Replica analysis of the generalized $p$-spin interaction glass model

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Abstract
We investigate the stability of replica symmetry breaking solutions in generalized $p$-spin models. It is shown that the kind of the transition to the one-step replica symmetry breaking state depends not only on the presence or absence of the reflection symmetry of the generalized $'p'$-operators $\hat{U}$ but on the number of interacting operators and their individual characteristics.

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(Some figures may appear in colour only in the online journal)

1. Introduction

In the study of spin glasses, the central role is played by the Sherrington–Kirkpatrick (SK) model [1]. It appeared as an attempt to describe unordered freezing of spins in dilute magnetic systems with disorder and frustration. This problem was soon solved at the mean-field level. It was demonstrated that the glassy phase in the SK model is characterized by the full replica symmetry breaking (FRSB) [2–4]. Later the $'p'$-spin model was introduced as a natural generalization of the SK model; $p$ spins interact in this model at each point of the lattice. It was shown in [5] that in this case of $p$-spin interactions when $(p \geq 3)$, the FRSB does not appear at the glass transition temperature $T_c$ and the 1-step replica symmetry breaking (1RSB) solution is stable in contrast to the SK model with 2-spin interaction. In addition, it was discovered that the glass order parameter in the $p$-spin model is discontinuous at $T_c$ contrary to the SK model.

The use of $p$-spin glass models as models for understanding structural glasses was pioneered in [6, 7]. It was shown that these models qualitatively describe many aspects of the glass transitions in liquids, e.g. two critical temperatures. The number of metastable states in these models is similar to that obtained in numerical modeling in liquids. The structure of
the dynamical equations for the correlation functions of supercooled liquids in mode-coupling theory and for the p-spin model is similar [8, 9].

Until now the p-spin glass model has remained the focus of intense investigations (see, e.g., the recent papers [10–15]) since it is a good starting point to understand the physics of real glasses. Some aspects are still far from being completely understood. We consider a generalization of the p-spin model of Ising spins where arbitrary diagonal operators \( \hat{U} \) stand instead of Ising spins [16, 17]. The operators have different meanings depending on the problem being studied. For example, Ising spin should be replaced with the molecule multipole moment if freezing of the orientational order is the target of the investigation [16–19].

An important development of the p-spin model is the study of the quadrupole system with \( J = 2 \) with multiparticle interaction, \( p = 3 [19] \). For the full molecule momentum, \( J = 2 \), and \( J_z = [0, \pm 1, \pm 2] \), the operator of the axial quadrupolar moment of a molecule has the form \( \hat{U} = \frac{1}{4} (3J_z^2 - 6) \). Using the operator \( \hat{U} \) instead of the spin in the p-spin model, it is possible to describe the orientational glass phase observed for high pressures in solid molecular ortho-\( \mathbf{H}_2 \) and para-\( \mathbf{H}_2 \). Moreover, it is possible to observe orientational transitions in such systems, which consist of initially spherically symmetric molecules with \( J = 0 \), because the probability of the transition, \( (J = 0) \rightarrow (J = 2) \), increases rapidly with the pressure. The computational results show that the glass state and the long-range orientational order coexist, which agrees of more than two particles play an important role.

We now consider a generalization of the p-spin model in the following way; the role of spins is played by the operators satisfying the condition of reflection symmetry, \( \text{Tr} \hat{U}^{(2k+1)} = 0 \) for any integer \( k \). Reflection symmetry of the operators \( \hat{U} \) leads to vanishing of a number of terms in the free energy, so that the replica symmetry (RS) solution for the order parameters is zero at high temperature. As a result, the behavior of 1RSB solution for the order parameters is like in the ordinary p-spin model of Ising spins with \( (p \geq 3) \) [5].

If operators \( \hat{U} \) do not have the reflection symmetry, \( \text{Tr} \hat{U}^{(2k+1)} \neq 0 \), then the glass freezing scenario is absolutely different from the Ising p-spin case. The characteristic properties of the system are already developed in the RS approximation. The nonlinear integral equation for the RS-glass order parameter simply has no trivial solutions at any temperature because the integrand is nonsymmetric due to the cubic terms in the free-energy expansion [16, 17]. There is a smooth increase in the order parameters (both glass and regular) as the temperature decreases. That is why 1RSB order parameters appear continually at the bifurcation point \( T_0 \). (In some sense, the 1RSB solution behaves here as in ordinary p-spin model with spins but in the external field [21, 22].)

We found \( m_1 \) analytically in the branching point and expressed the result through RS-order parameters. If \( m_1 \leq 1 \) in the branching point, then the 1RSB solution has physical meaning near \( T_0 \). The quadrupole glass with \( J = 1 \) and \( J_z = [0, \pm 1] \) (three-particle interaction) is the simplest example of the system without the reflection symmetry. In this case, \( \hat{U} = 3J_z^2 - 2 \) is the quadrupolar moment of the molecule (see figure 1 in [16]). Then, the 1RSB solution appears to be stable and it branches continuously at the bifurcation point \( T_0 = T_{\text{1RSB}} \) and smoothly on cooling.

If in the branching point \( m_1 > 1 \), then the 1RSB solution does not have a physical meaning in the vicinity of \( T_0 \). We suggest below an illustrative example of this conjecture. At other realizations of \( \hat{U} \) operators with \( \text{Tr} \hat{U}^{(2k+1)} \neq 0 \), the transition from RS to 1RSB does not take place at the bifurcation point \( T_0 \) where the 1RSB solution appears (see figure 1). Formally the 1RSB parameter \( m_1 > 1 \) at \( T_0 \); however, only \( m_1 \leq 1 \) have physical sense. While \( m_1 > 1 \) the 1RSB solution formally exists but it is unstable. The temperature \( T_{\text{1RSB}} \) where stable 1RSB appears coincides with the point where \( m_1 = 1 \). At this point, \( F_{\text{RS}} = F_{\text{1RSB}} \). When the
Figure 1. Order parameter evolution with temperature for the 3-quadrupole model in the subspace $J = 2$. For simplicity, only the physical solution for $\chi_{\text{1RSB}}$ is shown here. There are four characteristic temperatures in the model: $T_2$, $T_0$, $T_{\text{1RSB}}$, and $T^*$, where $\lambda_{\text{1RSB}} = 0$, $\lambda_{\text{RS}} = 0$, $m_{\text{1RSB}} = 1$ and the branches of 1RSB order parameters merge correspondingly. There is a region of temperatures in the graph, $T_0 < T < T^*$, where two different 1RSB solutions coexist with the RS solution. The physical solution should have the largest free energy. Our calculations show that the 1RSB solution (with $m_{\text{1RSB}} < 1$) has larger free energy than the RS solution when $T_0 < T < T_{\text{1RSB}}$. So this 1RSB solution is the physical one. The transition from the RS to the 1RSB solution occurs jumpwise at $T_{\text{1RSB}}$. At this point, $F_{\text{RS}} = F_{\text{1RSB}}$. So above $T_{\text{1RSB}}$, the RS solution is the physical one. When $T_2 < T < T_{\text{1RSB}}$ the 1RSB solution is also the physical one since it is stable with respect to small perturbations having 2RSB symmetry.

temperature is decreased, $m_1$ becomes smaller than 1 and the 1RSB solution leads to a larger (preferable) free energy than the RS solution.

Below we investigate the crossover from continuous to jumpwise behavior of the glass order parameters in generalized $p$-spin models using the bifurcation theory and analyze the stability. We show that in general, analytical progress can be made in the bifurcation region. We expand the boundaries of the ‘ordinary’ $p$-spin model and also consider 1RSB solutions for the pairwise interactions.

2. Generalized $p$-spin model

2.1. Main equations

The Hamiltonian of the $p$-spin model in general looks like

$$H = -\sum_{i_1 \leq i_2 \leq \ldots \leq i_p} J_{i_1 \ldots i_p} \hat{U}_{i_1} \hat{U}_{i_2} \ldots \hat{U}_{i_p},$$

(1)
where \( \hat{U} \) now is the arbitrary diagonal operator with \( \text{Tr}\hat{U} = 0 \), \( N \) is the number of sites on the lattice, \( i = 1, 2, \ldots, N \), and \( p \) is the finite integer giving the number of interacting particles. The coupling strengths are independent random variables with a Gaussian distribution

\[
P(J_{i_1 \cdots i_p}) = \frac{\sqrt{N^{p-1}}}{\sqrt{p! \pi J}} \exp \left[ -\frac{(J_{i_1 \cdots i_p})^2 N^{p-1}}{p! J^2} \right].
\]

Using the replica approach, we can write in general the free energy averaged over disorder:

\[
\langle F \rangle_J/NkT = \lim_{n \to 0} \frac{1}{n} \max \left\{ -\frac{t^2}{4} \sum_a (w^a)^p + \sum_a \mu^a w^a - \frac{t^2}{4} \sum_{a \neq \beta} (q^{a \beta})^p + \sum_{a \neq \beta} \lambda^{a \beta} q^{a \beta} - \ln \text{Tr}_{[U^{(1)}]} \exp \hat{\theta} \right\},
\]

where \( t = J/kT \) and

\[
\hat{\theta} = \sum_{a \neq \beta} \lambda^{a \beta} \hat{U}^a \hat{U}^\beta + \sum_a \mu^a (\hat{U}^a)^2.
\]

The extremum in equation (3) should be taken over the physical order parameters and over the corresponding Lagrange multipliers, \( \lambda^{a \beta} \) and \( \mu^a \). So the saddle point conditions give the glass order parameter \( q^{a \beta} \):

\[
q^{a \beta} = \frac{\text{Tr}[\hat{U}^a \hat{U}^\beta \exp(\hat{\theta})]}{\text{Tr}[\exp(\hat{\theta})]},
\]

the auxiliary order parameter \( w^a \) and the regular order parameter \( \chi^a \):

\[
w^a = \frac{\text{Tr}[(\hat{U}^a)^2 \exp(\hat{\theta})]}{\text{Tr}[\exp(\hat{\theta})]},
\]

\[
\chi^a = \frac{\text{Tr}[\hat{U}^a \exp(\hat{\theta})]}{\text{Tr}[\exp(\hat{\theta})]},
\]

and the parameters

\[
\lambda^{a \beta} = \frac{t^2}{4} p(q^{a \beta})^{(p-1)}, \quad \mu^a = \frac{t^2}{4} p(w^a)^{(p-1)}.
\]

To proceed with the RSB procedure, it is more convenient to rewrite equation (3) in the form

\[
\langle F \rangle_J/NkT = \lim_{n \to 0} \frac{1}{n} \max \left\{ (p - 1) \frac{t^2}{4} \sum_a (w^a)^p - \frac{t^2}{2} \sum_{a > \beta} (q^{a \beta})^p - \ln \text{Tr}_{[U]} \exp \hat{\theta} \right\},
\]

where

\[
\hat{\theta} = p \frac{t^2}{2} \sum_{a > \beta} (q^{a \beta})^{(p-1)} \hat{U}^a \hat{U}^\beta + \frac{t^2}{4} \sum_a (w^a)^{(p-1)} (\hat{U}^a)^2.
\]

Using the standard procedure (see, e.g., [4]), we perform the first stage of the RSB (\( n \) replicas are divided into \( n/m_1 \) groups with \( m_1 \) replicas in each) and obtain the expression for the free energy. Order parameters are denoted by \( q^{a \beta} = r_1 \) if \( a \) and \( \beta \) are from different groups and \( q^{a \beta} = 1 \) if \( a \) and \( \beta \) belong to the same group. So

\[
F_{\text{RSB}} = -NkT \left\{ m_1 t^2 (p - 1) \frac{p^p}{4} + (1 - m_1) (p - 1) t^2 \frac{(r_1 + v_1)^p}{4} - t^2 (p - 1) \frac{w_1^p}{4} + \frac{1}{m_1} \int \text{d}x^G \ln \int \text{d}x^G \text{Tr}[\exp(\hat{\theta}^{(m_1)})]^{m_1} \right\}.
\]
Here,
\[
\dot{\theta}_{\text{RS}} = \Delta T \sqrt{\frac{pr(\rho - 1)}{2} \hat{U} + 2, \quad \dot{\theta}_{\text{RS}}(t) \equiv \frac{1}{2} \dot{\theta}_{\text{RS}}(t) + H(t)}
\]
and
\[
\int dz^G = \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}} \exp \left( -\frac{z^2}{2} \right).
\]

The extremum conditions for \( F_{\text{RS}} \) yield equations for the glass order parameters \( r_1 \) and \( v_1 \), the additional order parameter \( w_1 \), the regular order parameter \( x_1 \) and the parameter \( m_1 \):

\[
r_1 = \int dz^G \left\{ \frac{\int dz^G [\text{Tr} \exp \hat{\theta}_{\text{RS}}(m_{1-1}) \text{Tr} \hat{U} \exp \hat{\theta}_{\text{RS}}]}{\int dz^G [\text{Tr} \exp \hat{\theta}_{\text{RS}}]} \right\}^2,
\]

\[
v_1 = \int dz^G \left\{ \frac{\int dz^G [\text{Tr} \exp \hat{\theta}_{\text{RS}}(m_{1-2}) \text{Tr} \hat{U} \exp \hat{\theta}_{\text{RS}}]}{\int dz^G [\text{Tr} \exp \hat{\theta}_{\text{RS}}]} \right\}^2 - \int dz^G \left\{ \frac{\int dz^G [\text{Tr} \exp \hat{\theta}_{\text{RS}}(m_{1-1}) \text{Tr} \hat{U} \exp \hat{\theta}_{\text{RS}}]}{\int dz^G [\text{Tr} \exp \hat{\theta}_{\text{RS}}]} \right\}^2,
\]

\[
w_1 = \int dz^G \left\{ \frac{\int dz^G [\text{Tr} \exp \hat{\theta}_{\text{RS}}(m_{1-1}) \text{Tr} \hat{U} \exp \hat{\theta}_{\text{RS}}]}{\int dz^G [\text{Tr} \exp \hat{\theta}_{\text{RS}}]} \right\}^2,
\]

\[
x_1 = \int dz^G \left\{ \frac{\int dz^G [\text{Tr} \exp \hat{\theta}_{\text{RS}}(m_{1-1}) \text{Tr} \hat{U} \exp \hat{\theta}_{\text{RS}}]}{\int dz^G [\text{Tr} \exp \hat{\theta}_{\text{RS}}]} \right\}.
\]

and

\[
m_1 \frac{(p-1)(r_1 + v)^{p-1} - (r_1)^{p-1}}{4} = \int dz^G \ln \left\{ \int dz^G [\text{Tr} \exp \hat{\theta}_{\text{RS}}] \right\}^2 + \int dz^G \left\{ \frac{\int dz^G [\text{Tr} \exp \hat{\theta}_{\text{RS}}]}{\int dz^G [\text{Tr} \exp \hat{\theta}_{\text{RS}}]} \right\}.
\]

If operators \( \hat{U} \) do not have the reflection symmetry, then the nontrivial solution for the regular order parameter \( x \) appears in spite of the fact that \( x \) is absent in \( \hat{\theta} \) (since \( J_0 = \langle J_{1-1} \rangle = 0 \) in (2)) [16, 17].

The heat capacity can be expressed through the glass order parameters:

\[
\frac{C_{v,\text{RS}}}{Nk} = \frac{d}{d(1/T)} \left[ \frac{m_1 r_1 + (1 - m_1)(r_1 + v)^p - (1)^p}{2} \right].
\]

The corresponding expressions for the RS approximation can be easily obtained from the preceding formulas (13)–(15) by setting \( v_1 = 0 \). For the glass order parameter \( q_{\text{RS}} \), we have

\[
q_{\text{RS}} = \int dz^G \left\{ \frac{\text{Tr} [\hat{U} \exp (\hat{\theta}_{\text{RS}})]}{\text{Tr} [\exp (\hat{\theta}_{\text{RS}})]} \right\}^2.
\]

Here,

\[
\dot{\theta}_{\text{RS}} = \Delta T \sqrt{\frac{p q_{\text{RS}}(p-1)}{2} \hat{U} + t^2 \left( \frac{w_{\text{RS}}(p-1)}{4} - q_{\text{RS}}(p-1) \right) \hat{U}^2}.
\]
2.2. Stability of the mean-field solutions

The stability of the saddle-point solution can be tested by the investigation of the Gaussian fluctuation contribution to the free energy near this solution. The mean-field (saddle-point) solution is stable while all the eigenmodes of the fluctuation propagator are positive. The most important mode is the so-called replicon mode [3, 17] since only its sign is usually very sensitive to the RSB degree and to the temperature. For example, the RS solution is stable unless the corresponding replicon mode energy \( \lambda_{\text{rep}} > 0 \). The RS solution can break at the temperature \( T_0 \) determined by the equation \( \lambda_{\text{rep}} = 0 \), where

\[
\lambda_{\text{rep}} = 1 + t^2 \left( \frac{p(p - 1)q_{\text{ex}}^{(p-2)}}{2} \int d\theta \left\{ \frac{\text{Tr}(\hat{U}^2 e^{\hat{\theta}})}{\text{Tr} e^{\hat{\theta}}} - \left[ \frac{\text{Tr} \hat{U} e^{\hat{\theta}}}{\text{Tr} e^{\hat{\theta}}} \right]^2 \right\}^2. \]  

(21)

The equation \( \lambda_{\text{rep}} = 0 \) can be obtained as the branching condition for equation (14), i.e. as a condition that a small solution with 1RSB can appear.

We see that the equation for the glass order parameter, equation (19), contains deviations \( \delta_q \). In order to get in general the 1RSB solution near the bifurcation point \( T_0 \neq 0 \), we can find that \( T_0 \neq 0 \). Then, the solutions with the unbroken symmetry may appear continuously not only for \( p = 2 \) but also at any integer \( p \). For example, the stable continuous 1RSB solution exists for \( p = 3 \) when \( \hat{U} = 3L^2 - 2 \) is the axial quadrupole moment in the subspace \( I = 1 \) with \( J_z \neq 0, \pm 1 \) (see figure 1 in [16]). Below we prove these statements.

In order to get in general the 1RSB solution near the bifurcation point \( T_0 \) where it is close to the RS solution, we expand the free energy (8)–(9) up to the third order, assuming that the deviations \( \delta q^p \rho^\delta \) from \( q_{\text{ex}} \) and \( \rho \) from \( w_{\text{ex}} \) are small.

We use the notation \( \Delta F \) for the difference of the free energy \( F_{\text{(ex)}} \) from its replica symmetry part \( F_{0,\text{(ex)}} \). So

\[
\frac{\Delta F}{NkT} = \frac{t^2 p(p - 1)q_{\text{ex}}^{(p-2)}}{2} \left\{ 1 - t^2 W \right\} - \frac{t^4}{2} L \left( r - (m_1 - 1)v_1 \right)^2 - \frac{t^6}{2} \left[ C \left( r - (m_1 - 1)v_1 \right)^3 \right] \\
+ D \left( r - (m_1 - 1)v_1 \right) v_1 m_1 (m_1 - 1) - B_3 v_1^3 m_1^2 (m_1 - 1) \\
+ B_4 v_1^3 m_1 (m_1 - 1)(2m_1 - 1) + \Psi(\rho) + \cdots, \]  

(22)

where \( t = t_0 + \Delta t, r_1 = q_{\text{ex}} + r, w_{\text{ex}} = w_{\text{ex}} + \rho \) and the expressions for \( W, L, C, D, B_3, B_4 \) and \( \Psi \) are some combinations of operators averaged over the RS solution (see appendices A and B). For example, the coefficient \( L \) enters \( \Delta F \) like \( \lim_{\rho \to 0} \frac{1}{\sum_{\rho^\delta} \delta^{p\delta}} \delta^{p\delta} \delta^{q^p} \).

First we consider the case \( \text{Tr} \hat{U}^{(2k+1)} \neq 0 \). Then, \( L_{k=0} \neq 0 \). Using extremum conditions for the free energy (22) and the inequality \( L_{k=0} \neq 0 \), we obtain the branching condition

\[
r - (m_1 - 1) v_1 = 0 + o(\Delta t)^2. \]  

(23)
This condition states that there is no linear term for the glass order parameters. There is no other linear term because \[ 1 - t^4 W \] at the branch point. Since the coefficients \( A_k \) of \( \lambda \rho \) in the expression for \( \Psi(\rho) \) (appendix B) are in general nonzero, then from the extremum condition, we get \( \varrho \sim [r - (m_1 - 1)v_1] \) and \( \Psi(\rho) = 0 + o(\Delta t)^2 \). Finally, we get

\[
2m_1 (1 - m_1) \Gamma \Delta t = 3u_0^2 m_1 (1 - m_1) [-B_4 + m_1 (-B_3 + 2B_4)] v_1 ,
\]

(24)

\[
(2m_1 - 1) \Gamma \Delta t = 3u_0^2 [ (2m_1 - 1) [-B_4 + m_1 (-B_3 + 2B_4)] \\
+ m_1 (m_1 - 1) (-B_3 + 2B_4)] v_1 ,
\]

(25)

where \( \Gamma \) is given explicitly in appendix B.

Here \( B_3, B_4 \) and \( \Gamma \) are taken at \( T = T_0 \). Then, we find from (24) and (25) (the cases \( m_1 = 0 \) and \( m_1 = 1 \) should be investigated separately, see, e.g., [25])

\[
m_1 = B_4 / B_3 ,
\]

(26)

at the branch point \( T_0 \) where the 1RSB solution appears and

\[
r_1 = q_\text{es} + (m_1 - 1)v_1 , \quad v_1 \sim \Delta t ,
\]

(27)

in the neighborhood of \( T_0 \).

The coefficient of proportionality (27) depends only on the RS solution at \( T_0 \):

\[
v_1 = \frac{p(p - 1)}{2} \frac{q_\text{es}^{(p-2)}}{6B_4(1 - m_1) u_0^2} \left\{ 1 + \frac{\gamma p(2 - p)}{2} \frac{q_\text{es}}{u_\text{es}} + \frac{\gamma^2 p^2 (p - 1)}{4} \frac{q_\text{es}^{(p-2)}}{u_\text{es}} \right\} \Delta t ,
\]

(28)

where

\[
\gamma = \left[ \left( \frac{w_\text{es}^{(p-1)}}{q_\text{es}} + \frac{\gamma p(2 - p)}{2} \frac{w_\text{es}^{(p-2)}}{q_\text{es}} \right) K_1 + \left( \frac{q_\text{es}^{(p-1)}}{w_\text{es}} + \frac{\gamma p(2 - p)}{2} \frac{q_\text{es}^{(p-2)}}{w_\text{es}} \right) K_2 \right] ,
\]

(29)

where \( q_\text{es} = \frac{\partial}{\partial \rho} q_\text{es} \) and \( w_\text{es} = \frac{\partial}{\partial \rho} w_\text{es} \). Expressions \( K_1 \) and \( K_2 \) are quite long and they are written in appendix B.

The 1RSB solution appears smoothly in most cases from the RS solution. However, if the order parameter in the branching point takes the unphysical values \( m_1 > 1 \), then at the temperature where \( m_1 = 1 \) and the physical (stable) 1RSB solution appears, it is drastically different from the RS solution and as a result the jump singularity appears in the heat capacity (e.g. [19]). The free energy does not have the discontinuity in this case (see equation (10)).

As an example, we consider the case of quadrupolar glass \( \hat{U} = \hat{Q} = \frac{1}{3} \left[ 3Jz^2 - 6 \right] \) [19] in the subspace \( J = 2 \) in detail and write the explicit solutions of equations (13) and (17) for the number of interacting particles \( p = 3 \) (see figure 1). The glass state and the state with the long-range orientational order coexist. The 1RSB solution appears smoothly from the RS solution. But in this case, the condition \( \lambda_{\text{RS,off}} = 0 \) does not determine the physical solution in the neighborhood of the branch point \( T_0 \), namely a transition to the nonphysical branch of the free energy takes place. In fact, the transition from the RS to the 1RSB solution occurs jumpwise at the point \( T_{\text{RS}} > T_0 \) determined by the condition \( m_1 = 1 \). At this point, \( F_{\text{RS}} = F_{\text{RSB}} \). The RS solution is stable above \( T_{\text{RSB}} \). When the temperature decreases, \( m_1 \) becomes smaller than 1 and the corresponding physical 1RSB solution corresponds to larger (preferable) free energy than the RS solution. An exceptionally important property of the model is that there exists a domain of stability where the 1RSB solution remains stable under further RSBs (see below).

Now we will consider the case \( \text{Tr} \hat{U}^{(2k+1)} = 0 \) and investigate the 1RSB solution near the branching point. Let us return to expression (22) for free energy at \( p = 2 \) since only in
this case the nontrivial 1RSB solution smoothly branches at finite \( t \). Note that in the case of zero RS solution for the glass order parameter, the expansion does not contain the terms where some indices occur only once. In the case of reflection symmetry operators, there are no terms where some indices occur an odd number of times, \( L_{1\text{RSB}} = 0 \) (see appendix B), and the branching condition (23) for 1RSB fails. Moreover, at the branching point, we have \( B_3 = B_4 = B_2 = B_0 = A_3 = A_{12} = A_{14} = 0; \ C = 2B_3, \ D = -3B_3 \). From the extremum condition, we find \( \Psi(\rho) = 0 + o(\Delta t)^4 \) and

\[
\begin{align*}
    m_1 &= 0, \\
    r_1 + v_1 = \frac{1 + \lambda_0^4 \left( w_{m_0} + \frac{3}{2} w_{m_0} \right) \left[ \left( \hat{U}_1^2 \hat{U}_2^2 \hat{U}_3^2 \right) - \left[ \hat{U}_1^2 \hat{U}_2^2 \hat{U}_3^2 \right] \right]}{\Delta t}.
\end{align*}
\]

It has been shown [26] that the Parisi FRSB scheme can be used not only for the SK model but also for any model with pair interactions and with reflection symmetry \( \text{Tr} \hat{U}^{(2k+1)} = 0 \), for example, such as spin glasses with arbitrary spin.

We break the RS once more and obtain the corresponding expressions for the free energy and the order parameters. The bifurcation condition for the 1RSB solution is 0 determining the temperature \( T = T_3 \) follows from the condition that a nontrivial small solution for the 2RSB glass order parameter appears as \( v_2 \to 0 \). We have

\[
\begin{align*}
    \lambda_{1\text{RSB,refl}} &= 1 - t^2 \frac{p(p-1)(r_1 + v_1)^{p-2}}{2}
    \times \frac{\int \text{d}z^G [\text{Tr} \exp(\hat{\theta}_{\text{1RSB}})]^{m_0} \left\{ \text{Tr} \left[ \hat{U}_1^2 \exp(\hat{\theta}_{\text{1RSB}}) \right] \right\} - \left\{ \text{Tr} \left[ \hat{U}_1 \exp(\hat{\theta}_{\text{1RSB}}) \right] \right\}^2 \right\}^2}{\int \text{d}z^G [\text{Tr} \exp(\hat{\theta}_{\text{1RSB}})]^{m_0}}.
\end{align*}
\]

Note that equation (32) depends only on the 1RSB solution. The expression for \( \lambda_{1\text{RSB,refl}} = 0 \) always has the solution for \( v_2 = 0 \), which determines the point \( T_0 \) and coincides with the solution of equation (21) \( \lambda_{1\text{RSB,refl}} = 0 \) (see figure 1).

Using the expressions for the glass order parameters obtained above near \( T_0 \), we can show that \( \frac{\partial}{\partial \lambda_{1\text{RSB,refl}}} |_{T_0} = 0 \) as for the two-body interaction in the presence of reflection symmetry and for the case \( \text{Tr} \hat{U}^{(2k+1)} \neq 0 \) and arbitrary \( p \).

In addition to the point \( T_0 \), one more bifurcation point \( \lambda_{1\text{RSB,refl}} = 0 \) (see, e.g., figure 1) may exist as \( v_2 \neq 0 \), and the 2RSB solution can appear at this point. At the point \( T_2 \), a transition to the FRSB state or to a stable 2RSB state may take place.

We now present several results that hold in the two cases above. They concern the general form of the expressions determining the stability of 1RSB solutions. We will investigate the stability of the 1RSB solution against small perturbations having 2RSB symmetry for any value of the inverse temperature \( t \). For the group of replicas of \( m_1 \) elements, we divide by \( m_1/m_2 \) groups of \( m_2 \) elements in order to find the 2RSB solution. The parameter \( q^{\alpha\beta} \) in the 2RSB case we shall denote as \( q^{\alpha\beta}_{2\text{RSB}} \), if replicas \( \alpha \) and \( \beta \) belong to the same subgroup and at the same time to the smallest subgroup (number of \( q^{\alpha\beta}_{2\text{RSB}} \) elements is equal to \( n(m_2 - 1)/2 \)). The elements of \( q^{\alpha\beta} \) will be labeled by \( q^{\alpha\beta}_{2\text{RSB}} \) if replicas belong to the same subgroup, but this subgroup is not the smallest one (there are \( n(m_1 - m_2)/2 \) replicas of that kind). Finally the notation \( q^{0\alpha}_{2\text{RSB}} \) will be used if the replicas \( \alpha \) and \( \beta \) belong to different subgroups (there are \( n(n - m_1)/2 \) replicas of this kind). We set \( q^{\alpha\beta}_{2\text{RSB}} = q^{\alpha\beta}_{2\text{RSB}} + \delta q^{\alpha\beta}_{2\text{RSB}} \) and assume that the deviations of the 1RSB solution from the 2RSB are small. We believe that one can neglect the changes of the order parameter \( w_{2\text{RSB}} \). Then, the free energy \( \Delta F_2 = F_{2\text{RSB}} - F_{1\text{RSB}} \) can be conveniently represented as follows:

\[
\Delta F_2/NkT = \frac{t^2}{16} \left\{ (\delta q^{2\text{RSB}}_0)^2 a + (\psi_2)^2 b + (v_2)^2 c \right\},
\]

(33)
where the following substitution was used

\[ \delta q_1^{\text{2RSB}} = \psi_2 - \frac{d}{b} \delta q_0^{\text{2RSB}} + (m_2 - 1)v_2, \] (34)

\[ \delta q_2^{\text{2RSB}} = \psi_2 - \frac{d}{b} \delta q_0^{\text{2RSB}} + (m_2 - m_1)v_2. \] (35)

Equation (33) was found in [27] for the pairwise quadrupole interaction \((J = 1, p = 2)\) without any assumptions about the behavior of the order parameter \(w_{\text{2RSB}}\).

Parameters \(a, b, c, d\) can be expressed through averages found on the 1RSB solution. For example,

\[ c = -2\lambda_{(\text{1RSB}_{\text{repl}}}(m_2 - m_1)(1 - m_1)(1 - m_2)p(p - 1)(r_1 + v_1)^{(p-2)}. \] (36)

Then, we find that the parameter \(c \leq 0\), if \(\lambda_{(\text{1RSB}_{\text{repl}}}> 0\), because from equations (13) and (14) it follows that \((r_1 + v_1) > 0\). Moreover, the RSB scenario [4] assumes that in the limit \(n \to 0\), we obtain \((m_2 - m_1) > 0, (1 - m_1) \geq 0\) and \((1 - m_2) \geq 0\). The explicit form of the other parameters, namely, \(a, b, d\), is given in appendix C. There it is shown that \(b < 0\) while \(\lambda_{(\text{1RSB}_{\text{repl}}}) > 0\).

It is shown in appendix C that while \(\hat{U}\) satisfies the reflection symmetry condition, the coefficient \(y_1 \neq 0\) and the stability is determined by the sign of \(\lambda_{(\text{1RSB}_{\text{repl}}}(a < 0\) if \(\lambda_{(\text{1RSB}_{\text{repl}}}> 0\).

The direct numerical calculation in the subspace: \(\hat{U}\)—quadrupole operator with \(J = 2, p = 3\), shows that \(a < 0\) if \(\lambda_{(\text{1RSB}_{\text{repl}}}> 0\) and at the same time \((m_1 - 1) < 0\), see figure 1. Thus, we can conclude that the 1RSB solution is stable indeed in this subgroup.

3. Conclusions

To conclude, we have demonstrated that the 1RSB solution in generalized \(p\)-spin models behaves differently depending on the symmetry of the operators \(\hat{U}\). If the operators \(\hat{U}\) have zero trace for all odd powers, \(\text{Tr}\hat{U}^{(2k+1)} = 0\) for all integer \(k\), then their trivial solution, \(q_\text{RS} = 0\), for the RS order parameter always exists. Therefore, the bifurcation condition, \(\lambda_{(\text{1RSB}_{\text{repl}}}) = 0\), can be satisfied for finite \(p \geq 3\) only at \(T_0 = 0\). In this case, the 1RSB solution smoothly branches only for \(p = 2\). When \(p \geq 3\) and \(\text{Tr}\hat{U}^{(2k+1)} = 0\), the 1RSB solution can only appear discontinuously at the temperature defined by the condition, \(m_1 = 1\) \((T_{\text{2RSB}} \neq 0)\) as in the usual \(p\)-spin model. The 1RSB solution is stable for \(\lambda_{(\text{1RSB}_{\text{repl}}}> 0\).

In contrast when \(\text{Tr}\hat{U}^{(2k+1)} \neq 0\), solutions with broken RS may smoothly appear not only in the case of pair interactions, \(p = 2\), but also at any finite \(p > 2\). The point is that in this case the trivial RS solution does not exist. The 1RSB branching condition \(\lambda_{(\text{1RSB}_{\text{repl}}}) = 0\) is satisfied at finite temperature \(T_0\). The solution with broken symmetry appears smoothly if \(m_1(T_{\text{1RSB}}) \leq 1\) otherwise the physically stable 1RSB solution appears discontinuously at temperature where \(m_1 = 1\). An algebraical expression for \(m_1(T_{\text{1RSB}})\) is obtained. It only depends on the RS solution at \(T_0\).

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Appendix A

The only nonzero sums

\[
\lim_{n \to 0} \frac{1}{n} \sum_{\alpha, \beta} (\delta q^{\alpha \beta})^2 = -[r - (m_1 - 1)v_1]^2 - m_1(1 - m_1)v_1^2; \\
\text{(A.1)}
\]

\[
\lim_{n \to 0} \frac{1}{n} \sum_{\alpha, \beta, \delta} \delta q^{\alpha \beta} \delta q^{\delta \beta} = [r - (m_1 - 1)v_1]^2; \\
\text{(A.2)}
\]

\[
\lim_{n \to 0} \frac{1}{n} \sum_{\alpha, \beta} (\delta q^{\alpha \beta})^3 = -[r - (m_1 - 1)v_1]^3 + 3m_1(m_1 - 1)[r - (m_1 - 1)v_1]v_1^2 \\
+ m_1(m_1 - 1)(2m_1 - 1)v_1^3; \\
\text{(A.3)}
\]

\[
\lim_{n \to 0} \frac{1}{n} \sum_{\alpha, \beta, \gamma} \delta q^{\alpha \beta} \delta q^{\beta \gamma} = 2[r - (m_1 - 1)v_1]^3 - 3m_1(m_1 - 1)[r - (m_1 - 1)v_1]v_1^2 \\
-m_1^2(m_1 - 1)v_1^3; \\
\text{(A.4)}
\]

\[
\lim_{n \to 0} \frac{1}{n} \sum_{\alpha, \beta, \gamma} (\delta q^{\alpha \beta})^2 \delta q^{\gamma \beta} = [r - (m_1 - 1)v_1]^3 - [r - (m_1 - 1)v_1]v_1^2; \\
\text{(A.5)}
\]

\[
\lim_{n \to 0} \frac{1}{n} \sum_{\alpha, \beta, \gamma, \delta} \delta q^{\alpha \beta} \delta q^{\beta \gamma} \delta q^{\gamma \delta} = \lim_{n \to 0} \frac{1}{n} \sum_{\alpha, \beta, \gamma, \delta} \delta q^{\alpha \beta} \delta q^{\beta \gamma} \delta q^{\gamma \delta} = -[r - (m_1 - 1)v_1]^3. \\
\text{(A.6)}
\]

The prime on the sum means that only the superscripts belonging to the same \(\delta q\) are necessarily different in \(\sum'\).

Appendix B

\[
W = \left[ \frac{p(p - 1)}{2} q^{(p-2)}_{ss} \right] \left[ \{\hat{U}_1^2 \hat{U}_2^2\} - 2\{\hat{U}_1^2 \hat{U}_2 \hat{U}_3\} + \{\hat{U}_1 \hat{U}_2 \hat{U}_3 \hat{U}_4\} \right]; \\
\text{(B.1)}
\]

\[
L = \left[ \frac{p(p - 1)}{2} q^{(p-2)}_{ss} \right]^2 \left[ \{\hat{U}_1^2 \hat{U}_2^2 \hat{U}_3\} - \{\hat{U}_1 \hat{U}_2 \hat{U}_3 \hat{U}_4\} \right]; \\
\text{(B.2)}
\]

\[
C = \left[ \frac{p(p - 1)}{2} q^{(p-2)}_{ss} \right]^3 \left[ -(B_2 + B'_2) + 2B_3 + B'_3 - B_4 \right]; \\
\text{(B.3)}
\]

\[
D = \left[ \frac{p(p - 1)}{2} q^{(p-2)}_{ss} \right]^3 \left[ -3B_3 - B'_3 + 3B_4 \right]; \\
\text{(B.4)}
\]
and

\[ B_2 = \left[ \frac{p(p-1)}{2} q_{\text{res}}^{(p-2)} \right]^3 \left\{ \frac{1}{2} \langle \hat{U}_1^2 \hat{U}_2 \hat{U}_3 \hat{U}_4 \rangle + \frac{1}{2} \langle \hat{U}_1 \hat{U}_2 \hat{U}_3 \hat{U}_4 \hat{U}_5 \rangle \right\} + \frac{1}{2} \langle \hat{U}_1^2 \hat{U}_2 \hat{U}_3 \hat{U}_5 \rangle ; \]

\[ B_3 = \left[ \frac{p(p-1)}{2} q_{\text{res}}^{(p-2)} \right]^3 \left\{ \frac{1}{3} \langle \hat{U}_1 \hat{U}_2 \hat{U}_3 \hat{U}_5 \hat{U}_6 \rangle - \frac{1}{2} \langle \hat{U}_1 \hat{U}_2 \hat{U}_3 \hat{U}_4 \hat{U}_6 \rangle + \frac{1}{6} \langle \hat{U}_1 \hat{U}_2 \hat{U}_3 \hat{U}_4 \hat{U}_6 \rangle \right\} ; \]

\[ B_4 = \left[ \frac{p(p-1)}{2} q_{\text{res}}^{(p-2)} \right]^3 \left\{ \frac{1}{6} \langle \hat{U}_1 \hat{U}_2 \hat{U}_3 \hat{U}_5 \hat{U}_6 \rangle - \frac{1}{2} \langle \hat{U}_1 \hat{U}_2 \hat{U}_3 \hat{U}_4 \hat{U}_6 \rangle + \frac{1}{6} \langle \hat{U}_1 \hat{U}_2 \hat{U}_3 \hat{U}_4 \hat{U}_6 \rangle \right\} ; \]

\[ \rho B_3 = \rho \left[ \frac{p(p-1)}{2} q_{\text{res}}^{(p-2)} \right]^3 \left\{ \frac{3}{2} \langle \hat{U}_1 \hat{U}_2 \hat{U}_3 \hat{U}_5 \hat{U}_6 \rangle - \frac{1}{2} \langle \hat{U}_1 \hat{U}_2 \hat{U}_3 \hat{U}_4 \hat{U}_6 \rangle + \frac{1}{6} \langle \hat{U}_1 \hat{U}_2 \hat{U}_3 \hat{U}_4 \hat{U}_6 \rangle \right\} - \frac{t^2 p^2 (p-1)^2 (p-2)}{8} q_{\text{res}}^{(p-2)} \left[ 1 - \frac{t^2}{2} W \right] ; \]

\[ K_1 = \langle \hat{U}_1^2 \hat{U}_2^2 \hat{U}_3^2 \hat{U}_4^2 \hat{U}_5^2 \hat{U}_6^2 \rangle - \langle \hat{U}_1^4 \hat{U}_2^2 \hat{U}_3^2 \hat{U}_4^2 \hat{U}_5^2 \hat{U}_6^2 \rangle - 2 \langle \hat{U}_1^2 \hat{U}_2^4 \hat{U}_3^2 \hat{U}_4^2 \hat{U}_5^2 \hat{U}_6^2 \rangle - 2 \langle \hat{U}_1 \hat{U}_2 \hat{U}_3 \hat{U}_4 \hat{U}_5 \hat{U}_6 \rangle \]

\[ + 3 \langle \hat{U}_1 \hat{U}_2 \hat{U}_3 \hat{U}_4 \hat{U}_5 \hat{U}_6 \rangle + 2 \langle \hat{U}_1 \hat{U}_2 \hat{U}_3 \hat{U}_4 \hat{U}_5 \hat{U}_6 \rangle + 2 \langle \hat{U}_1 \hat{U}_2 \hat{U}_3 \hat{U}_4 \hat{U}_5 \hat{U}_6 \rangle ; \]

\[ K_2 = \langle \hat{U}_1^2 \hat{U}_2^2 \hat{U}_3^2 \hat{U}_4^2 \hat{U}_5^2 \hat{U}_6^2 \rangle - 2 \langle \hat{U}_1^2 \hat{U}_2^2 \hat{U}_3^2 \hat{U}_4^2 \hat{U}_5^2 \hat{U}_6^2 \rangle + 2 \langle \hat{U}_1 \hat{U}_2 \hat{U}_3 \hat{U}_4 \hat{U}_5 \hat{U}_6 \rangle - 2 \langle \hat{U}_1 \hat{U}_2 \hat{U}_3 \hat{U}_4 \hat{U}_5 \hat{U}_6 \rangle - 2 \langle \hat{U}_1 \hat{U}_2 \hat{U}_3 \hat{U}_4 \hat{U}_5 \hat{U}_6 \rangle ; \]

\[ A_2 = \langle \hat{U}_1^2 \hat{U}_2^2 \hat{U}_3^2 \hat{U}_4^2 \hat{U}_5^2 \hat{U}_6^2 \rangle + \langle \hat{U}_1^2 \hat{U}_2^2 \hat{U}_3^2 \hat{U}_4^2 \hat{U}_5^2 \hat{U}_6^2 \rangle ; \]

\[ A_3 = \langle \hat{U}_1^2 \hat{U}_2^2 \hat{U}_3^2 \hat{U}_4^2 \hat{U}_5^2 \hat{U}_6^2 \rangle - \langle \hat{U}_1^2 \hat{U}_2^2 \hat{U}_3^2 \hat{U}_4^2 \hat{U}_5^2 \hat{U}_6^2 \rangle ; \]

\[ A_{11} = \langle \hat{U}_1 \hat{U}_2 \hat{U}_3 \hat{U}_4 \hat{U}_5 \hat{U}_6 \rangle - 3 \langle \hat{U}_1 \hat{U}_2 \hat{U}_3 \hat{U}_4 \hat{U}_5 \hat{U}_6 \rangle + 2 \langle \hat{U}_1 \hat{U}_2 \hat{U}_3 \hat{U}_4 \hat{U}_5 \hat{U}_6 \rangle ; \]

\[ A_{12} = \langle \hat{U}_1 \hat{U}_2 \hat{U}_3 \hat{U}_4 \hat{U}_5 \hat{U}_6 \rangle + 4 \langle \hat{U}_1 \hat{U}_2 \hat{U}_3 \hat{U}_4 \hat{U}_5 \hat{U}_6 \rangle - \langle \hat{U}_1 \hat{U}_2 \hat{U}_3 \hat{U}_4 \hat{U}_5 \hat{U}_6 \rangle - \langle \hat{U}_1 \hat{U}_2 \hat{U}_3 \hat{U}_4 \hat{U}_5 \hat{U}_6 \rangle ; \]

\[ A_{13} = - \langle \hat{U}_1 \hat{U}_2 \hat{U}_3 \hat{U}_4 \hat{U}_5 \hat{U}_6 \rangle + \langle \hat{U}_1 \hat{U}_2 \hat{U}_3 \hat{U}_4 \hat{U}_5 \hat{U}_6 \rangle + 3 \langle \hat{U}_1 \hat{U}_2 \hat{U}_3 \hat{U}_4 \hat{U}_5 \hat{U}_6 \rangle \]

\[ - 2 \langle \hat{U}_1 \hat{U}_2 \hat{U}_3 \hat{U}_4 \hat{U}_5 \hat{U}_6 \rangle - \langle \hat{U}_1 \hat{U}_2 \hat{U}_3 \hat{U}_4 \hat{U}_5 \hat{U}_6 \rangle - 2 \langle \hat{U}_1 \hat{U}_2 \hat{U}_3 \hat{U}_4 \hat{U}_5 \hat{U}_6 \rangle + 2 \langle \hat{U}_1 \hat{U}_2 \hat{U}_3 \hat{U}_4 \hat{U}_5 \hat{U}_6 \rangle ; \]

\[ A_{14} = 3 \langle \hat{U}_1 \hat{U}_2 \hat{U}_3 \hat{U}_4 \hat{U}_5 \hat{U}_6 \rangle - 2 \langle \hat{U}_1 \hat{U}_2 \hat{U}_3 \hat{U}_4 \hat{U}_5 \hat{U}_6 \rangle \]

\[ - \langle \hat{U}_1 \hat{U}_2 \hat{U}_3 \hat{U}_4 \hat{U}_5 \hat{U}_6 \rangle - 4 \langle \hat{U}_1 \hat{U}_2 \hat{U}_3 \hat{U}_4 \hat{U}_5 \hat{U}_6 \rangle + 4 \langle \hat{U}_1 \hat{U}_2 \hat{U}_3 \hat{U}_4 \hat{U}_5 \hat{U}_6 \rangle. \]
\[ \Psi(\rho) = \rho^2 \frac{t^2}{4} \frac{p(p-1)}{2} w_{\text{rs}}^{(p-2)} \left[ 1 + \frac{p(p-1)}{2} w_{\text{rs}}^{(p-2)} t^2 \right] 
+ \rho \left[ r - (m_1 - 1) v_1 \right] \left[ \frac{p(p-1)}{2} \right] w_{\text{rs}}^{(p-2)} q_{\text{rs}}^{(p-2)} t^2 \right] A_3 
+ \rho^3 p(p-1)(p-2) \frac{t^2}{4} \left[ 1 + \frac{3p(p-1)}{8} q_{\text{rs}}^{(p-2)} \right] A_2 
+ \rho^2 \left[ r - (m_1 - 1) v_1 \right] \left[ \frac{p(p-1)}{2} \right] (p-2) \frac{t^4}{2} w_{\text{rs}}^{(p-2)} q_{\text{rs}}^{(p-2)} A_3 
+ \rho \left[ r - (m_1 - 1) v_1 \right] (p-1) ^2 + v^2 m_1 (1 - m_1) \left[ \frac{p(p-1)}{2} \right] \frac{t^2}{2} w_{\text{rs}}^{(p-2)} q_{\text{rs}}^{(p-3)} A_3 
- \rho^3 \frac{t^6}{48} \left[ \frac{p(p-1)}{2} w_{\text{rs}}^{(p-2)} \right]^3 A_{11} 
- \rho^2 \left[ r - (m_1 - 1) v_1 \right] \frac{t^6}{8} \left[ \frac{p(p-1)}{2} \right] w_{\text{rs}}^{(p-2)} q_{\text{rs}}^{(p-2)} A_{12} 
+ \rho \left[ r - (m_1 - 1) v_1 \right] \frac{t^6}{4} \left[ \frac{p(p-1)}{2} \right] w_{\text{rs}}^{(p-2)} q_{\text{rs}}^{(p-2)} A_{13} 
+ \rho \left[ r - (m_1 - 1) v_1 \right] \frac{t^6}{4} \left[ \frac{p(p-1)}{2} \right] w_{\text{rs}}^{(p-2)} q_{\text{rs}}^{(p-2)} A_{14}; \]

\[ \Gamma = \frac{d}{dt} \left[ \frac{t^2}{4} \frac{p(p-1)}{2} d_{\text{rs}}^{(p-2)} \left[ 1 - t^2 W \right] \right]. \]

where

\[ \langle \hat{U}_1 \hat{U}_2 \ldots \rangle = \frac{\text{Tr}[\hat{U}_1 \hat{U}_2 \ldots \exp \hat{X}]}{\text{Tr} \exp \hat{X}} \]

and

\[ \hat{X} = \frac{t^2}{4} p w_{\text{rs}}^{(p-1)} \sum_a (\hat{U}_a)^2 + \frac{t^2}{4} p q_{\text{rs}}^{(p-1)} \sum_{a \neq \beta} (\hat{U}_a \hat{U}_\beta). \]

Appendix C

\[ b = -(1 - m_1) 2 \lambda (1_{\text{RSB}} \log p - 1) (r_1 + v_1)^{(p-2)} \]

\[ -(1 - m_1)^2 \frac{t^2}{2} p^2 (p-1)^2 (r_1 + v_1)^{(p-2)} \left\{ 4 \int dG^2 \frac{\int dG [\text{Tr} e^{\hat{U}_{\text{ras}}}] m_1 [\text{Tr} (\hat{U}_{\text{ras}} e^{\hat{U}_{\text{ras}}})]^{m_1} \left[ \frac{\text{Tr} e^{\hat{U}_{\text{ras}}}}{\text{Tr} e^{\hat{U}_{\text{ras}}}} \right]^{2} }{\int dG [\text{Tr} e^{\hat{U}_{\text{ras}}}] m_1} \right\}
+ (m_1 - 4) \int dG^2 \frac{\int dG [\text{Tr} e^{\hat{U}_{\text{ras}}}] m_1 [\text{Tr} (\hat{U}_{\text{ras}} e^{\hat{U}_{\text{ras}}})]^{m_1}}{\int dG [\text{Tr} e^{\hat{U}_{\text{ras}}}] m_1} \]

\[ - m_1 \int dG^2 \left\{ \frac{\int dG [\text{Tr} e^{\hat{U}_{\text{ras}}}] m_1 [\text{Tr} (\hat{U}_{\text{ras}} e^{\hat{U}_{\text{ras}}})]^{m_1}}{\int dG [\text{Tr} e^{\hat{U}_{\text{ras}}}] m_1} \right\}^{2}. \]
We will further use the following notations: \( \hat{\lambda}_{\text{RSB}} = \Theta \) and \( \langle J_0 \rangle = \int dx^G \langle \text{Tr} e^\Theta \rangle_{m_1} \).

It follows from the Cauchy–Schwarz inequality that the expression \((\text{Tr} \hat{U}^2 e^\Theta)(\text{Tr} e^\Theta) \geq (\text{Tr} \hat{U} e^\Theta)^2 \) follows from \((\sum_n a_n^2) (\sum_n b_n^2) \geq (\sum_n a_n b_n)^2 \). So

\[
4 \int \frac{dx^G}{\langle J_0 \rangle} \int dx^G (\text{Tr} e^\Theta)^{m_1-4} (\text{Tr} \hat{U} e^\Theta)^2 \left[ (\text{Tr} \hat{U}^2 e^\Theta) (\text{Tr} e^\Theta) - (\text{Tr} \hat{U} e^\Theta)^2 \right] \geq 0.
\]

Then, we find similarly that \( \int dx A(x)^2 \int dx B(x)^2 \geq [ \int dx A(x) B(x) ]^2 \) and

\[
m_1 \int \frac{dx^G}{\langle J_0 \rangle^2} \left[ \int dx^G (\text{Tr} e^\Theta)^{m_1} \int dx^G (\text{Tr} e^\Theta)^{m_1-4} (\text{Tr} \hat{U} e^\Theta)^4 \right. \\
- \left. \left[ \int dx^G (\text{Tr} e^\Theta)^{\frac{1}{2}} (\text{Tr} e^\Theta)^{\frac{1}{2}-2} (\text{Tr} \hat{U} e^\Theta)^2 \right]^2 \right] \geq 0. \tag{C.2}
\]

Then, it follows that \( b < 0 \) when \( \lambda_{\text{RSB}} > 0 \).

Here \( a = (\hat{a} b - d^2) / b \), where

\[
\hat{a} = -2m_1 p(p - 1) r_1 (p - 2) \left[ \frac{1}{2} + \frac{1}{2} p(p - 1) r_1 (p - 2) \left[ (y_1 + y_2 - y_3) + 4m_1 y_4 - 2m_1 y_3 - 3m_1 y_5 - 4m_1 (1 - m_1) y_6 + m_1 (2 - m_1) y_2 \right] \right]; \tag{C.3}
\]

\[
d = -m_1 (1 - m_1) p^2 (p - 1)^2 r_1 (p - 2) r_1 (p - 2) \left[ 2y_7 - m_1 y_6 - (2 - m_1) y_8 \right] \tag{C.4}
\]

and

\[
y_1 = \int \frac{dx^G}{\langle J_0 \rangle^3} \left[ \int dx^G (\text{Tr} e^\Theta)^{m_1-1} (\text{Tr} \hat{U}^2 e^\Theta) \right]^2; \\
y_2 = \int \frac{dx^G}{\langle J_0 \rangle^3} \left[ \int dx^G (\text{Tr} e^\Theta)^{m_1-2} (\text{Tr} \hat{U} e^\Theta)^2 \right]^2; \\
y_3 = \int \frac{dx^G}{\langle J_0 \rangle^3} \left[ \int dx^G (\text{Tr} e^\Theta)^{m_1-1} (\text{Tr} \hat{U}^2 e^\Theta) \right]^2; \\
y_4 = \int \frac{dx^G}{\langle J_0 \rangle^{\frac{1}{2}}} \left[ \int dx^G (\text{Tr} e^\Theta)^{m_1-1} (\text{Tr} \hat{U}^2 e^\Theta) \right]^2; \\
y_5 = \int \frac{dx^G}{\langle J_0 \rangle^4} \left[ \int dx^G (\text{Tr} e^\Theta)^{m_1-1} (\text{Tr} \hat{U} e^\Theta) \right]^4; \\
y_6 = \int \frac{dx^G}{\langle J_0 \rangle^3} \left[ \int dx^G (\text{Tr} e^\Theta)^{m_1-2} (\text{Tr} \hat{U} e^\Theta)^2 \right]^2; \\
y_7 = \int \frac{dx^G}{\langle J_0 \rangle^3} \left[ \int dx^G (\text{Tr} e^\Theta)^{m_1-1} (\text{Tr} \hat{U}^2 e^\Theta) \right]^2; \\
y_8 = \int \frac{dx^G}{\langle J_0 \rangle^3} \left[ \int dx^G (\text{Tr} e^\Theta)^{m_1-3} (\text{Tr} \hat{U} e^\Theta)^3 \right]^2.
\]

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