TORELLI THEOREM FOR MODULI SPACES OF SL(\(r, \mathbb{C}\))–CONNECTIONS ON A COMPACT RIEMANN SURFACE

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Abstract. Let \(X\) be any compact connected Riemann surface of genus \(g\), with \(g \geq 3\). For any \(r \geq 2\), let \(\mathcal{M}_X\) denote the moduli space of holomorphic SL(\(r, \mathbb{C}\))–connections over \(X\). It is known that the biholomorphism class of the complex variety \(\mathcal{M}_X\) is independent of the complex structure of \(X\). If \(g = 3\), then we assume that \(r \geq 3\). We prove that the isomorphism class of the variety \(\mathcal{M}_X\) determines the Riemann surface \(X\) uniquely up to an isomorphism. A similar result is proved for the moduli space of holomorphic GL(\(r, \mathbb{C}\))–connections on \(X\).

We also show that the Torelli theorem remains valid for the moduli spaces of connections, as well as those of stable vector bundles, on geometrically irreducible smooth projective curves defined over the field of real numbers.

1. Introduction

Let \(X\) be a compact connected Riemann surface of genus \(g\), with \(g \geq 3\). Fix an integer \(r \geq 2\). If \(g = 3\), then we will assume that \(r \geq 3\).

Let \(\mathcal{M}_X\) denote the moduli space parametrizing pairs of the form \((E, D)\) on \(X\) satisfying the following two conditions:

- \(E\) is a holomorphic vector bundle of rank \(r\) with determinant \(\wedge^r E \cong \mathcal{O}_X\), and
- \(D\) is a holomorphic connection on \(E\) such that the connection on \(\wedge^r E\) induced by \(D\) coincides with the trivial connection on \(\mathcal{O}_X\) given by the de Rham differential.

In other words, \(\mathcal{M}_X\) is the moduli space of holomorphic SL(\(r, \mathbb{C}\))–connections on \(X\). This moduli space \(\mathcal{M}_X\) is an irreducible normal quasiprojective variety, defined over \(\mathbb{C}\), of dimension \(2(r^2 - 1)(g - 1)\).

We prove the following Torelli type theorem:

**Theorem 1.1.** The isomorphism class of the Riemann surface \(X\) is uniquely determined by the isomorphism class of the variety \(\mathcal{M}_X\). In other words, if \(Y\) is another compact connected Riemann surface of genus \(g\), and \(\mathcal{M}_Y\) is the moduli space of holomorphic SL(\(r, \mathbb{C}\))–connections on \(Y\), then the two varieties \(\mathcal{M}_X\) and \(\mathcal{M}_Y\) are isomorphic if and only if the two Riemann surfaces \(X\) and \(Y\) are isomorphic.
Let \( \hat{\mathcal{M}}_X \) denote the moduli space of holomorphic \( \text{GL}(r, \mathbb{C}) \)-connections on \( X \). This moduli space \( \hat{\mathcal{M}}_X \) is an irreducible normal quasiprojective variety, defined over \( \mathbb{C} \), of dimension \( 2(r^2(g - 1) + 1) \).

We prove the following analog of Theorem 1.1:

**Theorem 1.2.** The isomorphism class of the Riemann surface \( X \) is uniquely determined by the isomorphism class of the variety \( \hat{\mathcal{M}}_X \).

The biholomorphism classes of the varieties \( \mathcal{M}_X \) and \( \hat{\mathcal{M}}_X \) depend only on \( g \) and \( r \). In particular, they are independent of the complex structure of \( X \) (see Proposition 2.4).

In [BM], similar Torelli type theorems were proved for some moduli spaces of logarithmic connections on \( X \) singular over a fixed point \( x_0 \in X \). More precisely, for \( r > 1 \) and \( d \) coprime to it, it was shown that the isomorphism class of each of the following two varieties determine \( X \) uniquely up to an isomorphism:

- The moduli space of logarithmic connections \((E, D)\) on \( X \), where \( E \) is a holomorphic vector bundle of rank \( r \) and degree \( d \) with \( \Lambda^r E \cong \mathcal{O}_X(dx_0) \), and \( D \) is a logarithmic connection on \( E \), singular exactly over \( x_0 \) with residue \(-\frac{4}{r} \text{Id}_{E_{x_0}}\), inducing the logarithmic connection on \( \mathcal{O}_X(dx_0) \) given by the de Rham differential.
- The moduli space of logarithmic connections \((E, D)\), where \( E \) is a holomorphic vector bundle of rank \( r \) and degree \( d \), and \( D \) is a logarithmic connection on \( E \) singular exactly over \( x_0 \) with residue \(-\frac{d}{r} \text{Id}_{E_{x_0}}\).

Both these moduli spaces have the advantage of being smooth quasiprojective varieties. On the other hand, the moduli spaces \( \mathcal{M}_X \) and \( \hat{\mathcal{M}}_X \) considered here are singular, which necessitates new inputs in the proofs. The geometric significance of \( \mathcal{M}_X \) and \( \hat{\mathcal{M}}_X \) makes them worth studying regardless of being singular.

In Section 6 we address Torelli type questions for moduli spaces of objects over smooth geometrically irreducible projective curves defined over the field of real numbers.

## 2. Preliminaries

Let \( X \) be a compact connected Riemann surface or, equivalently, an irreducible smooth projective curve defined over the field of complex numbers. The genus of \( X \) will be denoted by \( g \).

The Hodge type \((1, 0)\) cotangent bundle \((T^{1,0}X)^*\) will also be denoted by \( \Omega^{1,0}_X \), and the holomorphic line bundle on \( X \) defined by the sheaf holomorphic sections of \( \Omega^{1,0}_X \) will be denoted by \( K_X \). The \( C^\infty \) complex line bundle \((T^{0,1}X)^*\) will be denoted by \( \Omega^{0,1}_X \). The trivial holomorphic line bundle \( X \times \mathbb{C} \) over \( X \) will also be denoted by \( \mathcal{O}_X \).

**Definition 2.1.** A **holomorphic connection** on a holomorphic vector bundle \( E \) over \( X \) is a first order holomorphic differential operator

\[
\mathcal{D} : E \longrightarrow E \otimes K_X
\]
satisfying the Leibniz identity which says that $D(fs) = fD(s) + df \otimes s$, where $f$ (respectively, $s$) is a locally defined holomorphic function (respectively, holomorphic section of $E$).

Since a Riemann surface does not have nonzero $(2,0)$–forms, a holomorphic connection on a Riemann surface is automatically flat. If $D$ is a holomorphic connection on $E$, then $D + \overline{\partial}_E$ is a flat connection on $E$, where

$$\overline{\partial}_E : E \to E \otimes \Omega^{0,1}_X$$

is the Dolbeault operator for the holomorphic structure on $E$. Conversely, if $\nabla$ is a flat connection on a $C^\infty$ vector bundle $E$ over $X$, then the $(1,0)$–component $\nabla^{1,0}$ of $\nabla$ is a holomorphic connection with respect to the holomorphic structure on $E$ defined by the $(0,1)$–component $\nabla^{0,1}$. In particular, if a holomorphic vector bundle $V$ over $X$ admits a homomorphic connection, then degree$(V) = 0$.

**Remark 2.2.** A theorem due to Atiyah and Weil says that a holomorphic vector bundle $E$ over $X$ admits a holomorphic connection if and only if each indecomposable holomorphic direct summand of $E$ is of degree zero [At, p. 203, Theorem 10], [We1]. Let $E$ be a holomorphic vector bundle over $X$ that admits a holomorphic connection. If $D$ is a holomorphic connection on $E$ such that

$$\theta \in H^0(X, \text{End}(E) \otimes K_X),$$

then the differential operator $D + \theta : E \to E \otimes K_X$ is also a holomorphic connection on $E$; here $\text{End}(E) = E \otimes_{\mathcal{O}_X} E^*$. This operation defines an action of $H^0(X, \text{End}(E) \otimes K_X)$ on the space of all holomorphic connections on $E$. It is easy to see that this way the space of all holomorphic connections on $E$ is an affine space for the vector space $H^0(X, \text{End}(E) \otimes K_X)$.

As before, let $E$ be a holomorphic vector bundle over $X$ that admits a holomorphic connection, and let $r$ be its rank. Fix a holomorphic connection $D_0$ on the determinant line bundle $\wedge^r E$. Consider the space of all holomorphic connections on $E$ that induce the fixed connection $D_0$ on $\wedge^r E$. We will show that this space is nonempty. For this purpose, fix any holomorphic connection $D_E$ on $E$. Let $D'_E$ be the holomorphic connection on $\wedge^r E$ induced by $D_E$. So

$$\omega := D' - D_0 \in H^0(X, K_X).$$

Consider the holomorphic connection

$$D'_E := D_E - \text{Id}_E \otimes \frac{\omega}{r}$$

on $E$. The holomorphic connection on $\wedge^r E$ induced by $D'_E$ clearly coincides with $D_0$.

Given a holomorphic connection $D$ on $E$ that induces $E_0$ on $\wedge^r E$, for any

$$\theta \in H^0(X, \text{ad}(E) \otimes K_X),$$

where $\text{ad}(E) \subset \text{End}(E)$ is the subbundle of corank one given by the sheaf of trace zero endomorphisms, the holomorphic connection on $E$ given by the differential operator $D + \theta$
has the property that the induced connection on $\bigwedge^r E$ coincides with $D_0$. Conversely, if $D$ and $D'$ are two holomorphic connections on $E$ inducing the connection $D_0$ on $\bigwedge^r E$, then

$$D' - D \in H^0(X, \text{ad}(E) \otimes K_X).$$

Therefore, the space of all holomorphic connections on $E$ that induce the fixed connection $D_0$ on $\bigwedge^r E$ is an affine space for the vector space $H^0(X, \text{ad}(E) \otimes K_X)$. □

Given a holomorphic connection $(E, D)$ on $X$, a holomorphic subbundle $F$ of $E$ is said to be left invariant by $D$ if the differential operator $D$ sends any locally defined holomorphic section of $F$ to a section of $F \otimes K_X$. A holomorphic connection $(E, D)$ is called reducible if there is a holomorphic subbundle $F$ of $E$ with $1 \leq \text{rank}(F) < \text{rank}(E)$ which is left invariant by $D$. A holomorphic connection $(E, D)$ is called irreducible if it is not reducible.

**Remark 2.3.** Let $E$ be a holomorphic vector bundle over $X$ equipped with a holomorphic connection $D$. If a subbundle $F \subset E$ is left invariant by $D$, then $F$ admits a holomorphic connection. In that case we have

$$\text{degree}(E) = \text{degree}(F) = 0.$$  

Therefore, any holomorphic connection on $X$ is semistable [Si2, p. 88]. Furthermore, a holomorphic connection $D$ is stable if and only if it is irreducible [Si2, p. 88]. Two holomorphic connections on $X$ are called Jordan–Hölder equivalent if their semisimplifications are isomorphic [Si2, p. 90]. Therefore, any holomorphic connection is Jordan–Hölder equivalent to a unique, up to an isomorphism, direct sum of irreducible connections. A direct sum of irreducible connections is also called a polystable connection. □

Let $D_0$ denote the connection on the trivial line bundle $\mathcal{O}_X$ defined by the de Rham differential that sends any locally defined holomorphic function $f$ on $X$ to the holomorphic one–form $df$. For any integer $r \geq 1$, let $\mathcal{M}_X$ denote the moduli space of pairs $(E, D)$ of the following type:

- $E$ is a holomorphic vector bundle of rank $r$ over $X$ with $\bigwedge^r E \cong \mathcal{O}_X$,
- $D$ is a holomorphic connection on $E$, and
- the flat connection on $\bigwedge^r E$ induced by $D$ has trivial monodromy.

The last condition on $D$ is equivalent to the following condition: the connection on $\mathcal{O}_X$ given by $D$ coincides with $D_0$. We note that using an isomorphism $\bigwedge^r E \to \mathcal{O}_X$, a connection on $\bigwedge^r E$ gives a connection on $\mathcal{O}_X$ which is independent of the choice of the isomorphism between $\bigwedge^r E$ and $\mathcal{O}_X$.

Therefore, $\mathcal{M}_X$ is the moduli space of holomorphic $\text{SL}(r, \mathbb{C})$–connections on $X$. See [Si1], [Si2] for the construction of the moduli space $\mathcal{M}_X$. The scheme $\mathcal{M}_X$ is a reduced and irreducible normal quasiprojective variety defined over $\mathbb{C}$, and its (complex) dimension is $2(r^2 - 1)(g - 1)$, where $g$ is the genus of $X$ [Si2, p. 70, Theorem 11.1]. The closed points of the moduli space $\mathcal{M}_X$ are in bijection with all Jordan–Hölder equivalence classes of
holomorphic connections that satisfy the above three conditions (see Remark 2.3 for the definition of the Jordan–Hölder equivalence).

Let \( \hat{M}_X \) denote the moduli space of \( \text{GL}(r, \mathbb{C}) \)-connections on \( X \). Therefore, \( \hat{M}_X \) parametrizes Jordan–Hölder equivalence classes of holomorphic connections of rank \( r \). The moduli space \( \hat{M}_X \) is an irreducible normal quasiprojective variety defined over \( \mathbb{C} \), and its (complex) dimension is \( 2r^2(g - 1) + 2 \) (see [Si2, p. 70, Theorem 11.1]). Both \( M_X \) and \( \hat{M}_X \) are singular varieties.

**Proposition 2.4.** The biholomorphism class of \( M_X \) is independent of the complex structure of \( X \); it depends only on the integers \( g \) and \( r \). The same statement holds for \( \hat{M}_X \).

**Proof.** There is a canonical biholomorphism

\[
M_X \xrightarrow{\sim} R_{g,r} := \text{Hom}(\pi_1(X), \text{SL}(r, \mathbb{C}))/\text{SL}(r, \mathbb{C}),
\]

which sends any flat connection to its monodromy [Si2, p. 26, Theorem 7.1]; in \( R_{g,r} \), two equivalence classes of homomorphisms from \( \pi_1(X) \) to \( \text{SL}(r, \mathbb{C}) \) are identified if and only if their semisimplifications are isomorphic. The biholomorphism class of the complex variety \( \text{Hom}(\pi_1(X), \text{SL}(r, \mathbb{C}))/\text{SL}(r, \mathbb{C}) \) depends only on the isomorphism class of the group \( \pi_1(X) \) and the integer \( r \).

Similarly, the monodromy map gives a biholomorphism

\[
\hat{M}_X \xrightarrow{\sim} \hat{R}_{g,r} := \text{Hom}(\pi_1(X), \text{GL}(r, \mathbb{C}))/\text{GL}(r, \mathbb{C})
\]

[Si2, p. 26, Theorem 7.1]; as before, in \( \hat{R}_{g,r} \), two equivalence classes of homomorphisms from \( \pi_1(X) \) to \( \text{GL}(r, \mathbb{C}) \) are identified if their semisimplifications are isomorphic. Again, the biholomorphism class of \( \text{Hom}(\pi_1(X), \text{GL}(r, \mathbb{C}))/\text{GL}(r, \mathbb{C}) \) depends only on the isomorphism class of the group \( \pi_1(X) \) and the integer \( r \). \( \square \)

3. The second intermediate Jacobian of the moduli space

We continue with the notation of the previous section. Henceforth, we will assume that \( g = \text{genus}(X) \geq 3 \), and \( r \geq 2 \). If \( g = 3 \), then we will assume that \( r \geq 3 \).

Let

\[
(3.1) \quad U \subset M_X
\]

be the Zariski open subset parametrizing all holomorphic connections \( (E, D) \) such that the underlying vector bundle \( E \) is stable. The openness of this subset follows from [Ma, p. 635, Theorem 2.8], [Sh, p. 182, Proposition 10]. Let \( N_X \) denote the moduli space parametrizing all isomorphism classes of stable vector bundles \( E \) over \( X \) with \( \text{rank}(E) = r \) and \( \text{det}(E) \cong O_X \). This moduli space \( N_X \) is an irreducible smooth quasiprojective variety, of dimension \( (r^2 - 1)(g - 1) \), defined over \( \mathbb{C} \). Let

\[
(3.2) \quad \Phi : U \longrightarrow N_X
\]

be the forgetful morphism that sends any pair \( (E, D) \) to \( E \).
Any holomorphic vector bundle $E \in \mathcal{N}_X$ admits a unitary flat connection \[\text{NS}\], hence the projection $\Phi$ in Eqn. (3.2) is surjective. Any holomorphic connection on a stable vector bundle is irreducible. Hence from Remark 2.2 it follows that $\Phi$ makes $\mathcal{U}$ an affine bundle over $\mathcal{N}_X$. More precisely, $\mathcal{U}$ is a torsor over $\mathcal{N}_X$ for the holomorphic cotangent bundle $T^*\mathcal{N}_X$. This means that the fibers of the vector bundle $T^*\mathcal{N}_X$ act freely transitively on the fibers of $\Phi$.

As $\mathcal{M}_X$ is irreducible, and $\mathcal{U}$ is nonempty, the open subset $\mathcal{U} \subset \mathcal{M}_X$ is Zariski dense.

Lemma 3.1. Let $\mathcal{Z} := \mathcal{M}_X \setminus \mathcal{U}$ be the complement of the Zariski open dense subset $\mathcal{U}$ of $\mathcal{M}_X$ in Eqn. (3.1). The codimension of the Zariski closed subset $\mathcal{Z}$ of $\mathcal{M}_X$ is at least three.

Proof. Let

$$Y \subset \mathcal{M}_X$$

be the Zariski closed subset of the moduli space that parametrizes holomorphic connections that are not stable. Therefore, $Y$ is the complement of the stable locus in $\mathcal{M}_X$ (see Remark 2.3). Given any holomorphic connection on $X$, there is a canonically associated polystable holomorphic connection which is unique up to an isomorphism; see Remark 2.3. Two holomorphic connections on $X$ give the same point in the moduli space $\mathcal{M}_X$ if and only if their Jordan–Hölder equivalence classes coincide.

We will first show that the codimension of the subset $Y$ in Eqn. (3.3) is at least three. For that purpose, given any integer $1 \leq \ell \leq r$, let $\mathcal{M}_X^\ell$ denote the moduli space of all $\text{SL}(\ell, \mathbb{C})$–connections on $X$. So $\mathcal{M}_X = \mathcal{M}_X^r$. The moduli space of all $\text{GL}(1, \mathbb{C})$–connections on $X$ will be denoted by $\widehat{\mathcal{M}}_X^1$.

For each $1 \leq \ell < r$, we have a canonical morphism

$$f_\ell : \mathcal{M}_X^\ell \times \mathcal{M}_X^{r-\ell} \times \widehat{\mathcal{M}}_X^1 \rightarrow \mathcal{M}_X$$

defined by

$$((E_1, D_1), (E_2, D_2), (L, D_3)) \mapsto ((E_1 \otimes L^{\otimes (r-\ell)}) \oplus (E_2 \otimes (L^*)^{\otimes \ell}), D),$$

where $D$ is the connection on $((E_1 \otimes L^{\otimes (r-\ell)}) \oplus (E_2 \otimes (L^*)^{\otimes \ell})$ induced by $D_1$, $D_2$ and $D_3$. Clearly,

$$\text{image}(f_\ell) \subset Y,$$

where $Y$ is the subset in Eqn. (3.3). Furthermore,

$$Y = \bigcup_{\ell=1}^{r-1} \text{image}(f_\ell).$$

(3.4)
We have \( \dim \mathcal{M}_X^\ell = 2(\ell^2 - 1)(g-1) \), and \( \dim \hat{\mathcal{M}}_X^1 = 2g \), where \( g = \text{genus}(X) \). Hence

\[
\dim \text{image}(f_\ell) \leq 2(g-1)(r^2 - 2r\ell + 2\ell^2 - 1) + 2 \\
\leq 2(g-1)(r^2 - 2r + 1) + 2 \\
= \dim \mathcal{M}_X + 2 - 4r(g-1) \\
\leq \dim \mathcal{M}_X - 14.
\]

Therefore, using Eqn. (3.4) we conclude that the codimension of the Zariski closed subset \( Y \) of \( \mathcal{M}_X \) is at least three.

Consider the complement \( U' = \mathcal{M}_X \setminus Y \). Then \( U \subset U' \), and \( U \) is Zariski open in \( U' \). Let

\[
(3.5) \quad Z' := U' \setminus U \subset U'
\]

be the Zariski closed subset. Since the codimension of \( Y \subset \mathcal{M}_X \) is at least three, to complete the proof of the lemma it suffices to show that the codimension of \( Z' \) in \( U' \) is at least three.

We will first show that any \((E, D) \in U'\) is simple, i.e., \( \dim \text{Aut}(E, D) = 1 \). For that purpose, consider the automorphism of the underlying vector bundle \( E \) given by an automorphism \( T \) of \((E, D)\), which will also be denoted by \( T \). Let \( \lambda \in \mathbb{C} \) be an eigenvalue of \( T(x) \) for some \( x \in X \). Then \( T' := T - \lambda \cdot \text{Id}_E \) is also an endomorphism of \((E, D)\). Since the endomorphism \( T' \) of \( E \) fails to be an isomorphism over the above point \( x \), it follows that \( F := \ker(T') \neq 0 \). Indeed, the coherent subsheaf of \( E \) generated by the parallel translations of the subspace \( \ker(T'(x)) \subset E_x \) is contained in \( \ker(T') \). Clearly \( F \) is left invariant by \( D \). On the other hand \( D \) is irreducible. Therefore, we have \( F = E \). This implies that \( T' = 0 \), and hence \( T = \lambda \cdot \text{Id}_E \).

Fix a holomorphic vector bundle \( E \) which admits an irreducible holomorphic connection. Consider the space \( \mathcal{D}_E \) consisting of all irreducible holomorphic connections \( D \) on \( E \) such that \((E, D) \in U'\). From the openness of the stability condition it follows that \( \mathcal{D}_E \) is a Zariski open dense subset of an affine space for the vector space \( H^0(X, \text{ad}(E) \otimes K_X) \) (see Remark 2.2). Let

\[
(3.6) \quad \varphi : \mathcal{D}_E \longrightarrow U'
\]

be the obvious tautological map.

We have shown above that for the natural action of the global automorphism group \( \text{Aut}(E) \) on \( \mathcal{D}_E \), the isotropy at any point of \( \mathcal{D}_E \) is the subgroup defined by all nonzero scalar multiplications. Therefore, for the map \( \varphi \) in Eqn. (3.6),

\[
\dim \varphi(\mathcal{D}_E) = \dim H^0(X, \text{ad}(E) \otimes K_X) - (\dim \text{Aut}(E) - 1) \\
= \dim H^1(X, \text{ad}(E)) - \dim H^0(X, \text{ad}(E)) = (r^2 - 1)(g-1),
\]

where the last equality is the Riemann–Roch formula, and the middle equality follows from the Serre duality. Hence, to prove that \( Z' \) defined in Eqn. (3.5) has codimension
at least three in $\mathcal{U}'$ it suffices to show that the family of non-stable holomorphic vector bundles admitting an irreducible holomorphic connection has dimension no more than $(r^2 - 1)(g - 1) - 3$.

Let $\mathcal{F}$ denote the family of non-stable vector bundles admitting an irreducible connection.

First take any $E \in \mathcal{F}$ which is not semistable. Let 

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_{\ell-1} \subset E_{\ell} = E$$

be the Harder–Narasimhan filtration of $E$. We recall that the collection of pairs of integers $\{(\text{rank}(E_i), \text{degree}(E_i))\}_{i=1}^\ell$ is called the *Harder–Narasimhan polygon of* $E$ (see [Sh, p. 173]). The space of all isomorphism classes of holomorphic vector bundles over $X$, whose Harder–Narasimhan polygon coincides with that of the given vector bundle $E$, is of dimension at most $r^2(g - 1) - (r - 1)(g - 2)$ (this follows from [Bh, p. 247–248]; see also [BM]). Therefore the locus $\mathcal{F}_1$ of non-semistable vector bundles in $\mathcal{F}$ has dimension at most $r^2(g - 1) - (r - 1)(g - 2) - g$. Also, only finitely many Harder–Narasimhan polygons occur in a bounded family of vector bundles over $X$ [Sh, p. 183, Proposition 11]. Therefore,

$$\dim \mathcal{F}_1 \leq r^2(g - 1) - (r - 1)(g - 2) - g \leq (r^2 - 1)(g - 1) - 3.$$

Finally, take any $E \in \mathcal{F}$ which is semistable but not stable. Consider a Jordan–Hölder filtration of $E$ given by

$$(3.7) \quad 0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_{\ell-1} \subset E_{\ell} = E,$$

where $Q_i := E_i/E_{i-1}$, $1 \leq i \leq \ell$, is a stable vector bundle of degree zero. The vector bundle

$$Q := \bigoplus_{i=1}^\ell Q_i$$

is the *graduation* of $E$. The isomorphism class of $Q$ depends only on $E$. The rank of $Q_i$ will be denoted by $r_i$.

The dimension of the moduli space of all stable vector bundles over $X$ of rank $r_i$ is $r_i^2(g - 1) + 1$. Therefore, the dimension of the space of all graduations which are direct sum of stable vector bundles of ranks $r_1, r_2, \cdots, r_\ell$ is

$$(3.9) \quad \ell - g + \sum_{i=1}^\ell r_i^2(g - 1)$$

(note that $\bigotimes_{i=1}^\ell \Lambda^{r_i} Q_i \cong \mathcal{O}_X$).

We will now calculate the dimension of the space of all semistable vector with a fixed graduation.

Let $\mathcal{S}$ be the space of all isomorphism classes of semistable vector bundles of degree zero whose Jordan–Hölder filtration has the fixed graduation $Q$ in Eqn. (3.8). We may assume that isomorphism classes of all the $Q_i$ are distinct. Indeed, if some $Q_i$, is assumed
to be isomorphic to some $Q_i$, then the dimension of all possible graduations of that type becomes smaller than the number in Eqn. (3.9). Using this it is easy to see that it suffices to consider the case where all $Q_i$ are distinct.

For $2 \leq j \leq \ell$, the dimension of the spaces of equivalence classes of extensions

$$0 \longrightarrow E_{j-1} \longrightarrow F \longrightarrow Q_j \longrightarrow 0,$$

is

$$h^1(Q_j^* \otimes E_{j-1}) - 1 = r_j \left( \sum_{k<j} r_k \right) (g - 1) - 1;$$

this follows from the Riemann-Roch, the condition that degree($Q_i$) = 0 for all $i$, and the assumption that all the $Q_i$ are distinct. Therefore,

$$\dim S \leq -\ell + \sum_{i=1}^{\ell} \sum_{k<i} r_i r_k (g - 1). \tag{3.10}$$

Combining Eqn. (3.9) with Eqn. (3.10) we conclude that the dimension of the space of all semistable vector bundles having graduation of numerical type \{ $r_1, r_2, \cdots, r_\ell$ \} is maximum when $\ell = 2$ and \{ $r_1, r_2$ \} = \{ 1, r - 1 \}. Combining Eqn. (3.9) and Eqn. (3.10), the dimension of all semistable vector bundles with graduation type \{ 1, r - 1 \} is at most

$$2 - g + ((r - 1)^2 + 1)(g - 1) + (r - 1)(g - 1) - 1 = (r^2 - r)(g - 1) \leq (r^2 - 1)(g - 1) - 3.$$

This completes the proof of the lemma. \hfill \Box

For any complex algebraic variety $Y$, the torsion-free part $H^i(Y, \mathbb{Z})/\text{Torsion}$ is equipped with a mixed Hodge structure for all $i \geq 0$ [De1], [De2].

**Proposition 3.2.** Let $M_X^0$ be the smooth locus of $M_X$. Then the mixed Hodge structure $H^3(M_X^0, \mathbb{Z})/\text{Torsion}$ is isomorphic to the mixed Hodge structure $H^3(N_X, \mathbb{Z})/\text{Torsion}$, where $N_X$ is the moduli space in Eqn. (3.2).

**Proof.** Consider the diagram of morphisms

$$N_X \xleftarrow{\Phi} U \xrightarrow{\iota} M_X^0, \tag{3.11}$$

where $\Phi$ is the projection in Eqn. (3.2), and $\iota$ is the inclusion map. Since $U \xrightarrow{\Phi} N_X$ is a fiber bundle with contractible fibers, the induced homomorphism

$$\Phi^* : H^i(N_X, \mathbb{Z}) \longrightarrow H^i(U, \mathbb{Z}) \tag{3.12}$$

is an isomorphism for all $i \geq 0$. Therefore, $\Phi$ induces an isomorphism of the two mixed Hodge structures $H^i(N_X, \mathbb{Z})/\text{Torsion}$ and $H^i(U, \mathbb{Z})/\text{Torsion}$.

Let

$$H^3(M_X^0, U, \mathbb{Z}) \longrightarrow H^3(M_X^0, \mathbb{Z}) \xrightarrow{\iota^*} H^3(U, \mathbb{Z}) \longrightarrow H^4(M_X^0, U, \mathbb{Z}) \tag{3.13}$$
be the long exact sequence of relative cohomologies. We note that Eqn. (3.13) is an exact sequence of mixed Hodge structures \cite{De2} p. 43, Proposition (8.3.9)]. From Lemma 3.1 (and because $\mathcal{M}_X^\circ$ is smooth) we know that

\begin{equation}
H^i(\mathcal{M}_X^\circ, U, \mathbb{Z}) = 0
\end{equation}

for all $i \leq 4$. Therefore, the homomorphism $\iota^*$ in Eqn. (3.13) is an isomorphism. Consequently, the composition homomorphism

\[ (\iota^*)^{-1} \circ \Phi^* : H^3(\mathcal{N}_X, \mathbb{Z}) \longrightarrow H^3(\mathcal{M}_X^\circ, \mathbb{Z}), \]

where $\Phi^*$ is constructed in Eqn. (3.12), is the required isomorphism in the statement of the proposition. □

Intermediate Jacobians for mixed Hodge structures was introduced in \cite{Ca} (see \cite{Ca} p. 110). Let

\[ J^2(\mathcal{M}_X^\circ) := H^3(\mathcal{M}_X^\circ, \mathbb{C})/(F^2 H^3(\mathcal{M}_X^\circ, \mathbb{C}) + H^3(\mathcal{M}_X^\circ, \mathbb{Z})) \]

be the intermediate Jacobian of the mixed Hodge structure $H^3(\mathcal{M}_X^\circ)$. The intermediate Jacobian of any mixed Hodge structure is a generalized torus \cite{Ca} p. 111.

**Proposition 3.3.** The intermediate Jacobian $J^2(\mathcal{M}_X^\circ)$ is isomorphic to $J^2(\mathcal{N}_X)$, which is isomorphic to the Jacobian $\text{Pic}^0(X)$ of the Riemann surface $X$.

Given a smooth family $X_T \longrightarrow T$ of irreducible smooth projective curves of genus $g$, let $\mathcal{M}_T$ be the corresponding relative family of moduli spaces of holomorphic $\text{SL}(r, \mathbb{C})$-connections, and let $\mathcal{M}_T^\circ$ be the family consisting of the smooth locus of these moduli spaces. Then the relative family of Jacobians over $T$ for the family $X_T$ is isomorphic to the relative family of second intermediate Jacobians for $\mathcal{M}_T^\circ$.

**Proof.** There is a natural isomorphism of $J^2(\mathcal{N}_X)$ with $\text{Pic}^0(X)$ \cite{AS} p. 2, Theorem 1.0.2(b)]. Therefore, the first part of the proposition follows from Proposition 3.2.

To prove the second part we note that the construction of the isomorphism of $J^2(\mathcal{N}_X)$ with $\text{Pic}^0(X)$ in \cite{AS} p. 2, Theorem 1.0.2(b)] works over a family of curves. Also, the construction of the isomorphism in Proposition 3.2 evidently works over a family of curves. This completes the proof of the proposition. □

In the next section we will investigate the cohomology ring of $\mathcal{M}_X^\circ$.

### 4. Cohomology of moduli spaces

We start with a proposition.

**Proposition 4.1.** $H^2(\mathcal{M}_X^\circ, \mathbb{Z}) = \mathbb{Z}$.

**Proof.** Consider the long exact sequence of relative cohomologies

\[ H^2(\mathcal{M}_X^\circ, U, \mathbb{Z}) \longrightarrow H^2(\mathcal{M}_X^\circ, \mathbb{Z}) \xrightarrow{\iota^*} H^2(U, \mathbb{Z}) \longrightarrow H^3(\mathcal{M}_X^\circ, U, \mathbb{Z}) \]
given by \( \iota \) in Eqn. (3.11). Using Eqn. (3.14), from it we conclude that
\[ H^2(\mathcal{M}_X, \mathbb{Z}) = H^2(\mathcal{U}, \mathbb{Z}). \]
Hence, in view of Eqn. (3.12), to prove the proposition it suffices to show that
\[ H^2(\mathcal{N}_X, \mathcal{O}_{\mathcal{N}_X}) = \mathbb{Z}. \]

Let \( \text{NS}(\mathcal{N}_X) \) be the Néron–Severi group of \( \text{NS}(\mathcal{N}_X) \). Let
\[ \text{Br}'(\mathcal{N}_X) := H^2_{\text{ét}}(\mathcal{N}_X, \mathbb{G}_m) = H^2_{\text{ét}}(\mathcal{N}_X, \mathcal{O}_{\mathcal{N}_X}) \]
be the cohomological Brauer group. It is known that \( \text{NS}(\mathcal{N}_X) = \text{Pic}(\mathcal{N}_X) = \mathbb{Z} \) (see [DN, p. 55, Théorème B]). Therefore, using the exact sequence in [Gr, p. 145, (8.7)] it can be deduced that Eqn. (4.1) holds provided \( \text{Br}'(\mathcal{N}_X) \) is a finite group. More precisely, from [Gr, p. 145, (8.7)] we know that if \( \text{Br}'(\mathcal{N}_X) \) is a finite group, then
\[ H^2(\mathcal{N}_X, \mathbb{Z}/p\mathbb{Z}) = \text{NS}(\mathcal{N}_X) \otimes \mathbb{Z}/p\mathbb{Z} = \mathbb{Z}/p\mathbb{Z} \]
for all but finitely many primes \( p \). If \( H^2(\mathcal{N}_X, \mathbb{Z}/p\mathbb{Z}) = \text{NS}(\mathcal{N}_X) \otimes \mathbb{Z}/p\mathbb{Z} = \mathbb{Z}/p\mathbb{Z} \) for all but finitely many primes \( p \), then it is straightforward to deduce that
\[ H^2(\mathcal{N}_X, \mathbb{Z}) = \text{NS}(\mathcal{N}_X) = \mathbb{Z}. \]
Therefore, Eqn. (4.1) holds provided \( \text{Br}'(\mathcal{N}_X) \) is a finite group.

We will complete the proof of the proposition by showing that \( \text{Br}'(\mathcal{N}_X) \) is a finite group.

Although there is no universal vector bundle over \( X \times \mathcal{N}_X \), there is a canonical universal projective bundle
\[ f' : \hat{\mathbb{P}} \longrightarrow X \times \mathcal{N}_X \]
(see [BBNN, p. 6, Theorem 2.7]). Fix a point \( x_0 \in X \). Let
\[ f : \hat{\mathbb{P}}_{x_0} \longrightarrow \{x_0\} \times \mathcal{N}_X = \mathcal{N}_X \]
be the restriction of \( f' \). There is a Zariski open dense subset
\[ \mathcal{U}' \subset \hat{\mathbb{P}}_{x_0} \]
such that there is a universal vector bundle
\[ \mathcal{E} \longrightarrow X \times \mathcal{U}' \subset X \times \hat{\mathbb{P}}_{x_0} \]
satisfying the condition that for each point \( t \in \mathcal{U}' \), the restriction of \( \mathcal{E} \) to \( X \times \{t\} \) is in the isomorphism class of stable vector bundles represented by the point \( f(t) \in \mathcal{N}_X \). To construct \( \mathcal{E} \), consider the rational map
\[ \beta : \hat{\mathbb{P}}_{x_0} \longrightarrow \mathcal{N}_X(-x_0) \]
defined by Hecke transformation, where \( \mathcal{N}_X(-x_0) \) is the moduli space of stable vector bundles \( V \) over \( X \) of rank \( r \) with \( \bigwedge^r V = \mathcal{O}_X(-x_0) \). The open subset \( \mathcal{U}' \) in Eqn. (4.4) is contained in the domain of \( \beta \). Let \( \mathcal{V} \) denote the universal vector bundle over \( X \times \mathcal{N}_X(-x_0) \). The vector bundle \( \mathcal{E} \) in Eqn. (4.5) is obtained using the tautological Hecke transformation on \((\text{Id}_X \times \beta)^* \mathcal{V}\).
Let
\[ f^* : \text{Br}'(N_X) \longrightarrow \text{Br}'(\hat{P}_{x_0}) \]
be the pull back homomorphism for the projection \( f \) in Eqn. (4.3). The relative Picard group \( \text{Pic}(\hat{P}_{x_0}/N_X) \) for the projection \( f \) in Eqn. (4.3) is isomorphic to \( \mathbb{Z} \). The relative anticanonical line bundle for the projection \( f \) is nonzero in \( \text{Pic}(\hat{P}_{x_0}/N_X) \), and it restricts to \( \mathcal{O}(r) \) on each fiber of \( f \). Therefore, from the exact sequence in [Gr, p. 127, (5.9)] we conclude that the kernel of the homomorphism \( f^* \) in Eqn. (4.6) is a finite group. Consequently, in order to prove that \( \text{Br}'(N_X) \) is a finite group it suffices to show that
\[ f^* = 0. \]

There is a Zariski open subset
\[ (4.8) \quad \mathbb{P}^s_0 \subset \mathbb{P}^s_0 \]
of a complex projective space (following the notation of [DN], the superscript “s” stands for “stable”, and \( \mathbb{P}^s_0 \) is a complex projective space) together with a surjective morphism
\[ (4.9) \quad \phi : \mathbb{P}^s_0 \longrightarrow N_X \]
satisfying the following condition: If \( \mathcal{F} \longrightarrow X \times T \) is a family, parametrized by \( T \), of stable vector bundles on \( X \) of rank \( r \) and trivial determinant, then the corresponding classifying morphism
\[ T \longrightarrow N_X \]
lifts Zariski locally (in \( T \)) to maps to \( \mathbb{P}^s_0 \). (See [DN, p. 81, Sec. 7.2]; the variety \( \mathbb{P}^s_0 \) is defined in [DN, p. 86, Remarques 1].) In particular, for the vector bundle \( \mathcal{E} \) in Eqn. (4.5) we have a commutative diagram
\[ (4.10) \]
where \( U \) is a Zariski open dense subset of the variety \( U' \) in Eqn. (4.4), and \( \phi \) is the map in Eqn. (4.9).

We know that \( \text{Pic}(\mathbb{P}^s_0) = \mathbb{Z} \) (see [DN, p. 89, Proposition 7.13]). Therefore, the complement \( \mathbb{P}^s_0 \setminus \mathbb{P}^s_0 \) in Eqn. (4.8) is of codimension at least two. Consequently, from [Gr, p. 136, Corollaire (6.2)],
\[ (4.11) \quad \text{Br}'(\mathbb{P}^s_0) = \text{Br}'(\mathbb{P}^s_0) = 0. \]

Since \( U \) is a nonempty Zariski open subset of \( \hat{P}_{x_0} \), the pull back homomorphism
\[ \iota^* : \text{Br}'(\hat{P}_{x_0}) \longrightarrow \text{Br}'(U) \]
is injective, where \( \iota \) in the inclusion map in Eqn. (4.10); see [Gr, p. 136, Corollaire (6.2)].

Consider the diagram of Brauer groups associated to the diagram in Eqn. (4.10). In this diagram, the homomorphism \( \iota^* \) is injective, and Eqn. (4.11) holds. Therefore, we conclude that Eqn. (4.7) holds. This completes the proof of the proposition. \qed
Remark 4.2. We note that Eqn. (4.1) also follows from [Mu, Theorem 6.5].

By Lemma 3.1 and the fact that $\mathcal{U} \subset \mathcal{M}_X^o \subset \mathcal{M}_X$, we know that

$$\text{Pic}(\mathcal{M}_X) = \text{Pic}(\mathcal{M}_X^o) = \text{Pic}(\mathcal{U}) = \text{Pic}(\mathcal{N}_X) = \mathbb{Z}.$$ 

By Proposition 4.1, we know that $H^2(\mathcal{M}_X, \mathbb{Z}) = \text{Pic}(\mathcal{M}_X^o)$. There is a well-defined element

$$\gamma \in H^2(\mathcal{M}_X, \mathbb{Z})$$

which is the image of 1 under

$$\mathbb{Z} = \text{Pic}(\mathcal{M}_X) \rightarrow H^2(\mathcal{M}_X, \mathbb{Z}).$$

Note that this provides a splitting of the natural map

$$H^2(\mathcal{M}_X, \mathbb{Z}) \rightarrow H^2(\mathcal{M}_X^o, \mathbb{Z}).$$

Set $m := (r^2 - 1)(g - 1) - 3$. Let

$$F : \bigwedge^2 H^3(\mathcal{M}_X, \mathbb{Q}) \rightarrow H^{2(r^2-1)(g-1)}(\mathcal{M}_X, \mathbb{Q})$$

be the homomorphism defined by $(\alpha \wedge \beta) \mapsto \alpha \cup \beta \cup \gamma \otimes m$.

Consider the exact sequence defined by (4.14)

$$H^3(\mathcal{M}_X, \mathcal{M}_X^o) \rightarrow H^3(\mathcal{M}_X) \xrightarrow{q} H^3(\mathcal{M}_X^o).$$

We want to show that $F$ in Eqn. (4.13) induces a bilinear map on $H^3(\mathcal{M}_X^o)$. For this we need the following two lemmas.

Lemma 4.3. The homomorphism $q : H^3(\mathcal{M}_X) \rightarrow H^3(\mathcal{M}_X^o)$ in Eqn. (4.14) is surjective.

Proof. This map can be defined in families over the moduli space of smooth projective curves of genus $g$. If it is non-zero for a particular curve, then it is non-zero for a generic curve. For a generic curve $X$, $H^3(\mathcal{M}_X^o) \cong H^1(X)$ is an irreducible Hodge structure. Therefore the map $q$ is surjective for such $X$. But $q$ is a map of local systems over the moduli space of curves. Therefore if it is surjective for a particular point $X$, then it is surjective for all curves.

It remains to show that $q$ is non-zero for a particular $X$. We use the diffeomorphism provided by Eqn. (2.1). Under this diffeomorphism, $\mathcal{M}_X^o$ corresponds to the smooth locus $\mathcal{R}_{g,r}^o$ of $\mathcal{R}_{g,r}$. So we want to prove that the homomorphism

$$H^3(\mathcal{R}_{g,r}) \rightarrow H^3(\mathcal{R}_{g,r}^o)$$

is non-zero. By the description in Eqn. (2.1), $\mathcal{R}_{g,r}$ is the GIT quotient $V//\text{SL}(r, \mathbb{C})$, where $V = \text{Hom}(\pi_1(X), \text{SL}(r, \mathbb{C}))$. Therefore $H^3(\mathcal{R}_{g,r})$ is the $\text{SL}(r, \mathbb{C})$-invariant part of $H^3(V)$. But $\text{SL}(r, \mathbb{C})$ acts trivially on the cohomology, since it is a connected group. So the pull-back map yields an isomorphism

$$H^3(\mathcal{R}_{g,r}) \cong H^3(V).$$
Let $V^o$ be the preimage of $R_{g,r}^o$ under the natural map $V \to R_{g,r}$. We need to check that the homomorphism

$$H^3(V) \cong H^3(R_{g,r}) \to H^3(R_{g,r}^o)$$

is non-zero. For this it is enough to check that the map $H^3(V) \to H^3(V^o)$ is non-zero.

Actually, there is a universal bundle $E \to X \times N_X$ whose second Chern class $c_2(E)$ produces the required isomorphism

$$c_2(E) : H^1(X) \cong H^1(X).$$

We pull back $E$ to a bundle over $X \times R_{g,r}^o$, and then to a bundle $E' \to X \times V^o$. This bundle is clearly the restriction of the universal bundle $E'' \to X \times V$. The bundle $E'$ produces a map in cohomology $H^1(X) \to H^3(V^o)$. As $E'$ extends to the bundle $E''$, we have that the image of the map $H^3(V) \to H^3(V^o)$ contains the image of the homomorphism

$$H^3(R_{g,r}^o) \cong H^1(X) \to H^3(V^o).$$

We need to check that the map $H^3(R_{g,r}^o) \to H^3(V^o)$ is non-zero. But $V^o \to R_{g,r}^o$ is a PGL($r$, $\mathbb{C}$)-bundle, so that $H^3(R_{g,r}^o) \to H^3(V^o)$ is injective. The result is complete. □

Lemma 4.4. The kernel of $F$ defined in Eqn. (4.13) coincides with the image of the homomorphism

$$H^3(M_X, M_X^e) \to H^3(M_X).$$

Therefore $F$ descends to a pairing $\bar{F}$ on $H^3(M_X^e) \cong H^1(X)$.

Proof. Let $a \in H^3(M_X)$ be an element in the image of $H^3(M_X, M_X^e)$. We want to check that $a \cup \gamma^{m-3} \in H^{2m-3}(M_X)$ is the zero element.

For this, let us characterize a suitable representative of $\gamma \in H^2(M_X)$. By definition, this is the image of the generator of

$$\text{Pic}(M_X) = \text{Pic}(M_X^e) = \text{Pic}(N_X) = \mathbb{Z}.$$

Take a hyperplane section $H$ of $N_X$, consider its preimage $\Phi^*H$ in $U \subset M_X$ under the projection $\Phi$ of Eqn. (3.2). Denote by $\overline{H} \subset M_X$ the closure of this preimage. Now take $m-3$ generic such choices $\overline{H}_1, \ldots, \overline{H}_{m-3}$. The intersection

$$\overline{W} = \bigcap_{i=1}^{m-3} \overline{H}_i \subset M_X$$

satisfies that $\overline{W} \cap U = \Phi^*W$, where $W = \bigcap_{i=1}^{m-3} H_i \subset N_X$. By Eqn. (1.18) below, the codimension of the locus of strictly semistable bundles $\mathcal{Z}''$ in the moduli of semistable bundles $\overline{N}_X$ is at least 5, so the intersection of $m-3$ generic hyperplanes is a projective subvariety not intersecting $\mathcal{Z}''$. Hence $W$ is a 3-dimensional projective smooth variety. As $\overline{W}$ should be connected, it must be that $\overline{W} \subset U$. This element represents $\gamma^{m-3}$. 

As $a \cup \gamma^{m-3}$ is represented by a cycle which is the restriction of $a$ to $W$, and as we may take $a$ supported in a neighborhood of $\mathcal{M}_X - \mathcal{M}_X^\circ$, hence far away from $W$, we have that $a \cup \gamma^{m-3} = 0$. □

Lemmas 4.3 and 4.4 imply that the map in Eqn. (4.13) induce a map

\begin{equation}
F : \bigwedge^2 H^3(\mathcal{M}_X^0, \mathbb{Q}) \longrightarrow H^{2(r^2-1)(g-1)}(\mathcal{M}_X, \mathbb{Q}).
\end{equation}

We want to prove that $F$ is a polarization for $J^2(\mathcal{M}_X^0)$.

**Proposition 4.5.** The dimension of the image of the homomorphism $F$ in Eqn. (4.15) is one.

**Proof.** This is equivalent to prove that the dimension of the image of the homomorphism $F$ in Eqn. (4.13) is one.

We will use the properties the moduli space of Higgs bundles over $X$ which is naturally homeomorphic to $\mathcal{M}_X$. The moduli space of Higgs bundles is known as the *Dolbeault moduli space* (see [Si2]).

Let $\mathcal{H}_X$ denote the moduli space of all semistable Higgs bundles $(E, \theta)$ over $X$ of the following form:

- $E$ is a holomorphic vector bundle over $X$ of rank $r$ with $\bigwedge^r E \cong \mathcal{O}_X$, and
- $\theta : E \rightarrow E \otimes K_X$ is a Higgs field such that $\text{trace}(\theta) \in H^0(X, K_X)$ vanishes identically.

It is known that the moduli space $\mathcal{H}_X$ is naturally homeomorphic to $\mathcal{M}_X$ [Si2, p. 38, Theorem 7.18].

Since $\mathcal{H}_X$ and $\mathcal{M}_X$ are homeomorphic, we have that $H^2(\mathcal{H}_X, \mathbb{Q}) = H^2(\mathcal{M}_X, \mathbb{Q})$. Let $\gamma$ be the element of $H^2(\mathcal{H}_X, \mathbb{Q})$ corresponding to $\gamma \in H^2(\mathcal{M}_X, \mathbb{Q})$ defined in Eqn. (4.12). Let

\begin{equation}
\Gamma : \bigwedge^2 H^3(\mathcal{H}_X, \mathbb{Q}) \longrightarrow H^{2(r^2-1)(g-1)}(\mathcal{H}_X, \mathbb{Q})
\end{equation}

be the homomorphism defined by $(\alpha \bigwedge \beta) \mapsto \alpha \bigcup \beta \bigcup \gamma^{\otimes m}$, where $m = (r^2-1)(g-1)-3$.

Comparing the above homomorphism $\Gamma$ with $F$ defined in Eqn. (4.15) we conclude that the following lemma implies that $\dim \text{image}(F) \leq 1$.

**Lemma 4.6.** The dimension of the image of the homomorphism $\Gamma$, defined in Eqn. (4.16), is at most one.

**Proof.** To prove this lemma, we consider the *Hitchin map*

$$H : \mathcal{H}_X \longrightarrow \bigoplus_{i=2}^r H^0(X, K_X^{\otimes i})$$

defined by $(E, \theta) \mapsto \sum_{i=2}^r \text{trace}(\theta^i)$ [Hi1], [Hi2], [Si2, p. 20]. This map $H$ is algebraic and proper [Hi2, Ni, p. 291, Theorem 6.1], [Si2, p. 22, Theorem 6.11]. The fiber of $H$
over \((0, \cdots, 0)\) is known as the nilpotent cone. The nilpotent cone is a finite union of complete subvarieties of \(H_X\). It is in fact a Lagrangian subvariety of \(H_X\) \([\text{La}, \text{p. 648, Théorème (0.3)}]\). Hence each component of the nilpotent cone is a complete subvariety of dimension \((r^2 - 1)(g - 1)\).

The moduli space \(H_X\) is equipped with the following holomorphic action of \(\mathbb{C}^*\):
\[
\lambda \cdot (E, \theta) = (E, \lambda \cdot \theta),
\]
where \(\lambda \in \mathbb{C}^*\) and \((E, \theta) \in H_X\). Also, \(\mathbb{C}^*\) acts on \(\bigoplus_{i=2}^{r} H^0(X, K_X^{\otimes i})\) as
\[
\lambda \cdot \sum_{i=2}^{r} \omega_i = \sum_{i=2}^{r} \lambda^i \cdot \omega_i,
\]
where \(\omega_i \in H^0(X, K_X^{\otimes i})\). The Hitchin map \(H\) is equivariant with respect to these two actions of \(\mathbb{C}^*\).

We will construct a retraction of \(H_X\) to a neighborhood of the nilpotent cone.

Restrict the \(\mathbb{C}^*\)–action on \(H_X\) to the subgroup \(\mathbb{R}^+ \subset \mathbb{C}^*\). Consider the map
\[
B : \bigoplus_{i=2}^{r} H^0(X, K_X^{\otimes i}) \rightarrow \mathbb{R}_{\geq 0},
\]
defined by
\[
B\left(\sum_{i=2}^{r} \omega_i\right) := \sum_{i=2}^{r} ||\omega_i||^{1/i}.
\]
Clearly \(B\) is continuous, proper, and it vanishes only at the origin. Moreover,
\[
B(t \cdot \sum_{i=2}^{r} \omega_i) = t \cdot B\left(\sum_{i=2}^{r} \omega_i\right)
\]
for any \(t \in \mathbb{R}^+\). Hence for all \(\epsilon > 0\),
\[
V_{\epsilon} := B^{-1}([0, \epsilon])
\]
is a compact neighborhood of the origin, and the preimage of \(V_{\epsilon}\) by the Hitchin map,
\[
U_{\epsilon} := H^{-1}(V_{\epsilon})
\]
is a compact neighborhood of the nilpotent cone in \(H_X\). Any open neighborhood of \(0 \in \bigoplus_{i=2}^{r} H^0(X, K_X^{\otimes i})\) contains \(V_{\epsilon}\) whenever \(\epsilon\) is sufficiently small. Since the Hitchin map \(H\) is proper, this implies that any open neighborhood of \(H^{-1}(0)\) contains \(U_{\epsilon}\) provided \(\epsilon\) is sufficiently small.

We have a retraction of \(H_X\) onto \(U_{\epsilon}\) defined as follows
\[
R : H_X \times [0, 1] \rightarrow H_X
\]
\[
((E, \theta), t) \mapsto \begin{cases} (E, t \cdot \theta), & t \in [0, 1], \ t \geq \frac{\epsilon}{B(H(E, \theta))}, \\ (E, t_0 \cdot \theta), & t \in [0, 1], \ t \leq t_0 = \frac{\epsilon}{B(H(E, \theta))} \leq 1, \\ (E, \theta), & t \in [0, 1], \ B(H(E, \theta)) \leq \epsilon. \end{cases}
\]
Note that in the first two cases, either \( t \neq 0 \) or \( t_0 \neq 0 \) ensuring that the action is well defined. For any \((E, \theta) \in U_\epsilon\), we have \( R((E, \theta), t) = (E, \theta) \). Also, \( R((E, \theta), 1) = (E, \theta) \) for each \((E, \theta) \in \mathcal{H}_X\). For all \((E, \theta) \in \mathcal{H}_X\), it can be shown that \( R((E, \theta), 0) \in U_\epsilon \). Indeed, it is evident for all \((E, \theta)\) with \( B(H(E, \theta)) \leq \epsilon \). If \( B(H(E, \theta)) \geq \epsilon \), then it also holds because

\[
B(H(R((E, \theta), 0))) = B(H((E, t_0 \cdot \theta))) = t_0 \cdot B(H(E, \theta)) = \epsilon.
\]

Now the nilpotent cone \( N = H^{-1}(0) \) is a closed subvariety of \( \mathcal{H}_X \). Therefore there exists an analytic open neighborhood \( U \) of \( N \) (in the smooth topology) that retracts to \( N \), via a retraction \( R' \). Take \( \epsilon > 0 \) small enough so that \( U_\epsilon \subset U \). So the above retraction \( R \) followed by the retraction \( R' \) gives a retraction of \( \mathcal{H}_X \) onto the nilpotent cone, as required.

The nilpotent cone is a union of components which are complete subvarieties of (complex) dimension \((r^2 - 1)(g-1)\). Therefore, each component of the nilpotent cone defines an element of \( H_{2(r^2 - 1)(g-1)}(\mathcal{H}_X, \mathbb{Q}) \). Using the above retraction it follows that these elements together generate \( H_{2(r^2 - 1)(g-1)}(\mathcal{H}_X, \mathbb{Q}) \).

Now the proof of the lemma is completed by the argument involving monodromy in the proof of Lemma 4.3 in [BM]. However, one point should be clarified. Unlike in the coprime case, the monodromy of cohomology ring of moduli spaces of trivial determinant vector bundles does not factor through the symplectic group [CLM]. However, we have natural isomorphisms

\[
H^3(\mathcal{M}_X, \mathbb{Q}) = H^3(\mathcal{N}_X, \mathbb{Q}) = H^1(X, \mathbb{Q})
\]

that extend for a family of curves; see Proposition 3.3 and Proposition 3.2. Therefore, the monodromy of \( H^3(\mathcal{M}_X, \mathbb{Q}) \) for a family of curves factors through the symplectic group. \(\square\)

Continuing with the proof of Proposition 4.5 from Lemma 4.6 it follows that

\[
\dim \text{image}(F) \leq 1.
\]

We will complete the proof of the proposition by showing that \( \text{image}(F) \neq 0 \).

Let \( h \) be a sufficiently positive class of the smooth quasiprojective variety \( \mathcal{N}_X \) (defined in Section 3). By Eqn. (4.2), the image of \( h \) in \( H^2(\mathcal{N}_X, \mathbb{Q}) \),

\[
(4.17) \quad \gamma \in H^2(\mathcal{N}_X, \mathbb{Q}),
\]

is a generator of \( H^2(\mathcal{N}_X, \mathbb{Q}) = \mathbb{Q} \).

Denote by \( \overline{\mathcal{N}}_X \) the moduli space of \((\text{S--equivalence classes of})\) semistable vector bundles \( E \) on \( X \) of rank \( r \) and \( \bigwedge^r E = \mathcal{O}_X \). This moduli space \( \overline{\mathcal{N}}_X \) is an irreducible projective normal singular variety with \( \mathcal{N}_X \) as a Zariski open subset. We will show that the complement

\[
(4.18) \quad Z'' := \overline{\mathcal{N}}_X \setminus \mathcal{N}_X
\]

is of codimension at least five.
To prove this, for each \( \ell \in [1, r - 1] \), let \( \mathcal{N}_X^\ell \) be the moduli space of semistable vector bundles \( E \) over \( X \) of rank \( \ell \) and \( \bigwedge^\ell E = \mathcal{O}_X \). Consider the morphism

\[
f_\ell : \mathcal{N}_X^\ell \times \mathcal{N}_X^{r-\ell} \times \text{Pic}^0(X) \to Z''
\]

defined by \((E_1, E_2, L) \mapsto (E_1 \otimes L^{(r-\ell)} \oplus (E_2 \otimes (L^*)^{\ell}) \). We have

\[
\dim \text{image}(f_\ell) = (\ell^2 - 1)(g - 1) + ((r - \ell)^2 - 1)(g - 1) + g \leq \dim \mathcal{N}_X - 5,
\]

and also \( Z'' = \bigcup_{\ell=1}^{r-1} \text{image}(f_\ell) \). Hence, the codimension of \( Z'' \subset \mathcal{N}_X \) is at least five.

Since the codimension of \( Z'' \subset \mathcal{N}_X \) is at least five, from [DN, p. 55, Théorème B] and [DN, p. 76, Lemme 5.2] we know that \( \text{Pic}(\mathcal{N}_X) = \text{Pic}(\mathcal{N}_X) = \mathbb{Z} \).

Let \( h \in \text{Pic}(\mathcal{N}_X) \) be the ample class corresponding to \( h \). We assume that \( h \) is sufficiently positive so that \( h \) is very ample.

Let

\[
(4.19) \quad \overline{\gamma} \in H^2(\mathcal{N}_X, \mathbb{Q})
\]

be the cohomology class defined by \( \overline{h} \in \text{Pic}(\mathcal{N}_X) \). Hence \( \overline{\gamma} \) maps to \( \gamma \) in Eqn. \((4.17)\) by the map

\[
H^2(\mathcal{N}_X, \mathbb{Q}) \to H^2(\mathcal{N}_X, \mathbb{Q})
\]

Take

\[
(4.20) \quad m := (r^2 - 1)(g - 1) - 3
\]

(recall that \( \dim_C \mathcal{N}_X = (r^2 - 1)(g - 1) \)). Then a general \((m - 1)\)-fold intersection of hyperplanes from the complete linear system \( |\overline{h}| \) on \( \mathcal{N}_X \) does not intersect \( Z'' \), and furthermore, the intersection is a smooth projective 4-fold on \( \mathcal{N}_X \) because the codimension of \( Z'' \) in Eqn. \((4.18)\) is at least five. Let

\[
S \subset \mathcal{N}_X
\]

be a smooth complete 4-fold obtained by taking \((m - 1)\)-fold intersection of hyperplanes from \( |\overline{h}| \).

By the Lefschetz hyperplane section theorem, [BS, p. 215, Lefschetz Theorem (a)], we have

\[
H^2(S, \mathbb{Q}) = H^2(\mathcal{N}_X, \mathbb{Q})
\]

and

\[
(4.21) \quad H^3(S, \mathbb{Q}) = H^3(\mathcal{N}_X, \mathbb{Q}).
\]

Therefore, using the Hard Lefschetz theorem for the smooth projective variety \( S \), there exists

\[
\alpha, \beta \in H^3(S, \mathbb{Q})
\]

such that the element \( \alpha \cup \beta \cup \iota^* \overline{\gamma} \in H^8(S, \mathbb{Q}) = \mathbb{Q} \) is nonzero, where \( \iota : S \to \mathcal{N}_X \) is the inclusion map, and \( \overline{\gamma} \) is the cohomology class in Eqn. \((4.19)\). Let \( \overline{\alpha} \) (respectively, \( \overline{\beta} \)
be the element in \(H^3(\mathcal{N}_X, \mathbb{Q})\) corresponding to the element \(\alpha\) (respectively, \(\beta\)) in Eqn. (4.22) by the isomorphism in Eqn. (4.21). We have

\[(\bar{\alpha} \cup \bar{\beta} \cup \bar{\gamma}) \cap [S] \neq 0\]

because \(\alpha \cup \beta \cup \iota^*\gamma\) coincides with the left–hand side in Eqn. (4.23).

Since \(S\) is a complete intersection of hyperplanes on \(\mathcal{N}_X\) in the complete linear system \(|\mathcal{h}|\) on \(\mathcal{N}_X\), and the first Chern class of \(\mathcal{h}\) is \(\mathcal{y}\), it follows immediately that the left–hand side in Eqn. (4.23) is a positive multiple of \(\bar{\alpha} \cup \bar{\beta} \cup \bar{\gamma} m\), where \(m\) is defined in Eqn. (4.20). Therefore, from Eqn. (4.23) we have

\[\alpha \cup \beta \cup \gamma m \neq 0.\]

Let \(\delta : \mathcal{N}_X \to \mathcal{M}_X\) be the \(C^\infty\) embedding defined by associating to a polystable vector bundle the unique unitary flat connection on it (see [NS]). This embedding \(\delta\) corresponds to the embedding \(\mathcal{N}_X \to \mathcal{H}_X\) defined by \(E \mapsto (E, 0)\).

The homomorphism

\[(4.26) \quad \delta^* : H^3(\mathcal{M}_X, \mathbb{Q}) \to H^3(\mathcal{N}_X, \mathbb{Q})\]

sends \(\gamma\) to \(\mathcal{y}\), where where \(\mathcal{y}\) is the cohomology class in Eqn. (4.19), and \(\gamma\) is defined in Eqn. (4.12).

By Lemma 4.3 the morphism \(H^3(\mathcal{M}_X) \to H^3(\mathcal{M}_X^o) \cong H^3(\mathcal{N}_X)\) is surjective. Let

\[(4.27) \quad \alpha, \beta \in H^3(\mathcal{N}_X)\]

be the images of the classes \(\bar{\alpha}, \bar{\beta}\) under \(H^3(\mathcal{N}_X) \to H^3(\mathcal{N}_X)\). Take \(\hat{\alpha}, \hat{\beta} \in H^3(\mathcal{M}_X, \mathbb{Q})\) some preimages of the cohomology classes in Eqn. (4.26). Replace \(\bar{\alpha}\) and \(\bar{\beta}\) by the images of \(\hat{\alpha}\) and \(\hat{\beta}\) under \(\delta^*\) in Eqn. (4.25). Note that Eqn. (4.24) continues to hold since it only depends on the restriction of \(\alpha, \beta\) to \(S \subset \mathcal{N}_X\). From Eqn. (4.24) it follows immediately that

\[(\hat{\alpha} \cup \hat{\beta} \cup \hat{\gamma} m) \cap [\mathcal{N}_X] = \bar{\alpha} \cup \bar{\beta} \cup \bar{\gamma} m \neq 0.\]

Consequently, the homomorphism \(F\) in Eqn. (4.15) is nonzero. We have already shown that \(\dim \text{image}(F) \leq 1\). Therefore, the proof of the proposition is complete. \(\square\)

5. Torelli type theorems

In this section we will prove the Torelli theorem for both \(\mathcal{M}_X\) and \(\hat{\mathcal{M}}_X\).

**Theorem 5.1.** Let \(X\) and \(Y\) be two compact connected Riemann surfaces of genus \(g\), with \(g \geq 3\). Fix \(r \geq 2\). If \(g = 3\), then we assume that \(r \geq 3\). Let \(\mathcal{M}_X\) and \(\mathcal{M}_Y\) be the corresponding moduli spaces of holomorphic \(\text{SL}(r, \mathbb{C})\)–connections defined as in Section 2.
The two varieties $\mathcal{M}_X$ and $\mathcal{M}_Y$ are isomorphic if and only if the two Riemann surfaces $X$ and $Y$ are isomorphic.

Proof. First note that the smooth locus $\mathcal{M}_X^o \subset \mathcal{M}_X$ is determined by the algebraic structure of $\mathcal{M}_X$. In view of Proposition 3.3 to prove the theorem using the Torelli theorem for curves, [We2], we need to recover the principal polarization on $J^2(\mathcal{M}_X^o) \cong \text{Pic}^0(X)$.

The map $F$ in Eqn. (4.13) is well-defined since the element $\gamma$ is defined by the algebraic structure of $\mathcal{M}_X$. From Proposition 4.5 we know that $\dim \text{image}(F) = 1$. Consequently, fixing a generator of $\text{image}(F)$, the homomorphism $F$ gives a nonzero cohomology class $\theta \in \bigwedge^2 H^3(\mathcal{M}_X^o, \mathbb{Q})^* = H^2(J^2(\mathcal{M}_X^o), \mathbb{Q})$. The one–dimensional subspace

$$\hat{\theta} \subset H^2(J^2(\mathcal{M}_X^o), \mathbb{Q})$$

generated by the above nonzero element $\theta$ is independent of the choices of the generator of $\text{image}(F)$.

Consider the universal curve over the moduli space $\mathcal{M}_g^0$ parametrizing all smooth complex curves of genus $g$ which do not admit any nontrivial automorphisms. Let

$$\rho : \text{Pic}^0_{\mathcal{M}_g} \rightarrow \mathcal{M}_g^0$$

be the relative Jacobian for this family. It is known that the local system $R^2 \rho_* \mathbb{Q}$ has exactly one sub–local system of rank one, and this rank one sub–local system is generated by the canonical principal relative polarization. Indeed, this is an immediate consequence of the combination of the fact that the action of $\text{Sp}(2g, \mathbb{C})$ on $\bigwedge^2 \mathbb{C}^{2g}$ decomposes it into a direct sum of a one–dimensional $\text{Sp}(2g, \mathbb{C})$–module and an irreducible $\text{Sp}(2g, \mathbb{C})$–module of dimension $g(2g - 1)$, and the fact that the monodromy of $R^1 \rho_* \mathbb{C}$ is the full symplectic group (see [BN, p. 710, Theorem 4.2] for more details).

On the other hand, using the second part of Proposition 3.3 the line $\hat{\theta}$ in Eqn. (5.1) gives a sub–local system of rank one. Therefore, $\hat{\theta}$ must be a nonzero rational multiple of the canonical principal polarization.

Any principal polarization can uniquely be recovered from any nonzero rational multiple of it; see the proof of Theorem 4.4 in [BM]. This completes the proof of the theorem. $\square$

As in Section 2 let $\hat{\mathcal{M}}_X$ be the moduli space of holomorphic connections $(E, D)$ with $\text{rank}(E) = r$.

**Theorem 5.2.** The isomorphism class of the variety $\hat{\mathcal{M}}_X$ determines the Riemann surface $X$ uniquely up to an isomorphism.

Proof. In view of Theorem 5.1 the proof of the theorem is identical to that of Theorem 5.2 in [BM]. The only point to note is that there is no nonconstant algebraic map from a Zariski open subset of $\mathbb{CP}^1$ to an abelian variety. The variety $\mathcal{M}_X$ is unirational. Indeed,
using the diagram in Eqn. (3.11) it suffices to show that $N_X$ is unirational (recall that $\Phi$ in Eqn. (3.11) is an affine bundle). It is known that $N_X$ is unirational; the unirationality of $N_X$ follows from [Ne, p. 134, Lemma 5.2] and [Ne, p. 136, Remark].

6. Curves defined over real numbers

Fix $t = \sqrt{-1} a \in \mathbb{C}$, where $a$ is a positive real number. Let $\Lambda \subset \mathbb{C}$ be the $\mathbb{Z}$–module generated by $1$ and $t$. Let

$$C := \mathbb{C}/\Lambda$$

be the complex elliptic curve. Consider the anti–holomorphic involution

$$\sigma : C \to C$$

induced by the map $\mathbb{C} \to \mathbb{C}$ defined by

$$z \mapsto \bar{z} + \frac{1}{2}.$$

This involution $\sigma$ of $C$ clearly does not have any fixed points. Therefore, $(C, \sigma)$ is a smooth projective real curve of genus one without any real points. This real curve will be denoted by $C_t$.

Consider the real abelian variety $\text{Pic}^0(C_t)$. We note that there is a canonical isomorphism

$$\text{Pic}^0(C_t) \to \text{Pic}^0(\text{Pic}^0(C_t))$$

that sends any $L \in \text{Pic}^0(C_t)$ to the degree zero divisor on $\text{Pic}^0(C_t)$ defined by $L - \mathcal{O}_{C_t}$, where $\mathcal{O}_{C_t}$ is the trivial line bundle over $C_t$. Equivalently, the line bundle on $\text{Pic}^0(C_t) \times \text{Pic}^0(C_t)$ defined by the divisor $\Delta - \text{Pic}^0(C_t) \times \{\mathcal{O}_{C_t}\}$ gives the identification of $\text{Pic}^0(C_t)$ with $\text{Pic}^0(\text{Pic}^0(C_t))$, where $\Delta$ is the diagonal divisor.

On the other hand, $C_t$ is not isomorphic to $\text{Pic}^0(C_t)$. Indeed, $\text{Pic}^0(C_t)$ has a real point $\mathcal{O}_{C_t}$, while $C_t$ does not have any real points. Also, note that an abelian variety of dimension one has exactly one principal polarization.

Therefore, $\text{Pic}^0(C_t)$ and $\text{Pic}^0(\text{Pic}^0(C_t))$ are isomorphic as principally polarized abelian varieties, while $C_t$ and $\text{Pic}^0(C_t)$ are not isomorphic. Consequently, the Torelli theorem fails for real curves of genus one.

However the following is valid.

**Lemma 6.1.** Let $Y$ and $Z$ be two geometrically irreducible smooth real projective curves of genus $g_0$, with $g_0 \geq 2$, such that

- the real abelian variety $\text{Pic}^0(Y)$ is isomorphic to $\text{Pic}^0(Z)$, and
- there is an isomorphism $\text{Pic}^0(Y) \to \text{Pic}^0(Z)$ that takes the canonical principal polarization on $\text{Pic}^0(Y)$ to that on $\text{Pic}^0(Z)$.

Then the two real algebraic curves $Y$ and $Z$ are isomorphic.
Proof. Let $Y_C := Y \times_{\mathbb{R}} \mathbb{C}$ be the complexification of $Y$. We know that $\text{Pic}^0(Y_C)$ is the complexification of the real abelian variety $\text{Pic}^0(Y)$. Furthermore, the canonical principal polarization on $\text{Pic}^0(Y_C)$ is given by the canonical principal polarization on $\text{Pic}^0(Y)$.

Since the two principally polarized abelian varieties $\text{Pic}^0(Y)$ and $\text{Pic}^0(Z)$ are isomorphic, we know that $\text{Pic}^0(Y_C)$ is isomorphic, as a principally polarized variety, to the Jacobian of $Z_C := Z \times_{\mathbb{R}} \mathbb{C}$. Therefore, the Torelli theorem says that $Y_C$ is isomorphic to $Z_C$.

Since $Y_C$ is isomorphic to $Z_C$, we may, and we will, consider $Z$ as a real structure on the complex curve $Y_C$. In other words, the two real curves $Y$ and $Z$ are given by two anti–holomorphic involutions of $Y_C$. Let $\sigma_Y$ (respectively, $\sigma_Z$) be the anti–holomorphic involutions of $Y_C$ defining the real curve $Y$ (respectively, $Z$).

Let $\tau_Y$ (respectively, $\tau_Z$) be the anti–holomorphic involutions of $\text{Pic}^0(Y_C)$ induced by $\sigma_Y$ (respectively, $\sigma_Z$). Therefore, $(\text{Pic}^0(Y_C), \tau_Y)$ and $(\text{Pic}^0(Y_C), \tau_Z)$ define the real abelian varieties $\text{Pic}^0(Y)$ and $\text{Pic}^0(Z)$ respectively.

The anti–holomorphic involution of the Picard group of the complexification of a geometrically irreducible smooth projective real curve preserves the canonical polarization on it. In particular, both $\tau_Y$ and $\tau_Z$ preserve the element in $H^2(\text{Pic}^0(Y_C), \mathbb{Q})$ given by the canonical polarization.

Let

\begin{equation}
\eta : \text{Pic}^0(Y_C) \to \text{Pic}^0(Y_C)
\end{equation}

be the holomorphic automorphism that satisfies the identity

\begin{equation}
\eta = \tau_Z^{-1} \circ \tau_Y,
\end{equation}

where $\tau_Y$ and $\tau_Z$ are defined above.

Let $C$ be a compact connected complex curve of genus at least two. The Torelli theorem says that the group of all automorphisms of the complex Lie group $\text{Pic}^0(C)$ that preserve the canonical polarization on it is generated by $\text{Aut}(C)$ together with the inversion

\begin{equation}
\iota : \text{Pic}^0(C) \to \text{Pic}^0(C)
\end{equation}

defined by $L \mapsto L^*$; see [We2, p. 35, Hauptsatz]. We note that the inversion $\iota$ commutes with any automorphism of the Lie group $\text{Pic}^0(C)$. Therefore, we have a homomorphism from $\text{Aut}(\text{Pic}^0(C))$ to a quotient $Q$ of $\text{Aut}(C)$,

\begin{equation}
\text{Aut}(\text{Pic}^0(C)) \xrightarrow{\phi_C} Q \to 0
\end{equation}

The quotient $Q$ is identified with the subgroup of $\text{Pic}^0(C)$ generated by $\text{Aut}(C)$.

We also know that the natural homomorphism

\begin{equation}
p : \text{Aut}(C) \to \text{Aut}(\text{Pic}^0(C))
\end{equation}

is injective [FK, p. 287, Theorem]. (The Torelli theorem fails for real algebraic curves of genus one because the homomorphism $p$ in that case is not injective.) Therefore,
the quotient group \( Q \) in Eqn. (6.3) actually coincides with \( \text{Aut}(C) \). Furthermore, the homomorphism \( \phi_C \) in Eqn. (6.3) satisfies the identity

\[
\phi_C \circ p = \text{Id}_{\text{Aut}(C)},
\]

where \( p \) is the homomorphism in Eqn. (6.4).

Define \( \phi_Y \) as in Eqn. (6.3) by substituting the curve \( Y \) for \( C \). Consider \( \eta \) constructed in Eqn. (6.1). Since it satisfies Eqn. (6.2), using Eqn. (6.5) we conclude that the automorphism

\[
\phi_Y(\eta) : Y \longrightarrow Y
\]

satisfies the identity \( \phi_Y(\eta) = \sigma_Z^{-1} \circ \sigma_Y \).

In other words, \( \phi_Y(\eta) \) is an isomorphism between the two real curves \((Y_C, \sigma_Y)\) and \((Y_C, \sigma_Z)\). This completes the proof of the lemma. \( \square \)

In view of Lemma 6.1, we conclude that the Torelli theorems in [MN] and [NR] for the moduli space of stable vector bundles remain valid for curves defined over \( \mathbb{R} \). Consequently, Theorem 4.4 and Theorem 5.2 of [BM] remain valid for curves defined over \( \mathbb{R} \).

Similarly, [AS, p. 2, Theorem 1.0.2] remains valid for curves defined over \( \mathbb{R} \). Therefore, Theorem 5.1 and Theorem 5.2 proved here also remain valid for curves defined over \( \mathbb{R} \).

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