Slow dynamics at critical points: the field-theoretical perspective

Andrea Gambassi

Max-Planck-Institut für Metallforschung, Heisenbergstr. 3, D-70569 Stuttgart, Germany and Institut für Theoretische und Angewandte Physik, Universität Stuttgart, Pfaffenwaldring 57, D-70569 Stuttgart, Germany.

E-mail: gambassi@mf.mpg.de

Abstract. The dynamics at a critical point provides a simple instance of slow collective evolution, characterised by aging phenomena and by a violation of the fluctuation-dissipation relation even for long times. By virtue of the universality in critical phenomena it is possible to provide quantitative predictions for some aspects of these behaviours by field-theoretical methods. We review some of the theoretical results that have been obtained in recent years for the relevant (universal) quantities, such as the fluctuation-dissipation ratio, associated with the non-equilibrium critical dynamics.

1. Introduction

Non-equilibrium dynamics of statistical systems can still be regarded as one of the terrae incognitae of Physics. Equilibrium statistical mechanics successfully accounts for the connection between microscopic and macroscopic behaviour, providing a complete framework to describe complex collective phenomena such as, e.g., phase changes. On the other hand, an analogous general picture is missing for non-equilibrium behaviour (both static and dynamic) and therefore it is still worth focusing on very simplified models of actual physical systems. Unexpected dynamical behaviours, such as hysteresis an memory effects, a dramatic slowing-down of relaxation processes etc., have been predicted and then observed in disordered and complex systems [1, 2, 3]. One of the most striking of them is ageing: After a perturbation, the system persists in a non-equilibrium state for a macroscopically long time and therefore in experiments one can actually probe only this never-ending relaxation. Ageing in classic examples of glassy systems (e.g., spin glasses and supercooled liquids) has been discussed in detail during the Summer School on “Ageing and the Glass Transition” and therefore we do not discuss them here again, referring the reader to the corresponding proceedings [3]. On the other hand, as originally pointed out in [4, 5], ageing can also be observed in systems much “simpler” than those previously mentioned, such as non-disordered ferromagnets. Indeed, upon approaching their critical points (or within their ordered phases), slow-relaxing fluctuation modes keep them out of equilibrium. A deep understanding of the behaviour of statistical systems at their critical points (where they undergo second-order phase transitions) greatly facilitate the discussion of the corresponding ageing dynamics. Indeed, as in the case of static and dynamic (equilibrium) critical behaviour [6], one can naturally assume that the relevant properties of a critical system are determined by the fluctuations of the order parameter and therefore are, to some extent, independent of the actual
microscopic details of the system (universality). In turn, this simplification allows a detailed (quantitative) characterization of the ageing behaviour to an extent that, at present, is out of reach for the more general cases of glassy dynamics. In view of these facts, the non-equilibrium dynamics of systems at their critical points has been recently the subject of a renewed interest (for a review see [7]).

The aim of this short lecture is to discuss in some detail an explicit example in which the field-theoretical methods (as outlined, e.g., in the lecture by U. Täuber [8]) can be successfully applied in order to characterise the ageing behaviour associated with the slow dynamics at a critical point. What follows is meant to be a complement to [8], which the reader is referred to for further details and a general bibliography on the field-theoretic renormalization group.

2. Non-equilibrium relaxation

Consider the relaxation process of a sample, e.g., a ferromagnet – characterised by a critical temperature \( T_c \) – which has been prepared in an equilibrium state at high temperature \( T_0 > T_c \). At time \( t = 0 \) bring the sample in contact with a thermal bath of temperature \( T < T_0 \) and let it evolve. Two regimes are expected to characterise the ensuing relaxation process: (a) A transient one with non-equilibrium evolution, for \( t \ll t_{\text{eq}}(T) \), and (b) a stationary one with equilibrium evolution for \( t \gg t_{\text{eq}}(T) \), where \( t_{\text{eq}}(T) \) is a characteristic equilibration time\(^1\). During (a) the behaviour of the system depends on the specific initial condition (e.g., on \( T_0 \)) whereas during (b) time-translation invariance and time-reversal symmetry (in the absence of external fields) are recovered and such a dependence is lost; fluctuations are therefore described by the equilibrium dynamics. As long as \( t_{\text{eq}}(T) \) is finite equilibrium is eventually attained. This is always the case for \( T > T_c \) but generally not for \( T < T_c \), when phase ordering takes place [9, 10]. Indeed, consider as an example the Ising model with spin-flip dynamics (see below) quenched from a disordered state to \( T < T_c \). Just after the quench, domains of up (\(+\)) and down (\(-\)) spins of typical size \( \ell \) (locally minimizing the free energy density) form and grow each at the expenses of the others. Eventually a scaling regime is attained in which \( \ell(t) \sim t^{1/z_c} \) (\( z_c \) is the growth exponent [9, 10]) and the system is statistically invariant under rescaling by a factor \( \ell(t) \). In the thermodynamic limit and in the absence of fields breaking the \(+ \leftrightarrow -\) symmetry, none of the two possible domain signs prevails and therefore the completely ordered equilibrium state (either of up or down spins) is never realized, i.e., effectively, \( t_{\text{eq}}(T < T_c) = \infty \). In the following we focus on the case of an instantaneous quench from \( T_0 \) to the critical point. At \( T_0 \) (i.e., for \( t < 0 \)) the system is characterised by a small correlation length \( \xi \) between spatial fluctuations of the order parameter \( \varphi_x \) whereas \( \xi \) is expected to diverge at \( T_c \). Indeed, during the relaxation process, \( \xi(t) \sim t^{1/z_c} \), where \( z \) is the dynamic critical exponent [6, 8]. The dynamics of the system can be studied by looking at time-dependent quantities. The simplest ones depend only on the microscopic state of the system at a given time \( t \) and are therefore called one-time quantities. However, in the long-time limit, they usually approach an asymptotic value, no longer providing informations about the dynamics. Next, one may consider the so-called two-time quantities which depend on the state of the system at two distinct times. At variance with the previous case, they can be used to characterise the dynamics also in the long-time limit. The time-dependent correlation function \( C_x(t, t') \) of the order parameter and the linear response (susceptibility) \( R_x(t, t') \) to an external field are two natural dynamical quantities to look at. In particular the former is given by \( C_x(t, t') = \langle \varphi_x(t) \varphi_y(t') \rangle \), where \( \langle \cdot \rangle \) stands for the mean over the stochastic dynamics. The latter, instead, by \( R_x(t, t') = \delta \langle \varphi_x(t) \rangle / \delta h(t') \), where \( h \) is a small external field, conjugate to \( \varphi_x = 0 \) (e.g., if \( \varphi_x \) is a magnetic order parameter, \( h \) is the magnetic field), applied at time \( t' > 0 \) at the point \( x = 0 \). Note that causality implies \( R_x(t, t' > t) = 0 \) and that \( C_x(t, t') = C_x(t', t) \).

\(^1\) Generically one expects that the statistical correlation between a configuration of the system at time \( t \) and the initial one decays exponentially with time \( \sim e^{-\kappa(t)} \), i.e., that the system forgets its initial condition within a time \( \sim \kappa(T)^{-1} \), leading to a natural definition of \( t_{\text{eq}} \) as \( t_{\text{eq}}(T) = \kappa(T)^{-1} \).
in the bulk. According to the general picture of the relaxation process, one expects that for \( t > t' \gg t_{eq}(T) \) [i.e., in regime (b)] equilibrium is attained and therefore \( C_X(t, t') = C_X^{eq}(t - t') \) and \( R_X(t, t') = R_X^{eq}(t - t') \) where \( C_X^{eq} \) and \( R_X^{eq} \) are the corresponding equilibrium quantities, related by the fluctuation-dissipation theorem (FDT)

\[
R_X^{eq}(\tau > 0) = -\frac{1}{k_B T} \frac{dC_X^{eq}(\tau)}{d\tau},
\]

where \( k_B \) is Boltzmann’s constant and \( T \) the temperature of the thermal bath. In general one defines the so-called fluctuation-dissipation ratio (FDR) \([11]\)

\[
X_X(t, t') = k_B T \frac{R_X(t, t')}{\partial_x C_X(t, t')}
\]

and its long-time limit

\[
X^\infty = \lim_{t' \to \infty} \lim_{t \to \infty} X_X(t, t') .
\]

Because of the FDT, \( X^\infty = 1 \) whenever \( t_{eq}(T) < \infty \). On the other hand, when \( t_{eq}(T) = \infty \), the system never equilibrates and the correlation and response functions depend both on \( t \) and \( t' \). This behaviour is usually referred to as ageing [1, 2]. In this regime \( X^\infty \neq 1 \) is clear a signature of a non-equilibrium asymptotic dynamics. In addition it can be formally used to define an effective temperature \( T_{eff} = T/X^\infty \), which might have some features of a true thermodynamic temperature, e.g., controlling the direction of energy transfers and acting as a criterion for thermalization. This is indeed the case in some mean-field glass models where the construction of a sort of non-equilibrium thermodynamics has been attempted on this basis [12]. It is then natural to explore this possibility also in the case of simpler ageing dynamics like the critical one. In the following we focus on the field-theoretical computation of \( X_X(t, t') \) and \( X^\infty \), although other properties (such as universal scaling functions) of the non-equilibrium (ageing) dynamics have been analyzed along the same lines and with the same method [7].

What do we know in general about \( X^\infty \)? As a consequence of the fluctuation-dissipation theorem \( X^\infty(T > T_c) = 1 \). On the other hand, on the basis of general scaling arguments for the phase ordering regime [13], it has been shown that \( X^\infty(T < T_c) = 0 \). These results are expected to be actually independent of the specific system and of its microscopic details. For \( T = T_c \) there are no general arguments constraining the value of \( X^\infty \) and therefore it has to be determined for each specific model. In table 1 we report some of the values that has been found either by exact solutions or by means of Monte Carlo (MC) simulations (a more complete table can be found in [7]). Clearly, \( X^\infty(T = T_c) \) differs from \( X^\infty(T \neq T_c) \), depends on the model and on the space dimensionality \( d \). Nevertheless it has been argued on the basis of scaling arguments [13] that \( X^\infty(T = T_c) \) should be a universal quantity associated with the critical dynamics.

**Universality in critical phenomena**

Universality is a striking property of the collective behaviour of statistical systems close to critical points. Indeed, upon approaching them, the relevant physical scales are set by the growing correlation length \( \xi \) and correlation time \( \tau_c \sim \xi^z \), making the behaviour of the system actually independent of its microscopic details, at least at scales comparable with \( \xi \) and \( \tau_c \). This observation (originally suggested by experimental evidence) is nowadays understood within the framework of renormalization-group theory [8]. Some properties of the system, called universal (e.g., critical exponents, amplitude ratios, scaling functions etc), depend only on gross features such as symmetries, range of the microscopic interaction, conservation laws, space dimensionality etc. Therefore, in order to determine universal quantities, one may have recourse to very simplified models which, however, share with the original system such general features. Consider
Table 1. Values of $X^\infty$ in some models. $^a$Exact solution, $^b$Monte Carlo simulations. $^\dagger$2 < $d$ < 4.

| Model                              | Ref. | $T < T_c$ | $T = T_c$ | $T > T_c$ |
|------------------------------------|------|-----------|-----------|-----------|
| Random Walk$^a$                     | [4]  | —         | 1/2       | —         |
| Free Gaussian Field$^a$            | [4]  | —         | 1/2       | 1         |
| d-dim. Spherical Model$^a$         | [14] | 0         | 1 − 2/$d^\dagger$ | 1         |
| 1-dim. Ising–Glauber Model$^a$     | [15] | —         | 1/2       | 1         |
| 2-dim. Ising–Glauber Model$^b$     | [14] | 0.26(1)   |           |           |
|                                    | [16] | 0.340(5)  |           |           |
|                                    | [17] | 0.33(2)   |           |           |
|                                    | [18] | 0.330(5)  |           |           |
| 3-dim. Ising–Glauber Model$^b$     | [14] | 0.40      |           |           |

the case of the Ising model (nearest-neighbour ferromagnetic interaction) on a $d$-dimensional (hypercubic) lattice. The microscopic degrees of freedom are the spin variables $S_x \in \{+1, -1\}$ on the lattice sites $x \in \mathbb{Z}^d$. It is well-known that the static (i.e., time-independent) universal critical properties of this model are the same as those of the Landau-Ginzburg model with scalar order parameter $\varphi_x \in \mathbb{R}$ (some coarse-grained version of $S_x$), defined on the continuum $x \in \mathbb{R}^d$, which can be conveniently worked out by using a variety of different methods. In this sense one says that these two models belong to the same static universality class. Dynamics opens up different possible cases, given that the same equilibrium state can be realised by means of different dynamical processes. The lattice Ising model can be endowed with different dynamics: According to certain rates (determined by the microscopic Hamiltonian) one can either (G) flip a spin chosen at random (Glauber dynamics) or (K) exchange the position of two randomly chosen neighbouring spins (Kawasaki dynamics). In terms of conservation laws it is clear that (K) conserves the total magnetization of the sample $\sum_{x \in \mathbb{Z}^d} S_x$, whereas (G) does not. The resulting dynamic critical behaviour (e.g., the value of $z$) is different in the two cases, suggesting the presence of two distinct dynamic universality classes within the same static universality class. On the same footing the dynamics of the order parameter $\varphi_x(t)$ of the Landau-Ginzburg model can be assigned (in the form of a Langevin equation) in such a way that there are no conservation laws (Model A) or so that $\int d^d x \varphi_x = \text{const}$ (Model B) [6, 8]. As expected on the basis of universality, it turns out that Model A belongs to the same universality class as the Ising model with Glauber dynamics, whereas Model B to the same universality class as the Ising model with Kawasaki dynamics. Therefore the universal properties of the critical dynamics of lattice models can be conveniently studied by means of simplified field-theoretical models on the continuum. Accordingly, to give theoretical estimates of $X^\infty(T = T_c)$ in the Ising-Glauber model in $d$ dimensions (or in any other model belonging to its universality class) one can study the corresponding quantity in Model A dynamics of the Ginzburg-Landau model. In the following we focus on this specific example, although the analysis has been extended to different dynamic universality classes (for a summary see, e.g., [7]). Let us mention that the universality of $X^\infty(T = T_c)$ has also been directly confirmed by Monte Carlo simulations of different lattice models which are known to belong to the same dynamic universality class (see [18] for details).
3. Field-theoretical approach

3.1. The model

The critical properties of the relaxational dynamics of a system of $n$-component spins $S_x$ interacting through a short-range $O(n)$ symmetric potential on a $d$-dimensional lattice, are captured by Model A of [6]. This is specified in terms of an $n$-component field $\varphi(x, t)$ (the coarse-grained version of $S_x$) evolving according to the stochastic Langevin equation

$$\partial_t \varphi(x, t) = -D \frac{\delta H[\varphi]}{\delta \varphi(x, t)} + \zeta(x, t),$$  
(4)

where $D$ is a kinetic coefficient, $\zeta(x, t)$ a zero-mean stochastic Gaussian noise with $\langle \zeta_i(x, t) \zeta_j(x', t') \rangle = 2k_B T D \delta(x - x') \delta(t - t') \delta_{ij}$, and $H[\varphi]$ is the static Hamiltonian. The leading scaling behaviour is captured, near the critical point, by the Landau-Ginzburg form for $H[\varphi]$:

$$H[\varphi] = \int dx \left[ \frac{1}{2} \nabla \varphi^2 + \frac{r}{2} \varphi^2 + \frac{u}{4!} \varphi^4 \right],$$  
(5)

where $r \propto T$ ($r$ has to be tuned to a critical value $r_{\text{crit}}$ in order to approach the critical point for $T = T_c$) and $u > 0$ is the coupling constant. Note that in contrast to driven models, the Langevin equation (4) generically leads to an equilibrium state characterised by a probability distribution function $\sim e^{-H[\varphi]/k_B T}$. However, upon approaching the critical point, the time the system takes to converge to this state diverges and therefore the putative equilibrium state is never reached for dynamical reasons.\(^2\) Correlation and response functions can be obtained [8] from the field-theoretical action

$$A[\tilde{\varphi}, \varphi] = \int_0^\infty dt \int dx \left[ \tilde{\varphi} \partial_t \varphi + D \tilde{\varphi} \frac{\delta H}{\delta \varphi} - D k_B T \tilde{\varphi}^2 \right],$$  
(6)

where the additional field $\tilde{\varphi}$ has been introduced for convenience. Note that if an external field $h$ couples linearly to $\varphi$ in $H$ (as it is the case when $\varphi$ is the magnetization and $h$ the magnetic field), i.e., $H[\varphi; h] = H[\varphi] - \int dx h \varphi$ then it couples linearly to $\tilde{\varphi}$ in $A$, i.e., $A[\tilde{\varphi}, \varphi; h] = A[\tilde{\varphi}, \varphi] - D \int dt \int dx h \tilde{\varphi}$. In terms of $A$, the average over the stochastic dynamics induced by $\zeta$ [for a fixed initial condition $\varphi_0(x) = \varphi(x, t = 0)$] of a quantity $O[\varphi]$ depending on the order parameter field $\varphi$ is given by

$$\langle O[\varphi] \rangle = \int \left[ d\tilde{\varphi} d\varphi \right] O[\varphi] e^{-A[\tilde{\varphi}, \varphi]}.$$  
(7)

The effect of the external field $h$ on this average (denoted, for $h \neq 0$, by $\langle \ldots \rangle_h$) is accounted for by using $A[\tilde{\varphi}, \varphi; h]$ in the previous equation. Accordingly, the linear response of $O[\varphi]$ to the field $h$, given by

$$\frac{\delta \langle O[\varphi] \rangle_h}{\delta h(x, t)} \bigg|_{h=0} = D \langle \tilde{\varphi}(x, t) O[\varphi] \rangle,$$  
(8)

takes the form of a correlation between the field $\tilde{\varphi}$ and $O$. This is the reason why $\tilde{\varphi}$ is termed \textit{response field}. Finally, one has to account for the initial condition $\varphi_0$ [5]. In the case we are interested in, the initial state of the system corresponds to a high-temperature phase, characterised by a Gaussian distribution $\sim e^{-\mathcal{H}_0[\varphi_0]/k_B T_0}$ of the order parameter with

$$\mathcal{H}_0[\varphi_0] = \frac{\Delta}{2} \int dx \varphi_0^2(x).$$  
(9)

Accordingly, the averages over the possible realizations of the relaxation process we are interested in can be computed in term of the field-theoretical “action” $A[\tilde{\varphi}, \varphi] + \mathcal{H}_0[\varphi_0]/k_B T_0$.

\(^2\) This phenomenon can be regarded as the spontaneous symmetry breaking of the symmetries (time translation invariance and time reversal) associated with equilibrium dynamics [10].
3.2. Scaling forms
The renormalization-group analysis of the previous problem [8] leads to the following scaling functions for the Fourier transform of the two-point response and correlation functions \((t > t')\) at the critical point \(T = T_c\):

\[
R_{q=0}(t, t') = A_R (t - t')^a (t/t')^\theta F_R(t'/t),
\]
\[
C_{q=0}(t, t') = k_B T_c A_C (t - t')^a (t/t')^\theta F_C(t'/t),
\]
where \(a = (2 - \eta - z)/z\) and \(\theta\) is the initial-slip exponent (see, e.g., [5, 8]). The functions \(F_C\) and \(F_R\) are universal once the non-universal normalization constants \(A_R\) and \(A_C\) have been fixed so that \(F_1(0) = 1\). In [19] the quantity \(\hat{X}(t, t') = k_B T R_q(t, t')/\partial_t C_{q=0}(t, t')\) has been introduced, showing that \(\lim_{\nu \to \infty} \lim_{t \to \infty} \hat{X}(t, t') = X_\infty\) (see, however, [20]). Then, using equation (10) and (11), it is easy to see that \(X_\infty\) at criticality can be expressed as an amplitude ratio [14, 19]:

\[
X_\infty = A_R/[A_C(1 - \theta)].
\]

4. A simple one-loop computation
For the specific Hamiltonian in equation (5) one gets (in what follows \(D = 1\))

\[
\mathcal{A}[\hat{\varphi}, \varphi] = \int_0^\infty dt \int d^d x \left\{ \hat{\varphi} \partial_t - \nabla^2 + r \right\} \varphi - k_B T \hat{\varphi}^2 + \frac{u}{\delta h}(\hat{\varphi}_\varphi^2) \right\}.
\]

Diagrammatically we denote the field \(\varphi\) by \(\rightarrow\) whereas \(\hat{\varphi}\) by \(\leftrightarrow\). Accordingly, the interaction vertex \(\frac{u}{\delta h}(\hat{\varphi}_\varphi^2)\) is represented by \(\oint\). Let us consider first the case of the Gaussian model \(u = 0\). In this case the critical point correspond to \(r = r_{crit} = 0\). The response and correlation functions can be then easily computed in Fourier transform (taking into account the initial condition), finding [5]

\[
\frac{q}{p} \rightarrow \cdots \frac{q}{t} R_q(t, t') = \frac{\delta \langle \varphi_q(t) \rangle_h}{\delta h_{-q}(t')} \bigg|_{h=0} = \langle \varphi_q(t') \varphi_q(t) \rangle = \Theta(t - t') e^{-(q^2 + r)(t - t')} = R_q(t - t').
\]

The second equality follows the fact that \(h\) is the conjugate of \(\hat{\varphi}\) in \(\mathcal{A}\). \(\Theta(\tau)\) is the step function \([i.e., \Theta(\tau < 0) = 0 \text{ and } \Theta(\tau > 0) = 1]\), enforcing causality. For \(\Theta(0)\) we adopt the prescription \(\Theta(0) = 0\), which implies \(R_q(t, t') = 0\). Arrows in the diagrammatic representation of \(\varphi\)-fields eventually result in directed lines \([i.e., R_q(t, t')]\) which are non-vanishing only if the time variables attached to their endpoints increase in the direction indicated by the corresponding arrows. The correlation function is given by

\[
\frac{q}{p} \rightarrow \cdots \frac{q}{t} C_q(t, t') = \langle \varphi_{-q}(t) \varphi_q(t') \rangle = C_{q}^{\text{eq}}(t - t') - C_{q}^{\text{eq}}(t + t') + k_B T_0 \Delta^{-1} R_q(t, 0) R_q(t', 0),
\]
where

\[
C_{q}^{\text{eq}}(\tau) = \frac{k_B T}{q^2 + r} e^{-(q^2 + r)|\tau|}
\]
is the correlation function in equilibrium.\(^3\) The renormalization-group analysis of the problem [5, 8] leads to the conclusion that the leading scaling properties of the model can be worked out

\(^3\) Note that in the first line of equations (14) and (15) a delta function (corresponding to the volume in a finite system) has been factored out. Indeed \(\langle \varphi_q(t') \varphi_q(t) \rangle = (2\pi)^d \delta^{(d)}(q + q') R_q(t, t')\) and \(\langle \varphi_q(t') \varphi_q(t) \rangle = (2\pi)^d \delta^{(d)}(q + q') C_q(t, t')\).
Figure 1. Field-theoretical predictions for the fluctuation-dissipation ratio of the critical two-dimensional Ising model with purely dissipative dynamics (Model A). The dashed line refers to the Gaussian approximation \( u = 0 \), see equation (17) with \( r = r_{\text{crit}} = 0 \) and \( s \mapsto Ds \), whereas the solid one to the Padé approximants \([0, 1]\) and \([1, 0]\) of the one-loop result (see [19]), which are barely distinguishable on this scale. The inset provides a magnification of the main plot, highlighting the non-monotonic behaviour of the one-loop result, which, in contrast to the Gaussian approximation, becomes bigger than 1 for \( x \equiv 2Dq^2s \gtrsim 3.5 \) and attains its maximum at \( x \approx 5.1 \). The same qualitative behaviour is predicted for the three-dimensional Ising model with the maximum value \( \approx 1.004 \) attained for \( x \approx 6.1 \).

assuming that \( \Delta^{-1} = 0 \) (i.e., \( \Delta^{-1} \) is an irrelevant variable [5]). Accordingly the last term in equation (15) drops out and the resulting correlator satisfies the Dirichlet boundary condition at \( t = 0 \): \( C_q(t,0) = C_q(0,t') = 0 \). Note that, for \( t > s \), \( (kB T)^{-1}\partial_s C_q(t-s) = R_q(t-s) \), as stated by the fluctuation-dissipation theorem. Now, instead, using equation (15) one finds \( (kB T)^{-1}\partial_s C_q(t,s) = R_q(t,s) + R_q(t,-s) \) and therefore

\[
X_q(t,s) = (kB T)^{-1}\partial_s C_q(t,s) = \frac{1}{1 + e^{-2(q^2+r)s}},
\]

which depends only on the smaller time \( s \). Note that for \( r > r_{\text{crit}} = 0 \) (i.e., \( T > T_c \)), \( X_q \rightarrow 1 \) for long times. In figure 1 the dashed line refer to the dependence of \( X_q(t,s) \) on the scaling variable \( 2Dq^2s \), at the critical point \( T = T_c \) (i.e., \( r = r_{\text{crit}} = 0 \)). Accordingly, for \( q \neq 0 \) the long-time limit of the FDR is always 1. For \( q = 0 \), on the other hand, \( X_{q=0}(t,s) = \frac{1}{2} \). In the long-time limit the only fluctuation mode that does not equilibrate is the homogeneous one.

Non-Gaussian fluctuations are expected to influence the previous result. To go beyond the Gaussian approximation we have to include the effects of the interaction term \( u(\hat{\phi}\phi)^2 \). It is well-known (see, e.g., [8] and references therein) that on the continuum the perturbative series in \( u \) is plagued with divergences that have to be properly regularised and then removed in order to get a well-defined (renormalised) perturbative expansion. In what follows we employ the dimensional regularization to deal with divergent momentum integrals and then we perform the so-called \( \epsilon \)-expansion around the upper critical dimension, where \( \epsilon = 4 - d \) [8]. In this scheme the critical point is still given by \( r_{\text{crit}} = 0 \). The diagrams contributing at this order are reported in figure 2: \((r)\) for the response and \((c_{1,2})\) for the correlation. Given we are interested in the behaviour at the critical point we set \( r = r_{\text{crit}} = 0 \) in the following computation. The common
Figure 2. One-loop diagrams contributing to the response function \([r]\) and to the correlation function \([c_1]\) and \([c_2]\).
this is not the case for $R^{(1)}_{q=0}(t, s = 0)$. To render this quantity finite for $\epsilon \to 0$ one introduces a new renormalization constant $Z_0$ so that

$$\left[ R_q(t, s = 0) \right]_{\text{REN}} = \langle \varphi_q(t) \varphi_q(s = 0) \rangle_{\text{REN}} = Z_0^{1/2} \langle \varphi_q(t) \varphi_q(s = 0) \rangle, \quad (23)$$
i.e., in general,

$$\langle \varphi_q(s = 0) \rangle_{\text{REN}} = (Z_0 \tilde{Z})^{1/2} \langle \varphi_q(s = 0) \rangle,$$

where $\tilde{Z} = 1 + O(u^2)$ is the usual equilibrium renormalization constant for the field $\varphi$ [8]. $Z_0$ is therefore determined requiring $\left[ R_q(t, s = 0) \right]_{\text{REN}}$ to be finite for $\epsilon \to 0$. From equation (22) one finds

$$Z_0 = 1 - \frac{u_R}{\epsilon} K_4 \frac{n + 2}{3} + O(u_R^2), \quad (25)$$

where we have introduced the renormalised coupling constant $u_R$, given, at this order, by $u_R = u \mu + O(u^2)$. Here $\mu$ is the arbitrary momentum scale that has to be introduced in order to define the renormalised theory. The Wilson function associated with this new renormalization constant, entering the renormalization-group equations [8], is given by

$$\gamma_0(u_R) \equiv \mu \partial_\mu \ln Z_0|_{n=0} = \mu \partial_\mu u_R|_{n=0} \frac{\ln Z_0}{du_R} = \frac{n + 2}{3} K_4 u_R + O(u_R^2), \quad (26)$$

where the derivative is taken with fixed bare parameters and we used the fact that $\beta_n = \mu \partial_\mu u_R|_{n=0} = -c u_R + O(u_R^2)$ is the usual $\beta$-function associated with the coupling constant. The initial-slip exponent is defined as [5, 8]

$$\theta = \frac{\gamma_0(u_R^*)}{2z} \equiv \frac{n + 2 \epsilon}{n + 8} + O(\epsilon^2), \quad (27)$$

and can be obtained by using in equation (26) the stable fixed-point value of the (renormalised) coupling constant, given by $K_4 u_R^* = 3 \epsilon/(n + 8) + O(\epsilon^2)$ for $\epsilon > 0$. (For $d > 4$, i.e., $\epsilon < 0$, the stable fixed-point is $u_R^* = 0$ and therefore one recovers the results of the Gaussian model.) The response function is given by equation (21) expressed in terms of renormalised parameters and with $u_R = u_R^*$:

$$R^{(1)}_{q=0}(t, s) = 1 + \frac{n + 2 \epsilon}{n + 8} \ln \left( \frac{t}{s} \right) + O(\epsilon^2) = \left( \frac{t}{s} \right)^{\theta} + O(\epsilon^2), \quad (28)$$

where we have used, in the last equality, the information coming from the full renormalization-group analysis [see equation (10)], predicting the power-law behaviour that is perturbatively signalled by logarithmic terms. Comparing with equation (10) one finds $A^{(1)}_R = 1$ and $F^{(1)}_R(x) = 1$. For the correlation function one has ($t > s$)

$$(c) = (c_1) + (c_2) = \int_0^\infty dt \left[ R_{q=0}(t, t') B(t') C_{q=0}(t', s) + C_{q=0}(t, t') B(t') R_{q=0}(s, t') \right]. \quad (29)$$

Taking into account that $C_{q=0}(t_1, t_2) = 2 \min\{t_1, t_2\}$ the previous integrals are easily evaluated, leading to

$$C^{(1)}_{q=0}(t, s) = C_{q=0}(t, s) - u \frac{n + 2}{3!} \times (c) + O(u^2) = 2 k_B T_c s \left\{ 1 + u \frac{n + 2}{3} 2^{d/2} K_d \left[ \frac{2 s^{2-d/2}}{3 - d/2} + \frac{t^{2-d/2} - s^{2-d/2}}{2 - d/2} \right] + O(u^2) \right\}. \quad (30)$$
Note that: (i) as in the case of $R_{q=0}^{(1)}(t, s)$, this expression has no poles for $\epsilon \to 0$, in agreement with the renormalization-group analysis. (ii) $C_{q=0}^{(1)}(t, s = 0) = 0$, i.e., the Dirichlet boundary condition is preserved by the non-Gaussian fluctuations (indeed this is true to all orders in perturbation theory [5, 8]). (iii) $\partial_s C_{q=0}^{(1)}(t, s)_{s=0}$ has a pole for $\epsilon \to 0$. In principle one has to introduce an additional renormalization constant, as in equation (24), to renormalise $\partial_s \varphi_q(s)_{s=0}$. On the other hand one can prove that the poles in $\partial_s C_{q=0}^{(1)}(t, s)_{s=0}$ are related to those in $R_{q=0}^{(1)}(t, s = 0)$ given that $\partial_s \varphi_q(s)_{s=0} = 2k_B T c \tilde{\varphi}_q(s = 0)$ when inserted into correlation functions. Therefore no additional renormalization constant is actually required to make the correlation function and its derivatives finite for $\epsilon \to 0$. Expanding equation (30) in $\epsilon$, we find

$$C_{q=0}^{(1)}(t, s) = 2 k_B T c s \left\{ 1 + \frac{n + 2}{3} \left[ \frac{2}{4} + \ln \left( \frac{t}{s} \right) \right] + O(u^2, u\epsilon) \right\}$$

which, at the fixed point, becomes

$$C_{q=0}^{(1)}(t, s) = 2 k_B T c s \left\{ 1 + \frac{n + 2}{n + 8} \left[ 1 + \frac{n + 2}{n + 8} \ln \left( \frac{t}{s} \right) \right] + O(\epsilon^2) \right\}$$

$$= k_B T c A_C^{(1)} s \left( \frac{t}{s} \right)^\theta + O(\epsilon^2) ,$$

in agreement with the general scaling form (11), with $F_C^{(1)}(x) = 1$ and

$$A_C^{(1)} = 2 \left( 1 + \frac{n + 2}{n + 8} \right) .$$

Therefore

$$X_{q=0}^{(1)}(t, s) = k_B T c \frac{R_{q=0}^{(1)}(t, s)}{\partial_s C_{q=0}^{(1)}(t, s)}$$

$$= \frac{A_R^{(1)}}{A_C^{(1)}} \frac{1}{1 - \theta} = \frac{1}{2} \left( 1 - \frac{n + 2}{n + 8} \right) ,$$

which provides the field-theoretical prediction for the FDR in $\epsilon$-expansion. In principle, reliable numerical estimates of this quantity can only be obtained by higher-order computations. However, a rough estimate can be obtained by considering the Padé approximant [0, 1] and [1, 0] of the series (34). For the two-dimensional Ising model ($\epsilon = 2$, $n = 1$) one gets $X_{q=0}^{(1)}(t, s) = 0.423(6)$. Note that, up to this order, there is no dependence on $s/t$ and therefore $X^\infty[1\text{loop}] = 0.423(6)$. The previous computation can be generalised to non-vanishing momenta. We do not provide the details here, referring the reader to the original literature [19]. In figure 1 we report the Padé approximants [0, 1] and [1, 0] of the final result (solid line) for the two-dimensional Ising model, together with the Gaussian approximation (dashed line) given in equation (17). The two approximants are barely distinguishable and, as shown in the inset in figure 1, they are characterised by a non-monotonic behaviour of $X_{q}^{(1)}(t, s)$ as function of $x = 2Dq^2 s$ (up to $O(\epsilon)$ there is no dependence on $t$). Increasing the dimensionality $d$ towards $d = 4$ (i.e., decreasing $\epsilon \geq 0$) the qualitative picture does not change: The behaviour is still non-monotonic but the maximum approaches 1 as $\epsilon \to 0$ while the corresponding value of $x$ diverges in the same limit. Higher-order computations and MC simulations could provide a
Figure 3. Field-theoretical predictions for the asymptotic fluctuation-dissipation ratio $X^\infty$ of the $n$-vector model with purely dissipative dynamics (Model A), as a function of $\epsilon = 4 - d$. The different lines refer to the Padé approximants $[0, 2]$ and $[2, 0]$ of the two-loop field-theoretical result for the universality class of the Ising-Glauber model ($n = 1$, $\cdots$) and of the spherical model with dissipative dynamics ($n = \infty$, $\cdots$). For comparison we report also the exact solution, available (see table 1) in this case ($\cdots$). The full circle for $d = 3$ (i.e., $\epsilon = 1$), and the two vertical segments for $d = 2$ ($\epsilon = 2$) indicate the numerical estimates of $X^\infty$ that have been obtained by MC simulations of the Ising-Glauber model in the corresponding dimensions.

5. Conclusions and Perspectives
This section provides a brief survey of the questions that have been already addressed within the field-theoretical approach to non-equilibrium critical dynamics and of those deserving further investigations.

5.1. What has been done:
- The purely dissipative dynamics (Model A) of the Ginzburg-Landau Hamiltonian with $O(n)$ symmetry has been investigated up to two loops in the $\epsilon$-expansion [21]. This model provides a description of the universal aspects of the relaxation of actual anisotropic magnets and alloys for $n = 1$ (Ising-like symmetry) and, for generic $n$, of kinetic lattice spin models with short-ranged $O(n)$-symmetric interaction and Glauber dynamics. The comparison between the field-theoretical estimates of $X^\infty$ and the results of MC simulations for the two- and
three-dimensional Ising universality class and the three-dimensional XY model are quite satisfactory (see [7] for details).

- The dynamics with conserved order parameter (Model B) of the Ginzburg-Landau Hamiltonian for a scalar order parameter \( n = 1 \) has been investigated up to one loop in the \( \epsilon \)-expansion [7]. This model captures properly the universal critical properties of some uniaxial ferromagnets and of the lattice Ising model with Kawasaki dynamics (lattice gas). The scaling properties of the spin autocorrelation function \( C_{x=0}(t,s) \) of the one- and two-dimensional Ising model with Kawasaki dynamics have been investigated in some detail both theoretically and by means of MC simulations [22]. Unfortunately the long-time limit has not been accessed by the simulations of the two-dimensional model at criticality, although the conclusion \( X_{\text{Kawasaki}}^\infty > X_{\text{Glauber}}^\infty \) seems to be robust. This agrees with the fact that \( X_{\text{Model B}}^\infty = \frac{1}{2} + O(\epsilon^2) > X_{\text{Model A}}^\infty \) at least close to \( d = 4 \). Models A and B mentioned previously are usually too simple to be able to capture the relevant features of actual critical systems. Indeed in many cases the order parameter is not the only relevant slow variable of the system but there are additional quantities that have to be taken into account. For instance, in the case of a one-component fluid close to its critical point, the interaction of the conserved order parameter with three slow hydrodynamic modes affects the resulting dynamic critical behaviour. The so-called Model C is the simplest model in which a non-conserved order parameter \( \varphi(x,t) \) interacts with a (non-critical) conserved density \( \varepsilon(x,t) \): It captures the critical properties of some structural phase transitions in crystalline materials and of some uniaxial ferromagnets. The ageing properties of Model C have been investigated up to one loop in the \( \epsilon \)-expansion and the resulting \( X^\infty \) turns out to be the same as in the case of vanishing interaction between \( \varphi(x,t) \) and \( \varepsilon(x,t) \) [23]. Higher-order computation could clarify the extent to which this conclusion holds.

- Randomness may affect the critical properties of a statistical system, as it is the case of the three-dimensional Ising model when the spins in the lattice are diluted by non-magnetic impurities. The resulting critical properties are captured by the so-called random-temperature model. Its Model A dynamics has been studied up to one loop with different approaches. We refer the reader to [24, 20, 7] for details. Although the analytic expression of \( X^\infty \) in this case differ from that one of the pure model, it is not clear whether this leads to effective quantitative differences. Monte Carlo simulations could clarify this point.

- As mentioned in section 2, \( X^\infty \) can be formally used to introduce an effective temperature \( T_{\text{eff}} = T/X^\infty \). In equilibrium systems the temperature \( T \), determined via fluctuation-dissipation relations as equation (1), turns out to be actually independent of the specific quantities which response and correlation functions refer to. This property should also characterise any bona fide effective temperature \( T_{\text{eff}} \). For Model A dynamics of the \( O(n) \) Landau-Ginzburg Hamiltonian this issue has been addressed by computing, up to one loop, \( X^\infty \) associated with quantities that are quadratic in the order parameter [25]. It turns out that: (a) within the Gaussian model (i.e., \( u = 0 \)) \( X^\infty \) is indeed independent of the specific quantity used to compute it (b) this is no longer the case when the contribution of non-Gaussian fluctuations is included (\( u \neq 0 \)). Therefore \( X^\infty \) is not a good candidate for an effective temperature in non-equilibrium critical dynamics (see also [20, 26]), is spite of some suggestion coming from MC simulations [16].

5.2. What should be done:

- Awaiting experimental investigation of ageing and slow dynamics at critical points it would be very useful to consider more realistic models of dynamics such as, e.g., Models J and G for actual magnetic materials, or Model H for the liquid-vapour critical point.
• Most of the results previously mentioned have been obtained within the field-theoretical $\varepsilon$-expansion. Of course these findings can be improved by considering higher-order computations within the same scheme. On the other hand, it is also worth using other field-theoretical methods (e.g., non-perturbative ones) to get analytical estimates of the interesting universal quantities. Moreover, simple but important models like Model B surely deserve further investigation given the rich phenomenology that they display (see, e.g., [27]).

• The presence of surfaces in a system alters locally its static and dynamic critical behaviour and in particular, depending on some gross features of the surface, different surface universality classes emerge from the same bulk universality class [28]. Accordingly, one expects that also ageing properties (e.g., $X^\infty$ and exponents) at the surface will be affected, differing from those observed in a bulk system. MC simulations of the two- and three-dimensional Ising model with Glauber dynamics [29] and exact computation for the spherical model with dissipative dynamics [30] confirm this expectation. The field-theoretical methods have been successfully applied to study the effect of surfaces on critical behaviour. It would be interesting to extend such an analysis to ageing properties.

• The relaxation process discussed in section 2 is the result of a quench from the high-temperature phase. Alternatively, one can think of preparing the system in an equilibrium configuration corresponding to the low-temperature phase and then suddenly change the temperature of the thermal bath. This case has been recently considered in [31, 32, 26] for the purely dissipative dynamics of the spherical model, of some mean-field models, and of the $O(n>2)$ Landau-Ginzburg model in the dimensional expansion around two dimensions. In spite of these results the ageing properties of such a simple model as the Ising model when it is quenched from the ordered state to its critical point have to be fully investigated. The field-theoretical approach is expected to provide insight into this problem.

Acknowledgments
I thank Pasquale Calabrese for a careful reading of the manuscript and for the ongoing stimulating collaboration.

[1] Struik L C E 1978 Physical Aging in Amorphous Polymers and Other Materials (Amsterdam: Elsevier)
[2] Vincent E, Hammann J, Ocio M, Bouchaud J P and Cugliandolo L F 1997 Lect. Notes Phys. 492 184 (Preprint cond-mat/9607224)
Bochaud J P, Cugliandolo L F, Kurchan J and Mészár M 1998 in Spin Glasses and Random Fields, Directions in Condensed Matter Physics, vol 12 ed A P Young (Singapore: World Scientific) (Preprint cond-mat/9702070)
[3] Ageing and the Glass Transition (Lecture Notes in Physics vol ??) 2006 eds M Henkel M Pleimling and R Sanctuary (????: Springer)
[4] Cugliandolo L F, Kurchan J and Parisi G 1994 J. Phys. I (France) 4 1641 (Preprint cond-mat/9406053)
[5] Janssen H K, Schaub B and Schmiedmann B 1989 Z. Phys. B 73 539
Janssen H K 1992 in From Phase Transitions to Chaos – Topics in Modern Statistical Physics ed G Györgyi et al (Singapore: World Scientific)
[6] Hohenberg P C and Halperin B I 1977 Rev. Mod. Phys. 49 435
[7] Calabrese P and Gambassi A 2005 J. Phys. A: Math. Gen. 38 R133 (Preprint cond-mat/0410357)
[8] Täufer U C 2006 in [3] (Preprint cond-mat/0511743)
[9] Bray A J 1994 Adv. Phys. 43 357 (Preprint cond-mat/9501089) reprinted as 2002 Adv. Phys. 51 481
Gunton J D, San Miguel M and Sahni P S, 1983 in Phase Transitions and Critical Phenomena vol 8 ed C Domb and J L Lebowitz (London: Academic Press)
[10] Cugliandolo L F 2003 Slow relaxation and nonequilibrium dynamics in condensed matter Les Houches, École d’Été de Physique Théorique vol 77 ed J-L Barrat et al (Berlin: Springer) p 371 (Preprint cond-mat/0210312)
[11] Cugliandolo L F and Kurchan J 1993 Phys. Rev. Lett. 71 173 (Preprint cond-mat/9303036)
Cugliandolo L F and Kurchan J 1993 J. Phys. A: Math. Gen. 27 5749 (Preprint cond-mat/9311016)
Cugliandolo L F and Kurchan J 1995 Phil. Mag. B 71 50 (Preprint cond-mat/9403040)
[12] Cugliandolo L F, Kurchan J and Peliti L 1997 Phys. Rev. E 55 3898 (Preprint cond-mat/9611044)
[13] Godrèche C and Luck J M 2002 J. Phys.: Condens. Matter 14 1589 (Preprint cond-mat/0109212)
[14] Godrèche C and Luck J M 2000 J. Phys. A: Math. Gen. 33 9141 (Preprint cond-mat/0001264)
[15] Lippiello E and Zannetti M 2000 Phys. Rev. E 61 3369 (Preprint cond-mat/0001103)
[16] Mayer P, Berthier L, Garrahan J P and Sollich P 2003 Phys. Rev. E 68 016116 (Preprint cond-mat/0301493)
[17] Chatelain C 2003 J. Phys. A: Math. Gen. 36 10739 (Preprint cond-mat/0303545)
[18] Chatelain C 2004 J. Stat. Mech.: Theor. Exp. P06006 (Preprint cond-mat/0404017)
[19] Calabrese P and Gambassi A 2002 Phys. Rev. E 65 066120 (Preprint cond-mat/0203096)
    Calabrese P and Gambassi A 2002 Acta Phys. Slov. 52 311
[20] Schehr G and Paul R 2005 Phys. Rev. E 72 016105 (Preprint cond-mat/0412447)
    Schehr G and Paul R 2006 Proc. of the Int. Summer School “Ageing and the Glass Transition” to appear in
    J. Phys.: Conf. Series (Preprint cond-mat/0511571)
[21] Calabrese P and Gambassi A 2002 Phys. Rev. E 66 066101 (Preprint cond-mat/0207452)
[22] Majumdar S N, Huse D A and Lubachevsky B D 1994 Phys. Rev. Lett. 73 182
    Alexander F J, Huse D A and Janowsky S A 1994 Phys. Rev. B 50 663
    Majumdar S N and Huse D A 1995 Phys. Rev. E 52 270
    Godrèche C, Krzakala F and Ricci-Tersenghi F 2004 J. Stat. Mech.: Theor. Exp. P04007 (Preprint cond-mat/0401334)
    Sire C 2004 Phys. Rev. Lett. 93 130602 (Preprint cond-mat/0406333)
[23] Calabrese P and Gambassi A 2003 Phys. Rev. E 67 036111 (Preprint cond-mat/0211062)
[24] Calabrese P and Gambassi A 2002 Phys. Rev. B 66 212407 (Preprint cond-mat/0207487)
[25] Calabrese P and Gambassi A 2004 J. Stat. Mech.: Theor. Exp. P07013 (Preprint cond-mat/0406289)
[26] Annibale A and Sollich P 2005 Preprint cond-mat/0510731
[27] Krzakala F and Ricci-Tersenghi F 2006 Proc. of the Int. Summer School “Ageing and the Glass Transition”
    to appear in J. Phys.: Conf. Series (Preprint cond-mat/0512309)
[28] Diehl H W 1986 in Phase Transitions and Critical Phenomena vol 10 ed C Domb and J L Lebowitz (London: Academic)
    Pleimling M 2004 J. Phys. A: Math. Gen. 37 R79 (Preprint cond-mat/0402574)
[29] Pleimling M 2004 Phys. Rev. B 70 104401 (Preprint cond-mat/0404203)
[30] Baumann F and Pleimling M 2005 Preprint cond-mat/0509064
[31] Fedorenko A A and Trumper S 2005 Preprint cond-mat/0507112
[32] Garriga A, Sollich P, Pagonabarraga I and Ritort F 2005 Preprint cond-mat/0508243