Revisiting Spectral Graph Clustering with Generative Community Models

Pin-Yu Chen and Lingfei Wu
IBM Thomas J. Watson Research Center
Yorktown Heights, NY 10598, USA
Email: pin-yu.chen@ibm.com and wuli@us.ibm.com

Abstract—The methodology of community detection can be divided into two principles: imposing a network model on a given graph, or optimizing a designed objective function. The former provides guarantees on theoretical detectability but falls short when the graph is inconsistent with the underlying model. The latter is model-free but fails to provide quality assurance for the detected communities. In this paper, we propose a novel unified framework to combine the advantages of these two principles. The presented method, SGC-GEN, not only considers the detection error caused by the corresponding model mismatch to a given graph, but also yields a theoretical guarantee on community detectability by analyzing Spectral Graph Clustering (SGC) under GENerative community models (GCMs). SGC-GEN incorporates the predictability on correct community detection with a measure of community fitness to GCMs. It resembles the formulation of supervised learning problems by enabling various community detection loss functions and model mismatch metrics. We further establish a theoretical condition for correct community detection using the normalized graph Laplacian matrix under a GCM, which provides a novel data-driven loss function for SGC-GEN. In addition, we present an effective algorithm to implement SGC-GEN, and show that the computational complexity of SGC-GEN is comparable to the baseline methods. Our experiments on 18 real-world datasets demonstrate that SGC-GEN possesses superior and robust performance compared to 6 baseline methods under 7 representative clustering metrics.

I. INTRODUCTION

Community detection aims to assign community labels to nodes in a graph such that the nodes in the same community share higher similarity (better connectivity) than the nodes in different communities [1]. It is essentially an unsupervised learning problem since one is only provided with the information of graph connectivity. Despite its unsupervised nature, recent research developments have been able to identify the informational and algorithmic limits of community detection under certain generative community models (GCMs), especially for spectral graph clustering (SGC) algorithms, such as the use of eigenvectors of the graph Laplacian matrices [2] or the modularity matrix [3] for community detection. However, these analysis assuming that GCMs well match a graph may not hold in practice, which may often yield poor community detection results when there is a mismatch between the given graph and the underlying GCM. On the other hand, optimizing a designed objective function for community detection, such as normalized cut [4] or modularity [5], imposes no model assumption but is sensitive in community detection [6, 7].

Motivated by the advantages of the theoretical and objective principles, we propose SGC-GEN, a novel unified community detection framework that possesses the following features:

• **The power of community detectability.** Under GCMs, the theoretical analysis of community detectability allows us to assess the quality of communities by converting the theoretical guarantees to a loss function that quantifies the error in community detection.

• **The constraint to model mismatch.** By imposing an error metric on the level of inconsistency between a given graph and a GCM, one can confine the detection error due to model mismatch and hence improve community detection.

In particular, due to the extraordinary performance of SGC based on the normalized graph Laplacian matrix, a number of variants of SGC methods have been proposed to improve clustering performance in terms of scalability, robustness, and applicability. To provide a thorough analysis, in this paper we focus on the standard formulation of SGC based on the normalized graph Laplacian matrix introduced by the seminal works (see Sec. III-A) [8], [9], [2]. The main line of this paper is to demonstrate the effectiveness of SGC-GEN that combines standard SGC with GCMs [10] in an unified framework. Originated from the standard SGC formulation as presented in Sec. III-A, SGC-GEN can easily be generalized to many state-of-the-art SGC methods [11], [12], [13], [14]. By revisiting the standard formulation of SGC with GCMs, we establish a novel condition on correct community detection using SGC via the normalized graph Laplacian matrix under a GCM called the stochastic block model (SBM) [10]. We then convert this condition to a data-driven community detection loss function and apply it to SGC-GEN to develop effective and computationally-efficient community detection methods.

We highlight our contributions as following:

• **We propose SGC-GEN**, a unified community detection framework combining the principles of theoretical detectability and well-designed objective functions for improvement.

• **We establish a condition on the correctness of community detection using SGC under a SBM, which leads to a novel data-driven community loss function for SGC-GEN.** Moreover, since the loss function enables community quality assessment, the proposed SGC-GEN resembles the formulation of a supervised learning problem consisting of a loss function and a regularization function.

• **We present an algorithm for SGC-GEN and conduct rigor-
ous computational analysis showing that SGC-GEN could be implemented as efficient as other baseline methods.

- We compare the performance of community detection on 18 real-life graph datasets and use 7 representative clustering metrics to rank each method. The experimental results show that joint consideration of theoretical detectability and model mismatch using SGC-GEN can substantially improve community detection when compared to 6 baseline community detection methods of similar objective functions.

II. GENERAL FRAMEWORK

A. Notations

Throughout this paper bold uppercase letters (e.g., $\mathbf{X}$ or $\mathbf{X}_k$) denote matrices and $[\mathbf{X}]_{i,j}$ denotes the entry in the $i$-th row and the $j$-th column of $\mathbf{X}$, bold lowercase letters (e.g., $\mathbf{x}$ or $\mathbf{x}_k$) denote column vectors, the term $\mathbf{X}^T$ denotes matrix or vector transpose, italic letters (e.g., $x$, $x_k$ or $\mathbf{X}$) denote scalars, and calligraphic uppercase letters (e.g., $\mathcal{X}$ or $\lambda_i$) denote sets. The term $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ denotes a graph characterized by a node set $\mathcal{V}$ and an edge set $\mathcal{E} = \{(i,j) : i,j \in \mathcal{V}\}$. The number of nodes and edges in $\mathcal{G}$ are denoted by $n$ and $m$, respectively.

The convergence of a real rectangular matrix $\mathbf{X} \in \mathbb{R}^{n_1 \times n_2}$ is with respect to the spectral norm, which is defined as $\|\mathbf{X}\|_2 = \max_{\mathbf{z} \in \mathbb{R}^{n_2}} \|\mathbf{Xz}\|_2$, where $\|\mathbf{z}\|_2$ denotes the Euclidean norm of a vector $\mathbf{z}$. Based on the definition, $\|\mathbf{X}\|_2$ is equivalent to the largest singular value of $\mathbf{X}$. A matrix $\mathbf{X} \in \mathbb{R}^{n_1 \times n_2}$ is to say converges to another matrix $\mathbf{M}$ of the same dimension if $\|\mathbf{X} - \mathbf{M}\|_2$ approaches zero. For the convenience of notation, we write $\mathbf{X} \rightarrow \mathbf{M}$ if $\|\mathbf{X} - \mathbf{M}\|_2 \rightarrow 0$ as $n_1, n_2 \rightarrow \infty$.

B. Preliminaries

Throughout this paper, we consider the problem of non-overlapping community detection in a simple connected graph that is undirected, unweighted and contains no self-loops. Given a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and the number of communities $K$, non-overlapping community detection aims to assign each node a community label and divide the nodes into $K$ communities such that the nodes in the same community are better connected than nodes in different communities.

Spectral graph clustering (SGC). SGC is a widely used technique for community detection. It transforms a graph into a vector space representation via spectral decomposition of a matrix associated with a graph. Specifically, each node in the graph is represented by a low-dimensional vector using a common subset of eigenvectors of a matrix. Based on the vector space representation, K-means clustering is applied to obtain $K$ communities. One typical example of SGC is the normalized graph Laplacian matrix $\mathbf{L}$, where its $K$ smallest eigenvectors are used for community detection.

Generative community model (GCM). A GCM generates a graph that embeds community structures. A GCM can be either parametric or nonparametric. For example, the stochastic block model (SBM) is a parametric GCM that specifies a set of within-community and between-community edge connection probability parameters. The graphon model is a nonparametric GCM that generates a graph based on latent representations. Different GCMs are discussed in the survey paper.

SGC under GCMs. For graphs generated by certain GCMs, recent research findings suggest that the performance of community detection using SGC can be separated into two regimes: a detectable regime where the detected communities are consistent with the ground-truth communities, and an undetectable regime where the detected communities and the ground-truth communities are inconsistent. Moreover, the critical space that separates these two regimes can be specified. Consequently, the problem of evaluating the quality of detected communities can be converted to the problem of estimating to which regime the given graph belongs. More details are given in the related work section (Sec. VI).

C. Problem Formulation of SGC-GEN

Consider community detection in a graph $\mathcal{G}$ with an unknown number of communities. For each possible number of communities $K$, we can provide quantitative measures on community detectability and model mismatch for SGC under GCMs. Specifically, given a GCM $\mathcal{M}$ of $K$ communities and a set of communities $\{\mathcal{G}_k\}_{k=1}^K$ detected by a SGC method $\mathcal{F}$, the corresponding community detection loss function and model mismatch metric are as follows.

Community detection loss function. For any $\mathcal{F}$, $\mathcal{M}$ and $\{\mathcal{G}_k\}_{k=1}^K$, let $f(\{\mathcal{G}_k\}_{k=1}^K, \mathcal{F}, \mathcal{M})$ denote a nonnegative loss function that reflects the level of incorrect community detection using $\mathcal{F}$ under $\mathcal{M}$. Larger loss suggests the detected communities $\{\mathcal{G}_k\}_{k=1}^K$ are less reliable.

Model mismatch metric. Let $R(\{\mathcal{G}_k\}_{k=1}^K, \mathcal{M})$ be a real-valued function quantifying the difference between the detected communities $\{\mathcal{G}_k\}_{k=1}^K$ using $\mathcal{F}$ and the underlying GCM $\mathcal{M}$. Larger value of $R$ suggests the detected communities $\{\mathcal{G}_k\}_{k=1}^K$ are less consistent with the assumption of $\mathcal{M}$.

SGC-GEN. Inspired by the formation of supervised learning problems, community detection, albeit an unsupervised learning problem, can be formulated in a similar fashion by specifying a community detection loss function $f$ and a model mismatch metric $R$. Given a maximum number of communities $K_{\text{max}}$, a GCM method $\mathcal{F}$ and a GCM $\mathcal{M}$, the proposed community detection framework, called SGC-GEN, solves the following minimization problem

$$\min_{\{\mathcal{G}_k\}_{k=1}^K \in \mathcal{S}} f(\{\mathcal{G}_k\}_{k=1}^K, \mathcal{F}, \mathcal{M}) + \alpha \cdot R(\{\mathcal{G}_k\}_{k=1}^K, \mathcal{M}),$$

where $\mathcal{S} = \{\{\mathcal{G}_k\}_{k=1}^K : K = 2, \ldots, K_{\text{max}}\}$ denotes the set of candidate community detection results of different number of communities obtained by $\mathcal{F}$. Using terminology from supervised learning theory, $f$ is analog to the loss function, $R$ resembles the regularization function, and $\alpha \geq 0$ is the regularization parameter.

Many existing community detection methods can fit into the framework of SGC-GEN in (1). For example, objective-function-based algorithms specify a particular energy function $f$ for quality assessment and set $R = 0$. Greedy algorithms specify a model mismatch metric $R$ and set $f = 0$. 
and \( \alpha = 1 \). For example, the Louvain method [29] selects \( R \) to be the negative modularity, where modularity is a measure of relative difference between the detected communities and the corresponding configuration model [21].

III. Theoretical Foundation of SGC-GEN: Normalized Graph Laplacian Matrix and Stochastic Block Model

In this section we study the community detectability of SGC using the normalized graph Laplacian matrix under a stochastic block model (SBM). We establish a sufficient and necessary condition such that SGC is guaranteed to yield reliable community detection results for graphs generated by a SBM. The established condition will be used in Sec. [V] to devise a novel data-driven community detection loss function for the proposed SGC-GEN framework in [I]. For demonstration, we also provide a case study of the established condition under a simplified SBM. The proofs of the established theories are given in the supplementary material\[4\].

A. Normalized Graph Laplacian Matrix and Stochastic Block Model (SBM)

SGC using normalized graph Laplacian matrix. Let \( A \) denote the \( n \times n \) adjacency matrix of \( G \) and let \( D \) be the corresponding diagonal degree matrix. The normalized graph Laplacian matrix is defined as \( \frac{1}{2}D^{-\frac{1}{2}}(D-A)D^{-\frac{1}{2}} \). We denote the \( k \)-th smallest eigenpair of \( L_N \) by \((\lambda_k, y_k)\), where \( y_k \) is the eigenvector associated with the eigenvalue \( \lambda_k \) and \( \lambda_k \leq \lambda_{k+1} \). It is also known that \( \lambda_1 = 0 \) [2]. The standard SGC algorithm using \( L_N \) is summarized in Algorithm [I].

Let \( \hat{Y} = [y_1 \ldots y_K] \) be the matrix of eigenvectors \( \{y_k\}_{k=1}^K \). The matrix \( \hat{Y} \) is the solution of the minimization problem

\[
\min_{X \in \mathbb{R}^{n \times K}, \text{tr}(X^TL_NX)} \text{tr}(X^TL_NX),
\]

where \( I_K \) is the \( K \times K \) identity matrix, and the constraint \( X^TX = I_K \) imposes orthogonality and unit norm for the columns in \( X \). If \( G \) is a connected graph, then by the definition of \( L_N \), we have \( y_1 = \frac{1}{\sqrt{n}}[1 \ldots 1] \). Let \( \hat{Y} = [y_2 \ldots y_K] \) be the matrix after removing the first column \( y_1 \) from \( \hat{Y} \). Then (3) can be reformulated as

\[
\min_{X \in \mathbb{R}^{n \times (K-1)}, \text{tr}(X^TL_NX)} \text{tr}(X^TL_NX),
\]

where \( 1_n \) (0\( n \)) is the vector of 1’s (0’s) and \( \hat{Y} \) is the solution to (3). The minimization problem in (3) is a standard formulation of SGC based on the normalized graph Laplacian matrix [8], [9], [2], which is also a fundamental element of many state-of-the-art SGC methods [11], [12], [13], [14], and it will be the foundation of the theoretical results presented in Sec. [III-B].

Stochastic block model (SBM). SBM [10] is a fundamental GCM, and it has been the root of many other GCMs such as the degree-corrected SBM [22] and the random interconnection model [23]. SBM is a parametric GCM that assumes common edge connection probability for within-community and between-community edges. A graph \( G \) of \( K \) communities can be generated by a SBM as follows. The SBM first divides the \( n \) nodes into \( K \) groups, where each group has \( n_k \) nodes such that \( \sum_{k=1}^n n_k = n \). For each unordered node pair \((i, j)\), \( i \neq j \), an edge between \( i \) and \( j \) is connected with probability \( \rho_{ij} \), where \( \rho_{ij} \in \{1, \ldots, K\} \) denote the community labels of \( i \) and \( j \). Therefore, the SBM is parameterized by the number of communities \( K \) and the \( K \times K \) edge connection probability matrix \( P \), where \( P_{kk} = P_{kk} \) and \( P \) is symmetric. We denote the SBM with parameters \( K \) and \( P \) by \( \text{SBM}(K, P) \).

B. Theoretical Guarantees on Community Detectability

Here we analyze the performance of community detection on graphs generated by \( \text{SBM}(K, P) \) using \( L_N \). In particular, we establish a sufficient and necessary condition on correct community detection, where correct community detection means the detected communities using \( L_N \) match the oracle communities generated by \( \text{SBM}(K, P) \), up to some permutation in community labels. The condition of community detectability leads to a novel community detection loss function as will be discussed in Sec. [V]. Let \( n_{min} = \min_k n_k \), \( n_{max} = \max_k n_k \), and let \( \rho_k \) denote the limit value of \( \frac{\mu}{n_k} \) as \( n_k \to \infty \). The following lemma serves as a cornerstone that connects the dots between \( L_N \) and \( \text{SBM}(K, P) \).

Lemma 1. (matrix concentration under \( \text{SBM}(K, P) \))

Let \( A_{ij} \in \mathbb{R}^{n_i \times n_j} \) denote the adjacency matrix of edges between communities \( \mathcal{G}_i \) and \( \mathcal{G}_j \) of a graph generated by \( \text{SBM}(K, P) \), \( i, j \in \{1, \ldots, K\} \). The following holds almost surely as \( n_k \to \infty \), \( \forall k \in \{1, \ldots, K\} \) and \( \frac{\mu}{n} \to c > 0 \):

\[
\frac{\Delta_{ij}}{\mu} \to \sqrt{\rho_i \rho_j P_{ij}} \frac{1}{\sqrt{n_1 \mu_2}},
\]

Proof. We use the Latała’s theorem [24] and the Talagrand’s concentration inequality [25] to prove this lemma. The details are given in Appendix A of the supplementary material\[1\].

The matrix concentration result in Lemma 1 shows that the scaled adjacency matrix \( A_{ij} \) converges asymptotically to a constant matrix of finite spectral norm \( \sqrt{\rho_i \rho_j P_{ij}} \), which associates with the relative community size \( \rho_k \) and the edge connection probability \( P_{ij} \) under \( \text{SBM}(K, P) \). The condition \( c > 0 \) guarantees that all community sizes grow at a comparable rate. Note that Lemma 1 presumes each entry \( P_{ij} \) in \( P \) is a constant. In case of sparse graphs where \( P_{ij} = \frac{\mu}{n} \) or

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1Supplementary material can be downloaded from www.pinyuchen.com
\[ P_{ij} = \frac{b \log n}{n} \] for some positive constants \( a, b \), similar matrix concentration result holds with high probability under mild conditions via degree regularization techniques \([26, 27]\).

Since Algorithm \( \mathcal{A} \) is invariant to the permutation of node indices, for the purpose of analysis we treat the adjacency matrix \( A \) as a matrix of \( K \times K \) blocks \( \{A_{kj}\}_{k,j=1}^K \). Using Lemma \( \mathcal{L} \) we establish a sufficient and necessary condition on correct community detection using \( L_N \) for graphs generated by SBM(\( K, \mathcal{P} \)).

**Theorem 1.** (community detectability using \( L_N \) under SBM(\( K, \mathcal{P} \)))

For any graph \( \mathcal{G} \) generated by SBM(\( K, \mathcal{P} \)), let \( \theta = \sum_{k=2}^{K} 1 - \lambda_k \) and \( \check{Y} = [y_2 \ y_3 \ \ldots \ y_K] \), where \( (\lambda_k, y_k) \) is the \( k \)-th smallest eigenpair of \( L_N \). The following holds almost surely as \( n_k \to \infty \), \( \forall \ k \in \{1, \ldots, K\} \) and \( \frac{\bar{p}_{\max}}{\bar{p}_{\min}} \to c > 0 \):

The \( K \) communities in \( \mathcal{G} \) can be correctly detected using \( \check{Y} \) if and only if \( \theta > 0 \).

**Proof.** We provide a sketch of the proof below. The complete proof is given in Appendix B of the supplementary material.\(^1\)

**Step 1.** Specify the optimality condition of \( \check{Y} \) using \( \mathcal{Y} \).

**Step 2.** Show the distribution of the rows in \( \check{Y} \) can be separated into two regimes, detectable or undetectable, using Lemma \( \mathcal{L} \).

**Step 3.** If \( \check{Y} \) is in the undetectable regime, show the distribution of the rows in \( \check{Y} \) is inconsistent with the community structure.

**Step 4.** If \( \check{Y} \) is in the detectable regime, show the distribution of the rows in \( \check{Y} \) is consistent with the community structure.

**Step 5.** Show \( \check{Y} \) is in the detectable regime if \( \theta > 0 \). \( \Box \)

Note that Theorem \( \mathcal{A} \) provides a novel data-driven criterion for evaluating the quality of communities without the knowledge of the parameters \( \mathcal{P} \) in SBM(\( K, \mathcal{P} \)). In other words, for any graph generated by SBM(\( K, \mathcal{P} \)), for evaluating community detectability it suffices to compute the \( K-1 \) smallest nonzero eigenvalues \( \{\lambda_k\}_{k=2}^K \) of \( L_N \) and inspect the condition \( \theta > 0 \), which will be further explored in Sec. \( \mathcal{Y} \). In addition, Theorem \( \mathcal{A} \) also implies the feasibility of community detection using Algorithm \( \mathcal{A} \) since \( \check{Y} = [y_1 \ \check{Y}] \) and row normalization does not alter the sign of each entry in \( \check{Y} \).

**C. Case Study: SBM(\( 2, \mathcal{P} \))**

To investigate the implication of the sufficient and necessary condition for correct community detection in Theorem \( \mathcal{A} \) we study SBM(\( 2, \mathcal{P} \)), the case of SBM with two communities, and justify the condition via numerical experiments. Under SBM(\( 2, \mathcal{P} \)), we allow the size of the two communities, \( n_1 \) and \( n_2 \), to be arbitrary as long as their limit values \( \rho_1, \rho_2 > 0 \). We also simplify the notation of the edge connection matrix \( \mathcal{P} \) by defining \( P_{11} = p_1, \ P_{22} = p_2, \) and \( P_{12} = P_{21} = q \). The following corollary specifies the condition of community detectability in terms of \( p_1, p_2 \) and \( q \).

**Corollary 1.** (community detectability using \( L_N \) under SBM(\( 2, \mathcal{P} \)))

For any graph \( \mathcal{G} \) generated by SBM(\( 2, \mathcal{P} \)), let

\[ Y_2 = [Y_1^T \ Y_2^T]^T \]

denote the second smallest eigenvector of \( L_N \), where \( y_k, k = 1, 2, \) is the community-indexed block vector of \( Y_2 \). The following holds almost surely as \( n_1, n_2 \to \infty \) and \( \frac{\bar{p}_{\max}}{\bar{p}_{\min}} \to c > 0 \):

The two communities in \( \mathcal{G} \) can be correctly detected using \( \check{Y} \) if and only if \( q < \sqrt{p_1 p_2} \).

Furthermore, \( \check{Y}_1 \to \pm \beta_1 \sqrt{\frac{1}{\bar{p}_1}} \) and \( \check{Y}_2 \to \mp \beta_2 \sqrt{\frac{1}{\bar{p}_2}} \) for some \( \beta_1, \beta_2 > 0 \) if and only if \( q < \sqrt{p_1 p_2} \).

**Proof.** The results are induced from the condition \( \theta > 0 \) in Theorem \( \mathcal{A} \) under SBM(\( 2, \mathcal{P} \)). The proof is given in Appendix C of the supplementary material.\(^1\)

The established detectability condition in Corollary \( \mathcal{A} \) is universal in the sense that it does not depend on the ratio of the community sizes as long as its limit value \( c > 0 \). It is worth mentioning that the condition \( q < \sqrt{p_1 p_2} \) for correct community detection is also consistent with the condition using methods other than \( L_N \), such as the spectral modularity matrix \([28]\), the spectrum of modular matrix \([29]\), and the inference-based method \([30]\). In addition, when \( q < \sqrt{p_1 p_2} \), the results that \( \check{Y}_1 \to \pm \beta_1 \sqrt{\frac{1}{\bar{p}_1}} \) and \( \check{Y}_2 \to \mp \beta_2 \sqrt{\frac{1}{\bar{p}_2}} \) imply the nodes in the same community have identical yet community-wise distinct representation, as \( \check{Y}_1 \) and \( \check{Y}_2 \) are nonzero constant vectors with opposite signs. This guarantees that \( K \)-means clustering on \( y \) leads to correct community detection when \( q < \sqrt{p_1 p_2} \). In particular, when \( n_1 = n_2 \) and \( p_1 = p_2 = p, \) the parameters \( p \) and \( q \) reflect the expected number of within-community and between-community edges, respectively. The condition in Corollary \( \mathcal{A} \) then reduces to \( q < p \), which means the two communities can be correctly detected when there are more within-community edges than between-community edges.

Fig. \( \mathcal{A} \) displays two numerical examples of different community sizes to validate the detectability condition. It can be observed that in both cases when \( q < \sqrt{p_1 p_2} \), correct community detection can be achieved and \( \theta > 0 \). On the other hand, when \( q \geq \sqrt{p_1 p_2} \), correct community detection is impossible.
and $\theta$ is close to 0. Consequently, inspecting the data-driven parameter $\theta$ indeed reveals community detectability, which validates Theorem 1 and Corollary 1.

IV. COMMUNITY DETECTION ALGORITHMS USING SGC-GEN

A. SGC-GEN Meta Algorithm

The proposed SGC-GEN framework in [1] applies to any SGC method and any GCM. It is a meta algorithm that avails community detection by specifying the corresponding community detection loss function $f$ and the model mismatch metric $R$, in addition to the regularization parameter $\alpha$ and the maximum number of communities $K_{\text{max}}$. Algorithm 2 below summarizes SGC-GEN.

Algorithm 2 SGC-GEN meta algorithm

Input:
- graph $G$
- spectral graph clustering (SGC) method $F$
- generative community model (GCM) $M$
- maximum number of communities $K_{\text{max}}$
- regularization parameter $\alpha$
- community detection loss function $f$
- model mismatch metric $R$

Output: $K^*$ communities $\{G_k\}_{k=1}^{K^*}$ in $G$, $2 \leq K^* \leq K_{\text{max}}$

Step 1: Use $F$ to obtain the set $S$ of candidate community detection results. $S = \{\{G_k\}_{k=1}^{K_{\text{max}}} : K = 2, \ldots, K_{\text{max}}\}$

Step 2: Find $K^* = \arg \min_{\{G_k\}_{k=1}^{K_{\text{max}}}} \sum_{k=1}^K f(\{G_k\}_{k=1}^{K_{\text{max}}}, F, M) + \alpha \cdot R(\{G_k\}_{k=1}^{K_{\text{max}}}, M)$

Step 3: Output $\{G_k\}_{k=1}^{K^*}$

B. SGC-GEN via $L_N$ and SBM($K, P$)

Based on the theoretical analysis established in Sec. III, here we specify 2 SGC methods, the corresponding community detection loss function, and 4 model mismatch metrics for SGC-GEN. This yields 8 community detection methods originated from Algorithm 2. In particular, for these methods we select the GCM $M$ to be SBM($K, P$). These SGC-GEN-empowered community detection methods are summarized in Table I. The details are described as follows.

Two SGC methods.
- $F_1$: the first method is SGC using the normalized graph Laplacian matrix $L_N$ as described in Algorithm 1. To obtain the set $S$ in step 1 of Algorithm 2, one computes the $K_{\text{max}}$ smallest eigenvectors of $L_N$ and use Algorithm 1 to obtain the candidate communities $\{G_k\}_{k=1}^{K_{\text{max}}}$ of different $K$ in $S$.
- $F_2$: the second method is regularized SGC using the normalized graph Laplacian matrix $L_N$. It is similar to $F_1$ except that one replaces the matrix $D$ in step 1 of Algorithm 1 with $D + \lambda I$, where $\lambda$ is the average degree of the graph $G$.

Community detection loss function.

Since $F_1$ and $F_2$ are SGC methods via $L_N$, using Theorem 1 we set the community detection loss function $f$ to be

$$f = \exp \left( -\frac{\theta}{K-1} \right),$$

where $\theta = \sum_{k=2}^K 1 - \lambda_k$ and $\lambda_k$ is the $k$-th smallest eigenvalue of $L_N$. It is similar to the exponential loss function used in supervised learning problems. The denominator $K - 1$ serves the purpose of comparing different community detection results in $S$. When $\theta > 0$, the function $f$ is confined in the interval $[0, 1]$, and it favors the community detection results of small partial eigenvalue sum $\sum_{k=2}^K \lambda_k$, which is a measure of multiway cut in $G$ [2]. When $\theta \leq 0$, $f$ is greater than 1 and has exponential growth as $\theta$ decreases, which implies that $f$ imposes large loss on incorrect community detection results based on Theorem 1. Note that $f$ is a data-driven function since it only requires the knowledge of $\{\lambda_k\}_{k=2}^{K_{\text{max}}}$.

Four model mismatch metrics.
- $R_1$: spectral radius of the modular matrix with respect to SBM($K, P$). Define the $n \times n$ modular matrix $B$ with respect to SBM($K, P$) as $[B]_{ij} = [A]_{ij} - \hat{P}_{g_i g_j}$ if $i \neq j$ and $[B]_{ii} = 0$ if $i = j$, for all $i, j \in \{1, \ldots, n\}$, where $g_i$ denotes the community label of node $i$. The parameter $P_{g_i g_j}$ is the maximum likelihood estimator of $P_{g_i g_j}$. The largest eigenvalue of $P_{g_i g_j}$ under SBM($K, P$) is $\frac{\lambda_{g_i g_j}}{\mu_{g_i g_j}}$; the smallest eigenvalue of $P_{g_i g_j}$ under SBM($K, P$) is $\frac{-\lambda_{g_i g_j}}{\mu_{g_i g_j}}$.

It relates to the first-order eigenvalue approximation of the signed triangle counts [34], which is an effective statistic for testing latent structure in random graphs.
- $R_2$: negative modularity. Given communities $\{G_k\}_{k=1}^{K_{\text{max}}}$ in $G$, modularity is a measure of difference between $\{G_k\}_{k=1}^{K_{\text{max}}}$ and a random graph of the same degree sequence [5]. The modularity is defined as $Q = \sum_{k=1}^K \epsilon_{kk} - \mu^2$, where $\epsilon_{kk} = \frac{m_{kk}}{m}$ if $i \neq j$ and $\epsilon_{ij} = \frac{m_{ij}}{m}$ if $i = j$, for all $i, j \in \{1, \ldots, K\}$, and $b_i = \sum_{j=1}^K \epsilon_{ij}$. By defining $R_2 = -Q$, the model mismatch metric is small when the communities are distinct from the corresponding randomized graphs.
- $R_3$: AIC under SBM($K, P$). The Aikake information criterion (AIC) is a measure of the relative quality of statistical models for a given set of data. $R_3$ is defined as the AIC given communities $\{G_k\}_{k=1}^{K_{\text{max}}}$ under SBM($K, P$), which is $R_3 = K(K - 1) - 2\phi(\{G_k\}_{k=1}^{K_{\text{max}}})$, where $\phi$ denotes the log-likelihood of $\{G_k\}_{k=1}^{K_{\text{max}}}$ under SBM($K, P$). The closed-form expression of $\phi$ is given in [22].
- $R_4$: BIC under SBM($K, P$). The Bayesian information criterion (BIC) is another relative measure of data fitness to statistical models. $R_4$ is defined as the BIC of communities $\{G_k\}_{k=1}^{K_{\text{max}}}$ under SBM($K, P$), which is $R_4 = \ln m \cdot K(K - 1) - 2\phi(\{G_k\}_{k=1}^{K_{\text{max}}})$. 


TABLE I: Summary of 8 SGC-GEN-empowered methods (first 4 rows highlighted by brown color) and 6 comparative baseline approaches in Sec. V-B

| Method             | Algorithm | GCM         | f   | R               | Computational complexity                        |
|--------------------|-----------|-------------|-----|-----------------|-----------------------------------------------|
| SGC-EG (regSGC-EG) | $f_1$ ($f_2$) | SBM($K$, $P$) | (4) | $R_1$          | $O(K_{\text{max}}(m+n') + (K_{\text{max}} + K_{\text{max}})n)$ |
| SGC-MOD (regSGC-MOD) | $f_1$ ($f_2$) | SBM($K$, $P$) | (4) | $R_2$          | $O(K_{\text{max}}m + (K_{\text{max}} + K_{\text{max}})m)$ |
| SGC-AIC (regSGC-AIC) | $f_1$ ($f_2$) | SBM($K$, $P$) | (4) | $R_3$          | $O(K_{\text{max}}m + (K_{\text{max}} + K_{\text{max}})m)$ |
| SGC-BIC (regSGC-BIC) | $f_1$ ($f_2$) | SBM($K$, $P$) | (4) | $R_4$          | $O(K_{\text{max}}m + (K_{\text{max}} + K_{\text{max}})m)$ |
| SGC-MOD (regSGC-MOD) | $f_1$ ($f_2$) | SBM($K$, $P$) | (4) | $R_2$          | $O(K_{\text{max}}m + (K_{\text{max}} + K_{\text{max}})m)$ |
| SGC-AIC (regSGC-AIC) | $f_1$ ($f_2$) | SBM($K$, $P$) | (4) | $R_3$          | $O(K_{\text{max}}m + (K_{\text{max}} + K_{\text{max}})m)$ |
| SGC-BIC (regSGC-BIC) | $f_1$ ($f_2$) | SBM($K$, $P$) | (4) | $R_4$          | $O(K_{\text{max}}m + (K_{\text{max}} + K_{\text{max}})m)$ |
| SGC-MOD (regSGC-MOD) | $f_1$ ($f_2$) | SBM($K$, $P$) | (4) | $R_2$          | $O(K_{\text{max}}m + (K_{\text{max}} + K_{\text{max}})m)$ |
| SGC-AIC (regSGC-AIC) | $f_1$ ($f_2$) | SBM($K$, $P$) | (4) | $R_3$          | $O(K_{\text{max}}m + (K_{\text{max}} + K_{\text{max}})m)$ |
| SGC-BIC (regSGC-BIC) | $f_1$ ($f_2$) | SBM($K$, $P$) | (4) | $R_4$          | $O(K_{\text{max}}m + (K_{\text{max}} + K_{\text{max}})m)$ |
| SGC-MOD (regSGC-MOD) | $f_1$ ($f_2$) | SBM($K$, $P$) | (4) | $R_2$          | $O(K_{\text{max}}m + (K_{\text{max}} + K_{\text{max}})m)$ |
| SGC-AIC (regSGC-AIC) | $f_1$ ($f_2$) | SBM($K$, $P$) | (4) | $R_3$          | $O(K_{\text{max}}m + (K_{\text{max}} + K_{\text{max}})m)$ |
| SGC-BIC (regSGC-BIC) | $f_1$ ($f_2$) | SBM($K$, $P$) | (4) | $R_4$          | $O(K_{\text{max}}m + (K_{\text{max}} + K_{\text{max}})m)$ |

C. Computational Complexity Analysis

Here we analyze the computational complexity of the 8 SGC-GEN community detection methods listed in Table I. There are three main factors contributing to the computational complexity: (i) computation of the $K_{\text{max}}$ smallest eigenvectors of $L_N$, (ii) K-means clustering, and (iii) computation of the community detection loss function and the model mismatch metric. The overall computational complexity of each method is summarized in Table II.

For (i), computing the $K_{\text{max}}$ smallest eigenvectors of $L_N$ requires $O(K_{\text{max}}(m+n))$ operations using power iteration techniques [35], [36], [87], [38], where $m+n$ is the number of nonzero entries in $L_N$. For (ii), given any $K \leq K_{\text{max}}$, K-means clustering on the rows of the $K$ smallest eigenvectors of $L_N$ requires $O(nK^2)$ operations [40]. As a result, to obtain the set $S$ of candidate community detection results by varying $K$ from 2 to $K_{\text{max}}$ requires $O(nK_{\text{max}}^3)$ operations in total. For (iii), the complexity of computing the function $\theta$ and the loss function $f$ in (4) is negligible since they can be obtained in the process of (i). The computation of $R_1$ for a given $K$ requires $O(m)$ operations for computing $\{P_{k\ell}\}_{k,\ell=1}^{K}$ and $O(m' + n)$ operations for computing the spectral radius of $B$ using power iteration techniques, where $m'$ is the number of nonzero entries in $B$. Therefore, the overall computational complexity of $R_1$ in SGC-GEN is $O(K_{\text{max}}(m'' + m + n))$, where $m''$ is the maximum number of nonzero entries in $B$ ranging from $K = 2$ to $K = K_{\text{max}}$. The computation of $R_2$ for a given $K$ is $O(m)$, the same complexity for computing modularity [5]. The overall computational complexity of $R_2$ in SGC-GEN is $O(K_{\text{max}}m)$. For a given $K$, the computation of $R_3$ and $R_4$ requires $O(m)$ operations to compute the closed-form log-likelihood function $\phi$. The overall computational complexity of $R_3$ and $R_4$ in SGC-GEN is $O(K_{\text{max}}m)$. The computational complexity of $f_1$ and $f_2$ has the same order since the regularization step in $f_2$ simply adds $n$ entries to the degree matrix $D$. Similarly, the data storage of these methods require $O(K_{\text{max}}^2(m+n))$ space.

In summary, the overall computational complexity of SGC-GEN-enabled methods is linear in the number of nodes and edges ($n$ and $m$) and depends on $K_{\text{max}}$. In practice $K_{\text{max}}$ is a constant such that $K_{\text{max}} \ll n$ and $m$. Based on the computational analysis, the community detection methods based on SGC-GEN have the same order of complexity in $n$ and $m$ when compared with the baseline methods of similar objective functions described in Sec. V-B which suggests that utilizing SGC-GEN for community detection is computationally as efficient as these baseline methods.

V. PERFORMANCE EVALUATION

A. Dataset Description and Evaluation Metrics

**Dataset Description.** To compare the performance of community detection, we collected 18 real-life graph datasets from various domains, including online social, physical, biological, communication, collaboration, email, and publication networks. For each dataset, we extracted the largest connected component as the input graph $G$ for community detection. All input graphs are made undirected, unweighted and unlabeled. Among these datasets, 6 datasets are provided with additional community labels. If a node in the graph is provided with more than one community label, the most common label among its neighboring nodes is assigned to the node. The statistics of the collected graphs are summarized in Table III.

**Evaluation metrics.** We use 7 representative external and internal clustering metrics to evaluate the performance of different communication detection methods. External clustering metrics can be computed when the community labels are given. Internal clustering metrics evaluate the quality of communities in terms of connectivity, which can be computed without community labels.

- **external clustering metrics:**
  - Normalized mutual information (NMI) [40].
  - Rand index (RI) [40].
  - F-measure (FM) [40].

These external clustering metrics are properly scaled between 0 and 1, and larger value means better clustering performance.

- **internal clustering metrics:**
  - Conductance (COND) [8]: the averaged COND over all communities. Lower value means better performance.
  - Normalized cut (NC) [8]: the averaged NC over all communities. Lower value means better performance.
  - Average out-degree fraction (avg-ODF) [41]: the averaged avg-ODF over all communities. Lower value means better performance.
  - Modularity (MOD) [5]: MOD is defined in the model.
TABLE II: Statistics and descriptions of the collected graph datasets. “NA” stands for “not available”.

| Dataset        | Description                        | Node   | Edge   | # of nodes | # of edges | Community labels |
|----------------|------------------------------------|--------|--------|------------|------------|------------------|
| BlogCatalog    | online social network              | user   | friendship | 10312      | 333983     | 39 social groups |
| Youtube        | online social network              | user   | friendship | 22180      | 96092     | 47 social groups |
| PoliticalBlog  | online social network              | user   | blog reference | 1222      | 16714     | 2 political parties |
| Cora           | publication network                | paper  | citation | 2485       | 5069       | 7 research topics |
| Citeese         | publication network                | paper  | citation | 2110       | 3694       | 6 research topics |
| Pubmed         | publication network                | paper  | citation | 19717      | 44324     | 5 research topics |
| PrettyGoodPrivacy | communication network            | router | connection | 10680      | 24316          | NA                  |
| AS-Newman      | communication network              | router | connection | 22963      | 48436     | NA                |
| AS-SNAPE       | communication network              | router | connection | 6474       | 12572     | NA                |
| Facebook       | online social network              | user   | friendship | 4039       | 88234     | NA                |
| Email-Arenas   | email network                      | user   | communication | 1133      | 5451       | NA                |
| Email-Enron    | email network                      | user   | communication | 33696      | 180811     | NA                |
| MinnesotaRoad  | physical network                   | intersection | road   | 2640       | 3302       | NA                |
| PowerGrid      | physical network                   | power station | power line | 4941     | 6594       | NA                |
| Reactome       | biological network                 | protein | interaction | 5973      | 146385    | NA                |
| CA_AstroPh     | collaboration network              | researcher | coauthorship | 17903    | 197000     | NA                |
| CA_HepPh       | collaboration network              | researcher | coauthorship | 21363    | 91314     | NA                |
| CA_CondMat     | collaboration network              | researcher | coauthorship | 11204    | 117634    | NA                |

mismatch metric $R_2$ in Sec. IV-B. Larger value means better performance.

**Average rank score.** To combine multiple clustering metrics for performance evaluation of different community detection methods, we adopt the methodology proposed in [6], [7] and use the average rank score of all clustering metrics as the performance metric. For each dataset, we rank each community detection method for every clustering metric via standard competition rankings and obtain an average rank score of all clustering metrics. Therefore, lower average rank score means better community detection.

B. Baseline Comparative Methods

As summarized in Table I, we compare the performance of SGC-GEN methods with 6 baseline community detection methods similar to loss functions and model mismatch metrics:

- **SBM-AIC:** Given the number $K$ of communities, SBM-AIC uses Bayes’ inference techniques to evaluate the posterior distribution of community assignments given the graph $G$ under the SBM. We implemented the state-of-the-art package WSBM [31] and use the AIC to determine the final communities ranging from $K = 2$ to $K = K_{\text{max}}$.
- **SBM-BIC:** SBM-BIC is the same as SBM-AIC except that one uses the BIC to determine the final community detection results.
- **DCSBM-AIC:** DCSBM-AIC is the same as SBM-AIC except that one uses the degree-corrected SBM (DCSBM) [22] for inference.
- **DCSBM-BIC:** DCSBM-BIC is the same as SBM-BIC except that one uses the degree-corrected SBM (DCSBM) [22] for inference.
- **Self-Tuning:** Self-Tuning is a SGC algorithm that uses an energy function based on $L_A$ for basis rotation and finds the best community detection results among 2 to $K_{\text{max}}$ communities [19].
- **Louvain:** Louvain method is a greedy modularity maximization approach for community detection based on node merging [20].

C. The Effect of Regularization Parameter $\alpha$

Here we investigate the effect of the regularization parameter $\alpha$ in (1) on the performance of the eight SGC-GEN community detection methods listed in Table I. We set $K_{\max} = 50$ and use the six datasets with additional community labels in Table III to select $\alpha$ from the set $\{0, 10^{-6}, 10^{-5}, \ldots, 10^{2}\}$. For illustration, Fig. 2 displays the stacked average rank plot of these SGC-GEN methods separately ranked by different $\alpha$ in Youtube and Citeeseer datasets. The colors represent different methods and the width of each colored block represents average rank score based on the selected values of $\alpha$. It is observed that for each method, setting large $\alpha$ (i.e., underestimating the loss function) or neglecting the model mismatch metric (i.e., setting $\alpha = 0$) leads to the worst performance, which justifies the motivation of SGC-GEN. In addition, sweeping $\alpha$ within $\{10^{-6}, \ldots, 10^{-1}\}$ does not induce drastic changes in performance.

2http://socialcomputing.asu.edu/datasets/BlogCatalog3
3http://socialcomputing.asu.edu/datasets/YouTube2
4http://konect.uni-koblenz.de/networks/moreno-blogs
5http://www.cs.umd.edu/~snap/proj/data/cora.tgz
6http://www.cs.umd.edu/~snap/proj/data/citeeseer.tgz
7http://www.cs.umd.edu/projects/linugs/projects/lbc/Pubmed-Diabetes.tgz
8http://konect.uni-koblenz.de/networks/arenas-pgp
9http://www-personal.umiicht.edu/~mein/ndita.html
10http://snap.stanford.edu/data/as.html
11http://snap.stanford.edu/data/egonets-Facebook.html
12http://konect.uni-koblenz.de/networks/arenas-email
13http://snap.stanford.edu/data/email-Enron.html
14http://www.cise.ufl.edu/research/sparse/matrices/Gleich/minnesota.html
15http://konect.uni-koblenz.de/networks/opsohl-powergrid
16http://konect.uni-koblenz.de/networks/reactome
17http://snap.stanford.edu/data/ca-HepPh.html
18http://snap.stanford.edu/data/ca-Citeseer.html
19http://snap.stanford.edu/data/ca-CondMat.html
20http://tuvalu.santafe.edu/~arong/wsbm/
21http://www.vision.caltech.edu/Lihi/Libi/Demos/SelfTuningClustering.html
22https://perso.uluontain.be/vincent.blondel/research/louvain.html
the average rank score, which demonstrates the robustness of SGC-GEN. Based on the average rank score of these datasets, for the following experiments we assign $\alpha = 10^{-4}$ ($\alpha = 10^{-6}$) to the (regularized) SGC-GEN methods.

**D. Comparison to Baseline Methods**

Here we compare the 8 SGC-GEN methods to the 6 baseline methods in Sec. [C] For the Bayesian inference baseline methods we set $K_{\text{max}} = 20$, since we observe that larger $K_{\text{max}}$ does not improve their performance but significantly increases the computation time. For the SGC-GEN methods and Self-Tuning we set $K_{\text{max}} = 50$. For Louvain one does not need to specify $K_{\text{max}}$. All experiments are implemented by Matlab R2016 on a 16-core cluster with 128 GB RAM.

Table III displays the mean and standard deviation of average rank scores over all 18 graph datasets for each community detection method. Among these 14 methods, SGC-MOD and SGC-EIG have the best and second best mean average rank score over all datasets, which suggests that joint consideration of theoretical detectability and modular structure using the proposed SGC-GEN framework improves community detection. The results also suggest that the degree regularization technique does not necessarily guarantee better performance. For the baseline methods, it can be observed that Bayesian inference based approaches lead to poor performance, which can be explained by the fact the graph datasets may not comply with the assumption of the underlying generative community models. Louvain also yields poor performance since it is a greedy algorithm that only aims to maximize one single clustering metric (i.e., modularity). Self-Tuning performs better than some SGC-GEN methods but it does not prevail SGC-MOD and SGC-EIG, which can be explained by the fact that the energy function used in Self-Tuning does not exploit the discriminative power of community detectability. Since in Sec. [C] SGC-GEN is shown to be computationally as efficient as these baseline methods, we conclude that community detection via SGC-GEN yields superior performance without incurring additional computational costs.

**E. Comparison in graph domains and types**

For further analysis, we categorize the 18 graph datasets in Table III into 7 domains based on their descriptions. Fig. 3 displays the mean average rank score of each domain for 10 selected methods. It can be observed that no single community detection method outperforms others in all domains. For example, regSGC-MOD has superior performance in online social, publication and biological networks but has poor performance in email and communication networks. SBM-BIC has the best performance in communication networks but not in other domains. SGC-MOD has the best averaged performance over all datasets but it does not prevail others in every domain. The results suggest that considering the graph domain is essential for improving community detection.

We also separate the 18 datasets into two types: with community labels or without community labels. The corresponding average rank score is shown in Fig. 4. For the datasets with community labels, regSGC-MOD, regSGC-AIC and regSGC-BIC are outstanding, whereas for the datasets without community labels SGC-EIG and SGC-MOD prevail. Since community labels provide additional external clustering metrics, the results suggest that the external and internal clustering metrics have different evaluation criterion.

**VI. RELATED WORK**

Community detection and graph clustering have been an active research field in the past two decades. We refer readers to [1], [42] for an overview of community detection methods. In recent years, there has been a major breakthrough in analyzing both the informational and algorithmic limits of community detection under certain generative community models (GCMs). In this section we summarize the recent research findings in community detectability. Many informational and algorithmic limits of community detection have been analyzed under the stochastic block model (SBM) [10]. Abbe et al. analyzed the informational
In this paper we propose SGC-GEN, a new community detection framework that jointly exploits the discriminative power of community detectability under generative community models and confines the corresponding model mismatch. A novel condition on correct community detection is established for SGC-GEN, leading to effective and computationally efficient community detection methods.

Performance evaluation on 18 datasets and 7 clustering metrics shows that joint consideration of community detectability and modular structure via SGC-GEN outperforms 6 baseline approaches in terms of the average rank score. We also investigated the effect of graph domains and graph types on community detection.

The performance analysis established in this paper focuses on the standard formulation of SGC rooted in many advanced methods. Our future work involves developing scalable implementation of SGC-GEN to efficiently handle large-scale graphs and extending SGC-GEN to advanced community detection methods and models.

REFERENCES

[1] S. Fortunato, “Community detection in graphs,” Physics Reports, vol. 486, no. 3-5, pp. 75–174, 2010.
[2] U. Luxburg, “A tutorial on spectral clustering,” Statistics and Computing, vol. 17, no. 4, pp. 395–416, Dec. 2007.
[3] M. E. J. Newman, “Finding community structure in networks using the eigenvectors of matrices,” Phys. Rev. E, vol. 74, p. 036104, Sep 2006.
[4] S. White and P. Smyth, “A spectral clustering approach to finding communities in graph,” in SIAM International Conference on Data Mining (SDM), vol. 5, 2005, pp. 76–84.
[5] M. E. J. Newman, “Fast algorithm for detecting community structure in networks,” Phys. Rev. E, vol. 69, p. 066133, Jun 2004.
[6] J. Leskovec, J. K. Lang, and M. Mahoney, “Empirical comparison of algorithms for network community detection,” in ACM International Conference on World Wide Web (WWW), 2010, pp. 631–640.
[7] J. Yang and J. Leskovec, “Defining and evaluating network communities based on ground-truth,” Knowledge and Information Systems, vol. 42, no. 1, pp. 181–213, 2015.
[8] J. Shi and J. Malik, “Normalized cuts and image segmentation,” IEEE Trans. Pattern Anal. Mach. Intell., vol. 22, no. 8, pp. 888–905, 2000.
[9] A. Y. Ng, M. I. Jordan, and Y. Weiss, “On spectral clustering: Analysis and an algorithm,” in Advances in neural information processing systems (NIPS), 2002, pp. 849–856.
[10] P. W. Holland, K. B. Laskey, and S. Leinhardt, “Stochastic blockmodels: First steps,” Social Networks, vol. 5, no. 2, pp. 109–137, 1983.
[11] J. Liu, C. Wang, M. Danilevsky, and J. Han, “Large-scale spectral clustering on graphs,” in International Joint Conference on Artificial Intelligence. AAAI Press, 2013, pp. 1486–1492.
[12] F. Nie, X. Wang, and H. Huang, “Clustering and projected clustering with adaptive neighbors,” in ACM International Conference on Knowledge Discovery and Data Mining (KDD), 2014, pp. 977–986.
[13] Y. Li, J. Huang, and W. Liu, “Scalable sequential spectral clustering,” in AAAI, 2016, pp. 1809–1815.
[14] F. Nie, X. Wang, M. I. Jordan, and H. Huang, “The constrained laplacian rank algorithm for graph-based clustering,” in AAAI, 2016, pp. 1969–1976.
[15] A. Goldenberg, A. X. Zheng, S. E. Fienberg, and E. M. Airoldi, “A survey of statistical network models,” Foundations and Trends® in Machine Learning, vol. 2, no. 2, pp. 129–233, 2010.
[16] P. Diaconis and S. Janson, “Graph limits and exchangeable random graphs,” arXiv preprint arXiv:0712.2749, 2007.
[17] Y. Zhang, E. Levina, and J. Zhu, “Estimating network edge probabilities by neighborhood smoothing,” arXiv preprint arXiv:1509.08558, 2015.
[18] E. Abbe, A. S. Bandeira, and G. Hall, “Exact recovery in the stochastic block model,” IEEE Trans. Inf. Theory, vol. 62, no. 1, pp. 47–487, 2016.
[19] L. Zelnik-Manor and P. Perona, “Self-tuning spectral clustering,” in Advances in neural information processing systems (NIPS), 2004, pp. 1601–1608.

[20] V. D. Blondel, J.-L. Guillaume, R. Lambiotte, and E. Lefebvre, “Fast unfolding of communities in large networks,” Journal of Statistical Mechanics: Theory and Experiment, no. 10, 2008.

[21] M. E. J. Newman, “Modularity and community structure in networks,” Proc. National Academy of Sciences, vol. 103, no. 23, pp. 8577–8582, 2006.

[22] B. Karrer and M. E. J. Newman, “Stochastic blockmodels and community structure in networks,” Phys. Rev. E, vol. 83, p. 016107, Jan 2011.

[23] P.-Y. Chen and A. O. Hero, “Phase transitions and a model order selection criterion for spectral graph clustering,” arXiv preprint arXiv:1604.03159, 2016.

[24] R. Latala, “Some estimates of norms of random matrices,” Proc. Am. Math. Soc., vol. 133, no. 5, pp. 1273–1282, 2005.

[25] M. Talagrand, “Concentration of measure and isoperimetric inequalities in product spaces,” Publications Mathématiques de l’Institut des Hautes études Scientifiques, vol. 81, no. 1, pp. 73–205, 1995.

[26] C. M. Le and R. Vershynin, “Concentration and regularization of random graphs,” arXiv preprint arXiv:1506.00669, 2015.

[27] A. Joseph, B. Yu et al., “Impact of regularization on spectral clustering,” The Annals of Statistics, vol. 44, no. 4, pp. 1763–1791, 2016.

[28] P.-Y. Chen and A. O. Hero, “Universal phase transition in community detectability under a stochastic block model,” Phys. Rev. E, vol. 91, p. 032804, Mar 2015.

[29] T. P. Peixoto, “Eigenvalue spectra of modular networks,” Phys. Rev. Lett., vol. 111, p. 098701, Aug 2013.

[30] Y. Zhao, E. Levina, and J. Zhu, “Consistency of community detection in networks under degree-corrected stochastic block models,” The Annals of Statistics, vol. 40, no. 4, pp. 2266–2292, 08 2012.

[31] C. Aicher, A. Z. Jacobs, and A. Clauset, “Learning latent block structure in weighted networks,” Journal of Complex Networks, p. cnu026, 2014.

[32] K. Chaudhuri, F. C. Graham, and A. Tsiatas, “Spectral clustering of graphs with general degrees in the extended planted partition model,” in COLT, vol. 23, 2012, pp. 35–1.

[33] A. A. Amini, A. Chen, P. I. Bickel, E. Levina et al., “Pseudo-likelihood methods for community detection in large sparse networks,” The Annals of Statistics, vol. 41, no. 4, pp. 2097–2122, 2013.

[34] S. Bubeck, J. Ding, R. Eldan, and M. Z. Rácz, “Testing for high-dimensional geometry in random graphs,” Random Structures & Algorithms, 2016.

[35] O. E. Livne and A. Brandt, “Lean algebraic multigrid (la mg): Fast graph accurately computing singular triplets of large matrices,” arXiv preprint arXiv:1207.0396, 2012.

[36] L. Wu and A. Stathopoulos, “A preconditioned hybrid svd method for accurately computing singular triplets of large matrices,” SIAM Journal on Scientific Computing, vol. 37, no. 5, pp. S365–S388, 2015.

[37] L. Wu, J. Laeuchli, V. Kalantizis, A. Stathopoulos, and E. Gallopoulos, “Estimating the trace of the matrix inverse by interpolating from the diagonal of an approximate inverse,” Journal of Computational Physics, vol. 326, pp. 828–844, 2016.

[38] L. Wu, E. Romero, and A. Stathopoulos, “PrimeS_SVDS: A high-performance preconditioned svd solver for accurate large-scale computations,” arXiv preprint arXiv:1607.01404, 2016.

[39] M. J. Zaki and W. Meira Jr, Data mining and analysis: fundamental concepts and algorithms. Cambridge University Press, 2014.

[40] G. W. Flake, S. Lawrence, and C. L. Giles, “Efficient identification of web communities,” in ACM International Conference on Knowledge Discovery and Data Mining (KDD), 2000, pp. 150–160.

[41] S. Fortunato and D. Hric, “Community detection in networks: A user guide,” Physics Reports, vol. 659, pp. 1–44, 2016.

[42] E. Abbe and C. Sandon, “Community detection in general stochastic block models: fundamental limits and efficient recovery algorithms,” arXiv preprint arXiv:1503.00699, 2015.

[43] Y. Zhao and E. Levina, “Consistency of community detection in graphs with the bethe hessian,” in Advances in neural information processing systems, NIPS, 2014, pp. 406–414.

[44] J. Lei and A. Rinaldo, “Consistency of spectral clustering in stochastic block models,” Ann. Statist., vol. 43, no. 1, pp. 215–237, 02 2015.

[45] P.-Y. Chen and A. Hero, “Deep community detection,” IEEE Trans. Signal Process., vol. 63, no. 21, pp. 5706–5719, Nov. 2015.

[46] B. Zhang and M. A. Hasan, “Name disambiguation from link data in a collaboration graph,” in IEEE/ACM ASONAM, 2014, pp. 83–84.

[47] K. Saha, B. Zhang, and M. A. Hasan, “Name disambiguation from link data in a collaboration graph using temporal and topological features,” Social Network Analysis and Mining, vol. 5, no. 1, p. 11, 2015.

[48] T. K. Saha, B. Zhang, and M. A. Hasan, “Name disambiguation from link data in a collaboration graph using temporal and topological features,” Social Network Analysis and Mining, vol. 5, no. 1, p. 11, 2015.

[49] P.-Y. Chen and A. O. Hero, “Assessing and safeguarding network resilience to nodal attacks,” IEEE Commun. Mag., vol. 52, no. 11, pp. 138–143, Nov. 2014.

[50] M. Dhand, Q. Kou, B. Zhang, Y. He, and B. Rajwa, “Simplicity of kmeans versus deepness of deep learning: A case of unsupervised feature learning with limited data,” in IEEE ICMLA, 2015, pp. 883–888.

[51] P.-Y. Chen and A. O. Hero, “Local Fiedler vector centrality for detection of deep and overlapping communities in networks,” in IEEE ICASSP, 2014, pp. 1120–1124.

[52] X. Peng, R. S. Feris, X. Wang, and D. N. Metaxas, “A recurrent encoder-decoder network for sequential face alignment,” in ECCV, 2016, pp. 38–56.

[53] P.-Y. Chen and S. Liu, “Bias-variance tradeoff of graph laplacian regularizer,” IEEE Signal Processing Letters, vol. 24, no. 8, pp. 1118–1122, Aug 2017.

[54] X. Peng, J. Huang, Q. Hu, S. Zhang, A. Elgammal, and D. Metaxas, “From circle to 3-sphere: Head pose estimation by instance parameterization,” Computer Vision and Image Understanding, vol. 136, pp. 92–102, 2015.

[55] S. Liu, P.-Y. Chen, and A. O. Hero, “Accelerated distributed dual averaging over evolving networks of growing connectivity,” arXiv preprint arXiv:1704.05193, 2017.

[56] T. Qin and K. Rohe, “Regularized spectral clustering under the degree-corrected stochastic blockmodel,” in Advances in neural information processing systems (NIPS), 2013, pp. 3120–3128.

[57] G. Zhang, Z. Ma, A. Y. Zhang, and H. H. Zhou, “Community detection in degree-corrected block models,” arXiv preprint arXiv:1607.06993, 2016.

[58] P.-Y. Chen and A. O. Hero, “Phase transitions in spectral community detection,” IEEE Trans. Signal Process., vol. 63, no. 16, pp. 4339–4347, Aug 2015.

[59] ——, “Multi-layer spectral graph clustering via convex layer aggregation: Theory and algorithms,” IEEE Trans. Signal Inf. Process. Netw., 2017.
We separate the proof into two cases: (I) $i \neq j$, and (II) $i = j$. For case (I), notice that under $\text{SBM}(K, \bm{P})$ each entry in $\bm{A}_{ij}$ is an independent and identical Bernoulli random variable with success probability $P_{ij}$. Let $\Delta = \bm{A}_{ij} - \bar{\bm{A}}_{ij}$, where $\bar{\bm{A}}_{ij} = P_{ij} \mathbf{1}_{n_i} \mathbf{1}_{n_j}^T$. As a result, each entry in $\Delta$ is either $1 - P_{ij}$ with probability $P_{ij}$ or $-P_{ij}$ with probability $1 - P_{ij}$. The Latała’s theorem [24] states that for any random matrix $\bm{M}$ with statistically independent and zero mean entries, there exists a positive constant $c_1$ such that

$$
\mathbb{E}[\sigma_1(\bm{M})] \leq c_1 \left( \max_s \sqrt{\sum_i \mathbb{E}[\|\bm{M}\|_{2,i}^2]} + \max_{s,t} \sqrt{\sum_i \mathbb{E}[\|\bm{M}\|_{2,i}^4]} \right),
$$

(S1)

where $\sigma_1(\bm{M})$ is the largest singular value of $\bm{M}$. It is clear that each entry in $\Delta$ is independent and has zero mean. By replacing $\bm{M}$ with $\frac{\Delta}{\sqrt{n_i n_j}}$ in the Latała’s theorem, since $P_{ij} \in [0, 1]$, we have $\max_s \sqrt{\sum_i \mathbb{E}[\|\bm{M}\|_{2,i}^2]} = O\left(\frac{1}{\sqrt{n_i n_j}}\right)$, $\max_{s,t} \sqrt{\sum_i \mathbb{E}[\|\bm{M}\|_{2,i}^4]} = O\left(\frac{1}{n_i n_j}\right)$, and $\sqrt{\sum_s \mathbb{E}[\|\bm{M}\|_{2,i}^2]} = O\left(\frac{1}{n_i n_j}\right)$. Therefore, $\mathbb{E}[\sigma_1\left(\frac{\Delta}{\sqrt{n_i n_j}}\right)] \to 0$ as $n_i, n_j \to \infty$.

We then use the Talagrand’s concentration inequality, which is stated as follows. Let $h : \mathbb{R}^k \to \mathbb{R}$ be a convex and $1$-Lipschitz function. Let $\bm{x} \in \mathbb{R}^k$ be a random vector and assume that every element of $\bm{x}$ satisfies $||\bm{x}|| \leq C$ for all $i = 1, 2, \ldots, k$, with probability one. Then there exist positive constants $c_2$ and $c_3$ such that $\forall \epsilon > 0$,

$$
\Pr\left( |h(\bm{x}) - \mathbb{E}[h(\bm{x})]| \geq \epsilon \right) \leq c_2 \exp\left(-\frac{c_3 \epsilon^2}{C^2}\right).
$$

(S2)

Since $\sigma_1(\bm{M}) = \max_{x} h(\bm{x})$ [19], it is easy to check that $\sigma_1(\bm{M})$ is a convex and $1$-Lipschitz function. Therefore, applying the Talagrand’s inequality and substituting $\bm{M} = \frac{\Delta}{\sqrt{n_i n_j}}$ with the facts that $\mathbb{E}\left[\sigma_1\left(\frac{\Delta}{\sqrt{n_i n_j}}\right)\right] \to 0$ and $\frac{[\Delta]_{1,i}}{\sqrt{n_i n_j}} \leq \frac{[\Delta]_{1,i}}{\sqrt{n_i n_j}}$, we have

$$
\Pr\left( \sigma_1\left(\frac{\Delta}{\sqrt{n_i n_j}}\right) \geq \epsilon \right) \leq c_2 \exp\left(-c_3 n_i n_j \epsilon^2\right).
$$

(S3)

Note that, since $n_i n_j \geq \frac{n_i + n_j}{2}$ for any positive integer $n_i, n_j > 0$, we have $\sum_i n_i n_j c_2 \exp\left(-c_3 n_n j \epsilon^2\right) < \infty$. Hence, by the Borel-Cantelli lemma [71], $\sigma_1\left(\frac{\Delta}{\sqrt{n_i n_j}}\right) \xrightarrow{a.s.} 0$ when $n_i, n_j \to \infty$, where $\xrightarrow{a.s.}$ denotes almost sure convergence. Using the result from standard matrix perturbation theory [70] yields $\sigma_k(\bar{\bm{A}}_{ij} + \Delta) - \sigma_k(\bar{\bm{A}}_{ij}) \leq \sigma_1(\Delta)$ for all $k$, where $\sigma_k$ denotes the $k$-th largest singular value. By the fact that $\sigma_1\left(\frac{\Delta}{\sqrt{n_i n_j}}\right) \xrightarrow{a.s.} 0$, we have as $n_i, n_j \to \infty$,

$$
\sigma_1\left(\frac{\Delta}{\sqrt{n_i n_j}}\right) = \sigma_1\left(\frac{\bar{\bm{A}}_{ij} + \Delta}{\sqrt{n_i n_j}}\right) \xrightarrow{a.s.} \sigma_1\left(\frac{\bar{\bm{A}}_{ij}}{\sqrt{n_i n_j}}\right) = P_{ij};
$$

(S4)

$$
\sigma_k\left(\frac{\Delta}{\sqrt{n_i n_j}}\right) \xrightarrow{a.s.} 0, \forall k \geq 2,
$$

(S5)

which implies

$$
\frac{\operatorname{A}_{ij}}{\sqrt{n_i n_j}} \xrightarrow{a.s.} P_{ij} \frac{1}{\sqrt{n_i}} \frac{1}{\sqrt{n_j}}
$$

(S6)

as $n_i, n_j \to \infty$. Finally, for every $i, j \in \{1, \ldots, K\}, i \neq j,$

$$
\frac{\operatorname{A}_{ij}}{n} = \frac{\bar{\operatorname{A}}_{ij}}{\sqrt{n_i n_j}} \xrightarrow{a.s.} P_{ij} \frac{1}{\sqrt{n_i}} \frac{1}{\sqrt{n_j}}
$$

(S7)

as $n_i, n_j \to \infty$ and $\frac{n_{\max}}{n_{\min}} \to c > 0$, which completes the proof of case (I).

For case (II), since $\operatorname{A}_{ii}$ accounts for the adjacency matrix of edges within community $i$, it is a symmetric matrix with zeros on its main diagonal. Let $\tilde{\operatorname{A}}$ denote the matrix that has the same entries as $\operatorname{A}_{ii}$ in the upper diagonals and has zero entries in the lower diagonals. Then $\operatorname{A}_{ii} = \tilde{\operatorname{A}} + \tilde{\operatorname{A}}^T$. Applying the Latała’s theorem and the Talagrand’s concentration inequality to $\tilde{\operatorname{A}}_{ii}^{-1}$, we have

$$
\frac{\operatorname{A}_{ii}}{n} = \tilde{\operatorname{A}} + \frac{\tilde{\operatorname{A}}^T}{n} + \frac{P_i}{n} \mathbf{1}_{n_i} \frac{1}{\sqrt{n_i}} \frac{1}{\sqrt{n_i}}
$$

(S8)

as $n_i \to \infty$ due to the fact that $\frac{P_i}{n} \mathbf{1}_{n_i} \xrightarrow{a.s.} \mathbf{0}$, where $\mathbf{0}$ denotes the matrix of zeros. As a result, for every $i \in \{1, \ldots, K\},$

$$
\frac{\operatorname{A}_{ii}}{n} = \frac{\tilde{\operatorname{A}}_{ii}}{n} + \frac{n_i}{n} \xrightarrow{a.s.} \rho_i \frac{P_i}{n} \mathbf{1}_{n_i} \frac{1}{\sqrt{n_i}} \frac{1}{\sqrt{n_i}}
$$

(S9)

as $n_i \to \infty$ and $\frac{n_{\max}}{n_{\min}} \to c > 0$, which completes the proof of case (II).

**APPENDIX B**

**PROOF OF THEOREM 3.2**

A. Optimality condition of $\mathbf{Y}$: general case

Recall from Sec. 3.1-A that the eigenvector matrix $\mathbf{Y} = [\mathbf{y}_2 \ldots \mathbf{y}_K]$ of $\bm{L}_\mathcal{N}$ is the solution of the minimization problem

$$
\min_{\mathbf{X} \in \mathbb{R}^{n \times (K-1)}} \text{trace}(\mathbf{X}^T \bm{L}_\mathcal{N} \mathbf{X})
$$

(S10)

Using (S10), we can write the Lagrangian function $\Gamma(\mathbf{X})$ of the minimization problem as

$$
\Gamma(\mathbf{X}) = \text{trace}(\mathbf{X}^T \bm{L}_\mathcal{N} \mathbf{X}) - \nu^T \mathbf{X}^T \mathbf{D}^\perp \mathbf{1}_n - \text{trace} \left( \mathbf{U} (\mathbf{X}^T - \mathbf{I}_K) \right),
$$

(S11)

where $\nu \in \mathbb{R}^{K-1}$ and $\mathbf{U} \in \mathbb{R}^{(K-1) \times (K-1)}$ with $\mathbf{U} = \mathbf{U}^T$ are the Lagrange multiplier of the constraints $\mathbf{X}^T \mathbf{D}^\perp \mathbf{1}_n = 0_{K-1}$ and $\mathbf{X}^T = \mathbf{I}_K$, respectively.
Using the Karush-Kuhn-Tucker (KKT) conditions \[12\], \( Y \) satisfies the first-order optimality condition of \( \Gamma(X) \). That is, using matrix calculus and differentiating (S11) with respect to \( X \), we have
\[
\frac{d\Gamma(X)}{dX} = 2L_NX - D\frac{1}{n}\nu^T - 2XU, \tag{S12}
\]
which implies the optimality condition of \( Y \) is
\[
2L_NY - D\frac{1}{n}\nu^T - 2YU = O_{n\times(K-1)}, \tag{S13}
\]
where \( O_{n\times K} \) denotes the \( n \times K \) matrix of zeros. Furthermore, left multiplying (S13) by \((D\frac{1}{n})^T\), we obtain
\[
L_N^TY = YU. \tag{S15}
\]
Let \( \Lambda = \text{diag}([\lambda_2, \ldots, \lambda_K]) \) be a diagonal matrix of the eigenvalues \( \{\lambda_k\}_{k=2}^{K} \). Left multiplying (S15) by \( Y^T \), we obtain
\[
U = Y^TL_NY = \Lambda \tag{S16}
\]
due to the fact that \( Y^TY = I_{K-1} \).

To investigate the relationship between \( Y \) and the community structure, we denote the rows in \( Y \) indexed by the nodes in community \( k \) by \( Y_k \in \mathbb{R}^{n_k \times (K-1)} \) such that
\[
Y = [Y_1^T \ldots Y_K^T]^T. \tag{S17}
\]
We use similar notation for \( X \) such that \( X = [X_1^T \ldots X_K^T]^T \). Furthermore, the matrix \( L_N^T \) is partitioned into a \( K \times K \) block matrix, where the block \( L_{N_{ij}} \) is the \( n_i \times n_j \) submatrix of \( L_N \) indexed by the community labels \( i \) and \( j \), \( i, j \in \{1, \ldots, K\} \). Given the fact that \( \nu = 0_{K-1} \), the Lagrangian function in (S11) can be written in terms of \( \{X_k\}_{k=1}^{K} \) and \( \{L_{N_{ij}}\}_{i,j=1}^{K} \), which is
\[
\Gamma(X) = \sum_{k=1}^{K} \sum_{j=1}^{K} \text{trace}(X_k^TL_{N_{ij}}X_j) - \sum_{k=1}^{K} \text{trace}(UX_k^TU) + \text{trace}(U) \tag{S17}
\]
Differentiating \( \Gamma(X) \) with respect to \( X_k \), we have
\[
\frac{d\Gamma(X)}{dX_k} = 2L_N^{kk}X_k + 2 \sum_{j=1, j \neq k}^{K} L_{N_{kj}}X_j - 2X_kU, \tag{S18}
\]
which implies the optimality condition of \( Y \) in terms of \( \{Y_k\}_{k=1}^{K} \) is
\[
L_N^{kk}Y_k + \sum_{j=1, j \neq k}^{K} L_{N_{kj}}Y_j - Y_kU = O_{n_k \times (K-1)}, \quad \forall \ k \in \{1, \ldots, K\}. \tag{S19}
\]

### B. Optimality condition of \( Y \) under \( \text{SBM}(K,P) \)

Here we proceed to study the optimality condition of \( Y \) developed in Appendix [B-A] under the assumption of \( \text{SBM}(K,P) \). Unless specified, all convergence results are with respect to the condition when \( n_k \to \infty, \forall \ k \in \{1, \ldots, K\} \), and \( \frac{n_{\min}}{n_{\max}} \to c > 0 \).

Using the block representation \( \{A_{ij}\}_{i,j=1}^{K} \) for the adjacency matrix \( A \), define the block representation of the diagonal degree matrix \( D \) as \( D = [D_1^T \ldots D_K^T]^T \), where \( D_k = \text{diag}(\sum_{j=1}^{K} A_{kj}I_{n_j}) \). Using Lemma [1] we have for every \( k \in \{1, \ldots, K\} \),
\[
\frac{D_k}{n} = \frac{\text{diag}(\sum_{j=1}^{K} A_{kj}I_{n_j})}{n} \overset{a.s.}{\to} \sum_{j=1}^{K} \sqrt{\rho_k}P_{ij}I_{n_k}. \tag{S20}
\]
Similarly, the block representation of the unnormalized graph Laplacian matrix \( L = D - A \), denoted by \( \{L_{ij}\}_{i,j=1}^{K} \), has the following relation based on Lemma [1]
\[
\frac{L_{ij}}{n} = \begin{cases} \frac{D_i}{n} - \frac{A_{ii}}{n} \overset{a.s.}{\to} \sum_{k=1}^{K} \sqrt{\rho_k}P_{ik}I_{n_i} - \rho_iP_{ii}I_{n_i}, & \text{if } i = j; \\ -\frac{A_{ij}}{n} \overset{a.s.}{\to} -\sqrt{\rho_i}P_{ij}I_{n_i}, & \text{if } i \neq j. \end{cases} \tag{S21}
\]
Putting these pieces together, since by definition
\[
L_N^T = D^{-\frac{1}{2}}LD^{-\frac{1}{2}} = \left( \frac{D}{n} \right)^{-\frac{1}{2}}L\left( \frac{D}{n} \right)^{-\frac{1}{2}}, \tag{S22}
\]
the blocks \( \{L_{N_{ij}}\}_{i,j=1}^{K} \) of \( L_N^T \) satisfy
\[
\frac{L_{N_{ij}}}{n} = \left( \frac{D}{n} \right)^{-\frac{1}{2}}L_{ij}\left( \frac{D}{n} \right)^{-\frac{1}{2}} \overset{a.s.}{\to} \begin{cases} I_{n_i} - \rho_iP_{ii}I_{n_i}, & \text{if } i = j; \\ -\sqrt{\rho_i}P_{ij}I_{n_i}, & \text{if } i \neq j, \end{cases} \tag{S23}
\]
where \( a_k > 0 \) is defined as
\[
a_k = \sum_{j=1}^{K} \sqrt{\rho_kP_{kj}}, \quad \forall \ k \in \{1, \ldots, K\}. \tag{S25}
\]
Applying (S23) to the optimality condition in (S19) gives
\[
Y_k - \sum_{j=1}^{K} \sqrt{\rho_kP_{kj}}\frac{1}{\sqrt{n_j}}\frac{1}{\sqrt{n_i}}Y_j \overset{a.s.}{\to} O_{n_k \times (K-1)}, \quad \forall \ k \in \{1, \ldots, K\}. \tag{S26}
\]
As a result, (S26) is the asymptotic optimality condition of \( Y \) under \( \text{SBM}(K,P) \). In the sequel we will use (S26) to specify the feasibility of community detection using \( Y \).
C. Undetectable regime for community detection

Left multiplying the optimality condition \( \text{(S26)} \) of \( Y \) under \( \text{SBM}(K, P) \) by \( \frac{1_T}{\sqrt{n_k}} \), we have

\[
\frac{1_T}{\sqrt{n_k}} Y_k (I_{K-1} - U) - \sum_{j=1}^{K} \sqrt{\rho k_j P_{kj}} \frac{1_T}{\sqrt{n_j}} Y_j \xrightarrow{a.s.} 0_{K-1}, \quad \forall \ k \in \{1, \ldots, K\}. \tag{S27}
\]

A trivial solution for \( \{Y_k\}_k \) to satisfy \( \text{(S27)} \) is

\[
Y_k^T 1_{n_k} \xrightarrow{a.s.} 0_{K-1}, \quad \forall \ k \in \{1, \ldots, K\}. \tag{S28}
\]

Moreover, applying \( \text{(S28)} \) to \( \text{(S26)} \) gives

\[
Y_k (I_{K-1} - U) \xrightarrow{a.s.} O_{nk \times (K-1)}, \quad \forall \ k \in \{1, \ldots, K\}. \tag{S29}
\]

Since the orthogonality and unit-norm constraint \( Y^T Y = I_{K-1} \) is equivalent to \( \sum_{k=1}^{K} Y_k^T Y_k = I_{K-1} \), left multiplying \( \text{(S29)} \) by \( Y_k^T \) and summing over \( k = 1, \ldots, K \) gives

\[
U \xrightarrow{a.s.} I_{K-1} \tag{S30}
\]

when \( \text{(S28)} \) holds. Conversely, if \( \text{(S30)} \) holds, applying it to \( \text{(S26)} \) gives

\[
\sum_{j=1}^{K} \sqrt{\rho k_j P_{kj}} \frac{1_T}{\sqrt{n_j}} Y_j \xrightarrow{a.s.} O_{nk \times (K-1)}, \quad \forall \ k \in \{1, \ldots, K\}. \tag{S31}
\]

Since \( \text{(S31)} \) holds for any positive \( \{\rho k_j\}_j \), \( \{a_k\}_k \) and \( \{P_{kj}\}_{j=1}^{K} \), it implies the condition \( \text{(S28)} \). Consequently, we have established that \( Y_k^T 1_{n_k} \xrightarrow{a.s.} 0_{K-1}, \forall \ k \in \{1, \ldots, K\} \) if and only if \( U \xrightarrow{a.s.} I_{K-1} \).

Note that the condition in \( \text{(S28)} \) shows the rows of \( Y_k \) sum to a zero vector for each \( k \), which implies the row representation of nodes in the same community is incoherent. That is, for each column in \( Y_k \), the sum of nonzero entries is zero, which implies the nonzero entries in each column have alternating signs and hence the row representation is not identical for nodes in the same community. Furthermore, when one runs K-means clustering on the rows of \( Y \), the centroid of each community collapses to the same point due to \( \text{(S28)} \), which makes correct community detection impossible. Similar results can be concluded when one adopts the row normalization step and use \( Y \) for community detection as described in Algorithm \ref{Alg1} since the row normalization step does not alter the sign of each entry in \( Y \). As a result, community detection using \( L_N \) is said to be in the undetectable regime if \( \text{(S28)} \) holds.

Recall the definition \( \theta = \sum_{k=2}^{K} 1 - \lambda_k \) as defined in Theorem \ref{Thm1}. Taking the trace on both sides in \( \text{(S28)} \) and using \( \text{(S16)} \), we have

\[
\text{trace}(U) = \text{trace}(\Lambda) = \sum_{k=2}^{K} \lambda_k = \text{trace}(I_{K-1}) = K - 1, \tag{S32}
\]

which implies community detection using \( Y \) of \( L_N \) is undetectable if and only if \( \theta = 0 \).

D. Detectable regime for community detection

Appendix \ref{AppB-C} shows that the trivial solution \( \text{(S28)} \) to the optimality condition under \( \text{SBM}(K, P) \) in \( \text{(S27)} \) results in incorrect community detection. Here we investigate the other solution to \( \text{(S26)} \) and show that this nontrivial solution leads to correct community detection, which is called the detectable regime for community detection.

Using \( \text{(S26)} \), we can rewrite it as

\[
Y_k (I_{K-1} - U) \xrightarrow{a.s.} \sum_{j=1}^{K} \sqrt{\rho k_j P_{kj}} \frac{1_T}{\sqrt{n_j}} Y_j \times a_k a_j , \quad \forall \ k \in \{1, \ldots, K\}. \tag{S33}
\]

Left multiplying \( Y_k^T \) to \( \text{(S33)} \) and summing over \( k = 1, \ldots, K \) gives

\[
U \xrightarrow{a.s.} I_{K-1} - \sum_{k=1}^{K} \sum_{j=1}^{K} \sqrt{\rho k_j P_{kj}} Y_k^T \frac{1_T}{\sqrt{n_j}} Y_j \times a_k a_j. \tag{S34}
\]

Recall from \( \text{(S16)} \) that \( U = \Lambda \) and hence \( U \) is a diagonal matrix with positive entries on its diagonal. If the matrix \( I_{K-1} - U \) is invertible, then by \( \text{(S33)} \),

\[
Y_k \xrightarrow{a.s.} \sum_{j=1}^{K} \sqrt{\rho k_j P_{kj}} \frac{1_T}{\sqrt{n_j}} Y_j \times a_k a_j (I_{K-1} - U)^{-1}, \quad \forall \ k \in \{1, \ldots, K\}, \tag{S35}
\]

where the \((K-1) \times 1\) vector \( b_k \) is defined as

\[
b_k = (I_{K-1} - U)^{-1} \sum_{j=1}^{K} \sqrt{\rho k_j P_{kj}} Y_j^T \frac{1_T}{\sqrt{n_j}} a_k a_j. \tag{S37}
\]

The result of \( \text{(S36)} \) implies each block \( Y_k \) in \( Y \) has coherent row representation, which means the vector space representation of nodes in the same community is identical. The next step is to show that the row representation of each \( Y_k \) is distinct, and hence inspecting the distribution of rows in \( Y \) leads to correct community detection.

Using \( \text{(S36)} \) and \( \text{(S20)} \), the orthogonality and unit-norm constraints \( \sum_{k=1}^{K} Y_k^T Y_k = I_{K-1} \) and \( Y^T D \bar{Z} 1_n = 0_{K-1} \) yield

\[
\sum_{k=1}^{K} b_k b_k^T \xrightarrow{a.s.} I_{K-1}; \tag{S38}
\]

\[
\sum_{k=1}^{K} a_k b_k \xrightarrow{a.s.} 0_{K-1}, \tag{S39}
\]

where \( a_k > 0 \) is defined in \( \text{(S25)} \).
The result in (S38) imply that some \( b_k \) cannot be a zero vector since
\[
\sum_{k=1}^{K} |b_{k,j}|^2 = 1, \quad \forall j \in \{1, \ldots, K-1\}. \quad (S40)
\]
Furthermore, by (S38) and (S39), we have
\[
\sum_{k:|b_{k,j}|>0} a_k |b_{k,j}| = - \sum_{k:|b_{k,j}|<0} a_k |b_{k,j}|, \quad \forall j \in \{1, \ldots, K-1\}. \quad (S41)
\]
\[
\sum_{k:|b_{k,j}|,|b_{k,j}^i>|0} |b_{k,j}^i||b_{k,j}| = - \sum_{k:|b_{k,j}|,|b_{k,j}^i|<0} |b_{k,j}^i||b_{k,j}|, \quad \forall i, j \in \{1, 2, \ldots, K-1\}, i \neq j. \quad (S42)
\]
Combining the results in (S36), (S40), (S41) and (S42) leads to the following conclusion:
1) The columns of \( Y_k \) are constant vectors.
2) Each column of \( Y \) has at least two nonzero community-wise constant components, and these constants have alternating signs such that their weighted sum equals 0 (i.e., \( \sum_{k} a_k |b_{k,j}| = 0, \quad \forall j \in \{1, \ldots, K-1\} \)).
3) No two columns of \( Y \) have the same sign on the community-wise nonzero components.

As a result, we have proved that the rows in each \( Y_k \) have identical row representation, and the row representation of each \( Y_k \) is distinct. More importantly, the results suggest that in the vector space representation the within-cluster distance between any pair of row vectors in each \( Y_k \) is zero, whereas the between-cluster distance between any two row vectors of different clusters is nonzero. This suggests that the ground-truth communities are the optimal solution to K-means clustering, and hence K-means clustering on the rows of \( Y \) can yield correct communities.

Since \( U = A \) by (S10), comparing (S34) in the detectable regime to (S30) in the undetectable regime, one can see that the changes in the edge connection matrix \( K \) lead to changes in the eigenvalue matrix \( \Lambda \), and \( \theta = \text{trace}(I_{K-1} - U) = \sum_{k=2}^{K} 1 - \lambda_k > 0 \). As a result, the parameters \( P \) can also be separated into the detectable and undetectable regimes for community detection. Lastly, we have established correct community detection under SBM(\( K, P \)) provided that \( I_{K-1} - U \) is invertible, which implies \( \theta > 0 \). Notice that \( I_{K-1} - U \) is not invertible when \( U \xrightarrow{a.s.} I_{K-1} \), which leads to the undetectable regime as discussed in Sec. B-C. Consequently, the \( K \) communities under SBM(\( K, P \)) can be correctly detected using \( Y \) if and only if \( \theta > 0 \).

E. Summary: the distribution of the rows in \( Y \)

Here we summarize the established theoretical analysis for community detectability using the eigenvector matrix \( Y \) of the normalized graph Laplacian matrix \( L_N \) for graphs generated by \( \text{SBM}(K, P) \).

1) The community-indexed block matrix \( Y_k \) of \( Y \) satisfies the optimality condition in (S26).

E. Summary: the distribution of the rows in \( Y \)

Here we summarize the established theoretical analysis for community detectability using the eigenvector matrix \( Y \) of the normalized graph Laplacian matrix \( L_N \) for graphs generated by \( \text{SBM}(K, P) \).

1) The community-indexed block matrix \( Y_k \) of \( Y \) satisfies the optimality condition in (S26). Moreover, the distribution of the rows in \( Y \) is either in the detectable regime or the undetectable regime for community detection.

2) In the undetectable regime, the rows in each \( Y_k \) sum to a zero vector, resulting in incorrect community detection. In addition, \( Y \) is in the undetectable community detection regime if and only if \( \theta = \sum_{k=2}^{K} 1 - \lambda_k = 0 \).

3) In the detectable regime, the rows in each \( Y_k \) have identical representation, and each row representation of \( Y_k \) is distinct, resulting in correct community detection using K-means clustering on the rows of \( Y \). In addition, \( Y \) is in the detectable community detection regime if and only if \( \theta > 0 \).

APPENDIX C

PROOF OF COROLLARY 3.3

When restricted to the case of SBM(2, \( P \)), where \( P_{11} = p_1 \), \( P_{22} = p_2 \) and \( P_{12} = P_{21} = q \), the parameter \( a_k \) in (S25) can be simplified to
\[
\begin{align*}
a_1 &= \sqrt{\rho_1 p_1 + \sqrt{\rho_1 p_2 q};} \\
a_2 &= \sqrt{\rho_2 p_2 + \sqrt{\rho_1 p_2 q}}.
\end{align*}
\]

In addition, using (S20) and the orthogonality constraint \( y_1^T D y_2 = 0 \) of the second smallest eigenvector \( y_2 = \frac{y_1}{y_1^T y_2} \) gives
\[
\begin{align*}
a_1 \sqrt{\frac{p_1}{n_1}} + a_2 \sqrt{\frac{p_2}{n_2}} \rightarrow 0. \quad (S45)
\end{align*}
\]

Applying (S45) to (S44) gives
\[
\begin{align*}
U \xrightarrow{a.s.} 1 - \frac{\rho_1 p_1 (\sqrt{\frac{p_1}{n_1}})^2}{\frac{p_1}{n_1} a_1^2} + \frac{\rho_2 p_2 (\sqrt{\frac{p_2}{n_2}})^2}{\frac{p_2}{n_2} a_2^2} + 2 \sqrt{\rho_1 p_2 q} (\sqrt{\frac{p_1}{n_1}})^2 a_1^2
\end{align*}
\]
\[
= 1 - \left[ \left( \frac{p_2}{n_2} \right)^2 \rho_1 p_1 + 2 \left( \frac{p_2}{n_2} \right)^2 \sqrt{\rho_1 p_2 q} \right] \left( \frac{p_1}{n_1} \right)^2 a_1^2. \quad (S47)
\]

Substituting (S43) and (S44) to (S47) and factoring the resulting term, we have
\[
\begin{align*}
U \xrightarrow{a.s.} 1 - \frac{\rho_1 p_1 \left[ 2 \sqrt{\rho_1 p_2} + \rho_1 p_1 + \rho_2 p_2 \right] (\rho_1 p_1 - q^2) (\sqrt{\frac{p_1}{n_1}})^2}{\frac{p_1}{n_1} a_1^2}.
\end{align*}
\]

Comparing (S48) to (S30), SBM(2, \( P \)) is in the detectable regime for community detection if and only if \( q < \sqrt{p_1 p_2} \), and it is in the undetectable regime for community detection if and only if \( q \geq \sqrt{p_1 p_2} \).

Lastly, if \( q < \sqrt{p_1 p_2} \), then from (S37) we have
\[
\begin{align*}
y_1 \xrightarrow{a.s.} b_1 \frac{1}{\sqrt{n_1}}, \quad y_2 \xrightarrow{a.s.} b_2 \frac{1}{\sqrt{n_2}} \quad (S49)
\end{align*}
\]
for some $b_1$ and $b_2$. Applying (S49) to (S45) and the unit-norm constraint $\mathbf{y}_2^T \mathbf{y}_2 = \mathbf{y}_1^T \mathbf{y}_1 + \mathbf{y}_2^T \mathbf{y}_2 = 1$ gives

$$b_1 = \frac{\pm a_2}{\sqrt{a_1^2 + a_2^2}}; \quad b_2 = \frac{\mp a_1}{\sqrt{a_1^2 + a_2^2}}.$$  (S50)

Setting $\beta_1 = \frac{a_2}{\sqrt{a_1^2 + a_2^2}}$ and $\beta_2 = \frac{a_1}{\sqrt{a_1^2 + a_2^2}}$ completes the proof.