \cong \mathbb{Q} \).

(3.7) Lemma. The space \( U \) has trivial Chow groups \( CH^k \) for \( k \geq 2 \). The Chow ring of \( S^0 \) is trivial (i.e. \( \cong \mathbb{Q} \)). Furthermore, \( CH^k(\Xi) \cong \mathbb{Q} \) for \( k = 1, 2 \) and \( CH^k(\Xi) = 0 \) for \( k > 2 \). The Chow ring of \( \tilde{A}_3^{t=1} \) is generated by \( \lambda_1 \) and \( t \).

Proof. As in [F] we can parametrize an open subset of \( \tilde{A}_3^{t=1} \) by an open subset of the \( \mathbb{P}^5 - \text{Quadric of plane quartics with a node} \). Faber’s Lemma (1.14) implies the first statement. The second statement follows from the map \( \mathbb{A}^3 \rightarrow S^0 \), see above. The statements about \( \Xi \) follow from the above. The last statement follows from the decomposition \( \tilde{A}_3^{t=1} = U \cup \Xi \cup S \) and \( S \cong A_2 \). \[ \square \]

Summarizing, we find that \( CH^3(\tilde{A}_3) \) is generated by \( \lambda_1^3 \), by the class of \( S \), by the codimension 1 class \( (\lambda_1 t) \) on \( \Xi \) and by \( CH^1(\tilde{A}_3^{t\geq 2}) \).

The morphism \( \tilde{X}_2 \rightarrow \tilde{A}_3^{t\geq 1} \) induces a morphism \( \phi : \pi_2^{-1}(\tilde{A}_2^{t\geq 1}) \rightarrow \tilde{A}_3^{t\geq 2} \). The stratum \( \tilde{A}_2^{t\geq 2} \) is now the image of \( \pi_2^{-1}(\tilde{A}_2^{t\geq 1}) \) which is fibred over \( \tilde{A}_2^{t\geq 1} \), which itself is a quotient of \( \tilde{X}_1 \). The fibres are the image of a compactified \( \mathbb{G}_m \)-bundle over an elliptic
The stratum $\tilde{A}_3^{t=2}$ is thus the image of a compactified $\mathbb{G}_m$-bundle over $X_1 \times \mathcal{A}_t$, $\tilde{X}_1$. We call the divisor added to the $\mathbb{G}_m$-bundle $\Delta$. Since $\phi$ factors through $\text{Sym}^2(\mathcal{X}/\pm 1)$ we see that our stratum $\mathcal{A}_3^{t=2} - \phi(\Delta)$ is an affine bundle over a $\mathbb{P}^2$-bundle over $A^1$, the $j$-line, hence contributes only via its $CH^k \cong \mathbb{Q}$ for $k = 0, 1, 2$. The divisor $\Delta$ is also a $\mathbb{P}^2$-bundle over an affine line. Using these results on the structure of $\tilde{A}_3^{t=2}$ we find:

(3.8) Lemma. For $\tilde{A}_3^{t=2}$ we have $\text{rk}(CH^k) \leq 2$ for $k = 1, 2$ and $\text{rk}(CH^3) \leq 1$.

Collecting results, we get that $CH^2(\tilde{A}_3^{t=1})$ is generated by the class of $S$, the codimension 1 class $(\lambda_1 t)$ on $\Xi$ and $CH^1(\tilde{A}_3^{t=2})$. This gives as generators: $s$, $\lambda_1^2$, $\lambda_1 \sigma_1$, $\lambda_1 t$, and $\sigma_1 t$ and we have the bound $r_3 \leq 6$ for the rank $r_3$ of $CH^3(\tilde{A}_3)$.

The 3-dimensional toroidal stratum $\tilde{A}_3^{t=3}$ can be viewed using the map $\pi_2$ and $\phi$ as a quotient under a finite map of a surface bundle over a configuration of $\mathbb{P}^1$-s. The fibres are a configuration of $\mathbb{P}^1 \times \mathbb{P}^1$-s or a configuration of $\mathbb{P}^2$-s and $\mathbb{P}^2$-s blown up in three points, cf. Tsushima [T1, Lemma (4.4) and (7.1)]. Using this description one sees that $\text{rk}(CH^1(\tilde{A}_3^{t=3})) \leq 2$. Indeed, we find three strata that can contribute, but by writing down a rational function one gets a non-trivial relation between them. For codimension 2 we find $\text{rk}(CH^2(\tilde{A}_3^{t=3})) \leq 2$; it is however not difficult to see that the two 1-dimensional strata in $\tilde{A}_3^{t=3}$ generate a 1-dimensional space. But we do not need this latter estimate; it suffices to remark that all possible generators of $CH^5(\tilde{A}_3)$ lie in the ring generated by the $\lambda_i$ and the $\sigma_i$.

Collecting again, we find that $CH^4(\tilde{A}_3)$ is generated by a generator of $CH^2(\Xi)$, by the two generators of $CH^2(\tilde{A}_3^{t=2})$ and by the generators of $CH^1(\tilde{A}_3^{t=3})$. This gives the bound $r_4 \leq 5$ (and similarly $r_5 \leq 3$). Moreover, one sees that the generators we gave can be expressed using the $\sigma$’s and the $\lambda$’s.

Table 3b $3 \times 2$ on $\tilde{X}_2$.

| $3 \setminus 2$ | $\lambda_1^3$ | $\lambda_1^2 \sigma_1$ | $s$ | $\lambda_1 t$ | $t \sigma_1$ |
|-----------------|----------------|------------------------|-----|---------------|--------------|
| $\lambda_1$    | 0              | 0                      | 1/2880 | 0             | 0            |
| $\lambda_1^2 t$| 0              | 0                      | 0    | 1/1440        | 0            |
| $\lambda_1 s$  | 1/2880         | 0                      | 1/5760 | 0             | 0            |
| $\tilde{\sigma}_1 s$ | 0   | -1/24                  | 0    | 0             | 0            |
| $\tilde{\lambda}_1 \tilde{\sigma}_1 t$ | 0 | 0                      | 0    | -1/12        |              |

(3.9) Conclusion. The ranks $r_i = \text{rank}(CH^i(\tilde{A}_3^{t=1}))$ are bounded by $r_0 = r_5 = 1$, $r_1 = 3$, $r_2 \leq 5$, $r_3 \leq 5$ and $r_4 \leq 3$.

(3.10) Conclusion. The ranks of the Chow groups of $\tilde{A}_3$ satisfy: $r_0 = r_6 = 1$, $r_1 = 2$, $r_2 \leq 4$, $r_3 \leq 6$, $r_4 \leq 5$ and $r_5 \leq 3$. 

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(3.11) Lemma. We have $\lambda_3 \sigma_1 = 0$ and $\lambda_1^2 \sigma_2 = 0$.

Proof. Since the Hodge bundle restricted (to a component of) $\tilde{D}[\ell]$ possesses a subbundle of rank 2 on each irreducible component of $D[\ell]$ with trivial quotient its top Chern class vanishes there. The second statement follows from Lemma (1.2). \(\blacksquare\).

By the classical proportionality of Hirzebruch in cohomology we have $\deg \lambda_1^6 = \gamma/181440$ in level $\ell$. We interpret this simply as

$$\lambda_1^6 = \frac{1}{181440}$$

in level 1 omitting the degree. Note that $\tilde{A}_3$ is known to be unirational.

The work of Tsushima implicitly contains all the information about the intersection numbers on $\tilde{A}_3$ and $\tilde{D}$ that we need. We give the results in the form of tables and indicate how they can be obtained.

We begin with intersection numbers for the sigma-classes, cf. Lemma (7.7) of [T1] plus the correction in [T2].

Table 3c. The $\sigma$’s. (Multiply entries by $\gamma/\ell^6$ in level $\ell$.)

| $\sigma_6$ | $\sigma_5 \sigma_1$ | $\sigma_4 \sigma_2$ | $\sigma_4 \sigma_1^2$ | $\sigma_3^2$ | $\sigma_3 \sigma_2 \sigma_1$ | $\sigma_3 \sigma_1^3$ | $\sigma_2^3$ | $\sigma_2^2 \sigma_1^2$ | $\sigma_2 \sigma_1^4$ | $\sigma_1^6$ |
|------------|----------------------|----------------------|-----------------------|--------------|--------------------------|------------------|-------------|----------------------|---------------|------------------|
| $\frac{1}{48}$ | $-\frac{1}{16}$ | $\frac{3}{16}$ | $\frac{13}{48}$ | $\frac{41}{144}$ | $\frac{1}{16}$ | $-\frac{13}{48}$ | $-\frac{15}{16}$ | $-\frac{47}{48}$ | $-\frac{445}{144}$ | $\frac{1}{4103}$ |

Table 3d. Products $\lambda_1 \times$ a product of $\sigma$’s. (Multiply entries by $\gamma/\ell^5$ in level $\ell$.)

| $\lambda_1 \backslash \sigma_7$ | $\sigma_5$ | $\sigma_4 \sigma_1$ | $\sigma_3 \sigma_2$ | $\sigma_3 \sigma_1^2$ | $\sigma_2^3$ | $\sigma_2 \sigma_1^3$ | $\sigma_1^5$ |
|-----------------|---------|------------------|------------------|------------------|-------------|------------------|-------------|
| $\lambda_1$    | $0$     | $0$              | $1/48$           | $1/48$           | $-1/16$    | $-11/48$        | $-203/240$  |

Another intersection number that we need is

$$\lambda_1^3 \sigma_1^3 = \frac{1}{720}.$$

This follows easily from $\lambda_1^3 | D = (1/2880) F$ with $F$ a generic fibre of $\tilde{D} \to \tilde{A}_2$ and the expression for $\sigma_1 | D$ of Lemma (1.1). Indeed, under the pull-back to $\tilde{X}_2$ we have $\deg(j^*(\lambda_1^3 \sigma_1^3)) = \deg(4t^2/2880) = 1/360$. Dividing by the degree of $j$ gives the result.

The following tables can be constructed with this information. Together with the bounds on the ranks of the Chow groups these will show that we found the generators.

Table 3e. $1 \times 5$.

| $1 \backslash 5$ | $\lambda_1^5$ | $\lambda_1^3 \sigma_1^2$ |
|----------------|---------------|--------------------------|
| $\lambda_1$   | $1/181440$    | $0$                      |
| $\sigma_1$    | $0$           | $1/720$                  |
Table 3f. \(2 \times 4\)

| \(2 \times 4\) | \(\lambda_1^4\) | \(\lambda_1^3\sigma_1\) | \(\lambda_1^2\sigma_1^2\) | \(\lambda_1\sigma_2\sigma_1\) |
|----------------|----------------|----------------|----------------|----------------|
| \(\lambda_1^2\) | 1/181440       | 0              | 0              | 0              |
| \(\lambda_1\sigma_1\) | 0              | 0              | 1/720          | 0              |
| \(\sigma_1^2\) | 0              | 1/720          | 0              | -11/48         |
| \(\sigma_2\) | 0              | 0              | 0              | -1/16          |

Table 3g. \(3 \times 3\)

| \(3 \times 3\) | \(\lambda_3\) | \(\lambda_1^3\) | \(\lambda_1^2\sigma_1\) | \(\lambda_1\sigma_2\) | \(\lambda_1\sigma_1^2\) | \(\sigma_2\sigma_1\) |
|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| \(\lambda_3\) | 0              | 1/1451520      | 0              | 0              | 0              | 0              |
| \(\lambda_1^3\) | 1/1451520      | 1/181440       | 0              | 0              | 1/720          | 0              |
| \(\lambda_1\sigma_1\) | 0              | 0              | 0              | 0              | 0              | -1/16          |
| \(\lambda_1\sigma_2\) | 0              | 0              | 0              | 0              | 0              | -11/48         |
| \(\lambda_1\sigma_1^2\) | 0              | 0              | 1/720          | 0              | 0              | -11/48         |
| \(\sigma_2\sigma_1\) | 0              | 0              | 0              | -1/16          | -11/48         | -47/16         |

From these tables and the bounds for the ranks we get the relations:

\[
\sigma_3 = -40 \lambda_1^2\sigma_1 + \frac{44}{3} \lambda_1\sigma_2 - \frac{1}{3} \sigma_2\sigma_1,
\]

\[
\sigma_1^3 = 2016 \lambda_3 - 4 \lambda_2^2\sigma_1 - 24 \lambda_1\sigma_2 + \frac{11}{3} \sigma_2\sigma_1.
\]

Note that the class of the locus \(\beta_3\) is known: \(\lambda_1\sigma_2 = \frac{1}{4}\beta_3\).

We wish to determine the class of \(B_3\), that is, the \(Q\)-class of the cycle which is the closure of the cycle \(Z\) which is the image of the map

\[j: A_2 \rightarrow A_3, \quad [X] \mapsto [X \times E],\]

where \(E\) is a generic elliptic curve. This is a map of degree 2 in the sense of stacks: \(j_+(1) = 2[Z]\). Under pull back we find

Table 3h. Intersection with \(B_3\)

| deg \(j^*\) | \(\lambda_3\) | \(\lambda_1^3\) | \(\lambda_1^2\sigma_1\) | \(\lambda_1\sigma_2\) | \(\lambda_1\sigma_1^2\) | \(\sigma_2\sigma_1\) |
|-------------|---------------|----------------|----------------|----------------|----------------|----------------|
| 0           | 1/2880        | 0              | 0              | -1/24          | -1/4           |

Note that because of \(\lambda_2\sigma_1 = 0\) for \(g = 2\) we have in view of Table 2.2 \(j^*(\sigma_2\sigma_1) = (3/11)j^*(\sigma_1^3) = (3/11) \times (-11/12)\). This gives:

\[3.12\] Proposition. The \(Q\)-class of the closure of the locus of trivial extensions of abelian surfaces is given by

\[[B_3] = 252 \lambda_3 - 15 \lambda_2^2\sigma_1 + 2 \lambda_1\sigma_2.\]

We can also determine the class of \(\tilde{A}_{1,1,1}\).
(3.13) Proposition. The $Q$-class of the closure of the locus of products of three elliptic curves is given by

$$[\tilde{A}_{1,1,1}] = -35\lambda_3 + \frac{35}{2}\lambda_1^3 - \frac{25}{4}\lambda_1^2\sigma_1 + \frac{5}{8}\lambda_1\sigma_2 + \frac{5}{8}\lambda_1^2\sigma_1 - \frac{1}{12}\sigma_2\sigma_1$$

Proof. A priori we find (by restricting the Hodge bundle)

$$[\tilde{A}_{1,1,1}] = 35(\lambda_1\lambda_2 - \lambda_3) + r$$

with $r$ orthogonal to $\lambda_3$ and $\lambda_1^3$. We know that $(12\lambda_1 - \sigma_1)$ annihilates the class of $\tilde{A}_{1,1,1}$ and this suffices to determine $r$; alternatively one calculates the intersection numbers with our basis of $CH^3$ directly:

$$[1/82944, 1/13824, 1/1152, 1/192, 1/96, 1/16].$$

This implies the assertions in Theorem 3 about the Chow groups $CH^i$ for $i \leq 3$. As to $CH^4$, we know that its rank is $\leq 5$. Moreover it follows from the considerations above that $CH^4(\tilde{A}_3)$ is generated by elements of the subring generated by the $\lambda_i$ and the $\sigma_i$. In view of the relations above this leaves us with the following generators for $CH^4$:

$$\lambda_4, \lambda_3^2\sigma_1, \lambda_2^2\sigma_1, \lambda_1\sigma_1\sigma_2, \sigma_1^2\sigma_2, \sigma_2^2.$$  

I claim that we have two independent relations between these. Indeed, all these classes are represented by cycles on $\tilde{A}_{3}^{(\geq 2)}$ and as we observed above we have $CH^2(\tilde{A}_{3}^{(\geq 2)}) \leq 4$. Using the intersection numbers we find the relations expressing $\sigma_2^2$ and $\sigma_1^2\sigma_2$ in terms of the others. Similarly, in codimension 5 the classes generating the Chow group lie in the subring generated by the $\lambda_i$ and the $\sigma$'s. But the codimension 5 part of this subring has rank 2. So we get $r_5 = 2$. The unirationality of $\tilde{A}_3$ implies $r_6 = 1$. The completes the proof of Theorem 3.

§4 Comparison with the Chow ring of $\overline{M}_3$

Let $\overline{M}_3$ be the moduli space of stable curves of genus 3. The usual Torelli map $t: \mathcal{M}_3 \to \mathcal{A}_3$ can be extended to a morphism $\bar{t}: \overline{M}_3 \to \tilde{A}_3$ in the following way: associate to a curve $C$ the (coarse) moduli space $P$ of semi-stable torsion free rank 1 sheaves of degree $g - 1$ on $C$. This moduli space comes provided with a ‘theta’ divisor $B$ whose points correspond to the semi-stable sheaves with $h^0 \neq 0$. Such a pair $(P, B)$ defines in a natural way a point of the Delaunay-Voronoi compactification $\tilde{A}_3$. The Delaunay decomposition of $\mathbb{Z}^t$ with $t = \text{torus rank of } P$ corresponding to $C$ is obtained as follows. Let $\Gamma$ be the dual graph of the curve. The space $C_1(\Gamma)$ is provided with the standard Euclidean Delaunay decomposition, i.e. the standard cube, its faces and translates. It induces a decomposition on the linear subspace $H_1(\Gamma)$. This is the Delaunay decomposition of $\mathbb{Z}^t$, cf. [A, AN].

The Chow group $CH^1(\overline{M}_3)$ is generated by $t^*(\lambda_1), t^*(\sigma_1) = \delta_0$, and a new element denoted in [F] by $\delta_1$. It comes from the reducible curves with components of genus 1
and genus 2. The locus $\Delta_1$ of these reducible curves maps to the codimension 2 locus $\tilde{A}_{2,1}$. So in codimension 1 the map $t_* : CH^1(\overline{M}_3) \to CH^1(\tilde{A}_3)$ is given by

$$
t_*(\lambda_1) = 2\lambda_1,
$$
$$
t_*(\delta_0) = 2\sigma_1,
$$
$$
t_*(\delta_1) = 0.
$$

In codimension 2 the group $CH^2(\overline{M}_3)$ is generated by the four pull backs under $t^*$ of the classes $\lambda_1^2, \lambda_1\sigma_1, \sigma_1^2$ and $\sigma_2$ and three new generators. These are

$$
\lambda_1\delta_1, \sigma_1\delta_1, \delta_{1,1},
$$

where $\delta_{1,1}$ is the $Q$-class of the closure of the locus of genus 3 curves which are a string of elliptic curves, i.e. with dual graph

$$
\begin{array}{ccc}
\bullet & \bullet & \bullet
\end{array}
$$

These elements are mapped to zero under $t_*$ and with the notation $[a, b, c, d]$ for $a\lambda_1^2 + b\lambda_1\sigma_1 + c\sigma_1^2 + d\sigma_2$ the images under the map $t: \overline{M}_3 \to \tilde{A}_3$ are:

$$
t_*(\delta_{00}) = 2 \times [0, 0, 0, 1]
$$
$$
t_*(\xi_0) = 2 \times [0, 4, -1, 1]
$$
$$
t_*(\xi_1) = 2 \times [0, 5/2, 0, -1/2]
$$
$$
t_*(\eta_1) = 2 \times [63/2, -15/2, 3/8, 1/8] = 6 \times [\tilde{A}_{2,1}]
$$

For the reader’s convenience we also give

$$
t_*(\kappa_2) = 2 \times [41/2, -7/2, 1/8, 1/24]
$$
$$
t_*(\delta_1^2) = 2 \times [-21/2, 5/2, -1/8, -1/24] = -2 \times [\tilde{A}_{2,1}].
$$

We have (cf. [M2]): $t_*(\xi_0 + 2\xi_1) = 2 \times (9\lambda_1 - \sigma_1) \sigma_1 = [N_0]$. From [F, p. 368] we have the relation

$$
\delta_1^2 = 3\lambda_1\delta_1 - \frac{1}{3}\sigma_1\delta_1 - \frac{21}{2}\lambda_1^2 + \frac{5}{2}\lambda_1\sigma_1 - \frac{1}{8}\sigma_1^2 - \frac{1}{24}\sigma_2.
$$

Under $t_* : CH^3(\overline{M}_3) \to \tilde{A}_3$ the images of the classes $[(b)]_Q, [(d)]_Q, [(f)]_Q, [(g)]_Q$ are zero, while the image of $[(i)]_Q$ is $[\tilde{A}_{1,1,1}]$.

The following table gives the images of the generators of $CH^3(\overline{M}_3)$ under $t_*$. 

\begin{table}[h]
\begin{tabular}{|c|c|}
\hline
Generator & Image \\
\hline
$\kappa_2$ & $[41/2, -7/2, 1/8, 1/24]$ \\
$\delta_1^2$ & $[-21/2, 5/2, -1/8, -1/24]$ \\
$\xi_0 + 2\xi_1$ & $[9\lambda_1 - \sigma_1] \sigma_1$ \\
$\xi_0$ & $[0, 4, -1, 1]$ \\
$\xi_1$ & $[0, 5/2, 0, -1/2]$ \\
$\eta_1$ & $[63/2, -15/2, 3/8, 1/8]$ \\
$\delta_{00}$ & $[0, 0, 0, 1]$ \\
$\delta_0$ & $[0, 4, -1, 1]$ \\
$\delta_1$ & $[0, 5/2, 0, -1/2]$ \\
$\lambda_1$ & $2\lambda_1$ \\
$\sigma_1$ & $2\sigma_1$ \\
$\sigma_2$ & $0$ \\
\hline
\end{tabular}
\end{table}
Table 4a. Image under $t_*$

| $t_*(\cdot)/2$ | $\lambda_3$ | $\lambda_1^3$ | $\lambda_1^2\sigma_1$ | $\lambda_1\sigma_2$ | $\lambda_1\sigma_1^2$ | $\sigma_2\sigma_1$ |
|----------------|-------------|----------------|----------------|----------------|----------------|----------------|
| $t_*(\lambda_3)/2$ | 1           | 0              | 0              | 0              | 0              | 0              |
| $t_*(\lambda_3^2)/2$ | 0           | 1              | 0              | 0              | 0              | 0              |
| $t_*[(a)]Q/2$ | 0           | 0              | 0              | 4              | 0              | 0              |
| $t_*[(c)]Q/2$ | 0           | 0              | 30             | -6             | 0              | 0              |
| $t_*[(e)]Q/2$ | 0           | 0              | -40            | 32/3           | 0              | -1/3           |
| $t_*[(h)]Q/2$ | 0           | 0              | -25/2          | -5/2           | -5/4           | 1/4            |
| $t_*[(i)]Q/2$ | -35/2       | 35/4           | -25/8          | 5/16           | 5/16           | -1/24          |
| $t_*(\eta_0)/2$ | 756         | 0              | -45            | 6              | 0              | 0              |

A check on this is given by the relation

$$504 \lambda_3 = \frac{1}{2} \delta_{000} + \xi_{01} + \frac{2}{3} \eta_0$$

which can be deduced from [F], cf. [F2], p. 77. Here $\xi_{01}$ is the locus of curves consisting of an elliptic curve which intersects a degenerate elliptic curve in 2 points. Under $t_*$ this is translated into $504 \lambda_3 = 2 \lambda_1 \sigma_2 + t_*([(c)]_Q) + 2B_3$. So by the expression for $[B_3]$ we find $t_*(\xi_{01}) = 6\lambda_1(5\lambda_1 \sigma_1 - \sigma_2)$ which fits with the formula for $[(c)]_Q$ and the one for $[A_{1,1}]$ in genus 2. Another check is obtained from the formula

$$\delta_0^3 = \frac{47}{15}[(a)]_Q - \frac{54}{5}[(b)]_Q - \frac{54}{5}[(c)]_Q - \frac{89}{15}[(d)]_Q - 11[(e)]_Q + \frac{8}{5}[(f)]_Q + 8[(g)]_Q + \frac{8}{3} \eta_0$$

(cf [F, p. 411]). Under $t_*$ one finds our relation for $\sigma_1^3$.

§5 Concluding Remarks.

In general one cannot expect that the Chow ring of a suitable compactification $\bar{A}_g$ is generated by the tautological classes $\lambda_i$ and $\sigma_i$. But it makes sense to consider the subring generated by these elements. The relations satisfied by the $\lambda_i$ are known. I do not know whether a relation of the sort:

$$\sigma_1^g = \zeta(1 - 2g)\lambda_g + \text{classes on } \bar{A}_g^{\geq 2}$$

holds in the Chow ring. In cohomology it does. Another approach would be to study the Chow rings of the canonical partial compactification $\bar{A}_g^{(1)}$. For $g = 1, 2$ and 3 we find the following rings:

$g = 1$:

$$\mathbb{Q}[\lambda_1, \sigma_1]/(\lambda_1\sigma_1, \sigma_1 - 12\lambda_1)$$

$g = 2$:

$$\mathbb{Q}[\lambda_1, \sigma_1]/(\lambda_1\sigma_1, \sigma_1^2 + 120\lambda_1^2).$$

$g = 3$:

$$\mathbb{Q}[\lambda_1, \lambda_3, \sigma_1]/(\lambda_1^4 - 8\lambda_3\lambda_1, \lambda_1^2\sigma_1, \sigma_1^3 - 2016\lambda_3, \lambda_3\sigma_1).$$
As a final remark we observe that the Chow ring of $\tilde{X}_2$ is generated as an algebra over the Chow ring of $\tilde{A}_2$ (using the map $\pi_2^*$) by the element $t$. It satisfies the quadratic equation
\[ t^2 = -\tilde{\lambda}_1^2 + 2s + t\tilde{\sigma}_1/8. \]
Indeed, a priori, $t^2$ is a linear combination $a\tilde{\lambda}_1^2 + b\tilde{\lambda}_1\tilde{\sigma}_1 + cs + d\tilde{\lambda}_1t + et\tilde{\sigma}_1$. But in view of the $3 \times 2$-table for $\tilde{X}_2$ the relation $ts = 0$ implies that $b = 0$. Because $s^2 = \tilde{\lambda}_2s$ and $t^2$ has degree 2 in a general fibre we find $a = -1$ and $c = 2$. From [vdG] we know $(\pi_2)_*(t^3/3!) = \sigma_1/24$. This implies $d = 0$ and $e = 1/8$.

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