Metrics for the violation of detailed balance in microwave circuits: theory and experiment

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We propose a new approach to detailed balance violation in electrical circuits by relying on the scattering matrix formalism commonly used in microwave electronics. This allows to include retardation effects which are paramount at high frequencies. We define the spectral densities of phase space angular momentum, heat transfer and cross power, which can serve as criteria for detailed balance violation. We confirm our theory with measurements in the 4-8 GHz frequency range on several two port circuits of varying symmetries, in space and time. This validates our approach, which will allow to treat quantum circuits at ultra-low temperature.

I. INTRODUCTION

There has been recently a fast growing interest in the thermodynamics of ultimately simple, small systems, in particular through the study of heat engines [14] or particles driven by noise sources [5], such as brownian motion in a fluid [6]. A particular effort has been devoted to electrical circuits, in which the variables such as position or velocity of the brownian particle are replaced by macroscopic variables in the circuit, such as charge or voltage across a capacitor, and where the noise is the Johnson-Nyquist thermal noise of resistors [7, 8]. All these systems, from biological entities to electrical circuits, may indeed obey similar equations of motion.

The simplest and most intensively studied circuit consists of two capacitors coupled to two resistors through another capacitor. Even such a simple circuit shows non-trivial heat transport [10, 13], gyration and detailed balance violation [8, 9]. Most of these studies have been performed within the framework of classical physics. On the other hand there is a currently huge development of quantum technologies and circuits, understanding the thermodynamical properties of which is of utmost interest. It is thus crucial to extend the methods developed in classical circuits to quantum ones [14].

A mandatory condition to study circuits in the quantum regime is to work at frequencies, \( f \), such that \( hf \gtrsim k_B T \) with \( T \) the temperature. For context, a temperature of \( T = 1K \) corresponds to a frequency of \( f = k_B T / e = 21 \) GHz, as such, experiments are performed below 1K in the microwave domain. Since circuits are usually larger than the wavelength, retardation effects are paramount. Unfortunately, previous studies in classical circuits have focused to low frequencies where retardation effects are irrelevant and were neglected. It is the goal of the present paper to provide a new theoretical approach, based on the scattering matrix formalism, able to treat the case of circuits of any size and to validate the theory with experiments in the microwave regime.

The rest of the paper will be structured as follows, section II is the theory, where we introduce the scattering matrix formalism and express the metrics for detailed balance violation in terms of it. Section III goes over the experimental setup used to test our theoretical predictions in the microwave regime. Section IV presents the experimental data while we conclude in section V.

II. THEORY

A. Scattering matrix formalism

Previous work has focused on circuits without propagation, modeling them using time-domain differential equations between current and voltages in the circuit, and where noise sources appear are source terms [8, 9]. Including propagation within this formalism would be cumbersome. Since we deal with linear circuit, we find it more efficient to work in the frequency domain, where current and voltages are simply related by a frequency-dependent impedance matrix. Measuring voltages (respectively currents) requires high impedance (respectively low impedance) sensors, which are difficult to implement at high frequency. One would rather work with matched amplifiers, i.e. amplifiers which input impedance is the same as that of the transmission line connected to it, so that all the power sent to the amplifier is absorbed. Such amplifiers do not measure the voltage at a point in the circuit but the amplitude of the wave incoming to it. We will focus on matched amplifiers and discuss briefly the case of voltmeters. We chose to work, as usual in microwave electronics, with the scattering formalism [15]. In this formalism, a circuit connected to \( n \) ports is modeled by a \( n \times n \) matrix, the scattering matrix \( S \), which relates the amplitude of the voltage waves exiting each port to that entering each port. Sources appear as incoming waves and measurements can be performed on each outgoing wave.

In the following we will apply the scattering formalism to the simplest, yet nontrivial circuit, which contains only two ports, i.e. two noise sources and two measurements, as shown in Fig. 1. This situation is very close to the one usually considered at low frequency [8, 11]. This is the setup we have implemented experimentally, as we
show below. More complex circuits can be easily treated following the same lines. We suppose that the circuit is lossless and linear. Losses can be implemented by adding extra ports in which power exits the circuit. According to the fluctuation-dissipation theorem, losses must be accompanied by noise, i.e. extra noise sources must be added accordingly, which enter the circuit through the extra ports. Non-linearities would complexify the physics a lot, since different frequencies would be coupled, and the usual scattering formalism does not apply to such systems. We do not consider non-linearities, so voltage measured at a given frequency depends only on noise sources at the same frequency. As a consequence, we can deal with the spectral densities of the various quantities we are considering, and not necessarily their integral over a certain bandwidth.

Figure 1. (a) Two matched resistors of value \( R_0 \) at temperatures \( T_1 \) and \( T_2 \) and emitting voltage noises \( \eta_1 \) and \( \eta_2 \) respectively. Both noises act as inputs into a 2 port circuit represented by its four scattering parameters, while \( V_1 \) and \( V_2 \) correspond to the measured outputs.

Following Fig. 1 we note \( \eta_1(f) \) and \( \eta_2(f) \) the voltage amplitude of the waves emitted by the noise sources at frequency \( f \), and \( V_1(f) \), \( V_2(f) \) the measured amplitudes of the waves leaving the circuit. They are related by:

\[
V_1(f) = S_{11}(f)\eta_1(f) + S_{12}(f)\eta_2(f), \quad (1)
\]

\[
V_2(f) = S_{21}(f)\eta_1(f) + S_{22}(f)\eta_2(f), \quad (2)
\]

where \( S_{ij} \) are the frequency-dependent elements of the \( S \) matrix. For the sake of simplicity we suppose that the noise sources are uncorrelated and of spectral density \( \langle |\eta_i(f)|^2 \rangle = k_B T_i R_0 \). Here \( T_i \) is the (possibly frequency-dependent) noise temperature of the source \( i \) and \( R_0 \) the impedance of the sources, which are matched to the transmission lines connecting to the two ports of the circuit (for simplicity we take the same \( R_0 \) for both sources). If the noise sources are resistors in the classical regime, \( T_i \) is simply their thermodynamic temperature \( T_i \). If they are resistors in the quantum regime, the noise temperature \( T_i \) is related to the thermodynamical temperature \( T \) by \( T_i(f) = (\hbar f / 2k_B) \coth(\hbar f / 2k_B T) \). More generally, as for example if they are mesoscopic conductors at ultra-low temperature, \( T_i(f) = R_0 s_i / (2k_B) \) where \( s_i(f) \) is their voltage noise spectral density.

The temperature difference \( \Delta T = T_2 - T_1 \) is easily deduced from the measurement of the spectral densities \( \langle |V_i|^2 \rangle \) and the knowledge of the scattering matrix of the circuit, according to:

\[
\Delta T = \frac{\langle |V_1|^2 \rangle - \langle |V_2|^2 \rangle}{R_0 k_B (|S_{12}|^2 - |S_{22}|^2)} \quad (3)
\]

Which, unless in the particular case \( |S_{12}|^2 = |S_{22}|^2 \), allows to measure \( \Delta T \). It is important to note that to obtain this result the unitarity of the scattering matrix is used i.e \( SS^\dagger = 1 \). This implies in particular, \( |S_{11}| = |S_{22}| \) and \( |S_{12}| = |S_{21}| \). On an experimental point of view, this quantity suffers from its sensitivity to the unavoidable noise added by the amplifiers. An accurate determination of \( \Delta T \) would require the hard-to-achieve condition that the amplifiers have equal noise temperatures. In order to circumvent this technical difficulty, it is convenient to consider the cross-correlation between the measured voltages, namely \( \langle V_1(f) V_2^*(f) \rangle \). This quantity is immune no amplifier noise provided the averaging time is long enough. It also provides a measurement of \( \Delta T \):

\[
\Delta T = \frac{\langle V_1(f) V_2^*(f) \rangle}{R_0 k_B S_{11} S_{22}} \quad (4)
\]

If the noise sources were correlated, it would not be possible to express \( \Delta T \) in terms of the cross-correlation only. While expressing \( \Delta T \) in terms of the cross-correlation of measured voltages makes it immune to amplifier noise, it makes it sensitive to cable lengths, since it involves the phases of scattering amplitudes and not only their modulus as in Eq. (3).

Below we focus on two questions: i) how to compute interesting physical quantities introduced in previous work, such as heat transfers and fluctuations loops, using our approach? ii) can one find a better way to determine if the circuit is out of equilibrium?

### B. Heat transfers

A lot of thought has been devoted to the heat transfer between two capacitively coupled resistors \[10,11,13\]. Similar quantities can be calculated using the scattering formalism. Following \[11\] we note \( Q_1 \) the electrical power dissipated in the resistor at port 1. Since this quantity is nonlinear in voltage, it mixes frequencies: its spectral density involves a convolution in frequency space. However the spectral density of average power \( \langle q_1(f) \rangle \) is well defined, given by:

\[
\langle q_1(f) \rangle = \frac{1}{R_0} \left[ \langle |V_1(f)|^2 \rangle - \langle |\eta_1(f)|^2 \rangle \right] \quad (5)
\]

This has a clear interpretation: it corresponds to cooling the resistor by emission of a wave of amplitude \( \eta_1 \) and heating by the absorption of a wave of amplitude \( V_1 \). We find:

\[
\langle q_1 \rangle = k_B \Delta T |S_{12}|^2. \quad (6)
\]
This result is a generalization of what has been obtained at low frequency, see Eq. (16) of ref. [7]. At equilibrium there is no net power dissipated in either port.

C. Angular momentum and stochastic area

It was demonstrated in [8, 9] that fluctuation loops are observed in out of equilibrium circuits. These loops are closed trajectories in the \((V_1, V_2)\) plane, characterized by a stochastic area \(A\). The existence of fluctuation loops corresponds to an average rotation of \((V_1, V_2)\) to which is associated an angular momentum along the perpendicular axis given by:

\[
\langle L_z \rangle = \langle V_1(t)V_2(t) - V_2(t)V_1(t) \rangle \tag{7}
\]

It is simply related to the stochastic area by \(\langle L_z \rangle = 2\langle A \rangle\). This reads in Fourier space:

\[
\langle L_z \rangle = \int_{-\infty}^{+\infty} \langle l_z(f) \rangle \, df. \tag{8}
\]

with \(\langle l_z(f) \rangle\) the angular momentum spectral density. Experimentally the integral will have finite bounds due to the finite bandwidth of the circuit. We find:

\[
\langle l_z(f) \rangle = 4\pi f \Im \{\langle V_1 f V_2^* \rangle \} = 4\pi k_B \Delta T R_0 f \Im \{S_{12}S_{22}^*\}. \tag{9}
\]

The spectral density of the angular momentum also vanishes for \(\Delta T = 0\). However it can be both positive or negative depending on the scattering parameters, which means that measurement frequency and bandwidth must be chosen carefully. One can measure \(\langle L_z \rangle = 0\) even if \(T_1 \neq T_2\). In particular, a long cable length difference between the two ports would result in fast oscillations of \(\langle l_z(f) \rangle\) vs. frequency, thus a vanishing \(\langle L_z \rangle\). In contrast, \(\langle Q_1 \rangle\) is not sensitive to cable length since it involves only the modulus of \(S_{12}\).

D. Cross Power

The angular momentum, which is a direct measure of fluctuation loops, appears to be related to the cross-correlation \(\langle V_1 V_2^* \rangle\) although one quadrature of it, and with a weighing factor \(f\). It is clear that neither the absolute phase of \(\langle V_1 V_2^* \rangle\) nor the frequency-dependent weighing factor are essential to determine if the circuit is out of equilibrium. We thus define the cross correlation power spectral density as

\[
\langle p_{1,2} \rangle = \frac{1}{R_0} \langle |V_1V_2^*|^2 \rangle \tag{11}
\]

where we take the modulus of the cross correlation to remove the phase problem. We find:

\[
\langle p_{1,2} \rangle = k_B |\Delta T| |S_{12}S_{22}^*| \tag{12}
\]

This quantity is also a good metric of \(\Delta T\) that is immune to amplifier noise and cable lengths. However it does not allow to know which source is hotter than the other.

III. EXPERIMENTAL SETUP

A. General considerations

The experimental setup is shown on Fig. 2. All measurements have been performed at room temperature using a variable attenuator and a calibrated noise source as the hot source (\(T_1\) adjustable between 290K and 560K) and a 50Ω resistor at room temperature as the cold source (\(T_2 = 290K\)). We have chosen to work in the 4-8 GHz frequency range, which is similar to that of many experiments performed in the quantum regime at ultra low temperature. The separation between incoming and outgoing waves is achieved using two circulators: the signals emitted by the sources are injected in the circuit and not in the related amplifiers, while those leaving the circuits enter the amplifiers and are not lost in the sources. Moreover, the noise emitted by the amplifiers is absorbed by the sources which are matched to the microwave circuit. This minimizes parasitic cross-correlations. Given the noise temperature \(\sim 70K\) of the amplifiers and isolation of the circulators, we estimate a parasitic contribution of \(\sim 0.7K\) (more circulators could be used if needed). After amplification and filtering to keep the signal within a well defined bandwidth, the signals are digitized using a 20GHz, 40GS/s, 8 bit digital oscilloscope. The time series are acquired in batches of 2MS that are split in chunks of 2048 points. Spectra are then calculated using a discrete Fourier transform on each chunk. Then we calculated the auto- and cross-correlations using those spectra and average the over all the chunks. The size

\[
\text{Figure 2. (Top) Experimental setup. The dashed box represents the coupling circuit of scattering matrix S. (Bottom) All three coupling circuits used to test different S matrices.}
\]
of the chunk sets the frequency resolution, here 20MHz, which is enough for the circuits we considered. Our calculations suggest that the key ingredient to obtain nonzero heat transfer, angular momentum or cross-power is the temperature difference $\Delta T$ between the sources, regardless of spatial as well as time reversal symmetries. Note that while spatial symmetry have been already considered in previous work, time-reversal symmetry could not be probed since propagation times were neglected. In order to test this we have considered circuits with a various combinations of spatial and temporal symmetries as shown in Fig. 2. The circuits are made of pieces of coax cables of various lengths connected by T-junctions, and terminated by open or short circuits. The lengths $d, d'$ of 5.08 and 7.62 cm respectively, where chosen so that reflections of waves at the end of the cables and at the junctions provide rich interference patterns which result in strong frequency-dependence of the S-matrix, thus providing a deep test of our theoretical results, see dashed lines in Fig. 3. Here propagation times are essentials. Circuit (a) in Fig. 2 is symmetric upon exchange of ports 1 and 2 while circuit (b) is not. Circuit (c) contains a circulator in order to break time-reversal symmetry. In order to make a quantitative comparison between our predictions and the measurements it is necessary to know the $S$ matrix of all three circuits. This has been achieved using a vector network analyzer (VNA), see dashed lines in Fig. 3.

### IV. RESULTS

In Fig. 3 we show the spectral densities of the angular momentum $\langle l_z(f) \rangle$ (top) and transferred power $\langle \dot{q}_1(f) \rangle$ (bottom) as a function of frequency for the three circuits of Fig. 2. The two quantities exhibit strong oscillations vs. frequency which come from interferences occurring due to total reflections at the end of the cables and partial reflections at the junctions between them. We observe a very good quantitative agreement between the measurements and theoretical predictions of Eqs. (10) and (6), shown as dashed lines in Fig. 3. These have been obtained using the measured coefficients of the scattering matrix and the calibrated noise temperature of the hot noise source. We attribute the small differences between theory and experiment to experimental imperfections, in particular the lack of reproducibility of connections/disconnections between the VNA and noise measurements, and the losses in the cables and circulator. The slightly negative values of $\dot{q}$ are due to the presence of amplifier noise combined with the use of Eq. (5) where $|\eta_1|^2$ is obtained by inverting Eq. (1) and Eq. (2) using the measured voltages and the known scattering matrix. The dependence of the integrated angular momentum and cross-power on the temperature difference $\Delta T$ is shown in Fig. 4. These curves have been obtained by integrating the spectra of Fig. 3 over frequency between 7.3 and 7.5 GHz for the angular momentum and
the full 4-8 GHz bandwidth for the cross power. The data corresponding to $\Delta T < 0$ have been obtained by swapping the cold and hot sources. As predicted, $\langle L_z \rangle \propto \Delta T$, see Eq.10 and $\langle p_{1,2} \rangle \propto |\Delta T|$, see Eq.12, for all circuits, i.e. regardless of the spatial and/or time-reversal symmetries of the system. The relatively narrow bandwidth for $\langle L_z(f) \rangle$ has been chosen to avoid sign change leading to a vanishing $\langle L_z \rangle$. This has been possible thanks to the precise knowledge of the S matrix, measured using the VNA.

![Graph](image_url)

Figure 4. (Top) Total angular momentum in the frequency range 7.3–7.5 GHz for the three circuits of Fig.2 as a function of $\Delta T$. (Bottom) Total cross-power in the frequency range 4–8 GHz for the three circuits of Fig.2 as a function of $\Delta T$.

In an experiment at ultra-low temperature, it may be difficult to perform such a measurement, and choosing the right bandwidth might be difficult. In contrast, increasing the bandwidth will always lead to an increased cross-power. We notice that $\langle P_{1,2} \rangle$, which in theory should be proportional to $|\Delta T|$ is experimentally not exactly a perfectly even function of $\Delta T$. This comes from the impedances of the sources not being exactly 50 $\Omega$, so reversing the sources also slightly changes $S$. The same thing happens to $\langle L_z \rangle$ which is not perfectly odd, although it is less obvious. For $\Delta T = 0$ we find a residual cross-power which corresponds to what we expect for $\Delta T \sim 13K$. We understand this as amplifier noise ($\sim 70K$) being averaged a finite number $N = 100$ of times. This yields $\Delta T \approx 70/\sqrt{100} = 7K$, in reasonable agreement with the measurement. We also find a residual angular momentum corresponding to $\Delta T \sim 0.6K$, in good agreement with our prediction, see section III.

V. CONCLUSION

We have defined metrics for detailed balance violation in any linear circuit in steady state as spectral densities in the Fourier space: the heat transferred at any given port, the angular momentum in phase space and the cross power. We have calculated how these quantities are related to the temperature difference between two noise sources in the circuit via its scattering matrix. We have proven our predictions correct by measuring circuits of various symmetries in the 4-8 GHz range where propagation time of the signals is essential. Our approach is very general and could be used not only for microwaves but also for guided optics for example. In particular, it allows to treat circuits in the quantum regime $(\hbar \Omega \gtrsim k_B T)$ where the noise spectra of the sources have a frequency dependence which reflects the presence of vacuum fluctuations.

From our result one should be able to recover previously obtained results at low frequency where propagation times vanish. For this we have to consider voltmeters and not only matched amplifiers as we did. This can be done as follows: a voltmeter does not measure the outgoing voltage $V_{out}$ but the sum of the incoming $V_{in}$ and outgoing voltages, $V_{in} + V_{out}$. As a consequence, our approach is valid provided we replace the scattering matrix $S$ by $S + 1$. However, the latter matrix being non-unitary, simplifications of the formula cannot be carried out. A tedious but straightforward calculation indeed shows that this procedure applied on the previously studied circuit coincides with previous results.

We have focused on average quantities: average heat transfer, average angular momentum. Some previous works have considered fluctuations of these quantities and calculated their probability distribution. With our approach, it is possible to address fluctuations via their moments. One can for example calculate the variance of the angular momentum: it will involve moments of order 4 of the measured voltages, which will involve similar moments of the amplitude of the noise sources. This method applies also if the noise sources are not Gaussian.

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