Higher-order topological semimetal in phononic crystals

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The notion of higher-order topological insulator has opened up a new avenue
toward novel topological states and materials. Particularly, a 3D higher-order
topological insulator can host topologically protected 1D hinge states, referred to
as the second-order topological insulator, or 0D corner states, referred to as the
third-order topological insulator. Similarly, a 3D higher-order topological
semimetal can be envisaged, if it hosts states on the 1D hinges or 0D corners. Here,
we report the first realization of a second-order topological Weyl semimetal in a
3D phononic crystal, which possesses Weyl points in 3D momentum space, gapped
surface states on the 2D surfaces and gapless hinge states on the 1D hinges. The
1D hinge states in a triangle-shaped sample exhibit a dispersion connecting the
projections of the Weyl points. Our results extend the concept of the higher-order
topology from the insulator to the semimetal, which may play a significant role in
topological physics and produce new applications in materials.
The discovery of exotic topological states of matter is a thriving research topic in condensed matter physics and material science [1,2]. For a gapped phase, the conventional $d$-dimensional ($d$D) topological insulators are featured with $(d-1)$D gapless boundary states. Recently, the higher-order topological insulators exhibit an extended bulk-boundary correspondence, i.e., a $d$D $n$th-order topological insulator has $(d-n)$D boundary states [3-7]. For example, a 2D second-order one possesses the 0D corner states [3-5], while a 3D second-order topological insulator hosts the 1D hinge states [6,7]. Although the concept of higher-order topological insulators was first proposed in the electronic systems and implemented lately [8], the second-order or even third-order topological insulators have been extended and observed in the photonic crystals [9-12], phononic crystals (PCs) [13-20], and electric circuits [21,22], benefiting from their macroscopic scale and flexibility of fabrication.

For gapless phases, the topology of the nodal points in 3D momentum space gives rise to the concept of topological semimetal (TSM) [23], such as the Weyl [24] and Dirac [25] semimetals. Different from the topological insulator, the conventional 3D TSM is usually characterized by the 2D nonclosed surface-arc boundary states, in contrast to the closed surface-circle ones. A natural question arises: whether there exists the 3D higher-order TSM, which hosts the 1D hinge states or 0D corner states? Very recently, a few 3D higher-order TSMs have been proposed with the two-fold [5,26,27] or four-fold [28-30] degenerate nodal points, or two-fold degenerate nodal loops [30]. However, the higher-order TSMs are yet to be implemented in experiment.

In this work, we report the first realization of a 3D second-order TSM (SOTSM) in a PC, constructed by stacking a breathing Kagomé lattice with double-helix interlayer couplings. The SOTSM hosts the 2D gapped surface states and 1D gapless hinge states connecting the projections of the Weyl points, which result from the $k_z$-dependent polarization protected by the mirror and $C_3$ symmetries. We first illustrate the topological properties of the SOTSM by a tight-binding model, then present the experimental observation of the hinge dispersion. The theoretical, simulated and experimental results are in good agreement.
Let us first introduce the tight-binding model for the SOTSM. As shown in Fig. 1a, the lattice is constructed by stacking the breathing Kagomé lattice along the $z$ direction, in which a unit cell of each layer contains three sites denoted by A (red), B (blue), and C (green), respectively. The intralayer couplings contain the intracell hopping $t_a$ (gray) and the intercell hopping $t_b$ (cyan) in the $xy$ plane, while the interlayer interaction is the double-helix hopping $t_z$ (yellow), composed of two equal chiral interlayer couplings with clockwise and counterclockwise. On the basis of sublattices A-C, the Bloch Hamiltonian is written as

$$H(k) = \begin{pmatrix} 0 & h_{12} & h_{13} \\ h_{12}^* & 0 & h_{23} \\ h_{13}^* & h_{23}^* & 0 \end{pmatrix},$$

with $h_{12} = t_a + t_b' e^{-i(k_x/2 + \sqrt{3}k_y/2)}$, $h_{13} = t_a + t_b' e^{-i k_x}$, and $h_{23} = t_a + t_b' e^{i(k_x/2 + \sqrt{3}k_y/2)}$, where $t_b' = t_b + 2t_z \cos k_z$ and $k = (k_x, k_y, k_z)$ is the Bloch wavevector. The band gap closes when $t_a = t_b'$ with $(k_x, k_y) = (\pm 4\pi/3, 0)$, or $t_a = -t_b'$ with $(k_x, k_y) = (0, 0)$ [5]. Without loss of generality, here we focus on the case of $t_a = t_b'$, in which the two-fold degenerate points, or the Weyl points are located at $K_{\pm} = (4\pi/3, 0, \pm k_W)$ and their time-reversal counterparts $K_{\pm}'$ with $k_W = \arccos[(t_a - t_b)/2t_z]$, as shown in Fig. 1b. These four Weyl points carry topological charges $\pm 1$ (Fig. 1c) with linear dispersions along all three directions (see Supplementary Material I for detail).

The topological property can be characterized with the 2D topological index by considering $k_z$ as a parameter. The first-order topological index, i.e., the $k_z$-dependent Chern number, is zero, except for the closing bulk gap at $k_W$. The second-order topological index, the $k_z$-dependent polarization, is defined as

$$p_i(k_z) = \frac{1}{S} \iint_{RBZ} A_i d^2k,$$

where $d^2k$ is the area element in the reduced Brillouin zone with area $S$, $A_i = -i \langle u | \partial_i | u \rangle$ with $i = x, y$ is the Berry connection and $u$ is the Bloch function of the lowest band. The polarization $(p_x, p_y)$ for a fixed $k_z$ takes a quantized value,
because of the mirror and \( C_3 \) symmetries. As shown in Fig. 1d, the polarization is \((1/2, 1/2\sqrt{3})\) for \(|k_z| < k_W\) and \((0,0)\) for \(|k_z| > k_W\), corresponding to phase transition across the Weyl points. The nonzero polarization gives rise to the hinge states in a triangle-shaped sample with the dispersion connecting the projections of the Weyl points along the \( k_z \) direction, as shown by the distribution and dispersion of the hinge-arc states in Figs. 1e and 1f. The model unambiguously exhibits the bulk-hinge correspondence and identify itself a SOTSM.

We now implement the SOTSM in a PC. Figure 2a shows a unit cell of the PC, which is a double-helix layer-stacking structure with lattice constants \( a = 44 \) mm (\( xy \) plane) and \( h = 38.9 \) mm (\( z \) direction). The double-helix interlayer couplings are illustrated deliberately. There are three cylindrical cavities with diameter \( d_0 = 14 \) mm and height \( h_0 = 21.5 \) mm, separated by a distance \( d = 28.6 \) mm. The intralayer couplings are introduced by two types of rectangular tubes, whose widths and heights are \( d_1 = 5.6 \) mm and \( h_1 = 4.48 \) mm, \( d_2 = 2.7 \) mm and \( h_2 = 2.43 \) mm, respectively. The interlayer couplings are induced by double-helix tubes with radius \( r = 1.9 \) mm. As shown in Figs. 2b and 2c, there exist four Weyl points, linear crossings in all the three directions. All the four Weyl points reside at the same frequency, protected by the \( C_{2y} \) symmetry, i.e., the rotation along the \( y \) axis, and the time reversal symmetry, therefore this PC is an ideal Weyl semimetal.

It is known that there may exist Fermi arc surface states in the Weyl semimetals \([31,32]\). In our system, the surface states on the \( k_x-k_z \) plane have the dispersions as shown in Fig. 3. It can be noted that the surface states are gapped, although the bulk state dispersions close the gap at the Weyl points (Fig. 3c), different from those in the Weyl semimetals realized earlier \([31,32]\), where the surface states are gapless because of the nontrivial \( k_z \)-dependent Chern number. These results are consistent with those in the above lattice model, indicating the trivial first-order topological index, i.e., the zero \( k_z \)-dependent Chern number. Since the surface states are trivial, it is informative to further investigate the boundary states on farther lower dimension or hinges.

We consider a finite PC sample, which is triangle-shaped in the \( xy \) plane. The
experimental sample is fabricated by 3D printing, as shown in Fig. 4a. It is found that the hinge states exist at about 8 kHz, as demonstrated consistently by the simulated and measured dispersions in Fig. 4b. We also give all the eigenstates and the corresponding typical field distributions for $k_z = 0$, in Figs. 4c and 4d respectively. One can see that the bulk, edge, corner, and bulk states, as shown in panels I, II, III, and IV of Fig. 4d, respectively, resemble those in the second-order topological insulator [14,15]. The corner states with three-fold degeneracy at $k_z = 0$ indicate that the hinge states along the $k_z$ direction connecting the projected Weyl points are also three-fold degenerate. The hinge states are also observed in another SOTSM with different structural parameters, as presented in Supplementary Material II.

In conclusion, we have realized a 3D acoustic SOTSM, which hosts 1D gapless hinge states connecting the projected Weyl points and exhibits bulk-hinge correspondence. Our work extends the concept of the higher-order topology from the insulator to the semimetal, having fundamental significance and potential practical applications. In addition, with the flexibility in obtaining the opposite couplings, the layer-stacking method in our work may be used to realize other type higher-order TSMs requiring both positive and negative hoppings.

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Data availability
The data that support the plots within this paper and other findings of this study are available from the corresponding author upon reasonable request.

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Author contributions
All authors contributed extensively to the work presented in this paper.

Competing financial interests
The authors declare no competing financial interests.

Methods
**Numerical simulations.** All numerical simulations of phononic crystal are performed by the commercial COMSOL Multiphysics solver package. The phononic crystals are filled with air with a mass density \(1.18 \text{ kg m}^{-3}\) and sound velocity \(335 \text{ m s}^{-1}\) at room temperature. Due to the huge acoustic impedance mismatch compared with air,
the 3D printing plastic material was considered as hard boundary.

**Experimental measurement.** A subwavelength headphone with a diameter of 6 mm, and a microphone probe with a diameter of 1.5 mm are used to generate and measure acoustic waves, respectively. A network analyzer is used to send and record the acoustic signals. The projected dispersion of the triangle-shaped sample is obtained by Fourier transforming the scanned acoustic pressure field distributions on the hinge of the sample.
Figure 1 | Second-order topological semimetal and hinge-arc states for a 3D stacked breathing Kagomé lattice. a, Schematic of the lattice structure of three layers that exhibits the different intralayer and interlayer hoppings. b, The bulk band structure along the \( k_z \) direction with \((k_x, k_y) = (4\pi/3,0)\). The dashed blue line shows the position of the degenerate point. c, The first Brillouin zone and the distribution of the Weyl points. The hollow and solid spheres denote the Weyl points with opposite topological charges, respectively. d, Polarizations \((p_x, p_y)\) of the lowest band along the \( k_z \) direction. e, The distribution of the hinge-arc states. The vertical direction
represents the $k_z$ direction, and the horizontal directions denote real space. The linear crossing point is four-fold degenerate point projected by two Weyl points. Projected dispersion of a triangle-shaped sample along the $k_z$ direction. The hinge-arc states with three-fold degeneracy denote by the red solid line. The parameters are chosen as $t_a = -1$, $t_b = -2.4$, and $t_z = -1$ as unit.
Figure 2 | Three-dimensional PC with ideal Weyl points. a. Schematics of a unit cell, the side and front views. b. The bulk band structure as a function of $k_z$ with $(k_x, k_y) = (4\pi/3a, 0)$. There exists a pair of Weyl points at $k_z = \pm 0.383(2\pi/h)$ with the same frequency. c. The bulk band structure along the high-symmetry lines with $k_z = 0.383(2\pi/h)$, denoted as the dashed blue line in b.
Figure 3 | Two-dimensional gapped surface state dispersions of the PC. a-d. The projected dispersions along the $k_x$ direction for $k_z = 0, 0.25, 0.38$, and $0.5 (2\pi/h)$, respectively.
**Figure 4 | One-dimensional gapless hinge state dispersions of a triangle-shaped PC.**

a, A photo of the 3D sample. b, The projected dispersion along the $k_z$ direction. The color maps denote the experimental data, while the red and black circles represent the calculated results of the hinge states and other states. c, The simulated eigenfrequencies for $k_z = 0$. The red, blue and black spheres denote the corner, edge and bulk states, respectively. d, The simulated field distributions of the four parts I, II, III, and IV marked in b for $k_z = 0$. The corner states locate at part III, indicating they are the hinge states along the $k_z$ direction and connect the projected Weyl points.
Supplementary Materials for

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S-I. Weyl points and Fermi-arc in the lattice model

In the Main Text, we have shown the energy band of the Bloch Hamiltonian (1) possesses the degenerate points at $K_\pm$ and $K'_\pm$. Here we apply the degenerate perturbation theory to investigate the linear band dispersion and the loop distribution of Berry curvature in momentum space of the low-energy effective Hamiltonian expanding around these degenerate points. It will be demonstrated that these degenerate points are the Weyl points, which generate the so-called Fermi arc on the surface.

We first consider the degenerate point at $K_+$. Assuming $k = q + K_+$, where $q = (q_x, q_y, q_z)$ is the infinitesimal momentum, we construct the perturbation Hamiltonian around $K_+$ as

$$
\Delta H = H(q + K_+) - H(K_+).
$$

\text{(S1)}
Hence, the low-energy effective Hamiltonian is given by

$$ h_{K_+}(q) = \left( \langle \psi_1 | \Delta H | \psi_1 \rangle \langle \psi_1 | \Delta H | \psi_2 \rangle \right) $$

where $\psi_1$ and $\psi_2$ are the eigenfunctions of the lowest two bands of the Bloch Hamiltonian (1) of the main text. From the Hamiltonian (S2), we plot the band dispersion around $K_+$, as shown in Figs. S1a and S1b. The band dispersion around $K_-$ is similar to the case around $K_+$ and thus not plotted here. Obviously, the band dispersion around the degenerate points along any direction is linear. Furthermore, Berry curvature is given by

$$ F = \frac{\partial A_y}{\partial q_x} - \frac{\partial A_x}{\partial q_y} $$

where $A_\mu = -i(\phi|\nabla_\mu|\phi)$, with $\mu = x, y$ and $\phi(q)$ being its wavefunction, is Berry connection. Figures S1c and S1d show that the flux of the Berry curvature flows from $K_+$ to $K_-$, which is similar to the magnetic monopole in momentum space. So, $K_+$ and $K_-$ are a pair of the Weyl points with opposite charge, denoted by the hollow and solid spheres.

The Weyl points result in Fermi arcs, which are the equi-energy contours of the surface states at a fixed energy. Figure S2a shows the Fermi arcs at the energy of Weyl points. Because of the ideal property, i.e., all the four Weyl points are at the same energy, the Fermi arcs connect two Weyl points with opposite charges in the absence of coexisting with bulk states. However, these surface states are gapped, as shown in Figs. S2b-S2f. For a fixed $k_z$, the surface state does not connect the valence and conduction bands, indicating the trivial first-order topological index, i.e., the zero value of $k_z$-dependent Chern number. So it is needed to explore the second-order topological index to describe the topological property of this model.
Figure S1 | Band dispersion and Berry curvature around the Weyl points. a-b, Band dispersion around the Weyl point $K_+$ in the $q_x$-$q_y$ (a) and $q_x(q_y)$-$q_z$ (b) planes. c-d, Spatial distributions of Berry curvature around $K_+$ (c) and $K_-$ (d).
Figure S2 | Fermi arc and surface state dispersions. 

**a**, The equi-energy contours of the surface states at the energy of the Weyl points ($E = -1$). The hollow and solid spheres denote the Weyl points with opposite topological charges. 

**b-f**, The surface state dispersions along the $k_x$ direction for different $k_z$. The solid red line denotes the surface state dispersion and the dashed black line shows the position of $E = -1$. The coupling parameters are the same as those in Fig. 1 of the Main Text.
S-II. The SOTSM hosting both three-fold degenerate points and Weyl points

In the Main Text, we have shown the Weyl points locate at $K_\pm$ and $K'_\pm$ when $t_a = t_b'$. We now consider another coupling parameters to show that the SOTSM hosts the Weyl points at $K_\pm$ and $K'_\pm$, and meanwhile three-fold degenerate points at $\Gamma_\pm$. The coupling parameters are chosen as $t_a = -0.5$, $t_b = -0.6$, and $t_z = -1$. When $t_a = t_b'$, the Weyl points are still located at $K_\pm = (4\pi/3,0, \pm k_W)$ and their time-reversal counterparts $K'_\pm$ with $k_W = \arccos[(t_a - t_b)/2t_z]$. When $t_a = -t_b'$, the three-fold degenerate points occur at $\Gamma_\pm = (0,0, \pm k_R)$ with $k_R = \arccos[-(t_a + t_b)/2t_z]$, as shown in Figs. S3a and S3b. The distribution of these degenerate points is shown in Fig. S3c. In the case, the hinge state dispersion has a different configuration (Fig. S3d), where the hinge states connect the Weyl points or the three-fold degenerate points separately.

The acoustic analogy of this SOTSM can be constructed by tuning the structural parameters. Specifically, the parameters are $a = 44$ mm, $h = 40.5$ mm, $d_0 = 14$ mm, $h_0 = 22.5$ mm, $d = 28.6$ mm, $d_1 = 4.3$ mm, $h_1 = 3.23$ mm, $d_2 = 2.5$ mm, $h_2 = 3.25$ mm, and $r = 2.5$ mm. As shown in Figs. S4a and S4b, there exists a pair of three-fold degenerate points at $(k_x, k_y, k_z) = (0,0, \pm 0.352(2\pi/h))$. In Fig. S4c, we further calculate and measure the hinge state dispersion for a finite PC sample of this SOTSM. Although the hinge states connected the pair of three-fold degenerate points are merged with the edge states, they connect the projected Weyl points are clearly shown, with good agreement between the simulation and experimental results.
Figure S3 | Second-order topological semimetal hosting three-fold degenerate points and Weyl points in the lattice model. a, The bulk band structure along the \( k_z \) direction with \((k_x, k_y) = (0,0)\). The dashed blue line shows the position of the degenerate point. b, The bulk band structure along the high symmetry lines for \( k_z = k_T \). c, The first Brillouin zone and the distribution of the Weyl points. The hollow and solid blue spheres denote the Weyl points with opposite topological charges, and the orange spheres represent the three-fold degenerate points. d, Projected dispersion of a triangle-shaped sample along the \( k_z \) direction. The hinge-arc states with three-fold degeneracy denote by the red solid line. The parameters are chosen as \( t_a = -0.5 \), \( t_b = -0.6 \), and \( t_z = -1 \).
Figure S4 | Second-order topological semimetal hosting three-fold degenerate points and Weyl points in the PC with different structural parameters. a, The bulk band structure as a function of $k_z$ with $(k_x, k_y) = (0,0)$. There exists a pair of three-fold degenerate points at $k_z = \pm 0.352(2\pi/h)$. b, The bulk band structure along the high-symmetry lines with $k_z = 0.352(2\pi/h)$, denoted as the dashed blue line in a. c, The projected dispersion along the $k_z$ direction for a triangle-shaped PC. The color maps denote the experimental data, while the black circles represent the calculated results.