Rayleigh-Taylor Instability at Ionization Fronts: Perturbation Analysis

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ABSTRACT
The linear growth rate of the Rayleigh-Taylor instability (RTI) at ionization fronts is investigated via perturbation analysis in the limit of incompressible fluids. In agreement with previous numerical studies is found that absorption of ionizing radiation inside the H\textsc{ii} region due to hydrogen recombinations suppresses the growth of instabilities. In the limit of a large density contrast at the ionization front the RTI growth rate has the simple analytical solution \( n = -\nu_{\text{rec}} + (\nu_{\text{rec}}^2 + gk)^{1/2} \), where \( \nu_{\text{rec}} \) is the hydrogen recombination rate inside the H\textsc{ii} region, \( k \) is the perturbation’s wavenumber and \( g \) is the effective acceleration in the frame of reference of the front. Therefore, the growth of surface perturbations with wavelengths \( \lambda \gg \lambda_{\text{cr}} \equiv 2\pi g/\nu_{\text{rec}}^2 \) is suppressed by a factor \( \sim (\lambda_{\text{cr}}/4\lambda)^{1/2} \) with respect to the non-radiative incompressible RTI. Implications on stellar and black hole feedback are briefly discussed.

Key words: H\textsc{ii} regions radiative transfer - hydrodynamics instabilities methods: analytical – ISM: kinematics and dynamics ISM: jets and outflows – quasars: supermassive black holes

1 INTRODUCTION
Radiation feedback from stars and black holes is an ubiquitous physical process important for understanding the evolution of galaxies and the growth of black holes (BHs). Radiation-driven galactic winds may generate powerful gas outflows regulating the co-evolution of galaxies and the supermassive black holes at their centers \cite{King2003, Proga2006, DiMatteo2005, Murray2009, Murray2011, Hopkins2012}. Rayleigh-Taylor instability (RTI) can occur in these winds because a high-density shell is supported by a low-density fluid against a gravitational field or because the shell is accelerating outward \cite{Chandrasekhar1961, Stone2007}. Renewed interest on the growth of the RTI has followed these studies because the fragmentation of dense galactic supershells accelerated by thermal or radiation pressure, can reduce significantly the wind power, making this feedback loop less effective \cite{Krumholz2012, Jiang2013}. This problem has also prevented a definitive identification of the driving mechanisms for winds observed in a variety of systems \cite{Faucher2012}.

Among various stabilization mechanisms of the RTI, the presence of a D-type (density) ionization front (I-front) has recently been considered \cite{Park2013}. That I-fronts are stable to longitudinal perturbations has long been known \cite{Kahn1958, Axford1964}. The stabilizing mechanism is the gas opacity to ionizing radiation, that is typically due to hydrogen recombinations in the H\textsc{ii} region. Let’s consider a perturbation at the I-front displacing the front further from the ionizing source. The resulting increase of the column density of ionized gas between the radiation source and the I-front increases the gas opacity to ionizing radiation (due to the increased number of recombinations per unit time along the ray), thus decreases the ionizing flux at the I-front acting as a restoring mechanism of the perturbation. Similarly, for perturbations displacing the I-front toward the ionizing source the increase of the ionization flux at the I-front acts as a restoring mechanism of the perturbation. Based on a simple dimensional analysis is therefore expected that the growth rate of the RTI is reduced for

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perturbations with wavenumber \( k > k_{cr} \approx \nu_0^2 / g \). Here, \( \nu_0 \) is the hydrogen recombination rate inside the H ii region and \( g \) is the effective acceleration in the frame of reference of the I-front. However, a formal perturbation analysis of the I-front is necessary to derive the linear growth rate of the RTI including the effect of recombinations.

This paper presents a linear stability analysis of accelerating I-fronts and I-fronts in an external gravitational field. As far as we know this is the first published analytical perturbation analysis of accelerating I-fronts in which the RTI growth rate is derived including the stabilizing effect of the gas opacity to ionizing radiation (i.e., recombinations). This work builds on previous pioneering analytical studies, dating about 50 years ago, on the stability of non-accelerating I-fronts. A brief historical perspective is useful to summarize what is already known on the stability of I-fronts.

Vandervoort (1962) was the first to present a quantitative perturbation analysis of a non-accelerating plane I-front subject to appropriate jump and boundary conditions, assuming that the gas is compressible and isothermal. For simplicity the author derived the growth rate of the front perturbations in the limiting case when the temperature of the neutral gas is negligible with respect to the temperature of the ionized gas. The author considered the general case of incident radiation on the I-front with arbitrary inclination, but neglected hydrogen recombinations inside the H ii region. With these assumptions weak-D and D-critical type fronts were found to be unstable. Kahn (1958) had previously argued that absorption of the incoming radiation by neutral hydrogen atoms produced by recombinations in the ablation outflow could have a stabilizing effect because of the stronger absorption near the dimples of the front surface compared to the bumps. Axford (1964) building on the work of Vandervoort (1962), but assuming incident radiation perpendicular to the I-front, presented a quantitative study that showed that this stabilization mechanism is effective for perturbations with wavelengths larger than the recombinaton length. Newman & Axford (1967) extended this analysis to strong D-type and D-critical fronts. Saal (1966) extended previous analysis relaxing the simplifying assumption of a neutral gas much colder than the ionized gas and found a stronger suppression of the instabilities for the short-wavelength solutions previously found unstable. Sysoev (1997) provided more complete analysis and found numerical solutions where the growth of marginally stable solutions in the limit of cold neutral gas were unstable at longer-wavelengths for normally incident radiation. Gulliam (1973) and Williams (2002) presented analytical calculations and numerical simulations for the case of inclined incident radiation on the front showing that, in the limit of neutral gas much colder than the ionized gas, essentially all wavelength modes are unstable.

There is less work and is entirely based on numerical simulations, considering the RTI at accelerating I-fronts. Mizuta et al. (2005, 2007) studied the hydrodynamic instability of accelerating I-fronts using two-dimensional hydrodynamic simulations in which recombinations are either turned off or included. The authors conclude that the RTI can only grow when recombinations are turned off. In a series of papers Park & Ricotti (2011, 2012, 2013) focused on the problem of gas accretion onto BHs from galactic scales regulated by radiation feedback. In these simulations the ionizing radiation emitted by the BH produces a hot ionized bubble preceded by a denser neutral shell centered around the BH. The formation of the H ii region temporarily stops the feeding of the BH. However, the gas density inside the ionized bubble decreases with time and eventually the dense shell collapses onto the BH producing a burst of accretion and luminosity. These events repeat cyclically. The D-type I-front should be unstable to RTI, with perturbations at the smallest scale resolved in the simulations growing on timescales shorter than the period between two bursts, but instead the front appears remarkably stable at all scales. Park et al. (2013) have shown that the growth of the RTI at the I-front is stabilized by recombinations in the ionized gas. The RTI can grow only during a short phase of the cycle when the I-front is accelerating during a BH luminosity burst. The authors speculate that the RTI is suppressed for perturbations with wavenumber \( k > k_{cr} \), however a derivation of the growth rate of the RTI at I-fronts is not present in the literature. The goal of this paper is to fill the gap and provide such a derivation.

This paper is organized as follows. In §2 is presented a perturbation analysis of an accelerating I-front, and the characteristic equation for the growth rate of longitudinal perturbations is derived. In §3 analytical solutions of the characteristic equation in the relevant limits are presented, providing the dispersion relation for unstable surface modes. A summary of the results and a brief discussion on astrophysical applications is presented in §4.

## 2 LINEAR STABILITY ANALYSIS

Let’s consider a plane-parallel I-front and work in the frame of reference comoving with the I-front. The formalism adopted in this paper follows closely the one by Vandervoort (1962); Axford (1964) for non-accelerating I-fronts with two important modifications: (i) terms describing the acceleration \( \mathbf{g} \equiv \gamma \mathbf{m}_0 \) in the frame of reference of the I-front are introduced, where \( \mathbf{m}_0 \) is the unit vector normal to the unperturbed front; (ii) an incompressible equation of state for the gas is assumed (\( \nabla \cdot \mathbf{u} = 0 \)). The second assumption is justified for sake of simplicity given the increased complexity of the equations due to the additional terms arising because of the front acceleration. This approximation also allows to easily check the results against the well known growth rate of the incompressible RTI (i.e., \( n = \sqrt{gk} \)).

The methodology of the analysis can be summarized as follows. The governing hydrodynamic equations are linearized to first order for the neutral (quantities with subscripts “1”) and ionized gas (quantities with subscripts “2”) separately. These equation can be solved to derive the pressure perturbations as a function of the velocity and density perturbations. We
consider perturbations on the surface of the I-front in the form \( \xi = n_0 \xi \), where \( \xi = \xi_0 \exp(nt + ikz) \) and \( \xi_0 \) is the amplitude of the front deformation. The unperturbed (\( U \) velocity, \( \rho \) density and \( P \) pressure) and perturbed (\( u \) velocity, \( \delta \rho \) density and \( \delta P \) pressure) physical quantities must obey the jump conditions at the I-front derived from mass and momentum conservation and the perturbed quantities must vanish at large distance from the front (i.e., must obey the boundary conditions). Imposing these jump/boundary conditions results in a system of four linear equations in four unknowns: \( u_1, u_2, \delta P_2 \) and \( \xi_0 \) (it will be shown later that \( \delta P_1 = 0 \) in order to obey the boundary conditions at \( z \to \infty \) and \( \delta P \) can be expressed as a function of \( u \) and \( \delta \rho \)). By setting the determinant of the linear system to zero (to ensure non-trivial solutions) the characteristic equation for the growth rate of the perturbations, \( n \), is derived. Positive real solutions of the characteristic equations give the unstable growing modes of the system in the linear regime.

2.1 Governing Equations

A moving coordinate frame is chosen so that the I-front appears stationary, with \( z \)-axis perpendicular to the front and photons incident from \( z < 0 \) onto the I-front placed at \( z = 0 \) in the unperturbed state. The region \( z > 0 \) contains the neutral gas and the region \( z < 0 \) the ionized gas. The unperturbed velocity, density and pressure in the neutral (quantities with subscript “1”) and ionized regions (subscript “2”) are the constants \( U_1, U_2, \rho_1, \rho_2, P_1, P_2 \), respectively. An incident radiation field \( J \) directed along the \( z \)-axis is assumed. Given the symmetry of the system we can consider longitudinal perturbations of the I-front aligned along the \( x \)-axis without loss of generality, making the problem two-dimensional. Perturbations of the velocity \( U = U + u \), density \( \rho = \rho + \delta \rho \), and pressure \( P = P + \delta P \) fields will be considered on each side of the front. The following governing equations will allow to write \( \delta P \) as a function of the other perturbed quantities.

The linearized equations of continuity and motion for an inviscid gas with an external or fictitious acceleration \( g \) in the \( z \)-direction (i.e., orthogonal to the unperturbed I-front) pointing toward the ionized gas, are

\[
\left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial z} \right) \delta \rho + \rho (\nabla \cdot u) + u \cdot \nabla \rho = 0, \tag{1}
\]

\[
\rho \left( \frac{\partial}{\partial t} + U \frac{\partial}{\partial z} \right) u = -\nabla \delta P - \delta \rho g n_0. \tag{2}
\]

As usual, solutions of these equations are sought in the form \( A(x, z, t) = A_s(z) \exp(\psi t + ikx) \), where \( A \) is the perturbed quantity, \( n \) is the growth rate and \( k \) is the wavenumber of the perturbations along the \( x \)-axis. Equation (1), assuming incompressible gas (\( \nabla \cdot u = 0 \)) can be written as

\[
(n + U D) \delta \rho = -uD\delta \rho = 0, \tag{3}
\]

\[
D u + i ku_z = 0, \tag{4}
\]

where \( u \) and \( u_z \) are the \( x \)- and \( z \)-components of the perturbed velocity, respectively. The abbreviation \( D \equiv \partial/\partial z \) has been adopted. Similarly, Equation (2) becomes

\[
\rho(n + U D)u = -D\delta P - \delta \rho g, \tag{5}
\]

\[
\rho nu_z = -ik\delta P. \tag{6}
\]

Combining Equations (1) and (6) an expression for the pressure perturbation is easily obtained:

\[
\delta P = -\frac{\rho n Du}{k^2}. \tag{7}
\]

Equations (3) and (4) must satisfy the boundary conditions \( \delta \rho \to 0 \) and \( u \to 0 \) as \( |z| \to \infty \). Since \( U < 0 \) and unstable solutions with real part of \( n \) positive are sought, the solution for the perturbed density is

\[
\delta \rho = \begin{cases} 
0 & \text{if } z > 0, \\
\rho_2 \exp \left(-\frac{n}{U} z \right) \exp(\psi t + ikx) & \text{if } z < 0.
\end{cases} \tag{8}
\]

Hence, the assumption of incompressible gas leads to perturbations of the density in the ionized component while the neutral gas density remains unperturbed (\( \delta \rho_1 = 0 \)). Note that this is not the case when an isothermal gas is considered instead. Finally, Equation (6) can be solved in the neutral gas and ionized gas. For \( z > 0 \) (neutral gas) where \( \delta \rho = 0 \), the term proportional to \( g \) cancels out, thus the governing equation is

\[
D^2 u - \frac{k^2 U_1}{n} Du - k^2 u = 0. \tag{9}
\]

The general solution of this equation is \( u = u_1 \exp(p_1 z) \exp(\psi t + ikx) \), where

\[
p_1 = -\frac{k^2 U_1}{2n} \left(-1 - \sqrt{1 + \frac{2n}{kU_1}} \right) = -\frac{k}{n} f_1(\tilde{n}, \epsilon). \tag{10}
\]

Here, since \( U_2 < 0 \), the dimensionless growth rate \( \tilde{n} \equiv -n/kU_2 \) has the same sign as \( n \), hence \( p_1 \leq 0 \) for unstable modes (i.e.,
\( \delta > 0 \). The parameter \( \epsilon \equiv U_1/U_2 = \rho_2/\rho_1 \equiv \delta^{-1} \) is the inverse of the density contrast between the neutral and ionized gas, and \( f_1(\tilde{n}, \epsilon) \equiv \left[ 1 + (1 + 4(\tilde{n}/\epsilon)^2)^{1/2} \right]/2 \). In the limit \( \tilde{n} \gg \epsilon \), i.e., for large density contrasts \( \delta \gg 1 \), \( f_1 \to \tilde{n}/\epsilon \) (thus, \( \rho_1 = -k \)). In the limit \( \tilde{n} \ll \epsilon \), \( f_1 \to 1 \). Thus, from Equation (14) the pressure perturbation in the neutral gas is \( \delta P = \delta P_1 \exp (nt + kx) \) with
\[
\delta P_1 = -\frac{\rho_1 \rho_2 u_1}{k^2} = -\rho_1 U_1 f_1(\tilde{n}, \epsilon).
\]
For \( z < 0 \) (ionized gas) where \( \delta \rho \neq 0 \), the governing equation is
\[
D^2 u - \frac{k^2 U}{\tilde{n}} Du - k^2 u - \frac{gk^2}{\rho_2} \delta P_1 \exp \left( -\frac{n}{\tilde{n}} z \right) = 0
\]
(12)
The general solution of this equation is \( u = u(z) \exp (nt + kx) \) with \( u(z) = g_2 \exp (p_2 z) + \delta \rho_2 g/(\rho_2 \tilde{n}^2) \exp (-nz/U_2) \), where
\[
p_2 = -\frac{k^2 U_2}{2n} \left( -1 + \sqrt{1 + \left( \frac{2n}{kU_2} \right)^2} \right) = \frac{k}{n} f_2(\tilde{n}).
\]
(13)
Here, \( f_2(\tilde{n}) \equiv [-1 + (1 + 4\tilde{n}^2)^{1/2}]/2 \), and \( p_2 > 0 \) for unstable modes with \( \tilde{n} > 0 \). In the limit \( \tilde{n} \gg 1 \) (i.e., when \( n \gg k|U_2| \)), always valid for unstable modes when \( U_2 \to 0 \), \( f_2 \to \tilde{n} (p_2 = k) \). In the limit \( \tilde{n} \ll 1 \), \( f_2 \to \tilde{n}^2 \). The constant \( g_2 \) can be expressed in terms of \( g_2 = u(z) = 0 \) as \( g_2 = u_2 - \delta \rho_2 g/(\rho_2 \tilde{n}^2) \) and using Equation (11) the pressure perturbation in the ionized gas is \( \delta P = \delta P_2 \exp (nt + kx) \) with
\[
\delta P_2 = \rho_2 U_2 \left[ u_2 f_2(\tilde{n}) + \frac{q}{\rho_2} \left( 1 - \frac{f_2(\tilde{n})}{2\tilde{n}^2} \right) \right].
\]
(14)

2.2 Jump Conditions at the Front

Four additional equations are needed to close the system as the normalization constants \( u_1, u_2, \delta \rho_2 \) and \( \xi_0 \) are still undetermined. The equations that can be used to close the system are the jump conditions at the I-front: two energy conservation equations (for the \( x \) and \( z \) direction) and two momentum conservation equations in the \( z \)-direction for the top and bottom layers. The energy conservation jump condition for the unperturbed front is,
\[
\Delta [\rho U^2 + P] = 0,
\]
(15)
where the notation \( \Delta [A] \equiv A_1 - A_2 \) is adopted. The momentum conservation equations for the unperturbed quantities are:
\[
\rho_1 U_1 = \rho_2 U_2 = -\mu J_0,
\]
(16)
where \( J_0 \) is the number of ionizing photons that reach the unperturbed I-front per unit area and time, and \( \mu \) is the mean molecular mass. Let’s consider, as mentioned above, the deformation of the ionization front of the form \( \xi = n_0 \xi_0 \exp (nt + ikx) \), representing the displacement of the front with respect to the \( z = 0 \) steady state position. The perturbation of the velocity of the front is \( \delta \mathbf{V} = \partial \mathbf{v} / \partial t = n \xi \), and the unit vector normal to the front is \( \mathbf{n} = \mathbf{n}_0 + \delta \mathbf{n} \), where \( \delta \mathbf{n} = -\nabla \xi = -\mathbf{k} \xi \). The flux at the I-front is also perturbed due to the front deformation and the density perturbations in the ionized gas: \( J(\xi) = J_0 + \delta J(\xi) \exp (nt + kx) \). Is important to consider the effects of absorption in the H II region of ionizing radiation are it is know to stabilize the perturbations. The absorption of ionizing photons is described by the equation
\[
\frac{dJ}{dz} = -\nu_{\text{rec}} \frac{\rho_2}{\mu},
\]
(17)
where \( \nu_{\text{rec}} \equiv \rho_2 \alpha^{(2)}/\mu \) is the hydrogen recombination rate in the ionized region, with \( \alpha^{(2)} \approx 2.6 \times 10^{-13} \text{cm}^3 \text{s}^{-1} (T/10^4 \text{ K})^{-0.8} \) (Spitzer, 1962). Allowing for perturbations of the front position and density in the ionized gas, Equation (17) can be integrated to give
\[
\delta J(\xi) = -\nu_{\text{rec}} \frac{\rho_2}{\mu} \left( \xi_0 + 2 \int_{-\infty}^{0} \frac{\delta \rho_2}{\rho_2} dz \right) + O(\delta \rho_2^2).
\]
(18)
The first term on the right hand side of Equation (18) describes the front stabilizing mechanism proposed by Kahn (1958), while is not clear a priori whether the second term due to perturbations of the density in the ionized gas is stabilizing or destabilizing (it will be found to be a stabilizing term). The linearized perturbed Equations (15) and (16) are
\[
\Delta [\rho(\delta \mathbf{n} \cdot \mathbf{U}) \mathbf{U}] + \Delta [\delta \rho(n_0 \cdot \mathbf{U}) \mathbf{U}] + \Delta [\rho(\mathbf{n}_0 \cdot \mathbf{U})(\mathbf{u} - \delta \mathbf{V})] + \Delta [\rho(n_0 \cdot \mathbf{U})(\mathbf{u} - \delta \mathbf{V})] = 0,
\]
\[
+ \delta \mathbf{n} \Delta [P] + n_0 \Delta (\delta P) + n_0 g \int_{-\infty}^{\infty} dz \delta \rho = 0,
\]
(19)
\[
\rho(\delta \mathbf{n} \cdot \mathbf{U}) + \delta \rho(n_0 \cdot \mathbf{U}) + \mathbf{n}_0 \cdot (\mathbf{u} - \delta \mathbf{V}) = -\mu J \exp (nt + kx).
\]
(20)
Substituting the expressions for $\delta V$ and $\delta n$ two jump conditions at the front and two continuity equations for the neutral and ionized gas are obtained:

$$\Delta[\delta n^2] + 2\Delta[\rho U(u - n\xi)] + \Delta[\delta P] + g \int_1^1 dz \delta \rho = 0,$$

$$\Delta[U \delta \rho] + n\xi \Delta[P] = 0,$$

$$\delta \rho u + \rho (u - n\xi) = -\mu \delta J \exp (nt + kx).$$

The integral in Equations (20) and (22) can be easily evaluated across the front discontinuity:

$$\int_1^1 dz \delta \rho = -\frac{1}{n}(-U_2 \delta \rho_2 + u_1 \rho_1 - u_2 \rho_2),$$

while the ionization flux perturbation at the front location is obtained from Equation (18) using Equation (8) for $\delta p$:

$$-\mu \delta J = \rho_2 \nu_{rec} \left( \xi_0 - 2 \frac{U_2 \delta \rho_2}{n \rho_2} \right).$$

A few manipulations of Equations (22)-(24) give the following system of four equations in four unknowns:

$$\delta n^2 - 2\mu \delta J \Delta[U] + \Delta[\delta P] - g \Delta[\rho] \xi_0 = 0,$$

$$\Delta[U \delta \rho] - n \rho_1 U_1 \Delta[U] \xi_0 = 0,$$

$$\rho_1 u_1 = -\mu \delta J + n \rho_1 \xi_0,$$

$$\rho_2 u_2 = -\mu \delta J + n \rho_2 \xi_0 - U_2 \delta \rho_2,$$

where the relationship $\Delta[P] = -\Delta[\rho U^2] = -\rho_1 U_1 \Delta[U]$ has been used. Finally, Equations (11) and (14) allow to express the terms in $\delta P$ as:

$$\Delta[\delta P] = -(\rho_1 u_1 f_1 + \rho_2 u_2 f_2) - \frac{g}{n} U_2 \delta \rho_2 \left( 1 - \frac{f_2(\tilde{n})}{2\tilde{n}^2} \right),$$

$$\Delta[U \delta \rho] = -(\rho_1 u_1 f_1^2 + \rho_2 u_2 f_2^2) - \frac{g}{n} U_2^2 \delta \rho_2 \left( 1 - \frac{f_2(\tilde{n})}{2\tilde{n}^2} \right).$$

In order to keep the equations concise, is convenient to work using the dimensionless variables $\bar{n} \equiv -n/(kU_2)$, $\bar{v} \equiv -\nu_{rec}/(kU_2)$ and $\bar{g} \equiv g/(kU_2^2)$ and define the functions:

$$F_0 \equiv f_1 + f_2 \quad \left( \lim_{\epsilon \to 0} F_0 = \bar{n}/(\epsilon) \right),$$

$$F_1 \equiv \epsilon f_1 + f_2 \quad \left( \lim_{\epsilon \to 0} F_1 = 2\bar{n} \right),$$

$$F_2 \equiv \epsilon^2 f_1 + f_2 \quad \left( \lim_{\epsilon \to 0} F_2 = \tilde{n} \right),$$

$$G_0 \equiv \bar{g} \left( 1 - \frac{f_2(\tilde{n})}{2\tilde{n}^2} \right) \left( \lim_{\epsilon \to 0} G_0 = \bar{g} \right).$$

For convenience the limits of these functions for high density contrast $\delta \gg 1$ (i.e., $\epsilon \to 0$) are shown as well. With these definitions and substituting $u_1$ and $u_2$ from Equations (20)-(30) into Equations (27)-(28) gives the following system of two equations in two unknowns ($\delta \rho_2$ and $\xi_0$):

$$\left\{ 1 + f_2 + \frac{G_0}{n} + 2\bar{g} F_1 + 2(1 - \epsilon) \right\} \delta \rho_2 + k \rho_2 \left\{ \bar{n} F_0 + \bar{v} [F_1 + 2(1 - \epsilon)] - \frac{\Delta P}{\rho_2} \right\} \xi_0 = 0,$$

$$\left\{ f_2 + \frac{G_0}{n} + 2\bar{g} F_2 \right\} \delta \rho_2 + k \rho_2 \left\{ \bar{n} (F_1 - 1 + \epsilon) + \bar{v} F_2 \right\} \xi_0 = 0.$$

Non-trivial solutions for the dimensionless growth rate $\bar{n}$ are found by setting the discriminant of the linear system to zero:

$$[(1 + f_2)(F_1 - 1 + \epsilon) - f_2 F_0] \bar{n}^2 + \left\{ [(F_1 + 2(1 - \epsilon))(2F_1 - 2 + 2\epsilon - f_2) + F_2(1 + f_2 - 2F_0)] \bar{v} - \left[ G_0(F_0 + F_1 - 1 + \epsilon) - f_2 \bar{g} \left( \frac{1}{\epsilon} - 1 \right) \right] \right\} \bar{n} - \left[ 2F_2 \bar{g} \left( \frac{1}{\epsilon} - 1 \right) + G_0(F_1 + 2 - 2\epsilon - F_2) \right] \bar{v} + G_0 \bar{g} \left( \frac{1}{\epsilon} - 1 \right) = 0,$$

3 RESULTS

The general solution of the characteristic Equation (11) can be found numerically as a function of $\epsilon$, $\nu_{rec}$ and $\bar{g}$. However, in the limit of high density contrast across the I-front (i.e., $\delta \gg 1$), Equation (11) simplifies significantly and has an analytical
solution. The special cases of neglecting the effective gravity (i.e., stability of non-accelerating I-fronts) and neglecting recombinations will be considered separately to check whether known results on the classical RTI growth rate are recovered. Only density contrasts $\delta > 2$ (i.e., $\epsilon < 0.5$) should be considered as this condition is necessary for a D-type front to exist.

### 3.1 Limit 1: Neglecting Gravity

Setting both $g = 0$ and $\nu_{rec} = 0$, Equation (41) becomes

$$\left(1 + f_2\right)(F_1 - 1 + \epsilon) - f_2F_0 = 0. \quad (42)$$

The real roots of this equation are negative for any value $\epsilon < 0.4$, meaning that the I-front is stable for density contrast $\delta > 2.5$. Including recombinations (i.e., $\nu_{rec} > 0$) has minor effects on the stability. This is contrary to results of Vandervoort (1962) who found unstable solutions for weak D-type fronts when neglecting recombinations. However, Vandervoort (1962) assumed isentropic perturbations while the present result is derived in the limit of incompressible perturbations. Strong D-type fronts, that have $|U_2| > c_{is}$ where $c_{is}$ is the sound speed in the ionized gas, are instead stable even neglecting recombinations (Newman & Axford 1967). Is likely that the unstable modes found neglecting recombinations appear because of the effects of gas compressibility.

### 3.2 Limit 2: Neglecting Recombinations

In this subsection is shown that neglecting recombinations but including non-zero effective gravity the RTI for incompressible gas is recovered in the limit of large density contrast ($\epsilon \to 0$). Setting both $\nu_{rec} = 0$, Equation (41) becomes

\[
\left[(1 + f_2)(F_1 - 1 + \epsilon) - f_2F_0\right]\tilde{n}^2 - \left[G_0(F_0 + F_1 - 1 + \epsilon) - f_2\tilde{g}\left(\frac{1}{\epsilon} - 1\right)\right]n + G_0\tilde{g}\left(\frac{1}{\epsilon} - 1\right) = 0. \quad (43)
\]

This equation can be solved for $\tilde{n}$ only numerically, however, in the limit of large density contrast $\delta \gg 1$ ($\epsilon \to 0$) and $\tilde{n} \gg 1$, the expressions for the functions $f_1$ and $f_2$ become: $f_1 \to \tilde{n}/\epsilon$ and $f_2 \to \tilde{n}$. Hence: $F_0 \to \tilde{n}/\epsilon$, $F_1 \to 2\tilde{n}$, $F_2 \to \tilde{n}$. Substituting these limits in Equation (41) the following 4th order equation is obtained:

$$\tilde{n}^4 = \tilde{g}^2 \quad (44)$$

This equation has two real solutions $\tilde{n} = \pm \sqrt{\tilde{g}}$. Hence the growing mode in in physical units is $n = \sqrt{\tilde{g}/\rho}$ that indeed is the well known dispersion relation for the incompressible RTI in the limit $\delta \gg 1$. A numerical inspection of the Equation (43) shows that for values of $\epsilon \neq 0$ a better solution of the equation is $n = \sqrt{\tilde{g}A}$, where $A \equiv (\rho_1 - \rho_2)/(\rho_1 + \rho_2) = (1 - \epsilon)/(1 + \epsilon)$ is the Atwood number.

### 3.3 General Solution

The general solution of Equation (41) can only be found numerically. However, in the limit $\epsilon \to 0$ ($\delta \gg 1$) and $\tilde{n} \gg 1$ the following simplified 4th order equation is obtained:

$$\tilde{n}^4 + 2\tilde{n}^3 + 2\tilde{n}\tilde{g}n - \tilde{g}^2 = 0. \quad (45)$$

This equation has two complex solutions ($\pm \sqrt{-\tilde{g}}$) and two real solutions:

$$\tilde{n} = -\tilde{\nu} \pm \sqrt{\tilde{\nu}^2 + \tilde{g}} \quad (46)$$

Hence, in physical units the growing mode has a rate

$$n = -\nu_{rec} + \sqrt{\nu_{rec}^2 + gk}. \quad (47)$$

Therefore, at large scales $k \ll k_{cr} \equiv \nu_{rec}^2/g$ the growth of the RTI is $n \approx \sqrt{\tilde{g}A}(gk/4\nu_{rec}^2)^{1/2}$, suppressed by a factor $(k/4k_{cr})^{1/2}$ with respect to the classical RTI. A numerical inspection of the Equation (41) shows that for values of $\epsilon \neq 0$ a better approximation of the solution is $n = -\nu_{rec} + (\nu_{rec}^2 + gkA)^{1/2}$, where $A$ is the Atwood number.

### 3.4 Discussion on the Effects of Gas Compressibility

Let’s now discuss how the assumption of gas incompressibility adopted in this work may affect the results. The discussion will be guided by previous results on the stability of Rayleigh-Taylor modes and non-accelerating I-fronts assuming isothermal or isentropic perturbations.

The effect of compressibility on the linear growth of the RTI was considered by several authors assuming either isothermal or isentropic equilibrium states and perturbations (see Gauthier & Le Creurer 2010, for a review). Compressibility modifies the RTI growth rate with respect to the incompressible case as follows: the growth rate decreases as the “stratification” parameter, $g/k_{cr}^2$, increases and the adiabatic indices decrease. Stratification and compressibility effects are more important at small
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wavenumbers and the growth rates are larger when the light fluid is more compressible than the heavy one. Compressibility effects are larger at small Atwood numbers (i.e., low density contrast). For the isothermal case, compressibility stabilizes the RTI. However, the departure from incompressible type behavior is small in most circumstances. The correction term, \( g/kc^2 \), is much smaller than unity unless the effective acceleration at the I-front is \( g_{eff} \gg GM/R_s^2 \). It is easy to show that for \( g = GM/R_s \) the stabilizing term is of the order \( kR_{in}/R_s^2 \), that is much smaller than unity at wavelengths of interest \( \lambda < 2\pi R_s \), where \( R_{in} \ll R_s \) is the Bondi radius inside the HII region.

The stability of isentropic non-accelerating I-fronts depends on the recombination rate but also on the Mach number of the gas downstream the front. Supersonic flows (strong D-type and weak R-type) are stable even neglecting recombinations (Newman & Axford 1967). D-critical and weak D-type fronts with Mach number in the ionized gas \( M_2 \ll 1 \) (\( |U_2| \ll c_s \)) are also stable (Axford 1964). The stability of the front found in this paper may be understood in this context taking the limit \( \nu \approx \nu_{ion} \approx \nu_{S,B} \). The growth of the RTI in D-type ionization fronts has been investigated via perturbation analysis assuming incompressible \( \nu \), that gives \( M_2 \ll 1 \). Compressibility effects destabilize the front for weak D-type fronts with \( M_2 \gtrsim 1 \), but only on scales smaller than one tenth of the recombination scale in the ionized gas.

In light of the discussion above, the effects of compressibility on the stability of accelerating I-fronts should be small in many circumstances. Compressibility effects should be most relevant for weak D-type fronts with \( M_2 \sim 0.5 \). In this regime \( g \sim 4g/kc^2 \sim 1 \), and \( \lambda \lesssim 1 \). However, is unclear whether the instability growth rate is increased or reduced because Rayleigh-Taylor modes are typically stabilized while the I-front is slightly destabilized by compressibility effects.

4 SUMMARY AND CONCLUSIONS

The growth of the RTI in D-type ionization fronts has been investigated via perturbation analysis assuming incompressible gas. In the limit of a large density contrast at the ionization front the RTI growth rate has the simple analytical solution

\[ n = -\nu_{rec} + (\nu_{REC}^2 + gk)^{1/2}. \]

Therefore, recombinations in the ionized gas stabilize the RTI on scales larger than

\[ \lambda_{cr} = \frac{2\pi}{\nu_{REC} k}. \]

suppressing their growth for \( \lambda \gg \lambda_{cr} \) by a factor \(~(\lambda_{cr}/4\lambda)^{1/2} \) with respect to the non-radiative case. Contrary to previous analysis of isothermal non-accelerating I-fronts, the front is stable when gravity and recombinations are set to zero, suggesting that the assumption of gas incompressibility has a stabilizing effect. The stabilization is very effective because in most problems \( \lambda_{cr} \) is much smaller than the scales of interest. For instance, for non-accelerating fronts in a gravitational field produced by a point mass \( M \) with \( g = GM/r_s^2 \), where \( r_s \) is the location of the I-front,

\[ \theta_{cr} = \frac{\lambda_{cr}}{2\pi r_s} \equiv \frac{GM_{\nu_{REC} t_s^2}}{\nu_{REC}^3} \sim 4 \times 10^{-10} \frac{X}{\lambda}, \]

where \( r_s = (3S_0/4\pi n^2 c^2(t/3)^{1/3}) \) is the Str"omgren radius and \( S_0 = 5.7 \times 10^{48} s^{-1}(M/M_\odot)l/X \) is the number of hydrogen ionizing photons per unit time, expressed as a function of the Eddington ratio \( l \equiv L/L_{Edd} \) and the mean ionizing photon energy \( X \) in Rydbergs. In this example, at scales \( \theta \lesssim 10^{-3} \), the front becomes unstable to RTI only for very small Eddington ratios: \( l \lesssim 4 \times 10^{-7} X \) (see, Park et al. 2011). However, during a burst of luminosity when the front acceleration timescale \( t_{acc} = (de/dr)_{\nu_{REC}}^{-1/2} \) is comparable or shorter than the recombination timescale, the RTI can develop even for Eddington ratios of the order of unity. Another case in which the front is accelerating and the RTI may develop is when the front propagates toward regions of lower density, for instance if the ionizing source is at the center of a halo with a gas density profile \( \rho \propto r^{-2} \) (Whalen & Norman 2008).

This analytical work has been inspired by the results of numerical simulations (Park & Ricotti 2011, 2012, 2013) showing that recombinations in HII regions produced by BHs accreting from a neutral medium have a stabilizing effect on the growth of the RTI at the I-front. In these simulations the I-front is unstable only during two phases of the duty cycle: (i) during bursts of accretion onto the BH, when the outward acceleration of the front increases the effective gravity and (ii) just before a burst when the accretion luminosity reaches a minimum value triggering the collapse of the shell and the HII region onto the BH. During the burst the front fragments on the smallest resolved scales into knots that are optically thick, casting a shadow before dissolving (self-gravity is not included in the simulations). The phase when the I-front collapses onto the BH is characterized by denser fingers of gas protruding toward the BH. The results of the present study are in agreement with the observed phenomenology as discussed in detail in Park et al. (2013). Simulations of moving intermediate mass BHs with radiation feedback (Park & Ricotti 2013) also show the formation of a dense shell upstream the BH supported against gravity by less dense hot gas in a cometary-shaped HII region. Also in this case RTI at the front appears to be stabilized by recombinations.

The results of this work may also be relevant for the stability of supershells in AGN or galactic winds. In general, outflows or winds can be produced by pressure gradients as a result of thermal energy injection or transfer of momentum from the radiation field to the gas. Momentum driven winds can be produced by Compton scattering of photons on electrons, by photons scattered or absorbed by dust grains or photo-ionization of hydrogen and heavy ions by UV and X-rays. For instance,
radiation pressure on dust has been suggested as an important feedback mechanism in regulating star formation on galactic scales. However, for optically thick media and Eddington ratios close to unity (typical of many galaxies and star clusters), the gas and the radiation field are coupled producing phenomena such as photon bubbles and radiation RTI ([Turner et al. 2005; Jacquet & Krumholz 2011; Jiang et al. 2013]). In this regime [Krumholz & Thompson 2012; Jiang et al. 2013] showed that the transfer of IR radiation through the neutral gas triggers radiative RTI, driving turbulence but not a wind. Although these models do not include ionized bubbles, a realistic description of the wind involves supershells produced by OB associations near the galactic disk that may provide some stabilization during the initial phases of the wind launching process. A more direct application of the results of this work regards a possible driving mechanism of AGN winds based on photo-ionization of metal ions within 0.1 pc from the supermassive BH ([Proga et al. 2000; Debuhr et al. 2012; Novak et al. 2012]). Ionization fronts of metal ions of different elements are located at various distances near the AGN, and the ions may (or may not) be coupled to the hydrogen gas through Coulomb collisions ([Baskin & Laor 2012]). Momentum is deposited in shells at each ionization fronts, producing an accelerating wind subject to RTI. Ion recombinations should stabilize these shells at least on scales larger than a critical value that can be estimated using a modification of Equation (48) in which $\nu_{\text{rec}}$ is calculated for the appropriate metal ion.

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