Structural theorems on the distance sets over finite fields

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Abstract

Let $\mathbb{F}_q$ be a finite field of order $q$. Iosevich and Rudnev (2007) proved that for any set $A \subset \mathbb{F}_d^q$, if $|A| \gg q^{d+1/2}$, then the distance set $\Delta(A)$ contains a positive proportion of all distances. Although this result is sharp in odd dimensions, it is conjectured that the right exponent should be $d^2$ in even dimensions. During the last 15 years, only some improvements have been made in two dimensions, and the conjecture is still wide open in higher dimensions. To fill the gap, we need to understand more about the structures of the distance sets, the main purpose of this paper is to provide some structural theorems on the distribution of square and non-square distances.

1 Introduction

Let $\mathbb{F}_q$ be a finite field of order $q$ which is a prime power. The distance between two points $x$ and $y$ in the space $\mathbb{F}_d^q$ is defined by

$$||x - y|| := (x_1 - y_1)^2 + \cdots + (x_d - y_d)^2.$$ 

For $A \subset \mathbb{F}_d^q$, let $\Delta(A)$ denote the set of distances determined by pairs of points in $A$, i.e.

$$\Delta(A) = \{||x - y|| : x, y \in A\}.$$ 

Studying the magnitude of $\Delta(E)$ has been received much attention since 2007 due to its connection to the Falconer distance conjecture in Geometric Measure Theory, which says that for any compact set $A \subset \mathbb{R}^d$ of Hausdorff dimension greater than $d/2$, the distance set $\Delta(A)$ is of positive Lebesgue measure. Recent progress on this conjecture can be found in [2, 3, 4, 5].

In the finite field setting, Iosevich and Rudnev [8] proved that for $A \subset \mathbb{F}_d^q$, if $|A| \geq 4q^{d+1/2}$, then $\Delta(A) = \mathbb{F}_q$. The exponent $d+1/2$ is sharp in odd dimensions, namely, $d \equiv 3 \mod 4$ and $q \equiv 1 \mod 4$, or $d \equiv 1 \mod 4$, we refer the reader to [8] for constructions. In even dimensions, it is conjectured that the right exponent should be $d^2$ which is directly in line with the Falconer distance conjecture. In a very recent paper, Murphy, Petridis, Pham, Rudnev, and Stevens [17] obtained the exponent $d^2$ in the plane over prime fields, which improves the exponent $d^2$ by Chapman, Erdogan, Hart, Iosevich, and Koh in [1]. In particular, they showed that for $A \subset \mathbb{F}_p^2$ with $|A| \gg p^{5/4}$, then there exists $x \in A$ such that

$$|\Delta_x(A)| \gg p,$$ 

$$\text{(1)}$$

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where \( \Delta_x(A) := \{|\|x-a\|| : a \in A \} \). This matches the breakthrough dimensional threshold \( \frac{5}{4} \) on the Falconer distance conjecture given by Guth, Iosevich, Ou, and Wang in [5]. For \( 4p < |A| \leq p^{5/4} \), Murphy et al. also proved that there exists \( x \in A \) with

\[
|\Delta_x(A)| \gg \frac{|A|^{4/3}}{p^{2/3}}.
\]

This statement is similar to a result of Liu in [16] for the continuous setting which states that for any compact set \( A \subset \mathbb{R}^2 \), if the Hausdorff dimension of \( A \) is greater than 1, then

\[
dim_H(\Delta_x(A)) \geq \min\left\{ \frac{4}{3} \dim_H(A) - \frac{2}{3}, 1 \right\},
\]

for some \( x \in A \). We note that when the dimension of \( A \) is very close to 1, then Shmerkin [18] obtained better results, namely,

\[
dim_H(\Delta_x(A)) \geq \frac{29}{42},
\]

and

\[
dim_H(\Delta(A)) \geq \frac{40}{57}.
\]

Over finite fields, in higher even dimensions or \( d \equiv 3 \pmod{4} \) and \( q \equiv 3 \pmod{4} \), the best current exponent is still \( \frac{4d+4}{4d+3} \) due to Iosevich and Rudnev [8], which is far from the conjecture \( \frac{d}{2} \). In order to make further progress, we need to understand more about structures of the distance sets, and in this paper, we focus on the distribution of pairs of points of square and non-square distances in a given set \( A \subset \mathbb{F}_q^d \).

To state our main results, we need the following definition.

**Definition 1.1.** Let \( \eta \) denote the quadratic character of \( \mathbb{F}_q \) with \( \eta(0) = 0 \). For \( A \subset \mathbb{F}_q^d \), we define

\[
SQ(A) := |\{(x, y) \in A \times A : \eta(|\|x - y\||) = 1\}|,
\]

and

\[
ZR(A) := |\{(x, y) \in A \times A : \eta(|\|x - y\||) = 0\}|,
\]

as the sets of pairs of square distances and zero-distance, respectively.

For a given \( A \subset \mathbb{F}_q^d \), if we consider the zero distance as a square, then either the number of pairs of square distances or the number of pairs of non-square distances is at least \( |A|^2/2 \).

In the first theorem, we show that when the size of \( A \) is not small, most of the pairs of points in \( A \) are of non-square distances. Precisely, we have
Theorem 1.2. Let $A \subset \mathbb{F}^d_q$.

1. If $d \equiv 3 \mod 4$ and $q \equiv 3 \mod 4$, then
   \[ SQ(A) + ZR(A) \leq \frac{|A|^2}{2} + \frac{|A|^2}{2q} - \frac{|A|^2}{2q^{\frac{d}{2}}} + \frac{q^{\frac{d+1}{2}} |A|}{2}. \]

2. If $d \equiv 1 \mod 4$, or $d \equiv 3 \mod 4$ and $q \equiv 1 \mod 4$, then
   \[ SQ(A) + ZR(A) \leq \frac{|A|^2}{2} + \frac{|A|^2}{2q} + \frac{q^d |A|}{2} - \frac{q^{\frac{d-1}{2}} |A|}{2}. \]

3. If $d \equiv 2 \mod 4$ and $q \equiv 3 \mod 4$, then
   \[ SQ(A) + ZR(A) \leq \frac{|A|^2}{2} + \frac{|A|^2}{2q} + \frac{q^d |A|}{2} - \frac{q^{\frac{d-2}{2}} |A|}{2}. \]

4. If $d \equiv 0 \mod 4$, or $d \equiv 2 \mod 4$ and $q \equiv 1 \mod 4$, then
   \[ SQ(A) + ZR(A) \leq \frac{|A|^2}{2} + \frac{|A|^2}{2q} - \frac{q^d |A|}{2} + \frac{q^{\frac{d-2}{2}} |A|}{2}. \]

We say that a set $A \subset \mathbb{F}^d_q$ is a square distance set if for all $x, y \in A$, $||x - y||$ is either zero or a square number of $\mathbb{F}_q$. It is clear that for such a set, one has $SQ(A) + ZR(A) = |A|^2$ for any square distance set $A$ in $\mathbb{F}^d_q$. Hence, if $A \subset \mathbb{F}^d_q$ is a square distance set, then $|A|^2$ is dominated by the upper bound of $SQ(A) + ZR(A)$ given in Theorem 1.2. By solving the inequalities in terms of $|A|$, we recover the following results of Iosevich, Shparlinski, and Xiong [9]. We also refer the reader to [13, 10, 14, 9] for more discussions and the motivation of this problem.

Corollary 1.3 (Iosevich-Shparlinski-Xiong, [9]). Let $A$ be a square distance set in $\mathbb{F}^d_q$.

1. If $d \equiv 3 \mod 4$ and $q \equiv 3 \mod 4$, then
   \[ |A| \leq \frac{2q^{\frac{d+1}{2}}}{q - 1 + (q + 1)q^{-\left(\frac{d-1}{2}\right)}}. \]

2. If $d \equiv 1 \mod 4$, or $d \equiv 3 \mod 4$ and $q \equiv 1 \mod 4$, then
   \[ |A| \leq q^{\frac{d+1}{2}}. \]

3. If $d \equiv 2 \mod 4$ and $q \equiv 3 \mod 4$, then
   \[ |A| \leq q^d + \frac{2(q^d - q)}{q - 1 + 2q^{-\left(\frac{d-2}{2}\right)}}. \]

4. If $d \equiv 0 \mod 4$, or $d \equiv 2 \mod 4$ and $q \equiv 1 \mod 4$, then
   \[ |A| \leq q^d. \]
In the next two theorems, we are interested in the distribution of non-zero square distances in a
given set $A \subset \mathbb{F}_q^d$.

**Theorem 1.4** (odd dimensions). *Let $A$ be a subset of $\mathbb{F}_q^d$.  

1. Let $d \equiv 3 \mod 4$ and $q \equiv 3 \mod 4$.  
   If $|A| \geq (q^{(d+1)/2} + q)/(1 + q^{-(d-1)/2})$, then  
   $$\mathcal{SQ}(A) \leq \frac{|A|^2}{2} + q^{d-1}|A| - \frac{|A|^2}{2q} - \frac{|A|^2}{2q^{d-1}}.$$  
   If $|A| \leq (q^{(d+1)/2} + q)/(1 + q^{-(d-1)/2})$, then  
   $$\mathcal{SQ}(A) \leq \frac{|A|^2}{2} + q^{d-1}|A| - \frac{|A|^2}{2q} - \frac{|A|^2}{2}.  

2. Let $d \equiv 1 \mod 4$, or $d \equiv 3 \mod 4$ and $q \equiv 1 \mod 4$, then  
   $$\mathcal{SQ}(A) \leq \frac{|A|^2}{2} - \frac{q^{d-1}|A|}{2} - \frac{|A|^2}{2} + \min \left\{ \frac{q^{d+1}|A|}{2}, \frac{q^{d+1}|A|}{2} + \frac{|A|^2}{2}, \frac{|A|^2}{2} + \frac{q^{d+1}|A|}{2} - \frac{|A|^2}{2} \right\}.  

**Theorem 1.5** (even dimensions). *Let $A$ be a subset of $\mathbb{F}_q^d$.  

1. If $d \equiv 2 \mod 4$ and $q \equiv 3 \mod 4$, then  
   $$\mathcal{SQ}(A) \leq \frac{|A|^2}{2} + q^{d/2}|A| - \frac{|A|^2}{2q} - \frac{q^{d-2}|A|}{2}.$$  

2. Let $d \equiv 0 \mod 4$, or $d \equiv 2 \mod 4$ and $q \equiv 1 \mod 4$.  
   If $|A| \geq (q^{d/2} + q)/(1 + q^{-(d-2)/2})$, then  
   $$\mathcal{SQ}(A) \leq \frac{|A|^2}{2} + \frac{q^{d/2}|A|}{2} - \frac{|A|^2}{2q} + \frac{q^{d-2}|A|}{2}.$$  
   If $|A| \leq (q^{d/2} + q)/(1 + q^{-(d-2)/2})$, then  
   $$\mathcal{SQ}(A) \leq \frac{|A|^2}{2} + \frac{q^{d/2}|A|}{2} - \frac{|A|^2}{2q^2} - \frac{|A|}{2}.  

It has been proved in [9] that Corollary 1.3 (2) and (4) are sharp, so the upper bounds of Theorem 1.2 (2) and (4) are also best possible. It follows from Corollary 1.3 that if the set of non-zero distances is fully contained in the group of squares, then the size of $A$ can not be bigger than a certain threshold. After posting this paper to Arxiv, Prof. Igor Shparlinski raised the question of studying the case for an arbitrary subgroup of $\mathbb{F}_q \setminus \{0\}$. We hope to address this question in a sequel paper.
2 Discrete Fourier analysis and preliminary lemmas

In this section, we recall notations from Discrete Fourier analysis and properties. Let \( f \) be a complex valued function on \( \mathbb{F}_q^n \), then its Fourier transform denoted by \( \hat{f} \) is defined by

\[
\hat{f}(m) = q^{-n} \sum_{x \in \mathbb{F}_q^n} \chi(-m \cdot x) f(x),
\]

where \( \chi \) is the principal additive character of \( \mathbb{F}_q \).

With this definition, the Fourier inversion theorem reads as

\[
f(x) = \sum_{m \in \mathbb{F}_q^n} \chi(m \cdot x) \hat{f}(m),
\]

we also have the Plancherel theorem

\[
\sum_{m \in \mathbb{F}_q^n} |\hat{f}(m)|^2 = q^{-n} \sum_{x \in \mathbb{F}_q^n} |f(x)|^2,
\]

which can be proved easily by using the following orthogonal property

\[
\sum_{\alpha \in \mathbb{F}_q^n} \chi(\beta \cdot \alpha) = \begin{cases} 0 & \text{if } \beta \neq (0, \ldots, 0), \\ q^n & \text{if } \beta = (0, \ldots, 0). \end{cases}
\]

In this paper, the quadratic character of \( \mathbb{F}_q \) is denoted by \( \eta \) with a convention that \( \eta(0) = 0 \). For \( a \in \mathbb{F}_q^* \), the Gauss sum \( G_a \) is defined by

\[
G_a = \sum_{s \in \mathbb{F}_q^*} \eta(s) \chi(as),
\]

which can be rewritten as

\[
G_a = \sum_{s \in \mathbb{F}_q} \chi(as^2) = \eta(a)G_1.
\]

It is not hard to see that \( |G_a| = \sqrt{q} \). In the following lemma, we recall the explicit form of the Gauss sum \( G_1 \) from [15, Theorem 5.15].

**Lemma 2.1.** Let \( \mathbb{F}_q \) be a finite field with \( q = p^\ell \), where \( p \) is an odd prime and \( \ell \in \mathbb{N} \). Then we have

\[
G_1 = \begin{cases} (-1)^{\ell-1} q^{\frac{\ell}{2}} & \text{if } p \equiv 1 \mod 4 \\ (-1)^{\ell-1} i^\ell q^{\frac{\ell}{2}} & \text{if } p \equiv 3 \mod 4. \end{cases}
\]

**Corollary 2.2.** Let \( n \) be a positive integer.

1. If \( n \equiv 0 \mod 4 \) and \( q \equiv 3 \mod 4 \), then \( \eta(-1)G_1^n = -q^{\frac{n}{2}} \).
2. If \( n \equiv 2 \mod 4 \), or \( n \equiv 0 \mod 4 \) and \( q \equiv 1 \mod 4 \), then \( \eta(-1)G_1^n = q^{\frac{n}{2}} \).
3. If \( n \equiv 2 \mod 4 \) and \( q \equiv 3 \mod 4 \), then \( G_1^n = -q^{\frac{n}{2}} \).
4. If \( n \equiv 0 \mod 4 \), or \( n \equiv 2 \mod 4 \) and \( q \equiv 1 \mod 4 \), then \( G_1^n = q^{\frac{n}{2}} \).
The following formula, which can be proved by completing the square and using a change of
variables, will be used in our computations below

$$\sum_{s \in \mathbb{F}_q} \chi(as^2 + bs) = \eta(a)G_1\chi \left( \frac{b^2}{4a} \right),$$

where \( a \in \mathbb{F}_q^* \) and \( b \in \mathbb{F}_q \).

Let \( P \in \mathbb{F}_q[x_1, \ldots, x_n] \) be a polynomial, we define a variety \( V_P \) as

\[
V_P := \{ x \in \mathbb{F}_q^n : P(x) = 0 \}.
\]

In the next lemma, we count the number of pairs of points \((x, y)\) in a given set in \( \mathbb{F}_q^n \) such that \( x - y \in V_P \).

**Lemma 2.3.** Let \( E \) be a set in \( \mathbb{F}_q^n \). Then the number of pairs \((x, y)\) \( \in E \times E \) such that \( P(x - y) = 0 \) is equal to

\[
q^{2n} \sum_{m \in \mathbb{F}_q^n} \widehat{V}_P(m)\widehat{E}(m)^2.
\]

**Proof.** Let \( N \) be the number of pairs \((x, y)\) such that \( P(x - y) = 0 \), i.e.

\[
N = \sum_{x, y \in E} V_P(x - y).
\]

By identifying the variety \( V_P \) with the indicator function \( 1_{V_P} \) and using the Fourier inversion formula for the function \( V_P(x - y) \) one has

\[
N = \sum_{x, y \in \mathbb{F}_q^n} E(x)E(y) \sum_{m \in \mathbb{F}_q^n} \widehat{V}_P(m)\chi(m \cdot (x - y)).
\]

Then the lemma follows directly from the orthogonal property. \( \square \)

As we shall see in the proofs of our main results, given \( A \subset \mathbb{F}_q^d \) we shall relate the value \( S(Q(A) + ZR(A)) \) to the Fourier decay of the cone in \( \mathbb{F}_q^{d+1} \). For a positive integer \( n \geq 2 \), we recall that the cone \( C_n \) is defined as

\[
C_n := \{ x \in \mathbb{F}_q^n : x_n^2 = x_1^2 + x_2^2 + \cdots + x_{n-1}^2 \}.
\]

**Definition 2.4.** Let \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{F}_q^n \). We define

\[
\| x \|_{C_n} = x_1^2 + x_2^2 + \cdots + x_{n-1}^2 - x_n^2.
\]

Using the notation \( \| \cdot \|_{C_n} \), the cone \( C_n \) can be written by

\[
C_n = \{ x \in \mathbb{F}_q^n : \| x \|_{C_n} = 0 \}.
\]

The next lemma contains the Fourier transform formula for \( C_n \) which was computed explicitly in \([11] [12] \).
Lemma 2.5. Let \( C_n \) be the cone in \( \mathbb{F}_q^n \) defined in \( \bullet \). Then, for any \( m \in \mathbb{F}_q^n \), we have
\[
\widehat{C}_n(m) = q^{-1} \delta_0(m) + q^{-n-1} \eta(-1) G_1^n \sum_{s \neq 0} \eta^n(s) \chi \left( \frac{|m| c_n}{-4s} \right).
\]

3 Preliminary settings on estimates of \( SQ(A) \) and \( SQ(A) + ZR(A) \)

For efficient estimates, we decompose \( SQ(A) \) as the sum of \( SQ(A) + ZR(A)/2 \) and \(-ZR(A)/2\), and we also consider \( SQ(A) + ZR(A) \) as the sum of \( SQ(A) + ZR(A)/2 \) and \( ZR(A)/2 \). Therefore, we begin by doing delicate estimates on both \( SQ(A) + ZR(A)/2 \) and \( ZR(A)/2 \).

3.1 \( SQ(A) + ZR(A)/2 \) value

Let \( A \subset \mathbb{F}_q^d \). Recall that \( ZR(A) \) and \( SQ(A) \) denote the numbers of pairs \((x, y) \in A \times A\) such that \( ||x - y|| \) is zero and is a square number in \( \mathbb{F}_q^n \), respectively; namely \( ||x - y|| = 0 \) and \( ||x - y|| = r^2 \) for some \( r \in \mathbb{F}_q^n \). Since \( r^2 = (-r)^2 \) for \( r \in \mathbb{F}_q^d \), we can write
\[
SQ(A) = \frac{1}{2} \sum_{r \in \mathbb{F}_q} \sum_{x, y \in A: ||x - y|| = r^2} 1.
\]

Since \( ZR(A) = \sum_{x, y \in A: ||x - y|| = 0} 1 \), we have
\[
SQ(A) + \frac{ZR(A)}{2} = \frac{1}{2} \sum_{r \in \mathbb{F}_q} \sum_{x, y \in A: ||x - y|| = r^2} 1.
\]

Now notice that for each fixed \( r \in \mathbb{F}_q \), there are exactly \( q \) pairs \((s, s') \in \mathbb{F}_q \times \mathbb{F}_q\) such that \( r = s - s' \). This implies that
\[
SQ(A) + \frac{ZR(A)}{2} = \frac{1}{2q} \sum_{s, s' \in \mathbb{F}_q} \sum_{x, y \in A: ||x - y|| = (s - s')^2} 1
\]
\[
= \frac{1}{2q} \sum_{(x, s), (y, s') \in A \times \mathbb{F}_q: ||(x, s) - (y, s')||_{C_{d+1}} = 0} 1.
\]

Put \( E = A \times \mathbb{F}_q \subset \mathbb{F}_q^{d+1} \) and \( \widetilde{x} = (x, s), \widetilde{y} = (y, s') \in \mathbb{F}_q^{d+1} \). It follows
\[
SQ(A) + \frac{ZR(A)}{2} = \frac{1}{2q} \sum_{\widetilde{x}, \widetilde{y} \in E: \widetilde{x} - \widetilde{y} \in C_{d+1}} 1.
\]

We now use Lemma 2.3 with \( n = d + 1 \) to estimate the above summation. Then we see
\[
SQ(A) + \frac{ZR(A)}{2} = \frac{1}{2q} q^{2(d+1)} \sum_{\widetilde{m} \in \mathbb{F}_q^{d+1}} \widehat{C}_{d+1}(\widetilde{m}) |\widehat{E}(\widetilde{m})|^2.
\]

We replace the above value \( \widehat{C}_{d+1}(\widetilde{m}) \) by \( \bullet \) of Lemma 2.3. Then a direct computation gives us
the following:

$$S \mathcal{Q}(A) + \frac{Z \mathcal{R}(A)}{2} = \frac{|E|^2}{2q} + \frac{q^{d-1}\eta(-1)G_1^{d+1}}{2} \sum_{m \in \mathbb{F}_q^d} \sum_{s \neq 0} \eta^{d+1}(s) \chi \left( \frac{\|m\|C_{d+1}}{-4s} \right) |\hat{E}(\tilde{m})|^2.$$  

Notice that $|E| = |A|/q$ and $\hat{\mathbb{F}}_q(t) = 1$ for $t = 0$, and 0 otherwise. For $\tilde{m} = (m, t) \in \mathbb{F}_q^d \times \mathbb{F}_q$, we can check that

$$\hat{E}(\tilde{m}) = \hat{A} \times \hat{\mathbb{F}}_q(m, t) = \hat{A}(m)\hat{\mathbb{F}}_q(t).$$

Therefore, we obtain

$$S \mathcal{Q}(A) + \frac{Z \mathcal{R}(A)}{2} = \frac{|A|^2}{2} + \frac{q^{d-1}\eta(-1)G_1^{d+1}}{2} \sum_{m \in \mathbb{F}_q^d} \sum_{s \neq 0} \eta^{d+1}(s) \chi \left( \frac{\|m\|C_{d+1}}{-4s} \right) |\hat{A}(m)|^2.$$  

By a simple change of variables, $1/(-4s) \to s$, and the properties of $\eta,$

$$S \mathcal{Q}(A) + \frac{Z \mathcal{R}(A)}{2} = \frac{|A|^2}{2} + \frac{q^{d-1}\eta(-1)G_1^{d+1}}{2} \sum_{m \in \mathbb{F}_q^d} \sum_{s \neq 0} \eta^{d+1}(s) \chi (s\|m\|) |\hat{A}(m)|^2.  \quad (7)$$

For better exposition, we will use the following notations.

**Definition 3.1.** Let $A \subset \mathbb{F}_q^d$. We define that

$$\Omega^+(A) = \sum_{m \in \mathbb{F}_q^d : \eta(||m||) = 1} |\hat{A}(m)|^2,$$

$$\Omega^-(A) = \sum_{m \in \mathbb{F}_q^d : \eta(||m||) = -1} |\hat{A}(m)|^2,$$

and

$$\Omega^0(A) = \sum_{m \in \mathbb{F}_q^d : ||m|| = 0} |\hat{A}(m)|^2.$$  

It is clear that $\Omega^+(A), \Omega^-(A),$ and $\Omega^0(A)$ are non-negative real numbers. In addition, notice that if $A$ is a subset of $\mathbb{F}_q^d,$ then

$$\Omega^0(A) \geq |\hat{A}(0, \ldots, 0)|^2 = q^{-2d}|A|^2.  \quad (8)$$

Recall that the sphere $S_0$ in $\mathbb{F}_q^d$ with zero radius, centered at the origin, is defined by

$$S_0 := \{x \in \mathbb{F}_q^d : ||x|| = 0\}.$$  

With this notation and the definition of the Fourier transform, we have

$$\Omega^0(A) = \sum_{m \in S_0} |\hat{A}(m)|^2 = q^{-d} \sum_{x, y \in A} \hat{S}_0(x - y).$$

By the same argument as in the proof of Lemma 2.5, the Fourier transform of $S_0$, denoted by $\hat{S}_0,$
is given as follows (see, for example, [7]): for \( m \in \mathbb{F}_q^d \),

\[
\widehat{S}_0(m) = \frac{\delta_0(m)}{q} + q^{-d-1}\eta^d(-1)G_1^d \sum_{s \neq 0} \eta^d(s) \chi \left( \frac{||m||}{4s} \right).
\] (9)

In particular, when \( d \geq 3 \) is odd, one has

\[
\Omega^0(A) = q^{-d-1}|A| + q^{-2d-1}\eta(-1)G_1^{d+1} \sum_{x,y \in A} \eta(|x - y|).
\]

Since \( \Omega^0(A) \) is a real number and \( |G_1| = q^{1/2} \), we see that if \( d \geq 3 \) is odd, then

\[
\Omega^0(A) \leq q^{-d-1}|A| + q^{-\frac{3d+1}{2}}|A|^2.
\]

One the other hand, the Plancherel theorem gives the following estimate:

\[
\Omega^0(A) \leq \sum_{m \in \mathbb{F}_q^d} |\hat{A}(m)|^2 = q^{-d}|A|.
\]

In conclusion, for any odd \( d \geq 3 \), we get

\[
\Omega^0(A) \leq \min\{q^{-d}|A|, q^{-d-1}|A| + q^{-\frac{3d+1}{2}}|A|^2\}. \tag{10}\]

In odd dimensions \( d \geq 3 \), we obtain the following formula.

**Proposition 3.2.** Let \( A \) be a set in \( \mathbb{F}_q^d \).

1. If \( d \equiv 3 \mod 4 \) and \( q \equiv 3 \mod 4 \), then

\[
\mathcal{S}Q(A) + \frac{\mathcal{Z}R(A)}{2} = \frac{|A|^2}{2} - \frac{q^{d+1}}{2} \Omega^0(A) + \frac{q^{d+1}}{2} |A|.
\]

2. If \( d \equiv 1 \mod 4 \), or \( d \equiv 3 \mod 4 \) and \( q \equiv 1 \mod 4 \), then

\[
\mathcal{S}Q(A) + \frac{\mathcal{Z}R(A)}{2} = \frac{|A|^2}{2} + \frac{q^{d+1}}{2} \Omega^0(A) - \frac{q^{d+1}}{2} |A|.
\]

**Proof.** Since \( d \geq 3 \) is odd, the inequality (7) becomes

\[
\mathcal{S}Q(A) + \frac{\mathcal{Z}R(A)}{2} = \frac{|A|^2}{2} + \frac{q^d\eta(-1)G_1^{d+1}}{2} \sum_{m \in \mathbb{F}_q^d \setminus \{0\}} \chi(s||m||) |\hat{A}(m)|^2.
\]

By the orthogonality of \( \chi \), we compute the sum over \( s \neq 0 \). Then we get

\[
\mathcal{S}Q(A) + \frac{\mathcal{Z}R(A)}{2} = \frac{|A|^2}{2} + \frac{q^d\eta(-1)G_1^{d+1}}{2} \sum_{m \in \mathbb{S}_0} |\hat{A}(m)|^2 - \frac{q^{d-1}\eta(-1)G_1^{d+1}}{2} \sum_{m \in \mathbb{F}_q^d \setminus \mathbb{S}_0} |\hat{A}(m)|^2.
\]

Since \( \sum_{m \in \mathbb{F}_q^d} |\hat{A}(m)|^2 = q^{-d}|A| \), we have

\[
\mathcal{S}Q(A) + \frac{\mathcal{Z}R(A)}{2} = \frac{|A|^2}{2} + \frac{q^d\eta(-1)G_1^{d+1}}{2} \Omega^0(A) - \frac{\eta(-1)G_1^{d+1}|A|}{2q}. \tag{11}\]
To prove the first part (1) of the proposition, it suffices to show that $\eta(-1)G^{d+1}_1 = -q^{d+1}/2$. Since $d \equiv 3 \mod 4$ and $q \equiv 3 \mod 4$, the equation follows from Corollary 2.2 (1) with $n = d + 1$. To prove the second part (2) of the proposition, it is enough to show that $\eta(-1)G^{d+1}_1 = q^{d+1}/2$. Notice that this equation follows immediately from Corollary 2.2 (2) with $n = d + 1$.

In even dimensions $d \geq 2$, we have

**Proposition 3.3.** Let $A$ be a set in $F^d_q$.

1. If $d \geq 2$ is even, then
   \[ SQ(A) + \frac{ZR(A)}{2} = \frac{|A|^2}{2} + \frac{q^{d-1}G^{d+2}_1}{2} \Omega^+(A) - \frac{q^{d-1}G^{d+2}_1}{2} \Omega^-(A). \]

2. In particular, if $d \equiv 2 \mod 4$, then
   \[ SQ(A) + \frac{ZR(A)}{2} = \frac{|A|^2}{2} + \frac{q^{d}G^{d+2}_1}{2} \Omega^+(A) - \frac{q^{3d}G^{d+2}_1}{2} \Omega^-(A). \]

**Proof.** Since $d \geq 2$ even, (1) implies that
   \[ SQ(A) + \frac{ZR(A)}{2} = \frac{|A|^2}{2} + \frac{q^{d-1}G^{d+2}_1}{2} \Omega^+(A) - \frac{q^{d-1}G^{d+2}_1}{2} \Omega^-(A). \]

Note that the sum over $s \neq 0$ is $G_1 \eta(||m||)$. Since $\eta(||m||) = 0$ for $||m|| = 0$, the first statement (1) of the proposition follows. Proposition 3.3 (2) is a direct consequence of Proposition 3.3 (1) since $G_1^{d+2} = q^{(d+2)/2}$ for $d \equiv 2 \mod 4$. \hfill $\square$

### 3.2 $\frac{ZR(A)}{2}$ value

Recall that for $A \subset F^d_q$, we have

\[ ZR(A) = \sum_{x,y \in A: ||x-y||=0} 1. \]

By applying Lemma 2.3 with $n = d$ and $V = S_0$, we have

\[ \frac{ZR(A)}{2} = \frac{q^{2d}}{2} \sum_{m \in F^d_q} \hat{S}_0(m) |\hat{A}(m)|^2. \]

The following equality can be easily obtained by a direct computation after replacing the above value $\hat{S}_0(m)$ by the value in (9).

\[ \frac{ZR(A)}{2} = \frac{|A|^2}{2q} + \frac{q^{d-1} \eta(-1)G^d_1}{2} \sum_{m \in F^d_q} \left( \sum_{s \neq 0} \eta^d(s) \chi(s||m||) \right) |\hat{A}(m)|^2, \tag{12} \]

where we also used a simple change of variables, $1/(4s) \to s$, and the properties of the quadratic character $\eta$ of $F_q$.

In odd dimensions $d \geq 3$, the value $\frac{ZR(A)}{2}$ is written as follows.
Proposition 3.4. Let $A$ be a set in $\mathbb{F}_q^d$.

1. If $d \equiv 3 \pmod{4}$ and $q \equiv 3 \pmod{4}$, then
\[
\frac{Z\mathcal{R}(A)}{2} = \frac{|A|^2}{2q} - \frac{q^{\frac{d+1}{2}}}{2} \Omega^+(A) + \frac{q^{\frac{d-1}{2}}}{2} \Omega^-(A).
\]

2. If $d \equiv 1 \pmod{4}$, or $d \equiv 3 \pmod{4}$ and $q \equiv 1 \pmod{4}$, then
\[
\frac{Z\mathcal{R}(A)}{2} = \frac{|A|^2}{2q} + \frac{q^{\frac{d-1}{2}}}{2} \Omega^+(A) - \frac{q^{\frac{d+1}{2}}}{2} \Omega^-(A).
\]

Proof. Since $d$ is odd, $\eta^d = \eta$ and so the equality (12) becomes
\[
\frac{Z\mathcal{R}(A)}{2} = \frac{|A|^2}{2q} - \frac{q^{d-1} \eta(-1) G_1^d}{2} \sum_{m \in \mathbb{F}_q^d} \left( \sum_{s \neq 0} \eta(s) \chi(s||m||) \right) |\hat{A}(m)|^2
\]
\[
= \frac{|A|^2}{2q} + \frac{q^{d-1} \eta(-1) G_1^{d+1}}{2} \sum_{m \in \mathbb{F}_q^d} \eta(||m||) |\hat{A}(m)|^2.
\]

Since $\eta(||m||) = 0$ for $||m|| = 0$, it follows by the definitions of $\Omega^+(A)$ and $\Omega^-(A)$ that
\[
\frac{Z\mathcal{R}(A)}{2} = \frac{|A|^2}{2q} + \frac{q^{d-1} \eta(-1) G_1^{d+1}}{2} \Omega^+(A) - \frac{q^{d-1} \eta(-1) G_1^{d+1}}{2} \Omega^-(A).
\]

Applying Corollary 2.2 (1) and (2) with $n = d + 1$, we complete the proof. \qed

In even dimensions $d \geq 2$, we have

Proposition 3.5. Let $A$ be a set in $\mathbb{F}_q^d$.

1. If $d \equiv 2 \pmod{4}$ and $q \equiv 3 \pmod{4}$, then
\[
\frac{Z\mathcal{R}(A)}{2} = \frac{|A|^2}{2q} - \frac{q^{\frac{d}{2}}}{2} \Omega^0(A) + \frac{q^{\frac{d+2}{2}} |A|}{2}.
\]

2. If $d \equiv 0 \pmod{4}$, or $d \equiv 2 \pmod{4}$ and $q \equiv 1 \pmod{4}$, then
\[
\frac{Z\mathcal{R}(A)}{2} = \frac{|A|^2}{2q} + \frac{q^{\frac{d}{2}}}{2} \Omega^0(A) - \frac{q^{\frac{d+2}{2}} |A|}{2}.
\]

Proof. Since $d$ is even, $\eta^d \equiv 1$. Hence, the equality (12) and the orthogonality of $\chi$ yield
\[
\frac{Z\mathcal{R}(A)}{2} = \frac{|A|^2}{2q} + \frac{q^{d-1} G_1^d}{2} \sum_{m \in \mathbb{F}_q^d} \left( \sum_{s \neq 0} \chi(s||m||) \right) |\hat{A}(m)|^2
\]
\[
= \frac{|A|^2}{2q} + \frac{q^{d} G_1^d}{2} \Omega^0(A) - \frac{q^{d-1} G_1^d}{2} \sum_{m \in \mathbb{F}_q^d} |\hat{A}(m)|^2.
\]
Notice by the Plancherel theorem that \( \sum_{m \in \mathbb{F}_q} |\hat{A}(m)|^2 = q^{-d}|A| \). Then the proposition follows from the Gauss sum values given in Corollary 2.2 (3) and (4) with \( n = d \).

4 Proofs of results on \( SQ(A) + ZR(A) \) (Theorem 1.2)

To get a good upper bound of \( SQ(A) + ZR(A) \), we will use the previously established estimates of \( SQ(A) + ZR(A)/2 \) and \( ZR(A) \). In other words, we will find an upper bound of the sum of \( (SQ(A) + ZR(A)/2) \) and \( ZR(A)/2 \).

4.1 Proof of Theorem 1.2 (1)

Since we assume that \( d \equiv 3 \pmod{4} \) and \( q \equiv 3 \pmod{4} \), Propositions 3.2 (1) and 3.4 (1) play a key role in proving this theorem.

By inequality (8) and Proposition 3.2 (1), we see that

\[
(SQ(A) + ZR(A)/2) \leq |A|^2 - \frac{|A|^2}{2q} + \frac{q^{d-1}|A|}{2}.
\]

(13)

Ignoring a negative term, Proposition 3.4 (1) implies that

\[
ZR(A)/2 \leq \frac{|A|^2}{2q} + \frac{q^{d-1}|A|}{2} - \Omega^+(A).
\]

One can check that

\[
\Omega^-(A) \leq \sum_{m \in \mathbb{F}_q} |\hat{A}(m)|^2 - \Omega^0(A) \leq q^{-d}|A| - |\hat{A}(0, \ldots, 0)|^2 = q^{-d}|A| - q^{-2d}|A|^2.
\]

Thus, we obtain

\[
ZR(A)/2 \leq \frac{|A|^2}{2q} + \frac{q^{d-1}|A|}{2} - \frac{|A|^2}{2q^{d+1}}.
\]

Since \( SQ(A) + ZR(A) = (SQ(A) + ZR(A)/2) + ZR(A)/2 \), we complete the proof by adding the above two inequalities.

4.2 Proof of Theorem 1.2 (2)

Since \( d \equiv 1 \pmod{4} \) or \( d \equiv 3 \pmod{4} \) and \( q \equiv 1 \pmod{4} \), we are able to use the following result of Proposition 3.2 (2):

\[
SQ(A) + \frac{ZR(A)}{2} = \frac{|A|^2}{2} + \frac{q^{d+1}}{2} \Omega^0(A) - \frac{q^{d+1}|A|}{2}.
\]

Since \( \Omega^0(A) \) is dominated by \( \sum_{m \in \mathbb{F}_q} |\hat{A}(m)|^2 - \Omega^+(A) = q^{-d}|A| - \Omega^+(A) \), it follows that

\[
SQ(A) + \frac{ZR(A)}{2} \leq \frac{|A|^2}{2} + \frac{q^{d+1}}{2} (q^{-d}|A| - \Omega^+(A)) - \frac{q^{d+1}|A|}{2}.
\]
Proposition 3.4 (2) implies that
\[ \frac{ZR(A)}{2} \leq \frac{|A|^2}{2q} + \frac{q^{d-1}}{2} \Omega^+(A). \]

Adding the above two estimates, we have
\[ SQ(A) + ZR(A) \leq \frac{|A|^2}{2} + \frac{|A|^2}{2q} + \frac{q^{d+1}}{2} |A| - \frac{q^{d-1}}{2} |A| - \left( \frac{q^{d+1}}{2} - \frac{q^{d-1}}{2} \right) \Omega^+(A). \]

Since the term containing \( \Omega^+(A) \) value is negative or zero, the theorem follows.

4.3 Proof of Theorem 1.2 (3)
Since \( d \equiv 2 \mod 4 \) and \( q \equiv 3 \mod 4 \), the proof can use Propositions 3.3 (2) and 3.5 (1).

Since \(-\Omega^-(A) \leq 0\), Proposition 3.3 (2) implies that
\[ SQ(A) + ZR(A) \leq \frac{|A|^2}{2} + \frac{|A|^2}{2q} + \frac{q^{d+1}}{2} \Omega^0(A) + (A) - \frac{q^{d-1}}{2} |-A| - \Omega^0(A)). \]

Combining this estimate with the fact that \( \Omega^+(A) \leq q^{-d}|A| - \Omega^0(A) \), we obtain
\[ SQ(A) + ZR(A) \leq \frac{|A|^2}{2} + \frac{q^{d+1}}{2} \Omega^0(A) + (A) - \frac{q^{d-1}}{2} |-A| - \Omega^0(A)). \]

By Proposition 3.5 (1), we have
\[ \frac{ZR(A)}{2} = \frac{|A|^2}{2q} - \frac{q^{d-1}}{2} \Omega^0(A) + \frac{q^{d+1}}{2} |A|. \]

By adding the above two inequalities, we have
\[ SQ(A) + ZR(A) \leq \frac{|A|^2}{2} + \frac{|A|^2}{2q} - \frac{q^{d+1}}{2} \Omega^0(A) + \frac{q^{d-1}}{2} |A| + \frac{q^{d+1}}{2} |A|. \]

Since \( -\Omega^0(A) \leq -|\hat{A}(0, \ldots, 0)|^2 = -q^{-2d}|A|^2 \), we obtain the statement of the theorem.

4.4 Proof of Theorem 1.2 (4)
Since \( d \equiv 0 \mod 4 \), or \( d \equiv 2 \mod 4 \) and \( q \equiv 1 \mod 4 \), we may invoke Proposition 3.3 (1) and 3.5 (2).

First, Proposition 3.3 (1) tells us that for even integer \( d \geq 2 \),
\[ SQ(A) + ZR(A) = \frac{|A|^2}{2} + \frac{q^{d-1}}{2} \Omega^+(A) - \frac{q^{d-1}}{2} \Omega^-(A) := \frac{|A|^2}{2} + I + II. \]

Since \( d \) is even, Lemma 2.1 implies that \( G^{d+2} = \pm q^{(d+2)/2} \) which is a real number. Moreover, both \( \Omega^+(A) \) and \( \Omega^-(A) \) are non-negative real numbers. Hence, one of \( I \) and \( II \) is exactly non-negative.
and the other is non-positive. Therefore, in the case when \( G_1^{d+2} = q^{(d+2)/2} \), we have
\[
SQ(A) + \frac{Z R(A)}{2} \leq \frac{|A|^2}{2} + I = \frac{|A|^2}{2} + \frac{q^{d/2}}{2} \Omega^+(A).
\] (15)

On the other hand, when \( G_1^{d+2} = -q^{(d+2)/2} \), we have
\[
SQ(A) + \frac{Z R(A)}{2} \leq \frac{|A|^2}{2} + II = \frac{|A|^2}{2} + \frac{q^{d/2}}{2} \Omega^-(A).
\] (16)

In any case, we have
\[
SQ(A) + \frac{Z R(A)}{2} \leq \frac{|A|^2}{2} + \frac{q^{d/2}}{2} (q^{-d}|A| - \Omega^0(A)),
\]
since \( \max\{\Omega^+(A), \Omega^-(A)\} \leq \sum_{m \in \mathbb{F}_q^d} |\hat{A}(m)|^2 - \Omega^0(A) = q^{-d}|A| - \Omega^0(A) \).

By Proposition 3.5 (2), we have
\[
\frac{Z R(A)}{2} = \frac{|A|^2}{2q} + \frac{q^{d/2}}{2} |\Omega^0(A)| - \frac{q^{d/2}|A|}{2}.
\]

Hence, adding the above two estimates, we obtain the required upper bound of \( SQ(A) + Z R(A) \). \( \square \)

5 Proofs of results on \( SQ(A) \) in odd dimensions (Theorem 1.4)

5.1 Proof of Theorem 1.4 (1)

Since the assumption that \( d \equiv 3 \mod 4 \) and \( q \equiv 3 \mod 4 \) is the same as that of Theorem 1.2 (1), we will make use certain estimates given in the proof of Theorem 1.2 (1).

As in (13), we have
\[
(SQ(A) + \frac{Z R(A)}{2}) \leq \frac{|A|^2}{2} - \frac{|A|^2}{2q^{d/2}} + \frac{q^{d/2}}{2} |A|. \leq \frac{14}{2}
\] (17)

Since \( \Omega^-(A) \) is non-negative, Proposition 3.4 (1) implies that
\[
-\frac{Z R(A)}{2} \leq -\frac{|A|^2}{2q} + \frac{q^{d/2}}{2} \Omega^+(A).
\]

As before, we can check that \( \Omega^+(A) \) is bounded by \( q^{-d}|A| - q^{-2d}|A|^2 \). Thus, we have
\[
-\frac{Z R(A)}{2} \leq -\frac{|A|^2}{2q} + \frac{q^{d/2}}{2} (q^{-d}|A| - q^{-2d}|A|^2) = -\frac{|A|^2}{2q} + \frac{q^{d/2}}{2} |A| - \frac{|A|^2}{2q^{d/4}}.
\] (18)

By the definition of \( Z R(A) \), it is clear that \( Z R(A) \geq |A| \). Hence, we also have
\[
-\frac{Z R(A)}{2} \leq -|A|/2.
\] (19)

Since we can write \( SQ(A) = (SQ(A) + Z R(A)/2) - Z R(A)/2 \), the estimates (17) and (18) imply
that
\[ SQ(A) \leq \frac{|A|^2}{2} + q^{\frac{d-1}{2}}|A| - \frac{|A|^2}{2q} - \frac{|A|^2}{2q^{\frac{d-1}{2}}}. \]

Combining (17) and (19), we also obtain that
\[ SQ(A) \leq \frac{|A|^2}{2} + q^{\frac{d-1}{2}}|A| - \frac{|A|^2}{2q} - \frac{|A|^2}{2q^{\frac{d-1}{2}}}. \]

Comparing the above two estimates of \( SQ(A) \), we obtain the statement of Theorem 1.4 (1).

5.2 Proof of Theorem 1.4 (2)

Since \( d \equiv 1 \mod 4 \), or \( d \equiv 3 \mod 4 \) and \( q \equiv 1 \mod 4 \), we will make use of Propositions 3.2 (2) and 3.4 (2).

Combining inequality (10) and Proposition 3.2 (2), we see that
\[ SQ(A) + ZR(A) \leq \frac{|A|^2}{2} + q^{\frac{d+1}{2}}|A| - \frac{|A|^2}{2q} + \frac{q^{\frac{d+1}{2}}|A|^2}{2} - \frac{q^{d-1}|A|^2}{2}. \]  

As seen in (19), we have
\[ -ZR(A)/2 \leq -|A|/2. \]

Hence, the above two estimates imply that
\[ SQ(A) \leq \frac{|A|^2}{2} - \frac{q^{\frac{d-1}{2}}|A|}{2} - \frac{|A|^2}{2} + \min\left\{ \frac{q^{\frac{d+1}{2}}|A|^2}{2}, \frac{q^{\frac{d-1}{2}}|A|}{2} + \frac{|A|^2}{2} \right\}. \]

Next, we deduce another upper bound of \( SQ(A) \). Since \( \Omega^+(A) = \sum_{m \in \mathbb{F}_q^d} |\hat{A}(m)|^2 - \Omega^+(A) - \Omega^-(A) = q^{-d}|A| - \Omega^+(A) - \Omega^-(A) \leq q^{-d}|A| - \Omega^-(A) \), it follows by Proposition 3.2 (2) that
\[ SQ(A) + ZR(A) \leq \frac{|A|^2}{2} + \frac{q^{\frac{d+1}{2}}}{2}|A| - \frac{q^{\frac{d-1}{2}}|A|^2}{2}. \]

Since \( -\Omega^+(A) \leq 0 \), Proposition 3.4 (2) implies that
\[ -ZR(A)/2 \leq -|A|^2/2q + \frac{q^{\frac{d-1}{2}}}{2} \Omega^-(A). \]

Since \( -\Omega^-(A) \leq 0 \), adding the above two estimates gives us
\[ SQ(A) \leq \frac{|A|^2}{2} + \frac{q^{\frac{d+1}{2}}|A|}{2} - \frac{q^{\frac{d-1}{2}}|A|}{2} - \frac{|A|^2}{2q}. \]

Combining this estimate with inequality (20), we complete the proof of Theorem 1.4 (2).
6 Proofs of results on $SQ(A)$ in even dimensions (Theorem 1.5)

We begin by proving an upper bound of $SQ(A)$ for even $d \geq 2$. We will use some certain estimates appearing in the proof of Theorem 1.2 (4).

**Lemma 6.1.** Let $A$ be a subset of $\mathbb{F}_q^d$. If $d \geq 2$ is even, then we have

$$SQ(A) \leq \frac{|A|^2}{2} + \frac{q \frac{d}{2} |A| - |A|}{2q^\frac{d}{2}} - \frac{|A|}{2^2}.$$  

**Proof.** As observed in the proof of Theorem 1.2 (4), either the inequality (15) or the inequality (16) happens. Without loss of generality, we can assume that the inequality (15) holds. By the same method, we can deal with the other case. As seen before, we have

$$\Omega^+(A) \leq \sum_{m \in \mathbb{F}_q^d} |\hat{A}(m)|^2 = \frac{|A|^2}{2} + \frac{q \frac{d}{2} |A| - |A|}{2q^\frac{d}{2}} - \frac{|A|}{2^2}.$$  

(22)

Since $ZR(A) \geq |A|$ by the definition of $ZR(A)$, we have

$$-\frac{ZR(A)}{2} \leq -\frac{|A|}{2}.$$  

The lemma follows by adding the above two estimates. \qed

6.1 Proof of Theorem 1.5 (1)

Since $d \equiv 2 \mod 4$ and $q \equiv 3 \mod 4$, we can also use certain estimates which were already established in the proof of Theorem 1.2 (3). As seen in (14), we have

$$SQ(A) + \frac{ZR(A)}{2} \leq \frac{|A|^2}{2} + \frac{q \frac{d}{2} (q^{-d}|A| - q^{-2d}|A|^2)}{2} = \frac{|A|^2}{2} + \frac{q \frac{d}{2} |A|}{2q^\frac{d}{2}}.$$  

Using the fact that $\Omega^+(A) \leq q^{-d}|A| - \Omega^0A$, we obtain

$$SQ(A) + \frac{ZR(A)}{2} \leq \frac{|A|^2}{2} + \frac{q \frac{d}{2} (q^{-d}|A| - \Omega^0A)}{2}.$$  

By Proposition 3.5 (1), it follows that

$$-\frac{ZR(A)}{2} = -\frac{|A|^2}{2q} + \frac{q \frac{d}{2} \Omega^0(A) - q^\frac{d-2}{2} |A|}{2}.$$  

Adding the above two estimates gives the statement of the theorem. \qed

6.2 Proof of Theorem 1.5 (2)

Since $d \equiv 0 \mod 4$, or $d \equiv 2 \mod 4$ and $q \equiv 1 \mod 4$, we will use the estimate (22) and Proposition 3.5 (2).
Since \( d \) is even, it follows by (22) that
\[
\mathcal{S}\mathcal{Q}(A) + \frac{\mathcal{Z}\mathcal{R}(A)}{2} \leq \frac{|A|^2}{2} + \frac{q^d}{2}\frac{|A|}{2q^\frac{d}{2}}.
\] (23)

Since \(-\Omega^0(A) \leq -|\hat{A}(0, \ldots, 0)|^2 = -q^{-2d}|A|^2\), Proposition 3.5 (2) implies that
\[
-\frac{\mathcal{Z}\mathcal{R}(A)}{2} \leq -\frac{|A|^2}{2q} - \frac{|A|^2}{2q^\frac{d+2}{2}}\frac{|A|}{2q^\frac{d+2}{2}}.
\]

Thus, adding the above two estimates gives us an upper bound of \( \mathcal{S}\mathcal{Q}(A) \) as follows:
\[
\mathcal{S}\mathcal{Q}(A) \leq \frac{|A|^2}{2} + \frac{q^d}{2}\frac{|A|}{2q^\frac{d}{2}} - \frac{|A|^2}{2q^\frac{d+2}{2}}\frac{|A|}{2q^\frac{d+2}{2}}.
\]

Lemma 6.1 also gives the following upper bound of \( \mathcal{S}\mathcal{Q}(A) \):
\[
\mathcal{S}\mathcal{Q}(A) \leq \frac{|A|^2}{2} + \frac{q^d}{2}\frac{|A|}{2} - \frac{|A|^2}{2q^\frac{d+2}{2}}\frac{|A|}{2q^\frac{d+2}{2}}.
\]

The statement of the theorem follows by a direct comparison with two upper bounds of \( \mathcal{S}\mathcal{Q}(A) \). □

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