Chiral extrapolation of hadronic vacuum polarization and isospin-breaking corrections

Martin Hoferichter,\textsuperscript{a,\*} Gilberto Colangelo,\textsuperscript{a} Bai-Long Hoid,\textsuperscript{a} Bastian Kubis,\textsuperscript{b} Jacobo Ruiz de Elvira,\textsuperscript{c} Dominik Stamen\textsuperscript{b} and Peter Stoffer\textsuperscript{d,e}

\textsuperscript{a}Albert Einstein Center for Fundamental Physics, Institute for Theoretical Physics, University of Bern, Sidlerstrasse 5, 3012 Bern, Switzerland
\textsuperscript{b}Helmholtz-Institut für Strahlen- und Kernphysik (Theorie) and Bethe Center for Theoretical Physics, Universität Bonn, 53115 Bonn, Germany
\textsuperscript{c}Universidad Complutense de Madrid, Facultad de Ciencias Físicas, Departamento de Física Teórica and IPARCOS, Plaza de las Ciencias 1, 28040 Madrid, Spain
\textsuperscript{d}Physik-Institut, Universität Zürich, Winterthurerstrasse 190, 8057 Zürich, Switzerland
\textsuperscript{e}Paul Scherrer Institut, 5232 Villigen PSI, Switzerland

E-mail: hoferichter@itp.unibe.ch

By far the biggest contribution to hadronic vacuum polarization (HVP) arises from the two-pion channel. Its quark-mass dependence can be evaluated by combining dispersion relations with chiral perturbation theory, providing guidance on the functional form of chiral extrapolations, or even interpolations around the physical point. In addition, the approach allows one to estimate in a controlled way the isospin-breaking (IB) corrections that arise from the pion mass difference. As an application, we present an updated estimate of phenomenological expectations for electromagnetic and strong IB corrections to the HVP contribution to the anomalous magnetic moment of the muon. In particular, we include IB effects in the $\bar{K}K$ channel, which are enhanced due to the proximity of the $\bar{K}K$ threshold and the $\phi$ resonance. The resulting estimates make it unlikely that the current tension between lattice-QCD and data-driven evaluations of the HVP contribution is caused by IB corrections.

The 39th International Symposium on Lattice Field Theory,
8th-13th August, 2022,
Rheinische Friedrich-Wilhelms-Universität Bonn, Bonn, Germany

\*Speaker

© Copyright owned by the author(s) under the terms of the Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International License (CC BY-NC-ND 4.0).
1. Introduction

Understanding the tension between data-driven [1–8] and lattice-QCD determinations [9–11] of the hadronic-vacuum-polarization (HVP) contribution to the anomalous magnetic moment of the muon is of critical importance for the interpretation of the current 4.2σ discrepancy between experiment [12–16] and the prediction in the Standard Model [1–7,17–34], using $e^+e^- \rightarrow$ hadrons cross-section data for the latter. The comparison is well-defined for the total HVP contribution as well as for windows in Euclidean time [35], but, to some extent, even partial quantities evaluated in lattice QCD can be subject to further independent cross checks. In these proceedings, we focus on the role of isospin-breaking (IB) corrections. These have been estimated from phenomenology before [36,37], but with recent work on the dominant exclusive channels using dispersion relations and chiral perturbation theory (ChPT) [4, 21, 38–40] several estimates can be improved, most notably the impact of the pion-mass difference on the $2\pi$ channel. In addition, we include the $\bar{K}K$ channel [41], in which case IB effects in the kaon mass are enhanced due to the proximity of the $\bar{K}K$ threshold and the $\phi$ resonance. All numbers will be given in units of $10^{-10}$.

2. Pion-mass dependence of the two-pion channel

Throughout, we use a decomposition of the pion form factor

$$F^V_\pi(s) = \Omega^I_1(s) \times G_\omega(s) \times G_{\text{in}}(s),$$

(1)

where the three factors incorporate two-pion, three-pion, and higher intermediate states, respectively. The Omnès factor $\Omega^I_1(s)$ [42] does so in terms of the $P$-wave $\pi\pi$ scattering phase shift, $G_\omega(s)$ parameterizes $\rho-\omega$ mixing in terms of the residue $\epsilon_\omega$ at the $\omega$ pole, and $G_{\text{in}}(s)$ is expanded in a (conformal) polynomial, whose parameters can be matched onto the pion charge radius $\langle r^2_{\pi} \rangle$ and higher orders in the low-energy expansion of $F^V_\pi(s)$. Given that $G_\omega$ already represents an IB effect, it suffices to study the pion-mass dependence of the pure $I = 1$ correlator, denoted in Fig. 1 by $a^\mu_{\text{HVP}}[\pi\pi]$ to indicate that $\epsilon_\omega = 0$. To obtain the pion-mass dependence of the $\pi\pi$ phase shift and thus $\Omega^I_1(s)$ [43], we employ the inverse amplitude method (IAM) at one- (NLO) and two-loop (NNLO) order [44], with parameters determined from a combined fit to lattice QCD [45] and phenomenology [4]. For $G_{\text{in}}(s)$, we use the known two-loop expansion of $\langle r^2_{\pi} \rangle$ [46]. Here, the main uncertainty arises from a new low-energy constant $r'_{V_1} = 2.0 \times 10^{-5}$, which we estimate from resonance saturation and validate with lattice-QCD calculations of $\langle r^2_{\pi} \rangle$ at larger-than-physical pion masses [47, 48]. The resulting prediction for the pion-mass dependence in Fig. 1 reproduces the value at the physical point within uncertainties. Possible applications to lattice QCD are discussed in Ref. [39], ranging from a full fit of the $I = 1$ contribution to tests of the strength of infrared singularities in the relevant fit region [49]. In the application to IB, we find that the difference between charged and neutral pion mass gives

$$a^\mu_{\text{HVP}}[\pi\pi]|_{M_{\pi^\pm}} - a^\mu_{\text{HVP}}[\pi\pi]|_{M_{\pi^0}} = -7.67(4)_{\text{ChPT}}(3)_{\text{polynomial}}(4)_{\langle r^2_{\pi} \rangle}(21)_{r'_{V_1}}[22]_{\text{total}},$$

(2)

where the uncertainties refer to chiral convergence, comparison of a normal and conformal polynomial, and the uncertainties in $\langle r^2_{\pi} \rangle$, $r'_{V_1}$, respectively. This effect arises predominantly from the threshold region, in such a way that the resulting contribution is almost exclusively contained in the long-distance (LD) window.
Chiral extrapolation of HVP and isospin-breaking corrections

Martin Hoferichter

3. $\rho-\omega$ mixing and final-state radiation

Final-state radiation (FSR) is dominated by infrared enhanced effects, see Refs. [50, 51] for an explicit test in the context of the forward–backward asymmetry, and we will adopt the results from Ref. [40] obtained under this assumption, supplemented by small dispersive corrections from Ref. [52]

$$a_{\mu}^{\text{HVP}}[\pi\pi, \text{non-Born}] = 0.15\pi^{+}\pi^{-}\gamma + 0.03\pi^{0}\pi^{0}\gamma = 0.18(4).$$

With mixed higher-order terms $O(e^{2}\epsilon_{\omega})$ small, $\lesssim 0.1$, we have [40]

$$a_{\mu}^{\text{HVP}}[\pi\pi, \text{FSR, Born}] = 4.24(2), \quad a_{\mu}^{\text{HVP}}[\pi\pi, \rho-\omega] = 3.68(17).$$

While the FSR contribution represents a pure $O(e^{2})$ effect, it is less clear how to separate $\rho-\omega$ mixing into its QED, $O(e^{2})$, and strong, $O(\delta) = O(m_{u}-m_{d})$, parts. After removing a $\gamma-\omega$-mixing diagram that is subtracted in the bare cross section, leading-order vector-meson ChPT [53] suggests that the entire effect should be of $O(\delta)$, but later work showed that higher-order corrections are difficult to estimate [54, 55]. We will continue to book $\rho-\omega$ mixing in the $O(\delta)$ category, emphasizing that this ambiguity could potentially shift contributions between the two classes of IB.

4. Isospin breaking in the $\bar{K}K$ channel

The threshold region in the $\bar{K}K$ channel is dominated by the isoscalar form factor, and can thus be analyzed in terms of the $\phi$ resonance parameters [41]. The relevant IB effects arise from FSR, from IB in the kaon masses, and from IB in the $\phi$ residues

$$c_{\phi}^{K^{+}K^{-}} = 0.977(6), \quad c_{\phi}^{K^{0}\bar{K}^{0}} = 1.001(6).$$

The latter gives the dominant contribution to the uncertainty, about $\approx 0.8$ in the full HVP integral, as it is not clear which residue, or combination of the two, should be identified with the isospin limit. To define the kaon masses in the isospin limit, we use the charged-kaon self energy $(M_{K^{\pm}}^{2})_{\text{EM}} = 2.12(18) \times 10^{-3}$ GeV$^2$ from the Cottingham formula [41], leading to

$$M_{K^{\pm}} = (494.58 - 3.05\delta + 2.14e^{2}) \text{ MeV}, \quad M_{K^{0}} = (494.58 + 3.03\delta) \text{ MeV},$$
which is close to typical quark-mass-scheme decompositions in lattice-QCD [56, 57]. Using the $\phi$ spectral function from Ref. [41] and varying the kaon masses according to Eq. (6), we obtain

\[
\begin{align*}
\alpha_{\mu}^{HVP} [K^+ K^-, FSR] &= 0.75(4), \\
\alpha_{\mu}^{HVP} [K^+ K^-, e^2] &= -3.24(17), \\
\alpha_{\mu}^{HVP} [K^+ K^-, \delta] &= 4.98(26), \\
\alpha_{\mu}^{HVP} [K^+ K^-, e^2 \delta] &= -0.33(1), \\
\alpha_{\mu}^{HVP} [K^0 \bar{K}^0, e^2] &= -0.02(0), \\
\alpha_{\mu}^{HVP} [K^0 \bar{K}^0, \delta] &= -4.62(23), \\
\end{align*}
\]

so that, due to the resonance enhancement, IB effects as large as 30% are observed, and the mixed $O(e^2 \delta)$ contributions come out larger than in the $2\pi$ channel. While the $K^0$ self energy is negligible, indirect $O(e^2)$ effect from the $K^\pm$ contribution to the $\phi$ spectral function still produce a non-vanishing value in Eq. (7), and the remaining differences between isospin-limit $K^+ K^-$ (16.29) and $K^0 \bar{K}^0$ (16.47) are due to $c_\phi$ and the isovector form factor.

5. Phenomenological estimates of isospin-breaking effects in the HVP contribution

A summary of all effects is shown in Table 1, separately for $O(e^2, \delta)$ contributions and the decomposition into Euclidean windows. The comparison to Refs. [9, 35] indicates somewhat larger values in the intermediate window, especially for $O(e^2)$, but we emphasize that the quoted uncertainties in our phenomenological estimates do not include effects from the missing exclusive channels, which are expected to become more relevant in the intermediate and SD windows. For the full HVP contribution, both our estimate and the inclusive ChPT determination from Ref. [58] (with the critical low-energy constant $\delta C^{(1)}_{93}$ extracted from $\tau$ decays) indicate a larger $O(\delta)$ effect, albeit largely consistent within uncertainties. For $O(e^2)$ we observe good agreement with Ref. [9], which emerges as a result of substantial cancellations among several individually large effects.

6. Conclusions

In this contribution we collected phenomenological estimates of IB effects in the HVP contribution to the anomalous magnetic moment of the muon, improving especially the pion-mass correction in the $2\pi$ channel and adding an estimate of resonance-enhanced $\bar{K}K$ effects. In particular, we provided a breakdown into $O(e^2, \delta)$ components and Euclidean windows, see Table 1 for the main results. Given the limitations of the phenomenological approach to obtain inclusive numbers, there is reasonable agreement with current lattice-QCD calculations. In some cases, $O(e^2)$ for the intermediate window and $O(\delta)$ for the full HVP contribution, some differences are observed, but in both cases the result would increase further if the phenomenological estimates were adopted. We thus conclude that IB corrections are unlikely to play a relevant role in understanding the tension between $e^+ e^-$ data and lattice QCD.

---

1Estimating IB effects in subleading channels becomes increasingly challenging. In the $3\pi$ channel, threshold effects are strongly suppressed by phase space, while again IB in the residue $c_{\rho}^{\gamma} \gamma$ is hard to quantify. FSR effects should scale $\approx 0.4$ by naively comparing to the $2\pi$ channel, and the dependence of $\Gamma_{3\pi}$ on the pion mass [59] cancels out in the integral. Model-based estimates indicate $\approx -0.6$ from a $\rho \to 3\pi$ component [60, 61], but the underlying fit function cannot be reconciled with the analytic properties of the $\gamma^* \to 3\pi$ amplitude. QED corrections to the $R$-ratio are suppressed by $O(10^{-3})$ [62], which implies a correction $\leq 0.1$ in the HVP integral for the energy range in which perturbative QCD applies.
References

1. T. Aoyama et al., The anomalous magnetic moment of the muon in the Standard Model, Phys. Rept. 887 (2020) 1 [2006.04822].

2. M. Davier, A. Hoecker, B. Malaescu and Z. Zhang, Reevaluation of the hadronic vacuum polarisation contributions to the Standard Model predictions of the muon g – 2 and α(m_Z^2) using newest hadronic cross-section data, Eur. Phys. J. C 77 (2017) 827 [1706.09436].

3. A. Keshavarzi, D. Nomura and T. Teubner,Muon g – 2 and α(M_Z^2): a new data-based analysis, Phys. Rev. D 97 (2018) 114025 [1802.02995].

4. G. Colangelo, M. Hoferichter and P. Stoffer, Two-pion contribution to hadronic vacuum polarization, JHEP 02 (2019) 006 [1810.00007].

5. M. Hoferichter, B.-L. Hoid and B. Kubis, Three-pion contribution to hadronic vacuum polarization, JHEP 08 (2019) 137 [1907.01556].

6. M. Davier, A. Hoecker, B. Malaescu and Z. Zhang, A new evaluation of the hadronic vacuum

Table 1: Summary of IB effects from π^0γ, ηγ, 2π(γ), and K^0K(γ), separated into short-distance (SD), intermediate, and LD window, in comparison to the lattice-QCD calculations from Refs. [9, 35] and the ChPT estimate of the O(δ) contribution from Ref. [58].

Acknowledgments

We thank Alexander Keshavarzi for a cross check of the window decomposition of the ηγ channel, as well as Laurent Lellouch and Antonin Portelli for valuable discussions on the kaon mass in different isospin schemes. Support by the SNSF (Project Nos. 200020_175791, PCEFP2_181117, and PCEFP2_194272) and the DFG, through the funds provided to the Sino–German Collaborative Research Center TRR110 “Symmetries and the Emergence of Structure in QCD” (DFG Project-ID 196253076 – TRR 110), is gratefully acknowledged.

References

[1] T. Aoyama et al., The anomalous magnetic moment of the muon in the Standard Model, Phys. Rept. 887 (2020) 1 [2006.04822].

[2] M. Davier, A. Hoecker, B. Malaescu and Z. Zhang, Reevaluation of the hadronic vacuum polarisation contributions to the Standard Model predictions of the muon g – 2 and α(m_Z^2) using newest hadronic cross-section data, Eur. Phys. J. C 77 (2017) 827 [1706.09436].

[3] A. Keshavarzi, D. Nomura and T. Teubner, Muon g – 2 and α(M_Z^2): a new data-based analysis, Phys. Rev. D 97 (2018) 114025 [1802.02995].

[4] G. Colangelo, M. Hoferichter and P. Stoffer, Two-pion contribution to hadronic vacuum polarization, JHEP 02 (2019) 006 [1810.00007].

[5] M. Hoferichter, B.-L. Hoid and B. Kubis, Three-pion contribution to hadronic vacuum polarization, JHEP 08 (2019) 137 [1907.01556].

[6] M. Davier, A. Hoecker, B. Malaescu and Z. Zhang, A new evaluation of the hadronic vacuum
Chiral extrapolation of HVP and isospin-breaking corrections

Martin Hoferichter

polarisation contributions to the muon anomalous magnetic moment and to $\alpha(m_{\mu}^2)$, *Eur. Phys. J. C* 80 (2020) 241 [1908.00921].

[7] A. Keshavarzi, D. Nomura and T. Teubner, $g - 2$ of charged leptons, $\alpha(M_Z^2)$, and the hyperfine splitting of muonium, *Phys. Rev. D* 101 (2020) 014029 [1911.00367].

[8] G. Colangelo et al., Data-driven evaluations of Euclidean windows to scrutinize hadronic vacuum polarization, *Phys. Lett. B* 833 (2022) 137313 [2205.12963].

[9] Budapest-Marseille-Wuppertal collaboration, Leading hadronic contribution to the muon magnetic moment from lattice QCD, *Nature* 593 (2021) 51 [2002.12347].

[10] M. Cè et al., Window observable for the hadronic vacuum polarization contribution to the muon $g - 2$ from lattice QCD, 2206.06582.

[11] ETM collaboration, Lattice calculation of the short and intermediate time-distance hadronic vacuum polarization contributions to the muon magnetic moment using twisted-mass fermions, 2206.15084.

[12] Muon $g - 2$ collaboration, Final Report of the Muon E821 Anomalous Magnetic Moment Measurement at BNL, *Phys. Rev. D* 73 (2006) 072003 [hep-ex/0602035].

[13] Muon $g - 2$ collaboration, Measurement of the Positive Muon Anomalous Magnetic Moment to 0.46 ppm, *Phys. Rev. Lett.* 126 (2021) 141801 [2104.03281].

[14] Muon $g - 2$ collaboration, Magnetic-field measurement and analysis for the Muon $g - 2$ Experiment at Fermilab, *Phys. Rev. A* 103 (2021) 042208 [2104.03201].

[15] Muon $g - 2$ collaboration, Beam dynamics corrections to the Run-1 measurement of the muon anomalous magnetic moment at Fermilab, *Phys. Rev. Accel. Beams* 24 (2021) 044002 [2104.03240].

[16] Muon $g - 2$ collaboration, Measurement of the anomalous precession frequency of the muon in the Fermilab Muon $g - 2$ Experiment, *Phys. Rev. D* 103 (2021) 072002 [2104.03247].

[17] T. Aoyama, M. Hayakawa, T. Kinoshita and M. Nio, Complete Tenth-Order QED Contribution to the Muon $g - 2$, *Phys. Rev. Lett.* 109 (2012) 111808 [1205.5370].

[18] T. Aoyama, T. Kinoshita and M. Nio, Theory of the Anomalous Magnetic Moment of the Electron, *Atoms* 7 (2019) 28.

[19] A. Czarnecki, W.J. Marciano and A. Vainshtein, Refinements in electroweak contributions to the muon anomalous magnetic moment, *Phys. Rev. D* 67 (2003) 073006 [hep-ph/0212229].

[20] C. Gnendiger, D. Stöckinger and H. Stöckinger-Kim, The electroweak contributions to $(g - 2)_\mu$ after the Higgs boson mass measurement, *Phys. Rev. D* 88 (2013) 053005 [1306.5546].

[21] B.-L. Hoid, M. Hoferichter and B. Kubis, Hadronic vacuum polarization and vector-meson resonance parameters from $e^+e^- \rightarrow \pi^0\gamma$, *Eur. Phys. J. C* 80 (2020) 988 [2007.12696].

[22] A. Kurz, T. Liu, P. Marquard and M. Steinhauser, Hadronic contribution to the muon anomalous magnetic moment to next-to-next-to-leading order, *Phys. Lett. B* 734 (2014) 144.
Chiral extrapolation of HVP and isospin-breaking corrections

Martin Hoferichter

[1403.6400].

[23] K. Melnikov and A. Vainshtein, *Hadronic light-by-light scattering contribution to the muon anomalous magnetic moment revisited*, Phys. Rev. D 70 (2004) 113006 [hep-ph/0312226].

[24] P. Masjuan and P. Sánchez-Puertas, *Pseudoscalar-pole contribution to the $(g_\mu - 2)$: a rational approach*, Phys. Rev. D 95 (2017) 054026 [1701.05829].

[25] G. Colangelo, M. Hoferichter, M. Procura and P. Stoffer, *Rescattering effects in the hadronic-light-by-light contribution to the anomalous magnetic moment of the muon*, Phys. Rev. Lett. 118 (2017) 232001 [1701.06554].

[26] G. Colangelo, M. Hoferichter, M. Procura and P. Stoffer, *Dispersion relation for hadronic light-by-light scattering: two-pion contributions*, JHEP 04 (2017) 161 [1702.07347].

[27] M. Hoferichter, B.-L. Hoid, B. Kubis, S. Leupold and S.P. Schneider, *Pion-pole contribution to hadronic light-by-light scattering in the anomalous magnetic moment of the muon*, Phys. Rev. Lett. 121 (2018) 112002 [1805.01471].

[28] M. Hoferichter, B.-L. Hoid, B. Kubis, S. Leupold and S.P. Schneider, *Dispersion relation for hadronic light-by-light scattering: pion pole*, JHEP 10 (2018) 141 [1808.04823].

[29] A. Gérardin, H.B. Meyer and A. Nyffeler, *Lattice calculation of the pion transition form factor with $N_f = 2 + 1$ Wilson quarks*, Phys. Rev. D 100 (2019) 034520 [1903.09471].

[30] J. Bijnens, N. Hermansson-Truedsson and A. Rodríguez-Sánchez, *Short-distance constraints for the HLbL contribution to the muon anomalous magnetic moment*, Phys. Lett. B 798 (2019) 134994 [1908.03331].

[31] G. Colangelo, F. Hagelstein, M. Hoferichter, L. Laub and P. Stoffer, *Short-distance constraints on hadronic light-by-light scattering in the anomalous magnetic moment of the muon*, Phys. Rev. D 101 (2020) 051501 [1910.11881].

[32] G. Colangelo, F. Hagelstein, M. Hoferichter, L. Laub and P. Stoffer, *Longitudinal short-distance constraints for the hadronic light-by-light contribution to $(g_\mu - 2)_\mu$ with large-$N_c$ Regge models*, JHEP 03 (2020) 101 [1910.13432].

[33] RBC/UKQCD collaboration, *Hadronic Light-by-Light Scattering Contribution to the Muon Anomalous Magnetic Moment from Lattice QCD*, Phys. Rev. Lett. 124 (2020) 132002 [1911.08123].

[34] G. Colangelo, M. Hoferichter, A. Nyffeler, M. Passera and P. Stoffer, *Remarks on higher-order hadronic corrections to the muon $g-2$*, Phys. Lett. B 735 (2014) 90 [1403.7512].

[35] RBC/UKQCD collaboration, *Calculation of the hadronic vacuum polarization contribution to the muon anomalous magnetic moment*, Phys. Rev. Lett. 121 (2018) 022003 [1801.07224].

[36] Budapest-Marseille-Wuppertal collaboration, *Hadronic vacuum polarization contribution to the anomalous magnetic moments of leptons from first principles*, Phys. Rev. Latt.
Chiral extrapolation of HVP and isospin-breaking corrections
Martin Hoferichter

Lett. 121 (2018) 022002 [1711.04980].

[37] F. Jegerlehner, The Anomalous Magnetic Moment of the Muon, Springer Tracts Mod. Phys. 274 (2017) 1.

[38] G. Colangelo, M. Hoferichter and P. Stoffer, Constraints on the two-pion contribution to hadronic vacuum polarization, Phys. Lett. B 814 (2021) 136073 [2010.07943].

[39] G. Colangelo, M. Hoferichter, B. Kubis, M. Niehus and J. Ruiz de Elvira, Chiral extrapolation of hadronic vacuum polarization, Phys. Lett. B 825 (2022) 136852 [2110.05493].

[40] G. Colangelo, M. Hoferichter, B. Kubis and P. Stoffer, Isospin-breaking effects in the two-pion contribution to hadronic vacuum polarization, JHEP 10 (2022) 032 [2208.08993].

[41] D. Stamens, D. Hariharan, M. Hoferichter, B. Kubis and P. Stoffer, Kaon electromagnetic form factors in dispersion theory, Eur. Phys. J. C 82 (2022) 432 [2202.11106].

[42] R. Omnès, On the Solution of certain singular integral equations of quantum field theory, Nuovo Cim. 8 (1958) 316.

[43] F.-K. Guo, C. Hanhart, F.J. Llanes-Estrada and U.-G. Meißner, Quark mass dependence of the pion vector form factor, Phys. Lett. B 678 (2009) 90 [0812.3270].

[44] M. Niehus, M. Hoferichter, B. Kubis and J. Ruiz de Elvira, Two-Loop Analysis of the Pion Mass Dependence of the ρ Meson, Phys. Rev. Lett. 126 (2021) 102002 [2009.04479].

[45] C. Andersen, J. Bulava, B. Hörz and C. Morningstar, The $I=1$ pion-pion scattering amplitude and timelike pion form factor from $N_f=2+1$ lattice QCD, Nucl. Phys. B 939 (2019) 145 [1808.05007].

[46] J. Bijnens, G. Colangelo and P. Talavera, The Vector and scalar form-factors of the pion to two loops, JHEP 05 (1998) 014 [hep-ph/9805389].

[47] X. Feng, Y. Fu and L.-C. Jin, Lattice QCD calculation of the pion charge radius using a model-independent method, Phys. Rev. D 101 (2020) 051502 [1911.04064].

[48] $\chi$QCD collaboration, Lattice Calculation of Pion Form Factor with Overlap Fermions, Phys. Rev. D 104 (2021) 074502 [2006.05431].

[49] M. Golterman, K. Maltman and S. Peris, Chiral extrapolation of the leading hadronic contribution to the muon anomalous magnetic moment, Phys. Rev. D 95 (2017) 074509 [1701.08685].

[50] F. Ignatov and R.N. Lee, Charge asymmetry in $e^+e^-\to \pi^+\pi^-$ process, Phys. Lett. B 833 (2022) 137283 [2204.12235].

[51] G. Colangelo, M. Hoferichter, J. Monnard and J. Ruiz de Elvira, Radiative corrections to the forward-backward asymmetry in $e^+e^-\to \pi^+\pi^-$, JHEP 08 (2022) 295 [2207.03495].

[52] B. Moussallam, Unified dispersive approach to real and virtual photon-photon scattering at low energy, Eur. Phys. J. C 73 (2013) 2539 [1305.3143].
[53] R. Urech, \( \rho^0 - \omega \) mixing in chiral perturbation theory, *Phys. Lett. B* **355** (1995) 308 [hep-ph/9504238].

[54] J. Bijnens and P. Gosdzinsky, Electromagnetic contributions to vector meson masses and mixings, *Phys. Lett. B* **388** (1996) 203 [hep-ph/9607462].

[55] J. Bijnens, P. Gosdzinsky and P. Talavera, Vector meson masses in chiral perturbation theory, *Nucl. Phys. B* **501** (1997) 495 [hep-ph/9704212].

[56] L. Lellouch, private communication, 2022.

[57] A. Portelli, these proceedings, 2022.

[58] C.L. James, R. Lewis and K. Maltman, ChPT estimate of the strong-isospin-breaking contribution to the anomalous magnetic moment of the muon, *Phys. Rev. D* **105** (2022) 053010 [2109.13729].

[59] M. Dax, T. Isken and B. Kubis, Quark-mass dependence in \( \omega \to 3\pi \) decays, *Eur. Phys. J. C* **78** (2018) 859 [1808.08957].

[60] D. Boito, M. Golterman, K. Maltman and S. Peris, Evaluation of the three-flavor quark-disconnected contribution to the muon anomalous magnetic moment from experimental data, *Phys. Rev. D* **105** (2022) 093003 [2203.05070].

[61] BaBar collaboration, Study of the process \( e^+e^- \to \pi^+\pi^-\pi^0 \) using initial state radiation with BABAR, *Phys. Rev. D* **104** (2021) 112003 [2110.00520].

[62] R.V. Harlander and M. Steinhauser, rhad: A Program for the evaluation of the hadronic R ratio in the perturbative regime of QCD, *Comput. Phys. Commun.* **153** (2003) 244 [hep-ph/0212294].