Local Digital Algorithms for Estimating the Integrated Mean Curvature of $r$-Regular Sets

Anne Marie Svane

Received: 16 August 2012 / Revised: 7 January 2015 / Accepted: 11 May 2015 / Published online: 5 June 2015
© Springer Science+Business Media New York 2015

Abstract  Suppose an $r$-regular set is sampled on a random lattice. A fast algorithm for estimating the integrated mean curvature is to use a weighted sum of $2 \times \cdots \times 2$ configuration counts. We show that for a randomly translated lattice, no asymptotically unbiased estimator of this type exists in dimensions larger than two, while for stationary isotropic lattices, asymptotically unbiased estimators are plenty. The basis for this is a formula for the asymptotic behavior of hit-or-miss transforms of $r$-regular sets.

Keywords  Binary image · Design-based set-up · Configurations · Mean curvature · $r$-Regular sets · Hit-or-miss transform

Mathematics Subject Classification  94A08 · 28A75 · 60D05

1 Introduction

Suppose we are given a digital image of some geometric object. In many practical situations within science, one is mainly interested in certain geometrical characteristics of the underlying object, namely the so-called intrinsic volumes $V_i$. These include the volume $V_d$, surface area $2V_{d-1}$, integrated mean curvature $2\pi (d - 1)^{-1}V_{d-2}$, and Euler characteristic $V_0$. Therefore, a time-consuming reconstruction of the object is not of interest. Instead, we consider algorithms for estimating the intrinsic volumes based only on local information.
We model a digital image of a compact set \( X \subseteq \mathbb{R}^d \) as a binary image, i.e., as the set \( X \cap L \) where \( L \subseteq \mathbb{R}^d \) is some lattice. The vertices of each \( 2 \times \cdots \times 2 \) cell in the lattice may belong to either \( X \) or \( \mathbb{R}^d \setminus X \), yielding \( 2^{2d} \) possible configurations. We then estimate \( V_i \) as a weighted sum of the number of occurrences of each configuration. The advantage of such local algorithms is that they are efficiently implemented based on a linear filtering of the image, see [5] for more information on computational aspects.

We shall test these algorithms in the design-based setting where we sample a fixed compact set with a randomly translated lattice. Ideally, the estimator should be unbiased, at least asymptotically when the resolution goes to infinity.

Local estimators for \( V_{d-1} \) have already been widely studied. In [4], Kiderlen and Rataj proved a formula for the asymptotic behavior of such an estimator. This was later applied by Ziegel and Kiderlen in [12] to show that no asymptotically unbiased estimator for the surface area of the type described above can exist in dimension \( d = 3 \).

In this paper, we focus on the estimation of \( V_{d-2} \). For \( d = 2 \), \( V_{d-2} \) is the Euler characteristic.

Pavlidis showed in [6] that in 2D the approximation of \( X \) by the union of lattice squares with midpoint in \( X \cap L \) has the same Euler characteristic as \( X \) when \( X \) is a set with sufficiently ‘smooth’ boundary, more precisely an \( r \)-regular set, see Definition 2.1, and the lattice is sufficiently fine compared to \( r \). The Euler characteristic of the approximation can be computed by a local algorithm, which shows the existence of an asymptotically unbiased local algorithm for \( V_{d-2} \) when \( d = 2 \) and \( X \) is \( r \)-regular.

In contrast to this, Kampf has shown in [3] that there is no estimator for \( V_{d-2} \) in any dimension \( d \geq 2 \) which is asymptotically unbiased for all finite unions of polytopes. We shall consider the estimation of \( V_{d-2} \) in dimensions \( d > 2 \) when \( X \) is \( r \)-regular.

We first prove an extension to second order of Kiderlen and Rataj’s asymptotic result [4, Thm. 1]. From this, we obtain an asymptotic formula for the mean of a local estimator for \( V_{d-2} \) when \( d = 2 \) and \( X \) is \( r \)-regular.

Theorem 1.1 In dimension \( d > 2 \), no weighted sum of \( 2 \times \cdots \times 2 \) configuration counts with homogeneous weights defines an asymptotically unbiased estimator for \( V_{d-2} \) on the class of \( r \)-regular sets.

This is proved as Theorem 7.3. In fact, we show that no algorithm can be asymptotically unbiased on all sets of the form \( P \oplus B(r) \), where \( B(r) \) is the ball of radius \( r \) and \( P = \bigoplus_{i=1}^k [0, u_i] \) for orthonormal vectors \( u_1, \ldots, u_k \in \mathbb{R}^d \), \( [0, u_i] \) denoting the line segment and \( \oplus \) the Minkowski sum, see [7].

We formally define local algorithms in Sect. 2, and in Sect. 3 we introduce the design-based setting and review some known results. In Sect. 4, we state the main theoretical results on hit-or-miss transforms of \( r \)-regular sets. Their proofs are postponed to Sect. 8. As a corollary, we obtain asymptotic formulas for the mean estimator for \( V_{d-2} \) in Sect. 5 and apply these to find all asymptotically unbiased estimators in 3D under the extra assumption that \( L \) is isotropic. In Sect. 6, we rederive Pavlidis’ algorithm and show that it is unique in 2D. Theorem 1.1 is proved in Sect. 7.
2 Local Estimators for Intrinsic Volumes

Let \( C \) denote the unit square \([0, 1]^d\) in \( \mathbb{R}^d \) and let \( Z \) be the set of vertices in \( C \). The standard basis in \( \mathbb{R}^d \) will be denoted by \( e_1, \ldots, e_d \). We enumerate the elements of \( Z \) as follows: for \( x \in Z \) we write \( x = x_i \), where

\[
i = \sum_{k=1}^{d} 2^{k-1} \mathbb{1}_{\{\langle x, e_k \rangle = 1\}}.
\]

Here \( \mathbb{1}_{\{\langle x, e_k \rangle = 1\}} \) is the indicator function.

A \( 2 \times \cdots \times 2 \) configuration is a way of dividing \( Z \) into two disjoint sets \( B \) and \( W \). The elements of \( B \) are referred to as the ‘foreground’ or ‘black’ pixels, while the elements of the complement \( W = Z \setminus B \) are referred to as the ‘background’ or ‘white’ pixels.

There are \( 2^{2d} \) possible configurations. We denote these by \( (B_l, W_l) \) for \( l = 0, \ldots, 2^{2d} - 1 \) where the configuration \((B, W)\) is assigned the index

\[
l = \sum_{i=0}^{2^d - 1} 2^i \mathbb{1}_{\{x_i \in B\}}.
\]

Let \( \mathbb{Z}^d \) be the standard lattice in \( \mathbb{R}^d \). More generally, we shall consider orthogonal lattices \( L(c, \rho) = \rho(\mathbb{Z}^d + c) \), where \( c \in C \) is a translation vector and \( \rho \in \text{SO}(d) \) is a rotation. We let \( C(c, \rho), Z(c, \rho), \) and \( B_l(c, \rho) \) for the corresponding transformations of \( C, Z, \) and \( B_l \), respectively. We leave \( c \) and \( \rho \) out of the notation whenever it is clear from the context. We can change resolution by multiplying \( L \) by a factor \( a > 0 \). The results of Sect. 4 also apply to non-orthogonal lattices, but we leave this to the reader.

Let \( X \subseteq \mathbb{R}^d \) be the compact set we observe. We shall make the following smoothness assumption:

**Definition 2.1** A closed subset \( X \subseteq \mathbb{R}^d \) is called \( r \)-regular for \( r > 0 \) if for all \( x \in \partial X \) there exist two balls \( B^i(r) \) and \( B^a(r) \) of radius \( r \) both containing \( x \) such that \( B^i(r) \subseteq X \) and \( \text{int}(B^a(r)) \subseteq \mathbb{R}^d \setminus X \).

If \( X \) is \( r \)-regular, then for every \( x \in \partial X \) there is a unique outward pointing normal vector \( n(x) \). Moreover \( X = (X \ominus B(\varepsilon)) \oplus B(\varepsilon) \) whenever \( \varepsilon < r \), where \( \ominus \) is Minkowski subtraction given by \( A \ominus B = \{x \in \mathbb{R}^d \mid \forall b \in B : x - b \in A \} \). Noting that \( X \ominus B(\varepsilon) \) is a set of positive reach strictly greater than \( \varepsilon \), it follows from [1, 5.4] that \( \partial X \) is a \( C^1 \) manifold and \( n \) is Lipschitz and hence \( \mathcal{H}^{d-1} \)-almost everywhere differentiable.

Based on the digital image \( X \cap a L \), we want to estimate the intrinsic volumes \( V_i(X) \). Since we only consider the cases \( i = d, d - 1, d - 2, 0 \) in this paper, the definition of \( V_i \) given in the introduction will suffice. For more details when \( i = d - 2 \), see Sect. 4. A general definition for all \( i = 0, \ldots, d \) can be found in [1].
Let $\Phi_i(X; \cdot)$ denote the $i$th curvature measure, normalized as in [7]. Then

$$V_i(X) = \Phi_i(X; \mathbb{R}^d) = \sum_{z \in \mathbb{L}(0,\rho)} \Phi_i(X; aC_z(c, \rho)),$$

where $C_z(c, \rho) = z + \rho([0, 1]^d + c)$. Our approach is to estimate each term in the sum based on the only information available about $X \cap aC_z(c, \rho)$, namely the set $X \cap a(z + Z(c, \rho))$ where $z + Z(c, \rho), z \in \mathbb{L}(0, \rho)$, is a $2 \times \cdots \times 2$ block of lattice points in $\mathbb{L}$. If this equals $a(z + B_l(c, \rho))$, we estimate $\Phi_i(X; aC_z(c, \rho))$ by some $w_i^{(i)}(a) \in \mathbb{R}$, referred to as the weight, leading to an estimator of the form

$$\hat{V}_i(X \cap a\mathbb{L}) = \sum_{l=0}^{2^d-1} w_i^{(i)}(a) N_l(X \cap a\mathbb{L}),$$

(1)

where $N_l$ is the number of occurrences of the configuration $(B_l, W_l)$:

$$N_l(X \cap a\mathbb{L}(c, \rho)) = \sum_{z \in a\mathbb{L}(0,\rho)} 1_{\{z+aB_l(c,\rho) \subseteq X, z+aW_l(c,\rho) \subseteq \mathbb{R}^d \setminus X\}}.$$

As $V_i$ is homogeneous of degree $i$, i.e., $V_i(aX) = a^i V_i(X)$, we will require

$$\hat{V}_i(aX \cap a\mathbb{L}) = a^i \hat{V}_i(X \cap \mathbb{L}).$$

This is ensured by choosing $w_i^{(i)}(a) = a^i w_i^{(i)}$, where $w_i^{(i)} \in \mathbb{R}$ are constants.

Let $\mathcal{M}$ be the set of rigid motions and reflections preserving $aZ$. If $| \cdot |$ denotes cardinality,

$$\hat{V}'_i(X \cap a\mathbb{L}) = \frac{1}{|\mathcal{M}|} \sum_{m \in \mathcal{M}} \hat{V}_i(m(X \cap a\mathbb{L}))$$

is another estimator of the form (1) and the bias of $\hat{V}'_i(X \cap a\mathbb{L})$ is the average of the bias of $\hat{V}_i$ on the sets $m(X)$, since $V_i(X)$ is motion and reflection invariant. Hence the worst possible bias of $\hat{V}'_i$ on the sets $m(X)$ is smaller than that of $\hat{V}_i$. Thus, in the search for unbiased estimators, it is enough to consider estimators with weights satisfying $w_i^{(i)} = w_i^{(i)}$ whenever $B_{l_1} = m(B_{l_2})$ for some $m \in \mathcal{M}$.

Letting $\eta_j^{(i)}, j \in J$, denote the equivalence classes of the sets $B_l$ under the action of $\mathcal{M}$, we end up with an estimator of the form

$$\hat{V}_i(X \cap a\mathbb{L}) = a^i \sum_{j \in J} w_j^{(i)} \tilde{N}_j(X \cap a\mathbb{L}),$$

(2)

where $w_j^{(i)} \in \mathbb{R}$ and $\tilde{N}_j = \sum_{l : B_l \in \eta_j^{(i)}} N_l$.

One could of course also consider estimators based on $n \times \cdots \times n$ configurations. The formulas in Sect. 4 apply to this situation as well, but we treat only the case $n = 2$ in this paper.
3 The Design-Based Setting

In the design-based setting, we observe a fixed set $X \subseteq \mathbb{R}^d$ on a random lattice. If the lattice is of the form $\mathbb{L} = \rho(\mathbb{Z}^d + c)$ where $c \in C$ and $\rho \in \text{SO}(d)$ are both uniform random and mutually independent, we shall speak of a stationary isotropic lattice. If $\mathbb{L} = \rho(\mathbb{Z}^d + c)$ where the translation vector $c \in C$ is uniform random while $\rho \in \text{SO}(d)$ is fixed, we refer to it as a stationary non-isotropic lattice. In both cases, the estimator (2) is random with mean

$$
\mathbb{E} \hat{V}_i(X \cap a\mathbb{L}) = a^i \sum_{j \in J} w_j^{(i)} \mathbb{E} N_j(X \cap a\mathbb{L}).
$$

(3)

Ideally, this would equal $V_i(X)$. However, this is generally not true when $a$ is fixed. Instead, we consider the asymptotic behavior of $\mathbb{E} \hat{V}_i(X \cap a\mathbb{L})$ when $a \to 0$. By (3) it is enough to consider $a^i \mathbb{E} N_i(X \cap a\mathbb{L})$ when $a \to 0$.

Since $N_0$ is infinite, $w_0^{(i)}$ must equal zero in order for $\hat{V}_i$ to be well defined. All other $\mathbb{E} N_i$ are of order $O(a^{1-d})$, see (4) below, except $l = 2^d - 1$. In fact,

$$
\lim_{a \to 0} a^{d-l} \mathbb{E} \hat{V}_i(X \cap a\mathbb{L}) = \frac{w_0^{(i)}}{2^{2d-1}} V_d(X)
$$

for all $r$-regular sets $X$. Thus, for $i < d$, we must require $w_0^{(i)} = 0$; otherwise $\lim_{a \to 0} \mathbb{E} \hat{V}_i(X \cap a\mathbb{L})$ does not exist.

It was shown by Kiderlen and Rataj [4, Thm. 5] that for an $r$-regular set $X$ and $\mathbb{L}$ stationary non-isotropic,

$$
\lim_{a \to 0} a^{d-l} \mathbb{E} N_i(X \cap a\mathbb{L}) = \int_{\partial X} (-h(B_l \oplus \tilde{W}_i, n(x)))^+ d\mathcal{H}^d(X)
$$

(4)

when $l \neq 0$, $2^d - 1$. Here, if $S \subseteq \mathbb{R}^d$, then $h(S, u) = \text{sup}\{\langle s, u \rangle \mid s \in S\}$, $u \in S^{d-1}$, denotes the support function and $\tilde{S} = \{-s \mid s \in S\}$. Moreover, $x^+ = \text{max}\{0, x\}$ and $\mathcal{H}^k$ is the $k$-dimensional Hausdorff measure. In fact, the result is shown for $X$ belonging to the more general class of gentle sets, see [4] for the definition.

This result was later used by Ziegel and Kiderlen in [12] to prove that there is no asymptotically unbiased local estimator for $V_{d-1}$ in dimension $d = 3$.

Actually, Kiderlen and Rataj proved a much more general theorem, namely [4, Thm. 1], which we state here in a special case for later comparison. The set

$$(X \oplus a\tilde{B})(X \oplus a\tilde{W}) = \{z \in \mathbb{R}^d \mid z + aB \subseteq X, z + aW \subseteq \mathbb{R}^d \times X\}$$

is called the hit-or-miss transform of $X$.

If $\text{exo}(\partial X)$ denotes the set of points in $\mathbb{R}^d$ that do not have a unique closest point in $\partial X$, then $\xi_{\partial X}$ is the function $\xi_{\partial X} : \mathbb{R}^d \setminus \text{exo}(\partial X) \to \partial X$ that takes a point $z$ to the point in $\partial X$ closest to $z$. Finally, we write

$$
\delta_{B, W}(n) = 1_{\{h(B \oplus W, n) < 0\}}.
$$

Springer
Theorem 3.1 [4] Let $X \subseteq \mathbb{R}^d$ be a closed gentle set, $A \subseteq \mathbb{R}^d$ a bounded Borel set, and $B, W \subseteq \mathbb{R}^d$ two non-empty finite sets. Then

$$
\lim_{a \to 0} a^{-1} \mathcal{H}^d (\xi^{-1}_a (A) \cap (X \ominus a\tilde{B}) \setminus (X \ominus a\tilde{W})) = \int_{\partial X \cap A} (-h(B \oplus \tilde{W}, n(x))) + \mathcal{H}^d-1 (dx) = \int_{\partial X \cap A} ((-h(B, n(x))) - h(\tilde{W}, n(x))) \delta_{(B, W)}(n(x)) \mathcal{H}^d-1 (dx).
$$

(5)

In the third line, the integral has just been rewritten in a form similar to what we shall obtain later. Equation (4) follows from Theorem 3.1 and the observation, see [4], that for $i = 0, \ldots, d$

$$
a^i \mathbb{E}N_i (X \cap \mathbb{L}) = a^i \int_{\mathbb{C}} \sum_{z \in aL(c, \rho)} 1_{\{z+aB_i(\mathbb{L}(0, \rho)) \subseteq X, z+aW_i(\mathbb{L}(0, \rho)) \subseteq \mathbb{R}^d \setminus X\}} dc = a^{i-d} \mathcal{H}^d ((X \ominus a\tilde{B}_i) \setminus (X \ominus a\tilde{W}_i)).
$$

(6)

4 Main Results on Hit-or-Miss Transforms of $r$-Regular Sets

In this section, we state the main result of this paper, which is an extension of the asymptotic expansion (5) up to second order. Choosing $(B, W) = (B_i, W_i)$, Eq. (6) shows that this has implications for the asymptotics of $a^{d-2} \mathbb{E}N_i$ and thus for $\mathbb{E}V_{d-2}$.

We assume throughout that $X$ is $r$-regular. This is stronger than the gentleness in Theorem 3.1, but seems necessary in the proofs of the second order results.

Since the normal vector field $n$ of an $r$-regular set is $\mathcal{H}^d-1$-almost everywhere differentiable, the principal curvatures $k_i$ are defined almost everywhere as the eigenvalues of the differential $dn$ with principal directions $b_i$, generalizing the definition for $C^2$ manifolds. Note for later that all $k_i$ are bounded by $r^{-1}$.

Intrinsic volumes of $r$-regular sets can be defined via the principal curvatures, see [1]. In particular, $2\pi (d-1)^{-1} V_{d-2}$ is the integrated mean curvature

$$
V_{d-2}(X) = \frac{1}{2\pi} \int_{\partial X} (k_1(x) + \cdots + k_{d-1}(x)) \mathcal{H}^d-1 (dx).
$$

The principal curvatures also define a second fundamental form $II_x$ on the tangent space $T_x \partial X$ for $\mathcal{H}^d-1$-almost all $x \in \partial X$. For $\sum_{i=1}^{d-1} \alpha_i b_i \in T_x \partial X$, $II_x$ is given by

$$
II_x \left( \sum_{i=1}^{d-1} \alpha_i b_i \right) = \sum_{i=1}^{d-1} k_i(x) \alpha_i^2
$$

whenever $dn_x$ is defined. Note that $\text{Tr}(II) = k_1 + \cdots + k_{d-1}$.

When $X$ is $r$-regular, the orthogonal complement $N_x$ of $T_x \partial X$ is the line spanned by $n(x)$. Thus, we may define $Q$ to be the quadratic form given on $(\alpha, tn(x)) \in T_x \partial X \oplus N_x = \mathbb{R}^d$ by
\[ Q_x(\alpha, tn(x)) = -H_x(\alpha) + \text{Tr}(H_x) \alpha^2, \]

whenever \( H_x \) is defined.

Given a compact set \( S \subseteq \mathbb{R}^d \) and \( u \in S^{d-1} \), let

\[ S(u) = \{ s \in S \mid h(S, u) = \langle s, u \rangle \} \]

denote the support set. If \( x \in \partial X \) and \( H_x \) is defined, we define

\[ H^+_x(S) = \max \{ H_x(s) \mid s \in S(n(x)) \}, \]
\[ H^-_x(S) = \min \{ H_x(s) \mid s \in S(-n(x)) \}. \]

Here \( H_x(s) \) means \( H_x(\pi_x(s)) \) where \( \pi_x : \mathbb{R}^d \to T_x \partial X \) is the projection.

Choose \( s^+, s^- : \partial X \to S \) in such a way that \( s^\pm(x) \in S(\pm n(x)) \) and \( H^\pm_x(S) = H_x(s^\pm(x)) \) whenever \( H_x \) is defined. Then we may define

\[ Q^\pm_x(S) = Q_x(s^\pm(x)) = -H^\pm_x(S) + \text{Tr}(H_x)h(S, \pm n(x))^2, \]

when \( H_x \) is defined. The last equality holds since \( \langle s^\pm(x), n(x) \rangle = \pm h(S, \pm n(x)) \) and it shows that the definition is independent of the choice of \( s^\pm \).

We are now ready to state the second order version of Theorem 3.1:

**Theorem 4.1** Let \( X \subseteq \mathbb{R}^d \) be an \( r \)-regular set, \( A \subseteq \mathbb{R}^d \) a bounded Borel set, and \( B, W \subseteq \mathbb{R}^d \) two disjoint non-empty finite sets. Then

\[
\mathcal{H}^d(\xi_{\partial X}^{-1}(A) \cap (X \ominus a \tilde{B}) \setminus (X \oplus a \tilde{W})) \\
= a \int_{\partial X \cap A} (-h(B \oplus \tilde{W}, n(x)))^+ \mathcal{H}^{d-1}(dx) \\
\quad + \frac{a^2}{2} \left( \int_{\partial X \cap A} (Q^+_x(B) - Q^-_x(W)) \delta_{(B, W)}(n(x)) \mathcal{H}^{d-1}(dx) \right) \\
\quad + \int_{\partial X \cap A} (H^-_x(W) - H^+_x(B))^+ 1_{\{h(B \oplus \tilde{W}, n(x)) = 0\}} \mathcal{H}^{d-1}(dx) + o(a^2). \tag{7}
\]

Note how (7) resembles (5) but with a quadratic term instead of a linear one. The term (8) vanishes in particular if the \((d - 1)\)-th area measure \( S_{d-1}(X, \cdot) = \mathcal{H}^{d-1} \circ n^{-1} \) on \( S^{d-1} \), see [7] for convex \( X \), vanishes on all the great circles \( \{ u \in S^{d-1} \mid \langle b - w, u \rangle = 0 \} \) for \( b \in B, w \in W \). In particular, it vanishes for almost all rotations of \( X \).

We next state a version of Theorem 4.1 where the sets \( B, W \) are rotated by a uniform random element of \( \text{SO}(d) \) with respect to the Haar measure \( \nu_d \). First some notation: For a finite set \( S \), let

\[ D(S) = \bigcup_{s_1 \neq s_2 \in S} \{ u \in S^{d-1} \mid \langle s_1, u \rangle = \langle s_2, u \rangle \}. \]
This has measure zero in $S^{d-1}$. Whenever $u \notin D(S)$, the two sets $S(\pm u)$ contain exactly one point each. This defines $p^+_S, p^-_S : S^{d-1} \setminus D(S) \to S$, which we extend by zero to $p^+_S : S^{d-1} \to S \cup \{0\}$. These satisfy

$$p^+_S(n(x)) = s^+(x)1_{[n(x) \notin D(S)]},$$

$$p^+_S(u) = c_\rho p^+_S(u),$$

where $\rho \in \text{SO}(d)$ and $c_\rho p^+_S$ denotes the conjugation $c_\rho p^+_S(u) = \rho(p^+_S(\rho^{-1}(u)))$.

**Theorem 4.2** Let $X \subseteq \mathbb{R}^d$ be an r-regular set, $A \subseteq \mathbb{R}^d$ a bounded Borel set, and $B, W \subseteq \mathbb{R}^d$ two disjoint non-empty finite sets. Then

$$\int_{\text{SO}(d)} \mathcal{H}^d(\tilde{\xi}^{-1}_{dX}(A) \cap (X \ominus a\rho(\tilde{B})) \setminus (X \ominus a\rho(\tilde{W})))\nu_d(d\rho)$$

$$= a\Phi_d-1(X ; A)\frac{2}{\omega_d} \int_{S^{d-1}} \left(-h(B \oplus \tilde{W}, u)^+\mathcal{H}^{d-1}(du)\right)$$

$$+ a^2\Phi_d-2(X ; A)\frac{\pi}{\omega_d(d-1)} \int_{S^{d-1}} \left(d(h(B, u)^2 - h(\tilde{W}, u)^2)\right)$$

$$- (|p_B^+(u)|^2 - |p_W^-(u)|^2)\delta_{(B, W)}(u)\mathcal{H}^{d-1}(du) + o(a^2),$$

where $\omega_d = \mathcal{H}^{d-1}(S^{d-1})$. If $X$ is a smooth manifold, the convergence is $O(a)$.

Note how the resemblance to (5) is even better in the isotropic case. Taking $A = \mathbb{R}^d$, the second order term becomes $V_{d-2}(X)$ times a constant depending only on $B$ and $W$. We show in Corollary 5.2 that there exists a configuration for which this constant is non-zero.

The proofs of Theorems 4.1 and 4.2 are given in Sect. 8.

### 5 Application to Local Algorithms

In this section, we apply Theorems 4.1 and 4.2 to determine the asymptotic bias of local algorithms. In the isotropic case, we find that a local algorithm $\tilde{V}_{d-2}$ is asymptotically unbiased if and only if the weights satisfy two linear equations. This was already noted in 2D in [8] where the equations were also determined explicitly. Below, we give the explicit equations in 3D as well.

Introduce the following notation:

$$\tilde{\varphi}_j(X) = \sum_{B_l \in \eta^d_j} \int_{\partial X} (-h(B_l \oplus \tilde{W}_l, n(x)))^+\mathcal{H}^{d-1}(dx),$$

$$\psi_l = \frac{2}{\omega_d} \int_{S^{d-1}} (-h(B_l \oplus \tilde{W}_l, u))^+\mathcal{H}^{d-1}(du),$$

$$\tilde{\psi}_j = \sum_{l : B_l \in \eta^d_j} \psi_l.$$
\begin{align*}
\lambda_l(X) &= \frac{1}{2} \int_{\partial X} (Q^+_x(B_l) - Q^-_x(W_l)) \delta_{(B_l, W_l)}(n(x)) \mathcal{H}^{d-1}(dx) \\
&\quad - \frac{1}{2} \int_{\partial X} (II^+_x(B_l) - II^-_x(W_l)) + \mathbb{1}_{\{h(B_l \oplus W_l, n(x)) = 0\}} \mathcal{H}^{d-1}(dx), \\
\mu_l &= \frac{\pi}{\omega_d (d-1)} \int_{S^{d-1}} \left( d(h(B_l, u)^2 - h(\tilde{W}_l, u)^2) \\
&\quad - (|p^+_{B_l}(u)|^2 - |p^-_{W_l}(u)|^2) \delta_{(B_l, W_l)}(u) ight) d(B_l, W_l) du,
\end{align*}

\bar{\mu}_j = \sum_{l: B_l \in \eta^d_j} \mu_l.

Combining (6) with Theorems 4.1 and 4.2, we see that:

\begin{align*}
\lim_{a \to 0} (a^{d-2} \mathbb{E} N_l(X \cap a \mathbb{L}) - a^{-1} \lim_{a \to 0} a^{d-1} \mathbb{E} N_l(X \cap a \mathbb{L})) &= \lambda_l(X), \\
\lim_{a \to 0} (a^{d-2} \mathbb{E} N_l(X \cap a \mathbb{L}) - a^{-1} \lim_{a \to 0} a^{d-1} \mathbb{E} N_l(X \cap a \mathbb{L})) &= \mu_l V_{d-2}(X)
\end{align*}

if \( \mathbb{L} \) is stationary non-isotropic or stationary isotropic, respectively.

**Corollary 5.1** Suppose \( \hat{V}_{d-2} \) is a local estimator of the form (2). Then

\begin{align*}
\lim_{a \to 0} a \mathbb{E} \hat{V}_{d-2}(X \cap a \mathbb{L}) &= \sum_{j \in J} w^{(d-2)}_j \hat{\psi}_j(X), \\
\lim_{a \to 0} a \mathbb{E} \hat{V}_{d-2}(X \cap a \mathbb{L}) &= V_{d-1}(X) \sum_{j \in J} w^{(d-2)}_j \hat{\psi}_j
\end{align*}

in the non-isotropic and isotropic case, respectively. In particular, in both cases \( \lim_{a \to 0} \mathbb{E} \hat{V}_{d-2}(X \cap a \mathbb{L}) \) exists if and only if \( \lim_{a \to 0} a \mathbb{E} \hat{V}_{d-2}(X \cap a \mathbb{L}) = 0 \). In this case,

\begin{align*}
\lim_{a \to 0} \mathbb{E} \hat{V}_{d-2}(X \cap a \mathbb{L}) &= \sum_{j \in J} w^{(d-2)}_j \sum_{l: B_l \in \eta^d_j} \lambda_l(X), \\
\lim_{a \to 0} \mathbb{E} \hat{V}_{d-2}(X \cap a \mathbb{L}) &= V_{d-2}(X) \sum_{j \in J} w^{(d-2)}_j \bar{\mu}_j,
\end{align*}

respectively.

In the isotropic case, \( \hat{V}_{d-2} \) is asymptotically unbiased for all \( r \)-regular sets if and only if the weights satisfy the linear equations

\begin{equation*}
\sum_{j \in J} w^{(d-2)}_j \bar{\psi}_j = 0, \quad \sum_{j \in J} w^{(d-2)}_j \bar{\mu}_j = 1. \tag{9}
\end{equation*}

If \( X \) is smooth, the convergence rate is \( O(a) \).
Observe that not all \( \mu_l \) are zero, e.g., \( \mu_1 > 0 \) for all \( d \). If \( \eta_1^d \) and \( \eta_2^d \) denote the configuration classes of \( B_1 \) and \( B_2 \), respectively, this shows:

**Corollary 5.2** In the isotropic case, asymptotically unbiased estimators for \( V_{d-2} \) exist for all \( d \geq 2 \). For instance, one can take all weights equal to zero except

\[
\begin{align*}
\omega_1^{(d-2)} &= -\omega_{2d-1}^{(d-2)} = (2\bar{\mu}_1)^{-1}.
\end{align*}
\]

The coefficients \( \bar{\psi}_j \) and \( \bar{\mu}_j \) can in principle be computed directly for each configuration. However, the actual computations are tedious. In dimension \( d = 2 \) they were performed in [8]. We consider the case \( d = 3 \).

Note that \( \delta(B_l, W_l) \) vanishes if \( W_l \) and \( B_l \) cannot be strongly separated by a hyperplane, so we may ignore such configurations. Recall that we also ignore the configurations with \( l = 0, 255 \). In 3D, the remaining configurations fall into one of the eight equivalence classes pictured below:

![Configuration Classes](image)

The constants \( \bar{\psi}_j \) and \( \bar{\mu}_j \) are shown in Table 1 for these configurations. The computation details are omitted.

We conclude this section by noting some symmetries in the isotropic case. It is clear that

\[
\mu_l = -\mu_{2d-1-l}.
\]

| \( \bar{\psi}_j \) | \( \bar{\mu}_j \) |
|-----------------|-----------------|
| \( \bar{\psi}_1 = \bar{\psi}_7 \) | \( \bar{\mu}_1 = -\bar{\mu}_7 \) |
| \( \bar{\psi}_2 = \bar{\psi}_6 \) | \( \bar{\mu}_2 = -\bar{\mu}_6 \) |
| \( \bar{\psi}_3 = \bar{\psi}_5 \) | \( \bar{\mu}_3 = -\bar{\mu}_5 \) |
| \( \bar{\psi}_{4,1} \) | \( \bar{\mu}_{4,1} = 0 \) |
| \( \bar{\psi}_{4,2} \) | \( \bar{\mu}_{4,2} = 0 \) |

Table 1 The constants \( \bar{\psi}_j \) and \( \bar{\mu}_j \) in dimension \( d = 3 \) with \( \zeta = 3\sqrt{2 \arctan(\sqrt{2})} \).
Thus, we may as well choose $w_l^{(d-2)} = -w_{2^{d-1} - l}^{(d-2)}$. Since $\psi_l = \psi_{2^{d-1} - l}$, this also ensures that (9) is satisfied. Finally it ensures that interchanging foreground and background changes the sign of $\hat{V}_{d-2}$, which is desirable since $V_{d-2}$ has this property.

The last proposition of this section reduces the formula for $\tilde{\mu}_j$ in a way that resembles (5) and the formula for $\hat{\psi}_j$ even more.

**Proposition 5.3** If $B_l^1$ and $B_l^2$ belong to the same configuration class, then $\mu_l^1 = \mu_l^2$. Moreover,

$$
\tilde{\mu}_j = \frac{d\pi}{\omega_d(d-1)} \sum_{B_l \in \eta_j^d} \int_{S^{d-1}} (h(B_l, u)^2 - h(\tilde{W}_l, u)^2) \delta(B_l, W_l)(u) H^{d-1}(du).
$$

**Proof** Choose a rotation $\rho$ taking $C$ to $\tilde{C}$. For each configuration $B_l \in \eta_j^d$, also $B_l' = \rho(B_l) + (1, 1, 1) \in \eta_j^d$. The claim follows from the symmetries

$$
|p_{B_l}^+(u)|^2 = |p_{B_l'}(\rho(u))|^2,
$$
$$
|p_{W_l}^-(u)|^2 = |p_{W_l'}(\rho(u))|^2,
$$

and $\delta(B_l, W_l)(u) = \delta(B_l', W_l')(\rho(u))$. \qed

## 6 Unbiased Estimators for the Euler Characteristic in 2D

In this section, we assume that $\mathbb{L}$ is a stationary non-isotropic lattice in dimension $d = 2$. Then $V_{d-2}$ is simply the Euler characteristic. In this case, there exists a unique asymptotically unbiased estimator of the form (2). The existence goes back to Pavlidis [6] and the uniqueness may be derived from the results of [4]. We will show how this also follows from Corollary 5.1.

Let $\tilde{V}_{d-2}$ be a local estimator of the form (2). Again we ignore the configurations with $l = 0, 15$. Moreover, $\delta(B_l, W_l)$ vanishes for $l = 6, 9$. The remaining configurations fall into one of the following three equivalence classes:

$$
\eta_1^2 \quad \eta_2^2 \quad \eta_3^2
$$

In 2D, Theorem 4.1 simplifies to yield:

**Corollary 6.1** Let $X \subseteq \mathbb{R}^2$ be compact $r$-regular and $\mathbb{L}$ be stationary non-isotropic. Then for $l \neq 0, 15$,

$$
\lim_{a \to 0} (\mathbb{E} N_l(X \cap a\mathbb{L}) - a^{-1} \lim_{a \to 0} a \mathbb{E} N_l(X \cap a\mathbb{L})) = (2\pi)^{-1} \tilde{\mu}_l V_0(X).
$$
The set \( A = \{ u \in S^1 \mid h(B_l \oplus \tilde{W}_l, u) = 0 \} \) is finite. If \( n(x) \in A \) and \( n \) is differentiable at \( x \), then either \( dn_x = 0 \), in which case \( II_x = 0 \), or \( dn_x \neq 0 \) and thus there must be a neighborhood of \( x \) where \( n \notin A \). Thus (8) vanishes in 2D. Using the simple form of \( II \) in 2D and \( \Phi_0(X; \cdot) \circ n^{-1} = (2\pi)^{-1}V_0(X)H^1 \) as measures on \( S^1 \), the claim follows.

Writing \( w_j = w_j^{(0)} \), we first show the following convergence criterion:

**Proposition 6.2** For \( d = 2 \), \( \lim_{a \to 0} E \hat{V}_0(X \cap aL) \) exists for all \( r \)-regular \( X \) if and only if

\[
    w_2 = 0 \quad \text{and} \quad w_1 = -w_3. \tag{10}
\]

**Proof** By Corollary 5.1, \( \lim_{a \to 0} E \hat{V}_0(X \cap aL) \) exists if and only if

\[
    w_1 \bar{\phi}_1(X) + w_2 \bar{\phi}_2(X) + w_3 \bar{\phi}_3(X) = 0. \tag{11}
\]

Write \( n = (n_1, n_2) \in S^1 \subseteq \mathbb{R}^2 \). Then

\[
    \sum_{B_l \in n_j} (-h(B_l \oplus \tilde{W}_l, n))^+ = \begin{cases} 
    \min\{|n_1|, |n_2|\}, & j = 1, 3, \\
    \max\{|n_1|, |n_2|\} - \min\{|n_1|, |n_2|\}, & j = 2.
    \end{cases}
\]

Hence the left-hand side of Eq. (11) is

\[
    \int_{\partial X} ((w_1 + w_3 - w_2) \min\{|n_1(x)|, |n_2(x)|\} + w_2 \max\{|n_1(x)|, |n_2(x)|\}) H^1(dx).
\]

This vanishes for all \( X \) if \( w_1 + w_3 = w_2 = 0 \). On the other hand, this is also necessary, as one may realize by considering the sets \([0, (0, t)] \oplus B(r)\). \( \Box \)

**Theorem 6.3** If \( \hat{V}_0 \) satisfies (10), then \( \lim_{a \to 0} E \hat{V}_0(X \cap aL) = 4w_1V_0(X) \). Thus the estimator with weights

\[
    w_1 = -w_3 = \frac{1}{4} \quad \text{and} \quad w_2 = 0
\]

is the unique estimator for \( V_0 \) of the form (2) that is asymptotically unbiased in the non-isotropic setting for all \( r \)-regular sets.

**Proof** Under condition (10), \( \lim_{a \to 0} E \hat{V}_0(X \cap aL) \) is given by Corollary 6.1 if we can compute the coefficients \( \bar{\mu}_j \). This is done in [8, Sect. 8] and it yields

\[
    \lim_{a \to 0} E \hat{V}_0(X \cap aL) = 2(w_1 - w_3)V_0(X) = 4w_1V_0(X).
\]

\( \Box \)
7 Non-existence of Unbiased Estimators for \(V_{d-2}\) in Higher Dimensions

We now consider estimators of the form (2) for \(V_{d-2}\) in dimensions \(d \geq 3\) when \(L\) is stationary non-isotropic. Contrary to the \(d = 2\) case and the isotropic case, we shall see that no such estimators are asymptotically unbiased.

Let \(S_{d-k-1}(u_1, \ldots, u_k)\) denote the unit sphere in \(\text{span}(u_1, \ldots, u_k)\). We write \(w_j = w_j^{(d-2)}\) for simplicity.

The first lemma will allow us to ignore the term (8) in Theorem 4.1.

**Lemma 7.1** Let \((B_l, W_l)\) be a configuration, \(l \neq 0, 2^d - 1\). Let \(u_1, \ldots, u_k \in \mathbb{R}^d\) be orthonormal and \(Y = (\bigoplus_{l=1}^k [0, u_i]) \times S_{d-k-1}(u_1, \ldots, u_k)\). Then

\[
\int_Y (\Pi^+(B_l) - \Pi^-(W_l))^+ 1_{\{h(B_l \oplus W_l, n(x)) = 0\}} \mathcal{H}^{d-1}(dx) = 0.
\]

**Proof** If \(h(B_l \oplus W_l, n) = 0\), there are \(b \in B_l\) and \(w \in W_l\) with \(\Pi^+(B_l) = \Pi(b)\), \(\Pi^-(W_l) = \Pi(w)\), and \(b - w, n) = 0\). Clearly, \(v = b - w \neq 0\). For \(z \in \mathbb{R}^d\), write \(z = z_1 + z_2\), where \(z_1\) is the projection of \(z\) onto \(\text{span}(u_1, \ldots, u_k)\). Observe that \(n(x) = x_2\) for all \(x \in Y\). Thus the set \(\{x \in Y \mid \langle n, v \rangle = \langle x_2, v_2 \rangle = 0\}\) can only have positive \(\mathcal{H}^{d-1}\)-measure if \(v_2 = 0\), that is, if \(b_2 = w_2\). But then the claim follows since \(\Pi(b) = \Pi(b_2) = \Pi(w_2) = \Pi(w)\). \(\square\)

**Theorem 7.2** For \(d = 3\), there is no asymptotically unbiased estimator for \(V_1\) of the form (2) on the class of \(r\)-regular sets.

**Proof** Assume that \(\hat{V}_1\) is an estimator of the form (2) and that the weights are chosen so that \(\lim_{n \to 0} \mathbb{E} \hat{V}_1(X \cap aL) = V_1(X)\) for all \(r\)-regular sets \(X\). Clearly, we may assume that \(L\) is a random translation of \(\mathbb{Z}^3\).

In particular, this holds for \(X = B(r)\). Since \(B(r)\) is rotation invariant, a rotation of \(L\) does not change \(\mathbb{E} N_b(B(r) \cap aL)\). Thus \(\hat{\lambda}_1(X) = \mu_l V_{d-2}(B(r))\), so it follows from Table 1 that the weights must satisfy

\[
(3 - \sqrt{3})(w_1 - w_7) + (3\sqrt{3} - 3\sqrt{2})(w_2 - w_6) + (-3 + 6\sqrt{2} - 3\sqrt{3})(w_3 - w_5) = 1. \tag{12}
\]

Next consider three test sets of the form \(X_q = [0, v_q] \oplus B(r), q = 1, 2, 3\), where \(v_1 = (1, 0, 0), v_2 = \frac{1}{\sqrt{2}}(1, 1, 0)\), and \(v_3 = \frac{1}{\sqrt{3}}(1, 1, 1)\). Then

\[
V_1(X_q) = 1 + 4r = 1 + V_1(B(r)). \tag{13}
\]

Note that

\[
\partial X_q = (0 + rS^2 \cap H_{v_q}^- \cup (v_q + rS^2 \cap H_{v_q}^+) \cup ([0, v_q] \times rS^1(v_q)),
\]

where \(H_{v_q}^\pm\) denote the halfspaces \(\{z \in \mathbb{R}^3 \mid \pm \langle z, v_q \rangle \geq 0\}\). Thus by Lemma 7.1,

\[
\lambda_1(X_q) = \frac{1}{2} \int_{[0, v_q] \times rS^1(v_q)} (Q^+(B_l) - Q^-(W_l)) \delta_{(B_l, W_l)}(n(x)) \mathcal{H}^2(dx) + \lambda_1(B(r)).
\]

\(\odot\) Springer
Since $\hat{V}_1$ is asymptotically unbiased on both $B(r)$ and $X_q$, (13) shows that

$$h_q := \sum_{j \in J} w_j \sum_{B_l \in \eta_j} \frac{1}{2} \int_{[0,v_q] \times rS^1(v_q)} (Q_x^+(B_l) - Q_x^-(W_l))\delta_{(B_l,W_l)}(n(x))\mathcal{H}^2(dx)$$

must equal 1.

But $Q$ takes a very simple form on $[0,v_q] \times rS^1(v_q)$. Namely, for $t \in [0,1]$ and $u \in S^1(v_q)$,

$$Q_{tv_q + ru}(s) = \frac{1}{r} (\langle s, u \rangle^2 - \langle s, v_q \times u \rangle^2),$$

where $\times$ is the cross-product in $\mathbb{R}^3$. A straightforward computation now shows

$$h_1 = 2(w_2 - w_6),$$

$$h_2 = \sqrt{2}(w_1 - w_7) + \sqrt{2}(w_3 - w_5),$$

$$h_3 = \sqrt{3}(w_1 - w_7) + \sqrt{3}(w_2 - w_6) - \sqrt{3}(w_3 - w_5).$$

But no weights can satisfy the three equations $h_q = 1$ and (12) at the same time. \(\Box\)

**Theorem 7.3** There are no asymptotically unbiased estimators for $V_{d-2}$ of the form (2) in dimension $d \geq 3$.

**Proof** The idea is to generalize the proof for $d = 3$ by considering some example sets for which the computations reduce to the ones already performed in dimension 3. Again we assume that $L$ is a random translation of $\mathbb{Z}^d$.

Assume that an asymptotically unbiased estimator $\hat{V}_{d-2}$ is given. In particular, it is asymptotically unbiased on all sets of the form $X = P \oplus B(r)$, where $P = \bigoplus_{i=1}^k [0, u_i]$ for $k \leq d - 2$ orthonormal vectors $u_1, \ldots, u_k \in S^{d-1}$. We show that this leads to a contradiction.

Introduce the notation

$$Y(u_1, \ldots, u_k) = \left( \bigoplus_{i=1}^k [0, u_i] \right) \times rS^{d-k-1}(u_1, \ldots, u_k),$$

$$G_j(x) = \frac{1}{2} \sum_{B_l \in \eta_j} (Q_x^+(B_l) - Q_x^-(W_l))\delta_{(B_l,W_l)}(n(x)).$$

We first show by induction in $k$ that the weights must satisfy

$$\sum_{j \in J} w_j \int_{Y(u_1,\ldots,u_k)} G_j(x)\mathcal{H}^{d-1}(dx) = \frac{k_{d-k}}{\pi} \binom{d-k}{2} r^{d-k-2},$$

(14)

where $k_{d-k}$ is the volume of the unit ball in $\mathbb{R}^{d-k}$. This is obviously true for $k = 0$ since $\hat{V}_{d-2}$ is unbiased for $X = B(r)$. Consider $X = P \oplus B(r)$, where $P = \bigoplus_{i=1}^k [0, u_i]$.

\(\Box\) Springer
The relative open \( m \)-faces of \( P \) are the sets

\[
y + \bigoplus_{s=1}^{m} (0, u_{is})
\]

for

\[
y \in A(i_1, \ldots, i_m) = \left\{ \sum_{s \neq i_1, \ldots, i_m} \varepsilon_s u_s \mid \varepsilon_s \in \{0, 1\} \right\}.
\]

Let \( N(y, i_1, \ldots, i_m) \) be the normal cone of this face. Then \( \partial X \) can be divided into disjoint subsets of the form

\[
y + \left( \bigoplus_{s=1}^{m} (0, u_{is}) \right) \times (N(y, i_1, \ldots, i_m) \cap rS^{d-1})
\]

for \( y \in A(i_1, \ldots, i_m) \). Note that

\[
\bigcup_{y \in A(i_1, \ldots, i_m)} N(y, i_1, \ldots, i_m) \cap rS^{d-1} = rS^{d-m-1}(u_{i_1}, \ldots, u_{i_m})
\]

and for \( y_1 \neq y_2 \), \( N(y_1, i_1, \ldots, i_m) \cap N(y_2, i_1, \ldots, i_m) \cap rS^{d-1} \) has \( \mathcal{H}^{d-m-1} \)-measure zero in \( rS^{d-m-1}(u_{i_1}, \ldots, u_{i_m}) \). Thus,

\[
\sum_{y \in A(i_1, \ldots, i_m)} \int_{y + \left( \bigoplus_{s=1}^{m} (0, u_{is}) \right) \times (N(y, i_1, \ldots, i_m) \cap rS^{d-1})} G_j(x) \mathcal{H}^{d-1}(dx) = \int_{Y(u_1, \ldots, u_m)} G_j(x) \mathcal{H}^{d-1}(dx).
\]

But \( \lim_{a \to 0} \mathbb{E} \hat{V}_{d-2}(P \oplus B(r) \cap a \mathbb{L}) \) equals

\[
\sum_{j \in J} w_j \sum_{m=0}^{k} \sum_{1 \leq i_1 < \cdots < i_m \leq k, y \in A(i_1, \ldots, i_m)} \int_{y + \left( \bigoplus_{s=1}^{m} (0, u_{is}) \right) \times (N(y, i_1, \ldots, i_m) \cap rS^{d-1})} G_j(x) \mathcal{H}^{d-1}(dx)
\]

by Lemma 7.1, while on the other hand, the Steiner formula yields

\[
V_{d-2} (P \oplus B(r)) = \frac{1}{\pi} \sum_{m=0}^{d-2} \binom{d-m}{2} r^{d-m-2} \kappa_{d-m} \sum_{1 \leq i_1 < \cdots < i_m \leq k} 1,
\]

see [7, Eq. 4.23]. Since \( \hat{V}_{d-2} \) is asymptotically unbiased on \( X \), (15) and (16) must be equal, so (14) follows by induction.
In particular, (14) shows that
\[
\sum_{j \in J} w_j \int_{Y(q, e_4, \ldots, e_d)} G_j(x) \mathcal{H}^{d-1}(dx) = 1,
\] (17)
where \( v_q \in \text{span}(e_1, e_2, e_3) \) are defined as in Theorem 7.2 for \( q = 1, 2, 3 \).

If \( B_l \subseteq \text{span}(e_1, e_2, e_3) \cong \mathbb{R}^3 \) is a configuration in \( \mathbb{R}^3 \), we let \( B'_l \subseteq \mathbb{R}^d \) denote the configuration \( p^{-1}(B_l) \) where \( p : Z \rightarrow \text{span}(e_1, e_2, e_3) \) is the projection.

If \( B_{l_1} \) and \( B_{l_2} \) differ only by a rigid motion, so do \( B'_{l_1} \) and \( B'_{l_2} \). If the configuration classes \( \eta^d_j \) in \( \mathbb{R}^3 \) are indexed by \( j \in I \), we let \( \eta^d_j, j \in I \), denote the set of configurations \( p^{-1}(\eta^d_j) \).

For all \( x \in Y(v_q, e_4, \ldots, e_d) \), \( n(x) \in \text{span}(e_1, e_2, e_3) \) and therefore
\[
\delta((B_l, W_l))(n(x)) = \delta((p(B_l), p(W_l)))(n(x)).
\]
Thus only configurations of type \( \eta^d_j \) with \( j \in I \) can occur. Moreover, since all principal curvatures vanish in the directions \( v_q, e_4, \ldots, e_d \), (17) equals
\[
\sum_{j \in I} w_j \sum_{B_l \in \eta^d_j} \frac{1}{2} \int_{[0, v_q] \times S^1(v_q)} (Q^+(p(B_l)) - Q^-(p(W_l)))
\times \delta((p(B_l), p(W_l)))(n(x)) \mathcal{H}^2(dx) = h_q,
\]
where \( h_q \) is as in the proof of Theorem 7.2. Thus by (17) the weights must satisfy the equations \( h_q = 1 \).

Applying (14) to the \( k = d - 3 \) vectors \( e_4, \ldots, e_d \) shows that the weights must also satisfy (12). But then the \( w_j \) have to satisfy the same set of equations as in the proof of Theorem 7.2, which is impossible. \( \square \)

### 8 Proofs of the Main Theorems

This section contains the proofs of Theorems 4.1 and 4.2. The first lemma gives a convenient parametrization of \( \partial X \). For this, let
\[
T^r \partial X = \{ (x, \alpha) \in T \partial X \mid \alpha \in T_x \partial X, |\alpha| < r \}
\]
be the open \( r \)-disk bundle in the tangent bundle \( T \partial X \).

**Lemma 8.1** There is a function \( q : T^r \partial X \rightarrow \mathbb{R} \) such that \( x + \alpha + q(x, \alpha)n(x) \) is the point in \( \partial X \) closest to \( x + \alpha \) along the line parallel to \( n(x) \). Moreover, \( a^{-2} q(x, a\alpha) \) is uniformly bounded for \( x \in \partial X, \alpha \in T_x^R \partial X, \) and \( a \in [0, \frac{r}{R}] \) and
\[
\lim_{a \rightarrow 0} \frac{q(x, a\alpha)}{a^2} = -\frac{1}{2} I_x(\alpha)
\] (18)
whenever the right-hand side is defined.
Proof. Let \( x \in \partial X \) and let \( B^i = x - rn(x) + B(r) \) and \( B^0 = x + rn(x) + B(r) \) denote the inner and outer ball, respectively, as in Definition 2.1.

For \( \alpha \in T_X^x \partial X \), the line segment \( L_\alpha = [x+\alpha-rn, x+\alpha+rn] \) contains a boundary point \( y_\alpha = x+\alpha + q(x, \alpha)n \), as it hits both \( B_i \) and \( \text{int}(B_o) \). This point must be unique; otherwise choose \( \alpha_0 \) with \( |\alpha_0| \) minimal such that \( L_{\alpha_0} \) contains two different points \( p_1 \) and \( p_2 \). One of them, say \( p_1 \), must have a small neighborhood not containing any \( y_\alpha \) with \( |\alpha| < |\alpha_0| \) and thus the normal vector \( n(p_1) \) must be exactly \( -\frac{\alpha_0}{|\alpha_0|^2} \). But then the outer ball at \( p_1 \) must contain \( x \), which is a contradiction. Thus \( q \) is well defined. Moreover, \( a^{-2}|q(x, a\alpha)| \) is bounded by \( a^{-2}(r - \sqrt{r^2 - |a\alpha|^2}) \).

To show (18), let \( n \) be differentiable at \( x \). Then \( \gamma(a) = x + a\alpha + q(x, a\alpha)n(x) \) is a \( C^1 \) curve in \( \partial X \) with \( \gamma(0) = x \) and \( \gamma'(0) = \alpha \). Moreover \( q(x, a\alpha) = (n(x), \gamma(a) - x) \).

The claim now follows from l’Hôpital’s rule because

\[
\lim_{a \to 0} \frac{\langle n(x), \gamma'(a) \rangle}{a} = \lim_{a \to 0} \frac{\langle n(\gamma(0)) - n(\gamma(a)), \gamma'(a) \rangle}{a} = -dn_x(\alpha) = -II_x(\alpha). 
\]

\( \square \)

As in [4], the idea of proof is to use the generalized Weyl tube formula [2, Thm. 2.1]. For this, let

\[ f(z, a) = 1_{X \cap a\vec{B} \setminus (X \cup a\vec{W})}^{1}_{\xi_{\partial X}^{d-1}}(A). \]

Then \( f \) has support in \( \partial X \oplus B(r) \) whenever \( aR(B \cup W) \leq r \), where \( R(S) = \inf\{R > 0 \mid S \subseteq B(R) \} \) for any compact set \( S \). Therefore, [2, Thm. 2.1] expresses \( H^d(A \cap (X \cup a\vec{B}) \setminus (X \cup a\vec{W})) \) as

\[
\int_{\mathbb{R}^d} f(z, a)dz = \sum_{m=0}^{d-1} \int_{\partial X} \int_{-r}^r t^m f(x + tn(x), a)s_m(k(x))dtH^{d-1}(dx), \tag{19}
\]

where \( s_m(k) \) is the \( m \)-th symmetric polynomial in the principal curvatures \( k = (k_1, \ldots, k_{d-1}) \). In particular, \( s_1(k) = \text{Tr}(\mathcal{H}) \).

To study (19), we need to understand how \( f \) behaves along each normal line of the boundary. For \( x \in \partial X \) with \( n := n(x) \) and \( s \in \mathbb{R}^d \) with \( |s| \leq r \),

\[ x + as + tn \in X \quad \text{if and only if} \quad t \leq -a(s, n) + q(x, as - \langle as, n \rangle n) \]

for all \( t \in [-r, r] \). Thus,

\[ t(as) = -a(s, n) + q(x, as - a(s, n)n) \]

is the time \( x + as \) passes from \( X \) to \( \mathbb{R}^d \setminus X \) when moving in the normal direction. For a finite set \( S \), let

\[
t-(aS) = \max\{t(as) \mid s \in S\}, \\
t+(aS) = \min\{t(as) \mid s \in S\}.
\]
Although \( t_{\pm}(aS) \) depend on \( x \in \partial X \), we suppress this in the notation. With this notation, we obtain for \( aR(B \cup W) < r \):

\[
\int_{-r}^{r} t^{m} f(x + tn, a) dt = \frac{1}{m + 1} \left( t_{+}(aB)^{m+1} - t_{-}(aW)^{m+1} \right) \tau_{(B,W)}(x, a) \mathbb{1}_{\{x \in A\}},
\]

where

\[
\tau_{(B,W)}(x, a) = \mathbb{1}_{\{t_{+}(aB) > t_{-}(aW)\}}.
\]

The indicator function \( \tau_{(B,W)}(x, a) \) may not equal \( \delta_{(B,W)}(n(x)) \) everywhere, but the following lemma ensures that they do not differ too much.

**Lemma 8.2** Let \( B \) and \( W \) be two disjoint finite non-empty sets. There are constants \( M_{1} \) and \( \varepsilon_{1} \) depending only on \( R(B \cup W) \) such that

\[
|h(B \oplus \hat{W}, n(x))||\tau_{(B,W)}(x, a) - \delta_{(B,W)}(n(x))| \leq M_{1} a
\]

whenever \( a < \varepsilon_{1} \).

There are constants \( M_{2} \) and \( \varepsilon_{2} \) depending only on \( B \) and \( W \) such that

\[
v_{d}(\rho \in \text{SO}(d) \mid \tau_{(\rho(B),\rho(W))}(x, a) \neq \delta_{(\rho(B),\rho(W))}(n)) \leq M_{2} a
\]

whenever \( a < \varepsilon_{2} \).

**Proof** When \( \tau_{(B,W)} \neq \delta_{(B,W)} \), either \( t_{-}(aW) \geq t_{+}(aB) \) and \( h(B \oplus \hat{W}, n) < 0 \) or \( t_{-}(aW) < t_{+}(aB) \) and \( h(B \oplus \hat{W}, n) \geq 0 \).

In the first case, \( t_{-}(aW) \geq t_{+}(aB) \) implies that

\[
0 \leq t_{-}(aW) - t_{+}(aB) = -a\langle w, n \rangle + a\langle b, n \rangle + q(x, a\alpha_{1}) - q(x, a\alpha_{2})
\]

for some \( w \in W \) and \( b \in B \) and \( \alpha_{1}, \alpha_{2} \in T_{x}^{\alpha} \partial X \). Thus \( h(B \oplus \hat{W}, n) < 0 \) yields

\[
0 \leq -ah(B \oplus \hat{W}, n) \leq a\langle w, n \rangle - a\langle b, n \rangle \leq q(x, a\alpha_{1}) - q(x, a\alpha_{2}) \leq M_{1} a^{2},
\]

where \( M_{1} \) is independent of \( \alpha_{1}, \alpha_{2}, \) and \( x \) by Lemma 8.1.

In the second case, let \( b \in B(n) \) and \( w \in W(-n) \). The claim follows from

\[
0 \geq t_{-}(aW) - t_{+}(aB) \geq t(aw) - t(ab) = h(B \oplus \hat{W}, n) + q(x, a\alpha_{1}) - q(x, a\alpha_{2}).
\]

For the second statement, we see that

\[
v_{d}(\rho \in \text{SO}(d) \mid \tau_{(\rho(B),\rho(W))}(x, a) \neq \delta_{(\rho(B),\rho(W))}(n))
\]

\[
\leq v_{d}(\rho \in \text{SO}(d) \mid |h(\rho(B \oplus \hat{W}), n)| \in [0, M_{1} a])
\]

\[
\leq v_{d}(\rho \in \text{SO}(d) \mid \exists b \in B, w \in W : |\langle \rho(b - w), n \rangle| \leq M_{1} a)
\]

\[
\leq \omega_{d}^{-1}|B||W|H^{d-1}(u \in S^{d-1} \mid \langle u, n \rangle \leq Ma),
\]

which is bounded by \( M_{2} a \).
To find the limit of (20), we also need to know the asymptotic behavior of \(t_\pm(S)\). The following lemma is a fairly easy consequence of the definition \(t(as) = -a(s, n) + q(x, a\alpha)\) for some \(\alpha \in T^R(S)\partial X\) and the fact that \(q(x, a\alpha)\) is uniformly \(O(a^2)\) by Lemma 8.1. The details are left to the reader.

**Lemma 8.3** Let \(S\) be a finite set. For each \(x\), there is an \(\varepsilon > 0\) such that for all \(a < \varepsilon\), there are \(s_\pm \in S(\pm n(x))\) with

\[
\begin{align*}
t_+(as) &= t(as_+) = -ah(S, n) + q(x, a\alpha_1), \\
t_-(as) &= t(as_-) = ah(\tilde{S}, n) + q(x, a\alpha_2),
\end{align*}
\]

where \(|\alpha_1|, |\alpha_2| \leq R(S)\). Moreover, there are constants \(M\) and \(\varepsilon\) depending only on \(R(S)\) such that for all \(a < \varepsilon\)

\[
|t_+(as) + ah(S, n)|, |t_-(as) - ah(\tilde{S}, n)| \leq a^2M.
\]

With these lemmas, we are ready to prove Theorem 4.1. In the proofs, we will often leave the integration variable out of the integrals to simplify notation.

**Proof** (Theorem 4.1) We shall determine the second-order asymptotic behavior of the sum (19) when \(a \to 0\) by using (20).

First consider the terms with \(m \geq 1\). By Lemma 8.1,

\[
a^{-2}t(as)^m + 1 = a^{-2}(-a(s, n) + q(x, a\alpha))^{m+1}
\]

is bounded by some uniform constant for all \(s \in B \cup W\). When \(m > 1\), they converge to zero pointwise. Hence by dominated convergence,

\[
\lim_{a \to 0} a^{-2} \int_{\partial X \cap A} (t_+(as)^m - t_-(aw)^m) \tau(B, W)s_m(k)dtd\mathcal{H}^{d-1} = 0.
\]

For \(m = 1\), dominated convergence yields

\[
\lim_{a \to 0} a^{-2} \int_{\partial X \cap A} (t_+(as)^2 - t_-(aw)^2) \tau(B, W)s_1(k)d\mathcal{H}^{1} = \int_{\partial X \cap A} (h(B, n)^2 - h(\tilde{W}, n)^2)\delta(B, W)(n)s_1(k)d\mathcal{H}^{1}.
\]

This follows from the first part of Lemmas 8.2 and 8.3 since

\[
|h(B, n)^2 - h(\tilde{W}, n)^2||\tau(B, W)(x, a) - \delta(B, W)(n)| \leq 2R(B \cup W)M_1a. \tag{22}
\]

To handle the \(m = 0\) term, note first that if \(x \in \partial X\) with \(II_x\) defined,

\[
\lim_{a \to 0} a^{-2}(t_+(as) + ah(B, n)) = \lim_{a \to 0} a^{-2}\min\{t(ab) + a(b, n) \mid b \in B(n(x))\}
\]

\[
= -\frac{1}{2}II_x^+(B).
\]
This follows from the first part of Lemmas 8.1 and 8.3. Similarly, one may show
\[ \lim_{a \to 0} a^{-2}(t_+(aW) - ah(\tilde{W}, n)) = -\frac{1}{2}I_{\mathcal{X}}^-(W). \]

Consider
\begin{align*}
  a^{-2} & \int_{\partial X \cap A} ((t_+(aB) - t_-(aW))\tau_{(B,W)} + ah(B \oplus \tilde{W}, n)\delta_{(B,W)}(n))d\mathcal{H}^{d-1} \\
  &= \int_{\partial X \cap A} a^{-2}(t_+(aB) - t_-(aW)) + ah(B \oplus \tilde{W}, n)\delta_{(B,W)}(n)d\mathcal{H}^{d-1} \quad (23) \\
  &+ \int_{\partial X \cap A} a^{-2}(t_+(aB) - t_-(aW))(\delta_{(B,W)}(n) - \tau_{(B,W)}(n))d\mathcal{H}^{d-1}. \quad (24)
\end{align*}

By Lemma 8.3, the integrand in (23) is bounded, so dominated convergence yields the limit
\[ \frac{1}{2} \int_{\partial X \cap A} (I_{\mathcal{X}}^+(W) - I_{\mathcal{X}}^-(B))\delta_{(B,W)}(n)d\mathcal{H}^{d-1}. \quad (25) \]

The integrand in (24) is bounded according to Lemma 8.2, so dominated convergence applies again. Write
\[ \tau_{(B,W)} = \tau_{(B,W)}1_{\{h(B \oplus \tilde{W}, n) > 0\}} + \tau_{(B,W)}1_{\{h(B \oplus \tilde{W}, n) = 0\}} + \tau_{(B,W)}\delta_{(B,W)}(n). \]

The first term converges to zero and the last term converges to \( \delta_{(B,W)}(n) \). When \( h(B \oplus \tilde{W}, n) = 0 \),
\[ (t_+(aB) - t_-(aW))\tau_{(B,W)} = (t_+(aB) + ah(B, n) - (t_-(aW) - ah(\tilde{W}, n)))^+, \]
so (24) converges to
\[ -\frac{1}{2} \int_{\partial X \cap A} (I_{\mathcal{X}}^+(B) - I_{\mathcal{X}}^-(W))^+1_{\{h(B \oplus \tilde{W}, n) = 0\}}d\mathcal{H}^{d-1}. \quad (26) \]

The claim now follows by combining (21), (25), and (26). \( \square \)

The proof of Theorem 4.2 is similar:

Proof (Theorem 4.2) Tonelli’s theorem yields
\begin{align*}
  \mathcal{H}^{d-1}(\partial X \cap A)\omega_d^{-1} \int_{S^{d-1}} (-h(B \oplus \tilde{W}, u))^+ du \\
  = \int_{SO(d)} \int_{\partial X \cap A} (-h(\rho(B \oplus \tilde{W}), n))^+ d\mathcal{H}^{d-1} v_d(d\rho).
\end{align*}
Thus, we must compute the limit of
\[
\alpha^{-2} \int_{SO(d)} \int_{\partial X \cap A} \left( \sum_{m=0}^{d-1} \int_{t_m(\rho(W))}^{t_{m+1}(\rho(B))} t^m \tau_{\rho(B), \rho(W)} s_m(k) \, dt \right)
- a(-h(\rho(B \oplus \tilde{W}), n))^+ \right) \, d\mathcal{H}^{d-1} \, v_d(d\rho)
\]
when \(a \to 0\). Proving that this equals
\[
\frac{1}{2} \int_{\partial X \cap A} \int_{SO(d)} (Q^+(\rho(B)) - Q^-(\rho(\tilde{W}))) \delta_{(B, W)}(\rho^{-1}n) \, v_d(d\rho) \, d\mathcal{H}^{d-1} \quad (27)
\]
is done exactly as in the proof of Theorem 4.1, except that one now has to check that the limit also commutes with the integration over \(SO(d)\). This is true because the constants bounding the integrands are also uniform, depending only on \(R(B, W) = R(\rho(B), \rho(W))\). This yields the limit (27) plus the term
\[
- \frac{1}{2} \int_{\partial X \cap A} \int_{SO(d)} (\Pi^+(\rho(B)) - \Pi^-(\rho(W)))^+ 1_{\{h(\rho(B \oplus \tilde{W}), n) = 0\}} \, v_d(d\rho) \, d\mathcal{H}^{d-1}.
\]
But this vanishes since
\[
\{(x, \rho) \in \partial X \times SO(d) \mid h(\rho(B \oplus \tilde{W}), n) = 0\}
\]
has measure zero.

To simplify (27), let \(S \subseteq \mathbb{R}^d\) be finite and \(x \in \partial X\) fixed. Then
\[
\int_{SO(d)} Q^+_x(\rho(S)) \delta_{(B, W)}(\rho^{-1}n) \, v_d(d\rho)
= \int_{SO(d)} \int_{SO(d-1)} Q^+_x(\sigma \rho(S)) \delta_{(B, W)}((\sigma \rho)^{-1}n) \, v_{d-1}(d\sigma) \, v_d(d\rho)
= \int_{SO(d)} \int_{SO(d-1)} Q_x(c_{\sigma \rho} p^+_S(n)) \delta_{(B, W)}(\rho^{-1}n) \, v_{d-1}(d\sigma) \, v_d(d\rho), \quad (28)
\]
where \(SO(d-1)\) is the subgroup keeping \(n\) fixed. Note that \(c_{\sigma \rho} p^+_S = \sigma c_{\rho} p^+_S\). Hence
\[
\int_{SO(d-1)} Q_x(c_{\sigma \rho} p^+_S(n)) \, v_{d-1}(d\sigma)
= \int_{SO(d-1)} (-\Pi_x(c_{\sigma \rho} p^+_S(n))) \, v_{d-1}(d\sigma) + \text{Tr}(\Pi_x) |c_{\rho} p^+_S(n), n|^2
= \frac{1}{d-1} \text{Tr}(\Pi_x)(|c_{\sigma \rho} p^+_S(n), n|^2 - |c_{\rho} p^+_S(n)|^2) + \text{Tr}(\Pi_x) |c_{\rho} p^+_S(n), n|^2
= \frac{1}{d-1} \text{Tr}(\Pi_x)(d\langle p^+_S(\rho^{-1}n), \rho^{-1}n \rangle^2 - |p^+_S(\rho^{-1}n)|^2).
Thus (28) equals
\[
\frac{1}{ω_d(d - 1)} \int_{S^{d-1}} \text{Tr}(H_x)(dh(S, u)^2 - |p^+_{S}(u)|^2)\delta(B,W)(u)H^{d-1}(du).
\]

Applying this to (27) proves the first part of the theorem.

To prove the last claim, it is enough to show boundedness of
\[
a^{-3} \int_{SO(d)} \int_{\partial X \cap A} \left( \sum_{m=0}^{d-1} \frac{s_m(k)}{m+1} (t_+(aρ(B))^m+1 - t_-(aρ(W))^m+1)τρ(B),ρ(W)) - \frac{a^2}{2} (II^+(ρ(B)) - II^-(ρ(W))) \right.
\]
\[
\left. + \frac{a^2}{2} (h(ρ(B), n)^2 - h(ρ(W), n)^2)s_1(k)\right) δ(ρ(B),ρ(W))(n)dH^{d-1}ν_d(dρ).
\]

For \(m \geq 2\), \(a^{-3}t(as)^{m+1}\) is uniformly bounded whenever \(|s| \leq R(B \cup W)\).

For \(m = 1\), Lemma 8.3 shows that
\[
a^{-3}|t - (aρ(B))^2 + a^2h(ρ(B), n)^2|τρ(B),ρ(W))\]

is uniformly bounded. The \(W\)-terms are similar. Moreover, (22) shows boundedness of
\[
a^{-1}|h(ρ(B), n)^2 - h(ρ(W), n)^2| |δ(ρ(B),ρ(W)) - τρ(B),ρ(W))|.
\]

Let \(m = 0\). Since \(X\) is smooth, \(q : T'\partial X → \mathbb{R}\) is smooth. By compactness of \(X\),
\[
|q(x, aα) + \frac{1}{2}II_x(aα)| ≤ M_0|aα|^3
\]
for all \(x ∈ \partial X\) and \(a|α| < r\). It follows easily that
\[
a^{-3}|t_+(aρ(B)) + ah(ρ(B), n) + \frac{a^2}{2}II^+(ρ(B))|τρ(B),ρ(W))\]

is uniformly bounded. Finally, by Lemma 8.2
\[
(|h(ρ(B ⊕ \tilde{W}), n)| + a|II^+(ρ(B))|)|δ(ρ(B),ρ(W)) - τ(ρ(B),ρ(W))| ≤ M_1a
\]
uniformly for some \(M_1 > 0\) and it vanishes for all \(ρ\) outside a set of \(ν_d\)-measure \(M_2a\)
where \(M_2\) is uniform in \(x\). \(□\)

9 Discussion and Outlook

The results of this paper show that in dimension \(d ≥ 3\) there are no local algorithms based on \(2 × \cdots × 2\) configurations that are asymptotically unbiased for all \(r\)-regular sets. After the submission of the present paper, this was generalized to local algorithms for \(V_{d-1}\) and \(V_{d-2}\) based on larger \(n × \cdots × n\) pixel configurations in [10]. The proof
was based on Theorems 3.1 and 4.1. In this paper, the results by Kampf [3] were also generalized to show that there are no local algorithms for $V_k$, $k < d$, based on $n \times \cdots \times n$ configurations which are asymptotically unbiased for all convex polytopes.

All these results show that, with a few exceptions, local algorithms are not well suited for computing intrinsic volumes. Therefore, the research has now switched focus to other classes of algorithms. Local algorithms based on gray-scale images have shown much better asymptotic behavior, but still have not been tested in practice [11]. For global algorithms, the challenge is to ensure a small computation time. Such algorithms based on the Voronoi decomposition of $X \cap \mathbb{L}$ are currently being investigated and seem promising.

Acknowledgments The author was supported by the Centre for Stochastic Geometry and Advanced Bioimaging, funded by the Villum Foundation. The author is most thankful to Markus Kiderlen for helpful suggestions and proofreading.

References

1. Federer, H.: Curvature measures. Trans. Am. Math. Soc. 93, 418–491 (1959)
2. Hug, D., Last, G., Weil, W.: A local Steiner-type formula for general closed sets and applications. Math. Z. 246(1–2), 237–272 (2004)
3. Kampf, J.: A limitation of the estimation of intrinsic volumes via pixel configuration counts. Matematika 60(2), 485–511 (2014)
4. Kiderlen, M., Rataj, J.: On infinitesimal increase of volumes of morphological transforms. Mathematika 53(1), 103–127 (2006)
5. Ohser, J., Mücklich, F.: Statistical Analysis of Microstructures. Wiley, Chichester (2000)
6. Pavlidis, T.: Algorithms for Graphics and Image Processing. Computer Science Press, Rockville (1982)
7. Schneider, R.: Convex Bodies: The Brunn–Minkowski Theory, 2nd edn. Cambridge University Press, Cambridge, MA (2014)
8. Svane, A.M.: Local digital estimators of intrinsic volumes for Boolean models and in the design based setting. Adv. Appl. Probab. 46, 35–58 (2014)
9. Svane, A.M.: Local digital algorithms for estimating the integrated mean curvature of $r$-regular sets. arxiv:1309.3845v2 (2013)
10. Svane, A.M.: On multigrid convergence of local algorithms for intrinsic volumes. J. Math. Imaging Vis. 49(2), 352–376 (2014)
11. Svane, A.M.: Estimation of intrinsic volumes from digital grey-scale images. J. Math. Imaging Vis. 49(1), 148–172 (2014)
12. Ziegel, J., Kiderlen, M.: Estimation of surface area and surface area measure of three-dimensional sets from digitizations. Image Vis. Comput. 28, 64–77 (2010)