Optimal Multi-Dimensional Stochastic Harvesting with Density-dependent Prices

Luis H. R. Alvarez E. 1  Edward Lungu2  Bernt Øksendal3,4

30 June 2014

Abstract

We prove a verification theorem for a class of singular control problems which model optimal harvesting with density-dependent prices or optimal dividend policy with capital-dependent utilities. The result is applied to solve explicitly some examples of such optimal harvesting/dividend problems.

In particular, we show that if the unit price decreases with population density, then the optimal harvesting policy may not exist in the ordinary sense, but can be expressed as a "chattering policy", i.e. the limit as \( \Delta x \) and \( \Delta t \) go to 0 of taking out a sequence of small quantities of size \( \Delta x \) within small time periods of size \( \Delta t \).

Keywords: Optimal harvesting, interacting populations, Itô diffusions, singular stochastic control, verification theorem, density-dependent prices, chattering policies.

MSC(2010): Primary 60H10, 93E20. Secondary 91B70, 92D25.

1 Introduction

The determination of an optimal harvesting policy of a stochastically fluctuating renewable resource is typically subject to at least three key factors affecting either the intertemporal evolution of the resource stock or the incentives of a rational risk neutral harvester. First, the exact size of the harvested stock evolves stochastically due to environmental or demographical randomness. Second, the interaction between different populations has obviously a direct effect on the density of the harvested stocks. Third, most harvesting decisions are subject to density dependent costs and prices. The price of the harvested resource is typically decreasing as a function of the prevailing stock due to the decreasing marginal utility of consumption. The

---

1 Department of Accounting and Finance, Turku School of Economics, FIN-20014 University of Turku, Finland, e-mail: luis.alvarez@tse.fi
2 Department of Mathematics, University of Botswana, B.P. 0022 Gaborone, Botswana, e-mail: lungu@mopipi.ub.bw
3 Department of Mathematics, University of Oslo, Box 1053 Blindern, N–0316 Oslo, Norway, e-mail: oksendal@math.uio.no The research leading to these results has received funding from the European Research Council under the European Community’s Seventh Framework Program (FP7/2007-2013) / ERC grant agreement no [228087].
4 Norwegian School of Economics, Helleveien 30, N–5045 Bergen, Norway
more abundant a resource gets, the less consumers are prepared to pay from an extra unit of that particular resource and vice versa. In a completely analogous fashion the costs associated with harvesting depend typically on the abundance of the harvested resource. The scarcer a resource becomes, the higher are the costs associated with harvesting due to costly search or other similar factors. Our objective in this study is to investigate the optimal harvesting policy of a risk neutral decision maker facing all the three key factors mentioned above.

The problem of determining an optimal harvesting policy of a risk neutral decision maker can be viewed as a singular stochastic control problem. In an unstructured one-dimensional setting where the marginal profitability of a marginal unit of the harvested stock is a constant, the existing literature usually delineates circumstances under which the optimal harvesting policy is to deplete the entire resource stock immediately or to maintain it at all times below a critical threshold at which the expected present value of the cumulative yield is maximized ([A1, A3, AS, LES1, LES2, LO1]). As intuitively is clear, the optimal policy is altered as soon as the marginal profitability becomes state-dependent (cf. [A2]) or population interaction (cf. [LO2]) is incorporated into the analysis. In [A2] it is shown within a one-dimensional setting that the state dependence of the instantaneous yield from harvesting results into the emergence of circumstances under which the policy resulting into the maximal value constitutes a chattering policy which does not belong into the original class of admissible càdlàg-harvesting policies. On the other hand, in [LO2] it is shown that the presence of interaction between the harvested resource stocks leads to a harvesting strategy where the decision maker generically harvests only a single resource at a time.

In this paper we combine the approaches developed in [A2] and [LO2] and consider the problem of determining the optimal harvesting policy from a collection of interacting populations, described by a coupled system of stochastic differential equations, when the price per unit for each population is allowed to depend on the densities of the populations. In Section 2 we give a general verification theorem for such optimal harvesting problems (Theorem 2.1), and in Section 3 we study in detail some examples where the price is a decreasing function of the density and we show, perhaps surprisingly, that in such cases the optimal harvesting strategy may not exist in the ordinary sense, but can be described as a “chattering policy”. See Theorem 3.2 and Theorem 3.4.

2 The main result

We now describe our model in detail. This presentation follows [LO2] closely. Consider \( n \) populations whose sizes or densities \( X_1(t), \ldots, X_n(t) \) at time \( t \) are described by a system of \( n \) stochastic differential equations of the form

\[
\begin{align*}
\frac{dX_i(t)}{dt} &= b_i(t, X(t))dt + \sum_{j=1}^{m} \sigma_{ij}(t, X(t))dB_j(t); 0 \leq s \leq t \leq T \\
X_i(s) &= x_i \in \mathbb{R}; \quad 1 \leq i \leq n,
\end{align*}
\]

where \( B(t) = (B_1(t), \ldots, B_m(t)); t \geq 0, \omega \in \Omega \) is \( m \)-dimensional Brownian motion on a filtered probability space \( (\Omega, \mathcal{F}, \mathbb{P}) := \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P} \) and the differentials (i.e. the corresponding integrals) are interpreted in the Itô sense. We assume that \( b = (b_1, \ldots, b_n) : \mathbb{R}^{1+n} \to \mathbb{R}^n \) and \( \sigma = \sigma \)
The harvested family $X$ consists of an immediate harvest of size $\Delta \gamma$ at time $t = s$, it is necessary to distinguish between $X^{(\gamma)}(s)$ and $X^{(\gamma)}(s^-)$: Thus $X^{(\gamma)}(s^-)$ is the state right before harvesting starts at time $t = s$, while

$$X^{(\gamma)}(s) = X^{(\gamma)}(s^-) - \Delta \gamma$$

is the state immediately after, if $\gamma$ consists of an immediate harvest of size $\Delta \gamma$ at $t = s$.

Suppose that the price per unit of population number $i$, when harvested at time $t$ and when the current size/density of the vector $X^{(\gamma)}(t)$ of populations is $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$, is given by

$$\pi_i(t, \xi) = 0 \quad (t, \xi) \in S, \quad 1 \leq i \leq n,$$

where the $\pi_i : S \rightarrow \mathbb{R}; \quad 1 \leq i \leq n$, are lower bounded continuous functions. We call such prices density-dependent since they depend on $\xi$. The total expected discounted utility harvested from time $s$ to time $T$ is given by

$$J^{(\gamma)}(s, x) := E^{s,x} \left[ \int_{[s,T]} \pi(t, X^{(\gamma)}(t^-)) \cdot d\gamma(t) \right]$$
where \(\pi = (\pi_1, \ldots, \pi_n)\), \(\pi \cdot d\gamma = \sum_{i=1}^{n} \pi_i d\gamma_i\) and \(E^{s,x}\) denotes the expectation with respect to the probability law \(Q^{s,x}\) of the time-state process

\[
Y^{s,x}(t) = Y^{\gamma, s,x}(t) = (t, X^{(\gamma)}(t)) ; \quad t \geq s
\]

assuming that \(Y^{s,x}(s^{-}) = x\).

The optimal harvesting problem is to find the value function \(\Phi(s, x)\) and an optimal harvesting strategy \(\gamma^{*} \in \Gamma\) such that

\[
\Phi(s, x) := \sup_{\gamma \in \Gamma} J^{(\gamma)}(s, x) = J^{(\gamma^{*})}(s, x) .
\]

This problem differs from the problems considered in [A1], [A3], [AS], [LØ1] and [LØ2] in that the prices \(f_i(t, \xi)\) are allowed to be density-dependent. This allows for more realistic models. For example, it is usually the case that if a type of fish, say population number \(i\), becomes more scarce, the price per unit of this fish increases. Conversely, if a type of fish becomes abundant then the price per unit goes down. Thus in this case the price \(\pi_i(t, \xi) = \pi_i(t, \xi_1, \ldots, \xi_n)\) is a nonincreasing function of \(\xi_i\). One can also have situations where \(\pi_i(t, \xi)\) depends on all the other population densities \(\xi_1, \ldots, \xi_n\) in a similar way.

It turns out that if we allow the prices to be density-dependent, a number of new – and perhaps surprising – phenomena occurs. The purpose of this paper is not to give a complete discussion of the situation, but to consider some illustrative examples.

Remark  Note that we can also give the problem (2.12) an economic interpretation: We can regard \(X_i(t)\) as the value at time \(t\) of an economic quantity or asset and we can let \(\gamma_i(t)\) represent the total amount paid in dividends from asset number \(i\) up to time \(t\). Then \(S\) can be interpreted as the solvency set, \(T\) as the time of bankruptcy and \(\pi_i(t, \xi)\) as the utility rate of dividends from asset number \(i\) at the state \((t, \xi)\). Then (2.12) becomes the problem of finding the optimal stream of dividends. This interpretation is used in [JS] (in the density-independent utility case). See also [LØ2].

In the following \(H^0\) denotes the interior of a set \(H\), \(\overline{H}\) denotes its closure.

If \(G \subset \mathbb{R}^k\) is an open set we let \(C^2(G)\) denote the set of real valued twice continuously differentiable functions on \(G\). We let \(C^2_0(G)\) denote the set of functions in \(C^2(G)\) with compact support in \(G\).

If we do not apply any harvesting, then the corresponding time-state population process \(Y(t) = (t, X(t))\), with \(X(t)\) given by (2.1)–(2.2), is an Itô diffusion whose generator coincides on \(C^2_0(\mathbb{R}^{1+n})\) with the partial differential operator \(L\) given by

\[
Lg(s, x) = \frac{\partial g}{\partial s}(s, x) + \sum_{i=1}^{n} b_i(s, x) \frac{\partial g}{\partial x_i}(s, x) + \frac{1}{2} \sum_{i,j=1}^{n} (\sigma \sigma^T)_{ij}(s, x) \frac{\partial^2 g}{\partial s \partial x}(s, x)
\]

for all functions \(g \in C^2(S)\).

The following result is a generalization to the multi-dimensional case of Theorem 1 in [A2] and a generalization to density-dependent prices of Theorem 2.1 in [LØ2]. For completeness we give the proof.
Theorem 2.1. Assume that
\[ f(t, \xi) \text{ is nonincreasing with respect to } \xi_1, \ldots, \xi_n, \text{ for all } t. \]

a) Suppose \( \varphi \geq 0 \) is a function in \( C^2(S) \) satisfying the following conditions
\[
\begin{align*}
(i) & \quad \frac{\partial \varphi}{\partial x_i}(t, x) \geq \pi_i(t, x) \quad \text{for all } (t, x) \in S \\
(ii) & \quad L \varphi(t, x) \leq 0 \quad \text{for all } (t, x) \in S.
\end{align*}
\]
Then
\[ \varphi(s, x) \geq \Phi(s, x) \quad \text{for all } (s, x) \in S. \]

b) Define the nonintervention region \( D \) by
\[ D = \left\{ (t, x) \in S; \frac{\partial \varphi}{\partial x_i}(t, x) > \pi_i(t, x) \text{ for all } i = 1, \ldots, n \right\}. \]

Suppose that, in addition to (i) and (ii) above,
\[ L \varphi(t, x) = 0 \quad \text{for all } (t, x) \in D \]
and that there exists a harvesting strategy \( \hat{\gamma} \in \Gamma \) such that the following, (iv)–(vii), hold:
\[
\begin{align*}
(iv) & \quad X^{(i)}(t) \in D \quad \text{for all } t \in [s, T] \\
(v) & \quad \left( \frac{\partial \varphi}{\partial x_i}(t, X^{(i)}(t)) - \pi_i(t, X^{(i)}(t)) \right) \cdot d\hat{\gamma}^{(c)}_i(t) = 0; \quad 1 \leq i \leq n \quad \text{(i.e. } \hat{\gamma}^{(c)}_i \text{ increases only when } \frac{\partial \varphi}{\partial x_i} = \pi_i) \\
& \quad \text{and} \\
(vi) & \quad \varphi(t_k, X^{(i)}(t_k)) - \varphi(t_k, X^{(i)}(t_k^-)) = -\pi_i(t_k, X^{(i)}(t_k^-)) \cdot \Delta \hat{\gamma}(t_k) \\
& \quad \text{at all jumping times } t_k \in [s, T) \text{ of } \hat{\gamma}(t), \text{ where} \\
& \quad \Delta \hat{\gamma}(t_k) = \hat{\gamma}(t_k) - \hat{\gamma}(t_k^-) \\
& \quad \text{and} \\
(vii) & \quad E^{s,x}[\varphi(T_R, X^{(i)}(T_R))] \to 0 \quad \text{as } R \to \infty \\
& \quad \text{where} \\
& \quad T_R = T \land R \land \inf \{ t > s; |X^{(i)}(t)| \geq R \}; \quad R > 0.
\end{align*}
\]
Then
\[ \varphi(s, x) = \Phi(s, x) \quad \text{for all } (s, x) \in S \]
and
\[ \gamma^* := \hat{\gamma} \quad \text{is an optimal harvesting strategy.} \]
Let $\gamma \in \Gamma$ and $(s, x) \in S$. Then by Itô's formula for semimartingales (the Doléans-Dade-Meyer formula) [P, Th. II.7.33] we have

\[
E^{s, x}[\varphi(T_R, X^{(\gamma)}(T_R^-))] = E^{s, x}[\varphi(s, X^{(\gamma)}(s))]
\]
\[
+ E^{s, x}\left[ \int_s^{T_R} \frac{\partial \varphi}{\partial t}(t, X^{(\gamma)}(t))dt + \int_s^{T_R} \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i}(t, X^{(\gamma)}(t^-))dX_i^{(\gamma)}(t) \right. 
\]
\[
\left. + \sum_{i,j=1}^{n} \int_s^{T_R} \frac{1}{2} (\sigma \sigma^T)_{ij}(t, X^{(\gamma)}(t)) \frac{\partial^2 \varphi}{\partial x_i \partial x_j}(t, X^{(\gamma)}(t))dt \right]
\]
\[
(2.18)
\]
\[
+ \sum_{s < t_k < T_R} \left\{ \varphi(t_k, X^{(\gamma)}(t_k)) - \varphi(t_k, X^{(\gamma)}(t^-_k)) - \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i}(t_k, X^{(\gamma)}(t^-_k))\Delta X_i^{(\gamma)}(t_k) \right\},
\]
where the sum is taken over all jumping times $t_k \in (s, T_R)$ of $\gamma(t)$ and

\[
\Delta X_i^{(\gamma)}(t_k) = X_i^{(\gamma)}(t_k) - X_i^{(\gamma)}(t^-_k).
\]

Let $\gamma^{(c)}(t)$ denote the continuous part of $\gamma(t)$, i.e.

\[
\gamma^{(c)}(t) = \gamma(t) - \sum_{s \leq t_k \leq t} \Delta \gamma(t_k).
\]

Then, since $\Delta X_i^{(\gamma)}(t_k) = -\Delta \gamma_i(t_k)$ we see that (2.18) can be written

\[
E^{s, x}[\varphi(T_R, X^{(\gamma)}(T_R^-))] = \varphi(s, x)
\]
\[
+ E^{s, x}\left[ \int_s^{T_R} \left\{ \frac{\partial \varphi}{\partial t} + \sum_{i=1}^{n} b_i \frac{\partial \varphi}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{n} (\sigma \sigma^T)_{ij} \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \right\}(t, X^{(\gamma)}(t))dt \right]
\]
\[
(2.19)
\]
\[
- E^{s, x}\left[ \int_s^{T_R} \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i}(t, X^{(\gamma)}(t))dt \gamma_i^{(c)}(t) \right] + E^{s, x}\left[ \sum_{s \leq t_k < T_R} \Delta \varphi(t_k, X^{(\gamma)}(t_k)) \right]
\]

where

\[
\Delta \varphi(t_k, X^{(\gamma)}(t_k)) = \varphi(t_k, X^{(\gamma)}(t_k)) - \varphi(t_k, X^{(\gamma)}(t^-_k)).
\]

Therefore

\[
\varphi(s, x) = E^{s, x}[\varphi(T_R, X^{(\gamma)}(T_R^-))] - E^{s, x}\left[ \int_s^{T_R} L \varphi(t, X^{(\gamma)}(t))dt \right]
\]
\[
+ E^{s, x}\left[ \int_s^{T_R} \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i}(t, X^{(\gamma)}(t))dt \gamma_i^{(c)}(t) \right]
\]
\[
(2.20)
\]
\[
- E^{s, x}\left[ \sum_{s \leq t_k < T_R} \Delta \varphi(t_k, X^{(\gamma)}(t_k)) \right].
\]
Let $y = y(r); 0 \leq r \leq 1$ be a smooth curve in $U$ from $X^{(\gamma)}(t_k)$ to $X^{(\gamma)}(t_k^-) = X^{(\gamma)}(t_k) + \Delta \gamma(t_k)$. Then

$$\tag{2.21} - \Delta \varphi(t_k, X^{(\gamma)}(t_k)) = \int_0^1 \nabla \varphi(t_k, y(r)) dy(r).$$

We may assume that

$$dy_i(r) \geq 0 \quad \text{for all } i, r.$$

Now suppose that (i) and (ii) hold. Then by (2.20) and (2.21) we have

$$\varphi(s, x) \geq E^{s,x}[\int \sum_{i=1}^n \pi_i(t, X^{(\gamma)}(t))d\gamma_i^{(c)}(t)]$$

$$+ E^{s,x}[\sum_{s \leq t_k < T_R} \left( \int \sum_{i=1}^n \pi_i(t_k, y(r)) dy_i(r) \right)]$$

(2.22)

Since we have assumed that $\pi_i(t, \xi)$ is nonincreasing with respect to $\xi_1, \ldots, \xi_n$ we have

$$\pi_i(t_k, X^{(\gamma)}(t_k^-)) \leq \pi_i(t_k, y(r)) \leq \pi_i(t_k, X^{(\gamma)}(t_k))$$

for all $i, k$ and $r \in [0, 1]$. Hence

$$\tag{2.23} \int_0^1 \pi_i(t_k, y(r)) dy_i(r) \geq \pi(t_k, X^{(\gamma)}(t_k^-)) \cdot \Delta \gamma_i(t_k).$$

Combined with (2.22) this gives

$$\varphi(s, x) \geq E^{s,x}[\int_0^{T_R} \pi(t, X^{(\gamma)}(t))d\gamma^{(c)}(t) + \sum_{s \leq t_k < T} \pi(t_k, X^{(\gamma)}(t_k^-)) \cdot \Delta \gamma(t_k)]$$

$$= E^{s,x}\left[ \int_{[s, T_k]} \pi(t, X^{(\gamma)}(t^-))d\gamma(t) \right].$$

(2.24)

Letting $R \to \infty$ we obtain $\varphi(s, x) \geq J^{(\gamma)}(s, x)$. Since $\gamma \in \Gamma$ was arbitrary we conclude that (2.15) holds. Hence a) is proved.

b) Next, suppose that (iii)–(vii) also hold. Then if we apply the argument above to $\gamma = \hat{\gamma}$ we get in (2.20) the following:

$$\varphi(s, x) = E^{s,x}[\varphi(T_R, X^{(\hat{\gamma})}(T_R^-))]
+ E^{s,x}[\int_0^{T_R} \pi(t, X^{(\hat{\gamma})}(t))d\hat{\gamma}^{(c)}(t) + \sum_{s \leq t_k < T_R} \pi(t_k, X^{(\hat{\gamma})(t_k^-)}) \cdot \Delta \hat{\gamma}(t_k)]
= E^{s,x}[\varphi(T_R, X^{(\hat{\gamma})}(T_R^-))] + E^{s,x}\left[ \int_{[s, T_k]} \pi(t, X^{(\hat{\gamma})}(t))d\hat{\gamma}(t) \right]
\longrightarrow J^{(\hat{\gamma})}(s, x) \quad \text{as } R \to \infty.$$
Hence \( \varphi(s,x) = J^{(\gamma)}(s,x) \leq \Phi(s,x) \). Combining this with (2.14) from a) we get the conclusion (2.16) of part b). This completes the proof of Theorem 2.1.

If we specialize to the 1-dimensional case with just one population \( X^{(\gamma)}(t) \) given by

\[
\begin{cases}
  dX^{(\gamma)}(t) = b(t, X^{(\gamma)}(t))dt + \sigma(t, X^{(\gamma)}(t))dB(t) - d\gamma(t) \; ; \\
  X^{(\gamma)}(s^{-}) = x \in \mathbb{R}
\end{cases}
\]

then Theorem 2.1a) gets the form (see also \([A2, \text{Lemma 1}]\))

**Corollary 2.2. Assume that**

\( \xi \to \pi(t, \xi) ; \xi \in \mathbb{R} \) is nonincreasing for all \( t \in [0, T] \)

\( \varphi(t,x) \geq 0 \) is a function in \( C^2(S) \) such that

\[
\frac{\partial \varphi}{\partial x}(t,x) \geq \pi(t,x) \quad \text{for all } (t,x) \in S
\]

and

\( L\varphi(t,x) \leq 0 \quad \text{for all } (t,x) \in S . \)

Then

\( \varphi(s,x) \geq \Phi(s,x) \quad \text{for all } (s,x) \in S . \)

### 3 Examples

In this section we apply Theorem 2.1 or Corollary 2.2 to some special cases.

**Example 3.1.** Suppose \( X^{(\gamma)}(t) \) is given by

\[
\begin{cases}
  dX^{(\gamma)}(t) = \mu dt + \sigma dB(t) - d\gamma(t) \; ; \\
  X^{(\gamma)}(s) = x > 0
\end{cases}
\]

where \( \mu > 0 \) and \( \sigma \neq 0 \) are constants.

We want to maximize the total discounted value of the harvest, given by

\[
J^{(\gamma)}(s,x) = E^{s,x}\left[ \int_{[s,T]} e^{-\rho t} g(X^{(\gamma)}(t^-)) d\gamma(t) \right]
\]

where \( g : \mathbb{R} \to \mathbb{R} \) is a given nonincreasing function (the density-dependent price) and

\( T = \inf \{ t > s ; X^{(\gamma)}(t) \leq 0 \} \)

is the time of extinction, i.e. \( S = \{(t,x); x > 0\} \). The case with \( g \) constant was solved in \([JS]\). Then it is optimal to do nothing if the population is below a certain threshold \( x^* > 0 \) and then harvest according to local time of the downward reflected process \( \bar{X}(t) \) at \( \bar{X}(t) = x^* \).
Now consider the case when
\begin{equation}
(3.4) \quad g(x) = x^{-1/2}, \quad \text{i.e.} \quad \pi(t, x) = e^{-\rho t} x^{-1/2}; \quad x > 0.
\end{equation}
Then the price increases as the population size \( x \) goes to 0, so (2.24) holds. Suppose we apply the “take the money and run”-strategy \( \tilde{\gamma} \). This strategy empties the whole population immediately. It can be described by
\begin{equation}
(3.5) \quad \tilde{\gamma}(s) = X(s^{-}) = x.
\end{equation}
Such a strategy gives the harvest value
\begin{equation}
(3.6) \quad J(\tilde{\gamma})(s, x) = e^{-\rho s} x^{-1/2} = e^{-\rho s} \sqrt{x}; \quad x > 0.
\end{equation}
However, it is unlikely that this is the best strategy because it does not take into account that the price increases as the population size goes down. So we try the following “chattering policy”, denoted by \( \tilde{\gamma} = \tilde{\gamma}^{(m, \eta)} \), where \( m \) is a fixed natural number and \( \eta > 0 \):

At the times
\begin{equation}
(3.7) \quad t_k = \left( s + \frac{k}{m} \eta \right) \wedge T; \quad k = 1, 2, \ldots, m
\end{equation}
we harvest an amount \( \Delta \tilde{\gamma}(t_k) \) which is the fraction \( \frac{1}{m} \) of the current population. This gives the expected harvest value
\begin{equation}
(3.8) \quad J(\tilde{\gamma}^{(m, \eta)})(s, x) = E_{s, x} \left[ \sum_{k=1}^{m} e^{-\rho t_k} \left( X(\tilde{\gamma}^{(s, x)}(t_k^{-})) \right)^{-1/2} \right] \Delta \tilde{\gamma}(t_k),
\end{equation}
where we have used the notation
\[
x^+ = \max(x, 0); \quad x \in \mathbb{R}.
\]
This can be written
\begin{equation}
(3.9) \quad J(\tilde{\gamma}^{(m, \eta)})(s, x) = E_{s, x} \left[ \sum_{k=1}^{m} e^{-\rho t_k} \left( x - \tilde{\gamma}(t_k^{-}) \right)^{-1/2} \right] \Delta \tilde{\gamma}(t_k).
\end{equation}
Now let \( \eta \to 0 \). Then all the \( t_k \)'s converge to \( s \) and we get
\[
J(\tilde{\gamma}^{(m, 0)})(s, x) = \lim_{\eta \to 0} J(\tilde{\gamma}^{(m, \eta)})(s, x) = e^{-\rho s} \sum_{k=1}^{m} \left( x - \frac{k}{m} x \right)^{-1/2} \frac{1}{m} x
\]
\begin{equation}
(3.10) \quad = e^{-\rho s} \sum_{k=1}^{m} h(x_k) \Delta x_k,
\end{equation}
where \( h(y) = (x - y)^{-1/2}, x_k = \frac{k}{m} x \) and \( \Delta x_k = x_{k+1} - x_k = \frac{x}{m} \).

Now if \( \varepsilon > 0 \) is given we can find a natural number \( m \) such that
\begin{equation}
| \int_{0}^{x} (x - y)^{-1/2} dy - \sum_{k=1}^{m} h(x_k) \Delta x_k | < \varepsilon.
\end{equation}
Therefore, by choosing $m$ and $\eta$ properly we can obtain that

\begin{equation}
|J^{(\tilde{\gamma}(m,\eta))}(s, x) - e^{-\rho s} \int_0^x (x - y)^{-1/2} dy| < \varepsilon.
\end{equation}

We conclude that

\begin{equation}
\sup_{\gamma} J^{(\gamma)}(s, x) \leq e^{-\rho s} \int_0^x (x - y)^{-1/2} dy = e^{-\rho s} 2\sqrt{x}.
\end{equation}

We call this policy of applying $\tilde{\gamma}^{(m,\eta)}$ in the limit as $\eta \to 0$ and $m \to \infty$ the policy of immediate chattering down to 0. (This limit does not exist as a strategy in $\Gamma$.) From (3.13) we conclude that

\begin{equation}
\Phi(s, x) \geq 2e^{-\rho s} \sqrt{x}.
\end{equation}

On the other hand, let us check if the function

\begin{equation}
\varphi(s, x) := 2e^{-\rho s} \sqrt{x}
\end{equation}

satisfies the conditions (2.26)–(2.28) of Corollary 2.2: Condition (2.26) holds trivially, and since

\[
\frac{\partial \varphi}{\partial x}(s, x) = e^{-\rho s} x^{-1/2} = \pi(s, x),
\]

(2.27) holds.

Now

\[
L = \frac{\partial}{\partial s} + \mu \frac{\partial}{\partial x} + \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial x^2}
\]

and therefore

\[
L \varphi(s, x) = 2e^{-\rho s} \left[ -\rho x^{1/2} + \mu : \frac{1}{2} x^{-1/2} + \frac{1}{2} \sigma^2 \frac{1}{2} (-\frac{1}{2}) x^{-3/2} \right]
\]

\[
= -2\rho e^{-\rho s} x^{-3/2} \left[ x^2 - \frac{\mu}{2\rho} x + \frac{\sigma^2}{8\rho} \right].
\]

So (2.28) holds if $\mu^2 \leq 2 \rho \sigma^2$. By Corollary 2.2 we conclude that $\varphi = \Phi$ in this case.

We have proved part a) of the following result:

**Theorem 3.2.** Let $X^{(\gamma)}(t)$ and $T$ be given by (3.1) and (3.3), respectively.

a) Assume that

\begin{equation}
\mu^2 \leq 2 \rho \sigma^2.
\end{equation}

Then

\[
\Phi(s, x) := \sup_{\gamma \in \Gamma} E^{s, x} \left[ \int_{[s, T]} e^{-\rho t} \{X^{(\gamma)}(t^-)\}^{-1/2} d\gamma(t) \right] = 2e^{-\rho s} \sqrt{x}.
\]

This value is achieved in the limit if we apply the strategy $\tilde{\gamma}^{(m,\eta)}$ above with $\eta \to 0$ and $m \to \infty$, i.e. by applying the policy of immediate chattering down to 0.
b) Assume that 
\[ \mu^2 > 2\rho\sigma^2 \]

Then the value function has the form 
\[ \Phi(s, x) = \begin{cases} e^{-\rho s} C(e^{\lambda_1 x} - e^{\lambda_2 x}) ; & 0 \leq x < x^* \\ e^{-\rho s} (2\sqrt{x} - 2\sqrt{x^*} + A) ; & x^* \leq x \end{cases} \]

for constants \( C > 0 \), \( A > 0 \) and \( x^* > 0 \) specified by (3.26)-(3.29) below, where 
\[ \lambda_1 = \sigma^{-2} [-\mu + \sqrt{\mu^2 + 2\rho\sigma^2}] > 0 \], \( \lambda_2 = \sigma^{-2} [-\mu - \sqrt{\mu^2 + 2\rho\sigma^2}] < 0 \).

The corresponding optimal policy is the following:
\[ \begin{align*} 
\text{If } x > x^* \text{ it is optimal to apply immediate chattering from } x \text{ down to } x^*. \\
\text{if } 0 < x < x^* \text{ it is optimal to apply the harvesting equal to the local time of} \\
\text{the downward reflected process } \bar{X}(t) \text{ at } x^*. 
\end{align*} \]

Proof of b). First note that if we apply the policy of immediate chattering from \( x \) down to \( x^* \), where \( 0 < x^* < x \), then the value of the harvested quantity is 
\[ e^{-\rho s} \int_0^{x-x^*} (x - y)^{-1/2} dy = e^{-\rho s} \int_{x^*}^x u^{-1/2} du = 2e^{-\rho s} (\sqrt{x} - \sqrt{x^*}) . \]

This follows by the argument (3.7)–(3.12) above.

To verify (3.18)–(3.21) note that \( \lambda_1, \lambda_2 \) are the roots of the quadratic equation 
\[ -\rho + \mu \lambda + \frac{1}{2}\sigma^2 \lambda^2 = 0 . \]

Hence, with \( \varphi(s, x) \) defined to be the right hand side of (3.18) we have 
\[ L\varphi(s, x) = 0 \quad \text{for } x < x^* \]

and 
\[ \varphi(s, 0) = 0 . \]

We now require that \( \varphi \) is \( C^2 \) at \( x = x^* \). This gives the 3 equations 
\[ C(e^{\lambda_1 x^*} - e^{\lambda_2 x^*}) = A \]
\[ C(\lambda_1 e^{\lambda_1 x^*} - \lambda_2 e^{\lambda_2 x^*}) = (x^*)^{-1/2} \]
\[ C(\lambda_1^2 e^{\lambda_1 x^*} - \lambda_2^2 e^{\lambda_2 x^*}) = -\frac{1}{2}(x^*)^{-3/2} \]

Dividing (3.27) by (3.28) we get the equation 
\[ \frac{\lambda_1 e^{\lambda_1 x^*} - \lambda_2 e^{\lambda_2 x^*}}{\lambda_1^2 e^{\lambda_1 x^*} - \lambda_2^2 e^{\lambda_2 x^*}} = -2x^* . \]
Since the left hand side of (3.29) goes to $(\lambda_1 + \lambda_2)^{-1} < 0$ as $x^* \to 0^+$ and goes to $\lambda_1^{-1} > 0$ as $x^* \to \infty$ we see by the intermediate value theorem that there exists $x^* > 0$ satisfying this equation.

With this value of $x^*$ we define $C$ by (3.27) and then we define $A$ by (3.26). Then we have proved the existence of a solution $C > 0$, $A > 0$, $x^* > 0$ of the system (3.26)–(3.28). With this choice of $C$, $A$, $x^*$ the function $\varphi(s,x)$ becomes a $C^2$ function and one can verify that $\varphi$ satisfies conditions (i), (ii) of Theorem 2.1 (the details are left to the reader). Hence

$$\varphi(s,x) \geq \Phi(s,x) \quad \text{for all } s,x.$$  

Moreover, the nonintervention region $D$ given by (2.16) is seen to be

$$D = \{(s,x); 0 < x < x^*\}.$$  

Hence by (3.24) we know that (iii) holds.

Moreover, if $x \leq x^*$ it is well-known that the local time $\hat{\gamma}$ at $x^*$ of the downward reflected process $\hat{X}(t)$ at $x^*$ satisfies (iv)–(vi). (See e.g. [LO1] for more details.) And (vii) follows from (3.25). By Theorem 2.1 b) we conclude that if $x \leq x^*$ then $\gamma^* := \hat{\gamma}$ is optimal and $\varphi(s,x) = \Phi(s,x)$. Finally, if $x > x^*$ then it follows by (3.22) that immediate chattering from $x$ down to $x^*$ gives the value $2e^{-\rho s}(\sqrt{x} - \sqrt{x^*}) + \Phi(s,x^*)$. Hence

$$\Phi(s,x) \geq 2e^{-\rho s}(\sqrt{x} - \sqrt{x^*}) + \Phi(s,x^*) \quad \text{for } x > x^*.$$  

Combined with (3.30) this shows that

$$\varphi(s,x) = \Phi(s,x) \quad \text{for all } s,x$$  

and the proof of b) is complete.

**Example 3.3.** The Brownian motion example is perhaps not so good as a model of a biological stock, since Brownian motion is a poor model for population growth. Instead, let us consider a standard population growth model (in the sense that it can be generated from a classic birth-death-process), like the logistic diffusion considered in [AS]. That is, let us consider the problem

$$V(0,x) = V(x) = \sup_{\gamma \in \Gamma} \int_{[0,T]} e^{-\rho t} X^{-1/2}(t^-) d\gamma(t)$$  

subject to

$$dX(t) = \mu X(t)(1 - K^{-1}X(t)) dt + \sigma X(t) dB(t) - d\gamma(t), \quad X(0^-) = x > 0 ,$$  

where $\mu > 0$, $K^{-1} > 0$, and $\sigma > 0$ are known constants, $B(t)$ denotes a Brownian motion in $\mathbb{R}$, and $T = \inf\{t \geq 0 : X(t) \leq 0\}$ denotes the extinction time. We define the mapping $H : \mathbb{R}_+ \mapsto \mathbb{R}_+$ as

$$H(x) = \int_0^x y^{-1/2} dy = 2\sqrt{x}.$$  

12
The generator $A$ of $X(t)$ is given by

$$ A = \frac{1}{2} \sigma^2 x^2 \frac{d^2}{dx^2} + \mu x (1 - K^{-1}x) \frac{d}{dx} $$

and we find that

(3.34) \hspace{1cm} G(x) := ((A - \rho)H)(x) = \sqrt{x} \left[ \mu - 2\rho - \sigma^2/4 - \mu K^{-1}x \right].

Thus, if $\mu \leq 2\rho + \sigma^2/4$ then by the same argument as in Example 3.2 we see that the optimal policy is immediate chattering down to 0. We then have $T = 0$, and the value reads as

(3.35) \hspace{1cm} V(x) = 2\sqrt{x}.

However, if $\mu > 2\rho + \sigma^2/4$, then we see that the mapping $G(x)$ satisfies the conditions of Theorem 2 in [A2] and, therefore we find that there is a unique threshold $x^*$ satisfying the condition

(3.36) \hspace{1cm} x^* \psi''(x^*) + \frac{1}{2} \psi'(x^*) = 0,

where $\psi(x)$ denotes the increasing fundamental solution of the ordinary differential equation \((A - \rho)u(x) = 0\), that is, $\psi(x) = x^a M(\theta, 2\theta, 2\mu^{-1}x)$, where $\theta = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2\mu}{\sigma^2}}$, and $M$ denotes the confluent hypergeometric function. In this case, the value reads as

(3.37) \hspace{1cm} V(x) = \begin{cases} 2(\sqrt{x} - \sqrt{x^*}) + \sqrt{x^*(\mu(1 - K^{-1}x^*) - \sigma^2/4)/r}, & x \geq x^* \\ \sqrt{x^* \psi'(x^*)}, & x < x^*. \end{cases}

Especially, the value is a solution of the variational inequality

$$ \min\{(\rho - A)V(x), V'(x) - x^{-1/2}\} = 0. $$

We summarize this as follows:

**Theorem 3.4. a)** Assume that

(3.38) \hspace{1cm} \mu \leq 2\rho + \sigma^2/4.

Then the value function $V(x)$ of problem (3.31) is

(3.39) \hspace{1cm} V(x) = 2\sqrt{x}.

This value is obtained by immediate chattering down to 0.

**b)** Assume that

(3.40) \hspace{1cm} \mu > 2\rho + \sigma^2/4.

Then $V(x)$ is given by (3.37). The corresponding optimal policy is immediate chattering from $x$ down to $x^*$ if $x > x^*$, and local time at $x^*$ of the downward reflected process $\bar{X}(t)$ at $x^*$ if $x < x^*$, where $x^*$ is given by (3.36).
References

[A1] Alvarez, L.H.R. *Optimal harvesting under stochastic fluctuations and critical depensa-

[A2] Alvarez, L.H.R. *Singular stochastic control in the presence of a state-dependent yield struc-

[A3] Alvarez, L.H.R. *On the option interpretation of rational harvesting planning*, 2000, *Jour-

[AS] Alvarez, L.H.R. and Shepp, L.A. *Optimal harvesting of stochastically fluctuating popula-

[JS] Jeanblanc-Picqué, M. and Shiryaev, A. *Optimization of the flow of dividends*, 1995, *Rus-

[LES1] Lande, R. and Engen S. and Sæther B.-E. *Optimal harvesting, economic discounting and ex-

[LES2] Lande, R. and Engen S. and Sæther B.-E. *Optimal harvesting of fluctuating populations with a risk of extinction*, *The American Naturalist*, 1995, vol 145, 728–745.

[LØ1] Lungu, E. M. and Øksendal, B. *Optimal harvesting from a population in a stochastic crowed environment*, 1996, *Mathematical Biosciences*, vol. 145, 47–75.

[LØ2] Lungu, E. M. and Øksendal, B. *Optimal harvesting from interacting populations in a stochastic environment*, 2001, *BERNOULLI*, vol. 7, 527–539.

[P] Protter, P. *Stochastic Integration and Differential Equations*, 2004, Second Edition, Springer-Verlag.