Poincaré inequalities and $A_p$ weights on bow-ties

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Abstract. A metric space $X$ is called a bow-tie if it can be written as $X = X_+ \cup X_-$, where $X_+ \cap X_- = \{x_0\}$ and $X_\pm \neq \{x_0\}$ are closed subsets of $X$. We show that a doubling measure $\mu$ on $X$ supports a $(q, p)$–Poincaré inequality on $X$ if and only if $X$ satisfies a quasiconvexity-type condition, $\mu$ supports a $(q, p)$-Poincaré inequality on both $X_+$ and $X_-$, and a variational $p$-capacity condition holds. This capacity condition is in turn characterized by a sharp measure decay condition at $x_0$.

In particular, we study the bow-tie $X_{\mathbb{R}^n}$ consisting of the positive and negative hyper-quadrants in $\mathbb{R}^n$ equipped with a radial doubling weight and characterize the validity of the $p$-Poincaré inequality on $X_{\mathbb{R}^n}$ in several ways. For such weights, we also give a general formula for the capacity of annuli around the origin.

Key words and phrases: Bow-tie, capacity, doubling measure, metric space, Muckenhoupt $A_p$-weight, $p$-admissible weight, Poincaré inequality, radial weight, variational capacity.

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1. Introduction

The simplest example of a bow-tie is the union of the first and third closed quadrants in $\mathbb{R}^2$, which (when intersected with the diamond $\{(x_1, x_2) : |x_1| + |x_2| \leq 1\}$) resembles the shape of a bow-tie.

In the generality of metric spaces, we say that $X = (X, d)$ is a bow-tie if there is a point $x_0 \in X$ which splits it in the form $X = X_+ \cup X_-$, where $X_+ \cap X_- = \{x_0\}$ and $X_\pm \neq \{x_0\}$ are closed subsets of $X$. We do not require $X$ to be complete. $X$ is equipped with a positive complete Borel measure $\mu$ such that

$$0 < \mu(B) < \infty \quad \text{for all balls } B \subset X. \quad (1.1)$$

We also assume that $1 \leq p, q < \infty$.

Our aim is to characterize when Poincaré inequalities hold on $X$ in terms of their validity on $X_\pm$. It turns out that there are two extra conditions involved. Here $B_r = B(x_0, r)$ is the open ball centred at $x_0$ with radius $r$, and $B_r^\pm = B_r \cap X_\pm$. 

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Theorem 1.1. Assume that $\mu$ is doubling on $X$. Then $\mu$ supports a $(q,p)$-Poincaré inequality on $X$ if and only if the following conditions hold:

(i) $\mu$ supports a $(q,p)$-Poincaré inequality on $X_+$ and on $X_-;
(ii)$ there is $\Lambda$ so that $d(x_+, x_0) + d(x_0, x_-) \leq \Lambda d(x_+, x_-)$ for all $x_+ \in X_+$;
(iii) for all $0 < r < \frac{1}{4} \min \{\diam X_+, \diam X_-\}$,

$$\text{cap}^{X_+}_{p,\mu}(\{x_0\}, B_+^r) \simeq r^{-p} \mu(B_+^r) \quad \text{and} \quad \text{cap}^{X_-}_{p,\mu}(\{x_0\}, B_-^r) \simeq r^{-p} \mu(B_-^r). \quad (1.2)$$

If, in addition, $p > 1$ and there is a locally compact open set $G \ni x_0$, then condition (iii) can equivalently be replaced by

(iii') $p > \max\{Q_{x_0,R_0}^{X_+\mu}, Q_{x_0,R_0}^{X_-\mu}\}$, where $R_0 = \frac{1}{4} \min \{\diam X_+, \diam X_-\}$ and

$$Q_{x_0,R_0}^{X_+\mu} := \inf \left\{ Q > 0 : \frac{\mu(B_+^{2r})}{\mu(B_+^r)} \gtrsim \left(\frac{Q}{r}\right)^Q \quad \text{for all} \quad 0 < \rho < r < R_0 \right\}.$$ 

Here $\text{cap}^{X_{\pm}}_{p,\mu}$ denotes the capacity with respect to $X_{\pm}$ as the underlying space. Note that the dilation constants in the Poincaré inequalities on $X$ and $X_{\pm}$ are in general different, see Remark 3.3. It is well known that sets of capacity zero cannot separate a space supporting a Poincaré inequality, see [3, Lemma 4.6]. So requiring

$$\text{cap}^{X_{\pm}}_{p,\mu}(\{x_0\}, B_{\pm}^r) > 0 \quad \text{and} \quad \text{cap}^{X_{\pm}}_{p,\mu}(\{x_0\}, B_{\pm}^r) > 0 \quad \text{for small} \quad r > 0 \quad (1.3)$$

is a necessary condition for the validity of a $p$-Poincaré inequality on $X$. It has been folklore that it might also be sufficient (see e.g. Korte [28, p. 102]). However, Example 6.6 shows that the range of $p > 1$ so that (iii) holds can be considerably smaller than the range of $p$ for which (1.3) holds. In this example, $d\mu = w \, dx$ is doubling and supports a $1$-Poincaré inequality on $\mathbb{R}^2$ with a radial weight $w$ (and we consider the Euclidean bow-tie $X_{\mathbb{R}^2}$ as in Theorem 1.2 below). For $p = 1$ we do not know if (1.3) can hold while (iii) fails.

When specializing to bow-ties in $\mathbb{R}^n$ equipped with radial weights we obtain the following characterizations. Here $x_0 = (0, \ldots, 0)$ is the origin.

Theorem 1.2. Let $X_{\mathbb{R}^n} = X_{\mathbb{R}^n}^+ \cup X_{\mathbb{R}^n}^-$, where

$$X_{\mathbb{R}^n}^+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \pm x_j \geq 0, \ j = 1, \ldots, n\} \quad \text{and} \quad n \geq 1.$$ 

Also let $d\mu = w \, dx$ be a doubling measure on $\mathbb{R}^n$, where $w(x) = w(|x|)$ is a radial weight. Then the following are equivalent:

(a) $\mu$ supports a $p$-Poincaré inequality on $X_{\mathbb{R}^n}$;
(b) $\mu$ supports a $p$-Poincaré inequality on $\mathbb{R}^n$ and

$$\text{cap}^{X_{\mathbb{R}^n}}_{p,\mu}(\{0\}, B_r) \simeq r^{-p} \mu(B_r) \quad \text{for all} \quad r > 0; \quad (1.4)$$

(c) $\mu$ supports a $p$-Poincaré inequality on $X_{\mathbb{R}^n}^+$ and (1.4) holds;
(d) $w$ is an $A_p$-weight on $\mathbb{R}^n$ and (1.4) holds;
(e) $\tilde{w}(\rho) := |\rho|^{n-1} w(|\rho|)$ is an $A_p$-weight on $\mathbb{R}$.

Unweighted $\mathbb{R}^n$ with $1 \leq p \leq n$ shows that the capacity condition (1.4) is not redundant in Theorem 1.2 (a)$\Leftrightarrow$(b). Weighted $\mathbb{R}^n$ shows that (1.4) is not redundant in (b)$\Leftrightarrow$(d). More precisely, the weight $w(x) = |x|^a$ is doubling and supports a $p$-Poincaré inequality on $\mathbb{R}^n$, with $n \geq 2$, for all $p \geq 1$ and all $a > -n$, while it is an $A_p$ weight if and only if $-n < a < n(p-1)$ or $a = 0$, see Heinonen–Kilpeläinen–Martio [21, p. 10] for $p > 1$ and Proposition 7.2 and Corollary 7.3 for $p = 1$.

Note that for radial weights, the doubling condition for $\mu$ holds equivalently on $\mathbb{R}^n$, $X_{\mathbb{R}^n}^+$ and $X_{\mathbb{R}^n}^-$, see Lemma 5.6. Similarly, the capacity condition (1.4) holds simultaneously for $\text{cap}^{X_{\mathbb{R}^n}^+}_{p,\mu}$, $\text{cap}^{X_{\mathbb{R}^n}^-}_{p,\mu}$ and $\text{cap}^{\mathbb{R}^n}_{p,\mu}$, see Corollary 5.4.
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The following theorem characterizes when the capacity condition (1.2) holds. The implications

$$ p > Q_{x_0, R_0}^{x, \mu} \implies (1.5) \implies p \geq Q_{x_0, R_0}^{x, \mu} $$

appeared in Björn–Björn–Lehrbäck [9, Propositions 6.1 and 9.1] (under weaker assumptions on $\mu$ and covering also the case $p = 1$). As a somewhat surprising application of Theorem 1.1 and the deep self-improvement of Poincaré inequalities due to Keith–Zhong [26, Theorem 1.0.1], we now settle (in the negative) also the borderline case $p = Q_{x_0, R_0}^{x, \mu} > 1$. It is this characterization that makes it possible to replace (iii) by (iii$'$) in the last part of Theorem 1.1.

**Theorem 1.3.** Assume that $\mu$ is doubling and supports a $p$-Poincaré inequality on $X$, where $p > 1$. Let $0 < R_0 \leq \frac{4}{3} \text{diam } X$. Then

$$ \text{cap}^X_{p, \mu}(B_p, B_r) \gtrsim r^{-p} \mu(B_r) \quad \text{for } 0 < p < R_0 \quad (1.5) $$

if and only if

$$ p > Q_{x_0, R_0}^{x, \mu} := \inf \left\{ Q > 0 : \frac{\mu(B_p)}{\mu(B_r)} \gtrsim \left( \frac{p}{r} \right)^Q \quad \text{for } 0 < p < R_0 \right\}. $$

If, in addition, there is a locally compact open set $G \ni x_0$, then also the condition

$$ \text{cap}^X_{p, \mu}(\{x_0\}, B_r) \gtrsim r^{-p} \mu(B_r) \quad \text{for } 0 < R_0 $$

is equivalent to the two conditions above.

When $R_0 < \infty$, Lemma 4.1 and [9, Lemma 2.5] show that both (1.5) and $Q_{x_0, R_0}^{x, \mu}$ are independent of the particular choice of $R_0$. Example 4.2 illustrates the difference between finite and infinite $R_0$.

For $p = 1$ and radial weights on $\mathbb{R}^n$, each of (1.5) and (1.6) is equivalent to $\mu(B_p)/\mu(B_r) \gtrsim p/r$, i.e. $p = 1 \geq Q_{x_0, R_0}^{x, \mu}$, see Proposition 6.5. For other spaces and measures, the case $p = 1 = Q_{x_0, R_0}^{x, \mu}$ seems to be open. When $p = 1 > Q_{x_0, R_0}^{x, \mu}$, the equivalences follow from [9, Proposition 6.1] and Lemma 4.1.

Along the way, we obtain the following explicit formula for the capacity of annuli around the origin with respect to radial weights. (Note that here we do not require $\mu$ to be doubling, but (1.1) is required as before.)

**Proposition 1.4.** Assume that $d\mu = w \, dx$, where $w(x) = w(|x|)$ is a radial weight on $\mathbb{R}^n$. If $r > 0$, then

$$ \text{cap}^{\mathbb{R}^n}_{p, \mu}(\{0\}, B_r) = \begin{cases} \left( \int_0^r \hat{w}(\rho)^{1/(1-p)} \, d\rho \right)^{1-p} & \text{if } p > 1, \\ \text{ess inf}_{\hat{w} \hat{w}} \hat{w}(\rho) & \text{if } p = 1 \end{cases} \quad (1.7) $$

where $\hat{w}(\rho) := \omega_{n-1} w(\rho) \rho^{n-1}$ and $\omega_{n-1}$ is the surface area of the $(n-1)$-dimensional unit sphere in $\mathbb{R}^n$ (with $\omega_0 = 2$).

Similarly, if $0 < r' < r$, then

$$ \text{cap}^{\mathbb{R}^n}_{p, \mu}(B_{r'}, B_r) = \text{cap}^{\mathbb{R}^n}_{p, \mu}(B_{r'}, B_r) = \begin{cases} \left( \int_{r'}^r \hat{w}(\rho)^{1/(1-p)} \, d\rho \right)^{1-p} & \text{if } p > 1, \\ \text{ess inf}_{\hat{w} \hat{w}} \hat{w}(\rho) & \text{if } p = 1 \end{cases} $$
For $p > 1$ and radial weights supporting a $p$-Poincaré inequality on $\mathbb{R}^n$, the formula for $\text{cap}_{p,\mu}^{R_n}(B_r, B_r)$ was obtained in [9, Proposition 10.8]. Our proof here is shorter, more elementary and does not require any $p$-Poincaré inequality.

Poincaré inequalities are important tools in various applications. For example, in Heinonen–Kilpeläinen–Martio [21], an extensive nonlinear potential theory was developed for weighted $\mathbb{R}^n$ equipped with a weight $w$ supporting a $p$-Poincaré inequality and a doubling condition for the associated measure $d\mu = w \, dx$, a so-called $p$-admissible weight.

Bow-ties are the simplest examples of glueing two metric spaces together (at one point). Such constructions appeared in Heinonen–Koskela [22, Section 6.14], where Ahlfors $Q$-regular spaces supporting Poincaré inequalities were glued along various sets (not only at a single point). Bow-ties often provide (counter)examples of metric spaces with various interesting properties, see e.g. [2, Section 8], [3, Examples 5.6, 5.7, A.23 and A.24], [4, Example 6.2], [7, p. 51], [9, p. 1189], [10, Example 6.1], [11, Example 4.5], [12, Example 5.2], [24, p. 814], [27, Remark 5.2], [28, p. 102] and [29, Example 6.2].

Shanmugalingam [31, Example 4.3.1] seems to be the first who used the name bow-tie for such examples. Later, in [27] and [29], they were also called Gehring bow-ties, even though the example is not due to Gehring. Rather, the name stems from the fact that Fred Gehring (an analyst in Ann Arbor) was always wearing a bow-tie.

The outline of the paper is as follows: In Section 2 we discuss the necessary background from analysis on metric spaces. Section 3 is devoted to bow-ties in metric spaces and Theorem 1.1 is proved therein. The capacity conditions (1.5) and (1.6) are studied in Section 4, where the proof of Theorem 1.3 is given.

From Section 5 onwards we concentrate on weights on $\mathbb{R}^n$ and the bow-ties $X_{\mathbb{R}^n}$ (as given in Theorem 1.2). In Section 5 we develop the theory, as far as possible, for so-called $T$-invariant weights, while in Section 6 we turn to radial weights where more can be said and Theorem 1.2 and Proposition 1.4 are deduced.

Finally, in Section 7 we study when conditions (1.2) and (1.3) hold for logarithmic power weights. We leave it to the reader to draw conclusions using these facts together with Theorems 1.1 and 1.2.

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2. Preliminaries

In this section we introduce the necessary background notation on metric spaces and in particular on Sobolev spaces and capacities in metric spaces. See the monographs Björn–Björn [3] and Heinonen–Koskela–Shanmugalingam–Tyson [23] for more extensive treatments of these topics, including proofs of most of the results mentioned in this section.

We always assume that $1 \leq p < \infty$ and that $X = (X, d, \mu)$ is a metric space equipped with a metric $d$ and a positive complete Borel measure $\mu$ such that (1.1) holds. We say that $\mu$ is doubling if there exists a doubling constant $C > 0$ such that for all balls $B = B(x, r) := \{ y \in X : d(x, y) < r \}$ in $X$,

$$0 < \mu(2B) \leq C\mu(B) < \infty.$$  

Here and elsewhere we let $cB = B(x, cr)$.

A curve is a continuous mapping from an interval, and a rectifiable curve is a curve with finite length. We will only consider curves which are nonconstant, compact and rectifiable, and thus each curve can be parameterized by its arc length $ds$. 

Following Heinonen–Koskela [22], we introduce upper gradients as follows (called very weak gradients in [22]).

**Definition 2.1.** A Borel function $g : X \to [0, \infty]$ is an upper gradient of a function $f : X \to [-\infty, \infty]$ if for all curves $\gamma : [0, l_\gamma] \to X$,

$$|f(\gamma(0)) - f(\gamma(l_\gamma))| \leq \int_\gamma g \, ds,$$

where the left-hand side is considered to be $\infty$ whenever at least one of the terms therein is infinite.

The following version of Sobolev spaces on $X$ is from Shanmugalingam [32].

**Definition 2.2.** For a measurable function $f : X \to [-\infty, \infty]$, let

$$\|f\|_{N^{1,p}(X)} = \left(\int_X |f|^p \, d\mu + \inf_g \int_X g^p \, d\mu \right)^{1/p},$$

where the infimum is taken over all upper gradients $g$ of $f$. The Newtonian space on $X$ is

$$N^{1,p}(X) = \{f : \|f\|_{N^{1,p}(X)} < \infty\}.$$

The quotient space $N^{1,p}(X) / \sim$, where $f \sim h$ if and only if $\|f - h\|_{N^{1,p}(X)} = 0$, is a Banach space and a lattice, see [32]. In this paper we assume that functions in $N^{1,p}(X)$ are defined everywhere, not just up to an equivalence class in the corresponding function space. This is needed for the definition of upper gradients to make sense.

**Definition 2.3.** Let $1 \leq q < \infty$. We say that $X$ or $\mu$ supports a $(q, p)$-Poincaré inequality if there exist constants $C > 0$ and $\lambda \geq 1$ such that for all balls $B = B(x, r)$, all integrable functions $f$ on $X$, and all upper gradients $g$ of $f$,

$$\left(\frac{\int_B |f - f_B|^q \, d\mu}{\int_B g^p \, d\mu}\right)^{1/p} \leq C r \left(\frac{\int_B g^p \, d\mu}{\lambda B}\right)^{1/p},$$

where $f_B := \frac{\int_B f \, d\mu}{\mu(B)}$. If $q = 1$, we usually just say $p$-Poincaré inequality.

The Poincaré inequality holds equivalently for all measurable $f$, see [3, Proposition 4.13] and [7, p. 50]. See also [8, Lemma 2.6], [20, Theorem 3.2], [22, Lemma 5.15] [23, Theorems 8.1.49 and 8.1.53] and [25, Theorem 2] for further equivalent versions.

A weight $w$ on $\mathbb{R}^n$ is a nonnegative locally integrable function such that $d\mu = w \, dx$ is a Borel regular measure. If $\mu$ is doubling and supports a $p$-Poincaré inequality on $\mathbb{R}^n$, then $w$ is called a $p$-admissible weight. See Corollary 20.9 in [21] (which is only in the second edition) and Proposition A.17 in [3] for why this is equivalent to other definitions in the literature.

A weight $w$ on $\mathbb{R}^n$ is a (Muckenhoupt) $A_p$-weight if there exists $C > 0$ such that

$$\int_B w \, dx < C \begin{cases} \left(\frac{\int_B w^{1/(1-p)} \, dx}{\inf_B w}\right)^{1-p}, & \text{if } 1 < p < \infty, \\ \essinf_B w, & \text{if } p = 1, \end{cases}$$

for all balls $B \subset \mathbb{R}^n$. $A_p$-weights are $p$-admissible, see Heinonen–Kilpeläinen–Martio [21, Theorem 15.21] (for $p > 1$) and Björn [13, Theorem 4] (for $p = 1$).
Definition 2.4. Let $\Omega \subset X$ be open. The variational $p$-capacity of $E \subset \Omega$ with respect to $\Omega$ is

$$\text{cap}_{p,\mu}^X(E, \Omega) = \inf_u \int_\Omega g^p\,d\mu,$$

where the infimum is taken over all $u \in N^{1,p}(X)$, such that $u = 1$ in $E$ and $u = 0$ on $X \setminus \Omega$, and all upper gradients $g$ of $u$. We call such a function $u$ admissible for testing $\text{cap}_{p,\mu}^X(E, \Omega)$.

If $d\mu = w\,dx$ on $\mathbb{R}^n$ we also write $\text{cap}_{p,w}^X(E, \Omega) \equiv \text{cap}_{p,\mu}^X(E, \Omega)$. If $\mathbb{R}^n$ is equipped with a $p$-admissible weight $w$, then $\text{cap}_{p,w}^X$ is the usual variational capacity and $N^{1,p}(\mathbb{R}^n)$ and $N^{1,p}(\Omega)$ are the refined Sobolev spaces as in Heinonen–Kilpeläinen–Martio [21, p. 96], see Björn–Björn [4, Theorem 5.1] and [3, Appendix A.2].

Throughout the paper, we write $a \lesssim b$ if there is an implicit constant $C > 0$ such that $a \leq Cb$, where $C$ may depend on fixed parameters such as $X$, $\mu$ and $p$, but is independent of the essential parameters involved in $a$ and $b$. We also write $a \gtrsim b$ if $b \lesssim a$, and $a \simeq b$ if $a \lesssim b \lesssim a$.

3. Bow-ties in metric spaces

In this section we consider the metric space $X = X_+ \cup X_-$, where $X_+ \cap X_- = \{x_0\}$ is a fixed designated point and $X_+ \neq \{x_0\}$ are closed subsets of $X$. Note that $X$ is connected if and only if both $X_+$ and $X_-$ are connected.

The following condition (ii) from Theorem 1.1 will play a key role:

There is $\Lambda$ so that $d(x_+, x_0) + d(x_0, x_-) \leq \Lambda d(x_+, x_-)$ for all $x_+ \in X_+$. (3.1)

Since $X_+ \neq \{x_0\}$, we must have $\Lambda \geq 1$. We let $B_r = B(x_0, r)$ and $B_r^\pm = B_r \cap X_\pm$.

Proposition 3.1. Assume that $X$ is connected and that (3.1) holds. Then the measure $\mu$ is doubling on $X$ if and only if the following conditions hold:

(a) $\mu$ is doubling on $X_+$ and on $X_-;
(b) \mu(B_r) \simeq \mu(B_r^\pm)$ for $0 < r < \min\{\text{diam } X_+, \text{diam } X_-\}$.

Note that $X_+$ and $X_-$ may have very different diameters or may be unbounded. This complicates the use of (b) in the following proof.

Proof. Assume that $\mu$ is doubling on $X$ and let $0 < r < \min\{\text{diam } X_+, \text{diam } X_-\}$. Since $X$ is connected, the doubling condition implies that $\mu(\{x_0\}) = 0$, cf. [3, Corollary 3.9]. Since $X_+$ is connected we can find $\tilde{x} \in X_+$ with $d(\tilde{x}, x_0) = \frac{1}{2}r$. If $y \in X_-$, then by (3.1),

$$d(\tilde{x}, y) \geq \frac{1}{\Lambda}(d(\tilde{x}, x_0) + d(y, x_0)) \geq \frac{r}{2\Lambda}.$$

It thus follows that $\tilde{B} := B(\tilde{x}, r/2\Lambda) \subset B_r^+$. Since $B_r^- \subset 3\Lambda \tilde{B}$, the doubling property on $X$ yields

$$\mu(B_r^-) \leq \mu(3\Lambda \tilde{B}) \lesssim \mu(\tilde{B}) \leq \mu(B_r^+).$$

The reverse inequality is shown in the same way, so (b) holds.

To show (a), take any ball $B = B(x, r)$ with $x \in X_+$ and let $B_x = B \cap X_+$. We can assume that $r \leq \text{diam } X_+$, since otherwise $\mu(2B_x) = \mu(X_+) = \mu(B_x)$. Letting $B' := B(x, r/4\Lambda)$ and using the doubling condition on $X$, we get

$$\mu(2B_x) \leq \mu(2B) \lesssim \mu(B') = \mu(B' \cap X_+) + \mu(B' \cap X_-).$$
If \( B' \cap X_+ = \emptyset \), this immediately implies that \( \mu(2B_+) \lesssim \mu(B_+) \). If \( B' \cap X_- \neq \emptyset \), it follows from (3.1) that \( d(x,x_0) < \frac{1}{4}r \) and so

\[
\mu(B' \cap X_-) \leq \mu(B_{r/2}^-) \lesssim \mu(B_{r/2}^+),
\]

where the last inequality follows from (b) when \( \frac{1}{2}r < \text{diam} \ X_- =: d \), while

\[
\mu(B_{r/2}^+) \leq \mu(X_-) = \mu(2B_{2d}) \lesssim \mu(B_{d/2}) \simeq \mu(B_{r/2}^+ \cap B_{r/2}^-) \leq \mu(B_{r/2}^+) \quad (3.2)
\]
otherwise, by (b) and the doubling property on \( X \). Since \( B_{r/2}^+ \subset B_+ \), this proves the doubling property on \( X_+ \), and the argument for \( X_- \) is similar. Thus, (a) holds.

Conversely, assume that (a) and (b) hold. Again, the connectedness of \( X_\pm \) and the doubling property of \( \mu \) imply that \( \mu(\{x_0\}) = 0 \). Let \( B = B(x,r) \) be a ball in \( X \) and assume without loss of generality that \( x \in X_+ \). If \( d(x,x_0) \leq \frac{1}{4}r \), we see that \( B_{r/2} \subset B \) and \( 2B \subset B_{3r} \), and hence by the doubling property on \( X_+ \) and \( X_- \),

\[
\mu(2B) \leq \mu(B_{r/2}^+) + \mu(B_{3r}^-) \lesssim \mu(B_{r/2}^+) + \mu(B_{r/2}^-) \leq \mu(B).
\]

Assume therefore that \( d(x,x_0) > \frac{1}{4}r \). Using the doubling property on \( X_+ \) we have

\[
\mu(2B) = \mu(2B \cap X_+) + \mu(2B \cap X_-) \lesssim \mu(B \cap X_+) + \mu(2B \cap X_-). \quad (3.3)
\]

If \( 2B \cap X_- = \emptyset \), we are done. Otherwise, (3.1) implies that \( d(x,x_0) < 2\lambda r \) and \( 2B \cap X_- \subset B_{4\lambda r}^- \). Since \( \text{diam} \ X_\pm \geq d(x,x_0) > \frac{1}{4}r \), we see that

\[
\mu(2B \cap X_-) \leq \mu(B_{4\lambda r}^-) \lesssim \mu(B_{r/2}^-) \lesssim \mu(B_{r/2}^+), \quad (3.4)
\]

where the last inequality follows from (b) and from (3.2) with \( B_{d/2} \) replaced by \( B_{r/2}^- \). Inserting (3.4) into (3.3), and noting that

\[
\mu(B_{r/2}^+) \leq \mu(3\lambda B \cap X_+) \lesssim \mu(B \cap X_+) \leq \mu(B),
\]

shows that \( \mu(2B) \lesssim \mu(B) \) and concludes the proof. \( \square \)

We now turn to Theorem 1.1. Observe that

\[
cap_{p,\mu}^X(\{x_0\}, B_r) = \cap_{p,\mu}^{X_+}(\{x_0\}, B_r^+) + \cap_{p,\mu}^{X_-}(\{x_0\}, B_r^-).
\]

It follows from Lemma 4.1 below, that when both \( X_+ \) and \( X_- \) are bounded, condition (iii) in Theorem 1.1 can equivalently be replaced by the condition

\[
cap_{p,\mu}^{X_+}(\{x_0\}, B_r^+) \simeq r^{-p} \mu(B_r^+) \quad \text{for all } 0 < r < \frac{1}{4} \text{diam} \ X_+.
\]

On the other hand, Example 4.2 below shows that this is not the case when (exactly) one of \( X_+ \) and \( X_- \) is bounded.

The following lemma will make it possible to lift the Poincaré inequality from small to large sets when glueing \( X_+ \) and \( X_- \) together.

**Lemma 3.2.** (Björn–Björn [5, Lemma 4.11]) Let \( q \geq 1 \) and \( A, E \subset X \) be such that

\[
\mu(A \cap E) \geq \theta \mu(E)
\]

for some \( \theta > 0 \). Also assume that for some \( M \geq 0 \) and a measurable function \( u \),

\[
\|u - u_A\|_{L^q(A)} \leq M \quad \text{and} \quad \|u - u_E\|_{L^q(E)} \leq M.
\]

Then

\[
\|u - u_{A \cup E}\|_{L^q(A \cup E)} \leq 4(1 + \theta^{-1/q})M.
\]
Proof of Theorem 1.1. Assume that $\mu$ supports a $(q,p)$-Poincaré inequality on $X$ with dilation $\lambda$. We first obtain (ii). Let $x_+ \in X_+$, $r > d := d(x_+, x_-)$ and $B = B(x_+, r)$. If $d(x_+, x_0) \geq \lambda r$, then $u = \chi_{X_+}$ has 0 as an upper gradient in $\lambda B$ (as $X_+$ are closed subsets of $X$) and since $B \cap X_+ \neq \emptyset$, this contradicts the $(q,p)$-Poincaré inequality. Hence $d(x_+, x_0) < \lambda r$. Letting $r \to d$ shows that $d(x_+, x_0) \leq \lambda d$. Similarly, $d(x_0, x_-) \leq \lambda d$, and thus (ii) holds with $\Lambda = 2\lambda$.

By [3, Proposition 4.2], $X$ is connected, and so by Proposition 3.1, $\mu$ is doubling on $X_+$ and on $X_-$, and
\[
\mu(B_r^+) \simeq \mu(B_r^-) \quad \text{for } 0 < r < \min\{\text{diam } X_+, \text{diam } X_-.\}.
\] (3.5)

This also implies that $\mu(\{x_0\}) = 0$, see [3, Corollary 3.9].

We next wish to obtain (i). To do so, let $x_+ \in X_+$, $B = B(x_+, r)$ and $B_+ = B \cap X_+$. We already know that condition (3.1) referred to in (ii) holds with $\Lambda = 2\lambda$. Next, let $u$ be an integrable function on $X_+$ and $g$ be an upper gradient of $u$. We may assume that $u(x_0) = 0$. Extend $u$ and $g$ as 0 on $X_+ \setminus \{x_0\}$. It then follows that $g$ is an upper gradient of $u$ in $X$.

In order to obtain (i) we will consider different cases. We first note that when $\mu(B) \lesssim \mu(B_+)$, simple a application of the triangle inequality implies that
\[
\int_{B_+} |u-u_B|^q \, d\mu \leq 2^n \int_B |u-u_B|^q \, d\mu \leq \int_B |u-u_B|^q \, d\mu.
\] (3.6)

Applying the $(q,p)$-Poincaré inequality on $X$, using that $g = 0$ on $X_-$ and that $\mu(\lambda B_+) \leq \mu(\lambda B)$, we have
\[
\left(\int_{B_+} |u-u_B|^q \, d\mu \right)^{1/q} \lesssim \left(\int_{\lambda B_+} g^p \, d\mu \right)^{1/p} \leq r \left(\int_{\lambda B_+} g^p \, d\mu \right)^{1/p},
\] (3.7)

and combining this with (3.6) yields the desired $(q,p)$-Poincaré inequality for $B_+$.

Next we note that if $\Lambda r \leq d(x_0, x_+)$, then for any $x_- \in X_-$ we have $d(x_-, x_-) \geq r$ (using (ii)), and so $B = B_+$. Thus $\mu(B_+) = \mu(B)$ holds in this case, immediately yielding the $(q,p)$-Poincaré inequality for such balls. Suppose therefore that $\Lambda r > d(x_0, x_+)$. For any $x \in B_r^+$ we have
\[
d(x, x_+), d(x_0, x_+) \leq 2\Lambda r,
\]
showing that $B_r^+ \subset 2\Lambda B_r^-$. A similar estimate shows that $B \subset 2\Lambda B_r^-$. We now consider the following cases.

Case 1. $r \leq \min\{\text{diam } X_+, \text{diam } X_-.\}$. Using the last two inclusions, together with $\mu(B_{r/2}^+) \approx \mu(B_{r/2}^-)$ from (3.5), and the doubling property (on $X$ followed by $X_+$), we obtain that
\[
\mu(B) \leq \mu(2\Lambda B_r^-) \lesssim \mu(B_{r/2}^-) \approx \mu(B_{r/2}^-) \leq \mu(2\Lambda B_r^-) \lesssim \mu(B_r^+).
\]

Thus the $(q,p)$-Poincaré inequality for $B_+$ follows by (3.6) and (3.7) in this case.

Case 2. $d_- := \text{diam } X_- < r \leq \text{diam } X_+$. This time the above two inclusions, the doubling property on $X$, $X_+$ and $X_-$, (3.5) applied with the radius $\frac{1}{2}d_- < \min\{\text{diam } X_+, \text{diam } X_-\}$, and the equality $X_- = B_{2d_-}$ yield
\[
\mu(B) \leq \mu(2\Lambda B_r^+) + \mu(X_-) \lesssim \mu(B_r^+) + \mu(B_{r/2}^-) \approx \mu(B_r^+) \leq \mu(2\Lambda B_r^-) \lesssim \mu(B_r^+),
\]
so the $(q,p)$-Poincaré inequality for $B_+$ follows by (3.6) and (3.7) for this case as well.

Case 3. $d_- := \text{diam } X_- \leq \text{diam } X_+ < r$. Then $B_+ = X_+$ and $X_- = B_{2d_-}$. Thus
\[
\mu(B) \leq \mu(X_+) + \mu(X_-) \lesssim \mu(X_+) + \mu(B_{r/2}^-) \approx \mu(X_+) = \mu(B_+),
\]
so we get the \((q,p)\)-Poincaré inequality for \(B_+\) as before.

**Case 4.** \(d_+ := \text{diam } X_+ < \min\{r, \text{diam } X_-\}\). Let \(\rho = \min\{r, 2d_+\}\). In this case we note that

\[
\mu(B_+) = \mu(X_+) \geq \mu(B_{d_+/2}) \simeq \mu(B_{d_+}) \simeq \mu(B_+).
\]

Since also \(B_+ = X_+ \subset B_\rho\), we can apply the same reasoning as in (3.6)–(3.7), with \(B\) replaced by \(B_\rho\), to obtain the desired \((q,p)\)-Poincaré inequality for \(B_+\) with dilation \(\lambda\), which thus has been shown to hold on \(X_+\). Similarly \(\mu\) supports a \((q,p)\)-Poincaré inequality on \(X_-\), i.e. (i) holds.

To verify (iii), let \(0 < r < \frac{1}{4} \min\{\text{diam } X_+, \text{diam } X_-\}\) and let \(u \in N^{1,p}(X_+)\) be a function admissible for testing \(\text{cap}^{X_+}_{p,\mu}(\{x_0\}, B_\rho^+)\) and such that \(0 \leq u \leq 1\). In particular, \(u(x_0) = 1\) and \(u(x) = 0\) on \(X_+ \setminus B_\rho^+\). Consider the function \(v = 1 - u\), extended by 0 to \(X_-\). Then it is easily verified that \(g\), extended by 0 to \(X_- \setminus \{x_0\}\), is an upper gradient of \(v\). Testing the \(p\)-Poincaré inequality (which follows from the \((q,p)\)-Poincaré inequality and Hölder’s inequality) on \(B_{2r}\) with \(v\), and using (3.5) and the doubling property, shows that

\[
\int_{B_{2r}} |v - v_{B_{2r}}| \, d\mu \lesssim r \mu(B_{2r}) \left(\int_{\lambda B_{2r}} g^p \, d\mu\right)^{1/p} \lesssim r \mu(B_\rho^+)^{1 - 1/p} \left(\int_{B_\rho^+} g^p \, d\mu\right)^{1/p}.
\]

Now, depending on whether \(v_{B_{2r}} \geq \frac{1}{2}\) or \(v_{B_{2r}} \leq \frac{1}{2}\), the left-hand side is estimated as

\[
\int_{B_{2r}} |v - v_{B_{2r}}| \, d\mu \geq \frac{1}{2} \mu(B_\rho^+) \simeq \mu(B_\rho^+),
\]

using (3.5), or as

\[
\int_{B_{2r}} |v - v_{B_{2r}}| \, d\mu \geq \frac{1}{2} \mu(B_{2r} \setminus B_\rho^+) \simeq \mu(B_\rho^+),
\]

using the doubling property and the connectedness of \(X_+\), cf. [3, Lemma 3.7].

Taking infimum over all such \(u\) and \(g\) shows the lower bound for \(\text{cap}^{X_+}_{p,\mu}\) in (iii), while \(\text{cap}^{X_-}_{p,\mu}\) is treated similarly. The corresponding upper bounds follow from Lemma 4.1 below.

Conversely, assume that (i)–(iii) hold. Then \(X_+\) and \(X_-\) (and so \(X\)) are connected, by [3, Proposition 4.2]. As (ii) holds, \(\mu\) is doubling also on \(X_+\) and \(X_-\) and (3.5) holds, by Proposition 3.1. Let \(B = B(x,r)\) be a ball and \(B_\pm = B \cap X_\pm\).

Let \(u\) be an integrable function on \(X\) with upper gradient \(g\). Without loss of generality we may assume that \(u(x_0) = 0\). Let \(\lambda\) be a common dilation constant for the \(p\)-Poincaré inequalities on \(X_+\) and \(X_-\).

**Case 1.** \(B_+ = \emptyset\) or \(B_- = \emptyset\). The cases are similar so we may assume the latter, i.e. \(B \subset X_+\), in which case

\[
\left(\int_B |u - u_B|^q \, d\mu\right)^{1/q} \lesssim r \left(\int_{\lambda B_+} g^p \, d\mu\right)^{1/p} \leq \frac{r}{\mu(\lambda B_+)^{1/p}} \left(\int_{\lambda B_+} g^p \, d\mu\right)^{1/p}.
\]

Since \(\mu(AB) \lesssim \mu(B) \leq \mu(\lambda B_+)\), this concludes the proof of the Poincaré inequality for the ball \(B\).

**Case 2a.** \(x = x_0\) and

\[r < r_0 := \frac{1}{4} \min\{\text{diam } X_+, \text{diam } X_-\}\]

(Here \(r_0 = \infty\) is allowed.)
Assumption (iii) and Maz’ya’s inequality (see [3, Theorem 6.21]) for $\frac{1}{n}B_z$, applied in $X_+$ and $X_-$ separately, together with the doubling condition on $X_\pm$, yield
\[
\left( \int_{B_+} |u|^q \, d\mu \right)^{1/q} \leq \left( \frac{1}{\text{cap}_{p,\mu}(\{x_0\}, B_+)} \int_{\lambda B_+} g^p \, d\mu \right)^{1/p} \leq r \left( \int_{\lambda B_+} g^p \, d\mu \right)^{1/p}
\]
and, analogously,
\[
\left( \int_{B_-} |u|^q \, d\mu \right)^{1/q} \leq r \left( \int_{\lambda B_-} g^p \, d\mu \right)^{1/p}.
\]
Put together we have, using also the triangle inequality, that
\[
\int_B |u - u_B|^q \, d\mu \leq 2^q \int_B |u|^q \, d\mu \leq \int_B |u|^q \, d\mu + \int_B |u|^{2q} \, d\mu \leq \int_B |u|^q \, d\mu + \int_B |u|^q \, d\mu 
\]
\[
\leq r^q \left( \int_{\lambda B_+} g^p \, d\mu \right)^{q/p} + r^q \left( \int_{\lambda B_-} g^p \, d\mu \right)^{q/p}.
\]
where in the last step we used that $\mu(\lambda B_\pm) \approx \mu(B_\pm) \approx \mu(B)$, by (3.5).

Case 2b. \( x = x_0 \) and \( r \geq r_0 \). Then at least one of $X_+$ and $X_-$ is bounded and we may assume that $4r_0 = \text{diam} X_- \leq \text{diam} X_+$. Let $B' = B_{r_0/2}$ and $B'_+ = B' \cap X_+$. By (i) and case 2a, we already know that the \((q,p)\)-Poincaré inequality holds for $B_\pm$ and $B'$.

Let $A = B' \cup B_-$. As $\mu(B' \cap B_-) = \mu(B')$, by (3.5), it follows from Lemma 3.2 that
\[
\int_A |u - u_A|^q \, d\mu \leq \int_{B'} |u - u_B|^q \, d\mu + \int_{B_-} |u - u_{B_-}|^q \, d\mu.
\]
Similarly, $B = A \cup B_+$ and
\[
\mu(A \cap B_+) = \mu(B'_+) \approx \mu(X_+) \geq \mu(B_+),
\]
since $X_+ = 5B'_+$ is bounded. A second application of Lemma 3.2 then shows that
\[
\int_B |u - u_B|^q \, d\mu \leq \int_A |u - u_A|^q \, d\mu + \int_{B_+} |u - u_{B_+}|^q \, d\mu \leq \int_{B'} |u - u_{B'}|^q \, d\mu + \int_{B_-} |u - u_{B_-}|^q \, d\mu.
\]
The \((q,p)\)-Poincaré inequality for $B'$ (obtained in case 2a), together with the doubling property of $\mu$, yields
\[
\left( \int_{B'} |u - u_{B'}|^q \, d\mu \right)^{1/q} \leq r_0 \mu(B')^{1/q-1/p} \left( \int_{\lambda B'} g^p \, d\mu \right)^{1/p}.
\]
Similarly, using the \((q,p)\)-Poincaré inequality for $B_\pm$ (from (i)),
\[
\left( \int_{B_+} |u - u_{B_+}|^q \, d\mu \right)^{1/q} \leq r_0 \mu(B_+)^{1/q-1/p} \left( \int_{\lambda B_+} g^p \, d\mu \right)^{1/p}
\]
and
\[
\left( \int_{B_-} |u - u_{B_-}|^q \, d\mu \right)^{1/q} \leq r_0 \mu(B_-)^{1/q-1/p} \left( \int_{\lambda B_-} g^p \, d\mu \right)^{1/p}.
\]
Now, by the doubling property of $\mu$, the boundedness of $X_{+}$ and (3.5),

$$\mu(B') \simeq \mu(B'_{+}) \simeq \mu(B_{+}) \simeq \mu(B'_{-}) \lesssim \mu(B_{-}) \simeq \mu(B).$$

Inserting these estimates into (3.8) and dividing by $\mu(B) \simeq \mu(\lambda B)$ shows the $(q, p)$-Poincaré inequality for $B$ when $1/q - 1/p \geq 0$, i.e. for $q \leq p$.

When $q > p$, the last inequality is not enough to conclude the $(q, p)$-Poincaré inequality for $B$. However, in this case, the $(q, p)$-Poincaré inequality on $X_{-}$ and [3, Proposition 4.20] imply that

$$\frac{\mu(B')}{\mu(B_{-})} \gtrsim \left( \frac{r_0}{r} \right)^{q/(q-p)}$$

and hence in (3.8) we eventually get

$$r_0 \mu(B')^{1/q-1/p} \lesssim r \mu(B_{-})^{1/q-1/p} \simeq r \mu(B)^{1/q-1/p}.$$

**Case 3.** Both $B_{+}$ and $B_{-}$ are nonempty and $x \neq x_0$. Then $d(x, x_0) < \lambda r$, by (3.1) (with $y \in B_{+}$ if $x \in B_{+}$). Hence

$$B \subset B'' := \lambda B'' \subset \lambda' \lambda B,$$

where $\lambda' = \Lambda + \lambda(\Lambda + 1)$. Therefore, using the triangle inequality and case 2, applied to $B''$, we have

$$\int_{B'} |u - u_B|^q \, d\mu \leq 2^q \int_{B} |u - u_{B'}|^q \, d\mu \lesssim \int_{B''} |u - u_{B''}|^q \, d\mu \lesssim r \left( \int_{\lambda B''} g^p \, d\mu \right)^{q/p},$$

which shows the $(q, p)$-Poincaré inequality for $B$, with dilation constant $\lambda'$.

**Remark 3.3.** It follows from the proof above that if $\mu$ supports a $(q, p)$-Poincaré inequality on $X$ with dilation $\lambda$ in Theorem 1.1, then the $(q, p)$-Poincaré inequalities on $X_{+}$ also hold with dilation $\lambda$. The converse is not true, as can be seen by letting $X_{+} = \{(a, t, 0) : t \in \mathbb{R}\}$ and $X_{-} = \{t, (a, t) : t \in \mathbb{R}\}$ with small $a \neq 0$, since then there are disconnected balls in $X = X_{+} \cup X_{-}$.

If $q > p$ then the $(q, p)$-Poincaré inequality implies that $\mu$ is doubling, by Theorem 1 in Alvarado–Hajlasz [1]. Theorem 1.1 and Proposition 3.1 (and the fact that only connected spaces can support Poincaré inequalities) therefore give the following characterization of the $(q, p)$-Poincaré inequality, without presupposing that $\mu$ is doubling.

**Corollary 3.4.** If $q > p$, then $\mu$ supports a $(q, p)$-Poincaré inequality on $X$ if and only if conditions (i)–(iii) in Theorem 1.1 and (3.5) hold.

4. **The capacity condition (1.5) and Theorem 1.3**

In this section we study the capacity condition appearing in Theorem 1.1 (iii). We will consider it within the metric space $X$, but the results readily apply to $X_{\pm}$ in Section 3 as well. As before $x_0 \in X$ is a designated point and $B_r = B(x_0, r)$. 
Lemma 4.1. Assume that \( \mu \) is doubling on \( X \). Let \( 0 < R_0 \leq \infty \) and consider the following statements:

(a) \( \text{cap}^X_{p,\mu}(x_0, B_r) \simeq r^{-p} \mu(B_r) \) for all \( 0 < r < R_0 \);
(b) \( \text{cap}^\infty_{p,\mu}(x_0, B_r) \simeq r^{-p} \mu(B_r) \) for all \( 0 < r < R_0 \);
(c) \( \text{cap}^X_{p,\mu}(B_\rho, B_r) \simeq r^{-p} \mu(B_r) \) for all \( 0 < 2p < r < R_0 \);
(d) \( \text{cap}^\infty_{p,\mu}(B_\rho, B_r) \simeq r^{-p} \mu(B_r) \) for all \( 0 < p < r < R_0 \).

Then (a) \( \iff \) (b) \( \iff \) (c) \( \iff \) (d).

If there is a locally compact open set \( G \ni x_0 \), then (a)–(d) are equivalent.

If \( \mu \) supports a \( p \)-Poincaré inequality on \( X \), \( 0 < R_1, R_2 < \infty \) and \( R_1, R_2 \leq \frac{1}{7} \text{diam } X \) then each of the conditions (a)–(d) with \( R_0 = R_1 \) is equivalent to the same condition with \( R_0 = R_2 \).

Proof. Clearly (a) \( \Rightarrow \) (b), while the implications (b) \( \Rightarrow \) (d) and (c) \( \Rightarrow \) (d) follow directly from the monotonicity of \( \text{cap}^X_{p,\mu} \). The converse implications (b) \( \Rightarrow \) (a) and (d) \( \Rightarrow \) (c) follow by testing the capacity with \( u(x) = \min \{1, 2(1 - \text{dist}(x, x_0)/r)_{+}\} \).

Next, assume that there is a locally compact open set \( G \ni x_0 \). Theorems 1.5 and 1.9 in Eriksson-Bique–Sburlan [18], applied to a compact \( \overline{B} := B(x_0, \rho_0) \subset G \), then imply that Lipschitz functions are dense in \( N^{1,p} \). (For this, note that the doubling property of \( \mu \) implies that \( \overline{B} \) has finite Hausdorff dimension.) Theorem 5.29 in [3] then shows that all functions in \( N^{1,p} \) are quasicontinuous. In particular, if \( u \in N^{1,p}(X) \), then \( u \) is quasicontinuous in \( \overline{B} \). The proof of [3, Theorem 6.19 (vii)] then implies that

\[
\text{cap}^X_{p,\mu}(x_0, B_r) = \inf_{0 < \rho < r} \text{cap}^X_{p,\mu}(B_\rho, B_r).
\]

and hence (d) \( \Rightarrow \) (b).

Finally, the last part follows directly from Lemma 5.5 in Björn–MacManus–Shanmugalingam [15] (or [3, Lemma 11.22]). The proofs therein only require that \( \mu \) is doubling and supports a \( p \)-Poincaré inequality.

We are now ready to prove Theorem 1.3.

Proof of Theorem 1.3. If \( p > Q_{x_0, R_0} \), Proposition 6.1 (if \( R_0 = \infty \)) and Theorem 1.1 (if \( R_0 < \infty \)) in [9], together with the monotonicity of \( \text{cap}^X_{p,\mu} \), show that (1.5) holds.

Conversely, assume that (1.5) holds and let \( \tilde{X} \) be the completion of \( X \). The metric \( d \) extends directly to \( \tilde{X} \) and as in Björn–Björn [7] (see especially the corrigendum) we obtain a Borel regular measure \( \tilde{\mu} \) on \( \tilde{X} \) such that \( \tilde{\mu}(E \cap X) = \tilde{\mu}(E) \) for every \( \tilde{\mu} \)-measurable set \( E \subset \tilde{X} \).

It follows from Propositions 3.3 and 3.6 in [7] that \( \tilde{\mu} \) is doubling and supports a \( p \)-Poincaré inequality on \( \tilde{X} \). It is easy to see that

\[
\text{cap}^{\tilde{X}}_{p,\tilde{\mu}}(\tilde{B}_\rho, \tilde{B}_r) \geq \text{cap}^X_{p,\mu}(B_\rho, B_r),
\]

where \( \tilde{B}_r \) is the ball in \( \tilde{X} \) centred at \( x_0 \) and with radius \( r \). Thus, (1.5) holds with \( X \) replaced by \( \tilde{X} \). By [3, Proposition 3.1], \( \tilde{X} \) is locally compact, and thus condition (a) in Lemma 4.1 holds with \( X \) replaced by \( \tilde{X} \).

First, assume that \( R_0 = \frac{1}{7} \text{diam } X \) and create a new metric space \( X_0 = \tilde{X}_+ \cup \tilde{X}_- \) made of two copies \( \tilde{X}_\pm \) of \( \tilde{X} \) such that \( \tilde{X}_+ \cap \tilde{X}_- = \{x_0\} \) and \( d_{X_0}(x_+, x_-) = d(x_+, x_0) + d(x_0, x_-) \) if \( x_\pm \in \tilde{X}_\pm \). The measure \( \tilde{\mu} \) extends in an obvious way to \( X_0 \). Since (a) in Lemma 4.1 holds for \( \tilde{X} \), and \( X \) (and thus also \( \tilde{X} \)) is connected by [3, Proposition 4.2], it follows from Proposition 3.1 and Theorem 1.1 (with \( q = 1 \)) that \( \tilde{\mu} \) is doubling and supports a \( p \)-Poincaré inequality on \( X_0 \). (Note that we only use (i)–(iii) of Theorem 1.1, without (iii').) By
Keith-Zhong [26, Theorem 1.0.1], there is \( t < p \) such that \( \hat{\mu} \) supports a \( t \)-Poincaré inequality on \( X_0 \). Hence, by Theorem 1.1 again,
\[
\text{cap}^{G}_{t,\hat{\mu}}(\hat{B}_p, \hat{B}_r) \geq \text{cap}^{\hat{X}}_{t,\hat{\mu}}(\{x_0\}, \hat{B}_r) \simeq r^{-t} \hat{\mu}(\hat{B}_r) \quad \text{for } 0 < \rho < r < \frac{1}{t} \text{diam } \hat{X}.
\]

Testing \( \text{cap}^{G}_{t,\hat{\mu}}(\hat{B}_p, \hat{B}_r) \) with \( u(x) = \min\{1, (2 - \text{dist}(x, x_0)/\rho)_+\} \) shows that
\[
\text{cap}^{\hat{X}}_{t,\hat{\mu}}(\hat{B}_p, \hat{B}_r) \lesssim \frac{\hat{\mu}(\hat{B}_p)}{\rho^t} \quad \text{if } 0 < \rho \leq \frac{1}{t} r.
\]

Since \( \mu(B_r) = \hat{\mu}(\hat{B}_r) \) for all \( r > 0 \), comparing the last two estimates (and using that \( \mu(B_p) \simeq \mu(B_r) \) when \( \frac{1}{2} r < \rho < r \)) implies that \( p > t \geq Q_{x_0, R_0}^{\hat{X}, \hat{\mu}} = Q_{x_0, R_0}^{\hat{X}, \mu} \).

Next, assume that \( R_0 < \frac{1}{t} \text{diam } X \). In particular, \( R_0 < \infty \). If \( X \) is bounded, then the equivalence between (1.5) and \( p > Q_{x_0, R_0}^{\hat{X}, \hat{\mu}} \) follows from the case \( R_0 = \frac{1}{t} \text{diam } X \), together with the last part of Lemma 4.1.

If \( X \) is unbounded, then there is a bounded open connected set \( V \supset \hat{B}_{5R_0} \) in \( \hat{X} \) (since \( \hat{X} \) is quasiconvex, by e.g. [3, Theorem 4.32]). By Rajala [30, Theorem 1.1], there is a uniform domain \( G \) such that \( \hat{B}_{1R_0} \subset G \subset V \). By Björn–Shanmugalingam [16, Lemmas 2.5, 4.2 and Theorem 4.4], \( \hat{\mu} \) is doubling and supports a \( p \)-Poincaré inequality on \( G \). We refer to [16] or [30] for the definition of uniform domains.

If \( 0 < \rho < r < R_0 \), then (1.5) implies that
\[
\text{cap}^{G}_{p,\hat{\mu}}(\hat{B}_p, \hat{B}_r) = \text{cap}^{\hat{X}}_{p,\hat{\mu}}(\hat{B}_p, \hat{B}_r) \geq \text{cap}^{X}_{\rho,\mu}(B_p, B_r) \gtrsim r^{-p} \mu(B_r) = r^{-p} \hat{\mu}(\hat{B}_r),
\]
and so (1.5) holds with \( X \) replaced by \( G \). As \( R_0 \leq \frac{1}{t} \text{diam } G \), an application of the already settled bounded case to \( G \) instead of \( X \), shows that \( p > Q_{x_0, R_0}^{G, \hat{\mu}} = Q_{x_0, R_0}^{X, \mu} \).

The last part, concerning (1.6), follows in all cases directly from Lemma 4.1.

The following example shows that the range of \( p \) in Theorem 1.3 can differ between finite and infinite \( R_0 \).

**Example 4.2.** Let \( n \geq 2 \), \( -n < \alpha < 0 \) and \( d\mu = w \, dx \) on \( \mathbb{R}^n \), where
\[
w(x) = \begin{cases} |x|^\alpha, & \text{if } |x| \leq 1, \\ 1, & \text{if } |x| \geq 1, \end{cases}
\]
which is easily verified to be an \( A_\alpha \) weight (cf. Proposition 7.2) and thus 1-admissible. It is also rather straightforward to show
\[
Q_{0, R_0}^{\mathbb{R}^n, \mu} = \begin{cases} n + \alpha & \text{if } R_0 < \infty, \\ n & \text{if } R_0 = \infty. \end{cases}
\]

Since \( n + \alpha < n \), this shows that the range of \( p \) in Theorem 1.3 can be considerably larger for \( R_0 < \infty \) than for \( R_0 = \infty \).

### 5. Bow-ties in \( \mathbb{R}^n \) with \( T \)-invariant weights

From now on we consider bow-ties \( X_{\mathbb{R}^n} \) in \( \mathbb{R}^n \), \( n \geq 1 \), as in Theorem 1.2, i.e.
\[
X_{\mathbb{R}^n} = X_{\mathbb{R}^n}^+ \cup X_{\mathbb{R}^n}^-,
\]
where \( X_{\mathbb{R}^n}^\pm = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \pm x_j \geq 0, \ j = 1, \ldots, n\} \),
are the positive and negative hyperquadrants in \( \mathbb{R}^n \). We equip \( \mathbb{R}^n \) with a Borel regular measure \( d\mu = w \, dx \), where \( w \) is a weight on \( \mathbb{R}^n \). The fixed point will be the origin \( x_0 = 0 \).
In this section we deduce results valid for \( T \)-invariant weights, i.e. weights such that \( w \circ T = w \), where \( T : \mathbb{R}^n \to \mathbb{R}^n \) is given by \( T(x_1, \ldots, x_n) = (|x_1|, \ldots, |x_n|) \). We will also say that \( \mu \) is \( T \)-invariant if \( d\mu = w \, dx \), where \( w \) is \( T \)-invariant. In the next section we will turn to results for radial weights.

**Lemma 5.1.** If \( g : X^+_R \to [0, \infty) \) is an upper gradient of \( u : X^+_R \to [-\infty, \infty] \), then \( \hat{\gamma} = g \circ T \) is an upper gradient of \( \hat{u} = u \circ T \) in \( \mathbb{R}^n \).

**Proof.** Observe first that as \( g \) is a Borel function so is \( \hat{g} \). Let \( \gamma : [0, l_1] \to \mathbb{R}^n \) be a curve in \( \mathbb{R}^n \) and let \( \hat{\gamma} = T \circ \gamma \), which is a curve in \( X^+_R \). One can prove that \( \hat{\gamma} \) is arc length parameterized, but as we will not need this we omit such a proof. It is enough that the arc length of every subsegment of \( \gamma \) is not less than the arc length of the corresponding subsegment in \( \hat{\gamma} \), which follows directly from the 1-Lipschitzness of \( T \). Hence,

\[
|\hat{u}(\gamma(0)) - \hat{u}(\gamma(l_1))| = |u(\hat{\gamma}(0)) - u(\hat{\gamma}(l_1))| \leq \int_{\hat{\gamma}} g \, ds \leq \int_{\gamma} \hat{g} \, ds,
\]

showing that \( \hat{g} \) is an upper gradient of \( \hat{u} \).

**Lemma 5.2.** Assume that \( \mu \) is \( T \)-invariant on \( \mathbb{R}^n \). If \( r > 0 \), then

\[
cap_{p,\mu}(\{0\}, B_r) = \inf_u \int_{B_r} g^p \, d\mu,
\]

where the infimum is taken over all \( T \)-invariant functions \( u \) such that \( u(0) = 1 \) and \( u = 0 \) on \( \mathbb{R}^n \setminus B_r \), and all \( T \)-invariant upper gradients \( g \) of \( u \).

**Remark 5.3.** Corresponding results for the bow-tie \( X^+_R \) and for the hyperquad-

...rants \( X^+_R \) are proved in the same way. Similar identities hold also if \( \{0\} \) is replaced by \( B_\rho \) or \( \overline{B}_\rho \), where \( 0 < \rho < r \).

**Proof.** The statement follows from Definition 2.4 except for the assertion that it suffices to consider only \( T \)-invariant \( u \) and \( g \). To this end, let \( \varepsilon > 0 \) and let \( u \) be a function admissible for testing \( \cap_{p,\mu}(\{0\}, B_r) \), with an upper gradient \( g \) such that

\[
cap_{p,\mu}(\{0\}, B_r) + \varepsilon > \int_{B_r} g^p \, d\mu = \sum_{j=1}^{2^n} \int_{B_{r} \cap X_j} g^p \, d\mu,
\]

where \( X_1, \ldots, X_{2^n} \) are the \( 2^n \) closed coordinate hyperquad-

...rants in \( \mathbb{R}^n \).

There is thus \( k \) such that

\[
\int_{B_r \cap X_k} g^p \, d\mu \leq 2^{-n} \int_{B_r} g^p \, d\mu.
\]

Without loss of generality, \( X_k = X^+_R \). Next, \( v := u \circ T \) is also admissible for testing \( \cap_{p,\mu}(\{0\}, B_r) \) and \( \hat{g} := g \circ T \) is an upper gradient of \( v \), by Lemma 5.1. Since \( w \) is \( T \)-invariant, we get

\[
\int_{B_r} \hat{g}^p \, d\mu = 2^n \int_{B_r \cap X^+_R} g^p \, d\mu \leq \int_{B_r} g^p \, d\mu < \cap_{p,\mu}(\{0\}, B_r) + \varepsilon,
\]

and the conclusion follows after letting \( \varepsilon \to 0 \).

**Corollary 5.4.** Assume that \( \mu \) is \( T \)-invariant on \( \mathbb{R}^n \). If \( r > 0 \), then

\[
\cap_{p,\mu}(\{0\}, B_r) = 2^n \cap_{p,\mu}(\{0\}, B_r) = 2^{n-1} \cap_{p,\mu}(\{0\}, B_r).
\]
Similar identities hold also if \( \{0\} \) is replaced by \( B_\rho \) or \( \overline{B}_\rho \), where \( 0 < \rho < r \).

**Proof.** By Lemma 5.2 and Remark 5.3 it is enough to test all three capacities with \( T \)-invariant functions \( u \) and \( T \)-invariant upper gradients \( g \). Lemma 5.1 shows that \( T \)-invariant upper gradients of a \( T \)-invariant function are the same with respect to \( R^n \) and \( X^+_R \), and thus also with respect to \( X^+_R \) (after obvious use of restrictions and extensions). Hence the conclusions follow from the identities

\[
\int_{B_r} g^p \, d\mu = 2^n \int_{B_r \cap X^+_R} g^p \, d\mu = 2^{n-1} \int_{B_r \cap X^+_R} g^p \, d\mu \quad \text{if} \ g \text{ is } T \text{-invariant.}
\]

**Proposition 5.5.** Assume that \( \mu \) is \( T \)-invariant on \( R^n \). If \( \mu \) supports a \( (q,p) \)-Poincaré inequality on \( R^n \), then it also supports a \( (q,p) \)-Poincaré inequality on \( X^+_R \) with the same dilation constant \( \lambda \).

Note that we do not assume that \( \mu \) is doubling in this result. Thus, we cannot rely on Theorem 1 in Hajljasz–Koskela [19] (or [3, Proposition 4.48]) to know that one may assume that \( \lambda = 1 \).

Example 5.8 below shows that the converse implication is not true, not even if we assume that \( \mu \) is doubling. However, in Theorem 6.2 below we show that the converse implication is true for radial weights \( w \) such that \( d\mu = w \, dx \) is doubling.

**Proof.** Let \( B_r = B \cap X^+_R \), be an arbitrary ball in \( X^+_R \), where \( B = B(x,r) \) and \( x \in X^+_R \). For any integrable function \( f \) on \( X^+_R \) we see that

\[
\int_{B_r} f \, d\mu \leq \int_B (f \circ T) \, d\mu \leq 2^n \int_{B_r} f \, d\mu. \tag{5.1}
\]

In particular

\[
\mu(B_r) \simeq \mu(B). \tag{5.2}
\]

Let \( u \) be an integrable function on \( X^+_R \) with an upper gradient \( g \). Then \( \tilde{g} = g \circ T \) is an upper gradient of \( \tilde{u} = u \circ T \) by Lemma 5.1. Using the triangle inequality, (5.2) and (5.1), we get that

\[
\left( \int_{B_r} \left| u - u_{B_r} \right|^q \, d\mu \right)^{1/q} \leq 2 \left( \int_{B_r} \left| u - \tilde{u}_{B} \right|^q \, d\mu \right)^{1/q} \simeq \left( \int_{B_r} \left| \tilde{u} - \tilde{u}_{B} \right|^q \, d\mu \right)^{1/q} \leq r \left( \int_B \tilde{g}^p \, d\mu \right)^{1/p} \leq 2^n r \left( \int_{\lambda B_{r/2}} \tilde{g}^p \, d\mu \right)^{1/p}.
\]

**Lemma 5.6.** Assume that \( \mu \) is \( T \)-invariant on \( R^n \). Then \( \mu \) is doubling on \( X^+_R \) if and only if it is doubling on \( X^+_R \), which in turn happens if and only if it is doubling on \( X^+_R \).

**Proof.** As \( \mu(B(x,r)) = \mu(B(T(x),r)) \) this follows directly from (5.2) deduced in the proof of Proposition 5.5.

**Proposition 5.7.** Assume that \( \mu \) is a doubling \( T \)-invariant measure on \( R^n \). Then \( \mu \) supports a \( (q,p) \)-Poincaré inequality on \( X^+_R \) if and only if it supports a \( (q,p) \)-Poincaré inequality on \( X^+_R \) and

\[
\cap_M X^+_R (\{0\}, B_r) \simeq r^{-p} \mu(B_r) \quad \text{for all } r > 0. \tag{5.3}
\]

**Proof.** This is a direct consequence of Theorem 1.1.
By [3, Proposition 4.48] the dilation for the \((q, p)\)-Poincaré inequality on \(X_{R^n}\) can be chosen to satisfy \(\lambda \leq \sqrt{2}\), while on \(X_{R^n}^+\) it can be required to be 1. Lemma 5.6 shows that the doubling condition for \(\mu\) holds equivalently on \(R^n\), \(X_{R^n}\), and \(X_{R^n}^+\). Similarly, by Corollary 5.4, the capacity condition (5.3) holds simultaneously for \(cap^{X_{R^n}}_{p, \mu}, cap^{X_{R^n}^+}_{p, \mu}\), and \(cap^{R^n}_{p, \mu}\).

Next, we give an example showing that the converse implication in Proposition 5.5 can fail for doubling measures given by \(T\)-invariant weights.

**Example 5.8.** Let \(1 \leq p < \infty\) and \(n \geq 2\). We construct a \(T\)-invariant weight on \(R^n\) which is \(p\)-admissible on \(X_{R^n}^+\), but not on \(R^n\). In particular, it shows that Theorem 6.2 below does not hold for \(T\)-invariant weights, and it is essential that we only consider radial weights therein.

Let \(du = w \, dx\), where \(w(x_1, x_2, \ldots, x_n) = |x_1|^\alpha\) with \(\alpha > p - 1\), and let \(dv(x) = |x|^\alpha \, dx\) be a measure on \(Y = [0, \infty)\).

Then \(\mu\) is doubling on \(R^n\) and \(\nu\) on \(Y\). By Chua–Wheeden [17, Theorem 1.4], \(\nu\) supports a 1-Poincaré inequality for each interval \((a, b) \subset Y\) with \(\lambda = 1\) and the optimal constant

\[
C = \frac{4}{(b - a)\nu(a, b)} \frac{\|\nu((a, x))\nu((x, b))\|}{\|x^\alpha\|_{L^\infty(a, b)}} \leq \frac{4}{(b - a)} \frac{\|\nu((a, x))\|}{\|x^\alpha\|_{L^\infty(a, b)}} 
\]

where \(\xi \in (a, x)\) comes from the mean-value theorem. As this holds for all intervals \((a, b) \subset Y\), \(\nu\) supports a 1-Poincaré inequality on \(Y\) with constant 4. It then follows from Björn–Björn [6, Theorem 3 and Remark 4], that \(\mu\) supports a 1-Poincaré inequality on \(X_{R^n} = Y^n\).

To show that \(\mu\) is not \(p\)-admissible on \(R^n\), let \(K = \{0\} \times [0, 1]^{n-1}\) and \(u_t(x) = (1 - t \text{ dist}(x, K))_{+}, t \geq 1\). It then follows that the Sobolev capacity (see [3] or [21])

\[
C_{R^n, \mu}(K) \leq \|u_t\|_{W^{1,p}(R^n, \mu)} \leq \mu(\text{supp} \, u_t) + \int_{\text{supp} \, u_t} t^p \, d\mu \lesssim t^{-\alpha - 1} + t^{p - \alpha - 1},
\]

which tends to 0 as \(t \to \infty\). As sets of capacity zero cannot separate open sets in spaces supporting Poincaré inequalities (see [3, Lemma 4.6]), \(\mu\) does not support any \(p\)-Poincaré inequality on \(R^n\).

### 6. Bow-ties in \(R^n\) with radial weights

We say that a weight \(w\) is a radial weight if there is a function \(w : [0, \infty) \to [0, \infty]\) such that \(w(x) = w(|x|)\). By abuse of notation we denote both functions by \(w\), and treat other radial functions similarly. We will also say that \(\mu\) is radial if \(d\mu = w \, dx\), where \(w\) is radial.

**Lemma 6.1.** Assume that \(\mu\) is radial on \(R^n\). If \(r > 0\), then

\[
\text{cap}_{p, \mu}(\{0\}, B_r) = \inf_u \int_{B_r} g^p \, d\mu,
\]

where the infimum is taken over all radial functions \(u\) such that \(u(0) = 1\) and \(u = 0\) on \(R^n \setminus B_r\), and all radial upper gradients \(g\) of \(u\).

This was shown in the proof of Proposition 10.8 in Björn–Björn–Lehrbäck [9]. (The argument is valid also for \(p = 1\), as well as for \(n = 1\).) The corresponding result is proved in the same way for the bow-tie \(X_{R^n}^+\), as well as for \(X_{R^n}^+\). Similar identities hold also if \(\{0\}\) is replaced by \(B_p\) or \(\overline{B}_p\), where \(0 < p < r\).

For doubling radial weights, Proposition 5.5 admits a converse:
Theorem 6.2. Assume that $\mu$ is a doubling radial measure on $\mathbb{R}^n$. Then the following are equivalent:

(a) $\mu$ supports a $(q,p)$-Poincaré inequality on $X^{R_+}$;
(b) $\mu$ supports a $(q,p)$-Poincaré inequality on $X^{R_+}$ with dilation $\lambda = 1$;
(c) $\mu$ supports a $(q,p)$-Poincaré inequality on $\mathbb{R}^n$;
(d) $\mu$ supports a $(q,p)$-Poincaré inequality on $\mathbb{R}^n$ with dilation $\lambda = 1$.

The equivalences (a)$\iff$(b) and (c)$\iff$(d) are well known and due to Hajłasz–Koskela [19, Theorem 1], but it will be convenient to have them included here. Recall that the doubling condition for the radial measure $\mu$ holds equivalently on $\mathbb{R}^n$, $X^{R_+}$ and $X^{R_+}$, by Lemma 5.6.

Theorem 6.2 can directly be combined with Proposition 5.7 to give a characterization for the $(q,p)$-Poincaré inequality on $X^{R_+}$.

Proof. (a)$\iff$(b) and (c)$\iff$(d) As mentioned above, these equivalences follow from Hajłasz–Koskela [19, Theorem 1] (or [3, Proposition 4.48]).

(c)$\implies$(a) This follows from Proposition 5.5.

(b)$\implies$(c) By symmetry, any sector congruent to $X^{R_+}$ in $\mathbb{R}^n$, with vertex at the origin, also supports a $(q,p)$-Poincaré inequality with the same constants. In particular, this is true for any of the $2^n$ coordinate hyperquadrants.

Let $B = B(z,r)$ be a ball in $\mathbb{R}^n$ and $w$ be an integrable function on $\mathbb{R}^n$ with an upper gradient $g$. By symmetry, we can assume that $z_1 = z_2 = \ldots = z_n = 0$. If $r \leq z_1$ then $B \subset X^{R_+}$ and the $(q,p)$-Poincaré inequality holds for $B$. Assume therefore that $r > z_1 = |z|/\sqrt{n}$. In particular,

$$B \subset B(0,(1 + \sqrt{n})r) \subset (1 + 2\sqrt{n})B. \quad (6.1)$$

It therefore suffices to show that the $(q,p)$-Poincaré inequality holds on every ball $B_r$ centred at the origin, cf. (3.6)–(3.7). Consider the coordinate hyperquadrant

$$X' = \{(x_1, \ldots, x_n) : x_1 \leq 0 \text{ and } x_j \geq 0, \ j = 2, \ldots, n\},$$

neighbouring to $X^{R_+}$. Moreover, let

$$X'' = \{(x_1, \ldots, x_n) : x_2 \geq |x_1| \text{ and } x_j \geq 0, \ j = 3, \ldots, n\}$$

be a congruent sector intersecting both $X^{R_+}$ and $X'$. Note that, by the choice of $X''$ and because $w$ is radial,

$$\mu(B_r \cap X^{R_+} \cap X'') = \mu(B_r \cap X' \cap X'') = \frac{1}{2} \mu(B_r \cap X') = \frac{1}{2} \mu(B_r \cap X^{R_+}).$$

By assumption,

$$\left( \int_{B_r \cap X^{R_+}} |u - u_{B_r \cap X^{R_+}}|^q \, d\mu \right)^{1/q} \leq C(r) \left( \int_{B_r \cap X^{R_+}} g^p \, d\mu \right)^{1/p},$$

where $C(r) = C_{r\mu}(B_r \cap X^{R_+})^{1/q - 1/p}$. Similar $(q,p)$-Poincaré inequalities with the same $C(r)$ hold also with $X^{R_+}$ replaced by $X'$ and $X''$. Lemma 3.2, applied first to $B_r \cap X^{R_+}$ and then to $B_r \cap (X^{R_+} \cup X'')$ and $B_r \cap X'$, now implies that

$$\left( \int_{B_r \cap (X^{R_+} \cup X')} |u - u_{B_r \cap (X^{R_+} \cup X')}|^q \, d\mu \right)^{1/q} \leq C(r)(4(1 + 2^{1/q})^2 \left( \int_{B_r \cap (X^{R_+} \cup X')} g^p \, d\mu \right)^{1/p}. $$
Continuing in this way and aggregating further coordinate hyperquadrants using Lemma 3.2, we obtain
\[
\left( \int_{B_r} |u - u_{B_r}|^q d\mu \right)^{1/q} \leq C(r)(4(1 + 2^{1/q})^{2^{\omega + 1}} \left( \int_{B_r} g^p d\mu \right)^{1/p}.
\]
Since \( C(r) \approx r \mu(B_r)^{1/q-1/p} \) and in view of the inclusions (6.1), this implies a \((q,p)\)-Poincaré inequality with a dilation constant at most \( \lambda = 1 + 2^{\sqrt{n}} \) for every ball in \( \mathbb{R}^n \), which concludes the proof.

We are now ready to prove the explicit formula for the capacity of annuli around the origin with respect to radial weights.

**Proof of Proposition 1.4.** By Lemma 6.1, it suffices to test the capacity by radial functions \( u(x) = u(|x|) \) and their radial upper gradients \( g(x) = g(|x|) \). Since \( g \) is radial, it follows that \( \int_0^r g d\rho \geq 1 \).

Consider first the formula (1.7) for \( p = 1 \). Using spherical coordinates,
\[
\int_{B_r} g d\mu = \int_0^r g(\rho) \hat{\omega}(\rho) d\rho \geq \inf_{(0,r)} \hat{\omega}.
\]
Taking infimum over all such \( u \) and their radial upper gradients \( g \) shows one inequality in (1.7) for \( p = 1 \). To prove the reverse inequality, let
\[
A_\varepsilon := \{ \rho \in (0,r) : \hat{\omega}(\rho) \leq \inf_{(0,r)} (\hat{\omega} + \varepsilon) \}, \quad \varepsilon > 0,
\]
and consider the function
\[
u(x) = 1 - \frac{1}{|A_\varepsilon|} \int_{A_\varepsilon} \chi_{A_\varepsilon} d\rho,
\]
where \(|A_\varepsilon|\) stands for the Lebesgue measure of \( A_\varepsilon \). Then \( u \) is Lipschitz on \( \mathbb{R}^n \), admissible in the definition of \( \text{cap}^n_{(\hat{\omega},\mu)}(\{0\}, B_r) \) and has \( g(x) = |A_\varepsilon|^{-1} \chi_{A_\varepsilon}(|x|) \) as an upper gradient. (We may assume that \( \hat{\omega} \) is a Borel function, and thus \( A_\varepsilon \) is a Borel set.) Since \( \hat{\omega} \leq \inf_{(0,r)} (\hat{\omega} + \varepsilon) \) on \( A_\varepsilon \), it then follows that
\[
\text{cap}^n_{(\hat{\omega},\mu)}(\{0\}, B_r) \leq \frac{1}{|A_\varepsilon|} \int_{A_\varepsilon} \hat{\omega}(\rho) d\rho \leq \inf_{(0,r)} (\hat{\omega} + \varepsilon).
\]
Letting \( \varepsilon \to 0 \) concludes the proof of (1.7) for \( p = 1 \).

Next, we turn to the formula (1.7) for \( p > 1 \). Since \( w \) is locally integrable, \( \hat{\omega}(\rho) < \infty \) for a.e. \( \rho \). In this case, we get by Hölder’s inequality (with appropriate interpretation when \( \hat{\omega}(\rho) = 0 \)),
\[
1 \leq \int_0^r g \hat{\omega}^{1/p} d\rho \leq \left( \int_{B_r} g^p d\mu \right)^{1/p} \left( \int_0^r \hat{\omega}^{1/(1-p)} d\rho \right)^{(p-1)/p},
\]
and thus
\[
\left( \int_0^r \hat{\omega}^{1/(1-p)} d\rho \right)^{1-p} \leq \int_{B_r} g^p d\mu.
\]
Taking infimum over all such \( u \) and their radial upper gradients \( g \) shows one inequality in (1.7) for \( p > 1 \).

Conversely, let \( w_\varepsilon(\rho) = w(\rho) + \varepsilon \) and \( \hat{\omega}_\varepsilon(\rho) = \omega_{n-1} w_\varepsilon(\rho) \rho^{n-1} \). It then follows that
\[
0 < \int_0^r \hat{\omega}_\varepsilon^{p/(1-p)} d\rho < \infty.
\]
Let now \( g_\varepsilon(x) = g_\varepsilon(|x|) = c\varepsilon^{1/(1-p)} \), where \( c \) is chosen so that \( \int_0^r g_\varepsilon \, d\rho = 1 \). Also let
\[
u_\varepsilon(x) = 1 - \int_0^{\min\{r,|x|\}} g_\varepsilon(t) \, dt.
\]
We may again assume that \( \tilde{\nu} \) is a Borel function. Then \( g_\varepsilon \) is an upper gradient of \( \nu_\varepsilon \) in \( \mathbb{R}^n \). Considering when equality holds in Hölder’s inequality, we have
\[
1 = \int_0^r g_\varepsilon \tilde{\nu}_\varepsilon^{1/p} \tilde{\nu}_\varepsilon^{-1/p} \, d\rho = \left( \int_{B_r} g_\varepsilon^p \tilde{\nu}_\varepsilon \, dx \right)^{1/p} \left( \int_0^r \tilde{\nu}_\varepsilon^{1/(1-p)} \, d\rho \right)^{(p-1)/p}
\]
which shows that
\[
\text{cap}_{\rho,w}(\{0\}, B_r) \leq \text{cap}_{\rho,w}(\{0\}, B_r) \leq \left( \int_0^r \tilde{\nu}_\varepsilon(\rho)^{1/(1-p)} \, d\rho \right)^{1-p}.
\]
Letting \( \varepsilon \to 0 \) concludes the proof of the other inequality in (1.7) for \( p > 1 \). The last formula in the proposition is shown similarly.

**Theorem 6.3.** Assume that \( \mu \) is a doubling radial measure on \( \mathbb{R}^n \). If \( \mu \) supports a \( p \)-Poincaré inequality on \( X_{\mathbb{R}^n} \), then the weight \( \tilde{\nu}(\rho) := |\rho|^{n-1}w(|\rho|) \) is an \( A_p \)-weight on \( \mathbb{R}^n \).

When \( n = 1 \) this recovers Theorem 2 in Björn–Buckley–Keith [14] in the special case when \( \mu \) is radial.

**Proof.** Proposition 5.7 shows that \( \mu \) supports a \( p \)-Poincaré inequality on \( X_{\mathbb{R}^n} \) and that the capacity condition (5.3) is satisfied. Theorem 6.2 implies that \( \mu \) also supports a \( p \)-Poincaré inequality on \( \mathbb{R}^n \).

Hence, by Proposition 1.4, the capacity condition (5.3) for \( \text{cap}_{\rho,w}(\{0\}, B_r) \) is for \( p > 1 \) equivalent to
\[
\left( \int_0^r \tilde{\nu}_\varepsilon^{1/(1-p)} \, d\rho \right)^{1-p} \lesssim \frac{1}{r^p} \int_{B_r} w \, dx \lesssim \frac{1}{r^p} \int_0^r \tilde{\nu} \, d\rho.
\]
For \( p = 1 \) we instead get by Proposition 1.4 that (5.3) is equivalent to
\[
\text{ess inf}_{(0,r)} \tilde{\nu} \gtrsim \frac{1}{r} \int_0^r \tilde{\nu} \, d\rho.
\]
In both cases, this gives the \( A_p \)-condition for \( \nu \) and intervals in \( \mathbb{R} \) centred at the origin.

If \( I = (t-r, t+r) \) is an interval in \( \mathbb{R} \) such that \( r \geq \frac{1}{4}|t| \), then \( I \subset (-5r, 5r) \) and hence also (when \( p > 1 \))
\[
\frac{1}{|I|} \int_I \tilde{\nu} \, d\rho \left( \int_I \tilde{\nu}_\varepsilon^{1/(1-p)} \, d\rho \right)^{1-p} \lesssim \int_{-5r}^{5r} \tilde{\nu} \, d\rho \left( \int_{-5r}^{5r} \tilde{\nu}_\varepsilon^{1/(1-p)} \, d\rho \right)^{(p-1)} \lesssim (5r)^p.
\]
Thus, the \( A_p \)-condition for \( \nu \) holds for \( I \) when \( p > 1 \), and since
\[
\text{ess inf}_{I} \tilde{\nu} \gtrsim \text{ess inf}_{(0,5r)} \tilde{\nu} \gtrsim \frac{1}{5r} \int_0^{5r} \tilde{\nu} \, d\rho \gtrsim \frac{1}{|I|} \int_I \tilde{\nu} \, d\rho,
\]
also when \( p = 1 \).

Now assume that \( p > 1 \), \( I = (t-r, t+r) \) is an interval in \( \mathbb{R} \) and that \( r \leq \frac{1}{4}|t| \). We can assume that \( t > 0 \). Let \( \varepsilon > 0 \) be arbitrary and consider the function
\[
u(x) = \int_{t-r}^{\min\{t+r,|x|\}} (w(\rho) + \varepsilon)^{1/(1-p)} \, d\rho.
\]

(6.2)
Then \( u \) is Lipschitz on \( \mathbb{R}^n \), vanishes on \( B(0, t - r) \subset \mathbb{R}^n \) and
\[
u \equiv \int_{t-r}^{t+r} (w(\rho) + \varepsilon)^{1/(1-p)} \, d\rho =: a_{\varepsilon,t,r} \quad \text{on } \mathbb{R}^n \setminus B(0, t + r). \tag{6.3}
\]

We may assume that \( w \) is a Borel function. Then clearly,
\[
g = (w(|x|) + \varepsilon)^{1/(1-p)} \chi_{(t-r,t+r)}(|x|)
\]
is an upper gradient of \( u \). We shall apply the \( p \)-Poincaré inequality to \( u \) in the ball \( B(z, 2r) \), where \( z = (t, 0, \ldots, 0) \). Note that either
\[
|u - u_{B(z, 2r)}| = |u_{B(z, 2r)}| \geq \frac{1}{2} a_{\varepsilon,t,r} \quad \text{in } B(z, 2r) \cap B(0, t - r)
\]
or
\[
|u - u_{B(z, 2r)}| = |a_{\varepsilon,t,r} - u_{B(z, 2r)}| \geq \frac{1}{2} a_{\varepsilon,t,r} \quad \text{in } B(z, 2r) \setminus B(0, t + r).
\]

Moreover, because \( r < \frac{1}{4} t \) and \( \mu \) is doubling,
\[
\mu(B(z, 2r) \cap B(0, t - r)) \simeq \mu(B(z, 2r) \setminus B(0, t + r)) \simeq \mu(B(z, 2r)).
\]
The \( p \)-Poincaré inequality on \( B(z, 2r) \), together with the spherical coordinates, then implies that
\[
a_{\varepsilon,t,r}^p \lesssim r^p \int_{B(z, 2r)} g^p \, d\mu = r^p \int_{B(z, 2r)} (w(|x|) + \varepsilon)^{p/(1-p)} \chi_{(t-r,t+r)}(|x|)w(|x|) \, dx
\]
\[
\lesssim r^p \frac{r^{n-1}}{\mu(B(z, 2r))} \int_{t-r}^{t+r} (a_{\varepsilon,t,r}^p) \, dp = r^p \frac{r^{n-1}}{\mu(B(z, 2r))} a_{\varepsilon,t,r}. \tag{6.4}
\]

Since by spherical coordinates,
\[
\mu(B(z, 2r)) \simeq \mu(B(z, r)) \simeq r^{n-1} \int_{t-r}^{t+r} w \, dp, \tag{6.5}
\]
we conclude from (6.3) and (6.4) that
\[
\int_{t-r}^{t+r} w(\rho) \, d\rho \left( \int_{t-r}^{t+r} (w(\rho) + \varepsilon)^{1/(1-p)} \, d\rho \right)^{p-1} \lesssim r^p.
\]

Letting \( \varepsilon \to 0 \) and noting that \( \tilde{w}(\rho) \simeq t^{n-1} w(\rho) \) for all \( \rho \in I \) gives the \( A_\rho \)-condition (2.1) for \( I = (t - r, t + r) \) also when \( r < \frac{1}{4} |t| \), and concludes the proof when \( p > 1 \).

For \( p = 1 \), replace (6.2) by the function
\[
u(x) = \int_0^{|x|} \chi_{A_\varepsilon}(\rho) \, d\rho,
\]
where
\[
A_\varepsilon := \left\{ \rho \in I : w(\rho) \leq \text{ess inf}_I (w + \varepsilon) \right\}.
\]

Since \( \chi_{A_\varepsilon} \) is an upper gradient of \( u \) (because \( w \) can be assumed to be a Borel function), arguments similar to those for \( p > 1 \) lead to
\[
|A_\varepsilon| \lesssim r \int_{B(z, 2r)} \chi_{A_\varepsilon}(|x|)w(|x|) \, dx \lesssim \frac{r^n |A_\varepsilon| \text{ess inf}_I (w + \varepsilon)}{\mu(B(z, 2r))}
\]
Finally, using (6.5), dividing by \( |A_\varepsilon| > 0 \), and letting \( \varepsilon \to 0 \) concludes the proof also for \( p = 1 \). \( \square \)
The following result is the last link in proving Theorem 1.2.

**Theorem 6.4.** Assume that $w$ is radial on $\mathbb{R}^n$. If the weight $\tilde{w}(\rho) := |\rho|^{-n} w(|\rho|)$ is an $A_p$-weight on $\mathbb{R}$, then $w$ is an $A_p$-weight on $\mathbb{R}^n$ and the capacity condition (5.3) holds.

**Proof.** Assume that $\tilde{w}$ is an $A_p$-weight on $\mathbb{R}$ and let $B = B(z, r) \subset \mathbb{R}^n$ be a ball. If $r \leq \frac{1}{4}|z|$ then let $I = ([|z| - r, |z| + r)$ be the corresponding interval in $\mathbb{R}$. Note that $\rho \asymp |z|$ for all $\rho \in I$. Using spherical coordinates we therefore have

$$\int_B w \, dx \lesssim |z|^{1-n} r^{n-1} \int_I \tilde{w}(\rho) \, d\rho \quad (6.6)$$

and similarly (when $p > 1$),

$$\int_B w^{1/(1-p)} \, dx \lesssim |z|^{(n-1)/(p-1)} r^{n-1} \int_I \tilde{w}^{1/(1-p)}(\rho) \, d\rho.$$

Multiplying the last two estimates, cancelling $|z|$ and using the $A_p$-condition for $\tilde{w}$ on $I$ yields

$$\int_B w \, dx \left( \int_B w^{1/(1-p)} \, dx \right)^{p-1} \lesssim r^{p(n-1)p} = r^{np}, \quad (6.7)$$

i.e. the $A_p$-condition for $\tilde{w}$ holds on $B$ when $p > 1$. For $p = 1$ we instead combine (6.6) with the $A_1$-condition for $\tilde{w}$ and with

$$\operatorname{ess inf}_I \tilde{w} \lesssim |z|^{n-1} \operatorname{ess inf}_B w.$$

If $r > \frac{1}{4}|z|$, then $B \subset B(0, 5r)$ and it therefore remains to consider balls centred at the origin. For such balls, spherical coordinates show that (when $p > 1$)

$$\int_{B_r} w^{1/(1-p)} \, dx \simeq \int_0^r w(\rho)^{1/(1-p)} \rho^{n-1} \, d\rho \leq r^{(n-1)p/(p-1)} \int_0^r \tilde{w}(\rho)^{1/(1-p)} \, d\rho,$$

$$\int_{B_r} w \, dx \simeq \int_0^r \tilde{w}(\rho) \, d\rho \quad \text{and} \quad \operatorname{ess inf}_{B_r} \tilde{w} \leq r^{n-1} \operatorname{ess inf}_{B_r} w.$$

This and the $A_p$-condition for $\tilde{w}$ immediately give (6.7), as well as its analogue for $p = 1$, also for $B_r$. Thus $w$ is an $A_p$-weight on $\mathbb{R}^n$.

Finally, it remains to note that by Proposition 1.4, the lower bound in the capacity condition (5.3) is equivalent to the assumed $A_p$-condition for $\tilde{w}$ and $B_r$. The upper bound in (5.3) then follows from Lemma 4.1. \qed

We are now ready to deduce Theorem 1.2.

**Proof of Theorem 1.2.** (b)$\Rightarrow$(c) This follows from Proposition 5.5.

(c)$\Leftrightarrow$(a) This follows from Theorem 1.1 or Proposition 5.7.

(a)$\Rightarrow$(e) This follows from Theorem 6.3.

(e)$\Rightarrow$(d) This follows from Theorem 6.4.

(d)$\Rightarrow$(b) This follows from Heinonen–Kilpeläinen–Martio [21, Theorem 15.21] (for $p > 1$) and Björn [13, Theorem 4] (for $p = 1$). \qed

We end this section with the following result for $p = 1$, which is in contrast to the case $p > 1$ in Theorem 1.3.

**Proposition 6.5.** Assume that $\mu$ is a doubling radial measure on $\mathbb{R}^n$ which supports a 1-Poincaré inequality. Let $0 < R_0 \leq \infty$. Then the following conditions are equivalent:
\(a\) \(\mu(B_{2r}) \geq \frac{\rho}{r}\) for all \(0 < \rho < r < R_0\), i.e. \(Q_{0,R_0}^{\mathbb{R}^n,\mu} = Q_{0,R_0}^{\mathbb{R}^n,\mu} \leq 1\);

\(b\) \(\text{cap}^{\mathbb{R}^n}_{1,\mu}(B_{\rho}, B_{r}) \geq \rho^{-1}\mu(B_{r})\) for all \(0 < \rho < r < R_0\);

\(c\) \(\text{cap}^{\mathbb{R}^n}_{1,\mu}((x_0), B_r) \geq \rho^{-1}\mu(B_r)\) for all \(0 < \rho < R_0\).

Proof. \((b) \Rightarrow (c)\) This follows directly from Lemma 4.1.

\((b) \Rightarrow (a)\) For \(\rho \leq \frac{1}{2} r\), this follows from the simple estimate \(\text{cap}^{\mathbb{R}^n}_{1,\mu}(B_{\rho}, B_r) \lesssim \rho^{-1}\mu(B_{\rho})\), which is obtained by testing the capacity with

\[u(x) = \min\{1, (2 - |x|/\rho)\}.
\]

The monotonicity of \(\mu\) takes care of \(\frac{1}{2} r < \rho < r\).

\((a) \Rightarrow (b)\) Let \(\tilde{w}(t) := w(t)t^{n-1}, 0 < \varepsilon < r\) and

\[u(x) = \min\left(\frac{r + \varepsilon - |x|}{2\varepsilon}, 1\right).
\]

Then, by the doubling property of \(\mu\) and the 1-Poincaré inequality,

\[\mu(B_{r-\varepsilon}) \lesssim \int_{B_{2r}} |u - u_{B_{2r}}| \, d\mu \lesssim r \int_{r-\varepsilon}^{r+\varepsilon} \frac{1}{2\varepsilon} \tilde{w}(t) \, dt.
\]

Letting \(\varepsilon \to 0\) shows that

\[\frac{\mu(B_{r})}{r} \lesssim \tilde{w}(r) \quad \text{for a.e. } r > 0.
\]

On the other hand, by Proposition 1.4 and using \((a)\),

\[\text{cap}^{\mathbb{R}^n}_{1,\mu}(B_{\rho}, B_r) \leq \text{ess inf}_{\rho < \varepsilon < r} \frac{\mu(B_{\rho})}{\varepsilon} \leq \frac{\mu(B_r)}{r} \quad \text{if } 0 < \rho < r. \quad \Box
\]

We are now going to give an example of a radial 1-admissible weight such that the range of \(p\) for which the capacity condition (5.3) holds is considerably smaller than the range for which \(\text{cap}^{\mathbb{R}^n}_{p,\mu}(\{0\}, B_r) > 0\).

Example 6.6. We will follow the construction in Björn–Björn–Lehrbäck [9, Example 3.2]. Let \(\alpha_k = 2^{-2^k}\) and \(\beta_k = \alpha_k^{3/2} = 2^{-3 \cdot 2^{k-1}}, k = 0, 1, 2, \ldots,\) Note that \(\alpha_{k+1} = \alpha_k^2\). Consider the measure \(d\mu = w(|y|) \, dy\) on \(\mathbb{R}^n\), where

\[w(p) = \begin{cases} 
\alpha_{k+1}, & \text{if } \alpha_{k+1} \leq \rho \leq \beta_k, \quad k = 0, 1, 2, \ldots, \\
\rho^2/\alpha_k, & \text{if } \beta_k \leq \rho \leq \alpha_k, \quad k = 0, 1, 2, \ldots, \\
\rho, & \text{if } \rho \geq \frac{1}{2}.
\end{cases}
\]

It follows from Example 3.2 in [9] that \(w\) is 1-admissible, \(Q_{0,R_0}^{\mathbb{R}^n,\mu} = 4\) and

\[\mu(B_r) \gtrsim r^s \quad \text{for all } 0 < r < 1 \quad \text{if and only if } \quad s \geq \frac{\alpha}{2}.
\]

We thus conclude from Theorem 1.3 that the capacity condition (5.3) holds if and only if \(p > 4\). Proposition 8.2 in [9] shows that

\[\text{cap}^{\mathbb{R}^n}_{p,\mu}(\{0\}, B_r) > 0 \quad \text{if } p > \frac{40}{41} \quad \text{and } r > 0,
\]

so for \(p \in \left(\frac{40}{41}, 4\right]\), the weighted bow-tie does not support a \(p\)-Poincaré inequality, even though the gluing point has positive capacity. A straightforward calculation, using the formula in Proposition 1.4, shows that \(\text{cap}^{\mathbb{R}^n}_{p,\mu}(\{0\}, B_r) = 0\).

Given \(b > a > 1\), Example 3.4 in [9] provides similar radial weights, such that

\[Q_{0,R_0}^{\mathbb{R}^n,\mu} = b \quad \text{and hence the capacity condition (5.3) holds exactly when } p > b, \quad \text{while cap}^{\mathbb{R}^n}_{p,\mu}(\{0\}, B_r) > 0 \quad \text{for all } p > a.
\]
7. Logarithmic power weights on $\mathbb{R}^n$

Let from now on $\phi(\rho) = \max\{1, -\log \rho\}$. The following integral estimate is straightforward, but a bit tedious.

**Lemma 7.1.** It is true that

$$
\int_0^r \rho^{a-1} \phi(\rho)^b \, d\rho \simeq \begin{cases} 
\rho^a \phi(\rho)^b, & \text{if } a > 0 \text{ and } b \in \mathbb{R}, \\
\phi(\rho)^{b+1}, & \text{if } a = 0, b < -1 \text{ and } r \leq 1, \\
1 + \log r, & \text{if } a = 0, b < -1 \text{ and } r > 1, \\
\infty, & \text{if } a < 0, \text{ or } a = 0 \text{ and } b \geq -1.
\end{cases}
$$

**Proposition 7.2.** Let $w(|x|) = |x|^\alpha \phi(|x|)^\beta$ be a weight on $\mathbb{R}^n$, where $\alpha > -n$ and $\beta \in \mathbb{R}$. Then $w$ is an $A_1$-weight if and only if $0 < \alpha < 0$ or $\alpha = 0 < \beta$.

**Proof.** It follows directly from the definition that a continuous $A_1$-weight cannot vanish at any point. Hence, $\alpha > 0$ as well as purely logarithmic weights with $\beta < 0 = \alpha$ cannot give $A_1$-weights.

Conversely assume that $\alpha < 0$ or $\alpha = 0 \leq \beta$. Note that $\phi(r') \simeq \phi(r)$ if $r \leq r' \leq 2r$, and thus also $w(r') \simeq w(r)$ if $r \leq r' \leq 2r$. Moreover $w$ is approximately decreasing in the sense that

$$
w(r) \gtrsim w(r') \quad \text{whenever } 0 < r < r'.
$$

Let $B = B(x_0, r)$ be an arbitrary ball. If $r \leq \frac{1}{2}|x_0|$, then since $w$ is approximately decreasing we get that

$$
\int_B w(x) \, dx \lesssim w(|x_0| - r) \simeq w(|x_0| + r) \simeq \inf_{B} w.
$$

On the other hand, if $r > \frac{1}{2}|x_0|$, then $B \subset B_{3r}$. Hence using Lemma 7.1 we get that

$$
\int_B w \, dx \lesssim \frac{1}{r^n} \int_{B_{3r}} w \, dx \simeq \frac{1}{r^n} \int_0^{3r} \rho^{a+n-1} \phi(\rho)^\beta \, d\rho \simeq r^\alpha \phi(3r)^\beta.
$$

Moreover,

$$
\inf_{B} w \geq \inf_{B_{3r}} w \simeq w(3r) \simeq r^\alpha \phi(3r)^\beta,
$$

which shows that the $A_1$-condition is satisfied for all balls.

Using Proposition 7.2 we obtain the following result which is well known when $\beta = 0$, see Heinonen–Kilpeläinen–Martio [21, p. 10].

**Corollary 7.3.** Let $w(|x|) = |x|^\alpha \phi(|x|)^\beta$ be a weight on $\mathbb{R}^n$, $n \geq 2$, where $\alpha > -n$ and $\beta \in \mathbb{R}$. Then $w$ is $1$-admissible.

For $n = 1$ this is false, since any $1$-admissible weight on $\mathbb{R}$ is an $A_1$-weight by Theorem 2 in Björn–Buckley–Keith [14].

**Proof.** For $\alpha > 1 - n$ this follows from [9, Remark 10.6], while for $-n < \alpha \leq 1 - n$ it follows from Proposition 7.2.

We next characterize when these types of weights satisfy $\text{cap}_{p,p}^n(\{0\}, B_r) > 0$ resp. $\text{cap}_{p,p}^n(\{0\}, B_R) \simeq r^{-p}\mu(B_r)$. This again shows that the former condition can hold without the latter one, at least when $p > 1$, cf. Example 6.6.

**Proposition 7.4.** Let $w(|x|) = |x|^\alpha \phi(|x|)^\beta$ be a weight on $\mathbb{R}^n$, $n \geq 1$, with $\alpha > -n$ and $\beta \in \mathbb{R}$. Then $\text{cap}_{p,p}^n(\{0\}, B_r) > 0$ for some, or equivalently all, $r > 0$ if and only if one of the following conditions holds:

[The rest of the text contains additional propositions and theorems related to logarithmic power weights on $\mathbb{R}^n$.]
1. \( \alpha < p - n \),
2. \( \alpha = p - n \) and \( \beta > p - 1 \),
3. \( p = 1 \), \( \alpha = 1 - n \) and \( \beta \geq 0 \).

Moreover, with a fixed \( 0 < R_0 \leq \infty \), we have

\[
cap_{p,\mu}^R(\{0\}, B_r) \gtrsim \frac{\mu(B_r)}{r^p}
\]

for all \( 0 < r < R_0 \)

if and only if (a) or (c) holds.

Proof. Since \( \alpha + n > 0 \), Lemma 7.1 shows that \( r^{-p}\mu(B_r) \simeq r^{\alpha+n-p}\phi(r)^\beta \) for all \( \beta \in \mathbb{R} \) and \( r > 0 \). To estimate \( \cap_{p,\mu}^R(\{0\}, B_r) \) assume first that \( p > 1 \). It follows from Proposition 1.4 and Lemma 7.1 that

\[
\begin{align*}
\cap_{p,\mu}^R(\{0\}, B_r) &\simeq \left( \int_0^r \rho^{(\alpha+n-1)/(1-p)}\phi(p)^{\beta/(1-p)} \, dp \right)^{1-p} \\
&\mathrel{\overset{\text{def}}{=}} \begin{cases} 
  r^{\alpha+n-p}\phi(r)^\beta, & \text{if } \alpha < p - n, \\
  \phi(r)^{\beta+1-p}, & \text{if } \alpha = p - n, \, \beta > p - 1 \text{ and } r \leq 1, \\
  (1 + \log r)^{1-p}, & \text{if } \alpha = p - n, \, \beta > p - 1 \text{ and } r > 1, \\
  0, & \text{otherwise,}
\end{cases}
\end{align*}
\]

which proves the statement when \( p > 1 \).

Assume for the rest of the proof that \( p = 1 \). Proposition 1.4 implies that for all \( r > 0 \),

\[
\cap_{p,\mu}^R(\{0\}, B_r) \simeq \essinf_{0 < \rho < r} \rho^{\alpha+n-1}\phi(r)^\beta.
\]

Thus, \( \cap_{p,\mu}^R(\{0\}, B_r) > 0 \) if and only if either \( \alpha < 1 - n \) or \( \alpha = 1 - n \) and \( \beta \geq 0 \).

In this case also

\[
\cap_{p,\mu}^R(\{0\}, B_r) \simeq r^{\alpha+n-1}\phi(r)^\beta \simeq \frac{\mu(B_r)}{r},
\]

which concludes the proof. \( \square \)

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