A Dual Mesh Method for a Non-Local Thermistor Problem

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Abstract. We use a dual mesh numerical method to study a non-local parabolic problem arising from the well-known thermistor problem.

Key words: non-local thermistor problem; joule heating; box scheme method

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1 Introduction

In this work we propose a dual mesh numerical scheme for analysis of the following non-local parabolic problem coming from conservation law of electric charges:

\[ \frac{\partial u}{\partial t} - \nabla \cdot (k(u)\nabla u) = \lambda \frac{f(u)}{(\int_{\Omega} f(u) \, dx)^2} \quad \text{in} \quad \Omega \times ]0; T[, \]

\[ u = 0 \quad \text{on} \quad \partial \Omega \times ]0; T[, \quad u/t=0 = u_0 \quad \text{in} \quad \Omega, \]

where \( \nabla \) denotes the gradient with respect to the \( x \)-variables. The nonlinear problem (1) is obtained, under some simplificative conditions, by reducing the well-known thermistor problem (cf., e.g., [13, 14, 15]), which consists of the heat equation, with joule heating as a source, and subject to current conservation:

\[ \frac{d u}{d t} = \nabla \cdot (k(u)\nabla u) + \sigma(u)|\nabla \varphi|^2, \quad \nabla \cdot (\sigma(u)\nabla \varphi) = 0, \]

where the domain \( \Omega \subset \mathbb{R}^2 \) occupied by the thermistor is a bounded convex polygonal; \( \varphi = \varphi(x,t) \) and \( u = u(x,t) \) are, respectively, the distributions of the electric potential and the temperature in \( \Omega \); \( \sigma(u) \) and \( k(u) \) are, respectively, the temperature-dependant electrical and thermal conductivities; \( \sigma(u)|\nabla \varphi|^2 \) is the joule heating. The literature on problem (2) is vast (see e.g. [2, 3, 4, 5, 9, 10, 11, 16, 17]). With respect to numerical approximation results to problem (2) we are aware of [1, 11, 12, 18]: in [18] a numerical analysis of the non-steady thermistor problem by a finite element method is discussed; in [12] the authors study a spatially and completely discrete finite element model; in [11] a semi-discretization by the backward Euler scheme is given for the special case \( k = \text{Id} \); in [1] a box approximation scheme is presented and analyzed. A completely discrete scheme based on the backward Euler method with semi-implicit linearization to (2) is presented in [12] for the special case \( k(u) = 1 \). Existence and uniqueness of solutions to the problem [11] were proved in [10].
Finite volume methods emerged recently and seem to have a significant role on concrete applications, because they have very interesting properties in view of the subjacent physical problems: in particular in conservation of flows. An equation coming from a conservation law has a good chance to be correctly discretized by the finite volume method. We also recall that these schemes have been widely used to approximate solutions of the heat linear equation, semilinear or parabolic equations. Since we consider data \( f \) with lack of regularity when compared to previous work, we need a new way to discretize \( \mathbf{1} \). We present a dual mesh method capable of handling the non local term \( \frac{\lambda f(u)}{(I_h f(u))'} \) which is a noticeable feature of \( \mathbf{1} \), by generalizing the results of \( \mathbf{1} \). A box approximation scheme for discretizing \( \mathbf{1} \) with the case \( k \) being different from the identity is obtained. Speed of convergence is directly related with regularity of the continuous problem. When one increases regularity of the second term and data, the solution see its regularity increasing in parallel, and precise speed of convergence can be established. In the existing literature (see e.g. \[ 5, 12 \]) the error estimates for both the finite element or volume element method are usually derived for solutions that are sufficiently smooth. Because the domain is polygonal, special attention has to be paid to regularity of the exact solution. We give sufficient conditions in terms of data and the solution \( u \) that yield error estimates (see hypothesis (H1) below).

The text is organized as follows. In Section 2 we set up the notation and the functional spaces used throughout the paper. Section 3 introduces a box scheme model for problem \( \mathbf{1} \), and existence and uniqueness of the solution of the approximating problem \( \mathbf{12} \) is obtained from its regularity increasing in parallel, and precise speed of convergence can be established. In the previous work, we need a new way to discretize (1). We present a dual mesh method capable of handling the non local term \( \frac{\lambda f(u)}{(I_h f(u))'} \) which is a noticeable feature of (1), by generalizing the results of (1). A box approximation scheme for discretizing (1) with the case \( k \) being different from the identity is obtained. Speed of convergence is directly related with regularity of the continuous problem. When one increases regularity of the second term and data, the solution see its regularity increasing in parallel, and precise speed of convergence can be established. In the existing literature (see e.g. \[ 5, 12 \]) the error estimates for both the finite element or volume element method are usually derived for solutions that are sufficiently smooth. Because the domain is polygonal, special attention has to be paid to regularity of the exact solution. We give sufficient conditions in terms of data and the solution \( u \) that yield error estimates (see hypothesis (H1) below).

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## 2 Notation and functional spaces

Let \((\cdot, \cdot)\) and \(\| \cdot \|\) denote the inner product and norm in \(L^2(\Omega)\); \(H^1_0(\Omega) = \{ u \in H^1(\Omega), u/\partial \Omega = 0 \}\); \(\| \cdot \|_s, \| \cdot \|_{s,p}\) denote the \(H^s(\Omega)\) and the \(W^{s,p}(\Omega)\) norm respectively; \(T_h\) denote a triangulation of \(\Omega\); \(T^h_v\) be the set of vertices of a quasi-uniform triangulation \(T_h\); and \(\{S^0_h\}_{h>0}\) be the family of approximating subspaces of \(H^1_0(\Omega)\) defined by

\[
S^0_h = \{ v \in H^1_0(\Omega) : v/e \text{ is a linear function for all } e \in T_h \}.
\]

In the remainder of this paper we denote by \( c \) various constants that may depend on the data of the problem, and that are not necessarily the same at each occurrence. We assume that the family of triangulations is such that the following estimates \[ 4 \] hold for all \( v \in S^0_h \):

\[
\begin{align*}
\| v \|_{\beta,q} & \leq c h^{r-\beta-2\max\{0,1/p-1/q\}} \| v \|_{r,p}, & 0 \leq r \leq \beta \leq 1, & 1 \leq p, q \leq \infty, \\
\| v \|_{0,\infty} & \leq c \ln h^{\frac{1}{2}} \| v \|_1.
\end{align*}
\]

(3)

Let \( P_h : L^2(\Omega) \to S^0_h \) be the standard \( L^2 \)-projection. One has \[ 4 \]:

\[
\begin{align*}
\| v - P_h v \| + h \| v - P_h v \|_1 & \leq c h^2 \| v \|_2, \\
\| v - P_h v \|_{0,\infty} & \leq c h \| v \|_2, & \| P_h v \|_{1,\infty} & \leq c \| v \|_{1,\infty}.
\end{align*}
\]

(4)

We construct the box scheme \( B_h \) (dual mesh) employed in the discretization as follows. From a given triangle \( e \in T_h \), we choose a point \( q \in \pi \) as the intersection of the perpendicular bisectors of the three edges of \( e \). Then, we connect \( q \) by straight-line segments to the edge midpoints of \( e \). To each vertex \( p \in T^h_v \), we associate the box \( b_p \in B_h \), consisting of the union of subregions which have \( p \) as a corner (see Fig. 1). For the piecewise constant interpolation operator \( I_h \), defined by

\[
I_h : C(\Omega) \to L^2(\Omega), \quad I_h v = v(p), \quad \text{on } b_p \in B_h, \quad \forall p \in T^h_v,
\]

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\[
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\]
we have the following standard error estimates \[1, 3\]:

\[
\begin{align*}
    c^{-1} \|v\| & \leq \|I_h v\| \leq c\|v\|, \quad \forall \ v \in S_h^0, \\
    \|v - I_h v\| & \leq c h \|v\|_1, \quad \forall \ v \in S_h^0.
\end{align*}
\] (5)

We denote by \(N_h(p)\) the set of the neighboring vertices of \(p \in T_h v, \partial b = \bigcup_{p \in T_h b} \partial b_p, \partial b_p = \bigcup_{p* \in N_h(p)} \{\Gamma_{pp*}\}\) (see Fig. 1). Let \(l_{\partial b} : \partial b \to \mathbb{R}^+\) be defined as follows: for \(p \in T_h v\) and \(b_p \in B_h\),

\[
l_{\partial b}/\Gamma_{pp*} = |p - p*| \quad \text{for} \quad p* \in N_h(p).
\]

For \(b \in B_h\), we denote the jump in \(w\) across \(\partial b\) at \(x\) by \([w]_{\partial b}(x) = w(x+0) - w(x-0)\), where \(w(x \pm 0)\) are the outside and inside limit values of \(w(x)\) along the normal directions for \(\partial b\).

We now collect from the literature \[1, 3\] some important lemmas and trace results, that are needed in the sequel.

**Lemma 1.** Assume that \(B_h\) is a dual mesh. If \(v\) is a piecewise linear function, and \(x\) is not a vertex, then

\[
[I_h v]/\partial_{b_p}(x) = \frac{\partial v}{\partial n} l_{\partial b}/\Gamma_{pp*}, \quad x \in \Gamma_{pp*}, \quad \forall \ b \in B_h,
\]

where \(n\) is the unit outward normal vector on \(\partial b\).

The \(h\)-dependent norms are defined as follows:

\[
\|v\|_{1,h} = \left( \sum_{l \in \partial b} \|I_h v|_l|^2 \right)^{1/2} \quad \text{and} \quad \|v\|_{0,h} = \|I_h v\|.
\]

**Lemma 2.** There exists a constant \(c > 0\) such that

\[
\begin{align*}
    c^{-1} \|\nabla v\| & \leq \|v\|_{1,h} \leq c\|\nabla v\|, \quad \forall \ v \in S_h^0, \\
    c^{-1} \|v\| & \leq \|v\|_{0,h} \leq c\|v\|, \quad \forall \ v \in S_h^0.
\end{align*}
\]

**Lemma 3.** For any \(a \in C(\overline{\Omega})\) there exists a positive constant \(c\) such that

\[
\left| - \sum_{b \in B_h} \int_{\partial b} a \frac{\partial u}{\partial n} I_h v \right| \leq c \|u\|_1 \|v\|_1, \quad \forall \ u, v \in S_h^0.
\] (6)

Moreover, if there exists a constant \(a_0 > 0\) such that \(a \geq a_0\) in \(\Omega\), then

\[
\begin{align*}
    c^{-1} \|v\|_1^2 & \leq - \sum_{b \in B_h} \int_{\partial b} a \frac{\partial v}{\partial n} I_h v, \quad \forall \ v \in S_h^0.
\end{align*}
\] (7)
Let \( Q_h : H^2(\Omega) \to S^0_h \) be defined by \( Q_h u - i_h u \in S^0_h \), and
\[
- \sum_{b \in B_h} \int_{\partial b} a \frac{\partial (u - Q_h u)}{\partial n} I_h v = 0, \quad \forall v \in S^0_h,
\]
where \( i_h : C(\Omega) \to S^0_h \) is the Lagrangian interpolation operator and \( u \in H^2(\Omega) \).

**Lemma 4.** Assume that \( a \in L^\infty(\Omega) \), with \( a \geq a_0 \) for some constant \( a_0 > 0 \). Then, there exists \( c > 0 \) such that for \( u \in H^2(\Omega) \)
\[
\| u - Q_h u \|_1 \leq c h \| u \|_2.
\]
Moreover, if \( u \in H^2(\Omega) \cap W^{1,\infty}(\Omega) \), then
\[
\| Q_h u \|_{1,\infty} \leq c (\| u \|_{1,\infty} + \| u \|_2).
\]

**Lemma 5.** For each \( b \in B_h \) one has
\[
h^\frac{1}{2} \| v \|_{L^2(\partial b)} \leq c \left( \| v \|_{L^2(\partial b)} + h \| v \|_{H^1(\partial b)} \right), \quad \forall v \in H^1(b).
\]

Throughout this work, we assume that the following hypotheses on the solution and data of problem (12) are satisfied:

(H1) \( u \in L^\infty(H^1_0(\Omega) \cap H^2(\Omega)), u_t \in L^2(H^1(\Omega)); \)
(H2) \( c^{-1} \leq k(s) \leq c; \)
(H3) there exist positive constants \( c_1, c_2 \) and \( \nu \), such that \( \nu \leq f(\xi) \leq c_1|\xi| + c_2 \) for all \( \xi \in \mathbb{R}; \)
(H4) \( |f(\xi) - f(\xi')| + |k(\xi) - k(\xi')| \leq c|\xi - \xi'|. \)

### 3 Existence and uniqueness result for the box scheme method

Let \( u \) be the solution of (12). Integrating over an element \( b \) in \( B_h \) we obtain:
\[
\int_b u_t - \int_{\partial b} k(u) \frac{\partial u}{\partial n} = \frac{\lambda}{(\int_\Omega f(u) \, dx)^2} \int_b f(u), \quad \forall b \in B_h.
\]

We consider a box scheme defined as follows: find \( u_h \in S^0_h \) such that
\[
(I_h u_t^h, I_h v) - \sum_{b \in B_h} \int_{\partial b} k(u^h) \frac{\partial u^h}{\partial n} I_h v = \frac{\lambda}{(\int_\Omega f(u^h) \, dx)^2} (f(u^h), I_h v), \quad \forall v \in S^0_h,
\]
where \( u^h(0) = P_h u_0 \) and \( I_h \) is the interpolation operator.

**Theorem 1.** Let (H1)–(H4) be satisfied. Then, for each \( h > 0 \), there exists \( t_0(h) \) such that (12) possesses a unique solution \( u^h \) for \( 0 \leq t \leq t_0(h) \).

**Proof.** We begin by proving existence of solution. We define a nonlinear operator \( G \) from \( S^0_h \) to \( S^0_h \) as follows. For each \( u^h \in S^0_h \), \( w^h = G(u^h) \) is obtained as the unique solution of the following problem:
\[
(I_h w_t^h, I_h v) - \sum_{b \in B_h} \int_{\partial b} k(u^h) \frac{\partial w^h}{\partial n} I_h v = \frac{\lambda}{(\int_\Omega f(u^h) \, dx)^2} (f(u^h), I_h v), \quad \forall v \in S^0_h.
\]
We prove that equation (13) verified by this purpose, let $u$. We can easily see that $D$ such that there exists $\| I_h u \|_1 + 1 \| I_h u \|$. Thus, we have $\frac{d}{dt} \| I_h u \|^2 + c \| u \|^2 \leq c \| I_h u \| + c \| u \|_1 + c \| I_h u \|$. Thus, we have

$$
\frac{d}{dt} \| I_h u \|^2 + c \| u \|^2 \leq c \| u \|_1 + c.
$$

(14)

Integrating (14) with respect to $t$ and using the equivalency of $\| I_h \cdot \|$ and $\| \cdot \|$ in $S_h^0$ (see (5)) yields

$$
\| u \|^2 + c \int_0^t \| u \|^2 dt \leq c \| I_h P_h u_0 \|^2 + c \int_0^t \| u \|^2 dt + ct,
$$

$$
\| u \|^2 \leq c \| u_0 \|^2 + c \int_0^t \| u \|^2 dt + ct.
$$

Define now the following set

$$
D = \left\{ u_h \in S_h^0, \| u_h \|^2 + c \int_0^t \| u_h \|^2 dt \leq c (\| u_0 \|^2 + 1) \right\}.
$$

We can easily see that $D$ is closed subset of $L^\infty (0, t, L^2(\Omega))$ with its natural norm. We conclude that there exists $t > 0$ such that $G(D) \subset D$. To obtain that $G$ has a fixed point $u_h = G(u_h)$, we prove that $G$ is a contraction. Conclusion follows from Banach’s fixed point theorem. For this purpose, let $u_h^1$ and $u_h^2 \in S_h^0 \times S_h^0$ such that $Gu_h^1 = u_h^1$ and $Gu_h^2 = u_h^2$. We have, from the equation (13) verified by $u_h^1$ and $u_h^2$, that

$$
(I_h (u_h^1 - u_h^2), I_h v) - \sum_{b \in B_h} \int_{\partial b} k(u_h^1) \frac{\partial u_h^1}{\partial n} I_h v + \sum_{b \in B_h} \int_{\partial b} k(u_h^2) \frac{\partial u_h^2}{\partial n} I_h v
$$

$$
= \frac{\lambda}{(\int_\Omega f(u_h^1) dx)^2} (f(u_h^1), I_h v) - \frac{\lambda}{(\int_\Omega f(u_h^2) dx)^2} (f(u_h^2), I_h v).
$$

On the other hand, one has

$$
- \sum_{b \in B_h} \int_{\partial b} k(u_h^1) \frac{\partial u_h^1}{\partial n} I_h v + \sum_{b \in B_h} \int_{\partial b} k(u_h^2) \frac{\partial u_h^2}{\partial n} I_h v
$$

$$
= - \sum_{b \in B_h} \int_{\partial b} k(u_h^1) \frac{\partial (u_h^1 - u_h^2)}{\partial n} I_h v + \sum_{b \in B_h} \int_{\partial b} (k(u_h^1) - k(u_h^2)) \frac{\partial u_h^2}{\partial n} I_h v.
$$

Schwartz inequality implies that

$$
\sum_{b \in B_h} \int_{\partial b} (k(u_h^1) - k(u_h^2)) \frac{\partial u_h^2}{\partial n} I_h v \geq -c \| u_h^2 \|^2 \int_\Omega f(u_h^1)^2 dx (\sum_{b \in B_h} \| u_h^1 - u_h^2 \|_{0, \partial b}) \| v \|.
$$

By Lemma (5) we have

$$
h^{\frac{1}{2}} \sum_{b \in B_h} \| u_h^1 - u_h^2 \|_{0, \partial b} \leq c \sum_{b \in B_h} \left( \| u_h^1 - u_h^2 \|_{L^2(\Omega)} + h \| u_h^1 - u_h^2 \|_{H^1(\Omega)} \right) \leq c \| u_h^1 - u_h^2 \|_1.$$

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Thus, from the inverse estimate \( \text{(15)} \),

\[
\sum_{b \in B_h} \int_{\partial b} (k(u^h_1) - k(u^h_2)) \frac{\partial u^h_2}{\partial n} I_h v \geq -c(h)\|u^h_1\|_1 \|u^h_1 - u^h_2\|_1 \|v\|_1.
\]

(15)

On the basis of hypotheses (H1)–(H4), we have:

\[
\begin{align*}
\frac{\lambda}{(\int_{\Omega} f(u^h_1) \, dx)^2} (f(u^h_1), I_h v) - \frac{\lambda}{(\int_{\Omega} f(u^h_2) \, dx)^2} (f(u^h_2), I_h v) \\
+ \lambda \left( \frac{1}{(\int_{\Omega} f(u^h_1) \, dx)^2} - \frac{1}{(\int_{\Omega} f(u^h_2) \, dx)^2} \right) (f(u^h_2), I_h v)
\end{align*}
\]

\[
\leq c\|u^h_1 - u^h_2\|_2 \|v\| + \lambda (\int_{\Omega} f(u^h_2) - f(u^h_1)) (\int_{\Omega} f(u^h_2) + f(u^h_1)) (f(u^h_2), I_h v)
\]

\[
\leq c\|u^h_1 - u^h_2\|_2 \|v\| + c\|I_h v\|_{L^2(\Omega)} \|u^h_1 - u^h_2\|_{L^1(\Omega)}
\]

\[
\leq c\|u^h_1 - u^h_2\|_2 \|v\| \leq c\|u^h_1 - u^h_2\|_1 \|v\|_1.
\]

(16)

It follows from \( \text{(15)} \) and \( \text{(16)} \) that

\[
(\int_{\Omega} f(w^h_1 - w^h_2), I_h v) - \sum_{b \in B_h} \int_{\partial b} k(u^h_1) \frac{\partial (u^h_1 - u^h_2)}{\partial n} I_h v
\]

\[
\leq c(h)\|u^h_1 - u^h_2\|_1 \|v\|_1 - \sum_{b \in B_h} \int_{\partial b} (k(u^h_1) - k(u^h_2)) \frac{\partial u^h_2}{\partial n} I_h v \leq c(h)\|u^h_1 - u^h_2\|_1 \|v\|_1.
\]

(17)

Now, using \( v = w^h_1 - w^h_2 \) as a test function in \( \text{(17)} \), we obtain from \( \text{(7)} \):

\[
\frac{1}{2} \frac{d}{dt} \|I_h(w^h_1 - w^h_2)\|^2 + c\|w^h_1 - w^h_2\|^2 \leq c(h)\|u^h_1 - u^h_2\|_1 \|w^h_1 - w^h_2\|_1.
\]

(18)

With use of the Holder’s inequality and equivalency of \( \|I_h \cdot \| \) and \( \| \cdot \| \), integration of \( \text{(18)} \) with respect to time gives:

\[
\|(w^h_1 - w^h_2)\|^2 \leq c\|I_h(w^h_1 - w^h_2)\|^2 \leq c(h) \int_0^t \|(u^h_1 - u^h_2)\|^2_1 \, ds.
\]

Thus \( G \) is a contraction. We prove now uniqueness. Following the same arguments as before, we have

\[
(\int_{\Omega} f(u^h_1|t - u^h_2|t), I_h v) - \sum_{b \in B_h} \int_{\partial b} k(u^h_1) \frac{\partial (u^h_1 - u^h_2)}{\partial n} I_h v \leq c(h)\|u^h_1 - u^h_2\|_1 \|v\|.
\]

(19)

Choosing \( v = u^h_1 - u^h_2 \) as test function in \( \text{(19)} \), using again \( \text{(7)} \) and integrating, we obtain

\[
\|(u^h_1 - u^h_2)\|^2 \leq c(h) \int_0^t \|(u^h_1 - u^h_2)\|^2 \, ds,
\]

which gives, by Gronwall’s Lemma, uniqueness of solution.
4 Error analysis

In this section we prove error estimates under certain assumptions on regularity of the exact solution $u$.

**Theorem 2.** Under assumptions (H1)–(H4), if $(u, u^h)$ are solutions of \((11)\)–\((12)\) for $0 \leq t \leq t_0(h)$, then

$$
\|u^h - u\|_{L^\infty(L^2)} + \|u^h - u\|_{L^2(H^1)} \leq ch.
$$

**Proof.** From \((11)\) and \((12)\) we obtain

$$
(J_h(u^h - P_h u), I_h v) - \sum_{b \in B_h} \int_{\partial b} k(u^h) \frac{\partial (u^h - P_h u)}{\partial n} I_h v
$$

$$
= \frac{\lambda}{(\int_\Omega f(u^h) \, dx)^2} (f(u^h), I_h v) - \frac{\lambda}{(\int_\Omega f(u) \, dx)^2} (f(u), I_h v) + \sum_{b \in B_h} \int_{\partial b} k(u^h) \frac{\partial (P_h u - u)}{\partial n} I_h v
$$

$$
- \sum_{b \in B_h} \int_{\partial b} (k(u) - k(u^h)) \frac{\partial u}{\partial n} I_h v + ((I - P_h) u_t, I_h v) + ((I - I_h) P_h u_t, I_h v).
$$

(20)

We now estimate separately, the terms on the right-hand side of (20). We have from \((6)\) and \((11)\) that

$$
\left| \sum_{b \in B_h} \int_{\partial b} k(u^h) \frac{\partial (P_h u - u)}{\partial n} I_h v \right| \leq c\|P_h u - u\|_1 \|v\|_1 \leq ch\|u\|_2 \|v\|_1 \leq ch\|v\|_1,
$$

(21)

$$
\left| \sum_{b \in B_h} \int_{\partial b} (k(u) - k(u^h)) \frac{\partial u}{\partial n} I_h v \right|
$$

$$
\leq c\|v\|_1 \left( \sum_{b \in B_h} \left( \int_{\partial b} |u - u^h| \left| \frac{\partial u}{\partial n} \right| \right)^2 \right)^{1/2}
$$

$$
\leq ch^{1/2} \|v\|_1 \left( \sum_{b \in B_h} \|u - u^h\|_{0, \partial b}^2 \right)^{1/2}.
$$

(22)

By Lemma \(5\) inverse inequality \(8\) and \(11\), we have:

$$
\sum_{b \in B_h} \|u^h - u\|_{0, \partial b}^2 \leq 2 \sum_{b \in B_h} \|u^h - P_h u\|_{0, \partial b}^2 + 2 \sum_{b \in B_h} \|P_h u - u\|_{0, \partial b}^2
$$

$$
\leq c(h^2 + \|u^h - P_h u\|_2^2).
$$

(23)

Consequently, we obtain from \((23)\) that

$$
\left| \sum_{b \in B_h} \int_{\partial b} (k(u) - k(u^h)) \frac{\partial u}{\partial n} I_h v \right| \leq c(h + \|u^h - P_h u\|_1) \|v\|_1.
$$

(24)

Based on our earlier development in \((16)\), we also know:

$$
\frac{\lambda}{(\int_\Omega f(u^h) \, dx)^2} (f(u^h), I_h v) - \frac{\lambda}{(\int_\Omega f(u) \, dx)^2} (f(u), I_h v) \leq c\|u^h - u\|_1 \|v\|_1.
$$

(25)

Let $v = u^h - P_h u$ be a test function in \((20)\). Using Lemma \(3\) it follows from \((21)\)–\((26)\) that

$$
\frac{1}{2} \frac{d}{dt} \|I_h(u^h - P_h u)\|^2 + c\|u^h - P_h u\|_1^2
$$
Proof. From equations (11) and (12), we have:

\[ \text{Theorem 3.} \]

By properties (4) and Cauchy’s inequality, it follows:

Then, by the triangular inequality, we conclude with the intended result.

By properties \( \mathbb{H} \) and Cauchy’s inequality, it follows:

\[
\frac{d}{dt} \| I_h(u^h - P_h u) \|^2 + c \| u^h - P_h u \|^2 \leq c \| P_h u - u \|_1 \| u^h - P_h u \|_1 \\
+ c \{ h + \| (I - P_h)u_t \| + \| (I - I_h)P_h u_t \| \} \| u^h - P_h u \|_1 \\
\leq c \{ h^2 + \| (I - P_h)u_t \|^2 + \| (I - I_h)P_h u_t \|^2 + \| u^h - P_h u \|^2 \} + \frac{c}{2} \| u^h - P_h u \|^2 \\
\leq c \{ h^2 + h^2 \| u_t \|^2 + ch^2 \| P_h u_t \|_1^2 \} + c \| u^h - P_h u \|^2 + \frac{c}{2} \| u^h - P_h u \|^2.
\]

Hence,

\[
\frac{d}{dt} \| I_h(u^h - P_h u) \|^2 + c \| u^h - P_h u \|^2 \leq ch^2 + c \| u^h - P_h u \|^2.
\]

Integrating (26) and applying Gronwall Lemma and using again the equivalency of \( \| \cdot \| \) and \( \| I_h \cdot \| \), we get that

\[ \| u^h - P_h u \|^2 + c \int_0^t \| u^h - P_h u \|_1^2 \leq ch^2. \]

Then, by the triangular inequality, we conclude with the intended result. \( \blacksquare \)

Under more restrictive hypotheses on the data, it is possible to derive the following error estimate.

**Theorem 3.** Assume (H1)–(H4). If \( k(s) = 1 \) and \( u_0 \in H^1_0(\Omega) \cap H^2(\Omega) \), then

\[ \| u^h - u \|_{L^\infty(H^1)} \leq ch. \]

**Proof.** From equations (11) and (12), we have:

\[
(I_h u^h_t - u_t, I_h v) - \sum_{b \in B_h} \int_{\partial b} \frac{\partial u^h}{\partial n} I_h v + \sum_{b \in B_h} \int_{\partial b} \frac{\partial u}{\partial n} I_h v \\
= \frac{\lambda}{(\int_{\Omega} f(u^h) \, dx)^2} (f(u^h), I_h v) - \frac{\lambda}{(\int_{\Omega} f(u) \, dx)^2} (f(u), I_h v).
\]

Using the definition [3] of \( Q_h \), we get

\[
(I_h u^h_t - u_t, I_h v) - \sum_{b \in B_h} \int_{\partial b} \frac{\partial (u^h - Q_h u)}{\partial n} I_h v \\
= \frac{\lambda}{(\int_{\Omega} f(u^h) \, dx)^2} (f(u^h), I_h v) - \frac{\lambda}{(\int_{\Omega} f(u) \, dx)^2} (f(u), I_h v),
\]

and it follows that

\[
(I_h (u^h_t - Q_h u), I_h v) - \sum_{b \in B_h} \int_{\partial b} \frac{\partial (u^h - Q_h u)}{\partial n} I_h v
\]
We only obtain first-order estimates: higher order estimates are difficult to obtain due to the needs further developments.

In order to estimate the right side hand of the last inequality, we treat both terms separately. By similar arguments to those used in \(10\),

\[
\left| \frac{\lambda}{(\int_{\Omega} f(u^h) \, dx)^2} (f(u^h), I_h v) - \frac{\lambda}{(\int_{\Omega} f(u) \, dx)^2} (f(u), I_h v) \right| \leq c \| u^h - u \| \| v \|.
\]

Taking a function test \( v = (u^h - Q_h u)_t \) in \(27\), by \(9\) and \(10\) we have

\[
\| I_h (u^h - Q_h u)_t \|^2 - \sum_{b \in B_h} \int_{\partial b} \frac{\partial (u^h - Q_h u)_t}{\partial n} I_h (u^h - Q_h u)_t \\
\leq c \{ h + \| (I - I_h) Q_h u_t \| + \| (I - Q_h) u_t \| + \| Q_h u - u^h \| \} \| I_h (u^h - Q_h u)_t \| \\
\leq c \{ h + ch \| u_t \|_2 + ch \| Q_h u_t \|_1 + \| Q_h u - u^h \| \} \| I_h (u^h - Q_h u)_t \| \\
\leq c \{ h + \| Q_h u - u^h \| \} \| I_h (u^h - Q_h u)_t \| \\
\leq ch^2 + c \| Q_h u - u^h \|^2 + \frac{1}{2} \| I_h (u^h - Q_h u)_t \|^2.
\]

Integrating \(28\), we arrive to

\[
\| u^h - Q_h u \|_1^2 \leq c \left( h^2 + \int_0^t \| u^h - Q_h u \|^2 \right) \\
\leq c \left( h^2 + \int_0^t \| u^h - Q_h u \|^2 \right) = ch^2 + c \| u^h - Q_h u \|_{L^2(\Omega)}^2,
\]

and Theorem \(2\) gives

\[
\| u^h - Q_h u \|_1^2 \leq ch^2.
\]

On the other hand, by triangular inequality, \(9\) and the regularity of the exact solution \(u\), we have

\[
\| u^h - u \|_1^2 \leq 2 \| u^h - Q_h u \|_1^2 + 2 \| Q_h u - u \|_1^2 \leq ch^2 \| u \|_2^2 + ch^2 \leq ch^2.
\]

We conclude then with the desired error estimate. 

\[\blacksquare\]

5 Conclusion

In this paper a dual mesh numerical scheme was proposed for a nonlocal thermistor problem. We have showed the existence and uniqueness of the approximate solution via Banach’s fixed point theorem. We have also proved \(H^1\)-error bounds under minimal regularity assumptions. We only obtain first-order estimates: higher order estimates are difficult to obtain due to the nonstandard nonlocal term. Optimal error analysis to the present context, under appropriate smoothness assumptions on data, can be derived by application of the techniques of \(7\), but this needs further developments.
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[1] Allegretto W., Lin Y., Zhou A., A box scheme for coupled systems resulting from microsensor thermistor problems, *Dynam. Contin. Discrete Impuls. Systems*, 1999, V.5, N 1–4, 209–223.
[2] Antontsev S.N., Chipot M., The thermistor problem: existence, smoothness uniqueness, blowup, *SIAM J. Math. Anal.*, 1994, V.25, N 4, 1128–1156.
[3] Cai Z.Q., On the finite volume element method, *Numer. Math.*, 1991, V.58, N 7, 713–735.
[4] Ciarlet P.G., The finite element method for elliptic problems, Amsterdam, North-Holland, 1978.
[5] Chatzipantelidis P., Lazarov R.D., Thomée V., Error estimates for a finite volume element method for parabolic equations in convex polygonal domains, *Numer. Methods Partial Differential Equations*, 2004, V.20, Issue 5, 650–674.
[6] Cimatti G., On the stability of the solution of the thermistor problem, *Appl. Anal.*, 1999, V.73, N 3–4, 407–423.
[7] Cimatti G., Stability and multiplicity of solutions for the thermistor problem, *Ann. Mat. Pura Appl. (4)*, 2002, V.181, N 2, 181–212.
[8] El Hachimi A., Sidi Ammi M.R., Existence of weak solutions for the thermistor problem with degeneracy, in Proceedings of the 2002 Fez Conference on Partial Differential Equations, *Electron. J. Differ. Equ. Conf.*, 2002, V.9, 127–137.
[9] El Hachimi A., Sidi Ammi M.R., Thermistor problem: a nonlocal parabolic problem, in Proceedings of the 2004-Fez Conference on Differential Equations and Mechanics, *Electron. J. Differ. Equ. Conf.*, 2004, V.11, 117–128.
[10] El Hachimi A., Sidi Ammi M.R., Existence of global solution for a nonlocal parabolic problem, *Electron. J. Qual. Theory Differ. Equ.*, 2005, N 1, 9 pages.
[11] El Hachimi A., Sidi Ammi M.R., Semi-discretization for a non local parabolic problem, *Int. J. Math. Math. Sci.*, 2005, N 10, 1655–1664.
[12] Elliott C.M., Larsson S., A finite element model for the time-dependent Joule heating problem, *Math. Comp.*, 1995, V.64, N 212, 1433–1453.
[13] Lacey A.A., Thermal runaway in a non-local problem modelling Ohmic heating. I. Model derivation and some special cases, *European J. Appl. Math.*, 1995, V.6, N 2, 127–144.
[14] Lacey A.A., Thermal runaway in a non-local problem modelling Ohmic heating. II. General proof of blow-up and asymptotics of runaway, *European J. Appl. Math.*, 1995, V.6, N 3, 201–224.
[15] Tzanetis D.E., Blow-up of radially symmetric solutions of a non-local problem modelling Ohmic heating, *Electron. J. Differential Equations*, 2002, N 11, 26 pages.
[16] Xu X., Local and global existence of continuous temperature in the electrical heating of conductors, *Houston J. Math.*, 1996, V.22, N 2, 435–455.
[17] Xu X., Existence and uniqueness for the nonstationary problem of the electrical heating of a conductor due to the Joule–Thomson effect, *Int. J. Math. Math. Sci.*, 1993, V.16, N 1, 125–138.
[18] Yue X.Y., Numerical analysis of nonstationary thermistor problem, *J. Comput. Math.*, 2004, V.12, N 3, 213–223.