Correlation functions, free energies and magnetizations in the two-dimensional random-field Ising model

S. L. A. de Queiroza and R. B. Stinchcombe
a Instituto de Física, Universidade Federal do Rio de Janeiro, Caixa Postal 68528, 21945-970 Rio de Janeiro RJ, Brazil
b Department of Physics, Theoretical Physics, University of Oxford, 1 Keble Road, Oxford OX1 3NP, United Kingdom

Transfer-matrix methods are used to calculate spin-spin correlation functions \((G)\), Helmholtz free energies \((f)\) and magnetizations \((m)\) in the two-dimensional random-field Ising model close to the zero-field bulk critical temperature \(T_c\), on long strips of width \(L = 3 - 18\) sites, for binary field distributions. Analysis of the probability distributions of \(G\) for varying spin-spin distances \(R\) shows that describing the decay of their averaged values by effective correlation lengths is a valid procedure only for not very large \(R\). Connections between field- and correlation function distributions at high temperatures are established, yielding approximate analytical expressions for the latter, which are used for computation of the corresponding structure factor. It is shown that, for fixed \(R/L\), the fractional widths of correlation-function distributions saturate asymptotically with \(L^{-2.2}\). Considering an added uniform applied field \(h\), a connection between \(f(h), m(h)\), the Gibbs free energy \(g(m)\) and the distribution function for the uniform magnetization in zero uniform field, \(P_0(m)\), is derived and first illustrated for pure systems, and then applied for non-zero random field. From finite-size scaling and crossover arguments, coupled with numerical data, it is found that the width of \(P_0(m)\) varies against (non-vanishing, but small) random-field intensity \(H_0\) as \(H_0^{-3/7}\).

I. INTRODUCTION

The random-field Ising model (RFIM) has posed a number of challenges to researchers since its introduction as an apparently purely theoretical puzzle [1]. The later realization that it corresponds, give or take a few (hopefully irrelevant) details, to the experimentally realizable dilute Ising antiferromagnet in a uniform applied field [2], brought new insights, and new questions as well; among the latter, is the interpretation of experimental data in a suitable theoretical framework. This has proved to be rather an intricate subject, even down to basic aspects such as whether the lower critical dimensionality for the problem was \(d = 2\) or \(3\) [3,4]. Though by now this particular issue has been settled in favour of \(d = 2\) [5], several important aspects (such as the scaling behaviour near the destroyed phase transition in \(d = 2\), which will be of interest here) still require further elucidation [6].

In the present paper we deal with the two-dimensional RFIM, where long-range order is destroyed, and a zero-temperature, zero-field “anomalous” critical point appears [7]. The latter will not concern us directly, as we shall be working at high temperatures, close to the pure-system ferro-paramagnetic transition. We extend and complement our early work [8], making use of transfer-matrix (TM) methods on long, finite-width strips of a square lattice; we generate and analyze statistics of spin-spin correlation functions and uniform magnetizations. Wherever feasible, we attempt to draw connections between our numerical results and experimentally observable quantities. In what follows, we begin by briefly reviewing selected aspects of the numerical techniques used, and how they relate to the physical problem under study. We then recall the connection between structure factors and averaged correlations in random systems, and discuss the extraction of effective correlation lengths from our numerical data for correlation-function statistics. Next we exploit the connection between field- and correlation function distributions at high temperatures, in an attempt to derive approximate analytical expressions for the latter; such formulae are used in turn, in order to compute the corresponding structure factor. A short section is dedicated to a reanalysis of the asymptotic behaviour of the widths of correlation-function distributions, first presented in Ref. [9], and now complemented by additional data. In the next section, an additional uniform applied field is considered: free energies and uniform magnetizations are calculated on strips of both pure and RFIM systems. These quantities are used to calculate the corresponding Gibbs free energy which, in turn, gives the distribution function for the uniform magnetization in zero uniform field. Numerical data are then analyzed via finite-size scaling and crossover arguments. A final section summarizes our work.

II. NUMERICAL METHODS AND \(D = 2\) RFIM

We consider strips of a square lattice of ferromagnetic Ising spins with nearest-neighbour interaction \(J = 1\), of
width $3 \leq L \leq 18$ sites with periodic boundary conditions across. The random-field values $h_i$ are drawn for each site $i$ from the binary distribution:

$$p(h_i) = \frac{1}{2} \left[ \delta(h_i - H_0) + \delta(h_i + H_0) \right] .$$  

(1)

TM methods are used, on long strips of typical length $L_x = 10^6$ columns, as described at length in Ref. [8] and references therein, to generate representative samples of the quenched random fields. Along the strip, we calculate correlation functions (as explained in the next paragraph), as well as free energies and magnetizations (details in Section V).

Here we calculate the disconnected spin–spin correlation function $G(R) \equiv \langle \sigma_0 \sigma_R \rangle$, between spins on the same row (say, row 1), and $R$ columns apart. Related quantities, such as correlation lengths, are defined with connected correlations, $\langle \sigma_0 \sigma_R \rangle - \langle \sigma \rangle^2$, in mind; however, for the quasi-one-dimensional Ising systems under consideration (either pure or random) one is always in the paramagnetic phase, so the distinction between connected and disconnected correlations is unimportant. In Ref. [9] we explained why the ranges of spin–spin distance, temperature and random-field intensity of most interest for investigation by TM methods are, respectively, $R/L \sim 1$, $0 < T \lesssim T_{c,0} = 2.269 \cdots$ [we take $k_B \equiv 1$], $H_0 \lesssim 0.5$. Here we restrict ourselves to high $T \gtrsim 2.0$, and rather low fields, $H_0 \lesssim 0.1 - 0.15$. We use a linear binning for the histograms of occurrence of $G(R)$: usually the whole $[-1,1]$ interval of variation of $G(R)$ is divided into $10^3$ bins.

Since we shall be dealing with probability distributions, a word is in order about multifractality. Though multifractal behaviour has been found at the critical point of random-bond Potts systems [8,11], the available evidence strongly suggests that, off bulk criticality, correlation functions behave normally [10]. Thus, in the present case we expect that analysis of different moments of the probability distribution of $G$ will yield essentially the same results.

III. CORRELATION DECAY

The properties of correlation functions are usually incorporated into associated correlation lengths, whose basic definition is as (minus) the inverse slope of semi-logarithmic plots of correlation functions against distance. In this view, one assumes both that exponential decay can be well-defined at essentially all distances, and that a single length is enough to characterize such behaviour. In cases as the present, quenched randomness implies that configurational averages must be taken, and one must be careful in deciding what quantities are to be thus promediated. Recall that, e.g., in neutron scattering experiments, the intensity of the magnetic critical scattering is proportional to the average (over the crystal) of the scattering function $S(\vec{q})$, which is the Fourier transform of the correlation function for wave-vector transfer $\vec{q}$. With $G_R \equiv G(R)$, and wave vector $q$ in the row direction, $S$ becomes

$$[S(q)] = \int dR \, e^{iqR} \, G_R = \int dR \, e^{iqR} \, \langle G_R \rangle$$

(2)

where $[\cdots]$ stands for configurational average, and

$$\langle G_R \rangle = \int dG_R \, P(G_R) \, G_R \, ,$$

(3)

where $P(G_R)$ is the probability distribution for $G_R$. The last equality in Eq (2) depends only on the assumption that $P(G_R)$ is position-independent along the crystal.

The simplest assumption for $P(G_R)$ that incorporates both disorder and exponential decay given by a single length $\xi$ for all distances is a Gaussian distribution:

$$P(G_R) = \frac{1}{\sqrt{2\pi} \Delta(R)} \, e^{-y^2/2\Delta^2(R)} , \quad y \equiv G_R - e^{-R/\xi}$$

(4)

where distance–dependent widths $\Delta(R)$ allow for, e.g., (disorder-induced) larger uncertainties for larger spin–spin separations. However, using Eq. (4) in Eqs. (2,3) one obtains a width-independent Lorentzian form for the average structure factor:

$$[S(q)] = \frac{1}{q^2 + \xi^2} .$$

(5)

This coincides with the standard mean-field result for the disordered phase, and is deemed unsatisfactory upon comparison with experimental data [7,8].

![Fig. 1](image-url)

**FIG. 1.** Normalized histogram $P(G)$ of occurrence of $G$. Strip length $L_x = 10^6$ columns, binwidth $2 \times 10^{-3}$. Vertical bars located respectively at: $G_0$ (full line), $\langle G \rangle$ (dashed). Shaded region on horizontal axis from $\langle G \rangle - \bar{W}$ to $\langle G \rangle + \bar{W}$. 

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We now exhibit our numerical results, and compare their implications to those of Eqs. (3) and (4). For high temperatures and low random-field intensities, as specified above, we recall (see also Ref. [9]) the following main features found for the probability distribution \( P(G) \): (i) a clearly-identifiable, cusp-like peak, at some \( G_m \) below the zero-field value \( G_0 \equiv G(H_0 = 0) \); (ii) a short tail below the peak and a long one above it, such that (iii) all moments of order \( \geq 0 \) of the distribution are above \( G_0 \). In Figure 1, where the first moment \( \langle G \rangle \) is shown, one has \( G_m = 0.278 \), \( G_0 = 0.2853 \), \( \langle G \rangle = 0.321 \); the RMS width \( W \equiv \langle (G - \langle G \rangle)^2 \rangle^{1/2} = 0.071 \).

Therefore, the features depicted in Fig. 1, especially the asymmetric cusp, are at variance with the form Eq. (3). We now investigate what effects are carried over to the associated correlation lengths. We do so by mimicking the procedure outlined in Eqs. (2) and (3) above: first we average over randomness for a given spin-spin separation, and then study the variation of the averaged quantities over distance. The results are in Fig. 1, where our numerical data for \( \langle G \rangle \) are plotted against varying \( R \). \( H_0 = 0 \) data are also shown for comparison.

\[
\begin{align*}
\text{FIG. 2.} & \quad \text{Correlation decay along strips of widths} \ L = 5 \\
& \quad \text{and} \ 9. \ \text{Full lines:} \ H_0 = 0. \ \text{Points:} \ \langle G \rangle \ \text{for} \ H_0 = 0.1. \ \text{Dashed} \\
& \quad \text{lines: unweighted least-squares fits of} \ H_0 = 0.1 \ \text{data. Vertical bars give RMS} \\
& \quad \text{widths} \ W \ \text{of corresponding distributions.}
\end{align*}
\]

One sees from the respective slopes that, taking into account data for \( R \geq L \), the correlation length for \( H_0 \neq 0 \) would seem to be systematically larger than in zero field. This reflects the domain structure into which the system breaks down: at short distances the conditional probability for a spin to belong to the same domain as the one at the origin is larger than for \( H_0 = 0 \).

For longer distances, correlation functions start to show severe disorder-induced fluctuations, related to the crossing of domain walls. Contrary to the zero-field case, where temperature-induced domain walls are present but the respective sign changes in correlation functions average out to give an exponential fall off, here the domain wall configurations are essentially determined by the (quenched) accumulated random-field fluctuations. At large \( R \), such fluctuations play a very sensitive role even for very low random field intensities. One anticipates problems with defining correlation lengths from the corresponding data. The vertical bars in Fig. 1 show that, for fixed strip width \( L \), the width \( W \) of the distribution indeed grows apparently unbounded for increasing \( R \); though this is related to the crossing of domain walls just mentioned, it is also, and predominantly, an intrinsic feature of the quasi-one-dimensional systems used here. Thus, inferring two-dimensional behaviour from such trends may be risky. However, we now argue that in \( d = 2 \) one does actually run into problems for large \( R \), exactly as inferred above; only, the underlying reasoning is subtler.

In fact, see Ref. [9] and Section V below, a different analysis of correlation functions, at fixed \( R/L \), strongly suggests that the relative widths \( W \equiv \langle W \rangle/G \) grow as \( R/L \to \infty \) in \( d = 2 \), approaching a finite limiting value \( C \ H_0^2 \), \( C \simeq 2, \kappa \simeq 0.5 \). This means that, when one considers the dispersion of \( \ln \langle G \rangle \), the signal-to-noise ratio becomes of order one for large \( R, L \), and it is this latter fact that, in \( d = 2 \), must compromise attempts to extract correlation lengths in such range.

The effect of the above on fits of neutron-scattering data to lineshapes is that, since the latter rely on the idea that correlation lengths are always reliable quantities, they may be off the actual picture in the small-wavevector region.

We now attempt to derive approximate analytical expressions for \( P(G|\mathbf{R}) \); our ultimate goal is to predict a form for \( S(q) \) from Eqs. (2)–(3).
\[ h' + h'' = h_0 + h_1 + \frac{1}{2} \ln \left( \frac{\cosh 2(K + h_1)}{\cosh 2(K - h_1)} \right) \]

\[ 4K' = \ln \left( \frac{\cosh 2(K + h_1) \cosh 2(K - h_1)}{\cosh^2 2h_1} \right) . \] (6)

Iterating this procedure \( n \) times, one obtains a single renormalised bond \( \tilde{K} \) connecting sites 0 and \( \tilde{R} (R = 2^n) \), at which the respective rescaled fields are \( \tilde{h}_0, \tilde{h}_R \). The correlation function \( G(R) \equiv \langle \sigma_0 \sigma_R \rangle \) is therefore:

\[ G(R) = \frac{e^{2\tilde{K}} \cosh(\tilde{h}_0 + \tilde{h}_R) - \cosh(\tilde{h}_0 - \tilde{h}_R)}{e^{2\tilde{K}} \cosh(\tilde{h}_0 + \tilde{h}_R) + \cosh(\tilde{h}_0 - \tilde{h}_R)} . \] (7)

For low fields \( \tilde{H}_0 \ll 1 \), one uses \( \cosh(\tilde{h}_0 - \tilde{h}_R) \) to get

\[ G(R) \approx \tanh(\tilde{K} + \tilde{h}_0 \tilde{h}_R) . \] (8)

Then, provided also \( \tilde{H}_0 \ll \tilde{K} \), the distribution of \( G(R) \) is given by that of \( X \equiv \tilde{h}_0 \tilde{h}_R, \) since (to lowest order in \( \tilde{H}_0 \)) \( K' = \frac{1}{2} \ln \cosh 2K \) is field-independent. One has:

\[ P(X) = \int d\tilde{h}_0 \tilde{P}(\tilde{h}_0) \int d\tilde{h}_R \tilde{P}(\tilde{h}_R) \delta(X - \tilde{h}_0 \tilde{h}_R) . \] (9)

At low \( \tilde{H}_0 \), the scaling equations (3) give \( h'_0 \sim h_0 + h_1 \tanh 2\tilde{K} \). Repeated applications of this transformation give \( h_0 \sim \sum_{i=1}^{R} h_i \) if \( R \ll \xi \), where \( \xi \sim \ln(\tanh K)^{-1} \) is the correlation length at low \( \tilde{H}_0 \). Then \( \tilde{h}_0 \) (and similarly \( \tilde{h}_R \)) is the sum of \( N \) independent variables \( (N = R) \), so the individual distributions of \( \tilde{h}_0, \tilde{h}_R \) become (at large \( R, \xi \) ) Gaussians of width \( \Delta \equiv H_0 \sqrt{N}:

\[ \tilde{P}(\tilde{h}_0,\tilde{h}_R) \propto \exp \left( -\frac{(\tilde{h}_0,\tilde{h}_R)^2}{\Delta^2} \right) . \] (10)

For \( R \gg \xi \), the same form applies, but because the field accumulation under scaling is cut off by the decreasing \( \tanh K \), the relation for \( \Delta \) involves \( N \sim \xi \). So, in general, \( N \sim \min(R,\xi) \).

Making \( \tilde{h}_0 = s \cos \theta, \tilde{h}_R = s \sin \theta \),

\[ P(X) \propto \int_0^\infty ds s \int_0^{2\pi} d\theta e^{-s^2/\Delta^2} \delta(X - \frac{s^2}{2} \sin 2\theta) \] (11)

with the final result

\[ P(X) = \frac{a}{\Delta^2} e^{-y} \ln \left( 1 + \frac{1}{y} \right) , \quad y = \frac{2|X|}{\Delta^2} \] (12)

where \( a \) is an overall normalization constant. Strictly speaking, Eq. (12) is the asymptotic reduction of Eq. (11), valid for the regimes \( y \ll 1 \) (the relevant one for our purposes, as shown below) and \( y \gg 1 \).

Transforming back to \( P(G) \), one sees that the value \( G_m \) for which \( P(G) \) is maximum must correspond to \( X = 0 \), which maximizes \( P(X) \). Thus, from Eq. (8),

\[ y = \frac{2}{\Delta^2} |\tanh^{-1} G_m - \tanh^{-1} G| \] (13)

For \( G \) close to \( G_m \), linearization gives

\[ P(G) \sim \exp \left( -\frac{1}{\Delta^2} |G_m - G| \right) \frac{\Delta^2}{2\Delta^2 |1 - G^2|} \ln \left( 1 + \frac{\Delta^2}{G - G_m} \right) , \] (14)

with \( \Delta^2 = \frac{1}{2} \Delta^2 (1 - G_m^2) \).

The main feature exhibited by this form is a locally symmetric cusp, with infinite slope on either side, at \( G = G_m \). This is expected to carry over to more general contexts, provided that \( H_0 \ll 1 \). Indeed we have checked that a similar description (applying approximate Migdal-Kadanoff scaling calculations), with the prediction of a cusp, also applies on strips and in two dimensions (see Eqs. (15), (16) below, and related discussion). A quantitative test of Eq. (14) is shown in Figure 3, where only data for \( G \leq G_m \) are displayed (we shall deal with \( G > G_m \) immediately afterwards). The conditions are such that \( R \ll \xi, G_m \ll 1 \) (see Fig. 3), so the above low-field theory gives \( \Delta \sim H_0 \sqrt{R} \).

One sees that exponential decay against \( |G - G_m| / RH_0^2 \) is indeed the dominant behaviour, provided that \( H_0 \gtrsim 0.15 \); already for \( H_0 = 0.25 \), small departures show, which become more prominent for \( H_0 = 0.5 \).

As regards cusp asymmetry, not predicted by Eq. (14), we have found that although data for \( G > G_m \) still fall exponentially for small \( H_0 \), they do not collapse when plotted against \( |G - G_m| / RH_0^2 \). This is because the mutual reinforcement, between ferromagnetic spin-spin interactions and accumulated field fluctuations (responsible for
the long forward tail \[\text{[1]}\], is left out by the approximation above Eq. \[\text{[1]}\], namely that \(K'\) is \(H_0\)-independent.

Before calculating \([S(q)]\) from Eq. \[\text{[4]}\], we recall that Eqs. \[\text{[3]}-\text{[5]}\] are normally required for bulk systems, thus one must work out an approximate scheme to go from the \(d = 1\) regime of Eqs. \[\text{[4]}-\text{[5]}\] to \(d = 2\). We have done so via a Midgal-Kadanoff rescaling transformation at \(T \sim T_c\). As a consequence of the similarity of the corresponding recursion relations, to those for one dimension, one ends up, after \(m\) scalings such that \(2^m = R\), with a result very similar to Eq. \[\text{[5]}\]:

\[
G(R) = \tanh(\tilde{K} + \tilde{h}_0\tilde{h}_R),
\]

where again one assumes low fields, \(\tilde{h}_0, \tilde{h}_R\). For large \(R\) these have Gaussian distributions of width \(\Delta_R\) determined by the eigenvalue \(\lambda\) of the low-field scaling transformation of \(H_0\). For \(T \sim T_c\), \(\Delta_R \propto R^n H_0\) where \(\mu = \ln \lambda / \ln b \) (\(b = 2\)). Further,

\[
\tilde{K} \sim K_c - \frac{R}{\xi}.
\]

Since Eqs. \[\text{[3]}-\text{[12]}\] still apply, provided \(\Delta\) is replaced by \(\Delta_R\), one gets the dominant contribution to the scattering function as:

\[
[S(q)] \propto \text{Re} \int dR e^{iqR} e^{-R/\xi} (1 + C \Delta_R^2),
\]

where \(C\) is a constant of order unity. This can be transformed into:

\[
[S(q)] \propto \frac{1}{\xi^2 + q^2} + C H_0^2 (2\mu)! \text{Re} \left( \frac{1}{\xi - iq} \right)^{-(2\mu+1)}.
\]

If one assumes the form \(\Delta = H_0\sqrt{R}\), given for \(R \leq \xi\) in one dimension (see above and below Eq. \[\text{[10]}\]), and also by the Migdal-Kadanov scheme in \(d = 2\), then \(\mu = 1/2\) and Eq. \[\text{[18]}\] predicts the lineshape to be Lorentzian plus Lorentzian-squared, the mean-field form found when the disconnected contribution is taken into account \[\text{[14]}-\text{[15]}\]. On the other hand, if one goes by the saturation behaviour predicted in Ref. \[\text{[3]}\] and Section \[\text{[4]}\] below, and by the scaling approaches if \(R \gtrsim \xi\), then the result is \(\zeta = 0\), corresponding to a single Lorentzian in Eq. \[\text{[4]}\].

Though either of these final predictions is certainly open to challenge, in view of the number and severity of approximations involved in the course of their derivation, it is expected that the procedure described above will serve as a rough guide to attempts at connecting basic microscopic features (such as fluctuations of accumulated fields) to observable quantities, e.g. scattering functions.

V. WIDTHS OF \(G\)-DISTRIBUTION

In Ref. \[\text{[4]}\] we studied the variation of the RMS relative width \(W\) of the probability distribution of correlation functions, against field intensity and strip width, for fixed \(R/L\), high temperatures and small \(H_0\). We proposed the scaling form

\[
W = H_0^\kappa f(L H_0^u),
\]

and showed that, for \(R/L = 1, T = T_c\), good data collapse of \(y \equiv \ln [W H_0^\kappa]\) against \(x \equiv L H_0^u\) can indeed be obtained with \(\kappa \approx 0.43 - 0.50\) and \(u \approx 0.8\). We used \(L \leq 15\) and scanned \(0 < x \lesssim 2.8\); keeping \(\kappa = 0.45\) and \(u = 0.8\), we found for \(x > 1\) a satisfactory fit given by \(y = -0.3 - 5.3 \exp(-1.57x)\), which would imply an exponential saturation of the scaled width \(W h_0^{-\kappa}\) as \(x \to \infty\), with a limiting value \(\exp(-0.3) = 0.83\).

In Figure \[\text{[3]}\] we display again the data of Ref. \[\text{[4]}\], plus additional data for \(L = 15\) and 18, which enabled us to explore larger values of \(x\) (\(x \lesssim 4.0\)) while still keeping to relatively low \(H_0\). We then reanalyzed our full set of data, with the results that (i) we managed an excellent fit to the whole interval \(0 < x < 4\) by a single expression

\[
y = -2.2 \left\{ \ln \left( \frac{1.0}{x} + 0.6 \right) \right\} - 0.4025,
\]

(in which \(y\) and \(x\) involve the same values of the exponents \(\kappa, u\) earlier) and that (ii) this new fit, while still predicting saturation for \(x \gg 1\), implies that in the approach to two dimensions \(x \gg 1\), convergence of \(W h_0^{-\kappa}\) is power-law–like, giving a limiting scaled width \(W h_0^{-\alpha}\) as \(x \to \infty\), with a value \(\exp(-0.3) = 0.83\).

![FIG. 4. Semi-logarithmic scaling plot of RMS relative widths, \(W H_0^\kappa\) against \(L H_0^u\). Curve is fitting spline, given by Eq. \[\text{[20]}\).](image-url)
VI. MAGNETIZATIONS

In this section we examine the scaling properties of the uniform magnetization on strips of the $d = 2$ RFIM. For convenience we shall always keep $T = T_\alpha$.

We first outline our method, which involves a generalised Legendre transformation. Consider the Hamiltonian

$$\mathcal{H} = \mathcal{H}_0 (\{\sigma\}) - h \sum_i \sigma_i$$

(21)

where $\sigma_i$ are Ising spins, and $\mathcal{H}_0$ includes all interactions except that of the spins with the uniform field $h$. One has, for the corresponding partition function $Z(h)$:

$$\frac{Z(h)}{Z(0)} = \sum_M P_0(M) e^{\beta M h} , \quad \beta = \frac{1}{T}$$

(22)

where $P_0(M)$ is the probability of occurrence of the value $M$ for the magnetization, in zero uniform field. Assuming a system with $N \gg 1$ spins, with $f(h) \equiv$ negative free energy per site in units of $T$, and $m \equiv \beta M/N$,

$$e^{N(f(h) - f(0))} = N \int dm P_0(m) e^{N m h} .$$

(23)

In order for extensivity to be satisfied, one must have $N P_0(m) = \exp N g(m)$, where $g(m)$ is intensive and determined by

$$e^{N(f(h) - f(0))} = \int dm e^{N(g(m) + m h)} .$$

(24)

Assuming the usual sharp-peaked distribution around a thermodynamically averaged value $\overline{m}$, one sees that

$$f(h) - f(0) = g(\overline{m}) + \overline{m} h + \mathcal{O} \left( \frac{\ln N}{N} \right)$$

(25)

with $(dg/dm)\overline{m} = -h$. That is, $g$ is the standard Gibbs free energy per site. Substituting back in Eq. (23), one gets:

$$P_0(M) = \exp N g(m) ,$$

(26)

where terms of $\mathcal{O}(\ln N/N)$ have again been neglected.

Eq. (24), with $g(m)$ given through Eq. (25), is the natural starting point to study magnetization distributions by TM methods. Indeed, though one can get the thermodynamically averaged exact values of all moments of the distribution via TM [14,15], the distribution itself is not given directly. This contrasts with Monte-Carlo methods, which inherently incorporate readily-observable fluctuations around equilibrium, and have been widely used to study magnetization distributions at criticality, both in hypercubic geometries [17] and on planar lattices with various aspect ratios [18].

Recall that, on strips of width $L$ and length $L_x$, $L_x \gg L$ such as is the case here, the aspect ratio is essentially infinite, therefore $P_0(M)$ will be Gaussian, at least for pure systems [16,18]. Our purpose (as shown below) is to compare pure- and RFIM- results and explain their mutual differences, by using general theory of RF systems [15,16,21] coupled with finite-size scaling (FSS) [22].

A. Pure Ising systems

We start by illustrating the properties of $g(m)$ for pure Ising spins. One calculates $f(h), f(0), m(h)$ in Eq. (24) by standard numerical methods [23]: the first two by isolating the largest eigenvalue $\Lambda_0$ of the TM and using $f = L^{-1} \ln \Lambda_0$ (which is tantamount to assuming $L_x \rightarrow \infty$; more on this below), the third by calculating derivatives of $f$ relative to $h$. The latter is done here by perturbation theory [12,13,24,25], both for better numerical accuracy and because an adapted procedure proves convenient when dealing with the RF case, where samples over disorder must be accumulated.

![Fig. 5. Scaling plots of magnetization and excess free energy for pure Ising systems at criticality. Strip widths $L = 4$ (circles), 8 (crosses), 12 (triangles), 16 (squares). Normalized magnetizations ($= T_c \partial f / \partial h$) are used to avoid superposition of plots.](image-url)

At $t \equiv (T - T_c)/T_c = 0$, FSS [22] gives for the excess free energy: $\Delta f(h,L) \equiv f(t = 0,h,L) - f(0,0,L) = h^{1+\delta} F(L h^{1/\nu_b})$, with $\delta = 15$, $\nu_b = 15/8$. In Fig. 5 we show scaling plots of $\Delta f(h) h^{-(1+\delta)}$ and $m h^{-1/\nu}$ against $L h^{1/\nu_b}$. For low fields ($h \lesssim L^{-\nu_b}$), the slopes of both logarithmic plots are given by the (finite-size) initial susceptibility exponent $\gamma/\nu = 7/4$, as a consequence of the scaling relation $\nu_b(1 - 1/\delta) = \gamma/\nu$. 


For $t$ non-zero, but still for low fields, one generally expects $\Delta f(h) = a(t, L) h^\mu$ whence $m = a(t, L) h^{\mu-1}$, $g = \Delta f - mh = a(t, L)(1 - \mu) h^\mu$, implying $g \sim a(t, L)^{-1/(\mu-1)} m^{\mu/(\mu-1)}$. Subcases are:

(i) $t = 0$, $L = \infty$: $\Delta f \sim h^{1+1/\beta}$, so $g \sim m^{1+\beta}$.

(ii) $t$ small, $L = \infty$, $1 \gg t \gg m^{1/\beta}$: $\Delta f = a(t) h^2$, $a(t) = \frac{1}{2} \chi(t) \sim t^{-\gamma}$, so $g \sim t^{\gamma} m^2$.

(iii) $t = 0$, $L$ finite, $1 \ll L \ll m^{-\nu/\beta}$: $\Delta f = a(L) h^2$, $a(L) = \frac{1}{2} \chi(L) \sim L^{\gamma/\nu}$, so $g \sim L^{-\gamma/\nu} m^2$.

Case (iii) is depicted in Fig. 6. One sees that $-g L^{\gamma/\nu} \sim m^2$ as far as $m \approx 0.6$, which (through Eq. (21)) is consistent with the Gaussian behaviour predicted for $P_0(M)$ in this case. Close to $m = 1$ scaling breaks down, and the deviation from saturation magnetization must follow a single-spin-flip picture, $\varepsilon = 1 - m \sim \exp(-2/T_c)$. The effects of this on $g$ can be worked out from a high-field expansion, in which $H_0$ of Eq. (21) is taken as a perturbation on the field term $h \sum_i \sigma_i$ [26]. The result is:

$$\frac{dg}{d\varepsilon} = s_0 + \frac{1}{2} \ln \frac{1}{\varepsilon} + \mathcal{O}(\varepsilon)$$

where $s_0 = \frac{1}{2} (\ln 2 - 8/T_c) = -1.41617\ldots$.

Figure 7 shows that Eq. (27) is in excellent agreement with numerics already for $(1/2) \ln(1/\varepsilon) \approx 2.4$ ($\varepsilon \approx 10^{-2}$). This provides a rigorous check of our analytic and numerical procedures.

Returning to the connection between $g$ and magnetization deviations $\varepsilon$ against $(1/2) \ln(1/\varepsilon)$, points: numerically calculated derivatives from $L = 16$ data for $f$, $m$ ($L = 8$ already gives results indistinguishable from those displayed). Straight line: first two terms on RHS of Eq. (27).

![Graph](image-url)
where $2 - \eta = \gamma / \nu$ was used. Therefore the width of the Gaussian distribution is $W \sim g_0^{1/2} L^{(1-\eta)/2} / \sqrt{T_x}$.

### B. RFIM

We now include the term $\sum_i h_i \sigma_i$ in $H_0$ of Eq. (21), with the local fields $h_i$ distributed according to Eq. (3).

We first consider the application of FSS to the RFIM in zero uniform field. For bulk systems, theory predicts that the scaling behaviour of the RFIM depends on $H_0^2(t)^{-\phi}$ where $H_0$ is the random-field intensity, $t = (T - T_c(H_0))/T_c(H_0)$ is a reduced temperature, and the crossover exponent is $\phi = \gamma$, the pure Ising susceptibility exponent. For $d > 2$, $T_c(H_0)$ is the field-dependent temperature at which a sharp transition still occurs; in $d = 2$ the dominant terms still depend on the same combination, where now $T_c(H_0)$ denotes a pseudo-critical temperature marking, e.g., the location of the rounded specific-heat peak. In $d = 2$, specific heat and neutron-scattering data are in good agreement both with the choice of scaling variable as above, and with the exactly known $\gamma = 7/4$. For the excess free energy $\Delta^f f \equiv f(t, H_0) - f(t, 0)$ in two dimensions, an additive logarithmic correction arises [20-27]:

$$\Delta^f f = A^* t^2 \ln H_0 + H_0^{2 \nu / \phi} \Psi(t H_0^{-2 / \phi}) .$$

In Ref. [3], we showed that the appropriate FSS variable for the description of correlation functions in infinite RF systems at $t = 0$ is $x = L H_0^{-2 / \phi}$. While the second term on the RHS of Eq. (29) is in that way taken care of, the logarithm needs separate consideration. On the basis of renormalization-group arguments, in which $L^{-1}$ is seen as an additional relevant field [28], one realizes that the steps leading to the appearance of the $t^2 \ln H_0$ term in Eq. (29) also apply here. Indeed, since the respective eigenvalue [$y_T = 1$ in that case, $y_{L^{-1}} = 1$ here] divides the dimensionality $d = 2$ [21-27], a corresponding scenario obtains at $t = 0$ and $L^{-1} \rightarrow 0$, when $L^{-1 / \nu}$ is substituted for $t$. Therefore, we assume:

$$\Delta^f f(t = 0, L, H_0) = \tilde{A} L^{-2} \ln H_0 + H_0^{2 \nu / \phi} \tilde{\Psi}(L H_0^{2 \nu / \phi}) .$$

In Fig. 8, where $T = T_c$ (thus a small, $H_0$-dependent shift in $T_c(H_0)$ [20-27] has been neglected, which should not matter much for low RF intensities), are displayed results of a numerical test of Eq. (24), both with and without the logarithmic term.

We have found fits of a quality similar to that shown in Fig. 8(b), where $\tilde{A} = 10^{-4}$, for a wide range $10^{-5} \lesssim \tilde{A} \lesssim 10^{-2}$ along which the $\chi^2$ estimator remains approximately constant. At large $H_0 L^{\phi / 2 \nu}$, however, the fits deteriorate noticeably (not obvious from Fig. 8(b), because of the large vertical scale), no doubt owing to the incipient breakdown of the small-RF regime (where, e.g. the $H_0$-dependent shift in $T_c(H_0)$ is no longer negligible). Comparison with experimental data e.g. from Ref. [24] is not straightforward, as transforming from bulk scaling, Eq. (21), to FSS, Eq. (30), may involve numerical factors not immediately available.

![Figure 8](image-url)

**FIG. 8.** \(\Delta f \equiv (\Delta^f f(L, H_0) - \tilde{A} L^{-2} \ln H_0) H_0^{-2 \nu / \phi}\) plotted against $H_0 L^{\phi / 2 \nu}$, $\phi = 7/4$, $\nu = 1$. (a): $\tilde{A} = 0$. Bottom to top: $L = 4$, 6, 8, 10, and 12. (b): $\tilde{A} = 10^{-4}$, same notation.

Moving on towards incorporating both RF and uniform field effects, we again neglect the $H_0$-dependent shift in $T_c(H_0)$ and make $T = T_c \tilde{A} = 0$. In the presence of several relevant fields $u_1, u_2, \ldots$ with respective scaling powers $y_1, y_2, \ldots$, the singular part of the free energy scales as [27]:

$$f(u_1, u_2, \ldots) = |u_1|^{d / y_1} F \left( |u_2|^{y_2 / y_1}, |u_3|^{y_3 / y_1}, \ldots \right) .$$

Using $u_1 = H_0$, $u_2 = L^{-1}$, $u_3 = h$, one has in the case ($d = 2$): $y_1 = 2 \nu / \phi = 8/7$, $y_2 = 1$, $y_3 = y_4 = 15/8$, therefore

$$f(H_0, L, h) = H_0^{2 \nu / \phi} F(L H_0^{2 \nu / \phi}, h H_0^{-2 \nu / \phi}) .$$

Possible $\ln H_0$ corrections in the manner of Eq. (31) have been omitted, since our interest will focus on the calculation of the Gibbs free energy, which in the case depends on $\Delta^f f = f(H_0, L, h) - f(H_0, L, h = 0)$ (see...
Eqs. (21)–(23)); we are thus assuming that, at least for small enough \( h \), the logarithmic terms cancel in the subtraction.

Similarly to the pure case but always at \( t = 0 \) and \( L^{-1} \rightarrow 0 \), we investigate the small–\( h \) regime, in which one expects \( \Delta'' f = a(L, H_0) h^\mu \). From Eq. (32), this implies

\[
\Delta'' f = h^\mu H_0^{2\nu(1-\mu h)/\nu} F_1(L H_0^{2\nu/\nu}) .
\]

(33)

By assuming, as \( H_0 \rightarrow 0 \), a power-law dependence \( F_1(x) \sim x^\nu \), and demanding that, in this limit, (i) the \( H_0^{-1} \) dependence of \( \Delta'' f \) must vanish and (ii) the form \( h^2 L^{3/\nu} \) be reobtained, one gets \( \mu = 2 \), \( t = 7/4 \). Therefore, one has generally for small \( h \lesssim L^{-m} \):

\[
\Delta'' f = \left( \frac{h}{H_0} \right)^2 F_1(L H_0^{2\nu/\nu}) .
\]

(34)

whence \( a(L, H_0) = L^z H_0^{2\nu z/\nu - 2} \), yielding (see Subsection VI A):

\[
g(L, H_0, m) = -[a(L, H_0)]^{-1} m^2 = -L^{-z} H_0^{2-2\nu z/\nu} m^2 .
\]

(36)

Our data for \( g(L, H_0, m) \), displayed in Fig. 4, are consistent with \( z = 1 \), that is, \( g(L, H_0, m) \sim -L^{-1} H_0^{5/7} m^2 \).

One then has, using Eq. (20),

\[
P_0(M) = \exp \left[ -g_1 L_x H_0^{6/7} m^2 \right] ,
\]

(37)

with \( g_1 \approx 0.28 \) from the slope of the straight line in Fig. 4. Therefore the distribution is still Gaussian, with a width \( W \sim g_1^{-1/2} H_0^{-3/7}/\sqrt{L_x} \). Comparison with a corresponding pure system, see Eq. (28) and the arguments in the paragraph preceding it, gives:

\[
\frac{W(L, L_x, H_0)}{W(L, L_x, 0)} \sim \left( L H_0^{8/7} \right)^{-3/8} ,
\]

(38)

showing again that the FSS variable \( x \equiv L H_0^{2\nu/\nu} \) is the relevant one. For \( x \gg 1 \) where RFIM behaviour sets in, one sees that distribution widths are smaller for RFIM than in zero field.

VII. CONCLUSIONS

We have used TM methods to calculate spin-spin correlation functions, Helmholtz free energies and magnetizations on long strips of width \( L = 3 \rightarrow 18 \) sites of the two-dimensional RFIM, close to the zero-field bulk critical temperature.

Through analysis of the probability distributions of correlation functions for varying spin-spin distances \( R \), we have shown that fits to exponential decay of averaged values against \( R \) (for \( R \) not too large) give rise to effective correlation lengths larger than in zero field. This is because of the reinforcement of correlations within domains. At longer distances (i.e. across many domain walls, \( R/L \gg 1 \)), fits of exponential decay become unreliable, thus compromising definitions of effective correlation lengths.

We have worked out explicit connections between field–and correlation function distributions at high temperatures, yielding approximate analytical expressions for the latter. Such expressions account well for trends found in numerical data, namely the existence of peaked cusps and the functional dependence, on \( R \) and field intensity \( H_0 \), of data below the peak; above the peak, although agreement with numerics is not good, we have pinpointed that the responsibility for this lies in a truncation in our approximate scaling scheme, which decouples scaled nearest-neighbour interactions from the random field. We have discussed the use of analytical expressions, such as the ones found here, for computation of the corresponding.
structure factor. Though results as they stand are far from conclusive, we have established a rough guide to attempts at connecting basic microscopic features, such as fluctuations of accumulated fields, to experimentally observable quantities, e.g. scattering functions.

We have reanalyzed the asymptotic behaviour of the relative widths of correlation-function distributions, first presented in Ref. [3], and now complemented by additional data. While our earlier analysis seemed to point towards exponential saturation, the new set of data shows that, for fixed $R/L = 1$, the fractional widths of correlation-function distributions behave consistently with asymptotic power-law saturation, i.e. depending on $L^{-2.2}$, see Eq. (20). The scaling variables remain as given previously.

Considering a uniform applied field $h$, we have derived a connection between Helmholtz free energy $f(h)$, uniform magnetization $m(h)$, the Gibbs free energy $g(m)$, and the distribution function for the uniform magnetization in zero uniform field, $P_0(m)$, which is in principle applicable to any finite system. By working at the bulk zero-field critical temperature $T_c$0, we have illustrated our approach by showing that, for strips, one indeed gets a Gaussian distribution for $m$ not very close to saturation. Near $m = 1$, where scaling breaks down and a single-spin-flip picture holds, a perturbation expansion accounts for the properties of $g(m)$. Still at $T_c$0, in non-zero random field, we have found from finite-size scaling and crossover arguments, coupled with numerical data, that for strip geometries, $P_0(m)$ is still Gaussian, and its width varies against (non-vanishing, but small) random-field intensity $H_0$ as $h_0^{-3/7}$. This is again valid far from saturation (typically, for $m^2 \lesssim 0.4$, see Fig. [3]). The ratio between the width of $P_0(m)$ and the width of the corresponding distribution for a strip of same length and width in zero field varies as $(L H_0^{8/7})^{-3/8}$.

We expect that at least some of the features discussed here, for distributions of correlation functions and magnetizations on strips, translate also for other geometries. Considering, for instance, square systems: does a single-spin-flip picture hold, a perturbation expansion accounts for the properties of $g(m)$? For $m$ not very close to saturation, the new set of data presented in Ref. [9], and now complemented by additional data. While our earlier analysis seemed to point towards exponential saturation, the new set of data shows that, for fixed $R/L = 1$, the fractional widths of correlation-function distributions behave consistently with asymptotic power-law saturation, i.e. depending on $L^{-2.2}$, see Eq. (20). The scaling variables remain as given previously.

Finally, recalling possible connections with experimental data, that for strip geometries, $P_0(m)$ is still Gaussian, and its width varies against (non-vanishing, but small) random-field intensity $H_0$ as $h_0^{-3/7}$. This is again valid far from saturation (typically, for $m^2 \lesssim 0.4$, see Fig. [3]). The ratio between the width of $P_0(m)$ and the width of the corresponding distribution for a strip of same length and width in zero field varies as $(L H_0^{8/7})^{-3/8}$ as here?

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