On the monotonicity of the eigenvector method

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Niemand aber dürfte folgendes Prinzip bestreiten1

(Edmund Landau: Über Preisverteilung bei Spielturnieren)

Abstract

Pairwise comparisons are used in a wide variety of decision situations when the importance of different alternatives should be measured by numerical weights. One popular method to derive these priorities is based on the right eigenvector of a multiplicative pairwise comparison matrix. We introduce an axiom called monotonicity: increasing an arbitrary entry of a pairwise comparison matrix should increase the weight of the favoured alternative (which is in the corresponding row) by the greatest factor and should decrease the weight of the favoured alternative (which is in the corresponding column) by the greatest factor. It is proved that the eigenvector method violates this natural requirement. We also investigate the relationship between non-monotonicity and the Saaty inconsistency index. It turns out that the violation of monotonicity is not a problem in the case of nearly consistent matrices. On the other hand, the eigenvector method remains a dubious choice for inherently inconsistent large matrices such as the ones that emerge in sports applications.

Keywords: Decision analysis; pairwise comparisons; axiomatic approach; monotonicity; eigenvector method

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1 Introduction

Several decision-making methods involve the comparison of the criteria and the alternatives in pairs, making judgements, and the compilation of the results into multiplicative positive reciprocal pairwise comparison matrices. For instance, it is a crucial element of the

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1 “But nobody should deny the following principle” (Landau, 1914, p. 201)
popular Analytic Hierarchy Process (AHP), introduced by Thomas L. Saaty (Saaty, 1977, 1980). He has suggested deriving the priorities from such a matrix by its principal right eigenvector, which is called the eigenvector method. Since AHP has a wide variety of applications (Barker and Zabinsky, 2011; Ho, 2008; Saaty and Vargas, 2012; Subramanian and Ramanathan, 2012; Tam and Tummala, 2001; Vaidya and Kumar, 2006), a better understanding of this procedure seems to be an important research question. We will focus on some shortcomings caused by its mathematical properties.

Johnson et al. (1979) note that the use of the left eigenvectors is equally justified as long as the order is reversed, furthermore, the results from the two eigenvectors may disagree even when the matrix is nearly consistent. According to Genest et al. (1993), in the case of numerically coded ordinal preferences, the ranking obtained from the principal right eigenvector depends on the choice of parameter for the preferences. Bana e Costa and Vansnick (2008) prove that the right eigenvector can violate a condition of order preservation, which is fundamental in decision aiding according to the author’s opinion. Kulakowski (2015) examines the relationship between this property and the inconsistency index proposed by Saaty. Pérez and Mokotoff (2016) present an example where the alternative with the highest priority according to all decision-maker is not the best on the basis of their aggregated preferences. Csató (2017a) traces back the origin of this problem to the right-left asymmetry (Johnson et al., 1979), and provides a minimal counterexample with four alternatives.

According to Blanquero et al. (2006), the eigenvector solution is not necessarily Pareto efficient, in other words, there may exist a weight vector which is at least as good in approximating all elements of the pairwise comparison matrix, and strictly better in at least one position. However, Ábele-Nagy and Bozóki (2016) prove that this cannot occur if the pairwise comparison matrix differs from a consistent one only in one entry (and its reciprocal), while Ábele-Nagy et al. (2018) extend this result to double perturbed matrices, which can be made consistent by altering two elements and their reciprocals. On the other hand, the eigenvector method may lead to an inefficient weight vector for matrices with an arbitrarily small inconsistency (Bozóki, 2014). Finally, Bozóki and Fülöp (2018) propose linear programs to test whether a given weight vector is efficient or not, and Duleba and Moslem (2019) give the first examination of this property on real data.

The starting point of our paper is a remark by Saaty, who says that the priority vector has two meanings: “The first is a numerical ranking of the alternatives that indicates an order of preference among them. The other is that the ordering should also reflect intensity or cardinal preference as indicated by the ratios of the numerical values [...].” (Saaty, 2003, p. 86). In our view, this second interpretation suggests that the priority of a given alternative should be a monotonic function of its numerical comparisons with respect to any other alternatives. In other words, increasing an arbitrary entry of a pairwise comparison matrix should increase the weight of the favoured alternative (which is in the corresponding row) by the greatest factor, and should decrease the weight of the disfavoured alternative (which is in the corresponding column) by the greatest factor.

We will prove that – while the row geometric mean (logarithmic least squares) method trivially satisfies monotonicity – the eigenvector method violates it for certain pairwise comparison matrices. It is also investigated what is the probability that this problem emerges in the case of a randomly generated matrix as a function of its consistency ratio, the consistency measure suggested by Saaty (1977). The violation of monotonicity turns out to cause no problems in the case of nearly consistent matrices. On the other hand, the eigenvector method remains a dubious choice for inherently inconsistent large matrices.
such as the ones that emerge in sports applications.

The paper has the following structure. Section 2 outlines the topic of pairwise comparison matrices and defines the axiom of monotonicity. The eigenvector method is analysed in the view of this property in Section 3. Finally, Section 4 concludes.

2 The problem

In this section, the main notions around pairwise comparison matrices are briefly recalled, and a natural property is introduced.

2.1 Preliminaries: multiplicative pairwise comparison matrices

Let $N = \{1, 2, \ldots, n\}$ be a set of alternatives to be evaluated. Assume that their pairwise comparisons are known: $a_{ij}$ is a numerical answer to the question “How many times alternative $i$ is better than alternative $j$?”, that is, $a_{ij}$ measures the relative importance of alternative $i$ with respect to alternative $j$.

Let $\mathbb{R}_+^n$ and $\mathbb{R}_+^{n \times n}$ denote the set of positive (with all elements greater than zero) vectors of size $n$ and matrices of size $n \times n$, respectively.

The results of the comparisons are collected into a matrix whose elements below the diagonal are reciprocal to the corresponding elements above the diagonal.

**Definition 2.1.** Multiplicative pairwise comparison matrix: Matrix $A = [a_{ij}] \in \mathbb{R}_+^{n \times n}$ is a multiplicative pairwise comparison matrix if $a_{ji} = 1/a_{ij}$ for all $1 \leq i, j \leq n$.

In the following, the word “multiplicative” will be omitted for the sake of simplicity.

The set of all pairwise comparison matrices with $n$ alternatives is denoted by $\mathcal{A}_n \times n$.

Pairwise comparisons are carried out in order to get a priority vector $w$ such that the proportion of the weights $w_i$ and $w_j$ of the alternatives $i$ and $j$, respectively, approximates the value of their pairwise comparison $a_{ij}$. Thus the weights can be normalised arbitrarily.

**Definition 2.2.** Weight vector: Vector $w = [w_i] \in \mathbb{R}_+^n$ is a weight vector if $\sum_{i=1}^n w_i = 1$.

The set of weight vectors of size $n$ is denoted by $\mathcal{R}^n$.

**Definition 2.3.** Weighting method: Function $f: \mathcal{A}_n \times n \to \mathcal{R}^n$ is a weighting method.

The weight of alternative $i$ from the pairwise comparison matrix $A$ according to the weighting method $f$ is denoted by $f_i(A)$.

There exist many methods to estimate a suitable weight vector from a pairwise comparison matrix. Probably the most popular procedures are the row geometric mean (logarithmic least squares) method (Crawford and Williams, 1980, 1985; De Graan, 1980; de Jong, 1984; Rabinowitz, 1976), and the eigenvector method (Saaty, 1977). Although the latter suffers from a number of theoretical problems as mentioned in the Introduction, and there are sound axiomatic arguments in favour of the geometric mean (Fichtner, 1984; Barzilai et al., 1987; Barzilai, 1997; Lundy et al., 2017; Csató, 2018c; Bozóki and Tsyganok, 2019; Csató, 2019), the AHP methodology mainly uses the eigenvector method since the pioneering work of Saaty. This particular procedure will be our focus in the following.
Definition 2.4. Eigenvector method (Saaty, 1977): The eigenvector method associates the weight vector \( w_{EM}(A) \in \mathbb{R}^n \) for a given pairwise comparison \( A \in \mathcal{A}^{n \times n} \) such that

\[
Aw_{EM}(A) = \lambda_{\text{max}} w_{EM}(A),
\]

where \( \lambda_{\text{max}} \) denotes the maximal eigenvalue, also known as the principal or Perron eigenvalue, of the (positive) matrix \( A \).

There is a special case when all reasonable weighting methods give the same result.

Definition 2.5. Consistency: Let \( A = [a_{ij}] \in \mathbb{R}^{n \times n}_+ \) be a pairwise comparison matrix. It is called consistent if the condition \( a_{ik} = a_{ij} a_{jk} \) holds for all \( 1 \leq i, j, k \leq n \).

However, consistency is seldom observed in practice, pairwise comparison matrices are usually inconsistent. A variety of indices has been proposed to measure the level of inconsistency (see Brunelli (2018) for a survey), and recently some formal studies have appeared in the literature (Brunelli and Fedrizzi, 2015; Brunelli, 2017; Brunelli and Fedrizzi, 2019; Csató, 2018a,b; Koczkodaj and Szwarc, 2014; Koczkodaj and Urban, 2018). We will consider the oldest and by far the most popular Saaty inconsistency index (Saaty, 1977), which is closely related to the eigenvector method.

Definition 2.6. Consistency index (CI): Let \( A = [a_{ij}] \in \mathbb{R}^{n \times n}_+ \) be a pairwise comparison matrix. Its consistency index is:

\[
CI(A) = \frac{\lambda_{\text{max}} - n}{n - 1}.
\]

Saaty (1977) introduced the so-called random index \( RI_n \), that is, the average CI of a large number of \( n \times n \) pairwise comparison matrices with entries randomly generated from the scale \( \{1/9, 1/8, \ldots, 8, 9\} \). The proportion of CI and \( RI_n \) is called the consistency ratio \( CR \). This will be called the Saaty inconsistency index in the following.

Saaty (1977) considered a pairwise comparison matrix to be acceptable if \( CR \) does not exceed the threshold 0.1.

2.2 Monotonicity of the weights on single comparisons

The entry \( a_{ij} \) measures the dominance of alternative \( i \) over alternative \( j \). Thus it is expected that increasing \( a_{ij} \) is favourable for the weight of alternative \( i \) with respect to the weight of any third alternative \( k \). Otherwise, the counter-intuitive behaviour of the weights may lead to rank reversal.

The following axiom formalises this requirement.

Axiom 1. Monotonicity: Let \( A \in \mathcal{A}^{n \times n} \) be any pairwise comparison matrix and \( 1 \leq i, j \leq n \) be any two different alternatives. Let \( A' \in \mathcal{A}^{n \times n} \) be identical to \( A \) but \( a'_{ij} > a_{ij} \) (and \( a'_{ji} < a_{ji} \) because of the reciprocity property). Then weighting method \( f : \mathcal{A}^{n \times n} \rightarrow \mathbb{R}^n \) is called monotonic if \( f_i(A') / f_k(A') \geq f_i(A) / f_k(A) \) for all \( 1 \leq k \leq n \).

Monotonicity is a reformulation of analogous conditions that are widely used in social choice theory, for example, positive responsiveness (van den Brink and Gilles, 2009) and positive responsiveness to the beating relation (González-Díaz et al., 2014): if alternative \( i \) is ranked at least as high as alternative \( k \), then it should be ranked strictly higher when a comparison \( a_{ij} \) changes in favour of alternative \( i \).
A weaker version has been used in the axiomatic characterization of the row geometric mean (logarithmic least squares) ranking (Csató, 2018c) in order to get a stronger result: \( i \succeq j \) implies \( i > j \) whenever \( a_{ij} \) increases.

Landau (1914, p. 201) considers another principle for nonnegative tournament matrices, which also follows from Axiom 1:

\[
\frac{a_{ij}}{\sum_{k=1}^{n} f_k(A')} \leq \frac{f_i(A)}{\sum_{k=1}^{n} f_k(A)}. \tag{1}
\]

Brunelli and Fedrizzi (2015) have suggested an axiom with the same flavour called monotonicity on single comparisons in the context of inconsistency indices, which is satisfied by the inconsistency index \( CI \) (Aupertit and Genest, 1993). Brunelli and Fedrizzi (2015) also provide a short overview of the origin of this property.

3 Results

The row geometric mean (logarithmic least squares) method trivially meets monotonicity: a greater value of \( a_{ij} \) increases the weight of alternative \( i \), decreases the weight of alternative \( j \), while preserves the weights of all other alternatives, at least before normalization. The case of the eigenvector method will turn out to be more complicated.

3.1 The eigenvector method can be non-monotonic

Our point of departure is the following negative observation.

Proposition 3.1. The eigenvector method does not satisfy monotonicity.

Proof. It is sufficient to provide a counterexample.

Example 3.1. Consider the following parametric pairwise comparison matrix:

\[
A^\alpha = \begin{bmatrix}
1 & 8 & \alpha & 5 \\
1/8 & 1 & 3 & 7 \\
1/\alpha & 1/3 & 1 & 9 \\
1/5 & 1/7 & 1/9 & 1
\end{bmatrix}.
\]

The ratio of the weights of the first and the fourth alternatives is plotted in Figure 1 as a function of parameter \( \alpha \).

Note that \( w_1^{EM}(A^\alpha)/w_4^{EM}(A^\alpha) \) is not monotonically increasing around \( \alpha = 1 \). For instance, \( w_1^{EM}(A^1)/w_4^{EM}(A^1) > w_1^{EM}(A^{1.01})/w_4^{EM}(A^{1.01}) \), which shows the violation of Axiom 1 by the eigenvector method: increasing \( a_{13} \) is disadvantageous for the weight of the first alternative with respect to the weight of the fourth alternative.

However, it can be checked that both \( w_1^{EM}(A^\alpha)/w_2^{EM}(A^\alpha) \) and \( w_1^{EM}(A^\alpha)/w_3^{EM}(A^\alpha) \) are monotonically increasing around \( \alpha = 1 \). Furthermore, Example 3.1 does not violate condition (1) as \( w_1^{EM}(A^\alpha)/\sum_{k=1}^{4} w_k^{EM}(A^\alpha) \) is monotonically increasing around \( \alpha = 1 \).

Example 3.1 is minimal in the number of alternatives because the eigenvector method is equivalent to the row geometric mean (logarithmic least squares) method for \( n = 3 \) (Crawford and Williams, 1985), and the latter satisfies monotonicity.

On the domain of nonnegative tournament matrices of order \( n = 3 \), the principal right eigenvector violates even condition (1) (Landau, 1914), which is substantially weaker than Axiom 1. We have not found any such instances for reciprocal \( n \times n \) matrices.
3.2 A framework for analysing monotonicity

Although the eigenvector method violates monotonicity in certain cases, this in itself does not make the problem relevant in practice. In order to reveal this issue in depth, a computational technique will be used: a large number of pairwise comparison matrices will be considered and checked with respect to the monotonicity of the eigenvector.

The entries of the random pairwise comparison matrices are generated in two ways:

- **Discrete scale**: all entries \( a_{ij} \) above the diagonal \((i < j)\) are randomly chosen from the set
  \[
  \left\{ \frac{1}{9}; \frac{1}{8}; \frac{1}{7}; \frac{1}{6}; \frac{1}{5}; \frac{1}{4}; \frac{1}{3}; \frac{1}{2}; 1; 2; 3; 4; 5; 6; 7; 8; 9 \right\}
  \]
  with equal probability 1/17, and by setting \( a_{ji} = 1/a_{ij} \), as well as \( a_{ii} = 1 \).

- **Continuous scale**: a value is chosen randomly from the interval \([1, 10]\) according to a uniform distribution, and it is decided by an equal odds coin-toss whether this value or its reciprocal is written into the entry \( a_{ij} \) above the diagonal \((i < j)\).

Other elements of the pairwise comparison matrix are determined as above.

The discrete scale above is the standard proposed by Saaty. The continuous scale is examined because focusing on integers may hide some crucial features of the eigenvector method. Both of them have already been used in the literature, see, for example, Alonso and Lamata (2006); Bozóki and Rapcsák (2008).

Since we want to investigate the connection between monotonicity and the consistency ratio \( CR \), random indices \( RI_n \) should also be determined. They are reported in Table 1 for \( 4 \leq n \leq 9 \): for the discrete scale, the values given in Bozóki and Rapcsák (2008, Table 3) are used – they have been validated by our calculations, too –, while the random indices for the continuous scale have been computed from four million randomly generated matrices in each case.

Monotonicity of the principal right eigenvector is tested by perturbing a matrix element, and checking whether the condition required by Axiom 1 holds or not. Thus, one iteration of the computational process consists of the following steps:
1. A random pairwise comparison matrix $A$ of order $n$ is generated on the discrete/continuous scale.

2. Its consistency ratio $CR(A)$ and eigenvector $w^{EM}(A)$ is calculated, the number of matrices in the $m$th interval of consistency ratios, for which $\beta(m-1) \leq CR(A) < \beta m$, is increased by one (the actual value of $\beta$ will depend on the aim of the computation, see the discussion of figures later).

3. All entries above the diagonal are considered separately: $n(n-1)/2$ perturbed pairwise comparison matrices $A^{ij}$ are defined such that $A^{ij} = A$ except for its element in the $i$th row and $j$th column, $i < j$, which is given by $a_{ij}^{ij} = 1.01a_{ij}$, while reciprocity is preserved, so $a_{ji}^{ij} = (1/1.01)a_{ji}$.

4. Eigenvectors $w(A^{ij})$ are computed.

5. Fractions $w_i^{EM}(A)/w_k^{EM}(A)$ and $w_i^{EM}(A^{ij})/w_k^{EM}(A^{ij})$ are compared for all $i < j$ and $1 \leq k \leq n$.

6. The $m$th interval of consistency ratios with the flag of non-monotonicity, in which $CR(A)$ falls, is increased by one if $w_i^{EM}(A)/w_k^{EM}(A) > w_i^{EM}(A^{ij})/w_k^{EM}(A^{ij})$, that is, Axiom 1 is violated after increasing $a_{ij}$ by 1%.

7. The pairwise comparison matrix $A$, its consistency ratio $CR(A)$ and $i, j, k$ are saved as an example that violates monotonicity if $CR(A)$ is smaller than the consistency ratio of all previously generated pairwise comparison matrices with a non-monotonic eigenvector.

Steps 1-7 are repeated until the number of randomly generated pairwise comparison matrices reaches a predetermined limit.

### 3.3 The relationship between monotonicity and inconsistency

First, four million iterations of the process consisting of steps 1-7 have been made for all values of $n$ between 4 and 9. Figure 2 plots the proportion of pairwise comparison matrices for which the eigenvector method does not satisfy the axiom of monotonicity as a function of the consistency ratio $CR$. It clearly does not depend on whether the entries are drawn from a discrete or continuous scale, so only the former will be analysed in the following. Note also that $CR$ cannot be arbitrarily large: Aupetit and Genest (1993) derive a sharp upper bound on $\lambda_{max}$ when the responses are coded on a bounded scale applied here.
According to Figure 2, the probability of a non-monotonic right eigenvector gradually increases if the pairwise comparison matrix becomes more inconsistent. There is no violation of Axiom 1 for nearly consistent matrices ($CI < 0.15$), however, this problem is almost guaranteed to emerge for heavily inconsistent matrices in the case of at least six alternatives.

For $n = 4$, there are only six elements above the diagonal, and the total number of different matrices using the discrete scale for these entries is $17^6 = 24,137,569$. Although
some of them can be transformed into another by row/column permutations, we have decided to preserve all of them because of the nature of random matrix generation. Note that the random index $RI_4$ is determined without taking permutation filtering into account.

They are plotted in Figure 3 such that the pairwise comparison matrices are grouped according to their consistency ratios: matrix $A$ is registered in the $m$th interval if $CR(A)$ falls between $0.1(m - 1)$ and $0.1m$, while the last, 35th category contains all matrices with $CR \geq 3.5$. A matrix with the smallest consistency ratio that presents the non-monotonic behaviour of the eigenvector is the one presented in Example 3.1, where $CR \approx 0.4869$.

Since Saaty suggested a threshold of $0.1$ for the acceptable level of the consistency ratio $CR$, nearly consistent matrices prevail in certain applications, hence they are examined more thoroughly. This is highlighted in Figure 4 where the pairwise comparison matrices are classified more finely than before: matrix $A$ is registered in the $m$th interval if $0.02(m - 1) \leq CR(A) < 0.02m$.

Note that the running time of our simulations is significantly reduced by checking Axiom 1 only for matrices with a modest inconsistency of $CR < 0.4$, so the sample size of random matrices can be substantially increased. The eigenvector always remains monotonic on this domain if there are only four alternatives. The probability of non-monotonicity may emerge still around $CR \approx 0.2$ for $n \geq 7$, and it gradually increases as the pairwise comparison matrix becomes more inconsistent.

Until now the axiom of monotonicity has been checked with an increase of 1% in all entries (see Section 3.2). However, the property is defined on a continuous scale, so this choice may miss identifying some cases of non-monotonicity. Therefore a kind of sensitivity analysis has been implemented on the whole domain of $4 \times 4$ pairwise comparison matrices from the discrete scale such that entries have been increased by $0.1\%$ and $1\%$, too.

Figure 5 summarises these results, where matrix $A$ is registered in the $m$th interval if $0.01(m - 1) \leq CR(A) < 0.01m$, furthermore, the diagram is truncated at 3.49 because the remaining 600 pairwise comparison matrices are recognised similarly with $0.1\%$, $1\%$, $5\%$, $10\%$, or $20\%$.
Figure 4: Randomly generated pairwise comparison matrices with \( CR < 0.4 \) for which the eigenvector method does not satisfy monotonicity

Matrix size: \( n = 4 \); sample size: all \( (17^6) \)

Matrix size: \( n = 5 \); sample size: \( 2 \cdot 10^7 \)

Matrix size: \( n = 6 \); sample size: \( 4 \cdot 10^7 \)

Matrix size: \( n = 7 \); sample size: \( 7 \cdot 10^7 \)

Matrix size: \( n = 8 \); sample size: \( 15 \cdot 10^7 \)

Matrix size: \( n = 9 \); sample size: \( 35 \cdot 10^7 \)

and 10% change. On the right axis, the number of pairwise comparison matrices in the \( m \)th interval with a non-monotonic eigenvector is measured (normal blue line). On the left axis, the difference in the number of matrices presenting this problem with a 0.1% and 1% (dotted black line), as well as, with a 1% and 10% change (dashed red line) is plotted. For example, if \( 0.48 \leq CR < 0.49 \), then 240 matrices are found to have a
non-monotonic eigenvector with a 0.1% change, 192 matrices are found to have a non-monotonic eigenvector with a 1% change, and no matrix is found to have a non-monotonic eigenvector with a 10% change. The difference is 48 between the first and the second, while it is 192 between the second and the third.

Looking at Figure 1 reveals that the pairwise comparison matrix of Example 3.1 also appears in Figure 5: it is registered as problematic with a 0.1% and 1% change, but not with 10% because in the latter case, the decreasing part of the function $w^EM_4(A^\alpha)/w^EM_4(A^\alpha)$ is “jumped over”. It is worth mentioning that some types of non-monotonicity are recognised only with a rougher change (1% instead of 0.1%, or 10% instead of 1%) as both the red and the black lines go sometimes below zero. Furthermore, these interesting cases cluster around the smaller values of the Saaty inconsistency index, and the probability of their occurrence does not depend only on the number of matrices having a non-monotonic eigenvector. A deeper analysis of this anomaly remains the topic of future research.

4 Discussion

In the current paper, we have argued that monotonicity on the numerical comparisons is a key requirement for any weight vector derived from a pairwise comparison matrix. The eigenvector method is proved to violate this axiom. However, contrary to the right-left asymmetry (Johnson et al., 1979) and the Pareto inefficiency (Bozóki, 2014) of the eigenvector, as well as, to its violation of a condition of order preservation (Bana e Costa and Vansnick, 2008), the emergence of non-monotonicity seems to be avoidable for small values of the Saaty inconsistency index. On the other hand, arbitrary pairwise comparison matrices have a principal right eigenvector without the monotonicity property with a high probability, especially in the case of more than five alternatives, as Table 2 shows.
Table 2: The probability that the eigenvector does not satisfy monotonicity

| Matrix size | All values of $CR$ | $CR < 0.4$ |
|-------------|--------------------|------------|
| 4           | 0.3151             | 0.0000     |
| 5           | 0.6789             | 0.0313     |
| 6           | 0.8867             | 0.1665     |
| 7           | 0.9679             | 0.3388     |
| 8           | 0.9922             | 0.4913     |
| 9           | 0.9983             | 0.6168     |

Discrete scale; sample sizes: all possible matrices for $n = 4$, 4 million randomly generated matrix for $n \geq 5$ without restrictions on $CR$, see Figure 4 for $n \geq 5$ with $CR < 0.4$

Our results have useful implications for practitioners. First, the probability of non-monotonicity strongly depends on the inconsistency level of the pairwise comparison matrix, which emphasizes the need for inconsistency reduction methods (Abel et al., 2018; Bozóki et al., 2011, 2015; Ergu et al., 2011; Kocz kodaj and Szarek, 2010; Koczkodaj and Szybowski, 2016; Szybowski, 2018). Second, the possibly strange behaviour of the right eigenvector makes the use of this method questionable for inherently inconsistent large matrices such as the ones that emerge in sports applications (Csató, 2013; Bozóki et al., 2016; Csató, 2017b; Chao et al., 2018), because in this field, rewarding players or teams for poor performance is unfair (Csató, 2018d; Dagaev and Sonin, 2018; Kendall and Lenten, 2017; Vaziri et al., 2018). Finally, the analysis of monotonicity provides further arguments for the use of the row geometric mean (logarithmic least squares) method instead of the eigenvector method.

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