Entangling macroscopic oscillators exploiting radiation pressure

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It is shown that radiation pressure can be profitably used to entangle macroscopic oscillators like movable mirrors, using present technology. We prove a new sufficient criterion for entanglement and show that the achievable entanglement is robust against thermal noise. Its signature can be revealed using common optomechanical readout apparatus.

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The fundamental role of entanglement in quantum mechanics has been re-emphasized in recent years. In this context, an important point is to assess whether this peculiarity of the quantum world, i.e., the entanglement, could be applicable to macroscopic bodies and, moreover, measurable.

Literature focused on methods to prepare atoms in entangled states already exists. The recent experiment generating entanglement of two gas samples is a striking achievement. Then, a real challenge is to devise the possibility of applying similar arguments to macroscopic, massive oscillators. Here we propose an experiment, which could be realized with present technologies, to show that it is possible to entangle massive oscillators exploiting the radiation pressure force.

It is indeed usually believed that, being a superposition of states, the entanglement between massive, macroscopic, objects is practically impossible to detect because of the fast diagonalization of the system’s density matrix due to the coupling with the environment. On the contrary, by using a new sufficient criterion for entanglement, we shall derive the parameter region for which two massive, movable, cavity mirrors can be entangled by the radiation pressure exerted by a cavity mode, and we shall show how to measure the degree of entanglement. Beside foundation interest, the ability to place such oscillators in entangled states may even result useful in applications, as in high precision measurements.

To be concrete, as a specific model we consider two end mirrors of an optical cavity, which can both oscillate under the effect of radiation pressure force. Cavities with one movable mirror have already been studied, and a wide class of quantum states resulting from optomechanical coupling was proposed. Furthermore, due to recent technological developments in optomechanics, this area is now becoming experimentally accessible.

As pointed out in Ref., under the assumption that the measurement time is either less or of the order of the mechanical relaxation time, it is possible to consider a macroscopic oscillator, i.e., a movable mirror in our case, as a quantum oscillator. Then, for not too high oscillation frequency, with respect to the inverse round trip times of photons within the cavities, we can write the Hamiltonian of the system sketched in Figure as

\[
\mathcal{H} = \sum_{i=1}^{2} \hbar \omega_{i} a_{i}^\dagger a_{i} + \hbar \omega b^\dagger b + \hbar \Omega \sum_{i=1}^{2} \left( \frac{\omega_{i}^{2}}{2} + \frac{\omega^{2}}{2} \right) \\
-\hbar g a_{1}^\dagger b a_{2} + \hbar G b^\dagger b (q_{1} - q_{2}) \\
+i\hbar \sqrt{\gamma_{a}} \sum_{i=1}^{2} \left( \alpha^{in} e^{-i\omega_{a0} t} a_{i}^\dagger - \alpha^{in} e^{i\omega_{a0} t} a_{i} \right) \\
+i\hbar \sqrt{\gamma_{b}} \left( \beta^{in} e^{-i\omega_{b0} t} b^\dagger - \beta^{in} e^{i\omega_{b0} t} b \right),
\]

where \(a_{i}, a_{i}^\dagger\) are the destruction and creation operators of the electromagnetic fields corresponding to the meters mode and \(\omega_{i}\) their frequency (assumed equal for simplicity). Instead, \(b, b^\dagger\) are those of the entangler mode (the use of this terminology will become clear in the following) and \(\omega_{b}\) its frequency. Finally, \(q_{i}\) and \(p_{i}\) are the dimensionless position and momentum operators of the mirrors \(M_{i}\), both oscillating at frequency \(\Omega\), and having mass \(m\). The first row of equation simply represents the free Hamiltonian, whereas the second represents the effect of the radiation pressure force which causes the instantaneous displacement of the mirrors. The coupling constants are \(g = \tilde{g}/\sqrt{m\Omega}\) and \(G = G/\sqrt{m\Omega}\) where \(\tilde{g}, G\) are related to the cavity mode frequencies, to the equilibrium length of the cavities, and to the reflection angles. The last two rows represent the driving fields action in the usual rotating wave approximation. We assume that both modes \((a_{1}, a_{2})\) are driven at frequency \(\omega_{a0}\), while the entangler mode \(b\) is driven at frequency \(\omega_{b0}\); \(\alpha^{in}, \beta^{in}\) are the classical fields characterizing the input laser powers \(P_{a}^{in} = \hbar \omega_{a0} |\alpha^{in}|^{2}, P_{b}^{in} = \hbar \omega_{b0} |\beta^{in}|^{2}\), and \(\gamma_{a}, \gamma_{b}\) are the cavity linewidths.
fluctuations around steady state values. These are
\[ \langle q_j \rangle_{ss} = (-)^j [G|\beta|^2 - g|\alpha|^2]/\Omega, \]
\[ \langle p_j \rangle_{ss} = 0, \]
\[ \alpha \equiv \langle a_j \rangle_{ss} = \sqrt{\gamma_a}\alpha_{in}/[\gamma_a/2 - i\Delta_a], \]
\[ \beta \equiv \langle b \rangle_{ss} = \sqrt{\gamma_b} b_{in}[\gamma_b/2 - i\Delta_b]. \]

Moreover, \( \Delta_a \equiv \omega_a - \omega_a + g\langle q_1 \rangle_{ss}, \Delta_b \equiv \omega_b - G\langle q_1 \rangle_{ss} - \langle q_2 \rangle_{ss} \), are the radiation phase shifts due to the detuning and to the stationary displacement of the mirrors. Both radiation fields used as meters \( (a_1, a_2) \) are damped through output fixed mirrors at the same rate \( \gamma_a \), while the entangler mode \( b \) is damped at rate \( \gamma_b \). Furthermore, \( \Gamma \) is the mechanical damping rate for the mirrors Brownian motion. Without loss of generality, we choose \( \alpha \) real and \( \Delta_a = 0 \). The operators \( a_j^{in}(t) \) and \( b_j(t) \) represent the vacuum (white) noise operators at the cavity inputs. The noise operator for the quantum Brownian motion of the mirrors is \( \xi_j(t) \). The non-vanishing noise correlations are
\[ \langle a_j^{in}(t)a_k^{in\dagger}(t') \rangle = \delta(t - t') \delta_{j,k}, \quad j, k = 1, 2, \]
\[ \langle b_j^{in}(t)b_k^{in\dagger}(t') \rangle = \delta(t - t'), \]
\[ \langle \xi_j(t)\xi_k(t') \rangle = \delta_{j,k} \int \frac{d\omega}{2\Omega} \frac{\Gamma(\alpha h\omega/2k_B T - 1)}{e^{\omega(t-t')}}, \]
where \( k_B \) is the Boltzmann constant and \( T \) the equilibrium temperature (the two mirrors are considered in equilibrium with their respective bath at the same temperature). Notice that the used approach for the Brownian motion is quantum mechanical consistent at every temperature.\(^3\)

The unitary evolution under the linearized Hamiltonian leading to system of Eqs. \(^3\) gives entanglement, as in the non-linearized case discussed above. Hence, the main task is to see whether such quantum correlations are visible or blurred by noisy effects. To accomplish this task, we first solve the system \(^3\) in the frequency domain by introducing the pseudo Fourier transform \( \mathcal{O}(\omega) = \mathcal{F}^{-1/2} \int_{-\pi/2}^{\pi/2} dt e^{i\omega t} \mathcal{O}(t) \) for each operator \( \mathcal{O} \), where \( t \) is the measurement time assumed to be large compared to all the relevant timescales of the system dynamics. To solve this task, we first solve the system \(^3\) in the frequency domain by introducing the pseudo Fourier transform \( \mathcal{O}(\omega) = \mathcal{F}^{-1/2} \int_{-\pi/2}^{\pi/2} dt e^{i\omega t} \mathcal{O}(t) \) for each operator \( \mathcal{O} \), where \( t \) is the measurement time assumed to be large compared to all the relevant timescales of the system dynamics.

Let us now consider the measured current at each meter output. The boundary relations for the meters radiation fields \(^4\), i.e., \( a_j^{out} = \sqrt{\gamma_a} a_j - a_j^{in} \), yield the phase quadratures \( Y_j = -i(a_j - a_j^{\dagger}) \) at the output, namely
\[ Y_j^{out}(\omega) = \frac{2g\alpha\sqrt{\gamma_a}}{\gamma_a/2 - i\omega} q_j(\omega) + \frac{\gamma_a/2 + i\omega}{\gamma_a/2 - 2i\omega} Y_j^{in}(\omega). \]

Thus, the measurement of the output quadrature \( Y_j^{out} \), in the detection box \( D_j \), indirectly gives the mirror position.
in $q_j$. More precisely, in homodyne detections, the positive
and negative frequency components of the quadrature be-
ing measured are combined through a proper modulation,
in order to achieve the measurement of a hermitian op-
erator $a^\dagger a$. Then, it would be possible to indirectly mea-
sure either $[q_j(\omega) + q_j(-\omega)]$ or $i[q_j(-\omega) - q_j(\omega)]$, which
implies the possibility to measure position or momentum for
each macroscopic oscillator.

These measurements can be used to establish when the
two oscillating cavity mirrors are entangled. Sufficient
criteria for entanglement of continuous variable systems
already exist [15,16], but here we shall introduce a new
sufficient inseparability criterion, involving the product
of variances of continuous observables:

**Theorem.** If we define $u = q_1 + q_2$ and $v = p_1 - p_2$,
then, for any separable quantum state $\rho$, one has

$$\langle (\Delta u)^2 \rangle \langle (\Delta v)^2 \rangle \geq |\langle [q_1, p_1] \rangle|^2. \quad (6)$$

See the appendix for the proof.

This theorem allows us to establish a connection with Refs. [17],
which showed that when the inequality

$$\langle (\Delta u)^2 \rangle \langle (\Delta v)^2 \rangle < \frac{1}{4} |\langle [q_1, p_1] \rangle|^2, \quad (7)$$
is satisfied, an EPR-like paradox arises [15], based on
the inconsistency between quantum mechanics and local
realism. Notice that the sufficient condition for insepara-
bility of Eq. (6) is weaker than condition (7), but this is
not surprising, since entangled states are only a necessary
condition for the realization of an EPR-like paradox.

The theorem (6) can then be used to establish the condi-
tions under which the two massive oscillators are entan-
gled. In fact, defining the hermitian operator

$$R_{\{O\}}(\omega) = |\langle O(\omega) + O(-\omega) \rangle|/2 $$

for any operator $O(\omega)$ in the frequency
domain, and using Eq. (6), we define the degree of en-
tanglement $E(\omega)$ as

$$E(\omega) = \frac{\langle R_{\{u\}}^2(\omega) \rangle \langle R_{\{v\}}^2(\omega) \rangle}{|\langle [q_1, p_1] \rangle|^2}, \quad (8)$$

(we use the fact that $\langle u \rangle = \langle v \rangle = 0$ in our case) which is
a marker of entanglement whenever $E(\omega) < 1$. If, how-
over, it goes below 1/4, that indicates the presence of
EPR correlations.

To calculate the function $E(\omega)$ we evaluate the corre-
lations

$$\langle O(\omega)O(\pm \omega) \rangle = \int_{-\tau/2}^{\tau/2} \frac{dt}{\tau} \int_{-\infty}^{\infty} \frac{dt'}{\tau} e^{i\omega t'} \langle O(t)O(t' \mp t) \rangle, \quad (9)$$

and use the solutions of (3) and correlations (4) in the
frequency domain. In doing that, we require $G > g$ and
$P_b > P_a$ because a strong interaction between mirrors
and entangler is desirable. The strength of the system-
eter interaction, instead, has to guarantee only a suf-
icient measurement gain. This condition, by referring to
Eq. (3), corresponds to $g^2 a^2 \gg (\gamma a^2/4 + \omega^2)/4$.

In Fig. 2 we show the behavior of the degree of entan-
glement (4) as function of frequency and temperature
for massive oscillators with $m = 10^{-5}$ Kg and $\Omega = 10^5$
s$^{-1}$. The maximum entanglement is always obtained at
the frequency $\Omega$ of the oscillating mirrors where the
mechanical response is maximum. The useful bandwidth
becomes narrower and tends to disappear as the temper-
ature increases. Nevertheless, a large amount of entan-
glement is available at reasonable temperatures e.g.
$4^0 K$. It means to have purely quantum effects at macro-
scopic scale notwithstanding $k_B T \gg \hbar \Omega$. It is also worth
noting that the values of parameters here employed are
essentially those already used in experiments [18]. More-
over, considering single mode oscillators, as we have done
here, is not a restrictive assumption because the vari-
sious internal and external oscillating modes of the mirrors
differ different oscillation frequencies and they can be eas-
ily distinguished and addressed when measurements are
performed in the frequency domain. Other promising
candidates for the realization of entanglement between
two massive objects are given by mesoscopic resonators,
such as microfabricated cantilevers [19].

FIG. 2. Degree of entanglement $E$ as function of frequency
$\omega$ and temperature $T$. The plot has been cut at $E = 1$, and
the part of surface for which $0 \leq E < 1/4$ is black coloured.
The value of parameters are: $\gamma_a = \gamma_b = \Delta_b = 10^5$ s$^{-1}$;
$P_{\alpha} = 5 \times 10^{-4}$ W; $P_{\beta} = 5 \times 10^{-3}$ W; $\Omega = 10^5$ s$^{-1}$;
$m = 10^{-5}$ Kg; $\Gamma = 1$ s$^{-1}$; $g = 0.5$ s$^{-1}$; $G = 5$ s$^{-1}$.
With these param-
eters the cavity lengths are $\approx 10^{-2}$ m for the $b$ mode and
$\approx 10^{-1}$ m for the $a$ modes.

As can be evicted from Fig. 2 at low temperatures,
$E(\Omega)$ lies below the limit 1/4. Thus, the studied system
also provides an example of macroscopic EPR corre-
lations, though with the experimental set-up of Ref. [18].
In specific, a further condition, concerning the spatial separation
between the two systems, is required to test the paradox
[15]. However, other possible set-ups could be devised.
permitting even such test. Demonstration of entanglement is instead much less demanding.

In conclusion, we have exploited the ponderomotive force to entangle macroscopic oscillators. Reliable conditions to achieve this goal are established by also accounting for a measurement of the degree of entanglement. The obtained results appears quite robust against the thermal noise and could be challenging tested with current technologies opening new perspectives towards the use of Quantum Mechanics in macroscopic world. Moreover, the possibility to prepare entangled state at the macroscopic level may prove to be useful for high precision and metrology applications. For example, it is possible to see that a scheme similar to that of Fig. 1 can be used to improve the detection of weak forces [21].

Appendix

We prove the sufficient criterion for inseparability for the pair of continuous variable operators $u = |a|q_1 + \frac{1}{2}q_2$ and $v = |a|p_1 - \frac{1}{2}p_2$, where $a$ is an arbitrary (nonzero) real number. Assuming $\rho = \sum_i w_i \rho_{1i} \otimes \rho_{2i}$ and using the same first steps of the proof of Ref. [15], we have

$$
\langle (\Delta u)^2 \rangle \langle (\Delta v)^2 \rangle = \left\{ \sum_i w_i \left( a^2 \langle (\Delta q_1)^2 \rangle_i \right) + \frac{1}{a^2} \langle (\Delta p_2)^2 \rangle_i \right\} \times \left\{ \sum_i w_i \langle v_i \rangle_i^2 - \left( \sum_i w_i \langle v_i \rangle_i \right)^2 \right\}
$$

where the symbol $\langle \cdots \rangle_i$ denotes average over the product density operator $\rho_{1i} \otimes \rho_{2i}$. By applying the Cauchy-Schwarz inequality (sum $\sum_i w_i \langle u_i \rangle_i^2 \geq (\sum_i w_i \langle \langle u_i \rangle_i \rangle^2$), we can rewrite

$$
\langle (\Delta u)^2 \rangle \langle (\Delta v)^2 \rangle \geq \left\{ \sum_i w_i \left( a^2 \langle (\Delta q_1)^2 \rangle_i + \frac{1}{a^2} \langle (\Delta p_2)^2 \rangle_i \right) \right\} \times \left\{ \sum_i w_i \left( a^2 \langle (\Delta p_1)^2 \rangle_i + \frac{1}{a^2} \langle (\Delta q_2)^2 \rangle_i \right) \right\}.
$$

Then using the fact that $\alpha^2 + \beta^2 \geq 2\alpha\beta$, we have

$$
\langle (\Delta u)^2 \rangle \langle (\Delta v)^2 \rangle \geq 4 \left\{ \sum_i w_i \sqrt{\langle (\Delta q_1)^2 \rangle_i \langle (\Delta q_2)^2 \rangle_i} \right\} \times \left\{ \sum_i w_i \sqrt{\langle (\Delta p_1)^2 \rangle_i \langle (\Delta p_2)^2 \rangle_i} \right\}.
$$

We then use again the Cauchy-Schwarz inequality and get

$$
\langle (\Delta u)^2 \rangle \langle (\Delta v)^2 \rangle \geq 4 \left( \sum_i w_i \left[ \langle (\Delta q_1)^2 \rangle_i \langle (\Delta q_2)^2 \rangle_i \right] \right) \times \left( \sum_i w_i \left[ \langle (\Delta p_1)^2 \rangle_i \langle (\Delta p_2)^2 \rangle_i \right] \right)^{1/4} \geq \langle (\Delta p_1)^2 \rangle_i \langle (\Delta p_2)^2 \rangle_i \right\}^{1/4} \right)^2,
$$

which gives the final inequality of Eq. [8] when the Heisenberg uncertainty principle is applied.

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