Detection and Removal of B-mode Dust Foregrounds with Signatures of Statistical Anisotropy

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ABSTRACT

Searches for inflationary gravitational wave signals in the CMB B-mode polarisation are expected to reach unprecedented power over the next decade. A major difficulty in these ongoing searches is that galactic foregrounds such as dust can easily mimic inflationary signals. Though typically foregrounds are separated from primordial signals using the foregrounds’ different frequency dependence, in this paper we investigate instead the extent to which the galactic dust B-modes’ statistical anisotropy can be used to distinguish them from inflationary B-modes, building on the work of Kamionkowski and Kovetz (2014). In our work, we extend existing anisotropy estimators and apply them to simulations of polarised dust to forecast their performance for future experiments. Considering the application of this method as a null-test for dust contamination to CMB-S4, we find that we can detect residual dust levels corresponding to $r > 0.001$ at $2\sigma$, which implies that statistical anisotropy estimators will be a powerful diagnostic for foreground residuals (though our results show some dependence on the dust simulation used). Finally, considering applications beyond a simple null test, we demonstrate how anisotropy statistics can be used to construct an estimate of the dust B-mode map, which could potentially be used to clean the B-mode sky.

Key words: cosmology: cosmic microwave background - Galaxy: ISM - methods: data analysis - methods: statistical

INTRODUCTION

Though the cosmic microwave background (CMB) has already provided us with remarkable insights into the properties of our universe, a wealth of cosmological information still remains to be extracted from the polarisation of the CMB. In particular, the CMB B-mode polarisation signal contains much information about weak lensing (e.g. Hirata & Seljak 2003) and inflationary gravitational waves (IGWs) (e.g. Kamionkowski et al. 1997). As direct predictions of many of the simplest models of inflation, B-mode signals induced by IGWs allow physics to be probed at the highest energy scales and are thus of great interest to the fundamental physics community.

The amplitude of any IGW B-modes is expected to be small given current limits, with $r < 0.07$ at 95% confidence (BICEP2 Collaboration 2016). Therefore, precise techniques are likely required to distinguish between these primordial IGW B-modes and galactic foreground B-modes, which are typically much larger. At large angular scale, the main source of polarised CMB foregrounds is diffuse synchrotron emission (at a CMB frequency $\nu \lesssim 100$ GHz) and thermal dust emission (at higher $\nu$) (Planck Collaboration X 2016; Kamionkowski & Kovetz 2016). The latter arises from the infrared emission of Galactic dust grains (including silicates, polycyclic aromatic hydrocarbons and carbonaceous grains) which are heated by optical photons (Draine 2011; Thorne et al. 2017). This emission is polarised due to the preferential alignment of spinning aspherical grains’ rotation axes with the Galactic magnetic field (GMF), giving contributions to both CMB E- and B-modes (Draine & Fraisse 2009).

The canonical method for distinguishing dust from the underlying CMB signal makes use of the fact that the frequency dependence of the two components differs (e.g. Planck Collaboration Int. XXII 2015); making use of data from different frequency bands (e.g. Planck Collaboration Int. XLIV 2016), IGW constraints can thus be obtained despite foreground contamination. However, foreground separation methods can suffer from a number of complications, including incomplete knowledge of the dust frequency de-
pendence in parametric methods, for instance due to multiple components at different temperatures (cf. Thorne et al. 2017) or decorrelation, potentially arising from different dust structures being probed at different frequencies. In addition, the application of multifrequency foreground separation is more difficult for ground-based CMB experiments, where only a restricted frequency range is available due to atmospheric effects.

Here we consider the approach proposed by Kamionkowski & Kovetz (2014) (hereafter KK14): to search for foreground-specific signatures of statistical anisotropy in the two-dimensional power-spectra of the B-mode signal, resulting from a roughly coherent Galactic magnetic field (GMF) on few-degree scales, as supported by observations (e.g. Clark et al. 2015; Martin et al. 2015; Kalberla et al. 2016). In this paper, we focus on the use of these statistical anisotropy estimators as a null test to detect dust residuals in a poorly-cleaned CMB map, but also briefly discuss their use in mapping foregrounds and as a tool for dust-removal (‘dedusting’). We test the performance of these methods with reference to current and future CMB experiments, using simulated polarised thermal dust emission maps from Vansyngel et al. (2017, hereafter V17), as well as realistic noise and lensing implementations for futurist CMB experiments such as CMB-S4 (Abazajian et al. 2016). (In some cases, we will also compare with thermal dust simulations from Planck Collaboration XII (2016), hereafter FFP8.) A Python implementation of the code used in this analysis is freely available online.1

Although there is a range of previous work regarding the search for non-Gaussianity and statistical anisotropy in dust emission (stretching back to Komatsu et al. 2002), less research has been done on the applications of such a detection of anisotropy; however, the work of Rotti & Huf fenberger (2016) provides one such example. In this work, a bipolar-spherical harmonic basis is used to construct an anisotropy test for Planck 353 GHz maps, which is then utilised to assess the cleanest regions of the foreground sky. Here, we use high-resolution simulated data instead of Planck maps to emulate future experiments and focus on the use of anisotropy detection in the context of null-tests. In addition, much of the literature is concerned with detecting generalised anisotropy, whereas in this paper we consider only a very specific anisotropy signature typical of dust (or synchrotron emission) in a coherent GMF.

The paper is organised as follows. Sec. 2 describes the mathematical background to the work, including a description of the KK14 estimators, before the simulations are discussed in Sec. 3 and the full methodology is presented in Sec. 4. Sec. 5 details the application to null tests, and further results are presented in Sec. 6 through 7 (discussing spatial distributions, correlations and ‘dedusting’ respectively). Conclusions and a summary are then made in Sec. 8.

2 DUST ESTIMATORS

First, we describe the B-mode dust model used in this study. Following KK14, we assume that, across a sufficiently small section (‘tile’) of the CMB sky, the GMF should be roughly coherent, leading to the dust having a uniform polarisation direction. This gives rise to a particular hexadecapole anisotropy pattern in the two-dimensional B-mode power spectrum of that tile, $|B(l)|^2$ (for Fourier mode $B(l)$). For a spectral pixel at position vector $l$ and angle $\phi_l$ from the positive RA axis, the power is expected to take the form;

$$|B(l)|^2 = A f^\text{pol}_l [1 - f_s \cos 4 \phi_l - f_c \sin 4 \phi_l],$$

where the fiducial isotropic $l (= |l|)$ dependence is given by Planck Collaboration Int. XXX (2016) as $C_{l}^{\text{fid}} = l^{\gamma}$ with $\gamma = 2.42 \pm 0.02$, obtained by averaging over regions away from the galactic plane with no evidence for departure from this behaviour over 1% sky regions (BICEP2/Keck Array and Planck Collaborations 2015). The key model parameters in Eq. 1 are the monopole dust amplitude $A$ and sine (cosine) hexadecapole coefficients $A_f$, ($A_c$) which define the model anisotropy. $f_s$ and $f_c$ denote the fractional hexadecapole power, but will not be directly used in this analysis. From KK14, we note that $A_f$ and $A_c$ are proportional to $\sin 4\alpha$ and $\cos 4\alpha$ respectively, where $\alpha$ is the dust polarisation angle defined by

$$2\alpha = \arctan \frac{U}{Q}$$

in the IAU polarisation convention.2 This can be found from the measured hexadecapole parameters via

$$\tan 4\alpha = \frac{A_f}{A_c}.$$ (3)

In this analysis $A_f$ and $A_c$ are used rather than the fractional quantities ($f_s$ and $f_c$) due to their higher signal-to-noise ratio (since $f_s$, $f_c$) is influenced by errors in both $A_f$ ($A_c$) and $A$. A generalisation of this to a continuous description that allows for variations in the GMF direction is also presented in KK14 but we adopt the simpler procedure in this paper.

$A_f$ and $A_c$ may be computed for a specific tile using the following estimators (cf. KK14, Eq. 9-10):

$$\hat{A} = \frac{\sum l B^2(l) A_l^2 / C_{l}^{\text{fid}}}{\sum l A_l^2}$$

$$\hat{A}_f = \frac{\sum l B^2(l) A_l^2 \cos 4\phi_l / C_{l}^{\text{fid}}}{\sum l (A_l \cos 4\phi_l)^2}$$

$$\hat{A}_c = \frac{\sum l B^2(l) A_l^2 \sin 4\phi_l / C_{l}^{\text{fid}}}{\sum l (A_l \sin 4\phi_l)^2},$$ (4)

summing over all pixels at positions $l$. $B^2$ is the expected B-mode power due to dust alone;

$$B^2(l) = |B(l)|^2 - (C_{l}^{\text{ens}} + C_{l}^{\text{pol}})$$

for total 2D measured power $|B(l)|^2$, accounting for (isotropic) contamination from lensing and noise modes with known profile. This differs from that of KK14 (who just use $|B(l)|^2$ directly in the estimators), and is chosen to reduce bias in the low-dust limit. (We note that this is not used

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1 A Hexadecapolar Analysis for Dust Estimation in Simulations: http://github.com/oliverphilcox/HADES

2 We here adopt the IAU polarisation convention used by flipper, in contrast to the COSMO convention of Planck and HEALPix; this corresponds to a sign inversion of the U-map.
for cross-spectra between dust and isotropic Monte Carlo simulations where \( \langle B(I)^2 \rangle = 0 \) is expected.) In Eqs. 4, \( \Lambda_l \) is the signal-to-noise ratio for each pixel in the 2D power spectrum, chosen here as

\[
\Lambda_l = \frac{A_{l}^{\text{fid}}}{A_{l}^{\text{fid}} + C_{l}^{\text{noise}} - C_{l}^{\text{map}}}. \tag{6}
\]

(cf. KK14, Eq.12), which reduces to \( \Lambda_l = 1 \) in the noiseless limit, affording equal weight to all pixels. The dust-contributions to \( G_l \) are included in the denominator to account for errors in the parametrisation. We note that \( \Lambda_l \) depends on the monopole amplitude \( A \), hence \( \Lambda \) is computed iteratively, using a first estimate of \( A_0 = 1 \mu K^2 \) and stopping when the estimates for \( A \) converge to within 1% (this generally occurs within a few iterations, even for a far-off initial guess of \( A_0 = 1K^2 \)). In addition, once convergence is achieved, the same value of \( A \) is used in \( \Lambda_l \) for all applications of the estimators to this specific tile.

We note that we have assumed all components aside from dust and lensing to have an isotropic power spectrum. This may not be true in real data, for example if noise is anisotropic due to the survey scan strategy, or if a multifrequency component separation algorithm has been applied that does not preserve isotropy (such as a Needlet Internal Linear Combination algorithm). While the noise problem can easily be solved by relying on cross-spectra, anisotropic effects of component separation would need to be treated by subtracting off a simulated bias, or by a modification of the cleaning procedure. We defer such considerations to future work.

In addition to \( A, A_{f} \) and \( A_{c} \), we introduce the polarisation angle-independent hexadecapole amplitude, \( H \), via

\[
H^2 = (A_{f})^2 + (A_{c})^2. \tag{7}
\]

This quantity is motivated by the fact that, if we assume no prior knowledge of the dust polarisation direction, \( A_{f} \) and \( A_{c} \) are equally likely to be positive or negative; a quadratic statistic is thus needed. To compute the variance of such a measurement of \( H^2 \), under the null hypothesis of no anisotropy being present, we use a large set of Monte Carlo (MC) simulations created from random Gaussian realisations of the isotropic power spectrum for each tile. A further treatment of this, and the biases to \( H^2 \) arising from the non-negative nature of the statistic, are discussed in Sec. 4.3 and appendices A and B.

The debiased fractional anisotropy can also be defined as \( \varepsilon = \sqrt{f_{2}^2 + f_{3}^2} = H/A \), although we note that this suffers from the same signal-to-noise limitations as \( f_{2} \) and \( f_{3} \). For convenience, all model parameters and quantities derived from the estimators are summarised in tables 1 and 4.5 respectively.

3 SIMULATIONS

3.1 Noise & Lensing B-modes

To simulate realistic CMB experiments we must consider other (non-dust) sources of B-mode power, arising from noise and weak lensing of large scale structure. (At first, we will neglect contributions from IGWs, which are subdominant to lensing contributions over the \( l \)-ranges used here, given current constraints on the tensor-to-scalar ratio (BICEP Collaboration 2016); however, we explore this further in Sec. 5.2.)

For instrument noise, we take the commonly used parametrisation (e.g. Knox 1995; Zaldarriaga et al. 1997; Hu & Okamoto 2002)

\[
C_{l}^{\text{noise}} = \Delta_p^2 \exp \left( \frac{(l+1)\theta_{\text{FWHM}}^2}{8 \ln 2} \right) \tag{8}
\]

for noise power \( \Delta_p \) (in \( \mu K\text{-arcmin} \)) and beam full-width half-maximum (FWHM) \( \theta_{\text{FWHM}} \) (in radians, presented here in arcmin). The fiducial values we assume for each experiment are shown in table 2. This model assumes the noise to be isotropic, unlike the expected B-mode dust signal.

To simulate maps in the analysis below, noise modes are added separately to each tile cut out from the dust simulations, using Gaussian realisations of the above power spectrum. A more physical approach would be to generate a full-sky noise map which could be partitioned and added to the dust modes, taking into account large-scale correlations. However, as shown in appendix C, both methods give almost identical results, and hence the tile-based approach will be adopted here for computational efficiency.

In this study, we include lensing modes using a lensed CMB map from the Planck FFP10 simulation suite, created with a random realisation of the lensing potentials. Since there is considerably more power in lensed E-modes than B-modes, when we compute B-modes from small cut-outs of the Stokes Q and U maps, there can be significant E-to-B mode leakage. To account for this, the scalar E-modes were removed from the lensed full-sky spherical-harmonic data before the Stokes maps were computed, thus nulling the additional leakage power in our computed B-modes. Although this is unphysical, it can be justified by noting that we are here forecasting experimental performance, and a maximum likelihood approach or better E-B separation method (e.g. Grain et al. 2009) would be able to minimise E-B leakage to a significant extent. (Additionally, a continuous implementation of our method (c.f. KK14) would obviate the need for E-B decomposition on small tiles.)

Each individual lensing event is expected to have a small quadrupolar B-mode anisotropy signature (e.g. Kamionkowski & Kovetz 2016); however, the power derives from a sum over many lensing modes, leading to the power spectrum becoming almost isotropic when averaged over the \( l \)-regions used here. To approximate the process of ‘delensing’ (e.g. Knox & Song 2002; Kesden et al. 2002; Smith et al. 2012; Manzotti et al. 2017), we rescale the lensing power spectrum by a factor \( f_{\text{lens}} \), where \( f_{\text{lens}} = 0.1 \) would signify 90% delensing for example. Fiducial values for this are given in table 2.

Fig. 1 (left panel) shows the contributions to the isotropic \( C_{l}^{BB} \) power spectrum from the relevant components using two sets of noise parameters and delensing efficiencies. It is apparent, for the plotted dust-level, that a significant detection of anisotropy at \( l > 90 \) region available to this test (Sec. 4.2) will require low noise and, ideally, a substantially delensed map.

\footnote{Available from NERSC: \url{https://nim.nersc.gov/}}
Table 1. Model variables used in this study with their default values. For full descriptions see the text.

| Symbol | Definition | Fiducial Value |
|--------|------------|----------------|
| $\gamma$ | Fiducial isotropic map slope (Sec. 2) | 2.42 |
| $\Lambda_l$ | Estimator signal-to-noise ratio (Sec. 2) | Eq. 6 |
| $\Delta_p$ | Noise power (Eq. 8) in $\mu$K-arcmin | see tab. 2 |
| $\theta_{\text{FWHM}}$ | Noise FWHM (Eq. 8) in arcmin | see tab. 2 |
| $f_{\text{ens}}$ | Fractional contribution of lensing to $C_l^{BB}$ (Sec. 3.1) | see tab. 2 |
| $\nu$ | Simulated map frequency (Sec. 3.2) | 150 GHz |
| $\beta_d$ | Dust emissivity spectral index (Sec. 3.2) | 1.53 |
| $T_{\text{dust}}$ | Equilibrium dust grain temperature (Sec. 3.2) | 10.6 K |
| $\Delta \theta$ | Tile width in degrees (Sec. 4.1) | 3° |
| $W(x)$ | Real-space window function | - |
| $R_{\text{pad}}$ | Ratio of widths for zero padding (Sec. 4.2) | 2 |
| $l_{\text{min}}$ | Lowest $l$ used in statistical estimators (Eq. 4) | $360/(R_{\text{pad}} \Delta \theta)$ |
| $\Delta l$ | 1D spectrum binning width (Sec. 4.3) | $240/R_{\text{pad}} (\Delta \theta)$ |
| $N_{\text{sims}}$ | Number of Monte Carlo simulations (Sec. 4.3) | 500 |
| $N_{\text{bias}}$ | Number of bias simulations (Sec. 4.4) | 500 |
| $f_{\text{dust}}$ | Real-space fractional dust contribution (Sec. 5) | 1 |
| $r_{\text{eff}}$ | Effective tensor ratio for mean patch dust level (Sec. 5) | Eq. 14 |

Figure 1. Contributions to the B-mode power spectrum, $C_l^{BB}$. Left: A fiducial $l^{-2.42}$ dust power spectrum (red) at 353GHz (full lines) and 150GHz (dashed lines) with frequency scaling obtained via the Planck dust spectral energy distribution (Tibbs et al. 2012; Planck Collaboration 2018). The amplitude is taken from a representative tile in the 5% sky patch (Sec. 4.1) using the Vansyngel et al. (2017, V17) simulation. Also shown is the lensing B-mode spectrum (black, taken from a full-sky Planck FFP10 lensed scalar simulation) assuming 0% (full lines) and 90% delensing (dashed lines) efficiency. Noise modes (green) are displayed using the fiducial noise model, (Eq. 8, e.g. Knox 1995) with 5$\mu$K-CMB (full-lines) and CMB-S4 (dashed lines) experimental parameters (table 2). Regions with too low $l$ to be investigated via our methods (using 3° tile-widths) are shown in grey. Right: Binned B-mode power spectra of the V17 and FFP8 simulations extrapolated to 150 GHz, computed over the 5% sky mask for $l \in [50, 2000]$ using the PolSpice pseudo-$C_l$ estimator (Chon et al. 2004), with a maximum angle of 50° and binned with $\Delta l = 50$. No mask-deconvolution is performed here, which is not expected to have a large impact due to the wide mask used. A dashed line indicates the fiducial $l$-dependence (here calibrated to the V17 simulations at $l = 80$).

3.2 Dust Maps

The primary thermal-dust simulation used in this paper is from Vansyngel et al. (2017, V17). However, we also compare it to the simulation of Planck Collaboration XII (2016, FFP8). Both simulations cover the full-sky and make use of Planck $D_{353}$ dust intensity maps at 353 GHz (where the Stokes maps are dominated by thermally emitting dust). Each is given to a HEALPix 4 (Görska et al. 2005) resolution of $N_{\text{SIDE}} = 2048$; we use data up to $l = 2000$ in our analysis. B-mode power spectra for the two simulations are shown in Fig. 1 (right panel), computed for the 5% sky region used via the pseudo-$C_l$ estimator PolSpice 5 (Chon et al. 2004) which takes into account the effects of masking, using a apodisa-

4 http://healpix.sf.net
5 http://www2.iap.fr/users/hivon/software/PolSpice/
Polarised Dust Anisotropy

Table 2. Assumed noise and lensing parameters (as defined in Table 1) for current and future CMB experiments. The 5µK′-CMB experiment is included to emulate upcoming experiments with noise parameters similar to that of the Simons Observatory (SO). BICEP2, SO and CMB-S4 parameters are from BICEP2 Collaboration (2014); Suzuki et al. (2016); Abazajian et al. (2016) respectively. We note that the f\textsubscript{rms} values are rough approximations only.

| CMB Experiment | \(\Delta_P [\mu K']\) | \(\theta_p\) [\(\degree]\)] | \(f_{\text{rms}}\) |
|----------------|------------------|-----------------|-----------|
| Zero-noise     | 0                | 0               | 0         |
| BICEP2         | 5                | 30              | 1         |
| 5µK′-CMB       | 5                | 1.8             | 0.4       |
| CMB-S4         | 1                | 1.5             | 0.1       |

This simple scaling may not be perfectly accurate, since we neglect the existence of multi-temperature dust components and the (potentially frequency-dependent) spatial variation in \(T_{\text{dust}}\) and \(\beta_l\) (Thorne et al. 2017). Whilst these are an important consideration for multifrequency dust-cleaning techniques where high precision is needed, here we require only an estimate of the frequency dependence to generate simulations and hence forecast the applicability of our techniques to future experiments.

The simple foreground model leading to our anisotropy estimators (Eqs. 4) assumes similar effects in both E- and B-modes and hence that the power spectra are equal: \(C_{lEE} = C_{lBB}\). This contradicts the measurements of Planck Collaboration Int. XXX (2016), which have \(C_{lEE} \approx 2C_{lBB}\). The difference between E- and B-mode power is believed to arise from coupling of the filamentary structure of dust with the magnetic field, affecting both polarisation direction and level of intensity fluctuations (Clark et al. 2015; Planck Collaboration Int. XXXVIII 2016; Caldwell et al. 2017). This effect is included in V17 via their anisotropic realisations of the turbulent filamentary GMF with hyperparameters tuned to match Planck data, but not in the earlier FFP8 simulations, since they pre-date the Planck discovery of asymmetry.

In the context of this work, since our primary V17 simulations include these E-mode and B-mode effects in a physically motivated manner, the results presented below already take the asymmetry into account, with differences resulting in only a potential loss of significance due to the estimator being somewhat sub-optimal. By relying on more physically correct models of the dust B-modes in future work, one could potentially develop an improved (yet similar) estimator that would be even more sensitive to dust contamination.

4 IMPLEMENTATION

Below we describe the computation of the hexadecapole parameters (Sec. 2) from the full-sky simulated Stokes maps.

4.1 Cutting out Tiles

To match the simulated data to futuristic experiments, for the majority of this analysis we consider only parts of the entire sky, hereafter denoted ‘patches’. A simulation region of size 2150 deg\(^2\) is used primarily, covering 5% of the sky with Galactic co-ordinates [-110°, -20°] (RA), [-85°, -45°] (dec). This was chosen since it (a) encloses the BICEP region, and (b) is a region of minimal dust, as is desirable to search for IGWs. This mask was smoothed with a 2° FWHM Gaussian kernel in REALPIX to avoid edge effects, and is the default patch used in the analysis. For the Simons Observatory (SO), we note that the actual experimental region may be as large as 4000 deg\(^2\) (Suzuki et al. 2016), which would boost the significance of any detections of hexadecapolar anisotropy compared to that found by our mock 5µK′-CMB experiment.

The patch is then partitioned into tiles of width \(\Delta\theta\) (where \(\Delta\theta\) is small to ensure GMF coherency) by projecting each section of the relevant whole-sky (dust or lensing) Stokes map onto a flat grid in galactic co-ordinates. This partitioning is also applied to the mask map, which, when multiplied by a half-cosine apodisation window of width \(\Delta\theta/10\) (to
remove edge discontinuity effects), provides a window function for each tile. A discussion of the potential biases given by the projection onto the flat-sky is given in appendix D. We note that there is some small information loss due to the apodisation, since the unapodised window functions do not overlap (to avoid double counting modes), thus we lose some data at the edges of each tile. This may slightly reduce the significances of anisotropy detection.

Here, square tiles of $\Delta \theta = 3'$ are used, giving a sky fraction of 0.02% per tile.\footnote{Whilst larger tiles allow lower $l$ to be probed (where there are greater dust contributions), this reduces the coherency of the GMF necessary to observe the hexadecapole structure. $\Delta \theta = 3'$ was found to give the best performance in initial studies.} For studies investigating the spatial distribution of the hexadecapole parameters $a$ and $H^2$ across the patch (Sec. 6), the tiles are chosen to be overlapping, with centres separated by 0.5', to give increased resolution for display in figures. However, the overlapping tiling is not used to calculate any detection significances (Sec. 4.5).

4.2 Mock Data and Power Maps

From the cut-out Q and U lensing and dust tiles, we compute two-dimensional Fourier-space B-mode maps using \textit{flipper}\footnote{https://github.com/sudeepdas/flipper} (Das et al. 2009) and \textit{flipperPol}\footnote{https://github.com/amaurea/flipperpol} (Louis et al. 2013), via the latter’s ‘hybrid’ method, which uses spatial derivatives of the window function to avoid dust E- to B-mode leakage. These are used to create a combined Fourier-space map (hereafter referred to as the ‘mock data’):

$$B_{\text{tot}}(l) = \hat{f}_{\text{dust}} B_{\text{dust}}(l) + \sqrt{f_{\text{dust}}} B_{\text{dust}}(l) + B_{\text{noise}}(l)$$

where $f_{\text{dust}}$ is included to represent delensing and we generate a Fourier-space noise map, $B_{\text{noise}}(l)$, for each tile separately from a Gaussian realisation of the 1D noise spectrum (Eq. 8). A scaling factor $\hat{f}_{\text{dust}}$, corresponding to the level of dust residuals, is included to emulate a null test after incomplete foreground subtraction. The 2D power spectrum is given by $|B_{\text{tot}}(l)|^2/\langle W^2 \rangle$, where the angle bracket indicates a spatial average over the window function $W^2(x)$ (Namikawa & Takahashi 2014). We note that this procedure is only approximate, and a more complete treatment would perform full inversion of the mode-coupling matrix.

Since we only consider small tiles, there exists a limited resolution of the $l$-space pixels constraining the minimum $l$ that can be used in our estimators (given by $l_{\text{min}} \approx 360' / \Delta \theta$). To ameliorate this, we applied zero-padding to increase the effective resolution of the maps, using a padding ratio, $R_{\text{pad}}$ (the factor by which each tile width is increased), of 2.

Following this, the anisotropy estimators are applied to the combined power map, using $l \in [l_{\text{min}}, 000]$, with $l_{\text{min}} = 360 / (R_{\text{pad}} \Delta \theta) = 90$. It can be shown that there is a small additional bias caused by the square shape of the individual pixels, with, for example, the central nine pixels all satisfying $\sin 4\theta = 0$ (independently of the padding). This gives unphysical contributions to $A_f$ and $A_{fc}$, implying that the estimated quantities are not invariant under rotation of the pixel-grid of the power-map. To account for this, we performed 20 uniformly spaced rotations of the map, averaging over the rotationally corrected estimated quantities. Further details of this and zero-padding may be found in appendix E.

4.3 Monte Carlo Errors

To compute the variances in the hexadecapole parameters, we adopt a Monte Carlo (MC) procedure, based on the null hypothesis of there being no anisotropy present. We must therefore generate MC simulations which are isotropic, but have the same 1D power spectrum as is present on each tile. For this purpose, first the 1D power spectrum of the combined 2D power-space map is computed by binning in annuli with a width of $\Delta l = l_{\text{min}} = 60$ (equal to the pixel width). The $l$ co-ordinate for each annular bin was estimated via a simple analytic model of the expected 1D spectrum (using the fiducial $C_l$ and the computed monopole amplitude $A$), giving greater accuracy than assuming it to be in the centre of the $l$-range for each bin. The binned (logarithmic) data was then fit to a univariate spline curve, which well represents the data.

Gaussian random field maps drawn from this 1D power spectrum are generated as for the noise spectrum, applying zero-padding as previously. (The 2D power spectra of these maps will be isotropic on average, but, due to the stochastic nature of the limited-resolution maps, will still exhibit some random fluctuations of anisotropy.) The hexadecapole parameters are computed for $N_{\text{MC}}$ MC realisations, giving estimates for the variances and distributions of $A$, $A_f$, and $A_{fc}$.

Both the isotropic simulations and the pipeline were tested by comparing the estimated anisotropy parameters to those of input isotropic, dust-free maps and verifying that a null response was obtained. This was satisfied for sufficiently large $N_{\text{MC}}$, here set to 500.

4.4 Realisation Dependent Debiasing

Due to the non-negative nature of the $H^2$ statistic, there will be a significant biasing contribution from random fluctuations of isotropic maps, giving $(H^2_{\text{MC}}) \neq 0$ even for isotropic MC maps. Since $H^2$ is a four-point correlation function in $B$, this bias can be understood as a Gaussian or disconnected contribution to $H^2$. Naïvely, we may debias by subtracting $(H^2_{\text{MC}})$ from the mock data-derived $H^2$ estimate, but we use a more robust “realisation-dependent” approach here, which can correct for first-order errors in the estimation of the 1D power spectra. (A similar approach is typically used in measurements of the lensing power spectrum, cf. Namikawa & Takahashi 2014). Utilising estimates of $H^2$ obtained from power spectra of the MC Fourier-space maps (denoted $SS$) and cross-spectra of these with the mock data maps (denoted $DS$), we compute an isotropic bias term

$$\langle H^2 \rangle_{\text{bias,iso}} = 4\langle H^2 \rangle_{DS} - \langle H^2 \rangle_{SS}$$

as derived and explained in appendix A. (For computational efficiency, we approximate this full expression for the bias as $(H^2)_{\text{bias,iso}} = (H^2)_{SS}$ when using Monte Carlo simulations to obtain error bars and significances.) The debiased hexadecapole statistic is thus

$$H^2 = H^2 - \langle H^2 \rangle_{\text{bias,iso}}$$
which can be used to compare data and simulations, since it should (a) have zero mean for an isotropic map, and (b) be robust to small errors in the assumed power spectra of MC simulations. Here we use $N_{\text{bias}} = 500$ independent MC simulations for each tile (distinct from the previous set of simulations) to estimate a expectation value for this bias.

Since $\mathcal{H}^2$ derives from four copies of the B-mode Fourier map, we normalise it by dividing by $\sqrt{\langle W^2 \rangle}$ rather than $\langle W^2 \rangle$ (Namikawa & Takahashi 2014).

To compute the significance of hexadecapole detection for a single tile, we must consider the statistics of $\mathcal{H}^2$ for the isotropic MC simulations. Given a cumulative distribution function, $p(\mathcal{H}^2)$, we can compute the percentile value for the data-derived $\mathcal{H}^2$ estimate, allowing us to assess whether the data is indeed anisotropic (for $p \rightarrow 1$) or whether it is characteristic of random fluctuations of an isotropic spectrum (for $p \sim 0.5$). An analytic (exponential) CDF is derived in appendix B; here we adopt a simpler procedure, computing the statistical percentile from the $N_{\text{sims}} = 500$ MC simulations alone.

In addition, we define the equivalent significance, $\mathcal{P}_C(\mathcal{H}^2)$ of a detection of anisotropy, by converting the statistical percentile $p(\mathcal{H}^2)$ into an associated $\sigma$ value according to a standard Gaussian distribution CDF (e.g. $p = 95\%$ corresponds to $\mathcal{P}_C = 2$). This is used for greater clarity with $\langle \mathcal{P}_C \rangle \approx 0$ for isotropic tiles, and large $\mathcal{P}_C$ representing significant anisotropy.

### 4.5 Patch Anisotropy

To assess the likelihood of anisotropy over an entire patch of the sky, we define the patch anisotropy, $\Xi$, whose estimator is given by the mean of the hexadecapole power $\mathcal{H}^2$

$$\Xi = \frac{1}{N} \sum_{i=1}^{N} \mathcal{H}^2,$$

(12)

where $N$ is the number of non-overlapping tiles. Using the MC simulations, we can construct a probability distribution in $\Xi$ for isotropic dust realisations, which is well approximated as a Gaussian, $N(\mu_X, \sigma_X^2)$, from the Central Limit Theorem for large $N$ (appropriate for the 5% sky patch, with $N = 245$).

The mock data patch-hexadecapole value, $\Xi_0$, can thus be compared to the MC distribution via its significance:

$$S = \frac{\Xi_0}{\sigma_\Xi} - \langle S_{\text{bias}} \rangle,$$

(13)

indicating how well anisotropy can be detected in a particular experiment. We include a bias term to account for the small hexadecapole signature from lensing, though it is very small, corresponding to $\langle |S_{\text{bias}}| \rangle < 1.5\sigma_\Xi$ in all cases. (No bias is found when Gaussian isotropic realisations of the CMB spectrum are used instead of the FFP10 maps).

Due to long-wavelength fluctuations of lensing modes and large-scale power in dust, there will be some correlations in $\mathcal{H}^2$ between tiles even in the absence of anisotropy, meaning the tiles are not strictly independent (as assumed by the Central Limit Theorem). The effects of this are limited since the lowest measurable mode has $l = 90$, and the distribution was found to be well fit by a Gaussian.

### Table 3. Estimated quantities derived from the hexadecapole and other estimators (Eq. 4)

| Symbol | Definition |
|--------|------------|
| $A$ | Monopole dust B-mode amplitude (Eq. 1) |
| $A_{4s}, A_{4c}$ | Sine & cosine hexadecapole coefficients |
| $f_s, f_c$ | Fractional hexadecapole coefficients |
| $H^2$ | Hexadecapole strength (Eq. 7) |
| $\mathcal{H}^2$ | Debias Hexadecapole strength (eq. 11) |
| $\epsilon$ | Anisotropy Fraction (H/$A$) |
| $\alpha$ | Anisotropy angle (Eq. 9) |
| $\sigma_{A_{4s}}, \sigma_{A_{4c}}$ | MC error (mean of $\sigma_{A_{4s}}$ and $\sigma_{A_{4c}}$) |
| $p(H^2)$ | Anisotropy Likelihood (Eq. B3) |
| $\mathcal{P}_C(H^2)$ | Equivalent anisotropy significance (Sec. 4.3) |
| $\Xi$ | Patch Hexadecapole (Eq. 12) |
| $S$ | Significance of $\Xi$ (Eq. 13) |

### 5 NULL TESTS

#### 5.1 Quantifying the Power of Statistical Anisotropy Null Tests for Future Experiments

An important application of these dust anisotropy estimation techniques is to perform null tests to assess how well the hexadecapole estimators can detect residual dust after dust-subtraction using standard (multi-frequency) methods (e.g. Tegmark & Efstathiou 1996; Leach et al. 2008). We imagine an experiment which has performed dust cleaning using maps at multiple frequencies, uniformly reducing the real-space dust level to a fraction $f_{\text{dust}}$. How significantly can we then detect a dust residual in the 5% sky region? Here, this is achieved by rescaling the V17 and FFP8 dust maps by $f_{\text{dust}}$, and computing the associated significance of anisotropy detection ($S$), as described above.

To provide a physical scale to $f_{\text{dust}}$ (following Planck Collaboration Int. XXX 2016) we define the effective tensor-to-scalar ratio, $r_{\text{eff}}$, as the ratio of dust monopole amplitude to the expected B-mode power from tensor modes at $r = 1$. $r_{\text{eff}}$ is thus a useful quantity, since it corresponds to the level of a spurious $r$-detection that (potentially) could be investigated with our null test. We evaluate the dust amplitude $A$ at $l = 80$ (approximating the peak sensitivity of future ground-based experiments), giving

$$r_{\text{eff}}(f_{\text{dust}}) = \frac{\langle A_{80} \rangle}{\sigma_{A_{80}}(r = 1)} f_{\text{dust}}^2,$$

(14)

for expected tensor contribution $\epsilon_{\text{tensor}}(r = 1) = 7.31 \times 10^{-5} \mu K^2$ (from CMB, Lewis et al. 2000), and mean dust power at $l = 80$ $\langle A_{80} \rangle$ (averaged over all 245 tiles with no lensing or noise modes included). Using the 5% sky region, $\langle A_{80} \rangle$ is given by $3.9 \times 10^{-5} \mu K^2$ (1.5 $\times 10^{-4} \mu K^2$) in the V17 (FFP8) simulation.

Fig. 2 shows the results of this for a range of values of $f_{\text{dust}}$, using V17 simulations with noise and lensing parameters appropriate for the 5$\mu K$-CMB and CMB-S4 experiments (see table 2). The significance is plotted in units of $\sigma_{\Xi}$, showing the mean and standard error of $S$ for each $f_{\text{dust}}$ value obtained from 10 iterations of the test. (We expect some variation in $S$ due to the stochastic nature of the noise contributions to the mock data.) There is a similar shape to the plots for both experiments with greater $S$ for CMB-S4 due to the smaller noise levels and larger assumed delensing efficiency. As expected, $S$ rises monotonically with $r_{\text{eff}}$ due

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Figure 2. Significance, $S$ (in units of $\sigma_\Xi$), of a detection of dust statistical anisotropy via the patch-averaged hexadecapole, $\Xi$. This is given as a function of residual dust fraction, $f_{\text{dust}}$ and equivalent effective $r$ (Eq. 14) with green dashed lines representing 2, 5, 10 and 20$\sigma_\Xi$ significances. The large values of $S$ imply that this would be a valuable null test for future experiments. This uses the V17 dust simulation with noise and lensing parameters appropriate for the 5$\mu$K$'$-CMB and CMB-S4 experiments (tab. 2). Mock data were obtained from 245 3° width tiles of the 5% sky region, using $N_{\text{bias}} = N_{\text{sims}} = 500$ and error-bars represent the standard error obtained by re-running the analysis 10 times for each data-point. The significances are corrected for the small intrinsic MC error from lensing ($\sim 1\sigma_\Xi$) as described in Sec. 4.5.

Table 4. Projected upper bounds on $r_{\text{eff}}$ (Eq. 14) for given significance levels, $S$, for two forthcoming CMB experiments, using the V17 simulations across a 5% sky region split into 3° tiles with a padding ratio, $R_{\text{pad}}$, of 2.

| $S$ $|\sigma_\Xi|$ | 5$\mu$K$'$-CMB | CMB-S4 |
|---|---|---|
| 2 | 0.02 | 0.001 |
| 3 | 0.03 | 0.002 |
| 5 | 0.05 | 0.004 |
| 10 | 0.1 | 0.01 |

Fig. 3 compares the results from the FFP8 and V17 simulations using CMB-S4 noise and lensing parameters, and we note markedly (and surprisingly) different behaviour between the two dust models. (The $f_{\text{dust}}$ axis is not included here since $r_{\text{eff}}$ is calibrated with a different $\langle A_{80} \rangle$ for each simulation.)

First, we note that the maximum $r_{\text{eff}}$ (at $f_{\text{dust}} = 1$) is much greater for FFP8; a consequence of a larger mean dust amplitude in the 5% sky region, with $\langle A_{80} \rangle_{\text{FFP8}} / \langle A_{80} \rangle_{\text{V17}} \approx 3.9$. In addition, $S$ is much reduced across the range of $r_{\text{eff}}$ tested, with a 2$\sigma$ anisotropy detection corresponding to $r_{\text{eff}} \approx 0.02$ in this case. This is due to the different assump-
tions made in the creation of the two simulations; the older FFP8 maps have small-scale power added from an extension of the smoothed experimental Q/I and U/I power spectra, whereas V17 utilises newer Planck maps and a method for generating high-\(l\) power via realisations of the turbulent GMF, with hyperparameters optimised to reproduce Planck Collaboration Int. XXX (2016); Planck Collaboration Int. XLIV (2016) results. However, it is not immediately clear how the physical differences in the generation of the two simulations result in the trends found in Fig. 3. The different simulation choices also lead to the different \(C_l^{BB}\) spectra as noted in Fig. 1.

Due to the greater physical motivation in the mid-\(l\) regions probed in this paper from the inclusion of a GMF model, we affix greater credibility to the V17 simulation.

5.2 The Null Test in the Presence of Tensor Modes

An important check of our null test methodology is whether it avoids false positives when tensor modes are present instead of dust. As previously noted, IGW-induced spectra are expected to be isotropic (e.g. Krauss et al. 2010); therefore we do not expect them to contribute to the debiased hexadecapole statistic, \(H^2\), on which our null test is based (as any lower-order multipoles should vanish in the \(e^{-i\phi}\)-weighted summations of Eqs. 4).

To investigate our null test in the presence of tensor modes, we consider a full-sky Stokes map realisation of the \(r = 0.1\) tensor spectrum (motivated by the current constraints on \(r\), BICEP2 Collaboration (2016)). This map is obtained from a rescaled Planck FFP10 unlensed tensor mode simulation.\(^{11}\) The full analysis pipeline of Sec. 4 was reapplied to this map (instead of the V17 simulation) after the addition of noise and lensing modes, and the patch anisotropy \(\Xi\) was computed for the 5% sky region. (We use a whole-sky realisation of this spectrum rather than generating spectra individually for each tile to fully test for any additional sources of bias between the full-sky maps and the MC debiasing procedure.)

Computing \(\Xi\) using \(\Delta \theta = 3^\circ\) with \(N_{\text{bias}} = N_{\text{sims}} = 500\) for the predicted CMB-S4 noise and lensing parameters gives a significance of

\[
S_{f=0.1\,\text{bias},\text{tensor}} = -0.04 \pm 0.58 \sigma_{\Xi}
\]

where the error is obtained by rerunning the analysis 10 times with independent MC debiasing and noise modes. This is clearly consistent with zero, and far below the \(10\sigma\) detection found for 150 GHz V17 dust at this amplitude level. We hence conclude that our method is robust and capable of providing a null test for dust without false positives in the presence of tensor modes.

5.3 Anisotropy Detection Significances for Different Experimental Configurations

We will now briefly consider how the ability of our test to detect dust depends on the experimental configuration. To simplify the discussion, we will not simultaneously vary \(f_{\text{dust}}\), experiment parameters and sky areas, but will instead fix \(f_{\text{dust}} = 1\) and consistently analyse the 5% sky region. As detailed previously, we compute the patch-averaged hexadecapole, \(\Xi\), and derive its detection significance, \(S\), for different experimental noise levels. We thus obtain Fig. 4, in which we show the dust detection significance \(S\) as a function of noise parameters \(\Delta \theta\) and \(\theta_{\text{FWHM}}\) for both \(f_{\text{dust}} = 1\) and 0.1 (i.e. zero and 90% delensing efficiency). These plots use computations of \(S\) at 400 pairs of noise parameters.

The form of the plots is as expected, with high significances for low noise, which fall as \(\Delta \theta\) and \(\theta_{\text{FWHM}}\) increase and obscure the dust signal. The scatter seen in adjacent regions may be attributed to the fact that each data-point uses a finite number of different realisations of the noise. Comparing lensed to delensed plots, the significances are higher in the latter case for low-noise, but there is no significant difference for high noise parameters (where the significances are consistent with zero), since the lensing modes become subdominant.

For both cases, when using BICEP2 noise parameters (BICEP2 Collaboration 2014) on the 5% sky region, \(S\) is negligible, which is as expected since the BICEP2 experiment detected the dust monopole only at moderate significance, and the hexadecapole should be smaller still. Also marked are the assumed locations of the 5\(\mu\)K-CMB and CMB-S4 experiments, which have much higher detection significances; around 50\(r\) (35\(r\)) for CMB-S4 assuming \(f_{\text{dust}} = 0.1\) (\(f_{\text{dust}} = 1\)). (We expect \(f_{\text{dust}} \approx 0.4\) for 5\(\mu\)K-CMB, thus the delensed significances cannot be directly obtained

\(^{11}\) Also available from NERSC.
Figure 4. Significance of a detection of patch anisotropy, $\Xi$, (eq. 12) in units of $\sigma$ for the 5% sky region using a range of noise parameters (noise power $4\pi$ and FWHM $\theta_{\text{FWHM}}$). The panels assume (a) full lensing contributions to $C_\ell^{BB}$ and (b) 90% delensing efficiency (i.e. $0.1C_\ell^\text{CMB}$). The plots are generated via interpolation of 400 individual tests with the small MC bias subtracted and visualised using $\texttt{Scipy}$. Stars show the location of the BICEP2 (yellow), $5\mu K'\text{-CMB}$ (green) and CMB-S4 (black) experiments and significance contours are also marked. $\Xi$ was computed using $A = 3^\circ$ with $N_{\text{sim}} = N_{\text{bias}} = 500$.

6 VISUALISING DUST ANISOTROPY
6.1 Spatial Distributions

Following the methods of Sec. 4, we may also compute spatially-resolved maps of the hexadecapole parameters across the 5% sky patch, as shown in Fig. 5 (henceforth setting $f_{\text{dust}} = 1$). This uses lensing and noise parameters appropriate for CMB-S4 (table 2), and plots are given as Albers equal area conic projections, created using the Python skymapper package. Each individual pixel represents a 3° width tile, whose centres are separated by 0.5° to improve visibility, thus violating their independence.

The monopole amplitude $A$ (Fig. 5a) exhibits substantial variation across the map, with high amplitudes mostly concentrated towards lower latitudes, and hence closer to the galactic centre. We note that these amplitudes are still several orders of magnitude smaller than those closer to the Galactic plane. The region used by the BICEP experiment (shown by the dotted line) has low dust amplitude, as expected, though we note some higher amplitude regions, particularly towards smaller RA. Notably, the form of $A$ in the BICEP region is heuristically similar to that found in Rotti & Huijzenbergen (2016, fig. 7).

Due to the low dust amplitudes and considerable contributions from the stochastic noise spectra, we expect some scatter in $\mathcal{H}^2$ around its true value, leading to $\mathcal{H}^2 < 0$ for some tiles. This can be partially ameliorated by using a larger $N_{\text{bias}}$ to better constrain the bias term, although this adds considerable computation time. In Fig. 5b, $\mathcal{H}^2$ is displayed on a symmetric logarithmic scale (utilising a logarithmic scale for the majority of the data, with a linear region about zero to avoid infinities), thus including negative values, but we note that the biased quantity, $\mathcal{H}^2$, is used in Fig. 5c to define the anisotropy fraction $\epsilon = H/A$, else this is undefined for $\mathcal{H}^2 < 0$.

By eye, a correlation between $\mathcal{H}^2$ and $A$ is apparent, as confirmed by the values of the hexadecapole fraction $\epsilon$, which varies predominantly between 10% and 45% across the map. This will be further discussed in Sec. 6.2. In addition, we note that the anisotropy angle (Fig. 5d) seems to be coherent on scales larger than the pixel-separation (0.5°), as was originally assumed in construction of the estimators. (Even though the dust-contributions of overlapping tiles are not independent, the noise realisations are generated separately, thus we would not expect coherency on scales above 0.5° if the angle was significantly biased by noise).

The hexadecapole ‘equivalent significance’ (Fig. 5e, as defined in Sec. 4.4) shows that we detect anisotropy at levels exceeding $2\sigma$ for a number of regions across the patch, with highly anisotropic regions correlating with regions of high A. For most tiles however, we have small $\mathcal{P}_\ell$, implying largely insignificant detections of anisotropy on a single-tile level.

6.2 Full Sky Correlations

Using $A$ and $\mathcal{H}^2$ estimates from 3° tiles across the majority of the sphere, we can probe correlations over a range of angular scales as well as different levels of dust intensity. As noted in Sec. 3, the V17 simulations do not accurately predict the dust polarisation spectra close to the galactic plane, thus we use the Planck GAL80 unapodised mask to remove central galactic regions. Regions with declinations above 85° suffer from slight distortions due to the projection (as shown in appendix D) thus these are also excluded, and the overall

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12 https://github.com/pmelchior/skymapper
13 http://pla.esac.esa.int/pla/

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Figure 5. Plots showing the estimated anisotropy quantities (see tab. 4.5) for the 5% (2150 deg$^2$) sky patch, using CMB-S4 noise parameters (tab. 2) on ~9000 3' width tiles with centres separated by 0.5'. In (b), the debiased hexadecapole statistic is plotted on a symmetric logarithmic scale (using a logarithmic scale for both positive and negative data, with a small linear region encompassing zero) to include random negative fluctuations below the isotropic mean, with red regions indicating considerable anisotropy. For (c), we note that $\epsilon$ exhibits some bias, since it depends on the biased quantity $H^2$ rather than $H^2$. Plot (e) gives the significance of a hexadecapole detection for a single tile, compared to isotropic MC simulations. This is converted into equivalent Gaussian significances, with zero (white, $P_{\sigma} = 0$) and large (red) values representing isotropy and significant anisotropy respectively. RA (top) and declination (left) in Galactic co-ordinates are shown on the plots, and the BICEP region is indicated with a black dashed line. The panels are plotted as Albers equal area conic projections.
mask is apodised with a 3° FWHM Gaussian kernel to avoid edge effects.

$A$ and $\mathcal{H}^2$ are then computed across the remainder of the sky using tiles of $\Delta \theta = 3^\circ$ separated by $1^\circ$ to increase resolution, using CMB-S4 noise parameters and $N_{\text{bias}} = 100$ (to keep the computation time tractable). This gives $\sim 40,000$ measurements which are mapped into a HEALPix NSIDE = $64$ (50,000 pixels) map, using linear interpolation to compute the parameters at the centre of the HEALPix pixels. (We note that the two pixellation routines are not directly compatible since our cut-out technique natively takes square patches along the RA axis unlike the HEALPix tiling).

Fig. 6 shows Mollweide projections of the computed parameters, and we note that the majority of the $\mathcal{H}^2$ values are positive, indicating significant anisotropy detections. The monopole and hexadecapole plots are very similar in form across the scales probed here, especially in high-dust regions, thus it is instructive to consider their correlation coefficient.

PolSpice (Chon et al. 2004) is used to compute the necessary power-spectra, treating the HEALPix maps of $A$ and $\mathcal{H}^2$ as spin-zero fields, weighted by the apodised masks. From the output pseudo-$C_l$ spectra we can assess the correlation of $A$ and $\mathcal{H}^2$ on different scales via the cross-correlation;

$$\rho_{A\mathcal{H}^2}(l) = \frac{C_l^A C_l^{\mathcal{H}^2}}{\sqrt{C_l^{AA} C_l^{\mathcal{H}^2\mathcal{H}^2}}}.$$  

(16)

Here $\rho_{A\mathcal{H}^2} \rightarrow 1$ represents full correlation between variables.

This is shown in Fig. 7, with errors taken from the variances of the binned data. For the V17 mock data, there are clearly very strong correlations on large angular scales (small $l$) with $\rho_{A\mathcal{H}^2} \approx 0.9$ for $l \in [5,50]$ and $\rho_{A\mathcal{H}^2} > 0.7$ for $l < 150$. (We note that extremely low $l$ modes have considerable uncertainty due to cosmic variance and the application of the mask). At larger $l$, the correlation is reduced, most likely due to the small-scale fluctuations in $\mathcal{H}^2$ as a result of noise contamination. In addition, we cannot probe smaller scales than $l \sim 150$ with this dataset, since $A$ and $\mathcal{H}^2$ are only sampled at $1^\circ$ resolution.

To account for spurious correlations due to the possibility of bias terms in $\mathcal{H}^2$ correlating with $A$, we also plot $\rho_{A\mathcal{H}^2}$ using $\mathcal{H}^2$ estimates obtained from a single MC realisation of the isotropic $C_l$ spectrum for each tile. The correlation coefficient is clearly much smaller than that of the anisotropic data, thus the strong correlations seen for dust are (to a large extent) a real effect and not just an artefact of noise in $\mathcal{H}^2$.

The significant correlations between the monopole and hexadecapole powers confirm that by simply tracing $\mathcal{H}^2$ we can obtain clear bounds on $A$ and hence the level of dust. In addition, this correlation motivates us to consider a rudimentary method for cleaning the B-mode sky using statistical anisotropy estimators.

7 DEDUSTING

7.1 Methodology

Here we consider the possibility of using the above correlations to ‘dedust’ the B-mode sky; i.e. to construct an estimate of the dust B-mode map that can be subtracted off. In particular, from the measured hexadecapolar anisotropy, we can obtain both the polarisation angle and a proxy for the polarisation amplitude and thus generate such a template of dust B-modes; this can then be subtracted from the observed B-mode map in order to isolate any IGW signature. This technique of generating a template of large-scale B-modes based on the measured higher-order statistics on small scales is analogous to that used in delensing (e.g. Kesden et al. 2002, Smith et al. 2012, Sherwin & Schmittfull 2015). If it proves feasible, the approach will in principle be possible at only a single frequency, given data with sufficiently low noise and high delensing efficacy.

Throughout the above analysis we have assumed that the polarisation angle $\alpha$ is constant over small scales, allowing us to measure $A_f$ and $A_c$ (proportional to $\sin 4\alpha$ and $\cos 4\alpha$ respectively), which encode information on the ratio of the $Q$ and $U$ maps via Eq. 2. Whilst we still assume $\alpha$ to be constant over each tile, on larger scales, we promote $\alpha$ to be a function of spatial position, which we denote $\alpha(r)$. Here $r$ denotes the position of the centre of each tile. In addition, we assume that the position dependence of $\sqrt{Q^2 + U^2}$ may be written on these large scales as $R(r)I(r)$, where $I(r)$ is the familiar Stokes intensity map (also available at high resolution), and $R(r)$ is a function closely related to the polarisation fraction that is also taken to be constant over each tile. Thus

$$\bar{U}(r) \propto I(r)R(r)\sin(2\alpha(r))$$

$$\bar{Q}(r) \propto I(r)R(r)\cos(2\alpha(r)).$$

(17)

We allow $R(r)$ to vary across the sky following the empirical results of Planck Collaboration Int. XXX (2016). Here, we approximate this via the rescaling factor

$$R(r) = \left(\frac{\mathcal{H}^2(r)}{\langle I^4 \rangle_{\text{true}}(r)}\right)^{1/4}$$

(18)

where the average intensity is taken over 3° width tiles (as for $\mathcal{H}^2$) and $(I^4)$ is used since $\mathcal{H}^2$ is a 4-field quantity. The hexadecapole amplitude is a good proxy for the B-mode power (and hence the polarisation level), since it is highly correlated with the monopole amplitude (Sec. 6.2) and free from bias from other sources including IGWs (Sec. 5.2) and lensing.

We may obtain measurements of $\sin 2\alpha(r)$ and $\cos 2\alpha(r)$ on each tile from the measured parameters $A_f$ and $A_c$ via the relations

$$\sin 2\alpha = \pm \frac{\overline{f_2}}{\sqrt{2(f_2 + 1)}}\cos 2\alpha = \pm \frac{f_2 + 1}{2}$$

(19)

where both expressions have the same unknown sign, and $\overline{f_2}$ and $f_2$ represent $A_f$ and $A_c$ normalised by $\sqrt{(A_f)^2 + (A_c)^2}$ (these are equal to $\sin 4\alpha(r)$ and $\cos 4\alpha(r)$ respectively). Note that the unknown sign of $f_2$ is from information loss due to our estimators only being sensitive to the angle $\sin 2\alpha$. This (spatially dependent) ambiguity can be resolved by measuring a cross-correlation with the sky B-mode for each tile. We detail this procedure in App. F.

With expressions for $R$, $I$ and $2\alpha$ at each $r$, we may now
Figure 6. Full sky Mollweide projections of the dust monopole amplitude and hexadecapole power, computed at a spatial resolution of 1\arcmin using CMB-S4 noise and lensing parameters with the V17 simulation. Data is interpolated linearly onto the HEALPix grid from ~40,000 parameter estimates, applying the Planck GALB0 mask and additional masking at declinations δ satisfying |δ| > 85°. \( \mathcal{H}^2 \) is plotted on a symmetric logarithmic scale (to include negative fluctuations) and exhibits clear correlations with \( \mathcal{A} \). The BICEP region is shown as a dotted line.

Figure 7. Correlation coefficient (Eq. 16) between the monopole amplitude \( \mathcal{A} \) and debiased hexadecapole coefficient \( \mathcal{H}^2 \) for ~40,000 3° width tiles on the sphere (as for Fig. 6). Data is given for \( \mathcal{H}^2 \) estimates from the V17 dust simulation (red, assuming CMB-S4 noise and lensing) and a debiased Gaussian realisation of the isotropic \( C_l \) spectrum in each tile (blue). We note strong correlations between \( \mathcal{H}^2 \) and \( \mathcal{A} \) for the dust (with \( \rho \rightarrow 1 \) indicating full correlation), far in excess of those due to the isotropic bias. This uses the PolSpice emulator (Chon et al. 2004) to compute pseudo-\( C_l \) spectra, with errors obtained from the variances of the binned data.

We now consider the application and expected performance of this technique. To construct a cleaned B-mode map \( B_\text{C}(\mathbf{l}) = B(\mathbf{l}) - a_c(\mathbf{l})\tilde{B}(\mathbf{l}) \) for true dust B-mode Fourier map \( B(\mathbf{l}) \), we must first determine the cleaning coefficient \( a_c \) which minimises the cleaned (‘dedusted’) power-spectrum. This factor \( a_c \) will be equal to \( \eta \) in the limit of no noise. The power spectrum of the cleaned B-modes will be

\[
C_{l}^{B\text{C}B\text{C}} = C_{l}^{BB} - 2a_c C_{l}^{B\tilde{B}} + a_c^2 C_{l}^{\tilde{B}\tilde{B}}. \tag{20}
\]

Minimising \( C_{l}^{B\text{C}B\text{C}} \) with respect to the unknown \( a_c \), we obtain

\[
a_c(\mathbf{l}) = C_{l}^{B\tilde{B}} / C_{l}^{BB}. \tag{21}
\]

The reduction in dust B-mode power expected from our cleaning is thus entirely specified by the correlation coefficient, with \( \rho_{BB} \rightarrow 1 \) indicating perfect cleaning.

7.2 Application to Simulated Data

We now test this idea with simulated data. First, we compute the (phase-corrected) angle \( 2\alpha \) and ratio \( R \) as described previously for 3° width tiles in the 5% sky region, using CMB-S4 noise and delensing parameters.\(^\dagger\) This gives a set of values of \( 2\alpha(r) \) and \( R(r) \) (from tiles separated by 0.5\(^\circ\)), which are mapped via linear interpolation onto a HEALPix \( \text{NSIDE} = 128 \) partial sky grid (as in Sec. 6.2). The resolutions of these maps are upgraded to \( \text{NSIDE} = 1024 \) (on a higher resolution \( x \) coordinate grid, matching the intensity map), and smoothed with a Gaussian kernel of 0.5\(^\circ\) FWHM. (To correctly smooth angular data, we must apply Gaussian smoothing to \( \sin 2\alpha \) and \( \cos 2\alpha \) maps rather than a map of

\(^\dagger\) For simplicity, we assume the processes of dedusting and delensing are independent here, i.e. we may dedust an already delensed map.
2a.) We note that $R(x)$ is broadly consistent across the region, except for two isolated patches, which can be shown to be high-dust regions.

Here we apply the dedusting analysis primarily to a 1% sky region (RA $\in [-60^\circ, -20^\circ]$, dec $\in [-70^\circ, -45^\circ]$) in Galactic co-ordinates, which excludes the high-$R$ regions and is small enough that the flat-sky limit is appropriate. From $2\alpha(x)$ and $R(x)$, we construct estimates for the Stokes maps $\hat{Q}(x)$ and $\hat{U}(x)$ from the V17 high resolution intensity map $I(x)$ (via Eq. 17), and hence $\mathbf{C}_\ell^{BB}$ using flipperPol’s ‘hybrid’ B-mode estimator.

A comparison between the estimated and true B-mode dust maps is shown for the larger 5% region in Fig. 8, with $B(x)$ computed from the $\hat{B}_{lm}$ and $\hat{I}_{lm}$ spherical harmonic coefficients, showing only modes with $l \leq 60$. In addition, comparison between the true and estimated B-mode power spectra and the cross-spectrum gives the correlation coefficient $\rho_{BB}$, as plotted in Fig. 9 for the 1% region.

From the real-space maps, we observe clear correlations between $\hat{R}(x)$ and $\hat{B}(x)$ on large scales, in agreement with the correlation coefficient, which is markedly non-zero for $l \leq 80$, with a peak of 0.5 for $l \sim 20$. Recalling that the cleaned-power spectrum is proportional to $(1 - \rho_{BB}^2)$, this clearly demonstrates that the dedusting technique has partially worked in this case and would allow the removal of a significant fraction of low-$l$ dust.

It is important to note that this method is very approximate, since we use a constant 2a for each tile (an assumption partially alleviated by using overlapping tiles) and assume that $Q$ and $U$ have $l$ dependencies represented only by $I(l)$ and a simple position dependent factor $R(x)$. The method could be significantly improved through additional constraints from EB and EE power spectra or through a method for computing a continuous angle map (cf. KK14) instead of the discretely sampled technique used here. However, the fact that we can achieve distinctly non-zero correlation on large scales using our simple method suggests that, with further development, this technique could become useful for the subtraction of dust-induced B-modes for futuristic experiments, complementing usual approaches by using only single-frequency data. An advantage of our method is that no understanding of the dust frequency dependence is required. More generally, detailed knowledge of the dust properties is not needed for our method to work: adding our low-$l$ B-mode template to the bank of multifrequency data used by ILC techniques should yield improved dust mitigation, as long as the correlation coefficient between our B-mode template and the actual B modes is high.

8 CONCLUSIONS

Using measurements of the B-mode polarisation, future CMB experiments will be able to place strong bounds on the level of inflationary gravitational waves (IGWs) or detect them for the first time. However, robust measurements of IGWs require the true signal to be separated from thermal dust contamination and other foregrounds. While the canonical method for diagnosing and subtracting dust uses observations at different frequencies for foreground separation, in this paper we have explored complementary single-frequency methods.

We developed and tested methods for detecting dust via B-mode statistical anisotropy, following the suggestions of Kamionkowski & Kovetz (2014). Assuming a coherent Galactic magnetic field on few-degree scales, we expect a ‘hexadecapole’ pattern in B-mode power spectra, which we searched for with specialised estimators and Monte Carlo simulations. The technique was applied to realistic simulations of thermal dust emission (predominantly Vansyngen et al. 2017) to construct (a) a null-test for the presence of dust contamination on large scales, (b) spatial maps of the dust anisotropy, and (c) a proof-of-concept for a rudimentary technique for dust removal.

Applying our null test methodology to simulations, we found that such null tests can be a powerful tool for future experiments, giving the ability to detect dust residuals with an amplitude corresponding to a tensor-to-scalar ratio of only 0.001 at a 95% confidence level, assuming a CMB-S4-like survey. We caution that this result depends on the theoretical assumptions in the dust simulations, as seen in the significant differences when another type of simulation was tested. However, improved future simulation efforts and measurements of polarised emission from dust on small scales are expected to reduce the uncertainty arising from simulation inaccuracy.

We also found strong correlations between the dust monopole and anisotropy fields, with correlation coefficients approaching unity on small scales. This motivated an application of dust anisotropy statistics to the problem of single-frequency dust-subtraction: we derived and tested a rudimentary method for estimating the dust B-mode map from the computed hexadecapole angles and strengths. In simulations, this statistical-anisotropy-derived dust B-mode map was found to match the true B-mode map well on large scales, with significant correlation found.

There is much work to be done in this area in order to increase both the significances of anisotropy detection and the effectiveness of the new dust removal methods. The inclusion of $E$-mode data may contribute to these efforts, with the extra information that can be extracted from $C_l^{EE}$ and $C_l^{EB}$ spectra potentially allowing lower dust levels to be probed. Furthermore, the discrete sampling of angles used here could be generalised to a continuous field, which would allow greater resolution and the probing of higher multipoles in the 2D power spectra (as suggested by Kamionkowski & Kovetz 2014). Additional work could also involve combining these anisotropy tests with conventional multi-frequency approaches to obtain tighter constraints on residual dust levels and more efficient dust removal. Finally, although our work has focused on thermal dust emission, very similar anisotropy methods could also be constructed and implemented for synchrotron foreground emission - in fact, our method would presumably capture several types of statistical foreground anisotropy simultaneously.

The techniques presented here give a promising avenue into the detection and potential removal of polarised foregrounds via their anisotropy signature, and could, even in their current form, be used as a powerful null test in forthcoming CMB experiments.
Figure 8. Real-space maps of the (a) estimated and (b) true B-modes from dust, with estimates obtained solely from statistical anisotropy signatures (the hexadecapole parameters) as well as the dust intensity map (Eq. 17 & 18). $\hat{B}$ is defined up to an overall scaling factor $\eta$ which can be determined empirically. The plot uses symmetric logarithmic colour-bars and an Albers equal area conic projection; we filter out modes with $l > 60$. The 5% sky region is shown, with the BICEP patch indicated by a dotted line. We note marked similarities between the estimated and true maps on large scales, demonstrating that we have successfully estimated the dust B-modes from anisotropy statistics alone.

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APPENDIX A: CORRECTING FOR THE INTRINSIC BIAS OF $H^2$

We here derive the $H^2$ bias term used in the computation of $H^2$ in Eq. 11. To do this, we first consider the generalised estimator for $H^2$ obtained from two B-mode maps $X$ and $Y$. This will allow us to compute the realisation-dependent bias using $H^2$ estimates obtained both from MC maps and from their cross-spectra with the mock data.

A1 Deriving a Generalised Estimator for $H^2$

Consider Gaussian Random Fields (GRFs) $x^i(l)$ where $x$ represents the Fourier-space component of the B-mode CMB map with label $X$ representing either MC simulations or mock data ($x$ is treated as a real field for simplicity). From the definition of the power spectra, $|\langle l x^i(l) x^j(l) \rangle| = C_{ij}^{XY} \delta_{ij}$ for Kronecker delta $\delta_{ij}$. For the cross spectrum $X \otimes Y$, we have the generalised estimator for $A_{f_X}$ given two maps $X$ and $Y$:

$$\langle A_{f_X} \rangle_{XY} = \frac{1}{N} \sum_{ij} \frac{|x^i_y x^j_x|}{C_{ij}^{XY}} A_{f_X} A_{f_Y} \sin 4\phi$$

(cf. Eq 4) where the normalisation factor is $N = \sum_{ij} A_{f_X} A_{f_Y} \sin 4\phi$. ($A_{f_X}$ follows by symmetry.) To compute the isotropic power, $H_{iso}$, from the two maps we use

$$H_{iso,XY} = \langle A_{f_X}^2 \rangle_{XY} + \langle A_{f_Y}^2 \rangle_{XY}$$

(cf. Eq. 7) and note that $\langle A_{f_X} \rangle = \langle A_{f_Y} \rangle = 0$. By rotational invariance, the mean values satisfy $\langle A_{f_X}^2 \rangle_{XY} = \langle A_{f_Y}^2 \rangle_{XY}$, thus

\[ H_{iso,XY} = \sum_{ij} \left( \frac{|x^i_y x^j_x|}{C_{ij}^{XY}} A_{f_X} A_{f_Y} \sin 4\phi \right) \]

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APPENDIX B: PROBABILITY DISTRIBUTIONS FOR THE HEXADECAPOLE PARAMETERS

B1 Analytic Distributions

For an isotropic map, we expect $A_f$, $A_f^g$ and $A_f^c$ to be Gaussian distributed with the latter variables having zero mean and, by symmetry, equal variances (denoted $\sigma_{A_f}$). This follows from the central limit theorem, since they are computed from an appropriately weighted sum over many stochastic pixels (whose Fourier space components are $\chi^2$-distributed). Since $A_f^g$ and $A_f^c$ describe the B-mode power with respect to two orthogonal basis functions ($\sin 4\phi$ and $\cos 4\phi$), they are here assumed to be independently distributed. From this, we note that the angle $\alpha$ obtained from MC simulations is expected to be uniformly distributed in $(-45, 45]$ degrees. Its expected error, $\sigma_{\alpha}$, may thus be computed via

$$\sigma_{\alpha} = \left[ \frac{\partial \alpha}{\partial A_f^g} \frac{\sigma_{A_f}}{A_f^g} \right]^2 + \left[ \frac{\partial \alpha}{\partial A_f^c} \frac{\sigma_{A_f}}{A_f^c} \right]^2 \right]^{1/2},$$

where the factor of $180/\pi$ is to convert the angle to degrees.

To derive the statistics of $\mathcal{H}^2$ for the MC simulations, first consider

$$Q = \mathcal{H}^2 + \frac{\langle H^2 \rangle_{\text{bias,iso}}}{\sigma_{A_f}^2 W_{\text{corr}}} = \left( \frac{A_f^g}{\sigma_{A_f}} \right)^2 + \left( \frac{A_f^c}{\sigma_{A_f}} \right)^2.$$

where $W_{\text{corr}}$ is the window-correction factor $(W^2)^2/(\langle W^2 \rangle)$ and $\mathcal{H}^2_{\text{bias,iso}}$ is the isotropic simulation bias. Being the sum of two independent uniform Gaussians, $Q$ is distributed according to the $\chi^2$-distribution with two degrees of freedom and hence has probability density function (PDF) $f(Q) = \frac{1}{2}e^{-Q/2}$. This implies that the PDF and cumulative density function (CDF) for a measurement of $\mathcal{H}^2$ are

$$f(Q) = \frac{1}{2\sigma_{A_f}^2 W_{\text{corr}}} \exp \left( -\frac{\mathcal{H}^2 + \langle H^2 \rangle_{\text{bias,iso}}}{2\sigma_{A_f}^2 W_{\text{corr}}} \right),$$

$$p(Q) = 1 - \exp \left( -\frac{\mathcal{H}^2 + \langle H^2 \rangle_{\text{bias,iso}}}{2\sigma_{A_f}^2 W_{\text{corr}}} \right).$$

respectively. The analytic distribution $p(H^2)$ can be used to parametrise the probability that a particular tile is anisotropic for runs where it is not feasible to create many simulations, since only $\sigma_{A_f}$ need be computed.

B2 Testing the Model Statistics

To test our assumptions regarding the expected distributions of the monopole and hexadecapole parameters (appendix B1), we apply the hexadecapole estimators (Eq. 4) to $5 \times 10^5$ isotropic simulations of a representative tile from the 5% sky region, using CMB-S4 noise and lensing parameters, as shown in Fig. B1.

$A_f$ and $A_f^c$ are well represented by Gaussian curves (shown in red) with variances drawn from the data and zero-mean for $A_f^g$. Furthermore, the analytic PDF for $\mathcal{H}^2$ (Eq. B3) well represents this dataset, using the mean of $\sigma_{A_f}$.
Figure B1. Histograms of the probability distribution function (PDF) for $A$, $A_f$, and $H^2$, as labelled, obtained from $5 \times 10^5$ random isotropic MC simulations of a specific tile within the BICEP patch. The data is fit with Gaussians for $A$ and $A_f$ (the latter with zero-mean) with variances estimated from the data. For $H^2$, the expected PDF (Eq. B3) is plotted which closely represents the fit.

and $\sigma_{A_f}$ as the only distribution shape parameter. This hence supports our assumptions of Gaussianity in the parameters $A$, $A_f$, and $A_f$, and we note that the analytic PDF may be used interchangeably with the statistical percentile described in the text.

APPENDIX C: BIAS FROM NOISE APPROXIMATIONS

Throughout the above, we have assumed that the effects of noise may be incorporated solely via single-tile Gaussian realisations of the 1D noise power spectrum (Equation 8). Here we consider the biases resulting from this assumption by reapplying our analysis pipeline to the case where noise is generated instead from a full-sky map, which will include correlations on larger scales than the tile widths.

Using HEALPix, we created full-sky Q and U Stokes maps from a random realisation of the noise power spectra (via the synfast procedure). These were cut-out and transformed into Fourier space to compute a B-mode power spectrum for each tile as for the lensing modes, again utilising zero-padding. Comparison with the input $C_l$ showed that the correct spectra were indeed reproduced. The computed $B_{\text{noise}}(l)$ map was combined with the lensing and dust maps to give the ‘data’, which is a good approximation to a real experimental dataset. The remaining analysis proceeds as before, with the MC simulations being generated from the isotropic 1D power spectrum of the combined map.

Fig. C1 displays the analogue of Fig. 2 for CMB-S4 noise and lensing parameters, displaying the mean significance and $r_{\text{eff}}$ for analyses using both the fiducial (single-tile) noise generating technique (blue) and the full-sky noise maps (orange). Notably, the $r_{\text{eff}} - S$ curve is very similar for both datasets across the range of dust amplitudes tested, thus we conclude that there is no significant bias obtained by our generating noise purely on single-tile scales. This motivates the use of the fiducial method in our analysis, which is considerably less computationally expensive, especially for tests where the noise parameters are varied (e.g. Sec. 5.3).

APPENDIX D: DISTORTIONS DUE TO MAP PROJECTION

During the conversion of full-sky maps into small flat-sky tiles, we must project sections of a full HEALPix map onto a flat grid, which can give significant distortions. These are
seen most clearly at high galactic latitudes, since the polar co-ordinates become singular there, leading to unwanted stretching effects. This can affect both the real- and power-space data, and may have a significant impact on the derived hexadecapole quantities. The effect could be negated by altering the projection axis for each cut-out such that the desired tile always lies in equatorial regions, where it would experience the least distortion, but this was not implemented to avoid significant additional computational expense.

To explore this effect, we simulated a full-sky map with known $C_l^{EE}$ and $C_l^{BB}$ spectra, using HEALPix's synfast method. The spectra were derived from a dust-plus-contamination spectrum with dust amplitude appropriate for a tile in the BICEP region, and $C_l^{EE}$ was set to $2C_l^{BB}$, following Planck Collaboration Int. XXX (2016); Planck Collaboration Int. XLVIII (2016).

Focussing on the Southern Galactic pole (over an RA range of 100°), this was cut out into 3° width tiles with separation 1°, and the power spectrum was computed for each tile. In particular, we considered the observables $A$ and $H^2$ as a function of the declination, $\delta$, of the tile-centre. Fig. D1 shows the computed values for $\delta < -70°$, which should be constant with respect to $\delta$ in the absence of any distortions. Error-bars show the variation in the parameters for different tiles at this declination. We conclude that there is no significant biases to $H^2$ from any latitude, but a possible slight enhancement in $A$ at $\delta < -85°$.

An additional test is to ensure that there is no significant E- to B-mode leakage deriving from the projection onto small tiles at high Galactic latitudes. This is investigated in a similar way, instead using a simulated full-sky map with only T- and E-modes present (generated in HEALPix from a CAMB power spectrum of the unlensed scalar CMB). For each 3° tile in the same region as before, the ratio of B- to E-mode power was computed, denoted $R_{BE} = \langle C_l^{BB} \rangle / \langle C_l^{EE} \rangle$.

From Fig. D2, it is clear that there is non-zero E- to B-mode leakage at all tested latitudes, increasing as $\delta \rightarrow -90°$. However, $R_{BE} < 10^{-1.5}$ % for $\delta < -85°$, and, since we expect comparable powers in both modes for dust, this is not an important source of error in this paper. Combining both tests, we see that there are significant biases only for $|\delta| > 85°$, thus these regions have been excluded from all analyses.

**APPENDIX E: REDUCTION OF BIAS FROM PIXELLLATION**

In zero-padding, we insert extra zeros to the sides of all real-space cut-out tiles to increase the map width by a factor $R_{pad}$. This effectively boosts the resolution in Fourier space by interpolating between pixels (via convolution with a 2D sinc function). The number of pixels in the data is thus increased which reduces pixellation errors and allows smaller $l$ to be probed, increasing the significance of any detection of anisotropy. It is important to note that following zero-padding the Fourier-space pixels are no longer independent, thus when Gaussian realisations of known power spectra are created, they are generated in Fourier space using an unpadded template before zero-padding is added in real-space, to ensure that they have the same correlation properties as the data. In addition, we note that zero-padding cannot

**Figure D1.** Monopole and hexadecapole powers reproduced from 3° tiles generated from cutting out a full-sky map of known power spectra, satisfying $C_l^{BB} / C_l^{EE} = 0.5$. Values are given as a function of the declination of the map-centre, $\delta$, and error-bars are from the standard deviations of parameter values across different tiles at that latitude. Any deviation from uniformity with respect to $\delta$ indicates unwanted effects from projection-derived distortions.

**Figure D2.** Ratio of projected mean B- to E-mode power, $R_{BE}$, for 350 3° width tiles close to the South Galactic pole with centres at the stated declinations. This uses a simulated full-sky CMB map of unlensed scalar modes, created using CAMB and HEALPix, thus we expect no B-mode component in an ideal projection.
be applied in the high-noise limit (e.g. for noise parameters appropriate for BICEP2) since the Fourier maps are dominated by high-amplitude pixels at large \( l \), which give significant leakage into other (non-local) pixels due to the sinc interpolation.

We now consider the effects of rotating the power-map prior to applying the hexadecapole estimators. Naively, if the map is rotated by angle \( \chi \) (i.e. transforming \( \phi_l \rightarrow \phi_l + \chi \)), we expect the output hexadecapole parameters \( (A_{fs}', A_{fc}') \) to be related to their unrotated forms \( (A_{fs}, A_{fc}) \) via

\[
A_{fs} = A_{fs}' \cos(4\chi) - A_{fc}' \sin(4\chi) \\
A_{fc} = A_{fs}' \sin(4\chi) + A_{fc}' \cos(4\chi).
\]

(E1)

(The \( 4\chi \) dependence comes from the hexadecapolar nature of the estimators.) However, this also has the effect of changing the orientation of the Fourier-space data with respect to the pixel shape, resulting in an additional oscillation in \( A_{fs} \) and \( A_{fc} \) with respect to \( \chi \). Here, we average the corrected (derotated) estimates over 20 linearly-spaced values of \( \chi \in [0, \pi/2) \), which significantly diminishes the pixellation effects.

APPENDIX F: DETERMINING THE ANGLE SIGN VIA CROSS-CORRELATION

We now discuss how we can determine the sign of the quantities \( \sin 2\alpha \) and \( \cos 2\alpha \) used in the dedusting procedure in Sec. 7. Firstly, note that in the flat-sky approximation for each tile (appropriate for single-tile scales), we may use the canonical relation

\[
B(l) = \frac{1}{\sqrt{2}} [U(l) \cos 2\phi_l - Q(l) \sin 2\phi_l]
\]

(F1)

(e.g. Kamionkowski & Kovetz 2016, Eq. 30) to obtain an estimate of the single-tile B-mode Fourier dust spectrum, \( \tilde{B} \), as

\[
\eta \tilde{B}(l)_{\text{tile}} = \frac{\pm \eta(I) R_{\text{tile}}}{2\sqrt{f_c + 1}} \left[ f_s \cos 2\phi_l - (f_c + 1) \sin 2\phi_l \right].
\]

(F2)

(using Eqs. 17 & 19) where \( R_{\text{tile}} \) is the scaling ratio (Eq. 18, constant for each tile) and \( \eta > 0 \) is some unknown factor. \( \phi_l \) is the angle of \( I \) relative to the positive RA axis as before. We may compute the cross-spectrum between this estimate \( \tilde{B}(l) \) and the (known) true B-mode map \( B(l) \) \( (C^{B\tilde{B}}) \) which is expected to be positive if we have chosen the correct sign in Eq. F2. Measuring the sign of \( C^{B\tilde{B}} \) (by calculating the average value of the cross-spectrum for \( l \in [100, 500] \)) and applying this same sign to both \( \sin 2\alpha \) and \( \cos 2\alpha \) expressions, we resolve our spatially-dependent sign ambiguity and obtain complete knowledge about the angle \( 2\alpha \).

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