Parallel spinors on globally hyperbolic Lorentzian four-manifolds

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Abstract
We investigate the differential geometry and topology of globally hyperbolic four-manifolds \((M, g)\) admitting a parallel real spinor \(\varepsilon\). Using the theory of parabolic pairs recently introduced in [22], we first formulate the parallellicity condition of \(\varepsilon\) on \(M\) as a system of partial differential equations, the parallel spinor flow equations, for a family of polyforms on an appropriate Cauchy surface \(\Sigma \hookrightarrow M\). The existence of a parallel spinor on \((M, g)\) induces a system of constraint partial differential equations on \(\Sigma\), which we prove to be equivalent to an exterior differential system involving a cohomological condition on the shape operator of the embedding \(\Sigma \hookrightarrow M\). Solutions of this differential system are precisely the allowed initial data for the evolution problem of a parallel spinor and define the notion of parallel Cauchy pair \((\varepsilon, \Theta)\), where \(\varepsilon\) is a coframe and \(\Theta\) is a symmetric two-tensor. We characterize all parallel Cauchy pairs on simply connected Cauchy surfaces, refining a result of Leistner and Lischewski. Furthermore, we classify all compact three-manifolds admitting parallel Cauchy pairs, proving that they are canonically equipped with a locally free action of \(\mathbb{R}^2\) and are isomorphic to certain torus bundles over \(S^1\), whose Riemannian structure we characterize in detail. Moreover, we classify all left-invariant parallel Cauchy pairs on simply connected Lie groups, specifying when they are allowed initial data for the Ricci flat equations and when the shape operator is Codazzi. Finally, we give a novel geometric interpretation of a class of parallel spinor flows and solve it in several examples, obtaining explicit families of four-dimensional Lorentzian manifolds carrying parallel spinors.

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1 Introduction

Globally hyperbolic four-dimensional Lorentzian manifolds play a fundamental role in Lorentzian geometry and mathematical physics, especially in mathematical general relativity, where they provide a natural class of four-dimensional space-times for which the initial value problem of Einstein field equations is well-posed [21,25]. A natural geometric condition to impose on a globally hyperbolic spin four-manifold \((M, g)\) is the existence of a spinor parallel with respect to the Levi-Civita associated with \(g\). Despite the fact that the local structure of Lorentzian four-manifolds admitting a parallel spinor is well-known since the early days of mathematical general relativity and supergravity [48], see also [12], the more refined global differential geometric and topological aspects of such Lorentzian manifolds have been addressed in the literature only recently [5,7,35–37], see also [4,8,14,28,45] for related global problems in Lorentzian geometry. The global differential geometric and topological study of globally hyperbolic Lorentzian manifolds of special holonomy, of which manifolds admitting a parallel irreducible spinor constitute a particular class, was indeed proposed in [35] as a long-term research program in the study of Lorentzian manifolds of special geometric type. In the spirit of such proposal, the main goal of this article is to investigate the differential geometry and topology of connected, oriented and time-oriented globally hyperbolic Lorentzian four-manifolds \((M, g)\) carrying a real parallel spinor field \(\varepsilon \in \Gamma(S_g)\), understood as a section of a bundle of irreducible real Clifford modules over the bundle of Clifford algebras of \((M, g)\). In order to do this, we will exploit the theory of parabolic pairs, recently developed in [22], which provides an equivalent description of a parallel spinor as a pair of certain one-forms satisfying a specific system of first order partial differential equations. The theory of parabolic pairs is a particular case of a general framework developed in Op. Cit. to study irreducible real spinors satisfying a generalized Killing spinor equation.

The equivalent description of a parallel spinor as a parabolic pair allows us to give a novel formulation and characterization of the evolution problem and the corresponding constraint equations on a given Cauchy surface \(\Sigma \hookrightarrow M\). In particular, we show that specifying allowed initial data on \(\Sigma\) is equivalent to specifying a parallel Cauchy pair. The latter is defined as a pair \((\varepsilon, \Theta)\) where \(\varepsilon\) is a coframe on \(\Sigma\) and \(\Theta\) is a symmetric tensor of \((2, 0)\) type satisfying a specific exterior differential system. If such pair satisfies also the constraint equations corresponding to the four-dimensional Ricci-flatness problem, we say that \((\varepsilon, \Theta)\) is **constrained Ricci-flat**. On the other hand, if \(\Theta\) is a Codazzi tensor, we say that \((\varepsilon, \Theta)\) is Codazzi. By the resolution of the initial value problem of a parallel spinor [35,37], every parallel Cauchy pair \((\varepsilon, \Theta)\) on \(\Sigma\) admits a Lorentzian development equipped with a parallel spinor, a fact that we can exploit to study the existence of parallel spinors through the study of Cauchy pairs. Our first result in this direction is Theorem 3.11 which refines, in the specific case of four Lorentzian dimensions, Theorem 4 in [35], of which we provide an alternative proof using the framework of parallel Cauchy pairs; see Remark 3.13. We recall that, on the other hand, the results of [35] hold in every dimension.

Every parallel Cauchy pair \((\varepsilon, \Theta)\) defines a canonical transversely orientable codimension one foliation \(\mathcal{F}_\varepsilon \subset \Sigma\). Our next result is Theorem 3.18, which characterizes all Cauchy pairs and associated foliations \(\mathcal{F}_\varepsilon\) on compact Cauchy surfaces \(\Sigma\). Since the constraint equations of a parallel spinor correspond to a certain type of imaginary generalized Killing spinor...
equations [7], the previous theorems can be understood as classification results about three-manifolds admitting imaginary generalized Killing spinor equations.

Parallel Cauchy pairs \((e, \Theta)\) admit a natural notion of left-invariance when defined on three-dimensional Lie groups. We exploit this fact to provide a classification of all left-invariant Cauchy pairs on connected and simply connected Lie groups, specifying when they are in addition constrained Ricci-flat or Codazzi. This classification is presented in Theorem 4.9, which shows that there exists a plethora of allowed left invariant initial data for the problem of a parallel spinor on a Lorentzian four-manifold, some of which satisfy also the constraint equations of Ricci-flatness or the Codazzi condition on \(\Theta\), although not necessarily.

The evolution problem of a parallel Cauchy pair yields a complicated system of flow equations, which we show to admit a simple geometric interpretation for comoving globally hyperbolic Lorentzian manifolds in Theorem 5.4. The latter are defined as globally hyperbolic manifolds which admit a standard presentation in which \(g(\partial_t, \partial_t) = -1\), see Definitions 5.1 and 5.2. We exploit Theorem 5.4 to construct large families of Lorentzian four-manifolds admitting parallel spinors, which are obtained as explicit solutions of the evolution equations of a Cauchy pair. We believe that the formulation of the evolution problem given in the previous theorem might be useful in order to obtain a simplified proof of the well-posedness of the initial value problem of a parallel spinor [35,37].

1.1 Outline of the paper

In Sect. 2, we introduce the theory of parabolic pairs and parallel Cauchy pairs on globally hyperbolic Lorentzian four-manifolds, considering in addition the constrained Ricci-flat and Codazzi conditions. In Sect. 3, we characterize parallel Cauchy pairs on simply connected Cauchy surfaces, and we characterize all parallel Cauchy pairs and associated foliations in the compact case. In Sect. 4, we classify all left-invariant parallel Cauchy pairs on simply connected Lie groups. Finally, in Sect. 5, we study a particular parallel spinor flow which we characterize geometrically and solve explicitly in particular cases.

1.2 Conventions

We work in mostly plus signature, that is, Lorentzian metrics are always assumed to be of signature \((- , + , + , + )\), whence time-like vector fields have negative norm. Given a Lorentzian four-manifold \((M, g)\), every fiber of its bundle of Clifford algebras \(\text{Cl}(M, g)\) is isomorphic to the standard real Clifford algebra \(\text{Cl}(3, 1)\) defined with the + convention. That is, if \((e_0, e_1, e_2, e_3)\) is the standard orthonormal basis of four-dimensional Minkowski space \(\mathbb{R}^{3,1}\), with \(e_0\) time-like, then the following relations hold in \(\text{Cl}(3, 1)\):

\[
e_0^2 = -1, \quad e_1^2 = 1, \quad e_2^2 = 1, \quad e_3^2 = 1.
\]

Note that this convention is opposite to that of [11].

2 Parallel real spinors on Lorentzian four-manifolds

In this section, we develop the theory of parallel spinors on four-dimensional Lorentzian manifolds, assuming as the starting point of our investigation one of the main results of [22], which characterizes parallel spinors in terms of a certain type of distribution satisfying a prescribed system of partial differential equations.
2.1 General theory

Let \((M, g)\) be a four-dimensional space-time, that is, a connected, oriented and time oriented Lorentzian four-manifold equipped with a Lorentzian metric \(g\). We assume that \((M, g)\) is equipped with a bundle of irreducible real spinors \(S_g\). This is by definition a bundle of irreducible real Clifford modules over the bundle of Clifford algebras of \((M, g)\). Existence of such \(S_g\) is in general obstructed. The obstruction was shown in [33,34] to be equivalent to the existence of a spin structure \(Q_g\), in which case \(S_g\) can be considered to be a vector bundle associated with \(Q_g\) through the tautological representation induced by the natural embedding \(\text{Spin}_+(3, 1) \subset \text{Cl}(3, 1)\), where \(\text{Spin}_+(3, 1)\) denotes the connected component of the identity of the spin group in signature \((3, 1) = -+++\) and \(\text{Cl}(3, 1)\) denotes the real Clifford algebra in signature \((3, 1)\).

\[\text{Remark 2.1}\] The tautological representation of \(\text{Spin}_+(3, 1) \subset \text{Cl}(3, 1)\) is the representation obtained by restriction of the unique irreducible real Clifford representation \(\gamma : \text{Cl}(3, 1) \rightarrow \text{End}(\mathbb{R}^4)\) of \(\text{Cl}(3, 1)\). This representation is real of real type (the commutant of the image of \(\gamma\) in \(\text{End}(\mathbb{R}^4)\) is trivial) and \(\gamma\) is in fact an isomorphism of unital and associative algebras. In particular \(\mathbb{R}^4\) admits a skew-symmetric non-degenerate bilinear pairing which is invariant under \(\text{Spin}_+(3, 1)\) transformations [23,24] (note that this bilinear cannot be chosen to be symmetric).

We will assume, without loss of generality, that \((M, g)\) is spin and equipped with a fixed spin structure \(Q_g\). Then, the Levi-Civita connection \(\nabla^g\) on \((M, g)\) induces canonically a connection on \(S_g\), the spinorial Levi-Civita connection, which we denote for simplicity by the same symbol.

\[\text{Definition 2.2}\] A spinor field \(\varepsilon\) on \((M, g, S_g)\) is a smooth section \(\varepsilon \in \Gamma(S_g)\) of \(S_g\). A spinor field \(\varepsilon\) is said to be parallel if \(\nabla^g \varepsilon = 0\).

For every light-like one-form \(u \in \Omega^1(M)\) we define an equivalence relation \(\sim_u\) on the vector space of one-forms as follows. Given \(l_1, l_2 \in \Omega^1(M)\) the equivalence relation \(\sim_u\) declares \(l_1 \sim_u l_2\) to be equivalent if and only if \(l_1 = l_2 + fu\) for a function \(f \in C^\infty(M)\). We denote by:

\[\Omega^1_u(M) \overset{\text{def.}}{=} \Omega^1(M) / \sim_u,\]

the \(C^\infty(M)\)-module of equivalence classes defined by \(\sim_u\).

\[\text{Definition 2.3}\] A parabolic pair \((u, [l])\) on \((M, g)\) consists of a nowhere vanishing null one-form \(u \in \Omega^1(M)\) and an equivalence class of one-forms:

\([l] \in \Omega^1_u(M),\]

such that the following equations hold:

\[g(l, u) = 0, \quad g(l, l) = 1,\]

for some, and hence for all, representatives \(l \in [l]\).

The starting point of our analysis is the following result, which follows from [22, Theorems 4.26 and 4.32] and gives the characterization of parallel spinors on \((M, g)\) that will be most convenient for our purposes.
Proposition 2.4 A space-time four-manifold \((M, g)\) admits a parallel spinor field \(\epsilon \in \Gamma(S_g)\) for some bundle of irreducible spinors \(S_g\) over \((M, g)\) if and only if there exists a parabolic pair \((u, [l])\) on \((M, g)\) satisfying:

\[
\nabla^g u = 0, \quad \nabla^g l = \kappa \otimes u,
\]

for some representative (and hence for all) \(l \in [l]\) and a one-form \(\kappa \in \Omega^1(M)\).

Remark 2.5 More precisely, Reference [22] proves that a nowhere vanishing spinor \(\epsilon \in \Gamma(S_g)\) on \((M, g)\) defines a unique distribution of co-oriented parabolic two-planes in \(M\), which in turn determines uniquely both \(u\) and the equivalence class of one-forms \([l]\). Conversely, any such distribution determines a nowhere vanishing spinor on \((M, g)\), unique up to a global sign, with respect to a spin structure on \((M, g)\). Moreover, [22, Theorem 4.26] establishes a correspondence between a certain type of first-order partial differential equations for \(\epsilon\) and their equivalent as systems of partial differential equations for \((u, [l])\), of which Eq. (2.1) constitute the simplest case. The reader is referred to [22] for further details.

Remark 2.6 Given a parabolic pair \((u, [l])\), constructing its associated spinor field \(\epsilon \in \Gamma(S_g)\) can be difficult, since it requires computing the preimage of the polyform \(u + u \wedge l\) through the square spinor map [11, §IV]. This is, however, not problematic for our purposes, since we are not interested in the parallel spinor \(\epsilon \in \Gamma(S_g)\) per se but only in the geometric and topological consequences of its existence. In this context, the main point of Eq. (2.1) and the general formalism presented in [22] is to provide a framework to study spinorial differential equations without having to consider the spinorial geometry of the underlying pseudo-Riemannian manifold \((M, g)\). This point of view is motivated by the study of supersymmetric solutions supergravity, where \(\epsilon\) corresponds to the supersymmetry parameter, an auxiliary object that a priori bears no physical meaning and is only used to define mathematically the notion of supersymmetric solution.

We will say that a parabolic pair \((u, [l])\) is parallel if it corresponds to a parallel spinor field, that is, if it satisfies Eq. (2.1) for a representative \(l \in [l]\). The dual \(u^\sharp \in \mathfrak{X}(M)\) of \(u\) is a parallel vector field on \(M\) which is usually referred to as the Dirac current of \(\epsilon\) in the literature. The fact that the Dirac current of \(\epsilon\) is always null is specific (although not exclusive) of the type of irreducible real representation \(\gamma : \text{Cl}(3, 1) \rightarrow \text{End}(\mathbb{R}^4)\) that we have used to construct the spinor bundle \(S_g\). Indeed, it can be seen (see for instance [22, Proposition 3.22]) that the pseudo-norm of the Dirac current \(u^\sharp\) is given by the pseudo-norm of \(\epsilon\) computed with respect to the admissible bilinear pairing \(\mathcal{B}\) used to construct \(u^\sharp\). Admissible bilinear pairings were classified in [23, 24], from which it follows that in our case there exist two admissible pairings, both of them skew-symmetric. Therefore, \(\mathcal{B}(\epsilon, \epsilon) = 0\) automatically and \(u^\sharp\) is always null. It should be noted that the spinorial polyforms associated with the same spinor field \(\epsilon\) through the two different admissible bilinear pairings are related by Hodge duality.

Proposition 2.4 immediately implies that four-dimensional space-times admitting a parallel spinor field whose Dirac current is complete are particular instances of Brinkmann manifolds, which are precisely defined as space-times equipped with a complete and parallel null vector field [13]. Other well-known properties of space-times admitting a parallel spinor field, such as the special form of their Ricci tensor, are also immediate consequences of Proposition 2.4, which provides an adequate global and coordinate-independent framework to study the geometry and topology of four-dimensional space-times admitting parallel spinors. In particular, such framework seems to be specially well-adapted to prove splitting theorems in the spirit of [19], where the global geometry of Brinkmann space-times was investigated.
Recall that if a pair \((u, l)\), with \(l \in [l]\), satisfies Eq. (2.1) with respect to a given \(\kappa \in \Omega^1(M)\) then any other representative \(l' = l + fu\) satisfies again equation (2.1) with respect to the same null one-form \(u\) and a possibly different one-form \(\kappa'\) given by:

\[
\kappa' = \kappa + df.
\]

Rather than investigating the global geometry and topology of general space-times admitting parallel spinors, exploiting for instance the refined screen bundle construction that can be developed in the presence of a parabolic pair, we restrict the \textit{causality} of \((M, g)\), and we assume in the following that \((M, g)\) is globally hyperbolic as proposed in [5,35].

### 2.2 Globally hyperbolic \((M, g)\)

Let \((M, g)\) be a globally hyperbolic four-dimensional space-time. A celebrated theorem of Bernal and Sánchez [9,10] states that in this case \((M, g)\) has the following isometry type:

\[
(M, g) = (\mathbb{R} \times \Sigma, -\lambda_t^2 dt \otimes dt + h_t),
\]

where \(t\) is the canonical coordinate on \(\mathbb{R}\), \(\{\lambda_t\}_{t \in \mathbb{R}}\) is a smooth family of nowhere vanishing functions on \(\Sigma\) and \(\{h_t\}_{t \in \mathbb{R}}\) is a family of complete Riemannian metrics on \(\Sigma\). From now on, we consider the identification (2.2) to be fixed. We set:

\[
\Sigma_t \overset{\text{def}}{=} \{t\} \times \Sigma \hookrightarrow M, \\
\Sigma \overset{\text{def}}{=} \{0\} \times \Sigma \hookrightarrow M,
\]

and define:

\[
t_t = \lambda_t dt,
\]

to be the outward-pointing unit time-like one-form orthogonal to \(\Sigma_t\) for every \(t \in \mathbb{R}\). We will consider \(\Sigma \hookrightarrow M\), endowed with the induced Riemannian metric:

\[
h \overset{\text{def}}{=} h_0|_{T^*\Sigma \times T^*\Sigma},
\]

to be the Cauchy hypersurface of \((M, g)\). The \textit{shape operator} or scalar second fundamental form \(\Theta_t\) of the embedded manifold \(\Sigma_t \hookrightarrow M\) is defined in the usual way as follows:

\[
\Theta_t \overset{\text{def}}{=} \nabla^{h_t} t_t|_{T^*\Sigma_t \times T^*\Sigma_t},
\]

This definition can be seen to be equivalent to:

\[
\Theta_t = -\frac{1}{2\lambda_t} \partial_t h_t \in \Gamma(T^*\Sigma_t \otimes T^*\Sigma_t).
\]

Moreover, it can be seen that:

\[
\nabla^h \alpha|_{T^*\Sigma_t \times TM} = \nabla^{h_t} \alpha + \Theta_t(\alpha) \otimes t_t, \quad \forall \alpha \in \Omega^1(\Sigma_t),
\]

where \(\nabla^{h_t}\) denotes the Levi-Civita connection on \((\Sigma_t, h_t)\) and \(\Theta_t(\alpha) := \Theta_t(\alpha^{\otimes h_t})\) is by definition the evaluation of \(\Theta_t\) on the metric dual of \(\alpha\). Given a parabolic pair \((u, [l])\), we write:

\[
u = u^0_t t_t + u^\bot_t, \quad l = l^0_t t_t + l^\bot_t \in [l],
\]

where the superscript \(\bot\) denotes orthogonal projection to \(T^*\Sigma_t\) and where we have defined:

\[
u^0_t = g(u, t_t), \quad l^0_t = g(l, t_t),
\]

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Using the previous orthogonal splitting of $u$ and $l$, we can obtain an equivalent characterization of parallel spinors on a globally hyperbolic space-time in terms of tensor flow equations on $\Sigma$.

**Lemma 2.7** [5, Lemma 3.1] Let $u \in \Omega^1(M)$ be a null one-form on the globally hyperbolic manifold (2.2). Then, $\nabla^g u = 0$ if and only if:

$$
(\nabla^g v_1)(v_2) = 0,
$$

for every $v_1 \in \mathcal{X}(M)$ and every $v_2 \in \mathcal{X}(\Sigma_t)$.

**Proof** We compute:

$$
0 = g(\nabla^g u, u) = u^0_t g(\nabla^g u, t_1) + g(\nabla^g u, u^\perp_t) = u^0_t g(\nabla^g u, t_1),
$$

where we have used that the spatial projection of $\nabla^g u$ is zero by assumption. \hfill \Box

**Lemma 2.8** A globally hyperbolic four-manifold $(M, g) = (\mathbb{R} \times \Sigma, -\lambda^2 dt \otimes dt + h_t)$ admits a parabolic pair, and hence a parallel spinor field, if and only if there exists a family of orthogonal one-forms $\{u^\perp_t, l^\perp_t\}_{t \in \mathbb{R}}$ on $\Sigma$ satisfying the following equations:

$$
\partial_t u_t^\perp + \lambda_t \Theta_t(u_t^\perp) = u^0_t d\lambda_t, \quad u^0_t \partial_t l_t^\perp + \lambda_t u^0_t \Theta_t(l^\perp_t) + d\lambda_t(l^\perp_t) u_t^\perp = 0, \quad \nabla^h_t u_t^\perp + u^0_t \Theta_t = 0, \quad u^0_t \nabla^h_t l^\perp_t = \Theta_t(l^\perp_t) \otimes u_t^\perp,
$$

as well as:

$$
(u^0_t)^2 = |u_t^\perp|_{h_t}^2, \quad |l_t^\perp|_{h_t}^2 = 1.
$$

In particular, $\partial_t u^0_t = d\lambda_t(u_t^\perp)$ and $u^0_t + \Theta(u_t^\perp) = 0$. If Eqs. (2.3) and (2.4) are satisfied, the corresponding parabolic pair $(u, [l])$ is given by:

$$
u = u^0_t t_1 + u^\perp_t, \quad [l] = [l^\perp_t],
$$

where $|u_t^\perp|_{h_t}^2 = h_t(u_t^\perp, u_t^\perp)$ and $|l_t^\perp|_{h_t}^2 = h_t(l_t^\perp, l_t^\perp)$.

**Proof** Let $(u, [l])$ be a parabolic pair satisfying Eq. (2.1). Write $u = u^0_t t_1 + u^\perp_t$. We can find a representative $l \in [l]$ such that:

$$
l = l^\perp_t \in \Omega^1(\Sigma_t), \quad t \in \mathbb{R},
$$

that is, with $l$ purely spatial. Using this representative together with Lemma 2.7, it follows that Eq. (2.1) is equivalent to:

$$
\nabla^g_{\partial_t} u |_{T \Sigma_t} = 0, \quad \nabla^g u |_{T \Sigma_t} = 0, \quad \nabla^g_{u^\perp} l^\perp_t = \kappa(\partial_t) u, \quad \nabla^g_{u^\perp} l^\perp_t = \kappa(u_t^\perp) u, \quad \forall u_t \in T \Sigma_t.
$$

Denote by $\kappa^\perp_t$ the spatial projection of $\kappa \in \Omega^1(M)$. We compute:

$$
\nabla^g_{\partial_t} u |_{T \Sigma_t} = \partial_t u^\perp_t + \lambda_t \Theta_t(u^\perp_t) - u^0_t d\lambda_t, \quad \nabla^g u |_{T \Sigma_t \times T \Sigma_t} = \nabla^h_t u^\perp_t + u^0_t \Theta_t,
$$

$$
\nabla^g_{\partial_t} l_t^\perp = \partial_t l^\perp_t - d\lambda_t(l^\perp_t) t_1 + \lambda_t \Theta_t(l^\perp_t) = \kappa(\partial_t)(u^0_t t_1 + u^\perp_t),
$$

$$
\nabla^g l_t^\perp |_{T \Sigma_t \times T M} = \nabla^h_t l^\perp_t + \Theta_t(l^\perp_t) \otimes t_1 = \kappa^\perp_t \otimes (u^0_t t_1 + u^\perp_t).
$$

Isolating $\kappa$ in the previous equations, we obtain:

$$
\kappa(\partial_t) = -\frac{1}{u^0_t} d\lambda_t(l^\perp_t), \quad \kappa^\perp_t = \frac{1}{u^0_t} \Theta_t(l^\perp_t).
$$
Plugging these equations back into the expressions for the covariant derivatives of \( l^\perp_i \), we obtain all equations in (2.3) and (2.4). The fact that these equations imply \( \partial_t u^0_i = \delta \lambda_i (u^\perp_i) \) and \( du^0_i + \Theta (u^\perp_i) = 0 \) follows now by respectively manipulating the time and exterior derivatives of \( (u^0_i)^2 = |u^\perp_i|^2_h \). The converse follows directly by construction and hence we conclude. \( \square \)

Summarizing, Eqs. (2.3), (2.4) and (2.5) contain the necessary and sufficient conditions for a four manifold \( M \) to admit a parallel spinor field with respect to a globally hyperbolic metric in the form of flow equations on \( \Sigma \), to which we will refer as the parallel spinor flow equations.

**Definition 2.9** A parallel spinor flow on a direct product manifold \( M = \mathbb{R} \times \Sigma \), where \( \Sigma \) is an oriented three-manifold, is a tuple:

\[
(\{ \lambda_i \}_{i \in \mathbb{R}}, \{ h_i \}_{i \in \mathbb{R}}, \{ u^0_i \}_{i \in \mathbb{R}}, \{ u^\perp_i \}_{i \in \mathbb{R}}, \{ l^\perp_i \}_{i \in \mathbb{R}}) \]

satisfying Eqs. (2.3), (2.4) and (2.5).

**Remark 2.10** In the previous discussion, we have used informally the notion of family of tensors parametrized by \( \mathbb{R} \). This notion can be given a rigorous meaning as follows. A family of, say, one-forms \( \{ \alpha_i \}_{i \in \mathbb{R}} \) on \( \Sigma \) is by definition a smooth section \( \alpha: \mathbb{R} \times \Sigma \rightarrow \mathbb{R}^n(T^*\Sigma) \) of the pull-back of \( T^*\Sigma \) by the canonical projection \( p: \mathbb{R} \times \Sigma \rightarrow \Sigma \). Families of other types of tensors are defined similarly.

### 2.3 The constraint equations

Using Lemma 2.8 together with the resolution of the initial value problem for a parallel null spinor presented in [5,35,37], see also [1,6], we obtain the following characterization of parallel spinors on globally hyperbolic Lorentzian four-manifolds.

**Proposition 2.11** A globally hyperbolic four-manifold \((M, g)\) with Cauchy surface \( \Sigma \leftarrow M \) and second fundamental form \( \Theta \in \Gamma(T^*\Sigma \otimes T^*\Sigma) \) admits a parallel spinor \( \varepsilon \in \Gamma(S_g) \) if and only if \( \Sigma \) admits a pair of unit length orthonormal one-forms \( (e_u, e_l) \) on \( \Sigma \) satisfying the following equations:

\[
\nabla^h e_u + \Theta(e_u) \otimes e_u, \quad \nabla^h e_l = \Theta(e_l) \otimes e_u, \quad [(\Theta(e_u)]) = 0 \in H^1(\Sigma, \mathbb{R}), \quad (2.6)
\]

where \( h \) is the complete Riemannian metric induced by \( g \) on \( \Sigma \). Furthermore, \((M, g)\) is Ricci-flat only if \( h \) satisfies in addition the following equations:

\[
R_h = |\Theta|^2_h - Tr_h(\Theta)^2, \quad dTr_h(\Theta) = div_h(\Theta), \quad (2.7)
\]

on \( \Sigma \).

**Proof** The constraint equations for the evolution problem posed by Eq. (2.1) on the globally hyperbolic four-manifold \((M, g) = (\mathbb{R} \times \Sigma, -\lambda^2_t dt \otimes dt + h_1)\) are obtained by restriction of Eq. (2.4) together with Eq. (2.5) to the Cauchy surface \( \Sigma \leftarrow M \). The restriction is given by:

\[
(u^0)^2 = |u^\perp|^2_h, \quad |l^\perp|^2_h = 1, \quad \nabla^h u^\perp + u^0 \Theta = 0, \quad u^0 \nabla^h l^\perp = \Theta(l^\perp) \otimes u^\perp, \quad (2.8)
\]

where we have set:

\[
u^0 \overset{\text{def}}{=} u^0_0, \quad u^\perp \overset{\text{def}}{=} u^\perp_0, \quad l^\perp \overset{\text{def}}{=} l^\perp_0, \quad \Theta \overset{\text{def}}{=} \Theta_0.
\]
Defining now:
\[ e_u \overset{\text{def}}{=} \frac{u}{u^0}, \quad e_l \overset{\text{def}}{=} \frac{l}{l^0}, \]
the third and fourth equations in (2.8) are equivalently to:
\[ \nabla^h e_u + d \log(u^0) \otimes e_u + \Theta = 0, \quad \nabla^h l^\perp = \Theta(l^\perp) \otimes e_u. \]
On the other hand, taking the exterior derivative of the first equation in (2.8), we obtain:
\[ d \log(u^0) + \Theta(e_u) = 0, \]
which implies the third equation in (2.6). Combining this equation with Eq. (2.3) we obtain the first two equations in (2.6). Clearly we have \( |e_u|^2_h = |e_l|^2_h = 1 \) by construction. Conversely, assume that a pair of orthonormal one-forms \((e_u, e_l)\) satisfies Eq. (2.6) for a given Riemannian metric \( h \) and tensor \( \Theta \). Write:
\[ df = -\Theta(e_u), \tag{2.9} \]
for a function \( f \in C^\infty(\Sigma) \), which exists since \([\Theta(e_u)] = 0 \in H^1(\Sigma, \mathbb{R})\). Then, the triple:
\[ u^0 = e^f, \quad u^\perp = e^f e_u, \quad l^\perp = e_l, \]
is by construction a solution Eq. (2.8). Equation (2.9) determines \( u^0 \) modulo constant rescalings, in agreement with the fact that if \((u, [l])\) is a parallel parabolic pair then so is \((cu, [l])\) for every \( c \in \mathbb{R}^* \). Conversely, since the initial value problem of a parallel null spinor is well-posed by the results of \([5,35,37]\), and a parallel spinor is equivalent to a parallel parabolic pair (see Proposition 2.4), every solution to (2.6) admits a Lorentzian development carrying a parallel spinor and containing as Cauchy surface the submanifold \( (\Sigma, h) \) with associated second fundamental form \( \Theta \). The statement regarding the Ricci-flat condition follows from the celebrated resolution of the initial value problem of a Ricci-flat Lorentzian four-manifold; see \([20,25]\). \( \square \)

**Remark 2.12** The constraint equations corresponding to a parallel spinor on a globally hyperbolic Lorentzian manifold are well known to correspond to the imaginary generalized Killing spinor equation with respect to the shape operator of the Cauchy hypersurface \([3,7,35]\). Such type of characterization also applies to our problem, however we do not need to consider it thanks to the description of parallel spinors as parabolic pairs provided in Proposition 2.4.

In Reference [7] the authors study imaginary Codazzi spinors, which correspond to the constraint equations of a parallel spinor on a globally hyperbolic Lorentzian manifold of constant curvature. More recently, Reference [35] determines the local isometry type of the Cauchy surface of any Lorentzian manifold carrying a parallel spinor, showing that, in the four-dimensional case, corresponds to a certain warped product involving a family of two-dimensional flat metrics. Therefore, the results of this article can be considered as a continuation of those in Op. Cit. in the specific case of four Lorentzian dimensions. The system of Eq. (2.7) corresponds with the celebrated constraint equations of the initial value problem for Ricci-flat globally hyperbolic Lorentzian four manifolds and consequently has been intensively and extensively studied in the literature. The first equation in (2.7) is usually called the Hamiltonian constraint whereas the second equation in (2.7) is usually called the momentum constraint.
\textbf{Remark 2.13} Equation (2.6) contain a \textit{cohomological} condition, namely \([\Theta(e_u)] = 0\) which is automatically satisfied if \(H^1(\Sigma, \mathbb{R}) = 0\). However, it may restrict the discrete quotients to which a given solution descends, since an exact one-form on \(\Sigma\) may descend to a closed non-exact one-form on certain quotients of \(\Sigma\).

\subsection*{2.4 Parallel Cauchy pairs on \(\Sigma\)}

The variables of Eq. (2.6) corresponding to the restriction of a parallel spinor field to the Cauchy surface consist of a pair of orthonormal one-forms \((e_u, e_l)\) on \((\Sigma, h)\). However, in order to study the geometry and topology of Cauchy surfaces on Lorentzian four-manifolds equipped with a parallel spinor it is convenient to consider also the Riemannian metric \(h\) and the symmetric \((2, 0)\) tensor \(\Theta\) as variables of (2.6). We will refer to a symmetric tensor \(\Theta \in \Gamma(S^2 T^* \Sigma)\) on \(\Sigma\) simply as a \textit{shape operator}. Following standard usage in the literature, if the shape operator of a given solution \((h, \Theta, e_u, e_l)\) satisfies:

\begin{equation}
\nabla^h \Theta \in \Gamma(S^3 T^* \Sigma),
\end{equation}

we will say that \(\Theta\) is a \textit{Codazzi tensor} on \(\Sigma\). More explicitly, a shape operator \(\Theta\) is a Codazzi tensor if and only if:

\begin{equation}
(\nabla^h_{v_1} \Theta)(v_2, v_3) = (\nabla^h_{v_2} \Theta)(v_1, v_3),
\end{equation}

for every \(v_1, v_2, v_3 \in X(\Sigma)\). Denote by \(F(\Sigma)\) the principal bundle of oriented coframes of \(\Sigma\). In order to proceed further, we will first rewrite Eq. (2.6) in a more transparent geometric form.

\textbf{Lemma 2.14} \textit{There is a canonical one to one correspondence between tuples \((h, \Theta, e_u, e_l)\) as described above and pairs \((\epsilon, \Theta)\), where \(\epsilon: \Sigma \rightarrow F(\Sigma)\) is a section of \(F(\Sigma)\) and \(\Theta\) is a shape operator.}

\textbf{Proof} Given \((h, \Theta, e_u, e_l)\), set:

\[ \epsilon = (e_u, e_l, e_n) \text{ def.} = \ast_h (e_u \wedge e_l), \]

which is clearly a section of \(F(\Sigma)\). Conversely, given pair \((\epsilon, \Theta)\), write:

\[ \epsilon = (e_u, e_l, e_n), \]

and map \((\epsilon, \Theta)\) to the tuple \((h, \Theta, e_u, e_l)\), where:

\[ h = e_u \otimes e_u + e_l \otimes e_l + e_n \otimes e_n. \]

Such tuple is mapped back again to \((\epsilon = (e_u, e_l, e_n), \Theta)\) by the previous correspondence, and hence we obtain the desired one to one map. \(\Box\)

Therefore, in the following we will consider coframes and shape operators on \(\Sigma\) as variables of Eq. (2.6).

\textbf{Proposition 2.15} \textit{Equation (2.6) are equivalent to the following system of first-order partial differential equations:}

\begin{equation}
de \epsilon = \Theta(\epsilon) \wedge e_u, \quad [\Theta(e_u)] = 0. \tag{2.10}
\end{equation}

\(\text{for pairs } (\epsilon = (e_u, e_l, e_n), \Theta).\)
Remark 2.16 More explicitly, equation \( d\epsilon = \Theta(\epsilon) \wedge e_u \) corresponds to the following conditions:

\[
\begin{align*}
d\epsilon &= \Theta(e_u) \wedge e_u, \\
d\ell &= \Theta(e_l) \wedge e_u, \\
d\eta &= \Theta(e_n) \wedge e_u,
\end{align*}
\]

where \( \epsilon = (e_u, e_l, e_n) \). It should be noted that generalized Killing spinors on Riemannian three-manifolds, which differ from their imaginary version which is obtained in the Lorentzian framework considered in this article, can also be studied in terms of a global coframe satisfying a given exterior differential system, see [40] for more details.

Proof Suppose that \((\epsilon, \Theta)\) is a solution of Eq. (2.6) where \( h = e_u \otimes e_u + e_l \otimes e_l + e_n \otimes e_n \) and:

\[
\epsilon = (e_u, e_l, e_n = *_h(e_u \wedge e_l)).
\]

A direct computation shows that:

\[
\nabla^h e_n = \nabla^h *_h (e_u \wedge e_l) = *_h (\nabla^h e_u \wedge e_l) + *_h (e_u \wedge \nabla^h e_u) = *_h (\nabla^g e_u \wedge e_l) = \Theta(e_n) \otimes e_u.
\]

The skew-symmetrization of the previous equation together with the skew-symmetrization of the first two equations in (2.6) yields, together with the cohomological condition, Eq. (2.10). The converse follows easily by interpreting the first equation in (2.10) as the first Cartan structure equations for the coframe \( \epsilon \), considered as orthonormal with respect to the metric \( h = e_u \otimes e_u + e_l \otimes e_l + e_n \otimes e_n \).

Remark 2.17 Leaving aside the cohomological condition, Eq. (2.6) form a set of (a priori) nine independent equations. This is exactly the same number of (a priori) independent equations occurring in (2.10), reflecting thus the equivalence between Eqs. (2.6) and (2.10).

We will refer to Eq. (2.10) as the parallel Cauchy differential system, which yields the constraint equations of a parallel spinor field on a globally hyperbolic four-dimensional space-time and will be the main object of study in this article.

Definition 2.18 A Cauchy pair \((\epsilon, \Theta)\) consists of a coframe \( \epsilon \) and a symmetric \((2, 0)\) tensor \( \Theta \). A parallel Cauchy coframe with respect to \( \Theta \) is a coframe \( \epsilon \) on \( \Sigma \) such that \((\epsilon, \Theta)\) satisfies the parallel Cauchy differential system (2.10). A parallel Cauchy pair \((\epsilon, \Theta)\) is a Cauchy pair satisfying the parallel Cauchy differential system (2.10).

The Riemannian metric associated with a parallel Cauchy coframe \( \epsilon = (e_u, e_l, e_n) \), with respect to which Eq. (2.6) are satisfied is defined as follows:

\[
h_\epsilon \overset{\text{def.}}{=} e_u \otimes e_u + e_l \otimes e_l + e_n \otimes e_n.
\]

Remark 2.19 As explained above, a Cauchy pair \((\epsilon, \Theta)\) defines a Riemannian metric \( h_\epsilon \). Therefore, it is natural to impose the constraint equations of the four-dimensional Ricci-flatness problem on a pair \((h_\epsilon, \Theta)\) associated with a parallel Cauchy pair \((\epsilon, \Theta)\). Such data \((\epsilon, \Theta, h_\epsilon)\) would satisfy both the constraint equations of the parallel spinor and Ricci-flat problems. However, and to the best knowledge of the authors, this is not enough to guarantee that \((\Sigma, h_\epsilon)\) admits a Lorentzian development which at the same time is Ricci flat and admits a parallel spinor. The reason is that the evolution of the initial data prescribed by the parallel spinor and Ricci flat problems need not be isomorphic.

We provide now an example of a parallel Cauchy pair on a non-flat Riemannian three-manifold.
Example 2.20 Take $\Sigma = \tau_{3,\mu}$ to be the simply-connected non-unimodular Lie group $\tau_{3,\mu}$ where $-1 < \mu \leq 1$, $\mu \neq 0$, is a constant, see [27, Chapter 7] for its precise definition. On $\tau_{3,\mu}$ there exists a left-invariant co-frame $(e^1, e^2, e^3)$ satisfying:

$$de^1 = 0, \quad de^2 = \mu e^2 \wedge e^1, \quad de^3 = e^3 \wedge e^1.$$

Set:

$$e = (e_u, e_l, e_n) := (e^1, e^2, e^3), \quad h_\epsilon = e_u \otimes e_u + e_l \otimes e_l + e_n \otimes e_n, \quad \Theta := h + (\mu - 1) e_l \otimes e_l.$$

A direct computation shows that $(\epsilon, \Theta)$ defines a parallel Cauchy pair on $\tau_{3,\mu}$, that is, $(\epsilon, \Theta)$ is a solution of Eq. (2.10), or, equivalently, Eq. (2.6). Note that since $de_u = 0$ and $\tau_{3,\mu}$ is simply connected, the one-form $e_u = \Theta(e_u)$ is automatically exact. In particular, we have:

$$\nabla h e_u = -\mu e_l \otimes e_l - e_n \otimes e_n, \quad \nabla h e_l = \mu e_l \otimes e_u, \quad \nabla h e_n = e_n \otimes e_u,$$

conditions which are equivalent to Eq. (2.6). More explicitly, write $e_u = df$ for a real function $f \in C^\infty(\Sigma)$. Then $(\hat{e}_l = e^\mu_l e_l, \hat{e}_n = e^\mu_n e_n)$ defines a pair of closed nowhere vanishing one-forms. In particular, $\hat{e} = (e_u, \hat{e}_l, \hat{e}_n)$ is a closed global coframe on $\Sigma$. Set:

$$h_\epsilon \overset{\text{def.}}{=} e_u \otimes e_u + \hat{e}_l \otimes \hat{e}_l + \hat{e}_n \otimes \hat{e}_n.$$

to be the Riemannian metric defined by $\hat{\epsilon} \overset{\text{def.}}{=} (e_u, \hat{e}_l, \hat{e}_n)$. Since $d\hat{\epsilon} = 0$, the metric $h_\epsilon$ is flat and therefore:

$$h_\epsilon = e_u \otimes e_u + e^{-2\mu} e_l \otimes e_l + e^{-2\mu} e_n \otimes e_n,$$

is a warped product of flat metrics. Even more, since $\hat{\epsilon} = (e_u, \hat{e}_l, \hat{e}_n)$ is a closed coframe there exist local coordinates $(z, x, y)$ (global, if $\hat{\epsilon}$ is complete) such that:

$$e_u = df = d\hat{z}, \quad \hat{e}_l = dx, \quad \hat{e}_n = dy.$$

Therefore, the metric can be written as follows:

$$h_\epsilon = d\hat{z} \otimes d\hat{z} + e^{-2\mu} dx \otimes dx + e^{-2\mu} dy \otimes dy.$$

The scalar curvature of $h_\epsilon$ can be computed to be:

$$R_h = -2(1 + \mu + \mu^2).$$

which, together with the fact that $|\Theta|^2_h = 2 + \mu^2$ and $Tr_h(\Theta)^2 = (2 + \mu)^2$ shows that the Hamiltonian constraint is satisfied if and only if $1^1$:

$$\mu = 1.$$

Since the momentum constraint is clearly satisfied if and only if $\mu = 1$, we conclude that if $\mu \neq 1$ we obtain a solution to the constraint Eqs. (2.3) and (2.4) whose Lorentzian development yields a non Ricci-flat Lorentzian four manifold. On the other hand, if $\mu = 1$, the Riemannian three-manifold $(\Sigma, h_\epsilon)$ admits Lorentzian developments (not necessarily equal) which are either Ricci flat, admit a parallel spinor or both. In all these cases (with $\mu = 1$), $(\Sigma, h_\epsilon) \hookrightarrow (M, g)$ is a totally umbilical submanifold of $(M, g)$.

---

$1^1$ Recall that $-1 < \mu \leq 1$ and $\mu \neq 0$. 

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A Cauchy pair is said to be complete if $(\Sigma, h_\epsilon)$ is a complete Riemannian three-manifold. When necessary, the dual of a Cauchy coframe $\epsilon$ will be denoted by $\epsilon^\sharp = (e^\sharp_n, e^\sharp_\ell, e^\sharp_\gamma)$. Denote by:

$$\text{Conf}(\Sigma) \overset{\text{def}}{=} \Gamma(S^2T^*\Sigma) \times \Gamma(F(\Sigma)),$$

the configuration space of the Cauchy differential system that is, its space of variables. Likewise, denote by:

$$\text{Sol}(\Sigma) \subset \text{Conf}(\Sigma),$$

the subspace of solutions of the Cauchy differential system. We have a canonical map:

$$\text{Sol}(\Sigma) \rightarrow \text{Met}_c(\Sigma), \quad (\epsilon, \Theta) \mapsto h_\epsilon,$$

where $\text{Met}_c(\Sigma)$ denotes the space of complete Riemannian metrics on $\Sigma$. The image of the previous map, which we denote by $\text{Met}_c(\Sigma)$, is by definition the space of complete Riemannian metrics on $\Sigma$ that admit a solution to the Cauchy differential system for a shape operator $\Theta \in \Gamma(S^2T^*\Sigma)$. The group of orientation preserving diffeomorphisms $\text{Diff}(\Sigma)$ has a natural left action on $\text{Conf}(\Sigma)$ given by push-forward:

$$A_u : \text{Conf}(\Sigma) \rightarrow \text{Conf}(\Sigma), \quad (\epsilon, \Theta) \mapsto (u_\ast \epsilon, u_\ast \Theta).$$

For every $u \in \text{Diff}(\Sigma)$, define:

$$A_u : \text{Conf}(\Sigma) \rightarrow \text{Conf}(\Sigma), \quad (\epsilon, \Theta) \mapsto (u_\ast \epsilon, u_\ast \Theta).$$

Lemma 2.21 Let $u \in \text{Diff}(\Sigma)$. Then, $(\epsilon, \Theta) \in \text{Sol}(\Sigma)$ if and only if $A_u(\epsilon, \Theta) \in \text{Sol}(\Sigma)$.

**Proof** We compute:

$$du = \Theta(e_u) \wedge e_u \Leftrightarrow u_\ast du = u_\ast (\Theta(e_u) \wedge e_u) \Leftrightarrow du_\ast e_u = (u_\ast \Theta)(u_\ast e_u) \wedge u_\ast e_u,$$

and similarly for the remaining equations of the Cauchy differential system (2.10). Therefore, the orientation-preserving diffeomorphism group of $\Sigma$ has a well-defined action on the space of parallel Cauchy pairs and we can consider the quotient:

$$\mathfrak{M}(\Sigma) \overset{\text{def}}{=} \text{Sol}(\Sigma) / \text{Diff}(\Sigma),$$

defined by the action $A$. We call $\mathfrak{M}(\Sigma)$ the moduli space of parallel Cauchy pairs on $\Sigma$, which we plan to investigate in a separate publication. In the following, we will consider two parallel Cauchy pairs to be isomorphic if they are related by an orientation preserving diffeomorphism of $\Sigma$ as prescribed by the action $A$.

### 2.5 The momentum and Hamiltonian constraints

In this section, we consider the interplay between the Cauchy differential system and the constraint equations on $\Sigma$ induced from imposing Ricci-flatness in four dimensions. Recall that, in contrast with the situation occurring in Riemannian geometry, not every Lorentzian space-time admitting a parallel spinor is necessarily Ricci-flat, see for instance [12] for more details and explicit examples.

**Definition 2.22** A parallel Cauchy pair $(\epsilon, \Theta)$ is constrained Ricci-flat if it satisfies Eq. (2.7), that is, if $h_\epsilon$ satisfies the constraint equations corresponding to the initial value problem posed by the Ricci-flatness condition in four dimensions.
Lemma 2.23 Let \((ε, Θ)\) be a parallel Cauchy pair and write \(ε = (e_u, e_l, e_n)\). Then:
\[
\text{div}_h(Θ) \wedge e_u = \text{dTr}(Θ) \wedge e_u.
\]

Proof The statement is equivalent to:
\[
\text{div}_{h^ε}(Θ)(e_l) = \text{dTr}_{h^ε}(Θ)(e_l), \quad \text{div}_{h^ε}(Θ)(e_n) = \text{dTr}_{h^ε}(Θ)(e_n).
\]

Note that we indistinctly denote with the same symbol one-forms and their duals by the metric wherever no possible confusion may arise. Now we write:
\[
Θ = Θ_{ab} e_a \otimes e_b, \quad Θ_{ab} \in \mathcal{C}^∞(Σ), \quad a, b = u, l, n,
\]
where \(Θ_{ab} \in \mathcal{C}^∞(Σ)\) are smooth functions. Also, recall that by the definition of parallel Cauchy coframe, \(ε = (e_u, e_l, e_n)\) satisfies:
\[
\nabla^h e_b e_a = -δ_{au} Θ(b) + Θ(a, b)e_a.
\]

Using the previous equation, we compute:
\[
\text{div}_h(Θ)(e_l) = \sum_a (\nabla^h e_a(Θ)(e_a, e_l) = \sum_a e_a(Θ_{al}) - \sum_a Θ_{al}Θ_{aa},
\]
as well as:
\[
\text{dTr}(Θ)(e_l) = \sum_a e_l(Θ_{aa}).
\]

Hence:
\[
\text{dTr}_{h^ε}(Θ)(e_l) - \text{div}_h(Θ)(e_l) = -e_u(Θ_{ul}) - e_n(Θ_{ln}) + e_l(Θ_{uu}) + e_l(Θ_{nn}) + Θ_{ul}Tr_{h^ε}(Θ).
\]

Using now that \(d^2e_n = 0\) we obtain \(e_l(Θ_{nn}) - e_n(Θ_{ln}) = Θ_{ln}Θ_{nn} - Θ_{nn}Θ_{ul}\), which in turn implies \(\text{dTr}(Θ)(e_l) = \text{div}_h(Θ)(e_l)\). Similarly \(\text{div}_h(Θ)(e_n) = \text{dTr}(Θ)(e_n)\) and hence we conclude.

For further reference, we obtain the Ricci tensor and scalar curvature of the Riemannian metric \(h^ε\) associated with a parallel Cauchy pair \((ε, Θ)\).

Proposition 2.24 Let \((ε, Θ)\) be a parallel Cauchy pair. The Ricci curvature of \(h^ε\) is given by:
\[
\text{Ric}^ε = Θ \circ Θ - \text{Tr}_ε(Θ)Θ + (\text{dTr}_ε(Θ) - \text{div}_ε(Θ)) \otimes e_u + \nabla^ε e_a(Θ) - (\nabla^ε Θ)(e_u), (2.11)
\]

whereas the scalar curvature of \(h^ε\) reads:
\[
\text{R}^ε = |Θ|^2 - \text{Tr}_ε(Θ)^2 - 2(\text{div}_ε(Θ)(e_u^ε) - \text{dTr}_ε(Θ)(e_u^ε)) (2.12)
\]
where \(d^∇_h\) denotes the exterior covariant derivative associated with \(∇^h\).

Proof The result is proven through a direct computation using the fact that for a parallel Cauchy pair \((ε, Θ)\) we have:
\[
\nabla^h e_a^ε = Θ(e_a^ε) \otimes e_a^ε - δ_{aa} Θ^ε, \quad a = u, l, n
\]
as well as:
\[

abla^h (Θ(e_a^ε)) = \sum_b (dΘ_{ab} \otimes e_b + Θ_{ab}Θ_{b} \otimes e_u) - Θ_{aa}Θ
\]
where we have written \( \Theta = \Theta_{ab} e_a \otimes e_b \), \( a, b = u, l, n \). In our conventions, the Ricci curvature reads:

\[
\text{Ric} e = \sum_{c} e_{uc} \cdot d\nabla e (e_c \otimes \Theta_c) - \sum_a e_{ua} \cdot d\nabla e (\Theta_a \otimes e_u),
\]

where \( d\nabla e \) denotes the exterior covariant derivative for one-forms taking values on one forms. Expanding the desired result for the Ricci tensor follows. Taking the trace of Eq. (2.11), we obtain:

\[
R^h = |\Theta|^2 - \text{Tr}(\Theta)^2 - \text{div}_h(\Theta)(e_u^2) + d\text{Tr}(\Theta)(e_u^2) + \text{Tr}(\nabla e \Theta - (\nabla \Theta)(e_u)).
\]

The last term can be written as follows:

\[
\sum_a ((\nabla e_a \Theta)(e_u^2, e_a^2) - (\nabla e_a \Theta)(e_u^2, e_a^2)) = d\text{Tr}(\Theta)(e_u^2) - \text{div}(\Theta)(e_u^2),
\]

whence:

\[
R^e = |\Theta|^2 - \text{Tr}(\Theta)^2 - 2(\text{div}_h(\Theta)(e_u^2) - d\text{Tr}(\Theta)(e_u^2)),
\]

and we conclude.

\[\square\]

**Remark 2.25** If \( \Theta \) is Codazzi then Equation (2.11) simplifies to:

\[
\text{Ric} e = \Theta \circ \Theta - \text{Tr}_e(\Theta) \Theta,
\]

which matches [7, Proposition 5] modulo an unimportant constant factor.

**Proposition 2.26** A Cauchy pair \((e, \Theta)\) satisfies the Hamiltonian constraint, that is, the first equation in (2.7), if and only if \((e, \Theta)\) satisfies the momentum constraint, that is, the second equation in (2.7).

**Proof** Follows from the explicit expression (2.12) for the scalar curvature of \( h_e \) upon use of Lemma 2.23.

\[\square\]

**Proposition 2.27** A pair \((e, \Theta)\) \( \in \text{Conf}(\Sigma) \) is a constrained Ricci-flat parallel Cauchy pair if and only if:

\[
de_u = \Theta(e_u) \wedge e_u, \quad de_l = \Theta(e_l) \wedge e_u, \quad de_n = \Theta(e_n) \wedge e_u, \quad [\Theta(e_u)] = 0 \in H^1(\Sigma, \mathbb{R}), \quad R^{h_e} = |\Theta|^2 - \text{Tr}(\Theta)^2.
\]

where \( h_e \) is the Riemannian metric associated with \((e, \Theta)\). In particular, every Cauchy pair \((e, \Theta)\) whose shape operator \( \Theta \) is Codazzi is constrained Ricci-flat.

**Proof** By Proposition 2.26 we only need to prove that if \((e, \Theta)\) is a parallel Cauchy pair and \( \Theta \) is a Codazzi shape operator then the momentum constraint is automatically satisfied. Fix a point \( p \in \Sigma \) and an orthonormal (with respect to \( h_e \)) frame \( \{e_a\}, a = 1, 2, 3 \), such that \( \nabla_{h_e} e_a |_p = 0 \). We compute at \( p \in \Sigma \):

\[
d\text{Tr}(\Theta) |_p = \sum_a d(\Theta(e_a), e_a) |_p = \sum_a (\nabla_{h_e} \Theta)(e_a, e_a) |_p + 2 \sum_a \Theta(\nabla_{h_e} e_a, e_a) |_p
\]

\[
= \sum_a (\nabla_{e_a} \Theta)(e_a) |_p = \text{div}_{h_e}(\Theta) |_p,
\]

and hence we conclude.

\[\square\]
Remark 2.28 We will refer to a parallel Cauchy pair \((\varepsilon, \Theta)\) whose shape operator is Codazzi as a Codazzi parallel Cauchy pair.

Proposition 2.27 summarizes necessary conditions that a pair \((\varepsilon, \Theta)\) needs to satisfy in order for the Lorentzian development of \((\Sigma, h_\varepsilon)\) to be a Ricci-flat Lorentzian four-manifold admitting a parallel spinor field. These conditions are satisfied by all examples in [7].

3 The topology and geometry of Cauchy pairs

In this section, we investigate the diffeomorphism and isometry type of oriented three-manifolds \(\Sigma\) admitting a complete Cauchy pair \((\varepsilon, \Theta) \in \text{Sol}(\Sigma)\).

3.1 General considerations

Lemma 3.1 Let \((\varepsilon, \Theta)\) be a complete Cauchy pair on \(\Sigma\). The frame \(\varepsilon^\sharp = (\varepsilon_u^\sharp, \varepsilon_l^\sharp, \varepsilon_n^\sharp)\) dual of \(\varepsilon\) is complete, that is, each of its elements is a complete vector field on \(\Sigma\).

Proof Follows from the fact that \(h_\varepsilon\) is by assumption a complete Riemannian metric on \(\Sigma\) respect to which each of the elements of \(\varepsilon\) has unit norm, see [15, Page 154, Exercise 11].

Lemma 3.2 Let \(\varepsilon = (e_u, e_l, e_n)\) be a complete Cauchy coframe. The distribution \(\ker(e_u) \subset T\Sigma\) is integrable and defines a codimension one transversely orientable foliation in \((\Sigma, h_\varepsilon)\) whose leaves are complete and flat Riemann surfaces with respect to the metric induced by \(h_\varepsilon\).

Proof The first equation in the Cauchy differential system (2.10) immediately implies:

\[ e_u \wedge de_u = 0, \]

and thus Cartan’s criterion implies in turn that \(\ker(e_u) \subset T\Sigma\) defines an integrable transversely orientable codimension one distribution, whose associated foliation we denote by \(\mathcal{F}_\varepsilon\). Let \(p \in \Sigma\) and denote by \(\mathcal{F}_\varepsilon, p \subset \Sigma\) the maximal leaf of \(\mathcal{F}_\varepsilon\) passing through \(p\). The cotangent space of \(\mathcal{F}_\varepsilon, p\) is spanned over \(C^\infty(\mathcal{F}_\varepsilon, p)\) by the restriction of \(e_l\) and \(e_n\):

\[ T^*\mathcal{F}_\varepsilon, p = \text{Span}_{C^\infty(\mathcal{F}_\varepsilon, p)}(e_l|_{\mathcal{F}_\varepsilon, p}, e_n|_{\mathcal{F}_\varepsilon, p}). \]

Furthermore:

\[ h_\varepsilon|_{\mathcal{F}_\varepsilon, p} = e_l|_{\mathcal{F}_\varepsilon, p} \otimes e_l|_{\mathcal{F}_\varepsilon, p} + e_n|_{\mathcal{F}_\varepsilon, p} \otimes e_n|_{\mathcal{F}_\varepsilon, p}. \]

A direct computation, using the fact that \(\varepsilon\) is a parallel Cauchy coframe, shows that \((e_l|_{\mathcal{F}_\varepsilon, p}, e_n|_{\mathcal{F}_\varepsilon, p})\) is a flat coframe with respect to the Levi–Civita connection of the metric induced by \(h_\varepsilon\) whence \(h_\varepsilon|_{\mathcal{F}_\varepsilon, p}\) is flat. The fact that the leaves of \(\mathcal{F}_\varepsilon\) equipped with the metric induced by \(h_\varepsilon\) are complete manifolds follows from completeness of \(h_\varepsilon\) and is proved explicitly in [45, Proposition 1.26].

Since the leaves of \(\mathcal{F}_\varepsilon\) are complete and flat they must be isometric to either the euclidean plane, the euclidean cylinder or a flat torus. As we will see momentarily, this poses strong

2 Note however that there is a typo in Exercise 11, the correct condition being, using the notation of the exercise,

\[ |X(p)| < c \] rather than \([X(p)] > c\).
constraints on the differentiable topology of $\Sigma$. Given a Cauchy pair $(\epsilon, \Theta)$, the cohomological condition occurring in the Cauchy differential system (2.10) guarantees that there exists a function $f \in C^\infty(\Sigma)$ such that:

$$\Theta(\epsilon_u) = -df.$$ 

Therefore, by the first equation in (2.10), the one-form $\hat{\epsilon}_u := e\epsilon_u$ is closed and satisfies $\text{Ker}(\hat{\epsilon}_u) = \text{Ker}(e\epsilon_u)$, implying that we can consider $\mathcal{F}_\epsilon \subset \Sigma$ as a foliation defined by the kernel of the nowhere vanishing closed one-form $\hat{\epsilon}_u$, a type of foliation that has been extensively studied in the literature, see for example [17,49]. It can be easily seen that the metric $h_\epsilon$ will not be, in general, bundle-like with respect to $\mathcal{F}_\epsilon$. On the other hand, given a Cauchy pair $(\epsilon, \Theta)$, the following modified Riemannian metric:

$$h_\epsilon := \hat{\epsilon}_u \otimes \hat{\epsilon}_u + \epsilon_l \otimes \epsilon_l + \epsilon_n \otimes \epsilon_n,$$

is indeed bundle-like, that is, it satisfies the following condition:

$$\mathcal{L}_v h_\epsilon|_{T\mathcal{F}_\epsilon^\perp} = 0, \quad \forall \ v \in \Gamma(T\mathcal{F}_\epsilon).$$

In other words, $h_\epsilon|_{T\mathcal{F}_\epsilon^\perp}$ is a holonomy invariant transversal metric.

**Remark 3.3** By Lemma 3.1, $e\epsilon_u \in \mathcal{X}(\Sigma)$ is a complete vector field on $\Sigma$. However, the same statement may not hold for $\hat{\epsilon}_u \in \mathcal{X}(\Sigma)$, the metric dual of $\hat{\epsilon}_u$ with respect to $\hat{h}_\epsilon$.

**Definition 3.4** A Cauchy pair $(\epsilon, \Theta)$ is fully complete if it is complete and in addition $\hat{\epsilon}_u \in \mathcal{X}(\Sigma)$ is complete.

The notion of fully complete Cauchy pair is convenient to obtain global results about Cauchy pairs by using completeness of $\hat{\epsilon}_u$ to identify the leaves of $\mathcal{F}_\epsilon \subset \Sigma$.

**Proposition 3.5** Let $(\epsilon, \Theta)$ be a fully complete Cauchy pair on $\Sigma$ with associated foliation $\mathcal{F}_\epsilon \subset \Sigma$. The following holds:

1. All leaves are diffeomorphic to a model leaf given by either the plane $\mathbb{R}^2$, the cylinder or the torus.
2. Either all leaves are closed or all leaves are dense in $\Sigma$.
3. The Riemannian universal cover of $(\Sigma, h_\epsilon)$ is isometric to $(\mathbb{R}^3, \bar{h}_\epsilon)$ with metric $\bar{h}_\epsilon$ given by:

$$\bar{h}_\epsilon \overset{\text{def.}}{=} e^{2u}dx \otimes dx + h_x,$$

where $x$ is the first Cartesian coordinate of $\mathbb{R}^3$, $u \in C^\infty(\mathbb{R}^3)$ is a smooth function and, for every $x \in \mathbb{R}$, $h_x$ is a flat euclidean metric on $\{x\} \times \mathbb{R}^2 \subset \mathbb{R}^3$. If $(\epsilon, \Theta)$ is not fully complete, the previous characterization is only guaranteed to hold locally.

**Remark 3.6** Item (3) in the previous proposition recovers, in the specific case of four Lorentzian dimensions, items (1) and (2) in [35, Theorem 4]. Indeed, the function $e^{2u}$ can be shown to determine the norm of the one-form $u^\perp$ occurring in the original formulation of the constraint Eq. (2.8). The norm of $u^\perp$ is denoted by $u^2$ in Op. Cit.

**Proof** Bar over a symbol will denote lift to the universal cover of $\Sigma$, denoted by $\bar{\Sigma}$. We prove the proposition point by point:
Since $\mathcal{F}_e$ is defined by a closed nowhere-vanishing one-form the fact that all its leaves must be diffeomorphic is classical, see [17,39]. To motivate it, recall that the Lie derivative of $\hat{e}_u$ along $\hat{e}_u$ is zero (note that this is not true in general for $e_u$ and $e_u^2$). Hence, the flow $(\psi_t)_{t \in \mathbb{R}}$ defined by the complete vector field $\hat{e}_u$ preserves the leaves of $\mathcal{F}_e$, that is, maps leaves to leaves diffeomorphically. Furthermore, for every $p,q \in \Sigma$, there exists a $t_0 \in \mathbb{R}$ such that:

$$\psi_{t_0}|_{\mathcal{F}_e,p} : \mathcal{F}_e,p \to \mathcal{F}_e,q.$$ 

Hence, all leaves of $\mathcal{F}_e$ are diffeomorphic and by Lemma 3.2 they must be all diffeomorphic to either the plane, the cylinder or the torus.

(2) Follows from [18, Proposition 5.1].

(3) The fact that the universal cover $\tilde{\Sigma}$ is diffeomorphic to $\mathbb{R} \times \tilde{\mathcal{F}}_e$, where $\tilde{\mathcal{F}}_e$ denotes the universal cover of the typical leaf of $\mathcal{F}_e$ is proven in detail in [36, Proposition 8]. Furthermore, the foliation $\mathcal{F}_e \subset \Sigma$ lifts to the foliation whose leaves are given by $\{x\} \times \tilde{\mathcal{F}}_e \subset \mathbb{R} \times \tilde{\mathcal{F}}_e$ for $x \in \mathbb{R}$. Since the typical leaf of $\mathcal{F}_e$ is either the plane, the cylinder or the torus, then $\tilde{\mathcal{F}}_e = \mathbb{R}^2$ and therefore $\tilde{\Sigma} = \mathbb{R}^3$. The lift $\tilde{e}_u$ of $e_u$ to $\tilde{\Sigma}$ is orthogonal to $T^*\tilde{\mathcal{F}}_e \subset T^*\tilde{\Sigma}$, whence:

$$\tilde{e}_u = e^u dx, \quad u \in C^\infty(\mathbb{R}^3),$$

where $u$ is a function on $\mathbb{R}^3$ satisfying $\tilde{\Theta}(\tilde{e}_u) = -du$. Since the distribution $T\tilde{\mathcal{F}}_e \subset T\tilde{\Sigma}$ is defined by the kernel of $\tilde{e}_u$ we conclude that the lift of $h_e$ to $\tilde{\Sigma}$ can be written as follows:

$$\tilde{h}_e \overset{\text{def}}{=} e^{2u} dx \otimes dx + h_x,$$

for a family $\{h_x\}_{x \in \mathbb{R}}$ of two-dimensional metrics on $\mathbb{R}^2$, which must be flat by Lemma 3.2.

The leaves of the foliation $\mathcal{F}_e$ are all mutually diffeomorphic but a priori may not be mutually isometric since (the dual of) $\hat{e}_u$ which generates the flow that allows to identify different leaves of $\mathcal{F}_e$ may not be an isometry of $h_e$. We will refer to the type of any leaf of $\mathcal{F}_e$ as the typical leaf of $\mathcal{F}_e$, considered as a Riemann surface with the induced orientation. If the typical leaf of $\mathcal{F}_e$ is compact, we obtain the following result.

**Proposition 3.7** Let $(e, \Theta)$ be a fully complete Cauchy pair on $\Sigma$ with associated foliation $\mathcal{F}_e \subset \Sigma$. If the typical leaf of $\mathcal{F}_e$ is a flat torus, then either $\Sigma = \mathbb{R} \times T^2$ or $\Sigma$ admits the structure of a fiber bundle $\pi_e : \Sigma \to S^1$ inducing $\mathcal{F}_e$.

**Proof** Follows directly from [43, Corollary 8.6] by using the fact that every locally trivial fibration over $\mathbb{R}$ is trivial as well as the fact that if the leaves of $\mathcal{F}_e$ are compact then they must be diffeomorphic to the torus.

**Lemma 3.8** Let $(e, \Theta)$ be a complete Cauchy pair on $\Sigma$ with associated foliation $\mathcal{F}_e \subset \Sigma$. Then, $\Sigma$ admits a canonical locally free action of $\mathbb{R}^2$ whose orbits are the leaves of $\mathcal{F}_e$.

**Proof** Consider a Cauchy pair $(e, \Theta)$ and define the map:

$$\Psi : \mathbb{R}^2 \times \Sigma \to \Sigma, \quad (t_1, t_2, p) \mapsto \Phi_{e_{t_1}}^{t_2} \circ \Phi_{e_1}^{t_2} (p),$$
where $\Phi_{t_1}^t$ (respectively $\Phi_{e_{n_0}}^{t_2}$) denotes the flow generated by $e_i^\varphi$ (respectively $e_n^\varphi$) at the time $t_1$ (respectively $t_2$). Using that $e$ is a solution of the Cauchy differential system, we obtain:

$$[e_i^\varphi, e_n^\varphi] = \nabla^{e_i^\varphi} e_n^\varphi - \nabla^{e_n^\varphi} e_i^\varphi = 0,$$

hence $\Psi$ defines a smooth action of $\mathbb{R}^2$ on $\Sigma$, which, since both $e_l$ and $e_n$ are nowhere vanishing, is locally free. Furthermore, the fact that $e_i^\varphi$ and $e_l^\varphi$ are complete and span $T\mathcal{F}_e \subset T\Sigma$ implies that the orbits of $\Phi$ correspond to the leaves of $\mathcal{F}_e$. 

Locally free actions of the group $\mathbb{R}^2$ on three-manifolds have been extensively studied extensively in the literature, see [2,16,41,42] and references therein, especially in relation with the problem of finding the number of nowhere vanishing and everywhere linearly independent commuting vector fields on a compact three-manifold.

**Proposition 3.9** Let $(e, \Theta)$ be a fully complete Cauchy pair on $\Sigma$ such that the restriction of $\Theta$ to $T\mathcal{F}_e \subset T\Sigma$ vanishes, that is, $\Theta|_{T\mathcal{F}_e \times T\mathcal{F}_e} = 0$. Then, $\Sigma$ is diffeomorphic to $T^k \times \mathbb{R}^{3-k}$ for some integer $k \in \{0, 1, 2, 3\}$.

**Proof** Let $(e, \Theta)$ be a Cauchy pair such that $\Theta|_{T\mathcal{F}_e \times T\mathcal{F}_e} = 0$. Then, $\breve{e}^\varphi$ is a global frame of commuting vector fields, which can be used to define a smooth action of $\mathbb{R}^3$ on $\Sigma$ exactly as it occurred in the proof of Lemma 3.8 to define an action of $\mathbb{R}^2$. Since $\breve{e}$ is assumed to be complete, this action is transitive. The final step of the proof consist in showing that the stabilizer of the action is of the form $\mathbb{Z}^k \times \{0\} \subset \mathbb{R}^k \times \mathbb{R}^{3-k}$ acting naturally on $\mathbb{R}^3$. This is explicitly proven in [17, Chapter 4].

### 3.2 Complete Cauchy pairs on the universal Riemannian cover

Let $(e, \Theta)$ be a fully complete Cauchy pair on $\Sigma$. Proposition 3.5 states that universal Riemannian cover of $(\Sigma, h_e)$ is isometric to $\mathbb{R}^3$ when the latter is equipped with the metric:

$$\tilde{h}_e \overset{\text{def}}{=} e^{2u} dx \otimes dx + h_x, \tag{3.1}$$

where $h_x$ is a flat metric on $\{x\} \times \mathbb{R}^2 \subset \mathbb{R}^3$ for every $x \in \mathbb{R}$. The corresponding Cauchy coframe reads:

$$e_u = e^u dx, \quad e_l = e_l(x), \quad e_n = e_n(x), \tag{3.2}$$

where $e_l$ and $e_n$ depend only on the coordinate $x$. A quick computation shows that the exterior derivative of this frame is given by:

$$de_u = du \wedge e_u, \quad de_l = e^{-u} e_u \wedge \mathcal{L}_x e_l, \quad de_n = e^{-u} e_u \wedge \mathcal{L}_x e_n,$$

where the symbol $\mathcal{L}_x$ denotes Lie derivative with respect to $\partial_x$. Plugging the previous equations into the Cauchy differential system (2.10), we obtain the following lemma.

**Lemma 3.10** A pair $(e, \Theta) \in \text{Conf}(\mathbb{R}^3)$, where $e$ is given by the coframe (3.2), is a Cauchy pair if and only if the following equations are satisfied:

$$(du - \Theta(e_u)) \wedge e_u = 0, \quad (\Theta(e_l) + e^{-u} \mathcal{L}_x e_l) \wedge e_u = 0, \quad (\Theta(e_n) + e^{-u} \mathcal{L}_x e_n) \wedge e_u = 0.$$

The previous lemma is used in the following theorem to solve the shape operator of a parallel Cauchy pair $(e, \Theta)$ defined on a connected and simply connected three-manifold $\Sigma$ in terms of the Cauchy coframe $e$. 

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Theorem 3.11 A pair \((e, \Theta) \in \text{Conf}(\Sigma)\) is a parallel and fully complete Cauchy pair on a connected and simply connected three-manifold \(\Sigma\) if and only if there exist global coordinates \((x, y, z)\) identifying \(\Sigma = \mathbb{R}^3\) such that \(e\) satisfies:
\[
e = (e^u dx, e_l(x), e_n(x)), \quad (\mathcal{L}_x e_l)(e^u_n) = (\mathcal{L}_x e_n)(e^u_l),
\]
and in addition:
\[
\Theta = (\mathcal{F}(x) e^{-u} + \partial_x e^{-u}) e_u \otimes e_u + e_u \otimes du + du \otimes e_u - \frac{1}{2} e^{-u} \mathcal{L}_x h_x,
\]
where \(\mathcal{F} \in C^\infty(\mathbb{R})\) is a function of \(x\).

Remark 3.12 The second equation in (3.3) is non-trivial in general and hence restricts the type of coframes that can occur as part of a parallel Cauchy pair.

Proof Let \((e, \Theta)\) be a Cauchy pair on a connected and simply connected three-manifold \(\Sigma\). The fact that there exist global coordinates \((x, y, z)\) identifying \(\Sigma\) with \(\mathbb{R}^3\) respect to which \(e\) is given by:
\[
e = (e^u dx, e_l(x), e_n(x)),
\]
follows directly from Proposition 3.5. On the other hand, Lemma 3.10 implies:
\[
\Theta(e_u) = du + f_u e_u, \quad \Theta(e_l) = f_l e_u - e^{-u} \mathcal{L}_x e_l, \quad \Theta(e_n) = f_n e_u - e^{-u} \mathcal{L}_x e_n,
\]
for functions \(f_u, f_l, f_n \in C^\infty(\mathbb{R}^3)\). Symmetry of \(\Theta\) is equivalent to the following equations:
\[
f_l = du(e^u_l), \quad f_n = du(e^u_n), \quad (\mathcal{L}_x e_l)(e^u_n) = (\mathcal{L}_x e_n)(e^u_l).
\]
These conditions imply that \(\Theta\) must be of the form:
\[
\Theta = (f_u - du(e^u_u)) e_u \otimes e_u + e_u \otimes du + du \otimes e_u - \frac{1}{2} e^{-u} \mathcal{L}_x h_x,
\]
Furthermore, the fact that \(\Theta(e_u)\) must be closed, whence exact, is equivalent to:
\[
d(f_u e_u) = d(f_u e^u) \wedge dx = 0.
\]
Therefore, \(f_u e^u = \mathcal{F}(x)\) for a smooth function \(\mathcal{F}\) depending exclusively on the coordinate \(x\). Plugging this expression back in (3.5) we obtain (3.4). The converse follows by construction and can be verified explicitly by inserting (3.4) in the parallel Cauchy differential system (2.10).

Remark 3.13 Theorem 3.11 recovers [35, Theorem 4] in the language of parallel Cauchy pairs and in the specific case of four Lorentzian dimensions, refining it and providing an alternative proof of the result. The refinement is contained in the extra information provided by the Cauchy coframe \(e\), which needs to satisfy Eq. (3.3). On the other hand, equation (3.4) does not specify uniquely \(\Theta\) but allows the freedom of choosing the arbitrary function \(\mathcal{F}(x)\). This arbitrary function seems to be absent in [35, Theorem 4].

Example 3.14 Using the notation and framework established by Theorem 3.11, assume that:
\[
h_x = e^{2\varphi(x)}(dy \otimes dy + dz \otimes dz),
\]
where \((x, y, z)\) are the Cartesian coordinates of \(\mathbb{R}^3\) and \(\varpi(x)\) is a function on \(\mathbb{R}^3\) depending only on the coordinate \(x\). As defined above, \(h_x\) is clearly a family of flat metrics on \(\mathbb{R}^3\) parametrized by \(x \in \mathbb{R}\). The corresponding parallel Cauchy coframe reads:

\[
\epsilon = (e^u dx, e^\varpi(x) dy, e^\varpi(x) dz),
\]

One easily checks that the second equation in (3.3) is automatically satisfied. On the other hand, the corresponding parallel shape operator is given by:

\[
\Theta = (\tilde{\Theta} e^{-u} + \partial_x e^{-u}) e_u \otimes e_u + e_u \otimes du + du \otimes e_u - \partial_x \varpi(x) e^{-u} h_x.
\]

Using the previous expression, we compute:

\[
\text{Tr}_\epsilon(\Theta) = e^{-u}(\tilde{\Theta} + \partial_x u - 2\partial_x \varpi),
\]

\[
|\Theta|^2 = e^{-2u}((\tilde{\Theta} + \partial_x u)^2 + 2(\partial_x \varpi)^2) + 2e^{-2w}((\partial_y u)^2 + (\partial_z u)^2).
\]

In particular:

\[
|\Theta|^2 - \text{Tr}_\epsilon(\Theta)^2 = 2e^{-2u} \partial_x \varpi(x)(2(\tilde{\Theta}(x) + \partial_x u) - \partial_x \varpi(x)) + 2e^{-2w}((\partial_y u)^2 + (\partial_z u)^2),
\]

and since the scalar curvature of \(h_x\) is given by:

\[
R^x = e^{-2u}(4\partial_x \varpi \partial_x u - 4\partial_x^2 \varpi - 6(\partial_x \varpi)^2) - 2e^{-2w}((\partial_y u)^2 + (\partial_z u)^2 + \partial_y^2 u + \partial_z^2 u),
\]

we conclude that such parallel Cauchy pair \((\epsilon, \Theta)\) is constrained Ricci-flat if and only if:

\[
2e^{2w}(\tilde{\Theta} \partial_x \varpi + \partial_x^2 \varpi + (\partial_x \varpi)^2) + e^{2u}(2(\partial_y u)^2 + 2(\partial_z u)^2 + \partial_y^2 u + \partial_z^2 u) = 0.
\]

If the second term in the previous equation only depends on \(x\) and \(\partial_x \varpi \neq 0\) everywhere, then we can always solve it by choosing \(\tilde{\Theta}\) as follows:

\[
\tilde{\Theta} = -\frac{1}{\partial_x \varpi} \left(\partial_x^2 \varpi + (\partial_x \varpi)^2\right) - \frac{e^{2(u - w)}}{2\partial_x \varpi} \left(2(\partial_y u)^2 + 2(\partial_z u)^2 + \partial_y^2 u + \partial_z^2 u\right).
\]

### 3.3 Parallel Cauchy pairs on compact three-manifolds

In this section, we consider the isometry type of Cauchy pairs on closed three-manifolds, commenting briefly on the compact case with boundary.

**Proposition 3.15** Let \(\Sigma\) be an oriented closed three-manifold admitting a Cauchy pair \((\epsilon, \Theta)\). Then \(\Sigma\) is diffeomorphic to a torus bundle over \(S^1\), that is, it is diffeomorphic to the suspension \(X_k\) of \(T^2\) by an element \(k \in \text{SL}(2, \mathbb{Z})\).

**Proof** Let \((\epsilon, \Theta)\) be a Cauchy pair on \(\Sigma\). By Lemma 3.8 \(\Sigma\) admits locally free action of \(\mathbb{R}^2\). Reference [42] proves that \(\Sigma\) admits such an action if and only if \(\Sigma\) is diffeomorphic to a locally trivial torus bundle over \(S^1\), which can always be constructed as a suspension of \(T^2\) by an element \(k \in \text{SL}(2, \mathbb{Z})\) acting linearly on \(T^2\).

Since it will be of importance in the following, we briefly recall the suspension construction of a torus bundle over \(S^1\), which depends on a choice of orientation preserving diffeomorphism of \(T^2\) modulo homotopy equivalence. Since Diff \((T^2)\) is homotopy equivalent to \(\text{SL}(2, \mathbb{Z})\) acting linearly on \(T^2\), it is enough to consider elements in \(\text{SL}(2, \mathbb{Z})\). Let \(k \in \text{SL}(2, \mathbb{Z})\) and denote by \((k) \subset \text{SL}(2, \mathbb{Z})\) the cyclic group generated by the element \(k\).
There exists a natural properly discontinuous fixed point free action of \( \langle t \rangle \) on \( \mathbb{R} \times T^2 \) given by:

\[
\mathfrak{t} \cdot (z, \nu) = (z + 1, \mathfrak{t}(\nu)), \quad (z, \nu) \in \mathbb{R} \times T^2,
\]

where \( \mathfrak{t} \) acts linearly on \( \mathbb{R}^2 / \mathbb{Z}^2 \). The suspension of \( \mathbb{R} \times T^2 \) by \( \mathfrak{t} \in \text{SL}(2, \mathbb{Z}) \) is by definition the quotient:

\[
X_{\mathfrak{t}} = \frac{\mathbb{R} \times T^2}{\langle \mathfrak{t} \rangle},
\]

equipped with the projection:

\[
\pi : X_{\mathfrak{t}} \to S^1 = \mathbb{R} / \mathbb{Z}, \quad [z, \nu] \mapsto [z].
\]

Equivalently, \( X_{\mathfrak{t}} \) can be constructed by gluing \( \{0\} \times T^2 \) and \( \{1\} \times T^2 \) in \( [0, 1] \times T^2 \) through the diffeomorphism \( \mathfrak{t} : T^2 \to T^2 \). The element \( \mathfrak{t} \in \text{SL}(2, \mathbb{R}) \) determines completely the topology of \( X_{\mathfrak{t}} \) and in particular determines if a given foliation of \( X_{\mathfrak{t}} \) admits a bundle-like metric. Note that, given a Cauchy pair \((\epsilon, \Theta)\) on \( \Sigma = X_{\mathfrak{t}} \), the leaves of the foliation \( \mathcal{F}_\epsilon \subset X_{\mathfrak{t}} \) will not coincide in general with the fibers of \( X_{\mathfrak{t}} \). We summarize now two important methods for constructing foliations in \( X_{\mathfrak{t}} \).

- **Linear plane foliations on \( T^3 \).** Denote by \( \text{Diff}(S^1) \) the group of orientation preserving diffeomorphisms of \( S^1 \) and consider the three-manifold \( \mathbb{R}^2 \times S^1 \). Fix a representation:

\[
\rho = (\rho_a, \rho_b) : \pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z} \to \text{Diff}(S^1),
\]

such that the rotational numbers \( r_a \in S^1 \) and \( r_b \in S^1 \) of \( \rho_a(1) \) and \( \rho_b(1) \) are both irrational and rationally independent. Then, \( \rho_a(1) \) and \( \rho_b(1) \) generate a subgroup of the orientation preserving diffeomorphism group \( \text{Diff}(S^1) \), which we denote by:

\[
(\rho_a(1), \rho_b(1)) \subset \text{Diff}(S^1).
\]

There is a canonical fixed point free action of \( (\rho_a(1), \rho_b(1)) \) on \( \mathbb{R}^2 \times S^1 \) given by:

\[
\rho_a(1) \cdot (x_1, x_2, \theta) = (x_1 + 1, x_2, \rho_a(1)(\theta)) , \quad \rho_b(1) \cdot (x_1, x_2, \theta) = (x_1, x_2 + 1, \rho_b(1)(\theta)),
\]

on the generators \( \rho_a(1) \) and \( \rho_b(1) \). The quotient:

\[
X_\rho := \mathbb{R}^2 \times S^1 / (\rho_a(1), \rho_b(1)),
\]

of \( \mathbb{R}^2 \times S^1 \) by the previous action is diffeomorphic to \( T^3 \) and the plane foliation of \( \mathbb{R}^2 \times S^1 \) whose leaves are embedded planes \( \mathbb{R}^2 \times \{\theta\} \subset \mathbb{R}^2 \times S^1, \theta \in S^1 \), descends to a foliation by planes of \( \mathbb{R}^2 \times S^1 / (\rho_a(1), \rho_b(1)) \), which is called the *suspension foliation* defined by \( \rho \) and it is denoted by:

\[
\mathcal{F}_\rho \subset X_\rho = \mathbb{R}^2 \times S^1 / (\rho_a(1), \rho_b(1)).
\]

In particular, \( X_\rho \) admits the structure of a \( S^1 \) bundle over \( T^2 \) transverse to \( \mathcal{F}_\rho \), which is obtained by the standard associated bundle construction. Note that \( \rho_a(1) \) and \( \rho_b(1) \) may not be rotations of \( S^1 \) by a constant angle. In general, the foliation \( \mathcal{F}_\rho \) is only \( C^0 \) isomorphic to a foliation for which \( \rho_a(1) \) and \( \rho_b(1) \) are rotations, see [30] for an explicit counterexample. However, if \( \mathcal{F}_\rho \) is defined by a non-singular closed one-form then \( \mathcal{F}_\rho \) is at least \( C^1 \) isomorphic to a foliation for which \( \rho_a(1) \) and \( \rho_b(1) \) are rotations [30].
• **Cylinder foliations of circle bundles.** Consider the foliation $\mathcal{F}_0 \subset T^2 \times \mathbb{R}$ whose leaves are defined to be the embedded submanifolds $\{\theta_1\} \times S^1 \times \mathbb{R} \subset T^2 \times \mathbb{R} = S^1 \times S^1 \times \mathbb{R}$ for $\theta_1 \in S^1$. For every diffeomorphism $f: T^2 \to T^2$ preserving the foliation by standard circles $\{\theta_1\} \times S^1 \subset T^2 = S^1 \times S^1$ and such that its restriction to the first circle factor $f|_{S^1 \times \{\theta_2\}}: S^1 \to S^1$ has an irrational rotation number, we define a diffeomorphism of $T^2 \times \mathbb{R}$ as follows:

$$T^2 \times \mathbb{R} \to T^2 \times \mathbb{R}, \quad (\theta_1, \theta_2, x) \mapsto (f(\theta_1, \theta_2), x + 1). \quad (3.8)$$

By [16, Theorem 2] and [29, Page 254 Théorème 1] $f \in \text{SL}(2, \mathbb{Z})$ is conjugate to an element of the form:

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \in \text{SL}(2, \mathbb{Z}), \quad (3.9)$$

where $n \in \mathbb{Z}$ is an integer. Denote by $\langle f \rangle \subset \text{Diff}(T^2 \times \mathbb{R})$ the cyclic subgroup of $\text{Diff}(T^2 \times \mathbb{R})$ generated by the previous action, and define:

$$\mathcal{X}_f := \frac{\mathbb{R} \times T^2}{\langle f \rangle},$$

to be the quotient of $\mathbb{R} \times T^2$ by $\langle f \rangle$, which defines a fiber bundle $\pi_f: \mathcal{X}_f \to S^1$ with projection:

$$\pi_f([\theta_1, \theta_2, x]) = [x] \in S^1.$$

We see that the action of $\langle f \rangle$ preserves by construction $\mathcal{F}_0$, whence $\mathcal{F}_0$ descends to a foliation $\mathcal{F}_f \subset \mathcal{X}_f$ whose fibers are all diffeomorphic to the cylinder. More explicitly, the leaves of the foliation are given by:

$$p_f(\{\theta\} \times S^1 \times \mathbb{R}) \subset \mathcal{X}_f, \quad \theta \in S^1,$$

where $p_f: T^2 \times \mathbb{R} \to \mathcal{X}_f$ denotes the canonical projection.

**Proposition 3.16** Every codimension-one foliation of $\mathcal{X}_f$ defined by the kernel of a nowhere vanishing closed one-form whose leaves are all diffeomorphic to either the plane $\mathbb{R}^2$ or the cylinder $\mathbb{R}^2 \setminus \{0\}$ is isomorphic to one of the foliations defined above.

**Remark 3.17** By isomorphic foliations, we mean foliations for which there exists a $C^1$ diffeomorphism between their total spaces of the foliations mapping leaves to leaves diffeomorphically.

**Proof** The result is proven in [29] for the case of cylinder leaves and in [30] for the case of plane leaves.

**Theorem 3.18** Let $(\epsilon, \Theta)$ be a Cauchy pair on an oriented closed three-manifold $\Sigma$ with associated foliation $\mathcal{F}_{\epsilon} \subset \Sigma$ and Riemannian metric $h_{\epsilon}$. Then, one and only one of the following cases occur:

1. $\mathcal{F}_{\epsilon} \subset \Sigma$ is a foliation by plane leaves and there exists an isometry:

$$(\Sigma, h_{\epsilon}) = (\mathbb{R}^2 \times S^1, dx_1 \otimes dx_1 + dx_2 \otimes dx_2 + \epsilon^{2u}d\theta \otimes d\theta)/\langle \rho_a(1), \rho_b(1) \rangle,$$

where $\rho_a(1), \rho_b(1) \in \text{Diff}(S^1)$ are rotations of rationally independent constant irrational angle, respectively, and $u \in C^\infty(\mathbb{R}^2)$ is a function depending only on $x_1$ and $x_2$. In particular, $\Sigma$ is diffeomorphic to $T^3$ and $\mathcal{F}_{\epsilon}$ is isomorphic to the foliation $\mathcal{F}_\rho$ described above.
(2) \( \mathcal{F}_e \subset \Sigma \) is a foliation by cylinder leaves and there exists an isometry:
\[
(\Sigma, h_\epsilon) = (\mathbb{R}^2 \times S^1, dx_1 \otimes dx_1 + dx_2 \otimes dx_2 + e^{2u}d\theta \otimes d\theta)/\langle f \rangle ,
\]
where \( f \in \text{Diff}(T^2 \times \mathbb{R}) \) is as prescribed in (3.8) and (3.9) and \( u \in C^\infty(\mathbb{R}^2) \) is a function depending only on \( x_1 \) and \( x_2 \). In particular, \( \mathcal{F}_e \) is isomorphic to the foliation \( \mathcal{F}_1 \) described above.

(3) \( \mathcal{F}_e \subset \Sigma \) is a foliation by torus leaves and \( (\Sigma, h_\epsilon) \) is a conformal Riemannian submersion over \( S^1 \) with flat fibers and whose conformal factor is determined, modulo constant multiplicative factors, by:
\[
\Theta(e_\theta) = -df.
\]
In particular, \( \Sigma \) is diffeomorphic to a torus suspension by an element \( t \in \text{SL}(2, \mathbb{Z}) \).

**Proof** We prove the statement point by point.

(1) Let \( (\epsilon, \Theta) \) be a parallel Cauchy pair with associated foliation \( \mathcal{F}_e \subset \Sigma \) by planes. Then, and as explained above, \( \Sigma \) is diffeomorphic to \( T^3 \) (any compact connected 3-manifold with a foliation by planes is diffeomorphic to \( T^3 \)), \( (\Sigma, h_\epsilon) \) is covered by \( S^1 \times \mathbb{R}^2 \) and \( \mathcal{F}_e \) lifts to the plane foliation of \( S^1 \times \mathbb{R}^2 \) whose leaves are embedded planes \( \{\theta\} \times \mathbb{R}^2 \subset S^1 \times \mathbb{R}^2 \), \( \theta \in S^1 \). Hence, the lift of \( h_\epsilon \) to \( S^1 \times \mathbb{R}^2 \) reads:
\[
(S^1 \times \mathbb{R}^2, \tilde{h}_\epsilon = e^{2u}d\theta \otimes d\theta + h_\theta),
\]
where \( u \) is a function on \( S^1 \times \mathbb{R}^2 \), \( \theta \) is an angular coordinate on \( S^1 \) and \( h_\theta \) is a family of flat metrics on \( \mathbb{R}^2 \) parametrized by \( \theta \in S^1 \). Consequently, \( (\Sigma, h_\epsilon) \) has the following isometry type:
\[
(\Sigma, h_\epsilon) = (S^1 \times \mathbb{R}^2, e^{2u}d\theta \otimes d\theta + h_\theta)/\langle \rho_\alpha(1), \rho_\beta(1) \rangle,
\]
For the metric \( e^{2u}d\theta \otimes d\theta + h_\theta \) to descend to \( \Sigma \) through the previous quotient we must have:
\[
\rho_\alpha(1)^*(e^{2u}d\theta \otimes d\theta + h_\theta) = e^{2u}d\theta \otimes d\theta + h_\theta, \quad \rho_\beta(1)^*(e^{2u}d\theta \otimes d\theta + h_\theta) = e^{2u}d\theta \otimes d\theta + h_\theta,
\]
which immediately implies:
\[
u \circ \rho_\beta(1) = u, \quad h_\theta \circ \rho_\alpha(1) = h_\theta,
\]
for \( o = a, b \). Since \( \langle \rho_\alpha(1), \rho_\beta(1) \rangle \) generates a dense subgroup (recall that the action of any diffeomorphism \( \chi : S^1 \to S^1 \) with constant irrational rotation number has dense orbits) of \( S^1 \) this implies in turn that \( h_\theta \) and \( u \) are constant along \( S^1 \).

(2) Let \( (\epsilon, \Theta) \) be a parallel Cauchy pair with associated foliation \( \mathcal{F}_e \subset \Sigma \) by cylinder leaves. Then, and as explained above, \( (\Sigma, h_\epsilon) \) is covered by \( T^2 \times \mathbb{R} \) and \( \mathcal{F}_e \) lifts to the cylinder foliation of \( T^2 \times \mathbb{R} \) whose leaves are the embedded cylinders \( \mathbb{R} \times S^1 \times \{\theta\} \subset T^2 \times \mathbb{R} \), \( \theta \in S^1 \). Hence, the lift of \( h_\epsilon \) to \( S^1 \times \mathbb{R}^2 \) is given by:
\[
(S^1 \times \mathbb{R}^2, \tilde{h}_\epsilon = e^{2u}d\theta_1 \otimes d\theta_1 + h_{\theta_1}),
\]
where \( u \) is a function on \( S^1 \times S^1 \times \mathbb{R} \), \( (\theta_1, \theta_2) \) are angular coordinates on \( S^1 \times S^1 \) and \( h_{\theta_1} \) is a family of flat metrics on \( S^1 \times \mathbb{R} \) parametrized by \( \theta_1 \in S^1 \). Then:
\[
(\Sigma, h_\epsilon) = (S^1 \times S^1 \times \mathbb{R}, e^{2u}d\theta_1 \otimes d\theta_1 + h_{\theta_1})/\langle f \rangle.
\]
For the metric $e^{2u}d\theta \otimes d\theta + h_{\theta_1}$ to descend to $\Sigma$ the group we must have:

$$f^*(e^{2u}d\theta_1 \otimes d\theta_1 + h_{\theta_1}) = e^{2u}d\theta_1 \otimes d\theta_1 + h_{\theta_1},$$

which, since the rotation number of $f$ is irrational, immediately implies, as in the previous case, that neither $h_{\theta_1}$ nor $u$ depend on $\theta_1$.

(3) Let $(\epsilon, \Theta)$ be a parallel Cauchy pair with associated foliation $\mathcal{F}_\epsilon \subset \Sigma$ by torus leaves. Since $\mathcal{F}_\epsilon$ has trivial holonomy and $\Sigma$ is connected and compact, [43, Corollary 8.6] implies that $\mathcal{F}_\epsilon$ arises as the fibers of a fibration $\pi: \Sigma \rightarrow S^1$ and:

$$T \Sigma = H \oplus V,$$

where $V := \ker(d\pi)$ and $H$ is spanned by $e_i^\theta$, In particular, the vertical bundle $V$ is spanned by $e_i^\theta$ and $e_n^\theta$, so the fibers of $\pi$ are flat and we obtain a conformal submersion over $S^1$.

The fact that the conformal factor $e^f$ is as described in the statement follows from the first equation of the parallel Cauchy differential system, namely:

$$de_u = d f \wedge e_u,$$

which implies $d(e^f e_u) = 0$. Hence $e^f e_u$ is locally the exterior derivative of a coordinate $\hat{x}$ and the horizontal metric is locally $d\hat{x} \otimes d\hat{x}$.

\[\square\]

### 4 Left-invariant parallel Cauchy pairs on Lie groups

In this section we investigate left-invariant parallel Cauchy pairs on connected and simply connected three-dimensional Lie groups. In order to do this, we will exploit the classification of connected and simply connected three-dimensional Riemannian Lie groups developed in [38], together with the fact that every left-invariant Cauchy pair $(\epsilon, \Theta)$ defines a left-invariant metric $h_\epsilon$.

Let $(\epsilon, \Theta) \in \text{Conf}(\Sigma)$ be a left-invariant Cauchy pair on a three-dimensional connected and simply connected Lie group $\Sigma = G$, that is, $\epsilon$ is a left-invariant coframe and $\Theta$ is a left-invariant shape operator on $G$. Write:

$$\Theta = \sum_{a,b} \Theta_{ab} e_a \otimes e_b, \quad \Theta_{ab} \in \mathbb{R}, \quad a, b = u, l, n.$$ 

in terms of the left-invariant Cauchy coframe $\epsilon = (e_u, e_l, e_n)$. Using the previous expression for $\Theta$, the Cauchy differential system (2.10) evaluated on $(\epsilon, \Theta)$ is equivalent to:

$$de_u = (\Theta_{ul} e_l + \Theta_{un} e_n) \wedge e_u, \quad de_l = (\Theta_{ll} e_l + \Theta_{ln} e_n) \wedge e_u,$$

$$de_n = (\Theta_{nl} e_l + \Theta_{nn} e_n) \wedge e_u. \quad (4.1)$$

Taking the exterior derivative of the previous equations, we obtain the corresponding integrability conditions:

$$\Theta_{ll} \Theta_{un} - \Theta_{ln} \Theta_{ul} = 0, \quad \Theta_{ln} \Theta_{un} - \Theta_{nn} \Theta_{ul} = 0. \quad (4.2)$$

For further reference, we define the following quantities:

$$T := \Theta_{ll} + \Theta_{nn}, \quad \Delta := \Theta_{ll} \Theta_{nn} - \Theta_{ln}^2,$$

which respectively correspond to the trace and determinant of $\Theta$ restricted to the distribution defined by the kernel of $e_u$. 

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Proposition 4.1 A left invariant Cauchy pair \((e, \Theta)\) satisfies the cohomological condition \([\Theta(e_u)] = 0\) if and only if:
\[
(\Theta^2_{ul} + \Theta^2_{un})\text{Tr}_\epsilon(\Theta) = 0.
\]

Proof Since \(\Sigma\) is by assumption simply-connected we have \(H^1(\Sigma) = 0\) and it suffices to prove that \(\Theta(e_u)\) is closed. We impose:
\[
d\Theta(e_u) = \Theta_{uu}de_u + \Theta_{ul}de_l + \Theta_{un}de_n = 0.
\]

Using the parallel Cauchy differential system 4.1, the previous condition is equivalent to the following equations:
\[
\Theta_{uu}\Theta_{ul} + \Theta_{ul}\Theta_{ll} + \Theta_{un}\Theta_{ln} = 0, \quad \Theta_{uu}\Theta_{un} + \Theta_{ul}\Theta_{ln} + \Theta_{un}\Theta_{nn} = 0,
\]
which, upon the use of the integrability condition (4.2) of \((e, \Theta)\), are in turn equivalent to:
\[
\Theta_{ul}\text{Tr}_\epsilon(\Theta) = 0, \quad \Theta_{un}\text{Tr}_\epsilon(\Theta) = 0.
\]

These equations are satisfied if and only if \(\Theta_{ul} = \Theta_{un} = 0\) or \(\text{Tr}_\epsilon(\Theta) = 0\) (or both) hold. \(\Box\)

We consider now the case in which \(G\) is unimodular.

Lemma 4.2 Let \((e, \Theta) \in \text{Sol}(G)\) be a parallel Cauchy pair. Then, the simply connected three-dimensional group \(G\) is unimodular if and only if:
\[
T = \Theta_{ll} + \Theta_{nn} = 0, \quad \Theta_{un} = \Theta_{ul} = 0.
\] (4.3)

Proof A Lie group \(G\) is unimodular if and only if the adjoint map of the associated Lie algebra has vanishing trace. Since the parallel Cauchy coframe \(e = (e_u, e_l, e_n)\) is left-invariant, unimodularity of \(G\) is equivalent to:
\[
de_l(e_u, e_l) + de_n(e_u, e_n) = 0, \quad de_u(e_l, e_u) + de_n(e_l, e_n) = 0,
\]
\[
de_u(e_n, e_l) + de_l(e_n, e_l) = 0,
\]
which in turn is equivalent to:
\[
\Theta_{ll} + \Theta_{nn} = 0, \quad \Theta_{ul} = 0, \quad \Theta_{un} = 0,
\]
on the use of the parallel Cauchy differential system (4.1). \(\Box\)

Proposition 4.3 Let \((e, \Theta)\) be a left invariant Cauchy pair on an unimodular Lie group \(G\). Then, one and only one of the following holds:

- \(\Delta = 0\) and \((G, h_\epsilon)\) is isometric to the additive abelian Lie group \(\mathbb{R}^3\) equipped with its standard invariant flat Riemannian metric.
- \(\Delta \neq 0\) and \(\Sigma\) is isometric to the group \(E(1, 1)\) of rigid motions of two-dimensional Minkowski space equipped with a left-invariant Riemannian metric.

Proof We distinguish between the cases \(\Delta = 0\) and \(\Delta \neq 0\).

- \(\Delta = 0\). Since \(\Theta_{ll}\Theta_{nn} = \Theta_{ln}^2\) and we have \(\Theta_{ll} + \Theta_{nn} = 0\) by unimodularity, we obtain that \(\Theta_{ll} = \Theta_{nn} = \Theta_{ln} = 0\). Also, again by unimodularity, \(\Theta_{ul} = \Theta_{un} = 0\), so we conclude that \(de = 0\) and \(\Sigma\) is isomorphic to the abelian Lie group \(\mathbb{R}^3\).
• $\Delta \neq 0$. By unimodularity, see equation (4.3), we have $\Theta_{ll} = -\Theta_{nn}$ and hence $\Delta < 0$. The exterior derivative of the Cauchy coframe $\epsilon$ can be then written as follows:

$$de_u = 0, \quad dl = (\Theta_{ll} e_l + \Theta_{ln} e_n) \wedge e_u, \quad de_n = (\Theta_{ln} e_l - \Theta_{ll} e_n) \wedge e_u.$$  

If $\Theta_{ll} = 0$ the previous equations reduce to:

$$de_u = 0, \quad dl = \Theta_{ln} e_n \wedge e_u, \quad de_n = \Theta_{ln} e_l \wedge e_u.$$  

Since $\Delta < 0$, we have $\Theta_{ll} \neq 0$ and after rescaling $e_u$ by $\Theta_{ln}$ we obtain:

$$de'_u = 0, \quad dl = e_n \wedge e'_u, \quad de_n = e_l \wedge e'_u.$$  

Comparing with the classification of unimodular Riemannian Lie groups [38], see also Appendix A of [26] for a concise summary, existence of such left-invariant coframe implies that $G$ is isomorphic to the Lie group $E(1, 1)$. If $\Theta_{ll} \neq 0$ we consider following change of coframes:

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} \cos \beta & -\sin \beta & 0 \\ \sin \beta & \cos \beta & 0 \\ 0 & 0 & \sqrt{|\Delta|} \end{pmatrix} \begin{pmatrix} e_l \\ e_n \\ e_u \end{pmatrix}.$$  

where:

$$\sin \beta = \frac{\sqrt{2}}{2} \sqrt{1 - \frac{\Theta_{ln}}{\sqrt{|\Delta|}}}, \quad \cos \beta = \frac{\sqrt{2}}{2} \frac{\Theta_{ll}}{\sqrt{|\Delta|}} \sqrt{1 - \frac{\Theta_{ln}}{\sqrt{|\Delta|}}}.$$  

The exterior derivative of the transformed coframe $(e_1, e_2, e_3)$ reads:

$$de_1 = e_2 \wedge e_3, \quad de_2 = e_1 \wedge e_3, \quad de_3 = 0.$$  

By the classification of unimodular Riemannian Lie groups [38], existence of such left-invariant coframe implies that $G$ is again isomorphic to the Lie group $E(1, 1)$, and hence we conclude.

\[\Box\]

We consider now the case in which $G$ is non-unimodular.

**Proposition 4.4** Let $(\epsilon, \Theta)$ be a left invariant Cauchy pair on a non-unimodular Lie group $G$. Then, one and only one of the following holds:

• $\Delta = 0$ and $(G, h_\epsilon)$ is isometric to the Lie group $\tau_2 \oplus \mathbb{R}$ equipped with a left-invariant Riemannian metric.

• $\Delta \neq 0$ and $(G, h_\epsilon)$ is isometric to $\tau_3,\mu$ equipped with a left-invariant Riemannian metric, where $\mu$ is given by one of the following possibilities:

1. If $\Theta(e_l, e_n) \neq 0$, by:

$$\mu = \frac{T - \text{sign}(T)\sqrt{T^2 - 4\Delta}}{T + \text{sign}(T)\sqrt{T^2 - 4\Delta}}.$$  

2. If $\Theta(e_l, e_n) = 0$ and $|\Theta(e_l, e_l)| \geq |\Theta(e_n, e_n)|$, by:

$$\mu = \frac{\Theta(e_n, e_n)}{\Theta(e_l, e_l)}.$$
(3) If $\Theta(e_1, e_n) = 0$ and $|\Theta(e_n, e_n)| \geq |\Theta(e_1, e_1)|$, by:

$$\mu = \frac{\Theta(e_1, e_1)}{\Theta(e_n, e_n)}.$$ 

Recall that the possible values of $\mu$ satisfy $-1 < \mu \leq 1$, $\mu \neq 0$.

**Proof** We distinguish between the cases $\Delta = 0$ and $\Delta \neq 0$.

- $\Delta = 0$. Assume first that $T = \Theta_{ll} + \Theta_{nn} = 0$. Conditions $T = 0$ and $\Delta = 0$ can hold simultaneously if and only if $\Theta_{ll} = \Theta_{nn} = \Theta_{ln} = 0$. Hence:

$$de_u = \Theta_{ul} e_l \wedge e_u + \Theta_{un} e_n \wedge e_u, \quad dl = 0, \quad dn = 0, \quad du = 0,$$

where the last equation is equivalent to the one-form $\Theta(e_u)$ being exact. Since the coefficients $\Theta_{ul}$ and $\Theta_{un}$ cannot simultaneously vanish (otherwise $G$ would be unimodular) defining $e_1 = e_u, e_2 = \Theta_{ul} e_l - \Theta_{ul} e_n, e_3 = \Theta_{ul} e_l + \Theta_{un} e_n$ we conclude that $G$ is isomorphic to $\tau_2 \oplus \mathbb{R}$. If $T \neq 0$, then either $\Theta_{ll} \neq 0$ or $\Theta_{nn} \neq 0$ or both are non-vanishing. Assume $\Theta_{ll} \neq 0$ (completely analogue results hold if we consider $\Theta_{nn} \neq 0$). In this case, the integrability conditions (4.2) imply:

$$\Theta_{un} = \frac{\Theta_{ln}}{\Theta_{ll}} \Theta_{ul}.$$ 

This equation, together with condition $\Delta = 0$, implies:

$$de_u = \Theta_{ul} \left( e_l + \frac{\Theta_{ln}}{\Theta_{ll}} e_n \right) \wedge e_u, \quad dl = \Theta_{ll} \left( e_l + \frac{\Theta_{ln}}{\Theta_{ll}} e_n \right) \wedge e_u,$$

$$dn = \Theta_{ll} \left( e_l + \frac{\Theta_{ln}}{\Theta_{ll}} e_n \right) \wedge e_u,$$

which must be considered together with equation $(\Theta_{ll}^2 + \Theta_{un}^2)T e_u (\Theta) = 0$ to guarantee that $\Theta(e_u)$ is closed. We distinguish the following possibilities:

1. $\Theta_{ul} = \Theta_{ln} = 0$. In this case, it can be easily seen that $G$ is isomorphic to $\tau_2 \oplus \mathbb{R}$.
2. $\Theta_{ul} = 0$ and $\Theta_{ln} \neq 0$. In this case, we obtain:

$$de_u = 0, \quad dl = \Theta_{ll} \left( e_l + \frac{\Theta_{ln}}{\Theta_{ll}} e_n \right) \wedge e_u, \quad dn = \Theta_{ll} \left( e_l + \frac{\Theta_{ln}}{\Theta_{ll}} e_n \right) \wedge e_u.$$

Defining $e_1 := e_l + \frac{\Theta_{ln}}{\Theta_{ll}} e_n, e_2 := e_l - \frac{\Theta_{ln}}{\Theta_{ll}} e_n$ and $e_3 := T e_u$, we obtain:

$$de_1 = e_1 \wedge e_3, \quad de_2 = de_3 = 0,$$

Hence $G$ is isomorphic to $\tau_2 \oplus \mathbb{R}$.

3. $\Theta_{ln} = 0$, but $\Theta_{ul} \neq 0$. In this case, we obtain:

$$de_u = \Theta_{ul} e_l \wedge e_u, \quad dl = \Theta_{ll} e_l \wedge e_u, \quad dn = 0.$$

Defining $e_1 := e_l + \frac{\Theta_{ln}}{\Theta_{ul}} e_u, e_2 := e_l - \frac{\Theta_{ln}}{\Theta_{ul}} e_u$ and $e_3 := e_n$, we conclude that $G$ is isomorphic to $\tau_2 \oplus \mathbb{R}$ once we impose $\Theta_{ul} = -T$ in order to satisfy $|\Theta(e_u)| = 0$.

4. $\Theta_{ln} \neq 0$ and $\Theta_{ul} \neq 0$. Define $e_2 := e_l + \frac{\Theta_{ln}}{\Theta_{ul}} e_n$ and $e_3 := e_l - \frac{\Theta_{ln}}{\Theta_{ul}} e_n$. We obtain:

$$de_u = \Theta_{ul} e_2 \wedge e_u, \quad de_2 = T e_2 \wedge e_u, \quad de_3 = 0.$$

We redefine $\tilde{e}_2 = e_2 - \frac{T}{\Theta_{ul}} e_u$ and $e_1 = e_u$, we finally obtain:

$$de_1 = -\Theta_{ul} e_1 \wedge \tilde{e}_2, \quad \tilde{e}_2 = 0, \quad de_3 = 0,$$
implying that $G$ is isomorphic to $\tau_2 \oplus \mathbb{R}$ after imposing $\Theta_{uu} = -T$ in order to guarantee $\Theta(e_u)$ to be closed.

- $\Delta \neq 0$. Since $\Delta \neq 0$, the only possible solution to the integrability conditions (4.2) is $\Theta_{ul} = \Theta_{un} = 0$. Hence, non-unimodularity necessarily requires that $T = \Theta_{ll} + \Theta_{nn} \neq 0$ and the parallel Cauchy differential system reduces to:

$$\mathrm{de}_u = 0, \quad \mathrm{de}_l = (\Theta_{ll} e_l + \Theta_{ln} e_n) \wedge e_u, \quad \mathrm{de}_n = (\Theta_{ln} e_l + \Theta_{nn} e_n) \wedge e_u.$$

Assume $\Theta_{ln} \neq 0$ and define a global coframe $(e_1, e_2, e_3)$ in terms of the parallel Cauchy coframe $e$ as follows:

$$
\begin{pmatrix}
  e_1 \\
  e_2 \\
  e_3
\end{pmatrix} =
\begin{pmatrix}
  1 & \frac{\lambda - \Theta_{ll}}{\Theta_{ln}} & 0 \\
  0 & \frac{\mu - \Theta_{ll}}{\Theta_{ln}} & 0 \\
  0 & 0 & \lambda
\end{pmatrix}
\begin{pmatrix}
  e_l \\
  e_n \\
  e_u
\end{pmatrix},
$$

where:

$$\lambda = \frac{1}{2} (T + \text{sign}(T)\sqrt{T^2 - 4\Delta}), \quad \mu = \frac{1}{2} (T - \text{sign}(T)\sqrt{T^2 - 4\Delta}).$$

Note that $\lambda = \mu$ if and only if $\Theta_{ll} = \Theta_{nn}$ and $\Theta_{ln} = 0$, which is not possible since we are assuming $\Theta_{ln} \neq 0$. The exterior derivative of $(e_1, e_2, e_3)$ can be shown to be given by:

$$\mathrm{de}_1 = e_1 \wedge e_3, \quad \mathrm{de}_2 = \bar{\mu} e_2 \wedge e_3, \quad \mathrm{de}_3 = 0,$$

where we defined $\bar{\mu} = \frac{\mu}{\lambda}$. Note that $1 > |\bar{\mu}| > 0$, since $\Theta_{ln} \neq 0$ and $\Delta \neq 0$. Hence, $G$ is isomorphic to $\tau_{3,\bar{\mu}}$.

If $\Theta_{ln} = 0$, the exterior derivative of the Cauchy coframe $e$ reads:

$$\mathrm{de}_u = 0, \quad \mathrm{de}_l = \Theta_{ll} e_l \wedge e_u, \quad \mathrm{de}_n = \Theta_{nn} e_n \wedge e_u.$$

Assume first that $|\Theta_{ll}| \geq |\Theta_{nn}|$. Note that $\Theta_{ll} \neq -\Theta_{nn}$ by non-unimodularity. By rescaling $e_u$, we obtain:

$$\mathrm{de}_u = 0, \quad \mathrm{de}_l = e_l \wedge e_u, \quad \mathrm{de}_n = \Theta_{nn} e_n \wedge e_u.$$

Since $1 \geq \frac{\Theta_{nn}}{\Theta_{ll}} > -1$ and $\Theta_{nn} \neq 0$ (otherwise $\Delta = 0$), we conclude $\Sigma$ is isomorphic to $\tau_{3,\frac{\Theta_{nn}}{\Theta_{ll}}}$. An analogous conclusion holds if $|\Theta_{nn}| \geq |\Theta_{ll}|$.

$\square$

**Proposition 4.5** The shape operator $\Theta$ of a parallel Cauchy pair $(\epsilon, \Theta)$ on $G$ is Codazzi if and only if:

$$C_a \overset{\text{def.}}{=} e_u \otimes \Theta \circ \Theta(e_u) - \Theta(e_u) \otimes \Theta(e_u) - \delta_{ua} \Theta \circ \Theta + \Theta_{ua} \Theta = 0$$

for every $a = u, l, n$. 

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Proof We compute:
\[
\nabla^\epsilon_e \Theta = -\Theta(e_u) \otimes \Theta(e_a) - \Theta(e_a) \otimes \Theta(e_u) + \Theta \circ \Theta(e_a) \otimes e_u + e_u \otimes \Theta \circ \Theta(e_a). \tag{4.4}
\]

Similarly:
\[
(\nabla^\epsilon e \Theta)(e_a) = -\Theta(e_u, e_a) \Theta + \delta_{ua} \Theta \circ \Theta + \Theta \circ \Theta(e_a) \otimes e_u - \Theta(e_a) \otimes \Theta(e_u).
\]

Since \( \Theta \) is Codazzi if and only if \( \nabla^\epsilon_e \Theta = (\nabla^\epsilon e \Theta)(e_a) \) for all \( a = u, l, n \), matching the previous pair of equations we obtain:
\[
e_u \otimes \Theta \circ \Theta(e_a) - \Theta(e_u) \otimes \Theta(e_a) - \delta_{ua} \Theta \circ \Theta + \Theta_{ua} \Theta = 0.
\]

\( \qed \)

Remark 4.6 It is not hard to see that:
\[
C_a(e_b, e_d) = -C_b(e_a, e_d),
\]
for every \( a, b, c = u, l, n \). We will be use this identity momentarily.

Proposition 4.7 A parallel Cauchy pair \((\epsilon, \Theta)\) on \( G \) is constrained Ricci-flat if and only if:
\[
\Theta(e_u, e_u) \mathrm{Tr}_\epsilon \Theta = |\Theta|_\epsilon^2.
\]

Proof By Proposition 2.26, the Hamiltonian and momentum constraints for a Cauchy pair are equivalent. We consider the momentum constraint. We have \( \mathrm{dTr}_h(\Theta) = 0 \). Hence, by Lemma 2.23 the constraint Ricci-flatness condition for \((\Theta, \epsilon)\) is equivalent to:
\[
\mathrm{div}_\epsilon(\Theta)(e_u) = 0.
\]

Using Eq. (4.4) we compute:
\[
\mathrm{div}_\epsilon(\Theta)(e_u) = -\mathrm{Tr}_\epsilon(\Theta)(e_u) + |\Theta|_\epsilon^2 e_u,
\]
and therefore we conclude. \( \qed \)

Lemma 4.8 The shape operator \( \Theta \) of a parallel Cauchy pair \((\epsilon, \Theta)\) is Codazzi if and only if it satisfies one of the following conditions:
\[
\bullet \, \Theta_{ul} = \Theta_{un} = \Theta_{ln} = 0, \Theta_{ll} = \Theta_{lu} \Theta_{uu}, \Theta_{nn} = \Theta_{nn} \Theta_{uu}.
\]
\[
\bullet \, \Theta(e_u) = Te_u, \Delta = 0.
\]

Proof Let \( C_a \in \Gamma(T^*G \otimes T^*G) \) denote the tensor defined in Proposition 4.5. Remark 4.6 states that the only non-trivial and independent components are those corresponding to \( C_a(e_u, e_l), C_a(e_u, e_n) \) and \( C(e_l, e_n) \). Imposing these components to vanish we obtain:
\[
2\Theta^2_{ul} + \Theta^2_{ll} - \Theta_{uu} \Theta_{ll} = 0, \quad 2\Theta^2_{un} + \Theta^2_{nn} + \Theta^2_{ln} - \Theta_{uu} \Theta_{nn} = 0, \quad 2\Theta_{ul} \Theta_{un} + \Theta_{nn} \Theta_{ll} + \Theta_{lu} \Theta_{ll} - \Theta_{uu} \Theta_{ln} = 0, \tag{4.5}
\]

In order to solve them we impose the cohomological condition as stated in Proposition 4.1. Since the cohomological condition is satisfied if either \( \Theta_{ul} = \Theta_{un} = 0 \) or \( \Theta_{uu} = -T \), we distinguish between these two cases:
\[
\bullet \, \Theta_{ul} = \Theta_{un} = 0. \text{ Let us split this case into two subcategories:}
\]
\[- \Theta_{ln} = 0. \text{ One notices that the equations reduce directly to } \Theta_{uu} \Theta_{nn} = \Theta^2_{nn} \text{ and } \Theta_{uu} \Theta_{ll} = \Theta^2_{ll}.
\]
\(- \Theta_{in} \neq 0\). In such a case, from the last equation of \((4.5)\) one finds \(\Theta_{uu} = T\) and, upon substitution in the remaining equations they become linearly dependent and equivalent to the condition \(\Delta = 0\).

- \(\Theta_{uu} = -T\). In such a case, by summing the first and the second equations of \((4.5)\) and performing explicitly the substitution \(\Theta_{uu} = -T\), we find

\[
2 \Theta_{al}^2 + 2 \Theta_{an}^2 + (\Theta_{il} + \Theta_{nn})^2 + 2 \Theta_{in}^2 + \Theta_{il}^2 + \Theta_{nn}^2 = 0. \tag{4.6}
\]

This implies \(\Theta_{al} = \Theta_{an} = \Theta_{il} = \Theta_{nn} = \Theta_{in} = \Theta_{uu} = 0\), which brings us to the previous bullet-point.

\[\square\]

We elaborate now on the results of the previous discussion in order to obtain a full classification result about left-invariant parallel Cauchy pairs \((\epsilon, \Theta)\) on connected and simply connected three-dimensional Lie groups, characterizing those which are in addition Codazzi or constrained Ricci-flat. Collecting all results from Propositions 4.3 and 4.4 and bearing in mind Proposition 4.7 and Lemma 4.8, we obtain the following result.

**Theorem 4.9** A connected and simply-connected Lie group \(G\) admits left-invariant parallel Cauchy pairs (respectively constrained Ricci-flat parallel Cauchy pairs or a Codazzi parallel Cauchy pairs) if and only if \(G\) is isomorphic to one of the Lie groups listed in the Table below. If that is the case, a left-invariant shape operator \(\Theta\) belongs to a Cauchy pair \((\epsilon, \Theta)\) for certain left-invariant coframe \(\epsilon\) if and only if \(\Theta\) is of the form listed below when written in terms of \(\epsilon = (e_u, e_i, e_n)\): Regarding the case in which \(G \simeq \mathfrak{t}_3, \mu\):

| \(G\) | Cauchy parallel pair | Constrained Ricci-flat | Codazzi |
|------|---------------------|----------------------|---------|
| \(\mathbb{R}^3\) | \(\Theta = \Theta_{uu} e_u \otimes e_u\) | \(\Theta = \Theta_{uu} e_u \otimes e_u\) | \(\Theta = \Theta_{uu} e_u \otimes e_u\) |
| \(E(1,1)\) | \(\Theta = \Theta_{uu} e_u \otimes e_u + \Theta_{ij} e_i \otimes e_j\) | Not allowed | Not allowed |
| \(i, j = l, n, \quad \Theta_{il} = -\Theta_{nn}\) | \(\Theta = \Theta_{uu} e_u \otimes e_u + \Theta_{ij} e_i \otimes e_j\) | Not allowed | Not allowed |
| \(R \otimes \mathbb{R}\) | \(\Theta = \Theta_{uu} e_u \otimes e_u + \Theta_{ij} e_i \otimes e_j\) | \(\Theta = T e_u \otimes e_u + \Theta_{ij} e_i \otimes e_j\) | \(\Theta = T e_u \otimes e_u + \Theta_{ij} e_i \otimes e_j\) |
| \(i, j = l, n, \quad T \neq 0, \Delta = 0\) | \(\Theta = T e_u \otimes e_u + \Theta_{ij} e_i \otimes e_j\) | Not allowed | Not allowed |
| \(\mathfrak{t}_3, \mu\) | \(\Theta = \Theta_{uu} e_u \otimes e_u + \Theta_{ij} e_i \otimes e_j\) | \(\Theta = \Theta_{uu} e_u \otimes e_u + \Theta_{ij} e_i \otimes e_j\) | \(\Theta = \Theta_{uu} e_u \otimes e_u + \Theta_{ij} e_i \otimes e_j\) |
| \(i, j = l, n, \quad T \neq 0, \Delta = 0\) | \(\Theta = \Theta_{uu} e_u \otimes e_u + \Theta_{ij} e_i \otimes e_j\) | Not allowed | Not allowed |
| \(\mathfrak{t}_3, \mu\) | \(\Theta = \Theta_{uu} e_u \otimes e_u + \Theta_{ij} e_i \otimes e_j\) | \(\Theta = \Theta_{uu} e_u \otimes e_u + \Theta_{ij} e_i \otimes e_j\) | \(\Theta = \Theta_{uu} e_u \otimes e_u + \Theta_{ij} e_i \otimes e_j\) |
• If $\Theta_{ln} \neq 0$, then
\[
\mu = \frac{T - \text{sign}(T)\sqrt{T^2 - 4\Delta}}{T + \text{sign}(T)\sqrt{T^2 - 4\Delta}}.
\]
• If $\Theta_{ln} = 0$ and $|\Theta_{ll}| \geq |\Theta_{nn}|$, then
\[
\mu = \frac{\Theta_{nn}}{\Theta_{ll}}.
\]
• If $\Theta_{ln} = 0$ and $|\Theta_{nn}| \geq |\Theta_{ll}|$, then
\[
\mu = \frac{\Theta_{ll}}{\Theta_{nn}}.
\]

We hope that the previous theorem, together with Lemma 2.8, can serve as the basis of a formulation of the parallel spinor flow on Lie groups and homogeneous three-manifolds, in the spirit of [32].

5 Comoving parallel spinor flows

In this section we consider a specific type of parallel spinor flow which admits a particularly neat geometric description, with the goal of obtaining explicit time-dependent Lorentzian four-manifolds admitting parallel spinors.

5.1 Globally hyperbolic comoving spacetimes

We consider a particular class of parallel spinor flows defined by imposing the condition $\lambda_t = 1$ for all $t \in \mathbb{R}$.

**Definition 5.1** A parallel spinor flow $((\lambda_t)_{t \in \mathbb{R}}, (h_t)_{t \in \mathbb{R}}, \{u^0_t\}_{t \in \mathbb{R}}, \{u^1_t\}_{t \in \mathbb{R}}, \{l^1_t\}_{t \in \mathbb{R}})$ is comoving if $\lambda_t = 1$ for every $t \in \mathbb{R}$.

A comoving parallel spinor flow on a manifold of the form $M = \mathbb{R} \times \Sigma$, where $\Sigma$ is an oriented three-manifold, will be always understood as a parallel spinor flow on $\Sigma$ with respect to the cartesian coordinate of $\mathbb{R}$.

**Definition 5.2** A four-dimensional space-time $(M, g)$ is a comoving globally hyperbolic space-time if it is isometric to a model of the form:

$$
(M, g) = (I \times \Sigma, -dt \otimes dt + h_t),
$$

for a family $\{h_t\}_{t \in I}$ of complete Riemannian metrics on $\Sigma$, where $I \subset \mathbb{R}$ is an interval.

A metric of the type $g = -dt^2 + h_t$ will be called a comoving globally hyperbolic.

**Remark 5.3** The term comoving is motivated by the fact that the local metric of a comoving observer in a cosmological background is of comoving globally hyperbolic type. In particular, the time factor of the metric is constant.

**Theorem 5.4** An oriented four-manifold $(M, g)$ admits a comoving parallel spinor flow if and only if $\Sigma$ admits a family:

$$
\{e' := (e'_u, e'_l, e'_n) : \Sigma \to F(\Sigma)\}_{t \in \mathbb{R}}.
$$

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of sections of the oriented frame bundle $F(\Sigma)$ of $\Sigma$ satisfying the following system of partial differential equations:

$$
\partial_t e^t + \Theta_t(e^t) = 0, \quad de^t = \Theta_t(e^t) \wedge e^t_\lambda, \quad [\Theta_t(e^t_\mu)] = 0 \in H^1(\Sigma, \mathbb{R}), \quad \partial_t \Theta_t(e^t_\mu) = 0,
$$

(5.1)

where:

$$
\Theta_t = -\frac{1}{2} \partial_t(e^t_\mu \otimes e^t_\mu + e^t_\lambda \otimes e^t_\lambda + e^t_\nu \otimes e^t_\nu).
$$

If this is the case, the corresponding comoving globally hyperbolic metric is given by:

$$
g = -dt \otimes dt + h_{e^t}, \quad h_{e^t} = e^t_\mu \otimes e^t_\mu + e^t_\lambda \otimes e^t_\lambda + e^t_\nu \otimes e^t_\nu.
$$

**Proof** Consider a solution of Eqs. (2.3), (2.4) and (2.5) of the form:

$$(\lambda_t = 1)_{t \in \mathbb{R}}, \{h_t\}_{t \in \mathbb{R}}, \{u^0_t\}_{t \in \mathbb{R}}, \{u^0_t, l^0_t\}_{t \in \mathbb{R}}.$$  

To every such solution we canonically associate a family of sections $\{e^t\}_{t \in \mathbb{R}} : \Sigma \to F(\Sigma)$ defined as follows:

$$e^t = (e^t_\mu, e^t_\lambda, e^t_\nu) \overset{\text{def.}}{=} (u^0_t / u^t_\mu, l^0_t, *\Theta_t(e^t_\mu)) / u^t_\mu.
$$

where $*\Theta_t$ denotes the Hodge dual on $\Sigma$ with respect to $h_t$ and the fixed orientation. Clearly, the triple $\{(e^t_\mu, e^t_\lambda, e^t_\nu)\}_{t \in \mathbb{R}}$ yields a family of orthonormal coframes on $\Sigma$ with respect to $\{h_t\}_{t \in \mathbb{R}}$. Imposing $\lambda_t = 1$ for every $t \in \mathbb{R}$ in Eqs. (2.3) and (2.4) we obtain:

$$\partial_t u^\perp_t + \Theta_t(u^\perp_t) = 0, \quad \partial_t e^t_\lambda + \Theta_t(e^t_\lambda) = 0,$$

$$\nabla^{ht} u^\perp_t + u^0_t \Theta_t = 0, \quad \nabla^{ht} e^t_\lambda = \Theta_t(e^t_\lambda) \otimes e^t_\nu.$$

Taking the time derivative of the constraint $(u^0_t)^2 = |u^\perp_t|^2_{ht}$, we conclude $\partial_t u^0_t = 0$. Hence using the time independence of $u^0$ in the previous equations, we obtain:

$$\frac{1}{u^0_t} \nabla^{ht} u^\perp_t + \Theta_t = \nabla^{ht} e^t_\mu + d \log(u^0_t) \otimes e^t_\mu + \Theta_t = \nabla^{ht} e^t_\mu - \Theta_t(e^t_\mu) \otimes e^t_\mu + \Theta_t,$$

$$\partial_t e^t_\mu + \Theta_t(e^t_\mu) = 0.$$

Similar arguments yield the equation $\partial_t e^t_\nu + \Theta_t(e^t_\nu) = 0$. On the other hand:

$$\nabla^{ht} e^t_\lambda = \nabla^{ht} *\lambda_t (e^t_\mu \wedge e^t_\nu) = *\lambda_t (\nabla^{ht} e^t_\mu \wedge e^t_\nu) + *\lambda_t (e^t_\mu \wedge \nabla^{ht} e^t_\nu)$$

$$= *\lambda_t (\nabla^{ht} e^t_\nu \wedge e^t_\mu) = \Theta(e^t_\nu) \otimes e^t_\mu.$$

Hence, every parallel spinor flow of the form $\{(\lambda_t = 1)_{t \in \mathbb{R}}, \{h_t\}_{t \in \mathbb{R}}, \{u^0_t\}_{t \in \mathbb{R}}, \{u^0_t, l^0_t\}_{t \in \mathbb{R}}$ produces a canonical section of $F(\Sigma)$ satisfying (5.1). Conversely, assume that $\{e^t\}_{t \in \mathbb{R}}$ is a solution of (5.1). The third equation in (5.1) guarantees that there exist a family of functions $\{u^0_t\}_{t \in \mathbb{R}}$ such that:

$$d \log(u^0_t) = -\Theta_t(e^t_\mu).$$

Plugging this equation into the first equation of (5.1), we obtain:

$$\partial_t (u^0_t e^t_\mu) = \partial_t u^0_t e^t_\mu - u^0_t \Theta_t(e^t_\mu) = \partial_t \log(u^0_t) u^0_t e^t_\mu - \Theta_t(u^0_t e^t_\mu) = -\Theta_t(u^0_t e^t_\mu).$$
which yields the first equation in (2.3). The second equation in (2.3) follows similarly. Equations (2.4) and (2.5) follow by interpreting the second equation in (5.1) as the first Cartan structure equation for the orthonormal frame $e^\prime$ with respect to the metric:

$$h_{e^\prime} = e^\prime_u \otimes e^\prime_u + e^\prime_j \otimes e^\prime_j + e^\prime_n \otimes e^\prime_n,$$

and hence we conclude.

We will refer to Eq. (5.1) as the comoving parallel-spinor flow equations, and we will refer to its solutions as comoving parallel spinor flows. The general investigation of comoving parallel spinor flows is beyond the scope of this article and will be considered in a separate publication. Instead, we consider two particular important cases in detail.

### 5.2 A diagonal example on $\mathbb{R}^3$

Set $\Sigma = \mathbb{R}^3$ with Cartesian coordinates $(x, y, z)$ and consider comoving parallel spinor flows $\{e^\prime_t\}_{t \in \mathcal{I}}$ of the form:

$$e^\prime = (f^\prime_u dx, f^\prime_j dy, f^\prime_n dz),$$

for families of functions $\{f^\prime_u\}_{t \in \mathbb{R}}$, $\{f^\prime_j\}_{t \in \mathbb{R}}$ and $\{f^\prime_n\}_{t \in \mathbb{R}}$ on $\mathbb{R}^3$. Hence:

$$h_{e^\prime} = (f^\prime_u)^2 dx \otimes dx + (f^\prime_j)^2 dy \otimes dy + (f^\prime_n)^2 dz \otimes dz, \quad (e^\prime)^2 = \left(1 \frac{\partial}{\partial u}, 1 \frac{\partial}{\partial j}, 1 \frac{\partial}{\partial n}\right).$$

With these provisos in mind, we compute:

$$\Theta_t = -(f^\prime_u \partial_u f^\prime_u dx \otimes dx + f^\prime_j \partial_j f^\prime_j dy \otimes dy + f^\prime_n \partial_n f^\prime_n dz \otimes dz). \quad (5.2)$$

Therefore, equation $\partial_t e^\prime + \Theta_t(e^\prime) = 0$ is automatically satisfied. On the other hand, equations $[\Theta_t(e^\prime_u)] = 0$ and $\partial_t \Theta_t(e^\prime_u) = 0$ are equivalent to:

$$\partial_t df^\prime_u \wedge dx = 0, \quad \partial^2_t f^\prime_u = 0,$$

implying:

$$f^\prime_u = a + b t,$$

where $b = b(x)$ is a function of the coordinate $x$ and $a = a(x, y, z)$ is a function of all coordinates of $\mathbb{R}^3$. Note that, in order to have a well defined comoving parallel spinor flow, we must impose the constraint:

$$f^\prime_u(t, x, y, z) = a(x, y, z) + b(x) t \neq 0,$$

for every $t \in \mathcal{I}$ and $(x, y, z) \in \mathbb{R}^3$, which translates into a constraint in the allowed domain of definition $\mathcal{I} \subset \mathbb{R}$ of $t$. The only equations that remain to be solved for

$$e^\prime = ((a + b t) dx, f^\prime_j dy, f^\prime_n dz),$$

to be a comoving parallel spinor flow are $d e^\prime = \Theta(e^\prime) \wedge e^\prime_u$, which can be shown to be equivalent to:

$$da \wedge dx = 0, \quad (df^\prime_j - \partial_j f^\prime_u dx) \wedge dy = 0, \quad (df^\prime_n - \partial_n f^\prime_u dx) \wedge dz = 0.$$

These equations are in turn equivalent to:

$$a = a(x), \quad \partial_x f^\prime_j = f^\prime_u \partial_j f^\prime_u, \quad \partial_x f^\prime_n = f^\prime_u \partial_n f^\prime_n, \quad \partial_y f^\prime_n = 0, \quad \partial_y f^\prime_n = 0, \quad (5.3)$$
which do have explicit solutions, as we will show later in particular examples. On the other hand a direct computation shows that the Ricci curvature of the comoving globally hyperbolic Lorentzian metric $g = -dt \otimes dt + h_{ct}$ associated with such $e'$ vanishes if and only if the following condition holds:

$$b \left( \frac{\partial_i f'}{f'} + \frac{\partial_i f''}{f''} \right) - \frac{\partial_i \partial_j f'}{f'} - \frac{\partial_i \partial_j f''}{f''} = 0. \quad (5.4)$$

This condition will be explored in the examples below.

**Example 5.5** Suppose that both $a$ and $b$ are constants, with $b \neq 0$. With this assumption, a general solution of Eq. (5.3) is of the form:

$$f'_i = \mathcal{L}_t(x + \log |a + b t|/b, y), \quad f''_n = \mathcal{L}_n(x + \log |a + b t|/b, z),$$

for nowhere vanishing smooth functions $\mathcal{L}_t, \mathcal{L}_n \in C^\infty(\mathbb{R}^2)$. The corresponding coframe $e'$ reads:

$$e' = ((a + b t)dx, \mathcal{L}_t(x + \log |a + b t|/b, y)dy, \mathcal{L}_n(x + \log |a + b t|/b, z)dz),$$

which is well-defined in the intervals $t \in I_1 = (-\infty, -\frac{a}{b})$ or $t \in I_2 = (-\frac{a}{b}, \infty)$. The metric associated with the previous global coframe is given by:

$$g = -dt \otimes dt + (a + b t)^2 dx \otimes dx + \mathcal{L}_t(x + \log |a + b t|/b, y)^2 dy \otimes dy$$

$$+ \mathcal{L}_n(x + \log |a + b t|/b, z)^2 dz \otimes dz,$$

which provides a large family of four-dimensional Lorentzian metrics admitting a parallel spinor. If the induced Riemannian spatial metric:

$$h_{ct} = (a + b t)^2 dx \otimes dx + \mathcal{L}_t(x + \log |a + b t|/b, y)^2 dy \otimes dy$$

$$+ \mathcal{L}_n(x + \log |a + b t|/b, z)^2 dz \otimes dz,$$

on $\{t\} \times \mathbb{R}^3 \subset I_i \times \mathbb{R}^3$ (for $i = 1, 2$) is complete for all $t \in I_i$ we obtain a family of comoving globally hyperbolic metrics on $I_i \times \mathbb{R}^3$. Equation (5.4), implies now that $g$ is Ricci-flat if and only if:

$$\frac{b \partial_t \mathcal{L}_t - \partial_t \partial_x \mathcal{L}_t}{\mathcal{L}_t} + \frac{b \partial_t \mathcal{L}_n - \partial_t \partial_x \mathcal{L}_n}{\mathcal{L}_n} = 0,$$

where we have defined $\xi(t, x) := x + \log |a + b t|/b$. This Ricci-flatness condition is satisfied if the functions $\mathcal{L}_t(\xi, y)$ and $\mathcal{L}_n(\xi, z)$ take the form:

$$\mathcal{L}_t(\xi, y) = w_1(y)e^{b\xi} + w_2(y), \quad \mathcal{L}_n(\xi, z) = w_3(z)e^{b\xi} + w_4(z),$$

where $w_1, w_2, w_3, w_4$ are arbitrary smooth functions.

**Example 5.6** Assume that $a$ is a possibly non-constant strictly positive function and $b = 0$. With this assumption, the general solution of Eq. (5.3) is of the form:

$$f'_i = \mathcal{L}_t \left( t + \int_0^x a(\tau)d\tau, y \right), \quad f''_n = \mathcal{L}_n \left( t + \int_0^x a(\tau)d\tau, z \right),$$

for nowhere vanishing smooth functions $\mathcal{L}_t, \mathcal{L}_n \in C^\infty(\mathbb{R}^2)$. The corresponding coframe $e'$ reads:

$$e' = \left( a(x)dx, \mathcal{L}_t \left( t + \int_0^x a(\tau)d\tau, y \right) dy, \mathcal{L}_n \left( t + \int_0^x a(\tau)d\tau, z \right) dz \right).$$
which is well-defined for \( t \in \mathcal{I} = \mathbb{R} \). The metric associated with the previous global coframe is given, after a change and relabeling of coordinates, by the following expression:

\[
g = -dt \otimes dt + dx \otimes dx + \mathcal{L}_l(t + x, y)^2dy \otimes dy + \mathcal{L}_n(t + x, z)^2dz \otimes dz,
\]

which provides a large family of four-dimensional Lorentzian metrics admitting a parallel spinor. If the induced Riemannian spatial metric:

\[
h_{\xi_\nu} = dx \otimes dx + \mathcal{L}_l(t + x, y)^2dy \otimes dy + \mathcal{L}_n(t + x, z)^2dz \otimes dz,
\]
on \{t\} \times \mathbb{R}^3 \subset \mathcal{I} \times \mathbb{R}^3 is complete for all \( t \in \mathcal{I} \) we obtain a family of comoving globally hyperbolic metrics on \( \mathcal{I} \times \mathbb{R}^3 \). Implementing the change of coordinates:

\[
x^+ = \frac{t + x}{\sqrt{2}}, \quad x^- = -\frac{t + x}{\sqrt{2}},
\]

the metric \( g \) is given by:

\[
g = dx^+ \odot dx^- + \mathcal{L}_l(x^+, y)^2dy \otimes dy + \mathcal{L}_n(x^+, z)^2dz \otimes dz,
\]
after a suitable redefinition of the functions \( \mathcal{L}_l \) and \( \mathcal{L}_n \). This metric is a particular case of a Lorentzian metric expressed in Schimming coordinates [46], which exists in every Lorentzian manifold admitting a parallel null vector field. Equation (5.4) implies now that \( g \) is Ricci-flat if and only if:

\[
\frac{\partial_x+\partial_{x^+}\mathcal{L}_l}{\mathcal{L}_l} + \frac{\partial_x+\partial_{x^-}\mathcal{L}_n}{\mathcal{L}_n} = 0,
\]

Some simple solutions, and thus Ricci-flat examples, can be found just by setting \( \partial_x+\partial_{x^+}\mathcal{L}_l = \partial_x+\partial_{x^-}\mathcal{L}_n = 0 \):

\[
\mathcal{L}_l = w_1(y)x^+ + w_2(y), \quad \mathcal{L}_n = w_3(z)x^+ + w_4(z),
\]

where \( w_1, w_2, w_3, w_4 \) are arbitrary smooth functions.

### 5.3 An example in Schimming coordinates

In reference [46] it was proven that any four-dimensional space-time \((M, g)\) equipped with a parallel light-like vector field \( u^\alpha \in \mathfrak{X}(M) \) admits local coordinates \((x^+, x^-, y_1, y_2)\) in which the metric \( g \) and the vector field \( u^\alpha \) are written as follows:

\[
g = dx^+ \odot dx^- + k_{x^+}, \quad u^\alpha = \frac{\partial}{\partial x^\alpha},
\]

where:

\[
k_{x^+}(y_1, y_2) = k_{x^+ij}dy_i \otimes dy_j, \quad i, j = 1, 2.
\]
is a family of two-dimensional metrics parametrized by the coordinate \( x^+ \). A simple change of coordinates \( \sqrt{2}x^+ = t + x \) and \( \sqrt{2}x^- = x - t \) allows to write the previous metric \( g \) as:

\[
g = -dt \otimes dt + dx \otimes dx + k_{t+x}, \tag{5.5}
\]

whence we obtain a particular type of comoving globally hyperbolic spacetimes. Therefore, it is natural to study comoving parallel spinor flows adapted to the structure of the metric (5.5). Assume that the previous coordinate system is globally defined. Then, the Cauchy surface is given by \( \Sigma = \mathbb{R} \times X \), with \( X \) an oriented two-dimensional manifold, and the metric takes...
the form $h_t = dx \otimes dx + k_{t+x}$. Consequently, we assume that our comoving parallel spinor flow is of the form:

$$e' = (dx, e'_t(x), e'_n(x)),$$

where $k_{t+x} = e'_t \otimes e'_t + e'_n \otimes e'_n$. The comoving parallel spinor flow Eq. (5.1) reduce to:

$$\partial_t e'_t(x) + \Theta_t(e'_t(x)) = 0, \quad \partial_x e'_t(x) + \Theta_t(e'_t(x)) = 0, \quad de'_t(x)|_X = 0, \quad \Theta_t(\partial_x) = 0. \quad (5.6)$$

Hence, the comoving parallel spinor flow can be considered as a bi-parametric flow, parametrized by $t$ and $x$, for a family of closed oriented frames on $X$. In particular, $(X, k_{t+x})$ is flat for every $(t, x) \in \mathbb{R}^2$ and therefore isometric to euclidean space, the flat cylinder or a flat torus. Equations (5.6) immediately imply:

$$\partial_t e'_t(x) = \partial_x e'_t(x), \quad i = l, n.$$

Therefore, choosing coordinates $(y_1, y_2)$ on $X$, global for the plane and local for the torus and cylinder cases, every such family of solutions can be written as follows:

$$e'_t(x) = f'_1(t + x)dy_1 + f'_2(t + x)dy_2, \quad e'_n(x) = f''_1(t + x)dy_1 + f''_2(t + x)dy_2,$$

for functions $f'_1, f'_2, f''_1, f''_2 \in C^\infty(\mathbb{R})$ satisfying the following condition everywhere:

$$\delta = f''_1 f''_2 - f'_1 f'_2 \neq 0. \quad (5.7)$$

If this condition is satisfied, the dual frame is given by:

$$e'_t(x) = \frac{1}{\delta}(f''_1 \partial y_1 - f'_1 \partial y_2), \quad e'_n(x) = \frac{1}{\delta}(-f''_2 \partial y_1 + f'_1 \partial y_2).$$

In order to guarantee that Eq. (5.7) is satisfied, we assume the following ansatz:

$$f'_1 := e^{f_1}, \quad p := f'_2 = -f''_1, \quad f''_2 := e^{f_2}$$

in terms of functions $f_1, f_2, p \in C^\infty(\mathbb{R})$. This implies $\delta = e^{f_1 + f_2} + p^2 > 0$ and therefore Eq. (5.6) further reduce to:

$$\partial_x e'^+ = (\partial_x e'^+) ((e'^+) \ast (e'^+)^2) e'^+ + (\partial_x e'^+) ((e'^+) \ast (e'^+)^2) e'^+, \quad (5.8)$$

$$\partial_x e'^n = (\partial_x e'^+) ((e'^n) \ast (e'^n)^2) e'^n + (\partial_x e'^n) ((e'^n) \ast (e'^n)^2) e'^n, \quad (5.9)$$

where we have gone back to the coordinate $x^+ = t + x$ and written $e'^{\pm i} := e'_i(x), i = l, n.$ In particular, note that the condition $(\partial_x e'^+) ((e'^+) \ast (e'^+)^2) = (\partial_x e'^+) ((e'^n) \ast (e'^n)^2)$ is necessarily satisfied, as required by Theorem 3.11. By direct computation one finds that Eqs. (5.8) and (5.9) turn out to yield a single linearly independent equation which takes the form:

$$p(\partial_x e'^+ + \partial_x e'^n) = (e^{f_1} + e^{f_2}) \partial_x p.$$ 

The general solution of the previous equation is given by:

$$p = c (e^{f_1} + e^{f_2}),$$

for any real constant $c$. Therefore we are led to the following Lorentzian metric, which by construction admits a parallel light-like vector field given by $\partial_x^-$:

$$g = dx^+ \otimes dx^- + (e^{2 f_1} + c^2 (e^{f_1} + e^{f_2} g) dy_1 \otimes dy_1 + c^2 (e^{f_1} + e^{f_2} g) dy_1 \otimes dy_2$$

$$+ (e^{f_1} + c^2 (e^{f_1} + e^{f_2} g) dy_2 \otimes dy_2. \quad (5.10)$$
The Ricci tensor of the previous metric is given by:

\[
\text{Ric}^g = \left[ 2c^2 e^{f_1} ((\partial_x + f_1)^2 + \partial_y + \partial_x + f_1) + 2c^2 e^{f_n} ((\partial_x + f_n)^2 + \partial_y + \partial_x + f_n) \\
+ (1 + 2c^2)e^{f_1 + f_n} ((\partial_x + f_1)^2 + \partial_y + \partial_x + f_1 + (\partial_x + f_n)^2 + \partial_y + \partial_x + f_n) \right] \text{dx}^+ \otimes \text{dx}^+.
\]

which vanishes if the following conditions are satisfied:

\[(\partial_x + f_1)^2 + \partial_y + \partial_x + f_1 = 0, \quad (\partial_x + f_n)^2 + \partial_y + \partial_x + f_n = 0.\]

These ODEs are solved by:

\[f_1(x^+) = a + \log |x^+ - b|, \quad f_n(x^+) = c + \log |x^+ - d|,\]

for real constants \(a, b, c, d \in \mathbb{R}\). These solutions are well defined if \(x^+ \in (-\infty, \min(b, d))\) or if \(x^+ \in (\max(b, d), +\infty)\). However, although \(f_1\) and \(f_n\) present divergences whenever the argument of the logarithm vanishes, this is not problematic for the metric (5.10) as long as \(b \neq d\) (otherwise the metric would be degenerate at \(x^+ = b = d\)), since both functions \(f_1\) and \(f_n\) appear exponentiated. In addition, it can be checked that the space time \((\mathbb{R}^2 \times X, g)\) is a plane wave, since the Riemann curvature tensor \(R^g : \wedge^2 T(\mathbb{R}^2 \times X) \rightarrow \wedge^2 T(\mathbb{R}^2 \times X)\) satisfies \(R^g |_{(\partial_x)^\perp} \wedge (\partial_x)^\perp = 0\) and \(\nabla_V R^g = 0\) for all \(V \in (\partial_x)^\perp\).

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