Nonequilibrium steady state fluctuations in actively cooled resonators

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We analyze heat and work fluctuations in the gravitational wave detector AURIGA, modeled as a macroscopic electromechanical oscillator in contact with a thermostat and cooled by an active feedback system. The oscillator is driven to a steady state by the feedback cooling, equivalent to a viscous force. The experimentally measured fluctuations are in agreement with our theoretical analysis based on a stochastically driven Langevin system. The asymmetry of the fluctuations of the absorbed heat characterizes the oscillator’s nonequilibrium steady state and reveals the extent to which a feedback cooled system departs from equilibrium in a statistical mechanics perspective.

Cold damping feedback efficiently reduces the thermal noise motion of an oscillator by applying a viscous force. Since its first application in electrometers [1], it succeeded in a wide variety of devices [2,3], from nano to macroscopic resonators, and in a variety of implementations, including both optical and electrical forces. In basic research, the cold damping is considered in order to reduce below the level of intrinsic quantum fluctuations the uncertainty due to thermal noise of the position of macroscopic bodies [4], and to improve the behavior of gravitational wave detectors [5]. In this work we experimentally investigate the fluctuations of thermodynamic quantities of a cold-damped electromechanical oscillator: the resonant-bar gravitational wave detector AURIGA [6]. In particular we verify that they are consistent with recent theories of nonequilibrium phenomena.

After the seminal works of Ref. [7], which introduced the Fluctuation Relation (FR) concerning the Probability Density Function (PDF) of the entropy production rate in nonequilibrium systems, a large number of papers has been devoted to similar problems (see for instance Ref. [8] for a review). One finds that the FR for some properly identified observable (called dissipation function) is quite generally valid in systems of physical interest [9]. After the experimental evidence obtained for dragged colloidal particles [10], electrical circuits [11], and mechanical oscillators [12], the FR has become a standard tool to characterize nonequilibrium systems. Here, following Ref. [13] we focus on the FR for the PDF of the power necessary to maintain a dissipative system in a nonequilibrium steady state (NESS). A specific FR, obtained for the fluctuations of the injected power in a stochastically driven Langevin system [14], was recently confirmed in a simple electrical realization of that model [15]. Actually, this FR accurately fits the fluctuations of the injected power in wave turbulence as well [16]. We show that also the AURIGA detector, which is maintained in a NESS by an external driving in a feedback cooling scheme, can be described as a mechanical oscillator forced by a stochastic driving. We then analyze its behavior and demonstrate that: a) the statistics of its thermodynamic variables show a characteristic asymmetry between positive and negative fluctuations; b) the statistics of the injected power are in agreement with the FR of Ref. [14]. These results reveal the extent to which a feedback cooled system departs from equilibrium in a statistical mechanics perspective and prove the limits of usual assumption that cold-damped oscillators at temperature $T_0$ are equivalent to higher-loss ones, in thermodynamic equilibrium at a temperature $T_{eh} < T_0$.

AURIGA is based on a $2.2 \times 10^3$ kg, $3$ m long bar made of a low-loss aluminum alloy (Al5056), cooled to liquid helium temperature $T_0 = (4.6 \pm 0.2)$ K. The fundamental longitudinal mode of the bar, sensitive to gravitational waves, has effective mass $M = 1.1 \times 10^3$ kg and resonance frequency $\omega_0/2\pi \sim 900$ Hz. The bar resonator motion is detected by a capacitive transducer followed by a double stage dc-SQUID amplifier; the displacement sensitivity is about $5 \times 10^{-20}$ m/$\sqrt{\text{Hz}}$ over a $\sim 100$ Hz bandwidth around $\omega_0$, largely limited by thermal noise. The detector can be modeled by three coupled low-loss resonators: two mechanical ones (the bar and a plate of the capacitive transducer) and an LC electrical one [17,18]. Their dynamics is described by three normal modes at separate frequencies, $865.7, 914, 953$ Hz, with quality factors respectively of $1.2 \times 10^6, 0.88 \times 10^6, 0.77 \times 10^6$, determined by mechanical losses in the bar and the transducer and by dielectric losses in the electrical components. Each mode is modeled as a RLC series electrical oscillator with an effective inductance $L$, capacitance $C$ and resistance $R$, which assume different values for the 3 oscillators (Fig.1). These modes are electromechanical rather than purely...
The capacitor, quasi-harmonic approximation is obtained by sending back a current $I$.

The very high quality factor of the oscillator implies that the currents are negligible at moderate feedback gains, as those used in this experiment. The very many degrees of freedom of the thermal bath, and the feedback cooling concerns only 3 modes, out of the resonant mode around its resonance frequency, by a series-RLC circuit. The dc SQUID is represented as current amplifier. The observables is the current $I_s$ and the electronic feedback cooling is obtained by sending back a current $I_d$ which is a delayed copy of $I_s$ reduced by $G \ll 1$. The SQUID output voltage is $V_{out} = AI_s$ with $A = 2.6 \times 10^6 \, \Omega$.

Mechanical, but each one collects a significant fraction of the energy of the two mechanical resonators. From the detector calibration we estimate that the energy injected by an impulsive excitation of the longitudinal mode of the detector is $\frac{\gamma_0}{\Delta t}$ and $\frac{\Gamma}{\pi}$. Here $\Gamma = 2k_B T_0 R_0 \Gamma(t)$.

The quasiharmonic approximation is valid as long as the feedback current is not expected to significantly affect the thermal noise because of the constraints (1c) and (2): in the quasiharmonic approximation we have $I_s(t - t_d) \approx \omega_s q_s(t)$. Hence each oscillator obeys:

$$L \frac{dI_s(t)}{dt} + I_s(t) [R + R_d] + \frac{q_s(t)}{C} = \sqrt{2k_B T_0 R} \Gamma(t) \tag{3}$$

with $I_s(t) = \frac{dI_s(t)}{dt}$. Here $R_d = G \omega_s L_{in}$ expresses the viscous damping on the oscillator due to the feedback loop; the feedback efficiency is defined as $g = R_d/R$. The quasiharmonic approximation is valid as long as the feedback damped oscillator is still a low loss one. In Eq. (3) the driving is the same white process of Eq. (1): this is confirmed experimentally by the Lorentz-shaped power spectrum of the current $I_s$ around the resonance $\omega_s$. Equation (3) is not invariant under time reversal ($q_s^* = q_s$, $I_s^* = -I_s$, $t^* = -t$) and does not satisfy the Einstein relation. Nevertheless, it is formally identical to that describing an oscillator with damping $R + R_d$ in equilibrium at the fictitious “effective temperature” $T_{eff} = T_0/(1 + g)$. The discrepancy between $T_{eff}$ and the thermal bath temperature $T_0$ reveals the nonequilibrium nature of the phenomenon. Hence, the feedback cooled oscillator is usually treated as an equilibrium system, with $T_{eff}$ derived from the experimental value of $\langle I_s^2(t) \rangle = 2k_B T_{eff}$, even if no bath at $T_{eff}$ is present.

Multiplying Eq. (3) by $I_s(t)$ and integrating between $t$ and $t + \tau$, in the quasi-harmonic approximation we get an expression for the average power $P_\tau = \frac{1}{\tau} \int_{t}^{t+\tau} I_s(t')V_T(t')dt'$ injected by the stochastic thermal force during a time $\tau$:

$$P_\tau = \Delta U_r + \frac{R + R_d}{\tau} \int_{t}^{t+\tau} I_s^2(t')dt' \tag{4}$$

where $\Delta U_r = \frac{U(t+\tau) - U(t)}{\tau}$, $U(t)$ being the stored energy:

$$U(t) = \frac{1}{2} L I_s^2(t) + \frac{1}{2} \frac{q_s^2(t)}{C} = \frac{1}{2} L I_s^2(t) \tag{5}$$

The term proportional to $R$ represents the heat dissipated by the oscillator toward the bath while that proportional to $R_d$ is the work done by the oscillator on the feedback:

$$W_\tau = -\frac{1}{\tau} \int_{t}^{t+\tau} I_d(t') V_d(t')dt' = \frac{R_d}{\tau} \int_{t}^{t+\tau} I_s^2(t')dt' \tag{6}$$

Notice that the last identity is strictly valid only within the quasi-harmonic approximation, which relates both $I_s(t - t_d)$ and $dI_s(t)/dt$ to the instantaneous current $I_s(t)$.
Experimental error in and numerical data are within the uncertainty due to the experimental errors on the parameters, $\tau$. Further, if $\tau$ is highly separated from the other two and is thus our best approximation of a single oscillator. The sampled current $I_\ell(t)$ was processed via the AURIGA data analysis and integrated over the resonance in a 10 Hz bandwidth to obtain the current amplitude $\tilde{I}_\ell(t)$ in the harmonic approximation. From dedicated calibration of AURIGA we measure $L = (1.67 \pm 0.01) \times 10^{-4}$ H and $L_\text{in} = (1.48 \pm 0.01) \times 10^{-6}$ H. A first set of data covers a continuous 10 days time span acquired in March 2008. They yield $\omega / 2\pi = 865.7$ Hz, $T_{\text{eff}} = (21.1 \pm 0.2)$ mK and decay time $\tau_{\text{eff}} = (2.36 \pm 0.04)$ s; hence we estimate $g = 207 \pm 10$, $R = (6.8 \pm 0.5) \times 10^{-7}$ $\Omega$ and $G = (1.74 \pm 0.06) \times 10^{-2}$. The quality factor $\omega_r\tau_{\text{eff}}/2 \simeq 6.5 \times 10^3$ is high enough to justify the quasi-harmonic approximation leading to Eq. 3. Figure 2a and 2b show the PDF of the energy difference $\Delta U_\ell$ and of the work done by the oscillator $W_\ell$ averaged over growing times $\tau$: they are calculated from Eqs. 5 and 7 after dividing the experimental data in contiguous time intervals of duration $\tau$. Figure 2b shows the corresponding heat $Q_\ell$ exchanged by the oscillator with the bath averaged over the time $\tau$, computed via the energy conservation Eq. 8. The fluctuations of $Q_\ell$ are asymmetric, as expected for a NESS, Figure 2b shows also excellent agreement with numerical simulations of Eq. 3.

The PDF of $\Delta U_\ell$ is symmetric with respect to zero as for an equilibrium oscillator. It has exponential tails which decay faster for longer $\tau$. The PDF of $W_\ell$ is highly asymmetric. From Eq. 3 and $\langle \tilde{I}_\ell^2(t) \rangle = 2k_BT_{\text{eff}}/L$ we infer that $W_\ell$ is positive and has mean value $\pm 0.84$ $k_BT_{\text{eff}}/s$. Hence, $Q_\ell$ takes negative values only for short integration times, with the characteristic time scale given by the cold damped oscillator decay time $\tau_{\text{eff}} = 2L/(R + R_0)$. For $\tau \gg \tau_{\text{eff}}$ the contribution of the time averaged energy is negligible. So in the presence of feedback ($R_0 > 0$) there is a net heat transfer from bath to oscillator: this is the energy flux that feeds the NESS and makes the PDF of the heat asymmetric.

Further, if $\tau = N\frac{2\pi}{\omega}$, $N$ integer, in the same approximation we can also write:

$$W_\ell = \frac{1}{\tau} \int_0^{\tau} \tilde{I}_\ell^2(t') dt'$$

By energy conservation we obtain the heat $Q_\ell$, absorbed by the oscillator from the bath and averaged in a time $\tau$:

$$Q_\ell = \Delta U_\ell + W_\ell$$

To study nonequilibrium properties, we focused on the lowest frequency mode out of the 3, which is well separated in frequency from the other two and is thus our best approximation of a single oscillator. The sampled current $I_\ell(t)$ was processed via the AURIGA data analysis and integrated over the resonance in a 10 Hz bandwidth to obtain the current amplitude $\tilde{I}_\ell(t)$ in the harmonic approximation. From dedicated calibration of AURIGA we measure $L = (1.67 \pm 0.01) \times 10^{-4}$ H and $L_\text{in} = (1.48 \pm 0.01) \times 10^{-6}$ H. A first set of data covers a continuous 10 days time span acquired in March 2008. They yield $\omega / 2\pi = 865.7$ Hz, $T_{\text{eff}} = (21.1 \pm 0.2)$ mK and decay time $\tau_{\text{eff}} = (2.36 \pm 0.04)$ s; hence we estimate $g = 207 \pm 10$, $R = (6.8 \pm 0.5) \times 10^{-7}$ $\Omega$ and $G = (1.74 \pm 0.06) \times 10^{-2}$. The quality factor $\omega_r\tau_{\text{eff}}/2 \simeq 6.5 \times 10^3$ is high enough to justify the quasi-harmonic approximation leading to Eq. 3. Figure 2a and 2b show the PDF of the energy difference $\Delta U_\ell$ and of the work done by the oscillator $W_\ell$ averaged over growing times $\tau$: they are calculated from Eqs. 5 and 7 after dividing the experimental data in contiguous time intervals of duration $\tau$. Figure 2b shows the corresponding heat $Q_\ell$ exchanged by the oscillator with the bath averaged over the time $\tau$, computed via the energy conservation Eq. 8. The fluctuations of $Q_\ell$ are asymmetric, as expected for a NESS, Figure 2b shows also excellent agreement with numerical simulations of Eq. 3.

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On the contrary, if the feedback were switched off, we would have \( R_d = 0 \) and \( W_r = 0 \), hence \( Q_r = \Delta U_r \) by Eq. (8). In this case the PDF of \( Q_r \) would be symmetric with respect to its (zero) mean value, as in Fig. 2a, but with \( T_{\text{eff}} = T_0 \).

The PDF of the injected power \( P_r \) is essentially identical to that of \( Q_r \), shown in Fig. 2b, since \( P_r \approx Q_r \) when \( g \gg 1 \). Notice that of the two terms in Eq. (4), only \( \Delta g \) is responsible for the negative values of \( P_r \). Thus, large positive values of \( P_r \) are dominated by the contribution of the dissipated power [the integral in (14)] more than they are for small values of \( P_r \). The transition between these two regimes affects the shape of the PDF, which has been calculated in Ref. [14].

In a limit of large integration times it obeys:

\[
f(\tilde{\epsilon}_r) \equiv \lim_{\tau \to \infty} \frac{\ln \text{PDF}(\tilde{\epsilon}_r)}{\tau} = \begin{cases} -\gamma (1 - 2\tilde{\epsilon}_r) & \text{if } \tilde{\epsilon}_r \leq \frac{1}{3} \\ -\frac{\gamma}{\tilde{\epsilon}_r} (\tilde{\epsilon}_r - 1)^2 & \text{if } \tilde{\epsilon}_r \geq \frac{1}{3} \end{cases}
\]

where \( \tilde{\epsilon}_r = P_r L/(k_B T_0 R) \) is the reduced injected power and \( \gamma = (R + R_d)/L = 2/\tau_{\text{eff}} \). A remarkable singularity, located at \( \tilde{\epsilon}_r = 1/3 \), is present in the second derivative of \( f(\tilde{\epsilon}_r) \). In Fig. 3, we plot the quantity \( D_2(\tilde{\epsilon}_r) \equiv \frac{\partial^2 f(\tilde{\epsilon}_r)}{\partial \tilde{\epsilon}_r^2} \) evaluated from the output of AURIGA in the timespan May 2005/May 2008; here \( T_{\text{eff}} = (22 \pm 1) \) mK and \( \tau_{\text{eff}} = (2.4 \pm 0.2) \) s. A valley is clearly visible in the experimental data, which we interpret as precursor of the asymptotic singularity. The agreement with the asymptotic theory consistently improves as \( \tau/\tau_{\text{eff}} \) grows. This indicates that the asymptotic relation Eq. (9) holds even in presence of a harmonic pinning potential [14].

Eq. (9) easily leads to the FR for the injected power, i.e. to the ratio between the probability of positive and negative fluctuations of \( \tilde{\epsilon}_r \). If we define \( \rho(\tilde{\epsilon}_r) = \lim_{\tau \to \infty} \frac{1}{\tau} \ln \frac{\text{PDF}(\tilde{\epsilon}_r)}{\text{PDF}(\tilde{\epsilon}_r)} \), we have:

\[
\rho(\tilde{\epsilon}_r) = \begin{cases} 4\gamma \tilde{\epsilon}_r, & \text{if } \tilde{\epsilon}_r < \frac{1}{3} \\ \gamma \tilde{\epsilon}_r \left( \frac{2}{3} + \frac{3}{2\tilde{\epsilon}_r} - \frac{1}{\tilde{\epsilon}_r^2} \right), & \text{if } \tilde{\epsilon}_r \geq \frac{1}{3}. \end{cases}
\]

As shown in Fig. 3b, positive values of \( \tilde{\epsilon}_r \) are exponentially more probable than negative ones. Two conflicting features determine the details of the experimental curves: the agreement with the asymptotic theory improves with \( \tau/\tau_{\text{eff}} \), but the statistics blur at large values of \( \tilde{\epsilon}_r \) since negative events are rarer. For this reason a slope change is clearly seen for small values of \( \tau/\tau_{\text{eff}} \) where the precursor of the singularity occurs as shown in Fig. 3a, while it becomes barely visible at longer integration times.

In conclusion, we demonstrate that the actively cooled AURIGA detector is well described by the Langevin model of Eq. (8), which led us to evaluate the power \( P_r \) injected by the stochastic thermal force, the work \( W_r \) done on the feedback and the heat \( Q_r \) exchanged with the thermal bath. The statistics of \( P_r \) are consistent with Eq. (9), and with the consequent nonlinearity of the FR. The fluctuations of \( Q_r \) are asymmetric as expected for a NESS.

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