Mathematical modeling of creep and viscoplastic flow of a cylindrical layer material

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Abstract. Within the framework of a mathematical model of large elastoplastic deformations, taking into account viscous properties of materials, a boundary value problem of creep and viscoplastic flow of a material in the gap between two rigid coaxial cylindrical surfaces under rectilinear motion of the inner surface is considered. Irreversible deformations accumulate from the beginning of the deformation process and are initially associated with the creep process of the material, which is described using the Norton creep power law. With a subsequent increase in the speed of movement of the inner cylinder at some time point, the stress state will reach the yield surface and a viscoplastic flow will begin in the material. As a plastic potential, we use the Tresca yield criterion, generalized in the case of taking into account the viscous properties of the material. The viscoelastic deformation, the emergence and development of the viscoplastic flow under uniformly accelerated motion of the inner surface, the flow under constant speed, and the flow inhibition under equal slow motion of the surface to a full stop are considered. Stresses, reversible and irreversible deformations, displacements are calculated, stress relaxation after the complete stop of the cylinder is investigated.

1. Introduction
The most preferred method of processing some structural materials at the present time is a technology of cold forming, when large irreversible deformations are accumulated at the expense of the slow creep process. The difficulty in applying this technology lies in the fact that at impact a rigging to the moldable material in the places of their contact plastic areas are formed. Their presence substantially redistributes the stress field and, therefore, directly affects the creep process as a whole. Since in such industrial technologies of processing materials by pressure the materials significantly change the shape, therefore, they acquire large deformations. Estimated forecasting of processes in such technologies is based mainly on rigid-plastic analysis. The neglect of reversible deformations present in technological operations is absolutely not satisfied for technological practice, since exceptionally important effects are not taken into account: spring back and the formation of residual stresses, which significantly reduce the performance characteristics of metal structures. Thus, the development of mathematical modeling tools is an urgent task for calculating the optimal parameters of intensive deformation processes.

In [1], in the framework of the model of large elastoplastic deformations [2], it was proposed to divide the irreversible deformations acquired by the body on the deformations of creep and plastic flow by the mechanism of their production. Elastoplastic boundaries are surfaces on which the mechanism of accumulation of irreversible deformations changes from viscous to plastic and...
vice versa. The laws of creep and plastic flow must be harmonized so that on such surfaces a continuous growth of irreversible deformations takes place, which is achieved by an appropriate choice of plasticity conditions and creep laws. In [3], using the example of solving the boundary value problem of the theory of small deformations, an approach to such an agreed choice is given in the Norton creep power law [4] and in the von Mises yield criterion [5]. In the present paper, we consider the problem of deformation of a cylindrical layer and point out at this example the matching in the Norton creep law and Tresca yield criterion [6].

2. Basic model relations
Consider the basic model relations of the theory of large elastoplastic deformations. In a rectangular system of spatial Cartesian coordinates of Euler \( x_i \) the kinematics of a medium is given by the dependencies

\[
\begin{align*}
d_{ij} &= 0.5 \left( u_{i,j} + u_{j,i} - u_{k,i}u_{k,j} \right) = \varepsilon_{ij} + p_{ij} - 0.5\varepsilon_{ik}\varepsilon_{kj} - \varepsilon_{ik}p_{kj} - \varepsilon_{ik}p_{kj} + \varepsilon_{ik}p_{km}\varepsilon_{mj}, \\
\frac{De_{ij}}{Dt} &= \varepsilon_{ij} - \gamma_{ij} - 0.5 \left( (\varepsilon_{ik} - \gamma_{ik} + z_{ik})\varepsilon_{kj} + \varepsilon_{ik}(\varepsilon_{kj} - \gamma_{kj} - z_{kj}) \right), \\
\frac{Dp_{ij}}{Dt} &= \gamma_{ij} - p_{ik}\gamma_{kj} - \gamma_{ik}p_{kj}, \\
\frac{Dy_{ij}}{Dt} &= \frac{dy_{ij}}{dt} - r_{ik}y_{kj} + y_{ik}\gamma_{kj}, \\
\varepsilon_{ij} &= 0.5 \left( v_{i,j} + v_{j,i} \right), \\
v_i &= \frac{du_i}{dt} = \frac{\partial u_i}{\partial t} + u_{i,j}v_j, \\
u_{i,j} &= \frac{\partial u_i}{\partial x_j}, \\
r_{ij} &= w_{ij} + \varepsilon_{ij} \left( \varepsilon_{sk}, \varepsilon_{sk} \right), \\
w_{ij} &= 0.5 \left( v_{i,j} - v_{j,i} \right).
\end{align*}
\]

Here \( u_i, v_i \) are components of displacements and velocities of medium points; \( d_{ij} \) are components of the Almansi strain tensor; \( \varepsilon_{ij} \) and \( p_{ij} \) are their reversible and irreversible components; \( D/Dt \) is an operator of the used objective derivative of tensors with respect to time, which is written for an arbitrary tensor \( y_{ij} \); \( \gamma_{ij} \) and \( \varepsilon_{ij}^0 = \varepsilon_{ij} - \gamma_{ij} \) are rates of accumulation of irreversible and reversible deformations. The rotational tensor \( r_{ij} \) differs from the classical velocity vortex tensor \( w_{ij} \) by the presence of a nonlinear part \( z_{ij} \), which is completely given in [2].

Stresses in the medium are completely determined by irreversible deformations and for an incompressible medium are calculated from the relation

\[
\sigma_{ij} = -p\delta_{ij} + \frac{\partial W}{\partial e_{ik}} \left( \delta_{kj} - e_{kj} \right).
\]

In equation (2) \( \sigma_{ij} \) are components of the Euler–Cauchy stress tensor, \( p \) is unknown function of additional hydrostatic pressure, \( W \) is an elastic potential having a form

\[
W = -2\mu I_1 - \mu I_2 + b I_1^2 + (b - \mu)I_1 I_2 - \chi I_1^3 + ..., \\
I_1 = \varepsilon_{kk} - 0.5\varepsilon_{ks}\varepsilon_{sk}, \\
I_2 = \varepsilon_{ks}\varepsilon_{sk} - \varepsilon_{ks}\varepsilon_{st}\varepsilon_{tk} + 0.25\varepsilon_{ks}\varepsilon_{st}\varepsilon_{tn}\varepsilon_{nk}.
\]

Here \( \mu \) is the shear modulus, \( b, \chi \) are material constants.

The dissipative mechanism of irreversible deformation is associated with the rheological and plastic properties of the material. Farther we believe that irreversible deformations accumulate from the beginning of the deformation process and are initially associated with the creep process of the material. In areas where the stress state has not yet reached the yield surface, or where the plastic flow has been occurring but stopped, the corresponding dissipative mechanism of deformation we set in the form of Norton creep power law

\[
V (\sigma_{ij}) = B\Sigma^n (\sigma_1, \sigma_2, \sigma_3), \quad \Sigma = \max |\sigma_i - \sigma_j|, \quad \gamma_{ij} = \varepsilon_{ij} = \frac{\partial V (\Sigma)}{\partial \sigma_{ij}},
\]

(4)
wherein \( \sigma_1, \sigma_2, \sigma_3 \) are principal values of stress tensor, \( B, n \) are material creep parameters, \( \varepsilon_{ij}^v \) are components of tensor of creep strain rates.

Over time the stress state reaches the yield surface, the dissipative mechanism of deformation changes and plastic flow begins in the material. Without separation of irreversible deformations into components, we assume that irreversible deformations of creep accumulated by the time of the onset of plastic flow are initial values for plastic deformations that accumulate further in the flow region. In the case of taking into account the viscous properties of the medium during plastic flow, the coincidence of the rates of irreversible deformations is also required when the deformation mechanism changes from viscous to plastic.

According to the Mises maximum principle the relationship between plastic strain rates \( \varepsilon_{ij}^p \) and stresses is established by the associated law of plastic flow

\[
\alpha_{ij} = \lambda \frac{\partial F}{\partial \sigma_{ij}}, \quad F(\sigma_{ij}, \alpha_{ij}) = k, \quad \lambda > 0, \quad \alpha_{ij} = \varepsilon_{ij}^p - \varepsilon_{ij}^v, \tag{5}
\]

where \( F \) is a plastic potential, \( k \) is the yield strength, \( \varepsilon_{ij}^v \) are components of tensor of creep strain rates at the start time of plastic flow.

As a plastic potential we will use the Tresca yield criterion, generalized in the case of taking into account the viscous properties of the material

\[
\max |\sigma_i - \sigma_j| = 2k + 2\eta \max |\alpha_n|. \tag{6}
\]

Here \( \eta \) is a viscosity coefficient.

3. Boundary value problem statement and viscoelastic deformation

Consider the boundary value problem about creep and viscoplastic flow in an incompressible material located in the gap between two rigid cylindrical surfaces. The inner cylindrical surface of radius \( r = r_0 \) moves straightforward, and the outer surface of radius \( r = R \) is rigidly fixed. We consider that sticking conditions are satisfied on cylindrical walls. Then in the cylindrical coordinate system \( r, \varphi, z \) boundary conditions of the problem take the form

\[
v|_{r=R} = u|_{r=R} = 0, \quad v|_{r=r_0} = v_0, \quad u|_{r=r_0} = \int_0^t v_0 dt, \quad \sigma_{rr}|_{r=r_0} = a_0. \tag{7}
\]

Here \( v = v_z(r, t), u = u_z(r, t) \) are the only non-zero components of the velocity and displacement vectors, respectively; \( v_0 = v_0(t), a_0 = a_0(t) \) are given functions.

From relations (1) in this case we establish that the kinematics of the medium is described by the dependencies

\[
d_{rr} = -0.5u_{r,r}^2, \quad d_{rz} = 0.5u_{r,r}, \quad \varepsilon_{rz} = 0.5v_{r,r}. \tag{8}
\]

In (8) index after the comma denotes partial derivative with respect to the spatial coordinate.

To simplify the solution of the problem, we confine ourselves to the first-order terms along the diagonal components and the second-order terms along non-diagonal components. From relations (2) and (3) in this case we find the stresses in the medium

\[
\sigma_{rr} = -P + 2\mu \varepsilon_{rr}, \quad \sigma_{\varphi\varphi} = -P - 3\mu \varepsilon_{rz}^2, \quad \sigma_{zz} = -P + 2\mu \varepsilon_{zz}, \quad \sigma_{rz} = 2\mu \varepsilon_{rz}, \quad P = -(p + 2\mu) + 2b (\varepsilon_{rr} + \varepsilon_{zz}) + \mu \varepsilon_{rz}^2.
\]

In the framework of the quasistatic approximation we write the equilibrium equations for our problem

\[
\sigma_{rr,r} + \sigma_{rz,z} + (\sigma_{rr} - \sigma_{zz}) r^{-1} = 0, \quad \sigma_{rz,r} + \sigma_{zz,z} + \sigma_{rz} r^{-1} = 0. \tag{9}
\]
We integrate equations (9) under the assumption that the stresses are finite, i.e. $P_z = 0$:

$$\sigma_{rz} = c(t)r^{-1}, \quad e_{rz} = 0.5\mu^{-1}c(t)r^{-1}, \quad P = f(r, t). \tag{10}$$

We rewrite relations (4) for our problem

$$V(\sigma_{ij}) = B \left(4\sigma_{rr}^2 + (\sigma_{rr} - \sigma_{zz})^2\right)^{n/2}, \quad \varepsilon_{rr}^v = (-1)^{n-1}2^{n-1}Bn\sigma_{rr}^{n-1},$$

$$\varepsilon_{rr}^v = 0.5\varepsilon_{zz}^p (e_{rr} - e_{zz}) e_{rr}^{-1}. \tag{11}$$

Using the dependencies $\varepsilon_{rz} = \varepsilon_{rz}^p + \varepsilon_{rz}^v$, (8), (10), (11) and sticking conditions (7) on the surface $r = R$ we get the ratios for the velocity and displacement

$$v = F_1(\dot{c}, r, R) + F_2 (c^{n-1}, r, R), \quad u = F_1 (c, r, R) + F_2 (c_1, r, R)$$

$$F_1 (c, r, R) = \frac{c}{\mu} \ln \frac{r}{R}, \quad F_2 (c_1, r, R) = \frac{(-1)^{n-1}2^{n-1}Bn c_1}{2 - n} \left(\frac{1}{r^2} - \frac{1}{R^2}\right), \quad c_1(t) = \int_0^t c^{n-1}dt. \tag{12}$$

Overdot denotes the time derivative. Taking into account the boundary conditions (7) on the inner wall $r = r_0$, from (12) we obtain an ordinary differential equation for the unknown function $c = c(t)$

$$F_1 (\dot{c}, r_0, R) + F_2 (c^{n-1}, r_0, R) = v_0, \quad c(0) = 0. \tag{13}$$

Using the equation for changing the components of irreversible deformations from (1), we get that $\dot{p}_{rz} = \dot{\varepsilon}_{rz}^p$. Integrating this equation we find

$$p_{rz} = (-1)^{n-1}2^{n-1}Bn c_1 r^{n-1}. \tag{14}$$

With increasing speed of movement of the inner cylinder $v_0$, the obtained solution of the problem will be valid up to a time point $t = t_0$ at which on the inner wall $r = r_0$ the plasticity condition (6) will be fulfilled, in our case taking the form $\sigma_{rz}|_{r=r_0} = -k$. From the first relation (10), we obtain an equation for determining the time point $t = t_0$ at which the viscoplastic flow begins: $c(t_0) = -kr_0$.

4. Viscoplastic flow

The region of the viscoplastic flow $r_0 \leq r \leq m(t)$ developing from the time point $t = t_0$ is bounded by surfaces $r = r_0$ and $r = m(t)$. Viscoelastic deformation still occurs in the area $m(t) \leq r \leq R$. Surface $r = m(t)$ is a moving boundary of the viscoplastic flow region.

The dependencies of the previous section with an unknown integration function $c = c(t)$ remain true in the region $m(t) \leq r \leq R$.

Integrating the equilibrium equations (9) in the region of the viscoplastic flow $r_0 \leq r \leq m(t)$ and using the condition of continuity of the stress components on the elastoplastic boundary $r = m(t)$, we obtain that dependencies (10) also hold in this region.

From relations (5), (6) and (10) we find

$$\sigma_{rz} = -k + \eta (\varepsilon_{rz}^p - \varepsilon_{rz}^Tu) \quad \varepsilon_{rz}^p = \eta^{-1}(c_{rz}r^{-1} + k) + \varepsilon_{rz}^Tu. \tag{14}$$

From the condition of coincidence of the rates of irreversible deformations on the elastoplastic boundary $r = m(t)$ we find from (14) that $c(t) = -km(t)$.
In the flow region $r_0 \leq r \leq m(t)$ using relations $\varepsilon_{rz} = \varepsilon_{rz}^p + \varepsilon_{rz}^v$, (8), (10), (14) and boundary condition for the velocity on the surface $r = r_0$ (7) we find

$$v = F_1 (\dot{c}, r, r_0) + F_3 (c, r, r_0) + v_0, \quad F_3 (c, r, r_0) = 2\eta^{-1} \left( c (\ln (r r_0^{-1}) + k (r - r_0)) - 2^n k^{n-1} Bn (r - r_0). \right) \quad (15)$$

The position of the elastoplastic boundary $r = m(t)$ at each moment of time is given by an ordinary differential equation that follows from (12), (15) and the condition of continuity of the velocity of material points on this boundary

$$F_1 (-km, r_0, R) + F_2 ((-km)^{n-1}, m, R) - F_3 (-km, m, r_0) = v_0.$$

Integrating the rate of plastic deformations $\varepsilon_{rz}^p$ (14) on time and taking into account (10) and (11), we find the component of irreversible deformations $p_{rz}$ in the flow region

$$p_{rz} = \frac{1}{\eta} \left( \frac{c_2}{r} + k t \right) - 2^{n-1} k^{n-1} Bnt + g, \quad c_2 = \int_{t_0}^{t} c dt. \quad (16)$$

Here $g = g(r)$ is an unknown integration function, which we find from the condition of continuity of irreversible deformations (13) and (16) on the moving elastoplastic boundary

$$g = (-1)^n 2^{n-1} Bnc_1 (\zeta) r^{1-n} - \eta^{-1} (c_2 (\zeta) r^{-1} + k \zeta) + 2u^{-1} k^{n-1} Bn \zeta. \quad (17)$$

In dependency (17) the function $\zeta = \zeta(r)$ is determined from an ordinary differential equation

$$\zeta, \left( F_2 ((-kr)^{n-1}, r, R) - F_3 (-kr, r, r_0) - v_0 (\zeta) \right) = F_1 (k, r_0, R), \quad \zeta (r_0) = t_0.$$

Taking into account the relations (10), (16), equality $u_r = 2 (\varepsilon_{rz} + p_{rz})$ and the condition of continuity of displacements on the elastoplastic boundary $r = m(t)$, we find an expression for the displacements in the region of viscoplastic flow

$$u = F_1 (c, r, R) + F_2 (c_1, m, R) + t F_3 (t^{-1} c_2, r, m) + 2 \int_{m}^{r} g dr.$$

When performing calculations, the function $v_0$ was assumed linear: $v_0 = \xi_1 t$ in the interval $0 \leq t \leq t_1$, $v_0 = \xi_1 t_1$ in the interval $t_1 \leq t \leq t_2$, $v_0 = \xi_1 t_1 - \xi_2 (t - t_2)$ in the interval $t_2 \leq t \leq t_4$ and $v_0 = 0$ for $t \geq t_4$. The calculations were carried out in dimensionless variables $r R^{-1}, \tau = \xi_1 t^2 r_0^{-1}$ at constant values $k \mu^{-1} = 0.003, r_0 R^{-1} = 0.2, n = 3, B \mu^2 \sqrt{r_0 \xi_1^{-1}} = 3, \xi_1 \xi_2^{-1} = 0.5$. Fig. 1 shows the graphs of the elastoplastic boundary as a function of time in the deformation process (at time point $\tau_3$ the elastoplastic boundary has a maximum value; at time point $\tau_5$ the elastoplastic boundary has a minimum value $m = r_0$). The change of irreversible deformations in the interval $0 \leq \tau \leq \tau_5$ on the inner surface $r = r_0$ is shown in Fig. 2. Fig. 3 illustrates the distribution of displacements in the cylindrical layer at different time points and relaxation of the stress component $\sigma_{rz}$ from time point $\tau_5$ to $\tau = 1000$.

5. Conclusion

Here is given the solution to the problem of viscoelastic deformation and viscoplastic flow of the material at the increasing speed of movement of the internal cylinder. The following stages of deformation are considered, but not shown here: the viscoplastic flow at a constant speed of the inner cylinder, flow deceleration at a decreasing speed and medium unloading. We note that the ratios for these stages in general have the same form as the above dependencies.
Figure 1. Dependencies of the elastoplastic boundary on time.

Figure 2. Irreversible deformations depending on time at the point \( r = r_0 \).

Figure 3. Displacements and stresses at different time points.

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