Running Coupling Effects in BFKL Evolution

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Abstract

We resum the recently calculated second order kernel of the BFKL equation. That kernel can be viewed as the sum of a conformally invariant part and a running coupling part. The conformally invariant part leads to a corrected BFKL intercept as found earlier. The running coupling part of the kernel leads to a non-Regge term in the energy dependence of high energy hard scattering, as well as a $Q^2$-dependent intercept.

1 Introduction

Recently, the calculation of the second order kernel for BFKL evolution has been completed in two (partially) independent calculations. The second order kernel strongly modifies the leading order result for the BFKL intercept at all but the smallest QCD couplings. While this large intercept correction does not significantly alter our picture of BFKL dynamics, it suggests that even higher order corrections are important leaving us without a reliable theoretical calculation of the intercept which can, however, in principle be determined phenomenologically.

Of course, the corrected BFKL intercept governs single-scale high energy hard scattering only after a resummation has been done to bring that intercept into an exponential form. It is the purpose of the present paper to do a more complete resummation of the complete second order kernel for use at high energies.

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In Sec. 2, we develop a formalism which allows one to iterate the complete second order kernel an arbitrary number of times. It turns out to be convenient to separate the second order kernel into a conformally invariant part and a running coupling part. The resummation of the conformally invariant part leads to the result given in Refs. 3 and 4. In Sec. 3, after some calculation, we resum the running coupling part of the kernel, and it also exponentiates. The total result is given in (29) and (30). The running coupling part of the kernel has two effects. It determines the scale of the leading order intercept to be \( \frac{\Delta N_c}{\pi} \alpha_s(qq') \) where \( q \) and \( q' \) are the virtualities of the gluons which hook into the (compact) particles for which the scattering is being calculated, as given in (1). The running coupling part of the kernel also introduces a non-Regge behavior in the scattering, the \( \frac{b}{2} [\alpha(\alpha_P - 1)b] Y^3 \) term in (30). Indeed, this non-Regge term is necessary in order to be able to view evolution in a rapidity interval \( Y \) as built out of separate evolutions over regions \( y \) and \( Y - y \) as indicated in (32). Although this non-Regge term formally looks like a next-to-next-to-leading term, in the exponent, it does not appear possible to generate such a term from a third order kernel and so we believe the result given in (30) is solid. If one writes this term as \( c \alpha^5 Y^3 \), \( c \approx 5 \) for 3 flavors so that this term is reasonably small so long as \( \alpha \) is reasonably small and \( Y \) is not inordinately large. Parametrically, the Regge-type behavior in BFKL evolution begins to break down when \( Y \sim \alpha^{-5/3} \) a value where perturbation theory is still valid, where the next-to-leading conformal corrections to the intercept are (parametrically) small, but where unitarity corrections are expected to already be large. We note that E. Levin has previously arrived at \( Y \leq \alpha^{-5/3} \) as a criterion for the validity of the usual Regge-BFKL picture of high energy scattering. (See Eq. 82 of Ref. 5.)

In Sec. 4, we calculate the next-to-leading corrections to the anomalous dimension and coefficient functions at large orders of perturbation theory. The largest corrections to the coefficient function are determined completely by the shift of the branch point in the angular momentum plane from \( \omega = \alpha_P - 1 \) to the position of the corrected BFKL intercept, while the largest corrections to the anomalous dimension function are determined by the appearance of a pole, \( \gamma_\omega = \frac{-a(\alpha_P - 1)b/4}{\omega - (\alpha_P - 1)} \), as indicated in (41).

Finally, in Sec. 5, we investigate the limitations on using perturbation theory to calculate single-scale high energy hard scattering. We do this by examining the factorials which appear in perturbation theory and which determine the region where the QCD asymptotic expansion can be used.
Eq.46 confirms the result found in Ref.6 using a fixed coupling approach to the diffusion present in BFKL evolution.

2 Using the next-to-leading kernel to all orders

The total cross section for the scattering of two compact colorless particles (heavy quark) A and B can be written as

$$
\sigma_{AB}(s) = \int \frac{d^2q}{2\pi q^2} \int \frac{d^2q'}{2\pi q'^2} \Phi_A(q)\Phi_B(q') \int_{a-i\infty}^{a+i\infty} \frac{d\omega}{2\pi i}(\frac{s}{qq'})^{\omega}G_{\omega}(q, q')
$$  (1)

where we use the notation of Ref.3. $G_\omega$ obeys the equation

$$
\omega G_{\omega}(q, q') = \delta(q - q') + \int d^2\tilde{q}K(q, \tilde{q})G_{\omega}(\tilde{q}, q')
$$  (2)

with

$$
K(q, q') = K^{(1)}(q, q') + K^{(2)}(q, q')
$$  (3)

where the superscripts in (3) indicate the leading and next-to-leading BFKL kernels, respectively. If $G_{\omega}^{(1)}$ represents the leading BFKL result, the result of solving (2) taking only $K^{(1)}$ as the kernel of that equation, then $G_{\omega}$ can be written as

$$
G_{\omega}(q, q') = \sum_{N=0}^{\infty} G_{N\omega}(q, q')
$$  (4)

with

$$
G_{N\omega} = G_{\omega}^{(1)}[K^{(2)}G_{\omega}^{(1)}]^N
$$  (5)

where all of the products in (5) are understood to be convolutions as indicated in (2). It is convenient to write $G_{\omega}^{(1)}$ as

$$
G_{\omega}^{(1)}(q, q') = \frac{1}{\pi qq'} \int \frac{d\lambda}{2\pi i}(\frac{q}{q'})^{2\lambda-1} \frac{1}{\omega - \frac{\alpha_{N\omega}}{\pi} \chi(\lambda)}
$$  (6)

3
with $\chi(\lambda) = 2\psi(1) - \psi(\lambda) - \psi(1 - \lambda)$, and where we neglect the dependence on $\cos \phi = (q \cdot q'/qq')$ a dependence which disappears when $\sigma(s)$ is evaluated at large $s$. The $\lambda$-integration in (6) runs parallel to the imaginary axis with $\Re \lambda = 1/2$.

It is useful to express the kernel $K^{(2)}$ as a conformally invariant part and a “running coupling” part using the result [3]

$$
\int d^2q' K^{(2)}(q, q')(\frac{q'}{q})^{2\lambda-2} = k^{(2)}(q) = k_{conf}(\lambda) + k_{rc}(\lambda, q)
$$

(7)

with

$$
k_{conf}(\lambda) = -\frac{1}{4}(\frac{\alpha(\mu^2)N_c}{\pi})^2 c(\lambda)\chi(\lambda)
$$

(8a)

and

$$
k_{rc}(\lambda, q) = -\frac{2\alpha^2(\mu^2)N_c}{\pi} b\chi(\lambda) \ln \frac{q}{\mu}
$$

(8b)

where $c(\lambda)$ is defined in Ref.3, and where $b = \frac{11N_c - 2N_f}{12\pi}$. Define

$$
G_N(q, Y, q') = \int \frac{d\omega}{2\pi i} e^{\omega Y} G_{N\omega}(q, q'),
$$

(9)

along with similar definitions for $G$ and $G^{(1)}$. Then, substituting (6) in (5), using (9) as well as the identity

$$
\int \frac{d\omega}{2\pi i} \frac{e^{\omega Y}}{\omega - \frac{\alpha N_c}{\pi} \chi(\lambda)} \prod_{i=1}^{N} \frac{1}{\omega - \frac{\alpha N_c}{\pi} \chi(\lambda_i)} = \int \frac{d\omega}{2\pi i} \prod_{i=1}^{N} \frac{d\omega_i}{2\pi i} \frac{e^{\omega_i(y_i - y_{i-1})}}{\omega_i - \frac{\alpha N_c}{\pi} \chi(\lambda_i)} \frac{e^{\omega(Y - y_N)}}{\omega - \frac{\alpha N_c}{\pi} \chi(\lambda)};
$$

(10)

with $0 = y_0 \leq y_1 \leq y_2 \leq \cdots \leq y_N \leq Y$, one finds

$$
G_N(q, Y, q') = \int G^{(1)}(q, Y - y_N, q_N) d^2q_N d^2y_N k^{(2)}(q_N)
$$

$$
\prod_{i=1}^{N-1} G^{(1)}(q_i, y_{i+1} - y_i, q_i) d^2q_i d^2y_i k^{(2)}(q_i) G^{(1)}(q_1, y_1, q')
$$

(11)

where, in the saddle-point approximation,
\[ G^{(1)}(q, y, q') = \frac{e^{(\alpha_P - 1)y}}{2\pi qq'\sqrt{4\pi Dy}} \exp\left[-\frac{\ell n^2 q/q'}{4Dy}\right] \] (12)

with \( D = \frac{7\alpha_0(\mu^2)}{2\pi}N_c\zeta(3) \) and \( \alpha_P - 1 = \frac{4\alpha(\mu^2)N_c}{\pi}\ell n 2 \), where (12) is valid for large \( y \). Assuming \( 0 << y_1 << y_2 << \cdots << y_N << Y \) in (11) one gets

\[ G_N(q, Y, q') = \frac{e^{(\alpha_P - 1)Y}}{2\pi qq'\sqrt{4\pi DY}} \frac{1}{(4\pi D)^{N/2}} \]

\[ \int \prod_{i=1}^{N} \frac{dy_i du_i}{\sqrt{y_i - y_{i-1}}} k^{(2)}(u_i) \sqrt{\frac{Y}{Y - y_N}} \exp\left[-\frac{(u_i - u_{i-1})^2}{4D(y_i - y_{i-1})}\right] \exp\left[-\frac{(u - u_N)^2}{4D(Y - y_N)}\right] \] (13)

where \( u_i = \ell n q_i / \mu, u_0 = \ell n q'/\mu \) and \( u = \ell n q/\mu \). Now

\[ k^{(2)}(u_i) = k_{conf}(\frac{1}{2}) + k_{rc}(\frac{1}{2}, u_i) \] (14)

with

\[ k_{conf}(\frac{1}{2}) = -\frac{\alpha(\alpha_P - 1)N_c}{4\pi} c(\frac{1}{2}) \] (15a)

\[ k_{rc}(\frac{1}{2}, u_i) = -2\alpha(\alpha_P - 1)b u_i. \] (15b)

It is convenient to sum all orders of \( k_{conf} \) for a given order of \( k_{rc} \). Thus, one may write

\[ G(q, Y, q') = \exp\{\alpha_P - 1 \left[1 - \frac{a_N N_c}{4\pi} c(\frac{1}{2})\right]Y\} \sum_{N=0}^{\infty} I_N \] (16)

with

\[ I_N = \frac{1}{(4\pi D)^{N/2}} \int \prod_{i=1}^{N} \frac{dy_i du_i}{\sqrt{y_i - y_{i-1}}} k_{rc}(u_i) \sqrt{\frac{Y}{Y - y_N}} \exp\left[-\frac{(u_i - u_{i-1})^2}{4D(y_i - y_{i-1})}\right] \exp\left[-\frac{(u - u_N)^2}{4D(Y - y_N)}\right]. \] (17)
Choosing \( Y = \ell n \frac{\alpha}{\alpha'} \) and using (16) and (17) in (1) gives the result of summing the leading and next-to-leading kernels to all orders for the cross section \( \sigma_{AB} \). The first factor on the right-hand side of (16) is the answer given in Refs. 3 and 4.

### 3 Evaluating the running coupling contribution

In order to evaluate \( I_N \) one can introduce a factor \( \exp\{\sum_{i=1}^{N} J_i u_i\} \) in the integrand of (17), expressing each \( u_i \) coming from \( k_{rc}(u_i) \) as \( \frac{\partial}{\partial J_i} \) acting on the integral, and then setting the \( J_i = 0 \) at the end of the calculation

\[
I_N = \left[ -2\alpha(\alpha_P - 1)b \right]^N \frac{1}{(4\pi D)^{N/2}} \prod_{i=1}^{N} \frac{\partial}{\partial J_i} \int \frac{d y_i}{\sqrt{y_i - y_{i-1}}} d u_i \sqrt{\frac{Y}{Y - y_N}} 
\]

\[
\cdot \exp\left\{ -\frac{(u_i - u_{i-1})^2}{4D(y_i - y_{i-1})} + J_i u_i \right\} \exp\left\{ -\frac{(u - u_N)}{4D(Y - y_N)} \right\} \bigg|_{J=0}. \tag{18}
\]

It is useful to make the change of variables

\[
\begin{align*}
z_0 &= \frac{u_0}{\sqrt{Y}} \frac{1}{\sqrt{4D}} \\
z_i &= \frac{y_{i+1}(u_i - u_0) - y_i(u_{i+1} - u_0)}{\sqrt{y_i y_{i+1}(y_{i+1} - y_i)}} \frac{1}{\sqrt{4D}}, \quad i = 1, 2, \ldots, N \\
z &= \frac{u}{\sqrt{Y}} \frac{1}{\sqrt{4D}}
\end{align*}
\]

where \( y_{N+1} = Y, u_{N+1} = u \). The \( u_i \) are expressed in terms of the \( z_i \)'s by

\[
\begin{align*}
u_i - u_0 &= y_i \left[ \sum_{j=i}^{N} z_j \sqrt{\frac{y_{j+1} - y_j}{y_j y_{j+1}}} + \frac{z - z_0}{\sqrt{Y}} \right] \sqrt{4D} \tag{20a}

u - u_0 &= Y \frac{z - z_0}{\sqrt{Y}} \sqrt{4D}. \tag{20b}
\end{align*}
\]

After rescaling \( J_i \rightarrow \sqrt{4D} J_i \) we have
\[ I_N = [-2\alpha(\alpha P - 1)b]^N \exp\left(-\frac{1}{4DY} \ln^2 \frac{q}{q'}\right) \left(\frac{4D}{\pi}\right)^{N/2} \]

\[
\left[ \prod_{i=1}^{N} \frac{\partial}{\partial J_i} \int dy_i dz_i \exp\left\{ -z_i^2 + z_i \sqrt{\frac{y_{i+1} - y_i}{y_{i+1} y_i}} \sum_{j=1}^{i} J_j y_j \right\} \right] 
\cdot \exp\left\{ \frac{1}{\sqrt{4D}} \frac{\ln q'}{\mu} (J_1 + \cdots + J_N) + \frac{1}{\sqrt{4DY}} \frac{\ln q}{q'} (J_1 y_1 + \cdots + J_N y_N) \right\} \bigg|_{J=0} \tag{21} \]

where, as always, the \( y \)-integration is ordered \( 0 \leq y_1 \leq y_2 \leq \cdots \leq y_N \leq Y \) and where it is understood that all the \( \frac{\partial}{\partial J_i} \) factors are to be put to the left of all the \( J_i \) terms in the various exponents in (21). After performing the integration over \( dz_i \) and after some simple algebra formula (21) becomes

\[ I_N = [-2\alpha(\alpha P - 1)b]^N \exp\left(-\frac{1}{4DY} \ln^2 \frac{q}{q'}\right) \left(\frac{4D}{\pi}\right)^{N/2} \]

\[
\left[ \prod_{i=1}^{N} \frac{\partial}{\partial J_i} \int dy_i \exp\left\{ \frac{1}{2} \sum_{j=i+1}^{N} J_i J_j \frac{y_j (Y - y_j)}{Y} \right\} \right] 
\cdot \exp\left\{ \frac{1}{\sqrt{4D}} \frac{\ln q'}{\mu} (J_1 + \cdots + J_N) + \frac{1}{\sqrt{4DY}} \frac{\ln q}{q'} (J_1 y_1 + \cdots + J_N y_N) \right\} \bigg|_{J=0}. \tag{22} \]

Moving the terms linear in \( J \) in the exponent to the left of the \( \frac{\partial}{\partial J_i} \) factors gives

\[ I_N = [-2\alpha(\alpha P - 1)b]^N \exp\left(-\frac{1}{4DY} \ln^2 \frac{q}{q'}\right) \left(\frac{4D}{\pi}\right)^{N/2} \prod_{i=1}^{N} \]

\[
\int dy_i \left[ \frac{1}{\sqrt{4D}} \frac{\ln q'}{\mu} + \frac{1}{\sqrt{4DY}} \frac{y_i}{q'} \frac{\ln q}{q'} + \frac{\partial}{\partial J_i} \right] \exp\left\{ \frac{1}{2} \sum_{j=i+1}^{N} J_i J_j \frac{y_j (Y - y_j)}{Y} \right\} \bigg|_{J=0}. \tag{23} \]

Now consider the expression

\[ P_k = \prod_{i=1}^{k} \int dy_i \frac{\partial}{\partial J_i} \exp\left\{ \frac{1}{2} \sum_{j=i+1}^{k} J_i J_j \frac{y_j (Y - y_j)}{Y} \right\} \bigg|_{J=0}. \tag{24} \]

\( P_k \) is non-zero only for even \( k \) in which case
\[ P_k = \frac{1}{2^{k/2}} \frac{1}{(\frac{k}{2})!} \int dy_1 \cdots dy_k \sum_{\text{Perm}} \prod_{i<j} y_i (Y - y_j) \]  

(25)\]

where the sum goes over all permutations of the pairs \(i\) and \(j\). Eq. 25 is readily evaluated as

\[ P_k \equiv \frac{1}{2^{k/2}} \frac{1}{(\frac{k}{2})!} \left[ \int_0^Y dy_2 \int_0^{y_2} dy_1 \frac{y_1 (Y - y_2)}{Y} \right]^{k/2} = \frac{1}{(\frac{k}{2})!} \left[ \frac{Y^3}{2 \cdot 4} \right]^{k/2}. \]  

(26)\]

Rewriting (23) as

\[ I_N = \left[ -2\alpha (\alpha_P - 1)b \right]^N \exp \left\{ -\frac{1}{4DY} \ell n^2 \frac{q}{q'} \right\} (4D)^{k/2} \sum_{k=0}^{N} \frac{1}{(N-k)!} \frac{1}{2} \frac{Y \ell n \frac{qq'}{\mu^2}}{\mu^2} \]  

we obtain

\[ I_N = \exp \left\{ -\frac{1}{4DY} \ell n^2 \frac{q}{q'} \right\} \sum_{k=0}^{N} \frac{1}{(N-k)!} \left[ -\alpha (\alpha_P - 1)b \right] Y \ell n \frac{qq'}{\mu^2} \left[ \frac{1}{2} \frac{Y^3}{\mu^2} \right]^{N-k} \cdot \frac{1}{(\frac{k}{2})!} \]  

\[ \cdot \left[ \frac{1}{3} \left( \alpha (\alpha_P - 1)b \right)^2 Y^3 \right]^{k/2} \]  

(28)\]

where the sum goes over even values of \(k\). Using (28) in (16) gives

\[ G(q, Y, q') = e^{E(q, Y, q')}/(2\pi qq'\sqrt{4\pi DY}) \]  

(29)\]

with

\[ E(q, Y, q') = (\alpha_P(qq') - 1)(1 - \frac{\alpha N_c}{4\pi} c(\frac{1}{2}))Y + \frac{D}{3} (\alpha (\alpha_P - 1)b)^2 Y^3 - \frac{\ell n^2 q/q'}{4DY} \]  

(30)\]

where we have used

\[ (\alpha_P(\mu^2) - 1)(1 - \alpha (\mu^2)b(u_0 + u)) = \alpha_P(qq') - 1. \]  

(31)\]
The \(-{(\alpha_P - 1)\frac{\alpha N}{4\pi}c(\frac{1}{2})}\) term in (31) gives the next-to-leading correction to the BFKL intercept\[3, 4\]. This term comes from the conformally invariant part of the next-to-leading kernel, as given by (8a) and (15a). The running coupling part of the next-to-leading kernel gives the scale of the leading BFKL intercept, the \(\alpha_P(qg') - 1 = \frac{4N_c}{\pi}n2\alpha(qg')\) term in (30), and also the non-Regge term, the \(\alpha^5Y^3\) term in (30). The appearance of a \(Y^3\) term in the exponent signals a breakdown\[5\] of the usual picture of high energy scattering where powers of \(Y\) and the factorial denominators associated with them come only from a strongly ordered region of longitudinal phase space. The \(\alpha^5Y^3\) term in (30) is, however, purely perturbative and the overall coefficient is reasonably small, \(\frac{\alpha}{\pi}(\alpha_P - 1)b^2Y^3 \approx 5\alpha^5Y^3\), so that this term is likely not too important for present phenomenology. We note that the expression (29), with \(E\) given in (30), obeys the consistency condition

\[G(q,Y,q') = \int d^2k G(q,y,k)G(k,Y-y,q'),\] (32)

and that the \(\alpha^5Y^3\) term in \(E\) is crucial for (32) to hold. Indeed, one can view (32) as requiring the \(\alpha^5Y^3\) term, exactly as given in (30), once one has arrived at the scale \(qq'\) as the appropriate scale for the leading BFKL intercept.

4 The anomalous dimension and coefficient functions at large orders[7-10]

Certain higher order corrections to the gluon anomalous dimension have already been calculated\[3\]. Here we do a much simpler calculation. We shall calculate the dominant large order terms at next-to-leading level for both the coefficient function and the anomalous dimension function. These large order terms are insensitive to the scheme in which the energy scale is introduced.

To determine the coefficient function \(C_\omega(\alpha)\) we take \(q = q' = \mu\) in (29) and write

\[\pi qq'G(q,Y,q')|_{q=q'=-\mu} = \int \frac{d\omega}{2\pi i} C_\omega e^{\omega Y}.\] (33)

Keeping only the first next-to-leading order term one gets
\[
e^{(\alpha_P - 1)Y} \frac{1 - \alpha(\alpha_P - 1)N_c}{4\pi} e^{\frac{1}{2}Y} = \int \frac{d\omega}{2\pi i} (C_{0,\omega} + C_{1,\omega}) e^{\omega Y}
\]

(34)

where

\[
C_{0,\omega} = \sum_{N=0}^{\infty} \left( \frac{\alpha N_c}{\pi \omega} \right)^{N} \frac{1}{\omega} C_{0}^{(N)}, \quad C_{1,\omega} = \sum_{N=0}^{\infty} \left( \frac{\alpha N_c}{\pi \omega} \right)^{N} C_{1}^{(N)}.
\]

(35)

The \( C_{0}^{(N)} \) are given by

\[
\frac{C_{0}^{(N)}}{N \to \infty} \frac{1}{\sqrt{N}} \left( \frac{\alpha_P - 1}{\alpha N_c/\pi} \right)^{N+\frac{1}{2}} \frac{1}{\sqrt{56\pi \zeta(3)}}
\]

(36a)

while a simple calculation, using (34), leads to

\[
\frac{C_{1}^{(N)}}{N \to \infty} = -\frac{N c_{\frac{1}{2}}}{4} C_{0}^{(N-1)}.
\]

(36b)

In order to determine the anomalous dimension function at next-to-leading level one can take \( q = \mu, q' = Q \) and evaluate (29) to first order in \( \ell \ln Q^2/\mu^2 \). One finds

\[
-\frac{e^{(\alpha_P - 1)Y}}{2\sqrt{4\pi DY}} \alpha(\alpha_P - 1)bY \ell \ln Q/\mu = \int \frac{d\omega}{2\pi i} [C_{\omega}(\gamma - \frac{1}{2})]_1 \ell \ln Q^2/\mu^2 e^{\omega Y}
\]

(37)

where \([C_{\omega}(\gamma - \frac{1}{2})]_1\) indicates the next-to-leading corrections to the product \( C_{\omega}(\gamma - \frac{1}{2}) \). That is

\[
[C_{\omega}(\gamma - \frac{1}{2})]_1 = \sum_{N=0}^{\infty} [C(\gamma - \frac{1}{2})]_1^{(N)} \left( \frac{\alpha N_c}{\pi \omega} \right)^N.
\]

(38)

Using (37) it is straightforward to determine

\[
[C(\gamma - \frac{1}{2})]_1^{(N)} = -\left( \frac{\alpha_P - 1}{\alpha N_c/\pi} \right)^{N-\frac{1}{2}} \frac{b\pi}{2N c} \sqrt{\frac{N}{56\pi \zeta(3)\pi}}
\]

(39)

which, using (35) and
\[
\gamma_{0,\omega} = \sum_{N=0}^{\infty} \gamma_{0}^{(N)} \left( \frac{\alpha N_c}{\pi \omega} \right)^N, \quad \gamma_{1,\omega} = \sum_{N=1}^{\infty} \gamma_{1}^{(N)} \frac{\alpha N_c}{\pi} \left( \frac{\alpha N_c}{\pi \omega} \right)^{N-1}
\]

along with \( \gamma_{0}^{(N)} = \frac{1}{N} C_{0}^{(N)} \), leads to

\[
\gamma_{1}^{(N)} \sim \infty - \frac{\pi b}{4 N_c} \left( \frac{\alpha N_c}{\alpha P - 1} \right)^{3/2} \sqrt{56 \pi \zeta(3)} N^{3/2} \gamma_{0}^{(N)} = - \frac{b \pi}{4 N_c} \left( \frac{\alpha P - 1}{\alpha N_c / \pi} \right)^{N-1}
\]

The correction for \( C_{1}^{(N)} \) given in (36b) simply corresponds to a shift in the branch point in \( C_{\omega} \) from \( C_{\omega} = [16 D(\omega - (\alpha P - 1))]^{-\frac{1}{4}} \) to \( C_{\omega} = \{16 D[\omega - (\alpha P - 1)(1 - \frac{\alpha N_c}{4 \pi c(\frac{1}{2})})]\}^{-\frac{1}{4}} \). The \( N^{3/2} \) factor in (41) indicates the start of the appearance of a non-Regge term. The fact that \( C_{1}^{(N)} / C_{0}^{(N)} \) and \( \gamma_{1}^{(N)} / \gamma_{0}^{(N)} \) are large\([13]\) at large \( N \) does not in itself signal a breakdown of perturbation theory since these terms are known exactly and, at least in the case of \( \gamma_{1}^{(N)} \), lead to corrections which are not particularly large.

5 Limitations on the use of perturbation theory

Because of the diffusion inherent in BFKL dynamics it is clear that the perturbative approach to high energy single-scale short distance behavior must breakdown at sufficiently high energy. Estimates of the energies at which that breakdown occurs were given in Ref.6. We are now in a position to see this breakdown in a more detailed way than has been done previously. The idea is simple. Perturbation theory should itself indicate when it is breaking down by generating factorial terms which indicate where the asymptotic expansion, the perturbative expansion, is reliable. Suppose we take the \( N = 1 \) term in (5), but now instead of taking the running coupling correction to be given by (8b) we work to arbitrary order in running coupling corrections

\[
\tilde{k}_{rc}(\lambda, q) = \frac{\alpha(\mu^2) N_c}{\pi} \chi(\lambda) \sum_{n=1}^{\infty} (-2b\alpha(\mu^2) \ell n q/\mu)^n.
\]
For simplicity we take $q = q' = \mu$ in evaluating $G$. Thus, we replace $I_1$ in (16), (18), and (21) with

$$\tilde{I}_1 = \sum_{n=1}^{\infty} I_{1,n}$$

(43)

where

$$I_{1,n} = \left[ -2\alpha(\mu^2)b^n(\alpha P - 1) \right] \sqrt{4\pi D} \int_0^Y dy_1 \int_{\infty}^{-\infty} dz_1 e^{-\frac{z_1^2}{4}} \left[ y_1 z_1 \sqrt{\frac{Y - y_1}{y_1 Y}} \right]^n .$$

(44)

A simple calculation leads to

$$I_{1,n} = \frac{(\alpha P - 1)Yn}{2(n + 1)} (4\alpha^2b^2DY)^{n/2} \Gamma\left(\frac{n}{2}\right)$$

(45)

for $n$ even, and $I_{1,n} = 0$ for $n$ odd. Of course the sum indicated in (43) does not converge, and this is not surprising. So long as $4b^2\alpha^2DY << 1$, $I_{1,n}$ is small for small values of $n$ and the divergence of the sum in (43) is a standard renormalon problem. One can keep those terms in $n$ so long as $I_{1,n+1} \leq 1$, with higher orders in $n$ being discarded. However, when $4\alpha^2b^2DY \approx 1$ we are not allowed to safely use any of the terms in the perturbation series given in (45), and thus the whole perturbative approach breaks down. The criterion for being able to use perturbation theory without too much contamination from infrared regions then is

$$4\alpha^2Db^2Y \leq 1$$

(46a)

or

$$Y \leq \frac{\pi}{14N_c\zeta(3)b^2\alpha^3(\mu^2)}$$

(46b)

exactly as found in Ref.6. $\mu$ characterizes the scale of the single-scale hard process in question.
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