Aharony Dualities for 3d Theories with Adjoint Matter

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Abstract: We study Aharony dualities for 3d $N = 2$ gauge theories of classical gauge group with one adjoint and fundamental matters. We work out the 3d superconformal index for the dual pairs to find the perfect matchings. Along with it, we enumerate the independent monopole operators parametrizing the Coulomb branches and confirm the nonperturbative truncation of the chiral rings, consistent with the proposed dualities.
1. Introduction

Recently, there has been renewed interest in nonperturbative dualities between three dimensional theories such as mirror symmetry and Seiberg-like dualities. This is explained in part by the availability of sophisticated tools such as the partition function on $S^3$ and the superconformal index. Using these tools, one can give impressive evidence for various 3d dualities. Some of works in this area are [1]-[20]. One can also obtain the R-charge of the fields by maximizing the free energy of the theory of interest [21].

In this paper we continue this line of research and study Aharony dualities [22] for $\mathcal{N} = 2$ $d = 3$ gauge theories with classical gauge groups and matter both in the fundamental and the adjoint representations. Many of these proposed dualities can...
be motivated using the Hanany-Witten brane setup and brane moves passing through configurations with coincident NS5 branes [23, 24]. Similar dualities for $\mathcal{N} = 1 d = 4$ theories have been studied in the 90’s by Kutasov and collaborators [25, 26, 27] and others [28, 29, 30]. Previously in the Chern-Simons gauge theories, the Seiberg-like dualities with classical gauge groups and the tensor matters were worked out by [20]. On the other hand, from the examples of the theory with fundamentals, it’s known that Seiberg-like dualities for Chern-Simons gauge theories can be derived from the Aharony dualities for the gauge theories without the Chern-Simons terms. Chern-Simons terms are generated when fermions of the theories are integrated out. Thus for the gauge theories with tensor matters, it would be more desirable to work out the Aharony dualities for the gauge theories without Chern-Simons terms. In this paper, we explore various $\mathcal{N} = 2$ supersymmetric 3d gauge theories with classical groups $U/O/Sp$ and with one adjoint matter combined with fundamental representations and propose dual descriptions for them. Other tensor matters such as symmetric and antisymmetric representations can be easily incorporated following the method of the current paper. We give the various evidence for these dualities by working out the superconformal indices and analyzing chiral ring elements.

Important features of Aharony dualities with the adjoint is that such theory has generically multi-dimensional Coulomb branch so that we have to introduce multiple monopole operators in contrast with Chern-Simons gauge theories with tensor matters or Aharony dual pairs with just the fundamental representations. We propose the suitable form of monopole operators and subject this proposal to various tests.

The content of the paper is as follows. In section 2, we briefly review the superconformal index. In section 3, we handle Aharony dualities for $U(N)$ theories with one adjoint matter, matters in fundamental representations and with superpotential. We give evidences for the conjectured dualities by working out the superconformal index. In section 4 and 5 we work out the Aharony dualities for $O(N)$ and $Sp(2N)$ gauge theories with one adjoint and fundamental matters. In appendix, we work out the details of the superpotentials for the special values of $N$ and the flavor number $N_f$ and carry out the consistency checks. We explain the possible ambiguities in the determination of the superpotentials for some cases.

2. 3d superconformal index

Let us discuss the superconformal index for $\mathcal{N} = 2 d = 3$ superconformal field theories (SCFT). Here we closely follow [20]. The bosonic subgroup of the 3-d $\mathcal{N} = 2$ superconformal group is $SO(2,3) \times SO(2)$. There are three Cartan elements denoted by $\epsilon, j_3$ and $R$ which come from three factors $SO(2), SO(3)_{j_3}, SO(2)_R$ in the bosonic subgroup, respectively. The superconformal index for an $\mathcal{N} = 2 d = 3$
SCFT is defined as follows [38]:

\[ I(x, y) = \text{Tr}(-1)^F \exp(-\beta'\{Q, S\})x^{\epsilon + j_3} \prod_j y_j^{F_j} \]  

(2.1)

where \( Q \) is one supercharge with quantum numbers \( \epsilon = \frac{1}{2}, j_3 = -\frac{1}{2} \) and \( R = 1 \), and \( S = Q^t \). The trace is taken over the Hilbert space in the SCFT on \( \mathbb{R} \times S^2 \) (or equivalently over the space of local gauge-invariant operators on \( \mathbb{R}^3 \)). The operators \( S \) and \( Q \) satisfy the following anti-commutation relation:

\[ \{Q, S\} = \epsilon - R - j_3 := \Delta. \]  

(2.2)

As usual, only BPS states satisfying the bound \( \Delta = 0 \) contribute to the index, and therefore the index is independent of the parameter \( \beta' \). If we have additional conserved charges \( F_j \) commuting with the chosen supercharges \( (Q, S) \), we can turn on the associated chemical potentials \( y_j \), and then the index counts the algebraic number of BPS states weighted by their quantum numbers.

The superconformal index is exactly calculable using the localization technique [39, 40]. It can be written in the following form:

\[ I(x, y) = \sum_m \int da \frac{1}{|W_m|} e^{-S_{CS}^{(0)}(a, m) e^{ib_0(a, m)}} \prod_j y_j^{q_0(j)(m)} x^{\epsilon_0(m)} \exp \left[ \sum_{n=1}^{\infty} \frac{1}{n} f_{tot}(e^{ina}, y^n, x^n) \right] \]  

(2.3)

The origin of this formula is as follows. To compute the trace over the Hilbert space on \( S^2 \times \mathbb{R} \), we use path-integral on \( S^2 \times S^1 \) with suitable boundary conditions on the fields. The path-integral is evaluated using localization, which means that we have to sum or integrate over all BPS saddle points. The saddle points are spherically symmetric configurations on \( S^2 \times S^1 \) which are labeled by magnetic fluxes on \( S^2 \) and holonomy along \( S^1 \). The magnetic fluxes are denoted by \( \{m\} \) and take values in the cocharacter lattice of \( G \) (i.e. in \( \text{Hom}(U(1), T) \), where \( T \) is the maximal torus of \( G \)), while the eigenvalues of the holonomy are denoted by \( \{a\} \) and take values in \( T \). \( S_{CS}^{(0)}(a, m) \) is the classical action for the (monopole+holonomy) configuration on \( S^2 \times S^1 \), \( \epsilon_0(m) \) is the Casimir energy of the vacuum state on \( S^2 \) with magnetic flux \( m \), \( q_0(j)(m) \) is the \( F_j \)-charge of the vacuum state, and \( b_0(a, m) \) represents the contribution coming from the electric charge of the vacuum state. The last factor comes from taking the trace over a Fock space built on a particular vacuum state. \( |W_m| \) is the order of the Weyl group of the part of \( G \) which is left unbroken by the magnetic fluxes \( m \). These ingredients in the formula for the index are given by the following
explicit expressions:

\[ S_{CS}^{(0)}(a, m) = i \sum_{\rho \in R_F} k\rho(m)\rho(a), \tag{2.4} \]

\[ b_0(a, m) = -\frac{1}{2} \sum_{\Phi} \sum_{\rho \in R_\Phi} |\rho(m)|\rho(a), \]

\[ q_{0j}(m) = -\frac{1}{2} \sum_{\Phi} \sum_{\rho \in R_\Phi} |\rho(m)|F_j(\Phi), \]

\[ c_0(m) = \frac{1}{2} \sum_{\Phi} (1 - \Delta_\Phi) \sum_{\rho \in R_\Phi} |\rho(m)| - \frac{1}{2} \sum_{\alpha \in G} |\alpha(m)|, \]

\[ f_{\text{tot}}(e^{ia}, y, x) = f_{\text{vector}}(e^{ia}, x) + f_{\text{chiral}}(e^{ia}, y, x), \]

\[ f_{\text{vector}}(e^{ia}, x) = -\sum_{\alpha \in G} e^{i\alpha(a) |x|\alpha(m)}, \]

\[ f_{\text{chiral}}(e^{ia}, y, x) = \sum_{\rho \in R_\Phi} \sum_{\alpha \in G} \left[ e^{i\rho(a)} \prod_j y_j^{F_jx|\rho(m)|+\Delta_\Phi} \frac{1}{1 - x^2} - e^{-i\rho(a)} \prod_j y_j^{-F_jx|\rho(m)|+2-\Delta_\Phi} \frac{1}{1 - x^2} \right] \]

where \( \sum_{\rho \in R_F} \sum_{\Phi}, \sum_{\rho \in R_\Phi} \) and \( \sum_{\alpha \in G} \) represent summations over all fundamental weights of \( G \), all chiral multiplets, all weights of the representation \( R_\Phi \), and all roots of \( G \), respectively. For \( G = O(N) \) we need to carry out an additional \( \mathbb{Z}_2 \) projection corresponding to an element of \( O(N) \) whose determinant is \(-1\). This is explained in \[13\].

3. \( U(N) \) with an adjoint

For the \( \mathcal{N} = 2 \) \( U(N) \) gauge theory with an adjoint with \( N_f \) pairs of chiral and anti-chiral multiplets we propose the following dualities.

- Electric theory: \( U(N_c) \) gauge theory(without Chern-Simons term), \( N_f \) pairs of fundamental/anti-fundamental chiral superfields \( Q^a, \tilde{Q}_a \) (where \( a, b \) denote flavor indices), an adjoint superfield \( X \), and the superpotential \( W_e = \text{Tr} \: X^{n+1} \).

- Magnetic theory: \( U(nN_f - N_c) \) gauge theory(without Chern-Simons term), \( N_f \) pairs of fundamental/anti-fundamental chiral superfields \( q_a, \tilde{q}^a \), \( N_f \times N_f \) singlet superfields \( (M_j)_b^a, j = 0, \ldots, n - 1 \), \( 2n \) singlet superfields \( v_{0, \pm}, \ldots, v_{n-1, \pm} \), an adjoint superfield \( Y \), and a superpotential \( W_m = \text{Tr} \: Y^{n+1} + \sum_{j=0}^{n-1} M_j q^j Y^{n-1-j} + \sum_{i=0}^{n-1} (v_{i, +} \tilde{v}_{n-1-i, -} + v_{i, -} \tilde{v}_{n-1-i, +}) \).

where \( v_{0, \pm} \) and \( \tilde{v}_{0, \pm} \) are minimal bare monopoles of electric theory and magnetic theory, respectively. Those correspond to excitation of magnetic flux \((\pm1, 0, \ldots, 0)\). For the description of the monopole operators we had better use the operator state
correspondence to describe the operator as the corresponding state on \( \mathbb{R} \times S^2 \). When magnetic flux \((\pm 1, 0, \ldots, 0)\) is excited the gauge group \( U(N_c) \) is broken to \( U(1) \times U(N_c - 1) \). We denote the dressed monopole operator \( v_{i,\pm} \equiv \text{Tr}(v_{0,\pm}X^i), \)
\( i = 1, \ldots, n - 1 \) with the trace taken over \( U(1) \). More explanations of the monopole operators will follow shortly.

To motivate the number of independent monopole operators we consider the deformation of the superpotential \[ W = \sum_{j=0}^{n} \frac{s_j}{n + 1 - j} \text{Tr}X^{n+1-j}. \] (3.1)

For given \( \{s_j\} \) the superpotential has \( n \) distinct minima \( a_j \) related to the parameters in the superpotential

\[ W'(x) = \sum_{j=0}^{n} s_j x^{n-j} \equiv s_0 \prod_{j=1}^{n}(x - a_j). \] (3.2)

Vacua are labeled by sequences of integers \( (r_1, \cdots, r_n) \), where \( r_i \) is the number of eigenvalues of the matrix \( X \) residing in the \( l \)th minimum of the potential \( V = |W'(x)|^2 \). Thus, the set of \( \{r_j\} \) and \( \{a_j\} \) determines the expectation value of the adjoint field \( X \). When all \( \{a_j\} \) are distinct, the adjoint field is massive and the gauge group is broken:

\[ U(N_c) \to U(r_1) \times U(r_2) \times \cdots \times U(r_n). \] (3.3)

The theory splits in the infrared into \( n \) decoupled copies of \( \mathcal{N} = 2 \) \( U(r_i) \) theory with \( N_f \) flavors of quarks. In 3-dimensions, each \( U(r_i) \) has one pair of monopole operators, parametrizing the Coulomb branch. Thus the original theory should have at least \( n \) pairs of monopole operators. It turns out that the original theory has precisely \( n \) pairs of monopole operators.

The superpotential \( W = \text{Tr}X^{n+1} \) truncates the chiral ring, i.e., the operators involving \( X^j, j \geq n \) do not exist in the chiral ring. Thus the chiral ring generators are operators \( \text{Tr}X^j, M_j \) and \( v_{j,\pm} \) where \( j = 0, \ldots, n - 1 \). Due to superpotential in magnetic theory the operators \( \tilde{q}Y^{n-1-j}q \) and \( \tilde{v}_{i,\pm} \) are not chiral ring elements. Also the superpotential appearing in the above is of the generic form and for special values of \( N_c, N_f \) there will be additional superpotentials in the magnetic side. We will explicitly work out the complete superpotentials for several simple cases in appendix A.

We denote the superconformal R-charge of \( Q \) by \( R(Q) = r \). Due to the superpotential R-charges of both \( X \) and \( Y \) are \( \frac{2}{n+1} \). Global charges of minimal bare monopole operators are determined by counting the fermion zero modes \[ \text{[31]} \]. The quantum corrected global charged are given as follows.
Note that for \( n = 1 \), one can integrate out \( X \) and \( Y \), and the conjectured duality reduces to Aharony duality with only fundamental and anti-fundamental fields and minimal bare monopoles.

### 3.1 Chiral Ring elements and Monopole operators

The index formula we use counts the BPS states of the theory which is radially quantized and deformed to a weak coupling. The Hilbert space of the deformed theory is the direct sum of the states with different magnetic flux. Each vacua consists of bare monopole states and other BPS states are obtained by acting the creation operators of the fields on the bare monopole states. We would like to look for states corresponding to the chiral ring operators in the deformed theory. The chiral ring elements are BPS scalar states and the BPS condition is quite restrictive \([12]\). A bare monopole state \( |n_1, \ldots, n_{N_c}\rangle \) denote the background magnetic flux \((n_1, \ldots, n_{N_c})\).

The squarks \( Q^i \) and \( \tilde{Q}^i \) with gauge index \( i \) picks up anomalous spin \( |n_i|/2 \) when there is a non-zero magnetic charge \( n_i \) \([32, 33]\). So BPS scalar states are formed by a bare monopole with free squark modes \( Q^i, \tilde{Q}^i \) with gauge indices carrying no magnetic charges. Bare monopole states excited with the adjoint fields can also be BPS scalar states. The gauge group is broken to \( U(N_a) \times \cdots \times U(N_z) \subset U(N_c) \) by the magnetic flux. Then gauge invariant scalar states of adjoint fields come only from each factor of \( U(N_i) \) gauge group.

We list the counterparts of the chiral ring operators in the deformed theory by comparing quantum numbers of the BPS scalar states. We follow closely the argument of \([12]\). The bare monopole operator \( v_{0,+}^m v_{0,-}^n \) corresponds to \( |m, -n, 0, \ldots, 0\rangle \) where magnetic flux is \((m, -n, 0, \ldots, 0)\). With the bare monopole state \( |m, -n, 0, \ldots, 0\rangle \) the gauge group is broken to \( U(1)_+ \times U(1)_- \times U(N_c - 2) \) where the subscript of the gauge group denotes the sign of magnetic flux. So the chiral ring operator \( v_{0,+}^m v_{0,-}^n M_b^a \) corresponds to \( Q^a \tilde{Q}_b|m, -n, 0, \ldots, 0\rangle \) where the contracted gauge indices of squarks are in unbroken \( U(N_c - 2) \) gauge group. Note that the deformed theory does not
have a state corresponding to the operator $v_{0,+}^n v_{0,-}^n M_6^n$ if $N_c \leq 2$.

Now let us discuss the monopole operators involving the adjoint field. Let’s consider the background with a magnetic flux $|\pm 1, 0, \ldots\rangle$. In the state formalism the gauge group is broken to $U(1) \times U(N_c - 1)$. With respect to this unbroken subgroup, we define $v_{i,\pm} = \text{Tr}(v_{0,\pm} X^i)$ where $\text{Tr}$ is taken over $U(1)$ and propose that $v_{i,\pm}$, $i = 0, \ldots, N - 1$ describe $N$ dimensional coulomb branch for sufficiently large $n$. It means that $v_{i,\pm}$ are independent operators. In the magnetic side $v_{i,\pm}$ are additional singlet fields.

We would like to check how this is realized in the radially quantised and weakly deformed theory. Once magnetic flux $|\pm 1, 0, \ldots\rangle$ is turned on, scalar excitations of adjoint field is given by
\[
X = \begin{pmatrix} X_{11} & 0 \\ 0 & X' \end{pmatrix}
\tag{3.4}
\]
where $X'$ is an adjoint field of $U(N_c - 1)$ unbroken gauge group. Thus for the excitation of an adjoint field there are two independent states,
\[
v_{1,\pm} = X_{11}|\pm 1, 0, \ldots\rangle, \quad \text{Tr}X'|\pm 1, 0, \ldots\rangle. \tag{3.5}
\]
Note that we can also turn on $X_{11}$ since this does not carry charge under $U(1)$ so that with the excitation of $X_{11}$, one can satisfy the Gauss constraint in the state formalism. If we consider the operator product $v_{0,\pm} \text{Tr}X$, this is expected to have the nonzero overlap with $(X_{11} + \text{Tr}X')|\pm 1, 0, \ldots\rangle = \text{Tr}X|\pm 1, 0, \ldots\rangle$. When we consider the chiral ring structures, the natural operators are $v_{1,\pm}, v_{0,\pm} \text{Tr}X$. We saw that these elements can be generated by the basis elements of the monopole operators appearing in eq.\textbf{(3.5)}. On the magnetic side, these operators are mapped to $v_{1,\pm}, v_{0,\pm} \text{Tr} Y$.

Likewise, for the excitation of two adjoint fields there are four independent pairs of operators in the electric theory, $X_{11}^2|\pm 1, 0, \ldots\rangle$, $X_{11}\text{Tr}X'|\pm 1, 0, \ldots\rangle$, $(\text{Tr}X')^2|\pm 1, 0, \ldots\rangle$ and $\text{Tr}X'^2|\pm 1, 0, \ldots\rangle$. From these chiral ring elements $v_{0,\pm} \text{Tr} X^2$, $v_{0,\pm} (\text{Tr} X)^2$, $v_{1,\pm} \text{Tr} X$ and $v_{2,\pm}$ can be generated.

With the broken gauge group $U(1)_+ \times U(1)_- \times U(N_c - 2)$, we can have the following form of the adjoint field,\[
X = \begin{pmatrix} X_{11} & 0 & 0 \\ 0 & X_{22} & 0 \\ 0 & 0 & X' \end{pmatrix}
\tag{3.6}
\]

\footnote{Precisely speaking the operator product $v_{0,+,0,-}^m v_{0,-}^n M_6^n$ is different from the operator corresponding to the state $Q^a \bar{Q}_b|m, -n, 0, \ldots, 0\rangle$. However we expect naturally the nonzero overlap between the state $v_{0,+,0,-}^m v_{0,-}^n M_6^n$ and the operator corresponding to the state $Q^a \bar{Q}_b|m, -n, 0, \ldots, 0\rangle$. With this assumption when we count the number of operators such as $v_{0,+,0,-}^m v_{0,-}^n M_6^n$, we in fact count the operators corresponding to $Q^a \bar{Q}_b|m, -n, 0, \ldots, 0\rangle$.}
where $X'$ is an adjoint field of $U(N_c-2)$ unbroken gauge group. Thus one can consider the monopole operators corresponding to the states $X_{11}^i X_{22}^j \text{Tr} X'^k|m,-n,0,\ldots,0\rangle$. Later when we count the number of monopole operators we count these kinds of corresponding states on $\mathbb{R} \times S^2$.

With the definition $v_{i,\pm} = X_{11}^i|\pm 1,0,\ldots\rangle$, one can easily see that some of the monopole operators can be dependent if the gauge group is small enough. For example, if we consider the $U(2)$ gauge theory, we have the characteristic equation for

$$X^2 - X\text{Tr}X + \frac{(\text{Tr}X)^2 - \text{Tr}X^2}{2}I = 0 \quad (3.7)$$

where $I$ is the identity $2 \times 2$ matrix. From this one obtains

$$v_{2\pm} - v_{1\pm}\text{Tr}X + v_{0\pm} \frac{(\text{Tr}X)^2 - \text{Tr}X^2}{2} = 0. \quad (3.8)$$

Thus $v_2$ is expressed in terms of $v_0, v_1$. Similarly for $U(1)$ case, we have $v_1 = v_0\text{Tr}X$.

Let us reconsider a part of the magnetic superpotential

$$\sum_{i=0}^{n-1} (v_{i,\pm}\bar{v}_{n-1-i,-} + v_{i,-}\bar{v}_{n-1-i,\pm}) + \cdots. \quad (3.9)$$

and consider the possibility to use the different definition of $v_i, \tilde{v}_i$. For example we can consider $v_{i,\pm} = \text{Tr}X'^i|\pm 1,0,\ldots\rangle$ and similar definition of $\tilde{v}_{i,\pm}$. However this leads to the same theory. The equation of motion obtained by varying $v_i$ is given by $\tilde{v}_{j,\pm} = 0, \ j = 0 \cdots n - 1$. Thus the resulting equations of motion of $\tilde{v}_i$ is independent of the definition of $\tilde{v}_i$. This also holds quantum mechanically since the above magnetic superpotential gives rise to the delta functional $\delta \tilde{v}_{j,\pm}$. Also if we have small gauge group in either electric or magnetic side, we still write the relevant magnetic superpotential as in eq. (3.3) but understand that not all of the monopole operators are independent and they have similar relations like eq. (3.8) so that we can rewrite superpotential in terms of independent monopole operators.

Now we describe the matching of chiral ring generators of electric and magnetic theories. Duality is supposed to map chiral ring generators as follows.

$$\text{Tr} X^i \leftrightarrow \text{Tr} Y^i \quad (3.10)$$

$$Q^a X^j \bar{Q}_b \leftrightarrow (M_j)^a_b$$

$$\text{Tr} (v_{0,\pm} X^i) \leftrightarrow v_{i,\pm}. \quad (3.8)$$

The generalized meson $Q^a X^j \bar{Q}_b$ and the monopole operators $\text{Tr} (v_{0,\pm} X^i)$ in electric theory are mapped to the gauge singlet operators in magnetic theory.

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In a similar spirit of [22], one should not set $v_i = 0$ since $\tilde{v}_i$ is singular at the origin of the moduli space.
The chiral rings are constrained by characteristic equations of adjoint $X$ and $Y$. Classically, there are $N_c$ independent operators $\text{Tr} X^i$, $i = 1, \ldots, N_c$ due to characteristic equation of $X$ which is in $U(N_c)$ adjoint representation. With a superpotential $W = \text{Tr} X^{n+1}$ there are $a$ independent operators $\text{Tr} X^i$, $i = 0, \ldots, a$ where $a = \min(n-1, N_c)$. When the ranks of the gauge group of two theories are different $N_c \neq N'_c$ the numbers of independent $\text{Tr} X^j$ operators of two theories can be different.

For $N_c \leq n-1$ the mesonic operator $M_{N_c}$ of $U(N_c)$ theory can be written in terms of operators $\text{Tr} X^i$ and $M_i$ where $i \leq N_c-1$. For instances, there are classical relations $QX\bar{Q} = \bar{Q}\bar{Q}\text{Tr} X$ for $U(1)$ theory and $QX^2\bar{Q} = QX\bar{Q}\text{Tr} X - \frac{1}{2}Q\bar{Q}((\text{Tr} X)^2 - \text{Tr} X^2)$ for $U(2)$ theory. However there are always $n$ gauge singlet operators $M_i$ at the dual magnetic gauge theory. Thus classically there are different number of chiral ring generators.

In 4d analogue of the duality discussed here, it was proposed that there are additional relations in the chiral ring of the theory coming from characteristic equation of dual side. The trivial characteristic equation of one side becomes non-trivial quantum constraint in the other side. In the 3d duality with Chern-Simons term [20], it was explicitly checked that there are monopole operators which cancel the redundant operators which are present in one side but not in the other side. Thus chiral ring of dual pair turned out to be same.

The duality in this paper shows different mechanism which depend on gauge groups. If the gauge group of electric side is smaller than that of magnetic side, $N_c \leq N'_c$, the number of (classical) chiral ring generators of electric side is less than magnetic side. But the redundant chiral ring generators of magnetic side are cancelled by some monopole operators, similar to what happens in the dualities of Chern-Simons gauge theories with adjoint matter.

On the other hand, if $N_c > N'_c$, the electric theory seems to have more chiral ring generators than magnetic theory. But some non-trivial relation of monopole operators reduce the number of state so the chiral ring is again the same. We will show this explicitly by working out the index.

### 3.2 Negative R-charge

As noted in the paper by Dimofte, Gaiotto and Gukov [34] the quantity $E + j_3$ should be nonnegative in superconformal theories whose R-symmetry is not accidental. Some theories of interest have monopole operators whose $\epsilon + j_3$ is negative. The superconformal index takes the form of, $I(x) = \text{Tr} \left[(-1)^F x^{\epsilon + j_3}\right]$ where chemical potentials are ignored. If the quantity $\epsilon + j_3$ of some field is negative the superconformal index diverges. And the corresponding theory cannot be a superconformal theory.

Let us describe when such phenomena happen. The UV R-symmetry is mixed
with global $U(1)_A$ symmetry in IR so we put R-charge of squark of electric side as a free parameter $r$. Then R-charges of other fields can be written in terms of $r$. Nontrivial constraints come from two gauge invariant operators, mesons $M_j$, $\tilde{M}_j \equiv qY^{-1}\bar{q}$ and monopole operators $v_{j\pm}, \tilde{v}_j$. By requiring all these operators have positive conformal dimension, we obtain the constraints

$$0 < 2r + \frac{2j}{n+1} < 2, \quad 0 < -N_f r + N_f - \frac{2}{n+1}(N_c - 1) + \frac{2j}{n+1} < 2 \quad (3.11)$$

where $j = 0, \ldots, n-1$. Conditions for $r$ to have a solution satisfying both inequalities reduce to as follows.

$$|N_c - N'_c| < N_f + 2 \quad (3.12)$$

where $N'_c = nN_f - N_c$. We checked that whenever theories have values $(n, N_f, N_c)$ which do not satisfy the inequality, their superconformal index diverges.

3.3 The result of the index computations

We worked out the superconformal index for several low values of $n, N_f$ and $N_c$. The results are displayed in the Table 4.

| $n = 2, (N_f, N_c)$ | Electric $U(N_c)$ | Magnetic $U(2N_f - N_c)$ | Index |
|-------------------|------------------|----------------------|-------|
| (1,1)             | $U(1)$           | $U(1)$               | $1 + x^{2/3} - 2x^2 - x^{8/3} + 2x^{4-4r} + (2 + 2x^{2/3})x^{3-3r} + (2 + 2x^{2/3})x^{2-2r} + (2 + 2x^{2/3})x^{1-r} + (1 + x^{2/3})x^{2r} + x^{4r}$ |
| (2,1)             | $U(1)$           | $U(3)$               | $1 + x^{2/3} - 8x^2 + (2 + 2x^{2/3})x^{2-2r} + (4 + 4x^{2/3})x^{2r}$ |
| (1,2)             | $U(2)$           | $U(0)$               | $1 + 3x^{2/3} + 7x^{4/3} + 6x^2 + (1 + 4x^{2/3})x^4 + x^{3r}(2x^{1/3} + 6x) + x^{-r}(2x^{1/3} + 6x + 8x^{5/3}) + x^r(2x^{1/3} + (6 + 8x^{2/3})x) + x^{2r}(1 + 4x^{2/3} + 7x^{4/3}) + x^{-2r}(3x^{2/3} + 8x^{4/3} + 11x^2)$ |
| (2,2)             | $U(2)$           | $U(2)$               | $1 + x^{2/3} + 9x^{4/3} + 8x^2 + 11x^{8/3} + 10x^{4r} + x^{2r}(4 + 8x^{2/3} + 22x^{4/3}) + x^{-2r}(2x^{4/3} + 4x^{2} + 10x^{8/3} + 2x^{10/3}) + x^{-4r}(3x^{8/3} + 6x^{10/3} + 13x^4)$ |
| (3,2)             | $U(2)$           | $U(4)$               | $1 + x^{2/3} + x^{4/3} - 18x^2 + 18x^{8/3} + (9 + 18x^{2/3})x^{2r} + x^{-3r}(2x^{7/3} + 4x^{3})$ |
| (2,3)             | $U(3)$           | $U(1)$               | $1 + 9x^{2/3} + 54x^{4/3} + 186x^2 + (10 + 66x^{2/3})x^4 + 20x^{6r} + x^{2r}(4 + 28x^{2/3} + 134x^{4/3}) + x^{-2r}(2x^{2/3} + 16x^{4/3} + 78x^2) + x^{-4r}(3x^{1/3} + (23 + 104x^{2/3})x^3)$ |
| $n = 3, (N_f, N_c)$ | $U(N_c)$ | $U(3N_f - N_c)$ | Index |
|---------------------|-------------------------------|-------------------|--------|
| (3,3)               | $U(3)$                        | $U(3)$            | $1 + x^{2/3} + x^{4/3} - 17x^2 + 3x^{5/3} - 6r + 18x^{5/3} - (9 + 18x^{2/3})x^2r + 45x^{4r} + x^{-3r}(2x^{5/3} + 4x^{7/3})$ |
| (2,4)               | $U(4)$                        | $U(0)$            | Divergent |
| (3,4)               | $U(4)$                        | $U(2)$            | $1 + x^{2/3} + x^{4/3} + 477x^2 + 135x^{2-2r} + (18 + 54x^{2/3})x^{1-r} + (45 + 126x^{2/3})x^{4r} + (90 + 342x^{2/3})x^{1+r} + 330x^{1+3r} + x^{2r}(9 + 18x^{2/3} + 18x^{4/3}) + x^{-3r}((2 + 4x^{2/3})x + 4x^{7/3})$ |
| (4,4)               | $U(4)$                        | $U(4)$            | $1 + x^{2/3} + x^{4/3} + 241x^2 + 3x^{4-8r} + (2 + 4x^{2/3})x^{2-4r} + 32x^{2-2r} + (16 + 32x^{2/3})x^{2r} + 136x^{4r}$ |
| $n = 4, (N_f, N_c)$ | $U(N_c)$ | $U(4N_f - N_c)$ | Index |
| (1,1)               | $U(1)$                        | $U(2)$            | $1 + \sqrt{x} + x - 2x^2 + (2 + 2\sqrt{x})x^{2-2r} + x^{4r} + x^{2r}(1 + \sqrt{x} + x) + x^{-r}(2x + 2x^{3/2} + 2x^2)$ |
| (1,2)               | $U(2)$                        | $U(1)$            | $1 + \sqrt{x} + 4x + 5x^{3/2} + 7x^2 + x^{4r} + 2x^{4r} + x^{2r}(1 + 2\sqrt{x} + 5x) + x^{r}(2\sqrt{x} + (4 + 8\sqrt{x})x) + x^{-r}(2\sqrt{x} + 4x + 8x^{3/2} + 8x^2) + x^{-2r}((3 + 6\sqrt{x})x + (13 + 12\sqrt{x})x^2)$ |
| (2,2)               | $U(2)$                        | $U(4)$            | $1 + \sqrt{x} + 2x + 9x^{3/2} + 9x^2 + 3x^{3-4r} + 10x^{4r} + x^{2r}(4 + 8\sqrt{x} + 12x) + x^{-2r}(2x^{3/2} + (4 + 6\sqrt{x})x^2)$ |
| (1,3)               | $U(3)$                        | $U(0)$            | Divergent |
| (2,3)               | $U(3)$                        | $U(3)$            | $1 + \sqrt{x} + 10x + 26x^{3/2} + 71x^2 + 5x^{4-8r} + (4 + 10\sqrt{x})x^{3-6r} + 10x^{4r} + x^{2r}(4 + 8\sqrt{x} + 36x) + x^{-2r}(2x + 4x^{3/2} + 20x^2 + 44x^{5/2}) + x^{-4r}(3 + 7\sqrt{x})x^2 + 32x^3$ |
| (2,4)               | $U(4)$                        | $U(2)$            | $1 + 9\sqrt{x} + 56x + 255x^{3/2} + 940x^2 + x^{4r}(10 + 66\sqrt{x} + 327x) + x^{2r}(4 + 28\sqrt{x} + 148x) + 610x^{3/2} + x^{-2r}(2\sqrt{x} + 16x + 88x^{3/2} + 376x^2) + x^{-4r}(3x + 23x^{3/2} + 123x^2 + 503x^{5/2})$ |

- $\overline{\text{Index}}$
Table 2: Superconformal index for $U(N)$ gauge theories with an adjoint. Bold face letters denote the chiral ring elements discussed in the main text.

First note that some of the indices are divergent. As mentioned before, this is due to some operator which has negative conformal dimension. These theories can not be studied using superconformal index.

The chiral ring generators give identifiable contribution to the index. The generator $\text{Tr} \ X^j$ contributes a term $x^{\frac{r}{2} - j}$ to the index. The meson operators $M_j$ contribute $N_f^2 x^{2r+\frac{3}{2} j}$. The monopole operators $v_{j,\pm}$ contribute $2 x^{-N_f r + N_f - \frac{1}{2}(N_c-1-j)}$. All element of chiral ring, products of generators, also contribute to the index unless they are $Q$-exact. For example, $M_0 \text{Tr} X$ contributes a term $N_f^2 x^{2r+\frac{3}{2}}$ and $M_1$ has the same contribution. Thus for $N_c > 1$ the total contribution of the operators $M_1$ and $M_0 \text{Tr} X$ is $2 N_f^2 x^{2r+\frac{3}{2}}$.

Now let’s discuss the details of the chiral ring elements for several cases in conjunction with the index computation.

A. $n = 2$

For $n = 2$ the chiral ring generators contribute to the index as follows: $\text{Tr} \ X \sim x^{\frac{r}{2}}$, $(M_1, M_0 \text{Tr} X) \sim 2 N_f^2 x^{2r+\frac{3}{2}}$, $v_{0,\pm} \sim 2 x^{N_f - \frac{1}{2} N_c + \frac{3}{2} - N_f r}$, and $(v_{1,\pm}, v_{0,\pm} \text{Tr} X) \sim 4 x^{N_f - \frac{3}{4} N_c + \frac{3}{2} - N_f r}$. The bold face part of indices agree with the contributions of chiral ring generators specified above if theories have large gauge group rank $N_c > n - 1$. Those are shown in dualities $U(2)_E-U(2)_M$, $U(2)_E-U(4)_M$, $U(3)_E-U(3)_M$, $U(4)_E-U(2)_M$ and $U(4)_E-U(4)_M$.

When gauge group rank is small, $N_c \leq n - 1$, the number of generators of both theories seems at the classical level. Thus the chiral ring should receive nonperturbative effect. Let us consider two cases, $N_c \leq n N_f - N_c$ and $N_c > n N_f - N_c$. First consider $(n, N_f, N_c) = (2, 1, 1)$ $U(1)_E-U(1)_M$. For this case we have $U(1)$
gauge theory in the electric side. Hence \( M_1 \) and \( v_{1,\pm} \) are not independent state. Only \( M_0 \text{Tr } X \) and \( v_{0,\pm} \text{Tr } X \) state contribute to the index as shown in Table [2]. The corresponding terms are \( N_f^2 2x^{2r+\frac{d}{2}} \) and \( 2x^{N_f - \frac{d}{2}N_c + \frac{d}{2} - N_f r} \) respectively. However, at magnetic side, \( M_1 \) and \( v_{1,\pm} \) are present as singlets in addition to operators \( M_0 \text{Tr } Y \) and \( v_0 \text{Tr } Y \). Thus the operators \( M_1 \) and \( v_{1,\pm} \) must be paired up with monopole operators and disappear.

For \( (n, N_f, N_c) = (2, 1, 1) \) \( U(1)_M \) theory it is explicitly checked that the \( M_1 \) operator is canceled by one of operators \( \psi_{v_{1,\pm}} \tilde{v}_{0,\pm} \) and \( \psi_{v_{1,-}} \tilde{v}_{0,\pm} \). Here \( \psi_{v_{1,\pm}} \) is the fermionic partner of \( v_{1,\pm} \) since \( v_{1,\pm} \) is introduced as a singlet in the magnetic theory. For generic \( U(N) \) gauge group the operators \( (\tilde{v}_{0,+}, v_{0,-}, \psi_{v_{1,+}} \tilde{v}_{0,+}, \psi_{v_{1,+}} \tilde{v}_{0,-}, \psi_{v_{1,\pm}} \psi_{v_{1,-}}) \) are cancelled due to the superpotential \( v_{1,\pm} \tilde{v}_{0,\pm} \). But for \( U(1) \) gauge group the \( \tilde{v}_{0,+} \) state does not exist because \( \tilde{v}_{0,+} \) and \( \tilde{v}_{0,-} \) arise from monopole flux \( m = 1 \) and \( m = -1 \) respectively so that they can be paired up. Thus the nonperturbative truncation occurs to the \( M_1 \) operator.

Similarly, \( v_{1,\pm} \) operator is canceled by a \( \psi_{M_1} \tilde{v}_{0,\pm} \). In generic case the \( \psi_{M_1} \tilde{v}_{0,\pm} \) operator is supposed to pair up with a \( qq \tilde{v}_{0,\pm} \) operator due to the superpotential \( M_1 q \tilde{q} \). However \( U(1) \) gauge theory does not have scalar BPS state of the form \( qq \tilde{v}_{0,\pm} \). Thus due to the absence of the state \( qq \tilde{v}_{0,\pm} \), the state \( v_{1,\pm} \) is paired up with the state \( \psi_{M_1} \tilde{v}_{0,\pm} \).

In short, the size of gauge group of the electric theory restricts the number of chiral ring generator and the redundant operator of the magnetic theory is truncated by a monopole operator.

Next let us consider \( N_c > nN_f - N_c \) cases such as \( (n, N_f, N_c) = (2, 1, 2) \) \( U(2)_E - U(0)_M \) and \( (n, N_f, N_c) = (2, 2, 3) \) \( U(3)_E - U(1)_M \).

The \( U(0)_M \) magnetic theory does not have the adjoint field, while \( \text{Tr } X \) is an independent operator in \( U(2)_E \) electric theory. Furthermore, there is no monopole operator which cancel out the \( \text{Tr } X \) operator in \( U(2)_E \) electric theory. Thus the duality requires new counterpart of \( \text{Tr } X \) in the magnetic side. A term of index which corresponds to the energy level of \( \text{Tr } X \) is \( 3x^{2\frac{d}{2}} \). The three states are \( (\text{Tr } X, M_0 v_{0,+}^2, M_0 v_{0,-}^2) \) in electric theory. The index of magnetic theory has the same terms which comes from \( (M_0 v_{0,+} v_{0,-}, M_0 v_{0,+}^2, M_0 v_{0,-}^2) \). Thus the operator \( \text{Tr } X \) is mapped to the operator \( M_0 v_{0,+} v_{0,-} \). Note that the \( M_0 v_{0,+} v_{0,-} \) in the electric side has higher energy.

Other terms in the index also show the operator matching \( \text{Tr } X \leftrightarrow M_0 v_{0,+} v_{0,-} \). We define \( (l + 1) v_{0}^l = (v_{0,+}^l, v_{0,+}^{l-1} v_{0,-}^1, \ldots, v_{0,-}^1, v_{0,-}^0) \). The term \( 4x^{2r+\frac{d}{2}} \) comes from \( (M_1, M_0 \text{Tr } X, M_0^2 v_{0,\pm}^2)_{E} \) and \( (M_1, M_0^2 \cdot 3 v_{0,\pm}^2)_{M} \) and the term \( 6x^{1-r} \) comes from \( (v_{1,\pm}, v_0, \text{Tr } X, M_0 v_{0,\pm}^3)_{E} \) and \( (v_{1,\pm}, M_0 \cdot 4 v_{0,\pm}^3)_{M} \). In both cases, the operators are well matched by mapping \( \text{Tr } X \) to \( M_0 v_{0,+} v_{0,-} \).

Next, let us consider \( (n, N_f, N_c) = (2, 2, 3) \) \( U(3)_E - U(1)_M \) case. The operator \( (\text{Tr } X)^2 \) exists in the \( U(3)_E \) electric side while the operator \( (\text{Tr } Y)^2 \) become \( \text{Tr } Y^2 \) in
the $U(1)$ magnetic theory so it is truncated by superpotential. Thus at first sight two theories have different chiral ring. There are 10 states which have the same quantum numbers as $(\text{Tr } X)^2$ among $54x^4$. The operators are $((\text{Tr } X)^2, (4H_2-1)M_0^2 \cdot v_{0,+} v_{0,-})_E$ in the electric theory and $(4H_2M_0^2 \cdot v_{0,+} v_{0,-})_M$ in the magnetic theory where the coefficients of $M_j$ indicate the number of independent meson operators and $mH_1 = m+i-1C_l = \frac{(m+i-1)!}{i(m-1)!}$ is the combination with repetition. In electric theory the coefficient of $M_0^2$ comes from $M_0^{ab}M_0^{cd} = Q_i^a \tilde{Q}_j^b Q_j^c \tilde{Q}_j^d$ where the contracted gauge indices run over the gauge group corresponding to zero magnetic flux. In generic cases there are $4H_2 M_0^2$ operators. But in the presence of monopole flux $(1, -1, 0) \sim v_{0,+}v_{0,-}$, there is a constraint, $v_{0,+}v_{0,-} \text{det } M_0 = 0$, because the gauge group corresponding to zero magnetic flux is $U(1)$. i.e. the matrix $2 \times 2$ matrix $M_0$ have rank 1 in the presence of the monopole flux $(1, -1, 0)$. Thus there are $(4H_2-1)M_0^2 \cdot v_{0,+}v_{0,-}$ operators in electric theory. Therefore the operator $(\text{Tr } X)^2$ should be mapped to either $M^{11}M^{22}v_{0,+}v_{0,-}$ or $M^{12}M^{21}v_{0,+}v_{0,-}$. In other words a mapping of chiral ring generators make sense through two different constraints, $v_{0,+}v_{0,-} \text{det } M_0 = 0$ in electric theory and $(\text{Tr } Y)^2 - \text{Tr } Y^2 = 0$ in magnetic theory.

B. $n = 3$

For $n = 3$, there are more operators at each energy level of chiral ring generators:

$\text{Tr } X \sim x^{\frac{1}{2}}, (\text{Tr } X^2, (\text{Tr } X)^2) \sim 2x, M_0 \sim N_f^2 x^{2r}, (M_1, M_0 \text{Tr } X) \sim 2N_f^2 x^{2r+\frac{1}{2}}, (M_2, M_1 \text{Tr } X, M_0 \text{Tr } X^2, M_0(\text{Tr } X)^2) \sim 4N_f^2 x^{2r+1}, v_{0,\pm} \sim 2x^{N_f-\frac{1}{2}N_c+\frac{1}{2}-N_f}, (v_{1,\pm}, v_{0,\pm} \text{Tr } X) \sim 4x^{N_f-\frac{1}{2}N_c+1-N_f}$, and $(v_{2,\pm}, v_{1,\pm} \text{Tr } X, v_0,\pm \text{Tr } X^2, v_0,\pm(\text{Tr } X)^2) \sim 8x^{N_f-\frac{1}{2}N_c+\frac{3}{2}-N_f}$.

For $U(1)$ electric theory there are no independent $\text{Tr } X^{j+1}, M_j$ and $v_{j, j > 0}$ operators as shown in $(n, N_f, N_c) = (3, 1, 1), U(1)_E-U(2)_M$ example. There are nonperturbative truncations of chiral ring at magnetic side as described in $n = 2$ examples. The operators $\text{Tr } Y^2, M_1, M_2, v_{1,\pm}, v_{2,\pm}$ are paired up with monopole operators and disappear as shown in Table 2.

The $(n, N_f, N_c) = (3, 1, 2), U(2)_E-U(1)_M$ example shows the new mapping of chiral ring. In the $U(2)$ electric theory, $\text{Tr } X^2$ and $(\text{Tr } X)^2$ operators are independent but in the $U(1)$ magnetic theory there is a classical relation, $\text{Tr } Y^2 = (\text{Tr } Y)^2$. The new mapping of chiral ring is seen at $4x$ term of index. The $4x$ term comes from $(\text{Tr } X^2, (\text{Tr } X)^2, M_0v_{0,+}^2, M_0v_{0,-}^2)$ in electric side and $((\text{Tr } Y)^2, M_0v_{0,+}v_{0,-}, M_0v_{0,+}^2, M_0v_{0,-}^2)$ in magnetic side. $U(2)$ electric theory does not have the state $M_0v_{0,+}v_{0,-}$. Thus the new mapping is clear, $\text{Tr } X^2 \leftrightarrow M_0v_{0,+}v_{0,-}$.

4. $O(N)$ with an adjoint

For $\mathcal{N} = 2$ $O(N)$ gauge theory with an adjoint, we propose the following dualities.

- Electric theory: $O(N_c)$ gauge theory with $N_f$ fundamental chiral multiplets $Q^a$
with \( a = 1, \ldots, N_f \), and an adjoint chiral multiplet \( X \) with a superpotential \( W_e = \text{Tr} X^{2(n+1)} \).

- Magnetic theory: \( O((2n + 1)N_f + 2 - N_c) \) gauge theory with \( N_f \) fundamental chiral multiplets \( q_a \) with \( a = 1, \ldots, N_f \), an adjoint chiral multiplet \( Y \), color-singlet chiral multiplets \( M_{j}^{ab} \), \( j = 0, \ldots, 2n \), \( a, b = 1, \ldots, N_f \) which are symmetric (resp. anti-symmetric) for even (resp. odd) \( j \) and \( v_j, \ j = 0, \ldots, 2n \), and a superpotential \( W_m = \text{Tr} Y^{2(n+1)} + \sum_{j=0}^{2n} M_{j}^{ab} q_a Y^{2n-j} q_b + \sum_{j=0}^{2n} v_j \tilde{v}_{2n-j} \).

Note that for \( n = 0 \) the above duality is equivalent to the duality considered in [9]. The monopole operator of orthogonal gauge group is described in [19]. Due to superpotential all operator containing \( X_j \) (or \( Y_j \), \( j > 2n \)) are \( Q \)-exact. In magnetic theory the operators \( q_a Y_j q_b \) and \( v_j, \ j = 0, \ldots, 2n \) are \( Q \)-exact. Thus the classical chiral ring generators map as follows:

\[
\begin{align*}
\text{Tr} \, X^2 &\leftrightarrow \text{Tr} \, Y^2, \quad i = 1, \ldots, n \\
Q^a X^j Q^b &\leftrightarrow M_{j}^{ab}, \quad j = 0, \ldots, 2n \\
\text{Tr} \, (v_0 X^j) &\leftrightarrow v_j, \quad j = 0, \ldots, 2n.
\end{align*}
\]

The third equation is schematic and the precise meaning of \( \text{Tr} \) will be explained shortly.

The quantum corrected global charges are:

| \( Q \) | \( SU(N_f) \) | \( U(1)_A \) | \( U(1)_R \) |
|------|------|-------|-------|
| \( X \) | 1 | 0 | \( \frac{1}{n+1} \) |
| \( M_{2j} \) | \( \frac{N_f(N_f+1)}{2} \) | 2 | \( 2r + \frac{2j}{n+1} \) |
| \( M_{2j+1} \) | \( \frac{N_f(N_f-1)}{2} \) | 2 | \( 2r + \frac{2j+1}{n+1} \) |
| \( v_j \) | \( \frac{1}{N_f} \) | -1 | \( -N_f r + N_f - \frac{N_c - 2}{n+1} + \frac{j}{n+1} \) |
| \( q \) | \( \frac{1}{N_f} \) | 0 | \( -r + \frac{1}{n+1} \) |
| \( Y \) | 1 | \( N_f \) | \( N_f r - N_f + \frac{N_c}{n+1} + \frac{j}{n+1} \) |
| \( \tilde{v}_j \) | 1 | \( N_f \) | \( N_f r - N_f + \frac{N_c}{n+1} + \frac{j}{n+1} \) |

As in \( U(N) \) case, the \( R \)-charge of monopole operator can be negative in \( O(N) \) gauge theories. The constraints from meson and monopole operator are given by \( 0 < 2r + \frac{j}{n+1} < 2 \) and \( 0 < -N_f r + N_f - \frac{N_c - 2}{n+1} + \frac{j}{n+1} < 2 \) where \( j = 0, \ldots, 2n \). Conditions to have a solution \( r \) satisfying both inequalities are

\[
n N_f < N_c, \quad \text{and} \quad n N_f < N'_c. \quad (4.4)
\]

where \( N'_c = (2n + 1)N_f + 2 - N_c \). At the result of index computation we also list the examples that do not satisfy above inequalities, which are divergent.
To motivate the number of independent monopole operators, we consider the deformation of the superpotential \[ W = n \sum_{j=0}^{n} s_{2j}^2 (n+1-j) \text{Tr} X^2(n+1-j). \] (4.5)

For generic \( \{ s_{2j} \} \) the bosonic potential \( V \sim |W'|^2 \) has \( 2n + 1 \) distinct minima, one at the origin and \( n \) paired minima at \( \{ \pm a_j \} \)

\[ W'(x) = s_0 x \prod_{j=1}^{n} (x^2 - a_j^2). \] (4.6)

If \( r_0 \) eigenvalues of \( X \) sit at zero and \( r_j \) eigenvalues sit at \( \{ \pm a_j \} \), the gauge symmetry is spontaneously broken:

\[ O(N_c) \rightarrow O(r_0) \times U(r_1) \times U(r_2) \times \cdots \times U(r_n). \] (4.7)

The theory splits in the infrared into \( \mathcal{N} = 2 \) \( O(r_0) \) theory with \( 2N_f \) flavors of quarks and \( n \) decoupled copies of with \( \mathcal{N} = 2 \) \( U(r_i) \) theory with \( N_f \) flavors of quarks. In 3-dimensions, \( O(r_0) \) theory has one monopole operator and each \( U(r_i) \) has one pair of monopole operators, parametrizing the Coulomb branch. Thus the original theory should have at least \( 2n + 1 \) monopole operators. It turns out that the original theory has precisely \( 2n + 1 \) of monopole operators.

Let's discuss possible monopole operators involving the adjoint. Once magnetic flux \( |1, 0, \ldots \rangle \) is turned on scalar excitations of adjoint field is given by

\[ X = \begin{pmatrix} X_1 & 0 \\ 0 & X' \end{pmatrix} \] (4.8)

where \( X_1 \) and \( X' \) is an adjoint field of \( SO(2) \) and \( SO(N_c-2) \) gauge group respectively. Later we will consider the nontrivial \( Z_2 \) elements of \( O(N_c) \) and its action on the above state. The adjoint field of orthogonal gauge group is antisymmetric so the operators \( \text{Tr} X^{2i+1} \) having odd power are zero. However, with the magnetic flux there is a nontrivial state \( \text{Pf}X_1 |1, 0, \ldots \rangle \) and we identify this state as \( v_1 \). For this purpose we consider the nontrivial \( Z_2 \) element of \( O(N) \) nontrivially acting on \( SO(2) \) factor of the above. Under this \( Z_2 \), \( |m, 0, \ldots \rangle \) is mapped to \( |-m, 0, \ldots \rangle \). Furthermore \( \text{Pf}X_1 \) is projected out under \( Z_2 \). However, \( \text{Pf}X_1 |1, 0, \ldots \rangle \) is mapped to \( -\text{Pf}X_1 |1, 0, \ldots \rangle \). We denote \( Z_2 \)-invariant combination by \( v_1 \) and by the abuse of the notation we denote it as \( \text{Pf}X_1 |1, 0, \ldots \rangle \) with the understanding that for \( O(N) \) we can restrict the magnetic fluxes to be nonnegative due to the nontrivial \( Z_2 \) identification. Note that under the identification \( SO(2) = U(1) \), \( \text{Pf}X_1 \) is mapped to \( \text{Tr}X \) of \( U(1) \). In either \( SO(2) \) or \( U(1) \), \( \text{Pf}X_1 |1, 0, \ldots \rangle \) or \( \text{Tr}X |1, 0, \ldots \rangle \) can be obtained from the operator product of \( v_0 \) and \( \text{Tr}X \). However under the \( Z_2 \) action, \( \text{Tr}X \) is projected out and such product
structure is lost in either $O(2)$ or $U(1)/\mathbb{Z}_2$. This is how we obtain independent monopole operators $v_0, v_1$. Similarly, one can obtain the other independent monopole operators. We can define

\[
v_{2k} = \text{Tr}X_1^{2k}|1, 0, \ldots \rangle \\
v_{2k+1} = \text{Pf}X_1\text{Tr}X_1^{2k}|1, 0, \ldots \rangle.
\]  

Let us consider excitations of two adjoint fields with the magnetic flux. There are only two independent states $\text{Tr}X^2|1, 0, \ldots \rangle$ and $\text{Tr}X^2_1|1, 0, \ldots \rangle$ because $\text{Tr}X' = 0$. These states generate the chiral ring operators $v_0\text{Tr}X^2$ and $v_2$, which can be seen in magnetic theory. For the excitation of three adjoint fields there are two independent states $\text{Tr}X^2|1, 0, \ldots \rangle$ and $v_3 = \text{Pf}X_1\text{Tr}X^2_1|1, 0, \ldots \rangle$, which generate chiral ring operators $v_1\text{Tr}X^2$ and $v_3$. With four adjoint fields three independent states are given by $\text{Tr}X^4|1, 0, \ldots \rangle$, $(\text{Tr}X^2)^2|1, 0, \ldots \rangle$, $\text{Tr}X^2_1\text{Tr}X^2|1, 0, \ldots \rangle$ and $v_4 = \text{Tr}X^4_1|1, 0, \ldots \rangle$. From these, we obtain the corresponding chiral ring operators $v_0\text{Tr}X^4$, $v_0(\text{Tr}X^2)^2$, $v_2\text{Tr}X^2$ and $v_4$.

### 4.1 The result of the index computations

| $n = 1, (N_f, N_c)$ | Electric $O(N_c)$ | Magnetic $O(3N_f + 2 - N_c)$ | Index |
|---------------------|------------------|-----------------------------|-------|
| (1,1)               | $O(1)$           | $O(4)$                      | Divergent |
| (1,2)               | $O(2)$           | $O(3)$                      | $1 + x - x^2 + x^{4r} + x^{2r}(1 + x) + x^{-r}(x + x^{3/2} + x^2) + x^{-2r}(x^2 + x^{5/2})$ |
| (2,2)               | $O(2)$           | $O(6)$                      | Divergent |
| (1,3)               | $O(3)$           | $O(2)$                      | $1 + 2x + x^{3/2} + x^2 + x^{4r} + x^{7/2} + 3x^r(1 + 3x) + x^r(\sqrt{2}x + 2x^{3/2}) + x^{-r}(\sqrt{2}x + x^{3/2} + x^2) + x^{-2r}(x + x^{3/2} + 2x^2 + x^{5/2})$ |
| (2,3)               | $O(3)$           | $O(5)$                      | $1 + x + 3x^{3/2} - x^2 + x^{3-4r} + 6x^{4r} + x^{2r}(3 + \sqrt{2}x + 6x) + x^{-2r}(x^{3/2} + x^2 + x^{5/2})$ |
| (1,4)               | $O(4)$           | $O(1)$                      | Divergent |
| (2,4)               | $O(4)$           | $O(4)$                      | $1 + 4x + 4x^{3/2} + 10x^2 + 6x^{4r} + x^{2r}(3 + \sqrt{2}x + 12x) + x^{-2r}(x + x^{3/2} + 5x^2 + 5x^{5/2}) + x^{-4r}(x^2 + x^{5/2} + 6x^3)$ |
| (1,5)               | $O(5)$           | $O(0)$                      | Divergent |
| (2,5)               | $O(5)$           | $O(3)$                      | $1 + 3\sqrt{2}x + 11x + 29x^{3/2} + 64x^2 + 6x^{4r} + x^{2r}(3 + 7\sqrt{2}x + 25x) + x^{-2r}(\sqrt{2}x + 4x + 12x^{3/2} + 33x^2) + x^{-4r}(x + 4x^{3/2} + 13x^2 + 34x^{5/2})$ |
| \( n = 2, (N_f, N_c) \) | \( O(N_c) \) | \( O(5N_f + 2 - N_c) \) | Index |
|----------------|-------------|----------------|-------|
| \( (1,1) \)   | \( O(1) \)  | \( O(6) \)     | Divergent |
| \( (1,2) \)   | \( O(2) \)  | \( O(5) \)     | Divergent |
| \( (1,3) \)   | \( O(3) \)  | \( O(4) \)     | \( 1 + x + 2^{2/3} + 2x^{4/3} + x^{5/3} + x^{2r}(1 + 2x^{2/3} + 3x^{4/3}) + x^{-r}(x^{2/3} + x^{4/3} + x^{5/3} + 2x^2) + x^{-2r}(x^{4/3} + x^{5/3} + x^2 + x^{7/3}) \) |
| \( (1,4) \)   | \( O(4) \)  | \( O(3) \)     | \( 1 + 2x^{2/3} + x + 5x^{4/3} + 3x^{5/3} + 6x^2 + x^{2r}(1 + 3x^{2/3} + x + 6x^{4/3}) + x^{-r}(x^{1/3} + x^{2/3} + 3x + 3x^{4/3} + 6x^{5/3} + 5x^2) + x^{-2r}(x^{2/3} + x + 4x^{4/3} + 4x^{5/3} + 8x^2) \) |
| \( (1,5) \)   | \( O(5) \)  | \( O(2) \)     | Divergent |
| \( (1,6) \)   | \( O(6) \)  | \( O(1) \)     | Divergent |
| \( (2,6) \)   | \( O(6) \)  | \( O(6) \)     | \( 1 + 4x^{2/3} + 4x + 18x^{4/3} + 21x^{5/3} + 60x^2 + (6 + 3x^{1/3} + 26x^{2/3})x^{4r} + x^{2r}(3 + x^{1/3} + 12x^{2/3} + 11x + 47x^{4/3}) + x^{-2r}(x^{2/3} + x + 5x^{4/3} + 6x^{5/3} + 23x^2 + 28x^{7/3}) \) |

| \( n = 3, (N_f, N_c) \) | \( O(N_c) \) | \( O(7N_f + 2 - N_c) \) | Index |
|----------------|-------------|----------------|-------|
| \( (1,3) \)   | \( O(3) \)  | \( O(6) \)     | Divergent |
| \( (1,4) \)   | \( O(4) \)  | \( O(5) \)     | \( 1 + \sqrt{x} + x + x^{5/4} + 4x^{3/2} + 2x^{7/4} + 5x^2 + x^{2r}(1 + 2\sqrt{x} + 4x + x^{5/4} + 6x^{3/2}) + x^{-r}(\sqrt{x} + x^{3/4} + 2x + 2x^{5/4} + 4x^{3/2} + 4x^{7/4} + 6x^2) + x^{-2r}(x + x^{5/4} + 3x^{3/2} + 3x^{7/4} + 6x^2 + 6x^{9/4}) \) |
| \( (1,5) \)   | \( O(5) \)  | \( O(4) \)     | \( 1 + 2\sqrt{x} + x^{3/4} + 7x + 5x^{5/4} + 14x^{3/2} + 13x^{7/4} + 26x^2 + x^{2r}(1 + 3\sqrt{x} + x^{3/4} + 9x + 5x^{5/4} + 18x^{3/2}) + x^{-r}(x^{1/4} + \sqrt{x} + 4x^{3/4} + 4x + 10x^{5/4} + 11x^{3/2} + 21x^{7/4}) + x^{-2r}(\sqrt{x} + x^{3/4} + 4x + 4x^{5/4} + 10x^{3/2} + 11x^{7/4} + 21x^2 + 22x^{9/4}) \) |
| \( (1,6) \)   | \( O(6) \)  | \( O(3) \)     | Divergent |
Table 4: Superconformal index for $O(N)$ gauge theories with an adjoint and a superpotential. Bold face letters denote the chiral ring elements discussed in the main text.

As in $U(N)$ duality the chiral rings of both electric and magnetic theory are the same thanks to the nonperturbative effect. The generic contribution of chiral ring generators to index for $n = 1$ is as follows. $\text{Tr} \, x^2 \sim x$, $M_0 \sim \frac{N_f(N_f+1)}{2} x^{2r}$, $M_1 \sim \frac{N_f(N_f-1)}{N_f} x^{2r+1/2}$, $(M_2, \text{Tr} \, X^2 M_0) \sim N_f(N_f+1)x^{2r+1}$, $v_0 \sim x^{N_f-N_c/2+1-N_f}$, $v_1 \sim x^{N_f-N_c/2+3/2-N_f}$, $(v_2, \text{Tr} \, X^2 v_0) \sim 2x^{N_f-N_c/2+1-N_f}$. These contributions are seen at $(N_f, N_c) = (3, 5) \, O(5)_E-O(6)_M$ and $(N_f, N_c) = (3, 6) \, O(6)_E-O(5)_M$.

Let us consider the nonperturbative truncation which is seen at $(N_f, N_c) = (1, 2) \, O(2)_E-O(3)_M$ case. The operators $M_2$ and $v_2$ are not independent operator in $O(2)$ electric theory. On the other hand they exist as elementary fields in $O(3)$ magnetic theory. Thus they should be truncated for consistency. Indeed, $M_2$ is cancelled by a monopole operator $\psi v_0$ and $v_2$ is paired up with a monopole operator $\psi_M v_0$. Therefore, the chiral ring generators of both sides are the same, consistent with duality due to nonperturbative truncation.

For $(N_f, N_c) = (1, 3) \, O(3)_E-O(2)_M$ case the terms corresponding to the energy of chiral ring generators are as follows. $(\text{Tr} \, X^2, M_0 v_0^2)_{E,M} \sim 2x$, $(M_2, M_0 \text{Tr} \, X^2, M_0^2 v_0^2)_{E,M} \sim 3x^{2r+1}$. Let us consider the term $2x^{-r+3/2}$ of index. In electric theory it comes from $\text{Tr} X^2 |1\rangle$ and $QQ|3\rangle$. In magnetic side the first term correspond to a linear combination of $v_0 \text{Tr} \, X^2$ and $v_2$. One degree of freedom of the operators is truncated by a monopole operator $\psi_M v_0$. Thus chiral ring spectrum is consistent.

In $(N_f, N_c) = (2, 4) \, O(4)_E-O(4)_M$ case there are more states at the energy levels of chiral ring generators. They are simply products of chiral ring generators: $4x \sim (\text{Tr} \, X^2, 3M_0 v_0)_{E,M}$, $12x^{2r+1} \sim (3M_2, 3M_0 \text{Tr} \, X^2, 3H_2 M_0^2 v_0)_{E,M}$, $5x^{-2r+2} \sim (v_2, v_0 \text{Tr} \, X^2, 3M_0 v_0^2)_{E,M}$.

5. $Sp(2N)$ with an adjoint

For $\mathcal{N} = 2 \, Sp(2N)$ gauge theory with an adjoint, we propose the following dualities.

- Electric theory: $Sp(2N_c)$ gauge theory with $2N_f$ fundamental chiral multiplets $Q^a$, an adjoint chiral multiplet $X$, and a superpotential $W_e = \text{Tr} \, X^{2(n+1)}$.

- Magnetic theory: $Sp(2((2n+1)N_f-N_c-1))$ gauge theory with $2N_f$ fundamental chiral multiplets $qa$, an adjoint chiral multiplet $Y$, singlets $M_{j}^{ab} = Q^a X^j Q^b$, $j = 0, \ldots, 2n$ which are symmetric (resp. antisymmetric) in their flavor indices for odd (resp. even) $j$ and $v_j$, $j = 0, \ldots, 2n$, and a superpotential $W_m = \text{Tr} \, Y^{2(n+1)} + \sum_{j=0}^{2n} M_{j}^{ab} q_a Y^{2n-j} q_b + \sum_{j=0}^{2n} v_j v_{2n-j}$. 


Note that for \( n = 0 \) this duality reduces to the symplectic 3d Seiberg duality. All gauge indices are contracted with invariant antisymmetric tensor \( J \) in the product of the \( 2N \)-dimensional representation of \( Sp(2N) \). From this, one can see that \( \text{Tr} \, X^{2j+1} = 0 \) and the transformation property of \( M_j \) under flavor symmetry. Chiral ring generators map as follows:

\[
\begin{align*}
\text{Tr} \, X^i &\rightarrow \text{Tr} \, Y^i, \quad i = 1, \ldots, n \\
Q^a X^j Q^b &\rightarrow M^{ab}_j, \quad j = 0, \ldots, 2n \\
\text{Tr} \left( v_0 X^j \right) &\rightarrow v_j, \quad j = 0, \ldots, 2n.
\end{align*}
\]

The third equation is schematic and the precise definition is given shortly. The quantum corrected global charges are:

\[
\begin{array}{|c|c|c|}
\hline
\text{charge} & SU(2N_f) & U(1)_A \\ 
Q & 2N_f & 1 \\ 
X & 1 & 0 \\ 
M_{2j} & N_f(2N_f - 1) & 2 \\
M_{2j+1} & N_f(2N_f + 1) & 2 \\ 
v_j & 1 & -2N_f \\ 
q & \frac{2N_f}{2N_f} & -1 \\ 
Y & 1 & 0 \\ 
\tilde{v}_j & 1 & 2N_f \\
\hline
\end{array}
\]

The constraints on R-charge are given by \( 0 < 2r + \frac{j}{n+1} < 2 \) from mesons and \( 0 < -2N_f r + 2N_f - \frac{2N_r}{n+1} + \frac{j}{n+1} < 2 \) from monopole operators. Conditions to have a solution \( r \) satisfying both inequalities are

\[
nN_f < N_c, \quad \text{and} \quad nN_f < N_c'.
\]

where \( N_c' = (2n + 1)N_f - N_c - 1 \). At the result of index computation we also list the examples that do not satisfy above inequalities.

To motivate the number of independent monopole operators, we consider the deformation of the superpotential \[24\]

\[
W = \sum_{j=0}^{n} \frac{s_{2j}}{2(n + 1 - j)} \text{Tr} \, X^{2(n+1-j)}.
\]

For generic \( \{s_{2j}\} \) the bosonic potential \( V \sim |W'|^2 \) has \( 2n + 1 \) distinct minima, one at the origin and \( n \) paired minima at \( \{ \pm a_j \} \)

\[
W'(x) = s_0 x \prod_{j=1}^{n} (x^2 - a_j^2).
\]
If $2r_0$ eigenvalues of $X$ sit at zero and $r_j$ eigenvalues sit at $\{\pm a_j\}$, the gauge symmetry is spontaneously broken:

$$Sp(N_c) \rightarrow Sp(2r_0) \times U(r_1) \times U(r_2) \times \cdots \times U(r_n).$$

(5.7)

The theory splits in the infrared into $\mathcal{N} = 2$ $Sp(2r_0)$ theory with $N_f$ flavors of quarks and $n$ decoupled copies of with $\mathcal{N} = 2$ $U(r_i)$ theory with $N_f$ flavors of quarks. In 3-dimensions, $Sp(2r_0)$ theory has one monopole operator and each $U(r_i)$ has one pair of monopole operators, parametrizing the Coulomb branch. Thus the original theory should have at least $2n + 1$ monopole operators. It turns out that the original theory has precisely $2n + 1$ monopole operators.

Once magnetic flux $|1, 0, \ldots\rangle$ is turned on, scalar excitations of adjoint field is given by

$$X = \begin{pmatrix} X_1 & 0 \\ 0 & X' \end{pmatrix}$$

(5.8)

where $X_1$, $X'$ are an adjoint field of $Sp(2)$, $Sp(2N_c - 2)$ gauge group respectively. Let us explain where the gauge invariant $\text{Tr}X_1$ comes from. The adjoint of $Sp(2)$ is antisymmetric matrix, which can be written as a linear combination of a basis $I$, $i\sigma_1$ and $i\sigma_3$ where $I$ is an identity matrix and $\sigma_i$ are Pauli matrices. With the magnetic flux $|1, 0, \ldots\rangle$ the nontrivial scalar BPS state $\text{Tr}X_1$ comes from the identity matrix. We define $v_1$ to be the operator corresponding to the state $\text{Tr}X_1|1, 0, \ldots\rangle$.

In general we can define

$$v_j = \text{Tr}X_j^2|1, 0, \ldots\rangle.$$  

(5.9)

With two adjoint fields there are only two independent states $\text{Tr}X^2|1, 0, \ldots\rangle$ and $\text{Tr}X_1^2|1, 0, \ldots\rangle$, which generate chiral ring operators $v_0\text{Tr}X^2$ and $v_2$. For excitation of three adjoint fields two states $\text{Tr}X_1\text{Tr}X^2|1, 0, \ldots\rangle$ and $\text{Tr}X_2^3|1, 0, \ldots\rangle$ generate $v_1\text{Tr}X^2$ and $v_3$. Similarly, states with four adjoint fields, $\text{Tr}X^4|1, 0, \ldots\rangle$, $(\text{Tr}X^2)^2|1, 0, \ldots\rangle$, $\text{Tr}X_1^2\text{Tr}X^2|1, 0, \ldots\rangle$ and $\text{Tr}X_1^4|1, 0, \ldots\rangle$ generate chiral ring operators $v_0\text{Tr}X^4$, $v_0(\text{Tr}X^2)^2$, $v_2\text{Tr}X^2$ and $v_4$.

5.1 The result of the index computations

| $n = 1, (N_f, N_c)$ | Electric $Sp(2N_c)$ | Magnetic $Sp(2(3N_f - N_c - 1))$ | Index |
|---------------------|-------------------|-------------------------------|--------|
| (1,1)               | $Sp(2)$           | $Sp(2)$                       | $1 + x - 4x^2 + x^{4r} + x^{2r}(1 + 3\sqrt{x} + x) + x^{-2r}(x + x^{3/2} + x^2) + x^{-4r}(x^2 + x^{5/2} + x^3)$ |
| (1,2)               | $Sp(4)$           | $Sp(0)$                       | Divergent |

$^{3}$For $SU(2)$, any representation and its complex representation is equivalent, $(m, 0, \ldots)$ is identified with $(-m, 0, \ldots)$. 


Divergent

Table 6: Superconformal index for $Sp(N)$ gauge theories with an adjoint and a superpotential. Bold face letters denote the chiral ring elements discussed in the main text.

Chiral ring generators make identifiable contribution to the index: for $n = 1$, $\text{Tr} \, X^2 \sim x$, $M_0 \sim N_f(2N_f - 1)x^{2r}$, $M_1 \sim N_f(2N_f + 1)x^{2r+\frac{1}{2}}$, $(M_2, M_0 \text{Tr} \, X^2) \sim 2N_f(2N_f - 1)x^{2r+1}$, $v_0 \sim x^{-2N_f+r+2N_f-N_c}$, $v_1 \sim x^{-2N_f+r+2N_f-N_c+\frac{1}{2}}$. $(v_2, v_0 \text{Tr} \, X^2) \sim 2x^{-2N_f+r+2N_f-N_c+1}$. These contributions are shown exactly at the $(N_f, N_c) = (2,2)$ $Sp(4)_E$-$Sp(6)_M$ example.

Let us look at $(N_f, N_c) = (1,1)$ $Sp(2)_E$-$Sp(2)_M$ duality. At electric side the operators $M_2$ and $v_2$ are not independent operators while at magnetic side $M_2$ and $v_2$ exist as singlet in addition to $M_0 \text{Tr} \, Y^2$ and $v_0 \text{Tr} \, Y^2$ operators. Thus nonperturbative truncation should occur for $M_2$ and $v_2$ operators at magnetic side. Indeed the $M_2$ operator is canceled by a $\psi_{M_2} M_0 \tilde{v}_0$ operator. For generic high rank gauge group the $\psi_{M_2} M_0 \tilde{v}_0$ operator is supposed to cancel a $qq M_0 \tilde{v}_0$ operator because of the superpotential. But for $Sp(2)$ gauge group $qq$ gets angular momentum $j = 1$ in the presence of magnetic flux. Thus $qq M_0 \tilde{v}_0$ operator is absent at the energy $x^{2r+1}$ and the $\psi_{M_2} M_0 \tilde{v}_0$ operator cancel the $M_2$ operator instead of $qq M_0 \tilde{v}_0$. The $v_2$ operator
also disappear by nonperturbative effect. It is canceled by $\psi_M v_0 \tilde{v}_0$ operator which is expected to cancel $qqv_0 \tilde{v}_0$ operator. Therefore, the chiral ring generators of both theories are the same.

Lastly the $(N_f, N_c) = (2, 3)$ $Sp(6)_E$-$Sp(4)_M$ case shows that additional states exist at the various energy levels of chiral ring generators. But they are just products of chiral ring generators.

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A. Additional Superpotential

For special values of $N_c, N_f$ of Seiberg-like dual pairs, the magnetic theory can have the additional superpotential. For illustrative purposes, let us review the Aharony duality for $O(N_c)$ gauge theory with $N_f$ fundamentals $Q$ \cite{19}. The dual description is $O(N_f - N_c + 2)$ gauge theory with $N_f$ fundamentals $q$ and mesons $M \equiv QQ$ in the presence of superpotential $W = Mqq + v\tilde{v}$ where $v, \tilde{v}$ are monopole operators in original and dual theory respectively. The proper range of dualities is ”$R_v > 1$” where $R_v$ is UV R-charge of monopole operator $v$, $R_v = N_f - N_c + 2$. For $R_v = 0$ ($N_f = N_c - 1$) there is no dual gauge group and quantum moduli space can be obtained from the additional superpotential, which is given by ”$v^2 \det M + q^2 + 1 = 0$”. The quantum moduli space is smooth even though classical moduli space is singular at the origin. Finally for $R_v < 0$ ($N_f < N_c - 2$), the additional superpotential is ADS-like superpotential, $W_{\text{add}} = (v^2 \det M)^{1/R_v}$, so there is no supersymmetric vacuum. In this appendix, we explore analogues for $U(N)$ with an adjoint field. It turns out that $U(N)$ with an adjoint field is subtler. Some of the cases have ambiguities in the superpotentials, yet consistent with the index computation. At present, we do not know how to fix such ambiguities.

A.1 Instanton superpotential with an adjoint

The theory with an adjoint field has more additional superpotentials but shows similar behavior. The effective superpotential is constrained by symmetry, holomorphy and various limit of a theory. For example, in terms of operators $v_0,+, v_0,-$ and $M_0$ one can write additional superpotential which is consistent with global symmetries, $W \sim (v_0,+v_0,-\det M_0)^{1/R_0}$ where $R_0 = N_f - \frac{2}{n+1}(N_c - 1)$ is the UV R-charge of $v_0,\pm$ operator. When the moduli space includes the origin of fields space the effective superpotential is constrained to have a positive integer power because of holomorphy. Thus the additional superpotential can not be generated for $R_0 > 1$. This is applied to all other similar superpotentials, $W \sim (v_i,+,v_j,-\det M_k)^{1/R}$ which are absent for $R_0 > 1$ due to $R > R_0$.

The superpotentials are classified according to the UV R-charge of bare monopole $R_0$. For $R_0 > 1$ there is a duality with Aharony-type superpotential. For $0 < R_0 \leq 1$ the duality still holds with additional superpotentials. For $-1 < R_0 \leq 0$ there is smooth quantum moduli space for the examples we considered.

Let us describe the form of the additional superpotential for $0 < R_0 \leq 1$. The factors which are invariant under global symmetries except R-symmetry have the form of

$$v_i,+, v_j,- \cdot \epsilon_{a_1,a_2,\ldots,a_{N_f}} \epsilon_{b_1,b_2,\ldots,b_{N_f}} (M_{k_1})^{a_1,b_1} (M_{k_2})^{a_2,b_2} \ldots (M_{k_{N_f}})^{a_{N_f},b_{N_f}} \quad (A.1)$$
where all meson operators are contracted with $\epsilon$. For example if all $k_i$ are zero it is $v_{i,+}v_{j,-}\det M_0$. If only one of $k_i$ is non-zero it is $v_{i,+}v_{j,-}M_k\text{cof } M_0$ where cof is a cofactor of meson matrix. The factor (A.1) has R-charge 2($R_v=0$ and $R_v=1$). Thus the factors in the additional superpotential are determined by the condition that the superpotential has R-charge 2.

For $n = 2$ case, additional possible superpotentials are given as follows.

$$R_0 = 1 : W_{\text{add}} = v_{0,+}v_{0,-}\det M_0 \quad (A.2)$$

$$R_0 = \frac{2}{3} : W_{\text{add}} = (v_{1,+}v_{0,-} + v_{0,+}v_{1,-})\det M_0 + v_{0,+}v_{0,-}M_1\text{cof } M_0 \quad (A.3)$$

$$R_0 = \frac{1}{3} : W_{\text{add}} = v_{1,+}v_{1,-}\det M_0 + (v_{1,+}v_{0,-} + v_{0,+}v_{1,-})M_1\text{cof } M_0 \quad (A.4)$$

$$+ v_{0,+}v_{0,-}|(M_1)^2(M_0)^{N_f-2}|$$

$$+ v_{0,+}v_{0,-}\det M_0 \{ (v_{1,+}v_{0,-} + v_{0,+}v_{1,-})\det M_0 + v_{0,+}v_{0,-}M_1\text{cof } M_0 \}$$

$$+ (v_{0,+}v_{0,-}\det M_0)^3$$

where $|(M_1)^2(M_0)^{N_f-2}| = \epsilon_{a_1,a_2,\ldots,a_{N_f}}\epsilon_{b_1,b_2,\ldots,b_{N_f}}(M_1)^{a_1,b_1}(M_1)^{a_2,b_2}(M_0)^{a_3,b_3}\ldots(M_0)^{a_{N_f},b_{N_f}}.$

Note that in $R_0 = \frac{1}{3}$ case two and three instanton factors are used to form superpotential.  

A.2 Constraints from additional superpotential: $0 < R_0 \leq 1$

Now we would like to show that the quantum constraints are consistent in dual theories.

A. $R_0 = 1$

The theories in this range are $(N_f, N_c, N_c') = (1, 1, 1), (3, 4, 2), (5, 7, 3), \ldots$. Let us concentrate on $U(1)_1\cdot U(1)_{1'}$ duality where the subscript of gauge group indicates the flavor number $N_f$. The superpotential of the magnetic theory is given by

$$W = v_{0,+}v_{0,-}M_0 + Y^3 + M_0q\bar{q}Y + M_1q\bar{q} + v_{0,\pm}\bar{v}_{0,\mp}Y + v_{1,\pm}\bar{v}_{1,\mp} \quad (A.5)$$

where first two terms are additional superpotential. The equation of motions (EOM) for $Y, v_{1,\pm}$ and $M_1$ show that $Y^2 = \bar{v}_{0,\pm}q\bar{q} = 0$ which are consistent with electric side. Other EOM are given by

$$\partial_{v_{0,\pm}} W = \bar{v}_{0,\pm}Y + v_{0,\pm}M_0 = 0 \quad (A.6)$$

$$\partial_{M_0} W = q\bar{q}Y + v_{0,+}v_{0,-} = 0 \quad (A.7)$$

Using $\bar{v}_{0,\pm} = q\bar{q} = 0$, the constraints (A.11) and (A.12) are rewritten as $v_{0,\pm}M_0 = 0$ and $v_{0,+}v_{0,-} = 0$. These equations show that those states are Q-exact at each sector.

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^4For example, two instanton factors have the topological charge $U(1)_f \cdot 2$ compensated by the operators having the topological charge -2.
The quantum numbers \((\epsilon, T)\), energy and \(U(1)_J\) charge, of the states are \((r+1, 1)\) and \((-2r+2, 0)\) respectively. Indeed the result of index computation show there are no corresponding states. The term \(2x^{-2r+2}\) comes from \(v_{0,\pm}^2\). Manifestly, the deformed \(U(1)\) electric theory does not have such states as explained in the chiral ring section \[3.1\]. One can easily check that without the above additional superpotentials, the resulting equation of motion from the superpotential is not consistent with the chiral ring relations coming from the index computation.

However general cases have subtleties. For \(N'_c \geq 2\) the possible superpotential of the magnetic theory is given by

\[
W = v_{0,+}v_{0,-}\det M_0 + \text{Tr}Y^3 + M_0qY\bar{q} + M_1q\bar{q} + v_{0,0}\bar{v}_{1,\pm} + v_{1,\pm}\bar{v}_{0,\mp}
\]  

(A.8)

where the first term is additional superpotential. EOM are given by

\[
\partial_{v_{1,\pm}} W = \bar{v}_{0,\pm} = 0
\]  

(A.9)

\[
\partial_{M_1} W = q\bar{q} = 0
\]  

(A.10)

\[
\partial_{v_{0,\pm}} W = \bar{v}_{1,\pm} + v_{0,0}\det M_0 = 0
\]  

(A.11)

\[
\partial_{M_0} W = qY\bar{q} + v_{0,+}v_{0,-}\text{cof} M_0 = 0
\]  

(A.12)

From \((\text{A.11}), (\text{A.12})\) one can treat \(\bar{v}_{1,\pm}\) and \(qY\bar{q}\) dependent operators. Then the operators \(v_{i,\pm}, M_i\) are not constrained by the superpotential. In other words, they are independent operators as in electric theory. Thus even if additional superpotential is dropped for \(N'_c \geq 2\) the chiral ring of the theories are consistent. That is because \(v_{i,\pm}, M_i\) are not constrained.

**B. \(R_0 = \frac{2}{3}\)**

The theories in this range are \((N_f, N_c, N'_c) = (2, 3, 1), (4, 6, 2), (6, 9, 3), \ldots\) For \(N'_c \geq 2\) the superpotential of the magnetic theory is given as follows.

\[
W = v_{1,+}v_{0,-}\det M_0 + v_{0,+}v_{1,-}\det M_0 + v_{0,+}v_{0,-}M_1\text{cof}M_0 + \text{Tr}Y^3 + M_0qY\bar{q} + M_1q\bar{q} + v_{0,0}\bar{v}_{1,\pm} + v_{1,\pm}\bar{v}_{0,\mp}
\]  

(A.13)

where the first three terms of right-hand side are additional superpotential and all flavor indices of mesons are contracted. The EOM are given as follows.

\[
\partial_{M_1} W = q\bar{q} + v_{0,+}v_{0,-}\text{cof} M_0 = 0
\]  

(A.14)

\[
\partial_{v_{1,\pm}} W = \bar{v}_{0,\pm} + v_{0,0}\det M_0 = 0
\]  

(A.15)

\[
\partial_{M_0} W = qY\bar{q} + v_{1,+}v_{0,-}\text{cof} M_0 + v_{0,+}v_{1,-}\text{cof} M_0 + v_{0,+}v_{0,-}|(M_1)^1(M_0)^{N_f-2}| = 0
\]  

(A.16)

\[
\partial_{v_0} W = \bar{v}_{1,\pm} + v_{1,\pm}\det M_0 + v_{0,\pm}M_1\text{cof} M_0 = 0
\]  

(A.17)

One might wonder how much the result depends on the definition of \(v_i, \bar{v}_i\). Without the additional superpotential we previously show that equation of the motion is
independent of the definition of \( v_i, \tilde{v}_i \). The only ambiguity lies in \( v_1, \tilde{v}_1 \). If we redefine \( v_{1\pm} \to v_{1\pm} - v_{0\pm} \text{Tr}X \) so that \( v_{1\pm} = \text{Tr}X' | \pm 1, 0, \ldots \). Then we have
\[
\tilde{v}_{1\pm} = -(v_{1\pm} - v_{0\pm} \text{Tr}X) \det M_0 - v_{0\pm} M_1 \text{cof} M_0.
\] (A.18)

Since \( v_{0\pm} \text{Tr}X \det M_0 \) corresponds to \( \tilde{v}_{0\pm} \text{Tr}Y \), the redefinition of \( v_1 \) can be absorbed into the redefinition of \( \tilde{v}_1 \). Thus one can stick to the usual definition of \( v_i \).

As in \( R_0 = 1 \) case the fields \( v_{i\pm}, M_i \) are not constrained by the superpotential for \( N_c' \geq 2 \). Thus even if additional superpotential is dropped for \( N_c' \geq 2 \) the chiral ring of the theories are consistent.

C. \( R_0 = \frac{1}{2} \)

The theories in this range are \((N_f, N_c, N_c') = (1, 2, 0), (3, 5, 1), (5, 8, 2), \ldots\). We would like to describe \( U(2)_1 - U(0)_1 \) theory in detail. Naively the chiral ring structures of both theories look different. In the electric theory there is a chiral ring generator \( \text{Tr}X \) in addition to \( M_0, M_1, v_0 \) and \( v_1 \). But it seems that the \( U(0)_1 \) magnetic theory does not have the counterpart of \( \text{Tr}X \) operator. On the other hand, the operator \( v_{0\pm} v_{0\pm} M_0 \) exist in magnetic side while it has the higher energy in the \( U(2)_1 \) electric theory. Thus chiral rings of two theories seem different. However both states \( \text{Tr}X \) and \( v_{0\pm} v_{0\pm} M_0 \) have the same quantum numbers. Thus we propose a mapping \( \text{Tr}X \to v_{0\pm} v_{0\pm} M_0 \) under duality transformation. We will see this is consistent with states matching.

In \( U(0)_1 \) theory the superpotential is given by
\[
W = v_{1\pm} v_{1\pm} M_0 + v_{1\pm} v_{0\pm} M_1 + v_{0\pm} v_{1\pm} M_1
+ v_{0\pm} v_{0\pm} M_0 + v_{0\pm} v_{1\pm} M_0 + v_{0\pm} v_{0\pm} M_1
+ (v_{0\pm} v_{0\pm} M_0)^3.
\] (A.19)

The EOM are given by
\[
\partial_{v_{1\pm}} W = v_{1\pm} M_0 + v_{0\pm} M_1 + v_{0\pm} v_{0\pm} M_0^2 = 0 \quad \text{(A.20)}
\]
\[
\partial_{M_1} W = v_{1\pm} v_{0\pm} + v_{0\pm} v_{1\pm} + v_{0\pm} v_{0\pm} v_{0\pm} M_0 = 0 \quad \text{(A.21)}
\]
\[
\partial_{v_{0\pm}} W = v_{1\pm} M_1 + 2v_{0\pm} v_{1\pm} M_0^2 + v_{0\pm} M_0 (v_{0\pm} v_{1\pm} M_0 + 2v_{0\pm} v_{1\pm} M_0 + 3v_{0\pm} v_{0\pm} M_0^2) = 0 \quad \text{(A.22)}
\]
\[
\partial_{M_0} W = v_{1\pm} v_{1\pm} + v_{0\pm} v_{0\pm} (2v_{0\pm} v_{1\pm} M_0 + 2v_{1\pm} v_{0\pm} M_0 + v_{0\pm} v_{0\pm} M_1 + 3v_{0\pm} v_{0\pm} M_0^2) = 0 \quad \text{(A.23)}
\]

Let us check the EOM are consistent with the electric theory. The EOM (A.20) relate the three operators of the magnetic theory. On the other hand, the deformed \( U(2)_1 \) electric theory does not have a state corresponding to the operator \( v_{0\pm} v_{0\pm} M_0^2 \). But the new matching of the operator \( \text{Tr}X \leftrightarrow v_{0\pm} v_{0\pm} M_0 \) should be
considered. So the electric theory should have a constraint on the three operators $v_{1,±}M_0$, $v_{0,±}M_1$ and $\text{Tr} X v_{0,±}M_0$ for each sign. Those operators correspond to the states of the form, $XQ\bar{Q}|±1,0\rangle$ with broken gauge group $U(1)_{±1} × U(1)_0$. The squarks have $U(1)_0$ gauge index and the adjoint has either $U(1)_{±1}$ or $U(1)_0$ gauge index. Thus there are only two gauge invariant scalar states in both theories.

Let us consider the EOM (A.21) which reduces the number of independent operators three to two. As in EOM (A.20), the electric theory seems to have three operators, $v_{1,+}v_{0,-}$, $v_{0,+}v_{1,-}$ and $v_{0,+}v_{0,-}\text{Tr} X$. They correspond to the states of the form, $X|1,−1\rangle$ with broken gauge group, $U(1)_{+1} × U(1)_{−1}$. An adjoint field can be excited from each $U(1)$ to be gauge invariant scalar state. Thus two theories have two states at this sector.

The EOM (A.22) relates five operators but one can see that four operators except $v_{1,±}M_1$ are linearly dependent through three EOM of $M_1$, $v_{1,+}$ and $v_{1,−}$. Thus only one operator is independent among five operators due the the EOM (A.22) in the magnetic theory. In the electric theory the new matching $\text{Tr} X ↔ v_{0,+}v_{0,−}M_0$ leads to consider four operators $v_{1,±}M_1$, $v_{1,±}M_0\text{Tr} X$, $v_{0,±}M_1\text{Tr} X$ and $v_{0,±}M_0(\text{Tr} X)^2$. Those operators corresponds to the states of the form $X^2Q\bar{Q}|±1,0\rangle$ with broken gauge group $U(1)_{±1} × U(1)_0$. The squarks have only $U(1)_0$ gauge index to be BPS scalar states. The excitations of adjoint fields can be $X^2_1$, $X_1 · X_2$ and $X^2_2$ where subscript denotes a different factor of the gauge group. But the operators of the form $X^2$ are truncated due to the superpotential $\text{Tr} X^3$. Thus there is also only one state $X_1X_2Q\bar{Q}_2|±1,0\rangle$ in the electric theory.

Finally, the EOM (A.23) contains five operators. As in (A.22) the terms proportional to $v_{0,+}v_{0,-}$ are linearly dependent through three EOM of $M_1$, $v_{1,+}$ and $v_{1,-}$. So the EOM (A.23) implies that only one operator is independent among five operators. On the other hand, the electric theory has four relevant operators $v_{1,+}v_{1,-}$, $v_{1,+}v_{0,-}\text{Tr} X$, $v_{0,+}v_{1,-}\text{Tr} X$ and $v_{0,+}v_{0,-}(\text{Tr} X)^2$ from new matching of the operator $\text{Tr} X$. The corresponding states have a form $X^2|1,−1\rangle$ with broken gauge group $U(1)_1 × U(1)_{−1}$. As in previous example the only scalar BPS state is $X_1X_2|1,−1\rangle$. Therefore, both theories have one state at this sector.

Each equation of motion reduce the number of independent operators by one. It seems that if superpotential contains all operators $v_{i,±}$ and $M_i$ it still gives consistent result. For example, even though additional superpotential contain only first line of (A.20) the equation of motions reduce the number of independent operators consistently. But in other sector one can see additional requirement for BPS cancellation. Let us consider states which has charges $(E + j, T) = (2, 0)$ where $T$ is a topological charge. The electric theory has two fermionic states $Q^\dagger_{\psi_Q}$ and $\bar{Q}^\dagger_{\psi_{\bar{Q}}}$ at the sector. On the other hand, the magnetic theory has several states at the sector. There are
9 fermionic states and 7 bosonic states as follows.

\[
\text{Fermionic : } \psi_{M_0}^\dagger, \psi_{M_1}^\dagger, \psi_{v_{0,0}}, \psi_{v_{1,1}}, \\
\text{Bosonic : } v_{1,+}v_{1,-}, v_{0,+}v_{0,-}, v_{0,+}v_{0,-}M_1,
\]

Thus there should be 7 pair of BPS states which form the long representation in order to have a consistent BPS spectrum. Namely, all bosonic states must be paired up with some fermionic states. We focus on the operator \((v_{0,+}v_{0,-}M_0)^3\) and look for the superpotential which relate it to a fermionic operator. The terms in the second and the third line of (A.20) provide proper pairings. Terms in superpotential and corresponding fermionic operator which pair up with \((v_{0,+}v_{0,-}M_0)^3\) are as follows.

\[
W \sim v_{1,+}v_{0,+}v_{0,-}^2 M_0 : \psi_{v_{0,0}}^\dagger v_{0,+} v_{0,-} M_0 \\
W \sim v_{1,-}v_{0,+}v_{0,-}^2 M_0 : \psi_{v_{1,1}}^\dagger v_{0,+} v_{0,-} M_0 \\
W \sim v_{2,+}v_{0,+}^2 M_1 : \psi_{M_1}^\dagger v_{0,+} v_{0,-} M_0^2 \\
W \sim (v_{0,+}v_{0,-}M_0)^3 : \psi_{M_0}^\dagger M_0 \text{ or } \psi_{v_{0,0}}^\dagger v_{0,0}, v_{0,0}
\]

Therefore, superpotential have to include one of terms in (A.20) in order to reproduce the BPS spectrum properly.

### A.3 Theory with monopoles having negative R-charge

The IR R-charge of bare monopole field is given by \(R_0 - N_f r\) where \(r\) is the R-charge of quark. When the UV R-charge is non-positive, \(R_0 \leq 0\) the IR R-charge becomes negative due to \(r > 0\). The theories containing monopole operator which have negative R-charge are not superconformal so those cannot be studied by superconformal index. But some information can be obtained from the theories in \(R_0 > 0\). As integrating out one flavor \(N_f \rightarrow N_f - 1\) the value of \(R_0 = N_f - \frac{2}{n+1}(N_c - 1)\) decrease by one \(R_0 \rightarrow R_0 - 1\). The mass deformation of the 4d analogue of the duality was analysed in [26].

Let us analyse \(n = 2\) case. The theories in the range \(-1 < R_0 \leq 0\) are obtained from the theories with \(R_0 = 1, \frac{2}{3}, \frac{1}{3}\). First consider the theories which can be obtained from the \(R_0 = 1\) theory. With ordinary superpotential (A.2) and a mass term \(m M_0^{N_f} N_f\) the EOM for massive mesons are given by

\[
v_{0,+}v_{0,-} \det M_0 + q^{N_f} Y \tilde{q}^{N_f} + m = 0, \quad (A.25) \\
q^{N_f} \tilde{q}^{N_f} = 0 \quad (A.26)
\]
where flavor indices of meson fields run 1 to \(N_f - 1\). Generically the operator \(q^{N_f} Y \tilde{q}^{N_f}\) get non-zero vacuum expectation value so Higgs mechanism reduces the number of colors by 2. And solutions of the EOM for \(q^{N_f}\) and \(\tilde{q}^{N_f}\) are given by

\[
M_0^{N_f,i} = M_0^{i,N_f} = M_0^{N_f,N_f} = 0, \quad (A.27)
\]

\[
M_1^{N_f,i} = M_1^{i,N_f} = M_1^{N_f,N_f} = 0
\]

(A.28)

where flavor indices \(i\) of meson fields runs 1 to \(N_f - 1\). Thus all massive fields are integrated out and the number of flavors is reduced by one, \(N_f \rightarrow N_f - 1\).

Then the electric theory flows to \(U(N_c)\) theory with \(N_f - 1\) flavors and an adjoint field and the magnetic theory become \(U(2(N_f - 1) - N_c)\) theory with \(N_f - 1\) flavors and an adjoint field. These theories are not superconformal because R-charge of the bare monopoles is negative with the UV R-charge \(R_0 = 0\). A part of moduli space is given by \((A.27)\). It describes a smooth moduli space because it does not contain a singular point, the origin in this case, where all derivatives of the constraint vanish, \(d(v_{0,+}v_{0,-}\det M_0 + q^{N_f} Y \tilde{q}^{N_f} + m) = 0\).

In magnetic theory the superpotential can be written as

\[
W = \lambda(v_{0,+}v_{0,-}\det M_0 + m') + \Tr Y^3 + M_0 \tilde{q} Y q + M_1 \tilde{q} q + v_{0,\pm} \tilde{v}_{1,\mp} + v_{1,\pm} \tilde{v}_0, (A.29)
\]

where the first term is a Lagrange multiplier coming from \((A.25)\). The other terms are ordinary superpotentials which exist only for nontrivial magnetic gauge group.

Note that the superpotential include a constraint which come from \(R_0 = 1\) theory. There can be additional superpotentials which are not seen from mass deformation. Let us look for the additional superpotentials which are consistent with global symmetries. The general form of the additional superpotential is given by \((A.1)\). For \(R_0 = 0\) some terms are given by

\[
W_{\text{add}} = v_{1,+}v_{1,-} M_1 \text{cof} M_0 + (v_{1,+}v_{0,-} + v_{0,+}v_{1,-})(M_1)^2(M_0)^{N_f - 2} + v_{0,+}v_{0,-}(M_1)^3(M_0)^{N_f - 3} + (v_{1,+}v_{0,-}\det M_0 + v_{0,+}v_{1,-}\det M_0 + v_{0,}\pm v_{0,-}\text{cof} M_0) \\
\times (v_{1,+}v_{1,-}\det M_0 + v_{1,+}v_{0,-}M_1 \text{cof} M_0 + v_{0,}\pm v_{1,-}M_1 \text{cof} M_0 + v_{0,}v_{0,-}M_1^2(M_0)^{N_f - 2}) + (v_{1,+}v_{0,-}\det M_0 + v_{0,+}v_{1,-}\det M_0 + v_{0,}v_{0,-}M_1 \text{cof} M_0)^3. (A.30)
\]

Besides, any factor like \((v_{0,+}v_{0,-}\det M_0)^N\) can be multiplied to above operators because the R-charge of the operator \(v_{0,+}v_{0,-}\det M_0\) is zero. So it seems that there are infinitely many terms which are consistent with global symmetries in contrast to \(R_0 > 0\) cases. It’s not clear which terms are generated and which terms are not. It would be interesting to develop the explicit instanton calculus to confirm the precise form of the superpotential.

Secondly, one can start from the theories with \(R_0 = \frac{2}{3}\) with the superpotential \((A.3)\). The theories in this category show similar behavior to the previous case.
Theories with $R_0 = -\frac{1}{3}$ have smooth quantum moduli space,

$$v_{1,+}v_{0,-}\det M_0 + v_{0,+}v_{1,-}\det M_0 + v_{0,+}v_{0,-}M_1\text{cof } M_0 + q^{N_f} Y \tilde{q}^{N_f} + m = 0. \quad (A.31)$$

Finally, theories from $R_0 = \frac{1}{3}$ also show similar behavior. Thanks to the EOM of $q^{N_f}$ and $\tilde{q}^{N_f}$ the quantum moduli space of the theory with $R_0 = -\frac{2}{3}$ is described by

$$v_{1,+}v_{1,-}\det M_0 + (v_{1,+}v_{0,-} + v_{0,+}v_{1,-})M_1\text{cof } M_0 + v_{0,+}v_{0,-}|(M_1)^2(M_0)^{N_f-3}| + q^{N_f} Y \tilde{q}^{N_f} + m = 0 \quad (A.32)$$

where the third term exist only for $N_f \geq 3$. Thus all theories have smooth quantum moduli space for $-1 < R_0 \leq 0$ but there could be additional superpotentials, which we cannot fix them completely. It would be interesting to resolve this issue.

References

[1] A. Giveon and D. Kutasov, “Seiberg Duality in Chern-Simons Theory,” Nucl. Phys. B812 (2009) 1, [arXiv:0808.0360 [hep-th]].

[2] V. Niarchos, “Seiberg Duality in Chern-Simons Theories with Fundamental and Adjoint Matter,” JHEP 0811 (2008) 001, [arXiv:0808.2771 [hep-th]].

[3] V. Niarchos, “R-charges, Chiral Rings and RG Flows in Supersymmetric Chern-Simons-Matter Theories,” JHEP 0905 (2009) 054.

[4] A. Kapustin, B. Willett, I. Yaakov, “Nonperturbative Tests of Three-Dimensional Dualities,” [arXiv:1003.5694 [hep-th]].

[5] A. Kapustin, B. Willett, I. Yaakov, “Tests of Seiberg-like Duality in Three Dimensions,” [arXiv:1012.4021 [hep-th]].

[6] D. Bashkirov, A. Kapustin, “Dualities between N = 8 superconformal field theories in three dimensions,” JHEP 1105, 074 (2011). [arXiv:1103.3548 [hep-th]].

[7] C. Krattenthaler, V. P. Spiridonov, G. S. Vartanov, “Superconformal indices of three-dimensional theories related by mirror symmetry,” JHEP 1106 (2011) 008, [arXiv:1103.4075 [hep-th]].

[8] D. Jafferis and X. Yin, “Duality Appetizer,” [arXiv:1103.5700 [hep-th]].

[9] A. Kapustin, “Seiberg-like duality in three dimensions for orthogonal gauge groups,” [arXiv:1104.0466 [hep-th]].

[10] B. Willett and I. Yaakov, “N=2 Dualities and Z Extremization in Three Dimensions,” [arXiv:1104.0487 [hep-th]].
[11] A. Kapustin, B. Willett, “Generalized Superconformal Index for Three Dimensional Field Theories,” [arXiv:1106.2484 [hep-th]].
[12] D. Bashkirov, “Aharony duality and monopole operators in three dimensions,” [arXiv:1106.4110 [hep-th]].
[13] C. Hwang, H. Kim, K.-J. Park, J. Park, “Index computation for 3d Chern-Simons matter theory: test of Seiberg-like duality,” [arXiv:1107.4942 [hep-th]].
[14] D. Gang, E. Koh, K. Lee, J. Park, “ABCD of 3d $\mathcal{N}=8$ and 4 Superconformal Field Theories,” [arXiv:1108.3647 [hep-th]].
[15] D. Berenstein, M. Romo, “Monopole operators, moduli spaces and dualities,” [arXiv:1108.4013 [hep-th]].
[16] T. Morita, V. Niarchos, “F-theorem, duality and SUSY breaking in one-adjoint Chern-Simons-Matter theories,” [arXiv:1108.4963 [hep-th]].
[17] F. Benini, C. Clostet, S. Cremonesi, ”Comments on 3d Seiberg-like dualities,” [arXiv:1108.5373 [hep-th]].
[18] C. Hwang, K.-J. Park, J. Park, “Evidence for Aharony duality for orthogonal gauge groups,” [arXiv:1109.2828 [hep-th]].
[19] O. Aharony, I. Shamir, “On $O(N_c)$ d=3 $\mathcal{N}=2$ supersymmetric QCD Theories,” [arXiv:1109.5081 [hep-th]].
[20] A. Kapustin, H. Kim and J. Park, “Dualities for 3d Theories with Tensor Matter,” JHEP 1112, 087 (2011) [arXiv:1110.2547 [hep-th]].
[21] D. Jafferis, “The Exact Superconformal R-Symmetry Extremizes Z,” [arXiv:1012.3210 [hep-th]].
[22] O. Aharony “IR duality in d=3 $\mathcal{N}=2$ $Usp(2N_c)$ and $U(N_c)$ Gauge Theories” Nucl. Phys. B404 (1997) 71, [arXiv:hep-th/9703215].
[23] S. Elitzur, A. Giveon, D. Kutasov, “Branes and N=1 Duality in String Theory” Phys. Lett. B400 (1997) 269, [arXiv:hep-th/9702014].
[24] A. Giveon, D. Kutasov, “Brane Dynamics and Gauge Theory,” Rev.Mod.Phys. 71 (1999) 983, [arXiv:hep-th/9802067].
[25] D. Kutasov, “A Comment on Duality in N=1 Supersymmetric Non-Abelian Gauge Theories,” Phys. Lett. B351, 230-234 (1995). [hep-th/9503086].
[26] D. Kutasov, A. Schwimmer, “On Duality in Supersymmetric Yang-Mills Theory,” [arXiv:hep-th/9505004].
[27] D. Kutasov, A. Schwimmer, N. Seiberg, “Chiral Rings, Singularity Theory and Electric-Magnetic Duality,” [arXiv:hep-th/9510222].
[28] K. Intriligator, “New RG Fixed Points and Duality in Supersymmetric $SP(N_c)$ and $SO(N_c)$ Gauge Theories,” [arXiv:hep-th/9505051].

[29] R.G. Leigh, M.J. Strassler, “Duality of $Sp(2N)$ and $SO(N)$ Supersymmetric Gauge Theories with Adjunct Matter,” [arXiv:hep-th/9505088].

[30] K. Intriligator, R.G. Leigh, M.J. Strassler, “New Examples of Duality in Chiral and Non-Chiral Supersymmetric Gauge Theories,” [arXiv:hep-th/9506148].

[31] O. Aharony, A. Hanany, K. A. Intriligator, N. Seiberg and M. J. Strassler, “Aspects of N=2 supersymmetric gauge theories in three-dimensions,” Nucl. Phys. B 499, 67 (1997) [arXiv:hep-th/9703110].

[32] V. Borokhov, A. Kapustin and X. -k. Wu, “Topological disorder operators in three-dimensional conformal field theory,” JHEP 0211, 049 (2002) [arXiv:hep-th/0206054].

[33] V. Borokhov, A. Kapustin and X. -k. Wu, “Monopole operators and mirror symmetry in three-dimensions,” JHEP 0212, 044 (2002) [arXiv:hep-th/0207074].

[34] T. Dimofte, D. Gaiotto and S. Gukov, “3-Manifolds and 3d Indices,” arXiv:1112.5179 [hep-th].

[35] D. Jafferis, I. Klebanov, S. Pufu, B. Safdi, “Towards the F-Theorem: $N=2$ Field Theories on the Three-Sphere,” JHEP 1106 (2011) 102. [arXiv:1103.1181 [hep-th]].

[36] N. Hama, K. Hosomichi, S. Lee, “Notes on SUSY Gauge Theories on Three-Sphere,” JHEP 1103, 127 (2011).

[37] A. Kapustin, B. Willett, I. Yaakov, "Exact Results for Wilson Loops in Superconformal Chern-Simons Theories with Matter," JHEP 1003 (2010) 089, [arXiv:0909.4559 [hep-th]].

[38] J. Bhattacharya and S. Minwalla, “Superconformal Indices for $N = 6$ Chern Simons Theories,” JHEP, 0901 (2009) 014, [arXiv:0806.3251 [hep-th]].

[39] S. Kim, “The complete superconformal index for N=6 Chern-Simons theory,” Nucl. Phys. B821 (2009) 241, [arXiv:0903.4172 [hep-th]].

[40] Y. Imamura and S. Yokoyama, “Index for three dimensional superconformal field theories with general R-charge assignments,” [arXiv:1101.0557 [hep-th]].

[41] O. Aharony, O. Bergman, D. L. Jafferis, “Fractional M2-branes,” [arXiv:0807.4924 [hep-th]].

[42] E. Witten, “SL(2,Z) Action On Three-Dimensional Conformal Field Theories With Abelian Symmetry,” [arXiv:hep-th/0307041].
[43] E. Witten, “Supersymmetric Index Of Three-Dimensional Gauge Theory,” [arXiv:hep-th/9903005].