LOWER BOUNDS FOR MOMENTS OF QUADRATIC TWISTS OF MODULAR $L$-FUNCTIONS

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Abstract. We establish sharp lower bounds for the $2k$-th moment in the range $k \geq 1/2$ of the family of quadratic twists of modular $L$-functions at the central point.

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1. Introduction

The Birch–Swinnerton-Dyer conjecture motivates a great amount of work enriching our understanding of the central values of various $L$-functions. In [14], M. R. Murty and V. K. Murty obtained an asymptotic formula for the first moment of the derivative of quadratic twists of elliptic curve $L$-functions at the central point. Their result implies that there are infinitely many such $L$-functions with nonvanishing derivatives at that point of interest. The error term in this asymptotic formula was later improved by H. Iwaniec [6] in a smoothed version. A better error term had also been announced earlier by D. Bump, S. Friedberg, and J. Hoffstein in [1].

The result mentioned in [1] concerns more generally with all modular newforms. Subsequent work on the first moment of these modular $L$-functions or their derivatives at the central point can be found in [9, 11–13, 15, 16, 20]. Note that when the sign of the functional equation satisfied by an $L$-function equals $-1$, then the corresponding $L$-function has a vanishing central value. Hence its derivative at the central point is a natural object to study.

In [21, 22], M. P. Young developed a recursive method to improve the error terms in the first and third moment of quadratic Dirichlet $L$-functions. This method was adapted by Q. Shen [17] to ameliorate the error terms in the first moment of quadratic twists of modular $L$-functions. The results of Shen can be regarded as commensurate to a result of K. Sono [18] on the second moment of quadratic Dirichlet $L$-functions.

In [19], K. Soundararajan and M. P. Young obtained a formula for the second moment of quadratic twists of modular $L$-functions at the central point under the assumption of the generalized Riemann hypothesis (GRH). I. Petrow [15] computed the second moment of the derivative of quadratic twists of modular $L$-functions under GRH. In addition to the above mentioned work on moments of modular $L$-functions at the central point, there are now well-established conjectures for these moments due to J. P. Keating and N. C. Snaith [5], with subsequent contributions from J. B. Conrey, D. W. Farmer, J. P. Keating, M. O. Rubinstein and N. C. Snaith in [2] as well as from A. Diaconu, D. Goldfeld and J. Hoffstein [3].

In this paper, we are interested in obtaining lower bounds for the moments of modular $L$-functions at the central point of the conjectured order of magnitude. To state our results, we need some notation first. Let $f$ be a fixed holomorphic Hecke eigenform of weight $\kappa$ for the full modular group $SL_2(\mathbb{Z})$. The Fourier expansion of $f$ at infinity can be written as

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n)n^{(\kappa-1)/2}e(nz), \quad \text{where} \quad e(z) = \exp(2\pi iz).$$

Here the coefficients $\lambda_f(n)$ are real and satisfy $\lambda_f(1) = 1$ and $0 \neq |\lambda_f(n)| \leq d(n)$ for $n \geq 1$ with $d(n)$ being the number of divisors of $n$. We write $\chi_d = \left(\frac{d}{\cdot}\right)$ for the Kronecker symbol. For $\Re(s) > 1$, we define the twisted modular $L$-function $L(s, f \otimes \chi_d)$ as follows.

$$L(s, f \otimes \chi_d) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)\chi_d(n)}{n^s} = \prod_{p|d} \left(1 - \frac{\lambda_f(p)\chi_d(p)}{p^s} + \frac{1}{p^{2s}}\right)^{-1}.$$
Then \( L(s, f \otimes \chi_d) \) has analytic continuation to the entire complex plane and satisfies the functional equation

\[
\Lambda(s, f \otimes \chi_d) = \left( \frac{|d|}{2\pi} \right)^s \Gamma(s + \frac{k-1}{2}) L(s, f \otimes \chi_d) = i^n \epsilon(d) \Lambda(1 - s, f \otimes \chi_d),
\]

where \( \epsilon(d) = 1 \) if \( d \) is positive, and \( \epsilon(d) = -1 \) if \( d \) is negative.

We consider the family \( \{L(s, f \otimes \chi_{8d})\} \) with \( d \) positive, odd and square-free. Note that the sign of the functional equation for each member in this family is positive if \( \kappa \equiv 0 \pmod{4} \), in which case one can study moments of these central \( L \)-values. It is expected that for all positive real \( k \), we have

\[
\sum_{0 < d < X, (d, 2) = 1}^* |L(\frac{s}{2}, f \otimes \chi_{8d})|^k \sim C_k X (\log X)^{(k-1)}/2,
\]

where \( \Sigma^* \) means that the sum runs over square-free integers and the \( C_k \)'s are explicitly computable constants.

When \( \kappa \equiv 2 \pmod{4} \), the sign of the functional equation for each member of the above family is negative so that the corresponding central \( L \)-value is zero. In this case, one can study moments of the derivatives of these \( L \)-functions at \( s = 1/2 \). The expectation in this case is, for all positive real \( k \),

\[
\sum_{0 < d < X, (d, 2) = 1}^* |L'(\frac{s}{2}, f \otimes \chi_{8d})|^k \sim C'_k X (\log X)^{(k+1)}/2,
\]

with explicitly computable constants \( C'_k \).

The aim of this paper is to establish the lower bounds, of the conjectured order of magnitudes, for the moments in (1.1). Our results are as follows.

**Theorem 1.1.** With notations as above and let \( k \geq 1 \). For \( \kappa \equiv 0 \pmod{4} \) and \( \kappa \neq 0 \), we have

\[
(1.2) \quad \sum_{0 < d < X, (d, 2) = 1}^* |L(\frac{s}{2}, f \otimes \chi_{8d})|^k \gg_k X (\log X)^{(k-1)}/2.
\]

For \( \kappa \equiv 2 \pmod{4} \), we have

\[
(1.3) \quad \sum_{0 < d < X, (d, 2) = 1}^* |L'(\frac{s}{2}, f \otimes \chi_{8d})|^k \gg_k X (\log X)^{(k+1)}/2.
\]

Our proof of the above theorem is mainly based on the lower bounds principle of Heap and Soundararajan in \([5]\). As this method requires the evaluation of the twisted moments of modular functions, we shall also make crucial use of the results of Q. Shen \([17]\) on the twisted first moment of twisted modular functions.

**2. Plan of the Proof**

As the proofs of \((1.2)\) and \((1.3)\) are similar, we shall only consider the case \( \kappa \equiv 2 \pmod{4} \) in Theorem \((1.1)\) throughout the paper. We shall further replace \( k \) by \( 2k \) and assume that \( k > 1/2 \) since the case \( k = 1/2 \) follows from \([17]\) Theorems 1.1–1.2. Let \( \Phi \) be a smooth, non-negative function compactly supported on \([1/8, 7/8]\) such that \( \Phi(x) \leq 1 \) for all \( x \) and \( \Phi(x) = 1 \) for \( x \in [1/4, 3/4] \). Assuming that \( X \) is a large positive real number, we find that in order to prove \((1.3)\), it suffices to show that

\[
(2.1) \quad \sum_{(d, 2) = 1}^* |L'(\frac{s}{2}, f \otimes \chi_{8d})|^{2k} \Phi \left( \frac{d}{X} \right) \gg_k X (\log X)^{2k(2k+1)/2}.
\]

To this end, we recall the lower bounds principle of Heap and Soundararajan in \([5]\) for our situation. Let \( \{\ell_j\}_{1 \leq j \leq R} \) be a sequence of even natural numbers such that \( \ell_1 = 2 \lceil N \log \log X \rceil \) and \( \ell_{j+1} = 2 \lceil N \log \ell_j \rceil \) for \( j \geq 1 \), where \( N, M \) are two large natural numbers depending on \( k \) only and \( R \) is the largest natural number satisfying \( \ell_R > 10^M \). We shall choose \( M \) large enough to ensure \( \ell_j > \ell_{j+1}^2 \) for all \( 1 \leq j \leq R - 1 \). It follows that

\[
(2.2) \quad R \ll \log \log \ell_1 \quad \text{and} \quad \sum_{j=1}^{R} \frac{1}{\ell_j} \leq \frac{2}{\ell_R}.
\]
Let $P_1$ be the set of odd primes not exceeding $X^{1/2}$ and $P_j$ be the set of primes in the interval $(X^{1/j-1}, X^{1/j})$ for $2 \leq j \leq R$, we define

$$P_j(d) = \sum_{p \in P_j} \frac{\lambda_f(p)}{\sqrt{p}} \chi_{8d}(p),$$

and for any $\alpha \in \mathbb{R}$,

$$(2.3) \quad \mathcal{N}_j(d, \alpha) = E_{\ell_j}(\alpha P_j(d)) \quad \text{and} \quad \mathcal{N}(d, \alpha) = \prod_{j=1}^{R} \mathcal{N}_j(d, \alpha),$$

where for any integer $\ell \geq 0$ and any $x \in \mathbb{R}$,

$$E_{\ell}(x) = \sum_{j=0}^{\ell} \frac{x^j}{j!}.$$

Since $\lambda_f(p)$ is real, so is $P_j(d)$. Consequently, it follows from [10] Lemma 1 that the quantities defined in (2.3) are all positive. As $2k > 1$, Hölder’s inequality gives

$$(2.4) \quad \sum_{(d,2)=1}^* L'(\frac{1}{2}, f \otimes \chi_{8d}) \mathcal{N}(d, 2k - 1) \Phi\left(\frac{d}{X}\right) \\
\leq \left( \sum_{(d,2)=1}^* |L'(\frac{1}{2}, f \otimes \chi_{8d})|^{2k} \Phi\left(\frac{d}{X}\right) \right)^{1/(2k)} \left( \sum_{(d,2)=1}^* \mathcal{N}(d, 2k - 1)^{2k/(2k-1)} \Phi\left(\frac{d}{X}\right) \right)^{(2k-1)/(2k)}.$$

We deduce from (2.4) that in order to establish (2.1) and hence (1.3), it suffices to establish the following propositions.

**Proposition 2.1.** With notations as above, we have, for $\kappa \equiv 2 \pmod{4}$ and $k > 1/2$,

$$(3.1) \quad \sum_{(d,2)=1}^* L'(\frac{1}{2}, f \otimes \chi_{8d}) \mathcal{N}(d, 2k - 1) \Phi\left(\frac{d}{X}\right) \gg X(\log X)^{(2k^2+1)/2}.$$

**Proposition 2.2.** With notations as above, we have, for $\kappa \equiv 2 \pmod{4}$ and $k > 1/2$,

$$(3.2) \quad \sum_{(d,2)=1}^* \mathcal{N}(d, 2k - 1)^{2k/(2k-1)} \Phi\left(\frac{d}{X}\right) \ll X(\log X)^{(2k^2)/2}.$$

The remainder of the paper is thus devoted to the proofs of these propositions.

### 3. Preliminaries

We reserve the letter $p$ for a prime number and cite the following result on sum over primes.

**Lemma 3.1.** Let $x \geq 2$. We have, for some constant $b$,

$$(3.1) \quad \sum_{p \leq x} \frac{\lambda_f^2(p)}{p} = \log \log x + b + O\left(\frac{1}{\log x}\right).$$

Moreover, we have

$$(3.2) \quad \sum_{p \leq x} \frac{\log p}{p} = \log x + O(1).$$

**Proof.** The assertion in (3.1) follows from the Rankin-Selberg theory for $L(s, f)$, which can be found in [7] Chapter 5. The formula in (3.2) is given in [10] Lemma 2.7. \qed

Set $\delta_n = \square$ to be 1 if $n$ is a square and 0 otherwise. We note the following result that can be established in a manner similar to [10] Proposition 1.

**Lemma 3.2.** For large $X$ and any odd positive integer $n$, we have

$$(3.3) \quad \sum_{(d,2)=1}^* \chi_{8d}(n) \Phi\left(\frac{d}{X}\right) = \delta_{n=\square} \Phi(1) \frac{2X}{3\zeta(2)} \prod_{p|n} \left(1 - \frac{1}{p+1}\right) \Phi\left(\frac{p}{p+1}\right) + O(X^{1/2+\varepsilon} \sqrt{n}).$$
For any complex number $s$, let $\hat{\Phi}(s)$ be the Mellin transform of the function $\Phi$ described in Section 2 i.e.

$$\hat{\Phi}(s) = \int_0^\infty \Phi(x)x^s \frac{dx}{x}.$$

Applying [17, Theorem 1.4], we have the following asymptotic formulas for the twisted first moment of quadratic modular $L$-functions.

**Lemma 3.3.** Using the same notations as above and writing any odd $l$ as $l = l_1l_2^2$ with $l_1$ square-free, we have, for $\kappa \equiv 0 \pmod{4}$ and any $\varepsilon > 0$,

$$\sum_{(d,2)=1}^\ast L(\frac{1}{2}, f \otimes \chi_{sd})\chi_{sd}(l)\Phi\left(\frac{d}{X}\right) = 8C\pi^2\hat{\Phi}(1)\frac{\lambda_f(l_1)}{\sqrt{\zeta(1)}g(l)}L(1, \text{sym}^2 f)X + O\left(X^{3/4+\varepsilon l_1^{1/2+\varepsilon}}\right),$$

where $g(l)$ is a multiplicative function, i.e. $g(l) = \prod_{p^i||l} g(p^i)$. Here $g(p^i) = g(p) \geq 1$ for all $i \geq 1$ and $1/g(p) = 1 + O(1/p)$.

Moreover, for $\kappa \equiv 2 \pmod{4}$, we have

$$\sum_{(d,2)=1}^\ast L'(\frac{1}{2}, f \otimes \chi_{sd})\chi_{sd}(l)\Phi\left(\frac{d}{X}\right) = C\Phi(1)\frac{\lambda_f(l_1)}{\sqrt{\zeta(1)}g(l)}X\left(\log X\frac{1}{l_1} + C_2 + \sum_{p
mid l} \frac{C_2(p)}{p} \log p\right) + O\left(X^{3/4+\varepsilon l_1^{1/2+\varepsilon}}\right),$$

where $C$ is an absolute constant, $C_2$ is a constant whose value depends only on $\Phi$ and $C_2(p) \ll 1$ for all $p$.

**Proof.** We write $l = l_1l_2^2$ with $l_1$ square-free for any positive, odd integer $l$. For any complex number $\alpha$, we define

$$M(\alpha, l) = \sum_{(d,2)=1}^\ast L(\frac{1}{2} + \alpha, f \otimes \chi_{sd})\chi_{sd}(l)\Phi\left(\frac{d}{X}\right).$$

It follows from [17, Conjecture 1.3, Theorem 1.4] and the proofs of Theorems 1.1 and 1.2 in [17] that, for

$$|\Re(\alpha)| \ll (\log X)^{-1}, \ |\Im(\alpha)| \ll (\log X)^2,$$

we have

$$M(\alpha, l) = \frac{4X\hat{\Phi}(1)}{\pi^2l_1^{1/2+\alpha}}L(1+2\alpha, \text{sym}^2 f)Z(\frac{1}{2}+\alpha, l) + \frac{4\gamma_\alpha X^{1-2\alpha}\hat{\Phi}(1-2\alpha)}{\pi^2l_1^{1/2-\alpha}}L(1-2\alpha, \text{sym}^2 f)Z(\frac{1}{2} - \alpha, l) + O(l^{1/2+\varepsilon}X^{1/2+\varepsilon}),$$

where the implied constant depends on $\varepsilon, h$ and $\Phi$,

$$\gamma_\alpha = \left(\frac{8}{2\pi}\right)^{-2\alpha}\frac{\Gamma(\frac{1}{2} - \alpha)}{\Gamma(\frac{1}{2} + \alpha)}.$$

and $Z(\frac{1}{2}+\alpha, l)$ is defined below. Note that though dubbed a conjecture, [17, Conjecture 1.3] is proved therein. Moreover, $Z(\frac{1}{2} + \gamma, l)$ is analytic and absolutely convergent in the region $\Re(\gamma) > -\frac{1}{2}$ such that in this range, we write

$$L(1+2\alpha, \text{sym}^2 f)Z(\frac{1}{2}+\alpha, l) := \lambda_f(l_1) \prod_{(p,2)=1} Z_p(\frac{1}{2} + \alpha, l),$$

where

$$Z_p(\frac{1}{2} + \gamma, l) = \begin{cases} \lambda_f(p)^{-1}p^{1/2+\gamma}\left(\frac{p}{p+1}\right) \left(\frac{1}{2} \left( 1 - \frac{\lambda_f(p)}{p^{1/2+\gamma}} + \frac{1}{p^{1/2+\gamma}} \right)^{-1} - \frac{1}{2} \left( 1 + \frac{\lambda_f(p)}{p^{1/2+\gamma}} + \frac{1}{p^{1/2+\gamma}} \right)^{-1} \right), & \text{if } p|l_1, \\ \frac{p}{p+1} \left(\frac{1}{2} \left( 1 - \frac{\lambda_f(p)}{p^{1/2+\gamma}} + \frac{1}{p^{1/2+\gamma}} \right)^{-1} + \frac{1}{2} \left( 1 + \frac{\lambda_f(p)}{p^{1/2+\gamma}} + \frac{1}{p^{1/2+\gamma}} \right)^{-1} \right), & \text{if } p \nmid l_1, \ p|l_2, \\ \tilde{Z}_p(\frac{1}{2} + \gamma), & \text{if } (p, 2l) = 1, \end{cases}$$

and where

$$\tilde{Z}_p(\frac{1}{2} + \gamma) := 1 + \frac{p}{p+1} \left(\frac{1}{2} \left( 1 - \frac{\lambda_f(p)}{p^{1/2+\gamma}} + \frac{1}{p^{1/2+\gamma}} \right)^{-1} + \frac{1}{2} \left( 1 + \frac{\lambda_f(p)}{p^{1/2+\gamma}} + \frac{1}{p^{1/2+\gamma}} \right)^{-1} \right) - 1.$$
Note that we have \( L(1 + 2\alpha, \text{sym}^2 f) = \prod_p L_p(1 + 2\alpha, \text{sym}^2 f) \) with
\[
L_p(1 + 2\alpha, \text{sym}^2 f) := \left( 1 - \frac{\lambda_f(p)}{p^{1/2 + \alpha}} + \frac{1}{p^{1 + 2\alpha}} \right)^{-1} \left( 1 + \frac{\lambda_f(p)}{p^{1/2 + \alpha}} + \frac{1}{p^{1 + 2\alpha}} \right)^{-1} \left( 1 - \frac{1}{p^{1 + 2\alpha}} \right)^{-1}.
\]

If \( \kappa \equiv 0 \pmod{4} \), we set \( \alpha = 0 \) in \( \[5.6\] \) to compute \( Z(\frac{1}{2}, l) \) using \( \[5.7\] \) and the above to deduce \( \[5.8\] \) with \( C = \prod_p C_p \), where
\[
C_p = \begin{cases} 
L_2(1, \text{sym}^2 f)^{-1}, & p = 2, \\
\tilde{Z}_p(\frac{1}{2}) L_p(1, \text{sym}^2 f)^{-1}, & p > 2.
\end{cases}
\]
Note that it follows from the proof of \( [17, \text{Lemma 2.6}] \) that \( C_p = 1 + O(p^{-1-\varepsilon}) \) for some \( \varepsilon > 0 \), so that \( C \) is an absolute constant.

Moreover, it is readily seen that \( g(l) \) is a multiplicative function such that \( g(l) = \prod_{p \mid l} g(p) \), where for all \( i \geq 1 \),
\[
g(p^i) = Z_p(\frac{1}{2}, l)^{-1} \tilde{Z}_p(\frac{1}{2}).
\]
Simplifying the expressions in \( \[5.7\] \) and \( \[5.8\] \) leads to
\[
Z_p(\frac{1}{2}, l) = \begin{cases} 
\frac{1}{p+1} \left( 1 - \frac{\lambda_f(p)}{p^{1/2}} + \frac{1}{p} \right)^{-1} \left( 1 + \frac{\lambda_f(p)}{p^{1/2}} + \frac{1}{p} \right)^{-1}, & \text{if } p \mid l_1, \\
\frac{1}{p+1} \left( 1 - \frac{\lambda_f(p)}{p^{1/2}} + \frac{1}{p} \right)^{-1} \left( 1 + \frac{\lambda_f(p)}{p^{1/2}} + \frac{1}{p} \right)^{-1}, & \text{if } p \nmid l_1, p \mid l_2,
\end{cases}
\]
\[
\tilde{Z}_p(\frac{1}{2}) = \left( 1 - \frac{\lambda_f(p)}{p^{1/2}} + \frac{1}{p} \right)^{-1} \left( 1 + \frac{\lambda_f(p)}{p^{1/2}} + \frac{1}{p} \right)^{-1} \frac{1}{p+1}.
\]
Hence, we deduce that
\[
g(p^i) = \begin{cases} 
\frac{p+1}{p} \left( 1 + \frac{1}{p+1} \left( 1 + \frac{1}{p} \right)^2 - \frac{\lambda_f(p)}{p} \right), & \text{if } p \mid l_1, \\
\frac{1}{p+1} \left( 1 + \frac{1}{p} \right)^2 - \frac{\lambda_f(p)}{p}, & \text{if } p \nmid l_1, p \mid l_2,
\end{cases}
\]
One then checks from the above that \( g(p^i) \geq 1 \) using \( |\lambda_f(p)| < 2 \). Similarly, one shows that \( 1/g(p) = 1 + O(1/p) \) using \( \[3.9\] \), \( \[3.8\] \) and the above.

If \( \kappa \equiv 2 \pmod{4} \), we differentiate both sides of \( \[3.6\] \) with respect to \( \alpha \). The contribution of the derivative of the error term is still \( O(l^{1/2+\varepsilon}X^{1/2+\varepsilon}) \) using Cauchy’s integral formula and the observation that the error term in \( \[3.6\] \) is holomorphic on the disc centred at \( (0, 0) \) with radius \( \ll (\log X)^{-1} \) from the proof of \( [17, \text{Theorem 1.4}] \). Upon setting \( \alpha = 0 \), we obtain that
\[
\sum_{(d,2)=1}^* L'(\frac{1}{2}, f \otimes \chi_{8d}) \chi_{8d}(d) \Phi(d) X \left( \log \frac{X}{l_1} + \frac{2}{L(1, \text{sym}^2 f)} L(1, \text{sym}^2 f) \frac{Z'(\frac{1}{2}, l)}{Z(\frac{1}{2}, l)} + \log \frac{8}{2\pi} + \frac{\Gamma'(\frac{1}{2})}{\Gamma(\frac{1}{2})} + \frac{\Phi'(1)}{\Phi(1)} \right) + O(l^{1/2+\varepsilon}X^{1/2+\varepsilon}).
\]
Computing of \( Z(\frac{1}{2}, l) \) and \( Z'(\frac{1}{2}, l) \) using \( \[5.7\] \) then leads to \( \[3.8\] \) and completes the proof. \( \square \)

The following result is analogous to \( \[3\] \) Lemma 1 and is needed in the proof of Proposition \( \[2.2\] \).

**Lemma 3.4.** For \( 1 \leq j \leq R \), we have
\[
\mathcal{N}_j(d, 2k-1)^{2k/(2k-1)} \leq \mathcal{N}_j(d, 2k) \frac{(1 + e^{-\ell_j})^{2k/(2k-1)}}{(1 - e^{-\ell_j})^2} + Q_j(d),
\]
where
\[
Q_j(d) = \left( 124k^2 P_j(d) \right)^{2r_k \ell_j}, \quad \text{with} \quad r_k = 1 + \left\lfloor \frac{k}{2k-1} \right\rfloor.
\]
Proof. As in the proof of [3] Lemma 3.4, we have for $|z| \leq aK/20$ with $0 < a \leq 2$,
\begin{equation}
\sum_{r=0}^{K} \frac{z^r}{r!} - e^z \leq \frac{|z|^K}{K!} \leq \left(\frac{ae}{20}\right)^K.
\end{equation}

By taking $z = \alpha \mathcal{P}_j(d)$, $K = \ell_j$ and $a = \min(|\alpha|, 2)$ in (3.9), we see that when $|\mathcal{P}_j(d)| \leq \ell_j/(20(1 + |\alpha|))$,
\[ N_j(d, \alpha) = \exp(\alpha \mathcal{P}_j(d)) + \sum_{r=0}^{\ell_j} \frac{(\alpha \mathcal{P}_j(d))^r}{r!} - \exp(\alpha \mathcal{P}_j(d)) \]
\[ \leq \exp(\alpha \mathcal{P}_j(d)) \left(1 + \exp(|\alpha \mathcal{P}_j(d)|) \left(\frac{ae}{20}\right)^{\ell_j}\right) \leq \exp(\alpha \mathcal{P}_j(d)) (1 + e^{-\ell_j}). \]

Similarly, we have
\[ N_j(d, \alpha) \geq \exp(\alpha \mathcal{P}_j(d)) \left(1 - \exp(|\alpha \mathcal{P}_j(d)|) \left(\frac{ae}{20}\right)^{\ell_j}\right) \geq \exp(\alpha \mathcal{P}_j(d)) (1 - e^{-\ell_j}). \]

We apply the above estimations to $N_j(d, 2k - 1), \mathcal{N}_j(d, k)$ to get that if $k > 1/2$ and $|\mathcal{P}_j(d)| \leq \ell_j/(60k)$, then
\begin{equation}
N_j(d, 2k - 1) \leq \frac{2^{j\ell_j}}{\ell_j!} \leq \exp(2k \mathcal{P}_j(d)) (1 + e^{-\ell_j}) \leq \mathcal{N}_j(d, 2k)|^2 (1 + e^{-\ell_j}) \frac{2^{j\ell_j}}{(1 - e^{-\ell_j})^2}.
\end{equation}

Moreover, if $|\mathcal{P}_j(d)| > \ell_j/(60k)$,
\begin{equation}
|N_j(d, 2k - 1)| \leq \sum_{r=0}^{\ell_j} \frac{|(2k - 1) \mathcal{P}_j(d)|^r}{r!} \leq |2k \mathcal{P}_j(d)|^{\ell_j} \sum_{r=0}^{\ell_j} \left(\frac{60k}{\ell_j}\right)^{\ell_j - r} \frac{1}{r!} \leq \left(\frac{124k^2 |\mathcal{P}_j(d)|}{\ell_j}\right)^{\ell_j}.
\end{equation}

The assertion of the lemma now follows from (3.10) and (3.11) and the observation that $\mathcal{P}_j(d)$ is real and $\ell_j$ is even. \hfill $\square$

4. Proof of Proposition 2.1

We define functions $b_j(n), 1 \leq j \leq R$ such that $b_j(n) \in \{0, 1\}$ and $b_j(n) = 1$ if and only if $n$ has all prime factors in $P_j$ such that $\Omega(n) \leq \ell_j$, where $\Omega(n)$ denotes the number of primes dividing $n$. We also define $w(n)$ to be the multiplicative function such that $w(p^n) = a!$ for prime powers $p^n$. Also write $\check{\lambda}_j(n)$ for the completely multiplicative function defined on primes $p$ by $\check{\lambda}_j(p) = \lambda_j(p)$. Using these notations, we arrive at
\begin{equation}
\mathcal{N}_j(d, 2k - 1) = \sum_{n_j} \frac{\check{\lambda}_j(n_j)}{\sqrt{n_j}} \frac{(2k - 1)^{\Omega(n_j)}}{w(n_j)} b_j(n_j) \chi_{sd}(n_j), \quad 1 \leq j \leq R.
\end{equation}

Observe that each $\mathcal{N}_j(d, 2k - 1)$ is a short Dirichlet polynomial as $b_j(n_j) = 0$ unless $n_j \leq X^{1/\ell_j} = X^{1/\ell_j}$. It follows from this and (2.2) that $\mathcal{N}(d, 2k - 1)$ is also a short Dirichlet polynomial of length at most $X^{1/\ell_1 + \ldots + 1/\ell_R} < X^{2/10M}$.

We expand the term $\mathcal{N}(d, 2k - 1)$ using (4.1) and apply Lemma 3.3 to evaluate the left-hand side of (2.5). In this process, we may ignore the contribution from the error term in Lemma 3.3 as $\mathcal{N}(d, 2k - 1)$ is a short Dirichlet polynomial. Thus we focus on the main term contribution. Writing $n_j = (n_{j_1})_2$ with $(n_{j_1})_2$ square-free, we get that
\[ \sum_{(d,2)=1}^{*} L(\frac{1}{2}, f \otimes \chi_{sd}) \mathcal{N}(d, 2k - 1) \Phi \left(\frac{d}{X}\right) \gg X \sum_{n_1, \ldots, n_R} \prod_{j=1}^{R} \frac{\check{\lambda}_j(n_j) \check{\lambda}_j((n_{j_1})_2)}{\sqrt{n_j(n_{j_1})}} b_j(n_j) \frac{1}{g(n_{j_1})} \]
\[ \times \left(\log \left(\frac{X}{(n_{j_1})_2 \cdots (n_{j_1})_1}\right) + C_2 + \sum_{p \mid n_{j_1} \cdots n_{j_1}} \frac{C_2(p)}{p \log p}\right). \]

We shall concentrate on the terms involving $\log(X/((n_{j_1})_1 \cdots (n_{j_1})_1))$. The other terms involving
\[ C_2 + \sum_{p \mid n_{j_1} \cdots n_{j_1}} \frac{C_2(p)}{p \log p} \]

can be shown, using the same argument in this section, to be
\[ \ll X (\log X)^{(2k^2+1)/2-1} \]

and hence negligible. Thus we deduce that
\[ \sum_{(d,2)=1}^{*} L(\frac{1}{2}, f \otimes \chi_{sd}) \mathcal{N}(d, 2k - 1) \Phi \left(\frac{d}{X}\right) \gg S_1 - S_2, \]
where
\[ S_1 = X \log X \sum_{n_1, \ldots, n_R} \left( \prod_{j=1}^R \frac{\tilde\lambda_f(n_j) \tilde\lambda_f((n_j)_1)}{\sqrt{n_j(n_j)_1}} \frac{(2k - 1)^{\Omega(n_j)}}{w(n_j)} b_j(n_j) \frac{1}{g(n_j)} \right) \]
and
\[ S_2 = X \sum_{n_1, \ldots, n_R} \left( \prod_{j=1}^R \frac{\tilde\lambda_f(n_j) \tilde\lambda_f((n_j)_1)}{\sqrt{n_j(n_j)_1}} \frac{(2k - 1)^{\Omega(n_j)}}{w(n_j)} b_j(n_j) \frac{1}{g(n_j)} \right) \log \left( \prod_{i=1}^R (n_i)_1 \right) . \]

It remains to estimate \( S_1 \) from below and \( S_2 \) from above. We bound \( S_1 \) first by recasting it as
\[ S_1 = X \log X \prod_{j=1}^R \sum_{n_j} \left( \frac{\lambda_f(n_j) \lambda_f((n_j)_1)}{\sqrt{n_j(n_j)_1}} \frac{(2k - 1)^{\Omega(n_j)}}{w(n_j)} b_j(n_j) \frac{1}{g(n_j)} \right) . \]

We consider the sum above over \( n_j \) for a fixed \( j \). Note that the factor \( b_j(n_j) \) restricts \( n_j \) to have all prime factors in \( P_j \) such that \( \Omega(n_j) \leq \ell_j \). If we remove the restriction on \( \Omega(n_j) \), then we use the fact that \( g(p^r) > 0 \) from Lemma 3.3 and \( k > 1/2 \) that the sum becomes
\[
\prod_{p \in P_j} \left( \sum_{i=0}^\infty \frac{\lambda_f^2(p) (2k - 1)^{2i}}{p^i} \frac{1}{g(p^{2i})} \right) = \prod_{p \in P_j} \left( 1 + \frac{(2k - 1)^2}{2} \frac{\lambda_f^2(p)}{pg(p)} \right) \geq \exp \left( \sum_{p \in P_j} \log \left( 1 + \frac{(2k - 1)^2}{2} \frac{\lambda_f^2(p)}{pg(p)} \right) \right) \geq \exp \left( \left( \frac{(2k - 1)^2}{2} \frac{\lambda_f^2(p)}{p} \right) \right) \times D_j, \]

where the last inequality above follows from the observation that \( \log(1 + x) = x + O(x^2) \) for all \( x > 0 \) and \( 1/g(p) = 1 + O(1/p) \) from Lemma 3.3. Here
\[ D_j = \exp \left( \sum_{p \in P_j} O \left( \frac{1}{p^2} \right) \right) > 0. \]

On the other hand, using Rankin’s trick, noting that \( 2^{\Omega(n_1) - \ell_1} \geq 1 \) if \( \Omega(n_1) > \ell_1 \), we conclude that the error introduced this way does not exceed
\[
\sum_{n_j} \frac{\lambda_f(n_j) \lambda_f((n_j)_1)}{\sqrt{n_j(n_j)_1}} \frac{(2k - 1)^{\Omega(n_j)}}{w(n_j)} \frac{1}{g(n_j)} \leq 2^{-\ell_j} \prod_{p \in P_j} \left( 1 + \sum_{i=1}^\infty \frac{\lambda_f^2(p) (2k - 1)^{2i} 2^{2i}}{(2i)!} \frac{1}{g(p^{2i})} + \sum_{i=0}^\infty \frac{\lambda_f^{2i+2}(p) (2k - 1)^{2i+2} 2^{2i+2}}{(2i+1)!} \frac{1}{g(p^{2i+1})} \right) .
\]

Now using the well-known bound
\[ 1 + x \leq e^x, \quad \text{for all } x \in \mathbb{R}, \]
the expression in (4.3) is
\[
\leq 2^{-\ell_j} \exp \left( \sum_{p \in P_j} \left( \sum_{i=1}^\infty \frac{\lambda_f^2(p) (2k - 1)^{2i} 2^{2i}}{(2i)!} \frac{1}{g(p^{2i})} + \sum_{i=0}^\infty \frac{\lambda_f^{2i+2}(p) (2k - 1)^{2i+2} 2^{2i+2}}{(2i+1)!} \frac{1}{g(p^{2i+1})} \right) \right) \]
\[
\leq 2^{-\ell_j} \exp \left( (2(2k - 1)^2 + 2(2k - 1)) \sum_{p \in P_j} \frac{\lambda_f^2(p)}{p} + O \left( \sum_{p \in P_j} \frac{1}{p^2} \right) \right) .
\]

We may take \( M \) sufficiently large so that every \( \ell_j, \, 1 \leq j \leq R \) is large and we may also take \( N \) large enough so that by Lemma 3.1
\[ \frac{\ell_j}{4N} \leq \sum_{p \in P_j} \frac{\lambda_f(p)^2}{p} \leq \frac{2}{N} \ell_j, \quad 1 \leq j \leq R. \]

It follows from (4.5) and (4.6) that the error incurred in discarding the condition on \( \Omega(n_j) \) is
\[
\leq 2^{-\ell_j/2} \exp \left( \left( \frac{(2k - 1)^2}{2} + 2k - 1 \right) \sum_{p \in P_j} \frac{\lambda_f^2(p)}{p} \right) D_j .
\]
Combining (4.2) and (4.7), we infer that the sum over $n_j$ for each $j$, $1 \leq j \leq R$ in the expression of $S_1$ is

$$
\geq (1 - 2^{-\ell_i/2}) \exp \left( \left( \frac{(2k - 1)^2}{2} + 2k - 1 \right) \sum_{p \in P_i} \frac{\lambda_f^2(p)}{p} \right) D_j.
$$

Hence

$$
(4.8) \quad S_1 \geq X \log X \prod_{j=1}^{R} \left( (1 - 2^{-\ell_i/2}) \exp \left( \left( \frac{(2k - 1)^2}{2} + 2k - 1 \right) \sum_{p \in P_i} \frac{\lambda_f^2(p)}{p} \right) D_j \right).
$$

Now, we estimate $S_2$ by expanding $\log \left( \prod_{i=1}^{R} (n_i)_1 \right)$ as a sum of logarithms of primes dividing $\prod_{i=1}^{R} (n_i)_1$ to obtain that

$$
(4.9) \quad S_2 \leq X \sum_{q \in \bigcup P_j} \left( \sum_{l \geq 0} \log q \frac{(2k - 1)^{2l+1} \lambda_f^{2l+2}(q)}{q^{l+1} g(q^{2l+1})} \prod_{i=1}^{R} \left( \sum_{(n_i)_1 = 1} \frac{\tilde{\lambda}_f(n_i) \tilde{\lambda}_f((n_i)_1)}{\sqrt{n_i(n_i)_1}} (2k - 1)^{\Omega(n_i)} \frac{1}{g(n_i)} \right) \right)
$$

where we define $\tilde{b}_{i,t}(n_i) = b_{i}(n_i q^t)$ for the unique index $i$ ($1 \leq i \leq R$) such that $b_{i}(q) \neq 0$ and $\tilde{b}_{i,t}(n_i) = b_{i}(n_i)$ otherwise.

We fix a prime $q \in P_{i_0}$ to consider the sum

$$
S_q := \sum_{l \geq 0} \left( \sum_{i \neq i_0, l \neq 0} \left( \log q \frac{(2k - 1)^{2l+1} \lambda_f^{2l+2}(q)}{q^{l+1} g(q^{2l+1})} \prod_{i=1}^{R} \left( \sum_{(n_i)_1 = 1} \frac{\tilde{\lambda}_f(n_i) \tilde{\lambda}_f((n_i)_1)}{\sqrt{n_i(n_i)_1}} (2k - 1)^{\Omega(n_i)} \frac{1}{g(n_i)} \right) \right) \right)
$$

As above, if we remove the restriction of $\tilde{b}_{i,t}$ on $\Omega(n_i)$, then the sum over $n_i$ becomes

$$
\prod_{p \in P_i, (p,q) = 1} \left( \sum_{m=0}^{\infty} \frac{\lambda_f^{2m+2}(p)}{p^{m+1}} \frac{(2k - 1)^{2m+1}}{(2m + 1)!} \left( \frac{1}{g(p^{2m+1})} \right) \right)
$$

where the last estimation above follows (4.3) and

$$
E_i = \prod_{p \in P_i} \left( 1 + O \left( \frac{1}{p^2} \right) \right) > 0.
$$

Similar to our discussions above, we see that the error introduced in this process is

$$
\leq \begin{cases} 
2^{-\ell_i/2} \exp \left( \left( \frac{(2k - 1)^2}{2} + 2k - 1 \right) \sum_{p \in P_i} \frac{\lambda_f^2(p)}{p} \right) E_i, & \text{if } i \neq i_0, \\
2^{2l - \ell_i/2} \exp \left( \left( \frac{(2k - 1)^2}{2} + 2k - 1 \right) \sum_{p \in P_i} \frac{\lambda_f^2(p)}{p} \right) E_i, & \text{if } i = i_0.
\end{cases}
$$

We deduce from this that

$$
S_q \leq \prod_{i=1}^{R} \exp \left( \left( \frac{(2k - 1)^2}{2} + 2k - 1 \right) \sum_{p \in P_i} \frac{\lambda_f^2(p)}{p} \right) E_i \prod_{i=1}^{R} \left( 1 + 2^{-\ell_i/2} \sum_{l \geq 0} \frac{\log q (2k - 1)^{2l+1} \lambda_f^{2l+2}(q)}{q^{l+1} g(q^{2l+1})} \right) \left( 1 + 2^{2l - \ell_i/2} \right).
$$

Applying the bound $|\lambda_f(p)| \leq 2$, Lemma 3.1 implies that for some constant $A$ depending on $k$ only,

$$
(4.10) \quad S_q \leq A \left( \frac{\log q}{q} + O \left( \frac{\log q}{q^2} \right) \right) \exp \left( \left( \frac{(2k - 1)^2}{2} + 2k - 1 \right) \sum_{p \in P_i} \frac{\lambda_f^2(p)}{p} \right).
$$
We then conclude from (4.9), (4.10) and Lemma 3.1 that

\[ S_2 \leq AX \exp \left( \left( \frac{(2k-1)^2}{2} + 2k - 1 \right) \sum_{p \in P_r} \frac{\lambda_j^2(p)}{p} \right) \sum_{q \in \mathbb{Q}_j} \left( \log q + O \left( \frac{\log q}{q^2} \right) \right) \]

(4.11)

\[ \leq AX \exp \left( \left( \frac{(2k-1)^2}{2} + 2k - 1 \right) \sum_{p \in P_r} \frac{\lambda_j^2(p)}{p} \right) \left( \log X + O(1) \right). \]

Combining (4.8) and (4.11), we deduce that, upon taking \( M \) large enough,

\[ S_1 - S_2 \gg X \log X \prod_{i=1}^{R} \exp \left( \left( \frac{(2k-1)^2}{2} + 2k - 1 \right) \sum_{p \in P_i} \frac{\lambda_j^2(p)}{p} \right). \]

The proof of the proposition now follows from this and Lemma 3.1.

5. PROOF OF PROPOSITION 2.2

We apply Lemma 3.4 to get that

\[ \sum_{(d,2)=1}^{\ast} N(d, 2k - 1)^{2k/(2k-1)} \Phi \left( \frac{d}{X} \right) \leq \sum_{(d,2)=1}^{\ast} \left( \prod_{j=1}^{R} \left( N_j(d, 2k) \left( \frac{1 + e^{-\ell_j}}{1 - e^{-\ell_j}} \right) + Q_j(d) \right) \right) \Phi \left( \frac{d}{X} \right) \]

(5.1)

\[ \leq \prod_{j=1}^{R} \max \left( \frac{1 + e^{-\ell_j}}{1 - e^{-\ell_j}} , 1 \right) \sum_{(d,2)=1}^{\ast} \prod_{j=1}^{R} \left( N_j(d, 2k) + Q_j(d) \right) \Phi \left( \frac{d}{X} \right), \]

where the last estimation above follows by noting that

\[ \prod_{j=1}^{R} \max \left( \frac{1 + e^{-\ell_j}}{1 - e^{-\ell_j}} , 1 \right) \ll 1. \]

We use the notations in Section 4 and write for \( 1 \leq j \leq R, \)

\[ N_j(d, 2k - 1) = \sum_{n_j} \tilde{\lambda}_j (n_j) (2k - 1)^{\Omega(n_j)} \sqrt{n_j} \frac{w(n_j)}{b_j(n_j)} \chi_{8d}(n_j) \quad \text{and} \quad P_j(d)^{2r_k \ell_j} = \sum_{\Omega(n_j)=2r_k \ell_j} \frac{\tilde{\lambda}_j(n_j)(2r_k \ell_j)!}{\sqrt{n_j} w(n_j)} \chi_{8d}(n_j), \]

where \( r_k = 1 + \left\lceil \frac{k}{2k-1} \right\rceil. \)

Applying the bound

\[ \left( \frac{n}{e} \right)^n \leq n! \leq n \left( \frac{n}{e} \right)^n, \]

we deduce that

\[ \left( \frac{124k^2}{\ell_j} \right)^{2r_k \ell_j} \leq (2r_k \ell_j)! \leq 2r_k \ell_j \left( \frac{248k^2 r_k}{e} \right)^{2r_k \ell_j}. \]

Note further we have \( \tilde{\lambda}_j(n_j) \leq 2^{\Omega(n_j)} \) as \( \lambda_j(p) \leq 2 \) and \( b_j(n_j) \) restricts \( n_j \) to satisfy \( \Omega(n_j) \leq \ell_j. \) It follows from the above discussions that we can write \( N_j(d, 2k) + Q_j(d) \) as a Dirichlet polynomial of the form

\[ D_j(d) = \sum_{n_j \leq X^{2r_k / \ell_j}} \frac{a_{n_j, b_j(n_j)}}{\sqrt{n_j}} \chi_{8d}(n_j) \]

where for some constant \( B(k) \) whose value depends on \( k \) only,

\[ |a_{n_j}| \leq B(k)^{\ell_j}. \]

We then apply Lemma 3.4 to evaluate the last expression in (5.1) and deduce from (5.2) that, for large \( M \) and \( N, \) the contribution arising from the error term in (5.3) is

\[ \ll B(k)^{\sum_{i=1}^{R} \ell_j} X^{1/2+\varepsilon} X^{2r_k} \sum_{j=1}^{R} \ell_j \ll B(k)^{R \ell_1} X^{1/2+\varepsilon} X^{4r_k/\ell_1} \ll X^{1-\varepsilon}. \]
We may thus focus on the contributions arising from the main term in (5.3). Thus the last expression in (5.1) is

\[
\ll X \times \sum_{n_j=\square} \left( \prod_{j=1}^R \frac{a_{n_j} b_j(n_j)}{\sqrt{n_j}} \prod_{p|n_j} \frac{1}{(p+1)} \right) = X \times \prod_{j=1}^R \sum_{n_j=\square} \left( \frac{a_{n_j} b_j(n_j)}{\sqrt{n_j}} \prod_{p|n_j} \frac{1}{(p+1)} \right),
\]

where the summation condition \( m = \square \) restricts the summand \( m \) to squares. We note that

\[
\sum_{n_j=\square} \frac{a_{n_j} b_j(n_j)}{\sqrt{n_j}} \prod_{p|n_j} \left( \frac{p}{p+1} \right) = \sum_{n_j=\square} \frac{\tilde{\lambda}_f(n_j)}{\sqrt{n_j}} \prod_{p|n_j} \left( \frac{p}{p+1} \right)
\]

\[
+ \left( \frac{124k^2}{\ell_j} \right)^{2r_k \ell_j} (2r_k \ell_j)! \sum_{n_j=\square} \frac{\tilde{\lambda}_f(n_j)}{\sqrt{n_j}} \prod_{p|n_j} \left( \frac{p}{p+1} \right).
\]

Now, using \(|\lambda_f(p)| \leq 2\), we obtain

\[
\sum_{n_j=\square} \frac{\tilde{\lambda}_f(n_j)}{\sqrt{n_j}} \prod_{p|n_j} \left( \frac{p}{p+1} \right) \leq \prod_{p \in P_j} \left( 1 + \frac{\lambda_f^2(p)(2k)^2}{2p} \left( \frac{p}{p+1} \right) + \sum_{\ell \geq 2} \frac{\lambda_f^2(p)(2k)^{2\ell}}{p^{\ell}(2\ell)!} \left( \frac{p}{p+1} \right) \right)
\]

\[
\leq \prod_{p \in P_j} \left( 1 + \frac{\lambda_f^2(p)(2k)^2}{2p} \left( \frac{p}{p+1} \right) + e^{4k} \sum_{\ell \geq 2} \frac{\lambda_f^2(p)}{p^{\ell}(2\ell)!} \right)
\]

where (4.4) is again utilized to obtain the last bound above.

Upon replacing \( n \) by \( n^2 \) and noting that \( \tilde{\lambda}_f(n^2_j) = \tilde{\lambda}_f^2(n_j), 1/w(n^2) \leq 1/w(n) \), we deduce that

\[
\left( \frac{124k^2}{\ell_j} \right)^{2r_k \ell_j} (2r_k \ell_j)! \sum_{n_j=\square} \frac{\tilde{\lambda}_f(n_j)}{\sqrt{n_j}} \prod_{p|n_j} \left( \frac{p}{p+1} \right) \leq \left( \frac{124k^2}{\ell_j} \right)^{2r_k \ell_j} (2r_k \ell_j)! \left( \sum_{p \in P_j} \frac{\lambda_f^2(p)}{p} \right)^{r_k \ell_j}.
\]

Applying (4.6) and (5.2) to estimate the right side expression above to get that for some constant \( B_1(k) \) depending on \( k \) only,

\[
\left( \frac{124k^2}{\ell_j} \right)^{2r_k \ell_j} (2r_k \ell_j)! \sum_{n_j=\square} \frac{\tilde{\lambda}_f(n_j)}{\sqrt{n_j}} \prod_{p|n_j} \left( \frac{p}{p+1} \right) \ll B_1(k)^{\ell_j} e^{-r_k \ell_j \log(r_k \ell_j)} \left( \sum_{p \in P_j} \frac{\lambda_f^2(p)}{p} \right)^{r_k \ell_j}
\]

\[
\ll B_1(k)^{\ell_j} e^{-r_k \ell_j \log(r_k \ell_j)} e^{r_k \ell_j \log(2\ell_j/N)} \ll e^{-\ell_j} \left( \sum_{p \in P_j} \frac{\lambda_f^2(p)}{p} \right).
\]

Combining the above with (5.3) – (5.5), we get that the expression in (5.1) is

\[
\ll X \prod_{j=1}^R \left( 1 + e^{-\ell_j} \right) \exp \left( \frac{(2k)^2}{2} \sum_{p \in P_j} \frac{\lambda_f^2(p)}{p} \right) + \sum_{p \in P_j} \frac{e^{4k}}{p^2} \ll X (\log X)^{12k}.
\]

where the last bound emerges by using Lemma 5.1. The assertion of the proposition now follows from this.

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