TRUNCATED HEAT KERNEL AND ONE-LOOP DETERMINANTS FOR THE BTZ GEOMETRY

A. A. Bytsenko (a) and M. E. X. Guimarães (b)

(a) Departamento de Física, Universidade Estadual de Londrina
Caixa Postal 6001, Londrina-Paraná, Brazil

(b) Instituto de Física, Universidade Federal Fluminense,
Av. Gal. Milton Tavares de Souza s/n, Niterói-RJ, Brazil

Abstract

There is a special relation between the spectrum and the truncated heat kernel of the Euclidean BTZ black hole with the Patterson-Selberg zeta function. Using an orbifold description of this relation we calculate the on-shell corrections of the gravitational quantum fluctuations.

Keywords: ..........

abyts@uel.br
emilia@if.uff.br
1 Introduction

One of the most important issues in studying quantum gravity is the black hole physics, the origin of the Bekenstein-Hawking entropy and its quantum corrections. Recently quantum gravity partition functions in three dimensions have been analyzed in detail [1, 2]. The three dimensional case is quite simple (because of no propagating gravitons) but there is a common belief that it deserves attention as an useful laboratory as analog models for four dimensional physics. A simple geometrical structure of three-dimensional black hole (Bañados, Teitelboim, Zanelli (BTZ) black hole [3]) allows exact computations since its Euclidean counterpart is locally isomorphic to the constant curvature hyperbolic space $H^3$.

In physical literature usually assumes that the fundamental domain for the action of a discrete group $\Gamma$ has finite volume. On the other hand BTZ black hole has a Euclidean quotient representation $H_\Gamma = \Gamma \backslash H^3$ for an appropriate $\Gamma$, where the fundamental domain has infinite hyperbolic volume. For discrete groups of isometries of three-hyperbolic space with infinite volume of fundamental domain (i.e. for Kleinian groups) Selberg zeta functions and trace formulas, excluding fundamental domains with cusps, have been considered in [5], where results depends on previous works [6], [7]. Note that matters are difficult in case of infinite-volume setting due to the infinite multiplicity of the continuous spectrum and absence of a canonical renormalization of the scattering operator which makes it trace-class. However, for BTZ black hole one can by-pass much of the general theory and proceed more directly to define a Selberg zeta function attached to $H_\Gamma$ and establish a trace formula which is a special version of the Poisson formula for resonans (see for detail [4]).

In [8] the one-loop correction to the Bekenstein-Hawking entropy for the non-spinning black hole were studied by expressing the determinants in terms of the appropriate heat kernels, which were evaluated using a method of images (for the analogous derivation of the Selberg trace formula see [9]). However, this correction to the entropy is not completely corrected. Indeed, the procedure of regularization of the divergent volume of the fundamental domain, which has been made in [8], changes the group actions on the real hyperbolic three-space, and cannot be compatible with the origin structure of the cyclic groups. In fact, there is a special

---

3 For the non-spining black hole one can choose $\Gamma$ to be Abelian group generated by a single hyperbolic element [4].
relation between the spectrum and the truncated heat kernel of the Euclidean BTZ black hole with the Patterson-Selberg zeta function \([10]\). The main purpose of this work is to provide the corrected relation between the spectral functions and the truncated heat kernel for the BTZ black hole, and calculate the one-loop quantum correction to the partition function.

## 2 Three-dimensional black hole

The Euclidean three-dimensional black hole has an orbifold description \( H_{\Gamma(a,b)} = \Gamma(a,b) \backslash H^3 \) if we choose the suitable parameters \( a > 0, b \geq 0 \), where \( H^3 = \{(x, y, z) \in \mathbb{R}^3 \mid z > 0\} \) is an hyperbolic three-dimensional space (or simply a three-space) and \( \Gamma(a,b) \subset SL(2, \mathbb{C}) \) is a cyclic group of isometries. Space \( H_{\Gamma(a,b)} \) is a solution of the Einstein equations \( R_{ij} - \frac{1}{2} g_{ij} R_g - \Lambda g_{ij} = 0 \) with hyperbolic metric \( ds^2 = (\sigma/z)^2(dx^2 + dy^2 + dz^2) \) and negative cosmological constant \( \Lambda \), \( \sigma = (-\Lambda)^{-\frac{1}{2}} \), and constant scalar curvature \( R_g = 6\sigma^{-2} = -6\Lambda \). The original BTZ metric in the coordinates \((r, \phi, t)\) can indeed be transformed to the metric \( ds^2 \) by a specific change of variables \( (r, \phi, t) \rightarrow (x, y, z) \). It is the periodicity in the Schwarzschild variable \( x, y, z \) that allows for the above orbifold description. In fact, the parameters \( a, b \) are given as follows. For \( M > 0, J > 0 \) the black hole mass and angular momentum, and for \( r_+ > 0, r_- \in i\mathbb{R} \) \((i^2 = -1)\) the outer and inner horizons given by

\[
r^2_+ = \frac{Ma^2}{2} \left[ 1 + \left( 1 + \frac{J^2}{M^2\sigma^2} \right)^{\frac{1}{2}} \right], \quad r_- = -\frac{\sigma J_i}{2r_+}.
\]

One obtains \( a := \pi r_+ / \sigma, b := \pi |r_-| / \sigma \).

**Definition 2.1** \( \Gamma(a,b) \) is defined to be the cyclic subgroup of \( G = SL(2, \mathbb{C}) \) with generator

\[
\gamma(a,b) \overset{\text{def}}{=} \begin{bmatrix} e^{a+ib} & 0 \\ 0 & e^{-(a+ib)} \end{bmatrix}, \quad (2.2)
\]

\[
\Gamma(a,b) \overset{\text{def}}{=} \{ \gamma^n(a,b) \mid n \in \mathbb{Z} \}. \quad (2.3)
\]

The Riemannian volume element \( dV \) corresponding to the metric \( ds^2 \) is given by \( dV = \frac{\sigma^3}{8\pi} dx dy dz \). It is known that a fundamental domain \( F_{\Gamma(a,b)} \) for the action of \( \Gamma(a,b) \) on \( H^3 \) is given by \( F_{\Gamma(a,b)} = \{(x, y, z) \in H^3 \mid 1 < x^2 + y^2 + z^2 < e^{2a}\} \). It follows that \( \Gamma(a,b) \) is a Kleinian subgroup of \( G \) and

\[
\text{Vol}(F_{\Gamma(a,b)}) = \int_{F_{\Gamma(a,b)}} dV = \infty. \quad (2.4)
\]

**Scattering matrix and resonances.** Since \( F_{\Gamma(a,b)} \) has an infinite hyperbolic volume, the usual spectral theory for the Laplacian \( \Delta_{\Gamma(a,b)} \) of \( H_{\Gamma(a,b)} \) does not apply (as it does for finite volume orbifolds). We outline, briefly, a suitable spectral analysis of \(-\Delta_{\Gamma(a,b)}\) where a key notion is that of scattering resonances (see for more detail \([14]\)). Henceforth we shall write \( \Gamma \) for \( \Gamma(a,b) \); one notes that \( \Delta_{\Gamma} \) is given by

\[
\Delta_{\Gamma} = \frac{1}{\sigma^2} \left[ z^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) - z \frac{\partial}{\partial z} \right]. \quad (2.5)
\]

The space of square-integrable functions on the black hole \( H_{\Gamma} = \Gamma \backslash H^3 \), with respect to the measure \( dV \), has an orthogonal decomposition

\[
L^2(H_{\Gamma}, dV) = \bigoplus_{m,n \in \mathbb{Z}} H_{mn} \text{ with Hilbert space isomorphisms } H_{mn} \simeq L^2(\mathbb{R}_+, dt), \quad (2.6)
\]
for \( \mathbb{R}_+ \) the space of positive real numbers. A spectral decomposition becomes

\[- \sigma^2 \Delta \Gamma \simeq \sum_{m,n \in \mathbb{Z}} \oplus L_{mn}, \quad L_{mn} = -\frac{d^2}{dt^2} + 1 + V_{mn}(t), \quad (2.7)\]

where \( L_{mn} \) are the Schrödinger operators with Pöschel-Teller potentials

\[V_{mn}(t) = \left( k_{mn}^2 + \frac{1}{2} \right) \text{sech}^2 t + \left( m^2 - \frac{1}{4} \right) \cosh^2 t, \quad k_{mn} := -\frac{mb}{a} + \pi n. \quad (2.8)\]

For details of this and the following remarks the reader can consult [3] [11] [12], for example. The Schrödinger equation \( \Psi''(x) + [E - V_{mn}(x)] \Psi(x) = 0 \), which is the same as the eigenvalue problem \( L_{mn} \Psi = k^2 \Psi \) for \( E = k^2 - 1 \), has a known solution \( \Psi^+(x) \) (in terms of the hypergeometric function) with asymptotics

\[\Psi^+(x) \sim \frac{e^{ikx}}{T_{mn}(k)} + \frac{R_{mn}(k)}{T_{mn}(k)} e^{-ikx}, \quad (2.9)\]

for reflection and transmission coefficients \( T_{mn}(k), R_{mn}(k) \) respectively.

**Definition 2.2** For \( s \) defined by \( k = i(1 - s) \) one can form the scattering matrix \([\mathcal{S}_{mn}(s)] \) \( \equiv \) \( [R_{mn}(k)] \) of \( -\Delta \Gamma \), whose entries are quotients of gamma functions with ”trivial poles” \( s = 1 + j, j = 0, 1, 2, 3, \ldots \), and non-trivial poles

\[s_{mn}^\pm := -2j - |m| \pm i |k_{mn}| \quad (2.10)\]

for \( k_{mn} \) in (2.8), \( j = 0, 1, 2, 3, \ldots \). The \( s_{mn}^\pm \) are the scattering resonances.

### 3 Truncated heat kernel and the zeta function

In this section we briefly relate the result of the heat kernel trace (integration over the fundamental domain \( F = F_{(a,b)} \) along the diagonal) and a zeta function \( Z(\gamma) \) which has been discussed in [10]. The calculation is carried out conveniently with spherical coordinates: for \( \rho \geq 0, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi/2, x = \rho \sin \phi \cos \theta, y = \rho \sin \phi \sin \theta, z = \rho \cos \phi \).

**Notation 3.1** For \( p = (x,y,z) \in H^3 \), its image (i.e. its \( \Gamma \)-orbit) under the quotient map \( H^3 \rightarrow \Gamma \backslash H^3 = H_\Gamma \) will be denoted by \( \tilde{p} \). \( d(p_1,p_2) \) will denote the hyperbolic distance between two points \( p_1, p_2 = (x_1,y_1,z_1), (x_2,y_2,z_2) \) in \( H^3 \):

\[\cosh d(p_1,p_2) := 1 + \frac{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}{2z_1z_2}. \quad (3.1)\]

The heat kernel \( K^\Gamma_t (t > 0) \) of \( H_\Gamma \) is obtained by averaging over \( \Gamma \) the heat kernel \( K_t \) of \( H^3 \):

\[K^\Gamma_t (\tilde{p}_1, \tilde{p}_2) = \sum_{n \in \mathbb{Z}} K_t (p_1, \gamma^n p_2) = \sum_{n \in \mathbb{Z}} \frac{e^{-t - d(p_1, \gamma^n p_2)^2 / 4t}}{(4\pi t)^{\frac{3}{2}}} \frac{\sinh d(p_1, \gamma^n p_2)}{d(p_1, \gamma^n p_2)^{3/2}}, \quad (3.2)\]

where we write \( \gamma \) for \( \gamma_{(a,b)} \).

\(^4\) In particular, the proof of equation (2.7) is given in Section 3 of [3] - in equations (3.10)-(3.21) there.
Theorem 3.1 (A. A. Bytsenko, M. E. X. Guimarães and F. Williams [10]) For later purposes it is convenient to set \( \ell \overset{\text{def}}{=} 2a = 2\pi r_+ / \sigma \), \( \theta \overset{\text{def}}{=} 2b = 2\pi |r_-| / \sigma \).

Let \( K_t^{\Gamma(p_1, p_2)} \) denote the truncated heat kernel of \( H_\Gamma \), defined by restricting the sum over \( Z \) in (3.2) to the non-zero integers \( n \). The following result for the trace of \( K_t^{\Gamma(p_1, p_2)} \) holds:

**Definition 3.1** For the volume element \( dV = \frac{a^3}{2} dx dy dz, t > 0 \), and the theta-function

\[
\Theta_\Gamma(t) := \frac{\ell}{8\sqrt{4\pi t}} \sum_{n \in Z - \{0\}} \frac{e^{-(\frac{n^2 \ell^2}{4t})}}{[\sinh^2 (\frac{\ell n}{2}) + \sin^2 (\frac{\ell n}{2})]}. \tag{3.3}
\]

one has that

\[
\int \int K_t^{\Gamma(p_1, p_2)} dV = 2\sigma^3 \Theta_\Gamma(t). \tag{3.4}
\]

The following zeta function has been attached to the BTZ black hole \( H_\Gamma \) (see for detail [4] [13]):

\[
Z_\Gamma(s) := \prod_{k_1, k_2 \geq 0, k_1, k_2 \in \mathbb{Z}} \left[ 1 - (e^{i\theta})^{k_1} (e^{-i\theta})^{k_2} e^{-(k_1 + k_2 + s)\ell} \right]. \tag{3.5}
\]

\( Z_\Gamma(s) \) is an entire function of \( s \), whose zeros are precisely the following complex numbers \( \zeta_{n, k_1, k_2} = -(k_1 + k_2) + i (k_1 - k_2) \theta / \ell + 2\pi in / \ell \), and whose logarithmic derivative for \( \text{Re} s > 0 \) is given by

\[
\frac{d}{ds} \log Z_\Gamma(s) = \frac{\ell}{4} \sum_{n=1}^{\infty} \frac{e^{-n(\ell s - 1)}}{[\sinh^2 (\frac{\ell n}{2}) + \sin^2 (\frac{\ell n}{2})]} \tag{3.6}
\]

**Remark 3.1** It is remarkable fact that the set of scattering poles in (2.10) coincides with zeros of \( \zeta_{n, k_1, k_2} \), as it can be verified. Thus encoded in \( Z_\Gamma(s) \) is the spectrum of a BTZ black hole.

\( Z_\Gamma(s) \) is connected with the theta function \( \Theta_\Gamma(t) \) in (3.3), and hence with the heat kernel \( K_t^{\Gamma} \) via the following theorem:

**Theorem 3.2 (A. A. Bytsenko, M. E. X. Guimarães and F. Williams [10])** For \( \text{Re} s > 1 \) one has

\[
\int_0^\infty e^{-s(s-2)t} \Theta_\Gamma(t) dt = \frac{1}{2(s-1)} \frac{d}{ds} \log Z_\Gamma(s). \tag{3.7}
\]

4 Quantum corrections

We set up some more notation. For \( \sigma_p \) the natural representation of \( SO(2k - 1) \) on space of \( p \)-forms \( \Lambda^p \mathbb{C}^{2k-1} \) one has the corresponding Harish-Chandra-Plancherel density \( \mu_{\sigma_p(r)} \) given, for a suitable normalization of Haar measure \( dx \) on \( G \). Simplifying calculations we will take into account the case of smooth functions \( p = 0 \) or smooth vector bundle sections, and therefore the measure \( \mu(r) \equiv \mu_0(r) \) corresponds to the trivial representation of \( SO(n - 1) \). Let \( \chi_\sigma = \text{trace}(\sigma) \)
be the character of \( \sigma \). Since \( \sigma_0 \) is the trivial representation one has \( \chi_{\sigma_0} = 1 \). It follows that the scalar determinant on \( H_\Gamma \) has the form

\[
\log \det \triangle_\Gamma = -2\sigma^3 \int_0^\infty t^{-1} \text{Tr} K^\Gamma_t (\vec{p}_1, \vec{p}_2) \, dt = -2\sigma^3 \int_0^\infty t^{-1} \Theta_\Gamma(t) \, dt
\]

\[
= -2\sigma^3 \int_0^\infty \sum_{n=1}^{\infty} \frac{\ell}{4t \sqrt{2\pi \ell} \left[ \sinh^2 \left( \frac{tn}{2} \right) + \sin^2 \left( \frac{tn}{2} \right) \right]} \, dt \, e^{-(t + \frac{\ell^2}{4t})} \, dt \, e^{-(t + \frac{\ell^2}{4t})} \, dt . \tag{4.1}
\]

The calculation of the integral in (4.1) relies on the following representation for the Bessel function \( K_\nu(s) \),

\[
\int_0^\infty x^{\nu-1} e^{-\alpha x - \beta x} \, dx = 2 \left( \frac{\alpha}{\beta} \right)^{\nu/2} K_\nu(2\sqrt{\alpha \beta}), \quad \text{Re} \alpha > 0, \text{ Re} \beta > 0 . \tag{4.2}
\]

Taking into account that \( K_{\pm1/2}(s) = \sqrt{\pi/2s} \exp(-s) \) we finally get

\[
\log \det \triangle_\Gamma = -\frac{\sigma^3}{2} \sum_{n=1}^{\infty} \frac{e^{-n\ell}}{n \left[ \sinh^2 \left( \frac{tn}{2} \right) + \sin^2 \left( \frac{tn}{2} \right) \right]} = 2\sigma^3 \log Z_{\Gamma}(2) . \tag{4.3}
\]

### 4.1 The tensor kernel and spectral functions on p-forms

Quantum corrections coming from small fluctuations around metric extremum and from gauge-fixing can be computed by taking advantage of the of the connection between three-dimensional gravity and Chern-Simons theory [14, 15]. One-loop corrections can be given by the square root of the Ray-Singer torsion [16] (or equivalently the Reidemeister-Franz torsion) of the manifold \( H_\Gamma \). Let \( \xi \) be an irreducible representation of \( K \) on a complex vector space \( V_\xi \), and form the induced homogeneous vector bundle \( G \times_K V_\xi \) (the fiber product of \( G \) with \( V_\xi \) over \( K \)). Restricting the \( G \) action to \( \Gamma \) we obtain the quotient bundle

\[
E_\xi = \Gamma \backslash (G \times_K V_\xi) \longrightarrow H_\Gamma . \tag{4.4}
\]

The natural Riemannian structure on \( H_\Gamma \) induced by the Killing form (,.) of \( G \) gives rise to a connection Laplacian \( \triangle_\Gamma \) on \( E_\xi \). Let us briefly recall the definition of the Ray-Singer torsion. Suppose \( E = (E, \nabla) \) is a real or complex vector bundle \( E \) equipped with a flat connection \( \nabla \). Let \( g \) be a Riemannian metric on a manifold \( M \) and let \( \mu \) be a Hermitian metric on \( E \).

**Definition 4.1** The Ray-Singer torsion \( T_{an}(\nabla, g, \mu) \) is given by

\[
\log T_{an}(\nabla, g, \mu) \overset{\text{def}}{=} \frac{1}{2} \sum_p (-1)^{p+1} p \log [\det \triangle(p)]_{\text{reg}} . \tag{4.5}
\]

Here \( \triangle(p) \) denotes the Laplacian in degree \( p \) of the elliptic complex \( (\Omega(X; E), d_\nabla) \) equipped with the scalar product induced from the Riemannian metric \( g \) and the Hermitian metric \( \mu \), and \([\det \triangle(p)]_{\text{reg}} \) denotes its zeta regularized determinant.

The bundle (4.4) arises in the theory of geometric structures, and admits a natural flat connection with a holonomy group isomorphic to \( \Gamma \). If there are no ghost zero-modes, that is, if the de

---

5 It has been shown [14] that three-dimensional gravity can be rewritten as a Chern-Simons theory for the gauge group \( G \). In addition, extremum of the action determines a flat connection, with a corresponding bundle given by (4.4). For the Chern-Simons invariant of an irreducible flat connection on real hyperbolic three-manifold see [17].
Rham cohomology $H^0(H_\Gamma; E_\omega) = 0$ on forms $\omega$ of $H_\Gamma$ with values in the flat bundle $E_\omega$, then the Ray-Singer torsion is a topological invariant (is not dependent on the choice of auxiliary metric). When zero modes are present, they must be included in Eq. (15).

**Remark 4.1** A version of the trace formula for the heat kernel on $p$–forms, developed in [18] (see also [10]), leads to the identity $I_{\Gamma}^{(p)}$ and geodesic $\mathcal{G}_{\Gamma}^{(p)}$ terms. This decomposition reduces to divergences of identity terms, since they proportional to $\text{Vol}(H_\Gamma)$, and hence to divergent part of the effective action. Note that identity terms describe the renormalization of the cosmological constant at one-loop, and can be canceled by local counterterms.

### 4.2 Quantum contribution to the partition functions

**Remark 4.2** The effective action for the scalar field on three-dimensional BTZ black hole instanton can be calculated as follows. The non-divergent part of the effective action is given by [19]:

$$W_{\text{scalar}}^{\text{(non-divergent)}} = -\sum_{n=1}^{\infty} \frac{e^{-\sqrt{1-\xi}n\ell}}{4n[\sinh^2 \left(\frac{\theta_n}{2}\right) + \sin^2 \left(\frac{\theta_n}{2}\right)]},$$

where the constant $(1 - \xi)$ appearing in the heat equation formula. In the conformal invariant case of non-minimal coupling we have $\xi = 3/4$. Using the Selberg-like zeta function $Z_\Gamma(s)$ we get

$$W_{\text{scalar}}^{\text{(non-divergent)}} = \log Z_\Gamma(1 + \sqrt{1 - \xi}).$$

The divergent part of the effective action could be suppressed by introducing the local counterterms.

Corrections to the spectrum of three-dimensional gravity and the one-loop contribution to the partition function (the contribution of states of left- and right-moving modes) of the conformal field theory has been evaluated in [1, 2]. The later contribution has the form:

$$Z(\tau) = \left|q\bar{q}\right|^{-k} \prod_{n=2}^{\infty} \left|1 - q^n\right|^{-2},$$

where $24k = c_L = c_R = c$, and $c$ is the central charge of a conformal field theory. In [18] $q = \exp[2\pi i\tau] = \exp[2\pi(-\text{Im}\tau + i\text{Re}\tau)]$ such that $|q\bar{q}|^{-k} = \exp[4\pi k\text{Im}\tau]$ corresponds to the classical prefactor of the partition function. To make correspondence between models one can choose the pair $(a, b)$ in our definition (2.2), (2.3) as $(a = -\pi\text{Im}\tau, b = \pi\text{Re}\tau)$. Then since $k = (16\sigma G)^{-1}$ one gets

$$-\log Z_{\text{cl}}(\tau) = k\log|q\bar{q}| = 4\pi k\sigma r_+ = \frac{4\pi^2 r_+}{16\pi G}.$$  

The result (4.9) is the classical part of the contribution (see for example [8]). Eq. (4.8) is one-loop exact as has been claimed in [1].

**Modular invariance.** For three-dimensional gravity in locally Anti-de Sitter space-times the one-loop partition function has been calculate in [2], and the result is (Eqs. (4.27) and (4.28) of [2]):

$$Z_{\text{gravity}}^{\text{1-loop}}(\tau) = \prod_{m=2}^{\infty} \left|1 - q^m\right|^{-2}.$$  

7
If we let $\ell = 2\pi \text{Im} \tau, \theta = 2\pi \text{Re} \tau$, then
\[
\sinh^2 \left( \frac{\ell n}{2} \right) + \sin^2 \left( \frac{\theta n}{2} \right) = |\sin(n\pi \tau)|^2 = \frac{|1 - q^n|^2}{4|q|^n}. \tag{4.11}
\]
Using Eq. (4.11) we get
\[
\log \prod_{m=2}^{\infty} |1 - q^m|^{-2} = - \sum_{m=2}^{\infty} \log |1 - q^m|^2 = \sum_{n=1}^{\infty} \frac{q^{2n} + q^{-2n} - |q|^{2n}(q^n + q^{-n})}{n|1 - q^n|^2} = \sum_{n=1}^{\infty} \frac{e^{-2n\pi \text{Im} \tau} \cos(4n\pi \text{Re} \tau) - e^{-4n\pi \text{Im} \tau} \cos(2n\pi \text{Re} \tau)}{2n[\sinh^2(n\pi \text{Im} \tau) + \sin^2(n\pi \text{Re} \tau)]}. \tag{4.12}
\]
One can summarize the preceding formulas by getting the following result
\[
\log Z_{\text{gravity}}^{1-\text{loop}}(\tau) = \sum_{n=1}^{\infty} \frac{e^{-2n\pi \text{Im} \tau} \cos(4n\pi \text{Re} \tau) - e^{-4n\pi \text{Im} \tau} \cos(2n\pi \text{Re} \tau)}{2n[\sinh^2(n\pi \text{Im} \tau) + \sin^2(n\pi \text{Re} \tau)]} = \log \left[ \frac{Z_{\Gamma}(3 + i\text{Re} \tau)}{Z_{\Gamma}(3 - i\text{Re} \tau)} \frac{Z_{\Gamma}(3 - i\text{Re} \tau)}{Z_{\Gamma}(1 + i\text{Re} \tau)} \right]. \tag{4.13}
\]
Note that in [2] the one-loop partition function of three-dimensional gravity in hyperbolic spaces has been considered and stressed that such a geometry is also geometry of the Euclidean BTZ black hole. Thus if we let the modular transformation $\tau = \frac{1}{2\pi}(\theta + i\ell) \rightarrow -\frac{1}{\tau}$, i.e.
\[
\text{Re} \tau \rightarrow -\frac{\text{Re} \tau}{|\tau|^2}, \quad \text{Im} \tau \rightarrow \frac{\text{Im} \tau}{|\tau|^2}, \quad \text{Re} \tau \rightarrow -\frac{\text{Re} \tau}{\text{Im} \tau}, \quad \text{Im} \tau \rightarrow \frac{\text{Im} \tau}{\text{Im} \tau}, \tag{4.14}
\]
then the computation of the one-loop partition function of the three-dimensional gravity gives the one loop correction to black holes. It is easy seen that the last formula in (4.13) is invariant under the transformation $\tau \rightarrow 1 + \tau$. Therefore (we let $\sigma = 1$)
\[
Z_{\text{gravity}}^{1-\text{loop}}(\tau) = Z_{\text{(spinning BTZ)}}^{1-\text{loop}}(\ell, \theta). \tag{4.15}
\]
Finally, simplifying calculations one can consider a non-spinning black hole $(J=0)$; in this case $\theta = 0 (\text{Re} \tau = 0)$ and
\[
Z_{\text{(non-spinning BTZ)}}^{1-\text{loop}} = \left[ \frac{Z_{\Gamma}^{\theta=0}(3)}{Z_{\Gamma}^{\theta=0}(1)} \right]^2. \tag{4.16}
\]

5 Concluding remarks

In this paper we obtained the quantum correction to the BTZ black hole making use of three-dimensional gravity representation [1][2]. We have studied the quantum correction by expressing the determinants in terms of the appropriate heat kernels, which were evaluated using a method of images. This correction was expressed as a special value of the logarithms of the zeta functions. Earlier the first quantum correction for non-spinning black hole ($J=0$) has been obtained in [8], but it has error in the computation. In this paper we revised the correction using a special relation between the spectrum and the truncated heat kernel of the Euclidean BTZ spinning black hole with the Patterson-Selberg zeta function. It gives us possibility to provide the correct result for the quantum correction.
Acknowledgments

A. A. Bytsenko and M. E. X. Guimarães would like to thank the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) for support.

References

[1] A. Maloney and E. Witten, Quantum Gravity Partition Function In Three Dimensions, arXiv:0712.0155.

[2] S. Giombi, A. Maloney and X. Yin, One-loop Partition Functions of 3D Gravity, arXiv:hep-th/0804.1773.

[3] M. Bañados, C. Teitelboim and J. Zanelli, Black hole in three-dimensional spacetime, Phys. Rev. Letters 69 (1992) 1849-1851.

[4] P. Perry and F. Williams, Selberg zeta function and trace formula for the BTZ black hole, Internat. J. of Pure and Applied Math. 9 (2003) 1-21.

[5] P. Perry, A Poisson summation formula and lower bounds for resonances in hyperbolic manifolds, Int. Math. Res. Notes 34 (2003) 1837-1851.

[6] S. J. Patterson, The Selberg zeta-function of a Kleinian group, In Number Theory, Trace Formulas, and Discrete groups: Symposium in Honor of Atle Selberg, Oslo, Norway, July 14-21, 1987. New York, Academic press, 1989.

[7] S. J. Patterson and P. A. Perry, The devisor of the Selberg zeta function for Kleinian groups, with an appendix by Charles Epstein, Duke Math. J. 106 (2001) 321-390.

[8] A. A. Bytsenko, L. Vanzo, S. Zerbini, Quantum Correction to the Entropy of the (2+1)-Dimensional Black Hole, Phys. Rev. D 57 (1998) 4917-4924; arXiv:gr-qc/9710106.

[9] A. A. Bytsenko, L. Vanzo, S. Zerbini, Ray-Singer Torsion for a Hyperbolic 3-Manifold and Asymptotics of Chern-Simons-Witten Invariant, Nucl. Phys. B 505 (1997) 641-659; arXiv:hep-th/9704035.

[10] A. A. Bytsenko, M. E. X. Guimaraes and F. L. Williams, Spectral Functions for BTZ Black Hole Geometry, Lett. Math. Phys. 79 (2007) 203-211; arXiv:hep-th/0609102.

[11] J. Sjöstrand and M. Zworski, Lower bounds on the number of scattering poles, Comm. Partial Diff. Equations 18 (1993) 847-857.

[12] L. Guillopé and M. Zworski, Upper bounds on the number of resonances for non-compact Riemann surfaces, J. of Funct. Analysis 129 (1995) 364-389.

[13] F. Williams, A zeta function for the BTZ black hole, Internat. J. Modern Physics A 18 (2003) 2205-2209.

[14] E. Witten, 2+1 Dimensional gravity as an exactly soluble system, Nucl. Phys. B 311 (1988/89) 46-78.

[15] S. Carlip, The Sum over Topologies in Three-Dimensional Euclidean Quantum Gravity, Class. Quant. Grav. 10 (1993) 207-218; arXiv:hep-th/9206103.
[16] D. B. Ray and I. M. Singer, *Analytic Torsion For Complex Manifolds*, Annals Math. **98** (1973) 154-177.

[17] L. Bonora and A. A. Bytsenko, *Fluxes, Brane Charges and Chern Morphisms of Hyperbolic Geometry*, Class. Quant. Grav. **23** (2006) 3895-3916; [arXiv:hep-th/0602162](http://arxiv.org/abs/hep-th/0602162).

[18] D. Fried, *Analytic torsion and closed geodesics on hyperbolic manifolds*, Invent. Math. **84** (1986) 523-540.

[19] R. B. Mann and S. N. Solodukhin, *Quantum scalar field on three-dimensional (BTZ) black hole instanton: heat kernel, effective action and thermodynamics*, Phys. Rev. D **55** (1997) 3622-3632; [arXiv:hep-th/9609085](http://arxiv.org/abs/hep-th/9609085).