Cascades of Global Bifurcations and Chaos near a Homoclinic Flip Bifurcation: A Case Study

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July 29, 2022

Abstract

We study a specific homoclinic bifurcation called a homoclinic flip bifurcation of case C, where a homoclinic orbit to a saddle equilibrium with real eigenvalues changes from being orientable to nonorientable. This bifurcation is of codimension two, meaning that it can be found as a bifurcation point on a curve of homoclinic bifurcations in a suitable two-parameter plane. In fact, this is the lowest codimension for a homoclinic bifurcation of a real saddle to generate chaotic behavior in the form of (suspended) Smale horseshoes and strange attractors. We present a detailed numerical case study of how global stable and unstable manifolds of the saddle equilibrium and of bifurcating periodic orbits interact close to a homoclinic flip bifurcation of case C. This is a step forward in the understanding of the generic cases of homoclinic flip bifurcations, which started with the study of the simpler cases A and B. In a three-dimensional vector field due to Sandstede, we compute relevant bifurcation curves in the two-parameter bifurcation diagram near the central codimension-two bifurcation in unprecedented detail. We present representative images of invariant manifolds, computed with a boundary value problem setup, both in phase space and as intersection sets with a suitable sphere. In this way, we are able to identify infinitely many cascades of homoclinic bifurcations that accumulate on specific codimension-one heteroclinic bifurcations between an equilibrium and various saddle periodic orbits. Our computations confirm what is known from theory but also show the existence of bifurcation phenomena that were not considered before. Specifically, we identify the boundaries of the Smale–horseshoe region in the parameter plane, one of which creates a strange attractor that resembles the Rössler attractor. The computation of a winding number reveals a complicated overall bifurcation structure in the wider parameter plane that involves infinitely many further homoclinic flip bifurcations associated with so-called homoclinic bubbles.

1 Introduction

Homoclinic bifurcations lie at the heart of complicated dynamics in smooth vector fields. Apart from being interesting objects of study in their own right, homoclinic bifurcations appear in many applications, such as mathematical biology [22, 23], laser physics [39, 40] and electronic engineering [19]; more on their relevance for applications can be found, for example, in [14]. Indeed, the study of homoclinic bifurcations is important as a first step in the pursuit of understanding complex behavior that arises in mathematical models of different physical phenomena. Since the work of Shilnikov [35] on homoclinic bifurcations to a saddle focus (with complex eigenvalues), which is now called a Shilnikov bifurcation, a lot of attention has focused on the consequences of the existence of homoclinic bifurcations. Famously, the Shilnikov bifurcation, subject to certain eigenvalue conditions, is the only homoclinic bifurcation of codimension one that generates chaotic behavior in the form of (suspended) Smale–horseshoe dynamics; see [17] for a comprehensive review. This contrasts the situation of a codimension-one homoclinic bifurcation of a real saddle (with only real eigenvalues), which does not generate any nearby chaotic behavior. However, near the codimension-two homoclinic bifurcation of the real saddle, known as the homoclinic flip bifurcation, (suspended) Smale–horseshoes [15] and strange attractors [26, 27] arise when certain eigenvalue and geometric conditions are satisfied. More generally, homoclinic flip bifurcations can be of different cases (discussed below), and these cases act as organizing centers for the creation of periodic orbits and multi-pulse solutions [15]. Specifically, the existence

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of homoclinic flip bifurcations explains the creation of spiking behavior in mathematical models of neurons such as the Hindmarsh–Rose system [23].

This paper is about the homoclinic flip bifurcation of case C, which is the most complicated case that features chaotic dynamics. We consider here the defining situation of a three-dimensional vector field with a real saddle equilibrium. Without loss of generality, we assume that this equilibrium has two different real stable eigenvalues, and one unstable eigenvalue. Generically, at the moment of a codimension-one homoclinic bifurcation of a real saddle equilibrium, the two-dimensional stable manifold forms either a topological cylinder or a Möbius band as it is followed locally around the homoclinic orbit. The homoclinic flip bifurcation is characterized as the moment when this stable manifold changes from orientable to nonorientable [17, 18, 36, 43]. A homoclinic flip bifurcation has different unfoldings depending on the eigenvalues of the saddle equilibrium. In general, three generic cases have been identified, denoted A, B and C [17]. In case A a single attracting (or repelling) periodic orbit is created. The unfolding of case B involves saddle periodic orbits, and its unfolding features the main branch of homoclinic bifurcation, as well as only a single curve of saddle-node bifurcation, a single curve of period-doubling bifurcation and a single curve of homoclinic bifurcation of a period-doubled orbit [13, 17, 18]. Finally, the unfolding of case C gives rise to period-doubling cascades, n-homoclinic orbits, for any \( n \in \mathbb{N} \), a region of Smale–horseshoe dynamics [9, 15, 33], and strange attractors [20, 27]. Importantly, case C constitutes the homoclinic bifurcation of a real saddle with the lowest codimension that generates chaotic dynamics. The unfoldings for the three cases have been studied theoretically with different techniques including return maps [9, 15], Shilnikov variables [18] and Lin’s method [33]. The unfoldings for both cases A and B have been determined and proven for any vector field of dimension \( n \geq 3 \). The exact nature of the unfolding of case C is not as well understood as that for cases A and B; in particular, the complete unfolding is not known because infinitely many saddle periodic orbits are created and the interactions of their respective stable and unstable manifolds give rise to many other global bifurcations. This complexity is associated with the existence of Smale–horseshoes dynamics, which were conjectured to be part of the unfolding of a homoclinic flip bifurcation in [9] and subsequently proven in [15] to exist under specific eigenvalue and geometric conditions that define case C. More specifically, cascades of period-doubling and homoclinic bifurcations are featured in the unfolding to explain the annihilation process of the infinitely many saddle periodic orbits that exist in the Smale–horseshoe region [15]. However, the literature does not yet provide an understanding of the boundaries of the Smale–horseshoe region, the nature of additional bifurcations due to interactions of manifolds, or the implications of such interactions for the reorganization of basin of attraction in phase space. Moreover, all results concerning the dynamics and unfolding of case C have only been proven for three-dimensional vector fields. To conclude this discussion, in contrast to the other two cases, the understanding of case C is not as complete.

The existing theoretical results concerning the unfoldings of the homoclinic flip bifurcation of case C hold in a tubular neighborhood of the homoclinic orbit; they were obtained with well-known techniques by studying the family of local diffeomorphism defined on a Poincaré section. In this paper we adopt a complementary point of view. Thanks to advances in two-point boundary value problem (2PBVP) continuation techniques [38], we compute relevant invariant manifolds [20] and their interactions in a three-dimensional vector field introduced by Sandstede [34], which exhibits a homoclinic flip bifurcation of case C. In this way, we are able to present the overall dynamics near this bifurcation, beyond considering intersection sets on a Poincaré section. More specifically, our focus is on the role of global invariant manifolds of saddle equilibria and saddle periodic orbits in the organization of the three-dimensional phase space. These objects and surfaces rearrange themselves in sequences of global bifurcations that arise from the codimension-two homoclinic flip bifurcation point of case C. Our illustrations go well beyond the sketches of manifolds in earlier works (see, for example, [9, 16, 17]) in that we show these objects beyond a tubular neighborhood of the homoclinic orbit. Furthermore, to enhance the illustrated geometric properties of these objects, all our illustrations of the three-dimensional phase space are accompanied by animations. By studying the interactions of these manifolds in phase space, we find additional bifurcation phenomena in the specific case study of Sandstede’s model, which have been conjectured or not even considered before as part of the known literature of homoclinic flip bifurcation. In this way, our results contribute to the goal of finding the unfolding of the homoclinic flip bifurcation of case C, by presenting a set of relevant bifurcation phenomena that are involved.
As in previous work for cases A [1] and B [13], we consider the three-dimensional vector field

$$\begin{align*}
X^s(x,y,z) &= \begin{cases}
\dot{x} = P^1(x,y,z) := ax + by - ax^2 + (\bar{\mu} - \alpha z)x(2 - 3z) + \delta z, \\
\dot{y} = P^2(x,y,z) := bx + ay - \frac{3}{2}bx^2 - \frac{3}{2}axy - 2y(\bar{\mu} - \alpha z) - \delta z, \\
\dot{z} = P^3(x,y,z) := cz + \mu x + \gamma xz + \alpha \beta (x^2(1 - x) - y^2),
\end{cases}
\end{align*}$$

(1)

which was introduced by Sandstede in [34] as a model containing homoclinic flip bifurcations of all three different cases. We choose parameters as discussed in section 2.1, such that the equilibrium located at the origin 0 \(\in \mathbb{R}^3\) is a saddle with two different real negative (stable) and one positive (unstable) eigenvalues \(\lambda^{ss} < \lambda^s < 0 < \lambda^u\). Then the origin has a two-dimensional stable manifold \(W^s(0)\) and a one-dimensional unstable manifold \(W^u(0)\), which consist of the points in phase space that converge to 0 forward and backward in time, respectively. In particular, system (1) is at a homoclinic bifurcation when \(W^s(0) \cap W^u(0) \neq \emptyset\), that is, at the moment when there exists a trajectory that converges both to 0 forward and backward in time. System (1) exhibits a homoclinic bifurcation for \(\mu = 0\) [34], which is a homoclinic flip bifurcation of case C when \(\alpha \approx 0.3694818\). We denote this point by \(C_1\) throughout this paper, and we study the bifurcation diagram of system (1) in the \((\alpha, \mu)\)-plane near \(C_1\).

1.1 The codimension-two homoclinic flip orbit

Figure 1 shows phase portraits of system (1) at three different points along the homoclinic bifurcation curve given by \(\mu = 0\) in the \((\alpha, \mu)\)-plane. The individual panels show the homoclinic orbit together with the associated stable and unstable manifolds of 0, where the stable manifold \(W^s(0)\) is rendered in two shades of blue to illustrate its orientability; here, every trajectory that forms an orbit \(\Gamma\) given by \(\mu \approx 0.3694818\), where a homoclinic flip bifurcation takes place, which is called an inclination flip (IF). At this bifurcation point genericity condition (G3) is not fulfilled and \(W^s(0)\) does not close along \(W^{ss}(0)\); see fig. 1(c). It was proven in [2, 15, 18] that the inclination flip bifurcation is one mechanism by which one can transition between the orientable and nonorientable case of a homoclinic bifurcation of codimension one. It is not the only known mechanism: a transition in orientability of the homoclinic bifurcation may also occur via a so called orbit flip bifurcation, which corresponds to a homoclinic bifurcation that does not fulfill condition (G2). Both inclination and orbit flip are codimension-two homoclinic flip bifurcations whose unfolding can be of cases A, B and C.

For the purpose of this paper, we only consider the inclination flip bifurcation of case C, which occurs if one of the following eigenvalue conditions are satisfied [17]:

$$|\lambda^{ss}| < \lambda^u \quad \text{or} \quad 2|\lambda^s| < \lambda^u,$$

(2)

and the following geometric conditions are met:
Figure 1: The stable manifold $W^s(0)$ of system (1) in $\mathbb{R}^3$ at a codimension-one orientable and nonorientable homoclinic bifurcation in panels (a) and (b), respectively; panel (c) shows $W^s(0)$ at a codimension-two inclination flip bifurcation. Here, $W^s(0)$ is rendered transparent with one half colored dark blue and the other half light blue. The unstable manifold $W^u(0)$ is colored red and the homoclinic orbit is labeled $\Gamma_{\text{hom}}$. The one-dimensional strong stable manifold $W^{ss}(0)$ is highlighted by a blue curve. The common parameter values for all three panels are $(a, b, c, \beta, \gamma, \mu, \tilde{\mu}) = (0.7, 1, -2, 1, 2, 0, 0)$; furthermore, $\alpha = 0.2$ in panel (a), $\alpha = 0.5$ in panel (b) and $\alpha \approx 0.3694818$ in panel (c). See also the accompanying animation (GKO_Cflip_animatedFig1.gif).
(i) $|\lambda| \neq |\lambda^{ss}|/2$

(ii) If $|\lambda| < |\lambda^{ss}|/2$, the homoclinic orbit converges to 0 tangent to $e^{s}$ in a typical way. Geometrically, this means that the homoclinic orbit is not contained in the one-dimensional leading (weak) stable manifold tangent to 0, which exists uniquely as a smooth manifold under these eigenvalue conditions; for further details see [17].

(iii) If $|\lambda| > |\lambda^{ss}|/2$, the invariant manifold of the homoclinic orbit, which is tangent to the span\{\emph{e}^{s}, \emph{e}^{u}\} backward in time, has a nondegenerate quadratic tangency with $W^{s}(0)$ along the homoclinic orbit.

1.2 Bifurcation diagram of case C

The literature distinguishes between two different unfoldings of a homoclinic flip bifurcation of case C [15]. These are given by global conditions regarding the geometry of the stable manifold $W^{s}(0)$ during the creation of the Smale–horseshoe, and they are called an outward twist $C_{out}$ and an inward twist $C_{in}$ in the literature [9, 15]. Both twist cases contain the same codimension-one bifurcation curves but their relative positions differ. In particular, we find that the homoclinic flip bifurcation point $C_{I}$ found for system [1] corresponds to the outward twist case.

Figure 2 shows a sketch of what is known theoretically about the unfolding of a homoclinic flip bifurcation of case C for an outward twist $C_{out}$ [15] [16]. The unfolding consists of an orientable homoclinic bifurcation curve $H_{o}^{1}$ that becomes nonorientable (twisted) $H_{t}^{1}$ after transitioning through the codimension-two flip bifurcation point $C_{out}$. Furthermore, the following bifurcation curves emanate from $C_{out}$: a saddle-node bifurcation $SN^{1}$ of periodic orbits, an infinite sequence of period-doubling bifurcations labeled $PD^{n}$, $n = 1, 2, 4, 8, ..., n$ is a homoclinic bifurcations labeled $H_{t}^{n}$, $n = 2, 4, 8, ...,$ where $t$ indicates that these are codimension-one nonorientable (twisted) homoclinic bifurcations. The value of $n$ indicates the number of loops made by the homoclinic orbit before it connects back to the equilibrium point. As each homoclinic bifurcation occurs, a saddle periodic orbit with $n$ loops is created. The Smale–horseshoe region lies in between the period-doubling and homoclinic cascades; however, the boundaries of this region are not identified in the sketch. Note that the orientability of the saddle periodic orbits, labeled $1^{u}$ and $2^{u}$, is not explicitly stated in fig. 2 meaning that labels are not unique to each saddle periodic orbit.

To explain the overall structure of the unfolding, we start with the orientable saddle periodic orbit $1^{u}$ that bifurcates from the orientable homoclinic bifurcation $H_{o}^{1}$. This same label $1^{u}$ is also used for the nonorientable saddle periodic orbit that bifurcates from $H_{t}^{1}$. Note in the sketch that the orientable and nonorientable saddle periodic orbits persist throughout the cascades of period-doubling and homoclinic bifurcations, and...
Figure 3: Bifurcation diagram in the ($\alpha, \mu$)-plane near an homoclinic flip bifurcation $C_I$ of system (1). The inset shows an enlargement of the indicated area. Shown are the curves of homoclinic bifurcation $H_o$ and $H_t$ (brown), saddle-node bifurcations SNP of periodic orbits (dark-green), period-doubling bifurcations PD (red), heteroclinic bifurcations $Q^{\Gamma_o}_0$ from 0 to $\Gamma_o$ (magenta), heteroclinic bifurcation $Q^{\Gamma_t}_{0}$ from 0 to $\Gamma_t$ (purple) and codimension-one homoclinic bifurcation bifurcation $\text{Tan}_{\Gamma_o}$ of $\Gamma_o$ (violet). The Smale–horseshoe region $SH$ is the gray region. The other parameter values are $(a, b, c, \beta, \gamma, \bar{\mu}, \delta) = (0.7, 1, -2, 1, 2, 0, 0)$.

they disappear in the saddle-node bifurcation $SN^1$, after the nonorientable saddle periodic orbit becomes an attracting periodic orbit, labeled $1^s$, in the period-doubling bifurcation $PD^1$. Even though it is not illustrated in fig. 2, the unfolding is known to have infinitely many saddle-node bifurcation curves that also emanate from $C_{\text{out}}$ and occur along the period-doubling cascade [17].

Most of the literature on case $C$ does not delve into details about bifurcations related to interactions of manifolds of saddle periodic orbits. This is also the case in fig. 2 where the bifurcation diagram only shows local bifurcations of saddle periodic orbits and homoclinic bifurcations of the real saddle. For ease of exposition and in contrast to fig. 2, we label the orientable saddle periodic orbit that bifurcates from $H^1_o$ as $\Gamma_o$, and the nonorientable saddle periodic orbit that bifurcates from $H^1_t$ as $\Gamma_t$ throughout this paper.

1.3 Computing the bifurcation diagram near $C_I$

By computing the interaction of the manifolds of these saddle periodic orbits with other objects, we are able to present what is known about the unfolding of case $C$ in a new light, namely, as part of the bifurcation diagram in the ($\alpha, \mu$)-plane of system (1). Throughout this paper, we center our attention on the saddle periodic orbits $\Gamma_o$ and $\Gamma_t$, and their corresponding global manifolds. As a starting point, we show in fig. 3 the bifurcation diagram for system (1) in the ($\alpha, \mu$)-plane near the homoclinic flip point $C_I$ of case $C$. Note that we changed the orientation of the $\mu$-axis from top to bottom in the bifurcation diagram for ease of comparison. Figure 3 shows the principal homoclinic branch (brown curve) together with representative period-doubling PD (red curves) and saddle-node SNP (green curves) bifurcations of periodic orbits. To exemplify the relevance of the global bifurcations concerning saddle periodic orbits, we also compute representative codimension-one heteroclinic and homoclinic bifurcations in system (1) of the saddle periodic orbits $\Gamma_o$ and $\Gamma_t$, labeled $C^{\Gamma_o}_{0}$ and $Q^{\Gamma_o}_{0}[m\Gamma_o]$ for $m = 1, 2, 3, 4$. At first glance, the period-doubling and saddle-node bifurcation curves
are indiscernible near the inclination flip point $C_1$ in fig. 3. To visualize these curves, the inset shows an enlargement of a region of the bifurcation diagram that distinguishes the first three computed period-doubling bifurcation $PD_{r_1}$, $PD_{r_2}$, and $PD_{r_3}$ of a period-doubling cascade. Furthermore, it shows the saddle-node bifurcation curves $SNP_{r_1}$, $SNP_{r_2}$, and $SNP_{r_3}$ which are responsible for the disappearance of the orientable saddle periodic orbits created during the homoclinic cascade, as will be discussed in detail in section 5. In fact, by computing representative codimension-one homoclinic bifurcations close to $C_1$, with an implementation of Lin’s method in AUTO 9, we are able to obtain the bifurcating saddle periodic orbits; this allow us to find the subsequent saddle-node-bifurcation curves presented in fig. 3. For the remainder of this paper, the subindices in the label of each saddle-node bifurcation SNP of periodic orbits refers to the orientable saddle period orbit involved. The same applies to the period-doubling bifurcations PD, where the subindices refer to the corresponding nonorientable saddle periodic orbit involved. Note how the saddle-node bifurcation curves accumulate onto the codimension-one homoclinic bifurcation $Tan_{r_1}$ (violet curve) of $\Gamma_0$ where the stable and unstable manifolds of $\Gamma_0$ have a tangency. We identify the curve $Tan_{r_2}$ as one of the boundaries of the Smale–horseshoe region SH (gray region), as it delimits the region in the bifurcation diagram where there exist structurally stable homoclinic orbits of $\Gamma_0$.

Overall, the computed fig. 3 agrees well with the sketch in fig. 2 however, one of the most apparent differences are the curves $Q_0^{\alpha,\mu}[m\Gamma_0]$, with $m = 1, 2, 3, 4$, of heteroclinic connecting orbits between $\emptyset$ and $\Gamma_t$ (magenta curves). For each of these heteroclinic bifurcation curves, we find many homoclinic and heteroclinic bifurcation curves accumulating on it; these are not shown in fig. 3 because they lie very close to the curves $Q_0^{\alpha,\mu}[m\Gamma_0]$. We describe in more detail these accumulation cascades in section 5. Notice in fig. 3 how the sequence of heteroclinic bifurcation curves $Q_0^{\alpha,\mu}[m\Gamma_0]$ accumulates onto the heteroclinic bifurcation $Q_0^{\alpha,\mu}$ (purple curve) of a heteroclinic connecting orbit between $\emptyset$ and $\Gamma_0$, which corresponds to the other boundary of the Smale–horseshoe region SH in the bifurcation diagram. Both boundary curves, $Tan_{r_1}$ and $Q_0^{\alpha,\mu}$, were not included or studied in the theoretical unfolding as shown in fig. 2. Furthermore, the bifurcation $Tan_{r_1}$ acts as mechanism to destroy a strange attractor, which was conjectured to exist close to the inclination flip point due to the cascades of period-doubling and saddle-node bifurcation [26, 27]; this will be discussed in more detail in section 7.

The main purpose of this paper is to perform a detailed study of the role of representative invariant manifolds in the reorganization of phase space at different codimension-one bifurcations in the bifurcation diagram of system (1) near the point $C_1$. In this way, we obtain new insights into the bifurcation structures that must be expected as part of the unfolding of the inclination flip bifurcation of case C. Of particular interest is to understand the nature and roles of the infinitely many cascades of codimension-one homoclinic and heteroclinic bifurcations, the boundaries in parameter plane of the Smale–horseshoe regions, and the existence and annihilation of strange attractors, etc. To present our results, we choose parameter values in $(\alpha, \mu)$-plane close to $C_1$ and provide representative figures of the relevant objects in phase space, as well as their intersection sets with a suitable sphere. This allows us to show the consequences of different codimension-one bifurcations for the organization of phase space, and to present an overall picture of the unfolding through the bifurcation diagram of $C_1$ in the $(\alpha, \mu)$-plane. Motivated by the existence of infinitely many homoclinic and heteroclinic bifurcations, we define a winding number $\zeta$ as a topological invariant for system (1). We compute the value of $\zeta$ for system (1) as $\alpha$ and $\mu$ vary over a grid. Such a two-parameter sweep allows us to identify open regions in the $(\alpha, \mu)$-plane with constant winding number, both near $C_1$ and even further away from this central codimension-two point. The respective boundaries of these regions correspond to codimension-one homoclinic bifurcation curves. Hence, this parameter sweep contributes to obtaining a clearer picture of how the infinitely many homoclinic bifurcation are organized and accumulate onto heteroclinic bifurcations curves of the saddle periodic orbit $\Gamma_t$. Furthermore, by considering a larger parameter range in the $(\alpha, \mu)$-plane, we are able to identify and characterize a phenomenon where two different homoclinic bifurcation curves emanating from $C_1$ meet and create a structure referred to as a homoclinic bubble [16]. Such homoclinic bubbles do not arise as part of the unfolding in a small neighborhood of a codimension-two homoclinic flip bifurcation; however, they have been shown to be part of the unfolding of a codimension-three resonant homoclinic flip bifurcation, where they constitute a mechanism of transition between cases B and C [16, 28].

The organization of this paper is as follows. In section 2 we introduce some notation; here, we also present the parameter values used, and give the definition of the winding number $\zeta$. We present, in section 3 the bifurcation diagram of system (1) near $C_1$. Section 4 then focuses on the transition through the main
The parameter sweeping of ζ[10, 11] and its extension research.

We choose parameters such that the Jacobian arising in the bifurcation diagram of attractor that resembles the Rössler attractor. Finally, we characterize, in section 8, the bubble phenomenon on the Smale–horseshoe region, and section 7 on the period-doubling cascade and the existence of a strange codimension-one homoclinic bifurcation curve, section 5 on the homoclinic and heteroclinic cascades, section 6 of R surface foliated by trajectories that converge to 0.

We denote its basin of attraction as B(Γ). The manifold Ws(Γ) and Ws(Γ) are immersed manifolds in R³; they are as smooth as f and tangent to the linear spaces spanned by the stable and unstable eigenvectors of 0, respectively [31]. Furthermore, associated with λss, there is a unique one-dimensional strong stable manifold Wss(0) ⊂ Ws(0), defined as the subset of points on Ws(0) that converges to 0 tangent to the eigenvector eₚ.

System (1) has a second equilibrium q for the parameters chosen, which is a stable focus that lies near 0. We denote its basin of attraction as B(q). The set B(q) is an open connected set of R³ and consists of all points in phase space that converge to q. We also denote by Wss(q) ⊂ B(q) the subset of points that converge to q tangent to the eigenvector associated with the real eigenvalue of q, which is the largest eigenvalue in modulus.

Let Γ be a periodic orbit of system (1). We denote the two nontrivial Floquet multipliers of Γ by Λ₁, Λ₂ ∈ C; they are the eigenvalues of the variational equation of system (1) along Γ at time T, where T is the period of Γ. Note that the Floquet multipliers of Γ in a three-dimensional vector field are always such that their real parts have the same sign; each one has an associated eigenfunction that is referred as the Floquet bundle [37]. If Λ₁, Λ₂ ∈ R and 0 < |Λ₁| < 1 < |Λ₂| then one speaks of a saddle periodic orbit. A saddle periodic orbit has stable Ws(Γ) and unstable Ws(Γ) manifolds which consist of points that converge to Γ forward and backward in time, respectively. As for the saddle equilibrium case, the associated stable and unstable manifolds of a saddle periodic orbit are two dimensional immersed manifolds; they are tangent to the Floquet bundle of the periodic orbit associated with Λ₁ and Λ₂, respectively [31]. If 0 < Λ₁ < 1 < Λ₂, one speaks of an orientable saddle periodic orbit, which we denote by Γₒ, and its stable and unstable manifolds Ws(Γₒ) and Ws(Γₒ), respectively, are topological cylinders [29, 37]. Similarly, if Λ₂ < −1 < Λ₁ < 0, then the saddle periodic orbit is nonorientable, denoted Γᵣ, and Ws(Γᵣ) and Ws(Γᵣ) are locally topological Möbius bands [29, 37].

Let S₁, S₂, S₃ be hyperbolic saddle invariant objects of system (1) with manifolds Ws(S₁), Ws(S₂), Ws(S₂) and Ws(S₃) that are all two-dimensional. If there exist structurally stable heteroclinic orbits from S₁ to S₂ and from S₂ to S₃, that is, if Ws(S₁) ∩ Ws(S₂) and Ws(S₂) ∩ Ws(S₃) are non-empty transversal intersections, then we use the notation S₁ → S₂ → S₃ to represent this situation.

### 2.1 Sandstede’s Model

System (1) was constructed specifically to study different homoclinic flip bifurcations in three-dimensional vector fields [34]. It is a very convenient vector field, because its parameter can be chosen in such a way that either one of the cases A, B and C occur for both inclination and orbit flip bifurcations. System (1) has been used extensively, particularly to study numerically transitions between the three cases as codimension-three phenomena due to resonance [28], and to investigate the unfoldings of homoclinic flip bifurcations of cases A and B [1, 3].

Note that 0 is an equilibrium of $X^*$ for all parameter values. We use the parameter ranges found in [28] as a reference to study case C in the $(\alpha, \mu)$-plane and fix parameters as given in table 1. Since $\delta = 0$, the
| $a$ | $b$ | $c$ | $\beta$ | $\gamma$ | $\mu$ | $\delta$ |
|-----|-----|-----|--------|--------|------|------|
| 0.7 | 1.0 | -2.0 | 1.0    | 2.0    | 0.0  | 0.0  |

Table 1: Parameter values chosen for an inclination flip of case C in system (1).

Figure 4: Illustration of the change of the winding number $\zeta$ for system (1). Shown are $W^s(0)$ (red curve), $W^{ss}(0)$ (dark-blue curve) and $W^{ss}(q)$ (cyan curve), $\Sigma$ (green plane), the boundaries $\partial V$ (blue planes). The parameter values are $\mu = 0.001$, $\mu = 0.0$ and $\mu = -0.001$ for panels (a), (b) and (c), respectively. Furthermore, $\alpha = 0.5$ and the other parameter values are as given in table 1.

The value of $\zeta$ encodes the number of rotations that $W^u(0)$ makes around a tubular neighborhood of the homoclinic orbit that exists at the central singularity. In the present setting, $\zeta$ can be defined conveniently.
as the number of rotations around the one-dimensional manifold $W^{ss}(q)$, which always goes through ‘the hole’ of any such tubular neighborhood. To define the quantity $\zeta$ formally, let $V := \{p = (x, y, z) \in \mathbb{R}^3 \mid x \leq 0 \text{ and } y \leq 0\}$ and let $\partial V$ be its boundary. Since we set $b = 1 > 0$ and $\delta = 0$ in table 4, we have $x < 0$ on $\{x = 0\} \cap V \setminus \{y = 0\}$ and $y < 0$ on $\{y = 0\} \cap V \setminus \{x = 0\}$. Furthermore, the $z$-axis, $(x = 0) \cap \{y = 0\}$, is a subset of $V$ and it is invariant. Hence, $V$ is a positively invariant set for system 1, i.e., $\phi^t(V) \subset V$ for all $t \geq 0$ where $\phi^t$ is the flow defined by system 1. Therefore, any intersection of $W^u(0)$ with $\partial V$ must be transversal and $W^s(q)$ cannot intersect $\partial V$. We view $\zeta$, that is, the number of rotations that $W^u(0)$ makes around $W^{ss}(q)$, as a kind of linking number, which can only vary through bifurcation. Indeed, $W^{ss}(q)$ never intersects $\partial V$, and $W^u(0)$ cannot follow $W^{ss}(q)$ to undo itself by restriction of the flow of system 1; therefore, we may view these orbits segments as closed curves by identifying their endpoints.

Homoclinic bifurcations are a mechanism for $\zeta$ to change, as is illustrated in fig. 1 for system 1 with $\alpha = 0.5$ and three different values of $\mu$. In panel (a) the manifold $W^u(0)$ loops once around $W^{ss}(q)$ before reaching $\partial V$, that is, $\zeta = 1$. As $\mu$ decreases, system 1 goes through a homoclinic bifurcation at $\mu = 0$ in panel (b), after which $W^u(0)$ makes an extra turn around $W^{ss}(0)$ before intersecting $\partial V$, as shown in panel (c); hence, $\zeta$ increases to 2.

In practice, we calculate $\zeta$ by counting the number of intersections of $W^u(0)$ with $\Sigma := \{(x, y, z) \in \mathbb{R}^3 : x = q_x\}$, where $q_x$ is the $x$-component of $q$, and dividing this number by two; see fig. 1. The parameter sweeps of $\zeta$ shown in figs. 22 and 23 are performed on a $1000 \times 1000$ grid over the corresponding parameter ranges.

3 Bifurcation diagram near $C_1$

We now present more details of the bifurcation diagram shown in fig. 3. Namely, fig. 5 shows it over a slightly larger range of the $(\alpha, \mu)$-plane together with the parameter sweep of $\zeta$ where the colors indicate the value of the winding number $\zeta$; the inset shows an enlargement of the indicated region of the main panel. At first glance, we distinguish the curves of period-doubling, saddle-node and heteroclinic bifurcations presented in fig. 3. Recall that the subindices in the label of each saddle-node bifurcation SNP and period-doubling bifurcation PD of periodic orbits refers to the orientable and nonorientable saddle period orbit involved, respectively. Specifically for this bifurcation diagram, we only compute and label one of the infinitely many saddle-node and period-doubling curves that emanate from $C_1$, namely, the saddle-node bifurcation $\text{SNP}_{C_1}$ of $\Gamma_o$ (green curve) and period-doubling bifurcation $\text{PD}_{C_1}$ of $\Gamma_t$ (red curve).

We now discuss the overall features of the bifurcation diagram in fig. 5; further details will be presented in later sections. Note that, in between successive curves $Q_0^{\Gamma_t}[m\Gamma_o]$, with $m = 1, 2, 3, 4$, of heteroclinic connecting orbits between 0 and $\Gamma_t$ (magenta curves), there exist big regions of the $(\alpha, \mu)$-parameter plane with constant values of $\zeta$. The value of $\zeta$ in these regions increases as we approach the bifurcation $Q_0^{\Gamma_0^0}$ (purple) of a heteroclinic connecting orbit between 0 and $\Gamma_0$. This indicates the existence of more bifurcation curves $Q_0^{\Gamma_t}[m\Gamma_o]$, for $m > 4$. The labels of the codimension-one heteroclinic bifurcation are deliberately chosen to encode information of the corresponding heteroclinic orbit in phase space. Figure 6 shows, in phase space, representative heteroclinic orbits $Q_0^{\Gamma_t}$ and $Q_0^{\Gamma_0^0}$ (red curves) together with the saddle periodic orbits $\Gamma_t$ (purple curve) and $\Gamma_o$ (green curve) at the moment of the corresponding heteroclinic bifurcation. For simplicity, we use the same label of the heteroclinic bifurcation to refer to the corresponding orbit in phase space. The heteroclinic orbit is formed by one branch of $W^u(0)$, which is the branch shown in fig. 2. In panel (a1), the heteroclinic orbit $Q_0^{\Gamma_t}[\Gamma_o]$ makes an excursion close to the $(x, y)$-plane, and then follows $\Gamma_o$ for one rotation before accumulating onto $\Gamma_t$. In panels (a2)-(a5), the branch of $W^u(0)$ also accumulates onto $\Gamma_t$, but only after making two, three, four and five rotations near $\Gamma_o$, respectively. Hence, the number of rotations that the respective heteroclinic orbit of $Q_0^{\Gamma_t}$ makes in phase space around $\Gamma_o$ increases until it reaches its limiting case, where the number of rotations around $\Gamma_o$ has increased to infinity, at $Q_0^{\Gamma_0^0}$ as shown in panel (b). This accumulation is seen in fig. 5, where the cascade $Q_0^{\Gamma_t}[m\Gamma_o]$ accumulates on the final codimension-one heteroclinic bifurcation $Q_0^{\Gamma_0^0}$ of a connecting orbit between 0 and $\Gamma_o$ (dark-purple curve). Note also that the winding number $\zeta$ is at its computational maximum after this curve; this is due to the accumulation of the unstable manifold $W^u(0)$ onto $q$ after $Q_0^{\Gamma_0^0}$ is crossed, as will be detailed in section 6. This curve $Q_0^{\Gamma_0^0}$ represents the last moment where system 1 exhibits a homoclinic or heteroclinic bifurcation close to $C_1$, after which the dynamics is complicated due to the existence of structurally stable homoclinic orbits of $\Gamma_o$.

It was proved under the conditions of case C that a Hénon-like attractor unfolds from $C_1$ [20]. The
Figure 5: Bifurcation diagram in the \((\alpha, \mu)\)-plane near an inclination flip bifurcation \(C_I\) of case \(C\) of system \(1\), with coloring of regions according to the winding number \(\zeta\) as given by the colorbar. The inset shows an enlargement of the region indicated in the main panel. Shown are the curves of homoclinic bifurcation \(H_0\) and \(H_t\) (brown), non-principal homoclinic bifurcation (cyan), saddle-node bifurcation \(\text{SNP}_{\Gamma_o}\) and \(\text{SNP}^*\) of periodic orbits (dark green), Hopf bifurcation \(\text{HB}\) (light green), period-doubling bifurcation \(\text{PD}\) (red), fold bifurcation curve \(F\) of heteroclinic orbits from \(\Gamma_o\) to \(0\) (blue) and its extension \(F^*\) (blue dashed curve), heteroclinic bifurcation \(Q_{0}^{\Gamma}\) from \(0\) to \(\Gamma_t\) (magenta), heteroclinic bifurcation \(Q_{0}^{\Gamma}\) from \(0\) to \(\Gamma_o\) (purple) and codimension-one homoclinic bifurcation \(\text{Tan}_{\Gamma_o}\) of \(\Gamma_o\) (violet). The curves \(\text{SNP}_{\Gamma_o}\) and \(\text{SNP}^*\) meet at the cusp point \(\text{CP}\), and \(\text{SNP}^*\) ends at the generalized Hopf bifurcation point \(\text{GHB}\). The curve \(F\) meets \(Q_{0}^{\Gamma}\) at a codimension-two heteroclinic cycle point \(\text{Het}_{\text{Cycle}}\); the Smale–horseshoe region \(\text{SH}\) (gray region) is indicated. We denote by 1, 2, 3 and 4 four different regions close to the \(C_I\) point and the line \(\alpha = 0.5\) (dashed). The other parameters are as in table \(1\).
Smale–horseshoe region was conjectured to be bounded by a first quadratic tangency between the stable and unstable manifolds of a saddle periodic orbit \[ \Gamma_o \]. However, it was not even clear at the time that such a first tangency existed in the unfolding of an inclination flip point. We have found this first tangency and it corresponds to the bifurcation curve \( \text{Tan}_{\Gamma_o} \) (violet curve) in fig. 5 where the stable and unstable manifolds of \( \Gamma_o \) have a tangency. After leaving the Smale–horseshoe region through \( \text{Tan}_{\Gamma_o} \), we find a strange attractor in phase space whose corresponding branched manifold resembles that of the Rössler attractor. This attractor exists during a reverse period-doubling cascade which terminates at the bifurcation \( \text{PD}_{\Gamma_t} \), where the saddle periodic orbit \( \Gamma_t \) changes to an attracting periodic orbit \( \Gamma_a \) and the corresponding attracting periodic orbit \( 2\Gamma_a \) of twice the period disappears. Then \( \Gamma_t \) disappears in the saddle-node bifurcation \( \text{SNP}_{\Gamma_o} \) with \( \Gamma_o \).

Note in fig. 5 that we also find a bifurcation curve \( F \), similar to the ones found in \[1, 13\], that represents the moment when \( W^u(0) \) becomes tangent to \( W^s(\Gamma_o) \). If one follows the curve \( F \) in fig. 5, one sees that it intersects the curve \( Q_{\Gamma_o}^\Gamma \) at a point that we labeled \( \text{Het}_{\text{Cycle}} \) which is seen more clearly in the corresponding enlargement. At this point, there exists a codimension-two heteroclinic cycle between \( 0 \) and \( \Gamma_o \) in system (1). The unfolding of this cycle has been studied theoretically \[7, 24\] as an organizing center for the creation of chaotic regions in the parameter plane and the creation of multi-pulse homoclinic solutions that exhibit flip bifurcations close to it. This agrees with our computations, because the Smale–horseshoe region \( \text{SH} \) in fig. 5 also unfolds from the point \( \text{Het}_{\text{Cycle}} \); moreover, different homoclinic bifurcation branches in the bifurcation diagram exhibit inclination flip bifurcations, as will be discussed in detail in section 8.

Finally, note in fig. 5 that the additional stable equilibrium \( q \) of system (1) goes through a Hopf bifurcation labeled \( \text{HB} \) (light-green curve) and becomes an unstable saddle focus. On the curve \( \text{HB} \), there exists a generalized Hopf bifurcation point \( \text{GHB} \), which gives rise to a curve (dark-green) of saddle-node bifurcation \( \text{SNP}^* \) at which \( \Gamma_o \) and an attracting periodic orbit \( \Gamma^a \) are created. The curve \( \text{SNP}^* \) ends in a cusp bifurcation point \( \text{CP} \) with the curve \( \text{SNP}_{\Gamma_o} \); see the inset in fig. 5. The periodic orbits \( \Gamma_o \) and \( \Gamma_t \) disappear on the other side of \( \text{SNP}_{\Gamma_o} \) and only \( \Gamma^a \) persists; hence, the bifurcation curves \( Q_{\Gamma_o}^\Gamma \) and \( F \) disappear at the point \( \text{CP} \) and the curve \( \text{SNP}^* \), respectively. In particular, we find another curve \( F^* \) (dashed blue line), which

Figure 6: Configuration of the unstable manifold \( W^u(0) \) of system (1) in \( \mathbb{R}^3 \) at the codimension-one heteroclinic bifurcations. Shown are the heteroclinic branch of \( W^u(0) \) (red), and the periodic orbits \( \Gamma_o \) (purple) and \( \Gamma_t \) (green). Panels (a1)-(a5) illustrate connections from \( 0 \) to \( \Gamma_t \) and panel (b) shows the connection from \( 0 \) to \( \Gamma_o \). Here, we used \( \alpha = 0.5 \) and the respective \( \mu \)-values for each panel are given in table 3.
Table 2: Representative parameter values for the open regions 1–3 and the homoclinic bifurcations $H_o$ and $H_t$, as used in fig. 7; all other parameter values are as in table 1.

|     | 1  | 2  | 3  | $H_o$ | $H_t$ |
|-----|----|----|----|-------|-------|
| $\alpha$ | 0.200 | 0.500 | 0.200 | 0.500 | 0.500 |
| $\mu$   | 0.050 | 0.001 | -0.001 | -0.002 | 0.000 | 0.000 |

represents the moment when $W^s(0)$ becomes tangent to $W^u(q)$; this curve constitutes the extension of $F$ past its intersection with SNP$^*$.

4 Transition through the homoclinic bifurcations

In order to illustrate the difference between crossing the orientable homoclinic bifurcation $H_o$ and the nonorientable homoclinic bifurcation $H_t$ in fig. 5, we choose representative points in the regions 1 to 3 and at the bifurcations $H_t$ and $H_o$; see table 2 for the precise parameters chosen. In particular, we choose two points in region 1, labeled $1_o$ and $1_t$, which are close to $H_o$ and $H_t$, respectively. The phase portraits with the respective global invariant objects in $\mathbb{R}^3$ are shown in fig. 7.

Region 1 is characterized by the existence of the orientable saddle periodic orbit $\Gamma_o$, the saddle equilibrium $0$ and the stable-focus $q$. Panel $1_o$ of fig. 7 shows the corresponding phase portrait, which consists of the stable manifolds $W^s(0)$ (blue surface) and $W^s(\Gamma_o)$ (cyan surface), and the unstable manifolds $W^u(0)$ (red curve) and $W^u(\Gamma_o)$ (orange surface). Note that $W^u(\Gamma_o)$ is a topological cylinder that bounds the basin of attraction $S(q)$ of $q$. The manifold $W^s(0)$ accumulates from the outside onto $W^s(\Gamma_o)$ backward in time. One of the sheets that form $W^s(\Gamma_o)$ lies inside the topological cylinder $W^s(\Gamma_o)$ and, as such, it accumulates onto $q$; the other sheet is bounded by $W^u(0)$. The accumulation of $W^s(0)$ onto $W^s(\Gamma_o)$ and the fact that $W^u(0)$ bounds one sheet of $W^u(\Gamma_o)$ are due to the $\lambda$-lemma [31, 31]. The existence of a nontransversal intersection between $W^s(0)$ and $W^u(0)$ implies the existence of a structurally stable heteroclinic orbit from $\Gamma_o$ to $0$. Notice that $W^u(0)$ spirals once around $\Gamma_o$ before escaping to infinity, so that $\zeta = 1$ for this region. The boundary between regions $1_o$ to 2 is the codimension-one orientable homoclinic bifurcation $H_o$. At this bifurcation, the saddle periodic orbit $\Gamma_o$ becomes the homoclinic orbit $\Gamma_{hom}$ and the stable manifold $W^s(0)$ closes along its strong stable manifold $W^{ss}(0)$ in a topological cylinder; see panel $H_o$ of fig. 7. However, the moment one transitions into region 2, the homoclinic orbit $\Gamma_{hom}$ disappears, allowing $W^u(0)$ to accumulate onto $q$; see panel 2 of fig. 7. Hence, the $\zeta$-value in this region is infinite.

We now focus on the transition through the codimension-one nonorientable homoclinic bifurcation $H_t$. At the point $1_t$, system 1 is close to a nonorientable homoclinic bifurcation. Note that the computed manifolds in panels 1 and $1_t$ are topologically equivalent, but the way $W^s(0)$ approaches backward in time onto $W^{ss}(0)$ vastly differs. In particular, note in panel $1_t$ how the bottom part of the outer layer of $W^s(0)$ that accumulates onto $W^s(\Gamma_o)$ twists as it gets closer to $W^{ss}(0)$. In the transition from region $1_t$ to region 3, system 1 exhibits the bifurcation $H_t$ illustrated in panel $H_t$ of fig. 7. At this bifurcation, the stable manifold $W^s(0)$ closes along its strong stable manifold $W^{ss}(0)$ in a topological Möbius band. In contrast to $H_o$, the saddle periodic orbit $\Gamma_o$ persists and $\Gamma_{hom}$ bifurcated into the nonorientable saddle periodic orbit $\Gamma_t$; see panel 3 of fig. 7. Its stable manifold $W^s(\Gamma_t)$ (purple surface) accumulates onto $\Gamma_o$ backward in time, and $W^s(0)$ lies in between $W^s(\Gamma_o)$ and $W^s(\Gamma_t)$. Contrary to region 2, the unstable manifold $W^u(0)$ spirals twice around $W^s(\Gamma_o)$ and then escapes to infinity, so that $\zeta = 2$ for this region. Furthermore, the two-dimensional stable and unstable manifolds in region 3 have transverse intersections; in particular, we have a configuration of the form $\Gamma_o \rightarrow \Gamma_t \rightarrow 0$, that is, there exist heteroclinic orbits from $\Gamma_o$ to $\Gamma_t$ and from $\Gamma_t$ to $0$.

By applying the $\lambda$-lemma [31, 31] in a Poincaré section of $\Gamma_t$, we can show that $\Gamma_o \rightarrow \Gamma_t \rightarrow 0$ implies, in fact, the existence of infinitely many heteroclinic orbits from $\Gamma_o$ to $0$. Hence, we have the following:

**Remark 4.1.** The phase portrait in region 3 is topological equivalent to that in region 4 for case B if we contract $\Gamma_o$ to the point $q$; see [13].
Figure 7: Phase portraits of system (1) in regions 1–3 of the \((\alpha, \mu)\)-plane and along the bifurcations \(H_0\) and \(H_t\) near the inclination flip point \(C_1\). Shown are \(W^s(0)\) (blue surface), \(W^{ss}(0)\) (blue curve), \(W^u(0)\) (pink curve), \(W^s(\Gamma_o)\) (cyan surface) and \(W^s(\Gamma_t)\) (purple surface). The outer rims of the two-dimensional stable manifolds are highlighted in black. The \((\alpha, \mu)\)-values for each panel are given in table 2. See also the accompanying animation (GKO_Cflip_animatedFig7.gif).
\begin{figure}
\centering
\includegraphics[width=\textwidth]{Fig8}
\caption{Illustration of the intersection of the stable manifolds of $0$ and $\Gamma_0$ in system (1) with the sphere $S^*$ of radius 0.6 and center $(0.5,0,0)$; see also the accompanying animation (GKO_Cflip_animatedFig8.gif).}
\end{figure}

\section{4.1 Intersection sets with a sphere}

We study now the intersection sets of the stable manifolds with the sphere $S^* := \{x \in \mathbb{R}^3 : ||x - c|| = R\}$, where $c := (c_x, c_y, c_z) = (0.5,0,0)$ and $R = 0.6$. Since $S^*$ is a compact set, all intersections of the stable manifolds with $S^*$ are bounded. In particular, two-dimensional manifolds (in general position) lead to curves on $S^*$, while one-dimensional stable manifolds intersect $S^*$ in points. This is illustrated in fig. 8 for the invariant manifolds in region 1t.

Next, we stereographically project these intersection sets onto one of the tangent planes of $S^*$ to present them as two-dimensional pictures. More precisely, we first apply the transformation

$$\pi = \left(\frac{u}{|u|}, \frac{v}{|v|}, \frac{w}{|w|}\right),$$

formed by the vectors

$$u = (0.5706, 0.1854, 0)^T, v = (1, -(u_x + u_z)/u_y, 1)^T \text{ and } w = u \times v.$$

We rotate and translate points $(x, y, z) \in S^*$ to points $(x', y', z')$ on the sphere of radius $R = 0.6$ centered at the origin, that is, $(x', y', z') = \pi(x - c)$. This transformation rotates the stereographic point, the red dot in fig. 8, so that it corresponds to the south pole of the transformed sphere. We then use stereographic projection from the south pole onto the tangent plane at the north pole, via the transformation

$$(x', y', z') \mapsto \left(\frac{R x'}{R + z'}, \frac{R y'}{R + z'}\right).$$

This particular stereographic projection was chosen to improve the visibility of certain features of the intersection sets specific to case C.

Figure 9 shows the same series of phase portraits from fig. 7 in terms of their intersection sets of stable manifolds with $S^*$. The left column of fig. 9 illustrates the transition through $H_0$ and the right column through $H_{1t}$. Starting from panel 1o in fig. 9, note how the intersection set $\hat{W}^s(\Gamma_o)$ (cyan curve) is made up
Figure 9: Stereographic projections of the intersection sets of the invariant manifolds with $S^*$ in regions 1–3 and at the homoclinic bifurcations $H_o$ and $H_t$ near the inclination flip point $C_I$. The left and right columns show the transition through the homoclinic bifurcations $H_o$ and $H_t$, respectively. Shown are $\hat{W}^s(0)$ (dark-blue curves), $\hat{W}^{ss}(0)$ (light-blue dots), $\hat{W}^s(\Gamma_o)$ (cyan curve), $\hat{W}^s(\Gamma_t)$ (purple curve) and $\hat{B}(q)$ (shaded yellow region). See table 2 for the respective parameter values.
of two topological circles that bound the disconnected basin $\mathcal{B}(q)$, that is, $\partial\mathcal{B}(q) = \hat{W}^s(\Gamma_o)$. The intersection $\hat{W}^s(0)$ (blue curve) is a single curve that accumulates in a spiralling manner backward in time onto $\hat{W}^s(\Gamma_o)$. The accumulation is as expected: the $\lambda$-lemma implies that the transversal heteroclinic orbit from $\Gamma_o$ to 0 causes $\hat{W}^s(0)$ to spiral around $\hat{W}^s(\Gamma_o)$. Note that these phenomena are only observed if the sphere $S^*$ is chosen small enough. At the moment of the bifurcation, shown in panel $H_o$, the curve $\hat{W}^s(0)$ closes back on itself along $\hat{W}^{ss}(0)$ and $\hat{W}^s(\Gamma_o)$ disappears as $\Gamma_o$ becomes the homoclinic orbit $\Gamma_{hom}$; the basin $\mathcal{B}(q)$ is now bounded by a subset of $\hat{W}^s(0)$, that is, $\partial\mathcal{B}(q) \subset \hat{W}^s(0)$. Hence, at $H_o$, the boundary of the basin of attraction of $q$ is contained in $\hat{W}^s(0)$. As soon as we enter region 2, the entire intersection set $\hat{W}^s(0)$ becomes the boundary of $\mathcal{B}(q)$, which is now a connected region that is topologically equivalent to an open disk; see panel 2.

In panel $I_t$ of fig. 9 one sees the same topological configuration as in panel $I_o$, because both phase portraits are from the same region; however, note how only one end of $\hat{W}^s(0)$ gets close to both points in $\hat{W}^{ss}(0)$ in panel $I_t$. At the nonorientable homoclinic bifurcation $H_t$, the intersection set $\hat{W}^s(0)$ connects back on itself at $\hat{W}^{ss}(0)$ in a different way, as is illustrated in panel $H_t$ of fig. 9. Here, $\hat{W}^s(0)$ does not bound two open regions as in panel $H_o$. Instead, at $H_t$ the two segments that form $\hat{W}^s(0)$ accumulate onto the two topological circles $\hat{W}^s(\Gamma_o)$ due to the persistence of the heteroclinic orbit from $\Gamma_o$ to 0. The nature of $\mathcal{B}(q)$ and its boundary is unchanged from panel $I_o$. As one transitions to region 3, the homoclinic orbit $\Gamma_{hom}$ becomes the periodic orbit $\Gamma_t$ and the intersection set $\hat{W}^s(\Gamma_t)$ (purple curve) is a single curve that accumulates at both ends onto $\hat{W}^s(\Gamma_o)$ backward in time, due to the existence of a heteroclinic orbit from $\Gamma_o$ to $\Gamma_t$. As mentioned before, there are infinitely many heteroclinic orbits from $\Gamma_o$ to 0 in region 3. Since $S^*$ is a sufficiently small sphere that is transverse to $\hat{W}^s(0)$, there exist infinitely many curves in $\hat{W}^s(0)$. One curve in $\hat{W}^s(0)$ accumulates onto a single topological circle of $\hat{W}^s(\Gamma_o)$, and all other curves accumulate onto both topological circles of $\hat{W}^s(\Gamma_o)$. These sets of curves get arbitrary close to $\hat{W}^s(\Gamma_t)$ from both sides; we say that $\hat{W}^s(\Gamma_t)$ is a geometric accumulation curve of $\hat{W}^s(0)$. In panel 3 of fig. 9 we only show three of these infinitely many intersection curves of $\hat{W}^s(0)$, namely, the outer curve that accumulates onto the single topological circle of $\hat{W}^s(\Gamma_o)$, and two other curves that track $\hat{W}^s(\Gamma_t)$ on both sides. We remark that the intersection sets shown in fig. 9 are homotopic to the intersection sets shown in 13 for the transition through $H_o$ and $H_t$ in case B, provided one contracts $\hat{W}^{ss}(\Gamma_o)$ to $\hat{W}^s(q)$ through $H_o$ and blows up $\hat{W}^s(q)$ to a closed curve through $H_t$ in case B.

The fact that $\Gamma_o \rightarrow \Gamma_t \rightarrow 0$ implies the existence of infinitely many structurally stable heteroclinic orbits from $\Gamma_o$ to 0 is going to be a recurrent phenomenon for different equilibria and saddle periodic orbits. For ease of exposition, we summarize the consequences in general terms:

**Proposition 4.2.** Let A, B and C be hyperbolic saddle equilibria or periodic orbits of system (7). If $A \rightarrow B \rightarrow C$ then there exist infinitely many structurally stable orbits from A to C. Furthermore, if $\hat{W}^s(A)$ intersects $S^*$ transversally then there exist infinitely many curves in $\hat{W}^s(C)$. Each curve in $\hat{W}^s(C)$ accumulates onto $\hat{W}^s(A)$ backward in time, and $\hat{W}^s(B)$ is a geometric accumulation curve of $\hat{W}^s(C)$.

**Proof.** The proof follows from the $\lambda$-lemma and is a variation of the proof given in 13 for the case $q \rightarrow \Gamma_t \rightarrow 0$.

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**5 Cascades of homoclinic and heteroclinic bifurcations**

We now focus our attention on the invariant manifolds of system (1) during the cascades of homoclinic and heteroclinic bifurcations, which we study and illustrate along the line $\alpha = 0.5$ in the $(\alpha, \mu)$-plane.

**5.1 Transition through $^2H_t$**

Starting from region 3, we first encounter the codimension-one nonorientable homoclinic bifurcation $^2H_t$, which is shown in fig. 10 as a stereographic projection of the stable invariant objects on $S^*$ that exist in phase space. Panel (a) shows a sketch of the intersection sets in region 3 shown in fig. 9. The pink region indicates the relative location of the infinitely many curves in $\hat{W}^s(0)$ that accumulate geometrically onto $\hat{W}^s(\Gamma_t)$. At
Figure 10: Stereographic projection of the intersection sets of the invariant manifolds with $S^*$ before, at, and after the moment of the codimension-one homoclinic bifurcation $^2H_t$ in panels (a), (b1) and (b2), and (c), respectively. Panel (a) shows a sketch of the stereographic projection in region 3; panels (b1) and (b2) show a sketch and the computed projections of system (1) at $^2H_t$, respectively; and panel (c) shows a sketch after $^2H_t$. The color code and labeling of the regions is as in fig. 9. Panel (b2) is for parameter values $(\alpha, \mu) = (0.5, -0.002880267)$. 
the moment of the codimension-one homoclinic bifurcation $^2H_4$, the outer-most curve in $\hat{W}^s(0)$ that connects both topological circles of $\hat{W}^s(\Gamma_0)$ splits into two curves that close on $\hat{W}^s(0)$ at the two points $\hat{W}^s(0)$; see the sketch in panel (b1) of fig. 10 and the associated computed phase portrait on $S^*$ in panel (b2). The bifurcation $^2H_4$ gives rise to the saddle periodic orbit $\Gamma_t$, and we find the following transversal connections: $\Gamma_o \to \Gamma_0 \to 0$, $\Gamma_o \to \Gamma_t \to 2\Gamma_t$ and $\Gamma_o \to 2\Gamma_t \to \Gamma_t$. Panel (c) illustrates the situation by way of a representative sketch. Note how the two curves of $\hat{W}^s(0)$ that end at $\hat{W}^{s\ast}(0)$ in panel (b1) give rise to the two light-green regions, which represent the accumulation onto $\hat{W}^s(2\Gamma_t)$ (dark-cyan curves) of infinitely many curves in $\hat{W}^s(0)$ and $\hat{W}^s(\Gamma_t)$ due to the existence of $\Gamma_o \to 2\Gamma_t \to 0$ and $\Gamma_o \to 2\Gamma_t \to \Gamma_t$; we refer to proposition 4.2. Both circles in $\hat{W}^s(\Gamma_0)$ bound one of these two regions, but only one circle in $\hat{W}^s(\Gamma_o)$ bounds the other region. Furthermore, since $\Gamma_o \to \Gamma_t \to 0$ and $\Gamma_o \to \Gamma_t \to 2\Gamma_t$, the pink region must contain infinitely many curves of $\hat{W}^s(2\Gamma_t)$, and infinitely many curves in $\hat{W}^s(0)$ also accumulate geometrically onto these curves. The existence of both pink and light-green regions is important. As $\mu$ decreases, the outermost curve from the region that is closest to $\hat{W}^{s\ast}(0)$ will meet $\hat{W}^{s\ast}(0)$ at a certain value $\mu^*$; depending on whether this is a curve from $\hat{W}^s(0)$, $\hat{W}^s(\Gamma_t)$ or $\hat{W}^s(2\Gamma_t)$, system (1) exhibits a particular corresponding codimension-one homoclinic or heteroclinic bifurcation. Furthermore, as the infinitely many curves in $\hat{W}^s(0)$ accumulate onto either $\hat{W}^s(\Gamma_t)$ or $\hat{W}^s(2\Gamma_t)$, infinitely many homoclinic bifurcations must occur before a codimension-one heteroclinic bifurcation can take place. Whenever a homoclinic bifurcation occurs, the above sequence of different kinds of bifurcations occurs again; that is, an additional saddle periodic orbit is created, accompanied by two regions of accumulation with respect to the stable manifold of the new saddle periodic orbit. Moreover, an accumulation of intersection curves from this stable manifold is created inside the already existing regions. In addition, the closer $\hat{W}^s(0)$ lies to $\hat{W}^s(\Gamma_t)$ or $\hat{W}^s(2\Gamma_t)$, the higher the number of rotations are for both the homoclinic orbit and the bifurcating saddle periodic orbit.

5.2 Transition through $Q^{\Gamma}_0[\Gamma_o]$

We illustrate in fig. 11 the transition through the codimension-one heteroclinic bifurcation $Q^{\Gamma}_0[\Gamma_o]$ and its effect on the reorganization of the stable manifold $W^s(\Gamma_t)$ (purple surface) in phase space. The left column shows $W^s(\Gamma_t)$ in a tubular neighborhood of $\Gamma_t$, showing that $W^s(\Gamma_t)$ is topologically a Möbius band. The right column (2) shows a larger part of $W^s(\Gamma_t)$. Panels (a1) and (a2) show $W^s(\Gamma_t)$, at $\mu = -0.002$, before the bifurcation $Q^{\Gamma}_0[\Gamma_o]$ and the associated first homoclinic cascade; panels (b1) and (b2) are at the moment of the bifurcation, when $\mu \approx -0.002880324$; and panels (a3) and (b3) are for $\mu = -0.0035$, in region 4 past the first cascade; see fig. 3.

We first focus on the local behavior in a tubular neighborhood. In panel (a1), the unstable manifold $W^u(0)$ (red curve) makes one rotation close to $\Gamma_o$ and one close to $\Gamma_t$ and then escapes to infinity. As discussed before, a sequence of homoclinic bifurcations must occur before the bifurcation $Q^{\Gamma}_0[\Gamma_o]$ is exhibited by system (1). During this sequence, the unstable manifold $W^u(0)$ gains turns around $W^s(\Gamma_t)$, thus increasing $\zeta$, until its limiting case at the bifurcation $Q^{\Gamma}_0[\Gamma_o]$. However, note how the first loop of $W^u(0)$ is still close to $\Gamma_o$ before accumulating onto $\Gamma_t$; this situation is the same as in panel (a1) of fig. 6. After $Q^{\Gamma}_0[\Gamma_o]$, a sequence of homoclinic bifurcations occur which decreases the winding number of $W^u(0)$ around $W^s(\Gamma_t)$, thus decreasing $\zeta$. Finally in region 4, the unstable manifold $W^u(0)$ makes two rotation close to $\Gamma_o$ and one again close to $\Gamma_t$ before escaping to infinity; see panel (c1). In particular, the orientation of the homoclinic bifurcation of 0 that follows, depends on how $W^u(0)$ passes near $\Gamma_o$ and $\Gamma_t$.

We gain a deeper understanding by studying a larger portion of $W^s(\Gamma_t)$ as shown in the right column of fig. 11. Panel (a2) illustrates how $W^s(\Gamma_t)$ spirals around $W^s(\Gamma_o)$ due to the presence of heteroclinic orbits from $\Gamma_o$ to $\Gamma_t$; the outer rim of the computed part of $W^s(\Gamma_t)$ is highlighted as the black curve. Note how part of $W^s(\Gamma_t)$ gets close to $W^{s\ast}(0)$ before starting to spiral again. In panel (b2), this part of $W^s(\Gamma_t)$ has disappeared, and the existence of the heteroclinic orbit $Q^{\Gamma}_0[\Gamma_o]$ forces $W^s(\Gamma_t)$ to accumulate onto $W^{s\ast}(0)$ backward in time. After the bifurcation $Q^{\Gamma}_0[\Gamma_o]$, the manifold $W^s(\Gamma_t)$ no longer accumulates onto $W^{s\ast}(0)$; see panel (c2). Observe that $W^s(\Gamma_t)$ in panel (c2) seems identical to $W^s(\Gamma_t)$ in panel (b2), but it exhibits one additional loop. This extra loop would spread out if a much larger portion of $W^s(\Gamma_t)$ were computed, and it would get close to $W^{s\ast}(0)$ in the same way as in panel (a2).

Figure 12 illustrates the transition to region 4 on the level of the intersection sets of the invariant objects with $S^*$. Panels (a1) and (a2) are stereographic projections of the intersection sets with $S^*$ at the moment.
Figure 11: The stable manifold $W^s(\Gamma_t)$ (purple surface) of the nonorientable saddle periodic orbit $\Gamma_t$ (green curve), together with $W^u(0)$ (red curve) and $W^{ss}(0)$ (blue curve) before, at, and after the moment of the codimension-one heteroclinic bifurcation $Q_{\Gamma_t}^{\Gamma_0}$ of system (1) in rows (a), (b) and (c), respectively. The left column shows $W^s(\Gamma_t)$ in a tubular neighborhood of $\Gamma_t$, and the right column shows a larger part of $W^s(\Gamma_t)$. Rows (a), (b) and (c) are for $(\alpha,\mu) = (0.5, -0.002)$, $(\alpha,\mu) = (0.5, -0.0032880324)$ and $(\alpha,\mu) = (0.5, -0.0035)$, respectively. See also the accompanying animation (GKO_Cflip_animatedFig11.gif).
Figure 12: Stereographic projection of the intersection sets of the invariant manifolds with $S^*$ at the codimension-one heteroclinic bifurcation $Q_0^{\Gamma_o}$ in region 4, and at the codimension-one heteroclinic bifurcation $Q_0^{\Gamma_t}$ in rows (a) and (b), respectively. The left column shows the computed projections and the right column corresponding sketches. The color code and labeling of the regions is the same as in fig. 9. Panels (a1), and (b1) are for $(\alpha, \mu) = (0.5, -0.002880324)$ and $(\alpha, \mu) = (0.5, -0.0035)$, respectively.
of the first codimension-one heteroclinic bifurcation $Q_{0}^{s}[\Gamma_0]$ of $\Gamma_0$. At the moment of the bifurcation the curve $\hat{W}^s(\Gamma_1)$ (purple) splits into two curves that meet at the two points in $\hat{W}^{ss}(0)$; see panel (a1). This phenomenon is illustrated more clearly in the sketch shown in panel (a2), where the accumulation of curves in $\hat{W}^s(\Gamma_1)$ is represented by the pink shading. As mentioned before, each homoclinic bifurcation creates a saddle periodic orbit that gives rise to two new accumulation curves in the intersection set with $S^s$; these are shaded in panel (a2) with different tones of cyan and the label indicates the intersection curve of the principal stable manifold. Only two pairs of accumulation regions are indicated in panel (a2). The gray region represents the infinitely many regions created in further homoclinic bifurcations and specifies their relative location in the stereographic projection. It is clear from panels (a1) and (a2) of fig. 12 that, after crossing of $Q_{0}^{s}[\Gamma_0]$, there is another cascade of homoclinic bifurcations that terminates on the homoclinic bifurcation associated with the blue curve in $\hat{W}^s(0)$ that bounds the pink region in panel (a2); this last homoclinic bifurcation corresponds to the last bifurcation that system (4) exhibits before transitioning into region 4 in fig. 5. Panels (b1) and (b2) in fig. 12 show the situation in region 4 on the level of the stereographic projections. In particular, each accumulation region is represented by the main intersection curve of the stable manifold of the saddle periodic orbit that exists on it. Observe how one of the curves of $\hat{W}^s(\Gamma_0)$ and, thus, the corresponding accumulation region accumulate onto a single topological circle of $\hat{W}^s(\Gamma_0)$, while the other curve accumulates on both topological circles of $\hat{W}^s(\Gamma_0)$. This is true for each of the accumulation regions of the different saddle periodic orbits, while they accumulate at the same time onto $\hat{W}^s(\Gamma_1)$ (gray region).

5.3 Transition through $Q_{0}^{s}$

The final step at the end of all cascades is the heteroclinic bifurcation $Q_{0}^{s}$. The two-dimensional stable manifold $W^s(\Gamma_0)$ does not interact with any saddle invariant object, and it retains its topological properties during the cascades of homoclinic and heteroclinic bifurcations. However, this changes as soon as $W^s(\Gamma_0)$ intersects $W^u(0)$ at the bifurcation $Q_{0}^{s}$, which is illustrated in fig. 13. Specifically, the left column shows $W^s(\Gamma_0)$ (cyan surface), together with $W^u(0)$ (red curve), in a tubular neighborhood of $\Gamma_0$ and the right column shows larger parts of these manifolds. Panels (a1) and (a2) show, at $\mu = -0.002$, the situation before the bifurcation $Q_{0}^{s}$, that is, during the homoclinic and heteroclinic cascades; panels (b1) and (b2) are for $\mu \approx -0.004861805$ at the moment of the bifurcation $Q_{0}^{s}$; and panels (a3) and (b3) are for $\mu = -0.005$, after the bifurcation. It seems that, in the vicinity of $\Gamma_0$, the transition through $Q_{0}^{s}$ does not manifest itself as any topological change for $W^s(\Gamma_0)$, apart from its relative position with respect to $W^u(0)$. Hence, in panels (a1), (b1) and (c1), one sees $W^s(0)$ outside, at, and inside the topological cylinder $W^s(\Gamma_0)$, respectively. The difference becomes apparent only when we consider a larger portion of phase space. In panel (a2), the stable manifold $W^s(\Gamma_0)$ is still a topological cylinder, but note how part of $W^s(\Gamma_0)$ lies close to $W^{ss}(0)$. The moment $Q_{0}^{s}$ occurs, in panel (b2), one trajectory of the unstable manifold manifold $W^u(0)$ lies inside $W^s(\Gamma_0)$ which forces $W^s(\Gamma_0)$ to accumulate onto $W^{ss}(0)$ backward in time. After the bifurcation $Q_{0}^{s}$, as shown in panel (c2) of fig. 13, the lower half of $W^s(\Gamma_0)$ starts to spiral around its upper half. The spiralling accumulation is due to the existence of structurally stable homoclinic orbits of $\Gamma_0$. Note in panel (c2) that one branch of $W^u(0)$ lies in the basin $B(\mathcal{Q})$ and the other one lies in the positive invariant set $\mathcal{V}$, as defined in section 2.2. Hence, the winding number $\zeta$ is infinite and there cannot be any homoclinic or heteroclinic bifurcations occurring after $Q_{0}^{s}$. This implies that $Q_{0}^{s}$ marks the end of the homoclinic and heteroclinic cascade in the ($\alpha, \mu$)-plane.

5.4 Computation of bifurcation sequences

To determine different homoclinic and heteroclinic cascades, and to understand their organization in the parameter plane, we computed representative bifurcation points and their corresponding $\mu$-values along the line $\alpha = 0.5$. These are listed in table 3 which also lists the sequence of further bifurcations past these cascades. We first discuss the global bifurcations up to $Q_{0}^{s}$ and then the remaining bifurcations.

As $\mu$ decreases from the primary homoclinic bifurcation $H_4$ at $\mu = 0$, we encounter homoclinic and heteroclinic cascades in a particular order, followed by saddle-node and period-doubling bifurcations of periodic orbits. As mentioned before, a saddle periodic orbit is created for each homoclinic bifurcation that occurs as $\mu$ decreases; e.g., the saddle periodic orbits $\Gamma_1$ and $\Gamma_2$ are created from the homoclinic bifurcations $H_4$. 

22
Figure 13: The stable manifold $W^s(\Gamma_o)$ (cyan surface) of the orientable saddle periodic orbit $\Gamma_o$ (purple curve), together with $W^u(0)$ (red curve) and $W^{ss}(0)$ before, at, and after the moment of the codimension-one heteroclinic bifurcation $Q_{0}^{\Gamma_o}$ of system (1) in rows (a), (b) and (c), respectively. The left column shows $W^s(\Gamma_t)$ in a tubular neighborhood of $\Gamma_t$, and the right column shows a larger part of $W^s(\Gamma_t)$. Rows (a), (b) and (c) are for $(\alpha, \mu) = (0.5, -0.002)$, $(\alpha, \mu) = (0.5, -0.004861805)$ and $(\alpha, \mu) = (0.5, -0.005)$, respectively. See also the accompanying animation (GKO_Cflip_animatedFig13.gif).
and \(^2\mathbf{H}_t\), respectively. We now employ a similar notation as used for the heteroclinic bifurcations to denote the subsequent homoclinic bifurcations where the corresponding orbit makes rotations around certain saddle periodic orbits before accumulating backward and forward in time to 0. For ease of exposition, we append this information in their corresponding labels. For example, \(\mathbf{H}_o[n\Gamma_o, m\Gamma_t]\) represents the codimension-one orientable homoclinic bifurcation where the corresponding orbit makes \(n\) turns around \(\Gamma_o\) and then \(m\) turns around \(\Gamma_t\) before accumulating forward in time to 0. In particular, the homoclinic bifurcations \(\mathbf{H}_o[n\Gamma_o, m\Gamma_t]\) or \(\mathbf{H}_t[n\Gamma_o, m\Gamma_t]\), for \(n, m \in \mathbb{N}\), in table 3 create saddle periodic orbits that make \(n + m + 1\) loops (the first rotation cannot be assigned to any saddle periodic orbit) and have the same orientation as the homoclinic bifurcation. Heuristically, one can get information about the orientation of the homoclinic bifurcation by looking at the parity of \(m\), that is, the number of times the corresponding orbit of the homoclinic bifurcation rotates around \(\Gamma_t\) in phase space before converging to 0. In general the first turn changes the orientation of the homoclinic bifurcation to nonorientable, and every time a rotation occurs around \(\Gamma_t\), the orientation swaps; hence, for odd \(m\) the corresponding bifurcation is nonorientable and for even \(m\) it is orientable. For example, observe in table 3 that the homoclinic cascade of the form \(\mathbf{H}_o[\Gamma_o, m\Gamma_t]\), with \(m\) even, increases in the number of rotations around \(\Gamma_t\) until it accumulates onto \(Q_0^{r\Gamma} [\Gamma_o]\); the number of rotations then starts decreasing, with \(m\) odd, up to \(\mathbf{H}_o[\Gamma_o, \Gamma_t]\). In particular, \(\mathbf{H}_o[\Gamma_o, \Gamma_t]\) is the last bifurcation that occurs before transitioning into region 4. Note that between \(\mathbf{H}_o[\Gamma_o, \Gamma_t]\) and \(\mathbf{H}_t[2\Gamma_o]\) there is a larger gap in \(\mu\) compared to the previous homoclinic bifurcations. This is due to the termination and the start of new homoclinic and heteroclinic cascades, respectively. In particular, these two homoclinic bifurcations are related as they create the saddle periodic orbits \(3\Gamma_o\) and \(3\Gamma_t\), which disappear in the saddle-node bifurcation \(\text{SNP}_{3\Gamma_o}\). Because \(\zeta\) remains constant at \(\zeta = 3\), we also have numerical evidence that in this gap there cannot exist additional homoclinic or heteroclinic bifurcations.

We also present in table 3 the \(\mu\)-values of representative codimension-one homoclinic bifurcations from 0 to \(2\Gamma_t\). For example, at the heteroclinic bifurcation \(Q_0^{r\Gamma} [2\Gamma_o]\), system (1) exhibits a homoclinic orbit that connects 0 to \(2\Gamma_t\) while rotating twice around \(\Gamma_o\). This shows that there exist additional cascades inside the cascade \(\mathbf{H}_o[2\Gamma_o, m\Gamma_t]\) with \(m\) even, and the same holds for the cascade \(\mathbf{H}_t[2\Gamma_o, m\Gamma_t]\) with \(m\) odd, as our topological sketch in panel (b2) of fig. 12 suggests due to the \(\lambda\)-lemma. In general, there must be cascades of the form \(\mathbf{H}_o\{2\Gamma_o, m\Gamma^*\}, m \in \mathbb{N}\), where \(\Gamma^*\) is a different saddle periodic orbit, accumulating onto heteroclinic bifurcations \(Q_0^{r\Gamma} [2\Gamma_o]\). Therefore, there exist entire clusters of cascades. These clusters of cascades are repeated starting from \(\mathbf{H}_o[2\Gamma_o]\) and ending at \(\mathbf{H}_o[2\Gamma_o, \Gamma_t]\), and again for the pair \(\mathbf{H}_t[3\Gamma_o]\) and \(\mathbf{H}_o[3\Gamma_o, \Gamma_t]\), and so on. In general, the center of each cluster corresponds to the heteroclinic bifurcation \(Q_0^{r\Gamma}[n\Gamma_o]\), with \(n = 2, 3, 4, 5, ...\), which occurs only once in each cluster. For example, note from table 3 that the heteroclinic bifurcation \(Q_0^{r\Gamma} [2\Gamma_o]\) lies in between the bifurcations \(Q_0^{r\Gamma^*} [2\Gamma_o]\) and \(Q_0^{r\Gamma^*} [2\Gamma_o, \Gamma_t]\). Furthermore, the last orientable homoclinic bifurcation from the second cluster, i.e., \(\mathbf{H}_t[2\Gamma_o, \Gamma_t]\), is associated with the first nonorientable homoclinic bifurcation in the second cluster, i.e. \(\mathbf{H}_o[3\Gamma_o]\), because they create the saddle periodic orbits \(4\Gamma_o\) and \(4\Gamma_t\) that disappear in the saddle-node bifurcation \(\text{SNP}_{4\Gamma_o}\). These two homoclinic bifurcations also form a large region in the bifurcation diagram in fig. 5 with \(\zeta = 4\) constant. After \(\mathbf{H}_o[2\Gamma_o, \Gamma_t]\) occurs, the rotation of \(W^s(0)\) around \(\Gamma_t\) moves closer to \(\Gamma_o\) as \(\mu\) decreases, until it becomes \(\mathbf{H}_o[3\Gamma_o]\). Similarly, the third cluster of cascades ends with the orientable homoclinic bifurcation \(\mathbf{H}_o[3\Gamma_o, \Gamma_t]\), which is associated with the first homoclinic bifurcation \(\mathbf{H}_t[4\Gamma_o]\), because they create \(5\Gamma_o\) and \(5\Gamma_t\) which disappear in \(\text{SNP}_{5\Gamma_o}\); these two homoclinic bifurcations also form a large region in the bifurcation diagram fig. 5 now with \(\zeta = 5\).

In general, the last orientable homoclinic bifurcation of a cluster and the first nonorientable homoclinic bifurcation in the following cluster of cascades bound a large region in the bifurcation diagram fig. 5 with the number of their loops corresponding to the value of \(\zeta\). We are able to compute the homoclinic clusters up to \(\mathbf{H}_o[8\Gamma_o, \Gamma_t]\) and \(\mathbf{H}_t[9\Gamma_o]\), and detect the homoclinic bifurcations up to \(Q_0^{r\Gamma} [5\Gamma_o]\). The cascade ends on \(Q_0^{r\Gamma_o}\), which marks the start, as \(\mu\) decreases, of a regime with transversal homoclinic orbits of \(\Gamma_o\), that is, the existence of Smale–horseshoe dynamics [22].

### 5.5 Graphical representation of the bifurcation sequence

Figure 14 presents a sketch of the bifurcation sequence along the slice \(\alpha = 0.5\) from table 3. Here, we illustrate how the infinitely many homoclinic and heteroclinic bifurcations organize themselves according to our discussion in the previous paragraph. The intersection sets presented in fig. 12 give us a good explanation for the gap between \(\mathbf{H}_o[\Gamma_o, \Gamma_t]\) and \(\mathbf{H}_t[2\Gamma_o]\) in fig. 14 and the corresponding gap between the \(\mu\)-values in
| Bifurcation | $\mu \times 10^{-3}$ |
|-------------|---------------------|
| $H_t$       | 0.0                 |
| $2H_t$      | -2.880268           |
| $H_4[\Gamma_o, 2\Gamma_t]$ | -2.880324 |
| $H_4[\Gamma_o, 4\Gamma_t]$ | -2.880324 |
| $H_4[\Gamma_o, 6\Gamma_t]$ | -2.880324 |
| $H_4[\Gamma_o, 8\Gamma_t]$ | -2.880324 |
| $Q_0^+[\Gamma_o]$ | -2.880324 |
| $H_o[\Gamma_o, 7\Gamma_t]$ | -2.880324 |
| $H_o[\Gamma_o, 5\Gamma_t]$ | -2.880324 |
| $H_o[\Gamma_o, 3\Gamma_t]$ | -2.880324 |
| $H_o[2\Gamma_o]$ | -3.816057 |
| $Q_0^+[2\Gamma_o]$ | -3.816058 |
| $Q_0^+[2\Gamma_o, \Gamma_t]$ | -3.816233 |
| $Q_0^+[2\Gamma_o, \Gamma_t]$ | -3.816233 |
| $H_4[3\Gamma_o]$ | -4.249463 |
| $Q_0^+[3\Gamma_o]$ | -4.249465 |
| $Q_0^+[3\Gamma_o, \Gamma_t]$ | -4.249668 |
| $Q_0^+[3\Gamma_o, \Gamma_t]$ | -4.249669 |
| $H_o[3\Gamma_o, \Gamma_t]$ | -4.249669 |
| $H_4[4\Gamma_o]$ | -4.483178 |
| $Q_0^+[4\Gamma_o]$ | -4.483183 |
| $Q_0^+[4\Gamma_o]$ | -4.483359 |
| $Q_0^+[4\Gamma_o, \Gamma_t]$ | -4.483360 |
| $H_o[4\Gamma_o, \Gamma_t]$ | -4.483359 |

Table 3: Computed $\mu$-values at a selection from the infinitely many bifurcations points along the slice $\alpha = 0.5$ of system (1). The color of each bifurcation point matches the corresponding curve in fig. 5. We also show the heteroclinic bifurcation $Q_0^+\Gamma_t$ from 0 to $\Gamma_t$ (gray). The ordering and clustering of these bifurcations is illustrated in fig. 14.

Figure 14: Sketch of the bifurcation sequence along the slice $\alpha = 0.5$ for the bifurcation diagram of system (1) in fig. 5. Here, $\mu$ decreases to the right; see also table 3.
table \(3\), the curves in \(\hat{W}^s(0)\) involved in these two bifurcations are the boundaries of their respective clusters of accumulation, which are locally isolated. Notice that this is the case for all pairs of homoclinic bifurcations \(H_m(m\Gamma_\alpha, \Gamma_t)\) and \(H_m((m + 1)\Gamma_\alpha)\), as found for \(m = 1, 2, 3, 4, 5, 6, 7, 8, 9\) in table \(3\). Furthermore, each gap provides a parameter window during which the extra loop of \(W^s(0)\), created in the respective codimension-one heteroclinic bifurcation \(Q_0^{\Gamma_s}\), moves closer to \(\Gamma_\alpha\); recall that the next homoclinic bifurcation of \(\Gamma_\alpha\) will be nonorientable. During these gaps, the value of the winding number \(\zeta\) remains constant and is given by the number of loops of the homoclinic orbits that bound the region in \((\alpha, \mu)\)-plane. Note that after every cluster of accumulation, the unstable manifold \(W^u(0)\) gains a rotation that lies closer to \(\Gamma_\alpha\). This means that, as \(\mu\) decreases, the \(\zeta\) value in each gap increases until it reaches infinity at \(Q_0^{\Gamma_s}\). It was proven in \([15, 16, 17, 18]\) that these homoclinic bifurcations follow a specific symbolic ordering that is related to their orientation. In particular, the homoclinic bifurcations presented in table \(3\) follow this ordering, and the gaps in fig. \(14\) correspond to the isolation regions described in these theorems. We remark that this is the first time that this ordering of homoclinic bifurcation cascades has been computed in a specific vector field. Going beyond known results, we are also able to identify the codimension-one heteroclinic bifurcations \(Q_0^{\Gamma_s}\) as the limiting curves of these cascades. Moreover, we explain how \(Q_0^{\Gamma_s}\) acts as boundary curve of the region in the parameter plane where these homoclinic cascades can exist.

Figure \(14\) also illustrates the order of further bifurcations past \(Q_0^{\Gamma_s}\) that are computed and listed in table \(3\). There is a Smale–horseshoe region \(\text{SH}\) after transitioning through \(Q_0^{\Gamma_s}\); this region is discussed in more detail in section \(6\). Inside \(\text{SH}\), the stable manifold \(W^s(0)\) becomes tangent to the unstable manifold \(W^u(\Gamma_\alpha)\) of \(\Gamma_\alpha\) at the bifurcation \(\text{F}\), which plays the same role as the fold bifurcation curve of the homoclinic orbit from \(q\) to \(0\) for cases A and B in \([11, 13]\). The other boundary of the Smale–horseshoe region \(\text{SH}\) is the codimension-one homoclinic bifurcation \(\text{Tan}\Gamma_s\) of \(\Gamma_\alpha\), where \(W^u(\Gamma_\alpha)\) and \(W^s(\Gamma_\alpha)\) have a quadratic tangency. Past this bifurcation, there are no more homoclinic orbits of \(\Gamma_\alpha\), and one finds small regions of existence of chaotic attractors \([20]\), which are then destroyed via (reversed) period-doubling and saddle-node cascades of periodic orbits. This is discussed in more detail in section \(7\). The right-most period-doubling cascade in fig. \(14\) is the one ending with \(\text{PD}\Gamma_s\), changes \(\Gamma_t\) into an attracting periodic orbit \(\Gamma_t^{\ast}\) when \(2\Gamma_t^{\ast}\) disappears. The orbit \(\Gamma_t^{\ast}\) then disappears in the saddle-node bifurcation \(\text{SNP}\Gamma_s\), where it merges with the only other remaining periodic orbit \(\Gamma_\alpha\); this marks the entry into region \(2\).

6 Smale–horseshoe region

Figure \(15\) shows the transition from the bifurcation \(Q_0^{\Gamma_s}\) to the Smale–horseshoe region on the level of the intersection sets with the sphere \(S^1\). As before, the left column shows the numerically computed curves and the right column shows sketches where the gray and brown regions highlight the relative positions of the accumulation regions; each accumulation region is represented by the main intersection curve of the stable manifold of the saddle periodic orbit that exists on it. We present the stereographic projection at the codimension-one bifurcation \(Q_0^{\Gamma_s}\) in panels (a1) and (a2). Notice in panel (a1) how one of the topological circles in \(\hat{W}^s(\Gamma_\alpha)\) forms a connecting curve between the two intersection points in \(\hat{W}^s(0)\), as a consequence of \(W^s(\Gamma_\alpha)\) accumulating on \(W^s(0)\) backward in time; see fig. \(13\). In this limiting case, all the accumulation regions accumulate onto this single curve \(\hat{W}^s(\Gamma_\alpha)\), while spiraling around the other topological circle in \(\hat{W}^s(\Gamma_\alpha)\). The boundary of \(\hat{B}(q)\) is formed by \(\hat{W}^s(\Gamma_\alpha)\) and the segment of \(\hat{W}^s(0)\) in between the two points \(\hat{W}^s(0)\). Before \(\hat{W}^s(\Gamma_\alpha)\) connects the two points in \(\hat{W}^s(0)\) at \(Q_0^{\Gamma_s}\), infinitely many heteroclinic and homoclinic bifurcations must occur as the corresponding intersection sets accumulate in a spiraling manner onto \(\hat{W}^s(\Gamma_\alpha)\); see panels (b1) and (b2) of fig. \(12\). This explains why the cascades of heteroclinic bifurcations \(Q_0^{\Gamma_s}\) and homoclinic bifurcation accumulate onto the bifurcation curve \(Q_0^{\Gamma_s}\) in fig. \(6\).

In panels (b1) and (b2) of fig. \(15\), after the heteroclinic bifurcation \(Q_0^{\Gamma_s}\), the curve \(\hat{W}^s(\Gamma_\alpha)\) that connected the two points \(\hat{W}^s(0)\) now accumulates onto the topological circle in \(\hat{W}^s(\Gamma_\alpha)\) that persists through this bifurcation. This accumulation is a consequence of the existence of the homoclinic orbits of \(\Gamma_\alpha\) and the \(\lambda\)-lemma. Since there are infinitely many homoclinic orbits and heteroclinic cycles, there are infinitely many intersection curves in \(\hat{W}^s(\Gamma_\alpha), \hat{W}^s(\Gamma_t), \hat{W}^s(2\Gamma_t)\), etc., that accumulate onto the topological circle in \(\hat{W}^s(\Gamma_\alpha)\). Hence, the accumulation region in the intersection sets is more complicated; we illustrate this change by the brown shading in panel (b2). Note also that the boundary of the basin of attraction \(\partial\hat{B}(q)\) of \(q\) after the
Figure 15: Stereographic projection of the intersection sets of the invariant manifolds with $S^*$ at, and after the codimension-one heteroclinic bifurcation $Q_o^{\infty}$ in rows (a) and (b), respectively. Column (1) shows the computed projections and column (2) shows corresponding sketches. The color code and labeling of the regions is the same as in fig. 9. Panels (a1) and (b1) are for $(\alpha, \mu) = (0.5, -0.004861805)$ and $(\alpha, \mu) = (0.5, -0.0065)$, respectively.
Figure 16: Configuration in phase space of $W^s(\Gamma_0)$ (cyan surface) and $W^u(\Gamma_0)$ (orange surface) of system (1) in the Smale–horseshoe region. Panel (a) shows a big part of $W^s(\Gamma_0)$ and $W^u(\Gamma_0)$ in phase space; while panel (b) shows an enlargement of two transversal homoclinic orbits (maroon and lilac curves) of $\Gamma_0$. Panels (c1) and (c2) show the $x$-component of the transversal homoclinic orbits of $\Gamma_0$ separately; while panel (c3) shows them together. Also shown are $W^u(0)$ (red curve) and $W^{ss}(0)$ (blue curve). The panels are for $(\alpha, \mu) = (0.5, -0.0065)$. See also the accompanying animation (GKO_Clip_animatedFig16.gif).
bifurcation $Q_{0}^{\mu}$ in panels (b1) and (b2) has changed from two topological circles to a topological circle with two handles, that is, $\partial B(q) \subset W^{s}(0) \cup \hat{W}^{u}(\Gamma_{0})$.

6.1 Dynamics inside the region SH

Past the codimension-one heteroclinic bifurcation $Q_{0}^{\mu}$ of system (1), we find structurally stable homoclinic orbits of $\Gamma_{0}$; hence, there exist Smale–horseshoe dynamics in phase space [32]. Figure 16 (a) shows such homoclinic orbits (maroon and lilac curves) in more detail. The manifolds $W^{s}(\Gamma_{0})$ (cyan surface) and $W^{u}(\Gamma_{0})$ (orange surface) intersect along the homoclinic orbits $\Gamma_{0}$. To illustrate further how these manifolds interact, the accompanying animation shows the manifold $W^{s}(\Gamma_{0})$ from fig. 16 (a) as it is grown in size. Note how the outer boundary of the lower half of the topological cylinder $W^{s}(\Gamma_{0})$ (black curve) accumulates in a spiraling fashion onto the top half of $W^{s}(\Gamma_{0})$. Panel (b) shows an enlargement of panel (a) around the homoclinic orbits, where this accumulation is illustrated more clearly. The time series of the $x$-component of the two homoclinic orbits are shown in panels (c1) and (c2), and together in panel (c3). The lilac orbit moves faster away from $\Gamma_{0}$ than the maroon orbit. The existence of these homoclinic orbits implies infinitely many secondary homoclinic orbits of $\Gamma_{0}$ [32]. In addition, their existence also implies the appearance of heteroclinic cycles between different saddle periodic orbits, such as $\Gamma_{0}, \Gamma_{1}, \Gamma_{2}, \Gamma_{3}, \ldots$, due to the accumulation of their stable manifolds backward in time onto $W^{s}(\Gamma_{0})$ and the accumulation of $W^{u}(\Gamma_{0})$ onto their respective unstable manifolds.

6.2 Transition through $F$

The numerical results shown in table 3 and fig. 5 indicate that the fold bifurcation $F$, in the vicinity of $C_{1}$, occurs before the intersection between $W^{u}(\Gamma_{0})$ and $W^{s}(\Gamma_{0})$ becomes tangential, that is, before the bifurcation $\text{Tan}_{\Gamma_{0}}$. The left column of fig. 17 illustrates the crossing of $F$, that is, the moment $W^{u}(\Gamma_{0})$ (orange surface) is tangent to $W^{s}(0)$ (blue surface). Note the transversal intersection between $W^{s}(0)$ and $W^{u}(\Gamma_{0})$ in panel (a1) that persists since its creation in region 1. Since $\Gamma_{0} \rightarrow \Gamma_{1} \rightarrow 0$, there exist infinitely many heteroclinic orbits from $\Gamma_{0}$ to 0, of which we only show two in panel (a1) (green curves). As $F$ occurs, which is illustrated in panel (a2), the stable manifold $W^{s}(0)$ goes through a quadratic tangency with $W^{u}(\Gamma_{0})$; this is similar to cases A and B [1][13], where $W^{s}(0)$ exhibited a quadratic tangency with $W^{u}(q)$. In particular for case C, the infinitely many heteroclinic orbits merge in a tangent heteroclinic orbit, shown as the white curve in panel (a2). After the transition through $F$ shown in panel (a3), $W^{s}(0)$ no longer intersects $W^{u}(\Gamma_{0})$; hence, $W^{s}(0)$ no longer accumulates onto $W^{s}(\Gamma_{0})$ and the unstable manifold $W^{u}(0)$ no longer bounds $W^{u}(\Gamma_{0})$. In fact, we find that $W^{s}(0)$ no longer intersects any unstable manifold of any other saddle periodic orbit.

7 Transition to a strange attractor and period-doubling cascade

Past $\text{Tan}_{\Gamma_{0}}$ one finds chaotic attractors, and cascades of period-doubling and saddle-node bifurcations. This allows system (1) to transition back to region 2; see section 4.

7.1 Transition through $\text{Tan}_{\Gamma_{0}}$

The right column of fig. 17 illustrates the crossing of $\text{Tan}_{\Gamma_{0}}$, that is, the last tangency between $W^{u}(\Gamma_{0})$ (orange surface) and $W^{s}(\Gamma_{0})$ (cyan surface). After the transition through $F$ and before $\text{Tan}_{\Gamma_{0}}$ occurs, the intersection between $W^{s}(\Gamma_{0})$ and $W^{u}(\Gamma_{0})$ still persists, as shown in panel (b1). As $\mu$ is decreased, system (1) must exhibit a sequence of codimension-one homoclinic bifurcations of secondary tangencies between $W^{s}(\Gamma_{0})$ and $W^{u}(\Gamma_{0})$ [32], until it reaches the last homoclinic tangency at $\text{Tan}_{\Gamma_{0}}$; shown in panel (b2). The bifurcation $\text{Tan}_{\Gamma_{0}}$ represents the last intersection (blue curve) between $W^{s}(\Gamma_{0})$ and $W^{u}(\Gamma_{0})$. After the transition through $\text{Tan}_{\Gamma_{0}}$, shown in panel (b3), the manifold $W^{u}(\Gamma_{0})$ stops accumulating on itself, because it no longer intersects $W^{u}(\Gamma_{0})$.

Figure 18 shows additional details of the transition through $\text{Tan}_{\Gamma_{0}}$ by considering corresponding intersection sets with the sphere $S^{*}$. Row (a) shows the situation in between the bifurcations $F$ and $\text{Tan}_{\Gamma_{0}}$ were the intersection set $\hat{W}^{s}(0)$ is formed by a single closed curve; the left column shows the computed stereographic projections and the right column the corresponding sketches. Hence, the accumulation region (brown region)
Figure 17: Phase portraits of the transition through the bifurcation curves $F$ (left column) and $\mathrm{Tan}_{\Gamma_0}$ (right column) in system (1). Shown are the effect of the transitions with respect to $W^s(0)$ and $W^u(\Gamma_0)$ (left column), and with respect to $W^s(\Gamma_0)$ and $W^u(\Gamma_0)$ (right column). Shown are $W^s(0)$ (blue surface), $W^s(\Gamma_0)$ (cyan surface) and $W^u(\Gamma_0)$ (orange surface); also shown are the corresponding tangency orbits (white and blue curves), $W^u(0)$ (red curve), and $W^{ss}(0)$ (blue curve). Panels (a1), (a2) and (a3) are for $(\alpha, \mu) = (0.5, -0.0065)$, $(\alpha, \mu) = (0.5, -0.007054355)$ and $(\alpha, \mu) = (0.5, -0.00706)$, respectively. Panels (b1), (b2) and (b3) are for $(\alpha, \mu) = (0.5, -0.00706)$, $(\alpha, \mu) = (0.5, -0.007076705)$ and $(\alpha, \mu) = (0.5, -0.0071)$, respectively. See also the accompanying animation (GKO_Cflip_animatedFig17.gif).
Figure 18: Stereographic projection of the intersection sets of the invariant manifolds with $S^*$ after crossing the bifurcation curves $F$ (top row) and $\text{Tan}_{\Gamma_o}$ (bottom row). The left column shows the computed projections and the right column shows the corresponding sketches. The color code and labeling is the same as in fig. 9. Panels (a1) and (b1) are for $(\alpha, \mu) = (0.5, -0.00706)$ and $(\alpha, \mu) = (0.5, -0.0071)$, respectively.
that is sketched in panel (a2) does not contain any intersection curves of $\hat{W}^s(0)$. We find that the other intersection curves do not go through any significant changes for parameter values in between the bifurcations $F$ and $\Tan_{\Gamma_o}$. Panels (b1) and (b2) illustrate the consequence of crossing $\Tan_{\Gamma_o}$. In particular, the intersection set $\hat{W}^s(\Gamma_o)$ now consists of two single closed curves that do not accumulate onto any other intersection curve. Nevertheless, the existence of nontransversal intersections of the unstable manifolds of other saddle periodic orbits with $W^s(\Gamma_o)$ forces the other intersection curves to accumulate onto $\hat{W}^s(\Gamma_o)$. Since there are no homoclinic orbits of $\Gamma_o$ anymore, we color the accumulation region light gray in panel (b2). As we will see in the next section, this region may correspond to the basin of attraction of a strange attractor or of an attracting periodic orbit. Note also that $\hat{W}^s(\Gamma_o)$ is a topological annulus. The insets in panels (a1) and (b1) show enlargements around one of the topological circles of $W^s(\Gamma_o)$, illustrating the accumulation of $W^s(\Gamma_o)$ on itself and the lack of it before and after the transition through $\Tan_{\Gamma_o}$, respectively.

7.2 Evidence of the chaotic attractor

Figure 19 shows a chaotic attractor in the phase space of system (1) for $\mu = -0.007076768$. Panel (a) shows trajectories of three different initial conditions that agree up to five decimal places. The trajectories are shown after their transients have been discarded, to illustrate that they are attracted to a lower-dimensional object in phase space. In particular, the trajectories do not remain arbitrarily close to each other as time progresses, but they do remain arbitrarily close to $W^u(\Gamma_o)$, which is shown in panel (b). This evidence suggests the existence of a chaotic attractor, which would be contained in the closure of $W^u(\Gamma_o)$.

As further evidence, fig. 20 shows the successive returns $x^S_{i+1}$ to the plane $y = 0$ of the $x$-coordinates of each of these three trajectories, plotted as $x^S_{i+1}$ versus $x^S_i$, where $x^S_i$ is scaled to the interval $[0, 1]$. Clearly, the return map in fig. 20 is a unimodal map, which is effectively another clear indication that system (1) has a chaotic attractor. In fact, it was proven that the Poincaré return map close to an inclination flip bifurcation of case $C$, up to some $C^r$-rescaling, is $C^r$-close to the family of unimodal maps $\psi_\pi(u,v) = (0, 1 - \pi v^2)$ [26]. We infer from figs. 19 and 20 that the strange attractor contained in the closure of $W^u(\Gamma_o)$ can, therefore, be identified with a template that appears to be equivalent to the template of the Rössler attractor [12]. We conjecture that this strange attractor is destroyed in the tangency $\Tan_{\Gamma_o}$ between $W^u(\Gamma_o)$ and $W^s(\Gamma_o)$; however, the situation might be more complicated and involve a loss of hyperbolicity at a first period-doubling bifurcation, as considered in [2].
7.3 Transition through PD back to region 2

The saddle periodic orbits that are created during the homoclinic and heteroclinic cascades disappear in cascades of period-doubling and saddle-node bifurcations as $\mu$ is decreased. During this process the intersection sets of different stable manifolds disappear with the corresponding periodic orbits. In fact, the accumulation region alternates between being a basin of attraction of an attracting periodic orbit and that of a strange attractor. This is caused by the saddle-node bifurcations that create periodic windows and start different period-doubling cascades. These period-doubling and saddle-node cascades terminate at the final saddle-node bifurcation $\text{SNP}_{\Gamma_o}$, which marks the transition back to region 2.

Figure 21 illustrates the moment before and after the bifurcation $\text{SNP}_{\Gamma_o}$ on the level of the intersection sets of invariant manifolds with $S^*$; as before, the left column shows the computed stereographic projections and the right column the corresponding sketches. In panels (a1) and (a2), after the period-doubling bifurcation $\text{PD}_{\Gamma_a}$, all periodic orbits have disappeared with the exception of $\Gamma_o$ and $\Gamma_t$, which is now attracting periodic orbit $\Gamma_t$. Immediately after the period-doubling bifurcation, $\Gamma_t$ has a nonorientable strong stable manifold $W^{ss}(\Gamma_t)$; however, $\Gamma_t$ becomes an attracting periodic orbit $\Gamma_o$ with an orientable strong stable manifold before reaching $\text{SNP}_{\Gamma_o}$. This transition from having a nonorientable and an orientable strong stable manifold occurs via a crossing of the curves $\text{CC}^+_\Gamma_a$ and $\text{CC}^-\Gamma_a$ where the Floquet multipliers of $\Gamma_a$ ($\Gamma_o$) change from being real positive (negative) to complex conjugate. In between these curves, the attracting periodic orbit $\Gamma_o$ does not have a well-defined strong stable manifold. We discuss the existence of the curves $\text{CC}_{\Gamma_a}^\pm$ in more detail in section 8, where we study the bifurcation diagram of system (1) over an even larger parameter range. On the level of the intersection sets shown in fig. 21(a), the former accumulation region has now become the basin of attraction $\hat{\mathcal{B}}(\Gamma_t)$ of $\Gamma_t$ (green). Furthermore, the basin $\hat{\mathcal{B}}(\mathbf{q})$ is a disconnected region with boundaries $\tilde{W}^s(\Gamma_o)$ and $\tilde{W}^s(0)$. After, the saddle-node bifurcation $\text{SNP}_{\Gamma_o}$, which marks the disappearance of $\Gamma_a$ and $\Gamma_o$, we find ourselves again in region 2, as is illustrated in panels (b1) and (b2). Hence, these panels are topological equivalent to panel 2 of fig. 9.

8 Global picture in the $(\alpha, \mu)$-plane

Homoclinic flip bifurcations arise in different mathematical models that describe physical phenomena. Certainly from the applications point of view, it is of interest to study how the bifurcation curves that emanate from the homoclinic flip bifurcation point of case C are organized beyond a small neighborhood of this central codimension-two point. Hence, we now focus on the role of the homoclinic flip bifurcation point $C_1$ as an organizing center for dynamics and bifurcations in a larger region of the $(\alpha, \mu)$-plane. The overall structure of the bifurcation set in such a larger parameter range features further codimension-two bifurcation phenomena.
Figure 21: Stereographic projection of the intersection sets of the invariant manifolds with $S^*$ before (top row) and after (bottom row) the last saddle-node bifurcation $\text{SNP}_{C_a}$ of periodic orbits. The left column shows the computed projections and the right column shows the corresponding sketches. The color code and nomenclature of the regions is the same as given in fig. 9. Panels (a1) and (b1) are for $(\alpha, \mu) = (0.5, -0.0073)$ and $(\alpha, \mu) = (0.5, -0.0074)$, respectively.
Figure 22: Bifurcation diagram of system (1) for a larger region in the \((\alpha, \mu)\)-plane than shown in fig. 5. Panels (b) and (c) are enlargements as indicated in panel (a). The color code of the regions and labeling of the curves are the same as in fig. 5. We also show the curve \(Q^\Gamma_o\) (gray), and the curves \(CC^r_{\Gamma_a}\) and \(CC^{-}_{2\Gamma_a}\) (orange) of \(\Gamma^o\) and \(2\Gamma^a\), respectively.
that occur at some distance from $C_1$. As a particular type of bifurcation structure we find homoclinic bubbles, which are known to arise as a mechanism for transitioning between cases B and C in parameter slices near a codimension-three homoclinic flip bifurcation \[10\]. We show that homoclinic bubbles organize the different bifurcation curves that emerge locally from $C_1$ more globally in the $(\alpha, \mu)$-plane.

Figure 22 shows the numerically computed bifurcation diagram of $C_1$ for a much larger parameter range in panel (a), as well as two enlargements in panels (b) and (c). In particular, the parameter sweep of $\zeta$ now plays an even more prominent role in discerning small and subtle interactions of the different bifurcation curves. In Fig. 22 we recognize from Fig. 5 and Table 8 the principal homoclinic branch (brown curve) with $C_1$, as well as the curves of saddle-node bifurcation $\text{SNP}$ (dark-green), of period-doubling bifurcation $\text{PD}$ (red), of homoclinic bifurcation $\text{Tanr}_n$ (violet), of heteroclinic bifurcation $Q^0_\alpha$ (magenta), $Q^+_\alpha$ (purple) and $Q^+_\alpha[2\Gamma_o]$ (gray), and of homoclinic bifurcation $H_n[m\Gamma_o, n\Gamma_i]$ $H_i[m\Gamma_o, n\Gamma_i]$ (cyan curves). Note also the region $\text{SH}$ where Smale–horseshoe dynamics occurs, and the curve $\text{HB}$ of Hopf bifurcation (green curve) that transforms $q$ into an unstable saddle focus. Panel (a) shows that the curve $\text{HB}$ does not interact with any other bifurcation curve for $\mu$-values larger than that at the generalized Hopf bifurcation point $\text{GHB}$. We also observe more clearly the curve $F^+$ (dashed blue); recall that this curve represents the moment when $W^s(0)$ becomes tangent to $W^u(q)$, and it is the extension of the curve $F$ past its intersection with $\text{SNP}$, as shown in Fig. 5. At first glance, $F^+$ seems to bound the region where the homoclinic curves exists, but this is not the case, as we are going to discuss in more detail in the following section. We also show the curves $CC_{\Gamma^o}$ and $CC_{\Gamma^o}$ (orange) which represent the moment the Floquet multipliers of the attracting periodic orbit $\Gamma^o$ become complex conjugate with negative and positive real parts, respectively. In addition, we also present the curve $CC_{\Gamma^o}$ (dark orange) of the attracting periodic orbit $2\Gamma^o$; these are visible in panels (b) and (c).

The curves $Q^+_\alpha[2\Gamma_o]$ shown in Fig. 22 extend to the top part of the figure in panel (a). They represent the extensions of the heteroclinic bifurcation curves in Fig. 5 past the transition of $\Gamma$ to the attracting periodic orbit $\Gamma^o$, where $Q^+_\alpha$ corresponds to the intersection of the two-dimensional strong stable manifold $W^{ss}(\Gamma^o)$ with $W^u(0)$, which is also a codimension-one phenomenon. As such, these curves terminate at the respective curve $CC_{\Gamma^o}$, that is, at the moment when $W^{ss}(\Gamma^o)$ is no longer well defined. This also holds for the heteroclinic bifurcation $Q^+_\alpha[2\Gamma_o]$ (gray curve), which can be seen to terminate at $CC_{\Gamma^o}$ in Fig. 22(b).

8.1 Cascades of inclination flip bifurcations

The larger parameter range in Fig. 22(a) reveals how the homoclinic bifurcations, which were hard to discern in Fig. 5, fan out and are more distinguishable. Furthermore, we clearly see how each homoclinic bifurcation curve encloses a region in the $(\alpha, \mu)$-plane with a constant $\zeta$ value; indeed, the $\zeta$-value in each region increases as a heteroclinic bifurcation $Q^+_\alpha$ or $Q^+_\alpha$ is approached; see also section 5.

It is an important feature of Fig. 22 that many of the homoclinic, saddle-node and period-doubling bifurcation curves emanating from $C_1$ end at other codimension-two inclination flip bifurcation points. Notably, the principal homoclinic branch goes through a secondary inclination flip bifurcation $B^o_1$ of case B, which is the end point of the curves $2\text{H}_o$ and $\text{PD}_{\Gamma^o}$. Moreover, other homoclinic curves exhibit inclination flip bifurcations $C^3_1$ of case C, where $n$ represents the number of loops that the corresponding homoclinic orbit makes in phase space.

The bifurcation diagram near one of these inclination flip points, $C^3_1$, is shown in Fig. 22(b). Notice how $C^3_1$ is responsible for the transformation of the orientable homoclinic bifurcation $H_n[\Gamma_o, \Gamma_i]$ to the nonorientable homoclinic bifurcation $H_{n}[2\Gamma_o]$, both of which emanate from the initial inclination flip bifurcation $C_1$. Recall that these homoclinic bifurcations are related as they represent the last and the first bifurcation of their respective accumulation cluster, as discussed in section 5.4. Moreover, they create the saddle periodic orbits $3\Gamma_o$ and $3\Gamma_i$, respectively, and they bound a region in the parameter plane where $\zeta = 3$; see also Fig. 14. So it does not come as a surprise that $\text{SNP}_{3\Gamma^o}$ (green curve) ends at the bifurcation point $C^3_1$, as this bifurcation curve is responsible for the disappearance of $3\Gamma_o$ and $3\Gamma_i$; indeed $\text{SNP}_{3\Gamma^o}$ plays the same role as $\text{SNP}_{\Gamma^o}$ in the unfolding of $C_1$. In Fig. 22(c) we observe that $H_{n}[\Gamma_o, \Gamma_i]$ turns around before reaching $C^3_1$, and then the curve continues smoothly past $C^3_1$ as $H_{o}[2\Gamma_o]$ before returning to $C_1$. This same scenario can also be observed for the inclination flip points $C^5_1$ and $C^7_1$. Moreover, Fig. 22(c) shows that there exist infinitely many regions with constant $\zeta$ in between the two inclination flip points $C^3_1$ and $C^3_1$. The boundaries of each of these regions are homoclinic bifurcation curves emanating from $C_1$, which also exhibit an inclination
flip bifurcation of case C along them and then come back to $C_I$ as an homoclinic bifurcation curve of the opposite orientation. As a consequence, the respective saddle-node and period-doubling bifurcation curves extend from $C_I$ to this additional inclination flip bifurcation point.

### 8.2 Homoclinic bubbles

This type of overall bifurcation structure generated by a particular homoclinic curve is called a homoclinic bubble \cite{16}. Recall our discussion of the gaps between accumulation clusters along the slice $\mu = 0.5$, which we found in table 3 and sketched in fig. 14. Figure 22 (a) shows that the pairs of homoclinic bifurcations $H_0[m\Gamma_\alpha, \Gamma_{\ell}]$ and $H_k[(m + 1)\Gamma_\alpha]$, as found for $m = 1, 2, 3, 4, 5, 6, 7, 8, 9$ in table 3, form bubbles in the parameter plane. In particular, identified cascades of these bubbles explain the disappearance of the infinitely many codimension-one curves that emanate from the unfolding of case C during a transition to case B. In fact, two types of bubbles were conjectured \cite{16} and subsequently confirmed numerically \cite{28}: one is characterized by a homoclinic-doubling cascade and the other by an additional homoclinic flip bifurcation of case C. Figure 23 (a) shows the theoretical sketch of a single homoclinic $k$-bubble, for $k \in \mathbb{N}$, of the latter type, reproduced from \cite{16}. Notice how an orientable $k$-homoclinic bifurcation curve $H_k^0$ emanates from the principal homoclinic flip bifurcation point $C_{\text{out}}$ and then exhibits an inclination flip bifurcation $C_k^0$, after which it is the nonorientable $k$-homoclinic bifurcation curve $H_k^k$. In this way, the homoclinic and period-doubling bifurcation curves of the form $2^n k$, for $n \in \mathbb{N}$, emanating from $C_{\text{out}}$ disappear in the unfolding of $C_{\text{out}}$. For any $k \in \mathbb{N}$, there exists $k$-bubbles that annihilate all the corresponding $2^n k$ curves of $C_{\text{out}}$. Those are exactly the bifurcation structures we observe in the bifurcation diagram of fig. 22 in the transition between the inclination flip points $C_I$ and $B_I^\ast$.

Figure 23 (b) shows a different enlargement of the bifurcation diagram of $C_I$. Here, we use the coordinate transformation $\mu^* := \mu + 0.12460 - 0.06644$ and plot the bifurcation diagram in the ($\alpha, \mu^*$)-plane to improve visualization and comparison with the sketch in panel (a). The homoclinic curve $2H_k$ exhibits the inclination flip bifurcation $C_I^0$, which allows $2H_k$ to become $2H_k^0$ so it can disappear at $B_I^\ast$. This is one of the most important curves in the transition between the two cases \cite{16}. However, it does not form a bubble as presented in \cite{16} because both homoclinic curves do not emanate from the same point $C_I$. Nevertheless, the inclination flip point $C_I^0$ plays the same role as the homoclinic flip point $C_{\text{out}}^k$ of a $k$-bubble for $k = 2$ in terms of absorbing/generating the respective bifurcation curves. Notice in fig. 22 (b) how the cascade of nonorientable homoclinic bifurcation curves (cyan) of the form $H_n[\Gamma_\alpha, n\Gamma_{\ell}]$, for $n = 2, 4, 6, 8$, and the period-doubling bifurcation curves $PD_{\Gamma_\alpha}$ and $PD_{\Gamma_{\ell}}$ (red) connect the points $C_I$ and $C_I^0$ as sketched in panel (a).

The same phenomenon occurs at the points $C_I^0$, $C_I^1$, and $C_I^2$, but the curves are much closer together. Figure 23 (b) clearly shows the complicated structure of how some homoclinic bifurcation curves create bubbles and others connect to inclination flip points on bubbles. It also shows how infinitely many homoclinic bubbles accumulate on $Q_0^0[\Gamma_\alpha]$, which is illustrated by the increase of $C$. Furthermore, notice how the $C$-value also increases in the different regions in between bubbles, such as those of $C_I^0$, $C_I^1$, and $C_I^2$, each of which is associated with a codimension-one heteroclinic bifurcation of the type $Q_0^k$ for a suitable saddle periodic orbit $\Gamma^*$; see section 4. The accumulation of these homoclinic bifurcation curves and bubbles onto a heteroclinic bifurcation of the form $Q_0^k$ was not identified in either in \cite{16} or \cite{28}. Another interesting detail about the bubbles in Figure 22 (b) is how they seem to be bounded by a single smooth curve, where the inclination flip points $C_I^0$, $C_I^1$, and $C_I^2$ occur along it. At first glance, it appears that this curve is $F^*$, that is, the fold curve of a tangency between $W^s(q)$ and $W^u(0)$, as shown in panel (a) and (b) in fig. 22. However, fig. 23 (b) indicates that this is not the case. This boundary curve represents a boundary crisis \cite{30}, and infinitely many inclination flip bifurcations of case C occur along it. We believe that this curve of boundary crisis involves the tangency of an, as yet, unknown invariant object.

Finally, we remark that the eigenvalues of 0 do not change as $\alpha$ and $\mu$ vary in the bifurcation diagrams shown in fig. 22 and fig. 23; see section 2.1 This raises the question of the existence of $B_I^\ast$ in fig. 22 because it fulfills the eigenvalue conditions of case C. Preliminary work suggest that it also fulfills the necessary geometry condition for case C. On the other hand, our bifurcation analysis clearly identifies the point $B_I^\ast$ as an inclination flip bifurcation of case B. Further analysis of this codimension-two point is beyond the scope of this paper, and left for future work.
Figure 23: Comparison of a theoretical and numerical inclination flip bubble. Panel (a) shows the sketch of an inclination flip bubble from [16, Fig. 12]; panel (b) shows a region of fig. 22 in the $(\alpha, \mu^*)$-plane, where $\mu^* := \mu + 0.1246\alpha - 0.06644$. Color code of the regions and nomenclature of the curves are as in fig. 22 [The inset is reproduced from Journal of Dynamics and Differential Equations, Resonant Homoclinic Flip Bifurcations, 12(4), 2000, pages 807-850, A. J. Homburg and B. Krauskopf, ©Plenum Publishing Corporation 2000 with permission of Springer.]
9 Discussion

We conducted a detailed case study of a codimension-two inclination flip bifurcation point $C_1$ of case C in Sandstede’s model \cite{Aguirre2013}. More specifically, we presented the bifurcation set in the $(\alpha, \mu)$-parameter plane and illustrated the associated dynamics at representative parameter points in regions of distinct qualitative behavior. This required the computation of relevant global invariant manifolds of saddle objects in phase space, as well as many curves of homoclinic, heteroclinic, saddle-node and period-doubling bifurcations. Moreover, we calculated a winding number and identified the regions where it is constant.

Near the central point $C_1$ in the $(\alpha, \mu)$-plane we found a very complicated structure of bifurcations. As predicted by what is known about the theoretical unfolding \cite{Sandstede2002}, we identified cascades of local and global bifurcations, as well as regions with Smale–horseshoe dynamics and strange attractors. We clarified the precise arrangement of these different ingredients and, moreover, identified a number of new bifurcations near $C_1$. Specifically, we found that cascades of homoclinic bifurcations accumulate on codimension-one heteroclinic bifurcations between a nonorientable saddle periodic orbit $\Gamma_0$, and the central equilibrium $0$; different types of cascades can be distinguished by the number of rotations that the global orbit makes near the saddle periodic orbit. Moreover, we identified the boundaries of the Smale–horseshoe region: the codimension-one heteroclinic bifurcation $Q_0^{in}$, where there is a connecting orbit from $0$ to the orientable saddle periodic orbit $\Gamma_0$; and the codimension-one homoclinic bifurcation $\text{Tan}_{\Gamma_0}$ of $\Gamma_0$ (which was conjectured to exist in \cite{Sandstede2002}).

We proceeded by considering the bifurcation set in the $(\mu, \alpha)$-plane more globally, further away from the central point $C_1$, to determine the overall organization of the emanating curves of codimension-one bifurcations. First of all, we found that the region with Smale–horseshoe dynamics is, in fact, bounded, because the curves $Q_0^{in}$ and $\text{Tan}_{\Gamma_0}$ intersect to form a codimension-two heteroclinic cycle between $0$ and the orientable saddle periodic orbit $\Gamma_0$. Zooming out even more, revealed a complex overall picture involving the transition to an inclination flip bifurcation of case B. A prominent feature of it are bubbles formed by certain homoclinic bifurcation curves in parameter plane, which exhibit an additional homoclinic flip bifurcation of case C that changes the orientation of the respective homoclinic bifurcation. Such bubbles were proposed as a crucial ingredient in the transition between cases B and C in a codimension-three resonant homoclinic flip bifurcation \cite{Sandstede2002}, and then found numerically in \cite{Sandstede2002}. We identified infinitely many homoclinic bubbles in the $(\mu, \alpha)$-plane, clarified the role of heteroclinic bifurcations for their organization and described their accumulation on a specific boundary curve.

The detailed numerical investigation presented here identified several generic bifurcation phenomena that were not considered before, as part of the unfolding of a homoclinic flip bifurcation of case C or otherwise. Now that they are known and described, they can be studied theoretically by considering them in a more abstract and general setting. Hence, an advanced numerical study as presented here can contribute to a better understanding of theoretical constructs, especially in situations when the theory is very intricate. In this context it will be interesting to study, in the same spirit, the configuration of manifolds responsible for the inward twist case $C_{in}$, which features a related but different unfolding \cite{Sandstede2002}. However, at present, we are not aware of a concrete example of this codimension-two phenomenon in a three-dimensional vector field.

More generally, the results presented here can be seen as a showcase of the capabilities of advanced numerical methods based on two-point boundary value problems for the bifurcation analysis of a given system with complicated global bifurcations. In particular, homoclinic flip bifurcations have been found, for example, in the Hindmarsh–Rose model of a spiking neuron, where they explain the creation of large spiking excursions of periodic orbits in the presence of slow-fast dynamics \cite{Sandstede2002}. Further study of the role of global bifurcations in systems with multiple time scales is a promising direction for future research from both a theory and an applications point of view; for a recent example see \cite{Sandstede2002}.

References

[1] P. Aguirre, B. Krauskopf, and H. M. Osinga, Global invariant manifolds near homoclinic orbits to a real saddle: (Non)orientability and flip bifurcation, SIAM J. Appl. Dyn. Syst., 12 (2013), pp. 1803–1846.

[2] M. Asaoka, A natural horseshoe-breaking family which has a period doubling bifurcation as the first bifurcation, J. Math. Kyoto Univ., 37 (1997), pp. 493–511.
[3] R. Barrio, M. Lefranc, M. A. Martínez, and S. Serrano, *Symbolic dynamical unfolding of spike-adding bifurcations in chaotic neuron models*, EPL (Europhysics Letters), 109 (2015).

[4] R. Barrio and A. Shilnikov, *Parameter-sweeping techniques for temporal dynamics of neuronal systems: case study of hindmarsh-rose model*, J. Math. Neurosci., 1 (2011).

[5] R. Barrio, A. Shilnikov, and L. Shilnikov, *Kneadings, symbolic dynamics and painting Lorenz chaos*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., 22 (2012).

[6] R. C. Calleja, E. J. Doedel, A. R. Humphries, A. Lemos-Rodriguez, and E. B. Oldeman, *Boundary-value problem formulations for computing invariant manifolds and connecting orbits in the circular restricted three body problem*, Celest. Mech. Dyn. Astron., 114 (2012), pp. 77–106.

[7] A. R. Champneys, V. Kirk, E. Knobloch, B. E. Oldeman, and J. D. M. Rademacher, *Unfolding a tangent equilibrium-to-periodic heteroclinic cycle*, SIAM J. Appl. Dyn. Syst., 8 (2009), pp. 1261–1304.

[8] A. R. Champneys, Y. Kuznetsov, and B. Sandstede, *A numerical toolbox for homoclinic bifurcation analysis*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., 6 (1996), pp. 867–887.

[9] B. Deng, *Homoclinic twisting bifurcations and cusp horseshoe maps*, J. Dynam. Differential Equations, 5 (1993), pp. 417–467.

[10] E. J. Doedel, *AUTO: A program for the automatic bifurcation analysis of autonomous systems*, Congr. Numer., 30 (1981), pp. 265–284.

[11] E. J. Doedel and B. E. Oldeman, *AUTO-07p: Continuation and Bifurcation Software for Ordinary Differential Equations*, Department of Computer Science, Concordia University, Montreal, Canada, 2010. With major contributions from A. R. Champneys, F. Dercole, T. F. Fairgrieve, Y. Kuznetsov, R. C. Paffenroth, B. Sandstede, X. J. Wang and C. H. Zhang; available at [http://www.cmvl.cs.concordia.ca](http://www.cmvl.cs.concordia.ca/).

[12] R. Gilmore and M. Lefranc, *The Topology of Chaos: Alice in Stretch and Squeezeland*, Wiley-Interscience, 2002.

[13] A. Giraldo, B. Krauskopf, and H. M. Osinga, *Saddle invariant objects and their global manifolds in a neighborhood of a homoclinic flip bifurcation of case B*, SIAM J. Appl. Dyn. Syst., 16 (2017), pp. 640–686.

[14] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer-Verlag, New York, 1983.

[15] A. J. Homburg, H. Kokubu, and M. Krupa, *The cusp horseshoe and its bifurcations in the unfolding of an inclination-flip homoclinic orbit*, Ergodic Theory Dynam. Systems, 14 (1994), pp. 667–693.

[16] A. J. Homburg and B. Krauskopf, *Resonant homoclinic flip bifurcations*, J. Dynam. Differential Equations, 12 (2000), pp. 807–850.

[17] A. J. Homburg and B. Sandstede, *Homoclinic and heteroclinic bifurcations in vector fields*, in Handbook of Dynamical Systems, H. W. Broer, B. Hasselblatt, and F. Takens, eds., vol. 3, Elsevier, New York, 2010, pp. 381–509.

[18] M. Kisaka, H. Kokubu, and H. Oka, *Bifurcations to n-homoclinic orbits and n-periodic orbits in vector fields*, J. Dynam. Differential Equations, 5 (1993), pp. 305–357.

[19] M. Koper, *Bifurcations of mixed-mode oscillations in a three-variable autonomous Van der Pol-Duffing model with a cross-shaped phase diagram*, Phys. D, 80 (1995), pp. 72–94.
[20] B. Krauskopf and H. M. Osinga, *Computing invariant manifolds via the continuation of orbit segments*, in Numerical Continuation Methods for Dynamical Systems: Path Following and Boundary Value Problems, B. Krauskopf, H. M. Osinga, and J. Galán-Vioque, eds., Springer, The Netherlands, 2007, pp. 117–154.

[21] B. Krauskopf and T. Riess, *A Lin’s method approach to finding and continuing heteroclinic connections involving periodic orbits*, Nonlinearity, 21 (2008), pp. 1655–1690.

[22] Y. A. Kuznetsov, O. Feo, and S. Rinaldi, *Belyakov homoclinic bifurcations in a tritrophic food chain model*, SIAM J. Appl. Dyn. Syst., 62 (2001), p. 462–487.

[23] D. Linaro, A. Champneys, M. Desroches, and M. Storace, *Codimension-two homoclinic bifurcations underlying spike adding in the Hindmarsh-Rose burster*, SIAM J. Appl. Dyn. Syst., 11 (2012), pp. 939–962.

[24] A. Lohse and A. Rodrigues, *Boundary crisis for degenerate singular cycles*, Nonlinearity, 30 (2017), pp. 2211–2245.

[25] J. Mujica, B. Krauskopf, and H. M. Osinga, *Tangencies between global invariant manifolds and slow manifolds near a singular hopf bifurcation*, SIAM J. Appl. Dyn. Syst., 17 (2018), pp. 1395–1431.

[26] V. Naudot, *Strange attractor in the unfolding of an inclination-flip homoclinic orbit*, Ergodic Theory Dynam. Systems, 16 (1996), pp. 1071–1086.

[27] ———, *A strange attractor in the unfolding of an orbit-flip homoclinic orbit*, Dyn. Syst., 17 (2002), pp. 45–63.

[28] B. E. Oldeman, B. Krauskopf, and A. R. Champneys, *Numerical unfoldings of codimension-three resonant homoclinic flip bifurcations*, Nonlinearity, 14 (2001), pp. 597–621.

[29] H. M. Osinga, *Nonorientable manifolds in three-dimensional vector fields*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., 13 (2003), pp. 553–570.

[30] ———, *Locus of boundary crisis: Expect infinitely many gaps*, Phys. Rev. E, 74 (2006).

[31] J. Palis and W. de Melo, *Geometric Theory of Dynamical Systems*, Springer, New York, 1982.

[32] J. Palis and F. Takens, *Hyperbolicity and Sensitive Chaotic Dynamics at Homoclinic Bifurcations: Fractal Dimensions and Infinitely Many Attractors in Dynamics*, Cambridge University Press, 1995.

[33] B. Sandstede, *Verzweigungstheorie Homokliner Verdopplungen*, PhD thesis, University of Stuttgart, Stuttgart, Germany, 1993.

[34] ———, *Constructing dynamical systems having homoclinic bifurcation points of codimension two*, J. Dynam. Differential Equations, 9 (1997), pp. 269–288.

[35] L. P. Shilnikov, *A case of the existence of a denumerable set of periodic motions*, Sov. Math. Dokl., 6 (1965), pp. 163–166.

[36] ———, *On the generation of a periodic motion from trajectories doubly asymptotic to an equilibrium state of saddle type*, Mat. Sb. (N.S.), 77(119) (1968), pp. 461–472.

[37] L. P. Shilnikov, A. L. Shilnikov, D. V. Turaev, and L. O. Chua, *Methods of Qualitative Theory in Nonlinear Dynamics (Part I)*, vol. 4, World Scientific, Singapore, 1998.

[38] B. W., A. Champneys, E. Doedel, W. Govaerts, Y. Kuznetsov, and B. Sandstede, *Numerical continuation, and computation of normal forms*, in Handbook of Dynamical Systems, 1999.

[39] S. M. Wieczorek, *Global bifurcation analysis in laser systems*, in Numerical Continuation Methods for Dynamical Systems: Path Following and Boundary Value Problems, B. Krauskopf, H. M. Osinga, and J. Galán-Vioque, eds., Springer, The Netherlands, 2007, pp. 177–220.
[40] S. M. Wieczorek and B. Krauskopf, *Bifurcations of n-homoclinic orbits in optically injected lasers*, Nonlinearity, 18 (2005), pp. 1095–1120.

[41] S. Wiggins, *Introduction to Applied Nonlinear Dynamical Systems and Chaos*, Springer-Verlag, New York, 2nd ed., 2003.

[42] T. Xing, R. Barrio, and A. Shilnikov, *Symbolic quest into homoclinic chaos*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., 24 (2014).

[43] E. Yanagida, *Branching of double pulse solutions from single pulse solutions in nerve axon equations*, J. Differential Equations, 66 (1987), pp. 243–262.

[44] W. Zhang, B. Krauskopf, and V. Kirk, *How to find a codimension-one heteroclinic cycle between two periodic orbits*, Discrete Contin. Dyn. Syst., 32 (2012), pp. 2825–2851.