On characterization of balls via solutions to the Helmholtz equation

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A new analytical characterization of balls in the Euclidean space $\mathbb{R}^m$ is obtained. Previous results of this kind involved either harmonic functions or solutions to the modified Helmholtz equation (both have positive fundamental solutions), whereas solutions to the Helmholtz equation are used here. This is achieved at the expense of a restriction imposed on the size of admissible domains—a feature absent in the inverse mean value properties known previously.

1 Introduction and notation

As early as 1972 a simple proof of the following characterization of balls via harmonic functions was obtained in [1]:

Let $D$ be a domain (= connected open set) of finite (Lebesgue) measure in the Euclidean space $\mathbb{R}^m$ where $m \geq 2$. Suppose that there exists a point $P_0$ in $D$ such that, for every function $h$ harmonic in $D$ and integrable over $D$, the volume mean of $h$ over $D$ equals $h(P_0)$. Then $D$ is an open ball (disk when $m = 2$) centred at $P_0$.

The result was originally proved in [2] for a simply connected two-dimensional $D$. In the survey article [3], one finds its modification for a disconnected $D$ as well as a discussion of its applications and of possible similar results involving some kinds of average over $\partial D$, when $D$ is a bounded domain. Presumably, the paper [4] was the first one in which the quoted theorem was referred to as the inverse mean value property of harmonic functions; the term became widely accepted.

A great deal of other interesting material obtained in this area is reviewed in the recent survey [5]. In particular, other characterizations of balls as well as of other domains (strips, annuli etc.) via harmonic functions are described. Moreover, it contains a characterization of balls via panharmonic functions; see [6] for this convenient abbreviation for solutions to the modified Helmholtz equation

$$\nabla^2 u - \mu^2 u = 0, \quad \mu \in \mathbb{R} \setminus \{0\}. \quad (1.1)$$

Here and below, $\nabla = (\partial_1, \ldots, \partial_m)$, $\partial_i = \partial / \partial x_i$, denotes the gradient operator. This characterization discovered in 2021 (see [7]) is closely related to the result obtained in this note. Therefore, it is reasonable to formulate it here, but before that we introduce some notation.

Let $x = (x_1, \ldots, x_m)$ be a point in $\mathbb{R}^m$, $m \geq 2$, by $B_r(x) = \{ y \in \mathbb{R}^m : |y-x| < r \}$ we denote the open ball of radius $r$ centred at $x$ (just $B_r$, if centred at the origin). The ball is called admissible with respect to a domain $D \subset \mathbb{R}^m$ provided $\overline{B_r(x)} \subset D$. If $D$ has finite Lebesgue measure and a function $f$ is integrable over $D$, then

$$M(f, D) = \frac{1}{|D|} \int_D f(x) \, dx$$
is its volume mean value over $D$. Here and below $|D|$ is the domain’s volume (area if $D \subset \mathbb{R}^2$); the volume of $B_r$ is $|B_r| = \omega_m r^m$, where $\omega_m = 2 \pi^{m/2} / [m \Gamma(m/2)]$ is the volume of the unit ball; as usual $\Gamma$ denotes the Gamma function. A dilated copy of a bounded domain $D$ is $D_r = D \cup \{x \in \partial D | B_r(x)\}$. Thus, the distance from $\partial D_r$ to $D$ is equal to $r$.

Now we are in a position to give a corrected formulation of the assertion proved in [7].

**Theorem 1.1.** Let $D \subset \mathbb{R}^m$, $m \geq 2$, be a bounded domain such that its complement is connected, and let $r > 0$ be such that $|B_r| = |D|$. If for a point $x_0 \in D$ and some $\mu > 0$ the identity

$$
\Gamma \left( \frac{m}{2} + 1 \right) \frac{\Gamma_{m/2}(\mu r)}{(\mu r/2)^{m/2}} u(x_0) = M(u, D) \tag{1.2}
$$

is valid for every positive function $u$ panharmonic in $D_r$, then $D = B_r(x_0)$; $I_\nu$ stands for the modified Bessel function of order $\nu$.

In this note, we prove one more analytic characterization of balls based on the $m$-dimensional volume mean value identity, but involving metaharmonic functions instead of panharmonic ones. See [8], Appendix 2, where the term metaharmonic function was introduced as an abbreviation for a real-valued solution of the Helmholtz equation

$$
\nabla^2 u + \lambda^2 u = 0. \tag{1.3}
$$

Here $\lambda$ is an arbitrary nonzero real number, say positive.

## 2 Main result and discussion

Prior to formulating the main result, we recall the $m$-dimensional mean value formula for balls:

$$
a_m(\lambda r) u(x) = M(u, B_r(x)), \quad \text{where} \quad a_m(t) = \Gamma \left( \frac{m}{2} + 1 \right) \frac{J_{m/2}(t)}{(t/2)^{m/2}}. \tag{2.1}
$$

It is valid for every admissible ball $B_r(x)$ provided $u \in C^2(D) \cap L^1(D)$ is metaharmonic in $D$. As usual, $J_\nu$ denotes the Bessel function of order $\nu$; its $n$th positive zero is denoted by $j_{\nu,n}$ (this standard notation is used below). It is quite strange that identity (2.1) was obtained just recently (see [9], Corollary 2.1), whereas only the spherical mean value formula was known earlier (see, for example, [10], pp. 317–320). There is a detailed discussion of identity (2.1) in [9], and of the analogous formula for solutions of (1.1). Notice that $a_m(\lambda r)$ oscillates about the zero, whereas the similar expression in identity (2.1) is positive and increases monotonically with $r$. The role played by monotonicity in the proof of Theorem 1.1 suggests its following modification, which is our main result.

**Theorem 2.1.** Let $D \subset \mathbb{R}^m$, $m \geq 2$, be a bounded domain, and let $r > 0$ be such that $|B_r| = |D|$. Suppose that for some $\lambda > 0$ and a point $x_0 \in D$ the identity

$$
u(x_0) a_m(\lambda r) = M(u, D) \tag{2.2}
$$

is fulfilled for every function $u$ metaharmonic in $D_r$. If also

$$
D \subset B_{\lambda r_0}(x_0), \quad \text{where} \quad \lambda r_0 = j_{m/2,1}, \tag{2.3}
$$

then $D = B_r(x_0)$.  

Remark 2.1. For a given $\lambda$, this theorem is applicable only to domains, whose volume is less than or equal to $|B_{r_0}|$, where $\lambda r_0 = j_{m/2,1}$, because every such domain must lie within a ball of radius $r_0$.

Remark 2.2. In Theorem 2.1, there is no need to require that the complement of $D$ is connected. This essentially distinguishes it from Theorem 1.1.

Prior to proving this theorem, we introduce the following function:

$$U(x) = a_{m-2}(\lambda|x|), \quad x \in \mathbb{R}^m.$$  \hspace{1cm} (2.4)

Its main properties immediately follow from the representation:

$$U(x) = \frac{2\Gamma(m/2)}{\sqrt{\pi\Gamma((m-1)/2)}} \int_0^1 (1 - s^2)^{(m-3)/2} \cos(\lambda|x|s) \, ds,$$  \hspace{1cm} (2.5)

which is a consequence of Poisson’s integral for $J_\nu$ (see [11], p. 206). The latter expression takes particularly simple form for $m = 3$, namely,

$$U(x) = (\lambda|x|)^{-1} \sin \lambda|x|.$$  

It is clear that this function solves \text{(1.3)} in $\mathbb{R}^m$; indeed, \text{(2.5)} is differentiable easily, thus verifying the equation. Also, we have that $U(0) = 1$ and $U(x)$ decreases monotonically when $\lambda|x|$ belongs to the interval $(0, j_{m/2,1})$, and is positive on the smaller interval $(0, j_{(m-2)/2,1})$.

Proof of Theorem 2.1. Without loss of generality, we suppose that the domain $D$ is located so that $x_0$ coincides with the origin. If we assume that $D \neq B_r(0)$, then $G_i = D \setminus B_r(0)$ and $G_e = B_r(0) \setminus D$ are bounded open sets such that $|G_e| = |G_i| \neq 0$ in view of the definition of $r$. In order to obtain a contradiction from this assumption we write identity \text{(2.2)} for $U$ as follows:

$$|D| a_m(\lambda r) = \int_D U(y) \, dy;$$  \hspace{1cm} (2.6)

here the condition $U(0) = 1$ is taken into account. Since property \text{(2.1)} holds for $U$ over $B_r(0)$, we write it in the same way:

$$|B_r| a_m(\lambda r) = \int_{B_r(0)} U(y) \, dy.$$  \hspace{1cm} (2.7)

Subtracting \text{(2.7)} from \text{(2.6)} and using the definition of $r$, we obtain

$$0 = \int_{G_i} U(y) \, dy - \int_{G_e} U(y) \, dy < 0.$$  

Indeed, $U(y)$ monotonically decreases with $|y|$ in the whole $D$ because $D \subset B_{r_0}$. Therefore, the difference is negative in view that $U(y)$ is strictly greater than $[U(y)]_{|y|=r}$ in $G_e$ and strictly less than this value in $G_i$, whereas $|G_i| = |G_e|$. The obtained contradiction proves the theorem. \hfill \Box

Here, the argument is the same as in the proof of Theorem 1.1 given in [7]; both rely on monotonicity of a certain solution to the corresponding equation. However, there is an essential distinction between the two theorems, which concerns the size of a domain. Indeed, no restriction on the size is imposed in Theorem 1.1, because its proof involves the function analogous to $U$, but increasing monotonically for all $|x| > 0$. On the contrary, the radially symmetric function $U$
defined in (2.3) decreases monotonically near the origin, but only when \( \lambda |x| \) belongs to a bounded interval adjacent to zero. For this reason condition (2.3) is imposed in Theorem 2.1.

In the limit \( \lambda \to 0 \), equation (1.3) becomes the Laplace equation, whereas \( a_m(\lambda r) \to 1 \), and so the assumption about \( r \) becomes superfluous in the limiting form of Theorem 2.1. Thus, letting \( \lambda \to 0 \) this theorem turns into Kuran’s.

Furthermore, the integral \( \int_D u(y) \, dy \)—it appears in (2.2) in the formulation of Theorem 2.1—can be replaced by the flux integral \( \int_{\partial D} \partial u / \partial n_y \, dS_y \) provided \( \partial D \) is sufficiently smooth; here \( n \) is the exterior unit normal. Indeed, we have

\[
\int_D u(y) \, dy = -\lambda^{-2} \int_D \nabla^2 u(y) \, dy = -\lambda^{-2} \int_{\partial D} \partial u / \partial n_y \, dS_y .
\]

Let us evaluate how restrictive is condition (2.3), for which purpose we consider the following simple example: a square membrane \( D_s = \{ x \in \mathbb{R}^2 : x_1, x_2 \in (0, a) \} \), \( a > 0 \), fixed along the boundary. Its free oscillations are described by the following set of eigenfunctions:

\[
u_{ij} = \sin(i \pi x_1 / a) \sin(j \pi x_2 / a), \quad \nu_{ji} = \sin(j \pi x_1 / a) \sin(i \pi x_2 / a), \quad i, j = 1, 2, \ldots .
\]

They are linearly independent when \( i \neq j \) and satisfy equation (1.3) with

\[
\lambda_{ij} = \lambda_{ji} = (\pi / a) \sqrt{i^2 + j^2}, \quad i, j = 1, 2, \ldots .
\]

Let us check condition (2.3) for \( D_s \) with \( \lambda = \lambda_{21} = \lambda_{12} = \pi \sqrt{5} / a \). The reason to consider this particular value is that \( u_{21}(a / 2, a / 2) = u_{12}(a / 2, a / 2) = 0 \) and

\[
\int_{D_s} u_{21}(x) \, dx = \int_{D_s} u_{12}(x) \, dx = 0,
\]

that is, identity (2.2) is valid for these functions provided \( x_0 = (a / 2, a / 2) \). Of course, Theorem 2.1 is violated for \( D_s \), but what is the difference between \( \lambda r_0 \) and \( j_{1,1} \)? Since \( r_0 = a / \sqrt{2} \), we have that \( \lambda_{21} r_0 = \pi \sqrt{5 / 2} \approx 4.967294 \) for \( D_s \). It is clear that the right-hand side in equality (2.3) is less than this value, namely, \( j_{1,1} \approx 3.831706 \). However, the difference \( \lambda r_0 - j_{1,1} \) is not too large in this case.

In conclusion, we compare Theorem 2.1 with the result obtained in [12, 13] and concerning the so-called refined Schiffer’s conjecture. The latter also characterizes balls and is similar to the celebrated Serrin theorem [14] in this aspect. However, this conjecture involves equation (1.3) instead of Poisson’s that appears in [14]. For a simply connected two-dimensional domain with smooth boundary this conjecture was investigated in [15], p. 143 (see also [10] for another approach). It is worth mentioning that a clear description of the original Schiffer’s conjecture can be found in [17]; see also the review [18] for its discussion.

The next theorem was proved in the monograph [12], but it is more convenient to give its formulation that appeared in the brief note [13].

**Theorem 2.2.** Let \( D \subset \mathbb{R}^3 \) be a bounded \( C^2 \)-domain. If a nontrivial \( u \in C^2(D) \cap C^1(\overline{D}) \) satisfies equation (1.3) and the boundary conditions \( u = 0, \partial u / \partial n = c \) on \( \partial D \) with constant \( c \), then \( D \) is a ball of radius \( r \) such that \( \lambda r \) is a zero of \( j_0 \) — the spherical Bessel function of order zero.

In order to compare Theorems 2.2 and 2.1, we fix \( \lambda > 0 \). Since \( j_0(t) = t^{-1} \sin t \), we have that \( r = \pi k / \lambda, \; k = 1, 2, \ldots \), according to Theorem 2.2. Therefore, the existence of a solution to the
overdetermined boundary value problem assumed in this theorem guarantees that a domain is a ball provided its volume belongs to a discrete sequence of values tending to infinity. On the other hand, identity (2.2) (it is the key point of Theorem 2.1) implies that a domain $D \subset \mathbb{R}^3$ is a ball provided it lies within another ball, whose radius $r_0$ is defined by the equality $\lambda r_0 = j_{3/2,1} \approx 4.493409$. Therefore, the volume of a domain $D$ can be arbitrary, but within the interval $(0, |B_{r_0}|]$.

These considerations demonstrate clearly that the range of applicability of Theorem 2.1 is completely different from that of Theorem 2.2.

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