SELF-SIMILAR EQUILIBRIA OF SELF-GRAVITATING, MAGNETIZED, ROTATING ISOThermal SYSTEMS

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ABSTRACT

The self-similar equilibrium models of self-gravitating, rotating isothermal systems are investigated analytically. In these models, the rotation velocity is constant and the density varies as \( f(\theta, \varphi)/r^2 \), where \( r \) and \( \theta \) are the spherical radius and the colatitude, respectively. The nonaxisymmetric solutions contain three free parameters, one of which depends on the rotation velocity. These parameters determine the overall shape of the density distribution. By assuming that the dominant component of the magnetic field is purely toroidal and that the ratio of the purely toroidal magnetic pressure to the gas pressure, \( \alpha \), is spatially constant, the axisymmetric solutions were generalized so that the effect of magnetic field could be studied. We find that the equilibria of axially symmetric systems yield ellipsoids or spheres only when the ratio of rotation velocity to the sound speed is taken to be \((2\alpha)^{1/2}\).

Subject headings: galaxies: structure — hydrodynamics — ISM: structure — MHD — stars: formation — stars: kinematics

1. INTRODUCTION

Understanding the equilibrium structure of self-gravitating isothermal systems is a fundamental issue in astrophysics. Many structures have been identified in star-forming regions, from sheets and filaments to elongated clumps (Kulkarni & Heiles 1988; Jijina, Myers, & Adams 1999). The hydrostatic equilibrium of these structures has been studied previously with the use of some analytical results (Shu 1992; Ostriker 1964). Several authors, however, have investigated the problem using numerical techniques in order to include the effects of rotation and magnetic fields (see, e.g., Fiege & Poduriz 2000; Curry 2000; Tomisaka, Ikeuchi, & Nakamura 1987). Many models of newborn stars are built upon these equilibrium configurations (e.g., Kiguchi et al. 1987). In addition, there is a close analogy between gaseous self-gravitating equilibrium objects and collisionless stellar systems (see, e.g., Binney & Tremaine 1987). Thus the equilibria of isothermal systems can be used for determining the steady-state distribution function of collisionless stellar systems (Toomre 1982; Hayashi, Narita, & Miyama 1982).

With these considerations, recently Medvedev & Narayan (2000) hereafter MN classified and analytically derived self-similar axisymmetric equilibria of a self-gravitating isothermal system. They presented two families of equilibria: (1) cylindrically symmetric solutions, in which all quantities depend on the cylindrical radius only, and (2) axially symmetric solutions, in which the quantities vary as functions of the spherical radius and the colatitude. In fact, these axially symmetric solutions are very interesting. These equilibria form a two-parameter family of solutions. One of the parameters depends on the rotation velocity and the other controls the symmetry of the solution with respect to the equatorial plane. Nevertheless, this study, like most previous work, focused on the case of axisymmetric equilibria of self-gravitating systems. The absence in the literature of nonaxisymmetric equilibria of self-gravitating systems suggests that a fresh theoretical approach is necessary.

Our motivation for this work is to determine nonaxisymmetric equilibria of self-gravitating systems. We extend the work of MN by analytically obtaining the nonaxisymmetric self-similar equilibrium of a self-gravitating system. Because of the great complexity and intrinsically nonlinear nature of the problem, any analytical solution is unique and thus of great scientific value. Our nonaxisymmetric equilibrium configurations form a three-parameter family of solutions and, in the case of axisymmetric equilibria, reduce to the solutions of MN. Furthermore, we study the effect of magnetic fields on the axisymmetric self-similar equilibrium solutions. Since we are interested in treating the problem analytically, and for the sake of simplicity, we assume that the dominant component of the magnetic field is purely toroidal, and that the ratio of magnetic pressure to the gas pressure, \( \alpha \), is spatially constant. MN showed that the nonrotating, axially symmetric solutions form confocal ellipsoids or spheres. In contrast, we find that the nonrotating, magnetized, axially symmetric solutions tend to the cylindrically symmetric structures. However, the equilibria of axially symmetric systems with both rotation and a toroidal magnetic field yield ellipsoids or spheres only when the ratio of rotation velocity to the sound speed is taken to be \((2\alpha)^{1/2}\).

In § 2, we present the general formulation and basic assumptions. We derive a three-parameter family of nonaxisymmetric self-similar solutions in § 3 and investigate the properties of these solutions in § 4. We summarize our results in § 5.

2. GENERAL FORMULATION

Here we present the basic equations used to describe the equilibrium structure of a rotating self-gravitating system permeated by a toroidal magnetic field. We assume that the medium is an isothermal ideal gas and that the magnetic field is frozen into the gas. However, the assumption that the system is isothermal is good for low-density regions. For spherical coordinates \((r, \theta, \varphi)\), the basic equations are the magnetohydrostatic equation, the Poisson equation, and the equation of state:

\[
\frac{1}{\rho} \frac{\partial p}{\partial r} + \frac{\partial \Psi}{\partial r} + \frac{1}{4\pi \rho} \frac{\partial}{\partial r} \left( \frac{B_\varphi^2}{r} \right) = \frac{v^2}{r},
\]

(1)
where $\rho$, $p$, $v_\phi$, $c_s$, $B_\phi$, and $\Psi$ denote the gas density, pressure, toroidal component of the velocity due to the rotation, sound speed, toroidal magnetic field, and gravitational potential, respectively. We assume that the entire system is rotating around a common axis that specifies the axis of the spherical coordinate system. It is clear that if $\Omega$ denotes the angular velocity, then $v_\phi = \Omega r \sin \theta$. We note that a purely toroidal magnetic field under the cylindrical symmetry, $B_\phi(r, \theta)$, automatically satisfies $\nabla \times B = 0$. In fact, this configuration of the magnetic field requires a rather artificial current configuration, namely, that the current must flow along the rotation axis ($z$-axis). Although systems with such fields are not very common in astrophysics, our results provide an intuitive way to understand more complex systems. In addition, the rotation velocity, density, and gravitational potential are, in general, functions of $r$, $\theta$, and $\phi$.

Solving equations (1) to (5) is difficult work. Thus, in order to simplify the problem, we need another constraint. Observations for molecular clouds suggest that the magnetic field strength often varies with the density, roughly according to $B \sim \rho^{1/2}$ (e.g., Heiles et al. 1993). This implies that the ratio of magnetic pressure to the thermal pressure, $\alpha$, is spatially constant:

$$\alpha = B_\phi^2 / 8 \pi \rho,$$

where molecular cloud observations show that $0 < \alpha < 10$ (Heiles et al. 1993). For our problem, we use it as a free parameter. By changing the value of $\alpha$, the effect of a purely toroidal magnetic field on the equilibrium state of self-gravitating systems can be studied. It must be noted that, by using this constraint, we can investigate the effects of the magnetic field only on the axisymmetric solutions. In fact, our nonaxisymmetric solutions are nonmagnetized.

Now we introduce the dimensionless variables according to

$$\rho \to \hat{\rho} \rho, \quad r \to \hat{r} r, \quad \Psi \to \hat{\Psi} \Psi, \quad v_\phi \to \hat{v}_\phi v_\phi, \quad B_\phi \to \hat{B} B_\phi,$$

where

$$\hat{\rho} = \rho_0, \quad \hat{r} = (4 \pi G \rho_0)^{-1/2} c_s^2, \quad \hat{\Psi} = c_s^2, \quad \hat{v}_\phi = c_s, \quad \hat{B} = (4 \pi G \rho_0)^{1/2} c_s.$$  

By transforming to dimensionless variables and using equation (6), equations (1) through (4) are rewritten as the following:

$$\frac{\partial \Psi}{\partial \hat{r}} = \frac{v_\phi^2 - 2 \alpha}{r} - (\alpha + 1) \frac{\partial \ln \rho}{\partial \hat{r}}, \quad (9)$$

$$\frac{\partial \Psi}{\partial \hat{\theta}} = (v_\phi^2 - 2 \alpha) \cot \theta - (\alpha + 1) \frac{\partial \ln \rho}{\partial \hat{\theta}}, \quad (10)$$

$$\frac{\partial \Psi}{\partial \hat{\phi}} = -\frac{\partial \ln \rho}{\partial \hat{\phi}} - \frac{1}{2} \frac{\partial}{\partial \hat{\phi}} (v_\phi^2), \quad (11)$$

$$\frac{\partial^2 \Psi}{\partial \hat{r}^2} + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \hat{r}} \left( r^2 \frac{\partial \Psi}{\partial \hat{r}} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \hat{\theta}} \left( \sin \theta \frac{\partial \Psi}{\partial \hat{\theta}} \right) = 4 \pi G \rho,$$  

$$p = c_s^2 \rho.$$  

By substituting equations (9), (10), and (11) into equation (12), we obtain

$$\alpha + 1 \frac{\partial}{\partial \hat{r}} \left( r^2 \frac{\partial \ln \rho}{\partial \hat{r}} \right) + \alpha + 1 \frac{\partial}{\partial \hat{\theta}} \left( \sin \theta \frac{\partial \ln \rho}{\partial \hat{\theta}} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial^2 \ln \rho}{\partial \hat{\phi}^2} + \rho = K(r, \theta, \phi), \quad (13)$$

$$K(r, \theta, \phi) = \frac{1}{r^2 \hat{r}} \left( r^2 \frac{\partial v_\phi^2}{\partial \hat{r}} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \hat{\theta}} (v_\phi^2 \cos \theta) - \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \hat{\phi}^2} (v_\phi^2). \quad (14)$$

It would be difficult to study systematically the very large space of all possible solutions to these equations. In fact, we need some information about the density, $\rho$, and the rotation velocity, $v_\phi$. For example, by knowing the form of one of the variables, it is possible to find another variable. But here we are interested in self-similar solutions, which will be studied in the next section. These solutions are interpreted with much less effort than would be involved in a direct attempt to solve the full set of partial differential equations.

3. SELF-SIMILAR SOLUTIONS

We introduce the self-similar forms for the density and the rotation velocity as

$$\rho(r, \theta, \phi) = \frac{f(\theta, \phi)}{r^{\nu_1}}, \quad v_\phi = \frac{v_{0\phi}}{(r \sin \theta)^{\nu_2}}, \quad (15)$$

where $\nu_1$, $\nu_2$, and $f(\theta, \phi)$ need to be calculated. The same self-similar solutions for the axisymmetric case have been introduced by MN. Their angular part of the density distribution was a function of $\theta$ only. Since we are going to study the nonaxisymmetric solutions, the angular part of the density is a function of both $\theta$ and $\phi$. Substituting the self-similar forms into equations (13) and (14), we obtain

$$\nu_1 = 2, \quad \nu_2 = 0, \quad (16)$$

and $f(\theta, \phi)$ can be found by solving this equation:

$$-2(\alpha + 1) + \frac{\alpha + 1}{\sin \theta} \frac{\partial}{\partial \hat{\theta}} \left( \sin \theta \frac{\partial f}{\partial \hat{\theta}} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 f}{\partial \hat{\phi}^2} + f = 0. \quad (17)$$

This is the main equation that must be solved. We note that for the nonmagnetized and axisymmetric case, this equation reduces to the equation that has been solved by MN. To solve equation (17), we use transformations that are equivalent to Howard's transformation, used by Toomre (1982) and MN. First, we rewrite this equation as
follows:

$$\frac{a + 1}{\sin \theta} \frac{\partial}{\partial \theta} \left[ \sin \theta \frac{\partial}{\partial \theta} \ln (f \sin^2 \theta) \right] + \frac{1}{\sin^2 \theta} \frac{\partial^2 f}{\partial \varphi^2} + f = 0 .$$  \hspace{1cm} (18)

Upon introducing a new function, $w$, and a new independent variable, $\xi$, as follows:

$$w = \ln (f \sin^2 \theta), \quad \xi = \ln | \tan (\theta/2) |^{1/\sqrt{2 + 1}},$$  \hspace{1cm} (19)

equation (18) is greatly simplified and becomes

$$\frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \varphi^2} + e^w = 0 .$$  \hspace{1cm} (20)

This equation is similar to the Lane-Emden equation, which has been written in Cartesian coordinates. Schmid-Burgk (1967) and, independently, Stuart (1967) presented an interesting two-dimensional solution of the Lane-Emden equation in Cartesian coordinates. Thus, in analogy to their solution, we can write an analytical solution for equation (20),

$$w = \ln \left\{ \frac{2a^2(1 - C^2)}{[\cosh (a \xi + b) + b] + C \cos (a \varphi)]} \right\},$$  \hspace{1cm} (21)

where $a$, $b$, and $C$ are free parameters. Recalling the transformations in equation (19), we finally obtain an analytical solution,

$$f(\theta, \varphi) = \frac{1}{\sin^2 \theta} \times \frac{2a^2(1 - C^2)}{[\cosh (\ln | \tan (\theta/2) |^{1/\sqrt{2 + 1}} + b) + C \cos (a \varphi)]}. $$  \hspace{1cm} (22)

This solution can be simplified further, and the density becomes

$$\rho(r, \theta, \varphi) = \frac{2A^2B(1 - C^2)}{r^2 \sin^2 \theta} \times \frac{1}{[1 + B \tan (\theta/2)]^{A_{\varphi}/2}} \frac{| \tan (\theta/2) |^{A_{\varphi}/2}}{[1 + B \tan (\theta/2)]^{A_{\varphi}/2}} \frac{2C\sqrt{B} \tan (\theta/2)}{[1 + B \tan (\theta/2)]^{A_{\varphi}/2} \cos (A/2)} \right\},$$  \hspace{1cm} (23)

where $A = 2a$, $B = e^{2b}$, and $-1 < C < 1$. This solution describes the nonaxisymmetric density distribution of a self-gravitating system. It is clear that to obtain the nonmagnetized axisymmetric equilibria, we must set $a = 0$ and $C = 0$, and our solution reduces to the solution of MN. Equation (23) presents a three-parameter family of solutions (note that for nonaxisymmetric equilibria we must set $a = 0$) and is smooth and well behaved for all values of $\theta$ and $\varphi$ in the domains $0 < \theta < \pi$ and $0 \leq \varphi \leq 2\pi$. However, the equation is ill defined at $r = 0$ or $\theta = 0$. We can investigate whether a singular solution exists at the origin or on the axis. Using Gauss’ theorem, MN calculated these additional mass densities for the case of the nonmagnetized axisymmetric equilibria. The reader is referred to MN for a full derivation. Following MN, we find that our solution has no singular mass at the origin. But the requirement of a nonsingular density on the axis gives

$$A = 2 + v_{0\varphi}/\sqrt{a + 1} .$$  \hspace{1cm} (24)

This equation shows that the rotation velocity and the magnetic field determine the value of $A$. Thus the solution (23) becomes

$$\rho(r, \theta, \varphi) = \frac{2A^2B(1 - C^2)}{r^2 \sin^2 \theta} \times \frac{| \tan (\theta/2) |^{A_{\varphi}/2}}{[1 + B \tan (\theta/2)]^{A_{\varphi}/2}} \frac{2C\sqrt{B} \tan (\theta/2)}{[1 + B \tan (\theta/2)]^{A_{\varphi}/2} \cos (A/2)},$$  \hspace{1cm} (25)

where

$$A_{\varphi} = 2 + v_{0\varphi}/(a + 1) .$$  \hspace{1cm} (26)

Since the exponent of $\tan (\theta/2)$ determines the overall shape of the density distribution, equation (26) is very important. In the next section we study the properties of the solutions.

4. PROPERTIES OF THE SOLUTIONS

Since a real self-gravitating system is finite, we truncated the system by an external pressure. The external medium is assumed to be non-self-gravitating and of negligible density. In this way we can define the surface of the system. If $p_s$ represents the external pressure in a nondimensional form, we obtain the equation of the surface

$$r_s(\theta, \varphi) = \sqrt{\frac{2A^2B(1 - C^2)}{p_s} \frac{1}{\sin \theta}} \times \frac{| \tan (\theta/2) |^{A_{\varphi}/2}}{[1 + B \tan (\theta/2)]^{A_{\varphi}/2}} \frac{2C\sqrt{B} \tan (\theta/2)}{[1 + B \tan (\theta/2)]^{A_{\varphi}/2} \cos (A/2)},$$  \hspace{1cm} (27)

Notice also that for $a = 0$ and $C = 0$, this equation reduces to the solution that MN obtained. As equations (24) and (26) show, the rotation velocity, $v_{0\varphi}$, and the ratio of magnetic pressure to the gas pressure, $x$, determine the values of $A$ and $A_{\varphi}$. But it is possible to make another constraint: since we are interested in the solutions that have no discontinuity on the surface, we accept those values of $A$ for which

$$r_s(\theta, \varphi = 0) = r_s(\theta, \varphi = 2\pi).$$

Thus

$$A = 2v ,$$  \hspace{1cm} (28)

where $v$ is a positive integer. So only a limited number of values of $v_{0\varphi}$ and $a$ give acceptable values of $A$.

As mentioned previously, the value of $A_{\varphi}$ determines the overall shape of the system. For the nonmagnetized case, we have $A_{\varphi} = A$. Therefore, only the rotation velocity determines the value of $A$. As $v_{0\varphi}$ increases, the value of $A$ becomes larger. Equation (26) shows that as $x$ increases, the value of $A_{\varphi}$ decreases. It seems that a purely toroidal magnetic field with a constant ratio of the magnetic pressure to thermal pressure essentially cancels the effects of rotation. With these considerations, we focus the current discussion on nonmagnetized and magnetized solutions.

4.1. Nonmagnetized Equilibria

First, we study the nonrotating solutions: $A_{\varphi} = A = 2$. The solution to equation (27) reads

$$r_s(\theta, \varphi) = \lambda/(1 + \epsilon \cos \theta + \delta \sin \theta \cos \varphi) ,$$  \hspace{1cm} (29)
where

\[ \lambda = \frac{8(1 - C^2)}{\sqrt{p_1(1 + B)^2}} , \quad \epsilon = \frac{1 - B}{1 + B} , \quad \delta = \frac{2C\sqrt{B}}{1 + B} . \]  

Note that the case \( C = 0 \) and \( B \neq 1 \) corresponds to a prolate ellipsoid and that if \( B = 1 \), this ellipsoid reduces to a singular sphere. These axisymmetric equilibria have been studied extensively by MN. They showed that the solutions with \( B > 1 \) determine prolate configurations that are shifted upward along the \( z \)-axis, and those with \( B < 1 \) are shifted downward. Therefore, the parameter \( B \) controls the symmetry of the solutions with respect to the equatorial plane.

It can easily be verified that the general cases \( C \neq 0 \) and \( B \neq 1 \) represent a family of ellipsoids. By rotating the coordinate system \((x, y, z)\) about the \( y \)-axis from the positive direction of the \( z \)-axis to the positive direction of \( x \)-axis, we can define a new coordinate system \((X, Y, Z)\), and is invariant under the transformation \( r \to r - \theta \). Thus the equatorial plane is the symmetry plane, and projections of these ellipsoids onto the \( y - z \) plane are circles with centers at \(-\theta\). The general solution to equation (27) shows that \( r \to r - \theta \) and \( \phi \to \phi \). Thus the equatorial plane is also invariant under these transformations only for \( B = 1 \). Therefore, the parameter \( B \) in our nonaxisymmetric solution controls the symmetry of the solutions with respect to the equatorial plane.

As mentioned previously, we restrict our study only to the solutions that have no singular mass on the axis. This constraint gives \( A \geq 2 \) or \( v \geq 1 \); note that the case \( v = 1 \) has already been studied. For \( C = 0 \) and \( A > 2 \), equation (27) corresponds to a family of toroidal structures in which the parameter \( B \) determines the asymmetry of these equilibria with respect to the equatorial plane. In this case, as \( A \) increases, the systems become flattened and tend to a disk. It seems that in our solution, the parameter \( C \) alone determines the asymmetry of the system with respect to the \( z \)-axis, although it is true that the parameter \( A \) (or \( v \)) is also very important. We see that equation (27) is invariant under the transformations \( \theta \to \theta \) and \( \phi \to \pi + \phi \) only for even values of \( v \). Therefore, in the case \( C \neq 0 \), we may have configurations that are symmetric with respect to the \( z \)-axis. Figures 1 and 2 show families of solutions. The rotation velocity has a fundamental role in determining the overall shape of the density distribution. For small values of \( C \) and \( A > 2 \), the toroidal equilibrium structures are elongated in some directions. We refer to these elongated parts as “knobs.” As discussed in this section, the number of these knobs depends on the rotation velocity. For example, the case \( A = 8, (v = 4) \) represents a toroidal configuration with four knobs. As Figures 1 through 3 show, with increasing \( C \) and tending the parameter to the unity, these knobs become larger and the toroidal part becomes smaller.

4.2. Magnetized Equilibria

It is useful to study the effect of the magnetic field on our solutions. We postulated that the magnetic field structure is purely toroidal and corresponds to that of a constant ratio

![Fig. 1.—Isodensity surfaces for different values of the parameters](image)
of magnetic pressure to thermal pressure. Although purely toroidal fields are probably far from reality, this is the best way to analytically investigate the effects of the magnetic fields on the equilibria.

Next we consider the effects of the magnetic field on axisymmetric solutions: i.e., $C = 0$. Equation (26) shows that a nonrotating system always has $A_{ef} < 2$ and that the value of $A_{ef}$ decreases with increasing strength of the magnetic field. While the nonrotating and nonmagnetized systems are ellipsoids, the nonrotating and magnetized systems tend to the cylindrically symmetric configurations. In addition, if we consider rotation such that $v_0 r > (2a)^{1/2}$, the system has a cylindrically symmetric structure.

If $v_{0x} = (2a)^{1/2}$ we have $A_{ef} = 2$ and the equilibria are like those investigated in § 4.1 for the case $C = 0$. Finally, the case $v_{0x} > (2a)^{1/2}$ and $C = 0$ corresponds to a family of toroidal configurations. We know that as $v_{0x} \to \infty$, the equilibria tend to a disk. However, the magnetic field decreases the effect of rotation and in this case causes the system to become a toroidal structure, not a disk.

5. DISCUSSION AND SUMMARY

The self-similar equilibrium of a self-gravitating rotating system containing a purely toroidal magnetic field has been investigated. A three-parameter family of solutions for nonmagnetized self-gravitating systems has been found. By assuming that the ratio of the magnetic pressure to the gas pressure, $a$, is spatially constant, the axisymmetric solutions were generalized so that the effect of the magnetic field could be studied. We have shown that, depending on the values of these parameters, the overall behavior of the density distribution changes.

Recently, Galli et al. (2001) studied binary and multiple star formation by considering the nonaxisymmetric equilibrium of self-gravitating, magnetized, differentially rotating, completely flattened singular isothermal disks with critical
or supercritical ratios of mass to flux. They found that lop-sided configurations exist at any dimensionless rotation rate. In addition, multiple-lobed \((M = 2, 3, 4, \ldots)\) configurations correspond to rotations of the equilibrium configurations by multiples of \(\pi/M\), and bifurcations into sequences with \(M = 2, 3, 4, 5, \) and higher symmetry require considerably larger rotation rates. These results are in good agreement with our analysis, at least qualitatively. We must note that the flattening of our nonmagnetized equilibria are produced by rotation rather than by magnetic fields.

As equation (25) shows, the parameters \(A\) or \(A_{ef}\) determine the overall shape of the density distribution. We showed that only a limited number of values of \(v_{\phi 0}\) and \(\alpha\) give acceptable values for \(A\) or \(A_{ef}\). Since for non-magnetized solutions we have \(A = A_{ef}\), as the value of rotation velocity increases, the value of \(A\) becomes larger. Equation (25) shows that for \(A > 2\), the toroidal equilibrium structures have knobs, the number of which depends on the rotation velocity: as rotation velocity increases, the number of the knobs increases (see Figs. 1 through 3). It is interesting that in the case \(C \neq 0\), some values of \(A\) construct equilibrium configurations that are symmetric with respect to the \(z\)-axis (Fig. 3). We see that our multiple-lobed \((v = 1, 2, 3, \ldots)\) equilibria require larger rotation rates. The equilibria solutions become flattened as the rotation velocity increases, and they tend to a thin disk as \(A \to \infty\). Thus our nonaxisymmetric solution tends to a nonaxisymmetric thin disk as the rotation velocity increases. These configurations are equivalent to the non-axisymmetric solutions obtained by Galli et al. (2001).

Detailed observations of the gas disks associated with bipolar outflows indicate that the disks are rotating very fast (Kaufmann 1987). The rotation period is comparable to the free-fall timescale of the gas disks, and it means that the disk is near the rotational balance. Thus, in these systems, the effect of rotation should have been important in the formation of the central star. On the other hand, it is very important that the formation of multiple stellar systems could never result from any calculation that imposes, a priori, an assumption of axial symmetry. Therefore, as long as nonaxisymmetric disk solutions are the starting conditions, gravitational collapses seem to produce fragmentation in general.

For \(B = 1 \) and \(C = 0\), the solutions have been obtained by Toomre (1982) and Hayashi et al. (1982). In fact, the Toomre-Hayashi model is a good approximation to a centrifugally supported disk formed by collapse of a rotating gas cloud (see, e.g., Nomura & Mineshige 2000). Since the evolution of such systems may be driven by instabilities, Hanawa, Saigo, & Matsumoto (2001) discussed the stability of the Toomre-Hayashi model against axisymmetric perturbations, and Shu et al. (2000) analyzed the stability of this nonrotating and nonmagnetized systems form ellipsoids (or spheres). If a system rotates such that \(v_{\phi 0} < (2a)^{1/2}\), the system has a cylindrically symmetric structure. But if \(v_{\phi 0} = (2a)^{1/2}\), the systems form ellipsoids or spheres. For \(v_{\phi 0} > (2a)^{1/2}\), the axisymmetric equilibria corresponds to a family of toroidal configurations. In fact, the magnetic field decreases the effect of rotation.

MN proved that the axisymmetric solutions with \(B \neq 1\) are not force-free at two singular points, \(r = 0\) and \(r = \infty\). In fact, external forces hold the axisymmetric system in equilibrium. Are the nonaxisymmetric equilibria likely to be not force-free at singular points? Although we did not investigate this problem, it is likely that the non-axisymmetric solutions are also not force-free at singular points. However, we think such configurations represent legitimate states of equilibrium. In this regard, we would like to remember the explanation of Galli et al. (2001): "... In the ... solutions ..., the (infinite) gravitational force at the origin is exactly balanced by an (infinite) pressure gradient acting in concert perhaps with an (infinite) centrifugal force. This balance is qualitatively no different than at any other point in the system, and it would be an artificial restriction to rule out eccentric equilibria simply because they have a nontrivial balance of forces at the origin rather than a trivial one."

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