On the Supersymmetry of Bianchi attractors in Gauged supergravity

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Abstract

Bianchi attractors are near horizon geometries with homogeneous symmetries in the spatial directions. We construct supersymmetric Bianchi attractors in \( \mathcal{N} = 2, d = 4, 5 \) gauged supergravity. In \( d = 4 \) we consider gauged supergravity coupled to vector and hypermultiplets. In \( d = 5 \) we consider gauged supergravity coupled to vector multiplets with a generic gauging of symmetries of the scalar manifold and the \( U(1)_R \) gauging of the \( R \)-symmetry. Analyzing the gaugino conditions we show that when the fermionic shifts do not vanish there are no supersymmetric Bianchi attractors. This is analogous to the known condition that for maximally supersymmetric solutions, all the fermionic shifts must vanish. When the central charge satisfies an extremization condition, some of the fermionic shifts vanish and supersymmetry requires that the symmetries of the scalar manifold do not be gauged. This allows supersymmetric Bianchi attractors sourced by massless gauge fields and a cosmological constant. In five dimensions in the Bianchi I class we show that the anisotropic \( AdS_3 \times \mathbb{R}^2 \) solution is \( 1/2 \) BPS. We also construct a new class of \( 1/2 \) BPS Bianchi III geometries labeled by the central charge. When the central charge takes a special value the Bianchi III geometry reduces to the known \( AdS_3 \times \mathbb{H}^2 \) solution. For the Bianchi V and VII classes the radial spinor breaks all of supersymmetry. We briefly discuss the conditions for possible massive supersymmetric Bianchi solutions by generalizing the matter content to include tensor, hypermultiplets and a generic gauging on the \( R \) symmetry.

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I. INTRODUCTION

In recent years intensive research on extremal black holes in AdS space have unveiled relations between seemingly unrelated fields such as gravity and condensed matter systems. In AdS/CFT extremal black holes provide the bulk gravitational description of zero temperature ground states in strongly coupled field theories. Many condensed matter systems show novel and diverse phase structures. At the quantum critical point the field theory description is strongly coupled and exhibit phase transitions at zero temperature due to quantum fluctuations. The presence of diverse phases in the field theory predict an equally large number of dual extremal geometries in the bulk. It is an interesting program to identify and classify various possible extremal geometries. Some of the earlier work in this direction have identified extremal geometries that exhibit Lifshitz and hyperscaling violations. Of more recent research interest are extremal black branes dual to field theories with reduced symmetries. Some of these examples are anisotropic and display interesting phenomena such as violation of the KSS bound when the anisotropy becomes much larger than the temperature.

In five dimensions homogeneous anisotropic extremal black brane geometries have been constructed. The metrics display manifest homogeneous symmetries in three spatial directions. It is well known that the Killing vectors that generate these symmetries form algebras that are isomorphic to real Lie algebras in dimension three. These real Lie algebras have been well studied and are well known through the Bianchi classification. The five dimensional geometries that display manifest homogeneous symmetries in three spatial directions are referred to as the “Bianchi attractors”. These near horizon geometries are exact solutions to Einstein-Maxwell theories with massive/massless gauge fields and a cosmological constant.

Black holes in $N = 2$ supergravity exhibit a phenomenon known as the attractor mechanism. In a black hole background moduli fields flow to fixed point values at the horizon irrespective of their asymptotic values at spatial infinity. The fixed point values are determined entirely in terms of the charges carried by the black hole. As a result the Bekenstein-Hawking entropy of the black hole is determined in terms of its charges. Although initial studies have focused on supersymmetric black holes, it has been realized that the attractor mechanism is a consequence of extremality. Subsequently the attractor mechanism is generalized to non-supersymmetric extremal black holes.

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1 The terminology attractor is used because the horizon geometries solve the field equation exactly. Interpolating numerical solutions have been constructed, justifying the terminology. However analytic solutions are much harder to find.
In recent years enormous effort has gone into generalizing the attractor mechanism to gauged supergravities \[34–41\]. Significant progress has been made especially for dyonic \(AdS_4\) black holes \[42\]. Large \(N\) index computations in the dual twisted mass deformed ABJM theory find perfect matching of the microstate counting with the Bekenstein Hawking entropy of the black hole \[43, 44\].

It is interesting to ask if the attractor mechanism generalizes to black brane geometries in \(AdS\).

In this light, the first step is to embed these geometries in supergravity in order to study their properties such as supersymmetry and stability.

Some steps in this direction have been taken \[35, 45\] and explicit examples of Bianchi attractors in \(\mathcal{N} = 2\) gauged supergravity are constructed. However it turns out that the geometries are non-supersymmetric and are unstable under linearized fluctuations unless certain conditions are satisfied \[46, 47\]. The conditions are such that there must exist a critical point of the effective potential and the Hessian of the effective potential evaluated at the solution must have positive eigenvalues. For non-supersymmetric extremal black hole solutions the above two conditions are sufficient to guarantee a stable Bianchi attractor in gauged supergravity. However supersymmetric solutions always satisfy these conditions and guarantee stability.

In this work we look for supersymmetric Bianchi attractor geometries in \(\mathcal{N} = 2\) gauged supergravity. As a warmup, we study \(d = 4\) gauged supergravity coupled to vector and hyper multiplets with a generic gauging of the symmetries of the hyper Kähler manifold. In four dimensions the homogeneous symmetries are along the two spatial directions and the corresponding Lie algebras are of two types namely Bianchi I and Bianchi II. Bianchi I geometries such as \(AdS_5\) \[41\] and \(z = 2\) Lifshitz solution \[48, 49\] are well known solutions in this theory. In the Bianchi I case, we construct the \(AdS_2 \times \mathbb{R}^2\) geometry sourced by time like gauge fields. We analyze the Killing spinor equations and find that the radial spinor preserves 1/4 of the supersymmetry. The gaugino and hyperino conditions impose additional relations on the parameters of the theory. In the Bianchi II case, a \(\frac{1}{8}\) BPS \(AdS_2 \times \mathbb{H}^2\) solution sourced by magnetic fields has been found recently in \[50\]. We construct a \(AdS_2 \times \mathbb{H}^2\) solution sourced by time like gauge fields and find that the radial spinor breaks all of the supersymmetry. The Bianchi I and Bianchi II classes we studied in four dimensions correspond to the symmetries of \(\mathbb{R}^2\) and \(\mathbb{H}^2\). These are the only possible Bianchi classes of metric that one can construct in 3+1 dimensions with homogeneous symmetries in two spatial directions. Of course there exist more general manifolds like \(T^2\) \[34, 51\], however they do not belong to the Bianchi class and we do not consider them in our analysis.

In \(d = 5\) there exist a richer class of Bianchi attractor geometries. We consider the \(\mathcal{N} = 2\)
gauged supergravity coupled to vector multiplets with a generic gauging of both symmetries of the very special manifold and the $U(1)_R$ subgroup of the $SU(2)_R$ symmetry group. From the gaugino conditions we find that there are no supersymmetric Bianchi attractors when the fermionic shifts in the supersymmetry variations are non vanishing. This is in the same spirit as the general analysis for maximally supersymmetric solutions [41, 52]. This result holds for a generic gauging of the scalar manifold, is dependent on the choice of the gauge field configuration that sources the solution and is independent of the functional form of the Killing spinor. The basic argument is that the constant part of the Killing spinor should be a simultaneous eigenspinor of commuting matrices that can appear in the gaugino conditions. We find that for the known gauge field configurations that generate Bianchi type solutions, this does not happen in general. Independently we show that the radial Killing spinor for such solutions breaks all of supersymmetry.

When the central charge $Z$ of the solution satisfies an extremization condition, some of the fermionic shifts in the gaugino variations vanish. This is a reasonable condition to impose for any plausible geometry that can be an attractor solution. Given this condition, supersymmetry invariance then requires that the effective mass term at the attractor point vanish. 

This condition allows Bianchi attractor solutions sourced by massless gauge fields since at the attractor point the “effective mass terms” in gauged supergravity are proportional to $g^2$. There are no further conditions from the gaugino variations and hence the supersymmetry of the solutions are entirely determined by the Killing spinor equation that follows from the gravitino variation. It is crucial to observe that the Killing spinor equation depends only on the gauge coupling constant of the $R$ symmetry gauging, hence it follows that the Killing spinor integrability conditions (see eq 31 of [35]) do not depend on the gauging of the scalar manifold.

We construct Bianchi solutions sourced by massless gauge fields and a cosmological constant in the Bianchi I, Bianchi III, Bianchi V and Bianchi VII classes. In the Bianchi I case we find the anisotropic $AdS_3 \times \mathbb{R}^2$ geometry recently studied in [23, 53] to be 1/2 BPS. We also construct a supersymmetric class of 1/2 BPS Bianchi III geometries labeled by the central charge $Z$. When the central charge of the solution takes special values, the geometry reduces to the known $AdS_3 \times \mathbb{H}^2$ [54]. The Killing spinors in both these cases come in pairs where one spinor is purely radial and the other spinor depends on both radial and transverse coordinates other than $\mathbb{R}^2/\mathbb{H}^2$ directions. Moreover the constant part of the spinors are eigenspinors of the radial Dirac matrix in all the

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2 In gauged supergravity literature the supersymmetry variations in the gaugino and hyperino that are proportional to the gauge coupling constant are referred to as fermionic shifts.

3 One way to possibly avoid this is to consider tensor multiplets, we comment on this in §IV C.
above cases. For the Bianchi V and Bianchi VII classes we find that the radial spinor breaks all supersymmetry.

Finally, the presence of hyper and tensor multiplets can allow for some massive Bianchi attractor solutions in some special cases. In particular our results from the gaugino and Killing spinor conditions for the non-supersymmetric cases continues to hold even after including hypermultiplets and $SU(2)_R$ gauging as the Killing spinor equation is not affected seriously by this addition. However, addition of tensor multiplets will affect the analysis and depends crucially on the tensor field configuration in addition to new gaugino and hyperino conditions. We comment on the possibilities in §IV C leaving a detailed analysis for future work.

The paper is organized as follows. In §II we briefly describe homogeneous symmetries and motivate Bianchi attractors. Following this we present the analysis for the $d = 4$ Bianchi attractors in $\mathcal{N} = 2$ gauged supergravity in §III. We move on to the five dimensional case in §IV. In subsection §IVA we present our main argument for the absence of massive supersymmetric Bianchi attractors in gauged supergravity with $U(1)_R$ gauging and gauging of the symmetries of the very special manifold. Subsequently we analyze the Killing spinor equations for the massless cases in §IV B. In §IV C we comment on the possible generalizations and necessary conditions when hyper and tensor multiplets are included with generic gauging. We present our conclusions and summarize in §V. In appendix §A we provide useful supplementary material on spinors in $d = 4, 5$ and summarize our conventions. In appendices §B and §D we provide details about Bianchi type solutions in $d = 4$ and $d = 5$ respectively. The details of the Killing spinor equations for the massive cases are given in appendix §C.

II. HOMOGENEOUS SYMMETRIES AND BIANCHI ATTRACTORS

In this section we describe the homogeneous symmetries in two and three dimensions classified by the Bianchi classification of Lie algebras. Towards the end we describe the “Bianchi attractors”. These are proposed near horizon geometries of extremal black branes with homogeneous symmetries in the spatial directions $[13]$.

Consider a manifold $M$ endowed with a metric $g_{\mu\nu}$ that is invariant under a given set of isometries. The Killing vectors $X_i$ that generate the isometries close to form an algebra

$$[X_i, X_j] = C_{ij}^k X_k$$  \hspace{1cm} (1)
where $C_{ij}^k$ are structure constants and they obey the usual Jacobi identity. The symmetry group of the manifold is isomorphic to an abstract Lie group $G$ whose Lie algebra is generated by the algebra of Killing vectors.

A homogeneous manifold has identical metric properties at all points in space. Any two points on a homogeneous space are connected by a symmetry transformation. The symmetry group of a homogeneous space of dimension $d$ is isomorphic to the group corresponding to $d$ dimensional real Lie algebra \[24, 25\]. On the other hand given the real Lie algebra in a dimension $d$, it is possible to write the corresponding metric with manifest homogeneous symmetries as follows. First one finds a basis of invariant vectors $e_i$ that commute with the Killing vectors $X_i$

$$[X_i, e_i] = 0 ,$$

then the metric with homogeneous symmetries can be expressed in terms of one forms $\omega^i$ dual to the invariant vectors $e_i$ as

$$ds^2 = g_{ij} \omega^i \otimes \omega^j$$

where $g_{ij}$ are constants. The invariant one forms satisfy the relation

$$d\omega^k = \frac{1}{2} C_{ij}^k \omega^i \wedge \omega^j$$

where $C_{ij}^k$ are the same structure constants that appear in the algebra of the Killing vectors. The real Lie algebras of dimension three fall into nine classes and are given by the well known Bianchi classification. The structure constants and invariant one forms are listed in detail in \[25\] (or see Appendix A of \[13\]).

As an illustration let us consider Bianchi VII, it has the symmetries of a three dimensional euclidean space with translational symmetry along the $x$ direction and rotational symmetry along the $(y, z)$ plane. The Killing vectors have the form

$$X_1 = \partial_y , \quad X_2 = \partial_z , \quad X_3 = \partial_x + y \partial z - z \partial y .$$

These vectors close to form the Lie algebra

$$[X_1, X_2] = 0 , \quad [X_1, X_3] = X_2 , \quad [X_2, X_3] = -X_1 .$$
The structure constants are independent of spacetime coordinates. A nice way to see the homogeneous symmetries manifest is to construct invariant vector fields $e_i$ that commute with the Killing vectors

$$e_1 = \cos(x)\partial_y + \sin(x)\partial_z, \quad e_2 = -\sin(x)\partial_y + \cos(x)\partial_z, \quad e_3 = \partial_x.$$  

(7)

The invariant one forms

$$\omega^1 = \cos(x)dy + \sin(x)dz, \quad \omega^2 = -\sin(x)dy + \cos(x)dz, \quad \omega^3 = dx$$

(8)

satisfy the algebra

$$d\omega^1 = -\omega^2 \wedge \omega^3, \quad d\omega^2 = \omega^1 \wedge \omega^3, \quad d\omega^3 = 0.$$  

(9)

The three dimensional euclidean metric invariant under the Bianchi VII symmetry can be written in the form (3). For example, in a diagonal basis as

$$ds^2 = (\omega^1)^2 + (\omega^2)^2 + \lambda^2(\omega^3)^2$$

(10)

where $\lambda$ is a constant. For the case where $\lambda = 1$ the symmetry is enhanced to the usual translation and rotational symmetries in the $(x, y, z)$ directions. Note that the Bianchi I and Bianchi VII are sub algebras of the Poincaré algebra (see section §E).

To complete this discussion, it is useful to digress on the classification of two dimensional real Lie algebras. These are classified in an analogous classification and are of precisely two types. One is the simple Bianchi I algebra where all the generators commute. This is for instance the symmetry group of $\mathbb{R}^2$. Obviously the Bianchi I algebra in dimension 2 is a sub algebra of the Bianchi I algebra in dimension 3. While the only other non-trivial Bianchi algebra is the one corresponding to the symmetries of the manifold $\mathbb{H}^2$. The algebra

$$[X_1, X_2] = X_1$$

(11)

is generated by the Killing vectors $X_i, \; i = 1, 2$,

$$X_1 = \partial_y, \quad X_2 = \partial_x + y\partial_y.$$  

(12)
The invariant vector fields \( e_i \) that commute with the Killing vectors are defined as

\[
e_1 = e^x \partial_y, \quad e_2 = \partial_x, \quad [e_1, e_2] = -e_1.
\] (13)

The duals to the invariant one forms are given by

\[
\omega^1 = e^{-x} dy, \quad \omega^2 = dx, \quad d\omega^1 = \omega^1 \wedge \omega^2.
\] (14)

The metric invariant under this symmetry group takes the from

\[
ds^2 = (\omega^1)^2 + (\omega^2)^2.
\] (15)

Using the coordinate transformation \( x = \ln \rho \) it is easy to see that this is precisely the metric of Euclidean \( \text{AdS}_2 \). Note that the Bianchi II algebra corresponding to \( \mathbb{H}^2 \) in two dimensions is a sub algebra of the Bianchi III algebra in three dimensions [47].

In this work we investigate the supersymmetry conditions on various Bianchi attractor geometries described in [13]. The geometries have the general structure

\[
ds^2 = -g_{tt}(r) dt^2 + g_{ij}(r) d\omega^i \wedge d\omega^j + dr^2
\] (16)

where \( i = 1, 2 \) in case of four dimensions and the \( \omega^i \) are invariant one forms corresponding to the homogeneous symmetries of two dimensional real Lie algebras described above. In five dimensions \( i = 1, 2, 3 \) and the corresponding \( \omega^i \) are invariant one forms given by the usual Bianchi classification. The functions \( g_{tt}(r) \) and \( g_{ij}(r) \) have a general form \( e^{\beta r} \), where \( \beta \) are positive exponents. These metrics can be constructed as solutions to Einstein-Maxwell theories with massive/massless gauge fields and a cosmological constant. As long as the matter stress-tensor preserves the symmetries of the metric, explicit solutions can be constructed for a wide range of parameters of the theory of interest.

III. BIANCHI ATTRACTORS IN \( \mathcal{N} = 2, d = 4 \) GAUGED SUPERGRAVITY

In this section we describe \( \mathcal{N} = 2, d = 4 \) gauged supergravity with \( n_V \) vector and \( n_H \) hyper multiplets. We use the notations and conventions of [48], the relevant conventions are summarized in appendix A1. The gravity multiplet consists of a metric \( g_{\mu \nu} \), a graviphoton \( A^0_\mu \) and an \( SU(2) \)
doublet of gravitinos \((\psi^A_\mu, \psi^\dagger_\mu A)\) of opposite chirality, where \(A = 1, 2\) is an \(SU(2)\) index. The vector multiplet consists of a complex scalar \(z^i\), a vector \(A^i_\mu\), where \(i = 1, 2 \ldots n_V\) and an \(SU(2)\) doublet of gauginos \((\lambda^{iA}, \lambda^{\dagger i}_A)\) with opposite chirality. The hyper multiplets contain scalars \(q^X\), where \(X = 1, \ldots 4n_H\) and two hyperinos \((\zeta^\alpha, \zeta^\dagger_\alpha)\), \((\alpha = 1 \ldots 2n_H)\) of opposite chirality. The moduli space of the theory factorizes into a product of a special Kähler manifold and a quaternionic Kähler manifold \([55]\) \[\mathcal{M} = SK(n_V) \times Q(n_H)\].

For a Kähler manifold the metric is derived from a Kähler potential

\[g_{ij} = \partial_i \partial_j \mathcal{K}.\] (18)

Since the Kähler manifold is also special Kähler there exist local holomorphic sections \((X^A, F_\Lambda)\) where \(\Lambda = 0 \ldots n_V\) \(^4\), where \(F_\Lambda = \frac{dF}{dX^\Lambda}\) \((F\) is the holomorphic prepotential). The Kähler potential can be expressed in terms of the sections as

\[\mathcal{K} = -\ln(i(X^A F_\Lambda - X^\Lambda F_\Lambda^\dagger))\]. (19)

The Kähler manifold is also symplectic and hence one can introduce symplectic sections \((L^A(z, \bar{z}), M_\Lambda(z, \bar{z}))\) that satisfy

\[i(L^A M_\Lambda - L^\Lambda M_\Lambda) = 1\]. (20)

The symplectic sections are related to the usual sections via the relations

\[(X^A, F_\Lambda) = e^{-\frac{\mathcal{K}}{2}}(L^A, M_\Lambda)\]. (21)

All the matter couplings in the theory are completely determined in terms of the symplectic sections. Let \(K^i_\Lambda(z)\) be Killing vectors that generate isometries on the manifold \(SK\). Gauging the symmetries of the manifold amount to replacing ordinary derivatives with gauge covariant derivatives

\[D_\mu z^i = \partial_\mu z^i + K^i_\Lambda A^A_\mu\]. (22)

\(^4\) For \(\Lambda = 0\), \(A^0_\mu\) refers to the graviphoton.
For the rest of the discussion, the gauge group is abelian for simplicity.

The hyperscalars \( q^X, \, X = 1, \ldots, 4n_H \) parametrize the quaternionic Kähler manifold \( Q \). The metric on \( Q \) is defined by

\[
\text{ds}^2 = g_{XY} dq^X \otimes dq^Y 
\]

(23)
in suitable coordinates \( q^X \) on \( Q \). Since \( Q \) is also Kähler the metric can be derived from a suitable Kähler potential. The isometries on \( Q \) are generated by Killing vectors \( K^X_\Lambda \). Once again gauging is done by replacing ordinary derivatives with gauge covariant derivatives

\[
D_\mu q^X = \partial_\mu q^X + A^\Lambda_\mu K^X_\Lambda (q). 
\]

(24)

In the above we have set the gauge coupling constant to identity for simplicity. Notice that gauging \((22), (24)\) introduces additional terms in the theory. Supersymmetry invariance of the resultant action requires the addition of a potential

\[
\mathcal{V}(z, \bar{z}, q) = \left( (g_{i\bar{j}} K^i_\Lambda K^{\bar{j}}_\Sigma + 4g_{XY} K^X_\Lambda K^{Y}_{\bar{\Sigma}}) \bar{L}^\Lambda L^\Sigma + (g^{ij} f^A_\Lambda f^\Sigma_\Lambda - 3\bar{L}^\Lambda L^\Sigma) P^x_\Lambda P^x_\Sigma \right) 
\]

(25)

where \( f^A_\Lambda = (\partial_i + \frac{1}{2} \partial_i K) L^\Lambda \). The triplet \( P^x_\Lambda, x = 1, 2, 3 \) are real Killing prepotentials on the quaternionic Kähler manifold. The bosonic part of the Lagrangian of the \( \mathcal{N} = 2 \) theory takes the form

\[
\mathcal{L} = -\frac{1}{2} R + g_{ij} D^\mu z^i D_\mu \bar{z}^j + g_{XY} D_\mu q^X D^\mu q^Y + i(\bar{N}_{\Lambda\Sigma} F^-_{\mu\nu} F^\Sigma_{\mu\nu} - N_{\Lambda\Sigma} F^+_{\mu\nu} F^{+\Sigma\mu\nu}) \\
+ \mathcal{V}(z, \bar{z}, q) 
\]

(26)

where \( N_{\Lambda\Sigma} \) are the period matrices.\(^5\) The self/anti-self dual field strengths are defined as

\[
F^\pm_{\mu\nu} = \frac{1}{2} \left( F^\Lambda_{\mu\nu} \pm i F^\Lambda_{\mu\rho\sigma} F^{\Lambda\rho\sigma} \right) 
\]

(27)

where the usual field strength is defined as \( F^\Lambda_{\mu\nu} = \frac{1}{2} (\partial_\mu A^\Lambda_\nu - \partial_\nu A^\Lambda_\mu) \). The supersymmetry transfor-

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\(^5\) These are functions of \( z^i \) and can be expressed in terms of the sections \( M_\Lambda = N_{\Lambda\Sigma} L^\Sigma \).
mations of the fermionic fields are given by
\begin{align*}
\delta \psi_{\mu A} &= D_\mu \epsilon_A + i S_{AB} \gamma_\mu \epsilon^B + 2 i (\text{Im} N)_{A \Sigma} L^\Sigma F_{\mu \nu}^{-\Lambda} \gamma^\nu \epsilon_{AB} \epsilon^B \\
\delta \lambda^{i A} &= i D_\mu z^i \gamma_\mu \epsilon^A - g j^i j_j^i (\text{Im} N)_{A \Sigma} F_{\mu \nu}^{-\Lambda} \gamma^\nu \epsilon_{AB} \epsilon^B + W^{i AB} \epsilon^B \\
\delta \zeta_\alpha &= i U_X^{B \beta} D_\mu q^X \gamma_\mu \epsilon^A \epsilon_{AB} \epsilon_{\alpha \beta} + N^A_\alpha \epsilon_A 
\end{align*}
\hspace{1cm} (28)

where
\begin{align*}
S_{AB} &= \frac{i}{2} (\sigma^r)^A_\beta \epsilon_{BC} P^r_\alpha L^A \\
W^{i AB} &= \epsilon^{AB} k^i L^A + i (\sigma^r)_C^A \epsilon^C A^B g^i j^A j^A \\
N^A_\alpha &= 2 U_{\alpha X} A^X k^A L^A .
\end{align*}
\hspace{1cm} (29)

In the above $U_{\alpha X}^A$ are vielbeins on the quaternionic manifold. The covariant derivative on the spinor $\epsilon_A$ is defined as
\begin{align*}
D_\mu \epsilon_A &= \nabla_\mu \epsilon_A + i (\sigma^r)^B_\alpha \epsilon_{\alpha \beta} A^A_\mu P^r_\beta \epsilon_B 
\end{align*}
\hspace{1cm} (30)

where $\nabla_\mu$ is the covariant derivative defined with respect to the usual spin connection. For the rest of the discussion we assume a generic gauging of the symmetries of hypermultiplet manifold.

At the attractor point the scalars are independent of spacetime coordinates,
\begin{align*}
z^i &= \text{const} , q^X = \text{const} .
\end{align*}
\hspace{1cm} (31)

The supersymmetry variations \(^{28}\) at the attractor point then reduce to
\begin{align*}
\delta \psi_{\mu A} &= D_\mu \epsilon_A + i S_{AB} \gamma_\mu \epsilon^B + 2 i (\text{Im} N)_{A \Sigma} L^\Sigma F_{\mu \nu}^{-\Lambda} \gamma^\nu \epsilon_{AB} \epsilon^B \\
\delta \lambda^{i A} &= - g j^i j_j^i (\text{Im} N)_{A \Sigma} F_{\mu \nu}^{-\Lambda} \gamma^\nu \epsilon_{AB} \epsilon^B + W^{i AB} \epsilon^B \\
\delta \zeta_\alpha &= i U_X^{B \beta} \epsilon_{\alpha \beta} A^A_\mu \gamma_\mu \epsilon^A \epsilon_{AB} \epsilon_{\alpha \beta} + N^A_\alpha \epsilon_A .
\end{align*}
\hspace{1cm} (32)

Setting the gravitino variations to zero, we get the Killing spinor equation
\begin{align*}
\partial_\mu \epsilon_A + \frac{1}{4} \omega^a_{\mu} \gamma_{ab} \epsilon_{A \beta} + i (\sigma^r)_A^B P^r_\alpha A^A_\mu \epsilon_B + i S_{AB} \gamma_\mu \epsilon^B + 2 i (\text{Im} N)_{A \Sigma} L^\Sigma F_{\mu \nu}^{-\Lambda} \gamma^\nu \epsilon_{AB} \epsilon^B = 0 .
\end{align*}
\hspace{1cm} (33)

In the rest of the section, we evaluate the Killing spinor equation \(^{33}\), the gaugino and hyperino.
equations on the background of Bianchi geometries and derive the conditions for supersymmetry.

A. Bianchi I

Metrics with Bianchi I symmetry in the spatial directions have been studied in the gauged supergravity literature, the simplest of them being the supersymmetric $AdS_4$ solution [41]. A supersymmetric Lifshitz solution with exponent $z = 2$ has also been constructed earlier in gauged supergravity by [48, 49]. In this section following the analysis of [48] we present the supersymmetry conditions for a simple Bianchi I type - $AdS_2 \times \mathbb{R}^2$ solution. 6

The $AdS_2 \times \mathbb{R}^2$ metric has the form

$$ds^2 = \frac{R_0^2}{\sigma^2}(dt^2 - d\sigma^2) - R_0^2(dy^2 + d\rho^2) .$$ (34)

The Killing vectors along the spatial directions $X_1 = \partial_y, X_2 = \partial_\rho$ generate the Bianchi I algebra

$$[X_1, X_2] = 0 .$$ (35)

It is easy to construct this metric as a solution to the equations of motion that follow from the gauged supergravity action (26). It is supported by an electrically charged gauge field whose ansatz we choose to be

$$A^\Lambda = \frac{E^\Lambda}{\sigma} dt .$$ (36)

The scalar, gauge field and Einstein equations are listed in §B1. The Killing spinor equations (33) evaluated in the above background are

$$\frac{\gamma^0}{R_0}\partial_t \epsilon_A - \frac{\gamma^1}{2R_0}\epsilon_A + \frac{iG^B_A}{2R_0}\gamma^0_B\epsilon_B + \frac{iN}{2R_0}\gamma^{01}_{AB}\epsilon^B = 0$$ (37)

$$\frac{\gamma^1}{R_0}\partial_\sigma \epsilon_A + iS_{AB}\epsilon^B + \frac{iN}{2R_0}\gamma^{01}_{AB}\epsilon^B = 0$$ (38)

$$\frac{\gamma^2}{R_0}\partial_y \epsilon_A + iS_{AB}\epsilon^B - \frac{N}{2R_0}\gamma^{23}_{AB}\epsilon^B = 0$$ (39)

$$\frac{\gamma^3}{R_0}\partial_\rho \epsilon_A + iS_{AB}\epsilon^B - \frac{N}{2R_0}\gamma^{23}_{AB}\epsilon^B = 0$$ (40)

6 Magnetic $AdS_2 \times \mathbb{R}^2$ solutions and their stability have been well explored in the literature (see for instance [56, 58]).
where we have defined

\[ N = (\text{Im}N_{A \Sigma})L^\Sigma E^\Lambda, \quad G^R_A = (\sigma_x)_A^B P^x_\Lambda E^\Lambda \]

for brevity. We choose the following radial ansatz for the Killing spinor

\[ \epsilon_A = f(\sigma)\chi_A \]

where \( \chi_A \) is a constant spinor. The difference of (38) and (37) leads to

\[ \frac{\gamma^1}{R_0} \sigma \partial_\sigma \epsilon_A + \frac{\gamma^1}{2R_0} \epsilon_A - \frac{iG^R_A \gamma^0}{2R_0} \epsilon_B = 0. \]

The above equation has a simple solution

\[ f(\sigma) = \frac{1}{\sqrt{\sigma}} \]

provided we impose the condition

\[ E^\Lambda P^x_\Lambda = 0. \]

We note that this same condition has enabled a supersymmetric Lifshitz solution in 4d \( \mathcal{N} = 2 \) gauged supergravity [48]. Thus the Killing spinor equations reduce to the algebraic conditions

\[ -\frac{\gamma^1}{2R_0} \chi_A + iS_{AB} \chi^B + \frac{iN}{2R_0^2} \gamma^{01} \epsilon_{AB} \chi^B = 0 \]

\[ iS_{AB} \chi^B - \frac{iN}{2R_0^2} \gamma^{01} \epsilon_{AB} \chi^B = 0 \]

where we have substituted \( \gamma^{23} = -i\gamma^{01}\gamma_5 \) and used \( \gamma_5 \epsilon^A = -\epsilon^A \). It is straightforward to recast the above equations into the projection conditions

\[ \chi_A = \frac{2iN}{R} \epsilon_{AB} \gamma^0 \chi^B \]

\[ \chi_A = -4iRS_{AB} \gamma^1 \chi^B. \]

These projection conditions are very similar to the conditions obtained for the 4d Lifshitz case by
(c.f eq 67-68). Squaring the first projection condition we get
\[ |N| = \frac{R_0}{2}. \]  
\[ \text{(50)} \]

Mutual consistency of the two projectors leads to the equation
\[ \chi_A = 4R_0 S_{AB} \gamma^{10} \epsilon^{BC} \chi_C, \]  
\[ \text{(51)} \]
whose self consistency gives the condition
\[ \sum_{x=1}^{3}(P_x^\Lambda L^\Lambda)^2 = \frac{1}{4R_0^2}. \]  
\[ \text{(52)} \]

Note that the triplet of Killing prepotentials \( P_x^\Lambda \) are real functions on the quaternionic manifold. However, the symplectic sections \( L^\Lambda \) are complex functions in general. For simplicity we can choose the Killing prepotential to lie along the \( x = 3 \) direction. Thus the final projection conditions that follow from the gravitino Killing spinor equations are
\[ \chi_A = i \epsilon_{AB} \gamma^0 \chi^B \]
\[ \chi_A = (\sigma_3)_A^C \gamma^{10} \chi_C. \]  
\[ \text{(55)} \]

These are mutually self consistent projection conditions and together they preserve \( \frac{1}{4} \) of the supersymmetry. We now proceed to analyze the gaugino and hyperino conditions in (32).

Setting the Hyperino variation (32) to zero, we get the algebraic condition
\[ i \mathcal{U}^{A\beta} K^X_{\Lambda} \frac{E^\Lambda}{R_0} \gamma^0 \epsilon^{\beta \alpha} \epsilon_{BA} + 2 \mathcal{U}^{A}_{\alpha} K^X_{\Lambda} L^\Lambda \epsilon_A = 0. \]  
\[ \text{(56)} \]

We can use the \( \frac{1}{4} \) BPS projectors (55) to simplify the above expression to get
\[ \mathcal{U}^{A}_{\alpha} K^X_{\Lambda} \left( \frac{E^\Lambda}{R_0} + 2 L^\Lambda \right) \chi_A = 0. \]  
\[ \text{(57)} \]

Note that this sets
\[ P_x^\Lambda L^\Lambda = \frac{i}{2R_0}. \]  
\[ \text{(53)} \]

It is easy to check that this choice is consistent with the projection condition (48). Substituting the above in (49) we get
\[ \chi_A = i (\sigma_3)_A^D \epsilon_{BD} \gamma^1 \chi^B \]  
\[ \text{(54)} \]
that is self consistent.
An obvious way to solve the condition is to set $E^\Lambda = -2\bar{L} R_0$. In fact this leads to the correct equation of motion (second of (B7)). However this leads to an inconsistency with the known identity (see eq 4.38 of [55]) $\text{Im} N_{\Lambda \Sigma} L^\Lambda L^\Sigma = -\frac{1}{2}$ that is true for any $\mathcal{N} = 2$ supergravity. Note that this was also observed earlier in [48] for the 4d Lifshitz solution. However we can solve the hyperino conditions by choosing the Killing vectors to be degenerate on the quaternionic manifold. In other words,

$$K_X \left( \frac{E_\Lambda}{R_0} + 2\bar{L} \right) = 0 \ .$$

(58)

The gaugino conditions in (32) upon using the $\frac{1}{4}$ BPS projections have the very simple form

$$g^{ij} f^\Sigma_j \left( -\text{Im} N_{\Lambda \Sigma} \frac{E_\Lambda}{R_0} + i P_3^\Sigma \right) = 0 \ .$$

(59)

This concludes the set of conditions that follow from supersymmetry requirements. To summarize, the final set of conditions for a $\frac{1}{4}$ BPS $AdS_2 \times \mathbb{R}^2$ solution are

$$E^\Lambda P_3^\Lambda = 0 \ , \ \text{Im} N_{\Lambda \Sigma} L^\Lambda E^\Sigma = \frac{R_0}{2} \ , \ P_3^\Lambda L^\Lambda = \frac{i}{2 R_0} \ ,$$

$$K_X \left( \frac{E_\Lambda}{R_0} + 2\bar{L} \right) = 0 \ , \ g^{ij} f^\Sigma_j \left( -\text{Im} N_{\Lambda \Sigma} \frac{E_\Lambda}{R_0} + i P_3^\Sigma \right) = 0 \ .$$

(60)

In addition one has to impose the gauge field equations of motion (B3).

**B. Bianchi II**

In this section we discuss the supersymmetry conditions for a Bianchi II ($AdS_2 \times EAdS_2$) solution of the form

$$ds^2 = \frac{R_0^2}{\rho^2} (dt^2 - d\sigma^2) - \frac{R_0^2}{\rho^2} (dy^2 + d\rho^2) \ .$$

(61)

As discussed in (11) the symmetries along the spatial directions correspond to that of $EAdS_2$. Like the previous solution, the $AdS_2 \times EAdS_2$ solution can also be constructed using a time like gauge field (36) as source since it preserves the Bianchi II symmetry along the $(y, \rho)$ directions.

The equations of motion are presented in appendix (B2). The Killing spinor equations on this

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8 See [50] for magnetic black hole solutions interpolating between $AdS_2 \times \mathbb{H}^2$ and hyperscale violating solutions at infinity.
background are

\[
\frac{\gamma^0 \sigma}{R_0} \partial_t \epsilon_A - \frac{\gamma^1}{2R_0} \epsilon_A + \frac{iG^B}{2R_0} \gamma^0 \epsilon_B + \frac{iN}{2R_0} \gamma^0 \epsilon_A \epsilon_B = 0
\]

\[
\frac{\gamma^1 \sigma}{R_0} \partial_\sigma \epsilon_A + iS_{AB} \epsilon^B + \frac{iN}{2R_0} \gamma^0 \epsilon_A \epsilon^B = 0
\]

\[
\frac{\gamma^2 \rho}{R_0} \partial_y \epsilon_A - \frac{\gamma^3}{2R_0} \epsilon_A + iS_{AB} \epsilon^B - \frac{N}{2R_0} \gamma^23 \epsilon_A \epsilon^B = 0
\]

\[
\frac{\gamma^3 \rho}{R_0} \partial_\rho \epsilon_A + iS_{AB} \epsilon^B - \frac{N}{2R_0} \gamma^23 \epsilon_A \epsilon^B = 0
\]

(62)

where we have defined the quantities \( N \) and \( G^B \) in (41).

Taking the difference of the first and second equations of (62), and similarly the difference of the third and fourth equations in (62) we get the pair of differential equations

\[
\frac{\sigma}{R_0} (\gamma^0 \partial_t - \gamma^1 \partial_\sigma) \epsilon_A - \frac{\gamma^1}{2R_0} \epsilon_A + \frac{iG^B}{2R_0} \gamma^0 \epsilon_B = 0
\]

\[
\frac{\rho}{R_0} (\gamma^2 \partial_y - \gamma^3 \partial_\rho) \epsilon_A - \frac{\gamma^3}{2R_0} \epsilon_A = 0
\]

(63)

Since the \( AdS_2 \times EAdS_2 \) metric factorizes into a product form, with two radii \( \rho \) and \( \sigma \) we choose a Killing spinor ansatz of the form

\[
\epsilon_A = \frac{1}{\rho^m \sigma^n} \chi_A
\]

(64)

where \( \chi_A \) is a constant spinor, while \( m, n \) take real values. This form of the ansatz is also consistent with the Bianchi II symmetry of the metric. Substituting the above in (63) we get the conditions

\[
(2n - 1) \gamma^1 \epsilon_A + iG^B \gamma^0 \epsilon_B = 0
\]

\[
(2m - 1) \gamma^3 \epsilon_A = 0
\]

(65)

that can be solved by

\[
m = n = \frac{1}{2}, \quad E^\Lambda P^x_\Lambda = 0
\]

(66)

With the ansatz (64) and the condition (66) the remaining Killing spinor equations give the con-
dions

\((\gamma^1 + \gamma^3) \epsilon_A = 4iR_0 S_{AB} \epsilon^B\)

\((\gamma^1 - \gamma^3) \epsilon_A = \frac{2iN\gamma^{01}}{R_0} \epsilon_{AB} \epsilon^B\). \hspace{1cm} (67)

Unlike the \(AdS_2 \times \mathbb{R}^2\) case, these conditions are not as simple to work with. However we can simplify them by multiplying from the left by \(\gamma^1\) and writing in terms of the charge conjugate matrix \(C = \gamma^1 \gamma^3\) as

\((-1 + C) \chi_A = 4iR_0 S_{AB} \gamma^{01} C(\chi_B)^*\)

\((-C + 1) \chi_A = \frac{2iN}{R_0} \epsilon_{AB} (\chi_B)^*\)

where we have used \(\chi^B = -\gamma_0 C(\chi_B)^*\). We now show that the above condition breaks all of supersymmetry. Since \([\gamma_5, C] = 0\) (see §A1), it is convenient to use a decomposition of the spinor \(\chi_A\) in a basis of simultaneous eigenstates of \(\gamma_5\) and \(C\) as follows

\(\chi_A = \begin{pmatrix} 0 \\ C^+_A |+\rangle \\ 0 \\ C^-_A |\rangle \end{pmatrix} + \begin{pmatrix} 0 \\ C^+_A |\rangle \\ 0 \\ C^-_A |+\rangle \end{pmatrix}\)

where \(C^+_A\) and \(C^-_A\) are complex coefficients \(^9\) and \(|\pm\rangle\) are eigenstates of the Pauli matrices. Substituting in the second equation in (68) we obtain

\((1 - i) C^+_A |+\rangle + (1 + i) C^-_A |\rangle = \frac{2iN}{R_0} \epsilon_{AB} ((C^+_B)^* |\rangle + (C^-_B)^* |+\rangle) .\)

(70)

Linear independence of the states \(|+\rangle\) and \(|\rangle\) gives rise to the constraints

\((1 - i) C^+_A = \frac{2iN}{R_0} \epsilon_{AB} (C^-_B)^* \)

\((1 + i) C^-_A = \frac{2iN}{R_0} \epsilon_{AB} (C^+_B)^* .\)

(71)

It is straightforward to see that both of these constraints cannot be satisfied simultaneously as

\(^9\) Since \((A = 1, 2)\) there are 8 independent constants in \(\epsilon_A\) as it should be for a \(\mathcal{N} = 2\) spinor in four dimension.
their mutual consistency leads to

$$C^+_A (1 + \frac{2i|N|^2}{R^2_0}) = 0 \,.$$  \hfill (72)

Since $|N|^2$ is real, it follows that the only possible solution is that all the $C^+_A$ vanish and hence the metric (61) breaks all the supersymmetry. We now move on to the five dimensional case where there is a wider variety of solutions with Bianchi symmetries in the spatial directions.

**IV. BIANCHI ATTRAJECTORS IN $\mathcal{N} = 2, d = 5$ GAUGED SUPERGRAVITY**

We begin with a brief introduction to $\mathcal{N} = 2, d = 5$ gauged supergravity coupled to $n_V$ vector multiplets with a generic gauging of the very special manifold $S$ and the $U(1)_R$ subgroup of the $SU(2)_R$ symmetry group \textsuperscript{52}.\textsuperscript{10} The very special manifold is parametrized by $n_V + 1$ functions $h^I(\phi)$ subject to the constraint

$$N \equiv C_{IJK} h^I h^J h^K = 1$$  \hfill (73)

where $C_{IJK}$ are constant symmetric tensors. The $\phi^x, x = 1, 2, \ldots n_V$ are scalars in the $n_V$ vector multiplets. The $I, J$ indices can be raised or lowered using the ambient metric defined by

$$a_{IJ} = -\frac{1}{2} \frac{\partial}{\partial h^I} \frac{\partial}{\partial h^J} \ln N|_{N=1}.$$  \hfill (74)

The metric on the moduli space is obtained by pull back of the ambient metric into the moduli space

$$g_{xy} = a_{IJ} h^I_x h^J_y, \quad h^I_x \equiv \frac{\partial h^I}{\partial \phi^x}.$$  \hfill (75)

The Killing vectors $K^I_x$ generate isometries on the very special manifold. These isometries can be gauged by replacing the ordinary derivatives on the scalars with gauge covariant derivatives defined by

$$D_\mu \phi^x = \partial_\mu \phi^x + g A^I_\mu K^I_x(\phi)$$  \hfill (76)

\textsuperscript{10} Please note that in all of five dimensions we use the mostly plus metric signature.
where $A^I_\mu$ are the vectors in the vector multiplet and $g$ is the gauge coupling constant. For the purposes of this paper, we restrict ourselves to the case where the gauge group is abelian.

In addition to gauging the symmetries of the scalar manifold, one can also gauge the $U(1)_R$ subgroup of the $SU(2)_R$ symmetry of the $\mathcal{N} = 2$ supergravity.\textsuperscript{11} The R symmetry acts as a rotation on the fermions of the theory and gauging it replaces the usual covariant derivatives on the fermions by gauge covariant derivatives as

$$D_\mu \psi_{\nu i} = \nabla_\mu \psi_{\nu i} + g_R A^I_\mu P_{ij}^I \psi_{\nu j} \quad (77)$$

where $g_R$ is the $U(1)_R$ gauge coupling constant and

$$P_{ij} = -V_I \delta_{ij} \quad (78)$$

$V_I$ are the Fayet-Illioupoulos parameters. Note that in the above expression $\delta_{ij}$ does not play the role of $\epsilon_{ij}$ as a raising or lowering operator. The covariant derivative $\nabla$ is defined with the usual spin connection. Since the gauging (76) and (77) introduces new terms in the action, supersymmetric closure requires additional terms. These additional terms in the bosonic part of the Lagrangian take the form of a potential

$$\mathcal{V}(\phi) = -g_R^2 (2 P_{ij} P^{ij} - P^{aI} P^{aij}) \quad (79)$$

with the definitions

$$P_{ij} = h^I P_{Iij} \ , \ P^{aI} = h^{aI} P_{Iij} \ , \ h^{aI} = f^a_x h^x I \ , \ (80)$$

where $f^a_x$ are vielbeins on the very special manifold $\mathcal{S}$. Note that the potential is unaffected by the gauging of $\mathcal{S}$. Addition of hypermultiplets and tensor multiplets will change the shape of the potential, however to get $AdS$ vacuum it is sufficient to gauge the $U(1)_R$ symmetry.

With the various definitions as stated above the bosonic part of the Lagrangian reads as

$$\hat{e}^{-1} \mathcal{L} = -\frac{1}{2} R - \frac{1}{4} a_{IJ} F^I_{\mu \nu} F^J{}^{\mu \nu} - \frac{1}{2} g_{xy}(\phi) D_\mu \phi^x D^\mu \phi^y + \frac{\hat{e}^{-1}}{6 \sqrt{6}} C_{IJK} \epsilon^{\mu \nu \rho \sigma} F^I_{\mu \rho} F^J_{\rho \sigma} A^K - \mathcal{V}(\phi) \quad (81)$$

\textsuperscript{11} One can also gauge the $SU(2)_R$ symmetry by including hypermultiplets.
where $\hat{e} = \sqrt{-\det g_{\mu\nu}}$. The bosonic sector of the supersymmetry transformations are

\[
\delta \psi_{\mu i} = D_\mu \epsilon_i + \frac{i}{4\sqrt{6}} h_I F^{\nu \rho I} (\gamma_{\mu \nu \rho} - 4 g_{\mu \nu} \gamma_\rho) \epsilon_i + \frac{i}{\sqrt{6}} g_R \gamma_\mu \epsilon^j P_{ij}
\]

\[
\delta \lambda_i^a = -\frac{i}{2} f^a_x D_\mu \phi^x \gamma_\mu \epsilon_i + \frac{1}{4} h^a_I F^{\mu I} \gamma^{\mu \nu} \epsilon_i + g_R \epsilon^j P_{ij}.
\]

(82)

The $\lambda_i^a$ ($i = 1, 2$ and $a = 1, \ldots, n_V$) are gauginos in the vector multiplets and $\epsilon_i$ is a symplectic majorana spinor. The covariant derivative is defined as

\[
D_\mu \epsilon_i \equiv \partial_\mu \epsilon_i + \frac{1}{4} \omega_{\mu}^{ab} \gamma_{ab} \epsilon_i + g_R A_I^{\mu} P_{ij} \epsilon^j.
\]

(83)

See Appendix §A2 for our notations and conventions of 5d gamma matrices.

We are interested in Bianchi type near horizon solutions to (82) that satisfy attractor conditions. It is well known that at the attractor point the moduli are constants independent of spacetime coordinates

\[
\phi^x = \text{const}
\]

(84)

The field equations that follow from (81) are given in [35]. The supersymmetry transformations at the attractor point take the form

\[
\delta \psi_{\mu i} = D_\mu \epsilon_i + \frac{i}{4\sqrt{6}} h_I F^{\nu \rho I} (\gamma_{\mu \nu \rho} - 4 g_{\mu \nu} \gamma_\rho) \epsilon_i + \frac{i}{\sqrt{6}} g_R \gamma_\mu \epsilon^j P_{ij}
\]

\[
\delta \lambda_i^a = -\frac{i}{2} f^a_x A_I^{\mu} K^x_I \gamma_\mu \epsilon_i + \frac{1}{4} h^a_I F^{\mu I} \gamma^{\mu \nu} \epsilon_i + g_R \epsilon^j P_{ij}.
\]

(85)

In the following sections we evaluate the spinor conditions on the Bianchi attractor backgrounds. As discussed in §II the Bianchi type metrics have the generic form\(^\text{12}\)

\[
ds^2 = \eta_{ab} e^a e^b = L^2 \left( -e^{2\beta r} dt^2 + \eta_{ij}(r) \omega^i \otimes \omega^j + dr^2 \right)
\]

(86)

where $e^a, a = 0, \ldots, 4$, are one forms and $L$ is a positive constant that measures the size of the spacetime. The $\omega^i, i = 1, \ldots, 3$ are one forms manifestly invariant under the homogeneous symmetries described by the Bianchi classification.

\(^\text{12}\) In this coordinate system, the boundary of the Poincaré AdS metric lies at $r \to \infty$. 

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A. The gaugino conditions

In this section, we solve the gaugino conditions

\[
\delta \lambda_i^a = -\frac{i}{2} g f_z^a A_I A^I_\mu K_I \gamma^\mu \epsilon_i + \frac{1}{4} h_I^a F_{\mu \nu}^I \gamma^{\mu \nu} \epsilon_i - g_{R I} h_I^a V^I \delta_{ij} = 0 \tag{87}
\]

where we have substituted (78) and (80). In gauged supergravity literature the terms in the supersymmetry variations that are proportional to the gauge coupling constants are referred to as fermionic shifts. For maximal supersymmetry all of the fermionic shifts in the gaugino conditions must vanish [35, 52]. From the integrability conditions eq 31 of [35] it follows that the only maximally supersymmetric Bianchi type solution is \( \text{AdS}_5 \). Our first result will be to argue that the above result is also true for solutions with matter, in this case the Bianchi type geometries. Then we will require some of the fermionic shifts to vanish and explore conditions for supersymmetric solutions.

First we focus on the cases when none of the fermionic shifts vanish. Preserving some amount of supersymmetry from the gaugino and hyperino conditions require that the algebraic conditions on the constant part of the spinor \( \zeta_i \) be not too restrictive. In other words, the matrices that project out the various components of \( \zeta_i \) must commute with one another. The projection conditions that can appear on the spinor in the equations (87) are entirely dependent on the gauge field configurations. Typically the Bianchi type solutions are sourced by either timelike or spacelike massive gauge fields and a cosmological constant [13, 14, 35]. (In particular see appendix B of [13] for various choices of gauge field configurations that solve the equations of motion.) At the attractor point the scalars are constant and effective mass terms for the gauge fields

\[
g^2 K_{IJ}(\phi) A_I^\mu A_J^\mu \tag{88}
\]

appear due to the presence of the gauge covariant derivatives in the supergravity action (81). Here \( K_{IJ} \) is the Killing norm defined as \( g_{xy} K_I^x K_J^y \). The mass terms are proportional to the norm of the Killing vectors and to the square of the gauge coupling constant. We analyze two possible cases separately below.
1. Non vanishing fermionic shifts

To begin with we keep our analysis very generic with respect to the gauging of the scalar manifold (model independent) but specific only to the field content that generates the solution. By this, we mean that there are no specific conditions that the Killing vectors on $S$ are required to satisfy. We first consider the case where the gauge fields have only the time component turned on. The Bianchi metrics that have been constructed so far are sourced by time like or spacelike gauge fields. Time like gauge fields are of the form

$$ A = A(r)dt $$

$$ dA = \partial_r A(r)dr \wedge dt. $$ (89)

In order to solve the gaugino conditions (87) it is necessary to impose projection conditions on the constant part of the spinor $\epsilon_i$. From the time like gauge field configuration it is clear that the following conditions have to be imposed in (87)

$$ \gamma_0 \epsilon_i = \pm i \epsilon_i $$

$$ \gamma_0 A \epsilon_i = \pm \epsilon_i. $$ (90)

The first projector appears in the $A^\mu \gamma_\mu$ terms, while the second appears in the $F^{\mu \nu} \gamma_{\mu \nu}$ terms. While each of the projectors is well defined, it is clear that the two conditions are mutually incompatible since the projections (90) are mutually orthogonal. Thus when the fermionic shifts do not vanish all solutions sourced by time like gauge fields break supersymmetry. Thus with a time like gauge field, under gauging it is not possible to obtain supersymmetry preserving projection conditions. Note that this is completely independent of the functional dependence of the Killing spinor.

Let us now consider the case with gauge fields having spacelike components turned on. (For examples, see [13, 14, 35])

$$ A = A(x,r)\omega^i $$

$$ dA = \partial_r A(x,r)dr \wedge \omega^i + \partial_x^j A(x,r)dx^j \wedge \omega^i + \frac{1}{2} A(x,r)C_{ijk} \omega^j \wedge \omega^k $$ (91)

13 The spacetime coordinates are $x^\mu = (t, x^1, x^2, x^3, r)$, while the corresponding tangent space indices run over $a = 0, \ldots, 4$. 

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where \( x = x^i, i = 1, 2, 3 \) are the directions that have homogeneous symmetries. In this case, it is easy to see from \( \text{(87)} \) that the projections that can appear are

\[
\gamma_i \epsilon_i = \pm \epsilon_i \\
\gamma_{iA} \epsilon_i = \pm i \epsilon_i \\
\gamma_{ij} \epsilon_i = \pm i \epsilon_i .
\]

In any given configuration for the space like gauge field, the first projector always appears. Depending on the precise functional dependence the second/third or both second and third projectors can appear. In any case, we see that the first projector in \( \text{(92)} \) is mutually orthogonal to both the second and third. Thus even with a space like gauge field, under generic conditions it is not possible to obtain supersymmetry preserving conditions. Note that this too is completely independent of the functional dependence of the Killing spinor. Thus when the fermionic shifts do not vanish all massive Bianchi attractors are non-supersymmetric in gauged supergravity with generic gauging of the scalar manifold.

For all the solutions in this class, one can study the Killing spinor equations independently and find that the radial spinor breaks supersymmetry. We have summarized these results in appendix \( \text{\ref{sec:app}} \). The solutions constructed in \( \text{\cite{35}} \) are all of this type and are all non-supersymmetric.

2. Vanishing fermionic shifts

The other possibilities to solve \( \text{(87)} \) are situations where some of the fermionic shifts vanish in special cases. From the studies of the attractor mechanism for black holes in \( d = 5 \) ungauged supergravity, it is known that attractor solutions solve the gaugino conditions \( \text{\cite{54, 59}} \) with the extremization of central charge

\[
\partial_Z (Z) = \partial_Z (h^I Q_I) = 0 , \quad h^I V_I = 1 .
\]

Imposing the attractor conditions on \( \text{(87)} \)\(^{14} \), we find that the gaugino conditions reduce to

\[
\delta_\epsilon \lambda^a_f = - \frac{i}{2} gf^n A^I_{\mu} K_I^I \gamma^\mu \epsilon_i = 0 .
\]

\(^{14} \) The FI parameters \( V_I \) are arbitrary and can be scaled to satisfy this condition.
Note that the square of this fermionic shift term is proportional to
\[ g^2 g_{xy} A_I^I A_{\mu J}^J K_I^I K_J^J, \]  
the mass term discussed in the introduction of this section. Thus for preserving supersymmetry we have to set the effective mass term to zero. This can be achieved in two ways.

- The trivial choice is \( g = 0 \) or no gauging of the scalar manifold \( S \).
- The other more non-trivial possibility is to find a Killing direction in \( S \) that satisfies \( K_x^I Q_I = 0 \) at the attractor point.

Note that for the class of models discussed in [60, 61] studied earlier explicit solutions were constructed and analysed in [35, 46] and it can be checked that the condition \( K_x^I Q_I = 0 \) is not satisfied. However note that using this condition would kill the effective mass terms in the field equations of motion (see eq 18, eq 22 of [35]) which is problematic and would only lead to massless solutions.

We pause here to briefly summarize the conclusions of this section. Analyzing the gaugino conditions we have the results that in \( \mathcal{N} = 2 \) gauged supergravity with a generic gauging of the symmetries of scalar manifold and a \( U(1) \) gauging of the \( SU(2)_R \) symmetry,

- There are no massive Bianchi attractor solutions that preserve any amount of supersymmetry for a generic gauging when the fermionic shifts do not vanish.
- When the extremization condition is met \( \partial_x(Z) = 0 \), supersymmetry allows only massless Bianchi solutions. \(^{15}\)
- For massless Bianchi solutions, the gaugino conditions are completely solved by the attractor conditions \([93]\) and there are no additional projection conditions. The amount of supersymmetry preserved is completely determined by the Killing spinor equations.

Some examples of solutions with massless gauge fields are given in appendix \([11]\) These solutions can be easily constructed in Einstein-Maxwell theory with a cosmological constant, actually all of them can be also constructed easily, for instance in the \( U(1)_R \) gauged supergravity model studied in [47].

The last and final possibility for this section corresponds to vacuum solutions in the absence of matter. In this case the gaugino conditions are trivial. The supersymmetry conditions are

\(^{15}\) This possibility appears to be relaxed when tensor multiplets are included, we comment on this briefly in \([15, 20, 24]\)
completely determined by the Killing spinor equation that follows from the gravitino variation. The solution space includes the well known $AdS_5$ solution \[41, 52, 60\], Bianchi III $AdS_3 \times \mathbb{H}^2$ and Bianchi V $AdS_2 \times \mathbb{H}^3$ solutions sourced only by a cosmological constant (see appendix \[D\]). The results of this section can get modified by addition of tensor and hyper multiplets. We comment on this briefly in \[IV.C\].

B. The gravitino conditions: Killing spinor equation

We have already shown from the gaugino conditions that there are no possible supersymmetric Bianchi solutions sourced by massive gauge fields when the fermionic shifts do not vanish for the theory with a generic gauging of scalar manifold and a $U(1)_R$ gauging. For all these cases, one can show by analyzing the Killing spinor equation that a radial spinor independently gives rise to inconsistent projection conditions. We have summarized the results in \[C\].

In this section we analyze the gravitino Killing spinor equation for the Bianchi solutions sourced by massless gauge fields. For the $U(1)_R$ gauged supergravity \(78\), the Killing spinor equation we need to solve is of the form

\[
D_\mu \epsilon_i + \frac{i}{4\sqrt{6}} h_I F^{\mu\nu I} (\gamma_{\mu\nu} - 4g_{\mu\nu}\gamma_\rho) \epsilon_i + \frac{i}{\sqrt{6}} g_R \gamma_\mu \epsilon_i^k \epsilon_k = 0
\]  

where

\[
D_\mu \epsilon_i \equiv \partial_\mu \epsilon_i + \frac{1}{4} \omega^{ab}_\mu \gamma_{ab} \epsilon_i + g R A^I_\mu V_I \epsilon_i^k \epsilon_k
\]  

where we have used the attractor conditions \(93\). Note that we have used the notation $\epsilon_i^k \delta_{ij} = \epsilon_i^k$ where $\epsilon_i^k$ is numerically same as $-\epsilon_{ik}$. It follows that $\epsilon_i^k \epsilon_i^l = -\delta_i^l$. We need to remember that $\delta_{ij}$ is just one component of the general triplet in $P_{lij}$ and hence one cannot use $\delta_{ij}$ or $\epsilon_i^k$ to raise or lower the R symmetry index \([52, 61]\). In the following, we solve the Killing spinor equations for various Bianchi type geometries.

16 We thank Antoine Van Proeyen for useful communication regarding this issue.
1. **Bianchi I AdS$_5$**

As a warm up let us begin our analysis with the simplest known AdS$_5$ metric written in terms of the one forms

\[ e^0 = L e^r dt \], \[ e^1 = L e^r \omega^1 \], \[ e^2 = L e^r \omega^2 \], \[ e^3 = L e^r \omega^3 \], \[ e^4 = L dr \]  \hspace{1cm} (98)

where $L$ is the AdS scale. The invariant one forms

\[ \omega^i = dx^i, i = 1, 2, 3 \] \hspace{1cm} (99)

all commute with one another and satisfy $d \omega^i = 0$, characteristic of the Bianchi I algebra. Since we are discussing the $U(1)_R$ case, the gaugino conditions are trivial.$^{17}$

The Killing spinor equation (96) in the background (98) reads as,

\[
\begin{align*}
    e^{-r} \gamma_0 \partial_t \epsilon_i - \frac{1}{2} \gamma_4 \epsilon_i - \frac{i}{\sqrt{6}} L g_{R_k i}^k \epsilon_k &= 0 \\
    e^{-r} \gamma_1 \partial_{x^1} \epsilon_i + \frac{1}{2} \gamma_4 \epsilon_i + \frac{i}{\sqrt{6}} L g_{R_k i}^k \epsilon_k &= 0 \\
    e^{-r} \gamma_2 \partial_{x^2} \epsilon_i + \frac{1}{2} \gamma_4 \epsilon_i + \frac{i}{\sqrt{6}} L g_{R_k i}^k \epsilon_k &= 0 \\
    e^{-r} \gamma_3 \partial_{x^3} \epsilon_i + \frac{1}{2} \gamma_4 \epsilon_i + \frac{i}{\sqrt{6}} L g_{R_k i}^k \epsilon_k &= 0 \\
    \gamma_4 \partial_r \epsilon_i + \frac{i}{\sqrt{6}} L g_{R_k i}^k \epsilon_k &= 0.
\end{align*}
\] \hspace{1cm} (100)

The following equations can be obtained after some algebraic manipulations

\[
\begin{align*}
    \gamma_0 \partial_t \epsilon_i + \gamma_a \partial_{x^a} \epsilon_i &= 0 \\
    \gamma_a \partial_{x^a} \epsilon_i - \gamma_b \partial_{x^b} \epsilon_i &= 0 \\
    \gamma_4 \partial_r \epsilon_i + e^{-r} \gamma_0 \partial_t \epsilon_i - \frac{1}{2} \gamma_4 \epsilon_i &= 0 \\
    \gamma_4 \partial_r \epsilon_i - e^{-r} \gamma_a \partial_{x^a} \epsilon_i - \frac{1}{2} \gamma_4 \epsilon_i &= 0
\end{align*}
\] \hspace{1cm} (101)

$^{17}$ For more general $AdS$ critical points see [41, 60].
where $a = 1, 2, 3$. There are two independent solutions to the above equations

$$
\epsilon_i = e^{\frac{2}{3}} \zeta_i^+ \ , \ \gamma_4 \zeta_i^+ = \zeta_i^+ \quad (102)
$$

$$
\epsilon_i = \left( e^{-\frac{2}{3}} + e^{\frac{2}{3}} (x^m \gamma_m) \right) \zeta_i^- \ , \ \gamma_4 \zeta_i^- = -\zeta_i^- \quad . \quad (103)
$$

Each of the spinors (102) and (103) preserves $\frac{1}{2}$ the supersymmetry and the full solution enjoys a $\mathcal{N} = 2$ supersymmetry. Substituting the above in (100) we get the consistency condition

$$
\zeta_i^\pm = \mp \frac{2i}{\sqrt{6}} L g_R \epsilon_i^k \zeta_k^\pm \quad . \quad (104)
$$

It follows that (note that $\epsilon_i^k \epsilon_k^l = -\delta_i^l$)

$$
(1 - \frac{2}{3} L^2 g_R^2) \zeta_i^\pm = 0 \quad . \quad (105)
$$

This of course is the equation of motion for $AdS_5$ metric, thus we see that supersymmetry conditions automatically guarantee the equation of motion.

2. **Bianchi I: Anisotropic $AdS_3 \times \mathbb{R}^2$**

The anisotropic $AdS_3 \times \mathbb{R}^2$ solution can be easily constructed with magnetic fields and a cosmological constant (see §D).\footnote{See for example the isotropic solution in the $U(1)^3$ truncation of type II supergravity on $S^5$ by \cite{57}. For general geometries of the type $AdS_3 \times \Sigma_g$ in STU model of supergravity and their dual field theory interpretation see \cite{62,63}.} The metric has the simple form

$$
e^0 = e^r dt \ , \ e^1 = e^r \omega^1 \ , \ e^2 = \frac{|B|}{2} \omega^2 \ , \ e^3 = \frac{|B|}{2} \omega^3 \ , \ e^4 = dr \quad . \quad (106)
$$

The magnetic fluxes in the $x^2, x^3$ directions generate anisotropy but preserve the rotational symmetries of $\mathbb{R}^2$. The solution (106) has been of considerable interest in computations of shear viscosities in anisotropic phases \cite{23,53}. The invariant one forms

$$
\omega^i = dx^i \ , \ i = 1, 2, 3 \quad (107)
$$
all commute with one another and satisfy \( d\omega^i = 0 \) of the Bianchi I algebra. In (106) \(|B| = B^I B_I\) is the strength of the magnetic field. We choose our gauge field ansatz such that

\[
F_{x^2 x^3}^I = B^I .
\]  

(108)

The Killing spinor equations in the background are of the form

\[
\gamma_0 e^{-r} \partial_t \epsilon_i - \frac{1}{2} \gamma_4 \epsilon_i \left( \frac{Z}{2} \gamma_{23} \epsilon_i + g R \epsilon_i k \epsilon_k \right) = 0
\]  

(109)

\[
\gamma_1 e^{-r} \partial_{x^1} \epsilon_i + \frac{1}{2} \gamma_4 \epsilon_i \left( \frac{Z}{2} \gamma_{23} \epsilon_i + g R \epsilon_i k \epsilon_k \right) = 0
\]  

(110)

\[
\gamma_2 \partial_{x^2} \epsilon_i - \frac{i}{\sqrt{6}} \frac{|B|}{2} \left( - Z \gamma_{23} \epsilon_i + g R \epsilon_i k \epsilon_k \right) = 0
\]  

(111)

\[
\gamma_3 \partial_{x^3} \epsilon_i + \frac{i}{\sqrt{6}} \frac{|B|}{2} \left( - Z \gamma_{23} \epsilon_i + g R \epsilon_i k \epsilon_k \right) = 0
\]  

(112)

\[
\gamma_4 \partial_r \epsilon_i + \frac{i}{\sqrt{6}} \left( \frac{Z}{2} \gamma_{23} \epsilon_i + g R \epsilon_i k \epsilon_k \right) = 0
\]  

(113)

where \( Z = h_I B^I \) is the central charge. In the above we have chosen the following condition

\[
B^I V_I = 0 .
\]  

(114)

This condition is the five dimensional analogue of (115).

It is easy to obtain the following differential equations from the above set

\[
\gamma_0 \partial_t \epsilon_i + \gamma_1 \partial_{x^1} \epsilon_i = 0
\]  

\[
\gamma_2 \partial_{x^2} \epsilon_i - \gamma_3 \partial_{x^3} \epsilon_i = 0
\]  

\[
\gamma_4 \partial_r \epsilon_i + \gamma_0 e^{-r} \partial_t \epsilon_i - \frac{1}{2} \gamma_4 \epsilon_i = 0
\]  

\[
\gamma_4 \partial_r \epsilon_i - \gamma_1 e^{-r} \partial_{x^1} \epsilon_i - \frac{1}{2} \gamma_4 \epsilon_i = 0 .
\]  

(115)

Notice the similarity of the above equations to the ones we have obtained in \( AdS_5 \) case (111), except the second equation that suggests that the \( x^2, x^3 \) directions can scale differently as compared to the \( x^1 \) direction. Once again there are two independent solutions to the above equations

\[
\epsilon_i = e^{\frac{r}{2}} \zeta^+_i , \quad \gamma_4 \zeta^+_i = \zeta^+_i
\]  

(116)

\[
\epsilon_i = \left( e^{-\frac{r}{2}} + e^{\frac{r}{2}} (t \gamma_0 + x^1 \gamma_1 + \alpha (x^2 \gamma_2 + x^3 \gamma_3)) \right) \zeta^-_i , \quad \gamma_4 \zeta^-_i = - \zeta^-_i
\]  

(117)
where $\alpha$ is a real parameter. The projection due to the radial Dirac matrix has the same effect as in the $AdS$ case, namely the projector preserves one half of the supersymmetry in each of $\zeta^\pm$. Substituting the solution (117) in the $x^2, x^3$ equations (111)-(112) we find that $\alpha = 0$. Thus the Killing spinor (117) is independent of the $\mathbb{R}^2$ directions.

The remaining equations give rise to the conditions

$$\frac{1}{2} \gamma_i^\pm + \frac{i}{\sqrt{6}} \left( \frac{Z}{2} \gamma_{23} \zeta_i^\pm + g_R \epsilon_i^k \zeta_k^\pm \right) = 0$$

(118)

$$-Z \gamma_{23} \zeta_i^\pm + g_R \epsilon_i^k \zeta_k^\pm = 0.$$  

(119)

It is easy to see that the above two equations give rise to the conditions

$$\gamma_{23} \zeta_i^\pm = \epsilon_i^k \zeta_k^\pm$$

$$|Z| = |g_R| = \frac{\sqrt{6}}{3}.$$  

(120)

The projection above breaks half of the remaining supersymmetries in each of $\zeta^\pm$. As a result, each of $\zeta^\pm$ generate $\frac{1}{4}$ of the supersymmetry. Thus the solution (106) is a $\frac{1}{2}$ BPS solution.

3. Bianchi III and $AdS_3 \times \mathbb{H}^2$

In this section we construct a superymmetric Bianchi III type solution sourced by a massless gauge field

$$e^0 = Le^{\beta r} dt, \quad e^1 = L \omega^1, \quad e^2 = L e^{\beta r} \omega^2, \quad e^3 = L \omega^3, \quad e^4 = L dr,$$  

(121)

where the invariant one forms are

$$\omega^1 = e^{-x^1} dx^2, \quad \omega^2 = dx^3, \quad \omega^3 = dx^1.$$  

(122)

The spatial part of the metric has the symmetries of $\mathbb{H}^2 \times \mathbb{R}$. The symmetry algebra due to these Killing vectors form the Bianchi III algebra in the Bianchi classification in three dimensions. The sub algebra generated by the Killing vectors of $\mathbb{H}^2$ generate the Bianchi II algebra in two dimensions.
We choose the gauge field to have components along the $\omega^1$ direction

$$A^I = B^I e^1.$$  \hfill (123)

The Killing spinor equations evaluated in the background are ($Z = h_I B^I$)

$$e^{-\beta r} \gamma^0 \partial_t \epsilon_i - \frac{\beta}{2} \gamma_4 \epsilon_i - \frac{i}{\sqrt{6}} \left( \frac{1}{2} Z \gamma_{13} \epsilon_i + L g_R \epsilon_i^k \epsilon_k \right) = 0$$  \hfill (124)

$$e^{x^1} \gamma_1 \partial_{x^2} \epsilon_i - \frac{\gamma_3}{2} \epsilon_i + L g_R B^I V_I \gamma_1 \epsilon_i^k \epsilon_k + \frac{i}{\sqrt{6}} \left( -Z \gamma_{13} \epsilon_i + L g_R \epsilon_i^k \epsilon_k \right) = 0$$  \hfill (125)

$$e^{-\beta r} \gamma_2 \partial_{x^3} \epsilon_i + \frac{\beta}{2} \gamma_4 \epsilon_i + \frac{i}{\sqrt{6}} \left( \frac{1}{2} Z \gamma_{13} \epsilon_i + L g_R \epsilon_i^k \epsilon_k \right) = 0$$  \hfill (126)

$$\gamma_3 \partial_{x^1} \epsilon_i + \frac{i}{\sqrt{6}} \left( -Z \gamma_{13} \epsilon_i + L g_R \epsilon_i^k \epsilon_k \right) = 0$$  \hfill (127)

$$\gamma_4 \partial_r \epsilon_i + \frac{i}{\sqrt{6}} \left( \frac{1}{2} Z \gamma_{13} \epsilon_i + L g_R \epsilon_i^k \epsilon_k \right) = 0.$$  \hfill (128)

As before we can obtain the following equations from above

$$\gamma_0 \partial_t \epsilon_i + \gamma_2 \partial_{x^3} \epsilon_i = 0$$

$$e^{-\beta r} \gamma_0 \partial_t \epsilon_i - \frac{\beta}{2} \gamma_4 \epsilon_i + \gamma_4 \partial_r \epsilon_i = 0$$

$$e^{-\beta r} \gamma_2 \partial_{x^3} \epsilon_i + \frac{\beta}{2} \gamma_4 \epsilon_i - \gamma_4 \partial_r \epsilon_i = 0$$

$$e^{x^1} \gamma_{13} \partial_{x^2} \epsilon_i + \partial_{x^1} \epsilon_i + \frac{\epsilon_i}{2} + L g_R B^I V_I \gamma_{13} \epsilon_i^k \epsilon_k = 0.$$  \hfill (129)

The $AdS_3$ part of the Killing spinor will preserve some supersymmetry provided we assume that the Killing spinor does not depend on the $\mathbb{H}^2$ part. We get the following conditions from the above set of equations

$$\epsilon_i = e^{\frac{\beta r}{2} \xi_i^+}, \quad \gamma_{44} \xi_i^+ = \xi_i^+$$  \hfill (130)

$$\epsilon_i = \left( e^{\frac{\beta r}{2} \xi_i^+} + e^{\frac{\beta r}{2} (t \gamma_0 + x^3 \gamma_2)} \right) \xi_i^-, \quad \gamma_{44} \xi_i^- = -\xi_i^-$$  \hfill (131)

$$\gamma_{13} \xi_i^\pm = \epsilon_i^k \xi_k^\pm, \quad 4 L^2 g_R^2 (B_I V^I)^2 = 1.$$  \hfill (132)

As discussed in the previous sections, the two projectors above combine to break half of the total supersymmetries of the solution. Substituting the above relations in the Killing spinor equation
we find
\[ \frac{\beta}{2} \xi^\pm_i + \frac{i}{\sqrt{6}} \left( \left( \frac{Z}{2} + L_{GR} \right) \epsilon_i k \kappa^\pm_k \right) = 0 \quad (133) \]
\[ (-Z + L_{GR}) \epsilon_i k \kappa^\pm_k = 0 \quad . \quad (134) \]

Consistency of the above equations yield the conditions
\[ L_{GR} = Z, \quad \beta = \sqrt{\frac{3}{2}} Z, \quad 4L^2g_R^2(B_1V')^2 = 1 \quad . \quad (135) \]

Thus we have a one parameter family of 1/2 BPS Bianchi III solutions labeled by the central charge Z.

When the central charge takes the value (120) (the one corresponding to the $AdS_3 \times \mathbb{R}^2$ solution), it follows from (135) that
\[ L = 1, \quad \beta = 1 \quad . \quad (136) \]

This is the $AdS_3 \times \mathbb{H}^2$ solution constructed in [54].

It is also possible to construct the vacuum $AdS_3 \times \mathbb{H}^2$ solution (see (D9)). In this case the simplified equations (129) are
\[ \gamma_0 \partial_t \epsilon_i + \gamma_2 \partial_{x^3} \epsilon_i = 0 \]
\[ e^{-r} \gamma_0 \partial_t \epsilon_i - \frac{\gamma_4}{2} \epsilon_i + \gamma_4 \partial_r \epsilon_i = 0 \]
\[ e^{-r} \gamma_2 \partial_{x^3} \epsilon_i + \frac{\gamma_4}{2} \epsilon_i - \gamma_4 \partial_r \epsilon_i = 0 \]
\[ e^{x^1} \gamma_{13} \partial_{x^2} \epsilon_i + \frac{\epsilon_i}{2} + \partial_{x^1} \epsilon_i = 0 . \quad (137) \]

Any solution necessarily depends on the $\mathbb{H}^2$ coordinates and breaks supersymmetry. The $AdS_3$ part of the equations (first three of (137)) are solved by the usual
\[ \epsilon_i = e^{\frac{r}{2}} \xi^+_i, \quad \gamma_4 \xi^+_i = \zeta^+_i \]
\[ \epsilon_i = \left( e^{\frac{r}{2}} (\gamma_0 t + \gamma_2 x^3) + e^{-\frac{r}{2}} \right) \zeta^-_i, \quad \gamma_4 \zeta^-_i = -\zeta^-_i \quad . \quad (138) \]
whereas the $\mathbb{H}^2$ part of the equations (the last equation in (137)) are solved by

\[
\begin{align*}
\epsilon_i &= e^{-\frac{1}{2}x_1^1} \zeta_i^- , \quad \gamma_3 \zeta_i^- = -\zeta_i^- \\
\epsilon_i &= (e^{-\frac{1}{2}x_1^1} \gamma_1 x_2^2 + e^{\frac{1}{2}x_1^1}) \zeta_i^+ , \quad \gamma_3 \zeta_i^+ = \zeta_i^+ .
\end{align*}
\]

(139)

We see that $\zeta^\pm$ are required to be simultaneous eigenspinors of both $\gamma_3$ and $\gamma_4$ in order to solve the full set of equations (137). However, that is impossible since the matrices anti commute. Thus the product space in the vacuum case breaks all supersymmetry. In the charged case, we are able to avoid the spinor being an eigenspinor of $\gamma_3$ due to the condition (132). This is consistent with the conclusion from the integrability condition eq. 31 of [35] that $AdS_5$ is the unique maximally supersymmetric vacuum solution in the theory.

4. Bianchi V

The Bianchi V solution constructed in [13] is of the form

\[
e^0 = L e^{\beta x} dt , \quad e^1 = L \omega^1 , \quad e^2 = L \omega^2 , \quad e^3 = L \omega^3 , \quad e^4 = L dr
\]

(140)

where the invariant one forms are given by

\[
\omega^1 = e^{-x_1^1} dx^2 , \quad \omega^2 = e^{-x_1^1} dx^3 , \quad \omega^3 = dx^1 .
\]

(141)

The Bianchi V geometry in this case has the form of $AdS_2 \times \mathbb{H}^3$. The metric is sourced by a massless time like gauge field

\[
A^I = E^I e^0 .
\]

(142)

---

19 The general solution is a combination of (138) and (139).
The Killing spinor equations in this background take the form \((Z = E^I h_I)\)

\[
e^{-\frac{\beta t}{2}} \gamma_0 \partial_t \epsilon_i - \frac{\beta t}{2} \gamma_4 \epsilon_i + g_R L E^I V_I \gamma_0 \epsilon_i^k \epsilon_k + \frac{i}{\sqrt{6}} \left( \beta_t Z \gamma_0 \epsilon_i - L g_R \epsilon_i^k \epsilon_k \right) = 0
\]

\[
e^{x^1} \gamma_1 \partial_{x^2} \epsilon_i - \frac{1}{2} \gamma_3 \epsilon_i + \frac{i}{\sqrt{6}} \left( \beta_t \frac{1}{2} Z \gamma_0 \epsilon_i + L g_R \epsilon_i^k \epsilon_k \right) = 0
\]

\[
e^{x^1} \gamma_2 \partial_{x^3} \epsilon_i - \frac{1}{2} \gamma_3 \epsilon_i + \frac{i}{\sqrt{6}} \left( \beta_t \frac{1}{2} Z \gamma_0 \epsilon_i + L g_R \epsilon_i^k \epsilon_k \right) = 0
\]

\[
\gamma_3 \partial_{x^1} \epsilon_i + \frac{i}{\sqrt{6}} \left( \beta_t \frac{1}{2} Z \gamma_0 \epsilon_i + L g_R \epsilon_i^k \epsilon_k \right) = 0
\]

\[
\gamma_4 \partial_{x^1} \epsilon_i - \frac{i}{\sqrt{6}} \left( \beta_t \frac{1}{2} Z \gamma_0 \epsilon_i - L g_R \epsilon_i^k \epsilon_k \right) = 0. \tag{143}
\]

We can write down the following differential equations after some algebraic manipulations

\[
e^{-\frac{\beta t}{2}} \gamma_0 \partial_t \epsilon_i + \gamma_4 \partial_{x^1} \epsilon_i - \frac{\beta t}{2} \gamma_4 \epsilon_i + g_R L E^I V_I \gamma_0 \epsilon_i^k \epsilon_k = 0
\]

\[
\gamma_1 \partial_{x^2} \epsilon_i - \gamma_2 \partial_{x^3} \epsilon_i = 0
\]

\[
e^{x^1} \gamma_3 \partial_{x^2} \epsilon_i + \partial_{x^1} \epsilon_i + \frac{\epsilon_i}{2} = 0
\]

\[
e^{x^1} \gamma_2 \partial_{x^3} \epsilon_i + \partial_{x^1} \epsilon_i + \frac{\epsilon_i}{2} = 0. \tag{144}
\]

Following the arguments given in the previous section, we can solve the AdS\(_2\) part of the equations (first in \((144)\)) by

\[
\epsilon_i = e^{\frac{\beta t}{2} x^1} \zeta_i^+, \quad \gamma_4 \zeta_i^+ = \zeta_i^+
\]

\[
\epsilon_i = (e^{\frac{\beta t}{2} \gamma_0 t} + e^{-\frac{\beta t}{2}}) \zeta_i^- , \quad \gamma_4 \zeta_i^- = -\zeta_i^-
\]

\[
\text{provided we set } E^I V_I = 0. \quad \text{If } E^I V_I \neq 0 \text{ in this case, even the radial spinor breaks all supersymmetry.}^{20}
\]

Similarly the \(\mathbb{H}^3\) part of the equations (last three of \((144)\)) can be solved by

\[
\epsilon_i = e^{-\frac{x^1}{2}} \zeta_i^- , \quad \gamma_3 \zeta_i^- = -\zeta_i^-
\]

\[
\epsilon_i = (e^{-\frac{x^1}{2}} (\gamma_1 x^2 + \gamma_2 x^3) + e^{\frac{x^1}{2}}) \zeta_i^+ , \quad \gamma_3 \zeta_i^+ = \zeta_i^+. \tag{146}
\]

Once again, we see that the \(\zeta_{\pm}\) are required to be simultaneous eigenspinors of \(\gamma_3\) and \(\gamma_4\), that is impossible since the matrices do not commute.\(^{21}\) Thus the solution breaks all supersymmetry.

\(^{20}\)See \((C3)\) for some related details.

\(^{21}\)In this case too, the general solution of \((144)\) is a combination of \((145)\) and \((146)\).
The same arguments apply for the vacuum Bianchi V $AdS_2 \times \mathbb{H}^3$ solution (D9).

5. Bianchi VII

The Bianchi VII metric is expressed in terms of the following one forms

\[ e^0 = L e^\beta r dt, \quad e^1 = L dx^1, \quad e^2 = L e^\beta r (\cos(x^1) dx^2 + \sin(x^1) dx^3), \]
\[ e^3 = L \lambda e^\beta r (\sin(x^1) dx^2 + \cos(x^1) dx^3), \quad e^4 = L dr \] (147)

where $\lambda$ is a squashing parameter. The gauge field ansatz is of the form

\[ A^I = B^I e^2 \] (148)

where $B^I$ are constants. The Killing spinor equations in the above background take the form ($Z = h_I B^I$)

\[ e^{-\beta r} \gamma_0 \partial_t \epsilon_i - \frac{\beta}{2} \gamma_4 \epsilon_i + \frac{i}{\sqrt{6}} \left( \frac{Z}{2} \left( \beta \gamma_{24} - \frac{\gamma_{13}}{\lambda} \right) \epsilon_i - L g_R \epsilon_i^k \epsilon_k \right) = 0 \]
\[ \gamma_1 \partial_{x^1} \epsilon_i - \frac{1 + \lambda^2}{4\lambda} \gamma_{123} \epsilon_i - \frac{i}{\sqrt{6}} \left( \frac{Z}{2} \left( \beta \gamma_{24} + \frac{2\gamma_{13}}{\lambda} \right) \epsilon_i - L g_R \epsilon_i^k \epsilon_k \right) = 0 \]
\[ e^{-\beta r} \gamma_2 (\cos(x^1) \partial_{x^2} + \sin(x^1) \partial_{x^3}) \epsilon_i + \frac{(1 - \lambda^2)}{4\lambda} \gamma_{123} \epsilon_i + \frac{\beta}{2} \gamma_{4} \epsilon_i + L g_R B^I \gamma_{24} \epsilon_i^k \epsilon_k \]
\[ + \frac{i}{\sqrt{6}} \left( \frac{Z}{2} \left( \frac{\gamma_{13}}{\lambda} + 2\beta \gamma_{24} \right) \epsilon_i + L g_R \epsilon_i^k \epsilon_k \right) = 0 \]
\[ e^{-\beta r} \gamma_3 (\sin(x^1) \partial_{x^2} + \cos(x^1) \partial_{x^3}) \epsilon_i - \frac{(1 - \lambda^2)}{4\lambda} \gamma_{123} \epsilon_i + \frac{\beta}{2} \gamma_{4} \epsilon_i \]
\[ + \frac{i}{\sqrt{6}} \left( -\frac{Z}{2} \left( \beta \gamma_{24} + \frac{2\gamma_{13}}{\lambda} \right) \epsilon_i + L g_R \epsilon_i^k \epsilon_k \right) = 0 \]
\[ \gamma_{4} \partial_r \epsilon_i + \frac{i}{\sqrt{6}} \left( \frac{Z}{2} \left( \frac{\gamma_{13}}{\lambda} + 2\beta \gamma_{24} \right) \epsilon_i + L g_R \epsilon_i^k \epsilon_k \right) = 0. \] (149)

After a few algebraic steps we get the following differential equations

\[ e^{-\beta r} \gamma_3 (\sin(x^1) \partial_{x^2} + \cos(x^1) \partial_{x^3}) \epsilon_i - \gamma_1 \partial_{x^1} \epsilon_i + \frac{\lambda}{2} \gamma_{123} \epsilon_i + \frac{\beta}{2} \gamma_{4} \epsilon_i = 0 \]
\[ e^{-\beta r} \gamma_2 (\cos(x^1) \partial_{x^2} + \sin(x^1) \partial_{x^3}) \epsilon_i + \frac{(1 - \lambda^2)}{4\lambda} \gamma_{123} \epsilon_i - \gamma_4 \partial_r \epsilon_i + \frac{\beta}{2} \gamma_{4} \epsilon_i + L g_R B^I \gamma_{24} \epsilon_i^k \epsilon_k = 0 \] (150)
that are solved by the radial spinor

\[ B^I V_I = 0, \quad \gamma_{1234} \epsilon_i = \epsilon_i, \quad \beta = \lambda \]

\[ \epsilon_i = e^{\frac{\gamma_{24}}{4\beta}} \zeta_i . \]  

(151)

Substituting (151) back into (149) we obtain

\[ -\frac{\beta_t}{2} \gamma_4 \epsilon_i + \frac{i}{\sqrt{6}} \left( Z \frac{\beta^2 - 1}{\beta} \gamma_{24} \epsilon_i - L g R \epsilon_i^k \epsilon_k \right) = 0 \]

\[ -\left( \frac{1 + \beta^2}{4\beta} \right) \gamma_4 \epsilon_i + \frac{i}{\sqrt{6}} \left( Z \frac{\beta^2 + 1}{\beta} \gamma_{24} \epsilon_i - L g R \epsilon_i^k \epsilon_k \right) = 0 \]

\[ \frac{3\beta^2 - 1}{4\beta} \gamma_4 \epsilon_i + \frac{i}{\sqrt{6}} \left( Z \frac{2\beta^2 + 1}{\beta} \gamma_{24} \epsilon_i + L g R \epsilon_i^k \epsilon_k \right) = 0. \]  

(152)

As a simple check of the equations we see that \( \beta = 1, \beta_t = 1, Z = 0 \) correspond to the Bianchi I (AdS) solution (104). The equations (152) lead to the projections

\[ \gamma_4 \zeta_i = -i \epsilon_i^k \zeta_k, \quad (\beta_t + 2\beta)^2 = 6L^2 g_R^2 \]

\[ \gamma_2 \zeta_i = -i \zeta_i, \quad \left( \frac{\beta^2 - 1}{\beta^2 + 1} \right)^2 = \frac{3Z^2}{2}, \quad \beta_t = \frac{\beta^4 + 4\beta^2 - 1}{2\beta(1 + \beta^2)}. \]  

(153)

It is clear that the additional projection condition due to \( \gamma_2 \) breaks all of the supersymmetry. Thus the Bianchi VII solution (147) is non supersymmetric. However the Bianchi VII algebra is a sub algebra of the Poincaré algebra (see §E) and hence also part of the super Poincaré algebra. It is possible that there are more general solutions in this class that may be supersymmetric.

C. Including hyper and tensor multiplets

In this section, we briefly comment about the possibilities of new supersymmetric solutions due to addition of tensor or hypermultiplets. We will provide formal arguments as explicit solutions such as the ones constructed in [35] have not been explored yet in specific models with tensor/hypermultiplets. The addition of tensor/hyper multiplets modifies the supersymmetry transformations (82). Let us first consider the gravitino equation (52)

\[ \delta \psi_{\mu i} = D_\mu \epsilon_i + \frac{i}{4\sqrt{6}} h_{\gamma} H^{\gamma\rho}(\gamma_{\mu\nu} - 4g_{\mu\nu} \gamma_\rho) \epsilon_i + \frac{i}{\sqrt{6}} g R \gamma_\mu \epsilon^j P_{ij} \]  

(154)
where $H^\tilde{M}_{\mu\nu} = \{F^I_{\mu\nu}, B^J_{\mu\nu}\}$, $I = 0 \ldots n_V$ and $J = 1, \ldots, n_T$, $B^J_{\mu\nu}$ is an antisymmetric tensor that belongs to the tensor multiplet. The scalars $h^I_{ij} = \{h^I, h^J\}$ are similarly functions of scalars from the vector and tensor multiplets respectively. The addition of hypermultiplets allows more general R symmetry gauging of the full $SU(2)_R$ symmetry group,

$$P_{ij}(q) = h^I P_{ij}^I(q) = h^I P_{ij}^I(q)(\sigma_r)_{ij}$$  \hspace{1cm} (155)$$

where the potentials are now $SU(2)$ valued functions of the hyperscalars in the hypermultiplet.

First let us consider the case of hypermultiplets turned on, but no tensor multiplets. In this case, the only difference is that the quaternionic prepotential is a $SU(2)$ triplet function of the hyperscalars instead of a singlet for the $U(1)_R$ case. Hence for $\mathcal{N} = 2$ gauged supergravity with $SU(2)_R$ gauging, including vector and hypermultiplets, and a generic gauging of the symmetries of the very special manifold and the quaternionic Kähler manifold the Killing spinor results that pertain to non-supersymmetric solutions in §106 and §IV B continue to hold. 22 Of course, this does not affect the gaugino conditions, but in addition there are new conditions from hyperino equations. We will discuss them shortly.

With tensor multiplets turned on in addition there are more possibilities. If the tensor fields are oriented carefully there are possibilities of subtle cancellations that can potentially lead to interesting new solutions with supersymmetry preserving projection conditions. However, in the models that have been studied before in [35] we have not found any such possibility. Nevertheless this requires an independent analysis and it is helpful to obtain some conditions from gaugino and hyperino conditions first to aid in this direction.

The addition of tensor multiplets also changes the analysis of the gaugino conditions in an interesting way. The gaugino equations acquire an additional term due to tensor multiplets [52]

$$\delta_{\epsilon} \lambda_{\tilde{a}} = -\frac{i}{2} g A^I_{\mu} f^I_{\tilde{a} \tilde{x}} K_{I}^{\tilde{x} \gamma} \gamma^\mu \epsilon_i + \frac{1}{4} h^A_{\tilde{M}} \mathcal{H}_{\mu\nu}^\tilde{M} \gamma^{\mu\nu} \epsilon_i - g_R \epsilon_i P^I_{ij} + g \sqrt{6} h^I K^I_{\lambda} f^\lambda_{\tilde{a}} = 0$$  \hspace{1cm} (156)$$

where $\tilde{x} = 0, n_V + n_T$ labels the moduli $\phi_{\tilde{x}}$ in the vector and tensor multiplets. The vielbeins $f^I_{\tilde{a}}$ live on the tangent space corresponding to the very special manifold $S$. If we continue to impose

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22 For the supersymmetric solutions §106 and §IV B 3 addition of tensor and hypermultiplets imposes additional new relations from the hyperscalar equations and the tensor field equations of motion. Moreover the parameter space is also enhanced, so one can possibly find new such solutions. It will be interesting to see if the solutions §106 and §IV B 3 continue to remain supersymmetric in suitable models.
a straightforward generalization of the attractor conditions \[93\] 23

\[\partial_{\Xi}((Q_I h^I(\phi^*)) + B_J h^J(\phi^*)) = 0 , \ h^I(\phi^*)P^\Xi_I(q^*) = 1 \quad (157)\]

the gaugino equations reduce to

\[\delta_\epsilon \lambda^i = g f^\Xi_A K^\Xi_A \left( -iA^I_I \gamma^\mu + \frac{\sqrt{6}}{2} h^I \right) \epsilon_i = 0 \quad (158)\]

that can be solved for an electric solution by imposing the conditions

\[\gamma^0 \epsilon_i = \pm i \epsilon_i , \ g f^{\Xi_A}K^\Xi_A(\phi^*) \left( \pm Q^I + 2 \sqrt{6} h^I(\phi^*) \right) = 0 \quad (159)\]

or by imposing the conditions

\[Q^I K^\Xi_A(\phi^*) = 0 , \ h^I(\phi^*)K^\Xi_A(\phi^*) = 0 . \quad (160)\]

for either of electric or magnetic solutions. In addition one also has the hyperino conditions \[52\] at the attractor point

\[\delta \zeta^A = g f^A_X K^X_X \left( -iA^I_I \gamma^\mu + \frac{\sqrt{6}}{2} h^I \right) \epsilon_i = 0 \quad (161)\]

In the above, \(K^X_X\) are similarly Killing vectors on the Quaternioni manifold \(Q\), \(f^A_X\) are vielbeins on \(Q\), \(g\) is the gauge coupling constant for the gauging of the symmetries on \(Q\). Note that \(161\) is structurally similar to the gaugino condition \(158\) after imposing attractor like conditions \(157\). Thus for electric solutions we can impose

\[\gamma^0 \epsilon_i = \pm i \epsilon_i , \ g f^A_X(q^*)K^X_X(q^*) \left( \pm Q^I + 2 \sqrt{6} h^I(\phi^*) \right) = 0 \quad (162)\]

or by imposing the conditions

\[Q^I K^X_X(q^*) = 0 , \ h^I(\phi^*)K^X_X(q^*) = 0 . \quad (163)\]

for either of electric or magnetic solutions. It is interesting to note that the conditions in \(160\)

\[23\] Here \(\phi^*\) and \(q^*\) are constant attractor values of the moduli and \(B^I\) are the tensor charges.
appear in flow equations that preserve supersymmetry in AdS (see eq 2.60 of [41]). So it seems reasonable to impose the above conditions to find Bianchi attractor solutions that potentially flow to an asymptotic AdS geometry. However the conditions

\[ Q^I K^X_I (q^*) = 0, \ Q^I K^X_I (\phi^*) = 0 \]  

are problematic as they kill the effective mass terms in the field equations [35] and would still lead to massless solutions. Thus one possibility to find more interesting massive Bianchi solutions in the \( \mathcal{N} = 2 \) theory with vector, tensor and hypermultiplets with generic gauging is to consider solutions sourced by time like gauge fields. Then the gaugino and hyperino equations are satisfied by the attractor condition (157), the projections (159) and (162). However solving the Killing spinor equation would require great care in choosing the tensor field configuration, as we would require a projection condition on the spinor that would commute with that of (159) and (162). We have not found any such solution in the models considered earlier in [35, 46]. Perhaps instead of trying to find explicit solutions and then verifying supersymmetry it may be useful to analyse the Killing spinor integrability conditions carefully together with the flow conditions (164) to determine the possible supersymmetric Bianchi attractor solutions in this theory. We hope to report this in a future work.

V. SUMMARY

In this paper we analyzed the supersymmetry of Bianchi attractors in \( \mathcal{N} = 2 \) \( d = 4,5 \) gauged supergravity. In \( d = 4 \), we studied the supersymmetry of Bianchi I and II attractors sourced by electric fields. In the Bianchi I case, we studied an \( AdS_2 \times \mathbb{R}^2 \) metric sourced by a time like gauge field. We analyze the gaugino and Killing spinor equations and find that the radial spinor and its projection condition preserve 1/4 of the supersymmetry. In the Bianchi II case, we construct an electric \( AdS_2 \times \mathbb{H}^2 \) solution and find that the radial spinor breaks all supersymmetry.\(^{24}\) The main lesson we learnt from this exercise is that the radial spinor plays an important role in preserving

\(^{24}\) The magnetic \( AdS_2 \times \mathbb{H}^2 \) is known to be 1/8 BPS [50, 66].
supersymmetry. These results are special cases of the more general analysis of \[34, 51\].

In \(d = 5\) \(\mathcal{N} = 2\) gauged supergravity, we consider the theory with a generic gauging of symmetries of the scalar manifold and a \(U(1)_R\) gauging of the R symmetry. The Bianchi attractor geometries that can be constructed are sourced by massive or massless gauge fields. For a generic gauging of the scalar manifold and R symmetry, when the fermionic shifts in the gaugino and hyperino conditions do not vanish, the projection conditions that need to be imposed on the Killing spinor depend entirely on the gauge field/field strength configuration. We show that for the known field configurations that source the Bianchi type geometries, there are no supersymmetric projections possible. Independently we show that the radial spinor breaks supersymmetry for all metrics of this class. Thus for a generic gauging of the scalar manifold and when the fermionic shifts do not vanish there are no supersymmetric Bianchi attractors. This result for Bianchi type geometries is similar to the result for maximally supersymmetric solutions \[41, 52\].

When the central charge of the theory satisfies an extremization condition at the attractor point

\[ \partial_i Z = 0 \]  \hspace{1cm} (166)

some of the fermionic shifts vanish. Supersymmetry invariance of the resultant equations allow only massless solutions. This prompts the search for Bianchi type metrics sourced by massless gauge fields and cosmological constant. We construct new Bianchi I, Bianchi III, Bianchi V and Bianchi VII classes of solutions sourced by massless gauge fields and a cosmological constant. Since the gaugino conditions are completely solved in these cases, the supersymmetry preserved by the geometries are determined by the Killing spinor equation. In the Bianchi I class we construct an anisotropic 1/2 BPS \(AdS_3 \times \mathbb{R}^2\) solution where the anisotropy is generated by a magnetic field. The supersymmetry is entirely due to the \(AdS_3\) part and the Killing spinor does not depend on the \(\mathbb{R}^2\) directions. We also construct a one parameter family of 1/2 BPS Bianchi III geometries, labeled by the central charge. When the central charge of the Bianchi III geometry takes the same value as that of \(AdS_3 \times \mathbb{R}^2\), the solution reduces to the known 1/2 BPS \(AdS_3 \times \mathbb{H}^2\) solution \[54\]. For the Bianchi V and Bianchi VII classes the radial spinor breaks all supersymmetry and hence these are non-supersymmetric geometries. However, the parameters that characterize these solutions can be chosen in accordance with the stability criterion discussed in \[47\].

\[25\] In the study of the attractor mechanism in \(d = 5\) ungauged supergravity, It is well known that central charge satisfies an extremization condition at the attractor points \[59\].
Finally we also construct vacuum Bianchi III ($AdS_3 \times \mathbb{H}^2$) and Bianchi V ($AdS_2 \times \mathbb{H}^3$) geometries respectively. The solutions for the Killing spinor equations in both the cases necessarily require dependence on the $\mathbb{H}^2/\mathbb{H}^3$ coordinates respectively. The radial projection matrices for the $AdS$ and the $EAdS$ geometries do not commute and hence these geometries break all of supersymmetry. This is consistent with the results from integrability conditions in $\mathcal{N} = 2$ gauged supergravity [35].

In §IVC we explored the possible conditions to find more interesting Bianchi attractor geometries with massive gauge fields. Solutions with time like gauge fields and suitable tensor field configurations may give rise to supersymmetry preserving projection conditions in the Killing spinor equation. However, this has not worked so far in the models considered in [35, 46]. We hope to explore this more further in future works.

Having constructed some of the simplest supersymmetric Bianchi attractors it is interesting to find such solutions in theories with more supersymmetry. It will also be interesting to uplift these solutions to higher dimensional supergravity. The Killing spinor equations suggest that in most cases if the geometry has an $AdS_n$ part that factorizes, the corresponding Killing spinor is sufficient to preserve supersymmetry of the whole solution. Having an $AdS$ part may enable the construction of more general Bianchi attractor geometries. Finally, it will be most interesting to construct analytic solutions that interpolate to $AdS$. A related issue is the embeddability of the Bianchi algebra in the Poincaré or the conformal algebra. The Bianchi I and Bianchi VII algebras are sub algebras of the Poincaré algebra, the other Bianchi algebras have scaling type generators and may presumably be obtained from a truncation of the conformal algebra.

In this work we studied Bianchi attractors in $d = 4, 5$. Earlier works have constructed Bianchi attractors as generalized attractors in gauged supergravity [35, 45, 46]. In the studies of black holes in ungauged supergravity there have been studies on the 4d/5d correspondence where relation between the potential and critical points in $d = 4$ and $d = 5$ have been elucidated [67]. Similar studies have been performed for gauged supergravity relating black strings in $d = 5$ and $AdS_2 \times S^2$ in $d = 4$ [68]. It would be interesting to explore the relation between generalized attractor potentials in $d = 4$ and $d = 5$ and their critical points.

Acknowledgments

We would like to thank Prasanta Tripathy and Sandip Trivedi for several stimulating discussions throughout the course of this work. We would like to thank Arpan Saha, Prasanta Tripathy and Sandip Trivedi for collaboration during the initial stages of this work. BC would like to
thank Amitabh Virmani for discussions. K.I. would like to thank Antoine Van Proeyen for helpful correspondence. BC would like to acknowledge the hospitality of I.I.T. Gandhinagar and I.I.T. Kanpur while this work was in preparation. K.I. would like to acknowledge the hospitality of I.O.P. Bhubaneswar while this work was in preparation. The work of BC is partly supported by the D.S.T.-Max Planck Partner Group “Quantum Black Holes” between I.O.P. Bhubaneswar and A.E.I. Golm. The work of K.I. and R.S. was supported in part by Infosys Endowment for the study of the Quantum Structure of Space Time. We would all like to acknowledge our debt to the people of India for their generous and steady support for research in basic sciences.

APPENDIX A: CONVENTIONS

1. Gamma matrices and Spinors in four dimensions

The Clifford algebra in 4 space-time dimensions is

\[ \{\gamma_a, \gamma_b\} = 2\eta_{ab} \tag{A1} \]

with the metric convention \( \eta_{ab} = \{+,-,-,-\} \). The Dirac matrices in four dimensions can be chosen to be

\begin{align*}
\gamma^0 &= I_2 \otimes \sigma_1 \\
\gamma^1 &= i\sigma_1 \otimes \sigma_2 \\
\gamma^2 &= i\sigma_2 \otimes \sigma_2 \\
\gamma^3 &= i\sigma_3 \otimes \sigma_2 \tag{A2}
\end{align*}

where \( \sigma_i, i = 1, 2, 3 \) are the usual Pauli matrices and \( I_2 \) is the two dimensional unit matrix. We define the chirality matrix to be \( \gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3 \) and the charge conjugation matrix \( C = i\gamma^2\gamma^0 = \gamma^1\gamma^3 \). The charge conjugation matrix \( C \) has the property \( C^t = -C = C^{-1} \).

In four dimensions we can impose the weyl condition on a four component spinor such that

\begin{align*}
\gamma_5 \epsilon_A &= \epsilon_A \\
\gamma_5 \epsilon^A &= -\epsilon^A \tag{A3}
\end{align*}

\[ ^{26} \text{We follow the conventions of} \ [48] \text{for} \ N = 2, d = 4 \text{ gauged supergravity.} \]
where the conjugate spinor is defined as

\[ \epsilon^A = (\epsilon_A)^c = \gamma_0 C^{-1}(\epsilon_A)^* = -\gamma_0 C(\epsilon_A)^* . \] (A4)

We use the following decomposition of the spinors in some sections. Using the fact that \([\gamma_5, C] = 0\), we can decompose the spinor into simultaneous eigenstates of \(C\) and \(\gamma_5\) as follows

\[ \epsilon_A = \begin{pmatrix} 0 \\ C_A^+ |+\rangle \end{pmatrix} + \begin{pmatrix} 0 \\ C_A^- |-\rangle \end{pmatrix} \] (A5)

where \(C_A^+\) and \(C_A^-\) are complex coefficients. The two component states \(|+\rangle, |-\rangle\)

\[ |+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \quad |-\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} \] (A6)

are eigenstates of \(\sigma\) matrices

\[ \sigma^1|\pm\rangle = \pm i|\mp\rangle, \quad \sigma^2|\pm\rangle = \pm|\pm\rangle, \quad \sigma^3|\pm\rangle = |\mp\rangle . \] (A7)

2. Gamma matrices and Spinors in five dimensions

In this section, we summarize our notations and conventions for spinors in five dimensions. We mostly follow our conventions of \cite{52}. The Clifford algebra in 5 space-time dimensions is

\[ \{\gamma_a, \gamma_b\} = 2\eta_{ab} \] (A8)

where the metric signature that is mostly plus. The Dirac matrices in five dimensions are

\[
\begin{align*}
\gamma^0 &= -i\sigma_2 \otimes \sigma_3 \\
\gamma^1 &= -\sigma_1 \otimes \sigma_3 \\
\gamma^2 &= I_2 \otimes \sigma_1 \\
\gamma^3 &= I_2 \otimes \sigma_2 \\
\gamma^4 &= -i\gamma^0\gamma^1\gamma^2\gamma^3 = \sigma_3 \otimes \sigma_3
\end{align*}
\] (A9)
where \( \sigma_i, i = 1, 2, 3 \) are the usual Pauli matrices and \( I_2 \) is the two dimensional unit matrix. The charge conjugation matrix \( C \) has the property \( C^t = -C = C^{-1} \) and,

\[
C \gamma^a C^{-1} = (\gamma^a)^t
\]

(A10)

where \( C = B \gamma^0 \), with \( B = \gamma^3 \) such that \( B^* B = -1 \). The spinors in the theory carry an \( SU(2) \) index which is raised and lowered using \( \epsilon_{ij} \)

\[
X^j = \epsilon^{ji} X_i, \quad X_j = X^i \epsilon_{ij}
\]

(A11)

with \( \epsilon_{12} = \epsilon^{12} = 1 \).

Spinors in \( d = 5 \) satisfy a symplectic majorana condition. To apply this condition one needs \( B^* B = -1 \), even number of Dirac spinors \( \psi_i, i = 1, \ldots, 2n \) and an antisymmetric real matrix \( \Omega_{ij} \) with \( \Omega^2 = -1_{2n} \). The symplectic majorana condition on a generic spinor reads as

\[
\psi^*_i = \Omega_{ij} B \psi_j
\]

(A12)

or equivalently \([52]\) as

\[
\bar{\psi}^i \equiv (\psi^*_i)^t \gamma^0 = (\psi^i)^t C .
\]

(A13)

For \( \mathcal{N} = 2 \) supersymmetry \( i = 1, 2 \), and using \( \Omega_{ij} = \epsilon_{ij} \) \([12]\) reads as

\[
\psi_1^* = \gamma^3 \psi_2 .
\]

(A14)

Note that this condition does not reduce the degrees of freedom as compared to a single unconstrained Dirac spinor. This is because one needs at least a pair of Dirac spinors to apply the symplectic majorana condition \([12]\). However, it does make the R-symmetry manifest.

Antisymmetrization of indices in the Dirac matrices is done with the following convention

\[
\gamma_{a_1 a_2 \ldots a_n} = \gamma_{[a_1 a_2 \ldots a_n]} = \frac{1}{n!} \sum_{\sigma \in P_n} \text{Sign}(\sigma) \gamma_{a_{\sigma(1)} a_{\sigma(2)} \ldots a_{\sigma(n)}} .
\]

(A15)

In \( d = 5 \) only \( I, \gamma_a, \gamma_{ab} \) form an independent set, other matrices are related by the general identity
for $d = 2k + 3$

$$\gamma^\mu_1\mu_2...\mu_s = \frac{-i^{k+s(s-1)}}{(d-s)!}\epsilon^\mu_1\mu_2...\mu_s\gamma_{\mu_{s+1}...\mu_d}.$$  \hspace{1cm} (A16)

We also list some useful identities involving various Dirac matrices

\begin{align*}
[\gamma_a, \gamma_b] &= 2\gamma_{ab} \\
[\gamma_h, \gamma_{abc}] &= 2\gamma_{habc} \\
[\gamma_{abc}, \gamma_{eih}] &= \eta_{ef}\eta_{gp}\eta_{hk}(2\gamma_{abc} f^p k - 36\delta^{[f}_{[ab}\gamma_{c]}). \hspace{1cm} (A17)
\end{align*}

**APPENDIX B: BIANCHI SOLUTIONS IN 4D GAUGED SUPERGRAVITY**

In this section, we list the field equations of the Bianchi I (AdS$_2 \times \mathbb{R}^2$) and Bianchi II (AdS$_2 \times$ EAdS$_2$) solutions in $\mathcal{N} = 2, d = 4$ gauged supergravity. We are interested in an attractor type solution where the scalars $(z, q)$ are constants independent of spacetime coordinates and only the hypermultiplets are charged under abelian gauging. The field equations can be derived from an effective Lagrangian

$$\mathcal{L}_{eff} = -\frac{1}{2}R + \text{Im}N_{\Lambda \Sigma} F^\Lambda_{\mu \nu} F^\Sigma_{\mu \nu} - V(z, \bar{z}, q) + g_{XY} K_X^Y K^X_{\Lambda} A^{A}_\Lambda A^{\mu \Sigma}.$$  \hspace{1cm} (B1)

1. **Bianchi I: AdS$_2 \times \mathbb{R}^2$**

We write the AdS$_2 \times \mathbb{R}^2$ in a convenient coordinate system as

$$ds^2 = \frac{R_0^2}{\sigma^2}(dt^2 - d\sigma^2) - R_0^2(dy^2 + d\rho^2).$$  \hspace{1cm} (B2)

This metric can be easily supported by an electric gauge field, we choose our gauge field ansatz to be

$$A^\Lambda = \frac{E^\Lambda}{\sigma} dt.$$  \hspace{1cm} (B3)

The gauge field equations are

$$g_{XY} K_X^Y K^X_{\Lambda} E^A = 0.$$  \hspace{1cm} (B4)
There are $n_v + 1$ equations for the $n_v + 1$ variables $E_\Lambda$. At the attractor point the scalars are constants, as a result all the spacetime derivatives drop and the scalar field equations reduce to the extremization of an effective potential (attractor potential)

$$\frac{\partial}{\partial q^X} V_{eff} = 0, \quad \frac{\partial}{\partial z^i} V_{eff} = 0$$

$$V_{eff} = \mathcal{V}(z, \bar{z}, q) - g_{XY} K_\Lambda^X K_\Sigma^Y \frac{E_\Lambda E_\Sigma}{R_0^2} + \text{Im} N_{\Lambda\Sigma} \frac{E_\Lambda E_\Sigma}{2 R_0^4}.$$  (B5)

There are $n_v$ scalar equations for $z^i$ and $4n_H$ hyperscalar equations for $q^X$. The Einstein equations are

$$0 = R_0^2 V_{eff} + 2 g_{XY} K_\Lambda^X K_\Sigma^Y \frac{E_\Lambda E_\Sigma}{R_0^2} - \text{Im} N_{\Lambda\Sigma} \frac{E_\Lambda E_\Sigma}{R_0^2}$$

$$0 = -R_0^2 V_{eff} + \text{Im} N_{\Lambda\Sigma} \frac{E_\Lambda E_\Sigma}{R_0^2}$$

$$-\frac{1}{R_0^2} = V_{eff}.$$  (B6)

where $V_{eff}$ is defined in (B5). The above equations can be recast as the following conditions

$$\mathcal{V}(z, \bar{z}, q) = -\frac{1}{2R_0^2}$$

$$\frac{\text{Im} N_{\Lambda\Sigma} E_\Lambda E_\Sigma}{R_0^2} = -1$$

$$g_{XY} K_\Lambda^X K_\Sigma^Y E_\Lambda E_\Sigma = 0$$  (B7)

to be satisfied for a given specific model.

2. Bianchi II: $\text{AdS}_2 \times \text{EAdS}_2$ solution

As we have discussed before the Bianchi II symmetries are the isometries of a hyperbolic space $\mathbb{H}^2$ that is nothing but Euclidean AdS$_2$. In a suitable coordinate system a Bianchi II metric takes the following form

$$ds^2 = \frac{R_1^2}{\sigma^2}(dt^2 - d\sigma^2) - \frac{R_2^2}{\rho^2}(dy^2 + d\rho^2).$$  (B8)

Similar to the $\text{AdS}_2 \times \mathbb{R}_2$ solution discussed in the earlier section, this metric can also be supported by electric gauge fields. The gauge field ansatz is identical to the earlier case. The scalar field
equations and gauge field equations are same as (B5) and (B4) respectively. The Einstein equations take the form

\[
-\frac{R_2^2}{R_1^2} = R_1^2 V_{eff} + 2 g_{XY} K^X_A K^Y_\Sigma E^A E^\Sigma - \text{Im} N_{\Lambda \Sigma} \frac{E^A E^\Sigma}{R_1^2} \\
\frac{R_1^2}{R_2^2} = - R_1^2 V_{eff} + \text{Im} N_{\Lambda \Sigma} \frac{E^A E^\Sigma}{R_1^2} \\
V_{eff} = - \frac{1}{R_1^2}.
\] (B9)

The above equations can be recast in the form

\[
V(z, \bar{z}, q) = - \frac{1}{2} \left( \frac{1}{R_1^2} + \frac{1}{R_2^2} \right) \\
\text{Im} N_{\Lambda \Sigma} E^A E^\Sigma = \frac{R_1^2}{R_2^2} (R_1^2 - R_2^2) \\
g_{XY} K^X_A K^Y_\Sigma E^A E^\Sigma = 0.
\] (B10)

**APPENDIX C: KILLING SPINOR EQUATION FOR THE MASSIVE CASES**

In §IV A we demonstrated that the gaugino and hyperino conditions break all supersymmetry for Bianchi type solutions when \( g \neq 0 \) cases. In this section, we show that the massive solutions studied earlier do not solve the Killing spinor equation for a radial ansatz. The solutions studied in this section are sourced by time like gauge fields and a cosmological constant. These have been constructed earlier in [13, 35]. In this section, we show that (independent of the conclusions from §IV A) a radial Killing spinor (C1) breaks all supersymmetry conditions. For this section, we just assume the radial spinor ansatz

\[
\epsilon_i = f(r)\zeta_i
\] (C1)

where \( \zeta_i \) are constant symplectic majorana spinors. The Killing spinor equation has the form

\[
D_\mu \epsilon_i + \frac{i}{4\sqrt{6}} h_I E^{\mu \rho l} (\gamma_{\mu \nu \rho} - 4 g_{\mu \nu} \gamma_\rho) \epsilon_i + \frac{i}{\sqrt{6}} g R \gamma_\mu \epsilon_k^k \epsilon_k = 0
\] (C2)

where

\[
D_\mu \epsilon_i \equiv \partial_\mu \epsilon_i + \frac{1}{4} \omega^{ab}_\mu \gamma_{ab} \epsilon_i + g R A^I_\mu V_I \epsilon_i^k \epsilon_k.
\] (C3)
1. Bianchi I

We start with a simple Bianchi I type solution (see Appendix C of [13]) sourced by a magnetic
gauge field. The metric is written in terms of the one forms

\[ e^0 = L e^{\beta r} dt , \ e^1 = L e^{\beta_1 r} \omega_1 , \ e^2 = L e^{\beta r} \omega^2 , \ e^3 = L e^{\beta r} \omega^3 , \ e^4 = L dr \]  \hspace{1cm} (C4)

where \( \beta \geq 0, \beta_1 \geq 0 \) are the Lifshitz exponents. The invariant one forms

\[ \omega^i = dx^i , i = 1, 2, 3 \]  \hspace{1cm} (C5)

all commute with one another and satisfy \( d \omega^i = 0 \). We choose the gauge field to lie along the \( x^1 \) direction

\[ A^I = B^I e^1 \]  \hspace{1cm} (C6)

where \( B^I \) are constants. The Killing spinor equations in the above background have the form

\[
e^{-\beta r} \gamma_0 \partial_t \epsilon_i - \frac{\beta}{2} \gamma_4 \epsilon_i + \frac{i}{2 \sqrt{6}} \left( \frac{\beta_1}{2} B^I h_I \gamma_{14} \epsilon_i - L g_R \epsilon_i^k \epsilon_k \right) = 0
\]

\[
e^{-\beta_1 r} \gamma_1 \partial_{x^1} \epsilon_i + \frac{\beta_1}{2} \gamma_4 \epsilon_i + g_R L B^I \gamma_{1} V_I \epsilon_i^k \epsilon_k + \frac{i}{2 \sqrt{6}} \left( \frac{\beta_1}{2} B^I h_I \gamma_{14} \epsilon_i + L g_R \epsilon_i^k \epsilon_k \right) = 0
\]

\[
e^{-\beta r} \gamma_2 \partial_{x^2} \epsilon_i + \frac{\beta}{2} \gamma_4 \epsilon_i + \frac{i}{2 \sqrt{6}} \left( -\frac{\beta_1}{2} B^I h_I \gamma_{14} \epsilon_i + L g_R \epsilon_i^k \epsilon_k \right) = 0
\]

\[
e^{-\beta r} \gamma_3 \partial_{x^3} \epsilon_i + \frac{\beta}{2} \gamma_4 \epsilon_i + \frac{i}{2 \sqrt{6}} \left( -\frac{\beta_1}{2} B^I h_I \gamma_{14} \epsilon_i + L g_R \epsilon_i^k \epsilon_k \right) = 0
\]

Using the Killing spinor ansatz (C1) we get the following three independent equations

\[
-\frac{1}{2} \beta_1 \gamma_4 \epsilon_i + \frac{i}{2 \sqrt{6}} \beta_1 B^J h_J \gamma_{14} \epsilon_i - \frac{i}{2 \sqrt{6}} L g_R \epsilon_i^k \epsilon_k = 0
\]

\[
\frac{1}{2} \beta_1 \gamma_4 \epsilon_i + g_R L B^J \gamma_{1} V_J \epsilon_i^k \epsilon_k + \frac{i}{2 \sqrt{6}} \beta_1 B^J h_J \gamma_{14} \epsilon_i + \frac{i}{2 \sqrt{6}} L g_R \epsilon_i^k \epsilon_k = 0
\]

\[
\gamma_4 \partial_t \epsilon_i + \frac{i}{\sqrt{6}} \beta_1 B^J h_J \gamma_{14} \epsilon_i + \frac{i}{\sqrt{6}} L g_R \epsilon_i^k \epsilon_k = 0 . \hspace{1cm} (C8)
\]
Taking the difference of the last two equations from the above set we get

$$\partial_r \epsilon_i - \frac{1}{2} \beta_1 \epsilon_i = 0 \implies \epsilon_i = e^{\frac{2}{2r} \zeta_i}$$ \hspace{1cm} (C9)$$

where we have imposed the condition

$$B^I V_I = 0 \hspace{1cm} (C10)$$

Recollect that this same condition was used earlier for the 4d supersymmetric Lifshitz solution \([48]\) and for the $AdS_2 \times \mathbb{R}^2$ solution in \([III\)B\). Substituting (C9) and (C10) in (C8) we get the projection conditions

$$\gamma_{14} \zeta_i = X \epsilon_i \zeta_k \hspace{1cm} (C11)$$
$$\gamma_{4} \zeta_i = Y \epsilon_i \zeta_k \hspace{1cm} (C12)$$

where

$$X = \frac{2Lg_R (\beta - \beta_1)}{(\beta_1 + 2 \beta) \beta_1 (B^I h_I)}, \hspace{0.5cm} Y = \frac{\sqrt{6} Lg_R}{\beta_1 + 2 \beta}.$$ \hspace{1cm} (C13)$$

Consistency of (C11) and (C12) as projectors gives the conditions

$$(1 - X^2) \zeta_i = 0 \hspace{1cm} (C14)$$
$$(1 - Y^2) \zeta_i = 0 \hspace{1cm} (C15)$$

Note that there is no issue with either of the above projectors by themselves as both the conditions (C14) and (C15) can be individually met.

However mutual consistency of the projectors (C11) and (C12) together gives

$$\gamma_1 \zeta_i = -iXY \zeta_i \hspace{1cm} (C16)$$

that breaks supersymmetry. More explicitly squaring the projector we see that

$$(1 + X^2 Y^2) \zeta_i = 0 \hspace{1cm} (C17)$$

cannot be met. Hence it follows that the only solution to (C17) is that all the $\zeta_i$ vanish. This
implies that the projectors (C11) and (C12) together break all of the supersymmetry.

2. Bianchi II

The Bianchi II metrics are constructed out of the one forms

\[ e^0 = Le^{\beta_0} dt , \quad e^1 = Le^{(\beta_2 + \beta_3)} r \omega^1 , \quad e^2 = Le^{\beta_2} r \omega^2 , \quad e^3 = Le^{\beta_3} r \omega^3 , \quad e^4 = Ldr \]  

where the invariant one forms \( \omega^i \) are given by

\[ \omega^1 = dx^2 - x^1 dx^3 , \quad \omega^2 = dx^3 , \quad \omega^3 = dx^1 , \]  

and the exponents \( \beta_i \) are all positive. We choose the gauge field along the time direction

\[ A^I = E^I e^0 . \]  

The Killing spinor equations in this background with the spinor ansatz (C1) are given by

\[
\begin{align*}
e^{-\beta_0} \gamma_0 \partial_t \epsilon_i - \frac{1}{2} \beta_i \gamma_4 \epsilon_i + Lg_R E^I \gamma_0 V_I \epsilon_i \epsilon_k + \frac{i}{\sqrt{6}} \left( h_I E^I \beta_t \gamma_0 \epsilon_i - Lg_R \epsilon_i \epsilon_k \right) &= 0 \\
e^{-(\beta_2 + \beta_3)} r \gamma_1 \partial_x^2 \epsilon_i + \frac{1}{4} (2(\beta_2 + \beta_3) \gamma_4 - \gamma_{123}) \epsilon_i + \frac{i}{\sqrt{6}} \left( \frac{1}{2} h_I E^I \beta_t \gamma_0 \epsilon_i + Lg_R \epsilon_i \epsilon_k \right) &= 0 \\
e^{-\beta_2} \gamma_2 (x^3 + x^1 \partial_x^2) \epsilon_i + \frac{1}{4} (\gamma_{123} + 2\beta_2 \gamma_4) \epsilon_i + \frac{i}{\sqrt{6}} \left( \frac{1}{2} h_I E^I \beta_t \gamma_0 \epsilon_i + Lg_R \epsilon_i \epsilon_k \right) &= 0 \\
e^{-\beta_3} \gamma_3 \partial_x^{11} \epsilon_i + \frac{1}{4} (\gamma_{123} + 2\beta_3 \gamma_4) \epsilon_i + \frac{i}{\sqrt{6}} \left( \frac{1}{2} h_I E^I \beta_t \gamma_0 \epsilon_i + Lg_R \epsilon_i \epsilon_k \right) &= 0 \\
\gamma_4 \partial_t \epsilon_i - \frac{i}{\sqrt{6}} \left( h_I E^I \beta_t \gamma_0 \epsilon_i - Lg_R \epsilon_i \epsilon_k \right) &= 0 .
\end{align*}
\]  

(C21)
\(-\frac{1}{2}\beta_4\gamma_4\epsilon_i + g_R E^I \gamma_0 V_4 \epsilon_i^k \epsilon_k + \frac{i}{\sqrt{6}} \left( h_I E^I \beta_4 \gamma_0 \gamma_4 \epsilon_i - L g_R \epsilon_i^k \epsilon_k \right) = 0 \)  \(\text{(C22)}\)

\(\frac{1}{4} \left( 2(\beta_2 + \beta_3) \gamma_4 - \gamma_123 \right) \epsilon_i + \frac{i}{\sqrt{6}} \left( \frac{1}{2} h_I E^I \beta_4 \gamma_0 \gamma_4 \epsilon_i + L g_R \epsilon_i^k \epsilon_k \right) = 0 \)  \(\text{(C23)}\)

\(\frac{1}{4} \left( \gamma_123 + 2\beta_2 \gamma_4 \right) \epsilon_i + \frac{i}{\sqrt{6}} \left( \frac{1}{2} h_I E^I \beta_4 \gamma_0 \gamma_4 \epsilon_i + L g_R \epsilon_i^k \epsilon_k \right) = 0 \)  \(\text{(C24)}\)

\(\frac{1}{4} \left( \gamma_123 + 2\beta_3 \gamma_4 \right) \epsilon_i + \frac{i}{\sqrt{6}} \left( \frac{1}{2} h_I E^I \beta_4 \gamma_0 \gamma_4 \epsilon_i + L g_R \epsilon_i^k \epsilon_k \right) = 0 \)  \(\text{(C25)}\)

\(\gamma_4 \partial_r \epsilon_i - \frac{i}{\sqrt{6}} \left( h_I E^I \beta_4 \gamma_0 \gamma_4 \epsilon_i - L g_R \epsilon_i^k \epsilon_k \right) = 0 \)  \(\text{(C26)}\)

From (C23), (C24) and (C25) we immediately see that

\(\beta_2 = \beta_3 = \beta\)

\((\gamma_1234 + \beta)\zeta_i = 0\)  \(\text{(C27)}\)

It follows that \(\beta = +1\). The equation satisfied by the radial Killing spinor can be determined from (C26) and (C22)

\(\partial_r \epsilon_i - \frac{1}{2} \beta_4 \epsilon_i = 0 \implies \epsilon_i = e^{\frac{\beta_4}{2} \gamma} \zeta_i\)  \(\text{(C28)}\)

where we have imposed the condition \(E^I P^I_f = 0\) as before. The remaining conditions can be casted in the form of the projection conditions

\(\gamma_4 \zeta_i = -i Y \epsilon_i^k \zeta_k\)  \(\text{(C29)}\)

\(\gamma_0 \zeta_i = -i X \zeta_i\)  \(\text{(C30)}\)

where

\(Y = \frac{\sqrt{6} L g_R}{3 + \beta_t}\), \(X = \frac{\sqrt{6} \beta_t h_I E^I}{(3 - 2\beta_t)}\).  \(\text{(C31)}\)

The projectors (C29) and (C30) imply the following conditions respectively

\((1 - X^2)\zeta_i = 0 \implies X = \pm 1\)  \(\text{(C32)}\)

\((1 - Y^2)\zeta_i = 0 \implies Y = \pm 1\).  \(\text{(C33)}\)
However mutual consistency of the two projectors gives

\[ \gamma_{04}\zeta_i = -XY\epsilon_i^k\epsilon_k \implies (1 + X^2Y^2)\zeta_i = 0 \quad (C34) \]

that cannot be satisfied due to all the quantities \( X, Y \) being real. Hence the only possible solution is \( \zeta_i = 0 \) that breaks all supersymmetry.

3. Bianchi III, V, VI\(_h\)

The Bianchi III (\( h = 0 \)), V (\( h = 1 \)) and VI (\( h \neq 0, 1 \)) metrics are constructed out of the forms

\[ e^0 = Le^{\beta r} dt, \quad e^1 = Le^{\beta_1 r} \omega^1, \quad e^2 = Le^{\beta_2 r} \omega^2, \quad e^3 = L\omega^3, \quad e^4 = Ldr \quad (C35) \]

where the invariant one forms are given by

\[
\omega^1 = e^{-x^1} dx^2, \omega^2 = e^{-hx^1} dx^3, \omega^3 = dx^1. \quad (C36)
\]

In general the metrics of the Bianchi III, V, VI types are sourced by a time like gauge field \[13, 46\]

\[ A^I = E^I e^0. \quad (C37) \]

The few special examples that are highly symmetric We carry out the Killing spinor analysis for a general \( h \) and only set it to the appropriate values when needed.

\[
e^{-\beta r} \gamma_0 \partial_t \epsilon_i - \frac{\beta_1}{2} \gamma_4 \epsilon_i + g_R LE^I \gamma_0 V_I \epsilon_i^k \epsilon_k + \frac{i}{\sqrt{6}} \left( \beta_1 E^I h_I \gamma_{04} \epsilon_i - Lg_R \epsilon_i^k \epsilon_k \right) = 0
\]

\[
e^{-\beta_1 r + x^1} \gamma_1 \partial_{x^1} \epsilon_i - \frac{1}{2} \gamma_3 \epsilon_i + \frac{\beta_1}{2} \gamma_4 \epsilon_i + \frac{i}{\sqrt{6}} \left( \frac{1}{2} E^I h_I \gamma_{04} \epsilon_i + Lg_R \epsilon_i^k \epsilon_k \right) = 0
\]

\[
e^{hx^1 - \beta_2 r} \gamma_2 \partial_{x^3} \epsilon_i - \frac{h}{2} \gamma_3 \epsilon_i + \frac{\beta_2}{2} \gamma_4 \epsilon_i + \frac{i}{\sqrt{6}} \left( \frac{1}{2} E^I h_I \gamma_{04} \epsilon_i + Lg_R \epsilon_i^k \epsilon_k \right) = 0
\]

\[
\gamma_3 \partial_{x^1} \epsilon_i + \frac{i}{\sqrt{6}} \left( \frac{1}{2} E^I h_I \gamma_{04} \epsilon_i + Lg_R \epsilon_i^k \epsilon_k \right) = 0
\]

\[
\gamma_4 \partial_r \epsilon_i - \frac{i}{\sqrt{6}} \left( \beta_1 E^I h_I \gamma_{04} \epsilon_i - Lg_R \epsilon_i^k \epsilon_k \right) = 0 \quad (C38)
\]
Using the ansatz (C1) the Killing spinor equations take the form

\[ - \frac{\beta_t}{2} \gamma_4 \epsilon_i + g_R E^I V_I \gamma_0 \epsilon_i^k \epsilon_k + \frac{i}{\sqrt{6}} \left( \beta_t E^I h_I \gamma_{04} \epsilon_i - L g_R \epsilon_i^k \epsilon_k \right) = 0 \]  
(C39)

\[ - \frac{1}{2} \gamma_3 \epsilon_i + \frac{\beta_1}{2} \gamma_4 \epsilon_i + \frac{i}{\sqrt{6}} \left( \frac{\beta_t}{2} E^I h_I \gamma_{04} \epsilon_i + L g_R \epsilon_i^k \epsilon_k \right) = 0 \]  
(C40)

\[ - \frac{1}{2} h \gamma_3 \epsilon_i + \frac{\beta_2}{2} \gamma_4 \epsilon_i + \frac{i}{\sqrt{6}} \left( \frac{\beta_t}{2} E^I h_I \gamma_{04} \epsilon_i + L g_R \epsilon_i^k \epsilon_k \right) = 0 \]  
(C41)

\[ \frac{i}{\sqrt{6}} \left( \frac{\beta_t}{2} E^I h_I \gamma_{04} \epsilon_i + L g_R \epsilon_i^k \epsilon_k \right) = 0 \]  
(C42)

\[ \gamma_4 \partial_r \epsilon_i - \frac{i}{\sqrt{6}} \left( \beta_t E^I h_I \gamma_{04} \epsilon_i - L g_R \epsilon_i^k \epsilon_k \right) = 0 . \]  
(C43)

From (C43) and (C39) after imposing the condition \( E^I V_I = 0 \) as before, we get the radial equation

\[ \partial_r \epsilon_i - \frac{1}{2} \beta_t \epsilon_i = 0 \implies \epsilon_i = e^{\frac{\beta_t}{2} r} \zeta_i . \]  
(C44)

The remaining equations can be simplified to the following conditions

\[ \gamma_3 \zeta_i = \beta_1 \gamma_4 \zeta_i \]  
(C45)

\[ h \gamma_3 \zeta_i = \beta_2 \gamma_4 \zeta_i \]  
(C46)

\[ \beta_t E^I h_I \gamma_{04} \zeta_i = 2 L g_R \epsilon_i^k \zeta_k \]  
(C47)

\[ \beta_t \gamma_4 \zeta_i = i \sqrt{6} L g_R \epsilon_i^k \zeta_k . \]  
(C48)

We already see that the condition (C45) breaks all of the supersymmetry for Bianchi III, V and \( VI_h \) cases since

\[ \gamma_{34} \zeta_i = - \beta_1 \zeta_i \implies (1 + \beta_1^2) \zeta_i = 0 \]  
(C49)

cannot be satisfied as \( \beta_1 \) has to be real. For Bianchi V \( (h = 0) \), it is possible to avoid the equation (C46) by choosing \( \beta_2 = 0 \), however the rest of the conditions obviously break supersymmetry as
can be seen below. The remaining conditions from (C46)-(C48) are

\[
\left(1 + \frac{h^2}{\beta_2^2}\right) \zeta_i = 0 \quad (C50)
\]

\[
\left(1 + \left(\frac{2Lg_R}{\beta_i E^I h_I}\right)^2\right) \zeta_i = 0 \quad (C51)
\]

\[
\left(1 - \frac{6L^2g_R^2}{\beta_i^2}\right) \zeta_i = 0 . \quad (C52)
\]

The last condition (C52) in principle can be satisfied for all cases. However the first two conditions (C50), (C51) lead to the solution \( \zeta_i = 0 \) and break supersymmetry explicitly for all of Bianchi III, V and VI\(_h\) cases.

4. Bianchi IX

Our last example is Bianchi IX, the metric is written in terms of the one forms

\[
e^0 = Le^{\beta x} dt , \ e^1 = L\omega^1 , \ e^2 = L\omega^2 , \ e^3 = L\lambda\omega^3 , \ e^4 = Ldr \quad (C53)
\]

where \( \lambda \) is the squashing parameter as in the Bianchi VII case. The one forms invariant under the Bianchi IX symmetry are given by

\[
\omega^1 = -\sin x^3 dx^1 + \sin x^1 \cos x^3 dx^2 \\
\omega^2 = \cos x^3 dx^1 + \sin x^1 \sin x^3 dx^2 \\
\omega^3 = \cos x^1 dx^2 + dx^3 . \quad (C54)
\]

Following [26] we choose the gauge field ansatz to be

\[
A_1^I = E^I e^0 , \ A_2^I = B^I e^3 \quad (C55)
\]
where $E^I, B^J$ are constants and $I + J = n_V + 1$.

$$e^{-\beta r} \gamma_0 \partial_t \epsilon_i - \frac{\beta_t}{2} \gamma_4 \epsilon_i + L g_R E^I V_I \gamma_0 \epsilon_i^k \epsilon_k + \frac{i}{\sqrt{6}} \left( h_I E^J \beta_t \gamma_0^4 - \frac{\lambda h_J B^J}{2} \gamma_1^2 \right) \epsilon_i - L g_R \epsilon_i^k \epsilon_k = 0$$

$$\gamma_1 \left( - \sin x^3 \partial_x^1 + \frac{\cos x^3}{\sin x^1} \partial_x^2 - \frac{\cos x^3}{\tan x^1} \partial_x^3 \right) \epsilon_i + \frac{\lambda}{4} \gamma_1 \gamma_3 \epsilon_i^k \epsilon_k + \frac{i}{\sqrt{6}} \left( h_I E^J \beta_t \gamma_0^4 - \frac{\lambda h_J B^J}{2} \gamma_1^2 \right) \epsilon_i + L g_R \epsilon_i^k \epsilon_k = 0$$

$$\gamma_2 \left( \cos x^3 \partial_x^1 + \sin x^3 \partial_x^2 - \sin x^3 \partial_x^3 \right) \epsilon_i + \frac{\lambda}{4} \gamma_1 \gamma_3 \epsilon_i^k \epsilon_k + \frac{i}{\sqrt{6}} \left( h_I E^J \beta_t \gamma_0^4 - \frac{h_J B^J}{2} \gamma_1^2 \right) \epsilon_i + L g_R \epsilon_i^k \epsilon_k = 0$$

$$\gamma_3 \partial_{x^3} \epsilon_i + \frac{(2 - \lambda^2)}{4 \lambda} \gamma_1 \gamma_3 \epsilon_i + L g_R B^I V_I \gamma_3 \epsilon_i^k \epsilon_k + \frac{i}{\sqrt{6}} \left( h_I E^J \beta_t \gamma_0^4 + \frac{h_J B^J}{2} \gamma_1^2 \right) \epsilon_i + L g_R \epsilon_i^k \epsilon_k = 0$$

Using the ansatz (C1), the Killing spinor equations reduce to

$$\gamma_1 \left( - \sin x^3 \partial_x^1 + \frac{\cos x^3}{\sin x^1} \partial_x^2 - \frac{\cos x^3}{\tan x^1} \partial_x^3 \right) \epsilon_i + \frac{\lambda}{4} \gamma_1 \gamma_3 \epsilon_i^k \epsilon_k + \frac{i}{\sqrt{6}} \left( h_I E^J \beta_t \gamma_0^4 - \frac{h_J B^J}{2} \gamma_1^2 \right) \epsilon_i + L g_R \epsilon_i^k \epsilon_k = 0$$

$$\gamma_2 \left( \cos x^3 \partial_x^1 + \sin x^3 \partial_x^2 - \sin x^3 \partial_x^3 \right) \epsilon_i + \frac{\lambda}{4} \gamma_1 \gamma_3 \epsilon_i^k \epsilon_k + \frac{i}{\sqrt{6}} \left( h_I E^J \beta_t \gamma_0^4 - \frac{h_J B^J}{2} \gamma_1^2 \right) \epsilon_i + L g_R \epsilon_i^k \epsilon_k = 0$$

$$\gamma_3 \partial_{x^3} \epsilon_i + \frac{(2 - \lambda^2)}{4 \lambda} \gamma_1 \gamma_3 \epsilon_i + \frac{i}{\sqrt{6}} \left( h_I E^J \beta_t \gamma_0^4 + \frac{h_J B^J}{2} \gamma_1^2 \right) \epsilon_i + L g_R \epsilon_i^k \epsilon_k = 0$$

where we have set $E^I V_I = 0$ and $B^J V_J = 0$. The radial equation can be obtained by adding (C60) and (C57)

$$\partial_r \epsilon_i - \frac{\beta_t}{2} \epsilon_i = 0 \implies \epsilon_i = e^{\frac{\beta_t}{2} r} \zeta_i .$$

Subtracting (C59) and (C58) we get the equation

$$\frac{1 - \lambda^2}{\lambda} \gamma_3 \zeta_i + \frac{3i}{\sqrt{6}} \lambda h_J B^J \zeta_i = 0 .$$

This gives a projector of the type $\gamma_3 \zeta_i = i X \zeta_i$ that breaks all supersymmetry. We can avoid this condition by choosing $\lambda = 1$ (no squashing) and setting the corresponding source field $B^J = 0$.
The remaining independent equations are

\[-\frac{\beta_t}{2} \gamma_4 \zeta_i + \frac{i}{\sqrt{6}} \left( h_I E^I \beta_t \gamma_{04} \zeta_i - L g R e_i^k \zeta_k \right) = 0 \tag{C63} \]

\[\frac{1}{4} \gamma_{123} \zeta_i + \frac{i}{\sqrt{6}} \left( \frac{h_I E^I}{2} \beta_t \gamma_{04} \zeta_i + L g R e_i^k \zeta_k \right) = 0 \tag{C64} . \]

Adding the above equations and using \(\gamma_{123} = i \gamma_{04}\) we get the projection

\[\gamma_0 \zeta_i = i X \zeta_i , \quad X = \frac{1 + \sqrt{6} h_I E^I \beta_t}{2 \beta_t} . \tag{C65} \]

Squaring the above projector it follows that \(X = \pm 1\). We can solve for \(\beta_t\) to get

\[\beta_t = \pm \frac{1}{2 \pm \sqrt{6} E^I h_I} . \tag{C66} \]

Substituting (C65) in (C63) and (C64) we obtain

\[\gamma_4 \zeta_i = i Y e_i^k \zeta_k , \quad Y = \frac{\sqrt{6} L g R}{\beta_t \pm 1} . \tag{C67} \]

Squaring the above projector we get the condition

\[(1 - Y^2) \zeta_i = 0 \tag{C68} \]

that constrains the parameters in \(Y\). However as we have already seen in the previous cases we find that the projectors (C65) and (C67) together break all supersymmetry. Acting with (C65) on (C67) we get

\[\gamma_{04} \zeta_i = - X Y e_i^k \zeta_k \Rightarrow (1 + X^2 Y^2) \zeta_i = 0 \tag{C69} \]

that forces \(\zeta_i = 0\).
In this section, we list the Bianchi type solutions sourced by a massless gauge field and a cosmological constant. The action of the system we consider is

\[ S = \int d^5x \sqrt{-g} \left\{ R + \Lambda - \frac{1}{4} F^2 \right\} \]  

We note that in our conventions \( \Lambda > 0 \) corresponds to \( AdS \) space. The Einstein equations read as

\[ R_{\mu \nu} - \frac{1}{2} \delta_{\mu \nu} R = T_{\mu \nu}. \]  

with

\[ T_{\mu \nu} = \frac{1}{2} F^\mu_\lambda F^\lambda_\nu + \frac{1}{2} \delta_{\mu \nu} \left( \Lambda - \frac{1}{4} F^\rho_\sigma F_{\rho \sigma} \right) \]

We list the details of the various solutions below. All of the below solutions can also be constructed in the \( U(1)_R \) gauged supergravity model considered in [47].

1. **Bianchi I**: \( AdS_3 \times \mathbb{R}^2 \)

The metric is

\[ ds^2 = -e^{2r} dt^2 + dr^2 + e^{2r} dx^2 + cdy^2 + cdz^2. \]

where \( c \) is an undetermined constant. We choose the magnetic field along the \( \mathbb{R}^2 \) direction

\[ F_{yz} = B, \]

with \( B \) being a constant. It is easy to see that there are two independent equations

\[ B^2 - 2c^2 (-2 + \Lambda) = 0 \]  

\[ \frac{B^2}{2c} - \frac{1}{2} c (-6 + \Lambda) = 0 \]
that are solved by

\[ c = \frac{|B|}{2}, \quad \Lambda = 4. \]

### 2. Bianchi III $AdS_3 \times \mathbb{H}^2$ and Bianchi V $AdS_2 \times \mathbb{H}^3$

The $AdS_3 \times \mathbb{H}^2$ solutions have been constructed earlier with massless gauge fields \[47\]. Here we present a vacuum solution as well

\[ ds^2 = -e^{2r} dt^2 + dr^2 + e^{2r} dx^2 + dy^2 + e^{-2\lambda y} dz^2. \] (D8)

It is straightforward to check that the Einstein equations are solved by

\[ \lambda = \sqrt{2}, \quad \Lambda = 6. \]

Similarly, there is a vacuum $AdS_2 \times \mathbb{H}^3$ solution similar to the charged solution constructed in \[13\].

It is given by

\[ ds^2 = -e^{2\beta r} dt^2 + dr^2 + dx^2 + e^{-2x} dy^2 + e^{-2x} dz^2. \] (D9)

In this case too, the Einstein equations are solved by $\beta_t = \sqrt{2}, \Lambda = 6$.

### 3. Bianchi VII

The Bianchi VII metric is given by

\[ ds^2 = R^2[dr^2 - e^{2\beta r} dt^2 + (dx^1)^2 + e^{2\beta r} ((\omega^2)^2 + \lambda^2 (\omega^3)^2)]. \] (D10)

where the invariant one forms are defined as

\[ \omega^1 = dx^1; \quad \omega^2 = \cos(x^1)dx^2 + \sin(x^1)dx^3; \quad \omega^3 = -\sin(x^1)dx^2 + \cos(x^1)dx^3. \] (D11)

The gauge field configuration is

\[ A = e^{\beta r} \left( \sqrt{A_2} \omega^2 \right). \] (D12)
It follows that the gauge field equations of motion are

$$\lambda^2 (-2\beta (\beta + \beta_t)) + 2 = 0.$$  \hspace{1cm} (D13)

Note that the solution we seek has five parameters, $R, \beta_t, \beta, \lambda$, which enter in the metric and $\tilde{A}_2$, that determines the gauge field. These are all constants. The Einstein equations long the $tt$, $rr$, and $x^1 x^1$ directions read as

\[
\frac{2(1 + \tilde{A}_2)}{\lambda^2} + 2\lambda^2 + \tilde{A}_2 (2\beta^2) + 24\beta^2 - 4(1 + \Lambda) = 0 \hspace{1cm} (D14)
\]

\[
\frac{2(1 + \tilde{A}_2)}{\lambda^2} + 2\lambda^2 - \tilde{A}_2 (\beta^2) + 8\beta (\beta + 2\beta_t) - 4(1 + \Lambda) = 0 \hspace{1cm} (D15)
\]

\[
\frac{2(1 + \tilde{A}_2)}{\lambda^2} + 2\lambda^2 - \tilde{A}_2 (2\beta^2) - 8(3\beta^2 + 2\beta_t \beta + \beta_t^2) - 4(1 - \Lambda) = 0. \hspace{1cm} (D16)
\]

The components along the $x^2, x^3$ directions lead to

\[
\frac{2(3 + \tilde{A}_2)}{\lambda^2} - 2\lambda^2 - \tilde{A}_2 (2\beta^2) + 8(\beta^2 + \beta_t \beta + \beta_t^2) - 4(1 + \Lambda) = 0 \hspace{1cm} (D17)
\]

\[
\frac{2(1 + \tilde{A}_2)}{\lambda^2} - 6\lambda^2 - \tilde{A}_2 (2\beta^2) - 8(\beta^2 + \beta_t \beta + \beta_t^2) + 4(1 + \Lambda) = 0. \hspace{1cm} (D18)
\]

Counting eq. (D13) these are 6 equations in all. One can check that only 5 of these are independent. These 5 equations determine the 5 parameters $(\Lambda, \beta, \beta_t, \lambda, \tilde{A}_2)$, which then completely determines the solution. Solving for the 5 parameters gives:

\[
\lambda = 1.31964 \hspace{1cm} (D19)
\]

\[
\Lambda = 3.28755 \hspace{1cm} (D20)
\]

\[
\beta = 0.40893 \hspace{1cm} (D21)
\]

\[
\beta_t = 0.995299 \hspace{1cm} (D22)
\]

\[
\tilde{A}_2 = 5.73563 \hspace{1cm} (D23)
\]
APPENDIX E: BIANCHI ALGEBRAS FROM POINCARÉ ALGEBRA

In this section, we illustrate that the Bianchi I and VII algebras are easily embedded in the Poincaré algebra. We begin with the Conformal algebra in a $d$ dimensional spacetime

\[
\begin{align*}
[P_\mu, P_\nu] &= 0 \\
[L_\mu P_\rho, L_\mu P_\sigma] &= i (\eta_{\mu\rho} P_\sigma - \eta_{\mu\sigma} P_\rho) \\
[L_\mu L_{\mu\nu}, L_{\rho\sigma}] &= i (\eta_{\rho\mu} L_{\mu\nu} + \eta_{\mu\sigma} L_{\nu\rho} - \eta_{\mu\rho} L_{\nu\sigma} - \eta_{\nu\sigma} L_{\mu\rho}) \\
[D, P_\mu] &= i P_\mu \\
[D, K_\mu] &= -i K_\mu \\
[K_\mu, P_\nu] &= 2i (\eta_{\mu\nu} D - L_{\mu\nu}) \\
[K_\rho, L_{\mu\nu}] &= i (\eta_{\rho\mu} K_\nu - \eta_{\rho\nu} K_\mu).
\end{align*}
\] (E1)

The first three algebras form the Poincaré sub algebra of the conformal algebra.

Consider scaling the coordinates

\[(\lambda_1 x^1, \lambda_2 x^2, \lambda_3 x^3).\]

As a result the generators scale as

\[
\begin{align*}
P_i &\to \frac{1}{\lambda_i} P_i \\
L_{ij} &\to \frac{1}{\lambda_i \lambda_j} L_{ij} \\
D &\to D \\
\eta_{ij} &\to \frac{1}{\lambda_i \lambda_j} \eta_{ij} \\
K_i &\to \frac{1}{\lambda_i} K_i
\end{align*}
\] (E2)

The Bianchi I is generated by the usual translations

\[
\begin{align*}
\partial_1 &= P_1 \\
\partial_2 &= P_2 \\
\partial_3 &= P_3
\end{align*}
\] (E3)
and the Bianchi VII generators are the combination of translation and rotation generators

\[ \partial_2 = P_2 \]
\[ \partial_3 = P_3 \]
\[ \partial_1 - x_3 \partial_2 + x_2 \partial_3 = P_1 + L_{23} \]  \hspace{1cm} (E4)

Thus the Bianchi I and Bianchi VII algebras form a closed sub algebra of the Poincaré algebra. It follows that the algebras are also sub algebras of the super Poincaré algebra. The other Bianchi type algebras listed in \[13, 24, 25\] are not embedded in the Poincaré algebra in any obvious way.

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