A New Variant of $\mu - \tau$ Symmetry for One Generic Neutrino Mixing Angle: Analytical Study

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Abstract

We find a realization of the $Z_2$-symmetry behind the $\mu - \tau$ universality in the neutrino mass matrix able to impose a generic smallest mixing angle, in contrast to a zero-value predicted by the usual form of the $\mu - \tau$ symmetry. We extend this symmetry for the lepton sector within type-I seesaw scenario, and show it can accommodate the mixing angles, the mass hierarchies and the lepton asymmetry in the universe. We then study the effects of perturbing the specific form of the neutrino mass matrix imposed by the symmetry and compute the resulting mixing and mass spectrum. We trace back this “low-scale” perturbation to a “high-scale” perturbation, and find realizations of this latter one arising from exact symmetries with an enriched matter content.

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1 Introduction

Flavor symmetry is presumably behind the observed pattern of lepton flavor mixing. The benefits of flavor symmetry are not only limited to deciphering the true origin of neutrino masses and flavor structures, but it is also used to predict the nine free parameters of the light Majorana neutrino mass matrix $M_\nu$. These nine free parameters comprise the three masses ($m_1, m_2$ and $m_3$), the three mixing angles ($\theta_{12}, \theta_{23}$ and $\theta_{13}$) (commonly known as $\theta_{12}, \theta_{23}$ and $\theta_{13}$), the two Majorana-type phases ($\rho$ and $\sigma$) and the Dirac-type phase ($\delta$). One can set the symmetry at the Lagrangian level which would lead to specific textures of $M_\nu$ that one can test whether or not they can accommodate the experimental data summarized in Table 1.

The $\mu - \tau$ symmetry [1] is treated in many common mixing patterns such as tri-bimaximal mixing (TBM) [2], bimaximal mixing (BM) [3] and scenarios of $A_5$ mixing [4]. This symmetry is determined by fixing one of the two $Z_2$’s which $M_\nu$ respects to reflect exchange between the second and third families.

In [5, 6], we realized the $\mu - \tau$ symmetry by two textures, both of which led to a vanishing $\theta_{23}$ angle, and we extended the symmetry to the lepton sector showing it can accommodate the lepton mass hierarchies.

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However, in order to agree with a non-vanishing experimental value of $\theta_z$ we had to resort to “perturbing” the texture imposed by the symmetry. We studied possible forms of perturbations on $M_\nu$ originated from perturbations on the Dirac neutrino mass matrix $M_D$ at the high scale within type-I seesaw scenario, and showed how one can get by this way an experimentally acceptable value for $\theta_z$.

However, for an exact $\mu - \tau$ symmetry, nothing is special about $\theta_z = 0$. Had the experimental value of $\theta_z$ not been small enough ($\gtrsim 9^\circ$), one would not have been able to call in the perturbative approach in order to move $\theta_z$ from zero to its large value. Nonetheless, this does not mean that the exact symmetry cannot accommodate this new large value. The objective of this paper is to find a representation of the $\mu - \tau$ symmetry leading without perturbation to $\theta_z$ equal to an arbitrary value given in advance, and to study the consequences of this “deformed” symmetry, called henceforth $S^\pm$ in contrast to the “non-deformed” symmetry $S$ leading to $\theta_z = 0$. This issue is important since it negates the incorrect statement that exact $\mu - \tau$ symmetry forces a vanishing value for $\theta_z$ that one needs to perturb in order to agree with data.

After we find $S^\pm$, we extend it into the lepton sector and study the resulting mass hierarchies. As we shall see, $S^\pm$ predicts, in addition to $\theta_z$, the value $\theta_y = \frac{\pi}{4}$, and thus one can study perturbations on $S^\pm$ in order to account for $\theta_y$ slightly different from the acceptable value $\frac{\pi}{4}$. We carry out an analytical analysis of perturbing the $M_\nu$ form imposed by $S^\pm$ and compute the “perturbed” angles and masses in terms of the perturbation parameters. Moreover, we trace back, within type-I seesaw scenarios, this perturbation on $M_\nu$ to one affecting $M_D$ at the high scale and express the resulting spectrum in terms of the high-scale perturbation. As is considered probable [8], we neglect renormalization group effects when running between the two scales, and assume they will not affect the symmetry. In [6], we perturbed the symmetry $S$ and showed that considerable areas in the parameter space exist accommodating the data with many correlations between the $M_\nu$ parameters. We expect the same in the case of perturbing $S^\pm$ which is related “smoothly” to $S$. However, this needs to be confirmed by carrying out a complete numerical study scanning the perturbation parameters of $M_D$, which we intend to do in a future work.

In line with [6], the form of perturbation can be generated by assuming exact symmetries at the Lagrangian level, some of which are broken spontaneously by adding new matter fields, and we carry out the “non-trivial” work of finding this realization in the case of our deformed symmetry $S^\pm$.

The plan of the paper is as follows. In Section 2, we review the basic notation for the neutrino mass matrix. In Section 3, we find the new symmetry $S^\pm$ and compute the corresponding mixing and phase angles and the neutrino eigen masses. In Section 4, we implement $S^\pm$ into a type-I seesaw scenario. We address the charged lepton sector in Subsection 4.1, whereas we study the neutrino mass hierarchies in Subsection 4.2, and in Subsection 4.3, we comment on leptogenesis induced by $S^\pm$. In Section 5, we study deviations on $M_\nu$ caused by breaking $S^\pm$ in $M_D$, and find the effects on the angles and masses. In Section 6 we present a theoretical realization of the perturbed texture. We end by discussion and summary in Section 7. Technical details are reported in three appendices.

### 2 Notations

There are 3 lepton families in the Standard Model (SM). The charged-lepton mass matrix relating left-handed (LH) and right-handed (RH) components is arbitrary, but can always be diagonalized via a
bi-unitary transformation:

\[
V_L^\dagger M_i (V_R^\dagger)^\dagger = \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix}.
\] (1)

In the same manner, we can diagonalize the symmetric Majorana neutrino mass matrix by just one unitary transformation:

\[
V^\dagger M_\nu V^{\nu*} = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix},
\] (2)

with \(m_i\) (for \(i = 1, 2, 3\)) real and positive.

The mismatch between \(V^l\) and \(V^\nu\) leads to the observed neutrino mixing matrix

\[
V_{PMNS} = (V_L^l) V^\nu.
\] (3)

In the “flavor” basis, we have \(V_L^l = 1\) (the unity matrix) meaning that the charged lepton mass eigen states are the same as the current (gauge) eigen states. We assume that we are working in this basis implying that the measured mixing results only from neutrinos \(V_{PMNS} = V^\nu\). We justify this by noting that the deviations from \(V_L^l \neq 1\) are of order of the ratios of the hierarchical charged lepton masses which are small.

We shall adopt the parametrization of [9], related to other ones by simple relations [10], where the \(V_{PMNS}\) is given in terms of three mixing angles \((\theta_x, \theta_y, \theta_z)\) and three phases \((\delta, \rho, \sigma)\), as follows.

\[
\begin{align*}
P_{PMNS} & = \text{diag}(e^{i\rho}, e^{i\sigma}, 1), \\
U_{PMNS} & = R_y(\theta_y) R_z(\theta_z) \text{ diag }(1, e^{-i\delta}, 1) R_x(\theta_x) \\
& = \begin{pmatrix} c_x c_z & s_x c_z & s_x e^{-i\delta} s_y c_y \\ -c_x s_y s_z - s_x c_y e^{-i\delta} & -s_x s_y s_z + c_x c_y e^{-i\delta} & s_y c_z \\ -c_x s_y s_z + s_x c_y e^{-i\delta} & -s_x c_y s_z - c_x s_y e^{-i\delta} & c_y c_z \end{pmatrix}, \\
V_{PMNS} & = U_{PMNS} P_{PMNS} = \begin{pmatrix} c_x c_z e^{i\rho} & s_x c_z e^{i\sigma} & s_x e^{-i\delta} s_y c_y \\ -c_x s_y s_z - s_x c_y e^{-i\delta} e^{i\rho} & -s_x c_y s_z - c_x s_y e^{-i\delta} e^{i\sigma} & s_y c_z \\ -c_x s_y s_z + s_x c_y e^{-i\delta} e^{i\rho} & -s_x c_y s_z + c_x s_y e^{-i\delta} e^{i\sigma} & c_y c_z \end{pmatrix}
\end{align*},
\] (4)

where \(R_i(\theta_i) (i = x, y, z)\) is the rotation matrix around the \((i - 1)^{th}\)-axis \((x = 1, y = 2, z = 3 \text{ or } 0)\) by angle \(\theta_i\), and \(s_{\theta_i} \equiv \sin \theta_i\). . . . Note that in this adopted parametrization, the third column of \(V_{PMNS}\) is real which would be essential later to extract the parameters from the diagonalizing matrix. We write down in Appendix (A) the elements of the neutrino mass matrix in the flavor basis and in the adopted parametrization Eq. (124). This helps in viewing directly at the level of the mass matrix that the effect of swapping the indices 2 and 3 corresponds to the transformation \(\theta_y \rightarrow \frac{\pi}{2} - \theta_y\) and \(\delta \rightarrow \delta \pm \pi\). Hence, for a texture satisfying the \(\mu-\tau\) symmetry, one can check the correctness of any obtained formula by requesting it to be invariant under the above transformation.

3 The Deformed \(S^z\) versus the Non-deformed \(S\) Symmetries

In [6], we defined the \(S\) symmetry incorporating the \(\mu-\tau\) universality, and which leads to mixing angles in the first quadrant, by an orthogonal real matrix:

\[
S = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}
\] (5)

Another symmetry is possible by taking the \((1, 1)^{th}\) entry in \(S\) equal to 1 instead of \(-1\), but to fix the ideas, let us just restrict our study to \(S\) above. Requiring that \(M_\nu\) is invariant under \(S\):

\[
S^T M_\nu S = M_\nu
\] (6)
implies that $M_\nu$ has a specific form:

$$M_\nu = \begin{pmatrix} A_\nu & B_\nu & -B_\nu \\ B_\nu & C_\nu & D_\nu \\ -B_\nu & D_\nu & C_\nu \end{pmatrix}$$  \hspace{1cm} (7)$$

and that one can diagonalize simultaneously all the matrices $S$, $M_\nu$ and $M_\nu^* M_\nu$ by a unitary matrix $U$ of the form:

$$U = \begin{pmatrix} c_\varphi & s_\varphi & 0 \\ -s_\varphi e^{-i\xi} & \frac{c_\varphi e^{-i\xi}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ s_\varphi e^{-i\xi} & \frac{-c_\varphi e^{-i\xi}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix},$$  \hspace{1cm} (8)

such that

$$U^T M_\nu U = M_\nu^{\text{Diag}} = \text{Diag}(M_{\nu \nu}^{\text{Diag}}, M_{\nu \nu}^{\text{Diag}}, M_{\nu \nu}^{\text{Diag}}), \quad U^* M_\nu^* M_\nu U = \text{Diag}(m_1^2, m_2^2, m_3^2)$$  \hspace{1cm} (9)

The angles $\varphi$ and $\xi$ are determined by the requirement that $U^* M_\nu^* M_\nu U$ is a diagonal matrix and found to be equal to

$$\tan(2\varphi) = \frac{2\sqrt{2}|b_\nu|}{c_\nu - a_\nu - d_\nu}, \quad \xi = \text{Arg}(b_\nu),$$  \hspace{1cm} (10)

where we write the squared mass matrix in the form:

$$M_\nu^* M_\nu = \begin{pmatrix} a_\nu & b_\nu & -b_\nu \\ b_\nu^* & c_\nu & d_\nu \\ -b_\nu^* & d_\nu^* & c_\nu \end{pmatrix},$$  \hspace{1cm} (11)

The relations between the entries of $M_\nu^* M_\nu$ and those of $M_\nu$ are written in Eq. (126) in Appendix A. Also, we write there the mass spectrum corresponding to the eigenvalues of $M_\nu$ and $M_\nu^* M_\nu$ in Eq. (127) and Eq. (128) respectively. Multiplying $U$ by a diagonal phase matrix

$$Q = \text{Diag}\left\{\exp\left[-i\frac{1}{2}\text{Arg}(M_{\nu \nu}^{\text{Diag}})\right], \exp\left[-i\frac{1}{2}\text{Arg}(M_{\nu \nu}^{\text{Diag}})\right], \exp\left[-i\frac{1}{2}\text{Arg}(M_{\nu \nu}^{\text{Diag}})\right]\right\},$$  \hspace{1cm} (12)

in order to get rid of the phases of $M_\nu^{\text{Diag}}$, and rephasing the charged lepton fields so that to make the conjugate of $UQ$ in the same form as the adopted parametrization of $V_{\text{PMNS}}$ (a real third column) we find, in addition to $\theta_z = 0$, the following mixing and phase angles:

$$\theta_y = \pi/4, \quad \theta_x = \varphi, \quad \rho = \frac{1}{2}\text{Arg}(M_{\nu \nu}^{\text{Diag}}), \quad \sigma = \frac{1}{2}\text{Arg}(M_{\nu \nu}^{\text{Diag}}), \quad \delta = 2\pi - \xi.$$  \hspace{1cm} (13)

Thus the “non-deformed” symmetry $S$, which is diagonalized, as well as $M_\nu$, by $U = R_y(\pi/4) R_z(0) X$, where $X = \text{diag}(1,e^{-i\delta},1)$, $R_z(\theta_z)$, $\text{diag}(e^{i\rho}, e^{i\varphi}, 1)$ involves the phase angles and the rotation $R_x$, fixes two mixing angles ($\theta_y = \pi/4, \theta_x = 0$) whereas the other parameters are determined, without fine tuning, by $M_\nu$.

Now we see how one should proceed in order to seek a “deformed” symmetry $S^z$ leading directly to a non-vanishing value $\theta_z$, in that it should be diagonalized by $U^z = R_y(\pi/4) R_z(\theta_z) X$. The matrix $U^z$ can be written in a form related to $U$ as,

$$U^z = WU,$$  \hspace{1cm} (14)

provided $W$ is given by,

$$W = R_y(\pi/4) R_z(\theta_z) R_y^{-1}(\pi/4) = \begin{pmatrix} c_z & s_z & s_z \\ \frac{s_z}{\sqrt{2}} & \frac{c_z}{\sqrt{2}} & -s_z \frac{c_z}{\sqrt{2}} \\ -s_z \frac{c_z}{\sqrt{2}} & s_z \frac{c_z}{\sqrt{2}} & c_z \frac{c_z}{\sqrt{2}} \end{pmatrix},$$  \hspace{1cm} (15)
Starting from Eq. (9), we find that $U^z$ diagonalizes both the “rotated” neutrino mass matrix $M^z_\nu$ and its hermitian square $M^z_\nu M^z_\nu$ as,

$$
U^z T M^z_\nu U^z = M^\text{Diag}_\nu,
$$

$$
U^z \dagger M^z_\nu^* M^z_\nu U^z = M^\text{Diag}_\nu^* M^\text{Diag}_\nu,
$$

where

$$
M^z_\nu = W M_\nu W^T.
$$

We notice that the mass spectrum is the same for $M_\nu$ and $M^z_\nu$ expressing the fact that the eigenmasses are invariant under a change of basis.

The “deformed” symmetry, which again expresses exchange between the 2nd and 3rd families ($\mu$-$\tau$ symmetry), can be defined as

$$
S^z = W S W^T = \begin{pmatrix}
-c_{2z} & \sqrt{2} s_z & \sqrt{2} s_z \\
\sqrt{2} s_z & -s_z^2 & c_z^2 \\
\sqrt{2} s_z & c_z^2 & -s_z^2
\end{pmatrix}
$$

since we see that $M^z_\nu$ is form invariant under $S^z$:

$$
S^z T M^z_\nu S^z = M^z_\nu.
$$

The invariance of $M^z_\nu$ under the symmetry $S^z$ implies the following form:

$$
M^z_\nu = \begin{pmatrix}
A^z_\nu & B^z_\nu & B^z_\nu^* \\
B^z_\nu & C^z_\nu & D^z_\nu \\
B^z_\nu^* & D^z_\nu & C^z_\nu^*
\end{pmatrix},
$$

where $A^z_\nu$, $B^z_\nu$, $C^z_\nu$ and $D^z_\nu$ are arbitrary independent complex parameters, while $B^z_\nu$ and $C^z_\nu$ are dependent and given as

$$
B^z_\nu = -(1 - 2 t_z^2) B^z_\nu - \sqrt{2} t_z (A^z_\nu - C^z_\nu - D^z_\nu),
$$

$$
C^z_\nu = (1 - 2 t_z^2) C^z_\nu + 2 t_z^2 (A^z_\nu - D^z_\nu) + 2 \sqrt{2} t_z (1 - t_z^2) B^z_\nu.
$$

We state in Eq. (130) in Appendix A the expressions of the $M^z_\nu$ elements (Eq. (20)) in terms of the “non-deformed” $M_\nu$ elements (Eq. (7)). In addition, Eq. (132) there expresses the general form of the “deformed” square mass matrix $M^z_\nu M^z_\nu$ in terms of the “non-deformed” parameters of the square mass $M^2_\nu M_\nu$ defined in Eq. (126). Having established the relations between the deformed $\{M^z_\nu, M^z_\nu^* M^z_\nu\}$ and the non-deformed $\{M_\nu, M^2_\nu M_\nu\}$ mass matrices, we stress again that the mass spectra corresponding to both sets of $\{M^z_\nu, M^z_\nu^* M^z_\nu\}$ and $\{M_\nu, M^2_\nu M_\nu\}$ are the same, reflecting the fact that the eigenmasses are invariant under a change of basis.

Plugging Eqs. (8, 15) in Eq. (14) gives the most general unitary matrix diagonalizing the commuting matrices $S^z$, $M^z_\nu$ and $M^z_\nu^* M^z_\nu$. Alternatively one can start by assuming the form of $S^z$ given in Eq. (18), ignoring any relation between $S^z$ and $S$, and deduce from it the mixing and phase angles. Actually, one assumes the invariance given in Eq. (19), which implies that $S^z$ commutes with $M^z_\nu$ and $M^z_\nu^* M^z_\nu$ and consequently also with $M^z_\nu^* M^z_\nu$ and $M^z_\nu M^z_\nu^*$, then the diagonalizing matrices of $S^z$ includes the diagonalizing matrix $U^z$ present in Eq. (16). The eigenvalues of $S^z$ are: $\{-1, -1, 1\}$ corresponding respectively to the normalized eigen vectors,

$$
v_1 = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)^T, \quad v_2 = \left(c_z, -\frac{s_z}{\sqrt{2}}, -\frac{s_z}{\sqrt{2}}\right)^T, \quad v_3 = \left(s_z, \frac{c_z}{\sqrt{2}}, \frac{c_z}{\sqrt{2}}\right)^T.
$$

In constructing the general form (up to a diagonal phase matrix) of the unitary diagonalizing matrix of $S^z$ one should care about the two-fold degenerate eigenvalue $-1$, which entails the freedom for a unitary transformation defined by an angle $\varphi$ and phase $\xi$ in its eigenspace to get new eigen vectors in the following form:

$$
\overline{v}_1 = s_\varphi e^{-i \xi} v_1 + c_\varphi v_2, \quad \overline{v}_2 = -c_\varphi e^{-i \xi} v_1 + s_\varphi v_2, \quad \overline{v}_3 = v_3.
$$
The ambiguity in ordering the eigenvectors of $S^z$ is fixed by requiring that the mixing angles, contained in the unitary matrix $U^z$ diagonalizing $S^z$, should fall all in the first quadrant. The desired order turns out to correspond $\{−1,−1,1\}$ and thus the matrix $U^z$ assumes the following form:

$$U^z = [v_1, v_2, v_3] = \begin{pmatrix}
  c_z c_\varphi & c_z s_\varphi & s_z \\
  -\frac{1}{\sqrt{2}} (s_z c_\varphi + s_\varphi e^{-i\xi}) & \frac{1}{\sqrt{2}} (s_z s_\varphi - c_\varphi e^{-i\xi}) & \frac{c_z}{\sqrt{2}} \\
  -\frac{1}{\sqrt{2}} (s_z c_\varphi - s_\varphi e^{-i\xi}) & \frac{1}{\sqrt{2}} (s_z s_\varphi + c_\varphi e^{-i\xi}) & \frac{c_z}{\sqrt{2}}
\end{pmatrix}. \tag{24}$$

We can check that Eq. (24) is consistent with Eqs. (8, 14 and 15). The specific form of $U^z$ of Eq. (24) which diagonalizes also the hermitian matrix $M^z_{\nu} M^z_{\nu}$, which commutes with $S^z$, leads to the same mixing angle $\varphi$ and phase angle $\xi$ as determined in Eq. (10).

In order to get a positive mass spectrum we use the freedom of multiplying $U^z$ by a diagonal phase matrix $Q$ which turns out be the same as found in Eq. (12),

$$(U^z Q)^T M^z_{\nu} (U^z Q) = \text{Diag} (m_1, m_2, m_3). \tag{25}$$

Moreover, we re-phase the charged lepton fields in order to make real the 3$^{rd}$ column of the conjugate of $(U^z Q)$ in accordance with the adopted parametrization for $V_{PMNS}$ in Eq.(4), so that to identify the mixing and phase angles. We find that the new $\mu$-$\tau$ symmetry realized through $S^z$ entails the followings:

$$\begin{align*}
\theta_y &= \pi/4, \quad \theta_x = \varphi, \quad \theta_z = \theta_z, \\
\rho &= \frac{1}{2} \text{Arg} (M^\text{Diag}_{\nu 11} M^*_{\nu 33}), \quad \sigma = \frac{1}{2} \text{Arg} (M^\text{Diag}_{\nu 22} M^*_{\nu 33}), \quad \delta = 2\pi - \xi.
\end{align*} \tag{26}$$

These predictions are phenomenologically viable especially for predicting a non vanishing $\theta_z$, which is at our disposal through defining the symmetry $S^z$. Adjusting $\theta_x$ to accommodate the experimental value of $\theta_x \simeq 33.7^o$ does not require a special adjustment for the mass parameters $a_\nu, b_\nu, c_\nu, d_\nu$. There is still the need for a small deviation from the exact $S^z$ symmetry to shift $\theta_y$ to its experimental value $\theta_y \simeq 40^o$.

The various neutrino mass hierarchies can also be produced as can be seen from Eq.(128) and Eq.(10) where the three masses and the angle $\varphi$ are given in terms of four parameters $a_\nu, |b_\nu|, c_\nu, \text{ and } d_\nu$. Therefore, one can solve the four mixing equations to get $a_\nu, |b_\nu|, c_\nu,$ and $d_\nu$ in terms of the masses and the angle $\varphi$.

### 4 The seesaw mechanism and the $S^z$ symmetry

We extend now the $S^z$-symmetry in the Lepton sector at the Lagrangian level, then we use the type-I seesaw scenario to treat the effective neutrino mass matrix, with consequences on leptogenesis.

#### 4.1 The charged lepton sector

We start with the part of the SM Lagrangian which gives masses to the charged leptons:

$$\mathcal{L}_1 = Y^z_{\ell_i} \bar{L}_i \phi \ell_j^c, \tag{27}$$

where the SM Higgs field $\phi$ and the right handed (RH) leptons $\ell_j^c$ are assumed to be singlet under $S^z$, whereas the left handed (LH) leptons transform as:

$$L_i \to S^z_{ij} L_j. \tag{28}$$

Invariance under $S^z$ leads to:

$$S^{zT} Y^z = Y^z, \tag{29}$$

and this forces the Yukawa couplings to look like:

$$Y^z = \begin{pmatrix}
\sqrt{2} t_z a \\
\sqrt{2} t_z b \\
\sqrt{2} t_z c
\end{pmatrix} \begin{pmatrix}
a \\
b \\
c
\end{pmatrix}, \tag{30}$$
When $\phi$ gets a vev $v$, we have:

$$M_i^T M_i = v^2 \begin{pmatrix} 2t_z^2 & \sqrt{2}t_z & \sqrt{2}t_z \\ \sqrt{2}t_z & 1 & 1 \\ \sqrt{2}t_z & 1 & 1 \end{pmatrix} (|a|^2 + |b|^2 + |c|^2).$$

The eigenvalues of $M_i^T M_i$ are $2v^2(1+t_z)$ (with eigenvector $[s_z, c_z/\sqrt{2}, c_z/\sqrt{2}]^T$) and 0 (with an eigenspace spanned by the normalized eigenvectors $\sqrt{1+2t_z^2/(1+t_z^2)} [1, -t_z/\sqrt{2}, -t_z/\sqrt{2}]^T$) and $\frac{1}{\sqrt{2}} [0, -1, 1]^T$, then the charged lepton mass hierarchy cannot be produced, and the nontrivial diagonalizing matrix contradicts our assumption of being in the flavor basis. To remedy this, we introduce $SM$-singlet scalar fields $\Delta_k$ in addition to one $SM$-Higgs coupled to the lepton LH doublets through the dimension-5 operator (only one $SM$-Higgs field is chosen for purposes related to suppressing flavor-changing neutral currents):

$$L_2 = \frac{f_{ikr}}{\Lambda} T_i \phi_k \ell_r^c.$$  

We assume the $\Delta_k$’s transform under $S^z$ as:

$$\Delta_i \rightarrow S^z_{ij} \Delta_j.$$  

Invariance under $S^z$ implies,

$$S^z f_r^z S^z = f_r^z,$$

where $(f_r^z)_{ij} = f_{ijr}$,

and thus we can show that $f_r^z$ can be parameterized as

$$f_r^z = \begin{pmatrix} -\frac{1}{\sqrt{2}} t_z & F^r + G^r & \sqrt{2} t_z K^r + N^r & F^r + G^r - K^r & K^r \\ F^r & \sqrt{2} t_z (K^r - F^r) + N^r & P^r & N^r \\ G^r & \sqrt{2} t_z (K^r - G^r) + N^r & P^r & N^r \end{pmatrix}.$$  

We use this parametrization for $f_r^z$ in a way to put in its third column independent entries, so that when the fields $\Delta_k$ and the neutral component of the Higgs field $\phi^0$ acquire vevs ($\langle \Delta_k \rangle = \delta_k, v = \langle \phi^0 \rangle$) and when we assume having a hierarchy of the form: $\delta_1, \delta_2 \ll \delta_3$ then

$$\langle M^2 \rangle_{rr} \simeq \frac{v f_{3r}^2}{\Lambda} \delta_3 \simeq \frac{v \delta_3}{\Lambda} \begin{pmatrix} K^1 & K^2 & K^3 \\ N^1 & N^2 & N^3 \\ P^1 & P^2 & P^3 \end{pmatrix}.$$  

Interpreting each row of Yukawa couplings in the mass matrix Eq. (36) as a complex valued vector having norm defined in the standard way, and assuming that the ratio between the moduli of these Yukawa vectors matches the corresponding one between lepton masses as $|K| : |N| : |P| \sim m_e : m_\mu : m_\tau$, one can show, as was done in [11], that the LH charged lepton fields needs to be infinitesimally rotated in order to diagonalize the charged lepton mass matrix, which validates our assumption of working in the flavor basis to a good approximation.

On the other hand, one could have opted to work with many $SM$ Higgs doublets $\phi_i$ rather than many $SM$-singlet scalar fields $\Delta_i$. In this case, the lagrangian responsible for the mass of the charged leptons would assumes the form

$$L_2 = f_{ikr} T_i \phi_k \ell_r^c.$$  

The charged leptons acquire masses when the Higgs scalars get vevs which can be assumed to respect $v_1, v_2 \ll v_3$. We can follow the same procedure, described above when using many $SM$-singlet scalar fields $\Delta_i$, to get the diagonalized charged leptons while still working in the flavor basis to a good approximation. Regarding the flavor-changing neutral Yukawa interaction, it could be suppressed by properly adjusting the relevant Yukawa coupling combinations[12].
4.2 Neutrino mass hierarchies

The effective light LH neutrino mass matrix is generated through the seesaw mechanism formula

\[ M_\nu = M_D (M_R^\dagger)^{-1} M_D^T, \]  

(38)

The Dirac neutrino mass matrix \( M_D \), in case of single SM-Higgs, originates from the Yukawa term

\[ g^2 \mathcal{T}_i \, i \tau_2 \phi^* \nu_{Rj}, \]

(39)

after the Higgs field gets a vev, whereas the symmetric Majorana neutrino mass matrix \( M_R \) comes from a term

\[ \frac{1}{2} \nu_{Ri}^T C^{-1} (M_R)_{ij} \nu_{Rj}. \]

(40)

where \( C \) is the charge conjugation matrix.

We assume the RH neutrino to transform under \( S^z \) as:

\[ \nu_{Rj} \rightarrow S^z_{jr} \nu_{Rr}, \]

(41)

and by \( S^z \)-invariance we have

\[ S^{zT} \, g^z S^z = g^z, \quad S^{zT} M^z_R S^z = M^z_R. \]

(42)

Knowing the required textures for the above constraints in the case of \( S \) symmetry

\[ M_D = g \nu = \begin{pmatrix} A_D & B_D & -B_D \\ E_D & C_D & D_D \\ -E_D & D_D & C_D \end{pmatrix}, \quad M_R = \begin{pmatrix} A_R & B_R & -B_R \\ B_R & C_R & D_R \\ -B_R & D_R & C_R \end{pmatrix}, \]

(43)

one can find the necessary textures in the case of \( S^z \) symmetry by simply computing

\[ M_D = W M_D W^T, \quad M_R = W M_R W^T \]

(44)

Whereas the parametrization of the symmetric \( M_R \) is similar to \( M_\nu \) Eq. (20), the symmetry \( S^z \) dictates the following form of \( M_D^z \)

\[ M^z_D = \begin{pmatrix} A_D^z & B_D^z & B_D^{z*} \\ E_D^z & C_D^z & D_D^z \\ E_D^{z*} & D_D^{z*} & C_D^{z*} \end{pmatrix}, \]

(45)

where \( A_D^z, B_D^z, C_D^z, D_D^z \) and \( E_D^{z} \) are arbitrary independent complex parameters, while \( B_D^{z*}, C_D^{z*} \) and \( E_D^{z*} \) are dependent and given as

\[ B_D^{z*} = -B_D^z + \sqrt{2} t_z (D_D^z + C_D^z - A_D^z) + 2 t_z^2 E_D^z, \]
\[ E_D^{z*} = -(1 - 2 t_z^2) E_D^z + \sqrt{2} t_z (D_D^z + C_D^z - A_D^z), \]
\[ D_D^{z*} = D_D^z + \sqrt{2} t_z (E_D^z - B_D^z), \]
\[ C_D^{z*} = (1 - 2 t_z^2) C_D^z + \sqrt{2} t_z B_D^z + 2 t_z^2 (A_D^z - D_D^z) + \sqrt{2} t_z (1 - 2 t_z^2) E_D^z. \]

(46)

In the Appendix (A), we summarize in useful formulae (Eqs. (133) upto (139)) all the relevant relations between the entries corresponding to the set of mass matrices \( \{ M_D^z, M_D^{z\dagger}, M_D \} \) and those of \( \{ M_D, M_D^\dagger, M_D \} \).

The seesaw formula implies that

\[ \det (M_\nu^z M_\nu^z) = \det \left( M_D^{z\dagger} M_D \right)^2 \det (M_R^z M_R)^{-1}. \]

(47)
and since the determinant, or equivalently the product of eigenvalues, does not change when changing the basis, then the relevant spectra for $M_z^\nu M_\nu^\ast$, $M_R^\ast M_R$ and $M_D^\dagger M_D$, with the help of Eq. (128) and Eq. (135), can be written as,

$$\left\{ c_{v,R,D} + d_{v,R,D} - \frac{a_{v,R,D}}{2} \pm \frac{1}{2} \sqrt{(a_{v,R,D} + d_{v,R,D} - c_{v,R,D})^2 + 8 |b_{v,R,D}|^2} \right\}. \quad (48)$$

The mass spectrum and its hierarchy type are determined by the eigenvalues presented in Eq.(48). As one of the simplest realizations which can be envisaged from Eq.(47), one can adjust the spectrum of $M_z^\nu M_\nu^\ast$ so that to follow the same kind of hierarchy as $M_z^\nu M_z^\nu$. However, this does not necessitate that $M_D^\dagger M_D$ would behave in the same manner. Also, this is by no means an exclusive example, as there might be other possible realizations producing the same desired hierarchy, and what is mentioned is a mere simple possibility.

Later, we shall need the general forms of the symmetric and general matrices which are “sign-reversed”, i.e. multiplied by $-1$, under $S^z$. Therefore, for the sake of completeness and necessity, we state in the Appendix (A) all the constraints imposed by such kind of “sign-reversed” symmetry (Eq. (141) and Eq. (142)).

### 4.3 Leptogenesis

In [6] we showed that the $S$ symmetry can account for the lepton asymmetry observed in the universe. The relevant quantity in that calculation was the term

$$\left( \tilde{M}_D^\dagger \tilde{M}_D \right)_{ij} = \left( F_0^\dagger V_R^\dagger M_D^\dagger M_D V_R F_0 \right)_{ij} \quad (49)$$

where $\tilde{M}_D$ is the Dirac neutrino mass matrix in the basis where the RH neutrinos are mass eigenstates, $V_R$ is the diagonalizing matrix of $M_R$ and $F_0$ is a phase diagonal matrix so that the eigenvalues of $M_R$ are real positive.

Let’s look now at the expression above in the “rotated” basis defined by $W$:

$$V_R^\dagger M_D^\dagger M_D V_R^\dagger = V_R^\dagger M_D^\dagger M_D V_R. \quad (50)$$

Moreover, since the diagonal phase matrix $F_0$ depends on the mass spectrum of $M_R$, it remains the same upon going to “rotated” basis defined by $W$. Thus, the relevant discussion for leptogenesis remains the same in both “non-deformed” and “deformed” $\mu-\tau$ symmetries.

### 5 Perturbation on $S^z$

#### 5.1 Motivation and Preliminaries

In [5, 6], we introduced a perturbation on the $S$-symmetry in order to deal with its experimentally unacceptable zero value for $\theta_z$. In the case of $S^z$ symmetry, we can arrange to get $\theta_z$ equal any value given in advance. However, $S^z$ predicts that $\theta_y$ equals exactly $\pi/4$, which is, albeit experimentally allowable, not the best fit of $\theta_y$. Moreover, the value of $\theta_x$ is determined by the entries of $M_z^\nu$ and it would be good if we have a freedom in changing slightly the values of angles to account for possible new more precise measurements. This pushes us to consider the effects of perturbing the form of $M_z^\nu$ imposed by the $S^z$ symmetry. We carry out now a complete analytical analysis of $M_z^\nu$ perturbations. In the next section we shall present theoretical realizations justifying the possibility of taking the form of perturbations we are considering here. As to the numerical analysis, we shall report in a future work the results of scanning the free parameters of the model and determining the regions of parameter space consistent with data. We shall denote the predictions of $S^z$ symmetry (unperturbed) by an 0 upper index. Thus we have

$$M_z^\nu = M_z^{\nu0} + \delta M_z^\nu \quad (51)$$
and the diagonalizing matrix $U^z$ is defined such that
\[
U^z \quad \begin{bmatrix} M^0_{\nu} \end{bmatrix} \quad U^z \quad = \quad \text{Diag} \left( \left| M^0_{\nu 1} \right|^2, \left| M^0_{\nu 2} \right|^2, \left| M^0_{\nu 3} \right|^2 \right)
\]
whereas the corresponding matrix for the “unperturbed” $S^z$ case Eq. (14) or Eq. (24) is denoted by $U^{0z}$:
\[
U^{0z} \quad \begin{bmatrix} M^0_{\nu} \end{bmatrix} \quad U^{0z} \quad = \quad \text{Diag} \left( \left| M^{0z}_{\nu 1} \right|^2, \left| M^{0z}_{\nu 2} \right|^2, \left| M^{0z}_{\nu 3} \right|^2 \right)
\]
We define $I^z_\nu$ as
\[
I^z_\nu = \begin{pmatrix} 0 & \epsilon_1^z & \epsilon_2^z \\ -\epsilon_1^z & 0 & \epsilon_3^z \\ -\epsilon_2^z & -\epsilon_3^z & 0 \end{pmatrix}.
\]
Unitarity of $U^z$ and $U^{0z}$ means that $I^z_\nu$ is antihermitian:
\[
I^z_\nu = \begin{pmatrix} 0 & \epsilon_1^z & \epsilon_2^z \\ -\epsilon_1^z & 0 & \epsilon_3^z \\ -\epsilon_2^z & -\epsilon_3^z & 0 \end{pmatrix} = \begin{pmatrix} 0 & \epsilon_1^z & \epsilon_2^z \\ -\epsilon_1^z & 0 & \epsilon_3^z \\ -\epsilon_2^z & -\epsilon_3^z & 0 \end{pmatrix}^*.
\]
Working to first order in the perturbation $\delta M^z_\nu$, we get the condition:
\[
i, j \in \{1, 2, 3\}; i \neq j, \quad [I^z_\nu, \begin{bmatrix} M^{0z}_{\nu} \end{bmatrix}] = [U^{0z}]^T \left( \begin{bmatrix} M^{0z}_{\nu} \delta M^z_\nu + \delta M^{0z}_\nu \end{bmatrix} U^{0z} \right)_{ij}.
\]
We shall restrict our perturbations to those originating, within seesaw mechanism, from the following perturbation on $M^z_R$ (we assume that $M^z_R$ is form invariant under $S^z$):
\[
M^z_R = M^{0z}_R + \delta M^z_R, \quad \delta M^z_R = \alpha B^z_D \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]
Therefore, up to first order perturbation and after having denoted the generic non vanishing entries of $\delta M^z_R$ by $\alpha_{ij}$ with suitable indexing, we get $\delta M^z_\nu$ as
\[
\delta M^z_\nu = \delta M^z_R \quad \begin{pmatrix} M^z_R \end{pmatrix}^{-1} \quad \begin{bmatrix} M^{0z}_{\nu} \end{bmatrix} \quad \begin{pmatrix} M^z_R \end{pmatrix}^{-1} \quad \delta M^z_R = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{12} & 0 & 0 \\ \alpha_{13} & 0 & 0 \end{pmatrix}.
\]
We note here that the purpose of adding the perturbation $\delta M^z_\nu$ is to break the symmetry $S^z$, otherwise no new role can play $M^z_\nu$ that $M^{0z}_\nu$ can not. This non invariance can be easily verified by checking that we have
\[
S^{zT} \left( \begin{bmatrix} M^{0z}_\nu \end{bmatrix} \right) S^z \neq \delta M^z_\nu.
\]

5.2 The Perturbation Form and the Perturbed Mass Spectrum

We defined in Eq. (14) the relation between the deformed (“rotated”) unperturbed and the non-deformed (“non-rotated”) unperturbed diagonal bases for the diagonalizing matrices:
\[
U^{0z} = \quad W U^0.
\]
The expressions of $(U^0, U^{0z})$ are given respectively in Eqs. (8,24). We note that if the perturbations in the deformed and non-deformed cases are related by the same similarity transformation defined by $W$ Eq. (15), then $I^z_\nu$ is invariant with respect to change of basis, i.e. it is $z$-independent and equal to $I_\nu$ corresponding to perturbing the non-deformed diagonalizing matrix $U^0$. In fact, defining $U = W^T U^z$, and writing it in the form $U = U^0 (1 + I_\nu)$, we see from
\[
\delta M^z_\nu = W \delta M^z_\nu W^T, \quad M^{0z}_\nu = W M^{0z}_\nu W^T, \quad U^{0z} = W U^0,
\]
\[
\Rightarrow M^{0z}_{\nu \text{Diag}} + \delta M^z_\nu \quad \begin{pmatrix} M^{0z}_{\nu \text{Diag}} \end{pmatrix} \quad \begin{pmatrix} M^{0z}_{\nu \text{Diag}} \end{pmatrix} = \begin{pmatrix} M^{0z}_{\nu \text{Diag}} \end{pmatrix} \quad \begin{pmatrix} M^{0z}_{\nu \text{Diag}} \end{pmatrix},
\]
\[
U^{0z} \quad \begin{pmatrix} M^{0z}_{\nu} \delta M^z_\nu + \delta M^{0z}_\nu \end{pmatrix} \quad U^{0z} = \quad U^{0z} \quad \begin{pmatrix} M^{0z}_{\nu} \delta M^z_\nu + \delta M^{0z}_\nu \end{pmatrix} \quad U^{0z},
\]
and from Eq. (56), that \(I\) satisfies the same characterizing equation as \(I^z\). We thus deduce a method
to compute \(I^z\), which is necessary to evaluate the effects of the perturbation \(\delta M^z\) in the “rotated’ basis.
This method consists in writing down the corresponding perturbation in the “non-rotated’ basis in the
form \(\delta M_\nu = W^T \delta M^z W\), and then solving in this latter basis the defining equation:
\[
i, j \in \{1, 2, 3\}, i \neq j, \quad [I^z , M^{\text{Diag}}_\nu M^{\text{Diag}}_\nu]_{ij} = [U^{0z} \left( M^{0z}_\nu W^T \delta M^z W + W^T \delta M^{z*} W M^{0z}_\nu \right) U^{0z}]_{ij}.
\]  

(62)

Then the resulting \(I^z\) is also the correct answer in our rotated basis. We note in addition that the breaking
of the symmetry by the perturbation Eq. (59) is valid in both rotated and non-rotated bases.

We shall work out now an easier equivalent method to find \(I^z\), which originates from the generic
perturbation \(\delta M^z\), by diagonalizing \((M^z)\) rather than \((M^{z*} M^z)\). We have
\[
U^{\dagger} M^z U^z = \text{Diag} (M^{\text{Diag}}_{\nu 11}, M^{\text{Diag}}_{\nu 22}, M^{\text{Diag}}_{\nu 33}),
\]
and
\[
U^{0z} M^{0z} U^{0z} = \text{Diag} (M^{\text{Diag}}_{\nu 11}, M^{\text{Diag}}_{\nu 22}, M^{\text{Diag}}_{\nu 33}).
\]
(63)
(64)

Working now to first order in perturbation, we get
\[
U^{\dagger} M^z U^z = M^{0z} + I^{z*} M^{\text{Diag}} + M^{0z} M^{0z} I^z + U^{0z} \delta M^z U^{0z}.
\]

(65)

Expressing the fact that the LHS of Eq. (65) and its complex conjugate are diagonal, we obtain the
following equations which allow us to determine \(I^z\):^*
\[
i, j \in \{1, 2, 3\}, i \neq j, \quad \left\{
\begin{array}{l}
[I^z , M^{\text{Diag}} + U^{0z} \delta M^z U^{0z}]_{ij} = 0,
[I^z , M^{\text{Diag}} + U^{0z} \delta M^z U^{0z}]_{ij} = 0.
\end{array}
\right.
\]

(66)

The resulting linear system of six equations in the unknowns \((\epsilon^1, \epsilon^2, \epsilon^3, \epsilon^4, \epsilon^5, \epsilon^6)^*\) can be solved and
the solutions are reported in Appendix (B) Eq. (146). Substituting these solutions into Eq. (65), we get, for
the perturbation \(\delta M^z\) given in Eq. (58), the following (recalling \(M^{0z} = M^{\text{Diag}}\):
\[
M^{\text{Diag}}_{\nu 11} = M^{\text{Diag}}_{\nu 11} + \alpha_{11}^2 c^2 s^2 - \sqrt{2} (\alpha_{12}^2 + \alpha_{13}^2) c^2 s^2 + e^{-i\phi} \delta \alpha^2 c^2 s^2 c^2,
M^{\text{Diag}}_{\nu 22} = M^{\text{Diag}}_{\nu 22} + \alpha_{22}^2 c^2 s^2 + \sqrt{2} (\alpha_{12}^2 + \alpha_{13}^2) c^2 s^2 + e^{-i\phi} \delta \alpha^2 c^2 s^2 c^2,
M^{\text{Diag}}_{\nu 33} = M^{\text{Diag}}_{\nu 33} + \alpha_{33}^2 c^2 s^2 + \sqrt{2} (\alpha_{12}^2 + \alpha_{13}^2) c^2 s^2 + e^{-i\phi} \delta \alpha^2 c^2 s^2 c^2.
\]

(67)

The two procedures for determining \(I^z\) via Eq. (56) or Eq. (66) are shown, in Appendix (B), to be equivalent,
as expected.

5.3 Determining the Resulting Mixing and Phase Angles after Perturbation

Having now determined the matrix \(I^z\) (Eq. (146)), then the resulting mixing matrix, denoted by \(U_e\), would be
\[
U_e = U^{0z} (1 + I^z) \text{Diag} (e^{-i\phi_1}, e^{-i\phi_2}, e^{-i\phi_3}),
\]

(68)

where \(\phi_i = \frac{1}{2} \text{Arg} (M^{\text{Diag}}_\nu)\). The multiplication by the determined diagonal phase matrix is required
in order to make the eigenvalues of \(M_\nu\) real and positive. Before extracting the resulting mixing and phase
angles, one needs to carry out a further rephasing in order to make the third column of \(U_e\) real, so that
to be consistent with the adopted parametrization of Eq. (4). Thus, we rephase the fields of the charged
leptons as,
\[
e \rightarrow e^{i\psi_1} e, \quad \mu \rightarrow e^{i\psi_2} \mu, \quad \tau \rightarrow e^{i\psi_3} \tau, \quad \text{where} \quad \psi_i = \text{Arg} [U_e (i, 3)],
\]

(69)

^The second line is just the complex conjugate of the first line. One could limit oneself to the first line which represents
3 complex equations in 3 complex unknowns \(\epsilon_i^*, i = 1, 2, 3\). However, it is simpler to treat \(\epsilon_i^*\) as independent from \(\epsilon_i\), and
get a linear system of 6 equations in 6 unknowns. Eventually, we checked that the obtained solutions are consistent in that
\(\epsilon_i^*\) is indeed the complex conjugate of \(\epsilon_i\).
The full expressions for all entries of the matrix $U_\nu$ are presented in Eq. (147) in Appendix (B).

Identifying now $U_\nu$, after having suitably rephased the charge leptons, with $V^\nu_{\text{PMNS}}$, we can extract the angles. We list now the first order approximations for $\theta_x$, $\theta_y$ and $\theta_z$ whereas the full expressions are listed in the Appendix (B) Eq. (148),

$$
t_x \simeq t_\varphi \left| 1 + \frac{1}{t_\varphi} e^{s_\varphi} + s_\varphi \right|, \\
t_y \simeq \left| 1 - 2 s_\varphi e^{-i\xi} + 2 c_\varphi e^{-i\xi} \right|, \\
s_z \simeq s_{z_0} \left| 1 + \frac{c_\varphi}{t_{z_0}} e^{s_\varphi} + \frac{s_\varphi}{t_{z_0}} e^{i\xi} \right|, \\
(70)
$$

where $z_0$ is the angle which determines the deformed symmetry before perturbation, whereas $z$ corresponds to the deformed and perturbed symmetry.

As to the Majorana phase angles, which by convention belong to the first and second quadrants, they are determined to be:

$$
\rho = \pi - \text{Arg} \left[ \frac{c_{z_0} c_{\varphi} - c_{z_0} s_{\varphi} e^{s_{z_0} - s_{z_0} e^{s_{z_0}}}}{s_{z_0} + c_{z_0} c_{\varphi} e^{s_{z_0}} + c_{z_0} s_{\varphi} e^{s_{z_0}}} \right] - \frac{1}{2} \text{Arg} \left( M^z_{\nu_{13}} M^{\nu_{13}}_{\text{Diag}} \right), \\
\sigma = \pi - \text{Arg} \left[ \frac{c_{z_0} s_{\varphi} + c_{z_0} c_{\varphi} e^{s_{z_0} - s_{z_0} e^{s_{z_0}}}}{s_{z_0} + c_{z_0} c_{\varphi} e^{s_{z_0}} + c_{z_0} s_{\varphi} e^{s_{z_0}}} \right] - \frac{1}{2} \text{Arg} \left( M^z_{\nu_{33}} M^{\nu_{33}}_{\text{Diag}} \right). \\
(71)
$$

Finally, and after determining the mixing and Majorana phases, we can get the Dirac phase $\delta$ by solving an equation which results upon equating an entry of $V^\nu_{\text{PMNS}}$ involving $\delta$ with the corresponding one of $U_\nu$ (look for example at Eq. (149) in Appendix B).

6 Realization of perturbed deformed textures

It is important to find theoretical realizations for the perturbed textures, assuming at the level of the Lagrangian exact symmetries, some of which are broken spontaneously. We need to parameterize the perturbations on $M_{\nu}^0$, which we shall assume originating from perturbations on only $M^z_B$, so we need to parameterize the latter perturbations also. As to $M^z_R$ we shall assume that it is invariant under $S^z$. We find two parameters, $\chi$ and $\xi$ for the perturbations in $M^z_B$, that we shall dis-entangle in our future numerical work scanning the parameter space and contrasting to data, as was done in the case of $S$ symmetry [6]. Thus the question arises whether or not we can find a theoretical realization for the perturbed texture where one of the parameters, say $\chi$ only is present. In [6], we carried out this task for $S$-symmetry and we aim now to generalize this to $S^z$-symmetry, which, as we shall see, is not a trivial task.

6.1 Parameterizing the Perturbations

The general form for a symmetric matrix invariant under $S^z$ is given in Eqs. (20, 21). We rewrite them here for the unperturbed matrix $M^0_{\nu}:

$$
M_{\nu_{13}}^{0z} = (1 - 2 t_{z_0}^2) M_{\nu_{13}}^{0z} - \sqrt{2} t_{z_0} \left( M_{\nu_{11}}^{0z} - M_{\nu_{22}}^{0z} - M_{\nu_{23}}^{0z} \right), \\
M_{\nu_{33}}^{0z} = (1 - 2 t_{z_0}^2) M_{\nu_{33}}^{0z} + 2 t_{z_0} \left( M_{\nu_{11}}^{0z} - M_{\nu_{22}}^{0z} \right) + 2 \sqrt{2} t_{z_0} \left( 1 - t_{z_0}^2 \right) M_{\nu_{12}}^{0z}. \\
(72)
$$

As there are two constraints, Eq. (72), on the symmetric matrix obeying $S^z$ symmetry, we thus define two parameters which determine the perturbation by measuring the deviations from these two constraints:

$$
\chi = \frac{M_{\nu_{13}}^{z_0} - \left[ (1 - 2 t_{z_0}^2) M_{\nu_{13}}^{0z} - \sqrt{2} t_{z_0} (M_{\nu_{11}}^{0z} - M_{\nu_{22}}^{0z} - M_{\nu_{23}}^{0z}) \right]}{M_{\nu_{12}}^{0z}}, \\
\xi = \frac{M_{\nu_{33}}^{z_0} - \left[ (1 - 2 t_{z_0}^2) M_{\nu_{33}}^{0z} + 2 t_{z_0} (M_{\nu_{11}}^{0z} - M_{\nu_{22}}^{0z}) + 2 \sqrt{2} t_{z_0} (1 - t_{z_0}^2) M_{\nu_{12}}^{0z} \right]}{M_{\nu_{33}}^{0z}}, \\
(73)
$$

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The invariance of $M_D^{0z}$ under $S^z$ forces it to be parameterized in a such a way as presented through Eqs. (45–46). We rewrite it here in the following form:

\begin{align*}
M_D^{0z}_{13} & = -M_D^{0z}_{12} + \sqrt{2} t_{z0} \left( M_D^{0z}_{23} + M_D^{0z}_{22} - M_D^{0z}_{11} \right) + 2 t_{z0}^2 M_D^{0z}_{21}, \\
M_D^{0z}_{32} & = M_D^{0z}_{23} + \sqrt{2} t_{z0} \left( M_D^{0z}_{21} - M_D^{0z}_{12} \right), \\
M_D^{0z}_{33} & = (1 - 2 t_{z0}^2) M_D^{0z}_{22} - 2 t_{z0}^2 \left( M_D^{0z}_{23} - M_D^{0z}_{11} \right) + \sqrt{2} t_{z0} M_D^{0z}_{12} + \sqrt{2} t_{z0} \left( 1 - 2 t_{z0}^2 \right) M_D^{0z}_{21}, \\
M_D^{0z}_{31} & = - (1 - 2 t_{z0}^2) M_D^{0z}_{21} + \sqrt{2} t_{z0} \left( M_D^{0z}_{23} + M_D^{0z}_{22} - M_D^{0z}_{11} \right). \\
\end{align*}

Moreover, in Appendix (A), we restate this form of a general matrix invariant under $S^z$ (called there $M_D^z$) in Eq. (136), while the relations between the deformed and the non-deformed parameters are written in Eqs. (137–138). Having four constraints on $M_D^{0z}$ implied by $S^z$ invariance, as presented in Eq. (74), leads naturally to require four parameters in order to quantify the deviation from $S^z$ symmetry. However, for simplicity and flexibility purposes, one can work by taking the mass matrix $M_D^z$ as bearing only two perturbation parameters $\alpha$ and $\beta$ in the form:

\begin{equation}
M_D^z = \begin{pmatrix}
A_{D} & B_D^z(1+\alpha) & B_D^z \\
E_D^z(1+\beta) & C_D & D_D^z \\
E_D^z & D_D^z & C_D^z
\end{pmatrix},
\end{equation}

where the set of parameters $\{B_D^z, C_D, D_D^z, E_D^z\}$ are related to the set $\{A_D^z, B_D^z, C_D^z, D_D^z, E_D^z\}$ through the same relations given in Eq. (46). This fixes the parameters $\alpha, \beta$ for a given perturbed mass matrix $M_D^z$.

The “perturbed” $M_D^z$, as given by Eq. (75), satisfies the following relations, which clarify how the parameters $\alpha, \beta$ measure the deviations from the constraints of Eq. (74):

\begin{align*}
M_D^{z\alpha}_{13} + M_D^{z\alpha}_{12} + \sqrt{2} t_{z0} \left( M_D^{z\alpha}_{23} - M_D^{z\alpha}_{22} - M_D^{z\alpha}_{21} \right) - 2 t_{z0}^2 M_D^{z\alpha}_{21} = \alpha B_D^z - 2 \beta t_{z0} E_D^z, \\
M_D^{z\alpha}_{32} - (1 - 2 t_{z0}^2) M_D^{z\alpha}_{22} + 2 t_{z0}^2 \left( M_D^{z\alpha}_{23} - M_D^{z\alpha}_{11} \right) - \sqrt{2} t_{z0} M_D^{z\alpha}_{12} + \sqrt{2} t_{z0} \left( 1 - 2 t_{z0}^2 \right) M_D^{z\alpha}_{21} = \\
-\sqrt{2} t_{z0} \left[ \alpha B_D^z + \beta \left( 1 - 2 t_{z0}^2 \right) E_D^z \right], \\
M_D^{z\beta}_{32} - M_D^{z\beta}_{23} - \sqrt{2} t_{z0} \left( M_D^{z\beta}_{21} - M_D^{z\beta}_{12} \right) = \sqrt{2} t_{z0} \left( \alpha B_D^z - \beta E_D^z \right), \\
M_D^{z\beta}_{31} + M_D^{z\beta}_{21} + \sqrt{2} t_{z0} \left( M_D^{z\beta}_{11} - M_D^{z\beta}_{22} - M_D^{z\beta}_{23} \right) - 2 t_{z0}^2 M_D^{z\beta}_{21} = \beta E_D^z \left( 1 - 2 t_{z0}^2 \right). \\
\end{align*}

The perturbations induced by $\alpha$ and $\beta$ in $M_D^z$ would be transmuted into $M_D^z$ through the seesaw mechanism described in Eq. (38), and one can compute the corresponding perturbations parameters $\chi$ and $\xi$ which, to first order in $\alpha, \beta$ and $s_z$, turn out to be

\begin{align*}
\chi & = -\alpha B_D^z \left[ (C_D^0 - D_D^0) [A_{D}^0 (C_D^0 - D_D^0) - 2 B_{D}^0] - \beta E_D^z \left( C_{D}^0 + D_{D}^0 \right) [A_{D}^0 (C_D^0 - D_D^0) - 2 B_{D}^0 B_{D}^0] \right]/(C_{D}^0 + D_{D}^0) [B_{D}^0 A_{D} (D_{D}^0 - C_{D}^0) + E_D^0 A_{D}^2 (D_{D}^0 - C_D^0) + B_{D}^0 (2 B_{D}^2 E_D^0 + A_{D}^0 C_{D}^0 - A_{D}^0 D_{D}^0)]]; \\
\xi & = -\beta E_D^z \left( C_{D}^0 + D_{D}^0 \right) [E_D^0 (C_D^0 - D_D^0) + B_{D}^0 (D_{D}^0 - C_{D}^0)]/(A_{D}^2 C_{D}^0 - B_{D}^2)^2 + (C_{D}^0 + D_{D}^0) \left[ 2 B_{D}^2 E_D^0 (D_{D}^0 + C_{D}^0) + E_D^0 (C_{D}^0 - D_{D}^0) - 2 C_{D}^0 D_{D}^0 (A_{D}^0 D_{D}^0 + B_{D}^2)^2 \right] \right].
\end{align*}

We see directly, up to this given order, that when $\beta = 0$ then $\xi = 0$. As we seek in this section a realization for a dis-entangled perturbation parameterized solely by $\chi$, then we shall look for a realization of $M_D^z$ with $\beta = 0$.

In [6] we found a realization of the “dis-entangled” perturbation, due only to $\chi$ and not to $\xi$, assuming exact $S$-symmetry but at the expense of extending the symmetry and adding new matter. Here, we shall do the same but with the symmetry $S^z$. In order to find the $S^z$-transformations knowing the corresponding $S$-ones, we use the following rule of thumb:

\begin{equation}
\text{(deformed symmetry element)} = W \cdot \text{(non-deformed symmetry element)} \cdot W^T
\end{equation}

As in [6], we present two ways to get a perturbed $M_D^z$ with $\chi = 0$, the first one assuming a $S^z \times Z_2^z$ symmetry, whereas the symmetry in the other way is $S^z \times Z_8$.
6.2 \( S^z \times Z_2 \times Z'_2 \)-flavor symmetry

- Matter content and symmetry transformations

We have three SM-like Higgs doublets (\( \phi_i \), \( i = 1, 2, 3 \)) giving mass to the charged leptons and another three Higgs doublets (\( \phi'_i \), \( i = 1, 2, 3 \)) for the Dirac neutrino mass matrix. All the fields remain unchanged under \( Z'_2 \) except the fields \( \phi' \) and \( \nu_R \) which are multiplied by \(-1\), so that we assure that neither \( \phi \) can contribute to \( M_D \), nor \( \phi' \) to \( M_l \). We had in [6] the assignment of the fields under the \( S \)-symmetry, and so by the rule of thumb we get the following transformations.

The transformations under \( Z_2 \) are

\[
\begin{align*}
\nu_R & \xrightarrow{Z_2} W \text{ Diag}(1, -1, 1) W^T \nu_R, & \phi' & \xrightarrow{Z_2} W \text{ Diag}(1, -1, -1) W^T \phi', \\
L & \xrightarrow{Z_2} W \text{ Diag}(1, -1, 1) W^T L, & l^c & \xrightarrow{Z_2} W \text{ Diag}(1, 1, -1) W^T l^c, \\
\phi & \xrightarrow{Z_2} W \text{ Diag}(1, -1, -1) W^T \phi.
\end{align*}
\]

The transformation under \( S^z \) are

\[
\begin{align*}
\nu_R & \xrightarrow{S^z} S^z_{eR} \nu_R = S^z \nu_R, & \phi' & \xrightarrow{S^z} S^z_{\phi'} \phi' = W \text{ Diag}(1, 1, -1) W^T \phi', \\
L & \xrightarrow{S^z} S^z_L L = S^z L, & l^c & \xrightarrow{S^z} W \text{ Diag}(1, 1, 1) W^T l^c = l^c, \\
\phi & \xrightarrow{S^z} S^z_{\phi} \phi = S^z \phi.
\end{align*}
\]

- Charged lepton mass matrix-flavor basis

The Lagrangian responsible for \( M^l_i \) is given by:

\[
\mathcal{L}_2 = f^{ij}_{ik} \bar{T}_i \phi_k l^j
\]

(81)

The invariance of the Lagrangian under \( S^z \) implies the following for the Yukawa couplings \( f^{ij}_{ik} \):

\[
S^z_{\bar{L}i} f^{ij}_{ik} S^z_{\phi km} S^z_{ljm} = f^m_{lm}.
\]

(82)

In order to find \( f^{ij}_{ik} \), one can start from the known solutions in the case of \( S \)-symmetry:

\[
S^z_{\bar{L}i} f^j_{ik} S^z_{\phi km} S^z_{ljm} = f^m_{lm},
\]

(83)

and expressing the \( S^z \)'s in terms of the \( S^z \)'s in that \( S = W^T S^z W \), we get

\[
W^T_{i\alpha} S^z_{\bar{L}i} \phi \beta W^T_{\bar{L}j} S^z_{\phi \gamma} \nu \delta W^T_{\nu \gamma} S^z_{l\delta} W^T_{l\sigma} = f^m_{lm}.
\]

(84)

We find that a solution of Eq. (82) is given by:

\[
f^{ij}_{\beta \gamma} = W_{\beta i} f^j_{ik} W^T_{k\gamma} W^T_{j\sigma}.
\]

(85)

Defining the matrices \( f^{ij} \) and \( f^j \) as the matrices whose \((i, k)\)-th entries are respectively \( f^{z^j}_{ik} \) and \( f^{j}_{ik} \), then we can express Eq. (85) as

\[
f^{ij} = W f^j W^T (W^T)_{j\sigma},
\]

(86)

which means that the solution for the “rotated” basis is obtained by a similarity transformation applied onto the solution for “non-rotated” basis, followed by a linear combination weighted by \((W^T)_{j\sigma}\). Moreover we can re-express the symmetry constraint of Eq. (82) in matrix form as a weighted sum of similarity transformations:

\[
S^z_{\bar{L}i} f^{\sigma} S^z_{\phi} (S^z_{\nu})_{\sigma \Lambda} = f^{z^\Lambda}.
\]

(87)

Now, using the results of [6] where, taking into consideration the invariance under both \( S \) and \( Z_2 \)-symmetries, we obtained

\[
f^1 = \begin{pmatrix} A^1 & 0 & 0 \\ 0 & C^1 & D^1 \\ 0 & D^1 & C^1 \end{pmatrix}, \quad f^2 = \begin{pmatrix} A^2 & 0 & 0 \\ 0 & C^2 & D^2 \\ 0 & D^2 & C^2 \end{pmatrix}, \quad f^3 = \begin{pmatrix} 0 & B^3 & -B^3 \\ E^3 & 0 & 0 \\ -E^3 & 0 & 0 \end{pmatrix},
\]

(88)
and applying the similarity transformations by $W$:

$$\tilde{F}^{ij} = W f^j W^T$$  \hspace{1cm} (89)$$

we get the matrices $\tilde{F}^{ij}, j = 1, 2, 3$ whose expressions, in terms of new coefficients ($A^z_i, B^z_i, C^z_i, D^z_i, i = 1, 2$) and ($E^z_i, B^z_3$) related to the old coefficients, are given in Appendix C Eq. (150) and Eq. (151).

We follow this by the weighted sum $F^{z\sigma} = \tilde{F}^{ij} W_{jj}'$ using the expressions of $W$ in Eq. (15) to find finally

$$f^{z_1} = c_2 \tilde{f}^{z_1} + \frac{s_2}{\sqrt{2}} \tilde{f}^{z_2} + \frac{s_2}{\sqrt{2}} \tilde{f}^{z_3},$$

$$f^{z_2} = -\frac{s_2}{\sqrt{2}} \tilde{f}^{z_1} + c_2/2 \tilde{f}^{z_2} - s_2/2 \tilde{f}^{z_3},$$

$$f^{z_3} = -\frac{s_2}{\sqrt{2}} \tilde{f}^{z_1} - s_2/2 \tilde{f}^{z_2} + c_2/2 \tilde{f}^{z_3}. \hspace{1cm} (90)$$

When the Higgs fields $\phi^i$ acquire vevs, and assuming ($v_3 \gg v_1, v_2$) we get to lowest order in $s_z$:

$$M^{z}_l = v_3 \begin{pmatrix} 0 & 0 & -B^{z}_{3} \\ D^{z}_{i} & D^{z}_{3} & 0 \\ C^{z}_{i} & C^{z}_{3} & 0 \end{pmatrix} \Rightarrow M^{z}_l M^{z \dagger}_l = \frac{1}{v_3} \begin{pmatrix} |B^{z}|^2 & 0 & 0 \\ 0 & |D^{z}|^2 & D^{z} \cdot C^{z} \\ 0 & C^{z} \cdot D^{z} & |C^{z}|^2 \end{pmatrix}, \hspace{1cm} (91)$$

where $B^z = (0, 0, -B^z_3)^T$, $D^z = (D^z_1, D^z_2, 0)^T$ and $C^z = (C^z_1, C^z_2, 0)^T$, and where the dot product is defined as $D^z \cdot C^z = \sum_{i=1}^{3} D^z_i C^z_i$. Under the reasonable assumption that the magnitudes of the Yaukawa couplings come in ratios proportional to the lepton mass ratios as $|B^z| : |C^z| : |D^z| \sim m_e : m_\mu : m_\tau$, we can show, as was done in [5], that this form can be diagonalized by infinitesimal rotations applied onto the LH charged lepton fields, which justifies working in the flavor basis to a good approximation.

**Majorana neutrino mass matrix**

The mass term is directly present in the Lagrangian

$$\mathcal{L}_R = \frac{1}{2} \nu^T_R C^{-1} (M^z_R)_{ij} \nu_{Rj}. \hspace{1cm} (92)$$

The invariance under $Z^z_2$ is trivially satisfied while the one under $S^z \times Z_2$ is more involved. In [6], we found a form similar to, say, $f^1$ in Eq. (88), which was invariant under $S \times Z_2$, and so the corresponding form in the “deformed (rotated)” basis for $M^z_R$ would be similar to $\tilde{f}^1 = W f^1 W^T$, i.e. that $M^z_R$ would assume the following form,

$$M^z_R = \begin{pmatrix} A^{z}_{R} & B^{z}_{R} & B^{z}_{R} \\ B^{z}_{R} & C^{z}_{R} & D^{z}_{R} \\ B^{z}_{R} & D^{z}_{R} & C^{z}_{R} \end{pmatrix}, \text{ where } B^{z}_{R} = -\frac{t_{2z}}{2\sqrt{2}} (A^{z}_{R} - C^{z}_{R} - D^{z}_{R}). \hspace{1cm} (93)$$

One can check that $M^z_R$ above does satisfy the constraints of Eq. (20, 21) showing that $M^z_R$ is $S^z$-invariant.

**Dirac neutrino mass matrix**

The Lagrangian responsible for the neutrino mass matrix is

$$\mathcal{L}_D = g^{iz}_{k} \tilde{T}^{i}_{k} \tilde{\phi}'_{k} \nu_{Rj}, \text{ where } \tilde{\phi}' = i \sigma_2 \phi'^*.$$  \hspace{1cm} (94)$$

In a similar manner to our discussion in the above item about the charged lepton mass matrix, we find that invariance under $S^z$ implies the following constraint on Yukawa couplings (c.f. Eq. (82)):

$$S^{z}_{L,i} g^{iz}_{k} S^{z}_{\nu,n,j} m^{z}_{\phi/km} = g^{iz}_{lm}, \hspace{1cm} (95)$$
which can be written in an equivalent matrix form similar to Eq. (87) as

\[ S_L^{\tau} \mathbf{g} \mathbf{g}^{\sigma} S_{\nu R} (S_{\nu L}^\tau)_{\sigma \Lambda} = \mathbf{g} \mathbf{g}^{\Lambda}. \]  

(96)

where the matrix \( g^{zj} \) has \( g^{zj} \) at its \((i, k)\)-th entry.

Again, knowing the solution \( \mathbf{g} \) for “non-deformed” case we can get the corresponding one for the deformed case as,

\[ g^{zj} = W g^j W^T (W^T)^{j \sigma}, \]  

(97)

In [6], we found the expressions of \( g^j_{ik} \) taking into consideration the \( S \) and \( Z_2 \) symmetries:

\[ g^1 = \begin{pmatrix} A^1 & 0 & 0 \\ 0 & C^1 & D^1 \\ 0 & D^1 & C^1 \end{pmatrix}, \quad g^2 = \begin{pmatrix} 0 & B^2 & -B^2 \\ E^2 & 0 & 0 \\ -E^2 & 0 & 0 \end{pmatrix}, \quad g^3 = \begin{pmatrix} 0 & B^3 & B^3 \\ E^3 & 0 & 0 \\ E^3 & 0 & 0 \end{pmatrix}. \]  

(98)

We apply now the similarity transformations by \( W \):

\[ \tilde{g}^{zk} = W g^k W^T, \]  

(99)

and we get the matrices \( \tilde{g}^{zk}, k = 1, 2, 3 \) whose expressions, in terms of new coefficients \((A^i, B^i, C^i, D^i, E^i, F^i, i = 1, 2, 3)\) related to the old coefficients, are given in Appendix C (Eq. (152) and Eq.153)). We follow this by the weighted sum \( g^{z\sigma} = \tilde{g}^{zk} W_{k\sigma} \) (c.f. Eq. (90) replacing \( F^z \) by \( g^z \)).

Upon acquiring vevs \((v_i', i = 1, 2, 3)\) for the Higgs fields \((\phi_i')\), we get, up to leading order in \( s_z \), for Dirac neutrino mass matrix the form:

\[ M_D = g^{z\sigma}v_{\sigma}' = \begin{pmatrix} v_1' A_1 \quad v_2' B_2 + v_3' B_2 & -v_2' B_2 + v_3' B_2 \\ v_2' E_2 + v_3' E_3 \quad v_1' C_1 \\ -v_2' E_2 + v_3' E_3 \quad v_1' D_1 \end{pmatrix}, \]  

(100)

which can be matched, up to leading order of \( s_z \), with the form of Eq. (76) to yield,

\[ \alpha = \frac{2v_2'B_2}{v_2'B_2 - v_3'B_3}, \quad \beta = \frac{2v_3'E_3}{v_2'E_2 - v_3'E_3}. \]  

(101)

If the vevs satisfy \( v_3' \ll v_2' \) and the Yukawa couplings are of the same order, then we get perturbative parameters \( \alpha, \beta \ll 1 \). These perturbative parameters resurface as perturbative parameters for \( M_\nu^c \) Eq. (73). Although we do not get in general disentanglement of the perturbations (\( \xi = 0 \)), however, for specific choices of Yukawa couplings, for e.g. \( E_3' = 0 \) leading to \( \beta = 0 \) and hence \( \xi = 0 \), we get this disentanglement, where only \( \chi \) is not equal to zero and is given by Eq. (77) with \( B^z_R = 0 \) to lowest order.

6.3 \( S^2 \times Z_8 \)-flavor symmetry

Here, and as was the case in [6], we shall find a realization that gives \( \beta = 0 \) regardless of the Yukawa couplings values.

- Matter content and symmetry transformations

We have the left doublets \((L_i, i = 1, 2, 3)\), the RH charged singlets \((l_j^c, j = 1, 2, 3)\), the RH neutrinos \((\nu_{Rj}, j = 1, 2, 3)\) and the SM-Higgs three doublets \((\phi_i, i = 1, 2, 3)\) responsible for the charged lepton masses. We have also four Higgs doublets \((\phi'_j, j = 1, 2, 3, 4)\) leading to Dirac neutrino mass matrix, and two Higgs singlet scalars \((\Delta_k, k = 1, 2)\) related to Majorana neutrino mass matrix. We denote the octic root of the unity by \( \omega = e^{\frac{2\pi i}{8}} \). The fields transform according to the rule of thumb Eq. (78) as follows.
The transformations under $S^z$ are
\[ L \xrightarrow{S^z} S^z_L L = S^z L, \quad t^c \xrightarrow{S^z} W \text{Diag} \left( 1, 1, 1 \right) W^t t^c = t^c, \quad \phi \xrightarrow{S^z} S^z_\phi \phi = S^z \phi, \]

\[ \nu_R \xrightarrow{S^z} S^z_\nu_R \nu_R = S^z \nu_R, \quad \phi' \xrightarrow{S^z} W_{4\text{ext}} \text{Diag} \left( 1, 1, 1, -1 \right) W_{4\text{ext}}^\dagger \phi', \]

\[ \Delta \xrightarrow{S^z} W_{2\text{ext}} \text{Diag} \left( 1, 1 \right) W_{2\text{ext}}^\dagger \Delta = \Delta, \quad \phi' \xrightarrow{S^z} W_{4\text{ext}} \text{Diag} \left( 1, 1, 1, -1 \right) W_{4\text{ext}}^\dagger \phi'. \tag{102} \]

The transformation under $Z_8$ are
\[ L \xrightarrow{Z_8} W \text{Diag} \left( 1, -1, -1 \right) W^t L, \quad t^c \xrightarrow{Z_8} W \text{Diag} \left( 1, 1, -1 \right) W^t t^c, \]
\[ \phi \xrightarrow{Z_8} W \text{Diag} \left( 1, -1, -1 \right) W^t \phi, \quad \nu_R \xrightarrow{Z_8} W \text{Diag} \left( \omega, \omega^3, \omega^3 \right) W^t \nu_R, \]
\[ \phi' \xrightarrow{Z_8} W_{4\text{ext}} \text{Diag} \left( \omega, \omega^3, \omega^7, \omega^3 \right) W_{4\text{ext}}^\dagger \phi', \quad \Delta \xrightarrow{Z_8} W_{2\text{ext}} \text{Diag} \left( \omega^6, \omega^2 \right) W_{2\text{ext}}^\dagger \Delta, \]
\[ \phi' \xrightarrow{Z_8} W_{4\text{ext}} \text{Diag} \left( \omega^7, \omega^5, \omega, \omega^5 \right) W_{4\text{ext}}^\dagger \phi'. \tag{103} \]

We note here that we need to extend the symmetry $S^z$ to the case of two and four dimensional representations, in that we need to define the action of the element $W$ of the “rotations” group over the 2-dim $\Delta$-field and over the 4-dim $\phi'$-field. We also note that we use $W^\dagger$ rather than $W^T$, since it is the inverse $W^{-1}$ which is involved in the definition of the similarity transformation from the “non-rotated” to the “rotated” bases. For a unitary complex matrix, it is $W^\dagger$ which represents the inverse and not $W^T$.

The extension of the $W$-action from the fundamental representation of the rotations group acting on 3-dim space to 4-dim space is carried in the simplest way by embedding the 3-dim rotation into a 4-dim one by a canonical injection:
\[ R_{3 \times 3} \rightarrow \left( \begin{array}{cc} R_{3 \times 3} & 0 \\ 0 & 1 \end{array} \right) \Rightarrow W_{4\text{ext}} = \left( \begin{array}{cc} W & 0 \\ 0 & 1 \end{array} \right), \tag{104} \]

As to the extension of $W$, which is a rotation in $SO(3)$ into a 2-dim matrix, it is carried out by the 1-to-2 homomorphism between $SO(3)$ and its universal covering $SU(2)$, where every rotation will be mapped into an element of $SU(2)$ acting on 2-dim space. Denoting Pauli matrices by $\sigma$:
\[ \sigma_1 = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \quad \sigma_2 = \left( \begin{array}{cc} 0 & -i \\ i & 0 \end{array} \right), \quad \sigma_3 = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right), \tag{105} \]

we have the following correspondences:
\[ R_x \rightarrow \exp \left( -i \frac{\theta_x}{2} \sigma_3 \right), \quad R_y \rightarrow \exp \left( -i \frac{\theta_y}{2} \sigma_1 \right), \quad R_z \rightarrow \exp \left( -i \frac{\theta_z}{2} \sigma_2 \right), \tag{106} \]

and thus $W$ given by Eq. (15) would be extended into the $2 \times 2$ matrix:
\[ W_{2\text{ext}} = \exp \left( -i \frac{\pi}{8} \sigma_1 \right) \exp \left( -i \frac{\theta_z}{2} \sigma_2 \right) \exp \left( i \frac{\pi}{8} \sigma_1 \right), \]
\[ = \begin{pmatrix} c_{z/2} - \frac{i}{\sqrt{2}} s_{z/2} & -1 \frac{\sqrt{2}}{s_{z/2}} s_{z/2} \\ \frac{1}{\sqrt{2}} s_{z/2} & c_{z/2} + \frac{i}{\sqrt{2}} s_{z/2} \end{pmatrix}. \tag{107} \]

Thus we have
\[ \Delta \xrightarrow{Z_8} W_{2\text{ext}} \text{Diag} \left( \omega^6, \omega^2 \right) W_{2\text{ext}}^\dagger \Delta = \begin{pmatrix} -i c_{z/2} & -s_{z/2} - \frac{i}{\sqrt{2}} s_{z/2} \\ s_{z/2} - \frac{i}{\sqrt{2}} s_{z/2} & i c_{z/2} \end{pmatrix} \Delta. \tag{108} \]
• **Charged lepton mass matrix-flavor basis** The symmetry restrictions in constructing the charged lepton mass Lagrangian Eq. (81) is similar to what is obtained in the case of \((S^2 \times Z_2 \times Z_2')\). The similarity comes from the fact that the charges assigned to the fields \((L, \ell^c, \phi)\) for the factor \(Z_2\) (of \(S^2 \times Z_2 \times Z_2'\)) and for \(Z_8\) (of \(S^2 \times Z_8\)) are the same. Thus, the story repeats itself, and we end up, assuming a hierarchy in the Higgs \(\phi\)'s fields vevs \((v_3 \gg v_2, v_1)\), with a charged lepton mass matrix adjustable to be approximately in the flavor basis. Moreover, we showed in [6] that the \(Z_8\)-symmetry forbids the term \(\overline{L}_i \phi_k' \ell^c_j\), and this remains valid in our construction based on \(S^2 \times Z_8\).

• **Majorana neutrino mass matrix**

The mass term is generated from the Lagrangian

\[
\mathcal{L}_R = \frac{1}{2} h^z_{ij} \Delta_k \nu^T_{Ri} C^{-1} \nu_{Rj}. \tag{109}
\]

Again, comparing with Eq. (81) and doing the substitutions \((\overline{L}_i \rightarrow \nu^T_{Ri}, \phi_j \rightarrow \nu_{Rj} \text{ and } \ell^c_k \rightarrow \Delta_k)\) which should not be taken too much literally but must be considered as a mnemonic device. Therefore, the story is done over again, in that the Yukawa coupling \(h^z_{ij}\) \((i, j = 1, 2, 3; k = 1, 2)\) should satisfy the following constraint expressing the invariability of its components:

\[
S_{\nu R}^{\nu R} h^z_{ij} S_{\nu R}^{\nu R} S_{\Delta k} = h^z_{lm}, \tag{110}
\]

which can be written in an equivalent matrix form similar to Eq. (87) as

\[
S_{\nu R}^{\nu R} h^z_{ij} \sigma S_{\nu R} (S_{\Delta})_{\sigma \lambda} = h^z_{\lambda \lambda}. \tag{111}
\]

where the matrix \(h^z_{ij}\) has \(h^z_{ik}\) at its \((i, k)\)-th entry.

Again, knowing the solution \(h\) for “non-deformed” case we can get the corresponding one for the deformed case as,

\[
h^z_{ij} = W h^z W^T (W_{2_{\text{ext}}})_{ij}. \tag{112}
\]

We obtained in [6] considering both the \(S^2\) and \(Z_8\) symmetries the following

\[
h^1 = \begin{pmatrix} A_{R1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad h^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & C_{R2} & D_{R2} \\ 0 & D_{R2} & C_{R2} \end{pmatrix}. \tag{113}
\]

Making the similarity transformation by \(W\):

\[
\tilde{h}^z_{ik} = W h^z W^T, \tag{114}
\]

we get the matrices \(\tilde{h}^z_{ik}, k = 1, 2\) whose expressions, in terms of new coefficients related to the old coefficients, are given in Appendix C (Eq. (154) and Eq. (155)). We follow this by the weighted sum \(h^z_{ij} = \tilde{h}^z_{ik} (W_{2_{\text{ext}}})_{k\sigma}\):

\[
h^z_{ij} = \left( c_{z/2} + \frac{i}{\sqrt{2}} s_{z/2} \right) \tilde{h}^z_{i1} - \frac{1}{\sqrt{2}} s_{z/2} \tilde{h}^z_{i2},
\]

\[
h^z_{ij} = \frac{1}{\sqrt{2}} s_{z/2} \tilde{h}^z_{i1} + \left( c_{z/2} - i \sqrt{2} s_{z/2} \right) \tilde{h}^z_{i2}. \tag{115}
\]

We can verify explicitly that \(h^z_{i1}, h^z_{i2}\) satisfy the requirements of the \(S^2 \times Z_8\)-symmetry:

\[
S_{\nu R}^{\nu R} h^z_{ij} \sigma S_{\nu R} (S_{\Delta})_{\sigma \lambda} = h^z_{\lambda \lambda}. \tag{116}
\]

When the \(\Delta\)'s acquire vevs \((\Delta_{1,2}^0)\), then we get up to leading order in \(s_z\)

\[
M_{R \tilde{R}} = h^z_{ik} \Delta_{ik}^0 \approx \begin{pmatrix} A_{R1}^0 & 0 \\ 0 & A_{C_{R2}}^0 & D_{R2}^0 \\ 0 & D_{R2}^0 & C_{R2}^0 \end{pmatrix}, \tag{117}
\]
which is, to leading order, of the form of Eq. (20) with \( B_R^i = B_R^{i'} = 0 \), \( C_R^i = C_R^{i'} \). It is important to stress that the full expression, without any approximation, of the matrix \( M_R^{i'} \) fulfills the form requirement expressed in Eqs. (20–72. In case of approximating \( M_R^{i'} \) up to a certain order in \( s_z \), as is done in Eq. (117), then the relations expressed in Eq. (72) are still satisfied up to this certain order but there might be violations at the next order.

### Dirac neutrino mass matrix

The Lagrangian responsible for the Dirac neutrino mass matrix is given by Eq. (94). Following exactly as in the case of \( Z_2^* \times S^2 \)-symmetry, we find that Eqs. (95, 96) remain valid, and instead of Eq. (97) we have the solution as

\[
 g^{z\gamma} = W g^k W^T (W_{4\text{ext}}^T)_{k\gamma}. \tag{118}
\]

Note that for our extension, we have \( W_{4\text{ext}}^T = W_{4\text{ext}}^\dagger \) and the corresponding constraint of Eq. (95) that fits our case can be written in matrix form as:

\[
 S_L^z g^{z\sigma} S_R^z (S_{\phi}^* \sigma \Lambda) = g^{z\Lambda}. \tag{119}
\]

In [6], we found the expressions of \( g^i_{ik} \) taking into consideration the \( S \) and \( Z_8 \) symmetries:

\[
 g^1 = \begin{pmatrix} A^1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad g^2 = \begin{pmatrix} 0 & B^2 & -B^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad g^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & C^3 & D^3 \\ 0 & D^3 & C^3 \end{pmatrix}, \quad g^4 = \begin{pmatrix} 0 & B^4 & B^4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{120}
\]

Applying now the similarity transformations by \( W \):

\[
 g^{z\sigma} = W g^k W^T \tag{121}
\]

we get the matrices \( g^{z\sigma}, k = 1, 2, 3, 4 \) whose expressions, in terms of new coefficients related to the old ones, are given in Appendix C Eq. (156) and Eq. (157). We follow by the weighted sum \( g^{z\sigma} = g^{z\sigma} (W_{4\text{ext}}^T)_{k\sigma} \), and we checked that the obtained \( g^{z\sigma}, \sigma = 1, 2, 3, 4 \) do satisfy Eq. (119). When the \( \phi' \)'s get vevs \( (v_k', k = 1, 2, 3, 4) \), the expression of \( M^i_{D'} \) turns out to be quite complicated, but we have to leading order in \( s_z \):

\[
 M^i_{D'} = g^{z\sigma} v'_k = \begin{pmatrix} v'_1 A^1_1 + v'_2 B^2_2 + v'_4 B^4_1 & -v'_2 B^2_2 + v'_4 B^4_1 \\ 0 & v'_3 C^3_3 \\ 0 & v'_3 D^3_3 \end{pmatrix}. \tag{122}
\]

which can be put into the form of Eq. (45) with \( \xi^k_D = 0 \) and

\[
 \alpha = \frac{2v'_2 B^2_2 - v'_4 B^4_1}{v'_2 B^2_2 - v'_4 B^4_1}, \quad \beta = 0. \tag{123}
\]

If the vevs satisfy \( v'_2 \ll v'_4 \) and the Yukawa couplings are of the same order, then we get a perturbative parameter \( \alpha \ll 1 \). This perturbative parameter resurfaces as one perturbative parameter \( \chi \) for \( M^i_{U'} \) (Eq. 77 leading to \( \xi = 0 \)) which was the objective of this section.

### Discussion and summary

We determined \( S^2 \), the \( Z_2 \) symmetry behind the proposed new version of \( \mu-\tau \) neutrino universality which leads directly to a pre-given value for \( \theta_z \). We showed how the resulting texture can accommodate all the neutrino mass hierarchies. We implemented later the \( S^2 \)-symmetry in the whole lepton sector, and showed how it is able to account for the charged lepton mass hierarchies. We computed, within type-I seesaw, the neutrino mass hierarchies, and showed that \( S^2 \) can account for enough leptogenesis since it leads exactly to the same results as the symmetry \( S \) corresponding to a vanishing \( \theta_z \).
Whereas invoking perturbations was necessary to amend the experimentally unacceptable vanishing value of $\theta_z$, it is still an interesting issue to study the effects of perturbing $S^2$, at least to adopt, say, another value of $\theta_y$ which is predicted by the symmetry to be equal to $\pi/4$. We carry out this study and illustrate the connection between perturbing $S$ and perturbing $S^2$. We will report in a future work a complete numerical study contrasting the predictions of the symmetry and its perturbations to experimental data.

Finally, we presented a theoretical realization of the perturbed Dirac mass matrix, where the symmetry is broken spontaneously and the perturbation parameter originates from ratios of different Higgs fields vevs.

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**A The adopted parametrization for $M_\nu$ and constraints on $M_\nu$, $M_R$ and $M_D$ from the $S$ and $S^2$ symmetries**

- In the flavor basis and in the parametrization adopted in our work, the elements of the neutrino mass matrix are given by:

$$
M_{\nu 11} = m_1 c_x^2 e^{2i\rho} + m_2 s_x^2 e^{2i\sigma} + m_3 s_x^2,
$$

$$
M_{\nu 12} = m_1 \left( -c_z s_x c_y e^{2i\rho} - c_z c_x s_x c_y e^{i(2\rho-\delta)} \right) + m_2 \left( -c_z s_x s_y e^{2i\sigma} + c_z c_x s_x c_y e^{i(2\sigma-\delta)} \right) + m_3 c_z s_x s_y,
$$

$$
M_{\nu 13} = m_1 \left( -c_z s_x c_y e^{2i\rho} + c_z c_x s_x c_y e^{i(2\rho-\delta)} \right) + m_2 \left( -c_z s_x s_y e^{2i\sigma} - c_z c_x s_x s_y e^{i(2\sigma-\delta)} \right) + m_3 c_z s_x c_y,
$$

$$
M_{\nu 22} = m_1 \left( c_x s_x s_y e^{i\rho} + c_y s_x e^{i(\rho-\delta)} \right)^2 + m_2 \left( s_x s_y e^{i\sigma} - c_y c_x e^{i(\sigma-\delta)} \right)^2 + m_3 c_x^2 s_y^2,
$$

$$
M_{\nu 33} = m_1 \left( c_x s_x c_y e^{i\rho} - s_y s_x e^{i(\rho-\delta)} \right)^2 + m_2 \left( s_x c_y e^{i\sigma} + s_y c_x e^{i(\sigma-\delta)} \right)^2 + m_3 c_x^2 c_y^2,
$$

$$
M_{\nu 23} = m_1 \left( c_x c_y s_y s_x e^{2i\rho} + s_x c_x s_x \left( c_y^2 - s_y^2 \right) e^{i(2\rho-\delta)} - c_y s_y s_x^2 e^{2i(\rho-\delta)} \right) + m_2 \left( s_x c_y s_y s_x^2 e^{2i\sigma} + s_x c_x s_x \left( s_x^2 - c_y^2 \right) e^{i(2\sigma-\delta)} - c_y s_y c_x^2 e^{2i(\sigma-\delta)} \right) + m_3 s_y c_y c_x^2.
$$

- The invariance of the symmetric $M_\nu$ under the symmetry $S$ implies the following forms:

$$
M_\nu = \begin{pmatrix}
A_\nu & B_\nu & -B_\nu \\
B_\nu & C_\nu & D_\nu \\
-B_\nu & D_\nu & C_\nu
\end{pmatrix},
\quad
M_\nu^* M_\nu = \begin{pmatrix}
a_\nu & b_\nu & -b_\nu \\
b_\nu & c_\nu & d_\nu \\
-b_\nu & d_\nu & c_\nu
\end{pmatrix},
$$

where

$$
a_\nu = |A_\nu|^2 + 2|B_\nu|^2,
\quad
b_\nu = A_\nu^* B_\nu + B_\nu^* C_\nu - B_\nu^* D_\nu,
\quad
c_\nu = |B_\nu|^2 + |C_\nu|^2 + |D_\nu|^2,
\quad
d_\nu = -|B_\nu|^2 + C_\nu^* D_\nu + D_\nu^* C_\nu.
$$

(124)
As to the hermitian mass squared matrix terms of four independent complex parameters \{\nu^1, \nu^2, \nu^3, \nu^4\} and
\[ \nu^1 = \nu^2 = \nu^3 = \nu^4 = 31 \]

\[ z^2 M^2 M \nu^1 \nu^1 \nu^1 \nu^1 + \nu^2 \nu^2 \nu^2 \nu^2 + \nu^3 \nu^3 \nu^3 \nu^3 + \nu^4 \nu^4 \nu^4 \nu^4 = 3 \]

\[ z^2 = 1 \]

\[ M^2_{\nu^1} = C_\nu + D_\nu, \]

(127)

and

\[ m^2_1 = \frac{a_\nu + c_\nu - d_\nu}{2} + \frac{1}{2} \sqrt{(a_\nu + d_\nu - c_\nu)^2 + 8 |b_\nu|^2}, \]

\[ m^2_2 = \frac{a_\nu + c_\nu - d_\nu}{2} - \frac{1}{2} \sqrt{(a_\nu + d_\nu - c_\nu)^2 + 8 |b_\nu|^2}, \]

\[ m^2_3 = c_\nu + d_\nu. \]

(128)

- The invariance of the symmetric \( M^2_\nu \) under the symmetry \( S^2 \) implies that \( M^2_\nu \) can be written in terms of four independent complex parameters \( \{A_\nu^z, B_\nu^z, C_\nu^z, D_\nu^z\} \) as

\[ M^2_{\nu^1} = A_\nu^z, \quad M^2_{\nu^2} = B_\nu^z, \quad M^2_{\nu^3} = (1 - 2 t_2^2) C_\nu^z - \sqrt{2} t_2 (A_\nu^z + C_\nu^z - D_\nu^z), \]

\[ M^2_{\nu^4} = C_\nu^z, \quad M^2_{\nu^5} = D_\nu^z, \]

where the set of parameters \( \{A_\nu^z, B_\nu^z, C_\nu^z, D_\nu^z\} \) can be written in terms of the set \( \{A_\nu, B_\nu, C_\nu, D_\nu\} \) as,

\[ A_\nu^z = c_z^2 A_\nu + s_z^2 (D_\nu + C_\nu), \]

\[ B_\nu^z = c_z B_\nu - \frac{s_z}{2 \sqrt{2}} (A_\nu - C_\nu - D_\nu), \]

\[ C_\nu^z = \frac{1}{2} \left(1 + c_z^2\right) C_\nu - \sqrt{2} s_z B_\nu + \frac{s_z^2}{2} (A_\nu - D_\nu), \]

\[ D_\nu^z = \frac{1}{2} \left(1 + c_z^2\right) D_\nu + \frac{s_z^2}{2} (A_\nu - C_\nu). \]

(130)

The inverse relations can be expressed as,

\[ A_\nu = \frac{1}{c_z^2} A_\nu^z - \sqrt{2} t_2 (B_\nu^z - t_2^2 (C_\nu^z + D_\nu^z)), \]

\[ B_\nu = \frac{c_z}{c_z^2} B_\nu^z + \frac{t_2}{\sqrt{2} c_z} (A_\nu^z - C_\nu^z - D_\nu^z), \]

\[ C_\nu = \left(1 - \frac{t_2^2}{2}\right) C_\nu^z + \frac{t_2^2}{2} (A_\nu^z - D_\nu^z) + \sqrt{2} t_2 \left(1 - \frac{t_2^2}{2}\right) B_\nu^z, \]

\[ D_\nu = \left(1 + \frac{t_2^2}{2}\right) D_\nu^z + \frac{t_2^2}{\sqrt{2}} B_\nu^z - \frac{t_2^2}{2} (A_\nu^z - C_\nu^z). \]

(131)

As to the hermitian mass squared matrix \( M^* \cdot M^2_\nu \), it should have the form:

\[
(M^2_\nu M^*_{\nu^1})_{11} = a_\nu c_z^2 + s_z^2 (c_\nu + d_\nu), \\
(M^2_\nu M^*_{\nu^1})_{12} = b_\nu c_z - \frac{s_z}{\sqrt{2}} (a_\nu - c_\nu - d_\nu), \\
(M^2_\nu M^*_{\nu^1})_{13} = -b_\nu c_z + \frac{s_z}{\sqrt{2}} (a_\nu - c_\nu - d_\nu), \\
(M^2_\nu M^*_{\nu^1})_{14} = b_\nu c_z - \frac{s_z}{\sqrt{2}} (a_\nu - c_\nu - d_\nu), \\
(M^2_\nu M^*_{\nu^1})_{22} = \frac{1}{2} \left(1 + c_z^2\right) c_\nu + \frac{s_z}{\sqrt{2}} (a_\nu - d_\nu) - \sqrt{2} \text{Re}(b_\nu) s_z, \\
(M^2_\nu M^*_{\nu^1})_{23} = \frac{1}{2} \left(1 + c_z^2\right) d_\nu + \frac{s_z}{\sqrt{2}} (a_\nu - c_\nu) + i \sqrt{2} \text{Im}(b_\nu) s_z.
\]

21
\[(M^z_M^z)_{31} = -b^*_e c_z - \frac{s^2_{2z}}{2\sqrt{2}} (a_\nu - c_\nu - d_\nu),\]
\[(M^z_M^z)_{32} = \frac{1}{2} \left( 1 + c^2_z \right) d_\nu + \frac{s^2_{2z}}{2} (a_\nu - c_\nu) - i \sqrt{2} \text{Im} (b_\nu) s_z,\]
\[(M^z_M^z)_{33} = \frac{1}{2} \left( 1 + c^2_z \right) c_\nu + \frac{s^2_{2z}}{2} (a_\nu - d_\nu) + \sqrt{2} \text{Re} (b_\nu) s_z.\] (132)

- All results derived for \(M_\nu\) and \(M^z_\nu\) concerning symmetry properties under \(S\) and \(S^z\) and their mutual interrelations would equally apply to the case of \(M_R\) and \(M^z_R\). More precisely, we have all the formulae from Eq. (125) till Eq. (132) with the appropriate replacement of the subscript \(\nu\) into \(R\).

- The invariance of the Dirac neutrino mass matrix \(M_D\) under the symmetry \(S\) implies the following forms:
\[
M_D = \begin{pmatrix} A_D & B_D & -B_D \\ E_D & C_D & D_D \\ -E_D & D_D & C_D \end{pmatrix}, \quad M^\dagger_D M_D = \begin{pmatrix} a_D & b_D & -b_D \\ b^*_D & c_D & d_D \\ -b^*_D & d_D & c_D \end{pmatrix}, \tag{133}
\]
where
\[
a_D = |A_D|^2 + 2 |E_D|^2, \quad b_D = A^*_D B_D + E^*_D C_D - E^*_D D_D,
\]
\[
c_D = |B_D|^2 + |C_D|^2 + |D_D|^2, \quad d_D = -|B_D|^2 + C^*_D D_D + D^*_D C_D. \tag{134}
\]
The mass spectrum of \(M^\dagger_D M_D\) can be written as
\[
\left\{ c_D + d_D, \frac{a_D + c_D - d_D}{2} \pm \frac{1}{2} \sqrt{(a_D + d_D - c_D)^2 + 8 |b_D|^2} \right\}. \tag{135}
\]
- The invariance of \(M^z_D\) under the symmetry \(S^z\) implies that \(M^z_D\) can be written in terms of five independent complex parameters \(\{A^z_D, B^z_D, C^z_D, D^z_D, E^z_D\}\) as
\[
M^z_D_{11} = A^z_D, \quad M^z_D_{12} = B^z_D, \quad M^z_D_{13} = -B^z_D + \sqrt{2} t_z (D^z_D + C^z_D - A^z_D) + 2 t^2_z E^z_D,
\]
\[
M^z_D_{21} = E^z_D, \quad M^z_D_{22} = C^z_D, \quad M^z_D_{23} = D^z_D,
\]
\[
M^z_D_{31} = - (1 - 2 t^2_z) E^z_D + \sqrt{2} t_z (D^z_D + C^z_D - A^z_D), \quad M^z_D_{32} = D^z_D + \sqrt{2} t_z (E^z_D - B^z_D),
\]
\[
M^z_D_{33} = (1 - 2 t^2_z) C^z_D + \sqrt{2} t_z B^z_D + 2 t^2_z (A^z_D - D^z_D) + \sqrt{2} t_z (1 - 2 t^2_z) E^z_D. \tag{136}
\]
The set of parameter \(\{A^z_D, B^z_D, C^z_D, D^z_D, E^z_D\}\) can be written in terms of \(\{A_D, B_D, C_D, D_D, E_D\}\) as
\[
A^z_D = c^2_D A_D + s^2_z (D_D + C_D),
\]
\[
B^z_D = c_z B_D - \frac{s^2_{2z}}{2\sqrt{2}} (A_D - C_D - D_D),
\]
\[
C^z_D = \frac{1}{2} \left( 1 + c^2_z \right) C_D + \frac{s^2_z}{2} (A_D - D_D) - \frac{s_z}{\sqrt{2}} (B_D + E_D),
\]
\[
D^z_D = \frac{1}{2} \left( 1 + c^2_z \right) D_D + \frac{s^2_z}{2} (A_D - D_D) - \frac{s_z}{\sqrt{2}} (B_D - E_D),
\]
\[
E^z_D = c_z E_D - \frac{s^2_{2z}}{2\sqrt{2}} (A_D - D_D - C_D), \tag{137}
\]
while the inverse relations can written as
\[
A_D = (1 + t^2_z) A^z_D - \sqrt{2} t_z E^z_D - t^2_z (C^z_D + D^z_D),
\]
\[
B_D = \frac{1}{c_z} B^z_D + \frac{t_z}{\sqrt{2} c_z} (A^z_D - C^z_D - D^z_D) - \frac{t^2_z}{c_z} E^z_D,
\]
\[
C_D = \left( 1 - \frac{t^2_z}{2} \right) C^z_D - \frac{t^2_z}{2} (D^z_D - A^z_D) + \frac{t_z}{\sqrt{2}} B^z_D + \frac{1}{\sqrt{2}} t_z (1 - t^2_z) E^z_D,
\]
\[
D_D = \left( 1 + \frac{t^2_z}{2} \right) D^z_D + \frac{t^2_z}{2} (C^z_D - A^z_D) - \frac{t_z}{\sqrt{2}} B^z_D + \frac{t_z}{\sqrt{2} c^2_z} E^z_D,
\]
\[
E_D = \frac{1}{c_z} (1 - t^2_z) E^z_D + \frac{t_z}{\sqrt{2} c_z} (A^z_D - C^z_D - D^z_D). \tag{138}
\]
As to the hermitian mass squared matrix $M^2 \ast M^2$, it should have the form:

$$(M^2 \ast M^2)_{ij} = a_{ij} c^2 + s^2 (c_{2ij} + d_{2ij}),$$

$$(M^2 \ast M^2)_{12} = b_{12} c - \frac{s_{12}}{2 \sqrt{2}} (a_{12} - c_{12} - d_{12}),$$

$$(M^2 \ast M^2)_{13} = -b_{13} c + \frac{s_{13}}{2 \sqrt{2}} (a_{13} - c_{13} - d_{13}),$$

$$(M^2 \ast M^2)_{21} = b_{21} c - \frac{s_{21}}{2 \sqrt{2}} (a_{21} - c_{21} - d_{21}),$$

$$(M^2 \ast M^2)_{22} = \frac{1}{2} (1 + c_{22}) c + \frac{s_{22}}{2} (a_{22} - d_{22}) - \sqrt{2 \Re (b_{22})} s_{22},$$

$$(M^2 \ast M^2)_{23} = \frac{1}{2} (1 + c_{23}) d + \frac{s_{23}}{2} (a_{23} - c_{23}) + i \sqrt{2 \Im (b_{23})} s_{23},$$

$$(M^2 \ast M^2)_{31} = -b_{31} c + \frac{s_{31}}{2 \sqrt{2}} (a_{31} - c_{31} - d_{31}),$$

$$(M^2 \ast M^2)_{32} = \frac{1}{2} (1 + c_{32}) d + \frac{s_{32}}{2} (a_{32} - d_{32}),$$

$$(M^2 \ast M^2)_{33} = \frac{1}{2} (1 + c_{33}) c + \frac{s_{33}}{2} (a_{33} - d_{33}) + \sqrt{2 \Re (b_{33})} s_{33}. \quad (139)$$

The invariance of generic symmetric matrix $M = M^T$ under the $S^z$-sign-flipped symmetry defined as

$$S^z M S^z = -M, \quad (140)$$

leads to the fact that the symmetric matrix $M$ can be written in terms of two independent complex parameters $\{B, C\}$ as,

$$M_{11} = 2 \sqrt{2} t_2 B - 2 t_2^2 C, \quad M_{12} = B, \quad M_{13} = (1 - 2 t_2^2) B - \sqrt{2} t_2 (1 - t_2) C$$

$$M_{22} = C, \quad M_{23} = -\sqrt{2} t_2 B + t_2^2 C, \quad M_{33} = -2 \sqrt{2} t_2 B - (1 - 2 t_2^2) C. \quad (141)$$

The invariance as represented in Eq. (140) but for generic mass matrix $M$ implies that $M$ can be written in terms of four independent complex parameters $\{B, C, D, E\}$ as,

$$M_{11} = \sqrt{2} t_2 (B + E) - 2 t_2^2 C, \quad M_{12} = B, \quad M_{13} = B + \sqrt{2} t_2 (D - C),$$

$$M_{21} = E, \quad M_{22} = C, \quad M_{23} = D, \quad M_{31} = (1 - 2 t_2^2) E - 2 t_2^2 B - \sqrt{2} t_2 (1 - 2 t_2^2) C - \sqrt{2} t_2 D,$$

$$M_{32} = -D - \sqrt{2} t_2 (B + E) + 2 t_2^2 C, \quad M_{33} = -\sqrt{2} t_2 (B + E) - (1 - 2 t_2^2) C. \quad (142)$$

**B Perturbing $S^z$**

- For a proof that the two determinations of $I^z$ through Eq. (56) and Eq. (66) are equivalent, we start from the following two equations that force the diagonalization of $M^2$, namely,

$$\left[ -I^z M^0 \delta M U^0 \right]_{i \neq j} = 0,$$

$$\left[ -I^z M^0 \delta M U^0 \right]_{i \neq j} = 0. \quad (143)$$

Multiplying the first one from left by $M^0 \delta M^*$ while the second from right by $M^0 \delta M^*$, and then adding the resulting two equations we get

$$\left( [M^0 \delta M^*] M^0 \delta M^* I^z + M^0 \delta M^* U^0 T \delta M^* U^0 + U^0 T \delta M^* U^0 \right)_{i \neq j} = 0. \quad (144)$$

Upon using Eq. (64), the above equation gets transformed into,

$$\left( [M^0 \delta M^*] M^0 \delta M^* I^z + U^0 \delta M^* U^0 \right)_{i \neq j} = 0. \quad (145)$$

The above equation is nothing but the one that forces the diagonalization of the combination $M^z \ast M^z$ as written in Eq. (56). Conversely, one can easily go backward which means that Eq. (145) leads to Eq. (143).
Solving the linear system of Eqs. (66) we get:

\[ \epsilon_1 = \frac{1}{|M_{\nu 11}^{0\text{Diag}}|^2 - |M_{\nu 22}^{0\text{Diag}}|^2} \left( \frac{1}{\sqrt{2}} c_{2\varphi} c_{z_0} M_{\nu 11}^{0\text{Diag}*} (\alpha^{\ast}_{13} - \alpha^{\ast}_{12}) e^{-i\xi} + \frac{1}{\sqrt{2}} c_{2\varphi} c_{z_0} M_{\nu 22}^{0\text{Diag}*} (\alpha^{\ast}_{13} - \alpha^{\ast}_{12}) e^{i\xi} - M_{\nu 22}^{0\text{Diag}*} c_{z_0} s_{\varphi} c_{\varphi} \left[ -\sqrt{2} s_{z_0} (\alpha^{\ast}_{12} + \alpha^{\ast}_{13}) + c_{z_0} \alpha^{\ast}_{11} \right] \right), \]

\[ \epsilon_2 = \frac{1}{|M_{\nu 11}^{0\text{Diag}}|^2 - |M_{\nu 33}^{0\text{Diag}}|^2} \left( -\frac{1}{\sqrt{2}} s_{\varphi} s_{z_0} M_{\nu 11}^{0\text{Diag}*} (\alpha^{\ast}_{13} - \alpha^{\ast}_{12}) e^{-i\xi} - \frac{1}{\sqrt{2}} s_{\varphi} s_{z_0} M_{\nu 33}^{0\text{Diag}*} (\alpha^{\ast}_{13} - \alpha^{\ast}_{12}) e^{i\xi} - \frac{1}{2} M_{\nu 33}^{0\text{Diag}*} c_{\varphi} \sqrt{2} c_{z_0} (\alpha^{\ast}_{12} + \alpha^{\ast}_{13}) + s_{z_0} \alpha^{\ast}_{11} \right), \]

\[ \epsilon_3 = \frac{1}{|M_{\nu 22}^{0\text{Diag}}|^2 - |M_{\nu 33}^{0\text{Diag}}|^2} \left( \frac{1}{\sqrt{2}} c_{\varphi} s_{z_0} M_{\nu 22}^{0\text{Diag}*} (\alpha^{\ast}_{13} - \alpha^{\ast}_{12}) e^{-i\xi} + \frac{1}{2} M_{\nu 33}^{0\text{Diag}*} s_{\varphi} \left[ \sqrt{2} c_{z_0} (\alpha^{\ast}_{12} + \alpha^{\ast}_{13}) + s_{z_0} \alpha^{\ast}_{11} \right] \right). \]

The complete entries of the matrix \( U_\epsilon \) are

\[ U_\epsilon (1, 1) = (c_{z_0} c_{\varphi} - c_{z_0} s_{\varphi} \epsilon^{\ast}_{11} - s_{z_0} \epsilon^{\ast}_{22}) e^{-i\phi_1}, \]

\[ U_\epsilon (1, 2) = (c_{z_0} s_{\varphi} + c_{z_0} c_{\varphi} \epsilon^{\ast}_{11} - s_{z_0} \epsilon^{\ast}_{33}) e^{-i\phi_2}, \]

\[ U_\epsilon (1, 3) = (s_{z_0} c_{\varphi} + c_{z_0} c_{\varphi} \epsilon^{\ast}_{11} + s_{z_0} s_{\varphi} \epsilon^{\ast}_{11}) e^{-i\phi_3}, \]

\[ U_\epsilon (2, 1) = -\frac{1}{\sqrt{2}} (s_{z_0} c_{\varphi} + s_{\varphi} c_{\varphi} e^{-i\xi} - s_{z_0} s_{\varphi} \epsilon^{\ast}_{11} + c_{\varphi} \epsilon^{\ast}_{11} e^{-i\xi} + c_{z_0} \epsilon^{\ast}_{22}) e^{-i\phi_1}, \]

\[ U_\epsilon (2, 2) = -\frac{1}{\sqrt{2}} (s_{z_0} s_{\varphi} - c_{\varphi} c_{\varphi} e^{-i\xi} + s_{z_0} c_{\varphi} \epsilon^{\ast}_{11} + s_{\varphi} \epsilon^{\ast}_{11} e^{-i\xi} + c_{z_0} \epsilon^{\ast}_{33}) e^{-i\phi_2}, \]

\[ U_\epsilon (2, 3) = \frac{1}{\sqrt{2}} (c_{z_0} - c_{\varphi} s_{z_0} \epsilon^{\ast}_{22} - s_{\varphi} \epsilon^{\ast}_{22} e^{-i\xi} - s_{z_0} \epsilon^{\ast}_{22} e^{-i\xi} + s_{z_0} \epsilon^{\ast}_{33} + c_{\varphi} \epsilon^{\ast}_{33} e^{-i\xi}) e^{-i\phi_3}, \]

\[ U_\epsilon (3, 1) = -\frac{1}{\sqrt{2}} (s_{z_0} c_{\varphi} - s_{\varphi} c_{\varphi} e^{-i\xi} - s_{z_0} s_{\varphi} \epsilon^{\ast}_{11} - c_{\varphi} \epsilon^{\ast}_{11} e^{-i\xi} + c_{z_0} \epsilon^{\ast}_{22}) e^{-i\phi_1}, \]

\[ U_\epsilon (3, 2) = -\frac{1}{\sqrt{2}} (s_{z_0} s_{\varphi} + c_{\varphi} c_{\varphi} e^{-i\xi} + s_{z_0} c_{\varphi} \epsilon^{\ast}_{11} - s_{\varphi} \epsilon^{\ast}_{11} e^{-i\xi} + c_{z_0} \epsilon^{\ast}_{33}) e^{-i\phi_2}, \]

\[ U_\epsilon (3, 3) = \frac{1}{\sqrt{2}} (c_{z_0} - s_{\varphi} \epsilon^{\ast}_{11} + s_{\varphi} \epsilon^{\ast}_{11} e^{-i\xi} - s_{z_0} \epsilon^{\ast}_{33} + c_{\varphi} \epsilon^{\ast}_{33} e^{-i\xi}) e^{-i\phi_3}. \]

The corresponding mixing angles are the following.

\[ t_x = \frac{|U_\epsilon (1, 2)|}{|U_\epsilon (1, 1)|} = \frac{|c_{z_0} s_{\varphi} + c_{z_0} c_{\varphi} \epsilon^{\ast}_{11} - s_{z_0} \epsilon^{\ast}_{33}|}{|c_{z_0} c_{\varphi} - c_{z_0} s_{\varphi} \epsilon^{\ast}_{11} - s_{z_0} \epsilon^{\ast}_{22}|}, \]

\[ t_y = \frac{|U_\epsilon (2, 3)|}{|U_\epsilon (3, 3)|} = \frac{|c_{z_0} - c_{z_0} \epsilon^{\ast}_{22} s_{\varphi} - c_{z_0} \epsilon^{\ast}_{22} c_{\varphi} e^{-i\xi} - c_{z_0} s_{\varphi} + c_{z_0} \epsilon^{\ast}_{33} c_{\varphi} e^{-i\xi}|}{|c_{z_0} - c_{z_0} \epsilon^{\ast}_{22} s_{\varphi} + c_{z_0} \epsilon^{\ast}_{22} c_{\varphi} e^{-i\xi} - c_{z_0} s_{\varphi} - c_{z_0} \epsilon^{\ast}_{33} c_{\varphi} e^{-i\xi}|}, \]

\[ s_z = |U_\epsilon (1, 3)| = |s_{z_0} + c_{z_0} c_{\varphi} \epsilon^{\ast}_{11} + c_{z_0} s_{\varphi} \epsilon^{\ast}_{11}|. \]

An equation which allows to solve for the Dirac phase \( \delta \) is obtained, say, by equating the \( (2, 1)^{th} \) of \( U_\epsilon \), after suitably rephased by \( e^{-i\psi_2} \), with the corresponding one of \( V'^{\text{PINS}} \) to get:

\[ -\frac{1}{\sqrt{2}} (s_{z_0} c_{\varphi} + s_{\varphi} c_{\varphi} e^{-i\xi} - c_{\varphi} c_{\varphi} e^{-i\xi} + c_{z_0} s_{\varphi} + c_{z_0} \epsilon^{\ast}_{22}) e^{-i\phi_1} e^{-i\psi_2} = (-c_{z_0} s_{\varphi} s_{z_0} - s_x c_{\varphi} \epsilon^{\ast}_{11}) e^{-i\delta}. \]

Other equations, resulting from other entries, might be necessary to determine fully and consistently the Dirac phase.
C Realization

- Applying the similarity transformation Eq. (89) on the “non-deformed” matrices of Eq. (88), we get

\[ \tilde{f}^1 = \begin{pmatrix} A_\tilde{1} & B_\tilde{1} & B_{\tilde{1}} \\ B_\tilde{1} & C_\tilde{1} & D_{\tilde{1}} \\ B_{\tilde{1}} & D_{\tilde{1}} & C_{\tilde{1}} \end{pmatrix}, \quad \tilde{f}^2 = \begin{pmatrix} A_\tilde{2} & B_\tilde{2} & B_{\tilde{2}} \\ B_\tilde{2} & C_{\tilde{2}} & D_{\tilde{2}} \\ B_{\tilde{2}} & D_{\tilde{2}} & C_{\tilde{2}} \end{pmatrix}, \quad \tilde{f}^3 = \begin{pmatrix} 0 & B_{\tilde{3}} & -B_{\tilde{3}} \\ E_{\tilde{3}} & C_{\tilde{3}} & D_{\tilde{3}} \\ -E_{\tilde{3}} & -D_{\tilde{3}} & -C_{\tilde{3}} \end{pmatrix}, \quad (150) \]

where \( \{A_{\tilde{i}}, B_{\tilde{i}}, C_{\tilde{i}}, D_{\tilde{i}}\}, (i = 1, 2) \), and \( \{B_{\tilde{3}}, C_{\tilde{3}}, D_{\tilde{3}}, E_{\tilde{3}}\} \) are defined as

\[ A_{\tilde{i}} = c^2 A_i + s^2 (C_i + D_i), \quad B_{\tilde{i}} = -\frac{t_{2z}}{2\sqrt{2}} (A_{\tilde{i}} - C_{\tilde{i}} - D_{\tilde{i}}), \]

\[ C_{\tilde{i}} = \frac{s^2}{2} A_i + \frac{1}{2} (1 + c^2) C_i - \frac{s^2}{2} D_i, \quad D_{\tilde{i}} = \frac{s^2}{2} A_i + \frac{1}{2} (1 + c^2) D_i - \frac{s^2}{2} C_i, \]

\[ C_{\tilde{3}} = -\frac{t_z}{2\sqrt{2}} (B_{\tilde{3}}^3 + E_{\tilde{3}}^3), \quad D_{\tilde{3}} = \frac{t_z}{2\sqrt{2}} (B_{\tilde{3}}^3 - E_{\tilde{3}}^3), \quad B_{\tilde{3}} = c z B_3, \quad E_{\tilde{3}} = c z E_3. \quad (151) \]

- Applying the similarity transformation Eq. (89) on the “non-deformed” matrices of Eq. (98), we get

\[ \tilde{g}^1 = \begin{pmatrix} A_{\tilde{1}} & B_{\tilde{1}} & B_{\tilde{1}} \\ B_{\tilde{1}} & C_{\tilde{1}} & D_{\tilde{1}} \\ B_{\tilde{1}} & D_{\tilde{1}} & C_{\tilde{1}} \end{pmatrix}, \quad \tilde{g}^{(2)} = \begin{pmatrix} 0 & B_{\tilde{2}} & -B_{\tilde{2}} \\ E_{\tilde{2}} & C_{\tilde{2}} & D_{\tilde{2}} \\ -E_{\tilde{2}} & -D_{\tilde{2}} & -C_{\tilde{2}} \end{pmatrix}, \quad \tilde{g}^3 = \begin{pmatrix} A_{\tilde{3}} & B_{\tilde{3}} & B_{\tilde{3}} \\ E_{\tilde{3}} & C_{\tilde{3}} & D_{\tilde{3}} \\ -E_{\tilde{3}} & -D_{\tilde{3}} & -C_{\tilde{3}} \end{pmatrix}, \quad (152) \]

where

\[ B_{\tilde{i}} = -\frac{t_{2z}}{2\sqrt{2}} (A_{\tilde{i}} - C_{\tilde{i}} - D_{\tilde{i}}), \quad A_{\tilde{i}} = c^2 A_i + s^2 (C_i + D_i), \quad C_{\tilde{i}} = \frac{s^2}{2} (A_i - D_i) + \frac{1}{2} (1 + c^2) C_i, \]

\[ D_{\tilde{i}} = \frac{s^2}{2} (A_i - C_i) + \frac{1}{2} (1 + c^2) D_i, \quad C_{\tilde{2}} = -\frac{t_z}{2\sqrt{2}} (B_{\tilde{2}}^3 + E_{\tilde{2}}^3), \quad D_{\tilde{2}} = \frac{t_z}{2\sqrt{2}} (B_{\tilde{2}}^3 - E_{\tilde{2}}^3), \]

\[ B_{\tilde{2}} = c z B_2, \quad E_{\tilde{2}} = c z E_2, \quad A_{\tilde{3}} = \frac{t_z}{2\sqrt{2}} (B_{\tilde{3}}^3 + E_{\tilde{3}}^3), \quad B_{\tilde{3}} = c^2 B_3 - s^2 E_3, \quad E_{\tilde{3}} = c^2 E_3 - s^2 B_3. \quad (153) \]

- Applying the similarity transformation Eq. (114) on the “non-deformed” matrices of Eq. (113), we get the symmetric matrices

\[ \tilde{h}^1 = \begin{pmatrix} A_{\tilde{1}} & -\frac{t_z}{\sqrt{2}} A_{\tilde{1}} & -\frac{t_z}{\sqrt{2}} A_{\tilde{1}} \\ \left(\tilde{h}^1\right)_{12} & \frac{t_z^2}{2} A_{\tilde{1}} & \frac{t_z^2}{2} A_{\tilde{1}} \\ \left(\tilde{h}^1\right)_{13} & \left(\tilde{h}^1\right)_{23} & \frac{t_z^2}{2} A_{\tilde{1}} \end{pmatrix}, \quad (154) \]

where

\[ A_{\tilde{1}} = c^2 A_{R1}, \quad C_{\tilde{R2}} = \frac{1}{2} (1 + c^2) C_{R2} - s^2 D_{R2}, \quad D_{\tilde{R2}} = -\frac{s^2}{2} C_{R2} + \frac{1}{2} (1 + c^2) D_{R2}. \quad (155) \]
• Applying the similarity transformation Eq. (121) on the “non-deformed” matrices of Eq. (120), we get

\[
\tilde{g}^3 = \begin{pmatrix}
\frac{t_z}{\sqrt{2}} (C_3^2 + D_3^2) & \frac{t_z}{\sqrt{2}} (C_3^2 + D_3^2) & \frac{t_z}{\sqrt{2}} (C_3^2 + D_3^2) \\
\frac{t_z}{\sqrt{2}} (C_3^2 + D_3^2) & C_3^2 & D_3^2 \\
\frac{t_z}{\sqrt{2}} (C_3^2 + D_3^2) & D_3^2 & C_3^2
\end{pmatrix},
\]

\[
\tilde{g}^4 = \begin{pmatrix}
\sqrt{2} t_z B_1^3 & B_4^3 & B_1^3 \\
-t_z B_4^3 & -t_z B_2^3 & -t_z B_4^3 \\
-t_z B_4^3 & -t_z B_2^3 & -t_z B_4^3
\end{pmatrix},
\]

where

\[
A_1^3 = c_2^2 A_1, \quad B_2^3 = c_2 B_2, \quad C_3^2 = \frac{1}{2} (1 + c_2^2) C_3 - \frac{s_2^2}{2} D_3,
\]

\[
D_3^2 = -\frac{s_2^2}{2} C_3 + \frac{1}{2} (1 + c_2^2) D_3, \quad B_4^3 = c_2^2 B_4.
\]

References

[1] E. Ma and M. Raidal, Phys. Rev. Lett. 87, 011802 (2001); R. N. Mohapatra and S. Nasri, Phys. Rev. D 71, 033001 (2005); R. N. Mohapatra, S. Nasri and H. -B. Yu, Phys. Lett. B 615, 231 (2005); S. Nasri, Int. J. Mod. Phys. A 20, 6258 (2005); R. N. Mohapatra, S. Nasri and H. B. Yu, Phys. Lett. B 636, 114 (2006); R. N. Mohapatra, S. Nasri and H. B. Yu, Phys. Lett. B 639, 318 (2006); Z. -z. Xing, H. Zhang and S. Zhou, Phys. Lett. B 641, 189 (2006); T. Ota and W. Rodejohann, Phys. Lett. B 639, 322 (2006); Y. H. Ahn, S. K. Kang, C. S. Kim and J. Lee, Phys. Rev. D 73, 093005 (2006); I. Aizawa and M. Yasue, Phys. Rev. D 73, 015002 (2006); K. Fuki and M. Yasue, Phys. Rev. D 73, 055014 (2006); B. Adhikary, A. Ghosal and P. Roy, JHEP 0910 (2009) 040; Z. z. Xing and Y. L. Zhou, Phys. Lett. B 693, 584 (2010); S. -F. Ge, H. -J. He and F. -R. Yin, JCAP 1005, 017 (2010); I. de Medeiros Varzielas, R. González Felipe and H. Serodio, Phys. Rev. D 83, 033007 (2011); H. -J. He and F. -R. Yin, Phys. Rev. D 84, 033009 (2011); S. Gupta, A. S. Joshipura and K. M. Patel, JHEP 1309, 035 (2013); B. Adhikary, M. Chakraborty and A. Ghosal, JHEP 1310, 043 (2013).

[2] P. F. Harrison, D. H. Perkins and W. G. Scott, Phys. Lett. B 530, 167 (2002).

[3] V. D. Barger, S. Pakvasa, T. J. Weiler and K. Whisnant, Phys. Lett. B 437, 107 (1998); A. J. Baltz, A. S. Goldhaber and M. Goldhaber, Phys. Rev. Lett. 81, 5730 (1998).

[4] Y. Kajiyama, M. Raidal and A. Strumia, Phys. Rev. D 76, 117301 (2007); L. L. Everett and A. J. Stuart, Phys. Rev. D —79, 085005 (2009).

[5] E. I. Lashin, N. Chamoun, C. Hamzaoui and S. Nasri, Phys. Rev. D 89 093004 (2014).

[6] E. I. Lashin, N. Chamoun, C. Hamzaoui and S. Nasri, Phys. Rev. D 91 113014 (2015).

[7] F. Capozzi, G. L. Fogli, E. Lisi, A. Marrone, D. Montanino, and A. Palazzo Phys. Rev. D 89, 0953018 (2014).
[8] W. Grimus and L. Lavoura, Eur. Phys. J. C 39, 219 (2005); A. Dighe, S. Goswami and P. Roy, Phys. Rev. D 76, 096005 (2007); S. Luo and Z. -z. Xing, Phys. Rev. D 86, 073003 (2012).

[9] Z.Z. Xing; Phys. Lett. B 530 (2002), 159-166.

[10] E. I. Lashin and N. Chamoun, Phys. Rev. D 85, 113011 (2012).

[11] E. I. Lashin, N. Chamoun, E. Malkawi and S. Nasri, Phys. Rev. D 80 115013 (2009); E. I. Lashin, N. Chamoun, E. Malkawi and S. Nasri, Phys. Rev. D 83 013002 (2011).

[12] W. Grimus, A. S. Joshipura, L. Lavoura, and M. Tanimoto, Eur. Phys. J. C 36, 227 (2004); C. Hagedorn, J. Kersten, and M. Lindner, Phys. Lett. B 597, 63 (2004).