An effective action for monopoles and knot solitons in Yang-Mills theory

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Abstract

By comparison with numerical results in the maximal Abelian projection of lattice Yang-Mills theory, it is argued that the nonperturbative dynamics of Yang Mills theory can be described by a set of fields that take their values in the coset space SU(2)/U(1). The Yang-Mills connection is parameterized in a special way to separate the dependence on the coset field. The coset field is then regarded as a collective variable, and a method to obtain its effective action is developed. It is argued that the physical excitations of the effective action may be knot solitons. A procedure to calculate the mass scale of knot solitons is discussed for lattice gauge theories in the maximal Abelian projection. The approach is extended to the SU(N) Yang-Mills theory. A relation between the large N limit and the monopole dominance is pointed out.

1 Knot solitons and SU(2) Yang-Mills theory

The action

\[ S = \int d^4x \left\{ m^2 (\partial_\mu n)^2 + H_{\mu\nu}^2 \right\}, \] (1.1)

\[ H_{\mu\nu} = g^{-1} n \cdot (\partial_\mu n \times \partial_\nu n), \] (1.2)

where the field \( n(x) \) is a unit three-dimensional vector, \( n \cdot n = 1 \), \( m \) is a mass scale and \( g \) is a coupling constant, describes knot solitons in four-dimensional spacetime. The stability of solitons is due to the conservative charge known as the Hopf invariant \[4]. The knot solitons have a finite energy and, therefore, may be identified with particle-like excitations, provided a physical interpretation is given to the \( n \) field. An interesting relation between the action (1.1) and the SU(2) Yang-Mills theory action emerges if one takes the connection of the form \[2\]

\[ A_\mu = g^{-1} \partial_\mu n \times n \equiv A_\mu(n), \] (1.3)

where boldface letters are used to denote isovectors being elements of the adjoint representation of SU(2), and calculates the corresponding field strength

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g A_\mu \times A_\nu, \] (1.4)
one finds that
\[ F_{\mu\nu} = nH_{\mu\nu}. \] (1.5)

That is, the second term of the action (1.1) is the Yang-Mills action for the connection of the special form (1.3). It is therefore rather natural to conjecture that the first term can be generated by an interaction of the collective variable \( n \) with the other modes of the full Yang-Mills theory \[ [3, 4] \]. If such a conjecture is true, it would mean that the SU(2) quantum Yang-Mills theory has particle-like excitations being knot solitons. These excitations might be good candidates for glueballs. Since the position of a knot soliton is specified by a contour in space \[ [5, 6] \], an effective action for the field \( n \) would also provide a quantum field description of Polyakov’s strings.

In this letter we discuss a physical interpretation of the field \( n \). By analyzing recent developments in lattice gauge theories, we argue that the special Yang-Mills connections (1.3) describe the most relevant physical degrees of freedom of the Yang-Mills theory in the confinement phase. We develop a procedure to calculate their effective action. The mass scale \( m^2 \) is described in terms of expectation values of some functionals of \( A_\mu \). We also propose a numerical procedure to calculate an effective action of the field \( n \) using the Wilson ensemble of the lattice gauge theory in the maximal Abelian projection. The approach is extended to the SU(N) Yang-Mills theory, where an interesting relation between dynamics of the coset field and the large \( N \) limit is observed.

## 2 A general parameterization of the SU(2) connection

Consider a partition function of the SU(2) Yang-Mills theory
\[ Z \sim \int D A_\mu e^{-S(A)}. \] (2.1)

Here \( S(A) \) is the Yang-Mills action. We assume that some gauge fixing has been made to remove the divergence of the integral (2.1) caused by the gauge invariance of the action \( S \). For what follows the gauge choice is not important. In the integral (2.1) we want to make a change of integration variables
\[ A_\mu = g^{-1} \partial_\mu n \times n + C_\mu n + W_\mu, \] (2.2)

where the first two terms is the connection introduced by Cho \[ [2] \], which we denote by \( \alpha_\mu \), i.e., \( A_\mu = \alpha_\mu + W_\mu \), and the isovector \( W_\mu \) is perpendicular to \( n \), that is, \( n \cdot W_\mu = 0 \). The idea is then to integrate out \( C_\mu \) and \( W_\mu \) and obtain an effective action for \( n \). However, we observe that the number of independent field variables in the left hand side of Eq.(2.2) is 12, while in the right hand side is 14 (4 in \( C_\mu \), 2 in \( n \) and 8 in \( W_\mu \)). To make a change of variables, we have to impose two more conditions on \( W_\mu \).

Before doing so, let us analyze the gauge transformation law in the new variables. A gauge transformation of \( A_\mu \) contains three functional parameters
\[ \delta A_\mu = g^{-1} \partial_\mu \omega + A_\mu \times \omega \equiv g^{-1} \nabla_\mu (A) \omega. \] (2.3)
The variations $\delta C_\mu$, $\delta n$ and $\delta W_\mu$ that induce the gauge transformations (2.3) of the connection (2.2) should depend on five functional parameters because they may also involve variations under which $A_\mu$ does not change at all. Note that the number of the new variables exceeds that of the old variables exactly by 2. Consider a special subset of these five-parametric transformations which has the form

$$\delta n = n \times \omega, \quad \delta C_\mu = g^{-1} n \cdot \partial_\mu \omega, \quad \delta W_\mu = W_\mu \times \omega. \quad (2.4)$$

We have $\delta \alpha_\mu = g^{-1} \nabla_\mu (\alpha) \omega$ and, therefore, the transformations (2.4) induce the gauge transformations (2.3). The condition $n \cdot W_\mu = 0$ is invariant under the transformations (2.4). Hence, the gauge transformed configurations have the same form (2.2).

If we impose two additional conditions on $W_\mu$ which are covariant under the transformations (2.4), then the transformations (2.4) can be uniquely identified as the gauge transformations of the new variables. We propose the following conditions

$$\nabla_\mu (\alpha) W_\mu = \partial_\mu W_\mu + g C_\mu n \times W_\mu + n (\partial_\mu n \cdot W_\mu) = 0. \quad (2.5)$$

It is not hard to see that the covariant derivative $\nabla_\mu (\alpha) W_\mu$ transforms as an isovector under (2.4). Moreover, taking the dot product of the right hand side of Eq. (2.3) and $n$, we find $n \cdot \partial_\mu W_\mu + \partial_\mu n \cdot W_\mu = \partial_\mu (n \cdot W_\mu) \equiv 0$ because the isovectors $n$ and $W_\mu$ are perpendicular. Thus, the right hand side of Eq. (2.3) is an isovector perpendicular to $n$ and, therefore, the condition (2.3) implies only two independent conditions on $W_\mu$ as required.

So, we have obtained a change of variables and identified the gauge transformation law of the new variables. In principle one can inverse it and find the new variables as functional of $A_\mu$. We will discuss this later upon constructing the path integral measure for the new variables. One should point out that there are infinitely many ways to parameterize the Yang-Mills connection. Natural questions arise. Why is the parameterization we have chosen so special? Why is an effective action for the collective variable $n = n(A)$ has something to do with the dynamics of the Yang-Mills theory in the confinement phase? Let us discuss these important questions before we turn to constructing an effective action for $n$.

### 3 Properties of the new parameterization of the SU(2) connection

Eqs. (2.2) and (2.3) determine a complete parameterization of a generic SU(2) connection. The parameterization has several remarkable properties which we are going to discuss. Consider an isovector $b_\mu = b_\mu(n)$ which is constructed of the field $n$ and its derivatives so that $b_\mu \cdot n = 0$. Since $\partial_\mu n \cdot n = 0$, such an isovector can always be taken as a linear combinations $b_\mu = b_\mu, \partial_\mu n$ with the coefficients $b_\mu$ being functions of $n$ and its derivatives.

Let us construct a complex scalar field

$$\Phi = b_\mu \cdot W_\mu + i n \cdot (b_\mu \times W_\mu). \quad (3.1)$$

Consider gauge transformations that leave the field $n$ unchanged. These are rotations about $n$, i.e., $\omega = \xi n$ in (2.4). We get $\delta C_\mu = g^{-1} \partial_\mu \xi$ and $\delta \Phi = i \xi \Phi$. That is, the field $C_\mu$
is a Maxwell field with respect to the gauge subgroup U(1) which is a stationary group of $n$, while the scalar field $\Phi$ plays the role of a charged field. If for some reasons we would like to describe the SU(2) Yang-Mills theory as an effective QED, the set of fields $C_\mu$, $\Phi$ and $n$ is just enough to carry all physical degrees of freedom of the original SU(2) theory. This can be understood as follows. The field $W_\mu$ has six independent components in our parameterization of the SU(2) connection. Using the isotopic rotations we may impose two gauge conditions on $W_\mu$ to break SU(2) to U(1). Among the remaining four components of $W_\mu$ we can always select two (real) components which are to be identified with the complex scalar field $\Phi$ by a suitable choice of $b_\mu$. The other two components are uniquely fixed by the Gauss law $\delta S/\delta A_0 = 0$. Recall that the SU(2) gauge theory has three Lagrange multipliers $A_0$. If we impose a gauge condition on the dynamical variables $A_i$ ($i = 1, 2, 3$) with the aim to get a dynamical description only in terms of the physical degrees of freedom, then the Lagrange multipliers must be obtained by solving the Gauss law for $A_0$ in the gauge (or parameterization) chosen for $A_i$. The Gauss law imposes a restriction on possible parameterizations of connections $A_\mu$ via only physical variables. Since U(1) is left unbroken as the gauge group of the theory, only two equations in the Gauss law are to be solved so that the Lagrange multiplier $C_0$ associated with the U(1) symmetry does not get fixed. Thus, in such an effective Abelian gauge theory, which is dynamically equivalent to the non-Abelian SU(2) theory, the field $W_\mu$ carries only two physical degrees of freedom associated with the complex scalar field (3.1). Here only an effective dynamics of $n$ (an effective field theory for Polyakov’s strings) will be discussed and, therefore, an explicit parameterization of $W_\mu$ via physical variables is not relevant. Such a parameterization is important to formulate duality properties in Yang-Mills theory.

What is the physical meaning of the field $n$ in the effective Abelian gauge theory described above? To answer this question let us make an Abelian projection (in ’t Hooft’s terminology) of the Yang-Mills theory by imposing a gauge on the field $n$ rather than on $W_\mu$. By a suitable gauge rotation of the isovector $n$ we can always direct it along, say, the third coordinate axis in the isospace, i.e., $n = n_0 = (0, 0, 1)$. In this case the field $W_\mu$ would have four physical degrees of freedom (in the sense of the Hamiltonian formalism as described above). Moreover, since the conditions (2.5) are covariant under gauge transformations, the gauge transformed fields $C_\mu$, $\Phi$ and $n$ should also satisfy them. Setting $n$ equal to $n_0$ in (2.5), the latter turns into the condition which is well known in lattice Yang-Mills theories as the maximal Abelian gauge.

The reason of why the maximal Abelian gauge is so special in lattice Yang-Mills theory is the following. Suppose configurations $A_q^q$, $q = 1, 2, ..., Q$, are elements of the Wilson ensemble, i.e., they are generated by means of the Monte-Carlo method with the Boltzmann probability $\exp[-S(A)]$. An expectation value of any quantity $F(A)$ is $\langle F \rangle = Q^{-1} \sum_q F(A^q)$. For example, one can take $F = W_C$ to be the Wilson loop. From $\langle W_C \rangle$ one finds the string tension $\sigma_{su(2)}$ which is a coefficient in the linearly rising part of the potential between a heavy quark and antiquark. For every configuration $A_q^q$ one can find a gauge transformation $U = U(A^q)$ such that the gauge transformed configuration has the form $U A_q^q = n_0 U C_\mu^q + U W_\mu^q$ where $U W_\mu^q$ and $U C_\mu^q$ satisfy the condition (2.5)
with \( n = n_0 \). That is, every configuration from the Wilson ensemble have been gauge transformed to satisfy the maximal Abelian gauge. Now one takes only the Abelian parts \( n U^C q \mu \) of \( U^A q \mu \) and use them to calculate \( \langle W_C \rangle \) again. As \( U^C q \mu \) are Abelian configurations (the action is quadratic in \( C_\mu \)), one would expect the perimeter law corresponding to the Coulomb interaction of static sources rather than the area law, i.e., one would expect the corresponding string tension \( \sigma_{u(1)} \) to vanish. A surprising numerical result is that such a procedure gives the area law again, and \( \sigma_{u(1)} \) is nearly the same as the full string tension \( \sigma_{su(2)} \) (to be exact, it is 92 per cent of \( \sigma_{su(2)} \)). The phenomenon is called the Abelian dominance [10, 9]. It might look rather mysterious especially in view that \( W_C(A) \) is a gauge invariant quantity.

The paradox disappears if one observes that the gauge transformations \( U(A_q) \) that are used to implement the maximal Abelian gauge are not regular in spacetime (e.g., \( U^\dagger [\partial_\mu, \partial_\nu] U \neq 0 \)). Strictly speaking, they are not gauge transformations because the tensor quantities, like, e.g., the field strength, are no longer transformed homogeneously. For this reason the Abelian vector potential \( U^C q \mu = n_0 \cdot U^A q \mu \) has singularities (or defects) which have quantum numbers of Dirac magnetic monopoles with respect to the unbroken \( U(1) \) gauge group. Therefore the Maxwell vector potential \( U^C q \mu \) describes dynamics of photons and also that of Dirac magnetic monopoles which are physical degrees of freedom of the Wilson ensemble in the maximal Abelian gauge.

Thus, the singular transformations \( U(A_q) \) transfer some relevant physical degrees of freedom from the non-Abelian components (perpendicular to \( n_0 \)) to the Abelian ones (parallel to \( n_0 \)). These degrees of freedom have been identified as Dirac magnetic monopoles. Locations of monopoles can be found by studying the magnetic flux carried by \( U^C q \mu \) through surfaces around each dual lattice site [8]. Since regular (photon) configurations of \( U^C q \mu \) cannot provide the area law for the Wilson loop, the monopole part of \( U^C q \mu \) must be expected to give a major contribution to \( \sigma_{u(1)} \approx \sigma_{su(2)} \). If one removes the photon part from \( U^C q \mu \) and calculate \( \langle W_C \rangle \) using only the monopole part of \( U^C q \mu \), the corresponding string tension \( \sigma_m \) differs from \( \sigma_{u(1)} \) only by 5 per cent, i.e., \( \sigma_m \approx \sigma_{u(1)} \approx \sigma_{su(2)} \). This is known as the monopole dominance [11, 9].

The conclusion one obviously arrives at is that the nonperturbative Yang-Mills dynamics favors configurations which look like Dirac magnetic monopoles in the maximal Abelian gauge. An interpretation of these degrees of freedom as Dirac monopoles is gauge dependent, but the very fact of their existence is certainly gauge independent. These configurations capture the most relevant degrees of freedom of the Yang-Mills theory in the confinement phase. This is a nontrivial dynamical statement discovered in lattice gauge theories. Note that an explicit form of \( A_q \mu \) is determined by the full Yang-Mills action, and it is by no means obvious that the dynamics must be such that the configurations that saturate the area law should look like Dirac magnetic monopoles in some particular gauge.

In our parameterization of the SU(2) connection the maximal Abelian gauge is a simple algebraic gauge \( n = n_0 \). Therefore we expect that for some connections \( A_q \mu \) the gauge

\[^2\text{In fact, this is the case for any Abelian projection [9], but not for every Abelian projection the Abelian (or monopole) dominance holds [10].}\]
transformation which transforms \( n(A^q) \) to the special form \( n_0 \) is not regular, and the gauge transformed Abelian potential \( U C^q_\mu \) should have singularities which have quantum numbers of Dirac monopoles with respect residual gauge group U(1) (rotations about \( n_0 \)). As the condition (2.5) is covariant under the gauge transformations (2.4) and turns into the maximal Abelian gauge if we gauge transform \( n \rightarrow n_0 \), \( C_\mu \rightarrow U C^q_\mu = n_0 \cdot U A^q_\mu \) and \( W_\mu \rightarrow U W^q_\mu = U A^q_\mu - n_0 (n_0 \cdot U A^q_\mu) \), where \( U(A^q) \in SU(2)/U(1) \) is defined so that \( U A^q_\mu \) satisfies the maximal Abelian gauge, we conclude that our field \( n \) should have the form

\[
n(A^q) = \frac{1}{2} \text{tr} \left[ \tau U^\dagger(A^q) \tau_3 U(A^q) \right].
\] (3.2)

Here the components of the isovector \( \tau \) are the Pauli matrices, \( \text{tr} (\tau_a \tau_b) = 2 \delta_{ab} \). The relation (3.2) determines the configurations of the field \( n \) for the Wilson ensemble. The ensemble (3.2) contains all information about dynamics of the monopole degrees of freedom of the maximal Abelian projection by construction.

In the continuum theory it is also easy to give examples of \( n \) with such properties. If we set \( n(x, t) = g x/r \), where \( r = |x| \), then the configuration (3.3) is the famous non-Abelian monopole of Wu and Yang [12]. Observe that the Wu-Yang monopole corresponds to zero configurations of \( C_\mu \) and \( W_\mu \) in our parameterization. Now we can implement the maximal Abelian projection. We can always find an orthogonal matrix that transforms \( n \) to the special form \( n_0 = (0, 0, 1) \). This gauge transformation creates nonzero \( C_\mu \) according to (3.4). The gauge transformed configuration becomes purely Abelian, and \( C_\mu \) describes a Dirac magnetic monopole localized at the origin. Note that the orthogonal matrix used to direct \( n \) along the third coordinate axis is singular. This singularity generates a singularity of \( C_\mu \) on the Dirac string extended along the negative part of the third coordinate axis. In general, we can set \( n = \phi/\phi \), where \( \phi \) is the norm of an isovector field \( \phi \). If \( \phi(x) = 0 \), then a solution to this equation is a collection of worldlines \( x_\mu = x_\mu(s) \) which are identified with the worldlines of Dirac monopoles carried by \( C_\mu \) after the singular gauge transformation \( n \rightarrow n_0 \):

\[
g^{-1} \partial_\mu n \times n + C_\mu n \rightarrow \left[ g^{-1} n_0 \cdot (\partial_\mu \xi \times \xi) + C_\mu \right] n_0 = U C_\mu n_0 = (C^\xi_\mu + C_\mu) n_0.
\] (3.3)

where \( \xi = (\sin(\theta/2) \cos \varphi, \sin(\theta/2) \sin \varphi, \cos(\theta/2)) \) if \( n = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta) \).

The first term \( C^\xi_\mu \) in \( U C_\mu \) is the vector potential of magnetic monopoles in the maximal Abelian projection.

Thus, the collective variable \( n(A) \) captures the degrees of freedom responsible for the Abelian or monopole dominance discovered in lattice gauge theories. According to lattice simulations, in the confinement phase the monopole current (in the maximal Abelian projection) is dense, while in the deconfinement phase it is very dilute [9]. Frankly speaking, the monopoles condense in the confinement phase, and one needs a field (an order parameter) to describe their dynamics. A single Maxwell field \( U C_\mu \) cannot describe photons and monopoles simultaneously because it has no independent components available as the “monopole” field. Note that \( U C_\mu \) is a sum of the photon field \( C_\mu \) and the monopole field \( C^\xi_\mu \) in the maximal Abelian projection (3.3). Now we recall that the “monopole”
interpretation of the physical degrees of freedom of Yang-Mills theory which saturate the area law is associated with a specific gauge. But once these degrees of freedom have been identified as those of the field $n$, there is no need in that gauge anymore. Instead of imposing the (maximal Abelian) gauge $n = n_0$ and facing a hard problem to find a quantum field description of monopole defects in $U C_{\mu}$, we say that in the confinement phase the collective field $n = n(A)$ should be identified as the most relevant degree of freedom. Therefore its effective action should capture the main features of the nonperturbative Yang-Mills dynamics. This is a gauge invariant approach to the “monopole” dynamics because it uses only a reparameterization of the original Yang-Mills dynamical variables via new collective variables and no gauge fixing. The effective dynamics of $n$ can be studied in any Abelian projection (gauge) if so desired.

The action in the variables \eqref{2.2} has the form

$$S(A) = \frac{1}{4} \left\{ G_{\mu\nu}^2 + 2(\nabla_\mu W_\nu)^2 + 4(G_{\mu\nu} n + \nabla_\mu W_\nu) \cdot W_{\mu\nu} + W_{\mu\nu}^2 \right\}, \quad (3.4)$$

where $\nabla_\mu = \nabla_\mu(\alpha)$, $W_{\mu\nu} = W_\mu \times W_\nu$ and $G_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu - H_{\mu\nu}$. It is quadratic in $C_\mu$ so the Maxwell field can be integrated out, while the integral over $W_{\mu\nu}$ can only be done perturbatively, or in a stationary phase approximation by invoking instanton solutions, or numerically. As a point of fact, lattice simulations show a correlation between monopole-antimonopole loops and instantons [13]. So an instanton induced interaction of the field $n$ might be an important piece of the effective action of $n$. In this regard it is noteworthy [4] that Witten’s multi-instanton Ansatz [14] can be written in our parameterization of the SU(2) connection as follows

$$n = g x/r, \quad W_0 = 0, \quad W_i = \varphi_2(r, t) \partial_i n \times n + \varphi_1(r, t) \partial_i n, \quad C_i = B_1(r, t) n_i, \quad C_0 = B_2(r, t). \quad (3.5)$$

Here $B_{1,2}$ and $\varphi_{1,2}$ are, respectively, two-dimensional electromagnetic and scalar fields introduced by Witten. Note that the condition \eqref{2.3} is satisfied identically for the Ansatz \eqref{3.3}.

4 An effective action for the $n$ field

To construct an effective action for the field $n$, we observe that the change of variables proposed in section 2 allows us to find two equations which determine $n$ as an implicit functional of $A_\mu$. Indeed, we have $C_\mu = n \cdot A_\mu$. Therefore

$$W_\mu = A_\mu - n(A_\mu \cdot n) - g^{-1} \partial_\mu n \times n \equiv A_\mu - \alpha_\mu(A, n). \quad (4.1)$$

The field $W_\mu$ should satisfy the condition \eqref{2.3}, which leads to the desired equations for $n = n(A)$:

$$\chi(A, n) \equiv \nabla_\mu(A) \alpha_\mu(A, n) - \partial_\mu A_\mu = 0. \quad (4.2)$$

It is easy to verify that $n \cdot \chi \equiv 0$. So the isovector $\chi$ is always perpendicular to $n$ and, hence, Eq. \eqref{1.3} contains only two independent equations for two independent components of $n$. Consider the functional $\Delta = \Delta(A, n)$ defined by the equation

$$1 = \int Dn \Delta(A, n) \delta(\chi). \quad (4.3)$$
Substituting the identity (4.3) into (2.1) we find an equivalent representation of the Yang-Mills partition function

\[
Z \sim \int Dn e^{-S_{eff}(n)}, \quad (4.4)
\]

\[
S_{eff}(n) = -\ln \int DA \Delta(A, n) \delta(\chi) e^{-S(A)}. \quad (4.5)
\]

The integral over Yang-Mills fields in (4.5) could be done analytically only either by perturbation theory or by means of instantons. In the latter case the functional integral is replaced by an ordinary integral over the instanton moduli space, while \( A_\mu \) is replaced by an instanton configuration. To develop perturbation theory, one should write the functional \( \Delta \) as an integral over ghost fields. From the definition (4.3) of \( \Delta \) it follows that \( \Delta = \det(\delta \chi / \delta n) \). Introducing a set of complex ghost fields \( \eta \) such that \( \eta \cdot n = 0 \), we have

\[
\Delta(A, n) = \int D\eta^\dagger D\eta \exp \left\{ -S_{gh}(\eta^\dagger, \eta, A, n) \right\}, \quad (4.6)
\]

\[
S_{gh} = \int d^4x \eta^\dagger \cdot \nabla_\mu(A) [\eta(n \cdot A_\mu) + n(\eta \cdot A_\mu) + \partial_\mu n \times \eta - n \times \partial_\mu \eta]. \quad (4.7)
\]

The delta function of \( \chi \) in (4.3) can be written in the exponential form via the Fourier transform.

The ghosts introduced have nothing to do with a gauge fixing. As has been pointed out before, any gauge fixing condition can be assumed in (2.1) by an appropriate modification of the measure \( DA_\mu \). Note that the integral (4.3) is invariant under simultaneous gauge transformations of \( A_\mu \) and \( n \) (cf. (2.3) and (2.4)). If the ghost representation (4.6), (4.7) is used, then the ghost field is transformed as \( \delta \eta = \eta \times \omega \). By imposing a gauge on \( A_\mu \), this gauge freedom is removed. Therefore a gauge can be chosen to make computation of the integral (4.3) convenient (e.g., the Lorentz gauge for perturbation theory, or a background gauge if the instanton technique is used). Accordingly, there will also be a conventional set of ghost fields associated with the gauge chosen.

It is convenient to make a shift of the integration variables \( A_\mu \rightarrow A_\mu + g^{-1} \partial_\mu n \times n \) to extract the tree level contribution to the effective action (the second term in (4.1)). If the effective action (4.3) supports knot solitons as collective excitations, then in the gradient expansion of (4.3) there should exist the mass scale \( m^2 \neq 0 \) which is determined by the equation

\[
\left. \frac{\delta^2 S_{eff}(n)}{\delta n_a(x) \delta n_b(y)} \right|_{\delta n = 0} = -m^2 \delta_{a b} \delta_\mu^2 \delta(x - y). \quad (4.8)
\]

Substituting (4.3) into (4.8) we obtain the mass scale in terms of expectation values of certain operators in the Yang-Mills theory. As has been argued in [3], the mass scale term must be a leading term in the gradient expansion of the effective action \( S_{eff}(n) \). However, it may also appear to be zero. Since there is no monopole dominance in the perturbative regime, there must be a nonperturbative input into calculation of \( m^2 \).

One way to do so is to compute \( m^2 \) numerically. Suppose we have a Wilson ensemble of \( A_\mu \). For every configuration \( A_\mu \) one can calculate the group element \( U(A) \) such that \( UA_\mu \).
satisfies the maximal Abelian gauge. Next, by means of (3.2) one can obtain an ensemble of the field \( n \). The problem is to find the Boltzmann probability \( \exp(-S_{\text{eff}}(n)) \) which generates the ensemble of \( n \). Such a problem can be solved by the so-called inverse Monte-Carlo method \([15]\). This method has recently been applied to calculate an effective action of the monopole current in the maximal Abelian projection \([16]\). It would be interesting to apply this numerical technique to prove the existence of the mass scale \( m^2 \). The radiative corrections to the action (1.1) can be obtained by perturbation theory proposed above.

The mass scale term in (1.1) looks like a mass term in the (reduced) Yang-Mills theory of the special connection (1.3)

\[
m^2 \partial_\mu n \cdot \partial_\mu n = g^2 m^2 A^2_\mu(n) .
\]

(4.9)

This suggests also that the inverse Monte-Carlo method can be applied to the ensemble of \( A_\mu(n) \) directly. If one takes the ensemble of monopole connections \( n_0 C_\mu^k \) in the maximal Abelian projection (i.e., after the projection all off-diagonal components of the connections as well as the photon part in the diagonal components are set to zero), then the ensemble of \( A_\mu(n) \) can be obtained by gauge transformations \( U(A) \) of \( n_0 C_\mu^k \) because by construction (3.3) we have \( U(A)(n) = n_0 C_\mu^k \). Here \( U(A) \) are gauge group elements which are used to implement the maximal Abelian gauge on the Wilson ensemble.

## 5 A generalization to the SU(N) gauge group

A realistic theory has the gauge group SU(3). The monopole dominance has also been established for the SU(3) lattice gauge theory in the maximal Abelian projection (see, e.g., \([9]\) and references therein). Here we construct a parameterization of the SU(N) connection such that the “monopole” degrees of freedom in the maximal Abelian projection are described by a coset field \( SU(N)/[U(1)]^{N-1} \). The parameterization is similar, but not the same, to the parameterization proposed recently by Faddeev and Niemi \([17]\), and it also differs from the parameterization of Periwal \([18]\). To develop an effective action of the coset field, we follow the method discussed above for the SU(2) case, meaning that no explicit elimination of nonphysical degrees of freedom is made, and thereby a hard problem (unsolved in \([17, 18]\)) of solving the Gauss law is avoided. We shall point out an interesting relation emerging between the large N limit and the monopole dominance.

Let \( h_i, i = 1, 2, ..., N-1 \), be a basis of the Cartan subalgebra of the algebra \( su(N) \). We assume the basis to be orthonormal with respect to the Killing form \( (h_i, h_k) = \delta_{ik} \). Recall that for any two elements of a Lie algebra the Killing form is defined as \( (y, z) = \text{tr}(\hat{y}\hat{z}) \), where \( \hat{y}z = [y, z] \) and \([,] \) is a Lie product in the Lie algebra. In a matrix representation \( (y, z) = c \text{tr}(yz) \), where \( c \) depends on the Lie algebra \([15]\) \((c = 2N \text{ for } su(N))\). Consider an orthonormal Cartan-Weyl basis \([19]\) so that every Lie algebra element can be decomposed as

\[
\omega = \sum_{\beta>0} \left( \omega^c_{\beta}c_\beta + \omega^s_{\beta}s_\beta \right) + \sum_k \omega_k h_k .
\]

(5.1)

Here \( \beta \) ranges over all positive roots of the algebra, and the coefficients \( \omega^c_{\beta}, \omega^s_{\beta} \) are real. In what follows we will only need the commutation relations \([h, c_\beta] = i(h, \beta)s_{\beta} \) and \([h, s_\beta] =
\(-i(h, \beta)c_\beta\) (for any element \(h\) from the Cartan subalgebra). So the other commutation relations of the basis elements are omitted. They can be found in, e.g., the textbook [19].

Let \(U(x) \in SU(N)/[U(1)]^{N-1}\) and \(n_k = U^\dagger h_k U\). Since the Killing form is invariant under the adjoint action of the group, we conclude \((n_i, n_k) = \delta_{ik}\). We set (cf. [19])

\[
A_\mu = ig^{-1}N[\partial_\mu n_k, n_k] + n_k C^k_\mu + W_\mu \equiv \alpha_\mu + W_\mu ,
\]

where \((n_k, W_\mu) = 0\) and \(W_\mu\) also satisfies the following \(N^2 - N\) conditions

\[
\nabla_\mu (\alpha) W_\mu \equiv \partial_\mu W_\mu + ig C^k_\mu [n_k, W_\mu] - N [[\partial_\mu n_k, n_k], W_\mu] = 0 .
\]

The connection \(A_\mu\) has \(4(N^2 - 1)\) independent components which are now represented by \(N^2 - N\) independent functions in \(n_k\), \(4(N-1)\) functions in \(C^k_\mu\) and \(4(N^2-1)-(N^2-N)-4(N-1)\) functions in \(W_\mu\).

The gauge transformation law in the new variables reads

\[
\delta n_k = i[n_k, \omega] , \quad \delta W_\mu = i[W_\mu, \omega] , \quad \delta C^k_\mu = g^{-1}(n_k, \partial_\mu \omega) .
\]

To prove this, we have to show that the transformations (5.4) induce an infinitesimal gauge transformation of \(A_\mu\): \(\delta A_\mu = g^{-1}\nabla_\mu (A) \omega\). Comparing the gauge transformations of the left- and right-hand sides of Eq. (5.2), we find that the relation

\[
\partial_\mu \omega = N [n_k, [n_k, \partial_\mu \omega]] + n_k (n_k, \partial_\mu \omega)
\]

has to hold true for any \(\partial_\mu \omega\). Consider a local Cartan-Weyl orthonormal basis \(c^U_\beta = U^\dagger c_\beta U\), \(s^U_\beta = U^\dagger s_\beta U\), and \(n_k\). We decompose the element \(\partial_\mu \omega\) in this local basis and compute the commutators in the l.h.s. of Eq. (5.3) by means of the Cartan-Weyl commutation relations: \([n_k, n_k, s^U_\beta] = -i(h_\beta, \beta)[n_k, c^U_\beta] = (h_\beta, \beta)^2 s^U_\beta\) and, similarly, \([n_k, [n_k, c^U_\beta]] = (h_\beta, \beta)^2 c^U_\beta\), while \([n_k, n_j] = 0\). Relation (5.3) follows from the resolution of unity in the local Cartan-Weyl basis, provided \(\sum_k (h_\beta, \beta)^2 = 1/N\) for every root \(\beta\). As \(h_\beta\) form an orthonormal basis in the Cartan subalgebra with respect to the Killing form, we have \(\sum_k (h_\beta, \beta)^2 = (\beta, \beta) = tr\beta^2 = 1/N\). The latter means that all roots of \(su(N)\) should have the same norm, which is easily established from the Dynkin diagram for \(su(N)\). To compute the norm, we set \((\beta, \beta) = b\) for any root and calculate the matrix elements of \(\beta\) by applying it to the basis elements of the Cartan-Weyl basis, then from the relation \((\beta, \beta) = tr\beta^2\) and the root pattern it follows that \(b = 1/N\). Thus, relation (5.3) is a true identity. In a similar fashion, one can prove that \((n_k, \nabla_\mu (\alpha) W_\mu) = \partial_\mu (n_k, W_\mu) \equiv 0\). The idea is to compute the Abelian components of the last term in (5.3) in the local Cartan-Weyl basis. The Abelian components can only be produced by those \(c_\beta\) – and \(s_\beta\) – components of \(W_\mu\) and \([\partial_\mu n_k, n_k]\) that correspond to the same root because [19] \([c_\beta, s_\beta] = i\beta\), but \((h_\beta, [c_\beta, s_\beta]) = (h, [s_\beta, s_\beta]) = (h, [c_\beta, c_\beta]) = 0\) for any \(\beta \neq \beta'\) and any element \(h\) of the Cartan subalgebra. Thus, Eq. (5.3) indeed contains only \(N^2 - 1 -(N-1) = N^2 - N\) independent conditions on \(W_\mu\).

Now it is easy to see that in our parameterization the maximal Abelian gauge is an algebraic gauge \(n_k(x) = h_k\). Given a connection \(A_\mu\), one can find \(U(A) \in SU(N)/[U(1)]^{N-1}\).
such that $U A_{\mu}$ satisfies the maximal Abelian gauge. The Maxwell fields in the maximal Abelian projection $U_{\mu}^{Ck} = (h_k, U_{\mu})$ carry Dirac monopoles. By construction these “monopole” degrees of freedom are described by the coset fields $n_k(A) = U_1(A) h_k U(A)$ in our parameterization. If $\phi(x)$ is a local operator that transforms in the adjoint representation of the gauge group, then by a gauge transformation it can always be brought to the Cartan subalgebra. Such a gauge transformation will be singular at spacetime points where a special gauge invariant polynomial of $\phi$ has zeros [20]. This polynomial is the Jacobian of the change of variables $\phi = U^{-1} h U$ where $h$ is an element of the Cartan subalgebra. In particular, for the SU(N) group, the singularities form worldlines which can be identified with the worldlines of magnetic monopoles in the effective $[U(1)]^{N-1}$ Abelian gauge theory [20]. Therefore, if $n_k \sim \phi/(\phi, \phi)^{1/2}$, the gauge transformation $n_k \rightarrow h_k$ is singular, in general, and creates Dirac magnetic monopoles in the Maxwell fields $U_{\mu}^{Ck}$.

An effective action of the fields $n_k$ can be calculated similarly to the case of SU(2) discussed in section 4. A generalization is straightforward, so we omit the details. A few concluding remarks are in order. A remarkable fact that all roots of SU(N) have the same norm was crucial to establish a relation between the “monopole” configurations of the maximal Abelian projection and the parameterization (5.2) of the SU(N) connection. Clearly, for any other compact gauge group the parameterization (5.2), (5.3) is not applicable. This should not be regarded as a drawback. Note that only for SU(N), Abelian projections lead to monopole-like defects in gauge fields [7]. For instance, for SO(N) the defects would be objects extended in space, like Nielsen-Olsen strings [7]. So it is natural to expect a different form of the connection that captures such degrees of freedom.

It is also rather interesting that the “monopole” part of the SU(N) connection (5.2) has the factor $N$. So in the large $N$ limit it has to play the role of “dominant” degrees of freedom, which implies that there exists a relation between the large $N$ limit and the monopole dominance in gauge theories. Finally, we remark that an effective action for the fields $n_k$ may support solitons, provided the leading term (quadratic in derivatives $\partial_{\mu} n_k$) of the gradient expansion does not vanish [17]. The large $N$ expansion seems a natural analytical method to verify this conjecture.

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