Improved bounds for the mixing time of the random-to-random shuffle

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Abstract

We prove an upper bound of $1.5321n \log n$ for the mixing time of the random-to-random insertion shuffle, improving on the best known upper bound of $2n \log n$. Our proof is based on the analysis of a non-Markovian coupling.

Keywords: random-to-random shuffle; mixing time; non-Markovian coupling.

AMS MSC 2010: 60J10.

Submitted to ECP on November 29, 2014, final version accepted on November 26, 2015.

1 Introduction

How many shuffles does it take to mix up a deck of cards? Mathematicians have long been attracted to card shuffling problems. This is partly because of their natural beauty, and partly because they provide a testing ground for the more general problem of finding the mixing time of a Markov chain, which has applications to computer science, statistical physics and optimization.

Let $X_t$ be a Markov chain on a finite state space $V$ that converges to the uniform distribution. For probability measures $\mu$ and $\nu$ on $V$, define the total variation distance $||\mu - \nu|| = \frac{1}{2} \sum_{x \in V} |\mu(x) - \nu(x)|$, and define the $\varepsilon$-mixing time

$$T_{\text{mix}}(\varepsilon) = \min\{t : \|\Pr(X_t = \cdot) - \mathcal{U}\| \leq \varepsilon \text{ for all } x \in V\},$$

where $\mathcal{U}$ denotes the uniform distribution on $V$.

The random-to-random insertion shuffle has the following transition rule. At each step choose a card uniformly at random, remove it from the deck and then re-insert in to a random position. It has long been conjectured that the mixing time for the random-to-random insertion shuffle on $n$ cards exhibits cutoff at a time on the order of $n \log n$. That is, there is a constant $c$ such that for any $\varepsilon \in (0, 1)$, the $\varepsilon$-mixing time is asymptotic to $cn \log n$. It has further been conjectured (see [4]) that the constant $c = \frac{3}{4}$.

Uyemura-Reyes [9] proved a lower bound of $\frac{1}{2}n \log n$. This was improved by Subag [7] to the conjectured value of $\frac{3}{4}n \log n$. However, a matching upper bound has not been found. Diaconis and Saloff-Coste [5] used comparison techniques to prove an $O(n \log n)$ upper bound. The constant was improved by Uyemura-Reyes [9] and then by Saloff-Coste and Zuniga [8], who proved upper bounds of $4n \log n$ and $2n \log n$, respectively. The main

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Theorem 2.1. For any $\varepsilon \in (0, 1)$ we have $T^{(n)}_{\text{mix}}(\varepsilon) \lesssim 1.5321 n \log n$.

We think of a permutation $\pi$ in $S_n$ as representing the order of a deck of $n$ cards, with $\pi(i)$ = position of card $i$. Say $x$ and $x'$ are adjacent, and write $x \approx x'$, if $x' = (i, j) x$ for a transposition $(i, j)$. We prove Theorem 2.1 using a path coupling argument (see [1]) and the following lemma.

Lemma 2.2. If $n$ is sufficiently large and $x$ and $x'$ are adjacent permutations in $S_n$, then there exist positive constants $c$ and $\alpha$ such that

$$
||P^t(x, \cdot) - P^t(x', \cdot)|| \leq \frac{c}{n^{1+\alpha}} \quad \text{for all } t > 1.5321 n \log n.
$$

The proof of Lemma 2.2, which uses a non-Markovian coupling, is deferred to Section 3.

Proof of Theorem 2.1. Suppose that $t > 1.5321 n \log n$. By convexity of the $l^1$-norm, and since $U = \frac{1}{n!} \sum_{z \in S_n} P^t(z, \cdot)$, it follows that for any state $y$ we have

$$
||P^t(y, \cdot) - U|| \leq \max_{z} ||P^t(y, \cdot) - P^t(z, \cdot)||.
$$

(2.1)

Since any permutation in $S_n$ can be written as a product of at most $n - 1$ transpositions, by the triangle inequality the quantity on the right-hand side of (2.1) is at most

$$
(n - 1) \max_{z \approx x'} ||P^t(x, \cdot) - P^t(x', \cdot)||.
$$

(2.2)

By (2.1), (2.2), and Lemma 2.2, if $n$ is sufficiently large, there exist positive constants $c$ and $\alpha$ such that

$$
d(t) = \max_{y} ||P^t(y, \cdot) - U|| \leq \frac{c(n - 1)}{n^{1+\alpha}},
$$

which tends to zero as $n \to \infty$. \qed

3 Proof of Lemma 2.2

Recall that we think of a permutation $\pi$ in $S_n$ as representing the order of a deck of $n$ cards, with $\pi(i)$ = position of card $i$. Let $M_{i,j} : S_n \to S_n$ be the operation on permutations that removes the card of label $i$ from the deck and re-inserts it

\[
\begin{cases}
    \text{to the right of the card of label } j & \text{if } i \neq j; \\
    \text{to the leftmost position} & \text{if } i = j.
\end{cases}
\]
We call such operations shuffles. If \( \langle M_1, \ldots, M_k \rangle \) is sequence of shuffles, we write \( xM_1M_2 \cdots M_k \) for \( M_k \circ \cdots \circ M_1(x) \).

The transition rule for the random-to-random insertion shuffle can now be stated as follows. If the current state is \( x \), choose a shuffle \( M \) uniformly at random (that is, choose \( a \) and \( b \) uniformly at random and let \( M = M_{a,b} \)) and move to \( xM \).

We call the numbers in \( \{1, \ldots, n\} \) cards. If a shuffle \( M \) removes card \( c \) from the deck and then re-inserts it, we call \( M \) a \( c \)-move.

If \( \mathcal{P} = \langle M_1, M_2, \ldots \rangle \) is a sequence of shuffles, we write \( \langle \mathcal{P}x \rangle_t \) for the permutation \( xM_1 \cdots M_t \). Note that if \( \mathcal{P} \) is a sequence of independent uniform random shuffles, then \( \{\langle \mathcal{P}x \rangle_t : t \geq 0\} \) is the random-to-random insertion shuffle started at \( x \).

### 3.1 The Non-Markovian coupling

Fix a permutation \( x \) and \( i, j \in \{1, 2, \ldots, n\} \). The aim of this subsection is to define a coupling of the random-to-random insertion shuffle starting from \( x \) and \( \langle i, j \rangle \), respectively. Suppose that we couple the processes so that the same labels are chosen for each shuffle. Note that if there is an \( i \)-move (respectively, \( j \)-move) followed at some point by a \( j \)-move (respectively, \( i \)-move), then the processes will couple at the time of the \( j \)-move (respectively, \( i \)-move) provided that any cards placed to the right of card \( j \) (respectively, \( i \)) at any intermediate time (and any cards placed to the right of those cards, and so on) were subsequently removed. We keep track of these “problematic” cards using a process we call the queue.

For positive integers \( k \) we will call a sequence \( \langle M_1, \ldots, M_k \rangle \) of shuffles a \( k \)-path. For a \( k \)-path \( \mathcal{P} \), define the \( \mathcal{P} \)-queue (or, simply the queue) as the following Markov chain \( \{Q_t : t = 0, \ldots, k\} \) on subsets of cards. Initially, we have \( Q_0 = \emptyset \). If the queue at time \( t \) is \( Q_t \) and the shuffle at time \( t+1 \) is \( M_{a,b} \), the next queue \( Q_{t+1} \) is

\[
\begin{cases}
\{i\} & \text{if } a = j; \\
\{j\} & \text{if } a = i; \\
Q_t \cup \{a\} & \text{if } a \notin \{i, j\} \text{ and } b \in Q_t \setminus \{a\}.
\end{cases}
\]

We call a shuffle an \( i \)-or-\( j \) move if it is an \( i \)-move or a \( j \)-move. Note that at any time after the first \( i \)-or-\( j \) move the queue contains exactly one card from \( \{i, j\} \). Let \( \mathcal{P} = \langle M_1, \ldots, M_k \rangle \) be a \( k \)-path. For \( t < k \), we say that \( t \) is a good time of \( \mathcal{P} \) if

1. \( M_t \) is an \( i \)-or-\( j \) move;
2. there is a time \( t' \in \{t+1, \ldots, k\} \) such that
   
   (a) \( M_t \) is the next \( i \)-or-\( j \) move after \( M_t \);
   (b) the queue is a singleton at time \( t' - 1 \) (i.e., either \( \{i\} \) or \( \{j\} \));
   (c) the card moved at time \( t' \) is different from the card moved at time \( t \).

Define

\[
T = \max\{t < k : t \text{ is a good time of } \mathcal{P}\}, \quad \text{if there is a good time of } \mathcal{P},
\infty, \quad \text{otherwise.}
\]

and call \( T \) the last good time of \( \mathcal{P} \). Let \( \theta_{i,j} \mathcal{P} \) be the \( k \)-path obtained from \( \mathcal{P} \) by reversing the roles of \( i \) and \( j \) in each shuffle before time \( T \) (that is, by replacing shuffle \( M_{a,b} \) with \( M_{\sigma(a), \sigma(b)} \), where \( \sigma \) is a transposition of \( i \) and \( j \)). Note that \( \theta_{i,j} \mathcal{P} \) has \( i \)-or-\( j \) moves at the same times as \( \mathcal{P} \). Furthermore, since the queue is reset at the times of \( i \)-or-\( j \) moves, the \( \theta_{i,j} \mathcal{P} \)-queue will have the same values as the \( \mathcal{P} \)-queue at all times \( t \geq T \). It follows that the last good time of \( \theta_{i,j} \mathcal{P} \) is the same as the last good time of \( \mathcal{P} \), and hence
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\[ \theta_{i,j}(\theta_{i,j}(P)) = P. \] Since \( \theta_{i,j} \) is its own inverse, it is a bijection and hence if \( P \) is a uniform random \( k \)-path, then so is \( \theta_{i,j}P \).

Let \( x' = (i,j)x \). Let \( P_k \) be a uniform random \( k \)-path, and let \( T_k \) be the last good time of \( P_k \). Note that \( T_k < k \) or \( T_k = \infty \). For \( t \) with \( 0 \leq t \leq k \), define

\[ x_t = (P_kx)_t \quad x'_t = ((\theta_{i,j}P_k)x')_t. \]

It is clear that \( x_t \) and \( x'_t \) have distributions \( P^t(x,\cdot) \) and \( P^t(x',\cdot) \), respectively, for all \( t \leq k \).

Lemma 3.1. If \( x_k \neq x'_k \) then \( T_k = \infty \).

Proof. Assume that \( T_k < k \). Note that at any time \( t < T_k \), the permutation \( (P_kx)_t \) can be obtained from \( ((\theta_{i,j}P_k)x')_t \), by interchanging the cards \( i \) and \( j \). Suppose that the next \( i \)-or-\( j \) move after time \( T_k \) occurs at time \( T_k' \). Without loss of generality, there is an \( i \)-move at time \( T_k \) and a \( j \)-move at time \( T_k' \). We claim that for times \( t \) with \( T_k \leq t < T_k' \), the permutation \( x'_t \) can be obtained from \( x_t \) by moving only the cards in \( Q_t \), as shown in the diagram below. (In the diagram, the \( m \)th \( X \) in the top row represents the same card as the \( m \)th \( X \) in the bottom row, and \( Q \) represents all the cards in \( Q_t \).)

\[
\begin{align*}
\text{at } T_k & : \quad X \quad X \quad X \quad X \quad X \quad Q \quad X \quad X \quad X \\
\text{at } T_k' & : \quad X \quad X \quad X \quad X \quad Q \quad X \quad X \quad X \quad X 
\end{align*}
\]

To see this, note that it holds at time \( T_k \), when the queue is the singleton \( \{j\} \) (since at this time the \( i \)'s are placed in the same place), and the transition rule for the queue process ensures that if it holds at time \( t \) then it also holds at time \( t + 1 \). The claim thus follows by induction. This means that at time \( T_k' - 1 \) the permutations differ only in the location of card \( j \). That is, they are of the form:

\[
\begin{align*}
x_{T_k'-1} & : \quad X \quad X \quad X \quad X \quad X \quad j \quad X \quad X \quad X \\
x'_{T_k'-1} & : \quad X \quad X \quad j \quad X \quad X \quad X \quad X \quad X 
\end{align*}
\]

Thus at time \( T_k' \), when card \( j \) is removed and then re-inserted into the deck, the two permutations become identical, and they remain identical until time \( k \).

3.2 Tail estimate of the coupling time

Recall that \( T_k \) is the last good time of a uniform random \( k \)-path.

Lemma 3.2. Suppose that \( k > 1.5321n \log n \). Then there exist positive constants \( c \) and \( \alpha \) such that \( P(T_k = \infty) \leq \frac{n^c}{n^{\alpha n}} \) for sufficiently large \( n \).

Proof. Consider a process \( Y_t \in \{0,1,\ldots\} \cup \infty \) that is defined as follows. The process starts in state \( \infty \) and remains there until the first \( i \)-or-\( j \) move. From this point on, the value of \( Y_t \) is the size of the queue, until the first time that either

1. card \( i \) is moved when the queue is \( \{i\} \), or
2. card \( j \) is moved when the queue is \( \{j\} \).

At this point \( Y_t \) moves to state 0, which is an absorbing state. Note that \( T_k = \infty \) exactly when \( Y_k > 0 \).

For \( l = 1,2,\ldots \), define

\[
q(l) = \begin{cases} 
\frac{1}{n} & \text{if } l = 1, \\
\frac{3n-1}{n^2} & \text{if } l = 2, \\
\frac{(l-1)(n-l+1)}{n^2} & \text{if } l \geq 3;
\end{cases}
\]

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and define

\[ p(l) = \begin{cases} \frac{n-2}{n^2} & \text{if } l = 1, \\ \frac{2n-6}{n^2} & \text{if } l = 2, \\ \frac{l(l-1)}{n^2} & \text{if } l \geq 3. \end{cases} \]

It is easy to check that \( Y_t \) is a Markov chain with the following transition rule. If the current state is 0, the next state is 0. If the current state is \( \infty \), the next state is

\[ \begin{cases} 1 & \text{with probability } \frac{2}{n}; \\ \infty & \text{with probability } \frac{n-2}{n}. \end{cases} \]

If the current state is \( l \in \{1, 2, \ldots\} \), the next state is

\[ \begin{cases} l-1 & \text{with probability } q(l); \\ l+1 & \text{with probability } p(l); \\ 1 & \text{with probability } \frac{2}{n}, \text{if } l \geq 3; \\ l & \text{with the remaining probability.} \end{cases} \]

Let \( \tilde{Y}_t \) be the Markov chain on \( \{0, 1, \ldots, 8\} \cup \infty \) obtained from \( Y_t \) by replacing transitions to state 9 with transitions to \( \infty \). That is, if \( K \) and \( \tilde{K} \) denote the transition matrices of \( Y_t \) and \( \tilde{Y}_t \), respectively, then

\[ \tilde{K}(l, m) = \begin{cases} K(l, m) & \text{if } m \in \{0, 1, \ldots, 8\}; \\ K(8, 9) & \text{if } l = 8 \text{ and } m = \infty. \end{cases} \]

The possible transitions of \( Y_t \) and \( \tilde{Y}_t \) are indicated by the graph in Figure 1. We claim that if we start with \( \tilde{Y}_0 = Y_0 = \infty \) then the distribution of \( \tilde{Y}_t \) stochastically dominates the distribution of \( Y_t \) for all \( t \). To see this, note that \( Y_t \) changes state with probability less than \( \frac{1}{2} \) at each step, and when it changes state, it either makes a \( \pm 1 \) move or it transitions to 1. Since for \( m \in \{1, 2, \ldots\} \cup \infty \), the transition probability \( K(m, 1) \) is decreasing in \( m \), it follows that \( Y_t \) is a monotone chain. (That is, \( K(x, \cdot) \) is stochastically increasing in \( x \); see [3].) The claim follows since \( \tilde{Y}_t \) is obtained from \( Y_t \) by replacing moves to 9 with moves to the (larger) state of \( \infty \).

Let \( \tilde{K}_n \) be the value of the matrix \( \tilde{K} \) when the number of cards is \( n \), and \( \hat{K}_n \) the matrix obtained by deleting the first row and the first column of \( \tilde{K}_n \). If we write \( A_n \to A \) for a sequence of matrices \( A_n \) and a fixed matrix \( A \), it means that \( A_n \) converges to \( A \) component-wise as \( n \to \infty \).

Define \( C_n := n(\hat{K}_n - I) \), where \( I \) is the identity matrix. A straightforward calculation shows that \( C_n \to C \) where

\[
C = \begin{bmatrix}
-2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & -5 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & -7 & 3 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 3 & -9 & 4 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 4 & -11 & 5 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 5 & -13 & 6 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 6 & -15 & 7 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 & 7 & -17 & 8 \\
2 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 9
\end{bmatrix}_{9 \times 9}
\]
Figure 1: Graph indicating the possible transitions of $Y_t$ and $\tilde{Y}_t$. The dotted edge indicates a possible transition of $Y_t$ and the dashed edge indicates a transition of $\tilde{Y}_t$. (Self loops are not included.)

and that the eigenvalues of $C$ are real and distinct (and hence $C$ is diagonalizable), and negative. Denote the largest eigenvalue of $C$ by $-\lambda$, where $\lambda = 0.652703\ldots$ (We can improve the eigenvalue marginally by considering a Markov chain with more than 10 states. For example with 35 states we get an eigenvalue of $-0.6527363\ldots$ However, we can’t improve on this by more than $10^{-7}$ even if we use up to 100 states. Therefore, for simplicity we shall stick to our 10-state chain as a reasonable approximation to $Y_t$.)

Since $C^\top$ is diagonalizable, there exists an invertible $9 \times 9$ matrix $Q$ such that $Q^{-1}C^\top Q = D$, where $D$ is a diagonal matrix whose diagonal entries are the eigenvalues of $C$. Let $D_n = Q^{-1}C_n^\top Q$, and note that $D_n \to D$. For matrices $A$, let $\|A\|$ denote matrix norm induced by the $l^1$ norm on vectors. By continuity of the matrix exponential function and matrix norm, we have $\lim_{n \to \infty} \|e^{D_n}\| = \|e^D\| = e^{-\lambda}$. Since $\lambda > 0.6527$, it follows that $\|e^{D_n}\| \leq e^{-0.6527}$ for sufficiently large $n$. Since $k/n > 1.5321 \log n$, submultiplicativity of operator norms implies that for sufficiently large $n$ we have

$$\|e^{kD_n}\| \leq e^{-0.6527 \times 1.5321 \log n} \leq \frac{1}{n^{1+\alpha}} \quad \text{for some } \alpha > 0. \quad (3.1)$$

Since for any nonnegative integer $j$ we have $(C_n^\top)^j = Q D_n^j Q^{-1}$, it follows that

$$e^{\frac{k}{n}C_n^\top} = Q e^{\frac{k}{n}D_n} Q^{-1}. \quad (3.2)$$

Let $X$ be a Poisson random variable with mean $k$ that is independent of everything else. Then

$$e^{\frac{k}{n}C_n} = e^{k(\tilde{K}_n - I)} = \sum_{j=0}^\infty e^{-k\frac{j}{n}} \tilde{K}_j \approx \sum_{j=0}^\infty P(X = j) \tilde{K}_j. \quad (3.3)$$

Let $x_0 = (0, 0, \ldots, 0, 1) \in \mathbb{R}^9$. It follows from definition of $\tilde{Y}_t$ and (3.3) that

$$P(\tilde{Y}_t > 0) = \sum_{j=0}^\infty P(X = j) \left\| x_0 \tilde{K}_j \right\|_1 = \left\| \sum_{j=0}^\infty P(X = j) x_0 \tilde{K}_j \right\|_1 = \left\| x_0 e^{\frac{k}{n}C_n} \right\|_1.$$

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By (3.2) and (3.1), there exists some $c > 0$ independent of $n$ such that
\[
\|x_0e^{\frac{k}{n}C_n}\|_1 \leq \|e^{\frac{k}{n}C_n^T}\| = \|Qe^{\frac{k}{n}D_n}Q^{-1}\| \leq \frac{c}{2} \|e^{\frac{k}{n}D_n}\| \leq \frac{c}{2n^{1+\alpha}}.
\]

Since $Y_t$ is stochastically dominated by $\tilde{Y}_t$, we have
\[
P(Y_X > 0) \leq P(\tilde{Y}_X > 0) \leq \frac{c}{2n^{1+\alpha}}.
\]

Also, we have
\[
P(Y_X > 0) = \sum_{j=0}^{\infty} P(X = j)P(Y_j > 0)
\geq P(Y_k > 0) \sum_{j=0}^{k} P(X = j)
\geq \frac{1}{2} P(Y_k > 0),
\]

where the last line follows from the fact that the median of $X$ (defined as the least integer $m$ such that $P(X \leq m) \geq \frac{1}{2}$) equals $E[X] = k$ (see [2]). Therefore, we have
\[
P(T_k = \infty) = P(Y_k > 0) \leq 2P(Y_X > 0) \leq \frac{c}{n^{1+\alpha}} \quad \text{for sufficiently large } n.
\]

Proof of Lemma 2.2. Recall that for any two probability measures $\mu$ and $\nu$ on a probability space $\Omega$, we have
\[
\|\mu - \nu\| = \min \{P(X \neq Y) : (X,Y) \text{ is a coupling of } \mu \text{ and } \nu\}.
\]
The main lemma then follows immediately from Lemma 3.1 and Lemma 3.2.

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