Dispersion relation of finite amplitude Alfvén wave in a relativistic electron-positron plasma

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The linear dispersion relation of a finite amplitude, parallel, circularly polarized Alfvén wave in a relativistic electron-positron plasma is derived. In the nonrelativistic regime, the dispersion relation has two branches, one electromagnetic wave, with a low frequency cutoff at $\sqrt{1 + 2\omega_p^2/\Omega_p^2}$ (where $\omega_p = (4\pi n e^2/m)^{1/2}$ is the electron/positron plasma frequency), and an Alfvén wave, with high frequency cutoff at the positron gyrofrequency $\Omega_p$. There is only one forward propagating mode for a given frequency. However, due to relativistic effects, there is no low frequency cutoff for the electromagnetic branch, and there appears a critical wave number above which the Alfvén wave ceases to exist. This critical wave number is given by $c k_c/\Omega_p = a/\eta$, where $a = \omega_p^2/\Omega_p^2$ and $\eta$ is the ratio between the Alfvén wave magnetic field amplitude and the background magnetic field. In this case, for each frequency in the Alfvén branch, two additional forward propagating modes exist with equal frequency.

A simple numerical example is studied: by numerically solving the coupled system of fluid and Maxwell equations, normal incidence of a finite amplitude Alfvén wave on an interface between two electron-positron plasmas of different densities is considered.

\section{I. INTRODUCTION}

Electron-positron plasmas are different from electron-ion plasmas, because in the absence of a mass difference, there are no high or low natural frequency scales. Such plasmas are found in pulsar magnetospheres, models of primitive Universe, active galactic nuclei jets, and laboratory and tokamak plasmas. Relativistic effects are expected to play an important role in several of these systems. Understanding interactions between waves and relativistic electron-positron plasmas is relevant to proposed pulsar emission mechanisms and may give insight into structure formation in the early Universe.

Therefore, wave propagation in relativistic electron-positron plasmas has been the subject of many studies, either in the kinetic or the fluid treatments: linear waves, nonlinear waves, and nonlinear decays.

In this article we deal with an Alfvén wave propagating along a constant magnetic field in a pair plasma. When fully relativistic effects are considered in the particle motion, the dispersion relation exhibits unique features which, to our knowledge, have not been discussed before. We then outline the numerical strategies we are currently considering to examine the consequences of such features.

\section{II. DISPERSION RELATION}

We assume that the electron-positron plasma is described by the following set of equations:

\begin{equation}
\frac{\partial n_j}{\partial t} = -\vec{\nabla} \cdot (n_j \vec{v}_j),
\end{equation}

\begin{equation}
\left( \frac{\partial}{\partial t} + \vec{v}_j \cdot \vec{\nabla} \right) (\gamma_j \vec{v}_j) = \frac{q_j}{m_j} \left( \vec{E} + \frac{1}{c} \vec{v}_j \times \vec{B} \right),
\end{equation}

\begin{equation}
\vec{\nabla} \cdot \vec{E} = 4\pi \rho,
\end{equation}

\begin{equation}
\vec{\nabla} \times \vec{E} = \frac{1}{c} \frac{\partial \vec{B}}{\partial t},
\end{equation}

\begin{equation}
\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t},
\end{equation}
\[ \vec{J} = \sum_j q_j n_j \vec{v}_j , \]  
\[ \rho = \sum_j q_j n_j , \]  
\[ \gamma_j = \left( 1 - \frac{\vec{v}_j^2}{c^2} \right)^{-1/2} , \]  
where \( n_j \) is the density of each fluid, \( \vec{v}_j \) is the bulk velocity of each fluid, \( \vec{E} \) and \( \vec{B} \) are the electric and magnetic fields, respectively, \( \vec{J} \) is the total current, \( m \) is the particle mass, and \( c \) is the speed of light. \( j = p \) for positrons, and \( j = e \) for electrons.

We assume that a circularly polarized Alfvén wave propagates along the \( z \)-axis, as well as the existence of a constant magnetic field in the same direction, \( B_0 \hat{z} \). The wave electric and magnetic fields are given by:

\[ \vec{B} = B_0 \hat{z} \cos(kz - \omega t) + \hat{y} \sin(kz - \omega t) ] , \]  
\[ \vec{E} = \frac{\omega c k}{\hat{y}} [ \hat{x} \sin(kz - \omega t) - \hat{y} \cos(kz - \omega t) ] . \]  
Introducing (9) and (10) in the fluid equations, the transverse velocity for each species is obtained as 
\[ v_j c = \frac{\omega c k \Omega_j }{\omega (\gamma_j \omega - \Omega_j )} , \]  
where \( \Omega_j = q_j B_0 / m_j c \) is the cyclotron frequency, and \( \eta = B/B_0 \). The dispersion relation for the Alfvén wave is

\[ \frac{c^2 k^2}{\omega^2} = 1 - \sum_j \frac{\omega_j^2}{\omega (\gamma_j \omega - \Omega_j )} , \]  
where \( \omega_j = (4\pi n_{j0} q_j^2 / m_j)^{1/2} \) is the plasma frequency and \( n_{j0} \) is the rest density of species \( j \).

We want to numerically solve the dispersion relation, which is equivalent to simultaneously solving the set of equations (8), (11) and (12). For an electron positron plasma, \( \omega_p = \omega_e \). We define

\[ x = \frac{\omega}{\Omega_p} , \quad y = \frac{ck}{\Omega_p} , \quad a = \frac{\omega_p^2}{\Omega_p^2} , \quad u = \frac{v}{c} . \]  
Normalized equations (8), (11) and (12), are

\[ \gamma_j = (1 - u_j^2)^{-1/2} , \]  
\[ u_j = \frac{x \eta \pm 1}{y x} \gamma_j + 1 \]  
\[ y^2 = x^2 - \frac{ax}{x \gamma - 1} - \frac{ax}{x \gamma + 1} \]  
where the upper (lower) sign in (15) is for positrons (electrons).

Eliminating \( u_j \), the following equation is obtained for \( \gamma \):

\[ \gamma_j^4 + \gamma_j^3 \left( \pm \frac{2}{x} \right) + \gamma_j^2 \left( 1 + \frac{1}{x^2} - \frac{\eta^2}{y^2} \right) + \gamma_j \left( \pm \frac{2}{x} \right) - \frac{1}{x^2} = 0 . \]  
For a given \( y \), from (17) \( \gamma_j \) is calculated as a function of \( x \), and then the right hand size of Eq. (16) can be plotted. This is shown in Fig. 1. Since \( y \) is given, we plot the left side of Eq. (16) on the same graph, and the intersection points of the curves are the roots of the dispersion relation. The curve on the right corresponds to the light wave branch of the dispersion relation, and the curve on the left corresponds to the Alfvén branch. It is interesting to note that when \( y > y_c = a/\eta \) the Alfvén wave ceases to exist.

Thus dispersion relation (16) can be solved. The result is shown in Fig. 2. For \( \eta = 0 \) [Fig. 2(a)], the nonrelativistic result is recovered. There are two branches, one corresponds to the light wave, with a cutoff at \( \omega/\Omega_p = \sqrt{1 + 2a} \). The
other one is the Alfvén branch, which has a resonance at the positron gyrofrequency. For \( \eta = 0.1 \) [Fig. 2(b)], however, the light wave branch has no low frequency cutoff, and the Alfvén branch stops at \( y_c = a/\eta \). For even higher values of the wave amplitude \( \eta \) [see Fig. 2(c) for \( \eta = 1 \)], the Alfvén branch exists for a very short wavenumber range. The Alfvén branch is also constrained to a shorter frequency range. This can also be seen in Fig. 1. From Eqs. (14)–(16) it can be shown that the Alfvén branch in Fig. 1 has a frequency cutoff at the critical frequency

\[
x_c = \frac{\omega_c}{\Omega_p} = \left[ 1 + \left( \frac{\eta}{y} \right)^{2/3} \right]^{-3/2}.
\]

(18)

This cutoff depends on the wave amplitude \( \eta \), unlike the nonrelativistic result, where \( \omega_c = \Omega_p \).

FIG. 2: Dispersion relation (16) for (a) \( a = 1, \eta = 0 \); (b) \( a = 1, \eta = 0.1 \); (c) \( a = 1, \eta = 1 \). Since the relevant parameter is \( y_c = a/\eta \), Figs. (b) and (c) are reobtained if \( \eta = 0.1 \) is constant, whereas \( a \) is decreased from \( a = 1 \) to \( a = 0.1 \).

### III. OUTLINE OF THE NUMERICAL STUDY

Fig. 2 shows that, for a given frequency \( \omega \), the existence of Alfvén waves and the number of wave modes depend on the physical parameters of the plasma. This poses a number of new questions. For instance, we could consider the problem of normal incidence of a plane wave on a density interface. In the usual problem of a light wave in a dielectric medium, only one mode exists for a given frequency. Given only the boundary conditions that the fields must satisfy, it is possible to completely solve the problem and obtain explicit solutions for the field amplitude at both sides of the interface. However, now several modes exist for a given frequency. Moreover, if an Alfvén wave exists with a given frequency in one side of the interface, then if the density is small enough on the other side it could not exist. Does it become an evanescent wave? Is it converted to other modes?

In order to investigate this we are studying the system by means of a fluid simulation, and by direct time integration of the evolution equations.
For the fluid simulation, we notice that the system equations lead to the following wave equations for the electromagnetic field:

\[
\left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \vec{B} = \frac{4\pi}{c} e n_p \vec{v} \times (\vec{v}_p - \vec{v}_e) ,
\]

\[
\left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \vec{E} = -\frac{4\pi}{c} e n_p \frac{\partial}{\partial t} (\vec{v}_p - \vec{v}_e) .
\]

The wave equations for the fields (19) and (20), and the momentum equation (3), are discretized using time-centered and space-centered finite differences.

The second approach we are developing is time integration of the evolution equations (1)-(8) by means of the rationalized Runge-Kutta method.

Both procedures are under development/testing. A particularly simple problem is that of a pure Alfvén mode propagating in a relativistic plasma. Since in a pair plasma there is no harmonic generation, all oscillatory fields and velocities are purely transverse. We are currently working to obtain satisfactory results for this problem, in order to later consider incidence on a normal interface, where longitudinal oscillations may not be neglected.

**IV. CONCLUSIONS**

Propagation of a finite amplitude Alfvén wave in an electron-positron plasma has been studied. Full relativistic effects on the particle velocities in the wave field have been considered. The dispersion relation for propagation along a constant magnetic field has been obtained and numerically solved. Several features are different to the usual nonrelativistic result [Fig. 2(a)]. In the nonrelativistic case, there are two branches, an Alfvén wave and a light wave. The light wave has a low frequency cutoff at the frequency \( \omega/\Omega_p = \sqrt{1 + 2a} \), and the Alfvén wave has a high frequency cutoff at the positron gyrofrequency, \( \omega = \Omega_p \). However, in the relativistic case the light wave branch does not have a low frequency cutoff. As to the other branch, there are two Alfvén wave modes for any given frequency, and there is a high wavenumber cutoff at \( \omega c / \Omega_p = a/\eta \). Besides, the high frequency cutoff for the Alfvén wave depends on the plasma parameters. For large amplitude wave or small plasma density, the Alfvén branch is constrained to a small region of frequencies and wavenumbers [see Fig. 2(c)].

We intend to investigate the behavior of the system when a relativistic Alfvén wave of frequency \( \omega \) propagates through a pair plasma, and then finds a region where plasma parameters are such that such wave cannot propagate. The existence of more normal modes than in the usual nonrelativistic problem make this a nontrivial matter. As a first example, we are considering the simple problem of normal incidence of an Alfvén wave on a planar density discontinuity. Two approaches are being developed, namely a fluid simulation, and time integration of the evolution equations. Work on this is in progress, and we expect to present results elsewhere.

It is worth noting that for waves in the high-wavenumber end of the Alfvén branch in Fig. 2, particles have large Lorentz factors, and therefore synchrotron radiation emission will be important. Thus, the results presented here may be further modified when additional effects such like synchrotron loss are taken into account.

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