GEOMETRY OF DELIGNE COHOMOLOGY

PAWEL GAJER

The aim of this paper is to give a geometric interpretation of holomorphic and smooth Deligne cohomology. Before stating the main results we recall the definition and basic properties of Deligne cohomology.

Let $X$ be a smooth complex projective variety and let $\Omega^r_X$ be the sheaf of germs of holomorphic $r$-forms on $X$. The $q$th Deligne complex of $X$ is the complex of sheaves

$$Z(q)_D : \mathbb{Z}(q)_X \longrightarrow \Omega^0_X \longrightarrow \Omega^1_X \longrightarrow \cdots \longrightarrow \Omega^{q-1}_X,$$

where $\mathbb{Z}(q) = (2\pi \sqrt{-1})^q \mathbb{Z} \subset \mathbb{C}$, and $\mathbb{Z}(q)_X$ is the constant sheaf on $X$ associated with the group $\mathbb{Z}(q)$. The hypercohomology $H^\bullet(X, Z(q)_D)$ of the complex $Z(q)_D$ is called the Deligne cohomology of $X$. Our basic reference for Deligne cohomology is [EV].

One of the key properties of Deligne cohomology is that for every $p \geq 1$ the group $H^{2p}(X, \mathbb{Z}(p)_D)$ is the extension

$$0 \longrightarrow J^p(X) \longrightarrow \mathbb{H}^{2p}(X, \mathbb{Z}(p)_D) \longrightarrow H^{2p}_{\mathbb{Z}}(X) \longrightarrow 0 \tag{1}$$

of the group $H^{2p}_{\mathbb{Z}}(X)$ of integral $(p,p)$-classes of $X$ by the the $p$th intermediate Jacobian $J^p(X)$ of Griffiths. For $p = 1$ the group $\mathbb{H}^2(M, \mathbb{Z}(1)_D)$ is isomorphic to the first cohomology group $H^1(\mathcal{O}^\ast_X)$ of the sheaf $\mathcal{O}^\ast_X$ of germs of non-vanishing holomorphic functions on $X$, and the sequence (1) reduces in this case to the classical short exact sequence

$$0 \longrightarrow J(X) \longrightarrow H^1(X, \mathcal{O}^\ast_X) \longrightarrow H^{1,1}_{\mathbb{Z}}(X) \longrightarrow 0 \tag{2}$$

It is well known that the group $H^1(X, \mathcal{O}^\ast_X)$ is isomorphic to the group $CH^1(X)$ of divisors of $X$ modulo rational equivalence, the Jacobian $J(X)$ is isomorphic to the group $CH^1_{\text{hom}}(X)$ of rational equivalence classes of homologous to 0 divisors of $X$, and $H^{1,1}_{\mathbb{Z}}(X)$ is the image of the cycle map $CH^1(X) \rightarrow H^2(X; \mathbb{Z})$. One would like to have a similar cohomological description of the groups $CH^p(X)$ of codimension $p$ cycles of $X$ modulo rational equivalence, for $p > 1$, together with an analogous to (2) short exact sequence completely describing the image $H^{2p}_{\text{alg}}(X; \mathbb{Z})$ and the kernel $CH^p_{\text{hom}}(X)$ of the cycle homomorphism

$$CH^p(X) \longrightarrow H^{2p}_{\mathbb{Z}}(X). \tag{3}$$

Date: March 19, 2022.
Deligne cohomology can be thought as a step in this direction. Indeed, the cycle homomorphism lifts to a homomorphism

\[ CH^p(X) \to H^{2p}(X; \mathbb{Z}(p)_D), \]

so that the diagram

\[
\begin{array}{cccccc}
0 & \to & CH^p_{\text{hom}}(X) & \to & CH^p(X) & \to & H^{2p}_{\text{alg}}(X; \mathbb{Z}) & \to & 0 \\
AJ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & J^p(X) & \to & \mathbb{H}^{2p}(X, \mathbb{Z}(p)_D) & \to & H^p_{\text{alg}}(X; \mathbb{Z}) & \to & 0
\end{array}
\]

commutes, where \( AJ : CH^p_{\hom}(X) \to J^p(X) \) is the Abel-Jacobi homomorphism.

Another important property of Deligne cohomology is the existance of characteristic classes, called “regulators”

\[ c_{n,p} : K^0(X) \to \mathbb{H}^{2p-n}(X, \mathbb{Z}(p)_D) \]

from the algebraic \( K \)-groups of \( X \) into the Deligne cohomology of \( X \), which generalize the classical Chern classes of holomorphic vector bundles. Several important conjectures of arithmetic algebraic geometry are formulated in terms of these regulators [Bei, RSS].

The second degree Deligne cohomology groups \( \mathbb{H}^2(X; \mathbb{Z}(q)_D) \) have the following geometric interpretations.

- \( \mathbb{H}^2(X; \mathbb{Z}(0)_D) \) is the ordinary second cohomology group \( H^2(X; \mathbb{Z}) \) of \( X \) that can be identified with the group of smooth principal \( \mathbb{C}^* \)-bundles over \( X \).
- \( \mathbb{H}^2(X; \mathbb{Z}(1)_D) \) is isomorphic to the group \( H^1(O_X) \) of isomorphism classes of holomorphic principal \( \mathbb{C}^* \)-bundles over \( X \).
- \( \mathbb{H}^2(X; \mathbb{Z}(2)_D) \) is isomorphic to the group of isomorphism classes of holomorphic principal \( \mathbb{C}^* \)-bundles over \( X \) with holomorphic connections.
- For every \( q > 2 \) the group \( \mathbb{H}^2(X; \mathbb{Z}(q)_D) \) is isomorphic to the group of isomorphism classes of holomorphic principal \( \mathbb{C}^* \)-bundles with flat connections over \( X \).

Thanks to the above description of the groups \( \mathbb{H}^2(X; \mathbb{Z}(q)_D) \) the geometric structure of regulators, cycle homomorphisms, and Abel-Jacobi homomorphisms has been completely understood in the case of divisors. It is expected that a geometric interpretation of higher degree Deligne cohomology will lead to a better understanding of regulators, cycle homomorphisms, and Abel-Jacobi homomorphisms for cycles of codimension greater than one (see [Br2]). J-L. Brylinski and P. Deligne found a geometric description of the groups \( \mathbb{H}^p(X; \mathbb{Z}(3)_D) \) in terms of holomorphic gerbes with connective structure and curving (see [Br1]). The main drawback of this description is that it very difficult to generalize to higher degrees. In this paper we show that for every \( q \geq 0 \) and \( p \geq 2 \) the Deligne cohomology groups \( \mathbb{H}^p(X; \mathbb{Z}(q)_D) \) and the intermediate Jacobians \( J^p(X) \) have geometric interpretations naturally extending the geometric description of the groups \( \mathbb{H}^2(X; \mathbb{Z}(q)_D) \) and the classical Jacobian \( J(X) \) (see Theorem D). In particular, for every smooth complex projective variety \( X \) we
give a geometric interpretation of the short exact sequence (1) that generalizes the classical diagram.

\[
\begin{array}{cccccc}
0 & \longrightarrow & J(X) & \longrightarrow & H^1(X, \mathcal{O}_X^* ) & \longrightarrow & H^{1,1}_{\mathbb{Z}}(X) & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \\
0 & \longrightarrow & \text{iso. classes of topologically trivial holomorphic } \mathbb{C}^* \text{-bundles over } X & \longrightarrow & \text{iso. classes of holomorphic principal } \mathbb{C}^* \text{-bundles over } X & \longrightarrow & 0 \\
\end{array}
\]

The description of geometry of holomorphic Deligne cohomology will be preceded by a discussion of a geometric interpretation of a smooth counterpart of Deligne cohomology. The smooth Deligne cohomology of a smooth manifold \( M \) is the hypercohomology \( H^* (M, \mathbb{Z}(q)_{\mathbb{C}}^\infty ) \) of the \( q \)th smooth Deligne complex of \( M \), which is the complex of sheaves

\[
\mathbb{Z}(q)_{\mathbb{C}}^\infty \colon \mathbb{Z}(q)_M \longrightarrow \mathcal{A}^0_{M,\mathbb{C}} \longrightarrow \cdots \longrightarrow \mathcal{A}^{q-1}_{M,\mathbb{C}},
\]

where \( \mathcal{A}_M^n \) is the sheaf of germs of smooth differential \( n \)-forms on \( M \), and \( \mathcal{A}^n_{M,\mathbb{C}} = \mathcal{A}_M^n \otimes \mathbb{C} \). Smooth Deligne cohomology groups, similarly to ordinary Deligne cohomology groups, are extensions of ordinary cohomology groups. Moreover, they can be identified with Cheeger-Simons differential characters groups \([CS]\). Our basic reference for smooth Deligne cohomology and the pertinent homological algebra of sheaves is \([Br1]\).

Degree two smooth Deligne cohomology groups have similar to holomorphic Deligne cohomology groups geometric interpretations with smooth principal \( \mathbb{C}^* \)-bundles and smooth connections taking place of holomorphic line bundles and holomorphic connections. The short exact sequence (1) has the following two counterparts

\[
(4) \quad 0 \longrightarrow \frac{A^p_{\mathbb{C}}(M)}{A^p_{\mathbb{C}}(M)_0} \longrightarrow \mathbb{H}^p(M, \mathbb{Z}(q)_{\mathbb{C}}^\infty ) \longrightarrow H^p(M, \mathbb{Z}) \longrightarrow 0
\]

and for \( p < q \)

\[
(5) \quad 0 \longrightarrow \frac{H^{p-1}(M, \mathbb{C})}{H^{p-1}(M, \mathbb{Z})_{TF}} \longrightarrow \mathbb{H}^p(M, \mathbb{Z}(q)_{\mathbb{C}}^\infty ) \longrightarrow \text{Tors } H^p(M, \mathbb{Z}) \longrightarrow 0
\]

where \( A^p_{\mathbb{C}}(M) \) is the group of \( \mathbb{C} \)-valued \( (p-1) \)-forms on \( M \), \( A^p_{\mathbb{C}}(M)_0 \) is the subgroup of \( A^p_{\mathbb{C}}(M) \) consisting of closed \( (p-1) \)-forms with integral periods, the group \( H^{p-1}(M, \mathbb{Z})_{TF} \) is the image of \( H^p(M, \mathbb{Z}) \) in \( H^p(M, \mathbb{C}) \), and \( \text{Tors } H^p(M, \mathbb{Z}) \) is the torsion part of the group \( H^p(M, \mathbb{Z}) \). For \( p = 2 \) these sequences have the following geometric interpretations.
Proposition A. For every smooth manifold $M$ there is a commutative diagram

$$
\begin{array}{cccc}
0 & \longrightarrow & A^{1}_{C}(M) & \longrightarrow \ H^{2}(M, \mathbb{Z}(2)_{D}^{\infty}) & \longrightarrow & H^{2}(M, \mathbb{Z}) & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & \\
0 & \longrightarrow & \text{iso. classes of} & \text{iso. classes of} & \text{iso. classes of} & \longrightarrow & 0 \\
& & \text{connections on} & \text{smooth principal} & \text{smooth principal} & & \\
& & \mathbb{C}^{*} \times M \to M & \mathbb{C}^{*}\text{-bundles with connections} & \mathbb{C}^{*}\text{-bundles} & & \\
& & & \text{over } M & \text{over } M & & \\
\end{array}
$$

with exact rows and the vertical arrows isomorphisms.

Proposition B. For every smooth manifold $M$ and every $q > 2$ there exists a commutative diagram

$$
\begin{array}{cccc}
0 & \longrightarrow & H^{1}(M, \mathbb{C}) & \longrightarrow \ H^{2}(M, \mathbb{Z}(q)_{D}^{\infty}) & \longrightarrow & \text{Tors}H^{2}(M, \mathbb{Z}) & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & \\
0 & \longrightarrow & \text{iso. classes of} & \text{iso. classes of} & \text{iso. classes of} & \longrightarrow & 0 \\
& & \text{flat connections} & \text{smooth principal} & \text{smooth principal} & & \\
& & \mathbb{C}^{*}\text{-bundles with} & \mathbb{C}^{*}\text{-bundles} & \mathbb{C}^{*}\text{-bundles} & & \\
& & \text{flat connections} & \text{over } M & \text{over } M & & \\
& & & \text{admitting} & & & \\
& & & \text{flat connections} & & & \\
\end{array}
$$

with exact rows and the vertical arrows isomorphisms.

Smooth Deligne cohomology is a natural framework for formulation of the Weil-Kostant Integrality Theorem (see Theorems 2.2.14 and 2.2.15 in [Br1]), which is essentially equivalent to the following result.

Proposition C. For every smooth manifold $M$ there is a commutative diagram

$$
\begin{array}{cccc}
0 & \longrightarrow & H^{1}(M, \mathbb{C}^{*}) & \longrightarrow \ H^{2}(M, \mathbb{Z}(2)_{D}^{\infty}) & \longrightarrow & A^{2}_{C}(M)_{0} & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & \\
0 & \longrightarrow & \text{iso. classes of} & \text{iso. classes of} & \text{curvatures of} & \longrightarrow & 0 \\
& & \text{flat connections} & \text{smooth principal} & \text{connections} & & \\
& & \text{on smooth} & \mathbb{C}^{*}\text{-bundles} & \text{on smooth} & & \\
& & \text{principal} & \text{with connections} & \text{principal} & & \\
& & \mathbb{C}^{*}\text{-bundles} & \text{over } M & \mathbb{C}^{*}\text{-bundles} & & \\
& & & \text{over } M & \text{over } M & & \\
\end{array}
$$

with exact rows and the vertical arrows isomorphisms.

Later on we will describe higher order analogues of the Weil-Kostant Integrality Theorem.

Since $\mathbb{Z}(0)_{D}^{\infty} = \mathbb{Z}(0)_{D} = \mathbb{Z}_{M}$ both smooth and holomorphic Deligne cohomology specialize to ordinary cohomology. Therefore, a geometric interpretation of Deligne
cohomology induces a geometric interpretation of ordinary cohomology. Actually, it is natural to start from a geometric description of ordinary cohomology, and then enhance it to get a geometric model of smooth and holomorphic Deligne cohomology. We proceed as follows.

The interpretation of $H^2(M, \mathbb{Z})$ as the group of isomorphism classes of smooth principal $\mathbb{C}^*$-bundles over $M$ comes from the isomorphism

$$H^1(\mathbb{C}_M^*) \cong H^2(M, \mathbb{Z}),$$

given by the coboundary homomorphism in the cohomology long exact sequence associated with the exponential short exact sequence

$$0 \rightarrow \mathbb{Z}_M \xrightarrow{\cdot 2\pi i} \mathbb{C}_M \xrightarrow{\exp} \mathbb{C}_M^* \rightarrow 0$$

where for any Lie group $G$ the symbol $G_M$ stands for the sheaf of germs of smooth $G$-valued functions on $M$.

A geometric interpretation of the groups $H^p(M, \mathbb{Z})$, for $p > 2$, is derived from a generalized exponential sequence, which is constructed as follows. For every $s \geq 1$, the iterated classifying space of the group $\mathbb{C}^*$

$$B^s\mathbb{C}^* = B(\cdots B(\mathbb{C}^*) \cdots), \quad s \text{ times}$$

can be equipped with a differentiable space structure so that the homomorphisms in the short exact sequence

$$0 \rightarrow B^{s-1}\mathbb{C}^* \rightarrow EB^{s-1}\mathbb{C}^* \rightarrow B^s\mathbb{C}^* \rightarrow 0$$

are smooth maps. The composition of these short exact sequences induces an acyclic resolution of the group $\mathbb{Z}$

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{C} \rightarrow \mathbb{E}\mathbb{C}^* \rightarrow \mathbb{E}\mathbb{B}\mathbb{C}^* \rightarrow \mathbb{E}\mathbb{B}^2\mathbb{C}^* \rightarrow \cdots \rightarrow \mathbb{E}\mathbb{B}^n\mathbb{C}^* \rightarrow \cdots$$

which in turn induces an acyclic bar resolution of the sheaf $\mathbb{Z}_M$

$$0 \rightarrow \mathbb{Z}_M \rightarrow \mathbb{C}_M \rightarrow \mathbb{E}\mathbb{C}_M^* \rightarrow \mathbb{E}\mathbb{B}\mathbb{C}_M^* \rightarrow \mathbb{E}\mathbb{B}^2\mathbb{C}_M^* \rightarrow \cdots \rightarrow \mathbb{E}\mathbb{B}^n\mathbb{C}_M^* \rightarrow \cdots$$

where $\mathbb{E}\mathbb{B}^n\mathbb{C}_M^*$ stands for the sheaf of smooth $\mathbb{E}\mathbb{B}^n\mathbb{C}^*$-valued functions on $M$. The bar resolution of $\mathbb{Z}_M$ is a special case of a construction that assigns to a sheaf $\mathcal{F}$ an acyclic resolution of $\mathcal{F}$

$$0 \rightarrow \mathcal{F} \rightarrow \mathbb{E}\mathcal{F} \rightarrow \mathbb{E}\mathbb{B}\mathcal{F} \rightarrow \mathbb{E}\mathbb{B}^2\mathcal{F} \rightarrow \cdots \rightarrow \mathbb{E}\mathbb{B}^n\mathcal{F} \rightarrow \cdots$$
The truncation of the bar resolution of the sheaf $\mathcal{Z}_M$ in degree $p - 2$ induces the generalized exponential short exact sequence

$$
\begin{array}{c}
EB^{p-3}C^*_M \\ \downarrow \uparrow \\
EB^{p-2}C^*_M \\ \downarrow \uparrow \\
EB^pC^*_M \\ \downarrow \uparrow \\
EB^cC^*_M \\ \downarrow \uparrow \\
EB^p\Omega^c \\
0 \longrightarrow \mathcal{Z}_M \longrightarrow \mathcal{C}_M
\end{array}
$$

We show that the coboundary homomorphism

$$\delta : H^1(B^{p-2}C^*_M) = H^{p-1}(B^{p-2}C^*_M [-p + 2]) \longrightarrow H^p(\mathcal{Z}_M)$$

in the cohomology long exact sequence associated with the generalized exponential sequence is an isomorphism. Since $H^1(B^{p-2}C^*_M)$ is isomorphic to the group of isomorphism classes of smooth principal $B^{p-2}C^*$-bundles over $M$, we get the following result.

**Theorem H.** For every smooth manifold $M$ the group $H^p(M; \mathbb{Z})$ is isomorphic to the group of isomorphism classes of smooth principal $B^{p-2}C^*$-bundles over $M$.

Theorem H can be viewed as a generalization of J. Giraud’s result that identifies $H^3(M; \mathbb{Z})$ with the group of equivalence classes of gerbes bound by $\mathcal{C}_M$ (see [5] and [Br], Theorem 5.2.8). A correspondence between smooth principal $BC^*$-bundles and gerbes bound by $\mathcal{C}_M^*$ is explained in Appendix A.

It is natural to expect that the groups $\mathbb{H}^p(M, \mathcal{C}(q)|^*_D)$ have a description in terms of connections on smooth principal $B^{p-2}C^*$-bundles over $M$. Indeed, we show that for every $s \geq 1$ the group $B^sC^*$ is equipped with a $B^sC$-valued $B^sC^*$-equivariant connection 1-form that can be used to define connections on smooth principal $B^sC^*$-bundles. It turns out that for $q > p$ the group $\mathbb{H}^p(M, \mathcal{C}(q)|^*_D)$ is isomorphic to the group of isomorphism classes of smooth principal $B^{p-2}C^*$-bundles with flat connections over $M$.

A connection 1-form on a smooth principal $B^sC^*$-bundle over $M$ has higher degree analogues called $k$-connections, $k = 2, \ldots, s + 1$, which are defined inductively so that a relationship between the $(k + 1)$ and $k$-connections is analogous to the relation between a connection on a principal $B^sC^*$-bundle and a set of transition functions of this bundle. The relation of isomorphism of principal $C^*$-bundles with connections can be generalized to an equivalence relation on the set of $B^sC^*$-bundles with $k$-connections such that for every $p \geq 2$ the smooth Deligne cohomology group...
\( \mathbb{H}^p(M, \mathbb{Z}(p)_{\Sigma}) \) is isomorphic to the group of equivalence classes of smooth principal \( B^{p-2}\mathbb{C}^* \)-bundles with \( k \)-connections, \( k = 1, \ldots, p-1 \), over \( M \).

The following two theorems give a geometric interpretation of the short exact sequences (4) and (5).

**Theorem A.** For every smooth manifold \( M \) and every \( p \geq 2 \) there is a commutative diagram

\[
\begin{array}{cccccc}
0 & \to & \mathbb{A}^{p-1}_{C}(M) & \to & \mathbb{A}^{p-1}_{C}(M)_0 & \to & \mathbb{H}^p(M, \mathbb{Z}(p)_{\Sigma}) & \to & H^p(M, \mathbb{Z}) & \to & 0 \\
& & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
0 & \to & \left\{ \text{equivalence classes} \right\} & \to & \left\{ \text{equivalence classes} \right\} & \to & \left\{ \text{iso. classes of smooth principal} \right\} & \to & 0 \\
& & \left\{ \text{of } k \text{-connections} \right\} & \to & \left\{ \text{of smooth principal} \right\} & \to & \left\{ \text{of smooth principal} \right\} & \to & 0 \\
& & \left\{ \text{on the trivial} \right\} & \to & \left\{ \text{with } k \text{-connections} \right\} & \to & \left\{ \text{with } k \text{-connections} \right\} & \to & 0 \\
& & \left\{ \text{\( B^{p-2}\mathbb{C}^* \)-bundle} \right\} & \to & \left\{ \text{\( B^{p-2}\mathbb{C}^* \)-bundle} \right\} & \to & \left\{ \text{\( B^{p-2}\mathbb{C}^* \)-bundle} \right\} & \to & 0 \\
& & \left\{ \text{over } M \right\} & \to & \left\{ \text{over } M \right\} & \to & \left\{ \text{over } M \right\} & \to & 0 \\
\end{array}
\]

with exact rows and the vertical arrows isomorphisms.

Theorem A generalizes the description due to J-L. Brylinski and P. Deligne of the smooth Deligne cohomology group \( \mathbb{H}^3(M, \mathbb{Z}(3)_{\Sigma}) \) in terms of equivalence classes of gerbes bound by \( \mathbb{C}^* \) with connective structures and curving (see [Br1]). A procedure associating with a connection on a smooth principal \( BC^* \)-bundle \( E \to M \) a connective structure on the associated with \( E \to M \) gerbe is described in Appendix A.

**Theorem B.** For every smooth manifold \( M \) and every \( q > p \geq 2 \) there exists a commutative diagram

\[
\begin{array}{cccccc}
0 & \to & H^{p-1}(M, \mathbb{C}) & \to & \mathbb{H}^{p-1}(M, \mathbb{Z}_{\Sigma}) & \to & \mathbb{H}^p(M, \mathbb{Z}(q)_{\Sigma}) & \to & \text{TorsH}^p(M, \mathbb{Z}) & \to & 0 \\
& & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
0 & \to & \left\{ \text{iso. classes of} \right\} & \to & \left\{ \text{iso. classes of} \right\} & \to & \left\{ \text{iso. classes of} \right\} & \to & 0 \\
& & \left\{ \text{flat connections} \right\} & \to & \left\{ \text{smooth principal} \right\} & \to & \left\{ \text{smooth principal} \right\} & \to & 0 \\
& & \left\{ \text{on the trivial} \right\} & \to & \left\{ \text{with } k \text{-connections} \right\} & \to & \left\{ \text{with } k \text{-connections} \right\} & \to & 0 \\
& & \left\{ \text{\( B^{p-2}\mathbb{C}^* \)-bundle} \right\} & \to & \left\{ \text{\( B^{p-2}\mathbb{C}^* \)-bundle} \right\} & \to & \left\{ \text{\( B^{p-2}\mathbb{C}^* \)-bundle} \right\} & \to & 0 \\
& & \left\{ \text{over } M \right\} & \to & \left\{ \text{over } M \right\} & \to & \left\{ \text{over } M \right\} & \to & 0 \\
\end{array}
\]

with exact rows and the vertical arrows isomorphisms.

To an equivalence class \([E, \omega_1, \ldots, \omega_{p-1}]\) of a smooth principal \( B^{p-2}\mathbb{C}^* \)-bundle \( E \to M \) with \( k \)-connections \(-\omega_1, \ldots, (-1)^{p-1}\omega_{p-1}\) one can assign a scalar curvature

\[
s([E, \omega_1, \ldots, \omega_{p-1}]) = (-1)^{p-1}d\omega_{p-1},
\]

which is a \( \mathbb{C} \)-valued differential \( p \)-form on \( M \). We show that, if the scalar curvature of \([E, \omega_1, \ldots, \omega_{p-1}]\) is zero, then the sequence \((E, \omega_1, \ldots, \omega_{p-1})\) is equivalent to a
unique up to isomorphism sequence \((E, \omega', 0, \ldots, 0)\) with \(-\omega'\) being a flat connection on \(E \to M\). This gives a geometric interpretation of the short exact sequence

\[
0 \longrightarrow H^{p-1}(M, \mathbb{C}^*) \longrightarrow \mathbb{H}^p(M, \mathbb{Z}(p) \mathbb{D}) \longrightarrow A^p_{\mathbb{C}}(M)_0 \longrightarrow 0.
\]

**Theorem C.** For every smooth manifold \(M\) and every \(p \geq 2\) there is a commutative diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & H^{p-1}(M, \mathbb{C}^*) & \rightarrow & \mathbb{H}^p(M, \mathbb{Z}(p) \mathbb{D}) & \rightarrow & A^p_{\mathbb{C}}(M)_0 & \rightarrow & 0 \\
\uparrow & & \uparrow \cong & & \uparrow \cong & & \downarrow & & \\
0 & \rightarrow & \text{iso. classes of flat connections} & \rightarrow & \text{equivalence classes of smooth principal } B^{p-2}\mathbb{C}^*\text{-bundles} & \rightarrow & \text{scalar curvatures} & \rightarrow & 0 \\
& & \text{on smooth principal } B^{p-2}\mathbb{C}^*\text{-bundles over } M & & \text{with } k\text{-connections } k = 1, \ldots, p-1 & & \text{over } M & & \\
\end{array}
\]

with exact rows and the vertical arrows isomorphisms.

The above diagram shows that scalar curvatures are closed forms with integral periods, and that every closed form with integral periods is a scalar curvature of a connection on a smooth principal \(B^{s}\mathbb{C}^*\)-bundle. This generalizes the classical Weil-Kostant Integrality Theorem.

If one considers in the place of the smooth Deligne complex

\[
\mathbb{Z}(q)_M \rightarrow \mathbb{C}_M \xrightarrow{d} \mathcal{A}^1_{M, \mathbb{C}} \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{A}^{q-1}_{M, \mathbb{C}}
\]

the complex

\[
\mathbb{Z}(q)_M \rightarrow i\mathbb{R}_M \xrightarrow{d} i\mathcal{A}^1_{M, \mathbb{C}} \xrightarrow{d} \cdots \xrightarrow{d} i\mathcal{A}^{q-1}_{M, \mathbb{C}},
\]

where \(i = \sqrt{-1}\), then the hypercohomology groups of the last complex have essentially the same geometric interpretation as \(H^p(M, \mathbb{Z}(p) \mathbb{D})\), with the only difference being that one has to replace everywhere \(\mathbb{C}^*\) by the unit circle.

As was mentioned before the groups \(H^2(X, \mathbb{Z}(q)_D)\) and \(H^2(M, \mathbb{Z}(q) \mathbb{D})\) have similar geometric descriptions, with the only difference being that \(H^2(M, \mathbb{Z}(q) \mathbb{D})\) is described in terms of smooth principal \(\mathbb{C}^*\)-bundles and smooth connections and \(H^2(X, \mathbb{Z}(q)_D)\) is described in terms of holomorphic principal \(\mathbb{C}^*\)-bundles and holomorphic connections. Exactly the same phenomenon takes place in higher degrees. We define holomorphic principal \(B^{s}\mathbb{C}^*\)-bundles and holomorphic \(k\)-connections on them and prove the following holomorphic analogue of Theorem A.
**Theorem D.** For every smooth complex projective variety $X$ and every $p \geq 2$ and $q > 0$ the group $H^q(X, \mathbb{Z}(q)_D)$ is isomorphic to the group of equivalence classes of holomorphic principal $B^{p-2}\mathbb{C}^*$-bundles over $X$ with holomorphic $k$-connections, for $k = 1, 2, \ldots, q - 1$. Moreover, there is a commutative diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & J^p(X) \\
\downarrow \cong & & \downarrow \cong \\
\text{equivalence classes of holomorphic } & \text{equivalence classes of holomorphic principal } & \text{iso. classes of smooth principal } \\
k = 1, 2, \ldots, p - 1 & \text{principal } \mathbb{B}^{2(p-1)\mathbb{C}^*}\text{-bundles with } k\text{-connections} & \mathbb{B}^{2(p-1)\mathbb{C}^*}\text{-bundles over } X, \text{admitting } \\
on the topologically trivial, holomorphic & \text{over } X & \text{holomorphic structures} \\
\mathbb{B}^{2(p-1)\mathbb{C}^*}\text{-bundles over } X & \text{over } X & 0 \\
0 & \longrightarrow & \mathbb{H}^{2p}(X, \mathbb{Z}(p)_D) \\
\downarrow \cong & & \downarrow \cong \\
H^p_{\mathbb{Z}}(X) & \longrightarrow & 0
\end{array}
$$

with exact rows and the vertical arrows isomorphisms.

The paper is organized as follows. In Section 1 we define a differentiable space structure on $B^s\mathbb{C}^*$ for every $s \geq 1$. In Section 2 we discuss bar resolutions of sheaves. Section 3 is devoted to smooth principal $B^s\mathbb{G}$-bundles. There we show that for every smooth manifold $M$ the group $H^p(M; \mathbb{Z})$ is isomorphic to the group of isomorphism classes of smooth principal $B^{p-2}\mathbb{C}^*$-bundles over $M$. In Section 4 we study connections on smooth principal $B^s\mathbb{G}$-bundles and prove Theorem B. In Section 5 we define $k$-connections and prove Theorems A and C. In Section 6 we prove Theorem D. In Appendix A we discuss correspondence between $B\mathbb{C}^*$-bundles with connections and gerbes with connective structures. Appendix B has been included for the convenience of the readers not familiar with the geometric bar construction. In this appendix we review basic properties of the geometric bar construction, and discuss a geometric meaning of the relations appearing in the standard definition of the classifying space $BG$.

**Acknowledgments**

The inspiration to the my work on geometry of Deligne cohomology came from lectures of Paulo Lima-Filho on Deligne cohomology at Texas A&M University. Several conversations with Paulo were very helpful in the early development of this work for which I am grateful to him. An excellent introduction to the geometry of Deligne cohomology is Jean-Luc Brylinski’s book [Br]. The examples and ideas contained in this book served me as guiding principles in the studies on the structure of smooth and holomorphic principal $B^s\mathbb{C}^*$-bundles.

1. **Differentiable structures on $EB^s\mathbb{G}$ and $B^{s+1}\mathbb{G}$**

In this section we define for any abelian Lie group $G$ a differentiable space structure on the spaces $EB^s\mathbb{G}$ and $B^{s+1}\mathbb{G}$ for every $s \geq 1$. For the definition and basic properties of the geometric bar construction we refer the reader to Appendix B.
Let $M.$ be a simplicial smooth manifold. That is, $M.$ consists of a family $\{M_n\}_{n \in \mathbb{N}}$ of smooth manifolds, together with smooth maps
\[
\partial_i : M_n \to M_{n-1}, \quad s_i : M_n \to M_{n+1},
\]
where $i = 0, 1, \ldots, n,$ satisfying the identities
\[
\begin{align*}
\partial_i \partial_j &= \partial_j - \partial_{i-1} \quad &\text{for} & i < j, \\
s_i s_j &= s_i + 1 \quad &\text{for} & i = j, j + 1, \\
\partial_i s_j &= \begin{cases} 
\partial_j & \text{for} & i < j, \\
id_{|M_n} & \text{for} & i = j, j + 1, \\
\partial_{j-1} & \text{for} & i > j + 1
\end{cases}
\end{align*}
\]

The geometric realization $|M.|$ of $M.$ is the quotient space of the disjoint union
\[
\bigsqcup_{n \geq 0} \Delta^n \times M_n
\]
with respect to the equivalence relation $\sim$ generated by the relations
\[
\begin{align*}
(\partial^j x, m) &\sim (x, \partial_j m) \quad &\text{for} & (x, m) \in \Delta^{n-1} \times M_n, \\
(s^i x, m) &\sim (x, s_i m) \quad &\text{for} & (x, m) \in \Delta^{n+1} \times M_n,
\end{align*}
\]
where the maps $\partial^j : \Delta^{n-1} \to \Delta^n$ and $s^i : \Delta^{n+1} \to \Delta^n$ are defined in the baricentric coordinates by
\[
\begin{align*}
\partial^j(x_0, \ldots, x_{n-1}) &= (x_0, \ldots, x_{i-1}, 0, x_i, \ldots, x_{n-1}), \\
s^i(x_0, \ldots, x_{n+1}) &= (x_0, \ldots, x_{i-1}, x_i + x_{i+1}, x_{i+2}, \ldots, x_{n+1}).
\end{align*}
\]

A differentiable space structure on the geometric realization $|M.|$ of $M.$ consists of the class of all smooth $\mathbb{R}$-valued functions on $|M.|$. We say that a function $f : |M.| \to \mathbb{R}$ is smooth if the composition
\[
\bigsqcup_{n \geq 0} \Delta^n \times M_n \xrightarrow{q} |M.| \xrightarrow{f} \mathbb{R}
\]
is smooth$^1,$ where $q$ is the quotient map. Equivalently, a smooth $\mathbb{R}$-valued function on $|M.|$ is given by a family of smooth maps $f^n : \Delta^n \times M_n \to \mathbb{R}$ such that for every $n \geq 0$ the following two diagrams commute
\[
\begin{array}{ll}
\begin{array}{ccc}
\Delta^n \times M_n & \xrightarrow{f^n} & \mathbb{R} \\
\partial^i \times \text{id} & \downarrow & \quad \quad \quad \downarrow f^{n-1} \\
\Delta^{n-1} \times M_n & \xrightarrow{\partial^i \times \text{id}} & \Delta^{n-1} \times M_{n-1}
\end{array} & \\
\begin{array}{ccc}
\Delta^n \times M_n & \xrightarrow{f^n} & \mathbb{R} \\
s^i \times \text{id} & \downarrow & \quad \quad \quad \downarrow f^{n+1} \\
\Delta^{n+1} \times M_n & \xrightarrow{s^i \times \text{id}} & \Delta^{n+1} \times M_{n+1}
\end{array}
\end{array}
\]

$^1$A map $g : \Delta^n \times M_n \to \mathbb{R}$ is smooth at $x \in \partial \Delta^n \times M_n$, if there is an open neighborhood of $x$ in $H^n \times M_n$, where $H^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \sum x_i = 1\}$, and a smooth function $\tilde{g} : U \to \mathbb{R}$ which restricted to $U \cap (\Delta^n \times M_n)$ coincides with $g$. 
Let $M$ and $N$ are differentiable spaces (that is $M$ and $N$ are spaces equipped with the appropriately defined classes of smooth functions). Then a map $f : M \to N$ is called a smooth map if for every smooth function $g : N \to \mathbb{R}$ the composition $g \circ f : M \to \mathbb{R}$ is a smooth map.

It is easy to see that if $f : M \to N$ is a simplicial smooth map between simplicial smooth manifolds $M$ and $N$, then $f$ induces a smooth map $|f| : |M| \to |N|$ between the geometric realizations of $M$ and $N$ respectively.

**Example 1.1.** With every Lie group $G$ there are associate simplicial smooth manifolds $G$, $EG$, and $BG$ with simplicial smooth maps $G \to EG \to BG$. whose geometric realizations give a universal principal $G$-bundle $G \to EG \to BG$.

Thus, the inclusion $G \to EG$ and the projection $EG \to BG$ are smooth maps.

**Lemma 1.1.** Let $V$ be a vector space over a field $k$. Then $EV$ and $BV$, taken with respect to the additive group structure of $V$, are $k$-vector spaces with respect to the following multiplication by scalars

\[
\begin{align*}
    k \times EV &\to EV, \quad c \cdot [t_1, \ldots, t_n, v_0|v_1| \cdots |v_n]| = [t_1, \ldots, t_n, cv_0|cv_1| \cdots |cv_n]| \\
    k \times BV &\to BV, \quad c \cdot [t_1, \ldots, t_n, [v_1|v_1] \cdots [v_n]] = [t_1, \ldots, t_n, [cv_1|cv_1] \cdots [cv_n]]
\end{align*}
\]

Moreover, the projection $EV \to BV$ is a linear map.

The proof of Lemma 1.1 is an easy exercise which we leave for the reader.

**Example 1.2.** Let $V$ be a separable $\mathbb{C}$-vector space. It is easy to see that the homomorphism

\[
l : EV \to V, \quad l([x_0, \ldots, x_n, v_0, \ldots, v_n]) = \sum_{i=0}^{n} x_i v_i
\]

is a splitting of the short exact sequence

\[
0 \to V \to EV \to BV \to 0
\]

We will show that $l : EV \to V$ is a smooth map.

Let $e_0, \ldots, e_n, \ldots$ be a base of $V$ and let $\pi_n : V \to \mathbb{C}$ be the projection on the subspace span by $e_n$. To prove smoothness of $l : EV \to V$ it is enough to show that for every $k \geq 0$ the composition

\[
EV \xrightarrow{l} V \xrightarrow{\pi_k} \mathbb{C}
\]

is smooth. But

\[
\pi_k(l([x_0, \ldots, x_n, v_0, \ldots, v_n])) = \pi_k(\sum_{i=0}^{n} x_i v_i) = \sum_{i=0}^{n} x_i v_i, e_k = \sum_{i=0}^{n} x_i < v_i, e_k
\]

is a smooth map on $EV$. Hence $l : EV \to V$ is smooth.
Let $G$ be an abelian Lie group. A differentiable space structure on $EB^sG$ and $B^{s+1}G$, for $s \geq 1$, is defined by the following inductive procedure.

Suppose we have a notion of a smooth function on $B^sG$ as well as on each product $\Delta^k \times (B^sG)^m$ for $k, m \geq 0$. Then $f : EB^sG \to \mathbb{R}$ is smooth if the composition
\[
\prod_{n \geq 0} \Delta^n \times (B^sG)^{n+1} \xrightarrow{qE} EB^sG \xrightarrow{f} \mathbb{R}
\]
is smooth and $f : B^{s+1}G \to \mathbb{R}$ is smooth if the composition
\[
\prod_{n \geq 0} \Delta^n \times (B^{s+1}G)^n \xrightarrow{q_0} B^{s+1}G \xrightarrow{f} \mathbb{R}
\]
is smooth. A function $f : \Delta^k \times (B^{s+1}G)^m \to \mathbb{R}$ is smooth if the composition
\[
\Delta^k \times (\prod_{n \geq 0} \Delta^n \times (B^{s+1}G)^n)^m \xrightarrow{id \times (q_0)^m} \Delta^k \times (B^{s+1}G)^m \xrightarrow{f} \mathbb{R}
\]
is smooth.

Directly from the above definition of differentiable structures on $B^{s+1}G$ and $EB^sG$ it follows that all maps in the short exact sequence
\[
0 \to B^sG \to EB^sG \to B^{s+1}G \to 0
\]
are smooth. It is also not difficult to see that the map
\[
B^sG \times B^sG \to B^sG, \quad (g, h) \mapsto gh^{-1}
\]
is smooth.

A group $G$ carrying a differentiable space structure so that the map
\[
G \times G \to G, \quad (g, h) \mapsto gh^{-1}
\]
is smooth is called a differentiable group.

**Example 1.3.** For every differentiable group $G$, there is a smooth deformational retraction $r : EG \times I \to EG$ of $EG$ to $e \in EG$ which is a minor modification of the standard contraction from [Mm]. In particular, if $G = B^sC^*$, then for every $s \geq 1$ there is a smooth deformational retraction $r : EB^sC^* \times I \to EB^sC^*$.

The map $r$ is represented by the family of maps
\[
r_n : (EG)_n \times I \to (EG)_{n+1},
\]
where
\[
(EG)_n = qE\left(\prod_{i \leq n} \Delta^i \times G^{i+1}\right) \subset EG,
\]
and $r_n$ is defined by the formula
\[
r_n(t_1, \ldots, t_n, h_0[h_1| \ldots |h_n], t) = \left[\Phi(0, t), \Phi(t_1, t), \ldots, \Phi(t_n, t), [h_0|h_1| \ldots |h_n]\right],
\]
where $\Phi : [0,1]^2 \to [0,1]$ is the composition
\[
\Phi(x, t) = \phi(\min(1, x + t))
\]
with \( \phi : [0, 1] \to [0, 1] \) being a smooth nondecreasing function so that \( \phi(0) = 0 \) and \( \phi(1) = 1 \).

The contraction \( r : EG \times I \to EG \) is a smooth map, because for every smooth function \( g : EG \to \mathbb{R} \) the diagram

\[
\begin{array}{ccc}
(\Delta^n \times G^{n+1}) \times I & \xrightarrow{\tilde{r}_n} & \Delta^{n+1} \times G^{n+2} \\
q_n \times \text{id} & \downarrow & \downarrow q_B \\
EG \times I & \xrightarrow{r} & EG \xrightarrow{g} \mathbb{R}
\end{array}
\]

commutes, where

\[
\tilde{r}_n : (\Delta^n \times G^{n+1}) \times I \to \Delta^{n+1} \times G^{n+2}
\]

is a smooth map defined by the formula

\[
\tilde{r}_n(t_1, \ldots, t_n, h_0, h_1, \ldots, h_n, t) = (\Phi(0, t), \Phi(t_1, t), \ldots, \Phi(t_n, t), e, h_0, h_1, \ldots, h_n).
\]

**Lemma 1.2.** Suppose \( M \) is a smooth manifold and \( G \) is a differentiable group. If \( f : M \to BG \) is a map so that for every \( x \in M \) there is an open neighborhood \( U \) of \( x \) in \( M \) so that \( f \) restricted to \( U \) is of the form

\[
f = [f_0, f_1, \ldots, f_n, [g_1] \ldots [g_n]]
\]

with \( f_0, f_1, \ldots, f_n, g_1, \ldots, g_n \) being smooth maps, then \( f \) is a smooth map.

**Proof.** Suppose \( g : BG \to \mathbb{R} \) is a smooth map. Thus, for every \( n \geq 1 \) the composition

\[
\Delta^n \times G^n \xrightarrow{q_B} BG \xrightarrow{g} \mathbb{R}
\]

is smooth. Consider the commutative diagram

\[
\begin{array}{ccc}
M & \xrightarrow{f} & BG \xrightarrow{g} \mathbb{R} \\
\downarrow f & & \downarrow g \\
\Delta^n \times G^n & \xrightarrow{q_B} & \mathbb{R}
\end{array}
\]

where \( \tilde{f} = (f_0, f_1, \ldots, f_n, g_1, \ldots, g_n) \). Since both \( \tilde{f} \) and \( q_B \circ g \) are smooth, the composition \( f \circ g = \tilde{f} \circ q_B \circ g \) is smooth as well. Thus \( f : M \to BG \) is a smooth map.

\( \square \)

2. Bar resolutions of sheaves

The key to the geometric interpretations of the cohomology groups from Theorems A, B, C, and D is the following construction of a bar resolution of a sheaf.

Let \( G \) be an abelian group. The composition of the short exact sequences

\[
0 \to B^nG \to EB^nG \to B^{n+1}G \to 0
\]

(6)
induces the long exact sequence
\[
0 \to G \to EG \xrightarrow{\sigma} EBG \xrightarrow{\sigma} EB^2G \xrightarrow{\sigma} \cdots \xrightarrow{\sigma} EB^nG \xrightarrow{\sigma} \cdots
\]
where for every \( n \geq 0 \) the homomorphism
\[
\sigma : EB^nG \to EB^{n+1}G
\]
is the composition
\[
EB^nG \to B^{n+1}G \to EB^{n+1}G
\]
of the surjection \( EB^nG \to B^{n+1}G \) and the monomorphism \( B^{n+1}G \to EB^{n+1}G \).

If \( G \) is an abelian Lie (or differentiable) group, then, as we saw in Example 1.1, the short exact sequence \( (\mathbb{3}) \) is a smooth \( B^*G \)-extension of \( B^{*+1}G \) (that is both \( B^*G \to EB^*G \) and \( EB^*G \to B^{*+1}G \) are smooth homomorphisms). Hence, the long exact sequence \( (\mathbb{3}) \) induces the long exact sequence of sheaves
\[
0 \to G_M \to EG_M \xrightarrow{\sigma} EB_G M \xrightarrow{\sigma} EB^2G_M \xrightarrow{\sigma} \cdots \xrightarrow{\sigma} EB^nG_M \xrightarrow{\sigma} \cdots
\]
which will be called the bar resolution of the sheaf \( G_M \).

**Proposition 2.1.** The sequence \( (\mathbb{8}) \) is an acyclic resolution of the sheaf \( G_M \).

**Proof.** It is enough to show that for every differentiable group \( G \) the group \( H^i(EG_M) \) is trivial, for every \( i > 0 \).

Recall, that a sheaf \( F \) on \( X \) is soft if for every closed subset \( Z \) of \( X \) the restriction map \( F(X) \to F(Z) \) is a surjection. If \( X \) is a paracompact space and \( F \) is a soft sheaf on \( X \), then \( H^i(X; F) \cong 0 \) for all \( i > 0 \) (see [Br1, Theorem 1.4.6] ).

**Lemma 2.2.** For every differentiable group \( G \) the sheaf \( EG_M \) is soft.

**Proof.** Let \( Z \) be a close subset of \( M \) and let \( \sigma_Z \) be a section of \( EG_M \) over \( Z \). By the definition of a section of a sheaf over a closed set there is an open set \( U \supset Z \) and an extension \( \sigma_U \) of \( \sigma_Z \) to \( U \). Since \( M \) is paracompact, there is a neighborhood \( V \) of \( Z \) such that \( V \subset U \). The extension \( \sigma \) of \( \sigma_U \) (and hence also \( \sigma_Z \)) to a global section of \( EG_M \) is given by the formula
\[
\sigma(x) = r(\sigma_U(x), \psi(x)),
\]
where \( r : EG \times I \to EG \) is the deformational retraction from Example 1.1.B and \( \psi : M \to [0, 1] \) is a smooth function equal to 1 on \( V \) and equal to 0 on \( M - U \). \( \square \)

A bar resolution of the sheaf \( \mathbb{A}_M^k \) of germs of smooth differential \( k \)-forms on \( M \) is constructed as follows. Let \( \Lambda^kT^*M \) be the \( k \)th exterior power of the cotangent bundle \( T^*M \) of \( M \) and let \( EA^kT^*M \) and \( BA^kT^*M \) be the associated with \( \Lambda^kT^*M \) bundles with fibers over \( x \in M \) equal to \( E(\Lambda^kT_x^*M) \) and \( B(\Lambda^kT_x^*M) \) respectively. The groups \( E(\Lambda^kT_x^*M) \) and \( B(\Lambda^kT_x^*M) \) carry vector spaces structures as in Lemma 1.1.

Let \( EA^k_M \) and \( BA^k_M \) be the sheaves of germs of smooth sections of the vector bundles \( E\Lambda^kT^*M \) and \( B\Lambda^kT^*M \) respectively. A section \( \alpha \) of the sheaf \( EA^k_M \) over
$U \subset M$ is of the form
\[ \alpha = \left| f_0, \ldots, f_n, \alpha_0, \ldots, \alpha_n \right|, \]
and a section $\beta$ of the sheaf $BA^k_M$ over $U$ is of the form
\[ \beta = \left| f_0, \ldots, f_n, [\beta_0 : \cdots : \beta_n] \right|, \]
where $\alpha_0, \ldots, \alpha_n, \beta_0, \ldots, \beta_n$ are smooth differential $k$-forms on $U$ and $\{f_i\}_{i=0}^n$ is a smooth partition of unity on $U$. The group of sections of the sheaf $EA^k_M$ over an open set $U \subset M$ will be denoted by $\Gamma(U, EA^k_M)$. Similarly, $\Gamma(U, BA^k_M)$ stands for the group of sections of $BA^k_M$ over $U$.

Since the sequence of vector bundles
\[ 0 \rightarrow \Lambda^k T^* M \rightarrow E \Lambda^k T^* M \rightarrow BA^k T^* M \rightarrow 0 \]
is exact, the sequence of the groups
\[ 0 \rightarrow \Gamma(U, A^k_M) \rightarrow \Gamma(U, EA^k_M) \rightarrow \Gamma(U, BA^k_M) \rightarrow 0 \]
is exact, for every open subset $U$ of $M$. Hence, the sequence of sheaves
\[ 0 \rightarrow A^k_M \rightarrow EA^k_M \rightarrow BA^k_M \rightarrow 0 \]
is exact. Similarly, if $EB^{s-1}A^k_M$ and $B^sA^k_M$ are the sheaves of smooth sections of the vector bundles $EB^{s-1}\Lambda^k T^* M$ and $B^s\Lambda^k T^* M$ respectively, then the sequence of sheaves
\[ 0 \rightarrow B^{s-1}A^k_M \rightarrow EB^{s-1}A^k_M \rightarrow B^sA^k_M \rightarrow 0 \]
is exact. The composition of these sequences induces a long exact sequence
\[ 0 \rightarrow A^k_M \rightarrow EA^k_M \xrightarrow{\sigma} EBA^k_M \xrightarrow{\sigma} \cdots \xrightarrow{\sigma} EB^sA^k_M \xrightarrow{\sigma} \cdots \]
(9)
where
\[ \sigma : EB^sA^k_M \rightarrow EB^{s+1}A^k_M \]
is the composition
\[ EB^sA^k_M \rightarrow B^{s+1}A^k_M \rightarrow EB^{s+1}A^k_M \]
The sequence (9) will be called the bar resolution of the sheaf $A^k_M$.

A bar resolution of an arbitrary sheaf $F$ on a space $X$, which is not necessarily a smooth manifold, can be defined as follows.

Let $E^*F$ and $B^*F$ be the sheaves associated with the presheaves
\[ U \mapsto E(F(U)) \quad \text{and} \quad U \mapsto B(F(U)) \]
respectively. Since the stalks of $E^*F$ and $B^*F$ at $x \in X$ are $E(F_x)$ and $B(F_x)$ respectively, where $F_x$ is the stalk of the sheaf $F$ at $x$, and the sequence
\[ 0 \rightarrow F_x \rightarrow E(F_x) \rightarrow B(F_x) \rightarrow 0 \]
is exact, the sequence of sheaves
\[ 0 \rightarrow F \rightarrow E^*F \rightarrow B^*F \rightarrow 0 \]
is exact. Iterating the above bar constructions we get for every $s \geq 1$ the sheaves $E B^{s-1} \mathcal{F}$ and $B^s \mathcal{F}$ so that the sequence

$$0 \to B^{s-1} \mathcal{F} \to EB^{s-1} \mathcal{F} \to B^s \mathcal{F} \to 0$$

is exact. The composition of these sequences gives the bar resolution of $\mathcal{F}$

$$0 \to \mathcal{F} \to E \mathcal{F} \xrightarrow{\sigma} EB \mathcal{F} \xrightarrow{\sigma} EB^2 \mathcal{F} \xrightarrow{\sigma} \cdots \xrightarrow{\sigma} EB^n \mathcal{F} \xrightarrow{\sigma} \cdots$$

The complex of sheaves

$$B^*(\mathcal{F}) : E \mathcal{F} \xrightarrow{\sigma} EB \mathcal{F} \xrightarrow{\sigma} EB^2 \mathcal{F} \xrightarrow{\sigma} \cdots \xrightarrow{\sigma} EB^n \mathcal{F} \xrightarrow{\sigma} \cdots$$

will be called the bar complex of $\mathcal{F}$.

An easy modification of the proof of Lemma 2.2 shows that the bar resolution of $\mathcal{F}$ is an acyclic resolution of $\mathcal{F}$. Therefore, the cohomology of $\mathcal{F}$ is equal to the cohomology of the cochain complex

$$\Gamma(M, E \mathcal{F}) \xrightarrow{\sigma} \Gamma(M, EB \mathcal{F}) \xrightarrow{\sigma} \cdots \xrightarrow{\sigma} \Gamma(M, EB^n \mathcal{F}) \xrightarrow{\sigma} \cdots$$

The above complex will be called the bar cochain complex of $\mathcal{F}$ and we will denote it by $C^*_B(\mathcal{F})$.

Note, that the above construction applied to $G_M$ and $A^k_M$ produces resolutions of $G_M$ and $A^k_M$ that do not coincide with the resolutions (8) and (9). In the sequel, when referring to bar resolutions of $G_M$ or $A^k_M$ we will always mean the resolutions (8) or (9) respectively.

The bar cochain complex $C^*_B(G_M)$ of the sheaf $G_M$ is of the form

$$C^\infty(M, E G) \xrightarrow{\sigma} C^\infty(M, E B G) \xrightarrow{\sigma} \cdots \xrightarrow{\sigma} C^\infty(M, E B^n G) \xrightarrow{\sigma} \cdots$$

where

$$\sigma : C^\infty(M, E B^n G) \to C^\infty(M, E B^{n+1} G)$$

is the composition

$$C^\infty(M, E B^n G) \xrightarrow{\pi} C^\infty(M, B^{n+1} G) \xrightarrow{i} C^\infty(M, E B^{n+1} G)$$

with $\pi : E(B^n G) \to B(B^n G) = B^{n+1} G$ being the projection map of the universal principal $B^n G$-bundle and $i : B^{n+1} G \to E B^{n+1} G$ being the inclusion of the fiber into the total space of the universal principal $B^{n+1} G$-bundle.

In a sense, the bar cochain complex of the sheaf $Z_M$

$$C^\infty(M, E Z) \xrightarrow{\sigma} C^\infty(M, E B Z) \xrightarrow{\sigma} \cdots \xrightarrow{\sigma} C^\infty(M, E B^n Z) \xrightarrow{\sigma} \cdots$$

can be thought of as a smooth version of Karoubi’s complex (see [Kar]).

Map($X, AG(\mathbb{D}^1)$) $\xrightarrow{\sigma}$ Map($X, AG(\mathbb{D}^2)$) $\xrightarrow{\sigma}$ $\cdots$ $\xrightarrow{\sigma}$ Map($X, AG(\mathbb{D}^n)$) $\xrightarrow{\sigma}$ $\cdots$

of the topological non-commutative differential forms on a space $X$, where $AG(\mathbb{D}^n)$ is the free abelian group on the disk $\mathbb{D}^n$, and

$$\sigma_* : \text{Map}(X, AG(\mathbb{D}^n)) \to \text{Map}(X, AG(\mathbb{D}^{n+1}))$$
is a homomorphism induced by the composition of maps
\[ \mathbb{D}^n \to \mathbb{D}^n / \partial \mathbb{D}^n = S^n = \partial \mathbb{D}^{n+1} \to \mathbb{D}^{n+1}. \]

From the functoriality of the geometric bar construction it follows that the bar resolution of sheaves is functorial as well. Moreover, since for every short exact sequence of topological groups
\[ 0 \to K \to G \to H \to 0 \]
the sequences
\[ 0 \to EK \to EG \to EH \to 0 \]
and
\[ 0 \to BK \to BG \to BH \to 0 \]
are exact, every short exact sequence of sheaves
\[ 0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0 \]
induces a short exact sequence of complexes of sheaves
\[ 0 \to B^*(\mathcal{E}) \to B^*(\mathcal{F}) \to B^*(\mathcal{G}) \to 0 \]
Hence, every complex of sheaves \( \mathcal{F}^* \) has an acyclic resolution given by the total complex \( \text{Tot}^* (B^*(\mathcal{F}^*)) \) associated with the double complex \( B^*(\mathcal{F}^*) \). The cohomology of \( \mathcal{F}^* \) is equal to the cohomology of the total cochain complex \( \text{Tot}^* (C^*_B(\mathcal{F}^*)) \).

**Example 2.1.** The double cochain complex \( C^*_B(A^*_M) \) of the de Rham complex
\[ A^*_M : A^0_M \xrightarrow{d} A^1_M \xrightarrow{d} A^2_M \xrightarrow{d} \cdots \xrightarrow{d} A^n_M \xrightarrow{d} \cdots \]
is given by the diagram
\[
\begin{array}{ccc}
\Gamma(M, EA^2_M) & \xrightarrow{\sigma} & \Gamma(M, EBA^1_M) \\
\uparrow d & & \uparrow d \\
\Gamma(M, EA^1_M) & \xrightarrow{\sigma} & \Gamma(M, EBA^0_M) \\
\uparrow d & & \uparrow d \\
\Gamma(M, EA^0_M) & \xrightarrow{\sigma} & \Gamma(M, EBA^0_M) \\
\end{array}
\]
Note that the \( n \)-th column of this double complex is the complex of global sections of the acyclic resolution
\[ EB^{n-1}A^0_M \to EB^{n-1}A^1_M \to \cdots \to EB^{n-1}A^n_M \to \cdots \]
of the sheaf \( EB^{n-1}\mathbb{R}^\delta_M \), where \( \mathbb{R}^\delta \) is the group \( \mathbb{R} \) taken with the discrete topology. Therefore, the total complex of \( C^*_B(A^*_M) \) is an acyclic resolution of the de Rham complex of \( M \) and the bar complex
\[ EB\mathbb{R}^\delta_M \to EB\mathbb{R}^\delta_M \to EB^2\mathbb{R}^\delta_M \to \cdots \to EB^*\mathbb{R}^\delta_M \to \cdots \]
of $\mathbb{R}^d_M$. Thus, the bar complex of the de Rham complex $\mathcal{A}^*_M$ of $M$ plays a similar role to the Čech complex of $\mathcal{A}^*_M$ inducing an isomorphism between the de Rham cohomology of $M$ and the sheaf cohomology of the constant sheaf $\mathbb{R}^d_M$.

Remark 2.1. There is a close relationship between the bar and Čech cochain complexes of the sheaf $\mathcal{A}^*_M$. Actually, every smooth partition of unity $\{f_i\}_{i \in I}$ subordinated to an open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of a manifold $M$ induces a cochain homomorphism

$$\varphi^*: \check{C}^*(\mathcal{U}, \mathcal{A}^*_M) \to C^*_B(\mathcal{A}^*_M)$$

so that

$$\varphi^p: \check{C}^p(\mathcal{U}, \mathcal{A}^*_M) \to C^p_B(\mathcal{A}^*_M) = EB^p_*(\mathcal{A}^*_M)$$

is the composition

$$\check{C}^p(\mathcal{U}, \mathcal{A}^*_M) \to \check{C}^{p-1}(\mathcal{U}, B\mathcal{A}^*_M) \to \cdots \to \check{C}^0(\mathcal{U}, B^0_\mathcal{A}^*_M) \to EB^p_*(\mathcal{A}^*_M)$$

where for $r > 0$ the homomorphism

$$\varphi^{r,s}: \check{C}^r(\mathcal{U}, B^s_\mathcal{A}^*_M) \to \check{C}^{r-1}(\mathcal{U}, B^{s+1}_\mathcal{A}^*_M)$$

is defined for $\xi = \{\xi_{i_0, \ldots, i_r} \in B^s_\mathcal{A}^*_M(\bigcap_{j=0}^r U_{i_j})\}$ by the formula

$$\varphi^{r,s}(\xi)_{i_0, \ldots, i_{r-1}} = |f_{i_0}, \ldots, f_{i_n}, [\xi_{0, i_0, \ldots, i_{r-1}} : \cdots : \xi_{n, i_0, \ldots, i_{r-1}}]|$$

and the homomorphism

$$\varphi^{0,p}: \check{C}^0(\mathcal{U}, B^p_\mathcal{A}^*_M) \to EB^p_*(\mathcal{A}^*_M)$$

is given by

$$\varphi^{0,p}(\{\xi_i\}) = |f_{i_0}, \ldots, f_{i_n}, [\xi_0 : \cdots : \xi_n]|.$$

### 3. Smooth principal $B^s\mathbb{C}^*$-bundles

In this section we will show that if $M$ is a smooth manifold, then the group $H^k(M; \mathbb{Z})$ can be identified with the group of isomorphism classes of smooth principal $B^{k-2}S^1, B^{k-2}\mathbb{C}^*$, or $B^{k-1}\mathbb{Z}$ bundles over $M$.

Let $G$ be an abelian Lie group. A principal $B^sG$-bundle $E \to M$ over a smooth manifold $M$ is smooth if the transition functions of this bundle are smooth.

The proof of the following proposition, essentially due to tom Dieck [D], shows an explicit formula for a classifying map of a smooth principal bundle in terms of its transition functions.

**Proposition 3.1.** Let $G$ be a differentiable group. Then for every smooth principal $G$-bundle $\pi: E \to M$ there is a smooth map $\varphi: M \to BG$ such that $E \to M$ is the pull-back of the universal principal $G$-bundle by $\varphi$. 


Corollary 3.2. For every smooth principal $B^s\mathbb{C}^*$-bundle $E \to M$ there is a smooth map $\varphi : M \to B^{s+1}\mathbb{C}^*$ such that $E \to M$ is the pull-back of the universal principal $B^s\mathbb{C}^*$-bundle by $\varphi$.

Proof of Proposition 3.1. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open covering of $M$ so that for every $i \in I$ there is a trivialization

$$\psi_i : \pi^{-1}(U_i) \to U_i \times G.$$

Define $g_i : E \to G$ by the formula

$$g_i(x) = \begin{cases} \text{pr}_2(\psi_i(x)) & \text{for } x \in \pi^{-1}(U_i) \\ e & \text{for } x \notin \pi^{-1}(U_i) \end{cases}$$

where $e$ is the neutral element of $G$ and $\text{pr}_2 : U_i \times G \to G$ is the projection on the second factor.

Let $\{f_i\}_{i \in I}$ be a partition of unity subordinated to the covering $\mathcal{U}$ and let $\bar{\varphi} : E \to EG$ be the map

$$\bar{\varphi}(y) = |f_{i_0}(\pi(y)), f_{i_1}(\pi(y)), \ldots, f_{i_n}(\pi(y)), g_{i_0}(y), g_{i_1}(y), \ldots, g_{i_n}(y)|,$$

where $i_0, \ldots, i_n$ are the indices so that for each $i \in \{i_0, \ldots, i_n\}$ $f_i(\pi(y)) \neq 0$. It is easy to see that $\bar{\varphi}$ is a $G$-equivariant map and hence it induces a morphism of principal $G$-bundles

$$\begin{array}{ccc}
E & \xrightarrow{\varphi} & EG \\
\pi \downarrow & & \downarrow \\
M & \xrightarrow{\varphi} & BG
\end{array}$$

where the restriction of $\varphi : M \to BG$ to $U_j \subset M$ is given by the formula

$$\varphi(x) = |f_{i_0}(x), f_{i_1}(x), \ldots, f_{i_n}(x), [g_{i_0}(\sigma(x)): g_{i_1}(\sigma(x)): \ldots: g_{i_n}(\sigma(x))]|,$$

where $\sigma : U_j \to \pi^{-1}(U_j)$ is a smooth section of the restriction $\pi^{-1}(U_j) \to U_j$ of $\pi : E \to M$ to $\pi^{-1}(U_j)$. Note that $\varphi(x)$ does not depend on the choice of the section $\sigma$ because $g_i$s are $G$-equivariant maps. In the non-homogeneous coordinates

$$\varphi(x) = |f_{i_0}(x), f_{i_1}(x), \ldots, f_{i_n}(x), [g_{i_0}(x)|g_{i_1}(x)|g_{i_2}(x)| \ldots |g_{i_{n-1}}(x)|],$$

where

$$g_{ij}(x) = (g_i(\sigma(x)))^{-1} \cdot g_j(\sigma(x))$$

are the transition functions of the bundle $E \to M$ associated with the open covering of $M$ by the sets $\{x \in M | f_i(x) > 0\}$. Since $g_i$ is smooth on $\text{supp}(f_i) \cap U_j$ for every $i, j \in I$ and $\sigma$ is smooth on $U_j$, the map $\varphi$ is smooth on $U_j$ for every $j \in I$ and hence $\varphi$ is smooth on $M$. T. tom Dieck showed in [11] that $\varphi : M \to BG$ is the classifying map of the bundle $\pi : E \to M$ (tom Dieck works in the setting of Milnor’s bar construction, but all he does extends easily to the context of Milgram’s bar construction).
Example 3.1. The isomorphism $H^2(\mathbb{R}^3 - 0, \mathbb{Z}) \cong H^1(\mathbb{R}^3 - 0, \mathbb{C}^*_{\mathbb{R}^3 - 0})$ implies that every element of $H^2(\mathbb{R}^3 - 0, \mathbb{Z})$ corresponds to a unique isomorphism class of a smooth principal $\mathbb{C}^*$-bundle over $\mathbb{R}^3 - 0$. Let $L$ be a smooth principal $\mathbb{C}^*$-bundle over $\mathbb{R}^3 - 0$ representing a generator of $H^2(\mathbb{R}^3 - 0, \mathbb{Z}) \cong \mathbb{Z}$. The proof of Proposition 3.1 shows how to describe a smooth classifying map $\psi$ with associated with the generalized exponential sequence $\mathbb{R}^{\mathbb{R}^3}$.

Example 3.1. The isomorphism $H^2(\mathbb{R}^3 - 0, \mathbb{Z}) \cong H^1(\mathbb{R}^3 - 0, \mathbb{C}^*_{\mathbb{R}^3 - 0})$ implies that every element of $H^2(\mathbb{R}^3 - 0, \mathbb{Z})$ corresponds to a unique isomorphism class of a smooth principal $\mathbb{C}^*$-bundle over $\mathbb{R}^3 - 0$. Let $L$ be a smooth principal $\mathbb{C}^*$-bundle over $\mathbb{R}^3 - 0$ representing a generator of $H^2(\mathbb{R}^3 - 0, \mathbb{Z}) \cong \mathbb{Z}$. The proof of Proposition 3.1 shows how to describe a smooth classifying map $\psi_L : (\mathbb{R}^3 - 0) \to BC^*$ of $L$ in terms of some transition functions of $L$. Let $S^3 = \mathbb{R}^3 \cup \{\infty\}$ and consider the open subsets $U_0 = \mathbb{R}^3$ and $U_\infty = S^3 - \{0\}$ of $S^3$. Since $U_0 \cap U_\infty = \mathbb{R}^3 - 0$ we can think of the classifying map $\psi_L : (\mathbb{R}^3 - 0) \to BC^*$ as a transition function of a smooth principal $BC^*$-bundle $BL$ over $S^3$. From the proof of Proposition 3.1 it follows that the isomorphism class of $BL$ corresponds to the generator of $H^3(S^3, \mathbb{Z})$. Let $BL$ be the pull-back of $BL$ by the standard retraction $(\mathbb{R}^4 - 0) \to S^3$. The classifying map of the bundle $BL$ can be identified with a transition function of a smooth principal $B^2\mathbb{C}^*$-bundle over $S^4$, representing a generator of $H^4(S^4, \mathbb{Z})$. Iterating the above procedure we get a family of smooth principal $B^k\mathbb{C}^*$-bundle over $S^k$, representing generators of the groups $H^k(S^k, \mathbb{Z})$ for $k \geq 2$.

Proposition 3.3. Let $G$ be one of the group $S^1, \mathbb{C}^*$, or $B\mathbb{Z}$. Then for every smooth manifold $M$ and every $p \geq 2$ the group $H^p(M, \mathbb{Z})$ is isomorphic to:

(i) the group $L(B^{p-2}G)_M$ of isomorphism classes of smooth principal $B^{p-2}G$-bundles over $M$.

(ii) the group $[M, B^{p-1}G]^\infty$ of smooth homotopy classes of smooth maps from $M$ to $B^{p-1}G$.

Proof of part (i) of Proposition 3.3. Since for any abelian differentiable group $G$ the group of isomorphism classes of smooth principal $G$-bundles over $M$ is isomorphic to $H^1(G_M)$, we have to prove that there is an isomorphism

$$H^p(M, \mathbb{Z}) \cong H^1(B^{p-2}G_M).$$

Consider the cohomology long exact sequence

$$0 \to H^p(B^{p-2}G_M) \to H^1(B^{p-2}G_M) \to H^p(M, \mathbb{Z}) \to \cdots \to H^p(B^{p-2}G_M) \to 0$$

associated with the generalized exponential sequence

$$0 \to \mathbb{Z}_M \to B^{p-2}G_M \to B^{p-2}G_M[-p + 2] \to 0$$

where $B^{p-2}G_M$ is the complex

$$H_M \to E_G \to \cdots$$

with $H_M$ being equal to $\mathbb{R}, \mathbb{C}$, or $\mathbb{Z}$ for $G = S^1, \mathbb{C}^*$, or $B\mathbb{Z}$ respectively.

Since for every $s \geq 0$ the sheaf $B^{p-2}G_M$ is acyclic, the cohomology of the complex $B^{p-2}G_M$ is equal to the cohomology of the cochain complex

$$H^s(B^{p-2}G_M(M)) \to H^s(B^{p-2}G_M(M)) \to H^s(B^{p-2}G_M(M)) \to \cdots$$

with $H^s$ being equal to $\mathbb{R}, \mathbb{C}$, or $\mathbb{Z}$ for $G = S^1, \mathbb{C}^*$, or $B\mathbb{Z}$ respectively.
of the groups of global sections of the components of $\mathcal{E}\mathcal{B}^{p-2}G_M$. Therefore, for every $q > p - 2$

$$\mathbb{H}^q(\mathcal{E}\mathcal{B}^{p-2}G_M) \cong H^q(\mathcal{E}\mathcal{B}^{p-2}G_M(M)) \cong 0.$$ 

Hence, the coboundary homomorphism

$$\mathcal{H}^1(B^{p-2}G_M) \rightarrow H^p(M; \mathbb{Z})$$

in the cohomology long exact sequence associated with the isomorphism. □

Remark 3.1. Let $G$ be an arbitrary abelian Lie group. Replacing in the proof of Proposition 3.3 the sequence (10) by the appropriate short exact sequence associated with the bar resolution of $G_M$, we would get an isomorphism between $H^p(G_M)$ and the group $L(B^{p-2}G)_M$ of isomorphism classes of smooth principal $B^{p-2}G$-bundles over $M$.

Part (ii) of Proposition 3.3 is a straightforward consequence of the following lemma.

Lemma 3.4. Let $G$ be a differentiable group. Then the group of isomorphism classes of smooth principal $G$-bundles over $M$ is isomorphic to the group $[M, BG]_\infty$ of smooth homotopy classes of smooth maps from $M$ to $BG$.

Proof. Let $G$ be a differentiable group. We will show that there is an isomorphism

$$[M, BG]_\infty \cong H^1(G_M).$$

The beginning of the cohomology long exact sequence associated with the short exact sequence

$$0 \rightarrow G_M \rightarrow E_G M \rightarrow B_G M \rightarrow 0$$

is of the form

$$\cdots \rightarrow C^\infty(M, E_G) \xrightarrow{\pi_*} C^\infty(M, B_G) \rightarrow H^1(G_M) \rightarrow H^1(E_G M) \rightarrow \cdots$$

Since $H^1(E_G M) \cong 0$, we have the isomorphism

$$\frac{C^\infty(M, B_G)}{\pi_* C^\infty(M, E_G)} \cong H^1(G_M).$$

The image $\pi_* C^\infty(M, E_G)$ of the group $C^\infty(M, E_G)$ in $C^\infty(M, B_G)$ consists of those smooth maps from $M$ to $BG$ that lift to maps from $M$ to $EG$. It is easy to see that $f : M \rightarrow BG$ has a lift to $\tilde{f} : M \rightarrow EG$ if and only if $f$ is smooth homotopic to a constant map. Hence

$$\frac{C^\infty(M, B_G)}{\pi_* C^\infty(M, E_G)} \cong [M, BG]_\infty$$

□
Remark 3.2. Let $G$ be a topological group. Replacing in the proof of Lemma 3.4 the sheaves of smooth maps on $M$ by sheaves of continues maps on some space $X$, we get an isomorphism between the group of isomorphism classes of principal $G$-bundles over $X$ and the group $[X, BG]$ of homotopy classes of maps from $X$ to $BG$.

4. Flat Connections on Principal $B^s\mathbb{C}^*$-bundles

In this section we show that for every $s \geq 1$ the group $B^s\mathbb{C}^*$ is equipped with the canonical $B^s\mathbb{C}^*$-equivariant $B^s\mathbb{C}$-valued connection 1-form $B^s(z^{-1}dz)$. By analogy with the Lie group case, the form $B^s(z^{-1}dz)$ is used to define connections on smooth principal $B^s\mathbb{C}^*$-bundles. We show that for $q > p$ the smooth Deligne cohomology group $H^p(M, \mathbb{Z}(q)_n^s)$ is isomorphic to the group of isomorphism classes of smooth principal $B^q-2\mathbb{C}^*$-bundles with flat connections. Moreover, we prove Theorem B.

4.1. The canonical connection 1-forms on $B^s\mathbb{C}^*$ and $EB^s\mathbb{C}^*$. Let $M$ be a simplicial smooth manifold. A smooth $p$-form $\alpha$ on the geometric realization $|M|$ of $M$ is a family $\{\alpha^n\}$ of differential $p$-forms $\alpha^n$ on $\Delta^n \times M_n$ satisfying for every $0 \leq i \leq n$ the following compatibility conditions

\begin{align}
(\partial^i \times id)^*\alpha^n &= (id \times \partial_i)^*\alpha^{n-1} \\
(s^i \times id)^*\alpha^n &= (id \times s_i)^*\alpha^{n+1}
\end{align}

where $\partial^i \times id$, $id \times \partial_i$, $s^i \times id$, and $id \times s_i$ are the maps

$$
\Delta^{n-1} \times M_{n-1} \xrightarrow{id \times \partial_i} \Delta^{n-1} \times M_n \xrightarrow{\partial^i \times id} \Delta^n \times M_n
$$

$$
\Delta^{n+1} \times M_{n+1} \xrightarrow{id \times s_i} \Delta^{n+1} \times M_n \xrightarrow{s^i \times id} \Delta^n \times M_n
$$

with $\partial^i$ and $s^i$ being the coface and the codegeneracy maps on $\Delta^n$s and $\partial_i$, $s_i$ being the face and the degeneracy maps on $M_n$s.

Example 4.1.

(A) Let $G$ be a Lie group and let $g^{-1}dg$ be the canonical $g$-valued connection 1-form on $G$, where $g$ is the Lie algebra of $G$. The total space $EG$ of the universal principal $G$-bundle $EG \to BG$ carries a smooth $g$-valued form $\omega$ so that $\omega$ evaluated at $|x_0, \ldots, x_n, g_0, \ldots, g_n|$ is

$$
x_0 g_0^{-1} dg_0 + x_1 g_1^{-1} dg_1 + \cdots + x_n g_n^{-1} dg_n,
$$

where $x_0, \ldots, x_n$ are the barycentric coordinates in $\Delta^n$ and $g_i^{-1} dg_i = \pi^*_i (g^{-1}dg)$ for the projection $\pi_i : G^{n+1} \to G$ on the $i$th factor.

(B) The canonical connection 1-form $E(z^{-1}dz)$ on $E\mathbb{C}^*$ is defined by the family of $E\mathbb{C}$-valued 1-forms $E(z^{-1}dz)^n$ on $\Delta^n \times (\mathbb{C}^*)^{n+1}$ such that $E(z^{-1}dz)^n$ evaluated
GEOMETRY OF DELIGNE COHOMOLOGY 23

by the inductive formula

\[ |t_1, \ldots, t_n, z_0^{-1}v_0[z_1^{-1}v_1| \cdots |z_n^{-1}v_n]|. \]

In the sequel we will use the notation

\[ E(z^{-1}dz)^n_{t_1, \ldots, t_n,z_0[z_1| \cdots |z_n]|} = |t_1, \ldots, t_n, z_0^{-1}dz_0[z_1^{-1}dz_1| \cdots |z_n^{-1}dz_n]|. \]

Similarly, the canonical connection 1-form \( B(z^{-1}dz) \) on \( BC^* \) is defined by the family of \( BC^* \)-valued 1-forms \( B(z^{-1}dz)^n \) on \( \Delta^n \times (C^*)^n \), where

\[ B(z^{-1}dz)[t_1, \ldots, t_n, [z_1| \cdots |z_n]] = |t_1, \ldots, t_n, [z_1^{-1}dz_1| \cdots |z_n^{-1}dz_n]|. \]

The compatibility conditions \((11), (12)\) are easy to check calculations. It is also easy to see that \( E(z^{-1}dz) \) is a \( E\mathbb{C}^* \)-equivariant 1-form and \( B(z^{-1}dz) \) is a \( BC^* \)-equivariant 1-form.

Let \( G \) be an abelian Lie group. A smooth p-form on \( EB^sG \) and \( B^{s+1}G \), for \( s \geq 1 \), is defined by the following inductive procedure.

Suppose we have defined smooth p-forms on \( B^sG \) as well as on each product \( \Delta^k \times (B^sG)^m \) for \( k, m \geq 0 \). Then a smooth p-form \( \alpha \) on \( EB^sG \) consists of a family of p-forms \( \alpha^n \) on \( \Delta^n \times (B^sG)^n+1 \) satisfying the compatibility conditions \((11)\) and \((12)\). Similarly, a smooth p-form \( \alpha \) on \( B^{s+1}G \) consists of a family of p-forms \( \alpha^n \) on \( \Delta^n \times (B^sG)^{n+1} \) satisfying the compatibility conditions \((11)\) and \((12)\). A smooth p-form \( \alpha \) on \( \Delta^k \times (B^{s+1}G)^m \) consists of a family of p-forms \( \alpha^n \) on \( \Delta^k \times (\Delta^n \times (B^sG)^{n+1})^m \) satisfying the compatibility conditions

\[
(id_{\Delta^k} \times (\partial^i \times id)^m)^* \alpha^n = (id_{\Delta^k} \times (id \times \partial_i)^m)^* \alpha^{n-1}
\]

\[
(id_{\Delta^k} \times (s^i \times id)^m)^* \alpha^n = (id_{\Delta^k} \times (id \times s_i)^m)^* \alpha^{n+1}
\]

**Example 4.2.** Now, for every \( s > 0 \) we are going to construct \( EB^{s-1}C^* \)-valued differential 1-form \( EB^{s-1}(z^{-1}dz) \) on \( EB^{s-1}C^* \) and \( B^{s-1}C \)-valued differential 1-form \( B^s(z^{-1}dz) \) on \( B^s C^* \). Note that from Lemma \((1.1)\) we know that for every \( s > 0 \) the groups \( EB^{s-1}C \) and \( B^{s-1}C \) are \( \mathbb{C} \)-vector spaces. Thus, it make sense to talk about \( EB^{s-1}C \) or \( B^{s-1}C \)-valued differential forms.

The canonical connection 1-form \( EB^{s-1}(z^{-1}dz) \) on \( EB^{s-1}C^* \) is a 1-form on \( EB^{s-1}C^* \) so that \( EB^{s-1}(z^{-1}dz) \) evaluated at \( [t_1, \ldots, t_n, g_0[g_1| \cdots |g_n]] \) is given by the inductive formula

\[ |t_1, \ldots, t_n, B^s-1(g_0^{-1}dg_0)[B^s-1(g_1^{-1}dg_1)] \cdots |B^s-1(g_n^{-1}dg_n)|. \]

The canonical connection 1-form \( B^s(z^{-1}dz) \) on \( B^sC^* \) is a 1-form on \( B^sC^* \) so that \( B^s(z^{-1}dz) \) evaluated at \( [t_1, \ldots, t_n, g_1| \cdots |g_n] \) is given by

\[ |t_1, \ldots, t_n, [B^s-1(g_1^{-1}dg_1)] \cdots |B^s-1(g_n^{-1}dg_n)|. \]

where \( g_0, g_1, \ldots, g_n \in B^{s-1}C^* \) and \( B^{s-1}(g_i^{-1}dg_i) \) is the canonical connection 1-form \( B^{s-1}(z^{-1}dz) \) on \( B^{s-1}C^* \) evaluated at \( g_i \).
4.2. Connections on principal $B^s\mathbb{C}^*$-bundles. A connection on a smooth principal $B^s\mathbb{C}^*$-bundle $E \to M$ is a collection $\{\omega_i \in \Gamma(U_i, B^sA^1_{M,C})\}_{i \in I}$ of $B^s\mathbb{C}$-valued 1-forms, for some open covering $\{U_i\}_{i \in I}$ of $M$, such that for every $i, j \in I$ so that $U_i \cap U_j \neq \emptyset$

$$\omega_i - \omega_j = g_{ij}^* B^s(z^{-1}dz),$$

where $g_{ij} : U_i \cap U_j \to B^s\mathbb{C}^*$ is a transition function of the bundle $E \to M$.

Equivalently, a connection on a smooth principal $B^s\mathbb{C}^*$-bundle $E \to M$ is given by a $B^s\mathbb{C}^*$-equivariant global section of the sheaf $B^sA^1_{E,C}$.

The pull-back $g_{ij}^* B^s(z^{-1}dz)$ can be described explicitly by the formula

$$g_{ij}^* B^s(z^{-1}dz) = d\log(g_{ij}),$$

where $d\log(g_{ij})$ is defined by the induction on $s$ as follows. For any smooth function $f : U \to B^s\mathbb{C}^*$ given locally by the formula

$$f(x) = |t_1(x), \ldots, t_n(x), [f_1(x)] \cdots [f_n(x)]|,$$

where $f_i(x) : U \to B^{s-1}\mathbb{C}$, we define $d\log f \in \Gamma(U, B^sA^1_{M,C})$ by

$$d\log f(x) = |t_1(x), \ldots, t_n(x), [d\log f_1(x)] \cdots [d\log f_n(x)]|.$$

It is easy to see that if $f : U \to B^s\mathbb{C}^*$ and $d\log(f) = 0$, then $f : U \to B^s(\mathbb{C}^*)^\delta$, where $(\mathbb{C}^*)^\delta$ is the group $\mathbb{C}^*$ with the discreet topology.

Example 4.3.

(A) It is easy to see that the differential 1-form $\omega$ from Example 4.1 is $G$-equivariant. Hence, it is a connection 1-form on the universal principal $G$-bundle $EG \to BG$. We will call it the canonical connection 1-form of $EG \to BG$.

For $G = \mathbb{C}^*$ the form $\omega$ evaluated at $|x_0, x_1, \ldots, x_n, z_0, z_1, \ldots, z_n|$ is given by the formula

$$\omega_{|x_0,x_1,\ldots,x_n,z_0,z_1,\ldots,z_n} = \sum_{i=0}^{n} x_i \frac{dz_i}{z_i}.$$

Note that $\omega$ is the composition $l \circ E\omega$, where $E\omega$ is the canonical $E\mathbb{C}$-valued connection 1-form on $E\mathbb{C}^*$ and $l : E\mathbb{C} \to \mathbb{C}$ is the splitting of the short exact sequence

$$0 \to \mathbb{C} \to E\mathbb{C} \to B\mathbb{C} \to 0$$

given by the formula

$$l([x_0, x_1, \ldots, x_n, z_0, z_1, \ldots, z_n]) = \sum_{i=0}^{n} x_i z_i. \quad \text{(13)}$$

(B) The canonical connection 1-form on the universal $B^s\mathbb{C}^*$-bundle

$$B^s\mathbb{C}^* \to EB^s\mathbb{C}^* \to B^{s+1}\mathbb{C}^*$$
can be defined as the composition \( l \circ EB^s(z^{-1}dz) \), where \( l : EB^sC \to B^sC \) is the splitting of the short exact sequence
\[
0 \to B^sC \to EB^sC \to B^{s+1}C \to 0
\]
given by the formula (13), where now \( z_i \in B^sC \).

(C) From Corollary 3.2 and the above example it follows that every smooth principal \( B^sC^* \)-bundle carries a connection.

In particular, the smooth principal \( BC^* \)-bundle over \( S^3 \) from Example 3.1 can be equipped with the connection 1-form \( \omega \) that is the pull-back of the canonical connection 1-form \( l \circ E\omega \) on \( E\mathbb{C}^* \to BC^* \). Following J-L. Brylinski and P. Deligne (see [Br], Chapter 7) one can interpret \( \omega \) as the Dirac monopole.

The ordinary exterior derivative \( d : A^k_{M,C} \to A^{k+1}_{M,C} \) has an extension
\[
d : B^sA^k_{M,C} \to B^sA^{k+1}_{M,C}
\]
defined inductively by the formula
\[
d([t_1(x), \ldots, t_n(x), [\alpha_1(x)] \cdots [\alpha_n(x)])] = [t_1(x), \ldots, t_n(x), [d\alpha_1(x)] \cdots [d\alpha_n(x)]].
\]

Similarly, we define
\[
d : EB^sA^k_{M,C} \to EB^sA^{k+1}_{M,C}.
\]

Note that \( d(B^s(z^{-1}dz)) = 0 \), and hence,
\[
d\omega_i - d\omega_j =dg_{ij}^*(B^s(z^{-1}dz)) = g_{ij}^*(dB^s(z^{-1}dz)) = g_{ij}^*(B^s(d(z^{-1}dz))) = 0.
\]

Thus, the family \( \{d\omega_i\} \) defines a global section \( \Omega \) of the sheaf \( B^sA^2_{M,C} \), which is by definition the curvature of the connection \( \{\omega_i\} \).

**Proof of Theorem B.** The exponential short exact sequence
\[
(14) \quad 0 \to \mathbb{Z}_M \to \mathbb{C}_{\delta}^d_{M} \to (\mathbb{C}^*)_{M}^d \to 0
\]
induces the short exact sequence of the bar cochain complexes
\[
0 \to C^*_B(\mathbb{Z}_M) \to C^*_B(\mathbb{C}_{\delta}^d_{M}) \to C^*_B((\mathbb{C}^*)_{M}^d) \to 0
\]
For every abelian Lie group \( G \), the group \( H^p(M;G) \) is isomorphic to the group \( H^p(C^*_B(\mathbb{C}_M)) \), which in turn can be identified with the group \( L(M,B^{p-2}G) \) of isomorphism classes of smooth principal \( B^{p-2}G \)-bundles over \( M \) (see Remark 3.1). Therefore, the cohomology long exact sequence associated with the exponential short exact sequence (14) induces a commutative diagram
\[
\begin{array}{cccccc}
H^{p-1}(M;\mathbb{Z}) & \to & H^{p-1}(M;\mathbb{C}) & \to & H^{p-1}(M;\mathbb{C}^*) & \to & H^p(M;\mathbb{Z}) \\
\approx & & \approx & & \approx & & \\
\to & & \to & & \to & & \\
L(M,B^{p-2}\mathbb{Z}) & \to & L(M,B^{p-2}\mathbb{C}) & \to & L(M,B^{p-2}(\mathbb{C}^*)) & \to & L(M,B^{p-2}\mathbb{C}^*) \\
\end{array}
\]
where the isomorphism $H^p(M; \mathbb{Z}) \to L(M, B^{p-2}\mathbb{C}^*)$ is the composition of isomorphisms

$$H^p(M; \mathbb{Z}) \to L(M, B^{p-1}\mathbb{Z}) \to L(M, B^{p-2}\mathbb{C}^*)$$

and

$$f : L(M, B^{p-2}(\mathbb{C}^*)^\delta) \to L(M, B^{p-2}\mathbb{C}^*)$$

is the forgetful homomorphism induced by the homomorphism $B^{p-2}(\mathbb{C}^*)^\delta \to B^{p-2}\mathbb{C}^*$.

It is easy to see that the above diagram induces the following commutative diagram

$$
\begin{array}{cccccc}
0 & \to & H^{p-1}(M; \mathbb{C}) & \to & H^p(M; \mathbb{C}) & \to & \text{Tors} H^p(M; \mathbb{Z}) & \to & 0 \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
0 & \to & \ker(f) & \to & L(M, B^{p-2}(\mathbb{C}^*)^\delta) & \to & \text{im}(f) & \to & 0
\end{array}
$$

whose rows are exact sequences.

In order to finish the proof of Theorem B, we have to show the for every $q > p$ there is an isomorphism

(15) \[ \mathbb{H}^p(M, \mathbb{Z}(q)_{D}) \cong H^{p-1}(M; \mathbb{C}^*) \]

and that the group $L(M, B^{p-2}\mathbb{C}^*, \nabla_{\text{flat}})$ of isomorphism classes of flat connections on smooth principal $B^{p-2}\mathbb{C}^*$-bundles over $M$ is isomorphic to $L(M, B^{p-2}(\mathbb{C}^*)^\delta)$.

The isomorphism (15) follows from the fact that there is a quasi-isomorphism

(16) \[
\begin{array}{cccccccc}
\mathbb{Z}(q)_{M} & \to & A^0_{M, \mathbb{C}} & \xrightarrow{d} & A^1_{M, \mathbb{C}} & \xrightarrow{d} & \cdots & \xrightarrow{d} & A^{q-1}_{M, \mathbb{C}} \\
\downarrow \alpha & & \downarrow \alpha & & \downarrow \alpha & & \cdots & & \downarrow \alpha \\
0 & \to & \mathbb{C}^*_M & \xrightarrow{\text{dlog}} & A^1_{M, \mathbb{C}} & \xrightarrow{d} & \cdots & \xrightarrow{d} & A^{q-1}_{M, \mathbb{C}}
\end{array}
\]

where $\alpha(f) = \exp((2\pi \sqrt{-1})^{-1-q} \cdot f)$, between the smooth Deligne complex $\mathbb{Z}(q)_{D}^{\infty}$ and the complex $A^{\leq q}_{M, \mathbb{C}}(\text{dlog})[-1]$, where

$$A^q_{M, \mathbb{C}}(\text{dlog} : \mathbb{C}^*_M \to A^1_{M, \mathbb{C}} \to \cdots \to A^{q-1}_{M, \mathbb{C}}$$

is the truncation of the complex

$$A^q_{M, \mathbb{C}}(\text{dlog} : \mathbb{C}^*_M \to A^1_{M, \mathbb{C}} \to \cdots \to A^{q-1}_{M, \mathbb{C}}$$

which is an acyclic resolution of the constant sheaf $\mathbb{C}^*_M$. Therefore, for every $q > p$ there are isomorphisms

$$\mathbb{H}^p(M, \mathbb{Z}(q)_{D}) \cong \mathbb{H}^{p-1}(A^{\leq q}_{M, \mathbb{C}}(\text{dlog})) \cong \mathbb{H}^{p-1}(A^*_{M, \mathbb{C}}(\text{dlog})) \cong H^{p-1}(M; \mathbb{C}^*).$$
is a consequence of the following lemma.

**Lemma 4.1.** There is a one-to-one correspondence between flat connections on a smooth principal $B^s\mathbb{C}^\ast$-bundle $E \to M$ and reductions of the structure group of $E \to M$ to $B^{p-2}(\mathbb{C}^\ast)^\delta$.

**Proof.** Let $E \to M$ be a smooth principal $B^s\mathbb{C}^\ast$-bundle with a flat connection. We will show that the structure group of $E \to M$ can be reduced to $B^s(\mathbb{C}^\ast)^\delta$ or equivalently, that $E \to M$ has transition functions $\tilde{g}_{ij} : U_{ij} \to B^s(\mathbb{C}^\ast)^\delta$.

Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open covering of $M$ consisting of contractible subsets of $M$ and let $\{g_{ij} : U_{ij} \to B^s\mathbb{C}^\ast\}_{i,j \in I}$ be a family of transition functions of $E \to M$. Suppose, $\{\omega_i \in \Gamma(U_i, B^sA^1_{M,\mathbb{C}})\}_{i \in I}$ is a flat connection on $E \to M$. That is every $\omega_i$ is a closed form and for every $i, j$ so that $U_i \cap U_j \neq \emptyset$

$$\omega_i - \omega_j = d\log g_{ij}.$$  

It is easy to see (using the Poincare Lemma and the induction on $s$) that if $d\omega_i = 0$, then there is a $B^s\mathbb{C}$-valued function $f_i$ such that $df_i = \omega_i$.

For any smooth function $f : U \to B^s\mathbb{C}$ we define, by the induction on $s$, the $B^s\mathbb{C}^\ast$-valued function $\exp(f)$. If $f : U \to B^s\mathbb{C}$ is given locally by the formula

$$f(x) = \left|t_1(x), \ldots, t_n(x), |g_1(x)| \cdots |g_n(x)|\right|,$$

where $g_i : U \to B^{s-1}\mathbb{C}$, then

$$\exp f(x) = \left|t_1(x), \ldots, t_n(x), \exp g_1(x) \cdots \exp g_n(x)\right|.$$  

Since $d\log(\exp f) = df$,

$$d\log g_{ij} = \omega_i - \omega_j = d(f_i - f_j) = d\log(\exp(f_i - f_j)) = -d\log(\delta(\exp f)_{ij}).$$

Therefore, for $\tilde{g}_{ij} = g_{ij} + \delta(\exp f)_{ij}$

$$d\log \tilde{g}_{ij} = 0$$

and hence $\tilde{g}_{ij} : U_{ij} \to B^s(\mathbb{C}^\ast)^\delta$. The family $\{\tilde{g}_{ij}\}$ gives the required transition functions of $E \to M$.

Now suppose, $E \to M$ is a principal $B^s\mathbb{C}^\ast$-bundle with transition functions $g_{ij} : U_{ij} \to B^s(\mathbb{C}^\ast)^\delta$. A flat connection on $E \to M$ is given by the family $\{\omega_i\}$ of trivial (tautologicly equal to zero) 1-forms. Obviously, $d\omega_i = 0$ and $\omega_i - \omega_j = 0 = d\log g_{ij}$. □

5. **$k$-connections on principal $B^s\mathbb{C}^\ast$-bundles**

In this section we define $k$-connections, $k = 1, \ldots, s+1$, and scalar curvatures on smooth principal $B^s\mathbb{C}^\ast$-bundles and prove Theorems A and C. In particular, we show that for $p \geq 2$ the group $H^p(M, \mathbb{Z}(p)_{B}^\infty)$ is isomorphic to the group of equivalence classes of smooth principal $B^{p-2}\mathbb{C}^\ast$-bundles with $k$-connections for $k = 1, \ldots, p-1$. 
By definition, a 1-connection on smooth principal $B^\ast \mathbb{C}^\ast$-bundle is a connection on this bundle. To motivate a definition of a $k$-connection for $k \geq 2$, we will first reformulate the standard definition of a connection on smooth principal $\mathbb{C}^\ast$-bundle.

A smooth principal $\mathbb{C}^\ast$-bundle $E \to B$ is given either by a family
\[
\{g_{ij} : U_i \cap U_j \to \mathbb{C}^\ast\}_{i,j \in I}
\]
of transition functions associated with an open covering $U = \{U_i\}_{i \in I}$ of $M$ or by a smooth map $g : M \to B C^\ast$ so that $E \to B = g^\ast (EC^\ast \to BC^\ast)$. The map $g$ can be described in terms of the transition functions $\{g_{ij}\}$ by the formula
\[
g(x) = |f_{io}(x), f_{i1}(x), \ldots, f_{in}(x), [g_{ioi_1}(x)g_{i_1i_2}(x)\cdots g_{in-1i_n}(x)]|,
\]
wher $\{f_i\}_{i \in I}$ is a partition of unity subordinated to the covering $U$ (see the proof of Proposition 3.1).

Classically, a connection on $E \to B$ is given by a family of 1-forms $\{\omega_i \in \Gamma(U_i, A^1_M)\}_{i \in I}$ so that
\[
\omega_i - \omega_j = d\log g_{ij}, \quad \text{on } U_i \cap U_j \neq \emptyset.
\]

Alternatively, in terms of the map $g : M \to B C^\ast$, a connection on $E \to B$ is a global section $-\omega$ of the sheaf $EA^1_{M,\mathbb{C}}$ so that $\pi_*\omega = d\log g$, where $\pi_* : \Gamma(M, EA^1_{M,\mathbb{C}}) \to \Gamma(M, BA^1_{M,\mathbb{C}})$ is the homomorphism induced by the morphism of sheaves $\pi : EA^1_{M,\mathbb{C}} \to BA^1_{M,\mathbb{C}}$.

Indeed, if $\omega$ is given by
\[
\omega(x) = |f_{i_0}'(x), f_{i_1}'(x), \ldots, f_{i_n}'(x), \omega_{i_0}(x), \omega_{i_1}(x), \ldots, \omega_{i_n}(x)|,
\]
then
\[
\pi_*\omega(x) = |f_{i_0}'(x), f_{i_1}'(x), \ldots, f_{i_n}'(x), [\omega_{i_0}(x) : \omega_{i_1}(x) : \cdots : \omega_{i_n}(x)]|,
\]
and in the non-homogeneous coordinates
\[
\pi_*\omega(x) = |f_{i_0}'(x), \ldots, f_{i_n}'(x), [\omega_{i_1}(x) - \omega_{i_0}(x)] \cdots [\omega_{i_n}(x) - \omega_{i_{n-1}}(x)]|.
\]

Thus, the condition $\pi_*\omega = d\log g$ is equivalent to the system of equations
\[
\begin{cases}
  f_i' = f_i \\
  \omega_j - \omega_i = d\log g_{ij}
\end{cases}
\]
where the second equation holds for all $x \in M$ so that $f_i(x) \neq 0$ and $f_j(x) \neq 0$.

Since $\{f_i\}_{i \in I}$ is a partition of unity on $M$, the sets $U_i = \{x \in M \mid f_i(x) \neq 0\}$ form an open covering of $M$ and the family of 1-forms $\{-\omega_i \in \Gamma(U_i, A^1_M)\}_{i \in I}$ determines a connection on a smooth principal $\mathbb{C}^\ast$-bundle induced by the map $g : M \to BC^\ast$. 

In other words, the group $\tilde{L}(M, C^*, \nabla)$ of smooth principal $C^*$-bundles with connections over $M$ is the pull-back
\[
\begin{array}{ccc}
\tilde{L}(M, C^*, \nabla) & \longrightarrow & \Gamma(M, E\mathcal{A}_{M,C}^1) \\
\downarrow & & \downarrow_{\pi_*} \\
C^\infty(M, BC^*) & \xrightarrow{\text{dlog}} & \Gamma(M, B\mathcal{A}_{M,C}^1)
\end{array}
\]
of the projection $\pi_* : \Gamma(M, E\mathcal{A}_{M,C}^1) \to \Gamma(M, B\mathcal{A}_{M,C}^1)$ by the homomorphism $\text{dlog} : C^\infty(M, BC^*) \to \Gamma(M, B\mathcal{A}_{M,C}^1)$.

The group $L(M, C^*, \nabla)$ of the isomorphism classes of smooth principal $C^*$-bundles over $M$ is the quotient of $\tilde{L}(M, C^*, \nabla)$ by the action of $C^\infty(M, EC^*)$ given by the formula
\[
f \cdot (g, \omega) = (g + \pi_*(f), \omega + \text{dlog}(f)),
\]
where $\pi_* : C^\infty(M, EC^*) \to C^\infty(M, BC^*)$ is the homomorphism induced by the projection $\pi : EC^* \to BC^*$.

Essentially the same as above arguments show that a connection on a smooth principal $B^sC^*$-bundle induced by a map $g : M \to B^{s+1}C^*$ is given by a global section $-\omega$ of the sheaf $EB^s\mathcal{A}_{M,C}^1$ so that $\text{dlog} = \pi_*\omega$, where
\[
\pi_* : \Gamma(M, EB^s\mathcal{A}_{M,C}^1) \to \Gamma(M, B^{s+1}\mathcal{A}_{M,C}^1).
\]
Moreover, two pairs $(g, \omega), (g', \omega') \in C^\infty(M, B^{s+1}C^*) \oplus \Gamma(M, EB^s\mathcal{A}_{M,C}^1)$ determine isomorphic smooth principal $B^sC^*$-bundles with connections if and only if there is a smooth map $h : M \to EB^sC^*$ so that
\[
(g, \omega) = (g' + \pi_*h, \omega' + \text{dlog} h).
\]

Now, we are going to define a 2-connection of the isomorphism class $[E, \omega]$ of a smooth principal $BC^*$-bundle $E \to M$ with a connection $\omega$.

Let $E \to M$ be a smooth principal $BC^*$-bundle induced from the universal principal $BC^*$-bundle $EBC^* \to B^2C^*$ by a map $g : M \to B^2C^*$ and let $\omega \in \Gamma(M, E\mathcal{A}_{M,C}^1)$ be a connection on $E \to M$. That is, $\pi_*\omega = \text{dlog} g$, where $\pi_* : \Gamma(M, E\mathcal{A}_{M,C}^1) \to \Gamma(M, B^2\mathcal{A}_{M,C}^1)$ is the homomorphism induced by the morphism of sheaves $\pi : E\mathcal{A}_{M,C}^1 \to B^2\mathcal{A}_{M,C}^1$.

The curvature $-d\omega$ of the connection $-\omega$ is a global section of the sheaf $B\mathcal{A}_{M,C}^2$, because the sequence
\[
0 \longrightarrow \Gamma(M, B\mathcal{A}_{M,C}^2) \overset{i_*}{\longrightarrow} \Gamma(M, E\mathcal{A}_{M,C}^2) \overset{\pi_*}{\longrightarrow} \Gamma(M, B^2\mathcal{A}_{M,C}^2) \longrightarrow 0
\]
is exact and $\pi_*(d\omega) = d(\pi_*\omega) = d(\text{dlog} g) = 0$.

If $(g', \omega')$ determines an isomorphic to $(E, \omega)$ smooth principal $BC^*$-bundle with a connection, then $d\omega = d(\omega' + \text{dlog} h) = d\omega'$, and hence a curvature determines a
homomorphism

\[ d : L(M, BC^* \ni \nabla) \to \Gamma(M, BA^2_{M, \mathbb{C}}) \]

where \( L(M, BC^* \ni \nabla) \) is the group of isomorphism classes of smooth principal \( BC^* \)-bundles with connections over \( M \).

Consider the following pull-back diagram

\[
\begin{array}{ccc}
L(M, BC^*, \nabla) & \xrightarrow{d} & \Gamma(M, BA^2_{M, \mathbb{C}}) \\
\downarrow & & \downarrow -\pi_* \\
\tilde{L}(M, BC^*, \nabla_1, \nabla_2) & \to & \Gamma(M, E\mathcal{A}^2_{M, \mathbb{C}})
\end{array}
\]

The group \( \tilde{L}(M, BC^*, \nabla_1, \nabla_2) \) consists of elements \( ([g, \omega_1, \omega_2]) \), where \( g : M \to B^2C^* \) is a smooth map, \( [g, \omega_1] \) is the isomorphism class of a smooth principal \( BC^* \)-bundle \( g^*(EBC^* \to B^2C^*) \) with a connection \( -\omega_1 \), and \( \omega_2 \) is a global section of the sheaf \( E\mathcal{A}^2_{M, \mathbb{C}} \) so that \( -\pi_* \omega_2 = d\omega_1 \). The equation \( -\pi_* \omega_2 = d\omega_1 \) is an analogue of the connection condition \( -\pi_* \omega = d\log g \), therefore we will refer to \( \omega_2 \) as a 2-connection of the equivalence class \( [g, \omega_1] \) of the pair \( (g, \omega_1) \).

Note that there is an action

\[
\Gamma(M, E\mathcal{A}^1_{M, \mathbb{C}}) \times \tilde{L}(M, BC^*, \nabla_1, \nabla_2) \to \tilde{L}(M, BC^*, \nabla_1, \nabla_2)
\]

of \( \Gamma(M, E\mathcal{A}^1_{M, \mathbb{C}}) \) on \( \tilde{L}(M, BC^*, \nabla_1, \nabla_2) \) given by

\[ \alpha \cdot ([g, \omega_1], \omega_2) = ([g, \omega_1 - \sigma(\alpha)], \omega_2 + d\alpha) \]

where \( \sigma \) is the composition

\[ \Gamma(M, E\mathcal{A}^1_{M, \mathbb{C}}) \xrightarrow{\pi_*} \Gamma(M, BA^1_{M, \mathbb{C}}) \xrightarrow{\iota_*} \Gamma(M, EBA^1_{M, \mathbb{C}}) \]

The quotient

\[ L(M, BC^*, \nabla_1, \nabla_2) = \frac{\tilde{L}(M, BC^*, \nabla_1, \nabla_2)}{\Gamma(M, E\mathcal{A}^1_{M, \mathbb{C}})} \]

will be called the group of equivalence classes of smooth principal \( BC^* \)-bundles with 1 and 2-connections over \( M \).

For \( s \geq 1 \), a group

\[ L(M, B^sC^*, \nabla_1, \nabla_2, \ldots, \nabla_{s+1}) = L(M, B^sC^*, \{\nabla_i\}_{i=1}^{s+1}) \]

of equivalence classes of smooth principal \( B^sC^* \)-bundles with \( k \)-connections, \( k = 1, 2, \ldots, s + 1 \), over \( M \) will be defined by the following inductive procedure.

Suppose, we have already constructed the group \( L(M, B^sC^*, \{\nabla_i\}_{i=1}^{k}) \) of equivalence classes \( [g, \omega_1, \ldots, \omega_k] \) of smooth principal \( B^sC^* \)-bundles with \( j \)-connections, for \( 1 \leq j \leq k < s + 1 \), over \( M \). Then the group \( L(M, B^sC^*, \{\nabla_i\}_{i=1}^{k+1}) \) is defined as follows.
Consider the pull-back diagram
\[
\begin{array}{ccc}
\tilde{L}(M, B^s\mathbb{C}^*, \nabla_1, \ldots, \nabla_{k+1}) & \longrightarrow & \Gamma(M, EB^{s-k}A_{M,C}^{k+1}) \\
\downarrow & & \downarrow (-1)^k \pi_* \\
L(M, B^s\mathbb{C}^*, \nabla_1, \ldots, \nabla_k) & \longrightarrow & \Gamma(M, B^{s-k+1}A_{M,C}^{k+1}),
\end{array}
\]
where \(d([g, \omega_1, \ldots, \omega_k]) = d\omega_k\). There is an action
\[
\Gamma(M, EB^{s-k}A_{M,C}^k) \times \tilde{L}(M, B^s\mathbb{C}^*, \nabla_1, \ldots, \nabla_{k+1}) \longrightarrow \tilde{L}(M, B^s\mathbb{C}^*, \nabla_1, \ldots, \nabla_{k+1})
\]
of \(\Gamma(M, EB^{s-k}A_{M,C}^k)\) on \(\tilde{L}(M, B^s\mathbb{C}^*, \nabla_1, \ldots, \nabla_{k+1})\) given by
\[
\alpha \cdot ([g, \omega_1, \ldots, \omega_k], \omega_{k+1}) = ([g, \omega_1, \ldots, \omega_{k-1}, \omega_k + (-1)^k \sigma(\alpha)], \omega_{k+1} + d\alpha),
\]
where \(\sigma\) is the composition
\[
\Gamma(M, EB^{s-k}A_{M,C}^k) \xrightarrow{\pi_*} \Gamma(M, B^{s-k+1}A_{M,C}^{k+1}) \xrightarrow{\iota_*} \Gamma(M, EB^{s-k+1}A_{M,C}^k).
\]
We set
\[
L(M, B^s\mathbb{C}^*, \nabla_1, \ldots, \nabla_{k+1}) = \frac{\tilde{L}(M, B^s\mathbb{C}^*, \nabla_1, \ldots, \nabla_{k+1})}{\Gamma(M, EB^{s-k}A_{M,C}^k)}.
\]

The form \((-1)^{k+1} \omega_{k+1}\), where \(\omega_{k+1}\) is the component of an element \(([g, \omega_1, \ldots, \omega_k], \omega_{k+1})\) of \(\tilde{L}(M, B^s\mathbb{C}^*, \{\nabla_i\}_{i=1}^{k+1})\) is called a \((k+1)\)-connection of \([g, \omega_1, \ldots, \omega_k]\). The image of \(([g, \omega_1, \ldots, \omega_k], \omega_{k+1})\) in \(L(M, B^s\mathbb{C}^*, \{\nabla_i\}_{i=1}^{k+1})\) will be denoted by \([g, \omega_1, \ldots, \omega_{k+1}]\).

Iterating the above procedure we get the group \(L(M, B^s\mathbb{C}^*, \{\nabla_i\}_{i=1}^{s+1})\) of equivalence classes of smooth principal \(B^s\mathbb{C}^*\)-bundles with \(k\)-connections, \(k = 1, \ldots, s + 1\).

**Proposition 5.1.** For every \(p \geq 2\) there is an isomorphism
\[
\mathbb{H}^p(M, \mathbb{Z}(p)^\infty_D) \cong L(M, B^{p-2}\mathbb{C}^*, \{\nabla_i\}_{i=1}^{p-1})
\]

**Proof.** Consider the double complex

\[
\begin{array}{ccc}
\Gamma(M, E\mathcal{A}^2_{M,C}) & \longrightarrow & \Gamma(M, E\mathcal{A}^2_{M,C}) \\
\downarrow d & & \downarrow d \\
\Gamma(M, E\mathcal{A}^1_{M,C}) & \longrightarrow & \Gamma(M, E\mathcal{A}^1_{M,C}) \\
\downarrow d & & \downarrow d \\
\Gamma(M, E\mathcal{C}^*) & \longrightarrow & \Gamma(M, E\mathcal{C}^*) \\
\downarrow d\log & & \downarrow d\log \\
C^{\infty}(M, E\mathcal{C}^*) & \longrightarrow & C^{\infty}(M, E\mathcal{C}^*)
\end{array}
\]

of the bar cochain complexes of the components of \(\mathcal{A}_{M,C}^*(d\log)\).
There is a sequence of isomorphisms
\[ \mathbb{H}^p(M, \mathbb{Z}(p)_p^n) \cong \mathbb{H}^{p-1}(A_{M,C}^p(d\log)) \cong H^{p-1}(\text{Tot}^*(B^{s,\leq p}_M), D), \]
where \((\text{Tot}^*(B^{s,\leq p}_M), D)\) is the total complex of the double complex \(B^{s,\leq p}_M = \{B^{n,s}_M\}_{s<p}\) defined as follows

\[ \text{Tot}^m(B^{s,\leq p}_M) = \bigoplus_{n+s=m,s<p} B^{n,s}_M \]

\[ B^{n,s}_M = \begin{cases} C^\infty(M, EB^n\mathbb{C}^*) & \text{for } s = 0, n \geq 0 \\ \Gamma(M, EB^n A^s_{M,C}) & \text{for } s > 0, n \geq 0 \\ 0 & \text{for } s < 0 \text{ or } n < 0 \end{cases} \]

and

\[ D : \text{Tot}^m(B^{s,\leq p}_M) \longrightarrow \text{Tot}^{m+1}(B^{s,\leq p}_M) \]
is so that

\[ D = \begin{cases} d\log + \sigma & \text{on } B^{n,0}_M \\ d + (-1)^s \sigma & \text{on } B^{n,s}_M \text{ for } s > 0. \end{cases} \]

A \((p-1)\)-cocycle in \((\text{Tot}^*(B^{s,\leq p}_M), D)\) is a sequence \((g, \omega_1, \ldots, \omega_{p-1})\), where \(g \in C^\infty(M, EB^{p-1}\mathbb{C}^*)\) and \(\omega_i \in \Gamma(M, EB^{p-i-1} A^i_{M,C})\) so that

\[ \begin{cases} \sigma(g) = 0 \\ d\log g = \sigma(\omega_1) \\ d\omega_i = (-1)^i \sigma(\omega_{i+1}) \text{ for } 1 \leq i \leq p-2 \end{cases} \]

The condition \(\sigma(g) = 0\) means that \(g\) is a smooth map from \(M\) to \(B^{p-1}\mathbb{C}^*\), the condition \(d\log g = \sigma(\omega_1)\) means that \(-\omega_1\) is a connection on the smooth principal \(B^{p-2}\mathbb{C}^*\)-bundle over \(M\) induced by \(g\), and the conditions \(d\omega_i = (-1)^i \sigma(\omega_{i+1})\) mean that \((-1)^{i+1} \omega_{i+1}\) is a \((i+1)\)-connection of the sequence \((g, \omega_1, \ldots, \omega_i)\). It is easy to see that two cocycles \((g, \omega_1, \ldots, \omega_{p-1})\) and \((g', \omega'_1, \ldots, \omega'_{p-1})\) are cohomologous in \((\text{Tot}^*(B^{s,\leq p}_M), D)\) if and only if the corresponding principal bundles with connections are equivalent in \(L(M, B^{p-2}\mathbb{C}^*, \{\nabla_i\}_{i=1}^{p-1})\). Thus we get an isomorphism

\[ H^{p-1}(\text{Tot}^*(B^{s,\leq p}_M), D) \cong L(M, B^{p-2}\mathbb{C}^*, \{\nabla_i\}_{i=1}^{p-1}). \]

\[ \square \]

The rest of the section is devoted to proofs of Theorems A and C.

**Proof of Theorem C.** Let us start from a definition of a scalar curvature.

The **scalar curvature** of the element \([g, \omega_1, \omega_2, \ldots, \omega_{p-1}]\) of \(L(M, B^{p-2}\mathbb{C}^*, \{\nabla_i\}_{i=1}^{p-1})\) is the \(\mathbb{C}\)-valued \(p\)-form \((-1)^{p-1} d\omega_{p-1}\). Note, that a priori \(d\omega_{p-1}\) is a global section of the sheaf \(E A^p_{M,C}\), because \(\omega_{p-1} \in \Gamma(M, E A^p_{M,C})\). But \(\pi_*(\omega_{p-1}) = d\omega_{p-2}\), and
hence, \( \pi_*(d\omega_{p-1}) = d(\pi_*\omega_{p-1}) = d(d\omega_{p-2}) = 0 \). Therefore, \( d\omega_{p-1} \in A^p_\mathbb{C}(M) \). Actually, \( d\omega_{p-1} \) is a closed (but not necessarily exact) \( \mathbb{C} \)-valued \( p \)-form, because locally it is exact.

Thus, a scalar curvature induces a homomorphism

\[
s : L(M, B^{p-2}\mathbb{C}^*, \{\nabla_i\}_{i=1}^{p-1}) \longrightarrow A^p_\mathbb{C}(M)_{cl},
\]

where \( A^p_\mathbb{C}(M)_{cl} \) is the group of \( \mathbb{C} \)-valued closed \( p \)-forms on \( M \).

The form \( d\omega_{p-1} \) is a \( p \)-cocycle in \( \text{Tot}^*(B_M^{\ast}) \) which is cohomologous to zero in this complex, because \( d\omega_{p-1} = D(g, \omega, \omega_2, \ldots, \omega_{p-1}) \).

Since \( \text{Tot}^*(B_M^{\ast}) \) is the acyclic resolution of \( A^{\ast}_{M,\mathbb{C}}(d\log) \), which in turn is a resolution of the constant sheaf of the group \( \mathbb{C}^* \), the image of \( d\omega_{p-1} \) in

\[
H^p(\text{Tot}^*(B_M^{\ast})) \cong H^p(A^{\ast}_{M,\mathbb{C}}(d\log)) \cong H^p(M; \mathbb{C}^*)
\]

is zero. Therefore, because the diagram

\[
\begin{array}{ccc}
A^p_\mathbb{C}(M)_{cl} & \longrightarrow & H^p(M; \mathbb{C}^*) \\
\downarrow & & \downarrow \\
H^p(M; \mathbb{C}) & \longrightarrow & & \end{array}
\]

commutes and the sequence

\[
0 \longrightarrow H^p(M; \mathbb{Z})_{TF} \longrightarrow H^p(M; \mathbb{C}) \longrightarrow H^p(M; \mathbb{C}^*)
\]

is exact, the cohomology class of \( d\omega_{p-1} \) in \( H^p(M; \mathbb{C}) \) belongs to the image of \( H^p(M; \mathbb{Z}) \) in \( H^p(M; \mathbb{C}) \). That is \( d\omega_{p-1} \) is a closed form with integral periods. Thus, we showed that the image \( \text{im}(s) \) of the scalar curvature homomorphism

\[
s : L(M, B^{p-2}\mathbb{C}^*, \{\nabla_i\}_{i=1}^{p-1}) \longrightarrow A^p_\mathbb{C}(M)_{cl}
\]

is contained in the group \( A^p_\mathbb{C}(M)_{0} \) of \( \mathbb{C} \)-valued closed \( p \)-forms with integral periods on \( M \).

Consider the following “scalar curvature diagram”

\[
\begin{array}{ccc}
0 & \longrightarrow & L(M, B^{p-2}\mathbb{C}^*, \nabla^{\text{flat}}) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & H^{p-1}(M; \mathbb{C}^*) \\
\end{array}
\quad
\begin{array}{ccc}
& \longrightarrow & \mathbb{H}^p(M, \mathbb{Z}(p)_{\mathbb{C}}^{\infty}) \\
& \longrightarrow & A^p_\mathbb{C}(M)_{0} \\
\end{array}
\quad
\begin{array}{ccc}
0 & \longrightarrow & 0
\end{array}
\]

where the first vertical arrow is the isomorphisms from Theorem B the second vertical arrow is the isomorphisms from Propositions 5.1, and \( i : \text{im}(s) \rightarrow A^p_\mathbb{C}(M)_{0} \) is the inclusion homomorphism.

The lower row short exact sequence of the scalar curvature diagram is obtained from the cohomology long exact sequence

\[
0 \longrightarrow H^{p-1}(M; \mathbb{C}^*) \longrightarrow \mathbb{H}^{p-1}(A^{p}_{M,\mathbb{C}}(d\log)) \longrightarrow A^p_\mathbb{C}(M)_{cl} \longrightarrow H^p(M; \mathbb{C}^*) \longrightarrow
\]
associated with the short exact sequence of sheaves

\[ 0 \rightarrow C_M^* \rightarrow A_{M,C}^{p}(d\log) \xrightarrow{d} (A_{M,C}^{p})_{d[-p+1]} \rightarrow 0 \]

In order to prove the exactness of the upper row of the scalar curvature diagram one has to show that the kernel \( \ker(s) \) of the scalar curvature homomorphism \( s \) coincides with the group \( L(M, B^{p-2}C^*, \nabla^{\text{flat}}) \) of isomorphism classes of smooth principal \( B^{p-2}C^* \)-bundles with flat connections over \( M \).

If \((g, \omega_1, \ldots, \omega_{p-1})\) is a \((p-1)\)-cocycle in \( \text{Tot}^*(B^{s,<p}_M) \), then the condition \( d\omega_{p-1} = 0 \) holds if and only if \((g, \omega_1, \ldots, \omega_{p-1})\) is a \((p-1)\)-cocycle in \( \text{Tot}^*(B^{s}_M) \). That is \((g, \omega_1, \ldots, \omega_{p-1})\) represents an element of the group \( H^{p-1}(M; \mathbb{C}^*) \). By Theorem B the group \( H^{p-1}(M; \mathbb{C}^*) \) is isomorphic to \( L(M, B^{p-2}C^*, \nabla^{\text{flat}}) \). Hence

\[ \ker(s) \cong H^{p-1}(M; \mathbb{C}^*) \cong L(M, B^{p-2}C^*, \nabla^{\text{flat}}). \]

It is easy to see that the scalar curvature diagram commutes. Therefore, from 5-lemma it follows that the inclusion \( i : \text{im}(s) \rightarrow A_{C}^p(M)_0 \) is an isomorphism. This finishes the proof of Theorem C. \( \square \)

**Proof of Theorem A.** First we are going to show that there is a commutative diagram

\[
\begin{array}{ccc}
\mathbb{H}^p(M, Z(p)_D) & \rightarrow & \mathbb{H}^p(M, Z) \\
\downarrow \cong & & \downarrow \cong \\
L(M, B^sC^*,\{\nabla_i\}_{i=1}^{s+1}) & \rightarrow & L(M, B^sC^*)
\end{array}
\]

with the vertical arrows being the isomorphisms from Propositions 5.1 and 3.3.

For every \( s \geq 1 \) there is the forgetful homomorphism

\[ \varphi^L : L(M, B^sC^*,\{\nabla_i\}_{i=1}^{s+1}) \rightarrow L(M, B^sC^*) \]

that sends the element \([E, \omega_1, \omega_2, \ldots, \omega_{s+1}]\) of \( L(M, B^sC^*,\{\nabla_i\}_{i=1}^{s+1})\) to the isomorphism class of the bundle \( E \). The homomorphism \( \varphi^L \) is surjective, because every smooth principal \( B^sC^*\)-bundle carries a connection and for every \( i \geq 1 \) the homomorphism

\[ \pi_* : \Gamma(M, EB^{s-i}A_{M,C}^{i}) \rightarrow \Gamma(M, B^{s-i+1}A_{M,C}^{i}) \]

is surjective.

If \((g, \omega_1, \ldots, \omega_{p-1})\) is a cocycle of \( \text{Tot}^*(B^{s,<p}_M) \), then the assignment

\[ (g, \omega_1, \ldots, \omega_{p-1}) \mapsto \{g_{ij}\}, \]

where

\[ g(x) = |t_{i_1}(x), t_{i_2}(x), \ldots, t_{i_n}(x), [g_{i_0i_1}(x)g_{i_1i_2}(x)] \ldots |g_{i_n-i_n}(x)| \]

induces a homomorphism

\[ \varphi^H : \mathbb{H}^{p-1}(A^{<p}_{M,C}(d\log)) \rightarrow H^1(B^{p-2}C^*_M) \]
so that the diagram
\[
\begin{array}{ccc}
\mathbb{H}^{p-1}(A^p_{M,C}(d\log)) & \xrightarrow{\tilde{\varphi}^H} & H^1(B^{p-2}C^*_{M}) \\
\downarrow \cong & & \downarrow \cong \\
L(M, B^{p-2}C^*, \{\nabla_i\}^{p-1}_{i=1}) & \xrightarrow{\varphi^L} & L(M, B^{p-2}C^*)
\end{array}
\]
commutes.

Composing \(\tilde{\varphi}^H\) with the isomorphisms
\[
\mathbb{H}^p(M, \mathbb{Z}(p)_{\infty}^D) \rightarrow \mathbb{H}^{p-1}(A^p_{M,C}(d\log))
\]
and
\[
H^1(B^{p-2}C^*_{M}) \rightarrow H^p(M, \mathbb{Z})
\]
we get the homomorphism
\[
\varphi^H : \mathbb{H}^p(M, \mathbb{Z}(p)_{\infty}^D) \rightarrow H^p(M, \mathbb{Z})
\]
so that the diagram
\[
\begin{array}{ccc}
\mathbb{H}^p(M, \mathbb{Z}(p)_{\infty}^D) & \xrightarrow{\varphi^H} & H^p(M, \mathbb{Z}) \\
\downarrow \cong & & \downarrow \cong \\
L(M, B^{p-2}C^*, \{\nabla_i\}^{p-1}_{i=1}) & \xrightarrow{\varphi^L} & L(M, B^{p-2}C^*)
\end{array}
\]
commutes.

To finish the proof of Theorem A we have to show that there is a commutative diagram
\[
\begin{array}{cccc}
0 & \rightarrow & \ker(\varphi^L) & \rightarrow & L(M, B^{p-2}C^*, \{\nabla_i\}^{p-1}_{i=1}) & \xrightarrow{\varphi^L} & L(M, B^{p-2}C^*) & \rightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
0 & \rightarrow & A^{p-1}_{C} / A^{p-1}_{C}(M) & \rightarrow & \mathbb{H}^p(M, \mathbb{Z}(p)_{\infty}^D) & \rightarrow & H^p(M, \mathbb{Z}) & \rightarrow & 0
\end{array}
\]
with exact rows and the vertical arrows being isomorphisms.

We have already shown that the right square of the above diagram is commutative. Exactness of the upper row is obvious. Exactness of the lower row short exact sequence is derived from the cohomology long exact sequence associated with the short exact sequence
\[
0 \rightarrow A^{p-1}_{C} / A^{p-1}_{C}(M) \rightarrow \mathbb{Z}(p)_{\infty}^D \rightarrow \mathbb{Z}(p)_{M} \rightarrow 0
\]
For details the reader is referred to the proof of Theorem 1.5.3 in [Br1].

Now we will show that there is a homomorphism \(\ker(\varphi^L) \rightarrow A^{p-1}_{C} / A^{p-1}_{C}(M)\).
Suppose \((g, \omega_1, \omega_2, \ldots, \omega_{p-1})\) represents an element \(\Lambda\) of \(L(M, B^{p-2}C^*, \{\nabla_i\}_{i=1}^{p-1})\) which is in the kernel of the homomorphism \(\varphi^L\). That is, \(g\) is a smooth map from \(M\) to \(B^{p-1}C^*\) inducing a smooth principal \(B^{p-2}C^*\)-bundle isomorphic to the trivial \(B^{p-2}C^*\)-bundle over \(M\). Equivalently, \(g\) is homotopic to a constant map. Hence, it has a lift to a map \(h\) from \(M\) into \(EB^{p-2}C^*\). That is, \(\pi_*h = g\). Therefore, the cocycle
\[
(g, \omega_1, \omega_2, \ldots, \omega_{p-1}) = (\pi_*h, \omega_1, \omega_2, \ldots, \omega_{p-1})
\]
is cohomologous to a cocycle
\[
(0, \omega_1 - \text{dlog} \, h, \omega_2, \ldots, \omega_{p-1}) = (0, \omega_1', \omega_2, \ldots, \omega_{p-1})
\]

Since the rows in the double complex \(B_M^{*,*}\) are exact (everywhere except at the zero level), there is \(\beta_1 \in \Gamma(M, EB^{p-3}A_M^1, \mathbb{C})\) so that \(\sigma(\beta_1) = \omega_1'\). Hence, the sequence \((0, \omega_1', \omega_2, \ldots, \omega_{p-1})\) is cohomologous to the sequence \((0, 0, \omega_2 + d\beta_1, \ldots, \omega_{p-1})\).

Iterating the above process we get a representative of \(\Lambda\) which is of the form \((0, 0, 0, \ldots, 0, \omega_{p-1}')\). Since \(\pi_*(\omega_{p-1}') = 0\), \(\omega_{p-1}'\) is actually a \(\mathbb{C}\)-valued \((p-1)\)-form on \(M\).

If \((0, 0, 0, \ldots, 0, \omega_{p-1}'')\) is another representative of \(\Lambda\), then there is \((\beta_0, \ldots, \beta_{p-2}) \in \text{Tot}^{p-2}(B_M^{*,*})\) so that
\[
(0, 0, 0, \ldots, 0, \omega_{p-1}'') = (0, 0, 0, \ldots, 0, \omega_{p-1}''\rightarrow D(\beta_0, \ldots, \beta_{p-2}).
\]
The above equality means that \((\beta_0, \ldots, \beta_{p-2})\) is a cocycle in \(\text{Tot}^{p-2}(B_M^{*,<p-1})\) whose scalar curvature is \(\omega_{p-1}' - \omega_{p-1}''\). From Theorem C we know that scalar curvatures are closed forms with integral periods. Therefore, we get a homomorphism
\[
\ker(\varphi^L) \rightarrow A^{p-1}_C(M)/A^{p-1}_C(M)_0
\]
\[
[0, 0, 0, \ldots, 0, \omega_{p-1}] \mapsto [\omega_{p-1}],
\]
where \([\omega_{p-1}]\) is the class of the form \(\omega_{p-1}\) in the quotient \(A^{p-1}_C(M)/A^{p-1}_C(M)_0\). It is easy to see that this homomorphism makes the right square of the diagram of Theorem A commutes. Hence, by 5-lemma, it is an isomorphism.

6. Holomorphic Deligne cohomology

In this section we define holomorphic principal \(B^*C^*\)-bundles and holomorphic \(k\)-connections on them and prove Theorem D.

A smooth map \(f : X \rightarrow B^nC^*\) is called a holomorphic map if \(\bar{\partial}f = 0\), where for
\[
f(x) = |t_1(x), \ldots, t_n(x), [f_1(x)] \cdots [f_n(x)]|
\]
\(\bar{\partial}f\) is defined by the analogous to \(df\) inductive formula
\[
\bar{\partial}f(x) = |t_1(x), \ldots, t_n(x), [\bar{\partial}f_1(x)] \cdots [\bar{\partial}f_n(x)]|
\]
In a similar way we define \(EB^nC^*\)-valued holomorphic maps.
A smooth principal $B^n C^*$-bundle is called a holomorphic principal $B^n C^*$-bundle if its transition functions are holomorphic maps. It is easy to see that if $f : X \to B^{n+1} C^*$ is a holomorphic map, then the induced by $f$ principal $B^n C^*$-bundle over $X$ is a holomorphic principal $B^n C^*$-bundle. There is also an inverse to the above statement.

**Proposition 6.1.** For every holomorphic principal $B^s C^*$-bundle $E \to M$ there is a holomorphic map $f : M \to B^{s+1} C^*$ such that $E \to M$ is the pull-back of the universal principal $B^s C^*$-bundle by $f$.

The proof of Proposition 6.1 is essentially a similar as the proof of Proposition 3.1.

Let $B^n O^*_X$ and $EB^n O^*_X$ be the sheaves of germs of $B^n C^*$ and $EB^n C^*$-valued holomorphic maps on $X$. The composition of the short exact sequences

$$0 \to B^n O^*_X \to EB^n O^*_X \to B^{n+1} O^*_X \to 0$$

gives the bar resolution

$$E O^*_X \to EBO^*_X \to EB^2 O^*_X \to \cdots$$

of the sheaf $O^*_X$ of non-vanishing holomorphic functions on $X$.

Let $\Omega^r_X$ be the sheaf of holomorphic $r$-forms on $X$ and let $A^r,s_X$ be the sheaf of smooth $(r, s)$-forms on $X$. The sheaf $EB^n \Omega^r_X$ is the kernel of the sheaf morphism

$$\bar{\partial} : EB^n A^r,0_X \to EB^n A^r,1_X,$$

which assigns to a local section

$$|t_1(x), \ldots, t_n(x), \alpha_1(x) \ldots \alpha_n(x)|$$

of the sheaf $EB^n A^r,0_X$ the section

$$|t_1(x), \ldots, t_n(x), \bar{\partial}\alpha_1(x) \ldots \bar{\partial}\alpha_n(x)|$$

of the sheaf $EB^n A^r,1_X$. In the same way we define the sheaf $B^n \Omega^r_X$. The composition of the short exact sequences

$$0 \to B^n \Omega^r_X \to EB^n \Omega^r_X \to B^{n+1} \Omega^r_X \to 0$$

gives the bar resolution

$$E \Omega^r_X \to EBO^r_X \to EB^2 \Omega^r_X \to \cdots$$

of the sheaf $\Omega^r_X$.

**Lemma 6.2.** For every $n \geq 0$ the sheaves $EB^n O^*_X$ and $EB^n \Omega^r_X$ are soft.

The proof of Lemma 6.2 is essentially the same as the proof of Lemma 2.2.

We will denote by $L^{hol}(X, B^r C^*, \{\nabla_i\}_{i=1}^{q-1})$ the group of equivalence classes of holomorphic principal $B^r C^*$-bundles over $X$ with $k$-connections, for $k = 1, 2, \ldots, q-1$, which is defined by replacing everywhere in the definition of the group of equivalence classes of smooth principal $B^r C^*$-bundles with $k$-connections, the word “smooth” by the word “holomorphic”.
Proof of Theorem D. Let $\Omega_{X}^{\leq q}(d\log)$ be the complex
\[ \mathcal{O}_{X}^{d\log} \xrightarrow{\partial} \Omega_{X}^{1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Omega_{X}^{q-1} \]
with $\mathcal{O}_{X}$ placed in degree zero. There is a quasi-isomorphism between $\Omega_{X}^{\leq q}(d\log)[-1]$ and the Deligne complex $Z(q)_{D}$, which is a holomorphic analogue of the quasi-isomorphism \[ \mathbb{H}^{r}(X, Z(q)_{D}) \cong \mathbb{H}^{r-1}(\Omega_{X}^{\leq q}(d\log)). \]

Consider the bar resolution $\mathcal{B}(\Omega_{X}^{\leq q}(d\log))$

\[ \begin{array}{cccccccc}
\partial & \partial & \partial & \partial & \partial & \partial & \partial & \partial \\
E\Omega_{X}^{2} & \xrightarrow{\sigma} & EB\Omega_{X}^{2} & \xrightarrow{\sigma} & EB\Omega_{X}^{1} & \xrightarrow{\sigma} & EB\Omega_{X}^{1} & \xrightarrow{\sigma} \\
\partial & \partial & \partial & \partial & \partial & \partial & \partial & \partial \\
E\Omega_{X}^{1} & \xrightarrow{\sigma} & EB\Omega_{X}^{1} & \xrightarrow{\sigma} & EB\Omega_{X}^{1} & \xrightarrow{\sigma} & EB\Omega_{X}^{1} & \xrightarrow{\sigma} \\
d\log & d\log & d\log & d\log & d\log & d\log & d\log & d\log \\
E\mathcal{O}_{X}^{*} & \xrightarrow{\sigma} & E\mathcal{O}_{X}^{*} & \xrightarrow{\sigma} & E\mathcal{O}_{X}^{*} & \xrightarrow{\sigma} & E\mathcal{O}_{X}^{*} & \xrightarrow{\sigma} \\
\end{array} \]

of the complex $\Omega_{X}^{\leq q}(d\log)$. Since this is an acyclic resolution there is an isomorphism
\[ \mathbb{H}^{r-1}(\Omega_{X}^{\leq q}(d\log)) \cong H^{r-1}(\text{Tot}^{*}(B^{\ast, \leq q}_{X})), \]
where $B^{\ast, \leq q}_{X}$ is the global sections complex associated with $\mathcal{B}(\Omega_{X}^{\leq q}(d\log))$ and $\text{Tot}^{*}(B^{\ast, \leq q}_{X})$ is the total complex of $B^{\ast, \leq q}_{X}$.

A $(r-1)$-cocycle in $\text{Tot}^{*}(B^{\ast, \leq q}_{X})$ is a sequence $(g, \omega_{1}, \ldots, \omega_{q-1})$, where $g \in \Gamma(X, EB_{r-1}\mathcal{O}_{X}^{*})$ and $\omega_{i} \in \Gamma(X, EB_{r-i-1}\Omega_{X}^{*})$ so that
\[ \begin{cases} 
\sigma(g) = 0 \\
d\log g = \sigma(\omega_{1}) \\
\partial \omega_{i} = (-1)^{i}\sigma(\omega_{i+1}) \quad \text{for } 1 \leq i \leq r-2 
\end{cases} \]

The condition $\sigma(g) = 0$ means that $g$ is a holomorphic map from $X$ to $B^{r-1}\mathcal{C}^{*}$, the condition $d\log g = \sigma(\omega_{1})$ means that $-\omega_{1}$ is a connection on the smooth principal $B^{p-2}\mathcal{C}^{*}$-bundle over $X$ induced by $g$, and the conditions $\partial \omega_{i} = (-1)^{i}\sigma(\omega_{i+1})$ mean that $(-1)^{i+1}\omega_{i+1}$ is a $(i+1)$-connection of the sequence $(g, \omega_{1}, \ldots, \omega_{i})$. Two cocycles $(g, \omega_{1}, \ldots, \omega_{q-1})$ and $(g', \omega_{1}', \ldots, \omega_{q-1}')$ are cohomologous in $\text{Tot}^{*}(B^{\ast, \leq q}_{X})$ if and only if the corresponding principal bundles with connections are equivalent. This gives us an isomorphism
\[ H^{r-1}(\text{Tot}^{*}(B^{\ast, \leq q}_{X})) \cong L^{hol}(X, B^{r-2}\mathcal{C}^{*}, \{\nabla_{i}\}_{i=1}^{q-1}). \]
In order to get the commutative diagram from Theorem D consider the \( \text{bar} \) resolution

\[
0 \to B(\Omega_{X}^{<p}[-1]) \to B(\mathbb{Z}(p)_{D}) \to B(\mathbb{Z}(p)_{X}) \to 0
\]

the short exact sequence

\[
0 \to \Omega_{X}^{<p}[-1] \to \mathbb{Z}(p)_{D} \to \mathbb{Z}(p)_{X} \to 0
\]

Since

\[
\mathbb{H}^{2p}(\text{Tot}^{*}(B(\mathbb{Z}(p)_{D}))) \cong L^{hol}(X, B^{r}C^{*}, \{\nabla_{i}\}^{p-1}_{i=1}),
\]

and

\[
\mathbb{H}^{2p}(\text{Tot}^{*}(B(\mathbb{Z}(p)_{X}))) \cong L(X, B^{r}C^{*}).
\]

The hypercohomology long exact sequence associated with (17) induces the lower short exact sequence of the diagram from Theorem D. The quasi-isomorphisms

\[
\begin{align*}
\Omega_{X}^{<p}[-1] & \to \text{Tot}^{*}(B(\Omega_{X}^{<p}[-1])) \\
\mathbb{Z}(p)_{D} & \to \text{Tot}^{*}(B(\mathbb{Z}(p)_{D})) \\
\mathbb{Z}(p)_{X} & \to \text{Tot}^{*}(B(\mathbb{Z}(p)_{X}))
\end{align*}
\]

induce the vertical isomorphisms in this diagram.

\[\square\]

**Appendix A**

**Principal Bundles, Topological Extensions, and Gerbs**

In this appendix we show that there is an isomorphism between the group of isomorphism classes of smooth (or holomorphic) principal \( BC^{*} \)-bundles over a manifold \( M \) (or a complex projective variety \( X \)) and the group of equivalence classes of smooth (or holomorphic) gerbes bound by \( \mathbb{C}_{M}^{*} \) (or \( \mathcal{O}_{X}^{*} \)). This isomorphism is induced by a construction, described in [Br1], which assigns to a principal \( G \)-bundle \( \pi : E \to B \) and a topological central extension

\[
1 \to C \to K \to G \to 1
\]

a sheaf of groupoids \( \mathcal{G}_{\pi} \) measuring the obstruction to the existence of a reduction of the structure group of \( \pi : E \to B \) to \( K \) (see pp. 171-172 in [Br1]). In the case of smooth (or holomorphic) principal \( BC^{*} \)-bundles and the extension

\[
0 \to \mathbb{C}^{*} \to E\mathbb{C}^{*} \to BC^{*} \to 0
\]

the gerbe \( \mathcal{G}_{\pi} \) is equivalent to the gerbe of sections of the bundle. We will also describe a procedure which assigns to connection on a principal \( BC^{*} \)-bundle a connective structure on the associated gerbe (see pp. 169-170 in [Br1]).

Let us start with a definition of a gerbe. A *gerbe* on a space \( X \) is a sheaf of categories \( \mathcal{C} \) on \( X \) (for the precise definition of a sheaf of categories see Chapter 5 in [Br1]) satisfying the following three conditions
• For every open subset $U \subset X$ the category $\mathcal{C}(U)$ is a groupoid, that is, every morphism is invertible.
• Each point $x \in X$ has a neighborhood $U_x$ for which $\mathcal{C}(U_x)$ is non-empty.
• Any two objects $P_1$ and $P_2$ of $\mathcal{C}(U)$ are locally isomorphic. This means that each $x \in U$ has a neighborhood $V$ such that the restrictions of $P_1$ and $P_2$ to $V$ are isomorphic.

A gerbe $\mathcal{C}$ is said to be bound by a sheaf $\mathcal{A}$ of abelian groups on $X$, if for every open set $U \subset X$ and every object $P$ of $\mathcal{C}(U)$ there is an isomorphism of sheaves
$$\alpha : \text{Aut}(P) \longrightarrow \mathcal{A}|_U,$$
where $\mathcal{A}|_U$ is the restriction of the sheaf $\mathcal{A}$ to $U$, and $\text{Aut}(P)$ is the sheaf of automorphisms of $P$ so that for an open subset $V$ of $U$ the group $\text{Aut}(P)(V)$ is the group of automorphisms of the restriction $r_V(P)$ of $P$ to $V$. Such an isomorphism is supposed to commute with morphisms of $\mathcal{C}$ and must be compatible with restriction to smaller open sets.

Two gerbes $\mathcal{C}$ and $\mathcal{D}$ bound by $\mathcal{A}$ on a manifold $M$ are equivalent if the following two conditions are satisfied.

• For every open subset $U$ of $M$ there is an equivalence of categories $\phi(U) : \mathcal{C}(U) \rightarrow \mathcal{D}(U)$ so that for every object $P$ of $\mathcal{C}(U)$ there is a commutative diagram

$$\begin{array}{ccc}
\text{Aut}_{\mathcal{C}(U)}(P) & \longrightarrow & \text{Aut}_{\mathcal{D}(U)}(P) \\
@V\phi(U)VV & @VV\alpha_DV & \\
\Gamma(U, \mathcal{A}) & \longrightarrow & \Gamma(U, \mathcal{A}) \\
@V\alpha_CVV & @VV\alpha_DV & \\
\end{array}$$

• For every pair of open subsets $V, U$ of $M$ so that $V \subset U$ there is an invertible natural transformation
$$\beta : \phi(U) \circ r_D \longrightarrow r_C \circ \phi(V),$$
where
$$r_C : \mathcal{C}(U) \longrightarrow \mathcal{C}(V), \quad r_D : \mathcal{D}(U) \longrightarrow \mathcal{D}(V),$$
are the restriction natural transformations. It is required that for a triple of open set $V \subset U \subset W$ in $M$ some compatibility conditions are satisfied (see p. 200 in [Br1]).

With every principal $G$-bundle $\pi : E \rightarrow B$ and every central extension of topological groups
$$1 \rightarrow C \rightarrow K \rightarrow G \rightarrow 1$$
we can associate a gerbe $\mathcal{G}_\pi$ bound by $\mathcal{C}_M$ on $B$. The gerbe $\mathcal{G}_\pi$ is derived from the sheaf of sections of the bundle $\pi : E \rightarrow B$. For every open subset $U$ of $B$ the objects and morphisms of $\mathcal{G}_\pi(U)$ are defined as follows.
Every section \( s : U \to \pi^{-1}(U) \) of \( \pi^{-1}(U) \to U \) can be identified with a \( G \)-equivariant map
\[
t_s : \pi^{-1}(U) \to G
\]
so that for every \( \xi \in \pi^{-1}(U) \) we have \( t_s(\xi) \cdot s(\pi(\xi)) = \xi \). Let \( E_s \to \pi^{-1}(U) \) be the pull-back of principal \( C \)-bundle \( K \to G \) from \( G \) to \( \pi^{-1}(U) \), by the map \( t_s : \pi^{-1}(U) \to G \). It is clear that the composition \( \pi \circ t_s : E_s \to U \) is a principal \( K \)-bundle, and hence a reduction of the structure group of \( \pi^{-1}(U) \to U \) to \( K \). The objects of \( \mathcal{G}_\pi(U) \) are pairs \((E, f)\) of principal \( K \)-bundles \( \tilde{\pi} : E \to U \) and principal \( C \)-bundles \( f : E \to \pi^{-1}(U) \) so that the diagram
\[
\begin{array}{ccc}
E & \xrightarrow{f} & \pi^{-1}(U) \\
\downarrow{\tilde{\pi}} & & \downarrow{\pi} \\
U & & 
\end{array}
\]
commutes. A morphism from \((E, f)\) to \((E', f')\) is a morphism of principal \( K \)-bundles \( g : E \to E' \) so that the diagram
\[
\begin{array}{ccc}
E & \xrightarrow{g} & E' \\
\downarrow{f} & & \downarrow{f'} \\
\pi^{-1}(U) & & 
\end{array}
\]
commutes. The above condition implies that the group of automorphisms of any object \((E, f)\) of \( \mathcal{G}_\pi(U) \) is the group of maps from \( U \) to \( C \), which is the section of the sheaf \( \mathcal{C}_M \) over \( U \). Thus \( \mathcal{G}_\pi \) is the gerbe bound by \( \mathcal{C}_M \).

Note, that the gerbe \( \mathcal{G}_\pi \) has a global section if and only if there is a reduction of the structure group of \( \pi : E \to B \) to \( K \). In particular, if \( \pi : E \to B \) is a principal \( BC^* \)-bundle, and our extension is the universal extension
\[
0 \to \mathbb{C}^* \to EC^* \to BC^* \to 0
\]
then the associated with \( \pi : E \to B \) gerbe \( \mathcal{G}_\pi \) measures the obstruction for the existence of a reduction of the structure group of \( \pi : E \to B \) to \( EC^* \). Since \( EC^* \) is contractible, every principal \( EC^* \)-bundle is trivial. Thus, the gerbe \( \mathcal{G}_\pi \) has a global section if and only if \( \pi : E \to B \) is a trivial \( BC^* \)-bundle. The same property has the gerbe \( \mathcal{S}_\pi \) of local sections of the bundle \( \pi : E \to B \), which is defined as follows. For every open subset \( U \) of \( M \) the objects of \( \mathcal{S}_\pi(U) \) are sections of \( \pi : E \to B \) over \( U \). Every local section \( s : U \to \pi^{-1}(U) \) of \( \pi : E \to B \) induces a \( BC^* \)-equivariant map \( t_s : \pi^{-1}(U) \to BC^* \), which in turn gives a map \( \tau_s = s \circ t_s : U \to BC^* \). Let \( L_s \) be the principal \( C^* \)-bundle over \( U \) induced by the map \( \tau_s \). A morphism between the objects \( s, s' \in \mathcal{S}_\pi(U) \) is a morphism \( L_s \to L_{s'} \) of the corresponding principal \( C^* \)-bundles. It is clear that \( \mathcal{S}_\pi \) is a gerbe bound by \( \mathcal{C}_M^* \). It is not a difficult exercise to see that the natural transformation \( \mathcal{S}_\pi(U) \to \mathcal{G}_\pi(U) \) sending a section \( s \) to the pull-back \( E_s \) of the universal principal \( C^* \)-bundle by \( t_s \) is an equivalence of categories that extends to an equivalence of gerbes \( \mathcal{S}_\pi \to \mathcal{G}_\pi \).
The following theorem is an easy consequence of Theorem H (see the introduction) and Theorem 5.2.8 from [Br].

**Theorem A.1.** A map which sends to the isomorphism class of a principal $BC^*$-bundle $\pi : E \to B$ the equivalence class of the gerbe of section $S_\pi$ of $\pi : E \to B$ induces an isomorphism between the group of isomorphism classes of principal $BC^*$-bundles and the group of equivalence classes of gerbes bound by $\mathbb{C}^*$.

Let $\mathcal{G}$ be a gerbe on $M$ bound by $\Sigma^*_M$. A connective structure on $\mathcal{G}$ is an assignment to each object $P$ in $\mathcal{G}(U)$ a $A^1_{M,C^*}$-torsor $\text{Cop}_P$ on $U$. That is $\text{Cop}_P$ is a sheaf with an action of $A^1_{M,C^*}$ on $\text{Cop}_P$ such that every point has a neighborhood $U$ with the property that for each open set $V \subset U$ the group $\text{Cop}_P(V)$ is a principal homogeneous space under the group $\Gamma(V,A^1_{M,C^*})$. The assignment $P \mapsto \text{Cop}_P(U)$ should be functorial with respect to restriction of $U$ to smaller open set and should be so that for any morpism $\psi : P \to Q$ of objects of $\mathcal{G}(U)$ (necessarily an isomorphism since $\mathcal{G}$ is a gerbe), there is an isomorphism $\psi_* : \text{Cop}_P(U) \to \text{Cop}_Q(U)$ of $A^1_{M,C^*}$-torsors, which is compatible with composition of morphisms in $\mathcal{G}(U)$ and also compatible with restrictions to smaller open sets. If $\psi$ is an automorphism of $P$ induced by a $C^*$-valued function $g$, we require that $\psi_*$ be the automorphism $\nabla \mapsto \nabla - \frac{dg}{g}$ of the $A^1_{M,C^*}$-torsor $\text{Cop}_P(U)$. In a similar way one can define a holomorphic connective structure on a holomorphic gerbe bound by $O^*_X$.

A connection $\omega$ on a smooth principal $BC^*$-bundle $\pi : E \to M$ induces the following connective structure on $\mathcal{G}_\pi$. Let $U$ be an open subset of $M$ so that $\mathcal{G}_\pi(U)$ is non-empty and let $\omega_U$ be the restriction of $\omega$ to $\pi^{-1}(U)$. To every element $(E,f)$ of $\mathcal{G}_\pi(U)$ we assign a set $\text{Cop}_E^\omega(U)$ of connections on $E$ compatible with $\omega$. That is $\bar{\omega} \in \text{Cop}_E^\omega(U)$ if $q \circ \omega = f^*\omega$, where $q : E/C \to BC$ and $f : E \to \pi^{-1}(U)$ is the principal $C^*$-bundle. It is easy to see that the assignment $\omega \mapsto \text{Cop}_E^\omega$ is a connective structure on $\mathcal{G}_\pi$ (for detail see pp. 169-170 in [Br]). The equivalence of gerbes $S_\pi \to \mathcal{G}_\pi$ can be used to pull-back the connective structure from $\mathcal{G}_\pi$ to $S_\pi$. A similar to the above construction assigns to a holomorphic connection on a holomorphic principal $BC^*$-bundle $E \to X$ a holomorphic connective structure on the associated with $E \to X$ holomorphic gerbe.

**Theorem A.2.** A map which sends to the isomorphism class of a principal $BC^*$-bundle $\pi : E \to B$ with a connection $\omega$ the equivalence class of the gerbe of section $S_\pi$ of $\pi : E \to B$ with the connective structure on $S_\pi$ induced by $\omega$ induces an isomorphism between the group of isomorphism classes of principal $BC^*$-bundles with connection and the group of equivalence classes of gerbes bound by $\mathbb{C}^*$ with connective structures.

We leave the proof of this theorem as an exercise for the reader.

**Appendix B**
The Geometric Bar Construction

The objective of this appendix is twofold. First, we define and review basic properties of the geometric bar construction. Second, we explain how the geometric bar construction can be derived from the projective space construction. Our basic references for the geometric bar construction are [Mm] and the survey paper [Sta].

The geometric bar construction assigns to every topological group $G$ a sequence of principal $G$-bundles $E_n \to B_n$ so that for every $n \geq 0$ the space $E_n$ is contractible in $E_{n+1}$. The universal principal $G$-bundle $EG \to BG$ is the union $\bigcup_{n \geq 1} E_n \to \bigcup_{n \geq 1} B_n$ taken with the weak topology.

If $G$ is an abelian topological group, then $EG$ and $BG$ are abelian topological groups and the projection $EG \to BG$ is a continuous homomorphism with $G$ as the kernel.

The geometric bar construction is functorial and it preserves products. That is, every continuous homomorphism $f : G \to H$ induces continuous maps $Ef : EG \to EH$ and $Bf : BG \to BH$, which are homomorphisms for $G$ abelian, and

$$E(G \times H) = EG \times EH \quad B(G \times H) = BG \times BH,$$

where each product is taken with the compactly generated topology.

If $G$ is a countable CW-group, then $EG$ and $BG$ are countable CW-complexes. In this appendix $G$ is a countable CW-group. Actually, in the main body of the paper $G$ is $\mathbb{Z}$, $\mathbb{C}$, $\mathbb{C}^*$, $S^1$, or the abelian group of a separable $\mathbb{C}$-vector space. A technical advantage of working with countable CW-groups is that on the spaces appearing in the definitions of $EG$ and $BG$ one can take the product, versus compactly generated, topology.

The archetypes of the geometric bar construction are infinite real, complex, and quaternionic projective spaces. Actually, Milnor found a construction that associates with every topological group $G$ a principal $G$-bundle $E \Delta G \to B \Delta G$, which is a limit of a sequence of principal $G$-bundles $(E \Delta G)_n \to (B \Delta G)_n$, so that for $G = S^0$, $S^1$, and $S^3$ the bundle $(E \Delta G)_n \to (B \Delta G)_n$ is isomorphic to $S^n \to \mathbb{R}P^n$, $S^{2n+1} \to \mathbb{C}P^n$, and $S^{4n+3} \to \mathbb{H}P^n$ respectively.

---

2To every topological space $X$ one can assign a space $(X, k)$ with compactly generated topology so that a set is open in $(X, k)$ if and only if its intersection with every compact subset of $X$ is open.

3A topological group $G$ is called a countable CW-group if it is a countable CW-complex so that the map $g \mapsto g^{-1}$ of $G$ into itself and the product map $G \times G \to G$ are both cellular (that is, they carry the $k$-skeleton into the $k$-skeleton).
A drawback of Milnor’s construction is that for $G$ being an abelian topological group the spaces $E_\Delta G$ and $B_\Delta G$ are not abelian groups, so the construction cannot be iterated. The geometric bar construction is a “normalized version” of Milnor’s construction that fixes this problem.

There are several approaches to geometric bar construction (for a survey on this subject see [Sta]). Usually, the spaces $EG$ and $BG$ are defined as the quotients of the disjoint unions $\bigsqcup_{n \geq 0} \Delta^n \times G^{n+1}$ and $\bigsqcup_{n \geq 0} \Delta^n \times G^n$ respectively, by certain equivalence relations. To explain the geometric meaning of these relations we preceded the formal definition of geometric bar construction with the Milnor and the Dold-Lashof constructions [Mr], [DL].

B.1. The unnormalized geometric bar construction. Let $G$ be a countable CW-group. The join $G \star G$ is the quotient of the product $\Delta^1 \times G \times G$ of the standard 1-simplex

$$\Delta^1 = \{(x_0, x_1) \in \mathbb{R}^2 \mid x_0, x_1 \geq 0, \quad x_0 + x_1 = 1\}$$

with $G \times G$ by the equivalence relation

$$(0, 1, g_0, g_1) \sim (0, 1, e, g_1), \quad (1, 0, g_0, g_1) \sim (1, 0, g_0, e)$$

where $e$ is the neutral element of $G$. The equivalence class of the sequence $(x_0, x_1, g_0, g_1)$ will be denoted by $x_0 g_0 \oplus x_1 g_1$.

Let $I = [0, 1]$. The homeomorphism

$$\Delta^1 \times G \times G \rightarrow G \times I \times G, \quad (x_0, x_1, g_0, g_1) \mapsto (g_0, x_1, g_1)$$

induces a homeomorphisms between $G \star G$ and the quotient of the product $G \times C(G)$ of $G$ with the cone $C(G) = I \times G/0 \times G$ by the equivalence relation

$$(g_0, [1, g_1]) \sim (e, [1, g_1])$$

where $[t, g]$ is the image of the pair $(t, g)$ in $C(G)$.

The $(n + 1)$-fold join

$$G \star (n + 1) \star G = G \star (G \star \cdots (G \star G) \cdots)$$

which we will also denote by $(E_\Delta G)_n$, can be identified with the quotient of the product $\Delta^n \times G^{n+1}$ of the standard simplex

$$\Delta^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_i \geq 0, \quad \sum_{i=0}^n x_i = 1\}$$

and the $(n + 1)$-fold product $G^{n+1}$ of $G$ with itself, by the equivalence relation

$$(x_0, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n, g_0, \ldots, g_i, \ldots, g_n) \sim$$

$$\sim (x_0, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n, g_0, \ldots, e, \ldots, g_n)$$
Actually, if we denote by \(x_0g_0 \oplus \cdots \oplus x_ng_n\) the equivalence class of the sequence \((x_0, \ldots, x_n, g_0, \ldots, g_n)\), then the homeomorphism between \((E_{\Delta}G)_n\) and the quotient \((\Delta^n \times G^{n+1})/\sim\) is given by

\[
x_0g_0 \oplus (1 - x_0)\left(x_1g_1 \oplus (1 - x_1)(\cdots (x_{n-1}g_{n-1} \oplus (1 - x_{n-1})g_n) \cdots)\right) \mapsto
x_0g_0 \oplus (1 - x_0)x_1g_1 \oplus \cdots \oplus \prod_{i=0}^{n-1}(1 - x_i)g_n
\]

On the other hand, \((E_{\Delta}G)_n = G \ast (E_{\Delta}G)_{n-1}\) can be identified with the quotient of \(G \times C((E_{\Delta}G)_{n-1})\) by the equivalence relation

\[
(g, [1, x]) \sim (e, [1, x]).
\]

Note that there are inclusions

\[
(E_{\Delta}G)_{n-1} \overset{i}{\longrightarrow} C((E_{\Delta}G)_{n-1}) \overset{J}{\longrightarrow} (E_{\Delta}G)_n
\]
given by \(i(y) = [1, y]\) and \(j([t, y]) = (1 - t)e \oplus ty\).

Let

\[
E_{\Delta}G = \bigcup_{n \geq 2} (E_{\Delta}G)_n = \bigcup_{n \geq 2} C((E_{\Delta}G)_n).
\]

Since \(E_{\Delta}G\) is the union of cones, it is contractible.

There is a free action of \(G\) on \((E_{\Delta}G)_n\) given by

\[
g \cdot (x_0g_0 \oplus \cdots \oplus x_ng_n) = x_0(gg_0) \oplus \cdots \oplus x_n(gg_n)
\]
The orbit space of this action is denoted by \((B_{\Delta}G)_n\). For example, \((B_{\Delta}G)_0\) is a single point and \((B_{\Delta}G)_1\) is the suspension of \(G\).

Since the actions of \(G\) on \((E_{\Delta}G)_n\) and \((E_{\Delta}G)_{n+1}\) are compatible with the embedding \((E_{\Delta}G)_n \subset (E_{\Delta}G)_{n+1}\), there is a free action of \(G\) on \(E_{\Delta}G\). The quotient space \((E_{\Delta}G)/G\) is denoted by \(B_{\Delta}G\) and the natural map \(E_{\Delta}G \to B_{\Delta}G\) is Milnor’s universal principal \(G\)-bundle.

**Example B.1.** For \(G = S^0 = \mathbb{Z}/2\mathbb{Z} = \{\pm 1\}\) there are homeomorphisms

\[
(E_{\Delta}S^0)_n \cong S^n, \quad (B_{\Delta}S^0)_n \cong \mathbb{R}P^n
\]

induced by the map

\[
(E_{\Delta}S^0)_n \ni x_0\alpha_0 \oplus \cdots \oplus x_n\alpha_n \mapsto (\alpha_0\sqrt{x_0}, \ldots, \alpha_n\sqrt{x_n}) \in S^n
\]

Similarly, for \(G = S^1 = U(1), S^3 = SU(2), \text{ or } \mathbb{C}^*\) there are the following homeomorphisms

\[
(E_{\Delta}S^1)_n \cong S^{2n+1}, \quad (B_{\Delta}S^1)_n \cong \mathbb{C}P^n
\]
\[
(E_{\Delta}S^3)_n \cong S^{4n+3}, \quad (B_{\Delta}S^3)_n \cong \mathbb{H}P^n
\]
\[
(E_{\Delta}\mathbb{C}^*)_n \cong S^{2n+1} \times (\mathbb{R}_+)^{n+1}, \quad (B_{\Delta}\mathbb{C}^*)_n \cong \mathbb{C}P^n \times S^n_+
\]

where \(\mathbb{R}_+\) is the set of positive real numbers and \(S^n_+\) is the intersection \((\mathbb{R}_+)^{n+1} \cap S^n\).
Sometimes, it is convenient to replace the diagonal action \( \{x\} \) of \( G \) on \( (E \Delta G)_n \) by the action of \( G \) on the first component of \( (E \Delta G)_n \). This can be done by introducing a non-homogeneous coordinates on \( (E \Delta G)_n \)

\[
[x_0, \ldots, x_n, h_0[h_1] \cdots |h_n]|_\Delta = x_0 g_0 \oplus x_1 (g_0^{-1} g_1) \oplus \cdots \oplus x_n (g_0^{-1} g_n)
\]

where \((g_0, \ldots, g_n) \in G^{n+1}, h_0 = g_0, \) and \( h_i = g_{i-1}^{-1} g_i \) for \( i > 0 \).

The non-homogeneous coordinates on \((E \Delta G)_n\) lead to yet another model of \((E \Delta G)_n\), due to Dold and Lashof \([DL]\). For example, in the 2-fold join \( G \ast G \) the relations

\[
0 g_0 \oplus 1 g_1 = 0 e \oplus 1 g_1, \quad 1 g_0 \oplus 0 g_1 = 1 g_0 \oplus 0 e
\]
correspond, in the non-homogeneous coordinates, to the relations

\[
|0, 1, h_0|h_1|_\Delta = |0, 1, e|h_0 h_1|_\Delta, \quad |1, 0, h_0|h_1|_\Delta = |1, 0, h_0|e|_\Delta
\]

Thus, the symbols \([x_0, x_1, h_0|h_1]|_\Delta\) can be identified with the points of the space

\[
DL(G) = G \times C(G) \cup_\mu G = (G \times C(G) \sqcup G)/\sim
\]

where \( \mu : G \times G \to G \) is the group operation in \( G \) and \( \sim \) is an equivalence relation identifying \((h_0, [1, h_1])\) with \( \mu(h_0, h_1) = h_0 h_1 \).

Note, that there is an action of \( G \) on \( DL(G) \) given by

\[
g \cdot ([x_0, x_1, h_0|h_1]|_\Delta) = [x_0, x_1, g h_0|h_1]|_\Delta
\]

and hence we can apply the above construction to \( DL(G) \).

In general, to any space \( E \) with a \( G \)-action \( \mu : G \times E \to E \) we can associate the space

\[
DL(E) = G \times C(E) \cup_\mu G = (G \times C(E) \sqcup G)/\sim
\]

where \( \sim \) is an equivalence relation identifying \((h, [1, x])\) with \( \mu(h, x) \), and the action

\[
G \times DL(E) \to DL(E), \quad g \cdot (h|t|x) = (gh)|t|x
\]

where \( h|t|x \) is the equivalence class of the sequence \((h, [t, x]) \in G \times C(E)\) in \( DL(E) \).

The spaces \( DL(E) \) and \( G \ast E \) are \( G \)-equivariantly homeomorphic to each other with the \( G \)-equivariantly homeomorphisms given by

\[
DL(E) \to G \ast E, \quad h|t|y \mapsto th \oplus (1 - t)(hy)\\
G \ast E \to DL(E), \quad x_0 h \oplus x_1 y \mapsto h|x_0|h^{-1}y
\]

Therefore, the bundles \( DL(E) \to DL(B) \) and \( G \ast E \to (G \ast E)/G \) are isomorphic.

Applying \( n \) times the Dold-Lashof construction to a topological group \( G \), we get a principal \( G \)-bundle \( DL^n(G) \to DL^n(G)/G \) which is isomorphic to the bundle \((E \Delta G)_n \to (B \Delta G)_n \).
B.2. The geometric bar construction. In general, the spaces $E \Delta G$ and $B \Delta G$ have not group structure, but for $G$ abelian, some quotients of these spaces are groups.

The appropriate quotients are obtained by replacing the cone $C(E)$ in the Dold-Lashof construction $DL(E)$ by the reduced cone

$$\tilde{C}(E) = (I \times E)/(0 \times E \cup I \times e)$$

where $e$ is a base point of $E$. For example, for $(E, e) = (G, e)$ where $G$ is a topological group with the neutral element $e$ we define

$$\tilde{DL}(G) = G \times \tilde{C}(G) \cup E \mu G$$

The space $\tilde{DL}(G)$ is a quotient of $DL(G)$ by the equivalence relation

$$h|t|e = h|0|e.$$

The group action of $G$ on $DL(G)$ decents to a group action of $G$ on $\tilde{DL}(G)$. Thus, we can iterate this construction getting for every $n \geq 1$ a space $\tilde{DL}^n(G)$ with a free action of $G$ on itself. We set $(EG)_n = \tilde{DL}^n(G)$ and $(BG)_n = \tilde{DL}^n(G)/G$.

It is easy to see that $(EG)_n$ is the quotient of the disjoint union $\coprod_{m=0}^n \Delta^m \times \mathcal{G}^{m+1}$ by the equivalence relations

$$(x_0, \ldots, x_m, g_0, \ldots, g_m) \sim \\
\sim \begin{cases} 
(x_0, \ldots, x_i + x_{i+1}, \ldots, x_m, g_0, \ldots, \hat{g}_i, \ldots, g_m) & \text{for } g_i = g_{i+1} \text{ or } x_i = 0, 0 \leq i < m \\
(x_0, \ldots, x_{m-1} + x_m, g_0, \ldots, g_{m-1}) & \text{for } g_{m-1} = g_m \text{ or } x_m = 0
\end{cases}$$

In the non-homogeneous coordinates on $(EG)_n$ the above relations take the form

$$(t_1, \ldots, t_m, h_0[h_1] \cdots [h_m]) \sim \\
\sim \begin{cases} 
(t_2, \ldots, t_m, h_0h_1[h_2] \cdots [h_m]) & \text{for } t_1 = 0 \text{ or } h_0 = e \\
(t_1, \ldots, \hat{t}_i, \ldots, t_m, h_0[h_1] \cdots [h_{i+1}] \cdots [h_m]) & \text{for } t_i = t_{i+1} \text{ or } h_i = e \\
(t_1, \ldots, t_{m-1}, h_0[h_1] \cdots [h_{m-1}]) & \text{for } t_m = 1 \text{ or } h_m = e
\end{cases}$$

where $0 \leq t_1 \leq t_2 \leq \cdots \leq t_m \leq 1$ are non-homogeneous coordinates on $\Delta^m$ related with the baricentric coordinated $x_0, \ldots, x_n$ on $\Delta^n$ by the formula

$$t_i = x_0 + x_1 + \cdots + x_{i-1}.$$

The equivalence class of a sequence $(x_0, \ldots, x_m, g_0, \ldots, g_m)$ will be denoted by $|x_0, \ldots, x_m, g_0, \ldots, g_m|$ and the equivalence class of a sequence $(t_1, \ldots, t_m, h_0[h_1] \cdots [h_m])$ will be denoted by $|t_1, \ldots, t_m, h_0[h_1] \cdots [h_m]|$.

The space $EG$ is the quotient of the disjoint union $\coprod_{m=0}^\infty \Delta^m \times \mathcal{G}^{m+1}$ by the above equivalence relations.
Similarly, \((BG)_n\) is the quotient of the disjoint union \(\prod_{m=0}^n \Delta^m \times G^m\) by the equivalence relations
\[
(x_0, \ldots, x_m, [g_0 : \cdots : g_m]) \sim \begin{cases} 
(x_0, \ldots, x_i + x_{i+1}, \ldots, x_m, [g_0 : \cdots : \hat{g}_i : \cdots : g_m]) & \text{for } g_i = g_{i+1} \text{ or } x_i = 0, 0 \leq i < m \\
(x_0, \ldots, x_{m-1} + x_m, [g_0 : \cdots : g_{m-1}]) & \text{for } g_{m-1} = g_m \text{ or } x_m = 0
\end{cases}
\]
where \([g_0 : \cdots : g_m]\) is the equivalence class of the sequence \((g_0, \ldots, g_n) \in G^{m+1}\) by the equivalence relation
\[(g_0, \ldots, g_n) \sim (gg_0, gg_1, \ldots, gg_m)\]
for any \(g \in G\).

In the non-homogeneous coordinates on \((BG)_n\) the above relations take the form
\[
(t_1, \ldots, t_m, [h_1|\cdots|h_m]) \sim \begin{cases} 
(t_2, \ldots, t_m, [h_2|\cdots|h_m]) & \text{for } t_1 = 0 \text{ or } h_0 = e \\
(t_1, \ldots, t_{i+1}, t_m, [h_1|\cdots|h_i h_{i+1} | \cdots | h_m]) & \text{for } t_i = t_{i+1} \text{ or } h_i = e \\
(t_1, \ldots, t_{m-1}, [h_1|\cdots|h_{m-1}]) & \text{for } t_m = 1 \text{ or } h_m = e
\end{cases}
\]

The equivalence class of a sequence \((x_0, \ldots, x_m, [g_0 : \cdots : g_m])\) will be denoted by \(\langle x_0, \ldots, x_m, [g_0 : \cdots : g_m] \rangle\) and the equivalence class of a sequence \((t_1, \ldots, t_m, [h_1|\cdots|h_m])\) will be denoted by \(\langle t_1, \ldots, t_m, [h_1|\cdots|h_m] \rangle\).

The space \(BG\) is the quotient of the disjoint union \(\prod_{m=0}^\infty \Delta^m \times G^m\) by the above equivalence relations.

The projection \(EG \to BG\) is given by the formula
\[
\langle x_0, \ldots, x_m, g_0, \ldots, g_m \rangle \mapsto \langle x_0, \ldots, x_m, [g_0 : \cdots : g_m] \rangle
\]
or in the non-homogeneous coordinates by
\[
\langle t_1, \ldots, t_m, h_0[h_1|\cdots|h_m] \rangle \mapsto \langle t_1, \ldots, t_m, [h_1|\cdots|h_m] \rangle
\]

Sometimes it is convenient to write the elements of \(EG\) and \(BG\) in the form
\[
\langle m_0, \ldots, m_n x, h_0[h_1|\cdots|h_m] \rangle
\]
and
\[
\langle m_0, \ldots, m_n x, [h_1|\cdots|h_m] \rangle
\]
respectively, which is mixture of the baricentric coordinates on \(\Delta^m\) and homogeneous coordinates on \(G^{m+1}\) or \(G^m\).
B.3. A simplicial description of the geometric bar construction. The above definitions of $EG$ and $BG$ can be interpreted in terms of geometric realizations of some simplicial objects. Actually, to every topological group $G$ one can assign simplicial topological groups $EG$ and $BG$, defined as follows.

$EG_n = G^{n+1}$, the face homomorphisms $\partial_i : EG_n \to EG_{n-1}$ are given by the formula

$$\partial_i(g_0, \ldots, g_n) = (g_0, \ldots, g_{i-1}, \widehat{g}_i, g_{i+1}, \ldots, g_n)$$

or in the non-homogeneous coordinates by

$$\partial_i(h_0[h_1 | \cdots | h_n]) = \begin{cases} h_0 h_1[h_2|h_3| \cdots |h_n] & \text{for } i = 0 \\ h_0[h_1| \cdots | h_i \cdot h_{i+1}| \cdots |h_n] & \text{for } 0 < i < n \\ h_0[h_1| \cdots | h_{n-1}] & \text{for } i = n \end{cases}$$

The degeneracy homomorphism $s_i : EG_n \to EG_{n+1}$ are given by the formula

$$s_i(g_0, \ldots, g_n) = (g_0, \ldots, g_{i-1}, g_i, g_{i+1}, \ldots, g_n)$$

or in the non-homogeneous coordinates by

$$s_i(h_0[h_1 | \cdots | h_n]) = \begin{cases} h_0[e|h_1| \cdots |h_n] & \text{for } i = 0 \\ h_0[h_1| \cdots | h_i e|h_{i+1}| \cdots |h_n] & \text{for } 0 < i < n \\ h_0[h_1| \cdots | h_n e] & \text{for } i = n \end{cases}$$

$EG_n = G^n$, the face homomorphisms $\partial_i : BG_n \to BG_{n-1}$ in the homogeneous coordinates on $G^n = G^{n+1}/G$ is given by the formula

$$\partial_i([g_0 : \cdots : g_n]) = [g_0 : \cdots : g_{i-1} : \hat{g}_i : \cdots : g_n]$$

or in the non-homogeneous coordinates by

$$\partial_i([h_1 | \cdots | h_n]) = \begin{cases} [h_2|h_3| \cdots |h_n] & \text{for } i = 0 \\ [h_1| \cdots | h_i \cdot h_{i+1}| \cdots |h_n] & \text{for } 0 < i < n \\ [h_1| \cdots | h_{n-1}] & \text{for } i = n \end{cases}$$

The degeneracy homomorphisms $s_i : BG_n \to BG_{n+1}$ are given by the formula

$$s_i([g_0 : \cdots : g_n]) = [g_0 : \cdots : g_{i-1} : g_i : g_{i+1} : \cdots : g_n]$$

or in the non-homogeneous coordinates by

$$s_i([h_1 | \cdots | h_n]) = \begin{cases} [e|h_1| \cdots |h_n] & \text{for } i = 0 \\ [h_1| \cdots | h_i e|h_{i+1}| \cdots |h_n] & \text{for } 0 < i < n \\ [h_1| \cdots | h_n e] & \text{for } i = n \end{cases}$$
The geometric realization $|EG.|$ of the simplicial space $EG.$ is by definition the quotient space of the infinite disjoint union $\bigcup_{n=0}^{\infty} \Delta^n \times G^{n+1}$ by the equivalence relations

$$(\partial^i x, \bar{g}) \sim (x, \partial_i \bar{g}) \quad \text{for} \quad (x, \bar{g}) \in \Delta^{n-1} \times G^{n+1}$$

$$(s^i x, \bar{g}) \sim (x, s_i \bar{g}) \quad \text{for} \quad (x, \bar{g}) \in \Delta^{n+1} \times G^{n+1}$$

where the maps $\partial^i : \Delta^{n-1} \to \Delta^n$ and $s^i : \Delta^{n+1} \to \Delta^n$ are defined in the baricentric coordinates by

$$\partial^i(x_0, \ldots, x_n) = (x_0, \ldots, x_{i-1}, 0, x_i, \ldots, x_n)$$

$$s^i(x_0, \ldots, x_n) = (x_0, \ldots, x_{i-1}, x_i + x_{i+1}, x_{i+2}, \ldots, x_n)$$

and in the non-homogeneous coordinates by

$$\partial^i(t_0, \ldots, t_{n+1}) = (t_0, \ldots, t_i, t_{i+1}, \ldots, t_{n+1})$$

$$s^i(t_0, \ldots, t_{n+1}) = (t_0, \ldots, t_i, t_{i+1}, t_{i+2}, \ldots, t_{n+1})$$

Similarly, the geometric realization $|BG.|$ of the simplicial space $BG.$ is the quotient space of the disjoint union $\bigcup_{n=0}^{\infty} \Delta^n \times G^n$ by the equivalence relations

$$(\partial^i x, \bar{g}) \sim (x, \partial_i \bar{g}) \quad \text{for} \quad (x, \bar{g}) \in \Delta^{n-1} \times G^n$$

$$(s^i x, \bar{g}) \sim (x, s_i \bar{g}) \quad \text{for} \quad (x, \bar{g}) \in \Delta^{n+1} \times G^n$$

**B.4. Group structures on $EG$ and $BG.$** The usefulness of the non-homogeneous coordinates $t_1, \ldots, t_n$ on $\Delta^n$ comes from the fact that they supply a very simple formula

$$(t_1, \ldots, t_n) \times (t_{n+1}, \ldots, t_{n+m+1}) \mapsto (t_{\sigma(1)}, t_{\sigma(2)}, \ldots, t_{\sigma(n+m+1)})$$

for a homeomorphism pairing

$$\Delta^n \times \Delta^m \longrightarrow \Delta^{n+m}$$

where $\sigma$ is a permutation of the set $\{1, 2, \ldots, n + m + 1\}$ such that

$$t_{\sigma(1)} \leq t_{\sigma(2)} \leq \cdots \leq t_{\sigma(n+m+1)}.$$

Using this pairing we can define, for $G$ an abelian topological group, commutative, associative, and continuous pairings

$$|t_1, \ldots, t_n, h[h_1| \cdots |h_n]| + |t_{n+1}, \ldots, t_{n+m+1}, h'[h_{n+1}| \cdots |h_{n+m+1}]| =$$

$$= |t_{\sigma(1)}, \ldots, t_{\sigma(n+m)}, h \cdot h'[h_{\sigma(1)}| \cdots |h_{\sigma(n+m+1)}]|$$

and

$$|t_1, \ldots, t_n, [h_1| \cdots |h_n]| + |t_{n+1}, \ldots, t_{n+m+1}, [h_{n+1}| \cdots |h_{n+m+1}]| =$$

$$= |t_{\sigma(1)}, \ldots, t_{\sigma(n+m+1)}, [h_{\sigma(1)}| \cdots |h_{\sigma(n+m+1)}]|$$

on $EG$ and $BG$ respectively, which induce group structure on these spaces $[Mn].$
The above group pairings can be interpreted as the compositions

\[ EG \times EG = |EG| \times |EG| \xrightarrow{\phi} |EG \times EG| \xrightarrow{\psi} |E(G \times G)|. \]

\[ BG \times BG = |BG| \times |BG| \xrightarrow{\varphi} |BG \times BG| \xrightarrow{\psi} |B(G \times G)|. \]

where \( \bar{\varphi} \) and \( \varphi \) are the commutativity of geometric realization and product operations homeomorphisms, \( \psi \) and \( \psi \) are induced by the maps

\[ EG_n \times EG_n \rightarrow EG_n \]

\[ (g_0, \ldots, g_n) \times (g'_0, \ldots, g'_n) \mapsto ((g_0, g'_0), \ldots, (g_n, g'_n)) \]

and

\[ BG_n \times BG_n \rightarrow BG_n \]

\[ [g_0 : \cdots : g_n] \times [g'_0 : \cdots : g'_n] \mapsto [(g_0, g'_0) : \cdots : (g_n, g'_n)] \]

respectively, and \( |\bar{\mu}|, |\mu| \) are induced by the simplicial morphisms

\[ \bar{\mu} : E(G \times G). \rightarrow EG. \]

\[ \bar{\mu}((g_0, g'_0), \ldots, (g_n, g'_n)) = (g_0 g'_0, \ldots, g_n g'_n) \]

and

\[ \mu : B(G \times G). \rightarrow BG. \]

\[ \mu([(g_0, g'_0) : \cdots : (g_n, g'_n)]) = [g_0 g'_0 : \cdots : g_n g'_n] \]

which are well defined only when the multiplication pairing \( \mu : G \times G \rightarrow G \) is a homomorphism, or equivalently, when \( G \) is an abelian group.

References

[Bei] A. A. Beilinson, Higher regulators and values of L-functions, J. Soviet Math. 30 (1985), 2036–2070.

[Br1] J.-L. Brylinski, Loop spaces, characteristic classes, and geometric quantization, Birkhauser, 1992.

[Br2] J.-L. Brylinski, Holomorphic gerbs and the Beilinson regulator, preprint, 1993.

[CS] J. Cheeger and J. Simons, Differential characters and geometric invariants, Geometry and Topology (Berlin) (J. Alexander and J. Harer, eds.), Lecture Notes in Mathematics, vol. 1167, Springer Verlag, Berlin, 1985, pp. 50–80.

[DL] A. Dold and R. Lashof, Principal fibrations and fibre homotopy equivalence of bundles, Illinois J. Math. 3 (1959), 285–305.

[EV] H. Esnault and E. Viehweg, Deligne-Beilinson cohomology, Beilinson’s Conjectures on Special Values of L-functions (M. Rapoport, N. Shappacher, and P. Schneider, eds.), Perspectives in Mathematics, vol. 4, Academic Press, 1988, pp. 43–92.

[Gir] J. Giraud, Cohomologie non-abiétienne, Grundl., vol. 179, Springer-Verlag, 1971.

[Kar] M. Karoubi, Formes topologiques non commutatives, ESI Preprint no. 112 (1994). Available via anonymous ftp or gopher from ftp.esi.ac.at.

[Mr] J. W. Milnor, Construction of universal bundles, II, Ann. of Math. (2) 63 (1956), no. 3, 430–436.

[Mm] J. Milgram, The bar construction and abelian h-spaces, Illinois J. Math. 11 (1967), 242–250.
[RSS] M. Rapoport, N. Shappacher, and P. Schneider (eds.), Beilinson’s conjectures on special values of L-functions, Perspectives in Mathematics, vol. 4, Academic Press, 1988.

[Sta] J.D. Stasheff, H-spaces and classifying spaces: Foundations and recent developments, Proc. of Sympos. Pure Math., vol. 22, AMS, 1971, pp. 247–272.

[tD] T. tom Dieck, Klassifikation numerierbarer Bündel, Arch. Math. (Basel) 17 (1966), 395–399.

Department of Mathematics, Texas A&M University, College Station, TX 77843-3368

E-mail address: gajer@math.tamu.edu