ON THE SEMI-SIMPLICITY CONJECTURE FOR $\mathbb{Q}^{ab}$

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ABSTRACT. We show that the semi-simplicity conjecture for finitely generated fields follows from the conjunction of the semi-simplicity conjecture for finite fields and for the maximal abelian extension of the field of rational numbers.

1. Notation

1.1. If $R$ is a ring, for every $n \in \mathbb{Z}_{>0}$ we denote by $R[\zeta_n]$ the quotient $R[t]/(t^n - 1)$ and by $R[\zeta_\infty]$ the ring $\varprojlim_n R[\zeta_n]$. At the same time, if $\ell$ is a prime number we denote by $R[\zeta_\ell]$ the ring $\varprojlim_n R[\zeta_\ell^n]$. For every field $k$ we choose an algebraic closure $\overline{k}$ and we denote by $\Gamma_k$ the group $\text{Gal}(\overline{k}/k)$. We denote by $\mathbb{Q}^{ab}$ the maximal abelian extension of $\mathbb{Q}$ in $\overline{\mathbb{Q}}$.

1.2. Let $k$ be a field. A variety over $k$ will be for us a separated and reduced scheme of finite type over $k$. If $X$ is a scheme over $k$, we denote by $X_\mathbb{F}$ the scheme $X \otimes_k \mathbb{F}$, over $\mathbb{F}$. Let $\ell$ be a prime different from the characteristic of $k$. We denote by $\text{Rep}_{\mathbb{Q}_\ell}(\Gamma_k)$ the neutral Tannakian category of finite-dimensional $\ell$-adic $\mathbb{Q}_\ell$-linear representations of $\Gamma_k$. We will refer to objects in $\text{Rep}_{\mathbb{Q}_\ell}(\Gamma_k)$ simply as $\ell$-adic representations of $\Gamma_k$. Let $\mathcal{C}(k, \ell)$ be the smallest neutral Tannakian subcategory of $\text{Rep}_{\mathbb{Q}_\ell}(\Gamma_k)$, closed under subquotients, which contains all the $\ell$-adic representations of $\Gamma_k$ of the form $H^i_{\text{ét}}(X_\overline{k}, \mathbb{Q}_\ell)$, where $i$ is an integer, $X$ is a smooth and projective variety over $k$ and $X_\overline{k} := X \otimes_k \overline{k}$. We will say that an object in $\mathcal{C}(k, \ell)$ is an $\ell$-adic representation of $\Gamma_k$ coming from geometry.

1.3. For a field $K$ of characteristic 0 and $V$ a finite-dimensional $K$-vector space, we will say that a linear endomorphism $\varphi$ of $K$ is semi-simple if it is diagonalizable after a finite extension of $K$. Let $V_p$ be an $\ell$-adic representation of $\Gamma_F$ and $p \neq \ell$ a prime where $V_p$ is unramified, we will say that $\rho$ is semi-simple at $p$ if one (or equivalently any) Frobenius element at $p$ acts via a semi-simple automorphism.

2. Introduction

Let $k$ be a field, we consider the following statement:

$S(k)$: For every prime $\ell$ different from the characteristic of $k$, an $\ell$-adic representation of $\Gamma_k$ coming from geometry is semi-simple.

Grothendieck and Serre have conjectured that for a finitely generated field $k$, then $S(k)$ holds (see [Tat65]). This conjecture is commonly known as the semi-simplicity conjecture. Notice that the conjecture predicts that $S(k)$ is true even for fields $k$ which are infinite Galois extensions of a finitely generated field. Indeed, if $k'/k$ is a Galois extension, then $S(k)$ implies $S(k')$, because the restriction of a semi-simple representation to a normal subgroup is semi-simple. At the same time, we recall that $S(k)$ is false for arbitrary fields, for example it is false for $\mathbb{Q}_p$ and $\mathbb{C}((t))$.

In this article we prove that, if we assume $S(F_p)$ for every prime $p$, we can deduce $S(\mathbb{Q})$ from $S(\mathbb{Q}^{ab})$. This yields the following result.

Theorem 2.1 (Theorem 4.7). Let $k$ be a Galois extension of a finitely generated field. The conjunction of $S(F_p)$ for every prime $p$ and $S(\mathbb{Q}^{ab})$ implies $S(k)$.

Let us make a brief summary on what is already known about the conjecture. The first result has been obtained by Weil, who has proven in 1948, thanks to the positivity of the Rosati involution, the conjecture for abelian varieties (and hence for curves) over finite fields. In 1983, Faltings has proven the semi-simplicity conjecture for abelian varieties over number fields as an intermediate step for his proof.
Conjecture A. If an \(S\)-specified prime

\[\text{Theorem 2.5 (\cite{Ser00})}\]

semi-simplicity conjecture reduces to \(S\) and a Frobenius lift \(\hat{\phi}\) where

\[\text{By transport of structure, } \hat{\phi} \text{ is semi-simple, then } F \text{ is a semi-simple lisse sheaf on } X.\]

\[\text{Corollary 2.4. Let } k \text{ be a Galois extension of a finitely generated field of positive characteristic } p, \text{ then } S(F_p) \implies S(k).\]

In characteristic 0, thanks to Serre’s specialization argument, via Hilbert’s irreducibility theorem, the semi-simplicity conjecture reduces to \(S(Q)\).

\[\text{Theorem 2.5 (\cite{Ser00})}. \text{ Let } k \text{ be a Galois extension of a finitely generated field of characteristic } 0, \text{ then } S(Q) \implies S(k).\]

3. Some analogies

We intend to investigate in this article the following mixed-characteristic analogue of Theorem 2.3.

Conjecture A. If an \(\ell\)-adic representation of \(\Gamma_Q\) coming from geometry is semi-simple at some unramified prime \(p\) different from \(\ell\), then it is semi-simple as a representation of \(\Gamma_Q\).

We will show in the next section how to adapt Fu’s proof of Theorem 2.3 to prove that \(S(Q^{ab})\) implies Conjecture A.

Remark 3.1. We think \(S(Q^{ab})\) as the analogue of Theorem 2.2. We do not know whether it is possible to prove \(S(Q^{ab})\) via a suitable theory of weights for \(\Gamma_Q^{ab}\).

More concretely, let \(V_\rho\) be an \(\ell\)-adic representation coming from geometry of \(\Gamma_Q\) of weight 0. Let \(N\) be a positive multiple of every prime where \(V_\rho\) is ramified. We consider the vector space

\[H_\rho := H_{ab}(\text{Spec}(\mathbb{Z}[\ell \times \mathbb{Z}], V_\rho)).\]

By transport of structure, \(H_\rho\) is endowed with an action of the monoid

\[\text{End}(\mathbb{Z}[\ell \times \mathbb{Z}]) = (\hat{\mathbb{Z}}, \times),\]

where \((\hat{\mathbb{Z}}, \times)\) is the multiplicative monoid of the ring \(\hat{\mathbb{Z}}\). Let \(\varphi_\rho\) be the endomorphism on \(H_\rho\) induced by \(p \in (\hat{\mathbb{Z}}, \times)\). If we know that for every eigenvalue \(\alpha\) of \(\varphi_\rho\), there exists \(\iota: \mathbb{Q}_\ell \to \mathbb{C}\) such that \(|\iota(\alpha)| < 1\), then we also know that \(\varphi_\rho\) acts without fixed points. Thus we prove that there are no non-trivial extensions of \(\mathbb{Q}_\ell\) by \(V_\rho\) over \(\text{Spec}(\mathbb{Z}[\ell \times \mathbb{Z}])\) which descend to \(\text{Spec}(\mathbb{Z}[\ell \times \mathbb{Z}])\). For this strategy, the ring \(\mathbb{Z}[\ell \times \mathbb{Z}])\) can be also replaced by the Dedekind domain \(\mathbb{Z}[\ell \times \mathbb{Z}])\).

4. Our main result

We adapt Fu’s proof in [Fu90] in our situation. To do this we first introduce the notion of the Weil group of \(Q\) associated to a Frobenius element at a prime \(p\).

We take the natural exact sequence

\[1 \to \Gamma_{Q^{ab}} \to \Gamma_Q \xrightarrow{\delta} \hat{\mathbb{Z}}^\times \to 1,\]

where \(\hat{\mathbb{Z}}^\times = \text{Gal}(Q^{ab}/Q)\). We choose a closed embedding \(\Gamma_{Q_p} \subseteq \Gamma_Q\) induced by a field inclusion \(\mathbb{Q} \hookrightarrow \mathbb{Q}_p\) and a Frobenius lift \(F_p \in \Gamma_{Q_p} \subseteq \Gamma_Q\).
Definition 4.1. We define the Weil group of \( \mathbb{Q} \) with respect to \( F_p \) as the semi-direct product \( \mathbb{W}_{\mathbb{Q}, F_p} := \Gamma_{\mathbb{Q}_{ab}} \rtimes \mathbb{Z} \), where \( \mathbb{Z} \) is endowed with the discrete topology and the action of \( 1 \in \mathbb{Z} \) on \( \Gamma_{\mathbb{Q}_{ab}} \) is the adjoint action of \( F_p \in \Gamma_{\mathbb{Q}} \) on \( \Gamma_{\mathbb{Q}_{ab}} \).

We have a canonical injective continuous map \( \mathbb{W}_{\mathbb{Q}, F_p} \hookrightarrow \Gamma_{\mathbb{Q}} \) which sends \( 1 \in \mathbb{Z} \) to \( F_p \). Contrary to the Weil group of a connected scheme over a finite field, in this case for every choice of a Frobenius element \( F_p \), the subgroup \( \mathbb{W}_{\mathbb{Q}, F_p} \) is not dense in \( \Gamma_{\mathbb{Q}} \). Its closure is not even of finite index in \( \Gamma_{\mathbb{Q}} \). We will see in Proposition 4.5 how to overcome this problem.

We start by recalling a well-known fact on the quotients of \( \hat{\mathbb{Z}}^\times \). It is worth mentioning that this lemma is also one of the main ingredients of Moonen’s recent result on the semi-simplicity conjecture [Moo17].

Lemma 4.2. The profinite group \( \hat{\mathbb{Z}}^\times \) admits a unique continuous quotient isomorphic to \( \mathbb{Z}_\ell \).

Proof. The subgroup \( \bigoplus_{\ell'} \mathbb{Z}_{\ell'}^\times \subseteq \hat{\mathbb{Z}}^\times \) is dense, hence it is enough to prove the result on \( \bigoplus_{\ell'} \mathbb{Z}_{\ell'}^\times \). We know that \( \text{Hom}(\bigoplus_{\ell'} \mathbb{Z}_{\ell'}^\times, \mathbb{Z}_\ell) \) is isomorphic to \( \mathbb{Z}_\ell \). By an explicit computation

\[
\text{Hom}(\mathbb{Z}_{\ell'}^\times, \mathbb{Z}_\ell) = \begin{cases} 
1 & \text{for } \ell' \neq \ell \\
\mathbb{Z}_\ell & \text{for } \ell' = \ell.
\end{cases}
\]

This shows that \( \text{Hom}(\bigoplus_{\ell'} \mathbb{Z}_{\ell'}^\times, \mathbb{Z}_\ell) = \mathbb{Z}_\ell \), which implies the result. \( \square \)

We fix for every prime \( \ell \) a quotient \( \delta_\ell : \Gamma_{\mathbb{Q}} \twoheadrightarrow \mathbb{Z}_\ell \).

Lemma 4.3. If \( \ell \neq p \), the closure of the group generated by \( \delta_\ell(F_p) \) in \( \mathbb{Z}_\ell \) is an open subgroup.

Proof. Let \( K \) be the closure of the group generated by \( \delta_\ell(F_p) \) in \( \mathbb{Z}_\ell \). It is endowed with a natural \( \mathbb{Z}_\ell \)-module structure, hence it is an ideal of \( \mathbb{Z}_\ell \). We need to show that \( K \) is non-trivial.

Let \( \mathbb{Q}_p^{ur} \) be the maximal unramified extension of \( \mathbb{Q}_p \) in \( \overline{\mathbb{Q}}_p \). The map \( \Gamma_{\mathbb{Q}_p} \rightarrow \text{Gal}(\mathbb{Q}(\zeta_{\infty})/\mathbb{Q}) \) induced by the inclusion \( \Gamma_{\mathbb{Q}_p} \subseteq \Gamma_{\mathbb{Q}} \) we have chosen, factors through \( \text{Gal}(\mathbb{Q}^{ur}_p/\mathbb{Q}_p) \), because the extension \( \mathbb{Q}(\zeta_{\infty})/\mathbb{Q} \) is unramified at \( p \). By definition, \( F_p \) is sent to the unique Frobenius element in \( \text{Gal}(\mathbb{Q}^{ur}_p/\mathbb{Q}_p) \), thus its image in \( \text{Gal}(\mathbb{Q}(\zeta_{\infty})/\mathbb{Q}) \) is the unique automorphism of \( \mathbb{Q}(\zeta_{\infty}) \) which raises each root of unit to its \( p \)-th power. This automorphism has finite order in \( \text{Gal}(\mathbb{Q}(\zeta_{\infty})/\mathbb{Q}) \), thus its image \( \delta_\ell(F_p) \) \( \mathbb{Z}_\ell \)-quotient of \( \text{Gal}(\mathbb{Q}(\zeta_{\infty})/\mathbb{Q}) \) is non-trivial. In particular, is non-trivial, as we wanted. \( \square \)

Let \( \ell \) be a prime and \( \rho \) an \( \ell \)-adic representation of \( \Gamma_{\mathbb{Q}} \). We denote by \( \Pi \) the image of \( \Gamma_{\mathbb{Q}} \), by \( \Pi^0 \) the image of \( \Gamma_{\mathbb{Q}_{ab}} \), by \( \overline{\Pi} \) the quotient \( \Pi/\Pi^0 \) and by \( \pi \) the natural projection \( \pi : \Pi \rightarrow \overline{\Pi} \). We obtain the following commutative diagram with exact rows of profinite groups

\[
\begin{array}{cccccc}
1 & \rightarrow & \Gamma_{\mathbb{Q}_{ab}} & \rightarrow & \Gamma_{\mathbb{Q}} & \rightarrow & \hat{\mathbb{Z}}^\times & \rightarrow & 1 \\
\downarrow & & \downarrow & & \delta & & \downarrow & & \\
1 & \rightarrow & \Pi^0 & \rightarrow & \Pi & \rightarrow & \overline{\Pi} & \rightarrow & 1.
\end{array}
\]

Lemma 4.4. The group \( \overline{\Pi} \) is either a finite abelian group or it admits a surjective morphism \( \pi_\ell : \overline{\Pi} \twoheadrightarrow \mathbb{Z}_\ell \) with finite Kernel such that \( \delta_\ell = \pi_\ell \circ \pi \circ \rho \).

Proof. The group \( \Pi \) is a closed subgroup of the topological group \( \text{GL}(V_p) \), thus by [DSMS91, Theorem 9.6], it can be endowed with the structure of an \( \ell \)-adic analytic group. By [ibid., Theorem 8.32], it contains an open topologically finitely generated pro-\( \ell \)-subgroup. As \( \pi \) is surjective, the same is true for \( \overline{\Pi} \).

The group \( \overline{\Pi} \), being a quotient of \( \hat{\mathbb{Z}}^\times \), is an abelian group. Let \( \overline{\Pi}_\ell \subseteq \overline{\Pi} \) be the maximal pro-\( \ell \)-subgroup of \( \overline{\Pi} \). By the previous argument, \( \overline{\Pi}_\ell \) is open in \( \overline{\Pi} \) and it is topologically finitely generated. Hence \( \overline{\Pi}_\ell \) is a finitely generated \( \mathbb{Z}_\ell \)-module with respect to its natural \( \mathbb{Z}_\ell \)-module structure.

At the same time, the group \( \overline{\Pi}_\ell \) is a quotient of \( \hat{\mathbb{Z}}^\times \), thus by Lemma 4.2, it is a \( \mathbb{Z}_\ell \)-module of rank at most one. This means precisely that either \( \overline{\Pi} \) is finite or it is isomorphic to \( \Lambda \times \mathbb{Z}_\ell \), with \( \Lambda \) a finite group.
Notice that again, by the uniqueness of the $\mathbb{Z}_\ell$-quotients of $\hat{\mathbb{Z}}^\times$, we can choose a projection $\pi_\ell : \Pi \to \mathbb{Z}_\ell$ with finite Kernel such that $\pi_\ell = \pi_\ell \circ \pi \circ \rho$.

We can prove now the main technical result needed for the final theorem.

**Proposition 4.5.** An $\ell$-adic representation $\rho$ of $\Gamma_q$ is semi-simple if for some prime $p \neq \ell$ and for some choice of $F_p$, its restriction to $W_{\mathbb{Q}, F_p}$ is semi-simple.

*Proof.* Let $\Pi_p$ be the closure of the image of $W_{\mathbb{Q}, F_p}$ in $\Pi$. In light of [Fu90, Lemma 1], it is enough to show that $\Pi_p$ has finite index in $\Pi$. We notice that as $\Pi^0 \subseteq \Pi_p$, if we set $\Pi_p := \pi(\Pi_p)$, then $[\Pi : \Pi_p] = [\Pi : \Pi_p]$, thus we are reduced to show that $\Pi_p$ has finite index in $\Pi$.

By Lemma 4.4, we have two cases. If $\overline{\Pi}$ is finite the result holds trivially. If $\overline{\Pi}$ is infinite, we take $\pi_\ell : \Pi \to \mathbb{Z}_\ell$ as in the lemma. As $\pi_\ell$ has finite Kernel, it is enough to show that $[\mathbb{Z}_\ell : \pi_\ell(\Pi_p)]$ is finite. The profinite group $\Pi_p$ is topologically generated by $\rho(F_p)$, hence the group $\pi_\ell(\Pi_p)$ is topologically generated by $\delta_\ell(F_p)$. We conclude in virtue of Lemma 4.3.

**Theorem 4.6.** Let $\rho$ be an $\ell$-adic representation of $\Gamma_q$ which is semi-simple when restricted to $\Gamma_{q^{ab}}$. If there exists a prime $p \neq \ell$ and $F_p \in \Gamma_{q^{ab}}$, a Frobenius element at $p$, such that $\rho(F_p)$ is semi-simple, then $\rho$ is a semi-simple representation of $\Gamma_q$. In particular, $S(\mathbb{Q}^{ab})$ implies Conjecture A.

*Proof.* By Proposition 4.5, it is enough to check that $\rho$ is semi-simple when restricted to $W_{\mathbb{Q}, F_p} \rightarrow \Gamma_q$.

Let $G^0$ be the Zariski closure of $\Pi^0$ in $\text{GL}(V_p)$ and let $G_{F_p}$ be the semi-direct product $G^0 \times \mathbb{Z}$, where $\mathbb{Z}$ is endowed with the discrete topology and $1 \in \mathbb{Z}$ acts on $G^0$ as $\rho(F_p)$ acts on $G^0$ by conjugation.

We define $\bar{\rho} : W_{\mathbb{Q}, F_p} \rightarrow G_{F_p}$ as the only morphism making the following diagram commuting

\[
\begin{array}{cccccccc}
1 & \rightarrow & \Gamma_{q^{ab}} & \rightarrow & W_{\mathbb{Q}, F_p} & \rightarrow & \mathbb{Z} & \rightarrow & 1 \\
\downarrow{\rho} & & \downarrow{\bar{\rho}} & & \downarrow & & \downarrow & & \\
1 & \rightarrow & G^0 & \rightarrow & G_{F_p} & \rightarrow & \mathbb{Z} & \rightarrow & 1.
\end{array}
\]

At the same time, let $\sigma : G_{F_p} \rightarrow \text{GL}(V_p)$ be the representation which extends the tautological representation $G^0 \rightarrow \text{GL}(V_p)$ by sending $1 \in \mathbb{Z}$ to $\rho(F_p)$. The composition $\sigma \circ \bar{\rho} : W_{\mathbb{Q}, F_p} \rightarrow \text{GL}(V_p)$ is equal to $\rho$. As $\bar{\rho}$ has Zariski-dense image, to show that the restriction of $\rho$ to $W_{\mathbb{Q}, F_p}$ is semi-simple it is enough to show that $\sigma$ is semi-simple.

The group $G^0$ is reductive because, by assumption, $\rho$ is semi-simple when restricted to $\Gamma_{q^{ab}}$. In virtue of [Del80, Lemme 1.3.10], there exists a $\overline{\mathbb{Q}}_\ell$-point $g$ in the center of $G_{F_p}$ of the form $(g', d)$ where $g' \in G^0(\overline{\mathbb{Q}}_\ell)$ and $d \neq 0$. As $\rho(F_p)$ is semi-simple, we can apply [Fu90, Lemma 2] to the representation $\sigma$, obtaining thereby the desired result.

**Theorem 4.7.** Let $k$ be a Galois extension of a finitely generated field. The conjunction of $S(\mathbb{F}_p)$ for every prime $p$ and $S(\mathbb{Q}^{ab})$ implies $S(k)$.

*Proof.* In positive characteristic this is Fu’s result (Corollary 2.4). In characteristic zero, in virtue of $S(\mathbb{Q}^{ab})$, Theorem 4.6 implies Conjecture A. As we are assuming $S(\mathbb{F}_p)$ for every prime $p$, Conjecture A implies $S(\mathbb{Q})$. Finally Theorem 2.5 yields the desired result.

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