Sharp Reverse Isoperimetric Inequalities in Nonpositively Curved Cones

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Abstract
We prove a pair of sharp reverse isoperimetric inequalities for domains in nonpositively curved surfaces: (1) metric disks centered at the vertex of a Euclidean cone of angle at least $2\pi$ have minimal area among all nonpositively curved disks of the same perimeter and the same total curvature; (2) geodesic triangles in a Euclidean (resp. hyperbolic) cone of angle at least $2\pi$ have minimal area among all nonpositively curved geodesic triangles (resp. all geodesic triangles of curvature at most $-1$) with the same side lengths and angles.

Keywords Reverse isoperimetric inequalities · Euclidean cone · Nonpositive curvature · Geometric inequalities · Area comparison

Mathematics Subject Classification Primary 53C21; Secondary 49Q10

1 Introduction

Isoperimetric inequalities provide upper bounds on the area of domains in a surface with a fixed metric (typically of constant curvature) in terms of their boundary length; see [4] for an account on this classical subject.

Often the metric is fixed (flat, hyperbolic, or spherical), but there are a few instances where isoperimetric inequalities hold for large classes of metrics satisfying curvature...
bounds. Thus, Weil ([11], 1926) developed such inequalities for nonpositively curved planes, proving the Cartan–Hadamard isoperimetric conjecture in dimension two.

In this article, we establish a pair of sharp reverse isoperimetric inequalities providing lower bounds on the area of some domains in terms of their boundary length. These geometric inequalities hold for nonpositively curved surfaces. The optimal metrics for such reverse isoperimetric inequalities, as well as the extremal domains, can be described in terms of the total curvature of the domains under consideration. The boundary cases of equality in our optimal inequalities are attained by Euclidean cones with nonpositive total curvature. These inequalities provide bounds that are stronger than Euclidean ones.

It seems that such geometric inequalities should have been known already for some time, but we were unable to find them in the literature. A possible reason is that previous research focused more on isoperimetric inequalities for special homogeneous metrics regardless of the total curvature of the domains considered.

We will now present our two main results. As motivation, note that the area of a Euclidean cone of angle less than $2\pi$ can be easily decreased among nonnegatively curved metrics by smoothing off the tip of the cone. Our first result shows that this is impossible while preserving the nonpositive curvature condition; see Theorem 1.2 for a more general statement. Namely, one cannot decrease the area of a metric disk centered at the vertex of a Euclidean cone of angle at least $2\pi$ among all nonpositively curved disks of the same perimeter and the same total curvature. We will need the following definition.

**Definition 1.1** Given a surface with a complete Riemannian metric with Gaussian curvature function $K$, we define the nonnegative and nonpositive parts of $K$ as

$$K^+ = \max\{K, 0\} \quad \text{and} \quad K^- = \max\{-K, 0\}$$

so that $K = K^+ - K^-$. Our convention is consistent with the corresponding definitions of nonnegative and nonpositive parts of curvature measures introduced by Burago in [3]; see Definition 2.1. We can now state our first result.

**Theorem 1.2** Let $M$ be a surface with a complete Riemannian metric. Then every disk $D$ of radius $R$ and boundary length $L$ in $M$ satisfies the bound

$$\text{area}(D) \geq \text{area}(\hat{D}) \quad (1.1)$$

where $\hat{D}$ is the disk with the same boundary length $L$ as $D$, centered at the vertex of the Euclidean cone with total curvature $-\int_D K^- dA$.

A formula for the area of $\hat{D}$ is given in Sect. 2.

Comparison geodesic triangles play an important role in nonpositive curvature geometry. For our second result, we consider a geodesic triangle of Gaussian curvature at most $\lambda_0 \leq 0$ along with its comparison triangle (in the strong sense) in a cone $C^{\lambda_0}_{\theta}$ of constant nonpositive curvature $\lambda_0$ with angle $\theta$. Here, a comparison triangle is taken in
the following strong sense: both the angles and the side lengths of the two triangles are the same. The following theorem asserts that the area of the initial triangle is bounded from below by the area of its comparison triangle in the cone. (For simplicity, one can assume that $\lambda_0 = 0$.)

**Theorem 1.3** Let $\Delta$ be a geodesic (two-dimensional) triangle in a surface with a complete Riemannian metric of Gaussian curvature $K \leq \lambda_0$ for some constant $\lambda_0 \leq 0$. Suppose $\tilde{\Delta}$ is a geodesic (two-dimensional) triangle with the same side lengths and the same angles $\alpha, \beta, \gamma$ as $\Delta$ in the cone $\mathcal{C}_{\lambda_0}^0$ of constant curvature $\lambda_0$ with angle $\theta = 3\pi - (\alpha + \beta + \gamma)$. Then

$$\text{area}(\Delta) \geq \text{area}(\tilde{\Delta})$$

with equality if and only if $\Delta$ is isometric to $\tilde{\Delta}$.

Our proof of Theorem 1.3 also shows that the area of a geodesic triangle of Gaussian curvature at most $\lambda_0$ is bounded from below by the area of a comparison triangle having the same base length and the same adjacent angles in the plane $\mathbb{H}_{\lambda_0}$ of constant curvature $\lambda_0 \leq 0$; see Proposition 3.1.

We would like to provide some context for our study of reverse isoperimetric inequalities. We found the geometric inequalities of this paper while working on a proof that systolically extremal nonpositively curved surfaces are flat with finitely many conical singularities; see [7]. In that context, it was clear that it should be impossible to round off a conical singularity of angle greater than $2\pi$ in a nonsystolic region in order to decrease the area (while keeping the nonpositive curvature condition) by any cut-and-paste argument with metric disks of the same perimeter. This observation is formalized by our Theorem 1.2. Theorem 1.3 is a variation on this theme, while trying to cut-and-paste triangles instead of disks. Though we did not use these reverse isoperimetric inequalities in our argument, they confirmed our intuition that piecewise flat metrics with conical singularities should play a role in extremal systolic geometry through their local extremal features.

## 2 Reverse Isoperimetric Inequality for Metric Disks

The reverse isoperimetric inequality for metric disks established in this section shows the optimality of the nonpositively curved Euclidean cones for the area with respect to compact deformations keeping the same curvature sign.

Before proving this result, we need to extend the notion of curvature to singular spaces.

**Definition 2.1** Associated to a surface $M$ with a complete Riemannian metric $g$ of Gaussian curvature $K$ is the curvature measure $K \, dA$, where $dA$ is the area measure of $M$. The notion of curvature measure extends to piecewise flat surfaces with conical singularities (and more generally to Alexandrov surfaces), where it is denoted by $\omega$. The curvature measure $\omega$ is a signed measure which can be decomposed as $\omega = \omega^+ - \omega^-$, where $\omega^+$ and $\omega^-$ are the nonnegative and nonpositive parts of $\omega$, and are both nonnegative measures. For smooth metrics, we have $\omega^\pm = K^\pm \, dA$. 

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We refer to [1] and [9] for a precise definition of the curvature measure; see also [10] for a modern exposition on Alexandrov surfaces.

We will make use of the following result on bi-Lipschitz metric approximation, announced by Reshetnyak [8] and proved by Yu. Burago [3, Lemma 6], in the more general setting of Alexandrov surfaces; see also [4, Theorem 3.1.1], [10] and [7, Sect. 3].

**Proposition 2.2** (See [8] and [3, Lemma 6]) Let $M$ be a compact surface (possibly with boundary) with a Riemannian metric. Then there is a sequence $M_i$ of piecewise flat surfaces with conical singularities converging to $M$ in the Lipschitz topology such that the nonnegative and nonpositive parts $\omega_i^\pm$ of the curvature measure $\omega_i$ of $M_i$ weakly converge to their counterparts $\omega^\pm$ for the curvature measure $\omega$ on $M$.

Consider a sequence $(g_i)$ of piecewise flat metrics with conical singularities approximating a given complete Riemannian metric $g$ on a surface for every compact domain as in Proposition 2.2. Denote by $M$ the surface with the complete Riemannian metric $g$, and by $M_i$ the same surface with the piecewise flat metric $g_i$.

**Proposition 2.3** Fix $p \in M$. Let $D \subseteq M$ and $D_i \subseteq M_i$ be the disks of radius $R$ centered at the same point $p$. Then

1. the area of the symmetric difference $D \Delta D_i$ tends to zero (for every area measure). That is, $|D \Delta D_i| \to 0$;
2. $\lim \text{area}(D_i, g_i) = \text{area}(D, g)$;
3. $\lim \omega_i^\pm(D_i) = \omega^\pm(D)$;
4. $\lim \inf \text{length}(\partial D_i, g_i) \geq \text{length}(\partial D, g)$.

**Proof**

1. By bi-Lipschitz convergence of the metrics, the symmetric difference $D \Delta D_i$ is contained in the $\varepsilon$-tubular neighborhood $U_{\varepsilon}(\partial D)$ of $\partial D$ for $i$ large enough. Since $\partial D$ is 1-rectifiable, the area of this tubular neighborhood tends to zero (see [6, Theorem 3.2.39]), and the result follows.

2. By bi-Lipschitz convergence of the metrics, we have the inclusions

$$D - \varepsilon \subseteq D_i \subseteq D + \varepsilon$$

for $i$ large enough, where $D \pm \varepsilon \subseteq M$ are the balls of radius $R \pm \varepsilon$ centered at $p$. We also have weak convergence of the area measures. Taking the area with respect to $g_i$ in the previous double inclusion between $D_i$ and $D \pm \varepsilon$, and using the weak convergence of the area measure, we obtain

$$\text{area}(D - \varepsilon, g) - \varepsilon \leq \text{area}(D_i, g_i) \leq \text{area}(D + \varepsilon, g) + \varepsilon$$

for $i$ large enough. Since the area of the $\varepsilon$-tubular neighborhood $U_{\varepsilon}(\partial D)$ of $\partial D$ tends to zero, the result is immediate.

3. As in the proof of item (2), using the weak convergence of the curvature measure instead of the area measures, we obtain

$$\omega^\pm(D - \varepsilon) - \varepsilon \leq \omega_i^\pm(D_i) \leq \omega^\pm(D + \varepsilon) + \varepsilon$$
for $i$ large enough. Since the curvature measure $\omega^\pm = K^\pm \, dg$ is absolutely continuous with respect to the area measure, the curvature measure of the $\varepsilon$-tubular neighborhood $U_\varepsilon(\partial D)$ of $\partial D$ tends to zero. Hence the result.

(4) The flat distance between the one-cycles $\partial D$ and $\partial D_i$ is bounded by the mass of the 2-current defined as the difference $D - D_i$, see [6] for precise definitions. This mass is equal to the area of $\partial D \Delta D_i$. Thus, by (1), the sequence $\partial D_i$ converges to $\partial D$ in the flat topology. The desired result follows from the lower semicontinuity of the mass (here, the length); see [6].

We can now proceed to the proof of the following theorem. Recall that a disk of radius $R$ centered at the vertex of a Euclidean cone of angle $\theta$ has perimeter $L = \theta R$ and area $A = \frac{\theta}{2} R^2 = \frac{L^2}{2\pi}$. Thus, we have

$$A = \frac{L^2}{2(2\pi - K)} \quad (2.1)$$

where $K = 2\pi - \theta < 0$ is the total curvature of the Euclidean cone.

With this formula, Theorem 1.2 can be restated as follows.

**Theorem 2.4** Let $M$ be a surface with a complete Riemannian metric. Then every disk $D$ of radius $R$ of boundary length $L$ in $M$ satisfies

$$\text{area}(D) \geq \frac{L^2}{4\pi + 2K_D^-}$$

where $K_D^- = \int_D K^- \, dA \geq 0$ is the total mass of the nonpositive part of the curvature measure of $D$.

**Proof** Proposition 2.3 shows that it is sufficient to prove Theorem 2.4 for piecewise flat metrics with conical singularities. The desired result for Riemannian metrics will follow by piecewise flat metric approximation; see Proposition 2.2. This enables us to avoid regularity issues.

Let $p$ be the center of the disk $D \subseteq M$ of radius $R$. Consider the (closed) disk $D_t$ and the circle $C_t$ of radius $R - t$ centered at $p$. Note that $D_0 = D$ and $C_0 = \partial D$. Since the metric on $M$ is piecewise flat, the circle $C_t$ is a piecewise smooth curve (possibly with several connected components) bounding the metric disk $D_t$ in $M$. Moreover, the length function $t \mapsto L(C_t)$ is differentiable except for a finite number of values of $t$, and its derivative is given by the first variation formula; see [4, Lemma 3.2.3]. Namely, as long as $D_t$ is nonempty, we have

$$L'(C_t) = -\int_{C_t} \kappa(s) \, ds - S_t$$

for almost every $t$, where $\kappa$ is the geodesic curvature of the curve $C_t$ and $S_t$ is the sum of the angular difference of the tangent vectors at the corner points of $C_t$. 

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By the Gauss–Bonnet formula for polyhedral metrics (see [9, Theorem 5.3.2]), we derive

\[ L'(C_t) = -2\pi \chi(D_t) + \omega(D_t). \]

Since \( D_t \) is a connected region with boundary, its Euler characteristic \( \chi(D_t) \) is at most 1, that is, \( \chi(D_t) \leq 1 \). Since \( D_t \subseteq D \) and \( \omega = \omega^+ - \omega^- \), we have

\[ \omega(D_t) \geq -\omega^- (D). \]

Combining these two bounds, we deduce that

\[ L'(C_t) \geq -2\pi - \omega^- (D). \]

Integrating this relation leads to

\[ L(C_t) \geq L(\partial D) - (2\pi + \omega^- (D)) t. \quad (2.2) \]

In particular, the domain \( D_t \) is nonempty for every \( t < t_0 \), where

\[ t_0 = \frac{L(\partial D)}{2\pi + \omega^- (D)}. \]

By the coarea formula, integrating the inequality (2.2) between 0 and \( t_0 \), we obtain

\[ \text{area}(D) \geq L(\partial D) t_0 - \frac{1}{2} (2\pi + \omega^- (D)) t_0^2. \]

In other words, we have

\[ \text{area}(D) \geq \frac{L(\partial D)^2}{2(2\pi + \omega^- (D))} \]

where the right-hand side represents the area of the disk centered at the vertex of the Euclidean cone with total curvature \( K = -\omega^- (D) \) and with the same boundary length \( L(\partial D) \) as \( D \); see formula (2.1).

\[ \square \]

### 3 Area Comparison for Triangles with the Same Base

In order to prove Theorem 1.3, we will need the following result, which may be of independent interest. This result provides a lower bound on the area of a geodesic triangle of Gaussian curvature at most \( \lambda_0 \), in terms of the area of a comparison triangle having the same base length with the same adjacent angles in the plane \( \mathbb{H}_{\lambda_0} \) of constant curvature \( \lambda_0 \leq 0 \).
Proposition 3.1 Let $\Delta$ be a geodesic (two-dimensional) triangle with vertices $A$, $B$, $C$ in a surface $M$ with a complete Riemannian metric of Gaussian curvature $K \leq \lambda_0$ for some constant $\lambda_0 \leq 0$. Let $\bar{\Delta}$ be a geodesic (two-dimensional) triangle with distinct vertices $\bar{A}$, $\bar{B}$, $\bar{C}$ in the plane $\mathbb{H}_{\lambda_0}$ of constant curvature $\lambda_0$ such that

- the sides $\bar{A}\bar{B}$ and $AB$ have the same length;
- the angles at $\bar{A}$ and $A$ are the same;
- the angles at $\bar{B}$ and $B$ are the same.

Then

$$\text{area}(\Delta) \geq \text{area}(\bar{\Delta})$$ (3.1)

with equality if and only if $\Delta$ is isometric to $\bar{\Delta}$.

We first establish the proposition when one of the angles at $A$ or $B$, say $B$, is greater or equal to $\frac{\pi}{2}$; see Lemma 3.2. We will then derive the general result from this particular case.

Lemma 3.2 Let $\Delta$ and $\bar{\Delta}$ be as in Proposition 3.1. Let $\alpha$, $\beta$, $\gamma$ be the angles of $\Delta$ at $A$, $B$, $C$. Suppose that $\beta \geq \frac{\pi}{2}$. Then

$$\text{area}(\Delta) \geq \text{area}(\bar{\Delta})$$

with equality if and only if $\Delta$ is isometric to $\bar{\Delta}$.

Proof Let $\hat{\Delta}$ be a geodesic triangle of $\mathbb{H}_{\lambda_0}$ with the same side lengths as $\Delta$. Denote by $\hat{A}$, $\hat{B}$, $\hat{C}$ the vertices of $\hat{\Delta}$ and by $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$ their angles. We can (and will) assume that $\hat{A} = A$ and $\hat{B} = B$. By [2, Proposition II.1.7.(4)], we have

$$\begin{cases} \hat{\alpha} \geq \alpha \\
\hat{\beta} \geq \beta. \end{cases}$$ (3.2)

In particular, $\hat{\beta} \geq \frac{\pi}{2}$. Since the sum of the angles of a geodesic triangle in a nonpositively curved surface is at most $\pi$; see [2, Proposition II.1.7.(4)], we derive that $\hat{\gamma} \leq \frac{\pi}{2}$.

The geodesic triangle of $\mathbb{H}_{\lambda_0}$ with side $\hat{A}\hat{B}$, and angles $\alpha$ and $\beta$ at $\hat{A}$ and $\hat{B}$ can be isometrically identified to $\Delta$. It follows from the relations (3.2) that the triangle $\hat{\Delta}$ lies in $\Delta$; see Fig. 1. Since $\hat{\gamma} \leq \frac{\pi}{2}$, we deduce that

$$|\hat{A}\hat{C}| \leq |\hat{A}\hat{C}| = |AC|.$$ 

This relation holds for any point $D$ lying in the segment $BC$ by replacing $\hat{C}$ with $\hat{D}$ and $\hat{C}$ with $\hat{D}$ (note that $\hat{D}$ lies in the segment $\bar{B}\bar{C}$).

Consider the two exponential maps $\exp_{\hat{A}} : T_{\hat{A}}M \to M$ and $\exp_{\hat{A}} : T_{\hat{A}}\mathbb{H}_{\lambda_0} \to \mathbb{H}_{\lambda_0}$. Define $\sigma : \mathbb{H}_{\lambda_0} \to M$ as

$$\sigma = \exp_{\hat{A}} \circ I \circ \exp_{\hat{A}}^{-1}$$

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Proof of Proposition 3.1

Let \( \alpha, \beta, \gamma \) be the angles of \( \Delta \) at \( A, B, C \). The cases where \( \alpha \geq \frac{\pi}{2} \) or \( \beta \geq \frac{\pi}{2} \) are covered by Lemma 3.2. Thus, we can assume that \( \alpha < \frac{\pi}{2} \) and \( \beta < \frac{\pi}{2} \). This implies that the projection \( H \) of \( C \) to the segment \( AB \) strictly lies between \( A \) and \( B \). Furthermore, both angles \( \angle AHC \) and \( \angle BHC \) are equal to \( \frac{\pi}{2} \). Thus, the height \( CH \) decomposes \( \Delta \) into two right triangles \( \Delta' \) and \( \Delta'' \). Denote by \( \gamma' \) and \( \gamma'' \) the angles of \( \Delta' \) and \( \Delta'' \) at \( C \). Observe that \( \gamma = \gamma' + \gamma'' \). Since the angles of \( \Delta' \) and \( \Delta'' \) at \( H \) are right, we can apply Lemma 3.2 to the triangles \( \Delta' = AH'C' \) and \( \Delta'' = BH''C'' \), where \( H = H' = H'' \) and \( C = C' = C'' \). Thus, the areas of the triangles \( \Delta' \) and \( \Delta'' \) are bounded from below by the areas of the right triangles \( \tilde{\Delta}' = \tilde{A}\tilde{H}'\tilde{C}' \) and \( \tilde{\Delta}'' = \tilde{B}\tilde{H}''\tilde{C}'' \) in \( \mathbb{H}_{\lambda_0} \). We can glue together these two right triangles of \( \mathbb{H}_{\lambda_0} \) along their sides \( \tilde{H}'\tilde{C}' \) and \( \tilde{H}''\tilde{C}'' \) so that \( \tilde{H}' \) and \( \tilde{H}'' \) coincide. Since the angles at \( \tilde{H}' \) and \( \tilde{H}'' \) are right, the two segments \( \tilde{A}\tilde{H}' \) and \( \tilde{H}''\tilde{B} \) form a long segment \( \tilde{A}\tilde{B} \) of the same length as \( AB \); see Fig. 2.

If the heights \( \tilde{H}'\tilde{C}' \) and \( \tilde{H}''\tilde{C}'' \) have the same length, the union of these two triangles form a large triangle which satisfies the same geometric features as \( \tilde{\Delta} \) and so can be identified with \( \tilde{\Delta} \).

If one of these heights, say \( \tilde{H}'\tilde{C}' \), is shorter than the other, we extend the hypotenuse \( \tilde{A}\tilde{C}' \) until it intersects the other hypotenuse \( \tilde{B}\tilde{C}'' \) at some point \( \tilde{O} \); see Fig. 2. As previously, the triangle \( \tilde{A}\tilde{O}\tilde{B} \) can be identified with \( \tilde{\Delta} \). Moreover, the area of this triangle is bounded by the sum of the two triangles \( \tilde{\Delta}' \) and \( \tilde{\Delta}'' \).

In either case, we have

\[
\text{area}(\tilde{\Delta}) \leq \text{area}(\tilde{\Delta}') + \text{area}(\tilde{\Delta}'') \leq \text{area}(\Delta)
\]
with equality if and only if $\Delta$ is isometric to $\bar{\Delta}$.

\section{Area Comparison for Triangles with the Same Side Lengths and Angles}

We can now prove our second main result.

\begin{theorem}
Let $\Delta$ be a geodesic (two-dimensional) triangle in a surface with a complete Riemannian metric of Gaussian curvature $K \leq \lambda_0$ for some constant $\lambda_0 \leq 0$. Suppose $\bar{\Delta}$ is a geodesic (two-dimensional) triangle with the same side lengths and the same angles $\alpha$, $\beta$, $\gamma$ as $\Delta$ in the cone $C_{\lambda_0}^{\theta}$ of constant curvature $\lambda_0$ with angle $\theta = 3\pi - (\alpha + \beta + \gamma)$. Then
\begin{equation}
\text{area}(\Delta) \geq \text{area}(\bar{\Delta})
\end{equation}
with equality if and only if $\Delta$ is isometric to $\bar{\Delta}$.
\end{theorem}

\begin{remark}
Unlike the situation with Proposition 3.1, the comparison triangle $\bar{\Delta}$ in Theorem 4.1 does not necessarily exist in general.
\end{remark}

\begin{proof}[Proof of Theorem 4.1]
Let $A$, $B$, and $C$ be the vertices of $\Delta$, with angle $\alpha$ at the vertex $A$, angle $\beta$ at $B$, and angle $\gamma$ at $C$. Denote by $\bar{A}$, $\bar{B}$, $\bar{C}$ the corresponding vertices of $\bar{\Delta}$.

By [2, Proposition II.1.7.(4)], the angle $\theta$ of the conical singularity of the cone is at least $2\pi$. Moreover, we can assume that $\theta > 2\pi$ and that the conical singularity lies in $\Delta$. Otherwise, the sum $\alpha + \beta + \gamma$ of the angles of $\Delta$ and $\bar{\Delta}$ would be equal to $\pi$ and so the triangle $\Delta$ would be flat isometric to $\bar{\Delta}$ by [2, Proposition II.2.9].

The geodesic rays joining the vertices of $\bar{\Delta}$ to its conical singularity decompose each angle around the vertices of $\bar{\Delta}$ into two angles. In particular, the angle $\alpha$ splits into two angles $\alpha'$ and $\alpha''$, where $\alpha = \alpha' + \alpha''$. The same holds with $\beta$ and $\gamma$. Observe that these geodesic rays decompose $\bar{\Delta}$ into three small triangles.

We would like to carry out a similar construction for $\Delta$, except that there is no conical singularity to rely on. Instead, we consider the geodesic rays of $\Delta$ emanating from the vertices of $\Delta$ and splitting each angle as in $\bar{\Delta}$. These three geodesic rays do not necessarily intersect at a single point as in $\bar{\Delta}$. Nevertheless, they decompose $\Delta$ into three triangles $\Delta_a$, $\Delta_b$, $\Delta_c$ as in Fig. 3. Recall that two geodesic rays in a nonpositively curved surface intersect at most once. There may be a small triangular region
\begin{equation}
X = \Delta \setminus (\Delta_a \cup \Delta_b \cup \Delta_c)
\end{equation}

\end{proof}
lying in $\Delta$ which is not covered by the triangles $\Delta_a$, $\Delta_b$, $\Delta_c$.

Let $A'$ be the vertex of $\Delta_a$ different from $B$ and $C$. Denote by $\tilde{\Delta}_a$ the triangle of $\theta_{\lambda,0}$ with vertices $A'$, $\tilde{B}$, $\tilde{C}$ such that

- the sides $\tilde{B}\tilde{C}$ and $BC$ have the same length;
- the angles at $\tilde{B}$ and $B$ are the same (equal to $\beta''$);
- the angles at $\tilde{C}$ and $C$ are the same (equal to $\gamma'$).

Similarly, we define $\tilde{\Delta}_b$ and $\tilde{\Delta}_c$. By construction, the three triangles $\tilde{\Delta}_a$, $\tilde{\Delta}_b$ and $\tilde{\Delta}_c$ are isometric to the three smaller triangles forming $\tilde{\Delta}$ and delimited by the geodesic segments joining the conical singularity of $\tilde{\Delta}$ to its vertices. By Proposition 3.1, we have

\[ \text{area}(\tilde{\Delta}_a) \leq \text{area}(\Delta_a) \]

with equality if and only if $\Delta_a$ is isometric to $\tilde{\Delta}_a$. The same holds with $\tilde{\Delta}_b$ and $\tilde{\Delta}_c$.

Since the triangle $\tilde{\Delta}$ is partitioned into $\tilde{\Delta}_a$, $\tilde{\Delta}_b$ and $\tilde{\Delta}_c$, we derive that

\[
\begin{align*}
\text{area}(\tilde{\Delta}) &= \text{area}(\tilde{\Delta}_a) + \text{area}(\tilde{\Delta}_b) + \text{area}(\tilde{\Delta}_c) \\
&\leq \text{area}(\Delta_a) + \text{area}(\Delta_b) + \text{area}(\Delta_c) \\
&= \text{area}(\Delta) - \text{area}(X) \\
&\leq \text{area}(\Delta)
\end{align*}
\]

with equality if and only if $\Delta$ is isometric to $\tilde{\Delta}$.

\[ \Box \]

**Remark 4.3** It follows from the proof of Theorem 4.1 that the difference between \( \text{area}(\Delta) \) and \( \text{area}(\tilde{\Delta}) \) is bounded from below by the area of the small triangle $X = \Delta \setminus (\Delta_a \cup \Delta_b \cup \Delta_c)$ lying in $\Delta$.

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