RECOVERY OF SPARSEST SIGNALS VIA \( \ell^q \)-MINIMIZATION

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Abstract. In this paper, it is proved that every \( s \)-sparse vector \( x \in \mathbb{R}^n \) can be exactly recovered from the measurement vector \( z = Ax \in \mathbb{R}^m \) via some \( \ell^q \)-minimization with \( 0 < q \leq 1 \), as soon as each \( s \)-sparse vector \( x \in \mathbb{R}^n \) is uniquely determined by the measurement \( z \).

1. Introduction and Main Results

Define the norm \( \| x \|_q, 0 \leq q \leq \infty \), of a vector \( x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n \) by the number of its nonzero components when \( q = 0 \), the quantity \( (|x_1|^q + \cdots + |x_n|^q)^{1/q} \) when \( 0 < q < \infty \), and the maximum absolute value \( \max(|x_1|, \ldots, |x_n|) \) of its components when \( q = \infty \). We say that a vector \( x \in \mathbb{R}^n \) is \( s \)-sparse if \( \| x \|_0 \leq s \), i.e., the number of its nonzero components is less than or equal to \( s \).

In this paper, we consider the problem of compressive sensing in finding \( s \)-sparse solutions \( x \in \mathbb{R}^n \) to the linear system

\[
Ax = z
\]

via solving the \( \ell^q \)-minimization problem:

\[
\min \| y \|_q \quad \text{subject to } Ay = z
\]

where \( 0 < q \leq 1 \), \( 2 \leq 2s \leq m \leq n \), \( A \) is an \( m \times n \) matrix, and \( z \in \mathbb{R}^m \) is the observation data (1, 5, 7, 9, 12, 14).

One of the basic questions about finding \( s \)-sparse solutions to the linear system (1.1) is under what circumstances the linear system (1.1) has a unique solution in \( \Sigma_s \), the set of all \( s \)-sparse vectors.

Proposition 1.1. (12 [15]) Let \( 2s \leq m \leq n \) and \( A \) be an \( m \times n \) matrix. Then the following statements are equivalent:

(i) The measurement \( Ax \) uniquely determines each \( s \)-sparse vector \( x \).

(ii) There is a decoder \( \Delta : \mathbb{R}^m \mapsto \mathbb{R}^n \) such that \( \Delta(Ax) = x \) for all \( x \in \Sigma_s \).

(iii) The only \( 2s \)-sparse vector \( y \) that satisfies \( Ay = 0 \) is the zero vector.

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(iv) There exist positive constants \( \alpha_{2s} \) and \( \beta_{2s} \) such that

\[
\alpha_{2s} \|x\|_2 \leq \|Ax\|_2 \leq \beta_{2s} \|x\|_2 \quad \text{for all } x \in \Sigma_{2s}.
\]

The first contribution of this paper is to provide another equivalent statement:

(v) There exists \( 0 < q \leq 1 \) such that the decoder \( \Delta : \mathbb{R}^m \rightarrow \mathbb{R}^n \) defined by

\[
\Delta(x) := \arg\min_{y \in \mathbb{R}^n} \|y\|_q
\]

satisfies \( \Delta(Ax) = x \) for all \( x \in \Sigma_{s} \).

The implication from (v) to (ii) is obvious. Hence it suffices to prove the implication from (iv) to (v). For this, we recall the restricted isometry property of order \( s \) for an \( m \times n \) matrix \( A \), i.e., there exists a positive constant \( \delta \in (0, 1) \) such that

\[
(1 - \delta)\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1 + \delta)\|x\|_2^2 \quad \text{for all } x \in \Sigma_s.
\]

The smallest positive constant \( \delta \) that satisfies (1.3), to be denoted by \( \delta_s(A) \), is known as the restricted isometry constant \( [5, 7] \). Notice that given a matrix \( A \) that satisfies (1.3), its rescaled matrix \( B := \sqrt{2/(\alpha_{2s}^2 + \beta_{2s}^2)}A \) has the restricted isometry property of order 2\( s \) and its restricted isometry constant is given by \( (\beta_{2s}^2 - \alpha_{2s}^2)/(\alpha_{2s}^2 + \beta_{2s}^2) \). Therefore the implication from (iv) to (v) further reduces to establishing the following result:

**Theorem 1.2.** Let integers \( m, n \) and \( s \) satisfy \( 2s \leq m \leq n \). If \( A \) is an \( m \times n \) matrix with \( \delta_{2s}(A) \in (0, 1) \), then there exists \( 0 < q \leq 1 \) such that any \( s \)-sparse vector \( x \) can be exactly recovered by solving the \( \ell^q \)-minimization problem:

\[
\min \|y\|_q \quad \text{subject to } Ay = Ax.
\]

The above existence theorem about \( \ell^q \)-minimization is established in \([17]\) and \([9]\) under a stronger assumption that \( \delta_{2s+2}(A) \in (0, 1) \) and \( \delta_{2s+1}(A) \in (0, 1) \) respectively, as it is obvious that \( \delta_{2s}(A) \leq \delta_{2s+1}(A) \leq \delta_{2s+2}(A) \) for any \( m \times n \) matrix \( A \).

Given integers \( s, m \) and \( n \) satisfying \( 2s \leq m \leq n \) and an \( m \times n \) matrix \( A \), define

\[
q_s(A) := \sup \{ q \in [0, 1] \mid \text{any vector } x \in \Sigma_s \text{ can be exactly recovered by solving the } \ell^q - \text{minimization problem (1.6)} \}.
\]

Then \( q_s(A) > 0 \) whenever \( \delta_{2s}(A) < 1 \) by Theorem 1.2. It is also known that any \( s \)-sparse vector \( x \in \mathbb{R}^n \) can be exactly recovered by solving the \( \ell^q \)-minimization problem (1.6) whenever \( q < q_s(A) \) \([18]\). This establishes the equivalence among different \( q \in [0, q_s(A)] \) in recovering \( s \)-sparse solutions via solving the \( \ell^q \)-minimization problem (1.6). Hence in order to recover sparsest vector \( x \) from the measurement \( Ax \), one may solve the
\( \ell^q \)-minimization problem (1.6) for some \( 0 < q \leq 1 \) rather than the \( \ell^0 \)-minimization problem. Empirical evidence ([10, 22, 23]) strongly indicates that solving the \( \ell^q \)-minimization problem with \( 0 < q \leq 1 \) takes much less time than with \( q = 0 \).

The \( \ell^0 \)-minimization problem is a combinatorial optimization problem and NP-hard to solve [20], while on the other hand the \( \ell^1 \)-minimization is convex and polynomial-time doable to find local minimizer [19]. Various algorithms have been developed to solve the \( \ell^1 \)-minimization problem due to the nonconvexity and non-smoothness. In fact, it is NP-hard to find a global minimizer in general but polynomial-time doable to find local minimizer [19]. Various algorithms have been developed to solve the \( \ell^q \)-minimization problem (1.6), see for instance [8, 11, 14, 17, 21].

For any \( \delta \in (0, 1) \), define

\[
q_{\text{max}}(\delta; m, n, s) := \inf_{\delta_2(A) \leq \delta} q_s(A).
\]

Then given any positive number \( q < q_{\text{max}}(\delta; m, n, s) \) and any \( m \times n \) matrix \( A \) with \( \delta_2(A) \leq \delta \), any vector \( x \in \Sigma_s \) can be exactly recovered by solving the \( \ell^q \)-minimization problem (1.6). For any \( 0 < q \leq 1 \) and sufficiently small \( \varepsilon \), matrices \( A_\varepsilon \) of size \( (n - 1) \times n \) are constructed in [13] such that \( \delta_2(A_\varepsilon) < \frac{n}{2q - q - \eta_0} + \varepsilon \) and there is an \( s \)-sparse vector which cannot be recovered exactly by solving the \( \ell^q \)-minimization problem (1.6) with \( A \) replaced by \( A_\varepsilon \), where \( \eta_0 \) is the unique positive solution to \( \eta_0^{2/q} + 1 = 2(1 - \eta_0)/q \). The above construction of matrices for which the \( \ell^q \)-minimization fails to recover \( s \)-sparse vectors, together with the asymptotic estimate \( \eta_0 = 1 - qx_0 + o(q) \) as \( q \to 0 \), gives that

\[
\limsup_{\delta \to 1-} \frac{q_{\text{max}}(\delta; n - 1, n, s)}{1 - \delta} \leq \lim_{q \to 0^+} q(2 - q - \eta_0) = \frac{1}{2x_0 - 1} \approx 3.5911,
\]

where \( x_0 \) is the unique positive solution of the equation \( e^{-2x} = 2x - 1 \). The second contribution of this paper is a lower bound estimate for \( q_{\text{max}}(\delta; m, n, s) \) as \( \delta \to 1- \).
Theorem 1.3. Let $q_{\text{max}}(\delta; m, n, s)$ be defined as in (1.8). Then

\begin{equation}
\liminf_{\delta \to 1^-} \frac{q_{\text{max}}(\delta; m, n, s)}{1 - \delta} \geq \frac{e}{4} \approx 0.6796.
\end{equation}

Denote by $\mathbf{v}_S$ the vector which equals to $\mathbf{v} \in \mathbb{R}^n$ on $S$ and vanishes on the complement $S^c$ where $S \subset \{1, \ldots, n\}$. We say that an $m \times n$ matrix $\mathbf{A}$ has the null space property of order $s$ in $\ell^q$ if there exists a positive constant $\gamma$ such that

\begin{equation}
\| \mathbf{h}_S \|_q \leq \gamma \| \mathbf{h}_{S^c} \|_q
\end{equation}

hold for all $\mathbf{h}$ satisfying $\mathbf{A}\mathbf{h} = \mathbf{0}$ and all sets $S$ with its cardinality $\#S$ less than or equal to $s$ (12). The minimal constant $\gamma$ in (1.11) is known as the null space constant.

For $0 < q \leq 1$ and $\delta \in (0, 1)$, define

\begin{equation}
\alpha(q, \delta) := \inf_{0 < r_0 < 1} \max \left\{ \frac{1 + r_0\delta}{(1 + r_0\delta)^{1/q}}, \sup_{2 \leq y \leq 1} \frac{2y}{\sqrt{2(1 - r_0)\delta}} \right\},
\end{equation}

\begin{equation}
\alpha(q, \delta) := \sup_{2 \leq y \leq 1} \frac{3y}{(1 + y)^{1/q}}, \quad \sup_{1 \leq y \leq 1} \frac{2y}{(1 + y)^{1/q}} \right\}.
\end{equation}

The third contribution of this paper is the following result about the null space property of an $m \times n$ matrix.

Theorem 1.4. Let $q$ be a positive number in $(0, 1]$, integers $m, n$ and $s$ satisfy $2s \leq m \leq n$, $\mathbf{A}$ be an $m \times n$ matrix with $\delta_{2s}(\mathbf{A}) \in (0, 1)$, and set

\begin{equation}
\delta_1 := \left( \frac{1 - \delta_{2s}(\mathbf{A})}{1 + \delta_{2s}(\mathbf{A})} \right)^{1/2}.
\end{equation}

Then $\mathbf{A}$ has the null space property of order $s$ in $\ell^q$, and its null space constant is less than or equal to $\alpha(q, \delta_1)/\delta_1$.

The fourth contribution of this paper is to show that one can stably reconstruct a compressive signal from noisy observation under the hypothesis that

\begin{equation}
\alpha(q, \delta_1) < \delta_1.
\end{equation}

Theorem 1.5. Let $m, n$ and $s$ be integers with $2s \leq m \leq n$, $\mathbf{A}$ be an $m \times n$ matrix with $\delta_{2s}(\mathbf{A}) \in (0, 1)$, $\epsilon \geq 0$, $q \in (0, 1]$ satisfy (1.14) with $\delta_1$ given in (1.13), and $\mathbf{x}^*$ be the solution of the $\ell^q$-minimization problem:

\begin{equation}
\min_{\mathbf{x} \in \mathbb{R}^n} \| \mathbf{x} \|_q \text{ subject to } \| \mathbf{A}\mathbf{x} - \mathbf{y} \|_2 \leq \epsilon
\end{equation}

where $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{z}$ is the observation corrupted with unknown noise $\mathbf{z}$, $\| \mathbf{z} \|_2 \leq \epsilon$ and $\mathbf{x}$ is the object we wish to reconstruct. Then

\begin{equation}
\| \mathbf{x}^* - \mathbf{x} \|_2 \leq C_0 s^{1/2 - 1/q} \| \mathbf{x} - \mathbf{x}_s \|_q + C_1 \epsilon,
\end{equation}

and

\begin{equation}
\| \mathbf{x}^* - \mathbf{x} \|_q \leq C_2 \| \mathbf{x} - \mathbf{x}_s \|_q + C_3 s^{1/q - 1/2} \epsilon,
\end{equation}

where $\mathbf{x}_s$ is the solution of the $\ell^2$-minimization problem.
where \( x_s \) be the best \( s \)-sparse vector in \( \mathbb{R}^n \) to approximate \( x_0 \), i.e.,
\[
\|x_s - x\|_q = \inf_{x' \in \Sigma_s} \|x' - x\|_q
\]
and \( C_i, 0 \leq i \leq 3 \), are positive constants independent on \( \epsilon, x \) and \( s \).

The stable reconstruction of a compressive signal from its noisy observation is established under various assumptions on the restricted isometry constant, for instance, \( \delta_{3s}(A) + 3\delta_{4s}(A) < 2 \) and \( q = 1 \) in [5], and \( \delta_{2s}(A) < \sqrt{2} - 1 \) and \( q = 1 \) in [4], \( \delta_{2t}(A) < 2(\sqrt{2} - 1)(t/s)^{1/q} - 1/(1 + 2(\sqrt{2} - 1)(t/s)^{1/q} - 1/2) \) for some \( t \geq s \) and \( 0 < q \leq 1 \) in [17], and \( \delta_{ks}(A) + k^{2/p - 1} \delta_{k+1}s(A) < k^{3/q} - 1 \) for some \( k \in \mathbb{Z}/s \) and \( 0 < q \leq 1 \) in [22, 23].

As an application of Theorem 1.5, any \( s \)-sparse vector \( x \) can be exactly recovered by solving the \( \ell^q \)-minimization problem (1.6) when \( q \in (0, 1] \) satisfies (1.14).

**Corollary 1.6.** Let \( m, n \) and \( s \) be integers with \( 2s \leq m \leq n \), \( A \) be an \( m \times n \) matrix with \( \delta_{2s}(A) \in (0, 1) \), \( \epsilon \geq 0 \), \( q \in (0, 1] \) satisfy (1.14) with \( \delta_1 \) given in (1.13). Then any \( s \)-sparse vector \( x \) can be exactly recovered by solving the \( \ell^q \)-minimization problem (1.6).

Let
\[
q_{\text{succ}}(\delta) = \bar{q}_{\text{max}}\left(\sqrt{(1 - \delta)/(1 + \delta)}\right)
\]
where \( \bar{q}_{\text{max}}(\delta_1) = \sup\{q \in (0, 1]| a(q, \delta_1) < \delta_1\} \), and let \( q_{\text{fail}}(\delta) \) be the solution of the equation
\[
\left(\frac{(2 - q)\delta}{1 + \delta}\right)^{2/q} + 1 = \frac{2 - 2\delta + 2q\delta}{q + q\delta}
\]
if it exists and be equal to one otherwise. Then by Theorem 1.5 any \( s \)-sparse vector \( x \) can be exactly recovered by solving the \( \ell^q \)-minimization problem (1.6) when \( q < q_{\text{succ}}(\delta_{2s}(A)) \), while by (1.13) there exists a matrix \( A \) with \( \delta_{2s}(A) \leq \delta \) and an \( s \)-sparse vector \( x \) such that the vector \( x \) cannot be exactly recovered by solving the \( \ell^q \)-minimization problem (1.6) when \( q > q_{\text{fail}}(\delta) \). The functions \( q_{\text{succ}}(\delta) \) and \( q_{\text{fail}}(\delta) \) are plotted in Figure 1.

2. Proofs

In this section, we give the proofs of Theorems 1.2, 1.3, 1.4 and 1.5.

**2.1. Proof of Theorem 1.4.** To prove Theorem 1.4 we need three technical lemmas.

**Lemma 2.1.** Let \( 0 < q \leq 1, 0 \leq c \leq 1 \) and \( a, b > 0 \). Then
\[
(2.1) \quad a + \sum_{k=1}^{m} t_k \leq \max \left\{ \max_{1 \leq k \leq m} \frac{k + a}{(k + b)^{1/q}}, \frac{a + c}{(b + c^q)^{1/q}} \right\} \left( b + \sum_{k=1}^{m} t_k^q \right)^{1/q}
\]
holds for any \( (t_1, \ldots, t_m) \in [0, 1]^m \) with \( t_1 + \cdots + t_m \geq c \).
\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure1}
\caption{The function $q_{\text{succ}}(\delta)$ is plotted in continuous line, while the function $q_{\text{fail}}(\delta)$ is plotted in dashed line}
\end{figure}

\textbf{Proof.} Define

\begin{equation}
F_{q,a,b,c}(m, n) = \sup_{(t_1, \ldots, t_m) \in [0,1]^m, \ t_1 + \cdots + t_m \geq c} \frac{n + a + \sum_{k=1}^m t_k}{(n + b + \sum_{k=1}^m t_k^q)^{1/q}}.
\end{equation}

By the method of Lagrange multiplier, the function $(n + a + \sum_{k=1}^m t_k)(n + b + \sum_{k=1}^m t_k^q)^{-1/q}$ attains its maximum on the boundary or on those points $(t_1, \ldots, t_m)$ whose components are the same, i.e.,

\begin{equation}
F_{q,a,b,c}(m, n) = \max \left\{ F_{q,a,b,0}(m - 1, n + 1), F_{q,a,b,c}(m - 1, n), \sup_{c/m \leq t \leq 1} \frac{n + a + mt}{(n + b + mt^q)^{1/q}} \right\}.
\end{equation}

As the function $(n + a + mt)(n + b + mt^q)^{-1/q}$ has at most one critical point and the second derivative at that critical point (if it exists) is positive, we then have

\begin{equation}
F_{q,a,b,c}(m, n) = \max \left\{ F_{q,a,b,0}(m - 1, n + 1), F_{q,a,b,c}(m - 1, n), \frac{n + m + a}{(n + m + b)^{1/q}}, \frac{n + a + c}{(n + b + m^{1-q}c^q)^{1/q}} \right\}.
\end{equation}
Applying (2.3) iteratively we obtain

\[ F_{q,a,b,c}(m,n) = \max \left\{ F_{q,a,b,0}(m-2, n+2), F_{q,a,b,0}(m-2, n+1), \right\} \]

\[ = \max \left\{ F_{q,a,b,0}(m-2, n), \frac{n + 1 + a}{(n + 1 + b)^{1/q}}, \frac{n + m - 1 + a}{(n + m - 1 + b)^{1/q}}, \frac{n + m + a}{(n + m + b)^{1/q}}, \frac{n + a + c}{(n + b + (m - 1)^{1-q}c)^{1/q}} \right\} \]

\[ = \ldots \]

\[ = \max \left\{ F_{q,a,b,0}(1, n + m - 1), \cdots, F_{q,a,b,0}(1, n + 1), \right\} \]

\[ = \max \left\{ \max_{1 \leq k \leq m} \frac{n + k + a}{(n + k + b)^{1/q}}, \frac{n + a + c}{(n + b + (m - 1)^{1-q}c)^{1/q}} \right\}. \]

(2.4)

Then the conclusion (2.1) follows by letting \( n = 0 \) in the above estimate. \( \square \)

**Lemma 2.2.** Let \( 0 < q \leq 1, c_1, c_2 \in [0, 1] \) and \( a_i, b_i > 0 \) for \( i = 1, 2, 3 \). Then

\[ a_1 + a_2x + a_3y + \sum_{k=1}^{m} t_k \leq \max \left\{ \frac{a_1 + a_2}{(b_1 + b_2)^{1/q}}, \frac{a_1 + a_2 c_1}{(b_1 + b_2 c_1)^{1/q}}, \sup_{0 \leq t \leq m} \frac{a_1 + a_2 + (a_3 + t c_2)}{(b_1 + b_2 + (b_3 + t c_2)^{1/q})^{1/q}} \right\} \times \left( b_1 + b_2 x^q + b_3 y^q + \sum_{k=1}^{m} t_k^q \right)^{1/q} \]

holds for all \( 0 \leq t_1, \ldots, t_m \leq y, c_1 \leq x \leq 1 \) and \( 0 \leq y \leq c_2 \).

**Proof.** Note that the maximum values of the function \((a + bt)/(c + dt^q)^{1/q}\) on any closed subinterval of \([0, \infty)\) are attained on its boundary. Then

\[ \frac{a_1 + a_2 x + a_3 y + \sum_{k=1}^{m} t_k}{(b_1 + b_2 x^q + b_3 y^q + \sum_{k=1}^{m} t_k^q)^{1/q}} = \sup_{0 \leq t \leq m} \frac{a_1 + a_2 x + (a_3 + t) y}{(b_1 + b_2 x^q + (b_3 + t) y^q)^{1/q}} \]

\[ = \max \left\{ \frac{a_1 + a_2 x}{(b_1 + b_2 x^q)^{1/q}}, \sup_{0 \leq t \leq m} \frac{a_1 + a_2 + (a_3 + t) c_2}{(b_1 + b_2 + (b_3 + t) c_2^q)^{1/q}} \right\} \]

\[ \leq \max \left\{ \frac{a_1 + a_2}{(b_1 + b_2)^{1/q}}, \frac{a_1 + a_2 c_1}{(b_1 + b_2 c_1)^{1/q}}, \sup_{0 \leq t \leq m} \frac{a_1 + a_2 + (a_3 + t) c_2}{(b_1 + b_2 + (b_3 + t) c_2^q)^{1/q}} \right\}. \]
Lemma 2.3. Let $0 < q \leq 1$, $s \geq 1$ be a positive integer, and let $\{a_j\}_{j \geq 1}$ be a finite decreasing sequence of nonnegative numbers with

\[
\sum_{k \geq 1} \left( \sum_{i=1}^{s} a_{ks+i}^2 \right)^{1/2} \geq \delta \left( \sum_{i=1}^{s} |a_i|^2 \right)^{1/2}
\]

for some $\delta \in (0, 1)$. Then

\[
\sum_{k \geq 1} \left( \sum_{i=1}^{s} a_{ks+i}^2 \right)^{1/2} \leq a(q, \delta) s^{1/2 - 1/q} \left( \sum_{j \geq 1} a_j^q \right)^{1/q},
\]

where $a(q, \delta)$ is defined as in (1.12).

Proof. Clearly the conclusion (2.7) holds when $a_{s+1} = 0$ for in this case the left hand side of (2.7) is equal to 0. So we may assume that $a_{s+1} \neq 0$ from now on. Let $r_0$ be an arbitrarily number in $(0, 1)$. To establish (2.7), we consider two cases.

**Case I:** $\sum_{k \geq 2} a_{ks+1} \geq r_0 \delta a_{s+1}$.

In this case,

\[
\frac{\sum_{k \geq 1} \left( \sum_{i=1}^{s} a_{ks+i}^2 \right)^{1/2}}{\left( \sum_{j \geq 1} a_j^q \right)^{1/q}} \leq \frac{s^{1/2} \sum_{k \geq 1} a_{ks+1}}{s^{1/q} \left( \sum_{k \geq 1} a_{ks+1}^q \right)^{1/q}}
\]

\[
= s^{1/2 - 1/q} \frac{1 + \sum_{k \geq 2} a_{ks+1} / a_{s+1}}{(1 + \sum_{k \geq 2} (a_{ks+1} / a_{s+1})^q)^{1/q}}
\]

\[
\leq s^{1/2 - 1/q} \max \left\{ \frac{1 + r_0 \delta}{(1 + r_0^q \delta q)^{1/q}}, \max_{k \geq 1} \frac{k + 1}{(k + 1)^{1/q}} \right\}
\]

\[
= s^{1/2 - 1/q} (1 + r_0 \delta)(1 + r_0^q \delta^q)^{-1/q},
\]

where the first inequality holds because $\{a_j\}_{j \geq 1}$ is a decreasing sequence of nonnegative numbers, the second inequality follows from Lemma 2.1, and the last equality is true as $(1 + t)(1 + t^q)^{-1/q}$ is a decreasing function on $(0, 1)$.

**Case II:** $\sum_{k \geq 2} a_{ks+1} < r_0 \delta a_{s+1}$.

Let $s_0$ be the smallest integer in $[1, s]$ satisfying $a_{s+s_0+1} / a_{s+1} \leq (s_0 / s)^{1/2}$. The existence and uniqueness of such an integer $s_0$ follow from the decreasing property of the sequence $\{a_{s+s_0+1} / a_{s+1}\}_{s_0=1}$, the increasing property of the sequence $\{(s_0 / s)^{1/2}\}_{s_0=1}$, and $a_{s+s_0+1} / a_{s+1} \leq (s_0 / s)^{1/2}$ when $s_0 = s$. Then from the decreasing property of the sequence $\{a_j\}_{j \geq 1}$ and the definition of the integer $s_0$ it follows that

\[
\frac{a_{s+s_0}}{a_{s+1}} \geq \left( \frac{s_0 - 1}{s} \right)^{1/2}
\]
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\[ \sqrt{2}s_0^{1/2} a_{s+1} \geq (s_0 - s_0) a_{s+1}^2 + (s_0 - s_0) a_{s+1}^2 \geq \left( \sum_{i=1}^{s} a_{s+i}^2 \right)^{1/2} \]

\[ \geq \delta \left( \sum_{i=1}^{s} a_i^2 \right)^{1/2} - \sum_{k \geq 2} \left( \sum_{i=1}^{s} a_{k+i}^2 \right)^{1/2} \]

\[ \geq \delta s_0^{1/2} a_{s+1} - s_0^{1/2} \sum_{k \geq 2} a_{k+s+1} \geq (1 - r_0) \delta s_0^{1/2} a_{s+1}, \]

which implies that

(2.10) \[ s_0 \geq \frac{(1 - r_0)^2 \delta^2}{2} a. \]

Applying the decreasing property of the sequence \( \{a_j\} \) and using the inequality \((\theta a^2 + (1 - \theta)b^2)^{1/2} \leq \theta^{1/2} a + (1 - \theta^{1/2})b\) where \( a \geq b \geq 0 \) and \( \theta \in [0, 1] \), we obtain

\[ s^{-1/2} \sum_{k \geq 1} \left( \sum_{i=1}^{s} a_{k+i}^2 \right)^{1/2} \]

\[ \leq s^{-1/2} (s_0 - 1) a_{s+1}^2 + a^2_{s+s_0} + (s_0 - s_0) a^2_{s+s_0+1} \]

\[ + s^{-1/2} \sum_{k \geq 2} (s_0 a^2_{k+s+1} + (s_0 - s_0) a^2_{k+s_0+1})^{1/2} \]

\[ \leq \sqrt{s_0} \left( \frac{s_0 - 1}{s} a_{s+1}^2 + \frac{1}{s_0} a_{s+s_0}^2 \right)^{1/2} + \left( 1 - \sqrt{s_0} \right) a_{s+s_0+1} \]

\[ + \sum_{k \geq 2} \left( \sqrt{s_0} a_{k+s+1} + \left( 1 - \sqrt{s_0} \right) a_{k+s_0+1} \right) \]

(2.11) \[ \leq \sqrt{s_0 - 1} a_{s+1} + \sqrt{s_0 - s_0 - 1} a_{s+s_0} + \sum_{k \geq 1} a_{k+s_0+1}, \]

and

\[ \sum_{j \geq 1} a_j^q \geq (s + 1) a_{s+1}^q + (s_0 - 1) a_{s+s_0}^q \]

(2.12) \[ + a_{s+s_0+1}^q + s \sum_{k \geq 2} a_{k+s_0+1}^q. \]
Combining (2.11) and (2.12), recalling (2.9) and the definition of the integer $s_0$, and applying Lemma 2.2 with $c_1 = (s_0 - 1)/s$ and $c_2 = s_0/s$, we get

$$s^{1/q - 1/2} \sum_{k \geq 1} \left( \sum_{i=1}^s a_{ks+i}^2 \right)^{1/2} \left( \sum_{j \geq 1} a_j^q \right)^{1/q} \leq \frac{\sqrt{s_0 - 1/s} a_{s+1} + \sqrt{s_0 - 1/s} a_{s+s_0} + a_{s+s_0+1} + \sum_{k \geq 2} a_{ks+s_0+1}}{(1 + 1/s)a_{s+1}^q + (s_0 - 1)a_{s+s_0}^q + a_{s+s_0+1}^q + \sum_{k \geq 2} a_{ks+s_0+1}^q)^{1/q}} \leq \max \left\{ \frac{\sqrt{\frac{s_0}{s}}}{(1 + s_0/s)^{1/q}}, \frac{\sqrt{\frac{s_0}{s}}}{(1 + 1/s + ((s_0 - 1)/s)^{1+q/2})^{1/q}}, \frac{(l + 2)\sqrt{\frac{s_0}{s}}}{(1 + s_0/s + (l + 1/s)\sqrt{\frac{s_0}{s}})^{1/q}}, \frac{(l + 1)\sqrt{\frac{s_0}{s}} + \sqrt{s_0 - 1/s}(1 + \sqrt{s_0 - 1/s})}{(l + 1)\sqrt{s_0/s} + \sqrt{s_0 - 1/s}(1 + \sqrt{s_0 - 1/s})} \right\}.$$

(2.13) $\sup_{l \geq 0} \{(l + 1/s)\sqrt{s_0/s} + 1 + 1/s + ((s_0 - 1)/s)^{1+q/2})^{1/q}\}.$

Therefore

$$\sum_{k \geq 1} \left( \sum_{i=1}^s a_{ks+i}^2 \right)^{1/2} \left( \sum_{j \geq 1} a_j^q \right)^{1/q} \leq s^{1/2 - 1/q} \max \left\{ \frac{\sqrt{s_0/s}}{(1 + s_0/s)^{1/q}}, \frac{\sqrt{s_0/s}}{(1 + 2 - q/2(s_0/s)^{1+q/2})^{1/q}}, \frac{(l + 2)\sqrt{s_0/s}}{(1 + s_0/s + (l + 1/s)\sqrt{s_0/s} + 2 - q/2(s_0/s)^{1+q/2})^{1/q}}, \frac{(l + 2)\sqrt{s_0/s}}{(1 + 1/s + ((s_0 - 1)/s)^{1+q/2})^{1/q}} \right\} \leq s^{1/2 - 1/q} \max \left\{ \frac{2\sqrt{s_0/s}}{(1 + 2 - q/2(s_0/s)^{1+q/2})^{1/q}}, \sup_{l \geq 1} \frac{(l + 2)\sqrt{s_0/s}}{(1 + s_0/s)^{1/q}} \right\} \leq s^{1/2 - 1/q} \max \left\{ \sup_{\sqrt{2(1 - r_0)} \sqrt{2} \leq y \leq 1} \frac{2y}{(1 + 2 - q/2y^{2+q})^{1/q}}, \sup_{1 \leq y \leq \sqrt{2}} \frac{2y}{(1 + y)^{1/q}} \right\},$$

(2.14) $\sup_{\sqrt{2(1 - r_0)} \sqrt{2} \leq y \leq 1} \frac{2y}{(1 + 2 - q/2y^{2+q})^{1/q}}, \sup_{1 \leq y \leq \sqrt{2}} \frac{2y}{(1 + y)^{1/q}}$.

where the third inequality is valid by (2.10) and the first inequality follows from the following two inequalities:

$$\sqrt{\frac{t - 1}{s}} \left( 1 + \frac{\sqrt{t - 1}}{\sqrt{s}} \right) \leq \sqrt{\frac{t}{s}} \leq \sqrt{\frac{t - 1}{s}} \left( 1 + \frac{\sqrt{t - 1}}{\sqrt{s}} \right),$$

(2.15) $\sqrt{\frac{t - 1}{s}} \left( 1 + \frac{\sqrt{t - 1}}{\sqrt{s}} \right) \leq \sqrt{\frac{t}{s}}$ and

$$\frac{1}{s} + \left( \frac{t - 1}{s} \right)^{1+q/2} \geq \left( \frac{1}{s} \right)^{1+q/2} + \left( \frac{t - 1}{s} \right)^{1+q/2} \geq 2 - q/2 \left( \frac{t}{s} \right)^{1+q/2}, \quad 1 \leq t \leq s.$$

(2.16) $\frac{1}{s} + \left( \frac{t - 1}{s} \right)^{1+q/2} \geq \left( \frac{1}{s} \right)^{1+q/2} + \left( \frac{t - 1}{s} \right)^{1+q/2} \geq 2 - q/2 \left( \frac{t}{s} \right)^{1+q/2}, \quad 1 \leq t \leq s.$
The conclusion (2.7) follows from (2.8) and (2.14). □

Now we give the proof of Theorem 1.4.

Proof of Theorem 1.4. Let \( h \) satisfy

\[
Ah = 0
\]

and let \( S_0 \) be a subset of \( \{1, \ldots, n\} \) with cardinality \( \#S_0 \) less than or equal to \( s \). We partition \( S_0 \subset \{1, \ldots, n\} \) as \( S_0 = S_1 \cup \cdots \cup S_l \), where \( S_1 \) is the set of indices of the \( s \) largest components, in absolute value, of \( h \) in \( S_0 \), \( S_2 \) is the set of indices of the next \( s \) largest components, in absolute value, of \( h \) in \( (S_0 \cup S_1)^c \), and so on. Applying the parallelogram identity, we obtain from the restricted isometry property (1.5) that

\[
|\langle Au, Av \rangle| \leq \delta_2^2 s (A) \|u\|_2 \|v\|_2
\]

for all \( s \)-sparse vectors \( u, v \in \Sigma_s \) whose supports have empty intersection \([7]\). Combining (2.17) and (2.18) and using the restricted isometry property (1.5) yield

\[
(1 - \delta_2(A)) \left( \|h_{S_0}\|_2^2 + \|h_{S_1}\|_2^2 \right) \\
\leq \langle A(h_{S_0} + h_{S_1}), A(h_{S_0} + h_{S_1}) \rangle \\
\leq \langle A(\sum_{i \geq 2} h_{S_i}), A(\sum_{i \geq 2} h_{S_i}) \rangle \\
\leq \sum_{i,j \geq 2} \delta_2(A) \|h_{S_i}\|_2 \|h_{S_j}\|_2 + \sum_{j \geq 2} \|h_{S_j}\|_2^2 \\
= \delta_2(A) \left( \sum_{j \geq 2} \|h_{S_j}\|_2 \right)^2 + \sum_{j \geq 2} \|h_{S_j}\|_2^2 \\
\leq (1 + \delta_2(A)) \left( \sum_{j \geq 2} \|h_{S_j}\|_2 \right)^2 
\]

(2.19)

which implies that

\[
\sum_{j \geq 2} \|h_{S_j}\|_2 \geq \left( \frac{1 - \delta_2(A)}{1 + \delta_2(A)} \right)^{1/2} \|h_{S_1}\|_2.
\]

(2.20)

Applying Lemma 2.3 with \( \delta_1 = \left( \frac{1 - \delta_2(A)}{1 + \delta_2(A)} \right)^{1/2} \) gives

\[
\sum_{j \geq 2} \|h_{S_j}\|_2 \leq a(q, \delta_1) s^{1/2 - 1/q} \|h_{S_0}\|_q.
\]

(2.21)

Then substituting the above estimate for \( \sum_{j \geq 2} \|h_{S_j}\|_2 \) into the right hand side of the inequality (2.19) and recalling that \( h_{S_0} \) is an \( s \)-sparse vector lead to

\[
\|h_{S_0}\|_q \leq s^{1/q - 1/2} \|h_{S_0}\|_2 \leq s^{1/q - 1/2} (\delta_1)^{-1} \sum_{j \geq 2} \|h_{S_j}\|_2 \leq \frac{a(q, \delta_1)}{\delta_1} \|h_{S_0}\|_q,
\]

(2.22)
2.2. Proof of Theorem 1.5. We follow the argument in [1, 5]. Set $h = x^* - x$, and denote by $S_0$ the support of the vector $x_s \in \Sigma_s$, by $S_0^c$ the complement of the set $S_0$ in $\{1, \ldots, n\}$. Then

$$(2.23) \quad \|Ah\|_2 = \|Ax^* - Ax\|_2 \leq \|Ax^* - y\|_2 + \|z\|_2 \leq 2\epsilon$$

and

$$(2.24) \quad \|h_{S_0}\|_q^q \leq \|h_{S_0}\|_q^q + 2\|x - x_s\|_q^q,$$

since

$$\|x_s\|_q^q + \|x_{S_0}^c\|_q^q = \|x\|_q^q \geq \|x^*\|_q^q = \|x_s + h_{S_0}\|_q^q + \|x_{S_0}^c + h_{S_0}\|_q^q \geq \|x_s\|_q^q - \|h_{S_0}\|_q^q + \|h_{S_0}\|_q^q - \|x_{S_0}^c\|_q^q.$$ 

Similar to the argument used in the proof of Theorem 1.4, we partition $S_0^c \subset \{1, \ldots, n\}$ as $S_0^c = S_1 \cup \cdots \cup S_t$, where $S_1$ is the set of indices of the $s$ largest absolute-value components of $h$ in $S_0^c$. $S_2$ is the set of indices of the next $s$ largest absolute-value components of $h$ on $S_0^c$, and so on. Then it follows from (1.5), (2.19) and (2.23) that

$$(2.25) \quad (1 - \delta_{2s}(A))(\|h_{S_0}\|_2^2 + \|h_{S_1}\|_2^2)$$

$$\leq \langle A(h_{S_0} + h_{S_1}), A(h_{S_0} + h_{S_1}) \rangle$$

$$\leq \langle Ah - A(\sum_{i \geq 2} h_{S_i}), Ah - A(\sum_{j \geq 2} h_{S_j}) \rangle$$

$$\leq (2\epsilon + \sqrt{1 + \delta_{2s}(A)} \sum_{j \geq 2} \|h_{S_j}\|_2^2).$$

By the continuity of the function $a(q, \delta)$ about $\delta \in (0, 1)$ and the assumption (1.14), there exists a positive number $\eta$ such that

$$(2.26) \quad a(q, \delta_1/(1 + r)) < \delta_1/(1 + r).$$

If $\sum_{j \geq 2} \|h_{S_j}\|_2^2 \leq 2\epsilon/(r\sqrt{1 + \delta_{2s}(A)})$, then it follows from (2.24), (2.25) and the fact that $h_{S_0} \in \Sigma_s$ that

$$(2.27) \quad \|x^* - x\|_2 = \|h\|_2 \leq (\|h_{S_0}\|_2^2 + \|h_{S_1}\|_2^2)^{1/2} + \sum_{j \geq 2} \|h_{S_j}\|_2$$

$$\leq 2 \left( \frac{(1 + r)}{r\sqrt{1 + \delta_{2s}(A)}} + \frac{1}{r\sqrt{1 + \delta_{2s}(A)}} \right) \epsilon,$$

and

$$(2.28) \quad \|x^* - x\|_q^q \leq \|h_{S_0}\|_q^q + \|h_{S_0}^c\|_q^q \leq 2\|h_{S_0}\|_q^q + 2\|x - x_s\|_q^q$$

$$\leq 2s^{1-q/2}\|h_{S_0}\|_q^2 + 2\|x - x_s\|_q^q$$

$$\leq 2^{1+q}(1 + r)^q \sqrt{\epsilon^q} + 2\|x - x_s\|_q^q.$$
If \( \sum_{j \geq 2} \| h_{S_j} \|_2 \geq 2\varepsilon/(r \sqrt{1 + \delta_{2s}(A)}) \), then

\[
(2.29) \quad \delta_1 \left( \| h_{S_0} \|_2^2 + \| h_{S_1} \|_2^2 \right)^{1/2} \leq (1 + r) \sum_{j \geq 2} \| h_{S_j} \|_2
\]

by (2.25), where we set \( \delta_1 = \sqrt{(1 - \delta_{2s}(A))/(1 + \delta_{2s}(A))} \). Using (2.29) and applying Lemma 2.3 with \( \delta = \delta_1/(1 + r) \) gives

\[
(2.30) \quad \sum_{j \geq 2} \| h_{S_j} \|_2 \leq a(q, \delta_1/(1 + r)) s^{1/2 - 1/q} \| h_{S_0} \|_q.
\]

Noting the fact that \( h_{S_0} \in S_s \) and then applying (2.24), (2.29) and (2.30) yield

\[
\| h_{S_0} \|^q \quad \leq \quad s^{1-q/2} \| h_{S_0} \|^q \leq \left( \frac{a(q, \delta_1/(1 + r))}{\delta_1/(1 + r)} \right)^q \| h_{S_0} \|^q \leq \left( \frac{a(q, \delta_1/(1 + r))}{\delta_1/(1 + r)} \right)^q \| h_{S_0} \|^q + 2 \left( \frac{a(q, \delta_1/(1 + r))}{\delta_1/(1 + r)} \right)^q \| x - x_s \|^q.
\]

This, together with (2.26), leads to the following crucial estimate:

\[
(2.31) \quad \| h_{S_0} \|^q \leq \frac{2(a(q, \delta_1/(1 + r)))^q}{(\delta_1/(1 + r))^q - (a(q, \delta_1/(1 + r)))^q} \| x - x_s \|^q.
\]

Combining (2.24), (2.29), (2.30) and (2.31), we obtain

\[
\| x^* - x \|_2 \leq \left( \| h_{S_0} \|_2^2 + \| h_{S_1} \|_2^2 \right)^{1/2} + \sum_{j \geq 2} \| h_{S_j} \|_2
\]

\[
(2.32) \quad \leq \frac{2^{1/q}(1 + r + \delta_1)(a(q, \delta_1/(1 + r)))^2}{\delta_1((\delta_1/(1 + r))^q - (a(q, \delta_1/(1 + r)))^q)} s^{1/2 - 1/q} \| x - x_s \|_q,
\]

and

\[
\| x^* - x \|^q \leq 2\| h_{S_0} \|^q + 2\| x - x_s \|^q \leq \frac{2(a(q, \delta_1/(1 + r)))^q}{(\delta_1/(1 + r))^q - (a(q, \delta_1/(1 + r)))^q} \| x - x_s \|^q.
\]

The desired error estimates (1.16) and (1.17) follow from (2.27), (2.28), (2.32) and (2.33).

2.3. Proof of Theorem 1.2. The conclusion in Theorem 1.2 follows from Corollary 1.6 and the observation that

\[
(2.34) \quad \lim_{q \to 0^+} a(q, \delta) = 0
\]

for any \( \delta \in (0, 1) \).
2.4. Proof of Theorem 1.3

Let

\[
\tilde{q}_{\max}(\delta_1) = \sup \{ q \in (0, 1) \mid a(q, \delta_1) < \delta_1 \}.
\]

Take sufficiently small \( \epsilon > 0 \). Note that

\[
\sup \frac{2y}{\sqrt{2(1-r_{0})\delta_1/2} \leq y \leq 1} \left( 1 + 2^{-q/2}y^{2q} \right)^{1/q}
\]

\[
\begin{cases}
(1 + 2^{-1-q}((1-r_{0})\delta_1)^2 + q)_1^{1/q} & \text{if } q < 2^{-q}(1-r_{0})^{2+q}\delta_1^{2+q}, \\
q^{1/(2+q)}(1 + \frac{q}{2})^{-1/q}2^{(1+3q/2)/(2+q)} & \text{if } 1 \geq q \geq 2^{-q}(1-r_{0})^{2+q}\delta_1^{2+q}.
\end{cases}
\]

Then for any small \( q > (e/2 + \epsilon)\delta_1^2 \) and sufficiently small \( \delta_1 > 0 \), we have that \( q \geq 2^{-q}(1-r_{0})^{2+q}\delta_1^{2+q} \) for all \( r_{0} \in (0, 1) \). Then applying \([1.12]\) and \(2.36\) yields

\[
a(q, \delta_1) \geq \inf_{0<r_{0}<1} \sup_{\sqrt{2(1-r_{0})\delta_1/2} \leq y \leq 1} \frac{2y}{(1 + 2^{-q/2}y^{2q})^{1/q}} = q^{1/(2+q)}(1 + \frac{q}{2})^{-1/q}2^{(1+3q/2)/(2+q)} \geq (1 + \epsilon/e)^{1/2}\delta_1,
\]

where the last inequality holds since

\[
\lim_{q \to 0} q^{-q/(4+2q)} (1 + \frac{q}{2})^{-1/q}2^{(1+3q/2)/(2+q)} = (2/e)^{1/2}.
\]

Therefore

\[
(2.37) \quad \lim sup_{\delta_1 \to 0} \frac{\tilde{q}_{\max}(\delta_1)}{\delta_1^2} \leq \lim sup_{\delta_1 \to 0} \frac{(e/2 + \epsilon)\delta_1^2}{\delta_1^2} \leq \frac{e}{2} + \epsilon
\]

for any sufficiently small \( \epsilon > 0 \).

Take \( r_{0} = 1 - \sqrt{2}/4 \) and sufficiently small \( \epsilon > 0 \). Then for \( q \leq (e/2 - \epsilon)\delta_1^2 \) and sufficiently small \( \delta_1 > 0 \),

\[
(2.39) \quad \begin{cases}
(1 + r_{0}\delta_1)(1 + r_{0}^2\delta_1^2)_1^{1/q} \leq 2(3/2)_1^{1/q} \leq (1 - \epsilon/e)^{1/2}\delta_1, \\
\sup_{y \geq 1} y(1 + y)^{-1/q} \leq \sup_{y \geq 1} (1 + y)^{-1/q} \leq 2^{-1/q} \leq (1 - \epsilon/e)^{1/2}\delta_1/2, \\
\sup_{y \geq \sqrt{2(1-r_{0})\delta_1/2}} \frac{y}{(1+y)^{1/q}} = \frac{\sqrt{2(1-r_{0})\delta_1/2}}{(1+\sqrt{2(1-r_{0})\delta_1/2})^{1/q}} \leq (1 - \epsilon/e)^{1/2}\delta_1/3,
\end{cases}
\]

and

\[
(2.40) \quad \sup_{\sqrt{2}(1-r_{0})\delta_1/2 \leq y \leq 1} \frac{2y}{(1 + 2^{-q/2}y^{2q})^{1/q}} \leq (1 - \epsilon/e)^{1/2}\delta_1
\]

by \([1.12]\), \(2.36\) and \(2.37\). Therefore

\[
(2.41) \quad \lim inf_{\delta_1 \to 0} \frac{\tilde{q}_{\max}(\delta_1)}{\delta_1^2} \geq \lim sup_{\delta_1 \to 0} \frac{(e/2 - \epsilon)\delta_1^2}{\delta_1^2} \geq \frac{e}{2} - \epsilon
\]
by (2.39) and (2.40). Combining (2.38) and (2.41) and recalling that $\epsilon > 0$ is a sufficiently small number chosen arbitrarily, we have
\begin{equation}
\lim_{\delta_1 \to 0} \frac{\tilde{q}_{\max}(\delta_1)}{\delta_1^2} = \frac{e}{2}.
\end{equation}
By Corollary 1.6 we have
\begin{equation}
q_{\max}(\delta; m, n, s) \geq \tilde{q}_{\max}(\sqrt{(1 - \delta)/(1 + \delta)}).
\end{equation}
This together with (2.42) implies that
\begin{equation}
\lim_{\delta \to 1-} \frac{q_{\max}(\delta; m, n, s)}{1 - \delta} \geq \lim_{\delta \to 1-} \frac{\tilde{q}_{\max}(\sqrt{(1 - \delta)/(1 + \delta)})}{(1 - \delta)/(1 + \delta)} \times \frac{1}{1 + \delta} = \frac{e}{4},
\end{equation}
and hence completes the proof.

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