Inclusion-exclusion by ordering-free cancellation

Yin Chen, Jianguo Qian∗

School of Mathematical Sciences, Xiamen University, Xiamen 361005, P. R. China

Abstract

Whitney’s broken circuit theorem gives a graphical example to reduce the number of the terms in the sum of the inclusion-exclusion formula by a predicted cancellation. So far, the known cancellations for the formula strongly depend on the prescribed (linear or partial) ordering on the index set. We give a new cancellation method, which does not require any ordering on the index set. Our method extends all the ‘ordering-based’ methods known in the literatures and in general reduces more terms. As examples, we use our method to improve some relevant results on graph polynomials.

Keywords: inclusion-exclusion principle; ordering-free cancellation; graph polynomial; broken set

1 Introduction

Let (Ω, ℳ, μ) be a measure space, P be a finite index set and {A_p}_p∈P ⊆ ℳ be a family of measurable sets. The formula

$$
μ\left(\bigcap_{p∈P} A_p\right) = \sum_{I⊆P} (-1)^{|I|} μ\left(\bigcap_{i∈I} A_i\right)
$$

(1)

is known as the principle of inclusion-exclusion, where $\overline{A}_p$ denotes the complement of $A_p$.

*Corresponding author: jgqian@xmu.edu.cn.
The principle of inclusion-exclusion is a classic counting technique in combinatorics and has been extensively studied \[2, 5, 9, 10, 11, 13\]. Since the sum on the right side of Eq.(1) ranges over a large number of terms, it is natural to ask whether fewer terms would give the same result, that is, is it possible to reduce the number of terms by predicted cancellation? Lots of the answers to this question have been given by several authors. A well-known example is the one given by Whitney \[13\] in 1932 for chromatic polynomial of a graph, which states that the calculation of a chromatic polynomial can be restricted to the collection of those sets of edges which do not include any broken circuit as a subset.

Various cancellations for the inclusion-exclusion principle were given from the perspective of both combinatorics and graph theory in the literatures. In \[9\], Narushima presented a cancellation for the inclusion-exclusion principle, depending on a prescribed ordering on the index set \(P\). This result was later improved by Dohmen \[2\]. Using the same technique, Dohmen \[5\] also established an abstraction of Whitney’s broken circuit theorem, which not only applies to the chromatic polynomial, but also to other graph polynomials, see \[3, 4, 5, 8, 12\] for details.

So far, the known cancellation methods for inclusion-exclusion principle strongly depend on the prescribed (linear or partial) ordering on the index set \(P\). In this article we establish a new cancellation method, which does not require any ordering on \(P\). Our method extends all the ‘ordering-based’ methods given in the previous literatures and in general may reduce more terms. As examples, we use our ‘ordering-free’ method to improve the relevant results on the chromatic polynomial of hypergraphs, the independence polynomial and domination polynomial of graphs.

## 2 Inclusion-exclusion by predicted cancellations

For a subset \(B\) of a poset (partially ordered set) \(P\), let \(B’\) denote the set of upper bounds of \(B\) which are not in \(B\), that is,

\[
B’ = \{ p \in P : p > b \text{ for any } b \in B \}.
\]

In \[9\], Narushima presented a cancellation for the inclusion-exclusion principle on semilattices. This result was later extended to many forms. The following one was given by Dohmen \[2\]:

**Theorem 2.1.** \[2\] Let \((\Omega, \mathcal{A}, \mu)\) be a measure space, \(P\) be a poset and \(\{A_p\}_{p \in P} \subseteq \mathcal{A}\) be a family of measurable sets. If \(\mathcal{X}\) is a class of subsets of \(P\) such that

\[
\bigcap_{p \in B} A_p \subseteq \bigcup_{p \in B'} A_p
\]

(2)
for each \( B \in X \). Then

\[
\mu \left( \bigcap_{p \in P} \overline{A_p} \right) = \sum_{i \in 2^P \setminus \emptyset} (-1)^{|i|} \mu \left( \bigcap_{i \in i} A_i \right),
\]

(3)

where \( 2^P \) is the power set of \( P \) and \( \emptyset = \{ I \subseteq P : I \supseteq B \text{ for some } B \in X \} \).

Let \( \{B_1, B_1^2\}, \{B_2, B_2^2\}, \ldots, \{B_k, B_k^2\} \) be pairs of subsets of \( P \) with \( B_i \cap B_i^2 = \emptyset \) for every \( i \in \{1, 2, \ldots, k\} \). Denote

\[
\mathcal{B}_i = \{ I \subseteq P : I \supseteq B_i, I \nsubseteq B_j \setminus B_i^2 \text{ for } j < i \}
\]

and

\[
\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_k.
\]

(4)

We note that \( \mathcal{B}_i \) is empty when \( B_j \setminus B_i^2 \subseteq B_i^2 \) for some \( j < i \) since there is no \( I \) satisfies the requirement.

We now give our main result which does not require any ordering on \( P \).

**Theorem 2.2.** Let \((\Omega, \mathcal{A}, \mu)\) be a measure space, \( P \) be a set and \( \{A_p\}_{p \in P} \subseteq \mathcal{A} \) be a family of measurable sets. Let \( \{B_1, B_1^2\}, \{B_2, B_2^2\}, \ldots, \{B_k, B_k^2\} \) be pairs of subsets of \( P \). If \( B_i \cap B_i^2 = \emptyset \) and

\[
\bigcap_{p \in B_i} A_p \subseteq \bigcup_{p \in B_i^2} A_p
\]

(5)

for every \( i \in \{1, 2, \ldots, k\} \), then

\[
\mu \left( \bigcap_{p \in P} \overline{A_p} \right) = \sum_{i \in 2^P \setminus \emptyset} (-1)^{|i|} \mu \left( \bigcap_{i \in i} A_i \right).
\]

(6)

**Proof.** Let \( I \in \mathcal{B} \). Then \( I \in \mathcal{B}_i \) for some \( i \in \{1, 2, \ldots, k\} \). We claim that such \( \mathcal{B}_i \) is unique. In fact, suppose to the contrary that \( I \in \mathcal{B}_j \) and, with no loss of generality, that \( j < i \). Then by the definition of \( \mathcal{B}_i, I \nsubseteq B_j \). This contradicts that \( I \in \mathcal{B}_j \). As a result,

\[
\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_k
\]

are pairwise disjoint and therefore, (4) is a partition of \( \mathcal{B} \).

For \( I \in \mathcal{B}_i \), let \( I^* = I \setminus B_i^2 \). Since \( I \supseteq B_i \) and \( B_i \cap B_i^2 = \emptyset \), we have \( I^* \supseteq B_i \). We claim that \( I^* \cup D_i^* \in \mathcal{B}_i \) for any \( D_i^* \subseteq B_i^2 \).

Suppose to the contrary that \( I^* \cup D_i^* \notin \mathcal{B}_i \) for some \( D_i^* \subseteq B_i^2 \). Since \( I^* \cup D_i^* \supseteq I^* \supseteq B_i \), so by the definition of \( \mathcal{B}_i, I^* \cup D_i^* \supseteq B_j \setminus B_i^2 \) for some \( j < i \). Thus, \( I^* \supseteq B_j \setminus B_i^2 \) since \( D_i^* \subseteq B_i^2 \). Therefore,

\[
I \supseteq I^* \supseteq B_j \setminus B_i^2.
\]
Thus, So (6) follows directly, which completes our proof.

For $I \in \mathcal{B}_i$, let

$$
\langle I \rangle = \{ I' \cup D_i^* : D_i^* \subseteq B_i^* \}.
$$

Then

$$
\sum_{J \in \langle I \rangle} (-1)^{|J|} \mu \left( \bigcap_{p \in J} A_p \right) = \sum_{D_i^* \subseteq B_i^*} (-1)^{|I' \cup D_i^*|} \mu \left( \bigcap_{p \in I'} A_p \cap \bigcap_{p \in D_i^*} A_p \right)
$$

$$
= (-1)^{|I'|} \sum_{D_i^* \subseteq B_i^*} (-1)^{|D_i^*|} \mu \left( \bigcap_{p \in I'} A_p \cap \bigcap_{p \in D_i^*} A_p \right)
$$

$$
= (-1)^{|I'|} \mu \left( \bigcap_{p \in I'} A_p \cap \bigcap_{p \in B_i^*} \bar{A}_p \right),
$$

where the last equality holds by the principle of inclusion-exclusion. Notice that $\bigcap_{p \in B_i^*} \bar{A}_p$ is the complement of $\bigcup_{p \in B_i^*} A_p$. So by (5),

$$
\bigcap_{p \in I'} A_p \cap \bigcap_{p \in B_i^*} \bar{A}_p = \emptyset
$$

since $I' \supseteq B_i$. Therefore,

$$
\sum_{J \in \langle I \rangle} (-1)^{|J|} \mu \left( \bigcap_{p \in J} A_p \right) = 0. \quad (7)
$$

Finally, for any $I, J \in \mathcal{B}_i$, by the definition of $I'$ we can see that either $\langle J \rangle \cap \langle I \rangle = \emptyset$ or $\langle J \rangle = \langle I \rangle$. In other words, $\bigcup_{I \in \mathcal{B}_i} \langle I \rangle$ is a partition of $\mathcal{B}_i$, written by

$$
\mathcal{B}_i = \langle I_1 \rangle \cup \langle I_2 \rangle \cup \cdots \cup \langle I_r \rangle.
$$

Thus,

$$
\sum_{I \in \mathcal{B}_i} (-1)^{|I|} \mu \left( \bigcap_{i \in I} A_i \right) = \sum_{I \in \mathcal{B}_i} \sum_{i \in \mathcal{B}_i} (-1)^{|I|} \mu \left( \bigcap_{i \in I} A_i \right)
$$

$$
= \sum_{i=1}^{k} \sum_{J \in \langle I_i \rangle} (-1)^{|J|} \mu \left( \bigcap_{i \in I} A_i \right) = 0.
$$

So (6) follows directly, which completes our proof. \qed
Remark. Theorem 2.2 is an extension of Theorem 2.1 and may reduce more terms:

Firstly, let $\mathcal{X}$ be defined as in Theorem 2.1. Set $\{B_1, B_2, \cdots, B_k\} = \mathcal{X}$ and, for $i \in \{1, 2, \cdots, k\}$, set $B_i^* = B_i^\circ$ and let $b_i = \min B_i^\circ$ (the minimum element in $B_i^\circ$). Without loss of generality, we may assume that $b_1 \leq b_2 \leq \cdots \leq b_k$.

If $I \in \mathcal{I}$, say $I$ contains exactly $B_{i_1}, B_{i_2}, \cdots, B_{i_p}$ with $p > 0$ and $i_1 < i_2 < \cdots < i_p$, then we claim that $I \in \mathcal{B}_{i_1}$ and, therefore, $I \in \mathcal{B}$.

Suppose to the contrary that $I \notin \mathcal{B}_{i_1}$. Then there is $j < i_1$ such that $I \supseteq B_j \setminus B_i$. On the other hand, by the minimality of $i_1$, we have $I \notin B_j$ since $j < i_1$. This means that there is $b \in B_i$ such that $b \in B_j$. Therefore, $b < b_j$ since $b_j$ is an upper bound of $B_j$. This is a contradiction since $b_j \leq b_i \leq b$. Our claim follows.

Conversely, if $I \in \mathcal{B}$, say $I \in \mathcal{B}_{i_1}$, then we have $I \supseteq B_{i_1}$ and, therefore, $I \in \mathcal{I}$.

As a result, we have $\mathcal{I} = \mathcal{B}$. Thus, (6) implies (3).

Secondly, if $\{B, B^*\}$ is a pair such that $B$ differs from $B_1, B_2, \cdots, B_k$; $\{B, B^*\}$ satisfies (5); $\{B, B^*\}$ does not satisfy (2); $B_i \setminus B^* \notin \mathcal{B}$ for any $i = 1, 2, \cdots, k$. Then $\mathcal{B}$ can contain $B$ as an element while $\mathcal{X}$ and therefore $\mathcal{I}$ cannot contain $B$ as an element. This means that $\mathcal{B} \supseteq \mathcal{I}$, that is, (6) reduces more terms than (3) does. □

3 Examples in graph polynomials

As examples, in this section we apply Theorem 2.2 to chromatic polynomial of hypergraph, independence and domination polynomial of graph. We will see that the ordering-free method reduces more terms than the ordering-based method.

Let $P(G, x)$ be a graph polynomial of a graph $G$ represented in the form of inclusion-exclusion principle, i.e.,

$$P(G, x) = \sum_{F \subseteq E(G)} (-1)^{|F|} p(F, x),$$

where $E(G)$ is the edge set of $G$ and $p(F, x)$ is a polynomial in $x$ associated with $F \subseteq E(G)$. We specialize the index set $P$ to be $E(G)$ and, for any $F \subseteq E(G)$, set

$$\mu \left( \bigcap_{e \in F} A_e \right) = p(F, x).$$

For a pair $B, B^* \subseteq E(G)$ with $B \cap B^* = \emptyset$, if $B^*$ is a single-edge set, say $B^* = \{b\}$, then the condition

$$\bigcap_{e \in B} A_e \subseteq \bigcup_{e \in B^*} A_e,$$
i.e., $\bigcap_{e \in B} A_e \subseteq A_b$, is equivalent to

$$\bigcap_{e \in B} A_e = \bigcap_{e \in B \cup \{b\}} A_e.$$  

Combining with (8), we have

$$p(B, x) = p(B \cup \{b\}, x). \quad (9)$$

Thus, a pair $\{B, \{b\}\}$ (viewed as $\{B_i, B_i^*\}$) satisfies the requirement of Theorem 2.2 provided it satisfies (9). We refer to such pair $\{B, b\}$ as a broken pair of $P(G, x)$ and $B$ a broken set if $B$ is minimal (i.e., $B$ has no proper subset satisfying (9)). Further, given a linear ordering ‘$<$’ on $E(G)$, we call $B$ a broken pair with respect to ‘$<$’ if $\{b\} = B'$. By Theorem 2.2 we have the following corollary immediately.

**Corollary 3.1.** Let $\{B_1, B_1^*\}, \{B_2, B_2^*\}, \ldots, \{B_k, B_k^*\}$ be broken pairs of $P(G, x)$. Then

$$P(G, x) = \sum_{F \subseteq E(G) \setminus B} (-1)^{|F|} p(F, x).$$

**Chromatic polynomial of hypergraph.** The chromatic polynomial $\chi(H, x)$ of a simple hypergraph $H$ counts the number of the vertex colorings such that each (hyper) edge of cardinality at least two has two vertices of distinct colors \cite{1, 5}. The following inclusion-exclusion expression was given in \cite{5, 12}:

$$\chi(H, x) = \sum_{F \subseteq E(H)} (-1)^{|F|} \chi(F),$$

where $c(F)$ is the number of the components of the spanning subgraph of $H$ with edge set $F$.

Given a linear order ‘$<$’ on the edge set $E(H)$, Dohmen \cite{5} generalized the Whitney’s broken circuit theorem to hypergraph by extending the broken circuit defined on a cycle (see \cite{1} for the definition of a cycle), with a particular constraints that each edge of the cycle is included by the union of the other edges of that cycle. A set $F \subseteq E(H)$ is called a $\delta$-cycle if $F$ is minimal such that $c(F \setminus \{f\}) = c(F)$ for each $f \in F$. We note that every cycle with the above particular constraints is or contains a $\delta$-cycle while a $\delta$-cycle is not necessarily a cycle with this constraints. A set $B$ is called a broken cycle if $B$ is obtained from a $\delta$-cycle by deleting its maximum edge. In \cite{12}, Trinks generalized the Dohmen’s result by extending the broken circuit to broken cycle.

For $B \subseteq E(H)$ and $b \in E(H) \setminus B$, by (9) it can be seen that $B$ is a broken set of $\chi(H, x)$ provided $B$ is minimal such that

$$c(B) = c(B \cup \{b\}). \quad (10)$$
We can see that the notion ‘broken set’ for hypergraph is an extension of ‘broken cycle’. Moreover, in condition (10) there is no need to require \( b \) to be the maximum edge of \( B \cup \{ b \} \) for a broken set.

Let’s consider the hypergraph \( H = (V, E) \) with vertex set \( V = \{1, 2, 3, 4, 5, 6\} \) and edge set \( E = \{\{1, 2, 3\}, \{3, 4, 5\}, \{2, 3, 4\}, \{1, 2, 6\}\} \). We note that \( H \) contains neither broken circuit (with the particular constraints) nor broken cycle, no matter how to order its edges. This means that no terms in \( \chi(H, x) \) can be reduced by broken circuit or broken cycle.

For an edge \( \{i, j, k\} \) we write it simply as \( ijk \). By (10) it can be seen that \( H \) has two broken sets \( B_1 = \{123, 345\} \) with \( B_1' = \{b_1\} = \{234\} \) and \( B_2 = \{234, 126\} \) with \( B_2' = \{b_2\} = \{123\} \). Therefore, \( \mathcal{B}_1 = \{\{123, 345\}, \{123, 345, 234\}, \{123, 345, 126\}, \{123, 345, 234, 126\}\} \) and \( \mathcal{B}_2 = \{\{234, 126\}, \{234, 126, 123\}\} \).

Consider the edge ordering \( 123 < 345 < 234 < 126 \). Again by (10), \( H \) contains only one broken set with respect to ‘\(<\)’, i.e., \( B = \{123, 345\} \) with \( B' = \{234\} \). Thus, \( x = \{B\} \) (see Theorem 2.1) and \( \mathcal{X} = \{\{123, 345\}, \{123, 345, 234\}, \{123, 345, 126\}, \{123, 345, 234, 126\}\} = \mathcal{B}_1 \).

So by Theorem 2.1 and Corollary 3.1, the chromatic polynomial of \( H \) is

\[
\chi(H, x) = \sum_{F \in 2^E \ominus \mathcal{X}} (-1)^{|F|} x^{|F|} = \sum_{F \in 2^E \ominus (\mathcal{B}_1 \cup \mathcal{B}_2)} (-1)^{|F|} x^{|F|} = k^6 - 4k^4 + 3k^3 + k^2 - k.
\]

Moreover, we see that \(|2^E| = 16 > |2^E \ominus \mathcal{X}| = 12 > |2^E \ominus (\mathcal{B}_1 \cup \mathcal{B}_2)| = 10\).

Finally, it can be seen that \( H \) has at most one broken set with respect to ‘\(<\)’, no matter how to define the order ‘\(<\)’.

**Independence polynomial of graph.** For a graph \( G \), the independence polynomial \([6, 7] \) of \( G \) can be represented as the following inclusion-exclusion formula \([3]\):

\[
I(G, x) = \sum_{F \subseteq E(G)} (-1)^{|F|} x^{|G[F]|}(1 + x)^{n - |G[F]|}, \tag{11}
\]

where \(|G[F]|\) is the number of vertices in the subgraph of \( G \) induced by \( F \).

It was shown \([3]\) that the Whitney’s broken circuit theorem is also valid for independence polynomial. By (9) and (11), a set \( B \) of edges is a broken set provided \( B \) is minimal such that \( G[B] = G[B \cup \{ b \}] \) for some \( b \not\in B \). This means that \( B = \{e_1, e_2\} \) and \( e_1be_2 \) is a path or a cycle of length 3. We call such \( B \) a broken path. We note that every broken circuit includes a broken path as a subgraph.

Let’s consider the path \( G = e_1e_2e_3e_4 \) of length 4 with edge ordering \( e_1 < e_3 < e_2 < e_4 \). Similar to the previous example, we have \( B_1 = \{e_1, e_3\} \) with \( B_1' = \{e_2\} \) and
\[ B_2 = \{e_2, e_4\} \text{ with } B_2^* = \{e_3\}, \text{ and } \mathcal{X} = \{\{e_1, e_3\}\}. \text{ Therefore:} \\
\mathcal{B}_1 = \{\{e_1, e_3\}, \{e_1, e_2, e_3\}, \{e_1, e_3, e_4\}, \{e_1, e_2, e_3, e_4\}\}; \\
\mathcal{B}_2 = \{\{e_2, e_4\}, \{e_2, e_3, e_4\}\}; \text{ and} \\
\mathcal{Y} = \{\{e_1, e_3\}, \{e_1, e_2, e_3\}, \{e_1, e_3, e_4\}, \{e_1, e_2, e_3, e_4\}\} = \mathcal{B}_1. \\
\]

**Domination polynomial of graph.** For a graph \(G\) and \(W \subseteq V(G)\), denote by \(N[W]\) the closed neighbourhood of \(W\), i.e.,

\[ N[W] = W \cup \{v : v \text{ is adjacent to some vertex in } W\}. \]

Let \(d_i\) be the number of the sets \(W\) of \(i\) vertices such that \(N_G[W] = V(G)\). The domination polynomial \(D(G, x)\) is defined by \(D(G, x) = \sum_{i=1}^{n} d_i x^i\). The following form was given in [4],

\[ D(G, x) = \sum_{W \subseteq V(G)} (-1)^{|W|} (1 + x)^{n - |N[W]|}. \quad (12) \]

A set \(B\) is called broken neighbourhood if \(B = N(\nu)\) and \(\nu = \max N[\nu]\). In [4], Dohmen and Tittmann proved that the sum in (12) can be restricted to those subsets of vertices which do not contain any broken neighbourhood.

Due to (12), we replace the role of edges in (9) by vertices. For \(B \subseteq V(G)\) and \(b \in V(G) \setminus B\), by (9) it can be seen that \(B\) is a broken set of \(D(G, x)\) provided \(B\) is minimal such that \(|N[B]| = |N[B \cup \{b\}]|\), i.e.,

\[ N[b] \subseteq N[B]. \quad (13) \]

We can see that the ‘broken set’ of \(D(G, x)\) is an extension of ‘broken neighbourhood’.

Consider the path \(P = v_1v_2v_3v_4\) with vertex ordering \(v_1 < v_2 < v_3 < v_4\). Similarly, by (13) we have \(B_1 = \{v_1, v_3\}\) with \(B_1^* = \{v_2\}\), \(B_2 = \{v_1, v_4\}\) with \(B_2^* = \{v_2\}\) and \(B_3 = \{v_2, v_4\}\) with \(B_3^* = \{v_3\}\). Therefore:
\[ \mathcal{B}_1 = \{\{v_1, v_3\}, \{v_1, v_2, v_3\}, \{v_1, v_3, v_4\}, \{v_1, v_2, v_3, v_4\}\}; \]
\[ \mathcal{B}_2 = \{\{v_1, v_4\}, \{v_1, v_2, v_4\}\}; \]
\[ \mathcal{B}_3 = \{\{v_2, v_4\}, \{v_2, v_3, v_4\}\}; \text{ and} \]
\[ \mathcal{Y} = \{\{v_1, v_3\}, \{v_1, v_2, v_3\}, \{v_1, v_3, v_4\}, \{v_1, v_2, v_3, v_4\}, \{v_1, v_4\}, \{v_1, v_2, v_4\}\} = \mathcal{B}_1 \cup \mathcal{B}_2. \]

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