NEAR \( \omega \)-CONTINUOUS MULTIFUNCTIONS ON
BITOPOLOGICAL SPACES

E. ROSAS, C. CARPINTERO, N. RAJESH AND S. SHANTHI

ABSTRACT. In this paper, we introduce and study basic char-
acterizations, several properties of upper (lower) nearly \((i, j)\)-\(\omega\)-
continuous multifunctions on bitopological space.

1. Introduction

It is well known that various types of functions play a significant role
in the theory of classical point set topology. A great number of papers
dealing with such functions have appeared, and a good number of them
have been extended to the setting of multifunctions. This shows that
both, functions and multifunctions are important tools for studying
other properties of spaces and for constructing new spaces from previ-
ously existing ones. Generalized open sets play an important role in
General Topology and they are now the research topics of many topol-
ogists worldwide. Indeed a significant theme in General Topology and
Real analysis concerns the introduction of various modified forms of
continuity, separation axioms etc. by utilizing generalized open sets.
A generalization of closed sets, the notion of \(\omega\)-closed sets has been
introduced and studied by Hdeib [8]. Several characterizations and
properties of \(\omega\)-closed sets has been provided in [2, 4, 5, 6, 8, 9]. In this
paper, we introduce and study upper (lower) nearly \((i, j)\)-\(\omega\)-continuous
multifunctions on bitopological space.

2. Preliminaries

Throughout this paper, \((X, \tau_1, \tau_2)\) and \((Y, \sigma_1, \sigma_2)\) denote the bitopo-
logical spaces in which no separation axioms are assumed unless ex-
plicitly stated. Bitopological spaces and its different properties have
been investigated by Triparthy and Sarma ([11], [12], [14]), Sarma and
Triparthy [15], Tripathy and Acharjee [13], Acharjee and Tripathy [1],
Tripathy and Debnath [16], and others. For a subset \(A\) of \((X, \tau)\),
\(i \text{Cl}(A)\) (respectively \(i \text{Int}(A)\)) denote the closure of \(A\) with respect to
\(\tau_i\) (respectively the interior of \(A\) with respect to \(\tau_i\)). A point \(x \in X\)
is called a condensation point of \(A\) if for each \(U \in \tau\) with \(x \in U\), the
set \(U \cap A\) is uncountable. The set \(A\) is said to be \(\omega\)-closed [8] if it

2010 Mathematics Subject Classification. 54C10, 54C08, 54C05.
Key words and phrases. \((i, j)\)-regular open set, \(\omega\)-open sets, upper nearly \((i, j)\)-
\(\omega\)-continuous.
contains all its condensation points. The complement of an ω-closed set is said to be an ω-open set. It is well known that a subset \( W \) of a space \((X, τ)\) is ω-open if and only if for each \( x \in W \), there exists \( U \in τ \) such that \( x \in U \) and \( U \setminus W \) is countable. The family of all ω-open subsets of a topological space \((X, τ)\) forms a topology on \( X \) finer than \( τ \). The intersection of all ω-closed sets containing \( A \) is called the ω-closure[8] of \( A \) and is denoted by \( ω\text{Cl}(A) \). For each \( x \in X \), the family of all ω-open sets containing \( x \) is denoted by \( ωO(X, x) \). The family of all ω-open sets of \( X \) is denoted by \( ωO(X) \). A Multifunction \( F : X \to Y \) from a topological space \( X \) to a topological space \( Y \) is a point to set correspondence and is assumed that \( F(x) \neq \emptyset \) for all \( x \in X \). We denote the upper and lower inverse of a subset \( V \) of \( Y \) by \( F^+(V) \) and \( F^-(V) \) respectively; \( F^+(V) = \{ x \in X : F(x) \subseteq V \} \) and \( F^-(V) = \{ x \in X : F(x) \cap V \neq \emptyset \} \).

3. Upper (lower) nearly \((i, j)\)-ω-continuous multifunctions

Definition 3.1. [10] A subset \( A \) of a bitopological space \((X, τ_1, τ_2)\) is said to be \((i, j)\)-regular open if \( A = i\text{Int}(j\text{Cl}(A)) \). The complement of an \((i, j)\)-regular open set is called \((i, j)\)-regular closed set.

Definition 3.2. A subset \( A \) of a bitopological space \((X, τ_1, τ_2)\) is said to be \((i, j)\)-N-closed [10] if every cover of \( A \) by \((i, j)\)-regular open sets of \( X \) has a finite subcover.

Definition 3.3. A multifunction \( F : (X, τ_1, τ_2) \to (Y, σ_1, σ_2) \) is said to be:

1. upper nearly \((i, j)\)-ω-continuous at a point \( x \in X \) if for each \( σ_1\)-open set \( V \) containing \( F(x) \) and having \((i, j)\)-N-closed complement, there exists a \( τ_i\)-ω-open set \( U \) containing \( x \) such that \( U \subseteq F^+(V) \).
2. lower nearly \((i, j)\)-ω-continuous at a point \( x \in X \) if for each \( σ_1\)-open set \( V \) of \( Y \) meeting \( F(x) \) and having \((i, j)\)-N-closed complement, there exists a \( τ_i\)-ω-open set \( U \) of \( X \) containing \( x \) such that \( F(u) \cap V \neq \emptyset \) for each \( u \in U \).
3. \((i, j)\)-upper (resp. \((i, j)\)-lower) nearly ω-continuous on \( X \) if it has this property at every point of \( X \).

Example 3.4. Consider the set \( X = Y = \{a, b, c, d\} \) with topologies \( τ_1 = σ_1 = \{∅, X, \{a\}, \{b, c\}, \{a, b, c\}\} \) and \( τ_2 = σ_2 = \{∅, X, \{a\}, \{a, b\}, \{a, b, c\}\} \). Define the multifunction \( F : (X, τ_1, τ_2) \to (Y, σ_1, σ_2) \) as follows: \( F(a) = \{c\}, F(b) = \{a, b\}, F(c) = \{d\} \) and \( F(d) = \{a, b\} \). It is easy to see that the set \( \{b, c\} \) is \((i, j)\)-regular open and the multifunction \( F \) is \((i, j)\)-upper (resp. \((i, j)\)-lower) nearly ω-continuous on \( X \).

Theorem 3.5. For a multifunction \( F : (X, τ_1, τ_2) \to (Y, σ_1, σ_2) \), the following statements are equivalent:

1. \( F \) is upper nearly \((i, j)\)-ω-continuous.
Theorem 3.6. Let \( V \) be a \( \sigma_i \)-open set of \( Y \) having \((i, j)\)-\( N \)-closed complement. Then by (5), we have \( F^+(i \text{Int}(B)) \subset i \omega \text{Int}(F^+(B)) \) for every subset \( B \) of \( Y \) such that \( Y \setminus i \text{Int}(B) \) is \((i, j)\)-\( N \)-closed.

Proof. (1)\( \Rightarrow \) (2): Let \( x \in F^+(V) \) and \( V \) be a \( \sigma_i \)-open set of \( Y \) having \((i, j)\)-\( N \)-closed complement. From (1), there exists a \( \tau_i \)-open set \( U_x \) containing \( x \) such that \( U_x \subset F^+(V) \). It follows that \( F^+(V) = \bigcup_{x \in F^+(V)} U_x \). Since arbitrary union of \( \tau_i \)-open sets is \( \tau_i \)-open, \( F^+(V) \) is \( \tau_i \)-open in \((X, \tau_1, \tau_2)\).

(2)\( \Rightarrow \) (3): Let \( K \) be any \((i, j)\)-\( N \)-closed and \( \sigma_i \)-closed set of \( Y \). Then by (2), \( F^+(Y \setminus K) = X \setminus F^-(K) \) is an \( \tau_i \)-\( \omega \)-open set. Then it is obtained that \( F^-(K) \) is an \( \tau_i \)-\( \omega \)-closed set.

(3)\( \Rightarrow \) (4): Let \( B \) be any subset of \( Y \) having \((i, j)\)-\( N \)-closed \( \sigma_i \)-closure. By (3), we have \( F^-(B) \subset F^-(i \text{Cl}(B)) = i \omega \text{Cl}(F^-(i \text{Cl}(B))) \). Hence \( i \omega \text{Cl}(F^-(B)) \subset i \omega \text{Cl}(F^-(i \text{Cl}(B))) = F^-(\text{Cl}(B)) \).

(4)\( \Rightarrow \) (5): Let \( B \) be a subset of \( Y \) such that \( Y \setminus i \text{Int}(B) \) is \((i, j)\)-\( N \)-closed. Then by (4), we have \( Y \setminus i \omega \text{Int}(F^+(B)) = i \omega \text{Cl}(X \setminus F^+(B)) = i \omega \text{Cl}(F^-(Y \setminus B)) \subset F^-(i \text{Cl}(Y \setminus B)) \subset X \setminus F^+(i \text{Int}(B)) \). Therefore, we get \( F^+(i \text{Int}(B)) \subset i \omega \text{Int}(F^+(B)) \).

(5)\( \Rightarrow \) (1): Let \( x \in X \) and \( V \) be any \( \sigma_i \)-open set of \( Y \) containing \( F(x) \) and having \((i, j)\)-\( N \)-closed complement. Then by (5), \( x \in F^+(V) = F^+(i \text{Int}(V)) \subset i \omega \text{Int}(F^+(V)) \). There exists a \( \tau_i \)-\( \omega \)-open set \( U \) containing \( x \) such that \( U \subset F^+(V) \); and hence \( F(U) \subset V \). This shows that \( F \) is upper nearly \((i, j)\)-\( \omega \)-continuous.

Theorem 3.6. For a multifunction \( F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \), the following statements are equivalent:

1. \( F \) is lower nearly \((i, j)\)-\( \omega \)-continuous.
2. \( F^-(V) \) is \( \tau_i \)-\( \omega \)-open for each \( \sigma_i \)-open set \( V \) of \( Y \) having \((i, j)\)-\( N \)-closed complement.
3. \( F^+(K) \) is \( \tau_i \)-\( \omega \)-closed for every \( (i, j) \)-\( N \)-closed and \( \sigma_i \)-closed set \( K \) of \( Y \).
4. \( i \omega \text{Cl}(F^+(B)) \subset F^+(i \text{Cl}(B)) \) for every subset \( B \) of \( Y \) having \((i, j)\)-\( N \)-closed \( \sigma_i \)-closure.
5. \( F^-(i \text{Int}(B)) \subset i \omega \text{Int}(F^-(B)) \) for every subset \( B \) of \( Y \) such that \( Y \setminus i \text{Int}(B) \) is \((i, j)\)-\( N \)-closed.

Proof. The proof is similar to that of Theorem 3.5.

Corollary 3.7. A multifunction \( F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2) \) is upper nearly \((i, j)\)-\( \omega \)-continuous (resp. lower nearly \((i, j)\)-\( \omega \)-continuous) if
Let \( D \) we shall establish only the first equality and the last equality.

Conversely, let \( x \in Y \) then we have
\[
(\forall i,j \in \mathbb{N}) \setminus \{i,j\} \text{ -open set of } Y, \text{ having (i,j)-open sets } X
\]
then there exists \( F(x) \) containing \( F(x) \) and having \( (i,j) \)-N-closed complement such that \( x \in \text{ Int}(F(x)) \). Therefore, we have \( x \in F(x) \setminus \text{ Int}(F(x)) \subset \bigcup_{G \in \sigma NC} \{ F(G) \setminus \text{ Int}(F(G)) \} \).

\[ D^+_{n(i,j)}(F) = \bigcup_{G \in \sigma NC} \{ F(G) \setminus \text{ Int}(F(G)) \} \]

\[ = \bigcup_{B \in \sigma NC} \{ F(i \text{ Int}(B)) \setminus \text{ Int}(F(B)) \} \]

\[ = \bigcup_{B \in \sigma NC} \{ \text{ Int}(F^{-}(B)) \setminus F^{-}(i \text{ Cl}(B)) \} \]

\[ = \bigcup_{H \in \mathcal{F}} \{ \text{ Int}(F^{-}(H)) \setminus F^{-}(H) \}. \]

Where
\( \sigma NC \) is the family of all \( \sigma_i \)-open sets of \( Y \) having \((i,j)\)-N-closed complement,
\( iNC \) is the family of all subsets \( B \) of \( Y \) such that \( Y \setminus i \text{ Int}(B) \) is \((i,j)\)-N-closed,
\( NC \) is the family of all subsets \( B \) of \( Y \) having the \((i,j)\)-N-closed \( \sigma_i \)-closure,
\( \mathcal{F} \) is the family of all \( \sigma_i \)-closed and \((i,j)\)-N-closed sets of \( Y, \sigma_1, \sigma_2 \).

**Proof.** We shall establish only the first equality and the last equality since the proofs of other are similar to the first. Let \( x \in D^+_{n(i,j)}(F) \). Then there exists an \( \sigma_i \)-open set \( V \) of \( Y \) containing \( F(x) \) and having \((i,j)\)-N-closed complement such that \( x \in i \text{ Int}(F(x)) \). Therefore, we have \( x \in F(x) \setminus i \text{ Int}(F(x)) \subset \bigcup_{G \in \sigma NC} \{ F(G) \setminus i \text{ Int}(F(G)) \} \).

Conversely, let \( x \in \bigcup_{G \in \sigma NC} \{ F(G) \setminus i \text{ Int}(F(G)) \} \). Then there exists a \( \sigma_i \)-open set \( V \) of \( Y \) containing \( F(x) \) and having \((i,j)\)-N-closed complement such that \( x \in F(x) \setminus i \text{ Int}(F(x)) \). Hence \( x \in D^+_{n(i,j)}(F) \). We prove the last equality.

\[ \bigcup_{H \in \mathcal{F}} \{ \text{ Int}(F^{-}(H)) \setminus F^{-}(H) \} \subset \bigcup_{B \in \mathcal{NC}} \{ \text{ Int}(F^{-}(B)) \setminus F^{-}(i \text{ Cl}(B)) \} = D^+_{n(i,j)}(F). \]

Conversely, we have \( D^-_{n(i,j)}(F) = \bigcup_{B \in \mathcal{NC}} \{ i \text{ Cl}(F^{-}(B)) \setminus F^{-}(i \text{ Cl}(B)) \} \subset \bigcup_{H \in \mathcal{F}} \{ \text{ Int}(F^{-}(H)) \setminus F^{-}(H) \}. \]
Theorem 3.9. For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties hold:

$$D^-_{n(i,j)\omega} = \bigcup_{G \in \sigma_{NC}} \{F^-(G) \setminus \omega \text{Int}(F^-(G))\}$$
$$= \bigcup_{B \in \sigma_{NC}} \{F^-(i \text{Int}(B)) \setminus \omega \text{Int}(F^-(B))\}$$
$$= \bigcup_{B \in \sigma_{NC}} \{i \omega \text{Cl}(F^+(B)) \setminus F^+(i \text{Cl}(B))\}$$
$$= \bigcup_{H \in F} \{i \omega \text{Cl}(F^+(H)) \setminus F^+(H)\}.$$

Proof. The proof is similar to that of Theorem 3.8.

Definition 3.10. Let $(X, \tau)$ be a topological space and $A$ a subset of $X$. The $\omega$-frontier of $A$, $\omega\text{-Fr}(A)$, is defined by $\omega\text{-Fr}(A) = \omega \text{Cl}(A) \cap \omega \text{Int}(A)$.

Theorem 3.11. For a multifunction $F : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, $D^+_{n(i,j)\omega}(F)$ (resp. $D^-_{n(i,j)\omega}(F)$) is identical with the union of $\omega$-frontiers of the $(i, j)$-upper (resp. $(i, j)$-lower) inverse images of $\sigma_i$-open sets containing (resp. meeting) $F(x)$ and having $(i, j)$-$N$-closed complement.

Proof. We shall established the first case since the proof of the second can be established similar.

Let $x \in D^+_{n(i,j)\omega}(F)$. Then, there exists a $\sigma_i$-open set $V$ of $Y$ containing $F(x)$ and having $(i, j)$-$N$-closed complement such that $U \cap (X \setminus F^+(V)) \neq \emptyset$ for every $(i, j)$-open set $U$ containing $x$. Then we have $x \in \omega \text{Cl}(X \setminus F^+(V))$. On the other hand, since $x \in F^+(V) \subset \omega \text{Cl}(F^+(V))$ and hence $x \in \omega\text{-Fr}(F^+(V))$. Conversely, suppose that $F$ is upper nearly $(i, j)$-$\omega$-continuous at $x \in X$. Then for any $\sigma_i$-open set $V$ of $Y$ containing $F(x)$ and having $(i, j)$-$N$-closed complement, there exists $U \in \tau_i$-$\omega\text{Fr}(X)$ containing $x$ such that $F(U) \subset V$; hence $x \in U \subset F^+(V)$. Therefore, $x \in U \subset \omega\text{Int}(F^+(V))$. This contradicts to the fact that $x \in \omega\text{-Fr}(F^+(V))$.

References

[1] S. Acharjee, B. C. Tripathy, $p$-$j$-generator and $pl$-$j$-generator in bitopology; Boletim da Sociedade Paranaense de Matematica, 36 No 2 (2018),17-31.
[2] K. Al-Zoubi and B. Al-Nashef, The topology of $\omega$-open subsets, Al-Manarah 9 (2003), 169-179.
[3] K. Al-Zoubi, On generalized $\omega$-closed sets, Int. J. Math. Math. Sci. 13 (2005), 2011-2021.
[4] A. Al-Omari and M. S. M. Noorani, Contra-$\omega$-continuous and almost $\omega$-continuous functions, Int. J. Math. Math. Sci. 9, (2007), 169-179. doi:10.1155/2007/40469.
[5] A. Al-Omari, T. Noiri and M. S. M. Noorani, Weak and strong forms of $\omega$-continuous functions, Int. J. Math. Math. Sci. 9, (2009), 1-13. doi:10.1155/2009/174042.
[6] C. Carpintero, J. Pacheco, N. Rajesh and E. Rosas, Properties of nearly \(\omega\)-continuous multifunctions, \textit{Acta Univ. Sapientiae, Mathematic}, \textbf{9}, No. 1 (2017), 13-25. doi: 10.1515/ausm-2017-0002.

[7] E. Ekici, S. Jafari and S.P. Moshokoa, On a weak form of \(\omega\)-continuity, \textit{Annals Univ. Craiova, Math. Comp. Sci.} \textbf{37}. No 2 (2010), 38-46.

[8] H.Z. Hdeib, \(\omega\)-closed mappings, \textit{Revista Colombiana Mat.}, \textbf{16} (1982), 65-78.

[9] H.Z. Hdeib, \(\omega\)-continuous functions, \textit{Dirasat}, \textbf{16}. No. 2 (1989), 136-142.

[10] A. Richlewicz, On almost nearly continuity with reference to multifunctions in bitopological spaces, \textit{Novi Sad J. Math.}, \textbf{38}, No. 2 (2008), 5-14.

[11] B. C. Tripathy and D. J. Sarma, On \(b\)-locally open sets in bitopological spaces, \textit{Kyungpook Math. Journal}, \textbf{51}, No. 4 (2011), 429-433.

[12] B. C. Tripathy and D. J. Sarma, On weakly \(b\)-continuous functions in bitopological spaces, \textit{Acta Scientiarum Technology}, \textbf{35}, No. 3 (2013), 521-525.

[13] B. C. Tripathy and S. Acharjee, On \((\gamma, \delta)\)-Bitopological semi-closed set via topological ideal, \textit{Proyecciones J. Math.}, \textbf{33}, No. 3 (2014), 245-257.

[14] B. C. Tripathy and D. J. Sarma, Generalized \(b\)-closed sets in ideal bitopological spaces, \textit{Proyecciones J. Math.}, \textbf{33}, No. 3 (2014), 315-324.

[15] D. J. Sarma and B. C. Tripathy, Pairwise generalized \(b\)-\(R_0\)-spaces in bitopological spaces, \textit{Proyecciones J. Math.}, \textbf{36}, No. 4 (2017), 589-600.

[16] B. C. Tripathy and S. Debnath, Fuzzy \(m\)-structures, \(m\)-open multifunctions and bitopological spaces, \textit{Boletim da Sociedade Paranaense de Matematica}, \textbf{37}, No. 4 (2019), 119-128.