Theoretical justification and error analysis for slender body theory

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July 3, 2018

Abstract

Slender body theory facilitates computational simulations of thin fibers immersed in a viscous fluid by approximating each fiber as a one-dimensional curve of point forces. However, it has been unclear how well slender body theory actually approximates Stokes flow about a thin but truly three-dimensional fiber, in part due to the fact that simply prescribing data along a one-dimensional curve does not result in a well-posed boundary value problem for the Stokes equations in $\mathbb{R}^3$. Here, we introduce a well-posed PDE problem to which slender body theory (SBT) provides an approximation, thereby placing SBT on firm theoretical footing. Given only a 1D force density along a closed fiber, we show that the flow field exterior to the thin fiber is uniquely determined by imposing a fiber integrity condition: the surface velocity field on the fiber must be constant along cross sections orthogonal to the fiber centerline. Furthermore, a careful estimation of the residual, together with stability estimates provided by the PDE well-posedness framework, allow us to establish error estimates between the slender body approximation and the exact solution to the above problem. The error is bounded by an expression proportional to the fiber radius (up to logarithmic corrections) under mild regularity assumptions on the 1D force density and fiber centerline geometry.

Contents

1 Introduction .......................................................... 2
  1.1 Slender body geometry ........................................ 3
  1.2 Classical slender body theory ............................... 5
    1.2.1 The Stokeslet .......................................... 6
    1.2.2 Slender body theory ................................... 6
  1.3 Slender body PDE formulation .............................. 10

2 Well-posedness of slender body PDE ......................... 12
  2.1 Existence and uniqueness .................................. 15
  2.2 Higher regularity ........................................... 19
    2.2.1 Regularity away from $\Gamma_\epsilon$ ................. 19
    2.2.2 Tangential regularity up to $\Gamma_\epsilon$ ............ 24
    2.2.3 Normal regularity up to $\Gamma_\epsilon$ ............... 30

3 Geometry revisited ............................................. 32
  3.1 Trace inequality ............................................ 32

*This research was supported in part by NSF grant DMS-1620316 and DMS-1516978, awarded to Y.M., by NSF GRF grant 00039202 and a Torske Kubben Fellowship, awarded to L.O., and by NSF grant DMS-1516565, awarded to D.S. The authors also thank the IMA where most of this work was performed.
1 Introduction

Describing the motion of thin filaments immersed in a viscous fluid presents an important modeling problem in mathematical biology, engineering, and physics. Numerical simulations of slender fibers have been used to help explain the role of cilia in embryonic development [36], simulate microtubules forming the mitotic spindle during cell division [34], understand the rheology of fiber suspensions used in creating composite materials [15, 19, 30], and explore the dynamics of swimming microorganisms [18, 24, 29, 32, 33]. Models describing the interaction between thin structures and a viscous fluid also aid in the design and optimization of microfluidic devices [1, 3, 7, 12], though most such devices are still purely theoretical.

To handle the simulation of the large numbers of thin fibers arising in these models, many existing numerical methods rely on a classical approximation known as slender body theory. In essence, slender body theory reduces computational costs by exploiting the thin geometry of the objects being modeled.

To begin, we assume that the slender fibers are immersed in low Reynolds number flow, typified by any of the following: high viscosity, very slow (creeping) flow, or flow over very small length scales. Such flows are governed by the Stokes equations (1.1), where \( \mathbf{u} \) represents the fluid velocity, \( p \) is the pressure, and \( \mu \) is the viscosity:

\[
\begin{align*}
-\mu \Delta \mathbf{u} + \nabla p &= 0 \\
\text{div} \mathbf{u} &= 0
\end{align*}
\]

(1.1)

accompanied by appropriate boundary conditions. Stokes flow around solid objects in unbounded or semi-bounded domains can be represented succinctly via boundary integral equations over the surface of the object [31]. However, despite this explicit boundary integral representation of a solution to the Stokes system, solving integral equations over moving surfaces remains a computationally intensive task, especially when simulating tens or hundreds of individual objects.

Instead of treating a filament as a three-dimensional object and solving equations for its surface velocity, slender body theory approximates a thin filament as a one-dimensional curve of point
forces in three-dimensional space. The idea of modeling a thin fiber as a line distribution of fundamental singularities originated with Hancock [20], Cox [11], Batchelor [2], and Lighthill [26]. Later, Keller and Rubinow [22] and Johnson [21] included Stokeslets with doublet corrections along the fiber centerline to come up with the integral expression (1.15) that we regard as classical slender body theory. Since then, slender body theory has formed the basis for many numerical methods developed to model thin fibers in Stokes flow [5, 9, 10, 17, 35, 37].

Despite the many numerical results relying on this theory, there is a lack of rigorous error analysis for slender body theory itself. The theory is built on the assumption that a thin but inherently three-dimensional object is well approximated by a one-dimensional distribution of point forces along the object’s centerline. However, Stokes flow in three dimensions resulting from boundary data specified along a one-dimensional curve is inherently not a well-defined problem.

This difficulty manifests itself as a solvability issue: the classical notion of slender body theory gives rise to a singular integral equation for the force-per-unit-length \( f(s) \) along the fiber centerline that is not necessarily solvable without modification. These issues are explored in detail by Götz [17] and also addressed by Tornberg and Shelley [37]. The methods commonly employed to avoid these solvability issues – usually regularization of the integral kernel – still lack a rigorous theoretical justification.

Many of the foundational papers in slender body theory compute some notion of asymptotic accuracy of the slender body approximation [17, 21, 22]. Recently, Koenigs and Lauga [23] also derived the slender body expression as an asymptotic limit of the full boundary integral equations. However, given the solvability issues that arise when trying to compute the slender body approximation, it remains unclear how to compare the slender body approximation to the actual PDE solution for Stokes flow about a 3D fiber. It is therefore unclear how to rigorously prove that slender body theory is a good approximation to Stokes flow around a thin filament. Previous studies have numerically verified the convergence of the slender body approximation as the slender body radius tends to zero [5], but to what exactly the approximation is converging remains unclear.

To perform a rigorous error analysis, we need a different way of thinking about the question that slender body theory aims to resolve. In this paper, we address this question of what slender body theory approximates by giving meaning to a solution to the Stokes equations about a slender fiber in \( \mathbb{R}^3 \), given only one-dimensional force data and what we call the “fiber integrity constraint” (see Remark 1.2). Making sense of such a solution is important because it allows us to develop a new notion of “true solution” to the slender body problem against which we are actually able to compare the slender body approximation.

### 1.1 Slender body geometry

Before we can introduce the slender body approximation, we must precisely describe the slender geometries under consideration.

Let \( \mathbf{X} : \mathbb{T} \equiv \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}^3 \) denote the coordinates of a closed curve \( \Gamma_0 \subset \mathbb{R}^3 \), parameterized by arclength \( s \) with the length of \( \mathbf{X} \) normalized to 1. Let \( C^k(\mathbb{T}) \), \( k \in \mathbb{N} \), denote the space of \( k \)-continuously differentiable functions defined on \( \mathbb{T} \) (we will use the same notation, without confusion, for scalar or \( \mathbb{R}^3 \)-valued functions). We assume that \( \mathbf{X}(s) \in C^2(\mathbb{T}) \) so that its curvature \( \kappa(s) = \left| \frac{d^2 \mathbf{X}}{ds^2} \right| \) is well-defined.
We assume that $\Gamma_0$ is non-self-intersecting; in particular,

$$\inf_{s \neq t} \frac{|X(s) - X(t)|}{|s - t|} \geq c_\Gamma$$

for some constant $c_\Gamma > 0$.

For computational purposes, it will be convenient to construct a $C^1$ orthonormal frame along the slender body centerline $\Gamma_0$, periodic with respect to the arclength variable $s$. We begin by defining the tangent vector

$$e_t(s) = \frac{dX}{ds}.$$ 

We need to find a vector field $e_{n_1}(s) \in C^1(\mathbb{T})$ orthogonal to $e_t(s)$ at each $s \in \mathbb{T}$. We then define $e_{n_2} = e_t \times e_{n_1}$ so that $e_t, e_{n_1}, e_{n_2}$ form an orthonormal frame along $X(s)$. By orthonormality, any such $C^1$ moving frame must satisfy an ODE of the form

$$\frac{d}{ds} \begin{pmatrix} e_t(s) \\ e_{n_1}(s) \\ e_{n_2}(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1(s) & \kappa_2(s) \\ -\kappa_1(s) & 0 & \kappa_3(s) \\ -\kappa_2(s) & -\kappa_3(s) & 0 \end{pmatrix} \begin{pmatrix} e_t(s) \\ e_{n_1}(s) \\ e_{n_2}(s) \end{pmatrix},$$

where $\kappa_j, j = 1, 2, 3$ are continuous functions of $s$. Note that if $X$ is $C^3$ and the curvature $\kappa(s)$ is non-vanishing everywhere on $\mathbb{T}$, we can then use the simpler Frenet frame, where $e_{n_1}(s) = e_t'(s)/\kappa(s)$, $\kappa_1(s) = \kappa(s)$, $\kappa_2 \equiv 0$, and $\kappa_3 = \tau(s)$, the torsion of the curve $X(s)$. This is useful because the ODE satisfied by the basis vectors is simpler and the coefficients have a clear geometric meaning. However, to allow for more general $C^2$ curves with possibly vanishing curvature at some points, we must develop a frame that is well-defined when $\kappa(s) = 0$.

To this end, we state the following lemma, whose proof is contained in Appendix 6.1.
Lemma 1.1. There exist \( e_{n_1}(s), e_{n_2}(s) \in C^1(\mathbb{T}) \) which, together with \( e_t(s) \), form an orthonormal frame along \( X(s) \). Furthermore, \( \kappa_3 \) in (1.2) may be made to satisfy

\[ \kappa_3 \text{ does not depend on } s \text{ and } |\kappa_3| \leq \pi. \tag{1.4} \]

The orthonormal frame constructed here is almost the same as the Bishop frame [4], except that \( \kappa_3 \) cannot necessarily be made to vanish for a closed curve.

Although the geometric meaning of the general orthonormal frame coefficients \( \kappa_j \) is less clear than for the Frenet frame, we note that the curvature \( \kappa(s) \) of the fiber centerline always satisfies

\[ \kappa(s) = \sqrt{\kappa_1^2(s) + \kappa_2^2(s)}. \tag{1.5} \]

We define

\[ \kappa_{\max} = \max_{s \in \mathbb{T}} |\kappa(s)| \tag{1.6} \]

and note that, since \( X \) is a \( C^2 \) closed loop of length 1, we have \( 1/\pi < \kappa_{\max} < \infty \).

We also define the following cylindrical unit vectors with respect to the moving frame:

\[ e_\rho(s, \theta) := \cos \theta e_{n_1}(s) + \sin \theta e_{n_2}(s) \]
\[ e_\theta(s, \theta) := -\sin \theta e_{n_1}(s) + \cos \theta e_{n_2}(s). \]

Since the slender body is non-self-intersecting with \( C^2 \) centerline, we may parameterize points \( x \) with \( \text{dist}(x, X) < \frac{1}{2\kappa_{\max}} \) as a tube about the fiber centerline (see Figure 1.1):

\[ x = X(s) + \rho e_\rho(s, \theta). \tag{1.7} \]

For \( \epsilon < 1/(8\kappa_{\max}) \), we then define a slender body \( \Sigma_\epsilon \) with uniform radius \( \epsilon \) by

\[ \Sigma_\epsilon = \{ x \in \mathbb{R}^3 : x = X(s) + \rho e_\rho(s, \theta), \quad \rho < \epsilon \} \tag{1.8} \]

We parameterize the surface of the slender body, \( \Gamma_\epsilon = \partial \Sigma_\epsilon \), as

\[ \Gamma_\epsilon(s, \theta) = X(s) + \epsilon e_\rho(s, \theta). \tag{1.9} \]

The surface element on \( \Gamma_\epsilon \) is then given by

\[ dS = J_\epsilon(s, \theta) \, d\theta ds, \tag{1.10} \]

where we define

\[ J_\epsilon(s, \theta) := \epsilon \left( 1 - \epsilon (\kappa_1(s) \cos \theta + \kappa_2(s) \sin \theta) \right). \tag{1.11} \]

We also define the neighborhood

\[ \mathcal{O} = \left\{ x \in \Omega_\epsilon : x = X(s) + \rho e_\rho(s, \theta), \quad \epsilon < \rho < \frac{1}{2\kappa_{\max}} \right\} \tag{1.12} \]

of the slender body to refer to fluid points \( x \) near to the slender body.

1.2 Classical slender body theory

With the geometric constraints specified above, we now define the corresponding slender body approximation to Stokes flow about the thin fiber.
1.2.1 The Stokeslet

The essential building block of slender body theory is the Stokeslet, the free-space Green’s function for the Stokes equations (1.1). The Stokeslet represents the Stokes flow in $\mathbb{R}^3$ resulting from a point source at $x_0$ of strength $g$:

$$-\mu \Delta u + \nabla p = g \delta(x - x_0)$$
$$\text{div} \ u = 0$$
$$|u| \to 0 \quad \text{as} \quad |x| \to \infty,$$

where $\delta(x)$ denotes the Dirac delta at $x$. We define the Stokeslet and its associated pressure tensor as

$$S(\hat{x}) = \frac{I}{|\hat{x}|^3} + \frac{\hat{x} \hat{x}^T}{|\hat{x}|^3}, \quad p^S(\hat{x}) = \nabla \left( \frac{1}{|\hat{x}|} \right) = \frac{\hat{x}}{|\hat{x}|^3},$$

where $I$ is the identity matrix and $\hat{x} = x - x_0$ (see [31, 8] for a derivation). The solution to (1.13) is then given by

$$u = \frac{1}{8 \pi \mu} S(\hat{x}) g, \quad p = \frac{1}{4\pi} p^S(\hat{x}) \cdot g.$$

Since the singularly forced Stokes system (1.13) is linear, additional solutions may constructed by differentiating the Stokeslet and taking linear combinations of the Stokeslet and higher-order derivatives – dipoles, quadrupoles, octupoles, etc. Inclusion of these higher-order multipole terms in the expression of solutions to (1.13) can be useful especially in solving exterior problems, and is sometimes referred to as the method of singularities [31].

The higher-order term that plays the most important role in slender body theory, known as the doublet, is given by

$$D(\hat{x}) = \frac{1}{2} \Delta S(\hat{x}) = \frac{I}{|\hat{x}|^3} - 3 \frac{\hat{x} \hat{x}^T}{|\hat{x}|^5}.$$

1.2.2 Slender body theory

The idea of slender body theory is to approximate the velocity field around a thin filament in Stokes flow by integrating a superposition of Stokeslets, doublets, and possibly higher-order multipole terms along the centerline of the fiber. The slender body ansatz is given by the integral expression

$$u^\text{SB}(x) = u_\infty(x) + \frac{1}{8 \pi \mu} \int_T \left( S(x - X(t))g_1(t) + D(x - X(t))g_2(t) + \cdots \right) dt,$$

where $u_\infty$ is the undisturbed background fluid velocity, and the dots indicate the possibility of including higher-order multipole terms. The coefficients $g_i$ of the higher-order terms are chosen to best preserve the structural integrity of the fiber (see below).

The simplest prescription for $g_i$, $i = 1, 2, \ldots$ would be to set $g_1(t) = f(t), g_i = 0$ for $i \geq 1$, where $f(t)$ is the prescribed force density along the fiber centerline. The problem with this choice is that the surface velocity $u^\text{SB}|_{F}(s, \theta)$ has a strong $\theta$-dependence on each cross-section $s$ (see Figure 1.2). If the no-slip condition is satisfied on the fiber interface, this will lead to an instantaneous deformation of the fiber cross sectional geometry, destroying the structural integrity of the fiber. Setting $g_2(t) = \frac{\epsilon^2}{2} g_1(t)$ eliminates this $\theta$-dependence to leading order, so
that the surface velocity is almost constant along cross-sections. We term this the fiber integrity condition, to be formulated precisely below.

We note that this fiber integrity constraint ignores torque and does not allow the fiber to simply rotate about its centerline. The additional consideration of torque along the fiber is an extension to the classical slender body approximation \(1.15\) that will be addressed in future work.

Thus, the classical (non-local) slender body approximation to the fluid velocity at a point \(x\) away from the centerline is given by

\[
8\pi \mu (u^{SB}(x) - u_{\infty}(x)) = \int_{\Gamma} \left( S(R) + \frac{\epsilon^2}{2} D(R) \right) f(t) \, dt; \quad R = x - X(t),
\]

\[
S(R) = \frac{I}{|R|} + \frac{RR^T}{|R|^3}, \quad D(R) = \frac{I}{|R|^3} - 3 \frac{RR^T}{|R|^5}.
\] \hspace{1cm} (1.15)

Hereafter, we take \(u_{\infty} = 0\) and focus on the relationship between a prescribed force \(f(s)\) along the fiber centerline and the resulting slender body velocity \(u^{SB}\).

**Remark 1.2.** As justification for the choice of a doublet correction to the Stokeslet velocity, as well as its coefficient of \(\frac{\epsilon^2}{2}\), we consider the slender body approximation for a straight, infinite cylinder in 3D with constant force \(f^c\) prescribed along the fiber centerline, which we take to be the \(z\)-axis with respect to Cartesian coordinates \(x, y, z\). The resulting velocity field diverges logarithmically at infinity (due to Stokes’ paradox), but the following calculation will provide an instructive heuristic nonetheless. In this case, the velocity field within each plane orthogonal to the \(z\)-axis satisfies the 2D Stokes equations, with the constant force \(f^c\) projected onto its \(x\) and \(y\) components \(f_1^c\) and \(f_2^c\) (see Figure 1.3).

We may thus consider Stokes flow in \(\mathbb{R}^2\) due to a point source which we can take to be placed at the origin. Then the velocity due to the Stokeslet at each \(x \neq 0\) is given by

\[
u^S(x) = \frac{1}{4\pi} \left( -\log \frac{|x|}{|x|^2} + \frac{xx^T}{|x|^2} \right) \begin{pmatrix} f_1^c \\ f_2^c \end{pmatrix},
\]
where $\mathbf{I}$ is the 2D identity matrix. In polar coordinates $\mathbf{x} = (r \cos \theta, r \sin \theta)^T$, we can write the velocity from the Stokeslet as

$$
\mathbf{u}^S(r, \theta) = \frac{1}{4\pi} \left( -\log r \mathbf{I} + \frac{1}{2} \begin{pmatrix} 1 + \cos 2\theta & \sin 2\theta \\ \sin 2\theta & 1 - \cos 2\theta \end{pmatrix} \right) \left( \begin{pmatrix} f_1^c \\ f_2^c \end{pmatrix} \right).
$$

Now, to satisfy the slender body fiber integrity condition, we must eliminate the $\theta$-dependence on the cylinder surface at $r = \epsilon$. However, notice that

$$
\Delta \mathbf{u}^S(r, \theta) = \frac{\partial^2 \mathbf{u}^S}{\partial r^2} + \frac{1}{r} \frac{\partial \mathbf{u}^S}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \mathbf{u}^S}{\partial \theta^2}
= -\frac{1}{2\pi r^2} \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} \begin{pmatrix} f_1^c \\ f_2^c \end{pmatrix}.
$$

Thus, the $\theta$-dependence in the velocity due to the Stokeslet at $r = \epsilon$ can be canceled by adding a doublet term $(\frac{1}{2} \Delta \mathbf{u}^S)$ with coefficient $\epsilon^2/2$:

$$
\mathbf{u}^{SB} = \mathbf{u}^S + \frac{\epsilon^2}{4} \Delta \mathbf{u}^S.
$$

This heuristic shows that the doublet correction with coefficient $\frac{\epsilon^2}{2}$ is chosen to fully eliminate the $\theta$-dependence in the slender body velocity approximation about a straight, infinite fiber with constant prescribed force. Therefore, the addition of $\frac{\epsilon^2}{2} D$ is a natural choice for eliminating the leading-order $\theta$-dependence more generally if we consider the closed-loop fiber with varying force $f(s)$ as a perturbation about the straight centerline/constant force scenario. In fact, asymptotic calculations by Johnson [21] show that the doublet correction in (1.15) for a curved centerline $\mathbf{X}(s) \in C^2(\mathbb{T})$ allows the surface velocity $\mathbf{u}^{SB}|_{\Gamma_s}$ to satisfy the $\theta$-independence condition up to $O(\epsilon |\log \epsilon|)$.

From here, there are two different ways to view the slender body problem: either the force $f(s)$ is prescribed and we want to approximate $\mathbf{u}|_{\Gamma_s}$ and $\mathbf{u}(\mathbf{x})$, or the slender body velocity $\mathbf{u}|_{\Gamma_s}(s)$ is known and we want to approximate the slender body force $f(s)$ and subsequently solve for $\mathbf{u}(\mathbf{x})$ everywhere in the fluid. We refer to these two different perspectives as the forward and the inverse slender body problem, respectively, and write these two problems more precisely as follows.
Forward slender body problem (FP): Given the total force-per-cross-section \( f(s) \), find the (approximate) corresponding slender body velocity \( u^{SB}(s) \) as well as the corresponding Stokes flow \( u^{SB}(x), x \in \Omega_\varepsilon \).

Inverse slender body problem (IP): Given \( \theta \)-independent Dirichlet data \( \bar{u}(s) \) on \( \Gamma_\varepsilon \), find the (approximate) total force-per-cross-section \( f^{SB}(s) \) as well as the corresponding Stokes flow \( u^{SB}(x), x \in \Omega_\varepsilon \).

The main attraction of the inverse problem (IP) is that the true solution to which slender body theory is an approximation can be stated simply as a classical Dirichlet boundary value problem for the Stokes equations with boundary data that is required to be \( \theta \)-independent. The true solution to the inverse problem must satisfy

\[
-\mu \Delta u + \nabla p = 0, \quad \text{div} \ u = 0 \quad \text{in} \ \Omega_\varepsilon = \mathbb{R}^3 \setminus \Sigma_\varepsilon
\]

\[
u|_{\Gamma_\varepsilon} = \bar{u}(s),
\]

where \( \bar{u}(s) \) is a prescribed, \( \theta \)-independent function on \( \Gamma_\varepsilon \).

The existence, uniqueness, and regularity properties of solutions to (1.16) are well studied (see [16, 6] for an in-depth treatment). Hence, the classical slender body approximation [22, 21] was developed to treat this inverse problem. In the Keller-Rubinow approach [22], the expression (1.15) is evaluated at \( \rho = \epsilon \) and the resulting integral kernel \( S(s, \theta, t; \epsilon) + \frac{\epsilon^2}{2} D(s, \theta, t; \epsilon) \) is expanded asymptotically about \( \epsilon = 0 \) to obtain an integral equation on \( X(s) \) approximating \( f^{SB}(s) \) given \( u(s) \). For a periodic filament, the Keller-Rubinow formula (see [35, 10] for periodization of the original formula) is given by

\[
8\pi \mu u(s) = 2 \left[ (I - e_t e_t^T) - (I + e_t e_t^T) \log(\pi \epsilon / 4) \right] f^{SB}(s)
\]

\[
+ \int_T \left[ \left( \frac{1}{|R_0|} + \frac{R_0 R_0^T}{|R_0|^3} \right) f^{SB}(t) - \frac{I + e_t(s)e_t(s)^T}{|\sin(\pi(s-t))/\pi|} f^{SB}(s) \right] dt,
\]

where \( R_0(s,t) := X(s) - X(t) \). Although the underlying PDE framework for (IP) is well understood, and the Keller-Rubinow formulation leads to what appears to be a second-kind Fredholm integral equation for \( f^{SB}(s) \), the integral operator in (1.17) in fact has poor invertibility properties. These issues are explored in detail in [17]. In part due to this complication, it is not clear how to compare the resulting slender body approximation – obtained by plugging \( f^{SB} \) satisfying (1.17) into (1.15) – to the true solution obtained by solving the PDE (1.16).

However, for certain applications, the forward problem (FP) is the more relevant formulation of the slender body problem. In many cases (see [35, 37, 38, 25]), we are more interested in specifying a known elastic force \( f(s) \) along the filament and solving for the resulting fiber velocity \( u(s) \), rather than specifying the velocity and solving for the force. In fact, although many applications actually desire solutions to the forward slender body problem, approximations to \( u(s) \) are often based on the centerline integral equation (1.17), derived to treat the inverse problem.

The main difficulty in thinking about slender body theory in terms of the forward problem (FP) is that it is not immediately clear how to formulate the underlying true solution as a well-posed PDE. Given \( f(s) \), the question remains: to what, exactly, is the slender body expression (1.15) an approximation?
1.3 Slender body PDE formulation

From now on, we work with the forward slender body problem (FP). The difficulty here is that we want to reconstruct a Stokes flow in $\mathbb{R}^3$ given only one-dimensional force data $f(s)$, which we understand to be the total surface force exerted on the fluid per cross section $s$ of the slender body. This total force alone is not enough information to uniquely solve a Stokes boundary value problem, so we additionally impose a fiber integrity condition: we require that the surface velocity of the fiber at each cross section $s$ is independent of the angle $\theta$.

We thus formulate the slender body problem as a boundary value problem for the Stokes system over the fluid domain $\Omega_\epsilon = \mathbb{R}^3 \setminus \Sigma_\epsilon$. Note that by rescaling, we can take the viscosity $\mu \equiv 1$. Let $\sigma = \nabla u + (\nabla u)^T - pI$ denote the stress tensor and $n = \cos \theta e_{n_1}(s) + \sin \theta e_{n_2}(s) = e_\rho(s, \theta)$ denote the unit normal vector to the slender body surface at each point $(s, \theta) \in \Gamma_\epsilon$. We define the slender body Stokes PDE as follows:

$$
-\Delta u + \nabla p = 0, \quad \text{div} u = 0 \quad \text{in} \quad \Omega_\epsilon = \mathbb{R}^3 \setminus \Sigma_\epsilon,
$$

$$
\int_0^{2\pi} \sigma n J_\epsilon(s, \theta) d\theta = f(s) \quad \text{on} \quad \Gamma_\epsilon,
$$

$$
|u|_{\Gamma_\epsilon} = u(s) \quad \text{(unknown but independent of $\theta$)},
$$

$$
|u| \to 0 \text{ as } |x| \to \infty.
$$

(1.18)

Here we use the expression for the Jacobian factor $J_\epsilon(s, \theta)$ given by (1.11). In this formulation, the boundary data is specified as partial Neumann and partial Dirichlet information everywhere along the boundary $\Gamma_\epsilon$, giving rise to a semirigid theory of slender body motion. Fiber movements are constrained by the partial Dirichlet condition $u|_{\Gamma_\epsilon} = u(s)$, so the fiber may bend along its centerline, but cross sections maintain their circular shape and radius $\epsilon$ over time. Since the expression for $u|_{\Gamma_\epsilon}$ is not specified beyond the $\theta$-independence, an infinite family of flows $u$ satisfy this constraint. A unique solution is determined by specifying $f : \mathbb{T} \to \mathbb{R}^3$, a one-dimensional force along the fiber centerline. We define $f$ to be the total surface force $\sigma n|_{\Gamma_\epsilon}$ acting on the body over each cross section, weighted by the surface area of the fiber via $J_\epsilon(s, \theta)$: greater surface area contributes more to the total force along the centerline; smaller surface area contributes less. To close the system, we require that the velocity $u$ decays to 0 as $|x| \to \infty$.

As far as we know, this type of elliptic boundary value problem has not been explored in the literature. However, this formulation appears to be the natural PDE interpretation of the forward slender body problem, as any smooth enough solution to (1.18) satisfies the integral equation

$$
\int_{\Omega_\epsilon} 2|\mathcal{E}(u)|^2 \, dx = \int_{\Gamma_\epsilon} u(s) \sigma n J_\epsilon d\theta ds
$$

$$
= \int_{\mathbb{T}} u(s) f(s) ds; \quad \mathcal{E}(u) = \frac{\nabla u + (\nabla u)^T}{2};
$$

where $\mathcal{E}(u)$ is the strain tensor, or symmetric gradient. This expression has a natural physical interpretation: the dissipation per unit time due to viscosity (left hand side) balances the power exerted by the slender body (right hand side).

We show in Section 2 that the forward problem (FP) corresponding to (1.18) is well-posed in the homogeneous Sobolev space $D^{1,2}(\Omega_\epsilon)$ (see (2.1) for a definition). Using the definition of weak solution given by Definition 2 and (2.6), we show the following theorem:
Figure 1.4: In the forward formulation of the slender body problem, we specify a total force-per-cross-section \( f(s) \) everywhere along \( \Gamma_\epsilon \) and also require that the (unknown) fiber surface velocity \( u \mid \Gamma_\epsilon \) is independent of the angle \( \theta \).

**Theorem 1.3.** (Well-posedness of slender body PDE) Let \( \Omega_\epsilon = \mathbb{R}^3 \setminus \Sigma_\epsilon \) be the exterior of a slender body satisfying the geometric constraints in Section 1.1. Given \( f \in L^2(\mathbb{T}) \), there exists a unique weak solution \( (u, p) \in D^{1,2}(\Omega_\epsilon) \times L^2(\Omega_\epsilon) \) to (1.18) satisfying

\[
\| u \|_{D^{1,2}(\Omega_\epsilon)} + \| p \|_{L^2(\Omega_\epsilon)} \leq |\log \epsilon|^{1/2} c_\kappa \| f \|_{L^2(\mathbb{T})},
\]

where the constant \( c_\kappa \) depends only on the constants \( c_\Gamma \) and \( \kappa_{\text{max}} \) characterizing the shape of the fiber centerline.

Furthermore, if the slender body centerline \( X(s) \) is at least \( C^4 \) and the force \( f(s) \) is in \( H^{1/2}(\mathbb{T}) \), then \( (u, p) \) is a strong solution to (1.18); i.e. \( (u, p) \) is in \( D^{2,2}(\Omega_\epsilon) \times H^1(\Omega_\epsilon) \) and satisfies (1.18) pointwise almost everywhere. Furthermore, the strong solution pair \( (u, p) \) satisfies the estimate

\[
\| u \|_{D^{2,2}(\Omega_\epsilon)} + \| p \|_{H^1(\Omega_\epsilon)} \leq \epsilon^{-1} |\log \epsilon|^{1/2} c_\kappa \| f \|_{H^{1/2}(\mathbb{T})},
\]

where \( c_\kappa \) depends on \( c_\Gamma \), \( \kappa_{\text{max}} \), and the first and second derivatives of the moving frame coefficients \( \kappa_1(s) \) and \( \kappa_2(s) \) in (1.3).

The energy estimates (1.19) and (1.20) indicate that the PDE (1.18) is a natural interpretation of the slender body problem, as the specified one-dimensional slender body force \( f \) then controls the velocity and pressure \( (u, p) \) of the fluid. Comparing (1.19) and (1.20) suggests that it may be possible to relax the regularity assumption on \( f \) in (1.19) to \( f \in H^{-1/2}(\mathbb{T}) \); however, we do not explore this here. Furthermore, the higher regularity theory behind (1.20) gives rise to a solution satisfying the slender body PDE (1.18) in a classical sense (pointwise almost everywhere), and allows us to give meaning to the surface force \( \sigma n \mid \Gamma_\epsilon \) as a function in \( H^{1/2}(\Gamma_\epsilon) \).

The explicit \( \epsilon \)-dependence of the constant \( c_\kappa |\log \epsilon|^{1/2} \) is explored in detail in Section 3. We are ultimately interested in using the solution theory framework established for Theorem 1.3 to estimate the error between the true solution and the slender body approximation in terms of the slender body radius \( \epsilon \). Therefore it will be important to be able to control any \( \epsilon \)-dependence arising in the true problem itself. From a numerical analysis perspective, determining the \( \epsilon \)-dependence in the well-posedness theory is analogous to establishing the stability of a numerical
algorithm. To this end, in Section 3 we verify the $\epsilon$-dependence of the constants arising in the Korn inequality, trace inequality, and pressure estimate. These are each classical inequalities, but their dependence on the size of the radius in the exterior of a thin, flexible fiber may not have been well known previously. In particular, our trace inequality (Section 3.1) is genuinely new, as we rely on the fiber integrity constraint in an essential way. The Korn and pressure inequalities used here (Sections 3.2 and 3.4) apply to more general boundary value problems in the exterior of thin domains, but their dependence on the radius of the thin domain appears to not be well documented.

Most importantly, we can compare this true solution $u$ to the (FP) for (1.18) to the slender body approximation $u^{SB}$, defined by (1.15). Using the well-posedness result and energy bound (1.20) from Theorem 1.3, we show the following error bound:

**Theorem 1.4.** (Slender body theory error estimate) Let $\Omega = \mathbb{R}^3 \setminus \Sigma_\epsilon$ for $\Sigma_\epsilon$ satisfying the geometric constraints in Section 1.1 and with centerline $X(s) \in C^{2,\alpha}(\Gamma)$. Let $u$ be the true solution to the slender body PDE (1.18) and let $u^{SB}$ be the corresponding slender body approximation (1.15). Then the difference $u^{SB} - u$, $p^{SB} - p$ satisfies

$$\|u^{SB} - u\|_{D^{1,2}(\Omega_\epsilon)} + \|p^{SB} - p\|_{L^2(\Omega_\epsilon)} \leq \epsilon |\log \epsilon| c_\kappa \|f\|_{C^1(\Gamma)}.$$  \hfill (1.21)

where the constant $c_\kappa$ depends only on $c_T$, $\kappa_{\text{max}}$, and the $C^{2,\alpha}$ norm of the fiber centerline $X(s)$.

Furthermore, the $L^2$ trace of the error $u^{SB} - u$ along the slender body surface $\Gamma_\epsilon$, scaled by $|\Gamma_\epsilon|^{-1/2} \sim \frac{1}{\sqrt{\epsilon}}$ to account for the vanishing slender body surface area as $\epsilon \to 0$, satisfies

$$\frac{1}{|\Gamma_\epsilon|^{1/2}} \|\text{Tr}(u^{SB} - u)\|_{L^2(\Gamma_\epsilon)} \leq \epsilon |\log \epsilon|^{3/2} c_\kappa \|f\|_{C^1(\Gamma)}.$$  \hfill (1.22)

We note that the $C^{2,\alpha}$ requirement on the fiber centerline is not strictly necessary: we can derive a similar error estimate for $X(s) \in C^2$; however, the additional regularity allows us to get rid of an extra $|\log \epsilon|^{1/2}$ in the estimate (1.21).

Although the slender body PDE is well-posed for rough $f$, in order to actually obtain an error estimate, the force must be more regular. We will see that this is due to the fact that the error depends crucially on the change in the total force distribution along the fiber centerline. The other sources of error stem from the nonzero curvature of the fiber centerline as well as the finite length of the fiber. These error sources are identified in Section 4 by calculating the residual between the slender body approximation and the true force and velocity along $\Gamma_\epsilon$. Although slender body theory is a continuous approximation to a continuous problem, this step can be considered from a numerical analysis point of view as establishing the consistency of the slender body approximation. The exact form of the error estimates in Theorem 1.4 is then derived in Section 5 using the residuals from Section 4 in the PDE framework described in Section 2.

2 Well-posedness of slender body PDE

In this section and the following section, we prove Theorem 1.3. We begin by defining our notion of a weak solution to the PDE (1.18) and proceed to show existence, uniqueness, and higher regularity results for the solution. In Section 3 we complete the proof of Theorem 1.3 by verifying how the constants in the well-posedness theory depends on $\epsilon$ as $\epsilon \to 0$. 

12
We must first define the function space \( D^{1,2}(\Omega_\epsilon) \) for which the well-posedness result is stated. We seek a solution \( u \) to \( (1.18) \) defined over the exterior domain \( \Omega_\epsilon = \mathbb{R}^3 \setminus \Sigma_\epsilon \) such that \( u \) decays to 0 as \( |x| \to \infty \). However, we do not expect this decay to be especially fast. In particular, we expect that \( u \) solving \( (1.18) \) around a thin filament behaves like the Stokeslet far away from the slender body. Thus we expect \( |u| \) to decay like \( \frac{1}{|x|} \) as \( |x| \to \infty \); as such, we do not expect \( u \) to be in \( L^2(\Omega_\epsilon) \). Nevertheless, we do expect \( \nabla u \in L^2(\Omega_\epsilon) \), so we will consider functions in the homogeneous Sobolev space on \( \Omega_\epsilon \).

\[
D^{1,2}(\Omega_\epsilon) = \{ u \in L^6(\Omega_\epsilon) : \nabla u \in L^2(\Omega_\epsilon) \}, \quad (2.1)
\]

explored in detail in [16], Chapter II.6 - II.10. By the Sobolev inequality

\[
\|u\|_{L^6(\Omega_\epsilon)} \leq c_S \|\nabla u\|_{L^2(\Omega_\epsilon)}, \quad c_S > 0,
\]

valid in the exterior domain \( \Omega_\epsilon \subset \mathbb{R}^3 \), we have that

\[
\|u\|_{D^{1,2}(\Omega_\epsilon)} \equiv \|\nabla u\|_{L^2(\Omega_\epsilon)} \quad (2.3)
\]
is a norm on \( D^{1,2}(\Omega_\epsilon) \), and hence \( D^{1,2}(\Omega_\epsilon) \) is a Hilbert space arising naturally in the exterior domain \( \Omega_\epsilon \). We further note that any \( u \in D^{1,2}(\Omega_\epsilon) \) satisfies \( u \in H^1(\Omega_R) \) for any bounded domain \( \Omega_R = B_R \setminus \Sigma_\epsilon, R \gg \text{diam}(\Sigma_\epsilon) \).

We define \( D^0_{1,2}(\Omega_\epsilon) \) as the closure of \( C_0^\infty(\Omega_\epsilon) \) in \( D^{1,2}(\Omega_\epsilon) \). We denote the dual of \( D^0_{1,2}(\Omega_\epsilon) \) by \( D^{-1,2}(\Omega_\epsilon) \).

Since we are also interested in higher regularity, we define the spaces

\[
D^{k,2}(\Omega_\epsilon) = \{ u \in L^6(\Omega_\epsilon) : \nabla^l u \in L^2(\Omega_\epsilon), l = 1, \ldots, k \}
\]
along with the norm

\[
\|u\|_{D^{k,2}(\Omega_\epsilon)} = \sum_{l=1}^{k} \|\nabla^l u\|_{L^2(\Omega_\epsilon)},
\]

where \( \nabla^l \) denotes derivatives of order \( l \geq 1 \).

Equipped with these spaces, we define the notion of a weak solution to the slender body Stokes PDE. We begin by considering the variational formulation of \( (1.18) \). We define the space

\[
\mathcal{A}^\text{div}_\epsilon = \{ v \in D^{1,2}(\Omega_\epsilon) : \text{div} v = 0, v|_{\Gamma_\epsilon} = v(s) \}
\]

where the value of the function \( v(s) \) on the boundary \( \Gamma_\epsilon \) is unspecified but independent of the surface angle \( \theta \). We then have the following trace inequality on \( \mathcal{A}^\text{div}_\epsilon \): using that the trace of \( u \in \mathcal{A}^\text{div}_\epsilon \) satisfies \( \|\text{Tr}(u)\|_{L^2(\Gamma_\epsilon)} = \sqrt{2\pi\epsilon}\|\text{Tr}(u)\|_{L^2(\mathbb{T})} \), we have

\[
\frac{1}{\sqrt{2\pi\epsilon}}\|\text{Tr}(u)\|_{L^2(\Gamma_\epsilon)} = \|\text{Tr}(u)\|_{L^2(\mathbb{T})} \leq c_T\|\nabla u\|_{L^2(\Omega_\epsilon)}, \quad (2.4)
\]

where the \( \epsilon \)-dependence of the constant \( c_T \) will be explored in Section 3. The set \( \mathcal{A}^\text{div}_\epsilon \) is nonempty, as can be seen, for example, by taking any constant function on the surface \( \Gamma_\epsilon \) and solving the corresponding Stokes boundary value problem in \( \Omega_\epsilon \) with this boundary data (see [16], Chapter V.2 for treatment of the Stokes Dirichlet boundary value problem). Furthermore, \( \mathcal{A}^\text{div}_\epsilon \) is a closed subspace of \( D^{1,2}(\Omega_\epsilon) \), which can be shown by taking a sequence \( u_k \in \mathcal{A}^\text{div}_\epsilon \).
converging strongly to \( u \) in \( D^{1,2}(\Omega_e) \). By (2.4), we have strong convergence of the trace \( \text{Tr}(u_k) \) in \( L^2(\Gamma_e) \), and there exists a subsequence \( \text{Tr}(u_{k_j}) \to \text{Tr}(u) \) pointwise almost everywhere. Then the limit \( u \) satisfies the \( \theta \)-independence condition \( u|_{\Gamma_e} = u(s) \). Hence \( A^{\text{div}}_e \) is a Hilbert space with norm \( \| \nabla u \|_{L^2(\Omega_e)} \).

We then define a weak solution to \( (1.18) \) as follows:

**Definition 2.0** (Weak solution to slender body Stokes PDE) A weak solution \( u \in A^{\text{div}}_e \) to \( (1.18) \) satisfies

\[
\int_{\Omega_e} 2 \mathcal{E}(u) : \mathcal{E}(v) \, dx - \int_{\Gamma_e} v(s)f(s) \, ds = 0
\]

for any \( v \in A^{\text{div}}_e \).

**Remark 2.1.** To deepen our understanding of the apparent partial Dirichlet/partial Neumann boundary conditions specified in the slender body PDE \( (1.18) \), we note that the partial Dirichlet data, given by the fiber integrity condition \( u|_{\Gamma_e} = u(s) \), is enforced as part of the function space \( A^{\text{div}}_e \) (an essential boundary condition), whereas the partial Neumann data – the total force per fiber cross section equals \( f(s) \) – arises out of the variational formulation \( (2.5) \) itself (a natural boundary condition).

To formally verify that weak solutions of the slender body PDE \( (1.18) \) satisfy \( (2.5) \), we first note that away from \( \Gamma_e \), the Stokes equations can be rewritten in terms of the stress tensor \( \sigma = 2 \mathcal{E}(u) - pI \) as \( \text{div} \sigma = 0 \) in \( \Omega_e \). Assume \( u \in A^{\text{div}}_e \cap C^\infty(\Omega_e) \) satisfies the slender body PDE \( (1.18) \), where \( C^\infty(\Omega_e) \) denotes functions supported up to the slender body surface \( \Gamma_e \) but away from \( \infty \). Then the corresponding stress tensor satisfies \( \text{div} \sigma = 0 \) in \( \Omega_e \). Multiplying this equation by any \( v \in A^{\text{div}}_e \) and integrating by parts, we have

\[
0 = -\int_{\Omega_e} \text{div} \sigma \cdot v \, dx = \int_{\Omega_e} \sigma : \nabla v \, dx - \int_{\Gamma_e} v(s)\sigma n \, dS
= \int_{\Omega_e} (2 \mathcal{E}(u) : \nabla v - p \text{div} v) \, dx - \int_{\Gamma_e} \int_0^{2\pi} v(s)\sigma n J_e(s, \theta) \, d\theta ds
= \int_{\Omega_e} (\nabla u : \nabla v + \nabla u^T : \nabla v) \, dx - \int_{\Gamma_e} \int_0^{2\pi} \sigma n J_e(s, \theta) \, d\theta ds
= \int_{\Omega_e} 2 \mathcal{E}(u) : \mathcal{E}(v) \, dx - \int_{\Gamma_e} v(s)f(s) \, ds.
\]

Using this definition of a weak solution, we verify the existence and uniqueness claim of Theorem 1.3 Additionally, we must prove the following lemma showing that for any weak solution \( u \) satisfying \( (2.5) \), there is a corresponding weak pressure \( p \in L^2(\Omega_e) \):

**Lemma 2.2.** (Existence of pressure) Given a weak solution \( u \) satisfying \( (2.5) \), there exists a unique corresponding pressure \( p \in L^2(\Omega_e) \) satisfying

\[
\int_{\Omega_e} (2 \mathcal{E}(u) : \mathcal{E}(v) - p \text{div} v) \, dx - \int_{\Gamma_e} v(s)f(s) \, ds = 0
\]

for any \( v \in A_e = \{ v \in D^{1,2}(\Omega_e) : v|_{\Gamma_e} = u(s) \} \), where we have removed the divergence-free restriction on \( v \).
Note that if \((u, p) \in (A_\epsilon^\text{div} \cap C_0^\infty(\Omega_\epsilon)) \times C_0^\infty(\overline{\Omega}_\epsilon)\) satisfies (2.6), then, integrating by parts,

\[
0 = -\int_{\Omega_\epsilon} (2 \text{div}(\mathcal{E}(u)) \cdot v - \nabla p \cdot v) \, dx + \int_{\Gamma_\epsilon} (2 \mathcal{E}(u) n - \nabla p n) \cdot v \, dS - \int_T v(s) f(s) \, ds
\]

\[
= -\int_{\Omega_\epsilon} (\Delta u - \nabla p) \cdot v \, dx + \int_0^{2\pi} \sigma n \cdot v(s) J_c(s, \theta) \, d\theta ds - \int_T v(s) f(s) \, ds
\]

\[
= \int_{\Omega_\epsilon} (-\Delta u + \nabla p) \cdot v \, dx + \int_0^{2\pi} \sigma n J_c(s, \theta) \, d\theta - f(s) \, ds.
\]

Since this holds for any \(v \in A_\epsilon\), the pair \((u, p)\) in fact satisfies equation (1.18) pointwise almost everywhere. Therefore, any smooth enough solution pair \((u, p)\) satisfying the weak formulation (2.6) is a classical solution of (1.18). The higher regularity result of Theorem 1.3 justifies this type of calculation and shows that, given \(f\) regular enough, equation (1.18) is indeed satisfied in a classical sense.

We thus begin by showing the existence and uniqueness of weak solutions to (2.5) and hence to (2.6). We also show that the weak solution pair \((u, p)\) satisfies an estimate of the form (1.19) from Theorem 1.3 but do not consider the explicit \(\epsilon\)-dependence of the constant until Section 3.

### 2.1 Existence and uniqueness

To show existence of a weak solution \(u \in A_\epsilon^\text{div}\) to (2.5), we first show that the bilinear form

\[
B[u, v] := \int_{\Omega_\epsilon} 2 \mathcal{E}(u) : \mathcal{E}(v) \, dx
\]

is coercive on \(A_\epsilon^\text{div}\). To do so, we need two important lemmas from elasticity theory [13].

**Lemma 2.3.** (Rigid motion) Let \(\Omega \subset \mathbb{R}^3\) be any domain. If \(u : \Omega \to \mathbb{R}^3\) with \(\nabla u \in L^2(\Omega)\) satisfies \(\nabla u + (\nabla u)^T = 0\), then \(u\) is a rigid body motion: \(u(x) = Ax + b\) for some constant, antisymmetric \(A \in \mathbb{R}^{3 \times 3}\) and constant \(b \in \mathbb{R}^3\).

**Proof.** We prove the result for smooth functions; Lemma 2.3 follows by density. In coordinates, let \(u_{ij} = -u_{ji}, i, j = 1, 2, 3\), where the subscript “, i” denotes differentiation with respect to \(x_i\). Note that the diagonal elements \(u_{ii}\) (no summation) vanish. Then each entry satisfies

\[
u_{ij} = -u_{ji} = -u_{ji} = 0
\]

for each \(i, j = 1, 2, 3\) (again, no summation implied). Thus we in fact have that \(u_{ij} = u_{ij}(x_k)\), a function of \(x_k\) only, for each combination of \(i, j, k = 1, 2, 3\). In other words,

\[
\nabla u = \begin{pmatrix}
0 & u_{1,2}(x_3) & u_{1,3}(x_2) \\
-u_{1,2}(x_3) & 0 & u_{2,3}(x_1) \\
-u_{1,3}(x_2) & -u_{2,3}(x_1) & 0
\end{pmatrix},
\]

and therefore

\[
u_1 = a_1 x_2 x_3 + b_1 x_2 + c_1 x_3 + d_1
\]

\[
u_2 = a_2 x_1 x_3 + b_2 x_1 + c_2 x_3 + d_2
\]

\[
u_3 = a_3 x_1 x_2 + b_3 x_1 + c_3 x_2 + d_3.
\]

But the antisymmetry of \(\nabla u\) implies \(a_1 = -a_2 = a_3 = -a_1 = 0\), so \(u = Ax + b\) for some \(A = -A^T \in \mathbb{R}^{3 \times 3}\) and constant \(b \in \mathbb{R}^3\).

\[\square\]
Lemma 2.4. (Korn’s inequality) Fix a slender body radius $\epsilon > 0$ and let $\Omega_\epsilon = \mathbb{R}^3 \setminus \Sigma$. For any $u : \Omega_\epsilon \to \mathbb{R}^3$ with $u \in D^{1,2}(\Omega_\epsilon)$, the strain tensor $\mathcal{E}(u) = \frac{\nabla u + (\nabla u)^T}{2}$ satisfies

$$
\|\nabla u\|_{L^2(\Omega_\epsilon)} \leq c_K \|\mathcal{E}(u)\|_{L^2(\Omega_\epsilon)}
$$

for some constant $c_K > 0$.

Proof. We begin by showing that the above inequality holds for functions $v \in D^{1,2}(\mathbb{R}^3)$. We first consider test functions on $\mathbb{R}^3$, then take the closure to show the result for $D^{1,2}(\mathbb{R}^3)$. We have that $v \in C_0^\infty(\mathbb{R}^3)$ satisfies

$$
\int_{\mathbb{R}^3} |\mathcal{E}(v)|^2 \, dx = \int_{\mathbb{R}^3} 2|\nabla v|^2 + 2\nabla v : (\nabla v)^T \, dx
$$

$$
= \int_{\mathbb{R}^3} 2|\nabla v|^2 \, dx - 2\int_{\mathbb{R}^3} v \cdot \text{div}(\nabla v) \, dx
$$

$$
= \int_{\mathbb{R}^3} 2|\nabla v|^2 \, dx + 2\int_{\mathbb{R}^3} |\text{div} v|^2 \, dx
$$

$$
\geq \int_{\mathbb{R}^3} 2|\nabla v|^2 \, dx,
$$

where we have used integration by parts twice, as well as the fact that $v$ vanishes at $\infty$.

Now, let $u \in D^{1,2}(\Omega_\epsilon)$ for fixed slender body radius $\epsilon > 0$. Since for fixed $\epsilon > 0$ the surface $\Gamma_\epsilon$ is $C^2$, there exists a bounded linear operator $T_\epsilon : D^{1,2}(\Omega_\epsilon) \to D^{1,2}(\mathbb{R}^3)$ extending $u$ to the interior of the slender body and satisfying

1. $T_\epsilon u|_{\Omega_\epsilon} = u$ a.e.;
2. $\|\mathcal{E}(T_\epsilon u)\|_{D^{1,2}(\mathbb{R}^3)} \leq c_E \|\mathcal{E}(u)\|_{D^{1,2}(\Omega_\epsilon)}$.

In Section 3 we show that the constant $c_E$ is bounded independent of $\epsilon$ as $\epsilon \to 0$, but well-posedness for a fixed slender body radius does not rely on this result.

Using properties of the extension operator $T_\epsilon$, we then have

$$
\|\nabla u\|_{L^2(\Omega_\epsilon)} \leq \|\nabla (T_\epsilon u)\|_{L^2(\mathbb{R}^3)}
$$

$$
\leq \sqrt{2} \|\mathcal{E}(T_\epsilon u)\|_{L^2(\mathbb{R}^3)} \quad \text{by the Korn inequality on } \mathbb{R}^3,
$$

$$
\leq \sqrt{2} \|T_\epsilon \mathcal{E}(u)\|_{L^2(\mathbb{R}^3)} \quad \text{by linearity of the extension operator},
$$

$$
\leq \sqrt{2} c_E \|\mathcal{E}(u)\|_{L^2(\Omega_\epsilon)}.
$$

Taking $c_K = \sqrt{2} c_E$, we have (2.7). \qed

We proceed to verify the existence and uniqueness of solutions $u$ satisfying (2.5), keeping track of the constants that appear throughout the well-posedness proof but not yet establishing explicit $\epsilon$ dependence (see Section 3).

Proof of existence and uniqueness assertion in Theorem 1.3: Using Korn’s inequality (2.7), for any $u \in A_\epsilon^{\text{div}}$ we have

$$
\mathcal{B}[u, u] = \int_{\Omega_\epsilon} 2|\mathcal{E}(u)|^2 \, dx \geq \int_{\Omega_\epsilon} \frac{2}{c_K^2} |\nabla u|^2 \, dx = \frac{2}{c_K^2} \|\nabla u\|_{L^2(\Omega_\epsilon)}^2,
$$

where
so $B[\cdot, \cdot]$ is coercive on $\mathcal{A}_\epsilon^{\text{div}}$. Also, $B[\cdot, \cdot]$ is bounded, since

$$|B[u, v]| \leq \int_{\Omega_\epsilon} 2|\mathcal{E}(u)||\mathcal{E}(v)| \, dx \leq 2\|\mathcal{E}(u)\|_{L^2(\Omega_\epsilon)}\|\mathcal{E}(v)\|_{L^2(\Omega_\epsilon)} \leq 8\|\nabla u\|_{L^2(\Omega_\epsilon)}\|\nabla v\|_{L^2(\Omega_\epsilon)}.$$ 

Furthermore, for $f \in L^2(\mathbb{T})$ and $v \in \mathcal{A}_\epsilon^{\text{div}}$, the linear functional

$$L(f) := \int_{\mathbb{T}} f(s)v(s) \, ds$$

is bounded: using Cauchy-Schwarz and the trace inequality (2.4) in $\mathcal{A}_\epsilon^{\text{div}}$,

$$\int_{\mathbb{T}} v(s)f(s) \, ds \leq \|u\|_{L^2(\mathbb{T})}\|f\|_{L^2(\mathbb{T})} \leq c_T\|\nabla u\|_{L^2(\Omega_\epsilon)}\|f\|_{L^2(\mathbb{T})}.$$ 

Since the form $B[\cdot, \cdot]$ is bounded and coercive on $\mathcal{A}_\epsilon^{\text{div}}$ and the functional $L(\cdot)$ is bounded on $\mathcal{A}_\epsilon^{\text{div}}$, by the Lax-Milgram theorem there exists a unique solution $u \in \mathcal{A}_\epsilon^{\text{div}}$ to (2.3). Furthermore, taking $v = u$ in (2.3) and using Korn’s inequality (2.7), we obtain that this solution $u$ satisfies

$$\|\nabla u\|_{L^2(\Omega_\epsilon)}^2 \leq c_K^2\|\mathcal{E}(u)\|_{L^2(\Omega_\epsilon)}^2 \leq \frac{c_K^2}{2}\|f\|_{L^2(\mathbb{T})}^2\|u\|_{L^2(\mathbb{T})} \leq \frac{c_K^2}{2}\left(\frac{1}{4\delta}\|f\|_{L^2(\mathbb{T})}^2 + \delta\|u\|_{L^2(\mathbb{T})}^2\right) \leq \frac{c_K^2}{2}\left(\frac{1}{4\delta}\|f\|_{L^2(\mathbb{T})}^2 + \delta c_T^2\|\nabla u\|_{L^2(\Omega_\epsilon)}^2\right).$$

Taking $\delta = \frac{1}{c_T^2c_K^2}$, we have

$$\|\nabla u\|_{L^2(\Omega_\epsilon)} \leq C\|f\|_{L^2(\mathbb{T})}$$

(2.8)

where $C = \frac{1}{3\sqrt{2}c_K^2c_T}$.

The existence of a unique velocity $u \in D^{1,2}(\Omega_\epsilon)$ satisfying (2.3) can be used to show Lemma 2.2, the existence of a unique corresponding pressure $p \in L^2(\Omega_\epsilon)$. The existence of the pressure will also rely on the following lemma, the proof of which can be found in [16], Corollary III.5.1:

**Lemma 2.5.** (de Rham Theorem) Let $\Omega_\epsilon = \mathbb{R}^3 \setminus \Sigma_\epsilon$. Any bounded linear functional on $D_0^{1,2}(\Omega_\epsilon)$ identically vanishing on the divergence-free subspace $D_0^{1,2}(\Omega_\epsilon)$ is of the form

$$\int_{\Omega_\epsilon} p \text{div} \, w \, dx \quad \forall \, w \in D_0^{1,2}(\Omega_\epsilon)$$

for some uniquely determined $p \in L^2(\Omega_\epsilon)$.

**Proof of Lemma 2.2 (existence of pressure):** We begin by considering (2.5) away from $\Gamma_\epsilon$. Let $D_0^{1,2}(\Omega_\epsilon)$ represent the divergence-free subspace of $D_0^{1,2}(\Omega_\epsilon)$, where $D_0^{1,2}(\Omega_\epsilon)$ denotes the closure of test functions supported away from $\Gamma_\epsilon$ in the $D^{1,2}$-norm. Since $u$ is a weak solution to (2.5), we have

$$\int_{\Omega_\epsilon} 2\mathcal{E}(u) : \mathcal{E}(w) \, dx = 0 \quad \text{for all } w \in D_0^{1,2}(\Omega_\epsilon).$$

Using Lemma 2.5, we then have

$$\int_{\Omega_\epsilon} 2\mathcal{E}(u) : \mathcal{E}(w) \, dx = \int_{\Omega_\epsilon} p \text{div} \, w \, dx \quad \forall \, w \in D_0^{1,2}(\Omega_\epsilon).$$

(2.9)

Thus, removing the divergence-free restriction on $w \in D_0^{1,2}(\Omega_\epsilon)$, we recover $p$ in $\Omega_\epsilon$ away from the slender body surface $\Gamma_\epsilon$. We now must show that this $p$ satisfies the correct boundary
conditions for the total surface force over $\Gamma_\epsilon$ when integrated against arbitrary $b \in A = \{v \in D^{1,2}(\Omega_\epsilon) : v|_{\Gamma_\epsilon} = v(s)\}$.

Consider a solution $u \in A_\epsilon^{\text{div}}$ satisfying (2.5). For any $b \in A_\epsilon$ we write

$$b = w + \psi$$

where $\psi$ is the unique (weak) solution to the classical exterior Stokes boundary value problem

$$\begin{align*}
-\Delta \psi + \nabla \pi &= 0 \\
\text{div } \psi &= 0 \\
\psi|_{\Gamma_\epsilon} &= b(s) \\
\psi &\to 0 \quad \text{as } |x| \to \infty
\end{align*}$$

(2.10)

in the space $D^{1,2}_{\text{div}}(\Omega_\epsilon)$. Again the subscript “div” denotes the divergence-free subspace of $D^{1,2}(\Omega_\epsilon)$. We refer to [16], Chapter V.2 for details on the existence and uniqueness results for (2.10).

Thus $\psi$ is divergence-free and satisfies $\psi|_{\Gamma_\epsilon} = b(s)$, independent of $\theta$, so by Definition 2 we have

$$\int_{\Omega_\epsilon} 2 \mathcal{E}(u) : \mathcal{E}(\psi) \, dx = \int_T f(s)b(s) \, ds.$$  (2.11)

Furthermore, we then have that $w \in D^{1,2}_0(\Omega_\epsilon)$ satisfies

$$\int_{\Omega_\epsilon} 2 \mathcal{E}(u) : \mathcal{E}(w) \, dx = \int_{\Omega_\epsilon} p \text{div } w \, dx$$  (2.12)

by Lemma 2.5.

Adding (2.11) and (2.12) we therefore have that

$$\int_{\Omega_\epsilon} 2 \mathcal{E}(u) : \mathcal{E}(b) \, dx = \int_{\Omega_\epsilon} 2 \mathcal{E}(u) : \mathcal{E}(w) \, dx + \int_{\Omega_\epsilon} 2 \mathcal{E}(u) : \mathcal{E}(\psi) \, dx$$

$$= \int_{\Omega_\epsilon} p \text{div } w \, dx + \int_T f(s)b(s) \, ds.$$  

Hence the pressure $p$ from Lemma 2.5 satisfies the desired boundary condition on $\Gamma_\epsilon$, and therefore $(u, p) \in A^{\text{div}}_\epsilon \times L^2(\Omega_\epsilon)$ satisfies

$$\int_{\Omega_\epsilon} 2 \mathcal{E}(u) : \mathcal{E}(v) - p \text{div } v \, dx - \int_T v(s)f(s) \, ds = 0$$

(2.13)

for all $v \in A_\epsilon = \{v \in D^{1,2}(\Omega_\epsilon) : v|_{\Gamma_\epsilon} = v(s)\}$. We have thus removed the divergence-free constraint on $v$ to show the existence of a unique corresponding pressure $p \in L^2(\Omega_\epsilon)$.

Finally, from (2.13), we derive the form of the estimate (1.19) in Theorem 1.3. Here we show

$$\|u\|_{D^{1,2}(\Omega_\epsilon)} + \|p\|_{L^2(\Omega_\epsilon)} \leq C\|f\|_{L^2(T)}$$

(2.14)

for some constant $C$ whose $\epsilon$-dependence will be determined in Section 3.
Proof of estimate (2.14): Following [16], we first show that for \((u, p)\) satisfying (2.13), we have

\[
\|p\|_{L^2(\Omega)} \leq \hat{c}_p \|\mathcal{E}(u)\|_{L^2(\Omega)},
\]

(2.15)

for some constant \(\hat{c}_p > 0\). To show (2.15), we consider \(v \in D_0^{1,2}(\Omega_\epsilon)\) satisfying

\[
\text{div} v = p \quad \text{in } \Omega_\epsilon;
\]

\[
\|v\|_{D^{1,2}(\Omega_\epsilon)} \leq c_p \|p\|_{L^2(\Omega)}.
\]

(2.16)

The existence of such a \(v\) is guaranteed by [16], Theorem III.3.6. In Section 3.4 we reiterate the proof of this theorem to determine the dependence of the constant \(c_p\) on the slender body radius \(\epsilon\).

Now, substituting \(v\) satisfying (2.16) into (2.13), we have

\[
\int_{\Omega_\epsilon} p^2 \, dx = \int_{\Omega_\epsilon} 2 \mathcal{E}(u) : \mathcal{E}(v) \, dx \leq 2 \|\mathcal{E}(u)\|_{L^2(\Omega)} \|\mathcal{E}(v)\|_{L^2(\Omega)} \leq 2 \|\mathcal{E}(u)\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)}
\]

\[
\leq \frac{1}{\eta} \|\mathcal{E}(u)\|^2_{L^2(\Omega)} + \eta \|\nabla v\|^2_{L^2(\Omega)} \leq \frac{1}{\eta} \|\mathcal{E}(u)\|^2_{L^2(\Omega)} + \eta c_p^2 \|p\|^2_{L^2(\Omega)}, \quad \eta \in \mathbb{R}_+.
\]

Taking \(\eta = \frac{1}{2c_p}\), we obtain (2.15), with \(\hat{c}_p = 2c_p\).

Combining the pressure estimate (2.15) with the velocity estimate (2.8), we obtain (2.14), where the constant \(C = \frac{1}{2\sqrt{2} \sqrt{2}} c_K c_T (1 + 2c_p)\). The \(\epsilon\)-dependence of the Korn constant \(c_K\), the trace constant \(c_T\), and the pressure constant \(c_p\) will be verified in Section 3.

We therefore have shown the existence and uniqueness of a weak solution \((u, p) \in A_{\text{div}}^{\text{div}} \times L^2(\Omega_\epsilon)\) to the slender body PDE (1.18) satisfying an estimate of the form (1.19), where the explicit dependence of the constant on \(\epsilon\) will be verified in Section 3.

We now proceed to show the higher regularity result and estimate (1.20) of Theorem 1.3 again tracking constants that appear throughout but not verifying the explicit \(\epsilon\)-dependence in (1.20) until Section 3.

2.2 Higher regularity

We have shown the first half of Theorem 1.3: given \(f \in L^2(\mathbb{T})\), there exists a unique solution \((u, p) \in D^{1,2}(\Omega_\epsilon) \times L^2(\Omega_\epsilon)\) to the variational slender body Stokes problem (2.5). Furthermore, this solution pair satisfies the estimate (1.19). We now show that if, in addition to satisfying the geometric constraints of Section 11, the fiber centerline is at least \(C^4\) and the force \(f(s)\) is in \(H^{1/2}(\mathbb{T})\), we in fact have that \((u, p) \in D^{2,2}(\Omega_\epsilon) \times H^1(\Omega_\epsilon)\) and \((u, p)\) satisfies the estimate (1.20). The proof proceeds in three steps: 1. show higher regularity for \(u\) and \(p\) away from \(\Gamma_\epsilon\); 2. show higher tangential regularity up to \(\Gamma_\epsilon\); and 3. show higher regularity up to \(\Gamma_\epsilon\) in the normal direction.

2.2.1 Regularity away from \(\Gamma_\epsilon\)

We begin by showing higher regularity for \((u, p)\) away from the slender body surface \(\Gamma_\epsilon\). The following arguments closely follow [6].

We first make note of the following lemma, a version of a relation sometimes known as the Nečas inequality, valid in the homogeneous function space \(D^{1,2}(\Omega_\epsilon)\). The proof relies on the discussion surrounding the solution to the problem \(\text{div} v = p\) (2.16) at the end of Section 2.1.
Lemma 2.6 (Generalized Poincaré inequality). Let \( \Omega_\epsilon = \mathbb{R}^3 \setminus \Sigma_\epsilon \). For p \( \in L^1_{\text{loc}}(\Omega_\epsilon) \) with \( \nabla p \in D^{-1,2}(\Omega_\epsilon) \), we have
\[
\|p\|_{L^2(\Omega_\epsilon)} \leq c_P \|\nabla p\|_{D^{-1,2}(\Omega_\epsilon)}
\] (2.17)
where the constant \( c_P \) is the same constant arising in the \( \text{div} \, v = p \) estimate (2.16).

Proof. Choose \( f \in C_0^\infty(\Omega_\epsilon) \), and, by the discussion at the end of Section 2.1, let \( v \in C_0^\infty(\Omega_\epsilon) \) be a solution to
\[
\text{div} \, v = f
\]
\[
\|\nabla v\|_{L^2(\Omega_\epsilon)} \leq c_P \|f\|_{L^2(\Omega_\epsilon)}
\]
for some constant \( c_P > 0 \). The \( \epsilon \)-dependence of this constant will be explored in Section 3.4. Then
\[
\left| \int_{\Omega_\epsilon} p f \, dx \right| = \left| \int_{\Omega_\epsilon} p \text{div} v \right|
\]
\[
\leq \|\nabla p\|_{D^{-1,2}(\Omega_\epsilon)} \|v\|_{D_0^{1,2}(\Omega_\epsilon)}
\]
\[
\leq c_P \|\nabla p\|_{D^{-1,2}(\Omega_\epsilon)} \|f\|_{L^2(\Omega_\epsilon)},
\]
and therefore
\[
\|p\|_{L^2(\Omega_\epsilon)} \leq c_P \|\nabla p\|_{D^{-1,2}(\Omega_\epsilon)}.
\]

Remark 2.7. Lemma 2.6 holds in the exterior of the slender body, \( \Omega_\epsilon = \mathbb{R}^3 \setminus \Sigma_\epsilon \), with a constant \( c_P \) that is possibly \( \epsilon \)-dependent. In the following section, it will also be useful to note that the same type of inequality holds over all of \( \mathbb{R}^3 \), with a constant that clearly does not depend on \( \epsilon \). In particular, for \( q \in L^1_{\text{loc}}(\mathbb{R}^3) \) with \( \nabla q \in D^{-1,2}(\mathbb{R}^3) \), the generalized Poincaré inequality
\[
\|q\|_{L^2(\mathbb{R}^3)} \leq c_q \|\nabla q\|_{D^{-1,2}(\mathbb{R}^3)}
\] (2.18)
follows from solving the Poisson problem over \( \mathbb{R}^3 \).

We again seek \( v \in C_0^\infty(\mathbb{R}^3) \) satisfying
\[
\text{div} \, v = f
\]
\[
\|\nabla v\|_{L^2(\mathbb{R}^3)} \leq c_q \|f\|_{L^2(\mathbb{R}^3)}
\]
for some \( f \in C_0^\infty \). In \( \mathbb{R}^3 \), we can simply let \( v = \nabla \psi \), where \( \psi \in C_0^\infty(\mathbb{R}^3) \) is the solution to \( \Delta \psi = q \) in \( \mathbb{R}^3 \). Then by standard Poisson solution theory (see [16], Chapter II.11 for details),
\[
\|\nabla v\|_{L^2(\mathbb{R}^3)} = \|\nabla^2 \psi\|_{L^2(\mathbb{R}^3)} \leq c_q \|q\|_{L^2(\mathbb{R}^3)}.
\]
Then, following the proof of Lemma 2.6, we obtain (2.18). We leave the generalized Poicaré inequality over \( \mathbb{R}^3 \) as a separate remark since the constant \( c_q \) is clearly independent of the slender body radius \( \epsilon \), whereas the \( \epsilon \)-dependence of the constant \( c_P \) will need to be analyzed in greater detail in Section 3.

Lemma 2.6 and Remark 2.7 will be useful for estimates pertaining to the higher regularity of the pressure.

We now recall the local coordinates \((\rho, \theta, s)\) valid in the region \( \mathcal{O} \) of the fiber centerline (see (1.12)). Within \( \mathcal{O} \), we define the tubular region
\[
\mathcal{O}' = \{ x(\rho, \theta, s) : \rho < r \}.
\] (2.19)
for some $r > \epsilon$. Let
\[ \Omega_r = \mathbb{R}^3 \setminus \overline{\Omega} \subset \Omega_\epsilon. \]

We want to show that $(u, p) \in D^{2,2}(\Omega_r) \times H^1(\Omega_r)$ for any $r > \epsilon$; i.e. our solution $(u, p)$ is in fact more regular away from the slender body surface.

Following ([6], Proposition III.2.3), it suffices to show that $\overline{u \phi} \in D^{2,2}(\mathbb{R}^3)$ and $\overline{p \phi} \in H^1(\mathbb{R}^3)$ for any $\phi \in C_0^\infty(\Omega_\epsilon)$ supported away from the slender body surface $\Gamma_\epsilon$. Here the notation $\overline{\cdot}$ denotes the extension by zero to the interior of the slender body.

Let $v = \overline{u \phi}$ and $q = \overline{p \phi}$. We thus aim to show $(v, q) \in D^{2,2}(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$. For any vector $\psi \in C_0^\infty(\mathbb{R}^3)$, we have

\[
\int_{\mathbb{R}^3} (-\Delta v + \nabla q) \cdot \psi = \int_{\mathbb{R}^3} -v \Delta \psi - q \div \psi = \int_{\Omega_\epsilon} -u \phi \Delta \psi - p \phi \div \psi
\]

\[
= \int_{\Omega_\epsilon} -u(-\phi \Delta \psi - 2\nabla \phi \nabla \psi + \Delta(\phi \psi)) - p(-\nabla \phi \psi + \div(\phi \psi))
\]

\[
= \int_{\Omega_\epsilon} u(\psi \Delta \phi + 2\nabla \phi \nabla \psi) + p\nabla \phi \psi
\]

\[
= \int_{\Omega_\epsilon} (u \psi \Delta \phi - 2\nabla u \nabla \psi - 2u \Delta \phi \psi) + p\nabla \phi \psi
\]

\[
= \int_{\Omega_\epsilon} (-u \Delta \phi - 2\nabla u \nabla \phi + p \nabla \phi) \cdot \psi.
\]

Here we have used that
\[
\int_{\Omega_\epsilon} -u \Delta (\phi \psi) - p \div(\phi \psi) = 0
\]

since $\phi \psi \in C_0^\infty(\Omega_\epsilon)$ and $(u, p)$ solves Stokes distributionally in $\Omega_\epsilon$. Thus, in the sense of distributions, we have

\[-\Delta v + \nabla q = -u \Delta \phi - 2\nabla u \nabla \phi + p \nabla \phi \equiv m.\]

Note that $m \in L^2(\mathbb{R}^3)$ since $\phi \in C_0^\infty(\Omega_\epsilon)$. In particular, $m$ satisfies

\[
\|m\|_{L^2(\mathbb{R}^3)} \leq \|\Delta \phi\|_{L^3(\Omega_\epsilon)}\|u\|_{L^6(\Omega_\epsilon)} + 2\|\nabla \phi\|_{L^\infty(\Omega_\epsilon)}\|\nabla u\|_{L^2(\Omega_\epsilon)} + \|\nabla \phi\|_{L^\infty(\Omega_\epsilon)}\|p\|_{L^2(\Omega_\epsilon)}.
\]

Similarly, we have

\[
\div v = \nabla \phi \cdot \overline{u} + \phi \div \overline{u}
\]

\[
= \nabla \phi \cdot u
\]

\[
=: G(x),
\]

where $G(x) \in D^{1,2}(\mathbb{R}^3)$ satisfies

\[
\|
abla G\|_{L^2(\mathbb{R}^3)} \leq \|
abla^2 \phi\|_{L^3(\Omega_\epsilon)}\|u\|_{L^6(\Omega_\epsilon)} + \|\nabla \phi\|_{L^\infty(\Omega_\epsilon)}\|\nabla u\|_{L^2(\Omega_\epsilon)}.
\]

We now show that $(v, q) \in D^{2,2}(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ using finite difference operators. For a vector $h \in \mathbb{R}^3$ and a function $g$ defined on $\mathbb{R}^3$, we define the translation operator $\tau_h$ by

\[
\tau_h g(x) = g(x + h)
\]

21
and the difference operator $\delta_h$ by

$$
\delta_h g(x) = \tau_h g - g = g(x + h) - g(x).
$$

Clearly, for a function $g \in D^{k,2}(\mathbb{R}^3)$, we have $\delta_h g \in D^{k,2}(\mathbb{R}^3)$ also. Furthermore, we have that $\delta_h$ commutes with differentiation: $\delta_h (\nabla v) = \nabla v(x + h) - \nabla v(x) = \nabla (\delta_h v)$. We state two additional useful properties of finite difference operators, referring to [6] for proof. We note that these lemmas have been adapted to the $D^{k,2}$ setting, which follows easily from $u \in D^{k,2} \implies \nabla u \in H^{k-1}$. For ease of exposition, we define $D^{0,2}(\mathbb{R}^3) := L^2(\mathbb{R}^3)$ where applicable.

**Proposition 2.8. (Properties of finite difference operators)**

1. ([6], Lemma III.2.31): For $g \in D^{k,2}(\mathbb{R}^3)$, $k \geq 0$, and any $h \in \mathbb{R}^3$, we have

$$
\|\delta_h g\|_{D^{k-1,2}} \leq |h|\|\nabla g\|_{D^{k-1,2}} \leq |h|\|g\|_{D^{k,2}}.
$$

2. ([6], Proposition III.2.32): Let $(e_1, e_2, e_3)$ be the canonical basis of $\mathbb{R}^3$. For $g \in D^{k,2}(\mathbb{R}^3)$, $k \geq 0$, we define the norm

$$
|||g|||_{D^{k+1,2}} = \|g\|_{D^{k,2}} + \sum_{i=1}^3 \sup_{0<h<1} \frac{1}{h} \|\delta_h e_i g\|_{D^{k,2}},
$$

where $\delta_h e_i g = g(x + he_i) - g(x)$. The following equality holds:

$$
D^{k+1,2}(\mathbb{R}^3) = \{g \in D^{k,2}(\mathbb{R}^3) : |||g|||_{D^{k+1,2}} < \infty\},
$$

and

$$
\|\nabla g\|_{D^{k,2}} \leq |||g|||_{D^{k+1,2}} \quad \forall g \in D^{k+1,2}(\mathbb{R}^3).
$$

3. By linearity of $\delta_h$,

$$
\delta_h \mathcal{E}(u) = \frac{1}{2}(\delta_h \nabla u + \delta_h (\nabla u)^T) = \frac{1}{2}((\nabla (\delta_h u) + (\nabla (\delta_h u))^T) = \mathcal{E}(\delta_h u),
$$

and thus for the stress tensor $\sigma$ we have $\delta_h \sigma = \nabla (\delta_h u) + (\nabla (\delta_h u))^T - (\delta_h p)I$.

Therefore we have that $(v, q) \in D^{1,2}(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ satisfies

$$
-\Delta \delta_h v + \nabla \delta_h q = \delta_h m
\quad \text{div} \, \delta_h v = \delta_h G
$$

on $\mathbb{R}^3$ in the weak sense; i.e.

$$
\int_{\mathbb{R}^3} 2 \mathcal{E}(\delta_h v) : \mathcal{E}(w) \, dx = \int_{\mathbb{R}^3} (\delta_h q) \text{div} \, w \, dx + \int_{\mathbb{R}^3} (\delta_h m) w \, dx
$$

(2.21)

for all $w \in D^{1,2}_0(\mathbb{R}^3)$ with support away from the slender body $\Sigma_{\epsilon}$. Since $\phi \in C_0^\infty$ is supported away from $\Sigma_{\epsilon}$, by construction, $\delta_h v \in D^{1,2}_0(\mathbb{R}^3)$ is a suitable test function for [2.21]. We then have

$$
\|\nabla (\delta_h v)\|_{L^2(\mathbb{R}^3)}^2 = \frac{1}{2} \left( \int_{\mathbb{R}^3} (\delta_h q)(\delta_h G) \, dx + \int_{\mathbb{R}^3} (\delta_h m)(\delta_h v) \, dx \right)
$$
\[\leq \frac{1}{2}\left(\|\delta_h q\|_{L^2(\mathbb{R}^3)}\|\delta_h G\|_{L^2(\mathbb{R}^3)} + \|\delta_h m\|_{D^{-1,2}(\mathbb{R}^3)}\|\delta_h v\|_{D^{1,2}(\mathbb{R}^3)}\right)\]
\[\leq \frac{1}{2}\left(h\|\delta_h q\|_{L^2(\mathbb{R}^3)}\|\nabla G\|_{L^2(\mathbb{R}^3)} + |h|\|m\|_{L^2(\mathbb{R}^3)}\|\nabla (\delta_h v)\|_{L^2(\mathbb{R}^3)}\right).\]

Now, by Remark 2.7, \(\delta_h q \in L^2(\mathbb{R}^3)\) satisfies
\[\|\delta_h q\|_{L^2(\mathbb{R}^3)} \leq c_q\|\nabla (\delta_h q)\|_{D^{-1,2}(\mathbb{R}^3)}.\]

Using the equation (2.20), we have
\[\|\nabla (\delta_h q)\|_{D^{-1,2}(\mathbb{R}^3)} = \|\Delta (\delta_h v) + \delta_h m\|_{D^{-1,2}(\mathbb{R}^3)}\]
\[\leq \|\Delta (\delta_h v)\|_{D^{-1,2}(\mathbb{R}^3)} + \|\delta_h m\|_{D^{-1,2}(\mathbb{R}^3)}\]
\[\leq \|\nabla (\delta_h v)\|_{L^2(\mathbb{R}^3)} + |h|\|m\|_{L^2(\mathbb{R}^3)}.\]

Thus
\[\|\delta_h q\|_{L^2(\mathbb{R}^3)} \leq c_q \left(\|\nabla (\delta_h v)\|_{L^2(\mathbb{R}^3)} + |h|\|m\|_{L^2(\mathbb{R}^3)} \right),\]
and therefore
\[\|\nabla (\delta_h v)\|_{L^2(\mathbb{R}^3)}^2 \leq \frac{c_q}{2} \left(\|h\|\|\nabla (\delta_h v)\|_{L^2(\mathbb{R}^3)} + |h|^2\|m\|_{L^2(\mathbb{R}^3)}\right)\|\nabla G\|_{L^2(\mathbb{R}^3)}\]
\[\quad + \frac{1}{2}\|h\|\|m\|_{L^2(\mathbb{R}^3)}\|\nabla (\delta_h v)\|_{L^2(\mathbb{R}^3)}\]
\[\leq \frac{c_q}{2}|h|^2\|m\|_{L^2(\mathbb{R}^3)}\|\nabla G\|_{L^2(\mathbb{R}^3)} + \frac{|h|^2}{8\eta} \left(\|m\|_{L^2(\mathbb{R}^3)}^2 + c_q^2\|\nabla G\|_{L^2(\mathbb{R}^3)}^2\right)\]
\[\quad + \frac{\eta}{2}\|\nabla (\delta_h v)\|_{L^2(\mathbb{R}^3)}^2\]

for any \(\eta \in \mathbb{R}_+\). Taking \(\eta = 1\), we have
\[\|\nabla (\delta_h v)\|_{L^2(\mathbb{R}^3)}^2 \leq \frac{3|h|^2}{8} \left(\|m\|_{L^2(\mathbb{R}^3)}^2 + c_q^2\|\nabla G\|_{L^2(\mathbb{R}^3)}^2\right). \tag{2.22}\]

Since this inequality holds for arbitrary increment \(h\), by Proposition 2.8.2, we have \(\nabla v \in H^1(\mathbb{R}^3)\), and hence \(v \in D^{2,2}(\mathbb{R}^3)\).

The pressure term \(q\) then satisfies
\[\|\delta_h q\|_{L^2(\mathbb{R}^3)} \leq c_q \left(\|\nabla (\delta_h v)\|_{L^2(\mathbb{R}^3)} + |h|\|m\|_{L^2(\mathbb{R}^3)}\right)\]
\[\leq c_q |h| \left(\|\nabla^2 v\|_{L^2(\mathbb{R}^3)} + |m|\|m\|_{L^2(\mathbb{R}^3)}\right)\]

for any increment \(h\), and thus \(q \in H^1(\mathbb{R}^3)\). Recalling that \(v = \overline{u} \phi\) and \(q = \overline{p} \phi\), we therefore have \((u, p) \in D^{2,2}(\Omega_r) \times H^1(\Omega_r)\) for any \(r > \epsilon\).

In total, we have
\[\|\nabla^2 u\|_{L^2(\Omega_r)} + \|\nabla p\|_{L^2(\Omega_r)} \leq c_\phi \left(\|\nabla^2 v\|_{L^2(\mathbb{R}^3)} + \|\nabla p\|_{L^2(\mathbb{R}^3)}\right)\]
\[\leq c_\phi c_q \left(\|m\|_{L^2(\mathbb{R}^3)} + c_q\|\nabla G\|_{L^2(\mathbb{R}^3)}\right)\]
\[\leq \tilde{c}_q \left(\|\nabla u\|_{L^2(\Omega_r)} + |p|\|L^2(\Omega_r)\right)\]
\[\leq \tilde{c}_q c_q^2 c_T (1 + 2c_f)\|f\|_{L^2(\mathbb{T})},\]

where the constant \(\tilde{c}_q\) is independent of \(\epsilon\) since \(\phi \in C_0^\infty(\Omega_\epsilon)\) was arbitrary.
2.2.2 Tangential regularity up to $\Gamma_\epsilon$

We proceed with step 2 of the proof of higher regularity for Theorem 1.3: we show higher tangential regularity up to the slender body surface $\Gamma_\epsilon$. Recall that in the region $\mathcal{O} \in (1.12)$ near the slender body surface, a point $x$ in space is uniquely specified as

$$x(\rho, \theta, s) = X(s) + \rho e_\rho(s, \theta),$$

In addition, we define the region $\mathcal{O}' \subset \mathcal{O}$ of $\Gamma_\epsilon$ as

$$\mathcal{O}' = \left\{ x \in \Omega_\epsilon : x = x(\rho, \theta, s), \epsilon < \rho < \frac{1}{4\kappa_{\text{max}}} \right\}.$$  \hspace{1cm} (2.23)

For any $g \in H^k(\Omega_\epsilon)$, we denote

$$\nabla_N g = (\nabla g \cdot e_\rho) e_\rho \quad \text{in} \mathcal{O},$$

$$\nabla_T g = \nabla g - \nabla_N g \quad \text{in} \mathcal{O}.$$  

By Theorem III.3.14 in [6], since $u \in D^{1,2}(\Omega_\epsilon) \cap D^{2,2}(\Omega_\epsilon)$ for any $r > \epsilon$, we will have that $u \in D^{2,2}(\Omega_\epsilon)$ - in particular, up to $\Gamma_\epsilon$ - if and only if $\nabla_T u \in H^1(\mathcal{O})$ and $\nabla_N u \in H^1(\mathcal{O})$. We begin by showing $\nabla_T u \in H^1(\mathcal{O})$. To do so, we must make use of the additional regularity of the prescribed force $f \in H^{1/2}(T)$. In contrast to more traditional boundary value problems with Dirichlet, Neumann, or Robin boundary data specified pointwise along the surface, the data $f(s)$ is defined only in an average sense over each cross section of the slender body surface $\Gamma_\epsilon$. Thus we cannot rely on the often-used method of mapping open subsets of the surface $\Gamma_\epsilon$ to the plane in $\mathbb{R}^3$ and using the flat finite difference operators from Section 2.2.1 in the upper half-space $\mathbb{R}^3_+$ to show higher tangential regularity. Instead, each cross section $s$ of $\Gamma_\epsilon$ must be considered as a complete circle, $0 \leq \theta < 2\pi$, the entirety of which is needed to define and hence use the data $f(s)$. In order to show that $\nabla_T u \in H^1(\mathcal{O})$ while leaving each cross section $s$ of $\Gamma_\epsilon$ intact, we follow the construction in Boyer-Fabrie [6] and define translation operators $\tau^s$ and $\tau^\theta$ acting tangent to the surface $\Gamma_\epsilon$.

We note that, within the neighborhood $\mathcal{O}$, any vector field $\gamma \in C^2(\overline{\Omega_\epsilon})$ with $\gamma \cdot e_\rho = 0$ can be written as a linear combination of unit vectors in the $s$ and $\theta$ directions: $e_t(s)$ and $e_\theta(s) = -\sin \theta e_{n_s}(s) + \cos \theta e_{n_\theta}(s)$. To track possible dependence of any resulting constants on the slender body radius $\epsilon$, we decompose the tangential translation operators into the $\theta$ and $s$ directions and exploit the moving frame geometry of the slender body.

For $x \in \mathcal{O}'$, we define the tangential translation operators

$$\tau^\theta_h(x(\rho, \theta, s)) = x(\rho, \theta + 2\pi h, s)$$

$$\tau^s_h(x(\rho, \theta, s)) = x(\rho, \theta, s + h)$$

for $h \in [0, 1)$. To extend this definition of translation operator globally throughout $\Omega_\epsilon$, beyond the region $\mathcal{O}'$, for $j = \theta, s$ we define

$$\tau^j_h(x) = \tau^j_{\phi(\rho)h}(x)$$  \hspace{1cm} (2.24)

where $\phi(\rho)$ is a smooth cutoff function equal to 1 for $\rho < \frac{1}{8\kappa_{\text{max}}}$ and equal to zero beyond $\rho = \frac{1}{4\kappa_{\text{max}}}$ with $|\phi'(\rho)| \leq c\kappa_{\text{max}} = c_\kappa$. In other words, the translation “step size” varies smoothly from $h$ near the slender body surface to 0 in $\mathbb{R}^3 \setminus \mathcal{O}'$; in particular, we have

$$\tau^j_h(x) = \begin{cases} 
\tau^j_h(x), & \epsilon < \rho \leq \frac{1}{8\kappa_{\text{max}}}, \\
x, & x \in \mathbb{R}^3 \setminus \mathcal{O}'.
\end{cases}$$

24
Note that due to the periodicity in both the $s$ and $\theta$ directions, the tangential difference operators $\tau_{h}^{\theta}$ and $\tau_{h}^{s}$ are bijections preserving $\Gamma_{\epsilon}$ and the regions $\mathcal{O}, \mathcal{O}' \subset \Omega_{\epsilon}$.

Letting $\tau_{h}^{j}g = g \circ \tau_{h}^{j}(x)$, we define the tangential finite difference operator

$$\delta_{h}^{j}g = \tau_{h}^{j}g - g, \quad j = s, \theta$$

and note that by definition of $\tau_{h}^{j}$ this quantity vanishes outside of the region $\mathcal{O}'$ and hence outside of $\mathcal{O}$ as well. In particular, for $g \in D^{1,2}(\Omega_{\epsilon})$, we have $\delta_{h}^{j}g \in H^{1}(\mathcal{O})$.

The tangential translation and difference operators $\tau_{h}^{j}$ and $\delta_{h}^{j}$, $j = \theta, s$, behave similarly to the affine translation operator defined previously, but now translation does not commute with differentiation. For any function $g$ and differential operator $D$, we define the commutator

$$[D, \tau_{h}^{j}]g = D(\tau_{h}^{j}g) - \tau_{h}^{j}(Dg) = D(\delta_{h}^{j}g) - \delta_{h}^{j}(Dg),$$

and for any two functions $g_{1}$ and $g_{2}$, we define

$$\{g_{1}, g_{2}\}_{h}^{j} = (\tau_{h}^{j}g_{1})g_{2} - g_{1}(\tau_{h}^{j}g_{2}) = (\delta_{h}^{j}g_{1})g_{2} - g_{1}(\delta_{h}^{j}g_{2})$$

for $j = \theta, s$.

For $g \in H^{k}$, $k \geq 0$, we also define the norm

$$|||g|||_{T, H^{k+1}} = \sup_{0 < h < 1} \frac{1}{h} \left\| \frac{1}{\rho} \delta_{h}^{\theta}g \right\|_{H^{k}} + \sup_{0 < h < 1} \frac{1}{h} \left\| \delta_{h}^{s}g \right\|_{H^{k}}. \quad (2.27)$$

See Appendix 6 for additional properties and estimates related to the tangential finite difference operators.

The following higher regularity estimates will depend on $C^{4}$ regularity of the fiber centerline $X(s)$; in particular, we will require bounds on the first and second derivatives of the moving frame coefficients $\kappa_{1}(s)$ and $\kappa_{2}(s)$ from (1.3). We therefore define

$$m_{\kappa,1} := \max_{s \in \mathbb{T}^{1}} (|\kappa'_{1}(s)| + |\kappa'_{2}(s)|)$$

$$m_{\kappa,2} := \max_{s \in \mathbb{T}^{1}} (|\kappa''_{1}(s)| + |\kappa''_{2}(s)|), \quad (2.28)$$

and note that these constants will be used to show the commutator estimates in the Appendix 6.

In addition to the commutator estimates from the Appendix, we have the following property:

**Proposition 2.9.** For any $g \in \mathcal{A}_{\epsilon}$, $\delta_{h}^{j}g$ is also in $\mathcal{A}_{\epsilon}$ for $j = \theta, s$.

**Proof.** By Proposition 6.2.1, $\delta_{h}^{j}g$ preserves the regularity of $g$, so it remains to check that $\delta_{h}^{j}g|_{\Gamma_{\epsilon}}$, $j = \theta, s$, is independent of $\theta$. Since $g \in \mathcal{A}_{\epsilon}$, we have that for each $x \in \Gamma_{\epsilon}$, $g(x)$ is independent of $\theta$. But tangential translation preserves boundaries; i.e. if $x \in \Gamma_{\epsilon}$, then $\tau_{h}^{j}(x) \in \Gamma_{\epsilon}$. Thus for each $x \in \Gamma_{\epsilon}$, $g(\tau_{h}^{j}(x)) = \tau_{h}^{j}g$ is independent of $\theta$, and therefore $\delta_{h}^{j}g = \tau_{h}^{j}g - g$ is independent of $\theta$ for all $x \in \Gamma_{\epsilon}$. \qed

Since tangential translation does not commute with differentiation, taking $v \in \mathcal{A}^{\text{div}}_{\epsilon}$ does not imply that $\tau_{h}^{j}v \in \mathcal{A}^{\text{div}}_{\epsilon}$ for $j = s$ or $\theta$. However, the existence of a unique pressure in $L^{2}(\Omega_{\epsilon})$
corresponding to each solution \( u \in \mathcal{A}_{\text{div}}^\kappa \) allows us to make sense of the weak Stokes equation using test functions in \( \mathcal{A} = \{ v \in D^{1,2}(\Omega_\kappa) : v|_{\Gamma_\kappa} = v(s) \} \), via (2.13). For \( v \in \mathcal{A} \), we do have \( \tau_h^j v \in \mathcal{A}_\kappa \) due to Lemma 2.9.

Thus we can use \( \delta_h^j \delta_h^j u, j = \theta, s \), as a test function in (2.13). We have

\[
\int_{\Omega_\kappa} 2 \mathcal{E}(u) : \mathcal{E}(\delta_h^j \delta_h^j u) \, dx = \int_{\Omega_\kappa} p \, \text{div}(\delta_h^j \delta_h^j u) \, dx + \int_T f(s) \delta_h^j \delta_h^j u(s) \, ds. \tag{2.29}
\]

Using the commutators (2.25) and (2.26), we can use (2.29) to estimate \( \| \mathcal{E}(\delta_h^j u) \|_{L^2(\Omega_\kappa)} \). Recall that by definition of \( \delta_h^j u = \tau_h^j u - u, j = s, \theta \) (see (2.24)), we have that \( \delta_h^j u \) vanishes outside of \( \mathcal{O} \). We begin by rewriting the left hand side:

\[
\int_{\Omega_\kappa} 2 \mathcal{E}(u) : \mathcal{E}(\delta_h^j \delta_h^j u) \, dx = \int_{\mathcal{O}} 2 \mathcal{E}(u) : \mathcal{E}(\delta_h^j \delta_h^j u) \, dx
\]

We can then write (2.29) as

\[
\int_{\mathcal{O}} \mathcal{E}(\delta_h^j u) : \mathcal{E}(\delta_h^j u) \, dx = \frac{1}{2} \int_T f(s) \delta_h^j \delta_h^j u(s) \, ds + \frac{1}{2} \int_{\Omega_\kappa} p \, \text{div}(\delta_h^j \delta_h^j u) \, dx
\]

We now proceed to bound each term on the right hand side of (2.30) using Proposition 6.2 to obtain an estimate for \( \| \mathcal{E}(\delta_h^j u) \|_{L^2(\mathcal{O})} \). Throughout, we will label any constants depending only on the geometry of \( X(s) \) as \( c_{j,k}, j = s, \theta, k = 0, -1 \). The constants \( c_{0,0} \) depend only on \( \kappa_{\text{max}} \) and \( \kappa_\Gamma \), while the constants \( c_{\theta,1} \) and \( c_{s,0} \) depend on \( m_{\kappa,1} \) as well. The constants \( c_{s,-1} \) depend on \( \kappa_{\text{max}}, \kappa_\Gamma \), and both \( m_{\kappa,1} \) and \( m_{\kappa,2} \).

We begin by recalling the Sobolev-Slobodeckij characterization of the space \( H^{1/2} \). Let \( W \subset \mathbb{R}^n \) be any domain. A function \( g \in L^2(W) \) is in \( H^{1/2}(W) \) if the seminorm

\[
|g|_{H^{1/2}} = \int_W \int_W \frac{|g(x) - g(y)|^2}{|x - y|^{1+n}} \, dx \, dy \tag{2.31}
\]
is finite.

Using this definition, we note that for a prescribed force $f \in H^{1/2}(\mathbb{T})$, we have that $\delta_h^s f \in H^{1/2}(\mathbb{T})$ satisfies

$$\|\delta_h^s f\|_{L^2(\mathbb{T})}^2 = \int_{\mathbb{T}} (\delta_h^s f(s))^2 \, ds = \int_{\mathbb{T}} |f(s + h) - f(s)|^2 \, ds$$

$$= h^2 \int_{\mathbb{T}} \frac{|f(s + h) - f(s)|^2}{h^2} \, ds$$

$$\leq h^2 \int_{\mathbb{T}} \frac{|f(s + h) - f(s)|^2}{h^2} \, ds \, dh = h^2 |f|_{H^{1/2}(\mathbb{T})}^2,$$

since the integrand $|f(s + h) - f(s)|^2/h^2$ is clearly nonnegative.

We use this estimate to bound the first term in (2.30) for $j = s$. Note that this first term vanishes for $j = \theta$, by Proposition 6.2. For $j = s$, by (6.19), we have

$$\frac{1}{2} \int_{\mathbb{T}} f(s) \delta_h^s \delta_h^u(s) \, ds = \frac{1}{2} \int_{\mathbb{T}} \delta_h^s f(s) \delta_h^u(s) \, ds - \frac{1}{2} \int_{\mathbb{T}} \{f, \delta_h^s u\}_h \, ds$$

$$\leq \frac{1}{2} \|\delta_h^s f\|_{L^2(\mathbb{T})} \|\delta_h^u\|_{L^2(\mathbb{T})} + c_{s,0} \frac{|h|}{2} \|\delta_h^s u\|_{L^2(\mathbb{T})}$$

$$\leq \frac{|h|}{2} \|f\|_{H^{1/2}(\mathbb{T})} \|\delta_h^u\|_{L^2(\mathbb{T})} + c_{s,0} \frac{|h|}{2} \|\delta_h^s u\|_{L^2(\mathbb{T})}$$

$$\leq (1 + c_{s,0}) \|f\|_{H^{1/2}(\mathbb{T})} \|\nabla \delta_h^s u\|_{L^2(\mathbb{T})}$$

$$\leq c_T^2 (1 + c_{s,0})^2 \frac{|h|^2}{\eta} \|f\|_{H^{1/2}(\mathbb{T})}^2 + \eta \|\nabla \delta_h^s u\|_{L^2(\mathbb{T})}^2, \quad 0 < \eta \in \mathbb{R}.$$

Using Proposition 6.2, equation (6.17), for $j = s, \theta$, the third term on the right hand side of (2.30) can be estimated as

$$\int_{\mathbb{O}} \{\mathcal{E}(u), \mathcal{E}(\delta_h^s u)\}_h \, dx \leq c_{s,0} \|\nabla \mathcal{E}(u)\|_{L^2(\mathbb{O})} \|\mathcal{E}(\delta_h^s u)\|_{L^2(\mathbb{O})}$$

$$\leq \frac{c_T^2}{4\eta} |h|^2 \|\mathcal{E}(u)\|_{L^2(\mathbb{O})}^2 + \eta \|\mathcal{E}(\delta_h^s u)\|_{L^2(\mathbb{O})}^2$$

$$\leq \frac{c_T^2}{4\eta} |h|^2 \|\nabla u\|_{L^2(\mathbb{O})}^2 + \eta \|\nabla (\delta_h^s u)\|_{L^2(\mathbb{O})}^2.$$

By (6.20), for $j = s, \theta$, the fourth and fifth terms on the right hand side of (2.30) satisfy

$$\int_{\mathbb{O}} \mathcal{E}(u) : [\nabla, \tau_{-h}^s] (\delta_h^s u) \, dx + \int_{\mathbb{O}} \mathcal{E}(u) : \left([\nabla, \tau_h^s] (\delta_h^s u)\right)^T \, dx$$

$$\leq 2 \|\mathcal{E}(u)\|_{L^2(\mathbb{O})} \|\nabla, \tau_{-h}^s (\delta_h^s u)\|_{L^2(\mathbb{O})}$$

$$\leq 2 c_{j,0} |h| \|\mathcal{E}(u)\|_{L^2(\mathbb{O})} \|\nabla (\delta_h^s u)\|_{L^2(\mathbb{O})}$$

$$\leq \frac{c_T^2}{\eta} |h|^2 \|\mathcal{E}(u)\|_{L^2(\mathbb{O})}^2 + \eta \|\nabla (\delta_h^s u)\|_{L^2(\mathbb{O})}^2$$

$$\leq \frac{c_T^2}{\eta} |h|^2 \|\nabla u\|_{L^2(\mathbb{O})}^2 + \eta \|\nabla (\delta_h^s u)\|_{L^2(\mathbb{O})}^2.$$
Again by (6.20), the last two terms on the right hand side of (2.29) can be estimated as
\[
\int_{\Omega} \left( [\nabla, \tau_h^j] \mathbf{u} \right) : \mathcal{E}(\delta_h^j \mathbf{u}) \, dx + \int_{\Omega} \left( [\nabla, \tau_h^j] \mathbf{u} \right)^T : \mathcal{E}(\delta_h^j \mathbf{u}) \, dx \\
\leq 2 \| [\nabla, \tau_h^j] \mathbf{u} \|_{L^2(\Omega)} \| \mathcal{E}(\delta_h^j \mathbf{u}) \|_{L^2(\Omega)} \\
\leq c_{j,0} \| h \| \| \nabla \mathbf{u} \|_{L^2(\Omega)} \| \mathcal{E}(\delta_h^j \mathbf{u}) \|_{L^2(\Omega)} \\
\leq \frac{c_{j,0}}{\eta} \| h \|^2 \| \nabla \mathbf{u} \|_{L^2(\Omega)}^2 + \eta \| \mathcal{E}(\delta_h^j \mathbf{u}) \|_{L^2(\Omega)}^2 \\
\leq \frac{c_{j,0}}{\eta} \| h \|^2 \| \nabla \mathbf{u} \|_{L^2(\Omega)}^2 + \eta \| \nabla(\delta_h^j \mathbf{u}) \|_{L^2(\Omega)}^2.
\]

Finally, to estimate the remaining term involving the pressure, we note that by Lemma 2.6 we have
\[
\| \delta_h^j p \|_{L^2(\Omega)} = \| \delta_h^j p \|_{L^2(\Omega)} \leq c_P \| \nabla(\delta_h^j p) \|_{D^{-1,2}(\Omega)} \leq c_P c_r \| \nabla(\delta_h^j p) \|_{H^{-1}(\Omega)}.
\]  

(2.32)

**Remark 2.10.** The final inequality in (2.32) follows from the fact that $\delta_h^j p$ is supported only within the region $\Omega'$. We note that for any function $g \in D^{-1,2}(\Omega)$ with supp$(g) \subset \Omega'$, we have, for each $\psi \in C_0^{\infty}(\Omega)$,
\[
\left| \int_{\Omega} g \nabla \psi \, dx \right| = \left| \int_{\Omega'} g \nabla \psi \, dx \right| = \left| \int_{\Omega'} g \nabla(\psi \phi) \, dx \right|
\]
where we define $\phi$ to be the smooth cutoff equal to 1 in the region
\[
\Omega'' := \left\{ x \in \Omega : x = x(\rho, \theta, s), \epsilon < \rho < \frac{3}{\delta \kappa_{\text{max}}} \right\} \subset \Omega'
\]
and equal to zero outside of $\Omega$. Since $\kappa_{\text{max}}$ depends only on $\kappa$, the centerline curvature, the norm of the gradient of $\phi$ is bounded independent of $\epsilon$. We thus have $\| \phi \psi \|_{H_0^1(\Omega)} \leq \| \nabla \phi \|_{L^2(\Omega)} + \| \nabla \psi \|_{L^2(\Omega)} \| \psi \|_{L^6(\Omega)} \leq c_{\kappa} \| \psi \|_{D^{1,2}(\Omega)}$.

Then, provided $\nabla(\phi \psi)$ is non-vanishing, we have
\[
\frac{1}{\| \psi \|_{D^{1,2}(\Omega)}} \left| \int_{\Omega} g \nabla \psi \, dx \right| = \frac{1}{\| \psi \|_{D^{1,2}(\Omega)}} \left| \int_{\Omega'} g \nabla(\phi \psi) \, dx \right| \leq c_{\kappa} \frac{1}{\| \phi \psi \|_{H_0^1(\Omega)}} \left| \int_{\Omega'} g \nabla(\phi \psi) \, dx \right| \leq c_{\kappa} \| g \|_{H^{-1}(\Omega)},
\]
since functions of the form $\phi \psi$ are a subset of $H_0^1(\Omega)$. Taking the supremum over $\psi \in C_0^{\infty}(\Omega)$, we obtain
\[
\| g \|_{D^{-1,2}(\Omega)} \leq c_{\kappa} \| g \|_{H^{-1}(\Omega)}.
\]

Making use of the fact that $(\mathbf{u}, p)$ is a weak solution to the Stokes equations (1.1) and using Propositions 6.2 and 6.3, we have
\[
\| \nabla(\delta_h^j p) \|_{H^{-1}(\Omega)} \leq \| \nabla(\delta_h^j \mathbf{u}) \|_{H^{-1}(\Omega)} + \| \delta_h^j(\nabla p) \|_{H^{-1}(\Omega)} \\
\leq c_{j,-1} \| h \| \| p \|_{L^2(\Omega)} + \| \delta_h^j(\Delta \mathbf{u}) \|_{H^{-1}(\Omega)} \\
\leq c_{j,-1} \| h \| \| p \|_{L^2(\Omega)} + \| \Delta(\delta_h^j \mathbf{u}) \|_{H^{-1}(\Omega)} + \| \nabla(\delta_h^j \mathbf{u}) \|_{H^{-1}(\Omega)} \\
+ \| \nabla(\nabla, \tau_h^j \mathbf{u}) \|_{H^{-1}(\Omega)} \\
\leq c_{j,-1} \| h \| \| p \|_{L^2(\Omega)} + \| \nabla(\delta_h^j \mathbf{u}) \|_{L^2(\Omega)} + c_{j,-1} \| h \| \| \nabla \mathbf{u} \|_{L^2(\Omega)} + \| \nabla, \tau_h^j \mathbf{u} \|_{L^2(\Omega)}
\[ \leq c_{j,-1}|h| (||p||_{L^2(O)} + ||\nabla u||_{L^2(O)}) + ||\nabla (\delta_h^j u)||_{L^2(O)}. \]

Thus, using \((2.32)\), we have
\[ \|\delta_h^j p\|_{L^2(O)} \leq c_{pc, j,-1} \left( |h| (||p||_{L^2(O)} + ||\nabla u||_{L^2(O)}) + ||\nabla (\delta_h^j u)||_{L^2(O)} \right). \quad (2.33) \]

Using Proposition \(6.3\) and \((2.33)\), the pressure term on the right hand side of \((2.30)\) can be estimated as:
\[
\frac{1}{2} \int_O p \text{div} (\delta_h^j u) \, dx \leq \frac{1}{2} \int_O \left( \delta_h^j p \left( \text{div}, \tau_h^j \right) u - \{p, \text{div} (\delta_h^j u)\}_h + p[\text{div}, \tau_h^j] \delta_h^j u \right) \, dx \\
\leq \frac{1}{2} \left( ||\delta_h^j p||_{L^2(O)} ||\text{div}, \tau_h^j|| u ||\nabla u||_{L^2(O)} + c_{j,0}|h||p||_{L^2(O)} ||\nabla (\delta_h^j u)||_{L^2(O)} + ||p||_{L^2(O)} ||\text{div}, \tau_h^j|| \delta_h^j u ||L^2(O)|| \right) \\
\leq c_{j,0} \left( |h||\delta_h^j p||_{L^2(O)} ||\nabla u||_{L^2(O)} + |h||p||_{L^2(O)} ||\nabla (\delta_h^j u)||_{L^2(O)} \right) \\
\leq c_{j, -1} \left( |h||\nabla (\delta_h^j u)||_{L^2(O)} (c_p||\nabla u||_{L^2(O)} + ||p||_{L^2(O)}) \\
+ c_p|h|^2||\nabla u||_{L^2(O)} (||\nabla u||_{L^2(O)} + ||p||_{L^2(O)}) \right) \\
\leq c_{j,-1}(c_p + 1)^2 |h|^2 \left( ||\nabla u||_{L^2(O)} + ||p||_{L^2(O)} \right)^2 + \eta ||\nabla (\delta_h^j u)||_{L^2(O)}^2 \\
+ \eta(c_p + 1)^2 |h|^2 ||\nabla u||_{L^2(O)}^2. \quad (2.34) \]

Together, we have that \(\|E(\delta_h^j u)||_{L^2(O)}^2\) satisfies
\[ \|E(\delta_h^j u)||_{L^2(O)}^2 \leq \left( \frac{c_{h,-1}}{\eta} + \eta \right) \left( c_p + 1 \right)^2 |h|^2 \left( ||\nabla u||_{L^2(O)}^2 + ||p||_{L^2(O)}^2 \right) + 5\eta ||\nabla (\delta_h^j u)||_{L^2(O)}^2 \]
and
\[ \|E(\delta_h^j u)||_{L^2(O)}^2 \leq \left( \frac{c_{h,-1}}{\eta} + \eta \right) \left( c_p + 1 \right)^2 |h|^2 \left( ||\nabla u||_{L^2(O)}^2 + ||p||_{L^2(O)}^2 \right) \\
+ c_p^2 ||f||_{H^{1/2}(\mathbb{T})}^2 + 5\eta ||\nabla (\delta_h^j u)||_{L^2(O)}^2. \]

Using Korn’s inequality on \(\Omega\) \((2.7)\) and taking the parameter \(\eta = \frac{1}{3}\), we obtain
\[
||\nabla (\delta_h^j u)||_{L^2(O)}^2 \leq c_K(c_p + 1)^2 c_{h,-1}|h|^2 \left( ||\nabla u||_{L^2(O)}^2 + ||p||_{L^2(O)}^2 \right); \\
||\nabla (\delta_h^j u)||_{L^2(O)}^2 \leq c_K^2 c_{T,h}^2 ||\nabla u||_{L^2(O)}^2 + ||p||_{L^2(O)}^2 \quad (2.35) \\
+ c_K^2 (c_p + 1)^2 c_{s,-1}|h|^2 \left( ||\nabla u||_{L^2(O)}^2 + ||p||_{L^2(O)}^2 \right). \]

Since any tangent vector field \(\gamma\) with \(\gamma \cdot e_\rho = 0\) can be decomposed into \(\theta\) and \(s\) directions in \(O\), Theorem III.3.20 from \(\text{[6]}\) gives us that \(\nabla_T u \in H^1(O)\).

Furthermore, using \((2.35)\) in \((2.33)\), the pressure satisfies
\[ ||\delta_h^j p||_{L^2(O)} \leq c_{h,-1} c_K (c_p + 1)|h| \left( ||p||_{L^2(O)} + ||\nabla u||_{L^2(O)} \right) \quad \text{and} \]
\[ ||\delta_h^j p||_{L^2(O)} \leq c_{s,-1} c_K |h| \left( (c_p + 1)(||p||_{L^2(O)} + ||\nabla u||_{L^2(O)}) + c_T ||f||_{L^2(O)} \right) \]

Since this holds for any \(h \in [-1, 1]\), by Theorem III.3.20 in \(\text{[6]}\), we obtain \(\nabla_T p \in L^2(O)\) as well.
2.2.3 Normal regularity up to $\Gamma_\epsilon$

We complete the proof of higher regularity for Theorem 1.3 by using the interior and tangential regularity results of Sections 2.2.1 and 2.2.2 to show that $\nabla_N \mathbf{u} \in H^1(\mathcal{O})$ and $\nabla_N p \in L^2(\mathcal{O})$ and hence $(\mathbf{u}, p) \in D^{2,2}(\Omega_\epsilon) \times H^1(\Omega_\epsilon)$.

In the region $\mathcal{O}$, we can rewrite the Stokes equations with respect to the moving frame basis $\mathbf{e}_t$, $\mathbf{e}_\rho$, $\mathbf{e}_\theta$. We decompose $\mathbf{u} = u_\rho \mathbf{e}_\rho + u_\theta \mathbf{e}_\theta + u_t \mathbf{e}_t$, where $u_\rho = \mathbf{u} \cdot \mathbf{e}_\rho$, $u_\theta = \mathbf{u} \cdot \mathbf{e}_\theta$, and $u_t = \mathbf{u} \cdot \mathbf{e}_t$. Then, using the gradient and divergence with respect to the moving frame, we write the Stokes equations as:

$$-\Delta \mathbf{u} + \nabla p = -\Delta \mathbf{u} + \frac{\partial p}{\partial \rho} \mathbf{e}_\rho + \frac{1}{\rho} \frac{\partial p}{\partial \theta} \mathbf{e}_\theta + \frac{1}{1 - \rho \hat{\kappa}} \left(\frac{\partial p}{\partial s} - \kappa_3 \frac{\partial p}{\partial \theta}\right) \mathbf{e}_t = 0$$

$$\text{div} \mathbf{u} = \frac{1}{1 - \rho \hat{\kappa}} \left(\frac{1}{\rho} \frac{\partial (\rho(1 - \rho \hat{\kappa})u_\rho)}{\partial \rho} + \frac{1}{\rho} \frac{\partial ((1 - \rho \hat{\kappa})u_\theta)}{\partial \theta} + \frac{\partial u_t}{\partial s}\right) = 0.$$

Here $\hat{\kappa}(s, \theta) = \kappa_1(s) \cos \theta + \kappa_2(s) \sin \theta$ satisfies that each of $\|1 - \rho \hat{\kappa}\|_{L^\infty(\mathcal{O})}$, $\|\hat{\kappa}\|_{L^\infty(\mathcal{O})}$, and $\|\frac{\partial \hat{\kappa}}{\partial \rho}\|_{L^\infty(\mathcal{O})}$ are bounded by $c_\kappa$, a constant depending only on the fiber centerline geometry. Furthermore, by (6.8) and (6.15), both $\|1/(1 - \rho \hat{\kappa})\|_{L^\infty(\mathcal{O})} \leq c_\kappa$ and $\|\frac{\partial}{\partial s} \frac{1}{1 - \rho \hat{\kappa}}\|_{L^\infty(\mathcal{O})} \leq c_\kappa$, where the latter constant also depends on the first derivatives of the coefficients $\kappa_1$ and $\kappa_2$.

From the divergence-free condition on $\mathbf{u}$, after multiplying through by $\rho(1 - \rho \hat{\kappa})$ and differentiating once with respect to $\rho$, we obtain

$$\left\|\frac{\partial^2 u_\rho}{\partial \rho^2}\right\|_{L^2(\mathcal{O})} \leq c_\kappa \left(\left\|\frac{1}{\rho} \nabla \mathbf{u}\right\|_{L^2(\mathcal{O})} + \left\|\frac{1}{\rho} \right\|_{L^\infty(\mathcal{O})} \left|\mathcal{O}\right|^{1/3} \left\|\mathbf{u}\right\|_{L^6(\mathcal{O})} \right.$$  

$$+ \left\|\frac{1}{\rho} \frac{\partial}{\partial \theta} \left(\frac{\partial u_\theta}{\partial \rho}\right)\right\|_{L^2(\mathcal{O})} + \left\|\frac{\partial}{\partial s} \left(\frac{\partial u_t}{\partial \rho}\right)\right\|_{L^2(\mathcal{O})}\right).$$

$$\leq c_{\kappa,2} \left(\frac{1}{\epsilon} \left\|\nabla \mathbf{u}\right\|_{L^2(\mathcal{O})} + c_K c_T \left\|f\right\|_{H^{1/2}(\mathcal{T})} \right.$$  

$$+ c_K (c_P + 1) \left(\left\|\nabla \mathbf{u}\right\|_{L^2(\mathcal{O})} + \left\|p\right\|_{L^2(\mathcal{O})}\right),$$

using the tangential regularity properties of $\mathbf{u}$ and that $|\mathcal{O}|$ is bounded by a constant depending only on $\kappa_{\max}$ and not on $\epsilon$. Here we use $c_{\kappa,2}$ to denote a constant depending on $\kappa_{\max}$, $c_T$, $m_{\kappa,1}$ and $m_{\kappa,2}$. Then, using the estimate (2.14), we have

$$\left\|\frac{\partial^2 u_\rho}{\partial \rho^2}\right\|_{L^2(\mathcal{O})} \leq c_{\kappa,2} \frac{c_T}{\epsilon} c_K (1 + c_P)^2 (1 + c_K)^2 \left\|f\right\|_{H^{1/2}(\mathcal{T})}.$$

**Remark 2.11.** We note that the factor of $\frac{1}{\epsilon}$ in the above bound is necessary. As a heuristic, we consider an infinite straight cylinder of radius $\epsilon$ and take $\mathbf{u} = (\frac{1}{\rho} - \frac{1}{2}) \mathbf{e}_\theta$, where $\mathbf{e}_\theta$ is now the (constant) angular vector in straight cylindrical coordinates, and $p \equiv $ constant. Ignoring decay conditions toward infinity along the cylinder, $(\mathbf{u}, p)$ solves the Stokes equations with $\mathbf{u} = 0$ on the cylinder surface. Then

$$\left|\nabla^2 \mathbf{u}\right| = \left|\frac{\partial^2}{\partial \rho^2} \frac{1}{\rho^3}\right| = \left|\frac{2}{\rho^3}\right| = \frac{2}{\rho^3} \left|\nabla \mathbf{u}\right|,$$

and within the region $\epsilon < \rho \leq 2\epsilon$, we have $\left|\nabla^2 \mathbf{u}\right| \geq \frac{1}{\epsilon} \left|\nabla \mathbf{u}\right|.$
Furthermore, using the $e_\rho$ component of $-\Delta u + \nabla p = 0$, we have

$$
\frac{\partial p}{\partial \rho} = (\Delta u) \cdot e_\rho
$$

$$
= \frac{1}{\rho(1 - \rho \hat{\kappa})} \frac{\partial}{\partial \rho} \left( \rho(1 - \rho \hat{\kappa}) \frac{\partial u}{\partial \rho} \right) \cdot e_\rho + \frac{1}{\rho^2 (1 - \rho \hat{\kappa})} \frac{\partial}{\partial \theta} \left( (1 - \rho \hat{\kappa}) \frac{\partial u}{\partial \theta} \right) \cdot e_\rho
$$

$$
+ \frac{1}{1 - \rho \hat{\kappa}} \frac{\partial}{\partial s} \left( \frac{1}{1 - \rho \hat{\kappa}} \left[ \frac{\partial u}{\partial s} - \kappa \frac{\partial u}{\partial \theta} \right] \right) \cdot e_\rho
$$

$$
= \frac{1}{\rho(1 - \rho \hat{\kappa})} \frac{\partial}{\partial \rho} \left( \rho(1 - \rho \hat{\kappa}) \frac{\partial u}{\partial \rho} \right) + \frac{1}{\rho^2 (1 - \rho \hat{\kappa})} \frac{\partial}{\partial \theta} \left( (1 - \rho \hat{\kappa}) \frac{\partial u}{\partial \theta} \right) \cdot e_\rho
$$

$$
+ \frac{1}{1 - \rho \hat{\kappa}} \frac{\partial}{\partial s} \left( \frac{1}{1 - \rho \hat{\kappa}} \left[ \frac{\partial u}{\partial s} - \kappa \frac{\partial u}{\partial \theta} \right] \right) \cdot e_\rho,
$$

since $e_\rho(s, \theta)$ does not vary with $\rho$. Therefore, using the tangential regularity of $u$ and $p$ along with the bound on $\frac{\partial^2 u_j}{\partial \rho^2}$, we have

$$
\| \nabla p \|_{L^2(\Omega)} \leq c_{\kappa, 2} \frac{c_T}{\epsilon} c_K (1 + c_P)^2 (1 + c_K)^2 \| f \|_{H^{1/2}(\Gamma)},
$$

where again $c_{\kappa, 2}$ denotes a constant depending on $\kappa_{\text{max}}$, $c_T$, $m_{\kappa, 1}$ and $m_{\kappa, 2}$. Thus $p \in H^1(\Omega)$.

Finally, to estimate $\frac{\partial^2 u_j}{\partial \rho^2}$, $j = \theta, s$, we again use that

$$
\nabla p \cdot e_j = (\Delta u) \cdot e_j(s, \theta)
$$

$$
= \frac{1}{\rho(1 - \rho \hat{\kappa})} \frac{\partial}{\partial \rho} \left( \rho(1 - \rho \hat{\kappa}) \frac{\partial u_j}{\partial \rho} \right) + \frac{1}{\rho^2 (1 - \rho \hat{\kappa})} \frac{\partial}{\partial \theta} \left( (1 - \rho \hat{\kappa}) \frac{\partial u}{\partial \theta} \right) \cdot e_j
$$

$$
+ \frac{1}{1 - \rho \hat{\kappa}} \frac{\partial}{\partial s} \left( \frac{1}{1 - \rho \hat{\kappa}} \left[ \frac{\partial u}{\partial s} - \kappa \frac{\partial u}{\partial \theta} \right] \right) \cdot e_j, \quad j = \theta, s,
$$

since each of the basis vectors $e_l(s)$, $e_\rho(s, \theta)$ and $e_\theta(s, \theta)$ are independent of $\rho$. Then we have

$$
\left\| \frac{\partial^2 u_j}{\partial \rho^2} \right\|_{L^2(\Omega)} \leq c_{\kappa} \left( \frac{1}{\rho} \right) \| \nabla u \|_{L^2(\Omega)} + \left\| \frac{\partial^2 u}{\partial s^2} \right\|_{L^2(\Omega)}
$$

$$
+ \left\| \frac{\partial^2 u}{\partial s \partial \theta} \right\|_{L^2(\Omega)} + \left\| \frac{\partial^2 u}{\partial \theta^2} \right\|_{L^2(\Omega)} + \| \nabla p \|_{L^2(\Omega)}
$$

$$
\leq c_{\kappa, 2} \frac{c_T}{\epsilon} c_K (1 + c_P)^2 (1 + c_K)^2 \| f \|_{H^{1/2}(\Gamma)}, \quad j = \theta, s,
$$

where $c_{\kappa, 2}$ depends only on $\kappa_{\text{max}}$, $c_T$, $m_{\kappa, 1}$, and $m_{\kappa, 2}$.

Therefore, combining the interior, tangential, and normal estimates, we have that $(u, p) \in D^{2,2}(\Omega_\epsilon) \times H^1(\Omega_\epsilon)$ and satisfies the estimate

$$
\| \nabla^2 u \|_{L^2(\Omega_\epsilon)} + \| \nabla p \|_{L^2(\Omega_\epsilon)} \leq c_{\kappa, 2} \frac{c_T}{\epsilon} c_K (1 + c_P)^2 (1 + c_K)^2 \| f \|_{H^{1/2}(\Gamma)},
$$

(2.36)

where $c_{\kappa, 2}$ depends only on $\kappa_{\text{max}}$, $c_T$, $m_{\kappa, 1}$, and $m_{\kappa, 2}$.

Thus we have shown the higher regularity estimate (1.20) of Theorem 1.3 except for the explicit $\epsilon$-dependence of the constants. In the next section, we show that, of each of the constants $c_T$, $c_K$, and $c_P$, only $c_T$ depends on $\epsilon$ as $\epsilon \to 0$, growing as $|\log \epsilon|^{1/2}$.\"
3 Geometry revisited

So far we have proved the existence, uniqueness, and higher regularity claims in Theorem 1.3 for a fixed slender body radius $\epsilon > 0$. We now aim to prove the $\epsilon$-dependence in the estimate (1.20) as $\epsilon \to 0$, which will eventually allow us to derive the error estimate for slender body theory in Theorem 1.4. In particular, we show that the Korn constant $c_K$ and pressure constant $c_P$ are independent of $\epsilon$, while the constant $c_T$ in the trace inequality has a $|\log \epsilon|^{1/2}$ dependence as $\epsilon \to 0$. This $\epsilon$-dependence in the trace inequality is not surprising, as we expect that in the limit as $\epsilon \to 0$ the true solution will look something like the Stokeslet, which has unbounded velocity along the fiber centerline.

Throughout the following sections, we again use $c_\kappa$ to denote any constant depending only on $\kappa_{\text{max}}$ and $c_T$ via the moving frame coefficients $\kappa_i(s)$, $i = 1, 2, 3$ (1.3). In particular, the value of the constant $c_\kappa$ may change throughout the course of a computation, but is always independent of $\epsilon$.

3.1 Trace inequality

We must first establish the $\epsilon$-dependence in the $D^{1,2}(\Omega_\epsilon)$ trace inequality

$$\|\text{Tr}(u)\|_{L^2(\mathbb{T})} \leq c_T\|\nabla u\|_{L^2(\Omega_\epsilon)}.$$ 

Even though the slender body surface $\Gamma_\epsilon$ is codimension 1 and for $u \in D^{1,2}(\Omega_\epsilon)$ satisfies an $H^{1/2}(\Gamma_\epsilon)$ trace inequality, the trace inequality we need for our existence theory and error estimate is essentially a codimension 2 trace inequality. Indeed, for $A_\epsilon = \{ u \in D^{1,2}(\Omega_\epsilon) : u|_{\Gamma_\epsilon} = u(s) \}$ we have

$$\|\text{Tr}(u)\|_{L^2(\Gamma_\epsilon)}^2 = \int_{\mathbb{T}} \int_0^{2\pi} |(\text{Tr}(u))(s, \theta)|^2 J_\epsilon(s, \theta) \, d\theta ds = 2\pi \epsilon \int_{\mathbb{T}} |(\text{Tr}(u))(s)|^2 ds = 2\pi \epsilon \|\text{Tr}(u)\|_{L^2(\mathbb{T})}^2,$$

since $J_\epsilon(s, \theta) = \epsilon(1 - \epsilon(\kappa_1(s) \cos \theta + \kappa_2(s) \sin \theta))$. Thus the trace estimate on $\mathbb{T}$ appears to introduce an additional $1/\sqrt{\epsilon}$ that we must bound. However, we can show that the constant in the $L^2$ trace inequality grows only like $|\log \epsilon|^{1/2}$ as $\epsilon \to 0$.

Lemma 3.1. (Trace inequality) Let $\Omega_\epsilon = \mathbb{R}^3 \setminus \Gamma_\epsilon$ be as in Section 1.1. For $u \in A_\epsilon$, the $\theta$-independent trace of $u$ on $\Gamma_\epsilon$ satisfies

$$\|\text{Tr}(u)\|_{L^2(\mathbb{T})} \leq c_\kappa |\log \epsilon|^{1/2}\|\nabla u\|_{L^2(\Omega_\epsilon)},$$

(3.1)

where $\text{Tr} : D^{1,2}(\Omega_\epsilon) \to L^2(\mathbb{T})$ is the trace operator and the constant $c_\kappa$ depends on the constants $\kappa_{\text{max}}$ and $c_T$, but is independent of the fiber radius $\epsilon$.

Proof. Since the fiber centerline is $C^2$ — and hence the surface $\Gamma_\epsilon$ is $C^2$ — and the fiber does not self-intersect (1.2), we can cover the slender body by finitely many open neighborhoods $W_j$ where

$$W_j = \{ X(s) + \rho e_\rho(s, \theta) : 0 \leq \theta < 2\pi, 0 \leq \rho < 1/(2\kappa_{\text{max}}), a_j < s < b_j \}, \quad j = 1, \ldots, N < \infty.$$ 

Here $a_j$ and $b_j$ are chosen such that over each $W_j$, the fiber centerline can be considered as the graph of a $C^2$ function. Note that this choice of $a_j$ and $b_j$ depends only on the shape of the...
fiber centerline – in particular, \( \kappa_{\text{max}} \) and \( c^\Gamma \) – and not on the fiber radius.

Then, using a partition of unity \( \{ \phi_j \}_{j=1}^N \) subordinate to the cover \( \{ W_j \} \), there exist smooth, \( \epsilon \)-independent diffeomorphisms \( \psi_j \), \( j = 1, \ldots, N \) taking the curvature \( \kappa \) of the fiber centerline to zero on the set \( W_j \) while leaving the radius \( \epsilon \) intact.

![Diagram](image)

**Figure 3.1:** The slender body centerline can be straightened via \( \epsilon \)-independent diffeomorphisms \( \psi_j \); thus it suffices to consider functions \( u \) around a straight cylinder supported within the truncated cylindrical shell \( C_{\epsilon,a} \).

Let \( D_\rho \subset \mathbb{R}^2 \) denote the open disk of radius \( \rho \) in \( \mathbb{R}^2 \). Define the straight cylindrical surface \( \Gamma_{\epsilon,a} := \partial D_\epsilon \times [-a,a] \) and the cylindrical shell \( C_{\epsilon,a} := (D_1 \setminus \overline{D_\epsilon}) \times [-a,a] \) for some \( a < \infty \), parameterized in cylindrical coordinates \( (\rho, \theta, s) \). We define the function space

\[
\mathcal{A}_S := \{ v \in D^{1,2}(C_{\epsilon,a}) : v|_{\Gamma_{\epsilon,a}} = v(s); \ v|_{\partial C_{\epsilon,a} \setminus \Gamma_{\epsilon,a}} = 0 \}.
\]

Then \( \psi_j^*(\phi_j u) \in \mathcal{A}_S \), and to show Lemma 3.1 it suffices to prove the \( |\log \epsilon|^{1/2} \) dependence in the trace constant about a straight cylinder.

**Lemma 3.2.** Let \( u \in \mathcal{A}_S \). Then the \( \theta \)-independent trace of \( u \) on the straight cylinder \( \Gamma_{\epsilon,a} \) satisfies

\[
\| \text{Tr}(u) \|_{L^2([-a,a])} \leq \frac{1}{2\pi} |\log \epsilon|^{1/2} \| \nabla u \|_{L^2(\Omega_S)}.
\]  

**Proof.** We show the inequality \( 3.2 \) for \( u \in C^1(C_{\epsilon,a}) \cap C^0(\overline{C_{\epsilon,a}}) \); the proof for \( u \in D^{1,2}(C_{\epsilon,a}) \) then follows by density. At any point \( x = se_t + \epsilon e_\rho + \theta e_\theta \) along the cylinder surface \( \Gamma_{\epsilon,a} \) we have

\[
u(s, \theta, \epsilon) = -\int_1^\epsilon \frac{\partial u}{\partial \rho} d\rho.
\]
Therefore,

\[ |u(s, \theta, \epsilon)| \leq \int_{\epsilon}^{1} \left| \frac{\partial u}{\partial \rho} \right| d\rho = \int_{\epsilon}^{1} \frac{1}{\sqrt{\rho}} \sqrt{\rho} \left| \frac{\partial u}{\partial \rho} \right| d\rho \leq \left( \int_{\epsilon}^{1} \frac{1}{\rho} d\rho \right)^{1/2} \left( \int_{\epsilon}^{1} \left| \frac{\partial u}{\partial \rho} \right|^{2} \rho d\rho \right)^{1/2} \]

\[ = \sqrt{\log \epsilon} \left( \int_{\epsilon}^{1} \left| \frac{\partial u}{\partial \rho} \right|^{2} \rho d\rho \right)^{1/2}. \]

Thus any boundary value of \( u \) obeys

\[ |u|_{\Gamma, \epsilon}^{2} \leq |\log \epsilon| \int_{\epsilon}^{1} \left| \frac{\partial u}{\partial \rho} \right|^{2} \rho d\rho. \quad (3.3) \]

This holds for arbitrary \( u \in C^{1}(C_{\epsilon, a}) \cap C^{0}(C_{\epsilon, a}) \), but for \( u \in A_{S} \), since \( u|_{\Gamma, \epsilon} \) is independent of \( \theta \) we have

\[ \|u\|_{L^{2}([-a, a])}^{2} = \frac{1}{2\pi \epsilon} \|u\|_{L^{2}(\Gamma, \epsilon)}^{2} = \frac{1}{2\pi} \int_{-a}^{a} \int_{0}^{2\pi} |u|^{2} \epsilon d\theta ds = \frac{1}{2\pi} \int_{-a}^{a} \int_{0}^{2\pi} |u|^{2} d\theta ds. \]

Therefore, using (3.3), we have that

\[ \|u\|_{L^{2}([-a, a])}^{2} \leq \frac{1}{2\pi} \|\nabla u\|_{L^{2}}^{2} \leq \frac{1}{2\pi} \|\nabla u\|_{L^{2}}^{2} \leq \frac{1}{2\pi} \|\nabla u\|_{L^{2}}^{2} \leq \frac{1}{2\pi} \log \epsilon \|\nabla u\|_{L^{2}}^{2}. \]

This estimate holds for \( u \) defined around a straight cylinder; to return to a curved centerline, the diffeomorphisms \( \psi_{j}^{-1} \) result in an additional constant on each set \( W_{j} \) depending on \( \psi_{j} \) but not \( \epsilon \). Summing over the \( \phi_{j} \), we obtain the trace constant for any slender body \( \Sigma_{\epsilon} \) satisfying the geometric constraints in Section 1.1:

\[ c_{T} = c_{\kappa} |\log \epsilon|^{1/2}, \quad (3.4) \]

where \( c_{\kappa} \) depends on the shape of the fiber centerline – in particular, on the constants \( \kappa_{\text{max}} \) and \( c_{T} \) – but not on \( \epsilon \).

### 3.2 Korn inequality

The estimate [2.14] for the Stokes problem relies on a Korn inequality [2.4] bounding \( \nabla u \) by \( E(u) \), the symmetric part of the gradient. We show that the constant in the Korn inequality is bounded independent of \( \epsilon \) as \( \epsilon \to 0 \).

**Lemma 3.3.** (Korn inequality) Let \( \Omega_{\epsilon} = \mathbb{R}^{3} \setminus \Sigma_{\epsilon} \) be as in Section 1.1. There exists a constant \( c_{K} \) depending only on \( \kappa_{\text{max}} \) and \( c_{T} \) such that for all \( u \in D^{1,2}(\Omega_{\epsilon}) \), the Korn inequality holds:

\[ \|\nabla u\|_{L^{2}(\Omega_{\epsilon})} \leq c_{K} \|E(u)\|_{L^{2}(\Omega_{\epsilon})}. \quad (3.5) \]
The proof of Lemma 3.3 essentially relies on the existence of a linear operator \( T_\epsilon \) extending \( u \) to the interior of the slender body such that \( \mathcal{E}(T_\epsilon u) \) is bounded independent of \( \epsilon \) as \( \epsilon \to 0 \). We can then use the simple proof of the Korn inequality over all of \( \mathbb{R}^3 \) used in showing (2.4). The key is thus to show the following lemma:

**Lemma 3.4.** (Extension operator) Let \( \Omega_\epsilon = \mathbb{R}^3 \setminus \bigcup U \) be as in Section 1.1. For \( u \in D^{1,2}(\Omega_\epsilon) \), there exists a bounded linear operator \( T_\epsilon : D^{1,2}(\Omega_\epsilon) \to D^{1,2}(\mathbb{R}^3) \) extending \( u \) to the interior of the slender body and satisfying

1. \( T_\epsilon u|_{\Omega_\epsilon} = u \)
2. \( \|\mathcal{E}(T_\epsilon u)\|_{L^2(\mathbb{R}^3)} \leq \sqrt{2} c_E \|\mathcal{E}(u)\|_{L^2(\Omega_\epsilon)} \), where the constant \( c_E \) is independent of the slender body radius \( \epsilon \) as \( \epsilon \to 0 \).

Note that property 2 implies \( \|T_\epsilon u\|_{D^{1,2}(\mathbb{R}^3)} \leq \sqrt{2} c_E \|u\|_{D^{1,2}(\Omega_\epsilon)} \), since

\[
\begin{align*}
\|T_\epsilon u\|_{D^{1,2}(\mathbb{R}^3)} &= \|\nabla (T_\epsilon u)\|_{L^2(\mathbb{R}^3)} \\
&\leq \sqrt{2} \mathcal{E}(T_\epsilon u)\|_{L^2(\mathbb{R}^3)} \\
&\leq \sqrt{2} c_E \|\nabla u\|_{L^2(\Omega_\epsilon)} = \sqrt{2} c_E \|u\|_{D^{1,2}(\Omega_\epsilon)}.
\end{align*}
\]

In order to prove Lemma 3.4 we will need to show three additional lemmas. The first is an analogue of Lemma 3.1.2(1) in [28], adapted to use the symmetric gradient rather than the full gradient.

**Lemma 3.5.** (Extension-by-reflection scaling) Let \( \mathcal{D}, \mathcal{D}_e \) be bounded \( C^2 \) domains in \( \mathbb{R}^3 \) with \( \overline{\mathcal{D}}_e \subset \mathcal{D} \), and let \( \mathcal{D}_H = \mathcal{D} \setminus \overline{\mathcal{D}}_2 \) be a bounded \( C^2 \) domain with a hole. For the rescaled domains \( \mathcal{D}_{H,\epsilon} = \epsilon \mathcal{D}_H, \mathcal{D}_e = \epsilon \mathcal{D} \) (\( \epsilon \in \mathbb{R}_+ \)), there exists a linear extension operator \( T : H^1(\mathcal{D}_{H,\epsilon}) \to H^1(\mathcal{D}_e) \) satisfying

\[
\|Tu\|_{L^2(\mathcal{D}_\epsilon)} \leq c \|u\|_{L^2(\mathcal{D}_{H,\epsilon})}
\]

as well as the estimate

\[
\|\mathcal{E}(Tu)\|_{L^2(\mathcal{D}_\epsilon)} \leq c \left( \epsilon^{-1} \|u\|_{L^2(\mathcal{D}_{H,\epsilon})} + \|\mathcal{E}(u)\|_{L^2(\mathcal{D}_{H,\epsilon})} \right).
\]

**Proof.** For a function \( v \) defined in the upper half-space \( \mathbb{R}^3_+ \), we recall the standard extension-by-reflection \( E : \mathbb{R}^3_+ \to \mathbb{R}^3 \) across the boundary \( x_3 = 0 \) (see [28] or [14]):

\[
E v(x) = \begin{cases} 
  v(x), & v \in \mathbb{R}^3_+ \\
  v(x_1, x_2, -x_3), & v \notin \mathbb{R}^3_+.
\end{cases}
\]

For the domain-with-hole \( \mathcal{D}_H \subset \mathbb{R}^3 \), we cover a neighborhood of the inner boundary \( \partial \mathcal{D}_2 \) with finitely many balls \( B_i \) centered at points on \( \partial \mathcal{D}_2 \), choosing the cover such that \( \mathcal{D}|_{B_i} \) can be mapped via \( C^2 \) diffeomorphism to the half-ball \( B_i \cap \mathbb{R}^3_+ \). We then choose open sets \( U_j \subset \mathcal{D}_H \) such that \( \{B_i\} \cup \{U_i\} \) cover \( \mathcal{D}_H \). We define a partition of unity \( \{\varphi_i\} \cup \{\varphi_j\} \) subordinate to this cover, and define the usual extension operator \( T : \mathcal{D}_H \to \mathcal{D} \) by

\[
Tu = \sum_i E(\varphi_i u \circ \Phi_i) \circ \Phi_i^{-1} + \sum_j \varphi_j u.
\]

From this extension operator \( T \), we can directly estimate \( \|\mathcal{E}(Tu)\|_{L^2(\mathcal{D})} \):

\[
\|\mathcal{E}(Tu)\|_{L^2(\mathcal{D})} \leq c \sum_i \left\| \nabla \Phi_i^{-1}(\varphi_i \nabla u \nabla \Phi_i + \nabla \varphi_i u) + \nabla \Phi_i^{-T}(\varphi_i \nabla u \nabla \Phi_i + \nabla \varphi_i u) \right\|_{L^2(\mathcal{D}_H)}
\]

35
\[ + \sum_j \|\varphi_j \mathcal{E}(u)\|_{L^2(D)} + \sum_j \|\nabla \varphi_j u^T\|_{L^2(D)} \]
\[ \leq c \phi \|\nabla u + \nabla u^T\|_{L^2(D)} + c \phi \|u\|_{L^2(D)} + \|\mathcal{E}(u)\|_{L^2(D)} + c \|u\|_{L^2(D)} \]
\[ \leq c(\|u\|_{L^2(D)} + \|\mathcal{E}(u)\|_{L^2(D)}) \]

The above inequality, coupled with a scaling argument \((x \to cx)\) results in the desired \(c\)-dependent inequality \((3.7)\).

Again, let \(D\) be a bounded, \(C^2\) domain. On \(D\), we define the space of rigid motions
\[ \mathcal{R} = \{v \in H^1(D) : v = Ax + b \text{ for some } A = -A^T \in \mathbb{R}^{3 \times 3} \text{ and } b \in \mathbb{R}^3\}. \]

For \(u \in H^1(D)\), let \(P_{\mathcal{R}} u\) be the \(L^2\) projection of \(u\) onto the space of rigid motions, i.e.

\[ P_{\mathcal{R}} u = v \in \mathcal{R} \text{ such that } \|u - v\|_{L^2(D)} \leq \|u - w\|_{L^2(D)} \forall w \in \mathcal{R}. \]

**Lemma 3.6.** (Korn’s inequality for pure strain) Let \(D\) be a bounded Lipschitz domain and let \(\mathcal{R}\) be the space of rigid motions on \(D\). For any \(w \in H^1(D)\) with \(w \perp \mathcal{R}\), Korn’s inequality holds:

\[ \|\nabla w\|_{L^2(D)} \leq c\|\mathcal{E}(w)\|_{L^2(D)}. \]

**Proof.** The proof of Lemma 3.6 relies on the following Korn-type inequality for the bounded domain \(D\):

\[ \|u\|_{H^1(D)} \leq c(\|\mathcal{E}(u)\|_{L^2(D)} + \|u\|_{L^2(D)}). \] (3.8)

Since the domain dependence of the constant \(c\) does not need to be specified in Lemma 3.6, we refer to [13] for a proof of (3.8).

Now, assume Lemma 3.6 does not hold. Then there exists a sequence of functions \(\{w_k\} \subset H^1(D), k = 1, 2, 3, \ldots\), such that \(w_k \perp \mathcal{R}\) and

\[ \|\nabla w_k\|_{L^2(D)} > k\|\mathcal{E}(w_k)\|_{L^2(D)}. \]

Without loss of generality, \(\|w_k\|_{L^2(D)} = 1\), so by (3.8),

\[ \|\mathcal{E}(w_k)\|_{L^2(D)} < \frac{1}{k}\|\nabla w_k\|_{L^2(D)} \leq \frac{1}{k}\|w_k\|_{H^1(D)} \leq c(\|\mathcal{E}(w_k)\|_{L^2(D)} + 1). \]

Taking \(k\) sufficiently large (in particular, \(k > c\)), we have

\[ \left(1 - \frac{c}{k}\right)\|\mathcal{E}(w_k)\|_{L^2(D)} < \frac{c}{k}, \]

and thus \(\|\mathcal{E}(w_k)\|_{L^2(D)} \to 0\) as \(k \to \infty\). Again by the inequality (3.8),

\[ \|w_k\|_{H^1(D)} \leq c\left(\frac{c}{k - c} + 1\right), \]

so there exists a subsequence \(\{w_{k_j}\}\) such that \(w_{k_j} \to w\) in \(H^1(D)\). By Rellich compactness, \(w_{k_j} \to w\) in \(L^2\). Furthermore, \(\liminf_k \mathcal{E}(w_{k_j}) \geq \mathcal{E}(w)\), so \(\mathcal{E}(w) = 0\). Thus \(w \in \mathcal{R}\), but \(w_k \perp \mathcal{R}\) for all \(k\), and \(w_{k_j} \to w\) in \(L^2\), so \(w \equiv 0\). Thus \(w_{k_j} \to 0\) in \(L^2\), which contradicts \(\|w_k\|_{L^2(D)} = 1\) for all \(k\). \(\square\)
Lemma 3.7. (Korn-Poincaré inequality) Let D be a bounded, Lipschitz domain in \( \mathbb{R}^3 \). For any \( u \in H^1(D) \), we have
\[
\|u - P_R u\|_{L^2(D)} \leq c\|E(u)\|_{L^2(D)}
\]
for some constant \( c > 0 \).

Proof. Assume that inequality (3.9) does not hold. Then for each \( k = 1, 2, 3, \ldots \) there exists a sequence \( \{u_k\} \subset H^1(D) \) such that
\[
\|u_k - P_R u_k\|_{L^2(D)} > k\|E(u_k)\|_{L^2(D)}.
\]
Define \( w_k = u_k - P_R u_k \), so \( w_k \perp R \) for each \( k = 1, 2, 3, \ldots \) and \( E(w_k) = E(u_k) \). Without loss of generality \( \|w_k\|_{L^2(D)} = 1 \). Then
\[
1 = \|w_k\|_{L^2(D)} > k\|E(w_k)\|_{L^2(D)},
\]
so \( \|E(w_k)\|_{L^2(D)} < \frac{1}{k} \to 0 \) as \( k \to \infty \). Furthermore, since \( w_k \perp R \) for each \( k \), by Korn’s inequality for pure strain (Lemma 3.6) we have \( \|\nabla w_k\|_{L^2(D)} < \frac{c}{k} \). Thus \( w_k \) is uniformly bounded in \( H^1 \) and there exists a subsequence \( \{w_{k_i}\} \) such that \( w_{k_i} \rightharpoonup w \) in \( H^1 \) for some \( w \in H^1(D) \). By compactness, \( w_{k_i} \rightarrow w \) in \( L^2 \). Then, since \( \liminf\|E(w_{k_i})\|_{L^2(D)} \geq \|E(w)\|_{L^2(D)} \), we have that the limit \( w \) satisfies \( E(w) = 0 \), so \( w \in R \). But \( w_{k_i} \rightarrow w \) in \( L^2 \) and \( w_{k_i} \perp R \) for each \( k \), so we must have \( w \perp R \) as well. Thus \( w \equiv 0 \), so \( w_{k_i} \rightarrow 0 \) in \( L^2 \), which contradicts \( \|w_{k_i}\|_{L^2(D)} = 1 \).

With Lemmas 3.5 and 3.7 we are equipped to prove Lemma 3.4.

Proof of Lemma 3.4. Let \( D_r \) denote the disk in \( \mathbb{R}^3 \) of radius \( r \). Using the diffeomorphisms \( \psi_j \) defined in Section 3.1, it suffices to consider \( u \in D^{1,2}(\mathbb{R}^2 \setminus D_{\epsilon} \times \mathbb{R}) \) with \( \text{supp}(u) \subset \mathbb{R}^2 \setminus D_{\epsilon} \times [-a, a] \) for \( a < \infty \) and show that there exists an extension operator into the interior of the infinite cylinder \( D_{\epsilon} \times \mathbb{R} \subset \mathbb{R}^3 \) with symmetric gradient that is bounded independent of \( \epsilon \) as \( \epsilon \to 0 \).

First we define
\[
S_{\epsilon} = D_{2\epsilon} \times \mathbb{R} \quad \text{and} \quad G_{\epsilon} = (D_{2\epsilon} \setminus D_{\epsilon}) \times \mathbb{R} \subset \mathbb{R}^3.
\]
Since \( u \in D^{1,2}(\mathbb{R}^2 \setminus D_{\epsilon} \times \mathbb{R}) \) with \( \text{supp}(u) \subset \mathbb{R}^2 \setminus D_{\epsilon} \times [-a, a] \), we have \( u \in H^1(G_{\epsilon}) \). We show that we can in fact construct a linear extension operator extending \( u \in H^1(G_{\epsilon}) \) to \( H^1(S_{\epsilon}) \) whose symmetric gradient is bounded independent of \( \epsilon \).

Following [28], we begin by defining a cover \( \{Q_j\} \) of \( \mathbb{R} \):
\[
Q_j = \{s \in \mathbb{R} : |s - j| < 1\}, \quad j \in \mathbb{Z}.
\]
Let \( \{\eta_j\} \) denote a smooth partition of unity subordinate to \( Q_j \), where \( \eta_j \) can be written as \( \eta_j = \phi(s - j) \), translates of the same smooth cutoff function, such that \( \|
abla \eta_j\| \leq c \) for each \( j \). We define a sequence of cylinders and cylindrical layers
\[
S_{2}^{(j)} = D_2 \times Q_j \quad \text{and} \quad G_{2}^{(j)} = (D_2 \setminus D_1) \times Q_j \subset \mathbb{R}^3.
\]
and set \( S_{\epsilon}^{(j)} = \epsilon S_{2}^{(j)} \) and \( G_{\epsilon}^{(j)} = \epsilon G_{2}^{(j)} \). Then by Lemma 3.5, there exists a linear extension operator \( T_{\epsilon}^{(j)} : H^1(G_{\epsilon}^{(j)}) \to H^1(S_{\epsilon}^{(j)}) \) satisfying
\[
\|E(T_{\epsilon}^{(j)} u)\|_{L^2(S_{\epsilon}^{(j)})} \leq c \left( \epsilon^{-1}\|u\|_{L^2(G_{\epsilon}^{(j)})} + \|E u\|_{L^2(G_{\epsilon}^{(j)})} \right)
\]
and
\[
\|T_{\epsilon}^{(j)} u\|_{L^2(S_{\epsilon}^{(j)})} \leq c\|u\|_{L^2(G_{\epsilon}^{(j)})}.
\]
Let $P^{(j)}_{\mathcal{R}} u$ denote the projection of $u|_{G^{(j)}_{\epsilon}} \in H^1(G^{(j)}_{\epsilon})$ onto $\mathcal{R}$, the space of rigid motions on each $G^{(j)}_{\epsilon}$. Then, since $\mathcal{E}(w) = 0$ for any $w \in \mathcal{R}$, we have

$$
\|\mathcal{E}(u - P^{(j)}_{\mathcal{R}} u)\|_{L^2(G^{(j)}_{\epsilon})} = \|\mathcal{E}(u)\|_{L^2(G^{(j)}_{\epsilon})},
$$

By the Korn-Poincaré inequality (Lemma 3.7) and a scaling argument we also have

$$
\|u - P^{(j)}_{\mathcal{R}} u\|_{L^2(G^{(j)}_{\epsilon})} \leq c\|\mathcal{E}(u)\|_{L^2(G^{(j)}_{\epsilon})},
$$

(3.12)

Since $P^{(j)}_{\mathcal{R}} u \in \mathcal{R}$ on each cylindrical shell $G^{(j)}_{\epsilon}$, we can write $P^{(j)}_{\mathcal{R}} u = A_j x + b_j$ for $x \in G^{(j)}_{\epsilon}$. We then define the extension to each of the cylinders $S^{(j)}_{\epsilon}$ by

$$
\bar{P}^{(j)}_{\mathcal{R}} u = A_j x + b_j, \quad x \in S^{(j)}_{\epsilon}.
$$

(3.13)

With these tools in mind, we now define an extension operator from the cylindrical shell to the cylinder $S_{\epsilon}$. We take

$$
T_{\epsilon} u(x) = v(x) + w(x)
$$

(3.14)

where, for $x = x(\rho, \theta, s) \in S_{\epsilon}$ and $u_j = u|_{G^{(j)}_{\epsilon}}$,

$$
v(\rho, \theta, s) = \sum_{j \in \mathbb{Z}} \eta_j(s/\epsilon) \left( \bar{P}^{(j)}_{\mathcal{R}} u \right)(x)
$$

$$
w(\rho, \theta, s) = \sum_{j \in \mathbb{Z}} \eta_j(s/\epsilon) \left( T_{\epsilon}^{(j)}(u_j - P^{(j)}_{\mathcal{R}} u) \right)(x).
$$

Note that $T_{\epsilon} u|_{G^{(j)}_{\epsilon}} = u$. Furthermore, we show

$$
\|\mathcal{E}(T_{\epsilon} u)\|_{L^2(S_{\epsilon})} \leq c\|\mathcal{E}(u)\|_{L^2(G^{(j)}_{\epsilon})}
$$

(3.15)

where the constant $c$ does not depend on $\epsilon$ as $\epsilon \to 0$.

We begin by estimating $v$. Let

$$
\tilde{Q}_j = \{ s \in \mathbb{R} : 0 < s - j < 1 \}, \quad j \in \mathbb{Z}.
$$

Note that for each $j$ we have $\tilde{Q}_j \subset Q_j$ and $\tilde{Q}_j \subset Q_{j+1}$; in particular, $\eta_j(s) + \eta_{j+1}(s) = 1$ on $\tilde{Q}_j$.

Define

$$
\tilde{S}^{(j)}_{\epsilon} = \epsilon \left( D_2 \times \tilde{Q}_j \right) \quad \text{and} \quad \tilde{G}^{(j)}_{\epsilon} = \epsilon \left( (D_2 \setminus D_1) \times \tilde{Q}_j \right).
$$

On each $\tilde{S}^{(j)}_{\epsilon}$, $v$ can be rewritten as

$$
v(\rho, \theta, s) = \bar{P}^{(j)}_{\mathcal{R}} u + \eta_{j+1}(s/\epsilon)(\bar{P}^{(j+1)}_{\mathcal{R}} u - \bar{P}^{(j)}_{\mathcal{R}} u).
$$

Then, by the definition (3.13) and (3.15), we can bound the norm of $\bar{P}^{(j)}_{\mathcal{R}} u$ on each cylinder $\tilde{S}^{(j)}_{\epsilon}$ by its norm over the shell $\tilde{G}^{(j)}_{\epsilon}$: $\|\bar{P}^{(j)}_{\mathcal{R}} u\|_{L^2(\tilde{S}^{(j)}_{\epsilon})} \leq c\|P^{(j)}_{\mathcal{R}} u\|_{L^2(\tilde{G}^{(j)}_{\epsilon})}$. Using this, we bound the symmetric gradient of $v$:

$$
\|\mathcal{E}(v)\|_{L^2(\tilde{S}^{(j)}_{\epsilon})} = \|\nabla \eta_{j+1}(s/\epsilon)(\bar{P}^{(j+1)}_{\mathcal{R}} u - \bar{P}^{(j)}_{\mathcal{R}} u)^T + (\bar{P}^{(j+1)}_{\mathcal{R}} u - \bar{P}^{(j)}_{\mathcal{R}} u)\nabla \eta_{j+1}(s/\epsilon)^T\|_{L^2(\tilde{S}^{(j)}_{\epsilon})}
$$

$$
\leq c\epsilon^{-1}\|P^{(j+1)}_{\mathcal{R}} u - P^{(j)}_{\mathcal{R}} u\|_{L^2(\tilde{G}^{(j)}_{\epsilon})}
$$

38
\[ \leq c\varepsilon^{-1} \left( \| u - P_R^{(j+1)} u \|_{L^2(G_e^{(j+1)})} + \| u - P_R^{(j)} u \|_{L^2(G_e^{(j)})} \right), \]

where in the last step we have used that \( \tilde{G}_e^{(j)} \subset G_e^{(j+1)} \) and \( \tilde{G}_e^{(j)} \subset G_e^{(j)} \). Finally, using (3.12), we have

\[ \| \mathcal{E}(u) \|_{L^2(\tilde{S}_e^{(j)})} \leq c \left( \| \mathcal{E}(u) \|_{L^2(\tilde{G}_e^{(j)})} + \| \mathcal{E}(u) \|_{L^2(G_e^{(j+1)})} \right). \]

Summing over \( j \), we then have

\[ \| \mathcal{E}(u) \|_{L^2(\tilde{S}_e)} \leq c \| \mathcal{E}(u) \|_{L^2(G_e)} \]

where \( c \) is bounded independent of \( \varepsilon \) as \( \varepsilon \to 0 \).

We now bound the symmetric gradient \( w \). On each \( \tilde{S}_e^{(j)} \) we have

\[ \| \mathcal{E}(w) \|_{L^2(\tilde{S}_e^{(j)})} \leq \| \nabla \eta_j(s/\varepsilon) T_e^{(j)}(u_j(x) - P_R^{(j)} u)^T + T_e^{(j)}(u_j(x) - P_R^{(j)} u) \nabla \eta_j(s/\varepsilon)^T \|_{L^2(\tilde{S}_e^{(j)})} \]

\[ + \| \eta_j(s/\varepsilon) \mathcal{E}(T_e^{(j)}(u_j(x) - P_R^{(j)} u)) \|_{L^2(\tilde{S}_e^{(j)})} \]

\[ \leq c\varepsilon^{-1} \| T_e^{(j)}(u_j(x) - P_R^{(j)} u) \|_{L^2(\tilde{S}_e^{(j)})} + \| \mathcal{E}(T_e^{(j)}(u_j(x) - P_R^{(j)} u)) \|_{L^2(\tilde{S}_e^{(j)})}. \]

Using the inequalities (3.10), (3.11), and (3.12), we have

\[ \| T_e^{(j)}(u_j(x) - P_R^{(j)} u) \|_{L^2(\tilde{S}_e^{(j)})} \leq c \| u_j(x) - P_R^{(j)} u \|_{L^2(\tilde{G}_e^{(j)})} \]

and

\[ \| \mathcal{E}(T_e^{(j)}(u_j(x) - P_R^{(j)} u)) \|_{L^2(\tilde{S}_e^{(j)})} \leq c \left( \varepsilon^{-1} \| u_j(x) - P_R^{(j)} u \|_{L^2(\tilde{G}_e^{(j)})} + \| \mathcal{E}(u_j(x) - P_R^{(j)} u) \|_{L^2(\tilde{G}_e^{(j)})} \right) \]

\[ \leq c \| \mathcal{E}(u) \|_{L^2(\tilde{G}_e^{(j)})}. \]

Summing over \( j \), we have

\[ \| \mathcal{E}(w) \|_{L^2(\tilde{S}_e)} \leq c \| \mathcal{E}(u) \|_{L^2(G_e)}. \]

Therefore the extension operator \( T_e : G_e \to S_e \) (3.14) is bounded independent of \( \varepsilon \) as \( \varepsilon \to 0 \).

Defining \( T_e u = u \) in \( \mathbb{R}^3 \setminus S_e \) then gives the desired extension on all of \( \mathbb{R}^3 \). \( \square \)

**Proof of Lemma 3.3.** Using the extension operator \( T_e \) established in Lemma 3.4 to extend \( u \in D^{1,2}(\Omega_e) \) to all of \( \mathbb{R}^3 \), the proof of Lemma 3.3 is immediate. We refer to the proof of the Korn inequality over \( \mathbb{R}^3 \) (see Lemma 2.4) to show that the Korn inequality (3.3) holds independent of the slender body radius \( \varepsilon \) as \( \varepsilon \to 0 \). \( \square \)

### 3.3 Sobolev Inequality

Using the extension operator defined in the previous section, we show that the Sobolev inequality holds with bounds independent of \( \varepsilon \) as \( \varepsilon \to 0 \). We prove the following lemma:

**Lemma 3.8.** (Sobolev inequality) Let \( \Omega_e = \mathbb{R}^3 \setminus \Sigma_e \), the exterior of a slender body of radius \( \varepsilon \). For any \( u \in D^{1,2}(\Omega_e) \), we have

\[ \| u \|_{L^6(\Omega_e)} \leq c_S \| u \|_{L^2(\Omega_e)} \] (3.16)

with a constant \( c_S \) that is bounded independent of \( \varepsilon \) as \( \varepsilon \to 0 \).
Proof. We have
\[ \| \mathbf{u} \|_{L^6(\Omega_\epsilon)} \leq \| T_\epsilon \mathbf{u} \|_{L^6(\mathbb{R}^3)} \leq c_R \| \nabla (T_\epsilon \mathbf{u}) \|_{L^2(\mathbb{R}^3)} \]
\[ \leq c_R c_E \| \nabla \mathbf{u} \|_{L^2(\Omega_\epsilon)}, \quad \text{by Lemma 3.4} \]
where \( c_R \) is the constant in the Sobolev inequality on \( \mathbb{R}^3 \). Taking \( c_S = c_R c_E \), we obtain the desired result. \( \square \)

### 3.4 Pressure estimate

Finally, to complete the proof of the \( \epsilon \)-dependence in the estimate (1.19) of Theorem 1.3, we verify the \( \epsilon \)-independence of the constant \( c_P \) in the pressure inequality (2.15), which we recall here:
\[ \| p \|_{L^2(\Omega_\epsilon)} \leq c_P \| \mathcal{E}(\mathbf{u}) \|_{L^2(\Omega_\epsilon)}. \]

Following [16], Chapter III.3, we show the following lemma.

**Lemma 3.9.** (Solution to \( \text{div} \mathbf{v} = p \)) Let \( \Omega_\epsilon = \mathbb{R}^3 \setminus \Sigma_\epsilon \), the exterior of a slender body of radius \( \epsilon \). There exists a function \( \mathbf{v} \in D^{1,2}(\Omega_\epsilon) \) satisfying
\[ \text{div} \mathbf{v} = p \quad \text{in} \ \Omega_\epsilon; \]
\[ \| \mathbf{v} \|_{D^{1,2}(\Omega_\epsilon)} \leq c_P \| p \|_{L^2(\Omega_\epsilon)}, \]
where the constant \( c_P \) depends on \( \kappa_{\text{max}} \) and \( c_\Gamma \) but not on \( \epsilon \).

**Proof of Lemma 3.9.** We begin by taking a sequence \( \{ p_m \} \subset C_0^\infty(\Omega_\epsilon) \) approximating \( p \) in \( L^2(\Omega_\epsilon) \). For each \( m \in \mathbb{N} \), let \( \psi_m \) be the solution to the Poisson problem \( \Delta \psi_m = p_m \) in \( \mathbb{R}^3 \), where \( p_m \) denotes the extension by zero of \( p_m \) to the interior of \( \Sigma_\epsilon \); i.e. to all of \( \mathbb{R}^3 \). Then by standard solution theory for the Poisson problem ([16], Chapter II.11), we have the estimate
\[ \| \nabla^2 \psi_m \|_{L^2(\Omega_\epsilon)} \leq c_q \| p_m \|_{L^2(\mathbb{R}^3)} = c_q \| p_m \|_{L^2(\Omega_\epsilon)}, \quad \text{(3.17)} \]
where \( \nabla^2 \) denotes the matrix of second partial derivatives and the constant \( c_q \) is independent of \( \epsilon \).

We define
\[ \mathbf{v}_m := \nabla \psi_m + \mathbf{w}_m \]
where \( \mathbf{w}_m \in D^{1,2}(\Omega_\epsilon) \) is supported only within the neighborhood \( \mathcal{O} \) (1.12) of \( \Gamma_\epsilon \), and serves to correct for \( \nabla \psi_m \neq 0 \) on \( \Gamma_\epsilon \). To this end, \( \mathbf{w}_m \) can be considered as a function in \( H^1(\mathcal{O}) \) satisfying
\[ \text{div} \mathbf{w}_m = 0 \quad \text{in} \ \mathcal{O}; \]
\[ \mathbf{w}_m = -\nabla \psi_m \quad \text{on} \ \Gamma_\epsilon; \]
\[ \mathbf{w}_m = 0 \quad \text{on} \ \partial \mathcal{O} \setminus \Gamma_\epsilon, \quad \text{(3.18)} \]
which is then extended by zero to all of \( \Omega_\epsilon \). For each \( m \in \mathbb{N} \), such a function \( \mathbf{w}_m \) exists since \( \Delta \psi_m = 0 \) within \( \Sigma_\epsilon \) and therefore
\[ \int_{\Gamma_\epsilon} \nabla \psi_m \cdot \mathbf{n} = 0. \]

A solution to (3.18) can be constructed by considering the function \( \Psi_m = -\phi \nabla \psi_m \) where \( \phi \in C^\infty(\Omega_\epsilon) \) is a cutoff function satisfying
\[ \phi(\rho) = \begin{cases} 1, & \rho \leq \frac{1}{2 \kappa_{\text{max}}} \\ 0, & \rho > \frac{1}{2 \kappa_{\text{max}}}. \end{cases} \]
Then by [16], Theorem III.3.1, there exists a solution $w_m - \Psi_m \in H^1_0(O)$ satisfying
\[
\begin{align*}
\text{div}(w_m - \Psi_m) &= -\text{div} \Psi_m \quad \text{in } O; \\
\|\nabla(w_m - \Psi_m)\|_{L^2(O)} &\leq c_B\|\text{div} \Psi_m\|_{L^2(O)}.
\end{align*}
\tag{3.19}
\]

Since the slender body surface $\Gamma_\epsilon$ satisfies the geometric constraints in Section 1.1, the region $O$ satisfies an interior sphere condition with uniform radius $1/(4\kappa_{\text{max}})$. Then $O$ can be considered as the infinite union of balls of radius $1/8\kappa_{\text{max}}$. Following the construction in the proof of Lemma 2, Chapter 1.1.9 of [27], there exist a finite number of domains $O_k$, star-shaped with respect to balls of radius $1/4\kappa_{\text{max}}$, such that
\[
O = \bigcup_{k=1}^N O_k.
\]

Here $N$ depends only on $\kappa_{\text{max}}$. Then the domain dependence of the constant $c_B$ in (3.19) has an explicit formula ([16], equation III.3.27):
\[
c_B \leq c_0(\kappa_{\text{max}}\delta(O))^{n}(1 + \kappa_{\text{max}}\delta(O))
\]
where $\delta(O)$ is the radius of the region $O$ and $c_0$ depends on the radii of the domains $O_k$, each of which are bounded independent of $\epsilon$ as $\epsilon \to 0$.

Then, from (3.19), we have
\[
\|\nabla w_m\|_{L^2(\Omega_\epsilon)} \leq c_B\|\text{div} \Psi_m\|_{L^2(\Omega_\epsilon)} + \|\nabla \Psi_m\|_{L^2(\Omega_\epsilon)}
= c_B\|\text{div}(\phi \nabla \psi_m)\|_{L^2(\Omega_\epsilon)} + \|\nabla (\phi \nabla \psi_m)\|_{L^2(\Omega_\epsilon)}.
\tag{3.20}
\]

Therefore, using (3.17) and (3.20), we have
\[
\|\nabla w_m\|_{L^2(\Omega_\epsilon)} \leq (c_B + 1)(c_q\|p_m\|_{L^2(\Omega_\epsilon)} + c_\phi\|\nabla \psi_m\|_{L^2(O)}),
\]
where $c_\phi$ depends on $\nabla \phi$ but is independent of $\epsilon$. We then use the Sobolev inequality on $\mathbb{R}^3$ to obtain
\[
\|\nabla \psi_m\|_{L^2(O)} \leq |O|^{1/3}\|\nabla \psi_m\|_{L^6(O)}
\leq |O|^{1/3}\|\nabla \psi_m\|_{L^6(\Omega_\epsilon)}
\leq |O|^{1/3}c_S\|\nabla^2 \psi_m\|_{L^2(\Omega_\epsilon)}
\leq |O|^{1/3}c_S c_q\|p_m\|_{L^2(\Omega_\epsilon)}, \quad \text{using (3.17)}.
\]

Now, $|O| \leq c_\epsilon/\kappa_{\text{max}}^2$ is bounded independent of $\epsilon$, and the Sobolev constant $c_S$ is independent of $\epsilon$ (see Section 3.3). Thus
\[
\|\nabla w_m\|_{L^2(\Omega_\epsilon)} \leq c_W\|p_m\|_{L^2(\Omega_\epsilon)}
\]
for $c_W$ independent of $\epsilon$, and
\[
\|\nabla v_m\|_{L^2(\Omega_\epsilon)} \leq \|\nabla^2 \psi_m\|_{L^2(\Omega_\epsilon)} + \|\nabla w_m\|_{L^2(\Omega_\epsilon)} \leq (c_q + c_W)\|p_m\|_{L^2(\Omega_\epsilon)}.
\]
Passing to the limit we obtain the desired solution to the $\text{div} v = p$ problem of Lemma (3.9), as the constant $c_P = c_q + c_W$ is independent of $\epsilon$. 

\hfill \Box
Therefore we arrive at the final form of the estimate (1.19) in Theorem 1.3. Plugging the newly-verified \( \epsilon \)-dependence of the constants \( c_K, c_T, \) and \( c_P \) into (2.14), we have

\[
\|u\|_{D^{1,2}(\Omega_\epsilon)} + \|p\|_{L^2(\Omega_\epsilon)} \leq \frac{1}{2\sqrt{2}} c_K^2 c_T (1 + c_P) \|f\|_{L^2(T)} \leq c_\kappa |\log \epsilon|^{1/2} \|f\|_{L^2(T)}.
\]

The final form of the higher regularity estimate (1.20) also follows from tracking these same constants. Again using the form of the constants \( c_K, c_T, \) and \( c_P \) in (2.36), we have

\[
\|\nabla^2 u\|_{L^2(\Omega_\epsilon)} + \|\nabla p\|_{L^2(\Omega_\epsilon)} \leq c_\kappa \frac{|\log \epsilon|^{1/2}}{\epsilon} \|f\|_{H^{1/2}(T)}.
\]

We thus complete the proof of the estimates in Theorem 1.3.

4 Slender body residual calculations

Now that we have proved Theorem 1.3, we may proceed to the main aim of the paper: to compare the slender body approximation to the true solution and derive an error estimate in terms of the slender body radius \( \epsilon \). In this section, we calculate the residual for the slender body force and velocity approximations, which will then be used in the next section to prove the error bounds in Theorem 1.4.

4.1 Slender body calculations: setup

The proof of Theorem 1.4 requires knowledge of two expressions: the total surface force \( f^{SB}(s) \) exerted by the slender body approximation at each cross section \( s \) along the true surface \( \Gamma_\epsilon \), and the \( \theta \)-dependence in the slender body velocity \( u^{SB}(s) \). Although the true surface velocity \( u|_{\Gamma_\epsilon}(s) \) is unknown, we can measure the degree to which \( u^{SB} \) fails to satisfy the \( \theta \)-independence condition along \( \Gamma_\epsilon \). In analogy with finite element analysis, the \( \theta \)-dependence in \( u^{SB}|_{\Gamma_\epsilon}(s, \theta) \) can be considered as the \textit{non-conforming} error, as the slender body approximation \( u^{SB} \) therefore does not belong to the function space \( A^ {div} \) required by the well-posedness theory. The force residual \( f^{SB}(s) - f(s) \), on the other hand, can be considered as the conforming residual, as the slender body force approximation \( f^{SB} \) belongs to the same function space as the prescribed force \( f \).

For the computations that follow, we assume that \( f \in C^1(\mathbb{T}) \) and that the slender body satisfies the geometric constraints in Section 1.1. Although a solution to the slender body PDE (1.18) is guaranteed so long as \( f \) is in \( L^2(\mathbb{T}) \), this true solution can only be meaningfully compared to the slender body approximation if \( f \) is more regular; in particular, at least \( C^1 \). We recall that the slender body approximation is given by

\[
\begin{align*}
u^{SB}(x) &= \frac{1}{8\pi} \int_\mathbb{T} \left( S(R) + \frac{c_P^2}{2} D(R) \right) f(t) \, dt; \quad R = x - X(t), \quad \text{(4.1)} \\
S(R) &= \frac{1}{|R|} + \frac{RR^T}{|R|^3}, \quad D(R) = \frac{1}{|R|^3} - \frac{3RR^T}{|R|^5}, \quad \text{(4.2)} \end{align*}
\]

with the corresponding slender body pressure given by

\[
-p^{SB}(x) = -\frac{1}{4\pi} \int_\mathbb{T} \frac{R \cdot f(t)}{|R|^3} \, dt. \quad \text{(4.3)}
\]
Recall that within the neighborhood $O \{1.12\}$, any point $x$ can be written

$$x(\rho, \theta, s) = X(s) + \rho e_\rho(s, \theta).$$

Then, for $x \in O$, $R$ has the form

$$R(\rho, \theta, s; t) = X(s) - X(t) + \rho e_\rho(s, \theta).$$

Before we begin calculations to estimate $u^{SB}$ and $f^{SB}$, we note some useful facts. Using the moving frame ODE (1.3), we have

$$\frac{\partial R}{\partial \rho} = e_\rho(s), \quad (4.4)$$

$$\frac{1}{\rho} \frac{\partial R}{\partial \theta} = e_\theta(s), \quad (4.5)$$

$$\frac{1}{1 - \rho \hat{\kappa}} \left( \frac{\partial R}{\partial s} - \kappa_3 \frac{\partial R}{\partial \theta} \right) = e_t(s), \quad (4.6)$$

where

$$\hat{\kappa}(s, \theta) = \kappa_1(s) \cos \theta + \kappa_2(s) \sin \theta. \quad (4.7)$$

Next we note that, since $X$ is a $C^2$ function, for $s, t \in T$ we have

$$X(s) - X(t) = (s - t) e_t(s) + (s - t)^2 Q(s, t), \quad |Q(s, t)| \leq c_Q \equiv \kappa_{\max}^2. \quad (4.8)$$

Then, on the slender body surface $\Gamma_\epsilon$, we have

$$R = \bar{s} e_t(s) + \epsilon e_\rho(s) + \bar{s}^2 Q, \quad |Q| \leq c_Q, \quad \bar{s} = s - t, \quad (4.9)$$

where we have set $\rho = \epsilon$. It will often be convenient to view $R$ as a function of $\bar{s}$ and $s$ rather than $t$ and $s$. We may use this characterization of $R$ to obtain the following two simple estimates.

**Lemma 4.1.** Let $R$ be as in (4.9). Then, for sufficiently small $\epsilon$, we have:

$$\left| R - \sqrt{\bar{s}^2 + \epsilon^2} \right| \leq c_Q \bar{s}, \quad (4.10)$$

$$|R| \geq c_R \sqrt{\bar{s}^2 + \epsilon^2}, \quad (4.11)$$

where $|\bar{s}| \leq 1/2$ and $c_Q$ was defined in (4.8) and $c_R$ depends only on $c_T$ and $\kappa_{\max}$.

**Proof.** Note that

$$|\bar{s} e_t + \epsilon e_\rho| = \sqrt{\bar{s}^2 + \epsilon^2}. \quad (4.12)$$

Inequality (4.10) then follows from the triangle inequality applied to (4.9). To obtain (4.11), note from (4.10) that, if $|\bar{s}| \leq 1/(2c_Q)$,

$$|R| \geq \sqrt{\bar{s}^2 + \epsilon^2} - c_Q \bar{s}^2 \geq \frac{1}{2} \sqrt{\bar{s}^2 + \epsilon^2} + \frac{|\bar{s}|}{2} - c_Q \bar{s}^2 \geq \frac{1}{2} \sqrt{\bar{s}^2 + \epsilon^2}.$$

If $c_Q \leq 1$ we are done. Otherwise, suppose $1/(2c_Q) < |\bar{s}| \leq 1/2$. Then we have

$$|R| \geq |X(s) - X(t)| - \epsilon \geq c_T \bar{s} - \epsilon \geq \frac{c_T}{2c_Q} - \epsilon \geq \frac{c_T}{4c_Q}, \quad (4.13)$$

where we have used (1.2) in the second inequality and have taken $\epsilon \leq c_T/(4c_Q)$ in the last inequality. The above two estimates together imply (4.11). \qed
We also state an estimate on some definite integrals.

**Lemma 4.2.** Let $m, n$ be integers such that $m \geq 0$ and $n > 0$. Then, for $\epsilon$ sufficiently small, we have

$$\int_{-1/2}^{1/2} \frac{|s|^m}{(s^2 + \epsilon^2)^{n/2}} ds \leq \begin{cases} 3 |\log \epsilon| & \text{if } n = m + 1, \\ \pi \epsilon^{m+1-n} & \text{if } n \geq m + 2 \end{cases} \quad (4.14)$$

**Proof.** For $n = m + 1$, we have

$$\int_{-1/2}^{1/2} \frac{|s|^m}{s^2 + \epsilon^2} ds \leq \int_{-1/2}^{1/2} \frac{1}{s^2 + \epsilon^2} ds$$

$$= 2 |\log \epsilon| + 2 \log \left( \frac{1 + \sqrt{1 + 4\epsilon^2}}{2} \right) \leq 3 |\log \epsilon|$$

for $\epsilon$ sufficiently small. For $n \geq m + 2$, we have:

$$\int_{-1/2}^{1/2} \frac{|s|^m}{s^2 + \epsilon^2} ds \leq \int_{-1/2}^{1/2} \frac{1}{s^2 + \epsilon^2} ds$$

$$\leq \epsilon^{m+1-n} \int_{-\infty}^{\infty} \frac{1}{(s^2 + 1)(n-m)/2} ds \leq \epsilon^{m+1-n} \int_{-\infty}^{\infty} \frac{1}{s^2 + 1} ds = \pi \epsilon^{m+1-n}.$$ 

We now state a few estimates on integrals involving $R$.

**Lemma 4.3.** Let $R$ be as in (4.9). Suppose $m, n$ are integers such that $m \geq 0$ and $n > 0$. For $\epsilon$ sufficiently small, we have

$$\int_{-1/2}^{1/2} \frac{|s|^m}{|R|^n} ds \leq \begin{cases} c_n |\log \epsilon| & \text{if } n = m + 1, \\ c_n \epsilon^{m+1-n} & \text{if } n \geq m + 2, \end{cases} \quad (4.15)$$

where the constants $c_n$ depend only on $n$, $c_T$ and $\kappa_{\text{max}}$.

**Proof.** We have

$$\int_{-1/2}^{1/2} \frac{|s|^m}{|R|^n} ds \leq c_R^{-n} \int_{-1/2}^{1/2} \frac{|s|^m}{s^2 + \epsilon^2} ds \leq \begin{cases} 3c_R^{-n} |\log \epsilon| & \text{if } n = m + 1, \\ \pi c_R^{-n} \epsilon^{m+1-n} & \text{if } n \geq m + 2, \end{cases}$$

where we used (4.11) of Lemma 4.1 in the first inequality and Lemma 4.2 in the second. 

To state the next lemma, we introduce some notation. For scalar valued functions $w$ defined on $\mathbb{T}$, we set

$$\|g\|_{C^1(\mathbb{T})} = \|g\|_{C(\mathbb{T})} + \|g'\|_{C(\mathbb{T})}, \quad \|g\|_{C(\mathbb{T})} = \max_{s \in \mathbb{T}} |g(s)|, \quad (4.16)$$

where $g'$ is the derivative of $g$. For functions defined on $\mathbb{T}$ with values in $\mathbb{R}^3$, we set

$$\|g\|_{C^1(\mathbb{T})} = \|g\|_{C(\mathbb{T})} + \|g'\|_{C(\mathbb{T})}, \quad \|g\|_{C(\mathbb{T})} = \max_{s \in \mathbb{T}} |g(s)|. \quad (4.17)$$

**Lemma 4.4.** Let $R$ be as in (4.9). Suppose $m > 0$ is an odd integer and $n \geq m + 2$, and let $g \in C^1(\mathbb{T})$. Then, for sufficiently small $\epsilon$, we have

$$\int_{-1/2}^{1/2} \frac{|s|^m}{|R|^n} g'(s) ds \leq \begin{cases} c_{1,n} \|g\|_{C^1(\mathbb{T})} |\log \epsilon| & \text{if } n = m + 2, \\ c_{1,n} \|g\|_{C^1(\mathbb{T})} \epsilon^{m+2-n} & \text{if } n \geq m + 3, \end{cases} \quad (4.18)$$

where the constants $c_{1,n}$ depend only on $n$, $c_T$ and $\kappa_{\text{max}}$. 44
Proof. Let \( g = (g_1, g_2, g_3)^T \) and let \( g = g_i \) for some \( i = 1, 2, 3 \). First, we observe that

\[
\int_{-1/2}^{1/2} \frac{\bar{s}^m}{|R|^m} g(\bar{s}) d\bar{s} = I_1 + I_2,
\]

where we used Lemma 4.2 in the last inequality. Combining (4.20) and (4.22), and renaming \( I \), we turn to

\[
I = \int_{-1/2}^{1/2} \frac{1}{|R|^m} \left( 1 - \frac{1}{(\bar{s}^2 + \epsilon^2)^{n/2}} \right) g(0) d\bar{s},
\]

where we used the fact that \( m \) is odd in the last equality. We first estimate \( I_1 \). Note that

\[
|g(\bar{s}) - g(0)| \leq |\bar{s}| \left\| g' \right\|_{C(\mathbb{T})}.
\]

We have

\[
|I_1| \leq \int_{-1/2}^{1/2} \frac{\bar{s}^m+1}{|R|^m} \left\| g' \right\|_{C(\mathbb{T})} d\bar{s} \leq \left\{ \begin{array}{ll}
c_n \left\| g' \right\|_{C(\mathbb{T})} |\log \epsilon| & \text{if } n = m + 2, \\
c_n \left\| g' \right\|_{C(\mathbb{T})} \epsilon^{m+2-n} & \text{if } n \geq m + 3,
\end{array} \right.
\]

where we used Lemma 4.3. We turn to \( I_2 \). Note that

\[
\frac{1}{|R|^m} \left( 1 - \frac{1}{(\bar{s}^2 + \epsilon^2)^{n}} \right) = \left( \frac{1}{|R|} - \frac{1}{\sqrt{\bar{s}^2 + \epsilon^2}} \right) \sum_{l=0}^{n-1} \frac{1}{|R|^l (\sqrt{\bar{s}^2 + \epsilon^2})^{n-l}}.
\]

Using Lemma 4.1, we have

\[
\left| \frac{1}{|R|} - \frac{1}{\sqrt{\bar{s}^2 + \epsilon^2}} \right| = \left| \frac{|R| - \sqrt{\bar{s}^2 + \epsilon^2}}{|R| \sqrt{\bar{s}^2 + \epsilon^2}} \right| \leq \frac{c_Q \bar{s}^2}{c_R (\bar{s}^2 + \epsilon^2)}.
\]

Then, using Lemma 4.1 again, we have

\[
\left| \frac{1}{|R|^m} - \frac{1}{(\sqrt{\bar{s}^2 + \epsilon^2})^n} \right| \leq \frac{nc_Q \bar{s}^2}{c_R^{n+1} (\bar{s}^2 + \epsilon^2)^{n+1/2}}.
\]

Thus,

\[
|I_2| \leq \frac{c_Q c_R^{n+1}}{c_R} \left\| g \right\|_{C(\mathbb{T})} \int_{-1/2}^{1/2} \frac{\bar{s}^{m+2}}{(\bar{s}^2 + \epsilon^2)^{(n+1)/2}} d\bar{s}
\]

\[
\leq \left\{ \begin{array}{ll}
3c_Q c_R^{n+1} \left\| g \right\|_{C(\mathbb{T})} |\log \epsilon| & \text{if } n = m + 2, \\
\pi c_Q c_R^{n+1} \left\| g \right\|_{C(\mathbb{T})} \epsilon^{m+2-n} & \text{if } n \geq m + 3,
\end{array} \right.
\]

where we used Lemma 4.2 in the last inequality. Combining (4.20) and (4.22), and renaming constants, we obtain the scalar-valued version of inequality (4.18). The vector-valued version follows immediately. \( \Box \)

The final integral we estimate is the following.

**Lemma 4.5.** Suppose \( m \geq 0 \) is an even integer, \( n \) is an integer such that \( n \geq m + 3 \) and let \( g \in C^1(\mathbb{T}) \). Then, for sufficiently small \( \epsilon \), we have

\[
\left| \int_{-1/2}^{1/2} \frac{\bar{s}^m}{|R|^m} g(\bar{s}) d\bar{s} - \epsilon^{m+1-n} d_{mn} g(0) \right| \leq c_{0,n} \left\| g \right\|_{C^1(\mathbb{T})} \epsilon^{m+2-n},
\]

where

\[
d_{mn} = \int_{-\infty}^{\infty} \frac{\tau^m}{(\tau^2 + 1)^{n/2}} d\tau,
\]

and

\[
\lambda_{mn} = \int_{-1/2}^{1/2} \frac{\bar{s}^m}{|R|^m} g(\bar{s}) d\bar{s} - \epsilon^{m+1-n} d_{mn} g(0).
\]
where the constants \( c_{0,n} \) depend only on \( n \), \( c_T \) and \( \kappa_{\text{max}} \). For odd \( n \), we have

\[
d_{mn} = \sum_{k=0}^{m/2} (-1)^k \binom{m/2}{k} d_{0,n-k}, \quad d_{0n} = 2 \frac{(n-3)!!}{(n-2)!!}.
\]

(4.24)

For certain values of \( m \) and \( n \), this yields

\[
d_{03} = 2, \quad d_{05} = \frac{4}{3}, \quad d_{07} = \frac{16}{15}, \quad d_{25} = \frac{2}{3}, \quad d_{27} = \frac{4}{15}.
\]

(4.25)

Note that Lemma 4.5 immediately implies that, for \( g \in C^1(\Gamma_\epsilon) \), we have

\[
\left| \int_0^{2\pi} \int_{-1/2}^{1/2} \frac{s^m}{|R|^n} g(x, \theta) d\theta \right| \leq \frac{\max_{0 \leq \theta < 2\pi} \| g \|_{C^1(\Gamma)} \epsilon^{m+2-n}}{2}. \quad (4.26)
\]

Proof. As in the proof of Lemma 4.4, it suffices to prove the scalar version of the above result. First, note that

\[
I = \int_{-1/2}^{1/2} \frac{s^m}{|R|^n} d\bar{s} = g(0) \int_{-\infty}^{\infty} \frac{s^m}{(s^2 + \epsilon^2)^{n/2}} \, ds = I_1 + I_2 + I_3,
\]

\[
I_1 = \int_{-1/2}^{1/2} \frac{s^m}{|R|^n} (g(\bar{s}) - g(0)) \, d\bar{s},
\]

\[
I_2 = \int_{-1/2}^{1/2} \frac{1}{|R|^n} \left( \frac{1}{s^2 + \epsilon^2} \right) g(0) \, d\bar{s},
\]

\[
I_3 = 2g(0) \int_{1/2}^{\infty} \frac{s^m}{(s^2 + \epsilon^2)^{n/2}} \, ds.
\]

(4.27)

We may estimate \( I_1 \) and \( I_2 \) in exactly the same way as in the proof of Lemma 4.4. We find that

\[
|I_1| \leq c_n \| g' \|_{C(\Gamma)} \epsilon^{m+2-n}, \quad |I_2| \leq \pi c_Q c_R \cdot (n+1) \| g \|_{C(\Gamma)} \epsilon^{m+2-n}.
\]

For \( I_3 \), a simple estimation yields

\[
|I_3| \leq 2 \| g \|_{C(\Gamma)} \int_{1/2}^{\infty} \frac{s^m \, d\bar{s}}{(s^2 + \epsilon^2)^{n/2}} = 2 \epsilon^{m+1-n} \| g \|_{C(\Gamma)} \int_{1/(2\epsilon)}^{\infty} \frac{\tau^m \, d\tau}{(\tau^2 + 1)^{n/2}}
\]

\[
\leq 2 \epsilon^{m+1-n} \| g \|_{C(\Gamma)} \int_{1/(2\epsilon)}^{\infty} \frac{1}{\tau^{n-m}} \, d\tau = \frac{2^n}{n-m+1} \| g \|_{C(\Gamma)} \epsilon^{m+1-n} \epsilon^{-1}.
\]

Finally, we have

\[
\int_{-\infty}^{\infty} \frac{s^m}{(s^2 + \epsilon^2)^{n/2}} \, ds = \epsilon^{m+1-n} \int_{-\pi/2}^{\pi/2} \frac{\tau^m}{(\tau^2 + 1)^{n/2}} \, d\eta \equiv \epsilon^{m+1-n} d_{nm}.
\]

Combining all of the above, we obtain (4.23). Note that, since \( m \) is even,

\[
d_{mn} = \int_{-\infty}^{\infty} \frac{(\tau^2 + 1)^{m/2} \, d\tau}{(\tau^2 + 1)^{n/2}} = \sum_{k=0}^{m/2} (-1)^k \binom{m/2}{k} d_{0,n-k}.
\]
For \( n \) odd, we have
\[
d_{0n} = \int_{-\pi/2}^{\pi/2} \cos^{n-2} \varphi d\varphi = \frac{n-3}{n-2} \int_{-\pi/2}^{\pi/2} \cos^{n-4} \varphi d\varphi
= \cdots = \frac{(n-3)(n-5) \cdots 4 \cdot 2}{(n-2)(n-4) \cdot 3} \int_{-\pi/2}^{\pi/2} \cos \varphi d\varphi = 2\left(\frac{n-3}{n-2}\right)!.
\]

\[\square\]

### 4.2 Slender body velocity residual

The goal of this section is to obtain an estimate on the non-conforming error – the degree to which \( u^{SB}_{\mid \Gamma_\epsilon}(s, \theta) \) fails to satisfy the \( \theta \)-independence condition along the fiber surface \( \Gamma_\epsilon \).

With the aid of Lemmas 4.3, 4.4, and 4.5, we are now able to establish some estimates on \( u^{SB} \) and its derivatives along \( \Gamma_\epsilon \). The derivative estimates will be needed in Section 5 to obtain an actual error estimate between the slender body approximation \( u^{SB} \) and the true solution \( u \).

We show the following proposition.

**Proposition 4.6.** Consider \( u^{SB}(x) \) for \( x \in \Gamma_\epsilon \). For sufficiently small \( \epsilon \), we have
\[
\left| \frac{1}{\epsilon} \frac{\partial u^{SB}}{\partial \theta} \right| \leq c_\theta \| f \|_{C^1(T)} |\log \epsilon| \tag{4.28}
\]
where the constant \( c_\theta \) depends only on \( c_T \) and \( \kappa_{\max} \).

**Proof.** Write \( x = X(s) + \epsilon e_\rho \). Using (4.1), we have:
\[
\frac{8\pi}{\epsilon} \frac{\partial u^{SB}}{\partial \theta} = I_S + I_D;
\]
\[
I_S = \frac{1}{\epsilon} \frac{\partial}{\partial \theta} \int_{-1/2}^{1/2} S(R) f(s + \bar{s}) d\bar{s}, \tag{4.29}
\]
\[
I_D = \frac{1}{\epsilon} \frac{\partial}{\partial \theta} \int_{-1/2}^{1/2} \frac{c^2}{2} D(R) f(s + \bar{s}) d\bar{s}.
\]

We first consider \( I_S \). Using (4.2) and (4.5), we have
\[
I_S = I_{S,1} + I_{S,2};
\]
\[
I_{S,1} = - \int_{-1/2}^{1/2} \frac{R \cdot e_\theta}{|R|^3} f + 3 \frac{(R \cdot e_\theta)(R \cdot f)}{|R|^3} R \right) d\bar{s},
\]
\[
I_{S,2} = \int_{-1/2}^{1/2} \frac{(R \cdot f)e_\theta + (e_\theta \cdot f)R}{|R|^3} d\bar{s}.
\]

We estimate \( I_{S,1} \). First, note from (4.8) that
\[
|R \cdot e_\theta| = \bar{s}^2 |Q \cdot e_\theta| \leq c_Q \bar{s}^2.
\]

Thus, the integrand in \( I_{S,1} \) can be estimated as
\[
\left| \frac{R \cdot e_\theta}{|R|^3} f + 3 \frac{(R \cdot e_\theta)(R \cdot f)}{|R|^3} R \right| \leq 4 \frac{|R \cdot e_\theta|}{|R|^3} \| f \|_{C(T)} \leq \frac{4c_Q \bar{s}^2}{|R|^3} \| f \|_{C(T)}.
\]

47
Applying Lemma 4.3, we have
\[ |I_{S,1}| \leq 4c_3cQ \| f \|_{C(T)} |\log \epsilon|. \quad (4.30) \]

We now turn to \( I_{S,2} \). Note that \( (R \cdot f)e_\theta + (e_\theta \cdot f)R = \epsilon g_0 + \overline{s}_g + \overline{s}^2 g_2 \), where
\[
\begin{align*}
g_0(\overline{s}; s) &= e_\rho(s)(e_\theta(s) \cdot f(s + \overline{s})) + e_\theta(s)(e_\rho(s) \cdot f(s + \overline{s})), \\
g_1(\overline{s}; s) &= e_t(s)(e_\theta(s) \cdot f(s + \overline{s})) + e_\theta(s)(e_t(s) \cdot f(s + \overline{s})), \\
g_2 &= Q(e_\theta \cdot f) + e_\theta(Q \cdot f),
\end{align*}
\]
and we have written the explicit dependence of \( g_0 \) and \( g_1 \) on \( \overline{s} \) and \( s \). Applying Lemma 4.3 and (4.8), we have
\[
\left| \int_{-1/2}^{1/2} \frac{\overline{s}^2 g_2}{|R|^3} d\overline{s} \right| \leq \| g_2(\cdot; s) \|_{C(T)} \int_{-1/2}^{1/2} \frac{\overline{s}^2}{|R|^3} d\overline{s} \leq 2c_3cQ \| f \|_{C(T)} |\log \epsilon|. 
\]

Using Lemma 4.4, we have
\[
\left| \int_{-1/2}^{1/2} \frac{\overline{s}g_1(\overline{s}; s)}{|R|^3} d\overline{s} \right| \leq c_{1,3} \| g_1(\cdot; s) \|_{C^1(T)} |\log \epsilon| \leq 2c_{1,3} \| f \|_{C^1(T)} |\log \epsilon|. 
\]

Finally, using Lemma 4.5 with \( m = 0, n = 3 \), we have
\[
\left| \int_{-1/2}^{1/2} \frac{\epsilon g_0(\overline{s}; s)}{|R|^3} d\overline{s} - \frac{2}{\epsilon} g_0(0; s) \right| \leq c_{0,3} \| g_0(\cdot; s) \|_{C^1(T)} \leq 2c_{0,3} \| f \|_{C^1(T)};
\]
\[
\epsilon g_0(0; s) = e_\rho(s)(e_\theta(s) \cdot f(s)) + e_\theta(s)(e_\rho(s) \cdot f(s)) =: h(s). 
\]

Combining the above estimates, we obtain
\[
\left| I_{S,2} - \frac{2}{\epsilon} h(s) \right| \leq 2(c_3cQ + c_{1,3} + c_{0,3}) \| f \|_{C^1(T)} |\log \epsilon|. 
\]

Finally, combining (4.30) and (4.33), we have
\[
\left| I_S - \frac{2}{\epsilon} h(s) \right| \leq |I_{S,1}| + |I_{S,2} - \frac{2}{\epsilon} h(s)| \leq c_S \| f \|_{C^1(T)} |\log \epsilon|, 
\]
where the constant \( c_S \) depends only on \( c_T \) and \( \kappa_{\max} \).

We next consider \( I_D \) in (4.29). We write
\[
I_D = \frac{3\epsilon^2}{2} (I_{D,1} + I_{D,2}),
\]
\[
I_{D,1} = \int_{-1/2}^{1/2} \left( -\frac{R \cdot e_\theta}{|R|^3} f + \frac{5(R \cdot e_\theta)(R \cdot f)}{|R|} \right) d\overline{s},
\]
\[
I_{D,2} = -\int_{-1/2}^{1/2} \left( \frac{(R \cdot f)e_\theta + (e_\theta \cdot f)R}{|R|^3} \right) d\overline{s}. 
\]
In exactly the same way as in the above estimation of \( I_{S,1} \) and \( I_{S,2} \) in (4.30) and (4.33), we obtain
\[
|I_{D,1}| \leq 6c_5cQ \| f \|_{C(T)} \epsilon^{-2},
\]

48
Proof. First, note that where the constant $c$ small $\epsilon$

$I$ Proposition 4.7. Consider $u^{SB}(x)$ for $x \in \Gamma_{c}$. The following estimate holds for sufficiently small $\epsilon$:

$$\left| \frac{\partial}{\partial \theta} \left( \frac{\partial u^{SB}}{\partial s} - \kappa_3 \frac{\partial u^{SB}}{\partial \theta} \right) \right| \leq c_{s,\theta} \| f \|_{C^1(T)},$$

where the constant $c_{s,\theta}$ depends only on the constants $c_{\Gamma}$ and $\kappa_{\max}$.

Proof. First, note that

$$\frac{\partial}{\partial \theta} \left( \frac{\partial u^{SB}}{\partial s} - \kappa_3 \frac{\partial u^{SB}}{\partial \theta} \right) = \frac{1}{8\pi} \left( \frac{\partial I^{SB}}{\partial \theta} - \epsilon \frac{\partial \bar{I}^{SB}}{\partial \theta} \right);$$

$$I^{SB} = -\frac{8\pi}{1 - \epsilon\bar{\kappa}} \left( \frac{\partial u^{SB}}{\partial s} - \kappa_3 \frac{\partial u^{SB}}{\partial \theta} \right) = I_S + \frac{3\epsilon^2}{2} I_D,$$

$$I_S = \int_{-1/2}^{1/2} \left( -\frac{R \cdot e_t}{|R|^3} f + \frac{(R \cdot f) e_t + e_t \cdot (f \cdot R) R}{|R|^3} - 3 \frac{(R \cdot e_t)(R \cdot f)}{|R|^5} R \right) d\bar{s},$$

$$I_D = \int_{-1/2}^{1/2} \left( -\frac{R \cdot e_t}{|R|^3} f - \frac{(R \cdot f) e_t + e_t \cdot (f \cdot R) R}{|R|^5} + 5 \frac{(R \cdot e_t)(R \cdot f) R}{|R|^7} R \right) d\bar{s},$$

where we used (4.1) and (4.6) to obtain the expression for $I_S$ and $I_D$.

Let us estimate $I^{SB}$. We have

$$|I_S| \leq \int_{-1/2}^{1/2} \left| -\frac{R \cdot e_t}{|R|^3} f + \frac{(R \cdot f) e_t + e_t \cdot (f \cdot R)}{|R|^3} - 3 \frac{(R \cdot e_t)(R \cdot f)}{|R|^5} R \right| d\bar{s}$$

$$\leq \int_{-1/2}^{1/2} 6 \| f \|_{C^1(T)} \frac{d\bar{s}}{|R|^2} \leq 6c_2 \| f \|_{C^1(T)} \epsilon^{-1}.$$
where we used Lemma 4.3 in the last inequality. Likewise,
\[
|I_D| \leq \int_{-1/2}^{1/2} \left| \frac{-R \cdot e_t}{|R|^5} f - \frac{(R \cdot f)e_t + (e_t \cdot f)R}{|R|^5} + 5 \frac{(R \cdot e_t)(R \cdot f)}{|R|^5} R \right| d\bar{s}
\leq \int_{-1/2}^{1/2} \frac{8 \|f\|_{C(T)}}{|R|^4} d\bar{s} \leq 8c_4 \|f\|_{C(T)} \epsilon^{-3},
\]
where we again used Lemma 4.3 in the last inequality. Using the above estimates, we have
\[
|I_{SB}| \leq |I_S| + \frac{3c_2^2}{2} |I_D| \leq (6c_2 + 12c_4) \|f\|_{C(T)} \epsilon^{-1} =: c_I \|f\|_{C(T)} \epsilon^{-1}. \tag{4.39}
\]
We now estimate \( \partial I_{SB} / \partial \theta \). We have
\[
\frac{\partial I_S}{\partial \theta} = \epsilon (I_{S,1} + I_{S,2} + I_{S,3} + I_{S,4});
\]
\[
I_{S,1} = 3 \int_{-1/2}^{1/2} \frac{1}{|R|^5} \left( \left( \frac{R \cdot e_\theta}{|R|^5} \right)(R \cdot f)e_t - (R \cdot f)e_t - (e_t \cdot f)R \right) d\bar{s},
\]
\[
I_{S,2} = \int_{-1/2}^{1/2} \frac{(e_\theta \cdot f)e_t + (e_t \cdot f)e_\theta}{|R|^3} d\bar{s}, \tag{4.40}
\]
\[
I_{S,3} = -3 \int_{-1/2}^{1/2} \frac{1}{|R|^5} \left( (R \cdot f)e_\theta + (e_\theta \cdot f)R \right) d\bar{s},
\]
\[
I_{S,4} = 15 \int_{1/2}^{1/2} \frac{1}{|R|^3} \left( \frac{R \cdot e_t}{|R|^5} \right)(R \cdot f)(R \cdot e_\theta) R d\bar{s}.
\]
We estimate each term in turn. The integrand of \( I_{S,1} \) satisfies
\[
\left| \frac{(R \cdot e_\theta)}{|R|^5} \left( (R \cdot f)e_t - (R \cdot f)e_t - (e_t \cdot f)R \right) \right| \leq \frac{3c_Q s^2}{|R|^4} \|f\|_{C(T)},
\]
where we used (4.9). We thus have
\[
|I_{S,1}| \leq \int_{-1/2}^{1/2} \frac{9c_Q s^2}{|R|^4} \|f\|_{C(T)} d\bar{s} \leq 9c_4 c_Q \|f\|_{C(T)} \epsilon^{-1}, \tag{4.41}
\]
where we used Lemma 4.3.

To estimate \( I_{S,2} \) define \( g_1 \) as in (4.31). Using Lemma 4.5 with \( m = 0, n = 3 \), we have
\[
\left| I_{S,2} - \frac{2}{\epsilon^2} h(s) \right| \leq c_{0,3} \|g_1(\cdot; s)\|_{C^1(T)} \epsilon^{-1} \leq 2c_{0,3} \|f\|_{C^1(T)} \epsilon^{-1}; \tag{4.42}
\]
\[
h(s) = g_1(0; s) = (e_\theta(s) \cdot f(s))e_t(s) + (e_t(s) \cdot f(s))e_\theta(s).
\]
To estimate \( I_{S,3} \), let
\[
I_{S,3} = I_{S,31} + I_{S,32};
\]
\[
I_{S,31} = -3 \int_{-1/2}^{1/2} \frac{s^2(Q \cdot e_t)}{|R|^5} \left( (R \cdot f)e_\theta + (e_\theta \cdot f)R \right) d\bar{s},
\]
\[
I_{S,32} = -3 \int_{-1/2}^{1/2} \frac{s}{|R|^5} \left( (R \cdot f)e_\theta + (e_\theta \cdot f)R \right) d\bar{s}.
\]
Let us estimate $I_{S,31}$. The integrand may be estimated as

$$\left| \frac{s^2(Q \cdot e_t)}{|R|^3} ((R \cdot f) e_\theta + (e_\theta \cdot f) R) \right| \leq \frac{2cQs^2}{|R|^4} \|f\|_{C(\Gamma)}.$$ 

Thus,

$$|I_{S,31}| \leq \int_{-1/2}^{1/2} \frac{6cQs^2}{|R|^4} \|f\|_{C(\Gamma)} d\bar{s} \leq 6cQc_4 \|f\|_{C(\Gamma)} \epsilon^{-1},$$

where we used Lemma 4.3.

To estimate $I_{S,32}$, define $g_0, g_1, g_2$ as in (4.31). We first have

$$\left| \int_{-1/2}^{1/2} \frac{s^3 g_1}{|R|^5} d\bar{s} \right| \leq \int_{-1/2}^{1/2} \frac{2cQs^3}{|R|^5} d\bar{s} \leq 2cQc_5 \|f\|_{C(\Gamma)} \epsilon^{-1},$$

where we used (4.9) and Lemma 4.3. Next, we have

$$\left| \int_{-1/2}^{1/2} \frac{c\epsilon g_2}{|R|^6} d\bar{s} \right| \leq c_1 \|g_2(\cdot ; s)\|_{C^1(\Gamma)} \epsilon^{-1} \leq 2c_0.5 \|f\|_{C^1(\Gamma)} \epsilon^{-1},$$

where we used Lemma 4.5 with $m = 2, n = 5$ and $h(s)$ as defined in (4.32). For $g_2$, we have

$$\left| \int_{-1/2}^{1/2} \frac{c\epsilon g_2}{|R|^6} d\bar{s} \right| \leq c_1 \|g_2(\cdot ; s)\|_{C^1(\Gamma)} \epsilon^{-1} \leq 2c_1.5 \|f\|_{C^1(\Gamma)} \epsilon^{-1},$$

where we used Lemma 4.4. Combining the above estimates, we have

$$|I_{S,3} + \frac{2}{\epsilon^2} h(s)| \leq 6(c_Q(c_4 + c_5) + c_0.5 + c_1.5) \|f\|_{C^1(\Gamma)} \epsilon^{-1}. \quad (4.43)$$

Finally, we estimate $I_{S,4}$. The integrand satisfies

$$\left| \frac{\hat{R} \cdot e_t}{|R|^3} (R \cdot f) (R \cdot e_\theta) \right| \leq \frac{cQs^2}{|R|^4} \|f\|_{C(\Gamma)}.$$ 

Thus,

$$|I_{S,4}| \leq \int_{-1/2}^{1/2} \frac{15cQs^2}{|R|^4} \|f\|_{C(\Gamma)} d\bar{s} \leq 15cQc_4 \|f\|_{C(\Gamma)} \epsilon^{-1}. \quad (4.44)$$

Using the estimates (4.41), (4.42), (4.43) and (4.44) in (4.40), we obtain

$$\left| \frac{\partial I_S}{\partial \theta} \right| \leq c_S \|f\|_{C^1(\Gamma)}, \quad (4.45)$$

where the constant $c_S$ depends only on $\kappa_{\text{max}}$ and $c_T$. 

51
We may estimate $\partial I_D/\partial \theta$ in exactly the same way. We have

$$\frac{\partial I_D}{\partial \theta} = \epsilon (I_{D,1} + I_{D,2} + I_{D,3} + I_{D,4});$$

$$I_{D,1} = 5 \int_{-1/2}^{1/2} \left( \frac{(R \cdot e_\theta)(R \cdot e_t) + (R \cdot f)e_t + (e_t \cdot f)R}{|R|^7} \right) d\bar{s},$$

$$I_{D,2} = - \int_{-1/2}^{1/2} \frac{(e_\theta \cdot f)e_t + (e_t \cdot f)e_\theta}{|R|^5} d\bar{s},$$

$$I_{D,3} = 5 \int_{-1/2}^{1/2} \frac{R \cdot e_t}{|R|^7} ((R \cdot f)e_\theta + (e_\theta \cdot f)R) d\bar{s},$$

$$I_{D,4} = -35 \int_{1/2}^{1/2} \frac{(R \cdot e_t)(R \cdot f)(R \cdot e_\theta)}{|R|^9} R d\bar{s}. \quad (4.46)$$

The estimation of $I_{D,1}$ follows the same pattern as that for $I_{S,1}$ obtained in (4.41):

$$|I_{D,1}| \leq 15c_Qc_6 \| f \|_{C^1(T)} ^3 \epsilon^{-3}.$$ 

The estimation of $I_{D,2}$ is similar to (4.42):

$$\left| I_{D,2} + \frac{4}{3c^4} h(s) \right| \leq 2c_{0.5} \| f \|_{C^1(T)} \epsilon^{-3},$$

where we used Lemma 4.5 with $m = 0, n = 5$. We estimate $I_{D,3}$ following the steps of estimate (4.43). We obtain

$$\left| I_{D,3} - \frac{4}{3c^4} h(s) \right| \leq 10(c_Q(c_6 + c_7) + c_{0.7} + c_{1.7}) \| f \|_{C^1(T)} \epsilon^{-3},$$

where we used Lemma 4.5 with $m = 2, n = 7$. Finally, the estimation of $I_{D,4}$ is similar to (4.44). We have

$$|I_{D,4}| \leq 35c_Qc_6 \| f \|_{C^1(T)} \epsilon^{-3}.$$ 

Combining the above estimates, we obtain

$$\left| \frac{\partial I_D}{\partial \theta} \right| \leq c_D \| f \|_{C^1(T)} \epsilon^{-2}, \quad (4.47)$$

where the constant $c_D$ depends only on $c_1$ and $\kappa_{\max}$. 

Combining (4.45) and (4.47) and recalling the definition of $I_{SB}^*$ in (4.38), we have

$$\left| \frac{\partial I_{SB}^*}{\partial \theta} \right| \leq \left| \frac{\partial I_S}{\partial \theta} \right| + 3c^2 \left| \frac{\partial I_D}{\partial \theta} \right| \leq \left( c_S + \frac{3}{2} c_D \right) \| f \|_{C^1(T)} =: c_{I,\theta} \| f \|_{C^1(T)}. \quad (4.48)$$

We may finally use (4.39) and (4.48) together in (4.38) to obtain

$$\left| \frac{\partial}{\partial \theta} \left( \frac{\partial u_{SB}^*}{\partial s} - \kappa_3 \frac{\partial u_{SB}}{\partial \theta} \right) \right| \leq \left| \frac{1}{8\pi} \left( 1 + \epsilon |\bar{\zeta}| \right) \frac{\partial I_{SB}^*}{\partial \theta} \right| + \epsilon \left| \frac{\partial I_{SB}^*}{\partial \theta} \right| \| f \|_{C^1(T)} \leq \left| \frac{1}{8\pi} \left( \frac{5}{4} c_{I,\theta} + 2c_I\kappa_{\max} \right) \| f \|_{C^1(T)} \right|, \quad (4.49)$$

52
where, in the last inequality, we used that
\[ \epsilon |\hat{\kappa}| \leq 2\epsilon \kappa_{\text{max}} \leq \frac{1}{4}, \]
by (4.7) and (1.8), and
\[ \left| \frac{\partial \hat{\kappa}}{\partial \theta} \right| = | -\kappa \sin \theta + \kappa_2 \cos \theta | \leq 2 \sqrt{\kappa_1^2 + \kappa_2^2} = 2\kappa \leq 2\kappa_{\text{max}}, \]
by (4.7) and (1.5).

With Propositions 4.6 and 4.7, we are finally equipped to estimate the degree to which \( u^{SB} \) fails to satisfy the \( \theta \)-independence condition along \( \Gamma_\epsilon \). We define the residual \( u^r(s, \theta) \) as
\[ u^r(\theta, s) = u^{SB}(\epsilon, \theta, s) - \frac{1}{2\pi} \int_0^{2\pi} u^{SB}(\epsilon, \varphi, s) d\varphi. \] (4.50)

Note that the function \( u^r \) measures the deviation of \( u^{SB} \) from a \( \theta \)-independent function. We show the following estimates for \( u^r \).

**Proposition 4.8.** Consider the residual \( u^r \) defined in (4.50). For sufficiently small \( \epsilon \), we have
\[ |u^r| \leq c_r \| f \|_{C^1(T)} \epsilon |\log \epsilon|, \] (4.51)
\[ \left| \frac{1}{\epsilon} \frac{\partial u^r}{\partial \theta} \right| \leq c_{r,\theta} \| f \|_{C^1(T)} |\log \epsilon|, \] (4.52)
\[ \left| \frac{\partial u^r}{\partial s} \right| \leq c_{r,s} \| f \|_{C^1(T)}, \] (4.53)
where the constants \( c_r, c_{r,\theta} \) and \( c_{r,s} \) depend only on \( c_T \) and \( \kappa_{\text{max}} \).

**Proof.** Let \( u^r = (u^r_1, u^r_2, u^r_3) \) and likewise for \( u^{SB} \). We work component-wise. For each fixed \( s \), we can find \( \theta_0 \) satisfying
\[ u^{SB}_k(\epsilon, \theta_0, s) = \frac{1}{2\pi} \int_0^{2\pi} u^{SB}_k(\epsilon, \varphi, s) d\varphi. \]

Thus we can write
\[ u^r_k(\theta, s) = u^{SB}_k(\epsilon, \theta, s) - u^{SB}_k(\epsilon, \theta_0, s) = \int_{\theta_0}^{\theta} \frac{\partial u^{SB}_k}{\partial \theta}(\epsilon, \varphi, s) d\varphi. \]

Using Proposition 4.6, we have
\[ |u^r_k(\theta, s)| \leq \int_{\theta_0}^{\theta} \left| \frac{\partial u^{SB}_k}{\partial \varphi}(\epsilon, \varphi, s) \right| d\varphi \leq c_\theta |\theta - \theta_0| \| f \|_{C^1(T)} \epsilon |\log \epsilon| \]
\[ \leq c_\theta \pi \| f \|_{C^1(T)} \epsilon |\log \epsilon|, \]
where, in the last equality, we used the fact that \( \theta \) and \( \theta_0 \) are at most \( \pi \) apart. This establishes (4.51).

The estimate (4.52) is a direct consequence of Proposition 4.6.
We finally establish (4.53). For each fixed $s$, we find a $\theta_1$ satisfying
\[
\frac{\partial u_{k}^{\text{SB}}}{\partial s}(\epsilon, \theta_1, s) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial u_{k}^{\text{SB}}}{\partial s}(\epsilon, \varphi, s) \, d\varphi.
\]
Then we can write
\[
\frac{\partial u_{k}^{I}}{\partial s}(\theta, s) = \frac{\partial u_{k}^{\text{SB}}}{\partial s}(\epsilon, \theta, s) - \frac{\partial u_{k}^{\text{SB}}}{\partial s}(\epsilon, \theta_1, s) = \int_{\theta_1}^\theta \frac{\partial}{\partial \theta} \left( \frac{\partial u_{k}^{\text{SB}}}{\partial s} \right)(\epsilon, \varphi, s) \, d\varphi
\]
\[
= \int_{\theta_1}^\theta \frac{\partial}{\partial \theta} \left( \frac{\partial u_{k}^{\text{SB}}}{\partial s} - \kappa_3 \frac{\partial u_{k}^{\text{SB}}}{\partial \theta} \right)(\epsilon, \varphi, s) \, d\varphi
\]
\[
+ \kappa_3 \left( \frac{\partial u_{k}^{\text{SB}}}{\partial \theta}(\epsilon, \theta) - \frac{\partial u_{k}^{\text{SB}}}{\partial \theta}(\epsilon, \theta_1, s) \right)
\]
Thus, using Proposition 4.7 and Proposition 4.6 we have
\[
\left| \frac{\partial u_{k}^{I}}{\partial s}(\theta, s) \right| \leq c_{s, \theta} |\theta - \theta_1| \| f \|_{C^1(\Gamma)} + 2 |\kappa_3| c_\rho \| f \|_{C^1(\Gamma)} \epsilon \| \log \epsilon \|.
\]  
(4.55)

Noting that $|\theta - \theta_1| \leq \pi$ and $|\kappa_3| \leq \pi$ by Lemma 1.1 we obtain the desired estimate. \[\square\]

4.3 Slender body force residual

It remains to calculate the slender body approximation to the total force at each cross section $s \in \mathbb{T}$, given by
\[
f_{\text{SB}}^i(s) = \int_0^{2\pi} \left( -p_{\text{SB}} I + 2\mathcal{E}(\mathbf{u}^{\text{SB}}) \right) \mathbf{n} \mathcal{J}_\epsilon(s, \theta) \, d\theta.
\]  
(4.56)

From (4.56), calculating the slender body force requires two main components: the force due to the slender body pressure (4.3) and the force due to the surface strain $\mathcal{E}(\mathbf{u}^{\text{SB}})\mathbf{n}|_{\Gamma_\epsilon}$. With respect to the moving frame basis $\mathbf{e}_l(s), \mathbf{e}_\rho(s, \theta), \mathbf{e}_\theta(s, \theta)$, we can express the surface strain as
\[
2\mathcal{E}(\mathbf{u})\mathbf{n} = \frac{\partial \mathbf{u}}{\partial \rho} + \left( \frac{\partial \mathbf{u}}{\partial \theta} \cdot \mathbf{e}_\rho \right) \mathbf{e}_\rho + \frac{1}{\epsilon} \left( \frac{\partial \mathbf{u}}{\partial \theta} \cdot \mathbf{e}_\theta \right) \mathbf{e}_\theta + \frac{1}{1 - \epsilon \kappa} \left( \frac{\partial \mathbf{u}}{\partial s} - \kappa_3 \frac{\partial \mathbf{u}}{\partial \theta} \right) \cdot \mathbf{e}_l.
\]  
(4.57)

Remark 4.9. Before we estimate $f_{\text{SB}}^i$, we return to Remark 1.2 where we consider the (purely heuristic) slender body approximation about an infinitely long fiber with a straight centerline and constant total force $f^c$ over each cross section. Recall that, in this case, although the slender body velocity approximation diverges logarithmically at infinity, the velocity does exactly satisfy the $\theta$-independence condition on the the slender body surface due to the doublet correction with coefficient $\epsilon^2$. We can again consider this heuristic in relation to the slender body force approximation, noting that this is essentially the scenario for which slender body theory is designed to work.

Indeed, in the straight centerline/constant force scenario, the slender body force expression (4.56) also exactly recovers the prescribed force $f^c$. When $\kappa \equiv 0$, we have $\mathbf{R} = (s - t)\mathbf{e}_l + \epsilon \mathbf{e}_\rho(\theta)$, where the basis vectors no longer depend on the cross section $s$. We can then directly integrate the slender body approximation (1.15) in $t$ to obtain:
\[
\frac{\partial u_{k}^{\text{str}}}{\partial \rho} = \frac{1}{\epsilon 2\pi} \left[ f^c - \mathbf{e}_\rho(\mathbf{e}_\rho \cdot f^c) \right], \quad \left( \frac{\partial u_{k}^{\text{str}}}{\partial \rho} \cdot \mathbf{e}_\rho \right) \mathbf{e}_\rho = 0, \quad \frac{1}{\epsilon} \frac{\partial u_{k}^{\text{str}}}{\partial \theta} = 0, \quad \frac{\partial u_{k}^{\text{str}}}{\partial s} = 0.
\]  
(4.58)
Additionally, the slender body pressure contribution to the total force is given by
\[
-p_{\text{str}}^\text{SB}(s, \theta) = \frac{1}{\epsilon 2\pi} e_\rho \cdot f^c.
\]
(4.59)

Thus the slender body approximation to the constant force \(f^c\) prescribed along an infinite straight cylinder is given by
\[
\begin{align*}
\mathbf{f}_{\text{str}}^\text{SB} &= \int_0^{2\pi} \left[ -p_{\text{str}}^\text{SB} \mathbf{n} + 2 \mathbf{\mathcal{E}}(\mathbf{u}_{\text{str}}^\text{SB}) \mathbf{n} \right] \epsilon \, d\theta \\
&= \int_0^{2\pi} \left[ \frac{1}{2\pi} (e_\rho \cdot f^c) e_\rho + \frac{1}{2\pi} (f^c - e_\rho (e_\rho \cdot f^c)) \right] \, d\theta \\
&= \int_0^{2\pi} \frac{1}{2\pi} f^c \, d\theta = f^c,
\end{align*}
\]
(4.60)
so we exactly recover the force \(f^c\) at each cross section along the fiber.

Again, the straight centerline/constant force calculations are purely heuristic, but serve to show that the error in the slender body approximation to the total force, as well as the \(\theta\)-dependence in the slender body surface velocity, will arise due to the curvature of the fiber centerline, the finite fiber length, and variations in the prescribed force along the centerline.

Given a curved centerline and non-constant prescribed force \(f(s)\), we compute the slender body approximation to the force, \(\mathbf{f}_{\text{str}}^\text{SB}(s)\) using essentially the same perturbative argument as in the velocity estimation, where we relied on the straight centerline integrand to derive integral bounds for the curved centerline.

Although Lemmas 4.3, 4.4, and 4.5 are actually enough to obtain an \(O(\epsilon |\log \epsilon|)\) bound on the residual \(\mathbf{f}_{\text{str}}^\text{SB} - \mathbf{f}\), it turns out that we can use the \(\theta\) integration in the slender body force expression (4.56) to obtain a slightly stronger \(O(\epsilon)\) bound. However, in order to rely on these \(\theta\) integral cancellations, we must require slightly stronger differentiability on the filament centerline \(X\). Until now, we have required only \(C^2\) differentiability on \(X\), but to obtain the stronger estimate, we will require that the fiber centerline is at least \(C^{2, \alpha}\).

First we recall the following definition.

**Definition 4.3** We say that a function \(g : \mathbb{T} \to \mathbb{R}^3\) belongs to the Hölder space \(C^{k, \alpha}, 0 \leq k \in \mathbb{Z}, 0 < \alpha \leq 1\), if \(g \in C^k\) and
\[
\|g\|_{C^k(\mathbb{T})} + \sup_{s \neq t \in \mathbb{T}} \frac{|g^{(k)}(s) - g^{(k)}(t)|}{|s - t|^\alpha} < \infty,
\]
(4.61)
where \(g^k\) denote \(\frac{d^k g}{ds^k}\).

For a fiber centerline \(X \in C^{2, \alpha}\), we define the constant
\[
c_{\alpha} := \sup_{s \neq t \in \mathbb{T}} \frac{|X''(s) - X''(t)|}{|s - t|^\alpha}.
\]
(4.62)
Recall the notation
\[
\mathbf{R}_0(s, \bar{s}) = X(s) - X(s + \bar{s}).
\]
(4.63)
Given a $C^{2,\alpha}$ centerline $X$, we can write $R_0$ as

$$X(s) - X(s + \bar{s}) = -\bar{s}e_t + \bar{s}^{2+\alpha} q_1 e_t + \bar{s}^2 Q,$$

where $e_t \cdot Q = 0$ and $|q_1| \leq c_\alpha$. Then $R$ can be expanded as

$$R = -\bar{s}e_t + e_\rho + \bar{s}^{2+\alpha} q_1 e_t + \bar{s}^2 Q.$$

To justify the expansion (4.63), we first note that, using (1.3),

$$R_0(s, \bar{s}) = X(s) - X(s + \bar{s}) = -\int_0^1 X'(s + \bar{s}z) \bar{s} \, dz = -\int_0^1 e_t(s + \bar{s}z) \bar{s} \, dz.$$

Now, using the $C^{2,\alpha}$ regularity of $X(s)$, we can show that the remainder term $Q$ in (4.8) possesses additional structure. We have

$$Q(s, \bar{s}) = R_0(s, \bar{s}) + \bar{s}e_t(s) = -\int_0^1 (e_t(s + \bar{s}z) - e_t(s)) \bar{s} \, dz = -\int_0^1 \int_0^1 e_t'(\bar{s}z\bar{s} + s) \bar{s}^2 z \, d\bar{s} \, dz.$$

By (1.3), using that $e_t(s) \cdot e_t'(s) = 0$, we have

$$|e_t(s) \cdot Q(s, \bar{s})| = \left| \int_0^1 \int_0^1 e_t(s) \cdot (e_t'(\bar{s}z\bar{s} + s) - e_t'(s)) \bar{s}^2 z \, d\bar{s} \, dz \right| \leq c_\alpha \int_0^1 \int_0^1 |\bar{s}z\bar{s}|^\alpha \, |z|^2 \, d\bar{s} \, dz \leq c_\alpha |\bar{s}|^{2+\alpha},$$

where we have used (4.62) in the second inequality.

Using the expansion (4.65), we obtain a more refined upper bound for $R$.

**Lemma 4.10.** Let $R$ be as in (4.65). Then, for sufficiently small $\epsilon$, we have

$$|R| - \sqrt{\bar{s}^2 + \epsilon^2} \leq \bar{c}_Q(|\bar{s}|^{2+\alpha} + \epsilon|\bar{s}|),$$

where $|\bar{s}| \leq \frac{1}{2}$ and the constant $\bar{c}_Q$ depends only on $c_\Gamma$ and $\kappa_{\text{max}}$.

Note that the lower bound (4.11) for $R$ remains unchanged.

**Proof.** Using (4.65), we have

$$|R|^2 = (\bar{s} + \bar{s}^{2+\alpha} q_1)^2 + \epsilon^2 + \bar{s}^4 |Q_\perp|^2 + 2\epsilon \bar{s}^2 (e_\rho \cdot Q_\perp).$$

Then

$$|R| - \sqrt{\bar{s}^2 + \epsilon^2} = \frac{|R|^2 - (\bar{s}^2 + \epsilon^2)}{|R| + \sqrt{\bar{s}^2 + \epsilon^2}} \leq \frac{2|q_1||\bar{s}|^{3+\alpha} + q_1^2|\bar{s}|^{4+2\alpha} + \bar{s}^4 |Q_\perp|^2 + 2\epsilon \bar{s}^2 (e_\rho \cdot Q_\perp)}{(1 + c_R)\sqrt{\bar{s}^2 + \epsilon^2}} \leq 2\bar{c}_Q \frac{1 + c_Q(2+\alpha) + \epsilon|\bar{s}|}{(1 + c_R)\sqrt{\bar{s}^2 + \epsilon^2}} \leq 2\bar{c}_Q \frac{1 + c_Q}{1 + c_R} (|\bar{s}|^{2+\alpha} + \epsilon|\bar{s}|),$$

where we have used (4.11) in the second inequality, $|\bar{s}| \leq \frac{1}{2}$ in the third inequality, and $|\bar{s}|/\sqrt{\bar{s}^2 + \epsilon^2} \leq 1$ in the last inequality.

\hfill $\Box$
We use the smoother $C^{2,\alpha}$ centerline to show the following integral bound.

**Lemma 4.11.** Let $R$ be as in $[4.65]$. Suppose $m$ is a non-negative integer and $n = m + 1$ or $m + 2$. Furthermore, assume $g \in C(\mathbb{T})$. For $k \in \mathbb{Z}, k \neq 0$, $\theta_0 \in \mathbb{R}$ and $\epsilon > 0$ sufficiently small, we have

\[
\left| \int_{0}^{2\pi} \int_{-1/2}^{1/2} \frac{\overline{\sigma}^m g(\overline{\sigma})}{|R|^n} \cos(k(\theta + \theta_0)) d\overline{\sigma} d\theta \right| \leq \overline{c}_n \|g\|_{C(\mathbb{T})} \epsilon^{(m+2-n)/2},
\]

where the constant $\overline{c}_n$ depends only on $c_{\Gamma}, \kappa_{\text{max}}, m$, and $c_{\alpha}$.

**Remark 4.12.** Note that by plugging in the correct values of $k$ and $\theta_0$, Lemma 4.11 also covers integrands of the form $\overline{\sigma}^m e^\theta / |R|^n$ integrated against $\sin \theta$ or against odd triples $\sin^k \theta \cos^j \theta$, $k + j = 3$, $k, j \geq 0$, via the trigonometric identities

\[
\cos^3 \theta = \frac{1}{4} (3 \cos \theta + \cos(3\theta)), \quad \sin \theta \cos^2 \theta = \frac{1}{4} (\sin \theta + \sin(3\theta)),
\]

\[
\sin^3 \theta = \frac{1}{4} (3 \sin \theta - \sin(3\theta)), \quad \sin^2 \theta \cos \theta = \frac{1}{4} (\cos \theta - \cos(3\theta)).
\]

Note in particular that Lemma 4.11 applies to integrands of the form $\overline{\sigma}^m e^\theta (A(\overline{\sigma}) \cdot e^\rho)(B(\overline{\sigma}) \cdot e^\rho)$ and $\overline{\sigma}^m e^\theta (A \cdot e^\rho)(B \cdot e^\rho)$, where $A = (a_1, a_2, a_3)^T$ and $B = (b_1, b_2, b_3)^T$ are vector-valued functions that do not depend on $\theta$. We can expand these quantities as

\[
e^\rho (A \cdot e^\rho)(B \cdot e^\rho) = (a_2-b_2) \sin^3 \theta + (a_2-b_3) \sin \theta \cos \theta - b_2a_3 \cos^2 \theta \sin \theta) e_{n_1}(s)
\]

\[
+ (a_2-b_3) \sin \theta \cos^2 \theta \sin \theta) e_{n_2}(s),
\]

and, using the above trigonometric identities, apply Lemma 4.11 to each term.

**Proof of Lemma 4.11.** We split the integral into two parts:

\[
I + J = \int_{0}^{2\pi} \int_{-1/2}^{1/2} \frac{\overline{\sigma}^m g(\overline{\sigma})}{|R|^n} \cos(k(\theta + \theta_0)) d\overline{\sigma} d\theta,
\]

\[
I = \int_{0}^{2\pi} \int_{|\overline{\sigma}| \leq \sqrt{\epsilon}} \frac{\overline{\sigma}^m g(\overline{\sigma})}{|R|^n} \cos(k(\theta + \theta_0)) d\overline{\sigma} d\theta,
\]

\[
J = \int_{0}^{2\pi} \int_{\sqrt{\epsilon} \leq |\overline{\sigma}| \leq 1/2} \frac{\overline{\sigma}^m g(\overline{\sigma})}{|R|^n} \cos(k(\theta + \theta_0)) d\overline{\sigma} d\theta.
\]

Let us first estimate $I$. Note that

\[
I = \int_{0}^{2\pi} \int_{|\overline{\sigma}| \leq \sqrt{\epsilon}} \overline{\sigma}^m g(\overline{\sigma}) \left( \frac{1}{|R|^n} - \frac{1}{(\overline{\sigma}^2 + \epsilon^2)^{n/2}} \right) \cos(k(\theta + \theta_0)) d\overline{\sigma} d\theta.
\]

When $n = m + 1$, using (4.21) we have

\[
|I| \leq \int_{0}^{2\pi} \int_{|\overline{\sigma}| \leq \sqrt{\epsilon}} \frac{ncQ |\overline{\sigma}|^{m+2} |g(\overline{\sigma})|}{c_R^{m+2} (\overline{\sigma}^2 + \epsilon^2)^{(m+2)/2}} d\overline{\sigma} d\theta
\]

\[
\leq 4\pi \frac{ncQ}{c_R^{m+2}} \|g\|_{C(\mathbb{T})} \sqrt{\epsilon}.
\]

57
When \( n = m + 2 \), we need to use \( C^{2,\alpha} \) regularity of \( \mathbf{X} \). Using \([4.11]\) and Lemma \([4.10]\) we have

\[
|I| \leq \int_0^{2\pi} \int_{|\bar{s}| \leq \sqrt{\varepsilon}} |\bar{s}|^m \left| \mathbf{R} - \sqrt{\bar{s}^2 + \varepsilon^2} \right| |g(\bar{s})| \frac{c_R^{n+3} (\bar{s}^2 + \varepsilon^2)^{(m+3)/2}}{\bar{s}} \, d\bar{s} \, d\theta
\]

\[
\leq 4\pi \frac{\tilde{c}_Q}{c_R^{m+3}} \|g\|_{C(T)} \int_0^{\sqrt{\varepsilon}} \left| \bar{s} \right|^m \left| \left| \frac{\bar{s}^{2+\alpha} + \varepsilon}{\bar{s}^2 + \varepsilon^2} \right| \right| d\bar{s}
\]

\[
\leq 4\pi \frac{\tilde{c}_Q}{c_R^{m+3}} \|g\|_{C(T)} \left( \frac{1}{\alpha} + \frac{\varepsilon}{\bar{s}^2 + \varepsilon^2} \right) \frac{\pi}{\alpha c_R^{m+3}} \|g\|_{C(T)}.
\]

(4.70)

Note that \( \alpha > 0 \) is necessary for this inequality to hold, so a \( C^2 \) fiber centerline will not suffice.

To estimate \( J \), recalling the definition of \( \mathbf{R}_0 \) \([4.63]\), we note that for \( |\bar{s}| \geq \sqrt{\varepsilon} \), we have

\[
|\mathbf{R}_0| \geq c_T |\bar{s}| \quad |\mathbf{R}| \geq |\mathbf{R}_0| - \varepsilon \geq c_T |\bar{s}| - \varepsilon \geq \frac{c_T}{2} |\bar{s}|.
\]

(4.71)

where we used \([4.2]\). Now, note that

\[
J = \int_0^{2\pi} \int_{\sqrt{\varepsilon} \leq |\bar{s}| \leq 1/2} |\bar{s}|^m g(\bar{s}) \left( \frac{1}{|\mathbf{R}|^n} - \frac{1}{|\mathbf{R}_0|^n} \right) \cos(k(\theta + \theta_0)) \, d\bar{s} \, d\theta.
\]

We have

\[
\left| \frac{1}{|\mathbf{R}|^n} - \frac{1}{|\mathbf{R}_0|^n} \right| = \frac{|\mathbf{R} - |\mathbf{R}_0| |}{|\mathbf{R}| |\mathbf{R}_0| |} \sum_{\ell=0}^{n-1} \left| \frac{1}{|\mathbf{R}|^{n-\ell} |\mathbf{R}_0|^\ell} \right| \leq \frac{2^n \varepsilon}{c_T^{n+1} |\bar{s}|^{n+1}},
\]

where we used \([4.71]\). Thus,

\[
|J| \leq \int_0^{2\pi} \int_{\sqrt{\varepsilon} \leq |\bar{s}| \leq 1/2} \frac{2^n \varepsilon}{c_T^{n+1} |\bar{s}|^{n+1-m}} |g(\bar{s})| \, d\bar{s} \, d\theta \leq \frac{2^{n+2} \pi}{c_T^{n+1} |\bar{s}|^{n+1-m}} \|g\|_{C(T)} \varepsilon^{(m+2-n)/2}.
\]

The above, together with \([4.69]\) and \([4.70]\), gives the desired estimate. \(\square\)

We now proceed to estimate the slender body force \([4.56]\) for a fiber satisfying the geometric constraints of Section \([1.1]\) along with a \( C^{2,\alpha} \) centerline and true force \( \mathbf{f}(s) \) in \( C^1(\mathbb{T}) \). Since the stress tensor \( \sigma_{SB} = -p_{SB} \mathbf{I} + 2\mathcal{E}(\mathbf{u}_{SB}) \) with \( \mathcal{E}(\mathbf{u}_{SB}) \) given by \([4.57]\) essentially consists of five distinct terms, each of which in turn consists of derivatives of the slender body expression \([1.15]\), it will be convenient to estimate each of the components of \( \mathbf{f}_{SB} \) separately. We label the five components of the \( \mathbf{f}_{SB} \) expression as follows.

\[
\mathbf{f}_{SB} = \mathbf{f}_{p} + \mathbf{f}_{1} + \mathbf{f}_{2} + \mathbf{f}_{3} + \mathbf{f}_{4}.
\]

\[
\mathbf{f}_{p} := \int_0^{2\pi} -p_{SB} \mathcal{J}_t d\theta
\]

\[
\mathbf{f}_{1} := \int_0^{2\pi} \frac{\partial \mathbf{u}_{SB}}{\partial \rho} \mathcal{J}_t d\theta
\]

\[
\mathbf{f}_{2} := \int_0^{2\pi} \left( \frac{\partial \mathbf{u}_{SB}}{\partial \rho} \cdot \mathbf{e}_{\rho} \right) \mathcal{J}_t d\theta
\]

\[
\mathbf{f}_{3} := \int_0^{2\pi} \frac{1}{\varepsilon} \left( \frac{\partial \mathbf{u}_{SB}}{\partial \theta} \cdot \mathbf{e}_{\theta} \right) \mathcal{J}_t d\theta
\]

\[
\mathbf{f}_{4} := \int_0^{2\pi} \frac{1}{1 - \varepsilon \kappa} \left( \left( \frac{\partial \mathbf{u}_{SB}}{\partial \bar{s}} - \kappa_3 \frac{\partial \mathbf{u}_{SB}}{\partial \theta} \right) \cdot \mathbf{e}_{\rho} \right) \mathcal{J}_t d\theta
\]

(4.72)
We begin by estimating \( f_p^{SB} \), the contribution of the slender body pressure \( p^{SB} \) to the total force. We show the following proposition:

**Proposition 4.13.** Let the slender body \( \Sigma_c \) be as in Section 1.4 with \( C^{2,\alpha} \) centerline \( X(s) \). Given \( f \in C^1(\mathbb{T}) \), let \( f_p^{SB}(s) \) be the pressure component of the slender body force, defined in \((4.72)\). Then \( f_p^{SB} \) satisfies

\[
|f_p^{SB}(s) - \frac{1}{2} \left( (f(s) \cdot e_{n_1}(s))e_{n_1}(s) + (f(s) \cdot e_{n_2}(s))e_{n_2}(s) \right) | \leq \epsilon c_{p_0} \| f \|_{C^1(\mathbb{T})},
\]

where the constant \( c_{p_0} \) depends only on \( c_{\Gamma}, \kappa_{\text{max}}, \) and \( c_{\alpha} \).

**Proof.** As in the velocity residual computation, we will view \( R = R_0 + e \rho(s, \theta) \) as a function of \( \theta, s, \) and \( \bar{s} = -(s - t) \), rather than as a function of \( \theta, s, \) and \( t \). Then, using the expression \((4.3)\) for the pressure, along with \((1.11)\) and \((4.9)\), we calculate

\[
\begin{align*}
\dot{f}_p^{SB}(s) &= \frac{1}{4\pi} (F_1 + F_2 + F_3); \\
F_1 &= \int_0^{2\pi} \int_{-1/2}^{1/2} \frac{\epsilon e_{\rho} \cdot f(s + \bar{s})}{|R|^3} e_{\rho} \epsilon d\bar{s} d\theta \\
F_2 &= \int_0^{2\pi} \int_{-1/2}^{1/2} \frac{-\bar{s} e_1 \cdot f(s + \bar{s}) + \bar{s}^2 Q \cdot f(s + \bar{s})}{|R|^3} e_{\rho} \epsilon d\bar{s} d\theta \\
F_3 &= -\int_0^{2\pi} \int_{-1/2}^{1/2} \frac{R \cdot f(s + \bar{s})}{|R|^3} e_{\rho} \epsilon \kappa d\bar{s} d\theta.
\end{align*}
\]

First note that, using Lemma 4.3 and recalling that \( |\kappa| \leq 2\kappa_{\text{max}} \), we have that \( F_3 \) satisfies

\[
|F_3| \leq 2\pi \| f \|_{C^1(\mathbb{T})} \int_{-1/2}^{1/2} \frac{1}{|R|^2} \epsilon^2 |\kappa| d\bar{s} \leq \epsilon 4\pi \kappa_{\text{max}} c_3 \| f \|_{C^1(\mathbb{T})}.
\]

Next we estimate \( F_2 \). Recalling that \( e_{\rho}(s, \theta) = \cos \theta e_{n_1}(s) + \sin \theta e_{n_2}(s) \) while \( f(s + \bar{s}), e_1(s), \) and \( Q(s, \bar{s}) \) are all independent of \( \theta \), we can use Lemma 4.11 to show

\[
|F_2| \leq \epsilon 2c_3 (1 + \sqrt{\epsilon} c_Q) \| f \|_{C^1(\mathbb{T})}.
\]

Finally, using Lemma 4.5 with \( m = 0 \) and \( n = 3 \), we have that \( F_1 \) satisfies

\[
|F_1 - 2h_f(s)| \leq \epsilon c_{0.3} \| f \|_{C^1(\mathbb{T})};
\]

\[
h_f(s) := \int_0^{2\pi} e_{\rho}(s, \theta)(e_{\rho}(s, \theta) \cdot f(s)) d\theta = \pi \left( (f(s) \cdot e_{n_1}(s))e_{n_1}(s) + (f(s) \cdot e_{n_2}(s))e_{n_2}(s) \right).
\]

Combining these estimates, we obtain

\[
\left| f_p^{SB}(s) - \frac{1}{2\pi} h_f(s) \right| \leq \frac{1}{4\pi} \left( |F_1 - 2h_f(s)| + |F_2| + |F_3| \right) \leq \epsilon c_{p_0} \| f \|_{C^1(\mathbb{T})},
\]

where the constant \( c_{p_0} \) depends only on \( c_{\Gamma}, \kappa_{\text{max}}, \) and \( c_{\alpha} \). Recalling the definition of \( h_f(s) \) \((4.75)\), we obtain Proposition 4.13. \( \square \)

We now proceed to estimate \( f_1^{SB}(s) \), the next term in the expression \((4.72)\) for \( f^{SB} \). In particular, we show the following:
Proposition 4.14. Let $f_1^{SB}(s)$ be as defined in (4.72). Then $f_1^{SB}$ satisfies

\[
|f_1^{SB} - \frac{1}{2} \left( f(s) + (f \cdot e_1(s))e_1(s) \right) | \leq \epsilon c f_1 \|f\|_{C^1(T)} \tag{4.77}
\]

where the constant $c f_1$ depends only on $c_T$, $\kappa_{\text{max}}$, and $c_\alpha$.

Proof. Using the expression (4.72) for $f_1^{SB}(s)$ and recalling the slender body approximation (1.15), we consider $f_1^{SB}(s)$ as the sum of a Stokeslet and a doublet term. Again considering $R$ as a function of $\theta$, $s$, and $\bar{s}$, we can write

\[
f_1^{SB} = \frac{1}{8\pi} \left( F_{S,1} + \frac{3\epsilon^2}{2} F_{D,1} \right);
\]

\[
F_{S,1} := \int_0^{2\pi} \int_{-1/2}^{1/2} \frac{\partial}{\partial \rho} \Sigma(R) f(s + \bar{s}) e_\rho \epsilon d\bar{s} e(1 - \epsilon \kappa) d\theta
\]

\[
F_{D,1} := \int_0^{2\pi} \int_{-1/2}^{1/2} \frac{\partial}{\partial \rho} D(R) f(s + \bar{s}) e_\rho \epsilon d\bar{s} e(1 - \epsilon \kappa) d\theta.
\]

We begin by estimating $F_{S,1}$. Recalling the notation $R_0(s, \bar{s}) := X(s) - X(s + \bar{s})$, we have

\[
F_{S,1} = F_{S,11} + F_{S,12} + F_{S,13} + F_{S,14};
\]

\[
F_{S,11} = \int_0^{2\pi} \int_{-1/2}^{1/2} \left[ \frac{e f}{|R|^3} + \frac{3\epsilon R(R \cdot f)}{|R|^5} \right] d\bar{s} \epsilon d\theta
\]

\[
F_{S,12} = \int_0^{2\pi} \int_{-1/2}^{1/2} \left[ \frac{R_0 \cdot e_\rho f}{|R|^3} + \frac{3R(R \cdot f)(R_0 \cdot e_\rho)}{|R|^5} \right] d\bar{s} \epsilon d\theta
\]

\[
F_{S,13} = -\int_0^{2\pi} \int_{-1/2}^{1/2} \frac{e_\rho (R \cdot f) + R(R_0 \cdot e_\rho) + (R_0 \cdot e_\rho)}{|R|^3} d\bar{s} \epsilon d\theta
\]

\[
F_{S,14} = -\int_0^{2\pi} \int_{-1/2}^{1/2} \left[ \frac{R_0 \cdot e_\rho + \epsilon}{|R|^3} f - \frac{e_\rho (R_0 \cdot e_\rho) + R_0 \cdot e_\rho}{|R|^3} \right] d\bar{s} \epsilon d\theta
\]

First note that, using $(R_0 \cdot e_\rho) = \bar{s}^2 (Q \cdot e_\rho)$ by (4.8), the integrand of $F_{S,14}$ satisfies the bound

\[
\epsilon^2 |\kappa| \left| \frac{R_0 \cdot e_\rho + \epsilon}{|R|^3} f - \frac{e_\rho (R \cdot f) + R(R_\rho \cdot f)}{|R|^3} + \frac{3R(R_0 \cdot e_\rho + \epsilon)}{|R|^5} \right| \leq 2\epsilon^2 \kappa_{\text{max}} \left( \frac{4c_0 \bar{s}^2}{|R|^3} + \frac{2}{|R|^2} \right) \|f\|_{C(T)}
\]

and therefore, using Lemma 4.3 we have

\[
|F_{S,14}| \leq \epsilon 4\pi \kappa_{\text{max}} (c_3 (1 + 4c_Q \epsilon \|\log \epsilon\|) + 2\epsilon c_2) \|f\|_{C(T)}.
\]

Next, using (4.9), we can rewrite $F_{S,13}$ as

\[
F_{S,13} = F_{S,13a} + F_{S,13b};
\]

\[
F_{S,13a} := -\int_0^{2\pi} \int_{-1/2}^{1/2} e_\rho ((-\bar{s} e_t + \bar{s}^2 Q) \cdot f) + ((-\bar{s} e_t + \bar{s}^2 Q)(e_\rho \cdot f) d\bar{s} \epsilon d\theta
\]
Furthermore, using Lemma 4.5 with $m = 0$ and $n = 3$, we have

$$|F_{S,13}| \leq \epsilon c_3 (1 + \sqrt{c} Q) \|f\|_{C(T)}.$$  

Now we estimate $F_{S,12}$. Using (4.4) and the expansion (4.8) of $R_0 = X(s) - X(s + \bar{s})$, we can write

$$F_{S,12} = F_{S,12a} + F_{S,12b} + F_{S,12c};$$

$$F_{S,12a} := \int_{0}^{2\pi} \int_{-1/2}^{1/2} \epsilon \left[ \frac{\mathcal{S}^2(Q \cdot e_{\rho})}{|R|^3} f + \frac{3}{|R|^5} \mathcal{S}^4(e_{\rho} \cdot f)(Q \cdot e_{\rho}) \right] d\bar{s} d\theta;$$

$$F_{S,12b} := \int_{0}^{2\pi} \int_{-1/2}^{1/2} 3\epsilon \mathcal{S}^2(Q \cdot e_{\rho}) \left[ -\mathcal{S}^3(e_{\rho} \cdot f) - \epsilon c^2 Q(e_{\rho} \cdot f) + \epsilon e_{\rho}(e_{\rho} \cdot f) \right] d\bar{s} d\theta;$$

$$F_{S,12c} := \int_{0}^{2\pi} \int_{-1/2}^{1/2} \frac{\epsilon}{|R|^5} \left[ -\mathcal{S}^5(e_{\rho} \cdot f) + Q(e_{\rho} \cdot f) + \mathcal{S}^6(Q \cdot f) \right] d\bar{s} d\theta.$$

First, we have that $F_{S,12c}$ satisfies

$$|F_{S,12c}| \leq 2\pi \|f\|_{C(T)} \int_{-1/2}^{1/2} 3\epsilon \mathcal{S}^2 \left[ 2|\mathcal{S}|^2 + c_0 \mathcal{S}^4 + 2\epsilon \mathcal{S}^4 \right] d\bar{s}$$

$$\leq \epsilon 3\pi c^2_0 \|f\|_{C(T)} \left( c_0^{-1} \left( 4 + c_0 \right) + \int_{-1/2}^{1/2} 4\epsilon \mathcal{S}^4 d\bar{s} \right)$$

$$\leq \epsilon 3\pi c^2_0 \|f\|_{C(T)} \left( c_0^{-1} \left( 4 + c_0 \right) + 4\epsilon c_0 \epsilon \log \epsilon \right),$$

where we have used equation (4.11) and the fact that $|\mathcal{S}| \leq \frac{1}{2}$ to bound the first two terms, and we have used Lemma 4.3 to bound the third term.

Next, using Lemma 4.3, we have that $F_{S,12b}$ satisfies

$$|F_{S,12b}| \leq 2\pi \|f\|_{C(T)} \int_{-1/2}^{1/2} 6\epsilon \mathcal{S}^2 \left[ 2|\mathcal{S}|^2 + \epsilon \mathcal{S}^2 \right] d\bar{s} \leq \epsilon 36\pi c_0 c_3 \|f\|_{C(T)}.$$

Finally, we use Lemma 4.11 to show that $F_{S,12a}$ satisfies

$$|F_{S,12a}| \leq \epsilon^{3/2} c_0 (c_3 + 3c_0) \|f\|_{C(T)}.$$
Combining the above estimates, we obtain

$$\|F_{S,12}\| \leq \epsilon \left( 3\pi c_Q^2 (c_R^{-5}(4 + c_Q) + 4c_5\epsilon |\log \epsilon|) + 36\pi c_Q c_5 + \epsilon^{1/2} c_Q (\bar{c}_3 + 3\bar{c}_5) \right) \|f\|_{C(T)}. \quad (4.82)$$

It remains to estimate $F_{S,11}$. We have

$$F_{S,11} = F_{S,11a} + F_{S,11b} + F_{S,11c};$$

$$F_{S,11a} := \int_0^{2\pi} \int_{-1/2}^{1/2} \epsilon^2 \left[ f \frac{R}{|R|^3} + \frac{3[\bar{s}^2 e_t(e_t \cdot f) + \epsilon^2 e \cdot (e_t \cdot f)]}{|R|^5} \right] d\bar{s} d\theta$$

$$F_{S,11b} := -\int_0^{2\pi} \int_{-1/2}^{1/2} 3\epsilon^2 \bar{s} e_t (e_t \cdot f) + e \cdot (e_t \cdot f) \frac{d\bar{s}}{|R|^5} d\theta$$

$$F_{S,11c} := \int_0^{2\pi} \int_{-1/2}^{1/2} \frac{3\epsilon^2 [\bar{s}^3 (Q \cdot f) + Q(e_t \cdot f)] + \bar{s}^4 Q(Q \cdot f)}{|R|^5}$$

$$+ \frac{\epsilon \bar{s}^2 [e \cdot (Q \cdot f) + Q(e_t \cdot f)]}{|R|^5} \right] d\bar{s} d\theta.$$

First, using Lemma 4.3, we can bound $F_{S,11c}$ as

$$|F_{S,11c}| \leq 2\pi \|f\|_{C(T)} \int_{-1/2}^{1/2} 3\epsilon^2 \frac{2c_Q |\bar{s}|^3 + c_Q^2 \bar{s}^4 + 2\epsilon c_Q \bar{s}^2}{|R|^5} d\bar{s}$$

$$\leq \epsilon 6\pi c_5 (4c_Q + \epsilon |\log \epsilon| c_Q^3) \|f\|_{C(T)}$$

For $F_{S,11b}$, we use Lemma 4.4 to obtain

$$|F_{S,11b}| \leq \epsilon c_{1.5} \|f\|_{C^1(T)}.$$

Finally, to estimate $F_{S,11a}$, we use 4.5 to show

$$|F_{S,11a} - 2h_a(s) - 4h_f(s)| \leq \epsilon 2\pi (c_0.3 + 6c_0.5) \|f\|_{C^1(T)};$$

$$h_a(s) := \int_0^{2\pi} \left( f(s) + e_t(s)(e_t(s) \cdot f(s)) \right) d\theta$$

$$= 2\pi \left( f(s) + e_t(s)(e_t(s) \cdot f(s)) \right),$$

where $h_f(s)$ was defined in (4.75). Together, we have

$$|F_{S,11} - 2h_a(s) - 4h_f(s)| \leq \epsilon 2\pi (3c_5 (4c_Q + \epsilon |\log \epsilon| c_Q^3) + 6c_{1.5} + c_0.3 + 6c_0.5) \|f\|_{C^1(T)}. \quad (4.84)$$

Combining the estimates (4.80), (4.81), (4.82), and (4.84), we obtain

$$|F_{S,1} - 2h_a(s)| \leq \epsilon c_{S,1} \|f\|_{C^1(T)}, \quad (4.85)$$

where the constant $c_{S,1}$ depends only on $c_T$, $\kappa_{\text{max}}$, and $c_{\alpha}$.  

62
Now we estimate $F_{D,1}$. Following the same outline as in the $F_{S,1}$ estimate, we write

$$F_{D,1} = F_{D,11} + F_{D,12} + F_{D,13} + F_{D,14};$$

$$F_{D,11} = \int_{0}^{2\pi} \int_{-1/2}^{1/2} \left[ \frac{\epsilon f}{|R|^5} - \frac{5\epsilon R(R \cdot f)}{|R|^7} \right] d\bar{s} \epsilon d\theta$$

$$F_{D,12} = \int_{0}^{2\pi} \int_{-1/2}^{1/2} \left[ \frac{R_{0} \cdot e_{p} f - 5R(R \cdot f)(R_{0} \cdot e_{p})}{|R|^5} \right] d\bar{s} \epsilon d\theta$$

$$F_{D,13} = \int_{0}^{2\pi} \int_{-1/2}^{1/2} \frac{e_{p}(R \cdot f) + R(e_{p} \cdot f)}{|R|^5} d\bar{s} \epsilon d\theta$$

$$F_{D,14} = -\int_{0}^{2\pi} \int_{-1/2}^{1/2} \left[ \frac{R_{0} \cdot e_{p} + \epsilon}{|R|^5} f + \frac{e_{p}(R \cdot f) + R(e_{p} \cdot f)}{|R|^5} \right] d\bar{s} \epsilon^{2} \bar{\kappa} d\theta. \quad (4.86)$$

Again using Lemma 4.3, we can show

$$|F_{D,14}| \leq \epsilon^{2} 4\pi \kappa_{\text{max}} \|f\|_{C(T)} \int_{-1/2}^{1/2} \left[ \frac{6(cQ\bar{s}^{2} + \epsilon)}{|R|^5} + \frac{2}{|R|^7} \right] d\bar{s}$$

$$\leq \epsilon^{-1} 4\pi \kappa_{\text{max}} (6c_5(1 + \epsilon cQ) + 2c_4) \|f\|_{C(T)} . \quad (4.87)$$

To estimate $F_{D,13}$, we use (4.9) to write

$$F_{D,13} = F_{D,13a} + F_{D,13b} + F_{D,13c};$$

$$F_{D,13a} := \int_{0}^{2\pi} \int_{-1/2}^{1/2} \epsilon^{2} \frac{2e_{p}(e_{p} \cdot f)}{|R|^5} d\bar{s} d\theta$$

$$F_{D,13b} := -\int_{0}^{2\pi} \int_{-1/2}^{1/2} \epsilon^{2} \frac{\bar{s}}{|R|^5} \left[ e_{p}(e_{t} \cdot f) + e_{t}(e_{p} \cdot f) \right] d\bar{s} d\theta$$

$$F_{D,13c} := \int_{0}^{2\pi} \int_{-1/2}^{1/2} \epsilon^{2} \frac{\bar{s}}{|R|^5} \left[ e_{p}(Q \cdot f) + Q(e_{p} \cdot f) \right] d\bar{s} d\theta .$$

We then have

$$|F_{D,13}| \leq \epsilon^{2} \|f\|_{C(T)} \int_{-1/2}^{1/2} \frac{2cQ\bar{s}^{2}}{|R|^5} d\bar{s} \leq \epsilon^{-1} 4\pi cQC_{5} \|f\|_{C(T)} ,$$

by Lemma 4.3. Next, using Lemma 4.4, we can bound $F_{D,13b}$ as

$$|F_{D,13b}| \leq \epsilon^{-1} c_{1,5} 4\pi \|f\|_{C_{1}(T)} .$$

Finally, using Lemma 4.5, we can show

$$\left| F_{D,13a} - \epsilon^{-2} \frac{8}{3} h_{f}(s) \right| \leq \epsilon^{-1} 2\pi c_{0,5} \|f\|_{C_{1}(T)} ,$$

where $h_{f}(s)$ was defined in (4.75). Combining these three estimates, we obtain

$$\left| F_{D,13} - \epsilon^{-2} \frac{8}{3} h_{f}(s) \right| \leq \epsilon^{-1} 2\pi (2cQC_{5} + 2c_{1,5} + c_{0,5}) \|f\|_{C_{1}(T)} . \quad (4.88)$$
Next we estimate \( F_{D, 12} \). Using that \( R_0 \cdot e_\rho = \hat{z}^2 Q \), by Lemma 4.3 we have

\[
|F_{D, 12}| \leq \epsilon 2 \pi \| f \|_{C(T)} \int_{1/2}^{1/2} \frac{6c_Q \hat{z}^2}{|R|^5} d\bar{s} \leq \epsilon^{-1} 12 \pi c_Q c_5 \| f \|_{C(T)} .
\] (4.89)

Finally we estimate \( F_{D, 11} \). Using (4.9), we write

\[
F_{D, 11} = F_{D, 11a} + F_{D, 11b} + F_{D, 11c} ;
\]

\[
F_{D, 11a} := \int_{1/2}^{1/2} \frac{1}{2^2} \epsilon^2 \left[ \frac{f}{|R|^5} - \frac{3 \hat{z}^2 e_t(e_\eta \cdot f)}{|R|^7} + e_\eta(e_\rho \cdot f) \right] d\bar{s} d\theta 
\]

\[
F_{D, 11b} := \int_{1/2}^{1/2} \frac{1}{2^2} \epsilon^2 \left[ \frac{3 \hat{z}^2 e_t(e_\eta \cdot f)}{|R|^7} + e_\eta(e_\rho \cdot f) \right] d\bar{s} d\theta 
\]

\[
F_{D, 11c} := - \int_{1/2}^{1/2} \frac{1}{2^2} \epsilon^2 \left[ \frac{3 \hat{z}^2 (Q(f) + Q(e_\eta \cdot f))}{|R|^7} + \frac{\hat{z}^2 e_\eta(Q \cdot f)}{|R|^7} + e_\eta(e_\rho \cdot f) \right] d\bar{s} d\theta .
\]

Using Lemma 4.3 we obtain the bound

\[
|F_{D, 11c}| \leq 2 \pi \| f \|_{C(T)} \int_{1/2}^{1/2} 5c_Q \epsilon^2 \frac{2 \hat{z}^3 + c_Q \hat{z}^4 + 2 \hat{z}^2}{|R|^7} d\bar{s} \leq \epsilon^{-1} 10 \pi c_Q c_7 (4 + c_Q) \| f \|_{C(T)} .
\]

Furthermore, by Lemma 4.4 we have

\[
|F_{D, 11b}| \leq \epsilon^{-1} 20 \pi c_{1, 7} \| f \|_{C^1(T)} .
\]

To estimate \( F_{D, 11a} \), we use Lemma 4.5 with \( m = 0, 2, 0 \) and \( n = 5, 7, 7 \) for the first, second, and third term, respectively. Using that

\[
\int_0^{2\pi} \left( \frac{4}{3} f(s) - \frac{4}{3} e_t(s)(e_t(s) \cdot f(s)) - \frac{16}{3} e_\rho(s, \theta)(e_\rho(s, \theta) \cdot f) \right) d\theta 
\]

\[
= \frac{8\pi}{3} f(s) - \frac{8\pi}{3} e_t(s)(e_t(s) \cdot f(s)) - \frac{16\pi}{3} ((f(s) \cdot e_{n_1}(s)) e_{n_1}(s) + (f(s) \cdot e_{n_2}(s)) e_{n_2}(s)) 
\]

\[
= - \frac{8\pi}{3} ((f(s) \cdot e_{n_1}(s)) e_{n_1}(s) + (f(s) \cdot e_{n_2}(s)) e_{n_2}(s)) = - \frac{8}{3} h_f(s) ,
\]

where \( h_f(s) \) was defined in (4.75), we can show

\[
|F_{D, 11a} + \epsilon^{-2} \frac{8}{3} h_f(s)| \leq \epsilon^{-1} 2 \pi (c_{0, 5} + 10c_{0, 7}) \| f \|_{C^1(T)} .
\]

Together, we obtain an estimate for \( F_{D, 11} \):

\[
|F_{D, 11} + \epsilon^{-2} \frac{8}{3} h_f(s)| \leq \epsilon^{-1} 2 \pi (5c_Q c_7 (4 + c_Q) + 10c_{1, 7} + c_{0, 5} + 10c_{0, 7}) \| f \|_{C^1(T)} .
\] (4.90)

Combining estimates (4.87), (4.88), (4.89), and (4.90), we obtain

\[
|F_{D, 1}| \leq |F_{D, 11} + \epsilon^{-2} \frac{8}{3} h_f(s)| + |F_{D, 12}| + |F_{D, 13} - \epsilon^{-2} \frac{8}{3} h_f(s)| + |F_{D, 14}| \leq \epsilon^{-1} c_{D, 1} \| f \|_{C^1(T)} ,
\] (4.91)
where the constant \( c_{D,1} \) depends only on \( c_\Gamma \) and \( \kappa_{\max} \).

Finally, using the estimates (4.85) and (4.91), as well as the expression (4.78) for \( f_1^{SB} \), we obtain
\[
\left| f_1^{SB} - \frac{1}{4\pi} h_a(s) \right| \leq \frac{1}{8\pi} \left( |F_{S,1} - 2h_a(s)| + \frac{3\epsilon^2}{2} |F_{D,1}| \right) \leq \epsilon \left( c_{S,1} + \frac{3}{2} c_{D,1} \right) \|f\|_{C^1(\mathbb{T})},
\]
from which, using the form of \( h_a(s) \) in (4.83), we obtain Proposition 4.14.

Next we show the following bound for the component \( f_2^{SB}(s) \) of the slender body force, given by (4.72).

**Proposition 4.15.** Let the slender body \( \Sigma_\epsilon \) be as in Section 1.1 with \( C^{2,\alpha} \) centerline \( X(s) \). Given \( f \in C^1(\mathbb{T}) \), let \( f_2^{SB}(s) \) be defined as in (4.72). Then \( f_2^{SB} \) satisfies
\[
|f_2^{SB}| \leq \epsilon c_{f_2} \|f\|_{C^1(\mathbb{T})},
\]
where the constant \( c_{f_2} \) depends only on \( c_\Gamma, \kappa_{\max}, \) and \( c_\alpha \).

**Proof.** Using the \( f_1^{SB} \) computation as a guide, we again use (1.15) to consider \( f_2^{SB} \) as the sum of a Stokeslet and doublet term:
\[
f_2^{SB} = \frac{1}{8\pi} \left( F_{S,2} + \frac{3\epsilon^2}{2} F_{D,2} \right);\]
\[
F_{S,2} := \int_0^{2\pi} \int_{-1/2}^{1/2} e_\rho \left( \frac{\partial}{\partial \rho} S(R) f(s + \xi) \right) \cdot e_\rho \, d\xi \, \epsilon(1 - \epsilon \kappa) \, d\theta
\]
\[
F_{D,2} := \int_0^{2\pi} \int_{-1/2}^{1/2} e_\rho \left( \frac{\partial}{\partial \rho} D(R) f(s + \xi) \right) \cdot e_\rho \, d\xi \, \epsilon(1 - \epsilon \kappa) \, d\theta.
\]

As we did for \( f_1^{SB} \), we begin by estimating the Stokeslet term \( F_{S,2} \). We write \( F_{S,2} \) as
\[
F_{S,2} = F_{S,21} + F_{S,22} + F_{S,23} + F_{S,24};
\]
\[
F_{S,21} := \int_0^{2\pi} \int_{-1/2}^{1/2} \left[ \frac{\epsilon}{|R|^3} (f \cdot e_\rho) e_\rho + \frac{3\epsilon e_\rho (R \cdot e_\rho) (R \cdot f)}{|R|^5} \right] d\xi \, \epsilon \, d\theta
\]
\[
F_{S,22} := \int_0^{2\pi} \int_{-1/2}^{1/2} \left[ \frac{R_0 \cdot e_\rho (f \cdot e_\rho) e_\rho + 3\epsilon e_\rho (R \cdot e_\rho) (R_0 \cdot e_\rho)}{|R|^5} \right] d\xi \, \epsilon \, d\theta
\]
\[
F_{S,23} := -\int_0^{2\pi} \int_{-1/2}^{1/2} \left[ \frac{\epsilon}{|R|^3} \epsilon_\rho (f \cdot e_\rho) e_\rho + \frac{3\epsilon e_\rho (R \cdot e_\rho) (R_0 \cdot e_\rho) e_\rho}{|R|^5} \right] d\xi \, \epsilon \, d\theta
\]
\[
F_{S,24} := -\int_0^{2\pi} \int_{-1/2}^{1/2} \left[ \frac{R_0 \cdot e_\rho + \epsilon}{|R|^3} (f \cdot e_\rho) e_\rho - \frac{e_\rho (R_0 \cdot f) + e_\rho (R \cdot e_\rho) (R_0 \cdot e_\rho) e_\rho + \epsilon}{|R|^5} \right] d\xi \, \epsilon \, d\theta
\]
\[
+ \frac{3\epsilon e_\rho (R \cdot e_\rho) (R_0 \cdot e_\rho) e_\rho}{|R|^5} d\xi \, \epsilon^2 \kappa \, d\theta
\]

As before, we estimate \( F_{S,24} \) via Lemma 4.3. Using that \( R_0 \cdot e_\rho = \xi^2 Q \cdot e_\rho \), we have
\[
|F_{S,24}| \leq 2\pi \|f\|_{C(\mathbb{T})} \int_{-1/2}^{1/2} \epsilon^2 |\kappa| \left[ \frac{4 \xi^2 cQ + \epsilon}{|R|^3} + \frac{2}{|R|^2} \right] d\xi
\]
\[
\leq \epsilon \kappa_{\max} (2c_3 (1 + \epsilon |\log \epsilon| c_Q) + c_2) \|f\|_{C(\mathbb{T})}.
\]
Next we bound $F_{S,23}$. Following the same steps as in the $F_{S,13}$ estimate, we rewrite $F_{S,23}$ as

$$F_{S,23} = F_{S,23a} + F_{S,23b};$$

$$F_{S,23a} := - \int_0^{2\pi} \int_{-1/2}^{1/2} \frac{\epsilon_0((-\pi e_\theta + \pi^2 Q) \cdot f) + \pi^2 \epsilon_0(Q \cdot e_\theta)(e_\theta \cdot f)}{|R|^3} d\pi d\theta$$

$$F_{S,23b} := - \int_0^{2\pi} \int_{-1/2}^{1/2} \frac{2\epsilon_0(e_\theta \cdot f)}{|R|^3} d\pi d\theta.$$

As in the $F_{S,13a}$ estimate, to bound $F_{S,23a}$ we rely on the $\theta$-independence of $Q(s, \bar{s})$, $f(s + \bar{s})$, and $e_\theta(s)$, and use Lemma 4.111 - in particular, the remark about triple copies of $e_\theta$ to show

$$|F_{S,23a}| \leq 2\bar{c}_3(1 + 6\sqrt{c}Q) \|f\|_{C(T)}.$$

Noting that $F_{S,23b} = F_{S,13b}$, via Lemma 4.5 we again have

$$|F_{S,23} + 4h_f(s)| \leq \epsilon c_03 \|f\|_{C^1(T)};$$

where $h_f(s)$ was defined in (4.75). Altogether we obtain the estimate

$$|F_{S,23} + 4h_f(s)| \leq \epsilon(2\bar{c}_3(1 + 6\sqrt{c}Q) + c_03) \|f\|_{C^1(T)}.$$  \hfill (4.97)

Next we estimate $F_{S,22}$. We again decompose $F_{S,22}$ as

$$F_{S,22} = F_{S,22a} + F_{S,22b} + F_{S,22c};$$

$$F_{S,22a} := \int_0^{2\pi} \int_{-1/2}^{1/2} \frac{\pi^2(Q \cdot e_\theta)(f \cdot e_\theta)d\pi d\theta}{|R|^3}$$

$$F_{S,22b} := \int_0^{2\pi} \int_{-1/2}^{1/2} \frac{3\pi^2(Q \cdot e_\theta)[-\pi e_\theta(e_\theta \cdot f) + \epsilon e_\theta e_\theta e_\theta]}{|R|^3} d\pi d\theta$$

$$F_{S,22c} := \int_0^{2\pi} \int_{-1/2}^{1/2} \frac{\epsilon \pi^4[(Q \cdot f) + Q \cdot e_\theta)(e_\theta \cdot f)](Q \cdot e_\theta e_\theta)}{|R|^3} d\pi d\theta.$$

First, we have that $F_{S,22c}$ satisfies

$$|F_{S,22c}| \leq 2\pi c_2^2 \|f\|_{C(T)} \int_{-1/2}^{1/2} \frac{3\bar{s}^5 + c_2 \bar{s}^6 + 2\epsilon \bar{s}^4}{|R|^3} d\bar{s} \leq \epsilon 3\pi c_2^2 \|f\|_{C(T)} \left(c_5^5(4 + c_2^2) + 4c_5 \epsilon |\log \epsilon| \right).$$

This is the same bound as in the $F_{S,12c}$ estimate, where we relied on (4.111) as well as $|\bar{s}| \leq \frac{1}{2}$ to estimate the first two terms, and used Lemma 4.3 to bound the third term.

Next we use Lemma 4.3 to show

$$|F_{S,22b}| \leq 2\pi \|f\|_{C(T)} \int_{-1/2}^{1/2} \frac{3c_2 \bar{s}^2}{3c_2 \bar{s}^2} d\bar{s} \leq \epsilon 12c_2c_5 \|f\|_{C(T)}.$$

Furthermore, by Lemma 4.11 using the remark about integration against triples of the form $(A(\bar{s}) \cdot e_\theta)(B(\bar{s}) \cdot e_\theta)e_\theta$, we show

$$|F_{S,22a}| \leq \epsilon^{3/2} 6c_2c_3 \|f\|_{C(T)}.\]
The above three estimates together give
\[
|F_{S,22}| \leq \epsilon (3\pi c_Q^2 (c_R^{-5} (4 + c_Q) + 4c_Q \log \epsilon) + 12\pi c_Q c_5 + \epsilon^{1/2} 6c_Q c_5) \|f\|_{C(T)}. \tag{4.98}
\]

Finally we estimate \(F_{S,21}\). Using (4.9), we write
\[
F_{S,21} = F_{S,21a} + F_{S,21b} + F_{S,21c};
\]
\[
F_{S,21a} := \int_0^{2\pi} \int_{-1/2}^{1/2} \epsilon^2 \left[ \frac{(f \cdot e_\rho)e_\rho}{|R|^3} + \frac{3\epsilon e_\rho(e_\rho \cdot f)}{|R|^5} \right] d\bar{s} d\theta
\]
\[
F_{S,21b} := -\int_0^{2\pi} \int_{-1/2}^{1/2} 3\epsilon^2 e_\rho(e_\rho \cdot f) d\bar{s} d\theta
\]
\[
F_{S,21c} := \int_0^{2\pi} \int_{-1/2}^{1/2} 3\epsilon^2 \left[ \frac{\bar{s}^3(-e_\rho \cdot f) + \bar{s}(Q \cdot f)(Q \cdot e_\rho)e_\rho}{|R|^5} + \frac{\epsilon^2 (Q \cdot f) + (Q \cdot e_\rho)(e_\rho \cdot f)e_\rho}{|R|^5} \right] d\bar{s} d\theta.
\]

We first estimate \(F_{S,21c}\) using Lemma 4.3. We have
\[
|F_{S,21c}| \leq 2\pi \|f\|_{C(T)} \int_{-1/2}^{1/2} 3c_5 2\epsilon^2 \left[ \frac{\bar{s}^3 + c_Q \bar{s}^4 + 2\epsilon^2}{|R|^5} \right] d\bar{s} \leq 6\pi c_Q c_5 (3 + \epsilon |\log \epsilon| c_Q) \|f\|_{C(T)}.
\]

Next, using Lemma 4.4, we can estimate \(F_{S,21b}\) as
\[
|F_{S,21b}| \leq \epsilon 6\pi c_{1,5} \|f\|_{C^1(T)}.
\]

Lastly, by Lemma 4.5 we obtain the following estimate for \(F_{S,21a}\):
\[
|F_{S,21a} - 6h_f(s)| \leq \epsilon 2\pi (c_{0,3} + 3c_{0,5}) \|f\|_{C^1(T)},
\]

where \(h_f(s)\) was defined in (4.75). Altogether, we obtain the estimate
\[
|F_{S,21} - 6h_f(s)| \leq \epsilon 2\pi (3c_Q c_5 (3 + \epsilon |\log \epsilon| c_Q) + 3c_{1,5} + c_{0,3} + 3c_{0,5}) \|f\|_{C^1(T)}. \tag{4.99}
\]

Combining estimates (4.96), (4.97), (4.98), and (4.99), we obtain the following bound for \(F_{S,2}\):
\[
|F_{S,21} - 2h_f(s)| \leq \epsilon c_{5,2} \|f\|_{C^1(T)}, \tag{4.100}
\]

where the constant \(c_{5,2}\) depends only on \(c_T\), \(\kappa_{\text{max}}\), and \(c_\alpha\).

Now we estimate the doublet term of the expression (4.94) for \(f^2\). We have that \(F_{D,2}\) can be expressed as
\[
F_{D,2} = F_{D,21} + F_{D,22} + F_{D,23} + F_{D,24};
\]
\[
F_{D,21} := \int_0^{2\pi} \int_{-1/2}^{1/2} \epsilon \left[ \frac{\bar{s}^3 + c_Q \bar{s}^4}{|R|^5} + \frac{5\epsilon e_\rho (R \cdot e_\rho)(R \cdot f) e_\rho}{|R|^7} \right] d\bar{s} d\theta
\]
\[
F_{D,22} := \int_0^{2\pi} \int_{-1/2}^{1/2} \left[ \frac{R_0 \cdot e_\rho (f \cdot e_\rho) e_\rho - 5e_\rho (R \cdot e_\rho)(R \cdot f)(R_0 \cdot e_\rho)}{|R|^7} \right] d\bar{s} d\theta
\]
\[
F_{D,23} := \int_0^{2\pi} \int_{-1/2}^{1/2} \epsilon_\rho (R \cdot f) + e_\rho (R \cdot e_\rho)(e_\rho \cdot f) d\bar{s} d\theta
\]
\[
F_{D,24} := -\int_0^{2\pi} \int_{-1/2}^{1/2} \left[ \frac{R_0 \cdot e_\rho + \epsilon (f \cdot e_\rho) e_\rho + \frac{e_\rho (R \cdot f) + e_\rho (R \cdot e_\rho)(e_\rho \cdot f)}{|R|^5}}{5e_\rho (R \cdot e_\rho)(R \cdot f)(R_0 \cdot e_\rho) + \epsilon} \right] d\bar{s} \epsilon^2 \mathcal{K} d\theta.
\]
We estimate $F_{D,24}$ exactly as we did for $F_{D,14}$. Using Lemma 4.3, we have

$$|F_{D,24}| \leq 2\pi \|f\|_{C(T)} \int_{-1/2}^{1/2} \epsilon^2 \left[ \hat{\alpha} \left( \frac{6c_Q\bar{s}^2 + \epsilon}{|R|^5} + \frac{2}{|R|^4} \right) d\bar{s}\right]$$

$$\leq \epsilon^{-1}4\pi\kappa_{\text{max}}(c_5(1 + 6c_Q) + 2c_4) \|f\|_{C(T)}.$$  \hfill (4.102)

As in the $F_{D,13}$ estimate, we write $F_{D,23}$ as

$$F_{D,23} = F_{D,23a} + F_{D,23b} + F_{D,23c};$$

$$F_{D,23a} := \int_{0}^{2\pi} \int_{-1/2}^{1/2} \epsilon^2 \left[ \frac{2e\rho(e\cdot f)}{|R|^5} d\bar{s} d\theta \right]$$

$$F_{D,23b} := -\int_{0}^{2\pi} \int_{-1/2}^{1/2} \epsilon \left[ \frac{\bar{s}e\rho(e\cdot f)}{|R|^5} d\bar{s} d\theta \right]$$

$$F_{D,23c} := \int_{0}^{2\pi} \int_{-1/2}^{1/2} \epsilon \left[ \frac{\bar{s}^2[e\rho(Q\cdot f) + (Q\cdot e\rho)(e\cdot f)e\rho]}{|R|^5} d\bar{s} d\theta \right].$$

First we estimate $F_{D,23c}$. Lemma 4.3 gives

$$|F_{D,23c}| \leq 2\pi \|f\|_{C(T)} \int_{-1/2}^{1/2} \epsilon \left[ \frac{2c_Q\bar{s}^2}{|R|^5} d\bar{s}\right] \leq \epsilon^{-1}4\pi c_Q c_5 \|f\|_{C(T)}.$$  \hfill (4.103)

Furthermore, using Lemma 4.4 we obtain the bound

$$|F_{D,23b}| \leq \epsilon^{-1}2\pi c_{1,5} \|f\|_{C^1(T)}.$$  \hfill (4.104)

Finally, noting that $F_{D,23a} = F_{D,13a}$, by Lemma 4.5, we have

$$\left| F_{D,23a} - \epsilon^{-2}8\frac{3}{3}h_f(s) \right| \leq \epsilon^{-1}4\pi c_{0,5} \|f\|_{C^1(T)},$$

where $h_f(s)$ was defined in (4.75). Altogether we obtain the estimate

$$\left| F_{D,23} - \epsilon^{-2}8\frac{3}{3}h_f(s) \right| \leq \epsilon^{-1}2\pi(2c_Q c_5 + c_{1,5} + 2c_{0,5}) \|f\|_{C^1(T)}.$$  \hfill (4.105)

Next, by Lemma 4.3 we have

$$|F_{D,22}| \leq 2\pi \|f\|_{C(T)} \int_{-1/2}^{1/2} \epsilon \left[ \frac{6c_Q\bar{s}^2}{|R|^5} d\bar{s}\right] \leq \epsilon^{-1}2\pi c_Q c_5 \|f\|_{C(T)}.$$  \hfill (4.106)

Lastly we bound $F_{D,21}$, following the same steps as in the estimate of $F_{D,11}$. We first write

$$F_{D,21} = F_{D,21a} + F_{D,21b} + F_{D,21c};$$

$$F_{D,21a} := \int_{0}^{2\pi} \int_{-1/2}^{1/2} \epsilon^2 \left[ \frac{(f\cdot e\rho)e\rho}{|R|^5} - \frac{5e^2e\rho(e\cdot f)|R|^7}{|R|^4} d\bar{s} d\theta \right]$$

$$F_{D,21b} := \int_{0}^{2\pi} \int_{-1/2}^{1/2} 5e^3\bar{s}e\rho(e\cdot f)\frac{|R|^7}{|R|^7} d\bar{s} d\theta$$

$$F_{D,21c} := -\int_{0}^{2\pi} \int_{-1/2}^{1/2} 5e^2 \left[ \bar{s}^3[-(e\cdot f) + \bar{s}(Q\cdot f)](Q\cdot e\rho)e\rho \right]$$

$$\frac{|R|^7}{|R|^7} d\bar{s} d\theta.$$  \hfill (4.107)
Combining the estimates (4.102), (4.103), (4.104), and (4.105), we have that
\[ F \]
where
\[ h \]
We now show that a similar bound to Proposition 4.15 also holds for the next force component
obtain the bound
Finally, using the expression (4.94) for
\[ f \]
where the constant
We use Lemma 4.3 to estimate
Let the slender body
Proposition 4.16.
Next, by Lemma 4.4, we have that
\[ F \]
depends only on
\[ D \]
was defined in (4.75). In total we have that
\[ F \]
\[ c \]
where the constant
Given
\[ c \]
to consider
Proof. Following the same steps as in the calculations of \( f_1^{SB} \) and \( f_2^{SB} \), we use (1.15) in the expression (4.72) for \( f_3^{SB} \) to consider \( f_3^{SB} \) as the sum
\[ f_3^{SB} = \frac{1}{8\pi} \left( F_{S,3} + \frac{3\epsilon^2}{2} F_{D,3} \right); \]
\[ F_{S,3} := \int_0^{2\pi} \int_{-1/2}^{1/2} e_\theta \left( \frac{\partial}{\partial \theta} S(R)f(s + \bar{s}) \right) \cdot e_\rho \, d\bar{s} \, d\theta \epsilon (1 - \epsilon\hat{\kappa}) \, d\theta \]
As before, we begin by estimating $F_{S,3}$. We write

$$F_{S,3} = F_{S,31} + F_{S,32} + F_{S,33};$$

$$F_{S,31} = \int_{0}^{2\pi} \int_{-1/2}^{1/2} \left[ \frac{R_0 \cdot e_\theta (f \cdot e_\rho) e_\theta + 3(R \cdot e_\rho)(R \cdot f)(R_0 \cdot e_\theta) e_\theta}{|R|^5} \right] d\bar{s} \epsilon d\theta$$

$$F_{S,32} = -\int_{0}^{2\pi} \int_{-1/2}^{1/2} \frac{e_\theta(R \cdot e_\rho)(e_\theta \cdot f)}{|R|^3} d\bar{s} \epsilon d\theta$$

$$F_{S,33} = -\int_{0}^{2\pi} \int_{-1/2}^{1/2} \left[ \frac{R_0 \cdot e_\theta (f \cdot e_\rho) e_\theta - e_\theta(R \cdot e_\rho)(e_\theta \cdot f)}{|R|^3} \right.$$

$$+ \left. \frac{3(R \cdot e_\rho)(R \cdot f)(R_0 \cdot e_\theta) e_\theta}{|R|^5} \right] d\bar{s} \epsilon^2 R d\theta. \quad (4.110)$$

First, noting that $R_0 \cdot e_\theta = \bar{s}^2 Q \cdot e_\theta$, we can use Lemma 4.3 to bound $F_{S,33}$ as

$$|F_{S,33}| \leq 2\pi \|f\|_{C(T)} \int_{-1/2}^{1/2} \epsilon^2 |\bar{s}| \left[ \frac{4cQ\bar{s}^2}{|R|^3} + \frac{1}{|R|^2} \right] d\bar{s}$$

$$\leq \epsilon 4\pi \kappa_{\text{max}} (c_2 + \epsilon |\log \epsilon| 4cQc_3) \|f\|_{C(T)}. \quad (4.111)$$

Next we estimate $F_{S,32}$. Using (4.9), we write

$$F_{S,32} = F_{S,32a} + F_{S,32b};$$

$$F_{S,32a} := -\int_{0}^{2\pi} \int_{-1/2}^{1/2} \epsilon^2 \frac{e_\theta(e_\theta \cdot f)}{|R|^3} d\bar{s} d\theta$$

$$F_{S,32b} := -\int_{0}^{2\pi} \int_{-1/2}^{1/2} \epsilon \bar{s}^2 \frac{e_\theta(Q \cdot e_\rho)(e_\theta \cdot f)}{|R|^3} d\bar{s} d\theta. \quad (4.112)$$

Since $Q$, $e_\theta$, and $f$ are all independent of $\theta$, by Lemma 4.11 and the remark about integration against triples of the form $(A(\bar{s}) \cdot e_\rho)(B(\bar{s}) \cdot e_\theta)e_\theta$, we have

$$|F_{S,32b}| \leq \epsilon^{3/2} cQ \bar{s}^3 \|f\|_{C(T)}. \quad (4.113)$$

Also, by Lemma 4.5 we obtain the following estimate for $F_{S,32a}$:

$$|F_{S,32a} + 2h_b(s)| \leq \epsilon 2\pi c_{0,3} \|f\|_{C^1(T)};$$

$$h_b(s) := \int_{0}^{2\pi} e_\theta(s, \theta)(e_\rho(s, \theta) \cdot f(s)) d\theta$$

$$= \pi ((f(s) \cdot e_{n_1}(s))e_{n_1}(s) + (f(s) \cdot e_{n_2}(s))e_{n_2}(s)). \quad (4.114)$$

We note that in fact $h_b(s) = h_f(s)$, but we will not need to make use of this observation. Together, we have that $F_{S,32}$ satisfies

$$|F_{S,32} + 2h_b(s)| \leq \epsilon (2\pi c_{0,3} + \sqrt{\epsilon} cQ\bar{s}^3) \|f\|_{C^1(T)}. \quad (4.115)$$

Lastly, to estimate $F_{S,31}$, using (4.9) and that $(R_0 \cdot e_\theta) = \bar{s}^2(Q \cdot e_\theta)$, we write

$$F_{S,31} = F_{S,31a} + F_{S,31b} + F_{S,31c};$$

$$F_{S,31a} := \int_{0}^{2\pi} \int_{-1/2}^{1/2} \epsilon \left[ \frac{\bar{s}^2(Q \cdot e_\theta)(f \cdot e_\rho)e_\theta}{|R|^3} + \frac{3\epsilon^2 \bar{s}^2(Q \cdot e_\theta)(f \cdot e_\rho)e_\theta}{|R|^5} \right] d\bar{s} d\theta$$

70
\[ F_{S,31b} := -\int_0^{2\pi} \int_{-1/2}^{1/2} 3c^2 \bar{\pi}^3 (e_t \cdot f)(Q \cdot e_\theta) e_\theta d\bar{s} d\theta \]

\[ F_{S,31c} := \int_0^{2\pi} \int_{-1/2}^{1/2} 3c^2 \left[ \bar{\pi}^5 \left( - (Q \cdot e_\rho)(e_t \cdot f) + \bar{\pi}(Q \cdot f)(Q \cdot e_\theta)(Q \cdot e_\theta) e_\theta \right) \right] d\bar{s} d\theta. \]

We estimate \( F_{S,31c} \) in the same way that we estimated \( F_{S,12c} \) and \( F_{S,22c} \). In particular, we have

\[ |F_{S,31c}| \leq 2\pi \|f\|_{C(\tau)} \int_{-1/2}^{1/2} 3c^2 \bar{s}^5 + cQ \bar{s}^6 + 2c^4 \bar{s} d\bar{s} d\theta \]

\[ \leq 3\pi c^2 \|f\|_{C(\tau)} (c_R^{-5}(2 + c_Q) + \epsilon \|\log \epsilon\|2c_5), \]

where we have estimated the first two terms using the lower bound (4.11) on \( R \) along with \( |\bar{s}| \leq \frac{1}{2} \), and we have used Lemma 4.3 to bound the third term.

We estimate \( F_{S,31b} \) via Lemma 4.3, obtaining the bound

\[ |F_{S,31b}| \leq 2\pi \|f\|_{C(\tau)} \int_{-1/2}^{1/2} 3c^2 \bar{s}^3 \|\bar{s}^5 + cQ \bar{s}^6 + 2c^4 \bar{s}\| d\bar{s} d\theta \]

Finally, we can bound \( F_{S,31a} \) using Lemma 4.11, in particular, the remark about integration against triples of the form \((A(\bar{s}) \cdot e_\rho)(B(\bar{s}) \cdot e_\theta)e_\theta\). We then have

\[ |F_{S,31a}| \leq \epsilon^{3/2} 6cQ(c_3 + 3\bar{c}_5) \|f\|_{C(\tau)}. \]

Altogether we can estimate \( F_{S,3} \) as

\[ |F_{S,3}| \leq \epsilon (3\pi c^2 c_R^{-5}(2 + c_Q) + \epsilon \|\log \epsilon\|2c_5) + 6\pi cQ c_5 + \sqrt{6}cQ(c_3 + 3\bar{c}_5) \|f\|_{C(\tau)}. \] (4.114)

Combining the estimates (4.111), (4.113), and (4.114), we have that \( F_{S,3} \) satisfies

\[ |F_{S,3} + 2h_b(s)| \leq \epsilon c_{S,3} \|f\|_{C^1(\tau)}, \] (4.115)

where the constant \( c_{S,3} \) depends only on \( c_T, \kappa_{\text{max}}, \) and \( c_\alpha \).

Now we estimate the doublet term \( F_{D,3} \) in the expression (4.109) for \( f_3^{SB} \). As we did for the Stokeslet term, we decompose \( F_{D,3} \) as

\[ F_{D,3} = F_{D,31} + F_{D,32} + F_{D,33}; \]

\[ F_{D,31} = \int_0^{2\pi} \int_{-1/2}^{1/2} \left[ \frac{R_0 \cdot e_\theta}{|R|^5} (f \cdot e_\rho)e_\theta - \frac{5(R \cdot e_\rho)(R \cdot f)(R_0 \cdot e_\theta)e_\theta}{|R|^7} \right] d\bar{s} \epsilon d\theta \]

\[ F_{D,32} = \int_0^{2\pi} \int_{-1/2}^{1/2} \frac{e_\theta(R \cdot e_\rho)(e_\theta \cdot f)}{|R|^5} d\bar{s} \epsilon d\theta \] (4.116)

\[ F_{D,33} = -\int_0^{2\pi} \int_{-1/2}^{1/2} \left[ \frac{R_0 \cdot e_\theta}{|R|^5} (f \cdot e_\rho)e_\theta + \frac{e_\theta(R \cdot e_\rho)(e_\theta \cdot f)}{|R|^5} - \frac{5(R \cdot e_\rho)(R \cdot f)(R_0 \cdot e_\theta)e_\theta}{|R|^7} \right] d\bar{s} \epsilon^{2\kappa} d\theta. \]
As before, we can immediately estimate \( F_{D,33} \) via Lemma [4.3]. We have
\[
|F_{D,33}| \leq 2\pi \|f\|_{C(\Gamma)} \int_{-1/2}^{1/2} \epsilon^2 \left|\frac{6c_Q\bar{s}^2}{R^9} + \frac{1}{|R|^4}\right| d\bar{s} \\
\leq \epsilon^{-1} 4\pi \kappa_{\text{max}}(c_4 + 6c_Qc_5) \|f\|_{C(\Gamma)}.
\]

Next, as usual, we rewrite \( F_{D,32} \) as
\[
F_{D,32} = F_{D,32a} + F_{D,32b};
\]
\[
F_{D,32a} := \int_0^{2\pi} \int_{-1/2}^{1/2} \epsilon^2 e_\theta (e_\theta \cdot f) \frac{\bar{s}}{|R|^5} d\bar{s} d\theta 
\]
\[
F_{D,32b} := \int_0^{2\pi} \int_{-1/2}^{1/2} \epsilon \bar{s}^2 e_\theta (Q \cdot e_\rho)(e_\theta \cdot f) \frac{1}{|R|^5} d\bar{s} d\theta.
\]
Using Lemma [4.3], we have
\[
|F_{D,32b}| \leq 2\pi \|f\|_{C(\Gamma)} \int_{-1/2}^{1/2} \epsilon c_Q \bar{s} \frac{\bar{s}^2}{|R|^5} d\bar{s} \leq \epsilon^{-1} 2\pi c_Q c_5 \|f\|_{C(\Gamma)}.
\]
Also, by Lemma [4.5], we have that \( F_{D,32a} \) satisfies
\[
\left|F_{D,32a} - \epsilon^{-2} \frac{4}{3} h_b(s)\right| \leq \epsilon^{-1} 2\pi c_0,5 \|f\|_{C^1(\Gamma)},
\]
where \( h_b(s) \) was defined in (4.112). Together, we obtain
\[
\left|F_{D,32} - \epsilon^{-2} \frac{4}{3} h_b(s)\right| \leq \epsilon^{-1} 2\pi (c_Q c_5 + c_0,5) \|f\|_{C^1(\Gamma)}.
\]
Finally, we estimate \( F_{D,31} \). Using Lemma [4.3], we have
\[
|F_{D,31}| \leq 2\pi \|f\|_{C(\Gamma)} \int_{-1/2}^{1/2} \epsilon 6c_Q\bar{s}^2 \frac{1}{|R|^5} d\bar{s} \leq \epsilon^{-1} 12\pi c_Q \|f\|_{C(\Gamma)}.
\]
Altogether, the estimates (4.117), (4.118), and (4.119) yield
\[
\left|F_{D,3} - \epsilon^{-2} \frac{4}{3} h_b(s)\right| \leq \epsilon^{-1} c_{D,3} \|f\|_{C^1(\Gamma)},
\]
where the constant \( c_{D,3} \) depends only on \( c_\Gamma \) and \( \kappa_{\text{max}} \).

Using the estimates (4.115) and (4.120), together with the expression (4.109) for \( f_3^{\text{SB}} \), we obtain the bound
\[
|f_3^{\text{SB}}| \leq \frac{1}{8\pi} \left( |F_{S,3} + 2h_b(s)| + \frac{3c^2}{2} |F_{D,3} - \epsilon^{-2} \frac{4}{3} h_b(s)| \right) \leq \epsilon \left( c_{S,3} + \frac{3}{2} c_{D,3} \right) \|f\|_{C^1(\Gamma)},
\]
from which we immediately obtain Proposition 4.16.

It remains to estimate the final term \( f_4^{\text{SB}}(s) \) of the slender body force expression (4.72). We show that \( f_4^{\text{SB}}(s) \) satisfies the following proposition.

\[\]
Proposition 4.17. Let the slender body $\Sigma_\ell$ be as in Section 1.1 with $C^{2,\alpha}$ centerline $X(s)$. Given $f \in C^1(\mathbb{T})$, let $f_{4,SB}^L(s)$ be as defined in (4.72). We have that $f_{4,SB}^L$ satisfies the estimate
\[
|f_{4,SB}^L| \leq c f_4 \|f\|_{C^1(\mathbb{T})},
\] (4.122)
where the constant $c_{f_4}$ depends only on $c_T$, $\kappa_{\text{max}}$, and $c_\alpha$.

Proof. As with the previous slender body force components, we use the expression (4.72) for $f_{4,SB}^L(s)$ and (1.15) to write $f_{4,SB}^L(s)$ as the sum of a Stokeslet and a doublet term:
\[
f_{4,SB}^L = \frac{1}{8\pi} \left( F_{S,4} + \frac{3e^2}{2} F_{D,4} \right);
\]
\[
F_{S,4} := \int_0^{2\pi} \int_{-1/2}^{1/2} \frac{e_t}{1 - \epsilon \kappa} \left( \left. \frac{\partial S(R)}{\partial s} - \kappa_3 \frac{\partial S(R)}{\partial \theta} \right) f(s + \overline{s}) \cdot e_\rho d\overline{s} \epsilon (1 - \epsilon \kappa) d\theta \quad (4.123)
\]
\[
F_{D,4} := \int_0^{2\pi} \int_{-1/2}^{1/2} \frac{e_t}{1 - \epsilon \kappa} \left( \left. \frac{\partial D(R)}{\partial s} - \kappa_3 \frac{\partial D(R)}{\partial \theta} \right) f(s + \overline{s}) \cdot e_\rho d\overline{s} \epsilon (1 - \epsilon \kappa) d\theta.
\]

Following the same outline as in the previous calculations, we begin by estimating the Stokeslet term $F_{S,4}$. We again decompose $F_{S,4}$ into three terms:
\[
F_{S,4} = F_{S,41} + F_{S,42} + F_{S,43}.
\]
\[
F_{S,41} = \int_0^{2\pi} \int_{-1/2}^{1/2} \frac{R_0 \cdot e_t (f \cdot e_\rho)e_t}{|R|^3} + \frac{3e_t(R_0 \cdot e_\rho)(R_0 \cdot f)(R_0 \cdot e_t)}{|R|^5} d\overline{s} \epsilon d\theta \quad (4.124)
\]
\[
F_{S,42} = -\int_0^{2\pi} \int_{-1/2}^{1/2} \frac{R_0 \cdot e_\rho(e_t \cdot f)e_t}{|R|^3} d\overline{s} \epsilon d\theta
\]
\[
F_{S,43} = -\int_0^{2\pi} \int_{-1/2}^{1/2} \frac{R_0 \cdot e_t (f \cdot e_\rho)e_t}{|R|^3} - \frac{(R_0 \cdot e_\rho)(e_t \cdot f)e_t}{|R|^5} \left. + \frac{3e_t(R_0 \cdot e_\rho)(R_0 \cdot f)(R_0 \cdot e_t)}{|R|^5} \right] d\overline{s} \epsilon^2 \kappa d\theta.
\]

Again, using Lemma 4.3 along with the identity $R_0 \cdot e_t = -\overline{s} + \overline{s}^2 (Q \cdot e_t)$, we obtain
\[
|F_{S,43}| \leq 2\pi \|f\|_{C(\mathbb{T})} \int_{-1/2}^{1/2} \epsilon^2 \kappa \left[ \frac{|\overline{s}| + c_0 \overline{s}^2}{|R|^3} + \frac{1}{|R|^2} \right] d\overline{s}
\]
\[
\leq c_4 \kappa_{\text{max}} (4c_3 (1 + \epsilon |\log \epsilon| c_Q) + c_2) \|f\|_{C(\mathbb{T})}. \quad (4.125)
\]

For $F_{S,42}$, we again use (4.9) to rewrite the expression as
\[
F_{S,42} = F_{S,42a} + F_{S,42b};
\]
\[
F_{S,42a} := -\int_0^{2\pi} \int_{-1/2}^{1/2} \epsilon^2 (e_t \cdot f)e_t \overline{s} d\overline{s} d\theta
\]
\[
F_{S,42b} := -\int_0^{2\pi} \int_{-1/2}^{1/2} \epsilon^2 (Q \cdot e_\rho)(e_t \cdot f)e_t d\overline{s} d\theta.
\]

Now, noting the $\theta$-independence of $Q(s, \overline{s})$, $f(s + \overline{s})$, and $e_t(s)$, as well as the single copy of $e_\rho(s, \theta) = \cos \theta e_{n_1}(s) + \sin \theta e_{n_2}(s)$ in $F_{S,42b}$, we use Lemma 4.11 to obtain the estimate
\[
|F_{S,42b}| \leq c^3 \epsilon c_Q c_3 \|f\|_{C(\mathbb{T})}.
\]
Next, using Lemma 4.5, we have

\[ |F_{S,42} + 2h_c(s)| \leq \varepsilon 2\pi c_{0,3} \|f\|_{C^1(\mathbb{T})}; \]

\[ h_c(s) := \int_0^{2\pi} e_t(s)(e_t(s) \cdot f(s)) d\theta = 2\pi e_t(s)(e_t(s) \cdot f(s)). \]  

(4.126)

Together, we obtain the estimate

\[ |F_{S,42} + 2h_c(s)| \leq \varepsilon 2(\pi c_{0,3} + \sqrt{\varepsilon 4c_Qc_3}) \|f\|_{C^1(\mathbb{T})}. \]  

(4.127)

Lastly, we use \((R_0 \cdot e_t) = -\bar{s} + \bar{s}^2(Q \cdot e_t)\) and (4.9) to rewrite \(F_{S,41}\) as

\[ F_{S,41} = F_{S,41a} + F_{S,41b} + F_{S,41c} + F_{S,41d}; \]

\[ F_{S,41a} := - \int_0^{2\pi} \int_{1/2}^{1/2} \varepsilon \left( \frac{\bar{s}^2(f \cdot e_{\rho})e_t}{|R|^{3/2}} + 3\bar{s}^2(\frac{e_{\rho} \cdot f)}{|R|^{5/2}} \right) ds \, d\theta \]

\[ F_{S,41b} := \int_0^{2\pi} \int_{1/2}^{1/2} \varepsilon^2 \bar{s}^2(e_t \cdot f) e_t ds \, d\theta \]

\[ F_{S,41c} := \int_0^{2\pi} \int_{1/2}^{1/2} \varepsilon \left( \frac{\bar{s}^2(Q \cdot e_t)(f \cdot e_{\rho})e_t}{|R|^{3/2}} + 3\bar{s}^4(Q \cdot e_{\rho})(e_t \cdot f)e_t \right) ds \, d\theta \]

\[ F_{S,41d} := \int_0^{2\pi} \int_{1/2}^{1/2} \varepsilon \left( \frac{\bar{s}^6(Q \cdot e_{\rho})(Q \cdot f - \bar{s}^2(e_t \cdot f))e_t}{|R|^{3/2}} + 3\bar{s}^2(-\bar{s}(e_t \cdot f) + \varepsilon(e_{\rho} \cdot f))(Q \cdot e_t)e_t \right) ds \, d\theta \]

As in the estimates for \(F_{S,12c}, F_{S,22c},\) and \(F_{S,31c},\) we make use of the lower bound (4.11) and the fact that \(|\bar{s}| \leq \frac{1}{2}\) to obtain

\[ |F_{S,41d}| \leq 6c_Q \pi \|f\|_{C(\mathbb{T})} \int_{-1/2}^{1/2} \frac{\varepsilon^2 \bar{s}^6}{|R|^{5/2}} \left[ c_R^5(c_Q^5 + 2(1 + c_Q)) + \frac{1}{|R|^{5/2}} \frac{2c_Q \varepsilon \bar{s}^4 + 3\varepsilon |\bar{s}|^3 + \varepsilon^2 \bar{s}^2}{|R|^{5/2}} \right] ds \]

where we also used Lemma 4.3 in the last inequality. Next, noting the \(\theta\)-independence of \(Q, f,\) and \(e_t,\) and recalling \(e_{\rho} = \cos \theta e_{n_1}(s) + \sin \theta e_{n_2}(s),\) we use Lemma 4.11 to estimate \(F_{S,41c}:\)

\[ |F_{S,41c}| \leq \varepsilon 3c_Q \pi \|f\|_{C(\mathbb{T})} \left( c_R^5(c_Q^5 + 2(1 + c_Q)) + \frac{1}{|R|^{5/2}} \frac{2c_Q \varepsilon \bar{s}^4 + 3\varepsilon |\bar{s}|^3 + \varepsilon^2 \bar{s}^2}{|R|^{5/2}} \right) \]

Furthermore, using Lemma 4.5, we can estimate \(F_{S,41b}\) as

\[ |F_{S,41b} - 2h_c(s)| \leq \varepsilon 6\pi c_{0,5} \|f\|_{C^1(\mathbb{T})}. \]

Finally, by Lemma 4.4, we have the bound

\[ |F_{S,41a}| \leq \varepsilon (2\pi(c_{1,3} + 3c_{1,5}) \|f\|_{C^1(\mathbb{T})}. \]
Combining the above estimates, we obtain the following estimate for $F_{S,41}$:

$$
|F_{S,41} - 2h_c(s)| \leq \epsilon \left( c_Q 3\pi (c_R^{-5}(c_Q^2 + 2(1 + c_Q)) + c_5(\epsilon \log \epsilon |2c_Q + 4|)
+ \epsilon^{1/2} 2c_Q(\bar{c}_3 + 3\bar{c}_5) + 6\pi c_{0.5} + 2\pi (c_{1.3} + 3c_{1.5})\right) \|f\|_{C^1(\mathcal{T})}. 
$$

(4.128)

Using the estimates (4.125), (4.127), and (4.128), we can bound $F_{S,4}$ as

$$
|F_{S,4}| \leq \epsilon c_{S,4} \|f\|_{C^1(\mathcal{T})},
$$

(4.129)

where the constant $c_{S,4}$ depends only on $c_\Gamma$, $\kappa_{max}$, and $c_\alpha$.

We conclude with an estimate of the doublet component $F_{D,4}$ of $f_{SB}^4(s)$, defined in (4.123). As in previous doublet computations, we write $F_{D,4}$ as

$$
F_{D,4} = F_{D,31} + F_{D,32} + F_{D,33};
$$

$$
F_{D,41} = \int_0^{2\pi} \int_{-1/2}^{1/2} \left[ \frac{R_0 \cdot e_t}{|R|^5} (f \cdot e_\rho) e_t - \frac{5\epsilon t (R \cdot e_\rho)(R \cdot f)(R_0 \cdot e_t)}{|R|^7} \right] d\bar{s} \epsilon d\theta
$$

$$
= F_{D,42} = \int_0^{2\pi} \int_{-1/2}^{1/2} \left[ \frac{R_0 \cdot e_t}{|R|^5} (f \cdot e_\rho) e_t + \frac{(R \cdot e_\rho)(e_t \cdot f)e_t}{|R|^5}
- \frac{5\epsilon t (R \cdot e_\rho)(R \cdot f)(R_0 \cdot e_t)}{|R|^7} \right] d\bar{s} \epsilon^2 \hat{\eta} d\theta.
$$

(4.130)

Just as in the $F_{S,43}$ estimate, a bound for $F_{D,43}$ follows immediately from Lemma 4.3:

$$
|F_{D,43}| \leq 2\pi \|f\|_{C(\mathcal{T}))} \int_{-1/2}^{1/2} \epsilon^2 \hat{\eta} \left[ \frac{6 |\bar{s}| + c_Q \bar{s}^2}{|R|^5} + \frac{1}{|R|^4} \right] d\bar{s}
$$

$$
\leq \epsilon^{-1} 4 \pi \kappa_{max} (6c_5 (1 + \epsilon \log |c_Q| + c_4) \|f\|_{C(\mathcal{T}))}.
$$

(4.131)

To estimate $F_{D,42}$, we first use (4.9) to write

$$
F_{D,42} = F_{D,42a} + F_{D,42b};
$$

$$
F_{D,42a} := \int_0^{2\pi} \int_{-1/2}^{1/2} \epsilon^2 (e_t \cdot f)e_t d\bar{s} d\theta
$$

$$
= F_{D,42b} := \int_0^{2\pi} \int_{-1/2}^{1/2} \epsilon \bar{s}^2 (Q \cdot e_\rho)(e_t \cdot f)e_t d\bar{s} d\theta.
$$

(4.132)

Again by Lemma 4.3, we have that $F_{D,42b}$ satisfies the bound

$$
|F_{D,42b}| \leq 2\pi \|f\|_{C(\mathcal{T}))} \int_{-1/2}^{1/2} c_Q \epsilon \frac{\bar{s}^2}{|R|^5} d\bar{s} \leq \epsilon^{-1} 2\pi c_Q c_5 \|f\|_{C(\mathcal{T}))}.
$$

Furthermore, using Lemma 4.5, we estimate $F_{D,42a}$ as

$$
\left| F_{D,42a} - \epsilon^{-2} \frac{4}{3} h_c(s) \right| \leq \epsilon^{-2} 2\pi c_{0.5} \|f\|_{C^1(\mathcal{T})}.
$$

75
where \( h(s) \) was defined in [4.126]. Putting both of the above estimates together, we obtain

\[
|F_{D,42} - e^{-2 \frac{4}{3} h_e(s)}| \leq e^{-1} 2 \pi (c_Q c_5 + c_{0.5}) \|f\|_{C^1(T)}.
\]  
(4.132)

Finally we estimate \( F_{D,41} \). Noting that \((R_0 \cdot e_t) = -\bar{s} + \bar{s}_2 (Q \cdot e_t)\), we rewrite the expression for \( F_{D,41} \) as

\[
F_{D,41} = F_{D,41a} + F_{D,41b} + F_{D,41c} + F_{D,41d};
\]

\[
\begin{align*}
F_{D,41a} & := - \int_0^{2\pi} \int_{-1/2}^{1/2} \epsilon \left[ \frac{\bar{s}^5 (f \cdot e_t) e_t}{R^5} - 5 \epsilon^2 \bar{s} (e_{t \cdot f}) e_t \right] d\bar{s} \ d\theta \\
F_{D,41b} & := - \int_0^{2\pi} \int_{-1/2}^{1/2} \epsilon \frac{5 \epsilon^2 \bar{s}^2 (e_t \cdot f) e_t}{|R|^7} d\bar{s} \ d\theta \\
F_{D,41c} & := \int_0^{2\pi} \int_{-1/2}^{1/2} \epsilon \left[ \frac{\bar{s}^5 (Q \cdot e_t)(Q \cdot f) e_t - \bar{s}^4 (Q \cdot e_t)(e_t \cdot f) e_t}{|R|^7} ight. \\
& \quad \left. + \frac{\epsilon \bar{s}^3 [(Q \cdot e_t) (e_t \cdot f)] + (Q \cdot f)] e_t}{|R|^7} \right] d\bar{s} \ d\theta \\
F_{D,41d} & := \int_0^{2\pi} \int_{-1/2}^{1/2} \epsilon \left[ \frac{\bar{s}^4 (Q \cdot e_t)(f \cdot e_t) e_t}{|R|^5} - \frac{5 \epsilon^2 e_t (R_0 \cdot e_t)(R \cdot f)(Q \cdot e_t)}{|R|^7} \right] d\bar{s} \ d\theta.
\end{align*}
\]

First we bound \( F_{D,41d} \). By [4.3] we have

\[
|F_{D,41d}| \leq 2 \pi \|f\|_{C(T)} \int_{-1/2}^{1/2} 6 c_Q c_5 \bar{s}^2 d\bar{s} \leq e^{-1} 12 \pi c_Q c_5 \|f\|_{C(T)}.
\]

To bound \( F_{D,41c} \), we again use Lemma [4.3] to show

\[
|F_{D,41c}| \leq 2 \pi \|f\|_{C(T)} \int_{-1/2}^{1/2} 5 c_Q c_7 |\bar{s}|^5 + 4 |\bar{s}|^4 + \epsilon^2 |\bar{s}|^3 d\bar{s} \leq e^{-1} 10 \pi c_7 c_Q (3 + \epsilon c_Q) \|f\|_{C(T)}.
\]

We next estimate \( F_{D,41b} \). Using Lemma [4.5] we have

\[
|F_{D,41b} + e^{-2 \frac{4}{3} h_e(s)}| \leq e^{-1} 10 \pi c_{0.7} \|f\|_{C^1(T)},
\]

with \( h_e(s) \) as defined in [4.126].

Lastly, using Lemma [4.4] we can bound \( F_{D,41a} \) as

\[
|F_{D,41a}| \leq e^{-1} 2 \pi (c_{1.5} + 5 c_{1.7}) \|f\|_{C^1(T)}.
\]

Then, in total, the term \( F_{D,41} \) satisfies the estimate

\[
|F_{D,41} + e^{-2 \frac{4}{3} h_e(s)}| \leq e^{-1} 2 \pi (6 c_Q c_5 + 5 c_Q c_7 (3 + \epsilon c_Q) + 5 c_{0.7} + c_{1.5} + 5 c_{1.7}) \|f\|_{C^1(T)}.
\]  
(4.133)

Combining (4.131), (4.132), and (4.133), we thus obtain the following estimate for \( F_{D,4} \):

\[
\|F_{D,4}\| \leq e^{-1} c_{D,4} \|f\|_{C^1(T)},
\]  
(4.134)

where the constant \( c_{D,4} \) depends only on \( c_T \) and \( \kappa_{\text{max}} \).
Altogether, using the bounds (4.129) and (4.134) in the expression (4.123) for \( f^\text{SB}_4 \), we obtain the estimate
\[
|f^\text{SB}_4| \leq \frac{1}{8\pi} \left( |F_{S,4}| + \frac{3\epsilon^2}{2} |F_{D,4}| \right) \leq \epsilon \left( c_{S,4} + \frac{3}{2} c_{D,4} \right) \|f\|_{C^1(\mathbb{T})},
\]
from which follows Proposition 4.17. \( \square \)

Finally, we sum the estimates for the five force components defined in (4.72), resulting in the following estimate for the total slender body force \( f^\text{SB}(s) \).

**Proposition 4.18.** Let the slender body \( \Sigma_e \) be as in Section 1.1 with \( C^{2,\alpha} \) centerline \( X(s) \). Given \( f \in C^1(\mathbb{T}) \), let \( f^\text{SB}(s) \) be the corresponding slender body approximation, given by (4.56). Then \( f^\text{SB} \) satisfies
\[
|f^\text{SB}(s) - f(s)| \leq c_f \|f\|_{C^1(\mathbb{T})},
\]
where the constant \( c_f \) depends only on \( c_{\Gamma}, \kappa_{\max} \), and \( c_\alpha \).

**Proof.** First, we introduce some notation. Let \( f_1(s) := (f(s) \cdot e_1(s)), f_{n_1}(s) := (f(s) \cdot e_{n_1}(s)), \) and \( f_{n_2}(s) := (f(s) \cdot e_{n_2}(s)) \). Using the expression (4.72) for \( f^\text{SB} \), together with Propositions 4.13, 4.14, 4.15, 4.16, and 4.17, we have
\[
|f^\text{SB}(s) - f(s)| = \left| f^\text{SB}_1(s) - \frac{1}{2} f(s) - \frac{1}{2} (f_1(s)e_1(s) + f_{n_1}(s)e_{n_1}(s) + f_{n_2}(s)e_{n_2}(s)) \right|
\leq \left| f^\text{SB}_1(s) - \frac{1}{2} f(s) \right| + \left| \frac{1}{2} (f_1(s)e_1(s) + f_{n_1}(s)e_{n_1}(s) + f_{n_2}(s)e_{n_2}(s)) \right|
\leq \epsilon (c_{p_0} + c_{f_1} + c_{f_2} + c_{f_3} + c_{f_4}) \|f\|_{C^1(\mathbb{T})}.
\]

\( \square \)

## 5 Error estimate

Using the residual calculations for the surface velocity \( u^\text{SB}|_{\Gamma_e} \) and the total surface force \( f^\text{SB} \), we proceed to prove the error estimate (1.21) in Theorem 1.4.

Let \( u_e = u^\text{SB} - u, p_e = p^\text{SB} - p, \) and \( \sigma_e = -p_e I + 2\mathcal{E}(u_e) = \sigma^\text{SB} - \sigma \), where \( u, p, \) and \( \sigma = -pI + \nabla u + (\nabla u)^T \) correspond to the true solution to (1.18). Then the difference \( u_e \) satisfies
\[
-\Delta u_e + \nabla p_e = 0 \quad \text{in } \Omega_e
\]
\[
\text{div } u_e = 0 \quad \text{in } \Omega_e
\]
\[
\int_0^{2\pi} \sigma_e n \mathcal{J}_e(s, \theta) \, d\theta = f_e(s) \quad \text{on } \Gamma_e \quad (5.1)
\]
\[
u_{e}|_{\Gamma_e} = \bar{u}_e(s) + u^T(s, \theta)
\]
\[
u_e \to 0 \text{ as } |x| \to \infty
\]
where the boundary value \( \bar{u}_e(s) = (u^\text{SB} - u^T)|_{\Gamma_e(s)} - u|_{\Gamma_e(s)} \) is unknown (since \( u(s) \) is unknown) but independent of \( \theta \). Note that \( f_e(s) = f^\text{SB} - f \) and \( u^T(s, \theta) = u^\text{SB}(\epsilon, \theta, s) - \frac{1}{2\pi} \int_0^{2\pi} u^\text{SB}(\epsilon, \varphi, s) \, d\varphi \) are both completely known functions along \( \Gamma_e \).
For arbitrary \( \mathbf{w} \in D^{1,2}(\Omega_\epsilon) \), we can write (5.1) in variational form. The error \( \mathbf{u}_e \) satisfies
\[
\int_{\Omega_\epsilon} \left( 2 \mathcal{E}(\mathbf{u}_e) : \mathcal{E}(\mathbf{w}) - p_e \text{div} \mathbf{w} \right) \, dx = \int_{\Gamma_\epsilon} \sigma_\epsilon \mathbf{n} \mathbf{w} \, dS. 
\tag{5.2}
\]

Now, unless \( \mathbf{w} \in \mathcal{A}_\epsilon = \{ \mathbf{w} \in D^{1,2}(\Omega_\epsilon) : \mathbf{w}|_{\Gamma_\epsilon} = \mathbf{w}(s) \} \), i.e. \( \mathbf{w} \) additionally satisfies the \( \theta \)-independence condition on the slender body surface \( \Gamma_\epsilon \), we cannot make use of the known expression \( f_e(s) \) for the error in the total force. Note in particular that the function \( \mathbf{u}_e \) itself does not belong to the set \( \mathcal{A}_\epsilon \).

However, since \( (\mathbf{u}_e, p_e) \) satisfies (5.2), we can exactly follow the proof of the pressure estimate (2.15) to show that the pressure error \( p_e \) satisfies
\[
\|p_e\|_{L^2(\Omega_\epsilon)} \leq c_P \|\mathcal{E}(\mathbf{u}_e)\|_{L^2(\Omega_\epsilon)} 
\tag{5.3}
\]
where \( c_P \) is independent of the slender body radius \( \epsilon \).

To derive a \( D^{1,2}(\Omega_\epsilon) \) bound for \( \mathbf{u}_e \), we use (5.2) with a very specific choice of \( \mathbf{w} \). In particular, we take
\[
\mathbf{w} = \tilde{\mathbf{u}}_e := \mathbf{u}_e - \tilde{\mathbf{v}},
\tag{5.4}
\]
where \( \tilde{\mathbf{v}} \in D^{1,2}(\Omega_\epsilon) \) with \( \tilde{\mathbf{v}}|_{\Gamma_\epsilon} = \mathbf{u}^r(s, \theta) \). We explicitly construct such a \( \tilde{\mathbf{v}} \) in Section 5.1 that we can bound in terms of \( f(s) \), the true prescribed force.

We then have that \( \tilde{\mathbf{u}}_e|_{\Gamma_\epsilon} = \bar{\mathbf{u}}_e(s) \), where \( \bar{\mathbf{u}}_e(s) \) is unknown but independent of \( \theta \), so \( \bar{\mathbf{u}}_e \in \mathcal{A}_\epsilon \). Thus, using \( \bar{\mathbf{u}}_e \) in place of \( \mathbf{w} \) in (5.2), we obtain
\[
\int_{\Omega_\epsilon} \left( 2 \mathcal{E}(\mathbf{u}_e) : \mathcal{E}(\bar{\mathbf{u}}_e) - p_e \text{div} \bar{\mathbf{u}}_e \right) \, dx = \int_{\Gamma} f_e(s) \bar{\mathbf{u}}_e(s) \, ds. 
\tag{5.5}
\]

From (5.5) we will derive a \( D^{1,2}(\Omega_\epsilon) \) estimate for \( \mathbf{u}_e \) in terms of the prescribed force \( f(s) \).

### 5.1 Construction of \( \tilde{\mathbf{v}} \)

In order to use (5.5) to obtain an estimate for \( \mathbf{u}_e \) in terms of \( f(s) \), we must construct the function \( \tilde{\mathbf{v}} \in D^{1,2}(\Omega_\epsilon) \) with \( \tilde{\mathbf{v}}|_{\Gamma_\epsilon} = \mathbf{u}^r(s, \theta) \). Since \( \mathbf{u}^r \in H^1(\Gamma_\epsilon) \), it suffices to extend \( \mathbf{u}^r \) radially from \( \Gamma_\epsilon \) into the interior of \( \Omega_\epsilon \). We first define
\[
\mathbf{u}^\text{SB}_{\text{ext}}(\rho, \theta, s) = \begin{cases} 
\mathbf{u}^r(\theta, s) & \text{if } \rho < 4\epsilon, \\
0 & \text{otherwise}.
\end{cases}
\]

Let \( \phi(\rho) \) be a smooth cutoff function equal to 1 for \( \rho < 2\epsilon \) and equal to 0 for \( \rho > 4\epsilon \) with smooth decay between. We require this decay to satisfy
\[
\left| \frac{\partial \phi}{\partial \rho} \right| \leq \frac{c_\phi}{\epsilon} \tag{5.6}
\]
for some constant \( c_\phi > 0 \).

We define
\[
\tilde{\mathbf{v}}(\rho, \theta, s) = \phi(\rho) \mathbf{u}^\text{SB}_{\text{ext}}(\rho, \theta, s). \tag{5.7}
\]
Note that $\bar{v}(\rho, \theta, s)$ is supported in the region
\[
\mathcal{O}_e := \{ s e_t(s) + \rho e_\rho(s, \theta) + \theta e_\theta(s, \theta) : s \in \mathbb{T}, \epsilon \leq \rho \leq 4\epsilon, 0 \leq \theta < 2\pi \}
\] (5.8)
with $|\mathcal{O}_e| = c_\Omega^2 \epsilon^2$.

Now, obtaining a $D^{1,2}(\Omega_e)$ estimate for $u_e$ from [5.5] will require an $L^2(\Omega_e)$ bound for $\nabla \bar{v}$, so we consider
\[
\nabla \bar{v} = \phi \nabla u_{ext}^{SB} + (\nabla \phi)(u_{ext}^{SB})^T.
\] (5.9)
We have
\[
\phi \nabla u_{ext}^{SB} = \frac{\phi}{\epsilon} \frac{\partial u^T}{\partial \theta} e_\theta^T + \frac{\phi}{1 - \rho \kappa}(\frac{\partial u^T}{\partial s} - \kappa_3 \frac{\partial u^T}{\partial \theta}) e_t^T
\]
and
\[
(\nabla \phi)(u_{ext}^{SB})^T = \frac{\partial \phi}{\partial \rho} e_\rho (u^T)^T.
\]

Then, using Proposition 4.8 along with [5.6], (6.8), and Lemma 1.1, we have
\[
\| \nabla \bar{v} \|_{L^2(\Omega_e)} \leq \| \nabla \bar{v} \|_{L^\infty(\mathcal{O}_e)} \sqrt{|\mathcal{O}_e|}
\leq c_{\Omega} \varepsilon \left( \frac{1}{\varepsilon} \left\| \frac{\partial u^T}{\partial \theta} \right\|_{L^\infty(\Gamma_e)} + \left\| \frac{1}{1 - \rho \kappa} \left( \frac{\partial u^T}{\partial s} - \kappa_3 \frac{\partial u^T}{\partial \theta} \right) \right\|_{L^\infty(\Gamma_e)} + \frac{c_\phi}{\varepsilon} \left\| u^T \right\|_{L^\infty(\Gamma_e)} \right)
\leq \varepsilon \left( \log \varepsilon \| c_{\rho \kappa} c_\theta + c_{\rho \kappa} \theta \| + 4(c_{\rho \kappa} + \pi c_{\rho \kappa}) \| f \|_{C^1(\mathbb{T})} \right) \leq c_v \| f \|_{C^1(\mathbb{T})},
\] (5.10)
where $c_v$ depends only on $\kappa_{\max}$ and $c_T$.

### 5.2 Estimating the error

We now use [5.5] to obtain a $D^{1,2}(\Omega_e)$ bound for the error $u_e$. Recalling that $\bar{u}_e = u_e - \bar{v}$ and thus $\text{div} \bar{u}_e = -\text{div} \bar{v}$, we rewrite [5.5] as
\[
\int_{\Omega_e} 2 \mathcal{E}(u_e) \, dx = \int_{\Omega_e} \left( 2 \mathcal{E}(u_e) : \mathcal{E}(\bar{v}) - p_e \text{div} \bar{v} \right) \, dx + \int_{\mathbb{T}} f_e(s) \bar{u}_e(s) \, ds
\leq \left| \int_{\Omega_e} 2 \mathcal{E}(u_e) : \mathcal{E}(\bar{v}) \, dx \right| + \left| \int_{\Omega_e} p_e \text{div} \bar{v} \, dx \right| + \left| \int_{\mathbb{T}} f_e(s) \bar{u}_e(s) \, ds \right|.
\] (5.11)

Using Cauchy Schwarz, the first term on the right hand side of (5.11) satisfies
\[
\left| \int_{\Omega_e} 2 \mathcal{E}(u_e) : \mathcal{E}(\bar{v}) \, dx \right| \leq 2 \mathcal{E}(u_e) \mathcal{E}(\bar{v}) \| \nabla \bar{v} \|_{L^2(\Omega_e)}
\leq \eta \| \mathcal{E}(u_e) \|_{L^2(\Omega_e)}^2 + \frac{1}{\eta} \| \mathcal{E}(\bar{v}) \|_{L^2(\Omega_e)}^2
\leq \eta \| \mathcal{E}(u_e) \|_{L^2(\Omega_e)}^2 + \frac{2}{\eta} \| \nabla \bar{v} \|_{L^2(\Omega_e)}^2
\]
for any $\eta \in \mathbb{R}_+.$

By [5.3] and Cauchy Schwarz, the second term on the right hand side of (5.11) satisfies
\[
\left| \int_{\Omega_e} p_e \text{div} \bar{v} \, dx \right| \leq \| p_e \|_{L^2(\Omega_e)} \| \nabla \bar{v} \|_{L^2(\Omega_e)}
\]
\[ \leq c_P \| \mathcal{E}(u_e) \|_{L^2(\Omega_e)} \| \nabla \tilde{v} \|_{L^2(\Omega_e)} \]
\[ \leq \eta \| \mathcal{E}(u_e) \|_{L^2(\Omega_e)} + \frac{c_E^2}{4\eta} \| \nabla \tilde{v} \|_{L^2(\Omega_e)}^2. \]

Finally, the third term on the right hand side of (5.11) can be estimated using the trace inequality (3.1) on the admissible set \( \mathcal{A} \), the Korn inequality (3.5), and Cauchy Schwarz. We have

\[ \left| \int_T f_e(s) \tilde{u}_e(s) \, ds \right| \leq \| f_e \|_{L^2(T)} \| \tilde{u}_e \|_{L^2(T)} \leq c_T \| \nabla \tilde{u}_e \|_{L^2(\Omega_e)} \| f_e \|_{L^2(T)} \leq c_T c_K \| \mathcal{E}(\tilde{u}_e) \|_{L^2(\Omega_e)} \| f_e \|_{L^2(T)} \leq \eta \| \mathcal{E}(u_e) \|_{L^2(\Omega_e)}^2 + \frac{c_T c_K^2}{4\eta} \| f_e \|_{L^2(T)}^2 \]

again for any \( \eta \in \mathbb{R}_+ \).

Taking \( \eta = \frac{1}{3} \), we obtain the following estimate from (5.11):

\[ \| \mathcal{E}(u_e) \|_{L^2(\Omega_e)}^2 \leq \frac{3c_T^2 c_K^2}{4} \| f_e \|_{L^2(T)}^2 + \left( \frac{20}{3} + \frac{3c_T^2}{4} \right) \| \nabla \tilde{v} \|_{L^2(\Omega_e)}^2. \] (5.12)

Then using the Korn inequality (3.5), we have

\[ \| \nabla u_e \|_{L^2(\Omega_e)}^2 \leq \frac{3c_T^2 c_K^4}{4} \| f_e \|_{L^2(T)}^2 + \left( \frac{20c_T^2 c_K}{3} + \frac{3c_T^2 c_K^2}{4} \right) \| \nabla \tilde{v} \|_{L^2(\Omega_e)}^2. \] (5.13)

Recall that the Korn constant \( c_K \) (3.3) and the pressure constant \( c_P \) (2.15) are both independent of \( \epsilon \), while the trace constant \( c_T \) (3.4) satisfies \( c_T = c_\kappa | \log \epsilon |^{1/2} \). Also, from (5.10) and Proposition 4.18 we have

\[ \| \nabla \tilde{v} \|_{L^2(\Omega_e)} \leq | \log \epsilon | c_\kappa \| f \|_{C^1(T)} \]
\[ \| f_e \|_{L^2(T)} \leq c_f \| f \|_{C^1(T)}. \]

Therefore we have

\[ \| u_e \|_{D^{1,2}(\Omega_e)} \leq \epsilon (| \log \epsilon |^{1/2} + | \log \epsilon |) c_\kappa \| f \|_{C^1(T)} \]
\[ \leq \epsilon | \log \epsilon | c_\kappa \| f \|_{C^1(T)} \] (5.14)

where the constant \( c_\kappa \) depends only on the shape of the fiber centerline through \( \kappa_{\text{max}}, c_\Gamma \), and \( c_\alpha \). Since the pressure error \( p_e \) satisfies (5.3), we also obtain

\[ \| u_e \|_{D^{1,2}(\Omega_e)} + \| p_e \|_{L^2(\Omega_e)} \leq \epsilon | \log \epsilon | c_\kappa \| f \|_{C^1(T)}, \] (5.15)

where again, by Lemma 3.9, \( c_\kappa \) depends only on \( \kappa_{\text{max}}, c_\Gamma \), and \( c_\alpha \).

Furthermore, using the \( D^{1,2}(\Omega_e) \) bound on the error \( u_e = u^{SB} - u \) throughout the fluid domain \( \Omega_e \), we can obtain an \( L^2 \) bound for the trace of the error \( \text{Tr}(u_e) \) along the slender body surface \( \Gamma_e \), scaled by the square root of the slender body surface area \( | \Gamma_e |^{1/2} \). We scale the trace on \( \Gamma_e \) by \( | \Gamma_e |^{-1/2} = \frac{1}{\sqrt{\epsilon}} \) to distinguish the actual error from the fact that the surface area vanishes as \( \epsilon \to 0 \), so the \( L^2 \) trace on \( \Gamma_e \) always scales like \( \sqrt{\epsilon} \). We first write

\[ \| \text{Tr}(u_e) \|_{L^2(\Gamma_e)} \leq \| \tilde{u}_e(s) \|_{L^2(\Gamma_e)} + \| u' \|_{L^2(\Gamma_e)}. \]
Then, using the estimate \([4.51]\) for \(\mathbf{u}'\), we have
\[
\|\mathbf{u}'\|_{L^2(\Gamma_c)}^2 = \left( \int_\gamma \int_0^{2\pi} |\mathbf{u}'(s, \theta)|^2 \mathcal{J}_\epsilon(s, \theta) \, d\theta ds \right)^{1/2}
\leq 2\sqrt{\pi\epsilon} \|\mathbf{u}'\|_{L^\infty(\Gamma_c)} \leq \epsilon^{3/2} |\log \epsilon| c_1 \|\mathbf{f}\|_{C^1(\Gamma)}.
\]
Moreover, using the trace inequality \([3.1]\) and \([5.15]\), we have
\[
\|\mathbf{u}_e(s)\|_{L^2(\Gamma_c)} \leq \sqrt{2\pi\epsilon} \|\mathbf{u}_e\|_{L^2(\Omega)} \leq \sqrt{2\pi\epsilon} c_T \left( \|\nabla \mathbf{u}_e\|_{L^2(\Omega_s)} + \|\nabla \mathbf{v}\|_{L^2(\Omega_s)} \right)
\leq c_\epsilon (\epsilon |\log \epsilon|)^{1/2} \left( \|\mathbf{u}_e\|_{H^1,2(\Omega)} + \|\nabla \mathbf{v}\|_{L^2(\Omega)} \right)
\leq (\epsilon |\log \epsilon|)^{3/2} c_\epsilon \|\mathbf{f}\|_{C^1(\Gamma)}
\]
where the constant \(c_\epsilon\) still depends only on \(\kappa_{\max}\), \(c_T\), and \(c_\alpha\).

In total, scaling by \(|\Gamma_c|^{-1/2} = \frac{1}{\sqrt{t}}\), we obtain
\[
\frac{1}{|\Gamma_c|^{1/2}} \|\text{Tr}(\mathbf{u}_e)\|_{L^2(\Gamma_c)} \leq \epsilon |\log \epsilon|^{3/2} c_\epsilon \|\mathbf{f}\|_{C^1(\Gamma)}.
\]

6 Appendix

6.1 Proof of Lemma \(1.1\)

Here we show the existence of a \(C^1\) orthonormal frame along \(\Gamma_0\) satisfying the ODE \([1.3]\).

**Proof.** Let
\[
\Gamma_t = \{ \mathbf{p} \in S^2 : \mathbf{p} = \mathbf{e}_t(s) \text{ or } \mathbf{p} = -\mathbf{e}_t(s), \ s \in [0, T] \},
\]
where \(S^2 \subset \mathbb{R}^3\) is the two-sphere. The above is the trajectory of the Gauss map and its reflection through the origin. Note that \(\Gamma_t\) is a proper subset of \(S^2\).

Pick a point \(\mathbf{q} \in S^2 \setminus \Gamma_t\). Since \(\mathbf{e}_t(s)\) is a continuous function and \(\mathbf{q}\) is never equal to \(\pm \mathbf{e}_t(s)\),
\[
\max_{s \in [0, T]} |\mathbf{e}_t(s) \cdot \mathbf{q}| < 1.
\]

Let
\[
\mathbf{\bar{e}}_1(s) = \frac{\mathbf{q} - (\mathbf{e}_t(s) \cdot \mathbf{q})\mathbf{e}_t(s)}{|\mathbf{q} - (\mathbf{e}_t(s) \cdot \mathbf{q})\mathbf{e}_t(s)|}, \quad \mathbf{\bar{e}}_2(s) = \mathbf{e}_t(s) \times \mathbf{\bar{e}}_1(s).
\]

By \([6.1]\), the denominator in the expression for \(\mathbf{\bar{e}}_1(s)\) never vanishes. Thus, \(\mathbf{e}_t, \mathbf{\bar{e}}_1,\) and \(\mathbf{\bar{e}}_2\) define an orthonormal frame with \(C^1\) dependence on \(s\).

To prove the statement \([1.4]\), let
\[
\frac{d}{ds} \begin{pmatrix} \mathbf{e}_t \\ \mathbf{\bar{e}}_1 \\ \mathbf{\bar{e}}_2 \end{pmatrix} = \begin{pmatrix} 0 & \bar{\kappa}_1 & \bar{\kappa}_2 \\ -\bar{\kappa}_1 & 0 & \bar{\kappa}_3 \\ -\bar{\kappa}_2 & -\bar{\kappa}_3 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_t \\ \mathbf{\bar{e}}_1 \\ \mathbf{\bar{e}}_2 \end{pmatrix}
\]
be the ODE satisfied by the orthonormal frame. Take
\[
\bar{\kappa}_3 = \int_0^1 \bar{\kappa}_3(s) \, ds.
\]
and let \( k \) be the closest integer to \( \bar{\kappa}_3/2\pi \). Define

\[
\kappa_3 = \bar{\kappa}_3 - 2\pi k
\]

and let

\[
\varphi(s) = \int_0^s (\bar{\kappa}_3(\tau) - \kappa_3) \, d\tau.
\]

Note that, by construction,

\[
|\kappa_3| \leq \pi.
\]

Define

\[
\begin{pmatrix}
\mathbf{e}_{n_1}(s) \\
\mathbf{e}_{n_2}(s)
\end{pmatrix} = \begin{pmatrix}
\cos \varphi(s) & -\sin \varphi(s) \\
\sin \varphi(s) & \cos \varphi(s)
\end{pmatrix} \begin{pmatrix}
\mathbf{e}_1(s) \\
\mathbf{e}_2(s)
\end{pmatrix}.
\]

Since \( \varphi(1) = 2\pi k \), \( \mathbf{e}_{n_1}(s) \) and \( \mathbf{e}_{n_2}(s) \) are both in \( C^1(\mathbb{T}) \). It is also clear that \( \mathbf{e}_t(s), \mathbf{e}_{n_1}(s) \) and \( \mathbf{e}_{n_2}(s) \) define an orthonormal basis. A straightforward calculation shows that \( \mathbf{e}(s,\theta) = \cos \theta \mathbf{e}_{n_1}(s) + \sin \theta \mathbf{e}_{n_2}(s) \).

Then any function \( g \) defined in \( \mathcal{O} \) can be written

\[
g(\rho,\theta,s) = g(\mathbf{x}) = g(X(s)+\rho \mathbf{e}_\rho(s,\theta)).
\]

Let \( \nabla_M = e_\rho \frac{\partial}{\partial \rho} + e_\theta \frac{\partial}{\partial \theta} + e_t \frac{\partial}{\partial s} \) denote the gradient with respect to the cylindrical moving frame basis vectors \( \mathbf{e}_\rho(s,\theta), \mathbf{e}_\theta(s,\theta) = -\sin \theta \mathbf{e}_{n_1}(s) + \cos \theta \mathbf{e}_{n_2}(s), \mathbf{e}_t(s) \). We then have

\[
\nabla_M g(\rho,\theta,s) = (\nabla_M \mathbf{x}) \nabla_x g(\mathbf{x}) = \left[ e_\rho e_\rho^T + e_\theta e_\theta^T + (1 - \rho \bar{\kappa}) e_t e_t^T + \kappa_3 \rho e_t e_\theta^T \right] \nabla_x g(\mathbf{x}),
\]

where

\[
\bar{\kappa}(\theta,s) := \kappa_1(s) \cos \theta + \kappa_2(s) \sin \theta,
\]

and therefore

\[
\nabla_x g(\mathbf{x}) = \left[ e_\rho e_\rho^T + e_\theta e_\theta^T + \frac{1}{1 - \rho \bar{\kappa}} (e_t e_t^T - \kappa_3 \rho e_t e_\theta^T) \right] \nabla_M g(\rho,\theta,s) = A_s \nabla_M g.
\]

Here the matrix

\[
A_s := (\nabla_M \mathbf{x})^{-1} = e_\rho e_\rho^T + e_\theta e_\theta^T + \frac{1}{1 - \rho \bar{\kappa}} (e_t e_t^T - \kappa_3 \rho e_t e_\theta^T)
\]

\[\text{(6.7)}\]
is defined with respect to the basis vectors $e_\rho(s,\theta)$, $e_\theta(s,\theta)$, $e_t(s)$ at the point $s$ on the fiber centerline. By definition of $\kappa_{\text{max}}$ [1.6], within the region $\mathcal{O}$ we have

$$|1 - \rho \hat{\kappa}| \geq 1 - \rho|\kappa|(\cos \theta + \sin \theta)$$

$$ \geq 1 - \frac{1}{2\kappa_{\text{max}}} |\kappa| \sqrt{2} \geq 1 - \frac{\sqrt{2}}{2} \geq \frac{1}{4}. $$

Therefore

$$ \|A_s\|_{L^\infty(\mathcal{O})} \leq c_\kappa $$

for $c_\kappa$ independent of the slender body radius $\epsilon$.

For the higher regularity proof, we will require that the fiber centerline $X(s)$ is at least $C^4$, as we need bounds on both the first and second derivatives of the moving frame coefficients $\kappa_1(s)$ and $\kappa_2(s)$ [1.3]. Before we introduce the following proposition concerning tangential translation estimates, we recall the bounds [2.28]:

$$ m_{\kappa,1} := \max_{s \in \mathbb{T}^1} (|\kappa_1'(s)| + |\kappa_2'(s)|), \quad m_{\kappa,2} := \max_{s \in \mathbb{T}^1} (|\kappa_1''(s)| + |\kappa_2''(s)|). $$

We now show the following:

**Proposition 6.1.** (Estimates for translation operator Jacobian) We have

$$ \|\nabla x \tau^\theta_h - I\|_{L^\infty(\Omega)} = \|\nabla x \tau^\theta_h - I\|_{L^\infty(\mathcal{O})} \leq |h|c_{\theta,a} $$

(6.9)

$$ \|\det(\nabla x \tau^\theta_h) - 1\|_{L^\infty(\Omega)} = \|\det(\nabla x \tau^\theta_h) - 1\|_{L^\infty(\mathcal{O})} \leq |h|c_{\theta,b} $$

(6.10)

$$ \|\nabla x \det(\nabla x \tau^\theta_h)\|_{L^\infty(\Omega)} = \|\nabla x \det(\nabla x \tau^\theta_h)\|_{L^\infty(\mathcal{O})} \leq |h|c_{\theta,c}, $$

(6.11)

where $c_{\theta,a}$ and $c_{\theta,b}$ depend only on $\kappa_{\text{max}}$ and $c_\Gamma$, while $c_{\theta,c}$ also depends on $m_{\kappa,1}$. Also,

$$ \|\nabla x \tau^s_h - I\|_{L^\infty(\Omega)} = \|\nabla x \tau^s_h - I\|_{L^\infty(\mathcal{O})} \leq |h|c_{s,a} $$

(6.12)

$$ \|\det(\nabla x \tau^s_h) - 1\|_{L^\infty(\Omega)} = \|\det(\nabla x \tau^s_h) - 1\|_{L^\infty(\mathcal{O})} \leq |h|c_{s,b} $$

(6.13)

$$ \|\nabla x \det(\nabla x \tau^s_h)\|_{L^\infty(\Omega)} = \|\nabla x \det(\nabla x \tau^s_h)\|_{L^\infty(\mathcal{O})} \leq |h|c_{s,c}, $$

(6.14)

where $c_{s,a}$ and $c_{s,b}$ depend only on $\kappa_{\text{max}}$, $c_\Gamma$, and $m_{\kappa,1}$, while $c_{s,c}$ also depends on $m_{\kappa,2}$.

To prove Proposition 6.1, we will need $C^4$ regularity of the slender body centerline $X(s)$.

**Proof.** To show the inequalities (6.9) - (6.14), we follow the approach used in [6]. Defining the $C^2$ vector fields

$$ \Theta^\theta(x(\rho, \theta, s)) = \phi(\rho)(2\pi \rho e_\theta(s, \theta)), $$

$$ \Theta^s(x(\rho, \theta, s)) = \phi(\rho)((1 - \rho \hat{\kappa}) e_t(s) + \rho \kappa_3 e_\theta(s, \theta)), $$

we use the characterization of the tangential translation operators $\tau^\theta_h$ and $\tau^s_h$ as solutions to the ODE

$$ \frac{d}{dh} \tau^j_h(x) = \Theta^j(\tau^j_h(x)), \quad j = \theta, s $$

$$ \tau^0_0(x) = x. $$

Note that for the vector field $\Theta^s$ to be $C^2$, we need the coefficients $\kappa_j(s) \in C^2(\mathbb{T})$, $j = 1, 2$, and therefore we need the fiber centerline $X(s) \in C^4(\mathbb{T})$.  

83
We then have that $\nabla_x \tau_h(x)$ satisfies the ODE

$$\frac{d}{dh}(\nabla_x \tau_j^h(x)) = (\nabla_x \Theta^j)|_{\tau_j^h(x)}(\nabla_x \tau_j^h(x)), \quad j = \theta, s,$$

$$\nabla_x \tau_0^j(x) = I.$$

Within the region $O$, we have

$$|\nabla_x \Theta^\rho| = |A_s \nabla_M \Theta^\rho| \leq c_s|\nabla_M(\rho e_\theta(s, \theta))|, \quad \text{by} \quad (6.8),$$

$$= c_s|e_\rho - e_\theta + e_t((\kappa_1 \sin \theta - \kappa_2 \cos \theta)e_t - \kappa_3 e_\rho)| \leq c_{s, \beta},$$

where $c_{s, \beta}$ depends on $\kappa_{\text{max}}$ and $c_T$ but not on $\epsilon$, and

$$|\nabla_x \Theta^s| = |A_s \nabla_M \Theta^s|$$

$$\leq c_s|e_\rho - e_\theta + e_t((\kappa_1 \sin \theta - \kappa_2 \cos \theta)e_t - \kappa_3 e_\rho)|$$

$$\leq c_{s, \beta},$$

where $c_{s, \beta}$ depends only on $\kappa_{\text{max}}$, $c_T$, and $m_{s, T}$. Thus the difference $|\nabla_x \tau_j^h(x) - I|$ satisfies the differential inequality

$$\left|\frac{d}{dh}(\nabla_x \tau_j^h(x) - I)\right| \leq c_{j, \beta}|\nabla_x \tau_j^h(x) - I| + c_{j, \beta}, \quad j = \theta, s$$

$$|\nabla_x \tau_0^j(x) - I| = 0.$$

Using a Grönwall inequality, since $|h| < 1$, we obtain

$$|\nabla_x \tau_j^h(x) - I| \leq \int_0^h c_{j, \beta} dh' e^{c_{j, \beta}|h|} \leq c_{j, \beta}|h|,$$

giving (6.9) and (6.12).

Similarly, by [6], the Jacobian determinant of $\tau_j^h, j = \theta, s$, satisfies the ODE

$$\frac{d}{dh} \det \nabla_x \tau_j^h(x) = \text{div} \Theta^j|_{\tau_j^h(x)} \det \nabla_x \tau_j^h(x),$$

$$\det \nabla_x \tau_0^j(x) = 1.$$

Applying $\nabla_x$, we have

$$\frac{d}{dh} \nabla_x \det \nabla_x \tau_j^h(x) = \text{div} \Theta^j|_{\tau_j^h(x)} \nabla_x \det \nabla_x \tau_j^h(x)$$

$$+ \nabla_x \text{div} \Theta^j|_{\tau_j^h(x)} \cdot \nabla_x \tau_j^h(x) \det \nabla_x \tau_j^h(x),$$

$$\nabla_x \det \nabla_x \tau_0^j(x) = 0.$$

Now, for any vector field $\Theta$ defined in the region $O$, $\text{div} \Theta$ has the form

$$\text{div} \Theta = \frac{1}{1 - \rho \hat{\kappa}} \left( \frac{\partial}{\partial \rho} + \frac{\partial}{\partial \theta} + \frac{\partial}{\partial s} \right),$$
where $\Theta_\rho = \Theta \cdot e_\rho$, $\Theta_\theta = \Theta \cdot e_\theta$, $\Theta_s = \Theta \cdot e_t$.

Then
\[
\text{div } \Theta^\theta = \text{div}(\phi(\rho)2\pi \rho e_\theta) = \phi(\rho)2\pi \frac{\rho(k_1 \sin \theta - k_2 \cos \theta)}{1 - \rho \kappa},
\]
\[
\text{div } \Theta^s = \text{div}(\phi(\rho)((1 - \rho \kappa)e_t + \rho \kappa_3 e_\theta)) = \phi(\rho)\frac{\rho((k_1 \kappa_3 - k'_3) \sin \theta - (k_2 \kappa_3 + k'_1 \kappa) \cos \theta)}{1 - \rho \kappa},
\]

Note that, using (6.8), we have
\[
|\nabla_x(\frac{1}{1 - \rho \kappa(\theta, s)})| = \left| A_{s,\theta} \frac{\rho((k_1 \cos \theta + k_2 \sin \theta)e_\rho + (-k_1 \sin \theta + k_2 \cos \theta)e_\theta + \rho(k'_1 \cos \theta + k'_2 \sin \theta) + \kappa_3 \rho(k_1 \sin \theta - k_2 \cos \theta)e_t}{1 - \rho \kappa}\right| \leq c_{\kappa,1},
\]

where $c_{\kappa,1}$ depends on $\kappa_{\text{max}}$, $c_\Gamma$, and $m_{\kappa,1}$.

Furthermore, the cutoff function $\phi(\rho)$ by definition (see (2.24)) satisfies
\[
|\nabla_x \phi(\rho)| = |\phi'(\rho)e_\rho| \leq c_\kappa.
\]

Therefore, within the region $\mathcal{O}$, we can bound
\[
|\nabla_x \text{div } \Theta^\theta| \leq c_{\theta,\gamma}, \quad |\nabla_x \text{div } \Theta^s| \leq c_{s,\gamma},
\]

where the constant $c_{\theta,\gamma}$ depends only on $\kappa_{\text{max}}$, $c_\Gamma$, and $m_{\kappa,1}$, while $c_{s,\gamma}$ also depends on $m_{\kappa,2}$.

Thus we have that $\det \nabla_x \tau^j_h(x)$, $j = s, \theta$, satisfies the following differential inequalities:
\[
\left| \frac{d}{dh}(\det \nabla_x \tau^j_h(x) - 1) \right| \leq c_{j,\beta} |\det \nabla_x \tau^j_h(x) - 1| + c_{j,\beta},
\]
\[
|\det \nabla_x \tau^j_0(x) - 1| = 0,
\]

and
\[
\left| \frac{d}{dh} \nabla_x \det \nabla_x \tau^j_h(x) \right| \leq c_{j,\gamma} |\nabla_x \det \nabla_x \tau^j_h(x)| + c_{j,\gamma} |\nabla_x \tau^j_h(x) \det \nabla_x(\tau^j_h(x))|,
\]
\[
\nabla_x \det \nabla_x \tau^j_0(x) = 0.
\]

Again using a Grönwall inequality, from the first differential expression we obtain (6.10) and (6.13), and from the second we obtain (6.11) and (6.14). \hfill \Box

Using Proposition 6.1, we can derive the estimates for the commutators given in the following proposition. The proof of these statements follows ([6], Proposition III.3.19), relying on the estimates (6.9) - (6.14) for our specific tangential translation operators.

**Proposition 6.2.** (Tangential translation estimates)
The tangential translation operators $\tau^j_h$, $j = \theta, s$, both satisfy the following properties:

1. For $g \in H^k(\mathcal{O})$, $k = 0$ or $k = -1$, we have
\[
\sup_{0 < h < 1} \|\tau^j_h g\|_{H^k(\mathcal{O})} \leq c_{j,k} \|g\|_{H^k(\mathcal{O})}.
\]
2. For any \( g_1, g_2 \in L^2(\mathcal{O}) \), we have
\[
\left| \int_{\mathcal{O}} \{g_1, g_2\}_h^\theta dx \right| \leq c_{j,0} |h| \|g_1\|_{L^2(\mathcal{O})} \|g_2\|_{L^2(\mathcal{O})}.
\] (6.17)

Furthermore, for \( g_1 \in L^2(\mathcal{T}) \) and \( g_2 \in \mathcal{A}_\epsilon \), we have that the commutator along the fiber centerline satisfies
\[
\int_T \{g_1, g_2\}^{\theta}_h ds = 0,
\] (6.18)
and
\[
\int_T \{g_1, g_2\}^{s}_h ds \leq c_{s,0} |h| \|g_1\|_{L^2(\mathcal{T})} \|g_2\|_{L^2(\mathcal{T})}.
\] (6.19)

3. For \( g \in H^k(\mathcal{O}) \), \( k = 0 \) or \( k = -1 \), we have
\[
\sup_{0 < h < 1} \| [\nabla, \tau^\theta_h] g \|_{H^k(\mathcal{O})} \leq c_{j,k} |h| \|\nabla g\|_{H^k(\mathcal{O})}.
\] (6.20)

4. For any \( g \in H^k(\mathcal{O}) \), \( k = 0 \) or \( k = -1 \), we have
\[
\|g\|_{T,H^{k+1}(\mathcal{O})} \leq c_{s,k} \|g\|_{H^{k+1}(\mathcal{O})}.
\] (6.21)

In each estimate, the constants \( c_{\theta,0} \) depend only on \( \kappa_{\max} \) and \( \tau_\Gamma \), the constants \( c_{\theta,-1} \) and \( c_{s,0} \) depend on \( \kappa_{\max}, \tau_\Gamma, \) and \( m_{\kappa,1} \), and the constants \( c_{s,-1} \) depend on \( \kappa_{\max}, \tau_\Gamma, m_{\kappa,1}, \) and \( m_{\kappa,2} \).

**Proof.** 1. For \( k = 0 \), using (6.10), we have
\[
\int_{\mathcal{O}} g^2(\tau^\theta_h(x)) dx = \int_{\mathcal{O}} g^2(x) \left| \det(\nabla_x \tau^\theta_h(x)) \right| dx
\leq (1 + |h|c_{\theta,b}) \|g\|_{L^2(\mathcal{O})}^2.
\]
Using (6.13), a similar calculation holds for \( \tau^s_h \).

For the \( H^{-1} \) case, we proceed by duality, and show the result first for \( g \) smooth. The inequality for \( g \in H^{-1}(\mathcal{O}) \) then follows by density. For any \( \psi \in H^1_0(\mathcal{O}) \), by (6.10) and (6.11), we have
\[
\langle \tau^\theta_h g, \psi \rangle_{H^{-1},H^1_0} = \int_{\mathcal{O}} g(x) \psi(\tau^\theta_h(x)) \left| \det(\nabla_x \tau^\theta_h(x)) \right| dx
\leq \| \nabla_x \psi \|_{L^\infty(\mathcal{O})} \|g\|_{H^{-1}(\mathcal{O})} \|\psi\|_{L^2(\mathcal{O})}
+ \| \det(\nabla_x \tau^\theta_h(x)) \|_{L^\infty(\mathcal{O})} \|g\|_{H^{-1}(\mathcal{O})} \|\nabla_x \psi\|_{L^2(\mathcal{O})}
\leq (c_{\theta,a} + c_{\theta,b}) \|g\|_{H^{-1}(\mathcal{O})} \|\psi\|_{H^1_0(\mathcal{O})}.
\]

A similar computation holds for \( \tau^s_h g \) using (6.13) and (6.14).

2. By a change of variables \( x \to \tau_{-h}(x) \), we have
\[
\int_{\mathcal{O}} g_1(\tau^\theta_h(x))g_2(x) dx = \int_{\mathcal{O}} g_1(x)g_2(\tau^\theta_h(x)) \left| \det(\nabla_x \tau^\theta_h(x)) \right| dx.
\]
Therefore
\[
\int_{\mathcal{O}} \{g_1, g_2\}^\theta_h dx = \int_{\mathcal{O}} g_1(x)g_2(\tau^\theta_{-h}(x)) \left| \det(\nabla_x \tau^\theta_{-h}(x)) \right| - 1 dx
\]

86
where we have used (6.10) and (6.16). A similar calculation using (6.13) and (6.16) gives the result for $\tau_h^s$.

For $g_2 \in A_\epsilon$ and $x \in \Gamma_\epsilon$, we have that $g_2(\tau_h^\theta(x)) = g_2(x)$ and thus clearly

$$\int_{\Gamma_\epsilon} \{g_1, g_2\}^\theta_h \, dx = 0.$$  

Also, by (6.13),

$$\int_T \{g_1, g_2\}^\theta_h \, ds = \int_T g_1(s)g_2(s - h) \det(\nabla x \tau_h^\theta(x)) - 1 \, ds$$

$$\leq \|\det(\nabla x \tau_h^\theta(x)) - 1\|_{L^\infty(T)} \|g_1\|_{L^2(T)} \|g_2(s - h)\|_{L^2(T)}$$

$$\leq c_s,h \|g_1\|_{L^2(T)} \|g_2\|_{L^2(T)}.$$  

3. We begin with $k = 0$. We have that

$$\nabla x (g(\tau_h^\theta(x))) = (\nabla x \tau_h^\theta(x)) \nabla x g|_{\tau_h^\theta(x)}$$

and therefore

$$\|\nabla, \tau_h^\theta\|^2 \leq \|\nabla, \tau_h^\theta\|^2 \leq c_s,a \|\nabla x g\|_{L^2}.$$  

by (6.9). Thus we have

$$\sup_{0 < h < 1} \frac{1}{h} \|\nabla, \tau_h^\theta\|_{L^2} \leq c_s,a \|\nabla x g\|_{L^2}.$$  

The argument for $g(\tau_h^s(x))$ is similar, using (6.12).

For $g \in H^{-1}(O)$, we proceed by duality. For any $\psi \in H^1_0(O)$, we have

$$\langle \nabla(\tau_h^\theta g) - \tau_h^\theta(\nabla g), \psi \rangle_{H^{-1},H^1_0} = -\int_O (\tau_h^\theta g) \text{ div } \psi \, dx - \langle \nabla g, (\tau_h^\theta \psi) \text{ det } \nabla x \tau_h^\theta \rangle_{H^{-1},H^1_0}$$

$$= -\int_O g(x) (\tau_h^\theta (\text{ div } \psi)) \text{ det } \nabla x \tau_h^\theta \, dx$$

$$+ \int_O g(x) \text{ div } (\tau_h^\theta \psi) \text{ det } \nabla x \tau_h^\theta \, dx.$$  

Thus, using (6.10) and (6.11) for $j = \theta$ and (6.13) and (6.14) for $j = s$, we have

$$\langle \nabla(\tau_h^\theta g) - \tau_h^\theta(\nabla g), \psi \rangle_{H^{-1},H^1_0}$$

$$\leq \|g\|_{L^2(O)} \left( \|\tau_h^\theta (\text{ div } \psi)\|_{L^2(O)} \|\text{ det } \nabla x \tau_h^\theta\|_{L^\infty(O)} + \|\text{ div } (\tau_h^\theta \psi)\|_{L^2(O)} \|\text{ det } \nabla x \tau_h^\theta\|_{L^\infty(O)} \right)$$

\[87\]
Similarly, for $O$ By Taylor’s theorem,

$$4. \text{ We show the inequality for } k = 0 \text{ and } g \text{ sufficiently smooth. The } k = -1 \text{ case can be shown by a duality argument similar to the ones above.}

By Taylor’s theorem,

$$|\delta_h^0 g| = |g(\rho, \theta + 2\pi h, s) - g(\rho, \theta, s)|$$

$$\leq |h| \int_T |\nabla_M g(\tau_h^0(x)) \cdot (\rho e_\theta(s, \theta + th))| \, dt,$$

and thus

$$|\delta_h^0 g|^2 \leq |h|^2 \rho^2 \int_T |\nabla_M g(\tau_h^0(x))|^2 \, dt.$$

Integrating over $O$, we have, by (6.10),

$$\left\| \frac{1}{\rho} \delta_h^0 g \right\|_{L^2(O)}^2 \leq |h|^2 \int_T \int_O |\nabla_M g(\tau_h^0(x))|^2 \, dx \, dt$$

$$\leq |h|^2 \int_T \int_O |\nabla_M g(x)|^2 \, dx \, dt \, dx$$

$$\leq |h|^2 c_{\theta,b} \int_T \int_O |\nabla_M g(\tau_h^0(x))|^2 \, dx \, dt$$

$$= |h|^2 c_{\theta,b} \int_O |A_s^{-1} A_s \nabla_M g(x)|^2 \, dx$$

$$\leq |h|^2 c_{\theta,b} \int_O |\nabla_x g(x)|^2 \, dx.$$

Similarly, for $\delta_s^k g$, we have

$$|\delta_h^s g| = |g(\rho, \theta, s + h) - g(\rho, \theta, s)|$$

$$\leq |h| \int_T \left| \nabla_M g(\tau_h^s(x)) \left( (1 - \rho \kappa(s + th, \theta)) e_t(s + th) + \rho \kappa_3(s + th) e_\theta(s + th, \theta) \right) \right| \, dt.$$

Thus

$$|\delta_h^s g|^2 \leq |h|^2 c_{\kappa} \int_T |\nabla_M g(\tau_h^s(x))|^2 \, dt.$$

Integrating over $O$, we have, by (6.13),

$$||\delta_h^s g||_{L^2(O)}^2 \leq |h|^2 c_{\kappa} \int_T \int_O |\nabla_M g(\tau_h^s(x))|^2 \, dx \, dt$$

$$\leq |h|^2 c_{\kappa} \int_T \int_O |\nabla_M g(x)|^2 \, dx \, dt \, dx$$

$$\leq |h|^2 c_{\kappa} c_{s,b} \int_T \int_O |\nabla_M g(x)|^2 \, dx \, dt$$
\[
|h|^2 c_{\kappa,s,b} \int_{\Omega} |A_s^{-1} A_s \nabla_M g(x)|^2 \, dx \\
\leq |h|^2 c_{\kappa,s,b} \int_{\Omega} |\nabla_x g(x)|^2 \, dx.
\]

Adding the estimates for \( \|\frac{1}{\rho} \delta_h^0 g\|_{L^2(\Omega)} \) and \( \|\delta_h^s g\|_{L^2(\Omega)} \) we obtain 6.2.4.

From Proposition 6.2 we easily obtain the following properties:

**Proposition 6.3.** (Additional tangential translation properties) For \( g \in L^2(\Omega) \) and \( j = s, \theta \), we have

\[
\begin{align*}
\|\delta_h^j g\|_{H^{-1}} &\leq c_{j,-1} |h| \|g\|_{L^2} \\
\|[\text{div}, \tau_h^j] \nabla g\|_{H^{-1}} &\leq c_{j,-1} |h| \|g\|_{L^2} \\
\|\text{div}([\nabla, \tau_h^j] g)\|_{H^{-1}} \leq &\|\|\nabla, \tau_h^j]\|_{L^2} \leq c_{j,0} |h| \|\nabla g\|_{L^2},
\end{align*}
\]

where each \( c_{\theta,-1} \) depends only on \( \kappa_{\text{max}}, c\Gamma \), and \( m_{\kappa,1} \), while each \( c_{s,-1} \) also depends on \( m_{\kappa,2} \).

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