Density, short-range order and the quark-gluon plasma

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We study the thermal part of the energy density spatial correlator in the quark-gluon plasma. We describe its qualitative form at high temperatures. We then calculate it out to distances \( \approx 1.5/T \) in SU(3) gauge theory lattice simulations for the range of temperatures \( 0.9 \leq T/T_c \leq 2.2 \). The vacuum-subtracted correlator exhibits non-monotonic behavior, and is almost conformal by \( 2T \). Its broad maximum at \( r \approx 0.6/T \) suggests a dense medium with only weak short-range order, similar to a non-relativistic fluid near the liquid-gas phase transition, where \( \eta/s \) is minimal.

Hydrodynamics calculations \cite{1} successfully described the pattern of produced particles in heavy ion collisions at RHIC \cite{2}. This early agreement between ideal hydrodynamics and experiment has been refined in recent times. On the theory side, the dissipative effects of shear viscosity \( \eta \) have been included in full 3d hydrodynamics calculations \cite{2, 3, 4} and the sensitivity to initial conditions quantitatively estimated \cite{5} for the first time. On the experimental side, the elliptic flow observable \( v_2 \), which is sensitive to the value of \( \eta \) in units of entropy density \( s \), is now corrected for non-medium-generated two-particle correlations \cite{6}. The conclusion that \( \eta/s \) must be much smaller than unity has so far withstood these refinements of heavy-ion phenomenology \cite{6}.

The smallness of \( \eta/s \) was turned into the statement that the quark-gluon plasma (QGP) formed at RHIC is the “most perfect liquid known in nature” \cite{8}. A general question then comes to mind: what observable can be used to characterize the liquid nature of a system described by a quantum field theory \cite{9}? And secondly, what is the QCD prediction for that observable? This leads us to remind ourselves what the defining property of an ordinary liquid is. Surely the everyday-life notion that a liquid “has a definite volume, but no definite shape” is inadequate in the present context.

The two-body density distribution \( \rho(r_1, r_2) = g(r) \rho^2 \) of an ordinary substance (such as water) of density \( \rho \) behaves qualitatively differently in the solid, liquid and gas phase (see for instance \cite{10}). The radial distribution function \( g(r) \) characterizes the average density of particles at distance \( r \) from an arbitrarily chosen particle. In a dilute gas, \( g(r) \) is essentially equal to 1 for \( r \) greater than the size of a molecule. In a liquid on the other hand, \( g(r) \) vanishes at small \( r \), a reflexion of the short-distance repulsion between molecules. The function then rises and typically exhibits several gradually damped oscillations around unity. This reflects the “short-range order” in the fluid, namely the coherent motion of closely packed molecules up to distances a few times the molecule size. Over longer distances, this ordering is lost. Only a perfect crystal at low temperatures exhibits truly long-range order.

In quantum field theory, particle number is not (necessarily) conserved, so it is not immediately clear which spatial correlator is the closest analogue of the two-body density distribution in non-relativistic systems. In QCD, the only conserved quantities are energy, momentum, the quark numbers and the (non-singlet) axial charges. The energy density is an order parameter in the theoretical limit of a large number of colors \( N_c \), which puts it in natural correspondence with the density \( \rho \) of a non-relativistic fluid. Hence our strategy to study the spatial correlator of the energy density.

A peculiarity of quantum field theory is that energy density correlations are present even when the average energy density is zero, i.e. at \( T = 0 \) when the partition function is saturated by the quantum vacuum. This correlation has to be strong at short distance in an asymptotically free theory, \( \langle T_{00}(0)|T_{00}(r)\rangle \sim r^{-\delta} \), on dimensional grounds. Based on the Källen-Lehmann representation, it is monotonically decreasing in \( r \) at any \( T \).

Therefore, in order to isolate the thermal effects, we shall consider the subtracted correlator

\[
G_{ee}(T,r) = \langle T_{00}(0)|T_{00}(0,r)\rangle_T - \langle T_{00}(0)|T_{00}(0,r)\rangle_{T=0}.
\]

In the Euclidean SU\((N_c)\) gauge theory, the energy density operator,

\[
T_{00} = \theta_{00} + \frac{1}{4}\theta, \quad \theta = \frac{\beta(g)}{2g} F_{\mu\nu}^a F_{\mu\nu}^a, \quad \langle \theta \rangle = e - 3p, \quad \langle \theta_{00} \rangle = \frac{3}{4}(e + p) = \frac{3}{4}T s
\]

is composed of two terms which are separately scale-independent operators. Here \( e \) is the energy density, \( p \) the pressure and \( \beta(g) = -bg^3 + \ldots \) the beta function, with \( b = \frac{11N_c}{3(4\pi)^2} \). The expectation value of the ‘naive’ energy operator \( \theta_{00} \) is proportional to the entropy density, while the expectation value of the trace anomaly directly measures the deviation from the conformal limit.
where $e = 3p$ (we implicitly add a constant to $T_{00}$ such that $(T_{00})$ vanishes in the vacuum). We can thus investigate separately the $G_{aa}$ and $G_{bb}$ correlators, defined by replacing $T_{00}$ respectively by $\frac{3}{2}T_{00}$ and $\theta$ in Eq. 1.

### High-temperature behavior

The one-loop expression for the two correlators in $D$ space-time dimensions is

$$
\langle \theta_{00}(0)\theta_{00}(x) \rangle_{1L} = (8\pi b \alpha_s)^{-2} \langle \theta(0)\theta(x) \rangle_{1L} = \frac{d_A}{4\pi^4} \sum_{m,n \in \mathbb{Z}} \left( D - 8 + 16 \frac{(x_{[m]} \cdot x_{[n]})^2}{x_{[m]}^2 x_{[n]}^2} \right) \frac{1}{(x_{[m]}^2 x_{[n]}^2)^2}
$$

where $d_A = N_c^2 - 1$ and $x_{[n]} = (\frac{4\pi}{T} + x_0, \mathbf{x}, \mathbf{x})$, while $\langle \theta_{00}(0)\theta(0, r) \rangle_{1L}$ vanishes identically. The $m = n = 0$ term gives the zero-temperature expression.

Let us consider the high-temperature regime, $T \gg g^2(T)$: both the zero and high temperature correlators are well described by the one-loop formula, up to small radiative corrections. For $r \to 0$, $G_{ee}(T, r) \sim -\frac{34}{4\pi^4}$; for $Tr > 0.568$, $G_{ee} \sim e^2(T)$ (see Fig. 3).

$g^2(T) \ll T \ll T_c$: both the zero and high temperature correlators are still described by perturbation theory, $\frac{3d_A}{\pi^4 r^4}$, but the high-$T$ correlator is exponentially screened, so necessarily $G_{ee} \ll e^2(T)$.

$T r > T_c$: both the zero and high temperature correlators are exponentially screened. In the former case, the relevant mass is $M_4 \approx 3.3T_c$, corresponding to the lightest scalar glueball mass in $D = 4$, while at high temperatures, dimensional reduction takes places, and the screening mass is $(M_4/g_0^2) g^2(T)T$ with $M_4/g_0^2 \approx 2.4$ [13]. It is therefore clear that screening of the energy density is stronger at high temperatures, and hence $G_{ee}$ asymptotically approaches $e^2(T)$ from below at a rate $e^{-M_4 r}$.

In summary, at high temperatures $G_{ee}(T, r) \sim e^2(T)$ vanishes at least at two finite distances $r$: the first time at $Tr \approx 0.568$, and the second time at $Tr = O(g^2(T))$.

To investigate the function $G_{ee}(T, r)$ at temperatures accessible in heavy-ion colliders, we perform lattice simulations in the region $0.9T_c < T < 2T_c$.

### Operator product expansion

From expression (3) and from known results [14], we can obtain the leading terms in the operator-product ex-

\begin{align*}
\langle \theta(0)\theta(x) \rangle & \sim \frac{(8\pi b \alpha_s)^2}{\pi^4 r^8} \left[ 6A_h^2 \langle \theta_{00} \rangle - 32B_h^2 \langle \theta \rangle \right] \\
\langle \theta_{00}(0)\theta_{00}(x) \rangle & \sim \frac{3d_A}{\pi^4 r^8} - \frac{1}{3\pi^2} \langle \theta_{00} \rangle + O(\theta^4) \frac{\langle \theta \rangle}{r^4},
\end{align*}

where $x = (0, r)$ and $r^{-2}$ terms and softer have been omitted. The Wilson coefficients of the operators $1, \theta_{00}$ and $\theta$ are a least of $O(\alpha_s^2)$ for the product $\theta_{00}(0)\theta(0, r)$. The usefulness of the OPE in this context arises because of the exact cancellation of the $r^{-8}$ term in the difference between finite $T$ and $T = 0$ correlators. Using Eq. 4 we obtain the short-distance behavior

$$
G(T, r) \sim \frac{e + p}{(2\pi r)^2} + O(\alpha_s^4) \frac{e - 3p}{r^4} + O(\alpha_s r^{-4}, r^{-2})
$$

### Figure 1

FIG. 1: The correlators at $T = 0$ at different lattice spacings ($\tau_0 \approx 0.5$ [13]; the line is to guide the eye).

### Figure 2

FIG. 2: $G_{ee}$ (Eq. 1) across the deconfining phase transition, at $\beta = 6.018$. At $T = 1.24T_c$ we check for discretization errors by also showing data from $\beta = 6.200$ (filled squares).
discretization errors are under control.

Figure 2 shows the qualitative change of $G_{ee}$ across the deconfining phase transition. At $T = 1.24T_c$ the functions obtained from these two lattice spacings are in qualitative agreement, a non-trivial check, given the large cancellation between the finite and zero temperature correlators. The function $G_{ee}(r)$ is large and negative at short-distances, crosses the asymptotic value $e^2(T)$, reaches a maximum and presumably decreases monotonically after that. Although the signal becomes too small to tell beyond $Tr \simeq 1.5$, this is plausible in view of the small value of the thermal screening mass (see next section).

The $1/r^4$ short-distance divergence is not visible below $T_c$, a fact that the OPE and the smallness of $(e, p)$ in the confined phase easily account for. We note that $G_{ee}/e^2(T)$ reaches around $r = 0.6/T$ a maximum which is larger at $1.08$ than at $1.24T_c$. We understand this in terms of the larger fluctuations present near the (weakly) first order phase transition.

Figure 4 shows the temperature dependence of $G_{ee}$ up to $2.2T_c$. The position of the maximum remains $r_{\text{max}} \approx 0.6/T$, and $G_{ee}(r_{\text{max}}, T)/e^2(T)$ decreases slowly as the temperature rises. The curves at 1.65 and 2.20$L_c$ exhibit near-conformal behavior (i.e., $G_{ee}$ is essentially a function of $Tr$), however the asymptotic approach to $e^2(T)$ has the opposite sign, as a study of screening masses shows.

Figure 4 shows separately the entropy density correlator and the trace anomaly correlator. The former is qualitatively similar to the energy density correlator, while the latter has a rather featureless monotonic behavior.

**Screening masses**

The asymptotic large-$r$ behavior of $G_{ee}$ is dictated by the smallest screening mass that $T_{00}$ couples to. This is the state invariant under all the symmetries of a constant ‘z-slice’ $\bar{G}_{00}$, which has a volume $(1/T) \times L \times L$. From $D = 4$ simulations, we obtain directly

$$\frac{M(T)}{M_4} = 0.630(14), \quad 0.906(20), \quad 1.276(32)$$

respectively at $1.24$ ($N_f = 8$), 1.65, and 2.20$L_c$ ($N_f = 6$). So it is only at $T^* = 1.790(36)L_c$ that the thermal screening starts to exceed the $D = 4$ glueball mass. In particular, for $T > T^*$, $G_{ee}(r, T)$ approaches its asymptotic value from below, and therefore crosses $e^2(T)$ twice.

**Comparison with non-relativistic systems**

The radial distribution function $g(r)$ of a simple non-relativistic liquid, such as $^{36}$Ar at 85K [21], exhibits several very pronounced peaks above 1. In particular $g(r) - 1$ is of order unity at the first peak. However, it is

**Numerical results**

We now turn to a calculation of $G_{ee}(T, r)$ in the SU(3) gauge theory, i.e. in the plasma of gluons, using lattice Monte-Carlo techniques on a $(1/T) \times L^3$ lattice. We employ the (isotropic) Wilson action [15] and the ‘once-HYP-smeared’ ‘clover’ discretization of $\tau_{00}$ and $\theta$ developed in [16]. The variance of $(\theta_{00})$ was shown [17] to be reduced by almost two orders of magnitude as compared to the simplest ‘plaquette’ discretization. This technical improvement allows us to obtain a signal for $G_{ee} - e^2(T)$ out to $r \simeq 1.5/T$.

Figure 1 displays the relevant correlators at zero temperature. They fall off monotonically as $r^{-6}$ at short distance and exponentially at large distance. Note that the trace anomaly correlator, while $O(a^2)$ at short distances, dominates at large distances. For $r/a \geq 3$ comparison of the data obtained at three lattice spacings shows that

\[
G_{ee}(T, r) / (d_A T^8)
\]

FIG. 3: $G_{ee}$ at three temperatures, with $N_f = 6$. The curve corresponds to the correlator in the Stefan-Boltzmann limit $T \rightarrow \infty$.

\[
G_{SS}(d_A T^8)
\]

FIG. 4: The $N_f = 6$ entropy-entropy ($G_{SS}$) and action-action ($G_{00}$) correlators at 1.24 (circles), 1.65 (squares) and 2.20$L_c$ (triangles).
known \cite{22} that $\eta/s$ is minimal near the liquid-gas phase transition, and becomes large both at low and high temperature. A highly ordered mesoscopic scale favours the transport of momentum, because the holes between the closely packed molecules then play the role of quasiparticles with a long mean free path (an argument attributed to Enskog \cite{22}). Heating up the liquid has the effect of reducing the amplitude of the peaks in $g(r)$, until they disappear completely once the system is in a dilute gas phase. Thus the regime where $\eta/s$ is minimal is the one where $g(r)$ has only few, small oscillations around unity.

We have found that $G_{ee}$ has exactly one broad peak above $e^2(T)$, exceeding that value by about $5 - 15\%$ (the figure decreases slowly with temperature). By analogy with non-relativistic fluids, it is tempting to see a relation between this fact and the small value obtained for the shear viscosity \cite{23} in the same range of temperatures, $\eta/s < 1.0$.

#### Conclusion

We have calculated non-perturbatively the thermal part $G_{ee}$ of the energy-density spatial correlator in the plasma of SU(3) gluons, in the range of temperatures $0.9 \leq T/T_c \leq 2.2$, and found the qualitative high-$T$ behavior. It diverges as $-r^{-4}$ near the origin as dictated by the OPE, and above $T_c$ reaches a maximum at $r_{\text{max}} \approx 0.6/T$ typically $10\%$ above its asymptotic value $e^2(T)$. For $T < 1.8T_c$ it asymptotically approaches that value from above, and at higher temperatures it approaches from below. While the appearance of $G_{ee}$ is rather different from the radial distribution function of a typical liquid, we pointed out that it is precisely when the short-range order is weak that the ratio $\eta/s$ is minimal.

We note that the radial distribution function of monopoles has been computed in SU(2) gauge theory \cite{24}, and has a similar shape to $G_{ee}$. Color charge and monopole-antimonopole correlators that look alike have also been found in models \cite{24}. It would be interesting to see whether such models can reproduce $G_{ee}$.

A straightforward extension of this work is to consider the full spatial correlations of the energy momentum tensor $\langle T_{\mu\nu}\rangle$, in SU(3) gauge theory and in full QCD. As a benchmark it would be helpful to know the form of these correlators in the strongly coupled $N = 4$ SYM theory, which is known to be an excellent fluid from the smallness of $\eta/s$ \cite{26} and from its spectral functions \cite{27}.

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