Nonlinear open mapping principles, with applications to the Jacobian equation and other scale-invariant PDEs

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Abstract
For a nonlinear operator \(T\) satisfying certain structural assumptions, our main theorem states that the following claims are equivalent: i) \(T\) is surjective, ii) \(T\) is open at zero, and iii) \(T\) has a bounded right inverse. The theorem applies to numerous scale-invariant PDEs in regularity regimes where the equations are stable under weak\(^*\) convergence. Two particular examples we explore are the Jacobian equation and the equations of incompressible fluid flow.

For the Jacobian, it is a long standing open problem to decide whether it is onto between the critical Sobolev space and the Hardy space. Towards a negative answer, we show that, if the Jacobian is onto, then it suffices to rule out the existence of surprisingly well-behaved solutions.

For the incompressible Euler equations, we show that, for any \(p < \infty\), the set of initial data for which there are dissipative weak solutions in \(L^p_tL^2_x\) is meagre in the space of solenoidal \(L^2\) fields. Similar results hold for other equations of incompressible fluid dynamics.

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1 Introduction

The open mapping theorem is one of the cornerstones of functional analysis. When $X$ and $Y$ are Banach spaces and $L : X \to Y$ is a bounded linear operator, it asserts the equivalence of the following two conditions:

(i) Qualitative solvability: for all $f \in Y$ there is $u \in X$ with $Lu = f$, that is, $L(\mathbb{X}) = Y$;

(ii) Quantitative solvability: for all $f \in Y$ there is $u \in X$ with $Lu = f$ and $\|u\|_X \leq C\|f\|_Y$.

From a PDE perspective, the open mapping theorem justifies the method of a priori estimates [77, §1.7]. It is also a powerful tool in studying non-solvability of PDEs.

For applications to nonlinear PDE, one would like to have an analogue of the open mapping theorem in the case where $L$ is replaced by a nonlinear operator $T : X \to Y$. This is the main theme of the present paper. The search for a nonlinear open mapping principle has been spurred by the following problem of Rudin [70, page 67]:

**Question 1.1.** If $X_1$, $X_2$ and $Y$ are Banach spaces and $T$ is a continuous bilinear map of $X_1 \times X_2$ onto $Y$, does it follow that $T$ is open at the origin?

Following [44], we say that the open mapping principle holds for $T$ if $T$ is open at the origin. It is easy to see that, in general, $T$ is not open at all points. The origin plays a special role since, if $T$ is open at 0, then by scaling one obtains quantitative solvability: for all $f \in Y$ there exist $u_i \in X_i$ such that

$$T(u_1, u_2) = f$$

and

$$\|u_1\|_{X_1}^2 + \|u_2\|_{X_2}^2 \leq C\|f\|_Y.$$

The answer to Question 1.1 is negative. A first counter-example was given by Cohen in [20] and, shortly thereafter, a much simpler one was found by Horowitz [44], who considered the operator $T : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^4$ defined by

$$T(x, y) \equiv (x_1y_1, x_1y_2, x_1y_3 + x_3y_1 + x_2y_2, x_2y_2 + x_2y_1),$$

see also [10, 32]. These examples suggest that there is little hope of having a general nonlinear open mapping principle. Nonetheless, as the main result of this paper, we find natural sets of conditions under which the open mapping principle holds.

**Theorem** (Rough version). Consider a constant-coefficient system of PDEs with scaling symmetries, posed either over $\mathbb{R}^n$ or $\mathbb{R}^n \times [0, +\infty)$, which moreover is preserved by weak$^*$-convergence. The solution-to-data operator $T$ has a nonlinear open mapping principle.

For a precise version of the theorem we refer the reader to Theorem 5.2. Besides being genuinely nonlinear, this result has the important advantage of not requiring the operator $T$ to be defined over a vector space: as we will see below, this flexibility is very useful for applications to evolutionary PDEs.

The rest of this introduction is structured as follows. In the next subsection we present a precise version of the above theorem in a simple but important special case and then, in Section 1.2, we apply this version to the Jacobian equation. In Section 1.3 we discuss more general open mapping principles, suited for scale-invariant equations, and in Section 1.4 we expand on the applications of these generalised versions to the equations of incompressible fluid dynamics.
1.1 A first nonlinear open mapping principle

In this subsection we focus on a simple-to-state nonlinear open mapping principle.

**Theorem A.** Let $X$ and $Y$ be Banach spaces such that $B_{X^*}$ is sequentially weak* compact. We make the following assumptions:

(A1) $T: X^* \rightarrow Y^*$ is a weak*-to-weak* sequentially continuous operator.

(A2) $T(au) = a^s T(u)$ for all $a > 0$ and $u \in X^*$, where $s > 0$.

(A3) For $k \in \mathbb{N}$ there are isometric isomorphisms $\sigma_k^{X^*}: X^* \rightarrow X^*$, $\sigma_k^{Y^*}: Y^* \rightarrow Y^*$ such that

$$T \circ \sigma_k^{X^*} = \sigma_k^{Y^*} \circ T \quad \text{for all } k \in \mathbb{N}, \quad \sigma_k^{Y^*} f \overset{\text{w}}{\rightarrow} 0 \quad \text{for all } f \in Y^*.$$

Then the following conditions are equivalent:

(i) $T$ is onto: $T(X^*) = Y^*$;

(ii) $T$ is open at the origin;

(iii) For every $f \in Y^*$ there exists $u \in X^*$ such that $Tu = f$ and $\|u\|_{X^*}^s \leq C\|f\|_{Y^*}$.

Here and in the sequel $B_{X^*}$ denotes the unit ball of $X^*$. The hypothesis that $B_{X^*}$ is sequentially weak* compact holds, for instance, whenever $X$ is separable or reflexive. In general, the weak* topology on a dual space depends on the specific choice of predual: for instance, the spaces $c$ and $c_0$ induce different weak* topologies in their dual space, $\ell^1$. In fact it is only in this way that the spaces $X$ and $Y$ play a role in the statement of Theorem A. We also note that the simple example $T: \mathbb{R} \times L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$, $(t, f) \mapsto tf$ shows that there are operators satisfying the assumptions of Theorem A which are not open at all points [33].

The assumption (A1) is not always necessary, but it holds automatically in finite dimensional examples. An infinite dimensional case where it is not needed is the multiplication operator $(f, g) \mapsto fg: L^p \times L^q \rightarrow L^r$, where $1/p + 1/q = 1/r$: this operator does not satisfy (A1), although it verifies the open mapping principle [4, 5].

In Theorem A, instead of considering multilinear operators as in Question 1.1, we consider the larger class of positively homogeneous operators, that is, operators satisfying (A2). Nonetheless, many of the examples discussed in this paper are in fact multilinear.

The assumption (A3) may look somewhat mysterious. However, as Horowitz’s example shows, it cannot be omitted. It should be thought of as generalised translation invariance and indeed, when $T$ is a constant-coefficient partial differential operator and $X^*$ and $Y^*$ are function spaces on $\mathbb{R}^n$, natural choices of $\sigma_k^{X^*}$ and $\sigma_k^{Y^*}$ include translations

$$\sigma_k^{X^*} u(x) \equiv u(x - ke) \quad \text{and} \quad \sigma_k^{Y^*} f(x) \equiv f(x - ke), \quad \text{where } e \in \mathbb{R}^n \setminus \{0\}.$$

We note that the condition (A3) never holds if $Y$ is finite-dimensional and that moreover, when the target is two-dimensional, it is not needed: Downey has shown that, in this case, the answer to Question 1.1 is positive [33, Theorem 12].

In order for the reader to have a better grasp of the meaning of (A3), we briefly sketch the proof of Theorem A, explaining the role of this assumption in it. Suppose that, for all $f$ in some ball $B \subseteq Y^*$, one can solve the equation $Tu = f$. Since $T$ is weakly* continuous, it is not difficult to use the Baire Category Theorem to deduce that there is a sub-ball $B' \subseteq B$
such that one can actually solve $Tu = f$ quantitatively in $B'$: that is, there is a constant $C$ such that, for all $f \in B'$, there is $u \in X^*$ with $\|u\|_{X^*} \leq C$ and $Tu = f$. In other words: for weakly* continuous operators, qualitative solvability implies quantitative local solvability somewhere; in general, one cannot specify the location of $B'$. The assumption (A3) allows one to shift the centre of the ball $B'$ to the origin and, in combination with (A2), it upgrades the previous local statement to a global version.

Compensated compactness theory [64, 78] abounds with operators that satisfy the assumptions of Theorem A. The most famous examples are the Jacobian, the Hessian and the div-curl product; see [21] for numerous examples and [41, 42] for a systematic study. In fact, our motivation for Theorem A came from considering the Jacobian operator and the spaces

$$X^* = \dot{W}^{1, np}(\mathbb{R}^n, \mathbb{R}^n), \quad Y^* = \mathcal{H}^p(\mathbb{R}^n), \quad 1 \leq p < \infty,$$

see Question 1.2 below. The real-variable Hardy space $\mathcal{H}^p(\mathbb{R}^n)$ is defined by fixing any $\Phi \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \Phi(x) \, dx \neq 0$, denoting $\Phi_t(x) \equiv \Phi(x/t)/t^n$ for all $(x, t) \in \mathbb{R}^n \times (0, \infty)$ and setting

$$\mathcal{H}^p(\mathbb{R}^n) \equiv \left\{ f \in \mathcal{S}(\mathbb{R}^n) : \|f\|_{\mathcal{H}^p} \equiv \sup_{t > 0} |f * \Phi_t(x)| \|_{L^p} < \infty \right\}.$$ 

We refer the reader to the monograph [75] for the theory of $\mathcal{H}^p(\mathbb{R}^n)$. Here we just note that $\mathcal{H}^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$, with equivalent norms, whenever $p \in (1, \infty)$ and that moreover $\mathcal{H}^1(\mathbb{R}^n) \subseteq \{ f \in L^1(\mathbb{R}^n) : \int_{\mathbb{R}^n} f(x) \, dx = 0 \}$. Indeed, loosely speaking, elements of $\mathcal{H}^1(\mathbb{R}^n)$ have an extra logarithm of integrability [74]. In (1.1), the space $\dot{W}^{1, np}(\mathbb{R}^n, \mathbb{R}^n)$ is the usual homogeneous Sobolev space, seen as a Banach space, so that elements which differ by constants are identified.

### 1.2 Applications to the Jacobian equation

As discussed in the last subsection, we are interested in the Jacobian and, therefore, in the prescribed Jacobian equation

$$Ju \equiv \det Du = f \quad \text{in } \mathbb{R}^n.$$ (1.2)

This first-order equation appears naturally in Optimal Transport [14] and can be seen as the underdetermined analogue of the Monge–Ampère equation, see [39] for further discussion. It has a deep geometric content as, formally, one has the change of variables formula

$$\int_E Ju(x) \, dx = \int_{E_n} \# \left( u^{-1}(y) \cap E \right) \, dy, \quad E \subseteq \mathbb{R}^n \text{ is measurable.}$$ (1.3)

Thus, for a smooth solution of (1.2), $f$ measures the size of its image, counted with multiplicity.

When applied to the Jacobian, Theorem A reads as follows:

**Corollary B.** Fix $1 \leq p < \infty$. The following statements are equivalent:

(i) $J : \dot{W}^{1, np}(\mathbb{R}^n, \mathbb{R}^n) \to \mathcal{H}^p(\mathbb{R}^n)$ is surjective;

(ii) there is a bounded operator $E : \dot{W}^{1, np}(\mathbb{R}^n, \mathbb{R}^n) \to \mathcal{H}^p(\mathbb{R}^n)$ such that $J \circ E = \text{Id}$;

(iii) for all $f \in \mathcal{H}^p(\mathbb{R}^n)$ there is $u \in W^{1, np}(\mathbb{R}^n, \mathbb{R}^n)$ such that $Ju = f$ and

$$\|Du\|_{L^{np}(\mathbb{R}^n)} \lesssim \|f\|_{\mathcal{H}^p(\mathbb{R}^n)}.$$ (1.4)
We briefly describe the motivation behind Corollary B. The Jacobian benefits from improved integrability: after a remarkable result of Müller [61], Coifman, Lions, Meyer and Semmes proved in [22] that

\[ u \in \dot{W}^{1,n}(\mathbb{R}^n, \mathbb{R}^n) \implies J u \in \mathcal{H}^1(\mathbb{R}^n) \]

and that various other compensated compactness quantities also enjoy \( \mathcal{H}^1 \) integrability. Coifman & al. proceeded to ask whether compensated compactness quantities are surjective into the Hardy space \( \mathcal{H}^1(\mathbb{R}^n) \). The third author showed in [56] that the question must be formulated in terms of homogeneous Sobolev spaces, if it is to have a positive answer. The following problem remains open:

**Question 1.2.** For \( p \in [1, \infty) \) and \( f \in \mathcal{H}^p(\mathbb{R}^n) \), is there \( u \in \dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n) \) solving (1.2)?

We note that the case \( p > 1 \) was posed by Iwaniec, see also Conjecture 1.3 below for a stronger formulation of Question 1.2.

It is very difficult to predict the answer to Question 1.2, and in both directions, the currently available evidence is only tentative. A positive answer seems currently out of reach, as there is no systematic way of building solutions to (1.2) for a general discontinuous \( f \), although see [68] for some endpoint cases. This is in contrast with the case of Hölder continuous \( f \), where there is a well-posedness theory which goes back to the works of Dacorogna and Moser [25, 60], see also [24, 80] and the references therein.

In this subsection, we focus on progress towards a negative answer to Question 1.2. Our motivation for doing so comes, to some extent, from our results in [39]. There, we proved that a variant of Question 1.2—obtained by replacing \( \mathbb{R}^n \) with a bounded, smooth domain \( \Omega \) and additionally imposing Dirichlet boundary conditions on the solutions—is false.

The main difficulty in proving non-existence of solutions to (1.2) is the underdetermined nature of the equation: it implies that there is a multitude of possible solutions to rule out. This is the main reason why Question 1.2 is much harder than its analogue on a bounded domain, as the lack of boundary conditions makes the problem even more underdetermined. Indeed, the space \( \dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n) \) is extremely large, and so there is an abundance of possible solutions to consider, and moreover it contains many poorly-behaved maps, especially when \( p = 1 \). We now make precise this last point.

When \( p = 1 \), there are continuous maps in \( \dot{W}^{1,n}(\mathbb{R}^n, \mathbb{R}^n) \) which do not satisfy the change of variables formula (1.3), as they map a null set \( E \) into a set of positive measure, and hence

\[
0 = \int_E f \, dx < |u(E)| \leq \int_{\mathbb{R}^n} \#(u^{-1}(y) \cap E) \, dy.
\]

It is thus possible for (1.2) to hold a.e. in \( \mathbb{R}^n \), and hence also in the sense of distributions, and yet for its geometric information to be completely lost! Maps as above are said to violate the Lusin (N) property; their existence is classical and goes back to the work of Cesari [18], see also [58] for a more refined version.

We are interested in considering admissible solutions of (1.2), i.e. solutions for which the geometric information of the equation is preserved; here our choice of terminology is inspired by the fluid dynamics literature. Corollary B yields the quite surprising fact that

*the existence of rough solutions implies the existence of admissible solutions.*
This fact is made precise in the following result:

**Theorem C.** Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and take $f \in \mathcal{H}^1(\mathbb{R}^n)$ such that $f \geq 0$ in $\Omega$. Assume that $J : \dot{W}^{1,n}(\mathbb{R}^n, \mathbb{R}^n) \to \mathcal{H}^1(\mathbb{R}^n)$ is onto. Then there is a solution $u \in \dot{W}^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$ of (1.2) such that:

(i) $u$ is continuous in $\Omega$;
(ii) $u$ has the Lusin (N) property in $\Omega$;
(iii) $\int_{\mathbb{R}^n} |Du|^n \, dx \leq C\|f\|_{\mathcal{H}^1}$ with $C > 0$ independent of $f$.

In particular, $u$ satisfies the change of variables formula (1.3).

Moreover, if $n = 2$ and there is an open set $\Omega' \subseteq \Omega$ with $f = 0$ a.e. in $\Omega'$, then:

(iv) for any set $E \subset \Omega'$, we have $u(\partial E) = u(E)$;
(v) for $y \in u(\Omega')$, if $C$ denotes a connected component of $u^{-1}(y) \cap \Omega'$ then $C$ intersects $\partial \Omega'$.

Theorem C is proved through a regularisation argument: due to Corollary B, powerful tools from Geometric Function Theory become available. In the supercritical regime $p > 1$, the first part of Theorem C holds automatically, although one can still use the a priori estimate (1.4) to get solutions with additional structure, see Section 4.2 for further details. The second part of Theorem C also holds in any dimension if $p$ is taken to be sufficiently large.

Contrary to the approach discussed so far, it may be that the answer to Question 1.2 is positive. In this direction, Iwaniec suggested in [48] that one should prove (1.4) for all $p$-energy minimisers, that is, solutions of (1.2) which minimise $\int_{\mathbb{R}^n} |Du|^{np} \, dx$ under the constraint $Ju = f$. To achieve this goal, Iwaniec has proposed that one construct a Lagrange multiplier for every $p$-energy minimiser. This turns out to be a very difficult task since the standard methods fail, see Section 4.1 for further discussion. Nonetheless, when $p = 1$ and $n = 2$, uniformly bounded Lagrange multipliers were constructed in [55, 54] for a large class of $p$-energy minimisers, which then automatically satisfy the estimate (1.4); we also note that the methods of [55, 54] can be partly adapted to all the cases $n \geq 2$, $p \in [1, \infty)$.

In [48], see also [12], Iwaniec went further than Question 1.2 and posed the following:

**Conjecture 1.3.** For $p \in [1, \infty)$, there is a continuous map $E : \mathcal{H}^p(\mathbb{R}^n) \to \dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n)$ such that $J \circ E = \text{Id}$.

Conjecture 1.3 proposes the existence of a continuous right inverse for the Jacobian (abbreviated in this paragraph as r.i.). We have the following trivial chain of implications:

\[ \text{J has a bounded, F-differentiable r.i.} \implies \text{J has a bounded, continuous r.i.} \implies \text{J has a bounded r.i.} \implies \text{J has a r.i.} \]

Corollary B states the equivalence of the last two statements, while Corollary 4.11 below shows that the first statement is false. Finally, and concerning the second statement, Iwaniec pointed out in [48] that it is plausible that $p$-energy minimisers are unique up to rotations: if so, this would pave a road towards a solution of Conjecture 1.3. However, and as further negative evidence towards Question 1.2 and Conjecture 1.3, there are some data which admit uncountably many $p$-energy minimisers. This will be proved in our forthcoming work [38].
1.3 General nonlinear open mapping principles

In this subsection we formulate a more general open mapping principle. Precise statements, as well as multiple examples, can be found in Sections 5 and 6, but Theorem D below contains already a rough version of our main result.

In Theorem D, the positive homogeneity assumption (A2) from Theorem A is replaced by more general scaling symmetries; the assumption of translation invariance will be kept. To be precise, we assume that the equation $Tu(x, t) = f(x, t)$, $(x, t) \in \mathbb{R}^n \times [0, \infty)$ is invariant under a one-parameter group of scalings

$$
\tau^X_\lambda[u](x, t) \equiv \frac{1}{\lambda^\alpha} u \left( \frac{x}{\lambda^\beta}, \frac{t}{\lambda^\gamma} \right), \quad \tau^Y_\lambda[f](x, t) \equiv \frac{1}{\lambda^\delta} f \left( \frac{x}{\lambda^\epsilon}, \frac{t}{\lambda^\gamma} \right),
$$

(1.5)

where $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ are fixed and the group parameter is $\lambda > 0$. In our typical applications, $f$ is the initial datum of a Cauchy problem and $T$ is the solution-to-datum map.

Invariance under translations and scalings is an ubiquitous feature of physical processes. It expresses the covariance principle that the solutions of a PDE representing a physical phenomenon should not have a form which depends on the location of the observer or the units that the observer is using to measure the system [16]. For the computation of the symmetry groups of several representative PDEs we refer to [65, §2.4] and for the general role of scaling symmetries in physics and other sciences to [6].

It often happens that a PDE has several scaling symmetries. For instance, the positive $n$-homogeneity of the Jacobian operator can be expressed as symmetry of the equation $Ju = f$ under the scaling $u_\lambda = \lambda u$, $f_\lambda = \lambda^n f$ for all $\lambda > 0$, but the Jacobian equation also has the scaling symmetry $u_\lambda(x) = \lambda u(x/\lambda)$, $f_\lambda(x) = f(x/\lambda)$. For another example, concerning the incompressible Euler equations, see (1.6)–(1.9). An important theme in this work is that, whenever a PDE has several scaling symmetries, these symmetries must be compatible in order for the equation to be solvable for all data.

The next result encapsulates the two previous points: many scale-invariant PDEs satisfy a nonlinear open mapping principle and, for the equation to be solvable, the associated scalings need to be compatible.

**Theorem D (Rough version).** Consider a constant-coefficient system of PDEs, posed over $\mathbb{R}^n \times [0, \infty)$, which moreover is preserved under weak* convergence. Let $T$ be the solution-to-datum operator associated with the PDE.

Suppose that the equation $Tu = f$ is invariant under the scalings (1.5) and that the solutions and the data lie in homogeneous function spaces, which satisfy, for some $r, s \in \mathbb{R},$

$$
\|\tau^X_\lambda[u]\|_{X^*} \equiv \lambda^r \|u\|_{X^*}, \quad \|\tau^Y_\lambda[f]\|_{Y^*} \equiv \lambda^s \|f\|_{Y^*}, \quad \text{where } rs > 0.
$$

The following statements are then equivalent:

(i) For all $f \in Y^*$ there is $u \in X^*$ with $Tu = f$;

(ii) For all $f \in Y^*$ there is $u \in X^*$ with $Tu = f$ and $\|u\|_{X^*} \leq C \|f\|_{Y^*}$.

Moreover, suppose that $T$ is invariant under another pair of scalings $\tilde{\tau}^X_\lambda, \tilde{\tau}^Y_\lambda$, which satisfy

$$
\|\tilde{\tau}^X_\lambda[u]\|_{X^*} \equiv \tilde{\lambda}^r \|u\|_{X^*}, \quad \|\tilde{\tau}^Y_\lambda[f]\|_{Y^*} \equiv \tilde{\lambda}^s \|f\|_{Y^*}, \quad \text{where } \tilde{r}s > 0.
$$

Then solvability of the equation $Tu = f$ requires compatibility of the scalings, i.e.

$T$ is surjective $\implies$ $r/s = \tilde{r}/\tilde{s}$.
The first part of Theorem D yields a generalisation of Theorem A. We note that the assumption that \( r \) and \( s \) have the same sign, i.e. \( rs > 0 \), ensures that the norms in question are either subcritical or supercritical; the critical case \( r = s = 0 \) is beyond the scope of this work. We will expand on the second part of Theorem D in the next subsection. Solvability and a priori estimates in inhomogeneous function spaces are treated in Corollary 5.8.

Concerning the hypothesis of stability under weak* convergence, we note that it is typically satisfied by solutions above a certain regularity threshold: for instance, both the Navier–Stokes equations and the cubic wave equation in \( \mathbb{R}^3 \times [0, +\infty) \) are preserved under weak* convergence in the corresponding energy spaces. For further details, see Section 5.3. Moreover, by considering a relaxed version of the PDE the assumption of stability under weak* convergence can sometimes be bypassed, an idea which will be briefly discussed in the next subsection.

1.4 Applications to the equations of incompressible fluid flow

The validity of the assumptions of Theorem D is typically easy to check in practice. As such, the theorem gives rather immediate consequences on various physical PDEs. For further discussion of the merits and weaknesses of Theorem D, see Section 7.

In order to give a representative application of Theorem D we consider energy-dissipating solutions of the incompressible Euler equations

\[
\partial_t u + u \cdot \nabla u - \nabla P = 0, \tag{1.6}
\]

\[
\text{div } u = 0, \tag{1.7}
\]

\[
u(\cdot, 0) = u^0 \tag{1.8}
\]

in \( \mathbb{R}^n \times [0, \infty) \), for \( n \geq 2 \). Note that (1.6)–(1.8) are invariant under scalings of the form

\[
u_{\lambda}(x, t) \equiv \frac{1}{\lambda^n} u \left( \frac{x}{\lambda^\alpha}, \frac{t}{\lambda^{\alpha+\beta}} \right), \quad u_{\lambda}^0(x, t) \equiv \frac{1}{\lambda^n} u^0 \left( \frac{x}{\lambda^\alpha} \right), \quad P_{\lambda}(x, t) \equiv \frac{1}{\lambda^{2\alpha}} P \left( \frac{x}{\lambda^\alpha}, \frac{t}{\lambda^{\alpha+\beta}} \right) \tag{1.9}
\]

for any \( \alpha, \beta > 0 \). An interesting connection between (1.6)–(1.7) and (1.2) is detailed in [62].

Before proceeding further, we note that the incompressibility constraint (1.7) will often be codified in the appropriate function spaces through the subscript \( \sigma \); thus, for instance, we write \( L^2_\sigma \equiv \{ v \in L^2 : \text{div } v = 0 \} \).

Solutions of (1.6)–(1.8) which fail to conserve energy have been studied extensively in relation to the so-called Onsager conjecture, see [15, 27, 47] and the references therein. By a theorem of Székelyhidi and Wiedemann [76], for a dense set of initial data in \( L^2_\sigma \) there exist infinitely many admissible solutions \( u \in L^\infty_t L^2_{\sigma, x} \) of (1.6)–(1.8), that is, weak solutions which satisfy the energy inequality

\[
\int_{\mathbb{R}^n} |u(x, t)|^2 \, dx \leq \int_{\mathbb{R}^n} |u^0(x)|^2 \, dx \quad \text{for all } t \geq 0.
\]

Moreover, the data \( u^0 \) can be chosen to be \( C^\beta \) regular for any given \( \beta \in (0, 1/3) \), at least on the torus [26].

For some data \( u^0 \in L^2_\sigma \) there are even admissible compactly supported solutions. Schef-fer had already constructed in [71] solutions of the Euler equations which are compactly supported and square integrable in space-time, and a systematic study via convex integration.
was initiated by DE LELLIS and SZÉKELYHIDI in the groundbreaking works [27, 28]. Nevertheless, Theorem D easily implies that, for a Baire-generic datum $u^0 \in L^2_\sigma$, the kinetic energy $\frac{1}{2} \int_{\mathbb{R}^n} |u(x,t)|^2 \, dx$ of weak solutions cannot undergo an $L^q$-type decay, $q < \infty$:

**Corollary E.** Take $n \geq 2$ and $2 < p < \infty$. For every $M > 0$, the set of initial data for which there is a solution $u \in MB_{L^p_tL^2_x}$ of the Cauchy problem (1.6)-(1.8) is nowhere dense in $L^2_\sigma$. In particular, for a residual $G_\delta$ set of initial data in $L^2_\sigma$, the Cauchy problem (1.6)-(1.8) has no solution in $L^\infty_t L^2_x \cap \bigcup_{2<p<\infty} L^p_tL^2_x$.

**Corollary F.** Take $\tau > 0$. An admissible solution $u \in L^\infty_tL^2_{\sigma,x}$ with $\text{supp}(u) \subset \mathbb{R}^n \times [0, \tau]$ exists only for a nowhere dense set of data $u^0 \in L^2_\sigma$.

To deduce Corollary E from Theorem D we consider a linear relaxation of the equations (1.6)-(1.8); this is an idea in the spirit of TARTAR’s framework for studying oscillations and concentrations in conservation laws [78, 79]. Such a relaxation is used here in order to render the associated solution-to-datum operator weak*-to-weak*-continuous. Corollary E is proved in §6.2.

The proof of Corollary E also applies to many other models in fluid dynamics. For instance, concerning the Navier–Stokes equations, we prove in an elementary fashion upper bounds for the generic energy dissipation rate of weak solutions. Another example is given by the equations of ideal magnetohydrodynamics, for which the analogue of Corollary E holds true. In that context bounded solutions with compact support in space-time were constructed in [34]. On the torus $\mathbb{T}^3$, solutions in $L^\infty_t H^2_x$, for a small $\beta > 0$, violating magnetic helicity conservation were constructed in [8].

## 2 A nonlinear open mapping principle for positively homogeneous operators

The main goal of this section is to prove Theorem A. A related nonlinear uniform boundedness principle is proved in Proposition 2.3 and a precise statement concerning atomic decompositions in terms of $T$ is proved in Proposition 2.5.

In the case of the Jacobian, by adapting a standard proof of the standard Open Mapping Theorem to Question 1.2 one obtains the following statement: if $J(\dot{W}^{1, np}(\mathbb{R}^n, \mathbb{R}^n)) = \mathcal{H}^p(\mathbb{R}^n)$, then for every $f \in \mathcal{H}^p(\mathbb{R}^n)$ there exist $u, v \in \dot{W}^{1, np}(\mathbb{R}^n, \mathbb{R}^n)$ with

$$J_u + J_v = f \quad \text{and} \quad \int_{\mathbb{R}^n} (|Du|^{np} + |Dv|^{np}) \, dx \leq C \|f\|^p_{\mathcal{H}^p}. \quad (2.1)$$

Thus, quantitative control is gained at the expense of introducing an extra term $J_v$.

One could attempt to show the non-surjectivity of $J$ by disproving the a priori estimate in (2.1). However, the extra term $J_v$ makes this a formidable task since the equation $J_u + J_v = f$ admits much more pathological solutions than $J_u = f$. As a prototypical example, there exist Lipschitz maps $u, v: \mathbb{R}^2 \to \mathbb{R}^2$ vanishing in the lower half-plane and satisfying $J_u + J_v = 1$ in the upper half-plane [51, Lemma 5]. In Theorem A and Corollary B, the extra Jacobian $J_v$ is removed, leading to a genuinely nonlinear version of the Open Mapping Theorem.
2.1 The proof of Theorem A

Here we give a slightly more precise version of Theorem A:

**Theorem 2.1.** Let $X$ and $Y$ be Banach spaces such that $B_{X^*}$ is sequentially weak* compact. We make the following assumptions:

(A1) $T: X^* \to Y^*$ is a weak*-to-weak* sequentially continuous operator.

(A2) $T(au) = a^* T(u)$ for all $a > 0$ and $u \in X^*$, where $s > 0$.

(A3) For $k \in \mathbb{N}$ there are isometric isomorphisms $\sigma_k^{X^*}: X^* \to X^*$, $\sigma_k^{Y^*}: Y^* \to Y^*$ such that

$$T \circ \sigma_k^{X^*} = \sigma_k^{Y^*} \circ T \quad \text{for all } k \in \mathbb{N}, \quad \sigma_k^{Y^*} f \xrightarrow{\ell} 0 \quad \text{for all } f \in Y^*.$$  

Then the following conditions are equivalent:

(i) $T(X^*)$ is non-meagre in $Y^*$.

(ii) $T(X^*) = Y^*$.

(iii) $T$ is open at the origin.

(iv) For every $f \in Y^*$ there exists $u \in X^*$ such that

$$Tu = f, \quad \|u\|_X^* \leq C\|f\|_{Y^*}.$$  

A sufficient condition for $B_{X^*}$ to be sequentially weak* compact is that $X$ is a weak Asplund space [73, Theorem 3.5]. For instance, reflexive or separable spaces are weak Asplund [31].

**Proof of Theorem A.** We have (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) and so we just prove (i) $\Rightarrow$ (iv).

Assume that (i) holds. We may write $T(X^*)$ as a union $\bigcup_{k=1}^{\infty} K_k$, where

$$K_k = \left\{ f \in Y^*: \text{there exists } u \in X^* \text{ with } Tu = f \text{ and } \|u\|_X^* \leq \ell \|f\|_{Y^*} \right\}.$$  

Since balls in $X^*$ are sequentially weak* compact, by (A1), the sets $K_k$ are norm-closed. Now, by the Baire Category Theorem, some $K_k$ contains a closed ball $B_r(f_0)$.

Our aim is to solve (2.2) whenever $\|f\|_{Y^*} = r$; assumption (A2) then implies the claim. Suppose, therefore, that $\|f\|_{Y^*} = r$. For every $k \in \mathbb{N}$ we have $f_0 + (\sigma_k^{Y^*})^{-1} f \in B_r(f_0)$. Hence, we may choose $u_k \in X^*$ such that $Tu_k = f_0 + (\sigma_k^{Y^*})^{-1} f$ and

$$\|\sigma_k^{X^*} u_k\|_X^* = \|u_k\|_X^* \leq \ell \|f_0\|_{Y^*} + (\sigma_k^{Y^*})^{-1} f \|_{Y^*} \leq \ell (\|f_0\|_{Y^*} + r).$$  

Since balls in $X^*$ are sequentially weak* compact, after passing to a subsequence if need be, $\sigma_k^{X^*} u_k$ converges weakly* to some $u \in X^*$, so that $T(\sigma_k^{X^*} u_k) \xrightarrow{\ell} Tu$. By the lower semicontinuity of the norm we have

$$\|u\|_X^* \leq \liminf_{k \to \infty} \|\sigma_k^{X^*} u_k\|_X^* \leq \ell (\|f_0\|_{Y^*} + r).$$  

On the other hand, (A3) gives

$$T(\sigma_k^{X^*} u_k) = \sigma_k^{Y^*}(Tu_k) = \sigma_k^{Y^*} f_0 + f \xrightarrow{\ell} f,$$

so that, by (A1), $Tu = f$. Thus $u$ solves (2.2) and the proof is complete. □
The theory of Compensated Compactness provides many examples of nonlinear operators to which Theorem A applies. Here we give a general formulation in the spirit of [40], see also [64, 78], which we then illustrate with more concrete examples.

Example 2.2. Let $\mathcal{A}$ be an $l$-th order homogeneous linear operator, which for simplicity we assume to have constant coefficients; that is, for $v \in C^\infty(\mathbb{R}^n, V)$,

$$\mathcal{A}v = \sum_{|\alpha|=l} A_\alpha \partial^\alpha v, \quad A_\alpha \in \text{Lin}(V, \mathcal{W}),$$

where $V, \mathcal{W}$ are finite-dimensional vector spaces. For $p \in [1, +\infty)$ and $s \in \mathbb{N}$, $s \geq 2$, take

$$X^* = L^p_s(\mathbb{R}^n, V), \quad Y^* = \mathcal{H}^p(\mathbb{R}^n).$$

Here $L^p_s(\mathbb{R}^n, V)$ is the space of those $v \in L^p(\mathbb{R}^n, V)$ such that $\mathcal{A}v = 0$ in the sense of distributions. We will further need the following standard non-degeneracy assumption:

the symbol of $\mathcal{A}$, seen as a matrix-valued polynomial, has constant rank. (2.3)

Whenever (2.3) holds, we say that $\mathcal{A}$ has constant rank. We will not discuss this assumption here but it holds in all of the examples below; the reader may find other characterizations of constant rank operators in [41, 66].

Let $T : X^* \to Y^*$ be a homogeneous sequentially weakly continuous operator. Under the assumption (2.3), such operators were completely characterised in [40], and they are often called Compensated Compactness quantities. They can be realised as certain constant-coefficient partial differential operators and so they necessarily satisfy (A3) if one takes the isometries $\sigma_{kX^*}, \sigma_{kY^*}$ to be translations. The following are standard examples of such operators:

(i) $\mathcal{A} = \text{curl}$ and $T = J$. For this example, take $V = \mathbb{R}^{n \times n}$ and choose $\mathcal{A}$ in such a way that $\mathcal{A}v = 0$ if and only if $v = Du$, for some $u : \mathbb{R}^n \to \mathbb{R}^n$. For instance, we may take $(\text{curl } v)_{ijk} = \partial_k v_{ij} - \partial_j v_{ik}$. We also choose $s = n$ and so $X^* = \tilde{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n)$. The only positively $n$-homogeneous sequentially weakly continuous operator $X^* \to Y^*$ is the Jacobian, and in particular we recover Corollary B.

(ii) $\mathcal{A} = \text{curl}^2$ and $T = H$. Here $\mathcal{A}$ is chosen similarly to the previous example, but now $\mathcal{A}v = 0$ if and only if $v = D^2u$, for some $u : \mathbb{R}^n \to \mathbb{R}$. Again we take $s = n$ and so $X^* = \tilde{W}^{2,np}(\mathbb{R}^n, \mathbb{R}^n)$. We may take $T = H : X^* \to Y^*$ to be the Hessian, and Theorem A shows that it satisfies the open mapping principle.

The two previous examples admit a straightforward generalisation, where one considers $s$-th order minors (instead of the determinant) and a $j$-th order curl (instead of $j = 1, 2$).

(iii) $\mathcal{A} = (\text{div}, \text{curl})$ and $T = \langle \cdot, \cdot \rangle$. In this example, $s = 2$ and $T$ is the standard inner product acting on a pair $v \equiv (B, E) : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$; here, $B$ is thought of as a “magnetic field” and $E$ as an “electric field”. As before, Theorem A shows that $T$ satisfies the open mapping principle.

We conclude this subsection by comparing the above example with [22]. There, the authors address the problem of deciding whether Compensated Compactness quantities are surjective, particularly when $p = 1$. Thus Theorem A can be read as saying that openness at zero is a necessary condition for a positive answer to this problem.
2.2 A nonlinear uniform boundedness principle

We also present a nonlinear version of the Uniform Boundedness Principle in the spirit of Theorem A; under certain structural conditions, a family of operators which is pointwise bounded in a neighbourhood of the origin is uniformly bounded in a (possibly smaller) neighbourhood of the origin.

**Proposition 2.3.** Let $X$ and $Z$ be Banach spaces and let $I$ be an index set. Suppose the following conditions hold:

(i) For every $i \in I$, the mapping $T_i: X \to Z$ is such that $u \mapsto \|T_i u\|_Z: X \to \mathbb{R}$ is weakly sequentially lower semicontinuous.

(ii) There is $\varepsilon > 0$ such that $\sup_{i \in I} \|T_i(u)\|_Z < \infty$ whenever $\|u\|_X \leq \varepsilon$.

(iii) For $j \in \mathbb{N}$ there are isometric isomorphisms $\sigma_j^X: X \to X$ and $\sigma_j^Z: Z \to Z$ such that

\[
T_i \circ \sigma_k^X = \sigma_k^Z \circ T_i \quad \text{for all } i \in I \text{ and } k \in \mathbb{N},
\]

\[
\sigma_k^X u \to 0 \quad \text{for all } u \in X.
\]

Then there exists $\delta > 0$ such that

\[
\sup_{\|u\|_X \leq \delta} \sup_{i \in I} \|T_i u\|_Z < \infty.
\]

**Proof.** By (ii), we may write $\varepsilon \mathbb{B}_X = \bigcup_{\ell=1}^\infty C_\ell$, where $C_\ell \equiv \{ u \in \varepsilon \mathbb{B}_X: \sup_{i \in I} \|T_i u\|_Z \leq \ell \}$ and (i) shows that each $C_\ell$ is norm closed. Thus, by the Baire Category Theorem, some $C_\ell$ contains a closed ball $\bar{B}_\delta(u_0)$.

Let now $\|u\|_X \leq \delta$ and $i \in I$. By (iii), we have $u + \sigma_k^X u_0 = \sigma_k^X [u_0 + (\sigma_k^X)^{-1} u] \in \bar{B}(u_0, \delta)$ and moreover $u + \sigma_k^X u_0 \to u$. So by (i) and again (iii), we have

\[
\|T_i u\|_Z \leq \lim_{k \to \infty} \|T_i \sigma_k^X [u_0 + (\sigma_k^X)^{-1} u]\|_Z = \lim_{k \to \infty} \|\sigma_k^Z T_i [u_0 + (\sigma_k^X)^{-1} u]\|_Z \leq \ell.
\]

The proof is complete. 

We note that, in the linear case, it is possible to prove the Banach–Steinhaus Uniform Boundedness Principle without using Baire’s Category Theorem: the proof relies, instead, on the so-called “gliding hump method”. For an extension of the classical Uniform Boundedness Principle using this method, we refer the reader to [36].

2.3 Atomic decompositions in terms of $T$

The main motivation behind this subsection is Theorem 2.4. It establishes an analogue of the atomic decomposition of $\mathcal{H}^1(\mathbb{R}^n)$, giving a weak factorization on $\mathcal{H}^p(\mathbb{R}^n)$ in the spirit of the classical work of COIFMAN, ROCHBERG and WEISS [23]:

**Theorem 2.4.** Let $p \in [1, \infty)$. For every $f \in \mathcal{H}^p(\mathbb{R}^n)$ there are functions $u_i \in \dot{W}^{1,p} (\mathbb{R}^n)$ and real numbers $c_i$ such that

\[
f = \sum_{i=1}^\infty c_i J u_i, \quad \|u_i\|_{\dot{W}^{1,p}(\mathbb{R}^n)} \leq 1, \quad \sum_{i=1}^\infty |c_i| \lesssim \|f\|_{\mathcal{H}^p(\mathbb{R}^n)}.
\]

In particular, $\mathcal{H}^p(\mathbb{R}^n)$ is the smallest Banach space containing the range $J(\dot{W}^{1,p}(\mathbb{R}^n)).$
Theorem 2.4 was proved in [22] for $p = 1$, while the case $p > 1$ is much harder and was established only recently by Hytönen in [46]. It is conceivable that the operator $J: W^{1,p}(\mathbb{R}^n, \mathbb{R}^n) \rightarrow H^p(\mathbb{R}^n)$ is not surjective but (2.4) improves to a finitary decomposition of $H^p(\mathbb{R}^n)$ in terms of Jacobians. In Proposition 2.5, we formulate a rather precise classification of infinitary and finitary decompositions in the setting of Theorem A.

Take $\omega \in \mathbb{N} \equiv \mathbb{N} \cup \{\infty\}$. Given $T$ as in Theorem A, if every $f \in Y^*$ can be written as

$$f = \sum_{j=1}^{\omega} c_j T u_j, \quad c_j \in \mathbb{R}, \quad u_j \in B_{X^*}, \quad (2.5)$$

then, following [32], $T$ is said to be $1/\omega$-surjective. If, furthermore,

$$\sum_{j=1}^{\omega} |c_j| \lesssim \|f\|_{Y^*} \quad (2.6)$$

for all $f \in Y^*$, then $T$ is said to be $1/\omega$-open. Dixon [32] generalised Horowitz’s example by constructing, for every $m \in \mathbb{N}$, a continuous $1/m$-surjective bilinear map between Banach spaces which is not $1/m$-open. In fact, in Dixon’s notation, the constants $c_j$ are subsumed by the elements $u_j$. The formalism (2.5)–(2.6) is, however, more standard in the context of atomic decompositions.

In Proposition 2.5 we show that, for $\omega \in \mathbb{N}$, and under the assumptions of Theorem A, $1/\omega$-surjectivity implies $1/\omega$-openness.

**Proposition 2.5.** Suppose $X$, $Y$ and $T$ satisfy the assumptions of Theorem A. Let us define, for $\omega \in \mathbb{N}$, the sets

$$\Lambda_\omega \equiv \left\{ \sum_{j=1}^{\omega} c_j T u_j : u_j \in B_{X^*}, \quad c_j \in \mathbb{R} \text{ and } \sum_{j=1}^{\omega} |c_j| < \infty \right\}.$$ 

If $\Lambda_\infty$ is not meagre in $Y^*$, there is $\omega \in \mathbb{N}$ such that $\Lambda_\omega = Y^*$ and $\bigcup_{m<\omega} \Lambda_m$ is meagre in $Y^*$; moreover, $T$ is $1/\omega$-open.

**Proof.** We show that if $\bigcup_{m<\infty} \Lambda_m$ is not meagre in $Y^*$, then there is $m \in \mathbb{N}$ such that $\Lambda_m = Y^*$ and $\Lambda_{m-1}$ is meagre in $Y^*$. Note that, for each $m \in \mathbb{N}$, the set $\Lambda_m$ is closed; it follows from the Baire Category Theorem that one of the sets $\Lambda_m$ contains a ball. By using the $s$-homogeneity of $T$, we write $\Lambda_m = \{ \sum_{j=1}^{m} d_j T v_j : d_j \in \mathbb{R}, v_j \in X^* \}$. By applying Theorem A to the $(s+1)$-homogeneous operator

$$\tilde{T}: \mathbb{R}^m \times (X^*)^m \rightarrow Y^*, \quad \tilde{T}\left(\{d_j\}_{j=1}^{m}, \{v_j\}_{j=1}^{m}\right) \equiv \sum_{j=1}^{m} d_j T v_j,$$

we find that for each $f \in Y^*$ there are $d_j \in \mathbb{R}$ and $v_j \in X^*$ such that

$$\sum_{j=1}^{m} d_j T v_j = f, \quad \sum_{j=1}^{m} (|d_j|^{s+1} + \|v_j\|_{X^*}^{s+1}) \lesssim \|f\|_{Y^*}. \quad (2.7)$$

We set $c_j = d_j \|v_j\|_{X^*}^{s+1}$ and denote $u_j = v_j/\|v_j\|_{X^*}$ if $v_j \neq 0$ and $u_j = 0$ if $v_j = 0$. Thus $c_j T u_j = d_j T v_j$ for $j = 1, \ldots, m$. Consequently, through Young’s inequality, (2.7) yields

$$\sum_{j=1}^{m} c_j T u_j = f, \quad \sum_{j=1}^{m} |c_j| \lesssim \|f\|_{Y^*}, \quad u_j \in B_{X^*}. \quad (2.8)$$
It now suffices choose the smallest \( m \in \mathbb{N} \) such that \( T: X^* \to Y^* \) is \( 1/m \)-surjective; the \( 1/m \)-openness of \( T \) is given by (2.8).

We finally show that if \( \bigcup_{m<\infty} \Lambda_m \) is meagre but \( \Lambda_\infty \) is non-meagre, then in fact \( \Lambda_\infty = Y^* \) and \( T \) is \( 1/\infty \)-open. We denote \( V \equiv \{ \varepsilon Tu: \varepsilon = \pm 1, u \in \mathbb{B}_{X^*} \} \subset Y^* \). Now \( V \) is bounded and symmetric and, by assumption, \( \{ \sum_{j=1}^\infty c_j f_j : f_j \in V \text{ for all } j \text{ and } \sum_{j=1}^\infty |c_j| < \infty \} \) is non-meagre in \( Y^* \). By [56, Lemma 3.1], \( \{ \sum_{j=1}^\infty c_j T u_j : \sum_{j=1}^\infty |c_j| = 1, u_j \in \mathbb{B}_{X^*} \} \subset Y^* \) contains a ball centred at the origin. It immediately follows that given \( f \in Y^* \), conditions (2.5)–(2.6) can be satisfied with \( \omega = \infty \). \( \square \)

**Remark 2.6.** It is tempting to try and prove the last part of Proposition 2.5 by defining an auxiliary operator \( \tilde{T}: \ell^{s+1}(\mathbb{N}) \times \ell^{s+1}(\mathbb{N}; X^*) \to Y^* \) via \( T(\{ d_j \}_{j=1}^\infty, \{ v_j \}_{j=1}^\infty) \equiv \sum_{j=1}^\infty d_j T v_j \) and using Theorem A on \( \tilde{T} \), in analogy to the case \( \omega < \infty \). However, such an operator is never \( \text{weak}^* \)-to-\( \text{weak}^* \) continuous unless \( T \equiv 0 \). Indeed, suppose \( Tu \neq 0 \) and set \( d_{jk} = \delta_{jk} \) and \( v_{jk} = \delta_{jk} u \). Now \( \tilde{T}(\{ d_{jk} \}_{j=1}^\infty, \{ v_{jk} \}_{j=1}^\infty) = Tu \) for all \( k \in \mathbb{N} \) but \( \{ d_{jk} \}_{j=1}^\infty, \{ v_{jk} \}_{j=1}^\infty \xrightarrow{\ast} 0 \).

**Example 2.7.** Let us denote by \( H \) the Hilbert transform and by \( T: L^2(\mathbb{R}, \mathbb{R}) \to \mathcal{H}^1(\mathbb{R}) \) the operator \( T(\chi, \eta) \equiv H\chi \eta - \chi \eta \). The strong factorization \( \mathcal{H}^1(\mathbb{C}+) = \mathcal{H}^2(\mathbb{C}+) \cdot \mathcal{H}^2(\mathbb{C}+) \) of analytical Hardy spaces, see e.g. [54] for a proof, yields the surjectivity result

\[
\mathcal{H}^1(\mathbb{R}) = \left\{ T(\chi, \eta) : \chi, \eta \in L^2(\mathbb{R}) \right\}.
\] (2.9)

Thus, in this case, \( \Lambda_1 = \mathcal{H}^1(\mathbb{R}) \).

Another example is obtained by considering the operator \( J: W^{1, np}(\mathbb{R}^n, \mathbb{R}^n) \to \mathcal{H}^p(\mathbb{R}^n) \), where \( n \geq 2 \) and \( p \in [1, \infty) \); we emphasise that the Sobolev space is inhomogeneous. In this case, \( \Lambda_\infty \) is meagre in \( \mathcal{H}^p(\mathbb{R}^n) \), see [55] and Corollary 6.3. However, if we instead consider the Jacobian as defined on \( W^{1, np} \), then \( \Lambda_\infty = \mathcal{H}^p(\mathbb{R}^n) \) by the results of [46], although it is unclear whether this is optimal. We note that for \( J: W^{1, 2p}(\mathbb{R}^2, \mathbb{R}^2) \to \mathcal{H}^p(\mathbb{R}^2) \), the statement \( \Lambda_1 = \mathcal{H}^p(\mathbb{R}^2) \) is equivalent to

\[
\mathcal{H}^p(\mathbb{R}^2) = \left\{ |S \omega|^2 - |\omega|^2 : \omega \in L^{2p}(\mathbb{R}^2, \mathbb{R}^2) \right\},
\]

compare with (2.9). Here \( S \) is the Beurling–Ahlfors transform, which one may think of as the square of a complex Hilbert transform [49].

We are not aware of operators satisfying the assumptions of Theorem A and for which there is \( 1 < m \in \mathbb{N} \) such that \( \Lambda_m = Y^* \) but \( \bigcup_{m' < m} \Lambda_m' \) is meagre in \( Y^* \).

### 3 Tools from Geometric Function Theory

This section collects, for the convenience of the reader, useful known results about Sobolev maps and mappings of finite distortion. These results will only be used in relation to the Jacobian determinant in Section 4.

#### 3.1 The Lusin (N) property and the change of variables formula

The following notions are very relevant in relation to the change of variables formula:

**Definition 3.1.** Let \( u: \Omega \to \mathbb{R}^n \) be a continuous map which is differentiable a.e. in \( \Omega \). Then:
(i) $u$ has the Lusin (N) property if $|u(E)| = 0$ for any $E \subset \Omega$ such that $|E| = 0$;
(ii) $u$ has the (SA) property if $|u(E)| = 0$ for any open set $E \subset \Omega$ with $Ju = 0$ a.e. in $E$.

In the one-dimensional case, the Lusin (N) property is well understood: for instance, on an interval, a continuous function of bounded variation has the Lusin (N) property if and only if it is absolutely continuous. However, in higher dimensions, the situation is much more complicated, although we have the following characterisation, proved in [59]:

Proposition 3.2. Let $u \in W^{1,n}(\Omega, \mathbb{R}^n)$ be a continuous map with $Ju \geq 0$ in $\Omega$. Then $u$ has the Lusin (N) property if and only if it has the (SA) property.

We remark that Proposition 3.2 is in general false if $Ju \not\geq 0$, see [67] for a counterexample.

The following result, see [58], is also useful for our purposes:

Proposition 3.3. Let $u \in W^{1,n}(\Omega, \mathbb{R}^n)$ be a continuous map such that, for some $K \geq 1$,
\[ \text{diam}(u(B_r(x))) \leq K \text{diam}(u(\partial B_r(x))) \text{ for all } B_r(x) \Subset \Omega. \] (3.1)
Then $u$ has the Lusin (N) property.

The change of variables formula is closely related to the Jacobian determinant:

Theorem 3.4. Let $u \in C^0(\Omega, \mathbb{R}^n) \cap W^{1,n}(\Omega, \mathbb{R}^n)$ be a map with the Lusin (N) property. Then
\[ \int_E |Ju| \, dx = \int_{\mathbb{R}^n} \mathcal{N}(y, u, E) \, dy \quad \text{for all measurable sets } E \subset \Omega, \] (3.2)
where $\mathcal{N}$ is the multiplicity function, defined as $\mathcal{N}(y, u, E) \equiv \# \{ x \in E : u(x) = y \}$.

The reader may find the proof of Theorem 3.4, together with a wealth of information on geometric properties of Sobolev maps, in [35].

3.2 Mappings of finite distortion

In this subsection we recall some useful facts about mappings of finite distortion and, for simplicity, we focus on the planar case $n = 2$, see [2]. The reader can also find these and higher-dimensional results in [43, 49].

Definition 3.5. Let $u \in W^{1,1}(\Omega, \mathbb{R}^2)$ be such that $0 \leq Ju \in L^1_{\text{loc}}(\Omega)$. We say that $u$ is a map of finite distortion if there is a function $K : \Omega \to [1, \infty]$ such that $K < \infty$ a.e. in $\Omega$ and
\[ |Du(x)|^2 \leq K(x) Ju(x) \quad \text{for a.e. } x \in \Omega. \]

If $u$ has finite distortion, we can set $Ku(x) = \frac{|Du(x)|^2}{Ju(x)}$ if $Ju(x) \neq 0$ and $Ku(x) = 1$ otherwise; this function is the (optimal) distortion of $u$.

We note that, in Definition 3.5, $|\cdot|$ denotes the operator norm of a matrix.

We summarise here some of the key analytic and topological properties of mappings of finite distortion in the plane:

Theorem 3.6. Let $\Omega \subset \mathbb{R}^2$ and let $u \in W^{1,2}_{\text{loc}}(\Omega, \mathbb{R}^2)$ be a map of finite distortion. Then:
(i) $u$ has a continuous representative and, whenever $r < R$ and $B_R(x_0) \subset \Omega$,
\[
\left( \text{osc}_{B_r(x_0)} u \right)^2 \leq \frac{C}{\log(R/r)} \int_{B_R(x_0)} |Du|^2 \, dx;
\]

(ii) $u$ has the Lusin (N) property;

(iii) $u$ is differentiable a.e. in $\Omega$;

(iv) if $Ku \in L^1(\Omega)$ then $u$ is open and discrete;

(v) if $Ku \in L^1(\Omega)$ then for each $\Omega' \Subset \Omega$ there is $m = m(\Omega')$ such that
\[
N(y, u, \Omega') \leq m \quad \text{for all } y \in u(\Omega').
\]

Whenever $u$ is a map of finite distortion we always implicitly assume that $u$ denotes the continuous representative of the equivalence class in $W^{1,2}_{\text{loc}}(\Omega, \mathbb{R}^2)$. If $u$ is such that $Ku \in L^1(\Omega)$, we say that $u$ has integrable distortion; the theory of such maps was pioneered in [50].

We remark that the first three properties of Theorem 3.6 are a consequence of the fact that mappings of finite distortion are monotone in the sense of Lebesgue:

Proposition 3.7. Let $u \in W^{1,2}_{\text{loc}}(\Omega, \mathbb{R}^2)$ be a map of finite distortion; then (3.1) holds. In fact, if we measure the diameter in $\mathbb{R}^2$ with respect to the $\ell^\infty$ norm, we can take $K = 1$.

4 Applications to the Jacobian equation

This section expands on the relation between Theorem A and Question 1.2. We begin by discussing submersions, as it is well-known that they form a subclass of the class of open operators. We show that, for $p \in [1, 2)$, there are no submersions between the spaces in (1.1). As a consequence, we deduce in Corollary 4.11 the non-differentiability of an hypothetical right inverse of the Jacobian. In a different direction, we also combine the results from Section 3 with Theorem A in order to study solutions of the Jacobian equation, and in particular we prove Theorem C.

4.1 The Jacobian is a submersion nowhere

Let $X, Y$ be Banach spaces. We use the following terminology:

Definition 4.1. An operator $T: X \to Y$ is said to be a submersion at $x_0 \in X$ if $T$ is Gâteaux-differentiable at $x_0$ and $T'(x_0): X \to Y$ is onto. It is said to be a regular submersion at $x_0$ if additionally $\ker T'(x_0)$ is complemented in $X$.

We note that, in the literature, the word submersion often refers to a regular submersion. By analogy to the finite-dimensional case, if $T$ is a submersion at $x_0$ then it is open at $x_0$, see for instance [30, Corollary 15.2]:

Theorem 4.2. Let $T: X \to Y$ be a locally Lipschitz submersion at $x_0 \in X$. For all $R > 0$ sufficiently small, there is $r > 0$ such that $B_r(T(x_0)) \subseteq T(B_R(x_0))$. 

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The submersion condition also plays an important role in Lyusternik’s theory of constrained variational problems, through the existence of Lagrange multipliers. We remark that, in that setting, it is customary to deal with regular submersions. Here we do not discuss further the existence of Lagrange multipliers nor their properties, referring instead the interested reader to [81, §43] for their general theory. In the context of Question 1.2, Lagrange multipliers were considered in the third author’s doctoral thesis [55].

The main result of this subsection is Proposition 4.4, which shows that Theorem 4.2 does not apply to the Jacobian. We begin with the following straightforward lemma:

**Lemma 4.3.** Suppose \( T: X \to Y \) is Gâteaux-differentiable. If \( Y^* \) does not embed into \( X^* \) then \( T \) is a submersion at no point.

**Proof.** We prove the contrapositive. Suppose \( T \) is a submersion at some \( x_0 \in X \), that is, \( L \equiv T'(x_0): X \to Y \) is onto. By the classical open mapping theorem, \( L^*: Y^* \to X^* \) is bounded from below and is thus an isomorphism onto its image. Thus \( Y^* \) embeds into \( X^* \).

The main result of this section is the following:

**Proposition 4.4.** Let \( p \in [1, 2) \) and suppose \( T: W^{1,np}(\mathbb{R}^n, \mathbb{R}^n) \to H^p(\mathbb{R}^n) \) is Gâteaux-differentiable. Then \( T \) is a submersion at no point.

The range of \( p \) in Proposition 4.4 is optimal, see Remark 4.10.

**Proof.** The case \( p = 1 \) is simple: \((H^1(\mathbb{R}^n))^* = \text{BMO}(\mathbb{R}^n)\) is not reflexive and thus it cannot embed into a reflexive space, such as \( W^{1,n}(\mathbb{R}^n, \mathbb{R}^n)^* \).

For \( p \in (1, 2) \), we begin by using the isomorphism \((−\Delta)^{1/2}: W^{1,np}(\mathbb{R}^n, \mathbb{R}^n) \to L^{np}(\mathbb{R}^n, \mathbb{R}^n)\). Thus it suffices to show that \( L^{p'} \) does not embed into \( L^{(np)'} \) for \( p \in (1, 2) \), where \( q' \) denotes the Hölder conjugate of \( q \). Since \( p' > 2 \), we appeal to Lemma 4.5 below to finish the proof.

Thus, it remains to prove the next lemma, where \( H \) is a Hilbert space.

**Lemma 4.5.** Let \( p \in [1, 2], q \in [1, \infty) \). If \( L^q(\mathbb{R}^n) \) embeds into \( L^p(\mathbb{R}^n, H) \) then \( 1 \leq p \leq q \leq 2 \).

This result is well-known to the experts and a very complete statement can be found in [1, Theorem 6.4.20 and Proposition 12.1.10], which we quote here:

**Proposition 4.6.** Let \( p, q \in [1, \infty) \). Then \( L^q(\mathbb{R}^n) \) embeds into \( L^p(\mathbb{R}^n) \) if and only if one of the following conditions holds:

(i) \( 1 \leq p \leq q \leq 2 \),

(ii) \( 2 < p < \infty \) and \( q \in \{2, p\} \).

Moreover, if \( 1 < p, q \) then \( L^q \) embeds complementably into \( L^p \) if and only if \( q \in \{2, p\} \).

Lemma 4.5 is essentially deduced from Proposition 4.6, as the vector-valued \( L^p \) space poses only minor changes to the proof. We sketch a proof of Lemma 4.5 here, in order to improve the readability of the paper. The proof relies on the notions of (Rademacher) type and cotype of a Banach space:
Definition 4.7. Let \((\varepsilon_i)_{i=1}^{\infty}\) be a sequence of i.i.d. random variables such that

\[ P(\varepsilon_i = 1) = P(\varepsilon_i = -1) = \frac{1}{2}. \]

A Banach space \(X\) has type \(p\), \(p \in [1, 2]\) if there is a constant \(C\) such that

\[ \left( \mathbb{E} \left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\|^p \right)^{1/p} \leq C \left( \sum_{i=1}^{n} \|x_i\|^p \right)^{1/p} \]

for any vectors \(x_i \in X\). Likewise, \(X\) has cotype \(q\), \(q \in [2, +\infty]\), if there is \(C\) such that

\[ \left( \sum_{i=1}^{n} \|x_i\|^q \right)^{1/q} \leq C \left( \mathbb{E} \left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\|^q \right)^{1/q} \]

for any vectors \(x_i \in X\).

The range of \(p\) and \(q\) in the definitions of type and cotype are natural and are determined by Khintchine’s inequality. Moreover, if \(X\) is of type \(p\) then it is also of type \(r\) for any \(r < p\); if it is of cotype \(q\), it is also of cotype \(r\) for any \(r > q\).

Example 4.8. As before, \(X\) is a Banach space.

(i) A Hilbert space \(H\) has type and cotype 2: this follows from the parallelogram law.

(ii) If \(X\) has type \(p\) then \(X^*\) has cotype \(p'\), although the converse is not true.

(iii) If \(p \in [1, 2]\) then \(\ell^p\) has type \(p\) and if \(p \in [2, +\infty]\) then \(\ell^p\) has cotype \(p\). Moreover, these values are optimal, as can be seen by considering the standard basis.

(iv) If \(X\) has type \(p\) and cotype \(q\), the space \(L^r(\mathbb{R}^n, X)\) has type \(\min\{r, p\}\) and cotype \(\max\{r, q\}\).

The reader may find details and further examples in [1, 45].

Proof of Lemma 4.5. Clearly type and cotype are inherited by subspaces. Thus, if \(p \in [1, 2]\) and if \(L^q(\mathbb{R}^n)\) embeds into \(L^p(\mathbb{R}^n, H)\), then \(L^q(\mathbb{R}^n)\) must have type \(p\) and cotype 2. Since \(\ell^q\) embeds into \(L^q(\mathbb{R}^n)\), the same can be said for \(\ell^q\). Hence, the optimality in Example 4.8(iii) shows that \(p \leq q \leq 2\).

Remark 4.9. Inspection of the proof reveals that, in Proposition 4.4, the following stronger conclusion holds: for any \(u \in \dot{W}^{1, np}(\mathbb{R}^n, \mathbb{R}^n), T_u^* \colon \dot{W}^{1, np}(\mathbb{R}^n, \mathbb{R}^n) \to H^p(\mathbb{R}^n)\) does not have closed range. This condition also appears naturally in relation to the existence of Lagrange multipliers, see e.g. [30, §26.2].

Remark 4.10. Proposition 4.4 does not hold when \(p = 2\), even when \(T\) is linear. Indeed, consider the operator \(T = (-\Delta)^{1/2} \circ \pi\), where \(\pi \colon L^{2n}(\mathbb{R}^n, \mathbb{R}^n) \to L^2(\mathbb{R}^n)\) is the projection given by Proposition 4.6(ii). That \(\pi\) can taken to be continuous follows from the fact that \(L^2\) embeds complementably in \(L^{2n}\); hence \(T\) is continuous as well. The operator \(T\) is clearly surjective and, being linear, is a submersion.

Curiously, a weaker version of Proposition 4.4 holds in the case \(p > 2\): all operators \(T \colon \dot{W}^{1, np}(\mathbb{R}^n, \mathbb{R}^n) \to L^p(\mathbb{R}^n)\) are regular submersions nowhere. This follows from arguments similar to the ones above, using the last part of Proposition 4.6.
We note the following consequence of Proposition 4.4 and the previous remark, which should be compared with Conjecture 1.2:

**Corollary 4.11.** Fix \( p \in [1, +\infty) \setminus \{2\} \). If \( J : \dot{W}^{1,p}(\mathbb{R}^n, \mathbb{R}^n) \to \mathcal{H}^p(\mathbb{R}^n) \) is onto then all of its right-inverses are Fréchet-differentiable nowhere.

**Proof.** The statement follows from Proposition 4.4 together with the chain rule. For the case \( p > 2 \) we use the last part of Remark 4.10, noting that if \( J \) has a right-inverse which is Fréchet-differentiable at \( u_0 \) then \( J'_{u_0} \) has a complemented kernel. \( \square \)

We conclude this subsection by discussing results related to Proposition 4.4. In the finite-dimensional case, the class of submersions is a good approximation for the class of open operators. This is made precise by the following proposition, which follows immediately from the Morse–Sard theorem:

**Proposition 4.12.** Let \( F : X \to Y \) be a smooth, surjective operator between two finite-dimensional vector spaces. The set of points where \( F \) is a submersion is dense in \( X \).

Although there are versions of the Morse–Sard theorem in the infinite-dimensional setting, such a result fails completely in the absence of rather strong assumptions: for instance, a well-known version due to Smale [72] requires the operator to have Fredholm derivatives. In our context, the failure of the infinite-dimensional Morse–Sard theorem, and consequently of Proposition 4.12, is exemplified in a particularly striking way through the following result:

**Theorem 4.13.** Take \( p \in [1, +\infty) \) and let \( Y \) be a separable Banach space. There is a smooth, locally Lipschitz, surjective operator \( T : \dot{W}^{1,p}(\mathbb{R}^n, \mathbb{R}^n) \to Y \). If \( p > 1 \) then \( T \) can be taken to additionally satisfy \( \text{rank}(T'_u) \leq 1 \) for all \( u \in \dot{W}^{1,p}(\mathbb{R}^n, \mathbb{R}^n) \).

For the proof of Theorem 4.13 we refer the reader to the work of Bates [7], for the \( p > 1 \) case, as well as to [9, Proposition 11.25]. Our interpretation of Theorem 4.13 is that the possible lack of surjectivity of the operator \( J : \dot{W}^{1,p}(\mathbb{R}^n, \mathbb{R}^n) \to \mathcal{H}^p(\mathbb{R}^n) \) cannot be proved by very general Banach space geometrical considerations in the spirit of this subsection.

### 4.2 Existence of well-behaved solutions

In this subsection we focus on the case \( n = 2 \) for simplicity and we assume throughout that \( J : \dot{W}^{1,p}(\mathbb{R}^2, \mathbb{R}^2) \to \mathcal{H}^p(\mathbb{R}^2) \) is surjective. We are particularly interested in the case \( p = 1 \). Our goal is to illustrate the way in which Theorem A yields the following principle:

**the existence of rough solutions implies the existence of well-behaved solutions.**

The following is an example a rough solution, and something that we would like to avoid:

**Example 4.14 ([58]).** There is a map \( u \in W^{1,2}(\mathbb{R}^2, \mathbb{R}^2) \) such that

\[
Ju = 0 \text{ a.e. in } \mathbb{R}^2 \quad \text{and} \quad u([0, 1] \times \{0\}) = [0, 1]^2.
\]

In particular, \( u \) does not have the Lusin (N) property.

The main result of this subsection is the following theorem, which shows that in some sense it suffices to deal with non-pathological solutions.
Theorem 4.15. Let $\Omega \subset \mathbb{R}^2$ be a bounded open set and take $f \in \mathcal{H}^1(\mathbb{R}^2)$ such that $f \geq 0$ in $\Omega$. Assume that $J : \dot{W}^{1,2}(\mathbb{R}^2, \mathbb{R}^2) \to \mathcal{H}^1(\mathbb{R}^2)$ is onto. Then there is a solution $u \in \dot{W}^{1,2}(\mathbb{R}^2, \mathbb{R}^2)$ of (1.2) such that:

(i) $u$ is continuous in $\Omega$;
(ii) $u$ has the Lusin (N) property in $\Omega$.
(iii) $\int_{\mathbb{R}^2} |Du|^2 \, dx \leq C\|f\|_{\mathcal{H}^1}$ with $C > 0$ independent of $f$.

In particular, $u$ satisfies the change of variables formula (3.2). Moreover, let $\Omega' \subset \Omega$ be an open set such that $f = 0$ a.e. in $\Omega'$. Then:

(iv) for any set $E \subset \Omega'$, we have $u(\partial E) = u(\overline{E})$;
(v) for $y \in u(\Omega')$, if $C$ denotes a connected component of $u^{-1}(y) \cap \Omega$ then $C$ intersects $\partial \Omega'$.

Before proceeding with the proof, we note that (iv) is a type of degenerate monotonicity which had already appeared in the study of the hyperbolic Monge–Ampère equation [19, 52].

Proof. The point of the proof is to perturb $f$ appropriately; then the solution $u$ is obtained as a limit of mappings of integrable distortion.

Let $B^+$ be a ball containing $\Omega$ and let $B^-$ be another ball, disjoint from $\Omega$, and with the same volume as $B^+$. Consider the perturbations

$$f_\varepsilon \equiv f + \varepsilon a, \quad a \equiv \chi_{B^+} - \chi_{B^-},$$

which satisfy $f_\varepsilon > 0$ a.e. in $\Omega$. Clearly $a \in \mathcal{H}^1(\mathbb{R}^2)$, being bounded, compactly supported and with zero mean. Hence $f_\varepsilon \to f$ in $\mathcal{H}^1(\mathbb{R}^2)$ and, from Corollary B, we see that we can choose solutions $u_\varepsilon$ of $Ju_\varepsilon = f_\varepsilon$ such that $\int_{\mathbb{R}^2} |Du_\varepsilon|^2 \leq C\|f_\varepsilon\|_{\mathcal{H}^1}$ for all $\varepsilon > 0$. Since the maps $u_\varepsilon$ have finite distortion, we can apply Theorem 3.6(i) to conclude that the family $(u_\varepsilon)$ is equicontinuous. Hence, upon normalising the maps so that $u_\varepsilon(x_0) = 0$ for some fixed $x_0 \in \Omega'$, and up to taking subsequences, $(u_\varepsilon)$ converges both locally uniformly in $\Omega$ and weakly in $\dot{W}^{1,2}(\mathbb{R}^2, \mathbb{R}^2)$ to a limit $u$. This already proves (i) and (iii).

To prove (ii), we note that each $u_\varepsilon$ satisfies (3.1), c.f. Proposition 3.7. Since $u$ is the uniform limit of the sequence $(u_\varepsilon)$, $u$ also satisfies (3.1) and (ii) follows from Proposition 3.3.

For (iv), note that $\varepsilon \leq f_\varepsilon$ in $\Omega$ and so each map $u_\varepsilon$, having integrable distortion, is open; it follows that $\partial u_\varepsilon(E) \subseteq u_\varepsilon(\partial E)$. Suppose, for the sake of contradiction, that there is $y \in u(E) \setminus u(\partial E)$. On the one hand, there is some $\delta > 0$ such that, for all $\varepsilon$ small enough,

$$B_\delta(y) \cap \partial u_\varepsilon(\text{int} E) \subset B_\delta(y) \cap u_\varepsilon(\partial E) = \emptyset;$$
on the other hand, since $y \in u(\text{int} E)$, for all $\varepsilon$ small enough,

$$B_\delta(y) \cap u_\varepsilon(\text{int} E) \neq \emptyset.$$

It follows that $B_\delta(y) \subseteq u_\varepsilon(\text{int} E)$. We also have that $|u_\varepsilon(\text{int} E)| \to 0$ as $\varepsilon \to 0$: by the change of variables formula,

$$|u_\varepsilon(\text{int} E)| \leq \int_{u_\varepsilon(\text{int} E)} \mathcal{N}(y, u_\varepsilon, \text{int} E) \, dy = \int_E Ju_\varepsilon = \varepsilon |E| \to 0.$$

Thus, since $|B_\delta(y)| \leq |u_\varepsilon(E)|$, a contradiction is reached by sending $\varepsilon \to 0$.

Finally, (v) follows from (iv), as shown for instance in [52, Lemma 2.10].

\qed
In view of the change of variables formula, it is useful to control the multiplicity function. For the following proposition we again assume that the Jacobian is surjective.

**Proposition 4.16.** Let $\Omega \subset \mathbb{R}^2$ be an open set and let $Y := \{ f \in \mathcal{H}^p(\mathbb{R}^2) : f \geq c$ a.e. in $\Omega \}$, where $c > 0$. Suppose that $f_j \in Y$ is a sequence converging weakly to $f$ in $\mathcal{H}^p(\mathbb{R}^2)$. For any maps $u_j \in \dot{W}^{1;2p}(\mathbb{R}^2, \mathbb{R}^2)$ satisfying $J u_j = f_j$ and the a priori estimate (1.4), we have that

$$\sup_j \sup_{y \in u_j(\Omega')} N(y, u_j, \Omega') < \infty, \quad \text{whenever } \Omega' \subset \Omega.$$

**Proof.** We claim that the sequence $u_j$ is equicontinuous and converges to $u \in \dot{W}^{1,2p}(\mathbb{R}^2, \mathbb{R}^2)$, a solution of $J u = f$, uniformly in $\Omega'$. Once the claim is proved, the conclusion follows: $u$ has integrable distortion in $\Omega$ and so by Theorem 3.6(v) it is at most $m$-to-one in $\Omega'$, for some $m \in \mathbb{N}$. Thus, for all $j$ sufficiently large, $u_j$ is also at most $m$-to-one in $\Omega'$: if not, there are arbitrarily large $j$ and points $x_1^{(j)}, \ldots, x_{m+1}^{(j)} \in \Omega'$ such that $u_j(x_i^{(j)}) = y$ for some $y \in \mathbb{R}^n$ and all $i \in \{1, \ldots, m + 1\}$. By compactness, we can further assume that $x_i^{(j)} \to x_i$ for $i = 1, \ldots, m + 1$. However, there are at least two different points $y_1 \neq y_2$ such that

$$\{y_1, y_2\} \subset u(\{x_1, \ldots, x_{m+1}\});$$

for the sake of definiteness, say $u(x_1) = y_1, u(x_2) = y_2$. Let $\varepsilon < |y_1 - y_2|$ and take $j$ sufficiently large so that, for $i = 1, 2$,

$$|u_j(x_i^{(j)}) - u_j(x_i)| < \frac{\varepsilon}{4}, \quad |u_j(x_i) - u(x_i)| < \frac{\varepsilon}{4};$$

this is possible from equicontinuity of the sequence $u_j$ and the fact that it converges to $u$ uniformly. The triangle inequality gives $|y_1 - y_2| = |u(x_1) - u(x_2)| < \varepsilon$, a contradiction.

To prove the claim, we assume that the Jacobian is surjective and we use Corollary B. If $p > 1$ we appeal to Morrey’s inequality,

$$[u_j]_{C^{0,1-2/p}(\mathbb{R}^2)} \lesssim_p \|Du_j\|_{L^{2p}(\mathbb{R}^2)} \leq C,$$

while for $p = 1$ we use Theorem 3.6(i) instead. Either way, after normalizing the maps so that $u_j(x_0) = 0$, where $x_0 \in \Omega$, we see that the sequence $(u_j)$ is precompact in the local uniform topology over $\Omega'$.

Hence we may assume that $u_j$ converges to some map $u \in \dot{W}^{1,2p}(\mathbb{R}^2, \mathbb{R}^2)$ uniformly in $\Omega'$ and also weakly in $\dot{W}^{1,2p}(\mathbb{R}^2, \mathbb{R}^2)$. $\square$

## 5 A general nonlinear open mapping principle for scale-invariant problems

The main result of this section is Theorem 5.2, which is a generalisation of Theorem A to a wider class of translation-invariant, scaling-invariant PDEs. Section 5.3 illustrates the way in which Theorem 5.2 can be applied to some physical nonlinear equations: as particular examples, we consider the Navier–Stokes equations and the cubic wave equation.
5.1 A more general nonlinear open mapping principle

We begin by formulating a model problem abstractly as follows:

\[
\text{if } g - 1 \in Y^*, \text{ does there exist } v \text{ with } Jv = f \text{ and } v - \text{id} \in X^? \tag{5.1}
\]

Here \(X^\) and \(Y^\) are suitably chosen function spaces. In smooth domains \(\Omega \subseteq \mathbb{R}^n\), examples of (5.1) include the Dirichlet problem for the Jacobian equation, that is,

\[
\begin{cases}
Jv = g \text{ in } \Omega, \\
v = \text{id} \text{ on } \partial \Omega;
\end{cases} \tag{5.2}
\]

the condition \(g - 1 \in Y^\) is codified in the compatibility condition

\[
\int_\Omega (g - 1) \, dx = 0.
\]

We return to the abstract formulation (5.1). When \(n = 2\), denoting \(f \equiv g - 1\) and \(u \equiv v - \text{id}\) we get the following question, equivalent to (5.1):

\[
\text{if } f \in Y^*, \text{ does there exist } u \in X^\text{ with } Tu \equiv Ju + \text{div} u = f? \tag{5.3}
\]

The latter formulation has the advantage that \(X^\) and \(Y^\) are vector spaces, which makes the problem more amenable to scaling arguments.

In Example 5.1 we discuss a representative special case of (5.3). Here \(T\) does not map \(X^\) into \(Y^\) and we therefore need to choose a set \(D \subseteq X^\) as the domain of definition of \(T\).

**Example 5.1.** Let \(X^ = \dot{W}^{1,q}(\mathbb{R}^2, \mathbb{R}^2)\) and \(Y^ = L^p(\mathbb{R}^2)\) with \(p \in [2, \infty)\) and \(q \in [p, 2p]\). Since \(T = J + \text{div}\) does not map \(\dot{W}^{1,q}(\mathbb{R}^2, \mathbb{R}^2)\) into \(L^p(\mathbb{R}^2)\), it is natural to set

\[
D = \{ u \in \dot{W}^{1,q}(\mathbb{R}^2, \mathbb{R}^2): Tu \in L^p(\mathbb{R}^2) \}
\]

and study the range of

\[
T = J + \text{div}: D \rightarrow Y^\tag{5.4}
\]

Note that we may write \(T \circ \tau^D_\lambda = \tau^{Y^}_\lambda \circ T\) for all \(\lambda > 0\), where

\[
\tau^D_\lambda u(x) = u_\lambda(x) = \lambda u \left( \frac{x}{\lambda} \right), \quad \tau^{Y^}_\lambda f(x) = f_\lambda(x) = f \left( \frac{x}{\lambda} \right)
\]

give multiples of isometries:

\[
\|\tau^D_\lambda u\|_{\dot{W}^{1,q}} = \lambda^{2/q} \|u\|_{\dot{W}^{1,q}}, \quad \|\tau^{Y^}_\lambda f\|_{L^p} = \lambda^{2/p} \|f\|_{L^p}
\]

for all \(u \in D, f \in Y^\) and \(\lambda > 0\).

Since the set \(D\) contains the proper dense subspace \(C^\infty_c(\mathbb{R}^2, \mathbb{R}^2)\), it is neither weakly nor strongly closed in \(\dot{W}^{1,q}(\mathbb{R}^2, \mathbb{R}^2)\). This difficulty is reflected in the somewhat awkward assumption (A4) of Theorem 5.2 below.

Before formulating the result recall that when a direct sum of Banach spaces \(X = \oplus_{i=1}^I X_i\) is endowed with the norm \(\|w\|_X = \sum_{i=1}^I \|w_i\|_{X_i}\), the dual norm of \(X^* = \oplus_{i=1}^I X_i^*\) is of the form \(\|u\|_{X^*} = \max_{1 \leq i \leq I} \|u_i\|_{X_i^*}\).
Theorem 5.2. Let $X_1, \ldots, X_I$ and $Y_1, \ldots, Y_J$ be Banach spaces and denote $X = \bigoplus_{i=1}^I X_i$ and $Y = \bigoplus_{j=1}^J Y_j$. Suppose $B_{X^*}$ is sequentially weak* compact and $0 \in D \subset X^*$.

We make the following assumptions:

(\(\tilde{A}_1\)) $T : D \to Y^*$ has a weak*-to-weak* closed graph: if $u_j \to u$ weak* and $Tu_j \to f$ weak*, then $Tu = f$.

(\(\tilde{A}_2\)) For $\lambda > 0$, there exist bijections $\tau^D_\lambda : D \to D$ and $\tau^{Y^*}_\lambda : Y^* \to Y^*$ such that

$$
T \circ \tau^D_\lambda = \tau^{Y^*}_\lambda \circ T \quad \text{for all } \lambda > 0,
$$

$$
\| (\tau^D_\lambda u_i) \|_{X^*_i} = \lambda^i \| u_i \|_{X_i^*} \quad \text{for all } \lambda > 0, \ i = 1, \ldots, I, \ u_i \in X^*,
$$

$$
\| (\tau^{Y^*}_\lambda f)_j \|_{Y^*_j} = \lambda^j \| f_j \|_{Y^*_j} \quad \text{for all } \lambda > 0, \ j = 1, \ldots, J, \ f \in Y^*,
$$

where $0 < r_1 \leq \cdots \leq r_I$ and $0 < s_1 \leq \cdots \leq s_J$.

(\(\tilde{A}_3\)) There exist sequences of isometric bijections $\sigma^D_k : D \to D$ with $\sigma^D_k(0) = 0$ and isometric isomorphisms $\sigma^{Y^*}_k : Y^* \to Y^*$ such that

$$
T \circ \sigma^D_k = \sigma^{Y^*}_k \circ T \quad \text{for all } k \in \mathbb{N}, \quad \sigma^{Y^*}_k f \not\to 0 \quad \text{for all } f \in Y^*.
$$

(\(\tilde{A}_4\)) For $\ell \in \mathbb{N}$, the sets $D_\ell \equiv \{ u \in D : \| u \|_{X^*_\ell} \leq \ell, \| Tu \|_{Y^*_\ell} \leq \ell \}$ are weakly* sequentially closed in $X^*$.

The following conditions are then equivalent:

(i) $T(D)$ is non-meagre in $Y^*$.

(ii) $T(D) = Y^*$.

(iii) $T$ is open at the origin.

(iv) For every $f \in Y^*$ there exists $u \in D$ such that

$$
Tu = f, \quad \begin{cases} 
\sum_{i=1}^I \| u_i \|_{X^*_i}^{s_i/r_i} \leq C \| f \|_{Y^*}, \quad \| f \|_{Y^*} \leq 1, \\
\sum_{i=1}^I \| u_i \|_{X^*_i}^{s_i/r_i} \leq C \| f \|_{Y^*}, \quad \| f \|_{Y^*} > 1.
\end{cases} \tag{5.5}
$$

Proof. We first show (i) $\Rightarrow$ (iii), so assume (i) holds. Write $D = \bigcup_{\ell=1}^\infty D_\ell$ and note that $T(D) = \bigcup_{\ell=1}^\infty T(D_\ell)$. Since $B_{X^*}$ is weak* sequentially compact and $T : D \to Y^*$ has weak*-to-weak* closed graph, the sets $T(D_\ell)$ are closed in $Y$ and, therefore, complete. By the Baire Category Theorem, one of the sets $T(D_\ell)$ contains a ball $B_\eta(f_0)$. Clearly $\eta \leq \ell$. We first show that

$$
T(D \cap \ell \mathbb{B}_{X^*}) \supset \eta \mathbb{B}_{Y^*}. \tag{5.6}
$$

Suppose $f \in Y^*$ with $\| f \|_{Y^*} \leq \eta$. Since the maps $\sigma^{Y^*}_k : Y^* \to Y^*$ are isometries, we get $f_0 + (\sigma^{Y^*}_k)^{-1} f \in B_\eta(f_0) \subset T(D_\ell)$ for every $k \in \mathbb{N}$. For every $k \in \mathbb{N}$, choose $u_k \in D_\ell$ such that $Tu_k = f_0 + (\sigma^{Y^*}_k)^{-1} f$. By (\(\tilde{A}_3\)), each $\sigma^D_k$ maps $D_\ell$ into $D_\ell$. Thus, passing to a subsequence as in the proof of Theorem A, and using (\(\tilde{A}_4\)), $\sigma^D_k u_k \to u \in D_\ell$; by (\(\tilde{A}_1\)) we get $Tu = f$. Thus (5.6) is proved.

We are ready to show openness of $T$ at zero. Let $\varepsilon > 0$; our aim is to find $\delta > 0$ such that $T(D \cap \varepsilon \mathbb{B}_{X^*}) \supset \delta \mathbb{B}_{X^*}$. We first note that for each $\lambda > 0$ we have

$$
\tau^D_\lambda(D \cap \ell \mathbb{B}_{X^*}) = \{ u \in D : \| u_i \|_{X^*_i} \leq \lambda^i \ell \text{ for } i = 1, \ldots, I \}.
$$
By choosing $\lambda = \min_{1 \leq i \leq I} (\varepsilon/\ell)^{1/r_i}$ we get $\max_{1 \leq i \leq I} \lambda^r \ell \leq \varepsilon$ so that

$$T(D \cap \varepsilon B_{X^*}) \supset T(\tau^D_X (D \cap \varepsilon B_{X^*})) = \tau^{Y^*}_X T(D \cap \varepsilon B_{X^*}).$$

By using (5.6) and selecting $\delta = \min_{1 \leq i \leq I} \min_{1 \leq j \leq I} \eta (\varepsilon/\ell)^{s_j/r_i}$ we get

$$\tau^{Y^*}_X T(D \cap \varepsilon B_{X^*}) \supset \tau^{Y^*}_X (\eta B_{Y^*}) = \lambda^{s_i} \eta B_{Y^*_i} \times \cdots \times \lambda^{s_j} \eta B_{Y^*_j} \supset \delta B_{Y^*},$$

as wished.

We now prove (iii) $\Rightarrow$ (iv), so as above take some $\varepsilon > 0$ and get $\delta > 0$ in such a way that $\delta B_{Y^*} \subset T(D \cap \varepsilon B_{X^*})$. Assume, without loss of generality, that $\delta \leq 1$. Let $f \in Y^*$ and define $\lambda > 0$ via

$$\|f\|_{Y^*} = \mu = \min_{1 \leq i \leq I} \lambda^{s_i} \delta = \begin{cases} \lambda^{s_i} \delta, & \mu \leq \delta; \\ \lambda^{s_i} \delta, & \mu > \delta. \end{cases}$$

In either case, let $j_0$ be such that $\mu = \lambda^{s_{j_0}} \delta$. Then

$$f \in \tau^{Y^*}_X (\delta B_{Y^*}) \subset \tau^{Y^*}_X T(D \cap \varepsilon B_{X^*}) = T\tau^D_X (D \cap \varepsilon B_{X^*}) = T\{u \in D : \|u_i\|_{X^*_i} \leq \lambda^r \varepsilon \text{ for } i = 1, \ldots, I\}.$$ 

Suppose now $u \in D$ satisfies $\|u_i\|_{X^*_i} \leq \lambda^r \varepsilon$ for $i = 1, \ldots, I$. Then, for all $i$,

$$\|u_i\|^{s_{j_0}/r_i}_{X^*_i} \leq \lambda^{s_{j_0}} \varepsilon^{s_{j_0}/r_i} \leq \frac{\varepsilon^{s_{j_0}/r_i}}{\delta} \mu.$$

We conclude that

$$f \in T\{u \in D : \|u_i\|_{X^*_i} \leq \lambda^r \varepsilon \text{ for all } i\} \subset T\{u \in D : \sum_{i=1}^I \|u_i\|^{s_{j_0}/r_i}_{X^*_i} \leq C \mu \},$$

where

$$C = \sum_{i=1}^I \frac{\varepsilon^{s_{j_0}/r_i}}{\delta},$$

which yields (5.5) in the cases $\|f\|_{Y^*} \leq \delta$ and $\|f\|_{Y^*} > 1$. If $\|f\|_{Y^*} \in (\delta, 1]$, one obviously has $\lambda^{s_i} \approx \delta \lambda^{s_j}$ so that (5.5) holds for all $f$.

We conclude the proof of the theorem by noting that (iv) $\Rightarrow$ (ii) $\Rightarrow$ (i).

**Remark 5.3.** Inspection of the proof of Theorem 5.2 shows that, in the statement of the theorem, one may replace all occurrences of $Y^*$ with $K$, where $K \subset Y^*$ is a closed convex cone. Recall that $K$ is said to be a cone if $af \in K$ whenever $a > 0$ and $f \in K$.

Such a generalisation is occasionally useful, since it may be interesting to consider smaller data sets. For instance, in the case Question 1.2, natural examples include the set of radially symmetric data $K = \{f \in \mathcal{H}^p(\mathbb{R}^n) : f(x) \equiv f(|x|)\}$ and, when $p > 1$, the set of non-negative data $K = \{f \in L^p(\mathbb{R}^n) : f \geq 0\}$.

Returning to Example 5.1, it is easy to check that the assumptions of the theorem are satisfied, and so we may apply it to get the following:

**Corollary 5.4.** Let $p \in [2, \infty)$ and $q \in [p, 2p]$. The following claims are equivalent:

- (i) $f \in \mathcal{K}_2^p(\mathbb{R}^n)$
- (ii) $f \in \mathcal{K}_q(\mathbb{R}^n)$
- (iii) $S_\lambda f \in \mathcal{K}_q(\mathbb{R}^n)$
- (iv) $f \in \mathcal{K}_q(\mathbb{R}^n)$
(i) For all $f \in L^p(\mathbb{R}^2)$ there exists $u \in \dot{W}^{1,q}(\mathbb{R}^2,\mathbb{R}^2)$ such that $Ju + \text{div } u = f$.

(ii) For all $f \in L^p(\mathbb{R}^2)$ there exists $u \in \dot{W}^{1,q}(\mathbb{R}^2,\mathbb{R}^2)$ with

$$Ju + \text{div } u = f, \quad \|Du\|_{L^q}^q \leq C\|f\|_{L^p}^p.$$ 

**Remark 5.5.** When $\Omega \subset \mathbb{R}^n$ is a smooth, bounded domain, $n \leq p < \infty$, $X = W_0^{1,p}(\Omega, \mathbb{R}^2)$ and $Y^* = L^p(\Omega)/\mathbb{R}$, the question about surjectivity of the operator $T = \text{div} + J$ is closely related to [39, Question 1.2]. It would be interesting to find out whether Theorems A and 5.2 can be adapted to bounded domains.

### 5.2 A version for inhomogeneous function spaces

We are also interested in applying Theorem 5.2 to inhomogeneous function spaces. In order to achieve this we first recall some definitions from interpolation theory:

**Definition 5.6.** Suppose that $X_1$ and $X_2$ are Banach spaces embed into a topological vector space $Z$. We set

$$\|u\|_{X_1 \cap X_2} \equiv \max\{\|u\|_{X_1}, \|u\|_{X_2}\},$$

$$\|u\|_{X_1 + X_2} \equiv \inf\{\|u_1\|_{X_1} + \|u_2\|_{X_2} : u = u_1 + u_2, u_1 \in X_1, u_2 \in X_2\}.$$

If $X_1 \cap X_2$ is dense in both $X_1$ and $X_2$, then $(X_1, X_2)$ is called a *conjugate couple*.

The duals of spaces of the form $X_1 \cap X_2$ are well-known, c.f. [11, Theorem 2.7.1]:

**Theorem 5.7.** Let $(X_1, X_2)$ be a conjugate couple. Then, up to isometric isomorphism, $(X_1 \cap X_2)^* = X_1^* + X_2^*$ and $(X_1 + X_2)^* = X_1^* \cap X_2^*$.

Following the proof of Theorem 5.2 almost verbatim we obtain the following:

**Corollary 5.8.** For $i = 1, \ldots, I$, $j = 1, \ldots, J_i$, and $\mu = 1, \ldots, M$, $\nu = 1, \ldots, N_\mu$ let $X_{i,j}$ and $Y_{\mu,\nu}$ be Banach spaces. Consider $X^*, Y^*$ of the form

$$X^* = \bigoplus_{i=1}^I \left( \bigcap_{j=1}^{J_i} X_{i,j}^* \right), \quad Y^* = \bigoplus_{\mu=1}^M \left( \bigcap_{\nu=1}^{N_\mu} Y_{\mu,\nu}^* \right)$$

for some $I, M, J_i, N_\mu \in \mathbb{N}$.

Suppose assumptions $\text{(A1), (A3) and (A4)}$ of Theorem 5.2 hold. Suppose further that for $\lambda > 0$, there exist bijections $\tau^D_\lambda : D \rightarrow D$ and $\tau^{Y^*}_\lambda : Y^* \rightarrow Y^*$ such that

$$T \circ \tau^D_\lambda = \tau^{Y^*}_\lambda \circ T$$

for all $\lambda > 0$,

$$\|\tau^D_\lambda u_i\|_{X_{i,j}^*} = \lambda^{r_{i,j}} \|u_i\|_{X_{i,j}^*}$$

for all $\lambda > 0$, $i = 1, \ldots, I$, $j = 1, \ldots, J_i$, $u \in X^*$,

$$\|\tau^{Y^*}_\mu f\|_{Y_{\mu,\nu}^*} = \lambda^{s_{\mu,\nu}} \|f\|_{Y_{\mu,\nu}^*}$$

for all $\lambda > 0$, $\mu = 1, \ldots, M$, $\nu = 1, \ldots, N_\mu$, $f \in X^*$,

where $0 < r_{i,j}$ and $0 < s_{\mu,\nu} \leq s_2$.

Then the following conditions are equivalent:

(i) $T(D)$ is non-meagre in $Y^*$.

(ii) $T(D) = Y^*$.
The equations are invariant under the scalings $u \rightarrow \lambda u$, $P \rightarrow P\lambda$ and $u^0 \rightarrow u^0\lambda$.

$$Tu = f, \quad \begin{cases} \sum_{i=1}^{I} \sum_{j=1}^{J_i} \|u_i\|_{X_{r_{i,j}}^s}^{s_2/r_{i,j}} \leq C\|f\|_{Y^*}, & \|f\|_{Y^*} \leq 1, \\
\sum_{i=1}^{I} \sum_{j=1}^{J_i} \|u_i\|_{X_{r_{i,j}}^s}^{s_1/r_{i,j}} \leq C\|f\|_{Y^*}, & \|f\|_{Y^*} > 1. \end{cases}$$

5.3 Two model examples

In this subsection we illustrate the use of Theorem 5.2 in the model cases of the 3D Navier-Stokes equations and the 3D cubic wave equation.

Example 5.9. We illustrate the use of Theorem 5.2 in the model case of the homogeneous, incompressible Navier-Stokes equations in $\mathbb{R}^3 \times [0, \infty)$:

$$\begin{align*}
\partial_t u + u \cdot \nabla u - \nu \Delta u - \nabla P &= 0, \\
\text{div } u &= 0, \\
u u(\cdot,0) &= u^0,
\end{align*}$$

where $u$ is the velocity field, $P$ is the pressure, $\nu > 0$ is the viscosity and $u^0$ is the initial data.

The equations are invariant under the scalings $u \rightarrow u\lambda$, $P \rightarrow P\lambda$ and $u^0 \rightarrow u^0\lambda$.

$$u_\lambda(x,t) \equiv \frac{1}{\lambda} u \left( \frac{x}{\lambda}, \frac{t}{\lambda^2} \right), \quad P_\lambda(x,t) \equiv \frac{1}{\lambda^2} P \left( \frac{x}{\lambda}, \frac{t}{\lambda^2} \right), \quad u^0_\lambda(x) = \frac{1}{\lambda} u^0 \left( \frac{x}{\lambda} \right).$$

We divide the discussion into the following three steps: i) formally determining $T$; ii) choosing relevant ambient spaces $X^*$ and $Y^*$; iii) choosing the domain of definition $D$.

We begin by choosing the operator $T$ we wish to study. We incorporate (5.7)–(5.8) into the choice of the function spaces and choose, formally, $T(u) = u(\cdot,0)$. As the sought range we consider $Y^* = L^2_v = \{v \in L^2(\mathbb{R}^3, \mathbb{R}^3): \text{div } v = 0 \}$. We wish to choose the domain of definition $D$ to be a suitable set of functions which satisfy (5.7)–(5.9) for some $u^0 \in L^2_v$. We also need to determine the ambient space $X^*$.

In order for Theorem 5.2 to be applicable, we wish to consider regularity regimes where $T$ has a weak*–to-weak* closed graph and the sets $D_\ell = \{u \in D: \|u\|_X \leq \ell, \|Tu\|_{Y^*} \leq \ell \}$ are weakly* compact. It is natural to set $X^* = L^p_v(L^q_{\sigma,2})_x \cap L^r_\ell \tilde{W}^{1,s}_x$ for suitable $p, q, r, s \in [1, \infty]$.

For condition (A2) of Theorem 5.2 we compute, for all $p, q, r, s \in [1, \infty]$,

$$\begin{align*}
\|u^0_\lambda\|_{L^q_v} &= \lambda^{3/q-1} \|u^0\|_{L^q_v}, \\
\|u_\lambda\|_{L^p_v L^q_{\sigma,2}} &= \lambda^{2/p+3/q-1} \|u\|_{L^p_v L^q_{\sigma,2}}, \\
\|u_\lambda\|_{L^r_\ell \tilde{W}^{1,s}_x} &= \lambda^{2/r+3/s-2} \|u\|_{L^r_\ell \tilde{W}^{1,s}_x}.
\end{align*}$$

Thus (A2) requires the compatibility condition $2/p + 3/q - 1 = 2/r + 3/s - 2$ to hold.

For simplicity, we consider the most familiar choice of exponents, that is we consider $X^* = L^\infty_\ell L^2_{\sigma,2} \cap L^2_\ell \tilde{W}^{1,2}_x$. Recall that $u \in X$ is called a weak solution of (5.7)–(5.9) if $u$ satisfies

$$\int_0^T \langle u, \partial_t \varphi \rangle \, dt + \int_0^T \langle u \otimes u, D\varphi \rangle \, dt - \nu \int_0^T \langle Du, D\varphi \rangle \, dt + \langle u^0, \varphi(0) \rangle - \langle u(\tau), \varphi(\tau) \rangle = 0 \quad (5.10)$$
for all $\varphi \in C_c^\infty(\mathbb{R}^3 \times [0, \infty), \mathbb{R}^3)$ with $\text{div} \varphi = 0$ and almost every $\tau > 0$. In (5.10), $\langle \cdot, \cdot \rangle$ denotes the inner product in $L_2^\sigma$. This prompts us to set

$$D \equiv \{ u \in L_t^{\infty L_x^2} \cap L_t^{2 \dot{W}_x^{1,2}} : u \text{ is a weak solution of (5.7)–(5.9) for some } u^0 \in L_\sigma^{2} \},$$

$$T : D \to Y^* , \quad T(u) \equiv u^0 \text{ if (5.10) holds.}$$

We briefly indicate why $T : (D, \text{wk}^*) \to (Y^*, \text{wk}^*)$ has weak$^*$-to-weak$^*$ closed graph and why the sets $D_\ell$ are weakly$^*$ closed for all $\ell \in \mathbb{N}$. When $u \in D$, we have $\partial_t u \in L_t^{4/3}(0, \tau, (W_x^{1,2})^*)$ for all $\tau > 0$ (see [69, Lemma 3.7]). Thus, by using the Aubin–Lions lemma and a diagonal argument, if $u_j \xrightarrow{\text{wk}} u$ in $D$, then every subsequence has a subsequence converging strongly in $L_t^2(0, \tau, L_x^2(B_R, \mathbb{R}^3))$ for all $\tau, R > 0$. The strong convergence and (5.10) imply that every subsequence of $(u_j^0)_{j \in \mathbb{N}}$ has a subsequence converging weakly$^*$ to $u^0$. This implies the two claims made above.

Theorem 5.2 now says that solvability of (5.7)–(5.9) for all $u^0 \in L_\sigma^2$ is equivalent to solvability with the a priori estimate

$$\|u\|_{L_t^\infty L_x^2} + \|u\|_{L_t^2 \dot{W}_x^{1,2}} \leq C\|u(\cdot,0)\|_{L^2}.$$ 

Such an estimate is satisfied by Leray–Hopf solutions [69].

**Example 5.10.** Consider the cubic wave equation in $(1 + 3)$-dimensions

$$\begin{align*}
\partial_t u - \Delta u + u^3 &= 0 \text{ in } [0, +\infty) \times \mathbb{R}^3 , \\
(u(\cdot, 0), \partial_t u(\cdot, 0)) &= (u^0, u^1). 
\end{align*}$$

(5.11) (5.12)

We are interested in initial data in the energy space $Y^* = [\dot{H}_x^1(\mathbb{R}^3) \cap L^4(\mathbb{R}^3)] \times L^2(\mathbb{R}^3)$ and we look for solutions in the space

$$X^* = L_t^{\infty} \dot{H}_x^1 \cap L_t^{\infty} L_x^4 \cap L_t^\rho L_x^\sigma([0, \infty) \times \mathbb{R}^3), \quad \text{ where } \frac{1}{\rho} + \frac{3}{\sigma} = \frac{1}{2}. $$

In the notation of Corollary 5.8, we have $X_{1,1}^* = L_t^{\infty} \dot{H}_x^1$, $X_{1,2}^* = L_t^{\infty} L_x^4$, $X_{1,3}^* = L_t^\rho L_x^\sigma$ and $Y_{1,1}^* = \dot{H}_x^1$, $Y_{1,2}^* = L^4$ and $Y_{1,3}^* = L^2$. Recall that $u \in X^*$ is a weak solution of (5.11)–(5.12) if, for every test function $\varphi \in C_c^\infty(\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3)$,

$$\int_0^\tau u \partial_t \varphi + \langle Du, D\varphi \rangle \, dt + \langle u^3, \varphi \rangle \, dt = \langle u^0, \varphi(0, \cdot) \rangle - \langle u^1, \varphi(0, \cdot) \rangle$$

$$- \langle u(\tau, \cdot), \varphi(\tau, \cdot) \rangle + \langle u(\tau, \cdot), \varphi(\tau, \cdot) \rangle$$

(5.13)

for almost every $\tau > 0$. The equation is invariant under translations as well as the scalings $u \to u_\lambda$, $u^0 \to u^0_\lambda$ and $u^1 \to u^1_\lambda$ where

$$u_\lambda(t, x) \equiv \lambda u(\lambda t, \lambda x) \quad u^0_\lambda(x) \equiv \lambda u^0_\lambda(\lambda x) \quad u^1_\lambda(x) \equiv \lambda^2 u^1_\lambda(\lambda^2 x).$$

We formally define

$$D = \{ u \in X^* : u \text{ is a weak solution of (5.11)–(5.12) for some } (u^0, u^1) \in Y^* \},$$

$$T : C \to Y^*, \quad Tu = (u^0, u^1) \text{ if (5.13) holds.}$$
It is easy to compute
\[
\|u_0\|_{L^4} = \lambda^{\frac{2}{7}} \|u_0\|_{L^4}, \quad \|u\|_{L^4} = \lambda^{\frac{2}{7}} \|u\|_{L^4}, \quad \|u\|_{L^4} = \lambda^{\frac{2}{7}} \|u\|_{L^4}.
\]
Thus \(r_{11} = r_{13} = \frac{1}{2}, r_{12} = \frac{1}{4}\) and \(s_{11} = s_{21} = \frac{1}{2}, s_{12} = \frac{1}{4}\) in the notation of Corollary 5.8.

The subcritical nature of (5.11) implies that \(T: (D, \text{wk}^*) \to (Y^*, \text{wk}^*)\) has a closed graph and that the sets \(D_\ell\) are weakly* closed for all \(\ell \in \mathbb{N}\). Indeed, when \(u \in D\), it is not difficult to see that \(\partial_\ell u \in L^\infty(0, \tau; (\mathcal{H}^1 \cap L^4)^*)\) for all \(\tau > 0\): it suffices to test the weak formulation (5.13) against functions of the type \(\varphi(t, x) = \alpha(t)\phi(x)\) where both \(\alpha\) and \(\phi\) are test functions and \(\alpha \equiv 1\) on a given time interval \([t_1, t_2]\). Thus, by using the Aubin–Lions lemma and a diagonal argument, if \(u_j \rightharpoonup u\) in \(D\), then every subsequence has a subsequence converging strongly in \(L^2(0, \tau, L^4(B_R; \mathbb{R}^3))\) for all \(\tau, R > 0\). The strong convergence and (5.13) imply that every subsequence of \((u_j^n, u_j^n)_{j \in \mathbb{N}}\) has a subsequence converging weakly* to \((u^0, u^1)\). This in turn implies the two claims made above.

Corollary 5.8 now says that solvability of (5.11)–(5.12) for all \((u^0, u^1) \in [\mathcal{H}^1 \cap L^4] \times L^2\) is equivalent to solvability of the a priori estimate
\[
\begin{align*}
\|u\|_{L^{2}t_{1} \cap L^{4}x}^{\frac{1}{2}} + \|u\|_{L^{2}t_{1} \cap L^{4}x}^{\frac{2}{4}} \leq C \|u^0, u^1\|_{H^1 \cap L^4 \times L^2}, & \quad \text{if } \|u^0, u^1\|_{H^1 \cap L^4 \times L^2} \leq 1, \\
\|u\|_{L^{2}t_{1} \cap L^{4}x}^{\frac{1}{2}} + \|u\|_{L^{2}t_{1} \cap L^{4}x}^{\frac{2}{4}} \leq C \|u^0, u^1\|_{H^1 \cap L^4 \times L^2}, & \quad \text{if } \|u^0, u^1\|_{H^1 \cap L^4 \times L^2} > 1.
\end{align*}
\]

Taking powers and estimating the right-hand sides, we conclude in particular that solvability of the equation implies the more familiar-looking estimate
\[
\|u\|_{L^{2}t_{1} \cap L^{4}x}^{\frac{1}{2}} + \|u\|_{L^{2}t_{1} \cap L^{4}x}^{\frac{2}{4}} \leq C \left(\|u^0\|_{H^1}^{2} + \|u^0\|_{L^4}^{4} + \|u^1\|_{L^2}^{2}\right).
\]

The estimate in the Strichartz space \(L^{2}t_{1} L^{4}x\) is known from [37], and the reader may also find the stronger estimate
\[
\frac{1}{2} \|u\|_{L^{2}t_{1} \cap H^1}^{2} + \frac{1}{4} \|u\|_{L^{2}t_{1} \cap L^4}^{4} \leq \frac{1}{2} \|u^0\|_{H^1}^{2} + \frac{1}{4} \|u^0\|_{L^4}^{4}
\]
in [3, Theorem 8.41].

### 6 Non-surjectivity under incompatible scalings

Theorem 5.2 is useful in proving non-solvability of various problems which admit multiple scaling symmetries. The main result of this section, Theorem 6.1, encapsulates this idea. We then apply Theorem 6.1 to several examples, such as the Jacobian and the incompressible Euler and Navier–Stokes equations, proving in particular Corollary E. We note that, for these applications, the full nonlinear strength of Theorem 6.1 is not always needed, as often we can relax nonlinear PDEs into linear ones. Besides being useful to prove non-solvability, this strategy also gives an elementary way of proving upper bounds on the energy dissipation rates for Baire-generic initial data in evolutionary PDEs.
6.1 A general non-solvability result

In Theorem 5.2, surjectivity can only hold if all the scaling symmetries are compatible. The following result makes this precise.

**Theorem 6.1.** Consider the setup and assumptions of Theorem 5.2, with $J = 1$. Suppose, additionally, that there exists other bijections $\tilde{\tau}^D_\lambda$ and $\tilde{\tau}^Y_\lambda$ satisfying (A2):

\[
T \circ \tilde{\tau}^D_\lambda = \tilde{\tau}^Y_\lambda \circ T \\
|||\tilde{\tau}^D_\lambda(u_i)|||_{X^*_i} = \lambda^{\tilde{s}/\tilde{r}_i} |||u_i|||_{X^*_i} \quad \text{for all } \lambda > 0, \\
|||\tilde{\tau}^Y_\lambda f|||_{Y^*} = \lambda^s |||f|||_{Y^*} \quad \text{for all } \lambda > 0, f \in Y^*,
\]

where $0 < \tilde{r}_1 \leq \cdots \leq \tilde{r}_I$ and $\tilde{s} > 0$. If

\[
\frac{s}{\tilde{r}_i} > \frac{\tilde{s}}{\tilde{r}_i} \quad \text{for all } i,
\]

then $T(M\mathbb{B}_{X^*})$ is nowhere dense in $Y^*$ for every $M > 0$.

**Proof.** Seeking a contradiction, assume $T(M\mathbb{B}_{X^*})$ is not nowhere dense. By weak* sequential compactness of $\mathbb{B}_{X^*}$ and since $T$ has a weak*-to-weak* closed graph, the set $T(M\mathbb{B}_{X^*})$ is closed, and thus $T(M\mathbb{B}_{X^*})$ contains a ball $B_{\epsilon}(f_0)$. By Theorem 5.2, for every $f \in Y^*$ there exists $u \in X^*$ such that $Tu = f$ and $\sum_{i=1}^I |||u_i|||_{X^*_i}^{\tilde{s}/\tilde{r}_i} \leq C |||f|||_{Y^*}$ and $\tilde{u} \in X^*$ such that $T\tilde{u} = f$ and $\sum_{i=1}^I |||\tilde{u}_i|||_{X^*_i}^{\tilde{s}/\tilde{r}_i} \leq C |||f|||_{Y^*}$.

Fix $f \in Y^*$ with $|||f|||_{Y^*} = 1$; our aim is to show that $T0 = f$ and derive a contradiction. First note that $|||\tau^Y_\lambda f|||_{Y^*} = \lambda^s$. Choose $\tilde{u} \in X^*$ with $T\tilde{u} = \tau^Y_\lambda f$ and $\sum_{i=1}^I |||\tilde{u}_i|||_{X^*_i}^{\tilde{s}/\tilde{r}_i} \leq C \lambda^s$.

Write $\tilde{u} = \tau^D_\lambda u$, so that

\[
Tu = T(\tau^D_\lambda)^{-1}\tilde{u} = (\tau^Y_\lambda)^{-1}T\tilde{u} = f,
\]

\[
\sum_{i=1}^I \lambda^{\tilde{s}/\tilde{r}_i} |||u_i|||_{X^*_i}^{\tilde{s}/\tilde{r}_i} = \sum_{i=1}^I |||\tilde{u}_i|||_{X^*_i}^{\tilde{s}/\tilde{r}_i} \leq C \lambda^s.
\]

We conclude that $\sum_{i=1}^I \lambda^{\tilde{s}/\tilde{r}_i - s} |||u_i|||_{X^*_i}^{\tilde{s}/\tilde{r}_i} \leq \tilde{C}$. Thus, by letting $\lambda \to 0$ or $\lambda \to \infty$ we find a sequence of solutions $u^t$ of $Tu^t = f$ with $\sum_{i=1}^I |||u_i^t|||_{X^*_i}^{\tilde{s}/\tilde{r}_i} \to 0$. Now $|||u^t|||_{X^*} \to 0$ so that $u^t \not \to 0$, which yields $T0 = f$. Thus $1 = |||T0|||_{Y^*} = |||T\tau^D_\lambda 0|||_{Y^*} = |||\tau^Y_\lambda T0|||_{Y^*} = \lambda^s$ for all $\lambda > 0$. We have reached a contradiction. \hfill \Box

**Remark 6.2.** The conclusion of Theorem 6.1 also follows if $\tilde{s}/\tilde{r}_i \geq s/r_i$ for all $i$ and $\tilde{s}/\tilde{r}_{i_0} > s/r_{i_0}$ for some $i_0 \in \{1, \ldots, I\}$ such that $T(u_1, \ldots, u_{i_0-1}, 0, u_{i_0+1}, \ldots, u_I) \equiv 0$. The conclusion also follows if $X^* = \cap_{i=1}^I X^*_i$ instead of $X^* = \oplus_{i=1}^I X^*_i$ and $|||u_i|||_{X^*_i} = 0$ implies $Tu = 0$. We illustrate the latter point below by recovering the main result of [56] (albeit in a weaker form).

**Corollary 6.3.** If $n \geq 2$ and $p \in [1, \infty)$, the set $J(W^{1, np}(\mathbb{R}^n, \mathbb{R}^n))$ is meagre in $\mathcal{H}^p(\mathbb{R}^n)$.

**Proof.** Write $W^{1, np}(\mathbb{R}^n, \mathbb{R}^n) = L^{np}(\mathbb{R}^n, \mathbb{R}^n) \cap \tilde{W}^{1, np}(\mathbb{R}^n, \mathbb{R}^n)$. The scaling $u_\lambda = \lambda u(\cdot/\lambda)$, $f_\lambda = f(\cdot/\lambda)$, under which $J$ is invariant, gives

\[
|||u_\lambda|||_{L^{np}} = \lambda^{1+1/p} |||u|||_{L^{np}}, \quad |||u_\lambda|||_{\tilde{W}^{1, np}} = \lambda^{1/p} |||u|||_{\tilde{W}^{1, np}}, \quad |||f|||_{\mathcal{H}^p} = \lambda^{n/p} |||f|||_{\mathcal{H}^p},
\]

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so that \( s/r_1 = n/(p+1) \) and \( s/r_2 = n \), whereas the scalings \( \tilde{r}^D_\chi \equiv \lambda \text{id} \) and \( \tilde{r}^{\ast*}_\chi \equiv \lambda^n \text{id} \) give \( \tilde{s}/\tilde{r}_1 = \tilde{s}/\tilde{r}_2 = n \). The claim follows from Remark 6.2 since \( \|u\|_{L^0} = 0 \) and \( \|u\|_{W_1} < \infty \) yield \( J_u = 0 \).

As another example we consider the linear, homogeneous heat equation with \( L^2 \) data:

\[
\partial_t u - \nu \Delta u = 0 \quad \text{in } \mathbb{R}^3 \times (0, +\infty), \quad u(\cdot, 0) = u^0.
\]

Theorem 6.1 effortlessly yields the following essentially classical result:

**Corollary 6.4.** Let \( 1 < p < \infty \) and \( M > 0 \). The set of data \( u^0 \in L^2(\mathbb{R}^3) \) with a solution satisfying \( \|u\|_{L^p L^2} \leq M \) is nowhere dense in \( L^2(\mathbb{R}^3) \). In particular, a Baire-generic datum \( u^0 \in L^2(\mathbb{R}^3) \) does not have a solution \( u \in L^p L^2 \) for any \( p \in (1, \infty) \).

**Proof.** Let \( 1 < p < \infty \). We set \( D = \{ u \in L^p L^2 : u(\cdot, 0) = u^0 \text{ for some } u^0 \in L^2 \} \) and \( T(u) \equiv u^0 \). The claim follows from Theorem 6.1. As one scaling we use the parabolic one:

\[
uu_{\chi}(x, t) = \frac{1}{\chi} u \left( \frac{x}{\chi}, \frac{t}{\chi^2} \right),
\]

so that that \( \|u_{\chi}\|_{L^p L^2} = \lambda^{1/2 + 2/p} \|u\|_{L^p L^2} \) and \( \|u_{\chi}^0\|_{L^2} = \lambda^{1/2} \). As the other scaling we use \( \tilde{r}^D_\chi \equiv \lambda \text{id} \) and \( \tilde{r}^{\ast*}_\chi \equiv \lambda \text{id} \); now \( s/r = 1/(1 + 4/p) < 1/1 = \tilde{s}/\tilde{r} \). \( \square \)

### 6.2 The incompressible Euler equations and the proof of Corollary E

Our next aim is to prove Corollary E on the incompressible Euler equations in \( \mathbb{R}^n \times [0, \infty) \), \( n \geq 2 \). Recall that given \( u^0 \in L^2_\sigma \), a mapping \( u \in L^p_t L^2_{\sigma,x} \), \( 2 \leq p \leq \infty \), is a weak solution of the Cauchy problem (1.6)–(1.8) if

\[
\int_0^\infty \int_{\mathbb{R}^n} (u \cdot \partial_t \varphi + u \otimes u : D\varphi) \, dx \, dt + \int_{\mathbb{R}^n} u^0 \cdot \varphi(\cdot, 0) \, dx = 0 \quad \forall \varphi \in C^\infty_c(\mathbb{R}^n \times [0, \infty), \mathbb{R}^n). \tag{6.1}
\]

We cannot deduce Corollary E directly via Theorem 5.2. Indeed, the integral condition (6.1) leads to a well defined mapping \( T \) from a weak solution \( u \in L^p_t L^2_{\sigma,x} \) of (1.6)–(1.8) to the initial data \( u^0 \in L^2_\sigma \) but does not easily lend itself to a domain of definition \( D \subset L^p_t L^2_{\sigma,x} \) satisfying condition (A4) of Theorem 5.2. We therefore consider a relaxed problem where \( u \otimes u \in L^{p/2}_t L^1_x \) is replaced by a general matrix-valued mapping \( S \).

In order to apply Theorem 6.1 we embed \( L^2(\mathbb{R}^n, \mathbb{R}^{n \times n}) \) into the space of signed Radon measures \( M(\mathbb{R}^n, \mathbb{R}^{n \times n}) \) which is the dual of the separable Banach space \( C_0(\mathbb{R}^n, \mathbb{R}^{n \times n}) \). We endow \( M(\mathbb{R}^n, \mathbb{R}^{n \times n}) \) with the dual norm. In the relaxed problem we require \( u \in L^p_t L^2_{\sigma,x} \) and \( S \in L^{p/2} M_x \) to satisfy

\[
\int_0^\infty \int_{\mathbb{R}^n} (u \cdot \partial_t \varphi + S : D\varphi) \, dx \, dt + \int_{\mathbb{R}^n} u^0 \cdot \varphi(\cdot, 0) \, dx = 0 \quad \forall \varphi \in C^\infty_c(\mathbb{R}^n \times [0, \infty), \mathbb{R}^n). \tag{6.2}
\]

Unlike (6.1), due to linearity, condition (6.2) is stable under weak* convergence.

Relaxations such as (6.2) are studied in Tartar’s framework, where a system of nonlinear PDEs is decoupled into a set of linear PDEs (conservation laws) and pointwise constraints (constitutive laws) [78, 79]. Tartar’s framework has been very useful in convex integration.
both in the Calculus of Variations [62, 63], as well as in fluid dynamics [27, 28]. Specific constitutive laws do not play a role in the proof of Corollary E, and in fact, an analogous result holds for numerous other incompressible models of fluid mechanics. The result also trivially extends to subsolutions, that is solutions of the linear equations which take values in the so-called Λ-convex hull. Subsolutions can be interpreted as coarse-grained averages, see e.g. [17, 29].

Corollary E follows immediately from the next lemma, using interpolation and the fact that the class of residual $G_\delta$ sets is closed under countable intersections.

**Lemma 6.5.** Let $n \geq 2$, $M > 0$ and $p \in (2, \infty)$. The set of data $u^0 \in L^2$ with a solution $u \in M_{\mathbb{B}} L^p_{t}L^2_{\sigma,x}$ of (6.2) is nowhere dense in $L^2_{\sigma}$.

**Proof.** Denote $D = \{(u, S) \in L^p_t L^2_{\sigma,x} \times L^p_t L^2_{\sigma,x} : (6.2) \text{ holds for some } u^0 \in L^2_{\sigma}\}$ and define $T : D \to L^2_{\sigma}$ by $T(u, S) \equiv u^0$. Our intention is to verify the assumptions of Theorem 6.1.

Let $(u, S) \in D$. Given $\lambda > 0$ we set

$$u_\lambda(x,t) \equiv u \left( \frac{x}{\lambda}, \frac{t}{\lambda} \right), \quad S_\lambda(x,t) \equiv S \left( \frac{x}{\lambda}, \frac{t}{\lambda} \right), \quad u^0_\lambda(x,t) \equiv u^0 \left( \frac{x}{\lambda} \right).$$

(6.3)

Now (6.2)–(6.3) imply that $(u_\lambda, S_\lambda) \in D$ and $T(u_\lambda, S_\lambda) = u^0_\lambda$. We compute

$$\|u_\lambda\|_{L^p_t L^2_{\sigma}} = \lambda^{\frac{n}{p} + \frac{1}{2}} \|u\|_{L^p_t L^2_{\sigma}}, \quad \|S_\lambda\|_{L^p_t L^2_{\sigma,x}} = \lambda^{n + \frac{1}{p}} \|S\|_{L^p_t L^2_{\sigma,x}}, \quad \|u^0_\lambda\|_{L^2} = \lambda^{\frac{1}{2}} \|u^0\|_{L^2}.$$

Again we set $\tilde{\tau}^D_\lambda = \lambda \text{id}$ and $\tilde{\tau}^{\gamma^{*}}_\lambda = \lambda \text{id}$; Theorem 6.1 implies the claim.\[\Box\]

We conclude this subsection by briefly comparing Corollary E with the existing literature and we focus on the case $n = 2$, where the picture is more complete. Following [28], we say that an initial datum $u^0$ is wild if (1.6)–(1.8) admits infinitely many admissible weak solutions. Combining the results of [76] with [57, Theorem 4.2], we arrive at the following:

**Theorem 6.6.** When $n = 2$, the set of wild initial data is a dense, meagre $F_\sigma$ subset of $L^2_{\sigma}$.

We also note that some wild initial data admits compactly supported solutions [28], while Corollary E shows that such solutions exist only for a meagre $F_\sigma$ set of initial data.

### 6.3 Energy decay rate in the Navier–Stokes equations

We also illustrate the use of Theorem 6.1 in the presence of viscosity; we use the Navier–Stokes equations in $\mathbb{R}^n \times [0, \infty)$, $n \geq 2$, as an example. Given an initial datum $u^0 \in L^2_{\sigma}$, recall that weak solutions of (5.7)–(5.9) were defined in $L^\infty_t L^2_{\sigma,x} \cap L^2_t \dot{H}^1_x$ in §5.3. Furthermore, a weak solution is called a *Leray–Hopf solution* if it satisfies the energy inequality

$$\frac{1}{2} \int_{\mathbb{R}^3} |u(x,t)|^2 \, dx + \nu \int_t^\infty \int_{\mathbb{R}^3} |Du(x,\tau)|^2 \, dx \, d\tau \leq \frac{1}{2} \int_{\mathbb{R}^3} |u(x,s)|^2 \, dx$$

for all $t > s$ and for a.e. $s \in [0, \infty)$, including $s = 0$. LERAY showed in his milestone paper [53] that for every initial datum $u^0 \in L^2_{\sigma}$ there exists a Leray–Hopf solution $u \in L^\infty_t L^2_{\sigma,x} \cap L^2_t \dot{H}^1_x$ with $u(\cdot,0) = u^0$.

We briefly recall some of the pertinent results on energy decay of Leray–Hopf solutions and refer to the recent review [13] for more details and references.
LERAY asked in [53] whether \( E(t) = \frac{1}{2} \int_{\mathbb{R}^3} |u(x, t)|^2 \, dx \to 0 \) as \( t \to \infty \) for all Leray–Hopf solutions. An affirmative answer was given by theorems of KATO and MASUDA, see [13, Theorem 2–3]. SCHONBECK has shown that there is no uniform energy decay rate for general data \( u^0 \in L^2_\sigma \); more precisely, for every \( \beta, \varepsilon, T > 0 \) there exists \( u^0 \in \beta \mathbb{B}_{L^2_\sigma} \) such that a Leray–Hopf solution satisfies \( E(T) \geq (1 - \varepsilon) E(0) \). Furthermore, whenever \( u^0 \in L^2_\sigma \setminus \cup_{1 \leq p < 2} L^p \), the energy \( E(t) \) does not undergo polynomial decay. Several precise statements on the decay rate of \( E(t) \) under extra integrability assumptions on \( u^0 \in L^2_\sigma \) are given in [13].

In Corollary 6.8 below, we recover the lack of polynomial decay for a Baire-generic datum. The result applies to all \textit{distributional solutions} of (5.7)–(5.9), which we define as mappings \( u \in L^2_{\text{loc}, \tau} L^2_{\sigma, x}(\mathbb{R}^3 \times [0, \infty), \mathbb{R}^3) \) such that
\[
\int_0^\infty \langle u, \partial_t \varphi \rangle \, dt + \int_0^\infty \langle u_0 \otimes u, D\varphi \rangle \, dt + \nu \int_0^\infty \langle u, \Delta \varphi \rangle \, dt + \langle u_0, \varphi(0) \rangle = 0
\]
for all \( \varphi \in C_c^\infty(\mathbb{R}^3 \times [0, \infty), \mathbb{R}^3) \) with \( \text{div} \varphi = 0 \).

**Proposition 6.7.** Let \( p \in (2, \infty) \) and \( M > 0 \). The set of initial data for which (5.7)–(5.9) admits a distributional solution \( u \in M \mathbb{B}_{L^p L^2} \) is nowhere dense in \( L^2_\sigma \).

**Proof.** We consider the relaxed problem where we require \( u \in L^p_{\tau} L^2_{\sigma, x}, S^1 \in L^p_{\tau} M_x \) and \( S^2 \in L^p_{\tau} \dot{H}^{-1} \) to satisfy
\[
\int_0^\infty \langle u, \partial_t \varphi \rangle \, dt + \int_0^\infty \langle S^1, D\varphi \rangle \, dt + \nu \int_0^\infty \langle S^2, D\varphi \rangle \, dt + \langle u_0, \varphi(0) \rangle = 0
\]
for all \( \varphi \in C_c^\infty([0, \infty), \mathbb{R}^3) \) with \( \text{div} \varphi = 0 \). As before, denote by
\[
D \subset L^p_{\tau} L^2_{\sigma, x} \oplus L^p_{\tau} M_x \oplus L^p_{\tau} \dot{H}^{-1} \equiv X^*
\]
the set of triples \((u, S^1, S^2)\) such that (6.4) holds. One sets
\[
u_{\lambda}(x, t) = u \left( \frac{x}{\lambda}, \frac{t}{\lambda^2} \right), \quad S^i(x, t) = S^i \left( \frac{x}{\lambda}, \frac{t}{\lambda^2} \right);
\]

note that
\[
\| u_{\lambda} \|_{L^p_{\tau} L^2_{\sigma}} = \lambda^{n+1+1/p} \| u \|_{L^p_{\tau} L^2_{\sigma}}, \quad \| S^1_{\lambda} \|_{L^p_{\tau} M_x} = \lambda^{n+2/p} \| S^1 \|_{L^p_{\tau} M_x},
\]
\[
\| u_0 \|_{L^2} = \lambda^{n/2} \| u \|_{L^2}, \quad \| S^2 \|_{L^p_{\tau} \dot{H}^{-1}} = \lambda^{n/2+1+2/p} \| S^2 \|_{L^p_{\tau} \dot{H}^{-1}}.
\]
As before, we set \( \tilde{\tau}_{\lambda}^D = \lambda \text{id} \) and \( \tilde{\tau}_{\lambda}^{X^*} = \lambda \text{id} \). The claim now follows from Theorem 6.1. \( \square \)

**Corollary 6.8.** Let \( \varepsilon, C > 0 \). Consider the set \( X_{C, \varepsilon} \) of initial data \( u^0 \in L^2_\sigma \) such that a distributional solution of (5.7)–(5.9) satisfies
\[
(1 + t)^{\sigma} E(t) \leq C \quad \text{for almost every } t \in [0, \infty).
\]

The set \( X_{C, \varepsilon} \) is nowhere dense in \( L^2_\sigma \). In particular, for a Baire-generic \( u^0 \in L^2_\sigma \), distributional solutions satisfy
\[
\lim_{\tau \to \infty} \| t^{\varepsilon} E \|_{L^\infty(\tau, \infty)} = \infty \quad \text{for every } \varepsilon > 0.
\]
7 Concluding discussion

In this section, we discuss the advantages of the nonlinear open mapping principles proved in this paper, when compared to the classical Banach–Schauder theorem. We also point out some of the limitations of our results, as well as directions for future work.

We begin by recalling the standard proof of the Banach–Schauder open mapping theorem. If a bounded linear map \( L : X \rightarrow Y \) between Banach spaces is surjective, then the Baire category theorem yields a constant \( C > 0 \) and a ball \( B_r(f_0) \subset Y \) such that \( L(CB_X) \supseteq B_r(f_0) \), and the proof is completed as follows. First, by linearity, \( L(CB_X) = L(-CB_X) \supseteq -B_r(f_0) \), so that, by linearity again,

\[
L(2CB_X) = L(CB_X) - L(CB_X) \supseteq B_r(f_0) - B_r(f_0) = 2B_r(0).
\]

We notice that this proof uses in a fundamental way three properties:

(i) the linearity of the operator \( L \);

(ii) the vector space structure of the domain of definition of \( L \);

(iii) the symmetry of the range of \( L \).

Concerning (i), we note that if one attempts to generalise the above proof to nonlinear operators, then surjectivity only leads to “1/2-openness” and, more generally, 1/n-surjectivity leads to 1/2n-openness. To our knowledge, Theorem A and Proposition 2.5 give the first abstract results on Rudin’s problem which yield 1/n-openness from 1/n-surjectivity.

With respect to (ii), another key novelty of our work is that the domain of definition \( D \) of the operator \( T \) need not be a vector space. This is crucial when applying open mapping theorems to typical Cauchy problems in nonlinear evolutionary PDEs as is done in §5–6.

Finally, we note that (iii) is not needed for our results either. In fact, Theorems A and 5.2 apply when the target space is a closed convex cone such as \( \{ f \in L^p(\mathbb{R}^n) : f \geq 0 \text{ a.e.} \} \), for \( 1 < p < \infty \), c.f. Remark 5.3, and also when the symmetry of the range is non-trivial to check, as is the case for the Hessian operator \( H : \dot{W}^{1,2}(\mathbb{R}^2) \rightarrow \mathcal{H}^1(\mathbb{R}^2) \).

We now discuss some of the limitations of our work. From a PDE perspective, the main weak point of Theorem 5.2 is that assumption (A3) seems difficult to adapt to function spaces defined over the flat torus \( T^n \) or bounded domains. For instance, on \( T^n \), translations \( \sigma_k^Y f(x) = f(x - ke) \) typically fail the condition \( \sigma_k^Y f \overset{a}{\rightarrow} 0 \). On \( \mathbb{R}^n \), translations can often be replaced by scalings such as \( \sigma_k^Y f(x) = k^n f(kx) \), but such operators are of course not automorphisms on function spaces over the torus or bounded domains.

Despite the fact that Theorem 5.2 applies to many different equations, it would be interesting to look for generalisations, in order to account for other physical PDEs. One such situation concerns function spaces with critical scalings, i.e.

\[
\| \tau^D u \|_{X^*} = \| u \|_{X^*} \quad \text{and} \quad \| \tau^Y f \|_{Y^*} = \| f \|_{Y^*}.
\]

Note, however, that even if one does not assume that \( T \) is positively homogeneous, as in Theorem A, the proof of this theorem still provides \( \delta, M_\delta > 0 \) such that \( T(M_\delta B_{X^*}) \supset equal \delta B_{Y^*} \). It thus seems natural to ask whether one can achieve openness at the origin, i.e., whether one gets \( \lim_{\delta \searrow 0} M_\delta = 0 \). Another interesting problem is to decide whether the weak*-to-weak* closed graph assumption on the operators is an artifact of our proofs or a fundamental requirement for the validity of a nonlinear open mapping principle. We hope to address these questions in future work.
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