On The Diophantine Equation
\[ x^2 + 7^\alpha \cdot 11^\beta = y^n \]

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Abstract
In this paper, we give all the solutions of the Diophantine equation
\[ x^2 + 7^\alpha \cdot 11^\beta = y^n, \]
in nonnegative integers \( \alpha, \beta, x, y, n \geq 3 \) with \( x \) and \( y \) coprime, except for the case when \( \alpha \cdot x \) is odd and \( \beta \) is even.

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1 Introduction
The Diophantine equation
\[ x^2 + C = y^n, \quad n \geq 3 \] (1)
in positive integers \( x, y, n \) for given \( C \) has a rich history. In 1850, Lebesgue [27] proved that the above equation has no solutions when \( C = 1 \). The title equation is a special case of the Diophantine equation \( ay^2 + by + c = dx^n \), where \( a \neq 0, b, c \) and \( d 
eq 0 \) are integers with \( b^2 - 4ac \neq 0 \), which has at most finitely many integer solutions \( x, y, n \geq 3 \) (see [25]). In 1993, J.H.E. Cohn [19] solved the Diophantine equation (1) for several values of the parameter \( C \) in the range \( 1 \leq C \leq 100 \). The solution for the cases \( C = 74, 86 \) was completed by Mignotte and de Weger [36] which had not been covered by Cohn (indeed, Cohn solved these two equations of type (1) except for \( p = 5 \), in which case difficulties occur as the class numbers of the corresponding imaginary quadratic fields are divisible by 5). In [13], Bugeaud, Mignotte and Siksek improved modular methods to solve completely (1) when \( n \geq 3 \), for \( C \) in the range [1, 100]. So they covered the remaining cases.

Different types of the Diophantine equation (1) were studied also by various mathematicians. For effectively computable upper bounds for the exponent \( n \), we refer to [9] and [24]. However, these estimates are based on Baker’s theory of lower bounds for linear forms in logarithms of algebraic numbers, so they are quite impractical. In [39], Tengely gave a method to solve the equation

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Theorem 1

known results include the following theorem: There are three papers concerned with partial solutions for equation (2). The Diophantine equation $x^2 + C = 2y^n$ with $C$ a fixed positive integer and under the similar restrictions $n \geq 3$ and $\text{gcd}(x, y) = 1$ was studied. Recently, Luca, Tengely and Togbé studied the Diophantine equation $x^2 + C = 4y^n$ in nonnegative integers $x, y, n \geq 3$ with $x$ and $y$ coprime for various shapes of the positive integer $C$ in [33].

In recent years, a different form of the above equation has been considered, namely where $C$ is a power of a fixed prime. In [18], the equation $x^2 + 2^k = y^n$ was studied under some conditions by Arif and Muriefah. A conjecture of Cohn (see [18]) was verified saying that $x^2 + 2^k = y^n$ has no solutions with $x$ odd and even $k > 2$ by Le [20]. In [2], Abu Muriefah and Arif, gave all the solutions of $x^2 + 3^k = y^n$ with $k$ odd and, Luca [30], gave all the solutions with $k$ even. Again the same equation was independently solved in 2008 by Liqun in [28] for both odd and even $m$. All solutions of $x^2 + 5^k = y^n$ are given with $k$ odd in [9] and with $k$ even in [5]. Liqun solves the same equation again in 2009, in [29]. Recently, Bérczes and Pink [10], gave all the solutions of the Diophantine equation (11) when $C = p^k$ and $k$ is even, where $p$ is any prime in the interval $[2, 100]$.

The last variant of the Diophantine equation (11) in which $C$ is a product of at least two prime powers were studied in some recent papers. In 2002, Luca gave complete solution of $x^2 + 2^a \cdot 3^b = y^n$ in [33]. Since then, in 2006, all the solutions of the Diophantine equation $x^2 + 2^a \cdot 5^b = y^n$ were found by Luca and Togbé in [33]. In 2008, the equations $x^2 + 5^a \cdot 13^b = y^n$ and $x^2 + 2^a 5^b \cdot 13^c = y^n$ were solved in [6] and [23]. Recently, in [14] and [15], complete solutions of the equations $x^2 + 2^a 11^b = y^n$ and $x^2 + 2^a 3^b \cdot 11^c = y^n$ were found. In [10], the complete solution $(n, a, b, x, y)$ of the equation $x^2 + 5^a \cdot 11^b = y^n$ when $\text{gcd}(x, y) = 1$, except for the case when $xab$ is odd is given. In [38], Pink gave all the non-exceptional solutions (in the terminology of that paper) with $C = 2^a 3^b 5^c 7^d$. Note that finding all the exceptional solutions of this equation seems to be a very difficult task. A more exhaustive survey on this type of problems is [7].

Here, we study the Diophantine equation

$$x^2 + 7^n \cdot 11^3 = y^n, \quad \text{gcd}(x, y) = 1 \quad \text{and} \quad n \geq 3. \quad (2)$$

There are three papers concerned with partial solutions for equation (2). The known results include the following theorem:

**Theorem 1** (i) If $\alpha$ is even and $\beta = 0$, then the only integer solutions of the Diophantine equation

$$x^2 + 7^{2k} = y^n$$

are

$$n = 3 \quad (x, y, k) = (524 \cdot 7^{3\lambda}, 65 \cdot 7^{2\lambda}, 1 + 3\lambda),$$

$$n = 4 \quad (x, y, k) = (24 \cdot 7^{2\lambda}, 5 \cdot 7^{2\lambda}, 1 + 2\lambda) \text{ where } \lambda \geq 0 \text{ is any integer}.$$

(ii) If $\alpha = 1$ and $\beta = 0$, then the only integer solutions $(x, y, n)$ to the generalized Ramanujan–Nagell equation

$$x^2 + 7 = y^n$$

are

$$n = 3 \quad (x, y, k) = (524 \cdot 7^{3\lambda}, 1 + 3\lambda),$$

$$n = 4 \quad (x, y, k) = (24 \cdot 7^{2\lambda}, 5 \cdot 7^{2\lambda}, 1 + 2\lambda) \text{ where } \lambda \geq 0 \text{ is any integer}.$$
are 

\[(1, 2, 3), (181, 32, 3), (3, 2, 4), (5, 2, 5), (181, 8, 5), (11, 2, 7), (181, 2, 15).\]

(iii) If \(\alpha = 0\), then the only integer solutions of the Diophantine equation

\[x^2 + 11^\beta = y^n\]

are 

\[(x, y, \beta, n) = (2, 5, 2, 3), (4, 3, 1, 3), (58, 15, 1, 3), (9324, 443, 3, 3)\]

**Proof.** See [32], [13] and [14].

Our main result is the following.

**Theorem 2** The only solutions of the Diophantine equation \([2]\) are

\[n = 3: \quad (x, y, \alpha, \beta) \in \{(57, 16, 1, 2), (797, 86, 1, 2), (4229, 284, 3, 4), (3093, 478, 7, 2), (4, 3, 0, 1), (58, 15, 0, 1), (2, 5, 0, 2), (9324, 443, 0, 3), (1, 2, 1, 0), (181, 32, 1, 0), (524, 65, 2, 0), (13, 8, 3, 0)\};\]

\[n = 4: \quad (x, y, \alpha, \beta) \in \{(2, 3, 1, 1), (797, 86, 1, 2), (8343, 92, 5, 2), (3, 2, 1, 0), (24, 5, 2, 0)\};\]

\[n = 6: \quad (x, y, \alpha, \beta) = (57, 4, 1, 2);\]

\[n = 9: \quad (x, y, \alpha, \beta) = (13, 2, 3, 0);\]

\[n = 12: \quad (x, y, \alpha, \beta) = (57, 2, 1, 2);\]

When \(n \geq 5, n \neq 6, 9, 12\), the equation \([2]\) has no solutions \((x, y, \alpha, \beta)\) with at least one of \(\alpha, x\) even or with \(\beta\) is odd.

**Remark 3** For \(n \geq 5, n \neq 6, 9, 12\) the above theorem leaves out the solutions \((\alpha, \beta, x, y)\) when \(\alpha x\) is odd and \(\beta\) is even. These are exactly the exceptional solutions of the equation \([2]\) in the terminology of \([21]\); see also the remark \([8]\) at the end of this paper.

One can deduce from the Theorem \([1]\) and Theorem \([2]\) the following corollary.

**Corollary 4** The only integer solutions of the Diophantine equation \([2]\) are

\[n = 3: \quad (x, y, \alpha, \beta) \in \{(57, 16, 1, 2), (797, 86, 1, 2), (4229, 284, 3, 4), (3093, 478, 7, 2), (4, 3, 0, 1), (58, 15, 0, 1), (2, 5, 0, 2), (9324, 443, 0, 3), (1, 2, 1, 0), (181, 32, 1, 0), (524, 65, 2, 0), (13, 8, 3, 0)\};\]

\[n = 4: \quad (x, y, \alpha, \beta) \in \{(2, 3, 1, 1), (797, 86, 1, 2), (8343, 92, 5, 2), (3, 2, 1, 0), (24, 5, 2, 0)\};\]

\[n = 5: \quad (x, y, \alpha, \beta) = (57, 2, 1, 0), (181, 8, 1, 0);\]

\[n = 6: \quad (x, y, \alpha, \beta) = (57, 4, 1, 2);\]

\[n = 7: \quad (x, y, \alpha, \beta) = (11, 2, 1, 0);\]

\[n = 9: \quad (x, y, \alpha, \beta) = (13, 2, 3, 0);\]

\[n = 12: \quad (x, y, \alpha, \beta) = (57, 2, 1, 2);\]

\[n = 15: \quad (x, y, \alpha, \beta) = (181, 2, 1, 0).\]
2 The Proof of Theorem 2

We distinguish the cases $n = 3, 6, 9, 12, ~ n = 4$ and $n > 4$, devoting a subsection to the treatment of each case. We first treat the cases $n = 3$ and $n = 4$. This is achieved in Section 2.1 and Section 2.2, respectively. For the case $n = 3$, we transform equation (2) into several elliptic equations in Weierstrass’s form which we need to determine all their $\{7, 11\}$-integral points. In Section 2.2, we use the same method as in Section 2.1 to determine the solutions of (2) for $n = 4$. In the last section, we assume that $n > 4$ is prime and study the equation (2) under this assumption. Here we use the method of primitive divisors for Lucas sequences. All the computations are done with MAGMA [12] and with Cremona’s program mwrank.

2.1 The Cases $n = 3, 6, 9$ and $12$

Lemma 5 When $n = 3$, then only solutions to equation (2) are

$$\begin{align*}
(57, 16, 1, 2), (797, 86, 1, 2), (4229, 284, 3, 4), (3093, 478, 7, 2), \\
(4, 3, 0, 1), (58, 15, 0, 1), (2, 5, 0, 2), (9324, 443, 0, 3), \\
(1, 2, 1, 0), (181, 32, 1, 0), (524, 65, 2, 0), (13, 8, 3, 0);
\end{align*}$$

when $n = 6$, then only solution to equation (2) is $(57, 4, 1, 2)$; when $n = 9$, then only solution to equation (2) is $(13, 2, 3, 0)$; when $n = 12$, then only solution to equation (2) is $(57, 2, 1, 2)$.

Proof. Suppose $n = 3$. Writing $\alpha = 6k + \alpha_1$, $\beta = 6l + \beta_1$ in (2) with $\alpha_1, \beta_1 \in \{0, 1, 2, 3, 4, 5\}$, we get that

$$\left( \frac{x}{7^{3k}11^3}, \frac{y}{7^{2k}11^2} \right)$$

is an $S$-Integral point $(X, Y)$ on the elliptic curve

$$X^2 = Y^3 - 7^{\alpha_1} \cdot 11^{\beta_1},$$

where $S = \{7, 11\}$ with the numerator of $Y$ being coprime to 77, in view of the restriction $\gcd(x, y) = 1$. Now we need to determine all the $\{7, 11\}$-integral points on the above 36 elliptic curves. At this stage we note that in [22] Pethő, Zimmer, Gebel and Herrmann developed a practical method for computing all $S$-Integral points on Weierstrass elliptic curve and their method has been implemented in MAGMA [12] as a routine under the name SIntegralPoints. The subroutine SIntegralPoints of MAGMA worked without problems for all $(\alpha_1, \beta_1)$ except for $(\alpha_1, \beta_1) = (5, 5)$. MAGMA determined the appropriate Mordell-Weil groups except this case and we deal with this exceptional case separately. By computations done for equation (4) when $n = 3$, we obtain the following solutions for the $\{7, 11\}$-integral points on the
curves:

\[(1, 0, 0, 0), (3, 4, 0, 1), (15, 58, 0, 1), (5, 2, 0, 2), (11, 0, 0, 3), (443, 9324, 0, 3),
(2, 1, 1, 0), (32, 181, 1, 0), (478/49, 3093/343, 2), (11, 22, 1, 2), (16, 57, 1, 2),
(1899062/117649, 2338713355/40353607, 1, 2), (22, 99, 1, 2), (86, 797, 1, 2),
(88, 825, 1, 2), (638, 16115, 1, 2), (657547, 533200074, 1, 2), (242, 3751, 1, 4),
(65, 524, 2, 0), (7, 0, 3, 0), (8, 13, 3, 0), (14, 49, 3, 0), (28, 147, 3, 0),
(154, 1911, 3, 0), (77, 0, 3, 3), (242, 3025, 3, 4), (284, 4229, 3, 4),
(1435907/49, 1720637666/343, 3, 4).

We use the above points on the elliptic curves to find the corresponding solutions for equation \(\text{(4)}\). Identifying the coprime positive integers \(x\) and \(y\) from the above list, one obtains the solutions listed in \(\text{(1)}\) (note that not all of them lead to coprime values for \(x\) and \(y\)).

We give the details in case \((\alpha_1, \beta_1) = (5, 5)\) of equation \(\text{(1)}\). Observe that if \(Y\) is even, then \(X\) is odd and \(X^2 + 7^511^5 = 0 \pmod{8}\), and hence \(X^2 \equiv 3 \pmod{8}\), which is a contradiction. Therefore \(Y\) is always odd. We consider solutions such that \(X\) and \(Y\) are coprime.

Write \(K = \mathbb{Q}(i\sqrt{77})\). In this field, the primes 2, 7, 11 (all primes dividing \(d_K = 4d\)) ramify so there are prime ideals \(P_2, P_7, P_{11}\) such that \(2O_K = P_2^2, 7O_K = P_7^2, 11O_K = P_{11}^2\) respectively. Now, we show that the ideals \((X + 7^211^2\sqrt{77}i)O_K\) and \((X - 7^211^2\sqrt{77}i)O_K\) are coprime in the ring of integers \(O_K\). To show this, let us assume that the ideals \((X + 7^211^2\sqrt{77}i)O_K\) and \((X - 7^211^2\sqrt{77}i)O_K\) are not coprime. So, these ideals have a gcd that divides \(2.7^2.11^2\sqrt{77}i\). Hence there is an ideal \(P_2^a P_7^b P_{11}^c\) with \(a \leq 2\), and \(b, c \leq 5\). If \(b > 0\) then \(7 \mid X\). Hence \(7 \mid Y\), hence \(7^3 \mid X^2\), hence \(7^2 \mid X\), hence \(7^4 \mid Y^3\), hence \(7^2 \mid Y\), hence \(7^5 \mid X^2\), hence \(7^3 \mid X\). So, we have a contradiction as \(7^6 \mid X^2 - Y^3\). Thus \(b = 0\). Similarly we can prove that \(c = 0\).

Now let \((X + 7^211^2\sqrt{77}i)O_K = P_2^a P_7^b P_{11}^c\) (for some ideal \(\varphi\)) and \((X - 7^211^2\sqrt{77}i)O_K = P_2^a P_7^b P_{11}^c\) (for its conjugate ideal). If we take norms, then we get that \(\varphi^3 = 2^a N_K(\varphi)\varphi^3\), where \(N_K(\varphi)\) is odd. It follows that \(a = 0\) (as it could be at most 2). So, we showed that the ideals \((X + 7^211^2\sqrt{77}i)O_K\) and \((X - 7^211^2\sqrt{77}i)O_K\) are coprime. Equation \(\text{(4)}\) now implies that

\[
(X + 7^211^2\sqrt{77}i)O_K = \varphi^3 \quad \text{and} \quad (X - 7^211^2\sqrt{77}i)O_K = \varphi'^3
\]

for the ideals \(\varphi\) and \(\varphi'\). Let \(h(K)\) be the class number of the field \(K\), then \(\varphi^{h(K)}\) is principal for any ideal \(\delta\). Note that, \(h(K) = 8\) and so \((3, h(K)) = 1\). Thus since \(\varphi^3\) and \(\varphi'^3\) are principal, \(\varphi\) and \(\varphi'\) are also principal. Moreover, since the units of \(\mathbb{Q}(i\sqrt{77})\) are 1 and \(-1\), which are both cubes, we conclude that

\[
(X + 7^211^2\sqrt{77}i) = (u + \sqrt{77}iv)^3 \quad \text{(5)}
\]

\[
(X - 7^211^2\sqrt{77}i) = (u - \sqrt{77}iv)^3 \quad \text{(6)}
\]

for some integers \(u\) and \(v\). After subtracting the conjugate equation we obtain

\[
7^2 \cdot 11^2 = v(3u^2v - 77v^2). \quad \text{(7)}
\]
Since $u$ and $v$ are coprime, we have the following possibilities in equation (7)

\[ v = \pm 1; \ v = \pm 7^2; \ v = \pm 11^2; \ v = \pm 7^211^2 \]

All cases lead to the conclusion that no solution is obtained.

For $n = 6$, equation

\[ x^2 + 7^\alpha \cdot 11^\beta = y^6 \]

becomes equation

\[ x^2 + 7^\alpha \cdot 11^\beta = (y^2)^3. \]

Again, here we look in the list of solutions of equation (8) and observe that the only solution whose $y$ is a perfect square is $(57, 16, 1, 2)$. Therefore the only solution to equation (2) is $(57, 16, 1, 2)$.

For $n = 9$, equation

\[ x^2 + 7^\alpha \cdot 11^\beta = y^9 \]

becomes equation

\[ x^2 + 7^\alpha \cdot 11^\beta = (y^3)^3. \]

Again here, we look in the list of solutions of equation (8) and observe that only solution whose $y$ is a perfect cube is $(13, 8, 3, 0)$. Therefore the only solution to equation (2) is $(13, 8, 3, 0)$. This completes the proof of lemma.

If $(x, y, \alpha, \beta, n)$ is a solution of the Diophantine equation (2) and $d$ is any proper divisor of $n$, then $(x, y^d, \alpha, \beta, n/d)$ is also a solution of the same equation. Since $n > 3$ and we have already dealt with case $n = 3$, it follows that it suffices to look at the solutions $n$ for which $p \mid n$ for some odd prime $p$. In this case, we may certainly replace $n$ by $p$, and thus assume for the rest of the paper that $n \in \{4, p\}$.

2.2 The Case $n = 4$

**Lemma 6** The only solutions with $n = 4$ of the Diophantine equation (2) are given by

\[ (x, y, \alpha, \beta) = (2, 3, 1, 1), (57, 8, 1, 2), (8343, 92, 5, 2), (3, 2, 1, 0), (24, 5, 2, 0) \]

**Proof.** Suppose that $n = 4$. Rewrite equation (2) as

\[ 7^\alpha \cdot 11^\beta = (y^2 + x)(y^2 - x). \] (8)

From the equation (8), we have that

\[ y^2 + x = 7^{a_1}11^{b_1} \]
\[ y^2 - x = 7^{a_2}11^{b_2} \]

where $a_1, a_2, b_1, b_2 \geq 0$. Then we get that

\[ 2y^2 = 7^{a_1}11^{b_1} + 7^{a_2}11^{b_2} \]
from the sum of two equations. We multiply above equation by 2 and we can write the equation

\[ Z^2 = 2(7^{a_1} \cdot 11^{b_1} + 7^{a_2} \cdot 11^{b_2}) \] (9)

as

\[ 2U + 2V = Z^2 \] (10)

where \( Z = 2y, \ U = 7^{a_1} \cdot 11^{b_1} \) and \( V = 7^{a_2} \cdot 11^{b_2} \).

Let \( p_1, p_2, ..., p_s \) \((s \geq 1)\) be fixed distinct primes. The set of \( S \)-Units is defined as

\[ S = \{ \pm p_1^{x_1} p_2^{x_2} ... p_s^{x_s} \mid x_i \in \mathbb{Z}, \text{ for } i = 1,...k \} \]. Let \( a, b \in \mathbb{Q} - \{0\} \) be fixed.

In [20], B.M.M. de Weger dealt with the solutions of the Diophantine equation

\[ ax + by = z^2, \]

\[ a, b \in S, \ z \in \mathbb{Q}. \]

He showed that this equation has essentially only finitely many solutions. Moreover, he indicated how to find all the solutions of this equation for any given set of parameters \( a, b, p_1, ..., p_s \). The tools are the theory of p-adic linear forms in logarithms, and a computational p-adic diophantine approximation method. He actually performed all the necessary computations for solving (10) completely for \( p_1, ..., p_s = 2, 3, 5, 7 \) and \( a = b = 1 \), and reported on this elsewhere (see [21], Chapter 7). Then we can find all the solutions of the Diophantine equation (9). But this requires a lot of additional manual effort. To solve the equation \( x^2 + 7^\alpha \cdot 11^\beta = y^n \) instead of this method, we prefer using MAGMA (see [12]).

Writing in (2) \( \alpha = 4k + \alpha_1, \ \beta = 4l + \beta_1 \) with \( \alpha_1, \beta_1 \in \{0, 1, 2, 3\} \) we get that

\[ \left( \frac{x}{7^{2k} \cdot 11^l}, \frac{y}{7^{2k} \cdot 11^l} \right) \]

is an \( S \)-Integral point \((X, Y)\) on the hyperelliptic curve

\[ X^2 = Y^4 - 7^{\alpha_1} \cdot 11^{\beta_1}, \] (11)

where \( S = \{7, 11\} \) with the numerator of \( Y \) being prime to 77, in view of the restriction \( \gcd(x, y) = 1 \). We use the subroutine \texttt{SIntegralLjunggrenPoints} of MAGMA to determine the \( \{7, 11\} \)-integral points on the above hyperelliptic curves and we only find the following solutions

\[ (X, Y, \alpha_1, \beta_1) = \{(1, 0, 0, 0), (2, 3, 1, 0), (3, 2, 1, 1), (8, 57, 1, 2), (92/7, 8343/49, 1, 2), (5, 24, 2, 0)\} \]

With the conditions on \( x \) and \( y \) and the definition of \( X, Y \), one can obtain the solutions listed in the statement of the lemma.

\section*{2.3 The Case \( n > 4 \) and Prime}

\textbf{Lemma 7} The Diophantine equation (2) has no solutions with \( n > 4 \) prime except possibly for \( \alpha \) and \( x \) are odd and \( \beta \) even.

\textbf{Proof.} Since in section 2 we have finished the study of equation \( x^2 + 7^\alpha \cdot 11^\beta = y^n \) with \( n = 3 \), we can assume that \( n \) is a prime \( > 4 \). One can write the Diophantine equation (2) as \( x^2 + dz^2 = y^n \), where

\[ d \in \{1, 7, 11, 77\}, \ z = 7^{\alpha_1} \cdot 11^{\beta_1} \] (12)
the relation of $\alpha_1$ and $\beta_1$ with $\alpha$ and $\beta$, respectively, is clear. If $x$ is odd, then
by $z$ also being odd we have that $y$ is even, so $y^n \equiv 0 \pmod{8}$. As $x^2 = z^2 \equiv 1 \pmod{8}$ we have $1 + d \equiv 0 \pmod{8}$, so $d = 7$, implying $\alpha \equiv 1 \pmod{2}$ and $\beta \equiv 0 \pmod{2}$. This case is excluded in the lemma. Hence we have that $x$ is even, and $y$ is odd. We study in the field $K = \mathbb{Q}(i\sqrt{d})$. As gcd$(x, z) = 1$
standard argument tells us now that in $K$ we have
\begin{equation}
(x + i\sqrt{d}z)(x - i\sqrt{d}z) = y^n,
\end{equation}
where the ideals generated by $x + iz\sqrt{d}$ and $x - iz\sqrt{d}$ are coprime in $K$. Hence,
we obtain the ideal equation
\begin{equation}
\langle x + i\sqrt{d}z \rangle = \theta^n.
\end{equation}
Then, since the ideal class number of $K$ is 1 or 8, and $n$ is odd, we conclude that
the ideal $\theta$ is principal. The cardinality of the group of units of $\mathcal{O}_K$ is
2 or 4, all coprime to $n$. Furthermore, $\{1, i\sqrt{d}\}$ is always an integral base for
$\mathcal{O}_K$ except for when $d = 7$, and $d = 11$, in which cases an integral basis for $\mathcal{O}_K$ is
$\{1, (1 + i\sqrt{d})/2\}$. Thus, we may assume that
\begin{equation}
x + i\sqrt{d}z = \varphi^n, \varphi = \frac{u + i\sqrt{dv}}{2}
\end{equation}
the relation holds with some algebraic integer $\varphi \in \mathcal{O}_K$. The algebraic integers
in this number field are of the form $\varphi = \frac{u + i\sqrt{dv}}{2}$, where $u, v \in \mathbb{Z}$, with $u, v$ both
even, if $d = 1, 77$ and $u, v$ both odd if $d = 7, 11$. Note that
\begin{align*}
\varphi - \overline{\varphi} = vi\sqrt{d},& \quad \varphi + \overline{\varphi} = i\sqrt{dv}, \quad \varphi\overline{\varphi} = \frac{u^2 + dv^2}{4}.
\end{align*}
We thus obtain
\begin{equation}
\frac{2 \cdot 7^{\alpha_1} \cdot 11^{\beta_1}}{v} = \frac{2z}{v} = \frac{\varphi^n - \overline{\varphi}^n}{\varphi - \overline{\varphi}} \in \mathbb{Z}.
\end{equation}
Let $(L_m)_{m \geq 0}$ be the sequence with general term $L_m = (\varphi^m - \overline{\varphi^m})/(\varphi - \overline{\varphi})$ for
all $m \geq 0$. This is called a Lucas sequence. Note that
\begin{equation}
L_0 = 0, L_1 = 1 \text{ and } L_m = uL_{m-1} - \frac{u^2 + dv^2}{4}L_{m-2}, \quad m \geq 2.
\end{equation}
Following the nowadays standard strategy based on the important paper [11],
we distinguish two cases according as $L_n$ has or has not primitive divisors.
Suppose first that $L_n$ has a primitive divisor, say $q$. By definition, this means
that the prime $q$ divides $L_n$ and $q$ does not divide $(\mu - \overline{\mu})^2 L_1...L_{n-1}$, hence
\begin{equation}
q \nmid (\varphi - \overline{\varphi})^2 L_1...L_4 = (dv^2)u, \quad \frac{3u^2 - dv^2}{4}, \quad \frac{u^2 - dv^2}{2}.
\end{equation}
If $q = 2$, then (18) implies that $uv$ is odd, hence $d = 11$ or 77. If $d = 11$, then
third factor in the right hand-most side of (18) is even, a contradiction. If
If \( q = 7 \), then (18) implies that \( d = 1, 11 \) and 7 does not divide \( uv(3u^2 - dv^2)(u^2 - dv^2) \). It follows easily then that \( v^2 \equiv -da^2 \pmod{7} \), so that, by (17), \( L_m \equiv uL_{m-1} \pmod{8} \) for every \( m \geq 2 \). Therefore, \( q \mid L_n \), so that 7 cannot be a prime divisor of \( L_n \).

If \( q = 11 \), then by (18), \( d = 1 \) or 7. If \( d = 1 \) then we write \( u = 2v_1, v = 2v_1 \) with \( u_1, v_1 \in \mathbb{Z} \), so that \( \varphi = u_1 + i\sqrt{dv}v_1 \) and (18) becomes \( q \nmid u_1v_1(3u_1^2 - dv_1^2)(u_1^2 - dv_1^2) \). Moreover, \( L_m = 2u_1L_{m-1} - (u_1^2 + dv_1^2)L_{m-2} \) for \( m \geq 2 \). Note that \( \varphi \varphi = u_1^2 + dv_1^2 \neq 0 \pmod{8} \); therefore, by corollary 2.2 of (11), there exists a positive integer \( m_{11} \) such that \( 11 \mid L_{m_{11}} \) and \( m_{11} \mid m \) for every \( m \) such that \( 11 \mid L_m \). It follows then that \( 11 \mid \gcd(L_n, L_{m_{11}}) = L_{\gcd(n, m_{11})} \). Because of the minimality property of \( m_{11} \), we conclude that \( \gcd(n, m_{11}) \), hence, since \( n \) is a prime, \( m_{11} = n \). On the other hand, the Legendre symbol \( \left( \frac{\varphi - \varphi^2}{11} \right) = -1 \), hence by Theorem XII of (17) (or by theorem 2.2.4 (iv) of (34)), \( 11 \mid L_{12} \). Therefore \( m_{11} \mid 12 \), i.e. \( n \mid 12 \), a contradiction, since \( n \) is a prime \( \geq 5 \). If \( d = 7 \), then (18) implies \( 11 \nmid u_1v_1(3u_1^2 - dv_1^2)(u_1^2 - dv_1^2) \). Moreover, \( L_m = 2u_1L_{m-1} - (u_1^2 + dv_1^2)L_{m-2} \) for \( m \geq 2 \). Note that \( \varphi \varphi = u_1^2 + dv_1^2 \neq 0 \pmod{8} \); therefore, by corollary 2.2 of (11), there exists a positive integer \( m_{11} \) such that \( 11 \mid L_{m_{11}} \) and \( m_{11} \mid m \) for every \( m \) such that \( 11 \mid L_m \). It follows then that \( 11 \mid \gcd(L_n, L_{m_{11}}) = L_{\gcd(n, m_{11})} \). Because of the minimality property of \( m_{11} \), we conclude that \( \gcd(n, m_{11}) \), hence, since \( n \) is a prime, \( m_{11} = n \). On the other hand, the Legendre symbol \( \left( \frac{\varphi - \varphi^2}{11} \right) = 1 \), hence by Theorem XII of (17) (or by theorem 2.2.4 (iii) of (34)), \( 11 \mid L_{10} \). Therefore \( m_{11} \mid 10 \), i.e. \( n \mid 10 \). Since \( n \geq 5 \) is a prime, we get that \( n = 5 \).

We conclude that 11 is a primitive divisor for \( d = 7 \).

In particular, \( u \) and \( v \) are integers. Since \( 11 \) is coprime to \(-4dv^2 = -28v^2\), we get that \( v = \pm 7^{a_1} \). Since \( y = u^2 + 7v^2 \), we get that \( u \) is even.

In the case \( v = \pm 7^{a_1} \), equation (16) becomes

\[
\pm 11^{\beta_1} = 5u^4 - 70u^2v^2 + 49v^4.
\]

Since \( u \) is even, it follows that the right hand side of the last equation above is congruent to 1 (mod 8). So \( \pm 11^{\beta_1} = 1 \) (mod 8), showing that the sign on the left hand side is positive and \( \beta_1 \) is odd, or the sign on the left hand side is negative and \( \beta_1 \) is even.

Assume first that \( \beta_1 = 2\beta_0 + 1 \) be odd. We get

\[
11V^2 = 5U^4 - 70U^2 + 49,
\]

where \((U, V) = (u/v, 11^{\beta_0}/v^2)\) is a \{7\}-integral point on the above elliptic curve. We get that the only such points on the above curve are \((U, V) = (\pm 7, \pm 28)\). This does not lead to solutions of our original equation.

Assume now that \( \beta_1 = 2\beta_0 \) is even and we get that

\[
V^2 = 5U^4 - 70U^2 + 49,
\]
where \((U, V) = (u/v, 11^{\beta_0}/v^2)\) is a \(7\)-integral point on the above elliptic curve. With MAGMA, we get that the only such point on the above curve are \((U, V) = (0, 7)\). This does not lead to solutions of our original equation.

We now recall that a particular instance of the Primitive Divisor Theorem for Lucas sequences implies that, if \(n \geq 5\) is prime, then \(L_n\) always has a prime factor except for finitely many exceptional triples \((\varphi, \overline{\varphi}, n)\), and all of them appear in the Table 1 in [11] (see also [1]). These exceptional Lucas numbers are called defective.

Let us assume that we are dealing with a number \(L_n\) without primitive divisors. Then a quick look at Table 1 in [11] reveals that this is impossible. Indeed, all exceptional triples have \(n = 5, 7\) or \(13\). The defective Lucas numbers whose roots are in \(K = \mathbb{Q}(i\sqrt{d})\) with \(d = 7\) and \(n = 5, 7\) or \(13\) appearing in the list [12] is \((\varphi, \overline{\varphi}) = ((1 + i\sqrt{7})/2, (1 - i\sqrt{7})/2)\) for which \(L_7 = 7, L_{13} = -1\). Furthermore, with such a value for \(\varphi\) we get that \(y = |\varphi|^2 = 2\). However, this is not convenient since for us \(x\) and \(y\) are coprime so \(y\) cannot be even. For \(n = 5\) and \(d = 11\), we get \(L_5 = 1\) and \(y = 3\) with \((\varphi, \overline{\varphi}) = ((1 + i\sqrt{11})/2, (1 - i\sqrt{11})/2)\). Therefore the equation is \(x^2 + C = 3^5\), where \(C = 7^{n} \cdot 11^{\beta_3}\), with \(a\) even and \(b\) odd. Since \(11^3 > 3^5\), we have \(b = 1\), and next that \(a = 0\). But it doesn’t yield an integer value for \(x\). The proof is completed. ■

Remark 8 We mention here why the method applied for the proof of Lemma 7 does not apply when \(a\) and \(x\) are odd, \(\beta\) is even. In this case \(d = 7\), the class number of \(\mathbb{Q}(\sqrt{7i})\) is 1. With \(\omega = \frac{1 + \sqrt{7i}}{2}\) a prime dividing \(2\), and \(\omega'\) its conjugate, let us now write \((x + z\sqrt{7i}) = \omega^a_0 \varphi \xi\), where \(\xi\) is an integer in \(\mathbb{Q}(\sqrt{7i})\) of odd norm, not divisible by 7 and \(\xi'\) its conjugate. As both \(x\) and \(z\) are odd and they are coprime, we may take \(c = 1, b \geq 1\). Taking norms we get \(y^n = 2^{b+1} \xi'\), and it easily follows that \(\xi = c^n\) and \(b + 1 = k.n\). Now we take \(\varphi = 2^{k-1}c\), \(\overline{\varphi} = 2^{\omega n - 2}\), and then we have \(x + z\sqrt{7i} = \varphi \overline{\varphi}^{n}\). A way to look at the rest of argument why this case is essentially different from the primitive divisors in Lucas sequences thing: From \(x + z\sqrt{7i} = \varphi \overline{\varphi}^{n}\) and its conjugate it follows that

\[
z = \frac{\varphi \overline{\varphi}^{n} - \overline{\varphi} \varphi^{n}}{2\sqrt{7i}}
\]

If \(\varphi\) is in \(\mathbb{Q}\) then the right hand side is the \(n\)-th term of a Lucas sequence. As \(z\) has a very nice prime factorization \(7^p 11^q\) then theory of primitive divisors will work. But in our case \(\varphi\) is not in \(\mathbb{Q}\). Hence the right side, while it is the \(n\)-th term of a recurrence sequence, this is not a Lucas sequence, and does not have the nice divisibility properties of Lucas sequences. That’s why the method of [11] fails in our case.

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