FAMILIES OF CONFORMAL TORI OF REVOLUTION IN THE 3–SPHERE

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Abstract. For all positive integers \( n \) we construct a 1–parameter family of conformal tori of revolution in the 3–sphere with \( n \) bulges. These tori arise by Darboux transformations of constant mean curvature tori in the 3–sphere but do not have constant mean curvature in \( S^3 \).

1. Introduction

In a recent paper [2] it is shown that the multiplier spectral curve of a conformal torus \( f : T^2 \rightarrow S^4 \) is essentially given by the set of closed Darboux transforms of \( f \): to each multiplier on the spectral curve there exists a quaternionic holomorphic section with the given multiplier in the associated quaternionic holomorphic line bundle \( W \) of \( f \). The prolongation of the holomorphic section defines a new conformal torus \( \hat{f} \), and it turns out that \( f \) and \( \hat{f} \) satisfy a “weak enveloping” condition. Thus \( (f, \hat{f}) \) form a generalized Darboux pair; Classically, the Darboux transformation is defined for isothermic surfaces and a map \( \hat{f} : M \rightarrow \mathbb{R}^3 \) is called a classical Darboux transform [5] of an isothermic \( f \) if there exists a sphere congruence enveloping both \( f \) and \( \hat{f} \).

For every conformal torus the set \( H^0(W) \) of holomorphic sections with a given multiplier \( h \) is generically 1–dimensional, and at generic points the Darboux transformation preserves geometric properties: e.g., generic Darboux transforms of a constant mean curvature torus have constant mean curvature [4], and generic Darboux transforms of a Hamiltonian stationary torus are Hamiltonian stationary [8].

However, there exist examples, e.g. [8], of conformal tori which allow non–trivial multiplier on the spectral curve with high dimensional space of holomorphic sections. The existence of these singular multipliers should allow a deformation of the spectral curve: in the case of constant mean curvature tori in the 3–sphere of spectral genus zero [7] one can deform the spectral curve to obtain a family of Delaunay tori by removing this singularity of the spectral curve, and thus by adding geometric genus. By contrast the Darboux transformation preserves the geometric spectral genus [2] in the case when the Darboux transform is immersed but it may change geometric properties (e.g. break the constant mean curvature condition). In particular, the Darboux transformation at singular points is expected to allow to add or remove arithmetic genus of the spectral curve, and a thorough understanding of the singular points of the multiplier spectral curve may play an important role in understanding the reconstruction of conformal tori by their spectral data [6, 10, 11, 7] and the study of minimum energy tori in presence of a variational principle [11, 7].

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In this short note, we concentrate on the geometric properties of the Darboux transformation in the case when the conformal torus is a rectangular torus in the 3–sphere: in particular, we construct for each \( n \in \mathbb{N}, n \geq 2 \), a 1–parameter family of conformal tori of revolution in \( S^3 \) with \( n \) bulges which do not have constant mean curvature in \( S^3 \). Using a similar argument we also construct for each \( a \in \mathbb{R}, a > 0 \), a 1–parameter family of cylinder of revolution with non–constant mean curvature.

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2. Rectangular tori

In the following, we will apply the Darboux transformation on rectangular tori \( f : \mathbb{C}/\Gamma \rightarrow S^3 \subset \mathbb{R}^4 \) with lattice \( \Gamma = \frac{1}{u} \mathbb{Z} \oplus \frac{1}{v} \mathbb{Z} \), \( u,v > 0 \), to construct the families of conformal tori of revolution. We identify Euclidean 4–space \( \mathbb{R}^4 = \mathbb{H} \) with the quaternions and parametrize a rectangular torus with parameters \((u,v)\) by

\[
f(x,y) = \frac{uv}{u^2 + v^2} \left( \frac{1}{u} e^{2\pi jux} + \frac{1}{v} e^{2\pi jvy} \right).
\]

In particular, we will use the fact that a rectangular torus \( f : T^2 \rightarrow S^3 \) is Hamiltonian stationary, and use the methods and settings developed in \([8]\) to compute the Darboux transforms of \( f \). We can write \( f \) as

\[
f(x,y) = e^{\frac{ij}{2}} \left( \frac{1}{u} + \frac{1}{i} \right) \left( -\frac{j}{2\pi} \right) g,
\]

where the so–called Lagrangian angle \( \beta \) is given by

\[
\beta(z) = 2\pi \langle \beta_0, z \rangle \quad \text{with} \quad \beta_0 = u - vi \in \Gamma^* = u\mathbb{Z} \oplus v\mathbb{Z},
\]

and

\[
g = 2\pi \rho e^{\pi j (ux + vy)},
\]

with scale \( \rho = \frac{uv}{\sqrt{u^2 + v^2}} \in \mathbb{R} \). Moreover, The derivative of \( f \) can be written as

\[
df = 2\pi \rho e^{\pi j (ux - vy)} dz e^{\pi j (ux + vy)} = e^{\frac{ij}{2}} dz g,
\]

and \( f \) is a conformal immersion, that is \([3\text{ Sec. 2.2}]\)

\[
*df = N df = -df R.
\]

with left normal

\[
N = e^{ij} i = e^{2\pi j (ux - vy)} i.
\]

and right normal

\[
R = -g^{-1} i g = ie^{2\pi j (ux + vy)}.
\]

For every conformal immersion \( f : M \rightarrow \mathbb{H} \) with right and left normals \( R \) and \( N \) the normal bundle of \( f \) is given \([3\text{ Sec 2.2}]\) by

\[
\perp_f = \{ x \in \mathbb{H} | NxR = -x \}.
\]

In particular, if \( f : M \rightarrow S^3 \subset \mathbb{H} \) is a conformal map into the 3–sphere, then \( n = Nf = fR \) and \( f \) are unit normals. Thus, the second fundamental form \( \Pi_{\mathbb{H}} \) of \( f \) as a map into \( \mathbb{H} \) computes with

\[
(Xdf(Y))^\perp = -<df(Y),dn(X)> + n-<df(Y),df(X)>f
\]
\[ \Pi_{\mathbb{H}} = \Pi_{S^3} - |df|^2 f. \]

From this we see that the mean curvature vector \( \mathcal{H}_{\mathbb{H}} \) in \( \mathbb{R}^4 = \mathbb{H} \) relates to the mean curvature vector \( \mathcal{H}_{S^3} = H_{S^3} n \) via

\[ \mathcal{H}_{\mathbb{H}} = H_{S^3} n - f \]

where \( H_{S^3} \) is the mean curvature of \( f \) in \( S^3 \). We denote by

\[ (dN)' = \frac{1}{2} (dN - N \ast dN) \quad \text{and} \quad (dN)'' = \frac{1}{2} (dN + N \ast dN) \]

the \((1,0)\) and \((0,1)\)-parts of the derivative of \( N \) with respect to the complex structure \( N \), and define \( H \) by \( (dN)' = -dfH \). Then it is shown in [3, Sec. 7.2] that the mean curvature vector of a conformal immersion into \( \mathbb{R}^4 \) is given by

\[ (2.1) \quad \mathcal{H}_{\mathbb{H}} = NH. \]

Combining the previous equations, we see that the mean curvature \( H_{S^3} \) of a conformal immersion \( f : M \rightarrow S^3 \) of a Riemann surface \( M \) into \( S^3 \) is given by

\[ (2.2) \quad H_{S^3} = fH + N = \text{Re}(fH). \]

In particular, since \( H \) computes in the case of Hamiltonian stationary Lagrangians [8] to

\[ H = \pi g^{-1} \beta_0 e^{\beta_0 \theta} k \]

the constant mean curvature of a rectangular torus in \( S^3 \) is given by

\[ H_{S^3} = \frac{1}{2} \left( \frac{u}{v} - \frac{v}{u} \right). \]

3. The Darboux transformation

We will briefly recall the construction of Darboux transforms in the case when \( f : M \rightarrow \mathbb{R}^4 \) is a conformal immersion from a Riemann surface into Euclidean 4–space. For the general case of conformal immersions into the 4–sphere and details of the construction compare [2].

In our situation, the associated quaternionic holomorphic line bundle of the immersion \( f \) can be identified with the trivial quaternionic bundle \( \mathbb{H} \) equipped with the (quaternionic) holomorphic structure \( D \) given by

\[ D\alpha := \frac{1}{2} (d\alpha + N \ast d\alpha), \]

for \( \alpha \in \Gamma(\mathbb{H}) \) where \( N \) is the left normal of \( f : M \rightarrow \mathbb{R}^4 = \mathbb{H} \). We denote by \( H^0(\mathbb{H}) = \ker D \) the set of holomorphic sections of the holomorphic line bundle \( (\mathbb{H}, D) \). The prolongation of a local holomorphic section \( \alpha \in H^0(\mathbb{H}) \) is given by the local section

\[ \psi = \left( f\nu + \alpha \nu \right) \in \Gamma(\mathbb{H}^2) \]

of the trivial \( \mathbb{H}^2 \) bundle where \( \nu \) is defined by \( d\alpha = -df\nu \). Then \( \psi \) spans locally a quaternionic line bundle \( L \), and if \( d\alpha \) is nowhere vanishing, the corresponding map \( \hat{f} = f + \alpha\nu^{-1} \) is a branched conformal immersion into \( \mathbb{R}^4 \), a so–called a Darboux transform of \( f \). If we denote by \( T = \hat{f} - f \), then the derivative of \( \hat{f} \) is given away from the zeros of \( \alpha \) by

\[ d\hat{f} = -Td\nu\alpha^{-1} T. \]
From $d\alpha = -df \nu$ we see that $df \wedge d\nu = 0$, in other words, $*d\nu = -Rd\nu$ since $f$ has right normal $R$. In particular, $f$ has left normal

$$
(3.2) \quad \hat{N} = -TR^T^{-1}.
$$

To compute the mean curvature vector $\hat{H}$ of the Darboux transform $\hat{f}$, it remains to compute $\hat{H}$ using the defining equation $(d\hat{N})' = -d\hat{f} \hat{H}$. To this end, note that the derivative of $\hat{N}$ computes with $\hat{f} = \hat{f} + T$ as

$$
d\hat{N} = -*dT^{-1} + dT^{-1}\hat{N} - TR^T^{-1} + \hat{N}dfT^{-1} - *d\hat{T}T^{-1}
$$

so that the $(1,0)$-part of $d\hat{N}$ with respect to $\hat{N}$ is given by

$$
(3.3) \quad (d\hat{N})' = \hat{f}T^{-1}\hat{N} - T(dR)^T - *d\hat{T}T^{-1}
$$

where $(dR)^T = \frac{1}{2}(dR + R* dR)$. To obtain Darboux transforms which are globally defined, we consider holomorphic sections with multiplier, that is holomorphic sections of the trivial bundle $\tilde{M}$ over the universal cover $\tilde{M}$ of $M$ which satisfy

$$
\gamma^* \alpha = \alpha h_{\gamma}
$$

with $h_{\gamma} \in \mathbb{C}_\gamma$ for all $\gamma \in \pi_1(M)$. From $d\alpha = -df \nu$ and (3.1) we see that the prolongation $\psi$ of $\alpha$ has multiplier $h_\gamma$, that is $\gamma^* \psi = \psi h_{\gamma}$ for $\gamma \in \pi_1(M)$, so that $\psi$ defines a branched conformal immersion $\hat{f} : M \to \mathbb{H}$ if $\alpha$ is nowhere vanishing.

In the case when $f : T^2 \to S^4$ is a conformal 2–torus, the existence of global Darboux transforms is guaranteed by the link [2] between Darboux transforms and the multiplier spectral curve $\Sigma$ of $f$: to every multiplier $h \in \Sigma$ there exists at least one holomorphic section with multiplier $h$, and each such holomorphic section gives by prolongation a Darboux transform $\hat{f} : T^2 \to S^4$ of $f$. In other words, there is at least a Riemann surface worth of Darboux transforms of a conformal torus.

4. DARBOUX TRANSFORMS OF HAMILTONIAN STATIONARY LAGRANGIAN TORI

In the following we summarize notations and results of [8]. In the case of an Hamiltonian stationary Lagrangian torus $f : \mathbb{C}/\Gamma \to \mathbb{R}^4$ with Lagrangian angle $\beta = 2\pi \langle \beta_0, \cdot \rangle$, every multiplier of a holomorphic section is of the form

$$
h = h^{A,B} = e^{2\pi \langle (A, \cdot) - i(B, \cdot) \rangle}
$$

with $A, B \in \mathbb{C}^2$ such that

$$
\Gamma_{A,B}^* = \{ \delta \in \Gamma^* + \frac{\beta_0}{2} | \delta \text{ satisfies } |\delta - B|^2 - |A|^2 = \frac{|\beta_0|^2}{4} \text{ and } \langle \delta - B, A \rangle = 0 \}
$$

is not empty. A holomorphic section $\alpha \in H^0(\mathcal{V}/L)$ with multiplier $h^{A,B}$ is called monochromatic if it is given by a Fourier monomial, that is if

$$
(4.1) \quad \alpha = \alpha_\delta := e^{\beta_0/2} (1 - k\lambda_\delta) e_{\delta - B} e^{2\pi \langle A, \cdot \rangle}
$$

is given by a single frequency $\delta \in \Gamma_{A,B}^*$ where $\lambda_\delta := \frac{2}{\beta_0} (\delta - iA - B)$ and

$$
e_{\gamma}(z) := e^{2\pi i \langle \gamma, z \rangle}.
$$

A polychromatic holomorphic section with multiplier $h^{A,B}$ is given by a non–trivial linear combination $\alpha = \sum_{\delta \in \Gamma_{A,B}^*} m_\delta \alpha_\delta$ of monochromatic holomorphic sections, $m_\alpha \in \mathbb{C}$. 

**Definition 4.1.** A branched conformal immersion \( \hat{f} : M \to S^4 \) is called a *monochromatic Darboux transform* (respectively *polychromatic*) if it is given by the prolongation of a monochromatic (respectively polychromatic) holomorphic section.

In [8] it is shown that all monochromatic Darboux transforms of a rectangular torus are after reparametrization again a rectangular torus. Moreover, polychromatic holomorphic sections with multiplier \( h^{A,B} \neq 0 \), only occur if \( h^{A,B} \) is real. In particular, the corresponding Darboux transform coincides with a monochromatic Darboux transform. To obtain new tori we therefore have to consider polychromatic Darboux transforms with \( A = 0 \):

**Theorem 4.2** ([8]). Let \( f : \mathbb{C}/\Gamma \to \mathbb{R}^4 \) be a Hamiltonian stationary torus in \( \mathbb{R}^4 \) with Lagrangian angle \( \beta \) and \( df = e^{i\beta} dzg \). Then every non–constant, polychromatic Darboux transform \( \hat{f} : \mathbb{C}/\Gamma \to \mathbb{R}^4 \) of \( f \) with \( A = 0 \) is given by

\[
\hat{f} = f + e^{i\beta} \left( \sum_{s,t \in I_B} (1 + ke^{is})m_s \bar{m}_t e^{i\frac{\beta_0}{2}(e^{it} - e^{-it})} (1 + ke^{it}) \sin t \right) \frac{1}{R\pi \beta_0} g
\]

where the finite set

\[
I_B = \{ t \in [0,2\pi) \mid B - \frac{\beta_0}{2} e^{it} \in \Gamma^*_{0,B} \} \neq \emptyset,
\]

parametrizes the admissible frequencies and \( m_t \in \mathbb{C} \) are chosen so that the map

\[
R = \left| \sum_{t \in I_B} m_t \sin t e^{B - \frac{\beta_0}{2} e^{it}} \right|^2 + \left| \sum_{t \in I_B} m_t e^{it} \sin t e^{B - \frac{\beta_0}{2} e^{it}} \right|^2
\]

is nowhere vanishing.

5. **Families of conformal tori of revolution in \( S^3 \)**

In this section we discuss the Darboux transforms of a rectangular torus \( f : T^2 \to S^3 \). If the parameters \( (u, v) \) of \( f \) satisfy \( u \geq \sqrt{3v} \), we show that there exist polychromatic Darboux transforms which are conformal tori in the 3–sphere. Fix \( n \in \mathbb{N}, n \geq 2 \), and consider all rectangular tori \( f \) in the 3–sphere with parameters \( (u, v) \) satisfying \( u^2 + v^2(1 - n^2) \geq 0 \). The multiplier \( h = h^{0,B} \) with

\[
B = \frac{\beta_0}{2} + \frac{nvi}{2} - \frac{\sqrt{u^2 + v^2(1 - n^2)}}{2}
\]

has \( \dim H^0_{0,B} = |\Gamma^*_{0,B}| \geq 2 \) since

\[
\delta_+ = \frac{\beta_0}{2}, \quad \delta_- = \frac{\beta_0}{2} + nvi \in \Gamma^*_{0,B} = \{ \delta \in \Gamma^2 \mid |\delta - B|^2 = \frac{|\beta_0|^2}{4} \}
\]

which shows that \( f \) allows polychromatic Darboux transforms. In particular, we see that

\[
-\lambda_{\delta_\pm} = \frac{2}{\beta_0}(B - \delta_\pm) = c_\pm + is_\pm
\]

with

\[
c_\pm = \mp \frac{nvi^2 - us}{r^2}, \quad s_\pm = \pm \frac{uvn - sv}{r^2}
\]

and

\[
s := \sqrt{r^2 - v^2n^2} \quad \text{where} \quad r = |\beta_0| = \sqrt{u^2 + v^2}
\]
so that (4.1) all monochromatic holomorphic sections with multiplier \( h_{0,B} \) are given by
\[
\alpha_{\pm} := \alpha_{\delta_{\pm}} = e^{\pi j (ux - vy)} (1 + js_{\pm} + kc_{\pm}) e^{\pi i (sx \mp nvy)}.
\]
Using Theorem 4.2 we see that the corresponding polychromatic Darboux transforms are given by
\[
\hat{f} = e^{j \beta/2} \left( -\left( \frac{j}{u} + \frac{k}{v} \right) + \left( \sum_{i,j \in \{\pm\}} a_{ij} \right) \frac{u - iv}{R r^2} \right) \frac{g}{2\pi}
\]
where
\[
(5.2) \quad a_{ij} = (1 + ke^{it_i} m_i m_j e^{2\pi i (\delta_i - \delta_j)})(1 + ke^{it_j}) \sin t_j,
\]
and \( m_{\pm} \in \mathbb{C} \) have to be chosen so that
\[
2R(y) = |m_+ s_+ + m_- s_- e^{i\tilde{y}}|^2 + |m_+ s_+ e^{it_+} + m_- s_- e^{it_-} e^{i\tilde{y}}|^2
\]
is nowhere vanishing where
\[
\tilde{y} := 2\pi nvy.
\]
Using (5.1) and \(|z + w|^2 = |z|^2 + |w|^2 + 2Re(z\bar{w})\) the denominator \( R \) simplifies to
\[
R = |m_+|^2 s_+^2 + |m_-|^2 s_-^2 + \frac{2v^2(1 - n^2)}{r^4} \Re \left( m_+ \bar{m}_- (r^2 - n^2v^2 - isvn) e^{-i\tilde{y}} \right).
\]
In general, the above Darboux transforms will be conformal immersions into the 4–sphere. However, there exist constants \( m_+, m_- \in \mathbb{C}_* \) so that the polychromatic Darboux transform is a conformal immersion into the 3–sphere.

**Theorem 5.1.** For each \( n \in \mathbb{N}, n \geq 2 \), there exists a 1–parameter family of conformal tori of revolution in the 3–sphere with \( n \) bulges. Only one conformal torus in this family has constant mean curvature in \( S^3 \).

**Proof.** We consider the case that \( m_{\pm} = m \). In this case (5.1)
\[
s_+^2 + s_-^2 = \frac{2v^2}{r^4} \left( u^2(1 + n^2) + v^2(1 - n^2) \right)
\]
shows that
\[
\frac{R r^4}{2|m|^2 v^2} = u^2(1 + n^2) + v^2(1 - n^2) + (1 - n^2) \left( (r^2 - n^2v^2) \cos(\tilde{y}) - svn \sin(\tilde{y}) \right).
\]
From (5.2)
\[
a_{ij} = |m|^2 (1 + ke^{it_i}) e^{2\pi i (\delta_i - \delta_j)}(1 + ke^{it_j}) \sin t_j
\]
we see that \( \hat{f} \) is independent of the choice of \( m \in \mathbb{C} \). In particular, we may assume from now on that \( m = 1 \). Then
\[
a_{\pm,\pm} = 2ke^{it_\pm} s_{\pm} = 2(ke^{it_\pm} + js_{\pm}^2)
\]
is independent of \( z = x + iy \) while
\[
a_{\pm,\mp} = (1 + ke^{it_\mp}) s_{\pm} e^{\mp i\tilde{y}} + (ke^{it_\pm} - e^{it_\mp - it_\pm}) s_{\mp} e^{\pm i\tilde{y}}
\]
is independent of $x$. In particular, the Darboux transform is given by $\hat{f} = f + T$ where $T = e^{\frac{i\beta}{2}\tau}g$, $\tau = \tau_0 + i\tau_1$ with

\[
\begin{align*}
\tau_0(y) &= \frac{jun^2}{\pi R(y)} \in j\mathbb{R} \\
\tau_1(y) &= \frac{n(sv\cos \tilde{y} + s^2\sin \tilde{y}) + j(s^2 + (s^2\cos \tilde{y} - sv\sin \tilde{y}))}{\pi v R(y)} \in \mathcal{C} = \text{Span}\{1, j\}
\end{align*}
\]

and

\[
\hat{R}(y) = u^2(1 + n^2) + v^2(1 - n^2) + (1 - n^2)(s^2\cos \tilde{y} - sv\sin \tilde{y}).
\]

Writing $\hat{f} = e^{i\frac{\beta}{2}\hat{\tau}}g$ where

\[
(5.3) \quad \hat{\tau} = \tau + \sigma, \quad \sigma = -\frac{1}{2\pi} \left(\frac{j}{u} + \frac{k}{v}\right),
\]

a lengthy, but straightforward, computation shows that

\[
(5.4) \quad |\hat{\tau}|^2 = \frac{1}{4\pi^2\rho^2}.
\]

In other words, $\hat{f} : T^2 \to S^3$ is a conformal map into the 3-sphere. On the other hand, we have

\[
\hat{f} = e^{i\pi(ux - vy)}(\tau_0 - \frac{j}{2\pi u} + i(\tau_1 - \frac{j}{2\pi v}))2\pi \rho j e^{i\pi(ux + vy)} = e^{2i\pi ux\kappa_0 + 2i\pi vy\kappa_1}
\]

where $\kappa_0 = (2\pi\tau_0 + \frac{1}{u})\rho$ is a real valued function, and both $\kappa_0$ and $\kappa_1 = (2\pi\tau_1 + \frac{1}{v})\rho$ only depend on $y$, that is, $\hat{f}$ is a surface of revolution in $S^3$. Note that for $u = v\sqrt{n^2 - 1}$, the Darboux transform $\hat{f}$ is a rectangular torus with constant mean curvature in the 3-sphere. If $u \neq v\sqrt{n^2 - 1}$ then $\kappa_0$ is extremal at

\[
y_k = \frac{1}{2nv} \left(\frac{1}{\pi} \arctan\left(-\frac{vn}{\sqrt{u^2 + v^2(1 - n^2)}} + k\right)\right), \quad k = 0, \ldots, 2n - 1,
\]

in particular, $\hat{f}$ is a torus of revolution in $S^3$ with $n$ bulges. If $\hat{f}$ had constant mean curvature in $S^3$ and is not a rectangular torus, we would expect it to be a Delaunay torus and thus to be parametrized by elliptic functions. However, the Darboux transformation essentially preserves spectral genus [2], and in our case $\hat{f}$ is parametrized by (rational functions of) trigonometric functions. Indeed, we compute the mean curvature $\hat{H}_{S^3} = \ldots$
Figure 2. Non-embedded Darboux transform of the rectangular torus with parameter (2.6, 1)

Re (\(\hat{f}H\)) of \(\hat{f}\) in \(S^3\) explicitly when \(u \neq v\sqrt{n^2 - 1}\): first, \(\hat{H}\) is defined by \((d\hat{N})' = -d\hat{f}H\), that is \(3.3\)

\[
\hat{H} = -T^{-1} \hat{N} + \hat{f}^{-1}_x (Tr_x + \hat{N}\hat{f}_x) T^{-1}
\]

where \((dR)' = r_x dx + r_y dy\) with

\[
r_x = \pi g^{-1} j(u\hat{\i} - v)g.
\]

Moreover, we compute \(\hat{f}_x = e^{i\beta/2} 2\pi ju\hat{\i}g = e^{i\beta/2} qg\) with real valued \(q = 2\pi ju\hat{\i}0 = 1 - 2u^2n^2/R\).

The left normal of \(\hat{f}\) is given by \(3.2\)

\[
\hat{N} = e^{i\beta/2} \gamma \hat{\tau}^{-1} e^{-i\beta/2}
\]

so that \(\hat{f}\hat{H} = e^{i\beta/2} \hat{\gamma} e^{-i\beta/2}\) with \(\hat{\gamma} = \hat{\tau} \gamma \hat{\tau}^{-1}\) and

\[
\gamma = -i + q^{-1} \pi \tau j(u\hat{\i} - v) + \tau i \hat{\tau}^{-1}.
\]

Next observe that for any \(z, w \in \mathbb{H}\) we have \(Re(zw^{-1}) = Re w\) which shows that

\[
Re (\gamma) = q^{-1} \pi Im (v\tau_0 + u\tau_1),
\]

and the mean curvature of \(\hat{f}\) in \(S^3\) is, using \(\hat{\tau} = \tau + \sigma\), given by \(2.2\)

\[
\hat{H}_{S^3} = Re (\sigma \gamma \tau^{-1}) + \frac{\pi}{q} Im (v\tau_0 + u\tau_1),
\]

Furthermore, for \(\lambda = \lambda_0 + i\lambda_1 \in \mathbb{H}, \lambda_0, \lambda_1 \in \mathbb{C}\), we have with \(\hat{\tau} = -\tau\)

\[
Re (\sigma \lambda \tau^{-1}) = -\frac{1}{2\pi|\tau|^2} Im \left( \tau_0 \left( \frac{\lambda_0}{u} + \frac{\lambda_1}{v} \right) + \tau_1 \left( \frac{\bar{\lambda}_0}{v} - \frac{\bar{\lambda}_1}{u} \right) \right)
\]

and we get for both \(\lambda = -i\) and \(\lambda = \tau i \tau^{-1}\)

\[
Re (\sigma \lambda \tau^{-1}) = \frac{1}{2\pi|\tau|^2} Im \left( \frac{\tau_0}{v} - \frac{\tau_1}{u} \right).
\]

whereas for \(\lambda = q^{-1} \pi \tau j(u\hat{\i} - v)\)

\[
Re (\sigma \lambda \tau^{-1}) = \frac{1}{2q} \left( \frac{v}{u} + \frac{u}{v} \right) - \frac{1}{uvq|\tau|^2} (v Im \tau_0 + u Im \tau_1)^2.
\]

A straightforward computation shows that

\[
\pi uv|\tau|^2 = Im (v\tau_0 + u\tau_1)
\]
so that (5.5) simplifies to

\[ \hat{H}_{S^3} = \frac{1}{\pi |\tau|^2} \text{Im} \left( \frac{\tau_0}{\nu} - \frac{\tau_1}{u} \right) + \frac{1}{2q} \left( \frac{v}{u} + \frac{u}{v} \right). \]

For \( y = 0 \) one easily obtains with

\[ \tilde{R} := 2u^2 + v^2(1 - n^2)(2 - n^2) \quad \text{and} \quad \tilde{q} := 2r^2 - n^2v^2 \]

that

\[ \tau_0(0) = j \frac{n^2 u}{\pi \tilde{R}}, \quad \text{and} \quad \tau_1(0) = \frac{1}{\pi \nu \tilde{R}} \left( vn^2s + i(\tilde{q} - n^2v^2) \right) \]

so that (5.6) becomes

\[ \hat{H}_{S^3}(0) = \frac{r^2(v^2n^4 - \tilde{q}) - 2\tilde{q}(n^2 - 1)v^2}{2(n^2 - 1)uv\tilde{q}}. \]

Similarly, for \( y = \frac{1}{2n} \) we obtain with

\[ \tilde{R} := 2u^2 + v^2(1 - n^2) \]

that

\[ \tau_0 = j \frac{u}{\pi \tilde{R}}, \quad \text{and} \quad \tau_1 = -\frac{s}{\pi \tilde{R}} \]

which gives

\[ \hat{H}_{S^3}(\frac{1}{2n}) = \frac{2u^2v^2(n^2 - 1) - r^2\tilde{R}}{2(n^2 - 1)uv^3} \]

In particular, if \( \hat{f} \) has constant mean curvature in \( S^3 \) then \( \hat{H}_{S^3}(0) = \hat{H}_{S^3}(\frac{1}{2n}) \) gives

\[ r^2(v^2n^4 - \tilde{q})v^2 - 2\tilde{q}(n^2 - 1)v^4 = 2u^2v^2(n^2 - 1)\tilde{q} - r^2\tilde{R}\tilde{q} \]

which is equivalent to

\[ 4r^2(u^2 + v^2(1 - n^2))^2 = 0, \]

that is, \( u = v\sqrt{n^2 - 1} \). Finally, we notice that all rectangular tori are, up to reparametrization \( \tilde{x} = vz, z = x + iy \), rectangular tori with parameter \( (\frac{u}{v}, 1) \). Thus, we obtain for each \( n \in \mathbb{N}, n \neq 1 \), a 1–parameter family of tori of revolution in \( S^3 \) with \( n \) bulges, each torus given by a polychromatic Darboux transform of a rectangular torus with parameter \((u, 1), u \geq \sqrt{n^2 - 1} \). \qed
6. **Polychromatic Darboux Transforms of Cylinders**

We use similar methods to compute polychromatic Darboux transforms of a standard cylinder $f : M \to \mathbb{R}^3$. To stay close to the notations and computations in the previous sections, our maps $f$ will take values in $\text{Span}\{1, j, k\}$. Note that in this case $f$ has mean curvature $H$ given by

$$H_{\mathbb{R}^3} = -Hi$$

where $H$ is again given by $(dN)' = -dfH$. A standard cylinder

$$f(x, y) = \frac{1}{u} e^{2\pi jux} + 2\pi k y$$

is then a Hamiltonian stationary immersion with harmonic left and right normals

$$N = e^{2\pi jux} i \quad \text{and} \quad R = ie^{2\pi jux}$$

and Lagrangian angle $\beta(z) = 2\pi (\beta_0, z)$ with $\beta_0 = u$. Moreover, we have $df = e^{\frac{\beta_0}{2}} dzg$ with $g = 2\pi je^{\pi jux}$, and $f = e^{\frac{\beta}{2}} (-\frac{i}{2\pi} + \pi iy) \frac{2}{\pi}$. With the same methods as before (with the obvious adaptions to the situation of a cylinder), we consider for $a \in \mathbb{R}$ all cylinder with $u \geq a$, and obtain again for

$$B = \frac{1}{2}(u + ai - \sqrt{u^2 - a^2})$$

holomorphic sections with multiplier. The corresponding frequencies are

$$\delta_+ = \frac{u}{2}, \quad \delta_- = \frac{u}{2} + ai \in \Gamma_{\beta_0,B} = \{\delta \in u\mathbb{Z} + i\mathbb{R} + \frac{\beta_0}{2} | |\delta - B| = |\beta_0| \}$$

and the monochromatic holomorphic sections with multiplier $h^{0,B}$ are

$$\alpha_+ = \frac{1}{u} e^{j\pi ux}(u \pm ja - k\sqrt{u^2 - a^2})e^{\pi i(\sqrt{u^2 - a^2} + ay)}.$$
Again, we apply Theorem 4.2 with constants $m_+ = m_- = 1$, and obtain, after a similar computation as in the case of rectangular tori, the monochromatic Darboux transforms of a cylinder for $h_{(0,B)}$ as

$$\hat{f} = e^{i\frac{\beta}{2} (j\tau_0 + i\tau_1)} \frac{g}{\pi} = 2(-e^{2\pi jux} \tau_0 + k\tau_1)$$

with real valued functions

$$\tau_0(x,y) = \frac{1}{u}(-\frac{1}{2} + \frac{1}{R})$$

and

$$\tau_1(x,y) = \pi y + \frac{1}{aR} \left( \sin \tilde{y}(1 - \frac{a^2}{u^2}) + \cos \tilde{y}(\frac{a}{u^2} \sqrt{u^2 - a^2}) \right).$$

Here, we have with $\tilde{y} = 2\pi nvy$

$$\tilde{R}(y) = \frac{u^2}{4a^2} R = 1 - (1 - \frac{a^2}{u^2}) \cos \tilde{y} + \frac{a}{u^2} \sqrt{u^2 - a^2} \sin \tilde{y}.$$ 

and thus, both $\tau_0$ and $\tau_1$ only depend on $y$. In particular, $\hat{f}$ is a surface of revolution in the 3-space spanned by $1, j, k$, and obviously, $\hat{f}$ is a round cylinder whenever $u = a$ for $a \in \mathbb{R}$.

We now compute the mean curvature of $\hat{f}$. To that end, we observe that $\hat{f} = f + T$ with

$$T = \frac{2}{R} \left( -\frac{e^{2\pi jux}}{u} + \frac{k}{a} \left( \sin \tilde{y}(1 - \frac{a^2}{u^2}) + \cos \tilde{y}(\frac{a}{u^2} \sqrt{u^2 - a^2}) \right) \right),$$

and the left normal $\hat{N}$ of $\hat{f}$ is given by

$$\hat{N} = -\frac{ie^{-2\pi jux}(\kappa_0^2 - \kappa_1^2)}{\kappa_0^2 + \kappa_1^2} = \frac{2j\kappa_0\kappa_1}{\kappa_0^2 + \kappa_1^2}$$

with

$$\kappa_0 = -\frac{1}{u} \quad \text{and} \quad \kappa_1 = \frac{1}{u} \left( \sin \tilde{y}(1 - \frac{a^2}{u^2}) + \cos \tilde{y}(\frac{a}{u^2} \sqrt{u^2 - a^2}) \right)$$

real valued. As before, we compute with (5.5)

$$\hat{H} = i\left( \frac{\hat{R}\kappa_0}{\lambda} + \frac{1}{4\tau_0} \right),$$

and, by evaluating at $y = 0$ and $y = \frac{1}{2a}$, we see that $\hat{H}$ constant is equivalent to $a = u$. We summarize
Theorem 6.1. For all $a \in \mathbb{R}, a > 0$, the Darboux transformation gives a 1-parameter family of cylinder of revolution which are not constant mean curvature cylinder in the 3-space.

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