Supercoherent states approach to the SUSY harmonic oscillator

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Dedicated to Professor Bogdan Mielnik for his 50 years of scientific career

Abstract. The nonlinear supercoherent states, associated with a nonlinear generalization of the Kornbluth-Zypman (KZ) supersymmetric annihilation operator (SAO) of the supersymmetric harmonic oscillator, will be studied. We discuss as well the Heisenberg uncertainty relation \( \sigma_x^2 \sigma_p^2 \) for a special case which will allow us to compare our results with those obtained for the KZ linear supercoherent states.

1. Introduction
For the standard harmonic oscillator, the commutators between the Hamiltonian \( \hat{H} \) with the annihilation and creation operators \( \hat{a}, \hat{a}^\dagger \) generate the well known Heisenberg-Weyl algebra.

On the other hand, for the SUSY harmonic oscillator \([1, 2]\) the analysis of the commutation relations between the Hamiltonian \( \hat{H}_s \)

\[
\hat{H}_s = \omega \begin{pmatrix} \hat{a}^\dagger \hat{a} & 0 \\ 0 & \hat{a} \hat{a}^\dagger \end{pmatrix}
\]

and the corresponding creation \( \hat{A}^\dagger \) and annihilation operators \( \hat{A} \) of the system requires the knowledge of the explicit form of these operators \([3, 4]\).

Recently \([5]\), a quite general expression for \( \hat{A} \) was introduced and its eigenvectors \( |Z\rangle \) with complex eigenvalues \( z \) were found (called supercoherent states), which became expressed in terms of the standard harmonic oscillator coherent states \([6, 7]\).

Despite the expression proposed in \([5]\) for \( \hat{A} \) is very general, it is not unique. In this work, we will consider some specific deformations \( \hat{Q} \), which maintain the structure given in \([5]\). We will analyze also a particular deformation of the SAO, and we will find the explicit form of its eigenvectors \( |X\rangle \) with complex eigenvalues \( X \). Due to the deformation assumed, the eigenstates of \( \hat{Q} \) turn out to be expressed in terms of nonlinear coherent states associated to deformed (nonlinear) Lie algebras \([8–11]\), so they are called nonlinear supercoherent states.
2. Standard coherent states (CS)

A standard coherent state $|\alpha\rangle$ can be defined as an eigenstate of the annihilation operator $\hat{a}$ with complex eigenvalue $\alpha$.

In the Fock basis, the normalized coherent states read

$$ |\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle. $$

(2)

Other definitions can be used to build this type of quantum states, which are equivalent to each other for the harmonic oscillator; nevertheless, in this work we will use the previous definition.

3. Nonlinear coherent states (NLCS)

The nonlinear coherent states $|z\rangle_f$ can be defined as eigenstates of a deformed annihilation operator $\tilde{a} = f(\hat{N})\hat{a}$ with complex eigenvalue [12–15], where $f(\hat{N})$ is a well behaved real function of the number operator $\hat{N} = \hat{a}^{\dagger}\hat{a}$.

The structure of these states depends on which Fock states $|n\rangle$ are annihilated by $\tilde{a}$; this is determined by the explicit form of the function $f(\hat{N})$ since

$$ \tilde{a}|n\rangle = \sqrt{n}f(n-1)|n-1\rangle, \quad n = 0, 1, 2, \ldots $$

(3)

We will consider the following linear form $f(\hat{N}) = \hat{N} - k\hat{1}$, where $k \in \mathbb{N} \cup \{-1, 0\}$. So, the nonlinear coherent states $|\alpha\rangle_{(k)}$ associated to the operator $\tilde{a}_{(k)} = (\hat{N} - k\hat{1})\hat{a}$ turn out to be

$$ |\alpha\rangle_{(k)} = \left[ _0F_2(1, k + 2, r^2) \right]^{1/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\Gamma(n+1)\sqrt{\Gamma(n+k+2)}} |n+k+1\rangle, $$

(4)

where $r = |\alpha|$ and $_pF_q$ is the generalized hypergeometric function

$$ _pF_q(a_1, \ldots, a_p, b_1, \ldots, b_q; x) = \frac{\Gamma(b_1) \cdots \Gamma(b_q)}{\Gamma(a_1) \cdots \Gamma(a_p)} \sum_{n=0}^{\infty} \frac{\Gamma(a_1 + n) \cdots \Gamma(a_p + n)}{\Gamma(b_1 + n) \cdots \Gamma(b_q + n)} \frac{x^n}{n!}. $$

(5)

Note that in Eq. (4) the contribution of the lowest energy Fock states to the nonlinear coherent state depends on the parameter $k$, i.e., somehow it is possible to isolate the lowest energy eigenstates according to the value of $k$.

For instance, for $f(\hat{N}) = \hat{N} + 1$ ($k = -1$) it turns out that $f(n-1) = n \neq 0, \forall n = 1, 2, \ldots$ which implies that $\tilde{a}|0\rangle = 0$. Thus:

$$ |\alpha\rangle_{nl} \equiv |\alpha\rangle_{(-1)} = \left[ _0F_2(1, 1; r^2) \right]^{-1/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\Gamma(n+1)\sqrt{\Gamma(n+1)}} |n\rangle. $$

(6)

On the other hand, for $f(\hat{N}) = \hat{N}$ ($k = 0$) we have that $f(0) = 0$. This implies that $\tilde{a}|0\rangle = \tilde{a}|1\rangle = 0$, and the corresponding NLCS become now

$$ |\alpha\rangle_{NL} \equiv |\alpha\rangle_{(0)} = \left[ _0F_2(1, 2; r^2) \right]^{-1/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\Gamma(n+1)\sqrt{\Gamma(n+2)}} |n+1\rangle. $$

(7)

This result indicates that the contribution to $|\alpha\rangle_{NL}$ of the ground state is removed, since this eigenstate is isolated from the remaining ones, due to $\tilde{a}|0\rangle = \tilde{a}|1\rangle = 0$. 

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Note: The above text is a continuation of the previous discussion on nonlinear coherent states. It introduces the concept of standard coherent states, followed by the introduction of nonlinear coherent states, and explores their properties and behavior under specific conditions.
3.1. Heisenberg uncertainty relation

Knowing the explicit form of the standard coherent states \(|\alpha\rangle\) and the nonlinear ones \(|\alpha\rangle_{(k)}\), the corresponding Heisenberg uncertainty relations can be straightforwardly calculated.

For the standard coherent states, the mean values of the operators \(\hat{x}, \hat{p}, \hat{x}^2, \hat{p}^2\) become (with \(\hbar = m = \omega = 1\)):

\[
\langle \hat{x} \rangle = \sqrt{2} \text{Re}(\alpha), \quad \langle \hat{x}^2 \rangle = \frac{1}{2} + 2|\text{Re}(\alpha)|^2, \\
\langle \hat{p} \rangle = \sqrt{2} \text{Im}(\alpha), \quad \langle \hat{p}^2 \rangle = \frac{1}{2} + 2|\text{Im}(\alpha)|^2.
\]

Hence, the Heisenberg uncertainty relation turns out to be given by \((\sigma_x)_\alpha^2(\sigma_p)_\alpha^2 = 1/4\).

For the nonlinear coherent states \(|\alpha\rangle_{(k)}\), the mean values of the operators \(\hat{x}, \hat{p}, \hat{x}^2, \hat{p}^2\) are:

\[
\langle \hat{x} \rangle_{(k)} = \sqrt{2} \text{Re}(\alpha) \frac{\alpha F_2(2, k + 2; r^2)}{\alpha F_2(1, k + 2; r^2)}, \\
\langle \hat{p} \rangle_{(k)} = \sqrt{2} \text{Im}(\alpha) \frac{\alpha F_2(2, k + 2; r^2)}{\alpha F_2(1, k + 2; r^2)}, \\
\langle \hat{x}^2 \rangle_{(k)} = \frac{2k + 3}{2} + |\text{Re}(\alpha)|^2 \beta_{k,+}(r) + |\text{Im}(\alpha)|^2 \beta_{k,-}(r), \\
\langle \hat{p}^2 \rangle_{(k)} = \frac{2k + 3}{2} + |\text{Re}(\alpha)|^2 \beta_{k,-}(r) + |\text{Im}(\alpha)|^2 \beta_{k,+}(r),
\]

where

\[
\beta_{k,\pm}(r) = \frac{1}{k + 2} \frac{\alpha F_2(2, k + 3; r^2)}{\alpha F_2(1, k + 2; r^2)} + \frac{1}{2} \frac{\alpha F_2(3, k + 2; r^2)}{\alpha F_2(1, k + 2; r^2)}.
\]

The Heisenberg uncertainty relation becomes:

\[
(\sigma_x)_{\alpha,k}^2(\sigma_p)_{\alpha,k}^2 = \left[ \frac{2k + 3}{2} - |\text{Re}(\alpha)|^2 \tau_k(r) + |\text{Im}(\alpha)|^2 \beta_{k,-}(r) \right] \\
\times \left[ \frac{2k + 3}{2} - |\text{Im}(\alpha)|^2 \tau_k(r) + |\text{Re}(\alpha)|^2 \beta_{k,-}(r) \right],
\]

\[
\tau_k(r) = 2 \left[ \frac{\alpha F_2(2, k + 2; r^2)}{\alpha F_2(1, k + 2; r^2)} \right] - \beta_{k,+}(r).
\]

When \(k = -1\), the expression for the Heisenberg uncertainty relation is (see Figure 1):

\[
(\sigma_x)_{\alpha,k}^2(\sigma_p)_{\alpha}^2 = \left[ \frac{1}{2} - |\text{Re}(\alpha)|^2 \tau_{-1}(r) + |\text{Im}(\alpha)|^2 \beta_{-1,-}(r) \right] \\
\times \left[ \frac{1}{2} - |\text{Im}(\alpha)|^2 \tau_{-1}(r) + |\text{Re}(\alpha)|^2 \beta_{-1,-}(r) \right],
\]

where

\[
\tau_{-1}(r) = 2 \left[ \frac{\alpha F_2(1, 2; r^2)}{\alpha F_2(1, 1; r^2)} \right]^2 - \frac{\alpha F_2(2, 2; r^2)}{\alpha F_2(1, 1; r^2)} - \frac{1}{2} \frac{\alpha F_2(3, 1; r^2)}{\alpha F_2(1, 1; r^2)}.
\]

When \(k = 0\), the expression for the Heisenberg uncertainty relation is now (see Figure 2 and [16]):

\[
(\sigma_x)_{\alpha,k}^2(\sigma_p)_{\alpha}^2 = \left[ \frac{3}{2} - |\text{Re}(\alpha)|^2 \rho(r) \right] \left[ \frac{3}{2} - |\text{Im}(\alpha)|^2 \rho(r) \right],
\]

where

\[
\rho(r) = \frac{\alpha F_2(2, 2; r^2)}{\alpha F_2(1, 1; r^2)} - \frac{1}{2} \frac{\alpha F_2(3, 1; r^2)}{\alpha F_2(1, 1; r^2)}.
\]
where
\[ \rho(r) = 2 \left[ \frac{\psi_2(2, 2; r^2)}{\psi_1(2, 2; r^2)} \right]^2 - \frac{\psi_2(2, 3; r^2)}{\psi_1(1, 2; r^2)}. \]

For \( k = 1 \) and \( k = 2 \) we have shown the Heisenberg uncertainty relation in Figure 3 and Figure 4, respectively.

**Figure 1.** Heisenberg uncertainty relation \((\sigma_x)_k^2(\sigma_y)_k^2\) as function of \(\alpha\) for the nonlinear coherent states of Eq. (6) \((k = -1)\).

**Figure 2.** Heisenberg uncertainty relation \((\sigma_x)_0^2(\sigma_y)_0^2\) as function of \(\alpha\) for the nonlinear coherent states of Eq. (7) \((k = 0)\).

**Figure 3.** Heisenberg uncertainty relation \((\sigma_x)_1^2(\sigma_y)_1^2\) as function of \(\alpha\) for the nonlinear coherent states of Eq. (4) with \(k = 1\).

**Figure 4.** Heisenberg uncertainty relation \((\sigma_x)_2^2(\sigma_y)_2^2\) as function of \(\alpha\) for the nonlinear coherent states of Eq. (4) with \(k = 2\).

4. **Nonlinear supercoherent states**

A very general form for the SAO \(\hat{A}\) \([5]\) and its simplest deformation are respectively:

\[ \hat{A} = \begin{pmatrix} k_1\hat{a} & k_2 \\ k_3\hat{a}^2 & k_4\hat{a} \end{pmatrix}, \quad \hat{Q} = \begin{pmatrix} k_1f(\hat{N})\hat{a} & k_2 \\ k_3[f(\hat{N})\hat{a}]^2 & k_4f(\hat{N})\hat{a} \end{pmatrix}, \]

where \(k_i \in \mathbb{C}\) and we will take either \(f(\hat{N}) = \hat{N} + \hat{1}\) or \(f(\hat{N}) = \hat{N}\). We will label \(\hat{Q} = \hat{A}'\) when \(f(\hat{N}) = \hat{N} + \hat{1}\), and \(\hat{Q} = \hat{A}''\) when \(f(\hat{N}) = \hat{N}\). Note that \(\hat{Q} = \hat{A}\) when \(f(\hat{N}) = \hat{1}\).
The general solution of the matrix equation (16) leads to:

$$\hat{Q}|\mathcal{X}\rangle = X|\mathcal{X}\rangle, \quad |\mathcal{X}\rangle = \left( \sum_{n=0}^{\infty} a_n |n\rangle \right) \left( \sum_{n=1}^{\infty} c_n |n-1\rangle \right), \quad X \in \mathbb{C}. \quad (15)$$

From this equation an overall matrix relationship is found as

$$K \begin{pmatrix} \tilde{a}_{n+1} \\ \tilde{c}_{n+1} \end{pmatrix} = \begin{pmatrix} \tilde{a}_n \\ \tilde{c}_n \end{pmatrix}, \quad K = \begin{pmatrix} k_1 & k_2 \\ k_3 & k_4 \end{pmatrix}, \quad (16)$$

where the quantities $\tilde{a}_n$ and $\tilde{c}_n$ are defined as

$$\tilde{a}_n = \beta_n \sqrt{n!} X^{1-n} a_n, \quad \tilde{c}_n = \beta_{n-1} \sqrt{(n-1)!} X^{-n} c_n, \quad (17)$$

with

$$X = \begin{cases} z & \text{for } \hat{Q} = \hat{A} \\ Y & \text{for } \hat{Q} = \hat{A}', \quad \beta_n = \begin{cases} 1 & \text{for } \hat{Q} = \hat{A} \text{ and } n \geq 1 \\ n! & \text{for } \hat{Q} = \hat{A}' \text{ and } n \geq 1 \\ (n-1)! & \text{for } \hat{Q} = \hat{A}'' \text{ and } n \geq 2 \end{cases} \end{cases}. \quad (18)$$

As can be seen, both the linear and nonlinear supercoherent states depend on the eigenvalues $\psi_{\pm}$ of the matrix $K$. This induces a classification of the supercoherent states $|\mathcal{X}\rangle$ into three different families as follows: degenerate ($\psi_+ = \psi_- \equiv \psi \neq 0$); singular ($\psi_+ \psi_- = 0$); generic (everything else).

Furthermore, by rewriting $a_n$ and $c_n$ in terms of $\psi_{\pm}$, the eigenstates of $\hat{A}$ turn out to be expressed in terms of the CS in Eq. (2) while those of $\hat{A}'$ and $\hat{A}''$ are determined by the ones of Eq. (6) or Eq. (7), when choosing $f(\hat{N}) = \hat{N} + \hat{1}$ or $f(\hat{N}) = \hat{N}$ respectively. Then we will identify the respective supercoherent states as follows:

$$|\mathcal{X}\rangle = \begin{cases} |Z\rangle & \text{for } \hat{Q} = \hat{A} \\ |Y\rangle & \text{for } \hat{Q} = \hat{A}' \\ |Z\rangle & \text{for } \hat{Q} = \hat{A}'' \end{cases}. \quad (19)$$

### 4.1. Supercoherent states classification

#### 4.1.1. Generic

The general solution of the matrix equation (16) leads to:

$$|\mathcal{X}\rangle = \mathcal{B}_1 |\mathcal{X}_A\rangle + \mathcal{B}_2 |\mathcal{X}_C\rangle, \quad (20)$$

where

$$\mathcal{B}_1 = \begin{cases} a_0/k_1 & \text{for } \hat{Q} = \hat{A}, \hat{A}' \\ a_1/k_1 & \text{for } \hat{Q} = \hat{A}'' \end{cases}, \quad \mathcal{B}_2 = \begin{cases} c_1/(Xk_1) & \text{for } \hat{Q} = \hat{A}, \hat{A}' \\ c_2/(Xk_1) & \text{for } \hat{Q} = \hat{A}'' \end{cases}. \quad (21a)$$

$$|\mathcal{X}_A\rangle = \frac{1}{\psi_+ - \psi_-} G_A \left( |\chi_+\rangle \right), \quad |\mathcal{X}_C\rangle = \frac{1}{\psi_+ - \psi_-} G_C \left( |\chi_+\rangle \right), \quad (21b)$$

with

$$\chi_{\pm} \equiv X \psi_{\pm}^{-1}, \quad |\chi_{\pm}\rangle = \begin{cases} |\beta_{\pm}\rangle & \text{for } \hat{Q} = \hat{A} \\ |\varphi_{\pm}nl\rangle & \text{for } \hat{Q} = \hat{A}' \\ |\phi_{\pm}nl\rangle & \text{for } \hat{Q} = \hat{A}'' \end{cases}. \quad (22a)$$
Consider a superposition of states  

\[ G_A = \begin{pmatrix} \psi_+ (\psi_+ - k_4) & -\bar{\psi}_- (\bar{\psi}_- - k_4) \\ k_3 \chi & -k_3 \chi \end{pmatrix}, \]

\[ G_C = \begin{pmatrix} k_2 \psi_+ \bar{\psi}_- & -k_2 \psi_+ \bar{\psi}_- \\ X[k_1 \psi_+ - (k_1^2 + k_2 k_3)] & -X[k_1 \bar{\psi}_- - (k_1^2 + k_2 k_3)] \end{pmatrix}. \]

4.2. Superposition and uncertainties

Besides the set \( \{ |\mathcal{X}_A \rangle, |\mathcal{X}_C \rangle \} \), we can choose another set of supercoherent states formed by the elements

\[ |\mathcal{X}_\pm \rangle = \left( \frac{k_2 \psi_+ |\chi_\pm \rangle + \bar{\psi}_- |\chi_\pm \rangle}{\sqrt{\psi_+ - k_1} X |\chi_\pm \rangle} \right), \]

whereby it is possible to pass from a parameter space \( \{ k_1, k_2, k_3, k_4 \} \) to a new one, formed by \( \{ \psi_+, \psi_-, k_1, k_2 \} \).

4.1.2. Degenerate

The supercoherent states are explicitly

\[ |\mathcal{X} \rangle = B_1 |\mathcal{X}_A^d \rangle + B_2 |\mathcal{X}_C^d \rangle, \]

where

\[ |\mathcal{X}_A^d \rangle = G_A^{(d)} \left( |\chi \rangle \right), \quad |\mathcal{X}_C^d \rangle = G_C^{(d)} \left( |\chi \rangle \right), \]

with

\[ |\chi \rangle = \frac{d}{d\chi} |\chi \rangle, \]

\[ G_A^{(d)} = \begin{pmatrix} k_1 & -(\psi - k_4) |\chi \rangle \\ 0 & -k_3 \chi^2 \end{pmatrix}, \quad G_C^{(d)} = \psi \left( \begin{array}{c} 0 \\ k_1 \chi - (k_4 - k_1) \chi^2 \end{array} \right). \]

4.1.3. Singular

The corresponding supercoherent states become now

\[ |\mathcal{X}_s \rangle = \left( \frac{k_1 |\chi \rangle}{k_3 \chi |\chi \rangle} \right), \quad \chi = \frac{X}{(k_1 + k_4)}. \]

4.2. Superposition and uncertainties

Consider a superposition of states \( |\mathcal{X}_\pm \rangle \) with parameters \( \eta \) and \( \lambda \) as follows:

\[ |\mathcal{X}_m \rangle = \cos \eta |\mathcal{X}_+ \rangle + e^{i\lambda} \sin \eta |\mathcal{X}_- \rangle = \left( \begin{array}{c} \gamma_1^+ |\chi_+ \rangle + \gamma_1^- |\chi_- \rangle \\ \gamma_2^+ |\chi_+ \rangle + \gamma_2^- |\chi_- \rangle \end{array} \right), \]

where \( \gamma_{1\pm} \) and \( \gamma_{2\pm} \) are given by

\[ \gamma_1^+ = k_2 \psi_+ \cos \eta, \quad \gamma_1^- = k_2 \psi_- e^{i\lambda} \sin \eta, \]

\[ \gamma_2^+ = (\psi_+ - k_1) \cos \eta, \quad \gamma_2^- = (\psi_- - k_1) e^{i\lambda} \sin \eta. \]

The mean value of an arbitrary observable \( \hat{c} \) is then

\[ \langle \hat{c} \rangle = \frac{\langle \mathcal{X}_m | \hat{c} |\mathcal{X}_m \rangle}{\langle \mathcal{X}_m |\mathcal{X}_m \rangle}, \]

\[ \langle \mathcal{X}_m | \hat{c} |\mathcal{X}_m \rangle = \Gamma^+ \langle \mathcal{X}_+ | \hat{c} |\mathcal{X}_+ \rangle + \Gamma^- \langle \mathcal{X}_- | \hat{c} |\mathcal{X}_- \rangle + 2 \text{Re}(\Gamma^+ \langle \mathcal{X}_+ | \hat{c} |\mathcal{X}_- \rangle), \]

with

\[ \Gamma^+ = |\gamma_{1+}|^2 + |\gamma_{2+}|^2, \quad \Gamma^- = |\gamma_{1-}|^2 + |\gamma_{2-}|^2, \quad \Gamma^{+-} = \gamma_{1+}^* \gamma_{1-} + \gamma_{2+}^* \gamma_{2-} |X|^2. \]

This allows us to find straightforwardly the expressions for the uncertainties of the operators \( \hat{x} \) and \( \hat{p} \) and their squares.
4.2.1. Uncertainties for $\hat{Q} = \hat{A}$. The expressions for the uncertainties are [5]:

\[
\sigma_x^2 = \frac{\Gamma^+ [1 + (\beta_+ + \beta_+^*)] e^{\beta_+^*} + \Gamma^- [1 + (\beta_- + \beta_-^*)] e^{\beta_-^*} + 2 \text{Re}[\Gamma^+ - [1 + (\beta_+ + \beta_+^*)] e^{\beta_+^*} - 2 \text{Re}[\Gamma^+ - [1 + (\beta_+ + \beta_+^*)] e^{\beta_+^*} - \Gamma_- [1 + (\beta_- + \beta_-^*)] e^{\beta_-^*} + 2 \text{Re}[\Gamma^- - [1 + (\beta_- + \beta_-^*)] e^{\beta_-^*}]]}{2[\Gamma^+ e^{\beta_+^*} + \Gamma^- e^{\beta_-^*} + 2 \text{Re}[\Gamma^+ - e^{\beta_+^*} + \Gamma^- - e^{\beta_-^*}]]}
\]

\[
\sigma_p^2 = \frac{\Gamma^+ [1 - (\beta_+ - \beta_+^*)] e^{\beta_+^*} + \Gamma^- [1 - (\beta_- - \beta_-^*)] e^{\beta_-^*} + 2 \text{Re}[\Gamma^+ - [1 - (\beta_+ - \beta_+^*)] e^{\beta_+^*} - \Gamma_- [1 - (\beta_- - \beta_-^*)] e^{\beta_-^*}]]}{2[\Gamma^+ e^{\beta_+^*} + \Gamma^- e^{\beta_-^*} + 2 \text{Re}[\Gamma^+ - e^{\beta_+^*} + \Gamma^- - e^{\beta_-^*}]]}
\]

where the quantities in Eq. (31) are evaluated taking $X = z$.

4.2.2. Uncertainties for $\hat{Q} = \hat{A}'$. For this case, the expressions for the uncertainties are:

\[
\sigma_x^{(nl)} = \Delta_1^{-1} \left\{ \Gamma^+ \left[ \text{Re}(\varphi_+)^2 \beta_{-1,+}(\varphi_+) + \text{Im}(\varphi_+)^2 \beta_{-1,-}(\varphi_+) + \frac{1}{2} \right] + \Gamma^- \left[ \text{Re}(\varphi_-)^2 \beta_{-1,+}(\varphi_-) + \text{Im}(\varphi_-)^2 \beta_{-1,-}(\varphi_-) + \frac{1}{2} \right] + 2 \text{Re} \left( \frac{\Gamma^+}{2\sqrt{a F_2(1, 1; |\varphi_+|^2)} \sqrt{a F_2(1, 1; |\varphi_-|^2)} \left( (\varphi_+^2 + \varphi_-^2) \frac{a F_2(1, 3; \varphi_+^* \varphi_-) + 2 \varphi_+ \varphi_- a F_2(2, 2; \varphi_+^* \varphi_-) + a F_2(1, 1; \varphi_+^* \varphi_-)}{a F_2(1, 1; |\varphi_+|^2)} \right) \right) \right\}^2,
\]

\[
\sigma_p^{(nl)} = \Delta_1^{-1} \left\{ \Gamma^+ \left[ \text{Re}(\varphi_+)^2 \beta_{-1,+}(\varphi_+) + \text{Im}(\varphi_+)^2 \beta_{-1,+}(\varphi_+) + \frac{1}{2} \right] + \Gamma^- \left[ \text{Re}(\varphi_-)^2 \beta_{-1,+}(\varphi_-) + \text{Im}(\varphi_-)^2 \beta_{-1,+}(\varphi_-) + \frac{1}{2} \right] + 2 \text{Re} \left( \frac{\Gamma^+}{2\sqrt{a F_2(1, 1; |\varphi_+|^2)} \sqrt{a F_2(1, 1; |\varphi_-|^2)} \left( (\varphi_+^2 + \varphi_-^2) \frac{a F_2(1, 3; \varphi_+^* \varphi_-) + 2 \varphi_+ \varphi_- a F_2(2, 2; \varphi_+^* \varphi_-) + a F_2(1, 1; \varphi_+^* \varphi_-)}{a F_2(1, 1; |\varphi_+|^2)} \right) \right) \right\}^2.
\]

where $X = Y$ in Eq. (31), $\beta_{-1,\pm}(r)$ is given by Eq. (10) with $k = -1$ and

\[
\Delta_1 = \Gamma^+ + \Gamma^- + 2 \text{Re} \left( \frac{\Gamma^+ - a F_2(1, 1; \varphi_+^* \varphi_-)}{2\sqrt{a F_2(1, 1; |\varphi_+|^2)} \sqrt{a F_2(1, 1; |\varphi_-|^2)}} \right).
\]
4.2.3. Uncertainties for $\hat{Q} = \hat{A}''$. Similarly, the expressions for the uncertainties are now:

$$\sigma_x^{2(NL)} = \Delta_2^{-1} \left\{ \Gamma^+ \left[ \text{Re}(\phi_+) \right]^2 \frac{\partial F_2(2, 3; |\phi_+|^2)}{\partial F_2(1, 2; |\phi_+|^2)} + \frac{3}{2} \right\} + \Gamma^- \left[ \text{Re}(\phi_-) \right]^2 \frac{\partial F_2(2, 3; |\phi_-|^2)}{\partial F_2(1, 2; |\phi_-|^2)} + \frac{3}{2} \right\} + 2 \text{Re} \left( \Gamma^{+-} \right) \right\} \right\}

$$

$$\Delta_2^{-2} \left\{ \sqrt{2} \Gamma^+ \text{Re}(\phi_+) \frac{\partial F_2(2, 2; |\phi_+|^2)}{\partial F_2(1, 2; |\phi_+|^2)} + \sqrt{2} \Gamma^- \text{Re}(\phi_-) \frac{\partial F_2(2, 2; |\phi_-|^2)}{\partial F_2(1, 2; |\phi_-|^2)} + \sqrt{2} \text{Re} \left( \phi_+ + \phi_- \frac{\partial F_2(2, 2; |\phi_+|^2)}{\partial F_2(1, 2; |\phi_+|^2)} \right) \right\} \right\}

$$

$$\sigma_p^{2(NL)} = \Delta_2^{-1} \left\{ \Gamma^+ \left[ \text{Im}(\phi_+) \right]^2 \frac{\partial F_2(2, 3; |\phi_+|^2)}{\partial F_2(1, 2; |\phi_+|^2)} + \frac{3}{2} \right\} + \Gamma^- \left[ \text{Im}(\phi_-) \right]^2 \frac{\partial F_2(2, 3; |\phi_-|^2)}{\partial F_2(1, 2; |\phi_-|^2)} + \frac{3}{2} \right\} + 2 \text{Re} \left( \Gamma^{+-} \right) \right\} \right\}

$$

$$\Delta_2 = \Gamma^+ + \Gamma^- + 2 \text{Re} \left( \frac{\Gamma^{+-} \frac{\partial F_2(2, 2; |\phi_+|^2)}{\partial F_2(1, 2; |\phi_+|^2)} \frac{\partial F_2(2, 2; |\phi_-|^2)}{\partial F_2(1, 2; |\phi_-|^2)} + \sqrt{2} \frac{\partial F_2(2, 2; |\phi_+|^2)}{\partial F_2(1, 2; |\phi_+|^2)} \right) \right\}

$$

5. A particular case

To analyze the behavior of the supercoherent states (linear and nonlinear) the following particular values for the parameters $k_i$ are taken: $k_1 = k_4 = 1$, $k_2 = \cos \theta$ and $k_3 = \sin \theta$. Thus, the deformed SAO in Eq. (14) takes the generic form:

$$\hat{Q} = \begin{pmatrix} f(\hat{N})\hat{a} & \cos \theta \\ \sin \theta |f(\hat{N})\hat{a}|^2 & f(\hat{N})\hat{a} \end{pmatrix},$$

(40)

meanwhile the superposition (28) (with $\eta = \lambda = \pi/4$) becomes

$$|\lambda_0\rangle = \frac{1}{\sqrt{2}} (|\lambda_+\rangle + e^{i\pi/4}|\lambda_-\rangle).$$

(41)

On the other hand, the eigenvalues $\psi_{\pm}$ of the matrix $K$ of Eq. (16) become now:

$$\psi_{\pm, \theta} = 1 \pm \frac{1}{2} \sin(2\theta),$$

(42)

so that for $0 < \theta < \pi/2$ both $\psi_{\pm}$ are real while for $\pi/2 < \theta < \pi$ they turn out to be complex.

By substituting equations (41) and (42) in expressions (32)–(39), the Heisenberg uncertainty relation for each form of the operator $\hat{Q}$ is found.
5.1. Heisenberg uncertainty relation for $\hat{Q} = \hat{A}$.

The Heisenberg uncertainty relation associated with the linear supercoherent states of [5] is shown in Figures 5 and 6. For $z$ real (left plot, Figure 5) the uncertainty reaches a maximum value equal to $0.83$ at $|z| \sim 0.5$ for real eigenvalues $\psi_\pm (0 < \theta < \pi/2)$, while it shows a growing behavior as $|z|$ increases for complex eigenvalues $\psi_\pm (\pi/2 < \theta < \pi)$. For complex $z$ (right plot, Figure 6), the uncertainty relation behaves similarly as for real $z$.

5.2. Heisenberg uncertainty relation for $\hat{Q} = \hat{A}'$.

Figures 7 and 8 show the Heisenberg uncertainty relation $\sigma_x^2 \sigma_p^2 (Eqs. (32) and (33))$ for the nonlinear supercoherent states $|Y\rangle$ associated to the operator $\hat{A}'$. In case of $Y$ real (left plot, Figure 8), for both real ($0 < \theta < \pi/2$) as well as complex ($\pi/2 < \theta < \pi$) eigenvalues $\psi_\pm$, the uncertainty starts from a minimum value and grows slowly. In the region of degenerate eigenvalues, for $\theta = \pi/2$, the uncertainty reaches a maximum value. For complex $Y$ (right plot, Figure 8), the uncertainty relation behaves similarly as for real $Y$.

5.3. Heisenberg uncertainty relation for $\hat{Q} = \hat{A}''$.

Figures 9 and 10 show the uncertainty relations $\sigma_x^2 \sigma_p^2 (Eqs. (32) and (33))$ for the nonlinear supercoherent states $|Z\rangle$ associated to the operator $\hat{A}''$. In case of $Z$ real (left plot, Figure 9), for both real ($0 < \theta < \pi/2$) as well as complex ($\pi/2 < \theta < \pi$) eigenvalues $\psi_\pm$, the uncertainty starts from a value equal to $2.25$, then it decreases smoothly and grows again. In the region of degenerate eigenvalues $\psi_\pm$, for $\theta = \pi/2$, the uncertainty reaches a minimum value close to $0.25$. For complex $Z$ (right plot, Figure 10), the uncertainty relation behaves similarly as for real $Z$.

As shown in Figures 5–10, the behavior of the Heisenberg uncertainty relation for the linear supercoherent states $|Z\rangle$ becomes more involved than for the nonlinear ones, $|Y\rangle$ and $|Z\rangle$. Moreover, since the nonlinear supercoherent states $|Z\rangle$ are expressed in terms of the nonlinear coherent states $|\alpha\rangle_{NL}$ which isolate the eigenstate $|0\rangle$, the corresponding uncertainty relation starts from a different value at $Z = 0$ than its counterparts associated to $\hat{A}$ and $\hat{A}'$. 

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**Figure 5.** Heisenberg uncertainty relation $\sigma_x^2 \sigma_p^2 (Eqs. (32) and (33))$ for the supercoherent states $|Z\rangle$ with $z$ real.

**Figure 6.** Heisenberg uncertainty relation $\sigma_x^2 \sigma_p^2 (Eqs. (32) and (33))$ for the supercoherent states $|Z\rangle$ with $z$ complex.
Figure 7. Heisenberg uncertainty relation $\sigma_{x}^{2(nl)} \sigma_{p}^{2(nl)}$ (Eqs. (34) and (35)) for the nonlinear supercoherent states $|Y\rangle$ with $Y$ real.

Figure 8. Heisenberg uncertainty relation $\sigma_{x}^{2(nl)} \sigma_{p}^{2(nl)}$ (Eqs. (34) and (35)) for the nonlinear supercoherent states $|Y\rangle$ with $Y$ complex.

Figure 9. Heisenberg uncertainty relation $\sigma_{x}^{2(NL)} \sigma_{p}^{2(NL)}$ (Eqs. (37) and (38)) for nonlinear supercoherent states $|Z\rangle$ with $Z$ real.

Figure 10. Heisenberg uncertainty relation $\sigma_{x}^{2(NL)} \sigma_{p}^{2(NL)}$ (Eqs. (37) and (38)) for nonlinear supercoherent states $|Z\rangle$ with $Z$ complex.

6. Conclusions
The deformation of the supersymmetric annihilation operator $\hat{A}$ given in [5] allowed us to find interesting results related with the nonlinear supercoherent states. As it was noted, the form of the operator $\tilde{a} = f(\hat{N})\hat{a}$ in the corresponding nonlinear coherent states, enabled us to isolate the ground state contribution, which has interesting implications for the properties of the nonlinear supercoherent states and their associated Heisenberg uncertainty relation.

Let us remark that all the results obtained in this paper obey the Heisenberg uncertainty principle ($\sigma_x^2 \sigma_p^2 \geq 1/4$), which ensures that the states constructed here are consistent with the quantum theory.

Finally, the freedom we have for choosing the form of the function $f(\hat{N})$ will allow us to consider a more detailed algebraic study, focused on the commutation relations of the operators $\{\hat{H}_s, \hat{A}, \hat{A}^\dagger\}$ for the supersymmetric harmonic oscillator and the possible relation with the...
polynomial deformations of the Heisenberg algebra [17, 18], as it happens with the operators \( \{ \hat{H}, \hat{a}, \hat{a}^\dagger \} \) for the standard harmonic oscillator.

Acknowledgments
The authors acknowledge the support of Conacyt, Project 152574. EDB also acknowledges the Conacyt fellowship 290649.

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