Second Hankel determinant for tilted bi-starlike functions of order $\beta$

Shaharuddin Cik Soha*, Zammariyah Mustafa Kamalb,
Mohamad Huzaifah Mohd Dzubaidi, Noor Latiffah Adamd

$\text{a,b,c,dFaculty of Computer and Mathematical Sciences, Universiti Teknologi MARA, 40450 Shah Alam, Selangor, Malaysia}$

Abstract

Let $f(z)$ be analytic in the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$, and $S$ be the subclass of normalized univalent functions in $D$. We define the class of tilted bi-starlike functions $S_\beta(\beta, \delta)$, which satisfy the condition

$$\text{Re}\left\{e^{i\gamma}\frac{zf'(z)}{f(z)}\right\} > \beta$$

where $|\delta| < \pi$ and $\cos \delta > \beta$. The second Hankel determinant $|a_2a_4 - a_3^2|$ has been determined for $S_\beta(\beta, \delta)$. In this paper, we also found the coefficients bound for $|a_2|, |a_3|$ and $|a_4|$.

Keywords: Hankel Determinant, Bi-Starlike Functions, Unit Disk.

Introduction

Let $A$ denote the class of analytic functions $f$ such that,

$$f(z) = z + a_2z^2 + a_3z^3 + \cdots = z + \sum_{n=2}^{\infty} a_n z^n,$$

are defined on the open unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$. A function of the form (1.1) is said to be normalized by $f(0) = 0$ and $f'(0) = 1$. If $f(0)$ is univalent and has the form (1.1), it is called a normalized univalent function. The class of all normalized functions that are analytic and univalent in $D$ is denoted by $S$. Let $S^*$, and $CV$ denote the class of starlike functions and convex functions in $S$ respectively. A function $f \in S$ is in $S^*$ if and only if $\text{Re}\left\{zf'(z)/f(z)\right\} > 0$ for $z \in D$. In addition, a function $f \in S$ is in $CV$ if and only if $\text{Re}\left\{1 + (zf''(z)/f'(z))\right\} > 0$ for $z \in D$.

The class of bi-univalent functions was first introduced by Lewin in 1967 [9]. A function $f \in A$ is said to be bi-univalent function in unit disk, if both $f$ has an inverse $f^{-1}$ are univalent in $D$. Let $\sigma$ denote the class of bi-univalent functions defined in unit disk. In fact, the inverse function $f^{-1}$ can be written as follow

$$g(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots.$$  

In 1967, Lewin found the class of bi-univalent functions and discovered the bound for the second coefficients. Brannan and Taha (1985) obtained estimates on the initial coefficient of $a_2$ and $a_3$ for the class of bi-starlike and bi-convex functions. Deniz (2013) and Kumar et al. (2013) both extended the results by generalizing those classes using subordination. Deniz et al. (2015) managed to find the bounds for the second Hankel determinant for the classes of
bi-starlike and bi-convex functions.

In this paper, we determine the coefficients of $a_2$, $a_3$ and $a_4$ for new classes defined and obtain the second Hankel determinant for new classes define as functions $S^*_m(\beta, \delta)$, which satisfy the condition

$$\Re\left\{e^{i\delta}\frac{f'(z)}{f(z)}\right\} > \beta$$

(1.3)

where $|\delta| < \pi$ and $\cos \delta > \beta$. The consideration of a different condition for $\delta$ and $\beta$ have been studied earlier by Silverman and Silvia (1996), Mohamad (2001), Soh and Mohamad (2006), (2008), (2014), and Akbarally et al. (2011) for a class that was defined by Noshiro-Warschawski Theorem (Goodman, 1983).

In order to get our main results, we must recall here the following lemmas.

**Lemma 1.3.** [4] Let $p$ be the element of $D$ where $p$ is analytic in unit disc, $D$. If $p \in P$ is given by the series

$$p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \cdots + p_nz^n = 1 + \sum_{n=0}^{\infty} p_nz^n,$$

(1.4)

then

$$|a_n| \leq 2$$

holds for each $n$.

**Lemma 1.4.** [7] If the function $p \in P$ is given by the series (1.4), then

$$2c_2 = c_1^2 + x(4 - c_1^2)$$

(1.5)

for some $x$, $|x| \leq 1$ and

$$4c_3 = c_1^2 + 2(4 - c_1^2)c_1x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)t$$

(1.6)

or some $t$, $|t| \leq 1$.

**Initial Coefficient of Tilted Bi-Starlike Functions**

We will prove theorem 2.1 for our defined class of tilted bi-starlike functions.

**Theorem 2.1.** If $f(z) \in S^*_m(\beta, \delta)$ for $\cos \delta > \beta$ and $-\pi \leq \delta \leq \pi$, then

$$a_2 = \frac{c_1[\cos(\delta) - \beta]}{e^{i\theta}}$$

$$a_3 = \frac{c_1^2[\cos(\delta) - \beta]^2}{e^{i\theta}} + \frac{1}{4}[c_2 - d_2][\cos(\delta) - \beta]$$

and

$$a_4 = \frac{2}{3}c_1^3[\cos(\delta) - \beta]^3 + \frac{5}{8}c_1(c_2 - d_2)[\cos(\delta) - \beta]^2 + \frac{1}{6}[c_3 - d_2][\cos(\delta) - \beta]$$

$$e^{i\delta}\frac{f'(z)}{f(z)} - \beta = e^{i\delta}(1 + \sum_{n=1}^{\infty} h_nz^n) - \beta,$$

Next, by simplifying the equation, we obtained
\[
e^{\frac{id}{f(z)}} \frac{zf'(z)}{f(z)} - \beta - i \sin(\delta) = \cos(\delta) - \beta + e^{\frac{id}{\sum_{n=1}^{\infty} h_n z^n}},
\]

which results in
\[
\frac{e^{\frac{id}{f(z)}}}{\cos(\delta) - \beta} \frac{zf'(z)}{\cos(\delta) - \beta} = \frac{1 + e^{\frac{id}{\sum_{n=1}^{\infty} h_n z^n}}}{\cos(\delta) - \beta}.
\]

(2.1)

Therefore, from the left hand side of equation (2.1) and lemma 1.3, we get
\[
\frac{e^{\frac{id}{f(z)}}}{\cos(\delta) - \beta} = p(z),
\]

for some \( z \in D \), where \( p(z) \) has the series of equation (1.4) and \( p(z) \) in \( P \). Since, the function \( f(z) \) has the form (1.1), and further simplification for left hand side and right hand side of equation (2.2), we have the coefficients for
\[
a_2 = \frac{c_1[\cos(\delta) - \beta]}{e^{it}},
\]

(2.3)
\[
2a_3 - a_2^2 = \frac{c_1[\cos(\delta) - \beta]}{e^{it}},
\]

(2.4)
\[
3a_4 - 3a_2a_3 + a_2^3 = \frac{c_2[\cos(\delta) - \beta]}{e^{it}}.
\]

(2.5)

Next, we repeat the same process for \( g = f^{-1} \) which has the form (1.2). Therefore, we have the coefficients for
\[
a_2 = \frac{d_2[\cos(\delta) - \beta]}{e^{it}},
\]

(2.6)
\[
3a_2^2 - 2a_3 = \frac{d_2[\cos(\delta) - \beta]}{e^{it}},
\]

(2.7)
\[
-3a_4 + 12a_2a_3 - 10a_3^2 = \frac{d_4[\cos(\delta) - \beta]}{e^{it}}.
\]

(2.8)

From (2.3) and (2.6), we obtain
\[
e_1 = -d_4,
\]

(2.9)

and hence
\[
a_2 = \frac{c_1[\cos(\delta) - \beta]}{e^{it}}.
\]

(2.10)

Now, from equation (2.4), (2.7) and (2.10), we obtain solution for
\[
a_3 = \frac{c_2^2[\cos(\delta) - \beta]^2}{e^{2it}} + \frac{1}{4} \frac{c_2 - d_2)[\cos(\delta) - \beta]}{e^{it}}.
\]

(2.11)

Also, from (2.5) and (2.8), (2.10) and (2.11), we get
\[
a_4 = \frac{2 c_1^2[\cos(\delta) - \beta]^2}{e^{2it}} + \frac{5}{8} \frac{c_1(c_2 - d_2)[\cos(\delta) - \beta]^2}{e^{2it}} + \frac{1}{6} \frac{c_2 - d_2)[\cos(\delta) - \beta]}{e^{it}}.
\]

(2.12)

The proof of Theorem 2.2 is completed.
Second Hankel Determinant for Tilted Bi-Starlike Functions

This section deals with the proof of the second Hankel determinant theorem that correspond to the class $S_2^*(\beta, \delta)$.

Theorem 3.1. If $f(z) \in S_2^*(\beta, \delta)$ and if $A = \cos(\delta) - \beta$ for $-\pi \leq \delta \leq \pi$, then

$$|a_2a_4 - a_3^2| \leq \begin{cases} \frac{4}{3}A^2(4A^2 + 1), & A \in \left[0, \frac{3 + \sqrt{89}}{16}\right] \\ A^2 \left(13A^2 - 12A - 8\right) & A \in \left[\frac{3 + \sqrt{89}}{16}, 1\right] \end{cases}$$

Proof. From (2.10), (2.11) and (2.12) and letting $A = \cos \delta - \beta$, we have

$$a_2 = \frac{c_2A}{e^{2z}}, \quad a_3 = \frac{c_2A}{e^{2z}} + \frac{1}{4} [c_2(x - d_2)]^2,$$

and

$$a_4 = \frac{2c_2^3A^3}{3} + \frac{8}{8} \frac{c_2(x - d_2)^2}{e^{2z}} + \frac{1}{6} [c_2(x - d_2)]^2,$$

Hence, the functional $a_2a_4 - a_3^2$ will become

$$a_2a_4 - a_3^2 = -\frac{1}{3}A^4 + \frac{1}{8} \frac{c_2(x - d_2)^2}{e^{2z}} + \frac{1}{6} [c_2(x - d_2)]^2 - \frac{1}{16} [c_2(x - d_2)]^2,$$

According to Deniz et al. (2015), lemma 1.4 and equation (2.9), the equations are in the form of,

$$2c_2 = c_1^2 + x(4 - c_1^2),$$

$$2d_2 = d_1^2 + y(4 - d_1^2),$$

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)t,$$

and

$$4d_3 = d_1^3 + 2(4 - d_1^2)d_1y - d_1(4 - d_1^2)y^2 + 2(4 - d_1^2)(1 - |y|^2)s.$$

Therefore, we write

$$c_2 - d_2 = \frac{(x - y)(4 - c_1^2)}{2},$$

and

$$c_3 - d_3 = \frac{c_1^2}{2} + \frac{c_1(x + y)(4 - c_1^2)}{4} - \frac{c_1(x + y^2)(4 - c_1^2)}{4} \left[(1 - |x|^2)t + (1 - |y|^2)s\right].$$

By letting $\lambda = |x| \leq 1$, $\mu = |y| \leq 1$, $c_1 = c \in [0, 2]$ and $e^{i\beta} = 1$, then applying lemma 1.4 and triangular inequality for some $x, y, t$ and $s$ with $|x| \leq 1$, $|y| \leq 1$, $|t| \leq 1$, and $|s| \leq 1$, we have

$$|a_2a_3 - a_3^2| \leq \frac{1}{3} A^4 + \frac{1}{8} c_1^2 + c_1^2(4 - c_1^2)A^2 + \frac{1}{16} c_1^2(4 - c_1^2)A^2 + \frac{1}{12} c_1^2(4 - c_1^2)A^2 + \frac{1}{12} c_1^2(4 - c_1^2)A^2 (\lambda + \mu)$$

$$+ \frac{1}{12} c_1^2(4 - c_1^2)A^2 - \frac{1}{12} c_1^2(4 - c_1^2)A^2 \left(\lambda^2 + \mu^2\right) + \frac{1}{12} c_1^2(4 - c_1^2)A^2 (\lambda^2 + \mu^2)^2.$$
\[ T_3 = \frac{1}{24} c(4 - c^2)A^2(c - 2), \]

and
\[ T_4 = \frac{1}{64} (4 - c^2)^2 A^2. \]

Now we need to maximize \( F(\lambda, \mu) \) where \( 0 \leq \lambda \leq 1 \) and \( 0 \leq \mu \leq 1 \) using method of extrema function for multivariable. By differentiating the function \( F(\lambda, \mu) \) partially, we have
\[ \frac{\partial F}{\partial \lambda} = T_2 + 2T_3\lambda + 2T_4(\lambda + \mu) = 0, \] (3.1)
and
\[ \frac{\partial F}{\partial \mu} = T_2 + 2T_3\mu + 2T_4(\lambda + \mu) = 0. \] (3.2)

By equating (3.1) and (3.2), we obtain \( \lambda = \mu = \lambda = (T_2)/2(T_3 + 2T_4). \) Since, the function \( F(\lambda, \mu) \) cannot have a local maximum, we investigate the maximum of \( F(\lambda, \mu) \) on the boundary. For \( \lambda = 0 \) and \( 0 \leq \lambda \leq 1 \) (similarity \( \mu = 0 \) and \( 0 \leq \mu \leq 1 \)), we obtain \( F(0, \mu) = T_1 + T_2(\mu) + (T_3 + T_4)\mu^2 = G(\mu). \) We attained the interior point of \( 0 \leq \lambda \leq 2 \) for \( 0 \leq \mu \leq 1 \) when \( T_3 + T_4 \geq 0 \). The function \( G'(\mu) > 0 \) for \( \lambda > 0 \) indicate that \( F \) is an increasing function. Therefore, the upper bound for functional \( |\frac{1846}{2870}/\frac{1846}{2871}/\frac{1846}{2872}/\frac{1846}{2870}/\frac{1846}{2871}/\frac{1846}{2872}/\frac{1846}{2870}/\frac{1846}{2871}/\frac{1846}{2872} = \frac{1846}{2870} + 2 \frac{1846}{2871}/\frac{1846}{2872}/\frac{1846}{2870}/\frac{1846}{2871}/\frac{1846}{2872}/\frac{1846}{2870}/\frac{1846}{2871}/\frac{1846}{2872} = 0, \) (3.1)
and
\[ \frac{\partial F}{\partial \mu} = T_2 + 2T_3\mu + 2T_4(\lambda + \mu) = 0. \] (3.2)

Next, we looking for \( \lambda = 1 \) and \( 0 \leq \lambda \leq 1 \) (similarity \( \mu = 1 \) and \( 0 \leq \mu \leq 1 \)), we obtained
\[ F(1, \mu) = T_1 + T_2 + T_3 + T_4 + (T_2 + 2T_4)\mu + (T_3 + T_4)\mu^2 = H(\mu). \]

Similarly, to the above cases of \( T_3 + T_4 \) where \( \mu = 1 \), we get
\[ \max H(\mu) = H(1) = T_1 + 2T_2 + 2T_3 + 4T_4. \]

Since \( G(1) \leq H(1) \), we attained the interior point of \( c \in [0,2] \) where maximum of \( F \) occurs at \( \lambda = 1 \) and \( \mu = 1 \). Therefore, \( F(\lambda, \mu) = F(1,1) = T_1 + 2T_2 + 2T_3 + 4T_4 = K(c) \) in the function \( K(c) = [(16A^2 - 6A - 5)c^4 + (24A + 24)c^2 + 48]. \)

Assume that \( K(c) \) has a maximum value in an interior of \( c \in [0,2] \). By differentiating the function \( K(c) \) with respect to \( c \), we have
\[ K'(c) = \frac{d}{dc} \left[ 4c^3 (16A^2 - 6A - 5) + 48c(A + 1) \right]. \]

By letting \( 16A^2 - 6A - 5 \geq 0 \) that is
\[ A \in \left[ 0, \frac{-3 + \sqrt{17}}{16} \right]. \]

Therefore, \( K'(c) > 0 \) for \( c \in [0,2] \). Since \( K \) is increasing function in the interval \( c \in [0,2] \), maximum point of \( K \) be on the boundary where \( c = 2 \). Thus,
\[ \max K(c) = K(2) = \frac{4}{3} A^2 (4A^2 + 1). \]
Hence,
\[|a_2a_3 - a_3^2| \leq \frac{4}{3}A^2(4A^2 + 1).\]

Also, by letting \(16A^2 - 6A - 5 < 0\) that is
\[A \in \left[\frac{3 + \sqrt{97}}{16}, 1\right].\]

We observe that \(c_0 < 2\) that is \(c_0\) in the interval \([0, 2]\). Since \(K''(c_0) < 0\), the maximum value of \(K(c)\) occurs at \(c = c_0\). Therefore,
\[\max K(c_0) = K\left(\sqrt{\frac{-12A+11}{16A^2-6A-5}}\right) = A^2\left(\frac{13A^2-12A-8}{16A^2-6A-5}\right).\]

Hence,
\[|a_2a_3 - a_3^2| \leq A^2\left(\frac{13A^2-12A-8}{16A^2-6A-5}\right).\]

The proof of Theorem 3.1 is completed. By assuming \(\beta = 0\) and \(\delta = 0\), the results obtained are similar to the result of Deniz et al. (2015) as given in Corollary 3.2.

**Corollary 3.2.** Let \(f(z)\) given in (1.1) be in the class \(S^*_\alpha(\beta, \delta)\), and \(\beta = 0, \delta = 0\), then
\[|a_2a_3 - a_3^2| \leq \frac{20}{7}.\]

**Acknowledgment**

The authors would like to thank Faculty of Computer and Mathematical Sciences, Universiti Teknologi MARA, Malaysia for the facilities and financial support.

**References**

1. Akbarally, A., Mohamad, D., Soh, S. C. & Kaharudin, N. (2011). On the properties of a new class of \(\alpha\)-close–to-convex functions. Int. Journal of Math. Analysis, Vol. 5, No. 8, 387-396.
2. Braman, D. A., & Taha, T. S. (February 18-21, 1985). On some classes of bi-univalent functions. Mathematical Analysis and Its Applications, 3, 53-60.
3. Deniz, E., Çağlar, M., & Orhan, H. (2015). Second Hankel determinant for bi-starlike and bi-convex functions of order \(\beta\). Applied Mathematics and Computation, 271, 301-307.
4. Duren, P. L. (1983). Univalent Functions (Fundamental Principles of Mathematical Sciences (Vol. 259).
5. Deniz, E. (2013). Certain subclasses of bi-univalent functions satisfying subordinate conditions. Journal of Classical Analysis, 2(1), 49-60.
6. Goodman, A. W. (1983). Univalent functions (Vol. 1). Tampa, Florida : Mariner Publication Company.
7. Grenander, U., & Szego, G. (1958). Toeplitz forms and their applications. University of California Press.
8. Kumar, S. S., Kumar, V., & Ravichandran, V. (2013). Estimates for the initial coefficients of bi-univalent functions. Tamsui Oxford Journal of Information and Mathematical Sciences, 29(4) 487-504 Aletheia University.
9. Mohamad, D. (2001). On a Class Of Function Whose Derivatives Map the Unit Disc into a Half Plane. Bulletin Malaysian Mathematical Science Society (2ndSeries), 23, 163-171.
10. Rosihan, M. A., Lee, S. K., Ravichandran, V., & Supramaniam, S. (2012). Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions. Applied Mathematics Letters, 25(3), 344-351.
11. Silverman, H. and Silvia, E. M. (1996). On \(\alpha\)-close-to-convex functions. Publicationes Mathematicae Debrecen, 49, 532-537
12. Soh, S. C. and Mohamad, D. (2006). On Extremal Properties for a \(\alpha\)-close-to-convex Functions. Proceeding of the 2nd IMT-GT Regional Conference on Mathematics, Statistic and Application, University Sains Malaysia, 77-81
13. Soh, S. C. and Mohamad, D. (2008). Coefficient bounds for certain classes of close-to-convex functions. International Journal of Mathematical Analysis, Vol. 2, No 27, 1343-1351.
14. Soh, S. C., & Mohamad, D. (2014). Second hankel determinant for a class of close-to-convex functions with Fekete-Szego parameter. International Journal of Math. Analysis, Vol. 12, No. 8, 561-570.
