Twisted descent algebras and the Solomon-Tits algebra

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Introduction

The purpose of the present article is to define and study a new class of descent algebras, called twisted descent algebras. These algebras are associated to the Barratt-Joyal theory of twisted bialgebras in the same way than classical descent algebras are associated to classical bialgebras. The theory of twisted descent algebras is a refinement of the theory of descent algebras of twisted algebras, as introduced in [17]. The formal properties of twisted descent algebras seem particularly meaningful in view of applications to discrete probabilities, to the geometry of Coxeter groups and buildings, and to symmetric group combinatorics.

Let us survey briefly the classical theory, since most of the results that hold in that case have natural generalizations in the new setting. Recall first that classical shuffles and descent classes in the symmetric groups are one of the building blocks of modern algebraic combinatorics. Reutenauer’s

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monograph [19] contains an overview of the development until 1993, and
strongly indicates the impact of the so-called Solomon or descent algebra on
the theory. This classical descent algebra, introduced by Solomon in 1976 in
the more general setting of finite Coxeter groups, was originally constructed
as a noncommutative superstructure of the character ring of the underlying
group, and built upon the classical character theory encapsulated in the
celebrated Mackey formulae [22].

Shuffles and descents are in fact central in various fields of mathematics,
such as discrete probabilities (card shuffling, their asymptotic properties,
and so on), algebraic topology (where the shuffles were soon recognized as
the key ingredient for constructing products on the singular cohomology of
spaces; other operations such as Steenrod operations or operations on other
cohomology theories such as equivariant or generalized cohomologies can
often be described using symmetric group combinatorics and generalizations
of shuffles), and the theory of iterated integrals and special functions (see
[6 13 5 12 18] for some examples).

In the current point of view, descent algebras are part of a research
field including the theory of noncommutative symmetric functions, quasi-
symmetric functions, and their various generalizations [11 7 10]. Moreover,
a descent algebra can be associated to any commutative or cocommutative
bialgebra [15 14]. In this general setting, Solomon’s classical descent algebra
is recovered in the particular case when the underlying bialgebra is the shuffle
algebra over a countable alphabet.

Almost simultaneously to Solomon’s discovery, in 1977, Barratt intro-
duced a new class of algebras, twisted algebras, in order to study a subtle
class of invariants in the cohomology of topological spaces known as (gen-
eralized) Hopf invariants [1]. The theory of twisted algebras was developed
further by Joyal, who showed that the usual classes of algebras (Lie alge-
bras, enveloping algebras, bialgebras, and so on) can be defined consistently
in this new framework [9 23]. However, the importance of twisted bialge-
bras for algebraic combinatorics was uncovered only recently. One of the
most striking applications of the general theory so far is probably the con-
struction of a natural enveloping algebra structure on the direct sum of the
symmetric group algebras, deduced from the properties of the free twisted
algebra on one generator [17].

To combine the two notions of twisted algebras, and descent algebras,
was the motivation for the present article. In the first two sections, it is
shown that a twisted descent algebra can be associated naturally to any
twisted cocommutative or commutative bialgebra — that is, to a cocom-
mutative or commutative bialgebra in the monoidal symmetric category of (linear) tensor species. In Section 3, we show that each of the new twisted objects carries a second associative product, the composition product. The free twisted descent algebra is defined by generators and relations in Section 4, and maps onto any twisted descent algebra. More precisely, this free object behaves with respect to twisted descent algebras of twisted bialgebras exactly as the ordinary descent algebra behaves with respect to descent algebras of bialgebras. For example, it carries two products and a coproduct, just like the ordinary one.

Various combinatorial identities in twisted descent algebras are established along the way. This includes in particular a natural generalization (and explanation) of the crucial multiplicative reciprocity law for descent algebras, which (in the classical case) was originally discovered in [7] and links both products and, in the free case, the coproduct on these algebras (see Proposition 13 and Corollary 18). Besides, it turns out that the theory of twisted descent algebras is related to the Solomon-Tits algebra in the same way than the descent algebras of bialgebras are related to the classical descent algebra.

Recall that the Solomon-Tits algebra was introduced by Tits in an appendix to Solomon’s original paper [25]. As a vector space over the ground field, it is generated by the elements of the Coxeter complex of the symmetric group $S_n$, where $n$ is a fixed integer. The product of two elements of the Coxeter complex is defined in terms of the minimal galleries joining them (in Tits’ buildings terminology). The construction makes sense for general hyperplane arrangements and, in this generality, has been one of the leading tools in recent research on random walks associated to such arrangements and related topics, such as the Tsetlin library, or the random-to-top process in computer science [3, 4].

We feel that the heavy reference to geometry, together with the sometimes concise style of the appendix, was one of the reasons why Tits’ ideas concerning descents, unlike Solomon’s, have not yet been fully exploited by combinatorists. In that sense, our constructions and proofs unravel the combinatorial ideas hidden in Tits’ line of reasoning. This is further demonstrated in final Section 5. However, the objects defined and studied here are much richer than the original one: Tits associated to a Coxeter complex an associative algebra structure; our twisted descent algebras (the Solomon-Tits algebra of the symmetric group being a particular case of which) carry two different associative algebra structures, and, in the free case, a bialgebra structure. Besides, there is a fundamental rule linking all three structures.
1 Tensor species and twisted bialgebras

Recall that a (linear) tensor species is a functor from the category of finite sets $\text{Fin}$ (and set isomorphisms) to the category $\text{Vect}$ of vector spaces over a field $k$ (and linear isomorphisms) or, more generally, to the category of modules over an arbitrary commutative ring. Concretely, a tensor species $F$ associates to each finite set $S$ a vector space $F(S)$. A bijection from $S$ to $T$ in $\text{Fin}$ induces an isomorphism from $F(S)$ to $F(T)$. We will assume from now on that the finite sets we consider (and therefore the objects of $\text{Fin}$) are subsets of a given countable ordered set, for example the set $\mathbb{N}$ of positive integers. This is not a serious restriction, but a convenient one for notational purposes. We also assume that our tensor species are connected, that is: $F(\emptyset) = k$.

**Definition 1** A tensor species is a functor $F$ from $\text{Fin}$ to $\text{Vect}$ such that $F(\emptyset) = k$, where $k$ is the ground field.

**Proposition 2** The category $\text{Sp}$ of tensor species is a linear symmetric monoidal category for the tensor product defined by:

$$(F \otimes G)(S) := \bigoplus_{T \coprod U = S} F(T) \otimes G(U)$$

for all $F, G \in \text{Sp}$.

Here, $\coprod$ stands for disjoint union: that is, we have $T \cap U = \emptyset$ in the above formula.

The proof is straightforward and is left to the reader. The unit of the tensor product is the ground field $k$, identified with the tensor species (also denoted by $k$) with unique nontrivial component $k(\emptyset) := k$:

$$F \otimes k = k \otimes F = F$$

for all $F \in \text{Sp}$.

**Definition 3** A twisted algebra is an algebra in the linear symmetric monoidal category of tensor species.

See [9, 17] for further details on and references to Joyal’s theory of twisted algebras. Concretely, a twisted algebra is a tensor species $F$ provided with a product map (which is a map of tensor species: $F \otimes F \to F$). Associative algebras, commutative algebras, Lie algebras and so on, are defined accordingly.
For example, a twisted associative algebra with unit is a tensor species
\[ A \] provided with a product map
\[ A \otimes A \overset{m}{\to} A \]
such that associativity holds:
\[ m \circ (m \otimes A) = m \circ (A \otimes m). \]
Here, we write \( A \) for the identity morphism of \( A \). The unit condition is
defined in the same way: \( A \) has to be provided with a unit map
\[ \kappa \to A \]
satisfying the usual identities. The fundamental example of a twisted
associative algebra is the free twisted algebra \( S \) on one generator: if
\( k[1] \) denotes the tensor species defined by
\[ k[1](\{n\}) := k \]
for all \( n \in \mathbb{N} \), and
\[ k[1](S) := 0 \]
whenever \( S \) is not a singleton, then \( S = \bigoplus_{n \in \mathbb{N}} k[1]^{\otimes n} \). The
product map is the obvious one:
\[ m : k[1]^{\otimes n} \otimes k[1]^{\otimes m} \to k[1]^{\otimes n+m}. \]
The following examples illustrate the meaning of this construction and show
how to handle computations with twisted algebras in practice. Let us start
with the two-element set \( S = \{3, 5\} \). Then, by definition:
\[
S(S) = (k[1] \otimes k[1])(\{3, 5\}) = (k[1](\{3\}) \otimes k[1](\{5\})) + (k[1](\{5\}) \otimes k[1](\{3\})) = k \oplus k,
\]
that is, \( S(S) \) is isomorphic to the direct sum of two copies of the ground field
indexed by the two sequences \((3, 5)\) and \((5, 3)\). If \( U = \{2, 4\} \) and \( \phi \) is the
unique order preserving map from \( S \) to \( U \), then \( S(\phi) \) maps the component
of \( S(S) \) indexed by \((5, 3)\) to the component of \( S(U) \) indexed by \((4, 2)\), and
so on. In such a situation, we will say that \( S(\phi) \) is induced by a map on the
indices.

More generally, given any set \( S = \{s_1, \ldots, s_n\} \) of order \( n \), \( S(S) \) is isomorphic
to the direct sum of \( n! \) copies of the ground field, canonically indexed by all the
permutations of the sequence \((s_1, \ldots, s_n)\), that is by all the sequences
\((s_{\sigma(1)}, \ldots, s_{\sigma(n)})\), where \( \sigma \in S_n \), the symmetric group on \( \{1, \ldots, n\}\).
The product map \( m \) from \( S(S) \otimes S(T) \) to \( S(S \bigsqcup T) \) is induced by the map
on the indices:
\[
((s_{\sigma(1)}, \ldots, s_{\sigma(n)}), (t_{\tau(1)}, \ldots, t_{\tau(k)})) \mapsto (s_{\sigma(1)}, \ldots, s_{\sigma(n)}, t_{\tau(1)}, \ldots, t_{\tau(k)}),
\]
where \( \sigma \) and \( \tau \) run over all permutations in \( S_n \), respectively \( S_k \). Notice
that the product map in the twisted algebra \( S \) therefore bears an apparent
resemblance to the usual concatenation product of words. The fine algebraic structure of the free twisted algebra on one generator, however, is quite different from the algebraic structure of the algebra of words in an alphabet (the free associative algebra over this alphabet).

The notion of a twisted coalgebra is dual to the notion of a twisted algebra: a twisted coassociative coalgebra with counit is a tensor species $C$ provided with a coproduct map $\delta : C \rightarrow C \otimes C$ such that coassociativity holds:

$$(\delta \otimes C) \circ \delta = (C \otimes \delta) \circ \delta,$$

as well as the usual identities for the counit. Once again, the basic example is provided by the twisted algebra $S$, which has a natural coalgebra structure:

$$\delta : S(S) \rightarrow \bigoplus_{U \sqcup T = S} S(U) \otimes S(T).$$

(see [17] for another approach and details on the structure of $S$, and for generalities on twisted algebras or coalgebras.) As above, it is enough to specify the action of $\delta$ at the level of the indices: whenever $U \sqcup T = S$, the $(U, T)$-component of $\delta(S(S))$ is obtained by sending isomorphically the $(s_{\sigma(1)}, \ldots, s_{\sigma(n)})$-component of $S(S)$ to the component of $S(U) \otimes S(T)$ indexed by $u = (u_{\alpha(1)}, \ldots, u_{\alpha(p)})$ and $t = (t_{\tau(1)}, \ldots, t_{\tau(q)})$, where $u$ (respectively, $t$) is the unique ordered subsequence of $(s_{\sigma(1)}, \ldots, s_{\sigma(n)})$ composed of the elements of $U$ (respectively, of $T$). For example, if $S = \{2, 3, 5\}$, the coproduct $\delta$ on the $(5, 2, 3)$-component of $S(S)$ is induced by the following map of indices:

$$(5, 2, 3) \mapsto \{((5, 2, 3), \emptyset), ((5, 2), (3)), ((5, 3), (2)), ((2, 3), (5)), ((5), (2, 3)), ((2), (5, 3)), ((3), (5, 2)), (\emptyset, (5, 2, 3))\}.$$  

The coproduct $\delta$ is clearly cocommutative. It is related to the unshuffle coproduct on the tensor algebra in the same way than the product is related to the concatenation product. It is shown in [17] that $\delta$ induces a (new) coproduct on the direct sum of the symmetric group algebras, turning this direct sum into an enveloping algebra.
The tensor product (in the category of tensor species) of two twisted associative algebras $A$ and $A'$ is an associative algebra. The product map is defined, as in the usual case, using the symmetry isomorphism:

$$A \otimes A' \cong A' \otimes A,$$

that reads pointwise:

$$\left( A \otimes A' \right)[S] = \bigoplus_{T \prod_{U=S}} A[T] \otimes A'[U] \cong \bigoplus_{U \prod_{T=S}} A'[U] \otimes A[T] = (A' \otimes A)[S],$$

where the isomorphism in the middle is the usual symmetry isomorphism for the tensor product of vector spaces.

**Definition 4** A bialgebra in the category of tensor species, or twisted bialgebra, is a tensor species $B$ carrying a unital associative twisted algebra structure (with product written $m$) together with a coassociative counital coalgebra structure (with coproduct written $\delta$) such that $\delta : B \to B \otimes B$ is a map of twisted algebras.

The tensor species $S$ is provided with the structure of a twisted bialgebra, by the product $m$ and the coproduct $\delta$ defined above.

**2 Twisted descent algebras**

Recall that, given a set $I$, an $I$-graded vector space $V$ is a collection of vector spaces indexed by the elements of $I$:

$$V = \{V_i\}_{i \in I}, \quad V_i \in \text{Vect}.$$  

It is often convenient to identify $V$ with the direct sum of its graded components: $V = \bigoplus_{i \in I} V_i$. A morphism $\phi$ of graded vector spaces from $V$ to $W$ is a family of morphisms:

$$\phi_i : V_i \to W_i$$

or, equivalently, a morphism from $V = \bigoplus_{i \in I} V_i$ to $W = \bigoplus_{i \in I} W_i$ such that $\phi = \bigoplus_{i \in I} \phi_i$ with $\phi_i \in \text{Hom}(V_i, W_i)$. We denote by $\text{Hom}_I(V, W)$ the set of all these morphisms, and set $\text{End}_I(V) := \text{Hom}_I(V, V)$.  

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To each tensor species $F$ is canonically associated a $P$-graded vector space also written $F$, where $P$ denotes the set of finite subsets of $N$. For each $X \in P$, we set $F_X := F(X)$. The category $P$-Vect of $P$-graded vector spaces carries two tensor products $\otimes$ and $\otimes$: the homogeneous one, defined by

$$(V \otimes W)_X = V_X \otimes W_X$$

for all $X \in P$, and the linear species or "$P$-graded" one, defined by

$$(V \otimes W)_X = \bigoplus_{T \sqcup V = X} V_T \otimes W_U$$

for all $X \in P$, as in Section 1.

**Definition 5** Let $B$ be a twisted bialgebra. The convolution product on the vector space of $P$-graded linear endomorphisms of $B$ is defined by

$$f \ast g : B \overset{\delta}{\to} B \otimes B \overset{f \otimes g}{\to} B \otimes B \overset{m}{\to} B.$$

for all $f, g \in \text{End}_P(B)$.

Here, the tensor product $f \otimes g : B \otimes B \to B \otimes B$ is defined by:

$$(B \otimes B)_S = \bigoplus_{T \sqcup V = S} B_T \otimes B_V \rightarrow \bigoplus_{T \sqcup V = S} B_T \otimes B_V \rightarrow (B \otimes B)_S.$$

It is important to notice that we do not assume that $f$ and $g$ are endomorphisms of $B$ as a tensor species. Therefore, strictly speaking, the tensor product of $f$ and $g$ is not induced by the symmetric monoidal structure of $\text{Sp}$ (but by the one in $P$-Vect).

**Proposition 6** The convolution product is associative and unital. In particular, $(\text{End}_P(B), \ast)$ is an associative algebra with unit.

Indeed, for $f, g, h \in \text{End}_P(B)$, we have:

$$f \ast (g \ast h) = m \circ (f \otimes (g \ast h)) \circ \delta = m \circ ((f \otimes m) \circ (g \otimes h) \circ \delta) \circ \delta = (f \otimes g) \ast (B \otimes m) \circ (f \otimes g \otimes h) \circ (B \otimes \delta) \circ \delta$$

and, since $m$ (respectively, $\delta$) is associative (respectively, coassociative):

$$= m \circ (m \otimes B) \circ (f \otimes g \otimes h) \circ (\delta \otimes B) \circ \delta = (f \ast g) \ast h.$$
Definition 7 The characteristic map $1_S$ associated to $S \in \mathcal{P}$ is defined by:

$$1_S : B_S \rightarrow B_S, \quad \text{and}$$

$$1_S : B_{S'} \rightarrow 0 \quad \text{if } S \neq S'.$$

The unit $\eta$ of the convolution is the characteristic map $1_\emptyset$ associated to $\emptyset$, as can be checked easily.

Definition 8 The twisted descent algebra $\mathcal{T}_B$ of a twisted bialgebra $B$ is the convolution algebra generated by the characteristic maps of the finite subsets of $\mathbb{N}$.

The twisted descent algebra is generated, as a vector space, by $1_\emptyset$ and by the convolution products $1_{S_1} \ast \cdots \ast 1_{S_k}$, where $S_i \cap S_j = \emptyset \neq S_i$ for all $i, j$ such that $i \neq j$. For, if $T \cap U \neq \emptyset$, it follows from the definitions that $1_T \ast 1_U = 0$. We write from now on $1_{(S_1, \ldots, S_k)}$ for $1_{S_1} \ast \cdots \ast 1_{S_k}$. In general, the elements $1_{(S_1, \ldots, S_k)}$, even with the restrictions on the sets $S_i$ given above, are not linearly independent.

Since $m$ is associative, it induces a unique product map $m^{[3]}$ from $B^{\otimes 3}$ to $B$, given by: $m^{[3]} := m \circ (m \otimes B) = m \circ (B \otimes m)$. More generally, we write $m^{[k]}$ for the unique product map from $B^{\otimes k}$ to $B$ and (since the coproduct is coassociative) $\delta^{[k]}$ for the unique coproduct map from $B$ to $B^{\otimes k}$. Then, in particular, we have:

$$1_{(S_1, \ldots, S_k)} = m^{[k]} \circ 1^{\otimes}_{(S_1, \ldots, S_k)} \circ \delta^{[k]},$$

where $1^{\otimes}_{(S_1, \ldots, S_k)} := 1_{S_1} \otimes \cdots \otimes 1_{S_k}$.

Notice that the (ordinary) descent algebra of $B$ has already been defined in [77]: it is, in the language of the present article, the convolution algebra generated by the formal series $1_n := \sum_{S \in \mathcal{P}, |S|=n} 1_S \ (n \in \mathbb{N})$.

3 The fine algebraic structure of twisted descents

The twisted descent algebra of a twisted algebra $B$ carries, by its very definition, an associative product, the convolution product. In the present section, we show that it is also closed under the composition product in the category of $\mathcal{P}$-graded vector spaces, a property that extends to the twisted
setting one of the fundamental properties of the Solomon descent algebra
and, more generally, of descent algebras of commutative or cocommutative
bialgebras. Indeed, they also carry two products, that, together with certain
coalgebra properties, are the building blocks of the modern theory of descent
algebras. We restrict our attention to the case where $B$ is cocommutative.
Dual properties hold when $B$ is commutative.

**Theorem 9** The twisted descent algebra of a twisted cocommutative bial-
gebra $B$ is closed under the composition of $\mathcal{P}$-linear endomorphisms of $B$.
The composition product is given by:

$$1_{(S_1,\ldots,S_n)} \circ 1_{(T_1,\ldots,T_k)} = \begin{cases} 0 & \text{if } S_1 \sqcup \ldots \sqcup S_n \neq T_1 \sqcup \ldots \sqcup T_k, \\ 1_{(S_1 \cap T_1,\ldots,S_k \cap T_k,\ldots,S_n \cap T_1,\ldots,S_n \cap T_k),} & \text{otherwise.} \end{cases}$$

Note that, in the second part of the formula, possibly occurring empty
intersections $S_i \cap T_j$ may be omitted, since $1_{\emptyset}$ is the identity of $\mathcal{T}_B$. For
example, if $(S_1, S_2) = (\{3, 5\}, \{1, 4\})$ and $(T_1, T_2) = (\{5\}, \{1, 3, 4\})$, then

$$1_{(S_1, S_2)} \circ 1_{(T_1, T_2)} = 1_{(\emptyset, \emptyset)} = 1_{(\emptyset, \emptyset)}.$$ 

The two identities in the theorem are the most fundamental iden-
tities satisfied in twisted descent algebras. Since the first identity follows immedi-
ately from the definitions, we assume in the following that $S_1 \sqcup \ldots \sqcup S_n = T_1 \sqcup \ldots \sqcup T_k$. The proof of the second identity will be given in several steps.

Let us first introduce a useful notation. If $A$ is a twisted algebra with
product $m$ (respectively, a twisted coalgebra with coproduct $\delta$), $A^{\otimes 2}$ and,
more generally, $A^{\otimes k}$ is naturally provided with the structure of an algebra
(respectively, of a coalgebra). For example, the product on $A^{\otimes k}$ (denoted
by $m_k$) is defined by:

$$A^{\otimes k} \otimes A^{\otimes k} = (A_1 \otimes \cdots \otimes A_k) \otimes (A'_1 \otimes \cdots \otimes A'_k)$$

$$\cong (A_1 \otimes A'_1) \otimes \cdots \otimes (A_k \otimes A'_k)$$

$$m^{\otimes k} \rightarrow A^{\otimes k},$$

where we have written $A_i$ and $A'_i$ for copies of $A$. The corresponding co-
product on $A^{\otimes k}$ is denoted by $\delta^{\otimes k}$.

**Lemma 10** Let $B$ be a twisted bialgebra. Then, for all $l$ and all $j$, we have:

$$\delta^{[l]} \circ m^{[j]} = m^{[j]} \circ (\delta^{[l]})^{\otimes j}.$$ 

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The proof may, for example, be given by induction on $l$ and is essentially the same as the proof of the corresponding identity for classical bialgebras (see [15] or [14, Lem. II,8.]).

The preceding lemma implies:

$$ X := 1_{(S_1, \ldots, S_n)} \circ 1_{(T_1, \ldots, T_k)} $$

$$ = m[n] \circ 1_{(S_1, \ldots, S_n)} \circ \delta[n] \circ m[k] \circ 1_{(T_1, \ldots, T_k)} \circ \delta[k] $$

$$ = m[n] \circ 1_{(S_1, \ldots, S_n)} \circ m[k] \circ (\delta[n] \otimes \delta[k]) \circ 1_{(T_1, \ldots, T_k)} \circ \delta[k]. $$

**Lemma 11** We have, for $T \in \mathcal{P}$:

$$ \delta \circ 1_T = \sum_{U \sqcup V = T} (1_U \otimes 1_V) \circ \delta $$

and, more generally:

$$ \delta[n] \circ 1_T = \sum_{U_1 \sqcup \cdots \sqcup U_n = T} 1_{(U_1, \ldots, U_n)} \circ \delta[n]. $$

This follows from the definition of $\delta$: indeed, $\delta$ is a $\mathcal{P}$-graded map, thus maps the degree $T$ component of $B$ to the degree $T$ component of $B \otimes B$, that is to

$$ \bigoplus_{U \sqcup V = T} B(U) \otimes B(V). $$

The same argument applies to $m$ and $m[k]$. In particular:

**Lemma 12** We have:

$$ 1_{(S_1, \ldots, S_n)} \circ m[k]_n = m[k]_n \circ \sum_{V_1^i \sqcup \cdots \sqcup V^k_i = S_i} 1_{(V_1^{i_1}, \ldots, V_n^{i_1}, \ldots, V_1^{k_1}, \ldots, V_n^{k_1})}. $$

Combined with Lemma 11, this yields:

$$ X = m[n] \circ m[k] $$

$$ \circ \left( \sum_{V_1^i \sqcup \cdots \sqcup V^k_i = S_i} 1_{(V_1^{i_1}, \ldots, V_n^{i_1}, \ldots, V_1^{k_1}, \ldots, V_n^{k_1})} \right) $$

$$ \circ \left( \sum_{U_1^i \sqcup \cdots \sqcup U^k_i = T_i} 1_{(U_1^{i_1}, \ldots, U_n^{i_1}, \ldots, U_1^{k_1}, \ldots, U_n^{k_1})} \circ (\delta[n])^{\otimes k} \circ \delta[k]. $$

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Besides, since the coproduct is coassociative, we have: \((\delta^{[n]} \otimes k \circ \delta^{[k]}) = \delta^{[nk]}\) and, since the \(1_S\) are characteristic maps, we also have:

\[
1 \otimes (V_1^1, \ldots, V_n^1, \ldots, V_1^k, \ldots, V_n^k) = 0
\]

unless \(V_i^j = U_i^j\) for all \(i, j\). These identities are true if and only if \(V_i^j = U_i^j = \mathcal{S}_i \cap \mathcal{T}_j\). Therefore, the previous expression of \(X\) simplifies to:

\[
X = m^{[n]} \circ m^{[k]} \circ 1 \otimes (S_1 \cap T_1, \ldots, S_n \cap T_1, \ldots, S_1 \cap T_k, \ldots, S_n \cap T_k) \circ \delta^{[nk]}.
\]

The last point to notice is that, since the coproduct is cocommutative, we can change the order of the tensor products of characteristic maps freely in this expression, provided we modify accordingly the multiplication map on the left. For example, because of cocommutativity, the following identity holds:

\[
m_2^{[2]} \circ (1_U \otimes 1_V \otimes 1_W \otimes 1_Y) \otimes \delta^{[4]} = (m^{[2]} \otimes m^{[2]}) \circ (1_U \otimes 1_W \otimes 1_V \otimes 1_Y) \circ \delta^{[4]},
\]

as well as all the general identities that can be constructed on this pattern. In conclusion, we have

\[
X = m^{[n]} \circ (m^{[k]} \otimes n) \circ 1 \otimes (S_1 \cap T_1, \ldots, S_1 \cap T_k, \ldots, S_n \cap T_1, \ldots, S_n \cap T_k) \circ \delta^{[nk]}.
\]

The proof of the theorem is complete upon noting that associativity of \(m\) implies \(m^{[n]} \circ (m^{[k]} \otimes n) = m^{[nk]}\).

The same computation, or a duality argument, show that, in case commutativity of \(B\) was assumed, the second identity satisfied by the elements of the twisted descent algebra would read:

\[
1_{(T_1, \ldots, T_k)} \circ 1_{(S_1, \ldots, S_n)} = 1_{(S_1 \cap T_1, \ldots, S_1 \cap T_k, \ldots, S_n \cap T_1, \ldots, S_n \cap T_k)}.
\]

In other words, the structure of twisted descent algebras of cocommutative twisted bialgebras is dual to the one of twisted descent algebras of commutative twisted bialgebras.

We conclude this section with a remarkable identity for the linear generators of \(\mathcal{T}_B\), relating the two associative products on twisted descent algebras.

**Proposition 13** Let \(f, g, h, k\) be convolution products of characteristic maps in \(\mathcal{T}_B\) such that \((f \circ g) \ast (h \circ k) \neq 0\). Then we have:

\[
(f \circ g) \ast (h \circ k) = (f \ast h) \circ (g \ast k).
\]
Indeed, suppose that \( f = 1_{(S_1, \ldots, S_n)} \), \( g = 1_{(T_1, \ldots, T_m)} \), \( h = 1_{(U_1, \ldots, U_p)} \), \( k = 1_{(V_1, \ldots, V_q)} \). Then the assumption \((f \circ g) \ast (h \circ k) \neq 0\) implies \( S_1 \coprod \ldots \coprod S_n = T_1 \coprod \ldots \coprod T_m \), \( U_1 \coprod \ldots \coprod U_p = V_1 \coprod \ldots \coprod V_q \), and furthermore \( S_i \cap U_j = \emptyset \) and \( T_i \cap V_j = \emptyset \) for all \( i, j \). The claim thus follows from Theorem 9.

4 Solomon-Tits and the free twisted descent algebra.

Let \( (T, \ast) \) be the algebra defined by generators and relations as follows. The generators are the symbols \( 1_S, \ S \in \mathcal{P} \), subject to the relations:

\[
1_{S_1} \ast \cdots \ast 1_{S_n} = 0
\]

if \( S_i \cap S_j \neq \emptyset \) for some \( i \neq j \), and:

\[
1_{\emptyset} \ast 1_S = 1_S \ast 1_{\emptyset} = 1_S.
\]

Note that the twisted descent algebra \( T_B \) of any twisted bialgebra \( B \) is isomorphic to a quotient of \( T \), according to our previous computations.

**Proposition 14** The algebra \( T \) is, up to a canonical isomorphism, the twisted descent algebra of the twisted bialgebra freely generated, as a twisted algebra, by the primitive elements \( \alpha_S, \ S \in \mathcal{P} \).

Let us write \( B \) for the free twisted algebra on the generators \( \alpha_S, \ S \in \mathcal{P} \). A bijection from \( S \) to \( T \) maps \( \alpha_S \) to \( \alpha_T \) when \( B \) is viewed as a \( \mathcal{P} \)-graded vector space. Then, since \( B \) is free:

\[
B = \bigoplus_{n \in \mathbb{N} \cup \{0\}} \left( \bigoplus_{S \in \mathcal{P}} k(\alpha_S) \right)^{\otimes n},
\]

where we write \( k(\alpha_S) \) for the vector space generated by \( \alpha_S \). It follows that \( B(S) \) is spanned freely as a vector space by \( \alpha_S \) and by the products \( \alpha_{S_1} \cdots \alpha_{S_k} \), where \( S_1 \coprod \ldots \coprod S_k = S, S_i \neq \emptyset \), and where we write \( \alpha_{S_1} \cdots \alpha_{S_k} \) for the image in \( B(S) \) of \( \alpha_{S_1} \otimes \cdots \otimes \alpha_{S_k} \in B(S_1) \otimes \cdots \otimes B(S_k) \).

As a consequence, there is a unique coproduct \( \delta \) on \( B(S) \), defined by requiring that the \( \alpha_S \) are primitive elements:

\[
\delta(\alpha_{S_1} \cdots \alpha_{S_k}) = \sum_{\{i_1, \ldots, i_l\} \coprod \{j_1, \ldots, j_{k-l}\} = [k]} \alpha_{S_{i_1}} \cdots \alpha_{S_{i_l}} \otimes \alpha_{S_{j_1}} \cdots \alpha_{S_{j_{k-l}}},
\]

where \( [k] = \{1, \ldots, k\} \).
Since the twisted descent algebra $T_B$ of $B$ is a quotient algebra of $T$, the proposition is equivalent to the following: the convolution product $1_{S_1} \cdots 1_{S_k}$, $S_i \cap S_j = \emptyset \neq S_i$ are linearly independent in $T_B$.

To see this, note first the following consequence of the definition of the convolution product in $T_B$ and of the coproduct in $B$. If $l \geq k$, then

$$1_{(U_1, \ldots, U_l)}(\alpha_{S_1} \cdots \alpha_{S_k}) = 0$$

unless $l = k$ and $(U_1, \ldots, U_l)$ may be obtained by reordering $(S_1, \ldots, S_k)$; and in this case,

$$1_{(U_1, \ldots, U_k)}(\alpha_{S_1} \cdots \alpha_{S_k}) = \alpha_{U_1} \cdots \alpha_{U_k}.$$

Let us assume now that $f = \sum_{k=1}^{n} \left( \sum_{i=1}^{m_k} \lambda_{i,k} 1_{(S_1^{i,k}, \ldots, S_k^{i,k})} \right) = 0$ and that there exists an index $k$ such that $\lambda_{i,k} \neq 0$ for some $i$. Choose $k$ minimal with this property. Then

$$f(\alpha_{S_1}^{i,k} \cdots \alpha_{S_k}^{i,k}) = \lambda_{i,k} \alpha_{S_1}^{i,k} \cdots \alpha_{S_k}^{i,k} + \Gamma,$$

where $\Gamma$ is a linear combination of noncommutative monomials $\neq \alpha_{S_1}^{i,k} \cdots \alpha_{S_k}^{i,k}$ in the elements $\alpha_{S_j}^{j,k}$ ($j \in [k])$, thus linearly independent of $\alpha_{S_1}^{i,k} \cdots \alpha_{S_k}^{i,k}$. It follows that $\lambda_{i,k} = 0$, a contradiction. The proof of the proposition is complete.

We will refer to the algebra $T$ simply as “the” twisted descent algebra (or, sometimes, as we did in the introduction, as the free twisted descent algebra, when we want to emphasize that all descent algebras of cocommutative twisted bialgebras are quotients of $T$). Note that, alternatively (and better suited for certain applications), the twisted algebra $T$ may be described as the free associative algebra in the tensor category of connected $P$-graded vector spaces, with one generator $1_S$ in each degree. In general, we will call a $P$-graded algebra any algebra in the tensor category of connected $P$-graded vector spaces, so that $T$ is a free $P$-graded algebra. That is, if $K = \bigoplus_{S \in P, S \neq \emptyset} k$, we have:

$$T = \bigoplus_{n \in \mathbb{N} \cup \{0\}} K^\otimes n = \bigoplus_{(S_1, \ldots, S_n) \in P^n, S_i \neq \emptyset} \bigoplus_{S \in P, S \neq \emptyset} k.$$ 

The component of degree $S$ ($S \in P$) of $T$ is:

$$T_S = \bigoplus_{S_1 \cdots \bigotimes_{i=1}^{n} S_n = S, S_i \neq \emptyset} k.$$
and \( \mathcal{T}_0 = k^{\otimes 0} = k \). The \( \mathcal{P} \)-graded product is given on the indices by:

\[
((S_1, \ldots, S_n), (T_1, \ldots, T_l)) \mapsto (S_1, \ldots, S_n, T_1, \ldots, T_l).
\]

Recall that, by Theorem 9, there is a second associative product \( \circ \) on \( \mathcal{T} \), the composition product, which is homogeneous with respect to the \( \mathcal{P} \)-grading. That is,

\[1_{(S_1, \ldots, S_n)} \circ 1_{(T_1, \ldots, T_k)} = 0\]

if \( S_1 \cdots \cdots \cdots S_n \neq T_1 \cdots \cdots \cdots T_k \). In particular, the algebra \( (\mathcal{T}, \circ) \) splits as a product of its graded components. In other words, the graded component \( \mathcal{T}_S \) of \( \mathcal{T} \) of degree \( S \in \mathcal{P} \) is a two-sided ideal of \( \mathcal{T} \) with respect to \( \circ \). The algebraic structure of this ideal is studied in detail in [21]. Note that, if \( S, T \in \mathcal{P} \) such that \( |S| = |T| \), then any bijection from \( S \) to \( T \) induces an isomorphism of algebras from \( (\mathcal{T}_S, \circ) \) to \( (\mathcal{T}_T, \circ) \).

**Proposition 15** The algebra \( \mathcal{T}_{[n]} \) is isomorphic to the Solomon-Tits algebra of the symmetric group on \([n]\).

Recall briefly the definitions in Tits’ appendix to Solomon’s original paper [22]. Let \((W, S)\) be a Coxeter system (that is, \(W\) is a finite Coxeter group and \(S\) a simple system of generators, see e.g. [8]). For \( K \subset S \), the subgroup of \( W \) generated by \( K \) is written \( W_K \). The Coxeter complex \( \Sigma \) associated to \((W, S)\) is a simplicial complex. Its simplices are in bijection with the cosets \( W_K \cdot w \) with \( K \subset S \) and \( w \in W \). The fundamental geometrical idea, due to Tits [26], is that there is a product on \( \Sigma \), the associativity of which is proven in [25]. The product of \( A \) and \( B \) in \( \Sigma \) is, by definition, “the greatest common face of the first terms of all galleries of minimum length whose first and last term dominate, respectively, \( A \) and \( B \)” (recall that a gallery is a sequence of adjacent maximal simplices in the Coxeter complex).

We call the algebra \( \Sigma \), for this geometrical product, the Solomon-Tits algebra, since it was introduced by Tits in order to explain geometrically the meaning of Solomon’s constructions in [22], and to relate them to Tits’ previous results on buildings. The connection to the algebra \( \mathcal{T}_{[n]} \) is not obvious, but becomes so if one chooses the right parametrization of the simplices of the Coxeter complex of \( S_n \). We learned this connection in Brown [4], who also studies various generalizations and applications of Tits’ calculations on hyperplane arrangements.

The Coxeter complex can be viewed as a triangulation of the sphere associated to the hyperplane arrangement corresponding to \( S_n \), see [8] for details. This arrangement is obtained from the hyperplanes \( x_i = x_j, \ i \neq j \)
in $\mathbb{R}^n$ ($S_n$ is not essential relative to $\mathbb{R}^n$, leaving a 1-dimensional subspace invariant, but this does not matter at this point). The maximal cells of the corresponding triangulation of the sphere correspond to the $n$-dimensional simplicial cones in $\mathbb{R}^n$ obtained from the arrangement. That is, since the half-spaces associated to the hyperplanes are given by the condition $x_i \geq x_j$, there are $n!$ such cones, described by the following set of inequalities in the canonical coordinate system:

$$x_{\sigma(1)} \geq \cdots \geq x_{\sigma(n)},$$

where $\sigma \in S_n$. The faces of the cones are obtained by turning an arbitrary number of inequalities into equalities. For example, $x_3 \geq x_1 = x_2 \geq x_4$ is a face of both $x_3 \geq x_2 \geq x_1 \geq x_4$ and $x_3 \geq x_1 \geq x_2 \geq x_4$. It follows in particular from this description of $\Sigma$ that its elements are parametrized by the ordered partitions of $[n]$; for instance, the partition $(\{3\}, \{1, 2\}, \{4\})$ is associated to the face $x_3 \geq x_1 = x_2 \geq x_4$. Using this dictionary, it is an easy exercise to express Tits’ product on $\Sigma$ by using the defining formulas for $\circ$, see [4]. The proposition follows.

As a final structural ingredient, in the free case, we have:

**Proposition 16** The algebra $\mathcal{T}$ carries a coassociative and cocommutative coproduct, defined by:

$$\delta(1_{(S_1, \ldots, S_k)}) := \sum_{T_i \prod U_i = S_i} 1_{(T_1, \ldots, T_k)} \otimes 1_{(U_1, \ldots, U_k)}.$$

Empty sets are allowed in this summation formula. The coassociativity of the coproduct is straightforward. The same formula defines a cocommutative $\mathcal{P}$-graded coalgebra structure on $\mathcal{T}$.

**Theorem 17** The product $\ast$ and the coproduct $\delta$ turn $\mathcal{T}$ into a $\mathcal{P}$-graded bialgebra. The product $\circ$ and the coproduct $\delta$ turn $\mathcal{T}$ into a bialgebra.

Once again, this property parallels the fundamental properties of the classical descent algebra, see e.g. [10]. Notice, however, that the $\mathcal{P}$-graded hypothesis is necessary, since $(\mathcal{T}, \ast, \delta)$ is not bialgebra. For example, $\delta(1_{\{1, 2\}} \ast 1_{\{1, 2\}}) = \delta(0) = 0$. 

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On the other hand, we have:

\[ \delta(1_{\{1,2\}}) * 2 \delta(1_{\{1,2\}}) \]
\[ = (1_{\{1,2\}} \otimes 1_0 + 1_{\{1\}} \otimes 1_{\{2\}} + 1_{\{2\}} \otimes 1_{\{1\}} + 1_0 \otimes 1_{\{1,2\}}) \]
\[ * 2(1_{\{1,2\}} \otimes 1_0 + 1_{\{1\}} \otimes 1_{\{2\}} + 1_{\{2\}} \otimes 1_{\{1\}} + 1_0 \otimes 1_{\{1,2\}}) \]
\[ = 2 \cdot 1_{\{1,2\}} \otimes 1_{\{1,2\}} + 1_{\{1\},\{2\}} \otimes 1_{\{1\},\{2\}} + 1_{\{2\},\{1\}} \otimes 1_{\{1\},\{2\}}. \]

The first part of the theorem follows from the following identity, that holds for all mutually disjoint sets \( S_1, \ldots, S_n, V_1, \ldots, V_k \in \mathcal{P} \):

\[ \delta(1_{S_1, \ldots, S_n} \circ 1_{V_1, \ldots, V_k}) \]
\[ = \delta(1_{S_1, \ldots, S_n, V_1, \ldots, V_k}) \]
\[ = \sum_{T_i \prod U_i = S_i, W_j \prod Z_j = V_j} 1_{(T_1, \ldots, T_n, V_1, \ldots, V_k)} \otimes 1_{(U_1, \ldots, U_n, Z_1, \ldots, Z_k)} \]
\[ = \sum_{T_i \prod U_i = S_i, W_j \prod Z_j = V_j} [1_{(T_1, \ldots, T_n)} \otimes 1_{(U_1, \ldots, U_n)}] * 2 [1_{(W_1, \ldots, W_k)} \otimes 1_{(Z_1, \ldots, Z_k)}] \]
\[ = \delta(1_{S_1, \ldots, S_n}) * 2 \delta(1_{V_1, \ldots, V_k}). \]

Let us assume now that \( S_1 \prod \ldots \prod S_n = V_1 \prod \ldots \prod V_k \). Then, on the one hand, we have:

\[ A := \delta(1_{S_1, \ldots, S_n} \circ 1_{V_1, \ldots, V_k}) \]
\[ = \delta(1_{S_1 \cap V_1, \ldots, S_n \cap V_n}) \]
\[ = \sum_{U_{i,j} \prod T_{i,j} = S_i \cap V_j} 1_{(U_{1,1}, \ldots, U_{1,k}, \ldots, U_{n,1}, \ldots, U_{n,k})} \otimes 1_{(T_{1,1}, \ldots, T_{1,k}, \ldots, T_{n,1}, \ldots, T_{n,k})}. \]

On the other hand, we have:

\[ B := \delta(1_{S_1, \ldots, S_n}) \circ 2 \delta(1_{V_1, \ldots, V_k}) \]
\[ = [ \sum_{U_i \prod T_i = S_i} 1_{(U_1, \ldots, U_n)} \otimes 1_{(T_1, \ldots, T_n)} ] \circ 2 [ \sum_{U_j \prod T^j = V_j} 1_{(U^1, \ldots, U^k)} \otimes 1_{(T^1, \ldots, T^k)} \]
\[ = \sum_{U_i \prod T_i = S_i, U_j \prod T^j = V_j} [1_{(U_1, \ldots, U_n)} \circ (1_{(U^1, \ldots, U^k)}) \otimes [1_{(T_1, \ldots, T_n)} \circ 1_{(T^1, \ldots, T^k)}]. \]
Each term in the last sum is 0 unless $U_1 \coprod \ldots \coprod U_n = U_1 \coprod \ldots \coprod U^k$ and $T_1 \coprod \ldots \coprod T_n = T_1 \coprod \ldots \coprod T^k$. In particular, in this case, it follows that $U_i \cap U^j = U_i \cap V_j$. For otherwise, the elements of $U_i \cap V_j$ not in $U_i \cap U^j$ would lie in $T^j$ and would not belong to any $T_i$, and the term would be 0, a contradiction. In conclusion, setting $U_{i,j} := U_i \cap U^j$ and $T_{i,j} := T_i \cap T^j$, yields the requested identity, $A = B$, and completes the proof of Theorem 17.

The existence of the coproduct $\delta$ allows to derive the following variant of Proposition 18 for the algebra $\mathcal{T}$, now relating both products and the coproduct on the free twisted descent algebra.

**Corollary 18** Let $f, g, h \in \mathcal{T}$, then we have:

$$(f * g) \circ h = m((f \otimes g) \circ \delta(h)),$$

where $m : \mathcal{T} \otimes \mathcal{T} \to \mathcal{T}$ is the convolution product.

In more illustrative terms, using Sweedler’s notation $\delta(h) = \sum h^{(1)} \otimes h^{(2)}$ for the coproduct, we have

$$(f * g) \circ h = \sum (f \circ h^{(1)}) * (g \circ h^{(2)}).$$

For the proof, it suffices to consider $f = 1_{(S_1, \ldots, S_n)}$, $g = 1_{(T_1, \ldots, T_k)}$ and $h = 1_{(U_1, \ldots, U_p)}$, by linearity. Then, the right hand side

$$m((f \otimes g) \circ \delta(h))$$

$$= \sum \limits_{x_i \coprod \prod Y_i = U_i} m((1_{(S_1, \ldots, S_n)} \otimes 1_{(T_1, \ldots, T_k)}) \circ \delta_2 (1_{X_1, \ldots, X_p}) \otimes 1_{Y_1, \ldots, Y_p}))$$

$$= \sum \limits_{x_i \coprod \prod Y_i = U_i} (1_{(S_1, \ldots, S_n)} \circ 1_{X_1, \ldots, X_p}) * (1_{(T_1, \ldots, T_k)} \circ 1_{Y_1, \ldots, Y_p}))$$

does not vanish if and only if $S_i \cap T_j = \emptyset$ for all $i, j$ and $S \coprod T = U$, where $S = S_1 \coprod \ldots \coprod S_n$, $T = T_1 \coprod \ldots \coprod T_k$ and $U = U_1 \coprod \ldots \coprod U_p$. The same observation is true for the left hand side. And in this case, there is a unique summand $\neq 0$ in the above sum, indexed by $X_i = S \cap U_i$ and $Y_i = T \cap U_i$, that is:

$$m((f \otimes g) \circ \delta(h))$$

$$= (1_{(S_1, \ldots, S_n)} \circ 1_{(S \cap U_1, \ldots, S \cap U_p)}) * (1_{(T_1, \ldots, T_k)} \circ 1_{(T \cap U_1, \ldots, T \cap U_p)})$$

$$= 1_{(S \cap U_1, \ldots, S \cap U_p, \ldots, S \cap U_1, \ldots, S \cap U_p, \ldots, T_1 \cap U_1, \ldots, T_1 \cap U_1, \ldots, T_k \cap U_1, \ldots, T_k \cap U_1)}$$

$$= (f * g) \circ h$$

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as asserted.

Recall that the (ordinary) descent algebra of a twisted bialgebra \( B \) (as introduced in [17]) may be identified with the convolution algebra generated by the formal series \( 1_n := \sum_{S \in \mathcal{P}, |S| = n} 1_S, \ n \in \mathbb{N} \). Motivated by this remark, in the free case, we write \( D \) for the convolution subalgebra of \( \hat{T} := \prod_{S \in \mathcal{P}} T_S \) generated by the elements \( 1_n \in \hat{T}, \ n \in \mathbb{N} \).

**Theorem 19** The convolution algebra \( D \) is also a subalgebra of \( \hat{T} \) with respect to the composition product. More precisely, each graded component

\[
D_n := D \cap \prod_{|S| = n} T_S
\]

is closed under the composition \( \circ \) of endomorphisms of \( B \).

Moreover, \( (D_n, \circ) \) is isomorphic to Solomon’s classical descent algebra of the symmetric group on \( [n] \).

Notice that, as a particular consequence of the theorem, Corollary [13] is a generalization of the crucial reciprocity law for the descent algebra derived in [7, Proposition 5.2].

Concerning the proof, we observe that \( D_n \circ D_m = 0 \) whenever \( n \neq m \), and that the elements \( 1_{n_1,\ldots,n_k} := 1_{n_1} \circ \cdots \circ 1_{n_k} \) with \( n_1 + \cdots + n_k = n \) constitute a linear basis of \( D_n \). It remains to prove Solomon’s fundamental multiplication rule [22, Theorem 1] for the members of this basis, that is:

\[
1_{n_1,\ldots,n_k} \circ 1_{m_1,\ldots,m_l} = \sum 1_{a_1,\ldots,a_l,\ldots,a_k,\ldots,n_i},
\]

where the sum on the right is taken over all matrices \((a_i^j)\) of nonnegative integers such that \( \sum_{j=1}^l a_i^j = n_i \) and \( \sum_{i=1}^k a_i^j = m_j \) (see [19] [14]).

By definition, we have

\[
1_{n_i,\ldots,n_k} = \sum_{|S_i| = n_i, \ S_i \cap S_j = \emptyset} 1_{(S_1,\ldots,S_k)}
\]

and therefore:

\[
1_{n_1,\ldots,n_k} \circ 1_{m_1,\ldots,m_l} = \sum_{|S_i| = n_i, \ S_i \cap S_j = \emptyset} \sum_{|T_u| = m_u, \ T_u \cap T_v = \emptyset} 1_{(S_1,\ldots,S_k)} \circ 1_{(T_1,\ldots,T_l)}
\]
Writing $U_i^j$ for $S_i \cap T_j$, the last term reads:

$$= \sum_{j=1}^{l} \sum_{j=1}^{k} a_j^i = n_i \sum_{i=1}^{k} a_j^i = m_j$$

and the theorem is proven.

## 5 Applications to computations with shuffles

In this last section, we derive some important identities in the twisted descent algebra $T$. They hold in the twisted descent algebra of any twisted bialgebra $B$, since $T_B$ is a quotient of $T$. When these identities follow immediately from Theorem 9, the proofs are omitted.

**Lemma 20** Let $S = \{s_1, \ldots, s_n\} \in \mathcal{P}$. Then, for all $S_1, \ldots, S_k$ in $\mathcal{P}$ such that $S_1 \prod \ldots \prod S_k = S$, we have:

$$1_{\{(s_1), \ldots, (s_n)\}} \circ 1_{\{s_1, \ldots, s_k\}} = 1_{\{(s_1), \ldots, (s_n)\}}.$$  

The next lemma shows that the behavior of products in $(T, \circ)$ encodes the unshuffling of permutations, an important device in algebraic combinatorics and its applications (card shuffling, models of databases structure in computer science, and so on).

**Lemma 21** Let $S$ and $S_1, \ldots, S_k$ be as above. Then we have

$$1_{\{s_1, \ldots, s_k\}} \circ 1_{\{s_1, \ldots, s_k\}} = 1_{\{s_1, \ldots, s_k\}}$$

where $\beta(S)$ is the $(S_1, \ldots, S_k)$-relative unshuffling of $\{(s_1), \ldots, \{s_n\}\}$.

That is, $\beta(S)$ is the sequence of one-element subsets of $S$ obtained by selecting successively in the sequence $\{(s_1), \ldots, \{s_n\}\}$ the elements of $S_1, S_2, \ldots, S_k$. For example, we have:

$$1_{\{(1,3,5),\{2,4\}\}} \circ 1_{\{(3),\{4,5\},\{2\}\}} = 1_{\{(3),\{5\},\{1\},\{4\},\{2\}\}}.$$  

\[ From now on, we write $1_{\sigma}$ for $1_{\{\sigma(1)\}, \ldots, \{\sigma(n)\}}$ whenever $\sigma \in S_n$. \]
Corollary 22 Let \((S_1, \ldots, S_k)\) be an increasing partition of \([n]\). That is, we assume that \(s < s'\) for all \(s \in S_i, s' \in S_j\) whenever \(i < j\). Then,

\[1(S_1, \ldots, S_k) \circ 1_\sigma = 1_{\{1(1), \ldots, n\}}\]

if and only if \(\sigma\) is a shuffle of \(S_1, \ldots, S_k\).

Recall that a finite sequence \(C = (n_1, \ldots, n_k)\) of positive integers with sum \(n\) is a composition of \(n\). Recall furthermore that the descent set \(D(\sigma)\) of a permutation \(\sigma \in S_n\) is the subset of \([n - 1]\) consisting of all \(i\) such that \(\sigma(i) > \sigma(i + 1)\). We denote by \(D_C\) the sum of all permutations in \(S_n\) with descent set contained in \(\{n_1, n_1 + n_2, \ldots, n_1 + \cdots + n_{k-1}\}\).

The linear span of these elements in the integral group ring \(\mathbb{Z}[S_n]\) is the classical descent algebra of \(S_n\), in Solomon’s original setting. It is closed under composition of permutations, due to Solomon’s remarkable discovery. The elements \(D_C\) form a linear basis of Solomon’s algebra, and the corresponding structure coefficients (or, equivalently, those of the dual algebra) can be computed explicitly in terms of double coset representatives of Young subgroups of \(S_n\), or using Tits’ algebraic approach to the geometrical properties of Coxeter complexes. An equivalent combinatorial description already occured in the proof of Theorem 19.

If \((S_1, \ldots, S_k)\) is a partition of \([n]\), we say that the composition \(C = (|S_1|, |S_2|, \ldots, |S_k|)\) of \(n\) is the type of \((S_1, \ldots, S_k)\).

Proposition 23 Let \((S_1, \ldots, S_k)\) be an increasing partition of \([n]\) of type \(C\). Then the sum of all permutations \(\sigma \in S_n\) such that

\[1(S_1, \ldots, S_k) \circ 1_\sigma = 1_{\{1(1), \ldots, n\}}\]

is identical to \(D_C\), where \(*\) is the linear endomorphism of \(\mathbb{Z}[S_n]\) defined by \(\sigma^* := \sigma^{-1}\) for all \(\sigma \in S_n\).

This is another way of stating Corollary 22 since the inverse of a shuffle of \(S_1, \ldots, S_k\) is (when viewed as a word) a product \(u_1 \cdots u_k\) of increasing words \(u_i\) of length \(|S_i|\). In other terms, the mapping \(\sigma \mapsto \sigma^{-1}\) yields a bijection from the \((S_1, \ldots, S_k)\)-shuffles onto the set of all \(\sigma \in S_n\) such that \(D(\sigma) \subseteq \{|S_1|, |S_1| + |S_2|, \ldots, |S_1| + \cdots + |S_{k-1}|\}\), as asserted.

Recall that \(\mathcal{T}_n := \mathcal{T}_{[n]}\) (the graded component of \(\mathcal{T}\) of degree \([n]\)) is an algebra for the product \(\circ\), with linear basis consisting of the elements...
$1(s_1, \ldots, s_k)$ such that $S_1 \coprod \ldots \coprod S_k = [n]$ and $S_i \neq \emptyset$. There is a right action of $S_n$ on this basis, defined by:

$$1(s_1, \ldots, s_k) \cdot \sigma := 1(\sigma^{-1}(s_1), \ldots, \sigma^{-1}(s_k)),$$

which turns $T_n$ into a permutation module for $S_n$, by linearity.

**Proposition 24** The product $\circ$ on $T_n$ is equivariant with respect to the $S_n$-action.

The proposition follows from the definition of $\circ$ since, for any subsets $S$ and $T$ of $[n]$, we have:

$$\sigma^{-1}(T) \cap \sigma^{-1}(S) = \sigma^{-1}(T \cap S).$$

Note that $1(s_1, \ldots, s_k)$ and $1(t_1, \ldots, t_l)$ belong to the same $S_n$-orbit if and only if $(s_1, \ldots, s_k)$ and $(t_1, \ldots, t_l)$ are of the same type. In particular, the fixed space $F_n$ of $S_n$ in $T_n$ is linearly generated by the orbit sums

$$O_{n_1, \ldots, n_k} = \sum_{|S_i|=n_i} 1(s_1, \ldots, s_k),$$

where $n_1 + \cdots + n_k = n$. Furthermore, $S_n$-equivariance of the composition product on $T_n$ implies that $F_n$ is a subalgebra of $(T_n, \circ)$. The algebra $F_n$ is also isomorphic to Solomon’s classical descent algebra. This surprising observation is due to Bidigare (2, see also [4, 20]). The link to our considerations in the previous section is given by the following simple fact. Each linear generator $1_{n_1, \ldots, n_p}$ of $D_n$ is the sum of its graded components

$$1_{n_1, \ldots, n_k}(S) = \sum_{|S_i|=n_i, \prod S_i = S} 1(s_1, \ldots, s_k) \quad (|S| = n),$$

and we have:

**Proposition 25** The truncation mapping

$$1_{n_1, \ldots, n_k} \mapsto 1_{n_1, \ldots, n_k}([n]) = O_{n_1, \ldots, n_k}$$

is an isomorphism of algebras from $(D_n, \circ)$ onto $(F_n, \circ)$.
This is immediate from the multiplication rule \(1_{n_1,\ldots,n_k}(S) \circ 1_{m_1,\ldots,m_l}(T) = 0\) for all \(S \neq T\).

To conclude, observe that the stabilizer of \(1_{(S_1,\ldots,S_k)}\) in \(S_n\) is a parabolic subgroup (i.e. it is conjugated to a Young subgroup). If, in particular, \((S_1,\ldots,S_k)\) is an increasing partition of \([n]\) and \(n_i = |S_i|\), then the stabilizer of \(1_{(S_1,\ldots,S_k)}\) is the Young subgroup \(S_{n_1} \times \cdots \times S_{n_k} \subseteq S_n\). In other words, as an \(S_n\)-module, \(T_n\) is the direct sum of the Young modules arising from the action of \(S_n\) on the left cosets of \(S_{n_1} \times \cdots \times S_{n_k}\) in \(S_n\), \(n_1 + \cdots + n_k = n\). These cosets are well-known to be parametrized by \((S_1,\ldots,S_k)\)-shuffles, that is, by the summands of \(D^*_n\). More precisely, any permutation in \(S_n\) can be written uniquely as the composition of an element in the corresponding Young subgroup with a \((S_1,\ldots,S_k)\)-shuffle.

This property can be recovered easily using the twisted descent formalism: first, for any \(\sigma \in S_n\), the corresponding element of the Young subgroup is the permutation \(\beta\) defined by:

\[
1_{(S_1,\ldots,S_k)} \circ 1_\sigma = 1_\beta.
\]

The element \(\beta^{-1} \cdot \sigma\) is then a \((S_1,\ldots,S_k)\)-shuffle. Indeed, we have:

\[
1_{(S_1,\ldots,S_k)} \circ 1_{\beta^{-1} \cdot \sigma} = 1_{(S_1,\ldots,S_k)} \circ (1_\sigma \cdot \beta) = (1_{(S_1,\ldots,S_k)} \circ 1_\sigma) \cdot 1_\beta,
\]

by equivariance of the \(\circ\) product, and since \(\beta\) is an element of the Young subgroup stabilizing \(1_{(S_1,\ldots,S_k)}\),

\[
1_\beta \cdot \beta = 1_{\beta^{-1} \cdot \beta} = 1_{(\{1\},\ldots,\{n\})}.
\]

Now Corollary 22 implies the desired property.

It follows from these remarks, properties and identities, that there are straight connections between the multiple algebraic structure on \(T\) and the classical descent algebra \(D\). In particular, as already mentioned in the introduction, Solomon’s computations that involve fine computations with coset representatives of Young subgroups [22], can be rephrased and handled in the language of twisted descents in the same way that they were translated by Tits in the language of Coxeter complexes and buildings (see [22] pp.257 and [25]).
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