1. INTRODUCTION

In this work, we consider Rademacher Fourier series
\[ f(\theta) = \sum_{k \in \mathbb{Z}} \xi_k a_k e^{2\pi i k \theta} \]
where \( \xi_k \) are independent Rademacher random variables, which take the values \( \pm 1 \) with probability \( \frac{1}{2} \) each, and random Taylor series
\[ F(z) = \sum_{k \in \mathbb{N}_+} \zeta_k z^k \]
with independent symmetric complex-valued random variables \( \zeta_k \). Recall that the complex-valued random variable \( \zeta \) is called symmetric if \( -\zeta \) has the same distribution as \( \zeta \). In the Fourier case, the sequence of deterministic complex coefficients \( \{a_k\} \) belongs to \( \ell^2(\mathbb{Z}) \); in the Taylor case, we assume that the radius of convergence is almost surely (a.s., for short) positive.

1.1. Some motivation. There are several long-standing questions pertaining to the distribution of values of random Taylor series. For these questions, the Rademacher case already presents main difficulties. Moreover, in many instances, due to Kahane’s “reduction principle” [4, Section 1.7], the case of more general random symmetric coefficients can be reduced to the Rademacher case. Here, we explain the central rôle played by the logarithmic integrability of the Rademacher Fourier series in our approach to some of these questions.

Put \( T = \mathbb{R}/\mathbb{Z} \), and denote by \( m \) the normalized Lebesgue measure on \( T \). Consider a random Taylor series
\[ F(z) = \sum_{k \geq 0} \xi_k a_k z^k \]
with independent identically distributed complex-valued random coefficients \( \xi_k \) normalized by \( \mathbb{E}|\xi|^2 = 1 \). Let \( R, 0 < R \leq \infty \) denote the radius of convergence of this
Taylor series. Note that
\[ E \{|F(z)|^2\} = \sum_{k \geq 0} |a_k|^2 r^{2k} \]
for all \( z \) with \( |z| = r \). We denote the RHS by \( \sigma_F^2(r) \). We will always assume that \( \sigma_F(r) \to \infty \) as \( r \to R \).

Suppose we are interested in the asymptotics as \( r \to R \) of the random counting function \( n_F(r) \), which counts the number of zeroes of \( F \) in the disk \( \{|z| \leq r\} \). To simplify the notation, assume that \( a_0 = 1 \). Denote by
\[ N_F(r) = \int_0^r \frac{n_F(t)}{t} \, dt \]
the integrated counting function. Then, by Jensen’s formula,
\[ N_F(r) = \int_T \log |F(rt)| \, dm(t) - \log |F(0)| = \log \sigma_F(r) + \int_T \log |\hat{F}_r(t)| \, dm(t) - \log |\xi_0| \]
where \( \hat{F}_r(t) = F(rt)/\sigma_F(r) \). Note that
\[ \hat{F}_r(\epsilon 2\pi i \theta) = \sum_{k \geq 0} \xi_k \hat{a}_k(r) e^{2\pi i k \theta} \]
is a random Fourier series normalized by the condition \( \sum_{k \geq 0} |\hat{a}_k(r)|^2 = 1 \).

First, assume that the \( \xi_k \)'s are standard complex Gaussian random variables. Then, for every \( t \in \mathbb{T} \), the random variable \( \hat{F}_r(t) \) is again standard complex-valued and Gaussian, and \( E|\log |\hat{F}_r(t)|| \) is a positive numerical constant. Therefore,
\[ \sup_{r < R} E |N_F(r) - \log \sigma_F(r)| \leq C. \]
Since both \( N_F(r) \) and \( \log \sigma_F(r) \) are convex functions, we can derive from here that their derivatives are also close on average, i.e., that
\[ E \left| n_F(r) - \frac{d \log \sigma_F(r)}{d \log r} \right| \]
is relatively small outside a small exceptional set \( E \) of values of \( r \) where the derivative \( \frac{d \log \sigma_F(r)}{d \log r} \) changes too fast due to the irregular behaviour of \( a_k \)'s. Invoking an appropriate version of the Borel-Cantelli lemma, we can also establish an almost sure analogue of this result.

If we are interested in the angular distribution of zeroes, the same idea works, we only need to replace Jensen’s formula by its modification for angular sectors.

The same approach works for the Steinhaus coefficients \( \xi_k = e^{2\pi i \gamma_k} \), where \( \gamma_k \) are independent and uniformly distributed on \([0, 1]\). In this case, one needs to estimate the expectation of the modulus of the logarithm of the absolute value of a normalized linear combination of independent Steinhaus variables. This was done by Offord in [14]; twenty years later, Ullrich [17, 18] and Favorov [2, 3] independently rediscovered this idea and gave new applications.

A linear combination of Rademacher random variables \( x = \sum \xi_k a_k \) can vanish with positive probability. Then one cannot hope to estimate from below the logarithmic expectation \( E\{|\log |x||\} \). In [8], Littlewood and Offord invented ingenious and formidable
techniques to circumvent this obstacle. Later, these techniques were further developed by Offord in [13, 16], but unfortunately, they were not sufficiently powerful to arrive to the same conclusions as for the Gaussian and the Steinhaus coefficients. Still, there is a reserve: note that in order to estimate the error term in the Jensen formula we do not need to estimate $E\left|\log |\hat{F}_r(t)|\right|$ uniformly in $t \in T$. For our purposes, the integral estimate of $E\left\{\int_T |\log |\hat{F}_r(t)|| \, dm(t)\right\}$ is not worse than the uniform bound for $E|\log |\hat{F}_r(t)||$. To exploit this reserve, we employ some harmonic analysis techniques.

1.2. Logarithmic integrability of Rademacher Fourier series. Let $(\Omega, \mathcal{P})$ be the probability space on which the Rademacher random variables $\xi_k$ are defined. Denote by $L^2_\mathcal{P} \subset L^2(\Omega)$ the product measure space with the product measure $\mu = \mathcal{P} \times m$.

By $L^2_\mathcal{P} \subset L^2(\Omega)$ we denote the closed subspace whose elements are the Rademacher Fourier series (i.e., the closed linear span of $\xi_k e^{2\pi i k \theta}$), and $\|f\|_2$ always stands for the $L^2(\Omega)$-norm.

Our first result is a distributional inequality, which says that if a Rademacher Fourier series is small on a set $E \subset Q$ of positive measure, then it must be small everywhere on $Q$.

**Theorem 1.1.** For each $f \in L^2_\mathcal{P}$ and each set $E \subset Q$ of positive measure,

$$\int_Q |f|^2 \, d\mu \leq e^{C \log^6 \left(\frac{\mu(E)}{\mu(Q)}\right)} \int_E |f|^2 \, d\mu.$$  

The power 6 on the RHS is not the best possible, but we will show that it cannot be replaced by any number less than 2. Note that this does not contradict the possibility that the distributional inequality can be improved if one is ready to discard an event of small probability.

The proof of Theorem 1.1 is based on ideas from harmonic analysis developed by the first-named author in [11, 12] to treat lacunary Fourier series. It uses a Turán-type lemma from [11, Chapter 1], and the technique of small shifts introduced in [11, Chapter 3].

Theorem 1.1 immediately yields the following $L^p(\mu)$-bound for the logarithm of the Rademacher Fourier series.

**Corollary 1.2.** For each $f \in L^2_\mathcal{P}$ with $\|f\|_2 = 1$, and for each $p \geq 1$,

$$\int_Q |\log |f||^p \, d\mu \leq (Cp)^6 p.$$  

Note that even the case $p = 1$ of this corollary is already non-trivial and new.

1.3. The range of random Taylor series in the unit disk. One of the consequences of the logarithmic integrability is the answer to an old question from Kahane’s book [11] p.xii):

**Suppose that**

$$F(z) = \sum_{k \geq 0} \xi_k a_k z^k$$

**is a Rademacher Taylor series with the radius of convergence 1 and with**

$$\sum_{k \geq 0} |a_k|^2 = +\infty.$$
Does the range $F(\mathbb{D})$ fill the complex plane almost surely?

We will prove this, and even more.

**Theorem 1.3.** Suppose

$$F(z) = \sum_{k \geq 0} \zeta_k z^k,$$

where $\{\zeta_k\}_{k \geq 0}$ is a sequence of independent complex-valued symmetric random variables satisfying the conditions

$$\limsup_{k \to \infty} |\zeta_k|^{1/k} = 1 \quad \text{and} \quad \sum_{k \geq 0} |\zeta_k|^2 = +\infty \quad \text{a.s.}$$

Then, a.s.,

$$\sum_{w: F(w) = b} (1 - |w|) = \infty \quad \forall \ b \in \mathbb{C}.$$

Note that if the series $\sum_{k \geq 0} |\zeta_k|^2$ converges, then the function $F$ belongs to the Hardy space $H^2$, and therefore its $b$-points obey the Blaschke condition

$$\sum_{w: F(w) = b} (1 - |w|) < \infty.$$

Theorem [15] has some history. In 1972, Offord [15] proved this result in the case when $\zeta_k$ are uniformly distributed on the unit circle. The proof he gave also works for the Taylor series with Gaussian coefficients; see also Kahane [4, Section 12.3]. According to the “reduction principle” [4, Section 1.7], the special case $\zeta_k = \xi_k a_k$, where $\xi_k$ are independent Rademacher random variables and $a_k$ is a non-random sequence of complex numbers such that $\limsup_{k \to \infty} |a_k|^{1/k} = 1$ and $\sum_{k \geq 0} |a_k|^2 = \infty$, should yield the general case. In the Rademacher case, the result was known under some additional restrictions on the growth of the deterministic coefficients $a_k$. In 1981, Murai [10] proved it assuming that $\liminf |a_k| > 0$. Soon afterwards, Jacob and Offord [5] weakened this assumption to

$$\liminf_{N \to \infty} \frac{1}{\log N} \sum_{k=0}^N |a_k|^2 > 0.$$

To the best of our knowledge, since then there was no improvement.

Curiously enough, even in the case when $\zeta_k = \xi_k a_k$ with the standard complex Gaussian $\xi_k$’s, the question when $F(\mathbb{D}) = \mathbb{C}$ almost surely is not completely settled. Recall that in [9] Murai proved Paley’s conjecture, which states that if $F$ is a (non-random) Taylor series with Hadamard gaps and with the radius of convergence 1, then $F$ assumes every complex value infinitely often, provided that $\sum_{k \geq 0} |a_k|^2 = \infty$. Therefore, the same holds for random Taylor series with Hadamard gaps. Even the case of sequences $a_k$ with a regular behaviour remains open:

**Question 1.4.** Suppose that the non-random sequence $\{a_k\}$ is decaying regularly and satisfies

$$\sum_{k \geq 0} |a_k|^2 < \infty, \quad \sum_{k \geq 0} \frac{|a_k|}{\sqrt{k}} = \infty,$$

and suppose that $\xi_k$ are independent standard Gaussian complex-valued random variables. Does the range of the random Taylor series $F(z) = \sum_{k \geq 0} \xi_k a_k z^k$ fill the whole complex plane $\mathbb{C}$ a.s.?
Note that convergence of the first series in \( I \) yields that, a.s., the function \( F \) belongs to all Hardy spaces \( H^p \) with \( p < \infty \). Moreover, by the Paley-Zygmund theorem \([4, \text{Chapter 15}]\), a.s., we have \( e^{\lambda|\hat{F}|^2} \in L^1(T) \) for every positive \( \lambda \), where \( \hat{F} \) denotes the non-tangential boundary values of \( F \) on \( T \). On the other hand, by Fernique’s theorem \([3, \text{Chapter 15}]\), divergence of the second series in \( I \) yields that, a.s., \( F \) is unbounded in \( D \).

\[ * \quad * \quad * \]

It is worth mentioning that our techniques can be applied to some other questions about the distribution of zeroes of random Taylor series including the one about the angular distribution of zeroes of random entire functions in large disks. We plan to return to that question in a separate paper.

2. Proof of the distributional inequality for Rademacher Fourier series

2.1. List of notation.

\( T = \mathbb{R}/\mathbb{Z} \); we also identify \( T \) with the interval \([0, 1) \subset \mathbb{R}\);

\( m \) either the Lebesgue measure on \( T \) normalized by \( m(T) = 1 \), or the Lebesgue measure on \( \mathbb{R} \);

\( e(\theta) = e^{2\pi i\theta}, \theta \in T \);

\( \mathbb{R}_+ = (0, \infty) \);

\( (\Omega, \mathcal{F}, \mathbb{P}) \) a probability space;

\( \xi_k : \Omega \rightarrow \{ \pm 1 \}, k \in \mathbb{Z} \), independent Rademacher random variables;

\( (Q, \mu) = (\Omega \times T, \mathbb{P} \times m) \) product measure space, \( L^2(Q) = L^2(Q, \mu) \);

\( \varphi_k(\omega, \theta) = \xi_k(\omega)e(k\theta), k \in \mathbb{Z}, (\omega, \theta) \in Q \);

\( L^2_{RF} \subset L^2(Q) \) the subspace of Rademacher Fourier series \( f = \sum_{k \in \mathbb{Z}} a_k \varphi_k, \sum_{k \in \mathbb{Z}} |a_k|^2 < \infty \).

The system \( \{ \varphi_k \} \) is an orthonormal basis in the space \( L^2_{RF} \), and for \( f \in L^2_{RF} \), we have

\[
\| f \|_2^2 = \int_Q |f|^2 \, d\mu = \int_{\Omega} |f(\omega, \cdot)|^2 \, d\mathbb{P}(\omega) = \int_T |f(\cdot, \theta)|^2 \, d\mu(\theta) = \sum_{k \in \mathbb{Z}} |a_k|^2 = \| \{ a_k \} \|_{L^2(\mathbb{Z})}^2.
\]

For a set \( E \subset Q \), we denote its sections by \( E_\omega \overset{\text{def}}{=} \{ \theta \in T : (\omega, \theta) \in E \}, \omega \in \Omega \).

The set \( E \subset Q \) shifted by \( t \in T \) is denoted by \( E + t \overset{\text{def}}{=} \{ (\omega, \theta) : (\omega, \theta - t) \in E \} \).

Then \( E_\omega + t = \{ \theta : \theta - t \in E_\omega \} = (E + t)_\omega \).

We put \( \Delta_t(E) \overset{\text{def}}{=} \mu((E + t) \setminus E) \).

The function \( g \in L^2(Q) \) shifted by \( t \) is denoted by \( g_t : g_t(\omega, \theta) = g(\omega, \theta + t) \).

Note that for the indicator function of \( E \), we have \( \mathbb{1}_E \) is the identity function.

A measurable function \( b \) on \( Q \) that does not depend on \( \theta \) will be called a random constant.

We write \([x]\) for the integral part of \( x \).
2.2. The result. Here is the main result of this part of the paper. It shows that an arbitrary function \( f \) in \( L^2_{RF} \) cannot be too close to a random constant \( b \), provided that the uniform norm of \( b \) is small compared with the \( L^2 \)-norm of \( f \). The version we gave in the introduction corresponds to the case when \( b \) is the zero function. The extension below is needed for the proof of Theorem 1.3 on the range of random Taylor series in the unit disk.

**Theorem 2.1.** For each \( f \in L^2_{RF} \), for each random constant \( b \in L^\infty(\Omega) \) with \( \| b \|_\infty < \frac{1}{20} \| f \|_2 \), and for each set \( E \subset Q \) of positive measure,

\[
\int_Q |f|^2 \, d\mu \leq \exp \left( C \log^6 \left( \frac{2}{\mu(E)} \right) \right) \int_E |f - b|^2 \, d\mu.
\]

As an immediate corollary, we get

**Corollary 2.2.** For each \( f \in L^2_{RF} \) with \( \| f \|_2 = 1 \), for each \( b \in L^\infty(\Omega) \) with \( \| b \|_\infty < \frac{1}{20} \), and for each \( p \geq 1 \), we have

\[
\int_Q \left| \log |f - b| \right|^p \, d\mu \leq (Cp)^6p.
\]

We note that the condition on the function \( b \) is a technical one. Its purpose is to avoid degenerate cases, for example, the case when the functions \( f \) and \( b \) are both equal to \( \xi_0 \).

2.3. The basic tools. Here is the list of the tools we will be using in the proof of Theorem 2.1.

2.3.1. **Turán-type lemma** [11, Chapter I]. Suppose

\[
p(z) = \sum_{k=0}^{n} a_k e^{i \lambda_k t}, \quad a_k \in \mathbb{C}, \quad \lambda_0 < \ldots < \lambda_n \in \mathbb{R},
\]

is an exponential polynomial. Then for any interval \( J \subset \mathbb{R} \) and any measurable subset \( E \subset J \) of positive measure,

\[
\max_J |p| \leq \left( \frac{Cm(J)}{m(E)} \right)^n \sup_E |p|.
\]

We will also use the \( L^2 \)-bound that follows from this estimate, see [11, Chapter III, Lemma 3.3]. It states that under the same assumptions,

\[
\| p \|_{L^2(J)} \leq \left( \frac{Cm(J)}{m(E)} \right)^{n+\frac{1}{2}} \| p \|_{L^2(E)}.
\]

2.3.2. **Khinchin’s inequality.** Let \( \{ \xi_k \} \) be independent Rademacher random variables, and let \( \{ a_k \} \) be complex numbers. Then for each \( p \geq 2 \), we have

\[
\left( \mathbb{E} \sum_k a_k \xi_k \right)^{p \prime} \leq C \sqrt{p} \left( \sum_k |a_k|^2 \right)^{1/2}.
\]
2.3.3. Bilinear Khinchin’s inequality. Let \( \{ \xi_k \} \) be independent Rademacher random variables, and let \( \{ a_{k,\ell} \} \) be complex numbers. Then for each \( p \geq 2 \), we have
\[
\left( \mathbb{E} \left| \sum_{k \neq \ell} a_{k,\ell} \xi_k \xi_\ell \right|^p \right)^{1/p} \leq C_p \left( \sum_{k \neq \ell} |a_{k,\ell}|^2 \right)^{1/2}.
\]

A simple and elegant proof of this inequality can be found in a recent preprint by L. Erdös, A. Knowles, H.-T. Yau, J. Yin [11, Appendix B].

2.4. The class \( \text{Exp}_{1\text{oc}} \) of functions with almost linearly dependent small shifts. The proof of Theorem 2.1 uses the technique of small shifts developed in [11, Chapter III]. In this and the next two sections we will outline this technique.

Let \( \mathcal{H} \) be a Hilbert space. By \( L^2(\mathbb{T}, \mathcal{H}) \) we denote the Hilbert space of square integrable \( \mathcal{H} \)-valued functions on \( \mathbb{T} \) (in the sense of Bochner). Note that the space \( L^2(\mathbb{T}, L^2(\Omega)) \) can be identified with \( L^2(Q) \). To define the class of functions in \( L^2(\mathbb{T}, \mathcal{H}) \) with almost linearly dependent shifts, we introduce the following set of parameters:

- the order \( n \in \mathbb{N} \) (a large parameter);
- the localization parameter \( \tau > 0 \) (a small parameter);
- the error \( \varkappa > 0 \) (a small parameter).

Definition 2.2 (\( \text{Exp}_{1\text{oc}} \)). We say that a function \( g \in L^2(\mathbb{T}, \mathcal{H}) \) belongs to the class \( \text{Exp}_{1\text{oc}}(n, \tau, \varkappa, \mathcal{H}) \) if for each \( t \in (0, \tau) \) there exist complex numbers \( a_k = a_k(t), k \in \{0, \ldots, n\} \), with \( \sum_{k=0}^n |a_k|^2 = 1 \), such that
\[
\left\| \sum_{k=0}^n a_k g_{kt} \right\|_{L^2(\mathbb{T}, \mathcal{H})} < \varkappa.
\]

In the case \( \mathcal{H} = \mathbb{C} \), this class was introduced in [11, Chapter III]. “In small” (i.e., on intervals of length comparable with \( \tau \)), the functions from this class behave similarly to exponential sums with \( n \) frequencies and with coefficients in \( \mathcal{H} \). On the other hand, since the translations act continuously in \( L^2(\mathbb{T}, \mathcal{H}) \), for any given \( g \in L^2(\mathbb{T}, \mathcal{H}) \), \( n \in \mathbb{N} \), \( \varkappa > 0 \), one can choose the parameter \( \tau > 0 \) so small that \( g \in \text{Exp}_{1\text{oc}}(n, \tau, \varkappa, \mathcal{H}) \).

In the next three sections, we extend main results about this class (the spectral description, the local approximability by exponential sums with \( n \) terms, and the spreading lemma) from the scalar case to the case considered here. Since the proofs of these extensions are similar to the ones given in [11], we relegate them to the appendices.

2.5. Spectral description of the class \( \text{Exp}_{1\text{oc}} \). The first lemma shows that each function \( g \in \text{Exp}_{1\text{oc}}(n, \tau, \varkappa, \mathcal{H}) \) has an “approximate spectrum” \( \Lambda_g \), which consists of \( n \) frequencies so that the Fourier transform of \( g \) is small in the \( \ell^2 \)-norm away from these frequencies.

For \( m \in \mathbb{Z}, \Lambda \subset \mathbb{R} \), let
\[
\theta_\tau(m) = \min(1, |m|), \quad \Theta_{\tau, \Lambda}(m) = \prod_{\lambda \in \Lambda} \theta_\tau(m - \lambda).
\]

Lemma 2.1. Given \( g \in \text{Exp}_{1\text{oc}}(n, \tau, \varkappa, \mathcal{H}) \), there exists a set \( \Lambda = \Lambda_g \subset \mathbb{R} \) of \( n \) distinct frequencies such that
\[
\sum_{m \in \mathbb{Z}} \| \hat{g}(m) \|_2^2 \Theta_{\tau, \Lambda}^2(m) \leq (Cn)^4 \varkappa^2.
\]
The proof of this lemma will be given in Appendix A.

### 2.6. Local approximation by exponential sums with $n$ terms

Starting with this section, we assume that $\mathcal{H} = L^2(\Omega)$. Then $\text{Exp}_{1oc}(n, \tau, \kappa, L^2(\Omega)) \subset L^2(\Omega)$.

For a finite set $\Lambda \subset \mathbb{R}$, denote by $\text{Exp}(\Lambda, \omega)$ the linear space of exponential polynomials with frequencies in $\Lambda$ and with coefficients depending on $\omega$. The next lemma shows that, for a.e. $\omega \in \Omega$, the function $\theta \mapsto g(\omega, \theta)$, $g \in \text{Exp}_{1oc}(n, \tau, \kappa, L^2(\Omega))$, can be well approximated by exponential polynomials from $\text{Exp}(\Lambda, \omega)$, on intervals $J \subset [0, 1)$ of length comparable with $\tau$.

Suppose that $M > 1$ satisfies
\[ \ell = \frac{1}{\tau M} \in \mathbb{N}, \]
and partition $\mathbb{T}$ into $l$ intervals of length $M \tau$:
\[ \mathbb{T} = \bigcup_{k=0}^{l-1} \left[ \frac{k}{\ell}, \frac{k+1}{\ell} \right). \]

**Lemma 2.2.** Let $M$ be as above and let $g \in \text{Exp}_{1oc}(n, \tau, \kappa, L^2(\Omega))$. There exists a non-negative function $\Phi \in L^2(\Omega)$ with
\[ \|\Phi\|_2 \leq (Cn)^{2n} \kappa, \]
and with the following property: for every interval $J \subset \mathbb{T}$ in the partition, there exists an exponential polynomial $p^J \in \text{Exp}(\Lambda, \omega)$ such that, for a.e. $\omega \in \Omega$ and a.e. $\theta \in J$,
\[ \left| g(\omega, \theta) - p^J(\omega, \theta) \right| \leq M^n \Phi(\omega, \theta). \]

The proof of this lemma will be given in Appendix B.

### 2.7. Spreading Lemma

The next lemma is the crux of the proof of Theorem 2.1.

Given a set $E \subset \mathbb{Q}$ of positive measure, we put $\Delta_t(E) = \mu((E + t) \setminus E)$.

**Lemma 2.3.** Suppose $g \in \text{Exp}_{1oc}(n, \tau, \lambda, L^2(\Omega))$ and $E \subset \mathbb{Q}$ is a set of positive measure. There exists a set $\tilde{E} \supset E$ of measure $\mu(\tilde{E}) \geq \mu(E) + \frac{1}{2} \Delta_{\tau}(E)$ such that, for each $b \in L^2(\Omega)$,
\[ \int_E |g - b|^2 \, d\mu \leq \left( \frac{Cn^3}{\Delta_{\tau}^2(E)} \right)^{2n+1} \left( \int_E |g - b|^2 \, d\mu + \lambda^2 \right). \]

This lemma follows from the previous lemma combined with the Turán-type estimate (2.1). The proof of Lemma 2.3 will be given in Appendix C.

### 2.8. Starting the proof of Theorem 2.1

Zygmund’s premise and the operator $A_E$. Suppose that
\[ f = \sum_{k \in \mathbb{Z}} a_k \varphi_k, \quad \varphi_k(\omega, \theta) = \xi_k(\omega)e(\langle k\theta \rangle), \quad \{a_k\} \in \ell^2(\mathbb{Z}), \]
and that $b \in L^2(\Omega)$. Let $E \subset Q$ be a measurable set of positive measure. Then
\[
\int_E |f - b|^2 \, d\mu = \int_E \left[ \sum_{k,\ell} a_k \bar{a}_\ell \varphi_k \bar{\varphi}_\ell - 2 \Re (f \bar{b}) + |b|^2 \right] \, d\mu
\]
\[
\geq \int_E \left[ \sum_k |a_k|^2 |\varphi_k|^2 \right] \, d\mu + \int_E \left[ \sum_{k \neq \ell} a_k \bar{a}_\ell \varphi_k \bar{\varphi}_\ell \right] \, d\mu - 2 \Re \{ f, \mathbb{1}_E b \}
\]
\[
= \mu(E) \| f \|_2^2 + \langle A_E f, f \rangle - 2 \Re \{ f, \mathbb{1}_E b \},
\]
where $A_E$ is a bounded self-adjoint operator on $L^2_{RF}$, whose matrix $(A_E(k, \ell))_{k,\ell \in \mathbb{Z}}$ in the orthonormal basis $\{ \varphi_k \}$ is given by
\[
A_E(k, \ell) = \begin{cases} (\mathbb{1}_E, \varphi_k \bar{\varphi}_\ell), & k \neq \ell; \\ 0, & k = \ell. \end{cases}
\]
To estimate the Hilbert-Schmidt norm of $A_E$, we observe that the functions $\{ \varphi_k \bar{\varphi}_\ell \}_{k \neq \ell}$ form an orthonormal system in $L^2(Q)$, and that each function from this system is orthogonal to the function $\mathbb{1}$. Then
\[
\sum_{k \neq \ell} |A_E(k, \ell)|^2 + (|\mathbb{1}_E|)^2 \leq (\mathbb{1}_E) \| f \|_2^2 = \mu(E),
\]
and therefore,
\[
\| A_E \|_{HS} = \left( \sum_{k \neq \ell} |A_E(k, \ell)|^2 \right)^{1/2} \leq \sqrt{\mu(E) - \mu(E)^2}.
\]
This estimate is useful for sets $E$ of large measure.

2.9. **The sets $E$ of large measure.** For each $\mu \in (0, 1)$, let $D(\mu) \in (1, +\infty]$ be the smallest value such that the inequality
\[
\int_Q |f|^2 \, d\mu \leq D(\mu) \int_E |f - b|^2 \, d\mu
\]
is satisfied for every $E \subset Q$ with $\mu(E) \geq \mu$, for every $f \in L^2_{RF}$, and for every random constant $b \in L^\infty(\Omega)$ with $\|b\|_\infty < \frac{1}{\sqrt{2}}\|f\|_2$.

Using the estimates from the previous section, we get
\[
\int_E |f - b|^2 \, d\mu \geq (\mu(E) - \|A\|_2^2) \| f \|_2^2 - 2 \| \mathbb{1}_E b \|_2 \| f \|_2
\]
\[
\geq (\mu(E) - \sqrt{\mu(E) - \mu(E)^2} - \frac{1}{10}) \| f \|_2^2 \geq \frac{1}{2} \| f \|_2^2,
\]
provided that $\mu(E) \geq \frac{9}{10}$. That is, $D(\mu) \leq 2$ for $\mu \geq \frac{9}{10}$.

In order to get an upper bound for $D(\mu)$ for smaller values of $\mu$, first of all, we need to get a better bound for the Hilbert-Schmidt norm of the operator $A_E$.

2.10. **A better bound for the Hilbert-Schmidt norm of $A_E$.** Here, using the bilinear Khinchin inequalities $[2.3.3]$, we show that for each $p \geq 1$,
\[
\| A_E \|_{HS} \leq C_p \cdot \mu(E)^{1 - \frac{1}{2p}}.
\]
For sets $E$ of small measure, this bound is better than the one we gave in $[2.8]$. 

Proof: First, using duality and then Hölder’s inequality, we get
\[
\left(\sum_{k \neq \ell} |A_E(k, \ell)|^2\right)^{1/2} = \sup\left\{ \left| \sum_{k \neq \ell} A_E(k, \ell) g_{k, \ell} \right| : \sum_{k \neq \ell} |g_{k, \ell}|^2 \leq 1 \right\}
\]
\[
= \sup\left\{ \left\| \int_Q \mathbb{1}_E \bar{g} \, d\mu \right\| : g \in \text{span}\{ \varphi_{k, \ell} \}_{k \neq \ell}, \|g\|_2 \leq 1 \right\}
\]
\[
\leq \mu(E)^{1 - \frac{1}{2p}} \sup\left\{ \|g\|_2 : g \in \text{span}\{ \varphi_{k, \ell} \}_{k \neq \ell}, \|g\|_2 \leq 1 \right\}.
\]
Now, using the bilinear Khinchin inequality, we will bound \( \|g\|_2 \) by \( Cp\|g\|_2 \). Since \( g \in \text{span}\{ \varphi_{k, \ell} \}_{k \neq \ell} \),
\[
g(\omega, \theta) = \sum_{k \neq \ell} g_{k, \ell} \xi_k(\omega) \xi_\ell(\omega) e((k - \ell) \theta),
\]
whence,
\[
\int_Q |g|^{2p} \, d\mu = \int_T d\mu(\theta) \int_{\Omega} dP(\omega) \left| \sum_{k \neq \ell} g_{k, \ell} \xi_k(\omega) \xi_\ell(\omega) e((k - \ell) \theta) \right|^{2p}
\]
\[
\leq \int_T d\mu(\theta) (Cp)^{2p} \left( \sum_{k \neq \ell} |g_{k, \ell}| e((k - \ell) \theta) \right)^p
\]
\[
= (Cp)^{2p} \left( \sum_{k \neq \ell} |g_{k, \ell}|^2 \right)^{p/2} = (Cp)^2 \|g\|_2^2,
\]
completing the proof. \qed

2.11. The subspace \( V_{E,b} \). Let \( p \geq 1 \). We now show that there exists a positive numerical constant \( C' \) with the following property. If \( E \subset Q \) is a set of positive measure and \( b \in L^2(Q) \), then there exists a subspace \( V_{E,b} \subset L^2_{RF} \) of dimension at most
\[
n = \left\lfloor \frac{C' p^2}{\mu(E)^{1/p}} \right\rfloor
\]
such that for each function \( g \in L^2_{RF} \cap V_{E,b} \) and each \( b_1 = c \cdot b \) with \( c \in \mathbb{C} \), we have
\[
\int_Q |g|^2 \, d\mu \leq \frac{2}{\mu(E)} \int_E |g - b_1|^2 \, d\mu.
\]
Proof: This result is a rather straightforward consequence of the estimates from 2.8 and 2.10. We enumerate the eigenvalues of the operator \( A_E \) so that their absolute values form a non-increasing sequence: \( |\sigma_1| \geq |\sigma_2| \geq \ldots \). Let \( h_1, h_2, \ldots \) be the corresponding eigenvectors. Let \( m \in \mathbb{Z} \) and denote by \( \tilde{V}_E \) the linear span of \( h_1, \ldots, h_m \). Then the norm of the restriction \( A_E \) to \( L^2_{RF} \cap \tilde{V}_E \) equals \( |\sigma_{m+1}| \). Therefore, if the function \( g \in L^2_{RF} \cap \tilde{V}_E \), then \( |\langle A_E g, g \rangle| \leq |\sigma_{m+1}| \cdot \|g\|_2^2 \).

Next,
\[
\sigma^2_{m+1} \leq \frac{1}{m+1} \sum_{j=1}^{m+1} \sigma^2_j \leq \frac{1}{m+1} \sum_{j=1}^{\infty} \sigma^2_j
\]
\[
= \frac{1}{m+1} \|A_E\|_{HS}^2 \leq \frac{Cp^2}{m+1} \cdot \mu(E)^{2-\frac{1}{p}} < \frac{1}{4} \mu(E)^2,
\]
where
provided that
\[ m \geq \left\lceil \frac{C'p^2}{\mu(E)^{1/p}} \right\rceil - 1 \]
and \( C' \) is chosen large enough.

Denote by \( U_{E,b} \) the one-dimensional space spanned by the projection of the function \( \mathbb{1}_{E} \cdot b \) to \( L^2_{RF} \), and put \( V_{E,b} = \tilde{V} + U_{E,b} \). Then, assuming that \( g \in L^2_{RF} \cap V_{E,b} \subset L^2_{RF} \cap \tilde{V} \) and applying the estimate from 2.8, we get
\[
\int_{E} \left| g - b_1 \right|^2 \, d\mu \geq \mu(E)\|g\|^2_2 + \langle A_E g, g \rangle - 2 \Re \langle g, \mathbb{1}_{E} b_1 \rangle
\]
\[
\geq \mu(E)\|g\|^2_2 - \frac{1}{2} \mu(E)\|g\|^2_2 = \frac{\mu(E)}{2} \|g\|^2_2.
\]

Since \( \dim V_{E,b} \) is at most \( \left\lceil \frac{C'p^2\mu(E)^{-1/p}}{} \right\rceil \), the proof is complete. \( \square \)

Note that it suffices to take \( C' = 4C^2 + 1 \), where \( C \) is the constant that appears in the bilinear Khinchin inequality 2.3.3, though this is not essential for our purposes.

2.12. **Placing \( f \in L^2_{RF} \) in the class \( \text{Exp}_{1\text{oc}} \).** **Condition \( (C_r) \).** Introduce the function
\[ n(p, \mu) \overset{\text{def}}{=} \left\lceil \frac{C''p^2 \cdot \mu^{-1/p}}{} \right\rceil \]
where \( C'' > C' \) is a sufficiently large numerical constant. Fix \( p \geq 1 \) and let \( E \subset Q \) be a given set of positive measure. Put \( n = n(p, \frac{1}{2}\mu(E)) \) and choose the small parameter \( \tau \) so that, for every \( t \in (0, \tau) \),
\[ \mu\left( \bigcap_{k=0}^{n} (E - kt) \right) \geq \frac{1}{2} \mu(E). \]  

\( (C_r) \)

This is possible since the function \( t \mapsto \mu\left( (E - t) \cap E \right) \) is continuous and equals \( \mu(E) \) at 0.

Now we prove that given a set \( E \subset Q \) of positive measure, \( b \in L^2(Q) \), and \( p \geq 1 \), each function \( f \in L^2_{RF} \) belongs to the class \( \text{Exp}_{1\text{oc}}(n, \tau, \chi, L^2(\Omega)) \) with
\[ n = n(p, \frac{1}{2}\mu(E)), \quad \chi^2 = \frac{4(n + 1)}{\mu(E)} \int_{E} |f - b|^2 \, d\mu, \]
and arbitrary \( \tau \) satisfying condition \( (C_r) \).

**Proof:** To shorten the notation, we put
\[ E' = E'_{\tau} = \bigcap_{k=0}^{n} (E - kt). \]

Then for every \( k \in \{0, \ldots, n\} \),
\[ \int_{E'} |f_{kt} - b|^2 \, d\mu \leq \int_{E - kt} |f_{kt} - b|^2 \, d\mu = \int_{E} |f - b|^2 \, d\mu, \]

since \( b \) depends only on \( \omega \), and so, \( b_{kt} = b \).

Given \( t \in (0, \tau) \), we choose \( a_0, \ldots, a_n \in \mathbb{C} \) with \( \sum_{k=0}^{n} |a_k|^2 = 1 \) so that the function \( g = \sum_{k=0}^{n} a_k f_{kt} \) belongs to the linear space \( L^2_{RF} \cap V_{E',b} \). This is possible since
\[ \dim V_{E',b} \leq n(p, \mu(E')) \leq n(p, \frac{1}{2}\mu(E)) = n. \]
Since the function $g$ is orthogonal to the subspace $V_{E'}$, we can control its norm applying the estimate from [2.11] with $b_1 = b - \sum k a_k$:

\[
\int_Q |g|^2 \, d\mu \leq \frac{2}{\mu(E')} \int_{E'} |g - b_1|^2 \, d\mu
\]

\[
\leq \frac{4}{\mu(E)} \int_{E'} \sum_{k=0}^n a_k (f_{kt} - b) \, d\mu
\]

\[
\leq \frac{4}{\mu(E)} \int_{E'} \sum_{k=0}^n |f_{kt} - b|^2 \, d\mu \leq \frac{4(n+1)}{\mu(E)} \int_{E'} |f - b|^2 \, d\mu.
\]

That is,

\[
\left\| \sum_{k=0}^n a_k f_{kt} \right\|_2 \leq \varkappa,
\]

and we are done. \qed

2.13. **Spreading the $L^2$-bound.** Condition $(C_E)$. We apply the spreading Lemma 2.3 to the function $f$ and the set $E$. It provides us with a set $\tilde{E} \supset E$, such that $\mu(\tilde{E}) \geq \mu(E) + \frac{1}{2} \Delta_{n\tau}(E)$ and

\[
\int_{\tilde{E}} |f - b|^2 \, d\mu \leq \left(\frac{Cn^3}{\Delta_{n\tau}^2(E)}\right)^{2n+1} \left(\int_{E'} |f - b|^2 \, d\mu + \varkappa^2\right)
\]

\[
\leq \left(\frac{Cn^3}{\Delta_{n\tau}^2(E)}\right)^{2n+1} \cdot \frac{C(n+1)}{\mu(E)} \int_{E} |f - b|^2 \, d\mu,
\]

where $n = n(p, \frac{1}{2}\mu(E)) \leq 2Cn^2p^2 \cdot \mu(E)^{-\frac{1}{2}}$. There is not much value in this spreading until we learn how to control the parameter $\Delta_{n\tau}(E)$ in terms of our main parameters $\mu(E)$ and $p$. Clearly, the bigger $\Delta_{n\tau}(E)$ is, the better is our spreading estimate. Recall that till this moment, our only assumption on the value of $\tau$ has been condition $(C_{\tau})$ at the beginning of section 2.12.

Now we will need the following condition on our set $E$:

\[
\max_t \Delta_t(E) \geq \frac{1}{2n} \mu(E).
\]

Condition $(C_{E})$ holds then we can find $\tau > 0$ such that $\Delta_{n\tau}(E) = \frac{1}{2n} \mu(E)$, while for all $t \in (0, n\tau)$, $\Delta_t(E) < \frac{1}{2n} \mu(E)$.

Such $\tau$ will automatically satisfy condition $(C_{\tau})$ used in the derivation of the spreading estimate. Indeed,

\[
\mu\left(\bigcap_{k=0}^n (E - kt)\right) = \mu\left(E \setminus \bigcup_{k=1}^n (E \setminus (E - kt))\right)
\]

\[
\geq \mu(E) - \sum_{k=1}^n \mu(E \setminus (E - kt))
\]

\[
= \mu(E) - \sum_{k=1}^n \mu((E + kt) \setminus E)
\]

\[
= \mu(E) - \sum_{k=1}^n \Delta_{kt}(E) \geq \mu(E) - \sum_{k=1}^n \frac{\mu(E)}{2n} = \frac{1}{2} \mu(E).
\]
It is easy to see that there are sets $E \subset Q$ of arbitrary small positive measure that do not satisfy condition $(C_E)$. We assume now that condition $(C_E)$ is satisfied, putting aside the question “What to do with the sets $E$ for which $(C_E)$ does not hold?” till the next section.

Substituting the value $\Delta_{n\tau} = \frac{1}{2n} \mu(E)$ into the spreading estimate and taking into account that $n \leq 2C''p^2 \mu(E)^{-\frac{1}{2}}$, we finally get

\[
\int_E |f - b|^2 \, d\mu \leq \left( \frac{Cn^5}{\mu(E)^2} \right)^{2n+1} \frac{n+1}{\mu(E)} \int_E |f - b|^2 \, d\mu
\leq \left( \frac{Cp}{\mu(E)} \right)^{Cp^2 \mu(E)^{-1/p}} \int_E |f - b|^2 \, d\mu,
\]

while

\[
\mu(\tilde{E}) \geq \mu(E) + \frac{c}{p^2} \mu(E)^{1+\frac{1}{p}}.
\]

This is the spreading estimate that we will use for the sets $E$ satisfying condition $(C_E)$.

### 2.14. The case of sets $E$ that do not satisfy condition $(C_E)$.

Now, let us assume that $E \subset Q$ is a set of positive measure that does not satisfy condition $(C_E)$, that is, for each $t \in [0, 1]$, $\Delta_t(E) < \frac{1}{2n} \mu(E)$. The simplest example is any set of the form $E = \Omega_1 \times T$, $\Omega_1 \subset \Omega$. For these sets, $\Delta_t(E) = 0$ for every $t$. We will show that this example is typical, i.e., the sets $E$ that do not satisfy condition $(C_E)$ must have sufficiently many “long sections” $E_\omega$. More precisely, let

\[
\Omega_1 = \{ \omega \in \Omega : m(E_\omega) > 1 - \frac{1}{n} \}.
\]

We show that $\mathcal{P}\{\Omega_1\} > \frac{1}{2} \mu(E)$.

**Proof:** Let

\[
\Omega_2 = \Omega \setminus \Omega_1 = \{ \omega \in \Omega : m(E_\omega) \leq 1 - \frac{1}{n} \}.
\]

Since condition $(C_E)$ is not satisfied, we have

\[
\int_0^1 \Delta_t(E) \, dt < \frac{1}{2n} \mu(E).
\]

A straightforward computation shows that

\[
\int_0^1 m((E_\omega + t) \setminus E_\omega) \, dt = m(E_\omega) (1 - m(E_\omega)).
\]

Since $m(E_\omega) \leq 1 - \frac{1}{n}$ implies that $m(E_\omega) \leq nm(E_\omega)(1 - m(E_\omega))$, we get

\[
\int_{\Omega_2} m(E_\omega) \, d\mathcal{P}(\omega) \leq n \int_{\Omega_2} m(E_\omega) (1 - m(E_\omega)) \, d\mathcal{P}(\omega)
\]

\[
\leq n \int_{\Omega} m(E_\omega) (1 - m(E_\omega)) \, d\mathcal{P}(\omega)
\]

\[
= n \int_0^1 \Delta_t(E) \, dt < \frac{1}{2} \mu(E).
\]

Therefore,

\[
\mathcal{P}\{\Omega_1\} \geq \int_{\Omega_1} m(E_\omega) \, d\mathcal{P}(\omega) = \mu(E) - \int_{\Omega_2} m(E_\omega) \, d\mathcal{P}(\omega) > \frac{1}{2} \mu(E).
\]
Remark: Since \( n = n \left( p, \frac{1}{2} \mu(E) \right) \geq 2 \) if \( C'' \geq 2 \), we trivially have
\[
\mathcal{P}\{\Omega_1\} \leq \frac{n}{n-1} \int_{\Omega_1} m(E_\omega) \, d\mathcal{P}(\omega) \leq \frac{n}{n-1} \mu(E) \leq 2 \mu(E).
\]

2.15. Many “long sections”. Assume that the set \( E \) does not satisfy condition \((C_E)\). We will show that
\[
\int_Q |f|^2 \, d\mu \leq \frac{4}{\mu(E)} \int_E |f - b|^2 \, d\mu,
\]
where, as above, \( b = b(\omega) \) is a random constant, \( \|b\|_\infty < \frac{1}{20} \|f\|_2 \).

Let \( \mu = \mu(E) \) and \( \Omega_1 \) be as above. We have
\[
\int_E |f - b|^2 \, d\mu \geq \int_{\Omega_1} \left( \int_{E_\omega} |f - b|^2 \, dm \right) \, d\mathcal{P}(\omega)
\]
\[= \int_{\Omega_1} \int_T |f - b|^2 \, dm \, d\mathcal{P}(\omega) - \int_{\Omega_1} \int_{T \setminus E_\omega} |f - b|^2 \, dm \, d\mathcal{P}(\omega) = (I) - (II).
\]
Notice that by the result of section 2.13, we have \( 2\mu \geq \mathcal{P}\{\Omega_1\} \geq \frac{1}{2} \mu \).

Bounding integral \((I)\) from below is straightforward: we have
\[
\int_T |f - b|^2 \, dm \geq (\|f\|_2 - \|b\|_\infty)^2 \geq \frac{9}{10} \|f\|_2^2,
\]
whence,
\[ (I) \geq \frac{9}{10} \|f\|_2^2 \mathcal{P}\{\Omega_1\} \geq \frac{9}{20} \cdot \mu \|f\|_2^2.
\]

Now let us estimate the integral \((II)\) from above. We have
\[ (II) \leq 2 \int_{\Omega_1} \int_{T \setminus E_\omega} |f|^2 \, dm \, d\mathcal{P}(\omega) + 2 \int_{\Omega_1} \int_{T \setminus E_\omega} |b|^2 \, dm \, d\mathcal{P}(\omega) = (II_a) + (II_b).
\]

Estimating the second integral is also straightforward:
\[ (II_b) \leq \frac{4\mu}{n} \|b\|_\infty^2 < \frac{1}{10} \mu \|f\|_2^2
\]
(recall that \( n \geq 2 \) and \( \|b\|_\infty < \frac{1}{20} \|f\|_2 \)). Furthermore,
\[ (II_a) = 2 \int_{\Omega_1} \int_T \mathbb{I}_{T \setminus E_\omega} |f|^2 \, dm \, d\mathcal{P}(\omega) \leq 2 \left( \int_{\Omega_1} \int_T \mathbb{I}_{T \setminus E_\omega} \right) ^{\frac{1}{s}} \left( \int_{\Omega_1} \int_T |f|^{2s} \right) ^\frac{1}{s}
\]
with \( \frac{1}{r} + \frac{1}{s} = 1 \). By Khinchin’s inequality,
\[ \left( \int_{\Omega_1} \int_T |f|^{2s} \right) ^\frac{1}{s} \leq C_s \|f\|_2^2.
\]

Hence,
\[ (II_a) \leq \left( \frac{2\mu}{n} \right) ^\frac{1}{s} C_s \|f\|_2^2.
\]

Letting \( \frac{1}{r} = \frac{p}{p + 1}, \frac{1}{s} = \frac{1}{p + 1} \) and recalling that \( n \geq \frac{1}{2} C'' p^2 \mu^{-1/p} \) and that \( p \geq 1 \), we continue the estimate as
\[ (II_a) \leq \left( \frac{4\mu^{1+\frac{1}{p}}}{C'' p^2} \right) ^\frac{1}{p + 1} 2Cp \|f\|_2^2 < \frac{8C}{\sqrt{C''} \mu^{p-\frac{p-1}{p+1}}} \|f\|_2^2 < \frac{1}{10} \mu \|f\|_2^2,
\]

Thus, we obtain the desired inequality.
provided that the constant $C''$ in the definition of $n$ was chosen sufficiently big. Finally,

$$\int_E |f - b|^2 \geq (I) - (II_a) - (II_b) \geq \left(\frac{9}{20} - \frac{4}{20}\right) \mu \|f\|^2 = \frac{1}{4} \mu \|f\|^2,$$

completing the argument.

\[\square\]

2.16. End of the proof of Theorem 2.1: solving a difference inequality. Recall that by $D(\mu)$ we denote the smallest value such that the inequality

$$\int_Q |f|^2 \, d\mu \leq D(\mu) \int_E |f - b|^2 \, d\mu$$

holds for every $E \subset Q$ with $\mu(E) \geq \mu$, every $f \in L^2_{RF}$, and every random constant $b \in L^\infty(\Omega)$ satisfying $\|b\|_{\infty} < \frac{1}{20} \|f\|_2$.

By 2.9, $D(\mu) \leq 2$ for $\mu \geq \frac{9}{10}$, and by the estimates proven in 2.13 and 2.15, for $0 < \mu < \frac{9}{10}$ we have

$$D(\mu) \leq \max\left\{\left(\frac{Cp}{\mu}\right)^{Cp^2 \mu^{\frac{4}{9}}} D\left(\mu + \frac{c}{p^2} \mu^{1+\frac{2}{9}}\right), \frac{4}{\mu}\right\}.$$  

Increasing, if needed, the constant $C$ in the exponent, and taking into account that $\frac{p}{\mu} \geq \frac{1}{\sqrt[9]{10}} > 1$ and $D \geq 1$, we simplify this to

$$D(\mu) \leq \left(\frac{p}{\mu}\right)^{Cp^2 \mu^{\frac{4}{9}}} D\left(\mu + \frac{c}{p^2} \mu^{1+\frac{2}{9}}\right).$$

Put

$$\delta(\mu) = \frac{c}{p^2} \mu^{1+\frac{2}{9}}.$$  

Making the constant $c$ on the right-hand side small enough, we assume that $\delta\left(\frac{9}{10}\right) < \frac{1}{10}$ (it suffices to take $c < \frac{1}{10}$). Then, for $0 < \mu < \frac{9}{10},$

$$\log D(\mu) - \log D(\mu + \delta(\mu)) < Cp^2 \mu^{-\frac{4}{9}} \log\left(\frac{p}{\mu}\right) < C \delta(\mu)^{p^4 \mu^{-1-\frac{2}{9}}} \log\left(\frac{p}{\mu}\right).$$

To solve this difference inequality, we define the sequence $\mu_0 = \mu, \mu_{k+1} = \mu_k + \delta(\mu_k), k \geq 0$, and stop when $\mu_{s-1} < \frac{9}{10} \leq \mu_s$. Since we assumed that $\delta\left(\frac{9}{10}\right) < \frac{1}{10}$, the terminal value $\mu_s$ will be strictly less than 1. We get

$$\log D(\mu) = \log D(\mu_s) + \sum_{k=0}^{s-1} \left[\log D(\mu_k) - \log D(\mu_{k+1})\right]$$

$$< 1 + Cp^4 \sum_{k=0}^{s-1} \delta(\mu_k) \mu_k^{-\frac{4}{9}} \log\left(\frac{p}{\mu_k}\right)$$

$$< 1 + Cp^4 \log\left(\frac{p}{\mu}\right) \sum_{k=0}^{s-1} \delta(\mu_k) \mu_k^{-\frac{4}{9}} \mu_k.$$
Since $\mu_{k+1} = \mu_k + cp^{-2} \mu_k^{1+\frac{1}{p}} < C \mu_k$, we have $\mu_k^{-1-\frac{2}{p}} < C \mu_{k+1}^{-1-\frac{2}{p}}$. Therefore,

$$\sum_{k=0}^{s-1} \delta(\mu_k) \mu_k^{-1-\frac{2}{p}} < C \sum_{k=0}^{s-1} \delta(\mu_k) \mu_{k+1}^{-1-\frac{2}{p}} < C \int_\mu \frac{dx}{x^{1+\frac{2}{p}}} < C \int_\mu \frac{dx}{x^{1+\frac{2}{p}}} < C p \mu^{-\frac{2}{p}},$$

whence,

$$\log D(\mu) < 1 + C p^5 \mu^{-\frac{2}{p}} \log \left(\frac{p}{\mu}\right).$$

This holds for any $p \geq 1$. Letting $p = 2 \log\left(\frac{2}{\mu}\right)$, we finally get $\log D(\mu) < C \log^6\left(\frac{2}{\mu}\right)$.

This completes the proof of Theorem $2.1$. \hfill \Box

3. Proof of Theorem 1.3 on the range of random Taylor series

First, we prove the theorem in the special case when $\zeta_n = \xi_n a_n$, where $\xi_n$ are independent Rademacher random variables, and $\{a_n\}$ is a non-random sequence of complex numbers satisfying the conditions $\limsup_n |a_n|^{1/n} = 1$ and $\sum_n |a_n|^2 = \infty$. That part of the proof is based on the logarithmic integrability of the Rademacher Fourier series (Corollary 2.2 to Theorem 2.1) combined with Jensen’s formula. Then using “the principle of reduction” as stated in the Kahane book [4, Section 1.7], we get the result in the general case.

Let us introduce some notation. For $b \in \mathbb{C}$, $0 < r < 1$ we denote by $n_F(r, b)$ the number of solutions to the equation $F(z) = b$ in the disk $r \mathbb{D}$, the solutions being counted with their multiplicities. In this section it will be convenient to set

$$N_F(r, b) \overset{\text{def}}{=} \int_{1/2}^r \frac{n_F(t, b)}{t} \, dt.$$

By Jensen’s formula

$$(3.1) \quad N_F(r, b) = \int_T \log |F(re(\theta)) - b| \, dm(\theta) - \int_T \log |F(\frac{1}{2}e(\theta)) - b| \, dm(\theta).$$

We will prove that a.s. we have

$$\lim_{r \to 1} N_F(r, b) = \infty, \quad \forall b \in \mathbb{C},$$

which is equivalent to Theorem 1.3.

3.1. Proof of Theorem 1.3 in the Rademacher case. We define the functions $\sigma_F$ and $\hat{F}$ by

$$\sigma_F^2(r) \overset{\text{def}}{=} \sum_{n \geq 0} |a_n|^2 r^{2n}, \quad \hat{F}(z) \overset{\text{def}}{=} \frac{F(z)}{\sigma_F(|z|)},$$

and note that $\|\hat{F}(re(\theta))\|_{L^2(\mathbb{T})} = 1$.

Let $M \in \mathbb{N}$. For every $r \in \left(\frac{1}{2}, 1\right)$, the function $(\omega, b) \mapsto N_F(r, b)$ on $\Omega \times \mathbb{C}$ is measurable in $\omega$ for fixed $b$ and continuous in $b$ for fixed $\omega$. Therefore, we can find a measurable function $b^* = b^*(\omega)$ such that $|b^*| \leq M$ and

$$\inf_{|b| \leq M} N_F(r, b) \geq N_F(r, b^*) \geq N_F(r, b^*) - 1.$$
Then
\[
\inf_{|b| \leq M} N_F(r, b) \geq \int_{\mathbb{T}} \log |F(re(\theta)) - b^*| \, dm(\theta) - \int_{\mathbb{T}} \log |F(\frac{1}{M}e(\theta)) - b^*| \, dm(\theta) - 1
\]
\[
= (I_1) - (I_2) - 1.
\]

Note that
\[
(I_2) \leq \frac{1}{2} \log \left( \int_{\mathbb{T}} |F(\frac{1}{M}e(\theta)) - b^*|^2 \, dm(\theta) \right) \leq \frac{1}{2} \log \left( 2\sigma^2_F(\frac{1}{M}) + 2M^2 \right).
\]

For the integral \( (I_1) \), we have the following lower bound:
\[
(I_1) = \log \sigma_F(r) + \int_{\mathbb{T}} \log |\hat{F}(re(\theta)) - \sigma_F^{-1}(r) \cdot b^*| \, dm(\theta)
\]
\[
\geq \log \sigma_F(r) - \int_{\mathbb{T}} \log |\hat{F}(re(\theta)) - \sigma_F^{-1}(r) \cdot b^*| \, dm(\theta).
\]

If we assume that \( r \) is so close to 1 that \( \sigma_F(r) \geq 20M \), then, using our result on the logarithmic integrability of the Rademacher Fourier series (Corollary 2.2), we get
\[
\mathcal{P}\left\{ \int_{\mathbb{T}} \log |\hat{F}(re(\theta)) - \sigma_F^{-1}(r) \cdot b^*| \, dm(\theta) > T \right\}
\]
\[
\leq \frac{1}{T} \mathcal{E}\left( \int_{\mathbb{T}} \log |\hat{F}(re(\theta)) - \sigma_F^{-1}(r) \cdot b^*| \, dm(\theta) \right) \leq \frac{C}{T},
\]
for all \( T > 0 \).

Taking \( r = r_m \) so that \( \log \sigma_F(r_m) = 2m^2 \) and \( T = m^2 \), and applying the Borel-Cantelli lemma, we see that, for a.e. \( \omega \in \Omega \), there exists \( m_0 = m_0(\omega, M) \) such that, for each \( m \geq m_0 \),
\[
\int_{\mathbb{T}} \log |\hat{F}(r_m e(\theta)) - \sigma_F^{-1}(r_m) \cdot b^*| \, dm(\theta) < m^2,
\]
whence,
\[
\inf_{|b| \leq M} N_F(r_m, b) \geq m^2 - \frac{1}{2} \log \left( 2\sigma^2_F(\frac{1}{M}) + 2M^2 \right) - 1, \quad \forall m \geq m_0.
\]

Therefore, for every \( M \in \mathbb{N} \), there is a set \( A_M \subset \Omega \) with \( \mathcal{P}(A_M) = 1 \) such that, for every \( \omega \in A_M \) and every \( b \in \mathbb{C} \) with \( |b| \leq M \), we have
\[
(3.2) \quad \lim_{r \to 1} N_F(r, b) = \infty.
\]

Let \( A = \bigcap M A_M \). Then \( \mathcal{P}(A) = 1 \), and for every \( \omega \in A, b \in \mathbb{C} \), we have \( (3.2) \). Thus, the theorem is proved in the Rademacher case.

3.2. **Proof of Theorem 1.3 in the general case.** For every \( M \in \mathbb{N} \), consider the event
\[
B_M = \left\{ \omega: \lim_{r \to 1} \inf_{|b| \leq M} N_F(r, b) = +\infty \right\}.
\]

Given \( r \in \left( \frac{1}{M}, 1 \right) \), the function \( \inf_{|b| \leq M} N_F(r, b) \) is measurable in \( \omega \) (note that the infimum here can be taken over any dense countable subset of the disk \( \{|b| \leq M\} \)). Thus, the set \( B_M \) is measurable and so is the set \( B = \bigcap M B_M \), and for every \( \omega \in B, b \in \mathbb{C} \), we have \( (3.2) \). It remains to show that \( B \) holds almost surely.
To that end, we extend the probability space to $\Omega \times \Omega'$ and introduce a sequence of independent Rademacher random variables $\{\xi_n(\omega')\}, \omega' \in \Omega'$, which are also independent from the random variables $\{\zeta_n(\omega)\}, \omega \in \Omega$, and consider the random analytic function

$$G(z) = G(z; \omega, \omega') = \sum_{n \geq 0} \xi_n(\omega') \zeta_n(\omega) z^n, \quad (\omega, \omega') \in \Omega \times \Omega'.$$

By the previous section, for fixed $\zeta_n$'s (outside a set of probability zero in $\Omega$), the event

$$\left\{ \omega' \in \Omega': \lim_{r \to 1} \inf_{|b| \leq M} N_G(r, b) = +\infty \right\}
$$

occurs with probability 1. Hence, by Fubini’s theorem, the event $B_M$ occurs a.s. and so does the event $B$. Note that due to the symmetry of the distribution of $\zeta_n$'s, the random variables $\{\xi_n(\omega')\zeta_n(\omega)\}$ are equidistributed with $\{\zeta_n(\omega)\}$. This yields the theorem in the general case of symmetric random variables. \[\square\]

4. An example

In this section, we will present an example that shows that the constant 6 in the exponent on the RHS of the inequality proven in Theorem 1.1 cannot be replaced by any number smaller than 2.

Let $g_N(\theta) = (\sin(2\pi \theta))^2 = \left(\frac{e(\theta) - e(-\theta)}{2i}\right)^{2N} = \sum_{|n| \leq N} a_n e(2n\theta)$.

The function $g_N$ satisfies

$$(4.1) \quad |g_N(\theta)| \leq e^{-cN^2} \quad \text{for } |\theta| \leq e^{-CN},$$

provided that $C$ is large enough.

Now consider the Rademacher trigonometric polynomial $f_N(\theta) = \sum_{|n| \leq N} \xi_n a_n e(2n\theta)$, denote by $X_N$ the event that $\xi_n = +1$ for all $n \in \{-N, \ldots, N\}$, and put $E_N = X_N \times T_N$, where $T_N = [-e^{CN}, e^{CN}] \subset T$ is the set from (4.1). Then

$$\mu(E_N) \geq 2^{-(2N+1)} e^{-CN} \geq e^{-CN},$$

while

$$\int_{E_N} |f_N|^2 \, d\mu \leq e^{-cN^2} \mu(E_N) \leq e^{-cN^2},$$

and

$$\int_{\Omega \times T} |f_N|^2 \, d\mu = \int_T |g_N|^2 \, dm.$$

It is not difficult to see that the integral on the RHS is not less than $\frac{c}{N}$, for some constant $c > 0$. Recalling that $|\log \mu(E_N)| \leq CN$, we see that for every $\varepsilon > 0$, $C > 0$, the inequality

$$\int_Q |f_N|^2 \, d\mu \leq e^{C|\log \mu(E_N)|^{2-\varepsilon}} \int_{E_N} |f_N|^2 \, d\mu$$

fails when $N \geq N_0(\varepsilon, C)$. This shows that one cannot replace 6 by any number less than 2. \[\square\]
APPENDIX A. PROOF OF THE APPROXIMATE SPECTRUM LEMMA 2.1

The proof of Lemma 2.1 with small modifications, follows [11 Section 3.1]. We start with the following observation: if \( g \in L^2(T, \mathcal{H}) \) and \( a_0(t), \ldots, a_n(t) \) are complex numbers, then the \( m \)-th Fourier coefficient of the function

\[
x \mapsto \sum_{k=0}^n a_k(t)g_{kt}(x) = \sum_{k=0}^n a_k(t)g(x + kt)
\]
equals

\[
\hat{g}(m) \cdot \sum_{k=0}^n a_k(t)e(ktm) = \hat{g}(m) \cdot q_t(e(tm)),
\]

where \( q_t(z) = \sum_{k=0}^n a_k(t)z^k \). Slightly perturbing the coefficients \( a_k(t) \), we may assume without loss of generality that the coefficients \( a_0(t) \) and \( a_n(t) \) do not vanish for \( 0 < t < \tau \) (so that, for every \( t \) in this range, the polynomial \( q_t \) is exactly of degree \( n \) and does not vanish at the origin) and that the arguments of the roots of \( q_t \) are all distinct.

By Parseval’s theorem,

\[
(A.1) \quad \int_T \left( \left\| \sum_{k=0}^n a_k(t)g_{kt}(x) \right\|_{\mathcal{H}}^2 \right) dx = \sum_{m \in \mathbb{Z}} \left\| \hat{g}(m) \right\|_{\mathcal{H}}^2 \left| q_t(e(tm)) \right|^2.
\]

If \( g \in \text{Exp}_{\text{loc}}(n, \tau, \mathcal{H}) \), then we can choose \( a_0, \ldots, a_k \) so that the LHS of (A.1) will be small for each \( t \in (0, \tau) \). On the other hand, whenever the norm of \( \hat{g}(m) \) is large, the RHS of (A.1) can be small only when \( q_t(e(tm)) \) is small. The proof of Lemma 2.1 will be based on two facts. The first is that, on average, \( \left| q_t(e(tm)) \right| \) is relatively large outside some exceptional set, which can be covered by at most \( n \) intervals of length \( \frac{1}{4n(n+1)\tau} \). The second is that there exists a \( t_0 \) such that \( q_{t_0}(e(tm)) \) can be effectively bounded from below on this exceptional set.

We start with a lemma on arithmetic progressions.

**Lemma A.1.** Given a measurable set \( G \subset \mathbb{R}_+ \), put

\[
V_G = \left\{ t \in \left( \frac{1}{2}\tau, \tau \right) : \exists k \in \mathbb{N} \text{ s.t. } \frac{k}{t} \in G \right\}.
\]

Then \( m(V_G) < \tau^2 m(G) \).

This lemma shows that if \( m(G) < \frac{1}{2\tau} \), then there are significantly many points \( t \in \left( \frac{1}{2}\tau, \tau \right) \) such that no point \( k/t, k \in \mathbb{N}, \) belongs to \( G \).

**Proof of Lemma A.1.** We have

\[
\sum_{k \in \mathbb{N}} 1_{G} \left( \frac{k}{t} \right) \geq 1_{V_G}(t).
\]

Integrating over \( t \in \left( \frac{1}{2}\tau, \tau \right) \), we get

\[
m(V_G) \leq \int_{\tau/2}^{\tau} \sum_{k \in \mathbb{N}} 1_{G} \left( \frac{k}{t} \right) dt = \sum_{k \in \mathbb{N}} k \int_{k/\tau}^{2k/\tau} 1_{G}(s) \frac{ds}{s^2}
\]

\[
= \int_0^\infty 1_{G}(s) \left( \sum_{s\tau/2 < k < s\tau} k \right) \frac{ds}{s^2} < \tau^2 \int_0^\infty 1_{G}(s) ds = \tau^2 m(G),
\]

where \( 1_{G}(s) \) is the indicator function of \( G \).
because \( \sum_{s \tau/2 < k < s \tau} k < \tau^2 s^2 \).

The following lemma shows that the Fourier coefficients \( \hat{g}(m) \) are small outside \( n \) intervals of controlled length. Put

\[
\delta = \frac{1}{8n(n + 1)}.
\]

This choice of \( \delta \) will stay fixed till the end of the proof of Lemma 2.1.

**Lemma A.2.** There exist \( n \) intervals \( I_1, \ldots, I_n \) of length \( \frac{2\delta}{\tau} \) such that

\[
\sum_{m \in \mathbb{Z} \setminus \bigcup I_j} \|\hat{g}(m)\|_{\mathcal{H}}^2 < \left( \frac{C}{\delta} \right)^{2n} \chi^2.
\]

**Proof of Lemma A.2.** By the continuity of the shift in \( L^2(\mathbb{T}, \mathcal{H}) \), we can assume that the coefficients \( a_k(t) \) are piecewise constant functions of \( t \), and hence measurable. Then, we can integrate Parseval’s formula (A.1) over the interval \((0, \tau)\). Recalling that the LHS of (A.1) is less than \( \chi^2 \), we get

\[
\sum_{m \in \mathbb{Z}} \|\hat{g}(m)\|_{\mathcal{H}}^2 \rho^2(m) < \chi^2,
\]

where

\[
\rho^2(m) = \frac{1}{\tau} \int_0^\tau |q_t(e(tm))|^2 \, dt.
\]

Introduce the set

\[
S = \left\{ m \in \mathbb{Z} : \rho^2(m) < \frac{1}{4(n + 1)} \left( \frac{\delta}{A} \right)^{2n} \right\}.
\]

Here and elsewhere in this section, \( A \) is the positive numerical constant from the RHS of the Turán-type Lemma 2.3.1. Then Lemma A.2 will follow from the following claim:

(A.2) \( S \) cannot contain \( n + 1 \) integers \( m_1 < \ldots < m_{n+1} \) such that \( m_{j+1} - m_j > \frac{2\delta}{\tau} \), \( \forall j \in \{1, \ldots, n\} \).

Indeed, this condition yields that the set \( S \) can be covered by at most \( n \) intervals \( I_1, \ldots, I_n \) of length \( 2\delta/\tau \) and

\[
\rho^2(m) \geq \frac{1}{4(n + 1)} \left( \frac{\delta}{A} \right)^{2n}, \quad m \in \mathbb{Z} \setminus \bigcup I_j,
\]

whence

\[
\sum_{m \in \mathbb{Z} \setminus \bigcup I_j} \|\hat{g}(m)\|_{\mathcal{H}}^2 \leq 4(n + 1) \left( \frac{A}{\delta} \right)^{2n} \chi^2 < \left( \frac{C}{\delta} \right)^{2n} \chi^2
\]

with some numerical constant \( C \). Thus, we need to prove claim (A.2).

Suppose that (A.2) does not hold, i.e., there are \( n + 1 \) integers \( m_1 < \ldots < m_{n+1} \) with \( m_{j+1} - m_j > 2\delta/\tau \) that belong to the set \( S \). Then

(A.3)

\[
\int_{\tau/2}^{\tau} \sum_{j=1}^{n+1} |q_t(e(tm_j))|^2 \, dt < \frac{\tau}{4} \left( \frac{\delta}{A} \right)^{2n}.
\]
We call the value \( t \in \left( \frac{1}{2} \tau, \tau \right) \) bad if
\[
\sum_{j=1}^{n+1} |q_t(e(tm_j))|^2 < \left( \frac{\delta}{A} \right)^{2n}.
\]
Otherwise, the value \( t \) is called good. By (A.3), the measure of good \( t \)'s is less than \( \tau/4 \). In the rest of the proof we will show that the measure of bad \( t \)'s is also less than \( \tau/4 \), and this will lead us to a contradiction, which will prove Lemma [A.3].

We will use the following

Claim A.4. Let \( q(z) = \sum_{k=0}^{n} a_k z^k \) with \( \sum_{k=0}^{n} |a_k|^2 = 1 \). Given \( \Delta \in (0, 1) \), let
\[
U = \left\{ s \in \mathbb{T} : |q(e(s))| < \left( \frac{\Delta}{A} \right)^n \right\}.
\]
Then the set \( U \) is a union of at most \( n \) intervals of length at most \( \Delta \) each.

Proof of Claim A.4. \( U \) is an open subset of \( \mathbb{T} \) which consists of open intervals (since \( \Delta < 1 \) and \( A \geq 1 \) we have that \( U \neq \mathbb{T} \)). The boundary points of these intervals satisfy the equation \( |q(e(s))|^2 = \left( \frac{\Delta}{A} \right)^{2n} \), which can be rewritten as
\[
\left( \sum_{k=0}^{n} a_k z^k \right) \left( \sum_{k=0}^{n} \bar{a}_k z^{-k} \right) = \left( \frac{\Delta}{A} \right)^{2n}, \quad z = e(s).
\]
The LHS of this equation is a rational function of degree at most \( 2n \), and therefore the number of solutions is at most \( 2n \). Hence \( U \) consists of \( l \leq n \) intervals \( J_1, \ldots, J_l \), \( l \leq n \).

Next, note that since the sum of squares of the absolute values of the coefficients of \( q \) equals 1, we have \( \max_{s \in \mathbb{T}} |q(e(s))| \geq 1 \). Then, applying Lemma 2.3.1 to the exponential polynomial \( s \mapsto q(e(s)) \), we get
\[
1 \leq \sup_{s \in \mathbb{T}} |q(e(s))| \leq \left( \frac{A}{m(J_i)} \right)^n \sup_{s \in J_i} |q(e(s))| \leq \left( \frac{\Delta}{m(J_i)} \right)^n.
\]
Hence, \( m(J_i) \leq \Delta \), proving the claim. \( \square \)

Note that in the proof of this claim we did not use the full strength of Turán’s lemma. For instance, we could have used the much simpler Remez’ inequality.

Now for \( t \in \left( \frac{1}{2} \tau, \tau \right) \) consider the set
\[
S_t = \left\{ m \in \mathbb{Z} : |q_t(e(tm))| < \left( \frac{\delta}{A} \right)^n \right\}.
\]
By the previous claim (applied with \( \Delta = \delta \)), there are points \( \xi_1, \ldots, \xi_n \in \mathbb{R} \) (centers of the intervals \( J_i \)) such that, for each \( m \in S_t \), there exist \( i \in \{1, \ldots, n\} \) and \( l \in \mathbb{Z} \) such that
\[
(A.5) \quad |tm - l - \xi_i| < \frac{1}{2} \delta.
\]
Suppose that the value \( t \) is bad. Then the \( n + 1 \) integers \( m_1, \ldots, m_{n+1} \) belong to the set \( S_t \), and by the Dirichlet box principle, there are two of these integers, say \( m_{j'} \) and \( m_{j''} \) with \( j' < j'' \), that satisfy (A.5) with the same value \( i \). Then for this pair \( |t(m_{j''} - m_{j'}) - k| < \delta \), with some non-negative integer \( k \). Thus,
\[
\frac{k}{l} - \left( \frac{m_{j''} - m_{j'}}{l} \right) < \frac{\delta}{\tau} < \frac{2\delta}{\tau}.
\]
Note that since \( m_{j''} - m_{j'} > \frac{2\delta}{\tau} \), the integer \( k \) must be positive. We conclude that the set of bad values \( t \) is contained in the set \( V_G \), where \( G \) is the union of \( \frac{1}{2}n(n+1) \) intervals of length \( \frac{4\delta}{\tau} \) centered at all possible differences \( m_{j''} - m_{j'} \) with \( j'' > j' \). The measure of the set \( G \) is \( \frac{n(n+1)}{2} \cdot \frac{4\delta}{\tau} \), which, due to the choice of \( \delta \), equals \( \frac{1}{16} \). By Lemma A.1, \( m(V_G) < \tau^2m(G) \leq \frac{1}{16} \tau \). Thus, the measure of the set of bad \( t \)'s is also less than \( \frac{1}{16} \tau \), which finishes off the proof of Lemma A.2.

Proof of Lemma A.2. We need to find a set \( \Lambda = \Lambda_g \subset \mathbb{R} \) of \( n \) frequencies such that

\[
\sum_{m \in \mathbb{Z}} \| \hat{g}(m) \|_\mathcal{H}^2 \Theta_{r,\Lambda}^2(m) \leq (Cn)^4n^2,
\]

where

\[
\Theta_{r,\Lambda}(m) = \prod_{\lambda \in \Lambda} \theta_r(m - \lambda), \quad \theta_r(m) = \min(1, |r|m!).
\]

By Lemma A.2 there exists a collection of \( n \) intervals \( \{I_j\} \), each of length \( \frac{2\delta}{\tau} \), such that

\[
\sum_{m \in \mathbb{Z} \setminus \bigcup_j I_j} \| \hat{g}(m) \|_\mathcal{H}^2 \Theta_{r,\Lambda}^2(m) \leq \sum_{m \in \mathbb{Z} \setminus \bigcup_j I_j} \| \hat{g}(m) \|_\mathcal{H}^2 \leq (Cn)^4n^2.
\]

Therefore, it remains to estimate the sum

\[
\sum_{m \in \bigcup_j I_j} \| \hat{g}(m) \|_\mathcal{H}^2 \Theta_{r,\Lambda}^2(m).
\]

By Parseval’s identity (A.1), for every \( t \in (0, \tau) \),

\[
\sum_{m \in \bigcup_j I_j} \| \hat{g}(m) \|_\mathcal{H}^2 |q_t(e(tm))|^2 < \tau^2.
\]

Hence, it suffices to show that there exist a value \( t_0 \in (0, \tau) \) and a set \( \Lambda \) of \( n \) real numbers such that \( |q_{t_0}(e(t_0m))| \geq \delta^n \Theta_{r,\Lambda}(m) \) for every \( m \in \bigcup_j I_j \).

First, we bound the absolute value of the polynomial \( q_t \) from below by the absolute value of another polynomial \( p \) whose zeroes are obtained from the zeroes of \( q_t \) by the radial projection to the unit circle.

Claim A.6. Let \( z_j \neq 0 \) for \( 1 \leq j \leq n \), and let \( g(z) = c \cdot \prod_{j=1}^{n}(z - z_j) \) be a polynomial of degree \( n \) such that \( \sup_{|z|=1} |g(z)| \geq 1 \). Let \( h(z) = \prod_{j=1}^{n}(z - \zeta_j) \), where \( \zeta_j = z_j/|z_j| \). Then, for every \( z \in \mathbb{T} \),

\[
|h(z)| \leq 2^n |g(z)|.
\]

Proof of Claim A.6. The ratio \( \frac{|z - \zeta_j|}{|z - z_j|} \) attains its maximum on \( \{|z| = 1\} \) at the point \( z = -\zeta_j \), where it is equal to \( \frac{2}{1 + |z_j|} \). Therefore,

\[
\frac{|h(z)|}{|g(z)|} \leq \frac{1}{|c|} \prod_{j=1}^{n} \frac{2}{1 + |z_j|}.
\]
By our assumption, there is some \( z' \), \( |z'| = 1 \), such that \( |g(z')| \geq 1 \). Hence,

\[
1 \leq |c| \prod_{j=1}^{n} |z' + z_j| \leq |c| \prod_{j=1}^{n} (1 + |z_j|) .
\]

Overall, we have

\[
|h(z)| \leq 2^n |g(z)| \cdot \prod_{j=1}^{n} \frac{1}{1 + |z_j|} \leq 2^n |g(z)| ,
\]

proving the claim. \( \square \)

Recall that \( \sup |q_t(z)| \geq 1 \). Hence, applying Claim A.6 we conclude that \( |q_t(z)| \geq 2^{-n} |p_t(z)| \) for \( |z| = 1 \), where \( p_t \) is a monic polynomial of degree \( n \) with all its zeroes on the unit circle.

To choose \( t_0 \), we consider \( n \) intervals \( \tilde{I}_j \) of length \( 4\delta \tau^{-1} \) with the same centers as the intervals \( I_j \) of Lemma A.2 and put \( \tilde{S} = \bigcup_j \tilde{I}_j \). Let \( \tilde{G} = \bar{S} - \tilde{S} \) be the difference set, with \( m(\tilde{G}) \leq 8\delta \tau^{-1} \cdot n^2 \). We call the value \( t \in \left( \frac{4}{7} \tau, \tau \right) \) bad if there exists an integer \( k \neq 0 \) such that \( k/t \in \tilde{G} \). Since the set \( \tilde{G} \) is symmetric with respect to 0, we can estimate the measure of bad \( t \)'s by applying Lemma A.1 to the set \( \tilde{G} \cap \mathbb{R}_+ \). Then the measure of bad values of \( t \) is less than \( \tau^2 \cdot \frac{1}{2} m(\tilde{G}) \leq 4\delta \tau \cdot n^2 < \frac{1}{2} \tau \), since \( \delta \cdot 8n^2 < 1 \). Therefore, there exists at least one good value \( t_0 \in \left( \frac{4}{7} \tau, \tau \right) \) for which every arithmetic progression with difference \( t_0^{-1} \) has at most one point in \( \tilde{S} \). We fix this value \( t_0 \) till the end of the proof.

To simplify notation, we put \( p = p_{t_0} \). The zero set of the function \( x \mapsto p(e(t_0 x)) \) consists of \( n \) arithmetic progressions with difference \( t_0^{-1} \). By the choice of \( t_0 \), at most \( n \) zeroes of this function belong to the set \( \tilde{S} \). We denote these zeroes by \( \lambda_1, \ldots, \lambda_\ell, \ell \leq n \). If \( \ell < n \), we choose \( n - \ell \) zeroes \( \lambda_{t+1}, \ldots, \lambda_n \) in \( \mathbb{R} \setminus \tilde{S} \) so that \( \{ e(t_0 \lambda_j) \}_{1 \leq j \leq n} \) is a complete set of zeroes of the algebraic polynomial \( p \); we recall that these zeroes are all distinct.

It remains to define a set \( \Lambda \) of \( n \) numbers, and to estimate from below \( |p(e(t_0 m))| \) when \( m \in \bigcup_j I_j \). Denote by \( d_j(m) \) the distance from the integer \( m \) to the nearest point in the arithmetic progression \( \{ \lambda_j + k t_0^{-1} \}_{k \in \mathbb{Z}} \). We have

\[
|p(e(t_0 m))| = 2^n \prod_{j=1}^{n} |\sin(\pi t_0 (m - \lambda_j))| \geq 2^n \prod_{j=1}^{n} (2t_0 d_j(m)) \geq 2^n \tau^n \prod_{j=1}^{n} d_j(m).
\]

We put \( \Lambda = \{ \lambda_j \}_{1 \leq j \leq n} \). Recall that here \( m \in \bigcup_j I_j \), \( \tilde{S} = \bigcup_j \tilde{I}_j \), and that the arithmetic progression \( \{ \lambda_j + k t_0^{-1} \}_{k \in \mathbb{Z}} \) either misses the set \( \tilde{S} \), or has at most one element in \( \tilde{S} \). In the first case, we get \( d_j(m) \geq \delta \tau^{-1} \), while in the second case, \( d_j(m) \geq \min \{ \frac{\delta}{\tau}, |m - \lambda_j| \} \). Therefore, in both cases,

\[
d_j(m) \geq \min \{ \frac{\delta}{\tau}, |m - \lambda_j| \} \geq \frac{\delta}{\tau} \min \{ 1, \tau |m - \lambda_j| \} = \frac{\delta}{\tau} \cdot \theta_{\tau}(m - \lambda_j).
\]
Tying the ends together, we get

\[
|q(t_0(e(t_0m))| \geq 2^{-n} |p(e(t_0m))| \geq 2^{-n} \cdot 2^n \tau^n \prod_{j=1}^{n} d_j(m) \\
\geq \tau^n \cdot \left( \frac{\delta}{\tau} \right)^n \Theta_{\tau,\Lambda}(m) = \delta^n \Theta_{\tau,\Lambda}(m).
\]

This completes the proof of Lemma 2.1.

Appendix B. Proof of the lemma 2.2 on the local approximation

The proof of Lemma 2.2 is very close to the proof of the corresponding result in [11, Section 3.2]. We start with a lemma on solutions of ordinary differential equations (cf. Lemma 3.2 in [11]).

**Lemma B.1.** Let

\[
D = \prod_{j=1}^{n} e(\lambda_j x) \frac{d}{dx} e(-\lambda_j x) \quad \lambda_1, \ldots, \lambda_n \in \mathbb{R}, \quad \lambda_i \neq \lambda_j \text{ for } i \neq j
\]

be a differential operator of order \( n \geq 1 \), and let \( J \subset [0, 1] \) be an interval. Suppose that \( f \in L^2(\Omega \times J) \) and, for a.e. \( \omega \in \Omega, \ x \mapsto f(\omega, x) \) is a \( C^n(J) \)-function satisfying the differential equation \( Df = h \) with \( h \in L^2(\Omega \times J) \).

Then there exists an exponential polynomial \( p \) with spectrum \( \lambda_1, \ldots, \lambda_n \), such that, for a.e. \( \omega \in \Omega, \)

\[
\sup_{x \in J} |f(\omega, x) - p(\omega, x)| \leq m(J)^n \frac{1}{m(J)} \int_{J} |h(\omega, x)| \, dx.
\]

**Proof of Lemma B.1.** Let \( \varphi \) be a particular solution of the equation \( D\varphi = h \) constructed by repeated integration:

\[
\varphi = \left( \prod_{j=1}^{n} e(\lambda_j x) J e(-\lambda_j x) \right) h
\]

where \( J \) is the integral operator

\[
(J \psi)(\omega, x) = \int_a^x \psi(\omega, t) \, dt
\]

and \( a \) is the left end-point of the interval \( J \). Then, for a.e. \( \omega, \)

\[
|\varphi(\omega, x)| \leq m(J)^n \frac{1}{m(J)} \int_{J} |h(\omega, x)| \, dx.
\]

The function \( f - \varphi \) satisfies the homogeneous equation \( D(f - \varphi) = 0 \). Hence, \( p = f - \varphi \) is an exponential polynomial with coefficients depending on \( \omega, \)

\[
p(\omega, x) = \sum_{j=1}^{n} c_j(\omega) e(\lambda_j x).
\]

Now we turn to the proof of Lemma 2.2. We fix a function \( g \in \text{Exp}^{\text{loc}}(n, \tau, \kappa, L^2(\Omega)) \). By Lemma 2.1, this function has an “approximate spectrum” \( \Lambda = \Lambda_g = \{\lambda_j\}_{1 \leq j \leq n} \) so that

\[
\sum_{m \in \mathbb{Z}} \|\hat{g}(m)\|^2 \Theta^2_{\tau,\Lambda}(m) \leq (Cn)^{4n} \kappa^2,
\]
with
\[ \Theta_{\tau,\Lambda}(m) = \prod_{\lambda \in \Lambda} \theta_{\tau}(m - \lambda), \quad \theta_{\tau}(m) = \min(1, \tau | m |). \]

We fix \( M > 1 \) so that \( 1/(M \tau) \) is a positive integer, and partition \( \mathbb{T} \) into intervals \( J \) of length \( M \tau \).

Put
\[ I_k = \left( \lambda_k - \frac{1}{2}, \lambda_k + \frac{1}{2} \right), \quad \tilde{I}_k = \left( \lambda_k - \frac{2}{2}, \lambda_k + \frac{2}{2} \right), \quad E_0 = \mathbb{R} \setminus \bigcup_{k=1}^{n} I_k, \quad E_k = I_k \setminus \bigcup_{j=1}^{k-1} I_j. \]

The sets \( E_k, 0 \leq k \leq n \), form a partition of the real line. Accordingly, we decompose \( g \) into the sum \( g = \sum_{k=0}^{n} g_k \), where \( g_k \) is the projection of \( g \) onto the closed subspace of \( L^2(Q) \) that consists of functions with spectrum contained in \( E_k \). For each \( k = 0, \ldots, n \), we have
\[ \sum_{m \in \mathbb{Z}} \| \hat{g}_k(m) \|_{L^2(\Omega)}^2 \Theta_{\tau,\Lambda}^2(m) < (Cn)^{\kappa} \chi^2 \equiv \tilde{\chi}^2. \]

Since, for \( m \in E_0 \), \( \Theta_{\tau,\Lambda}^2(m) = 1 \), we get \( \| g_0 \|_{L^2(Q)} \leq \tilde{\chi} \).

Now let \( 1 \leq k \leq n \). Let \( n_k \) denote the number of points \( \lambda_j \) lying in \( \tilde{I}_k \). We define a differential operator \( D_k \) of order \( n_k \) by
\[ D_k \overset{\text{def}}{=} \prod_{\lambda_j \in \tilde{I}_k} e(\lambda_j x) \frac{d}{dx} e(-\lambda_j x). \]

The function \( g_k(x) \) is a trigonometric polynomial with coefficients depending on \( \omega \), hence, for a.e. \( \omega \), it is an infinitely differentiable function of \( x \). We set \( h_k \overset{\text{def}}{=} D_k g_k \).

Note that this is a trigonometric polynomial with the same frequencies as \( g_k \):
\[ \hat{h}_k(\omega, m) = (2\pi)^{n_k} \hat{g}_k(\omega, m) \prod_{\lambda_j \in \tilde{I}_k} (m - \lambda_j). \]

Consequently,
\[ |\hat{h}_k(\omega, m)| = (2\pi)^{n_k} |\hat{g}_k(\omega, m)| \prod_{\lambda_j \in \tilde{I}_k} |m - \lambda_j|. \]

In the product on the RHS, \( m \in E_k \subset I_k \) and \( \lambda_j \in \tilde{I}_k \). Recalling the definition of the function \( \theta_{\tau} \), we see that
\[ |m - \lambda_j| \leq \frac{2}{\tau} \theta_{\tau}(m - \lambda_j) \quad \text{for} \quad m \in I_k, \lambda_j \in \tilde{I}_k. \]

Therefore,
\[ |\hat{h}_k(\omega, m)| \leq \left( \frac{6\pi}{\tau} \right)^{n_k} |\hat{g}_k(\omega, m)| \prod_{\lambda_j \in \tilde{I}_k} \theta_{\tau}(m - \lambda_j). \]

Note that for \( m \in E_k \) and for \( \lambda_j \in \mathbb{Z} \setminus \tilde{I}_k \), we have \( \theta_{\tau}(m - \lambda_j) = 1 \). Thus,
\[ |\hat{h}_k(\omega, m)| \leq \left( \frac{6\pi}{\tau} \right)^{n_k} |\hat{g}_k(\omega, m)| \Theta_{\tau,\Lambda}(m), \quad \omega \in \Omega, \]
whence, recalling estimate (B.1), we obtain
\[ \| h_k \|_{L^2(Q)} \leq \left( \frac{6\pi}{\tau} \right)^{n_k} \tilde{\chi}. \]
Applying Lemma 3.1 to an interval $J$ of length $M\tau$, we obtain an exponential polynomial $p_k^J$ with spectrum consisting of frequencies $\lambda_j \in \widehat{I}_k$ and with coefficients depending on $\omega$, such that, for every $x \in J$ and almost every $\omega \in \Omega$,

$$|g_k(\omega, x) - p_k^J(\omega, x)| \leq (M\tau)^{n_k} \cdot \frac{1}{M\tau} \int_J |h_k(\omega, t)| \, dt.$$  

We denote by

$$\mathfrak{M} f(\omega, x) = \sup_{L, x \in L} \frac{1}{m(L)} \int_L |f(\omega, t)| \, dt$$

the Hardy-Littlewood maximal function. The supremum is taken over all intervals $L \subset [0, 1]$ containing $x$, but it is easy to see that it is enough to restrict ourselves to the intervals with rational endpoints, which allows us to rewrite $\mathfrak{M} f$ as $\sup \{ F_{\alpha,\beta} : \alpha, \beta \in \mathbb{Q} \}$, where

$$F_{\alpha,\beta}(\omega, x) = \mathbb{1}_{[\alpha,\beta]}(x)G_{\alpha,\beta}(\omega) \quad \text{and} \quad G_{\alpha,\beta}(\omega) = \frac{1}{\beta - \alpha} \int_\alpha^\beta |f(\omega, t)| \, dt.$$  

By the Fubini theorem, $G_{\alpha,\beta}$ are measurable functions on $\Omega$, so $F_{\alpha,\beta}$ are measurable functions on $\mathbb{Q}$ and, thereby, $\mathfrak{M}$ is measurable on $\mathbb{Q}$ as well.

Let $\tilde{h}_k = \tau^{n_k}h_k$. Then

$$|g_k(\omega, x) - \tilde{p}_k^J(\omega, x)| \leq M^{n_k} \cdot \mathfrak{M}\tilde{h}_k(\omega, x).$$

Using the classical estimate for the $L^2$-norm of the maximal function, we get, for a.e. $\omega$,

$$\int_T \left[ \mathfrak{M}\tilde{h}_k(\omega, x) \right]^2 \, dx \leq C \int_T |\tilde{h}_k(\omega, x)|^2 \, dx.$$  

Recalling that $\|\tilde{h}_k\|_{L^2(\mathbb{T})} < C^n \tilde{\tau}$, we obtain

$$\|\mathfrak{M}\tilde{h}_k\|_{L^2(\mathbb{T})}^2 = \int_{\Omega \times \mathbb{T}} \left[ \mathfrak{M}\tilde{h}_k(\omega, x) \right]^2 \, dP(\omega) \leq C \|\tilde{h}_k\|_{L^2(\mathbb{T})}^2 \leq C^{2n} \tilde{\tau}^2.$$  

We now set $p^J \overset{\text{def}}{=} \sum_{k=1}^n p_k^J$. Notice that all the frequencies of the polynomial $p^J$ belong to the set $\Lambda_g$. Then, for every $x \in J$,

$$|g(\omega, x) - p^J(\omega, x)| \leq |g_0(\omega, x)| + \sum_{k=1}^n |g_k(\omega, x) - p_k^J(\omega, x)|$$

$$\leq |g_0(\omega, x)| + M^n \sum_{k=1}^n \mathfrak{M}\tilde{h}_k(\omega, \theta)$$

$$\leq M^n \left( |g_0(\omega, x)| + \sum_{k=1}^n \mathfrak{M}\tilde{h}_k(\omega, x) \right) \overset{\text{def}}{=} M^n \Phi(\omega, x).$$

It remains to bound the norm of the “error function” $\Phi$:

$$\|\Phi\|_{L^2(\mathbb{Q})} \leq \|g_0\|_{L^2(\mathbb{Q})} + \sum_{k=1}^n \|\mathfrak{M}\tilde{h}_k\|_{L^2(\mathbb{Q})} \leq \tilde{\tau} + \sum_{k=1}^n C^{n_k} \tilde{\tau} \leq C^n \tilde{\tau} \leq (Cn)^{2n} \tilde{\tau}.$$  

This proves the desired result. \qed
Appendix C. Proof of the spreading lemma

Till the end of this section, we fix the function \( g \in \text{Exp}_{\text{loc}}(\tau, \omega, L^2(\Omega)) \), the set \( E \subset Q \) of positive measure, and the “random constant” \( b \in L^2(\Omega) \).

We will use two parameters, \( M > 1, \frac{1}{M} \in \mathbb{N} \) and \( \gamma \in (0, 1) \); their specific values will be chosen later in the proof.

**Definition:** Let \( J \) be an interval of length \( M \tau \) in the partition of \( \mathbb{T} \). The interval \( J \) is called \( \omega \)-white if \( m(J \cap E_\omega) \geq \gamma m(J) \); otherwise it is called \( \omega \)-black.

Given \( \omega \), the union of all \( \omega \)-white intervals will be denoted by \( W_\omega \). By \( W \subset Q \) we denote the union of all sets \( W_\omega \). Similarly, we denote by \( B_\omega \) the union of all \( \omega \)-black intervals and by \( B \subset Q \) the union of all sets \( B_\omega \). Since we can write the set \( W \) as

\[
\bigcup \{ \omega : m(J \cap E_\omega) \geq \gamma m(J) \} \times J
\]

and the function \( \omega \mapsto m(J \cap E_\omega) \) is measurable on \( \Omega \) for every interval \( J \) in the partition, we see that \( W \) and \( B = Q \setminus W \) are measurable subsets of \( Q \).

Let \( \Phi \) be the error function given by the Local Approximation Lemma. The next lemma enables us to extend our estimates for \( g - b \) from the set \( E \) to the set \( W \).

**Lemma C.1.** We have

\[
\int_W |g - b|^2 \, d\mu \leq \left( \frac{C}{\gamma} \right)^{2n+1} \left[ \int_{W \cap E} |g - b|^2 \, d\mu + M^{2n+1} \int_W \Phi^2 \, d\mu \right].
\]

**Proof of Lemma C.1.** Let \( J \) be one of the \( \omega \)-white intervals of length \( M \tau \). By Lemma 2.2 for almost every \( \omega \in \Omega \) and every \( \theta \in J \), we have

\[
|(g(\omega, \theta) - b(\omega)) - (p^J(\omega, \theta) - b(\omega))| = |g(\omega, \theta) - p^J(\omega, \theta)| \leq M^n \Phi(\omega, \theta),
\]

where \( \Phi(\omega, \theta) \) is a exponential polynomial with \( n \) frequencies and coefficients depending on \( \omega \). Therefore,

\[
(C.1) \quad \int_J |g - b|^2 \, d\theta \leq 2 \left( \int_J |p^J - b|^2 \, d\theta + M^{2n} \int_J \Phi^2 \, d\theta \right).
\]

Applying the \( L^2 \)-version of the Turán-type lemma to the exponential polynomial \( p^J - b \), which has at most \( n + 1 \) frequencies, we get

\[
\int_J |p^J - b|^2 \, d\theta \leq \left( \frac{C m(J)}{m(J \cap E_\omega)} \right)^{2n+1} \int_{J \cap E_\omega} |p^J - b|^2 \, d\theta \leq \left( \frac{C}{\gamma} \right)^{2n+1} \int_{J \cap E_\omega} |p^J - b|^2 \, d\theta.
\]

Plugging this into (C.1), we find that

\[
\int_J |g - b|^2 \, d\theta \leq \left( \frac{C}{\gamma} \right)^{2n+1} \int_{J \cap E_\omega} |p^J - b|^2 \, d\theta + 2M^{2n} \int_J \Phi^2 \, d\theta.
\]

Summing these estimates over all \( \omega \)-white intervals \( J \), and using that

\[
|p^J - b| \leq |g - b| + |g - p_J| \leq |g - b| + M^n \Phi,
\]

we get

\[
\int_{W_\omega} |g - b|^2 \, d\theta \leq \left( \frac{C}{\gamma} \right)^{2n+1} \left[ \int_{W_\omega \cap E_\omega} |g - b|^2 \, d\theta + M^{2n} \int_{W_\omega} \Phi^2 \, d\theta \right].
\]
Integrating over $\omega$, we get the result. □

The effectiveness of this lemma depends on the size of the set $W \cap E^c$. The following lemma is very similar to Lemma 3.4 from [11]. For the reader’s convenience, we reproduce its proof. Recall that $\Delta_{n\tau}(E) = \mu((E + n\tau) \setminus E)$.

**Lemma C.2.** For $\gamma < \frac{1}{2}$,

$$
\mu(W \cap E^c) \geq \Delta_{n\tau}(E) - \left( \gamma + \frac{n}{M} \right).
$$

**Proof of Lemma C.2.** We have

$$
m((E_\omega + n\tau) \setminus E_\omega) = m((E_\omega + n\tau) \cap E_\omega^c) = m((E_\omega + n\tau) \cap E_\omega^c \cap W_\omega) + m((E_\omega + n\tau) \cap E_\omega^c \cap B_\omega) \leq m(W_\omega \cap E_\omega^c) + m((E_\omega + n\tau) \cap E_\omega^c \cap B_\omega).
$$

We need to estimate the second term on the RHS.

If the interval $J$ is $\omega$-black, then

$$
m(J \cap E_\omega^c \cap (E_\omega + n\tau)) \leq m(J \cap (E_\omega + n\tau)) \leq m(J \setminus (J + n\tau)) + m((E_\omega + n\tau) \cap (J + n\tau)) \leq n\tau + m(E_\omega \cap J) < n\tau + n\gamma m(J) = \left( \frac{n\tau}{m(J)} + \gamma \right) m(J).
$$

Summing this inequality over all $\omega$-black intervals $J$, and recalling that $m(J) = M\tau$, we obtain

$$
m((E_\omega + n\tau) \cap E_\omega^c \cap B_\omega) \leq \left( \frac{n\tau}{M\tau} + \gamma \right) m(B_\omega) \leq \frac{n}{M} + \gamma.
$$

Integrating over $\Omega$ we get the required result. □

**Proof of Lemma C.3.** We write $\Delta = \Delta_{n\tau}(E)$ and put

$$
M_1 = \frac{8n}{\Delta}.
$$

We consider two cases, according to whether $M_1\tau \leq 1$ or not.

In the first case, we choose $M \in [M_1, 2M_1]$, so that $1/(M\tau)$ is an integer. Notice that $M > 1$. We set $\gamma = \frac{1}{8} \Delta < \frac{1}{2}$ and let $\bar{E} = E \cup (W \cap E^c) = E \cup W$, where $W$ is the union of the corresponding white intervals. By Lemma C.2

$$
\mu(W \cap E^c) \geq \Delta - \left( \gamma + \frac{n}{M} \right) \geq \Delta - \left( \frac{\Delta}{8} + \frac{\Delta}{8} \right) > \frac{\Delta}{2}.
$$

Furthermore, using Lemma C.1, we get

$$
\int_W |g - b|^2 \, d\mu \leq \left( \frac{C}{\gamma} \right)^{2n+1} \left[ \int_{W \cap E} |g - b|^2 \, d\mu + M^{2n} \int_W \Phi^2 \, d\mu \right].
$$

Plugging in the values of the parameters $\gamma$ and $M$ and taking into account the bound on the norm of $\Phi$, we find that the RHS is

$$
\leq \left( \frac{C}{\Delta} \right)^{2n+1} \left[ \int_{W \cap E} |g - b|^2 \, d\mu + \left( \frac{Cn^2}{\Delta} \right)^{2n} \int_W \Phi^2 \, d\mu \right]
\leq \left( \frac{C}{\Delta} \right)^{2n+1} \left[ \int_E |g - b|^2 \, d\mu + \left( \frac{Cn^2}{\Delta} \right)^{2n} \mu^2 \right]
\leq \left( \frac{Cn^2}{\Delta^2} \right)^{2n+1} \left[ \int_E |g - b|^2 \, d\mu + \mu^2 \right].
$$
Now we consider the second case, when $M_1 \tau > 1$. We set $M = \frac{1}{\tau}$ (that is, there is only one interval in the ‘partition’) and note that

$$M = \frac{1}{\tau} < M_1 = \frac{8n}{\Delta}.$$  

We set $\gamma = \frac{\Delta}{2}$, and once again $\tilde{E} = E \cup (W \cap E^c) = E \cup W$. Similarly to the first case, Lemma [C.1] gives us

$$\int_W |g - b|^2 \, d\mu \leq \left( \frac{Cn^3}{\Delta^2} \right)^{2n+1} \left[ \int_E |g - b|^2 \, d\mu + \gamma^2 \right].$$

We now show that there are sufficiently many $\omega$-white intervals that contain a noticeable portion of $E^c$. We define the function $\delta(\omega) = m((E_\omega + n\tau) \setminus E_\omega)$ and notice that

$$\int_{\Omega} \delta(\omega) \, dP(\omega) = \Delta.$$  

Let $L = \{ \omega \in \Omega : \delta(\omega) > \frac{\Delta}{2} \}$. It is clear that

$$\int_L \delta(\omega) \, dP(\omega) \geq \frac{\Delta}{2}.$$  

For $\omega \in L$ we have that $m(E^c_\omega), m(E_\omega) \geq \delta(\omega) > \frac{\Delta}{2} = \gamma$, and therefore $L \times T \subseteq W$. Thus $(L \times T) \cap E^c \subseteq W \cap E^c$ and

$$m(W \cap E^c) \geq m((L \times T) \cap E^c) = \int_L m(E^c_\omega) \, dP(\omega) \geq \int_L \delta(\omega) \, dP(\omega) \geq \frac{\Delta}{2},$$

proving the lemma.  \[\square\]

References

[1] L. Erdős, A. Knowles, H.-T. Yau, J. Yin, Delocalization and diffusion profile for random band matrices. arXiv:1205.5669v1
[2] S. Yu. Favorov, Growth and distribution of the values of holomorphic mappings of a finite-dimensional space into a Banach space. Siberian Math. J. 31 (1990), 137–146.
[3] S. Yu. Favorov, On the growth of holomorphic mappings from a finite-dimensional space into a Banach space. Mat. Fiz. Anal. Geom. 1 (1994), 240–251.
[4] J. P. Kahane, Some Random Series of Functions. 2-nd edition. Cambridge Univ. Press, Cambridge, 1985.
[5] M. Jacob, A. C. Offord, The distribution of the values of a random power series in the unit disk. Proc. Roy. Soc. Edinburgh Sect. A 94 (1983), 251–263.
[6] B. Ya. Levin, Distribution of Zeros of Entire Functions. (Russian) Gosudarstv. Isdat. Tehn.-Teor. Lit., Moscow, 1956. English translation: Amer. Math. Soc., Providence, R.I., 1980.
[7] J. E. Littlewood, A. C. Offord, On the number of real roots of random algebraic equation. (I) J. London Math. Soc. 13 (1938), 288-295; (II) Proc. Cambridge Phil. Soc. 35 (1939), 133-148.
[8] J. E. Littlewood, A. C. Offord, On the distribution of zeros and $a$-values of a random integral function. (II). Ann. of Math. (2) 49, (1948), 885–952; errata 50, (1949), 990–991.
[9] T. Murai, The value-distribution of lacunary series and a conjecture of Paley. Ann. Inst. Fourier (Grenoble) 31 (1981), 135–156.
[10] T. Murai, The value-distribution of random Taylor series in the unit disk. J. London Math. Soc. (2) 24 (1981), 480–494.
[11] F. Nazarov, Local estimates for exponential polynomials and their applications to inequalities of the uncertainty principle type. (Russian) Algebra & Analiz 5 (1993), no. 4, 3–66; English translation in St. Petersbourg Math. J. 5 (1994), 663–717.
[12] F. Nazarov, Summability of large powers of logarithm of classic lacunary series and its simplest consequences. Preprint, 1995.
[13] A. C. Offord, The distribution of the values of an entire function whose coefficients are independent random variables. (I) Proc. London Math. Soc. (3) 14a (1965), 199–238.
[14] A. C. Offord, The distribution of zeros of power series whose coefficients are independent random variables. Indian J. Math. 9 (1967), 175–196.
[15] A. C. Offord, The distribution of values of a random function in the unit disk. Studia Math. 41 (1972), 71–106.
[16] A. C. Offord, The distribution of the values of an entire function whose coefficients are independent random variables. (II). Math. Proc. Cambridge Phil. Soc. 118 (1995), 527–542.
[17] D. C. Ullrich, An extension of the Kahane-Khinchine inequality in a Banach space. Israel J. Math. 62 (1988), 56–62.
[18] D. C. Ullrich, Khinchin’s inequality and the zeroes of Bloch functions. Duke Math. J. 57 (1988), 519–535.

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