On Algebraic Properties of Sets of Banach Ideal Function Spaces

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Dedicated to the memory of S. Banach.

Abstract. It is shown that a set \( \mathcal{J}(\mu) \) of Banach lattices of real-valued measurable functions, defined on a measure space \((\Omega, \Sigma, \mu)\), may be equipped with a some natural ordering under which it becomes a distributive lattice, which is Dedekind complete provided \( \mu \) is a probability measure. Moreover, some natural operations on considered spaces are in Galois connexion. These results are of most interest for symmetric Banach spaces.

1. Introduction

This paper is devoted to study algebraic properties of a set \( \mathcal{J}(\mu) \) of Banach ideal spaces of real valued \( \mu \)-measurable functions. Namely, it will be shown that a quite natural ordering "\( \subset^1 \)" on \( \mathcal{J}(\mu) \) makes this set to be a lattice; some restrictions on spaces from \( \mathcal{J}(\mu) \) mark out sublattices of \( \langle \mathcal{J}(\mu), \subset^1 \rangle \) having nice algebraic properties. Compositions of some natural operations on Banach ideal spaces (such as either the operation of conversion of a given space \( E \) into its dual \( E' \) or the operation to pick out all elements of \( E \) having an absolutely continuous norm to generate a new space \( E_0 \)) may be chosen in a such way that they will be in the Galois connexion.

Section 2 is devoted to recall some definitions and notations that touch on Banach ideal spaces. The commonly used terminology is widely changed from one paper to another. Below mainly will be used the terminology of reviews [1] and [2]. For all results that are mentioned below without proofs the reader refers to these reviews.

Original results are contained in sections 3, 4 and 5.

2. Definitions

Let \((\Omega, \Sigma, \mu)\) be a measure space, i.e., an abstract set \( \Omega \) with a \( \sigma \)-algebra \( \Sigma \) of its subsets and a countably additive function (measure) \( \mu \), defined on \( \Sigma \) with the range in \( \mathbb{R}_+ \).

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Let \( L_0(\mu) = L_0(\Omega, \Sigma, \mu) \) be the set of (classes of) \( \mu \)-measurable real valued functions, defined on \( \Omega \). Certainly, \( L_0(\mu) \) is a vector space under usual operations of addition of functions and multiplication by a scalar. \( L_0(\mu) \) is also a lattice under a natural partial order \( (x(\omega) \leq y(\omega) \) means that \( x(\omega) \leq y(\omega) \) a.e.).

An ideal Banach function space (shortly: BIS) \( E(\mu) \) is a vector subspace of \( L_0(\mu) \), which is equipped with a Banach norm \( \| \cdot \|_E \) (i.e. \( E(\mu) \) is complete in the norm topology), which is monotone, i.e., such that

\[
y(\omega) \in E(\mu), \ x(\omega) \in L_0(\mu) \text{ and } |x(\omega)| \leq |y(\omega)| \implies \|x\|_E \leq \|y\|_E.
\]

Classical examples of BIS are Lebesgue-Riesz spaces \( L_p(\mu) \), \( 1 \leq p \leq \infty \). However there are examples of such measure space \( (\Omega, \Sigma, \mu) \) (which does not have the direct sum property; definitions see below) that the space \( L_p(\mu) \) is not complete (i.e., is not a Banach space). Certainly, it may be completed (by the usual procedure of completion) and the resulting Banach space will be of kind \( L_0(\mu') \) as well. However the measure space \( (\Omega', \Sigma', \mu') \), where the complete space \( \overline{L_p(\mu)} = L_p(\mu') \) will be defined differs from the initial one.

So, it is necessary to put some restrictions on the measure space \( (\Omega, \Sigma, \mu) \).

**Definition 1.** A measure space \( (\Omega, \Sigma, \mu) \) is said to be admissible if it satisfies the following conditions:

- If \( A \in \Sigma; \mu(A) = 0 \) and \( B \subset A \) then \( B \in \Sigma \) and \( \mu(B) = 0 \) (the measure \( \mu \) is complete).
- If \( A \subset \Omega \) and every \( B \in \Sigma \) with \( \mu(B) < \infty \) is so that \( A \cap B \in \Sigma \), then \( A \in \Sigma \).
- If \( A \in \Sigma \) and \( \mu(A) = \infty \) then there exists \( B \subset A, B \in \Sigma \) such that \( \mu(B) < \infty \) (the measure \( \mu \) is semifinite).
- There exists a set \( \{A_i : i \in I\} \) of pairwise disjoint subsets of \( \Omega \) with \( \mu(A_i) < \infty \) for every \( i \in I \) so that
  - Every \( B \in \Sigma \) of finite measure \( \mu(B) < \infty \) may be represented as
  \[
  B = \cup \{B \cap A_i : i \in I_0\} \cup A_0,
  \]
  where \( I_0 \) is a countable subset of \( I \) and \( \mu(A_0) = 0 \) (the measure \( \mu \) has the direct sum property).

It is known (see e.g. [1] and [2]) that for every admissible measure space \( (\Omega, \Sigma, \mu) \) each BIS \( E(\mu) \) is conditionally Dedekind complete, and a set of all integral functionals on \( E(\mu) \) is total over it.

For any \( A \in \Sigma \) the triple \((A, \Sigma_A, \mu_A)\), where \( \Sigma_A = \{B \cap A : B \in \Sigma\} \) and \( \mu_A(B) = \mu(B \cap A) \) is a restriction of \( \mu \) is an admissible set provided \((\Omega, \Sigma, \mu)\) is admissible.

It will be said that BIS \( E(\mu) \) is of maximal width in \((\Omega, \Sigma, \mu)\) if

\[
\{z \in S(\mu) : zy = 0 \text{ for all } y \in E(\mu)\} = \emptyset.
\]

**Definition 2.** Let \((\Omega, \Sigma, \mu)\) be an admissible measure space. A set \( J(\mu) \) is the set of all BIS \( E(\mu) \) that are of maximal width in \((\Omega, \Sigma, \mu)\).

So, \( E(\mu_A) \in J(\mu) \) if and only if \( \mu(\Omega \setminus A) = 0 \).

Let \( I = \langle I, \ll \rangle \) be a partially ordered set. It will be said that \( I \) is a directed set if for any \( i_1, i_2 \in I \) there exists \( i \in I \) such that \( i_1 \ll i \) and \( i_2 \ll i \).
A sequence \( \{x_i : i \in I\} \), which is indexed by elements of the directed set \( I = (I, \preceq) \) will be called a net. It will be written \( x_i \downarrow \) if \( i \preceq j \) implies \( x_i \geq x_j \). If \( x_i \downarrow \) and \( \inf_{i \in I} (x_i) = x_0 \), we shall write \( x_i \downarrow x_0 \).

Let \( E = E(\mu) \) be a BIS; \( x \in E \). It will be said that the norm of \( x \) is order continuous (shortly: (o)-continuous) provided the condition \( |x| \geq x_i \downarrow 0 \) implies that \( \lim_t \|x_t\|_E = 0 \).

The set of all elements of \( E \) having the (o)-continuous norm is denoted by \( E_0 \).

Certainly, \( E_0 \) is a closed Banach subspace of \( E \) and is an ideal in \( E \):

\[
\text{if } x \in E_0; \ y \in E \text{ and } |y| \leq |x| \text{ then } y \in E_0.
\]

Notice that \( E_0 \) need not to be of maximal width in \((\Omega, \Sigma, \mu)\); moreover it may be trivial. E.g., \((L_\infty [0, 1])_0 = \{0\}\).

Recall that a subset \( F \) of a BIS \( E \) is said to be a foundation in \( E \) if it is an ideal in \( E \) and is of maximal width in \((\Omega, \Sigma, \mu)\).

Let \( E(\mu) \in J(\mu) \). A dual space \( E' \) is the space of all elements \( f(t) \in L_0(\mu) \) such that

\[
\|f\|_{E'} = \sup \{ \int_\Omega f(t)x(t)\,d\mu : \|x\|_E = 1 \} < \infty.
\]

\( E' \) may be identified with a subset of the conjugate space \( E^* \): every element \( f \in E' \) generates the (integral) functional \( f \in E^* \) by the rule:

\[
\langle f, x \rangle = \int_\Omega f(t)x(t)\,d\mu.
\]

Certainly, \( E' \) is a Banach space under the norm \( \|\cdot\|_{E'} \) and is a BIS (of maximal width), which belongs to \( J(\mu) \).

**Remark 1.** There may be situations when \( E_0 \) is a foundation in \( E \) but \( (E')_0 \) is not a foundation in \( E' \). E.g., \( E = E_0 = L_1[0, 1]; E' = L_\infty[0, 1] \) and \( (E')_0 = \{0\} \).

The paper [3] contains an example of such BIS \( E \) that \( E_0 = (E')_0 = \{0\} \).

**Definition 3.** Let \((\Omega, \Sigma, \mu)\) be an admissible measure space. A set \( J_0(\mu) \) is the set of all Banach ideal spaces \( E = E(\mu) \) such that \( E_0 = E_0(\mu) \) is a foundation in \( E(\mu) \) and \( (E')_0 \) is a foundation in \( E' \).

Let \((\Omega, \Sigma, \mu)\) be an admissible measure space.

Let \( E \subset F \) be BIS. Define an operator \( i(E, F) : E \to F \), which asserts to any \( x \in E \) the same function \( x \in F \). This operator is called the inclusion operator. Its norm (that is the infimum of all possible constants \( c(E, F) \)) is called the inclusion constant.

The relation \( E \subset^1 F \) means that the inclusion constant \( c(E, F) = 1 \).

The class \( J(\mu) \) is partially ordered by the relation \( E \subset^1 F \)

**Definition 4.** \( \langle J(\mu), \subset^1 \rangle \) is a partially ordered set, in which two BIS \( E \) and \( F \) are identified if and only if \( E \subset^1 F \) and \( F \subset^1 E \). This means that \( E \) and \( F \) are identical as sets, as vector lattices, as topological vector spaces and as Banach spaces.

On \( J(\mu) \) the more general relation \( E \subset^c F \) may be defined. It means that the inclusion constant \( c(E, F) \) is bounded. This relation partially orders the quotient set \( J^\approx(\mu) = J(\mu)/\approx \), where the equivalence relation \( E \approx F \) means that \( E \subset^c F \) and \( F \subset^c E \).

Below it will be regarded only the partially ordered set \( \langle J(\mu), \subset^1 \rangle \). The study of the set \( \langle J^\approx(\mu), \subset^c \rangle \) is reserved to readers.
Remark 2. It is worthwhile to note that the definition of relations \( \subset^1 \) and \( \subset^c \) in a general case has some defects (which are eliminated while symmetric Banach spaces will be considered).

E.g., spaces \( E = L_p[0,1/2] \oplus L_q[1/2,1] \) and \( F = L_q[0,1/2] \oplus L_p[1/2,1] \) for \( p \neq q \) must be considered as different. Indeed, they are not compatible neither in the sense of \( \subset^1 \) nor in the sense of \( \subset^c \).

The following summary assembles all results about the order \( \subset^1 \) and operations \( E \hookrightarrow E_0; \quad E \hookrightarrow E' \) that will be needed later.

Summary 1. Let \((\Omega, \Sigma, \mu)\) be an admissible set; \( \mathcal{J}(\mu) \) - the corresponding set of Banach ideal spaces of \( \mu \)-measurable functions defined on \( \Omega \) that are of maximal width. Let \( E, F \in \mathcal{J}(\mu) \). Then

1. \( E_0 \subset^1 E \);
2. \( E \subset^1 F \) implies that \( E_0 \subset^1 F_0 \);
3. \( (E_0)_0 = E_0 \);
4. \( E \subset^1 E'' \);
5. \( E' = E''' \);
6. If \( E \subset^1 F \) then \( F' \subset^1 E' \).

3. \( \langle \mathcal{J}(\mu), \subset^1 \rangle \) as a lattice

Consider an admissible measure space \((\Omega, \Sigma, \mu)\) and the corresponding set \( \mathcal{J}(\mu) \) equipped with the partial order \( \subset^1 \).

Let \( E, F \in \mathcal{J}(\mu) \). According to [4] define a pair of BIS: \( E \cap F \) and \( E + F \).

\( E \cap F \) consists of all functions \( x \), that are common to \( E \) and \( F \): \( f \in E \cap F \) and is equipped with the norm

\[
\|f\|_{E \cap F} = \max\{\|f\|_E, \|f\|_F\}.
\]

\( E + F \) is formed by functions of kind \( f = u + v, \ u \in E; \ v \in F \), such that

\[
\|f\|_{E + F} = \inf\{\|u\|_E + \|v\|_F : u + v = f\} < \infty;
\]

Define on \( \langle \mathcal{J}(\mu), \subset^1 \rangle \) binary operations, \( \vee \) and \( \wedge \). Namely, put

\[
E \vee F := E + F; \quad E \wedge F := E \cap F;
\]

Theorem 1. \( \langle \mathcal{J}(\mu), \vee, \wedge \rangle \) is a lattice.

Proof. Recall that an algebraic structure \( A \), endowed with a pair \( \vee, \wedge \) of binary mappings

\[
\vee : A^2 \to A; \quad \wedge : A^2 \to A
\]

is a lattice, if for any \( a, b, c \in A \)

\[
a \vee b = b \vee a; \quad a \wedge b = b \wedge a
\]

\[
a \vee (b \vee c) = (a \vee b) \vee c; \quad a \wedge (b \wedge c) = (a \wedge b) \wedge c
\]

\[
a \vee (a \wedge b) = a \wedge (a \vee b) = a.
\]

It is an easy exercise to verify lattice axioms for \( \langle \mathcal{J}(\mu), \vee, \wedge \rangle \). \( \square \)

Remark 3. In a general case \( \langle \mathcal{J}(\mu), \subset^1 \rangle \) is not Dedekind complete.

Indeed, consider a BIS \( E \) and a sequence \( \langle E, \|\cdot\|_n \rangle_{n \in \mathbb{N}} \), where \( \|x\|_n = n \|x\| \) for all \( n \in \mathbb{N} \). Certainly, such a sequence does not have any upper bound.
This difficulty may be overcome if $\mu$ is a probability measure, say, $\mathbb{P}$, which does not contain atoms of positive measure.

Let $\mathcal{J}(\mathbb{P})$ be a set of all BIS that satisfies the norming condition $\|\chi_\Omega(t)\| = 1$, where $\chi_A(t)$ is the indicator function of $A \in \Sigma$:

$$\chi_A(t) = 1 \text{ for } t \in A; \quad \chi_A(t) = 0 \text{ for } t \notin A$$

and are of maximal width on the probability space. The following theorem is valid.

**Theorem 2.** The lattice $\langle \mathcal{J}(\mathbb{P}), \lor, \land \rangle$ is Dedekind complete.

**Proof.** Indeed, if $E(\mathbb{P})$ satisfies the norming conditions then

$$L_\infty(\mathbb{P}) \subset \overline{E}(\mathbb{P}) \subset L_1(\mathbb{P}).$$

The greatest lower bound of a family $\{E_i\}_{i \in I}$ of BIS that satisfy norming conditions is the space $\oplus E_i$, which consists of all elements that are common to all $E_i$'s. Its norm is given by

$$\|x\|_{\oplus E_i} = \sup\{\|x\|_{E_i} : i \in I\}.$$ 

Since every set of BIS that satisfy the norming condition is bounded (by $L_1(\mathbb{P})$), the least upper bound of $\{E_i\}_{i \in I}$ may be obtained as the greatest lower bound of a family of all upper bounds of $\{E_i\}_{i \in I}$.

Another way is to define (according to [4]) a space $\mathbb{P}E_i$ as an ideal in $L_0(\mathbb{P})$ that consists of all $x \in L_0(\mathbb{P})$, which has the representation of kind

$$x = \sum_{i \in I} u_i \text{ where } u_i \in E_i; \quad \sum_{i \in I} \|u_i\|_{E_i} < \infty.$$ 

It is equipped with the norm

$$\|x\|_{\mathbb{P}E_i} = \inf\{\sum_{i \in I} \|u_i\|_{E_i} : x = \sum_{i \in I} u_i; \quad u_i \in E_i\}.$$ 

From results of [4] it follows that $\sup_{i \in I} (E_i) = \mathbb{P}E_i$. □

Now return to an arbitrary admissible measure space $(\Omega, \Sigma, \mu)$ and consider the lattice $\langle \mathcal{J}(\mu), \lor, \land \rangle$.

Let $E, F \in \mathcal{J}(\mu)$; $E \subset \subset F$. An order interval $[E, F]_{\subset \subset}$ is given by

$$[E, F]_{\subset \subset} := \{H \in \mathcal{J}(\mu) : E \subset \subset H \subset \subset F\}.$$ 

It will be denoted by $[E, F]$ for simplicity. For any order interval $[E, F]$ there may be defined mappings $\lambda = \lambda_{E,F}$ and $\rho = \rho_{E,F}$ as follows. Put for $H \in \mathcal{J}(\mu)$

$$\lambda_{E,F}(H) = (H \cap E) + F; \quad \rho_{E,F}(H) = (H + F) \cap E.$$ 

The following result is obvious.

**Proposition 1.** For any $W \in \mathcal{J}(\mu)$

$$\lambda_{E,F}(W) \in [E, F]; \quad \rho_{E,F}(W) \in [E, F];$$

Then for any $H \in [E, F]$

$$\lambda(H) = H; \quad \rho(H) = H.$$ 

So $\lambda_{E,F}$ and $\rho_{E,F}$ may be regarded as projections of $\mathcal{J}(\mu)$ on the interval $[E, F]$. It is clear that $\lambda^2 = \rho \lambda = \lambda; \quad \rho^2 = \lambda \rho = \rho$.

Recall that lattices, which have the property $\lambda = \rho$, are said to be modular.

**Theorem 3.** The lattice $\langle \mathcal{J}(\mu), \lor, \land \rangle$ is modular.
Proof. From \((H \cap E) \subset F \subset H\) it follows that for all \(H \in \mathcal{J}(\mu)\)
\[
(H \cap E) + F \subset (H + F) \cap E,
\]
i.e., \(\lambda(H) \subset F\) for all \(H \in \mathcal{J}(\mu)\).

Let \(x \in (H + F) \cap E\). Recall that \([E, F]\) is an order segment, i.e. \(E \subset F\). Then \(x \in H + F\) and, hence, \(x = u + v\), where \(v \in H\) and \(u \in F\). Its norm \(\|x\|_{H + F} = \inf\{\|v\|_H + \|u\|_F\}\). Hence, the norm \(\|x\|^1\) of \(x\), which is regarded as an element of \((H + F) \cap E\), is equal to
\[
\|x\|^1 = \max\{\inf\{\|v\|_H + \|u\|_F : v + u = x\} : \|v + u\|_E\}. 
\]
The norm \(\|x\|^2\) of \(x\), when \(x\) is regarded as element of \((H \cap E) + F\), is equal to
\[
\|x\|^2 = \inf\{\|u\|_F, \max\{\|v\|_H, \|v\|_E\} : u + v = x\}. 
\]
Clearly,
\[
\|x\|^1 = \max\{\inf\{\|v\|_H + \|u\|_F : \|x\|_E = v + u\} : \|v + u\|_E\} 
\geq \max\{\inf\{\|v\|_H + \|u\|_F : \|v\|_E = v + u\} : \|v + u\|_E\} 
\geq \inf\{\|u\|_F, \max\{\|v\|_H, \|v\|_E\} : u + v = x\} = \|x\|^2.
\]
Hence, \((H + F) \cap E \subset (H \cap E) + F\) and, consequently, \(\lambda_{E,F} = \rho_{E,F}\) for any interval \([E, F]\). \(\square\)

From the property of \(\langle \mathcal{J}(\mu), \vee, \wedge \rangle\) to be modular follows the first theorem of uniqueness.

Theorem 4. Let \(E, F \in \mathcal{J}(\mu); E \subset F\). If there exists such \(G \in \mathcal{J}(\mu)\) that \(G \cap E = G \cap F\) and \(G + E = G + F\) then \(E = F\).

Proof. Let \(G \cap E = G \cap F = X\) and \(G + E = G + F = Y\). Then \(X \subset F \subset Y\) and \((G \cap F) + E = E \subset (G + E) \cap F = F\). If \(E \neq F\) then \(\mathcal{J}(\mu)\) is not modular. \(\square\)

In fact, the more powerful result is true.

Recall that a lattice \(A\) is said to be distributive if for any \(a, b, c \in A\) the following equalities hold:
\[
a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) ; \quad a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) .
\]
It is well known that these equalities are not independent; any of them is a consequence of other one. Both of them are equivalent to the inequality: for any \(a, b, c \in A\)
\[
(a \vee b) \wedge c \leq a \vee (b \wedge c)
\]
(a \leq b means that \(a \wedge b = a\)). From the last inequality it follows that every distributive lattice is modular.

Theorem 5. The lattice \(\langle \mathcal{J}(\mu), \vee, \wedge \rangle\) is distributive.

Proof. It is sufficient to show that for any \(E, F, G \in \mathcal{J}(\mu)\) the following inequality holds:
\[
(E + F) \cap G \subset (E + G) \cap (F + G) .
\]
Let \(w \in (E + F) \cap G\). Its norm is
\[
\|w\|^1 = \max\{\inf\{\|u\|_E + \|v\|_F : u + v = w\}, \|w\|_G\} 
= \max\{\|w\|_{E + F}, \|w\|_G\}.
\]
Assume that \( \|w\|_{E+F} \leq \|w\|_G \). Since \( \|w\|_{E+G} \leq \|w\|_G \), the norm \( \|w\|^2 \) of the same element \( w \in E + (F \cap G) \) may be estimated as follows:

\[
\|w\|^2 = \inf \{ \|u\|_E + \max \{ \|v\|_F, \|v\|_G \} : u + v = w \} \\
\leq \inf \{ \max \{ \|u\|_E + \|v\|_F, \|u\|_E + \|v\|_G \} : u + v = w \} \\
\leq \max \{ \|w\|_{E+F}, \|w\|_{E+G} \} = \|w\|_G^1.
\]

If we assume that \( \|w\|_{E+F} \geq \|w\|_G \), then \( \|w\|_{E+F} \geq \|w\|_{E+G} \) as well. Hence,

\[
\max \{ \|w\|_{E+F}, \|w\|_{E+G} \} = \|w\|_{E+F} = \|w\|_G^1.
\]

Certainly, this implies the desired inequality. \( \square \)

As a corollary we have the second theorem of uniqueness.

**Theorem 6.** Let \( E, F, G \in J(\mu) \) be such that \( E \cap G = F \cap G; E + G = F + G \). Then either \( G = E \) or \( F = E \) or \( G = F \).

**Proof.** Let \( E \neq G; F \neq G \) and \( E + G = F + G \); \( E \cap G = F \cap G \). Then from distributivity it follows that

\[
E = E \cap (F + G) = (E \cap F) + (E \cap G) = E \cap F.
\]

Hence \( E \subseteq F \) and either \( E = F \) or \( J(\mu) \) is not modular (cf. theorem 4). Since every distributive lattice is modular, \( E = F \). \( \square \)

**Corollary 1.** A pair of BIS \( E \) and \( F \) is uniquely determined by their sum \( E + F \) and intersection \( E \cap F \).

**Remark 4.** According to the M. Stone's theorem [5] every distributive lattice \( A \) is isomorphic (as lattice) to a some ring of sets. Moreover, as this ring of sets it may be chosen the ring of compact open sets of the so-called Stonian space \( S(A) \) of the lattice \( A \) - the topological \( T_0 \)-space, which is uniquely (up to a homeomorphism) determined by \( A \) and has the following properties:

- The base of open sets of \( S(A) \) forms compact open sets;
- Intersection of two compact open sets is compact;
- If \( K \subseteq S(A) \) is closed then \( \cap \{ U_i : i \in I \} \cap K \neq \emptyset \) for an arbitrary set of compact open sets \( \{ U_i : i \in I \} \) \( (I \neq \emptyset) \) of \( S(A) \) so that
  - For any \( i, j \in I \) there is \( l \in I \) such that \( U_l \subseteq U_i \cap U_j \);
  - \( U_i \cap K \neq \emptyset \) for all \( i \in I \).

If, in addition, \( A \) has the maximal element, then \( S(A) \) is compact itself.

E.g., in the aforementioned case \( J(\mathbb{P}) \), the Stonian space \( S(J(\mathbb{P})) \) is compact. Other examples may be given by using Banach symmetric spaces (see the concluding section).

4. **Closure operators and Galois connexions on \( J(\mu), \subseteq^1 \)**

Recall some algebraic definitions.

**Definition 5.** Let \( \langle L, \langle \rangle \rangle \) be a lattice. A mapping \( \pi : L \rightarrow L \) is said to be a closure operator, if for all \( a, b \in L \)

- \( a < b \) implies that \( \pi(a) < \pi(b) \);
- \( a < \pi(a) \);
- \( \pi \circ \pi(a) = \pi(a) \).
\textbf{Definition 6.} (Cf. [6]). Let $\langle L, \prec \rangle$ and $\langle L', \prec' \rangle$ be lattices; $k : L \to L'$ and $k' : L' \to L$ be mappings. The pair $(k, k')$ is said to be the Galois connexion between $L$ and $L'$ if

\begin{itemize}
  \item $a \prec b \Rightarrow k(a) \prec' k(b)$ for $a, b \in L$;
  \item $a' \prec' b' \Rightarrow k'(a') < k'(b')$ for $a', b' \in L'$;
  \item $k' \circ k(a) < a$ for $a \in L$;
  \item $k \circ k'(a') < a'$ for $a' \in L'$.
\end{itemize}

Below it will be needed the following simple result.

\textbf{Theorem 7.} Let $E \in \mathcal{J}_0(\mu)$. Then

\begin{enumerate}
  \item $(E_0)^\circ \supset \subset (E_0)^\mathcal{P} \supset \subset (E_0)'' \supset \subset E''$;
  \item $E'' \subset \subset ((E')_0)'$;
  \item $((E_0)'_0)' \subset \subset ((E'_0)'_0)'$;
  \item $(E_0)'_0 \subset \subset (E_0)' \Rightarrow (E_0)'_0'' \subset \subset (E'_0)'_0''$.
\end{enumerate}

\textbf{Proof.} Since we assume that $E \in \mathcal{J}_0(\mu)$, both $E_0$ and $(E')_0$ are nontrivial. So,

1. $E_0 \subset \subset E \Rightarrow E' \subset \subset (E_0)' \Rightarrow (E_0)^\circ \subset \subset E''$.
2. $(E')_0 \subset \subset E' \Rightarrow E'' \subset \subset ((E')_0)'$.
3. $E_0 \subset \subset E \Rightarrow E' \subset \subset (E_0)' \Rightarrow (E_0)'_0 \subset \subset ((E'_0)'_0)'$.
4. $((E_0)'_0)' \subset \subset (E_0)' \Rightarrow (E_0)'_0'' \subset \subset (E'_0)'_0''$. \qed

Consider the lattice $\mathcal{J}_0^\ast(\mu)$, equipped with the inverse order $\prec: E \prec F$ is the same as $F \subset \subset E$ for $E, F \in \mathcal{J}_0(\mu)$. Put $\mathcal{J}_0 := \langle \mathcal{J}_0(\mu), \subset \rangle; \mathcal{J}_0^* := \langle \mathcal{J}_0(\mu), \prec \rangle; \mathcal{J} := \langle \mathcal{J}(\mu), \subset \rangle$.

\textbf{Theorem 8.} The mapping $\langle (\_ \_ \_ \_ \_ \_ \_ 0) : \mathcal{J}_0^* (\mu) \to \mathcal{J}_0^* (\mu) \rangle$ is a closure operator on $\mathcal{J}_0^*$. The mapping $(\_ \_ \_ \_ \_ \_ \_ 0) : \mathcal{J}_0 (\mu) \to \mathcal{J}_0 (\mu) \rangle$ is a closure operator on $\mathcal{J}$.

\textbf{Proof.} The proof is an obvious consequence of definitions and the summary. \qed

It may be defined the most important sublattices of the $\mathcal{J}(\mu)$

\textbf{Definition 7.} The lattice $\mathcal{J}_00(\mu)$ consists of all BIS $E$ with the absolute continuous norm (i.e., such that $E = E_0$).

The lattice $\mathcal{J}'(\mu)$ contains all BIS $E$ of kind $E = F'$ for some BIS $F$.

\textbf{Corollary 2.} Lattices $\mathcal{J}_00(\mu)$ and $\mathcal{J}'(\mu)$ both are distributive (and, hence, modular). If the measure $\mu$ is a probability measure $\mathbb{P}$ and all BIS $E$ from $\mathcal{J}(\mathbb{P})$ satisfy the norming condition (i.e. if $\mathcal{J}(\mathbb{P})$ is Dedekind complete) then sublattices $\mathcal{J}_00(\mathbb{P})$ and $\mathcal{J}'(\mathbb{P})$ both are Dedekind complete too.

\textbf{Proof.} As it is well known, the set of fixed points of a closure operator that acting on the distributive Dedekind complete lattice $A$ is a sublattice of $A$ that holds these properties. Clearly, the set of fixed points of $(\_ \_ \_ \_ \_ \_ \_ (\_ \_ \_ \_ \_ \_ \_ 0)$ is exactly $\mathcal{J}_00(\mu)$. To show that $\mathcal{J}'(\mu)$ coincides with the set of fixed points of the closure operator $(\_ \_ \_ \_ \_ \_ \_ 0)$ it is enough to notice that $E = F'$ for some BIS $F$ if and only if $E = E''$. Certainly, if $E = E''$ then $E = F'$ for $F = E'$. Conversely, if $E = F'$ then $E'' = F'' = (F')'' = F'$. \qed
Correspondences $E \rightsquigarrow E_0$ and $E \rightsquigarrow E'$ may be regarded as mappings. It will be written $(\alpha) : E \to E_0$; $(\beta) : E \to E'$.

Using the mappings $(\alpha)$ and $(\beta)$ it may be constructed a pair of mappings, say $k$ and $k'$, which are in the Galois connexion.

Let $k : \mathcal{J}_0(\mu) \to \mathcal{J}_0^*(\mu)$ and $k' : \mathcal{J}_0^*(\mu) \to \mathcal{J}_0(\mu)$ are given by

$$kE = (E_0)^{\dagger}; \quad k'E = (E'_0).$$

**Theorem 9.** The pair $(k, k')$ is the Galois connexion between $\mathcal{J}_0^*$ and $\mathcal{J}_0$.

**Proof.** Let us check up properties from the definition 5.

- $E \in F \Rightarrow E_0 \subset F_0 \Rightarrow (F_0)^{\dagger} \subset (E_0)^{\dagger} \Rightarrow (E_0)^{\dagger} \subset (F_0)^{\dagger}$.
- $E \subset F \Rightarrow F_0 \subset E \Rightarrow (E'_0) \subset (F'_0)$.
- $((F''_0)_0)^{\dagger} = ((F''_0)_0)^{\dagger}$. By the theorem 4, $F'' \subset (F''_0)^{\dagger}$ and, hence,
  $$F \subset (F''_0)^{\dagger}.$$  
- $((E_0)^{\dagger})_0 = ((E_0)^{\dagger})_0 \subset E$. Hence, $E \subset (E_0)^{\dagger}$.

**Corollary 3.** Compositions $k \circ k'$ and $k' \circ k$ are closure operators on $\mathcal{J}_0^*$ and $\mathcal{J}_0$ respectively.

**Proof.** This is a obvious consequence of definitions.

**Remark 5.** So, we obtain some more closure operators on $\mathcal{J}_0(\mu)$. Notice that $k \circ k' : E \to ((E'_0)^{\dagger}$ and $k' \circ k : E \to ((E_0)^{\dagger})_0$. The second mapping is coincide with the usual $(\alpha) : E \to E_0$. However the first one pick out from $\mathcal{J}_0'(\mu)$ those BIS that are dual to BIS having the absolutely continuous norm.

### 5. Symmetric Banach spaces

Results from previous sections are of the most interest when a special class of BIS - the class of symmetric Banach spaces is considered.

Recall the definition.

**Definition 8.** A Banach ideal space $E$ of (classes of) measurable real functions, which are defined on the admissible measure space $(\Omega, \Sigma, \mu)$ is said to be symmetric if for any functions $x = x(t)$ and $y = y(t)$ of $E$ the following condition holds:

- If $x \in E$ and functions $|y(t)|$ and $|x(t)|$ are equimeasurable then $y \in E$ and $\|y\|_E = \|x\|_E$.

Let $\mathcal{S}(\Omega, \Sigma, \mu)$ be a class of all symmetric Banach spaces (they in the future will be referred to as symmetric spaces). This class contains Lebesgue-Riesz spaces $L_p(\mu)$; Orlicz spaces $L_M(\mu)$ and so on. Usually properties 1 and 2 are supplemented with the following norming condition:

$$\|x\|_E = 1 \text{ for any set } e \in \Sigma \text{ of the measure } \mu(e) = 1.$$  

This condition implies that $\mathcal{S}(\Omega, \Sigma, \mu)$ is a Dedekind complete distributive lattice because of the known theorem of inclusion:

$$L_1(\mu) \cap L_\infty(\mu) \subset L_1(\mu) \subset L_1(\mu) + L_\infty(\mu).$$
For a probability measure \( P \) these inclusions looks like
\[
L_\infty (P) \subset E (P) \subset L_1 (P).
\]

For a purely atomic measure (with mass of every point is equal to 1) we obtain so called symmetric discrete (or sequence) spaces defined on an arbitrary set. The only characteristic that distinguishes corresponding classes of discrete spaces is the cardinality of \( \Omega \). For \( \text{card} \, \Omega = \infty \) the class \( S (\Omega, \Sigma, \mu) \) will be denoted by \( S (\infty) \). Inclusions in this case looks like:
\[
l_1 (\infty) \subset E (\infty) \subset l_\infty (\infty).
\]

Notice that in the case of symmetric spaces the finiteness of \( \mu \) implies that it is non-atomic.

As it follows from the preceding consideration, all lattices of symmetric Banach spaces may be participate into three parts:

- \( S^{(1)} \) - lattices of symmetric spaces, defined on a probability (non atomic) space;
- \( S^{(\infty)} \) - lattices of symmetric spaces, defined on a non atomic space of infinite measure;
- \( S^{(D)} \) - lattices of symmetric sequence spaces:
\[
S = S^{(1)} \cup S^{(\infty)} \cup S^{(D)}.
\]

Our nearest aim is to show that all lattices \( \langle S (\mu) \rangle \) from a given class \( S (?) \), where \( ? \in \{1, \infty, D\} \) are pairwise lattice isomorphic, i.e. that there are at most three different lattices amongst all of kind \( S (\mu) \).

**THEOREM 10.** Lattices \( S (\mu) \) and \( S (\nu) \) that belong to the same class \( S (?) \), where \( ? \in \{1, \infty, D\} \) are lattice-isomorphic.

**Proof.** The one-to-one correspondence between members of these lattices may be shown by using the operation of replanting of symmetric spaces from one measure to another. Such operation was suggested by A.A. Mekler [7].

Namely, let \( E (\mu) \in S (\mu) \). To \( x (t) \in E (\mu) \) corresponds its distribution function
\[
n_x (s) = \text{mes} \{ \{ t \in [0, 1] : x (t) > s \} \}
\]
and its non-increasing rearrangement
\[
x^* (t) = \inf \{ s \in [0, \infty) : n_{|x|} (s) < t \}.
\]

Obviously, \( x^* (t) \) is defined either on \([0, 1]\) or on \([0, \infty)\) (both with the Lebesgue measure) or at \( \mathbb{N} \) (with mass 1 in every point) and, hence is an element of the corresponding vector lattice \( L_0 [0, 1] \) (resp., \( L_0 [0, \infty] \) or \( L_0 [\mathbb{N}] \)). Notice that usually \( L_0 [\mathbb{N}] \) is denoted by \( s \).

Let, for distinctness, \( x^* (t) \in L_0 [0, 1] \).

It is clear that the set
\[
E [0, 1] = \{ x^* (\sigma t) : x (t) \in E (\mu) ; \, \sigma \in \text{Aut} \}
\]
(where \( \text{Aut} \) denotes the set of all preserving measure automorphisms of \([0, 1]\)) is an ideal in \( L_0 [0, 1] \), which, being equipped with the norm
\[
\| x^* (t) \|_E := \| x \|_{E(\mu)} ; \quad \| y (t) \|_E := \| y^* (t) \|_E,
\]
becomes a Banach symmetric function space on \([0, 1]\).

Moreover, this space is uniquely determined by \( E (\mu) \).
Notice that for any probability measure $\nu$ the space $\tilde{E} [0, 1]$ in the same way uniquely defines the corresponding space $\tilde{E} (\nu)$.

It will be said that the space $\tilde{E} (\nu)$ is obtained from $E (\mu)$ by the \textit{replanting Mekler’s procedure}:

$$Mekl_{\mu, \nu} : S (\mu) \to S (\nu) ;$$

$$E (\mu) \mapsto \tilde{E} (\nu)$$

So, every symmetric space $E = E (\mu)$ generates a tower

$$[E] = \{ \tilde{E} (\Omega, \Sigma, \nu) = Mekl_{\mu, \nu} (E (\mu)) \}$$

where $(\Omega, \Sigma, \nu)$ runs all probability spaces.

Obviously, the procedure $Mekl_{\mu, \nu}$ holds the relation $\subset$ and lattice operations:

$$E \subset F \Rightarrow Mekl_{\mu, \nu} (E) \subset Mekl_{\mu, \nu} (F) ;$$

$$Mekl_{\mu, \nu} (E \cap F) = Mekl_{\mu, \nu} (E) \cap Mekl_{\mu, \nu} (F) ;$$

$$Mekl_{\mu, \nu} (E + F) = Mekl_{\mu, \nu} (E) + Mekl_{\mu, \nu} (F) ;$$

$$Mekl_{\mu, \nu} (E_0) = (Mekl_{\nu} (E))_0 ; \quad Mekl_{\mu, \nu} (E') = (Mekl_{\mu, \nu} (E))^\prime$$

and, hence, generates the lattice isomorphism between lattices $S (\mu)$ and $S (\nu)$. Similarly for lattices that are belong to $S (?)$, where either $? = \infty$, or $? = D$. \hfill $\square$

\textbf{Remark 6.} It is not clear: whether are lattice-isomorphic lattices of different types (e.g., from $S (1)$ and $S (\infty)$). It may be shown that lattices $S (1)$ and $S (D)$ are isomorphic to quotient lattices of $S (\infty)$

\textbf{Summary 2.} There exists (up to lattice-isomorphism) at most three lattices $S = S (?)$ ($? \in \{ 1, \infty, D \}$) of symmetric spaces.

All of them are Dedekind complete, distributive (and, hence, modular).

Operations $kE = (E_0)^\prime$ and $k' E = (E')_0$ interrelate the "fundamental part" $S_0$ (see the definition 3) of each of these lattices with itself in the reverse order (i.e., with $S_0^*$ by the given notation):

The pair $(k, k')$ is the Galois connexion between $S_0^*$ and $S_0$.

\textbf{Remark 7.} For the class $S$ of all symmetric spaces, classes $S_{00}$ and $S'$ (see the definition 6) are just those, whose elements are usually called rearrangement invariant Banach spaces

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