Pairs of equiperimeter and equiareal triangles
whose sides are perfect squares
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Abstract

In this paper we consider the problem of finding pairs of triangles
whose sides are perfect squares of integers, and which have a common
perimeter and common area. We find two such pairs of triangles,
and prove that there exist infinitely many pairs of triangles with the
specified properties.

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squared sides.

1 Introduction

Ever since the discovery of right-angled triangles with integer sides, there has
been considerable interest in finding triangles as well as polygons with certain
gEometric properties and all of whose sides are given by integers. Several
mathematicians have also considered diophantine problems pertaining to a
pair of triangles or other geometric objects (see for instance, [2], [4], [6],
[7], [8], [10], [13]). Considerable attention has been given to the problem of
finding two triangles with the same perimeter and the same area and such
that all the sides and the common area of the two triangles are given by
integers (see [1], [3], [9], [12]). In fact, Choudhry [5] has described a method
of generating an arbitrarily large number of scalene rational triangles with
the same perimeter and the same area.

In this paper we consider the problem of finding two triangles with the
same perimeter and the same area and such that all the sides of the two
triangles are perfect squares of integers.
2 Equiperimeter and equiareal triangles with squared sides

Let us consider two triangles whose sides are \( a^2, b^2, c^2 \) and \( d^2, e^2, f^2 \) where \( a, b, \ldots, f \), are all integers. The two triangles will have the same perimeter if the integers \( a, b, \ldots, f \), satisfy the following condition:

\[
    a^2 + b^2 + c^2 = d^2 + e^2 + f^2. \tag{2.1}
\]

Further, on using Heron’s well-known formula for the area of a triangle, the condition that the two triangles will have the same area may be written as follows:

\[
    (a^2 + b^2 + c^2)(a^2 + b^2 - c^2)(a^2 - b^2 + c^2)(-a^2 + b^2 + c^2)
    = (d^2 + e^2 + f^2)(d^2 + e^2 - f^2)(d^2 - e^2 + f^2)(-d^2 + e^2 + f^2). \tag{2.2}
\]

Since both equations (2.1) and (2.2) are homogeneous, any rational solution of these equations will yield, on appropriate scaling, a solution in integers. It therefore suffices to obtain rational solutions of (2.1) and (2.2).

We note that, in view of the condition (2.1), equation (2.2) reduces to

\[
    (a^2 + b^2 - c^2)(a^2 - b^2 + c^2)(-a^2 + b^2 + c^2)
    = (d^2 + e^2 - f^2)(d^2 - e^2 + f^2)(-d^2 + e^2 + f^2). \tag{2.3}
\]

To solve the simultaneous diophantine equations (2.1) and (2.3), we write

\[
    a = pu + q + r, \quad b = qu - p - r, \quad c = ru - p + q,
    d = pu - q - r, \quad e = qu + p + r, \quad f = ru + p - q, \tag{2.4}
\]

where \( p, q, r \) and \( u \) are arbitrary parameters.

With these values of \( a, b, \ldots, f \), it is readily verified that equation (2.1) is identically satisfied. Further, equation (2.3) reduces to

\[
    16u(u - 1)(u + 1)(q + r)(p + r)(p - q)
    \times \{(p^3 + p^2q - p^2r + pq^2 - 2pqr + pr^2 + q^3 - q^2r + qr^2 - r^3)u^2
    + 2p^2q - 2p^2r + 2pq^2 + 12pqr + 2pr^2 - 2q^2r + 2qr^2\} = 0. \tag{2.5}
\]

We note that when the parameters \( p, q, r \) and \( u \) satisfy the condition,

\[
    u(u - 1)(u + 1)(q + r)(p + r)(p - q)(p + q - r) = 0, \tag{2.6}
\]

we get a trivial solution of equations (2.1) and (2.2). Accordingly, we must find rational numbers \( p, q, r \) and \( u \) such that the above condition is not satisfied and the last factor of (2.5) becomes 0. Thus there must exist an integer
such that
\[ t^2 = -\left(p^3 + p^2q - p^2r + pq^2 - 2pqr + pr^2 + q^3 - q^2r + qr^2 - r^3\right) \times \left(2p^2q - 2p^2r + 2pq^2 + 12pqr + 2pr^2 - 2q^2r + 2qr^2\right). \] (2.7)

Further, to obtain actual triangles with a common perimeter and common area, we need to find solutions of the simultaneous equations (2.1) and (2.2) such that the sides \(a^2, b^2, c^2\) and \(d^2, e^2, f^2\) of the two triangles satisfy the triangle inequalities. A computer search in the range \(|p| + |q| + |r| \leq 700\) yielded two sets of values of \(p, q, r\) such that all conditions are satisfied.

The first pair of triangles is obtained by taking \((p, q, r) = (14, -27, -25)\) when we get two triangles with sides
\[ 661^2, 1498^2, 1515^2 \quad \text{and} \quad 921^2, 1310^2, 1553^2, \] (2.8)
such that the two triangles have a common perimeter and common area.

A second example is obtained with \((p, q, r) = (46, 73, 371)\) when again we get two triangles with a common perimeter and common area, the sides of the two triangles being
\[ 71297^2, 77895^2, 97154^2 \quad \text{and} \quad 67005^2, 81926^2, 96893^2. \] (2.9)

We will now show that there exist infinitely many pairs of triangles with a common perimeter and common area and such that all the sides of both the triangles are given by perfect squares of integers.

On writing
\[ q = -p(1 - mx), \quad r = px, \quad t = p^3xy, \] (2.10)
the condition (2.7) may be written as follows:
\[ y^2 = 2(m - 1)^2(m^2 + 1)mx^4 - 2m(m - 1)(m^3 + 10m^2 + m + 8)x^3 \]
\[ + (6m^4 + 50m^3 - 18m^2 + 14m - 16)x^2 - 8m^2(m + 8)x \]
\[ + 4m(m + 8). \] (2.11)

Any rational solution of equation (2.11) yields, on using the relations (2.10), a solution of the equation (2.7). Moreover, the solution \((p, q, r) = (14, -27, -25)\) of condition (2.7) corresponds to the solution
\[ (m, x, y) = (13/25, -25/14, -339/245) \]
of equation (2.11).

Accordingly, we fix \(m = 13/25\) in equation (2.11) when we get
\[ y^2 = \left(2972736/9765625\right)x^4 + \left(55402464/9765625\right)x^3 \]
\[ - \left(2389884/390625\right)x^2 - \left(287976/15625\right)x + 11076/625. \] (2.12)
Now equation (2.12) represents a quartic model of an elliptic curve, and the birational transformation given by

\[
\begin{align*}
    x &= -25(2048250X + 4633Y - 1188723263160) \\
        &\quad \times (140122182X - 23657Y - 532179246194760)^{-1}, \\
    y &= 5061168(137842X^3 - 73907924100X^2 - 68921Y^2 \\
        &\quad - 182461764476160Y + 3026923795437314317841600) \\
        &\quad \times \{5(140122182X - 23657Y - 532179246194760)\}^{-2}
\end{align*}
\]  

(2.13)

and

\[
\begin{align*}
    X &= 4(1311254697763x^2 + 2523717076250x + 1621155375000y \\
        &\quad - 3370847328125)(577x - 2825)^{-2}, \\
    Y &= 60734016(223993723304x^3 + 1368674740750x^2 \\
        &\quad + 364901515625xy - 2217844262500x + 133349609375y \\
        &\quad - 372517031250)(577x - 2825)^{-3}
\end{align*}
\]  

(2.14)

reduces the quartic curve (2.12) to the Weierstrass form given by

\[
Y^2 = X^3 - 21151030877616X + 31685265497576201600. 
\]  

(2.15)

Using the software SAGE, we quickly found the rank on the elliptic curve (2.15) to be 2, with the generators being

\[
(6008706700/1681, 91230882238080/68921),
\]

and \((7840706250956/1168561, -17496345598032878080/1263214441)\).

Now we already know a rational point \((x, y) = (-25/14, -339/245)\) on the quartic curve (2.12), and corresponding to this rational point, we readily find a point \(P\) on the cubic curve (2.15), the coordinates of \(P\) being as follows:

\[
(-7450305309428/4661281, -78862809542759294976/10063705679). 
\]  

(2.16)

It is clear from the manner in which we have obtained the elliptic curve (2.15) that corresponding to each rational point on the curve (2.15), there exists a rational solution of the simultaneous equations (2.1) and (2.2). The point \(P\) on the curve (2.15) corresponds to the solution \((p, q, r) = (14, -27, -25)\) of equation (2.7) and yields the two triangles mentioned at (2.8).

Since the curve (2.15) has positive rank, there are infinitely many rational points on the elliptic curve (2.15) and it follows from a theorem of Poincare and Hurwitz [11, Satz 11, p. 78] that there exist infinitely many rational points on the curve (2.15) in the neighbourhood of the point \(P\), and
these points will yield infinitely many rational solutions of the simultaneous equations (2.1) and (2.2) such that \( a^2, b^2, c^2 \) and \( d^2, e^2, f^2 \) satisfy the triangle inequalities, and we can thus generate infinitely many pairs of triangles whose sides are perfect squares of integers, and which have a common perimeter and common area.

3 Concluding remarks

The aforementioned theorem of Poincare and Hurwitz does not give any method of generating infinitely many rational points on the curve (2.15) in the neighbourhood of the point \( P \), and hence is of no help in actually finding pairs of triangles whose sides are perfect squares of integers, and which have a common perimeter and common area. It would be of interest to find an effective method of generating infinitely many pairs of triangles with the desired properties.

Finally we note that while the triangles we have obtained have a common perimeter and common area, the area of our triangles is not an integer. It is very unlikely that there exist pairs of triangles whose sides are perfect squares of integers, and which have a common perimeter and common area which is also an integer.

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