DEEP ELASTIC PROCESSES OF
COMPOSITE PARTICLES IN FIELD THEORY
AND ASYMPTOTIC FREEDOM*

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This is an English translation of my 1977 Russian preprint. It contains the first explicit definition of the pion distribution amplitude (DA), the expression for the pion form factor asymptotics in terms of the pion DA, and formulates the pQCD parton picture for hard exclusive processes.

Abstract of the original paper:
The large $Q^2$ behavior of the pion electromagnetic form factor is explicitly calculated in the non-Abelian gauge theory to demonstrate a field-theoretical approach to the deep elastic processes of composite particles. The approach is equivalent to a new type of parton model.
At present, it is usually accepted that the hadrons are complex objects composed of quarks interacting through a mediating gluonic field. The approximate scaling in deep inelastic scattering indicates that the effective coupling of the quark-gluon interaction is small in the region of large transferred momenta. Hence, there is some justification for the hope that the quantum-field calculations based on the use of perturbation theory can be applied in the region of large momentum transfers. An efficient technique based on the use of the operator expansions and the renormalization group (RG) methods was applied to the study of the deep inelastic scattering processes [1]. Incorporating the recent studies of the pion form factor behavior at large $Q^2$ with the help of the operator expansions [2] and the analysis of the Feynman graphs in the $\alpha$-representation [3], we investigate deep elastic processes involving composite particles with the help of a technique analogous to that used in [1].

Consider the pion electromagnetic form factor, treating the pion as a bound state of spin 1/2 quarks. The form factor of such a system can be written in terms of the Bethe-Salpeter wave function $\chi$ as shown in Fig. 1a. Due to property $\chi = K \otimes \chi$, where $\chi$ is the Bethe-Salpeter kernel, we can rewrite $F_\pi$ as shown in Fig. 1b.

![Fig. 1. Electromagnetic form factor of a composite pion.](image)

Each graph can be formally written in the $\alpha$-representation (see, e.g., [4], [5]):

$$R^\mu(P, P') = g^2 \left( \frac{g^2}{16\pi^2} \right)^{N_{\text{loop}}} \int_0^\infty \frac{d\alpha}{D_\alpha} G^\mu(\alpha, P, P') \exp \left\{ i q^2 A(\alpha) + i I(\alpha, m_\pi^2) \right\},$$

where the functions $D, G, A$ and $I$ are determined by the topological structure of the diagram. Note that $\chi, \bar{\chi}$ vertices should be treated as subgraphs rather than points. One of the most important results of the analysis of the Feynman graphs in the $\alpha$-representation is that the large-$Q^2$ behavior in a theory with dimensionless coupling constant is determined by integration over a region of the $\alpha$-parameters belonging to subgraphs the contraction of which into point eliminates the dependence of the graph on the large variable $Q^2$. For the pion electromagnetic form factor, integration over small $\lambda_V$ ($\lambda_V = \sum_{\sigma \in V} \alpha_\sigma$) gives the leading contribution $F_V^{\text{lead}}(Q^2) \sim Q^{-\Sigma t_i + 2}$, where $t_i$ is the twist (equal dimension minus spin) of the field describing the $i$-th external line of the subgraph $V$, and $Q = \sqrt{-q^2}$.

Taking into account that, in a theory with vector gluons, $t = 1$ for quark fields, but $t = 0$ for vector fields (both for gluons and the photon), one may conclude that the leading contribution can be given only by subgraphs having 4 external quark lines and an arbitrary number of gluon lines (Fig. 2).

Using the Mellin representation

$$F_\pi(Q^2) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \Phi(J)(Q^2)^J dJ,$$

one can say that integration over the region $0 \leq \lambda_V \leq 1/m^2$ gives a pole $(J + 1)^{-1} (1/m^2)^{J+1} \bar{f} \otimes E \otimes f$, where $E$ is the contribution due to the subgraph $V$ and $f, \bar{f}$ are the contributions of the left and right weakly connected parts resulting from the contraction of $V$ into point.
The functions $f, \bar{f}$ do not depend on $J$ because the contraction of $V$ into point eliminates the dependence on $Q^2$. Summation over all possible subgraphs looks like a perturbative series, with $E(Q^2/\mu^2, g(\mu^2))$ describing short-distance interactions while $f$ and $\bar{f}$ correspond to the properly normalized quark-pion vertices:

$$F_\pi(Q^2) = \frac{1}{Q^2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \bar{f}_m^I(\mu^2, g, P') E_{mn}^{IK}(Q^2/\mu^2, g) f_n^K(\mu^2, g, P) \ ,$$

where $I, K$ denote the spinor indices. The leading contribution can be obtained only when $f$ and $\bar{f}$ are projected on the axial structure [2,3,6]. The functions $f^A_n = a_n$ are defined through matrix elements of the twist-2 operators $\bar{\psi}\gamma_5\gamma_\mu D_\mu \cdots D_{\mu+n+1} \psi$:

$$2^n i^{-n-1} \langle 0 | \bar{\psi}(0) \gamma_5 \{ \gamma_\mu D_\mu \cdots D_{\mu+n+1} \} | P \rangle = \{P_{\mu_1}, \ldots, P_{\mu_{n+1}}\} a_n(\mu^2, g(\mu^2)) \frac{1+(-1)^n}{2} ,$$

where $D_\mu$ is the covariant derivative acting on the quark fields, $\{ \}$ means that the corresponding function is symmetric in $\mu_1, \ldots, \mu_{n+1}$ and all its contractions with $g^{\mu_\lambda \mu_\lambda}$ are zero, and $\mu^2$ serves as the renormalization parameter.

Applying $\mu d/d\mu$ to both sides of Eq.(4), we obtain a renormalization group equation

$$\sum_{n=n'} \sum_{m=m'} \left\{ \left[ \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} \right] a_n(\mu^2, g(\mu^2)) \right\} E_{mn} = 0 ,$$

where

$$\left[ \frac{\partial}{\partial \mu} + \beta(g) \frac{\partial}{\partial g} \right] a_n(\mu^2, g(\mu^2)) = \sum_{n'=0}^n z_{nn'} a_{n'} .$$

According to (6), there is mixing between operators with different spin (but the same twist).

Let us illustrate the method of calculations on the example of the lowest-order diagram (Fig. 3a). The $\sigma$-representation for the leading contribution of such a diagram in a non-Abelian gauge theory has the following structure:

$$g^2 C_2(R) \sum_{V_L, V_R} \left( \frac{g^2}{16\pi^2} \right)^{N_L+N_R} \int \frac{\prod d\alpha^{(L)}(V_L)}{D^2(V_L)} e^{iH(\alpha^{(L)}, m^2)} \frac{1}{4} \text{Sp} \{\gamma_5 \gamma_\lambda G_L\} \int \frac{\prod d\alpha^{(R)}(V_R)}{D^2(V_R)} e^{iH(\alpha^{(R)}, m^2)} \frac{1}{4} \text{Sp} \{\gamma_5 \gamma_\lambda G_R\} \int d\alpha^{(L)}(V_L) e^{i\frac{2}{\beta} D(V_R)} e^{i\frac{2}{\lambda} D(V_L)} e^{i\frac{2}{\gamma} D(V_R)} \ ,$$

where $\alpha^{(L)}_\sigma \in V_L, \alpha^{(R)}_\sigma \in V_R$. We also have taken into account that, for the leading contribution, the function $G_L(G_R)$ depends only on $P \ (P')$ and that $D = D(V_L)D(V_R) + O(\lambda)$, where $\lambda = \alpha + \beta$. The meaning of the 2-trees $L, L', R, R'$ is clear from Fig. 3b.
Since \( |R_+/D| \leq 1 \), the function vanishes for \(|\xi| \geq 1\).

In other words, the function \( \tilde{\varphi}(\xi, \mu^2) \) is such a function, the \( n \)-th moment of which is equal to \( a_n(\mu^2) \) \{3\}:

\[
\int_{-1}^{1} d\xi \xi^n \tilde{\varphi}(\xi, \mu^2) = a_n(\mu^2) \frac{1 + (-1)^n}{2} .
\]

Hence, \( \tilde{\varphi}(\xi) = \tilde{\varphi}(-\xi) \).

The function \( \tilde{\varphi} \) can be normalized by noticing that the magnitude of the matrix element of the axial current is known from the \( \pi \to \mu \nu \) process:

\[
\langle 0 | \bar{\psi}(0) \gamma_5 \gamma_\mu \psi(0) | P \rangle = i \sqrt{2} f_\pi P_\mu ,
\]

with \( f_\pi \approx 100 \) MeV. Since the axial current has zero anomalous dimension, the relation (11) is valid for any \( \mu^2 \). That is why we take \( \tilde{\varphi}(\xi, \mu^2) = \frac{\xi}{\sqrt{2}} f_\pi a(\xi, \mu^2) \). The final result then has the following form:

\[
F_\pi^{\text{lead}}(Q^2) = 16\pi G_F(Q^2) \frac{f_\pi^2}{Q^2} C_2(R) \frac{|\gamma(Q^2)|^2}{N_c} ,
\]

where

\[
\gamma(Q^2) = \int_0^1 a(\xi, Q^2) \frac{d\xi}{1 - \xi^2} , \quad \left| \int_0^1 a(\xi, Q^2) d\xi \right| = 1 .
\]

The representation (3) is equivalent to the following parton picture for deep elastic scattering. The splitting of the pion into two quarks with momenta \( x_1 P, x_2 P \) is described by the wave function \( \varphi(x_1, x_2, \mu^2) \), \( (1-x_1-x_2) \), while the fusion of the quarks with momenta \( y_1 P', y_2 P' \) into the pion is described by the wave function \( \varphi^*(y_1, y_2, \mu^2) \delta(1-y_1-y_2) \). The amplitude for \( x_1 P, x_2 P \to y_1 P', y_2 P' \) transition is constructed according to the usual rules:

\[
F_\pi(Q^2) = \int_0^1 dx \int_0^1 dy \varphi^*(y, 1-y, \mu^2) E(x, y, P, P', \mu^2) \varphi(x, 1-x, \mu^2) .
\]

Fig. 3. Structure of the lowest-order diagram.
For the diagram 3a \{4\}

\[ P_\mu E = \frac{1}{xyq^2} \frac{1}{xq^2} \text{Sp} \left\{ \gamma^\mu \gamma_5 \hat{P} \gamma^\alpha \gamma_5 \hat{P}' \gamma_\alpha (\hat{P}' - xP) \right\}, \]

the wave functions are symmetric \( \varphi(x_1, x_2) = \varphi(x_2, x_1) \) and \( \bar{\varphi}(\xi, \mu^2) \) is maximal for \( \xi = 0 \) and vanishes for \( \xi = 1 \). Taking for simplicity \( |a(\xi, \mu^2)| = \frac{3}{2} (1 - \xi^2) \) \{5\} (where \( 3/2 \) is determined by the normalization condition) and incorporating that the analysis of scaling violation in deep inelastic scattering under the assumption that the asymptotic freedom takes place, i.e., that \( \alpha_s(Q^2) = \frac{3}{2 \ln(Q^2/\Lambda^2)} \), gives \( \Lambda = 0.5 \text{ GeV} \) \{7\–9\}, we obtain that \( \alpha_s(Q^2) = 2 \text{ GeV}^2 = 0.7 \), and, taking into account that \( C_2(R) = 4/3, N_e = 3, \) we find \( F_\pi(Q^2 = 2) = 0.18 \). The formula \( F_\pi(Q^2) = (1 + Q^2/0.47)^{-1} \) (the extrapolation of the \( \rho \)-meson pole well describing experimental data) gives \( F_\pi(Q^2 = 2) = 0.19 \). The factor \( m_\pi^2/Q^2 \) determining the magnitude of the pion mass corrections is small for \( Q^2 \geq 2 \). Namely, \( m_\pi^2/Q^2 < 1\% \). Thus, the corrections of the \( \xi \)-scaling type \{10\} may be neglected. There remains the contribution of the twist-3 operators \( \bar{\psi}_\gamma D \ldots \bar{D} \psi \). It can be calculated in the same way as the axial contribution. One should also include the next term in the expansion of the function \( E(1, \bar{g}(Q^2)) \) having the order of magnitude of \( \alpha_s(Q^2)/\pi \approx 20\% \).

Thus, we observe that the agreement of quantum chromodynamics for \( \alpha_s(2) = 0.7 \) with experimental data is not bad, at least in the region \( 1 \leq Q^2 \leq 3 \text{ GeV}^2 \).

The parton formulation of our approach can be easily applied to other hadrons. The proton in deep inelastic scattering is described by the wave function \( \varphi(x_1, x_2, x_3, \mu^2) \delta(1 - x_1 - x_2 - x_3) \) related to operators of the \( \psi_1 C \gamma_5 \gamma_\mu \bar{\psi}_2 \bar{D}(d_3\bar{D}_2) \bar{D}(d_1) \bar{D}(d_3) N_1 \psi_3 \) type, where \( C \) is the charge conjugation matrix. Unfortunately, there is no a priori information about normalization of the function \( \varphi(x_1, x_2, x_3, \mu^2) \). It can be extracted from the data on the proton form factor, and then the parton wave functions can be used to fit the data on deep elastic wide-angle hadron scattering:

\[ M(s,t)|_{AB \rightarrow CD} = \int \prod dx_k^{(A)} dx_l^{(B)} dx_m^{(C)} dx_n^{(D)} \delta \left( 1 - \sum x_k^{(A)} \right) \delta \left( 1 - \sum x_l^{(B)} \right) \delta \left( 1 - \sum x_m^{(C)} \right) \delta \left( 1 - \sum x_n^{(D)} \right) \]
\[ \times \varphi^*(x_m^{(C)}) \varphi(x_n^{(D)}) E \left( x_k^{(A)}, x_l^{(B)}, x_m^{(C)}, x_n^{(D)}, P_A, P_B, P_C, P_D, \mu^2 \right) \varphi(x_k^{(A)}) \varphi(x_l^{(B)}). \]

The lowest-order approximation for \( E \) reproduces the quark counting rules \{11,12\} \( da/dt|_{AB \rightarrow CD} \sim s^{-N+2} \), where \( N = n_A + n_B + n_C + n_D \). In addition, there appears a factor \( |a_s(ut/s)|^{N-2} \) that gives an extra dependence on \( s \). In the region \( ut/s \sim m_\pi^2 \) one should take into account the proton mass corrections. The function \( Q^2 F_\pi^2(Q^2) \) behaves like \( \alpha_s(Q^2) \) even for moderately large \( Q^2 \). For this reason, the investigation of the pion electromagnetic form factor seems to be the best tool for the experimental study of the nature of the quark-gluon coupling constant renormalization. The wave functions \( \varphi(x, Q^2) \) also depend on \( Q^2 \) due to the anomalous dimensions. However, preliminary estimates show that the change of \( \gamma(Q^2) \) with \( Q^2 \) cannot compensate the decrease of \( \alpha_s(Q^2) \) in a non-Abelian asymptotically free gauge theory, and the absence of the extra logarithmic dependence in the region \( Q^2 > m_\pi^2 \) would be a serious argument against the asymptotic freedom \( \alpha_s(Q^2) \sim 1/\ln(Q^2/\Lambda^2) \) in favor of the assumption \{5\} that \( \alpha_s(Q^2) = \text{const} \) for \( Q^2 \geq 2 \text{ GeV}^2 \).

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NOTES ADDED

{1} This paper was finished in June 1977, when I was A.V. Efremov’s graduate student at Moscow University. The original version is reproduced above without changes: just a few typos in formulas were fixed and references are added to short notes presented below. They contain updating and clarifying remarks.

{2} The paper was submitted to Physics Letters B, but was not accepted. After completing my Ph.D. in February 1978, we started to write papers containing a detailed description of our approach to pQCD factorization both for inclusive and exclusive processes. The first paper [13] presented the outline of the forthcoming cycle [14–17], of which the paper on the pion form factor [17] was intended to be the last. It heavily relied on the content of the preceding three papers of the cycle. A self-contained paper [18] on factorization for the pion form factor was prepared for the Alushta 1979 conference. At the conference, I learned from H.D. Politzer about the paper [19] by S. J. Brodsky and G. P. Lepage. Their approach was rather different from ours, but they mentioned our papers [13,17]. After their subsequent papers [20,21] appeared, we submitted a shortened version [22] of [18] to Physics Letters B. It was published there in 1980. A more detailed discussion was published in the same year as the second part of the review in Rivista del Nuovo Cimento [23]. Our 1978 paper [17] was eventually also published by Theor. Math. Fiz. in 1980 (in fact, before the first three papers [14–16] of the cycle).

{3} Nowadays, $\tilde{\phi}(\xi,\mu^2)$ is usually called the pion distribution amplitude (DA). This terminology was introduced in [24] where the pion DA is defined through a $k_\perp$ integral of the light-cone wave function $\psi(x,k_\perp)$. Our definition (10) relating the moments of DA to matrix elements of local operators is widely used in modern applications, like higher-order calculations, in lattice simulations, etc.

{4} This simple formula and Eq. (13) were far from being obvious even at the end of 1977, see e.g. Appendix B of [25] for an illustration. From my discussions with V. L. Chernyak in the fall of 1977, I learned that he derived an equivalent of Eq.(13) well before me, but the relevant publication [26] appeared only in December 1977.

{5} This form coincides with the asymptotic $Q^2 \to \infty$ limit of the pion distribution amplitude (with integration over $\xi$ reduced to $0 \leq \xi \leq 1$). The explicit statement that the asymptotic form of the pion DA in QCD is given by $\frac{1}{3} f_\pi (1 - \xi^2)$ was made in our “advertisement” paper [13]. Implicitly, this result is also contained in the paper by V.L. Chernyak, A.R. Zhitnitsky and V.G. Serbo [26] as a statement about the asymptotic limit of their analogue of $\gamma(q^2)$, Their derivation was outlined in [27]. Our solution of the evolution equation (6) was given in [17]. Under simplifying assumptions (that are valid for spin-0 gluons but not in QCD) the solution of the evolution equation was also obtained by Farrar and Jackson [28,29]. The (pseudo)scalar gluon model for the pion form factor was studied earlier in our paper [3] (English preprint: JINR E2-9717, April 1976). As explained in my review [30], its final result contains the expression amounting to the solution of the evolution equation for the pion DA in this model. It is equivalent to the solution given in Refs. [28,29]. A very efficient way of solving the DA evolution equations was developed by G. P. Lepage and S. J. Brodsky [20].

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