EIGENVALUES OF MAJORIZED HERMITIAN MATRICES
AND LITTLEWOOD-RICHARDSON COEFFICIENTS.

WILLIAM FULTON

September 21, 2000

Abstract. Answering a question raised by S. Friedland, we show that the possible
eigenvalues of Hermitian matrices (or compact operators) $A, B,$ and $C$ with $C \leq A + B$ are given by the same inequalities as in Klyachko’s theorem for the case where
$C = A + B$, except that the equality corresponding to $\text{tr}(C) = \text{tr}(A) + \text{tr}(B)$ is
replaced by the inequality corresponding to $\text{tr}(C) \leq \text{tr}(A) + \text{tr}(B)$. The possible
types of finitely generated torsion modules $A, B,$ and $C$ over a discrete valuation
ring such that there is an exact sequence $B \to C \to A$ are characterized by the same
inequalities.

Introduction

A. Klyachko [10], cf. [13], [6], [7], has shown that the possible eigenvalues $\alpha, \beta, \gamma$
of Hermitian $n$ by $n$ matrices $A, B, C$ with $C = A + B$ are characterized by a certain
list of inequalities, together with the trace equality $\sum \gamma_i = \sum \alpha_i + \sum \beta_i$. Combined
with the solution of the saturation conjecture by A. Knutson and T. Tao [11], cf.
[2], [3], one can show that this list of inequalities is exactly that conjectured by A.
Horn [8]. P. Belkale [1] showed that that a smaller list of inequalities is sufficient,
and Knutson, Tao, and C. Woodward [12] have announced that this smaller list is
actually minimal. This story, with more of the history and references, can be found
in [7].

Eigenvalues (and all $n$-tuples of real numbers) are always written in descending
order, so we start with the $3n - 3$ inequalities

$$\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n, \quad \beta_1 \geq \beta_2 \geq \ldots \geq \beta_n, \quad \gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_n.$$  

S. Friedland [4] subsequently gave a characterization of the inequalities satisfied by
the eigenvalues $\alpha, \beta, \gamma$ of Hermitian $n$ by $n$ matrices $A, B,$ and $C$ with $C \leq A + B$,
i.e., such that $A + B - C$ is positive semidefinite. His inequalities included those of
Klyachko, together with the trace inequality

$$\sum_{i=1}^{n} \gamma_i \leq \sum_{i=1}^{n} \alpha_i + \sum_{i=1}^{n} \beta_i.$$  

1991 Mathematics Subject Classification. 15A42, 22E46, 14M15, 05E15, 13F10, 47B07.

Key words and phrases. Hermitian, eigenvalues, Littlewood-Richardson, Schubert calculus.

The author was partly supported by NSF Grant #DMS9970435

Typeset by $\LaTeX$
but they also included some other inequalities that are less easy to describe. For $n = 2$ and $n = 3$, however, he showed that these extra inequalities are superfluous.

In this paper we show that these extra inequalities are always superfluous, so that the natural generalization of Klyachko et al. to the majorization situation is valid.

The inequalities of Horn or Klyachko have the form

$$(IJK)\quad \sum_{k \in K} \gamma_k \leq \sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j$$

where $(I, J, K)$ vary over certain subsets of the same cardinality $r$ of the index set $[n] = \{1, \ldots, n\}$, $1 \leq r \leq n - 1$. If, for $I = \{i_1 < \ldots < i_r\}$ we set

$$\lambda(I) = (i_r - r, \ldots, i_2 - 2, i_1 - 1),$$

then $(I, J, K)$ is on the Horn/Klyachko list exactly when the Littlewood-Richardson coefficient $c_{\lambda(I) \lambda(J)}^{\lambda(K)}$ of the three corresponding partitions is positive. The minimal list of inequalities of Belkale-Knutson-Tao-Woodward consists of those $(IJK)$ for which $c_{\lambda(I) \lambda(J)}^{\lambda(K)} = 1$, together with the trace equality and the inequalities $(\dagger)$, for $n \geq 3$. (For $n = 2$ the inequalities (\dagger) follow from the others.)

**Theorem 1.** A triple $\alpha, \beta, \gamma$ satisfying (\dagger) occurs as the eigenvalues of $n$ by $n$ Hermitian matrices $A, B, C$ with $C \leq A + B$ if and only if they satisfy (\dagger\dagger) and (\dagger$IJK$) for all $(I, J, K)$ of cardinality $r < n$ such that $c_{\lambda(I) \lambda(J)}^{\lambda(K)} = 1$.

In fact, granting the minimality assertion of [12], we show that these inequalities (\dagger), (\dagger\dagger), and (\dagger$IJK$) are a minimal list of inequalities for this majorization problem, for all $n \geq 1$.

As in [4], this generalizes to give an infinite list of inequalities for eigenvalues of compact self-adjoint nonnegative operators $A, B, C$ with $C \leq A + B$.

As in [6] and [7], the same assertions are valid for real symmetric matrices (or operators) or for quaternionic Hermitian matrices. All the results extend to sums of $m$ rather than two factors; these generalizations are given in Section 2.

As explained in [7], Klyachko’s theorem has analogues in several other areas of mathematics, such as representation theory, algebra, and combinatorics. The same is true of the stronger theorems presented here. In particular, the inequalities in Theorem 1, when restricted to the case where $\alpha, \beta, \gamma$ are partitions, are equivalent to the conditions that $\gamma$ is contained in some partition $\tilde{\gamma}$ such that the Littlewood-Richardson coefficient $c_{\alpha\beta}^{\gamma}$ is positive, and also to the condition that $\alpha$ and $\beta$ contain partitions $\tilde{\alpha}$ and $\tilde{\beta}$ such that $c_{\tilde{\alpha}\tilde{\beta}}^{\gamma}$ is positive. A consequence is a simple criterion for there to be homomorphisms putting three given finite abelian $p$-groups in an exact sequence. More generally, recall that the type of a finitely generated torsion module $A$ over any discrete valuation ring $R$ is the partition $\alpha : \alpha_1 \geq \ldots \geq \alpha_n$ such that $A$ is isomorphic to $R/p^{\alpha_1} \oplus \ldots \oplus R/p^{\alpha_n}$, where $p$ is the maximal ideal of $R$. Then we have:

\[A \quad \text{partition} \quad \tilde{\gamma} \quad \text{contains} \quad \gamma \text{ if } \tilde{\gamma}_i \geq \gamma_i \text{ for all } i, \text{i.e., the Young diagram of } \tilde{\gamma} \text{ contains that of } \gamma. \text{ In this case we write } \tilde{\gamma} \supset \gamma. \]
**Theorem 2.** For any discrete valuation ring $R$, there is an exact sequence

$$B \rightarrow C \rightarrow A$$

of $R$-modules of types $\beta$, $\gamma$, and $\alpha$ (all partitions of lengths at most $n$) if and only if they satisfy the inequalities (†), (††), and (†IJK) for all $(I,J,K)$ of cardinality $r < n$ such that $c_{\lambda(I)}^{\lambda(J)} = 1$.

This is a minimal list of inequalities. More general results appear in Section 2.

One naturally expects Theorem 1 to be a consequence of Klyachko’s theorem, since an inequality $C \leq A(1) + \ldots + A(m)$ is equivalent to the existence of an $A(m+1)$ with nonpositive eigenvalues such that $C = A(1) + \ldots + A(m+1)$. This means that the polyhedral cone describing the majorization problem is a projection of a polyhedral cone describing the equality problem with one more factor. Computing explicit inequalities to describe the projection of a polyhedral cone, however, is seldom easy, and it is exactly this that leads to the extraneous and inexplicit inequalities of [4].

Our proof does depend on the Klyachko case of equality, but not by a projection. The essential point is to show that if some inequality (†IJK) is an equality, then the triple $(\alpha, \beta, \gamma)$ splits into two triples, one $(\alpha', \beta', \gamma')$ of length $r$ (consisting of those $\alpha_i$ for $i \in I$, $\beta_j$ for $j \in J$ and $\gamma_k$ for $k \in K$), and one $(\alpha'', \beta'', \gamma'')$ of length $n-r$ (consisting of the others). We show that $(\alpha', \beta', \gamma')$ satisfies the conditions to be the eigenvalues of $r$ by $r$ Hermitian matrices $A', B', C'$ with $C' = A' + B'$, and, by induction, that $(\alpha'', \beta'', \gamma'')$ satisfies the conditions to be the eigenvalues of $n-r$ by $n-r$ Hermitian matrices $A'', B'', C''$ with $C'' \leq A'' + B''$. The direct sums $A = A' \oplus A''$, $B = B' \oplus B''$, and $C = C' \oplus C''$ then do the trick.

To prove that these inductive conditions are satisfied involves a little Schubert calculus, which is carried out in the first section. This deduces some nonzero intersections in a Grassmann variety $\text{Gr}(r,n)$ of $r$-planes in $\mathbb{C}^n$ from nonzero intersections in $\text{Gr}(p,r)$ or $\text{Gr}(p,n-r)$ for smaller $p$. This can be regarded as a contribution from the Schubert calculus side of the general problem, emphasized in [7], of understanding why all the conditions in the variations of Klyachko’s theorem are inductive, i.e., their answers for given $n$ are determined by knowing the answers to the same questions for smaller $n$.

We thank H. Derksen, J. Weyman, J. Harris, and A. Buch for stimulating discussions that led to the proof given here, and S. Friedland for raising the question and suggesting improvements.

**Section 1. Preliminaries from Schubert calculus**

We start by fixing some notation.

A subset $I$ of $[n] = \{1, \ldots, n\}$ is always written in increasing order, so that $I = \{i_1 < i_2 < \ldots < i_r\}$. For a subset $P$ of $[r]$ of cardinality $x$, set

$$I_P = \{i_p \mid p \in P\} = \{i_{p_1} < \ldots < i_{p_x}\},$$

a subset of $[n]$ of cardinality $x$. For $P$ a subset of $[n-r]$ of cardinality $y$, set

$$I_P^+ = I \cup (I^c)_P,$$
where \( I^c \) denotes the complement of \( I \) in \([n]\); this is a subset of \([n]\) of cardinality \( r+y \).

For \( m \)-tuples \( \mathcal{I} = (I(1), \ldots, I(m)) \) and \( \mathcal{P} = (P(1), \ldots, P(m)) \) of such subsets, we write \( \mathcal{I}_\mathcal{P} \) for \( (I(1)_{P(1)}, \ldots, I(m)_{P(m)}) \), and \( \mathcal{I}_{\mathcal{P}^r} \) for \( (I(1)_{P(1)}^r, \ldots, I(m)_{P(m)}^r) \).

For subsets \( H \) and \( I \) of \([n]\) of the same cardinality \( r \), we write \( H \preceq I \) if \( H = \{h_1 < \ldots < h_r\} \) and \( I = \{i_1 < \ldots < i_r\} \) with \( h_a \leq i_a \) for \( 1 \leq a \leq r \). Equivalently, \(|H \cap [k]| \geq |I \cap [k]| \) for \( 1 \leq k \leq n \).

Let \( V \) be an \( n \)-dimensional complex vector space, and let

\[
F_\bullet : 0 = F_0 \subset F_1 \subset \ldots \subset F_n = V
\]

be a complete flag of subspaces of \( V \). For a subset \( I \) of \([n]\) of cardinality \( r \), the Schubert variety \( \Omega_I(F_\bullet) \) is the subvariety of the Grassmannian \( \text{Gr}(r, V) \) of \( r \)-planes in \( V \) defined by

\[
\Omega_I(F_\bullet) = \{L \in \text{Gr}(r, V) \mid \dim(L \cap F_{i_k}) \geq k \text{ for } 1 \leq k \leq r\}.
\]

The class of \( \Omega_I(F_\bullet) \) in the cohomology ring \( H^*(\text{Gr}(r, n)) = H^*(\text{Gr}(r, V)) \), which is independent of choice of the flag \( F_\bullet \), is denoted \( \omega_I \). If \( \lambda = (\lambda_1, \ldots, \lambda_r) \), with \( \lambda_k = n-r+k-i_k \); this class is also denoted \( \sigma_{\lambda} \); the complex codimension of \( \Omega_I(F_\bullet) \) is \(|\lambda| = \sum_{i=1}^r \lambda_i \), so we have

\[
\sigma_{\lambda} = \omega_I = [\Omega_I(F_\bullet)] \in H^{2|\lambda|}(\text{Gr}(r, n)).
\]

The Schubert variety \( \Omega_I(F_\bullet) \) is the closure of the Schubert cell \( \Omega^0_I(F_\bullet) \), which is the set of subspaces \( L \) such that the jumps in the sequence

\[
0 = \dim(L \cap F_0) \leq \dim(L \cap F_1) \leq \ldots \leq \dim(L \cap F_n) = r
\]

occur exactly at the integers in \( I \); that is, \( I = \{i \in [n] \mid L \cap F_i \neq L \cap F_{i-1}\} \). For a fixed flag, every \( r \)-dimensional subspace \( L \) of \( V \) belongs to a unique Schubert cell. We use the well-known fact (cf. [5], §310) that

\[
\Omega_I(F_\bullet) = \bigcup_{J \subseteq I} \Omega^0_J(F_\bullet).
\]

**Lemma 1.** Let \( H \) and \( I \) be subsets of \([n]\) of cardinality \( r \) with \( H \preceq I \).

- (i) If \( P \) and \( Q \) are subsets of \([r]\) of cardinality \( x \) with \( P \leq Q \), then \( H_P \leq I_Q \) (as subsets of \([n]\) of cardinality \( x \)).
- (ii) If \( P \) and \( Q \) are subsets of \([n-r]\) of cardinality \( y \) with \( P \leq Q \), then \( H_P^+ \leq I_Q^+ \) (as subsets of \([n]\) of cardinality \( r+y \)).

**Proof.** The proof of (i) is trivial: \( h_{p_a} \leq h_{q_a} \leq i_{q_a} \). With the assumptions of (ii), it follows from (i) that \( I_P^+ \leq I_Q^+ \), so it suffices to show that \( H_P^+ \leq I_Q^+ \). Any \( H \preceq I \) can be obtained from \( I \) by a succession of moves, each of which decreases one integer, leaving the others alone. We may therefore assume that, for some \( k \in [r] \) and some \( m \in [n] \), \( h_a = i_a \) for \( a \neq k \), and \( h_k = m-1 \), \( i_k = m \). The complementary sequences satisfy \( h_a^+ = i_a^+ \) for \( a \neq m-k \), while \( i^+_{m-k} = m-1 \) and \( h^+_{m-k} = m \). Therefore \( H_P^+ = I_P^+ \) if \( P \) contains \( m-k \), and \( H_P^+ < I_P^+ \) otherwise.
Let $U$ be a subspace of $V$ of dimension $r$. A complete flag $F_\bullet$ on $V$ determines a complete flag $F_i U$ on $U$ and a complete flag $F_i W$ on $W = V/U$, which are defined as follows. Let $I = \{i_1 < \ldots < i_r\}$ be the set such that $U$ is in $\Omega^p_I(F_\bullet)$. Set

$$F_k U = F_{i_k} \cap U, \quad 1 \leq k \leq r.$$ 

With $I^c = \{i_1^c < \ldots < i_n^c\}$, set

$$F_k W = (U + F_i)/U, \quad 1 \leq k \leq n - r.$$ 

From the isomorphism $(U + F_i)/U \cong F_i/F_i \cap U$ it follows that $F_i \cap U = F_{i - 1} \cup U$ if and only if $U + F_i \neq U + F_{i - 1}$.

**Lemma 2.** Let $U$ be in a Schubert variety $\Omega_I(F_\bullet)$ in $Gr(r, V)$.

(i) If $X$ is a subspace of $U$ of dimension $x$, with $X$ in $\Omega_P(F_i U)$ in $Gr(x, U)$, then $X$ is in $\Omega_P(F_i)$ in $Gr(x, V)$.

(ii) If $Y$ is a subspace of $W$ of dimension $y$, with $Y$ in $\Omega_P(F_i W)$ in $Gr(y, W)$, and $Y = Z/U$, then $Z$ is in $\Omega_P(F_i)$ in $Gr(r + y, V)$.

**Proof.** For (i), if $U \in \Omega^p_H(F_\bullet)$ and $X \in \Omega^p_P(F_i U)$, then by Lemma 1(i), $H_Q \leq I_P$, so $\Omega_{H_Q}(F_\bullet) \subset \Omega_{I_P}(F_i)$. So we may assume $U \in \Omega^p_{I_P}(F_i)$ and $X \in \Omega^p_{I_P}(F_i U)$. Then the jumps in the sequence $\dim(U \cap F_i)$, $1 \leq i \leq n$, occur at $i \in I$, and the jumps of the sequence $\dim(X \cap F_i)$ occur at $i \in I_P$, so $X \in \Omega^p_{I_P}(F_i)$, which proves (i).

For (ii), using Lemma 1(ii) similarly, we may assume $U \in \Omega^p_{I_P}(F_i)$ and $Y \in \Omega^p_{I_P}(F_i W)$. The jumps in the sequence $\dim(Y \cap F_i W)$, $1 \leq k \leq n - r$, occur at $k$ in $P$. So the jumps in the sequence

$$\dim(Z \cap (F_i + U)), \quad 1 \leq i \leq n,$$

occur at $i$ in $I_P$. It follows that the jumps in the sequence $\dim(Z \cap F_i)$, $1 \leq i \leq n$, must occur at $i$ in $I \cup I_P$. For if $i \notin I \cup I_P$, then $U \cap F_i = U \cap F_{i - 1}$ and $Z \cap (F_i + U) = Z \cap (F_{i - 1} + U)$, and these imply that $Z \cap F_i = Z \cap F_{i - 1}$. Therefore $Z$ is in $\Omega^p_{I_P}(F_i)$, which proves (ii).

**Proposition 1.** Suppose $I(1), \ldots , I(m)$ are subsets of $[n]$ of cardinality $r$, with $\prod_{s=1}^m \omega_{I(s)} \neq 0$ in $H^*(Gr(r, n))$.

(i) If $P(1), \ldots, P(m)$, subsets of $[r]$ of cardinality $x$, have $\prod_{s=1}^m \omega_{P(s)} \neq 0$ in $H^*(Gr(x, r))$, then $\prod_{s=1}^m \omega_{I(s) \cup P(s)} \neq 0$ in $H^*(Gr(x, n))$.

(ii) If $P(1), \ldots, P(m)$, subsets of $[n - r]$ of cardinality $y$, have $\prod_{s=1}^m \omega_{P(s)} \neq 0$ in $H^*(Gr(y, n - r))$, then $\prod_{s=1}^m \omega_{I(s) \cup P(s)} \neq 0$ in $H^*(Gr(r + y, n))$.

**Proof.** We use the fact (cf. [7], §4) that for $m$ general flags $F_\bullet(1), \ldots, F_\bullet(m)$, the intersection $\cap_{s=1}^m \Omega_{I(s)} F_\bullet(s)$ is not empty if and only if the product $\prod_{s=1}^m \omega_{I(s)}$ of their classes is not zero. Moreover, for arbitrary flags $F_\bullet(s)$, if the product of the classes is not zero, then the intersection $\cap_{s=1}^m \Omega_{I(s)} F_\bullet(s)$ cannot be empty.

Take general flags $F_\bullet(1), \ldots, F_\bullet(m)$. There is an $r$-dimensional subspace $U$ that is in $\cap_{s=1}^m \Omega_{I(s)} F_\bullet(s)$. In case (i), there is an $X$ in $\cap_{s=1}^m \Omega_{P(s)} F_\bullet(s) U$. By Lemma 2(i), $X$ is in $\cap_{s=1}^m \Omega_{I(s) \cup P(s)} F_\bullet(s)$, which proves (i). In case (ii), there is a $Y = Z/U$ in $\cap_{s=1}^m \Omega_{P(s)} F_\bullet(s) W$, with $W = V/U$, and by Lemma 2(ii), $Z$ is in $\cap_{s=1}^m \Omega_{I(s) \cup P(s)} F_\bullet(s)$; this proves (ii).
Remark. In case (i), even if both products are the classes of a point, that is, if $\prod_{s=1}^{n} \omega_{I(s)} = \omega_{[r]}$ and $\prod_{s=1}^{n} \omega_{P(s)} = \omega_{[x]}$, it does not follow that $\prod_{s=1}^{n} \omega_{I(s)} \cdot \omega_{P(s)} = \omega_{[r]} \cdot \omega_{[x]}$, or even that the codimension of this class must be $x(n-x)$. For example, with $n = 4$, $r = 2$, $x = 1$, and $I = \{(2,4), \{2,4\}, \{2,3\}\}$, $P = \{(2), \{2\}, \{1\}\}$, we have $I_P = \{(4), \{4\}, \{2\}\}$, whose product has codimension 2, not 3. Similarly in case (ii), one can take $I = \{(2,4), \{2,4\}, \{1,4\}\}$, $P = \{(2), \{2\}, \{1\}\}$, with $I_P^+ = \{(2,3,4), \{2,3,4\}, \{1,2,4\}\}$, again with the product of codimension 2, not 3.

We will use standard notation and facts about Littlewood-Richardson coefficients $c_{\alpha \beta}^\gamma$, for which the discussion in [7], §3 should suffice. Note that if the lengths of the three partitions are at most $n$, and their widths (their first entries) are at most $N - n$, then $c_{\alpha \beta}^\gamma$ is the coefficient of the class $\sigma_\gamma$ in the product $\sigma_\alpha \cdot \sigma_\beta$ in $H^*(\text{Gr}(n, N))$. We will also need the following lemma.

**Lemma 3.** Let $\alpha$, $\beta$, $\gamma$ be partitions, with $c_{\alpha \beta}^\gamma > 0$. Let $\tilde{\gamma}$ be a partition with $\tilde{\gamma} \subset \gamma$. Then there are partitions $\tilde{\alpha} \subset \alpha$ and $\tilde{\beta} \subset \beta$ with $c_{\tilde{\alpha} \tilde{\beta}}^{\tilde{\gamma}} > 0$.

**Proof.** We deduce this from a theorem of Green and Klein [9], which says that for some (or any) discrete valuation ring, there is a finitely generated torsion module $C$ of type $\gamma$, with a submodule $B$ of type $\beta$, whose quotient module $A = C/B$ has type $\alpha$, if and only if $c_{\alpha \beta}^\gamma$ is positive. Choose a submodule $\tilde{C}$ of $C$ of type $\tilde{\gamma}$. Then $\tilde{B} = B \cap \tilde{C}$ has type $\tilde{\beta} \subset \beta$, and, since $A = \tilde{C}/\tilde{B} \hookrightarrow A$, $\tilde{A}$ has type $\tilde{\alpha} \subset \alpha$. By the Green-Klein theorem again, $c_{\tilde{\alpha} \tilde{\beta}}^{\tilde{\gamma}} > 0$.

**Section 2. General theorems and proofs**

In the statements of theorems in this section, phrases in brackets indicate alternative versions, meaning that the theorem is true with or without any of all of these bracketed phrases.

For $1 \leq r \leq n$ and $m \geq 1$, as in [7] we let $S_m^r(n)$ be the set of $m$-tuples $I = (I(1), \ldots, I(m))$ of subsets of cardinality $r$ in $[n] = \{1, \ldots, n\}$ such that the product of the corresponding classes $\omega_{I(s)}$ in $H^*(\text{Gr}(r, n))$ does not vanish:

$$S_m^r(n) = \{I = (I(1), \ldots, I(m)) \mid m = \prod_{s=1}^{m} \omega_{I(s)} \neq 0\}.$$ 

Let $R_m^r(n)$ be the set of $m$-tuples $I$ whose product is the class of a point in the Grassmannian, with coefficient 1:

$$R_m^r(n) = \{I = (I(1), \ldots, I(m)) \mid m = \prod_{s=1}^{m} \omega_{I(s)} = \omega_{[r]} \in H^{2r(n-r)}(\text{Gr}(r, n))\}.$$ 

We augment these lists for $r = n$, so that $R_m^r(n) = S_m^r(n)$ consists of the set $[n]$ repeated $m$ times. We set

$$S_n^r(n) = \bigcup_{1 \leq r \leq n} S_r^r(n), \quad \text{and} \quad R_n^r(n) = \bigcup_{1 \leq r \leq n} R_r^r(n).$$

The symmetric version of Theorem 1, for any number of factors, is:

---

2We did not find this fact in the literature, although, when asked, Buch, S. Fomin, J. Stembridge, Tao, and A. Zelevinski quickly produced different proofs of a more combinatorial flavor. Buch shows in fact that if $\alpha \subset \tilde{\gamma}$, then one can find $\tilde{\beta} \subset \beta$ with $c_{\alpha \beta}^{\tilde{\gamma}} > 0$. 
Theorem 3. Let \( \alpha(1), \ldots, \alpha(m) \) be sequences of \( n \)-tuples of real numbers, with \( \alpha(s) = (\alpha_1(s) \geq \alpha_2(s) \geq \ldots \geq \alpha_n(s)) \). There are complex Hermitian [real symmetric] quaternionic Hermitian matrices \( A(1), \ldots, A(m) \) with eigenvalues \( \alpha(1), \ldots, \alpha(m) \) and \( A(1) + \ldots + A(m) \leq 0 \) if and only if

\[
(\forall \mathcal{I}) \quad \sum_{s=1}^{m} \sum_{i \in I(s)} \alpha_i(s) \leq 0
\]

for all \( \mathcal{I} = (I(1), \ldots, I(m)) \) in \( S^n(m) [R^n(m)] \).

Proof. The necessity of the conditions follows immediately from Klyachko’s theorem. Indeed, if \( \sum_{s=1}^{m} A(s) \leq 0 \), there is a positive semidefinite \( A(m+1) \) with \( \sum_{s=1}^{m+1} A(s) = 0 \). For any \( \mathcal{I} = (I(1), \ldots, I(m)) \) in \( S^n(m) \), the \((m+1)\)-tuple \( \mathcal{I}' = (I(1), \ldots, I(m), \{n-r+1, \ldots, n\}) \) is in \( S^n(m+1) \) (since \( \omega_{\{n-r+1, \ldots, n\}} = 1 \)), and the inequality \((\forall \mathcal{I}')\) implies \((\forall \mathcal{I})\) since all the eigenvalues of \( A(m+1) \) are nonnegative. Note that for \( r = n \), the inequality says that \( \sum_{s=1}^{m} \text{tr}(A(s)) = \text{tr}(\sum_{s=1}^{m} A(s)) \leq 0 \).

We prove the converse by induction on \( n \), the case \( n = 1 \) being trivial. Assume \( \alpha(1), \ldots, \alpha(m) \) satisfy the inequalities \((\forall \mathcal{I})\) for all \( \mathcal{I} \in S^n(m) \). We consider first the case where there is some \( \mathcal{I} = (I(1), \ldots, I(m)) \) in some \( S^n(m) \) for some \( 1 \leq r \leq n \) for which the inequality \((\forall \mathcal{I})\) is satisfied with equality: \( \sum_{s=1}^{m} \sum_{i \in I(s)} \alpha_i(s) = 0 \). Define \( r \)-tuples \( \alpha'(s), 1 \leq s \leq m, \) by taking the \( r \) values \( \alpha_i(s) \) for \( i \in I(s) \). In symbols, if \( I(s) = \{i_1(s) < \ldots < i_r(s)\} \), then \( \alpha'(s) \) is

\[
\alpha_{i_1(s)}(s) \geq \ldots \geq \alpha_{i_r(s)}(s).
\]

Define similarly \( \alpha''(s), 1 \leq s \leq m, \) by taking the \( n-r \) values \( \alpha_i(s) \) for \( i \notin I(s) \).

Claim. The \( m \)-tuple \( \alpha'(1), \ldots, \alpha'(m) \) satisfies the inequalities \((\forall \mathcal{P})\) for all \( \mathcal{P} \in S^r(m) \), and the \( m \)-tuple \( \alpha''(1), \ldots, \alpha''(m) \) satisfies the inequalities \((\forall \mathcal{P})\) for all \( \mathcal{P} \in S^{n-r}(m) \).

To prove the claim, for any \( \mathcal{P} \in S^r(m) \), Proposition 1(i) implies that the \( m \)-tuple \( \mathcal{T}_\mathcal{P} \) is in \( S^r(n) \). The inequality \((\forall \mathcal{T}_\mathcal{P})\) for \( \alpha(1), \ldots, \alpha(m) \) is exactly the inequality \((\forall \mathcal{P})\) for \( \alpha'(1), \ldots, \alpha'(m) \). Similarly by Proposition 1(ii), \( \mathcal{T}^+_\mathcal{P} \) is in \( S^n(m) \), and the inequality \((\forall \mathcal{T}^+_\mathcal{P})\) for \( \alpha(1), \ldots, \alpha(m) \) is equivalent to the inequality \((\forall \mathcal{P})\) for \( \alpha''(1), \ldots, \alpha''(m) \), as one sees by adding the number \( \sum_{s=1}^{m} \sum_{i \in I(s)} \alpha_i(s) = 0 \) to the left side.

By the claim, the \( m \)-tuple \( \alpha'(1), \ldots, \alpha'(m) \) satisfies all the inequalities \((\forall \mathcal{P})\) for \( \mathcal{P} \in S^r(m) \) of Klyachko’s theorem. By the real version of this theorem ([6] Thm.

\[3\] It also follows almost as easily from the proof of the easy half of this theorem: Choose flags \( F_\mathcal{P}(s) \) with \( F_\mathcal{P}(s) \) spanned by eigenvectors of \( A(s) \) corresponding to its first \( k \) eigenvalues. Since \( \prod_{s=1}^{m} \omega(I(s)) = 0 \), there is an \( L \) in \( \cap_{s=1}^{m} \Omega(I(s))(F_\mathcal{P}(s)) \). If \( R_\mathcal{A}(L) \) denotes the Rayleigh trace (the trace of \( L \to \mathbb{C}^n \to \mathbb{C}^n \to L \), where the first map is the inclusion, the second is given by \( A \), and the third is orthogonal projection onto \( L \)), then by the easy Hersch-Zwahlen Lemma (cf. [7], Prop. 1), with \( A = \sum_{s=1}^{m} A(s) \),

\[
\sum_{s=1}^{m} \sum_{i \in I(s)} \alpha_i(s) \leq \sum_{s=1}^{m} R_\mathcal{A}_s(L) = R_\mathcal{A}(L) \leq 0,
\]

the last inequality since \( A \) is negative semidefinite.
4, of [7] §10.7, there are real symmetric $r$ by $r$ matrices $A'(1), \ldots, A'(m)$ with eigenvalues $\alpha'(1), \ldots, \alpha'(m)$ and $\sum_{s=1}^m A'(s) = 0$. Similarly, the $\alpha''(1), \ldots, \alpha''(m)$ satisfy all the inequalities ($\tilde{t}_P$) for $P \in S^{n-r}(m)$. By induction, since $n - r < n$, there are $n - r$ by $n - r$ real symmetric matrices $A''(1), \ldots, A''(m)$ with eigenvalues $\alpha''(1), \ldots, \alpha''(m)$ and $\sum_{s=1}^m A''(s) \leq 0$. The direct sum matrices $A(s) = \begin{pmatrix} A'(s) & 0 \\ 0 & A''(s) \end{pmatrix}$ are then real symmetric $n$ by $n$ matrices whose eigenvalues are the given $\alpha(1), \ldots, \alpha(m)$, with $A(1) + \ldots + A(m) \leq 0$.

Now suppose that all the inequalities ($\tilde{t}_I$) are strict, for all $I \in S^n(m)$. Since this is a finite list, we may find a positive $\epsilon$ such that, after replacing each $\alpha_i(s)$ by $\tilde{\alpha}_i(s) = \alpha_i(s) + \epsilon$, all the inequalities ($\tilde{t}_I$) are satisfied for $\tilde{\alpha}(1), \ldots, \tilde{\alpha}(m)$, but at least one holds with equality. By the case just proved, there are real symmetric $\tilde{A}(1), \ldots, \tilde{A}(m)$ with eigenvalues $\tilde{\alpha}(1), \ldots, \tilde{\alpha}(m)$ and $\sum_{s=1}^m \tilde{A}(s) \leq 0$. The matrices $A(s) = \tilde{A}(s) - D(\epsilon)$, where $D(\epsilon)$ is the diagonal matrix with diagonal entries $\epsilon$, satisfy the required conditions.

To complete the proof of the theorem, we must show that the inequalities ($\tilde{t}_I$) for $I \in S^n(m)$ follow from those for $I \in R^n(m)$. This is proved by the method of Belkale and Woodward. In fact, the proof of Proposition 10 in [7], §7 shows that, for any weakly decreasing sequences $\alpha(1), \ldots, \alpha(m)$ of $n$ real numbers,

$$\max_{I \in S^n(m)} \sum_{s=1}^m \sum_{i \in I(s)} \alpha_i(s) = \max_{I \in R^n(m)} \sum_{s=1}^m \sum_{i \in I(s)} \alpha_i(s).$$

**Corollary.** Given weakly decreasing $n$-tuples $\alpha(1), \ldots, \alpha(m)$, $\gamma$, there are complex Hermitian [real symmetric] [quaternionic Hermitian] $n$ by $n$ matrices $A(1), \ldots, A(m)$, $C$ with eigenvalues $\alpha(1), \ldots, \alpha(m)$, $\gamma$ with $C \leq A(1) + \ldots + A(m)$ if and only if

$$\sum_{k \in K} \gamma_k \leq \sum_{s=1}^m \sum_{i \in I(s)} \alpha_i(s)$$

for all $I(1), \ldots, I(m)$, $K$ of cardinality $r$ in $[n]$ such that $\sigma_{\lambda(K)}$ occurs in the product $\prod_{s=1}^m \sigma_{\lambda(I(s))}$ [with coefficient 1] in $H^*(\mathrm{Gr}(r, n))$, for all $r \leq n$.

**Proof.** The theorem for $m + 1$ factors is applied to the situation

$$-A(1) - A(2) - \ldots - A(m) + C \leq 0$$

which changes signs and reverses the order of the eigenvalues of each $A(s)$. One uses duality in the Schubert calculus ($\sigma_{\lambda(I)}$ is dual to $\omega_I$) to make the translation. For details on this translation, see [7], §10.1. Note that the inequality $\sum_{i=1}^n \gamma_i \leq \sum_{s=1}^m \sum_{i=1}^n \alpha_i(s)$ is that for $I(1) = \ldots = I(m) = K = [n]$, with $r = n$.

Theorem 1 from the introduction is a special case of this corollary.

If $A$ is a compact, self-adjoint, and positive semidefinite, linear operator on a separable Hilbert space, there is an orthonormal basis $e_1, e_2, \ldots$ so that $A e_i = \alpha_i e_i$ for all $i$, with $\alpha_1 \geq \alpha_2 \geq \ldots$, and $\lim \alpha_i = 0$; these $\alpha_i$ are the eigenvalues of $A$. The operator is said to be of trace class if $\sum_{i=1}^\infty \alpha_i < \infty$. The proof given by Friedland in [4], §5 applies without change to give the following stronger result:
Theorem 4. Let $\alpha(1), \ldots, \alpha(m)$ and $\gamma$ be weakly decreasing infinite sequences of nonnegative numbers that converge to zero. These occur as eigenvalues of compact self-adjoint operators $A(1), \ldots, A(m)$ and $C$ on a real or complex separable Hilbert space, with $C \leq A(1) + \ldots + A(m)$, if and only if they satisfy the inequalities $(\hat{I}_K^{\gamma})$ for all $I(1), \ldots, I(m), K$ of cardinality $r$ in $[n]$ such that $\sigma_{\lambda(K)}$ occurs in the product $\prod_{s=1}^{m} \sigma_{\lambda(I(s))}$ with coefficient 1 in $H^*(Gr(r, n))$, for all $1 \leq r \leq n < \infty$. Moreover, if each sum $\sum_{i=1}^{\infty} \alpha_i(s)$ is finite and $\sum_{i=1}^{\infty} \gamma_i = \sum_{s=1}^{m} \sum_{i=1}^{\infty} \alpha_i(s)$, then these inequalities are necessary and sufficient for the existence of operators $A(1), \ldots, A(m)$ and $C$ with these eigenvalues, and $C = A(1) + \ldots + A(m)$.

Note that an inequality $(\hat{I}_K^{\gamma})$ depends only on $I$ and $K$, and not on $n$; to get a list without repeats, one may take $n$ to be the largest integer in $K$. In contrast to the case for matrices, as Friedland points out, the resulting list is far from minimal. Indeed, any finite number of the inequalities $(\hat{I}_K^{\gamma})$ can be omitted. For $m = 2$, the fact that $(\hat{I}_K^{\gamma})$ is redundant follows from the fact that for $(I, J, K) \in S^r$, and any $a > n$ and $b > n$, the triple $(I \cup \{a\}, J \cup \{b\}, K \cup \{c\})$ is in $S^{r+1}_c$, where $c = a + b - r - 1$. Indeed, the equality

$$c_{\lambda(K \cup \{c\})}^{\lambda(I \cup \{a\}, \lambda(J \cup \{b\})} = c_{\lambda(I) \lambda(J)}$$

follows from a simple general fact about Littlewood-Richardson coefficients: if $\lambda$, $\mu$, and $\nu$ are partitions, with $\lambda_1 + \mu_1 = \nu_1$, then $c_{\lambda \mu}^{\nu}$ is equal to the Littlewood-Richardson coefficient for the three partitions obtained from $\lambda$, $\mu$, and $\nu$ by removing each of their first entries (cf. [5], Chapter 5). Letting $a$, $b$, and therefore $c$ tend to infinity, one sees that $(\hat{I}_K^{\gamma})$ follows from the inequalities $(\hat{I}_{\cup \{a\}}^{\gamma_{\cup \{a\}}})_{K \cup \{c\}}$.

Proposition 2. Let $\alpha(1), \ldots, \alpha(m)$ be weakly decreasing sequences of $n$ integers. The following are equivalent:

1. The inequality $(\hat{I}_I)$ is valid for all $I$ in $S^n(m)$ $[R^n(m)]$.
2. There are weakly decreasing sequences $\alpha(1), \ldots, \alpha(m)$ of integers such that $\alpha_i(s) \geq \alpha_i(s)$ for $1 \leq s \leq m$, $1 \leq i \leq n$, with $\sum_{s=1}^{m} \sum_{i=1}^{\infty} \alpha_i(s) = 0$, such that $\alpha(1), \ldots, \alpha(m)$ satisfies $(\hat{I}_I)$ for all $I$ in $S^n(m)$ $[R^n(m)]$.
3. The same assertion as in (2), except that $\alpha(s) = \alpha(s)$ for all $s \neq s_0$, for any choice of $s_0 \in \{1, \ldots, m\}$.

Proof. The assertions (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are clear. To prove that (1) $\Rightarrow$ (3), it suffices to show that if $\sum_{s=1}^{m} \sum_{i=1}^{\infty} \alpha_i(s) < 0$, we can increase one of the integers $\alpha_i(s_0)$ by 1, leaving the others unchanged, so that all the inequalities $(\hat{I}_I)$ remain valid; repeating this until the full sum becomes 0 produces the required $\alpha(s_0)$. If all the $(\hat{I}_I)$ are strict inequalities, we can simply replace $\alpha_1(s_0)$ by $\alpha_1(s_0) + 1$. Otherwise take $r$ maximal such that $(\hat{I}_I)$ is an equality for some $I \in S^r(m)$. As in the proof of Theorem 3, this partitions the $m$-tuple $\alpha = (\alpha(1), \ldots, \alpha(m))$ into two $m$-tuples $\alpha'$ and $\alpha''$, satisfying the conditions $(\hat{P})$ for $P$ in $S^r(m)$ and $S^{n-r}(m)$ respectively, and with $\sum_{s=1}^{m} \sum_{i=1}^{\infty} \alpha_i'(s) = 0$. By the maximality of $r$, all the inequalities $(\hat{P})$ for $\alpha'$ must be strict; otherwise $(\hat{P}_{\alpha'})$ would be an equality for $\alpha'$, contradicting the maximality of $r$.

Let $\alpha''$ be obtained from $\alpha''$ by increasing $\alpha''(s_0)$ by 1, leaving all other values unchanged. By Theorem 3, $\alpha'$ and $\alpha''$ are eigenvalues of Hermitian matrices $A'(1), \ldots, A'(m)$ and $A''(1), \ldots, A''(m)$, with $\sum_{s=1}^{m} A'(s) = 0$ and $\sum_{s=1}^{m} A''(s) \leq 0$. The direct sums $A(s) = A'(s) \oplus A''(s)$ also have $\sum_{s=1}^{m} A(s) \leq 0$, so by Theorem...
3 their eigenvalues satisfy the inequalities \((\dagger)\) for all \(I \in S^n(m)\). The eigenvalues of \(A(s)\) are \(\alpha(s)\) for \(s \neq s_0\), while the eigenvalues of \(A(s_0)\) are obtained from \(\alpha(s_0)\) by increasing one \(\alpha_i(s_0)\) by 1 (this \(i\) is the smallest integer in \([n]\) not in \(I(s_0)\)).

**Theorem 5.** Let \(\alpha(1), \ldots, \alpha(m)\) and \(\gamma\) be partitions of lengths at most \(n\). The following are equivalent:

1. The inequalities \((\dagger)\) are valid for all subsets \(I(1), \ldots, I(m), K\) of \([n]\) of cardinality \(r\), \(1 \leq r \leq n\), such that \(\sigma_{\lambda(K)}\) occurs in \(\prod_{s=1}^{m} \sigma_{\lambda(I(s))}\) [with coefficient 1] in \(H^*(\text{Gr}(r, n))\).
2. There are complex Hermitian [real symmetric] [quaternionic Hermitian] matrices \(A(1), \ldots, A(m)\) and \(C\) with eigenvalues \(\alpha(1), \ldots, \alpha(m)\) and \(\gamma\) such that \(C \leq A(1) + \ldots + A(m)\).
3. There is a partition \(\bar{\gamma} \supset \gamma\) such that \(\sum_{i=1}^{n} \bar{\gamma}_i = \sum_{s=1}^{m} \sum_{i=1}^{n} \alpha_i(s)\) and the inequalities \((\dagger)\) hold for the same \(I, K\) as in (1) for all \(r < n\).
4. There are partitions \(\bar{\alpha}(s) \subset \alpha(s)\) so that \(\sum_{i=1}^{n} \bar{\gamma}_i = \sum_{s=1}^{m} \sum_{i=1}^{n} \bar{\alpha}_i(s)\) and the inequalities \((\dagger)\) hold for the same \(I, K\) as in (1) for all \(r < n\).

**Remark.** The partitions \(\alpha(1), \ldots, \alpha(m)\) and \(\bar{\gamma}\) in (3), as well as the partitions \(\bar{\alpha}(1), \ldots, \bar{\alpha}(m)\) and \(\gamma\) of (4), satisfy the Klyachko conditions, so all the equivalent conditions of \([7]\), \S 10, Thm. 17 can be substituted, and the list continued.

For example, with \(V(\lambda)\) denoting the irreducible (polynomial) representation of \(\text{GL}(n, \mathbb{C})\) of highest weight \(\lambda\), condition (3) and (4) are equivalent to:

5. There is a partition \(\bar{\gamma} \supset \gamma\) such that the representation \(V(\bar{\gamma})\) occurs in \(V(\alpha(1)) \otimes \ldots \otimes V(\alpha(m))\).
6. There are partitions \(\bar{\alpha}(s) \subset \alpha(s)\) such that the representation \(V(\gamma)\) occurs in \(V(\bar{\alpha}(1)) \otimes \ldots \otimes V(\bar{\alpha}(m))\).

There are also equivalent conditions involving invariant factors:

7. For some [every] discrete valuation ring \(R\), there exist \(n\) by \(n\) matrices \(A(1), \ldots, A(m)\) with entries in \(R\) and invariant factors \(\alpha(1), \ldots, \alpha(m)\) such that the product \(A(1) \cdot \ldots \cdot A(m)\) has invariant factors \(\bar{\gamma}\) for some \(\bar{\gamma} \supset \gamma\).
8. For some [every] discrete valuation ring \(R\), there is an \(R\)-module \(C\) of type \(\gamma\), with a filtration \(0 = C(0) \subset C(1) \subset \ldots \subset C(m) = C\) of submodules such that \(C(s)/C(s-1)\) is isomorphic to a submodule of a module of type \(\alpha(s)\), for \(1 \leq s \leq m\).

**Proof.** The equivalence of (1) and (2) is a case of the Corollary to Theorem 3. Similarly, the equivalence of (1) and (3) of Proposition 2, applied with \(m + 1\) factors and with \(s_0 = m + 1\) corresponding to \(\gamma\), shows that (1) and (3) are equivalent. Since (4) clearly implies (1), it remains to show that (3) implies (4). This amounts to the following assertion. We are given a partition \(\bar{\gamma} \supset \gamma\) such that \(\sigma_{\gamma}\) appears in the product \(\prod_{s=1}^{m} \sigma_{\alpha(s)}\) in \(H^*(\text{Gr}(n, N))\), for any \(N\) with \(N - n \geq \bar{\gamma}_1\). We must find partitions \(\bar{\alpha}(s) \subset \alpha(s)\) so that \(\sigma_{\gamma}\) appears in \(\prod_{s=1}^{m} \sigma_{\bar{\alpha}(s)}\). We argue by induction on \(m\), the case \(m = 2\) being Lemma 3 in Section 1. For \(m > 2\), there is some \(\sigma_{\beta}\) occurring in \(\prod_{s=1}^{m} \sigma_{\alpha(s)}\) such that \(\sigma_{\gamma}\) appears in \(\sigma_{\alpha(1)} \cdot \sigma_{\beta}\). By the case for \(m = 2\), we can find \(\bar{\alpha}(1) \subset \alpha(1)\) and \(\bar{\beta} \subset \beta\) so that \(\sigma_{\gamma}\) occurs in \(\sigma_{\bar{\alpha}(1)} \cdot \sigma_{\bar{\beta}}\). By the case for \(m - 1\), there are \(\bar{\alpha}(s) \subset \alpha(s)\), \(2 \leq s \leq m\), so that \(\sigma_{\beta}\) occurs in \(\prod_{s=1}^{m} \sigma_{\bar{\alpha}(s)}\). Therefore \(\sigma_{\gamma}\) occurs in \(\prod_{s=1}^{m} \sigma_{\bar{\alpha}(s)}\), as required.
Theorem 2 of the introduction follows from this theorem. Indeed $\alpha, \beta, \gamma$ satisfy the conditions of Theorem 2 exactly when there are $\tilde{\alpha} \subset \alpha$ and $\tilde{\beta} \subset \beta$ such that the Littlewood-Richardson coefficient $c^{\gamma}_{\tilde{\alpha}, \tilde{\beta}}$ is positive. By the Green-Klein theorem [9] this is equivalent to the existence of a short exact sequence of modules $0 \to \widetilde{B} \to C \to A \to 0$, with $A$ of type $\tilde{\alpha}$, $\tilde{B}$ of type $\tilde{\beta}$, and $C$ of type $\gamma$. Taking $A$ of type $\alpha$ and $B$ of type $\beta$, there is an epimorphism $B \to \widetilde{B}$ and a monomorphism $\tilde{A} \to A$, and this gives the required exact sequence $B \to C \to A$.

We now turn to the question of the minimality of these lists of inequalities. Knutson, Tao, and Woodward [12] have announced that, for $n \geq 3$, the inequalities $(\dagger)$ and $(\dagger IJK)$, for $(I, J, K) \in R^n_r$ and $r < n$, when restricted to the hyperplane defined by $\sum_{i=1}^n \gamma_i = \sum_{i=1}^n \alpha_i + \sum_{i=1}^n \beta_i$, are independent: if any one is left out, the cone defined by the others is strictly larger. It is not hard to deduce from this the independence of our inequalities, where the hyperplane condition is replaced by an inequality to be on one side of it:

**Theorem 6.** For any $n \geq 1$, the inequalities $(\dagger)$ and $(\dagger IJK)$ for $(I, J, K) \in R^n_r$ and $1 \leq r \leq n$, are independent.

**Proof.** For $n = 1$ we have only one inequality $\gamma_1 \leq \alpha_1 + \beta_1$. For $n = 2$ there are seven inequalities $\alpha_1 \geq \alpha_2, \beta_1 \geq \beta_2, \gamma_1 \geq \gamma_2, \gamma_1 \leq \alpha_1 + \beta_1, \gamma_2 \leq \alpha_2 + \beta_1, \gamma_2 \leq \alpha_1 + \beta_2$, and $\gamma_1 + \gamma_2 \leq \alpha_1 + \alpha_2 + \beta_1 + \beta_2$, which are easily verified to be independent. For $n \geq 3$, we know that none of the chamber inequalities $(\dagger)$ or the inequalities $(\dagger IJK)$ for $(I, J, K) \in R^n_r$ and $r < n$ can be omitted; indeed, if one is omitted, the others, even when restricted to the hyperplane $\sum_{i=1}^n \gamma_i = \sum_{i=1}^n \alpha_i + \sum_{i=1}^n \beta_i$, define a larger cone. Finally, the last inequality $(\dagger IJK)$, for $I = J = K = [n]$, cannot be omitted, since for example the triple $\alpha = \beta = \gamma = (n - 1, n - 3, \ldots, -n + 3, -n + 1)$ is in the boundary hyperplane but all the other inequalities are strict (cf. [6], Lemma 2).

Granting the corresponding assertion from [12], the inequalities $(\dagger)$ and $(\dagger I)$ for $\mathcal{I} \in R^n(m)$ in Theorem 3 are independent. Again the special case $n = 2$, with its two inequalities, is handled directly. The same holds for the inequalities $(\dagger)$ and $(\dagger I)$ (for those $\mathcal{I}, K$ such that $\sigma_{\lambda(K)}$ appears in $\prod_{s=1}^m \sigma_{\lambda(I(s))}$) in Theorem 5.

There are similar results when the majorization is taken from the other side. Since no new ideas are required for the proof (Lemma 3 is not even needed), we leave the proof, as well as the statements of variations as in the remark after Theorem 5, and the corresponding assertions about the minimality of the list of inequalities, to the reader.

**Theorem 7.** Given weakly decreasing sequences $\alpha(1), \ldots, \alpha(m)$ and $\gamma$ of real numbers of length $n$, the following are equivalent:

1. $\sum_{s=1}^m \sum_{i \in I(s)} \alpha_i(s) \leq \sum_{k \in K} \gamma_k$ for all $I(1), \ldots, I(m), K$ of cardinality $r \leq n$ such that $\omega_K$ appears [with coefficient 1] in $\prod_{s=1}^m \omega_{I(s)}$.

2. There are Hermitian [real symmetric] [quaternionic Hermitean] $n$ by $n$ matrices $A(1), \ldots, A(m), C$ with $A(1) + \ldots + A(m) \leq C$.

If all $\alpha_i(s)$ and $\gamma_i$ are integers, these are equivalent to:

3. There are integral $\tilde{\alpha}(s)$ with $\alpha_i(s) \geq \alpha_i(s)$ for $1 \leq i \leq n$ and for all $s$ [with $\tilde{\alpha}(s) = \alpha(s)$ for all $s \neq s_0$, for any given $s_0$], such that $V(\gamma)$ occurs in $V(\tilde{\alpha}(1)) \otimes \ldots \otimes V(\tilde{\alpha}(m))$. 

If all $\alpha(s)$ and $\gamma$ are partitions, these are equivalent to:

(4) There is a partition $\tilde{\gamma} \subset \gamma$ such that $V(\tilde{\gamma})$ occurs in $V(\alpha(1)) \otimes \ldots \otimes V(\alpha(m))$.

References

[1] P. Belkale, Local systems on $\mathbb{P}^1 \setminus S$ for $S$ a finite set, Ph.D. thesis, University of Chicago, 1999.
[2] A. Buch, The saturation conjecture (after A. Knutson and T. Tao), to appear in l’Enseignement Math., math.CO/9810180.
[3] H. Derksen and J. Weyman, Semi-invariants of quivers and saturation for Littlewood-Richardson coefficients, J. Amer. Math. Soc. 13 (2000), 467–479.
[4] S. Friedland, Finite and infinite dimensional generalizations of Klyachko’s theorem, Linear Algebra Appl., this volume.
[5] W. Fulton, Young Tableaux, Cambridge University Press, 1997.
[6] ———, Eigenvalues of sums of Hermitian matrices (after A. Klyachko), Séminaire Bourbaki 845, June, 1998, Astérisque 252 (1998), 255–269.
[7] ———, Eigenvalues, invariant factors, highest weights, and Schubert calculus, Bull. Amer. Math. Soc. 37 (2000), 209–249, math.AG/9908013.
[8] A. Horn, Eigenvalues of sums of Hermitian matrices, Pacific J. Math. 12 (1962), 225–241.
[9] T. Klein, The multiplication of Schur-functions and extensions of p-modules, J. London Math. Society 43 (1968), 280–284.
[10] A. A. Klyachko, Stable bundles, representation theory and Hermitian operators, Selecta Math. 4 (1998), 419–445.
[11] A. Knutson and T. Tao, The honeycomb model of $GL_n(\mathbb{C})$ tensor products I: proof of the saturation conjecture, J. Amer. Math. Soc. 12 (1999), 1055–1090, math.RT/9807160.
[12] A. Knutson, T. Tao and C. Woodward, Honeycombs II: facets of the Littlewood-Richardson cone, to appear.
[13] B. Totaro, Tensor products of semistables are semistable, Geometry and Analysis on complex Manifolds, World Sci. Publ., 1994, pp. 242–250.