Learning computationally efficient dictionaries and their implementation as fast transforms
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Supplementary material

A. Anonymous

1 Projection operator proof

We want to find the projection operator onto the following set:

\[ \mathcal{E} := \{ A \in \mathbb{R}^{n \times n} : \| A \|_0 \leq p, \| A \|_F = 1 \} \]  

with \( p \in \mathbb{N}^* \). We are interested in the projection of some matrix \( S \in \mathbb{R}^{n \times n} \) onto the set \( \mathcal{E} \), namely we want to find \( S^* \) such that:

\[ S^* = P_{\mathcal{E}}(S) = \arg\min_U \{ \| U - S \|_2^2 : U \in \mathcal{E} \} \]  

Proposition 1.1. Projection operator formula.

\[ P_{\mathcal{E}}(S) = B_1(T_p(S)) \]

where

\[ T_p(S) := \arg\min_V \{ \| V - S \|_2^2 : \| V \|_0 \leq p \} \]

and

\[ B_1(R) := \arg\min_W \{ \| W - R \|_2^2 : \| W \|_F = 1 \} \]

Proof. Let us denote by \( P \) the set of indices corresponding to the \( p \) greatest entries (in absolute value) of \( S \), and by \( S_P \) the matrix we get by setting all the other entries of \( S \) to 0. Let us also introduce \( S_{\bar{P}} = S - S_P \), the matrix we get by keeping only the \( n^2 - p \) littlest entries. We have:

\[ \| S \|_F^2 = \| S_P \|_F^2 + \| S_{\bar{P}} \|_F^2 \]  

and

\[ S_P = \arg\min_V \{ \| V - S \|_F^2 : \| V \|_0 \leq p \} = T_p(S) \]  

and

\[ P = \arg\max_J \{ \| S_J \|_F : \text{card}(J) \leq p \} \]

The operator \( B_1 \) is simply the projection onto the Euclidean sphere of radius 1, that is:

\[ B_1(X) = \frac{1}{\| X \|_F} X. \]
Let us now denote by $I$ the support of any $U \in \mathcal{E}$, we have:

$$\|U - S\|_F^2 = \|S_I\|_F^2 + \|U - S_I\|_F^2$$

so the problem can be rewritten:

$$\begin{align*}
(S^*, I^*) = & \arg \min_{(U, I)} \{ \|S_I\|_F^2 + \|U - S_I\|_F^2 : \|U\|_F = 1, \text{card}(I) \leq p \} \\
\end{align*}$$

Isolating $U$ we have:

$$S^* = \arg \min_U \{ \|U - S_I^*\|_F^2 : \|U\|_F \leq q \} = B_1(S_{I^*})$$

with:

$$\begin{align*}
I^* &= \arg \min_U \{ \|S_I\|_F^2 + \inf_U \{ \|U - S_I\|_F^2 : \|U\|_F = 1 \} : \text{card}(I) \leq p \} \\
I^* &= \arg \min_I \{ \|S_I\|_F^2 + \|B_1(S_I) - S_I\|_F^2 : \text{card}(I) \leq p \} \\
I^* &= \arg \min_I \{ \|S_I\|_F^2 + (1 - \frac{1}{\|S_I\|_F^2})^2 \|S_I\|_F^2 : \text{card}(I) \leq p \} \\
I^* &= \arg \max_I \{ \|S_I\|_F : \text{card}(I) \leq p \} \\
I^* &= \text{arg max} \{ \|S_I\|_F : \text{card}(I) \leq p \}
\end{align*}$$

So we have: $S^* = B_1(S_P) = B_1(T_p(S))$.

**2 Lipschitz moduli proof**

Let us look at the Lipschitz moduli of the gradient of the smooth part of the objective:

$$\begin{align*}
\left\| \nabla_{S_I} H(S_I^{i+1} \ldots S_i \ldots S_P, \lambda^I) - \nabla_{S_I} H(S_I^{i+1} \ldots S_2 \ldots S_P, \lambda^I) \right\|_F &= \left\| \lambda^I L^T(\lambda^I L S_1 R - X)R^T - \lambda^I L^T(\lambda^I L S_2 R - X)R^T \right\|_F \\
&= (\lambda^I)^2 \left\| L^T L (S_1 - S_2) R R^T \right\|_F \\
&= (\lambda^I)^2 \left\| (R R^T) \otimes (L^T L) \vec{S}_1 - \vec{S}_2 \right\|_2 \\
&\leq (\lambda^I)^2 \left\| (R R^T) \otimes (L^T L) \right\|_2 \|S_1 - S_2\|_F \\
&= (\lambda^I)^2 \|X\|^2_2 \cdot \|L\|^2_2 \|S_1 - S_2\|_F.
\end{align*}$$

So we can say that the following quantity is a Lipschitz modulus: $L_j(L, R, \lambda^I) = (\lambda^I)^2 \|X\|^2_2 \cdot \|L\|^2_2$. 

\[\square\]