BLP DISSIPATIVE STRUCTURES IN PLANE

A.V. Yurov

ABSTRACT. We study the Darboux and Laplace transformations for the Boiti-Leon-Pempinelli equations (BLP). These equations are the (1+2) generalization of the sinh-Gordon equation. In addition, the BLP equations reduced to the Burgers (and anti-Burgers) equation in a one-dimensional limit. Localized nonsingular solutions in both spatial dimensions and (anti) “blow-up” solutions are constructed. The Burgers equation’s "dressing" procedure is suggested. This procedure allows us to construct such solutions of the BLP equations which are reduced to the solutions of the dissipative Burgers equations when $t \to \infty$. These solutions we call the BLP dissipative structures.

1. INTRODUCTION

Localized structures are frequently associated with dispersion and nonlinearity. A large class of nonlinear evolution equations with dispersion in two spatial dimensions have been studied. Some of them are physically relevant equations, for example the Davey-Stewartson (DS) and the Kadomtsev-Petviashvili (KP) equations. The DS equations are the (1+2)-dimensional integrable generalizations of the nonlinear Schrödinger equation and the KP equations are the same for the KdV equation. Recently, interest has appeared [1-2] into the (1+2)-dimensional Boiti-Leon-Pempinelli (BLP) equation. Their integro-differential equation reduced either to the sine-Gordon or to the sinh-Gordon equation in a one-dimensional limit. In [1] Boiti, Leon and Pempinelli used the integrable structure of their equation to extract a Bäcklund transformation for soliton solution. In [2] it is shown that the considered equation is hamiltonian and some soliton’s type solutions are constructed.

There are two purposes of this paper. The first is the study of Darboux and Laplace transformations to construct a set of the exact solutions of the BLP equation. The second is connected with the extremely simple tie between the BLP equation and the Burgers equation. In the special case the (1+2) BLP equation is reduced to the (1+1) Burgers or “anti-Burgers” (i.e., Burgers equation with the time reversed) equations. Thus we have the interesting question: whether the BLP equation has dissipative solutions in plane? The positive answer is set forth in the last Section of this paper but here it is necessary to define more exactly our terminology. We’ll do it in the next paragraph.

The Burgers equation is the known model that describes the nonlinear and dissipative (diffusion) processes. The Burgers equation’s solutions we shall call the Burgers dissipative structures. The situation with the BLP equations is more complex because one is reduced both to the Burgers and to the anti-Burgers equations. In the Section 5 we show the way to construct such solutions of the BLP equation which are reduced to the solutions of the dissipative Burgers equation when $t \to \infty$. These solutions we call the BLP dissipative structures.
The plan of the paper is as follows. In Section 2 we review the BLP equation (with the reduction to the Burgers equation) and the BLP’s set of discrete symmetries – the Darboux (DT) and Laplace (LT) transformations. The result of the multiple DT and LT is discussed in Section 3. In Section 4 we use the DT for constructing exact solutions of the BLP equation. Section 5 is devoted to the Burgers equation’s ”dressing procedure” via LT. We leave some final comments to the last section.

2. THE BLP EQUATIONS

The integro-differential BLP equation appeared for the first time in [1] and can be write as the system of two equations ([1], [2]),

\[ a_{ty} + (a^2 - a_x)_{xy} + 2b_{xx} = 0, \]
\[ b_t + (b_x + 2ab)_x = 0, \]

(1)

where \( a = a(t, x, y) \), \( b = b(t, x, y) \). We can introduce new field \( p = p(t, x, y) \) which satisfies \( b = p_y \) and rewrite (1) in the following form

\[ a_t + 2aa_x + (2p - a)_{xx} = 0, \]
\[ p_{yt} + (p_{yx} + 2ap_y)_x = 0. \]

(2)

We’ll call (1) and (2) the BLP equations. These equations has \([L, A]\) pair and the Hamilton structure [2]. On the other hand, the reduction achieved by imposing \( p = 0 \) maps the BLP equations (2) onto the differentiated Burgers equation

\[ a_t + 2aa_x - a_{xx} = 0. \]

(3)

If we choose \( p = a \) then the eq. (2) reduced ”anti-Burgers equation” which turn into (3) with \( t \to -t, x \to -x \).

Thus the BLP equations contain the dissipative (Burgers) and anti-dissipative equation. We have no conflict with this fact and the BLP’s Hamilton structure. It may be shown that if \( b = 0 \) then the Hamiltonian \( H = 0 \). Moreover, the fact of Hamiltonian’s existence (and the fact of existence of two additional conservation laws [2]) is right only for special case asymptotic behavior of \( a \) and \( b \): \( a \to 0 \) and \( b \to const \) if \( x^2 + y^2 \to \infty \) [2]. If these conditions are not realized then the BLP’s (anti) dissipative solutions are admissible. We’ll demonstrate such solutions in Sections 4 and 5.

Eq. (1) and (2) is the compatibility condition of the linear system of equations \([L, A]\) pair

\[ \psi_{xy} + a\psi_y + p_y\psi = 0, \quad \psi_t = \psi_{xx} + 2p_x\psi. \]

(4)

Let \( \phi \) and \( \psi \) be solutions of (4). We define two functions \( \tau = \partial_x(\ln \phi) \) and \( \rho = (\partial_y(\ln \phi))^{-1} \). Eq. (4) is covariant with respect to the two types of DT,

\[ \psi \to \psi[1] = \rho\psi_y - \psi, \quad a \to a[1] = a - \partial_x \ln \rho, \]

(5)

\[ p \to p[1] = p - \rho p_y, \]

and

\[ \psi \to \psi[1] = \psi_x - \tau\psi, \quad a \to a[1] = a - \partial_x \ln(a + \tau), \]

(6)

\[ p \to p[1] = p + \tau. \]
In addition, eq. (4) is covariant with respect to the LT

\[ \psi \rightarrow \psi[1] = \frac{\psi_y}{p_y}, \quad a \rightarrow a[1] = a + \partial_x \ln p_y, \]
\[ p \rightarrow p[1] = p + a + \partial_x \ln p_y, \]

and the inverse transformation being given by the formulas

\[ \psi \rightarrow \psi[-1] = \partial_x \psi + a \psi, \quad a \rightarrow a[-1] = a - \partial_x \ln (p - a)_y, \]
\[ p \rightarrow p[-1] = p - a. \]

Remark 1. The DT (5)-(6) and the LT (7)-(8) are mutually complementary with each other. It means that it is not possible to convert DT into LT and inversely. In fact, for the DT we get \( \psi[1] = 0 \) if \( \phi = \psi \) (see (5)-(6)). But if \( \psi[-1] = 0 \) (see LT (8)) then we get \( p = a \) (see eq. (4)) and \( a[-1] \) can’t be calculated from (8). It is easy to verify the analogous conclusion for the LT (7).

Remark 2. Let one defines three functions \( u, v, \chi, \)

\[ a = -\partial_x \ln u, \quad b = p_y = -uv, \quad \chi = \frac{\partial_x \psi}{u} \]

and cone variables \( \xi \) and \( \eta, \)

\[ \partial_y = \partial_\eta - \partial_\xi, \quad \partial_x = \partial_\eta + \partial_\xi, \]

then the L-equation from (4) can be rewritten in the Zakharov-Shabat equation’s form

\[ \partial_\eta \Phi = \sigma_3 \partial_\xi \Phi + U \Phi, \]

where

\[ \Phi = \begin{pmatrix} \psi \\ \chi \end{pmatrix}, \quad U = \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

In Ref. [3], we applied the DT for (9) to construct exact solutions to the Dirac equations with (1+1) potentials and to the DS equations. It is not difficult to check that the DT from [3] is the superposition formula for the two simpler Darboux transformations given by formulas (5) and (6).

Remark 3. Eq. (9) is the spectral problem for the DS equations. LT produce an explicitly invertible Bäcklund autotransformations for the DS equations [4]. In Ref. [5] we showed that these transformations allow one to construct solutions to the DS equations that fall off in all directions in the plane according to exponential and algebraic law.

3. CRUM LAW

In Ref. [3], we obtained the extended Crum law for the Eq. (9). In this Section we will consider this procedure for the DT (5)-(6) and LT (7)-(8).

Let us consider the DT (5) and define \( Q_N \psi \equiv \partial_y^{N+1} \partial_x \psi \). It is easy to check that if \( \psi \) is a solution of (4) then \( Q_N \) can be written as

\[ Q_N = -a \partial_y^{N+1} - (p + Na)_y \partial_y^N + \sum_{k=1}^{N-1} c_{N,k} \partial_y^k - \partial_y^{N+1} p, \]
where $c_{N,k} = c_{N,k}(a, p_y; a_x, p_{xy}; a_y, p_{yy}; \ldots)$.

Let $\{\psi, \psi_1, \ldots, \psi_N\}$ are $N + 1$ particular solutions of (4). After $N$-time DT (5) we get

$$\psi[N] = -\psi + \sum_{k=1}^{N} \alpha_{N,k} \partial_y^k \psi.$$  \hspace{1cm} (11)

Plugging (11) into

$$\psi_{xy}[N] + a[N] \psi_y[N] + p_y[N] \psi[N] = 0$$  \hspace{1cm} (12)

and using (10) we get

$$a[N] = a - \partial_x \ln \alpha_{N,N}, \quad p[N] = p - \sum_{k=1}^{N} \alpha_{N,k} \partial_y^k p.$$  \hspace{1cm} (13)

To compute $\alpha_{N,k}$ we take into account that

$$\sum_{k=1}^{N} \alpha_{N,k} \partial_y^k \psi_j = \psi_j, \quad j = 1, \ldots, N,$$

therefore

$$\psi[N] = \frac{D_N(1 | 1)}{D_N(1, N + 2 | 1, 2)}, \quad a[N] = a - \partial_x \ln \frac{D_N(1, 2 | 1, 2)}{D_N(1, N + 2 | 1, 2)},$$

$$p[N] = p + (-1)^N \frac{D_N(1 | 2)}{D_N(1, N + 2 | 1, 2)},$$  \hspace{1cm} (13)

where $D_N(i, j | k, m)$ are determinants which can be obtained by the deletion of columns $i, j$ and rows $k, m$ from the $(N + 2) \times (N + 2)$ determinant $D_N$

$$D_N = \begin{vmatrix}
\partial^{N+1} p & \partial^{N} p & \partial^{N-1} p & \ldots & \partial p & 0 \\
\partial^{N+1} \psi & \partial^{N} \psi & \partial^{N-1} \psi & \ldots & \partial \psi & \psi \\
\partial^{N+1} \psi_1 & \partial^{N} \psi_1 & \partial^{N-1} \psi_1 & \ldots & \partial \psi_1 & \psi_1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\partial^{N+1} \psi_N & \partial^{N} \psi_N & \partial^{N-1} \psi_N & \ldots & \partial \psi_N & \psi_N \\
\end{vmatrix}.$$  \hspace{1cm} (14)

In (14) for brevity $\partial_y \equiv \partial$.

Now, let us consider the DT (6). We define $P_N \psi = \partial_x^{N+1} \partial_y \psi$, then

$$P_N = -p_y \partial_x^N + \sum_{k=0}^{N-1} \beta_{N,k} \partial_x^k + \lambda_N \partial_y,$$  \hspace{1cm} (15)

As far as $P_{N+1} = \partial_x P_N$, therefore

$$\beta_{N+1,N} = \beta_{N,N-1} - p_{xy}, \quad \beta_{N+1,k} = \partial_x \beta_{N,k} + \beta_{N,k-1}, \quad 1 \leq k \leq N - 1,$$

$$\beta_{N+1,0} = \partial_x \beta_{N,0} - p_y \lambda_N, \quad \lambda_{N+1} = \partial_x \lambda_N - a \lambda_N.$$  \hspace{1cm} (16)
After $N$-time DT (6) we get
\[ \psi[N] = \partial_x^N \psi + \sum_{k=0}^{N-1} \alpha_{N,N-k} \partial_x^k \psi. \] (17)

Substituting (17) into (12) and taking (15), (16) into account we get
\[ a[N]p_y[N] = ap_y + \partial_y \left( \partial_x (\alpha_{N,1} - Np) + \alpha_{N,2} - \frac{1}{2} \alpha_{N,1}^2 \right), \] (18)
\[ p[N] = p - \alpha_{N,1}. \]

To compute $\alpha_{N,k}$ we take into account that
\[ \partial_x^N \psi_j + \sum_{k=0}^{N-1} \alpha_{N,N-k} \partial_x^k \psi_j = 0, \] (19)
therefore
\[ \psi[N] = \frac{W_N}{W_N(1 \mid 1)}, \quad \alpha_{N,1} = -\frac{W_N(2 \mid 1)}{W_N(1 \mid 1)}, \quad \alpha_{N,2} = \frac{W_N(3 \mid 1)}{W_N(1 \mid 1)}, \] (20)
where
\[ W_N = \begin{vmatrix}
\partial_x^N \psi & \partial_x^{N-1} \psi & \ldots & \psi \\
\partial_x^N \psi_1 & \partial_x^{N-1} \psi_1 & \ldots & \psi_1 \\
\vdots & \vdots & \ddots & \vdots \\
\partial_x^N \psi_N & \partial_x^{N-1} \psi_N & \ldots & \psi_N \\
\end{vmatrix} \] (21)
and $\partial_x \equiv \partial$.

With the help of (7) and (8) one can find now the result of $N$-fold LT $a[N]$ ($a[-N]$) and $b[N]$ ($b[-N]$) (it is convenient to use fields $b$ instead of fields $p$ in these cases). Let us consider the LT (7). After $N$-time transformations we get for the $\psi[N]$ (compare with the (11))
\[ \psi[N] = \sum_{k=1}^{N} \alpha_{N,k} \partial_y^k \psi. \] (22)

It is convenient to redesignate $a[N] \equiv a_N$, $b[N] \equiv b_N$. Substituting (22) into (12) we get
\[ a_N = a - \partial_x \ln \alpha_{N,N}, \quad b_N = b + \partial_y \left( Na - \partial_x \ln(\alpha_{N-1,N-1} \alpha_{N,N}) \right) \] (23)
with the supplementaty condition
\[ \sum_{k=1}^{N} \alpha_{N,k} \partial_y^{k-1} b = 0. \] (24)

Thus it is necessary to find coefficients $\alpha_{N,k}$ and to use (22)-(23). After the calculation we get
\[ \alpha_{N,N} = \left( \prod_{j=1}^{N} b_{j-1} \right)^{-1}, \] (25)
so (23) is the nonlinear superposition formula for the BLP equations. To compute \( \psi[N] \) we must obtain the rest coefficients \( \alpha_{N,k} \). To do it we introduce the operators

\[
\mathbf{f}_j = \frac{1}{b_j} \partial_y, \quad \mathbf{F} = \mathbf{f}_{N-1} \mathbf{f}_{N-2} \ldots \mathbf{f},
\]

where \( j = 0, \ldots, N - 1, \ b_0 = b, \ f_0 = f \). It’s clear that

\[
\psi[N] = \sum_{k=1}^{N} \alpha_{N,k} \partial^k_y \psi = \mathbf{F} \psi. \tag{26}
\]

The coefficients \( \alpha_{N,k} \) from the (26) can be calculated by the Crum law’s manner if to find \( N \) functions \( \theta_j \) \( (j = 1, \ldots, N) \) such that \( \mathbf{F} \theta_j = 0 \). It is easy to obtain these functions in the following way: for an arbitrary index \( j \) we construct the sequence of equations for the \( j \) functions \( \theta_j \) \( (k = 1, \ldots, j) \),

\[
f_{k-1} \theta_j^{(k-1)} = \theta_j^{(k)}, \quad \theta_j^{(0)} = \theta_j, \quad \theta_j^{(j-1)} = 1, \tag{27}
\]

The system (27) can be solved and we get \( N \) functions \( \theta_j \),

\[
\theta_j = <b < b_1 < \ldots < b_{j-1} > \ldots > y .
\]

In what follows \( < … >_{x,y} \) denote the integration by the \( x, y \)-variables,

\[
< S >_x = \frac{1}{2} \int dz \ sgn(x - z) S(z, y, t), \quad < S >_y = \frac{1}{2} \int dz \ sgn(y - z) S(x, z, t).
\]

We express the \( \alpha_{N,1} \) from (24), then

\[
\mathbf{F} \theta_j = \sum_{k=2}^{N} M_{kj} \alpha_{N,k} = 0, \quad M_{kj} = \partial^k_y \theta_j - \frac{\partial^{k-1} b}{b} \partial_y \theta_j. \tag{28}
\]

It is obvious that \( M_{k1} = M_{k2} = 0 \). Plugging (25) into (28) we get the nonhomogeneous system of linear equations (for desired coefficients) which can be solved by the Kramer’s formulae.

The result of \( N \)-fold LT (8) can be obtained by analogy with LT (7). It is easy to see that \( \psi[-N] \) has the form (17) with new coefficients \( \alpha_{N,k} \). It is means that \( b[-N] \) and \( a[-N] \) can be computed by the (18). Therefore our task is reduced to the search of new coefficients \( \alpha_{N,k} \). These coefficients may be expressed in terms of determinants (20), (21) with the change \( \psi_k \rightarrow \theta_k \) where these functions \( (\theta_k) \) satisfy the equation (19).

To find \( \theta_k \) we rewrite (19) as

\[
(\partial_x + a_{N-1}) (\partial_x + a_{N-2}) \ldots (\partial_x + a) \theta_j = 0, \quad j = 1, \ldots, N \tag{29}
\]

where we designate \( a[-k] \equiv a_k, b[-k] = b_k \) for the convenience. The (29) can be rewritten as a system,

\[
\begin{cases}
(\partial_x + a_j) \theta_m^{(j)} = \theta_m^{(j+1)}, & \text{for } j = 0, \ldots, m - 2, \\
(\partial_x + a_{m-1}) \theta_m^{(m-1)} = 0,
\end{cases} \tag{30}
\]
for any $\theta_m$. In (30) $\theta_m^{(0)} = \theta_m$, $m = 2, ..., N$, and
$$\theta_1 = \exp (-a_x).$$
By solving (30) we get the required functions
$$\theta_m = \theta_1 \Phi_m, \quad \Phi_m = b_1 < b_2 < ... < b_{m-1} > ... > x.$$

4. SOLITONS ON THE VACUUM BACKGROUND

In Ref. [2], author applied the Bäcklund transformations to construct singular solutions that fall off according to rational law in all directions in the plan with $a \to 0$ and $b \to \text{const}$ if $x^2 + y^2 \to \infty$. In this Section we show that the DT (5), (13) and (6), (18) allow one to obtain a rich set of exact solutions of the BLP equations (2). We’ll consider two examples: (i) the localized nonsingular solution $b$ falling off according to the exponential law as a function of $x$ and according to the rational law as a function of $y$; (ii) the ”blow-up” solution which is nonsingular when $t > 0$ and which has singularity when $t < 0$ (may be it is more correctly to call such solutions ”anti-blow-up” or ”white hole” solutions).

Let $a = b = 0$. To construct the solution (i) we choose the solution of (4) in the form
$$\phi = \exp (\mu^2t) \cosh(\mu x) + B(y),$$
where $B = B(y)$ is an arbitrary differentiable function. Taking (6) into account we get
$$b[1] = -\frac{\mu B' \sinh(\mu x) \exp (\mu^2t)}{(B + \cosh(\mu x) \exp(\mu^2t))^2}, \quad a[1] = -\frac{\mu \exp (\mu^2t) + B \cosh(\mu x)}{\sinh(\mu x) (B + \cosh(\mu x) \exp(\mu^2t))}. \quad (31)$$
If we choose the function $B(y)$ increasing according to the exponential law ($y \to \pm \infty$) that the solution (31) has nonclosed level curves, so we choose one as an even polynomial (the property of being even guarantees the nonsingular behavior of (31)). Thus, assuming $B(y) = \nu^2 y^{2N} + \lambda^2$ ($\nu$ and $\lambda$ are const and $3\nu = 3\lambda = 0$) we get the desired solution
$$b[1] = -\frac{2\mu N \nu^2 y^{2N-1} \sinh(\mu x) \exp (\mu^2t)}{(\lambda^2 + \nu^2 y^{2N} + \cosh(\mu x) \exp(\mu^2t))^2}. \quad (32)$$
The regular solution (32) falling off according to the exponential law when $x \to \pm \infty$ and according to the rational law $(1/y^{2N+1})$ when $y \to \pm \infty$, at the same time the singular solution $a[1]$ has level curves along $y$-line.

To construct (anti) blow-up solutions of BLP (ii) we shall apply two DT (5) and (6) on the vacuum background ($a = b = 0$). As a result we get
$$b[2] = \frac{W_2(1 \mid 1) D_2(1, 4 \mid 1, 2)}{D_2^2(1, 1 \mid 1, 1).} \quad (33)$$
Two functions $\psi_k$ ($k = 1, 2$) from the determinants $D$ and $W$ (see (14) and (21)) are the solutions of (4). We choose ones in the form
$$\psi_1 = C_1 \exp (\lambda^2t) \cosh(\lambda x) + \cosh(\alpha y), \quad \psi_2 = C_2 \exp (-\mu^2t) \sin(\mu x) + \sinh(\beta y).$$
Then \( D_2(1, 1 | 1, 1) = D_1(y) + D_2(x, y, t) \) where

\[
D_1(y) = \beta \cosh(\beta y) \cosh(\alpha y) - \alpha \sinh(\beta y) \sinh(\alpha y),
\]

\[
D_2(x, y, t) = \beta C_1 \exp(\lambda^2 t) \cosh(\lambda x) \cosh(\beta y) - \alpha C_2 \exp(-\mu^2 t) \sinh(\alpha y) \sin(\mu x).
\]

If \( \beta > \alpha > 0 \) then \( D_1(y) > 0 \) for the \( y \in (-\infty, +\infty) \).

Let us consider \( D_2(x, y, t) \). It is clear that if \( C_1 > 0 \) then \( D_2(x, 0, t) > 0 \). So the solution (33) is nonsingular by the \( y = 0 \). A singularity can appear if \( D_2(x, y, t) < 0 \). It is possible if such values \( x', y', t' \) of variables \( x, y, z \) exist that \( D_2(x', y', t') = 0 \), i.e.

\[
\rho \exp\left((\lambda^2 + \mu^2)t'\right) \cosh(\beta y') \cosh(\lambda x') = \sinh(\alpha y') \sin(\mu x'),
\]

(34)

when \( \rho \equiv \beta C_1(\alpha C_2)^{-1} \). It is obvious that the condition (34) is not right by the \( y' = 0 \). We shall change (34) on,

\[
\rho \exp\left((\lambda^2 + \mu^2)t'\right) |\theta_1(y')| = |\theta_2(x')|,
\]

(35)

with

\[
\theta_1(y) = \frac{\cosh(\beta y)}{\sinh(\alpha y)}, \quad \theta_2(x) = \frac{\sin(\mu x)}{\cosh(\lambda x)}.
\]

We are interested in selection of parameters such that (35) is not right because this choice guarantees the nonsingular behavior of (33). Let \( y_0 \) and \( x_0 \) are solutions of equations

\[
\tanh(\beta y_0) \tanh(\alpha y_0) = \frac{\alpha}{\beta}, \quad \tanh(\lambda x_0) \tan(\mu x_0) = \frac{\mu}{\lambda},
\]

and \( x_0 \) is the maximum point of \( \theta_2(x) \). Since \( \exp\left((\lambda^2 + \mu^2)t'\right) \geq 1 \) by the \( t' \geq 0 \) then for the such values of \( t \) we can to choose

\[
\rho > \frac{\theta_2(x_0)}{\theta_1(y_0)}.
\]

(36)

Thus the condition (36) guarantees the nonsingular behavior of (33) by the \( t \geq 0 \). It is possible to choose parameters such that (33) is singular (blow-up) solution in a defined region and by the \( t < 0 \).

The solution (33) has oscillation by the variable \( x \). We can obtain a blow-up solution without one. To construct it we choose \( \psi_2 \) as

\[
\psi_2 = C_2 \exp(\mu^2 t) \sinh(\mu x) + \sinh(\beta y).
\]

The solution (33) will be regular by the \( t \geq 0 \) (and has blow-up in a defined bounded region by the \( t < 0 \) and by the special choice of parameters) if \( \lambda > \mu \) and

\[
\rho > \frac{\tilde{\theta}_2(\tilde{x}_0)}{\tilde{\theta}_1(y_0)}, \quad \tilde{\theta}_2(x) = \frac{\sinh(\mu x)}{\cosh(\lambda x)},
\]

where \( \tilde{x}_0 > 0 \) is maximum point of the function \( \tilde{\theta}_2(x) \).

5. THE BURGERS EQUATION’S DRESSING PROCEDURE
In Section 2 we showed that the reduction achieved by imposing \( p = 0 \) maps the BLP equations (2) onto the differentiated Burgers equation (3). This fact allows us to use solutions of the Burgers equations in the capacity of the “supplier” of the BLP’s solutions. Namely, using an arbitrary solution of the Burgers equation which has parameters depending on \( y \) it is possible to construct exact solutions of (2) with the help of the LT (8) (we can’t apply the LT (7) by the \( p = 0 \)).

It is convenient to redefine variables by the following way

\[
x = \frac{\xi}{\sqrt{\nu}}, \quad y = \frac{\eta}{\nu\sqrt{\nu}}, \quad a = \frac{A(\xi, \eta, t)}{2\sqrt{\nu}}, \quad p = \frac{P(\xi, \eta, t)}{\nu\sqrt{\nu}}.
\]

Here \( \nu > 0 \) and \( \mu \) is arbitrary constant. As a result the BLP equations take the form

\[
\begin{align*}
A_t + AA_\xi - \nu A_{\xi\xi} &= -4P_\xi, \\
P_{t\eta} + (AP_\eta + \nu P_{\xi\eta})_\xi &= 0.
\end{align*}
\]

(37)

\( \nu \) is the parameter that may be called the coefficient of viscosity. If we shall assume that \( P = 0 \) and \( A \) is a solution of the Burgers equation that (8) allows us to construct solutions of the BLP equations (37),

\[
P[-1] = -\frac{\nu}{2} A, \quad A[-1] = A - 2\nu \partial_\xi \ln (A_\eta).
\]

(38)

This is the procedure that we call the the Burgers equation’s ”dressing procedure”. Let us consider a simple example for the shock wave of the Burgers equation [6],

\[
A = A_1 + \frac{A_2 - A_1}{1 + \exp \left\{ \frac{A_2 - A_1}{2\nu}(\xi - Ut) \right\}},
\]

where

\[
A_1 = A_1(\eta), \quad A_2 = A_2(\eta), \quad U = \frac{A_1 + A_2}{2}.
\]

Assuming that \( A_1 = 0 \) and taking (38) into account we get

\[
A[-1] = \frac{A_2 \left[ G^2 A_2(\xi - A_2t) - 2\nu(2G^2 + 3G + 1) \right]}{(1 + G) \left[ GA_2(\xi - A_2t) - 2\nu(1 + G) \right]},
\]

\[
P[-1] = -\frac{\nu A_2}{2(1 + G)};
\]

\[
G \equiv \exp \left\{ \frac{A_2(2\xi - A_2t)}{4\nu} \right\}.
\]

(39)

It is interesting to consider the behavior of (39) by the \( \nu \to 0 \). We suppose that \( A_2 > 0 \) for all values of the \( \eta \). Then for the \( 2\xi \neq A_2t \), we get

\[
P[-1] \to 0, \quad A[-1] \to A_2,
\]

and for the \( 2\xi = A_2t \)

\[
P[-1] \to 0, \quad A[-1] \to \frac{A_2}{2}.
\]
Using other well known solutions of the Burgers equation (see for example the remarkable monograph [6]) it is easy to construct a rich set of the exact solutions of the BLP equations via the Burgers equation’s ”dressing procedure”. Let $A$ be the dissipative solution of the Burgers equations, i.e. $A \to 0$ if $t \to \infty$. Using the (38) we get dissipative solution $P[-1]$ therefore we get a tending to zero (when $t \to \infty$) right term in the first equation (37) and $A[-1] \to A_B$, where $A_B$ is solution of the Burgers equation. We call such BLP’s solutions the BLP dissipative structures (see Section 1).

For example we consider the dissipative solution of the BLP equations,

$$A = -2\nu \partial_\xi \ln f, \quad f = \alpha + \sqrt{\frac{\beta}{t}} \exp \left( -\frac{\xi^2}{4\nu t} \right), \quad P = 0,$$

(40)

where $\alpha = \alpha(y)$ and $\beta = \beta(y)$ are some arbitrary functions (if $\alpha$ and $\beta$ are constants then (40) is the so called N-waves solution of the dissipative Burgers equation [6]). Substituting (40) into (38) we get

$$A[-1] = -\frac{2\nu}{\xi} + \frac{\alpha \xi}{t f}, \quad P[-1] = -\frac{\nu \xi (f - \alpha)}{2tf}.$$

It easy to see that if $t \to +\infty$ then

$$P[-1] \to 0, \quad A[-1] \to A_B = -\frac{2\nu}{\xi} + \frac{\xi}{t},$$

and $A_B$ is the solution of the Burgers equation.

6. CONCLUSION

In this paper we have studied in detail the Darboux and Laplace transformations for the BLP equations. These transformations represents a really fruitful mathematical tool in the search for exact solutions of integrable nonlinear evolution equations in two spatial dimensions. The physical interpretation of the BLP equations within the framework of theory of dissipative processes is not entirely clear to me because the BLP equation reduced to a differented Burgers equation and to a anti-Burgers equation too. Therefore only parth of BLP exact solutions has dissipative behavior. In this work we have showed how these solutions (BLP dissipative structures) associated with the dissipative Burgers equation may be obtained.

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236041, Theoretical Physics Department, Kaliningrad State University, Al.Nevsky St., 14, Kaliningrad, Russia

*E-mail address: yurov@ freemail.ru*