Constancy of curvature and conformal-projective flatness of statistical manifolds

Chol Rim Min, Won Hak Ri, Kum Hyok Kwak
Faculty of Mathematics, Kim Il Sung University, Democratic Peoples’ Republic of Korea
June, 2016

Abstract

An identity of conformal-projective curvature tensor of a statistical manifold \((M, g, \nabla)\) is studied in this paper. The relation between the constancy of curvature and conformal-projective flatness of statistical manifolds is also discussed.

Keywords: statistical manifold of constant curvature, conformal-projective flatness of statistical manifolds, conformal-projective curvature tensor

1 Introduction

Conformal-projective equivalence of statistical manifolds can be considered as a natural generalization of conformal equivalence of Riemannian metrics, which was introduced by Matsuzoe [4].

Conformal equivalence of Riemannian metrics and projective equivalence of affine connections are combined or generalized to lead up to conformal-projective equivalence or \(\alpha\)-conformal equivalence for statistical manifolds (See \([2, 4, 5]\)). Conformal-projective curvature tensor in a statistical manifold plays an important role as Weyl conformal curvature tensor does in Riemannian geometry.

Let \(M\) be an \(n\)-dimensional manifold, \(\nabla\) a torsion-free affine connection on \(M\), and \(g\) a Riemannian metric on \(M\).

We denote by \(\Gamma(E)\) the set of smooth sections of a vector bundle \(E \to M\). So \(\Gamma(TM)\) means the set of smooth vector field on \(M\) and \(\Gamma(TM^{(r,s)})\) means the set of tensor fields of type \((r,s)\) on \(M\).

The curvature tensor of \(\nabla\) is denoted by \(R \in \Gamma(TM^{(1,3)})\).

Two statistical manifold \((M, g, \nabla)\) and \((\bar{M}, \bar{g}, \bar{\nabla})\) are said to be conformally-projectively equivalent (or generalized conformal equivalent) if there exist two functions \(\varphi, \psi \in C^\infty(M)\) satisfying that

\[
\bar{g}(X,Y) = e^{\varphi+\psi}g(X,Y)
\]

\[
\bar{\nabla}_X Y = \nabla_X Y + d\varphi(X)Y + d\varphi(Y)X - g(X,Y)\text{grad}_g\psi
\]

for all \(X,Y \in \Gamma(TM)\) (See [4]).

Conformal-projective curvature tensor \(W\) is defined by

\[
W(X,Y)Z = R(X,Y)Z + \frac{1}{n(n-2)}\{Y[(n-1)Ric(X,Z) + \overset{*}{Ric}(X,Z)] - X[(n-1)Ric(Y,Z) + \overset{*}{Ric}(Y,Z)] + (n-1)\overset{*}{Ric}(Y) + Ric^2(Y)]g(X,Z) - [(n-1)\overset{*}{Ric}(X) + Ric^2(X)]g(Y,Z) + \frac{\sigma}{(n-1)(n-2)}[Xg(Y,Z) - Yg(X,Z)]
\]

(1.1)

(1.1)
and an \( n(\geq 4) \)-dimensional statistical manifold \((M, g, \nabla)\) is conformally-projectively flat if and only if the conformal-projective curvature tensor vanishes everywhere on \( M \), where \( R, \text{Ric} \) and \( \sigma \) are curvature tensor field, Ricci tensor field and scalar curvature of \((M, g, \nabla)\), respectively, and the Ricci operator \( \text{Ric}^\sharp \) of \((M, g, \nabla)\) is the \((1, 1)\)-tensor field determined by

\[
g(\text{Ric}^\sharp(X), Y) = \text{Ric}(X, Y)
\]

and the corresponding quantities for \( \tilde{\nabla} \) which is a dual connection of \( \nabla \) are denoted with * (See [3]).

On the other hand, for any \( \alpha \in \mathbb{R} \), two statistical manifolds \((M, g, \nabla)\) and \((M, \tilde{g}, \tilde{\nabla})\) are said to be \( \alpha \)-conformally equivalent if there exists a function \( \varphi \in C^\infty(M) \) satisfying that

\[
\begin{align*}
\tilde{g}(X, Y) &= e^{\alpha \varphi} g(X, Y) \\
\tilde{g}(\tilde{\nabla}_X Y, Z) &= g(\nabla_X Y, Z) - \frac{1 + \alpha}{2} d\varphi(Z) g(X, Y) + \frac{1 - \alpha}{2} [d\varphi(X) g(Y, Z) + d\varphi(Y) g(X, Z)]
\end{align*}
\]

for all \( X, Y, Z \in \Gamma(TM) \) (See [2]).

It is easily verified that if two statistical manifolds \((M, g, \nabla)\) and \((M, \tilde{g}, \tilde{\nabla})\) are 1-conformal equivalent then they are conformal-projective equivalent and so if a statistical manifold \((M, g, \nabla)\) is 1-conformal flat then it is conformal-projective flat.

We study some properties of conformal-projective curvature tensor of a statistical manifold and a sufficient condition for a statistical manifold to be conformal-projective flat in this paper.

In section 2, we show some properties of conformal-projective curvature tensor of a statistical manifold.

In section 3, we show that a statistical manifold of constant curvature is conformal-projective flat.

### 2 Conformal-projective curvature tensor of a statistical manifold

In this section, we give an expression of conformal-projective curvature tensor \( W \) of a statistical manifold \((M, g, \nabla)\) and show the relationship between conformal-projective curvature tensor \( W \) of a statistical manifold \((M, g, \nabla)\) and one \( \tilde{W} \) of a dual statistical manifold \((M, g, \tilde{\nabla})\).

We first give the following fact, which has been obtained independently by Zhang [6]. We quote the fact from his paper with a suitable modification for later use.

**Proposition 2.1** ([6]) Let \( \sigma \) and \( \tilde{\sigma} \) be the scalar curvature of a statistical manifold \((M, g, \nabla)\) and a dual statistical manifold \((M, g, \tilde{\nabla})\), respectively. Then we have

\[
\sigma = \tilde{\sigma} \tag{2.1}
\]

**Lemma 2.1** The conformal-projective curvature tensor \( W \) of a statistical manifold \((M, g, \nabla)\) can be expressed as follows:

\[
W(X, Y) Z = R(X, Y) Z + Y L(X, Z) - X L(Y, Z) + \tilde{\sigma}^\sharp(Y) g(X, Z) - \tilde{L}^\sharp(Y) g(X, Z) \tag{2.2}
\]

for all \( X, Y, Z \in \Gamma(TM) \), where \( L, \tilde{\sigma}^\sharp \) and \( \tilde{L}^\sharp \) are tensor fields of type \((0, 2)\) and \((1,1)\), respectively, given by
\[ L(X,Y) = \frac{1}{n-2} \{ \frac{1}{n} [(n-1) \text{Ric}(X,Y) + \text{Ric}^\sharp(X,Y)] - \frac{\sigma}{2(n-1)} g(X,Y) \} \]
\[ \hat{L}(X,Y) = \frac{1}{n-2} \{ \frac{1}{n} [(n-1) \text{Ric}^\sharp(X,Y) + \text{Ric}(X,Y)] - \frac{\hat{\sigma}}{2(n-1)} g(X,Y) \} \]
\[ g(L^\sharp(X,Y)) = \hat{L}(X,Y) \]

**Proof.** Using Eq. (2.1), we can express Eq. (1.1) as follows:
\[
W(X,Y)Z = R(X,Y)Z + \frac{1}{n(n-2)} \{ Y[(n-1)\text{Ric}(X,Z) + \text{Ric}^\sharp(X,Z)] - X[(n-1)\text{Ric}(Y,Z) + \text{Ric}^\sharp(Y,Z)] + [(n-1)\text{Ric}^\sharp(X) + \text{Ric}(X)] g(X,Z) -
\]
\[
- [(n-1)\text{Ric}(X) + \text{Ric}^\sharp(X)] g(Y,Z) \} + \frac{\sigma + \hat{\sigma}}{2(n-1)(n-2)} [X g(Y,Z) - Y g(X,Z)]
\]
\[
= R(X,Y)Z + \frac{1}{n-2} Y \{ \frac{1}{n} \text{Ric}(Y,Z) + \text{Ric}^\sharp(Y,Z) \} - \frac{\sigma}{2(n-1)} g(Y,Z) -
\]
\[
+ \frac{1}{n-2} X \{ \frac{1}{n} [(n-1)\text{Ric}(Y,Z) + \text{Ric}^\sharp(Y,Z)] - \frac{\sigma}{2(n-1)} g(Y,Z) +
\]
\[
+ \frac{1}{n-2} \{ \frac{1}{n} [(n-1)\text{Ric}^\sharp(X) + \text{Ric}(X)] - \frac{\hat{\sigma}}{2(n-1)} g(Y,Z) \}
\]

On the other hand, since \( g(\text{Ric}^\sharp(X),Y) = \text{Ric}^\sharp(X,Y) \) and \( g(\text{Ric}(X),Y) = \text{Ric}(X,Y) \) hold for all \( X,Y \in \Gamma(TM) \), we have
\[ g((n-1)\text{Ric}^\sharp(X) + \text{Ric}(X) - \frac{\hat{\sigma}}{2(n-1)} X, Y) = \hat{L}(X,Y) \]
for all \( X,Y \in \Gamma(TM) \). So
\[ \text{L}^\sharp(X) = (n-1) \text{Ric}^\sharp(X) + \text{Ric}(X) - \frac{\hat{\sigma}}{2(n-1)} X \]
holds for all \( X \in \Gamma(TM) \). Therefore we have
\[ W(X,Y)Z = R(X,Y)Z + YL(X,Z) - XL(Y,Z) + \text{L}^\sharp(Y)g(X,Z) - \text{L}^\sharp(X)g(Y,Z) \]
for all \( X,Y,Z \in \Gamma(TM) \). \( \Box \)

**Theorem 2.1** Let \( W^\sharp \) and \( W \) be the conformal-projective curvature tensors of a statistical manifold \((M,g,\nabla)\) and a dual statistical manifold \((M,\hat{g},\hat{\nabla})\), respectively. Then we have
\[ g(W(X,Y)Z, U) + g(W^\sharp(X,Y)U, Z) = 0 \] (2.3)
for all \( X,Y,Z,U \in \Gamma(TM) \).
Proof. From Eq. (2.2), we have
\[
\begin{align*}
g(W(X,Y)Z,U) &= g(R(X,Y)Z,U) + g(Y,U)L(X,Z) - g(X,U)L(Y,Z) + \\
&\quad + L(Y,U)g(X,Z) - L(X,U)g(Y,Z)
\end{align*}
\]
for all \(X,Y,Z,U \in \Gamma(TM)\). Since \(g(R(X,Y)Z,U) + g(\tilde{R}(X,Y)U,Z) = 0\) holds for all \(X,Y,Z,U \in \Gamma(TM)\), Eq. (2.3) holds. □

Eq. (2.3) shows that a statistical manifold \((M,g,\nabla)\) is conformally-projectively flat if an only if the dual statistical manifold \((M,g,\nabla^*)\) is conformally-projectively flat.

3 Constancy of curvature and conformal-projective flatness of a statistical manifold

It is known that if an \(n(\geq 4)\)-dimensional Riemannian manifold is of constant curvature, it is conformal flat and that an \(n(\geq 3)\)-dimensional Riemannian manifold is projective flat if and only if it is of constant curvature.

The conformal-projective equivalence of a statistical manifold is a generalization of conformal equivalence and projective equivalence of a Riemannian manifold. So constancy of curvature of a statistical manifold has a close relationship to the conformal-projective equivalence of a statistical manifold. It is shown that an \(n(\geq 2)\)-dimensional statistical manifold \((M,g,\nabla)\) is of constant curvature if and only if the tangent bundle \(TM\) over \(M\) with complete lift statistical structure \((g^c,\nabla^c)\) is conformally-projectively flat by Hasegawa [1].

The following theorem shows that the relationship between constancy of curvature and conformal-projective flatness of a statistical manifold \((M,g,\nabla)\).

Theorem 3.1 If an \(n(\geq 4)\)-dimensional statistical manifold \((M,g,\nabla)\) is of constant curvature, it is conformally-projectively flat.

Proof. From Eq. (2.2), we have
\[
\begin{align*}
W(X,Y)Z &= R(X,Y)Z + Y L(X,Z) - X L(Y,Z) + \tilde{L}^* (Y)g(X,Z) - \tilde{L}^* (X)g(Y,Z)
\end{align*}
\]
for all \(X,Y,Z \in \Gamma(TM)\).

Since a statistical manifold \((M,g,\nabla)\) is of constant curvature,
\[
Ric = \hat{Ric}
\]
holds and so we have
\[
L(X,Y) = \hat{L}(X,Y) = \frac{1}{n-2} \{Ric(X,Y) - \frac{\sigma}{2(n-1)}g(X,Y)\}
\]
for all \(X,Y \in \Gamma(TM)\). On the other hands, since
\[
R(X,Y)Z = K\{g(Y,Z) - g(X,Z)\}
\]
for all \(X,Y,Z \in \Gamma(TM)\).
holds for all $X, Y, Z \in \Gamma(TM)$, we have

$$Ric(Y,Z) = \text{tr}\{X \mapsto R(X,Y)Z\} = (n - 1)Kg(Y,Z)$$

for all $Y,Z \in \Gamma(TM)$. Since from the above equation,

$$\sigma = \text{tr}_g\{(Y,Z) \mapsto Ric(Y,Z)\} = n(n - 1)K$$

holds, we have

$$L(X,Y) = \text{\^}L(X,Y) = \frac{K}{2}g(X,Y)$$

for all $X,Y \in \Gamma(TM)$ and so

$$L^\sharp(X) = \text{\^}L^\sharp(X) = \frac{K}{2}X$$

holds for all $X \in \Gamma(TM)$. Therefore we have

$$W(X,Y)Z = R(X,Y)Z + YL(X,Z) - XL(Y,Z) + \text{\^}L(Y)g(X,Z) - \text{\^}L(X)g(Y,Z)$$

$$= K\{Xg(Y,Z) - Yg(X,Z)\} + \frac{K}{2}Yg(X,Z) - \frac{K}{2}Xg(Y,Z) +$$

$$+ \frac{K}{2}Yg(X,Z) - \frac{K}{2}Xg(Y,Z)$$

$$= 0$$

for all $X,Y,Z \in \Gamma(TM)$ . So the proof is finished. \qed

If a statistical manifold is a self-dual statistical manifold , that is, a Riemannian manifold, conformal-projective flatness of a statistical manifold becomes to conformal flatness of a Riemannian manifold. So theorem 3.1 shows that if a Riemannian manifold is of constant curvature, it is conformal flat, which is well known in Riemannian geometry.

Consequently, theorem 3.1 generalizes the fact that an $n(\geq 4)$-dimensional Riemannian manifold of constant curvature is conformal flat to case of a statistical manifold.

Theorem 3.1 and theorem in [1] give the following:

**Corollary 3.1** Let $(M, g, \nabla)$ be an $n(\geq 4)$-dimensional statistical manifold. If the tangent bundle $(TM, g^c, \nabla^c)$ over $M$ with complete lift statistical structure is conformally-projectively flat, $(M, g, \nabla)$ is conformally-projectively flat.

**References**

[1] I. Hasegawa , K. Yamauchi, Conformal-projective flatness of tangent bundle with complete lift statistical structure, Differential Geometry-Dynamical Systems, 10, 148-158, 2008.

[2] T. Kurose, On the divergence of 1-conformally flat statistical manifolds, Tôhoku Math. J., 46, 427-433, 1994.

[3] T. Kurose, Conformal-Projective geometry of Statistical Manifolds, Interdisciplinary information Sciences, 8, 1, 89-100, 2002.

[4] H. Matsuzoe, On realization of conformally-projectively flat statistical manifolds and the divergences, Hokkaido Math. J., 27, 409-421, 1998.
[5] K. Uohashi, On $\alpha$--conformal equivalence of statistical manifolds, J. Geom., 75, 179-184, 2002.

[6] J. Zhang, A note on curvature of $\alpha$--connections of a statistical manifold, AISM, 59, 161-170, 2007.