Open-Closed String Topology via Fat Graphs

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Abstract

Given a smooth closed manifold $M$ with a family $\{L_i\}$ of closed submanifolds, we consider the free loop space $LM$ and the spaces $PM(L_i,L_j)$ of open strings (paths $\gamma : [0, 1] \to M$ with $\gamma(0) \in L_i, \gamma(1) \in L_j$). We construct string topology operations resulting in an open-closed TQFT on the family $(h_*(LM), \{h_*(PM(L_i,L_j))\}_{i,j \in B})$ which extends the known string topology TQFT on $h_*(LM)$. Here, $h_*$ is a multiplicative generalized homology theory supporting orientations for $M$ and the $L_i$. To construct the operations, we introduce the notion of open-closed fat graph, generalizing fat graphs to the open-closed setting.

1 Introduction

The area of string topology began with a construction by M. Chas and D. Sullivan [CS] of previously undiscovered algebraic structure on the homology of the free loop space of a closed oriented manifold $M$. This is is the space $LM$ of all continuous maps from the circle to $M$. Among other results, Chas and Sullivan found that the homology of $LM$, with its grading suitably shifted, is a graded commutative algebra, much like the homology of an oriented manifold does by virtue of Poincaré duality.

This operation, the string loop product, can be understood by considering the space $\text{Map}(P, M)$ of maps from a pair-of-pants surface $P$ to $M$, where $P$ is regarded as a cobordism from a disjoint union of two circles to a single circle. $P$ is a model for the basic interaction in string theory, in which two strings merge to form a single string.

There is a diagram

$$LM \times LM \leftarrow \text{Map}(P, M) \rightarrow LM,$$

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where \( i \) and \( j \) are restriction maps to the incoming and outgoing boundary of \( P \).

The product is then constructed as the composition

\[
H_*(LM) \otimes H_*(LM) \xrightarrow{i^!} H_*(\text{Map}(P,M)) \xrightarrow{j_*} H_*(LM),
\]

where \( i^! \) is an umkehr map.

Since the spaces involved are infinite-dimensional, the existence of \( i^! \) is not immediate. However, it can be constructed by observing that \( P \) is homotopy equivalent to a “figure-eight” space \( \Gamma = S^1 \lor S^1 \), so that we may replace (1) by

\[
LM \times LM \xleftarrow{i} \text{Map}(\Gamma,M) \xrightarrow{j} LM,
\]

where the maps are finite codimension embeddings in an appropriate sense. Using essentially diagram (2), Chas and Sullivan constructed the operation using transversality of smooth chains in \( LM \). There have been other constructions since then, notably including the homotopy-theoretic approach by R. Cohen and J. Jones [CJ02] in which \( i^! \) is defined using a Pontrjagin-Thom collapse.

R. Cohen and V. Godin [CG04] showed later that the pair-of-pants \( P \) may be replaced by any oriented connected cobordism \( \Sigma \) between one-manifolds, provided that it has nonempty outgoing boundary. The result is a family of operations

\[
\mu_\Sigma : H_*(LM)^{\otimes p} \to H_*(LM)^{\otimes q}
\]

compatible with gluing of cobordisms. This is a form of topological quantum field theory (TQFT). The construction exploits the fact that any surface with boundary may be represented as a fat graph, which is a finite graph \( \Gamma \) endowed with extra data that determines a surface with boundary having \( \Gamma \) as a deformation retract.

The appearance of fat graphs reflects a result, due in its various forms to Harer [Har86], Penner [Pen87], and Strebel [Str84], that moduli spaces of punctured Riemann surfaces are homotopy equivalent to spaces of metric fat graphs.

### 1.1 Open-closed string topology

Open-closed string topology generalizes string topology by allowing the strings to be “open,” that is, paths in a manifold \( M \) which need not be loops. The endpoints of open strings are constrained to lie in certain distinguished submanifolds \( L_b \subseteq M \).

The basic spaces of open strings considered are then of the form \( PM(L_1,L_2) \), standing for the space of paths \( \gamma : [0,1] \to M \) such that \( \gamma(0) \in L_1 \) and \( \gamma(1) \in L_2 \). These submanifolds (or typically objects with more structure) are called D-branes in string theory, and we will shorten it to “branes.”

The idea of open string topology was introduced by Sullivan in [Sul04], using transversality of smooth chains. The prototype construction, in homotopy-theoretic terms, is as follows. Consider the diagram

\[
\begin{array}{ccc}
PM(L_1,L_2) \times PM(L_2,L_3) & \xrightarrow{i} & PM(L_1,L_2,L_3) \xrightarrow{j} PM(L_1,L_3) \\
\downarrow \text{ev}_1 \times \text{ev}_0 & & \downarrow \text{ev}_{1/2} \\
L_2 \times L_2 & \xrightarrow{\Delta} & L_2.
\end{array}
\]
Here, we define $PM(L_1, L_2, L_3) = \{ \gamma \in PM(L_1, L_3) : \gamma(1/2) \in L_2 \}$. The map $i$ takes a path in $PM(L_1, L_2, L_3)$ and splits it at the middle into two paths, $j$ is the inclusion, and the vertical maps are evaluation maps. We may define a composition operation

$$H_*(PM(L_1, L_2)) \otimes H_*(PM(L_2, L_3)) \to H_*(PM(L_1, L_3))$$

as $j_* \circ i^!$. When $M$ and $L_2$ are oriented, the umkehr map $i^!$ exists by the Pontrjagin-Thom collapse, because $i$ is a finite codimension inclusion in a suitable sense. See Section 6 for a general statement.

Our aim is to extend the string topology TQFT of [CG04] to the open-closed setting. Here the one-manifolds considered may have endpoints, which carry boundary conditions. Thus each endpoint is labeled by an element of a set $B$ indexing a family $\{ L_b \}_{b \in B}$ of branes. Cobordisms between closed one-manifolds are replaced accordingly by the natural cobordisms between one-manifolds with labeled boundary. We call the resulting structures open-closed TQFTs, or $B$-TQFTs to make explicit the dependence on $B$; see Definition 6. The main result is as follows.

**Theorem A** Let $M$ be a closed smooth manifold, and let $\{ L_b \}_{b \in B}$ be a family of smooth closed submanifolds. Suppose that $h_*$ is a multiplicative generalized homology theory whose coefficient ring $h_*(\ast)$ is a graded field, and suppose that $M$ and the $L_b$ are oriented with respect to $h_*$. Then, the family

$$(h_*LM, \{ h_*PM(L_a, L_b) \}_{a, b \in B})$$

supports a positive-boundary $B$-TQFT structure over the coefficient ring. This extends the known string topology TQFT on $h_*(LM)$. $\square$

The qualifier “positive boundary” means that $B$-TQFT operations only exist for those cobordisms having nonempty outgoing boundary on each connected component. This restriction is also present in the closed case.

The main tools for constructing the operations will be open-closed fat graphs (Definition 12). Open-closed fat graphs are a generalization of fat graphs. They have a well-defined notion of “string boundary,” which is a graph isomorphic to a one-manifold with $B$-labeled boundary. They also have a “fattening” operation, which yields an associated “open-closed surface.” These two properties generalize the corresponding properties of fat graphs.

The content of this paper is essentially part of the author’s PhD thesis [Ram05]. We have recently learned of work by E. Harrelson ([Harb], [Hara]) that overlaps with the present paper. We hope that this will be the subject of future collaboration.

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2 Open-closed surfaces and cobordisms

Assume given an arbitrary set $\mathcal{B}$ of “formal branes,” which we will eventually use to index the actual branes.

**Definition 1** A $\mathcal{B}$-labeled one-manifold $C$ is an oriented one-manifold with boundary together with a function $\beta : \partial C \to \mathcal{B}$ (the $\mathcal{B}$-labeling). An isomorphism between two $\mathcal{B}$-labeled one-manifolds is a diffeomorphism that preserves the orientation and the $\mathcal{B}$-labeling. Let $C^*$ stand for $C$ with the opposite orientation. Given $a, b \in \mathcal{B}$, let $I_{a,b}$ be a copy of the unit interval, oriented in the direction from 0 to 1, with $\beta(0) = a, \beta(1) = b$; note that $I_{a,b}^*$ is isomorphic to $I_{b,a}$.

An open-closed surface (with brane labels drawn from $\mathcal{B}$) is a smooth oriented surface with boundary $S$, together with distinguished embedded one-dimensional submanifolds with boundary $\partial^S S, \partial^f S \subseteq \partial S$ (the string boundary and free boundary, respectively) and a locally constant function $\partial^f S \to \mathcal{B}$ (the brane labeling) such that:

1. $\partial S = \partial^S S \cup \partial^f S$, and,
2. $\partial(\partial^S S) = \partial(\partial^f S) = \partial^S S \cap \partial^f S$.

The restriction of $\beta$ to $\partial(\partial^S S)$ makes $\partial^S S$ a $\mathcal{B}$-labeled one-manifold. □

**Example 1** Figure 1(a) shows an open-closed surface with genus two, five string boundary components (only one of which is closed), and seven free boundary components, drawn dotted, with brane labeling in $\mathcal{B} = \{1, 2, \ldots \}$.

**Definition 2** Given $\mathcal{B}$-labeled one-manifolds $C_-, C_+$, an open-closed cobordism from $C_-$ to $C_+$ is a $\mathcal{B}$-surface $S$ together with a decomposition $\partial^S S = \partial^- S \sqcup \partial^+ S$ and orientation-preserving isomorphisms $\imath_- : C_- \to \partial^- S, \imath_+: C_+^* \to \partial^+ S$ of $\mathcal{B}$-labeled one-manifolds. We call $\partial^- S, \partial^+ S$ the incoming and outgoing boundary, respectively. □
Example 2 Figure 1(b) shows an open-closed cobordism $S$, where the incoming boundary is at the bottom and the outgoing boundary at the top. Notice that a single topological boundary component may contain both incoming and outgoing string boundary subintervals.

3 Open-closed TQFT

Recall that an ordinary TQFT consists of a vector space $V$ together with a homomorphism $\mu_\Sigma : V^\otimes p \to V^\otimes q$ for every oriented cobordism $\Sigma$ from $\bigsqcup_p S^1$ to $\bigsqcup_q S^1$. The maps $\mu_\Sigma$ are required to be diffeomorphism-invariant and to satisfy natural compatibility conditions with respect to disjoint union and composition of cobordisms. This can be described succinctly using the language of PROPs: a TQFT is an algebra over $\text{Cob}$, which is a PROP of oriented cobordisms between circles (see, for example, Voronov [Vor]). We will introduce an analogous notion to describe TQFTs involving open and closed strings.

Definition 3 Define $M_B$ to be the free abelian monoid on the symbols $S^1$ and $I_{a,b}$ for $a, b \in B$. Define a $B$-PROP to be a symmetric strict monoidal category having $M_B$ as its monoid of objects.

This can be elaborated as follows. The symmetric monoidal axioms [ML98] imply that each object

$$x = nS^1 + \sum_{a,b \in B} m_{a,b}I_{a,b} \in M_B$$

of a $B$-PROP $\mathcal{C}$ carries an action $\Sigma_x : \text{Aut}_x(x)$, where the group

$$\Sigma_x = \Sigma_n \times \prod_{a,b \in B} \Sigma_{m_{a,b}}$$

is a permutation group associated to $x$. It follows that each set $\mathcal{C}(x, y)$ has an action by $\Sigma_x$ on the right and a commuting action by $\Sigma_y$ on the left. Moreover, for any $x, y, z \in \text{Ob}(\mathcal{C})$, the composition map $\circ : \mathcal{C}(x, y) \times \mathcal{C}(y, z) \to \mathcal{C}(x, z)$ is equivariant with respect to the $\Sigma_x$ and $\Sigma_z$ actions, and it satisfies $p \circ (\sigma q) = (p \sigma) \circ q$ for every $p \in \mathcal{C}(y, z), q \in \mathcal{C}(x, y), \sigma \in \Sigma_y$.

Definition 4 An algebra over a $B$-PROP $\mathcal{C}$ is defined to be a monoidal functor from $\mathcal{C}$ to the symmetric monoidal category of $R$-modules for some ring $R$.

In more detail, an algebra over $\mathcal{C}$ is specified by giving:

1. a $B$-family $\mathcal{V} = (V, \{V_{a,b}\}_{a,b \in B})$ of $R$-modules, and,

2. for every $x, y \in M_B$, a map $\mathcal{C}(x, y) \to \text{Hom}(\mathcal{V}(x), \mathcal{V}(y))$, where we define

$$\mathcal{V}(nS^1 + \sum_{a,b \in B} m_{a,b}I_{a,b}) := V^\otimes n \otimes \bigotimes_{a,b \in B} V_{a,b}^\otimes m_{a,b}.$$
These maps are required to satisfy the necessary functoriality and equivariance conditions.

**Definition 5** Given any $x = nS^1 + \sum_{a,b \in B} m_{a,b} I_{a,b} \in M_B$, let $|x|$ be the $B$-labeled one-manifold $(S^1)^{\text{lin}} \sqcup \bigsqcup_{a,b \in B} I_{a,b}$ (in particular, $|0| = \emptyset$).

Define the $B$-PROP $\text{Cob}_B$ (the cobordism $B$-PROP) by letting a morphism from $x$ to $y$ be an equivalence class of triples $(S, L_-, L_+)$, where:

1. $S$ is a $B$-cobordism with $\partial^- S \cong |x|$ and $\partial^+ S \cong |y|^*$, together with a choice of parametrization for each string boundary component by either $I$ or $S^1$; this parametrization is orientation-preserving on the incoming components and orientation-reversing on the outgoing components.

2. $L_-$ is a function $\pi_0(\partial^- S) \to \mathbb{N}$ which restricts to a bijection with $\{1, \ldots, m_{a,b}\}$ on incoming components of type $I_{a,b}$ and to a bijection with $\{1, \ldots, n\}$ on incoming components of type $S^1$. Similarly for $L_+: \pi_0(\partial^+ S) \to \mathbb{N}$.

3. Two such triples are considered equivalent if they are related by an isomorphism of $B$-cobordisms which preserves the boundary parametrizations and the orderings $L_\pm$ of boundary components.

The composition of $(S, L_-, L_+) \in \text{Cob}_B(x, y)$ and $(S', L'_-, L'_+) \in \text{Cob}_B(y, z)$ is given by $(S \cup_{|y|} S', L_-, L'_+) \in \text{Cob}_B(x, z)$, where $S \cup_{|y|} S'$ is the $B$-cobordism that results from identifying each outgoing component $c$ of $\partial^+ S$ to the unique incoming component $c' \subset \partial^- S'$ of the same type such that $L_+(c) = L_-(c')$. The identification is with respect to the boundary parametrizations which are part of the data. The monoidal structure is given by letting $(S, L_-, L_+) \otimes (S', L'_-, L'_+) := (S \sqcup S', L_-, L'_+)$, where each labeling $\tilde{L}_\pm$ is given by ordering the boundary components of $S'$ after those of $S$ in each $B$-labeling type. The group $\Sigma_y \times \Sigma_{y'}$ then acts on $(S, L_-, L_+) \in \text{Cob}_B(x, y)$ by letting $\Sigma_x$ permute the labeling $L_-$ and letting $\Sigma_y$ permute $L_+$. □

**Definition 6** An open-closed topological quantum field theory with branes $B$ (which we will abbreviate $B$-TQFT) is an algebra over the $B$-PROP $\text{Cob}_B$. □

This definition restricts to the usual definition of a TQFT when $B = \emptyset$.

The string topology operations do not yield a whole $B$-TQFT, since there are no operations associated to $B$-cobordisms to the empty one-manifold. The appropriate variant is as follows.

**Definition 7** Let $\text{Cob}_B^+$ be the subcategory of $\text{Cob}_B$ consisting of $B$-cobordisms in which every connected component has nonempty outgoing boundary. This category inherits a $B$-PROP structure from $\text{Cob}_B$. A positive boundary $B$-TQFT is an algebra over $\text{Cob}_B^+$. □

**Remark 1** We regard this definition chiefly as an ad-hoc device, and we do not claim that it is the most adequate definition of an open-closed field theory. For other treatments, we refer the reader to A. D. Lauda and H. Pfeiffer [LP] and to K. Costello [Cos]. □
4 Graphs

Fat graphs have been used extensively to study moduli spaces of punctured and bordered Riemann surfaces. A fat graph consists of a finite graph together with a cyclic ordering of the edges (more precisely, half-edges) incident at every given vertex. This extra structure specifies a canonical “fattening” of the graph having the form of an oriented surface with boundary with the graph as a deformation retract, in which the punctures correspond to certain cycles of oriented edges.

We will define the notion of open-closed fat graph, which is a fat graph with extra structure, which will induce an open-closed surface structure on the fattening. To fix notation, we recall some basic definitions.

**Definition 8** A graph $\Gamma = (V(\Gamma), H(\Gamma), s, r)$ consists of

- a set $V(\Gamma)$ of vertices,
- a set $H(\Gamma)$ of half-edges,
- a “source” map $s : H(\Gamma) \to V(\Gamma)$ taking a half-edge to the vertex that it attaches to, and,
- a “reversal” involution $r$ of $H(\Gamma)$, having no fixed points, understood to take a half-edge to the opposite half-edge of their common edge.

The edges $E(\Gamma)$ are the orbits of $r$. We introduce the notation $H(v) := s^{-1}(v)$ for the set of half-edges incident with a given vertex. The degree (or valence) of a vertex $v \in V(\Gamma)$ is defined to be the number $\#H(v)$.

Define $t$ as the composition $s \circ r$, which is the “target” map taking a half-edge to its destination vertex (the source of its reversal).

Say that a graph is discrete if it has no edges; any set (in particular, $B$) can then be regarded as a discrete graph, and we will do so without mention.

Given two graphs $\Gamma_1, \Gamma_2$, define a morphism $\varphi : \Gamma_1 \to \Gamma_2$ of graphs to be a pair $(\varphi_V, \varphi_H)$ of functions $\varphi_V : V(\Gamma_1) \to V(\Gamma_2), \varphi_H : H(\Gamma_1) \to H(\Gamma_2) \sqcup V(\Gamma_2)$, such that

1. $\varphi_V(s(e)) = s(\varphi_H(e))$ for all $e \in H(\Gamma_1)$, and,
2. $\varphi_H(r(e)) = r(\varphi_H(e))$ for all $e \in H(\Gamma_1)$,

where we extend the structure maps $s$ and $r$ to $V(\Gamma_2)$ as the identity. When unambiguous, we will refer to both maps $\varphi_V, \varphi_H$ simply by $\varphi$. Define $\mathcal{G}raph$ to be the category of finite graphs, with these morphisms.

Define a subgraph of a graph $\Gamma$ to be a graph $\Gamma'$ with $H(\Gamma') \subseteq H(\Gamma), V(\Gamma') \subseteq V(\Gamma)$ and for which the edge reversal and source maps are given by restriction from those of $\Gamma$; we write $\Gamma' \subseteq \Gamma$.

All our graphs (possibly excluding $B$) will be finite.
Remark 2 A half-edge \( e \in H(\Gamma) \) can be equally regarded as an oriented edge, oriented (for definiteness) in the direction that points away from its source vertex \( s(e) \). We will use this point of view when convenient. \( \square \)

Given \( \Gamma' \subseteq \Gamma \), we would like a complement graph \( \Gamma \setminus \Gamma' \). This is obtained naively by removing from \( \Gamma \) all vertices and edges belonging to \( \Gamma' \). After this, though, every \( e \in H(\Gamma) \setminus H(\Gamma') \) attached to a vertex \( s(e) \in V(\Gamma') \) ends up with no source vertex. We repair the result by formally introducing a new vertex for each such \( e \). Precisely:

Definition 9 Given \( \Gamma' \subseteq \Gamma \), define the complement graph \( \Gamma \setminus \Gamma' \) as follows:

- **Vertices:** \( V(\Gamma \setminus \Gamma') := (V(\Gamma) \setminus V(\Gamma')) \cup \delta(\Gamma \setminus \Gamma') \), where \( \delta(\Gamma \setminus \Gamma') \) is the set of half-edges attached to a vertex of \( \Gamma' \) but not lying in \( \Gamma' \).

- **Half-edges:** \( H(\Gamma \setminus \Gamma') := H(\Gamma) \setminus H(\Gamma') \), with edge-reversal involution given by restriction from that of \( \Gamma \).

- **Incidence of half-edges:** Define \( s_{\Gamma \setminus \Gamma'}(e) := \begin{cases} e, & \text{if } e \in s_{\Gamma}^{-1}(V(\Gamma')) \\ s_{\Gamma}(e), & \text{otherwise.} \end{cases} \)

Remark 3 There is a natural map \( \Gamma \setminus \Gamma' \rightarrow \Gamma \) which is injective on edges but not necessarily on vertices. It is easy to see that there is a pushout square in \( \text{Graph} \). Topologically, \( \Gamma \setminus \Gamma' \) is the complement in \( \Gamma \) of an open neighborhood of \( \Gamma' \). \( \square \)

Any other graph-related notions we use (such as tree, geometric realization and connected components) will be assumed well-known.

4.1 Fat graphs

We will use the usual definition of fat graph, except for the restriction on vertices to be at least trivalent.

Definition 10 A fat graph is a finite graph \( \Gamma \) equipped with a cyclic ordering on each of the sets \( H(v), v \in V(\Gamma) \). We will encode this as a permutation \( \sigma \) of \( H(\Gamma) \) with disjoint cycle decomposition given by the \( H(v) \), and we will use the notation \( \hat{\sigma} \) for the composition \( \sigma \circ r : H(\Gamma) \rightarrow H(\Gamma) \), where \( r \) is edge reversal. The boundary cycles of \( \Gamma \) are the cycles of \( \hat{\sigma} \). \( \square \)

It will be useful to represent the boundary of \( \Gamma \) as an associated graph \( \partial \Gamma \) having a natural morphism \( \partial \Gamma \rightarrow \Gamma \).
Definition 11 Given a fat graph $\Gamma$, define $\partial \Gamma$ as follows. Let the vertices of $\partial \Gamma$ be $V(\partial \Gamma) := H(\Gamma)$. Let the half-edges be given by $H(\partial \Gamma) := H(\Gamma) \times \{0, 1\}$, with edge-reversal involution $r(e, i) := (e, 1 - i)$. Define the attachment of half-edges by the source map $s(e, 0) := e$, $s(e, 1) := \hat{\sigma}(e)$. Define a morphism $\iota : \partial \Gamma \to \Gamma$ by letting $\iota(e) := s(e)$ on vertices and $\iota(e, 0) := e$, $\iota(e, 1) := r(e)$ on half-edges.

It is immediate that the boundary cycles of $\Gamma$ are in bijection with the connected components of $\partial \Gamma$, each of which is a cyclic graph. Moreover, $\partial \Gamma$ has a natural orientation induced by $\hat{\sigma}$.

4.2 Open-closed fat graphs

Now we extend the notion of fat graph to include free boundary with labels in $B$.

Definition 12 An open-closed fat graph (with brane labels drawn from $B$) is a fat graph $\Gamma$ together with:

1. a distinguished free boundary subgraph $\partial_f \Gamma \subseteq \partial \Gamma$ such that the restriction of $\iota : \partial \Gamma \to \Gamma$ is an embedding $\iota_f : \partial_f \Gamma \hookrightarrow \Gamma$, and,

2. a labeling $\beta : \partial_f \Gamma \to B$ assigning an element of $B$ to each connected component of $\partial_f \Gamma$.

Given an open-closed fat graph, we define its string boundary $\partial_s \Gamma$ as $\partial \Gamma \setminus \partial_f \Gamma$ (Definition 9), and we let $\iota_s : \partial_s \Gamma \to \Gamma$ be the restriction of $\iota$.

Each of $\partial_f \Gamma$ and $\partial_s \Gamma$ is necessarily a disjoint union of linear and cyclic graphs. In the case of $\partial_f \Gamma$, the definition includes the possibility of components that are isolated vertices. Moreover, the two graphs $\partial_f \Gamma$ and $\partial_s \Gamma$ intersect by definition only on the set $\delta \partial_s \Gamma$ of vertices which are endpoints of linear components of $\partial_s \Gamma$, and the restriction of $\beta$ then makes $|\partial_s \Gamma|$ a $B$-labeled one-manifold.

Example 3 Consider the fat graph

\[
\begin{array}{c}
\Gamma = \\
\end{array}
\]

with the usual convention that the cyclic ordering at each vertex is counterclockwise. Its boundary graph is
We can specify an open-closed structure on $\Gamma$ by choosing a subgraph $\partial \Gamma$ of $\partial \Gamma$ with a $B$-labeling of its components. This can be represented as follows:

where the dotted lines and the hollow vertex are $\partial \Gamma$ and the numbers indicate the labeling in $B = \{1, 2, \ldots\}$. This gives $\Gamma$ the structure of a genus one open-closed fat graph, having string boundary of type $I_{3,3} \sqcup I_{1,1} \sqcup S^1 \sqcup S^1$.

Example 4 Another example of an open-closed fat graph, using the same conventions, is

With a boundary partitioning, this represents the “composition” operation from the introduction.

As is well-known, given two graphs $\Gamma_1, \Gamma_2$ with $\Gamma_1$ a fat graph, a morphism $\varphi : \Gamma_1 \to \Gamma_2$ induces a fat graph structure on $\Gamma_2$ if it is simple. A simple morphism is one that can be written as a composition of “edge collapses,” that is, morphisms whose effect on the $B$-graph, up to isomorphism, is to collapse a single non-loop edge down to a vertex, leaving the rest of the graph intact. More precisely:
Definition 13 Call a morphism in $\mathcal{Graph} \varphi: \Gamma_1 \to \Gamma_2$ simple if the inverse image subgraph of any vertex of $\Gamma_2$ is a tree and $\varphi_H$ is injective on $\varphi_H^{-1}(H(\Gamma_2))$. A morphism $\varphi: \Gamma_1 \to \Gamma_2$ of fat graphs is a simple morphism of graphs which takes the fat graph structure of $\Gamma_1$ to that of $\Gamma_2$.

Given open-closed fat graphs $\Gamma_1, \Gamma_2$, define a morphism from $\Gamma_1$ to $\Gamma_2$ to be a morphism $\varphi: \Gamma_1 \to \Gamma_2$ of the underlying fat graphs such that the induced morphism $\partial_1 \Gamma_1 \to \partial_2 \Gamma_2$ restricts to a labeling-preserving simple morphism $\partial_1 \Gamma_1 \to \partial_2 \Gamma_2$. Let $\mathcal{Fat}_B$ be the resulting category of open-closed fat graphs.

Definition 14 A partitioning of an open-closed fat graph is a decomposition of $\partial s \Gamma$ as a disjoint union of graphs $\partial^- \Gamma$ (incoming) and $\partial^+ \Gamma$ (outgoing). We denote by $\iota \pm$ the restriction to $\partial \pm \Gamma$ of the morphism $\iota: \partial \Gamma \to \Gamma$. 

5 $B$-labeled graphs and their mapping spaces

Now we will let the set $B$ index actual branes $\{L_b\}_{b \in B}$ in a manifold $M$. Then, the $B$-labels carried by an open-closed fat graph $\Gamma$ (and by its boundary graph) may be used to specify constraints on maps from $|\Gamma|$ to $M$. This can be generalized as follows.

Definition 15 A $B$-labeled graph (or $B$-graph for short) is a diagram

$$\Gamma \xleftarrow{\theta} B(\Gamma) \xrightarrow{\beta} B$$

of graphs in which $\theta$ is an embedding. The $B$-graphs form a category $\mathcal{Graph}_B$, in which a morphism $\varphi: \Gamma_1 \to \Gamma_2$ of $B$-graphs is defined to be a pair $\varphi: \Gamma_1 \to \Gamma_2, \varphi_B: B(\Gamma_1) \to B(\Gamma_2)$ of morphisms in $\mathcal{Graph}$ making the diagram

$$\begin{array}{ccc}
\Gamma_1 & \xleftarrow{\theta_1} & B(\Gamma_1) \\
\Gamma_2 & \xrightarrow{\theta_2} & B(\Gamma_2)
\end{array}$$

commute. □

This is another way of saying that a $B$-graph is a graph $\Gamma$ together with a distinguished family $B(\Gamma)$ of disjoint connected subgraphs, each labeled by an element of $B$. We put it in this way to clarify the morphisms. We have two examples of $B$-graphs.

Example 5 Given an open-closed fat graph $\Gamma$, and letting $B(\Gamma) = \partial_1 \Gamma$, $\Gamma$ is naturally a $B$-graph via the diagram $\Gamma \xleftarrow{\iota \partial_1} \partial_1 \Gamma \xrightarrow{\beta} B$. □

Example 6 The string boundary graph $\partial_0 \Gamma$ of an open-closed fat graph becomes a $B$-graph by letting $B(\partial_0 \Gamma)$ be the discrete subgraph $\delta \partial_0 \Gamma$ consisting of the endpoints of the linear components of $\partial_0 \Gamma$. There is a morphism $\iota_0: \partial_0 \Gamma \to \Gamma$ of $B$-graphs. □
Given a $B$-graph $\Gamma$ and a family of closed submanifolds $L_b$ of a given smooth closed manifold $M$, we may consider maps from $|\Gamma|$ to $M$ which respect the $B$-labeling.

**Definition 16 (mapping spaces of $B$-graphs)** We will call $M = (M, \{L_b\}_{b \in B})$ a $B$-brane system in $M$. Given a $B$-graph $\Gamma \xleftarrow{\theta} B(\Gamma) \xrightarrow{\beta} B$, define $[[\Gamma]]_M$ to be the space of continuous maps $f : |\Gamma| \to M$ such that for every connected component $C$ of $B(\Gamma)$, the composition $f \circ |\theta| : |C| \to M$ has image in $L_{\beta(C)} \subseteq M$.

For a fixed $M$, this is a contravariant functor from $\text{Graph}_B$ to topological spaces. We will denote its action on a morphism $\varphi$ by $[[\varphi]]_M$. We will write $[\varphi]$ instead of $[[\varphi]]_M$ when it is clear by context.

### 6 Generalized Pontrjagin-Thom collapse

The construction of the open-closed string topology operations will make use of a homology-level umkehr homomorphism for the restriction map $[[\Gamma]]_M \xrightarrow{[\cdot]} [[\partial-\Gamma]]_M$, where $\Gamma$ is a partitioned open-closed fat graph. As is the case in the closed case, this homomorphism will arise from a Pontrjagin-Thom collapse. Here, we gather without proof some properties of a generalized form of the Pontrjagin-Thom collapse. In the following section, we specialize to the case of the maps that we are interested in.

**Basic fact** Assume given a homotopy pullback square

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
p & & \downarrow q \\
P & \xrightarrow{g} & Q
\end{array}
\]

such that $P$ and $Q$ are smooth closed manifolds, and $g$ is a smooth map. There is a stable backwards map $f^! : \Sigma^\infty Y \to X^{TQ-TP}$, where by abuse of notation $X^{TQ-TP}$ stands for the Thom spectrum of the virtual bundle $p^*g^*TQ - p^*TP$ on $X$.

If $h_*$ is a generalized homology theory and the manifolds $P$ and $Q$ are $h_*$-oriented, we may apply the Thom isomorphism theorem $\tilde{h}_*(X^\xi) \xrightarrow{\simeq} h_{*-\dim \xi}(X)$ to obtain a homomorphism $h_*(Y) \to h_{*-d}(X)$, also denoted $f^!$, where $d = \dim Q - \dim P$.

#### 6.1 Sketch of the construction

This construction and its properties do not seem to be published in this generality; however, an upcoming article by R. Cohen and J. Klein [CK] will include full derivations. We include only a sketch of the construction.

First assume that the square is a pullback square with $g$ an embedding and $q$ a locally trivial fiber bundle. In that case, we may take a tubular neighborhood $U_g$
for \( g \) and pull it back to an open set \( U_f \subseteq Y \). Then, \( U_f \) will be a tubular neighborhood for \( f \), in the sense that the pair \((U_f, U_f \setminus f(X))\) is homeomorphic to the pair \((p^*\nu_f, p^*\nu_f \setminus X)\), where \( \nu_f \) is the normal bundle of \( Q \) and \( X \) includes in \( p^*\nu_f \) as the zero section. The desired map \( f^! \) then comes about as usual, by collapsing the complement of \( U_f \) to a point and mapping the rest homeomorphically.

If \( g \) is not an embedding, then we choose an embedding \( i \) of \( P \) into a high-dimensional Euclidean space \( \mathbb{R}^N \) and replace \( g \) and \( q \) respectively by \((g, i) : P \to Q \times \mathbb{R}^N \) and \( q \times id : Y \times \mathbb{R}^N \to Q \times \mathbb{R}^N \). Different choices of (sufficiently large) \( N \) yield target Thom spaces that differ by a suspension, and the result is a stable map into the Thom spectrum.

Finally, if \( q \) is not locally trivial, it may be replaced by a Serre fibration using the standard mapping path space construction (see, e.g., [May99]). This fibration has sufficient structure to allow a tubular neighborhood for \( g \) to be lifted to one for \( f \) using parallel transport along paths.

### 6.2 Naturality properties

The generalized Pontrjagin-Thom collapse satisfies the following two naturality properties.

**Proposition 1 (Functoriality)** Consider a diagram

\[
\begin{array}{ccc}
X & \overset{f_1}{\longrightarrow} & Y \overset{f_2}{\longrightarrow} Z \\
\downarrow{p_1} & & \downarrow{p_2} \\
P & \overset{g_1}{\longrightarrow} & Q \overset{g_2}{\longrightarrow} R,
\end{array}
\]

where the \( g_i \) are smooth maps between \( h^* \)-oriented closed manifolds and both squares are homotopy pullbacks.

Then, the umkehr homomorphisms satisfy \((f_2 \circ f_1)^! = f_1^! \circ f_2^!\). \( \square \)

**Proposition 2 (Compatibility with induced maps on homology)** Consider a commutative diagram

\[
\begin{array}{ccc}
P_1 \overset{g_1}{\longrightarrow} Q_1 & & \\
\downarrow{s_1} & & \downarrow{s_2} \\
X_1 & \overset{f_1}{\longrightarrow} & Y_1 \overset{f_2}{\longrightarrow} Z_1 \\
\downarrow{u_1} & & \downarrow{u_2} \\
X_2 & \overset{p_2}{\longrightarrow} & Y_2 \overset{p_1}{\longrightarrow} Z_2 \\
\end{array}
\]

such that each \( p_i \) is a homotopy pullback of \( g_i \) via \( q_i \), the \( P_i \) and \( Q_i \) are \( h^* \)-oriented manifolds, and the virtual bundles \( s^*(g_2^*TQ_2 - TP_2) \) and \( g_1^*TQ_1 - TP_1 \) are stably
equivalent. Then the diagram
\[
\begin{array}{ccc}
h_*(X_1) & \xrightarrow{f'_1} & h_*(X_2) \\
\downarrow u_* & & \downarrow v_* \\
h_*(Y_1) & \xrightarrow{f'_2} & h_*(Y_2)
\end{array}
\]
commutes. □

7 Umkehr maps induced by morphisms of $\mathcal{B}$-graphs

We need a good family of morphisms $\varphi : \Gamma_1 \to \Gamma_2$ of $\mathcal{B}$-graphs for which the induced map $[\varphi] : [\Gamma_2] \to [\Gamma_1]$ admits a homology umkehr map.

First, observe that pushout squares in $\mathcal{G}raph_{\mathcal{B}}$ become pullback squares of mapping spaces after applying $[-]_{\mathcal{M}}$. The proof is standard and we omit it.

Next, we identify a class of morphisms $\varphi : \Gamma_1 \to \Gamma_2$ in $\mathcal{G}raph_{\mathcal{B}}$ with the property that the induced map $[\varphi]_{\mathcal{M}}$ fibers naturally over a smooth map of manifolds. For this, we introduce the following construction.

**Definition 17** Given a $\mathcal{B}$-graph $\Gamma$ define its *vertex $\mathcal{B}$-graph* $V(\Gamma)$ as the $\mathcal{B}$-graph that results from removing all the edges of $\Gamma$, keeping only the vertices and their labels. Formally, if $\Gamma$ is given by a diagram $\Gamma \xleftarrow{\theta} B(\Gamma) \xrightarrow{\beta} \mathcal{B}$, define $V(\Gamma)$ by the diagram
\[
\begin{array}{ccc}
V(\Gamma) & \xleftarrow{\theta_{V(B(\Gamma))}} & V(B(\Gamma)) \\
& \xrightarrow{\beta_{V(B(\Gamma))}} & \mathcal{B}
\end{array}
\]
This defines a functor $\mathcal{G}raph_{\mathcal{B}} \to \mathcal{G}raph_{\mathcal{B}}$ having a natural inclusion $V(\Gamma) \hookrightarrow \Gamma$.

**Proposition 3** Let $\varphi : \Gamma_1 \to \Gamma_2$ be a morphism in $\mathcal{G}raph_{\mathcal{B}}$ such that

1. $\varphi_H$ is a bijection $H(\Gamma_1) \to H(\Gamma_2)$, and,
2. $\varphi$ carries unlabeled edges to unlabeled edges; that is, $\varphi$ takes edges not in the image of $B(\Gamma_1)$ to edges not in the image of $B(\Gamma_2)$.

Then, the diagram $\begin{array}{ccc}
\Gamma_1 & \xrightarrow{\varphi} & \Gamma_2 \\
\downarrow V(\Gamma_1) & & \downarrow V(\Gamma_2)
\end{array}$ is a pushout square in $\mathcal{G}raph_{\mathcal{B}}$. □

**Proof** The hypothesis says that, up to isomorphism compatible with $\varphi$, $\Gamma_2$ is obtained from $\Gamma_1$ by identifying together vertices with a common preimage and then adjoining (possibly labeled) isolated vertices corresponding to $V(\Gamma_2) \setminus \varphi(V(\Gamma_1))$. But this is equivalent to this square being a pushout. ■
Note that \([\mathcal{V}(\Gamma)]_M \cong \prod_{v \in V(\Gamma)} L_v\), where \(L_v\) is the submanifold of \(M\) corresponding to the unique label carried by the vertex \(v\), or \(M\) if \(v\) is unlabeled. In particular, \([\mathcal{V}(\Gamma)]_M\) is a smooth manifold. In addition, if \(\varphi : \Gamma_1 \to \Gamma_2\) is a morphism in \(\text{Graph}_B\), then \(\varphi\) induces a smooth map \([\mathcal{V}(\Gamma_2)]_M \to [\mathcal{V}(\Gamma_1)]_M\); in fact, this map is a cartesian product of coordinate projections of the form \(L \times L' \to L\) and diagonal inclusions of the forms \(L \hookrightarrow L^p \times M^q\) and \(M \hookrightarrow M^p\).

In view of this and Proposition 3, we can make the following definition.

**Definition 18** Let \(\varphi : \Gamma_1 \to \Gamma_2\) be a morphism of \(B\)-graphs satisfying the hypothesis of Proposition 3. Suppose given a \(B\)-brane system \(M = (M, \{L_b \subseteq M\}_{b \in B})\) which is \(h^*\)-oriented; that is, such that \(M\) and each of the \(L_b\) is oriented with respect to a multiplicative homology theory \(h^*\).

Define 
\[ [\varphi]^*_M : h^*([\Gamma_1]_M) \to h^*([\Gamma_2]_M) \]

as the umkehr homomorphism associated by the generalized Pontrjagin-Thom collapse to the pullback square

\[
\begin{array}{ccc}
[\Gamma_1]_M & \xleftarrow{[\varphi]^*_M} & [\Gamma_2]_M \\
\downarrow & & \downarrow \\
[\mathcal{V}(\Gamma_1)]_M & \xleftarrow{[\mathcal{V}(\Gamma_2)]_M} & [\mathcal{V}(\Gamma_2)]_M.
\end{array}
\]

This uses that the normal bundle of \(L\) in \(M\) is \(h^*_s\)-oriented if both \(TM\) and \(TL\) are. □

### 7.1 Enlarging the class of morphisms: the category \(\text{Graph}_B^I\)

Thus a morphism of \(B\)-graphs which is a bijection on edges and preserves unlabeled edges induces an umkehr map in a natural way. However, we will need umkehr homomorphisms for a larger class of \(B\)-graph morphisms:

**Definition 19** Let \(\text{Graph}_B^I\) be the subcategory of \(\text{Graph}_B\) consisting of morphisms \(\varphi : \Gamma_1 \to \Gamma_2\) of \(B\)-graphs such that:

1. \(\varphi_H\) induces an injection \(H(\Gamma_1) \to H(\Gamma_2)\) of half-edges, and,

2. \(\varphi\) carries unlabeled edges to unlabeled edges in the sense of Proposition 3. □

We will construct the desired homomorphisms by showing that morphisms in \(\text{Graph}_B^I\) can be naturally factored up to homotopy into morphisms satisfying the hypothesis of Proposition 3.

Let \(\varphi : \Gamma_1 \to \Gamma_2\) be a morphism in \(\text{Graph}_B^I\). Let
\[
\Xi_\varphi = \bigsqcup_{e \in E(\Gamma_2) \setminus \varphi(E(\Gamma_1))} \hat{e},
\]
where each \(\hat{e}\) is a \(B\)-graph consisting of two vertices joined by a single edge, which is labeled by the same label carried by \(e\) in \(\Gamma_2\), or unlabeled if \(e\) is unlabeled. Let
be the $B$-graph obtained from $\Xi_\varphi$ by collapsing each $\hat{e}$ to a vertex carrying the same label, if any. Then, $\varphi$ decomposes as

$$\Gamma_1 \xrightarrow{\varphi'} \Gamma_1 \sqcup \xi_\varphi \xrightarrow[\varphi'']{\Xi_\varphi} \Gamma_2,$$

where $\varphi'$ is the natural inclusion, $\varphi''$ extends the defining quotient map $\Xi_\varphi \to \xi_\varphi$ by the identity on $\Gamma_1$, and $\varphi'''$ is the morphism which extends $\varphi$ by taking $\hat{e}$ to $e$.

The morphisms $\varphi'$ and $\varphi'''$ are readily seen to satisfy the hypothesis of Proposition 3. Moreover, $\varphi''$ induces a homotopy equivalence $[\Gamma \sqcup \xi_\varphi]_M \to [\Gamma \sqcup \Xi_\varphi]_M$, and $[\varphi]_M$ is homotopic to $[\varphi']_M \circ [\varphi'']_M \circ [\varphi''']_M$.

Example 7 We illustrate this for simplicity when $B(\Gamma_1) = B(\Gamma_2) = \emptyset$. Consider the morphism $\varphi\colon \bullet \to \bullet\rightarrow \bullet\rightarrow \bullet$, which clearly lies in $\mathcal{G}raph_B$. In this case, the factorization takes the form

$$\varphi' \to \bullet \xrightarrow[\varphi'']{\bullet\rightarrow \bullet}
\xrightarrow[\varphi''']{\bullet\rightarrow \bullet} \bullet.$$

We may now make the following definition.

**Definition 20** Given a morphism $\varphi \in \mathcal{G}raph_B$ and an $h_*$-oriented $B$-brane system $\mathcal{M}$ we define the umkehr homomorphism

$$[\varphi]_M : h_*(\Gamma_1)_M \to h_*(\Gamma_2)_M$$

as the composition $[\varphi'']_M \circ [\varphi''']_M \circ [\varphi']_M$, where the homomorphisms $[\varphi']_M$ and $[\varphi''']_M$ are given by Definition 18.

**Remark 4** Since $\xi_\varphi$ is discrete, the map $[\varphi]_M$ is a projection $[\Gamma_1]_M \times N \to [\Gamma_1]_M$ with $N$ a closed $h_*$-oriented manifold. Its corresponding umkehr map is simply crossing with the fundamental class of $N$. In these terms, the map $[\varphi''']_M$ is the inclusion $[\Gamma_1]_M \times N \to [\Gamma_1]_M \times PN$, where $PN$ stands for the space of arbitrary continuous paths in $N$, with $N$ included as the constant paths.

Finally, we observe that this homomorphism behaves well under the appropriate notion of simple morphism for $B$-graphs, as well as under pushouts of $B$-graph embeddings.

**Definition 21** Say that a morphism $\varphi = (\varphi, \varphi_B) : \Gamma_1 \to \Gamma_2$ of $B$-graphs is simple if each of $\varphi : \Gamma_1 \to \Gamma_2$ and $\varphi_B : B(\Gamma_1) \to B(\Gamma_2)$ is a simple morphism of graphs.

A morphism of open-closed fat graphs is in particular a simple morphism of the underlying $B$-graphs, and it induces a simple morphism of the associated string boundary $B$-graphs.
Remark 5 A simple morphism of $B$-graphs is of course a composition of edge collapses. However, a non-loop edge may be collapsed only if it is “wholly monochromatic,” that is, if the subgraph consisting of the edge and its two endpoints is either disjoint from $B(\Gamma)$ or contained in $B(\Gamma)$.

Proposition 4 Let $\gamma_1 \xrightarrow{\alpha} \gamma_2$ be a commutative diagram in $Graph_B$. Suppose that $\alpha$ and $\beta$ lie in $Graph_B^l$, and that either:

1. the diagram is a pushout and $\alpha$ is an embedding, or,
2. the $\gamma_i$ are simple.

Then, the diagram

$$h_*(\Gamma_1 \mathcal{M}) \xrightarrow{[\alpha]^*} h_*(\Gamma_2 \mathcal{M})$$

commutes.

Proof Case 1: the diagram is a pushout and $\alpha$ is an embedding. In this case, the $\gamma_i$ induce a commutative diagram

$$\Gamma_1 \xrightarrow{\alpha'} \Gamma_1 \cup \xi \xleftarrow{\alpha''} \Gamma_1 \cup \Xi \xrightarrow{\alpha'''} \Gamma_2$$

where $\xi = \xi_\alpha \cong \xi_\beta$, $\Xi = \Xi_\alpha \cong \Xi_\beta$, and in which the leftmost and rightmost squares are pushouts in $Graph_B$. It follows that these squares become pullbacks upon applying $[\mathcal{V}(\_)]_\mathcal{M}$. Moreover, each of these pullbacks fibers, in the sense of Proposition 1, over the corresponding pullback square of manifolds obtained by applying $[\mathcal{V}(\_)]_\mathcal{M}$. The result follows because the hypothesis on normal bundles of Proposition 1 is easily verified to hold for the latter squares.

Case 2: $\gamma_1$ and $\gamma_2$ are simple. By induction we can assume that $\gamma_2$ collapses a single edge. If the collapsed edge is in the image of $\alpha$, the result reduces to case 1. Assume then that $\gamma_2$ collapses a single edge $e$ which is not in the image of $\alpha$. We have up to isomorphism that $\tilde{\Gamma}_2 = \Gamma_2/e$ and $\tilde{\Gamma}_1 = \Gamma_1$. Let us change notation for clarity, writing $\Xi$ and $\xi$ for $\Xi_\beta$, $\xi_\beta$ respectively. We clearly have $\Xi_\alpha \cong \Xi \cup \ast_e$ and $\xi_\alpha \cong \xi \sqcup \ast_e$, where $\ast_e$ stands for a one-vertex graph carrying the same label as $e$, if any. Also write $\gamma$ for $\gamma_2$; $\gamma_1$ becomes the identity in this case.
We have the diagram

\[
\begin{array}{c}
\Gamma_1 \xrightarrow{\alpha'} \Gamma_1 \sqcup \xi \sqcup \ast_e \xrightarrow{\alpha''} \Gamma_1 \sqcup \Xi \sqcup \hat{\ast} \xrightarrow{\alpha'''} \Gamma_2 \\
\downarrow \beta' \quad \downarrow \gamma \\
\Gamma_1 \sqcup \xi \xleftarrow{\beta''} \Gamma_1 \sqcup \Xi \xleftarrow{\beta'''} \Gamma_2/\ast
\end{array}
\]

and we are to show that \([\alpha''']^1 \circ [\alpha'']^* \circ [\alpha']^1 = [\gamma]^*_* \circ [\beta''']^1 \circ [\beta'']^* \circ [\beta']^1\). For this, complete the diagram as follows:

\[
\begin{array}{c}
\Gamma_1 \xrightarrow{\alpha'} \Gamma_1 \sqcup \xi \sqcup \ast_e \xrightarrow{\alpha''} \Gamma_1 \sqcup \Xi \sqcup \hat{\ast} \xrightarrow{\alpha'''} \Gamma_2 \\
\downarrow \beta' \quad \downarrow \gamma \\
\Gamma_1 \sqcup \xi \xleftarrow{\beta''} \Gamma_1 \sqcup \Xi \xleftarrow{\beta'''} \Gamma_2/\ast
\end{array}
\]

Here, \(\gamma'\) collapses \(\hat{\ast} \) to \(*_e\), \(\varphi\) extends \(\beta''\) by the identity on \(*_e\), \(\psi\) extends \(\beta'''\) by mapping \(*_e\) to the vertex to which \(e\) is collapsed, and \(\delta\) and \(\delta'\) are the obvious inclusions. Note that \(\delta, \delta'\) and \(\psi\) satisfy the hypothesis of Definition 18.

By Remark 5 above, \(e\) is wholly monochromatic; this ensures that \([\delta]^*\) and \([\delta']^*\) have the same normal bundle data (in the sense of Proposition 2) when \([-\rightarrow\] is applied (the stable normal bundle is a pullback of \(-TL\) for both, where \(L\) is either \(M\) or the brane submanifold corresponding to the label carried by \(e\)). This is also true for the pair \([\alpha''']^1, [\psi]^1\).

The equality of the two homomorphisms then follows by applying the naturality properties:

\[
[\gamma]_* \circ [\beta''']^1 \circ [\beta'']^* \circ [\beta']^1 = ([\gamma]_* \circ [\psi]^1) \circ ([\delta']^1 \circ [\beta'']^* \circ [\beta']^1) = ([\alpha''']^1 \circ [\gamma]_* \circ [\varphi]^1) \circ ([\delta']^1 \circ [\beta']^1) = ([\alpha''']^1 \circ [\gamma]_* \circ [\varphi]^1) \circ [\alpha']^1 = [\alpha''']^1 \circ [\alpha'']^* \circ [\alpha']^1,
\]

as desired.

\[\square\]

8 Definition of the operations

With the constructions of the previous section, the definition of the string topology operation associated to an open-closed fat graph is straightforward.
**Definition 22** Say that a partitioning of an open-closed fat graph $\Gamma$ is *admissible* if the morphism $\iota_- : \partial^- \Gamma \to \Gamma$ lies in $\mathcal{G}raph_{\mathcal{B}}$. If this is the case, we say that $\Gamma$ is *well-partitioned*.

Define a category $\mathcal{Kat}_{\mathcal{B}}$ as follows. An object of $\mathcal{Kat}_{\mathcal{B}}$ is a well-partitioned open-closed fat graph. The morphisms from $\Gamma_1$ to $\Gamma_2$ are the morphisms $\varphi : \Gamma_1 \to \Gamma_2$ of open-closed fat graphs which respect the partitioning, in the sense that the induced morphism $\partial_0 \Gamma_1 \to \partial_0 \Gamma_2$ takes $\partial^- \Gamma_1$ to $\partial^- \Gamma_2$ and $\partial^+ \Gamma_1$ to $\partial^+ \Gamma_2$.

**Definition 23** Let $\mathcal{M}$ be an $h_\ast$-oriented $\mathcal{B}$-brane system, and let $\Gamma$ be a well-partitioned open-closed fat graph. Define the homomorphism $\Gamma_\ast$ as

$$
\Gamma_\ast := [\iota_+]_\ast \circ [\iota_-]^1 : h_\ast(\partial^- \Gamma)_\mathcal{M} \to h_\ast(\partial^+ \Gamma)_\mathcal{M}.
$$

This homomorphism is the *open-closed string topology operation* corresponding to $\Gamma$.

The operations $\Gamma_\ast$ are invariant under morphisms of open-closed fat graphs:

**Proposition 5** If $\Gamma_1, \Gamma_2 \in \mathcal{Kat}_{\mathcal{B}}$ and there is a morphism $\varphi : \Gamma_1 \to \Gamma_2$, then $(\Gamma_2)_\ast = [\partial^+ \varphi]_\ast^{-1} \circ (\Gamma_1)_\ast \circ [\partial^- \varphi]_\ast$.

**Proof** In the commutative diagram

$$
\begin{array}{ccc}
\partial^+ \Gamma_1 & \xrightarrow{\partial^+ \varphi} & \partial^+ \Gamma_2 \\
\downarrow^{\iota_1^2} & & \downarrow^{\iota_2^2} \\
\partial^- \Gamma_1 & \xrightarrow{\partial^- \varphi} & \partial^- \Gamma_2
\end{array}
$$

of $\mathcal{B}$-graphs, the morphisms $\iota_1^1, \iota_2^1 \in \mathcal{G}raph_{\mathcal{B}}$ and the horizontal morphisms are simple. Then, with an application of Proposition 4, we have that

$$
(\Gamma_2)_\ast = [\iota_2^1]_\ast \circ [\iota_1^2]^1 = ([\partial^+ \varphi]_\ast^{-1} \circ [\iota_1^1]_\ast \circ [\varphi]_\ast) \circ ([\varphi]_\ast^{-1} \circ [\iota_1^1]_\ast \circ [\partial^- \varphi]_\ast) = ([\partial^+ \varphi]_\ast^{-1} \circ (\Gamma_1)_\ast \circ [\partial^- \varphi]_\ast),
$$

as desired.

---

**9 Gluing**

Now we describe the combinatorial counterpart to gluing of cobordisms.

Suppose given $\Gamma_1, \Gamma_2 \in \mathcal{Kat}_{\mathcal{B}}$ together with isomorphisms $\partial^+ \Gamma_1 \xrightarrow{\zeta_1} \Delta \xrightarrow{\gamma_1} \partial^- \Gamma_2$, where $\Delta$ is a $\mathcal{B}$-graph. Assume that $\gamma_2 \circ \gamma_1^{-1}$ is orientation-reversing. We may construct a $\mathcal{B}$-graph $\Gamma_1 \# \Gamma_2$ by identifying the outgoing boundary of $\Gamma_1$ with...
the incoming boundary of $\Gamma_2$ according to their common identification with $\Delta$. More precisely, we can define $\Gamma_1 \# \Gamma_2$ by the pushout diagram

\[
\begin{array}{c}
\Gamma_1 \# \Gamma_2 \\
\downarrow \alpha_1 \\
\Gamma_1
\end{array}
\begin{array}{c}
\Gamma_2 \\
\downarrow \alpha_2
\end{array}
\begin{array}{c}
\Delta,
\end{array}
\]

in $\mathcal{G}raph_B$, where $\alpha_1 = \iota_1 \circ \gamma_1$ and $\alpha_2 = \iota_2 \circ \gamma_2$.

While the pushout $\Gamma_1 \# \Gamma_2$ exists, it does not necessarily inherit an open-closed fat graph structure having the correct isomorphism type. For that, we need an extra condition on the partitioning:

**Definition 24** Say that a partitioning of an open-closed fat graph $\Gamma \in \mathcal{F}at_B^\star$ is **very admissible** if the inclusion $\iota_- : \partial^- \Gamma \to \Gamma$ is an embedding of $B$-graphs (in that case, $\Gamma$ is very well-partitioned). □

**Remark 6** This condition is somewhat analogous to the chord diagram constraint of [CG04]. However, we don’t require the complement of the incoming boundary to be a forest; we may do this because the factorization described in Section 7.1 makes it unnecessary to collapse this complement. □

**Lemma 6** Suppose given $\Gamma_1, \Gamma_2 \in \mathcal{F}at_B^\star$, with $\Gamma_2$ very well-partitioned. The $B$-graph $\Gamma_1 \# \Gamma_2$, as defined by pushout diagram (4), has an induced partitioned open-closed fat graph structure with $\partial^- \Gamma_1 \cong \partial^-(\Gamma_1 \# \Gamma_2)$, $\partial^+ \Gamma_2 \cong \partial^+(\Gamma_1 \# \Gamma_2)$ as $B$-graphs. □

**Proof** To describe the fat graph structure permutation $\sigma$ of $\Gamma_1 \# \Gamma_2$, we may first describe its associated permutation $\hat{\sigma}$ (Definition 10), and then verify that, if we define $\sigma$ as $\hat{\sigma} \circ r$, the result is a fat graph.

Describing $\hat{\sigma}$ is equivalent to constructing a graph $\partial \Gamma$ and a morphism $\iota : \partial \Gamma \to \Gamma$ in $\mathcal{G}raph$, taking edges to edges, such that:

1. $\partial \Gamma$ is a disjoint union of cyclic graphs, each with a chosen orientation, and,
2. every oriented edge $e \in H(\Gamma)$ is the orientation-preserving image of exactly one edge in $\partial \Gamma$.

In the process, we will construct a $B$-labeled free boundary subgraph $\partial_b(\Gamma_1 \# \Gamma_2) \subseteq \partial(\Gamma_1 \# \Gamma_2)$ and the partitioning $\partial_b(\Gamma_1 \# \Gamma_2) = \partial^-(\Gamma_1 \# \Gamma_2) \sqcup \partial^+(\Gamma_1 \# \Gamma_2)$.

Since pushouts in $\mathcal{G}raph_B$ can be constructed setwise, we have that $B(\Gamma_1 \# \Gamma_2) = B(\Gamma_1) \cup_{B(\Delta)} B(\Gamma_2)$. The graph $B(\Gamma_1 \# \Gamma_2)$ is a disjoint union of cyclic and linear (possibly degenerate) graphs because each of the $B(\Gamma_i)$ is by definition one such graph, and the discrete graph $B(\Delta)$ identifies pairs of endpoints of linear components. Thus we may identify the discrete graph $\delta B(\Gamma_1 \# \Gamma_2)$ as consisting of the
endpoints of the linear components of $B(\Gamma_1 \# \Gamma_2)$, with multiplicity two for the degenerate ones. Define $\partial(\Gamma_1 \# \Gamma_2)$ by the pushout

$$
\begin{array}{c}
\partial(\Gamma_1 \# \Gamma_2) \\
B(\Gamma_1 \# \Gamma_2)
\end{array} \leftarrow \begin{array}{c}
\partial^- \Gamma_1 \sqcup \partial^+ \Gamma_2 \\
\delta B(\Gamma_1 \# \Gamma_2)
\end{array}
$$

(this is a case of the pushout in Remark 3). It is easy to see that the resulting graph is a disjoint union of cyclic graphs. It inherits a preferred orientation from the orientations on the $\partial \Gamma_i$, contains $B(\Gamma_1 \# \Gamma_2)$ as a subgraph, and satisfies $\partial(\Gamma_1 \# \Gamma_2) \setminus B(\Gamma_1 \# \Gamma_2) \cong \partial^- \Gamma_1 \sqcup \partial^+ \Gamma_2$.

It remains to verify that the permutation $\sigma := \hat{\sigma} \circ r$ implicit in this construction has exactly one disjoint cycle $H(v)$ per vertex of $\Gamma_1 \# \Gamma_2$. Here, we use the fact that $\Gamma_2$ is very well partitioned. This implies that the map $V(\Gamma_1) \to V(\Gamma_1 \# \Gamma_2)$ is injective, and hence that the vertices of the pushout $\Gamma_1 \# \Gamma_2$ are of only two types:

1. vertices $u$ corresponding to those $v \in V(\Gamma_2)$ not in the image of $V(\Delta)$, and,

2. vertices $u$ corresponding to $w \in V(\Gamma_1)$, resulting as the identification of $w$ with each of the vertices $\alpha_2(\alpha_1^{-1}(w)) \subseteq V(\Gamma_2)$.

In the first case, we have $H(u) \cong H(v)$, and the cyclic ordering is given by the cyclic ordering in $H(v)$.

In the second case, for each $v \in \alpha_2(\alpha_1^{-1}(w))$, $H(v)$ has a preferred linear ordering given by opening its cyclic ordering at the unique $e \in H(v)$ lying in $\partial^- (\Gamma_2)$; it is unique since otherwise $\partial^- \Gamma_2 \to \Gamma_2$ would not be injective. Then, $H(u)$ is obtained from $H(w)$ by inserting the half-edges $H(v)$ in this linear order in spaces between half-edges in $H(w)$ determined by the image of $\Delta$ in $\Gamma_1$. This yields a cyclic ordering on $H(u)$ given by “cyclically splicing” the linear orders on the $H(v)$ into the cyclic ordering on $H(w)$.

These cyclic orders are directly seen to agree with the ones induced by $\partial(\Gamma_1 \# \Gamma_2)$ above. □

**Remark 7** Verifying that $\sigma$ gives a cyclic ordering to each $H(u)$, $u \in V(\Gamma_1 \# \Gamma_2)$ is a necessary step. If $\Gamma_2$ is not very admissible, then, while we may still construct the requisite $\partial(\Gamma_1 \# \Gamma_2) \to \Gamma_1 \# \Gamma_2$, it may induce a permutation $\sigma$ for which a single $H(u)$ contains multiple cycles of $\sigma$. □

**9.1 Compatibility with gluing**

We will show that the open-closed string operations are compatible with the gluing construction, in the sense that $(\Gamma_1 \# \Gamma_2)_* = (\Gamma_2)_* \circ (\Gamma_1)_*$, appropriately interpreted, whenever the $\Gamma_i$ are very well-partitioned open-closed fat graphs that fit in a gluing setting.
Proposition 7 Let gluing data \((\Gamma_1, \Gamma_2, \partial^+ \Gamma_1 \xleftarrow{\gamma_1} \Delta \xrightarrow{\gamma_2} \partial^- \Gamma_2)\) be given with \(\Gamma_2\) very well-partitioned, and let be formed accordingly. Then,

\[
(\Gamma_1 \# \Gamma_2)_* = (\Gamma_2)_* \circ [\gamma_1^{-1} \circ \gamma_2]_* \circ (\Gamma_1)_*.
\]

\[\square\]

Proof We have the diagram

\[
\begin{array}{ccc}
\partial^- \Gamma_1 & \xrightarrow{\iota_1} & \Gamma_1 \\
& \downarrow{\delta_1} & \downarrow{\alpha_2} \\
\Gamma_1 \# \Gamma_2 & \xrightarrow{\delta_2} & \Gamma_2 \\
& \downarrow{\iota_2} & \\
\partial^+ \Gamma_2 & & \\
\end{array}
\]

where the upper-right square is the defining pushout. Proposition 4 (case 1) implies that this square has the property

\[
[[\alpha_2]]^! \circ [[\alpha_1]]_* = [[\delta_2]]_* \circ [[\delta_1]]^!
\]

in homology. Then we have:

\[
(\Gamma_1 \# \Gamma_2)_* = [[\iota_+]_* \circ [[\iota_-]]^! = [[\iota_2^+]_* \circ [[\delta_2]]_* \circ [[\delta_1]]^! \circ [[\iota_-]]^! = [[\iota_2^+]_* \circ [[\delta_2]]_* \circ [[\delta_1]]^! \circ [[\iota_-]]^! = [[\iota_2^+]_* \circ [[\delta_2]]_* \circ [[\delta_1]]^! \circ [[\iota_-]]^! = (\Gamma_2)_* \circ [\gamma_1^{-1} \circ \gamma_2]_* \circ (\Gamma_1)_*.
\]

10 The string topology \(\mathcal{B}\)-TQFT

Let \(h_*\) be a multiplicative homology theory whose coefficient ring \(h_*\) is a graded field, that is, a graded ring in which all nonzero homogeneous elements are invertible. Given an \(h_*\)-oriented \(\mathcal{B}\)-brane system \(\mathcal{M} = (M, \{L_b\}_{b \in \mathcal{B}})\), we have a \(\mathcal{B}\)-family

\[
\mathcal{V}_\mathcal{M} = (h_*LM, \{h_*PM (L_a, L_b)\}_{a, b \in \mathcal{B}})
\]

over the coefficient ring of \(h_*\). Here \(LM\) is the free loop space of \(M\). The constraints on \(h_*\) have the effect that there is a product map \(h_* (X) \otimes_{h_*} h_* (Y) \to h_* (X \times Y)\) (because \(h_*\) is multiplicative) which is moreover an isomorphism of graded \(h_*\)-modules (because the graded field condition on \(h_*\) makes the Künneth spectral sequence collapse).

In this section, we will describe the positive-boundary \(\mathcal{B}\)-TQFT structure on \(\mathcal{V}_\mathcal{M}\) arising from the open-closed string topology operations from Definition 23.
Definition 25 Recall that, given a fat graph $\Gamma$, there is an associated oriented surface $S(\Gamma)$ having $|\Gamma|$ as a deformation retract, and having an identification $\partial S \cong |\partial \Gamma|$. If $\Gamma$ is additionally an open-closed fat graph, $S$ becomes naturally an open-closed surface by decreeing the image of $|\partial \Gamma|$ in $\partial S$ to be the free boundary,$^1$ with the induced $B$-labeling. A partitioning of $\Gamma$ then makes $S(\Gamma)$ into an open-closed cobordism.

Proposition 8

- If two partitioned open-closed fat graphs are related by a morphism in $\mathcal{F}at^*_B$, then their corresponding open-closed cobordisms are isomorphic

- Gluing of partitioned open-closed fat graphs translates, up to isomorphism, into gluing of the corresponding open-closed cobordisms.

Proof sketch The first statement is clear from the corresponding result for fat graphs. For the second statement, we may observe that the defining pushout diagram (4) for gluing realizes to a homotopy pushout square equivalent to the counterpart diagram that arises when gluing cobordisms; this determines the Euler characteristic of $S(\Gamma_1 \# \Gamma_2)$. By a comparison of boundary components, also the genus and $B$-labeling structure are seen to correspond.

We will make use of the following lemma, whose proof we defer to the appendix.

Lemma 9 The connected components of $\mathcal{F}at^*_B$ are in one-to-one correspondence with isomorphism types of open-closed cobordisms $S$ for which $\partial^+ S$ intersects every connected component of $S$.

Using the notation of Definition 5, we first show how a morphism $S : x_\rightarrow \rightarrow x_+$ in $\text{Cob}^+_B$ (with $x_\pm \in \mathcal{M}_B$) produces a homomorphism $\mu_S : \mathcal{V}_M(x_-) \rightarrow \mathcal{V}_M(x_+)$. We may assume that $S$ is connected, and induce the remaining operations by tensor product.

By Lemma 9, we can represent the morphism $S$ by a well-partitioned open-closed fat graph $\Gamma \in \mathcal{F}at^*_B$ having $|\partial^- \Gamma| \cong |x_-|$ and $|\partial^+ \Gamma| \cong |x_+|$ as $B$-labeled one-manifolds, together with:

1. a choice of orientation-preserving parametrization of each component of $\partial \Gamma$,

2. a choice $L_-$ of linear orderings of the incoming boundary components in each $B$-labeling type, and a similar choice of $L_+$ for the outgoing boundary components.

A boundary parametrization may be specified by a starting point on each closed string boundary component; it is then determined by starting at that point and

$^1$Strictly speaking we take a small closed neighborhood of $|\partial_1 \Gamma|$ in $\partial S$ in order to deal with degenerate components of $\partial_1 \Gamma$. 

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parameterizing each edge in a piecewise-linear fashion at constant speed. String boundary intervals have a natural parametrization in the same way.

The parametrizations and linear orderings (together with our restrictions on $h_*$) give isomorphisms $h_*(\partial^\pm \Gamma) \cong V_M(x_{\pm})$, and therefore the operation $\Gamma_*$ defines a homomorphism $V_M(x_-) \to V_M(x_+)$, which we call $\mu_S$.

**Lemma 10** The homomorphism $\mu_S$ is well-defined; that is, it is independent of the choice of representing well-partitioned open-closed fat graph and the choice of boundary parameterization.

**Proof** We will temporarily use $\mu_\Gamma$ for the operation defined with a particular choice of boundary-parametrized open-closed fat graph $\Gamma$. Define a category $\mathcal{F}at_B^*$ in which an object is a well-partitioned open-closed fat graph together with a choice of starting point in each string boundary cycle, with morphisms being those morphisms of partitioned open-closed fat graphs which preserve the choices of starting points. We can argue as in [CG04] to show that the obvious forgetful functor $\mathcal{F}at_B^* \to \mathcal{F}at_B^*$ is a torus fibration on each component, and thus the connected components of $\mathcal{F}at_B^*$ are also in one-to-one correspondence with isomorphism types of open-closed cobordisms. Thus, it is enough to show that if $\varphi : \Gamma_1 \to \Gamma_2$ is a morphism in $\mathcal{F}at_B^*$ with $\Gamma_1$ and $\Gamma_2$, then $\mu_{\Gamma_1} = \mu_{\Gamma_2}$.

Consider the diagrams

The $\gamma_i^\pm$ are isomorphisms of $B$-labeled one-manifolds induced by the choices of starting points on the boundary cycles. The two triangles commute up to homotopy relative to the boundary. It follows that, when we apply the functor $[\_\_]_M$, the induced homomorphisms on homology satisfy

$$[\gamma_2^\pm]_* = [\gamma_1^\pm]_* \circ [\partial^\pm \varphi]_*.$$

By definition, we have

$$\mu_{\Gamma_2} = [\gamma_2^+]_* \circ (\Gamma_2)_* \circ [\gamma_2^-]_*^{-1},$$

and therefore

$$\mu_{\Gamma_2} = [\gamma_1^+]_* \circ [\partial^\pm \varphi]_* \circ (\Gamma_2)_* \circ [\partial^- \varphi]_*^{-1} \circ [\gamma_1^-]_*^{-1}.$$

By Proposition 5, we have that $(\Gamma_1)_* = [\partial^\pm \varphi]_* \circ (\Gamma_2)_* \circ [\partial^- \varphi]_*^{-1}$, and thus

$$\mu_{\Gamma_2} = [\gamma_1^+]_* \circ (\Gamma_1)_* \circ [\gamma_1^-]_*^{-1}.$$

But this is equal to $\mu_{\Gamma_1}$ by definition. 

\[24\]
Lemma 11 The assignment $S \mapsto \mu_S$ is functorial.

Proof This is directly implied by Proposition 7.

As a corollary, we have our theorem.

Theorem A If $h_*$ is a multiplicative generalized homology theory for which the coefficient ring is a graded field, then, letting $\mathcal{V}_M$ be the $\mathcal{B}$-family

$$\{h_*LM, \{h_*PM(L_a, L_b)\}_{a,b \in \mathcal{B}}\}$$

over $h_*(\ast)$, there is a positive-boundary $\mathcal{B}$-TQFT structure on $\mathcal{V}_M$ which extends the known positive-boundary string topology TQFT structure on $h_*(LM)$.

A Connected components of $\mathcal{F}at^*_\mathcal{B}$

Here we turn to the proof of Lemma 9. We will show that in each connected component of $\mathcal{F}at^*_\mathcal{B}$ there is, after making a few controlled choices, a graph in a particular “normal” form, and that this form is uniquely determined, up to these choices, by the isomorphism type of its associated open-closed cobordism. This will show that the connected components of $\mathcal{F}at^*_\mathcal{B}$ are as desired. Our proof of this relies heavily on the corresponding construction for chord diagrams in the closed string case, as presented by R. Cohen and V. Godin [CG04].

Remark 8 We include this proof for completeness, but it will be superseded by a result stating that the category $\mathcal{F}at^*_\mathcal{B}$ realizes to a space homotopy equivalent to an appropriate moduli space of open-closed Riemann surfaces; this will recover Lemma 9 upon applying $\pi_0$. 

A.1 Isomorphism invariants of open-closed surfaces

There are a few invariants which together determine the isomorphism type of a connected open-closed surface. They are as follows:

- The genus $g$ of $S$.
- The subset $\partial_{\text{cl}} S$ of $\pi_0(\partial S)$ consisting of closed string boundary components.
- The subset $\partial_{\text{int}} S$ of $\pi_0(\partial S)$ consisting of string boundary intervals.
- The function $\xi : \partial_{\text{int}} S \to \mathcal{B}$ assigning to a string boundary interval the label of its final endpoint, where “final” is with respect to the orientation.
- A permutation $\psi \in \text{Sym}(\partial_{\text{int}} S)$ taking a string boundary interval $c$ to the one following $c$ in the same boundary component, in the direction induced by the orientation.
- The subset $\partial_w S$ of $\pi_0(\partial S)$ consisting of closed components (“windows”).
- The $B$-labeling $\beta : \partial_w S \to B$ induced from $\beta : \partial I S \to B$

If $S$ is an open-closed cobordism, we can further identify:
- The partitions $\partial_{S^1} S = \partial_{S^1}^- S \sqcup \partial_{S^1}^+ S$, $\partial I S = \partial I^- S \sqcup \partial I^+ S$.

Denote by $X(S)$ the tuple

$$X(S) := (g, \partial S_{1} S, \partial I S, \xi, \psi, \partial_w S, \beta).$$

For an open-closed cobordism, denote by $Y(S)$ the tuple

$$Y(S) := (g, \partial S_{1} S, \partial I S, \xi, \psi, \partial_w S, \beta, \partial_{\pm} S_{1}, \partial_{\pm} I S).$$

**Definition 26** In the absence of an open-closed surface or cobordism $S$, we consider tuples

$$(g, \partial S_{1}, \partial I, \xi, \psi, \partial_w, \beta)$$

where $g \geq 0$ is an integer, $\partial S_{1}$, $\partial I$ and $\partial_w$ are arbitrary finite sets, $\xi : \partial I \to B$ and $\beta : \partial_w \to B$ are arbitrary functions, and $\psi$ is a permutation of $\partial I$. We call these open-closed data tuples. A partitioned open-closed data tuple is one of the form $(X, \partial_{\pm} S_{1}, \partial_{\pm} I S)$, where $X = (g, \partial S_{1}, \partial I, \ldots)$ is an open-closed data tuple and the $\partial_{\pm} S_{1}$, $\partial_{\pm} I$ are partitionings of $\partial S_{1}$, $\partial I$.

Two such tuples (partitioned or not) are isomorphic if they have the same $g$ and there are bijections of the corresponding sets which preserve the $B$-labelings, the permutation $\psi$, and the partitions if present.

**Proposition 12**

1. Two open-closed surfaces (resp. cobordisms) are isomorphic if and only if their data tuples (resp. partitioned data tuples) are isomorphic.

2. Given an arbitrary (resp. partitioned) open-closed data tuple, there is an open-closed surface $S$ with $X(S)$ isomorphic to it (resp. an open-closed cobordism with $Y(S)$ isomorphic to it).

**Proof sketch** This is mostly clear, so we only provide a sketch of the construction for the second statement, which should make the first statement obvious.

Choose an ordinary oriented surface $S$ of genus $g$ having its boundary components in bijection with $\partial S_{1} \sqcup \partial I \sqcup \partial_w$. Label each entire component corresponding to $w \in \partial_w$ by $\beta(w)$, making it part of the free boundary. Given a cycle $c = (x_1, \ldots, x_k)$ of $\xi$, let $A_c \subseteq \partial S_{1}$ be the corresponding boundary component. Choose an embedding $\{x_1, \ldots, x_k\} \times [0, 1] \hookrightarrow A_c$ which is orientation-preserving such that the cyclic order of the $x_i$’s induced from the orientation of $A_c$ coincides with the cyclic order given by $\xi$. Declare the image of this embedding to belong to the string boundary. Decree the components of $\overline{A_c \setminus \{(x_1, \ldots, x_k) \times [0, 1]\}}$ to be in the free boundary, and label them according to the rule that the component coming after $\{x\} \times [0, 1]$ in the cyclic order carries the label $\xi(x)$. The boundary components corresponding to $\partial S_{1}$ are decreed to be part of the string boundary. We omit the easy verification that this yields an open-closed surface having $X(S) \cong X$. ■
Now, given an open-closed fat graph $\Gamma$, $X(S(\Gamma))$ is entirely determined by $\Gamma$, and in fact we can write $X(\Gamma)$ for an open-closed data tuple obtained directly from $\Gamma$, as follows:

$$X(\Gamma) := (g(\Gamma), \partial_S \Gamma, \partial_I \Gamma, \xi, \psi, \partial_w \Gamma, \beta|_{\partial_w \Gamma}),$$

where

- $\partial_I \Gamma, \partial_S \Gamma \subseteq \pi_0(\partial_s \Gamma)$ are the sets of free boundary intervals and cycles, respectively.
- $\partial_w \Gamma \subseteq \pi_0(\partial_l \Gamma)$ is the set of closed free boundary components.
- $g(\Gamma) := 1 - \frac{\#\pi_0(\partial_l \Gamma) + \chi(\Gamma)}{2}$.
- For $c \in \partial_I \Gamma$, $\xi(c)$ is the $B$-label carried by the final endpoint of $c$.
- $\psi$ takes a string boundary interval $c$ to the one appearing immediately after it in the component of $\partial \Gamma$ containing $c$.

If $\Gamma$ has a partitioning, we can further define $Y(\Gamma) = (X(\Gamma), \partial^+_S \Gamma, \partial^+_I \Gamma)$. The following is clear.

**Proposition 13** $Y(S(\Gamma)) \cong Y(\Gamma)$. \qed

### A.2 Preliminary reductions

Let $\Gamma \in \mathcal{Fat}_G^*$ and let $b \subseteq \partial_l \Gamma$ be a linear component of the free boundary.

**Definition 27** Denote by $b_+ \subseteq \partial_s \Gamma$ (resp. $b_-$) the string boundary component that appears immediately after $b$ (resp., before $b$) in $\partial_l \Gamma$ according to the orientation. We can classify $b$ as belonging to one of four types:

- say that $b \in B^-(\Gamma)$ if $b_- \subseteq \partial^- \Gamma$ and $b_+ \subseteq \partial^- \Gamma$,
- say that $b \in B^+(\Gamma)$ if $b_- \subseteq \partial^+ \Gamma$ and $b_+ \subseteq \partial^+ \Gamma$,
- say that $b \in B^{++}(\Gamma)$ if $b_- \subseteq \partial^- \Gamma$ and $b_+ \subseteq \partial^+ \Gamma$,
- say that $b \in B^{+-}(\Gamma)$ if $b_- \subseteq \partial^+ \Gamma$ and $b_+ \subseteq \partial^- \Gamma$.

We now identify a convenient class of graphs containing a representative of each connected component.

**Definition 28** Call $\Gamma$ *special* if the following conditions hold:

1. Each linear component of $\partial_l \Gamma$ consists of a single vertex.
2. Each cyclic component of $\partial_l \Gamma$ has exactly one edge.
3. The image loop in $\Gamma$ of each cyclic free boundary component $b$ is attached to a trivalent vertex; this vertex is therefore incident with the two half-edges forming the loop and with a third edge; the loop is then a balloon attached to $\Gamma$ by this edge.

4. Each $b \in B^{-+}(\Gamma) \cup B^{+-}(\Gamma) \cup B^+(\Gamma)$ is attached to a univalent vertex of $\Gamma$.

5. Each $b \in B^-(\Gamma)$ is attached to a bivalent vertex of $\Gamma$.

6. There are no other bivalent or univalent vertices.

A special open-closed fat graph $\Gamma$ then has distinguished $B$-labeled bivalent vertices, leaves with $B$-labeled endpoint, and $B$-labeled balloons.

**Proposition 14** Any $\Gamma \in \mathcal{Fat}_B$ is connected to a special graph by a sequence of morphisms in $\mathcal{Fat}_B$.

**Proof** The first two and last conditions can be attained by collapsing the image in $\Gamma$ of a maximal forest in $\partial \Gamma$; after that the three other conditions can be attained by expanding suitable vertices into trees.

**Remark 9** We cannot have a $b \in B^-(\Gamma)$ attached to a univalent vertex—this vertex would in turn be the endpoint of an edge having both its orientations in $\partial^- \Gamma$, and the partitioning would not be admissible.

### A.3 Normal forms

In view of previous section, it is enough to show that two special graphs $\Gamma, \Gamma' \in \mathcal{Fat}_B$ with $Y(\Gamma) = Y(\Gamma')$ are in the same connected component.

Our strategy for finding suitable normal forms will be to use the algorithm by R. Cohen and V. Godin in [CG04], henceforth referred to as “algorithm V.” The idea is, roughly, to run $\Gamma$ through this algorithm, having it work on the underlying fat graph $U(\Gamma)$. However, it will not be quite this simple, since we have to be careful for two reasons:

- The algorithm in [CG04] assumes that the starting graph is a chord diagram. Since we use the laxer admissibility condition from Definition 22, we will work to achieve this form; see Lemma 17 below.

- Open-closed fat graphs may have incoming and outgoing string boundary intervals in addition to cycles.

- Whenever the algorithm expands an edge of $U(\Gamma)$, there is at least one way to expand the corresponding edge of $\Gamma$ while keeping $\Gamma$ special; however, when the algorithm collapses an edge of $U(\Gamma)$, the corresponding edge of $\Gamma$ may not be collapsable, since it may result in joining two components of the image of $\partial \Gamma$. 

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To describe the different cases that arise when finding the normal forms, we introduce some terms.

**Definition 29** Say that \( \Gamma \in \mathcal{Fat}_B^* \) is **clean** if it is special and, additionally, \( \partial I \Gamma = B^{-+}(\Gamma) \cup B^{+-}(\Gamma) \).

Given a special \( \Gamma \in \mathcal{Fat}_B^* \), let \( w(\Gamma) \) be the open-closed fat graph obtained from \( \Gamma \) by

1. removing \( B^+(\Gamma) \cup B^-(\Gamma) \) from \( \partial I \Gamma \),
2. removing all the cyclic components of \( \partial I \Gamma \), as well as their image loops in \( \Gamma \),
3. removing from \( \Gamma \) any leaves or bivalent vertices created by the previous two steps.

Notice that \( w(\Gamma) \) inherits a boundary partitioning from \( \Gamma \), since the only removed free boundary intervals lie between string boundary components on the same side of the partitioning of \( \Gamma \).

We define a **weak string boundary component** of \( \Gamma \) to be a string boundary component of \( w(\Gamma) \), and we let \( \partial w^\pm \Gamma := \partial^w \pm(\Gamma) \).

The result we aim for is as follows.

**Lemma 15** Let \( \Gamma \in \mathcal{Fat}_B^+ \) be connected.

- **Case 1.** Suppose that \( \Gamma \) has a weak incoming boundary cycle. Then, \( \Gamma \) is connected in \( \mathcal{Fat}_B^* \) to an open-closed fat graph of the following form:

![Diagram](image)

- **Case 2.** Suppose that \( \Gamma \) has no weak incoming boundary cycles, but it has more than one topological boundary component. Then, \( \Gamma \) is connected to a graph of the form:
Case 3. Suppose that $\Gamma$ has no weak incoming boundary cycles, and that $\Gamma$ has exactly one topological boundary component. Then, $\Gamma$ is connected to a graph of the form:

The symbols used in the pictures are as follows:

- A serrated portion of an edge \( \bullet \bullet \bullet \bullet \) stands for a (possibly empty) sequence of bivalent vertices carrying elements of $B^-(\Gamma)$; the triangles point towards the topological boundary component containing them.

- The symbol \( \leadsto \bullet \bullet \bullet \bullet \leadsto \) represents a (possibly empty) sequence in which where each element \( \leadsto \bullet \bullet \bullet \bullet \leadsto \) stands for a structure of the form

Here, $c$ stands for an element of $B^{+-}(\Gamma)$, $d$ for one of $B^{-+}(\Gamma)$, and each $a_i$ stands for an element of $B^+(\Gamma)$ (the meaning of the serrated edge is as before).
The symbol \[ \cdots \] stands for a (possibly empty) sequence of balloons of the form

\[
\begin{array}{c}
\vdots \\
w_2 \\
w_1
\end{array}
\]

(that is, loops in the image of \( \partial \Gamma \) labeled by elements \( w_i \in B \)).

Note that those boundary components that contain a serrated edge are weak incoming boundary components, and the ones that contain a \[ \cdots \] are topological boundary components which contain part of the outgoing boundary.

In the first two cases, we will use terminology partially adapted from [CG04], as follows:

- The component marked \( c_0 \) in the pictures will be called the outer component (this is called the big incoming circle in [CG04], but it is an outgoing component in our case 2).

- The topological boundary components in the top- and bottom-right quadrants in cases 1 and 2 will be called simple outgoing cycles.

- The topological boundary component in the top-right quadrant will be called the complicated outgoing cycle.

- The weak incoming cycles obtained by going clockwise around the small circles on the lower left of case 1 will be called small incoming cycles.

The uniqueness of these normal forms is as follows:

- In case 1 we may choose an arbitrary weak incoming cycle \( c_0 \) to be the outer component, and we may choose an arbitrary topological boundary component containing part of the outgoing boundary to be the complicated outgoing cycle. Moreover, we can specify arbitrarily the order in which the simple outgoing cycles and the small incoming cycles occur in the cyclic ordering around the central vertex.

- In case 2 we may choose an arbitrary topological boundary component \( c_0 \) (necessarily containing part of the outgoing boundary) to be the outer component, and one to be the complicated outgoing cycle. As in case 1, we can specify the order of the simple outgoing cycles arbitrarily.

- In all cases, we can choose an arbitrary linear ordering (consistent with the underlying cyclic order) for the \( B \)-labels to appear in each \[ \cdots \] or \[ \cdots \] cluster.
In all cases, we can choose an arbitrary order for the labels $w_i$ of the balloons appearing in the $\cdots \cdots$.

In each case, the normal form is uniquely determined, after the corresponding choices have been made, by the combinatorial data carried by the invariant $Y(\Gamma)$, and therefore uniquely determined by the isomorphism type of $S(\Gamma)$.

Lemma 16 If $\Gamma \in \mathcal{F}at_B^*$ has nonempty $\partial^- \Gamma$, then it is connected by morphisms to a special graph $\Gamma'$ for which every edge of $w(\Gamma')$ has exactly one of its orientations in the incoming string boundary.

Proof Assume without loss of generality that $\Gamma$ is connected and special. We aim to get rid of edges of $w(\Gamma)$ having two outgoing orientations (since $\Gamma$ is well-partitioned, there are no edges having two incoming orientations). We will do this inductively, by showing that we can reduce the number of such “bad” edges via morphisms that introduce only “good” edges.

Suppose there is at least one bad edge. Since $\Gamma$ is connected, there is a vertex not in the image of $\partial \Gamma$, and incident to both a bad edge and to at least one edge taking part in the incoming boundary. The cyclic ordering at this vertex hence looks like:

$$\cdots \oplus \oplus \cdots,$$

where the symbols $\ominus$, $\oplus$ denote incoming and outgoing boundary, respectively. We can modify $\Gamma$ in two steps, as follows:

$$\cdots \ominus \oplus \ominus \rightarrow \ominus \ominus \oplus \ominus \cdots,$$

The graph in the middle maps to the other two graphs by obvious morphisms in $\mathcal{F}at_B^*$; the one on the left is $\Gamma$ and the one on the right has one less bad edge than $\Gamma$. We leave it to the reader to verify that this works even if the bad edge is a loop.

Lemma 17 If $\Gamma \in \mathcal{F}at_B^*$ has nonempty $\partial^- \Gamma$, then it is connected by morphisms in $\mathcal{F}at_B^*$ to a special $\Gamma' \in \mathcal{F}at_B^*$ for which $\partial^- w(\Gamma')$ is embedded in $w(\Gamma)$ in such a way that the complement graph is a forest.
Proof We may assume that $\Gamma$ is special and satisfies the condition in the conclusion of Lemma 16. So the set of edges of $\partial^- w(\Gamma)$ is already embedded in the set of edges of $w(\Gamma)$; it remains to make the vertices embedded too. For every vertex $v$ not in the image of $\partial w(\Gamma)$, the angles around $v$ must alternate between incoming and outgoing when traversed according to the cyclic ordering at $v$. Because of this, the following transformation (illustrated in the case that $v$ has valence 6) may be used to replace the vertex $v$ by a tree, resulting in $\Gamma' \in \mathcal{F} \mathcal{T}_2^g$ mapping to $\Gamma$ by a morphism:

\[
\begin{array}{c}
\Gamma \\
\leftarrow \\
\Gamma'
\end{array}
\]

It is clear that applying this transformation to every vertex achieves the desired condition.

With these preliminary result in place, we are ready to prove our main lemma.

Proof of Lemma 15 We may assume that $\Gamma$ is special. Notice that we may disregard the balloons throughout, since any two special open-closed fat graphs that differ only on the location of balloons along the outgoing boundary are connected by a sequence of morphisms; we leave this as an exercise. Thus, we can let them move around the graph arbitrarily, and we can collect them at the end to form a single at the correct location.

Assume first that $\Gamma$ is clean, so that $\Gamma = w(\Gamma)$. By Lemma 17, we may assume that $\partial^- \Gamma$ is embedded in $\Gamma$ with a forest complement. Following [CG04], we will call the edges of the complement “ghost edges.”

We can apply much of Algorithm $V$ to $\Gamma$, by using an arbitrary (open or closed) incoming boundary component $c_0$ of $\Gamma$ instead of the “big incoming component,” and we may treat the other incoming boundary intervals much of the way as if they were incoming boundary cycles. We leave it to the reader to verify that $\Gamma$ can be transformed, following the first steps of Algorithm $V$, into a form in which exactly one non-univalent vertex $v_0$ is incident with more than one ghost edge, and in which every incoming boundary component other than $c_0$ is connected by exactly one ghost edge to $v_0$.

The next step in Algorithm $V$ is to push the “small incoming cycles” all the way to the “right” in the cyclic ordering at $v_0$. This relies on these boundary components being cycles; thus we are not able to do it with the incoming boundary intervals. However, after the previous step, we may collapse the unique ghost edge attached to each of the incoming boundary intervals (other than $c_0$). After doing this, we create a structure of type $\Rightarrow \cdots \Rightarrow$ for each incoming interval. Call the resulting graph $\text{State } \star$. 

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Every time a structure of type \( \rightsquigarrow \rightarrow \leftarrow \leftarrow \) is created, we will treat it thereafter as a single unit, so its constituent edges will not be collapsed. We will call edges that are not in any of these structures “active;” we will call a vertex active if all its incident edges are active.

Now, suppose that \( c_0 \) is an incoming boundary cycle. Then, we can in fact continue applying Algorithm \( V \) to the end, and we obtain, after relocating structures of type \( \rightsquigarrow \rightarrow \leftarrow \leftarrow \) by means of expansion/collapse pairs if necessary, precisely the normal form in case 1.

Suppose that we are in case 2, so that \( c_0 \) must be a incoming boundary interval. After State \( \star \) there are no incoming boundary cycles, and we have collapsed all the incoming weak boundary intervals but one, namely \( c_0 \). To proceed, collapse \( c_0 \) except for its two endpoints (which carry labels in \( B^{-} \Gamma \) and \( B^{+} \Gamma \) respectively); this is possible since \( c_0 \) is still embedded in \( \Gamma \). This creates a structure of type \( \rightsquigarrow \rightarrow \leftarrow \leftarrow \). After this, we collapse a maximal active subtree of \( \Gamma \). The result is a special open-closed fat graph with all its active half-edges belonging to \( \partial \Gamma \), having more than one topological boundary cycle and exactly one active vertex.

Change notation, letting \( c_0 \) be any topological boundary cycle, which will act as our “big cycle” from now on. Using Lemma 16, we can replace \( \Gamma \) by one in which each edge has exactly one of its orientations in \( c_0 \) (to do this, we introduce a temporary boundary partitioning on \( \Gamma \) which makes \( c_0 \) incoming and the remaining components outgoing, and then apply the lemma; here it is essential that \( \Gamma \) has more than one topological boundary component). Then, as we did in Lemma 16 for the incoming boundary, we can further replace \( \Gamma \) by a graph in which \( c_0 \rightarrow \Gamma \) is an embedding and the complement of \( c_0 \) in \( \Gamma \) is a forest. From this point on, we apply Algorithm \( V \), treating \( c \) as if it were the “big incoming cycle,” resulting in the required normal form.

In case 3, we have after State \( \star \) and after collapsing the remaining incoming interval and a maximal active tree that \( \Gamma \) has one active vertex, a single topological component and all its active half-edges in \( \partial^{+} \Gamma \). Since \( \partial \Gamma \) is connected, we can ensure that all the labeled univalent vertices are contiguous by using a sequence of expansions/collapse pairs. The normal form for this case then follows directly from Lemma 18 below on the structure of fat graphs with a single vertex and a single boundary component. (This case does not arise in [CG04] because they do not consider fat graphs with empty incoming boundary.)

If \( \Gamma \) is not clean, then we apply the preceding procedure to \( w(\Gamma) \). Every time our algorithm expands a vertex of \( w(\Gamma) \) into an edge, the corresponding operation may be carried out on \( \Gamma \). When the algorithm contracts an edge of \( w(\Gamma) \), we must be careful because this edge might come from a sequence of edges in \( \Gamma \) separated by bivalent vertices carrying labels in \( B^{-} \Gamma \). However, we can still collapse the corresponding edges in \( \Gamma \) provided that we first move those bivalent vertices further along in the boundary of \( \Gamma \); this can be done by a straightforward sequence of expansion/collapse pairs. At the end, we are in a state for which \( w(\Gamma) \) is in one of the special forms; we can then use expansion/collapse pairs so that all the \( B^{-} \Gamma \) labels are contiguous (forming serrated edges), and so that all the univalent vertices carrying \( B^{+} \Gamma \) labels are also contiguous, and form part of a structure of type.
We omit the proof of the uniqueness statement, which follows from a computation of the open-closed data tuple of each of the normal form and from observing that they cover distinct isomorphism types.

Lemma 18 Let $\Gamma$ be a fat graph having a single boundary component and a single vertex. Then, $\Gamma$ it is connected by a sequence of morphisms to a fat graph having single vertex $v$ in which the cyclic ordering of half-edges at $v$ is of the form

$$(e_1, e_2, r(e_1), r(e_2), e_3, e_4, r(e_3), r(e_4), \ldots, e_{2k-1}, e_{2k}, r(e_{2k-1}), r(e_{2k}))$$

(where as usual $r$ stands for the edge-reversal involution on $\Gamma$).

The proof is an easy induction.

Lemma 9 is now direct corollary of Lemma 15.

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