Abstract. We define a class of sites such that the associated topos is equivalent to the category of smooth sets (representations) of some monoid. This is a generalization of the fact that the topos associated to the étale site of a scheme is equivalent to the category of sets with continuous action by the étale fundamental group.

We then define a subclass of sites such that the topos is equivalent to the category of discrete sets with a continuous action of a locally profinite group.

1. Introduction

1.1. Our result may be regarded as a generalization of the following statements. Let $X$ be a connected noetherian scheme. Let $C$ be the category of schemes finite and étale over $X$ whose morphisms are $X$-morphisms. Take a geometric point $x \in X$. Then the fiber functor and the category satisfies the axioms for Galois category [SGA1] Exp.V.4, p.98] and gives rise [SGA1] Thm.4.1, p.104] to the étale fundamental group $\pi_1^\text{ét}(X, x)$ of $X$. One can put topology on $C$ so that the category of sheaves is equivalent to the category of discrete $\pi_1^\text{ét}(X, x)$-sets. (This can be done in a manner similar to that in [SGA4] Exp. VIII Prop 2.1] where the case $X$ is the spectrum of a field is treated.)

We give a set of conditions for a site (a pair of a category and a Grothendieck topology). We call a site satisfying these conditions a $Y$-site. Our theorem says that if a $Y$-site satisfies a cardinality condition (e.g., the hom sets are finite), then the category of sheaves on the site is equivalent to the category of smooth sets (to be defined in this paper) of some monoid. If we assume further that the Grothendieck topology is atomic, then our second theorem says that there exists a locally profinite group such that the category of sheaves on this site is equivalent to the category of discrete sets with continuous action of this group. In case the site has a final object, the locally profinite group is profinite. In the opposite direction, given a locally profinite group, we can construct a $Y$-site meeting a cardinality condition, which gives rise by the procedure above to the given group. Our theorem is a characterization theorem in this sense.

There is no direct way to compare Grothendieck’s axioms and our conditions because his axioms are on the pair of a category and a functor while ours are on...
the site (a category and a topology). We note that we have common examples. The main examples of Galois categories are the étale sites of schemes, giving rise to the étale fundamental groups of schemes. Our theorem is a generalization in the sense that the étale site of a scheme with a connectedness assumption (see Example 4.2.2) meets our conditions. Our conditions hold true for such a site and by our theorem we obtain a locally profinite group, which in this case is the (profinite) étale fundamental group.

1.2. Let us give an outline of the contents of this article leading to the statement of our main theorem in this subsection. We have to give many new conditions and many new definitions. We wish to explain the motivation and ideas behind them here in the introduction. Those in italics are to be defined in this article. The phrases in quotations ‘-’ are ideas intended to help the readers, which may not be rigorous.

We start this article by defining semi-localizing collections. (We note that the term collection is used to avoid set theoretic complications only. See Section 2.1.) A semi-localizing collection is a collection of morphisms in a category satisfying 3 conditions (Definition 2.3.1). As remarked in Remark 2.3.2, the 3 conditions are the first 3 of the 4 conditions of Gabriel and Zisman [GZ] to admit ‘right calculus of fractions’. From a semi-localizing collection \( \mathcal{T} \), we can construct (Lemma 2.3.4) a Grothendieck topology \( J_{\mathcal{T}} \). We call topology of such form an A-topology (Definition 2.4.1). We arrived at this notion when considering a class of Grothendieck topologies such that one needs to look only at the coverings of the form \( \{ X \to Y \} \) and not at those of the form \( \{ X_i \to Y \}_{i \in I} \) with cardinality of \( I \) greater than 1. Thus in this topology, the coverings are captured within the category. We will be using the collection \( \mathcal{T}(J) = \mathcal{\hat{T}} \supset \mathcal{T} \) (see Definition 2.3.5 for \( \mathcal{\hat{T}} \)) which is a ‘saturation’ in some sense, again semi-localizing, giving the same topology. The idea is that the collection \( \mathcal{T}(J) \) is ‘the set of all coverings’. The atomic topology is an example of A-topology (see Section 2.4.1). This is the case when \( \mathcal{T} \) is the collection of all morphisms of the given category. We believe that by this restriction to A-topology we do not lose much generality.

We define a Galois covering (Definition 3.1.2) to be a morphism \( X \to Y \) in a category \( \mathcal{C} \) such that there exists a group \( G \) for which \( \text{Hom}_\mathcal{C}(Z,Y) \to \text{Hom}_\mathcal{C}(Z,X) \) is a pseudo \( G \)-torsor for each object \( Z \) (see Definition 3.1.1). We say that a site equipped with an A-topology associated to a semi-localizing collection \( \mathcal{T} \) has enough Galois coverings (Definition 3.1.4) if \( \mathcal{T} = \mathcal{\hat{T}} \) where \( \mathcal{T} \) is the collection of all Galois coverings.

An \( E \)-category is a category where all the morphisms are epimorphisms (Definition 4.1.1). We then define a B-site to be a site \( (\mathcal{C}, J) \) where the underlying category \( \mathcal{C} \) is an \( E \)-category, whose topology \( J \) is an A-topology, and when the following condition is satisfied: For any diagram \( Z \xrightarrow{g} Y \xrightarrow{f} X \) in \( \mathcal{C} \), the composite \( g \circ f \) belongs to \( \mathcal{T}(J) \) if and only if \( f \) and \( g \) belong to \( \mathcal{T}(J) \). This condition may not seem pleasant. This notion may be described better if the category has coproducts or is equipped with the notion of ‘\( \pi_0 \)’, but we have not assumed so. Note that the usual finite étale site of a scheme does not satisfy this condition, but if we impose that all schemes are connected, it does. One outcome of these definitions is the following: If we assume moreover that there are enough Galois coverings, the
sheafification functor from the category of presheaves on a $B$-site can be described using Galois coverings (see Section 4.4).

Our principal objects of study are $Y$-sites (Definition 5.4.2). A $Y$-site is defined to be a $B$-site satisfying two more conditions (a set-theoretic one is not discussed in this paragraph). One is that: Given any two objects $X_1, X_2$ in the underlying category, there exist morphisms $f_1: Y \to X_1$ and $f_2: Y \to X_2$ both in $T(J)$. This condition can be thought of as a form of the existence of an initial object, or the existence of a ‘universal covering’ if the site has some topological meaning. The other is that $T(J)$ has enough Galois coverings.

We regard partially ordered sets (posets) as categories in a natural manner. We introduce quasi-posets as those categories that are equivalent to posets (Definition 5.2.1). These are used only in the proof of the existence of grids.

A grid $(C_0, \iota_0)$ of a $Y$-site is a pair of a poset $C_0$ and a functor $\iota_0$ from this poset to the underlying category of the site satisfying some conditions (Definition 5.5.3). An edge object is defined to be an object of $C_0$ such that every morphisms to it is mapped to $T(J)$ by $\iota_0$ (Definition 5.5.2). Using the endomorphisms of the poset $C_0$ and some natural isomorphisms, we construct a monoid $M_{(C_0, \iota_0)}$, which we call the absolute Galois monoid (Section 5.6.1). For each edge object $X$, we also have a subgroup $K_X$ of the monoid (Section 5.6.2). Roughly speaking, it is defined as the ‘stabilizer’ of the object. The term ‘grid’ may not be a good one. In the easiest case, a grid is something like ‘the set of all finite field extensions of a field, that is contained in a fixed separable closure’. It forms a lattice in this case, and we wanted to call our grid a lattice, but we have not done so since technically it may not be a lattice in general.

This absolute Galois monoid and its subgroups let us define the notion of smooth sets. A smooth $M_{(C_0, \iota_0)}$-set is defined to be a set with the action of the absolute Galois monoid satisfying the following condition: For an element in the set, the stabilizer contains the group of the form $K_X$ for some edge object $X$ (Definition 5.7.7).

We may also equip the absolute Galois monoid with a structure of a topological monoid using those subgroups (Section 10.1). Then by definition, the category of smooth $M_{(C_0, \iota_0)}$-sets is canonically equivalent to the category of discrete sets with continuous action of the topological monoid $M_{(C_0, \iota_0)}$.

When the topology is the atomic topology, $M_{(C_0, \iota_0)}$ is a group and all objects are edge objects.

We took this term ‘smooth’ from the representation theory of locally profinite groups, which is known for its applications in number theory (see Remark 5.7.8). Examples of locally profinite groups include profinite groups, the finite adele valued points of an algebraic group, and the nonarchimedean local field valued points of an algebraic group. One of the original motivations was to describe the categories of smooth representations of these groups using sheaf theory.

Another thing that we construct from a grid is the fiber functor $\omega_{(C_0, \iota_0)}$. It is a functor from the topos of sheaves on the $Y$-site to the category of sets, which factors through the category of smooth sets. A sheaf is sent via this fiber functor to the colimit of sections over the edge objects, which is actually isomorphic to the colimit over the whole grid. This action of the absolute Galois monoid may be thought of as a generalization of the action of the finite adeles on the limit of elliptic modular curves or of Drinfeld modular varieties. We will consider these examples in a future paper.
Before coming to our theorem, we need to mention one more thing: cardinality conditions (Section 5.8.1). There are two kinds of cardinality conditions. Cardinality Condition (1) is that the hom sets of the underlying category of the site are finite. There is also another type of cardinality condition. The first kind is used primarily in the form: the projective limit of finite sets with surjective transitive maps is non empty and the limit surjects onto each finite set.

One important proposition (Proposition 6.2.1) says: Under certain cardinality conditions, there exists a grid. The proof can be divided into two parts. The first half (Proposition 6.1.1) says that there exists what we call pregrid. For the proof, we follow the proof of the existence of an algebraic closure of a field. The second half seems to be new.

Now we come to our main theorem. Suppose we are given a $Y$-site. Assume certain cardinality condition holds. Note that then there exists a grid by the proposition and we can construct the absolute Galois monoid and its subgroups. Hence we have smooth sets and we also have the associated fiber functor. Our theorem (see Theorem 5.8.1 for the precise statement) says:

**Theorem 1.2.1.** Under these assumptions, the fiber functor gives an equivalence between the topos and the category of smooth sets.

1.3. Let us restrict ourselves to atomic topologies and assume cardinality Condition (1), that is, the finiteness of the hom sets, holds. Then as an application of our theorem, we obtain a 'reconstruction' theorem (Theorem 10.2.1) as follows.

**Theorem 1.3.1.** Suppose given a $Y$-site whose topology is the atomic topology. Assume that Condition (1) of the cardinality conditions holds. Then there exists a locally profinite group $G$ such that the topos is equivalent to the category of discrete sets with continuous $G$-action.

When the topology is the atomic topology, the absolute Galois monoid becomes a group. If there exists a grid, we can equip the group with the structure of a topological group such that the subgroups $\{K_X\}$ for objects $X$ of the grid is a fundamental system of neighborhoods of the unit.

If cardinality Condition (1) holds, then there exists a grid as remarked above. We can further show that the topological group is locally profinite. By definition, the category of smooth sets is equal to the category of discrete sets with continuous action of this topological group. Hence Theorem 1.3.1 follows from Theorem 1.2.1.

1.4. The question that what kind of monoids appear as the absolute Galois monoid seems difficult to us. For groups, we have the following examples.

We have a construction of a $Y$-site and a grid, starting from a topological group $G$ of certain type. These give examples of grids where the cardinality conditions may not hold true. The site is essentially that constructed by MacLane and Moerdijk in [MM]. We will see that any complete separated prodiscrete group appear as the absolute Galois group.

1.5. There are many other generalizations of Galois theory or of Grothendieck’s Galois theory [SGA1]. We do not know all of the generalizations. Let us mention some of the differences.

In the works covered by Dubuc in [DF], they consider axioms for topoi (rather than sites or categories) and the Galois group is a localic group. We note that we impose certain “cardinality conditions” which imply that certain projective limit
is non-trivial. We believe that this is the place to use locales (so that the limit is non-trivial without such hypotheses) but we have not pursued this point further. The cardinality conditions are satisfied in the cases of our interest, namely, for the étale sites of schemes and for the case where the Galois group becomes the general linear group of finite adeles.

Another point is the appearance of monoids as ‘Galois groups’. A Galois group is usually conceived as the automorphism group of some fiber functor from a topos, while our absolute Galois monoid is constructed using more structure of the underlying site. We have not seen statements describing topoi as classifying spaces of some monoid elsewhere.

The Galois theory of Joyal and Tierney ([JT]) states roughly that any topos is equivalent to the classifying topos of a localic group. In particular, given a Y-site, we obtain some ‘Galois group’ using their theory. However, it is difficult to see what exactly the Galois group is from their theory. Our aim is to give axioms so that the Galois group is computable from the given data.

Recall that in Grothendieck’s Galois theory, the existence of a final object, fiber products, and finite sums follow from the axioms. We do not impose such conditions on our underlying categories of the Y-sites. This is because, in application, we would like to have ‘fewer objects’ to work with so that the computation is not so complicated. Note that because we lack objects such as final objects, fiber products, etc., we think that the formulation of the axioms in terms of the fiber functor becomes complicated.

1.6. Let us give the list of contents of each section. More technical details are found at the beginning of each section.

In Section 2 we first recall some general definitions and constructions from sheaf theory to make this paper self-contained. Basic notions such as Grothendieck topology, sieve, and sheaf are recalled. We then define what it is for a collection of morphisms in a category to be semi-localizing. It is shown that a Grothendieck topology is associated to a semi-localizing collection, which we call A-topology. After showing certain general properties of A-topology, we give a fairly explicit criterion for a presheaf to be a sheaf for A-topology.

In Section 3 we define Galois covering and what it means for a site to have enough Galois coverings. We spend few pages on the generality on quotient objects in a category. This will appear to be useful in our future paper, where we consider sheaves with values in a category other than the category of sets.

In Section 4 we define B-sites (Definition 4.2.1) and give some of the properties. In Section 4.4 we give an explicit description of the sheafification functor on B-sites when there are enough Galois coverings. This will be used in the proof of our main theorem.

The aim of Section 5 is to state our main theorem (Theorem 5.8.1). We define Y-site as a B-site with some more conditions. We define cardinality conditions, grid, the absolute Galois monoid, the smooth sets, and the fiber functor \( \omega(\mathcal{C}_0, \iota_0) \) associated to a grid. Theorem 5.8.1 says that the fiber functor induces an equivalence of categories between the topos and the category of smooth representations of the absolute Galois monoid under the cardinality conditions.

In Section 6 we prove the existence of grid under the cardinality conditions. We first prove the existence of the pregrid, and then of the grid. The proof for the pregrid follows the same line as for the proof of the existence of an algebraically
closed field of a field. What is new appears in the construction of the grid. We note that when the topology is the atomic topology, the pregrid is already a grid.

In Section 7, we start the proof of our main theorem. We show that the fiber functor is fully faithful. In Section 8, we show the essential surjectivity, finishing the proof of Theorem 5.8.1.

In Section 9, we show that the fiber functor has a left adjoint, thereby showing that we have a point of the topos. As an application of our main theorem, we see that the topos has enough points.

In Section 10, we show how to equip the absolute Galois monoid with the structure of a topological monoid. Then we give a precise form of Theorem 1.3.1.

In Section 11, we give examples of Y-sites and grids, which do not meet the cardinality conditions, such that the absolute Galois monoids are locally prodiscrete groups.

We give more examples of Y-sites with grids in Section 12. The simplest example where the absolute Galois monoid is the monoid of non-negative integers is given in Section 12.1. The example in Section 12.2 gave us the motivation to write this article. Its details and an application will be given in a future paper.

1.7. We give a remark concerning the use of universes. Usually, an author fixes a universe $\mathcal{U}$ and suppresses the appearance by declaring that everything belongs to the fixed universe. However, the reader will find occasionally in this paper the phrase “essentially $\mathcal{U}$-small”. This is because the primary example is the étale site of a scheme in $\mathcal{U}$, the underlying category of which is essentially $\mathcal{U}$-small. There is a standard technique of changing the universe to a larger one so that proving statements in an arbitrary fixed universe suffices (which enables one to suppress the appearance of the universe). However, it was not clear to us if such a technique should eliminate the phrase.

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2. A-topology

A collection of morphisms in a category is defined (Definition 2.3.1) to be semi-localizing, when it satisfies the first three of the four axioms for the collection to admit ‘right calculus of fractions’ in the sense of Gabriel and Zisman [GZ]. We arrived at the definition when considering the class of Grothendieck topologies that is generated by coverings of the form $\{X_i \to Y\}_{i \in I}$ where the cardinality of $I$ is one.

2.1. Presheaves. Throughout the paper we fix once for all a Grothendieck universe $\mathcal{U}$. 

Recall from [SGA4, EXPOSE I, 1.0, and Définition 1.1] that a set $X$ is called $\mathfrak{U}$-small if $X$ is isomorphic to an element in $\mathfrak{U}$, and that a category $\mathcal{C}$ is called an $\mathfrak{U}$-category if for any objects $X, Y$ of $\mathcal{C}$, the set $\text{Hom}_\mathcal{C}(X, Y)$ is $\mathfrak{U}$-small. From now on, unless otherwise stated, a set is assumed to be $\mathfrak{U}$-small and a category is assumed to be a $\mathfrak{U}$-category. We use the terminology “collection” to refer to a set which is not necessarily $\mathfrak{U}$-small. A category $\mathcal{C}$ is called $\mathfrak{U}$-small if the collection of objects of $\mathcal{C}$ is a set.

2.1.1. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. We call a contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ a presheaf on $\mathcal{C}$ with values in $\mathcal{D}$. When $\mathcal{D}$ is the category of $\mathfrak{U}$-small sets (resp. $\mathfrak{U}$-small abelian groups, resp. $\mathfrak{U}$-small rings with units), a presheaf on $\mathcal{C}$ with values in $\mathcal{D}$ is called a presheaf (resp. an abelian presheaf, resp. a presheaf of rings) on $\mathcal{C}$.

In this article, we will only consider presheaves of sets. In our future article, we will use some more general categories.

2.1.2. Let $\mathcal{C}$ be a category and let $X$ be an object of $\mathcal{C}$. We let $\mathfrak{h}_\mathcal{C}(X) = \text{Hom}_\mathcal{C}(\cdot, X)$ denote the presheaf on $\mathcal{C}$ which associates, to each object $Y$ of $\mathcal{C}$, the set $\text{Hom}_\mathcal{C}(Y, X)$. The presheaf $\mathfrak{h}_\mathcal{C}(X)$ on $\mathcal{C}$ is called the presheaf represented by $X$.

We denote by $\mathfrak{h}_\mathcal{C} : \mathcal{C} \to \text{Presh}(\mathcal{C})$ denote the functor which associates, to each object $X$ of $\mathcal{C}$, the presheaf $\mathfrak{h}_\mathcal{C}(X) = \text{Hom}_\mathcal{C}(\cdot, X)$ represented by $X$ on $\mathcal{C}$. It follows from Yoneda’s lemma that the functor $\mathfrak{h}_\mathcal{C}$ is fully faithful.

2.1.3. A category $\mathcal{C}$ is called essentially $\mathfrak{U}$-small if $\mathcal{C}$ is equivalent to a $\mathfrak{U}$-small category, or equivalently, if there exists a set $S$ of objects of $\mathcal{C}$ such that any object of $\mathcal{C}$ is isomorphic to an object which belongs to $S$. Let $\mathcal{C}$ be an essentially $\mathfrak{U}$-small category and let $\mathcal{D}$ be a category. Then the presheaves on $\mathcal{C}$ with values in $\mathcal{D}$ form a category, which we denote by $\text{Presh}(\mathcal{C}, \mathcal{D})$. When $\mathcal{D}$ is the category of $\mathfrak{U}$-small sets, we simply write $\text{Presh}(\mathcal{C})$ for the category $\text{Presh}(\mathcal{C}, \mathcal{D})$.

2.1.4. Let $\mathcal{C}$ be an essentially $\mathfrak{U}$-small category and let $F$ be a presheaf on $\mathcal{C}$. Let $X$ be an object of $\mathcal{C}$. We denote by $y_{F,X}$ the map

$$y_{F,X} : \text{Hom}_{\text{Presh}(\mathcal{C})}(\mathfrak{h}_\mathcal{C}(X), F) \to F(X)$$

which sends a morphism $\phi : \mathfrak{h}_\mathcal{C}(X) \to F$ of presheaves on $\mathcal{C}$ to the image of $\text{id}_X \in \text{Hom}_\mathcal{C}(X,X)$ under the map $\phi(X) : \text{Hom}_\mathcal{C}(X,X) \to F(X)$. It follows from Yoneda’s lemma that the map $y_{F,X}$ is bijective.

2.1.5. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F : \mathcal{C} \to \mathcal{D}$ be a covariant functor. For an object $X$ of $\mathcal{D}$, we denote by $I_X^F$ the following category. The objects of $I_X^F$ are the pairs $(Y, f)$ of an object $Y$ of $\mathcal{C}$ and a morphism $f : F(Y) \to X$ in $\mathcal{D}$. For two objects $(Y_1, f_1)$ and $(Y_2, f_2)$ of $I_X^F$, the morphisms from $(Y_1, f_1)$ to $(Y_2, f_2)$ in $I_X^F$ are the morphisms $g : Y_1 \to Y_2$ in $\mathcal{C}$ satisfying $f_1 = f_2 \circ F(g)$.

2.1.6. Let $\mathcal{C}$ be an essentially $\mathfrak{U}$-small category. Let us choose a full subcategory $\mathcal{C}' \subset \mathcal{C}$ such that $\mathcal{C}'$ is $\mathfrak{U}$-small and that the inclusion functor $\mathcal{C}' \hookrightarrow \mathcal{C}$ is an equivalence of categories. We call such a full subcategory $\mathcal{C}'$ of $\mathcal{C}$ a $\mathfrak{U}$-small skeleton of $\mathcal{C}$. We denote by $\mathfrak{h}_{\mathcal{C}'} : \mathcal{C}' \to \text{Presh}(\mathcal{C})$ the composite of the inclusion functor $\mathcal{C}' \hookrightarrow \mathcal{C}$ with the functor $\mathfrak{h}_\mathcal{C}$.

For an object $G$ of $\text{Presh}(\mathcal{C})$, let $I_G$ denote the category $I_G^{\mathfrak{h}_\mathcal{C}}$. By definition, the objects of $I_G$ are the pairs $(X, \xi)$ of an object $X$ of $\mathcal{C}'$ and a morphism $\xi : \mathfrak{h}_\mathcal{C}(X) \to G$ in $\text{Presh}(\mathcal{C})$. 
For two objects \((X_1, \xi_1)\) and \((X_2, \xi_2)\) of \(I_G\), the morphisms from \((X_1, \xi_1)\) to \((X_2, \xi_2)\) in \(I_G\) are the morphisms \(f : X_1 \to X_2\) in \(C\) satisfying \(\xi_1 = \xi_2 \circ h(f)\).

This shows that the category \(I_G\) is \(\bullet\)-small. For an object \((X, \xi)\) of \(I_G\), we let \(y_{G,X}(\xi)\) denote the element of \(G(X)\) which is the image of \(\xi\) under the bijection \(y_{G,X} : \text{Hom}_{\text{Presh}(C)}(h_C(X), G) \xrightarrow{\sim} G(X)\) in \((2.1)\).

Let \(g : G \to H\) be a morphism in \(\text{Presh}(C)\). For an object \((X, \xi)\) of \(I_G\), let \(g_{X,\xi} \in H(X)\) denote the image of \(y_{G,X}(\xi) \in G(X)\) under the map \(G(X) \to H(X)\) given by \(g\). By varying \((X, \xi)\), we obtain an element \((g_{X,\xi})\) in the limit \(\lim_{\xi \in \text{Obj } I_G} H(X)\). It then can be checked easily that the map

\[
(2.2) \quad \text{Hom}_{\text{Presh}(C)}(G, H) \to \lim_{(X, \xi) \in \text{Obj } I_G} H(X)
\]

which sends \(g\) to \((g_{X,\xi})\) is bijective.

2.2. Sieves and Grothendieck topologies.

2.2.1. Let us recall the notion of sieve (cf. [SGA4], EXPOSE I, Définition 4.1, Arcata, (6.1)). Let \(C\) be a category and let \(X\) be an object of \(C\). A sieve on \(X\) is a full subcategory \(R\) of the overcategory \(C_{/X}\) satisfying the following condition: let \(f : Y \to X\) be an object of \(C_{/X}\) and suppose that there exist an object \(g : Z \to X\) of \(R\) and a morphism \(h : Y \to Z\) in \(C\) satisfying \(f = g \circ h\). Then \(f\) is an object of \(R\).

For a sieve \(R\) on \(X\), we denote by \(h_C(R)\) the following subpresheaf of \(h_C(X)\): for each object \(Y\) of \(C\), the subset \(h_C(R)(Y) \subset h_C(X)(Y) = \text{Hom}_C(Y,X)\) consists of the morphisms \(f : Y \to X\) in \(C\) such that \(f\) is an object of \(R\).

2.2.2. Let \(C\) and \(D\) be categories, let \(X\) be an object of \(C\), and let \(Y\) be an object of \(D\). Suppose that a covariant functor \(F : C_{/X} \to D_{/Y}\) is given. For a sieve \(R\) on \(Y\), we denote by \(F^*R\) the full subcategory of \(C_{/X}\) whose objects are those objects \(f : Z \to X\) of \(C_{/X}\) such that \(F(f)\) is an object of \(R\). It is then easy to check that \(F^*R\) is a sieve on \(X\).

Let \(G : C \to D\) be a covariant functor. Suppose that \(G(X) = Y\) and that \(F\) is equal to the covariant functor \(C_{/X} \to D_{/Y}\) induced by \(G\). In this case we denote the sieve \(F^*R\) on \(X\) by \(G^*R\).

Let \(f : X \to Z\) be a morphism in \(C\). Suppose that \(C = D\), \(Y = Z\), and \(F\) is equal to the covariant functor \(C_{/X} \to C_{/Z}\) which sends an object \(g : W \to X\) of \(C_{/X}\) to the object \(f \circ g\) of \(C_{/Z}\). In this case we denote the sieve \(F^*R\) on \(Y\) by \(R_{X,Z}\) and call it the pullback of \(R\) with respect to the morphism \(f\).

2.2.3. For a morphism \(f : Y \to X\) in a category \(C\), we let \(R_f\) denote the full subcategory of \(C_{/X}\) whose objects are the morphisms \(g : Z \to X\) in \(C\) such that \(g = f \circ h\) for some morphism \(h : Z \to Y\) in \(C\). It is then easy to check that \(R_f\) is a sieve on \(X\).

More generally, suppose that \(X\) is an object of a category \(C\) and that a family \((f_i : Y_i \to X)_{i \in I}\) of objects of \(C_{/X}\) indexed by a set \(I\) is given. We then let \(R_{(f_i)_{i \in I}}\) denote the full subcategory of \(C_{/X}\) whose objects are the morphisms \(g : Z \to X\) in \(C\) such that \(g = f_i \circ h_i\) for some \(i \in I\) and for some morphism \(h_i : Z \to Y_i\) in \(C\). It is then easy to check that \(R_{(f_i)_{i \in I}}\) is a sieve on \(X\).
2.2.4. Let us recall the notion of Grothendieck topology (cf. [SGA4 EXPOSE II, Définition 1.1], [SGA4 Arcata, (6.2)]). Let \( C \) be a category.

**Definition 2.2.1.** A Grothendieck topology \( J \) on \( C \) is an assignment of a collection \( J(X) \) of sieves on \( X \) to each object \( X \) of \( C \) satisfying the following conditions:

1. For any object \( X \) of \( C \), the overcategory \( C/_{X} \) is an element of \( J(X) \).
2. For any morphism \( f : Y \to X \) in \( C \) and for any element \( R \) of \( J(X) \), the sieve \( R \times_X Y \) on \( Y \) is an element of \( J(Y) \).
3. Let \( X \) be an object of \( C \), and let \( R, R' \) be two sieves on \( X \). Suppose that \( R \) is an element of \( J(X) \) and that for any object \( f : Y \to X \) of \( R \), the sieve \( R' \times_X Y \) on \( Y \) is an element of \( J(Y) \). Then \( R' \) is an element of \( J(X) \).

Let \( J \) be a Grothendieck topology on \( C \) and let \( X \) be an object of \( C \). We say that a morphism \( f : F \to h_{C}(X) \) of presheaves on \( C \) is a covering of \( X \) with respect to \( J \) if the image of \( f \) is equal to the subpresheaf \( h_{C}(R) \) of \( h_{C}(X) \) for some sieve \( R \) on \( X \) which belongs to \( J(X) \). We say that a morphism \( f : F \to G \) of presheaves on \( C \) is a covering with respect to \( J \) if for any object \( X \) of \( C \) and for any element \( \xi \in G(X) \), the first projection from the fiber product \( h_{C}(X) \times_{G} F \) of the diagram

\[
\begin{array}{ccc}
  h_{C}(X) & \xrightarrow{y_{G,X}(\xi)} & G \\
  \downarrow{f} & & \downarrow{F} \\
  & & 
\end{array}
\]

of \( X \) with respect to \( J \). When \( G = h_{C}(X) \) for some object \( X \) of \( C \), it follows from Condition (2) in Definition 2.2.1 that \( f \) is a covering with respect to \( J \) if and only if \( f \) is a covering of \( X \) with respect to \( J \). Let \( (f_{i} : Y_{i} \to X)_{i \in I} \) be a family of objects of \( C/_{X} \) indexed by a set \( I \). We say that \( (f_{i})_{i \in I} \) is a family covering \( X \) with respect to \( J \) if the sieve \( R_{(f_{i})_{i \in I}} \) on \( X \) belongs to \( J(X) \).

2.3. Semi-localizing collections.

**Definition 2.3.1.** Let \( C \) be a category. We say that a collection \( T \) of morphisms in \( C \) is semi-localizing if it satisfies the following conditions:

1. For any object \( X \) of \( C \), the identity morphism \( \text{id}_X \) belongs to \( T \).
2. The collection \( T \) is closed under composition.
3. Let \( Y_{1} \xrightarrow{f_{1}} X \xrightarrow{f_{2}} Y_{2} \) be a diagram in \( C \). Suppose that \( f_{1} \) belongs to \( T \). Then there exist an object \( Z \) in \( C \) and morphisms \( g_{1} : Z \to Y_{1} \) and \( g_{2} : Z \to Y_{2} \) such that \( g_{2} \) belongs to \( T \) and \( f_{1} \circ g_{1} = f_{2} \circ g_{2} \).

**Remark 2.3.2.** The three conditions above are taken from [GZ, I.2.2]. In their book, Gabriel and Zisman give a list of 4 conditions on a collection of morphisms in a category. They say that a collection admits a right calculus of fractions when the 4 conditions are met. In [GM, III.2.6], Gelfand and Manin call such collections "localizing". Our conditions are the first 3 conditions of the 4 conditions. Therefore we say that the collection is "semi-localizing". We note also that in Definition 10.3.4 of the textbook [W], Condition (3) is called the Õre condition.

2.3.1.

**Definition 2.3.3.** Let \( C \) be a category and let \( T \) be a collection of morphisms in \( C \). For an object \( X \) of \( C \), we let \( J_{T}(X) \) denote the collection of sieves \( R \) on \( X \) such that there exists an object \( f : Y \to X \) of \( R \) which belongs to \( T \).
Lemma 2.3.4. Let $\mathcal{T}$ be a semi-localizing collection of morphisms in a category $\mathcal{C}$. Then the assignment $J_{\mathcal{T}}$ of the collection $J_{\mathcal{T}}(X)$ to each object $X$ in $\mathcal{C}$ is a Grothendieck topology on the category $\mathcal{C}$.

Proof. We prove that $J_{\mathcal{T}}$ satisfies the three conditions in Definition 2.2.1.

By the definition of a sieve, $\mathcal{C}_X$ is a sieve. Since id$X$ belongs to $\mathcal{C}_X$ and $\mathcal{T}$, the sieve $\mathcal{C}_X$ is an element of $J_{\mathcal{T}}(X)$. Hence Condition (1) in Definition 2.2.1 is satisfied.

Let $X$ be an object of $\mathcal{C}$. It follows from Condition (1) in Definition 2.3.1 that the identity morphism id$X : X \to X$ in $\mathcal{C}$ belongs to $\mathcal{T}$. This shows that the sieve $\mathcal{C}_X$ of $X$ belongs to $J_{\mathcal{T}}(X)$. Hence $J_{\mathcal{T}}$ satisfies Condition (1) in Definition 2.2.1.

Let $f : Y \to X$ be a morphism in $\mathcal{C}$ and let $R$ be a sieve on $X$ which belongs to $J_{\mathcal{T}}(X)$. By the definition of $J_{\mathcal{T}}(X)$, there exist an object $Z$ of $\mathcal{C}$ and a morphism $g : Z \to Y$ such that $g$ belongs to $\mathcal{T}$ and that $g$ is an object of $R$. It follows from Condition (3) in Definition 2.3.1 that there exist an object $W$ of $\mathcal{C}$ and morphisms $g' : W \to Y$ and $f' : W \to Z$ in $\mathcal{C}$ such that $g' \in \mathcal{T}$ and such that $g = g \circ f'$. Since $g \circ f' = g \circ f'$ is an object of $R$, the morphism $g'$ is an object of $\mathcal{R}_X$. Since $g'$ belongs to $\mathcal{T}$, the sieve $\mathcal{R}_X$ on $X$ belongs to $J_{\mathcal{T}}(Y)$. This shows that $J_{\mathcal{T}}$ satisfies Condition (2) in Definition 2.2.1.

Let us turn to the proof of (3). Suppose $R$ belongs to $J_{\mathcal{T}}(X)$. Then there exists an object $f : Y \to X$ of $\mathcal{R}$ which belongs to $\mathcal{T}$. Since $R' \times_X Y$ belongs to $J_{\mathcal{T}}(Y)$, there exists a morphism $g : Z \to Y$ in $\mathcal{C}$ such that $g$ belongs to $\mathcal{T}$ and that the composite $f \circ g$ is an object of $R'$. It follows from Condition (2) in Definition 2.3.1 that $f \circ g$ belongs to $\mathcal{T}$. This shows that $R'$ belongs to $J_{\mathcal{T}}(X)$. Hence $J_{\mathcal{T}}$ satisfies Condition (3) in Definition 2.2.1. This completes the proof. □

2.3.2.

Definition 2.3.5. Let $\mathcal{C}$ be a category. For a collection $\mathcal{T}$ of morphisms in $\mathcal{C}$, we let $\hat{\mathcal{T}}$ denote the set of morphisms $f : Y \to X$ in $\mathcal{C}$ such that there exists a morphism $g : Z \to Y$ satisfying $f \circ g \in \mathcal{T}$.

Lemma 2.3.6. Let $\mathcal{T}$ be a semi-localizing collection of morphisms in $\mathcal{C}$. Then the collection $\hat{\mathcal{T}}$ contains $\mathcal{T}$ and is semi-localizing.

Proof. Since the identity morphisms are contained in $\mathcal{T}$, we see that $\mathcal{T} \subset \hat{\mathcal{T}}$ holds. In particular, Condition (1) is satisfied.

Let $f : Y \to X$ and $g : Z \to Y$ be morphisms in $\mathcal{C}$ which belong to $\hat{\mathcal{T}}$. Then there exist objects $Y'$, $Z'$ of $\mathcal{C}$, and morphisms $f' : Y' \to Y$ and $g' : Z' \to Z$ in $\mathcal{C}$ such that the composites $f \circ f'$ and $g \circ g'$ are morphisms in $\mathcal{C}$ which belong to $\mathcal{C}$. Since $\mathcal{T}$ is semi-localizing, there exist an object $W$ of $\mathcal{C}$ and morphisms $g'' : W \to Y'$ and $f'' : W \to Z'$ in $\mathcal{C}$ such that $g''$ belongs to $\mathcal{T}$ and such that $f' \circ g'' = g \circ g' \circ f''$. Since $\mathcal{T}$ is semi-localizing, there exist an object $V$ of $\mathcal{C}$ and a morphism $h : V \to W$ such that the composite $f \circ f' \circ g'' \circ h$ is a morphism in $\mathcal{C}$ which belongs to $\mathcal{T}$. Since $(f \circ g) \circ (g' \circ f'' \circ h) = f \circ f' \circ g'' \circ h$ is a morphism in $\mathcal{C}$ which belongs to $\mathcal{T}$, it follows from the definition of $\hat{\mathcal{T}}$ that we have $f \circ g \in \hat{\mathcal{T}}$. This shows that Condition (3) is satisfied.

Let $Y_1 \xrightarrow{f_1} X \xrightarrow{f_2} Y_2$ be a diagram in $\mathcal{C}$ and suppose that $f_1$ belongs to $\hat{\mathcal{T}}$. Let us take a morphism $f_3 : Y_3 \to Y_1$ in $\mathcal{C}$ such that $f_1 \circ f_3$ belongs to $\mathcal{T}$. Since the collection $\mathcal{T}$ satisfies Condition (3) in Definition 2.3.1 there exist an object $Z$ in $\mathcal{C}$ and morphisms $g_2 : Z \to Y_2$ and $g_3 : Z \to Y_3$ such that $g_2$ belongs to $\mathcal{T}$ and
Since $T \subset \hat{T}$, the morphism $g_2$ belongs to $\hat{T}$ and we have $f_1 \circ (f_3 \circ g_3) = f_2 \circ g_2$. This shows that Condition (3) in Definition 2.3.1 is satisfied for $\hat{T}$.

2.3.3.

**Lemma 2.3.7.** Let $T$ be a collection of morphisms in a category $C$. Then for any object $X$ of $C$, we have $J_T(X) = J_{\hat{T}}(X)$.

**Proof.** Let $X$ be an object of $C$. Since $T \subset \hat{T}$, we have $J_T(X) \subset J_{\hat{T}}(X)$. Hence it suffices to prove $J_{\hat{T}}(X) \subset J_T(X)$. Let $R$ be a sieve which belongs to $J_{\hat{T}}(X)$. Then there exists an object $f : Y \to X$ of $R$ which belongs to $\hat{T}$. It follows from the definition of $\hat{T}$ that there exists a morphism $g : Z \to Y$ in $C$ such that the composite $f \circ g$ belongs to $T$. Since $f \circ g$ is an object of $R$, it follows that the sieve $R$ belongs to $J_T(X)$. This proves that $J_{\hat{T}}(X) \subset J_T(X)$. This completes the proof.

2.4. $A$-topologies.

**Definition 2.4.1.** We say that a Grothendieck topology $J$ on a category $C$ is an $A$-topology if there exists a semi-localizing collection $T$ of morphisms in $C$ such that $J = J_T$. Such a collection $T$ is called a basis of the $A$-topology $J$.

**Definition 2.4.2.** For a Grothendieck topology $J$ on a category $C$, we let $T(J)$ denote the collection of morphisms $f : Y \to X$ in $C$ such that $R_f$ belongs to $J(X)$.

**Proposition 2.4.3.** Let $J = J_T$ be an $A$-topology on a category $C$. Then $T(J) = \hat{T}$ and it is a basis of the $A$-topology.

**Proof.** It follows from the definition of $T(J)$ that we have $T(J) = \hat{T}$. Hence from Lemma 2.3.6 we conclude that $T(J)$ is semi-localizing. Using Lemma 2.3.7 we have $J = J_T(J)$. This proves the claim.

It follows immediately from the definition that any basis of an $A$-topology $J$ is contained in $T(J)$.

2.4.1. Semi-cofiltered, atomic topology.

**Definition 2.4.4.** We say that a category $C$ is semi-cofiltered if the collection $\text{Mor}(C)$ of the morphisms in $C$ is semi-localizing.

Set $\mathcal{T} = \text{Mor}(C)$. Then Conditions (1)(2) of Definition 2.3.1 are satisfied automatically. Hence a category $C$ is semi-cofiltered if and only if $\mathcal{T}$ satisfies Condition (3). In [A, A.2.1.11 (h)] it is suggested to use the terminology “the right Ore condition” for this Condition (3).

When $C$ is semi-cofiltered, we call, following [BD] and [MM, p. 115], the Grothendieck topology $J_{\text{Mor}(C)}$ on $C$ the atomic topology on $C$.

2.5. Sheaves for $A$-topology.
2.5.1. Let \( \mathcal{C} \) be an essentially \( \mathcal{U} \)-small category and let \( J \) be a Grothendieck topology on \( \mathcal{C} \). Let \( F \) be a presheaf on \( \mathcal{C} \). For an object \( X \) of \( \mathcal{C} \) and for an element \( R \) of \( J(X) \), we let

\[
c_{F,X,R} : F(X) \xrightarrow{\psi_{F,X}} \text{Hom}_{\text{Presh}(\mathcal{C})}(\mathcal{h}_{C}(X), F) \to \text{Hom}_{\text{Presh}(\mathcal{C})}(\mathcal{h}_{C}(R), F)
\]
denote the map given by the composition with the inclusion \( \mathcal{h}_{C}(R) \to \mathcal{h}_{C}(X) \).

We say that a presheaf \( F \) on \( \mathcal{C} \) is \( J \)-separated (resp. a \( J \)-sheaf, or simply a sheaf if the topology is clear from the context) if the map \( c_{F,X,R} \) is injective (resp. bijective) for every object \( X \) of \( \mathcal{C} \) and for every element \( R \) of \( J(X) \). We denote by \( \text{Shv}(\mathcal{C}, J) \subset \text{Presh}(\mathcal{C}) \) the full subcategory of \( J \)-sheaves on \( \mathcal{C} \). We will use separated presheaves in our future paper.

**Proposition 2.5.1.** Let \( \mathcal{T} \) be a semi-localizing collection of morphisms in an essentially \( \mathcal{U} \)-small category \( \mathcal{C} \). Then a presheaf \( F \) on \( \mathcal{C} \) is a \( J- \)sheaf if and only if the following condition is satisfied:

\((*): For any object \( X \) of \( \mathcal{C} \) and for any morphism \( f : Y \to X \) in \( \mathcal{C} \) which belongs to \( \mathcal{T} \), the map \( c_{F,X,R_f} \) is bijective.

**Proof.** The “only if” part is easy, since for any \( f : Y \to X \) in \( \mathcal{T} \), it follows from the definition of \( J_{\mathcal{T}} \) that the sieve \( R_f \) belongs to \( J_{\mathcal{T}} \).

We prove the “if” part. Let \( F \) be a presheaf on \( \mathcal{C} \) that satisfies the condition \((*)\). Let \( X \) be an object of \( \mathcal{C} \) and let \( R \) be an element of \( J_{\mathcal{T}}(X) \). We prove that the map \( c_{F,X,R} \) is bijective. It follows from the definition of \( J_{\mathcal{T}}(X) \) that there exist an object \( Y \) of \( \mathcal{C} \) and a morphism \( f : Y \to X \) in \( \mathcal{C} \) such that \( f \) belongs to \( \mathcal{T} \) and that \( f \) is an object of \( R \). Then \( R_f \) is a full subcategory of \( R \), and hence \( \mathcal{h}_{C}(R_f) \) is a subpresheaf of \( \mathcal{h}_{C}(R) \).

Let

\[
c : \text{Hom}_{\text{Presh}(\mathcal{C})}(\mathcal{h}_{C}(R), F) \to \text{Hom}_{\text{Presh}(\mathcal{C})}(\mathcal{h}_{C}(R_f), F)
\]
denote the map given by the composition with the inclusion \( \mathcal{h}_{C}(R_f) \to \mathcal{h}_{C}(R) \). We then have \( c_{F,X,R_f} = c \circ c_{F,X,R} \). By assumption, the map \( c_{F,X,R_f} \) is bijective. Hence it suffices to prove that the map \( c \) is injective.

Suppose that the map \( c \) is not injective. Then there exist two elements \( \alpha_1, \alpha_2 \in \text{Hom}_{\text{Presh}(\mathcal{C})}(\mathcal{h}_{C}(R), F) \) such that \( \alpha_1 \neq \alpha_2 \) and \( c(\alpha_1) = c(\alpha_2) \). For \( i = 1, 2 \) and for an object \( Z \) of \( \mathcal{C} \), we let \( \alpha_i(Z) : \text{Hom}_{\mathcal{C}}(Z, \mathcal{h}_{C}(R))(Z) \to F(Z) \) denote the map induced by \( \alpha_i \) on the sections over \( Z \). Since \( \alpha_1 \neq \alpha_2 \), there exist an object \( g : Z \to X \) in \( R \) such that \( \alpha_1(Z)(g) \neq \alpha_2(Z)(g) \). Since \( \mathcal{T} \) is semi-localizing, there exists an object \( W \) in \( \mathcal{C} \) and morphisms \( f' : W \to Z \) and \( g' : W \to Y \) such that \( f' \) belongs to \( \mathcal{T} \) and that \( g \circ f' = f \circ g' \).

Since \( g \circ f' = f \circ g' \), the composite \( g \circ f' \) is an object of \( R_f \). Since \( c(\alpha_1) = c(\alpha_2) \), the two elements \( \alpha_1(Z)(g), \alpha_2(Z)(g) \in F(Z) \) are mapped to the same element of \( F(W) \) under the pullback map \( F(Z) \to F(W) \) with respect to \( f' \). This shows that the images of \( \alpha_1(Z)(g), \alpha_2(Z)(g) \in F(Z) \) under the map \( c_{F,Z,R_f} \) coincide. Since \( f' \) belongs to \( \mathcal{T} \), the map \( c_{F,Z,R_f} \) is bijective. Hence we have \( \alpha_1(Z)(g) = \alpha_2(Z)(g) \), which leads to a contradiction. This completes the proof. \( \square \)

**Lemma 2.5.2.** Let \( \mathcal{C} \) be a category and let \( f : Y \to X \) be a morphism in \( \mathcal{C} \). Let \( \text{un} \) consider the sieve \( R_f \) on \( X \). Let \( e_f : \mathcal{h}_{C}(Y) \to \mathcal{h}_{C}(R_f) \) denote the morphism of presheaves on \( \mathcal{C} \) defined as follows: for each object \( Z \) of \( \mathcal{C} \), the map \( e_f(Z) : \mathcal{h}_{C}(Y)(Z) = \text{Hom}_{\mathcal{C}}(Z,Y) \to \mathcal{h}_{C}(R_f)(Z) \) sends \( g \in \text{Hom}_{\mathcal{C}}(Z,Y) \) to \( f \circ g \in \mathcal{h}_{C}(R_f)(Z) \). Then for any presheaf \( F \) on \( \mathcal{C} \), the map \( \text{Hom}_{\text{Presh}(\mathcal{C})}(\mathcal{h}_{C}(R_f), F) \to \))
\[ \text{Corollary 2.5.3.} \text{ Let } \mathcal{C} \text{ be a category and let } T \text{ be a semi-localizing collection of morphisms in } \mathcal{C}. \text{ Then a presheaf } F \text{ on } \mathcal{C} \text{ is a } J_T\text{-sheaf if and only if for any morphism } f : Y \to X \text{ in } \mathcal{C} \text{ which belongs to } T, \text{ the map } F(f) : F(X) \to F(Y) \text{ is injective and its image is equal to the equalizer of } F(Y) \overset{F(f)}{\leftarrow} \overset{F(f)}{\rightarrow} \text{Hom}_{\text{Presh}(\mathcal{C})}(h_C(Y), F) \Rightarrow \text{Hom}_{\text{Presh}(\mathcal{C})}(h_C(Y), F) \times h_C(X) \overset{h_C(Y) \to h_C(Y)}{\rightarrow} h_C(Y). \]

\[ \text{Proof.} \text{ It follows from the definition that the subpresheaf } h_C(R_f) \text{ of } h_C(X) \text{ is equal to the image of the morphism } h_C(f) : h_C(Y) \to h_C(X) \text{ and the induced epimorphism } h_C(Y) \to h_C(R_f) \text{ is equal to } e_f. \text{ From this we see that the inclusion } h_C(Y) \times h_C(X) \overset{h_C(Y) \to h_C(Y)}{\to} h_C(Y) \times h_C(Y) \text{ is an equivalence relation in } \mathcal{C} \text{ in the sense of } \text{SGA}3 \text{ EXPOSE IV, 3.1}, \text{ and that the object } h_C(R_f) \text{ together with morphism } e_f \text{ is the quotient object (in the category Presh(}\mathcal{C}\text{)) of } h_C(Y) \text{ by this equivalence relation. Hence the claim follows.} \]

\[ \text{Corollary 2.5.3.} \text{ Let } \mathcal{C} \text{ be a category and let } T \text{ be a semi-localizing collection of morphisms in } \mathcal{C}. \text{ Then a presheaf } F \text{ on } \mathcal{C} \text{ is a } J_T\text{-sheaf if and only if for any morphism } f : Y \to X \text{ in } \mathcal{C} \text{ which belongs to } T, \text{ the map } F(f) : F(X) \to F(Y) \text{ is injective and its image is equal to the equalizer of } F(Y) \overset{F(f)}{\leftarrow} \overset{F(f)}{\rightarrow} \text{Hom}_{\text{Presh}(\mathcal{C})}(h_C(Y), F) \Rightarrow \text{Hom}_{\text{Presh}(\mathcal{C})}(h_C(Y), F) \times h_C(X) \overset{h_C(Y) \to h_C(Y)}{\rightarrow} h_C(Y). \]

\[ \text{Proof.} \text{ This follows from Proposition 2.5.4 and Lemma 2.5.2.} \]

3. Galois coverings

We define what it means for a morphism in a category to be a Galois covering. Then we define what it means for a topology to have enough Galois coverings. In Section 3.2, we collect some general nonsense concerning quotient objects in a category. It is necessary to make this notion precise since we deal with automorphisms of an object.

3.1. Galois coverings.

3.1.1. Let \( X \) be an object of a category \( \mathcal{C} \). Let \( Y_1 \) and \( Y_2 \) be objects of \( \mathcal{C} \) and suppose that morphisms \( f_1 : Y_1 \to X \) and \( f_2 : Y_2 \to X \) are given. We say that a morphism (resp. an isomorphism) \( g : Y_1 \to Y_2 \) in \( \mathcal{C} \) is a morphism (resp. an isomorphism) over \( X \) if \( f_2 \circ g = f_1 \). In other words, \( g \) is a morphism (resp. an isomorphism) over \( X \) if it is a morphism (resp. an isomorphism) from \( f_1 \) to \( f_2 \) in the overcategory \( \mathcal{C}/X \). The set of morphisms (resp. an isomorphism) from \( Y_1 \) to \( Y_2 \) over \( X \) is denoted by \( \text{Hom}_X(Y_1, Y_2) \) (resp. by \( \text{Isom}_X(Y_1, Y_2) \)). For a morphism \( f : Y \to X \) in \( \mathcal{C} \), we write \( \text{End}_X(Y) \) (resp. \( \text{Aut}_X(Y) \)) for \( \text{Hom}_X(Y, Y) \) (resp. \( \text{Isom}_X(Y, Y) \)). The set \( \text{End}_X(Y) \) forms a monoid with respect to the composition of morphisms, and \( \text{Aut}_X(Y) \) is equal to the group of invertible elements of \( \text{End}_X(Y) \).

\[ \text{Definition 3.1.1.} \text{ Let } S \text{ be a set on which a group } G \text{ acts from the left. We say that a map } \phi : S \to S' \text{ of sets is a pseudo } G\text{-torsor if the following three conditions are satisfied:} \]

\[ \begin{align*}
(1) \text{ We have } \phi(gs) = \phi(s) \text{ for any } g \in G \text{ and for any } s \in S. \\
(2) \text{ The group } G \text{ acts freely on } S. \\
(3) \text{ The map } G\backslash S \to S' \text{ induced by } \phi \text{ is injective.}
\end{align*} \]

\[ \text{Definition 3.1.2.} \text{ Let } \mathcal{C} \text{ be a category and let } f : Y \to X \text{ be a morphism in } \mathcal{C}. \text{ We say that } f \text{ in } \mathcal{C} \text{ is a Galois covering if there exists a group } G \text{ and a homomorphism } \rho : G \to \text{Aut}_X(Y) \text{ of groups such that the following condition is satisfied: for any} \]

\[ \begin{align*}
(1) \text{ We have } \phi(gs) = \phi(s) \text{ for any } g \in G \text{ and for any } s \in S. \\
(2) \text{ The group } G \text{ acts freely on } S. \\
(3) \text{ The map } G\backslash S \to S' \text{ induced by } \phi \text{ is injective.}
\end{align*} \]
object $Z$ of $C$, the map $\text{Hom}_C(Z,Y) \to \text{Hom}_C(Z,X)$ given by the composition with $f$ is a pseudo $G$-torsor. Here each $g \in G$ acts on $\text{Hom}_C(Z,Y)$ by the composition with $\rho(g)$.

3.1.2. Let $f : Y \to X$ be a Galois covering. The fiber product of $f$ and $f$ has the following description. Let $\rho : G \to \text{Aut}_X(Y)$ be a homomorphism as in Definition 3.1.2. Then it can be checked easily that the diagram

\[
\begin{array}{ccc}
\prod_{g \in \rho(G)} h_C(Y) & \xrightarrow{p_2} & h_C(Y) \\
\downarrow p_1 & & \downarrow h_C(f) \\
h_C(Y) & \xrightarrow{h_C(f)} & h_C(X)
\end{array}
\]

is cartesian, which induces an isomorphism from the coproduct $\prod_{g \in \rho(G)} h_C(Y)$ to the fiber product $h_C(Y) \times_{h_C(X)} h_C(Y)$. Here $p_1$ (resp. $p_2$) is the morphism $\prod_{g \in \rho(G)} h_C(Y) \to h_C(Y)$ whose component $g \in \rho(G)$ is the morphism $h_C(g)$ (resp. the identity morphism on $h_C(Y)$).

**Lemma 3.1.3.** Let $C$ be a category and let $f : Y \to X$ in $C$ be a Galois covering in $C$. Let $\rho : G \to \text{Aut}_X(Y)$ be the group homomorphism satisfying the condition in Definition 3.1.2. Then we have $\text{Aut}_X(Y) = \text{End}_X(Y)$ and $\rho$ is an isomorphism.

**Proof.** Let $\phi : \text{Hom}_C(Y,Y) \to \text{Hom}_C(Y,X)$ denote the map given by the composition with $f$. Then $\phi$ is a pseudo $G$-torsor. Since $\phi^{-1}(f) = \phi^{-1}(\phi(id_Y))$ is non-empty, the group $G$ acts simply transitively on $\phi^{-1}(f)$. Since $G \subset \text{Aut}_Y(X) \subset \phi^{-1}(f)$, it follows that $G = \text{Aut}_Y(X) = \phi^{-1}(f)$. This proves the claim. \qed

3.1.3. Enough Galois coverings.

**Definition 3.1.4.** Let $C$ be a category and $T$ be a collection of morphisms in $C$. Let $\mathcal{C} \subset C$ be the collection of Galois coverings. We say that $T$ has enough Galois coverings if $\mathcal{T} = \mathcal{T}$. We say that an $A$-topology $J$ on $C$ has enough Galois coverings if $\mathcal{T}(J)$ has enough Galois coverings.

**Corollary 3.1.5.** Let $C$ be a category and let $J$ be an $A$-topology on $C$. Suppose that $J$ has enough Galois coverings. Then a presheaf $F$ on $C$ is a sheaf if and only if for any object $X$ of $C$ and for any Galois covering $f : Y \to X$ in $C$ which belongs to $T(J)$, the map $F(f) : F(X) \to F(Y)$ is injective and its image is equal to the $\text{Aut}_X(Y)$-invariant part $F(Y)^{\text{Aut}_X(Y)}$ of $F(Y)$.

**Proof.** This follows from Lemma 3.1.3 Corollary 2.5.3 and the remark in Section 3.1.2. \qed

3.2. Quotient objects. In this paragraph we recall the notion of quotient object by an action of a group in a general category and prove some of the basic properties.

**Definition 3.2.1.** Let $C$ be a category, $Y$ an object in $C$, and $G$ a subgroup of $\text{Aut}_C(Y)$. A quotient object $X$ of $Y$ by $G$ is an object in $C$ equipped with a morphism $c : Y \to G \backslash Y$ in $C$ satisfying the following universal property: for any object $Z$ in $C$ and for any morphism $f : Y \to Z$ in $C$ satisfying $f \circ g = f$ for all $g \in G$, there exists a unique morphism $f : X \to Z$ such that $f = f \circ c$. In other words, a quotient object $X$ is an object in $C$ which co-represents the covariant functor from $C$ to the category of sets which associates, to each object $Z \in C$, the $G$-invariant part $\text{Hom}_C(Y,Z)^G$ of the set $\text{Hom}_C(Y,Z)$. We call the morphism $c : Y \to X$ the canonical quotient morphism.
3.2.1. A quotient object of $Y$ by $G$ is unique up to unique isomorphism in the following sense. Suppose that both $Y'_1$ and $Y'_2$ are quotient objects of $Y$ by $G$. We denote by $c_1 : Y \to Y'_1$ and $c_2 : Y \to Y'_2$ the canonical quotient morphisms. Then there exists a unique isomorphism $\alpha : Y'_1 \cong Y'_2$ satisfying $\alpha \circ c_1 = c_2$. This claim follows easily from the universality of quotient objects. We use the symbol $G \setminus Y$ to denote any quotient object of $Y$ by $G$.

3.2.2. There is another equivalent way of defining a quotient object. Let $*_G$ denote the category such that $*_G$ has only one object $*$, that the set $\text{Hom}_{*_G}(*,*)$ is equal to $G$, and the composite of morphisms and the identity morphism are given by the group structure of $G$. Then the quotient object of $Y$ by $G$ is nothing but a colimit in the category $C$ of the diagram $*_G \to C$ which sends $*$ to $Y$ and which sends $g : * \to *$ to $g : Y \to Y$ for all $g \in G$.

Lemma 3.2.2. Let the notation as in Definition 3.2.1. Then the canonical quotient morphism $c : Y \to G \setminus Y$ is an epimorphism.

Proof. Suppose that there exist an object $Z \in C$ and morphisms $f_1, f_2 : G \setminus Y \to Z$ satisfying $f_1 \circ c = f_2 \circ c$. We prove that $f_1 = f_2$.

We set $f = f_1 \circ c = f_2 \circ c$. It follows from the definition of $G \setminus Y$ that there exists a unique morphism $\overline{f} : G \setminus Y \to Z$ such that $f = \overline{f} \circ c$. By the uniqueness of $\overline{f}$, we have $f_1 = f_2 = \overline{f}$. This proves the claim. \hfill \Box

Lemma 3.2.3. Let $C$ be a category and let $C' \subset C$ be a full subcategory. Let $Y$ be an object in $C'$, and $G$ be a subgroup of $\text{Aut}_C(Y)$. Suppose that a quotient object $G \setminus Y$ of $Y$ by $G$ in $C$ exists and that $G \setminus Y$ is isomorphic in $C$ to an object $Z$ in $C'$. Then $Z$ is a quotient object of $Y$ by $G$ in $C'$.

Proof. The universality of $Z$ can be checked easily. \hfill \Box

Lemma 3.2.4. Let $C$ be a category and let $J$ be a Grothendieck topology on $C$. Suppose that any representable presheaf on $C$ is a $J$-sheaf. Let $f : Y \to X$ be a Galois covering in $C$ such that the sieve $R_f$ on $X$ belongs to $J(X)$. Then the object $X$ of $C$ together with the morphism $f$ is a quotient object of $Y$ by $\text{Aut}_X(Y)$.

Proof. Let $Z$ be an arbitrary object of $C$. It suffices to show that the map $\text{Hom}(X, Z) \to \text{Hom}(Y, Z)$ given by the composition with $f$ is injective and its image is equal to the $\text{Aut}_X(Y)$-invariant part $\text{Hom}(Y, Z)$. Since $\mathfrak{h}_C(Z)$ is a $J$-sheaf and $R_f$ belongs to $J(X)$, this follows from Corollary 3.1.5. \hfill \Box

4. $B$-sites

We define $B$-sites in this section and study their properties. The reader will find in Section 4.2 that the basic statements from Galois theory also hold true in our setting. Section 4.3 contains a technical proposition and its corollary. This is one place where it is very different from the classical Galois theory. In the classical case, the underlying category of the site contains a final object. In that case the proofs of those statements are much easier. In Section 4.4 we assume that the $B$-site has enough Galois coverings, and give an explicit description of the sheafification functor in terms of the Galois coverings. This will be used in Section 4.7.

We will use the convention for the terminology “poset” as in Section 5.1.
4.1. \textit{E-categories.}

\textbf{Definition 4.1.1.} We say that a category $\mathcal{C}$ is an \textit{E-category} if every morphism in $\mathcal{C}$ is an epimorphism.

\textbf{Lemma 4.1.2.} Let $\mathcal{C}$ be a category. Let $Y_1 \xrightarrow{f_1} X \xrightarrow{f_2} Y_2$ be a diagram in $\mathcal{C}$. Suppose that $f_2$ is a Galois covering in $\mathcal{C}$. Then for any two morphisms $h_1, h_2 : Y_1 \to Y_2$ which are over $X$, there exists a unique element $g \in \text{Aut}_X(Y_2)$ satisfying $h_1 = g \circ h_2$.

\textit{Proof.} Let $\alpha : \text{Hom}_C(Y_1, Y_2) \to \text{Hom}_C(Y_1, X)$ denote the map given by the composition with $f_2$. We have $\alpha(h_1) = \alpha(h_2) = f_1$. Since $f_2$ is a Galois covering, it follows from Lemma 3.1.3 that $\alpha$ is a pseudo $\text{Aut}_X(Y_2)$-torsor, i.e., the group $\text{Aut}_X(Y_2)$ acts simply transitively on the set $\alpha^{-1}(f_1)$. This proves the claim. $\square$

\textbf{Lemma 4.1.3.} Let $\mathcal{C}$ be an \textit{E-category}. Let $f : Y \to X$ be a Galois covering in $\mathcal{C}$. Suppose that $f$ is written as the composite $f = f_1 \circ f_2$ of two morphisms in $\mathcal{C}$. Then $f_2$ is a Galois covering in $\mathcal{C}$.

\textit{Proof.} Let $X'$ denote the target of the morphism $f_2$. The group $\text{Aut}_X(Y)$ is a subgroup of the group $\text{Aut}_X(Y)$. It suffices to show that, for any commutative diagram

$$
\begin{array}{ccc}
Z & \xrightarrow{h} & Y \\
\downarrow \alpha' & & \downarrow f_2 \\
Y & \xrightarrow{f_2} & X',
\end{array}
$$

there exists a unique automorphism $g \in \text{Aut}_X(Y')$ satisfying $h' = g \circ h$. Since $f$ is a Galois covering, there exists a unique automorphism $g \in \text{Aut}_X(Y)$ satisfying $h' = g \circ h$. Hence to prove the claim, it suffices to prove that $g \in \text{Aut}_X(Y)$. We have $f_2 \circ g \circ h = f_2 \circ h' = f_2 \circ h$. Since $h$ is an epimorphism, we have $f_2 \circ g = f_2$. Hence we have $g \in \text{Aut}_X(Y)$, which completes the proof. $\square$

4.2. \textit{B-sites.} We write $(\mathcal{C}, J)$ to denote a site whose underlying category is $\mathcal{C}$ and whose Grothendieck topology is $J$.

\textbf{Definition 4.2.1.} A \textit{B-site} $(\mathcal{C}, J)$ is a site satisfying the following conditions:

1. $\mathcal{C}$ is an \textit{E-category}.
2. $J$ is an \textit{A-topology}
3. For any diagram $Z \xrightarrow{\beta} Y \xrightarrow{f} X$ in $\mathcal{C}$, the composite $g \circ f$ belongs to $T(J)$ if and only if $f$ and $g$ belong to $T(J)$.

It follows from Proposition 2.4.3 that if $g \circ f \in T(J)$ then $g \in T(J)$. Hence we may replace Condition(3) above by the weaker condition: If $g \circ f \in T(J)$ then $f \in T(J)$.

Let $(\mathcal{C}, J)$ be a B-site. We say that a morphism $f : Y \to X$ in $\mathcal{C}$ is a Galois covering in $T(J)$ if $f$ belongs to $T(J)$ and $f$ is a Galois covering in $\mathcal{C}$.

\textbf{Example 4.2.2.} Here we give a basic example of a B-site. Let $S$ be a connected noetherian scheme. Let us consider the full subcategory $\text{EtConn}_S$ of the category of schemes over $S$ whose objects are connected $S$-schemes which are finite étale over $S$. Then the category $\text{EtConn}_S$ is semi-cofiltered, and the pair of $\text{EtConn}_S$ and the atomic topology is B-site.

If we do not assume them to be connected, Condition (3) is not satisfied.
**Lemma 4.2.3.** Let \((C, J)\) be a \(B\)-site. Let \(Y_1 \xrightarrow{f_1} X \xleftarrow{f_2} Y_2\) be a diagram in \(C\). Suppose that \(f_1\) is a Galois covering in \(T(J)\) and that there exists a morphism \(h : Y_1 \to Y_2\) over \(X\). Then

1. For any commutative diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{f'_2} & Y_1 \\
\downarrow{f'_1} & & \downarrow{f_1} \\
Y_2 & \xrightarrow{f_2} & X
\end{array}
\]

in \(C\), there exists an automorphism \(g \in \text{Aut}_X(Y_1)\) such that \(f'_1 = h \circ g \circ f'_2\). Moreover the morphism \(h \circ g\) is uniquely determined by the commutative diagram above.

2. The group \(\text{Aut}_X(Y_1)\) acts transitively on the set \(\text{Hom}_X(Y_1, Y_2)\) and the diagram

\[
\begin{array}{ccc}
\prod_{h \in \text{Hom}_X(Y_1, Y_2)} & h_C(Y_1) & \xrightarrow{f'_2} & h_C(Y_1) \\
\downarrow{f'_1} & \downarrow{h_C(f_1)} & & \downarrow{h_C(f_1)} \\
& h_C(Y_2) & \xrightarrow{h_C(f_2)} & h_C(X)
\end{array}
\]

in \(\text{Presh}(C)\) is cartesian. Here \(f'_1\) denotes the morphism whose component at \(h\) is the identity map for every \(h\), and \(f'_2\) denotes the morphism whose component at \(h\) is equal to \(h_C(h)\).

**Proof.** Let the notation be as in the claim (1). Observe that by Condition (3) of Definition 4.2.1 the morphism \(h\) belongs to \(T(J)\). Since \(T(J)\) is semi-localizing, there exist an object \(Z'\) of \(\mathcal{C}\) and morphisms \(h' : Z' \to Z\) and \(f'_{1} : Z' \to Y_1\) in \(\mathcal{C}\) such that \(h'\) belongs to \(T(J)\) and that the diagram

\[
\begin{array}{ccc}
Z' & \xrightarrow{h'} & Z \\
\downarrow{f''_1} & & \downarrow{f'_1} \\
Y_1 & \xrightarrow{h} & Y_2
\end{array}
\]

is commutative.

Set \(h'' = f'_2 \circ h' : Z' \to Y_1\). Since both \(h''\) and \(f''_1\) are morphisms over \(X\) and \(f_1\) is a Galois covering, it follows from Lemma 4.1.2 that there exist an element \(g \in \text{Aut}_X(Y_1)\) satisfying \(f''_1 = g \circ h''\). Hence we have \(f''_1 \circ h' = h \circ f''_1 = h \circ g \circ h'' = h \circ g \circ f'_2 \circ h'\). Since \(\mathcal{C}\) is an \(E\)-category, the morphism \(h'\) is an epimorphism. Hence we have \(f'_1 = h \circ g \circ f'_2\). The uniqueness of \(h \circ g\) follows since \(f'_2\) is an epimorphism. This proves the claim (1).

Let the notation be as in the claim (2). Let \(h_2 : Y_1 \to Y_2\) be any morphism in \(\mathcal{C}\) over \(X\). We apply the claim (1) to the commutative diagram

\[
\begin{array}{ccc}
Y_1 & \xrightarrow{\text{id}_{Y_1}} & Y_1 \\
\downarrow{h_2} & & \downarrow{f_1} \\
Y_2 & \xrightarrow{f_2} & X
\end{array}
\]
There exists an automorphism $g \in \text{Aut}_X(Y_1)$ such that $h_2 = h \circ g$. This proves that the action of $\text{Aut}_X(Y_1)$ on $\text{Hom}_X(Y_1, Y_2)$ is transitive. The rest of the claim (2) follows as an immediate consequence of the claim (1).

**Corollary 4.2.4.** Let $(C, J)$ be a $B$-site. Let $Y_1 \xrightarrow{f_1} X \xleftarrow{f_2} Y_2$ be a diagram in $C$. Suppose that $f_1$ is a Galois covering in $T(J)$ and that there exists a morphism $h : Y_1 \to Y_2$ over $X$. Then for any morphism $f' : Z \to Y_1$ in $C$, the map $\text{Hom}_X(Y_1, Y_2) \to \text{Hom}_X(Z, Y_2)$ given by the composite with $f'$ is bijective.

**Proof.** The injectivity follows since $f'$ is an epimorphism. We prove the surjectivity. Let $h' : Z \to Y_2$ be a morphism over $X$. It follows from Lemma 4.2.3 (1) that there exists an automorphism $g \in \text{Aut}_X(Y_1)$ such that $h' = h \circ g \circ f'$. This proves the surjectivity. □

4.2.1. Let $f : Y \to X$ be a morphism in a category $C$. Let $\tilde{\alpha} : Y \xrightarrow{\sim} Y$ and $\alpha : X \xrightarrow{\sim} X$ be automorphisms in $C$. We say that $\tilde{\alpha}$ descends to $\alpha$ via $f$, or that $\alpha$ ascends to $\tilde{\alpha}$, if the diagram

$$
\begin{array}{c}
Y & \xrightarrow{\tilde{\alpha} \cong} & Y \\
\downarrow{f} & & \downarrow{f} \\
X & \xrightarrow{\alpha \cong} & X
\end{array}
$$

is commutative. We say that an automorphism $\tilde{\alpha} \in \text{Aut}_C(Y)$ descends to $X$ via $f$ if it descends to some element in $\text{Aut}_C(X)$. Suppose that the morphism $f$ is an epimorphism. Then if $\tilde{\alpha} \in \text{Aut}_C(Y)$ descends to $X$ via $f$, then it descends to a unique element in $\text{Aut}_C(X)$ via $f$.

**Lemma 4.2.5.** Let $C$ be a category. Let $Y \xrightarrow{f} X \xleftarrow{f'} W$ be a diagram in the category $C$. Suppose that $f'$ is a Galois covering in $C$.

(1) Any automorphism $\bar{g} \in \text{Aut}_W(Y)$ descends to a unique element in $g \in \text{Aut}_W(X)$ via $f$.

(2) The map $\text{Aut}_W(Y) \to \text{Aut}_W(X)$ which sends $\bar{g}$ to $g$ is a group homomorphism.

**Proof.** We apply Lemma 4.1.2 to the two morphisms $f, f \circ \bar{g} : Y \to X$ over $W$. There exists a unique element $g \in \text{Aut}_W(X)$ satisfying $f \circ \bar{g} = g \circ f$. Hence $\bar{g}$ descends to $g$ via $f$. This proves the claim (1).

The claim (2) follows from the uniqueness of $g \in \text{Aut}_C(X)$ to which $\bar{g}$ descends via $f$. □

**Lemma 4.2.6.** Let $(C, J)$ be a $B$-site. Let $Y \xrightarrow{f} X \xleftarrow{f'} W$ be a diagram in $C$ such that the composite $f' \circ f$ is a Galois covering in $T(J)$.

(1) Any automorphism in $\text{Aut}_W(X)$ ascends to an automorphism in $\text{Aut}_W(Y)$ via $f$.

(2) $f'$ is a Galois covering if and only if any automorphism in $\text{Aut}_W(Y)$ descends to an automorphism in $\text{Aut}_W(X)$ via $f$.

(3) Suppose that $f'$ is a Galois covering. Then the group homomorphism $\text{Aut}_W(Y) \to \text{Aut}_W(X)$ in Lemma 4.2.5 induces a short exact sequence

$$1 \to \text{Aut}_X(Y) \to \text{Aut}_W(Y) \to \text{Aut}_W(X) \to 1$$
of groups.

Proof. The claim (1) follows from Lemma 4.2.3

Suppose that any automorphism in Aut_W(Y) descends to an automorphism in Aut_W(X). Let f'' : W'' → W be a morphism in C. Since C is an E-category, the group Aut_W(X) acts freely on the set Hom_W(W'', X). Hence to prove that f' is a Galois covering, it suffices to prove that the group Aut_W(X) acts transitively on the set Hom_W(W', X). Let h_1, h_2 : W' → X be morphisms over W. Since f belongs to T(J) and T(J) is semi-localizing, there exist an object W'' of C and morphisms h_1' : W'' → Y and f_2' : W'' → W' such that f_2' belongs to T(J) and that f ○ h_1' = h_1 ○ f_2'. It then follows from Lemma 4.2.3 (1) that there exists an automorphism g ∈ Aut_W(Y) satisfying h_2 ○ f_2' = f ○ g ○ h_1'. By assumption, the automorphism g descends to an automorphism g ∈ Aut_W(X) via f. Hence h_2 ○ f_2' = f ○ g ○ h_1' = g ○ f ○ h_1' = g ○ h_1 ○ f_2'. Since f_2' is an epimorphism, we have h_2 = g ○ h_1. Hence the group Aut_W(X) acts transitively on the set Hom_W(W', X). This proves that f' is a Galois covering if any automorphism in Aut_W(Y) descends to an automorphism in Aut_W(X). The converse follows immediately from Lemma 4.2.3. This proves the claim (2).

It follows from the claim (1) that the homomorphism Aut_W(Y) → Aut_W(X) is surjective. It is easy to see that the kernel of this homomorphism is equal to Aut_X(Y). This proves the claim (3). □

4.3. Cofinality.

4.3.1. For a poset P, we denote by P^{op} the dual of the poset P, regarded as a Ω-small category. Let C be a category and let F : P^{op} → C be a covariant functor. Let (X, (f_y)_{y ∈ P}) be a pair of an object X of C and a morphism f_y : X → F(y) in C for each y ∈ P. We say that the pair (X, (f_y)_{y ∈ P}) is an object of C over F if for any y_1, y_2 ∈ P with y_1 ≤ y_2, we have f_{y_1} = F(f_{y_2}) ◦ f_y, where g denotes the unique morphism from y_2 to y_1 in P^{op}.

Proposition 4.3.1. Let C be a category which is semi-cofiltered. Let P be a non-empty finite poset. Suppose that P is a rooted tree, which means that P has the bottom element and for any element y ∈ P, the subset {z ∈ P | z ≤ y} is a totally ordered set. Let P^{op} denote the dual of the poset P, regarded as a Ω-small category. Let F : P^{op} → C be a covariant functor from P^{op} to C. Then there exists an object (X, (f_y)_{y ∈ P}) of C over F.

Proof. We proceed by induction on the cardinality of P. If P consists of a single element y, then X = F(y) and f_y = id_X satisfies the desired property. Suppose that P consists of more than two elements. Since P is a finite poset, there exists a maximal element z ∈ P. Set P' = P \ {z}. Then P' is again a tree. Let F : P^{op} → C be a covariant functor. We denote by F' the restriction of F to P'^{op}. By the induction hypothesis, there exists an object (X', (f'_y)_{y ∈ P'}) in C over F'. We let z' ∈ P' denote the maximal element of the finite totally ordered set \{y ∈ P' | y ≤ z\}. Let f : z → z' denote the unique morphism in P^{op}. Let us consider the diagram

\[
F(z) \xrightarrow{F(f)} F(z') \xleftarrow{f^{-1}} X'
\]

in C. (We wrote X' for the source of f'_{z'}.) Since C is semi-cofiltered, there exist an object X of C and morphisms f' : X → F(z) and g' : X → X' in C satisfying
$F(f) \circ f' = f'_y \circ g'$. We set $f_y = f'_y \circ g'$ for $y \in P'$ and $f_z = f'$. It is then straightforward to check that the pair $(X, (f_y)_{y \in P})$ is an object in $C$ over $F$. This proves the claim.

4.3.2. When $T$ is a semi-localizing collection of morphisms in $C$, recall that we have $\hat{T} = T(J)$ where $J$ is the $A$-topology associated to $T$. We let $C(T(J))$ denote the following subcategory of $C$. The objects of $C(T(J))$ are the objects of $C$. For two objects $X, Y$ of $C$, the morphisms from $X$ to $Y$ in $C(T(J))$ are the morphisms from $X$ to $Y$ in $C$ which belong to $\hat{T}$. It follows from Lemma 2.3.6 that the composition of morphisms in $C(T(J))$ is well-defined.

**Corollary 4.3.2** (cofinality). Let $(C, J)$ be a $B$-site. Suppose that $T(J)$ has enough Galois coverings. Let $P$ be a finite poset. Suppose that $P$ is a rooted tree. Let $P^{\text{op}}$ denote the dual of the poset $P$ regarded as a small category. Let $F : P^{\text{op}} \to C$ be a covariant functor such that $F(h)$ belongs to $T(J)$ for any morphism $h$ in $P^{\text{op}}$. Then there exists an object $(X, (f_y)_{y \in P})$ of $C$ over $F$ such that the morphism $f_y : X \to F(y)$ is a Galois covering in $T(J)$ for each $y \in P$.

**Proof.** Set $\hat{T} = T(J)$. Recall that $\hat{T} = T$. It follows from the definition that $\hat{T}(\hat{T}^\text{op})$ is semi-cofiltered.

We regard $F$ as a covariant functor from $P^{\text{op}}$ to $C(\hat{T}(J))$. It follows from Proposition 4.3.1 that there exists an object $(X', (f'_{y'})_{y' \in P'})$ of $C(\hat{T}(J))$ over $F$. Let $y_0 \in P$ denote the bottom element. Since $T(J)$ has enough Galois coverings, there exist an object $X$ of $C$ and a morphism $h : X \to X'$ in $C$ such that $h$ belongs to $T(J)$ and that the composite $f'_{y_0} \circ h$ is a Galois covering. For each $y \in P$, we set $f_y = f'_{y} \circ h$. Then $(X, (f_y)_{y \in P})$ is an object in $C$ over $F$. Since the morphism $f_{y_0}$ factors through $f_y$, it follows from Lemma 4.1.3 that $f_y$ is a Galois covering in $T(J)$. This proves the claim. □

4.4. **An explicit construction of the sheafification functor.** Let $(C, J)$ be a $B$-site. Suppose that $C$ is essentially $\mathcal{U}$-small. In the paragraphs below, we give an explicit description of the sheafification functor $a_J : \text{Presh}(C) \to \text{Shv}(C, J)$ when there are enough Galois coverings.

4.4.1. Let us fix a set $G$ of objects of $C$ such that any object of $C$ is isomorphic to an object which belongs to $G$. Let $F : C^{\text{op}} \to (\text{Sets})$ be a presheaf on $C$. In the following paragraphs we give a description of the sheaf $a_J(F) : C^{\text{op}} \to (\text{Sets})$ associated to $F$.

4.4.2. For an object $X$ of $C$, we let $\text{Gal}/X$ denote the following category. The objects in $\text{Gal}/X$ are the pairs $(Y, f)$ of an object $Y$ of $C$ which belongs to $G$ and a morphism $f : Y \to X$ in $C$ which is a Galois covering in $T(J)$. For two objects $(Y_1, f_1)$ and $(Y_2, f_2)$ of $\text{Gal}/X$, the set of morphisms from $(Y_1, f_1)$ to $(Y_2, f_2)$ in $\text{Gal}/X$ is the set $\text{Aut}_X(Y_2) \backslash \text{Hom}_X(Y_1, Y_2)$. It is clear that the category $\text{Gal}/X$ is $\mathcal{U}$-small. It follows from Lemma 4.3.2 that for any two objects $(Y_1, f_1)$ and $(Y_2, f_2)$ of $\text{Gal}/X$, there exists at most one morphism from $(Y_1, f_1)$ to $(Y_2, f_2)$ in $\text{Gal}/X$. Hence the composition of two morphisms is well-defined. It follows from Lemma 4.3.2 that the category $\text{Gal}/X$ is cofiltered.

Let $X$ be an object of $C$. For an object $(Y, f)$ of $\text{Gal}/X$, we set $F_{/X}(Y, f) = F(Y)^{\text{Aut}_X(Y)}$. Let $(Y_1, f_1)$ and $(Y_2, f_2)$ be two objects of $\text{Gal}/X$ and let $h : Y_1 \to Y_2$ be a morphism over $X$. Then it follows from Lemma 4.1.2 that the map $F(h) :
$F(Y_2) \to F(Y_1)$ sends an element in the $\text{Aut}_X(Y_2)$-invariant part $F_{/X}(Y_2, f_2) \subset F(Y_2)$ to an element in $F_{/X}(Y_1, f_1)$. The induced map $F_{/X}(h) : F_{/X}(Y_2, f_2) \to F_{/X}(Y_1, f_1)$ depends only on the class of the class of $h$ in $\text{Aut}_X(Y_2) \setminus \text{Hom}_X(Y_1, Y_2)$. We obtain a contravariant functor $F_{/X}$ from $\text{Gal}/X$ to the category of sets.

For an object $(Y, f)$ of $\text{Gal}/X$, Lemma 2.5.2 gives a bijection $F_{/X}(Y, f) \cong \text{Hom}_{\text{Presh}(\mathcal{C})}(\theta_{\mathcal{C}}(R_f), F)$ which is functorial in $(Y, f)$. It follows from Definition 2.3.1 and from that there are enough Galois coverings, the set $\{R_f \mid f \in \text{Obj} \text{Gal}/X\}$ is cofinal in the collection $J(X)$. Hence it follows from the construction of the sheafification functor, given in Section 3 of [SGA4, EXPOSE II], that we have a bijection

\[(4.2) \quad a_J(F)(X) \cong \lim_{\to} F_{/X} := \lim_{(Y,f)} F_{/X}(Y,f)\]

where the colimit is taken over the objects in the $\mathcal{U}$-small category $\text{Gal}/X$.

4.4.3. Let $f : X \to X'$ be a morphism in $\mathcal{C}$. We give a description of the restriction map $a_J(F)(f) : a_J(F)(X') \to a_J(F)(X)$ via the bijection (4.2). Let $\text{Gal}/f$ denote the full subcategory of $\text{Gal}/X$ whose objects are the pairs $(Y, h)$ in $\text{Gal}/X$ such that the composite $f \circ h : Y \to X'$ is a Galois covering in $\mathcal{C}$. It follows from Lemma 4.3.2 that the subcategory $\text{Gal}/f$ is cofinal in $\text{Gal}/X$.

To each object $(Y, h)$ in $\text{Gal}/f$, we associate the object $(Y, f \circ h)$ in $\text{Gal}/X'$. It is easy to check that this defines a functor $\text{Gal}/f \to \text{Gal}/X'$.

**Lemma 4.4.1.** The functor $\text{Gal}/f \to \text{Gal}/X'$ is fully faithful, and its image is cofinal in $\text{Gal}/X'$.

**Proof.** First we prove that the functor is fully faithful. Let $f'_1 : Y_1 \to X$ and $f'_2 : Y_2 \to X$ be Galois coverings such that both $f \circ f'_1$ and $f \circ f'_2$ are Galois coverings, and let $h : Y_1 \to Y_2$ be a morphism over $X'$. It suffices to prove that there exists an automorphism $g \in \text{Aut}_{X'}(Y_2)$ such that $g \circ h$ is a morphism over $X$. We apply Lemma 1.2.3(1) to the commutative diagram

\[
\begin{array}{ccc}
Y_1 & \xrightarrow{h} & Y_2 \\
\downarrow{f'_1} & & \downarrow{f \circ f_2} \\
X & \xrightarrow{f} & X'.
\end{array}
\]

There exists $g \in \text{Aut}_{X'}(Y_2)$ such that $f'_1 = f'_2 \circ g \circ h$. This proves that the functor is fully faithful.

The cofinality of the image of this functor follows from Corollary 4.3.2. □

4.4.4. We define the map $f^* : \lim_{\to} F_{/X'} \to \lim_{\to} F_{/X}$ as follows. Since $\text{Gal}/f$ is cofinal in $\text{Gal}/X$, the set $\lim_{\to} F_{/X}$ is equal to the colimit

\[\lim_{\to} F_{/X}|_{\text{Gal}/f} := \lim_{(Y,h) \in \text{Obj} \text{Gal}/f} F_{/X}(Y, h)\]

of the restriction of $F_{/X}$ to $\text{Gal}/f$. It follows from Lemma 4.4.1 that $\lim_{\to} F_{/X'}$ is equal to the colimit

\[\lim_{\to} F_{/X'}|_{\text{Gal}/f} := \lim_{(Y,h) \in \text{Obj} \text{Gal}/f} F_{/X'}(Y, f \circ h)\]
of the restriction of $F_{/X'}$ to $\text{Gal}/f$. The inclusion map $F_{/X'}(Y, f \circ h) = F(Y)^\text{Aut}_{X'}(Y) \hookrightarrow F(Y)^\text{Aut}_X(Y) = F_{/X}(Y, f)$ for each object $(Y, h)$ in $\text{Gal}/f$ gives a natural transform $F_{/X'}|_{\text{Gal}/f} \to F_{/X}|_{\text{Gal}/f}$. Passing to the colimit we obtain the desired map $f^* : \varprojlim F_{/X'} \to \varprojlim F_{/X}$.

It then follows from the construction of $a_J(F)$ given in Section 3 of [SGA4, EXPOSE II] that via the bijections (4.2) for $X$ and $X'$, the map $f^*$ gives the restriction map $a_J(f) : a_J(F)(X) \to a_J(F)(X')$.

5. Grid, fiber functor, Galois monoid of a $Y$-site

The main aim of this section is to state our main theorem (Theorem 5.8.1).

We define what we call a $Y$-site, which is a $B$-site satisfying some more conditions. We define a pregrid and a grid of such a site. When we are considering the étale site of the spectrum of a field, the pregrid and the grid are analogues of the algebraically closure of the field. A pregrid may also be regarded as an analogue of the maximal tree associated to a graph.

To a grid, we associate an analogue of the fiber functor (in the sense of Galois category theory). We also have an analogue of the absolute Galois group, which turns out to be a monoid (not necessarily a group) with certain family of specified submonoids.

We prove in Section 6 that under certain conditions on cardinality, a grid of a $Y$-site exists.

The main theorem states that, under some cardinality assumptions, when a grid exists, we have the equivalence of categories of sheaves on the site and the smooth sets of the absolute Galois monoid.

In Sections 5.1 and 5.2 we give some convention on the term poset and define the term quasi-poset. This is only to fix terminology: we regard a partially ordered set naturally as a category and we call that poset in this article. A quasi-poset is a category which is similar to and almost a poset. In Section 5.3 we have a lemma on the non-emptiness of certain projective limit. This will be used in Section 6 where we construct a pregrid of a $Y$-site assuming certain cardinality conditions.

5.1. Convention: Posets. By a partially ordered set, or a poset, we mean a set equipped with a partial order. We assume throughout that the partial order is antisymmetric.

Let $P$ be a poset. Let $C_P$ denote the following category. The objects of $C_P$ are the elements of $P$. For two elements $x, y$ of $P$, there exists at most one morphism from $x$ to $y$ in $C_P$ and there exists a morphism from $x$ to $y$ if and only if $x \leq y$. The assignment that sends a partially ordered set $P$ to the category $C_P$ is a functor from the category of partially ordered sets to the category of categories. This functor is faithful.

In what follows, we regard a poset as a category by the functor above. By abuse of notation, we say that a category is a poset if it lies in the image of the functor above.

5.2. Quasi-posets.

Definition 5.2.1. We say that a category $C$ is a quasi-poset if the following two conditions are satisfied:

(1) The category $C$ is non-empty and essentially $\aleph$-small.
(2) The category $C$ is thin, i.e., for any two objects $X, Y$ of $C$, there exists at most one morphism from $X$ to $Y$ in $C$.

Let $C$ be a quasi-poset satisfying the following condition.

(3) The category $C$ is skeletal, i.e., any two objects of $C$ which are isomorphic are equal.

Then $C$ is a poset.

5.2.1. Let $C$ be a quasi-poset. Then there exists a poset subcategory (i.e., a subcategory which is a poset) $C_1$ of $C$ such that the inclusion functor $i : C_1 \hookrightarrow C$ is an equivalence of categories. The subcategory $C_1$ is unique in the following sense: for any other such poset subcategory $i' : C'_1 \hookrightarrow C$, there exists a unique isomorphism $\alpha : C_1 \cong C'_1$ of categories such that the functor $i$ is naturally isomorphic to the composite functor $i' \circ \alpha$. We call the poset subcategory $C_1$ a poset skeleton of $C$.

The following lemma can be checked easily:

**Lemma 5.2.2.** Let $C$ and $C'$ be quasi-posets and let $C_1$ and $C'_1$ be their poset skeletons. Then for any functor $F : C \to C'$, there exists a unique functor $F_1 : C_1 \to C'_1$ such that the diagram

$$
\begin{array}{ccc}
C_1 & \xrightarrow{F_1} & C'_1 \\
\downarrow i & & \downarrow i' \\
C & \xrightarrow{F} & C',
\end{array}
$$

where $i$ and $i'$ denote the inclusion functors, is commutative up to natural isomorphisms. $\square$

**Corollary 5.2.3.** Any equivalence of categories between two posets is an isomorphism of categories. $\square$

It follows from the condition (3) that for any category $C$ and for any poset $C'$, the natural isomorphisms between two functors from $C$ to $C'$ are the identities. In particular, the 2-category of posets (the full subcategory of the category of $U$-small categories where the objects are posets) can be regarded as a 1-category.

5.3. **Torsors under pro-groups.** Let $I$ be a directed partially ordered set and let $G = (G_i)_{i \in I}$ a projective system of groups indexed by the elements of $I$. For $i, j \in I$ with $i \leq j$, we denote by $\phi_{j,i} : G_j \to G_i$ the transition homomorphism in the projective system $G$. A (left) $G$-torsor is a projective system $(S_i)_{i \in I}$ of sets equipped with a left action of the group $G_i$ on $S_i$ for each $i \in I$ such that $S_i$ is a left $G_i$-torsor for each $i \in I$ and that for any two $i, j$ with $i \leq j$ and for any $g \in G_j$, $s \in S_j$, we have $f_{j,i}(g \cdot s) = \phi_{j,i}(g) \cdot f_{j,i}(s)$, where $f_{j,i} : S_j \to S_i$ denotes the transition map.

We say that the projective system $G$ has trivial first non-abelian cohomology if for any $G$-torsor $(S_i)_{i \in I}$, the limit $\lim \leftarrow_{i \in I} S_i$ of sets is non-empty.

**Lemma 5.3.1.** Let $I$ be a directed partially ordered set and let $G = (G_i)_{i \in I}$ be a projective system of groups. Suppose that at least one of the following conditions are satisfied:

1. For every $i \in I$, $G_i$ is a finite group.
2. $I$ is a finite set.
3. $I$ is a countable set, and the transition maps are surjective.
Then $G$ has trivial first non-abelian cohomology.

Proof. In the case when Condition (1) is satisfied, the claim follows since any filtered limit of non-empty finite sets is non-empty. In the other cases the claim can be checked more directly. □

We can rephrase the condition that $G$ has trivial first non-abelian cohomology by introducing the notion of the first non-abelian cohomology $R^1\lim_{i\in I} G_i$. It is a pointed set defined as follows. Let $J$ denote the set of pairs $(i, j) \in I \times I$ with $i \leq j$. Let $Z^1\lim_{i\in I}(G_i)$ denote the set of elements $(g_{i,j})_{(i,j)\in J} \in \prod_{(i,j)\in J} G_i$ satisfying $g_{i,j}\phi_{j,i}(g_{j,k}) = g_{i,k}$ for any $i, j, k \in I$ with $i \leq j \leq k$. We say that two elements $(g_{i,j})$ and $(g'_{i,j})$ in $Z^1\lim_{i\in I}(G_i)$ are equivalent if there exists an element $(h_i) \in \prod_{i\in I} G_i$ satisfying $g'_{i,j} = h_i^{-1}g_{i,j}\phi_{j,i}(h_j)$ for any $(i, j) \in J$. We define $R^1\lim_{i\in I} G_i$ to be the quotient of $Z^1\lim_{i\in I}(G_i)$ under the equivalence relation above. We regard $R^1\lim_{i\in I} G_i$ as the set pointed at the class of $(1)_{(i,j)\in J} \in Z^1\lim_{i\in I}(G_i)$.

Lemma 5.3.2. Let $G = (G_i)_{i\in I}$ be a projective system of groups. Then $G$ has trivial first non-abelian cohomology if and only if $R^1\lim_{i\in I} G_i$ consists of one point.

Proof. Let $J$ denote the set of pairs $(i, j)$ with $i \leq j$. For $(i, j) \in J$, let $\phi_{j,i} : G_j \to G_i$ denote the transition map.

For an element $z = (g_{i,j})_{(i,j)\in J}$ of $Z^1\lim_{i\in I} G_i$, we associate a left $G$-torsor $(S_{z,i})_{i\in I}$ as follows. For $i \in I$ we set $S_{z,i} = G_i$, and for $(i, j) \in J$, let $f_{j,i} : S_{z,j} \to S_{z,i}$ denote the map which sends $g \in G_j$ to $\phi_{j,i}(g)g_{i,j}^{-1}$. Then $(S_{z,i})_{i\in I}$ with the system $(f_{j,i})_{(i,j)\in J}$ of transition maps is a left $G$-torsor which we denote by $S_z$. On then can check easily that the map which sends $z \in Z^1\lim_{i\in I} G_i$ to the isomorphism class of $S_z$ induces a bijection from $R^1\lim_{i\in I} G_i$ to the set of isomorphism classes of left $G$-torsors. Thus the claim follows. □

5.4. $\Lambda$-sites.

Definition 5.4.1. We say that a category $\mathcal{C}$ is $\Lambda$-connected if for any two objects $X, Y$ in $\mathcal{C}$, there exist an object $Z$ in $\mathcal{C}$ and morphisms $Z \to X$ and $Z \to Y$ in $\mathcal{C}$.

Definition 5.4.2. Let $(\mathcal{C}, J)$ be a $B$-site. Recall that $\mathcal{T}(J) = \widehat{\mathcal{T}(J)}$. We say that a $B$-site $(\mathcal{C}, J)$ is a $Y$-site if the following properties hold:

1. $\mathcal{C}$ is essentially $1$-small.
2. $\mathcal{C}(\mathcal{T}(J))$ is $\Lambda$-connected.
3. $\mathcal{T}(J)$ has enough Galois coverings.

5.5. Pregrids.

Definition 5.5.1. A pregrid of a $\Lambda$-connected category $\mathcal{C}$ is a pair $(\mathcal{C}_1, \iota_1)$ of a $\Lambda$-small category and a covariant functor $\iota_1 : \mathcal{C}_1 \to \mathcal{C}$ satisfying the following conditions:

1. The category $\mathcal{C}_1$ is a poset and is $\Lambda$-connected.
2. The functor $\iota_1$ is essentially surjective.
3. For any object $X$ of $\mathcal{C}_1$, the functor $\mathcal{C}_{1,X/} \to \mathcal{C}_{/\iota_1(X)}$ between the overcategories induced by $\iota_1$ is essentially surjective.
4. For any object $X$ of $\mathcal{C}_1$, the functor $\mathcal{C}_{1,X/} \to \mathcal{C}_{/\iota_1(X)}$ between the undercategories induced by $\iota_1$ is an equivalence of categories.
Let \((\mathcal{C}, J)\) be a \(Y\)-site. Let \((\mathcal{C}_0, \iota_0)\) be a pair of a category \(\mathcal{C}_0\) and a functor \(\iota_0 : \mathcal{C}_0 \rightarrow \mathcal{C}\).

**Definition 5.5.2.** (edge objects) A morphism \(f\) of \(\mathcal{C}_0\) is called of type \(J\) if the morphism \(\iota_0(f)\) belongs to \(\mathcal{T}(J)\). An object \(X\) of \(\mathcal{C}_0\) is called an edge object of \(\mathcal{C}_0\) if for any morphism \(f : Y \rightarrow X\) in \(\mathcal{C}_0\) is of type \(J\).

For any object \(X\) of \(\mathcal{C}_0\), we let \(\iota_{0,X'}\) denote the functor \(\iota_{0,X'} : \mathcal{C}_{0,X'} \rightarrow \mathcal{C}_{\iota_0(X')}\) between undercategories induced by \(\iota_0\).

**Definition 5.5.3.** (grids) Let \((\mathcal{C}, J)\) be a \(Y\)-site. We define a grid of \((\mathcal{C}, J)\) to be a pair \((\mathcal{C}_0, \iota_0)\) of a \(\Lambda\)-small category \(\mathcal{C}_0\) and a functor \(\iota_0 : \mathcal{C}_0 \rightarrow \mathcal{C}\) satisfying the following properties:

1. The category \(\mathcal{C}_0\) is a poset and is \(\Lambda\)-connected.
2. For any object \(X'\) of \(\mathcal{C}\), there exists an edge object \(X\) of \(\mathcal{C}_0\) such that \(\iota_0(X)\) is isomorphic to \(X'\) in \(\mathcal{C}\).
3. For any object \(X\) of \(\mathcal{C}_0\) and for any morphism \(f : Y \rightarrow \iota_0(X)\) in \(\mathcal{C}\) which belongs to \(\mathcal{T}(J)\), there exist a morphism \(f' : Y' \rightarrow X\) in \(\mathcal{C}_0\) and an isomorphism \(\alpha : Y \xrightarrow{\cong} \iota_0(Y')\) in \(\mathcal{C}\) satisfying \(f = \iota_0(f') \circ \alpha\).
4. For any object \(X\) of \(\mathcal{C}_0\), the functor \(\iota_{0,X'} : \mathcal{C}_{0,X'} \rightarrow \mathcal{C}_{\iota_0(X')}\) is an equivalence of categories.

**5.6. The absolute Galois monoid associated to a grid.** Let \((\mathcal{C}_0, \iota_0)\) be a grid of a \(Y\)-site \((\mathcal{C}, J)\).

**5.6.1. The absolute Galois monoid \(M(\mathcal{C}_0, \iota_0)\).** We denote by \(M(\mathcal{C}_0, \iota_0)\) the set of pairs \((\alpha, \gamma_\alpha)\) of an endomorphism \(\alpha : \mathcal{C}_0 \rightarrow \mathcal{C}_0\) of the category \(\mathcal{C}_0\) and a natural isomorphism \(\gamma_\alpha : \iota_0 \cong \iota_0 \circ \alpha\). For two elements \((\alpha, \gamma_\alpha)\) and \((\beta, \gamma_\beta)\) of \(M(\mathcal{C}_0, \iota_0)\) we define the composite \((\alpha, \gamma_\alpha) \circ (\beta, \gamma_\beta) \in M(\mathcal{C}_0, \iota_0)\) to be

\[
(\alpha, \gamma_\alpha) \circ (\beta, \gamma_\beta) = (\alpha \circ \beta, (\beta^* \gamma_\alpha) \circ \gamma_\beta)
\]

where \(\beta^* \gamma_\alpha\) denotes the natural isomorphism \(\iota_0 \circ \beta \cong \iota_0 \circ \alpha \circ \beta\) induced by \(\gamma_\alpha\). The binary operation \(- \circ -\) on \(M(\mathcal{C}_0, \iota_0)\) gives a structure of monoid on the set \(M(\mathcal{C}_0, \iota_0)\) whose unit element of \(M(\mathcal{C}_0, \iota_0)\) is equal to \((\text{id}_{\mathcal{C}_0}, \text{id}_{\iota_0})\). We call this monoid the absolute Galois monoid associated to the grid \((\mathcal{C}_0, \iota_0)\).

**5.6.2. The associated submonoids.** Let \(X\) be an object of \(\mathcal{C}_0\). We define the submonoid \(\mathbb{K}_X \subset M(\mathcal{C}_0, \iota_0)\) to be

\[
\mathbb{K}_X = \{ (\alpha, \gamma_\alpha) \in M(\mathcal{C}_0, \iota_0) \mid \alpha(X) = X, \gamma_\alpha(X) = \iota_0(\alpha(X)) \cong \iota_0(X) \}
\]

We use this monoid only when \(X\) is an edge object. We will see later (Lemma 5.7.1) that when \(X\) is an edge object, \(\mathbb{K}_X\) is a group.

**5.7. The fiber functor associated to a grid.**

**5.7.1. A grid is cofiltered.**

**Lemma 5.7.1.** Let \(\mathcal{C}'\) be a category which is non-empty, thin, and \(\Lambda\)-connected. Then the category \(\mathcal{C}'\) is cofiltered.

**Remark 5.7.2.** Here we assume that the notion “cofiltered” is defined as in I.2.7 of SGA4. There are several equivalent definitions for this notion. For some choice of the definition, the statement of Lemma 5.7.1 below may be an immediate consequence of the definition.
Proof. Since $\mathcal{C}$ is non-empty and thin, it suffices to show that $\mathcal{C}'$ is semi-cofiltered. Let $Y_1 \xrightarrow{f_1} X \xleftarrow{f_2} Y_2$ be a diagram in $\mathcal{C}'$. Since $\mathcal{C}'$ is $\Lambda$-connected, there exists a diagram $Y_1 \xrightarrow{g_1} Z \xrightarrow{g_2} Y_2$ in $\mathcal{C}'$. By using that $\mathcal{C}'$ is thin again, we have $f_1 \circ g_1 = f_2 \circ g_2$, which proves the claim.

Corollary 5.7.3. Let $(\mathcal{C}_0, \iota_0)$ be a grid of a $\Lambda$-site $(\mathcal{C}, J)$ such that $\mathcal{C}$ is non-empty. Then $\mathcal{C}_0$ is cofiltered.

Proof. The claim follows immediately from the definition.

5.7.2. The fiber functor associated to a grid. Let $(\mathcal{C}_0, \iota_0)$ be a grid of a $\Lambda$-site $(\mathcal{C}, J)$ such that $\mathcal{C}$ is non-empty. Let $\mathcal{C}_1$ denote the full subcategory of $\mathcal{C}_0$ whose objects are the edge objects of $\mathcal{C}_0$.

Lemma 5.7.4. Let $X$ be an edge object of $\mathcal{C}_0$ and let $f : Y \to X$ be a morphism in $\mathcal{C}_0$. Then $Y$ is an edge object of $\mathcal{C}_0$.

Proof. Let $g : Z \to Y$ be an arbitrary morphism in $\mathcal{C}_0$. Since $X$ is an edge object of $\mathcal{C}_0$, the morphisms $f$ and $f \circ g$ are of type $J$. Hence it follows from Condition (3) of Definition 4.2.1 that morphism $g$ is of type $J$. This shows that $Y$ is an edge object of $\mathcal{C}_0$.

Lemma 5.7.5. The edges objects are cofinal in $\mathcal{C}_0$.

Proof. Let $X$ be an object of $\mathcal{C}_0$. Let us choose an edge object $Y$ of $\mathcal{C}_0$. Then $\mathcal{C}_0$ is $\Lambda$-connected, there exists an object $Z$ of $\mathcal{C}_0$ and morphisms $Z \to X$ and $Z \to Y$.

It follows from Lemmas 5.7.3 that $Z$ is an edge object of $\mathcal{C}_0$. We thus obtain a morphism $Z \to X$ from an edge object of $\mathcal{C}_0$ to $X$. Since $X$ is arbitrary, this shows that the edge objects are cofinal in $\mathcal{C}_0$.

Lemma 5.7.6. The category $\mathcal{C}_1$ is cofiltered.

Proof. The category $\mathcal{C}_1$ is non-empty and thin since it is a full subcategory of $\mathcal{C}_0$. Hence it suffices to show that $\mathcal{C}_1$ is semi-cofiltered. Let $Y \xrightarrow{f} X \xleftarrow{g} X'$ be a diagram in $\mathcal{C}_1$. It follows from Corollary 5.7.3 that there exist an object $Y'$ of $\mathcal{C}_0$ and morphisms $f' : Y' \to X'$ and $g' : Y' \to Y$ in $\mathcal{C}_0$ satisfying $g \circ f' = f \circ g'$. Since it follows from Lemma 5.7.4 that $Y'$ is an object of $\mathcal{C}_1$, this shows that the category $\mathcal{C}_1$ is semi-cofiltered, which proves the claim.

For a presheaf $F$ on $\mathcal{C}$, we define $\omega_{(\mathcal{C}_0, \iota_0)}$ to be the filtered colimit

$$\omega_{(\mathcal{C}_0, \iota_0)}(F) = \lim_{X \in \text{Obj}(\mathcal{C}_1)} F(\iota_0(X)).$$

We note that, since the objects of $\mathcal{C}_1$ are cofinal in $\mathcal{C}_0$ by Lemma 5.7.5, the natural map

$$\omega_{(\mathcal{C}_0, \iota_0)}(F) \to \lim_{X \in \text{Obj}(\mathcal{C}_0)} F(\iota_0(X))$$

is bijective. By associating $\omega_{(\mathcal{C}_0, \iota_0)}(F)$ to each presheaf $F$ on $\mathcal{C}$, we obtain the functor $\omega_{(\mathcal{C}_0, \iota_0)}$ from $\text{Presh}(\mathcal{C})$ to the category of sets. By abuse of notation, we denote by the same symbol $\omega_{(\mathcal{C}_0, \iota_0)}$ the restriction of $\omega_{(\mathcal{C}_0, \iota_0)}$ to the full subcategory $\text{Shv}(\mathcal{C}, J) \subset \text{Presh}(\mathcal{C})$. 
5.7.3. The action of the absolute Galois monoid. Let $F$ be a presheaf on $C$. We define the action of the monoid $M_{(\mathcal{C}_0,\omega_0)}$ on the set $\omega_{(\mathcal{C}_0,\omega_0)}(F)$. Let $(\alpha, \gamma_\alpha) \in M_{(\mathcal{C}_0,\omega_0)}$. We define the map $(\alpha, \gamma_\alpha)_* : \omega_{(\mathcal{C}_0,\omega_0)}(F) \to \omega_{(\mathcal{C}_0,\omega_0)}(F)$ to be the composite

$$\omega_{(\mathcal{C}_0,\omega_0)}(F) = \lim \frac{F(t_0(X))}{X} \to \lim \frac{F(t_0(\alpha(X)))}{X} \to \lim \frac{F(t_0(Y))}{Y \in \text{Obj} (\mathcal{C}_0)} \cong \omega_{(\mathcal{C}_0,\omega_0)}(F),$$

where $X$ runs over the edge objects of $\mathcal{C}_0$ and $j$ is the map induced by the inclusion

$$\{\alpha(X) \mid X \in \text{Obj} (\mathcal{C}_1)\} \subset \text{Obj} (\mathcal{C}_0).$$

**Definition 5.7.7.** We say that a left $M_{(\mathcal{C}_0,\omega_0)}$-set $S$ is smooth if for any $s \in S$ there exists an edge object $X$ of $\mathcal{C}_0$ such that $s = gs$ holds for any $g \in \mathbb{K}_X$. We denote by $(M_{(\mathcal{C}_0,\omega)}$-set)_{sm} the category of smooth $M_{(\mathcal{C}_0,\omega_0)}$-sets.

**Remark 5.7.8.** We give a remark concerning the use of the term smooth above. A locally profinite group is defined to be a locally compact Hausdorff group such that the compact open subgroups form a basis for the neighborhoods of the unit. A smooth representation of such groups is defined to be a representation where each vector has an open isotropy subgroup. We refer to Casselman’s article [C] for generalities. Smooth representations of locally profinite groups are of interest in the theory of automorphic forms. Examples of such groups include GL$_n(\mathbb{A}_f)$ or GL$_n(F)$ where $\mathbb{A}_f$ denotes the ring of finite adeles of some global field, and $F$ is a nonarchimedian local field.

**Remark 5.7.9.** We will see later in Section 10.1 that the absolute Galois monoid is naturally equipped with the structure of a topological monoid such that the category of smooth sets is equivalent to the category of discrete sets with continuous action for this structure.

We note that if $F$ is a presheaf on $C$, then for each edge object $X$ of $\mathcal{C}_0$, the image of the map $F(X) \to \omega_{(\mathcal{C}_0,\omega_0)}(F)$ given by the universality of colimit is contained in the $\mathbb{K}_X$-invariant part $\omega_{(\mathcal{C}_0,\omega_0)}(F)^{\mathbb{K}_X}$ of $\omega_{(\mathcal{C}_0,\omega_0)}(F)$, with respect to the action of $M_{(\mathcal{C}_0,\omega_0)}$ defined as above. It follows that $\omega_{(\mathcal{C}_0,\omega_0)}(F)$ is a smooth $M_{(\mathcal{C}_0,\omega_0)}$-set. It is straightforward to check that the action of $M_{(\mathcal{C}_0,\omega_0)}$ on $\omega_{(\mathcal{C}_0,\omega_0)}(F)$ is functorial on $F$. Hence the functor $\omega_{(\mathcal{C}_0,\omega_0)}$ factors through the category of smooth $M_{(\mathcal{C}_0,\omega_0)}$-sets. By abuse of notation, we denote by the same symbol $\omega_{(\mathcal{C}_0,\omega_0)}$ the latter functor and its restriction to the full subcategory $\text{Shv}(\mathcal{C}, J) \subset \text{Presh}(\mathcal{C})$.

5.8. The statement of the main theorem.

5.8.1. Conditions on cardinality. The main result of this article is the following. For a category $\mathcal{D}$, let us consider the following conditions:

1. For any object $X, Y$ of $\mathcal{D}$, the set $\text{Hom}_\mathcal{D}(X, Y)$ is a finite set.
2. There exists a set $S$, whose cardinality is at most countable, of objects of $\mathcal{D}$ such that to any object $X$ of $\mathcal{D}$, there exists a morphism from an object of $\mathcal{D}$ which belongs to $S$.

These conditions appear in the statement of our main theorem below. Technically, these will be appear only at the following two points:

1. In the proof of the existence of a pregrid, where we use Lemma 5.3.1 and
2. In the proof of Lemma 7.6.1.
5.8.2. The statement.

**Theorem 5.8.1.** Let \((\mathcal{C}, J)\) be a \(Y\)-site. Let \((\mathcal{C}_0, \tau_0)\) be a grid. Then the functor
\[
\omega_{(\mathcal{C}_0, \tau_0)} : \text{Shv}(\mathcal{C}, J) \to (M(\mathcal{C}_0, \tau))_{\text{sm}}
\]
is faithful.

If we moreover assume that, for each object \(X\) of \(\mathcal{C}\), the overcategory \(\mathcal{C}(\mathcal{T}(J))/X\) satisfies at least one of the two conditions in Section 5.8.7 then the functor \(\omega_{(\mathcal{C}_0, \tau_0)}\) is an equivalence of categories.

6. Existence of a grid of a \(Y\)-site

Assuming a certain cardinality condition, in this section, we prove the existence of a grid of a \(Y\)-site.

6.1. Existence of a pregrid.

6.1.1. Existence of a pregrid.

**Proposition 6.1.1.** Let \((\mathcal{C}, J)\) be a \(Y\)-site. Suppose that there exists an object \(X_0\) of \(\mathcal{C}(\mathcal{T}(J))\) such that the overcategory \(\mathcal{C}(\mathcal{T}(J))/X_0\) satisfies at least one of the two cardinality conditions in Section 5.8.7. Then there exists a pregrid of \(\mathcal{C}(\mathcal{T}(J))\).

6.1.2. Construction step 1. Set \(\mathcal{D} = \mathcal{C}(\mathcal{T}(J))\) for short.

We use the category \(\text{Gal}/X_0\) introduced in Section 4.4.2. We use the following terminology. For an object \((Y, f)\) of \(\text{Gal}/X_0\), we denote by \(G(Y, f)\) the Galois group of \(f\). For an object \((Y, f)\) of \(\text{Gal}/X_0\) and for an object \(h\) of \(\mathcal{C}/X_0\), we say that \((Y, f)\) dominates \(h\) if there exists a morphism from \(f\) to \(h\) in \(\mathcal{C}/X_0\).

Since \(\mathcal{C}\) is essentially \(\mathcal{U}\)-small, we can take a \(\mathcal{U}\)-small set \(V\) of objects of \(\mathcal{D}/X_0\) such that any object of \(\mathcal{D}/X_0\) is isomorphic to an object which belongs to \(V\). When \(\mathcal{C}\) satisfies Condition (2), we may and will assume that \(V\) is at most countable. Let \(S \subset V\) be a subset. For each object \((Y, f)\) of \(\text{Gal}/X_0\) we set
\[
\tilde{U}_S(Y, f) = \prod_{s \in S} \text{Hom}_{\mathcal{C}/X_0}(f, s)
\]
and \(U_S(Y, f) = G(Y, f)\backslash \tilde{U}_S(Y, f)\), where the quotient is taken with respect to the diagonal action of \(G(Y, f)\). (When \(S = \emptyset\) is an empty set, we understand that the set \(\tilde{U}_0(Y, f)\) consists of a single element.) Let \((Y, f)\) and \((Y', f')\) be two objects of \(\text{Gal}/X_0\). Let \(h : Y' \to Y\) be a morphism from \(f'\) to \(f\) in \(\mathcal{C}/X_0\). The composition with \(h\) gives a map \(U_S(h) : U_S(Y, f) \to U_S(Y', f')\). It can be checked easily that the map \(U_S(h)\) depends only on the class of \(h\) in \(\text{Hom}_{\text{Gal}/X_0}((Y', f'), (Y, f))\). Hence by associating \(U_S(Y, f)\) to each object \((Y, f)\) of \(\text{Gal}/X_0\), we obtain a presheaf on \(\text{Gal}/X_0\) which we denote by \(U_S\). Since the topology has enough Galois coverings, there exists an object \((Y, f)\) of \(\mathcal{D}/X_0\) which dominates \(s\) for every \(s \in S\). For such an object \((Y, f)\) the set \(U_S(Y, f)\) is non-empty. Let \((Y, f)\) and \((Y', f')\) be two objects of \(\text{Gal}/X_0\) such that both \(U_S(Y, f)\) and \(U_S(Y', f')\) are non-empty. Let \(h\) be a morphism from \(f'\) to \(f\) in \(\mathcal{D}/X_0\). It follows from Lemma 4.2.4 that the morphism \(h\) induces a surjective homomorphism \(G(Y', f') \to G(Y, f)\) of groups. It follows from Corollary 4.2.4 that the composition with \(h\) give a bijection \(\text{Hom}_{\mathcal{D}/X_0}(f, s) \to \text{Hom}_{\mathcal{D}/X_0}(f', s)\). This shows that the map \(U_S(h) : U_S(Y, f) \to U_S(Y', f')\) is bijective.
Let us fix a $\mathcal{U}$-small subcategory $\text{Gal}'/X_0$ of $\text{Gal}/X_0$ which is equivalent to $\text{Gal}/X_0$. For a finite subset $S \subset V$, we set

$$U(S) := \lim_{(Y,f)} U_S(Y,f)$$

where the colimit is taken over the objects in the category $\text{Gal}'/X_0$. We note that, for any object $(Y,f)$ of $\text{Gal}'/X_0$, the map $U_S(Y,f) \to U(S)$ is bijective if $U_S(Y,f)$ is non-empty. Let $S$ and $S'$ be two finite subsets of $V$ with $S' \subset S$. The projection $\tilde{U}_S(Y,f) \to \tilde{U}_{S'}(Y,f)$ for each object $(Y,f)$ of $\text{Gal}/X_0$ gives a morphism $U_S \to U_{S'}$ of presheaves on $\text{Gal}/X_0$. The collection $(U_S)_{S \subset V}$ forms a projective system of presheaves on $\text{Gal}/X_0$ indexed by the finite subsets of $V$. Hence $(U(S))_{S \subset V}$ forms a projective system of sets indexed by the finite subsets of $V$.

**Lemma 6.1.2.** The projective limit $U = \lim_{\to S} U(S)$ is non-empty.

**Proof.** It is easy to see that $U(S)$ is non-empty for any finite subset $S \subset V$ and the transition map $U(S) \to U(S')$ is surjective for any finite subsets $S, S' \subset V$ with $S' \subset S$. Recall that we have assumed Condition (1) or (2) on $C$. If $\mathcal{C}$ satisfies Condition (1), then $U(S)$ is a finite set for every finite subset $S \subset V$. If $\mathcal{C}$ satisfies Condition (2), then $V$ is at most countable. In the first case, one can use the fact that the projective limit of non-empty finite sets is non-empty to conclude. In the second case, one can see directly that the limit $U = \lim_{\to S} U(S)$ is non-empty. \qed

6.1.3. Construction step 2. Let us fix an element $(x_S) \in U$. For each finite subset $S \subset V$, let us fix an object $(Y_S, f_S)$ of $\text{Gal}'/X_0$ such that $U_S(Y_S, f_S)$ is non-empty. Let $y_S$ denote the element of $U_S(Y_S, f_S)$ which is mapped to $x_S$ via the bijection $U_S(Y_S, f_S) \to U(S)$. Let us take a representative $\tilde{y}_S = (\tilde{y}_{S,s})_{s \in S} \in \tilde{U}_S(Y_S, f_S)$ of $y_S$. By definition, $\tilde{y}_{S,s}$ is a morphism from $f_S$ to $s$ in $D/X_0$ for each $s \in S$. Let $C_{1,S} = C_{1,S,\tilde{y}_S}$ denote the full subcategory of the undercategory $D_{Y_S}/f_S$ whose objects are the morphisms $f : Y_S \to Z$ in $D$ satisfying $f = g \circ \tilde{y}_{S,s}$ for some $s \in S$ and for some morphism $g$ in $D$. Since $D$ is essentially $\mathcal{U}$-small and all morphisms in $D$ are epimorphisms, it follows that the category $C_{1,S}$ is a quasi-poset. Let us choose a poset skeleton $C_{1,1}$ of $C_{1,1}$.

For each pair $(S_1, S_2)$ of finite subsets of $V$ with $S_1 \subset S_2$, we construct a fully faithful functor $C_{1,S_1} \to C_{1,S_2}$ as follows. Let us choose an object $(W,f)$ of $\text{Gal}/X_0$ which dominates both $f_{S_1}$ and $f_{S_2}$. For $i = 1, 2$, let us choose a morphism $h_i$ from $f$ to $f_i$ in $D/X_0$. Let $\tilde{y}_{S_i} \in \tilde{U}_{S_i}(W, f)$ denote the image of $\tilde{y}_{S_i}$ under the map $\tilde{U}_{S_i}(Y_{S_i}, f_{S_i}) \to \tilde{U}_{S_i}(W, f)$ given by the composition with $h_i$. Since $x_{S_2}$ is mapped to $x_{S_1}$ under the map $U(S_2) \to U(S_1)$, there exists an element $\alpha \in G(W, f)$ such that $\alpha \cdot \tilde{y}_{S_1}$ is equal to the image of $\alpha \circ \tilde{y}_{S_2}$ under the projection $\tilde{U}_{S_1}(W, f) \to \tilde{U}_{S_1}(W, f)$. We have a diagram

$$D_{Y_{S_1}} \xrightarrow{-\circ h_1} D_{W} \xrightarrow{-\circ \alpha} D_{W} \xleftarrow{-\circ h_2} D_{Y_{S_2}}$$

of undercategories, which induces a diagram

$$C_{1,S_1,\tilde{y}_{S_1}} \xrightarrow{1} C_{1,S_1,\tilde{y}_{S_1}} \xrightarrow{(2)} C_{1,S_2,\alpha \tilde{y}_{S_2}} \xrightarrow{(3)} C_{1,S_2,\tilde{y}_{S_2}} \xleftarrow{(4)} C_{1,S_2,\tilde{y}_{S_2}}$$

**Lemma 6.1.3.** The functors $(1), (3)$, and $(4)$ are equivalences of categories and the functor $(2)$ is fully faithful.
Proof. It can be checked easily that the functor (2) is fully faithful, that the functor (3) is an equivalence of categories, and that the functors (1) and (4) is essentially surjective. Since the morphisms 
\( h_1 \) and \( h_2 \) in \( D \) are epimorphisms, it follows that the functors (1) and (4) are fully faithful, which proves the claim.

By taking the composite of (1), (2), (3) and the inverse of (4), we obtain a fully faithful functor \( i_{S_1,S_2} : C_{1,S_2} \to C_{1,S_2} \) between posets. When we fix \((W,f)\), \( h_1 \), and \( h_2 \), the element \( \alpha \in G(W,f) \) is not uniquely determined and the functor \( - \circ \alpha : D_{W/} \to D_{W/} \) may depend on the choice of \( \alpha \). However the functor (3) induced by \( - \circ \alpha \) is independent of the choice of \( \alpha \). Using this observation, one can check straightforwardly that the functor \( i_{S_1,S_2} \) does not depend on the choice of \((W,f)\), \( h_1 \), \( h_2 \), and \( \alpha \). For three finite subsets \( S_1,S_2,S_3 \subset V \) with \( S_1 \subset S_2 \subset S_3 \), we have \( i_{S_1,S_3} = i_{S_2,S_3} \circ i_{S_1,S_2} \). Hence the pair \(([C_{1,S}]_{S \subset V}, (i_{S_1,S})_{S \subset S})\) forms an inductive system in the category of poset categories We define \( C_1 \) to be the colimit of this inductive system. (The notation \( C_1 \) has already appeared in Section 6.2.)

The conflict is explained in Remark 6.2.7.) By taking a limit of the composite \( C_{1,S} \to C_{1,S} \to D \), we obtain a functor \( i_1 : C_1 \to D \), which is uniquely determined up to natural equivalences.

6.1.4. Proof of Proposition 6.1.1. We claim that the pair \((C_1,i_1)\) is a pregrid of \( D \). By definition the category \( C_1 \) is a poset. We show that \( C_1 \) is \( \Lambda \)-connected. Let \( Z_1 \) and \( Z_2 \) be two objects of \( C_1 \). For \( i = 1, 2 \), take a finite subset \( S_i \subset V \) such that \( Z_i \) belongs to the image of the functor \( C_{1,S_i} \to C_1 \). We set \( S = S_1 \cup S_2 \). For \( i = 1, 2 \), there exists an object \( z_i \) of \( C_{1,S_i} \) whose image under the functor \( C_{1,S_i} \to C_1 \) is equal to \( Z_i \). Then \( z_i \) is, by definition, a morphism from from \( Y_S \) to \( Z_i \). We set \( T = S \cup \{ f \} \). Observe that the morphism \( \tilde{y}_{T,f} : Y_T \to W \) is an object of \( C_{1,T} \).

Hence the diagram

\[
\begin{array}{ccc}
Z_1' & \xleftarrow{z_1 \circ \alpha} & W \\
\downarrow{z_2 \circ \alpha} & & \downarrow{z_2 \circ \alpha} \\
Z_2' & \xleftarrow{z_2 \circ \alpha} & Z_2
\end{array}
\]

in \( D \) gives a diagram \( z_1 \circ \alpha \circ \tilde{y}_{T,f} \leftarrow \tilde{y}_{T,f} \to z_2 \circ \alpha \circ \tilde{y}_{T,f} \) in \( C_{1,T} \). The last diagram induces a diagram of the form \( Z_1' \leftarrow Z' \to Z_2 \) for some \( Z' \) in \( C_1 \). This proves that the category \( C_1 \) is \( \Lambda \)-connected. Next we prove that the functor \( i_1 \) is essentially surjective. Let us take an arbitrary object \( X \) of \( D \). Since \( D \) is \( \Lambda \)-connected, there exists a diagram

\[
\begin{array}{ccc}
X_0 & \xleftarrow{f_0} & Y \\
\downarrow{f} & & \downarrow{f} \\
X & \xrightarrow{j} & X
\end{array}
\]

in \( D \). We may choose this diagram in such a way that \( f_0 \) belongs to \( V \). Let \( S = \{ f_0 \} \). Then \( \tilde{y}_{S,f_0} \), regarded as a morphism in \( D \), is an object of \( C_{1,S} \). Let \( X' \) denote the object of \( C_1 \) given by the object \( f \circ \tilde{y}_{S,f_0} \) of \( C_{1,S} \). It then follows from the definition of \( i_1 \) that the object \( i_1(X') \) of \( D \) is isomorphic to \( X \), which proves that the functor \( i_1 \) is essentially surjective.

Lemma 6.1.4. The pair \((C_1,i_1)\) satisfies the condition (3) in Definition 5.5.1.

Proof. Let \( X \) be an object of \( C_1 \) and let \( f : Y \to i_1(X) \) be a morphism in \( D \). Let us choose a finite subset \( S \subset V \) such that \( X \) belongs to the image of an object \( h \) of \( C_{1,S} \) under the functor \( C_{1,S} \to C_1 \). The object \( h \) of \( C_{1,S} \) is, by definition, a morphism from \( Y_S \) to an object \( X' \) in \( D \) and there exists an element \( s \in S \) such that the morphism \( h \) is the composite of the morphism \( \tilde{y}_{S,s} \) and a morphism \( h_1 : Y_s \to X' \), where \( Y_s \) denotes the domain of \( s \). Observe that \( X' \) and \( i_1(X) \) are isomorphic in
\[ D \]. Let us fix an isomorphism \( \alpha : \iota_1(X) \xrightarrow{\sim} X' \). Since \( D \) is semi-cofiltered, there exists a commutative diagram

\[
\begin{array}{ccc}
W & \xrightarrow{h'} & Y \\
\downarrow f' & & \downarrow \alpha \circ f \\
Y_S & \xrightarrow{h} & X'
\end{array}
\]

in \( D \). We may and will assume that the composite \( f_S \circ f' \) is a Galois covering and belongs to \( V \). We set \( T = S \cup \{ f_S \circ f' \} \). Since \( x_T \) is mapped to \( x_S \) under the map \( U(T) \to U(S) \), there exists an element \( \beta \in G(Y_T, f_T) \) such that \( \tilde{y}_{S,S} \circ f' \circ \tilde{y}_{T,f_S \circ f'} \) is equal to \( \tilde{y}_{T,S} \circ \beta \) and that the image of \( h \) under the functor \( C_{1,S} \to C'_{1,T} \) is isomorphic to \( h \circ f' \circ \tilde{y}_{T,f_S \circ f'} \circ \beta^{-1} \). Since \( f_S \circ f' \) is a Galois covering, there exists an element \( \beta' \in G(W; f_S \circ f') \) satisfying \( \tilde{y}_{T,f_S \circ f'} \circ \beta = \beta' \circ \tilde{y}_{T,f_S \circ f'} \). Hence \( h \circ f' \circ \tilde{y}_{T,f_S \circ f'} \circ \beta^{-1} = \alpha \circ f' \circ \beta' \circ \tilde{y}_{T,f_S \circ f'} \). Hence the morphism \( \alpha \circ f \) can be regarded as a morphism from \( h \circ f' \circ \tilde{y}_{T,f_S \circ f'} \) to \( h \circ f' \circ \tilde{y}_{T,f_S \circ f'} \circ \beta^{-1} \) in \( C_{1,T} \).

This morphism induces an object in the overcategory \( C_{1,X} \) whose image under the functor \( C_{1,X} \to D_{1,X} \) induced by \( \iota_1 \) is isomorphic to \( f \). This proves that the pair \( (C_{1}, \iota_1) \) satisfies Condition (3) in Definition 5.5.1. \( \square \)

**Lemma 6.1.5.** The pair \( (C_{1}, \iota_1) \) satisfies Condition (4) in Definition 5.5.1

**Proof.** Let \( X \) be an object of \( C_{1} \). Let us choose a finite subset \( S \subset V \) such that \( X \) belongs to the image of an object \( h \) of \( C_{1,S} \) under the functor \( C_{1,S} \to C_{1} \). For any finite subset \( T \subset V \) with \( S \subset T \), let \( h_T \) denote the image of \( h \) under the functor \( i_{S,T} : C_{1,S} \to C_{1,T} \). The object \( h_T \) of \( C_{1,T} \) is, by definition, a morphism from \( Y_T \) to an object \( X_T \) in \( D \) Then it can be checked easily that the inclusion functor \( C_{1,T} \to D_{Y_T} \) induces an equivalence \( C_{1,T,h_T/} \xrightarrow{\sim} D_{X_T/} \) of undercategories. This shows that the functor \( \iota_1 \) induces an equivalence \( C_{1,X/} \xrightarrow{\sim} D_{\iota_1(X)/} \) of undercategories. This completes the proof. \( \square \)

6.2. **Existence of a grid.** The goal of this section is to prove the following proposition. Its proof is given at the end of this subsection.

**Proposition 6.2.1.** Let \( (C, J) \) be a \( Y \)-site. Suppose that there exists an object \( X_0 \) of \( C(T(J)) \) such that the overcategory \( C(T(J))/X_0 \) satisfies at least one of the two cardinality conditions in Section 7.8.7. Then there exists a grid of \( (C, J) \).

6.2.1. **Construction of a grid from a pregrid.** We construct a pair \( (C_0, \iota_0) \) of a \( \mathcal{U} \)-small category \( C_0 \) and a covariant functor \( \iota_0 : C_0 \to C \), from a pregrid \( (C_{1}, \iota_1 : C_{1} \to D = C(T(J)), \) as follows.

For each object \( X \) of \( C_{1} \), the undercategory \( D_{\iota_1(X)/} \) is a quasi-poset, since \( C \) is an \( E \)-category and essentially \( \mathcal{U} \)-small. Let us choose a poset skeleton \( C_{0,X} \) of \( D_{\iota_1(X)/} \). When \( f : Y \to X \) is a morphism in \( C_{1} \), the functor \( D_{\iota_1(X)/} \to D_{\iota_1(Y)/} \) given by the composition with \( f \) is fully faithful. Hence it induces a fully faithful functor \( C_{0,X} \to C_{0,Y} \) between posets. We define \( C_0 \) to be the colimit \( C_0 = \lim_{X \in \text{Obj}C_{1}} C_{0,X} \).

(Consider the diagram over \( X \in \text{Obj}C_{1} \) in the category of partially ordered sets, take the colimit as a partially ordered set, and regard it as a category using the functor in Section 5.1.)
By associating, to each object $Y$ of $\mathcal{C}_1$, the image of the initial object of $\mathcal{C}_{0,Y}$ under the functor $\mathcal{C}_{0,Y} \rightarrow \mathcal{C}_0$, we obtain a functor $j : \mathcal{C}_1 \rightarrow \mathcal{C}_0$. One can check easily that the functor $j$ is fully faithful. By patching the functors $\mathcal{C}_{0,X} \rightarrow \mathcal{D}_{t_1(X)/} \rightarrow \mathcal{C}$ for various $X$, we obtain a functor $\iota_0 : \mathcal{C}_0 \rightarrow \mathcal{C}$ satisfying $\varphi \circ \iota_1 = \iota_0 \circ j$, where $\varphi : \mathcal{D} \rightarrow \mathcal{C}$ denotes the inclusion functor.

6.2.2.

**Lemma 6.2.2.** The pair $(\mathcal{C}_0, \iota_0)$ constructed above is a grid.

We check below that the conditions of a grid (Definition 5.5.3) are satisfied.

**Lemma 6.2.3.** Conditions (1) is satisfied.

**Proof.** It follows from the construction of $\mathcal{C}_0$ that given an object $X$ of $\mathcal{C}_0$, there exist an object $X'$ of $\mathcal{C}_1$ and a morphism $j(X') \rightarrow X$. Note that the category $\mathcal{C}_1$ is $\Lambda$-connected since $(\mathcal{C}_1, \iota_1)$ is a pregrid. Hence $\mathcal{C}_0$ is $\Lambda$-connected. This shows that Condition (1) is satisfied.

**Lemma 6.2.4.** Condition (3) is satisfied.

**Proof.** Let $X$ be an object of $\mathcal{C}_0$ and let $f : Y \rightarrow \iota_0(X)$ be a morphism in $\mathcal{C}(\mathcal{T}(J))$. Let us choose an object $X'$ in $\mathcal{C}_1$ and a morphism $g : j(X') \rightarrow X$ in $\mathcal{C}_0$. Since $\mathcal{T}(J)$ is semi-localizing, there exist an object $Y'$ of $\mathcal{C}$ and morphisms $f' : Y' \rightarrow \iota_1(X')$ and $g' : Y' \rightarrow Y$ in $\mathcal{C}$ such that $f'$ belongs to $\mathcal{T}(J)$ and that the equality $f \circ g' = \iota_0(g) \circ f'$ holds. Since $(\mathcal{C}_1, \iota_1)$ is a pregrid, there exist an object $Y'_1$ of $\mathcal{C}_1$, a morphism $f'_1 : Y'_1 \rightarrow X'$ in $\mathcal{C}_1$, and an isomorphism $\alpha : Y' \xrightarrow{\cong} \iota_1(Y'_1)$ in $\mathcal{C}(\mathcal{T}(J))$ satisfying $f' = \iota_1(f'_1) \circ \alpha$. Since the inclusion functor $\mathcal{C}_{0,Y'_1} \rightarrow \mathcal{C}_{\iota_1(Y'_1)/}$ and the functor $\mathcal{C}_{\iota_1(Y'_1)/} \rightarrow \mathcal{C}_{Y'/}$ given by the composition with $\alpha$ are equivalences of categories, there exist an object $\iota_1(Y'_1) \rightarrow Y_1$ of $\mathcal{C}_{0,Y'_1}$ and an isomorphism $\beta : Y_1 \xrightarrow{\cong} Y$ such that the map $g'$ is equal to the composite $Y' \xrightarrow{\alpha} \iota_1(Y'_1) \rightarrow Y_1 \xrightarrow{\beta} Y$. Let $Y_2$ denote the image of the object $\iota_1(Y'_1)$ in $\mathcal{C}_{0,Y}$ under the functor $\mathcal{C}_{0,Y'_1} \rightarrow \mathcal{C}_{0,Y}$ to $\mathcal{C}_0$. Then there exist a morphism $f_2 : Y_2 \rightarrow X$ in $\mathcal{C}_0$ and the isomorphism $\beta$ induces an isomorphism $\beta_2 : \iota_0(Y_2) \xrightarrow{\cong} Y$ satisfying $\iota_0(f_2) = f \circ \beta_2$. This shows that the pair $(\mathcal{C}_0, \iota_0)$ satisfies the property (3).

**Lemma 6.2.5.** Condition (4) is satisfied.

**Proof.** We prove that the pair $(\mathcal{C}_0, \iota_0)$ satisfies the property (3). It remains to prove that the pair $(\mathcal{C}_0, \iota_0)$ satisfies the property (4). Let $X$ be an object of $\mathcal{C}_0$. Let us choose an object $X'$ in $\mathcal{C}_1$ and an object $g$ in $\mathcal{C}_{0,X'}$ whose image under the functor $\mathcal{C}_{0,X'} \rightarrow \mathcal{C}_0$ is equal to $X$. We regard $g$ as a morphism $g : j(X') \rightarrow \iota_0(X)$ in $\mathcal{C}$. Then the inclusion functor $\mathcal{C}_{0,X'} \rightarrow \mathcal{C}_{\iota_1(X')/}$ induces equivalences of categories $\mathcal{C}_{0,X',g/} \xrightarrow{\cong} (\mathcal{C}_{\iota_0(X')/})_{\iota_0(g)/} \cong \mathcal{C}_{\iota_0(X)/}$. For any morphism $f : Y \rightarrow X'$ in $\mathcal{C}_1$, the functor $f^* : \mathcal{C}_{0,X'} \rightarrow \mathcal{C}_{0,Y}$ induced by the composition with $f$ gives an equivalence of categories from $\mathcal{C}_{0,X',g/}$ to $\mathcal{C}_{0,Y,f^*(g)/}$. Passing to the colimit, we see that the functor $\mathcal{C}_{0,X} \rightarrow \mathcal{C}_0$ induces an equivalence of categories from $\mathcal{C}_{0,X',g/}$ to $\mathcal{C}_{0,X'/}$. Hence the functor $\mathcal{C}_{0,X'/} \rightarrow \mathcal{C}_{\iota_0(X)/}$ induced by $\iota_0$ is an equivalence of categories. This shows that that the pair $(\mathcal{C}_0, \iota_0)$ satisfies the property (4), which completes the proof.

**Lemma 6.2.6.** Condition (2) is satisfied.
Proof. Via the functor $j$ we regard $C_1$ as a full subcategory of $C_0$. Note that the restriction of $\iota_0$ to $C_1$ is equal to $\iota_1$. Since $\iota_1$ is essentially surjective, it suffices to show that any object of $C_1$ is an edge object of $C_0$. Let $X$ be an object of $C_1$ and let $f : Y \to X$ be a morphism in $C_0$. It follows from the construction of the category $C_0$ that there exists an object $Z$ of $C_1$ and morphisms $g : Z \to X$ and $h : Z \to Y$. Since $C_0$ is thin, we have $g = f \circ h$. Since $g$ is a morphism in $C_1$, the morphism $\iota_0(g)$ belongs to $T(J)$. Hence $\iota_0(f)$ belongs to $T(J)$. Since $f$ is arbitrary, this shows that $X$ is an edge object of $C_0$. This completes the proof. \hfill \Box

This completes the proof of Lemma 6.2.2.

Remark 6.2.7. More strongly, one can show that an object of $C_0$ is an edge object if and only if it belongs to $\text{Obj}(C_1)$. The “if” part follows from the argument of the proof of the lemma above. The “only if” part can be proved as follows. Let $X$ be an edge object of $C_0$. It follows from the construction of $C_0$ that there exists an object $Y$ of $C_1$ and a morphism $f : Y \to X$ in $C_0$. Since $X$ is an edge object, $f$ is of type $J$. Since $(C_0, \iota_0)$ is a grid of $(C, J)$ and $(C_1, \iota_1)$ is a pregrid of $C(T(J))$, the functor $\iota_0(X) : C_0 \to C_1$ is an equivalence of categories and its restriction to $C_{1, Y}$ induces an equivalence $C_{1, Y} \cong C(T(J))_{\iota_0(Y)}$ of categories. Hence if we regard $f$ as an object of $C_0$, then $f$ is isomorphic to an object of $C_{1, Y}$ in $C_{0, Y}$. Since $C_0$ is skeletal, this shows that $f$ is a morphism in $C_1$. Hence $X$ is an object of $C_1$. 

Proof of Proposition 6.2.1 This follows from Proposition 6.1.1 and Lemma 6.2.2. \hfill \Box

7. The Fiber Functor is Fully Faithful

7.1. Properties of a Grid. In this subsection, we will supply some preliminaries required in our proof of Theorem 5.8.1, most of which follow from Lemma 6.2.2.

Let $(C, J)$ be a $J$-site and let $(C_0, \iota_0)$ be a grid of $(C, J)$.

Lemma 7.1.1. Let $Y_1 \xrightarrow{f_1} X \xleftarrow{f_2} Y_2$ be a diagram in $C_0$. Suppose that $\iota_0(f_1)$ is a Galois covering and that there exists a morphism from $\iota_0(Y_1)$ to $\iota_0(Y_2)$ over $\iota_0(X)$. Then there exists a morphism from $Y_1$ to $Y_2$ over $X$.

Proof. The category $C_0$ is $\Lambda$-connected (Lemma 6.2.2). Hence there exist an object $Z$ of $C_0$ and morphisms $g_1 : Z \to Y_1$ and $g_2 : Z \to Y_2$ which make the diagram

$$
\begin{array}{ccc}
Z & \xrightarrow{g_1} & Y_1 \\
\downarrow{g_2} & & \downarrow{f_1} \\
Y_2 & \xleftarrow{f_2} & X
\end{array}
$$

commutative. Let us apply the functor $\iota_0$ to this diagram. It then follows from Lemma 4.2.3(1) that there exists a morphism in the undercategory $C_{\iota_0(Y_1)}$ from $\iota_0(g_1)$ to $\iota_0(g_2)$. Hence it follows from (4) of Lemma 6.2.2 that there exists a morphism in the undercategory $C_{0, Y_1}$ from $g_1$ to $g_2$. In particular there exists a morphism $h : Y_1 \to Y_2$ in $C_0$. Since $C_0$ is thin, any diagram in $C_0$ is commutative. Hence $h$ is a morphism over $X$, which proves the claim. \hfill \Box

Corollary 7.1.2. Let $X$ be an object of $C_0$ and let $f_1 : Y_1 \to X$ and $f_2 : Y_2 \to X$ be two morphisms in $C_0$ such that $\iota_0(f_1)$ and $\iota_0(f_2)$ are Galois coverings in $C$. Suppose

\hfill
that there exists an isomorphism \( \iota_0(Y_1) \cong \iota_0(Y_2) \) over \( \iota_0(X) \) in \( \mathcal{C} \). Then we have \( Y_1 = Y_2 \) and \( f_1 = f_2 \).

Proof. It follows from Lemma 7.1.3 that there exist a morphism from \( Y_1 \) to \( Y_2 \) and a morphism from \( Y_2 \) to \( Y_1 \). Since \( \mathcal{C}_0 \) is a poset, the claim follows.

Lemma 7.1.3. Let \( X \) and \( X' \) be objects of \( \mathcal{C}_0 \) and let \( \beta : \iota_0(X) \cong \iota_0(X') \) be an isomorphism in \( \mathcal{C} \). Then for any morphism \( f : X \rightarrow Y \) in \( \mathcal{C}_0 \), there exists a unique morphism \( f' : X' \rightarrow Y' \) in \( \mathcal{C}_0 \) satisfying the following property: there exists an isomorphism \( \beta' : \iota_0(Y) \cong \iota_0(Y') \) in \( \mathcal{C} \) which makes the diagram

\[
\begin{array}{ccc}
\iota_0(X) & \xrightarrow{\beta} & \iota_0(X') \\
\iota_0(f) \downarrow & & \downarrow \iota_0(f') \\
\iota_0(Y) & \xrightarrow{\beta'} & \iota_0(Y')
\end{array}
\]

commutative. (We note that such an isomorphism \( \beta' \) is unique since \( f \) is an epimorphism.)

Proof. From the given data, we obtain a morphism \( \iota_0(f) \circ \beta^{-1} : \iota_0(X') \rightarrow \iota_0(Y) \). Using that the functor \( \mathcal{C}_0, X' \rightarrow \mathcal{C}_{\iota_0(X')} \) induced by \( \iota_0 \) is an equivalence of categories (Lemma 6.2.2(4)), we obtain morphisms \( f' \) and \( \beta' \) which make the diagram commutative. The uniqueness of \( f' \) follows since \( \mathcal{C}_0 \) is skeletal.

Lemma 7.1.4. Let \( X \) and \( X' \) be objects of \( \mathcal{C}_0 \) and let \( \beta : \iota_0(X) \cong \iota_0(X') \) be an isomorphism in \( \mathcal{C} \). Then for any morphism \( f : Y \rightarrow X \) in \( \mathcal{C}_0 \) such that \( \iota_0(f) \) is a Galois covering in \( \mathcal{T} \), there exists a unique morphism \( f' : Y' \rightarrow X' \) in \( \mathcal{C}_0 \) satisfying the following property: there exists a (not necessarily unique) isomorphism \( \beta' : \iota_0(Y) \cong \iota_0(Y') \) in \( \mathcal{C} \) which makes the diagram

\[
\begin{array}{ccc}
\iota_0(Y) & \xrightarrow{\beta'} & \iota_0(Y') \\
\iota_0(f) \downarrow & & \downarrow \iota_0(f') \\
\iota_0(X) & \xrightarrow{\beta} & \iota_0(X')
\end{array}
\]

commutative.

Proof. The existence of \( f' \) follows from (3) of Lemma 6.2.2. We prove the uniqueness. Suppose that the two morphisms \( f'_1 : Y'_1 \rightarrow X' \) and \( f'_2 : Y'_2 \rightarrow X' \) satisfy the property of the lemma. Since both \( \iota_0(Y'_1) \) and \( \iota_0(Y'_2) \) are isomorphic to \( \iota_0(Y) \) over \( X \), there exists an isomorphism from \( \iota_0(Y'_1) \) to \( \iota_0(Y'_2) \) over \( \iota_0(X) \). Hence it follows from Lemma 7.1.3 that there exist morphisms \( Y'_1 \rightarrow Y'_2 \) and \( Y'_2 \rightarrow Y'_1 \) over \( X \). Since the category \( \mathcal{C}_0 \) is a poset, it follows that \( Y'_1 = Y'_2 \) and \( f'_1 = f'_2 \), which proves the claim.

Lemma 7.1.5. Let \( X \) be an object of \( \mathcal{C}_0 \) and let \( \iota_0(X) \xleftarrow{f} Z \xrightarrow{g} Y \) be a diagram in \( \mathcal{C} \) such that \( f \) belongs to \( \mathcal{T} \). Then there exist a diagram \( X \leftarrow Z' \rightarrow Y' \) in \( \mathcal{C}_0 \) and
isomorphisms $ι_0(Z') \cong Z$ and $ι_0(Y') \cong Y$ which make the diagram
\[
ι_0(X) \xleftarrow{f} ι_0(Z') \xrightarrow{=} ι_0(Y')
\]
(7.1)
in $\mathcal{C}$ commutative.

Proof. It follows from (3) of Lemma 6.2.2 that there exist an object $Z'$ of $\mathcal{C}_0$ and the isomorphism $ι_0(Z') \cong Z$ in $\mathcal{C}$ which make the left square of (7.1) commutative. It then follows from (4) of Lemma 6.2.2 that there exist an object $Y'$ of $\mathcal{C}_0$ and the isomorphism $ι_0(Y') \cong Y$ in $\mathcal{C}$ which make the right square of (7.1) commutative. This proves the claim. \qed

7.2. Proof of Theorem 5.8.1: the functor $\omega(\mathcal{C}_0,ι_0)$ is faithful.

Lemma 7.2.1. Let $F$ be a sheaf on $(\mathcal{C}, J)$. Then for any edge object $X$ of $\mathcal{C}_0$, the map $F(X) → \omega(\mathcal{C}_0,ι_0)(F)$ given by the universality of colimit is injective.

Proof. Let $𝑓 : Y → X$ be a morphism in $\mathcal{C}_0$. Then $ι_0(𝑓)$ belongs to $\mathcal{T}$. Since $F$ is a sheaf on $(\mathcal{C}, J)$, it follows from Corollary 6.1.1(a) that the map $F(𝑥) → F(𝑓)$ is injective. Hence the map $F(𝑥) → \omega(\mathcal{C}_0,ι_0)(F)$ is injective. \qed

Proof of Theorem 5.8.1: faithfulness. We prove the faithfulness of the functor (5.1). Let $F$ and $F'$ be sheaves on $(\mathcal{C}, J)$ and let $𝑓, g : F → F'$ be two morphisms of sheaves on $(\mathcal{C}, J)$. Suppose that $ω(\mathcal{C}_0,ι_0)(𝑓) = ω(\mathcal{C}_0,ι_0)(𝑔)$. We show that $𝑓 = 𝑔$. Let us take an arbitrary object $X$ of $\mathcal{C}$. It suffices to show that $𝑓(𝑥) = 𝑔(𝑥)$. Since $ι_0$ is essentially surjective, there exist an object $X'$ of $\mathcal{C}_0$ and an isomorphism $ι_0(X') → X$ in $\mathcal{C}$. Hence the claim follows from the commutativity of the diagram
\[
\begin{array}{ccc}
F(𝑥) & \xrightarrow{α^*} & F(ι₀(X')) \\
\downarrow{𝑓(𝑥),𝑔(𝑥)} & & \downarrow{}
\end{array}
\]
\[
\begin{array}{ccc}
G(𝑥) & \xrightarrow{α^*} & G(ι₀(X')) \\
\downarrow{𝑔(𝑥)} & & \downarrow{}
\end{array}
\]
and Lemma 7.2.1. \qed

7.3. The category $I_X$. Let $X$ be an edge object of $\mathcal{C}_0$. We denote by $I_X$ the full-subcategory of the overcategory $\mathcal{C}_{0/X}$ whose objects are the morphisms $𝑓 : Y → X$ in $\mathcal{C}_0$ such that $ι_0(𝑓)$ is a Galois covering in $\mathcal{T}$. For an object 𝑓 : $Y → X$ of $I_X$, we write $Gal(𝑓)$ for $Gal(ι₀(𝑓)).$

Lemma 7.3.1. The category $I_X$ is cofiltered and the objects of $I_X$ are cofinal in $\mathcal{C}_{0/X}$.

Proof. Since $\mathcal{C}_0$ is $Ą$-connected and thin, the overcategory $\mathcal{C}_{0/X}$ is $Ą$-connected and thin. It follows from Lemma 5.7.1 that $\mathcal{C}_{0/X}$ is cofiltered. Hence it suffices to prove that the objects of $I_X$ are cofinal in $\mathcal{C}_{0/X}$. Let $f : Y → X$ be an object of $\mathcal{C}_{0/X}$ and let us regard it as a morphism in $\mathcal{C}_0$. Since $\mathcal{T}(𝐽)$ has enough Galois coverings, there exists a morphism $g' : Z' → ι₀(Y)$ in $\mathcal{C}$ such that $g'$ belongs to $\mathcal{T}(𝐽)$ and the composite $ι₀(𝑓) ∘ g'$ is a Galois covering in $\mathcal{C}$. It follows from the property (3) of the grid $(\mathcal{C}_0,ι_0)$ that there exist a morphism $g : Z → Y$ in $\mathcal{C}_0$ and an isomorphism $α : ι₀(Z) \xrightarrow{≅} Z'$ in $\mathcal{C}$ satisfying $ι₀(g) = g' ∘ α$. The morphism $f ∘ g$ in $\mathcal{C}_0$, regarded
as an object of $C_{0/Z}$ is an object of $I_X$, since $t_0(f \circ g) = t_0(f) \circ g' \circ \alpha$ is a Galois covering in $C$. This proves the claim.

Lemma 7.3.2. Let $X$ be an edge object of $C_0$ and let $Z$ be an object of $C_0$. Then there exist an object $f : Y \to X$ of $I_X$ and a morphism $Y \to Z$ in $C_0$.

Proof. Since $C_0$ is $\Lambda$-connected, there exist an object $Y'$ of $C_0$ and morphisms $h_1 : Y' \to X$ and $h_2 : Y' \to Z$ in $C_0$. It follows from Lemma 7.3.1 that there exists a morphism $h : Y \to Y'$ such that the composite $f = h_1 \circ h : Y \to X$ is an object of $I_X$. This proves the claim, since $h_2 \circ h$ is a morphism from $Y$ to $Z$ in $C_0$.

Corollary 7.3.3. Let $X$ be an edge object of $C_0$. Then the functor

$$\lim_{(f: Y \to X) \in \text{Obj} I_X} C_{0,Y/} \to C_0$$

given by the forgetful functors $C_{0,Y/} \to C_0$ is an isomorphism of categories.

Proof. This follows immediately from the previous lemma.

7.4. The functor $\theta_{Z,Z',h}$. Let $Z, Z'$ be objects of $C_0$ and let $h : t_0(Z) \to t_0(Z')$ be a morphism in $C$. Let $\sim h : C_{0/Z'}/ \to C_{0/Z}/$ denote the functor given by the composition with $h$. Since $C$ is an $\mathcal{E}$-category, the two undercategories $C_{0/Z}/$ and $C_{0/Z'}/$ are thin and the functor $\sim h$ is fully faithful. Let us consider the diagram

$$ C_{0/Z'}/ \xrightarrow{t_0, Z'/} C_{0/Z}/ \xleftarrow{\sim h} C_{t_0(Z')}/ \xrightarrow{t_0, Z/} C_0/Z/ $$

of categories and functors. Since the two functors in this diagram other than $\sim h$ are equivalences of categories, this diagram gives a functor $C_{0/Z'}/ \to C_{0/Z}/$ which is fully faithful. We denote this functor by $\theta_{Z,Z',h}$. Since the category $C_{0/Z}/$ is skeletal, the following statement holds (although it is a simple observation, we state it as a lemma since we will refer to the statement several times):

Lemma 7.4.1. Let $Z, Z'$ and $h$ be as above. Then $\theta_{Z,Z',h}$ is the unique functor from $C_{0/Z}$ to $C_{0/Z'}$ which makes the diagram

$$ C_{0/Z}/ \xrightarrow{\theta_{Z,Z',h}} C_0/Z/ $$

commutative up to natural equivalence.

Corollary 7.4.2. Let $Z''$ is another object of $C_0$ and $h' : t_0(Z') \to t_0(Z'')$ is a morphism in $C$, then we have $\theta_{Z,Z'',h' \circ h} = \theta_{Z,Z',h} \circ \theta_{Z',Z'',h'}$.

Corollary 7.4.3. Suppose that $h$ is an isomorphism in $C$. then the functor $\theta_{Z,Z',h}$ is an isomorphism of categories.

Let $f' : Z' \to Y'$ be an object of $C_{0/Z'}/$, and denote the object $\theta_{Z,Z',h}(f')$ of $C_{0/Z}/$ by $f : Z \to Y$. It follows from Lemma 7.4.1 that there exists an isomorphism
Lemma 7.4.4. Let the notation be as above. Then the functor \( \vartheta' : C_{0,Y'} / \to C_{0,Y} / \) induced by the functor \( \vartheta_{Z,Z'} : h \) is equal to the functor \( \vartheta_{Y,Y',\alpha} \).

Proof. It follows from the definition of the functor \( \vartheta' \) that the diagram (7.2) induces a diagram

\[
\begin{array}{ccc}
C_{0,Y'} / & \xrightarrow{\vartheta'} & C_{0,Y} / \\
\xrightarrow{\iota_{0,Y'}} & & \xrightarrow{\iota_{0,Y}} \\
C_{\iota_0(Y')} / & \xrightarrow{-\alpha} & C_{\iota_0(Y)} /
\end{array}
\]

of categories and functors, which is commutative up to natural isomorphisms. Hence the claim follows from Lemma 7.4.1.

Let \( Z, Z' \) and \( h : \iota_0(Z) \to \iota_0(Z') \) be as in Lemma 7.4.1. Then the natural equivalence from the composite \( C_{0,Z'} / \xrightarrow{\vartheta_{Z,Z',h}} C_{0,Z} / \xrightarrow{\iota_{0,Z}} C_{\iota_0(Z)} / \) to the composite \( C_{0,Z'} / \xrightarrow{\iota_{0,Z'}} C_{\iota_0(Z') /} \xrightarrow{-h} C_{\iota_0(Z) /} \) is unique, since the category \( C_{\iota_0(Z)} / \) is thin. We denote this natural equivalence by \( \xi_{Z,Z',h} \).

7.5. Some properties of the monoids \( M_{(C_0,a,\iota_0)} \) and \( \mathbb{K}_X \).

Lemma 7.5.1. Let \( (\alpha, \gamma_\alpha) \) be an element of the monoid \( M_{(C_0,a,\iota_0)} \). Then for any object \( X \) of \( C_0 \), the functor \( \alpha' : C_{0,X} \to C_{0,a(X)} / \) induced by \( \alpha \) on the undercategories is equal to the isomorphism \( \theta_{\alpha(X),X,\gamma_\alpha(X)^{-1}} \) of categories.

Proof. We set \( \beta = \gamma_\alpha(X) : \iota_0(X) \xrightarrow{\sim} \iota_0(a(X)) \). Let \( - \circ \beta^{-1} : C_{\iota_0(X)} / \to C_{\iota_0(a(X)) /} \) denote the functor given by the composition with \( \beta^{-1} \). Then the natural isomorphism \( \gamma_\alpha \) induces a natural isomorphism from the composite \( C_{0,X} \xrightarrow{\iota_{0,X}} C_{\iota_0(X)} / \xrightarrow{-\circ \beta^{-1}} C_{\iota_0(a(X)) /} \) to the composite \( C_{0,X} \xrightarrow{\alpha'} C_{0,a(X)} / \xrightarrow{\iota_{0,a(X)} /} C_{\iota_0(a(X)) /} \). Hence the claim follows from Lemma 7.4.1.

Lemma 7.5.2. Let \( (\alpha, \gamma_\alpha) \) be an element of the monoid \( M_{(C_0,a,\iota_0)} \). Then the functor \( \alpha : C_0 \to C_0 \) is fully faithful.

Proof. Let \( X_1 \) and \( X_2 \) be objects of \( C_0 \) and suppose that there exists a morphism \( f : \alpha(X_1) \to \alpha(X_2) \) in \( C_0 \). Since the category \( C_0 \) is thin, it suffices to show that there exists a morphism from \( X_1 \) to \( X_2 \) in \( C_0 \). Since \( C_0 \) is \( \Lambda \)-connected, there exist an object \( Y \) of \( C_0 \) and morphisms \( h_1 : Y \to X_1 \) and \( h_2 : Y \to X_2 \). By applying the functor \( \alpha \), we obtain the diagram \( \alpha(X_1) \xleftarrow{\alpha(h_1)} Y \xrightarrow{\alpha(h_2)} \alpha(X_2) \) in \( C_0 \). Since the category \( C_0 \) is thin, we have \( \alpha(h_2) = f \circ \alpha(h_1) \) and hence \( f \) can be regarded as a morphism from \( \alpha(h_1) \) to \( \alpha(h_2) \) in \( C_{0,a(Y)} / \). It follows from Lemma 7.5.2 that the functor \( \alpha' : C_{0,Y} / \to C_{0,a(Y)} / \) induced by \( \alpha \) is equal to \( \theta_{\alpha(Y),Y,\gamma_\alpha(Y)^{-1}} \). Note that \( \alpha' \) sends the object \( h_1 \) of \( C_{0,Y} / \) to the object \( \alpha(h_1) \) of \( C_{0,a(Y)} / \). Since \( \theta_{\alpha(Y),Y,\gamma_\alpha(Y)^{-1}} \)
is an isomorphism of categories, it follows that there exists a morphism from $h_1$ to $h_2$ which is sent to the morphism $f$ by $\theta_{\alpha(Y), Y, \gamma_\alpha(Y)^{-1}}$. This proves that $\alpha$ is fully faithful.

Lemma 7.5.3. Let $X$ be an edge object of $C_0$ and let $(\alpha, \gamma_\alpha)$ be an element of $K_X$. Let $f : Y \to X$ be a morphism in $C_0$ such that $\iota_0(f)$ is a Galois covering in $T$. Then we have $\alpha(Y) = Y$, $\alpha(f) = f$, and $\gamma_\alpha(Y) : \iota_0(Y) \cong \iota_0(Y)$ is an element of the Galois group of $\iota_0(f)$.

Proof. Since $\alpha(X) = X$ and $\gamma_\alpha(X)$ is the identity, we have the commutative diagram

$$
\begin{array}{ccc}
\iota_0(Y) & \xrightarrow{\gamma_\alpha(Y)} & \iota_0(\alpha(Y)) \\
\downarrow{\iota_0(f)} & & \downarrow{\iota_0(f)} \\
\iota_0(X) & \xrightarrow{\iota_0(\alpha(f))} & \iota_0(X).
\end{array}
$$

In particular, $\iota_0(\alpha(f))$ is a Galois covering in $T$. Hence by Corollary 7.1.2 we have $\alpha(Y) = Y$ and $\alpha(f) = f$, from which the claim follows.

Lemma 7.5.4. Let $X$ be an edge object of $C_0$. Then the submonoid $K_X \subset M_{(C_0, \iota_0)}$ introduced in Section 5.6.1 is a group.

Proof. Let $(\alpha, \gamma_\alpha)$ be an element of $K_X$. It suffices to show that the functor $\alpha : C_0 \to C_0$ is an isomorphism of categories. Since the category $C_0$ is skeletal, it then suffices to prove that the functor $\alpha$ gives an equivalence of categories. It follows from Lemma 7.3.2 that the functor $\alpha$ is fully faithful. We prove that $\alpha$ is essentially surjective. Let us take an arbitrary object $Z$ of $C_0$. It follows from Lemma 7.3.2 that there exist a morphism $f : Y \to X$ in $C_0$ and a morphism $h : Y \to Z$ in $C_0$ such that $\iota_0(f)$ is a Galois covering in $T$. By Lemma 7.5.3 we have $\alpha(Y) = Y$. We set $\beta = \gamma_\alpha(Y)^{-1}$, which is an automorphism of the object $\iota_0(Y)$. Since $\theta_{Y,Y,\beta}$ is an automorphism of the category $C_{0,Y'/f}$, there exists an object $h' : Y \to Z'$ of $C_{0,Y'/f}$ which is sent to $h$ by the automorphism $\theta_{Y,Y,\beta}$. It then follows from Lemma 7.3.2 that we have $\alpha(Z') = Z$, which proves that the functor $\alpha$ is essentially surjective. This completes the proof.

7.6. The isomorphisms $\psi_X$ and $\phi_X$.

7.6.1. Let $X$ be an edge object of $C_0$. Let $f_1, f_2$ be two objects of $I_X$ and suppose that there exists a morphism $g$ from $f_2$ to $f_1$ in $I_X$. Note that $\iota_0(f_1)$ and $\iota_0(f_2)$ are Galois coverings in $C$. It follows from Lemma 4.2.6 (1) that for any $\alpha \in \text{Gal}(f_2)$ there exists a unique $\alpha' \in \text{Gal}(f_1)$ satisfying $\iota_0(g) \circ \alpha = \alpha' \circ \iota_0(g)$. By sending $\alpha$ to $\alpha'$ we obtain a map $\text{Gal}(f_2) \to \text{Gal}(f_1)$. It follows from Lemma 4.2.6 (2) that this map is a homomorphism of groups. It follows from Lemma 4.2.6 that this homomorphism is surjective. We set

$$
H_X = \lim_{f \in \text{Obj} I_X} \text{Gal}(f).
$$

Lemma 7.6.1. Let the notation be as above. Suppose that one of the conditions on cardinality in Section 5.8.1 is satisfied for the category $C(T(J))_{/X}$. Then the canonical map

$$
\rho : H_X \to \text{Gal}(f)
$$

where $f \in I_X$, is surjective.
Lemma 7.1.1 implies that we have either that \( \text{Gal}(H) \) is a finite group for any \( f' \in \text{Obj} I_X \) or that the set \( I_X \) is at most countable. Hence the natural projection \( H_X \rightarrow \text{Gal}(f) \) is surjective for any \( f \in I_X \). 

7.6.2. It follows from Lemma 7.5.4 that \( \mathbb{K}_X \) is a group. Let us construct a homomorphism \( \psi_X : H_X \rightarrow \mathbb{K}_X \) of groups. Let \( \beta = (\beta_f)_{f \in \text{Obj} I_X} \in H_X \). For each object \( f : Y \rightarrow X \) of \( I_X \), we let \( \alpha(\beta)_Y \) denote the automorphism \( \theta_{Y,Y,\beta_f^{-1}} : C_{0,Y} \xrightarrow{\cong} C_{0,Y} \) of categories. Let \( f' : Y' \rightarrow X \) be another object of \( I_X \) and let \( h : Y' \rightarrow Y \) be a morphism in \( I_X \). Then it follows from Lemma 7.4.4 that the diagram

\[
\begin{array}{ccc}
C_{0,Y}/ & \xrightarrow{\alpha(\beta)_Y} & C_{0,Y}/ \\
\downarrow{\sim} & & \downarrow{\sim} \\
C_{0,Y'}/ & \xrightarrow{\alpha(\beta')_Y} & C_{0,Y'}/
\end{array}
\]

is commutative. By taking the colimit with respect to \( f \) and by using Corollary 7.3.3 we obtain an isomorphism \( C_0 \xrightarrow{\cong} \mathcal{C}_0 \) of categories which we denote by \( \alpha(\beta) \).

7.6.3. For each object \( f : Y \rightarrow X \) of \( I_X \), we denote by \( E_{1,Y} \) the composite \( E_{1,Y} : C_{0,Y}/ \xrightarrow{\iota_0,Y/} C_{\alpha(Y)/} \xrightarrow{\sim} C_{0,Y}/ \) and by \( E_{2,Y} \) the composite \( E_{2,Y} : C_{0,Y}/ \xrightarrow{\alpha(\beta)_Y} C_{0,Y'/} \xrightarrow{\iota_0,Y'/} C_{\alpha(Y)'/} \). We set \( \gamma(\alpha(\beta)) : = \xi^{-1}_{Y,Y,\beta_f^{-1}} \), which is a natural isomorphism from the composite \( E_{1,Y} \) to the composite \( E_{2,Y} \).

7.6.4. Let \( f' : Y' \rightarrow X \) be another object of \( I_X \) and let \( h : Y' \rightarrow Y \) be a morphism in \( I_X \). Let us consider the functors \( \iota_0 \circ h : C_{0,Y'/} \rightarrow C_{0,Y'/} \) and \( \iota_0 \circ h : C_{\alpha(Y)/} \rightarrow C_{\alpha(Y)'/} \) given by the compositions with \( h \) and \( \iota(h) \), respectively.

For \( i = 1, 2 \), the functor \( E_{i,Y'} \circ (- \circ h) : C_{0,Y'/} \rightarrow C_{\alpha(Y)'/} \) is equal to the functor \( (- \circ \iota_0(h)) \circ E_{i,Y} : C_{0,Y}/ \rightarrow C_{\alpha(Y)'/} \). It follows from the proof of Lemma 7.4.4 that the natural isomorphism \( \gamma(\alpha(\beta))' \circ (- \circ h) \circ E_{1,Y'} \circ (- \circ h) \circ E_{2,Y'} \circ (- \circ h) \) is equal to the natural isomorphism \( \gamma(\alpha(\beta))' \circ (- \circ h) \circ E_{1,Y'} \circ (- \circ h) \circ E_{2,Y'} \circ (- \circ h) \). By taking the colimit with respect to \( f \), we obtain a natural isomorphism \( \gamma(\alpha(\beta)) : = \iota_0 \xrightarrow{\cong} \iota_0 \circ \alpha(\beta) \). Thus we obtain an element \( \alpha(\beta), \gamma(\alpha(\beta)) \) in \( M(\mathbb{C}_0, \mathbb{K}_X) \).

It is easy to check that the element \( \alpha(\beta), \gamma(\alpha(\beta)) \) belongs to \( \mathbb{K}_X \) and that the map \( \psi_X : H_X \rightarrow \mathbb{K}_X \) which sends \( \beta \in H_X \) to \( \alpha(\beta), \gamma(\alpha(\beta)) \) is a homomorphism of groups.

7.6.5. Let us construct a homomorphism \( \phi_X : \mathbb{K}_X \rightarrow H_X \) as follows. Let \( (\alpha, \gamma(\alpha)) \in \mathbb{K}_X \). For any object \( f : Y \rightarrow X \) of \( I_X \), it follows from Lemma 7.5.3 that we have \( \alpha(Y) = Y \) and \( \gamma(\alpha) \in \text{Gal}(f) \). Let \( f' : Y' \rightarrow X \) be another object of \( I_X \) and let \( h : Y' \rightarrow Y \) be a morphism in \( I_X \). Since \( \gamma(\alpha) \) is functorial in \( Y \), we have \( h \circ \gamma(\alpha)(Y') = \gamma(\alpha)(Y) \circ h \). This shows that \( (\gamma(\alpha))(f) : Y \rightarrow X \) is an element of \( H_X \). We define \( \phi_X : \mathbb{K}_X \rightarrow H_X \) to be the map which sends \( \alpha, \gamma(\alpha) \in \mathbb{K}_X \) to the element \( (\gamma(\alpha))(f) : Y \rightarrow X \) of \( H_X \). One can check easily that \( \phi_X \) is a homomorphism of groups.

**Lemma 7.6.2.** The homomorphism \( \psi_X : H_X \rightarrow \mathbb{K}_X \) is an isomorphism.
Proof. It is clear from the construction of \( \psi_X \) that the composite \( \phi_X \circ \psi_X \) is equal to the identity. Hence to prove that \( \psi_X \) is bijective, it suffices to prove that \( \phi_X \) is injective. 

Let \((\alpha, \gamma_0) \in K_X \) and suppose that \( \phi_X((\alpha, \gamma_0)) = 1 \). Then for any object \( f : Y \to X \) of \( I_X \), we have \( \alpha(Y) = Y \) and \( \gamma_0(Y) = \text{id}_{I_0(Y)} \). It follows from Lemma 7.5.1 that the automorphism \( C_{0,Y} \to C_{0,Y} \) induced by \( \alpha \) is equal to the identity. Hence it follows from Corollary 7.3.3 that the automorphism \( \gamma \) is equal to the identity. Since the category \( C_{0,Y} \) is thin, any natural auto-equivalence \( \gamma \) of the functor \( C_{0,Y} \to C_{0,Y} \) induced by \( \gamma_0 \) is equal to the identity. This shows that the natural auto-equivalence \( \gamma_0 \) is equal to the identity for any object \( f : Y \to X \) of \( I_X \). Hence it follows from Corollary 7.3.3 that the natural auto-equivalence \( \gamma_0 \) is equal to the identity, which proves that the homomorphism \( \phi_X \) is injective. This proves the claim. \( \square \)

Corollary 7.6.3. Let \( F \) be a sheaf on \((C,J)\). Then for any object \( X \) of \( C_0 \), the map \( F(I_0(X)) \to \omega_{(C_0,t_0)}(F) \) induces a bijection \( F(I_0(X)) \cong \omega_{(C_0,t_0)}(F)^{K_X} \).

Proof. For any object \( f : Y \to X \) of \( I_X \), the pullback map \( F(I_0(X)) \to F(I_0(Y)) \) induces an isomorphism \( F(I_0(X)) \to F(I_0(Y))^{\text{Gal}(f)} \). Passing to the inductive limit with respect to \( f \), we can see that the map \( F(I_0(X)) \to \omega_{(C_0,t_0)}(F) \) induces an isomorphism \( F(I_0(X)) \cong \omega_{(C_0,t_0)}(F)^{H_X} \). Hence the claim follows from Lemma 7.6.2 \( \square \)

7.7. The monoid \( M_{(C_0,t_0)} \) has sufficiently many elements. From now on until the end of Section 9 we assume that, for any object \( X \) of \( C \), the category \( C_{/X} \) satisfies at least one of the two conditions in Section 5.8.1.

Lemma 7.7.1. Let \( X \) be an edge object of \( C_0 \), \( X' \) an object of \( C_0 \), and \( \beta : I_0(X) \cong I_0(Y) \) an isomorphism in \( C \). Then there exists an element \((\alpha, \gamma) \in M_{(C_0,t_0)} \) which satisfies \( \alpha(X) = X' \) and \( \gamma_0(X) = \beta \).

Proof. For any object \( f : Y \to X \) of \( I_X \), let \( S_f \) denote the set of pairs \((f', \beta') \) of a morphism \( f' : Y' \to X' \) in \( C_0 \) and an isomorphism \( \beta' : I_0(Y) \cong I_0(Y') \) in \( C \) which make the diagram

\[
\begin{array}{ccc}
I_0(Y) & \xrightarrow{\beta'} & I_0(Y') \\
\downarrow_{I_0(f)} & & \downarrow_{I_0(f')} \\
I_0(X) & \xrightarrow{\beta} & I_0(X')
\end{array}
\]

commutative. For \((f', \beta') \in S_f \) and for \( \sigma \in \text{Gal}(f) \), we set \( \sigma \cdot (f', \beta) = (f', \beta \circ \sigma^{-1}) \). This gives an action from the left of the group \( \text{Gal}(f) \) on the set \( S_f \). It follows from Lemma 7.1.4 that the set \( S_f \) is non-empty. It follows from Corollary 7.1.2 that for any two elements \((f', \beta'), (f'', \beta'') \) of \( S_f \), we have \( f' = f'' \). This shows that the set \( S_f \) is a left \( \text{Gal}(f) \)-torsor. Let \( g : Z \to X \) be another object of \( I_X \) and let \( h : Z \to Y \) be a morphism in \( I_X \). Let \((g' : Z' \to X', \beta'') \) be an element of \( S_{g'} \). Let us consider the isomorphism \( \theta_{Z',Z,\beta''-1} \) of categories. Let us regard \( f \) as a morphism from \( h \) to \( g \) in \( C_{0,Z} \) and set \( f' = \theta_{Z',Z,\beta''-1}(f) \) and \( \beta' = \xi_{Z',Z,\beta''-1} \). Then \((f', \beta') \) is an element of \( S_f \). Let us write \( f' : Y' \to X' \). It then follows from Lemma 7.1.3 that the isomorphism \( \beta' : I_0(Y) \to I_0(Y') \) in \( C \) has the following characterization:
\( \beta' \) is the unique isomorphism \( \iota_0(Y) \xrightarrow{\sim} \iota_0(Y') \) such that there exists a morphism \( h' : Z' \to Y' \) which makes the diagram

\[
\begin{array}{ccc}
\iota_0(Z) & \xrightarrow{\beta'} & \iota_0(Z') \\
\downarrow \iota_0(h) & & \downarrow \iota_0(h') \\
\iota_0(Y) & \xrightarrow{\beta'_{\equiv}} & \iota_0(Y')
\end{array}
\]

commutative. By sending \( (f'', \beta'') \) to \( (f', \beta') \) we obtain a map \( S_{f''} \to S_f \) which we denote by \( S(h) \). The characterization of the isomorphism \( \beta' \) given above shows that we have \( S(h)(\sigma \cdot (f'', \beta'')) = h_{\ast}(\sigma) \cdot S(h)((f'', \beta'')) \) for any \( \sigma \in \text{Gal}(g) \), where \( h_{\ast} : \text{Gal}(g) \to \text{Gal}(f) \) denotes the homomorphism in (2) of Lemma \ref{lem:iso-characterization}, and that we have \( S(h \circ h') = S(h) \circ S(h) \) for any composable morphisms \( h, h' \) in \( I_X \).

Let \( \bar{H}_X \) denote the projective system \( (\text{Gal}(f))_{f \in \text{Obj } I_X} \) of groups. Then \( (S_f)_{f \in \text{Obj } I_X} \) is a left \( \bar{H}_X \)-torsor. Since \( C_{f_X} \) satisfies at least one of the conditions in Section \ref{sec:6.1} the partially ordered set corresponding to the poset category \( I_X \) satisfies at least one of the conditions in Lemma \ref{lem:6.3.1}. Hence the limit \( \lim_{f \in \text{Obj } I_X} S_f \) is non-empty. Let us choose an element \( (h_f, \beta_f) \in \text{Obj } I_X \) of \( \text{lim}_{f \in \text{Obj } I_X} S_f \). For an object \( f : Y \to X \) of \( I_X \), let \( Y' \) denote the domain of \( h_f \). Let us consider the isomorphism \( \theta_{Y', Y, \beta_f^{-1}} : C_{0, Y'} \xrightarrow{\approx} C_{0, Y'} \) of categories. Let \( g : Z \to X \) be another object of \( I_X \) and let \( h : Z \to Y \) be a morphism from \( g \) to \( f \) in \( I_X \). Let \( Z' \) denote the domain of \( h_f \). It follows from the definition of the transition map of the projective system \( (S_f) \) that there exists a morphism \( h' : Z' \to Y' \) in \( C_0 \) which makes the diagram

\[
\begin{array}{ccc}
\iota_0(Z) & \xrightarrow{\beta_f} & \iota_0(Z') \\
\downarrow \iota_0(h) & & \downarrow \iota_0(h') \\
\iota_0(Y) & \xrightarrow{\beta_f_{\equiv}} & \iota_0(Y')
\end{array}
\]

commutative. It follows from Lemma \ref{lem:6.4.3} that \( \theta_{Z', Z, \beta_f^{-1}} \) sends the object \( h \) of \( C_{0, Z'} \) to the object \( h' \) of \( C_{0, Z} \). Hence it follows from Lemma \ref{lem:7.4.4} that the functor \( C_{0, Y'} \to C_{0, Y'} \) induced by the isomorphism \( \theta_{Z', Z, \beta_f^{-1}} \) of categories is equal to the isomorphism \( \theta_{Y', Y, \beta_f^{-1}} \) of category. Hence by taking the colimit with respect to \( f \), we obtain functors

\[
C_0 = \lim_{f \in \text{Obj } I_X} C_{0, Y'} \to \lim_{f \in \text{Obj } I_X} C_{0, Y'} \to C_0,
\]

whose composite we denote by \( \alpha : C_0 \to C_0 \). We have the natural isomorphism \( \xi_{Y', Y, \beta_f^{-1}}^{-1} \) for each object \( f : Y \to X \) of \( I_X \). By taking the colimit with respect to \( f \), we obtain a natural isomorphism \( \gamma_\alpha \) from \( \iota_0 \) to \( \iota_0 \circ \alpha \). We thus obtain an element \( (\alpha, \gamma_\alpha) \in M(C_{\iota_0}) \). One can check easily that the element \( (\alpha, \gamma_\alpha) \) satisfies the desired properties. \( \square \)

**7.8. Proof of Theorem 5.8.1** the functor \( \omega(C_{\iota_0}) \) is full. Let us prove that the functor \( \omega(C_{\iota_0}) \) is full. Let \( F_1 \) and \( F_2 \) be sheaves on \((C, J)_\iota\), and let \( t : \omega(C_{\iota_0})(F_1) \to \omega(C_{\iota_0})(F_2) \) be a morphism of left \( M(C_{\iota_0}) \)-sets. For each object \( X \) of \( C \), let us choose an edge object \( E_X \) of \( C_0 \) and an isomorphism \( \beta_X : \iota_0(E_X) \xrightarrow{\sim} X \) in
By Corollary 7.6.3 we have isomorphisms $F_1(t_0(E_X)) \cong \omega(c_{0,t_0})(F_1)^{K_X}$ and $F_2(t_0(E_X)) \cong \omega(c_{0,t_0})(F_2)^{K_X}$. We define the map $t_X : F_1(X) \to F_2(X)$ to be the unique map which makes the diagram commutative. Let $f : X \to Y$ be a morphism in $\mathcal{C}$. It follows from (4) of Lemma 6.2.2 that there exists a unique morphism $f' : E_X \to E_Y$ in $\mathcal{C}_0$ and an isomorphism $\beta' : t_0(Y') \cong Y$ in $\mathcal{C}$ such that the diagram

\[
\begin{array}{ccc}
t_0(E_X) & \xrightarrow{\beta_X} & X \\
\downarrow t_0 & & \downarrow f \\
t_0(Y') & \xrightarrow{\beta'} & Y
\end{array}
\]

is commutative. We set $\beta = \beta_0^{-1} \circ \beta_Y : t_0(E_Y) \cong t_0(Y')$. It follows from Lemma 7.7.1 that there exists an element $(\alpha, \gamma_\alpha)$ of $M(c_{0,t_0})$ satisfying $\alpha(E_Y) = Y'$ and $\gamma_\alpha(E_Y) = \beta$. It then follows from the definition of the action of $(\alpha, \gamma_\alpha)$ on $\omega(c_{0,t_0})(F_i)$ that the diagram

\[
\begin{array}{ccc}
F_1(Y) & \xrightarrow{\beta_Y} & F_1(t_0(E_Y)) \\
\downarrow f^* & & \downarrow (t_0(f') \circ \beta') \\
F_1(X) & \xrightarrow{\beta_X} & F_1(t_0(E_X))
\end{array}
\]

is commutative for $i = 1, 2$. Hence the diagram

\[
\begin{array}{ccc}
F_1(Y) & \xrightarrow{t_Y} & F_2(Y) \\
\downarrow f^* & & \downarrow f^* \\
F_1(X) & \xrightarrow{t_X} & F_2(X)
\end{array}
\]

is commutative. Thus the collection of maps $(t_X : F_1(X) \to F_2(X))_{X \in Obj \mathcal{C}}$ gives a morphism $t' : F_1 \to F_2$ of sheaves on $(\mathcal{C}, J)$ such that the map $\omega(c_{0,t_0})(F_1) \to \omega(c_{0,t_0})(F_2)$ induced by $t'$ is equal to $t$. This proves that the functor $\omega(c_{0,t_0})$ is full.

### 8. Proof of Theorem 7.8.1

The fiber functor $\omega(c_{0,t_0})$ is essentially surjective.

Until the end of Section 8 we assume that, for any object $X$ of $\mathcal{C}$, the category $\mathcal{C}/X$ satisfies at least one of the two conditions in Section 5.8.1.

### 8.1. Lemmas on edge objects.

**Lemma 8.1.1.** Let $(\alpha, \gamma_\alpha)$ be an element of $M(c_{0,t_0})$. Then for any morphism $f : Y \to X$ in $\mathcal{C}_0$ of type $J$, the morphism $\alpha(f)$ is of type $J$. 
Proof. In the commutative diagram

\[
\begin{array}{ccc}
t_0(Y) & \xrightarrow{t_0(f')} & t_0(X) \\
\gamma_\alpha(Y) & \downarrow & \gamma_\alpha(X) \\
t_0(\alpha(Y)) & \xrightarrow{t_0(f)} & t_0(\alpha(X)),
\end{array}
\]

the upper horizontal arrow belongs to \( \mathcal{T}(J) \) and the vertical arrows are isomorphisms. Hence the lower horizontal arrow belongs to \( \mathcal{T}(J) \), which proves the claim. \( \square \)

Lemma 8.1.2. Let \((\alpha, \gamma_\alpha)\) be an element of \( M(C_0, t_0) \). Let \( X \) be an edge object of \( C_0 \) and set \( X' = \alpha(X) \). Then for an object \( Y' \) of \( C_0 \), the following two conditions are equivalent.

1. There exists an object \( Y \) of \( C_0 \) satisfying \( Y' = \alpha(Y) \).
2. There exists an object \( Z' \) of \( C_0 \) and a diagram \( X' \xleftarrow{f'} Z' \rightarrow Y' \) such that \( f' \) is of type \( J \).

Proof. First we prove that the condition (1) implies the condition (2). Suppose that the condition (1) is satisfied. Let us choose an object \( Y \) of \( C_0 \) satisfying \( Y' = \alpha(Y) \). It follows from Lemma 7.3.2 that there exists an object \( Z \) of \( C_0 \) and a diagram \( X \xleftarrow{f} Z \rightarrow Y \) in \( C_0 \) such that \( f \) is of type \( J \). We set \( Z' = \alpha(Z) \) and \( f' = \alpha(f) \). By applying \( \alpha \) to the diagram above, we obtain a diagram \( X' \xleftarrow{f'} Z' \rightarrow Y' \). It follows from Lemma 8.1.1 that the morphism \( f' \) is of type \( J \). Hence the condition (2) is satisfied.

Next we prove that the condition (2) implies the condition (1). Suppose that the condition (2) is satisfied. Let us choose an object \( Z' \) of \( C_0 \) and a diagram \( X' \xleftarrow{f'} Z' \xrightarrow{g'} Y' \) such that \( f' \) is of type \( J \). By replacing \( f' \) with its composite with a suitable morphism \( Z'' \rightarrow Z' \) in \( C_0 \), may assume that \( t_0(f') \) is a Galois covering in \( C \). Let us consider the isomorphism \( t_0(X) \xrightarrow{\gamma_\alpha(X)} t_0(X') \) in \( C \) and the morphism \( f' : Z' \rightarrow X' \) in \( C_0 \). It follows from Lemma 7.3.3 that there exist a morphism \( f : Z \rightarrow X \) in \( C_0 \) and an isomorphism \( \gamma' : t_0(Z) \cong t_0(Z') \) which make the diagram

\[
\begin{array}{ccc}
t_0(X) & \xrightarrow{t_0(f)} & t_0(Z) \\
\gamma_\alpha(X) & \downarrow & \gamma' \\
t_0(X') & \xrightarrow{t_0(f')} & t_0(Z')
\end{array}
\]

commutative. Since \( t_0(f') \circ \gamma' = \gamma_\alpha(X) \circ t_0(f) = t_0(\alpha(f)) \circ \gamma_\alpha(Z) \), we have a commutative diagram

\[
\begin{array}{ccc}
t_0(\alpha(Z)) & \xrightarrow{\gamma' \circ \gamma_\alpha(Z)^{-1}} & t_0(Z') \\
t_0(\alpha(f)) & \downarrow & \downarrow \gamma' \\
t_0(X') & \xrightarrow{t_0(f')} & t_0(Z')
\end{array}
\]

in \( C \). Since we have assumed that \( t_0(f') \) is a Galois covering, \( t_0(\alpha(f)) \) is a Galois covering. Hence it follows from Corollary 7.3.2 that \( Z' = \alpha(Z) \) and \( f' = \alpha(f) \). Let us consider the isomorphism \( \gamma_\alpha(Y) : t_0(Y) \xrightarrow{\cong} t_0(Y') \) and the morphism \( g' : Y' \rightarrow Y' \).
$Z'$ in $C_0$. It follows from Lemma 8.1.3 that there exist a morphism $g : Z \to Y$ in $C_0$ and an isomorphism $\gamma' : t_0(Y) \cong t_0(Y')$ which make the diagram

$$
t_0(Z) \xrightarrow{t_0(g)} t_0(Y) \\
\gamma'_a(Z) \downarrow \quad \downarrow \gamma' \\
t_0(Z') \xrightarrow{t_0(g')} t_0(Y')
$$

commutative. Since $\gamma'^{-1} \circ t_0(g') = t_0(g) \circ \gamma_a(Z)^{-1} = \gamma_a(Y)^{-1} \circ t_0(a(g))$, we have a commutative diagram

$$
t_0(Z') \xrightarrow{t_0(a(g))} t_0(Z') \\
\downarrow t_0(a(g)) \quad \uparrow t_0(a(g')) \\
t_0(a(Y)) \xrightarrow{\gamma'_a(Y)} t_0(Y')
$$
in $C$. Since the functor $t_0_{Z/I} : C_{0, Z/I} \to C_{a(Y)}$ is an equivalence of categories and $C_0$ is skeletal, we have $Y' = a(Y)$ and $g' = a(g)$. This in particular shows that the condition (1) is satisfied.

**Lemma 8.1.3.** Let $X$ be an edge object of $C_0$ and let $(\alpha, \gamma_a)$ be an element of $K_X$. Then for any morphism $f : X \to Y$ in $C_0$, we have $a(Y) = Y$ and $\gamma_a(Y) = \text{id}_{t_0(Y)}$.

**Proof.** We have a commutative diagram

$$
t_0(X) \xrightarrow{t_0(f)} t_0(X) \\
\downarrow t_0(f) \quad \downarrow t_0(a(f)) \\
t_0(Y) \xrightarrow{\gamma_a(Y)} t_0(a(Y))
$$
in $C$. Since the functor $t_0_{X/I} : C_{0, X/I} \to C_{a(Y)}$ is an equivalence of categories and $C_0$ is skeletal, we have $a(Y) = Y$ and $f = a(f)$. Hence $t_0(f) = \gamma_a(Y) \circ t_0(a(f))$. Since $t_0(f)$ is an epimorphism, we have $\gamma_a(Y) = \text{id}_{t_0(Y)}$. This proves the claim.

**Lemma 8.1.4.** Let $(\alpha, \gamma_a)$ and $(\alpha', \gamma_{a'})$ be two elements of $M_{(C_0, a_0)}$. Suppose that for any object $X$ of $C_0$, there exists an object $X'$ of $C_0$ satisfying $a'(X) = a(X')$. Then there exists a unique element $(\alpha'', \gamma_{a''})$ of $M_{(C_0, a_0)}$ satisfying $(\alpha', \gamma_{a'}) = (\alpha, \gamma_a) \circ (\alpha'', \gamma_{a''})$.

**Proof.** Let $X$ be an arbitrary object of $C_0$. By assumption, there exists an object of $C_0$ whose image under the functor $a$ is equal to $a'(X)$. We denote this object by $a''(X)$. It follows from Lemma 7.5.2 that the object $a''(X)$ is uniquely determined by this property and that if $X' \to X$ is a morphism in $C_0$, then there exists a morphism from $a''(X')$ to $a''(X)$ in $C_0$. Hence by associating $a''(X)$ to each object $X$ of $C_0$, we obtain a functor $a'' : C_0 \to C_0$. By construction we have $\alpha' = a \circ a''$. For any object $X$ of $C_0$, we have the diagram

$$
t_0(X) \xrightarrow{\gamma_{a''}(X)} t_0(a'(X)) \\
\cong \\
t_0(a''(X)) \xrightarrow{\gamma_{a''}(a''(X))} t_0(a \circ a''(X))$$
in $C$. Hence there exists a unique isomorphism $\gamma_{\alpha''}(X) : t_0(X) \cong t_0(\alpha''(X))$ such that the equality $\gamma_{\alpha'}(\alpha''(X)) \circ \gamma_{\alpha''}(X) = \gamma_{\alpha}(X)$ holds. Since $\gamma_{\alpha}$ and $\gamma_{\alpha'}$ are isomorphisms of functors, it follows that the isomorphism $\gamma_{\alpha''}(X)$ is functorial in $X$ in the following sense: for any morphism $f : X' \to X$ in $C$, the diagram

$$
\begin{array}{ccc}
t_0(X') & \xrightarrow{t_0(f)} & t_0(X) \\
\gamma_{\alpha''}(X') & \downarrow & \gamma_{\alpha''}(X) \\
t_0(\alpha''(X')) & \xrightarrow{t_0(\alpha''(f))} & t_0(\alpha''(X))
\end{array}
$$

is commutative. Hence the isomorphisms $\gamma_{\alpha''}(X)$ for the objects $X$ of $C_0$ give an isomorphism $\gamma_{\alpha''} : t_0 \cong t_0 \circ \alpha''$ of functors and the pair $(\alpha'', \gamma_{\alpha''})$ is an element of $M_{(C_0, t_0)}$ satisfying $(\alpha', \gamma_{\alpha'}) = (\alpha, \gamma_0) \circ (\alpha'', \gamma_{\alpha''})$. The uniqueness of $(\alpha'', \gamma_{\alpha''})$ follows from the uniqueness of $\alpha''(X)$ for each object $X$ of $C_0$. This completes the proof. \hfill \Box

**Lemma 8.1.5.** Let $X$ be an edge object of $C_0$. Let $(\alpha, \gamma_0)$ and $(\alpha', \gamma_{\alpha'})$ be two elements of $M_{(C_0, t_0)}$. Suppose that $\alpha(X) = \alpha'(X)$ and $\gamma_0(X) = \gamma_{\alpha'}(X)$. Then there exists an element $(\alpha'', \gamma_{\alpha''})$ of $K_X$ satisfying $(\alpha', \gamma_{\alpha'}) = (\alpha, \gamma_0) \circ (\alpha'', \gamma_{\alpha''})$.

**Proof.** Let $Y$ be an arbitrary object of $C_0$. It follows from Lemma 8.1.4 that there exists a diagram

$$
X \xleftarrow{f} Z \to Y
$$

in $C_0$ such that $f$ is of type $J$. By applying $\alpha'$, we have a diagram

$$
\alpha'(X) \xrightarrow{\alpha'(f)} \alpha'(Z) \to \alpha'(Y).
$$

It follows from Lemma 8.1.1 that $\alpha'(f)$ is of type $J$. Hence it follows from Lemma 8.1.2 that there exists an object of $C_0$ whose image under the functor $\alpha$ is equal to $\alpha'(Y)$. Hence it follows from Lemma 8.1.4 that there exists a unique element $(\alpha'', \gamma_{\alpha''})$ of $M_{(C_0, t_0)}$ satisfying $(\alpha', \gamma_{\alpha'}) = (\alpha, \gamma_0) \circ (\alpha'', \gamma_{\alpha''})$.

To prove the claim, it remains to show that $(\alpha'', \gamma_{\alpha''})$ is an element of $K_X$. Since $\alpha' = \alpha \circ \alpha''$ and $\alpha(X) = \alpha'(X)$, we have $\alpha(\alpha''(X)) = \alpha(X)$. Hence it follows from Lemma 7.5.2 that we have $\alpha''(X) = X$. By applying the equality $\gamma_0(\alpha''(Y)) \circ \gamma_{\alpha''}(Y) = \gamma_0(Y)$ to $Y = X$, we have $\gamma_0(X) \circ \gamma_{\alpha''}(X) = \gamma_{\alpha'}(X)$ for each object $X$ of $C_0$. Hence we have $\gamma_{\alpha''}(X) = \text{id}_{t_0(X)}$, which proves that $(\alpha'', \gamma_{\alpha''})$ is an element of $K_X$. This completes the proof. \hfill \Box

**Remark 8.1.6.** In the proof of the previous lemma, a stronger statement is proved: There exists a unique element $(\alpha'', \gamma_{\alpha''})$ of $M_{(C_0, t_0)}$ satisfying $(\alpha', \gamma_{\alpha'}) = (\alpha, \gamma_0) \circ (\alpha'', \gamma_{\alpha''})$. Moreover the element $(\alpha'', \gamma_{\alpha''})$ belongs to $K_X$. As we do not need this statement, the details are suppressed.

**Lemma 8.1.7.** Let $(\alpha, \gamma_0)$ be an element of $M_{(C_0, t_0)}$ and $X$ be an edge object of $C_0$. Suppose that $\alpha(X)$ is an edge object of $C_0$. Then $(\alpha, \gamma_0)$ is an invertible element of $M_{(C_0, t_0)}$.

**Proof.** By Lemma 8.1.4 we are reduced to proving that the functor $\alpha : C_0 \to C_0$ is an isomorphism of categories. It follows from Lemma 7.5.2 that $\alpha$ is fully faithful. Since the category $C_0$ is skeletal, it suffices to show that $\alpha$ is essentially surjective.
Let $Y$ be an arbitrary object of $C_0$. Since $\alpha(X)$ is an edge object of $C_0$, it follows from Lemma 7.3.2 that there exists a diagram

$$\alpha(X) \xleftarrow{f} Z \to Y$$

in $C_0$ such that $f$ is of type $J$. Hence it follows from Lemma 8.1.2 that there exists an object $Y'$ of $C_0$ satisfying $Y = \alpha(Y')$. This shows that $\alpha$ is essentially surjective, which proves the claim. \hfill \Box

**Lemma 8.1.8.** Let $Y$ be an edge object of $C_0$ and let $f : Y \to X$ be a morphism in $C_0$ of type $J$. Then $X$ is an edge object of $C_0$.

**Proof.** Let $g : X' \to X$ be an arbitrary morphism in $C_0$. Since $C_0$ is semi-cofiltered (see proof of Lemma 5.7.1), there exist an object $Y'$ of $C_0$ and morphisms $f' : Y' \to X'$ and $g' : Y' \to Y$ satisfying $f \circ g' = g \circ f'$. Since $Y$ is an edge object, we have $g'$ is of type $J$. Hence $f \circ g'$ is of type $J$. It follows from Proposition 2.4.3 that $g$ is of type $J$. This proves that $X$ is an edge object of $C_0$. \hfill \Box

8.2. **Proof of Theorem 5.8.1** the fiber functor is essentially surjective.

8.2.1. **Proof:** Step 1. Let $T$ be an arbitrary smooth left $M_{(C_0, t_0)}$-set. For each object $X$ of $C$, let us choose an edge object $E_X$ of $C_0$ and an isomorphism $\beta_X : t_0(E_X) \cong X$ in $C$. We set $F_T(X) = T^{K_{E_X}}$. One can check, by modifying the argument of the paragraph below, that $F_T(X)$ is independent of the choice of the element $(E_X, \beta_X)$ up to canonical isomorphisms. However we will not use this independence in the proof of Theorem 5.8.1 given below.

Let $f : X \to Y$ be a morphism in $C$. In this paragraph, we define a map $f^* : F_T(Y) \to F_T(X)$. We use the following notation introduced in Section 7.8. It follows from (4) of Lemma 6.2.2 that there exist a unique morphism $\alpha : t_0(Y') \to t_0(Y)$ in $C_0$ and an isomorphism $\beta' : t_0(Y') \cong Y$ in $C$ such that the diagram

$$
\begin{array}{ccc}
t_0(E_X) & \xrightarrow{\beta_X} & X \\
\downarrow t_0(f') & & \downarrow f \\
t_0(Y') & \xrightarrow{\beta'} & Y \\
\end{array}
$$

is commutative. We set $\beta = \beta'^{-1} \circ \beta_Y : t_0(E_Y) \cong t_0(Y')$. It follows from Lemma 7.7.1 that there exists an element $(\alpha, \gamma_0)$ of $M_{(C_0, t_0)}$ satisfying $\alpha(E_Y) = Y'$ and $\gamma_0(E_Y) = \beta$. We now show for any element $(\alpha', \gamma_0')$ of $K_{E_X}$, there exists a unique element $(\alpha'', \gamma_0'')$ of $K_{E_Y}$ satisfying $(\alpha', \gamma_0') \circ (\alpha, \gamma_0) = (\alpha'', \gamma_0'')$. Let $(\alpha', \gamma_0')$ be an element of $K_{E_X}$. It follows from Lemma 8.1.3 that we have $\alpha'(Y') = Y'$ and $\gamma_0'(Y') = \text{id}_{t_0(Y')}$. Hence it follows from Lemma 8.1.3 that there exists an element $(\alpha'', \gamma_0'')$ of $K_{E_Y}$ satisfying $(\alpha', \gamma_0') \circ (\alpha, \gamma_0) = (\alpha'', \gamma_0'')$. Hence the map $T \to T$ given by the multiplication by $(\alpha, \gamma_0)$ induces a map $F_T(Y) = T^{K_{E_Y}} \to T^{K_{E_X}} = F_T(X)$ which we denote by $f^*$.

8.2.2. **Proof:** Step 2. We now show that the map $f^*$ is independent of the choice of the element $(\alpha, \gamma_0)$. Suppose that $(\alpha_1, \gamma_0_1)$ is another choice of an element $M_{(C_0, t_0)}$ satisfying $\alpha_1(E_Y) = Y'$ and $\gamma_0_1(E_Y) = \beta$. Then it follows from Lemma 8.1.3 that there exists an element $(\alpha''', \gamma_0''')$ of $K_{E_Y}$ satisfying $(\alpha_1, \gamma_0_1) = (\alpha'', \gamma_0'') \circ (\alpha''', \gamma_0''')$. This implies that the map $f^*$ is independent of the choice of the element $(\alpha, \gamma_0)$. \hfill \Box
8.2.3. Proof: Step 3. We now show that, for any morphism \( g : Y \rightarrow Z \) in \( \mathcal{C} \), we have \((g \circ f)^* = f^* \circ g^*\). It follows from (4) of Lemma 6.2.2 that there exist a unique morphism \( g' : E_Y \rightarrow Z' \) in \( \mathcal{C}_0 \) and an isomorphism \( \beta_1' : t_0(Z') \cong Z \) in \( \mathcal{C} \) such that the diagram

\[
t_0(E_Y) \xrightarrow{\beta_Y} Y \\
\downarrow \quad \quad \downarrow g \\
t_0(Z') \xrightarrow{\beta_1'} Z
\]

is commutative. We set \( \beta_1 = \beta_1' \circ \beta : t_0(E_Z) \xrightarrow{\cong} t_0(Z') \). It follows from Lemma 7.7.1 that there exists an element \( (\alpha, \gamma) \) of \( M_{(\mathcal{C}_0, \iota_0)} \) satisfying \( \alpha(E_Z) = Z' \) and \( \gamma(E_Y) = \beta_1 \). We set \( Z'' = \alpha(Z) \) and \( h = \alpha(g') \circ f' \). Then \( (h, \beta_2') \) is the unique pair of a morphism \( h : E_X \rightarrow Z'' \) and an isomorphism \( \beta_2 : t_0(Z'') \cong Z \) such that the diagram

\[
t_0(E_X) \xrightarrow{\beta_X} X \\
\downarrow \quad \quad \downarrow g \circ f \\
t_0(Z'') \xrightarrow{\beta_2} Z
\]

is commutative. We set \( \beta_2 = \beta_2' \circ \beta : t_0(E_Z) \xrightarrow{\cong} t_0(Z'') \). Then the element \( (\alpha_2, \gamma_{a_2}) = (\alpha, \gamma_a) \circ (\alpha_1, \gamma_{a_1}) \) of \( M_{(\mathcal{C}_0, \iota_0)} \) satisfies \( \alpha_2(E_Z) = Z'' \) and \( \gamma_a(E_Y) = \beta_2 \). It follows from the definition that the map \((g \circ f)^* \) is given by the multiplication by the element \( (\alpha_2, \gamma_{a_2}) \). Since the maps \( f^* \), \( g^* \) are given by the multiplication by the elements \( (\alpha, \gamma_a) \) and \( (\alpha_1, \gamma_{a_1}) \), respectively, we have the desired equality \( f^* \circ g^* = (g \circ f)^* \). Thus we obtain a presheaf \( F_T \) on \( \mathcal{C} \). Since the action of \( \hat{M}_{(\mathcal{C}_0, \iota_0)} \) on \( T \) is smooth, we have \( \omega_{(\mathcal{C}_0, \iota_0)}(F_T) = T \).

8.2.4. Proof: Step 4. Suppose that \( f : X \rightarrow Y \) is a Galois covering in \( T \). Let \( f' : E_X \rightarrow Y' \), \( \beta' : t_0(Y') \xrightarrow{\cong} Y \), and \( \beta : t_0(E_Y) \xrightarrow{\cong} t_0(Y') \) be as in Section 8.2.3 above. Then it follows from Lemma 8.1.8 that \( Y' \) is an edge object of \( \mathcal{C}_0 \). Hence it follows from Lemma 8.1.7 that the element \( (\alpha, \gamma_a) \in M_{(\mathcal{C}_0, \iota_0)} \) is invertible. Since \( f \) is a Galois covering in \( T \), the morphism \( t_0(f') \) is a Galois covering in \( T \). It follows from Lemma 8.1.3 that \( \mathbb{K}_{E_X} \) is a subgroup of \( \mathbb{K}_{Y'} \). Let \( i : \mathbb{K}_{E_X} \rightarrow \mathbb{K}_{Y'} \) denote the inclusion. Let \( \rho : \mathbb{K}_{Y'} \rightarrow \text{Gal}(f') \) denote the composite of \( \phi_{Y'} : \mathbb{K}_{Y'} \rightarrow H_{Y'} \) with the projection map \( H_{Y'} \rightarrow \text{Gal}(f') \). Let us consider the sequence

\[
1 \rightarrow \mathbb{K}_{E_X} \xrightarrow{i} \mathbb{K}_{Y'} \xrightarrow{\rho} \text{Gal}(f') \rightarrow 1.
\]

It is obvious that the map \( i \) is injective. It follows from the definition of \( \mathbb{K}_{E_X} \) that the kernel of \( \rho \) is equal to the image of \( i \). We have shown in Lemma 7.6.1 that \( \rho \) is surjective, when \( \mathcal{C}_{/Y} \) satisfies one of the conditions in Section 6.1. Hence the sequence (8.1) is exact. Since \( (\alpha, \gamma_a) \in M_{(\mathcal{C}_0, \iota_0)} \) is invertible, this implies that the map \( f'^* : F_T(Y) \rightarrow F_T(X) \) induces a bijection \( F_T(Y) \rightarrow F_T(X)^{\text{Gal}(f')} \). This shows that \( F_T \) is a sheaf on \( (\mathcal{C}, J) \).

Let \( X \) be an edge object of \( \mathcal{C}_0 \). Applying Lemma 7.7.1 to the isomorphism \( \beta_{t_0(X)} : t_0(E_{t_0(X)}) \xrightarrow{\cong} t_0(X) \), we can choose an element \( (\alpha_{X}, \gamma_{a_{X}}) \) of \( M_{(\mathcal{C}_0, \iota_0)} \) satisfying \( \alpha_X(E_{t_0(X)}) = X \) and \( \gamma_{a_{X}} = \beta_{t_0(X)} \). It follows from Lemma 8.1.7 that \( (\alpha_X, \gamma_{a_{X}}) \) is an invertible element of \( M_{(\mathcal{C}_0, \iota_0)} \). Hence the action of \( (\alpha_X, \gamma_{a_{X}}) \) on
$T$ induces a bijection $F_T(X) = T^{\text{Ker}(\alpha)(X)} \cong T^{\mathbb{S}}$ which we denote by $\epsilon_X$. Let $f : X \to Y$ be a morphism in $\mathcal{C}_1$. By Lemma 8.1.3 we have the inclusion map $T^{\mathbb{S}} \subset T^{\mathbb{S}}$. We set $(\alpha, \gamma_0) = (\alpha_X, \gamma_{\alpha_X})^{-1} \circ (\alpha_Y, \gamma_{\alpha_Y})$. We set $f' = \alpha_X^{-1}(f)$. It is a morphism in $\mathcal{C}_0$ whose domain is equal to $E_{\alpha(X)}$. Let $Y'$ denote the codomain of the morphism $\alpha_X^{-1}(f)$ and set $\beta = \gamma_{\alpha_X}(Y') : \tau_0(Y') \cong \tau_0(Y')$. We then have $\tau_0(f) \circ \gamma_{\alpha_0(X)} = \beta' \circ \tau_0(f')$. Hence it follows from the argument in Section 8.2.2 that the map $f^* : F_T(Y) \to F_T(X)$ is given by the multiplication by $(\alpha, \gamma_0)$. This shows that the diagram

$$
\begin{array}{ccc}
F_T(Y) & \xrightarrow{f^*} & F_T(X) \\
\epsilon_Y & \cong & \epsilon_X \\
T^{\mathbb{S}} & \subset & T^{\mathbb{S}}
\end{array}
$$

is commutative. Hence the bijections $\epsilon_X$ give an bijection $\omega(\mathcal{C}_0, \tau_0)(F_T) \cong T$. It is straightforward to check that this bijection is an isomorphism of $M(\mathcal{C}_0, \tau_0)$-sets. Therefore the functor $\omega(\mathcal{C}_0, \tau_0)$ is essentially surjective. This completes the proof of Theorem 5.8.1

9. THE TOPOS HAS ENOUGH POINTS

In this section, we show that the topos associated to a $Y$-site under cardinality conditions has enough points. We show that the fiber functor $\omega(\mathcal{C}_0, \tau_0)$ has a left adjoint, hence the fiber functor is a point of the topos $\text{Shv}(\mathcal{C}, J)$. Then we obtain as a corollary to Theorem 5.8.1 that the topos has enough points.

Until the end of this section we assume that, for any object $X$ of $\mathcal{C}$, the category $\mathcal{C}(\mathcal{T}(\mathcal{J}))/_X$ satisfies at least one of the two conditions in Section 5.8.1

9.1. Lemma 9.1.1. The functor $\omega(\mathcal{C}_0, \tau_0) : \text{Presh}(\mathcal{C}) \to (\text{Sets})$ commutes with finite limits and arbitrary colimits.

Proof. Recall that we have defined, for any presheaf $F$ on $\mathcal{C}$, the set $\omega(\mathcal{C}_0, \tau_0)(F)$ to be a filtered colimit of sections of $F$. Observe that in the category $\text{Presh}(\mathcal{C})$, limits and colimits can be taken sectionwise. Hence the claim (1) follows from that filtered colimits of sets commute with finite limits and arbitrary colimits in the following sense: for any filtered poset $I$, for any finite poset $J$ (resp. for any poset $J'$), and for any functor $S$ from $I \times J^{op}$ (resp. $I \times J$) to the category of sets, the natural map $\lim_{i \in I} \lim_{j \in J} S(i, j) \to \lim_{(i, j) \in I \times J} S(i, j)$ (resp. $\lim_{i \in I} \lim_{j \in J} S(i, j) \to \lim_{(i, j) \in I \times J} S(i, j)$) is a bijection. This proves the claim. □

Lemma 9.1.2. Let $F$ be a presheaf on $\mathcal{C}$ and $\alpha_J(F)$ its associated sheaf on $(\mathcal{C}, J)$. Then the adjunction morphism $F \to \alpha_J(F)$ of presheaves induces a bijection $\omega(\mathcal{C}_0, \tau_0)(F) \cong \omega(\mathcal{C}_0, \tau_0)(\alpha_J(F))$.

Proof. Let $\mathcal{C}_1$ denote the full subcategory of $\mathcal{C}_0$ whose objects are the edge objects of $\mathcal{C}_0$. Let $X$ be an edge object of $\mathcal{C}_0$. Since $\mathcal{C}_0$ is a poset, the functor $\mathcal{C}_0/X \to \mathcal{C}_0$ which associates, to each object $f : Y \to X$ of $\mathcal{C}_0/X$, the object $Y$ of $\mathcal{C}_0$ is fully faithful. It follows from Lemma 5.7.3 that this functor induces a fully faithful functor $\mathcal{C}_0/X \to \mathcal{C}_1$. Via this functor we regard $\mathcal{C}_0/X$ as a full subcategory of $\mathcal{C}_1$. Lemma 5.7.6 shows that $\mathcal{C}_1$ is $\Lambda$-connected. Hence the objects of $\mathcal{C}_0/X$ are cofinal
in $C_1$, it then follows from Lemma 7.3.1 that the objects of $I_X$ are cofinal in $C_1$. Hence the natural map

$$\lim_{(f:Y \to X) \in \text{Obj} I_X} F(i_0(Y)) \to \omega_{(\mathcal{C}_0, i_0)}(F)$$

is bijective.

**Lemma 9.1.3.** The functor $i_0$ induces a functor $I_X \to \text{Gal} / i_0(X)$ which we denote by $j_X$. The functor $j_X$ is an equivalence of categories.

**Proof.** It follows from Condition (3) of Definition 5.5.3 that the functor $j_X$ is essentially surjective. Let $f_1 : Y_1 \to X$ and $f_2 : Y_2 \to X$ be two objects of $I_X$. Suppose that there exists a morphism from $j_X(f_1)$ to $j_X(f_2)$ in $\text{Gal} / i_0(X)$. Then there exists a morphism $g : i_0(Y_1) \to i_0(Y_2)$ in $C$ satisfying $i_0(f_1) = i_0(f_2) \circ g$. It follows from Condition (4) of Definition 5.5.3 that there exist an object $Y_2'$ of $\mathcal{C}_0$, morphisms $g' : Y_1 \to Y_2'$ and $f_2' : Y_2' \to X$, and an isomorphism $\beta : i_0(Y_2) \simeq i_0(Y_2')$ satisfying $i_0(g') = \beta \circ g$ and $i_0(f_2) = i_0(f_2') \circ \beta$. Since $i_0(f_2)$ is a Galois covering in $C$, it follows that $i_0(f_2')$ is a Galois covering in $C$. Hence it follows from Corollary 7.1.2 that we have $Y_2 = Y_2'$ and $f_2 = f_2'$. This shows that $\beta = \text{id}_{i_0(Y_2)}$ and $g = i_0(g')$. Hence $g'$ gives a morphism from $f_1$ to $f_2$ in $I_X$. Since both $I_X$ and $\text{Gal} / i_0(X)$ are thin, this shows that the functor $j_X$ is fully faithful. This completes the proof of the claim that $j_X$ is an equivalence of categories. □

Therefore the bijection (4.12) gives a bijection

$$a_f(F)(i_0(X)) \cong \lim_{(f:Y \to X) \in \text{Obj} I_X} F(i_0(Y))^\text{Gal}(f).$$

Since (4.11) is bijective, we have a bijection

$$a_f(F)(i_0(X)) \cong \omega_{(\mathcal{C}_0, i_0)}(F)^{H_X}$$

where $H_X$ acts on $\omega_{(\mathcal{C}_0, i_0)}$ via the homomorphism $\psi_X$. Hence it follows from Lemma 7.6.2 that we have a bijection

$$a_f(F)(i_0(X)) \cong \omega_{(\mathcal{C}_0, i_0)}(F)^{K_X}.$$

By composing the inverse of this bijection with the bijection in Corollary 7.0.3 we obtain a bijection

$$\delta_{F,X} : \omega_{(\mathcal{C}_0, i_0)}(F)^{K_X} \to \omega_{(\mathcal{C}_0, i_0)}(a_f(F))^{K_X}.$$

It is then straightforward to check that the diagram

$$\begin{array}{ccc}
\omega_{(\mathcal{C}_0, i_0)}(F)^{K_X} & \xrightarrow{\delta_{F,X}} & \omega_{(\mathcal{C}_0, i_0)}(a_f(F))^{K_X} \\
\downarrow & & \downarrow \\
\omega_{(\mathcal{C}_0, i_0)}(F) & \xrightarrow{\omega_{(\mathcal{C}_0, i_0)}(a_f(F))} & \omega_{(\mathcal{C}_0, i_0)}(a_f(F)),
\end{array}$$

where the vertical arrows are inclusions and the lower horizontal arrow is a map induced by the adjunction morphism $F \to a_f(F)$, is commutative. Since $\omega_{(\mathcal{C}_0, i_0)}(F)$ and $\omega_{(\mathcal{C}_0, i_0)}(a_f(F))$ are smooth $\mathcal{M}_{(\mathcal{C}_0, i_0)}$-sets, this shows that the map $\omega_{(\mathcal{C}_0, i_0)}(F) \to \omega_{(\mathcal{C}_0, i_0)}(a_f(F))$ is bijective, which proves the claim. □
9.2. Let us consider the functor $\omega_{(C_0, i_0)}$ restricted to the full subcategory $\text{Shv}(C, J)$ of sheaves in $\text{Pres}(C)$. Let us denote it by $F^* : \text{Shv}(C, J) \to \text{(Sets)}$. Lemma 9.1.1 shows that $F^*$ commutes with fiber products. Let us show that the functor $F^*$ has a right adjoint. For a set $Y$, we construct a presheaf $F_*(Y)$ on $C$ by setting $F_*(Y)(X) = \text{Map}(\omega_{(C_0, i_0)}(\mathfrak{h}_C(X)), Y)$ for each object $X$ of $C$.

**Lemma 9.2.1.** The presheaf $F_*(Y)$ is a sheaf on $(C, J)$.

**Proof.** Let $f : X' \to X$ in $C$ be an arbitrary Galois covering which belongs to $\mathcal{T}$. Let us consider the map $\omega_{(C_0, i_0)}(\mathfrak{h}_C(f)) : \omega_{(C_0, i_0)}(\mathfrak{h}_C(X')) \to \omega_{(C_0, i_0)}(\mathfrak{h}_C(X))$. By definition, the map $\omega_{(C_0, i_0)}(\mathfrak{h}_C(f))$ is equal to the map

$$\lim_{Z \in \text{Obj}_C} \text{Hom}_C(i_0(Z), X') \to \lim_{Z \in \text{Obj}_C} \text{Hom}_C(i_0(Z), X)$$

induced by the composition with $f$. The map $\omega_{(C_0, i_0)}(\mathfrak{h}_C(f))$ is a pseudo $\text{Gal}(f)$-torsor since it is a filtered colimit of pseudo $\text{Gal}(f)$-torsors. It follows from Condition (3) of Definition 2.3.1 and Condition (3) of Definition 5.5.3 that the map $\omega_{(C_0, i_0)}(\mathfrak{h}_C(f))$ is surjective. This shows that the set $\omega_{(C_0, i_0)}(\mathfrak{h}_C(X))$ together with the map $\omega_{(C_0, i_0)}(\mathfrak{h}_C(f))$ is a quotient object of $\omega_{(C_0, i_0)}(\mathfrak{h}_C(X'))$ by $\text{Gal}(f)$ in the category of sets. Hence the pullback map $F_*(Y)(X) \to F_*(Y)(X')$ induces a bijection $F_*(Y)(X) \cong F_*(Y)(X')^{\text{Gal}(f)}$. This completes the proof of the claim that $F_*(Y)$ is a sheaf on $(C, J)$. □

Let $Y$ be a set. Using Lemma 9.2.1 above, we can apply $F^*$ to $F_*(Y)$.

**Lemma 9.2.2.** Let $Y$ be a set. Then we have a bijection

$$F^*(F_*(Y)) \cong \lim_{Z \in \text{Obj}_C} \text{Map}(M_{(C_0, i_0)}/K_Z, Y)$$

which is functorial in $Y$.

**Proof.** By definition, we have

$$F^*(F_*(Y)) = \lim_{Z \in \text{Obj}_C} \text{Map}(\omega_{(C_0, i_0)}(\mathfrak{h}_C(i_0(Z))), Y).$$

For any sheaf $G$ on $(C, J)$, we have isomorphisms

$$\text{Hom}_{M_{(C_0, i_0)}}(\omega_{(C_0, i_0)}(\mathfrak{h}_C(i_0(Z))), \omega_{(C_0, i_0)}(G)) \cong \text{Hom}_{\text{Shv}(C, J)}(\omega_{(C_0, i_0)}(a_J(\mathfrak{h}_C(i_0(Z)))), \omega_{(C_0, i_0)}(G)) \cong \text{Hom}_{\text{Presh}(C)}(\mathfrak{h}_C(i_0(Z)), G) \cong G(i_0(Z)) \cong \omega_{(C_0, i_0)}(G)_{K_Z} \cong \text{Hom}_{M_{(C_0, i_0)}}(M_{(C_0, i_0)}/K_Z, \omega_{(C_0, i_0)}(G)),$$

where (1) and (2) are isomorphisms given by Lemma 9.1.2 and Theorem 5.8.1 respectively. Hence by Yoneda's lemma, we have an isomorphism

$$\omega_{(C_0, i_0)}(\mathfrak{h}_C(i_0(Z))) \cong M_{(C_0, i_0)}/K_Z.$$

of smooth $M_{(C_0, i_0)}$-sets. From this we obtain an isomorphism

$$F^*(F_*(Y)) = \lim_{Z \in \text{Obj}_C} \text{Map}(M_{(C_0, i_0)}/K_Z, Y).$$
Lemma 9.2.3. Let $H$ be a sheaf on $(C, J)$ and let $Y$ be a set. Then we have a bijection
\[ \text{Map}(F^*(H), Y) \cong \text{Hom}_{\text{Shv}(C, J)}(H, F_*(Y)) \]
which is functorial in $H$ and $Y$.

Proof. It follows from Theorem 5.8.1 that $\text{Hom}_{\text{Shv}(C, J)}(H, F_*(Y))$ is isomorphic to $\text{Hom}_{M(C_{0, 0})}(F^*(H), \omega_{(0, 0)}(F_*(Y)))$. Let us consider the map
\[ j : \text{Map}(F^*(H), Y) \rightarrow \text{Map}(M(C_{0, 0}) \times F^*(H), Y) \]
given by the composition with the action $M(C_{0, 0}) \times F^*(H) \rightarrow F^*(H)$ of $M(C_{0, 0})$ on $F^*(H)$. Let us regard the target $\text{Map}(M(C_{0, 0}) \times F^*(H), Y)$ of this map as the set $\text{Map}(F^*(H), \text{Map}(M(C_{0, 0}), Y))$. Then the image of the map $j$ is contained in the subset $\text{Hom}_{M(C_{0, 0})}(F^*(H), \text{Map}(M(C_{0, 0}), Y))$ of $\text{Map}(F^*(H), \text{Map}(M(C_{0, 0}), Y))$. One can check easily that the map $\text{Map}(F^*(H), Y) \rightarrow \text{Hom}_{M(C_{0, 0})}(F^*(H), \text{Map}(M(C_{0, 0}), Y))$ induced by $j$ is bijective. We have seen in the last paragraph that $\omega_{(0, 0)}(F_*(Y))$ is equal to the smooth part of the set $\text{Map}(M(C_{0, 0}), Y)$. Since $F^*(H)$ is a smooth $M(C_{0, 0})$-set, we have
\[ \text{Hom}_{M(C_{0, 0})}(F^*(H), \text{Map}(M(C_{0, 0}), Y)) = \text{Hom}_{M(C_{0, 0})}(F^*(H), \omega_{(0, 0)}(F_*(Y))). \]
Thus we have bijections
\[ \text{Map}(F^*(H), Y) \cong \text{Hom}_{M(C_{0, 0})}(F^*(H), \omega_{(0, 0)}(F_*(Y))) \cong \text{Hom}_{\text{Shv}(C, J)}(H, F_*(Y)). \]

It is straightforward to check that these bijections are functorial with respect to $H$ and $Y$. □

9.3. Theorem 9.3.1. The pair $(F_*, F^*)$ of functors gives a point of the topos $\text{Shv}(C, J)$.

Proof. Lemma 9.2.3 implies that the functor $F^*$ is a right adjoint to the functor $F_*$. Thus the pair $(F_*, F^*)$ of functors gives a point of the topos $\text{Shv}(C, J)$. □

Corollary 9.3.2. The topos $\text{Shv}(C, J)$ has enough points.

Proof. Let $f : F_1 \rightarrow F_2$ be a morphism of sheaves. Using Theorem 5.8.1 we know that $f$ is an isomorphism if and only if $\omega_{(0, 0)}(f)$ is an isomorphism of smooth $M(C_{0, 0})$-sets. This is an isomorphism if it is an isomorphism of (the underlying) sets. This implies the claim. □

10. On locally profinite groups

Suppose we are given a $Y$-site and a grid. Then our absolute Galois monoid $M(C_{0, 0})$ came with the set of subgroups indexed by the edge objects in the grid. We show in Section 10.4 that $M(C_{0, 0})$ is naturally equipped with the structure of a topological monoid such that the category of smooth $M(C_{0, 0})$-sets is canonically equivalent to the category of discrete sets with continuous action of the topological monoid $M(C_{0, 0})$.

In Section 10.2 we give one of our main theorems. It states that the topos associated to a $Y$-site, whose topology is atomic, that satisfies the cardinality condition (1) is equivalent to the category of discrete sets with continuous action of some locally profinite group. This may be regarded as a reconstruction theorem for locally topological groups.
10.1. **The absolute Galois monoid as a topological monoid.** Let \((\mathcal{C}, J)\) be a \(Y\)-site. Suppose we are given a grid \((\mathcal{C}_0, t_0)\) for this \(Y\)-site. Let us equip the associated Galois monoid \(M_{(\mathcal{C}_0, t_0)}\) with the structure of a topological monoid as follows.

For each element \(m \in M_{(\mathcal{C}_0, t_0)}\), consider the set

\[ \mathfrak{U}_m = \{ mK_X | X \text{ an edge object} \} \]

of subsets of \(M_{(\mathcal{C}_0, t_0)}\). Let \(\mathfrak{U} = (\mathfrak{U}_m)_{m \in M_{(\mathcal{C}_0, t_0)}}\).

**Lemma 10.1.1.** The set \(\mathfrak{U}\) is a fundamental system of neighborhood for some topology on (the underlying set of) \(M_{(\mathcal{C}_0, t_0)}\).

*Proof.* We check below only some of the axioms for the set of subsets to be a fundamental system of neighborhoods. The rest is left to the reader.

Let \(m, m_1, m_2 \in M_{(\mathcal{C}_0, t_0)}\) and \(X_1, X_2\) be edge object. Let us show that if \(m \in m_1K_{X_1} \cap m_2K_{X_2}\) then there exists an edge object \(Y\) such that \(Y \subset m_1K_{X_1} \cap m_2K_{X_2}\). We can write \(m = m_1k_1 = m_2k_2\) for some \(k_1 \in K_1\) and \(k_2 \in K_2\). We want to show that there exists an edge object \(Y\) such that \(k_1K_Y \subset K_{X_1}\) and \(k_2K_Y \subset K_{X_2}\). It suffices to find \(Y\) such that \(K_Y \subset K_{X_1} \cap K_{X_2}\).

Using the \(\Lambda\)-connectedness of the grid, we see that for any \(X, X' \in \mathcal{C}_0\), there exists an object \(Y \in \mathcal{C}_0\) such that there are morphisms \(Y \to X\) and \(Y \to X'\). Hence we have \(K_Y \subset K_X \cap K_{X'}\). By Lemma 5.7.3, \(Y\) is an edge object, hence the claim follows. \(\square\)

**Lemma 10.1.2.** The product map \(M_{(\mathcal{C}_0, t_0)} \times M_{(\mathcal{C}_0, t_0)} \to M_{(\mathcal{C}_0, t_0)}\) is continuous when \(M_{(\mathcal{C}_0, t_0)}\) is equipped with the topology as above.

*Proof.* We will prove the following claim: for any \(m = (\alpha, \gamma_\alpha) \in M_{(\mathcal{C}_0, t_0)}\) and any edge object \(X\), there exists an edge object \(Y\) such that \(K_Y m \subset mK_X\). From Lemma 8.1.5, it follows that the set \(mK_X\) equals

\[ \{(\beta, \gamma_\beta) | \beta(X) = \alpha(X), \gamma_\beta(X) = \gamma_\alpha(X)\} \]

From Lemma 5.7.3, it follows that there exists an edge object \(Y\) such that there is a morphism \(Y \to \alpha(X)\). Now take \((\delta, \gamma_\delta) \in K_Y\). Then from Lemma 8.1.3 it follows that \(\delta(\alpha(X)) = \alpha(X)\) and \(\gamma_\delta(\alpha(X)) = id_{\alpha(X)}\). Let \((\beta, \gamma_\beta) = (\delta, \gamma_\delta) \circ m\). Then we have \(\beta(X) = \alpha(X)\) and \(\gamma_\beta(X) = \gamma_\alpha(X)\). Hence \((\beta, \gamma_\beta) \in mK_X\) and the claim follows. \(\square\)

**Corollary 10.1.3.** The absolute Galois monoid is a topological monoid for the topology constructed as above.

*Proof.* Immediate from the previous lemma. \(\square\)

**Remark 10.1.4.** The category of smooth \(M_{(\mathcal{C}_0, t_0)}\)-sets defined in Section 5.7.7 is canonically equivalent to the category of discrete sets with continuous action of the topological monoid \(M_{(\mathcal{C}_0, t_0)}\). This follows from the definitions. See also [MM, p.151].

10.2. **Locally profinite groups.** As an application of our main theorem, we obtain a ‘reconstruction’ theorem as follows.

**Theorem 10.2.1.** Let \((\mathcal{C}, J)\) be a \(Y\)-site. Suppose that the topology is atomic and suppose that Condition (1) of the cardinality conditions holds true. Then there
exists a locally profinite group \( G \) such that the topos \( \text{Shv}(\mathcal{C}, J) \) is equivalent to the category of discrete sets with continuous action of \( G \).

If moreover there exists a final object in \( \mathcal{C} \), then the locally profinite group is profinite.

**Proof.** Since the cardinality Condition (1) holds true, by Proposition 6.2.1, there exists a grid \((\mathcal{C}_0, \iota_0)\) for this \( \mathcal{Y} \)-site. From Theorem 5.8.1, it follows that the topos is equivalent to the category of discrete sets with continuous action of \( G \).

We can use the procedure in Section 10.1 to equip \( M_{(\mathcal{C}_0, \iota_0)} \) with a structure of a topological group. We noted in Remark 10.1.4 that the category of smooth \( M_{(\mathcal{C}_0, \iota_0)} \)-sets is canonically equivalent to the category of discrete sets with continuous action of \( M_{(\mathcal{C}_0, \iota_0)} \) for this structure of topological group. The claim then follows from the following Lemma 10.2.2.

**Lemma 10.2.2.** Let the setup be as above. The absolute Galois monoid \( M_{(\mathcal{C}_0, \iota_0)} \), when equipped with the structure of a topological group as in Section 10.1 is locally profinite. If moreover there exists a final object in \( \mathcal{C} \), then \( M_{(\mathcal{C}_0, \iota_0)} \) is profinite.

**Proof.** Let \( X \in \mathcal{C}_0 \). Note that under the cardinality condition, \( H_X \) is by definition a profinite group. Let us equip \( \mathbb{K}_X \) with the structure of a profinite group via the isomorphism \( \psi_X : H_X \to \mathbb{K}_X \) (see Section 7.6.2). To prove the proposition, it suffices to show that the inclusion \( \mathbb{K}_X \subset M_{(\mathcal{C}_0, \iota_0)} \) is a continuous and open map of topological groups.

First, for any morphism \( Y \to X \), one can check that the induced inclusion \( \mathbb{K}_X \to \mathbb{K}_Y \) is a continuous open map. Second, given an open subgroup \( \mathbb{K}' \subset \mathbb{K}_X \), one can find a morphism \( Y \to X \) such that \( \mathbb{K}_Y \subset \mathbb{K}' \). These two statements can be used to prove the claim.

Now suppose that there exists a final object. Since \( \iota_0 \) is essentially surjective, there is an object \( X \in \mathcal{C}_0 \) that is sent to a final object in \( \mathcal{C} \). Let \( (\alpha, \gamma) \in M_{(\mathcal{C}_0, \iota_0)} \). Then since \( \mathcal{C}_0 \) is thin, we have \( \alpha(X) = X \) and \( \gamma_\alpha(X) = \text{id}_{X} \). This means that the inclusion \( \mathbb{K}_X \subset M_{(\mathcal{C}_0, \iota_0)} \) is an equality. We saw above that the inclusion is a homeomorphism onto its image. As \( \mathbb{K}_X \) is profinite, the claim follows.

### 11. \( \mathcal{Y} \)-sites and grids for locally prodiscrete groups

Suppose we are given a locally prodiscrete group \( G \), which is complete and separated. Then the main aim here is to construct a \( \mathcal{Y} \)-site and a grid such that the absolute Galois monoid is isomorphic to \( G \).

In [MM] p.150, Section 9, given a topological group \( G \), MacLane and Moerdijk construct a site such that the associated topos is equivalent to the category of discrete sets with continuous \( G \)-action. We use essentially the same site and view their fiber functor as a guide to the construction of our grid. Since the equivalence is already proved in [MM], the emphasis of this section is on the computation of the absolute Galois monoid.

The problem of the construction of a \( \mathcal{Y} \)-site and a grid giving rise to a given topological monoid seems difficult.

#### 11.1. The construction of a \( \mathcal{Y} \)-site and a grid.
11.1.1. A certain class of topological groups. We consider the following class of topological groups, which is more general than the class of locally prodiscrete groups.

Let $G$ be a topological group. Consider the set of open subgroups $\mathcal{V} = \{H \subset G\}$ where $H$ satisfies the following property: For any open subgroup $U \subset G$, there exists an open subgroup $K$ such that $K \subset U \cap H$ and $K$ is a normal subgroup of $H$. We consider those topological groups such that the set $\mathcal{V}$ is non-empty.

11.1.2. We can construct a $Y$-site and a grid, starting from a topological group as in Section 11.1.1. Let us construct the site. Let $G$ be a topological group as in Section 11.1.1. Let $\mathcal{C}$ be the category of discrete left $G$-sets consisting of a single $G$-orbit, isomorphic to the $G$-set of the form $G/H$ for some $H \in \mathcal{V}$. Then the category $\mathcal{C}$ is semi-cofiltered and equipped with the atomic topology $J$ is a $Y$-site.

We can construct a grid of the $Y$-site in the following manner. Let $\mathcal{P}_G$ denote the set of open subgroups which belongs to $\mathcal{V}$. We regard $\mathcal{P}_G$ as a partially ordered set with respect to the inclusions. We denote by $\mathcal{C}_0$ the poset (viewed as a category) $\mathcal{C}_{\mathcal{P}_G}$ associated with the partially ordered set $\mathcal{P}_G$. The group $G$ acts from the left on the set $\mathcal{P}_G$ by the conjugation, i.e., $g \cdot K := g K g^{-1}$. By associating $G/K$ to each element $K$ of $\mathcal{P}_G$, we obtain a functor $\iota_0 : \mathcal{C}_0 \to \mathcal{C}$. Then the pair $(\mathcal{C}_0, \iota_0)$ is a grid for the $Y$-site $(\mathcal{C}, J)$.

11.1.3. Given a topological group $G$ as in Section 11.1.1, we construct a locally prodiscrete group $\hat{G}$ as follows.

First, we give the definition of a locally prodiscrete group.

**Definition 11.1.1.** By a prodiscrete group, we mean a topological group which is a filtered limit of discrete groups in the category of topological groups. A locally prodiscrete group is a topological group such that there exists an open subgroup which is a prodiscrete group.

We set $\hat{G} = \lim_{H \in \mathcal{V}} G/H$ to be the limit of discrete sets $G/H$ in the category of topological spaces. We can equip $\hat{G}$ with the structure of a topological group as follows. For two elements $g_1 = (g_{1,H})_{H \in \mathcal{V}}, g_2 = (g_{2,H})_{H \in \mathcal{V}} \in \hat{G}$, we define the product $g_1 g_2 \in \hat{G}$ as follows. We set the $H$-component of $g_1 g_2$ to be $\tilde{g}_{1,H} g_{2,H} H \in G/H$. Here, first take a lift $\tilde{g}_{1,H} \in G$ of the element $g_{1,H} \in G/H$, set $g' = \tilde{g}_{2,H} \tilde{g}_{1,H}^{-1}$, and take a lift $\tilde{g}_{1,H'} \in G$ of $g_{1,H'} \in G/H'$. The resulting element does not depend on the choices of the lifts.

Let us construct the inverse $s = (s_H)_{H \in \mathcal{V}}$ of $g = (g_H)_{H \in \mathcal{V}}$ as follows. For $H \in \mathcal{V}$, we take a lift $\tilde{g}_H \in G$ of $g_H \in G/H$. Take $K \in \mathcal{V}$. By the definition of $\mathcal{V}$ (Section 11.1.1), there exists a normal subgroup $K'$ of $K$ such that $K' \subset \tilde{g}_K^{-1} K \tilde{g}_K \cap K$. Then we set $s_K = \tilde{g}_K^{-1} K \in G/K$.

The topological group $\hat{G}$ is locally prodiscrete. Take an open subgroup $H \subset G$ which belongs to $\mathcal{V}$. Consider its image, which is open, in $\hat{G}$. Since $H$ belongs to $\mathcal{V}$, the image is a prodiscrete group. This in turn shows also that $\hat{G}$ is a topological group.

There is a canonical morphism of topological groups $G \to \hat{G}$. We note that this morphism induces an equivalence of categories from the category of discrete $\hat{G}$-sets to the category of discrete $G$-sets.
Lemma 11.1.2. When $G$ is separated and complete, the morphism $G \to \hat{G}$ is an isomorphism.

Proof. Separatedness implies $\bigcap_{H \in G} H = \{1\}$. This implies that the map is injective. We use completeness to obtain surjectivity.

In particular, if $G$ is locally profinite, then $G \to \hat{G}$ is an isomorphism.

11.1.4. We compute the absolute Galois monoid for the grid above.

Lemma 11.1.3. The absolute Galois monoid $M_{(C_0, \iota_0)}$ associated to the grid constructed above is isomorphic to $\hat{G}$.

Proof. We construct an isomorphism $\hat{G} \to M_{(C_0, \iota_0)}$ as follows. Let $g = (g_H)_{H \in \mathcal{W}} \in \hat{G}$. For an object $H$ of $P_G$, we set $\alpha_g H = \tilde{g}_H H \tilde{g}_H^{-1}$ for some lift $\tilde{g}_H$ of $g_H \in G/H$. We have $\iota_0(H) = G/H$ and $\iota_0(\alpha_g(H)) = G/(gHg^{-1})$. We can construct a map $\iota_0(H) \to \iota_0(\alpha_g(H))$ by sending the coset $hH$ to the coset $hg^{-1} \cdot gHg^{-1}$. Then one can check that these form a natural isomorphism $\gamma_{\alpha_g}$. Thus we have a map that sends $g \in \hat{G}$ to $\alpha_g \in M_{(C_0, \iota_0)}$. The proof that this map is an isomorphism is left to the readers.

As a corollary, we obtain the following.

Corollary 11.1.4. Let $G$ be a locally prodiscrete group which is separated and complete. Then there exist a $Y$-site and a grid such that the associated Galois monoid $M_{(C_0, \iota_0)}$ is isomorphic to $G$.

Proof. This follows from Lemmas 11.1.2 and 11.1.3.

11.2. For the associated fiber functor and the topos, we have the following claim.

Proposition 11.2.1. Let $G$ be a locally prodiscrete group. Then there exists a $Y$-site and a grid such that the fiber functor associated to the grid induces an equivalence of the sheaves and the category of discrete sets with continuous $G$-action.

Proof. This proposition essentially follows from [MM, p.154, Theorem 2]. The site constructed above is essentially that of loc. cit. and one can also check that the fiber functor is essentially the functor considered there. Their theorem says that the functor induces an equivalence. Using that the category of discrete $G$-sets and $\hat{G}$-sets are equivalent, we obtain the claim.

12. Examples

We give two examples. The examples in Section 12.1 are the simplest examples where the Galois groups are the abelian group of integers and the monoid of natural numbers. The example in Section 12.2 served as the motivation to write this paper. In a future paper, we will consider a sheaf on the site with values in the category of simplicial schemes.
12.1. The simplest examples. Let $\mathcal{C}$ be the following category. The objects of $\mathcal{C}$ are the sets $[0]$, $[1]$, $[2]$, ... where $[n]$ denote the set $\{0, 1, \ldots, n-1\}$ for $n \geq 0$. For two integers $m, n \geq 0$, the morphisms from $[m]$ to $[n]$ in $\mathcal{C}$ are the maps $f : [n] \to [m]$ satisfying $f(i+1) = f(i) + 1$ for $i = 0, \ldots, n$. (This is not a typo. The morphisms go in the “opposite” direction.) The category $\mathcal{C}$ is $\mathfrak{U}$-small, $A$-connected, semi-cofiltered, and is an $E$-category. One can check that any morphism in $\mathcal{C}$ is a monomorphism. Hence any morphism in $\mathcal{C}$ is a Galois covering whose Galois group is isomorphic to $\{1\}$.

Let $\mathcal{T} = \text{Mor}(\mathcal{C})$ and let $\mathcal{T}_+$ denote the set of morphisms $f$ in $\mathcal{C}$ satisfying $f(0) = 0$. Then both $\mathcal{T}$ and $\mathcal{T}_+$ are semi-localizing collections of morphisms in $\mathcal{C}$. The pairs $(\mathcal{C}, J_T)$ and $(\mathcal{C}, J_{T_+})$ are $B$-sites which have enough Galois coverings. We note that $\mathcal{T} = \mathcal{T}(J_T)$ and $\mathcal{T}_+ = \mathcal{T}(J_{T_+})$. In particular the notation $\mathcal{C}(\mathcal{T}_+)$ makes sense. Since the set $\text{Hom}_\mathcal{C}([m], [n])$ is a finite set for any $m, n \geq 0$, it follows from Lemma 6.2.2 that both $(\mathcal{C}, J_T)$ and $(\mathcal{C}, J_{T_+})$ admit grids. We can explicitly construct grids as follows.

Let $\mathcal{C}_0$ be the following category: the objects are the finite sets $S$ of the form $S = \{a, a+1, \ldots, b\}$ for some integers $a, b \in \mathbb{Z}$ with $a \leq b$. The morphisms in $\mathcal{C}_0$ are the opposite of the inclusions, i.e., the category $\mathcal{C}_0$ is thin and for any two objects $S_1, S_2$ of $\mathcal{C}$, there exists a morphism from $S_1$ to $S_2$ in $\mathcal{C}$ if and only if $S_1 \supset S_2$. Let $\mathcal{C}'_{+,0}$ denote the full subcategory of $\mathcal{C}_0$ whose objects are $[0]$, $[1]$, $[2]$, .... Let $\iota : \mathcal{C}_0 \to \mathcal{C}$ denote the functor which sends $\{a, a+1, \ldots, b\}$ to $[b-a]$. Then one can check easily that the pair $(\mathcal{C}_0, \iota)$ is a grid for $\mathcal{C}$. The pair $(\mathcal{C}'_{+,0}, \iota|_{\mathcal{C}'_{+,0}})$ is a pregrid for $(\mathcal{C}, J_{T_+})$. Let $\mathcal{C}_{+,0}$ denote the full subcategory of $\mathcal{C}_0$ whose objects are the sets of the form $\{a, a+1, \ldots, b\}$ for some $a, b \in \mathbb{Z}$ with $0 \leq a \leq b$. Then the pair $(\mathcal{C}_{+,0}, \iota|_{\mathcal{C}_{+,0}})$ is a grid of $(\mathcal{C}, J_{T_+})$.

Let $\alpha : \mathcal{C}_0 \to \mathcal{C}$ denote the isomorphism of categories which sends $\{a, a+1, \ldots, b\}$ to $\{a+1, a+2, \ldots, b+1\}$. The isomorphism $\alpha$ induces the functor $\mathcal{C}_{+,0} \to \mathcal{C}_{+,0}$ which we denote by $\alpha_\ast$. We have $\iota = \iota \circ \alpha$ and $\iota|_{\mathcal{C}_{+,0}} \circ \alpha_\ast = \iota|_{\mathcal{C}_{+,0}}$. Hence the pairs $(\alpha, \iota)$ and $(\alpha_\ast, \iota)$ are elements of the monoids $M_{(\mathcal{C}_0, \iota)}$ and $M_{(\mathcal{C}_{+,0}, \iota|_{\mathcal{C}_{+,0}})}$, respectively. We then have isomorphisms $\mathbb{Z} \cong M_{(\mathcal{C}_0, \iota)}$ and $\mathbb{Z}_{\geq 0} \cong M_{(\mathcal{C}_{+,0}, \iota|_{\mathcal{C}_{+,0}})}$ which sends 1 to $(\alpha, \iota)$ and $(\alpha_\ast, \iota)$, respectively.

12.1.1. We give another example which is essentially the same as the one above. Below is the least ad hoc, ‘coordinate-free’ version. Note that the underlying category is essentially $\mathfrak{U}$-small and not $\mathfrak{U}$-small in general, while in the previous example it was $\mathfrak{U}$-small.

Let $\mathcal{C}$ denote the following category. The objects are finite well-ordered sets. The set of morphisms is the set of maps of sets that sends the successor (if it exists) to the successor. We set $\mathcal{T} = \text{Mor}(\mathcal{C})$ and $\mathcal{T}_+$ to be those morphisms that sends the least element to the least element.

We regard the totally ordered sets $[n]$ objects of $\mathcal{C}$ in a natural manner. Note that each object of $\mathcal{C}$ is isomorphic to the well-ordered set $[n]$ for some $n$. The pairs $(\mathcal{C}_0, \iota_0)$ and $(\mathcal{C}'_{+,0}, \iota|_{\mathcal{C}'_{+,0}})$ defined above make sense in this setup and form the grids for $(\mathcal{C}, J_T)$ and $(\mathcal{C}, J_{T_+})$ respectively. The absolute Galois monoids are hence $\mathbb{Z}$ and $\mathbb{Z}_{\geq 0}$ respectively as above.

12.2. Our starting example. The following example is the starting point of this project. The Galois group is the finite adele valued points of the general linear group. We will come back to this in a future paper.
Let \( d \geq 1 \) be an integer. We define the category \( \mathcal{C}^d \) as follows. An object in \( \mathcal{C}^d \) is a finite abelian group which is generated by at most \( d \) elements. For two objects \( N \) and \( N' \) in \( \mathcal{C}^d \), the set \( \text{Hom}_{\mathcal{C}^d}(N, N') \) of morphisms from \( N \) to \( N' \) is the set of isomorphism classes of diagrams

\[
\xymatrix{ N' & N'' & N' \ar[l] \ar[r] & N \\
\ | & \ar[d]_{\cong} & | \\
N' & N''' \ar[r] & N' 
\}
\]

in the category of abelian groups where the left arrow is surjective and the right arrow is injective. Here two diagrams \( N' \xrightarrow{\sim} N'' \xleftarrow{\sim} N' \) and \( N' \xrightarrow{\sim} N''' \xleftarrow{\sim} N' \) are considered to be isomorphic if there exists an isomorphism \( N'' \xrightarrow{\sim} N''' \) of abelian groups such that the diagram

\[
\xymatrix{ N' & N'' & N' \ar[l] \ar[r] & N \\
\ | & \ar[d]_{\cong} & | \\
N' & N''' \ar[r] & N' 
\}
\]

is commutative. The composition of two morphisms \( N' \xrightarrow{g} M \xleftarrow{f} N \) and \( N'' \xrightarrow{g'} M' \xleftarrow{f'} N' \) is seen in the following diagram:

\[
\xymatrix{ N \ar[d] & \\
N' \ar[d] & M \ar[l] \ar[d] & M' \ar[l] \\
N'' & M' \ar[l] \ar[d] & M \ar[l] \\
\}
\]

where the square means that the square is cartesian. This definition of morphisms is due to Quillen (1973) except that here we take morphisms in the opposite direction.

Let \( \mathcal{T} = \text{Mor}(\mathcal{C}^d) \) and let \( \mathcal{T}_+ \) denote the set of morphisms in \( \mathcal{C} \) represented by diagrams \( N' \xrightarrow{\sim} N'' \xleftarrow{\sim} N \) of abelian groups with \( i \) bijective. Then both \( \mathcal{T} \) and \( \mathcal{T}_+ \) are semi-localizing collections of morphisms in \( \mathcal{C}^d \). In a future paper, we shall show that the pairs \( (\mathcal{C}, \mathcal{T}) \) and \( (\mathcal{C}, \mathcal{T}_+) \) are \( B \)-sites which have enough Galois coverings. We note that \( \mathcal{T} = \mathcal{T}(\mathcal{J}_T) \) and \( \mathcal{T}_+ = \mathcal{T}(\mathcal{J}_{T+}) \). In particular the notation \( \mathcal{C}^d(\mathcal{T}_+) \) makes sense. Since the set \( \text{Hom}_{\mathcal{C}^d}(M, N) \) is a finite set for any object \( M, N \) of \( \mathcal{C}^d \), it follows from Lemma 5.2.2 that both \( \mathcal{C}^d \) and \( \mathcal{C}^d(\mathcal{T}_+) \) admit grids. We can explicitly construct grids for \( (\mathcal{C}^d, \mathcal{J}_T) \) and for \( (\mathcal{C}^d, \mathcal{J}_{T+}) \) as follows.

Let \( \mathbf{Lat}^d \) denote the set of \( \mathbb{Z} \)-submodules of \( \mathbb{Q}^{\oplus d} \) which are free of rank \( d \). We regard \( \mathbf{Lat}^d \) as a partially ordered with respect to the inclusions. We let \( \mathbf{Pair}^d \) denote the following poset. The elements of \( \mathbf{Pair}^d \) are the pairs \( (L_1, L_2) \) of elements in \( \mathbf{Lat}^d \) with \( L_1 \leq L_2 \). For two elements \( (L_1, L_2) \) and \( (L'_1, L'_2) \) in \( \mathbf{Pair}^d \), we have \( (L_1, L_2) \leq (L'_1, L'_2) \) if and only if \( L'_1 \leq L_1 \leq L_2 \leq L'_2 \). Let \( \mathbf{C}^d_0 \) denote the poset category corresponding to the order dual of \( \mathbf{Pair}^d \). Let \( \mathbf{C}^d_{+,0} \) denote the full subcategory of \( \mathbf{C}^d_0 \) whose objects are the pairs \( (L_1, L_2) \) with \( L_2 = \mathbb{Z}^{\oplus d} \). (We regard \( \mathbb{Z}^{\oplus d} \subset \mathbb{Q}^{\oplus d} \) as the standard lattice.)

Let \( \iota : \mathbf{C}^d_0 \rightarrow \mathbf{C}^d \) denote the functor which sends an object \( (L_1, L_2) \) of \( \mathbf{C}^d_0 \) to \( L_2/L_1 \) and which sends a morphism from \( (L_1, L_2) \) to \( (L'_1, L'_2) \) in \( \mathbf{C}^d_0 \) to a morphism in \( \mathbf{C}^d \) represented by the diagram \( L'_2/L_1 \xrightarrow{\iota} (L'_2/L_1) \xrightarrow{\iota} L_2/L_1 \). In our future paper, we shall show that the pair \( (\mathbf{C}^d_0, \iota) \) is a grid for \( (\mathbf{C}^d, \mathcal{J}_T) \) and the pair \( (\mathbf{C}^d_{+,0}, \iota|_{\mathbf{C}^d_{+,0}}) \) is a pregrid for \( (\mathbf{C}^d, \mathcal{J}_{T+}) \). Let \( \mathbf{C}^d_{+,0} \) denote the full subcategory of \( \mathbf{C}^d_0 \) whose objects are the pairs \( (L_1, L_2) \) satisfying \( L_2 \subset \mathbb{Z}^{\oplus d} \). Then the pair \( (\mathbf{C}^d_{+,0}, \iota|_{\mathbf{C}^d_{+,0}}) \) is a grid for \( (\mathbf{C}^d, \mathcal{J}_{T+}) \).
Let $\hat{\mathbb{Z}} = \varprojlim_{n \geq 1} \mathbb{Z}/n\mathbb{Z}$ be the profinite completion of $\mathbb{Z}$ and let $\mathbb{A}^\infty = \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$ denote the ring of finite adeles over $\mathbb{Q}$. Let us consider the group $GL_d(\mathbb{A}^\infty)$. It is a locally compact, totally disconnected topological group. We set $\text{Mat}^- = \{g \in GL_d(\mathbb{A}^\infty) \mid g^{-1} \in \text{Mat}_d(\hat{\mathbb{Z}})\}$. We have inclusions $GL_d(\hat{\mathbb{Z}}) \subset \text{Mat}^- \subset GL_d(\mathbb{A}^\infty)$ of monoids. Let $M$ be $GL_d(\mathbb{A}^\infty)$ or $\text{Mat}^-$. We say that a left $M$-set $S$ is smooth if for any $s \in S$, the $GL_d(\hat{\mathbb{Z}})$-orbit of $s$ in $S$ is a finite set. If $M = GL_d(\mathbb{A}^\infty)$ and $S$ is a left $\mathbb{Z}[M]$-module, this coincides with the usual notion of smoothness given e.g. in [V, I, 4.1]. When $M = GL_d(\mathbb{A}^\infty)$ (resp. when $M = \text{Mat}^-$), we shall construct in a future paper an isomorphism $M \cong M_{(C, \iota)}$ (resp. $M \cong M(C^+, \iota)$) of monoids and show that this isomorphism induces a one-to-one correspondence between smooth $M$-modules and smooth $M_{(C, \iota)}$-sets (resp. smooth $M(C^+, \iota)$-sets). Therefore, Theorem [5.8.1] gives an equivalence from the category $\text{Shv}(C^d, J_T)$ (resp. $\text{Shv}(C^d, J_T^\circ)$) to the category of smooth left $M$-sets.

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