How to compose Lagrangian?

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Abstract

A method for constructing Lagrangians for the Lie transformation groups is explained. As examples, the Lagrangians for real plane rotations and affine transformations of the real line are constructed.

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1 Introduction and outline of the paper

It is a well-known problem in physics and mechanics how to construct Lagrangians for mechanical systems via their equations of motion. This inverse variational problem has been investigated for some types of equations of motion in [4].

In [1, 5], the plane rotation group $SO(2)$ was considered as a toy model of the Hamilton-Dirac mechanics with constraints. By introducing a Lagrangian in a particular form, canonical formalism for $SO(2)$ was developed. The crucial idea of this approach is that the Euler-Lagrange and Hamilton canonical equations must in a sense coincide with Lie equations of the Lie transformation group.

In this paper, the method for constructing such a Lagrangian is proposed. It is shown, how it is possible to find the Lagrangian, based on the Lie equations of the Lie transformation group.

By composing a Lagrangian, it is possible to describe given Lie transformation group as a mechanical system and to develop the corresponding Lagrange and Hamilton formalisms.

2 General method for constructing Lagrangians

Let $G$ be an $r$-parametric Lie group with unit $e \in G$ and let $g^i$ ($i = 1, \ldots, r$) denote the local coordinates of an element $g \in G$ from the vicinity of $e$. Let $\mathcal{X}$ be an $n$-dimensional manifold and denote the local coordinates of $X \in \mathcal{X}$ by $X^\alpha$ ($\alpha = 1, \ldots, n$). Consider a (left) differentiable action of $G$ on $\mathcal{X}$ given by

$$\mathcal{X} \ni X' = S_gX \in \mathcal{X}$$

Let $gh$ denote the multiplication of $G$. Then

$$S_gS_h = S_{gh}, \quad g, h \in G$$
By introducing the auxiliary functions $u^i_j$ and $S^\alpha_j$ by

$$(gh)^i = h^i + u^i_j(h)g^j + \ldots$$

$$(S_g X)^\alpha = X^\alpha + S^\alpha_j(X)g^j + \ldots$$

the Lie equations read

$$\varphi_j^\alpha(X; g) = u^i_j(g) \frac{\partial (S_g X)^\alpha}{\partial g^i} - S^\alpha_j(S_g X) = 0$$

The expressions $\varphi_j^\alpha$ are said to be constraints for the Lie transformation group $(\mathcal{X}, G)$. Then we search for such a vector Lagrangian $L = (L_1, \ldots, L_r)$ with components

$$L_k = \sum_{\alpha=1}^{n} \sum_{s=1}^{r} \lambda^s_{\alpha k} \varphi_s^\alpha, \quad k = 1, 2, \ldots, r$$

and such Lagrange multipliers $\lambda^s_{\alpha k}$ that the Euler-Lagrange equations in a sense coincide with the Lie equations.

The notion of a vector Lagrangian was introduced and developed in [2, 6].

**Definition 2.1** (weak equality). The functions $A$ and $B$ are called weakly equal, if

$$\left. (A - B) \right|_{\varphi_j^\alpha = 0} = 0 \quad \forall j = 1, 2, \ldots, r, \quad \forall \alpha = 1, 2, \ldots, n$$

In this case we write $A \approx B$.

By denoting

$$X^{\alpha i} = \frac{\partial X^{\alpha i}}{\partial g^i}$$

the conditions for the Lagrange multipliers read as the weak Euler-Lagrange equations

$$L_{\alpha k} = \frac{\partial L_k}{\partial X^{\alpha i}} - \sum_{i=1}^{r} \frac{\partial}{\partial g^i} \frac{\partial L_k}{\partial X^{\alpha i}} \approx 0$$

Finally, one must check by direct calculations that the Euler-Lagrange equations $L_{\alpha k} = 0$ imply the Lie equations of the Lie transformation group.

### 3 Lagrangian for $SO(2)$

First consider the 1-parameter Lie transformation group $SO(2)$, the rotation group of the real two-plane $\mathbb{R}^2$. In this case $n = 2$ and $r = 1$. Rotation of the plane $\mathbb{R}^2$ by an angle $g \in \mathbb{R}$ is given by the transformation

$$\begin{cases}
(S_g X)^1 = X'^1 = X'^1(X^1, X^2, g) = X^1 \cos g - X^2 \sin g \\
(S_g X)^2 = X'^2 = X'^2(X^1, X^2, g) = X^1 \sin g + X^2 \cos g
\end{cases}$$

We consider the rotation angle $g$ as a dynamical variable and the functions $X'^1$ and $X'^2$ as field variables for the plane rotation group $SO(2)$.

Denote

$$X^{\alpha i} = \frac{\partial X^{\alpha i}}{\partial g}$$
The infinitesimal coefficients of the transformation are
\[
\begin{align*}
S^1(X^1, X^2) &= \dot{X}^1(X^1, X^2, e) = -X^2 \\
S^2(X^1, X^2) &= \dot{X}^2(X^1, X^2, e) = X^1 \\
\end{align*}
\]
and the Lie equations read
\[
\begin{align*}
\dot{X}^1 &= S^1(X^1, X^2) = -X^2 \\
\dot{X}^2 &= S^2(X^1, X^2) = X^1 \\
\end{align*}
\]
Rewrite the Lie equations in implicit form as follows:
\[
\begin{align*}
\varphi_1 &= \dot{X}^1 + X^2 = 0 \\
\varphi_2 &= \dot{X}^2 - X^1 = 0 \\
\end{align*}
\]
We search a Lagrangian of SO(2) in the form
\[
L_1 = \sum_{a=1}^{2} \sum_{s=1}^{1} \lambda^s_a \varphi_s^a = \lambda^1_1 \varphi_1^1 + \lambda^1_2 \varphi_2^2
\]
It is more convenient to rewrite it as follows:
\[
L = \lambda^1_1 \varphi_1^1 + \lambda^2_2 \varphi_2^2
\]
where the Lagrange multipliers \(\lambda_1\) and \(\lambda_2\) are to be found from the weak Euler-Lagrange equations
\[
\frac{\partial L}{\partial X^1} - \frac{\partial}{\partial g} \frac{\partial L}{\partial X^1} \approx 0, \quad \frac{\partial L}{\partial X^2} - \frac{\partial}{\partial g} \frac{\partial L}{\partial X^2} \approx 0
\]
Calculate
\[
\begin{align*}
\frac{\partial L}{\partial X^1} &= \frac{\partial}{\partial X^1} \left[ \lambda^1_1 (X^1 + X^2) + \lambda^1_2 (\dot{X}^2 - X^1) \right] \\
&= \frac{\partial \lambda^1_1}{\partial X^1} \varphi^1 + \frac{\partial \lambda^1_2}{\partial X^1} \varphi^2 - \lambda^1_2 \approx -\lambda^1_2 \\
\frac{\partial L}{\partial X^2} &= \frac{\partial}{\partial X^2} \left[ \lambda^1_1 (X^1 + X^2) + \lambda^1_2 (\dot{X}^2 - X^1) \right] \approx \lambda^1_1 \\
\frac{\partial}{\partial g} \frac{\partial L}{\partial X^1} &= \frac{\partial \lambda^1_1}{\partial X^2} \dot{X}^1 + \frac{\partial \lambda^1_1}{\partial X^2} \dot{X}^1 \approx -\frac{\partial \lambda^1_1}{\partial X^1} X^2 + \frac{\partial \lambda^1_1}{\partial X^2} X^1
\end{align*}
\]
from which it follows
\[
\frac{\partial L}{\partial X^1} - \frac{\partial}{\partial g} \frac{\partial L}{\partial X^1} \approx 0 \quad \iff \quad -\lambda^1_2 + \frac{\partial \lambda^1_1}{\partial X^1} X^2 - \frac{\partial \lambda^1_1}{\partial X^2} X^1 \approx 0
\]
Analogously calculate
\[
\begin{align*}
\frac{\partial L}{\partial X^2} &= \frac{\partial}{\partial X^2} \left[ \lambda^2_1 (X^1 + X^2) + \lambda^2_2 (\dot{X}^2 - X^1) \right] \\
&= \frac{\partial \lambda^2_1}{\partial X^2} \varphi^1 + \frac{\partial \lambda^2_2}{\partial X^2} \varphi^2 + \lambda^1_1 \approx \lambda^1_1 \\
\frac{\partial L}{\partial X^2} &= \frac{\partial}{\partial X^2} \left[ \lambda^1_1 (X^1 + X^2) + \lambda^1_2 (\dot{X}^2 - X^1) \right] \approx \lambda^2_2 \\
\frac{\partial}{\partial g} \frac{\partial L}{\partial X^2} &= \frac{\partial \lambda^2_2}{\partial X^1} \dot{X}^1 + \frac{\partial \lambda^2_2}{\partial X^1} \dot{X}^2 \approx -\frac{\partial \lambda^2_2}{\partial X^1} X^2 + \frac{\partial \lambda^2_2}{\partial X^2} X^1
\end{align*}
\]
from which it follows
\[
\frac{\partial L}{\partial X^2} - \frac{\partial}{\partial \dot{X}^2} \frac{\partial L}{\partial X^2} \approx 0 \iff \lambda_1 + \frac{\partial \lambda_2}{\partial X^1} X^2 - \frac{\partial \lambda_2}{\partial X^2} X^1 \approx 0
\]

So the calculations imply the following system of differential equations for the Lagrange multipliers:
\[
\begin{cases}
\frac{\partial \lambda_1}{\partial X^1} X^2 + \frac{\partial \lambda_1}{\partial X^2} X^1 \approx -\lambda_2 \\
\frac{\partial \lambda_2}{\partial X^1} X^2 + \frac{\partial \lambda_2}{\partial X^2} X^1 \approx \lambda_1
\end{cases}
\]

We are not searching for the general solution for this system of partial differential equations, but the Lagrange multipliers are supposed to be a linear combination of the field variables \(X^1\) and \(X^2\),
\[
\begin{cases}
\lambda_1 = \alpha_1 X^1 + \alpha_2 X^2 \\
\lambda_2 = \beta_1 X^1 + \beta_2 X^2
\end{cases}, \quad \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}
\]

By using these expressions, one has
\[
\begin{cases}
-\alpha_1 X^2 + \alpha_2 X^1 \approx -\beta_1 X^1 - \beta_2 X^2 \\
-\beta_1 X^2 + \beta_2 X^1 \approx \alpha_1 X^1 + \alpha_2 X^2
\end{cases} \iff \begin{cases}
(\alpha_2 + \beta_1) X^1 + (\beta_2 - \alpha_1) X^2 \approx 0 \\
(\beta_2 - \alpha_1) X^1 - (\alpha_2 + \beta_1) X^2 \approx 0
\end{cases}
\]

This is a homogeneous system of two linear equations of four unknowns \(\alpha_1, \alpha_2, \beta_1, \beta_2\). The system is satisfied, if
\[
\begin{cases}
\alpha_2 + \beta_1 = 0 \\
\beta_2 - \alpha_1 = 0
\end{cases} \iff \begin{cases}
\beta_1 = -\alpha_2 \\
\beta_2 = \alpha_1
\end{cases}
\]

The parameters \(\alpha_1, \alpha_2\) are free. Thus,
\[
\begin{cases}
\lambda_1 = \alpha_1 X^1 + \alpha_2 X^2 \\
\lambda_2 = -\alpha_2 X^1 + \alpha_1 X^2
\end{cases}
\]

and the desired Lagrangian for \(SO(2)\) reads
\[
L = \alpha_1 (X^1 \dot{X}^1 + X^2 \dot{X}^2) + \alpha_2 \left[ X^2 \dot{X}^2 + (X^2)^2 - X^1 \dot{X}^1 + (X^1)^2 \right] \quad (3.1)
\]

with free real parameters \(\alpha_1, \alpha_2\). Thus we can propose the

**Theorem 3.1.** The Euler-Lagrange equations for the Lagrangian \((3.1)\) coincide with the Lie equations of \(SO(2)\).

**Proof.** Calculate
\[
\frac{\partial L}{\partial X^1} = \frac{\partial}{\partial \dot{X}^1} \left[ \alpha_1 (X^1 \dot{X}^1 + X^2 \dot{X}^2) + \alpha_2 \left( X^2 \dot{X}^1 + (X^2)^2 - X^1 \dot{X}^2 + (X^1)^2 \right) \right] = \alpha_1 \dot{X}^1 - \alpha_2 \dot{X}^2 + 2\alpha_2 X^1
\]

\[
\frac{\partial L}{\partial X^2} = \frac{\partial}{\partial \dot{X}^2} \left[ \alpha_1 (X^1 \dot{X}^1 + X^2 \dot{X}^2) + \alpha_2 \left( X^2 \dot{X}^1 + (X^2)^2 - X^1 \dot{X}^2 + (X^1)^2 \right) \right] = \alpha_1 X^1 + \alpha_2 X^2 \quad \Rightarrow \quad \frac{\partial}{\partial g} \frac{\partial L}{\partial \dot{X}^1} = \alpha_1 \dot{X}^1 + \alpha_2 \dot{X}^2
\]
from which it follows

\[
\frac{\partial L}{\partial X'^1} - \frac{\partial}{\partial g} \frac{\partial L}{\partial X'^1} = 0 \iff 2\alpha_2 X'^1 - 2\alpha_2 \dot{X}'^2 = 0 \iff \dot{X}'^2 = X'^1
\]

Analogously calculate

\[
\begin{align*}
\frac{\partial L}{\partial X'^2} &= \frac{\partial}{\partial X'^2} \left[ \alpha_1 (X'^1 \dot{X}'^1 + X'^2 \dot{X}'^2) + \alpha_2 \left( X'^2 \dot{X}'^1 + (X'^2)^2 - X'^1 \dot{X}'^2 + (X'^1)^2 \right) \right] \\
&= \alpha_2 \dot{X}'^1 + 2\alpha_2 \dot{X}'^2 + \alpha_1 \ddot{X}'^2 \\
\frac{\partial L}{\partial X'^2} &= \frac{\partial}{\partial X'^2} \left[ \alpha_1 (X'^1 \dot{X}'^1 + X'^2 \dot{X}'^2) + \alpha_2 \left( X'^2 \dot{X}'^1 + (X'^2)^2 - X'^1 \dot{X}'^2 + (X'^1)^2 \right) \right] \\
&= -\alpha_2 X'^1 + \alpha_1 X'^2
\end{align*}
\]

from which it follows

\[
\frac{\partial L}{\partial X'^2} - \frac{\partial}{\partial g} \frac{\partial L}{\partial X'^2} = 0 \iff 2\alpha_2 \dot{X}'^1 + 2\alpha_2 X'^2 = 0 \iff \dot{X}'^1 = -X'^2 \quad \Box
\]

4 Physical interpretation

While the Lagrangian \( L \) of SO(2) contains two free parameters \( \alpha_1, \alpha_2 \), particular forms of it can be found taking into account physical considerations. In particular, if \( \alpha_1 = 0 \) and \( \alpha_2 = -1/2 \), then the Lagrangian of SO(2) reads

\[
L(X'^1, X'^2, \dot{X}'^1, \dot{X}'^2) = \frac{1}{2} (X'^1 \ddot{X}'^2 - \dot{X}'^1 X'^2) - \frac{1}{2} [(X'^1)^2 + (X'^2)^2]
\]

By using the Lie equations one can easily check that

\[
X'^1 \ddot{X}'^2 - \dot{X}'^1 X'^2 = (\dot{X}'^1)^2 + (\dot{X}'^2)^2
\]

The function

\[
T = \frac{1}{2} \left[ (\dot{X}'^1)^2 + (\dot{X}'^2)^2 \right]
\]

is the kinetic energy of a moving point \((X'^1, X'^2) \in \mathbb{R}^2\), meanwhile

\[
l = X'^1 \ddot{X}'^2 - \dot{X}'^1 X'^2
\]

is its kinetic momentum with respect to origin \((0, 0) \in \mathbb{R}^2\).

This relation has a simple explanation in the kinematics of a rigid body \[3\]. The kinetic energy of a point can be represented via its kinetic momentum as follows:

\[
\frac{1}{2} \left[ (\dot{X}'^1)^2 + (\dot{X}'^2)^2 \right] = T = \frac{l}{2} = \frac{1}{2} \left[ X'^1 \ddot{X}'^2 - \dot{X}'^1 X'^2 \right]
\]

Thus we can conclude, that for the given Lie equations (that is, on the extremals) of SO(2) the Lagrangian \( L \) gives rise to a Lagrangian of a pair of harmonic oscillators.
5 Lagrangian for the affine transformations of the line

Now consider the affine transformations of the real line. The latter may be represented by

\[
\begin{align*}
X'^1 &= X'^1(X^1, X^2, g^1, g^2) \doteq g^1 X^1 + g^2 \\
X'^2 &= X'^2(X^1, X^2, g^1, g^2) \doteq 1, \quad 0 \neq g^1, g^2 \in \mathbb{R}
\end{align*}
\]

Thus \( r = 2 \) and \( n = 2 \). Denote

\[
\epsilon = (1, 0), \quad g^{-1} = \frac{1}{g^1}(1, -g^2)
\]

First, find the multiplication rule

\[
(X'^1) \doteq (X'^1)' = S_{gh} X^1 = S_g(S_h X^1) = S_g(h^1 X^1 + h^2) = g^1 (h^1 X^1 + h^2) + g^2 = (g^1 h^1) X^1 + (g^1 h^2 + g^2)
\]

Calculate the infinitesimal coefficients

\[
\begin{align*}
S^1_1(X^1, X^2) &\doteq X'^1 \bigg|_{g=\epsilon} = X_1 \\
S^1_2(X^1, X^2) &\doteq X'^2 \bigg|_{g=\epsilon} = 1 \\
S^2_1(X^1, X^2) &\doteq X'^1 \bigg|_{g=\epsilon} = 0 \\
S^2_2(X^1, X^2) &\doteq X'^2 \bigg|_{g=\epsilon} = 0
\end{align*}
\]

and auxiliary functions

\[
\begin{align*}
u^1_1(g) &\doteq \frac{\partial(S_{gh} X^1)}{\partial g^1} \bigg|_{h=g^{-1}} = \frac{1}{g^1} \\
u^1_2(g) &\doteq \frac{\partial(S_{gh} X^1)}{\partial g^2} \bigg|_{h=g^{-1}} = 0 \\
u^2_1(g) &\doteq \frac{\partial(S_{gh} X^2)}{\partial g^1} \bigg|_{h=g^{-1}} = -\frac{g^2}{g^1} \\
u^2_2(g) &\doteq \frac{\partial(S_{gh} X^2)}{\partial g^2} \bigg|_{h=g^{-1}} = 1
\end{align*}
\]

Next, write Lie equations and find constraints

\[
\begin{align*}
X'^1_1 &= \frac{1}{g^1} X'^1 - \frac{g^2}{g^1} \\
X'^1_2 &= 1 \\
X'^2_1 &= 0 \\
X'^2_2 &= 0
\end{align*} \quad \iff \quad \begin{align*}
\varphi^1_1 &\doteq X'^1_1 - \frac{1}{g^1} X'^1 - \frac{g^2}{g^1} \\
\varphi^1_2 &\doteq X'^1_2 - 1 \\
\varphi^2_1 &\doteq X'^2_1 \\
\varphi^2_2 &\doteq X'^2_2
\end{align*}
\]

We search for a vector Lagrangian \( L = (L_1, L_2) \) as follows:

\[
L_k = \sum_{a=1}^{2} \sum_{s=1}^{2} \lambda_{k a} \varphi^a_s = \lambda^1_{k1} \varphi^1_1 + \lambda^2_{k1} \varphi^1_2 + \lambda^1_{k2} \varphi^2_1 + \lambda^2_{k2} \varphi^2_2
\]

\[
= \lambda^1_{k1} \left( X'^1_1 - \frac{1}{g^1} X'^1 - \frac{g^2}{g^1} \right) + \lambda^2_{k2} \left( X'^2_2 - 1 \right) + \lambda^2_{k2} X'^2_1 + \lambda^2_{k2} X'^2_2, \quad k = 1, 2
\]
By substituting the Lagrange multipliers $\lambda_{k\alpha}$ into the weak Euler-Lagrange equations

$$\frac{\partial L_k}{\partial X^{\alpha}} - \sum_{i=1}^{2} \frac{\partial}{\partial g^i} \frac{\partial L_k}{\partial X^{\alpha}_i} \approx 0$$

we get the following PDE system

$$\begin{cases}
(X^1 - g^2) \frac{\partial \lambda^1_1}{\partial X^1} + g^1 \frac{\partial \lambda^2_1}{\partial X^1} + \lambda^1_{1k} \approx 0 \\
(X^1 - g^2) \frac{\partial \lambda^1_2}{\partial X^2} + g^1 \frac{\partial \lambda^2_2}{\partial X^2} \approx 0, \quad k = 1, 2
\end{cases}$$

We find some particular solutions for this system. For example,

$$k = 1 : \begin{cases}
\lambda^1_{11} = 0 \\
\lambda^2_{11} = \psi^1_{11}(X^2) \\
\lambda^1_{12} = \psi^1_{12}(X^2) \\
\lambda^2_{12} = \psi^2_{12}(X^2)
\end{cases} \quad \text{and} \quad k = 2 : \begin{cases}
\lambda^1_{21} = \psi^1_{21}(X^2) \\
\lambda^2_{21} = -\frac{X^1}{g^2} \psi^1_{21}(X^2) \\
\lambda^1_{22} = 0 \\
\lambda^2_{22} = 0
\end{cases}$$

with $\psi^1_{21}(X^2), \psi^1_{11}(X^2), \psi^1_{12}(X^2), \psi^2_{12}(X^2)$ as arbitrary real valued functions of $X^2$.

Thus we can define the Lagrangian $L = (L_1, L_2)$ with

$$\begin{align*}
L_1 &= \psi^1_{11}(X^2) (X^1_1 - 1) + \psi^1_{12}(X^2) X^2_1 + \psi^2_{12}(X^2) X^2_2 \\
L_2 &= \psi^1_{21}(X^2) \left( X^1_1 - \frac{1}{g^2} X^1 + \frac{g^2}{g^2} \right)
\end{align*}$$

(5.1)

and propose the

**Theorem 5.1.** The Euler-Lagrange equations for the vector Lagrangian $L = (L_1, L_2)$ with components (5.1) coincide with the Lie equations of the affine transformations of the real line.

**Proof.** Calculate

$$\begin{align*}
\frac{\partial L_1}{\partial X^1} &= \frac{\partial}{\partial X^1} \left[ \psi^1_{11}(X^2) (X^1_1 - 1) + \psi^1_{12}(X^2) X^2_1 + \psi^2_{12}(X^2) X^2_2 \right] = 0 \\
\frac{\partial}{\partial g^1} \frac{\partial L_1}{\partial X^1} &= 0 \\
\frac{\partial}{\partial g^2} \frac{\partial L_1}{\partial X^2_1} &= \frac{\partial \psi^2_{11}(X^2)}{\partial g^2} X^2_2
\end{align*}$$

from which it follows

$$\frac{\partial L_1}{\partial X^1} - \sum_{i=1}^{2} \frac{\partial}{\partial g^i} \frac{\partial L_1}{\partial X^1_i} = 0 \quad \Leftrightarrow \quad \frac{\partial \psi^2_{11}(X^2)}{\partial X^2_2} X^2_2 = 0 \quad \Rightarrow \quad X^2_2 = 0$$

Analogously calculate

$$\begin{align*}
\frac{\partial L_1}{\partial X^2} &= \frac{\partial}{\partial X^2} \left[ \psi^1_{11}(X^2) (X^1_1 - 1) + \psi^1_{12}(X^2) X^2_1 + \psi^2_{12}(X^2) X^2_2 \right] \\
&= \frac{\partial \psi^1_{11}(X^2)}{\partial X^2} (X^1_1 - 1) + \frac{\partial \psi^1_{12}(X^2)}{\partial X^2} X^2_1 + \frac{\partial \psi^2_{12}(X^2)}{\partial X^2} X^2_2 \\
\frac{\partial}{\partial g^1} \frac{\partial L_1}{\partial X^1} &= \frac{\partial \psi^1_{12}(X^2)}{\partial g^1} = \frac{\partial \psi^1_{12}(X^2)}{\partial X^2} X^2_1 \\
\frac{\partial}{\partial g^2} \frac{\partial L_1}{\partial X^2} &= \frac{\partial \psi^2_{12}(X^2)}{\partial g^2} = \frac{\partial \psi^2_{12}(X^2)}{\partial X^2} X^2_2
\end{align*}$$
from which it follows

\[
\frac{\partial L_1}{\partial X'^2} - \sum_{i=1}^{2} \frac{\partial}{\partial g_i} \frac{\partial L_1}{\partial X'_i} = 0 \quad \iff \quad \frac{\partial \psi_1^2(X'^2)}{\partial X'^2} (X'^2_1 - 1) = 0 \quad \iff \quad X'^2_1 - 1 = 0
\]

Now we differentiate the second component of the Lagrangian \( L \). Calculate

\[
\frac{\partial L_2}{\partial X'^2} = \frac{\partial}{\partial X'^2} \left[ \psi_1^2(X'^2) \left( X'^2_1 - \frac{1}{g'^2} X'^1 \right) + \frac{g^2}{g'} \right] - \frac{1}{g'} X'^1 \psi_1^2(X'^2)(X'^2_2 - 1) = - \frac{1}{g'} \psi_1^2(X'^2) - \frac{1}{g} \psi_1^2(X'^2) X'^2_2 + \frac{1}{g'} \psi_1^2(X'^2) = - \frac{1}{g'} \psi_1^2(X'^2) X'^2_2
\]

\[
\begin{align*}
\frac{\partial}{\partial g^1} \frac{\partial L_2}{\partial X'^1} &= \frac{\partial \psi_2^1(X'^2)}{\partial X'^2} \quad \frac{\partial \psi_1^2(X'^2)}{\partial X'^2} X'^2_1 \\
\frac{\partial}{\partial g^2} \frac{\partial L_2}{\partial X'^2} &= \frac{\partial \psi_2^1(X'^2)}{\partial X'^2} \left( - \frac{1}{g'} X'^1 \psi_1^2(X'^2) \right) = \frac{1}{g'} \left( \psi_1^2(X'^2) X'^2_2 + X'^1 \frac{\partial \psi_1^2(X'^2)}{\partial X'^2} X'^2_2 \right)
\end{align*}
\]

from which it follows

\[
\frac{\partial L_2}{\partial X'^2} - \sum_{i=1}^{2} \frac{\partial}{\partial g_i} \frac{\partial L_2}{\partial X'_i} = 0 \quad \iff \quad \frac{\partial \psi_1^2(X'^2)}{\partial X'^2} \left( X'^2_1 - \frac{1}{g'} X'^1 X'^2_2 \right) = 0
\]

\[
\quad \iff \quad X'^2_1 - \frac{1}{g'} X'^1 X'^2_2 = 0
\]

Analogously calculate

\[
\frac{\partial L_2}{\partial X'^2} = \frac{\partial}{\partial X'^2} \left[ \psi_1^2(X'^2) \left( X'^1 - \frac{1}{g'^2} X'^2 \right) + \frac{g^2}{g'} \right] - \frac{1}{g'} X'^2 \psi_1^2(X'^2)(X'^2_1 - 1) = \frac{\partial \psi_1^2(X'^2)}{\partial X'^2} \left( X'^1 - \frac{1}{g'} X'^2 \right) + \frac{g^2}{g'} \frac{\partial \psi_1^2(X'^2)}{\partial X'^2} X'^2_1 (X'^2_1 - 1)
\]

\[
\begin{align*}
\frac{\partial}{\partial g^1} \frac{\partial L_2}{\partial X'^1} &= \frac{\partial \psi_1^2(X'^2)}{\partial X'^2} \frac{\partial \psi_1^2(X'^2)}{\partial X'^2} 0 = 0 \\
\frac{\partial}{\partial g^2} \frac{\partial L_2}{\partial X'^2} &= \frac{\partial \psi_1^2(X'^2)}{\partial X'^2} \frac{\partial \psi_1^2(X'^2)}{\partial X'^2} 0 = 0
\end{align*}
\]

from which it follows

\[
\frac{\partial L_2}{\partial X'^2} - \sum_{i=1}^{2} \frac{\partial}{\partial g_i} \frac{\partial L_2}{\partial X'_i} = 0 \quad \iff \quad \frac{\partial \psi_1^2(X'^2)}{\partial X'^2} \left( X'^1 - \frac{1}{g'} X'^1 X'^2_2 + \frac{g^2}{g'} \right) = 0 \\
\quad \iff \quad X'^1 - \frac{1}{g'} X'^1 X'^2_2 + \frac{g^2}{g'} = 0
\]

Thus, the Euler-Lagrange equations read

\[
\begin{align*}
X'^2_2 &= 0 \\
X'^2_1 - 1 &= 0 \\
X'^1 - \frac{1}{g'} X'^1 X'^2_2 &= 0 \\
X'^1 - \frac{1}{g'} X'^1 X'^2_2 + \frac{g^2}{g'} &= 0
\end{align*}
\]

It can be easily verified, that the latter is equivalent to the system of the Lie equations. \( \blacksquare \)

**Remark 5.2.** While the Lagrangian \( L \) contains four arbitrary functions, particular forms of it can be fixed by taking into account physical considerations.
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