THE BRAIDING OF CHIRAL VERTEX OPERATORS
WITH CONTINUOUS SPINS IN 2D GRAVITY

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Abstract

Chiral vertex-operators are defined for continuous quantum-group spins $J$ from free-field realizations of the Coulomb-gas type. It is shown that these generalized chiral vertex operators satisfy closed braiding relations on the unit circle, which are given by an extension in terms of orthogonal polynomials of the braiding matrix recently derived by Cremmer, Gervais and Roussel. This leads to a natural extension of the Liouville exponentials to continuous powers that remain local.
1 INTRODUCTION

Recently there has been considerable progress in understanding the structure of two-dimensional gravity from the continuum point of view[1]-[3]. It was brought about by the realization of the underlying quantum group structure of the theory which fully determines the chiral operator algebra[1],[2],[6],[8]. The basic chiral fields from which the local Liouville operators \( \exp(-\alpha J \Phi) \) are constructed (\( \alpha \) is one of the screening charges) were seen to fall into standard representations of \( U_q(sl(2)) \) for (positive) half-integer spin \( J \). They yield a comprehensive description of the ”minimal” sector of the theory consisting only of degenerate primaries. Correspondingly, the derivation of refs.[2],[8] was largely based on the null-vector decoupling-equations for the simplest degenerate primaries, which arise naturally upon quantization of the Liouville dynamics. Further progress was made[3],[6],[7] by putting forward the symmetry \( J \to -J - 1 \) between positive and negative spins which already leads out of the ”minimal” sector and allows to construct the three-point function of minimal matter coupled to gravity[7], as well as to derive unitary-truncation theorems in strong coupling theories[1],[11],[12]. However, for a full understanding of the integrability structure of the theory, and, in particular, for the construction of arbitrary Liouville exponentials \( e^{\lambda \phi} \) (and thus the Liouville field itself as \( \phi = \frac{d}{d\lambda}|_{\lambda=0} e^{\lambda \phi} \)) it is necessary to consider the general case of continuous spins. This will provide information about the structure of Liouville theory as a local conformal theory in its own right — not just as a ”gravitational dressing” for some particular matter theory — which seems hardly accessible from the matrix model approach. Such insights appear particularly useful to understand 2 dimensional gravity, and its W generalizations in their strong-coupling regimes where, so far, only the quantum-group approach has led to significant progress[3],[6],[11],[12]. In this paper we shall take a first step in this direction by establishing control over the exchange algebra of chiral primaries with continuous spins, which is the essential ingredient for the construction of general Liouville exponentials. In the absence of an underlying null-vector decoupling equation, we shall rely exclusively on free field techniques, pushing further previous attempts[14] which were confined to the lowest orders in the cosmological constant and

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3To simplify, we only deal with irrational theories explicitly, so that these representations are in a one-to-one correspondence with those of \( sl(2) \)
did not disentangle the chiral algebra which is ultimately responsible for the
locality properties of the Liouville field. The emerging picture will be in
accord with the naive expectation that, in the general situation, the alge-
bra of the chiral primaries should be governed by the universal $R$-matrix of
$SL(2)_q$, suitably generalized to (infinite-dimensional) highest/lowest-weight
representations with continuous spins. Our notations are identical to those of
refs.[2]-[8]. The chiral primaries $V^{(J)}_m$ in the Bloch-wave basis are the fields
with diagonal monodromy — as opposed to the fields $\xi^{(J)}_M$ of the quantum-
group basis, which transform as multiplets of spin $J$ under quantum-group
action\[5\]. In this letter, we make use of one of the two screening charges only\[5\],
so that we start from $V^{(J)}_m$, and not from the most general fields $V^{(J,J')}_m$. In
ref.[7], it was shown that, for $2J$ a positive integer, the exponential of the
Liouville field is given by an expression of the form

$$e^{-J\alpha_\omega \Phi(\sigma, \tau)} = \sum_{m=-J}^{J} B^{(J)}_m(\omega) V^{(J)}_m(x_+) \nabla^{(J)}_m(x_-)$$  (1)

where $\sigma$, and $\tau$ are world-sheet variables, $x_\pm = \sigma \mp i\tau$, $\omega$ is the Liouville
zero mode\[6\], and $\nabla^{(J)}_m$ are the antiholomorphic components. The function
$B^{(J)}_m(\omega)$ may be deduced from the expression given in ref.[7] — it is not
explicitly needed at this point. It is determined from the requirement that
two Liouville exponentials commute at different $\sigma$, for equal $\tau$, as required
by locality. This neatly follows from the braiding relations of the $V$ fields which are given by\[8\]:

$$V^{(J)}_m(\sigma) V^{(J')}_{m'}(\sigma') = \sum_{m_1 m_2} \hat{R}(J, J'; \omega)^{m_2 m_1}_{m' m} V^{(J')}_{m_2}(\sigma') V^{(J)}_{m_1}(\sigma)$$  (2)

$$\hat{R}(J, J'; \omega)^{m_2 m_1}_{m' m} = q^{m m' + m'^2 - m_m^2 + \omega (m' - m)} g_{J,x+M}^{x+m_2} g_{J',x+M}^{x+m_2} \{ J_x x \, \mid x+m_2 \} \{ J'_{x'} x' \, \mid x_m + m \}$$

$$M = m + m' = m_1 + m_2, \quad x := (\omega - \omega_0)/2, \quad \omega_0 := 1 + \pi/\hbar, \quad q := e^{i\hbar},$$

$$\hbar = \frac{\pi}{12} (C - 13 - \sqrt{(C - 25)(C - 1)}),$$

\[4\]In the language of integrable lattice models, the former lead to interaction-around-
the-face models, while the latter describe interactions of the vertex type.
\[5\]of course, this does not matter since we will eventually take $J$ to be continuous.
\[6\]appropriately rescaled
where \( \{ J, J', x + M \}_{x, \bar{x}} \) denotes the quantum 6-j symbol, \( C \) is the central charge, and the \( g \)'s are coupling constants whose expression will be recalled below when needed. In the approach of ref.\[7\], one takes the zero modes of the left-moving and right-moving Liouville modes to commute, so that \( V \) and \( \bar{V} \) commute. Thus the equations just written, together with their right-moving counterparts are the first step to ensure the locality properties. Since one considers the braiding at equal \( \tau \) one lets \( \tau = 0 \) once and for all, and works on the unit circle \( u = e^{i\sigma} \). In the above equations and in the following we discuss the braiding of the holomorphic components assuming that \( \sigma \) and \( \sigma' \) range between 0 and \( \pi \) for concreteness. Our task in the present letter is to generalize Eqs.2 and 3 to continuous \( J \). To construct the chiral primaries, one introduces two equivalent free fields[16]

\[
\phi_j(\sigma) = q_0^{(j)} + p_0^{(j)} \sigma + i \sum_{n \neq 0} \frac{e^{-i n \sigma}}{n} p_n^{(j)} / n, \quad j = 1, 2, (5)
\]

such that (primes mean derivatives)

\[
\left[ \phi'_1(\sigma_1), \phi'_2(\sigma_2) \right] = \left[ \phi'_2(\sigma_1), \phi'_1(\sigma_2) \right] = 2\pi i \delta'(|\sigma_1 - \sigma_2|),
\]

\[
p_0^{(1)} = -p_0^{(2)} = -i\sqrt{\hbar/2\pi \varpi}, (6)
\]

Chiral primaries \( V_{m}^{(J)} \) with arbitrary \( m \) and \( J \) can then be defined as fusions of exponentials of \( \phi_1 \) and \( \phi_2 \):

\[
V_{m}^{(J)} = K_{m}^{(J)}(\varpi) N^{(1)}(e^{\sqrt{\hbar/2\pi} (J-m) \phi_1}) \bullet N^{(2)}(e^{\sqrt{\hbar/2\pi} (J+m) \phi_2}), (7)
\]

Here \( N^{(j)} \) denotes normal ordering with respect to the modes of \( \phi_j \), and \( \bullet \) abbreviates the renormalized short-distance product (fusion) of the two factors[13]. \( K_{m}^{(J)} \) is a normalization factor such that the ground state matrix elements of \( V_{m}^{(J)} \) are canonically normalized:

\[
\langle \varpi | V_{m}^{(J)}(\tau = \sigma = 0) | \varpi + 2m \rangle = 1 (8)
\]

In particular, one may see that \( V_{\pm J}^{(J)} \) are expressible as simple exponentials:

\[
V_{-J}^{(J)} = N^{(1)}(e^{\sqrt{\hbar/2\pi} 2J \phi_1}), \quad V_{J}^{(J)} = N^{(2)}(e^{\sqrt{\hbar/2\pi} 2J \phi_2}). (9)
\]

They are simple “tachyon” vertex operators. One may of course verify that, for \( J \pm m = 0 \), and \( J' \pm m' = 0 \), Eq.\[3\] does reduce to the simple braiding of exponentials of free fields.

\[^{7}\text{For an earlier derivation in a different form, see also \,[3]}\]
2 Derivation of the chiral algebra for continuous J

For the reconstruction of the Liouville exponentials with arbitrary $J$ generalizing Eq.1, one needs only the chiral primaries with $J + m$ (positive) integer, or equivalently, those with $J - m$ integer. This can be seen already from the classical formula[5] and will be shown in detail in a later publication. We shall only discuss the case of integer $J + m$ explicitly here. The other case may be obtained by the trivial substitution $\phi_1 \to \phi_2$ and $\varpi \to -\varpi$ in all the formulae. In this situation, the Bloch-wave operator $V_{m}^{(J)}$ may be re-expressed in terms of $\phi_1$ alone by means of a Coulomb-gas type integral representation. It has the form[15]

$$V_{m}^{(J)} = (I_{m}^{(J)}(\varpi))^{-1}U_{m}^{(J)}, \quad (10)$$

where $I_{m}^{(J)}(\varpi)$ are normalization constants (independent of $\sigma$), and

$$U_{m}^{(J)}(\sigma) = V_{-J}^{(J)}(\sigma)[S(\sigma)]^{J+m} \quad (11)$$

$$S(\sigma) = e^{2ih(\varpi+1)} \int_{0}^{\sigma} dxV_{1}^{(-1)}(x) + \int_{\sigma}^{2\pi} dxV_{1}^{(-1)}(x). \quad (12)$$

Indeed, first it is easily seen that $V_{1}^{(-1)}$ is one of the two screening operators. Second, using the explicit expression for the Liouville zero-mode $\varpi = i\pi_0^{(1)}\sqrt{2\pi/h}$ (see Eq.8), one verifies that

$$V_{m}^{(J)}\varpi = (\varpi + 2m)V_{m}^{(J)} \quad (13)$$

as expected. Third the formula for $S(\sigma)$ is such that

$$S(\sigma + 2\pi) = e^{2ih(1+\varpi)}S(\sigma). \quad (14)$$

From this, one deduces that

$$U_{m}^{(J)}(\sigma + 2\pi) = e^{2ihm\varpi+2ihm^2}U_{m}^{(J)}(\sigma) \quad (15)$$

which gives the correct monodromy eigenvalue for the fields $V_{m}^{(J)}$. Since the above expressions make sense also for continuous $2J$, provided that $J + m$ is a non-negative integer, we take them as the definition of the generalized
vertex operators. The only potential problem comes from divergences of the product of $S(\sigma)$. However it is easily seen that $[S]^{J+m}$ makes sense for small enough $h$. The singularities which appear when $h$ increases are to be handled by analytic continuation (more about this below). We now turn to the derivation of the algebra of the fields $U_m^{(J)}$. The quantum group picture leads us to expect that there should in fact exist a closed exchange algebra for these operators which is related to the braiding of highest (lowest) weight representations of $U_q(sl(2))$ with continuous spin. Hence we start with an ansatz of the form

$$U_m^{(J)}(\sigma)U_m^{(J')}\left(\sigma'\right) = \sum_{m_1,m_2} R(J, J'; \omega)_{m_1 m_2}^{m_1 m_2} U_m^{(J)}(\sigma')U_m^{(J)}(\sigma)$$

for $\pi > \sigma' > \sigma > 0$,

and

$$U_m^{(J)}(\sigma)U_m^{(J')}\left(\sigma'\right) = \sum_{m_1,m_2} \tilde{R}(J, J'; \omega)_{m_1 m_2}^{m_1 m_2} U_m^{(J)}(\sigma')U_m^{(J)}(\sigma)$$

for $\pi > \sigma' > \sigma > 0$, \hspace{1cm} (16)

with the sums extending over non-negative integer $J + m_1$ resp. $J' + m_2$.

Comparing the monodromy properties of both sides of Eq.16, we conclude that, as in the half-integer spin case, the braiding matrix is nonzero only when

$$m_1 + m_2 = m + m' =: M$$

Since there are no null-vector decoupling equations for continuous $J$, we have to rely exclusively on the free field techniques just summarized. Fortunately, it turns out that the exchange of two $U_m^{(J)}$ operators can be mapped into an equivalent problem in one-dimensional quantum mechanics, and becomes just finite-dimensional linear algebra. Given the fact that the $U_m^{(J)}$ consist only of the ”tachyon operators” $V_{-J}^{(J)}$ resp. $V_1^{(-1)}$, the essential observation is to remember that

$$V_{-J}^{(J)}(\sigma)V_{-J'}^{(J')}\left(\sigma'\right) = e^{-i\frac{1}{2}J J'h(\sigma - \sigma')}V_{-J'}^{(J')}\left(\sigma'\right)V_{-J}^{(J)}(\sigma)$$

where $\epsilon(\sigma - \sigma')$ is the sign of $\sigma - \sigma'$. This means that when commuting the tachyon operators in $U_m^{(J)}(\sigma')$ through those of $U_m^{(J)}(\sigma)$, one only encounters phase factors of the form $e^{\pm i\alpha \beta h}$ resp. $e^{\pm \alpha \beta h}$, with $\alpha$ equal to $J$ or $-1$, $\beta$ equal to $J'$ or $-1$, since we take $\sigma, \sigma' \in [0, \pi]$. Hence we are led to decompose the integrals defining the screening charges $S$ into pieces which commute with
each other and with $V^{(J)}_{\sigma} (\sigma), V^{(J')}_{\sigma'} (\sigma')$ up to one of the above phase factors. We consider explicitly only the case $0 < \sigma < \sigma' < \pi$ and write

$$S(\sigma) = S_{\sigma \sigma'} + S_\Delta, \quad S(\sigma') = S_{\sigma \sigma'} + k(\varpi) S_\Delta \equiv S_{\sigma \sigma'} + \hat{S}_\Delta,$$

$$S_{\sigma \sigma'} := k(\varpi) \int_0^\sigma V^{(-1)}_1(\tilde{\sigma}) d\tilde{\sigma} + \int_{\sigma'}^{2\pi} V^{(-1)}_1(\tilde{\sigma}) d\tilde{\sigma},$$

$$S_\Delta := \int_\sigma^\sigma V^{(-1)}_1(\tilde{\sigma}) d\tilde{\sigma}, \quad k(\varpi) := e^{2i\varpi(\pi+1)} \tag{19}$$

Using Eq.\ref{eq:18}, we then get the following simple algebra for $S_{\sigma \sigma'}, S_\Delta, \hat{S}_\Delta$:

$$S_{\sigma \sigma'} S_\Delta = q^{-2} S_\Delta S_{\sigma \sigma'}, \quad S_{\sigma \sigma'} \hat{S}_\Delta = q^2 \hat{S}_\Delta S_{\sigma \sigma'}, \quad S_\Delta \hat{S}_\Delta = q^4 \hat{S}_\Delta S_\Delta, \tag{20}$$

and their commutation properties with $V^{(J)}_{\sigma} (\sigma), V^{(J')}_{\sigma'} (\sigma')$ are given by

$$V^{(J)}_{\sigma} (\sigma) S_{\sigma \sigma'} = q^{-2J} S_{\sigma \sigma'} V^{(J)}_{\sigma} (\sigma), \quad V^{(J')}_{\sigma'} (\sigma') S_{\sigma \sigma'} = q^{-2J'} S_{\sigma \sigma'} V^{(J')}_{\sigma'} (\sigma'),$$

$$V^{(J)}_{\sigma} (\sigma) S_\Delta = q^{-2J} S_\Delta V^{(J)}_{\sigma} (\sigma), \quad V^{(J')}_{\sigma} (\sigma) \hat{S}_\Delta = q^{-2J} \hat{S}_\Delta V^{(J')}_{\sigma} (\sigma),$$

$$V^{(J')}_{\sigma'} (\sigma') S_\Delta = q^{2J'} S_\Delta V^{(J')}_{\sigma'} (\sigma'), \quad V^{(J')}_{\sigma'} (\sigma') \hat{S}_\Delta = q^{-2J} \hat{S}_\Delta V^{(J')}_{\sigma'} (\sigma'). \tag{21}$$

Finally, all three screening pieces obviously shift the zero mode in the same way:

$$\begin{align*}
S_{\sigma \sigma'} & \\
S_\Delta & \\
\hat{S}_\Delta &
\end{align*} \quad \varpi = (\varpi + 2) \begin{cases} 
S_{\sigma \sigma'} \\
S_\Delta \\
\hat{S}_\Delta
\end{cases} \tag{22}$$

Using Eqs.\ref{eq:21} we can commute $V^{(J)}_{\sigma} (\sigma)$ and $V^{(J')}_{\sigma'} (\sigma')$ to the left on both sides of Eq.\ref{eq:16} so that they can be cancelled. Then we are left with

$$\left( q^{-2J} S_\Delta + q^{2J'} S_{\sigma \sigma'} \right)^{J+J'} \left( \hat{S}_\Delta + S_{\sigma \sigma'} \right)^{J'+J''} =$$

$$\sum_{m_1, m_2} R(J, J'; \varpi + 2(J + J')) m_1^{m_1} m_2^{m_2} q^{-2J} q^{-2J'} (q^{2J} S_{\sigma \sigma'} + q^{-2J} \hat{S}_\Delta)^{J+J'}.$$  

It is apparent from Eq.\ref{eq:23} that the braiding problem of the $U^{(J)}_m$ operators is governed by the Heisenberg-like algebra Eq.\ref{eq:20}, characteristic of one-dimensional quantum mechanics. However, to see this structure emerge, we had to decompose the screening charges $S(\sigma), S(\sigma')$ in a way which depends
on both positions $\sigma, \sigma'$; hence the embedding of this Heisenberg algebra into the 1+1 dimensional field theory is somewhat nontrivial. To evaluate Eq.\[23\] we could sort both sides of the equation in powers of $S_{\sigma\sigma'}, S_{\Delta}$ and then compare coefficients. This indeed can be carried out straightforwardly, upon observing that $S_{\sigma\sigma'}$ and $S_{\Delta}$ resp. $\hat{S}_{\Delta}$ behave like the components $a, b$ of a vector in the quantum plane, with $ba = abq^2$, so that one can make use of the q-binomial formula

$$(a + b)^N = \sum_{\nu=0}^{N} \binom{N}{\nu} q^{(N-\nu)\nu} a^\nu b^{N-\nu}, \quad \binom{N}{\nu} := \frac{[N]!}{[N-\nu]![\nu]!}$$

$$(\binom{N}{\nu})$$ is a q-deformed binomial coefficient, with

$$[\nu]! = [1][2] \cdots [\nu], \quad [x] := \sin(hx)/\sin(h)$$

denoting q-factorials resp. q-numbers. For our purposes, however, another form of the equations is better suited, which is obtained by choosing the following simple representation of the algebra Eq.\[20\] in terms of one-dimensional quantum mechanics ($y$ and $y'$ are arbitrary complex numbers):

$$S_{\sigma\sigma'} = y' e^{2Q}, \quad S_{\Delta} = y e^{2Q-P}, \quad \hat{S}_{\Delta} = y e^{2Q+P}, \quad [Q, P] = ih. \quad \text{(25)}$$

The third relation in Eq.\[23\] follows from the second one in view of $\hat{S}_{\Delta} = k(\varpi) S_{\Delta}$ (cf. Eq.\[13\]). This means we are identifying here $P \equiv ih\varpi$ with the zero mode of the original problem. Using $e^{2Q+P} = e^{P} e^{2Q} q^{c}$ we can commute all factors $e^{2Q}$ to the right on both sides of Eq.\[23\] and then cancel them. This leaves us with

$$\prod_{j=1}^{J+m} (y' q^{2J' + yq^{-2J+2j-1}}) \prod_{\ell=1}^{J'+m'} (y' + yq^{\varpi-2J'+2m+2\ell-1}) =$$

$$\sum_{m_1} R(J, J'; \varpi)^{m_2 \text{m}_1} q^{-2JJ'} \times$$

$$\prod_{j=1}^{J'+m_2} (y' q^{2J'} + yq^{\varpi+4J-2J'+2j-1}) \prod_{\ell=1}^{J+m_1} (y' + yq^{-(\varpi-2J+2m_2+2\ell-1)}) \quad \text{(26)}$$

where we have shifted back $\varpi + 2(J + J') \rightarrow \varpi$ compared to Eq.\[23\]. Since the overall scaling $y \rightarrow \lambda y, y' \rightarrow \lambda y'$ only gives back Eq.\[17\], we can set
$y' = 1$. By putting $y$ equal to the zeros of the first or the second product on the RHS of Eq.26, plus one other arbitrary value, we obtain a linear system of equations for $R$ in triangular form, which shows that the solution of Eq.26 is unique for any fixed $J, J', m, m'$. We will now demonstrate that it is given by a straightforward extension of the R-matrix of ref.[8] (see Eq.3). This extension can be written in terms of orthogonal polynomials and we prove its correctness by showing that Eq.26 is equivalent to the orthogonality relations for these polynomials. According to the beginning of this section, the braiding matrices of the $V$ and $U$ fields are related by

$$R(J, J'; \omega)^{m_2m_1}_{m_m} = \hat{R}(J, J'; \omega)^{m_2m_1}_{m_m}$$

(27)

The normalizations $I^{(J)}_m$ can be computed by making use of the Fateev-Dotsenko integration formulae[17]. One finds, letting $n = J + m$,

$$I^{(J)}_m(\omega) = i^n \prod_{l=1}^{n} \left\{ e^{i\pi\beta(l-1)}(1 - e^{2\pi i(\gamma + \beta(l-1))}) \right\} \times$$

$$\prod_{l=1}^{n} \left\{ \frac{\Gamma(1 - \beta)\Gamma(1 + \gamma + (l - 1)\beta)\Gamma(1 + \alpha + (l - 1)\beta)}{\Gamma(1 - l\beta)\Gamma(2 + \gamma + \alpha + (n - 2 + l)\beta)} \right\}$$

$$\alpha = 2J\frac{h}{\pi}, \quad \beta = -\frac{h}{\pi}, \quad \gamma = \frac{h}{\pi}(\omega + 2m - 1) - 1.$$  

(28)

This relation is valid for arbitrary $J, \omega$ and $J + m = n$ a non-negative integer. We remark that the divergence of the product $[S]^{J+m}$ appears as the first pole in $h$ in the formula just written. It shows where the integral representation Eq.[[19]] breaks down. However, Eq.[28] has meaning beyond this point by the usual analytic continuation of the gamma function. Indeed, the general formula Eq.[27] is valid for arbitrary $h$ and will give rise to a 3-point function which is analytic in $h$. Hence Eq.[28] is valid for all $h$. Next we consider the formula for the coupling constant $g^{J_1J_2}_{J_1J_2}$ of ref.[8]. It is given by (letting $F(z) \equiv \Gamma(z)/\Gamma(1 - z)$)

$$g^{J_1J_2}_{J_1J_2} = \prod_{k=1}^{J_1+J_2-J_{12}} \sqrt{F(1 + (2J_k - k + 1)h/\pi)} \times$$

8 details will be given elsewhere.
\[
\left[ F(1 + (2J_2 - k + 1)h/\pi)F(-1 - (2J_{12} + k + 1)h/\pi) \right]^{F(1 + kh/\pi)} \tag{29}
\]

It is easily seen that for the general \( J \) case, \( J_1 + J_2 - J_{12} \) remains an integer, so that this expression directly makes sense. Thus we only need to specify the proper continuation of the 6j-symbol. Using the relation between the 6j-symbol and the \( 4F_3 \) q-hypergeometric function\(^9\), one can write

\[
\{J \atop J' \atop x \atop x} \{x + m_2 \atop x + m \} = \sqrt{[2x + 2m_2 + 1][2x + 2m + 1]} \times \\
\Delta(J, x + M, x + m_2)\Delta(J, x, x + m)\Delta(J', x, x + m_2)\Delta(J', x + m, x + M) \times \\
[2x + N + 1]! [2x + m_2 + m - J - J']! [J + m_1]! [J - m]! [J' + m']! [J' - m_2]! \times \\
\frac{1}{[m_2 - m']!} 4F_3 \left( m - J, -m_2 - J', -m_2 + J - m_1, -J - m_2, -J' + m'; q, 1 \right) \tag{30}
\]

with

\[
N = J + J' + M, \quad \Delta(a, b, c) = \sqrt{\frac{[a - b + c]![a + b]![a + b - c]!}{[a + b + c]![a + b + c + 1]!}}
\]

The RHS of Eq.(30) makes sense for arbitrary \( J \). The q-hypergeometric function is defined as

\[
4F_3(a, b, c, d; q, \rho) = \sum_{n=0}^{\infty} \frac{[a]_n [b]_n [c]_n [d]_n}{[e]_n [f]_n [g]_n [n]!} \rho^n,
\]

\[
[a]_n := [a][a+1] \cdots [a+n-1], \quad [a]_0 := 1 \tag{31}
\]

In the present context, we have

\[
a = m - J, \quad b = m_2 - J', \quad c = -J - m_1, \quad d = -n' = -(J' + m'), \quad e = -2x - N - 1,
\]

\[
f = m_2 - m' + 1, \quad g = 1 + a + b + c + d - e - f \tag{32}
\]

\(^9\)our conventions for q-hypergeometric functions coincide with the ones of ref.\[^8\].
To make contact with orthogonal polynomials, we bring the $\mathbf{4F3}$ into a different form by means of the q-version\(^{[10]}\) of a well-known transformation formula for balanced $\mathbf{4F3}$ functions:

$$\mathbf{4F3}\left(\begin{array}{c}
a, b, c, \\
e, f, 1+a+b+c+d-e-f\end{array}\right), n^\prime \right) = \frac{[f-c]_{n^\prime} [f+e-a-b]_{n^\prime}}{[f]_{n^\prime} [f+e-a-b-c]_{n^\prime}} \times \mathbf{4F3}\left(\begin{array}{c}
e-a, e-b, c, \\
e+e-a-b, 1+c-f-n^\prime\end{array}\right), q, 1 \right)$$

(33)

where $n^\prime$ is a non-negative integer. The $\mathbf{4F3}$ on the RHS of Eq.33 can now be identified with an Askey-Wilson (or Racah) polynomial\(^{[22]}\):

$$p_{n^\prime}(\mu(z); \alpha, \beta, \gamma, \delta; q) = \mathbf{4F3}\left(\begin{array}{c}
e-a, e-b, c, \\
e+e-a-b, 1+c-f-n^\prime\end{array}\right), q, 1 \right)$$

where

$$\mu(z) = q^{-2z} + q^{2(z+c+e-b)}, z = -c = J + m_1,$$

and

$$\alpha = e - 1, \beta = d - a, \gamma = c + d - f, \delta = e + f - b - d - 1.$$  \hspace{1cm} (34)

$n^\prime = J + m'$ is the degree in the variable $\mu(z)$. Note that $p_{n^\prime}(\mu(z))$ really depends on $z$ only via $\mu(z)$; the other coefficients are functions only of $J, J'$ and $M$. The Askey-Wilson polynomials satisfy orthogonality relations of the form \(^{[22]}\):

$$\sum_{z=0}^{N} p_n(\mu(z)) p_m(\mu(z)) w(z) = 0 \quad \text{for } m \neq n.$$  \hspace{1cm} (35)

Here $N = J + J' + M$. The weight function $w(z)$ is defined as

$$w(z; \alpha, \beta, \gamma, \delta; q) = \frac{[2z+1+\gamma+\delta][\gamma+\delta+1]z[\alpha+1]z[\beta+\delta+1]z[\gamma+1]z}{[\gamma+\delta+1][z][\gamma+\delta-\alpha+1]z[\gamma-\beta+1]z[\delta+1]z}.$$  \hspace{1cm} (36)

where $\alpha, \beta, \gamma, \delta$ are the same coefficients as in the definition of $p_{n^\prime}$.

\(^{10}\)It can be proven\(^{[10]}\) exactly along the same lines using the method explained, for instance in ref.\(^{[20]}\) of the classical formula\(^{[21]}\)
Thus for any fixed $J, J', m, m'$ we can represent the braiding matrix $R(J, J'; m, m')$ defined by Eqs.3 and 27 – 34 by the Askey-Wilson polyno-
mial $p_{n'}(\mu(z))$, up to a prefactor which is independent of $z$. Inserting into Eq.26, and putting $y$ equal to one of the zeros of the LHS,

$$y = -q^{\varphi + 2J' - 2J + 2j_0 - 1}, \quad j_0 = 1 \cdots J + m$$

resp.

$$y = -q^{(\varphi - 2J' + 2m + 2l_0 - 1)}, \quad l_0 = 1 \cdots J' + m'$$

(37)

we arrive at the following homogeneous relations:

$$0 = \sum_{z=0}^{N} [2x + j_0 + 1]_{N-z} [J' - J - M + j_0 - \frac{1}{2}]_{z} \frac{(-1)^z}{[-2x - 2M]_z} p_{n'}(\mu(z)) w(z),$$

resp.

$$0 = \sum_{z=0}^{N} [J' - m - M - 2x - l_0 - 1]_{z} [J - m - l_0 + 1]_{N-z} \frac{(-1)^z}{[-2x - 2M]_z} p_{n'}(\mu(z)) w(z),$$

with

$$j_0 = 1 \cdots J + m, \quad l_0 = 1 \cdots J' + m', \quad N = J + J' + M, \quad n' = J' + m'$$

(38)

Next, one proves by induction in $n' = J' + m' (J, J' and N fixed)$ that the conditions Eqs.38 are equivalent to the relations Eqs.35. Considering the first relation in Eq.38, we have to show that the q-products in front of $p_{n'}(\mu(z)) w(z)$ are given by a linear combination of $p_k(\mu(z))$ with $k \neq n'$. Indeed we shall prove that

$$[2x + j_0' - n' + 1]_{N-z} [J' - M - m' + j_0' - 1]_{z} \frac{(-1)^z}{[-2x - 2M]_z} = \sum_{k=n'+1}^{N} a_{j_0'k} p_k(\mu(z)),$$

with

$$j_0' = j_0 + n' = n' + 1 \cdots N$$

(39)

We start at $n' = N - 1$ and go downward. For $n' = N - 1$, where $j_0' = N$ is the only allowed value, the q-product on the left must be proportional to
\( p_N(\mu(z)) \). The latter is given by Eq.34 with the \(_4F_3\) collapsing into a \(_3F_2\) since two of the coefficients are equal:

\[
p_N(\mu(z)) = {3F_2 \left( \begin{array}{l}
-(J' - J + M + 2x + 1),
-n',
-(2J + 2M + 2x + 1 - z) \\
-(N + 2x + 1),
-2M - 2x,
\end{array} \right) ; q, 1}
\]  

(40)

By means of a well-known relation for the \(_3F_2\) functions\[21\], this can be further reduced to a single product of \(q\)-numbers, so that

\[
p_N(\mu(z)) = \text{const} \cdot \frac{(2x + 2N - z) \cdots (J' - J - M + 1)_z (2J)_{-z} \cdots (2M + 2x + 1)_z}{(2x + 2N - z) \cdots (J' - J - M + 1)_z}
\]

(41)

where the constant is independent of \(z\). In view of Eq.39, this proves the induction start. Consider now the induction step \(m' \to m' - 1, m \to m + 1\). This amounts to changing \(j'_0 \to j'_0 + 1\) in Eq.39. The case \(j'_0 \leq N - 1\) is covered by the induction hypothesis. When \(j'_0 = N\), we have to show that

\[
\mu(z)p_n(\mu(z)) = A_n p_{n+1}(\mu(z)) + B_n p_n(\mu(z)) + C_n p_{n-1}(\mu(z))
\]

(43)

Since \(A_N = 0\), the induction step is complete. The proof of the second relation in Eq.38 is completely analogous. So we know now that our conjecture for \(R\) is correct up to normalization. The latter can be easily checked by putting \(y = -q^{\sigma + 2M - 1}\) in Eq.36. Then the products on the RHS vanish except when \(J + m_1 = 0\). Inserting for \(R\) as before and using \(p_n(\mu(0)) = 1 \forall n\), it is then a straightforward exercise to check that Eq.36 is also fulfilled in this inhomogeneous case. Thus we have established that the braiding of the \(V_m(J)\) resp. \(U_m(J)\) operators with general \(J\), and \(J + m\) a non-negative integer, is given by the extension Eq.30 or 34 of the R-matrix of \[8\]. In particular, of course, we reproduce the result for the half-integer case, without having taken recourse to the null-vector decoupling equation.
As an important consequence of this fact, one obtains that the construction presented in [7] for the Liouville exponentials Eq.1 with positive half-integer $J$ as local operators possesses an immediate continuation to arbitrary $J$: 

$$e^{-J\alpha_\Phi(\sigma, \tau)} = \sum_{n=0}^{\infty} \frac{B_{n-J}(\varpi)}{I_{n-J}(\varpi)I_{n-J}^{(j)}(\varpi)} U_{n-J}(x_+) U_{n-J}^{(j)}(x_-),$$  

where now one does not stop, since starting from $m = -J$, one does not reach $m = J$ after integer steps. One may see that there is no problem with extending the expression for $B_{n-J}(\varpi)$ to non-integer $J$, and this defines general exponentials of the Liouville field. Locality is guaranteed, just as in the half-integer case, by the fact that the $R$-matrix for the right-moving chiral fields is essentially the inverse of the $R$-matrix for the left movers, as can easily be shown also for our extension to general $J$. Details will be given elsewhere. Finally, we remark that our results can be rephrased in terms of the fields $\xi_{M}^{(J)}$ defined in ref.[2] which form representations of the quantum group, and which differ from the $U_{m}^{(J)}$ by a linear transformation:

$$\xi_{M}^{(J)} = \sum_{m} |\vec{J}, \varpi\rangle_{M}^{m} U_{m}^{(J)}$$

with suitable coefficients $|\vec{J}, \varpi\rangle_{M}^{m}$. Again, the basis transformation $|\vec{J}, \varpi\rangle_{M}^{m}$ can be extended straightforwardly to general $J$. Consequently, also the formula given in [7] for the Liouville exponentials in terms of the quantum group basis possesses an immediate extension.

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