Mixed Monotonic Programming for Fast Global Optimization

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Abstract—While globally optimal solutions to convex programs can be computed efficiently in polynomial time, this is, in general, not possible for nonconvex optimization problems. Therefore, locally optimal approaches or other efficient suboptimal heuristics are usually applied for practical implementations. However, there is also a strong interest in computing globally optimal solutions of nonconvex problems in offline simulations in order to benchmark the faster suboptimal algorithms. Global solutions often rely on monotonicity properties. A common approach is to reformulate problems into a canonical form of a monotonic optimization problem, where the monotonicity becomes evident, but this often comes at the cost of nested optimizations, increased numbers of variables, and/or slow convergence. The framework of mixed monotonic programming (MMP) proposed in this paper is a more direct approach that exploits hidden monotonicity properties without performance-deteriorating reformulations. By means of a wide range of application examples from the area of signal processing for communications (including energy efficiency for green communications, resource allocation in interference networks, scheduling for fairness and quality of service, as well as beamformer design in multiantenna systems), we demonstrate that the novel MMP approach leads to tremendous complexity reductions compared to state-of-the-art methods for global optimization.

Index Terms—Resource allocation, global optimization, interference networks, monotonic optimization, branch-and-bound

I. INTRODUCTION

In point-to-point communication systems without interference, the optimization of various performance metrics can be formulated as convex programs (e.g., rate maximization [3] or mean square error minimization [4]). More complicated objective functions in the context of energy efficiency optimization can at least be shown to be pseudoconvex or quasiconvex [5]. Even in advanced scenarios with, e.g., multiple antennas or parallel transmission on orthogonal carriers, these optimization problems can be solved with efficient methods from convex optimization [4] or fractional programming [5], and sometimes even in closed form [5]. However, in multi-terminal scenarios with interfering users, performance optimization typically involves nonconvex problems. This is, e.g., due to interference terms that make the rate equations nonconcave or due to product operations contained in multiuser utility functions.

Apart from special cases, where efficient solutions exist, performance optimization in interference networks is thus usually tackled by locally optimal approaches or suboptimal heuristics. Examples are gradient ascent algorithms [8]–[11], successive allocation methods [12], [13], successive (pseudo-)convex approximation [14], [15], alternating optimization [16]–[19], distributed interference pricing [20], or game-theoretic methods [21]–[23]. Such heuristics are good candidates for practical implementation due to their low computational complexity and/or the possibility of distributed implementation. However, to assess the fundamental limitations of the considered multiuser communication systems and to have benchmarks for the heuristic methods, there is also a strong interest in globally optimal solutions.

In order to obtain such global solutions, researchers have applied methods from the field of monotonic optimization (e.g., [25]–[28]) to optimization problems in various communication systems. For instance, monotonic programming was applied in interference channels [7], [29]–[36], in broadcast channels with linear transceivers [11], [37]–[40], in interfering broadcast channels [41], in relaying scenarios [33], [34], and in satellite systems [42], with the aim of maximizing weighted sum rates [7], [29]–[31], [33], fairness-based performance metrics [32], [33], [36]–[38], [41], or the energy efficiency [23], [35], [39], [40] as well as minimizing the required sum transmit power [11], [39], [42]. Some of these applications include solutions for multiantenna systems [7], [11], [29], [30], [36]–[39], [41], allow to average data rates over several time slots [32], [33], [36], [37], [39], and/or incorporate additional robustness considerations [41]. Moreover, monotonic optimization can also be applied for optimizations on the medium access control layer, e.g., for the transmit probabilities in the slotted ALOHA protocol [33]. A wider overview with further application examples can be found in [33], [34], [43].

A common approach is to reformulate the objective func-

1E.g., in multiple-input/multiple-output (MIMO) broadcast channels with dirty paper coding [6], or for rate balancing problems in multiple-input/single-output (MISO) interference channels with interference treated as noise [7].
tion as a difference $f^+(x) - f^-(x)$ of nondecreasing functions $f^+$ and $f^-$. The resulting difference-of-monotonic (DM) problem can then be further reformulated to a canonical monotonic optimization problem, where a nondecreasing function is maximized over a normal set. For instance, instead of maximizing $f^+(x) - f^-(x)$ over a box $[r, s]$, we can maximize the nondecreasing function $f^+(x) + t$ under the additional constraints $f^-(x) + t \leq 0$ and $t \in [-f^-(s), -f^-(r)]$, which constitutes a normal set. The resulting canonical monotonic optimization problem can then be solved with the so-called polyblock algorithm (PA) [28, Sec. 11.2], as was done, e.g., in [29, 30]. An important drawback is that the number of optimization variables is increased by introducing $t$, which strongly affects the convergence speed since the complexity of the PA is exponential in the number of variables [43].

As an alternative, DM problems can be solved by means of branch-reduce-and-bound (BRB) or branch-and-bound (BB) techniques as described in [27]. This approach, which was pursued, e.g., in [11, 39], avoids the overhead of the additional optimization variable $t$, but still suffers from a drawback, which will also be observed in Sec. IV-A. Just like the PA, BRB methods rely on calculating utopian bounds to the objective function, and the convergence speed heavily depends on the quality of these bounds. Unfortunately, DM bounds are generally not very tight [27].

Therefore, several authors have proposed to get better convergence properties by reparameterizing an optimization in terms of a new set of variables. For instance, [31–33] use the signal-to-interference-plus-noise ratio (SINR) values of the users as optimization variables instead of their transmit powers, while [7, 11, 36–39] use the achievable rates, and [42] uses the received interference powers. The resulting monotonic or DM problems can then be solved by means of the PA [31–33], [39–38] or a BRB algorithm [11, 39, 42]. However, the change of variables usually makes the evaluation of the objective and constraint functions more costly. For instance, the SINR values and achievable rates can be calculated analytically when the transmit powers are used as optimization variables, but a numerical solver (e.g., a fixed point iteration) is necessary to compute the transmit powers if the SINR values or the achievable rates are used as variables (see, [31–33] and [7, 11, 36–39], respectively). The change of variables might then reduce the number of iterations required in the applied monotonic programming method, but at the cost of increasing the computational complexity of each iteration.

Moreover, not all optimization problems can be conveniently rewritten in terms of monotonic functions or differences of monotonic functions. For instance, in the context of energy-efficient communications, we encounter objective functions that can be written as fractions of differences of monotonic functions. For this type of problems, the fractional monotonic programming method proposed in [23, 34, 35] uses a monotonic programming approach as an inner solver inside Dinkelbach’s method for fractional programs. This combination has the drawback that a highly complex monotonic programming algorithm has to be executed not only once, but repeatedly in each iteration of the outer algorithm. Moreover, it is no longer possible to obtain a rigorous guarantee that the obtained solution is indeed optimal, i.e., that it is no more than a given constant $\eta$ away from the exact globally optimal solution.

In this paper, we propose the framework of mixed monotonic programming (MMP) (Sec. II), which avoids all these drawbacks since it neither requires a reformulation of the objective function nor a change of variables. Instead, the main idea is that a function defined by several terms might have different monotonicity properties in each term and variable. Thus, the MMP approach does not consider whether the whole function is monotonic in a variable, but takes the monotonicity for each occurrence of a variable separately into account by formulating a so-called mixed monotonic (MM) function. If such an MM function can be constructed for a given optimization problem, the problem can be solved by the BRB technique discussed in Sec. III. In Sec. IV we show that a wide variety of optimization problems (including the difficult fractional monotonic problems mentioned above) can be solved with the MMP approach, and we demonstrate significant advantages compared to state-of-the-art solutions using the C++ implementation available at [44].

Note that there are several existing approaches that can be considered as special cases of the MMP framework, the most prominent being DM formulations. However, the MMP approach is much more general and can be used to find solution methods that are faster than the DM approach. This will become clear after the formal definition of an MMP problem in Sec. III. Moreover, some specialized solution methods developed for particular optimization problems can be identified to fall into the more general MMP framework. For instance, [40] exploited a structure with a fraction of nonnegative nondecreasing functions of a scalar variable, and [45, 46] consider an optimization in a two-user interference channel that can be identified to be a two-dimensional special case of the MMP framework. An implementation of the BRB algorithm for MMP problems can thus be readily applied to any of these special cases.

**Notation:** We use $\mathbf{0}$ for the zero vector, $\mathbf{1}$ for the all-ones vector, and $\mathbf{I}_L$ for the identity matrix of size $L$. Vectors are written in bold-face lowercase, matrices in bold-face uppercase. Inequalities between vectors are meant component-wise, and $[r,s] = \{x \mid r \leq x \leq s\}$ denotes a box (hyperrectangle). We use shorthand notations of the form $(\bullet_k)_{\forall k} = (\bullet_1, \ldots, \bullet_k)$, and we write $\mathcal{C}N(0,1)$ for the circularly symmetric Gaussian distribution with zero mean and unit variance.

**II. MIXED MONOTONIC PROGRAMMING**

Consider the optimization problem

$$\max_{x \in \mathcal{D}} f(x) \tag{P}$$

with continuous objective function $f : \mathbb{R}^n \to \mathbb{R}$ and compact feasible set $\mathcal{D} \subseteq \mathbb{R}^n$. For now, we do not need any further

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2Similar reformulations can be applied to the constraints if needed.

3A set $\mathcal{G} \subset \mathbb{R}^n$ is called normal if $\{0 : x \leq \mathcal{G} \} \subseteq \mathcal{G}$ for all $x \in \mathcal{G}$ [60].

4Please refer to [23, Thm. 11.1] for more details.
where monotonic programming requires the combination of mixed monotonic programming (MMP) (P) is said to be a MMP framework. The most prominent among them are DM problems. As mentioned before, some well established problem formulations can be identified as special cases of the novel MMP framework. The most prominent among them are DM programs [26], i.e.,

$$\max_{x \in X} f^+(x) - f^-(x)$$

where $f^+$ and $f^-$ are nondecreasing functions. A MMP representation of that objective is $F(x, y) = f^+(x) - f^-(y)$.

However, the MMP approach is much more versatile. For example, consider the fraction

$$\frac{p^+(x) - p^-(x)}{q(x)}$$

with nondecreasing $p^+$, $p^-$, and $q$, where we assume $p^+(x) - p^-(x) \geq 0$ and $q(x) > 0$ for all $x$. Maximizing this function with monotonic programming requires the combination of Dinkelbach’s algorithm [42] as outer and monotonic programming as inner solver [35]. This approach has the drawbacks that the inner global optimization problem needs to be solved several times and the stopping criterion does not guarantee an $\eta$-optimal solution. Instead, it is easily verified that

$$F(x, y) = \frac{p^+(x) - p^-(y)}{q(y)}$$

is an MMP representation of (4). Thus, (4) can be optimized directly by the proposed algorithm.

Observe that there might exist more than one MMP representation of $f$. For example, consider maximizing the throughput in a wireless interference network [35]

$$\max_{0 \leq p \leq P} \sum_{i=1}^{K} \log \left(1 + \frac{\alpha_i p_i}{\sigma_i^2 + \sum_{j=1}^{\beta_{ij} x_j}}\right)$$

with positive constants $\alpha_i$, $\sigma_i$, and nonnegative $\beta_{ij}$. Conventionally, (6) is converted into a DM program [31] with

$$f^+(x) = \sum_{i=1}^{K} \log(\alpha_i x_i + \sigma_i^2 + \sum_{j=1}^{\beta_{ij} x_j})$$

and $f^-(x) = \sum_{i=1}^{K} \log(\sigma_i^2 + \sum_{j=1}^{\beta_{ij} x_j} x_j)$. This yields the MMP representation in the text below (3). A more direct approach to obtain $F$ from (6) is

$$F(x, y) = \sum_{i=1}^{K} \log \left(1 + \frac{\alpha_i x_i}{\sigma_i^2 + \beta_{ij} x_i + \sum_{j \neq i} \beta_{ij} y_j}\right).$$

This example will be continued in Section IV-A. An important aspect discussed there is that the precise choice of $F$ directly impacts the convergence speed of the obtained algorithm.

To conclude this section, we state some useful properties of MMP functions. Let $F_i(x, y)$ be MMP functions for $i = 1, \ldots, K$, $g(x)$ a real-valued, nondecreasing function, and $h(x)$ a real-valued, nonincreasing function. Then,

$$(x, y) \mapsto \sum_{i=1}^{K} F_i(x, y),$$

$$(x, y) \mapsto \max_{i=1,\ldots,K} F_i(x, y),$$

$$(x, y) \mapsto \min_{i=1,\ldots,K} F_i(x, y),$$

$$(x, y) \mapsto g(F_i(x, y)),$$

$$(x, y) \mapsto h(F_i(x, y)).$$

are MMP functions. In particular, it follows from (10) that

$$(x, y) \mapsto -F_i(x, y)$$

and

$$(x, y) \mapsto 1/F_i(x, y)$$

are MMP if $F_i(x, y)$ is a positive MMP function. If in addition $F_i(x, y) \geq 0$ for all $i = 1, \ldots, K$ and $x, y \in X$ for some $X \subseteq \mathbb{R}^n$, then

$$(x, y) \mapsto \prod_{i=1}^{K} F_i(x, y)$$

is an MMP function on $X$.

III. GLOBAL OPTIMAL SOLUTION OF (P)

We design a BB algorithm to determine a global $\eta$-optimal solution of (P), i.e., a feasible point $x \in X$ such that $f(x) \geq f(x) - \eta$ for all $x \in X$. The core idea of BB is to relax the feasible set $D$ and subsequently partition it such that upper bounds on the objective value can be determined easily. This is where the MMP representation $F$ of the objective function $f$ comes in handy as it is well suited to compute upper bounds over rectangular sets. Let $M = [r, s]$ be a box in $\mathbb{R}^n$. Then,

$$\max_{x \in M \cap D} f(x) \leq \max_{x \in M} f(x, x) \leq \max_{x \in M} f(x, y) = F(s, r)$$

(12)

gives an upper bound $U(M) = U([r, s]) = F(s, r)$ on the optimal value of $f(x)$ on $M \cap D$. Thus, rectangular subdivision [28, Sec. 6.1.3], where a box $M$ is partitioned along a hyperplane parallel to one of its facets, is an excellent choice to partition $D$. Given a point $v \in M$ and index $j \in \{1, 2, \ldots, n\}$, we divide $M$ along the hyperplane $x_j = v_j$. The resulting partition sets are the subrectangles

$$M^- = \{x | r_j \leq x_j \leq v_j, r_i \leq x_i \leq s_i (i \neq j)\}$$

(13a)

$$M^+ = \{x | v_j \leq x_j \leq s_j, r_i \leq x_i \leq s_i (i \neq j)\}.$$ (13b)

This is referred to as a partition via $(v, j)$ of $M$. A partition of $M$ via $(\frac{1}{3}s + r, j)$ where $j \in \arg\max_j s_j - r_j$ is called a bisection of $M$.

We say that $\{M_k\}$ is a decreasing sequence of sets if, for all $k$, $M_{k+1} \subseteq M_k$, i.e., $M_{k+1}$ is a descendent of $M_k$. The following proposition is an important property for the convergence of BB methods.

Lemma 1 [28, Corollary 6.2.1]: Let $\{M_k\}$ be a decreasing sequence of sets such that $M_{k+1}$ is a descendent of $M_k$ in a bisection. Then, the diameter $\text{diam}(M_k)$ of $M_k$ tends to zero as $k \to \infty$.

Besides the subdivision procedure and computation of bounds, the selection of the next box (or branch) for further
partitioning is crucial for the convergence and implementation of our BB procedure. A widely used selection criterion is
\[ M_k \in \arg\max\{U(M) \mid M \in \mathcal{B}_{k-1}\}. \]  
(14)
due to the finiteness of \( \mathcal{B}_k \), every set \( M \in \mathcal{B}_k \) will be deleted or selected after finitely many iterations \([25, \text{p. 130]}\).

The final BB procedure is stated in Algorithm 1. Note that the optional reduction step is discussed separately in \[\text{Sec. III-B}.\]

The convergence of Algorithm 1 is established below.

Algorithm 1 [BBB] Algorithm for MM Problems

- **Step 0 (Initialization)** Choose \( M_0 \supseteq D \) and \( \eta > 0 \). Let \( k = 1 \) and \( \mathcal{R}_0 = \{M_0\} \). If available or easily computable, find \( \hat{x}_0 \in D \) and set \( \gamma_0 = f(\hat{x}_0) \). Otherwise, set \( \gamma_0 = -\infty \).

- **Step 1 (Branching)** Select a box \( M_k = [r^k, s^k] \in \mathcal{R}_{k-1} \) and bisect \( M_k \) via \( (\frac{r^k + s^k}{2}, j) \) with \( j \in \arg\max_{s^k \in M_k} (s^k - r^k)_+ \).

- **Step 2 (Reduction)** For each \( M \in \mathcal{R}_k \), replace \( M \) by \( M' \) such that \( M' \subseteq M \) and
\[ \{(M \setminus M') \cap \{x \in D \mid f(x, x) > \gamma_k\} = \emptyset\}. \]  
(16)

- **Step 3 (Incumbent)** For each \( M \in \mathcal{R}_k \), find \( \bar{x} \in M \cap D \) and set \( \alpha_M = f(\bar{x}) \). If \( M \cap D = \emptyset \), set \( \alpha_M = -\infty \). Let \( \alpha_k = \max \{\alpha_M \mid M \in \mathcal{R}_k\} \). If \( \alpha_k > \gamma_{k-1} \), set \( \gamma_k = \alpha_k \) and let \( \hat{x}_k \in D \) such that \( \alpha_k = f(\hat{x}_k) \). Otherwise, let \( \gamma_k = \gamma_{k-1} \) and \( \hat{x}_k = \hat{x}_{k-1} \).

- **Step 4 (Pruning)** Delete every \( M = [r, s] \in \mathcal{R}_k \) with \( M \cap D = \emptyset \) or \( f(s, r) \leq \gamma_k + \eta \). Let \( \mathcal{R}_k' \) be the collection of remaining sets and set \( \mathcal{R}_k = \mathcal{R}_k' \cup (\mathcal{R}_{k-1} \setminus \{M_k\}) \).

- **Step 5 (Termination)** Terminate if \( \mathcal{R}_k = \emptyset \) or, optionally, if \( \{r, s\} \in \mathcal{R}_k \mid f(s, r) > \gamma_k + \eta \} = \emptyset \). Return \( \hat{x}_k \) as a global-\( \eta \)-optimal solution. Otherwise, update \( k \leftarrow k + 1 \) and return to Step 1.

Theorem 1: Algorithm 1 converges towards a global-\( \eta \)-optimal solution of (P) if the selection is bound improving.

**Proof:** In Step 2 let \( D' = \{x \in D \mid f(x, x) > \gamma_k\} \subseteq D \) and observe that \( D \setminus D' \) does not contain any solutions better than the current best solution. Thus, if \( M' \) satisfies (15), no solutions better than the current incumbent are lost and the reduction does not affect the solution of (P).

3We combine the convergence proof from \[\text{[25, Prop. 5.6]}\] and \[\text{[28, Prop. 6.1]}\] with the idea of a general selection criterion from \[\text{[25, Thm. IV.3]}\].

If the algorithm terminates in Step 3 and iteration \( K \), then \( F(s, r) \leq \gamma_K + \eta \) for all \( [r, s] \in \mathcal{R}_K \) and, since \( F(x, y) > -\infty \), \( \gamma_K > -\infty \). Hence, \( \hat{x}_K \) is feasible and
\[ \gamma_K = f(\hat{x}_K) \geq F(r, s) - \eta \]  
(17)
for every \( M \in \mathcal{R}_K \). Now, for every \( x \in D \), either \( x \in D \cap \bigcap_{M \in \mathcal{R}_K} M \) or \( x \in D \setminus \bigcap_{M \in \mathcal{R}_K} M \). In the first case, \( f(x) - \eta \leq f(\hat{x}_K) \) due to (17). In the latter case, \( x \in M' \) for some \( M' = [r', s'] \in \mathcal{R}_K \) and some \( M_0 \cap \bigcap_{M \in \mathcal{R}_K} M \). Because \( \{\gamma_k\} \) is nondecreasing and due to Step 4 \( f(x) \leq F(s', r') \leq \gamma_K + \eta \). Hence, for every \( x \in D \), \( f(x) \leq f(\hat{x}_K) + \eta \).

It remains to show that Algorithm 1 is finite. Suppose this is not the case. Then, due to the bound improving selection, there exists an infinite decreasing subsequence of \( \{\gamma_k\} \) such that \( M_{k_q} \in \arg\max\{F(s, r) \mid [r, s] \in \mathcal{R}_K\} \). Because \( M_{k_q} \cap \bigcap_{M \in \mathcal{R}_K} M \neq \emptyset \), there exists an \( x_{k_q} \in M_{k_q} \cap \bigcap_{M \in \mathcal{R}_K} M \). Due to Lemma (32) \( \text{diam} \cap \mathcal{R}_K \rightarrow 0 \) as \( q \rightarrow \infty \). Thus, \( x_{k_q}, s_{k_q}, r_{k_q} \) all converge towards a common limit, and, together with (15), \( F(s_{k_q}, r_{k_q}) \rightarrow f(x_{k_q}) \). Since \( (17) \leq \sup_{x \in [r, s]} f \leq F(s, r) \leq F(s_{k_q}, r_{k_q}) = f(x_{k_q}) \), hence, \( F(s_{k_q}, r_{k_q}) = \gamma_{k_q} + \delta_{k_q} \) with \( \delta_{k_q} = 0 \) and \( \lim_{q \rightarrow \infty} \delta_{k_q} = 0 \). Thus, there exists a limit such that \( \delta_k \leq \eta \) for all \( k \geq K \), and \( f(s_{k_q}, r_{k_q}) \leq \gamma_k + \eta \) for all \( [r, s] \in \mathcal{R}_K \) and \( k > K \). Then, either the algorithm is directly terminated in Step 3 or the remaining sets in \( \mathcal{R}_K \) are successively pruned in finitely many iterations until \( \mathcal{R}_K = \emptyset \).

Remark 1 (Relative Tolerance): Algorithm 1 determines an \( \eta \)-optimal solution of (P), i.e., a feasible solution \( \hat{x} \in \mathcal{R}_K \) that satisfies \( f(\hat{x}) \geq f(x) - \eta \) for all \( x \in D \). Instead, by replacing all occurrences of \( \gamma_K + \eta \) in Algorithm 1 with \( (1 + \eta)\gamma_K \), the convergence becomes relative to the optimal value and the algorithm terminates if the solution satisfies \( (1 + \eta)\gamma_{k_q} \) for all \( x \in D \). The necessary modifications of Theorem 1 are straightforward.

A. Properties of D and Implementation of the Feasibility Check

To implement the [BBB] method as described in Algorithm 1, it is necessary to have means to perform the feasibility check in Step 3 and Step 4. Let us first discuss cases in which this can be easily done. Afterwards, we comment on workarounds that can be used if no conclusive feasibility check is available.

A conclusive feasibility test based solely on the properties of MM functions is not possible. Consider a feasible set
\[ D = \{x \mid G_i(x, x) \leq 0, \ i = 1, \ldots, m\} \]  
(18)
where \( G_i \) satisfies (24) and (25). These properties lead to the following sufficient conditions for (in-)feasibility of \( \mathcal{M} \).

**Proposition 1:** Let \( \mathcal{M} = [r, s] \) and \( D \) as in (18). Then,
\[ \forall i \in \{1, \ldots, m\} : G_i(s, r) \leq 0 \Rightarrow \mathcal{M} \cap D = \emptyset \]  
(19a)
and
\[ \exists i \in \{1, \ldots, m\} : G_i(r, s) > 0 \Rightarrow \mathcal{M} \cap D = \emptyset. \]  
(19b)

**Proof:** From (24) and (25), \( G_i(x, x) \leq G_i(s, r) \) and \( G_i(x, x) \geq G_i(r, s) \) for all \( x \in \mathcal{M} \). Thus, if \( G_i(s, r) \leq 0 \)
for all $i = 1, \ldots, l$, then $G_i(x, x) \leq 0$ for all $i$ and $x \in M$. Hence, (19a). Similarly, if $G_i(r, s) > 0$ for some $i = 1, \ldots, l$, then also $G_i(x, x) > 0$ for this $i$ and all $x \in M$. Thus, $x \notin D$ and (19b) holds.

In general, there exist boxes for which neither (19a) nor (19b) holds, so that it remains open whether $M$ contains a feasible point. However, we could consider the special case where

$$G_i \left( \sum_{j \in I_i} x_j e_j, \sum_{k \in I_i} y_k e_k \right) = G_i(x, y), \quad \forall x, y \in \mathbb{R}^n$$

(20)

for some index set $I \subseteq \{1, \ldots, n\}$ and all $i = 1, \ldots, m$ where $I^c = \{1, \ldots, n\} \setminus I$. That is, each function $g_i(x)(x, x)$ is nondecreasing in the variables $x_j$, $j \in I$, and nonincreasing in the remaining variables $x_k$, $k \in I^c$. In this case, the following proposition is a simple feasibility test based on MM properties.

**Proposition 2:** Let $M = [r, s]$ and $D$ be defined as in (15) by MM-functions $G_i(x, y)$ satisfying (2a), (2b) and (20). Then, $M \cap D \neq \emptyset$ if and only if $G_i(r, s) \leq 0$ for all $i = 1, \ldots, m$.

In that case, $\sum_{j \in I_i} r_j e_j + \sum_{k \in I_i} s_k e_k \in M \cap D$ with $I$ and $I^c$ as in (20).

**Proof:** Let $\xi = (r_1, \ldots, r_n, s_{n+1}, \ldots, s_{2n})^T$. Then, for all $i$ and due to (23), $G_i(\xi, \xi) = G_i(r, s)$. Thus, if $G_i(r, s) \leq 0$, then $\xi \in D$. Since, trivially, $\xi \in M$, $\xi \in D \cap M \neq \emptyset$. Finally, from (19b) follows $M \cap D \neq \emptyset \Rightarrow G_i(r, s) \leq 0$. ■

**Corollary 1:** Let $M = [r, s]$ and $D$ be a normal set, i.e.,

$$D = \{ x \mid g_i(x) \leq 0, \quad i = 1, \ldots, m \}$$

(21)

with $g_i$ being nondecreasing functions. Then, $D \cap M \neq \emptyset$ if and only if $g_i(r) \leq 0$ for all $i = 1, \ldots, m$.

**Corollary 2:** Let $M = [r, s]$ and $D$ be a conormal set, i.e.,

$$D = \{ x \mid h_i(x) \geq 0, \quad i = 1, \ldots, m \}$$

(22)

with $h_i$ being nonincreasing functions. Then, $D \cap M \neq \emptyset$ if and only if $h_i(r) \geq 0$ for all $i = 1, \ldots, m$.

Proposition 2 and Corollaries 1 and 2 cover a wide range of feasible sets. However, none of these properties is necessary as long as we have other means to perform a feasibility check. For instance, consider the case where we can express $D$ by

$$g_i(x) \leq 0, \quad i = 1, \ldots, m, \quad h_j(x) = 0, \quad j = 1, \ldots, l$$

(23)

where $g_i$ are convex functions and $h_j$ are affine functions. In this case, $D$ is a closed convex set and the feasibility check can be solved with polynomial complexity by standard tools from convex optimization [44, 50]. In particular, (23) includes polyhedral sets where $g_i$ are affine functions.

Let us now discuss workarounds for cases where a feasibility test as described above is not available, but the constraints can be written as MM-functions as in (15). The first possible workaround is to alter Algorithm 1 such that, in Step 3, a feasible point is only required if available and, in Step 4, boxes are only pruned if we are met. Due to this modification, we can use Proposition 2 instead of a fully conclusive feasibility test. Then, by a similar argument as in [27, Prop. 7.4] and according to the proof of Theorem 1 there exists an infinite decreasing sequence of sets $\{M_k\}_{k=1}^\infty$ such that $G_i^k(r^k, s^k) \leq 0$, for all $i$ and $q = 1, 2, \ldots$. Since diam $M_k \to 0$, $r^k$ and $s^k$ approach a common limit point $x$. Due to the continuity of $G_i$, this point satisfies $G_i(x, x) \leq 0$ for all $i$. Altering Theorem 1's proof accordingly, it can be shown that the modified algorithm is infinite and, whenever it generates an infinite sequence $\{x^k\}$, every accumulation point of this sequence is a global optimum.

In practice, an infinite algorithm often converges in finite time (see the numerical example in Section 8.2), but there are no theoretical guarantees for this, and for some problem instances, the resulting algorithm can have very slow convergence.

Another widely accepted workaround is to accept an $\eta$-optimal point that is approximately feasible as solution, i.e., a point $x$ satisfying $f(x) \geq f(x) - \eta$ for all $x \in D$ and $G_i(x, x) \leq \varepsilon$ for all $i = 1, \ldots, m$ and some $\varepsilon > 0$. Such a point is called $(\varepsilon, \eta)$-approximate optimal solution.

This second method restores finite convergence, but gives rise to numerical problems. If $\varepsilon$ is not chosen sufficiently small, the $(\varepsilon, \eta)$-approximate optimal solution might be far from the true optimum. The issue is that it is usually unclear how small is "sufficient" to guarantee a good approximate solution [28, Sec. 7.5]. Even worse, if the true optimum is an isolated point, any change in the tolerances $\varepsilon, \eta$ can lead to drastic changes in the $(\varepsilon, \eta)$-approximate optimal solution [52, Sec. 4]. We thus generally do not recommend the $(\varepsilon, \eta)$-approximate approach.

A more suitable method for optimization problems with such "hard" feasible sets is the successive incumbent transferring scheme from [52], which algorithmically excludes all isolated feasible points and provides an elegant solution to the feasibility check issues. An optimization framework based on this scheme is published in [51] along with source code, and could be combined with the MM-property concept. Besides its numerical stability, this scheme also improves efficiency for problems that are only nonconvex due to some of their variables [51].

We stress the fact that none of the above workarounds is required if a fully conclusive feasibility test can be implemented (preferably with low computational complexity), so that the unmodified algorithm as stated in Algorithm 1 can be used.

B. The Reduction Procedure in Step 2 of Algorithm 1

In Step 2 of Algorithm 1 each box $M \in \mathcal{P}_k$ is replaced by a smaller box $M'$ that still contains all feasible points that might improve the current best known solution. This step speeds up the convergence since smaller boxes result in tighter bounds. However, it also increases the computation time per iteration and, thus, slows down the algorithm. Ultimately, it depends on the problem at hand, especially the structure of the feasible set $D$, and the implementation of the reduction procedure whether Step 2 speeds up Algorithm 1 or not. Hence, an important observation is that Step 2 is entirely optional since choosing $M' = M$ satisfies the above condition. Moreover, note that (16) is also satisfied if $M'$ satisfies $(M \setminus M') \cap D = \emptyset$.

A feasible point is called isolated if it is at the center of a ball containing no other feasible points. Please refer to [51] for a numerical example showing the existence of isolated feasible points in a radio resource allocation problem.
For $D$ convex (or even linear), we refer the reader to the vast literature on convex (or linear) optimization regarding possible implementations of the reduction. Here, we just mention the most straightforward approach, namely to solve the convex (linear) optimization problems
\[
 r_i' = \min_{x \in M' \cap D} x_i \quad s_i' = \max_{x \in M' \cap D} x_i
\]
for all $i = 1, \ldots, n$, and let $M' = [r', s']$.

For a feasible set defined by $MM$ constraints as in [18], the reduction can be carried out in a similar fashion as for DM programming problems [28, Sec. 11.2.1]. Let $M = [r, s]$ and observe from (19b) that if, for some $i = 1, \ldots, m$, $G_i(r, s) > 0$, then $M \cap D = \emptyset$ and $M' = \emptyset$. Moreover, if $F(s, r) \leq \gamma_k$, then $\{x \in M | F(x, x) > \gamma_k\} = \emptyset$ and $M' = \emptyset$ satisfies (16). Otherwise, i.e., if $G_i(r, s) \leq 0$ for all $i$ and $F(s, r) \geq \gamma_k$, let $M' = [r', s']$ with
\[
r' = s - \sum_{i=1}^{n} \alpha_i(s_i - r_i)e_i, \quad s' = r' + \sum_{i=1}^{n} \beta_i(s_i - r_i)e_i
\]
and, for all $i = 1, \ldots, n$,
\[
\alpha_i = \sup \left\{ \alpha \in [0, 1] | F(s - \alpha(s_i - r_i)e_i, r) > \gamma_k \right\},
\]
\[
G_i(r, s - \alpha(s_i - r_i)e_i) \leq 0, \quad j = 1, \ldots, m \}
\]
\[
\beta_i = \sup \left\{ \beta \in [0, 1] \right\} F(s, r' + \beta(s_i - r_i)e_i) \geq \gamma_k,
\]
\[
G_i(r' + \beta(s_i - r_i)e_i) \leq 0, \quad j = 1, \ldots, m \}
\]

The proof that (16) holds for this reduction procedure is an extension of [28, Lem. 11.1] and is omitted due to space constraints. Equations (26a) and (26b) can be implemented efficiently by a low precision bisection. It is important though that the obtained solutions are greater (or equal) than the true $\alpha_i$, $\beta_i$. Otherwise, feasible solutions might be lost.

IV. APPLICATION EXAMPLES

To demonstrate the usefulness and exceptional performance of the proposed $MM$ approach, we consider examples of various applications in the area of signal processing for communications. Where available, existing globally optimal approaches are discussed and compared to the proposed framework, and run time comparisons show tremendous gains over the state-of-the-art solutions. For the example considered in Sec. IV-A, we even provide an analytical justification why the proposed method outperforms previous $DM$ formulations.

The complete source code is available on GitHub [44]. All reported performance results were obtained on Intel Haswell nodes with Xeon E5-2680 v3 CPUs running at 2.50GHz.

The presented applications are only meant as examples: we are aware of further optimization problems for which the $MM$ framework can be useful, and we are convinced that further applications can also be identified in other research areas.

A. Weighted Sum Rates in the K-User Interference Channel

As a first application example, we consider weighted sum rate maximization in a $K$-user interference channel (IC) under the assumption that the input signals are proper Gaussian and that interference is treated as noise. Letting $\alpha_k$ denote the gains of the intended channels, and $\beta_{kj}$ with $j \neq k$ the gains of the unintended channels, we can write the achievable rates as
\[
r_k = \log_2 \left( 1 + \frac{\alpha_k p_k}{\sigma^2 + \sum_{j=1}^{K} \beta_{kj} p_j} \right)
\]
where $\sigma^2$ is the noise variance, and $p_k$ is the transmit power of user $k$. Due to the possibility of modeling self-interference or hardware impairments by choosing $\beta_{kk} \neq 0$, this formulation is more general than in some of the previous works mentioned below. Note that there are several other system models for which the rate expressions can be brought to a form equivalent to (27), e.g., certain massive MIMO, cellular, and relay-aided scenarios [14], [53].

The weighted sum rate maximization problem with minimum rate constraints is
\[
\max \quad \sum_{k=1}^{K} \frac{w_k r_k}{s.t. \quad r_k \geq R_{\min, k}, \quad k = 1, \ldots, K.}
\]

For this problem, various approaches can be found in the literature. In the MAPE framework [31], [33], the problem is parametrized in terms of SINRs as
\[
\max \quad \sum_{(\gamma_k)_{k=1}^{K}} \frac{w_k \log_2(1 + \gamma_k)}{s.t. \quad \gamma_k \geq \gamma_{\min, k}, \forall k}
\]

where the set of possible SINR combinations $\mathcal{G}$ is approximated from the outside by means of the PA [26] until the global optimal solution is found. Instead, the authors of [7] formulate the problem as
\[
\max \quad \sum_{k=1}^{K} \frac{w_k r_k}{s.t. \quad P_k \geq P_{\min, k}, \quad k = 1, \ldots, K}
\]

where $R$ is the achievable rate region due to (27) and the power constraints. The rate region is then approximated by the PA. This special case of the framework [36], [37] is termed as “Ratespace PA” in the numerical results below. A disadvantage of both methods is that in every iteration an inner problem with considerable computational complexity has to be solved to project points from outside the feasible set onto its boundary.

Another approach to apply the monotonic optimization framework [26] is to rewrite the rates as $DM$ functions
\[
r_k = \log_2 \left( \alpha_k p_k + \frac{\sum_{j=1}^{K} \beta_{kj} p_j}{\sigma^2 + \sum_{j=1}^{K} \beta_{kj} p_j} \right)
\]

This problem can either be solved via the PA by introducing an auxiliary variable [26], [54] (termed as $PA$) or directly via the $BRB$ method for $DM$ problems [27] (“BRB DM”). Among the state-of-the-art, this $BRB$ approach is most closely related to the proposed $MM$ framework. Indeed, finding an $MM$ representation of (34) is straightforward by replacing all powers in the first log-term of (34) by nondecreasing variables $x_i$, and all powers in the second log-term by nonincreasing variables $y_i$. However, we will show below in (35) that this leads to looser bounds and, thus, slower average convergence speed than the new $MM$ representation proposed below.
By calculating the partial derivatives of $r_k$ in (27), it is easy to verify that $r_k$ is nondecreasing in $p_{ik}$ (regardless of the value of $p_{jk}$) and nonincreasing in $p_{j}$ for $j \neq k$ (regardless of value of $p_{ik}$). Thus, the MMP function

$$R_k(x,y) = \log_2 \left(1 + \frac{\alpha_k x_k}{\sigma^2 + \beta_{kk} x_k + \sum_{j \neq k} \beta_{kj} y_j} \right)$$

is an MMP representation of (27). Using (8) and (10), an MMP representation of the objective of (28) is obtained as

$$F(x,y) = \sum_{k=1}^{K} w_k R_k(x,y).$$

An MMP formulation of the feasible set is given by (18) with

$$G_k(x,y) = R_{\min,k} - R_k(y,x), \quad k = 1, \ldots, K. \quad (34)$$

The average convergence speed of BB methods depends strongly on the quality of the bounds, i.e., tighter bounds lead, in general, to faster convergence (e.g., (27)). Consider a single rate $r_k$ and the bounds obtained by (31) and (32), labeled as “MMP, “ to solve (28) with best-first selection (14) and MM function (32), labeled as “MMP, “ requires only slightly more

iterations to converge than the best-first rule. The benefits of the oldest-first selection will be further evaluated below.

A comparison based on iterations works well for algorithms with similar computational complexity per iteration. However, when comparing algorithms as different as the polyblock and BB algorithms, comparing the number of iterations is meaningless: the PA, typically requires much less iterations but each iteration takes much longer than in a BB algorithm. Thus, we resort to comparing the average run time of the algorithms in the C++ implementation available at [44]. We have taken great care to implement the state-of-the-art algorithms with the same rigor and amount of code optimization as the proposed method to make this benchmark as fair as possible. The average run time and memory consumption of all discussed approaches is displayed in Figs. 2 and 3, respectively. The same parameters as in the computation of Fig. 1 were used.

First, observe that all algorithms scale both in run time and memory consumption exponentially in the number of variables. Since problem (28) is NP hard [55, Thm. 1], better asymptotic complexity is most likely not achievable. However, it is obvious that the computational complexity still may have very different slope and some algorithms are significantly more efficient than others. The proposed MMP framework solves problem (28) in considerably less time and memory requirements than all other state-of-the-art methods. The Polyblock based methods all consume more memory than the BB based

Fig. 1. Average number of iterations in Algorithm 1 to solve (28) with bounds obtained by (32) with best- and oldest-first selection, respectively, and (31) with best-first selection. Results are averaged over 100 independent channel realizations.

Fig. 2. Average run time required to solve (28) with different algorithms. Results are averaged over 100 independent channel realizations.
methods starting from three optimization variables. In terms of run time, they are already outperformed by at least 1.5 orders of magnitude in the number of variables and soon reach our run time limit of 8 h. For the BB methods, the observations from Fig. 1 continue to hold in Fig. 2. From the memory consumption, it can be observed that good bounds are not only critical for fast convergence but also for memory efficiency. In this example, the MMP method was able to solve problems more than twice the size of the DM BB method within a memory limit of 2.5 GB.

Finally, observe that the best-first approach consumes considerably more memory than the oldest-first rule, e.g., 3.4× or 546 MB more at 18 variables. Further, observe that while requiring less iterations, the best-first rule has longer run times than the oldest-first rule. This can be explained from Fig. 3 since the memory consumption is directly proportional to the number of boxes in $R_k$. The best-first rule is the mathematical description of a priority queue. While accessing the top-element in a priority queue has complexity $O(1)$, insertion has worst-case complexity $O(\log n)$ [56, pp. 148–152], where $n$ is the number of elements in the data structure. Compared to the other operations during each iteration of the algorithm, which have polynomial complexity in the number of variables, $O(\log n)$ is extremely small except when the size of the queue is very large. Instead, the implementation of the oldest-first rule is a queue, i.e., a first in, first out (FIFO) list. Here, the insertion, deletion, and access to the front element all require constant time $O(1)$ and do not grow with the number of stored elements.

### B. Energy Efficiency Optimization

The global energy efficiency (GEE) is a key performance metric for 5G and beyond networks measuring the network energy efficiency [53, 14, 34]. It is defined as the benefit-cost ratio of the total network throughput in a time interval $T$ and the energy necessary to operate the network during this time:

$$\text{GEE} = \frac{TB \sum_{k=1}^{K} r_k}{\sum_{k=1}^{K} \phi \theta r_k p + P_c} = \frac{B \sum_{k=1}^{K} r_k}{\phi \theta p + P_c} \left[ \frac{\text{bit}}{\text{J}} \right],$$  

(36)

where $r_k$ is the achievable rate of link $k$, $B$ is the bandwidth, $\phi \geq 1$ contains the inverses of the power amplifier efficiencies and $P_c$ is a constant modeling the constant part of the circuit power consumption.

Maximizing the GEE for interference networks with treating interference as noise, i.e., where $r_k$ is as in (27), results in the nonconvex fractional programming problem [34, 57]

$$\max_{0 \leq p \leq P} \sum_{k=1}^{K} r_k \left( \frac{1}{\phi \theta p + P_c} \right)$$  

(37)

where we have omitted the inessential constant $B$ and minimum rate constraints that are already discussed in Section IV-A. As the objective includes the sum rate as a special case for $\phi = 0$ and $P_c = 1$, this problem is also NP-hard due to [55, Thm. 1]. As already mentioned below [4], the state-of-the-art approach to solve (37) is to combine Dinkelbach’s Algorithm [34, 27] with monotonic programming. This was first proposed in [23] and subsequently developed into the fractional monotonic programming framework in [33]. Dinkelbach’s Algorithm solves (37) as a sequence of auxiliary problems

$$\max_{0 \leq p \leq P} \sum_{k=1}^{K} r_k - \lambda \phi \theta p + P_c$$  

(38)

with non-negative parameter $\lambda$. Problem (38) can be solved by monotonic programming much in the same way as discussed in Section IV-A. While most works use the Polyblock Algorithm to solve (38) (e.g., [23], [35]), we have already demonstrated above that the BB with DM bounds from [27] outperforms the classical Polyblock Algorithm [26].

The MMP framework even allows to solve (37) without the need of Dinkelbach’s Algorithm. An MMP representation of (36) can be obtained similar to (4). Specifically, with (32) and the identities in (8), (10) and (11), we obtain

$$F(x, y) = \frac{B \sum_{k=1}^{K} R_k(x, y)}{\phi y + P_c}$$  

(39)

with $R_k(x, y)$ as in (32).

The run time performance of both algorithms is evaluated in Fig. 4 where $\phi = 5$, for all $k$, and $P_c = 1$. The remaining parameters were chosen as in Section IV-A. It can be observed that MMP requires significantly less time to solve (37) than the legacy approach employing Dinkelbach’s Algorithm. For example, with $K = 6$ variables, MMP is on average almost five orders of magnitude faster than fractional monotonic programming. The memory consumption (not displayed) scales almost identically to the run time with MMP using four orders of magnitude less memory for six variables. Besides showing much better run time and memory performance, the MMP method also guarantees an $\eta$-optimal solution. By contrast, Dinkelbach’s Algorithm does not provide any guarantees on the solution quality since an inaccuracy of $\eta$ in the inner solver might propagate to larger inaccuracies in the overall results.

Other energy efficiency (EE) metrics can be maximized with the MMP framework in a similar manner. For example, in interference networks with rate function (27), the weighted minimum EE (WMEE) has the objective (e.g., [35])

$$\text{WMEE} = \min_{k=1,...,K} w_k \frac{B r_k}{\phi_k p_k + P_{c,k}}$$  

(40)
with nonnegative weights \( w_1, \ldots, w_K \) and MMP representation
\[
F_{\text{WMEE}}(x, y) = \min_{k=1, \ldots, K} \frac{B_R(x, y)}{\phi_k y_k + P_{c,k}}.
\]
The weighted sum EE (WSEE) has the objective (e.g., [35])
\[
\text{WSEE} = \sum_{k=1}^{K} w_k \frac{B_{R_k}}{\phi_k y_k + P_{c,k}}
\]
with MMP representation
\[
F_{\text{WSEE}}(x, y) = \sum_{k=1}^{K} w_k \frac{B_{R_k}(x, y)}{\phi_k y_k + P_{c,k}}.
\]
The WMEE can be maximized similarly to the GEE with a combination of the Generalized Dinkelbach Algorithm and monotonic programming [35]. Instead, optimizing the WSEE with monotonic optimization is much more challenging since neither Dinkelbach’s Algorithm nor its generalization are applicable. In [35], it is proposed to transform (42) into a single fractional program, i.e.,
\[
\sum_{k=1}^{K} w_k B_{R_k} \prod_{j \neq k}(\phi_j p_j + P_{c,j})
\]
\[
\prod_{k=1}^{K}(\phi_k p_k + P_{c,k})
\]
and then apply fractional monotonic programming. While this works in theory, it is shown in [33] that this approach has very poor convergence. Instead, the MMP framework allows to directly optimize both metrics without cumbersome transformations and without using Dinkelbach’s Algorithm or its generalized version.

C. Proportional Fair Rate Optimization with Scheduling

The weighted sum rate utility in (38) can also be replaced by a utility function that accounts for the fairness between users, such as the proportional fair utility
\[
U(r_1, \ldots, r_K) = \sum_{k=1}^{K} \ln r_k.
\]
In this context, a common approach (e.g., [32], [33], [36], [37]) is to increase the flexibility in the optimization by scheduling different transmit strategies in multiple time slots and averaging the data rates, i.e.,
\[
\max_{(0 \leq p_k(t) \leq p_k)_{k \forall t}} U (\bar{r}_1, \ldots, \bar{r}_K) \text{ s.t. } \bar{r}_k \geq \rho_k, \forall k
\]
with
\[
\bar{r}_k = \sum_{t=1}^{L} r_k^{(t)}, \quad r_k^{(t)} = \log_2 \left( 1 + \frac{\alpha_k p_k(t)}{\sigma^2 + \sum_{j \neq k} \beta_{kj} p_j^{(t)}} \right).
\]
In this application example, we restrict ourselves to the proportional fair utility [46] since this problem with \( L \geq 3 \) was shown to be NP hard [55] even though the utility is concave in the per-user rates.

In [32], [33], an algorithm called S-MAPEL for nondecreasing utility functions was proposed. The approach is based on the PA and makes use of the reformulation [29] as well as of the observation that the rates are nondecreasing functions of the time fractions \( \tau_{r} \). By arguing that no more than \( L = K + 1 \) strategies are necessary due to the Carathéodory theorem, the approach from [32], [33] uses \( L K = (K + 1) K \) optimization variables in total. This leads to a significant computational complexity. A second disadvantage of this approach is as follows. The optimizer of (46) is not unique since any re-indexing of the time index \( \ell \) leads to an optimal solution as well, but when directly solving (46), this inherent symmetry is not exploited. The authors of [32], [33] thus proposed an accelerated algorithm called A-S-MAPEL, which employs a heuristic (with an additional tolerance parameter \( \varepsilon_{\text{tol}} \)) to exploit the symmetry, but the resulting strategy is no longer guaranteed to be \( \eta \)-optimal.

We focus on the following alternative method for concave utility functions from [36], [37], which avoids increasing the number of variables at the cost of having to solve a series of monotonic optimization problems. To obtain an efficient algorithm, we combine this approach with the MMP framework.

We rewrite problem (46) as
\[
\max_{(0 \leq p_k(t) \leq p_k)_{k \forall t}} U (\rho_1, \ldots, \rho_K) \text{ s.t. } \bar{r}_k \geq \rho_k, \forall k
\]
and consider the Lagrangian dual problem
\[
\max_{\mu \geq 0} \min_{(0 \leq p_k(t) \leq p_k)_{k \forall t}} U (\rho_1, \ldots, \rho_K) + \sum_{k=1}^{K} \mu_k (\bar{r}_k - \rho_k)
\]
where \( \bar{r}_k \) depends on the optimization variables via (47). Since averaging the rates can be interpreted as optimizing over the convex hull of the achievable rate region, [48] can be rewritten as a convex program to show that strong duality holds [36], [37], i.e., [49] has the same optimal value as [48].

In fact, the number of strategies can be reduced to \( L = K \) due to an extension to the Carathéodory Theorem discussed in [38], yielding a total number of \( L K = K^2 \) variables.
We note that $p^{(\ell)}$ can be optimized separately for each $\ell$, and that these inner problems are all equivalent, i.e.,

$$\max_{(0 \leq p^{(\ell)}_k \leq P_k)_{\forall k}} \mu^T \sum_{\ell=1}^{L} \tau^{(\ell)} r^{(\ell)}$$

which implies that the choice of $L$ and $\tau$ in the dual problem is arbitrary. Thus, the dual problem (50) can be rewritten as

$$\min_{\mu \geq 0} u_\mu(p^* (\mu)) + v_\mu(p^* (\mu))$$

(51)

where

$$p^* (\mu) = \arg \max_{(\rho \geq R_{min, k})_{\forall k}} u_\mu(\rho), \quad u_\mu(\rho) = U(\rho) - \mu^T \rho$$

(52a)

and

$$r^* (\mu) = \arg \max_{(0 \leq p_k \leq P_k)_{\forall k}} v_\mu(p), \quad v_\mu(p) = \mu^T r.$$  

(52b)

In total, we have to solve three optimization problems in (51). The outer minimization is a convex problem in the dual variables $\mu$ and can, e.g., be solved by the cutting plane method (59), (60), which successively refines outer approximations

$$\min_{\mu \geq 0, z \in \mathbb{R}} z$$

s. t. $z \geq u_\mu(p^{(\ell)} (\mu)) + v_\mu(p^{(\ell)} (\mu)) \quad \forall \ell \in \{1, \ldots, L\}$.

(53a)\hspace{1cm} (53b)

For given constant vectors $p^{(\ell)}$ and $r^{(\ell)}$, this is a linear program in $\mu$ and $z$. By solving for the optimal $p^*$, setting

$$(\rho^{(L+1)}(\mu^*), p^{(L+1)}(\mu^*)) = (p^* (\mu^*), p^* (\mu^*)),$$  

and incrementing $L$, a refined approximation is obtained. In every iteration, a feasible approximate solution to the primal problem (48) can be recovered by solving the dual linear program of (53), and these solutions converge from below to the global optimum (see, e.g., [59] Sec. 6.5). Note that primal recovery implicitly performs the convex hull operation corresponding to the rate averaging in (47) if needed [37] Sec. 3.3.2. In addition, each iteration delivers a feasible value of the dual problem in (51), which acts as an upper bound to the global optimum of (48). As a termination criterion, we thus check whether the difference of these values is below a predefined accuracy threshold $\varepsilon_{CP}$.

In each iteration of the cutting plane method, evaluating $p^*$ and $r^*$ requires solving the inner problems (52). The first maximization problem (52a) is a convex program due to the assumption of a concave utility. In the special case of the proportional fair utility (49), it can even be solved in closed form.

The challenging nonconvex problem (52b) is a weighted sum rate maximization, which can be tackled by any of the methods discussed in Sec. IV-A. In [56], [37], it was proposed to apply the PA with the rates as optimization variables. Motivated by the run time comparison from Sec. IV-A we instead use the BRB algorithm for MMP problems. Combining this approach with the cutting plane method for the outer problem, we get the guarantee that the obtained solution lies at most $\eta + \varepsilon_{CP}$ away from the global optimum, i.e., it is $(\eta + \varepsilon_{CP})$-optimal.

For a run time comparison using the implementation in [44], we reconsider the example from [32] with $K = 4$ interfering links and channel gains derived from a path loss model for the network topology given in [32] Fig. 5. As in [32], we maximize the proportional fair utility (49) without any constraints on the per-user rates. Since the original S-MAPEL algorithm did not converge within a reasonable amount of time, we use the A-S-MAPEL heuristic with accuracy $\eta = 10^{-2}$ and $\varepsilon_{CP} = 10^{-3}$. Unlike S-MAPEL, this accelerated heuristic cannot give a rigorous guarantee for the quality of the obtained solutions [32], but we can use its run time of 3146 seconds as a (very loose) lower bound for the actual run time of S-MAPEL. The proposed combination of the cutting plane algorithm and the MMP framework with a guarantee of $\eta + \varepsilon_{CP} = 9 \cdot 10^{-4} + 1 \cdot 10^{-3} = 10^{-2}$ converged in only 1.77 seconds.

D. Coded Time-Sharing and Rate Balancing

The combination of a Lagrangian dual approach and the MMP framework can be extended to solve various further problems. For instance, we can consider coded time-sharing (e.g., [61]), where not only the rates but also the transmit powers may be averaged. In this case, we have to dualize the resulting average power constraints $\sum_{\ell=1}^{L} \tau P^{(\ell)}_k \leq P_k$ in addition to the rate constraints. Moreover, we could replace the fairness optimization by a so-called rate balancing problem, which can be used to characterize the Pareto boundary of the rate region (rate profile method [62]) and to guarantee the quality of service of all users.

As an example, let us combine both mentioned modifications in an IC under the assumptions of Gaussian inputs, coded time-sharing, and treating interference as noise. The resulting rate balancing problem with coded time-sharing reads as

$$\max_{(p^{(\ell)}(\rho) \geq 0)_{\forall \rho, \ell}} R$$

s. t. $\sum_{\ell=1}^{L} \tau \rho^{(\ell)}_k \geq \rho_k R, \quad \forall k$  

(54a)

$\sum_{\ell=1}^{L} \tau P^{(\ell)}_k \leq P_k, \quad \forall k$  

(54b)

for given relative rate targets $\rho_k, k = 1, \ldots, K$, and with $\tau_k$ from (47). After introducing dual variables $\lambda$ and $\mu$ for the rate constraints and power constraints, respectively, and performing some reformulations similar to the ones in Sec. IV-C the dual problem of (54) can be written as [46]

$$\min_{\mu \geq 0, \lambda \geq 0} \left( \sum_{k=1}^{K} \lambda_k P_k + \max_{(p^{(\ell)}(\rho) \geq 0)_{\forall \rho, \ell}} (\mu^T r - \lambda^T p) \right).$$

(55)

The inner maximization is no longer a pure weighted sum rate problem, but using the MMP function

$$F(x, y) = \sum_{k=1}^{K} (\lambda_k R_k(x, y) - \lambda_k y_k)$$

(56)

it can still be solved via the MMP framework.
where (Sec. IV-A), the energy efficiency optimizations (Sec. IV-B), with MM function (56). For further details and numerical simulations for the rate balancing problem, the reader is referred to [46].

E. Multiantenna Interference Channels

The MMP framework can also be used in multiantenna scenarios. In the single-input/multiple-output (SIMO) [IC]
\[
y_k = \sum_{j=1}^{K} h_{kj} x_j + \sigma_k^2
\]  
(57)
the achievable rates (with Gaussian codebooks, interference treated as noise, and assuming no self-interference) can be expressed as (e.g., [8])
\[
r_k = \log_2 \left( 1 + \frac{p_k h_{kk}^H (I_{M_k} + \sum_{j \neq k} p_j h_{kj} h_{kj}^H)^{-1} h_{kk}}{\sigma_k^2 + p_j \beta_j (\xi)} \right)
\]  
(58)
where \(M_k\) is the number of antennas at receiver \(k\) and \(I_{M_k}\) is the identity matrix of this size. By replacing the rate (27) by (58), we can formulate the weighted sum rate maximization (Sec. IV-A), the energy efficiency optimizations (Sec. IV-B), the scheduling problem (Sec. IV-C), and the rate balancing problem (Sec. IV-D) for the SIMO [IC].

By calculating the partial derivatives with respect to \(x\) and \(y\) in order to study monotonicity, it can be verified that
\[
R_k(x, y) = \log_2 \left( 1 + x_k h_{kk}^H (I_{M_k} + \sum_{j \neq k} y_j h_{kj} h_{kj}^H)^{-1} h_{kk} \right)
\]  
(59)
is a corresponding MMP function. We can thus directly apply the MMP framework to solve all the above-mentioned problems in the SIMO [IC].

For the MISO [IC] with multiple antennas at the transmitter side, the optimization is more involved since transmit covariance matrices or beamforming vectors need to be designed instead of transmit powers. In the following, we present a beamformer-based method for the two-user MISO [IC]
\[
y_k = h_{kk}^H x_k + h_{kj}^H x_j + \eta_k
\]  
(60)
with \(j = 3 - k\). The transmit signals \(x_k = \sqrt{p_k} b_k s_k\) are generated from scalar Gaussian inputs \(s_k \sim \mathcal{CN}(0,1)\), where \(b_k\) is a normalized beamforming vector with \(\|b_k\| = 1\).

The approach is based on [30], which uses parameters \(k\) to construct a convex combination of the maximum ratio transmission (MRT) beamformer and the zero-forcing (ZF) beamformer, which is provably sufficient to parameterize all Pareto-optimal transmit strategies in the considered scenario. We thus use the beamforming vectors
\[
b_k = b_k \|b_k\|^{-1}, \quad b_k = k b_{MRT} + (1 - k) b_{ZF},
\]  
(61a)
\[
b_{MRT} = h_{kk} \|h_{kk}\|^{-1}, \quad b_{ZF} = \Pi_{h_{kk}}^+, h_{kk} \|\Pi_{h_{kk}}^+ h_{kk}\|^{-1}
\]  
(61b)
where \(\Pi_{h_{kk}}^+ = I_{M_k} - h_{kj} h_{kk}^H h_{kj} h_{kk}^+\) is the orthogonal projection onto the orthogonal complement of the span of \(h_{kj}\), and \(k = 1, 2\) are auxiliary variables that need to be optimized. The achievable rates (with Gaussian codebooks, interference treated as noise, and assuming no self-interference) can then be expressed as (e.g., [30])
\[
r_k = \log_2 \left( 1 + \frac{p_k h_{kk}^H b_k^2}{\sigma_k^2 + p_j \beta_j (\xi)} \right) = \log_2 \left( 1 + \frac{p_k \alpha_k (\zeta)}{\sigma^2 + p_j \beta_j (\xi)} \right)
\]  
(62)
where
\[
\alpha_k (\zeta) = |h_{kk}^H b_k|^2 = \frac{(\zeta k_{y} + (1 - \zeta) k_{y})^2}{1 - 2\zeta k_{y} (1 - \zeta)(1 - \gamma_{kk})} \geq 0,
\]  
(63a)
\[
\beta_k (\zeta) = |h_{kj}^H b_k|^2 = \frac{\zeta k_{y}^2 r_k^2 (\gamma_{kk} - 2)}{1 - 2\zeta k_{y} (1 - \zeta)(1 - \gamma_{kk})} \geq 0
\]  
(63b)
with \(\gamma_{kk} = \|h_{kk}\|^2, \gamma_{kj} = \|\Pi_{h_{kk}}^+ h_{kk}\|, \) and \(\delta_{kj} = |h_{kk}^H h_{kj}|^2\). Since \(\alpha_k (\zeta)\) and \(\beta_k (\zeta)\) are nondecreasing in both components of \(\zeta\) (see [30] for a proof), we can use (56), (61) and (62) to establish the MMP rate expression
\[
R_k (\zeta) = \log_2 \left( 1 + \frac{p_k \alpha_k (\zeta)}{\sigma^2 + q_j \beta_j (\zeta)} \right)
\]  
(64)
We can thus optimize the global energy efficiency in the two-user MISO [IC] by replacing \(R_k\) in (39) by (64) with \(x = [\xi^T, p^T]^T\) and \(y = [\eta^T, q^T]^T\). This means that we apply the BRB algorithm in a four-dimensional space.

In a similar manner, all other optimization problems for the single-antenna interference channel that could previously be formulated by means of the MMP rate expression \(R_k\) from (32) can be easily extended to the two-user MISO interference channel by using the MMP rate expression (64) instead. An example for this is the rate balancing problem (54) in the MISO [IC] which we considered in [2] (for further details and numerical results, see therein). For the special case of a weighted sum rate maximization (28) without minimum rate constraints, the problem can even be simplified since it is then optimal that both users exploit their full power budget [21, Proposition 1], so that we only have to apply the BRB algorithm for the two auxiliary variables \(\zeta\).

The MMP framework can also be applied to nonconvex optimization problems in other multiantenna scenarios, such as, e.g., the \(K\)-user MISO broadcast channel with linear transceivers. An example is the method in [63, Sec. 7.3.1.2], which is in fact a special case of the MMP framework.

F. Probability Optimization for Slotted ALOHA

To demonstrate that the proposed MMP framework can also be useful for solving problems on the medium access control layer, we study the problem from [33, Ch. 7], where the transmission probabilities in the slotted ALOHA protocol with \(K\) users were optimized, i.e.,
\[
\max\limits_{0 \leq \theta \leq 1} U (r_1 (\theta), \ldots, r_K (\theta))
\]  
(65a)
s.t. \(r_k (\theta) \geq R_{\min, k} \forall k\)
(65b)
with an increasing (not necessarily concave) utility function \(U\), and average per-user throughput
\[
r_k (\theta) = c_k \theta_k \prod_{j \in \mathcal{I}_k} (1 - \theta_j).
\]  
(66)
Here, \( \theta = [\theta_1, \ldots, \theta_K]^T \) contains the probabilities \( \theta_k \) that user \( k \) attempts to transmit a packet in any time-slot, and \( \mathcal{I}(k) \) contains the indices of all users that cause interference to receiver \( k \). The data rates \( r_k \) are given by the product of the data rate \( c_k \) of a successful transmission and the probability of a collision-free transmission.

The first solution approach in [33, Ch. 7] transforms the problem to a canonical monotonic optimization problem

\[
\begin{align*}
\max_{\theta \geq 0, \theta \geq 0} & \ U \left( \hat{r}_1(\theta, \tilde{\theta}), \ldots, \hat{r}_K(\theta, \tilde{\theta}) \right) \\
\text{s.t.} & \ \hat{r}_k(\theta, \tilde{\theta}) \geq R_{\text{min},k} \quad \forall k \\
& \ \theta + \tilde{\theta} \leq 1 \\
& \end{align*}
\]

(67a)

with

\[
\hat{r}_k(\theta, \tilde{\theta}) = c_k \theta_k \prod_{j \in \mathcal{I}(k)} \tilde{\theta}_j
\]

(68)

and solves it by means of the PA algorithm. As an alternative, this problem could also be solved with the BRB algorithm for DM problems from [27, Sec. 7]. However, no matter which algorithm is applied, the formulation in (67) suffers from the doubled dimensionality of the optimization problem, which has drastic consequences [33, Ch. 7] since the worst-case complexity of the polyblock or BRB algorithms grows exponentially in the number of variables [43].

Therefore, a second approach

\[
\max_{\psi \geq 0} U(\psi_1 v_1, \ldots, \psi_K v_K)
\]

was proposed in [33, Ch. 7], where

\[
\mathcal{Y} = \{ \psi \ | \ c_k \psi_k \geq R_{\text{min},k}, \forall k \text{ and } 0 \leq \psi \leq 1 : c_k \psi_k = r_k(\theta), \forall k \}.
\]

(70)

As a result, the PA algorithm can be implemented with \( K \) variables only, but this comes at the cost that a geometric program (for details see [33, Ch. 7]) has to be solved to perform the projection to \( \psi \in \mathcal{Y} \) in each iteration of the PA method.

To avoid the drawbacks of both methods, we reformulate (65) in terms of the MMP framework with MM objective and MM constraint functions given as

\[
\begin{align*}
F(x, y) &= U \left( R_1(x, y), \ldots, R_K(x, y) \right), \\
G_k(x, y) &= R_{\text{min},k} - R_k(y, x),
\end{align*}
\]

(71a)

(71b)

with

\[
R_k(x, y) = c_k x_k \prod_{j \in \mathcal{I}(k)} (1 - y_j).
\]

(71c)

Note that these constraints do not fulfill the additional requirements in (22). Thus, Proposition 2 is not applicable and (65) with MMP representations (71) needs to be solved with the modified, infinite version of Algorithm 1 described in Section II-A. However, although the algorithm is infinite in theory, it turns out to have very fast convergence in practice.

It is important to note that the auxiliary variables \( y \) in the MMP method are used only as a vehicle to compute bounds, without considering them as additional optimization variables. Thus, unlike the canonical monotonic reformulation (67), the MMP method does not increase the dimensionality of the problem. Moreover, the MMP formulation avoids an additional inner solver as needed in the geometric-programming-based formulation (69).

### Table I

Mean and median run times of various solution methods for (65).

|                           | 3 Users Mean | 3 Users Median | 4 Users Mean | 4 Users Median |
|---------------------------|--------------|----------------|--------------|----------------|
| 67 & PA (no reduction)    | > 23 h       |                |              |                |
| 67 & BRB (no reduction)   | 88.250 s     | 21.305 s       |              |                |
| 67 & BRB (reduction)      | 23.182 s     | 5.919 s        |              |                |
| 69 & PA                   | 3.629 s      | 0.961 s        | 13.4 h       | 22.935 s       |
| MMP LP (no reduction)     | 0.769 s      | 0.172 s        | 150.1 s      | 4.838 s        |
| MMP LP (reduction)        | 1.413 s      | 0.364 s        | 256.0 s      | 8.139 s        |

For the numerical results in Table I, we have used a three-user system with proportional fair utility (45). Full interference \( \mathcal{I}(k) = \{ j \ | \ j \neq k \} \), and \( c_k = \log_2(1 + |\alpha_k'|^2) \) with \( \alpha_k' \sim \mathcal{CN}(0, 1) \) for all \( k \). To create a variety of challenging scenarios in which some of the minimum rate constraints are active, we have generated \( R_{\text{min},k} = c_k \chi_k \) with \( \chi_k \sim \mathcal{N}((K-1)^2, 0.05^2) \) since \( R_{\text{min},k} = (K-1)^2 c_k \forall k \) would be the boundary to infeasibility in case of full interference. All infeasible scenarios among the generated ones have been discarded, and the results are averaged over 100 feasible scenarios. As all algorithms have fundamentally different per-iteration complexities, it does not make sense to count iterations. We thus again fall back to comparing computation times of the C++ implementations [44].

In addition to the significantly lower run time of the MMP method, another remarkable aspect can be observed. The reduction step (25) reduces the run time of the DM approach while it increases the run time of the MMP approach. As stated before, it depends on the problem under consideration whether or not performing a reduction leads to an overall gain in computation time. An example where the reduction step proves to be very helpful in combination with the MMP method is the rate balancing problem with time-sharing in [2].

### V. DISCUSSION

The mixed monotonic programming (MMP) framework that we propose in this article can directly exploit hidden monotonicity of single terms in a function expression even if the function as a whole is neither monotonic nor a difference of monotonic functions. This allows us to derive bounds that are tighter than previously used difference-of-monotonic (DM) bounds, leading to faster convergence of algorithms for monotonic optimization such as the branch-reduce-and-bound (BRB) algorithm. Moreover, the MMP framework enables us to derive bounds even for a wide range of problems for which no DM reformulation exists, so that we can avoid previously proposed nested algorithms, e.g., for fractional monotonic problems. Due to these advantages, solutions based on the new MMP framework achieve tremendous reductions of run time and memory consumption compared to state-of-the-art solutions in all numerical examples that we considered.

An interesting theoretical aspect of the MMP framework is that it can be considered as a generalization of the DM approach and of other special cases previously studied in the literature. From a practical perspective, we have discussed the
oldest-first selection rule and a reduction method for MMP problems as additional methods to speed up the implementation in specific scenarios. In the code repository [44], we provide a C++ implementation of the proposed algorithm for MMP which can be easily adapted to arbitrary MMP problems, as well as the simulation code for all numerical
tation in specific scenarios. In the code repository [44], we

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