Perfect observables for the hierarchical non-linear $O(N)$-invariant $\sigma$-model

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Abstract

We compute moving eigenvalues and the eigenvectors of the linear renormalization group transformation for observables along the renormalized trajectory of the hierarchical non-linear $O(N)$-invariant $\sigma$-model by means of perturbation theory in the running coupling constant. Moving eigenvectors are defined as solutions to a Callan-Symanzik type equation.
1 Introduction

Hasenfratz and Niedermayer calculated in [HN] the asymptotic form of the renormalized trajectory called perfect action of the two dimensional $O(N)$-model with non-linear block spin. They observed that the block spin transformation could be treated classically in the asymptotically free domain to an excellent approximation. In [WX] we introduced an improvement scheme in which the zeroth order approximation is the perfect action. This scheme is a combination of perturbation theory in the running coupling with the idea of scaling. We worked it out for the $O(N)$-model with linear block spin in the hierarchical approximation. The improved action is not yet the complete improved theory. The complete improved theory should also include improved observables. In this paper we show how the improvement program in [WX] can be extended to observables for the hierarchical $O(N)$-model.

In Wilson’s renormalization group [WK] a Euclidean quantum field theory is thought of in terms of the flow of effective interactions $V(\phi)$ generated by a block spin transformation $\mathcal{R}$. In the following $\phi$ will denote an $N$-component real valued field on a two dimensional unit lattice. The interactions will be invariant under global $O(N)$ transformations. $\mathcal{R}$ will respect global $O(N)$ invariance. The fixed points $\mathcal{R}V^* = V^*$ of $\mathcal{R}$ describe scale invariant (continuum) field theories. The eigenvalues of the linearized block spin transformation at a fixed point determine the associated critical exponents. The two dimensional $O(N)$-invariant non-linear $\sigma$-model is expected to possess only two fixed points for $N > 2$ (and $N > 1$ in the hierarchical approximation), an asymptotic, unstable ultraviolet fixed point $V_{UV}$ and a stable infrared fixed point $V_{IR}$. The model is asymptotically free in the ultraviolet: $V_{UV}$ is a theory of $N - 1$ massless free fields. In the hierarchical case $V_{UV}$ is better thought of from the orthogonal point of view as a high temperature fixed point of a single component radial field. The fixed points are expected to be connected by a one dimensional renormalized trajectory which is stable under block spin transformations. The points on the renormalized trajectory describe (continuum) theories with broken scale invariance. Interactions on the renormalized trajectory are said to scale. The renormalized trajectory is both of principal and of practical interest since it admits the description of a true continuum theory (not a lattice approximation) in terms of a lattice theory. It is the ultimate improved action in the sense of Symanzik [S].

Let the renormalized trajectory be parametrized in terms of a local coordinate $f$. Recall that it is a curve in the space of $O(N)$ invariant interactions. The block spin transformation then becomes

$$\mathcal{R}V_{RT}(\phi|f) = V_{RT}(\phi|\beta(f)).$$

(1)
A natural local coordinate in the vicinity of the asymptotically free fixed point is the running coupling. The running coupling will be defined as the leading trilinear $\sigma$-$\pi$-$\pi$-vertex. In the hierarchical approximation it is identical with the inverse radius. The complete dynamics of the renormalization group on the renormalized trajectory is encoded in the flow of the running coupling. We call $\beta(f)$ Callan-Symanzik function since it defines this discrete flow.

A block spin transformation comes together with a linear mapping of observables which preserves their expectation values. This mapping is the linearized renormalization group transformation. We shall consider this linear mapping $\mathcal{LR}$ to the block spin transformation over the renormalized trajectory. An observable will be called an eigenvector with eigenvalue $\epsilon(f)$ if it satisfies the equation

$$\mathcal{L}_{RT} \mathcal{R} \mathcal{O}(\phi|f) = \epsilon(\beta(f)) \mathcal{O}(\phi|\beta(f)).$$

(2)

In accordance with the terminology for the interaction we will speak of this property as scaling. The space of eigenvectors defines a moving frame in the tangent space over the renormalized trajectory. The set of eigenvalues will be called the spectrum of the theory. It is exactly calculable at the fixed point. Moving away from the fixed point on the renormalized trajectory it becomes perturbed. In the vicinity of the ultraviolet fixed point it is therefore natural to perform a perturbation expansion in the running coupling to determine the spectrum from (2). In this paper we address the question of computing these perturbative corrections. At the fixed point (2) becomes the eigenvalue equation for the critical exponents. Away from the fixed point the spectrum governs the rate of attraction towards the renormalized trajectory. The set of eigenvectors forms a basis of the linear space of observables. Associated with this basis is a set of fusion rules. We will give a general expression for correlation functions of eigenvectors on the renormalized trajectory in terms of the spectrum and the fusion rules. The investigation of the spectrum and the moving frame complete the analysis begun in [WX].

We should mention that the two dimensional hierarchical $O(N)$-model has been rigorously studied by Gawedzki and Kupiainen in [GK], where both the existence of an ultraviolet limit and the existence of a renormalized trajectory has been established beyond perturbation theory. This result has also been obtained by Pordt and Reisz in [PR] using a slightly different technology. Both groups however do not address the question of the spectrum. A rigorous study of the full $O(N)$-model both in the ultraviolet and more importantly in the infrared domain is still an outstanding problem.
2 Hierarchical $O(N)$ model

Let us consider the renormalization group flow generated by the hierarchical block spin transformation $^{1}$

$$e^{-\mathcal{R}V(\psi)} = \mathcal{N} \left( \int d\mu_\gamma(\zeta)e^{-V(\psi+\zeta)} \right)^2. \quad (3)$$

in $D = 2$ dimensions with scale parameter $L = \sqrt{2}$. The interaction $V(\phi)$ is taken to be a function of a real $N$-component variable $\phi$.

$$d\mu_\gamma(\zeta) = (2\pi\gamma)^{-\frac{N}{2}} e^{-\frac{\zeta^2}{2\gamma}} d^N \zeta \quad (4)$$

is the Gaussian measure on $\mathbb{R}^N$ with mean zero and covariance $\text{diag}(\gamma, \ldots, \gamma)$. The subspace of $O(N)$ invariant interactions is stable under (3). We will restrict our attention to this subspace. $O(N)$ invariance requires $V(\phi)$ to be a function of the modulus $|\phi|$. The normalization constant in (3) is conveniently chosen such that the interaction is always zero at its minimum. We will denote this minimum by $r$ and call it the radius of the potential.

Let us define the renormalized trajectory as the curve $V_{RT}(\phi|f)$ in the space of $O(N)$ invariant potentials, parametrized by the inverse radius $f = \frac{1}{r}$, with the following two properties:

1) $V_{RT}(\phi|f)$ is stable under the block spin transformation $\mathcal{R}$. It follows that there exists a function $\beta(f)$ such that

$$\mathcal{R}V_{RT}(\phi|f) = V_{RT}(\phi|\beta(f)). \quad (5)$$

In other words a block spin transformation acts on the renormalized trajectory by a transformation of the coordinate given by a $\beta$-function.

2) The asymptotic behavior of $V_{RT}(\phi|f)$ as the running coupling $f$ goes to zero is given by

$$V_{RT}(\phi|f) = V_{RT}^{(1)}(\phi|f) + O(f^2),$$

$$V_{RT}^{(1)}(\phi|f) = \frac{1}{2\gamma} \left(|\phi| - \frac{1}{f} \right)^2. \quad (6)$$

Up to corrections of second order in the running coupling the renormalized trajectory coincides with the perfect action.

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$^1$ Note that (3) differs from the transformation used in [WX]. We have exchanged the integration with the rescaling step.
At least in terms of perturbation theory in the running coupling (5) and (6) have a unique solution as has been shown in [WX]. Let us remark that (6) is compatible with (5) since it reproduces itself under (3) according to

\[ RV_{RT}^{(1)}(\phi|f) = V_{RT}^{(1)}(\phi|\beta(f)) + O(f^2), \]
\[ \beta(f) = f + O(f^2). \] (7)

up to second order corrections. The general form of the solution to (5) and (6) in terms of perturbation theory is

\[ V_{RT}(\phi|f) = \sum_{n=2}^{\infty} P_n(f) \left( \left| \phi \right| - \frac{1}{f} \right)^n, \]
\[ P_2(f) = \frac{1}{2\gamma} + \sum_{m=0}^{\infty} c_{2,2m+2} f^{2m+2}, \]
\[ P_n(f) = \sum_{m=0}^{\infty} c_{n,2m+n} f^{2n+m}, \]
\[ \beta(f) = f + \sum_{m=0}^{\infty} b_{2m+3} f^{2m+3}. \] (8)

The coefficients up to fifth order in the running coupling turn out to be given by

\[ c_{2,2} = -\frac{3(N-1)}{4}, \]
\[ c_{2,4} = \left( \frac{9(N-1)^2}{8} - \frac{61(N-1)}{28} \right) \gamma, \]
\[ c_{3,3} = \frac{7(N-1)}{18}, \]
\[ c_{3,5} = \left( -\frac{5(N-1)^2}{8} + \frac{257(N-1)}{180} \right) \gamma, \]
\[ c_{4,4} = -\frac{15(N-1)}{56}, \]
\[ c_{5,5} = \frac{31(N-1)}{150}, \]
\[ b_3 = \frac{(N-1)\gamma}{2}, \]
\[ b_5 = \left( \frac{3(N-1)^2}{8} + \frac{13(N-1)}{12} \right) \gamma^2. \] (9)
It follows that the model is asymptotically free for $N > 1$. It is custom to use $g = f^2$ instead of $f$ as running coupling in discussions on asymptotic freedom. From the $\beta$-function to fifth order in $f$ it follows that the recursion relation to third order in $g$ is given by

$$
\bar{\beta}(g) = g + \sum_{m=2}^{\infty} b_m g^m,
\bar{b}_2 = (N - 1) \gamma,
\bar{b}_3 = \left( (N - 1)^2 + \frac{13(N - 1)}{6} \right) \gamma^2.
$$

(10)

Thus $g$ is marginally relevant for $N > 1$.

This definition of a renormalized trajectory does not refer to a continuum limit procedure. It is nevertheless identical with the continuum limit effective potential of models in the $O(N)$ universality class in the hierarchical renormalization scheme. To prove this we can perform the continuum limit using (8) as bare interaction. Define the bare coupling to be the $n$-fold preimage $\beta^{-n}(f)$ of a renormalized value $f$. By construction of the renormalized trajectory it follows that

$$
R^n V_{RT}(\phi | \beta^{-n}(f)) = V_{RT}(\phi | f)
$$

(11)

for all numbers $n$ of renormalization group steps. The continuum limit $n \to \infty$ is immediately performed since the right hand side of (11) is independent of $n$. All that is needed is an analysis of the recursion relation defined by the $\beta$-function or rather its inverse. This is a comparatively easy task. See for instance [GK]. For $N > 1$ it follows from (11) that the bare coupling tends to zero as $n$ goes to infinity. (One needs to take into account logarithmic corrections piled up by the term of third order in $g$, fifth order in $f$.) It follows that the perturbation expansion in the running coupling is valid. To make contact with the hierarchical real world one should also assign a scale, for instance in form of a lattice spacing $a$, to the point on the renormalized trajectory where the running coupling is given by the renormalized value $f$. The bare cutoff in (11) is then $L^{-n}a$ with $L$ the block scale.

The sceptical reader may worry to what extent this construction is connected with the continuum limit of a bare theory defined by his favorite $O(N)$ invariant interaction. Consider for instance the standard interaction of the linear $O(N)$-model defined by

$$
V(\phi | f) = \lambda f^2 \left( \phi^2 - \frac{1}{f^2} \right)^2.
$$

(12)

Its continuum limit is constructed as the result of the infinite iteration

$$
\lim_{n \to \infty} R^n V(\phi | f^{-n}(f)) = V_{\text{cont}}(\phi | f).
$$

(13)
Here the bare coupling $f_{-n}(f)$ is tuned such that the minimum of $\mathcal{R}^n V(\phi|f_{-n}(f))$ is located at the renormalized radius $|\phi| = \frac{1}{f}$. The continuum limit is universal as we have learnt from the work of Wilson [WK]. Therefore the connection to the above definition is simply

$$V_{\text{cont}}(\phi|f) = V_{\text{RT}}(\phi|f).$$

(14)

A rigorous proof of the existence of (13) and, inbetween the lines, also of (14) has been given by Gawedzki and Kupiainen [GK]. More generally, the set of all bare interactions sharing (8) as their common continuum limit defines the universality class of the hierarchical $O(N)$-model. (12) is known to belong to this class and so is the original model with sharp constraint. So what we do, when we define the renormalized trajectory by the above two conditions, is to completely disentangle the admitingly also interesting question if, how, and at what pace the continuum limit is reached by some particular bare model.

In practice we may not be able to compute (8) to all orders of perturbation theory in the running coupling for the models we are really interested in, for example the full nonlinear $O(N)$-model in terms of block spin transformation on a unit lattice. In this situation the best one can do is to take the highest order approximation to (8) accessible as bare interaction. In the hierarchical model for instance already the second approximant

$$V^{(2)}_{\text{RT}}(\phi|f) = \left( \frac{1}{2\gamma} - \frac{3(N-1)}{4} f^2 \right) \left( |\phi| - \frac{1}{f} \right)^2$$

(15)

turns out to be an excellent starting point for a numerical study of the renormalized trajectory. Its main property is of course

$$\mathcal{R}V^{(2)}_{\text{RT}}(\phi|f) = V^{(2)}_{\text{RT}}(\phi|\beta(f)) + O(\beta(f)^3).$$

(16)

It therefore coincides with the renormalized trajectory up to corrections of third order in $f$. Let us finally mention that the third order approximant would be ideally suited for a rigorous construction along the lines of Gawedzki and Kupiainen [GK] and of Pordt and Reisz [PR]. The perturbative part is trivial since the action reproduces itself to third order. The construction therefore reduces to the proof of a stability bound which controls non perturbative corrections. This bound is already implicitly contained in the rigorous work on the $O(N)$-model with standard bare interaction (12).

### 3 Perfect Observables

Let us consider the hierarchical renormalization group transformation for local observables corresponding to (8) in the case of $D = 2$ dimensions with scale parameter
\( L = \sqrt{2} \). It is given by the linear transformation

\[
\mathcal{L}_V \mathcal{R} \mathcal{O}(\psi) = \frac{\int d\mu_\gamma(\zeta) e^{-V(\psi + \zeta)} \mathcal{O}(\psi + \zeta)}{\int d\mu_\gamma(\zeta) e^{-V(\psi + \zeta)}}. \tag{17}
\]

The block volume is \( L^D = 2 \). (17) is the linearization of (3) divided by the block volume

\[
\mathcal{L}_V \mathcal{R} \mathcal{O}(\psi) = \frac{1}{2} \frac{\partial}{\partial z} \mathcal{R}(V + z \mathcal{O})|_{z=0}. \tag{18}
\]

In other words the transformation of observables is the tangent map of the transformation of the potential.

The setup of perturbation theory for the linear transformation (17) is as follows. Since the model is \( O(N) \)-invariant we can choose the block spin to be given by \( \psi = (r + \Psi)\hat{e} \) with \( \hat{e} \) any \( N \) component unit vector. That is, we can trade \( \psi \) for \( \Psi = |\psi| - r \). Observables will also be assumed \( O(N) \)-invariant. The shift by \( r \) serves to place us into the minimum of the potential. We then decompose orthogonally the fluctuation field into a radial and a tangential part with respect to the direction of the block spin \( \psi \). The decomposition is \( \zeta = \sigma \hat{e} + \pi \). The one component variable \( \sigma \) is the radial fluctuation field. The \( (N-1) \)-component variable \( \pi \) is the tangential fluctuation field. The Gaussian measure factorizes into

\[
d\mu_\gamma(\zeta) = d\mu_\gamma(\sigma) d\mu_\gamma(\pi). \tag{19}\]

We then insert this decomposition of \( \zeta \) in the formula for the potential on the renormalized trajectory and extract the terms of order zero in the running coupling. The result is

\[
V_{RT}((r + \Psi + \sigma)\hat{e} + \pi|f) = \frac{(\Psi + \sigma)^2}{2\gamma} + V_{RT}^{(1)}(\Psi + \sigma, \pi|f). \tag{20}\]

The zeroth order terms cannot be treated as perturbations. They are fortunately only linear and quadratic in the radial fluctuation field. They and can therefore be accomodated for using the identity

\[
d\mu_\gamma(\sigma) e^{-\frac{(\Psi + \sigma)^2}{2\gamma}} = d\mu_\frac{\hat{\gamma}}{2}(\sigma + \frac{\Psi}{2}) \frac{1}{\sqrt{2\gamma}} e^{-\frac{\Psi^2}{2\gamma}}. \tag{21}\]

We then shift the radial fluctuation field to \( \xi = \sigma + \frac{\Psi}{2} \). After these standard manipulations the transformation (17) becomes

\[
\mathcal{L}_{RT} \mathcal{R} \mathcal{O}((r + \Psi)\hat{e}) = \frac{\int d\mu_\frac{\hat{\gamma}}{2}(\xi) \int d\mu_\gamma(\pi) e^{-V_{RT}^{(1)}(\Psi + \xi, \pi|f)} \mathcal{O}((r + \Psi + \xi)\hat{e} + \pi)}{\int d\mu_\frac{\hat{\gamma}}{2}(\xi) \int d\mu_\gamma(\pi) e^{-V_{RT}^{(1)}(\Psi + \xi, \pi|f)}}. \tag{22}\]
At this point perturbation theory is applicable. The expansion parameter is the inverse radius $f = \frac{1}{r}$. For the observables we have in mind we also have to perform an expansion in terms of the running coupling.

An observable is called a moving eigenvector to the moving eigenvalue $\epsilon(f)$ if it satisfies the renormalization group equation

$$\mathcal{L}_{RT} \mathcal{RO}(\phi|f) = \epsilon(\beta(f)) \mathcal{O}(\phi|\beta(f)).$$

Solutions to (23) are perfect observables. A parametrization well suited for perturbation theory is

$$\mathcal{O}(\phi|f) = \sum_{n=0}^{\infty} Q_n(f) \left(|\phi| - \frac{1}{f}\right)^n.$$  

(24)

Here the coefficients $Q_n(f)$ are taken to be power series in the running coupling $f$. (The coefficients are not expected to exhibit any singularity at zero coupling.) We will organize the solutions of (23) according to their zeroth order coefficients. The zeroth order observables are simply normal ordered monomials. Let us also perform the perturbation expansion for the observables of the form (24). Define

$$W(\phi|f) = |\phi| - \frac{1}{f}.$$  

(25)

As in the potential we find a term of order zero in the running coupling. Separating it off we obtain

$$W(\left(\frac{1}{f} + \Psi + \xi\hat{e} + \pi|f\right) = \frac{\Psi}{2} + \xi + W^{(1)}(\frac{\Psi}{2} + \xi, \pi|f).$$

(26)

In the limit when the radius becomes infinite the transformation (17) reduces to the convolution with the radial Gaussian measure. This transformation is identical with that of a one component scalar field in two dimensions at the high temperature fixed point. We are therefore immediately lead to normal ordered monomials. Put $\gamma' = \frac{2n}{3}$. Then it follows that

$$\int d\mu_{\frac{2}{3}}(\xi) : (\frac{\Psi}{2} + \xi)^n : \gamma' = \frac{1}{2^n} : \Psi^n : \gamma'.$$  

(27)

Therefore the eigenvectors are normal ordered monomials to zeroth order as expected. The zeroth order spectrum is $\epsilon_n = \frac{1}{2^n}$. Let us write the normal ordered monomials in the form

$$: \Psi^n : \gamma' = \sum_{m=0}^{n} h_{n,m} \Psi^m.$$  

(28)
They are of course rescaled Hermite polynomials in the variable $\Psi$. We then write the associated sequence of observables defined as solutions of the scaling equation (23) in the form

$$O_n(\phi|f) = \sum_{m=0}^{\infty} Q_{n,m}(f) \left( |\phi| - \frac{1}{f} \right)^m.$$  \hfill (29)

The coefficients $Q_{n,m}(f)$ are given by the power series expansions

$$Q_{n,m}(f) = \sum_{l=0}^{\infty} d_{n,m}^{(l)} f^l$$  \hfill (30)

with zero order coefficients of the normal ordered form

$$Q_{n,m}^{(0)} = h_{n,m}.$$  \hfill (31)

That is, the observables are perturbations of normal ordered monomials. The perturbative form of the moving spectrum is

$$\epsilon_n(f) = \sum_{m=0}^{\infty} \epsilon_n^{(m)} f^m$$  \hfill (32)

with zero order coefficients

$$\epsilon_n^{(0)} = \frac{1}{2^n}.$$  \hfill (33)

This completes the setup for the improvement program for observables. Let us then turn to the question of how to compute the higher coefficients in the expansions (30) and (32) for the moving eigenvectors and observables. The strategy is an adaptation of the improvement program for the potential. Let us choose the first (nontrivial) observable $O_1(\phi|f)$ as an example and perform a third order computation to some detail. The zeroth order argument provides us with the information that

$$O_1(\phi|f) = W(\phi|f) + O(f),$$  \hfill (34)

$$\epsilon_1(f) = \frac{1}{2} + O(f).$$  \hfill (35)

Let us denote the zeroth order approximation by $O(\phi|f) = W(\phi|f)$. As we will see this observable already scales to second order. We immediately perform a perturbation expansion to third order for the effective observable of the zeroth order approximation. The result is

$$(\mathcal{L}_{HT}\mathcal{R}) O(\phi|f) = \epsilon(f') \left( Q_0(f') + W(\phi|f') + Q_2(f')W(\phi|f')^2 \right) + O((f')^4),$$

9
\[ \epsilon(f') = \frac{1}{2} + \frac{(N - 1)\gamma}{4}(f')^2, \]
\[ Q_0(f') = -\frac{5(N - 1)\gamma^2}{3}(f')^3, \]
\[ Q_2(f') = -\frac{(N - 1)\gamma}{6}(f')^3, \]

where the old running coupling \( f \) is expressed in terms of the new running coupling \( f' = \beta(f) \). (\( L_{RT}R \) is understood as an operator which is applied to the function \( O(\phi|f) \).) This change of coupling prepares in particular the ground for further iterations. Let us emphasize that not only the coefficients but also the coordinate functions depend on the running coupling and have to be adjusted. From this we conclude that the observable already scales to second order. The eigenvalue is therefore already correct to second order. To third order both a constant and a term quadratic in \( W(\phi|f') \) are generated. Therefore the observable does not reproduce its dependence on the field to third order. To find the observable which scales to third order we make the ansatz

\[ O(\phi|f) = d_0f^3 + W(\phi|f) + d_2f^3W(\phi|f)^2. \]

The ansatz involves two improvement parameters \( d_0 \) and \( d_2 \). To determine their value one again computes the effective observable starting from (37). The expansion gives

\[ (L_{RT}R)O(\phi|f) = \epsilon(f') \left( Q_0(f') + W(\phi|f') + Q_2(f')W(\phi|f')^2 \right) + O((f')^4), \]
\[ \epsilon(f') = \frac{1}{2} + \frac{(N - 1)\gamma}{4}(f')^2, \]
\[ Q_0(f') = \left( -\frac{5(N - 1)\gamma^2}{3} + d_2\gamma + 2d_0 \right)(f')^3, \]
\[ Q_2(f') = \left( -\frac{(N - 1)\gamma}{6} + \frac{d_2}{2} \right)(f')^3. \]

The improvement parameters are then determined by the simple linear equations

\[ d_2 = -\frac{(N - 1)\gamma}{6} + \frac{d_2}{2}, \]
\[ d_0 = -\frac{5(N - 1)\gamma^2}{3} + d_2\gamma + 2d_0. \]

The solution is

\[ d_2 = -\frac{(N - 1)\gamma}{3}, \quad d_0 = 2(N - 1)\gamma^2. \]
To third order we therefore find

\[ O_1(\phi|f) = 2(N-1)\gamma^2 f^3 + W(\phi|f) - \frac{(N-1)\gamma}{3} f^3 W(\phi|f)^2 + O(f^4) \]  

(41)

together with the eigenvalue

\[ \epsilon_1(f) = \frac{1}{2} + \frac{(N-1)\gamma}{4} f^2 + O(f^4). \]  

(42)

Note that the eigenvalue does not have a term of third order in the running coupling. This scheme is iterated in the obvious manner. Suppose that we have computed the observable to order \( s \) in the running coupling

\[ O_1(\phi|f) = O_1^{(s)}(\phi|f) + O(f^{s+1}). \]  

(43)

Suppose that the order \( s \) improved observable is given by the general form

\[
O_1^{(s)}(\phi|f) = \sum_{n=0}^{s-1} Q_{1,n}^{(s)}(f) W(\phi|f)^n, \\
Q_{1,0}^{(s)}(f) = \sum_{r=0}^{s} d_{1,0}^{(r)} f^r, \\
Q_{1,1}^{(s)}(f) = 1, \\
Q_{1,n}^{(s)}(f) = \sum_{r=n+1}^{s} d_{1,n}^{(r)} f^r.
\]  

(44)

Then by induction it follows that the effective observable to order \( s + 1 \) is again of this general form. It follows that to every order of perturbation theory only finitely many powers of the coordinate functions \( W(\phi|f) \) appear. The sums turn out to involve either even or odd powers in the running coupling respectively. We assume that \( O_1^{(s)}(\phi|f) \) scales to order \( s \). That is,

\[ (\mathcal{L}_{RT}) O_1^{(s)}(\phi|f) = O_1^{(s)}(\phi|f') + O((f')^{s+1}). \]  

(45)

Then we take an ansatz of the form (44) to order \( s + 1 \) treating the coefficients of order \( s + 1 \) in the running coupling as improvement parameters. We compute the effective observable. It depends linearly on the improvement parameters. To order \( s + 1 \) it has no other choice. Then we claim invariance to obtain a linear system of equations for the improvement coefficients. This system turns out always to have a unique solution: the improved observable.
4 Hierarchical fusion rules

The outcome of our analysis is a system of observables \( O_n(\phi|f) \) on the renormalized trajectory parametrized by \( f \). Under a hierarchical renormalization group transformation \( O_n(\phi|f) \) is multiplied by the moving eigenvalue \( \epsilon_n(\beta(f)) \) and the coordinate is changed to \( \beta(f) \). This scheme is obviously iterable which is the reason why we introduced it from the beginning. To compute general correlation functions we need one more ingredient which is the notion of hierarchical fusion rules. The general form of our observables in terms of \( \Psi = |\phi| - r \) is

\[
O_n(\phi|f) = \sum_{r=0}^{\infty} O_n^{(r)}(\Psi) f^r,
\]

\[
O_n^{(0)}(\Psi) = : \Psi^n :,
\]

\[
O_n^{(1)}(\Psi) = 0,
\]

\[
O_n^{(2)}(\Psi) = \sum_{m=0}^{[n-2]} O_{n,n-2-2m}^{(2)} : \Psi^{n-2-2m} :,
\]

\[
O_n^{(r)}(\Psi) = \sum_{m=0}^{[n+r-2]} O_{n,n+r-2-2m}^{(r)} : \Psi^{n+r-2-2m} :.
\]

The normal ordering covariance is \( \gamma' = \frac{2}{3}\gamma \). Let us put \( \gamma = 1 \) to simplify the notation. To zeroth order in \( f \) we rediscover normal ordered monomials in \( \Psi \). Their perturbations along the renormalized trajectory prove to have no first order terms in the running coupling \( f \). To every order of perturbation theory in \( f \) we find only finitely many normal ordered powers of \( \Psi \). The highest power is \( n+r-2 \) for \( r > 2 \).

Associated with this system of observables is a system of fusion rules defined by

\[
O_n(\phi|f)O_m(\phi|f) = \sum_{l=0}^{\infty} N_{n,m;l}(f)O_l(\phi|f),
\]

\[
N_{n,m;l}(f) = \sum_{r=0}^{\infty} N_{n,m;r}^{(r)} f^r.
\]

Furthermore from the fusion rules we obtain a symmetric bilinear form on the linear space of observables. It is defined by

\[
(\mathcal{O}_n(\phi|f), \mathcal{O}_m(\phi|f)) = N_{n,m;0}(f).
\]

The physical significance of this bilinear form is that in the thermodynamic limit only the overlap of an observable with the constant term is expected to survive.
Since correlation functions are expected to decrease with distance the spectrum consists of eigenvalues strictly smaller than one on the renormalized trajectory. To zeroth order of perturbation theory we recapture the fusion rules of normal ordered products

\[
N_{n,m;l}^{(0)} = \frac{n!m!(\gamma')}{}{n'!m'!(\gamma')}^l, \\
n' = \frac{1}{2}(l + m - n), \\
m' = \frac{1}{2}(n + l - m), \\
l' = \frac{1}{2}(n + m - l), \tag{49}
\]

for \(|n - m| \leq l \leq n + m\) and \(n + m - l \in 2\mathbb{Z}\), zero else. Furthermore to zeroth order the observables are orthogonal with respect to the bilinear form \(N_{n,m;l}^{(0)} = \delta_{n,m} m! (\gamma')^m\). \(\tag{50}\)

The simple pattern \((49), (50)\) becomes perturbed as one moves away from the ultraviolet fixed point on the renormalized trajectory. The perturbation expansion for \(N_{1,1,l}(f)\) to fifth order in \(f\) is for instance

\[
N_{1,1,0}(f) = \frac{2}{3} + \frac{5(N - 1)}{9} f^2 - \left(\frac{20(N - 1)^2}{27} - \frac{233(N - 1)}{63}\right) f^4 + O(f^6),
\]

\[
N_{1,1,1}(f) = -2(N - 1) f^3 + \left(\frac{17(N - 1)^2}{3} - 26(N - 1)\right) f^5 + O(f^7),
\]

\[
N_{1,1,2}(f) = 1 + O(f^6),
\]

\[
N_{1,1,3}(f) = 24(N - 1) f^5 + O(f^7). \tag{51}
\]

The perturbation expansion for \(N_{1,2,l}(f)\) to fifth order in \(f\) is given by

\[
N_{1,2,0}(f) = \frac{40(N - 1)^2}{21} f^5 + O(f^7),
\]

\[
N_{1,2,1}(f) = \frac{4}{3} + \frac{10(N - 1)}{9} f^2 - \left(\frac{40(N - 1)^2}{27} - \frac{598(N - 1)}{63}\right) f^4 + O(f^6),
\]

\[
N_{1,2,2}(f) = -4(N - 1) f^3 + \left(\frac{34(N - 1)^2}{3} - \frac{404(N - 1)}{7}\right) f^5 + O(f^7),
\]

\[
N_{1,2,3}(f) = 1 - \frac{4(N - 1)}{21} f^4 + O(f^6),
\]

\[
N_{1,2,4}(f) = \frac{48(N - 1)}{35} f^5 + O(f^7). \tag{52}
\]
All other fusion rules are zero to fifth order. We observe that orthogonality is violated to fifth order for the first and second observable.

With the fusion rules we can compute correlation functions of our observables. Let us consider for example a general two point function. It depends on the hierarchical distance $k$ and the hierarchical lattice size $k'$ of the system. The explicit formula is

$$\langle O_n(\phi|f)O_m(\phi|f) \rangle = \prod_{j=1}^{k} \left( \epsilon_n \left( \beta^j(f) \right) \epsilon_m \left( \beta^j(f) \right) \right) \sum_{l=0}^{\infty} N_{n,m;l} \left( \beta^{k'}(f) \right) \prod_{i=1}^{k'-k} \epsilon_l \left( \beta^{k+i}(f) \right)$$

for the two point function on the point of the renormalized trajectory parametrized by $f$. This formula simply says that each observable is renormalized independently $k$-times. Each renormalization step produces a factor given by the eigenvalue of the corresponding observable at the location on the renormalized trajectory. After $k$ steps the observables land in the same block and are there fused together. The result of fusion is then renormalized $(k' - k)$-times to obtain the value of the two point function. The thermodynamic limit corresponds to $k' = \infty$. The formula (53) is still true in this limit provided that one changes to a different parametrization of the renormalized trajectory at the point where the running coupling $f$ diverges.

5 Numerical results

When dealing with perturbation theory it is natural to question its validity. To tackle this problem we have determined our perfect observables and their corresponding eigenvalues numerically. The main technical task is to compute the transformations

$$\mathcal{R}V(\psi) = -2 \ln \left( \int d\mu_\gamma(\zeta) e^{-V(\psi+\zeta)} \right),$$

$$\mathcal{L}_V \mathcal{R}O(\psi) = \frac{\int d\mu_\gamma(\zeta) e^{-V(\psi+\zeta)} O(\psi + \zeta)}{\int d\mu_\gamma(\zeta) e^{-V(\psi+\zeta)}}.$$

Both equations can be reduced to integrals of the type

$$\mathcal{I}_F(\psi) = \int d\mu_\gamma(\zeta) F(|\psi + \zeta|)$$

with certain scalar functions $F(\varphi)$. A shift $\zeta \rightarrow \psi + \zeta$ in the fluctuation field yields

$$\mathcal{I}_F(|\psi|) = (2\pi\gamma)^{N/2} \int d^N \zeta \exp \left( -\frac{1}{2\gamma}[\zeta^2 + \psi^2] + \frac{1}{\gamma}\psi\zeta \right) F(|\zeta|).$$
By using polar coordinates and integrating out the polar angles we find for \( N = 3 \)
\[
\mathcal{I}_F(|\psi|) = \mathcal{N} \int_0^\infty dR \exp \left( -\frac{1}{2\gamma} [R^2 + |\psi|^2] \right) \sinh \left( \frac{|\psi|R}{\gamma} \right) \frac{RF(R)}{|\psi|}. \tag{58}
\]

In this form the integral can be evaluated by standard numerical methods.

Equipped with integrators for (54) and (55) the strategy goes as follows: The first step is to determine \( V_{RT} \). For this purpose we start with the perturbatively improved perfect action \( V_{RT}^{\text{pert}}(\phi|r_0) \) at a given radius \( r_0 \) as bare potential and iterate the RG transformation (54). In each step the resulting potential is driven closer and closer towards the RT. After 10 steps we end up with a good approximation of \( V_{RT}(r) \) at some radius \( r = r(r_0) \). In this iterative process the potential \( V(\varphi) \) is represented as a cubic spline with \( N_\varphi \) equidistant knots \( \varphi_i \) in the range \( I_\varphi = [\varphi_{\text{min}}, \varphi_{\text{max}}] \). For each iteration one has to evaluate \( V' = \mathcal{R}V \) at these points. The interval \( I_\varphi \) must be chosen in such a way that for the computation of \( V'(\varphi \not\in I_\varphi) \) contributions \( V(\varphi \not\in I_\varphi) \) are numerically negligible in (54).

In the second step the eigenvalues and eigenoperators of the linearized RG transformation \( \mathcal{L}_{RT} \) at this very potential \( V_{RT}(r) \) are computed. The operator \( \mathcal{L}_{RT} \) acts on the infinite dimensional space of observables. Naturally a computer can only handle the restriction of \( \mathcal{L}_{RT} \) to a finite subspace. Our program uses the space spanned by the operators \( B_m(\phi|r) = (|\phi| - r)^m \) with \( m = 0, \ldots, M \). To obtain the representation matrix \( L_{n,m} \) of \( \mathcal{L}_{RT} \) in this basis the image of every \( B_m(\phi|r) \) under \( \mathcal{L}_{RT} \) is numerically expanded by a finite difference method in terms of \( B_n(\phi|r') \)
\[
(\mathcal{L}_{RT} \mathcal{R}) B_m(\phi|r) = \sum_{n=0}^M L_{n,m} B_n(\phi|r'). \tag{59}
\]

Finally the desired eigenvectors \( O_m(r') = \sum_{n=0}^M Q_{n,m}(r') B_n(r') \) and their eigenvalues \( \epsilon_n(r') \) are calculated from \( L_{n,m} \). Fig. 1 shows the flow of the largest eigenvalues \( \epsilon_m(r') \). As in the case of the potential the perturbative predictions are in excellent agreement with the numerical results down to a radius of about \( r' \approx 4 \). Then non-perturbative effects show up forcing the eigenvalues to become smaller. A similar behaviour can be found for the expansions coefficients \( Q_{n,m}(r') \) except that deviations already show up at \( r' \approx 10 \).

At \( r = r_{\text{cr}} \approx 2.04 \) the effective radius \( r'(r) \) vanishes and \( r \) ceases to be an appropriate parametrization of the RT but the eigenvalues and eigenvectors continue to flow against their fixpoint values at the HT-fixpoint.
A Hierarchical renormalization group

In a hierarchical lattice field theory in the form advocated by Gallavotti [G] and by Gawedzki and Kupiainen [GK] the massless free covariance $(-\Delta)^{-1}$ is replaced by the hierarchical approximation

$$v(x, y) = \gamma \sum_{m=0}^{n} L^{m(2-D)} \delta_{[L^{-m}x], [L^{-m}y]}.$$ (60)

Its parameters are a positive real number $\gamma$ and the positive integer valued block size $L$. The bracket $[L^{-m}x]$ denotes the integer part of $L^{-m}x$. The lattice is $\{0, 1, \ldots, L^n - 1\}^D$ with $n \in \mathbb{N} \cup \{\infty\}$. The main property of (60) is the hierarchical splitting

$$v = L^{2-D} C^T v C + \Gamma,$$ (61)

$C$ being the unnormalized block average $C(x, y) = \delta_{x, [L^{-1}y]}$ and $\Gamma$ the diagonal part $\Gamma(x, y) = \gamma \delta_{x, y}$. The hierarchical covariance on the right hand side is understood to live on a lattice shrunk by the blocking scale $L$. The splitting (61) induces a decomposition of the lattice field $\phi$ into a fluctuation field $\zeta$ and a background field $\psi$. The associated block spin transformation is

$$e^{-R V(\psi)} = \mathcal{N} \int d\mu_{\Gamma}(\zeta) e^{-V(L^{1-D} C^T \psi + \zeta)}.$$ (62)

Here $d\mu_{\Gamma}(\zeta)$ is the Gaussian measure on field space with mean zero and covariance $\Gamma$. The $n$-fold iteration of (62) computes the hierarchical partition function (when the normalization is $\mathcal{N} = 1$). The transformation (62) preserves locality. For $V(\phi) = \sum_x V(\phi(x))$ we find $\mathcal{R} V(\phi) = \sum_x \mathcal{R} V(\phi(x))$ in a selfexplanatory notation. ($\mathcal{R}$ is not meant to be linear.) (62) then reduces to the non-linear transformation

$$e^{-\mathcal{R} V(\psi)} = \mathcal{N} \left( \int d\mu_{\Gamma}(\zeta) e^{-V(L^{1-D} C^T \psi + \zeta)} \right)^{L^D}.$$ (63)

on functions of a single variable $\phi$. Note that $C^T \psi$ is constant on blocks and that $L^D$ is the number of sites in one block. The linear block spin transformation for observables is

$$L_y \mathcal{R} \mathcal{O}(\psi) = \frac{\int d\mu_{\Gamma}(\zeta) e^{-V(L^{1-D} C^T \psi + \zeta)} \mathcal{O}(L^{1-D} C^T \psi + \zeta)}{\int d\mu_{\Gamma}(\zeta) e^{-V(L^{1-D} C^T \psi + \zeta)}}.$$ (64)

The expectation value of an effective observable in the effective theory equals that in the original theory. (65) preserves the property of factorization provided that the
interaction is local. A local observable $\mathcal{O}(\phi(x))$ is defined as a function of the field $\phi$ at a single site $x$. Local observables are again transformed into local observables according to

$$L_V \mathcal{O} = \int d\mu(\gamma) e^{-V(L^{-\frac{D}{2}}x\psi+\zeta)} O(L^{-\frac{D}{2}}x\psi+\zeta).$$

(65)

The transformation (64) is the linearization of (62) at the interaction $V(\phi)$. The local transformation (65) is the linearization of (63) at the local interaction $V(\phi)$ divided by the block volume $L^D$. The expectation value of a local observable is the $n$-fold iteration of (65) evaluated at zero external field. The thermodynamic limit $n = \infty$ follows from an infinite iteration.

The computation of higher correlation functions requires another ingredient called fusion. Let us consider for instance a bilocal observable of the product form $O_1(\phi(x_1))O_2(\phi(x_2))$ depending on the field $\phi$ at two sites $x_1$ and $x_2$. The hierarchical distance of $x_1$ and $x_2$ is given by $\text{dist}(x_1, x_2) = L^k(x_1, x_2)$, where $k(x_1, x_2)$ is the smallest positive integer such that $[L^{-k(x_1, x_2)}x_1] = [L^{-k(x_1, x_2)}x_1]$. Both observable are then transformed independently $k(x_1, x_2)$-times according to the local transformation (65) until $x_1$ and $x_2$ land in the same block, whereupon the effective observables are multiplied together. We call this process fusion. The result is then again transformed $n - k(x_1, x_2)$-times according to the local rule (65) to give the hierarchical two point function. In the thermodynamic limit $n = \infty$ the observable has to be renormalized an infinite number of steps after fusion. The process can be visualized as a fork.

Both (63) and (65) make sense when the dimension parameter $D$ and the scale parameter $L$ are continued to real values.

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Figure 1: Flow of the eigenvalues $\epsilon(m,r')$ and the expansion coefficients $Q_{n,m}(r')$ along the renormalized trajectory. From top to down $\epsilon_1, \epsilon_2, \epsilon_3$ and $Q_{0,1}, Q_{0,3}, Q_{0,2}$. Solid lines represent numerical data, dashed lines perturbative results. Calculations were done with $N_\phi = 200, \varphi_{\text{min}} = 0, \varphi_{\text{max}} = 30, M = 10$. 

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