Tachyon Condensation on Orbifolds and McKay Correspondence

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Abstract

In this short note we present a mathematical interpretation of tachyon condensation on (three dimensional) orbifolds within the framework of boundary string field theory (BSFT). We explicitly show that important parts of decay modes in brane-antibrane systems with $\mathcal{N}=2$ boundary supersymmetry can be interpreted as the McKay correspondence described as complexes. This will give an example of the recent interpretation of D-branes as derived category. We also discuss the $\mathcal{N}=4$ boundary supersymmetry as a more refined structure.

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1 Introduction

The studies of tachyon condensation in open string theory [1] have revealed important aspects of D-branes. One of these is the classification of D-brane charges as $K$-theory [3]. Brane-antibrane systems can be considered topologically as the difference of two bundles and this fact just corresponds to the mathematical definition of $K$-theory. As a next step it is natural to ask whether we can go beyond this topological argument by assuming more refined structures such as world-sheet extended supersymmetries. Recently the interpretation of a D-brane on complex manifolds as a complex of coherent sheaves has been discussed (for example see [4, 5, 6, 7]). A coherent sheaf $\mathcal{E}$ is a more general and refined notion than a vector bundle (locally free sheaf). For example, it admits ‘almost rank=0’ or point-like instanton configurations. It includes its cokernel $\mathcal{E}''$ defined by the exact sequence $0 \to \mathcal{E} \to \mathcal{E}' \to \mathcal{E}'' \to 0$ and this can be regarded as tachyon condensation on a brane $\mathcal{E}$ and an antibrane $\mathcal{E}'$ into the smaller brane $\mathcal{E}''$. These considerations eventually lead to the interpretation of D-branes as derived category [6, 7, 4].

In this paper we would like to discuss such an interpretation for tachyon condensation on brane-antibrane systems in orbifold theories. In particular we describe the tachyon condensation in the framework of boundary string field theory (BSFT) [8, 9, 10, 11, 12]. This theory is very useful because it is believed that infinitely many massive fields are not relevant for tachyon condensation. Furthermore we require the boundary $\mathcal{N} = 2$ (B-type) supersymmetry [13, 14] so as to study the algebra geometric aspects of D-branes. The general strategy of applying BSFT to orbifold theories has already been discussed in the previous paper [15]. Here we investigate especially brane-antibrane systems on three dimensional orbifolds in detail. From this analysis we will find that the tachyon condensation can be analyzed in a group theoretical way and the essential decay processes are described by the complexes which describe the McKay correspondence [16] proved by Ito and Nakajima (see [17] for the original paper and also [18] for a review). We will also discuss the $\mathcal{N} = 4$ boundary supersymmetry as a more refined structure and see that this symmetry leads to a certain quaternionic constraint on tachyon fields.

\[2\text{For earlier work on tachyon condensation see [4].}\]
2 Tachyon Condensation on Orbifolds as McKay Correspondence

Here we discuss the interpretation of tachyon condensation on orbifolds as the McKay correspondence \[16, 17, 18\]. Though our arguments below can be easily generalized into arbitrary dimensional abelian orbifolds, we describe the results for D6-D6 systems in three dimensional abelian orbifolds \( \mathbb{C}^3/\Gamma \). This is not only because these are the most physically interesting but because the corresponding mathematical results on the resolution of orbifold singularities have not been obtained in higher dimensions.

Let us first review the outline of the analysis in \[15\]. Since the physics of tachyon condensation involves off-shell string theory, one should consider within the framework of string field theory. We apply the boundary string field theory (BSFT) \[8\] for brane-antibrane systems \[11, 12\] to tachyon condensation on orbifolds.

We begin with the disk world-sheet action in flat background with boundary interactions which preserve \( \mathcal{N} = 2 \) (B-type) boundary supersymmetry \[14\]. We denote the ten dimensional complex coordinates of the target space as \((Z_1, \bar{Z}_1), \cdots, (Z_5, \bar{Z}_5)\). The explicit expression of the boundary interactions is given as follows:

\[
I_B = -\int_{\partial \Sigma} d\tau d\theta d\bar{\theta} \sum_i \Gamma_i \bar{\Gamma}_i + \int_{\partial \Sigma} d\tau d\theta \frac{1}{\sqrt{2\pi}} \sum_i \Gamma_i T_i(Z) + (\text{h.c.}),
\]  

where we have employed \( \mathcal{N} = 2 \) boundary superspace \((\tau, \theta, \bar{\theta})\); the boundary fermionic chiral and antichiral superfields \( \Gamma_i, \bar{\Gamma}_i \) are defined in our conventions as

\[
\Gamma_i = -\frac{i}{\sqrt{2}} \eta_i + \theta F_i - \frac{i}{\sqrt{2}} \theta \bar{\theta} \partial_\tau \eta_i,
\]

\[
\bar{\Gamma}_i = \frac{i}{\sqrt{2}} \bar{\eta}_i + \theta \bar{F}_i - \frac{i}{\sqrt{2}} \theta \bar{\theta} \partial_\tau \bar{\eta}_i.
\]

The fermions \( \eta_i, \bar{\eta}_i \) are called boundary fermions and the scalar fields \( F_i, \bar{F}_i \) are auxiliary fields. Since we are interested in the decay of a D6-D6 system into D0-branes below, we need \( 2^3/2 = 4 \) pairs of D6-D6. We can express this brane-antibrane system by using three fermi superfields \( \Gamma_i \) \((i = 1, 2, 3)\).

Note that the world-sheet \( \mathcal{N} = 2 \) supersymmetry requires the form of tachyon fields \( T_i(Z) \) to be holomorphic and this means that we consider a certain subspace of the field space of BSFT. This restriction is essential in our arguments not only because one can apply non-renomalization theorem but because it matches with complex geometry.

The spacetime action \( S \) of BSFT for brane-antibrane system is defined to be equal to the disk partition function \( Z \) \[13, 10\]. For example, if one considers the following tachyon
fields on the $D6$-$\overline{D6}$ system

$$T_1 = (Z_1)^p, \quad T_2 = (Z_2)^q, \quad T_3 = (Z_3)^r,$$

(2.3)

then we obtain $pqr$ D0-branes after the tachyon condensation. This can be shown by computing spacetime action $S$ employing the non-renomalization theorem [14] or calculating the RR-couplings [13] expressed by Quillen’s superconnection [20, 11, 12]. In particular the tachyon field $p = q = r = 1$ is known as Atiyah-Bott-Shapiro configuration [21], which carries the unit $K$-theory charge. This configuration will also play an very important role in the orbifold theories discussed below.

The above arguments of tachyon condensation on flat spaces can be generalized to orbifold theories. In the previous paper [15], we defined the BSFT for brane-antibrane systems on (abelian) orbifolds following the general framework of [22] and analyzed the examples in two dimensional orbifolds $\mathbb{C}^2/\mathbb{Z}_N$ in detail. From the results in BSFT one can determine how many fractional D0-branes [23] are generated after the tachyon condensation, but it is difficult to see what kinds of fractional D0-branes they are. Therefore we were also needed to apply the conservation law of twisted RR-charges.

Now let us discuss the tachyon condensation on a three dimensional orbifold $\mathbb{C}^3/\Gamma$. We assume that the discrete group $\Gamma$ is abelian because it is difficult to describe non-abelian orbifold actions in the boundary interaction (2.1). For example, if $\Gamma = \mathbb{Z}_N$, then the action on the coordinate $(z^1, z^2, z^3)$ of $\mathbb{C}^3$ is defined as follows.

$$g : (z^1, z^2, z^3) \rightarrow (e^{\frac{2\pi i a_1}{N}} z^1, e^{\frac{2\pi i a_2}{N}} z^2, e^{\frac{2\pi i a_3}{N}} z^3),$$

(2.4)

where $(a_1, a_2, a_3)$ are integers which satisfy $a_1 + a_2 + a_3 = 0 \pmod{N}$. We define such a three dimensional fundamental representation as $Q$. Generally, D-branes in orbifold theories are classified by the representations of the group $\Gamma$. Here we are interested in fractional D-branes and these correspond to the irreducible representations. For the orbifold $\Gamma = \mathbb{Z}_N$, they are given by one dimensional representations $\{\rho_{\alpha}\}$ ($\alpha = 0, 1 \cdot \cdot \cdot, N-1$), which are defined by the multiplication of the phase factor $e^{\frac{2\pi i \alpha}{N}}$. Therefore we will express each type of these fractional D-branes by $\rho_{\alpha}$ below. Though the following discussion can be applied to any abelian orbifold action, we mainly consider $\Gamma = \mathbb{Z}_N$ case.

Then let us examine the orbifold action on boundary fields. For this purpose it is useful to rewrite the interaction (2.1) into non-abelian tachyon fields $T$ by employing the

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3 If one considers more general holomorphic configurations than eq. (2.3), then some brane-antibrane systems will be generated after the tachyon condensation as observed in [13].
interpretation of $\Gamma_i, \Gamma_i$ as $\gamma$-matrices $\gamma_i^+, \gamma_i^-:

\[ T = \sum_{i=1}^{3} (\gamma_i^+ T_i + \gamma_i^- T_i) = \begin{pmatrix} T_3 & 0 & T_2 & T_1 \\ 0 & T_3 & -\overline{T}_1 & -\overline{T}_2 \\ -\overline{T}_2 & T_1 & -T_3 & 0 \\ T_1 & -2 \overline{T}_2 & 0 & -\overline{T}_3 \end{pmatrix} \]

From this we can consider the orbifold action on the Chan-Paton factors \cite{ref22} for the tachyon fields (2.3). The invariance under this action requires us that the original D6-D6 system should consist of four D6-branes $\rho_{\alpha}, \rho_{\alpha+pa_1+qa_2}, \rho_{\alpha+qa_2+ra_3}, \rho_{\alpha+ra_3+pa_1}$ (corresponding to each row of the matrix (2.5)) and four anti D6-branes $\rho_{\alpha+ra_3}, \rho_{\alpha}, \rho_{\alpha+qa_2}, \rho_{\alpha+pa_1}$ (corresponding to each column of the matrix (2.5)). In this way the requirement of boundary $\mathcal{N}=2$ supersymmetry determines the types of 6-branes before tachyon condensation up to an arbitrary integer $\alpha$.

The orbifold action on the boundary fields is given by

\[ g : \Gamma_i \rightarrow \Gamma_i e^{-2\pi i a_i}, \quad (2.6) \]

and by this action we define the BSFT for orbifolded brane-antibrane systems \cite{ref15}. Then we obtain $pqr$ fractional D0-branes after the tachyon condensation. In order to determine the types of fractional D0-branes one needs to examine the twisted RR-charges.

Here let us discuss twisted RR-charges in a group theoretical way. The value of these can be read \cite{ref24} from the explicit forms of boundary states in twisted sectors as shown in \cite{ref25, ref15}. Then the RR-charges $Q_{\alpha}^{Dp}(g)$ in $g \in \Gamma$ twisted sector for a D$p$-brane $\rho_{\alpha}$ are given by (with an appropriate normalization)

\[ Q_{\alpha,k}^{D6} = \frac{1}{|\Gamma|} \chi_{\alpha}(g) \sqrt{\text{Tr}(g)}, \]
\[ Q_{\alpha,k}^{D0} = \frac{1}{|\Gamma|} \chi_{\alpha}(g), \quad (2.7) \]

where $\chi_{\alpha}(g)$ denotes the character for the irreducible representation $\rho_{\alpha}$ of $\Gamma$. The factor $\text{Tr}(g)$ comes from the trace of the bosonic zeromodes in open string sector\cite{ref4} and is given

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\( \gamma_i^+, \gamma_i^- \) such that $\{ \gamma_i^+, \gamma_j^- \} = \delta_{ij}, \{ \gamma_i^+, \gamma_j^+ \} = \{ \gamma_i^-, \gamma_j^- \} = 0$.  

Here we have normalized the bosonic zeromodes as $\langle z | z' \rangle = \delta^2(z - z')$.  

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by

\[ \text{Tr}(g) = \int (dz)^3 (dz)^3 \langle z_i | g | z_i \rangle \]
\[ = (\det(1 - g))^{-2} \]
\[ = (1 - \chi_Q(g) + \chi_Q \wedge Q(g) - \chi_Q \wedge Q(g))^{-2}, \tag{2.8} \]

where \( \chi_Q \) denotes the character for the fundamental representation \( Q \) and the symbol \( \wedge \) means wedge product of representations. Thus we obtain the following result using the formula \( \chi_Q(g) \chi_\alpha(g) = \chi_{Q \otimes \rho_\alpha}(g) \)

\[ Q^{D0}_\alpha(g) = (1 - \chi_Q(g) + \chi_Q \wedge Q(g) - \chi_Q \wedge Q(g)) Q^{D6}_\alpha(g) \]
\[ = \sum_{\beta} (a^{(0)}_{\beta \alpha} - a^{(1)}_{\beta \alpha} + a^{(2)}_{\beta \alpha} - a^{(3)}_{\beta \alpha}) Q^{D6}_\beta(g), \tag{2.9} \]

where the integers \( a^{(i)}_{\beta \alpha} \) are defined by the following decomposition of representations

\[ \wedge^i Q \otimes \rho_\alpha = \sum_{\beta} a^{(i)}_{\beta \alpha} \rho_\beta. \tag{2.10} \]

This result (2.9) shows that the twisted RR charges (or \( K \)-theory charges) of the fractional D0-brane \( \rho_\alpha \) are equal to that of the brane-antibrane system which consists of \( \sum_{\beta} (a^{(0)}_{\beta \alpha} + a^{(3)}_{\beta \alpha}) \rho_\beta \) D6-branes and \( \sum_{\beta} (a^{(1)}_{\beta \alpha} + a^{(3)}_{\beta \alpha}) \rho_\beta \) antiD6-branes. Note that the above results hold for any (generally non-abelian) discrete group \( \Gamma \in \text{SL}(3, \mathbb{C}) \).

If we return the case \( \Gamma = \mathbb{Z}_N \) again, it is easy to see

\[ a^{(0)}_{\beta \alpha} = a^{(3)}_{\beta \alpha} = 1, \quad \sum_{\beta} a^{(1)}_{\beta \alpha} \rho_\beta = \rho_{\alpha + a_1} \oplus \rho_{\alpha + a_2} \oplus \rho_{\alpha + a_3}, \]
\[ \sum_{\beta} a^{(2)}_{\beta \alpha} \rho_\beta = \rho_{\alpha + a_1 + a_2} \oplus \rho_{\alpha + a_2 + a_3} \oplus \rho_{\alpha + a_3 + a_1}. \tag{2.11} \]

In particular if we consider the specific tachyon fields \( p = q = r = 1 \) in eq. (2.3), then the above results (2.9) and (2.11) show that a fractional D0-brane \( \rho_\alpha \) will be generated and this result is consistent with the previous result from BSFT. Even though the calculation of twisted RR-charges seems very powerful to determine the decay product at first sight, it should be noted that this cannot distinguish '0' from '\( Q^{D6}_\alpha(g) - Q^{D6}_\alpha(g) \)'. As we have seen, this ambiguity is fixed by the consideration of BSFT. For general \( p = p_1, \ q = p_2, \ r = p_3 \) one can show the following identity:

\[ \sum_{\beta_i=0}^{p_i-1} Q^{D0}_{\alpha + a_{1} \beta_1 + a_2 \beta_2 + a_3 \beta_3}(g) = Q^{D6}_\alpha(g) - \left( Q^{D6}_{\alpha + p_{a_1}}(g) + Q^{D6}_{\alpha + p_{a_2}}(g) + Q^{D6}_{\alpha + p_{a_3}}(g) \right) \]
\[ + \left( Q^{D6}_{\alpha + q_{a_1} + q_{a_2}}(g) + Q^{D6}_{\alpha + q_{a_2} + p_{a_3}}(g) + Q^{D6}_{\alpha + p_{a_3} + p_{a_1}}(g) \right) - Q^{D6}_\alpha(g) \tag{2.12} \]
Thus we can conclude the \( pqr = p_1p_2p_3 \) fractional D0-branes which are represented by left-handed side of eq.(2.12) are generated after tachyon condensation on the D6-\( \overline{D6} \) system defined by the right-handed side.

In this way we have determined the decay processes of brane-antibrane systems on orbifolds. We have also found that the tachyon condensation with boundary \( \mathcal{N} = 2 \) supersymmetry sees ‘holomorphy’ beyond the familiar \( K \)-theory charges. As argued in [4, 5, 6, 7] the tachyon condensation on brane-antibrane systems in complex manifolds such as Calabi-Yau manifolds can be represented by a complex of coherent sheaves. More precisely, one should identify two complexes which have the same cohomology complex (quasi-isomorphic) and we eventually obtain the notion of the derived category [26] as discussed in [4, 6, 7]. Let us briefly review this idea. For a given (bounded) complex,

\[
\mathcal{E}_n \xrightarrow{d_{n-1}} \mathcal{E}_{n-1} \xrightarrow{d_{n-2}} \mathcal{E}_{n-2} \xrightarrow{d_{n-3}} \cdots \xrightarrow{d_1} \mathcal{E}_1,
\]

it has been proposed to interpret the coherent sheaves \( \mathcal{E}_{even} \) as branes and \( \mathcal{E}_{odd} \) as antibranes because their cohomologies, which have essential information for any complex, roughly mean the ‘subtraction’ of adjacent modules. The tachyon fields between the brane-antibrane system correspond to the arrows of the complex. This can also be seen from the \( K \)-theory charge of the complex, which is given by \([ \oplus_n H^{2n} ] \ominus [ \oplus_n H^{2n+1} ]\). The simplest example of this will be the Koszul complex which describes tachyon condensation on flat space [14, 7]. Below we would like to relate our BSFT description of tachyon condensation on orbifolds to the complex which defines McKay correspondence [16] as a more non-trivial example. If we blow up the orbifold singularities, we can consider brane-antibrane systems in such a space as an object of the derived category of coherent sheaves. For earlier discussions of this complex in physical context see for example [27, 28]. See also [29] for descriptions of some other complexes.

The fundamental complex of McKay correspondence in three dimension is given by

\[
\mathcal{R} \xrightarrow{d_3} Q \otimes \mathcal{R} \xrightarrow{d_2} \wedge^2 Q \otimes \mathcal{R} \xrightarrow{d_1} \wedge^3 Q \otimes \mathcal{R} = \mathcal{R}.
\]

The bundle \( \mathcal{R} \) denotes \(|\Gamma|\) dimensional bundle on the (resolved) orbifold space \( X \sim \mathbb{C}^3/\Gamma \) (tautological bundle) and is defined by \( \mathcal{R} = P \times_{GL_G(R)} \mathbb{C}^{[\Gamma]} \). The fiber \( \mathbb{C}^{[\Gamma]} \) is charged under the (complex) quiver gauge group \( GL_G(R) \) so as to belong to the regular representation. The fibration \( P \) over \( X \) is equivalent to the quiver theory for a bulk D0-brane (regular representation) [22] such that \( P/GL_G(R) = X \). The action \( Q \) denotes the three dimensional representation given by the inclusion \( \Gamma \in \text{SL}(3, \mathbb{C}) \). The boundary operator \( d_i \) is defined by \( d_i = B \wedge \), where \( B \) denotes the multiplication of the coordinate functions \((z_1, z_2, z_3)\). Note that it is easy to see the nilpotency \( d_id_{i+1} = 0 \). Then the
bundle $\mathcal{R}$ can be decomposed into $|\Gamma|$ line bundles corresponding to each irreducible representations:

$$\mathcal{R} = \bigoplus \mathcal{R}_\alpha \otimes \rho_\alpha,$$

and each of them can be naturally interpreted as a fractional D6-brane $\rho_\alpha$. Then we can divide the complex (2.14) into $|\Gamma|$ parts:

$$S_\alpha : - \left[ \mathcal{R}_\alpha \xrightarrow{d_3} \bigoplus_\beta a_{\alpha\beta}^{(1)} \mathcal{R}_\beta \xrightarrow{d_2} \bigoplus_\beta a_{\alpha\beta}^{(2)} \mathcal{R}_\beta \xrightarrow{d_1} \bigoplus_\beta a_{\alpha\beta}^{(3)} \mathcal{R}_\beta = \mathcal{R}_\alpha \right].$$

One of the results shown in [16] is that the $K$-group (Grothendick group) of bounded complex $S_\alpha$ has a support on the exceptional locus and defines a basis of $K^c(X)$, where $K^c(X)$ is the $K$-group for complexes on the exceptional locus. Furthermore the intersection between $\mathcal{R}_\alpha$ and $S_\beta$ is shown to be $\langle \mathcal{R}_\alpha, S_\beta \rangle = \delta_{\alpha\beta}$. From this and the complex (2.16) one obtains the intersection between $S_\alpha$ as follows

$$\langle S_\alpha, S_\beta \rangle = a_{\alpha\beta}^{(2)} - a_{\alpha\beta}^{(1)} = a_{\beta\alpha}^{(1)} - a_{\alpha\beta}^{(1)}.$$

Then in string theory one can identify $S_\alpha$ as a fractional D0-brane $\rho_\alpha$.

We argue that this complex (2.16) just represents the tachyon condensation of D6-branes $\mathcal{R}_\alpha \oplus_\beta a_{\alpha\beta}^{(2)} \mathcal{R}_\beta$ and anti D6-branes $\mathcal{R}_\alpha \oplus_\beta a_{\alpha\beta}^{(1)} \mathcal{R}_\beta$. Indeed for the orbifold $\Gamma = \mathbb{Z}_N$ this is the same as the previous result for the tachyon field $p = q = r = 1$ in BSFT with $\mathcal{N} = 2$ boundary supersymmetry. One can understand the interpretation of the arrows in the complex as tachyon fields in detail. Let us remember the explicit form of non-abelian tachyon field (2.5). Then it is easy to see that the multiplication of the tachyon fields $T_1 = Z_1$, $T_2 = Z_2$, $T_3 = Z_3$ transform a D6-brane $\rho_\alpha$ into three anti D6-branes $\rho_{\alpha+a_1} \oplus \rho_{\alpha+a_2} \oplus \rho_{\alpha+a_3}$ and so on. This is equivalent to the arrows (the boundary operator $d_i$) defined before. In other words, the wedge product of $B$ corresponds to the multiplications of the operator $\sum_i \Gamma_i T_i$. This explicitly shows its nilpotency. We can also consider more general tachyon fields for any $p, q, r$. From the previous analysis we obtain the following complex:

$$\sum_{\beta_i=0}^{p_i-1} S_{\alpha+\beta_1 a_1+\beta_2 a_2+\beta_3 a_3} : \mathcal{R}_\alpha \rightarrow \bigoplus_1 \mathcal{R}_{\alpha+p_1 a_i} \rightarrow \bigoplus_k \mathcal{R}_{\alpha+p_1 a_i+p_2 a_j} \rightarrow \mathcal{R}_{\alpha+p_1 a_1+p_2 a_2+p_3 a_3}.$$

Note that this complex can be obtained from the linear combination of (2.16) with appropriate 'brane-antibrane annihilation' (or more precisely up to quasi-isomorphism).
For non-abelian orbifolds we cannot show this correspondence explicitly since it is difficult to construct the appropriate boundary interaction. However it is natural from the above results to believe that any tachyon condensation which corresponds to the complex (2.10) preserves $\mathcal{N} = 2$ boundary supersymmetry. In this way the tachyon condensation in orbifold theory gives an interesting example which relates the string field theory restricted by $\mathcal{N} = 2$ boundary supersymmetry to mathematics on complex manifolds.

3 Comment on $\mathcal{N} = 4$ Boundary Supersymmetry

As we have seen, $\mathcal{N} = 2$ supersymmetry in tachyonic boundary interactions enables a refined mathematical viewpoint. Thus it will also be interesting to consider $\mathcal{N} = 4$ supersymmetry further. The world-sheet $\mathcal{N} = 4$ supersymmetry constrains the target space $\mathcal{M}$ to be hyperkahler as is well-known [30] and thus we assume that the dimension of $\mathcal{M}$ is a multiple of four. Here we would like to discuss world-sheet with a boundary which preserves $\mathcal{N} = 4$ boundary supersymmetry. In particular we discuss the Neumann boundary condition $\partial_1 \Phi_i|_{\partial \Sigma} = 0$, $(\psi^i_L - \psi^i_R)|_{\partial \Sigma} = 0$ since we are interested in brane-antibrane systems wrapping on the whole manifold $\mathcal{M}$ with no flux. The generalization of our results to other boundary conditions is obtained by the $SO(3)$ rotation (see [33]) directly. Since the appropriate $\mathcal{N} = 4$ off-shell superspace is not known, we work within $\mathcal{N} = 1$ superspace or by using its component expressions.

The $\mathcal{N} = 4$ non-linear sigma model is defined by the following action [30]:

$$I_0 = \frac{1}{4!} \int_{\Sigma} (d\sigma)^2 (d\theta)^2 g_{ij}(\Phi) \overline{D} \Phi^i D \Phi^j,$$

(3.1)

where $\Phi^i(\sigma) = \Phi^i(\sigma) + \theta \psi^i(\sigma) + \cdots$ denote the $\mathcal{N} = 1$ superfields on the world-sheet $\Sigma$. Then this bulk action is invariant under the $\mathcal{N} = 4$ supersymmetry. Its restriction on the boundary (along $x^0$) of the world-sheet is given by

$$\delta \phi^i = i \epsilon \psi^j f^{(a)i}_j$$
$$\delta \psi^i = -h^{(a)}_j \epsilon (\partial_0 \phi^j) - i \Gamma^i_{kl} f^{(a)l}_j \epsilon \psi^j \psi^k,$$

(3.2)

where $f^{(a)l}_j = (h^{(a)-1})^l_j$, $(a = 1, 2, 3)$ denotes three independent complex structures and this defines a hyperkahler structure. In other words these satisfy

$$f^{(a)l}_j f^{(b)j}_k + f^{(b)j}_j f^{(a)l}_k = -2 \delta_{ab} \delta^i_k, \quad g_{ij} f^{(a)l}_k f^{(a)j}_l = g_{kl}, \quad \nabla_i f^{(a)}_j = 0.$$

(3.3)

The first equation shows the correct algebra of 2D $\mathcal{N} = 4$ supersymmetry. The second and third are required by the invariance of the action (3.1) by the transformations (3.2).
The boundary interaction for tachyon fields is given by

\[ I_B = \int_{\partial\Sigma} (d\tau)(d\theta)(\Gamma^A D\Gamma^A + T^A(\Phi)\Gamma^A). \]  

(3.4)

Note that here we have employed \( \mathcal{N} = 1 \) boundary superspace and thus the boundary fermionic superfields \( \Gamma^A = \eta^A + \theta F^A \) are real. We would like for the combined action \( I_0 + I_B \) to be invariant under the \( \mathcal{N} = 4 \) boundary supersymmetry (3.2) and

\[ \delta\eta^A = \epsilon F^B f_B^{(a)A}, \]

\[ \delta F^A = -i\epsilon \partial_\tau \eta^B h_B^{(a)A}. \]  

(3.5)

This is possible if the following conditions are satisfied

\[ f_B^{(a)A} h_C^{(a)B} = \delta^A_C, \]  

(3.6)

\[ f_B^{(a)A} f_C^{(b)B} + f_B^{(b)A} f_C^{(a)B} = -2\delta^{ab} \delta^A_C, \]  

(3.7)

\[ f_B^{(a)A} = h_C^{(a)B}, \]  

(3.8)

\[ \partial_\tau f_B^{(a)A} = 0, \]  

(3.9)

\[ f_j^{(a)i}(\partial_i T^A) - (\partial_j T^B) f_B^{(a)A} = 0. \]  

(3.10)

The equations (3.6) and (3.7) ensure the correct algebra of the boundary \( \mathcal{N} = 4 \) supersymmetry. The others are needed for the invariance of the action \( I_0 + I_B \) under this supersymmetry. As can be seen from these requirements the three matrices \( f_B^{(a)A} \in O(4) \) (\( a = 1, 2, 3 \)), which is constant due to eq. (3.9), define a hyperkahler structure on the vector space \( V \) along the boundary superfields \( \Gamma^A \). Thus the dimension of this vector space should be a multiple of four again. Finally, the last equation (3.10) constrains the tachyon fields so as to preserve the extended supersymmetry. If one requires only \( \mathcal{N} = 2 \) supersymmetry (\( a = 1 \)), then this means that the tachyon fields are holomorphic and reproduce the result in [14]. In the \( \mathcal{N} = 4 \) case now considered, this gives a more strong restriction. For simplicity let us assume \( \text{dim}(V) = \text{dim}(\mathcal{M}) = 4 \) and \( \mathcal{M} \) is flat. By using \( O(4) \) rotation we can set \( f^{(a)} = f^{(a)} \) and the explicit form can be given using the Pauli matrices as follows:

\[ f^{(1)} = (i\sigma_2) \otimes 1, \quad f^{(2)} = \sigma_1 \otimes (i\sigma_2), \quad f^{(3)} = \sigma_3 \otimes (i\sigma_2). \]  

(3.11)

Then it is easy to find the following linear solutions to eq. (3.10):

\[ T^A(\Phi) = \sum_{i=1}^{4} M_i^A \Phi_i, \]  

(3.12)
where the matrix $M$ is any linear combination of both the identity and the following three matrices, which represent the quaternionic algebra:

$$I = 1 \otimes (-i\sigma_2), \quad J = (-i\sigma_2) \otimes \sigma_3, \quad K = (-i\sigma_2) \otimes \sigma_1. \quad (3.13)$$

If we apply this result to the previous arguments on tachyon condensation on two dimensional orbifolds, we find that these allowed configurations include the decay of $D4 - \overline{D4}$ into a fractional $D0$-brane which corresponds to the following complex similar to (2.16):

$$\mathcal{S}_\alpha : \mathcal{R}_\alpha \xrightarrow{d_2} \bigoplus_{\beta} a^{(1)}_{\alpha\beta} \mathcal{R}_\beta \xrightarrow{d_1} \mathcal{R}_\alpha. \quad (3.14)$$

For more general configurations of tachyon fields we have not obtained any definite solutions of eq.(3.10).

If gauge fields on D-branes have a non-trivial configuration, then this space $V$ is twisted and the boundary superfields $\Gamma^A$ should be viewed as a special kind of section (satisfying a restriction like eq.(3.10)) of a vector bundle on the hyperkahler manifold $\mathcal{M}$. It would also be interesting to construct $\mathcal{N} = 4$ boundary interactions including gauge fields and clarify the meaning of the ‘quaternionic’ constraint (3.10).

4 Conclusions

In this paper we have studied a mathematical aspect of tachyon condensation on brane-antibrane systems in the (three dimensional) orbifold theories. We have applied to this system the boundary string field theory with boundary $\mathcal{N} = 2$ supersymmetry. As a result we have observed that the essential part of the tachyon condensation can be understood as the McKay correspondence. This example shows that the $\mathcal{N} = 2$ supersymmetry enables us to employ the language of homological algebra in algebraic geometry. One can regard this as an example of the recent interpretation of D-branes as derived category. We have also discussed the boundary $\mathcal{N} = 4$ supersymmetry and observed this symmetry leads to a certain quaternionic constraint on tachyon fields.

Acknowledgments

I am very grateful to T. Eguchi and Y. Matsuo for valuable discussions and encouragements. I also thank T. Muto for explaining useful facts on orbifolds and thank S. Terashima, M. Hamanaka and H. Kajiura for helpful discussions. This work is supported by JSPS Research Fellowships for Young Scientists.
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