Mapping threefolds onto three-quadrics

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Abstract

We prove that the degree of a nonconstant morphism from a smooth projective 3-fold $X$ with Néron-Severi group $\mathbb{Z}$ to a smooth 3-dimensional quadric is bounded in terms of numerical invariants of $X$. In the special case where $X$ is a 3-dimensional cubic we show that there are no such morphisms. The main tool in the proof is Miyaoka’s bound on the number of double points of a surface.

1 Introduction

During the last years, a lot of progress has been made concerning the classification of algebraic varieties of dimension larger than 2. In order to complete the picture, it seems interesting to study the possible morphisms between these varieties too. Here, only a few results are known. For instance, Remmert and Van de Ven showed in [R-V] that the 2-dimensional complex projective space $\mathbb{P}^2$ cannot be mapped onto any other smooth complex variety of dimension larger than 0, and conjectured that the same is true for complex projective spaces of arbitrary dimension. This conjecture was proven by Lazarsfeld (see [La]), who used Mori’s characterization of $\mathbb{P}^n$ as the only smooth, $n$-dimensional variety with ample tangent bundle. Later, Paranjape and Srinivas proved that every nonconstant morphism from a complex homogeneous space, which is not a projective space, to itself is an isomorphism (see [P-S]). Furthermore, they showed that the complex, $n$-dimensional quadric can only be mapped onto $\mathbb{P}^n$ and itself. This last result has been generalised to arbitrary characteristic except 2 by Cho and Sato (see [C-S]).

In this paper we work only with varieties and morphisms over the complex numbers. We will study morphisms from smooth projective threefolds with...
Néron-Severi group $\mathbb{Z}$ to smooth quadric hypersurfaces in $\mathbb{P}^4$. The main new tool is the use of Miyaoka’s inequality, bounding the number of double points of a surface with numerically effective dualizing sheaf in terms of numerical invariants of the surface (see Theorem 3). In fact, it will be shown that the degree of a finite morphism from a smooth projective threefold $X$ with Néron-Severi group $\mathbb{Z}$ to a smooth quadric of dimension 3 is bounded in terms of numerical invariants of $X$. Before giving an exact statement of this result, we will introduce some definitions for the purpose of this paper.

Let $X$, $Y$ be smooth projective threefolds with Néron-Severi group $\mathbb{Z}$. Denote the ample generator of $NS(X)$ respectively $NS(Y)$ by $H_X$ respectively $H_Y$. The numerical index of $X$ is defined to be the integer $k$ such that the canonical divisor of $X$, $K_X$, is numerically equivalent to $kH_X$. Let $f$ be a morphism from $X$ to $Y$, which is not constant. It follows that $f$ is finite (see Lemma 2). The positive integer $d$ which satisfies $f^*H_Y \equiv dH_X$ will be called the generator degree of $f$. As usual, the number of points, mapping to a fixed general point in $Y$ under $f$, will be called the degree of $f$.

The main result of this paper is the following:

**Theorem 1** Let $X$ be a smooth projective threefold with Néron-Severi group $\mathbb{Z}$, $Q$ a smooth quadric hypersurface in $\mathbb{P}^4$ and $f$ a nonconstant morphism from $X$ to $Q$. Then the degree of $f$ is bounded in terms of the Chern numbers $c_3^1(X)$ and $c_2^1c_2(X)$ and the numerical index of $X$.

This theorem will be proven in Section 2. We remarked before, that the main tool, used in the proof, is Miyaoka’s inequality. As Miyaoka’s inequality is only valid for surfaces in characteristic 0, this proof cannot be generalized to other characteristics.

In Section 3, the special case where $X$ is a cubic hypersurface in $\mathbb{P}^4$ will be treated. We will prove the following result:

**Theorem 2** Let $X$ be a smooth cubic hypersurface in $\mathbb{P}^4$ and $Q$ a smooth quadric hypersurface in $\mathbb{P}^4$. There are no nonconstant morphisms from $X$ to $Q$.

In the proof we will use the results from Section 2, which imply that there are no morphisms from a cubic to a quadric of generator degree larger than 2. It will be shown by means of an ad hoc argument, based on a theorem
of Lazarsfeld about morphisms to projective spaces (see Theorem 3), that morphisms of generator degree 2 cannot occur either.

I thank Prof. Van de Ven for helpful conversations and especially for suggesting to use [Mi]. I thank Johan de Jong and Endre Szabó for stimulating discussions on this subject.

2 Proof of Theorem 1

The proof of Theorem 1 is based upon the following result of Miyaoka (see [Mi]):

**Theorem 3** Let $S$ be a complex projective surface with only ordinary double points and numerically effective dualizing sheaf $K_S$. Let $\tilde{S}$ be the minimal resolution of $S$. Then

$$\# \{\text{double points}\} \leq \frac{2}{3}(c_2(\tilde{S}) - \frac{1}{3}K_S^2).$$

Another important ingredient of the proof is the following lemma, which will be proven later on. Here the tangent hyperplane to a smooth quadric $Q$ of dimension 3 at a point $p \in Q$ is denoted by $T_pQ$.

**Lemma 1** Let $X$ be a smooth, projective variety of dimension 3 and $f$ a finite morphism from $X$ to $Q$. Then there is a dense open subset $U$ in $Q$ such that $f^*(T_pQ \cap Q)$ has no singularities away from $f^{-1}(p)$ for all $p \in U$.

Before starting the proof of Theorem 1, let us state the following lemma, which has an elementary proof.

**Lemma 2** Let $X, Y$ be smooth projective threefolds and $f : X \to Y$ a morphism. Assume that $X$ has Néron-Severi group $\mathbb{Z}$. Then the following two statements are equivalent:

i) $f$ is nonconstant;

ii) $f$ is finite.

**Proof of Theorem 1:** Suppose $f : X \to Q$ is a nonconstant morphism of generator degree $d$. By Lemma 2, $f$ is finite. The degree of $f$ is equal to $H_X^3d^3/2$, where $H_X$ is the ample generator of the Néron-Severi group of $X$. 

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For every point \( p \) on \( Q \), the hyperplane section \( H_p = T_pQ \cap Q \) is a quadric with one ordinary double point, namely \( p \). So, if \( p \) is not contained in the branch divisor of \( f \), then the surface \( f^*(H_p) \) contains \( H_X^3d^3/2 \) ordinary double points, which map to \( p \) under \( f \). From Lemma [1] it follows that \( f^*(H_p) \) has no other singularities for general \( p \).

Now fix a point \( p \) such that \( f^*(H_p) \) has exactly \( H_X^3d^3/2 \) ordinary double points and no other singularities and denote \( f^*(H_p) \) by \( S_p \). It will be shown that, if \( d \) is large enough, Theorem 3 can be applied to \( S_p \) and provides an upper bound on the number of ordinary double points of \( S_p \) which is smaller than \( H_X^3d^3/2 \). In order to compute this bound, we have to compute \( c_2(\tilde{S}_p) \) and \( K_{\tilde{S}_p}^2 \), where \( \tilde{S}_p \) is the minimal resolution of \( S_p \).

By Bertini’s Theorem (see [Jo], Théorème 6.10), there is a hyperplane section \( H \) of \( Q \) such that the surface \( f^*(H) \) is nonsingular. Denote this surface by \( S \). The surfaces \( S \) and \( \tilde{S}_p \) are homeomorphic (see [A], Theorem 3), so \( c_2(\tilde{S}_p) = c_2(S) \) and \( c_1^2(\tilde{S}_p) = c_1^2(S) \). As \( S_p \) has only ordinary double points, it follows that \( K_{\tilde{S}_p}^2 = K_S^2 \). Using \( K_{\tilde{S}_p} \equiv kH_{\tilde{S}_p} \), where \( k \) is the numerical index of \( X \), the adjunction formula gives:

\[
K_S \equiv (K_X + S)|_S \equiv (k + d)H_X|_S,
\]

so

\[
K_{\tilde{S}_p}^2 = (k + d)^2H_X^3|_S = (k + d)^2dH_X^3.
\]

The second Chernclass of \( S \) can easily be computed by means of adjunction:

\[
c_2(S) = dc_2(X)H_X + d^2(d + k)H_X^3.
\]

Thus the expression \( \frac{4}{9}c_2(\tilde{S}_p) - \frac{4}{9}K_{\tilde{S}_p}^2 \), which equals \( \frac{4}{9}(c_2(S) - \frac{1}{3}K_S^2) \) by previous remarks, becomes the following polynomial expression in \( d \):

\[
\frac{4}{9}H_X^3d^3 + \frac{2}{9}kH_X^3d^2 + \frac{2}{3}(c_2(X)H_X - \frac{1}{3}k^2H_X^3)d.
\]

We can apply Theorem [3] to \( S_p \) if \( S_p \) has only double points and \( K_{S_p} \) is nef. As we remarked before, the first condition is certainly satisfied because of Lemma [1]. As for the second one, using the adjunction formula, it follows that \( K_{S_p} \) is linearly equivalent to \( (K_X + S_p)|_{S_p} \). So \( K_{S_p} \) is nef if and only if \( (K_X + S_p)|_{S_p} \) is nef. As \( (K_X + S_p)|_{S_p} \equiv (k + d)H_X|_{S_p} \), this is certainly true if \( d \geq -k \). Notice that this condition is empty if \( k \geq -1 \). If \( k = -2 \),
then the condition becomes \( d \geq 2 \), which is also an empty condition as there are no morphisms of degree 1 between a variety of numerical index -2 and the quadric \( Q \), which has numerical index -3. The only smooth threefolds with numerical index less than -2 are \( \mathbb{P}^3 \), which has numerical index -4 and \( Q \) (see [K-O]). So, the only cases in which we cannot apply Theorem 3 to \( S_p \) are when \( X = \mathbb{P}^3 \) and \( d \leq 3 \) or \( X = Q \) and \( d \leq 2 \). In the other cases, it tells us that the number of ordinary double points on \( S_p \) is restricted by the expression (1). However, we remarked that \( S_p \) contains exactly \( 4H_X^3X^3/2 \) double points. As the leading term of (1), \( 4H_X^3X^3/9 \), is smaller than \( H_X^3X^3/2 \) for large \( d \). So we obtain a contradiction if \( d \) is larger than the largest positive zero of the polynomial \( H_X^3X^3/2 - \) (1) (if \( X = \mathbb{P}^3 \) resp. \( Q \), then we moreover have to require \( d \geq 4 \) resp. \( d \geq 3 \)). We conclude that the generator degree \( d \) of \( f \) and thus also the degree of \( f \) is bounded in terms of the coefficients of this polynomial. Since \( H_X = -c_1(X)/k \), the statement of the theorem follows.

Proof of Lemma 4: Denote the 4-dimensional dual projective space by \( \mathbb{P}^4^\vee \). Let \( Q^\vee \) be the dual variety of \( Q \). It is isomorphic to \( Q \) via the following isomorphism:

\[
\begin{align*}
Q & \longrightarrow Q^\vee, \\
p & \mapsto T_pQ.
\end{align*}
\]

Given \( T_pQ \in Q^\vee \), denote the surface \( f^*(T_pQ \cap Q) \subset Y \) by \( S_p \). We will study the singularities of \( S_p \), using the following criterion:

\[
x \in S_p \text{ is a singularity of } S_p \Leftrightarrow T_pQ \supset f_*T_xX. \tag{\textit{*}}
\]

In order to describe \( f_*T_xX \), we introduce some notation. For \( i \in \{0, 1, 2, 3\} \), the set

\[
X_i := \{ x \in X \mid f \text{ has rank at most } i \text{ at } x \}
\]

is an algebraic subset of \( X \) of dimension \( i \). So its image under the finite map \( f \) is an algebraic subset of \( Q \) of dimension \( i \). Denote the surface \( f(X_2) \) by \( B \) (this is just the branch locus of \( f \)), the curve \( f(X_1) \) by \( C \) and the finite set \( f(X_0) \) by \( R \). Furthermore, denote the union of all irreducible curves along which \( B \) is singular by \( \Gamma \) and the set of isolated singularities of \( B \) by \( \Sigma \). Finally, the singular locus of the curve \( C \) respectively \( \Gamma \) is denoted by \( Sing(C) \) respectively \( Sing(\Gamma) \). Let us now examine when a point \( x \in X \) is a singularity of \( S_p \).

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If \( x \in X_i \setminus X_{i+1} \) \((i \in \{1, 2, 3\})\) and \( f(x_i) \) is a smooth point of \( f(X_i) \), then \( f \ast T_x X = T_{f(x)} f(X_i) \). So, according to criterion \((*)\), \( x \) is a singularity of \( S_p \) if and only if \( T_p Q \) contains \( T_{f(x)} f(X_i) \), in other words if and only if \( T_p Q \) is tangent to \( f(X_i) \) at the point \( f(x) \). Especially, taking \( i = 3 \), it follows that \( x \in X_3 \setminus X_2 \) is a singularity of \( S_p \) if and only if \( f(x) = p \).

If \( x \in X_2 \setminus X_1 \) and \( f(x) \in \Gamma \setminus \text{Sing} \( \Gamma \) \), then \( f \ast T_x X \) contains \( T_{f(x)} \Gamma \). So in this case we see from \((*)\) that if \( x \) is a singularity of \( S_p \) then \( T_p Q \) contains \( T_{f(x)} \Gamma \), which means that \( T_p Q \) is tangent to \( \Gamma \) at \( f(x) \).

Finally, if \( x \in X_0 \), then \( f \ast T_x X = 0 \), so by \((*)\) the point \( x \) is certainly a singularity of \( S_p \).

Combining these observations, it follows that \( S_p \) has no singularities away from \( f^{-1}(p) \) if the hyperplane \( T_p Q \) lies in the intersection of the following 4 subsets of \( Q \lor \):

\[
U_1 := \{ H \in Q^\lor \mid H \cap (R \cup \text{Sing}(C) \cup \text{Sing}(\Gamma) \cup \Sigma) = \emptyset \},
U_2 := \{ H \in Q^\lor \mid H \text{ intersects } C \text{ transversally} \},
U_3 := \{ H \in Q^\lor \mid H \text{ intersects } \Gamma \text{ transversally} \},
U_4 := \{ H \in Q^\lor \mid H \cap B \setminus (\Gamma \cup \Sigma) \text{ is nonsingular} \}.
\]

Thus, in order to prove the lemma, it is sufficient to show that the sets \( U_i \) \((i \in \{1, 2, 3, 4\})\) are Zariski-open in \( Q^\lor \) and nonempty.

As \( R \cup \text{Sing}(C) \cup \text{Sing}(\Gamma) \cup \Sigma \) is a finite set, this is certainly true for \( U_1 \).

As for \( U_2 \), notice that in order to show that a general hyperplane \( H \in Q^\lor \) intersects \( C \) transversally, it is sufficient to show that this is true for every irreducible component of \( C \). So we can without loss of generality assume that \( C \) is irreducible.

Let \( Z \) denote the following closed subscheme of \( Q \times Q^\lor \):

\[
Q \times Q^\lor \supset Z := \{ (x, H) \in Q \times Q^\lor \mid x \in H \}.
\]

Consider the scheme-theoretic intersection \( (C \times Q^\lor) \cap Z \) and the restriction of the projections from \( Q \times Q^\lor \) onto \( Q \) respectively \( Q^\lor \) to this subscheme of \( Q \times Q^\lor \):

\[
q : (C \times Q^\lor) \cap Z \to C,
\]

\[
r : (C \times Q^\lor) \cap Z \to Q^\lor.
\]

The fiber of \( r \) over \( H \in Q^\lor \) is the scheme-theoretic intersection of \( C \) and \( H \). So in order to show that \( U_2 \) is open and dense in \( Q^\lor \), we have to prove
that the general fiber of $r$ is reduced. Notice that all fibers of $q$ are singular quadric surfaces, so they have constant Hilbert polynomial. It follows that $q$ is flat (see [Ha], Chapter III, Theorem 9.9). As all fibers of $q$ are reduced and $q$ itself is flat, we conclude that $U := q^{-1}(C \setminus \text{Sing}(C))$ is reduced. Now $U$ is a dense open subscheme of $(C \times Q^\vee) \cap Z$, so the general fiber of $r$, which is finite, is contained in $U$. Restricting to those fibers which are contained in the smooth part of $U$ and have empty intersection with the ramification locus of $r$, we see that the general fiber of $r$ is reduced. It follows that $U_2$ is Zariski-open in $Q^\vee$ and nonempty.

Replacing $C$ by $\Gamma$ and repeating the above reasoning proves that $U_3$ is also Zariski-open in $Q^\vee$ and nonempty.

In order to prove that $U_4$ is open and dense in $Q^\vee$, we will show that, for a general element $H \in Q^\vee$, the intersection of $H$ and $B$ is smooth away from the singular locus $\Gamma \cup \Sigma$ of $B$. As it is sufficient to show that this holds for every irreducible component of $B$, we may without loss of generality assume that $B$ is irreducible.

Assume first that $B$ is a hyperplane section of $Q$, corresponding to an element $H_B$ in $\mathbf{P}^4$. Then the hyperplane sections of $B$ correspond to the lines in $\mathbf{P}^4$ through $H_B$. If $H_B$ is not contained in $Q^\vee$, then every line through $H_B$ intersects $Q^\vee$ in 2 (not necessarily distinct) points. So every hyperplane section of $B$ can be written as the intersection of $B$ and an element of $Q^\vee$. If $H_B$ is an element of $Q^\vee$, then every line through $H_B$ which is not contained in the tangent space to $Q^\vee$ at $H_B$ intersects $Q^\vee$ in exactly one point different from $H_B$. So in this case the elements of a dense open subset of the space of hyperplane sections of $B$ can uniquely be written as the intersection of $B$ and an element of $Q^\vee$. In both cases we conclude from Bertini’s Theorem (see [G-H], page 137), applied to $B$, that the intersection of $B$ and a general element of $Q^\vee$ is smooth away from $\Gamma \cup \Sigma$. So $U_4$ is Zariski-open in $Q^\vee$ and nonempty.

From now on, assume that $B$ is not a hyperplane section of $Q$ and interpret $Q^\vee$ as a quadric system contained in the space $\mathbf{P}(H^0(B, \mathcal{O}_B(1)))$ of all hyperplane sections of $B$. In order to prove that the intersection of $H$ and $B$ is smooth away from $\Gamma \cup \Sigma$ for general $H$ in $Q^\vee$, we will use the following special case of Bertini’s Theorem (see [G-H], page 137). Here a one-dimensional linear system is called a pencil.

**Lemma 3** Let $X \subset \mathbf{P}^N$ be a projective variety and $\Lambda \subset \mathbf{P}(H^0(X, \mathcal{O}_X(1)))$
a pencil. Then the general element of $Λ$ is smooth away from the base locus of $Λ$ and the singular locus of $X$.

We will show that $Q^\vee$ contains enough pencils to globalise Lemma 3, which holds for any of these pencils, to a Bertini type theorem for the whole space $Q^\vee$.

For every element $H$ of $Q^\vee$ there is a one-dimensional family of pencils in $Q^\vee$ containing $H$, parametrised by a plane quadric curve. By Lemma 3, the general element of a pencil is smooth away from the base locus of the pencil and $Γ \cup Σ$. We will call such an element general for that pencil. Denote the base locus in $B$ of a pencil $Λ$ in $Q^\vee$ by $B_Λ$. Denote the isomorphism from $Q$ to $Q^\vee$, mapping $p ∈ Q$ to the hyperplane $T_p Q$, by $T$. An easy computation shows that

$$B_Λ = T^{-1}(Λ) \cap B.$$ (2)

By a dimension argument, there is a Zariski-open subset $V$ of $Q^\vee$ such that every element $H$ of $V$ is general for almost all pencils containing $H$. We will show that $V \setminus (V \cap T(B))$ is contained in $U_4$.

Let $T_p Q$ be an element of $V$. By (2) the intersection $B_Λ \cap B_Λ'$ of the base loci of any 2 pencils $Λ$ and $Λ'$ in $Q^\vee$ is empty if $p \notin B$ and equals $p$ if $p \in B$. As $T_p Q$ is an element of $V$, it follows that the hyperplane section $T_p Q \cap B$ is smooth away from $Γ \cup Σ$ if $p \notin B$. So the nonempty Zariski-open subset $V \setminus (V \cap T(B))$ of $Q^\vee$ is contained in $U_4$. We conclude that $U_4$ is open and dense in $Q^\vee$.

**Remark 1** Notice that, if $X$ is $P^3$ respectively a smooth three-dimensional quadric $Q$, then the statement of Theorem 1 follows also from Theorem 5 respectively the following theorem (see [P-S]):

**Theorem 4** Let $Y$ be a smooth quadric hypersurface of dimension at least 3, $X$ a smooth projective variety and $f : Y \to X$ a surjective morphism. Then either $f$ is an isomorphism or $X$ is isomorphic to a projective space.

Conversely, we conclude from Theorem 1 that the degree of a nonconstant morphism from $Q$ to itself is bounded. So every such morphism must have degree 1, as otherwise we could produce nonconstant selfmaps of $Q$ of arbitrarily high degree by composition.
More generally, let $X$ be a smooth threefold with Néron-Severi group $\mathbb{Z}$. From Theorem 1 it follows that, if $X$ does allow some nonconstant morphism to the quadric (for example if $X$ is a Fermat hypersurface of even degree in $\mathbb{P}^4$), then every nonconstant morphism from $X$ to itself has degree 1, so is an isomorphism.

**Remark 2** Replacing $Q$ by another smooth threefold with Néron-Severi group $\mathbb{Z}$, I did not manage to prove an analogue of Theorem 1. For instance, let $Y$ be a smooth complete intersection of $N−3$ hyperplanes of degrees $m_1, \ldots, m_{N−3}$ in $\mathbb{P}^N$. Let $X$ be as in Theorem 1 and $f : X \to Y$ a finite morphism of generator degree $d$. Then the degree of $f$ equals $H^3_X d^3 / (\prod_{i=1}^{N−3} m_i)$. So, if $H$ is a hyperplane section of $Y$ with $n$ ordinary double points, none of which lie in the branch locus of $f$, then $f^*(H)$ has $nH^3_X d^3 / (\prod_{i=1}^{N−3} m_i)$ ordinary double points. As the leading term of the Miyaoka bound for $f^*(H)$ in Theorem 3 is equal to $\frac{4}{9} H^3_X$, it is clear that the idea of the proof of Theorem 1 can only work in this case if

$$n \geq \frac{4}{9} \prod_{i=1}^{N−3} m_i.$$  \hspace{1cm} (3)

The point is that, in order to apply Theorem 3, it still has to be checked that for at least one such hyperplane section $H$ the surface $f^*(H)$ has only ordinary double points. If $Y$ is a cubic in $\mathbb{P}^4$ or the intersection of two quadrics in $\mathbb{P}^5$, it follows from (3) that we have to study hyperplane sections of $Y$ with at least 2 ordinary double points. In both cases I did not succeed in proving an analogue of Lemma 1. If $Y$ is a cubic, the system of hyperplane sections with 2 ordinary double points has only dimension 2 and I couldn’t apply any Bertini type argument; if $Y$ is an intersection of 2 quadrics, this system has dimension 3, but I could not get any grip on it.

Another possibility is to try to prove something weaker than Lemma 1. If one can prove that $f^*(H)$ has only mild singularities apart from the ordinary double points lying over the double points of $H$, then one may try to apply Theorem 3 after blowing up these singularities. Of course this is only possible if one knows these singularities well.

If one can prove that $f^*(H)$ contains only quotient singularities, then one may try to apply a more general version of Theorem 3 valid for surfaces with only quotient singularities (see [Mi]).
However, these approaches seem quite hard and I did not try them seriously up to now.

3 Proof of Theorem 2

Look at the special case where \( X \) is a smooth hypersurface of degree \( m \) in \( \mathbb{P}^4 \) and \( f : X \to Q \) a nonconstant morphism of generator degree \( d \). As \( X \) has Néron-Severi group \( \mathbb{Z} \), we can apply Theorem 1 which gives that the generator degree of \( f \) is bounded. To estimate this bound, consider expression (1). As

\[
\begin{align*}
H^3_X &= m, \\
H_Xc_2(X) &= m^3 - 5m^2 + 10m
\end{align*}
\]

and the index of \( X \) is equal to \(-5 + m\), this expression becomes:

\[
\frac{4}{9}md^3 + \frac{2}{9}m^2 - \frac{10}{9}md^2 + \left( \frac{4}{9}m^3 - \frac{10}{9}m^2 + \frac{10}{9}m \right)d.
\] (4)

The degree of \( f \) is equal to \( md^3/2 \). Thus, in this case, the upper bound on the generator degree of \( f \) we obtained in the proof of Theorem 1 is given by the maximal positive integer \( d \) for which (4) \( \geq \) \( md^3/2 \). Denote this integer by \( d_m \). A calculation shows that, for \( m \gg 0 \), the bound \( d_m \) we obtain in this way on the generator degree of \( f \) grows approximately linearly with \( m \):

\[d_m \sim (2 + 2\sqrt{3})m + \text{constant}.
\]

However, the generator degree of \( f \) is also bounded from below. Choose coordinates \((x_0 : \cdots : x_4)\) on the projective space \( \mathbb{P}^4 \) containing \( X \) and coordinates \((y_0 : \cdots : y_4)\) on the projective space \( \mathbb{P}^4 \) containing \( Q \) such that \( Q \) is given by the equation \( \sum_{i=0}^4 y_i^2 = 0 \). Then \( f \) is given by

\[
\begin{align*}
f : X &\longrightarrow Q, \\
(x_0 : \cdots : x_4) &\longmapsto (\phi_0(x_0 : \cdots : x_4) : \cdots : \phi_4(x_0 : \cdots : x_4))
\end{align*}
\]

where the \( \phi_i \) are homogeneous polynomials of degree \( d \), defined on \( X \). As the natural map \( H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(d)) \to H^0(X, \mathcal{O}_X(d)) \) is surjective, these polynomials can be extended to polynomials of degree \( d \), defined on \( \mathbb{P}^4 \) (but not necessarily in a unique way). The extensions will also be denoted by \( \phi_i \). Let \( X \subset \mathbb{P}^4 \) be given by the equation \( F_X = 0 \), where \( F_X \) is a homogeneous
polynomial of degree $m$. As $\sum_{i=0}^{4} \phi_i^2 = 0$ on $X$, it follows that

$$\sum_{i=0}^{4} \phi_i^2 = F_X G,$$

where $G$ is a homogeneous polynomial of degree $2d - m$. Thus, the generator degree of $f$ must be larger than or equal to $m/2$.

In fact, if $m$ is even, then there exist hypersurfaces $X$ of degree $m$ and morphisms of generator degree $m/2$ from $X$ to $Q$, for instance if $X$ is the Fermat hypersurface of degree $m$ in $\mathbb{P}^4$. From (5) it follows that, if $m$ is even, there is a morphism of generator degree $m/2$ from $X$ to $Q$ if and only if $F_X$ can be written as the sum of 5 squares of homogeneous polynomials of degree $m/2$, having no common zeroes on $X$.

Now consider the case $m = 3$, where $X$ is a smooth cubic. In order to prove Theorem 2, we will use the following theorem of Lazarsfeld (see [La]):

**Theorem 5** Let $X$ be a smooth projective variety of dimension at least 1 and let $f : \mathbb{P}^n \to X$ be a surjective morphism. Then $X \cong \mathbb{P}^n$.

*Proof of Theorem 5:* Let $f : X \to Q$ be a morphism of generator degree $d$. A computation shows that the upper bound $d_3$ on the generator degree of $f$, which was introduced above, is equal to 3. Morphisms of generator degree 3 cannot occur as the expression $3d^3/2$ which should be equal to the degree of such a morphism is not integer for $d = 3$. So, all that remains to be proven is that there are no morphisms of generator degree 2 between cubics and quadrics.

Assume $f$ has generator degree 2. As above, choose coordinates $(x_0 : \ldots : x_4)$ and $(y_0 : \ldots : y_4)$ on $\mathbb{P}^4$, and let $\phi_0, \ldots, \phi_4$ be homogeneous polynomials of degree 2, defining $f$. In this case we get, as in (5):

$$\sum_{i=0}^{4} \phi_i^2 = F_X L,$$

where $L$ is a homogeneous linear polynomial, defining a hyperplane in $\mathbb{P}^4$. This hyperplane will also be denoted by $L$, for convenience.

We claim that the $\phi_i$ do not have any common zeroes on the hyperplane $L$. As the $\phi_i$ do not have any common zeroes on $X$, the claim follows for
points in $X \cap L$. Now let $p$ be a point in $L \setminus (X \cap L)$. If $\phi_i(p) = 0$ for all $i \in \{0, \ldots, 4\}$, equation (3) implies that

$$\frac{\partial F_X L}{\partial x_i}(p) = 0, \text{ for all } i \in \{0, \ldots, 4\}.$$ 

As $L(p) = 0$ and $F_X(p) \neq 0$ by assumption, we get:

$$\frac{\partial L}{\partial x_i}(p) = 0 \text{ for all } i \in \{0, \ldots, 4\}.$$ 

But this is impossible because $L$, being a hyperplane, is nonsingular. This proves the claim.

Thus, the $\phi_i$ define a morphism from $L$ to $Q$. Restricted to the surface $L \cap X$, this morphism equals $f|_{L \cap X}$, so it is not constant. This is a contradiction by Theorem 3.

Remark 3 Notice that the argument in the proof of Theorem 2 works for every hypersurface $X$ of degree $m$ in $\mathbb{P}^4$ with a morphism $f : X \to Q$ of generator degree $d$ such that $2d - m = 1$. So, if $m$ is odd, there are no morphisms of generator degree $(m + 1)/2$ from $X$ to $Q$ (if $m \equiv 1 \mod 4$, this is also clear from the fact that the expression $md^3/2$ which should be equal to the degree of such a morphism is not integer in this case).

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