ON UNITARITY OF A LINEARIZED YANG-MILLS FORMULATION FOR MASSLESS AND MASSIVE GRAVITY WITH PROPAGATING TORSION

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A perturbative regime based on contortion as a dynamical variable and metric as a (classical) fixed background, is performed in the context of a pure Yang-Mills formulation for gravity in a 2 + 1 dimensional space-time. In the massless case we show that the theory contains three degrees of freedom and only one is a non-unitary mode. Next, we introduce quadratical terms dependent on torsion, which preserve parity and general covariance. The linearized version reproduces an analogue Hilbert-Einstein-Fierz-Pauli unitary massive theory plus three massless modes, two of them represents non-unitary ones. Finally we confirm the existence of a family of unitary Yang-Mills-extended theories which are classically consistent with Einstein’s solutions coming from non massive and topologically massive gravity. The unitarity of these YM-extended theories is shown in a perturbative regime. A possible way to perform a non-perturbative study is remarked.

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1. Introduction

There were some contributions on the exploration of classical consistency of a pure Yang-Mills (YM) type formulation for gravity, including the cosmological extension[1,2] (and the references therein), among others. In those references, Einstein’s theory is recovered after the imposition of torsion constraints.

Unfortunately, the path to a quantum version (if it is finally possible) is not straightforward. For example, it is well known that the Lagrangian of a pure YM theory based on the Lorentz group $SO(3,1) \simeq SL(2,C)$ leads to a non-positive Hamiltonian (due to non-compactness of the aforementioned gauge group) and, then the canonical quantization procedure fails. However, there is a possible way out if it is considered an extension of the YM model thinking about a theory like Gauss-Bonnet with torsion[3] and this is confirmed because the existence of a possible family of quadratical curvature theories from which can be recovered unitarity[4].

A first aim of this work is to expose, with some detail, a similar (and obvious)
situation about non-unitarity in a YM formulation for gravity in both massless and massive theories. There is an interest focused in the study of massive gravity and propagating torsion, among others. Particularly, the massive versions that we will explore here arise, on one hand from some quadratical terms set ($T^2$-terms) preserving parity which depends on torsion (the old idea about considering $T^2$-terms in a dynamical theory of torsion has been considered in the past and, at a perturbative regime they give rise to a Fierz-Pauli's massive term (in analogy with the recently BHT model). On the other hand, we remark some aspects of the topologically massive version of the YM gravity which do not preserves parity and how is the way to reach unitarity in a perturbative level, at least.

Whatever the model considered (massive or non massive), throughout this work we follow the spirit of Kibble’s idea, treating the metric as a fixed background, meanwhile the torsion (contortion) will be considered as a dynamical field and it would be thought as a quantum fluctuation around a classical fixed background.

This paper is organized as follows. The next section is devoted to a brief review on notation of the cosmologically extended YM formulation in $N$-dimensions and its topologically massive version in 2+1 dimension. In section 3, we consider the scheme of linearization of the massless theory around a fixed Minkowskian background, allowing fluctuations on torsion. Next, the Lagrangian analysis of constraints and construction of the reduced action is performed, showing that this theory does propagate degrees of freedom, including a ghost. We end this section with some remarks about the counting of degrees of freedom in a non perturbative level. In section 4, we introduce an appropriate $T^2$-terms, which preserve parity, general covariance, and its linearization gives rise to a Fierz-Pauli mass term. There, the non-positive definite Hamiltonian problem gets worse: the Lagrangian analysis shows that the theory has more non-unitary degrees of freedom and we can’t expect other thing. Gauge transformations are explored in section 5. Although $T^2$-terms provide mass only to some spin component of contortion, the linearized theory loses the gauge invariance and there is no residual invariance. This is clearly established through a standard procedure for the study of possible chains of gauge generators. In section 6 we confirm the well known fact that there exists a family of theories which can cure the ghost problem, at least at perturbative level. Those theories are classically consistent when it is shown that the set of solutions contains the Einsteinian’s ones. We end up with some concluding remarks.

2. A pure Yang-Mills formulation for gravity: massless and topological massive cases

Let $M$ be an $N$-dimensional manifold with a metric, $g_{\mu\nu}$ and coordinates transformations, $U$ provided. A (principal) fiber bundle is constructed with $M$ and a 1-form connection is given, $(A_\lambda)^\mu_\nu$ which will be non metric dependent. The affine connection transforms as $A_\lambda' = UA_\lambda U^{-1} + U\partial_\lambda U^{-1}$ with $U \in GL(N, R)$. Torsion and curvature tensors are $T^\nu_{\lambda\nu} = (A_\lambda)^\mu_\nu - (A_\nu)^\mu_\lambda$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. 
respectively. The contortion tensor is defined by
\[ K_{\lambda \mu \nu} \equiv \frac{1}{2} (T^\lambda_{\mu \nu} + T^\mu_{\lambda \nu} + T^\nu_{\lambda \mu}) . \]
Components of the Riemann-Cartan tensor are
\[ R_{\sigma \alpha \mu \nu} \equiv (F_{\nu \mu})^\sigma_{\alpha} . \]
The gauge invariant action with cosmological contribution is
\[ S(N)_0 = \kappa^2 (4 - N) \langle - \frac{1}{4} \text{tr} F_{\alpha \beta} F_{\alpha \beta} + q(N) \lambda^2 \rangle , \]
where \( \kappa^2 \) is in length units, \( \langle ... \rangle \equiv \int d^N x \sqrt{-g} (...), \) \( \lambda \) is the cosmologic constant and the parameter \( q(N) = \frac{2(4 - N)}{(N - 2)(N - 1)} \) depends on dimension. The field equations are
\[ T^g_{\alpha \beta} = -\kappa^2 g_{\alpha \beta} \lambda^2 \]
where \( T^g_{\alpha \beta} \equiv \kappa^2 \text{tr} [F_{\alpha \sigma} F_{\beta \sigma} - \frac{1}{4} F_{\mu \nu} F_{\mu \nu}] \) is the energy-momentum tensor of gravity, and equation coming from variation of connection is
\[ \sqrt{-g} \partial_\alpha (\sqrt{-g} F^{\alpha \lambda}) + [A_\alpha, F_{\alpha \lambda}] = 0 , \]
which can be rewritten as follows
\[ \nabla_\mu R_{\sigma \lambda} - \nabla_\lambda R_{\sigma \mu} = 0 , \]
and the trace \( \sigma - \lambda \) gives the expected condition \( R = \text{constant} . \)

It is well known that the introduction of a Chern-Simons lagrangian term (CS) in the Hilbert-Einstein formulation of gravity provides a theory which describes a massive excitation of a graviton in 2+1 dimension. If a cosmological term is included, the cosmologically extended topological massive gravity (TMG\( \lambda \)) arises.

So, the study of consistence of a Yang-Mills type formulation for topological massive gravity with cosmological constant has been performed. There, it is verifying the existence of causal propagation and the fact that the standard TMG\( \lambda \) can be recovered from the aforementioned model at the torsionless limit. The model is
\[ S = S^{(3)}_0 + \frac{m \kappa^2}{2} \langle e^{\nu \rho \lambda} \text{tr} (A_\mu \partial_\nu A_\lambda + \frac{2}{3} A_\mu A_\nu A_\lambda) \rangle , \]
which does not preserve parity and \( S^{(3)}_0 \) is given by (11) for \( N = 3 \).

Now, the torsionless limit of (3) is explored by introducing nine torsion’s constraints through the new action \( S' = S + \kappa^2 \int d^3 x \sqrt{-g} b_{\alpha \beta} \epsilon^{\beta \lambda \sigma} (A_\lambda)^\alpha_{\sigma} \), where the nine auxiliary fields \( b_{\alpha \beta} \) can be seen as Lagrange multipliers. Variation on connection and metric gives rise the following field equations
\[ \nabla_\mu R_{\sigma \lambda} - \nabla_\lambda R_{\sigma \mu} - m e^{\nu \rho} \epsilon_{\sigma} (g_{\lambda \nu} R_{\rho \mu} - g_{\mu \nu} R_{\lambda \rho} - \frac{2}{3} R g_{\nu \mu} g_{\lambda \rho}) = 0 , \]
\[ R_{\sigma \mu} R^{\sigma \nu} - R R_{\mu \nu} + \frac{g_{\mu \nu}}{4} R^2 - g_{\mu \nu} \lambda^2 = 0 , \]
and Lagrange multipliers are
\[ b_{\mu \nu} = \frac{m R}{6} g_{\mu \nu} . \]

The trace \( \sigma - \lambda \) of (11) leads to the following consistency condition
\[ R = \text{constant} , \]
and due to this condition on the Ricci scalar, we can test particular solutions of the type

\[ R_{\mu\nu} = \frac{R}{2} g_{\mu\nu}, \]

by plugging them in (3), and this gives

\[ R = \pm 6 | \lambda |, \]

verifying the existence of (Anti) de Sitter solutions.

Finally, at the torsionless limit, the TMG model is recovered from this YM one if we take the mass value \( m \) as the mass of the Chern-Simons model and the consistency condition (7) is fixed as (8).

### 3. Linearization of the massless theory

With a view on the performing of a perturbative study of the massive model, we wish to note some aspects of the variational analysis of free action (1) in 2+1 dimensions. As we had said above, the connection will be considered as a dynamical field whereas the space-time metric would be a fixed background, in order to explore (in some sense) the isolated behavior of torsion (contortion) and avoid higher order terms in the field equations. For simplicity we assume \( \lambda = 0 \).

Then, let us consider a Minkowskian space-time with a metric \( \text{diag}(-1, 1, 1) \) provided and, obviously with no curvature nor torsion. The notation is

\[ g_{\alpha\beta} = \eta_{\alpha\beta}, \]

\[ F^{\alpha\beta} = 0, \]

\[ T^{\lambda}_{\mu\nu} = 0. \]

It can be observed that curvature \( F^{\alpha\beta} = 0 \) and torsion \( T^{\lambda}_{\mu\nu} = 0 \), in a space-time with metric \( g_{\alpha\beta} = \eta_{\alpha\beta} \) satisfy the background equations,

\[ \frac{1}{\sqrt{-g}} \partial_{\alpha} (\sqrt{-g} F^{\alpha\lambda}) + \{ \mathcal{A}_{\alpha}, F^{\alpha\lambda} \} = 0 \text{ and } \mathcal{T}_{g \alpha\beta} = 0, \]

identically.

Thinking in variations

\[ A_\mu = \mathcal{A}_\mu + a_\mu, \quad |a_\mu| \ll 1, \]

for this case \( \mathcal{A}_\mu = 0 \). Then, action (11) takes the form

\[ S^{(3)}_0 = \kappa^2 \left\{ - \frac{1}{4} tr f^{\alpha\beta}(a) f_{\alpha\beta}(a) \right\}, \]

where \( f_{\alpha\beta}(a) = \partial_\alpha a_\beta - \partial_\beta a_\alpha \) and (13) is gauge invariant under

\[ \delta a_\mu = \partial_\mu \omega, \]

with \( \omega \in G = SO(1, 2) \).

In order to describe in detail the action (13), let us consider the following decomposition for perturbed connection

\[ (a_\mu)_\alpha^\beta = \epsilon_\alpha^\gamma k_{\mu\gamma} + \delta_\alpha^\beta v_\mu - \eta_\mu\beta v_\alpha, \]

(15)
where \( k_{\mu\nu} = k_{\nu\mu} \) and \( v_\mu \) are the symmetric and antisymmetric parts of the rank two perturbed contortion (i.e., the rank two contortion is \( K_{\mu\nu} \equiv -\frac{1}{2} \epsilon^{\sigma\rho\nu} K_{\sigma\mu\rho} \)), respectively. It can be noted that decomposition (15) has not been performed in irreducible spin components and explicit writing down of the traceless part of \( k_{\mu\nu} \) would be needed. This component will be considered when the study of reduced action will be performed. Using (15) in (13), we get

\[
S^{(3)}(L)_0 = \kappa^2 \langle k_{\mu\nu} \Box k^{\mu\nu} + \partial_\mu k^{\nu\rho} \partial_\nu k^{\rho\sigma} - 2 \epsilon^{\sigma\alpha\beta} \partial_\alpha v_\beta \partial_\nu k^{\rho\sigma} - v_\mu \Box v^\mu + (\partial_\mu v^\mu)^2 \rangle ,
\]

which is gauge invariant under the following transformation rules (induced by (14))

\[
\delta k_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu ,
\]

\[
\delta v_\mu = -\epsilon^{\sigma\rho\mu} \partial_\sigma \xi_\rho ,
\]

with \( \xi_\mu \equiv \frac{1}{4} \epsilon^{\beta\alpha\mu} w^{\alpha\beta} \). These transformation rules clearly show that only the antisymmetric part of \( w \) is needed (i.e.: only three gauge fixation would be chosen).

Now, let us study the system of Lagrangian constraints in order to explore the number of degrees of freedom. A possible approach consists in a 2 + 1 decomposition of the action (16) in the way

\[
S^{(3)}(L)_0 = \kappa^2 \langle [-k_{0i} + 2\partial_i k_{00} - 2\partial_n k_{ni} - 2\epsilon_{in} v_n + 2\epsilon_{in} \partial_n v_0] k_{0i} + k_{ij} k_{ij} + [2 \epsilon_{ij} \partial_n k_{00} + 2 \epsilon_{ij} \partial_m k_{nm} - \dot{v}_j - 2 \dot{\partial}_j v_0] \dot{v}_j + 2(\dot{v}_0)^2 + k_{00} \Delta k_{00} - 2 k_{0i} \Delta k_{0i} + k_{ij} \Delta k_{ij} - (\partial_i k_{0i})^2 + \partial_n k_{ni} \partial_m k_{mi} - 2 \epsilon_{ij} \partial_i v_j \partial_n k_{ni} - 2 \epsilon_{in} \partial_m v_0 \partial_n k_{ni} + v_0 \Delta v_0 - v_i \Delta v_i + (\partial_n v_n)^2 \rangle
\]

and using a Transverse-Longitudinal (TL) decomposition\(^4\) with notation

\[
k_{00} \equiv n ,
\]

\[
h_{i0} = h_{0i} \equiv \partial_i k^L + \epsilon_i \partial_i k^T ,
\]

\[
k_{ij} = k_{ji} \equiv (\eta_{ij} \Delta - \partial_i \partial_j) k^{TT} + \partial_i \partial_j k^{LL} + (\epsilon_{ik} \partial_k \partial_j + \epsilon_{jk} \partial_k \partial_i) k^{TL}
\]

\[
v_0 \equiv q ,
\]

\[
v_i \equiv \partial_i v^L + \epsilon_{ii} \partial_i v^T ,
\]
where $\Delta \equiv \partial_i \partial_i$, eq. (19) can be rewritten as follows
\begin{equation}
S^{(3)L}_0 = \kappa^2 \langle \dot{k}^L \Delta \dot{k}^L + \dot{k}^T \Delta \dot{k}^T + \dot{v}^L \Delta \dot{v}^L + \dot{v}^T \Delta \dot{v}^T \\
+ 2\dot{v}^L \Delta \dot{k}^T - 2\dot{v}^T \Delta \dot{k}^L + (\Delta \dot{k}^{TT})^2 + (\Delta \dot{k}^{LL})^2 \\
+ 2(\Delta \dot{k}^{TL})^2 + 2(\dot{q})^2 - 2n\Delta \dot{k}^L + 2n\Delta \dot{v}^T \\
+ 2q\Delta \dot{v}^L - 2q\Delta \dot{k}^T + 2\Delta k^{LL} \Delta \dot{k}^L + 2\Delta k^{TL} \Delta \dot{k}^T \\
+ 2\Delta k^{LL} \Delta \dot{v}^T - 2\Delta \dot{k}^{TT} \Delta \dot{v}^L + q\Delta q + n\Delta n \\
+(\Delta k^L)^2 + 2(\Delta k^T)^2 + 2(\Delta v^L)^2 + (\Delta v^T)^2 \\
+ 2\Delta v^T \Delta \dot{k}^L + 2q\Delta^2 \dot{k}^{TT} + \Delta \dot{k}^{TT} \Delta^2 \dot{k}^{TT} \\
+ \Delta \dot{k}^{TL} \Delta^2 \dot{k}^{TL} \rangle .
\end{equation}

Primary Lagrangian constraints, joined to some links among accelerations, can be obtained through an inspection on field equations, which arise from (25). A "Coulomb" gauge (i.e., $\partial_i k_{\mu} = 0$) is considered. Then, we get the following set of twelve Lagrangian constraints
\begin{equation}
n = \dot{n} = \dot{v} = \dot{v}^T = k^L = \dot{k}^L = k^{LL} = \dot{k}^{LL} = k^{TL} = \dot{k}^{TL} = 0 ,
\end{equation}
\begin{equation}
\dot{k}^T - \dot{v}^L + \dot{q} = 0 ,
\end{equation}
\begin{equation}
\Delta \dot{k}^T - \Delta v^L + \dot{q} = 0 ,
\end{equation}
and all accelerations are solved. Then, there are three degrees of freedom, and the constraint system give rise to reduced action
\begin{equation}
S^{(3)L*}_0 = \kappa^2 \langle \dot{k}^T \Delta \dot{k}^T + 4(\Delta \dot{k}^T)^2 + 4(\dot{q})^2 \\
+ 4q\Delta q + (\Delta \dot{k}^{TT})^2 + \Delta k^{TT} \Delta^2 \dot{k}^{TT} \rangle .
\end{equation}

Introducing notation
\begin{equation}
Q \equiv 2q ,
\end{equation}
\begin{equation}
Q^T \equiv 2(-\Delta \dot{k}^T) ,
\end{equation}
\begin{equation}
Q^{TT} \equiv \Delta k^{TT} ,
\end{equation}
the reduced action is rewritten as follows
\begin{equation}
S^{(3)L*}_0 = \kappa^2 \langle Q \Box Q - Q^T \Box Q^T + Q^{TT} \Box Q^{TT} \rangle ,
\end{equation}
showing two unitary and one non-unitary modes, then the Hamiltonian is not positive definite. This study could also have considered from the point of view of the exchange amplitude procedure, in which is considered the coupling to a (conserved) energy-momentum tensor of some source, through Lagrangian terms $\kappa k_{\mu\nu} T^{\mu\nu}$ and $\chi v_\mu J^\mu$.

Some features about the degrees of freedom’s counting of this model at a non perturbative regime can be remarked. If one keep in mind a physical system where...
the metric is considered as a non dynamical field (otherwise we get a new problem with additional degrees of freedom coming from fluctuations of the metric), the essential problem to face up here is related to the Lagrangian (or Hamiltonian) analysis of a second order action with potentials which depend on the fourth-order power of the contortion field. In detail, the Riemann-Cartan curvature, \( F_{\mu \nu} (A) \) can be decomposed in terms of the Riemann-Christoffel, \( F_{\mu \nu} (\Gamma) \) and contortion, this means

\[
F_{\mu \nu} (A) = F_{\mu \nu} (\Gamma) + F_{\mu \nu} (K) + [\Gamma_\mu, K_\nu] + [K_\mu, \Gamma_\nu] ,
\]

where the components of the matrix \( \Gamma_\mu \) and \( K_\mu \) are the Christoffel’s symbols and contortion components, respectively. Then, the action (1) for \( N = 3 \) and \( \lambda = 0 \) is rewritten for any given metric as follows

\[
S^{(3)}_0 = \kappa^2 \text{tr} \left< -\frac{1}{4} F^{\alpha \beta} (K) F_{\alpha \beta} (K) + P_1^{\alpha \beta} (K, \Gamma) F_{\alpha \beta} (K) + P_2 (K, \Gamma) \right> ,
\]

where \( P_1^{\alpha \beta} (K, \Gamma) \) and \( P_2 (K, \Gamma) \) are polynomials of order one and two in contortion, respectively and they are identically null when \( \Gamma_\mu = 0 \). One can start considering a space-time with a Minkowskian metric provided (i. e., a Weitzenb"{o}ck space), hence action (35) is rewritten as

\[
S^{(3)}_{W_0} = \kappa^2 \text{tr} \left< -\frac{1}{4} F^{\alpha \beta} (K) F_{\alpha \beta} (K) \right> ,
\]

which perturbative regime has been studied above. Next, we will show that this theory contains three degrees of freedom as in the linearized level. For this purpose, there is a way inspired in the well known first order formalism for Yang-Mills theories\cite{13}, reducing derivatives and potential’s powers, simultaneously. In 2 + 1 dimension, particularly one can consider a rank three auxiliary field with notation \( f_{\mu} \). Then, the first order version of the action (36) is introduced through

\[
S_{W_0} = \kappa^2 \text{tr} \left< \frac{1}{2} f^{\mu} f_\mu \right> ,
\]

After its 2+1 splitting and using Lagrangian constraints (removing \( K_0 \) and \( f_0 \)), the reduced action is

\[
S^*_{W_0} = \kappa^2 \text{tr} \left< \epsilon_{ij} f_i \dot{K}_j + \frac{f_i f_j}{2} \right> ,
\]

where \( \epsilon_{ij} \equiv \epsilon^{0}_{ij} \). The Lagrangian constraints joined with the "Coulomb" gauge fixation (i. e., \( \partial_i K_i = 0 \)) constitute a set of forty two constraints

\[
\epsilon_{ij} \dot{K}_j + f_i = 0 ,
\]

\[
\dot{f}_j = 0 ,
\]

\[
\partial_i K_i = 0 ,
\]

\[
\partial_i \dot{K}_i = 0 ,
\]
for forty eight fields and velocities, confirming the existence of three degrees of freedom.

The main question is about the behavior of the theory for any given metric. Again we resort to auxiliary fields $f_\mu$ and the first order version of the action (1) for $N = 3$ and $\lambda = 0$, is $S_0 = \kappa^2 \text{tr} \left( \frac{1}{2} f^\mu f_\mu - \frac{\varepsilon^{\mu\nu\lambda}}{2} f_\lambda f_{\mu\nu}(A) \right)$. Using (34), this action can be written in explicit terms of the dynamical variables

$$S_0 = \kappa^2 \text{tr} \left( \frac{1}{2} f^\mu f_\mu - Q^\mu(g) f_\mu - \varepsilon^{\mu\nu\lambda} f_\lambda \left( \frac{1}{2} F_{\mu\nu}(K) + [\Gamma_\mu, K_\nu] \right) \right),$$

where $Q^\mu(g)$ is Poincare’s dual of Riemann-Christoffel’s curvature given by

$$Q^\sigma(g) = \varepsilon^{\mu\nu\sigma} \left( \delta^\alpha_\nu R_{\beta\mu} - g_{\beta\nu} R^\alpha_\mu - \frac{1}{2} \delta^\alpha_\nu g_{\beta\mu} \right).$$

It is possible to show, under certain conditions a narrow analogy between this theory and that of the Weitzenböck, given by action (37). Thinking about the anti-symmetric property of the contortion ($K^{\alpha\mu\nu} = -K^{\mu\alpha\nu}$), the first step is to consider a symmetric-antisymmetric decomposition of all fields involved in the action (42). In other words, let us introduce the next decomposition

$$\Gamma_\mu = \Gamma_\mu + \tilde{\Gamma}_\mu,$$

$$f_\mu = f_\mu + \tilde{f}_\mu,$$

where $\Gamma_\mu$ and $\tilde{\Gamma}_\mu$ are the symmetric and antisymmetric parts of the Christoffell’s symbols, respectively (i.e., $(\tilde{\Gamma}_\mu)_\sigma^\rho = \frac{1}{2} \partial_\rho g_{\sigma\mu}$ and $(\Gamma_\mu)_\sigma^\rho = \frac{1}{2} \left( \partial_\rho g_{\sigma\mu} - \partial_\sigma g_{\rho\mu} \right)$). The same idea is reflected in notation of (45). Using these definitions in (42) and performing a $2 + 1$ splitting, the set of Lagrange constraints allow us to remove the non dynamical fields (this means $\tilde{f}_0$ and $\tilde{f}_\mu$), then the reduced action is

$$S_0^* = \kappa^2 \text{tr} \left( \varepsilon^{ij} \tilde{f}_i (K_j + [\tilde{\Gamma}_0, K_j] - Q^*_j(g)) + \frac{g^{ij}}{2} \tilde{f}_i \tilde{f}_j \right).$$

with $Q^i(g) = \varepsilon^{ij} Q^*_j(g)$.

Next, a new antisymmetric variable (in the sense of definitions (44) and (45)) is introduced

$$q_j = K_j - Q^*_j(g),$$

where $Q^*_j(g)$ is a solution of the non homogeneous and first order differential equation $\dot{Q}^*_j(g) + [\tilde{\Gamma}_0, Q^*_j(g)] = Q^i(g)$. This suggests the definition of the following operator which acts on any object with matrix representation (i.e., a subgroup of $GL(3, R)$), $\chi$, in other words

$$\tilde{\nabla}_\mu \chi \equiv \partial_\mu \chi + [\tilde{\Gamma}_\mu, \chi]$$

and on a real function $h$

$$\tilde{\nabla}_\mu h \equiv \partial_\mu h,$$
On unitarity of a Yang-Mills formulation for...

for example $\tilde{\nabla}_\mu \varepsilon^{ij} = \partial_\mu \varepsilon^{ij}$, $\tilde{\nabla}_\mu g^{ij} = \partial_\mu g^{ij}$, etc.

Let $\chi$ and $\xi$ be objects with matrix representation, some properties of $\tilde{\nabla}_\mu$ are

$$\tilde{\nabla}_\mu (\xi \chi) = \xi \tilde{\nabla}_\mu \chi + (\tilde{\nabla}_\mu \xi) \chi, \quad (50)$$

$$[\tilde{\nabla}_\mu, \tilde{\nabla}_\nu] \chi = [F_{\mu\nu}(\tilde{\Gamma}), \chi]. \quad (51)$$

It must be pointed out that $\tilde{\nabla}_\mu$ is not a covariant derivative for an arbitrary background.

Using (47) and (48) in (46), we write the reduced action as follows

$$S^*_0 = \kappa^2 \text{tr} \left( \varepsilon^{ij} \tilde{f}_i \tilde{\nabla}_0 g_{j} + \frac{g^{ij}}{2} \tilde{f}_i \tilde{f}_j \right). \quad (52)$$

and it gives rise a set of twelve constraints

$$\phi_i \equiv \tilde{\nabla}_0 \tilde{f}_i = 0, \quad (53)$$

$$\psi_i \equiv \tilde{\nabla}_0 q_i + \varepsilon_{0ij} \tilde{f}_j = 0, \quad (54)$$

whose preservation give the relations for accelerations

$$\dot{\phi}_i \simeq \tilde{\nabla}_0^2 \tilde{f}_i = 0, \quad (55)$$

$$\dot{\psi}_i \simeq \tilde{\nabla}_0^2 q_i + \partial_0 \varepsilon_{0ij} \tilde{f}_j = 0. \quad (56)$$

However, the complete Lagrangian analysis depends on a gauge fixation. In order to illustrate the remaining Lagrangian process we find some similarities with the Weitzenböck case if certain conditions are demanded on the background. Therefore, let the metric be static-stationary, this means $\partial_0 g_{\mu\nu} = 0$ and $g_{0i} = 0$ (i. e., Schwarzschild background), for instance relation (56) gives $\tilde{\nabla}_0^2 q_i = 0$.

Now, one can explore a gauge fixation. For example, the axial gauge provide six additional constraints

$$\varphi^A \equiv q_2 = 0, \quad (57)$$

$$\dot{\varphi}^A \simeq \tilde{\nabla}_0 q_2 = 0, \quad (58)$$

and joined to (53) and (54), say that there are three degrees of freedom. An equivalent procedure can be developed if one perform a "Coulomb" gauge fixation

$$\varphi^C \equiv \tilde{\nabla}_i q_i + [a_i, q_i] = 0, \quad (59)$$

where $a_i$ satisfies the differential equation $\tilde{\nabla}_0 a_i = F_{0i}(\tilde{\Gamma})$. Preservation of (59) provide another three constraints

$$\dot{\varphi}^C \simeq \tilde{\nabla}_i \tilde{\nabla}_0 q_i + [a_i, \tilde{\nabla}_0 q_i] = 0, \quad (60)$$

and the procedure is finished (preservation of (60) is identically satisfied). Again, the constraint system shows three degrees of freedom.
4. YM gravity with parity preserving massive term

It can be possible to write down a massive version which respect parity and we introduce a possible model as follows

\[ S^{(3)}_m = S^{(3)}_0 - \frac{m^2 \kappa^2}{2} \left< T^\sigma\sigma T^\rho\rho - T^\lambda\mu\nu T_\mu\lambda\nu - \frac{1}{2} T^\lambda\mu\nu T_{\lambda\mu\nu} \right>. \]  

(61)

In a general case, two types of field equations can be obtained if independent variations on metric and connection are allowed. On one hand, variations on metric give rise to the expression of the gravitacional energy-momentum tensor, \( T^\alpha_\beta \equiv \kappa^2 t \left[ F^{\alpha\sigma} F_{\beta\sigma} - \frac{g^{\alpha\beta}}{2} F_{\mu\nu} F^{\mu\nu} \right] \), in other words

\[ T^\alpha_\beta = - T_t^\alpha_\beta - \kappa^2 g^{\alpha\beta} \lambda^2, \]

(62)

where \( T_t^\alpha_\beta \equiv -m^2 \kappa^2 \left[ 3t^\alpha_\sigma t^\beta_\sigma + 3t^\alpha_\sigma t^\beta_\sigma - t^\alpha_\sigma t^\beta_\sigma - (t^\alpha_\beta + t^\beta_\alpha) t^\sigma_\sigma - \frac{5g^{\alpha\beta}}{2} t_{\mu\nu} t_{\mu\nu} + 3g^{\alpha\beta} (t_\sigma)^2 \right] \) is the torsion contribution to the energy-momentum distribution and \( T_t^\alpha_\beta \equiv \frac{\kappa}{2} T^\beta_{\mu\nu} \). This says, for example, that the quest of possible black hole solutions must reveal a dependence on parameters \( m^2 \) and \( \lambda^2 \).

On the other hand, variations on connection provide the following equations

\[ \frac{1}{\sqrt{-g}} \partial_\alpha (\sqrt{-g} F^{\alpha\lambda}) + [A_\alpha, F^{\alpha\lambda}] = J^\lambda, \]

(63)

where the current is \( (J^\lambda)^\nu_\sigma = m^2 (\delta^\lambda_\nu \delta^K_\rho - \delta^K_\sigma K^\rho_\nu + 2K^K_\nu_\lambda) \). We can observe in [63] that contortion and metric appear as sources of gravity, where the cosmological contribution is obviously hide in space-time metric. In a weak torsion regime, equation (63) takes a familiar shape, this means \( \nabla_\alpha F^{\alpha\lambda} = J^\lambda \).

Now we explore the perturbation of the massive case given at (61) and with the help of (15), the linearized action is

\[ S^{(3)}_{L_m} = \kappa^2 \left< k_{\mu\nu} \square k^{\mu\nu} + \partial_\mu k^{\mu\nu} \partial_\nu k^{\nu\sigma} - \frac{2\epsilon^{\sigma\alpha\beta}}{\delta} \partial_\sigma \partial_\beta \partial_\nu k^{\nu\sigma} - v_\nu \square v_\nu + (\partial_\mu v^\mu)^2 - m^2 (k_{\mu\nu} k^{\mu\nu} - k^2) \right>. \]

(64)

Using a TL-decomposition defined by (20)-(24), we can write (64) in the way

\[ S^{(3)}_{L_m} = \kappa^2 \left< k^L \Delta k^L + k^T \Delta k^T + \dot{v}^L \Delta \dot{v}^L + \dot{v}^T \Delta \dot{v}^T + 2\dot{v}^L \Delta \dot{k}^T - 2\dot{v}^T \Delta \dot{k}^L \right. \]

\[ + (\Delta k^L)^2 + (\Delta k^L)^2 + 2(\Delta k^T)^2 + 2(\dot{v}^T)^2 - 2n \Delta k^L + 2n \Delta \dot{v}^T \]

\[ + 2q \Delta \dot{v}^L - 2q \Delta k^T + 2k^L \Delta k^L + 2k^T \Delta k^T + 2k^{LL} \Delta \dot{v}^L \]

\[ - 2k^{TT} \Delta \dot{v}^T + q \Delta q + n \Delta n + (k^L)^2 + (k^T)^2 + 2(\dot{v}^L)^2 \]

\[ + (\Delta v)^2 + 2\Delta v \Delta k^L + 2q \Delta^2 k^L + \Delta k^{TT} \Delta k^{TT} + \Delta k^{TT} \Delta k^{TL} + \Delta k^{TT} \Delta k^{LL} + \Delta k^{TT} \Delta k^{TT} \]

\[ + m^2 [-2k^L \Delta k^L - 2k^T \Delta k^T - 2(\Delta k^{TL})^2 - 2n (\Delta k^{TT} + \Delta k^{LL})^2 \]

\[ + 2\Delta k^{TT} \Delta k^{LL}] \right>. \)

(65)

Here, there is no gauge freedom (as it will be confirmed in next section) and field equations provide primary constraints and some accelerations. The preservation
The field equations are
\[ \Delta k^{LL} - \Delta k^L - \Delta v^T + m^2k^L = 0 , \]
\[ \Delta k^{TT} - \Delta k^L - \Delta v^T + m^2k^T = 0 , \]
\[ \Delta k^{LL} - \Delta k^L - \Delta v^T + m^2(k^{TT} + k^{LL}) = 0 , \]
\[ \dot{k}^L + \Delta k^{TT} - n = 0 , \]
\[ \dot{k}^T - \Delta k^{TL} = 0 , \]
\[ \dot{v}^T + \Delta k^{TT} + m^2(k^{TT} + k^{LL}) - 2m^2\Delta^{-1}n = 0 , \]
\[ \dot{n} - \Delta k^L = 0 , \]

which says that this massive theory get five degrees of freedom. In order to explore the physical content, we can take a short path to this purpose and it means to start with a typical transverse-traceless (Tt) decomposition instead the TL-decomposition one. Notation for the Tt-decomposition of fields is
\[ k_{\mu
u} = k^{Tt}_{\mu
u} + \hat{\partial}_\mu \theta^T_{\nu} + \hat{\partial}_\nu \theta^T_{\mu} + \hat{\partial}_\mu \hat{\partial}_\nu \psi + \eta_{\mu\nu} \phi , \]
\[ v_{\mu} = v^T_{\mu} + \hat{\mu} v , \]

with the subsidiary conditions
\[ k^{Tt}_{\mu
u} = 0 , \quad \partial^\mu k^{Tt}_{\mu
u} = 0 , \quad \partial^\mu \theta^T_{\mu} = 0 , \quad \partial^\mu v^T_{\mu} = 0 , \]

where we use the operator \( \hat{\partial}_\sigma \equiv \square - \frac{1}{2} \partial_\sigma \) defined in reference [15]. Action (64) is
\[ S^{(3)}_m = \kappa^2 (k^{Tt}_{\mu
u}(\square - m^2)k^{Tt}_{\mu\nu} - \theta^T_{\mu}(\square - 2m^2)\theta^T_{\mu} - 2\epsilon^\alpha\beta\partial_\alpha v^T_{\beta} \square \theta^T_{\sigma} - v^T_{\mu} \square v^T_{\mu} + 2v\square v + 2\phi\square \phi + 4m^2\psi\phi + 6m^2\phi^2) \]

A new transverse variable, \( a^{T}_{\mu} \) is introduced through
\[ \theta^T_{\mu} = \epsilon^{\alpha\beta}_\mu \hat{\partial}_\alpha a^{T}_{\beta} , \]

and the action (77) is rewritten as
\[ S^{(3)}_m = \kappa^2 (k^{Tt}_{\mu\nu}(\square - m^2)k^{Tt}_{\mu\nu} - a^T_{\mu}(\square - 2m^2)a^T_{\mu} - 2a^T_{\mu} \square v^T_{\mu} - v^T_{\mu} \square v^T_{\mu} + 2v\square v + 2\phi\square \phi + 4m^2\psi\phi + 6m^2\phi^2) . \]

The field equations are
\[ (\square - m^2)k^{Tt}_{\mu\nu} = 0 , \]
\[ \square v^T_{\mu} = 0 , \]
and reduced action is

$$S^{(3)} L^* = \kappa^2 \langle \kappa T^{\mu \nu} (\Box - m^2) k T^{\mu \nu} + 2v \Box v - v T^{\mu} \Box v T^{\mu} \rangle ,$$

saying that the contortion propagates two massive helicities ±2, one massless spin-0 and two massless ghost vectors. Then, there is not positive definite Hamiltonian. This observation can be confirmed in the next section when we will write down the Hamiltonian density and a wrong sign appears in the kinetic part corresponding to the canonical momentum of $v_i$ (see eq. (97)).

5. Gauge variance at the linearized regime

The quadratical Lagrangian density dependent in torsion and presented in (61), has been constructed without free parameters, with the exception of $m^2$, of course. It has a particular shape which only gives mass to the spin 2 component of the contortion, as we see in the perturbative regime. Let us comment about de non existence of any possible "residual" gauge invariance of the model. The answer is that the model lost its gauge invariance and it can be shown performing the study of symmetries through computation of the gauge generator chains. For this purpose, a 2 + 1 decomposition of (64) is performed, this means

$$S^{(3)} L^*_m = \kappa^2 \langle \{-k_{0i} + 2\partial_0 k_{00} - 2\partial_0 k_{ni} - 2\epsilon_{in} \dot{v}_n + 2\epsilon_{in} \partial_0 v_0\} \dot{k}_{0i} + \dot{k}_{ij} \dot{k}_{ij}$$

$$+ [2\epsilon_{ij} \partial_0 k_{00} + 2\epsilon_{ij} \partial_0 k_{nm} - \dot{v}_j - 2\partial_0 v_0] \dot{v}_j + 2(\dot{v}_0)^2 + k_{00} \Delta k_{00}$$

$$- 2k_{0i} \Delta k_{0i} + k_{ij} \Delta k_{ij} - (\partial_0 k_{00})^2 + \partial_0 k_{ni} \partial_0 k_{ni} - 2\epsilon_{ij} \partial_0 v_0 \partial_0 k_{00}$$

$$- 2\epsilon_{nm} \partial_0 v_0 \partial_0 k_{ni} + v_0 \Delta v_0 - v_i \Delta v_i + (\partial_0 v_0)^2$$

$$+ m^2 [2k_{0i} k_{0i} - k_{ij} k_{ij} - 2k_{00} k_{ii} + (k_{ii})^2] \rangle ,$$

where $\Delta \equiv \partial_i \partial_i$.

Next, the momenta are

$$\Pi \equiv \frac{\partial L}{\partial \dot{k}_{00}} = 0 ,$$

$$\Pi^i \equiv \frac{\partial L}{\partial \dot{k}_{0i}} = -2k_{0i} - 2\epsilon_{in} \dot{v}_n + 2\partial_0 k_{00} - 2\partial_0 k_{0i} + 2\epsilon_{in} \partial_0 v_0 ,$$

$$\Pi^{ij} \equiv \frac{\partial L}{\partial \dot{k}_{ij}} = 2\dot{k}_{ij} ,$$

$$P \equiv \frac{\partial L}{\partial \dot{v}_0} = 4\dot{v}_0 ,$$

$$\Box v = 0 ,$$

$$a T^\mu = 0 ,$$

$$\psi = \phi = 0 ,$$

and reduced action is

$$S^{(3)} L^* = \kappa^2 \langle k T^{\mu \nu} (\Box - m^2) k T^{\mu \nu} + 2v \Box v - v T^{\mu} \Box v T^{\mu} \rangle ,$$

saying that the contortion propagates two massive helicities ±2, one massless spin-0 and two massless ghost vectors. Then, there is not positive definite Hamiltonian. This observation can be confirmed in the next section when we will write down the Hamiltonian density and a wrong sign appears in the kinetic part corresponding to the canonical momentum of $v_i$ (see eq. (97)).

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$$S^{(3)} L^*_m = \kappa^2 \langle [-k_{0i} + 2\partial_0 k_{00} - 2\partial_0 k_{ni} - 2\epsilon_{in} \dot{v}_n + 2\epsilon_{in} \partial_0 v_0] \dot{k}_{0i} + \dot{k}_{ij} \dot{k}_{ij}$$

$$+ [2\epsilon_{ij} \partial_0 k_{00} + 2\epsilon_{ij} \partial_0 k_{nm} - \dot{v}_j - 2\partial_0 v_0] \dot{v}_j + 2(\dot{v}_0)^2 + k_{00} \Delta k_{00}$$

$$- 2k_{0i} \Delta k_{0i} + k_{ij} \Delta k_{ij} - (\partial_0 k_{00})^2 + \partial_0 k_{ni} \partial_0 k_{ni} - 2\epsilon_{ij} \partial_0 v_0 \partial_0 k_{00}$$

$$- 2\epsilon_{nm} \partial_0 v_0 \partial_0 k_{ni} + v_0 \Delta v_0 - v_i \Delta v_i + (\partial_0 v_0)^2$$

$$+ m^2 [2k_{0i} k_{0i} - k_{ij} k_{ij} - 2k_{00} k_{ii} + (k_{ii})^2] \rangle ,$$

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$$\Pi \equiv \frac{\partial L}{\partial \dot{k}_{00}} = 0 ,$$

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and reduced action is

$$S^{(3)} L^* = \kappa^2 \langle k T^{\mu \nu} (\Box - m^2) k T^{\mu \nu} + 2v \Box v - v T^{\mu} \Box v T^{\mu} \rangle ,$$

saying that the contortion propagates two massive helicities ±2, one massless spin-0 and two massless ghost vectors. Then, there is not positive definite Hamiltonian. This observation can be confirmed in the next section when we will write down the Hamiltonian density and a wrong sign appears in the kinetic part corresponding to the canonical momentum of $v_i$ (see eq. (97)).
and we establish the following commutation rules

\begin{align}
\{k_{00}(x), \Pi(y)\} & = \{v_0(x), P(y)\} = \delta^2(x - y) , \\
\{k_0(x), \Pi^j(y)\} & = \{v_i(x), P_j(y)\} = \delta^j_i \delta^2(x - y) , \\
\{k_{ij}(x), \Pi^{\alpha \beta}(y)\} & = \frac{1}{2} (\delta^\alpha_i \delta^\beta_j + \delta^\alpha_j \delta^\beta_i) \delta^2(x - y) .
\end{align}

It can be noted that (87) is a primary constraint that we name

\[ G^{(K)} = \Pi , \]

where \( K \) means the initial index corresponding to a possible gauge generator chain, provided by the algorithm developed in reference [10]. Moreover, manipulating (89) and (91), other primary constraints appear

\[ G^{(K)}_i = \partial_i k_{ni} - \epsilon_{in} \partial_n v_0 - \frac{\epsilon_{in}}{4} P^m + \frac{1}{4} \Pi^i , \]

and we observe that \( G^{(K)} \) and \( G^{(K)}_i \) are first class.

The preservation of constraints requires to obtain the Hamiltonian of the model. First of all, the Hamiltonian density can be written as

\[ \mathcal{H}_0 = \Pi^i \dot{h}_{0i} + \Pi^j \dot{h}_{ij} + P^i \dot{v}_i + \mathcal{L} , \]

in other words

\[ \mathcal{H}_0 = \frac{\Pi^i \Pi^j}{4} + \frac{P^i P^j}{8} + \epsilon_{ij} \partial_m k_{nm} P^j + v_0 [\partial_i P^i + 4 \epsilon_{in} \partial_n \partial_n k_{ni}] \\
+ k_{00} [2 \partial_m \partial_n k_{mn} - \epsilon_{nm} \partial_i P^m + 2 m^2 k_{ii}] + 2 k_{0i} \Delta k_{0i} - k_{ij} \Delta k_{ij} \\
+ (\partial_i k_{0i})^2 - 2 \partial_n k_{0i} \partial_m k_{mi} + 2 \epsilon_{ij} \partial_i v_j \partial_n k_{ni} + v_0 \Delta v_i - (\partial_n v_n)^2 \\
- m^2 [2 k_{0i} k_{ni} - k_{ij} k_{ij} + (k_{ii})^2] . \]

Then, the Hamiltonian is

\[ \mathcal{H}_0 = \int dy^2 \mathcal{H}_0(y) \equiv \langle \mathcal{H}_0 \rangle_y \]

and the preservation of \( G^{(K)} \), defined in (95) is

\[ \{ G^{(K)}(x), \mathcal{H}_0 \} = - 2 \partial_m \partial_n k_{mn}(x) + \epsilon_{nm} \partial_n P^m(x) - 2 m^2 k_{ii}(x) . \]

The possible generators chain is given by the rule: "\( G^{(K-1)} + \{ G^{(K)}(x), \mathcal{H}_0 \} \) = combination of primary constraints", then

\[ G^{(K-1)}(x) = 2 \partial_m \partial_n k_{mn}(x) - \epsilon_{nm} \partial_n P^m(x) + 2 m^2 k_{ii}(x) \\
+ \langle a(x,y) G^{(K)}(y) + b^i(x,y) G^{(K)}_{ji}(y) \rangle_y . \]

The preservation of \( G^{(K)}_i \), defined in (96), is

\[ \{ G^{(K)}_i(x), \mathcal{H}_0 \} = \frac{\partial_i \Pi^m(x)}{2} - \frac{\epsilon_{in}}{4} \partial_n P^m(x) + \frac{\epsilon_{in}}{2} \Delta v_m(x) + \frac{\epsilon_{in}}{2} \partial_n \partial_m v_m(x) \\
+ \frac{\epsilon_{nm}}{2} \partial_i \partial_n v_m(x) - (\Delta - m^2) k_{0i}(x) , \]

\[ \{ G^{(K)}_i(x), \mathcal{H}_0 \} = \frac{\partial_i \Pi^m(x)}{2} - \frac{\epsilon_{in}}{4} \partial_n P^m(x) + \frac{\epsilon_{in}}{2} \Delta v_m(x) + \frac{\epsilon_{in}}{2} \partial_n \partial_m v_m(x) \\
+ \frac{\epsilon_{nm}}{2} \partial_i \partial_n v_m(x) - (\Delta - m^2) k_{0i}(x) , \]

\[ \{ G^{(K)}_i(x), \mathcal{H}_0 \} = \frac{\partial_i \Pi^m(x)}{2} - \frac{\epsilon_{in}}{4} \partial_n P^m(x) + \frac{\epsilon_{in}}{2} \Delta v_m(x) + \frac{\epsilon_{in}}{2} \partial_n \partial_m v_m(x) \\
+ \frac{\epsilon_{nm}}{2} \partial_i \partial_n v_m(x) - (\Delta - m^2) k_{0i}(x) , \]
then
\[
G_i^{(K-1)}(x) = -\frac{\partial_i \Pi^{\mu}(x)}{2} + \frac{\epsilon_{in}}{4} \partial_n P(x) - \frac{\epsilon_{in}}{2} \Delta v_n(x) - \frac{\epsilon_{in}}{2} \partial_n \partial_m v_m(x) - \frac{\epsilon_{mn}}{2} \partial_l \partial_m v_m(x) + (\Delta - m^2)k_0(x)
\]
\[
+ \langle a^i(x, y)G^{(K)}(y) + b^i_j(x, y)G^{(K)}(y) \rangle_y . \tag{101}
\]

The undefined objects \( a(x, y) \), \( b^i(x, y) \), \( a^i(x, y) \) and \( b^i_j(x, y) \) in expressions (99) and (101), are functions or distributions. If it is possible, they can be fixed in a way that the preservation of \( G^{(K-1)}(x) \) and \( G_i^{(K-1)}(x) \) would be combinations of primary constraints. With this, the generator chains could be interrupted and we simply take \( K = 1 \). Of course, the order \( K - 1 = 0 \) generators must be first class, as every one. Next, we can see that all these statements depend on the massive or non-massive character of the theory.

Taking a chain with \( K = 1 \), the candidates to generators of gauge transformation are (95), (96), (99) and (101). But, the only non null commutators are
\[
\{ G^{(1)}_i(x), G^{(0)}_j(y) \} = \frac{m^2}{4} \eta_{ij} \delta^2(x - y) , \tag{102}
\]
\[
\{ G^{(0)}(x), G^{(0)}_i(y) \} = m^2 (\partial_i \delta^2(x - y) + \frac{b^i(x, y)}{4}) , \tag{103}
\]
saying that the system of "generators" is not first class. Moreover, the unsuccessful conditions (in the \( m^2 \neq 0 \) case) to interrupt the chains, are
\[
\{ G^{(0)}(x), H_0 \} = m^2 (\Pi^{\mu\nu}(x) - 2\partial_\mu k_{0\nu}(x)) , \tag{104}
\]
\[
\{ G^{(0)}_i(x), H_0 \} = m^2 (\partial_n k_{in}(x) + \partial_0 k_{00}(x) - \partial_ik_{nn}(x)) , \tag{105}
\]
where we have fixed
\[
a(x, y) = 0 , \tag{106}
\]
\[
b^i(x, y) = -2\partial_i \delta^2(x - y) , \tag{107}
\]
\[
a^i(x, y) = 0 , \tag{108}
\]
\[
b^i_j(x, y) = 0 . \tag{109}
\]

All this indicates that in the case where \( m^2 \neq 0 \) there is not a first class consistent chain of generators and, then there is no gauge symmetry.

However, if we revisit the case \( m^2 = 0 \), conditions (104) and (105) are zero and the chains are interrupted. Now, the generators \( G^{(1)} \), \( G^{(1)}_i \), \( G^{(0)} \) and \( G^{(0)}_i \) are first class. Using (106)-(107), the generators are rewritten again
\[
G^{(1)} \equiv \Pi , \tag{110}
\]
On unitarity of a Yang-Mills formulation for...  15

\[ G^{(1)}_i = \partial_n k_{ni} - \epsilon_{in} \partial_n v_0 - \frac{\epsilon_{in}}{4} P^n + \frac{1}{4} \Pi^i, \quad (111) \]

\[ G^{(0)} = -\frac{\epsilon_{nm}}{2} \partial_n P^m - \frac{\partial_n \Pi^n}{2}, \quad (112) \]

\[ G^{(0)}_i = -\frac{\partial_n \Pi^n}{2} + \frac{\epsilon_{in}}{4} \Delta v_n - \frac{\epsilon_{in}}{2} \partial_n \partial_m v_m - \frac{\epsilon_{nm}}{2} \partial_i \partial_n v_m + \Delta k_{0i}. \quad (113) \]

Introducing the parameters \( \varepsilon(x) \) and \( \varepsilon^i(x) \), a combination of (110)-(113) is taken into account in the way that the gauge generator is

\[ G(\dot{\varepsilon}, \dot{\varepsilon}^i, \varepsilon, \varepsilon^i) = \langle \dot{\varepsilon}(x) G^{(1)}_i(x) + \dot{\varepsilon}^i(x) G^{(1)}_i(x) + \varepsilon(x) G^{(0)}_i(x) + \varepsilon^i(x) G^{(0)}_i(x) \rangle \quad (114) \]

and with this, for example the field transformation rules (this means, \( \delta(...) = \{(...), G\} \)) are written as

\[ \delta k_{00} = \dot{\varepsilon}, \]

\[ \delta k_{0i} = \frac{\dot{\varepsilon}^i}{4} + \frac{\partial_i \varepsilon}{2}, \]

\[ \delta k_{ij} = \frac{1}{4} (\partial_i \varepsilon_j + \partial_j \varepsilon_i), \]

\[ \delta v_0 = \frac{\epsilon_{nm}}{4} \partial_n \varepsilon_m, \]

\[ \delta v_i = \frac{\epsilon_{in}}{4} \dot{\varepsilon}_n - \frac{\epsilon_{in}}{2} \partial_i \varepsilon, \]

and, redefining parameters as follows: \( \varepsilon \equiv 2 \xi_0 \) and \( \dot{\varepsilon}^i \equiv 4 \xi^i \), it is very easy to see that these rules match with (17) and (18), as we expected.

6. YM-extended formulation

Here we review a possible quadratical term family which allows to eliminate non-unitary propagations in the contortion (torsion) perturbative regime in 2 + 1 dimension, at least in a perturbative regime. The most general shape of a Lagrangian counter terms set is

\[ S^{(3)}_0 = \kappa^2 \langle -\frac{1}{4} (F^{\mu\nu})^\rho (F_{\mu\nu})^\rho + a_1 (F_{\mu\nu})^\sigma (F^\mu_{\sigma})^\rho + a_2 (F_{\mu\nu})^\sigma (F^\rho_{\sigma})^\mu \]

\[ + a_3 (F_{\mu\nu})^\sigma (F^\rho_{\nu})^\mu + a_4 (F_{\mu\sigma})^\rho (F^\nu_{\rho})^\sigma + a_5 ((F_{\mu\nu})^{\mu\nu})^2 \rangle, \quad (120) \]

where \( a_1, a_2, a_3, a_4 \) and \( a_5 \) are real parameters.

A na"ive try to reach unitarity at a linearized level consists to perform a direct matching between the perturbative action coming from (120) and the linearized Hilbert-Einstein one, given by

\[ S_{\text{HE}}^L = -2\kappa^2 \langle h_{\mu\nu} G^L_{\mu\nu} \rangle = \kappa^2 \langle h_{\mu\nu} \Box h^{\mu\nu} + 2 \partial_\mu h^{\mu\sigma} \partial_\sigma h^{\nu} + 2 h \partial_\mu \partial_\nu h^{\mu\nu} - h \Box h \rangle, \quad (121) \]
where \( h_{\mu\nu} \) is the metric perturbation and \( G_{\mu\nu} \) is the linearized Einstein’s tensor. Then, under perturbations of the contortion (torsion), one can use again eq. (15), this time in (120). Next, making comparison between this result and (121), a linear equation’s system for parameters \( a_1, a_2, a_3, a_4 \) and \( a_5 \) arise and only two of them remain free (i.e., \( a_3 \equiv \alpha \) and \( a_5 \equiv \beta \)). This means that for any \( \alpha \) and \( \beta \) one can get an unitary (linearized) theory which contains massless spin 2 in 2+1 dimension, in other words, we demand that linearized version of (120) must be proportional to

\[
\langle k_{\mu\nu} \partial^2 + 2 \partial^\mu k_{\rho\sigma} \partial^\nu k^\rho \sigma + 2 k^\partial^\rho \partial^\nu k_{\mu\nu} - k \square k \rangle.
\]

These family of theories labeled by free parameters \( \alpha \) and \( \beta \), are

\[
S_{(\alpha,\beta)} = \kappa^2 \left( -\frac{1}{2} (F_{\mu\nu})^\rho \partial_\rho (F_{\mu\nu})^\sigma - (1 + \alpha)(F_{\mu\nu})^\rho \partial_\rho (F^\mu_\nu)^\nu \sigma + (\frac{5}{8} + \alpha + 4\beta)(F_{\mu\nu})^\rho \partial_\rho (F^\mu_\nu)^\nu \sigma + \alpha (F_{\mu\nu})^\rho \partial_\rho (F^\mu_\nu)^\nu \sigma - (\frac{1}{2} + \alpha + 4\beta)(F_{\mu\nu})^\rho \partial_\rho (F^\mu_\nu)^\nu \sigma + \beta ((F_{\mu\nu})^{\mu\nu})^2 \right) .
\]

Another illustrative shape of this action can be obtained using again (34) and decomposing the contortion in a symmetric\((K_{\mu\sigma} = K_{\sigma\mu})\)-antisymmetric\((V_\beta)\) parts as follows

\[
K_{\mu\beta} = (K_{\mu})^\alpha_{\beta} = \varepsilon^{\sigma\alpha\beta} K_{\mu\sigma} + \delta^\alpha_{\mu} V_{\beta} - g_{\mu\beta} V^\alpha , \tag{123}
\]

then, action (122) is rewritten in the next manner

\[
S_{(\alpha,\beta)} = \kappa^2 \left( K_{\mu\nu} \square K_{\mu\nu} + 2 \nabla_\mu K^{\mu\sigma} \nabla_\nu K^{\nu\sigma} + 2 K \nabla_\mu \nabla_\nu K_{\mu\nu} - K \square K + q^{(4)}(K, V) + p^{(3)}(\Gamma, K, V) \right) , \tag{124}
\]

where \( \nabla_\mu \) is the Levi-Civita derivative and \( \square \equiv \nabla_\mu \nabla^\mu \). In (124), \( q^{(4)}(K, V) \) means a polynomial of fourth order in \( K_{\mu\nu} \) and \( V_\mu \) and first order in derivatives of these fields, in other words

\[
q^{(4)}(K, V) \equiv -\frac{1}{2} (f_{\mu\nu})^\rho \partial_\rho (K_{\mu\nu})^\sigma \sigma - 2(1 + \alpha)(f_{\mu\nu})^\rho \partial_\rho (K_{\mu\nu})^\sigma \sigma + (\frac{5}{4} + \alpha + 2\beta)(f_{\mu\nu})^\rho \partial_\rho (K_{\mu\nu})^\sigma \sigma + 2\alpha (f_{\mu\nu})^\rho \partial_\rho (K_{\mu\nu})^\sigma \sigma - (1 + 2\alpha + 8\beta)(f_{\mu\nu})^\rho \partial_\rho (K_{\mu\nu})^\sigma \sigma + 2\beta (f_{\mu\nu})^\rho \partial_\rho (K_{\mu\nu})^\sigma \sigma - \frac{1}{4} [K_{\mu\nu} (K_{\mu\nu})^\rho \partial_\rho (K_{\mu\nu})^\sigma \sigma - (1 + \alpha) \{K_{\mu\nu}, K_{\sigma\nu}\}^\rho \partial_\rho (K_{\mu\nu})^\sigma \sigma + (\frac{5}{8} + \alpha + 4\beta) [K_{\mu\nu}, K_{\rho\sigma}]^\sigma \partial_\sigma (K_{\mu\nu})^\rho \partial_\rho (K_{\mu\nu})^\sigma \sigma - (\frac{1}{2} + \alpha + 4\beta) [K_{\mu\nu}, K_{\sigma\nu}]^\rho \partial_\rho (K_{\mu\nu})^\sigma \sigma + \beta ([K_{\mu\nu}, K_{\rho\sigma}]^\sigma \partial_\sigma (K_{\mu\nu})^\rho \partial_\rho (K_{\mu\nu})^\sigma \sigma)^2 , \tag{125}
\]

where \( (K_{\mu})^\alpha_{\beta} \) is evaluated on eq. (123) and \( (f_{\mu\nu})_{\alpha\beta} \equiv 2\varepsilon_{\alpha\beta} \nabla_\mu K_{\sigma\nu} + 2g_{[\mu\sigma} \nabla_{\nu]} V_{\beta} - 2g_{[\beta\sigma} \nabla_{\mu] V_{\alpha} \) (symbol \([\mu\nu]\) means antisymmetrization). The object \( p^{(3)}(\Gamma, K, V) \) is a third order polynomial in fields and first order in derivatives of these ones and even though its shape is awful, however it identically vanishes when Christofell’s
symbols are null. The expression (124) clearly explains the demanded behavior of
the perturbative regime in a background with a flat metric provided.

There are two possible massive cases. On one hand, the topological massive
model (3) can be considered, which is sensitive under parity. On the other hand,
there is a "Fierz-Pauli" model (61), whose mass vanishes when one take a null
torsion. Our main purpose in this section is to study the classical consistence
of field equations (we assume that the torsionless limit must be consistent with
Einstein’s theory), and then focusing the attention at the massless and topological
massive cases.

In the massless theory with cosmological constant, $\lambda$ in 2 + 1 dimension, we
introduce a cosmological term as follows

$$S^{(3)}(\alpha, \beta, \lambda) = S^{(3)}(\alpha, \beta) + \kappa^2 \langle q(\alpha, \beta) \lambda^2 \rangle,$$

(126)

where $q(\alpha, \beta)$ is a (unknown) real function of family’s parameters. Next, in order to
consider classical consistence at the torsionless regime, we take into account some
auxiliary fields (Lagrange multipliers), $b_{\mu\nu}$ and the action with torsion constraints
is given by

$$S'(3)(\alpha, \beta, \lambda) = S^{(3)}(\alpha, \beta) + \kappa^2 \langle q(\alpha, \beta) \lambda^2 \rangle + \kappa^2 \langle b_{\alpha\beta} \varepsilon^{\beta\lambda\sigma} (A_{\lambda})^\alpha_{\sigma} \rangle,$$

(127)

where arbitrary variations on fields $b_{\mu\nu}$, obviously provide the condition
$T^{\alpha}_{\mu\nu} = 0$.

Then, the field equation coming from variations of connection is

$$\nabla_{\mu} (F_{\sigma \rho}) + b_{\mu \nu} \varepsilon^{\mu \nu \sigma} = 0,$$

(128)

where $F_{\mu \nu}$ is defined in terms of the Yang-Mills curvature, $F_{\mu \nu}$ in the way

$$(F_{\mu \nu})_{\rho} = (F_{\mu \nu})^\sigma_{\rho} + \left(\frac{5}{4} + \alpha + 2 \beta\right) ((F_{\rho}^\sigma)^{\nu\mu} - (F_{\rho}^\sigma)^{\mu\nu})$$

+ $(1 + 2 \alpha) ((F_{\rho}^\nu)^{\lambda \mu} - (F_{\rho}^\nu)^{\mu \lambda}) + 2 \alpha ((F_{\nu}^\lambda)^{\lambda \rho \sigma} - (F_{\nu}^\lambda)^{\rho \lambda \sigma})$

+ $(1 + 2 \alpha + 8 \beta) ((F_{\lambda}^\nu)^{\lambda \rho} - (F_{\lambda}^\nu)^{\rho \lambda}) + 2 \beta (F_{\lambda\kappa}^\nu)^{\lambda \nu (g^{\mu \sigma} \delta_{\rho}^\nu - g^{\rho \sigma} \delta_{\mu}^\nu)},$

(129)

and now, we can match the YM curvature with the Riemann-Christoffel one (i. e.,
$(F_{\mu \nu})_{\alpha \beta} = R_{\alpha \beta \nu \rho}$), which satisfies the well known algebraic properties and Bianchi
identities, recalling as follows

**Symmetry:** $R_{\alpha \beta \nu \mu} = R_{\nu \rho \alpha \beta},$

(130)

**Antisymmetry:** $R_{\alpha \beta \nu \mu} = -R_{\beta \alpha \nu \mu} = R_{\beta \alpha \mu \nu} = -R_{\alpha \beta \mu \nu},$

(131)

**Cyclicity:** $R_{\alpha \beta \nu \mu} + R_{\alpha \mu \beta \nu} + R_{\nu \alpha \beta \mu} = 0,$

(132)

**Bianchi identities:** $\nabla_\sigma R_{\alpha \beta \nu \mu} + \nabla_\mu R_{\alpha \beta \sigma \nu} + \nabla_\nu R_{\alpha \beta \mu \sigma} = 0.$

(133)
In 2+1 dimension, the curvature tensor can be written in terms of Ricci’s tensor \((R_{\mu\nu}^\lambda = R^{\lambda}_{\mu\lambda\sigma})\) and its trace \((R = R_{\lambda\lambda})\) in the way
\[
R_{\lambda\mu\nu\sigma} = g_{\lambda\nu} R_{\mu\sigma} - g_{\lambda\sigma} R_{\mu\nu} - g_{\mu\nu} R_{\lambda\sigma} + g_{\mu\sigma} R_{\lambda\nu} - \frac{R}{2} (g_{\lambda\nu} g_{\mu\sigma} - g_{\lambda\sigma} g_{\mu\nu}).
\]
So, the object defined in (129) takes the shape
\[
(F_{\sigma\nu})_{\lambda\mu} = \left(\frac{3}{2} + 4\beta\right) R_{\lambda\mu\nu\sigma} + (1 + 8\beta)(g_{\mu\nu} R_{\lambda\sigma} - g_{\mu\sigma} R_{\lambda\nu}) + 2\beta R (g_{\lambda\nu} g_{\mu\sigma} - g_{\lambda\sigma} g_{\mu\nu}),
\]
which do not depend on parameter \(\alpha\). Moreover, if \(\beta\) is fixed as
\[
\beta = -\frac{1}{8},
\]
then, relation (134) leads to
\[
(F_{\sigma\nu})_{\lambda\mu} \big|_{\beta=-\frac{1}{8}} = R_{\lambda\mu\nu\sigma} - \frac{R}{4} (g_{\lambda\nu} g_{\mu\sigma} - g_{\lambda\sigma} g_{\mu\nu}),
\]
and one satisfies all symmetry properties of a curvature, showing in relations (130)-(132) with the exception of the Bianchi identities, (133). It can be noted that the trace of (136), this means \((F_{\sigma\lambda})^\lambda_{\mu}\) is the Einstein’s tensor.

Next, some discussion on the critical value (135) will be performed when the connection’s field equation is taking into account. With the help of symmetry properties, Bianchi’s identities, and relationship between Riemann-Christoffel and Ricci tensor, the field equation (128) can be rewritten as follows
\[
\left(\frac{1}{2} - 4\beta\right) \nabla_\rho R_{\nu\sigma} - \left(\frac{3}{2} + 4\beta\right) \nabla_\sigma R_{\nu\rho} + \left(\frac{1}{2} + 2\beta\right) g_{\nu\rho} \partial_\sigma R + 2\beta g_{\nu\sigma} \partial_\rho R + b_{\rho\mu} \varepsilon^{\mu\nu\sigma} = 0,
\]
and with some algebraic computation on this last equation, it can be shown (for all \(\beta\)) the next symmetry property
\[
b_{\nu\mu} = b_{\mu\nu},
\]
and
\[
(\beta - \frac{5}{8}) b_{\mu\nu} = 0.
\]
Consistence condition (139) establishes that the work out of Lagrange multipliers depends on the following restriction
\[
\beta \neq -\frac{5}{8},
\]
then, \(b_{\mu\nu} = 0\). This last result means that one can consistently replace \(b_{\mu\nu} = 0\) inside the action (127) and, then the torsionless limit can be recovered through the condition \(T^{\lambda}_{\mu\nu} = 0\) imposed on the new field equations.
Condition (141) induces a wide set of possible vacuum’s solutions, including non-Einsteinian ones beside (A)dS, because eq. (140) becomes an identity when it is evaluated on the critical $\beta$ given by (135). This fact is confirmed when $\beta = -\frac{1}{8}$ is introduced in eq. (137), in other words

$$\nabla_\rho R_{\nu\sigma} - \nabla_\sigma R_{\nu\rho} = 0 ,$$

(142)

where notation means

$$R_{\mu\nu} \equiv R_{\mu\nu} - g_{\mu\nu} 4 R .$$

(143)

It can be observed that equation (142) looks like eq. (2), but here, as one can expect the trace $\sigma - \lambda$ of (142) is an identity.

In order to conclude the comments on the massless theory, next we consider the field equation which comes from variations on metric of the action (127) and it can be written in terms of Ricci’s tensor and Ricci’s scalar as follows

$$(\frac{3}{2} - \alpha + 12 \beta) R_{\sigma\nu} R^\sigma{}_{\nu} - (\frac{1}{2} - \alpha + 6 \beta) RR_{\mu\nu} - (1 - \alpha + 4 \beta) R^\sigma{}_{\rho} R_{\sigma\rho} g_{\mu\nu}$$

$$+ (\frac{5}{16} - \frac{\alpha}{2} + 2 \beta) R^2 g_{\mu\nu} + \frac{q}{2} \lambda^2 g_{\mu\nu} = 0 .$$

(144)

Immediately, the consistence with (A)dS solutions is evaluated by replacing the contractions of $R_{\rho\mu\nu\sigma} = \lambda (g_{\rho\sigma} g_{\mu\nu} - g_{\rho\nu} g_{\mu\sigma})$ in (144). This gives

$$q(\alpha) = \frac{3}{2} - 4 \alpha ,$$

(145)

and this indicates that if $\alpha = \frac{3}{8}$ is introduced in action (127) get implicit (A)dS solutions from its field equations.

Now we take a look on the gauge formulation of topological massive gravity with cosmological constant, considering the YM-extended action at the torsionless limit, this means

$$S' = S^{(3)}_{(\alpha, \beta)} + \frac{m\kappa^2}{2} \left( \varepsilon^{\mu\nu\lambda} tr \left( A_\mu \partial_\nu A_\lambda + \frac{2}{3} A_\mu A_\nu A_\lambda \right) \right) + \kappa^2 \left( q(\alpha) \lambda^2 \right)$$

$$+ \kappa^2 \left( b_{\alpha\beta} \varepsilon^{\beta\sigma\lambda} (A_\lambda)^\alpha{}_{\sigma} \right) ,$$

(146)

where $q(\alpha)$ is defined by (145) then, this action is consistent with (A)dS solutions when $m = 0$. Variations on the metric conduce to the known equations (144). So, the connection field equation is

$$\nabla_\mu (F^{\mu\nu})^\sigma{}_{\rho} + \frac{m}{2} \varepsilon^{\alpha\beta\nu} (F_{\alpha\beta})^\sigma{}_{\rho} + b_{\rho\mu} \varepsilon^{\mu\nu\sigma} = 0 ,$$

(147)

and $(F^{\mu\nu})^\sigma{}_{\rho}$ is defined in (129). Recalling that $(F_{\mu\nu})_{\alpha\beta} = R_{\alpha\beta\mu\nu}$ in a torsionless space-time, equation (147) can be rewritten in terms of Ricci’s tensor as follows

$$\left( \frac{1}{2} - 4 \beta \right) \nabla_\rho R_{\nu\sigma} - \left( \frac{3}{2} + 4 \beta \right) \nabla_\sigma R_{\nu\rho} - \left( \frac{1}{2} + 2 \beta \right) g_{\nu\rho} \partial_\sigma R + 2 \beta g_{\nu\sigma} \partial_\rho R$$

$$- m \varepsilon^{\alpha\beta\nu} (g_{\alpha\sigma} R_{\beta\rho} - g_{\alpha\rho} R_{\beta\sigma} - R g_{\alpha\sigma} g_{\beta\rho}) + b_{\rho\mu} \varepsilon^{\mu\nu\sigma} = 0 .$$

(148)
Performing some algebraic manipulation on this last equation, conditions (138) and (140), which establish the symmetry property of Lagrange multipliers and the indetermination of scalar curvature when $\beta = -\frac{1}{8}$, rise again in a similar way that they do in the massless theory.

Then, using condition (141), the Lagrange multipliers are given by

$$b_{\mu \nu} = 2 \left( \frac{\beta + \frac{1}{2}}{\beta - \frac{5}{8}} \right) m R_{\mu \nu} - \left( \frac{\beta + \frac{1}{2}}{\beta - \frac{5}{8}} \right) \frac{m R}{2} g_{\mu \nu},$$

and if (135) is fixed, the result (6) is recovered. So, evaluating the theory on $\beta = -\frac{1}{8}$, the action (148) becomes in a similar form as in (4), this means

$$\nabla_\mu R_{\sigma \lambda} - \nabla_\lambda R_{\sigma \mu} - m \varepsilon^{\rho \sigma} (g_{\lambda \nu} R_{\mu \rho} - g_{\mu \nu} R_{\lambda \rho} - \frac{2}{3} R g_{\lambda \nu} g_{\mu \rho}) = 0,$$

where $R_{\mu \nu}$ is defined as in (143) and the trace $\sigma - \lambda$ is an identity, as one can expect.

7. Conclusion

A perturbative regime based on arbitrary variations of the contortion and metric as a (classical) fixed background, is performed in the context of a pure Yang-Mills formulation of the $SO(1,2)$ gauge group. There, we analyze in detail the physical content and the well known fact that a variational principle based on the propagation of torsion (contortion), as dynamical and possible candidate for a quantum canonical description of gravity in a pure YM formulation gets serious difficulties.

In the $2 + 1$ dimensional massless case we show that the theory contains three massless degrees of freedom, one of them a non-unitary mode, considering a non dynamical background’s metric. Then, introducing appropiate quadratical terms dependent on torsion, which preserve parity and general covariance, we can see that the linearized limit do not reproduces an equivalent pure Hilbert-Einstein-Fierz-Pauli massive theory for a spin-2 mode and, moreover there is other non-unitary modes. Roughly speaking, at first sight one can blame it on the kinetic part of YM formulation because the existence of non-positive Hamiltonian connected with non-unitarity problem. Nevertheless there are other possible $F^2$ models (or simply YM-extended) which could solve the unitarity problem.

Exploring the massless and the topological massive gravity models in $2 + 1$ dimension, the well known existence of a YM-extended theories family is noted. This family is labeled with two free parameters, $\alpha$ and $\beta$ and can cure non-unitary propagations in the linearized level. Moreover, when the classical consistence between these type of theories and the Einstein’s one is tackled, what we have mentioned as torsionless limit, the relationship between parameter $\alpha$ and the shape of the coupling of the cosmological constant in the action, is shown.

Meanwhile, the parameter $\beta$ get two types of critical values. On one side, the number $\beta = \frac{5}{8}$ is connected to the classical consistence requirement which demands
the introduction of torsion’s Lagrangian constraints with solvable Lagrange multipliers. On the other side, the value $\beta = \frac{-1}{8}$ establishes a wide set of theories, including the Einstein’s solutions after the imposition of a auxiliary condition $R = \text{constant}$ and non-Einsteinian ones when the Ricci scalar became an arbitrary function in an empty space-time. But, even though the Lagrangian extension of the YM formulation for gravity conduces to the well known fact that there exists unphysical classical solutions, the same occurs (in a much less severe way) without these corrections and one can recall the YM pure formulation gives rise a set of solutions for the massless and topological massive gravity with the property $R = \text{constant}$ and only Einsteinian results can be obtained if the auxiliary condition $R = -6\lambda$ is fixed.

A generalization of the study of physical content of the YM-extended model in a non perturbative level, including a dynamical metric would be considered elsewhere.

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