On the Bosonization of $L$-Operators for Quantum Affine Algebra $U_q(\hat{\mathfrak{sl}}_2)$

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Abstract

Some relations between different objects associated with quantum affine algebras are reviewed. It is shown that the Frenkel-Jing bosonization of a new realization of quantum affine algebra $U_q(\hat{\mathfrak{sl}}_2)$ as well as bosonization of $L$-operators for this algebra can be obtained from Zamolodchikov-Faddeev algebras defined by the quantum $R$-matrix satisfying unitarity and crossing-symmetry conditions.

1 Introduction

Recently there have been arised a great interest to the free field realization of the Zamolodchikov-Faddeev (ZF) algebras [1] appearing in the context of quantum integrable models [2, 3]. These realizations are important for calculations of the correlation function and form factors in these models. It was shown in [2, 4] that, if $R$-matrix defining the ZF algebra is intertwiner of finite-dimensional representations of the quantum affine algebra $U_q(\hat{\mathfrak{sl}}_2)$, then one can naturally realize the corresponding ZF algebras in terms of operators that intertwine the infinite-dimensional highest or lowest weight modules over $U_q(\hat{\mathfrak{sl}}_2)$.

The algebra $U_q(\hat{\mathfrak{sl}}_2)$ was introduced by Drinfeld and Jimbo [5, 6] using deformed relations between Chevalley generators. Lately in the framework of quantum inverse scattering method or in “the RLL formalism” this algebra was formulated in terms of $L$-operators [7]. Quite recently, it has been understood that these operators can be expressed via operators satisfying ZF algebras associated with the quantum $R$-matrix acting in the tensor product of the finite-dimensional representation spaces of $U_q(\hat{\mathfrak{sl}}_2)$ [8].

So called “new realization of quantum affine algebras” was introduced in [9] which appeared to be a quantum analog of the loop realization of affine algebras. I. Frenkel and...
J. Ding have found an isomorphism between the R-matrix formulation of $U_q(\hat{\mathfrak{sl}}_n)$ and the new realization for this algebra [11].

Fixing level of $U_q(\hat{\mathfrak{sl}}_2)$ to be unity, one we can easily obtain the free field realization of all these objects starting from Frenkel-Jing bosonization of level one $U_q(\hat{\mathfrak{sl}}_2)$ [11]. Using Hopf structure of the algebra $U_q(\hat{\mathfrak{sl}}_2)$ one can bosonize the corresponding ZF algebras following [3]. Then, using Miki’s formulas (9), the bosonization of $L$-operators can be immediately obtained.

The aim of this note is to show that it is possible to reconstruct the non-abelian symmetry algebra of the model starting from the corresponding ZF algebras. In this letter we consider only the case of ZF algebras related to the XXZ model in anti-ferroelectric regime on the infinite lattice. The symmetry algebra for this model is the quantum affine algebra $U_q(\hat{\mathfrak{sl}}_2)$ with $-1 < q < 0$. Let us formulate the main observation of this note.

A new realization of the quantum affine algebra $U_q(\hat{\mathfrak{sl}}_2)$ at level one follows from the free field realization of Zamolodchikov-Faddeev algebras associated with R-matrix which satisfies the conditions of unitarity and crossing-symmetry.

Let us note that a similar approach to the bosonization of massive integrable models in the field theory was developed by S. Lukyanov in [12]. For $SU_q(n)$ invariant spin chain on the finite lattice the problem of reconstruction of the non-abelian symmetry algebra from Yang-Baxter operators have been solved in [13].

The paper is organized as follows. In Sect. 2 we will give a short review of all objects that we mentioned in Introduction. Sect. 3 is devoted to the free field realization of ZF algebras. In conclusion we will list the problems, which the approach developed in this letter might be also applied to.

2 Algebra $U_q(\hat{\mathfrak{sl}}_2)$ in Different Realizations

We start with definition of $U_q(\hat{\mathfrak{sl}}_2)$ following [3]. Let $P$ and $P^*$ will be weight and dual weight lattices generated respectively by $\{\Lambda_0, \Lambda_1, \delta\}$ and $\{h_0, h_1, d\}$ with the canonical pairing

$$\langle \Lambda_i, h_j \rangle = \delta_{ij}, \quad \langle \Lambda_i, d \rangle = 0, \quad \langle \delta, h_i \rangle = 0, \quad \langle \delta, d \rangle = 1.$$ 

Let $\alpha_0 = 2\Lambda_0 - 2\Lambda_1 + \delta$ and $\alpha_1 = 2\Lambda_1 - 2\Lambda_0$ be the simple roots. Algebra $U_q(\hat{\mathfrak{sl}}_2)$ is generated by the symbols $t_i, t_i^{-1}, e_i, f_i, q^d$ $i = 0, 1$ subjected to the following relations

$$t_i t_j = t_j t_i, \quad t_i e_j t_i^{-1} = q^{\langle \alpha_j, h_i \rangle} e_j, \quad t_i f_j t_i^{-1} = q^{-\langle \alpha_j, h_i \rangle} f_j,$$

$$[e_i, f_j] = \delta_{ij} \frac{t_i - t_i^{-1}}{q - q^{-1}}, \quad q^d e_j q^{-d} = q^{\delta_{ij} d} e_j, \quad q^d f_j q^{-d} = q^{-\delta_{ij} d} f_j,$$

$$\sum_{k=0}^{3} (-1)^k \frac{[3]!}{[k]! [3-k]!} e_i^k e_j e_i^{3-k} = 0, \quad \sum_{k=0}^{3} (-1)^k \frac{[3]!}{[k]! [3-k]!} f_i^k f_j f_i^{3-k} = 0.$$ 

Throughout the paper we will use the standard notation $[n] = (q^n - q^{-n})/(q - q^{-1})$ and $[k] = [1][2] \cdots [k]$. 


Let $V = \mathbb{C} v_+ + \mathbb{C} v_-$ be a two-dimensional space and $V_z = V \otimes \mathbb{C}[z, z^{-1}]$ be its affinization. Define the action of the generators $t_i, t_i^{-1}, e_i, f_i$ on the space $V_z$ as follows

$$
\pi_z(e_i)v_\varepsilon z^n = \pi(e_i)v_\varepsilon z^{n+\delta_{i0}}, \quad \pi_z(f_i)v_\varepsilon z^n = \pi(f_i)v_\varepsilon z^{n-\delta_{i0}}, \quad \pi_z(t_i)v_\varepsilon z^n = \pi(t_i)v_\varepsilon z^n,
$$

where the action on the space $V$ is given by

$$
\pi_z(e_0) = \pi_z(f_0) = \pi_z(e_1) = \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right), \quad \pi_z(f_0) = \pi_z(e_1) = \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right), \quad \pi_z(t_0^{-1}) = \pi_z(t_1) = \left(\begin{array}{ccc} q & 0 & 0 \\ 0 & q^{-1} \end{array}\right)
$$

These formulas define the fundamental (vector) representation of algebra $U_q(\hat{sl}_2)$.

$U_q(\hat{sl}_2)$ is a Hopf algebra. Using the Hopf structure we can consider the tensor products of fundamental representations and calculate the intertwining operator ($R$-matrix) between these tensor products

$$
\overline{R}_{12}(z_1/z_2) : V_{z_1} \otimes V_{z_2} \to V_{z_2} \otimes V_{z_1}.
$$

With normalization $\overline{R}_{12}(z)v_+ \otimes v_+ = v_+ \otimes v_+$ $R$-matrix reads as follows

$$
\overline{R}_{12}(z) = \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & b(z) & c(z) & 0 \\ 0 & \bar{c}(z) & b(z) & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)
$$

where

$$
c(z) = \bar{c}(z)z = \frac{(1-q^2)z}{1-q^2z}, \quad b(z) = \frac{(1-z)q}{1-q^2z}.
$$

In what follows we will define ZF algebras with $R$-matrix satisfying additional conditions of unitarity

$$
R_{12}(z)R_{21}(z^{-1}) = 1
$$

and crossing-symmetry

$$
R_{12}(z^{-1}) = (\sigma^x \otimes 1)R_{12}(q^2z)(\sigma^x \otimes 1),
$$

where

$$
\sigma^x = \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right).
$$

These conditions along with the requirement that $R$-matrix is an analytic function for $q^2 \leq |z| \leq q^{-2}$ uniquely determine two normalization factors

$$
r^\pm(z) = \pm \frac{1}{\sqrt{z}} \frac{(q^4z^{-1})_\infty(q^2z)_\infty}{(q^4z^2z^{-1})_\infty}, \quad (a)_\infty = \prod_{n=0}^{\infty}(1-aq^{4n})
$$

in front of $\overline{R}_{12}(z)$, so we can define two ZF algebras

$$
\Phi(z_2)\Phi(z_1) = r^+(z)\overline{R}_{12}(z_1/z_2)\Phi(z_1)\Phi(z_2), \quad \Psi(z_1)\Psi(z_2) = r^-(z)\overline{R}_{12}(z_1/z_2)\Psi(z_2)\Psi(z_1).
$$
In components these relations are
\[
\Phi_{\varepsilon_2}(z_2)\Phi_{\varepsilon_1}(z_1) = r^+(z)\tilde{R}_{\varepsilon_1\varepsilon_2}^\dagger(z_1/z_2)\Phi_{\varepsilon_1}(z_1)\Phi_{\varepsilon_2}(z_2),
\]
\[
\Psi_{\varepsilon_1}(z_1)\Psi_{\varepsilon_2}(z_2) = r^-(z)\tilde{R}_{\varepsilon_1\varepsilon_2}(z_1/z_2)\Psi_{\varepsilon_2}(z_2)\Psi_{\varepsilon_1}(z_1),
\]
where indices \(\varepsilon_i\) take the values \(\pm\) and we assume the summation over repeated indices. We will see in the next section that the operators \(\Phi\) and \(\Psi\) will correspond to type I and type II operators in terms of the paper [3].

Let us formulate the algebra \(U_q(\mathfrak{sl}_2)\) in \(R\)-matrix approach following [7]. Let
\[
\tilde{R}(z) = \rho(z)R(z)
\]
be an universal \(R\)-matrix for the algebra \(U_q(\mathfrak{sl}_2)\) evaluated on the tensor product of two vector representations of \(U_q(\mathfrak{sl}_2)\). Obviously it should coincide with \(R\)-matrix [7] up to normalization factor. This normalization factor \(\rho(z)\) can be calculated following [4] and is equal to
\[
\rho(z) = \frac{1}{\sqrt{q}(z)\infty(q^2z)\infty}.
\]
Algebra \(U_q(\mathfrak{sl}_2)\) at level one is generated by the coefficients of the matrix series
\[
L^{(\pm)}(z) = \sum_{n=0}^{\infty} L^{(\pm)}_n z^n,
\]
satisfying the relations
\[
\tilde{R}_{12}(z_1/z_2)L^{(\pm)}_1(z_1)L^{(\pm)}_2(z_2) = L^{(\pm)}_2(z_2)L^{(\pm)}_1(z_1)\tilde{R}_{12}(z_1/z_2),
\]
\[
\tilde{R}_{12}(z_1q/z_2)L^{(\pm)}_1(z_1)L^{(-)}_2(z_2) = L^{(-)}_2(z_2)L^{(+)}_1(z_1)\tilde{R}_{12}(z_1q^{-1}/z_2).
\]
In components these relations read as follows
\[
\tilde{R}_{\varepsilon_1\varepsilon_2}^\dagger(z_1/z_2)L^{(\pm)}_{\varepsilon_1\nu_1}(z_1)L^{(\pm)}_{\varepsilon_2\nu_2}(z_2) = L^{(\pm)}_{\varepsilon_2\nu_2}(z_2)L^{(\pm)}_{\varepsilon_1\nu_1}(z_1)\tilde{R}^\nu_{\varepsilon_1\nu_2}(z_1/z_2),
\]
\[
\tilde{R}_{\varepsilon_1\varepsilon_2}(z_1q/z_2)L^{(\pm)}_{\varepsilon_1\nu_1}(z_1)L^{(-)}_{\varepsilon_2\nu_2}(z_2) = L^{(-)}_{\varepsilon_2\nu_2}(z_2)L^{(+)}_{\varepsilon_1\nu_1}(z_1)\tilde{R}^{\nu}_{\varepsilon_1\nu_2}(z_1q^{-1}/z_2).
\]
In fact, the relations (5) and (7) define algebra \(U_q(\mathfrak{sl}_2)\). In order to obtain \(U_q(\mathfrak{sl}_2)\) we have to impose two more relations that fix a quantum determinant of the operators \(L^{(\pm)}(z)\)
\[
q\text{-det}L^{(\pm)} = -q. \tag{8}
\]
The statement that quantum determinants \(q\text{-det}L^{(\pm)}\) belong to the center of the algebra can be easily checked using the standard arguments of quantum inverse scattering method and the fact that, after a proper rescaling \(R\)-matrix [4] in the point \(z = q^{-2}\) has a simple form
\[
\frac{1}{\sqrt{q}(q^4z)\infty} \left| \begin{array}{c} 1 \\ 1 - zq^2 \tilde{R}_{12}(z) \end{array} \right|_{z = q^{-2}} = \left( \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 1 & -q^{-1} & 0 \\ 0 & -q & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right).
\]

\footnote{By the quantum determinant of the operator \(\begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix}\) we understand the following combination \(a(zq^{-2})d(z) - qb(zq^{-2})c(z)\).}
Our aim now is to relate the ZF algebras (3) and (4) to algebra (9), (10). It can be done following [9]. The bilinear combinations of the ZF operators \( \Phi_\varepsilon(z) \) and \( \Psi_\nu(z) \) \((\varepsilon, \nu = \pm)\)
\[
L^{(\pm)}_{\varepsilon\nu} = \sqrt{z}g\Psi_\nu(z)\Phi_\varepsilon(zq) \\
L^{(-)}_{\varepsilon\nu} = \sqrt{z}g\Phi_\nu(z)\Psi_\varepsilon(zq) \\
g = (q^2)_{\infty}/(q^4)_{\infty}
\] will satisfy the relations (3) and (7) provided the operators \( \Psi_\nu(z) \) and \( \Phi_\varepsilon(z) \) commute with the scalar function \( \tau(z) \)
\[
\Psi_\nu(z_1)\Phi_\varepsilon(z_2) = \tau((z_1/z_2)\Phi_\varepsilon(z_2)\Psi_\nu(z_1)
\]
which is defined by
\[
\tau^2(z) = \frac{r^+(zq)}{r^-(zq)}\frac{\rho(zq^{-1})}{\rho(z)} = \frac{1}{z^2(q^3z^{-1})^2_{\infty}(qz)^2_{\infty}}
\]
and satisfies the relation
\[
\tau(zq)\tau(zq^{-1}) = -1.
\]
Let us describe now a relation between algebra (3), (7) and the new realization of quantum affine algebra \([\hat{\text{sl}}_2]\) following [10]. Let
\[
L^\pm(z) = \begin{pmatrix} 1 & e^\pm(z) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} k^\pm_1(z) & 0 \\ 0 & k^\pm_2(z) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ f^\pm(z) & 1 \end{pmatrix}
\]
be the decomposition of \( L \)-operators. Then the operators
\[
\psi(z) = k_2^-(zq)k_1^+(zq)^{-1}, \quad \phi(z) = k_2^+(zq)k_1^-(zq)^{-1}
\]
\[
x^+(z) = \frac{e^+(zq^{-1/2}) - e^-(zq^{1/2})}{q - q^{-1}}, \quad x^-(z) = \frac{f^+(zq^{1/2}) - f^-(zq^{-1/2})}{q - q^{-1}}
\]
satisfy the commutation relations of the algebra \( U_q(\hat{\text{sl}}_2) \) at level 1 in the new realization
\[
\psi(z)\phi(w) = \frac{(z - wq^3)(z - wq^{-3})}{(z - wq)(z - wq^{-1})}\phi(w)\psi(z) \quad (11)
\]
\[
\psi(z)x^+(w) = q^{\pm 2}z - q^{\mp 5/2}w
\quad x^+(w)\psi(z) \quad (12)
\]
\[
\phi(z)x^+(w) = q^{\pm 2}z - q^{\mp 3/2}w
\quad x^+(w)\phi(z) \quad (13)
\]
\[
x^+(z)x^+(w) = \frac{(zq^{\pm 2} - w)}{(z - w^{\pm 2})}x^+(w)x^+(z) \quad (14)
\]
\[
[x^+(z), x^-(w)] = \frac{(zw)^{-1}}{q - q^{-1}}(\psi(wq^{1/2})\delta(z/qw) - \phi(zq^{1/2})\delta(w/qz)) \quad (15)
\]
where \( \delta(z) = \sum_{n \in \mathbb{Z}} z^n \) is the delta function.
If we introduce the operators $K, a_n, a_{-n}, n \in \mathbb{Z}_+, x_m^\pm, m \in \mathbb{Z}$ by means of the formulas

$$
\psi(z) = \sum_{n=0}^{\infty} \psi_n z^{-n} = K \exp \left( (q-q^{-1}) \sum_{n=1}^{\infty} a_n z^{-n} \right),
$$

$$
\phi(z) = \sum_{n=0}^{\infty} \phi_n z^n = K^{-1} \exp \left( -(q-q^{-1}) \sum_{n=1}^{\infty} a_{-n} z^n \right),
$$

$$
x^\pm(z) = \sum_{n \in \mathbb{Z}} x_n^\pm z^{-n-1}
$$

then we can deduce from (11)–(15) the commutation relations

$$
[a_n, a_m] = \delta_{n,-m} \frac{[2n][n]}{n}, \quad [a_n, K] = 0,
$$

$$
[a_n, x_m^\pm] = \pm \frac{[2n]}{n} q^{-|n|/2} x_m^\pm, \quad K x_m^\pm K^{-1} = q^{+2} x_m^\pm,
$$

$$
x_{m+1}^+ x_m^- - q^{-2} x_m^+ x_{m+1}^- = q^{+2} x_m^+ x_{m+1}^- - x_m^+ x_{m+1}^-,
$$

$$
[x_n^+, x_m^-] = \frac{1}{q - q^{-1}} (q^{(n-m)/2} \psi_{n+m} - q^{(m-n)/2} \phi_{n+m}).
$$

We see that operators $a_n$ are free boson operators. Using commutation relations (16) one can find the bosonized expression for the currents $x^\pm(z)$ (11). But we would like to do the different things. Namely, we want to show that the new realization of the level one quantum affine algebra $U_q(\hat{sl}_2)$ follows from the free field representation of the ZF algebras (2) and (3). As a by-product of this approach we obtain formulas for the Frenkel-Ding isomorphism between different realizations of $U_q(\hat{sl}_2)$.

3 **Bosonization of Zamolodchikov-Faddeev Algebras**

Let us start with bosonization of algebras (2) and (3). First note that due to (11) commutation relations for the operators $\Phi_\varepsilon(z_1), \Phi_\varepsilon(z_2)$ and $\Psi_\nu(z_1), \Psi_\nu(z_2)$ with the same isotopic indexes $\varepsilon, \nu$ are simple

$$
\Phi_\varepsilon(z_2) \Phi_\varepsilon(z_1) = r^+(z_1/z_2) \Phi_\varepsilon(z_1) \Phi_\varepsilon(z_2)
$$

(17)

$$
\Psi_\nu(z_2) \Psi_\nu(z_1) = r^-(z_1/z_2) \Psi_\nu(z_2) \Psi_\nu(z_1).
$$

(18)

In order to solve (17) and (18) let us introduce infinite dimensional Heisenberg algebra

$$
[a_n, a_m] = \delta_{n+m,0} s_n, \quad [\partial_\alpha, \alpha] = s
$$

where $s_n$ and $s$ are complex numbers and we would like to consider them as parameters. Also define the Fock space $\mathcal{F}$, where the operators $\Phi_\varepsilon(z)$ and $\Psi_\nu(z)$ act

$$
\mathcal{F} = \text{linear span} \left\{ \prod_{j_k \geq \cdots \geq j_1 > 0} a_{-j_k} \cdots a_{-j_1} \right\} \otimes e^{x\alpha}, \quad x \in \mathbb{C}.
$$

(19)
In accordance with this definition of the Fock space, we understand the normal ordered operator as an operator where all the negative modes \(a_{-n}\) and \(\alpha\) are put on the left hand side of the positive modes \(a_n\) and \(\partial_\alpha\).

Fix somehow \(\varepsilon, \nu\) and define the vertex operators

\[
\Phi_\varepsilon(z) = \exp \left( \sum_{n=1}^\infty c_{-n} a_{-n} z^n \right) \exp \left( \sum_{n=1}^\infty c_n a_n z^{-n} \right) e^{c\alpha} z^c \partial_\alpha
\]

(20)

\[
\Psi_\nu(z) = \exp \left( \sum_{n=1}^\infty d_{-n} a_{-n} z^n \right) \exp \left( \sum_{n=1}^\infty d_n a_n z^{-n} \right) e^{d\alpha} z^d \partial_\alpha
\]

(21)

where \(c_{\pm n}, d_{\pm n}, c, c', d, d'\) are parameters. Let us note first that normalization factors in (17) and (18) can be written as follows

\[
r^+(z) = \exp \left( -\sum_{k=1}^\infty \frac{q^k [k]}{2k} (z^k - z^{-k}) \right) z^{-1/2},
\]

(22)

\[
r^-(z) = \exp \left( \sum_{k=1}^\infty \frac{q^{-k} [k]}{2k} (z^k - z^{-k}) \right) z^{1/2}.
\]

(23)

In order to obtain (22) and (23) we used the formula

\[
(z)_{\infty} = \prod_{n=0}^\infty (1 - z q^{4n}) = \exp \left( -\sum_{k=1}^\infty \frac{1}{k} \frac{z^k}{1 - q^{4k}} \right).
\]

(24)

On the other hand, the commutation relation for operators (20) and (21) are

\[
\Phi_\varepsilon(z_2) \Phi_\varepsilon(z_1) = \exp \left( \sum_{n=1}^\infty s_n c_n c_{-n} (z^n - z^{-n}) \right) z^{-scc'} \Phi_\varepsilon(z_1) \Phi_\varepsilon(z_2),
\]

(25)

\[
\Psi_\nu(z_1) \Psi_\nu(z_2) = \exp \left( \sum_{n=1}^\infty s_n d_n d_{-n} (z^n - z^{-n}) \right) z^{sd'd'} \Psi_\nu(z_2) \Psi_\nu(z_1).
\]

(26)

Identifying logarithms of the normalization factors \(r^\pm(z)\) and of those arisen in (25) and (26), we obtain the following relations between parameters

\[
s_n c_n c_{-n} = -\frac{q^n [n]}{n [2n]}, \quad scc' = \frac{1}{2},
\]

(27)

\[
s_n d_n d_{-n} = -\frac{q^{-n} [n]}{n [2n]}, \quad sd'd' = \frac{1}{2}.
\]

(28)

It is clear now that we cannot use the same representation (20) and (21) for the operators \(\Phi_{-\varepsilon}(z)\) and \(\Psi_{-\nu}(z)\) because in this case the relations

\[
\Phi_{-\varepsilon}(z_2) \Phi_{-\varepsilon}(z_1) = r^+(z_1/z_2) (c(z_1/z_2) \Phi_{-\varepsilon}(z_1) \Phi_{-\varepsilon}(z_2)) + b(z_1/z_2) \Phi_{-\varepsilon}(z_1) \Phi_{-\varepsilon}(z_2),
\]

(29)

\[
\Phi_{-\varepsilon}(z_2) \Phi_{-\varepsilon}(z_1) = r^+(z_1/z_2) (c(z_1/z_2) \Phi_{-\varepsilon}(z_1) \Phi_{-\varepsilon}(z_2)) + b(z_1/z_2) \Phi_{-\varepsilon}(z_1) \Phi_{-\varepsilon}(z_2)
\]

(30)

and

\[
\Psi_{-\nu}(z_2) \Psi_{-\nu}(z_1) = r^-(z_1/z_2) (c(z_1/z_2) \Psi_{-\nu}(z_1) \Psi_{-\nu}(z_2) \Psi_{-\nu}(z_1)) + b(z_1/z_2) \Psi_{-\nu}(z_1) \Psi_{-\nu}(z_2),
\]

(31)

\[
\Psi_{-\nu}(z_2) \Psi_{-\nu}(z_1) = r^-(z_1/z_2) (c(z_1/z_2) \Psi_{-\nu}(z_1) \Psi_{-\nu}(z_2) \Psi_{-\nu}(z_1)) + b(z_1/z_2) \Psi_{-\nu}(z_1) \Psi_{-\nu}(z_2).
\]

(32)
are not satisfied. But it is still obvious that operators $\Phi_{-\varepsilon}(z)$ and $\Psi_{-\nu}(z)$ should be somehow proportional to the operator $\Phi_{\varepsilon}(z)$ and $\Psi_{\nu}(z)$ respectively in order to cancel the common normalization factors $r^+(z_1/z_2)$ and $r^-(z_1/z_2)$. To this end we introduce two more vertex operators

\[ X(w) = \exp \left( \sum_{n=1}^{\infty} x_{-n} a_{-n} w^n \right) \exp \left( \sum_{n=1}^{\infty} x_n a_n w^{-n} \right) e^{x\alpha} w^{x'\partial_\alpha} \]
\[ Y(w) = \exp \left( \sum_{n=1}^{\infty} y_{-n} a_{-n} w^n \right) \exp \left( \sum_{n=1}^{\infty} y_n a_n w^{-n} \right) e^{y\alpha} w^{y'\partial_\alpha} \]

and try to satisfy the commutation relations (29), (30) and (31), (32) by the linear combination

\[ \Phi_{-\varepsilon}(z) = (\Phi_{-\varepsilon}(z,w))_m, \quad \Phi_{-\varepsilon}(z,w) = \Phi_{\varepsilon}(z) X(w) + AX(w)\Phi_{\varepsilon}(z), \tag{33} \]
\[ \Psi_{-\nu}(z) = (\Psi_{-\nu}(z,w))_m, \quad \Psi_{-\nu}(z,w) = \Phi_{\nu}(z) Y(w) + BY(w)\Psi_{\nu}(z), \tag{34} \]

where $A, B$ are parameter to be determined and notation $(O(w))_m$ means $O_m$ if $O(w) = \sum_{m \in \mathbb{Z}} O_m w^{-m-1}$.

Some comment is in order now. Throughout this section we adopt the “formal series” point of view. This means that two formal series (in spectral parameters) are equal if and only if all their pairwise coefficients coincide.

Let us concentrate now on solving (29) and (30). Relations (31) and (32) can be treated analogously. In order to simplify the notations, let us introduce the new set of parameters

\[ \alpha_{-n} = -\frac{x_{-n}}{c_{-n}} \frac{q^n[n]}{2n}, \quad \alpha_n = -\frac{x_n}{c_n} \frac{q^n[n]}{2n}, \quad a = \frac{x}{2c}, \quad a' = \frac{x'}{2c'} \]

such that the normal ordering products of the operators $\Phi_{\varepsilon}(z)$ and $X(w)$ can be written as follows

\[ \Phi_{\varepsilon}(z) X(w) = \exp \left( \sum_{n=1}^{\infty} \frac{\alpha_{-n}}{n} \left( \frac{w}{z} \right)^n \right) z^\alpha \Phi_{\varepsilon}(z) X(w):, \tag{35} \]
\[ X(w) \Phi_{\varepsilon}(z) = \exp \left( \sum_{n=1}^{\infty} \frac{\alpha_n}{n} \left( \frac{z}{w} \right)^n \right) w^{\alpha'} X(w) \Phi_{\varepsilon}(z):. \tag{36} \]

Our strategy now is to find the relation between parameters $\alpha_{\pm n}$ which fulfils (29), (30) and

\[ \Phi_{-\varepsilon}(z_2) \Phi_{-\varepsilon}(z_1) = r^+(z_1/z_2) \Phi_{-\varepsilon}(z_1) \Phi_{-\varepsilon}(z_2). \tag{37} \]

First note that the zero mode part of operators $\Phi_{\varepsilon}(z)$ and $\Phi_{-\varepsilon}(z)$ will be equal $e^{\varepsilon\alpha}$ and $e^{(-\varepsilon\alpha)}$ respectively. It is natural to require that these operators has opposite zero modes, namely, to require that $x + c = -c$. It fixes the parameter $a$

\[ a = -1. \]

Now consider the equation (29). Substitute there the operators $\Phi_{-\varepsilon}(z_1,w)$ and $\Phi_{\varepsilon}(z_2)$ and normal order all the operator products using (33) and (34). It can be easily seen that the
yields the following unique solutions for the parameters and ε
(two terms) action on the tensor products of two representations of
generators that are proportional to corresponding Chevalley generators and have simplest
m
leads to the same solution (40) but with
gets the unique solution
nontrivial equations for the coefficients 𝑈
As far as these coefficients are symmetric functions of the spectral parameters
Comparing (38) and (39), one fixes
A
and put
The analogous consideration of (31), (32) and
With these solution for the parameters
Some comment is in order now on ansatz (33). It is clear that we were succesful solving
in the product Φ
−ε
w
= Φ
−ε
z
in order to obtain (up to function that arose
Comparing (38) and (39), one fixes A = −q and, solving these equations by induction, one
gets the unique solution
α
n
= α
−n
= (−q)
(40)
Some comment is in order now on ansatz (33). It is clear that we were succesful solving equations (38) and (33) only due to this ansatz. In fact, looking at these equations, one can note that the other ansatz
\[ \Phi_{-\varepsilon}(z) = (\Phi_{-\varepsilon}(z, w))_m, \quad \Phi_{-\varepsilon}(z, w) = z^{-1}(\Phi_{\varepsilon}(z)X(w) + AX(w)\Phi_{\varepsilon}(z)) \] (41)
leads to the same solution (40) but with A = −q⁻¹. The existence of these simplest ansatz is due to the fact that in the new realization of quantum affine algebra \( U_q(\hat{sl}_2) \) there are four generators that are proportional to corresponding Chevalley generators and have simplest (two terms) action on the tensor products of two representations of \( U_q(\hat{sl}_2) \).

With these solution for the parameters \( \alpha_{\pm} \) one can easily calculate the coefficients at \( (wv)^{-m} \) in the product \( \Phi_{-\varepsilon}(z_2, w)\Phi_{-\varepsilon}(z_1, v) \) in order to obtain (up to function that arose from normal ordering of the operators \( \Phi_{\varepsilon}(z_2)\Phi_{\varepsilon}(z_1) \) and cancels the factor \( r^+(z_1/z_2) \))
\[ (-q)^{2m-1}, \quad \text{for } m > 0, \quad (-q)^{-2m-1}(z_1z_2)^{m-1}, \quad \text{for } m < 0 \]
and
\[ \frac{q^2[2]z_1z_2 - q(z_1 + z_2)^2}{z_1^2z_2^2}, \quad \text{for } m = 0. \]
As far as these coefficients are symmetric functions of the spectral parameters \( z_1 \) and \( z_2 \), the equation (37) is automatically satisfied by solution (40) for any \( m \) in (33).

The analogous consideration of (31), (33) and
\[ \Psi_{-\nu}(z_2)\Psi_{-\nu}(z_1) = r^-(z_1/z_2)\Psi_{-\nu}(z_1)\Psi_{-\nu}(z_2) \]
yields the following unique solutions for the parameters
\[ \beta_{-n} = \frac{y_n q^{-n}[n]}{d_{-n} [2n]} = (-q)^{-n}, \quad \beta_n = \frac{y_n q^{-n}[n]}{d_n [2n]} = (-q)^{-n}, \]
b = \frac{y}{2d} = -1, \quad b' = \frac{y'}{2d'} = -1 \quad (42)

with \( B = -q \) in (34) or with \( B = -q^{-1} \) in the ansatz

\[
\Psi_{-\epsilon}(z) = (\Psi_{-\epsilon}(z, w))_m, \quad \Psi_{-\epsilon}(z, w) = z(\Psi_{\epsilon}(z)Y(w) + BY(w)\Psi_{\epsilon}(z)). \quad (43)
\]

Now we are in position to calculate the operator product of the operators \( X(w)X(v) \) and \( Y(w)Y(v) \)

\[
X(w)X(v) = \exp \left( -\sum_{n=1}^{\infty} \frac{(1 + q^{-2n})\alpha_{n}\alpha_{-n}}{n} \left( \frac{v}{w} \right)^n \right) w^{-2a'}X(w)X(v):
\]

\[
= (w - v)(w - q^2v):X(w)X(v):
\]

\[
Y(w)Y(v) = \exp \left( -\sum_{n=1}^{\infty} \frac{(1 + q^{2n})\beta_{n}\beta_{-n}}{n} \left( \frac{v}{w} \right)^n \right) w^{-2b'}Y(w)Y(v):
\]

\[
= (w - v)(w - q^{-2v}):Y(w)Y(v):
\]

It follows from these products that the vertex operators \( X(w) \) and \( Y(w) \) will satisfy the relations

\[
X(v)X(w) = \frac{(vq^{-2} - w)}{(v - w^{-2})}X(w)X(v)
\]

\[
Y(v)Y(w) = \frac{(vq^2 - w)}{(v - w^2)}Y(w)Y(v)
\]

that coincide with (14).

In order to calculate the commutation relations between operators \( X(w) \) and \( Y(v) \) we have to use the relation (10). Comparising logarithm of the factor in the commutation relation

\[
\Psi_{\epsilon}(z_1)\Phi_{\nu}(z_2) = \exp \left( \sum_{n=1}^{\infty} s_n(c_{-n}d_nz^{-n} - c_n d_{-n}z^n) \right) z_1^{scd'} z_2^{-sc'd} \Phi_{\nu}(z_2)\Psi_{\epsilon}(z_1)
\]

with logarithm of the function \( \tau(z) \)

\[
\tau(z) = \exp \left( -\sum_{k=1}^{\infty} \frac{1}{k} \frac{[k]}{[2k]} (z^k - z^{-k}) \right) z^{-1/2}, \quad z = z_1/z_2
\]

gives the relations

\[
s_n c_{-n} d_n = s_n c_n d_{-n} = \frac{1}{n} \frac{[n]}{[2n]}, \quad scd' = sc'd = -\frac{1}{2}. \quad (44)
\]

These relations along with (27), (28), (10) and (12) allow one to express all parameters only in terms of the parameters \( c_{\pm n}, c \) and \( c' \)

\[
d_n = -c_n q^{-|n|}, \quad x_n = (-)^n c_n \frac{[2n]}{[n]}, \quad y_n = (-q)^{-|n|} c_n \frac{[2n]}{[n]}, \quad n = \pm 1, \pm 2, \ldots
\]
\[ -x = y = -2d = 2c, \quad -x' = y' = -2d' = 2c' \]

It follows from (13) that the operators
\[ [\Phi_{\nu}(z), Y(w)] = [\Psi_{\varepsilon}(z), X(w)] = 0 \]
commute, so the rest three relations in (14) are satisfied.

Commutation relation for operators \( X(w) \) and \( Y(v) \) can be easily calculated now
\[ [Y(v), X(w)] = \frac{(wv)^{-1}}{q-q^{-1}} (\psi(w)\delta(vqw) - \phi(v)\delta(wqv)) \]
where we introduced the operators
\[
\psi(w) = X(w)Y(qw) = q^{2\varepsilon}\partial_{\alpha} \exp \left( \sum_{n=1}^{\infty} -(q-q^{-1})a_n w^{-n} c_n [2n](-q)^{-n} \right)
\]
\[
\phi(w) = X(qw)Y(w) = q^{-2\varepsilon}\partial_{\alpha} \exp \left( \sum_{n=1}^{\infty} -(q-q^{-1})a_{-n} w^n c_{-n} [2n](-)^{-n} \right)
\]

In addition, the simple calculation shows that the commutation relations between operators
\( \psi(z), \phi(z) \) and \( \psi(z), \phi(z), X(w), Y(w) \) are
\[
\psi(z)\phi(w) = \frac{(z-wq^3)(z-wq^{-3})}{(z-wq)(z-wq^{-1})} \phi(w)\psi(z)
\]
\[
\psi(z)Y(w) = q^{2z-q^{-3}w}Y(w)\psi(z)
\]
\[
\psi(z)X(w) = q^{-2z-q^{2}w}X(w)\psi(z)
\]
\[
\phi(z)Y(w) = q^{2z-q^{-2}w}Y(w)\phi(z)
\]
\[
\phi(z)X(w) = q^{-2z-q^{2}w}X(w)\phi(z)
\]
and coincide with the defining relation of \( U_q(\hat{sl}_2) \) in the new realization given by (11)–(13) up to some rescaling of the spectral parameters.

Let us summarize. From the bosonization of normalization factors in commutation relations of ZF algebras (2), (3), (10) and from requirements that operator equations (29)–(32) to be identified as formal power series with respect to all the spectral parameters, one obtains the commutation relations of the level one quantum affine algebra \( U_q(\hat{sl}_2) \) in the new realization.

Now we are ready to obtain formulas for the Frenkel-Ding isomorphism between different realization of \( U_q(\hat{sl}_2) \). Consider a following operator valued \( 2 \times 2 \) matrices (indices \( \nu \) and \( \varepsilon \) are fixed)
\[
L^{(+)}(z, w, v) = \begin{pmatrix} 1 & E^{(+)}(z) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} P^{(+)}(z) & 0 \\ 0 & K^{(+)}(z) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ F^{(+)}(z) & 1 \end{pmatrix}
\]
\[
= \sqrt{z}g \begin{pmatrix} \Psi_{-\nu}(z, w)\Phi_{-\varepsilon}(z, v) & \Psi_{-\nu}(z, w)\Phi_{\varepsilon}(z) \\ \Psi_{\nu}(z)\Phi_{-\varepsilon}(z, v) & \Psi_{\nu}(z)\Phi_{\varepsilon}(z) \end{pmatrix}
\]
\[
L^{(-)}(z, w, v) = \begin{pmatrix}
1 & E^{(-)}(z) \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
P^{(-)}(z) & 0 \\
0 & K^{(-)}(z)
\end{pmatrix}
\begin{pmatrix}
1 & F^{(-)}(z) \\
0 & 1
\end{pmatrix}
= \sqrt{z}g
\begin{pmatrix}
\Phi_{-\varepsilon}(z, w)\Psi_{-\nu}(qz, v) & \Phi_{-\varepsilon}(z, w)\Psi_{\nu}(qz) \\
\Phi_{\varepsilon}(z)\Psi_{-\nu}(qz, v) & \Phi_{\varepsilon}(z)\Psi_{\nu}(qz)
\end{pmatrix}
\]

Simple calculations give the following relations
\[
K^{(+)}(z) = q^{\varepsilon}\partial_\alpha \exp \left( \sum_{n=1}^{\infty} c_{-n}(q^n - q^{-n})a_{-n}z^n \right)
\]
\[
K^{(-)}(z) = q^{-\varepsilon}\partial_\alpha \exp \left( \sum_{n=1}^{\infty} q^{-n}c_{n}(q^n - q^{-n})a_nz^n \right)
\]

Using commutativity of the operators \(\Phi(z), Y(w)\) and \(\Psi(z), X(w)\), one can write off-diagonal elements of \(L\)-operators as follows
\[
E^{(+)}(z, w)K^{(+)}(z) = K^{(+)}(z)Y(w) - qY(w)K^{(+)}(z) \quad (46)
\]
\[
K^{(+)}(z)F^{(+)}(z, v) = K^{(+)}(z)X(v) - qX(v)K^{(+)}(z) \quad (47)
\]
\[
K^{(-)}(z)F^{(-)}(z, v) = K^{(-)}(z)X(v) - qX(v)K^{(-)}(z) \quad (48)
\]
\[
E^{(-)}(z, w)K^{(-)}(z) = K^{(-)}(z)Y(w) - qY(w)K^{(-)}(z) \quad (49)
\]

Using the property of the delta function
\[
f(z)\delta(w/z) = f(w)\delta(w/z)
\]
and the variant of the formula (24)
\[
\exp \left( \sum_{n=1}^{\infty} \frac{q^n - q^{-n}}{n} z^n \right) = 1 + q^{-1}(q - q^{-1}) \sum_{n=1}^{\infty} (qz)^{n}
\]
on one can obtain from (46)-(49) the Frenkel-Ding formulas which relate the currents \(X(w)\) and \(Y(w)\) to elements of the \(L\)-operators.
\[
\frac{E^{(+)}(-zq^{-1}, w) - E^{(-)}(-zq^{-2}, w)}{q - q^{-1}} = Y(z)\delta(z/w)
\]
\[
\frac{F^{(-)}(-zq^{-1}, w) - F^{(+)}(-zq^{-2}, w)}{q - q^{-1}} = qX(z)\delta(z/w)
\]

Using the commutation relations for the currents \(X(z)\) and \(Y(z)\) one can show that the ratio \(P^{(\pm)}(z, w, v)/K^{(\pm)}(z)\) is related to operators \(\phi(z)\) and \(\psi(z)\) as follows
\[
P^{(+)}(-zq^{-1}, w, v)(K^{(+)}(-zq^{-1}))^{-1} = -\frac{1}{z^2}\delta(v/qz)\delta(w/z)\phi(z), \quad (50)
\]
\[
P^{(-)}(-zq^{-1}, w, v)(K^{(-)}(-zq^{-1}))^{-1} = -\frac{1}{z^2}\delta(v/z)\delta(w/qz)\psi(z). \quad (51)
\]

Until now we did not fix the integer number \(m\) in the ansatz for the operators \((\Psi_{-\nu}(z, w))_m\) and \((\Phi_{-\varepsilon}(z, w))_m\). In order to fix it we have to use the relation which fixes quantum determinants of the operators \(L^{(\pm)}(z)\). Let us calculate \(q\)-det\(L^{+}(z, w, v)\). Using obvious equality
\[
K^{(+)}(-zq^{-1})K^{(+)}(-zq) = \phi(z)^{-1}
\]
and the commutation relation
\[ F^{(+)}(zq^{-2}, v)K^{(+)}(z) = qK^{(+)}(z)F^{(+)}(z, v) \]
that follows from definition of \( F^{(+)}(z, v) \) we obtain the relation
\[ q \text{-det}L^+(z, w, v) = -\frac{1}{q^2z^2}\delta \left(-\frac{v}{zq^2}\right)\delta \left(-\frac{w}{zq}\right) \]

Now it is clear that only the quantum determinant of the coefficient at \( (wv)^{-1} \) of the operator \( L^+(z, w, v) \) satisfies (8). It means that in order to reconstruct a proper \( L \) operators from operators \( \Psi(z) \) and \( \Phi(z) \) using Miki’s formulas (9) we have to set \( m = 0 \) in ansatz (33) and (34). Similar arguments show that fixing the quantum determinants along with ansatz (41) for the operator \( \Phi_{-\epsilon}(z, w) \) gives \( m = 1 \) and doing this along with ansatz (43) gives \( m = -1 \).

Now the operators \( \Phi_{\pm\epsilon}(z) \) and \( \Psi_{\pm\nu}(z) \) are completely determined and can be interpreted as intertwiners of the corresponding highest weight modules over quantum affine algebra \( U_q(\widehat{sl}_2) \) at level one. The Fock space (19) decomposes into two subspaces irreducible with respect the action of \( U_q(\widehat{sl}_2) \)
\[ \mathcal{F}_i = \text{linear span} \left\{ \prod_{j_k \geq \cdots \geq j_1 > 0} a_{-j_k} \cdots a_{-j_1} \right\} \otimes e^{(2n+i)\alpha}, \quad n \in \mathbb{Z}, \quad i = 0, 1 \]
which can be identified with these modules.

4 Conclusion

To conclude, let us discuss possible extensions of the approach developed in this paper. First of all, this is the case of higher levels for quantum affine algebra \( U_q(\widehat{sl}_2) \). The commutation relations of ZF algebras are to be more complicated \cite{13}, since they include the face \( R \)-matrices. It is obvious that, in order to bosonize these ZF algebras, one has to introduce, besides the free bosonic field, a \( \beta\gamma \)-system similar to bosonization of arbitrary level algebra \( U_q(\widehat{sl}_n) \) developed in \cite{16, 17}. In this case we meet the problem of decomposing the huge Fock space to irreducible pieces that was solved recently in \cite{18} using some deformation of the Fedler construction.

Another possible generalization is to extend this analysis to \( U_q(\widehat{sl}_n) \). In \cite{19} the \( R \)-matrices which intertwine different finite dimensional fundamental representations have been calculated and corresponding ZF algebras have been defined. At level one, these algebras were bosonized in \cite{20}. The main problem here is to find a proper generalization of Miki formulas for the \( L \)-operators. It can be easily seen that naive generalization of (9) to the case of algebra \( U_q(\widehat{sl}_n) \) leads to incorrect commutation relations between type I and type II vertex operators.

But the most interesting new development of our approach would be to construct a free field representation of the elliptic generalization of the affine algebra \( \widehat{sl}_2 \), which has been recently formulated in \cite{21, 22} within the \( R \)-matrix approach. Bosonization of the corresponding ZF algebras should allow one to calculate the correlation functions and the form factors in the XYZ model.
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References

[1] A.B. Zamolodchikov and Al.B. Zamolodchikov. Factorized S-matrices in two dimensions as the exact solutions of certain relativistic quantum field theory models. *Ann. Phys. (N.Y.)*, 120 253–291, (1979).

[2] B. Davies, O. Foda, M. Jimbo, T. Miwa and A. Nakayashiki. Diagonalization of the XXZ hamiltonian by vertex operators. *Comm. Math. Phys.*, 151 89–153, (1993).

[3] M. Jimbo, K. Miki, T. Miwa and A. Nakayashiki. Correlation functions of the XXZ model for $\Delta < -1$. *Phys. Lett.*, A 168 256–263, (1992).

[4] I.B. Frenkel and N.Yu. Reshetikhin. Quantum affine algebras and holonomic difference equations. *Comm. Math. Phys.*, 146 1–60, (1992).

[5] V.G. Drinfeld. *Quantum groups. In Proceedings of the International Congress of Mathematicians*. Berkley, 1987.

[6] M. Jimbo. A $q$-difference analogue of $U(\hat{g})$ and Yang-Baxter equation. *Lett. Math. Phys.*, 10 63–69, (1985).

[7] N.Yu. Reshetikhin and M.A. Semenov-Tian-Shansky. Central extension of quantum current groups. *Lett. Math. Phys.*, 19 133–142, (1990).

[8] K. Miki. Creation/annihilation operators and form factors of the XXZ model. *Phys. Lett.*, A 186 217–224, (1994).

[9] V.G. Drinfeld. A new realization of Yangians and quantum affine algebras. *Soviet Math. Doklady*, 36 212–216, (1988).

[10] I.B. Frenkel and J. Ding. Isomorphism of two realization of quantum algebra $U_q(\hat{gl}(n))$. *Comm. Math. Phys.*, 156 277–300, (1993).

[11] I.B. Frenkel and N.H. Jing. Vertex representations of quantum affine algebras. *Proc. Nat’l. Acad. Sci. USA*, 85 9373–9377, (1988).

[12] S. Lukyanov. Free field representation for massive integrable models. RU-93-30, hep-th/9307196, 1993.
[13] H.J. de Vega and A. González-Ruiz. The highest weight property for the $SU_q(n)$ invariant spin chains. *Preprint L.P.T.H.E.*, 1994.

[14] S.-J. Kang, M. Kashiwara, K. Misra, T. Miwa, T. Nakashima and A. Nakayashiki. Affine crystals and vertex models. *Inter. J. Modern Phys., A* 7(Suppl. 1A) 449–484, (1992).

[15] M. Idzumi, K. Iohara, M. Jimbo, T. Miwa, T. Nakashima and T. Tokihiro. Quantum affine symmetry in vertex models. *Intern. J. Modern Phys., A* 8 1479–1511, (1993).

[16] A. Matsuo. A $q$-deformation of Wakimoto modules, primary fields and screening operators. *Comm. Math. Phys.,* 160 33–48, (1994).

[17] J. Shiraishi. Free boson representation of $U_q(\widehat{sl}_2)$. *Phys. Lett., A* 171:243–248, (1992).

[18] H. Konno. BRST cohomology in quantum affine algebra $U_q(\widehat{sl}_2)$. *RIMS-927*, 1993.

[19] E. Date and M. Okado. Calculation of excitation spectra of the spin models related with the vector representation of the quantized affine algebra of type $A_n^{(1)}$. *Intern. J. Modern Phys., A* 9 399–417, (1994).

[20] Y. Koyama. Staggered polarization of vertex models with $U_q(\widehat{sl}_n)$-symmetry. *Comm. Math. Phys.,* 164 277–291, 1993.

[21] O. Foda, K. Iohara, M. Jimbo, R. Kedem, T. Miwa and H. Yang. An elliptic quantum algebra for $\widehat{sl}_2$. *RIMS-974*, 1994.

[22] O. Foda, K. Iohara, M. Jimbo, R. Kedem, T. Miwa and H. Yang. Notes on highest weight modules of the elliptic algebra $A_{q,p}(\widehat{sl}_2)$. *RIMS-979*, 1994.