Towards a Verified Tableau Prover for a Ground
Fragment of Set Theory

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Abstract
Using Isabelle/HOL, we verify the state-of-the-art decision
procedure for multi-level syllogistic with singleton (MLSS
for short), which is a ground fragment of set theory. We form-
alise the syntax and semantics of MLSS as well as a sound
and complete tableau calculus for it. We also provide an ab-
stract specification of a decision procedure that applies the
rules of the calculus exhaustively and prove its termination.

1 Introduction
In contrast to pen-and-paper proofs where we expect the
reader to fill in trivial details, interactive theorem provers
(ITPs) require us to justify each proof step rigorously. Manu-
ally reducing each proof step down to the axioms would be
impractical, though; hence, ITPs usually offer a wide array
of automated methods to deal with the proof goals that arise
during interactive use.

In Isabelle/HOL, there are specialised procedures for deal-
ing with e.g. natural numbers, linear arithmetic, and metric
spaces. Some of these procedures have been verified in Isa-
belie/HOL such as a procedure for Presburger arithmetic [10]
that was later extended to mixed real-integer arithmetic [9].
This procedure, though, uses reflection to work on goals
in Isabelle/HOL, which, during execution, either sacrifices
speed by going through the simplifier or soundness by trust-
ing generated code. More recently, Stevens and Nipkow [24]
presented a verified decision procedure for orders that pro-
duces certificates. This approach offers efficient execution by
using generated code as well as soundness as the certificates
are replayed through Isabelle’s inference kernel.

The focus of this paper is another ubiquitous structure
in mathematics, namely sets. In particular, we consider a
ground fragment of set theory which Cantone and Zarba [7]
call multi-level syllogistic with singleton (MLSS). The frag-
ment includes the usual set operations of union, intersection,
difference, membership, equality and, in addition, it allows
the construction of singleton sets. Due to the omnipresence
of sets, we think that automation for this fragment will be-
efit the users of Isabelle, especially when integrated into
the simplifier like the aforementioned order solver. We also
expect that the generation of certificates will be straightfor-
ward since MLSS admits a tableau calculus.

1.1 Contributions
In this paper, we present a formalisation in Isabelle/HOL of
a tableau calculus for MLSS due to Cantone and Zarba [7][5,
Chapter 14]. We prove soundness and completeness of the
calculus and give an abstract specification of a decision pro-
cedure that applies the rules of the calculus exhaustively.
To obtain total correctness of the procedure, we prove its
termination. The formalisation follows the paper closely but
gives a more thorough account of some important details:

• We deliver the omitted proof of Lemma 2 in the pa-
per [7], which is a key building block for the complete-
ness proof of the calculus.
• The formal proof of completeness lead to the discovery
that the calculus was missing a rule for eliminating
double negation.
• We derive an explicit upper bound for the number of
formulae in a branch of the tableau.

1.2 Related Work
Since the literature on decidable fragments of set theory
is vast, we only focus on MLSS here. The fragment was
first shown to be decidable by Ferro et al. [12]. Subsequent
work [4] found the decision problem to be NP-complete. To
obtain a practical decision procedure, Cantone [2] proposed
a tableau calculus, which was later improved by Beckert and
Hartmer [1]. Both of these procedures construct a model dur-
ing execution that is used to guide the proof search. Beckert
and Hartmer also cover an extension of the calculus with
uninterpreted functions, which was revisited by Cantone
and Zarba [8] who avoided the construction of a model dur-
ing execution. In this paper, we consider a version of the
latter procedure due to Cantone and Zarba [7] that is special-
ised to MLSS and where the branching rules of the calculus
are set up in a way to guarantee mutual exclusivity of the
branches. Later extensions of the calculus added certain in-
terpreted functions such as monotone functions [6] and the
inverse of a function [3]. The latter extension notably in-
cludes the Cartesian product. Those extensions, though, did
not improve upon the tableau calculus for MLSS.

While a lot of work has gone into the verification of de-
cision procedures, seemingly little has been put towards the
verification of tableau calculi. In Lean, there is a verified
tableau calculus for basic modal logic [17]. In Coq, there is a
formalisation of a tableau calculus for ecumenical logic [18]; it is, however, missing a completeness proof. Concerning Isabelle/HOL, there is formalisation of a sound and complete tableau calculus for hybrid logic [16] available in the Archive of Formal Proofs [14]. The termination proof is part of ongoing work [15].

1.3 Notation
Isabelle/HOL [20] conforms to everyday mathematical notation for the most part. We establish notation and in particular some essential data types together with their primitive operations that are specific to Isabelle/HOL.

We write \( t :: \text{'a} \) to specify that the term \( t \) has the type \( 'a \) and \( 'a \Rightarrow 'b \) for the space of total functions from type \( 'a \) to type \( 'b \).

Sets with elements of type \( 'a \) have the type \( 'a \) set. The cardinality of a set \( A \) is denoted by \( \text{card} \ A \) and the image of \( A \) under \( f \) by \( f \cdot A \).

We use \( 'a \) list to describe the type of lists, which are constructed using the empty list \([\] \) constructor or the infix cons constructor \&, and are appended with the infix operator \@. The function set converts a list into a set.

We remark that \( \longleftrightarrow \) is equivalent to \( = \) on the type of Booleans \( \text{bool} \) and \( \equiv \) is definitional equality of the meta-logic of Isabelle/HOL, which is called Isabelle/Pure. Meta-implication is denoted by \( \Rightarrow \) and a chain of implications

\[
A_1 \Rightarrow \cdots \Rightarrow A_k \Rightarrow C
\]
can be abbreviated by

\[
[ A_1; \ldots; A_k ] \Rightarrow C.
\]

2 Syntax and Semantics of MLSS
2.1 Syntax
At the heart of MLSS, we have the type of set terms which is the disjoint union of the empty set and variables as well as the operations union, intersection, set difference, and the singleton set represented by the constructor \( \text{Single} \). We keep the type of variables abstract by making it a parameter of the set term data type. The only restriction on the type of variables is that it needs to be infinite. Isabelle/HOL’s data type package automatically defines a function that gives us the set of variables in a set term, which we name \( \text{vars} \).

In what follows, we will overload the function \( \text{vars} \) to also work on set atoms, formulae, and branches.

\[
\text{datatype} \ (\text{vars: 'a}) \ \text{pset Atom} = \\
\begin{align*}
& 0 \mid \text{Var} 'a \mid \text{Single} ('a \text{ psetTerm}) \\
& 'a \text{ psetTerm} \cup 'a \text{ psetTerm} \\
& 'a \text{ psetTerm} \cap 'a \text{ psetTerm} \\
& 'a \text{ psetTerm} \setminus 'a \text{ psetTerm}
\end{align*}
\]

We can combine two set terms to form a set atom by using the membership or the equality operator.

\[
\text{datatype} \ (\text{vars: 'a}) \ \text{pset Term} = \\
\begin{align*}
& 0 \mid \text{Var} 'a \mid \text{Single} ('a \text{ psetTerm}) \\
& 'a \text{ psetTerm} \cup 'a \text{ psetTerm} \\
& 'a \text{ psetTerm} \cap 'a \text{ psetTerm} \\
& 'a \text{ psetTerm} \setminus 'a \text{ psetTerm}
\end{align*}
\]

With the above operators we can also represent the subset operator \( \subseteq \) and enumerate finite sets: \( s \subseteq A \) is equivalent to \( s \cup \emptyset \supseteq s \) and a finite set of elements \( \{ t_1, \ldots, t_k \} \) can be expressed by \( \text{Single} \ t_1 \cup \ldots \cup \text{Single} \ t_k \).

We use the propositional fragment of formulae due to Nipkow [19] with set atoms as propositional atoms to form the unquantified fragment MLSS of set theory.

\[
\text{datatype} \ (\text{atoms: 'a}) \ \text{fm} = \\
\begin{align*}
& A 'a \mid \neg ('a \text{ fm}) \\
& 'a \text{ fm} A 'a \text{ fm} | 'a \text{ fm} \lor 'a \text{ fm}
\end{align*}
\]

\[
\text{type_synonym} \ 'a \text{ pset fm} = 'a \text{ pset Atom fm}
\]

We will often drop the atom constructor \( A \) to reduce clutter. Additionally, we use the abbreviations \( s \not\in t \) and \( s \neq t \) to denote \( \neg A \ (s \in s \ t) \) and \( \neg A \ (s = s \ t) \), respectively.

Like the function \( \text{vars} \), the automatically defined function \( \text{atoms} \) is a set retrieves all set atoms in a formula. We combine these functions to extract all the variables occurring in a set formula.

\[
\text{definition} \ \text{vars} :: 'a \text{ pset fm} \Rightarrow 'a \text{ set where} \\
\text{vars} \phi \equiv \bigcup (\text{vars} 'a \text{ atoms} \phi)
\]

Similarly to \( \text{vars} \), we introduce the overloaded constant \( \text{subterms} \).

\[
\text{fun} \ \text{subterms} :: 'a \text{ pset Term} \\
\Rightarrow 'a \text{ pset Term set where} \\
\text{subterms} \emptyset = \{0\} \\
| \text{subterms} (\text{Var} x) = \{\text{Var} x\} \\
| \text{subterms} (s \cup t) = \{s \cup t\} \cup \text{subterms} s \cup \text{subterms} t \\
| \text{subterms} (s \cap t) = \{s \cap t\} \cup \text{subterms} s \cup \text{subterms} t \\
| \text{subterms} (s \setminus t) = \{s \setminus t\} \cup \text{subterms} s \cup \text{subterms} t \\
| \text{subterms} (\text{Single} t) = \{\text{Single} t\} \cup \text{subterms} t
\]

The subterms of an atom are then the subterms of its arguments.

\[
\text{fun} \ \text{subterms} :: 'a \text{ pset Atom} \\
\Rightarrow 'a \text{ pset Term set where} \\
\text{subterms} (s \subseteq t) = \text{subterms} s \cup \text{subterms} t \\
| \text{subterms} (s \subseteq t) = \text{subterms} s \cup \text{subterms} t \\
| \text{subterms} (s \subseteq t) = \text{subterms} s \cup \text{subterms} t
\]

Finally, we lift this to formulae by using atoms again.

\[
\text{definition} \ \text{subterms} :: 'a \text{ pset fm} \Rightarrow 'a \text{ pset Term set where} \\
\text{subterms} \phi \equiv \bigcup (\text{subterms} 'a \text{ atoms} \phi)
\]
Lastly, we consider the subformulae of a formula and define a function that computes them.

```haskell
fun subfms :: 'a fm ⇒ 'a fm set where
  subfms (A a) = {A a}
| subfms (p ∧ q) = {p ∧ q} ∪ subfms p ∪ subfms q
| subfms (p ∨ q) = {p ∨ q} ∪ subfms p ∪ subfms q
| subfms (¬ q) = {¬ q} ∪ subfms q
```

**Literals** are a type of subformulae of particular importance, so we define a predicate for them.

```haskell
fun is_literal :: 'a fm ⇒ bool where
  is_literal (A _) = True
| is_literal (¬ (A _)) = True
| is_literal _ = False
```

### 2.2 Semantics

We base the semantics of MLSS on the von Neumann hierarchy $\mathcal{V}$ of sets that is defined inductively as

\[
\begin{align*}
\mathcal{V}_0 &= \emptyset, \\
\mathcal{V}_{α+1} &= \mathcal{P}(\mathcal{V}_α) \quad \text{for each ordinal } α, \\
\mathcal{V}_λ &= \bigcup_{α<λ} \mathcal{V}_α \quad \text{for each limit ordinal } λ, \\
\mathcal{V} &= \bigcup_α \mathcal{V}_α,
\end{align*}
\]

where $\mathcal{P}(S)$ is the powerset of $S$. The sets in $\mathcal{V}$ are well-founded, i.e., there can be no membership cycle in $\mathcal{V}$.

In Isabelle/HOL, which is simply typed, this definition is not accepted, though. A way to work around this limitation is provided by Lawrence in an entry [22] of the *Archive of Formal Proofs*, which adds $\mathcal{V}$ to Isabelle/HOL by way of axiomatisation. Two articles [11, 23] have been written that provide further context to this entry. The entry declares a type $\mathcal{V}$ that comes with the following functionality:

- The function `elts :: $\mathcal{V} ⇒ \mathcal{V}$ set maps a set to its elements. Note that even though we use the typed sets provided by Isabelle/HOL here, `elts` actually returns a class (relative to $\mathcal{V}$). This class only constitutes a set (relative to $\mathcal{V}$) if the class is small.
- The predicate `small :: 'a set ⇒ bool`, which is polymorphic in 'a, indicates whether a class is small.
- The function `vset :: 'a set ⇒ $\mathcal{V}$ turns a class into a set of type $\mathcal{V}$ if it is small.
- The usual set operations such as equality (=$\equiv$), union ($\sqcup$), intersection ($\sqcap$), and difference ($\setminus$) are defined.
- Finally, the empty set coincides with the ordinal 0, so it is denoted by $\emptyset :: \mathcal{V}$.

Equipped with the above, we define the interpretation function $I_{st}$ for set terms that interprets a set term with respect to a valuation function $M :: 'α ⇒ \mathcal{V}$ for variables.

```haskell
fun I_{st} :: ('α ⇒ $\mathcal{V}$) ⇒ 'α pset_term ⇒ $\mathcal{V}$ where
  I_{st} M Φ = 0
| I_{st} M (Var x) = M x
| I_{st} M (Single s) = vset (I_{st} M s)
| I_{st} M (s ⊆ t) = I_{st} M s ⊆ I_{st} M t
| I_{st} M (s ∩ t) = I_{st} M s ∩ I_{st} M t
| I_{st} M (s − t) = I_{st} M s − I_{st} M t
```

The interpretation function $I_{sa}$ for set atoms is straightforward as well.

```haskell
fun I_{sa} :: ('α ⇒ $\mathcal{V}$) ⇒ 'α pset_atom ⇒ bool where
  I_{sa} M (s ∈ s t) = I_{st} M s ∈ elts (I_{st} M t)
| I_{sa} M (s = s t) = I_{st} M s = I_{st} M t
```

We write $M ⊨ φ$ for the judgement that the formula $φ$ holds under the valuation function $M$. The implementation of $⊨$ coincides with the interpretation function of Nipkow [19]. As usual, a formula $φ$ is called **satisfiable** if there exists a model $M$ with $M ⊨ φ$. Otherwise, $φ$ is called **unsatisfiable**.

### 3 A Tableau Calculus for MLSS

We formalise the tableau calculus for MLSS as described by Cantone and Zarba [7]. Inspired by the formalisation of a tableau calculus for hybrid logic by From [14], we simply use a list to represent a branch of the tableau tree. Note that formulae are added to the front of the list during branch expansion, so the last branch $b$ is always the formula that we are trying to disprove with the tableau.

```haskell
definition vars :: 'a branch ⇒ 'a set
  vars b = ∪(vars ⊥ set b)

definition subterms ::
  'a branch ⇒ 'a pset_term set where
  subterms b = ∪(subterms ⊥ set b)
```

In the standard tableau calculus for propositional logic as Fitting [13] describes it, a branch is called **closed** if it contains both the negation of a formula and the formula itself; conversely, it is called **open** if it is not closed. For MLSS, we extend the notion of closedness with three additional rules; the first two are straightforward while the last one states that a branch is closed when the branch contains a membership cycle

\[
t_0 \in_s t_1, t_1 \in_s t_2, \ldots, t_k \in_s t_0.
\]

```haskell
inductive bclosed :: 'a branch ⇒ bool where
  [\[ φ ∈ set b; ¬ φ ∈ set b ]] ⇒ bclosed b
| (t ∈_s ∅) ∈ set b ⇒ bclosed b
| (t ≠_s t) ∈ set b ⇒ bclosed b
| [\[ member_cycle cs; set cs ⊆ set b]] ⇒ bclosed b
```
The mechanics of the decision procedure are typical for a procedure based on a tableau calculus: it decides the satisfiability of a given formula \( \phi \) by determining whether the formula has a closed tableau. More specifically, it initialises the tableau with the singleton branch \([\phi]\) and checks whether this branch can be expanded to a closed tableau.

We abstractly specify the behaviour of the decision procedure, leaving an executable specification from which code can be generated to future work. The implementation as displayed in Figure 1 uses a couple features of Isabelle/HOL’s function package: instead of defining the function via pattern matching, we specify the equations of the function as conditional rewrite rules. This requires us to prove that the assumptions of the equations are non-overlapping, which is done by automation. The other concern is that Isabelle/HOL requires functions to be total, so a recursive function needs to terminate in order for it to be well-defined; nevertheless, the termination proof is separated from the definition of the function for modularity. The function package maintains the soundness of the definition by introducing a so-called domain predicate \texttt{mlss_proc_branch_dom} which characterises for which arguments the function terminates. Each equation of the function is guarded by an assumption that the predicate holds for the argument. To be able to reason about this predicate, we pass the \texttt{domintros} flag that instructs the function package to derive introduction rules for it. In Section 8, we will use these rules to discharge the guarding assumptions for the context that \texttt{mlss_proc_branch} is called in. Before we go into more detail on how the termination is proved, we discuss its definition.

The purpose of the function is to determine whether a given branch can be expanded to a closed tableau. As stated
Propositional Rules

| Rule                | Implication |
|---------------------|-------------|
| $p \land q$         | $p, q$      |
| $\neg (p \lor q)$   | $\neg p, \neg q$ |
| $p \lor q, \neg p$  | $q$         |
| $p \lor q, \neg q$  | $p$         |
| $\neg (p \lor q), p$ | $\neg q$     |
| $\neg (\neg p)$    | $p$         |

Rules for $\sqcup_s$

| Rule                | Implication |
|---------------------|-------------|
| $s \in_s t_1 \sqcup_s t_2$ | $s \in_s t_1, s \in_s t_2$ |
| $s \notin_s t_1$     | $s \notin_s t_1 \sqcup_s t_2$ |
| $s \notin_s t_2$     | $s \notin_s t_1 \sqcup_s t_2$ |
| $s \notin_s t_1, s \notin_s t_2$ | $s \notin_s t_1 \sqcup_s t_2$ |

Rules for $\Pi_s$

| Rule                | Implication |
|---------------------|-------------|
| $s \in_s t_1 \Pi_s t_2$ | $s \in_s t_1, s \in_s t_2$ |
| $s \notin_s t_1$     | $s \notin_s t_1 \Pi_s t_2$ |
| $s \notin_s t_2$     | $s \notin_s t_1 \Pi_s t_2$ |
| $s \notin_s t_1, s \notin_s t_2$ | $s \notin_s t_1 \Pi_s t_2$ |

Rules for Single

| Rule                | Implication |
|---------------------|-------------|
| $s \in_s \text{Single } t$ | $s =_s t$ |
| $s \notin_s \text{Single } t$ | $s \neq_s t$ |

Rules for $=$

| Rule                | Implication |
|---------------------|-------------|
| $t_1 =_s t_2, 1$    | $1 \{t_2/t_1\}$ |
| $s_1 \in_s t, s_2 \notin_s t$ | $s_1 \neq_s s_2$ |

Table 1. Linear expansion rules.

| Rule                     | Precondition               | Subsumption condition |
|--------------------------|---------------------------|-----------------------|
| $p \land q$              | $p \lor q \in \text{set } b$ | $p \in \text{set } b \lor \neg p \in \text{set } b$ |
| $\neg p$                 | $\neg (p \land q) \in \text{set } b$ | $\neg p \in \text{set } b \lor p \in \text{set } b$ |
| $s \in_s t_1 \land_s t_2$ | $(s \in_s t_1 \land_s t_2) \in \text{set } b$ | $\neg p \in \text{set } b \lor p \in \text{set } b$ |
| $s \notin_s t_1$         | $(s \notin_s t_1) \in \text{set } b$ |
| $s \notin_s t_2$         | $(s \notin_s t_2) \in \text{set } b$ |
| $s \in_s t_1 \land_s t_2$ | $(s \in_s t_1 \land_s t_2) \in \text{subterms (last } b)$ | $\neg p \in \text{set } b \lor p \in \text{set } b$ |
| $s \notin_s t_1$         | $(s \notin_s t_1) \in \text{set } b$ |
| $s \notin_s t_2$         | $(s \notin_s t_2) \in \text{set } b$ |
| $s \in_s t_1 \land_s t_2$ | $(s \in_s t_1 \land_s t_2) \in \text{subterms (last } b)$ | $\neg p \in \text{set } b \lor p \in \text{set } b$ |
| $s \notin_s t_1$         | $(s \notin_s t_1) \in \text{set } b$ |
| $s \notin_s t_2$         | $(s \notin_s t_2) \in \text{set } b$ |
| $s \in_s t_1 \land_s t_2$ | $(s \in_s t_1 \land_s t_2) \in \text{subterms (last } b)$ | $\exists s. (s \in_s t_1) \in \text{set } b \land (s \notin_s t_2) \in \text{set } b$ |
| $s \notin_s t_1$         | $(s \notin_s t_1) \in \text{set } b$ |
| $s \notin_s t_2$         | $(s \notin_s t_2) \in \text{set } b$ |

Table 2. Branching expansion rules.
function (domintros) mlss_proc_branch :: 'a branch ⇒ bool where
  ¬ lin_sat b ⇒ mlss_proc_branch (b)
  mlss_proc_branch b = mlss_proc_branch ((SOME b’. b’ ⊃ b ∧ set b ⊂ set (b’ ⊕ b)) ⊕ b)
  | [ lin_sat b; bclosed b ] ⇒ mlss_proc_branch b = True
  | [ ¬ sat b; bopen b; lin_sat b ]
  ⇒ mlss_proc_branch b = (∀ b’ ∈ (SOME bs. bs ⊃ b). mlss_proc_branch (b’ ⊕ b))
  | [ lin_sat b; sat b ] ⇒ mlss_proc_branch b = bclosed b

definition mlss_proc :: 'a pset_fm ⇒ bool where
mlss_proc ϕ = mlss_proc_branch [ϕ]

Figure 1. Definition of the decision procedure through mlss_proc_branch and mlss_proc.

before, we first use linear expansion rules in order to prevent premature branching; to this end, we recursively expand the branch with linear rules until the branch is linearly saturated. Note that we use Hilbert’s $e$-operator in the form of SOME to choose some rule that actually adds new newsubterms to the branch. As soon as the branch is linearly saturated, we terminate if the branch is closed as shown in the second equation. Otherwise, we choose an applicable branching rule and recursively check whether all newly created branches can be closed. The final equation applies once no further branch expansion is possible, in which case we just test for closedness of the branch.

The procedure mlss_proc then calls mlss_proc_branch with a singleton branch $[ϕ]$ to determine the satisfiability of a given formula $ϕ$.

This means that mlss_proc_branch is only applied to branches that result from applying the expansion rules. We call this kind of branches well-formed. In the definition below, the expression $b’ ⊃ b$ denotes that $b’$ is one of the branches that results from applying (potentially zero) expansion rules to $b$.

definition wf_branch b ≡ $∃ φ. b ⊃ [ϕ]$

In particular, we will use this notion in Section 7 where we derive an upper bound for the cardinality of well-formed branches. This is then used in Section 8 to justify the termination of the decision procedure. Before we come to that, though, we prove soundness and completeness in Section 6 and 5, respectively. In Section 8, we also show that both properties easily transfer to mlss_proc which, together with termination, establishes that it is a decision procedure.

5 Completeness of the Calculus

For completeness of the calculus, we need to show that every unsatisfiable formula has a closed tableau or, conversely, that the formula is satisfiable if there is a saturated and open branch in the tableau. To facilitate inductive reasoning, we show a stronger statement by constructing a model $M$ such that $M ∪ ϕ$ for all $ϕ ∈ \text{set } b$. At the core of the model, there is a realisation function that maps set terms to sets of type $V$. A subset of the witnesses, which we call pure witnesses, receives special treatment from the realisation function for reasons that will become apparent in Section 5.1. The set terms of a branch can thus be partitioned into two collections as defined below.

definition wits’ :: 'a branch ⇒ 'a set where
wits’ b ≡
{ c ∈ wits b. ∀ t ∈ \text{subterms } (\text{last } b).
AT (\text{Var } c = s t) \notin \text{set } b ∧
AT (t = s \text{ Var } c) \notin \text{set } b}

definition subterms’ ::
'a branch ⇒ 'a pset_term set where
subterms’ b ≡
\text{subterms } (\text{last } b) \cup
\text{Var }' (\text{wits } b - \text{wits’ } b)

We aim to construct a syntactic model that is derived from the membership literals $s ∈ t$ in the branch. To this end, we construct a graph whose vertices are the disjoint union of the sets above and there is an edge from $s$ to $t$ in the graph if, and only if, $s ∈ t$ in $b$. Note that we use Noschinski’s graph library [21] that represents a graph as a record of vertices, arcs (directed edges), and two functions tail and head that map an arc to its source respectively target vertex.

definition bgraph b ≡
let vs = Var ' wits' b ∪ subterms' b
in ( verts = vs,
  arcs = {(s, t). (s ∈ s t) ∈ set b},
  tail = fst, head = snd )

The realisation function is defined relative to this graph. As mentioned before, the pure witnesses $\text{Var }' \text{ wits’ } b$ and the rest of the set terms $\text{subterms’ } b$ are treated differently by the realisation function. Terms in the latter set are evaluated in accordance to the structure of the graph, i.e. the realisation of a vertex is defined as the union of the realisations of the parent vertices. For the former set, we choose
a function \( \text{realise} \) that assigns the witnesses in \( \text{Var} \setminus \text{wits}\text{' } b \) pairwise distinct sets with cardinality greater than that of the vertices. We can always choose such a function since we assume an infinite universe of variables. Then, we return the singleton set \( \{I x\} \), which, together with the cardinality constraint, guarantees that realisations are distinct between pure witnesses themselves as well as between pure witnesses and set terms. The notation \( u \rightarrow_G s \) in the definition below indicates that there is an edge from \( u \) to \( s \) in the graph \( G \).

**abbreviation** parents \( G \ s = \{u. \ u \rightarrow_G s\} \)

**function** \( \text{realise} :: \ 'a \ \text{pset_term} \Rightarrow \text{V} \) where

\[
| x \in \text{subterms}\text{' } b \implies \text{realise} x = \text{vset} \{I x\}
\]

| \( x \notin \text{verts} \ G \implies \text{realise} x = \emptyset \)

Again, we need to ensure that the assumptions of the equations are non-overlapping and that the function terminates. The former is taken care of by automation, leaving us to prove termination. The assumption that \( b \) is open implies that there are no membership cycles and thus \( bgraph \text{'} b \) is acyclic. Now, consider the ancestors \( av \) of a vertex \( v \), which are those vertices from which we can reach \( v \). Due to acyclicity, it holds that the ancestors \( ap \) of a parent \( p \) of \( v \) are a subset of \( av \). They are also a proper subset of \( av \) since \( p \) is an ancestor of \( v \) but not of \( p \) itself. Together with the fact that \( bgraph \text{'} b \) is finite, it follows that the cardinality of the ancestors decreases in each recursive call, thus proving the termination of \( \text{realise} \).

Before we prove that the realisation function constitutes a model in Section 5.2, we will first explain the significance of the pure witnesses.

### 5.1 Characterisation of the Pure Witnesses

Recall that the pure witnesses of a branch \( b \) are those witnesses that are not related to other subterms in \( \text{last } b \) by equality. In the context of a well-formed branch, this characterisation can be strengthened to any set term and, in addition, we also get that there is no membership literal where a pure witness is on the right-hand side. Intuitively speaking, the realisation of a pure witness does not depend on the realisation of any other set term.

**lemma** lemma_2:

- **assumes** \( \text{wf_branch } b \)
- **assumes** \( c \in \text{wits}\text{' } b \)
- **shows** \( (\text{Var } c =_s t) \neq \text{set } b \)
  - \( (t =_s \text{Var } c) \neq \text{set } b \)
  - \( (t \in_s \text{Var } c) \neq \text{set } b \)

So why are pure witnesses treated differently? According to the definition of \( \text{realise} \), the pure witnesses would be evaluated to the empty set \( \emptyset :\ V \), were they not treated separately. To see that this is a problem, consider the branch

\[
b = [\text{Var } s \neq_\text{Var } t, \text{Var } t \neq_\text{Var } u]
\]

which expands to several open and saturated branches, one of which is

\[
[\text{Var } x \neq_\text{Var } y, \text{Var } x \in_s \text{Var } s, \text{Var } x \notin_\text{Var } s \text{Var } t, \text{Var } y \in_s \text{Var } t, \text{Var } y \notin_\text{Var } s \text{Var } u] \subseteq b
\]

for some fresh \( x \) and \( y \). Assigning both \( \text{Var } x \) and \( \text{Var } y \) a value of \( \emptyset \) would contradict the literal \( \text{Var } x \neq_\text{Var } y \). To prevent this, we assign the pure witnesses pairwise different values.

The proof of lemma_2 is more technical than interesting so we refer the reader to the formalisation.

### 5.2 Realisation of an Open Branch

Remember that for completeness, we need to show that the realisation function for an open and saturated branch \( b \) actually constitutes a model for all formulae in the branch. We start by verifying that the realisation function models all literals in the branch; more formally, the following propositions hold:

(i) We have \( \text{realise} s \in \text{elts} (\text{realise } t) \) if it holds that \( s \in_s t \) is in \( b \).

(ii) We have \( \text{realise } s = \text{realise } t \) if \( s =_s t \) is in \( b \).

(iii) We have \( \text{realise } s \neq \text{realise } t \) if \( s \neq_\text{Var } t \) is in \( b \).

(iv) We have \( \text{realise } s \notin \text{elts} (\text{realise } t) \) if it holds that \( s \notin_\text{Var } t \) is in \( b \).

To illustrate the usefulness of lemma_2, we prove Proposition (ii). The proofs of all propositions translate well into Isabelle, so we refer to the original paper [7] for the remaining proofs.

**Proof of Proposition (ii).** Assume that \( s =_s t \) is in \( b \). If there exists a \( c \in \text{wits}\text{' } b \) where \( s = \text{Var } c \) or \( t = \text{Var } c \), we arrive at a contradiction due to lemma_2. Therefore, both \( s \in \text{subterms}\text{' } b \) and \( t \in_s \text{subterms}\text{' } b \) must hold. Now, assume for contradiction that

\[
\text{realise } s \neq \text{realise } t
\]

which implies that

\[
\text{elts} (\text{realise } s) \neq \text{elts} (\text{realise } t).
\]

Without loss of generality — the other case is symmetric — we obtain an \( e \) such that

\[
e \in \text{elts} (\text{realise } s) \land e \notin \text{elts} (\text{realise } t).
\]

Considering the fact that \( s \in \text{subterms}\text{' } b \) and the definition of \( \text{realise} \), we obtain an \( u \) where \( e = \text{realise } u \) and \( u \rightarrow bgraph \text{'} b \ s \). This, in turn, yields that \( u \in_s s \) must be in \( b \). Together with the assumption \( (s =_s t) \in s \) and the saturation of \( b \), it follows that \( u \in_s t \) must also be in \( b \). But then we have
We therefore have

\[ u \in \text{elts (realise } t) \]

using Proposition (i), which stands in contradiction to the assumption \( e \not\in \text{elts (realise } t) \). \( \square \)

The results on literals can now be lowered to set terms.

(a) It holds that realise \( \emptyset = \emptyset \).

(b) Let \( s \in \{L, -, \ln \} \). If the term \( s \star t \) occurs in subterms \( b \), then

\[ \text{realise } (s \star t) = \text{realise } s \star \text{realise } t. \]

(c) If \( \text{Single } t \in \text{subterms } b \), then

\[ \text{realise } (\text{Single } t) = \text{vset } \{\text{realise } t\}. \]

\[ \text{Proof of Proposition (a). By definition, realise } \emptyset \text{ is equal to } \emptyset \text{ if } \emptyset \text{ is not a vertex of bgraph } b. \text{ Otherwise, } \emptyset \text{ must be in subterms } b \text{ as it is not a witness. Moreover, since } b \text{ is open, we know } s \in \emptyset \text{ if } \emptyset \text{ is not in } b \text{ for any } s \text{, implying that the term } \emptyset \text{ has no parents in } b \text{. Ultimately, we obtain realise } \emptyset = \emptyset. \] \( \square \)

\[ \text{Proof of Proposition (b) for } \star = -_. \text{ For equality, it suffices to prove subset inclusion in both directions.} \]

\textbf{Direction } \subseteq. \text{ Fix some } e \in \text{elts } (\text{realise } (s - s \ t)). \text{ Again, we obtain an } u \text{ with}

\[ e = \text{realise } u \land u \rightarrow_{b \text{graph } b} (s - s \ t). \]

We therefore have \((u \in \emptyset \ s - s \ t) \in \text{set } b\) which, due to \( b \) being saturated, implies that \( u \in \emptyset \ s \) is in \( b \). Furthermore, the fifth branching rule (see Table 2) must have its subsumption condition met due to saturation of \( b \), meaning that either one of \( u \in \emptyset \ t \) or \( u \not\in \emptyset \ t \) must be in the branch. Only the latter, however, can be the case since the branch is open. By applying Proposition (i) and (iv) we obtain

\[ \text{realise } u \in \text{realise } s \land \text{realise } u \not\in \text{realise } t \]

and together with \( e = \text{realise } u \) we arrive at the goal

\[ e \in \text{elts (realise } s - \text{realise } t) \].

\textbf{Direction } \supseteq. \text{ Let } e \in \text{elts } (\text{realise } s - \text{realise } t) \text{ for some fixed } e. \text{ As before, we obtain an } u \text{ such that } e \text{ is the realisation of } u, \text{i.e. } e = \text{realisation } u, \text{ and}

\[ u \rightarrow_{b \text{graph } b} s \land u \rightarrow_{b \text{graph } b} t. \]

An immediate consequence is that \( u \in \emptyset \ s \) is in \( b \). Moreover, we claim that \( u \not\in \emptyset \ t \) is also in \( b \). To see that this is true, recall that \( b \) is saturated and thus the subsumption condition of the fifth branching rule must hold. This implies that either \( u \in \emptyset \ t \) or \( u \not\in \emptyset \ t \) is in \( b \). But the latter being in \( b \) would be a contradiction to \( u \rightarrow_{b \text{graph } b} t \) meaning that the former must hold. Ultimately, we infer that \( u \in \emptyset \ s - s \ t \) is in \( b \) by invoking saturation again. The last step is to apply Proposition (i) in order to obtain the goal

\[ \text{realise } u \in \text{elts (realise } (s - s \ t)) \]

\[ = e \in \text{elts (realise } (s - s \ t)). \]

The final step to obtain a proper model is to connect the realisation function to the semantics as defined in Section 2. For set terms, we can use the Propositions (a)–(c) to prove the lemma below by induction on \( t \).

\textbf{lemma}

\[ \text{assumes } t \in \text{subterms } b \]

\[ \text{shows } I_{\text{st}} (\lambda x. \text{realise } (\text{Var } x))) t = \text{realise } t \]

Lifting the above result to formulae yields the coherence of \( b \) as the original paper [7] calls it. The proof is a tedious but straightforward induction on the the size of the formulæ.

\textbf{lemma} coherence:

\[ \text{assumes } \phi \in \text{set } b \]

\[ \text{shows } (\lambda x. \text{realise } (\text{Var } x)) \models \phi \]

The coherence property finishes the proof of completeness of the calculus as it gives us a model for every formula in an open and saturated branch.

\section{Soundness of the Calculus}

A tableau calculus is sound if for any closed tableau, the corresponding formula is unsatisfiable. We prove the following two properties to establish soundness.

(i) \text{ A closed branch contains an unsatisfiable formula.}

(ii) \text{ The expansion rules maintain satisfiability.}

We formalise the first property in Isabelle as follows.

\textbf{lemma} bclosed_sound:

\[ \text{assumes } \text{bclosed } b \text{ shows } \exists \phi \in \text{set } b. M \not\models \phi \]

\textbf{Proof}. Assume for contradiction that \( b \) is closed but we have a model, i.e. it holds that \( \forall \phi \in \text{set } b. M \models \phi \). We perform a case analysis on \( \text{bclosed } b \). It is clear that, for any \( s \), neither does \( M \) model \( s \in \emptyset \) nor \( s \not\in \emptyset \). Furthermore, no model can satisfy both \( \phi \) and \( \neg \phi \) at the same time. This leaves us with the case where \( b \) contains a membership cycle

\[ t_0 \in \text{elts } t_1, t_1 \in \text{elts } t_2, \ldots, t_k \in \text{elts } t_0. \]

By assumption, each atom of the membership cycle holds under the model \( M \); in other words, we have

\[ I_{\text{st}} M t_1 \in \text{elts } (I_{\text{st}} M t_1 + 1 \mod k + 1) \]

for each \( i \in \{0, \ldots, k\} \). Therefore, the transitive closure of the membership relation

\[ [(x, y). x \in \text{elts } y)]^* \]

contains a cycle. This, however, is a contradiction because the membership relation of \( V \) is well-founded (see Section 2) and well-foundedness is closed under transitive closure. \( \square \)

We are left with showing that both linear and branching expansion rules preserve satisfiability. As for the linear rules, a straightforward proof by case analysis on \( b^+ \) that \( b^+ \) suffices to obtain the lemma below.
We still need to give a termination proof to ascertain total
bound holds.

Proof. We perform a case analysis on the derivation
such that it maps

\[ x \in \text{elts } (I_{st} M s) \land y \notin \text{elts } (I_{st} M t) \lor y \in \text{elts } (I_{st} M t) \land y \notin \text{elts } (I_{st} M s). \]

We now update the assignment \( M \) such that it maps \( x \) to \( y \) to obtain the assignment \( M' \). Note that values of both \( s \) and \( t \) are unchanged under this update as \( x \notin \text{vars } s \) and \( x \in \text{vars } t \). Considering our assumption about \( y \) and the fact that \( I_{st} M s \neq I_{st} M t \) because \( M \) is a model. This inequality manifests itself through some \( y \) with

\[ \exists b' \in bs'. \forall \psi \in set b'. M' \models \psi. \]

In other words, due to the disjunction, it holds that \( M' \) is a model for one \( b' \in bs \). Thus, we are left with proving that \( M' \) is a model for \( b \). This is indeed the case since the assumption \( M \models \psi \) is equivalent to \( M' \models \psi \) for all \( \psi \in \text{set } b \) given that \( x \) is fresh with respect to \( b \).

\[ \boxdot \]

**7 Bounding the Cardinality of a Branch**

We still need to give a termination proof to ascertain total
correctness. The core of our argument is that the number of distinct formulæ that can be derived with the expansion rules is finite. In particular, we claim that the following upper bound holds.

\[
\text{card } (\text{set } b) = \text{card } \{ \psi \in \text{set } b. \neg \text{is_literal } \psi \} + \\
\text{card } \{ \psi \in \text{set } b. \text{is_literal } \psi \} \\
\leq 2 \times \text{card } (\text{subfms } (\text{last } b)) + \\
\text{card } \{ \psi \in \text{set } b. \text{is_literal } \psi \} \\
\leq 2 \times \text{card } (\text{subfms } (\text{last } b)) + \\
16 \times (\text{card } (\text{subterms } (\text{last } b)))^4
\]

We treat both inequalities separately, starting with the first one whose justification hinges on an invariant: every new composite subformula that is introduced by an expansion rule is the negation of a subformula that already occurs in last \( b \). Written as a formula, we have

\[
(\psi \in \text{set } b. \neg \text{is_literal } \psi) \\
\subseteq \text{subfms } (\text{last } b) \cup \neg \text{ subfms } (\text{last } b).
\]

Confirming that this is an invariant is an easy exercise of going through each of the propositional rules in Table 1 and the first two branching rules in Table 2. Thus, we have

\[
\text{card } \{ \psi \in \text{set } b. \neg \text{is_literal } \psi \} \\
\leq \text{card } (\text{subfms } (\text{last } b)) + \\
\text{card } (\neg \text{ subfms } (\text{last } b)) \\
\leq 2 \times \text{card } (\text{subfms } (\text{last } b))
\]

which is the first inequality we asserted.

Concerning the second inequality, first observe that each literal contains an atom, implying that the set of literals is at most twice as large as the set of atoms in a branch. In addition, an atom is formed by combining two subterms occurring in the branch with either \( e \) or \( e \); hence, we have

\[
\text{card } \{ \psi \in \text{set } b. \text{is_literal } \psi \} \\
\leq 2 \times \text{card } (\bigcup (\text{atoms } \setminus \text{set } b)) \\
\leq 2 \times 2 \times (\text{card } (\text{subterms } (\text{last } b)))^2.
\]

Since no expansion rule introduces new subterms, we know the cardinality of the subterms in a branch \( b \) is limited by the subterms of last \( b \) plus the number of witnesses added to the branch. Note that the last branching rule requires \( s \neq t \) to be in \( b \) for some \( s, t \in \text{subterms } (\text{last } b) \). The subsumption condition then prohibits that the rule can be applied again for the combination of \( s \) and \( t \), thus yielding an upper bound of \( (\text{card } (\text{subterms } (\text{last } b)))^2 \) for the witnesses. This allows us to finish the calculation that justifies the second inequality:

\[
4 \times \text{card } (\text{subterms } b)^2 \\
\leq 4 \times \text{card } (\text{subterms } (\text{last } b)) + \text{card } (\text{wits } b)^2 \\
\leq 4 \times \text{card } (\text{subterms } (\text{last } b)) + (\text{card } (\text{subterms } (\text{last } b))^2)^2 \\
\leq 4 \times 4 \times \text{card } (\text{subterms } (\text{last } b)))^4.
\]

All in all, we get the following lemma.

**Lemma card_wf_branch_ub:**

\[
\text{assumes } \text{wf_branch } b \\
\text{shows } \text{card } (\text{set } b) \\
\leq 2 \times \text{card } (\text{subfms } (\text{last } b)) + \\
16 \times (\text{card } (\text{subterms } (\text{last } b)))^4
\]
8 Total Correctness of the Decision Procedure

We first prove the termination of the procedure for well-formed branches, i.e. every well-formed branch is in the domain of mlss_proc_branch.

**Lemma** mlss_proc_branch_dom_if_wf_branch:
- **Assumes** wf_branch b
- **Shows** mlss_proc_branch_dom b

**Proof.** We proceed by strong induction on the difference between the upper bound due to card_wf_branch_ub and card b.

We only have one inductive case where the induction hypothesis holds for all well-formed branches b' where the difference between the upper bound and the cardinality is smaller than that of b. We strengthen the assumptions of the induction hypothesis to obtain

\[
\begin{align*}
\{ b' \triangleright b; \text{set } b \subset \text{set } b' \} & \implies \text{mlss_proc_branch_dom } b' \\
\end{align*}
\]

together with

\[
\text{mlss_proc_branch_dom } b
\]

for all b'. The former strengthening is admissible since, by definition, every expansion of a well-formed branch is again well-formed while the latter ensures that the cardinality of b' is larger than the cardinality of b, implying that the difference to the upper bound decreases.

Now, consider the case where b is saturated. Then, the procedure terminates without recursive calls and we immediately get mlss_proc_branch_dom b using the corresponding introduction rule for the domain predicate.

If b is not saturated, we can expand b using a linear or a branching rule, which adds new formulae to the resulting branch(es). In the first case, we can thus choose a branch b' with some that linearly expands b. Moreover, the domain predicate holds for b' @ b since the assumptions of the modified induction hypothesis are fulfilled. Ultimately, we use the corresponding introduction rule

\[
\begin{align*}
\{ \sim \text{lin_sat } b; \text{mlss_proc_branch_dom } (b' @ b) \} & \implies \text{mlss_proc_branch_dom } b \\
\end{align*}
\]

for the domain predicate to obtain mlss_proc_branch b.

The case of a branching rule is analogous. □

The above lemma allows us to use the computation induction rule of mlss_proc_branch on well-formed branches, which we do to prove soundness and completeness, starting with completeness.

**Lemma** mlss_proc_branch_complete:
- **Fixes** b :: 'a branch
- **Assumes** wf_branch b
- **Assumes** ¬ mlss_proc_branch b
- **Assumes** infinite (UNIV :: 'a set)
- **Shows** ∃M. M |= last b

**Proof.** We proceed to prove the completeness by computation induction on b.

The cases of the first and third equation, where the branch can be expanded, are immediately resolved with the induction hypothesis.

We evaluate mlss_proc_branch b to True in the case of the third equation, yielding a contradiction with the assumption ¬ mlss_proc_branch b.

This leaves us with the final case where we assume that the branch is saturated and mlss_proc_branch b evaluates to bclosed b. The branch, however, must be open due to the assumption ¬ mlss_proc_branch b. This allows us to use the Lemma coherence to obtain the model

\[
(\lambda x. \text{realise } (\text{Var } x)) \models \text{last } b.
\]

□

With completeness out of the way, only soundness remains to be proven.

**Lemma** mlss_proc_branch_sound:
- **Assumes** wf_branch b
- **Assumes** ∀ψ ∈ set b. M |= ψ
- **Shows** ¬ mlss_proc_branch b

**Proof.** We proceed by assuming mlss_proc_branch b and deriving False. Using the assumptions, we can prove that there exists b' such that

\[
b' \triangleright b \land \\
(\exists M. \forall \psi \in \text{set } b'. M |= \psi) \land \text{bclosed } b'
\]

by computation induction on b.

In the case where we choose a branching rule to expand b with the branches in bs', we use Lemma bexpands_sound to obtain b' ∈ bs' and M' with

\[
\forall \psi \in \text{set } (b' @ b). M' |= \psi.
\]

Together with the induction hypothesis, we obtain a b'' where

\[
b'' \triangleright (b' @ b) \land \\
(\exists M. \forall \psi \in \text{set } b''. M |= \psi) \land \text{bclosed } b''.
\]

Due to the transitivity of ⊳', the above also holds if we replace b' @ b with b, which is our goal.

The case of a linear expansion is dealt with in an analogous manner using Lemma lexpands_sound.

The two remaining cases are trivial as the branch is already closed, finishing the proof of the proposition.

This proposition, however, is a contradiction to the statement of Lemma bclosed_sound: no closed branch can have a model. □

To finish of the proof of total correctness, note that every singleton branch is trivially well-formed; thus, termination, completeness, and soundness easily transfer to mlss_proc.
In Automated Deduction in Classical and Non-Classical Logics, Selected Papers (Lecture Notes in Computer Science, Vol. 1761), Ricardo Caferra and Gernot Salzer (Eds.). Springer, 126–136. https://doi.org/10.1007/3-540-46508-1_8

[8] Domenico Cantone and Calogero G. Zarba. 1999. A Tableau-Based Decision Procedure for a Fragment of Set Theory Involving a Restricted Form of Quantification. In Automated Reasoning with Analytic Tableaux and Related Methods (Lecture Notes in Computer Science, Vol. 1617), Neil V. Murray (Ed.). Springer, 97–112. https://doi.org/10.1007/3-540-48754-9_12

[9] Amine Chaieb. 2006. Verifying Mixed Real-Integer Quantifier Elimination. In International Joint Conference on Automated Reasoning (Lecture Notes in Computer Science, Vol. 4130), Ulrich Furbach and Natarajan Shankar (Eds.). Springer, 528–540. https://doi.org/10.1007/11814771_43

[10] Amine Chaieb and Tobias Nipkow. 2005. Verifying and Reflecting Quantifier Elimination for Presburger Arithmetic. In Logic for Programming, Artificial Intelligence, and Reasoning (Lecture Notes in Computer Science, Vol. 3835), Geoff Sutcliffe and Andrei Voronkov (Eds.). Springer, 367–380. https://doi.org/10.1007/11591912_26

[11] Mirna Džamonja, Angeliki Koutsoukou-Argyraki, and Lawrence C. Paulson. 2022. Formalizing Ordinal Partition Relations Using Isabelle/HOL. Experimental Mathematics 31, 2 (2022), 383–400. https://doi.org/10.1080/10586458.2021.1980464

[12] Alfredo Ferro, Eugenio G. Omodeo, and Jacob T. Schwartz. 1980. Decision procedures for elementary sublanguages of set theory. I. Multi-level syllogistic and some extensions. Communications on Pure and Applied Mathematics 33, 5 (1980), 599–608. https://doi.org/10.1002/cpa.316030503

[13] Melvin Fitting. 1996. Semantic Tableaux and Resolution. Springer New York, New York, NY. https://doi.org/10.1007/978-1-4612-2360-3_3

[14] Asta Halkjær From. 2019. Formalizing a Seligman-Style Tableau System for Hybrid Logic. In Archive of Formal Proofs (December 2019). https://isa-afp.org/entries/Hybrid_Logic.html, Formal proof development.

[15] Asta Halkjær From. 2022. Formalizing a Seligman-Style Tableau System for Hybrid Logic. In Isabelle Workshop. https://files.sketis.net/Isabelle_Workshop_2022/Isabelle_2022_paper_13.pdf

[16] Asta Halkjær From, Patrick Blackburn, and Jørgen Villadsen. 2020. Formalizing a Seligman-Style Tableau System for Hybrid Logic. In Automated Reasoning, Nicolas Peltier and Viorica Sofronie-Stokkermans (Eds.). Springer International Publishing, Cham, 474–481.

[17] Melvin Gattinger. 2022. A Verified Proof of Craig Interpolation for Basic Modal Logic via Tableaux in Lean. https://malv.in/2022/AiML2022-basic-modal-interpolation-lean.pdf, Short presentation.

[18] Renato R Leme, Giorgio Venturi, and Bruno Lopes. 2022. Coq Formalization of a Tableau for the Classical-Intuitionistic Propositional Fragment of Ecumenical Logic. In Anais do III Workshop Brasileiro de Lógica. SBC, 25–32.

[19] Tobias Nipkow. 2008. Linear Quantifier Elimination. In Automated Reasoning, Alessandro Armando, Peter Baumgartner, and Gilles Dowek (Eds.). Springer, 18–33.

[20] Tobias Nipkow, Lawrence C. Paulson, and Markus Wenzel. 2002. Isabelle/HOL – A Proof Assistant for Higher-Order Logic. LNCS, Vol. 2283. Springer.

[21] Lars Noschinski. 2013. Graph Theory. Archive of Formal Proofs (April 2013). https://isa-afp.org/entries/Graph_Theory.html, Formal proof development.

[22] Lawrence C. Paulson. 2019. Zermelo Fraenkel Set Theory in Higher-Order Logic. Archive of Formal Proofs (October 2019). https://isa-afp.org/entries/ZFC_in_HOL.html, Formal proof development.
[23] Lawrence C. Paulson. 2022. Wetzel: Formalisation of an Undecidable Problem Linked to the Continuum Hypothesis. In *Intelligent Computer Mathematics*, Kevin Buzzard and Temur Kutsia (Eds.). Springer International Publishing, Cham, 92–106.

[24] Lukas Stevens and Tobias Nipkow. 2021. A Verified Decision Procedure for Orders in Isabelle/HOL. In *Automated Technology for Verification and Analysis*, Zhe Hou and Vijay Ganesh (Eds.). Springer International Publishing, 127–143.