On Equivariant Gromov–Witten Invariants of Resolved Conifold with Diagonal and Anti-Diagonal Actions

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Abstract

We propose two conjectural relationships between the equivariant Gromov-Witten invariants of the resolved conifold under diagonal and anti-diagonal actions and the Gromov-Witten invariants of $\mathbb{P}^1$, and verify their validity in genus zero approximation. We also provide evidences to support the validity of these relationships in genus one and genus two.

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1 Introduction

Let $g, d, n$ be non-negative integers, and

$$(\tau_{k_1}(\phi_{\alpha_1}) \cdots \tau_{k_n}(\phi_{\alpha_n}))_{g,d}^{\mathbb{P}^1}$$

be the Gromov–Witten (GW) invariants of $\mathbb{P}^1$ of genus $g$ and degree $d$, where $\alpha_1, \ldots, \alpha_n \in \{1, 2\}$ and the integers $k_1, \ldots, k_n \geq 0$. The generating function

$$F_{\mathbb{P}^1}(s; q; \epsilon) := \sum_{g \geq 0} \epsilon^{2g-2} \sum_{n,d \geq 0} \frac{s^{\alpha_1,k_1} \cdots s^{\alpha_n,k_n}}{n!} q^d \langle \tau_{k_1}(\phi_{\alpha_1}) \cdots \tau_{k_n}(\phi_{\alpha_n}) \rangle_{g,d}^{\mathbb{P}^1}$$ (1.1)
of these numbers is called the free energy of the GW invariants of \( \mathbb{P}^1 \). Here \( s := (s^{\alpha,k})_{\alpha=1,2;k\geq 0} \) is an infinite vector of indeterminates, \( \epsilon \) is an indeterminate called the string coupling constant, and summation over repeated upper and lower Greek indices is assumed. For the definition of GW invariants see \([2, 29, 30, 35, 38]\). The free energy has the following form of genus expansion:

\[
F_{g}^{\mathbb{P}^1}(s; q, \epsilon) = \sum_{g \geq 0} \epsilon^{2g-2} F_{g}^{\mathbb{P}^1}(s; q),
\]

(1.2)

where \( F_{g}^{\mathbb{P}^1}(s; q) \) is called the genus \( g \) free energy. The exponential of the free energy

\[
Z_{\mathbb{P}^1}(s; q, \epsilon) := e^{F_{0}^{\mathbb{P}^1}(s; q, \epsilon)}
\]

is called the partition function of the GW invariants of \( \mathbb{P}^1 \).

The restriction of the genus zero free energy to the small phase space yields the potential

\[
F_{0}^{\mathbb{P}^1} = \frac{1}{2} (v^1)^2 v^2 + q e^{v^2}
\]

(1.4)

of a two-dimensional Frobenius manifold \([9]\), which is called the \( \mathbb{P}^1 \)-Frobenius manifold denoted by \( M_{\mathbb{P}^1} \). We know from \([9]\) that the genus zero free energy \( F_{0}^{\mathbb{P}^1}(s; q) \) can be reconstructed in terms of the so-called topological solution to the Principal Hierarchy of the Frobenius manifold \( M_{\mathbb{P}^1} \) (see Section 2 for the definition of the Principal Hierarchy of a Frobenius manifold). Moreover, the higher genus free energies \( F_{g}^{\mathbb{P}^1}(s; q) \), \( g \geq 1 \), can be obtained by solving the loop equation of the Frobenius manifold \([21]\), and the partition function \( Z_{\mathbb{P}^1}(s; q; \epsilon) \) is a particular tau-function of the extended Toda hierarchy \([22]\) (cf. also \([6, 26, 33, 34, 39]\)).

Now let us consider the equivariant GW invariants of the target

\[
X = (\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)) \cdot \mathbb{T},
\]

where \( \mathbb{T} \simeq \mathbb{C}^* \) is the torus action with two characters \( (\kappa_1, \kappa_2) \) on the fibers. Denote by

\[
\langle \tau_{k_1}(\phi_{\alpha_1}) \cdots \tau_{k_n}(\phi_{\alpha_n}) \rangle_{g,d}^{X,\text{di/ad}}
\]

the genus \( g \) and degree \( d \) equivariant GW invariants of \( X \) under diagonal or anti-diagonal action, which corresponds respectively to the case with \( \kappa_1 = 1 \) or to the case \( \kappa_1 = -1 \) (cf. \([1]\)). Define the partition function of the equivariant GW invariants of \( X \) with diagonal or anti-diagonal action by

\[
Z_{X,\text{di/ad}}(t; q; \epsilon) = e^{F_{X,\text{di/ad}}(t; q; \epsilon)},
\]

(1.5)

where \( t = (t^{\alpha,k})_{\alpha=1,2;k\geq 0} \), and

\[
F_{X,\text{di/ad}}(t; q; \epsilon) := \sum_{g \geq 0} \sum_{d \geq 0} \sum_{n \geq 0} \frac{t^{\alpha_1,k_1} \cdots t^{\alpha_n,k_n}}{n!} \langle \tau_{k_1}(\phi_{\alpha_1}) \cdots \tau_{k_n}(\phi_{\alpha_n}) \rangle_{g,d}^{X,\text{di/ad}}
\]

(1.6)

is the free energy of the equivariant GW invariants of \( X \) with diagonal or anti-diagonal action. The restrictions of the genus zero free energies to the small phase space yield the primary genus zero free energies \([3, 4]\)

\[
F_{X,\text{di}} = \frac{1}{2} (u^1)^2 u^2 + \frac{1}{3} (u^1)^3 + \text{Li}_3(q e^{u^2}),
\]

(1.7)
which serve as potentials of two Frobenius manifolds. Here \( \text{Li}_3 \) denotes a special polylogarithmic function, i.e.,
\[
\text{Li}_k(z) := \sum_{n \geq 1} \frac{z^n}{n^k}, \quad k \in \mathbb{Z}.
\]
We denote these two Frobenius manifolds by \( M_{X,di} \) and \( M_{X,ad} \) respectively, and we note that these two Frobenius manifolds do not possess Euler vector fields. Before proceeding, we introduce the following notations that will be used later:
\[
b_{1,n}^{\alpha,m,di} = \frac{n^m}{n!} (\delta^{\alpha,1} + 2m\delta^{\alpha,2}), \quad b_{2,n}^{\alpha,m,di} = \frac{(n+1)^m}{n!} \delta^{\alpha,2},
\]
\[
b_{1,n}^{\alpha,m,ad} = \frac{(-1)^{n-1}n^m}{n!} (\delta^{\alpha,1} + (2m+1)\delta^{\alpha,2}), \quad b_{2,n}^{\alpha,m,ad} = \frac{(-1)^n(n+1)^m}{n!} \delta^{\alpha,2},
\]
where \( \alpha = 1, 2 \) and \( m, n \) are non-negative integers.

Let us proceed and propose two conjectural relationships between the equivariant GW invariants of the resolved conifold under diagonal and anti-diagonal actions and the GW invariants of \( \mathbb{P}^1 \) via the theory of Frobenius manifold.

For the diagonal case, the conjectural relationship can be interpreted by observing the almost duality between the Frobenius manifolds \( M_{\mathbb{P}^1} \) and \( M_{X,di} \). From the definition of almost duality introduced by Dubrovin in [10] it follows that the potential of the almost dual \( \hat{M}_{\mathbb{P}^1} \) of \( M_{\mathbb{P}^1} \) coincides with (1.7) if we choose the flat coordinates of \( \hat{M}_{\mathbb{P}^1} \) as follows:
\[
\begin{align*}
\alpha_1 &= \log \left( v^1 + \sqrt{(v^1)^2 - 4qe^{v^2}} \right) - \log 2, \\
\alpha_2 &= \log \left( v^2 - 2\sqrt{(v^1)^2 - 4qe^{v^2}} \right) + 2\log 2.
\end{align*}
\]

For the definition of \( \hat{M}_{\mathbb{P}^1} \) see Section 2. It then follows from a general principle (cf. Lemma (2.1) of Section 2) that tau-symmetric hamiltonian densities of the Principal Hierarchy of \( \hat{M}_{\mathbb{P}^1} \) are linear combinations of those of the Principal Hierarchy of \( M_{\mathbb{P}^1} \). Motivated by the Hodge-GUE correspondence [14, 17] (cf. also [21, 24]), we then propose an explicit conjectural relationship between \( Z_{X,di}(t; q; \epsilon) \) and \( Z_{\mathbb{P}^1}(s; q; \epsilon) \).

**Conjecture 1.1** The identity
\[
Z_{\mathbb{P}^1}(s; q; \epsilon) = \exp \left( \frac{A_{di}(s)}{\epsilon^2} \right) Z_{X,di}(t_{di}(s); q; \epsilon)
\]
holds true in
\[
V_{di,\epsilon} := \mathbb{C}(\epsilon) \otimes \mathbb{C}[[s^{1,0}, s^{2,0}, s^{1,1}, s^{2,1}, \ldots ; q]].
\]
Here \( A_{di}(s) \) and \( t_{di}(s) \) are defined by
\[
A_{di}(s) := \sum_{k, \ell \geq 0} \frac{s^{1,k}s^{2,\ell}}{(k + \ell + 1)k!\ell!} - \sum_{k \geq 0} \frac{s^{2,k}}{(k + 2)k!},
\]
\[
t_{di}^{\alpha,k}(s) := \sum_{\ell \geq 0} s^{\beta,\ell} s^{\beta,\ell} - b_{1,1}^{\alpha,k,di} + \delta^{\alpha,1}\delta^{k,1}, \quad \alpha = 1, 2, k \geq 0.
\]
Let us now consider the anti-diagonal case. It was found in [3, 4] (see also [11, 12]) that the function \( F_{X, \text{ad}} \) is the potential of the almost dual \( \widehat{M}_{\text{AL}} \) of the Frobenius manifold \( M_{\text{AL}} \) associated with the Ablowitz–Ladik (AL) hierarchy. The potential of \( M_{\text{AL}} \) is given by

\[
F_{\text{AL}} = \frac{1}{2} (t_1)^2 t_2^2 + q t_1 e^{t_2} + \frac{1}{2} (t_1)^2 \log t_1,
\]

(1.18)

where \( t_1, t_2 \) are related to \( u_1, u_2 \) by

\[
t_1 = \sqrt{-1} e^{u_1} (1 - q e^{u_2}), \quad t_2 = u_1 + u_2.
\]

(1.19)

In [3] Brini conjectured that the generating function of the equivariant GW invariants of \( X \) with anti-diagonal action is the logarithm of a tau-function of the AL hierarchy. The main motivation that leads to his conjecture is the fact that the quasi-trivial transformation of the AL equation gives the genus expansion of the free energy of \( X \) with anti-diagonal action up to genus one, as well as some evidences of validity of his conjecture at genus two. According to the general approach of Dubrovin–Zhang (DZ) relating GW invariants with integrable hierarchies, the anti-diagonal equivariant GW invariants of \( X \) should be related with the topological deformation of the Principal Hierarchy of \( \widehat{M}_{\text{AL}} \). More precisely, the partition function of the anti-diagonal equivariant GW invariants of \( X \) should be a tau-function of the DZ hierarchy of \( \widehat{M}_{\text{AL}} \), and the genus expansion of the free energy of \( X \) with anti-diagonal action should be given by the quasi-trivial transformation relating the Principal Hierarchy of \( \widehat{M}_{\text{AL}} \) with this DZ hierarchy.

In Brini’s approach [3], he used the quasi-trivial transformation of the AL equation to arrive at the conjectural genus expansion of the free energy of \( X \) with anti-diagonal action. The reason, to our understanding, is that the AL equation should be obtained by the Miura transformation of a certain non-trivial infinite linear combination of the flows of the DZ hierarchy of \( \widehat{M}_{\text{AL}} \) (cf. Lemma 2.1), so the flow given by the AL equation commutes with the flows of the DZ hierarchy of \( \widehat{M}_{\text{AL}} \), thus the quasi-trivial transformation of the AL hierarchy coincides with that of the DZ hierarchy of \( \widehat{M}_{\text{AL}} \). On the other hand, the Frobenius manifold \( M_{\text{AL}} \) is related to the Frobenius manifold \( M_{\text{P}^1} \) by a Legendre-type transformation (see Appendix B of [2]):

\[
v_1 = \frac{\partial^2 F_{\text{AL}}}{\partial t_1^1 \partial t_2^2}, \quad v_2 = \frac{\partial^2 F_{\text{AL}}}{\partial t_1^1 \partial t_1^1},
\]

and

\[
\frac{\partial^2 F_{\text{P}^1}}{\partial v^\alpha \partial v^\beta} = \frac{\partial^2 F_{\text{AL}}}{\partial t_1^\alpha \partial t_1^\beta}, \quad \alpha, \beta = 1, 2.
\]

Since the tau-functions of the DZ hierarchies of \( M_{\text{AL}} \) and \( M_{\text{P}^1} \) are equivalent in a certain sense, (cf. [6]; see also [37]), it is then more convenient for us to use the partition function of \( M_{\text{P}^1} \) to give an explicit relationship between the equivariant GW invariants of \( X \) with anti-diagonal action and the extended Toda hierarchy which governs the GW invariants of \( \text{P}^1 \) [22, 26, 39]. Motivated by Brini’s conjecture, by the Hodge-GUE correspondence [14], and by the above-mentioned Legendre-type transformation between \( M_{\text{AL}} \) and \( M_{\text{P}^1} \), we are to propose an explicit conjectural relationship between \( Z_{X, \text{ad}}(t; q; \epsilon) \) and \( Z_{\text{P}^1}(s; q; \epsilon) \) in Conjecture 1.2.

**Conjecture 1.2** The identity

\[
Z_{\text{P}^1}(s; q; \epsilon) = \exp \left( \frac{A_{\text{ad}}(s)}{\epsilon^2} \right) Z_{X, \text{ad}}(t_{\text{ad}}(s); q; \sqrt{-1} \epsilon)
\]

(1.20)
where \( F \) anti-diagonal action.

We would like to thank Paolo Lorenzoni for helpful discussions on topological recursion relations. In Section 5, we give some further remarks.

From the definitions (1.13) and (1.15) we see that the conjectural identity (1.14) is equivalent to the following identities:

\[
\sum_{k,\ell \geq 0} \frac{(\sqrt{-1})^{k+\ell+1}}{(k+\ell+1)k!} s^{1,k} s^{2,\ell} + \sum_{k \geq 0} \frac{(\sqrt{-1})^{k}}{(k+2)k!} s^{2,k},
\]

(1.22)

\[
\epsilon_{\alpha,k} := \sum_{\ell \geq 0} \epsilon_{\alpha,k,\ell}, \quad \epsilon_{1,1} = 0, \quad \epsilon_{1,2} = 0, \quad \epsilon_{2,1} = 0, \quad \epsilon_{2,2} = 0.
\]

(1.23)

**Remark 1.3** Since we know that \( F_{g} = \log Z_{g}(s; q) \) belongs to the space \( \mathbb{C}(s) \otimes \mathbb{C}[s^{1,0}, s^{2,0}, s^{1,1}, s^{2,1}, \ldots] \), the conjectural identities (1.14) and (1.20) imply that the logarithms of the right-hand sides belong to this space.

From the definitions (1.3) and (1.5) we see that the conjectural identity (1.14) is equivalent to the following identities:

\[
\mathcal{F}_{g}^{p1}(s; q) = \mathcal{F}_{g}^{X,di}(t_{di}(s); q) + A_{di}(s)\delta_{g,0}, \quad g \geq 0,
\]

(1.24)

\[
\mathcal{F}_{g}^{di}(s; q) = (-1)^{g-1}\mathcal{F}_{g}^{X,ad}(t_{ad}(s); q) + A_{ad}(s)\delta_{g,0}, \quad g \geq 0,
\]

(1.25)

where \( \mathcal{F}_{g}^{X,di}(t; q) \) denotes the genus \( g \) free energy of the equivariant GW invariants of \( X \) with diagonal action. Similarly, the conjectural identity (1.20) is equivalent to

For \( g = 0 \), there are explicit expressions for both sides of the conjectural identities (1.24) and (1.25). We will verify the validity of these identities in Section 3.

For \( g \geq 1 \), the verification is more involved because, as far as we know, there are no efficient algorithms to compute the right-hand sides of (1.24) and (1.25). However, since the free energies \( \mathcal{F}_{g}^{X,di}(t; q) \), \( \mathcal{F}_{g}^{X,ad}(t; q) \) and \( \mathcal{F}_{g}^{p1}(s; q) \) can be represented in terms of the so-called jet variables \([7, 21, 22, 24]\), from the conjectural identities (1.24) and (1.25) we can determine the functions \( \mathcal{F}_{g}^{X,di}(t; q), \mathcal{F}_{g}^{X,ad}(t; q) \) in terms of the function \( \mathcal{F}_{g}^{p1}(s; q) \). We are then to give supports of the validity of the identities (1.24) and (1.25) for \( g = 1,2 \) by proving that the functions \( \mathcal{F}_{g}^{X,ad}(t; q) \) and \( \mathcal{F}_{g}^{X,di}(t; q) \) determined by these identities satisfy the genus one and genus two topological recursion relations given in [1, 27, 28].

The paper is organized as follows. In Section 2 we recall the definition of the Principal Hierarchy of a Frobenius manifold. In Section 3 we give a proof of the identities (1.24) and (1.25) for \( g = 0 \). In Section 4 we derive the expressions of the free energies \( \mathcal{F}_{g}^{X,di/ad}(t; q) \) in terms of \( \mathcal{F}_{g}^{p1}(s; q) \) for \( g = 1,2 \), and show that they satisfy the genus one and genus two topological recursion relations. In Section 5 we give some further remarks.

**Acknowledgements** We would like to thank Paolo Lorenzoni for helpful discussions on Lemma 2.4. This work is partially supported by NSFC No.12171268, No.11725104 and No.12061131014.
2 The Principal Hierarchy of a Frobenius manifold

In this section, we recall the definitions of the Principal Hierarchy and almost duality of a Frobenius manifold (for details see [9, 10, 21]).

Let \((M, \cdot, \eta, e, E)\) be an \(n\)-dimensional Frobenius manifold of charge \(d\), where \(\cdot\) denotes the operation of multiplication on the tangent spaces of \(M\), \(\eta\) is the invariant flat metric, \(e\) is the unit vector field and \(E\) is the Euler vector field. Fix a system of flat coordinates \(v^1, \ldots, v^n\) of the metric \(\eta\) such that \(e = \frac{\partial}{\partial v^1}\). In these coordinates we have

\[
\frac{\partial}{\partial v^\alpha} \cdot \frac{\partial}{\partial v^\beta} = c^\gamma_{\alpha\beta}(v) \frac{\partial}{\partial v^\gamma}.
\]

Here and below, free Greek indices always take \(1, \ldots, n\). Denote \(\eta_{\alpha\beta} = \eta(\frac{\partial}{\partial v^\alpha}, \frac{\partial}{\partial v^\beta})\), \(c_{\alpha\beta\gamma}(v) := \eta_{\alpha\sigma} c^\sigma_{\beta\gamma}\), then the potential \(F(v) = F(v^1, \ldots, v^n)\) of the Frobenius manifold \(M\) is defined by

\[
c_{\alpha\beta\gamma} = \frac{\partial^3 F(v)}{\partial v^\alpha \partial v^\beta \partial v^\gamma}.
\]

An important geometric object of a Frobenius manifold is the deformed flat connection \(\tilde{\nabla}\) defined by

\[
\tilde{\nabla}_a b = \nabla_a b + za \cdot b, \quad a, b \in TM.
\] (2.1)

One can find a system of deformed flat coordinates of the form \((\tilde{v}_1(v; z), \ldots, \tilde{v}_n(v; z)) = (\theta_1(v; z), \ldots, \theta_n(v; z))z^R\) which satisfy the equations

\[
\tilde{\nabla} d\tilde{v}_\alpha(v; z) = 0.
\] (2.2)

Here the constant matrices \(\mu = \text{diag}(\mu_1, \ldots, \mu_n)\), \(R = R_1 + \cdots + R_m\) are the monodromy data of the Frobenius manifold at \(z = 0\), and \(\theta_\alpha(v; z)\) have the expressions

\[
\theta_\alpha(v; z) = \sum_{k \geq 0} \theta_{\alpha,k}(v)z^k.
\] (2.3)

In terms of the functions \(\theta_{\alpha,p}\), the equations (2.2) can be rewritten as

\[
\frac{\partial^2 \theta_{\gamma,p+1}}{\partial v^\alpha \partial v^\beta} = \frac{\xi}{\gamma_{\alpha\beta}} \frac{\partial \theta_{\gamma,p}}{\partial v^\gamma}, \quad p \geq 0.
\] (2.4)

We can also require that these deformed flat coordinates satisfy the following normalization conditions:

\[
\theta_\alpha(v; 0) = \eta_{\alpha\beta} v^\beta, \quad \frac{\partial \theta_\alpha(v; z)}{\partial v^1} = z \theta_\alpha(v; z) + \eta_1, \quad \frac{\partial \theta_{\alpha}(v; z)}{\partial v^\gamma} \eta^{\alpha\sigma} \frac{\partial \theta_\alpha(v; z)}{\partial v^\sigma} = \eta_{\alpha\beta}, \quad \text{with } (\eta^{\alpha\beta}) := (\eta_{\alpha\beta})^{-1},
\] (2.5) (2.6)
and the quasi-homogeneity condition

$$
\partial_v \theta_{\alpha, p}(v) = \left( p + \frac{2-d}{2} + \mu_\alpha \right) \theta_{\alpha, p}(v) + \sum_{k=1}^{p} \theta_{\xi, p-k}(v) (R_k)_{\alpha}^\xi + \text{constant.} \tag{2.7}
$$

For the Frobenius manifold $M_{P1}$, we have

$$
\mu_1 = -\mu_2 = -\frac{1}{2}, \quad R = R_1 = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}.
$$

From the conditions (2.5)–(2.7) it follows that the functions $\theta_\alpha(v; z)$ can be chosen as \cite{21,22}

$$
\begin{align*}
\theta_{1}^{p}(v; q; z) &= -2e^{zv^1} \left( K_0 \left( 2z\sqrt{qe^{v^2}} \right) + (\log z + \gamma + \frac{1}{2} \log q)I_0 \left( 2z\sqrt{qe^{v^2}} \right) \right) \\
&= -2e^{zv^1} \sum_{k \geq 0} \left( \gamma - \frac{1}{2}v^2 + \psi(k + 1) \right) q^k e^{kv^1} \frac{z^{2k}}{(k!)^2}, \\
\theta_{2}^{p}(v; q; z) &= z^{-1}e^{zv^1} I_0 \left( 2z\sqrt{qe^{v^2}} \right) - z^{-1} \\
&= z^{-1} \left( \sum_{k \geq 0} q^k e^{kv^2 + zv^1} \frac{z^{2k}}{(k!)^2} - 1 \right),
\end{align*}
$$

where $I_0$ and $K_0$ are the modified Bessel functions of the first and second kinds, $\gamma$ is the Euler’s constant, $\psi(z)$ is the digamma function, and we also indicate the dependence of the functions $\theta_{\alpha, p}$ on the parameter $q$.

The Principal Hierarchy of $M$ is defined as the following system of evolutionary Hamiltonian PDEs of hydrodynamic type:

$$
\frac{\partial v^\alpha}{\partial s^{p+1}} = \eta^{\alpha \gamma} \partial_x \frac{\partial \theta_{\beta, p+1}}{\partial v^\gamma}, \quad p \geq 0. \tag{2.10}
$$

Since the flow $\frac{\partial}{\partial s^{1,0}}$ reads

$$
\frac{\partial v^\alpha}{\partial s^{1,0}} = \frac{\partial v^\alpha}{\partial x},
$$

we identify the variable $s^{1,0}$ with the spatial variable $x$. It is shown in \cite{9} that the Principal Hierarchy is integrable. In particular, the Hamiltonian densities $\theta_{\alpha, p}$, $p \geq 0$, are conserved densities for each flow of the Principal Hierarchy, and therefore they satisfy the following second order linear PDEs \cite{21}:

$$
c_{\rho \alpha}^\beta \frac{\partial^2 h}{\partial v^\beta \partial v^\alpha} = c_{\rho \alpha}^\sigma \frac{\partial^2 h}{\partial v^\beta \partial v^\sigma}. \tag{2.11}
$$

The topological solution $v_{\text{top}}(s)$ to the Principal Hierarchy is specified by the initial condition

$$
v_{\text{top}}^\alpha \big|_{s^{3, k} = \eta^{3,1} \delta^k, x} = \delta^{\alpha, 1} x. \tag{2.12}
$$
It can be solved uniquely from the following genus zero Euler–Lagrange equation:

\[ \sum_{p \geq 0} \tilde{s}^{\alpha,p} \frac{\partial \theta_{\alpha,p}(v)}{\partial v^\beta} = 0, \]  

(2.13)

where \( \tilde{s}^{\alpha,p} = s^{\alpha,p} - \delta^{\alpha,1} \delta^{p,1} \).

We call the following power series

\[ F_0 = \frac{1}{2} \sum_{k,\ell \geq 0} \tilde{s}^{\alpha,k} \tilde{s}^{\beta,\ell} \Omega_{\alpha,k;\beta,\ell}(v_{\text{top}}(s)) \in \mathbb{C}[[s]] \]  

(2.14)

the genus zero free energy of the Frobenius manifold, where the functions \( \Omega_{\alpha,k;\beta,\ell}(v) \) are defined by

\[ \sum_{k,\ell \geq 0} \Omega_{\alpha,k;\beta,\ell}(v) z^k w^\ell := \eta(\nabla \theta_{\alpha}(v; z), \nabla \theta_{\beta}(v; w)) - \eta_{\alpha\beta}. \]  

(2.15)

Now let us recall the definition of the almost dual \( \hat{M} \) of a Frobenius manifold \( M \) introduced in [10]. Let \( g \) be the intersection form of \( M \) defined by

\[ g(v)(a,b) := i_E(a \cdot b), \quad a, b \in T^*_v M, \]  

(2.16)

where the multiplication operation on the cotangent space is induced from the one defined on tangent space by the metric \( \eta \). Then we can introduce a Frobenius manifold structure on \( \hat{M} = M \setminus \{ v \in M \mid g(v) \text{ is degenerate on } T^*_v M \} \) by defining a new product

\[ a \ast b = E^{-1} \cdot a \cdot b, \quad a, b \in T_v M. \]  

(2.17)

The flat metric and the unit vector field of \( \hat{M} \) are given by \( \hat{\eta} = g^{-1} \) and \( E \) respectively. We note that on the almost dual \( \hat{M} \) there is no Euler vector field in general. Choose a system of local flat coordinates \( u^1, \ldots, u^n \) for the metric \( \hat{\eta} \), then there exists a function \( \hat{F} \) satisfying the equations

\[ \frac{\partial^2 \hat{F}}{\partial u^\alpha \partial w^\beta \partial u^\gamma} = \hat{\eta}_{\alpha\xi} \hat{\eta}_{\beta\xi} \frac{\partial u^\xi}{\partial u^\sigma} \frac{\partial u^\xi}{\partial v^\rho} \frac{\partial v^\delta}{\partial u^\gamma} \frac{\partial v^\delta}{\partial u^\rho}. \]  

(2.18)

We call \( \hat{F} \) the almost dual potential. The structure constants for \( \hat{M} \) will be denoted by \( \hat{c}^\beta_{\alpha\gamma} \).

The following lemma which connects the conserved densities of the Principal Hierarchies of \( M \) and \( \hat{M} \) will be used later.

**Lemma 2.1** If \( h(v) \) is a conserved density for the Principle Hierarchy of \( M \), then \( h(v(u)) \) is a conserved density of the Principle Hierarchy of \( \hat{M} \).

**Proof** By using (2.11), (2.18) and the fact that \( u^1, \ldots, u^n \) are the flat coordinates of \( \hat{\eta} \), it can be verified that \( h \) satisfies the following second order PDEs:

\[ \hat{c}^\beta_{\rho\sigma} \frac{\partial^2 h(v(u))}{\partial w^\beta \partial u^\sigma} = \hat{c}^\beta_{\rho\sigma} \frac{\partial^2 h(v(u))}{\partial w^\beta \partial u^\alpha}. \]  

(2.19)

The lemma is then proved. \( \square \)
As we mentioned in the Introduction, the almost dual $\widehat{M}_{p_1}$ of the Frobenius manifold $M_{p_1}$ coincides with the Frobenius manifold associated with the resolved conifold $X$ with diagonal action. It has the potential $\hat{F}^{p_1} = F^{X,\text{di}}$ given by (1.7), and the flat coordinates $u^1, u^2$ are given by (1.12), (1.13).

Denote $\alpha_k := a(a+1) \cdots (a+k-1)$. Recall that the deformed flat coordinates for $X$ with diagonal action can be chosen as follows:

$$\theta_{1,\text{di}}^X(u; q; z) = e^{zu^1} \sum_{d \geq 0} q^d e^{du^2} \frac{(-z)^d d(-z)}{d!^2} \left( u^2 + 2z^{-1} - 2 \sum_{k=1}^{d} \frac{z^k + 1}{k(z - k + 1)} \right) - 2z^{-1},$$

(2.20)

$$\theta_{2,\text{di}}^X(u; q; z) = z^{-1}e^{zu^1} \sum_{d \geq 0} q^d e^{du^2} \frac{(-z)^d d(-z)}{d!^2} - z^{-1}.$$

(2.21)

It can be verified that $\theta^X_{\alpha,k}(u; q; z)$ coincide with the calibrations given by Brini in [3]. Obviously, $\theta_{\alpha,k}^X(u; q)$ belong to $\mathbb{Q}[u^1, u^2, e^{u^2}] [[q]]$. More precisely, they have the form

$$\theta_{1,k}^X = \sum_{d \geq 0} \sum_{0 \leq i \leq |k|/2} \sum_{0 \leq j \leq |k|/2} c_{i,j,k} (u^1)^{k-2i} (q e^{u^2})^d + u^2 \sum_{d \geq 0} \sum_{0 \leq i \leq |k|/2} a_{i,k} (u^1)^{k-2i} (q e^{u^2})^d,$$

(2.22)

$$\theta_{2,k}^X = \sum_{d \geq 0} \sum_{0 \leq i \leq |k+1|/2} a_{i,k} (u^1)^{k+1-2i} (q e^{u^2})^d,$$

(2.23)

where $a_{i,k}, c_{i,k} \in \mathbb{Q}$. We note that $c_{i,k}$ vanish when $d < i$.

We also mentioned in the Introduction that the almost dual $\widehat{M}_{AL}$ of the Frobenius manifold $M_{AL}$ associated with the Ablowitz-Ladik hierarchy coincides with the Frobenius manifold associated with the resolved conifold $X$ with anti-diagonal action. The almost dual potential is given by (1.8), and the relation of the flat coordinates of the Frobenius manifold $\widehat{M}_{AL}$ with those of $M_{AL}$ is given by (1.19). The deformed flat coordinates can be chosen as follows:

$$\theta_{1,\text{ad}}^{\text{X}}(u; q; z) = e^{zu^1} \sum_{d \geq 0} q^d e^{du^2} \frac{z d(-z)}{d!^2} \left( u^2 + 2 \sum_{k=0}^{d-1} \frac{k + z^2}{(k+1)(k^2 - z^2)} \right),$$

(2.24)

$$\theta_{2,\text{ad}}^{\text{X}}(u; q; z) = z^{-1}e^{zu^1} \sum_{d \geq 0} q^d e^{du^2} \frac{z d(-z)}{d!^2} - z^{-1}.$$

(2.25)

It can also be verified that $\theta^{\text{X},\text{ad}}_{\alpha,k}(u; q; z)$ coincide with the calibrations given by Brini in [3]. Obviously, $\theta^{\text{X},\text{ad}}_{\alpha,k}(u; q)$ belong to $\mathbb{Q}[u^1, u^2, e^{u^2}] [[q]]$, and they have the form

$$\theta_{1,k}^{\text{X},\text{ad}} = \sum_{d \geq 0} \sum_{0 \leq i \leq |k|/2} c_{i,k} (u^1)^{k-2i} (q e^{u^2})^d + u^2 \sum_{d \geq 0} \sum_{0 \leq i \leq |k|/2} a_{i,k} (u^1)^{k-2i} (q e^{u^2})^d,$$

(2.26)

$$\theta_{2,k}^{\text{X},\text{ad}} = \sum_{d \geq 0} \sum_{0 \leq i \leq |(k+1)/2|} a_{i,k} (u^1)^{k+1-2i} (q e^{u^2})^d,$$

(2.27)

where $a_{i,k}, c_{i,k} \in \mathbb{Q}$. We note that $c_{i,k}, a_{i,k}$ vanish when $d < i$. 

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In what follows, we will use \( \Omega_{P_1}^{\alpha,k;\beta,\ell} \), \( \Omega_{X,di}^{\alpha,k;\beta,\ell} \), and \( \Omega_{Ad,di}^{\alpha,k;\beta,\ell} \) to denote the genus zero two-point correlation functions defined by (2.15) for the Frobenius manifolds \( M_{P_1}, \hat{M}_{P_1} \), and \( \hat{M}_{AL} \). We also denote by \( v_{top}(s;q) \), \( u_{top,di}(t;q) \) and \( u_{top,ad}(t;q) \) the topological solutions to the Principal Hierarchies of these three Frobenius manifolds.

3 Verification for the genus zero case

In this section, we will verify the genus zero part of Conjecture 1.1.

3.1 The diagonal case

Let us first prove the following lemma.

**Lemma 3.1** Under the coordinate transformation

\[
v^1 = e^{u_1} \left( 1 + qe^{u_2} \right), \quad v^2 = 2u^1 + u^2,
\]

the following identities hold true:

\[
\theta_{P_1}^{\alpha,k}(v;q) = \sum_{\ell \geq 0} b_{\beta,\ell}^{\alpha,k} \theta_{X,di}^{\beta,\ell}(u(v;q);q) + \frac{\delta_{\alpha,2}}{(k+1)!}, \quad k \geq 0.
\]

Here the coefficients \( b_{\beta,\ell}^{\alpha,k} \) are given by (1.10).

**Proof** From the expression (1.10) for \( b_{\beta,\ell}^{\alpha,k} \), we know that the identity (3.2) can be equivalently written as

\[
\theta_{P_1}^{\alpha,k}(v(q);q) = \frac{1}{k!} \theta_{X,di}^{\alpha,k}(u(q);k) + \frac{2z}{k!} \left. \theta_{X,di}^{\alpha,k}(u(q);z) \right|_{z=k},
\]

where

\[
\theta_{P_1}^{\alpha,1}(v(q);q) = \frac{1}{k!} \theta_{X,di}^{\alpha,1}(u(q);k+1) + \frac{1}{(k+1)!}.
\]

We know from Lemma 2.1 that both sides of (3.4) satisfy the second-order linear PDE (2.19). By using the formulae (2.9) and (3.1) as well as (2.21), we know that both sides of (3.4) belong to \( e^{(k+1)u^1} \mathbb{C}[qe^{u^2}] \). It then follows from an elementary exercise that the identity (3.4) can be reduced to a finite number of simple identities. This proves the validity of (3.4). In a similar way we can prove (3.3). The lemma is proved. □

Let us now prove the following two lemmas.

**Lemma 3.2** For all \( k, \ell \geq 0 \), we have

\[
\Omega_{P_1}^{\alpha,k;\beta,\ell}(v;q) = \sum_{m,n \geq 0} b_{\alpha,k}^{\beta,m,\ell}^{\alpha,n} \Omega_{X,di}^{\beta,m,\ell;\alpha,n}(u(v;q);q) + \frac{n_{\alpha,\beta}}{(k+\ell+1)k!}.\]

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Proof Let us first prove (3.5) for \((\alpha, \beta) = (2, 2)\). From (2.8), (2.9), we know that the formula (2.7) for \(\mathbb{P}^1\) reads
\[
E^\theta_\alpha = (k + \alpha - 1)\theta_\alpha + 2\delta \theta_{\alpha-1}.
\] (3.6)
Then by using the formula (2.15), one can verify that
\[
E \left( \eta(\nabla^\theta(v; q; z_1), \nabla^\theta(v; q; z_2)) \right)
\]
\[
= z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2} + 2 \left( \eta(\nabla^\theta(v; q; z_1), \nabla^\theta(v; q; z_2)) \right)
\]
\[
= \sum_{k, \ell \geq 0} (k + \ell + 2) \Omega_{2, k; 2, \ell} z_1^k z_2^\ell.
\] (3.8)
On the other hand, it follows from (2.4) that (3.7) can also be written as
\[
\frac{E^\theta}{z_1 + z_2} \left( \partial^2 \theta^\alpha_\beta(v; q; z_1) \eta_{\alpha\beta} \partial \theta^\beta_\alpha(v; q; z_2) + \partial \theta^\alpha_\beta(v; q; z_1) \eta_{\alpha\beta} \partial \theta^\beta_\alpha(v; q; z_2) \right)
\]
\[= \eta_{\alpha\beta} \partial \theta^\alpha_\beta(v; q) \partial \theta^\beta_\alpha(v; q; z_2),
\]
where the coefficients \(g^\alpha_\beta\) are given by the intersection form (2.16) for \(\mathbb{P}^1\):
\[
\begin{pmatrix}
g^{\alpha_\beta} \\
\end{pmatrix} = \begin{pmatrix}
2q^{e_{v^2}} & v^1 \\
v^1 & 2
\end{pmatrix}.
\] (3.9)
Hence we obtain that
\[
\Omega_{2, k; 2, \ell}^\alpha(v; q) = \frac{1}{k + \ell + 2} \left( \partial \theta^\alpha_\beta(v; q) \partial \theta^\beta_\alpha(v; q; z_1) \right),
\]
\[
\Omega_{2, k; 2, \ell}^\alpha(v; q) = \frac{1}{k + \ell + 2} \left( \partial \theta^\alpha_\beta(v; q) \partial \theta^\beta_\alpha(v; q; z_2) \right),
\]
(3.10)
It follows from (3.10) and Lemma 3.1 that
\[
\Omega_{2, k; 2, \ell}^\alpha(v; q) = \frac{1}{k + \ell + 2} \left( \partial \theta^\alpha_\beta(v; q) \partial \theta^\beta_\alpha(v; q; z_1) \right),
\]
\[
\Omega_{2, k; 2, \ell}^\alpha(v; q) = \frac{1}{k + \ell + 2} \left( \partial \theta^\alpha_\beta(v; q) \partial \theta^\beta_\alpha(v; q; z_2) \right),
\]
Here, to derive the last equality we used the definition (2.13) for \(X\) with diagonal action. So the identities (3.5) hold true for \((\alpha, \beta) = (2, 2)\).
Next let us prove (3.5) for \((\alpha, \beta) = (1, 2)\). Similar to the proof of (3.10), from the formulae (2.4), (2.7) and (2.13) it follows that
\[
\Omega_{1, k; 2, \ell}^{\alpha_\beta}(v; q) = \frac{g^{\alpha_\beta}}{k + \ell + 1} \left( \partial \theta^\alpha_\beta(v; q) \partial \theta^\beta_\alpha(v; q; z_1) \right) - \frac{2g^{\alpha_\beta}}{(k + \ell + 1)^2} \left( \partial \theta^\alpha_\beta(v; q) \partial \theta^\beta_\alpha(v; q; z_2) \right).
\] (3.11)
By using Lemma 3.1 we have
\[
\Omega_{1, k; 2, \ell}^\alpha(v; q) = \frac{\eta^{\alpha_\beta, X, di}}{k! \ell!(k + \ell + 1)} \left( \partial \theta^X_1(v; q; k) \partial \theta^X_2(v; q; k + \ell + 1) \right),
\]
Proposition 3.4

The \( g = 0 \) part of the identity (1.24) holds true.
3.2 The anti-diagonal case

Similarly to the diagonal case, we are going to prove the $g = 0$ part of the identity (1.25).

**Lemma 3.5** Under the coordinate transformation

\[ v^1 = \sqrt{-1}e^{u^1}(1 - 2qe^{u^2}), \quad v^2 = 2u^1 + u^2 + \log(1 - qe^{u^2}), \]  

(3.16)

the following identities hold true:

\[
\frac{\partial \theta^\alpha_{\alpha,k}}{\partial v^\beta}(v; q) = M^\gamma_\beta(u; q) \sum_{\ell \geq 0} b_{\alpha,k}^{\sigma,\ell,\text{ad}} \frac{\partial \theta^X_{\alpha,\ell,\text{ad}}}{\partial u^\gamma}(u; q; q), \quad k \geq 0.
\]  

(3.17)

Here the coefficients $M^\alpha_\beta = M^\gamma_\beta(u; q)$ are defined by

\[
(M^\alpha_\beta)_{\alpha,\beta=1,2} = \begin{pmatrix}
  e^{-u^1} & \sqrt{-1}qe^{u^2} \\
  -e^{-u^1} & \sqrt{-1}(1 - qe^{u^2})
\end{pmatrix},
\]  

(3.18)

and the coefficients $b_{\alpha,k}^{\beta,\ell,\text{ad}}$ are given by (1.11).

**Proof** From the explicit expression (1.11) for $b_{\alpha,k}^{\beta,\ell,\text{ad}}$ we can rewrite (3.17) in the form

\[
\frac{\partial \theta^\alpha_{1,k}}{\partial v^\beta}(v; q) = \frac{(\sqrt{-1})^k}{k!} M^\gamma_\beta \frac{\partial}{\partial u^\gamma} \left( \theta^X_{1,\text{ad}}(u; q; k) + (2k\partial_z + 1)\theta^X_{2,\text{ad}}(u; q; z)|_{z=k} \right),
\]  

(3.19)

\[
\frac{\partial \theta^\alpha_{2,k}}{\partial v^\beta}(v; q) = \frac{(\sqrt{-1})^k}{k!} M^\gamma_\beta \frac{\partial \theta^X_{2,\text{ad}}(u; q; k+1)}{\partial u^\gamma}.
\]  

(3.20)

For $k = 0$, it is easy to see that both sides of (3.19) are equal to $\eta^\alpha_{1,\beta}$, and both sides of (3.20) are equal to $\eta^\alpha_{2,\beta}$. For $k \geq 1$, it follows from the relation (3.6) that (3.19), (3.20) can be further rewritten as

\[
\theta^\alpha_{1,k}(v; q; q) = \frac{(\sqrt{-1})^k}{k!} \sum_{\alpha=1}^2 \frac{\partial}{\partial u^\alpha} \left( \theta^X_{1,\text{ad}}(u; q; k) + (2k\partial_z - 1)\theta^X_{2,\text{ad}}(u; q; z)|_{z=k} \right),
\]  

(3.21)

\[
\theta^\alpha_{2,k}(v; q; q) = \frac{(\sqrt{-1})^{k+1}}{(k+1)!} \sum_{\alpha=1}^2 \frac{\partial \theta^X_{2,\text{ad}}(u; q; k+1)}{\partial u^\alpha}.
\]  

(3.22)

By using the map (3.16), equation (2.11) and Lemma 2.1, we find that both sides of (3.21) satisfy the following second-order linear PDE:

\[
\frac{qe^{u^2}}{1 - qe^{u^2}} \frac{\partial^2 f(u)}{\partial u^1 \partial u^1} + \frac{\partial^2 f(u)}{\partial u^2 \partial u^2} + \frac{qe^{u^2}}{1 - qe^{u^2}} \left( \frac{\partial f(u)}{\partial u^1} - \frac{\partial f(u)}{\partial u^2} \right) = 0.
\]  

(3.23)

From the formula (2.8), (3.16) as well as (2.25), we know that both sides of (3.22) belong to $e^{(k+1)u^1}C[[qe^{u^2}]]$. It then follows from an elementary exercise that the identity (3.22) can be reduced to a finite number of simple identities. This proves validity of (3.22). In a similar way one can prove (3.21). The lemma is proved.

Let us proceed to prove the following two lemmas.
Lemma 3.6 For all $k, \ell \geq 0$, we have
\[ \Omega_{\alpha,k;\beta,\ell}(v; q) = - \sum_{m,n \geq 0} b_{\alpha,m}^{\sigma} t_{\beta,n}^{\sigma} \Omega_{\alpha,m;\beta,n}(u(v; q); q) + \eta_{\alpha,\beta}^{(1)} \frac{(\sqrt{-1})^{k+\ell+1}}{(k+\ell+1)!} \] (3.24)
where $u(v; q)$ is given by (3.16).

Proof The proof is similar to that of Lemma 3.2, so we omit the details. \qed

Lemma 3.7 Let $t_{\text{ad}}(s)$ be defined as in (1.23). The identities
\[ v_{\text{top}}^{1}(s; q) = \sqrt{-1}e^{u_{\text{top},\text{ad}}(t_{\text{ad}}(s); q)} \left( 1 - 2 q e^{u_{\text{top},\text{ad}}^{2}(t_{\text{ad}}(s); q)} \right), \] (3.25)
\[ v_{\text{top}}^{2}(s; q) = 2u_{\text{top},\text{ad}}^{1}(t_{\text{ad}}(s); q) + u_{\text{top},\text{ad}}^{2}(t_{\text{ad}}(s); q) + \log \left( 1 - q e^{u_{\text{top},\text{ad}}^{2}(t_{\text{ad}}(s); q)} \right), \] (3.26)
hold true in $V_{\text{ad}} := \mathbb{C}[[s^{1.0} - \sqrt{-1}, s^{2.0}, s^{1.1}, s^{2.1}, \ldots]]$.

Proof Similarly to the proof of Lemma 3.3, we can prove that both sides of (3.25), (3.26) belong to $V_{\text{ad}}$. By using the genus zero Euler–Lagrange equation (2.13), the formula (1.23) and Lemma 3.5, we obtain that
\[ 0 = \sum_{k \geq 0} b_{\alpha,k}^{\sigma} \frac{\partial \theta_{\alpha,k}^{X_{\text{ad}}}}{\partial u^{3}} (u_{\text{top},\text{ad}}(t_{\text{ad}}(s); q); q) \]
\[ = \sum_{k,\ell \geq 0} b_{\sigma,k,\ell}^{\gamma} \frac{\partial \theta_{\alpha,k}^{X_{\text{ad}}}}{\partial u^{3}} (u_{\text{top},\text{ad}}(t_{\text{ad}}(s); q); q) \]
\[ = - \sqrt{-1} \left( M^{-1} \right)^{\beta}_{\gamma} (u(v; q); q) \sum_{k \geq 0} s_{\alpha,k}^{\gamma} \frac{\partial \theta_{\alpha,k}^{S_{1}^{1}}}{\partial v^{3}} (\tilde{v}(s; q); q), \]
where $\tilde{v}(s; q)$ denotes the right-hand side of (3.13), (3.25). Hence $v_{\text{top}}(s; q) = \tilde{v}(s; q)$. The lemma is proved. \qed

From the formula (2.14), Lemma 3.6 and Lemma 3.7, we arrive at the following proposition.

Proposition 3.8 The genus zero part of the identity (1.25) holds true.

4 Evidence for the genus one and genus two cases

In this section, we provide evidence for the genus one and genus two parts of the conjectural identities (1.24) and (1.25).

4.1 Review on topological recursion relations in genus one and genus two

Let us recall the genus one and genus two topological recursion relations given in [23, 28]. For a smooth algebraic variety $Y$, take a homogeneous basis $\phi_{1}, \ldots, \phi_{n}$ of the cohomology ring of $Y$. Denote by $F_{g}(t)$ ($g \geq 0$) the genus $g$ free energy for the GW invariants of $Y$, and
\[ \langle \tau_{i_{1}}(\phi_{\alpha_{1}}) \cdots \tau_{i_{k}}(\phi_{\alpha_{k}}) \rangle_{g} := \frac{\partial^{k} F_{g}}{\partial t_{\alpha_{1},i_{1}} \cdots \partial t_{\alpha_{k},i_{k}}}, \quad U_{\alpha_{i}}^{\beta} := \langle \phi_{\alpha} \phi_{\beta} \rangle_{0}, \] (4.1)
where $\phi^\alpha$ is the dual of $\phi_\alpha$ with respect to the Poincaré paring. Define a family of operators through the generating series

$$
\sum_{m \geq 0} D_{\alpha, m} z^m := \sum_{p \geq 0} \left( \left( \frac{\partial}{\partial U} + z U \right) U \right)^{\beta} \frac{\partial}{\partial v^{\beta, p}},
$$

where $v^{\beta, p} := \langle \langle \phi^{\beta+1}_1 \phi^{\beta}_p \rangle \rangle_0$ are the jet variables. It is proved in [23] that

$$
D_{\alpha, k} F_1 = \begin{cases}
\frac{1}{24} \frac{\partial}{\partial v^{\alpha, m}} \text{Tr} (U), & k = 1, \\
0, & k > 1,
\end{cases}
$$

$$
D_{\alpha, k} F_2 = R_{\alpha, k}, \quad k \geq 2,
$$

$$
(D_{\alpha, 1} D_{\beta, 1} - 3 \langle \langle \phi_\alpha \phi_\beta \phi^\gamma \rangle \rangle_0 D_{\gamma, 1}) F_2 = R_{\alpha, 1; \beta, 1},
$$

where $R_{\alpha, k}$ and $R_{\alpha, 1; \beta, 1}$ are given by

$$
R_{\alpha, 2} = \langle \langle \phi_\alpha \phi_\beta \phi^\gamma \rangle \rangle_0 \left( \frac{7}{10} \langle \langle \phi^{\beta} \rangle \rangle_1 \langle \langle \phi^\gamma \rangle \rangle_1 + \frac{1}{10} \left( \langle \langle \phi^{\beta} \phi^\gamma \rangle \rangle_1 + \frac{1}{240} \langle \langle \phi_\alpha \phi_\beta \phi^\gamma \rangle \rangle_0 \langle \langle \phi^{\beta} \rangle \rangle_1 \right) - \frac{1}{240} \langle \langle \phi_\alpha \phi^{\beta} \phi^{\gamma} \rangle \rangle_1 \langle \langle \phi_\beta \phi^\gamma \rangle \rangle_0 + \frac{1}{960} \langle \langle \phi_\alpha \phi_\beta \phi^{\gamma} \phi^\gamma \rangle \rangle_0 \right),
$$

$$
R_{\alpha, 3} = \langle \langle \phi_\alpha \phi_\beta \phi^\gamma \rangle \rangle_0 \left( \frac{1}{20} \langle \langle \phi^{\beta} \rangle \rangle_1 \langle \langle \phi^\sigma \phi_\sigma \rangle \rangle_0 + \frac{1}{480} \langle \langle \phi^{\beta} \phi^\sigma \phi^\sigma \phi_\sigma \rangle \rangle_0 \right) + \frac{1}{1152} \langle \langle \phi_\alpha \phi^{\beta} \phi^{\gamma} \phi_\gamma \rangle \rangle_0 \langle \langle \phi_\beta \phi^\sigma \phi_\sigma \rangle \rangle_0,
$$

$$
R_{\alpha, 4} = \frac{1}{1152} \langle \langle \phi_\alpha \phi^{\beta} \phi^{\gamma} \rangle \rangle_0 \langle \langle \phi_\beta \phi^\sigma \phi_\sigma \rangle \rangle_0 \langle \langle \phi_\alpha \phi^{\beta} \phi^{\sigma} \phi_\sigma \rangle \rangle_0, \quad R_{\alpha, k} = 0 (k > 4),
$$

$$
R_{\alpha, 1; \beta, 1} = \frac{13}{10} \langle \langle \phi_\alpha \phi_\beta \phi^\gamma \phi_\sigma \rangle \rangle_0 \langle \langle \phi^\gamma \rangle \rangle_1 \langle \langle \phi^\sigma \rangle \rangle_1 + \frac{4}{5} \left( \langle \langle \phi_\alpha \phi^\gamma \rangle \rangle_1 \langle \langle \phi^\sigma \rangle \rangle_1 + \frac{1}{24} \langle \langle \phi_\alpha \phi^\gamma \phi^\sigma \rangle \rangle_1 \langle \langle \phi^\sigma \rangle \rangle_0 \right) \langle \langle \phi_\beta \phi^\gamma \phi^\sigma \rangle \rangle_0 + \frac{4}{5} \langle \langle \phi_\alpha \phi^{\gamma} \phi^\sigma \rangle \rangle_0 \left( \langle \langle \phi_\beta \phi^\gamma \phi_\sigma \rangle \rangle_1 + \frac{1}{24} \langle \langle \phi_\beta \phi^\gamma \phi_\sigma \rangle \rangle_1 \right) + \frac{1}{48} \langle \langle \phi_\alpha \phi_\beta \phi^\gamma \phi^\sigma \rangle \rangle_0 \langle \langle \phi_\beta \phi^\gamma \phi^\sigma \rangle \rangle_1 - \frac{4}{9} \langle \langle \phi_\alpha \phi_\beta \phi^\gamma \phi_\sigma \rangle \rangle_0 \left( \langle \langle \phi^\gamma \phi_\sigma \rangle \rangle_1 + \frac{1}{24} \langle \langle \phi^\gamma \phi_\sigma \phi^\sigma \rangle \rangle_1 \right) + \frac{1}{48} \langle \langle \phi_\alpha \phi^\gamma \phi_\sigma \rangle \rangle_1 \langle \langle \phi_\beta \phi^\gamma \phi_\sigma \rangle \rangle_0 + \frac{23}{240} \langle \langle \phi_\alpha \phi_\beta \phi^\gamma \phi_\sigma \phi^\sigma \rangle \rangle_0 \langle \langle \phi^\gamma \rangle \rangle_1 - \frac{1}{80} \langle \langle \phi_\alpha \phi_\beta \phi^\gamma \phi_\sigma \rangle \rangle_1 \langle \langle \phi^\gamma \phi^\sigma \phi_\sigma \rangle \rangle_0 + \frac{7}{30} \langle \langle \phi_\alpha \phi_\beta \phi^\gamma \phi_\sigma \phi^\sigma \rangle \rangle_0 \langle \langle \phi^\sigma \rangle \rangle_1 + \frac{1}{576} \langle \langle \phi_\alpha \phi_\beta \phi^\gamma \phi_\sigma \phi^\sigma \phi^\sigma \rangle \rangle_0.
$$

We are going to show that for the resolved conifold $X$ with diagonal and anti-diagonal actions, the genus one and genus two free energies obtained from Conjectures [14] [12] satisfy the above topological recursion relations [4.3] [4.5]. This will give evidences for the validity of these two conjectures. We know from [7] [21] [22] [24] the following lemma.

**Lemma 4.1** For $g \geq 1$, there exist functions $F_{g}^{\phi_1}(v_0, v_1, \ldots, v_{3g-2}; q)$, where $v_k = (v^1_k, v^2_k)$ are pairs of indeterminates, such that

$$
F_{g}^{\phi_1}(s; q) = F_{g}^{\phi_1}(v_{0 \text{top}}(s; q), \frac{\partial v_{0 \text{top}}(s; q)}{\partial s^{1,0}}, \ldots, \frac{\partial^{3g-2} v_{0 \text{top}}(s; q)}{\partial (s^{1,0})^{3g-2}}; q).
$$

(4.6)
Similarly, for \( g \geq 1 \), there exist functions \( F^X_{g, di/ad}(u_0, u_1, \ldots ; q) \), where \( u_k = (u^1_k, u^2_k) \) are pairs of indeterminates, such that

\[
F^X_{g, di/ad}(t; q) = F^X_{g, di/ad}\left(u_{top, di/ad}(t; q), \frac{\partial u_{top, di/ad}(t; q)}{\partial t^1, 0} ; \ldots ; q\right).
\]

(4.7)

### 4.2 Evidence for the diagonal case

Let us now consider the diagonal case. It follows from Lemma 4.1 that both sides of (1.24) admit jet variable representations. As we mentioned in the Introduction, unlike the non-invertibility of the transformation (1.17), the jet variables between the two models are invertible. To be precise, denote

\[
v^\alpha = v^\alpha_0 := v^\alpha_{top}, \quad u^\alpha = u^\alpha_0 := u^\alpha_{top, di}, \quad v^\alpha_k := \frac{\partial^k v^\alpha_{top}}{\partial (s^1, 0)^k}, \quad u^\alpha_k := \frac{\partial^k u^\alpha_{top, di}}{\partial (t^1, 0)^k}, \quad k \geq 1,
\]

(4.8)

then Lemma 3.3 reads

\[
v^1 = e^{u^1} \left(1 + qe^{u^2}\right), \quad v^2 = 2u^1 + u^2.
\]

(4.9)

By applying \( \frac{\partial}{\partial s^1} \) to both sides of the above equations and by using (1.17), we obtain that

\[
v^1_1 = e^{u^1} \left((1 + qe^{u^2})u^1_1 + qe^{u^2}u^2_1\right), \quad v^2_1 = 2u^1_1 + u^2_1.
\]

(4.10)

In general, we have the recursion relation

\[
v^\alpha_k = \sum_{\ell \geq 1} u^\alpha_{\ell} \frac{\partial u^\alpha_{k-1}}{\partial u^\alpha_{k-1}}, \quad k \geq 1.
\]

(4.11)

It follows from (4.6) that for \( g \geq 1 \), the left-hand side of (1.24) can be represented in terms of the jet variables \( v^\alpha_k \). Then substituting (4.9) and (4.11) into this representation, we obtain the predicted expressions of \( F^X_{g, di} \) \((g \geq 1)\) in terms of jet variables \( u^\alpha_k \). For example, the genus one free energy \( F^X_{1, di} \) obtained from (1.24) has the following expression:

\[
F^X_{1, di} = \frac{1}{24} \log D_{di} - \frac{1}{12} \text{Li}_1(qe^{u^2}) - \frac{1}{24} u^2.
\]

(4.12)

Here

\[
D_{di} := (u^1_1)^2 - 2\xi u^1_1 u^2_1 - \xi (u^2_1)^2, \quad \xi := \frac{qe^{u^2}}{1 - qe^{u^2}};
\]

(4.13)

and we used the explicit expression for \( F^p_1 \) [19, 21]:

\[
F^p_1 = \frac{1}{24} \log \left((v^1_1)^2 - qe^{u^2}(v^2_1)^2\right) - \frac{v^2_1}{24}.
\]

(4.14)

Similarly, from the explicit expression for \( F^p_2 \) (cf. [16, 19, 21]) and the \( g = 2 \) case of (1.24), we arrive at

\[
F^X_{2, di} = \frac{\xi^4((\xi + 1)^4 (64\xi^3 + 80\xi^2 + 24\xi + 1) (u^1_1)^{10}}{90D_{di}^4}
\]

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In general, we have the recursion relation

\[ F = \frac{4\xi^4(\xi + 1)^4 (16\xi^3 + 24\xi^2 + 10\xi + 1) u_1^2 (u_2^2)^0}{45D_{dl}^1} \]

\[ + \frac{\xi^3(\xi + 1)^3 (496\xi^4 + 13888\xi^3 + 12240\xi^2 + 3160\xi + 121) (u_1^2)^8}{5760D_{dl}^3} + \ldots. \]  

(4.15)

Here we omit the explicit expressions of the remaining 98 terms.

By using the explicit expressions (4.12), (4.15) of \( F_1^{X,dl} \) and \( F_2^{X,dl} \), we arrive at the following proposition.

**Proposition 4.2** The genus one and genus two free energies \( F_1^{X,dl} \) and \( F_2^{X,dl} \) obtained from (1.24) satisfy the topological recursion relations (4.3)–(4.5).

By using these expressions, we obtain in particular the primary genus one and genus two free energies:

\[ F_1^{X,dl} |_{\alpha,k = \delta k, 0} = -\frac{1}{24} \log \xi - \frac{1}{12} \log (1 - \xi), \]  

(4.16)

\[ F_2^{X,dl} |_{\alpha,k = \delta k, 0} = -\frac{1}{2880} \log (1 - \xi) + \frac{1}{2880} \log (1 + \xi) + \frac{1}{240} \log (1 - \xi). \]  

(4.17)

### 4.3 Evidence for the anti-diagonal case

Let us consider the anti-diagonal case. Denote

\[ v^\alpha = v^\alpha_0 := v^\alpha_{top}, \ u^\alpha = u^\alpha_0 := u^\alpha_{top, ad}, \ v^\alpha_k := \frac{\partial^k v^\alpha_{top}}{\partial \xi^k}, \ u^\alpha_k := \frac{\partial^k u^\alpha_{top, ad}}{\partial \xi^k}, \ k \geq 1, \]  

(4.18)

then Lemma 3.7 reads

\[ v^1 = \sqrt{1 - 2\xi^2} \left( 1 - 2\xi^2 \right), \quad v^2 = 2u^1 + u^2 + \log \left( 1 - \xi^2 \right). \]  

(4.19)

By applying \( \frac{\partial}{\partial \xi^k} \) to both sides of the above equations and by using (1.23), we obtain

\[ v^1_1 = e^{u^1_1} (1 - 4\xi^2 u^2_1) u^1_1 - e^{u^1 + u^2} \frac{3 - 4\xi u^2}{1 - \xi u^2} u^2_1, \quad v^2_1 = \frac{3 - 4\xi u^2}{1 - \xi u^2} u^1_1 + \frac{1 - 4\xi u^2}{1 - \xi u^2} u^2_1. \]  

(4.20)

In general, we have the recursion relation

\[ v^\alpha_k = \sum_{\ell \geq 1} u^\ell_0 \frac{\partial v^\alpha_{k-1}}{\partial u^\beta_{\ell-1}} + \sum_{m \geq 0} \left( \sum_{\ell \geq 1} u^\ell_0 \frac{\partial}{\partial u^\beta_{\ell-1}} \right) \left( \frac{\partial v^\alpha_{k-1}}{\partial u^\beta_{m}} \right) \left( \frac{\partial v^\beta_{k-m}}{\partial u^\gamma_{k-1}} \right) \left( \frac{\partial v^\gamma_{m}}{\partial u^\ell_{\ell-1}} \right) \left( \frac{\partial v^\ell_{m}}{\partial u^\alpha_{\ell-1}} \right), \]  

(4.21)

By substituting (4.19) and (4.21) into the jet representation of the left-hand side of (1.25), we obtain the predicted expressions of \( F_1^{X,ad} \) \( (g \geq 1) \) in terms of jet variables \( u^\alpha_k \). For example, the genus one and genus two free energies \( F_1^{X,ad}, F_2^{X,ad} \) obtained from (1.25) have the following expressions:

\[ F_1^{X,ad} = \frac{1}{24} \log D_{ad} - \frac{1}{12} \log (1 - \xi^2) - \frac{1}{24} \log v^2, \]  

(4.22)
\[ F_{2,ad} = -\frac{\xi^4(\xi + 1)^3(u_1^2)^{10}}{90D_{ad}^4} - \frac{\xi^4(\xi + 1)^2(u_2^2)^8u_2^2}{15D_{ad}^4} + \cdots, \] (4.23)

where
\[ D_{ad} := (u_1^1)^2 + \xi(u_2^1)^2, \quad \xi := \frac{qe^u}{1 - qe^u}, \] (4.24)

and we omit the explicit expressions of the remaining 51 terms in (4.23).

By using the explicit expressions (4.22), (4.23) of \( F_{1,ad} \) and \( F_{2,ad} \), we arrive at the following proposition.

**Proposition 4.3** The genus one and genus two free energies \( F_{1,ad} \) and \( F_{2,ad} \) obtained from (1.25) satisfy the topological recursion relations (4.3) – (4.5).

In particular, by taking \( t_{\alpha,k} = t_{\alpha,k}0 = 0 \), we obtain the primary parts of \( F_{1,ad} \) and \( F_{2,ad} \)
\[
\begin{align*}
F_{1,ad}^{\alpha,k=0} &= -\frac{1}{24}t_2^2 + \frac{1}{24}\log(1 - qe^u) - \frac{1}{12}\text{Li}_1(qe^u), \\
F_{2,ad}^{\alpha,k=0} &= \frac{1 + 10qe^u + q^2e^{2u}}{2880(1 - qe^u)^2} = \frac{1}{2880} + \frac{1}{240}\text{Li}_1(qe^u),
\end{align*}
\]
which agree with the ones given in [3]:
\[
\begin{align*}
F_{g}^{X,ad}|_{t_{\alpha,k}=t_{\alpha,k}0} &= \frac{|B_{2g}|}{2g(2g - 2)!}\text{Li}_{3-2g}(qe^u) + \frac{|B_{2g}B_{2g-2}|}{2g(2g - 2)(2g - 2)!}, \quad g \geq 2. \tag{4.25}
\end{align*}
\]

5 Further remarks

We know that the genus one free energy can be represented in the following form (cf. [7, 20, 27]):
\[
F_1 = \left( \frac{1}{24}\log \det(c_{ij}^\alpha v_2^\alpha) + G_1(v) \right) \bigg|_{v=v_{top}}, \tag{5.1}
\]
and the genus two free energy can be represented in the form [15]:
\[
F_2 = \left( \sum_{i=1}^{16} c_iQ_i + G_2(v, v_x, v_{xx}) \right) \bigg|_{v=v_{top}}, \tag{5.2}
\]

where \( c_i \) are certain constants, \( Q_i \) are given by some genus zero and genus one correlation functions corresponding to the dual graphs of some stable algebraic curves, and \( G_1(v), G_2(v, v_x, v_{xx}) \) are called the genus one and genus two \( G \)-functions respectively.

Our first remark is that from the explicit expressions (4.12), (4.15), (4.22) and (4.23), it follows that the corresponding genus one and genus two \( G \)-functions have the expressions
\[
\begin{align*}
G_1^{X,di} &= \frac{1}{12}\log(1 - qe^u) - \frac{1}{24}u^2, \\
G_2^{X,di} &= \frac{12\xi^2 + 12\xi - 1}{2880}D_{di} + \frac{2\xi(1 + \xi)(1 + 2\xi)}{2880}u_1^1u_2^2,
\end{align*}
\]

\[ \tag{4.23} \]
\[-\frac{\xi(1 + \xi)(1 + 62\xi + 64\xi^2)}{2880} (u_1^2)^2 + \frac{\xi(1 + \xi)}{80} u_1^2 - \frac{\xi(1 + \xi)(4 + 17\xi)}{720} u_2^2\]
\[-\frac{\xi^2(1 + \xi)^2(1 + 2\xi)}{180} u_1^2 u_1^2 - \frac{\xi(1 + \xi)(1 + 2\xi)}{320} u_1 u_1 u_2^2 + \frac{\xi(1 + \xi)(1 + 24\xi + 32\xi^2)}{1440} u_1 u_2 u_2^2\]
\[-\frac{\xi(1 + \xi)(1 + 8\xi)}{2880} u_2^2 + \frac{\xi^2(1 + \xi)(1 + 2\xi)}{320} u_2^2 - \frac{\xi^2(1 + \xi)(3 + 11\xi + 8\xi^2)}{1440} (u_2^2)^2\]
\[G_1^{X, ad} = -\frac{1}{12} \log(1 - qe^{u^2}) - \frac{1}{24} u^2,\]
\[G_2^{X, ad} = \frac{\xi(1 + \xi)u_2^2}{1440D_{ad}} \left(2\xi(\xi + 1)(u_2^2)^3 + 3\xi u_2 u_2^4 + 3u_1 u_2^3\right)\]
\[-\frac{\xi}{2880} \left(16\xi(\xi + 1) u_2^2 + (26\xi^2 + 25\xi + 1)(u_1^2)^2\right) + \frac{12\xi^2 + 12\xi + 1}{2880} D_{ad}.\]

Our second remark is that the genus one and genus two free energies \(\mathcal{F}_{X,1,2}^{X, ad}\) and \(\mathcal{F}_{X,1,2,2}^{X, ad}\) obtained from [1,24] and [1,25] also satisfy the following Belorousski–Pandharipande equations [1] up to 0 ≤ \(k_1, k_2, k_3 ≤ 1\):

\[-2\langle \phi_0 \rangle_2 \langle \phi^a \phi_0 \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \tau_k (\phi_0) \rangle_0\]
\[+ 2 \left(\langle \tau_1 (\phi_0) \rangle_2 \langle \phi^a \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 - \langle \phi_0 \rangle_2 \langle \phi^a \phi_0 \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 \right)\]
\[+ 3 \left(\langle \phi_0 \tau_k (\phi_0) \rangle_2 \langle \phi^a \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 - \langle \phi_0 \tau_k (\phi_0) \rangle_0 \langle \phi^a \phi_0 \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 \right)\]
\[+ 3 \left(\langle \tau_{k_1} (\phi_0) \rangle_2 \langle \phi^a \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 - \langle \phi_0 \tau_k (\phi_0) \rangle_0 \langle \phi^a \phi_0 \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 \right)\]
\[+ \frac{1}{5} \langle \phi_0 \rangle_1 \langle \phi^a \phi_0 \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 - \frac{6}{5} \langle \phi_0 \rangle_1 \langle \phi^a \phi_0 \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 \rangle_1\]
\[+ \frac{12}{5} \langle \phi_0 \rangle_1 \langle \phi^a \phi_0 \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 - \frac{18}{5} \langle \phi_0 \rangle_1 \langle \phi^a \phi_0 \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 \rangle_1\]
\[+ \frac{6}{5} \langle \phi_0 \rangle_1 \langle \phi^a \phi_0 \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 - \frac{9}{5} \langle \phi_0 \rangle_1 \langle \phi^a \phi_0 \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 \rangle_1\]
\[+ \frac{6}{5} \langle \phi_0 \rangle_1 \langle \phi^a \phi_0 \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 - \frac{1}{5} \langle \phi_0 \rangle_1 \langle \phi^a \phi_0 \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 \rangle_1\]
\[+ \frac{3}{40} \langle \phi_0 \rangle_1 \langle \phi^a \phi_0 \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 \rangle_1\]
\[+ \frac{1}{120} \langle \phi_0 \rangle_1 \langle \phi^a \phi_0 \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 \rangle_1\]
\[+ \frac{3}{10} \langle \phi_0 \rangle_1 \langle \phi^a \phi_0 \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 \rangle_1\]
\[- \frac{1}{20} \langle \phi_0 \rangle_1 \langle \phi^a \phi_0 \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 \langle \phi^b \tau_k (\phi_0) \rangle_0 \rangle_1\]
\[+(1 \leftrightarrow 2 \leftrightarrow 3) = 0.\]

Here (1 ↔ 2 ↔ 3) denotes the other terms obtained by permuting \((\alpha_1, k_1), (\alpha_2, k_2), (\alpha_3, k_3)\).
A Some related calculations

The first several coefficients of the deformed coordinates are given in Table A

| (α, k) | θ^a_s | θ^{X,di}_{a,k} | θ^{X,ad}_{a,k} |
|--------|--------|----------------|----------------|
| (1, 0) | v^2    | 2u^1 + u^2     | u^2            |
| (2, 0) | v^1    | u^1            | u^1            |
| (1, 1) | v^1u^2 | (u^1)^2 + u^1u^2 | u^1u^2         |
| (2, 1) |   | (u^1)^2 + v^2   | (u^1)^2 - Li_2(qe^u^2) |
| (1, 2) |   | (u^1)^2v^2 + qe^v^2(v^2 - 2) | 1/3(u^1)^3 + 1/2(u^1)^2u^2 + u^2Li_2(qe^u^2) - 2Li_3(qe^u^2) + 2Li_3(qe^u^2) |
| (2, 2) |   | (u^1)^3 + qv^1e^v^2 | (u^1)^3 - u^1Li_2(qe^u^2) |

Table 1: First several θ_{α,k}

We also list some two-point correlations functions in genus zero as follows:

\[
\Omega_{1,1,1,1}^{P_1} = (v^1)^2v^2 + 2qe^{u^2} - 2qv^2e^{v^2} + q(v^2)^2e^{v^2},
\]

\[
\Omega_{1,2,2,1}^{P_1} = \frac{1}{3}(u^1)^3 + qv^1v^2e^{v^2},
\]

\[
\Omega_{2,1,2,1}^{P_1} = q(v^1)^2e^{v^2} + \frac{1}{2}q^2e^{2v^2},
\]

\[
\Omega_{1,1,1,1}^{X,di} = \frac{2}{3}(u^1)^3 + (u^1)^2u^2 + 2Li_3(qe^u^2) - 2u^2Li_2(qe^u^2) + (u^2)^2Li_1(qe^u^2),
\]

\[
\Omega_{1,1,1,2,1}^{X,di} = \frac{1}{3}(u^1)^3 + q^1v^2Li_1(qe^u^2) - (Li_1(qe^u^2))^2,
\]

\[
\Omega_{2,1,2,1}^{X,di} = \left((u^1)^2 - 2u^1Li_1(qe^u^2)\right)Li_1(qe^u^2) - (Li_1(qe^u^2))^3 + \int_{-\infty}^{u^2} \frac{Li_1(qe^y)^2dy}{1 - qe^y},
\]

\[
\Omega_{1,1,1,1}^{X,ad} = (u^1)^2u^2 - 2Li_3(qe^u^2) + 2u^2Li_2(qe^u^2) - (u^2)^2Li_1(qe^u^2),
\]

\[
\Omega_{1,1,1,2,1}^{X,ad} = \frac{1}{3}(u^1)^3 - u^1u^2Li_1(qe^u^2),
\]

\[
\Omega_{2,1,2,1}^{X,ad} = -(u^1)^2Li_1(qe^u^2) + \int_{-\infty}^{u^2} (Li_1(qe^y))^2dy.
\]

By using the genus zero Euler–Lagrange equation (2.13) and the data in Table A we obtain the first several terms in the topological solution \(v_{top}\) as follows:

\[
v_{top}^1 = s^{1,0} + qs^{2,1} + s^{1,0}s^{1,1} + qs^{1,1}s^{2,1} + qs^{2,0}s^{2,1} + s^{1,0}(s^{1,1})^2 + q(s^{1,1})^2s^{2,1} + 2qs^{1,1}s^{2,0}s^{2,1} + \frac{q}{2}(s^{2,0})^2s^{2,1} + qs^{1,0}(s^{2,1})^2 + q^2s^{2,0}(s^{2,1})^3 + \cdots,
\]

\[
v_{top}^2 = s^{2,0} + s^{1,0}s^{2,0} + s^{1,0}s^{2,1} + q(s^{2,1})^2 + (s^{1,1})^2s^{2,0} + 2s^{1,0}s^{1,1}s^{2,1} + 2qs^{1,1}(s^{2,1})^2 + qs^{2,0}(s^{2,1})^2 + \cdots.
\]
Similarly, the topological solutions $u_{\text{top,di}}$ and $u_{\text{top,ad}}$ have the form

$$
\begin{align*}
    u_{\text{top,di}}^1 &= t^{1,0} - \log(1 - q) t^{2,1} + t^{1,0} t^{1,1} - \log(1 - q) t^{1,1} t^{2,1} + \frac{q}{1 - q} t^{2,0} t^{2,1} \\
    &\quad + \frac{2q^3 q \log(1 - q)}{1 - q} (t^{2,1})^2 + t^{1,0} (t^{1,1})^2 - \log(1 - q) (t^{1,1})^2 t^{2,1} + \frac{2q}{1 - q} t^{1,1} t^{2,0} t^{2,1} \\
    &\quad + \frac{q}{2(1 - q)^2} (t^{2,0})^2 t^{2,1} + \frac{4q \log(1 - q)}{1 - q} t^{1,1} (t^{2,1})^2 + \frac{q}{1 - q} t^{1,0} (t^{2,1})^2 \\
    &\quad + \frac{2q(1 - q) \log(1 - q)}{(1 - q)^2} (t^{2,0})^2 t^{2,1} + \frac{q \log(1 - q)(2 \log(1 - q) - 1 - 3q)}{(1 - q)^2} (t^{2,1})^3 + \ldots ,
\end{align*}
$$

$$
\begin{align*}
    u_{\text{top,di}}^2 &= t^{2,0} + 2 \log(1 - q) t^{2,1} + t^{1,1} t^{2,0} + t^{1,0} t^{2,1} + 2 \log(1 - q) t^{1,1} t^{2,1} \\
    &\quad - \frac{2q}{1 - q} t^{2,0} t^{1,1} - \frac{(1 + 3q) \log(1 - q)}{1 - 1} (t^{2,1})^2 + \frac{(t^{1,1})^2 t^{2,0} + 2 t^{1,0} t^{1,1} t^{2,1}}{1 - q} \\
    &\quad + 2 \log(1 - q) (t^{1,1})^2 t^{2,1} - \frac{4q}{1 - q} t^{1,1} t^{2,0} t^{2,1} - \frac{q}{(1 - q)^2} (t^{2,0})^2 t^{2,1} - \frac{2q}{1 - q} t^{1,0} (t^{2,1})^2 \\
    &\quad - \frac{2(1 + 3q) \log(1 - q)}{1 - q} t^{1,1} (t^{2,1})^2 + \frac{(1 + 3q - 4 \log(1 - q)) t^{2,0} (t^{2,1})^2}{(1 - q)^2} \\
    &\quad + \frac{4q (1 + q - \log(1 - q)) \log(1 - q) (t^{2,1})^3}{(1 - q)^2} + \ldots ,
\end{align*}
$$

$$
\begin{align*}
    u_{\text{top,ad}}^1 &= t^{1,0} + \log(1 - q) t^{2,1} + t^{1,0} t^{1,1} + \log(1 - q) t^{1,1} t^{2,1} - \frac{q}{1 - q} t^{2,0} t^{2,1} \\
    &\quad + \log(1 - q) (t^{1,1})^2 t^{2,1} - \frac{2q}{1 - q} t^{1,1} t^{2,0} t^{2,1} - \frac{q}{2(1 - q)^2} (t^{2,0})^2 t^{2,1} \\
    &\quad - \frac{q \log(1 - q)}{1 - q} (t^{2,1})^3 + t^{1,0} (t^{1,1})^2 - \frac{q}{1 - q} t^{2,0} (t^{2,1})^2 + \ldots ,
\end{align*}
$$

$$
\begin{align*}
    u_{\text{top,ad}}^2 &= t^{2,0} + t^{1,1} t^{2,0} + t^{1,0} t^{2,1} + \log(1 - q) (t^{2,1})^2 + 2 t^{1,0} t^{1,1} t^{2,1} \\
    &\quad + 2 \log(1 - q) t^{1,1} (t^{2,1})^2 + t^{2,0} (t^{1,1})^2 - \frac{q}{1 - q} t^{2,0} (t^{2,1})^2 + \ldots .
\end{align*}
$$

### B Partial correlation functions

The genus $g$ correlation functions

$$
\langle \langle \tau_1(\phi_{\alpha_1}) \cdots \tau_k(\phi_{\alpha_k}) \rangle \rangle_g
$$

(B.1)
evaluated at $t^{a,p} = t_a \delta^{p,0}$, $p \geq 0$ are called the partial correlation functions (cf. [18]), and we still use the same notations to denote these partial correlation functions.

For the $\mathbb{P}^1$ model, we list the following genus 0, 1, 2 partial correlation functions:

$$
\begin{align*}
    \langle \langle \tau_1(\phi) \rangle \rangle_0^{\mathbb{P}^1} &= \frac{1}{2} t_2^2 + q (t_2 - 2) e_2^2, \\
    \langle \langle \tau_1(\phi_2) \rangle \rangle_0^{\mathbb{P}^1} &= \frac{1}{6} (t_1)^3 + q t_1 e_2^2, \\
    \langle \langle \tau_1(\phi_1) \tau_1(\phi_1) \rangle \rangle_0^{\mathbb{P}^1} &= (t_2)^2 t_2 + q ((t_2)^2 - 2 t_2 + 2) e_2^2, \\
    \langle \langle \tau_1(\phi_1) \tau_1(\phi_2) \rangle \rangle_0^{\mathbb{P}^1} &= \frac{1}{3} (t_1)^3 + q t_1 e_2^2, \\
    \langle \langle \tau_1(\phi_2) \tau_1(\phi_2) \rangle \rangle_0^{\mathbb{P}^1} &= q (t_1)^2 e_2^2 + \frac{1}{2} q^2 e_2 t_2, \\
    \langle \langle \tau_1(\phi_1) \rangle \rangle_1^{\mathbb{P}^1} &= -\frac{1}{12} t_2 + \frac{1}{12}, \\
    \langle \langle \tau_1(\phi_2) \rangle \rangle_1^{\mathbb{P}^1} &= -\frac{1}{24} t_1.
\end{align*}
$$
\[ \langle \tau_1(\phi_1) \rangle^{p_1}_1 = -\frac{1}{12} t_2 + \frac{1}{12}, \quad \langle \tau_1(\phi_1) \rangle^{p_1}_1 = -\frac{1}{12} t_1, \quad \langle \tau_1(\phi_2) \rangle^{p_1}_1 = 0, \quad \langle \tau_1(\phi_2) \rangle^{p_1}_2 = \langle \tau_1(\phi_1) \tau_1(\phi_2) \rangle^{p_1}_2 = 0. \]

Denote
\[ \omega := \frac{q e^{t_2}}{1 - q e^{t_2}}, \]
then for the resolved conifold \( X \) with diagonal action, we have
\[ \langle \tau_1(\phi_1) \rangle^X_{0, \text{d}i} = -2 \text{Li}_3(q e^{t_2}) + t_2 \text{Li}_2(q e^{t_2}) + \frac{1}{3} t_1^3 + \frac{1}{2} t_1^2 t_2, \]
\[ \langle \tau_1(\phi_2) \rangle^X_{0, \text{d}i} = (\text{Li}_1(q e^{t_2}))^2 + t_1 \text{Li}_2(q e^{t_2}) - t_2 \text{Li}_1(q e^{t_2}) + \frac{1}{6} t_1^3, \]
\[ \langle \tau_1(\phi_1) \rangle^X_{0, \text{d}i} = 2 \text{Li}_3(q e^{t_2}) + \frac{2}{3} t_1^3 - 2 t_2 \text{Li}_1(q e^{t_2}) + t_1^2 t_2 + \frac{1}{6} \text{Li}_1(q e^{t_2}), \]
\[ \langle \tau_1(\phi_1) \rangle^X_{0, \text{d}i} = - (\text{Li}_1(q e^{t_2}))^2 + \frac{1}{3} t_1^3 + t_1 t_2 \text{Li}_1(q e^{t_2}), \]
\[ \langle \tau_1(\phi_2) \rangle^X_{0, \text{d}i} = \int_{-\infty}^{t_2} \frac{(\log(1 - q e^y))^2 dy}{1 - q e^y} - (\text{Li}_1(q e^{t_2}))^3 - 2 t_2 (\text{Li}_1(q e^{t_2}))^2 + t_1^2 \text{Li}_1(q e^{t_2}), \]
\[ \langle \tau_1(\phi_1) \rangle^X_{1, \text{d}i} = \langle \tau_1(\phi_2) \rangle^X_{1, \text{d}i} = \frac{1}{12} \log(1 + \omega) + t_1 \left( -\frac{1}{24} + \frac{1}{12} \omega \right), \]
\[ \langle \tau_1(\phi_1) \rangle^X_{1, \text{d}i} = \langle \tau_1(\phi_2) \rangle^X_{1, \text{d}i} = \frac{1}{12} + t_2 \left( -\frac{1}{12} + \frac{1}{6} \omega \right) + t_2 \left( -\frac{1}{4} \omega^2 - \frac{1}{6} \omega(1 + \omega) \log(1 + \omega) \right) + \frac{1}{12} t_1 t_2 \omega(1 + \omega), \]
\[ \langle \tau_1(\phi_1) \rangle^X_{1, \text{d}i} = \langle \tau_1(\phi_2) \rangle^X_{1, \text{d}i} = \omega \left( \frac{1}{12} + \frac{1}{6} \omega \right) - \left( \frac{1}{12} - \frac{1}{6} \omega - \omega^2 \right) \log(1 + \omega) + \frac{1}{3} \omega(1 + \omega) \log(1 - \omega) \]
\[ + t_1 \left( -\frac{1}{2} \omega^2 - \frac{1}{3} \omega(1 + \omega) \log(1 + \omega) \right) + \frac{1}{12} t_1^2 \omega(1 + \omega), \]
\[ \langle \tau_1(\phi_1) \rangle^X_{2, \text{d}i} = \langle \tau_1(\phi_2) \rangle^X_{2, \text{d}i} = \frac{-1 + 12 \omega + 12 \omega^2}{1440} + \frac{1}{240} t_2 \omega(1 + \omega)(1 + 2 \omega), \]
\[ \langle \tau_1(\phi_1) \rangle^X_{2, \text{d}i} = \langle \tau_1(\phi_2) \rangle^X_{2, \text{d}i} = \frac{-\omega^2(15 + 14 \omega)}{1440} - \frac{1}{120} \omega(1 + \omega)(1 + 2 \omega) \log(1 + \omega) + \frac{1}{240} t_1 \omega(1 + \omega)(1 + 2 \omega), \]
\[ \langle \tau_1(\phi_1) \rangle^X_{2, \text{d}i} = \langle \tau_1(\phi_2) \rangle^X_{2, \text{d}i} = \frac{-1 + 12 \omega + 12 \omega^2}{480} + \frac{1}{40} t_2 \omega(1 + \omega)(1 + 2 \omega) + \frac{1}{240} t_2 \omega(1 + \omega)(1 + 6 \omega + 6 \omega^2), \]
\[ \langle \tau_1(\phi_1) \rangle^X_{2, \text{d}i} = \langle \tau_1(\phi_2) \rangle^X_{2, \text{d}i} = \frac{-\omega^2(15 + 14 \omega)}{480} - \frac{1}{40} \omega(1 + \omega)(1 + 2 \omega) \log(1 + \omega) + \frac{1}{60} t_1 \omega(1 + \omega)(1 + 2 \omega) \]
For the resolved conifold $X$ with anti-diagonal action, we have

$$
\langle \tau_1(\phi_2) \rangle_2^{X,\text{ad}} = \frac{1}{120} t_1 \omega(1 + \omega)(\omega(7 + 11\omega) + 2(1 + 6\omega + 6\omega^2)) + \frac{1}{240} t_1 t_2 \omega(1 + \omega)(1 + 6\omega + 6\omega^2),
$$

$$
\langle \tau_1(\phi_2) \rangle_2^{X,\text{di}} = \omega^2(61 + 74\omega + 16\omega^2) + \frac{\omega(1 + \omega)(1 + 16\omega + 22\omega^2) \log(1 + \omega)}{120} + \frac{\omega(1 + \omega)(1 + 6\omega + 6\omega^2)(\log(1 + \omega))^2}{60} + \frac{1}{240} t_1^2 \omega(1 + \omega)(1 + 6\omega + 6\omega^2)
$$

$$
- \frac{1}{120} t_1 \omega(1 + \omega) \left(\omega(7 + 11\omega) + 2(1 + 6\omega + 6\omega^2) \log(1 + \omega)\right).
$$

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