Universal coding for classical-quantum channel

Masahito Hayashi
Graduate School of Information Sciences, Tohoku University, Sendai, 980-8579, Japan.
E-mail: hayashi@math.is.tohoku.ac.jp

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Abstract: We construct a universal code for stationary and memoryless classical-quantum channel as a quantum version of the universal coding by Csiszár and Körner. Our code is constructed by the combination of irreducible representation, the decoder introduced through quantum information spectrum, and the packing lemma.

1. Introduction

The channel coding theorem for a stationary and memoryless classical-quantum channel has been established by combining the direct part shown by Holevo [1] and Schumacher-Westmoreland [2] with the (weak) converse (impossible) part which goes back to 1970’s works by Holevo[3,4]. Its strong converse part has been shown by Ogawa and Nagaoka [5] and Winter [6]. This theorem is a fundamental element of quantum information theory[7]. After their achievement, Ogawa and Nagaoka [8] and Hayashi and Nagaoka [9] constructed other codes attaining the capacity. However, since the existing codes depend on the form of the channel, they are not robust against the disagreement between the sender’s frame and receiver’s frame. In the classical system, Csiszár and Körner [10] constructed a universal channel coding, whose construction does not depend on the channel and depends only on the mutual information and the ‘type’ of the input system, i.e., the empirical distribution of code words, whose precise explanation will be explained in Section 3. Such a universal code for the quantum case was also constructed for variable-length source coding[11,12] and fixed-length source coding[10].

Concerning the quantum system, Jozsa et al. [13] constructed a universal fixed-length source coding, which depends only on the compression rate and

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1 Throughout the paper, a stationary memoryless channel without using entangled input states is simply referred to as a stationary memoryless channel.
attains the minimum compression rate. Hayashi [14] discussed the exponential decreasing rate of its decoding error. Further, Hayashi and Matsumoto [15] constructed a universal variable-length source coding in the quantum system. However, any universal coding for classical-quantum channel was not constructed. In fact, the universal coding is required when the receiver cannot synchronize his frame with the sender’s frame.

In the present paper, we construct a universal coding for a classical-quantum channel, which attains the quantum mutual information and depends only on the coding rate and the ‘type’ of the input system. In the proposed construction, the following three methods play essential roles. One is the decoder given by the proof of the information spectrum method. In the information spectrum method, the decoder is constructed by the square root measurement of the projectors given by the quantum analogue of the likelihood ratio between the signal state and the mixture state [9,10].

The second method is the irreducible decomposition of the dual representation of the special unitary group and the permutation group. The method of irreducible decomposition provides the universal protocols in quantum setting [13]. However, even in the classical case, the universal channel coding requires the conditional type as well as the type [10]. In the present paper, we introduce a quantum analogue of the conditional type, which is the most essential part of the present paper.

The third method is the packing lemma, which yields a suitable combination of the signal states independent of the form of the channel in the classical case [10]. This method plays the same role in the present paper.

The remainder of the present paper is organized as follows. In section 2 we give the notation herein and the main result including the existence of a universal coding for classical-quantum channel. In this section, we presented the exponential decreasing rate of the error probability of the presented universal code. In section 3, the notation for group representation theory is presented and a quantum analogue of conditional type is introduced. In section 4, we give a code that well works universally. In section 5, the exponential decreasing rate mentioned in section 2 is proven by using the property given in section 3.

2. Main Result

In the classical-quantum channel, we focus on the set of input alphabets $\mathcal{X} := \{1, \ldots, k\}$ and the representation space $\mathcal{H}$ of the output system, whose dimension is $d$. Then, a classical-quantum channel is given as the map from $\mathcal{X}$ to the set of densities on $\mathcal{H}$ with the form $i \mapsto W(i)$. The $n$-th discrete memoryless extension is given as the map from $\mathcal{X}^n$ to the set of densities on the $n$-th tensor product system $\mathcal{H}^{\otimes n}$. That is, this extension maps the input sequence $i = (i_1, \ldots, i_n)$ to the state $W_n(i_n) := W(i_1) \otimes \cdots \otimes W(i_n)$. Sending the message $\{1, \ldots, M_n\}$ requires an encoder and a decoder. The encoder is given as a map $\varphi_n$ from the set of messages $\{1, \ldots, M_n\}$ to the set of alphabets $\mathcal{X}^n$, and the decoder is given by a POVM $Y^n = \{Y^n_i\}_{i=1}^{M_n}$. Thus, the triplet $\Phi_n := (M_n, \varphi_n, Y)$ is called a code. Its performance is evaluated by the size $|\Phi_n| := M_n$ and the average error
probability given by
\[
\varepsilon[\Phi_n, W] := \frac{1}{M_n} \sum_{i=1}^{M_n} \text{Tr} W_n(\varphi_n(i))(I - Y_i^n).
\]

As mentioned in the following main theorem, there exists an asymptotically optimal code that depends only on the coding rate.

**Theorem 1** For any distribution \( p = \{p_i\}_{i=1}^k \) on the set of input alphabets \( \mathcal{X} := \{1, \ldots, k\} \) and any real number \( R \), there is a sequence of codes \( \{\Phi_n\}_{n=1}^{\infty} \) such that
\[
\lim_{n \to \infty} \frac{-1}{n} \log \varepsilon[\Phi_n, W] \geq \max_{0 \leq t \leq 1} \frac{\phi_{W,p}(t) - tR}{1 + t}
\]
\[
\lim_{n \to \infty} \frac{1}{n} \log |\Phi_n| = R
\]
for any classical-quantum channel \( W \), where \( \phi_{W,p}(t) \) is given by
\[
\phi_{W,p}(t) := -(1 - t) \log \text{Tr}(\sum_{i=1}^{k} p_i W(i)^{1-t})^\frac{1}{1-t}.
\]

Note that the code \( \{\Phi_n\}_{n=1}^{\infty} \) does not depend on the channel \( W \), and depends only on the distribution \( p \) and the coding rate \( R \).

The derivative of \( \phi_{W,p}(t) \) is given as
\[
\phi'_{W,p}(0) = I(p, W) := \sum_{i=1}^{k} p_i \text{Tr} W(i)(\log W_i - \log W_p)
\]
\[
W_p := \sum_{i=1}^{k} p_i W(i).
\]

When the transmission rate \( R \) is smaller than the mutual information \( I(p, W) \),
\[
\max_{0 \leq t \leq 1} \frac{\phi_{W,p}(t) - tR}{1 + t} > 0
\]
because there exists a parameter \( t \in (0, 1) \) such that \( \phi_{W,p}(t) - tR > 0 \). That is, the average error probability \( \varepsilon[\Phi_n, W] \) goes to zero.

3. Group representation theory

In this section, we focus on the dual representation on the \( n \)-fold tensor product space by the the special unitary group \( SU(d) \) and the \( n \)-th symmetric group \( S_n^2 \). For this purpose, we focus on the Young diagram and the ‘type’. The former is a key concept in group representation theory and the latter is that in information theory\[10\]. When the vector of integers \( n = (n_1, n_2, \ldots, n_d) \) satisfies
\[2\] Christandl\[22\] contains a good survey of representation theory for quantum information.
the condition $n_1 \geq n_2 \geq \ldots \geq n_d \geq 0$ and $\sum_{i=1}^{d} n_i = n$, the vector $\mathbf{n}$ is called the Young diagram (frame) with size $n$ and depth $d$, the set of which is denoted as $Y^d_n$.

When the vector of integers $\mathbf{n}$ satisfies the condition $n_i \geq 0$ and $\sum_{i=1}^{d} n_i = n$, the vector $\mathbf{p} = \frac{n}{d}$ is called the ‘type’ with size $n$, the set of which is denoted as $T^d_n$. Further, for $\mathbf{p} \in T^d_n$, the subset of $X^n$ is defined as:

$$T_p := \{ x \in X^n | \text{The empirical distribution of } x \text{ is equal to } \mathbf{p} \}.$$ 

The numbers of these sets are evaluated as follows:

$$|Y^d_n| \leq |T^d_n| \leq (n + 1)^{d-1}$$

$$(n + 1)^{-d-nH(\mathbf{p})} \leq |T_p|,$$

where $H(\mathbf{p}) := -\sum_{i=1}^{d} p_i \log p_i$. Using the Young diagram, the irreducible decomposition of the above representation can be characterized as follows:

$$\mathcal{H}_n^{\otimes d} = \bigoplus_{\mathbf{n} \in Y^d_n} \mathcal{U}_n \otimes \mathcal{V}_n,$$

where $\mathcal{U}_n$ is the irreducible representation space of $SU(d)$ characterized by $\mathbf{n}$, and $\mathcal{V}_n$ is the irreducible representation space of $n$-th symmetric group $S_n$ characterized by $\mathbf{n}$. Here, the representation of the $n$-th symmetric group $S_n$ is denoted as $V : s \in S_n \mapsto V_s$. For $\mathbf{n} \in Y^d_n$, the dimension of $\mathcal{U}_n$ is evaluated by

$$\dim \mathcal{U}_n \leq n^{\frac{d(d-1)}{2}}.$$ 

Then, denoting the projection to the subspace $\mathcal{U}_n \otimes \mathcal{V}_n$ as $I_n$, we define the following.

$$\rho_n := \frac{1}{\dim \mathcal{U}_n \otimes \mathcal{V}_n} I_n$$

$$\rho_{\mathcal{U},n} := \sum_{\mathbf{n} \in Y^d_n} \frac{1}{|Y^d_n|} \rho_n.$$ 

Any state $\rho$ and any Young diagram $\mathbf{n} \in Y^d_n$ satisfy the following:

$$\dim \mathcal{U}_n \rho_n \geq I_n \rho^{\otimes n}.$$ 

Thus, (1), (3), and (5) yield the inequality

$$n^{\frac{d(d-1)}{2}} |Y^d_n| \rho_{\mathcal{U},n} \geq \rho^{\otimes n}.$$ 

Next, we focus on two systems $\mathcal{X}$ and $\mathcal{Y} = \{1, \ldots, l\}$. When the distribution of $\mathcal{X}$ is given by a probability distribution $\mathbf{p} = (p_1, \ldots, p_d)$ on $\{1, \ldots, d\}$, and the conditional distribution on $\mathcal{Y}$ with the condition on $\mathcal{X}$ is given by $\mathbf{V}$, we denote the joint distribution on $\mathcal{X} \times \mathcal{Y}$ by $\mathbf{pV}$ and the distribution on $\mathcal{Y}$ by $\mathbf{p} \cdot \mathbf{V}$. When the empirical distribution of $\mathbf{x} \in \mathcal{X}^n$ is $(\frac{n_1}{n}, \ldots, \frac{n_d}{n})$, the sequence of types $\mathbf{V} = (v_1, \ldots, v_d) \in \mathcal{T}_{n_1} \times \cdots \times \mathcal{T}_{n_d}$ is called a conditional type for $\mathbf{x}$.
We denote the set of conditional types for \( x \) by \( V(x, \mathcal{Y}) \). For any conditional type \( V \) for \( x \), we define the subset of \( \mathcal{Y}^n \):

\[
T_V(x) := \left\{ y \in \mathcal{Y}^n \mid \text{The empirical distribution of } \{(x_1, y_1), \ldots, (x_n, y_n)\} \text{ is equal to } pV. \right\},
\]

where \( p \) is the empirical distribution of \( x \).

We define the state \( \rho_x \) for \( x \in \mathcal{X}^n \). For this purpose, we consider a special element \( x' = (1, \ldots, 1, 2, \ldots, 2, \ldots, k, \ldots, k) \). The state \( \rho_{x'} \) is defined as

\[
\rho_{x'} := \rho_{U,m_1} \otimes \rho_{U,m_2} \otimes \cdots \otimes \rho_{U,m_k}.
\]

For a general element \( x \in \mathcal{X}^n \), we choose a permutation \( s \in S_n \) such that \( x = sx' \). Then, we define the state \( \rho_x \) is defined as \( \rho_x := U_s \rho_{x'} U_s^\dagger \), where \( U_s \) is the unitary representation of \( S_n \). This state plays a similar role as the conditional type in the classical case. Using the inequality [6], we have

\[
n^{-\frac{d(d-1)}{2}} |Y_n^d|^k \rho_x \geq W_n(x). \tag{7}
\]

For \( n_1 \in Y_{m_1}^d, n_2 \in Y_{m_2}^d, \ldots, n_k \in Y_{m_k}^d \), the density \( \rho_{n_1} \otimes \rho_{n_2} \otimes \cdots \otimes \rho_{n_k} \) is commutative with the projector \( I_n \) for \( n \in Y_n^d \). This fact implies that the density \( \rho_x \) is commutative with the density \( \rho_{U,n} \). This property is essential for the construction of the proposed decoder.

4. Construction of code

According to Csiszár and Körner[10], the proposed code is constructed as follows.

**Lemma 1** For a positive number \( \delta > 0 \), a type \( p \in T_n^d \) and a real positive number \( R < H(p) \), there exist \( M_n := e^{n(R-\delta)} \) distinct elements \( \mathcal{M}_n := \{x_1, \ldots, x_{M_n}\} \subset T_p \) such that their empirical distributions are \( p \) and

\[
|T_V(x) \cap (\mathcal{M}_n \setminus \{x\})| \leq |T_V(x)| e^{-n(H(p)-R)}
\]

for \( x \in \mathcal{M}_n \subset T_p \) and \( V \in V(x, \mathcal{X}) \).

This lemma can be shown by substituting the identical map into \( \hat{V} \) in Lemma 5.1 in Csiszár and Körner[10]. Since Csiszár and Körner proved Lemma 5.1 using the random coding method, we can replace \( \delta \) by \( 1/n \). That is, there exist \( M_n := e^{nR-\sqrt{\pi}} \) distinct elements \( \mathcal{M}_n := \{x_1, \ldots, x_{M_n}\} \subset T_p \) such that their empirical distributions are \( p \) and

\[
|T_V(x) \cap (\mathcal{M}_n \setminus \{x\})| \leq |T_V(x)| e^{-n(H(p)-R)} \tag{8}
\]

for \( x \in \mathcal{M}_n \subset T_p \) and \( V \in V(x, \mathcal{X}) \). Now, we transform the property [8] to a more useful form.

Using the encoder \( \mathcal{M}_n \), we can define the distribution \( P_{\mathcal{M}_n} \) as

\[
p_{\mathcal{M}_n}(x) = \begin{cases} \frac{1}{|\mathcal{M}_n|} & x \in \mathcal{M}_n \\ 0 & x \notin \mathcal{M}_n. \end{cases}
\]
For any $x \in X^n$, we define the invariant subgroup $S_x \subset S_n$:

$$S_x := \{ s \in S_n | s(x) = x \}.$$  

Since $x' \in T_p$ implies that

$$p^n(x') = e^{-nH(p)},$$

any element $x' \in TV(x) \cap M_n \subset T_p$ satisfies

$$\sum_{s \in S_x} \frac{1}{|S_x|} p_{M_n} \circ s(x') = \frac{|TV(x) \cap M_n|}{|TV(x)||M_n|} = \frac{|TV(x) \cap (M_n \setminus \{x\})|}{|TV(x)||M_n|} \leq e^{-nH(p)} e^{\sqrt{n}} = p^n(x') e^{\sqrt{n}}$$ (9)

when the conditional type $V$ is not identical. Relation (9) holds for any $x' (\neq x) \in M_n$ because there exists a conditional type $V$ such that $x' \in TV(x)$ and $V$ is not identical.

Next, for any $x \in X^n$ and any real number $C_n$, we define the projection

$$P(x) := \{ \rho_x - C_n \rho_{U,n} \geq 0 \};$$

where \( \{ X \geq 0 \} \) presents the projection $\sum_{i:x_i \geq 0} E_i$ for a Hermitian matrix $X$ with the diagonalization $X = \sum_i x_i E_i$. Remember that the density $\rho_x$ is commutative with the other density $\rho_{U,n}$. Using the projection $P(x)$, we define the decoder:

$$Y_{x'} := \sqrt{\sum_{x \in M_n} P(x)}^{-1} P(x') \sqrt{\sum_{x \in M_n} P(x)}^{-1}.$$ 

In the following, the above-constructed code $(e^{nR-\sqrt{n}}, M_n, \{Y_x \}_{x \in M_n})$ is denoted by $\Phi_{U,n}(p, R)$.

5. Exponential evaluation

Hayashi and Nagaoka [9] showed that

$$I - Y_{x'} \leq 2(I - P(x')) + 4 \sum_{x (\neq x') \in M_n} P(x).$$

Then, the average error probability of $\Phi_{U,n}(p, R)$ is evaluated by

$$\frac{1}{|M_n|} \sum_{x' \in M_n} \text{Tr} W_n(x')(I - Y_{x'})$$

$$\leq \frac{2}{|M_n|} \sum_{x' \in M_n} \text{Tr} W_n(x')(I - P(x')) + \frac{4}{|M_n|} \sum_{x' \in M_n} \text{Tr} W_n(x') \sum_{x (\neq x') \in M_n} P(x)$$

$$= \frac{2}{|M_n|} \sum_{x \in M_n} \text{Tr} W_n(x)(I - P(x))$$

$$+ 4 \text{Tr} \left[ \sum_{x \in M_n} P(x) \left( \frac{1}{|M_n|} \sum_{x' (\neq x) \in M_n} W_n(x') \right) \right].$$ (10)
Since the density $\rho_x$ is commutative with the density $\rho_{U,n}$, we have

$$(I - P(x)) = \{\rho_x - C_n \rho_{U,n} < 0\} \leq \rho_x^{-t} C_n^t \rho_{U,n}^t$$

for $0 \leq t \leq 1$. Since the density $\rho_x$ is commutative with the density $W_n(x)$, $W_n(x) \rho_x^{-t}$ is a Hermite matrix and (11) implies that

$$W_n(x) \rho_x^{-t} \leq n^{\frac{ktd(d-1)}{2}} |Y_n^d k^t C_n^t| W_n(x)^{1-t}. \quad (12)$$

Using (11) and (12), we have

$$\text{Tr} \; W_n(x)(I - P(x)) \leq \text{Tr} \; W_n(x) \rho_x^{-t} \rho_{U,n}^t C_n^t$$

$$\leq n^{\frac{ktd(d-1)}{2}} |Y_n^d k^t C_n^t| \text{Tr} \; W_n(x)^{1-t} \rho_{U,n}^t. \quad (13)$$

Since the quantity $\text{Tr} \; W_n(x)(I - P(x))$ is invariant for the action of the permutation and the relation (12) implies that

$$p^n(x) = e^{-nH(p)} \geq \left(\frac{n+1}{|T_p|}\right)^d$$

for $x \in T_p$, we obtain

$$\text{Tr} \; W_n(x)(I - P(x)) = \frac{1}{|T_p|} \sum_{x' \in T_p} \text{Tr} \; W_n(x')(I - P(x'))$$

$$\leq (n+1)^d \sum_{x' \in X^n} p^n(x') \text{Tr} \; W_n(x')(I - P(x'))$$

$$\leq (n+1)^d n^{\frac{ktd(d-1)}{2}} |Y_n^d k^t C_n^t| \text{Tr} \left( \sum_{x' \in X^n} p^n(x') W_n(x')^{1-t} \right) \rho_{U,n}^t$$

$$\leq (n+1)^d n^{\frac{ktd(d-1)}{2}} |Y_n^d k^t C_n^t| \max_{\sigma} \text{Tr} \left[ \sum_{x \in X^n} p(x) W_n(x)^{1-t} \right] \sigma^t$$

$$\leq (n+1)^d n^{\frac{ktd(d-1)}{2}} |Y_n^d k^t C_n^t| \left( \text{Tr} \left( \sum_{x \in X^n} p(x) W_n(x)^{1-t} \right)^{1-t} \right)^{1-t}$$

$$= (n+1)^d n^{\frac{ktd(d-1)}{2}} |Y_n^d k^t C_n^t| e^{-n \phi_{W,p}(t)}$$

where (15), (16), and (17) follow from (14), (13), and Lemma 2 in Appendix, respectively.
Next, we evaluate the second term of (10) using the invariant property of $S_x$:

\[
\begin{align*}
\text{Tr} & \left[P(x) \left( \frac{1}{|M_n|} \sum_{x' \neq x \in M_n} W_n(x') \right) \right] \\
& = \text{Tr} \left[ P(x) \sum_{x' \neq x \in M_n} p_{M_n}(x') W_n(x') \right] \\
& = \text{Tr} \left[ P(x) \sum_{s \in S_x} \frac{1}{|S_x|} \sum_{x' \neq x \in M_n} p_{M_n}(x') V_s W_n(x') V_s^* \right] \\
& \leq \text{Tr} \left[ P(x) \sum_{x' \neq x \in M_n} \sum_{s \in S_x} \frac{1}{|S_x|} p_{M_n} \circ s^{-1}(x') W_n(x') \right] \\
& \leq e^{\sqrt{n}} \text{Tr} \left[ P(x) W_p^{\otimes n} \right] \\
& \leq e^{\sqrt{n}} \text{Tr} \left[ P(x) n^{d(d-1)/2} |Y|^d |\rho_{U,n} \right] \\
& \leq e^{\sqrt{n}} \text{Tr} \left[ P(x) n^{d(d-1)/2} |Y|^d |C_n^{-1} \rho_{x} \right] \\
& \leq e^{\sqrt{n}} \text{Tr} \left[ n^{d(d-1)/2} |Y|^d |C_n^{-1} \rho_{x} \right] = e^{\sqrt{n}} n^{d(d-1)/2} |Y|^d |C_n^{-1}, \tag{19}
\end{align*}
\]

where (19), (20), and (21) follow from (9), (6), and the inequality $P(x)(\rho_{U,n} - C_n^{-1} \rho_{x}) \leq 0$.

For any $t \in (0, 1)$ and $R > 0$, we choose $|M_n| := e^{nR-\sqrt{n}}$, $C_n := e^{n(R+r(t))}$, and $r(t) := \frac{\phi_{W,p}(t)-tR}{1+t}$. Since $r(t) = \phi_{W,p}(t) - t(R+r(t))$, from (10), (18) and (22), the exponential decreasing rate of the average error probability is evaluated as

\[
\lim_{n \to \infty} \frac{-1}{n} \log \epsilon(\Phi_{U,n}(p, R), W) \geq \min \{ \phi_{W,p}(t) - t(R+r(t)), r(t) \} = \frac{\phi_{W,p}(t) - tR}{1+t}.
\]

That is, when we choose $t_0 := \arg\max_{t \in (0, 1)} \frac{\phi_{W,p}(t)-tR}{1+t}$, $|M_n| := e^{nR-\sqrt{n}}$, and $C_n := e^{n(R+r(t_0))}$, we obtain

\[
\lim_{n \to \infty} \frac{-1}{n} \log \epsilon(\Phi_{U,n}(p, R), W) \geq \max_{t \in (0, 1)} \frac{\phi_{W,p}(t) - tR}{1+t}
\]

for any channel $W$. Therefore, we obtain Theorem [10].

6. Discussion

We have constructed a universal code attaining the quantum mutual information based on the combination of information spectrum method, group representation...
theory, and the packing lemma. The presented code well works because any tensor product state $\rho^\otimes n$ is close to the state $\rho_{U,n}$. Indeed, Krattenthaler and Slater [23] demonstrated the existence of the state $\sigma_n$ such that 

$$
1/n \| \rho^\otimes n - \sigma_n \|_1 \to 0
$$

for any state $\rho$ in the qubit system as a quantum analogue of Clarke and Barron’s result [24]. Its $d$-dimensional extension is discussed in another paper [25].

Further, Hayashi [26] derived an exponential decreasing rate of error probability in classical-quantum channel, which is

$$
\max_{t:0 \leq t \leq 1} - (\log \sum_i p_i \text{Tr}[W(i)^{1-t}W_p^t]) - tR.
$$

Since

$$
e^{-\phi_{W,P}(t) - (R + r(t))} \geq e^{tR} \left( \sum_i p_i W(i)^{1-t} \right) \left( \sum_i p_i W(i) \right)^t = e^{-\left( \sum_i p_i \text{Tr}[W(i)^{1-t}W_p^t] \right) - tR},
$$

we obtain

$$
\max_{t:0 \leq t \leq 1} - (\log \sum_i p_i \text{Tr}[W(i)^{1-t}W_p^t]) - tR \geq \max_{t:0 \leq t \leq 1} \frac{\phi_{W,P}(t) - tR}{1 + t}.
$$

That is, the obtained exponential decreasing rate is smaller than that of Hayashi [26]. However, according to Csiszár and Körner [10], the exponential decreasing rate of the universal coding is the same as the optimal exponential decreasing rate in the classical case when the rate is close to the capacity. Hence, if a more sophisticated evaluation is applied, a better exponential decreasing rate can be expected. Such an evaluation is left as a future problem.

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A. Maximization

The following lemma is used for the derivation in Section 5.

**Lemma 2** When $X$ is a positive semi-definite, we have

$$
\max_{\sigma} \text{Tr} X \sigma^t = (\text{Tr} X \frac{1}{1+t})^{1-t}
$$

for $0 \leq t \leq 1$, where $\sigma$ is a density matrix.

**Proof:** First, we prove

$$
\max_{q_i \geq 0, \sum_i q_i = 1} \text{Tr} X \sum_i q_i |i\rangle\langle i| = \left( \sum_i \langle i|X|i\rangle \frac{1}{1+t} \right)^{1-t}
$$

(24)
by the Lagrange multiplier method. Let $\lambda$ be the Lagrange multiplier. Then,

$$0 = \sum_i (\langle i | X | i \rangle t q_i^{t-1} + \lambda) \delta q_i$$

Thus,

$$0 = \langle i | X | i \rangle t q_i^{t-1} + \lambda.$$

That is,

$$-\frac{t}{X} \langle i | X | i \rangle = q_i^{1-t}.$$

Then, when the maximizing $q_i$ has the form $C\langle i | X | i \rangle^{1-t}$ with the normalizing constant $C$, the constant $C$ has the form $C = \frac{1}{\sum_j \langle j | X | j \rangle^{1-t}}$. Substituting $\langle i | X | i \rangle^{1-t}$ into $q_i$, we obtain (24).

Since

$$\left(\sum_i \langle i | X | i \rangle^{1-t}\right)^{1-t} = \text{Tr} X \left(\sum_i \langle i | \frac{1}{\text{Tr} X} X | i \rangle^{1-t}\right)^{1-t},$$

the maximum $\max_{\sigma} \text{Tr} X \sigma^t$ is given when we choose the basis $\{|i\}$ maximizing $\sum_i \langle i | \frac{1}{\text{Tr} X} X | i \rangle^{1-t}$. Since the function $x \mapsto x^{1-t}$ is a convex function, $\langle i | \frac{1}{\text{Tr} X} X | i \rangle^{1-t} \leq \langle i | (\frac{1}{\text{Tr} X} X)^{1-t} | i \rangle$. Therefore,

$$\left(\sum_i \langle i | X | i \rangle^{1-t}\right)^{1-t} \leq (\text{Tr} X^{1-t})^{1-t}.$$

The equality holds when we choose the basis $\{|i\}$ as the eigenvectors of $X$. Therefore, we obtain (23).

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