Gauge-invariant metric fluctuations from NKK theory of gravity: de Sitter expansion

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Abstract

In this paper we study gauge-invariant metric fluctuations from a Noncompact Kaluza-Klein (NKK) theory of gravity in a de Sitter expansion. We recover the well known result \( \delta \rho / \rho \simeq 2\Phi \), obtained from the standard 4D semiclassical approach to inflation. The spectrum for these fluctuations should be dependent of the fifth (spatial-like) coordinate.

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I. INTRODUCTION

The relativistic theory of cosmological perturbations is a cornerstone in our understanding of the early universe as it is indispensable in relating early universe scenarios, such as inflation, to cosmological data such as the Cosmic Microwave Background (CMB) anisotropies. The inflationary model [1] solves several difficulties which arise from the standard cosmological model, such as the horizon, flatness and monopole problems, and it provides a mechanism for the creation of primordial density of fluctuations, needed to explain the structure formation [5]. The most widely accepted 4D approach assumes that the inflationary phase is driving by a quantum scalar field \( \varphi \) related to a scalar potential \( V(\varphi) \) [2]. Within this perspective, the semiclassical approach to inflation proposes to describe the dynamics of this quantum field on the basis of two pieces: the spatially homogeneous (background) and inhomogeneous components [6]. Usually the homogeneous one is interpreted as a classical...
field $\varphi_b(t)$ that arises from the vacuum expectation value of the quantum field. The inhomogeneous component $\delta \varphi(\vec{R}, t)$ are the quantum fluctuations. These quantum field fluctuations are responsible for metric fluctuations around the background Friedmann-Robertson-Walker (FRW) metric [7].

The two current versions of 5D gravity theory are membrane theory [8] and induced-matter theory [9]. In the former, gravity propagates freely into the bulk, while the interactions of particle physics are confined to a hypersurface (the brane). The induced-matter theory in its simplest form is the basic Kaluza-Klein (KK) theory in which the fifth dimension is not compactified and the field equations of general relativity in 4D follow from the fact that the 5D manifold is Ricci-flat. Thus the large extra dimension is responsible for the appearance of sources in 4D general relativity. Hence, the 4D world of general relativity is embedded in a 5D Ricci-flat manifold. There has recently been an uprisng interest in finding exact solutions of the Kaluza-Klein field equations in 5D, where the fifth coordinate is considered as noncompact. This theory reproduces and extends known solutions of the Einstein field equations in 4D. Particular interest revolves around solutions which are not only Ricci flat, but also Riemann flat. This is because it is possible to have a flat 5D manifold which contains a curved 4D submanifold, as implied by the Campbell theorem. So, the universe may be “empty” and simple in 5D, but contain matter of complicated forms in 4D [10]. This paper is devoted to study a 4D de Sitter expansion of the universe from a NKK theory of gravity, taking into account the scalar metric fluctuations, which are gauge-invariant. In particular, we are aimed to describe the 4D dynamics of the gauge-invariant scalar metric fluctuations from the NKK theory of gravity. To make it we shall consider the action

$$I = -\int d^4x \sqrt{-g} \left[ \frac{R_{(5)}}{16\pi G} + \mathcal{L}(\varphi, \varphi, A) \right],$$

(1)

for a scalar field $\varphi$, which is minimally coupled to gravity. Since we are aimed to describe a manifold in apparent vacuum the Lagrangian density $\mathcal{L}$ in (1) should be only kinetic in origin

$$\mathcal{L}(\varphi, \varphi, A) = \frac{1}{2} g^{AB} \varphi_{,A} \varphi_{,B},$$

(2)

where $A, B$ can take the values 0, 1, 2, 3, 4 and the perturbed line element $dS^2 = g_{AB} dx^A dx^B$ is given by

$$dS^2 = \psi^2 (1 + 2\Phi) dN^2 - \psi^2 (1 - 2\Psi) e^{2N} dr^2 - (1 - Q) d\psi^2.$$  

(3)

Here, the fields $\Phi, \Psi$ and $Q$ are functions of the coordinates $[N, \vec{r}(x, y, z), \psi]$, where $N, x, y, z$ are dimensionless coordinates and $\psi$ has spatial units. Note that $\bar{R}$ in the action (1) is the Ricci scalar evaluated on the background metric $(dS^2) = \bar{g}_{AB} dx^A dx^B$. In our case we shall consider the background canonical metric

$$(dS^2)_b = \psi^2 dN^2 - \psi^2 e^{2N} dr^2 - d\psi^2,$$

(4)

which is 3D spatially isotropic, homogeneous and flat [3]. Furthermore, the metric (4) is globally flat (i.e., $\bar{R}_{BCD} = 0$) and describes an apparent vacuum: $\bar{G}_{AB} = 0$.

The energy-momentum tensor is given by

$$T_{AB} = \varphi_{,A} \varphi_{,B} - \frac{1}{2} g_{AB} \varphi_{,C} \varphi_{,C}.$$  

(5)
II. FORMALISM

In order to describe scalar metric fluctuations we must consider the covariant energy-momentum tensor $T_{AB}$ to be symmetric. In such a case we obtain that $\Psi = \Phi$ and $Q = 2\Phi$, so that the line element (3), now holds

$$dS^2 = \psi^2 (1 + 2\Phi) dN^2 - \psi^2 (1 - 2\Phi) e^{2N}dr^2 - (1 - 2\Phi) d\psi^2,$$

(6)

where the field $\Phi(N, \vec{r}, \psi)$ is the scalar metric perturbation of the background 5D metric (4). For the metric (6), $|\bar{g}| = \psi^8 e^{6N}$ is the absolute value of the determinant for the background metric (4) and $|\bar{g}_0| = \psi_0^8 e^{6N_0}$ is a constant of dimensionalization, for the constants $\psi_0$ and $N_0$. Furthermore, $G = M_p^{-2}$ is the gravitational constant and $M_p = 1.2 \times 10^{19}$ GeV is the Planckian mass. In this work we shall consider $N_0 = 0$, so that $|\bar{g}_0| = \psi_0^8$. Here, the index “0” denotes the value at the end of inflation.

On the other hand, the contravariant metric tensor, after a $\Phi$-first order approximation, is

$$g^{AB} = \begin{pmatrix}
\frac{(1-2\Phi)}{\psi^2} & 0 & 0 & 0 & 0 \\
0 & -\frac{e^{-2N(1+2\Phi)}}{\psi^2} & 0 & 0 & 0 \\
0 & 0 & -\frac{e^{-2N(1+2\Phi)}}{\psi^2} & 0 & 0 \\
0 & 0 & 0 & -\frac{e^{-2N(1+2\Phi)}}{\psi^2} & 0 \\
0 & 0 & 0 & 0 & -(1 + 2\Phi)
\end{pmatrix},$$

(7)

which can be written as $g^{AB} = \bar{g}^{AB} + \delta g^{AB}$, being $\bar{g}^{AB}$ the contravariant background metric tensor. The dynamics for $\varphi$ and $\Phi$ are well described by the Lagrange and Einstein equations, which we shall study in the following subsections.

A. Lagrange equations

The Lagrange equations for the fields $\varphi$ and $\Phi$ are respectively given by

$$\frac{\partial^2 \varphi}{\partial N^2} + 3 \frac{\partial \varphi}{\partial N} - e^{-2N} \nabla_r^2 \varphi - \psi \left( \frac{\partial^2 \varphi}{\partial \psi^2} + 4 \frac{\partial \varphi}{\partial \psi} \right) - 2\Phi \left[ \frac{\partial^2 \varphi}{\partial N^2} + 3 \frac{\partial \varphi}{\partial N} - e^{-2N} \nabla_r^2 \varphi + \psi \left( \frac{\partial^2 \varphi}{\partial \psi^2} + 4 \frac{\partial \varphi}{\partial \psi} \right) \right] - 2 \left( \frac{\partial \varphi}{\partial N} \frac{\partial \Phi}{\partial N} + \psi^2 \frac{\partial \Phi}{\partial \psi} \frac{\partial \varphi}{\partial \psi} \right) = 0,$$

(8)

$$\left( \frac{\partial \varphi}{\partial N} \right)^2 + e^{-2N} (\nabla \varphi)^2 + \psi^2 \left( \frac{\partial \varphi}{\partial \psi} \right)^2 = 0.$$  

(9)

Now we can make the following semiclassical approximation: $\varphi(N, \vec{r}, \psi) = \varphi_b(N, \psi) + \delta \varphi(N, \vec{r}, \psi)$, such that $\varphi_b$ is the solution of eq. (8) in absence of the inflaton and metric fluctuations [i.e., for $\Phi = \nabla_r \varphi_b = 0$]. Hence, the Lagrange equations for $\varphi_b$ and $\delta \varphi$ are

$$\frac{\partial^2 \varphi_b}{\partial N^2} + 3 \frac{\partial \varphi_b}{\partial N} - \psi \left[ \frac{\partial^2 \varphi_b}{\partial \psi^2} + 4 \frac{\partial \varphi_b}{\partial \psi} \right] = 0,$$

(10)
\[
\frac{\partial^2 \delta \phi}{\partial N^2} + 3 \frac{\partial \delta \phi}{\partial N} - e^{-2N} \nabla^2_r \delta \phi - \psi \left[ 4 \frac{\partial \delta \phi}{\partial \psi} + \psi \frac{\partial^2 \delta \phi}{\partial \psi^2} \right] \\
- 2 \frac{\partial \phi_b}{\partial N} \frac{\partial \Phi}{\partial N} - 2 \psi^2 \left[ \frac{\partial \phi_b}{\partial \psi} \frac{\partial \Phi}{\partial \psi} + \left( \frac{\partial^2 \phi_b}{\partial \psi^2} + 4 \frac{\partial \phi_b}{\partial \psi} \frac{\partial \Phi}{\partial \psi} \right) \Phi \right] = 0. 
\]

(11)

Note that for \( \Phi = \nabla_r \phi_b = 0 \), the equation (9) holds

\[
\left( \frac{\partial \phi_b}{\partial N} \right)^2 + \psi^2 \left( \frac{\partial \phi_b}{\partial \psi} \right)^2 = 0, 
\]

(12)

which will be useful later.

\section*{B. 5D Einstein equations}

The diagonal perturbed first order 5D Einstein equations \( \delta G_{AA} = -8\pi G \delta T_{AA} \), are

\[
9 \frac{\partial \Phi}{\partial N} - 9 \psi \frac{\partial \Phi}{\partial \psi} - 3 \psi^2 \frac{\partial^2 \Phi}{\partial \psi^2} - 3 e^{-2N} \nabla^2_r \Phi + 12 \Phi = -16\pi G \psi^2 \Phi \left( \frac{\partial \phi_b}{\partial \psi} \right)^2, 
\]

(13)

\[
3 \psi^2 e^{-2N} \frac{\partial^2 \Phi}{\partial \psi^2} - 36 e^{-2N} \Phi + 2 \nabla^2_r \Phi - 30 e^{-2N} \frac{\partial \Phi}{\partial N} + 3 \psi \frac{\partial \Phi}{\partial \psi} = 48\pi G e^{-2N} \Phi \left( \frac{\partial \phi_b}{\partial N} \right)^2, 
\]

(14)

\[
3 \frac{\partial^2 \Phi}{\partial N^2} - e^{-2N} \nabla^2_r \Phi + 24 \Phi + 15 \frac{\partial \Phi}{\partial N} - 6 \psi \frac{\partial \Phi}{\partial \psi} = -16\pi G \Phi \left( \frac{\partial \phi_b}{\partial N} \right)^2, 
\]

(15)

for the components \( NN, \; rr \) and \( \psi \psi \), respectively. Furthermore, the non-diagonal 5D Einstein equations (which are symmetric with respect to indices permutation): \( \delta G_{AB} = -8\pi G \delta T_{AB} \) (with \( A \neq B \)), are

\[
\frac{\partial \Phi}{\partial x^i} + 3 \frac{\partial^2 \Phi}{\partial x^i \partial N} = 0, 
\]

(16)

\[
\psi \frac{\partial^2 \Phi}{\partial \psi \partial N} + 2 \frac{\partial \Phi}{\partial \psi} - \frac{\partial \Phi}{\partial N} = 0, 
\]

(17)

\[
3 \frac{\partial \Phi}{\partial x^i} - \psi \frac{\partial^2 \Phi}{\partial x^i \partial \psi} = 0, 
\]

(18)

for the components \( Nx^i, \; N\psi \) and \( x^i \psi \), respectively (latin indices can take values 1, 2, 3).

After some algebra, from the equations (13), (14) and (15), we obtain

\[
\frac{\partial^2 \Phi}{\partial N^2} + 3 \frac{\partial \Phi}{\partial N} - e^{-2N} \nabla^2_r \Phi - 2 \psi^2 \frac{\partial^2 \Phi}{\partial \psi^2} + \frac{16\pi G}{3} \Phi \left[ \left( \frac{\partial \phi_b}{\partial N} \right)^2 + \psi^2 \left( \frac{\partial \phi_b}{\partial \psi} \right)^2 \right] = 0. 
\]

(19)

From eq. (12), the eq. (19) holds

\[
\frac{\partial^2 \Phi}{\partial N^2} + 3 \frac{\partial \Phi}{\partial N} - e^{-2N} \nabla^2_r \Phi - 2 \psi^2 \frac{\partial^2 \Phi}{\partial \psi^2} = 0, 
\]

(20)

which is the equation of motion for the 5D scalar metric fluctuations \( \Phi(N, r, \psi) \).
C. Normalization of $\Phi$ in 5D

We consider the following separation of the 5D metric fluctuations: $\Phi(N, \vec{r}, \psi) = \Phi_1(N)\Phi_2(\vec{r})\Phi_3(\psi)$. The equation (20) can be rewritten as three differential equations

$$\psi^2 \frac{d^2 \Phi_3}{d\psi^2} = k^2 \psi^2 \Phi_3, \quad (21)$$
$$\nabla^2 \Phi_2 = -k^2 \Phi_2, \quad (22)$$
$$\frac{d^2 \Phi_1}{dN^2} + 3 \frac{d\Phi_1}{dN} - \left(2k^2 \psi^2 + e^{-2N} k^2 r^2\right) \Phi_1 = 0, \quad (23)$$

where $k^2 \psi^2 > 0$.

Now we consider the transformation $\Phi(N, \vec{r}, \psi) = e^{-3N/2} \frac{\psi}{\sqrt{\psi}} \chi(N, \vec{r})$. This transformation also can be physically justified in the sense that we can make observations on any hypersurface with constant $\psi$. The equation of motion for $\chi$ is

$$\frac{\partial^2 \chi}{\partial N^2} - \left(e^{-2N} \nabla^2 r + 2k^2 \psi^2\right) \chi = 0, \quad (24)$$

where $\chi$ can be written as a Fourier expansion

$$\chi(N, \vec{r}) = \frac{1}{(2\pi)^3/2} \int d^3k_r \int dk_\psi \left[ a_{k_r k_\psi} e^{i\vec{k}_r \cdot \vec{r}} \xi_{k_r k_\psi}(N) + a_{k_r k_\psi}^\dagger e^{-i\vec{k}_r \cdot \vec{r}} \xi^*_{k_r k_\psi}(N) \right], \quad (25)$$

and the asterisk denotes the complex conjugate and $(a_{k_r k_\psi}, a_{k_r k_\psi}^\dagger)$ are, respectively, the annihilation and creation operators which satisfy the algebra

$$\left[a_{k_r k_\psi}, a_{k_r' k_\psi'}^\dagger\right] = \delta^{(3)}(\vec{k}_r - \vec{k}_r') \delta(\vec{k}_\psi - \vec{k}_\psi'), \quad \left[a_{k_r k_\psi}, a_{k_r' k_\psi'}\right] = \left[a_{k_r k_\psi}^\dagger, a_{k_r' k_\psi'}^\dagger\right] = 0.$$

The equation of motion for the $N$-dependent modes $\xi_{k_r k_\psi}$ is

$$\frac{d^2 \xi_{k_r k_\psi}}{dN^2} + \left[e^{-2N} k^2 r^2 - 2k^2 \psi^2\right] \xi_{k_r k_\psi} = 0. \quad (26)$$

The general solution for this equation is

$$\xi_{k_r k_\psi}(N) = C_1 \mathcal{H}_\nu^{(1)}[x(N)] + C_2 \mathcal{H}_\nu^{(2)}[x(N)], \quad (27)$$

where $\nu = \sqrt{2} k \psi$ is a constant and $x(N) = k_r e^{-N}$. Using the generalized Bunch-Davies vacuum [4], we obtain that

$$\xi_{k_r k_\psi}(N) = i \sqrt{\frac{4}{\pi}} \mathcal{H}_\nu^{(2)}[x(N)], \quad (28)$$

which are the normalized $N$-dependent modes of $\chi$.

III. EFFECTIVE 4D DE SITTER EXPANSION

In this section we shall study the effective 4D $\Phi$-dynamics in an effective 4D de Sitter background expansion of the universe, which is considered 3D (spatially) flat, isotropic and homogeneous.
A. Ponce de Leon metric

We consider the transformation [11]

\[ t = \psi_0 N, \quad R = \psi_0 r, \quad \psi = \psi. \quad (29) \]

Hence, the 5D background metric (4) becomes

\[ (dS^2)_b = \left( \frac{\psi}{\psi_0} \right)^2 \left[ dt^2 - e^{2t/\psi_0} dR^2 \right] - d\psi^2, \quad (30) \]

which is the Ponce de Leon metric [12], that describes a 3D spatially flat, isotropic and homogeneous extended (to 5D) Friedmann-Robertson-Walker metric in a de Sitter expansion. Here, \( t \) is the cosmic time and \( R^2 = X^2 + Y^2 + Z^2 \). This Ponce de Leon metric is a special case of the separable models studied by him, and is an example of the much-studied class of canonical metrics \( dS^2 = \psi^2 g_{\mu\nu} dX^\mu dX^\nu - d\psi^2 \) [13]. Now we can take a foliation \( \psi = \psi_0 \) in the metric (30), such that the effective 4D metric results

\[ (dS^2)_b \rightarrow (ds^2)_b = dt^2 - e^{2t/\psi_0} dR^2, \quad (31) \]

which describes a 4D expansion of a 3D spatially flat, isotropic and homogeneous universe that expands with a constant Hubble parameter \( H = 1/\psi_0 \) and a 4D scalar curvature \((4)^{\mathcal{R}} = 12H^2\). Hence, the effective 4D metric of (6) on hypersurfaces \( \psi = 1/H \), is

\[ dS^2 \rightarrow ds^2 = (1 + 2\Phi) dt^2 - (1 - 2\Phi) e^{2Ht} dR^2, \quad (32) \]

where the metric (32) describes the perturbed 4D de Sitter expansion of the universe, where \( \Phi(\vec{R},t) \) is gauge-invariant.

B. Dynamics of \( \Phi \) in an effective 4D de Sitter expansion

In order to study the 4D dynamics of the gauge-invariant scalar metric fluctuations \( \Phi(\vec{R},t) \) in a background de Sitter expansion we can take the equation (20) with the transformations (29), for \( \psi = \psi_0 = 1/H \)

\[ \frac{\partial^2 \Phi}{\partial t^2} + 3H \frac{\partial \Phi}{\partial t} - e^{-2Ht} \nabla^2_R \Phi - 2 \frac{\partial^2 \Phi}{\partial \psi^2} \bigg|_{\psi=H^{-1}} = 0, \quad (33) \]

where \( \frac{\partial^2 \Phi}{\partial \psi^2} \bigg|_{\psi=H^{-1}} = k^2 \psi_0^2 \Phi \). To simplify the structure of this equation we propose the redefined quantum metric fluctuations \( \chi(\vec{R},t) = e^{3Ht/2} \Phi(\vec{R},t) \), so that \( \chi \) complies with the following equation of motion

\[ \ddot{\chi} - e^{-2Ht} \nabla^2_R \chi - \left[ \frac{9}{4} H^2 + 4 k^2 \psi_0 \right] \chi = 0, \quad (34) \]

where the redefined field \( \chi(\vec{R},t) \) can be expanded as
\[ \chi(\vec{R}, t) = \frac{1}{(2\pi)^{3/2}} \int d^3k_R \int dk_\psi \left[ a_{k_Rk_\psi} e^{i\vec{k}_R \cdot \vec{R}} \xi_{k_Rk_\psi}(t) + \text{c.c.} \right] \delta (k_\psi - k_{\psi_0}). \] (35)

Here, the operators \( a_{k_Rk_\psi} \) and \( a_{k_Rk_\psi}^\dagger \) comply

\[ \left[ a_{k_Rk_\psi}, a_{k_R'k_\psi}' \right] = \delta^{(3)}(\vec{k}_R - \vec{k}_R') \delta \left( k_\psi - k_\psi' \right), \quad \left[ a_{k_Rk_\psi}, a_{k_R'k_\psi}' \right] = 0. \]

and the time dependent modes \( \xi_{k_Rk_\psi}(t) \) are given by the equation

\[ \ddot{\xi}_{k_Rk_\psi}(t) + \left[ k_\psi^2 e^{-2Ht} - \left( \frac{9}{4} H^2 + 4k_\psi^2 \right) \right] \xi_{k_Rk_\psi}(t) = 0. \] (36)

The general solution for this equation is

\[ \xi_{k_Rk_\psi}(t) = A_1 H^1[\mu y(t)] + A_2 H^2[\mu y(t)], \] (37)

where \( \mu = \sqrt{9+16k_\psi^2/H^2} \) and \( y(t) = k_\psi e^{-Ht} \). Using the Bunch-Davies vacuum [4], we obtain

\[ \xi_{k_Rk_\psi}(t) = i \sqrt{\frac{\pi}{4H^2}} H^2[\mu y(t)], \] (38)

which are the normalized time dependent modes of \( \chi(\vec{R}, t) \).

**C. Energy density fluctuations**

In order to obtain the energy density fluctuations on the effective 4D FRW metric, we must calculate

\[ \frac{\delta \rho}{\langle \rho \rangle} = \frac{\delta T_{NN}}{\langle T_{NN} \rangle} \bigg|_{t=\psi_0, R=\psi_0, \psi=1/H}, \] (39)

where \( \delta T_{NN} = -\frac{1}{2} \delta g_{NN} \varphi \varphi_L \varphi^L \) is linearized and the brackets \( <> \) denote the expectation value on the 3D hypersurface \( R(X,Y,Z) \). Using the semiclassical expansion \( \varphi(\vec{R}, t) = \varphi_b(t) + \delta \varphi(\vec{R}, t) \), after some algebra we obtain

\[ \frac{\delta \rho}{\langle \rho \rangle} \approx 2\Phi \left\{ 1 - \frac{\langle (\delta \varphi)^2 + e^{-2Ht} (\nabla_R \delta \varphi)^2 + 2V(\delta \varphi) \rangle}{(\varphi_b)^2 + 4H^2 (\varphi_b)^2} \right\} \approx 2\Phi, \] (40)

where we have considered the following approximation:

\[ \frac{\langle (\delta \varphi)^2 + e^{-2Ht} (\nabla_R \delta \varphi)^2 + 2V(\delta \varphi) \rangle}{(\varphi_b)^2 + 4H^2 (\varphi_b)^2} \ll 1, \] (41)

being \( V(\delta \varphi) = V(\varphi) - V(\varphi_b) \)
\[ V(\delta \varphi) = -\frac{1}{2} \left( \bar{g}^{\psi \psi} \left( \frac{\partial \varphi}{\partial \psi} \right)^2 \bigg|_{\psi = H^{-1}} - \bar{g}^{\psi \psi} \left( \frac{\partial (\varphi_b)}{\partial \psi} \right)^2 \bigg|_{\psi = H^{-1}} \right), \]

with

\[ V(\varphi_b) = -\frac{1}{2} \bar{g}^{\psi \psi} \left( \frac{\partial \varphi_b}{\partial \psi} \right)^2 \bigg|_{\psi = H^{-1}} = 2H^2 (\varphi_b)^2. \quad (42) \]

The approximation (41) is valid during inflation on super Hubble scales (on the infrared sector), on which the inflaton field fluctuations are very “smooth”. Finally, we can compute the amplitude for the 4D gauge-invariant metric fluctuations for a de Sitter expansion on the infrared sector \((k_R \ll e^{Ht})\)

\[ \langle \Phi^2 \rangle = \frac{e^{-3Ht}}{(2\pi)^3} \int_0^{e^{Ht}} d^3k_R \xi_{k_R} \xi^{*}_{k_R}, \quad (43) \]

where \(\epsilon \approx 10^{-3}\) is a dimensionless constant. The squared \(\Phi\)-fluctuations has a power-spectrum \(P(k_R)\)

\[ P(k_R) \sim k_R^{-\sqrt{9+16k^2_{\psi_0}/H^2}}, \quad (44) \]

which is nearly scale invariant for \(k^2_{\psi_0} \psi_0^2 = k^2_{\psi_b}/H^2 \ll 1\). In other words, the 3D power-spectrum of the gauge-invariant metric fluctuations depends on the wavenumber \(k_{\psi_0}\) related to the fifth coordinate on the hypersurface \(\psi = \psi_0 \equiv H^{-1}\).

It is well known from experimental data [14] that

\[ n_s = 0.97 \pm 0.03, \quad (45) \]

where \(n_s = 4 - \sqrt{9+16k^2_{\psi_0}/H^2}\) is the energy perturbation spectral index. From the experimental condition (45), we obtain

\[ 0 \leq k_{\psi_0} < 0.15 \, H, \quad (46) \]

which is the main result of this paper.

**IV. FINAL COMMENTS**

In this paper we have studied 4D gauge-invariant metric fluctuations from a NKK theory of gravity. In particular we have examined these fluctuations in an effective 4D de Sitter expansion for the universe using a first-order expansion for the metric tensor. A very important result of this formalism is the confirmation of the well known 4D result \(\delta \rho/\rho \simeq 2\Phi\) [7], during inflation. Furthermore, the spectrum of the energy fluctuations depends on the fifth coordinate. More exactly, the result (46) can be written as \((k_{\psi_0} \psi_0)^2 < (0.15)^2\), being \((k_{\psi_0} \psi_0)^2\) the degenerated eigenvalue of the equation (21):

\[ \psi^2 \frac{\partial^2 \Phi}{\partial \psi^2} \bigg|_{\psi = \psi_0} = k^2_{\psi} \psi^2 \Phi \bigg|_{\psi = \psi_0}, \]

with \(t = \psi_0 N, R = \psi_0 r\) on the hypersurface \(\psi = \psi_0 = 1/H\). Of course, this formalism could be extended to other inflationary and cosmological models where the expansion of the universe is governed by a single scalar field.
REFERENCES

[1] A. H. Guth, Phys. Rev. D23, 347 (1981).
[2] for a review about inflation, the reader can see for example: A.D.Linde, Particle Physics and Inflationary Cosmology (Harwood, Chur, Switzerland, 1990) and references therein.
[3] see, for example: E. M. Aguilar, M. Bellini, Eur. Phys. J. C38, 123 (2004).
[4] T. S. Bunch and P. C. W. Davies, Proc. Roy. Soc. A360, 117 (1978).
[5] A.A.Starobinsky, in Current Topics in Field Theory, Quantum Gravity, and Strings, ed. by H.J. de Vega and N.Sánchez, Lecture Notes in Physics 226 (Springer, New York, 1986).
[6] S.Habib and M.Mijic, UBC report 1991 (unpublished); S.Habib, Phys. Rev. D46, 2408 (1992); M. Bellini, H. Casini, R. Montemayor, P. Sisterna. Phys. Rev. D54, 7172 (1996).
[7] V. F. Mukhanov, H. A. Feldman, R. Brandenberger, Phys. Rept. 215, 203 (1992); P. H. Lyth, A. Riotto, Phys. Rept. 314, 1 (1999); A. Riotto. E-print: hep-ph/0210162.
[8] N. Akdani-Hamed, S. Dimopoulos, G. Dvali, Phys. Lett. B429, 263 (1998); I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, G. Dvali, Phys. Lett. B436, 257 (1998); L. Randall, R. Sundrum, Phys. Rev. Lett. 83, 4690 (1999); D. Youm, Phys. Rev. D62, 084002 (2000); P. S. Wesson, B. Mashhoon, H. Liu, W. N. Sajko, Phys. Lett. B456, 34 (2001); J. Ponce de Leon, Phys. Lett. B523, 311 (2001).
[9] P. S. Wesson, Mod. Phys. Lett. A7, 921 (1992); J. M. Overduin, P. W. Wesson, Phys. Rept. 283, 303 (1997).
[10] the reader can see, for example: P. S. Wesson, Space-Time-Matter, World Scientific, Singapore, 1999.
[11] Mauricio Bellini. Physics Letters B609, 187 (2005).
[12] J. Ponce de Leon, Gen.Rel. Grav. 20, 539 (1988).
[13] the reader can see, for example: M. Mashhoon, H. Liu, P. Wesson, Phys. Lett. B331, 305 (1994).
[14] Review of Particle Physics: Phys. Lett. B592, 207 (2004).