Hidden Algebra of Three-Body Integrable Systems

Alexander Turbiner†

Theoretical Physics Institute, University of Minnesota, Minneapolis, MN 55455, USA
and
Instituto de Ciencias Nucleares, UNAM, Apartado Postal 70–543, 04510 Mexico D.F., Mexico

Abstract

It is shown that all 3-body quantal integrable systems that emerge in the Hamiltonian reduction method possess the same hidden algebraic structure. All of them are given by a second degree polynomial in generators of an infinite-dimensional Lie algebra of differential operators.

†On leave of absence from the Institute for Theoretical and Experimental Physics, Moscow 117259, Russia
E-mail: turbiner@tpi1.hep.umn.edu
turbiner@xochitl.nuclecu.unam.mx
The Hamiltonian reduction method (see, for example, [1]) has provided several few-parameter families of integrable many-body potentials. The goal of this Letter is to show that all three-body, rational and trigonometric integrable $A_2, BC_2, G_2$ Hamiltonians possess the same hidden algebra. They are given by a second degree polynomial in generators of some infinite-dimensional algebra. The information about their Hamiltonian reduction origin and the coupling constants is coded in the coefficients of this polynomial.

1. $A_2$ integrable system (rational case).

The Hamiltonian of the three-body Calogero model or, in other words, the rational $A_2$ integrable model is defined by

$$H_{\text{Cal}} = \frac{1}{2} \sum_{i=1}^{3} \left[ -\frac{\partial^2}{\partial x_i^2} + \omega^2 x_i^2 \right] + g \sum_{i<j}^{3} \frac{1}{(x_i - x_j)^2} ,$$  \hspace{1cm} (1)

where $g = \nu(\nu - 1) > -\frac{1}{4}$ is the coupling constant and $\omega$ is the harmonic oscillator frequency. This Hamiltonian describes a system of three identical particles on the line with pairwise interaction. The ground state eigenfunction is given by

$$\Psi_0^{(c)}(x) = \Delta^\nu(x)e^{-\omega X^2} ,$$  \hspace{1cm} (2)

where $\Delta(x) = \prod_{i<j} |x_i - x_j|$ is the Vandermonde determinant and $X^2 = \sum_i x_i^2$. The Hamiltonian (1) is $Z_2$-invariant, $x \to -x$, which leads to two families of eigenstates: even and odd. Throughout the paper we will deal with even eigenstates only. The odd eigenstates can be treated similarly and nothing conceptually new appears.

Basically, the internal dynamics of the system is defined by relative motion. To study this relative motion let us introduce the center-of-mass coordinate $Y = \sum_{j=1}^{3} x_j$ and the translation-invariant relative coordinates (Perelomov coordinates) \cite{2},

$$y_i = x_i - \frac{1}{3} Y , \quad i = 1, 2, 3 ,$$  \hspace{1cm} (3)

which obey the constraint $y_1 + y_2 + y_3 = 0$. To incorporate permutation symmetry and translational invariance we consider the coordinates \cite{2}

$$\tau_2 = -y_1^2 - y_2^2 - y_1 y_2 , \quad \tau_3 = -y_1 y_2 (y_1 + y_2) ,$$  \hspace{1cm} (4)

or \cite{2},

$$\lambda_1 = \tau_2 , \quad \lambda_2 = \tau_3^2 .$$  \hspace{1cm} (5)

It is worth mentioning that the coordinates (5) are $Z_2$ symmetrical, $\lambda_{1,2}(-x) = \lambda_{1,2}(x)$. Performing a gauge rotation of the Hamiltonian (1),

$$h_{\text{Cal}} = -2(\Psi_0^{(c)}(x))^{-1}H_{\text{Cal}}\Psi_0^{(c)}(x) ,$$  \hspace{1cm} (6)
and rewriting the resulting operator in the coordinates (5), we get the following differential operator with polynomial coefficients,

\[ h_{\text{Cal}}(\lambda_1, \lambda_2) = -2\lambda_1 \partial^2_{\lambda_1} - 12\lambda_2 \partial^2_{\lambda_2} + \frac{8}{3} \lambda_1^2 \lambda_2 \partial^2_{\lambda_2} - \left[ 4\omega \lambda_1 + 2(1 + 3\nu) \right] \partial_{\lambda_1} - \left( 12\omega \lambda_2 - \frac{4}{3} \lambda_1^2 \right) \partial_{\lambda_2}. \]  

(7)

It can be called the algebraic form of the A\textsubscript{2} Calogero model.

Following the philosophy of (quasi)-exact-solvability (see [5]) let us try to find the hidden algebra of (1) as an origin of solvability of this model. It can be shown that the operator (7) can be rewritten in terms of the generators of some infinite-dimensional Lie algebra of differential operators generated by the eight operators

\[ L^1 = \partial_{\lambda_1} \quad (-1, 0), \quad L^2 = \lambda_1 \partial_{\lambda_1} - \frac{n}{3} \quad (0, 0), \]

\[ L^3 = 2\lambda_2 \partial_{\lambda_2} - \frac{n}{3} \quad (0, 0), \quad L^4 = \lambda_1^2 \partial_{\lambda_1} + 2\lambda_1 \lambda_2 \partial_{\lambda_2} - n\lambda_1 \quad (+1, 0), \]

\[ L^5 = \partial_{\lambda_2} \quad (0, -1), \quad L^6 = \lambda_1 \partial_{\lambda_2} \quad (+1, -1), \]

\[ L^7 = \lambda_1^2 \partial_{\lambda_2} \quad (+2, -1), \quad T = \lambda_2 \partial^2_{\lambda_1} \quad (-2, +1), \]  

(8)

where the numbers in brackets \((a_1, a_2)\) mean the grading of the generator, \(A : \lambda_1^{k_1} \lambda_2^{k_2} \mapsto \lambda_1^{k_1+a_1} \lambda_2^{k_2+a_2}\). This algebra was introduced at the first time in [4] and was called there \(g^{(2)}\). The first seven generators \(L_i\) form the \(gl_2 \ltimes \mathbb{R}^3\)-algebra. If \(n\) is a non-negative integer number, the finite-dimensional irreducible representation (8) appears with the invariant subspace

\[ W_n = (\lambda_1^{n_1} \lambda_2^{n_2} | 0 \leq (n_1 + 2n_2) \leq n). \]  

(9)

The generators of the \(gl_2 \ltimes \mathbb{R}^3\)-algebra, \(L_1, \ldots L_7\), act on \(W_n\) reducibly, having the invariant subspace

\[ \tilde{W}_n = (\lambda_1^p | 0 \leq p \leq n). \]  

(10)

It is worth mentioning that at \(n = 0\) the \(gl_2 \ltimes \mathbb{R}^3\)-algebra becomes the algebra of vector fields, which act on 2-Hirzebruch surface, \(\Sigma_2\) and the modules are the sections of holomorphic line bundles over this surface (see [6] and references therein).

Finally, when the operator (7) is written in terms of the generators (8) it becomes,

\[ h_{\text{Cal}} = -2L^2 L^1 - 6L^3 L^1 + \frac{4}{3} L^7 L^3 - 2(1 + 3\nu)L^1 - 4\omega L^2 - 6\omega L^3 - \frac{4}{3} L^7, \]  

(11)

where the parameter \(n\) is equal to zero, \(n = 0\). This is called the Lie-algebraic form of the Calogero model. The representation (11) contains the generators of the Borel subalgebra of \(gl_2 \ltimes \mathbb{R}^3\)-algebra only, preserving the infinite flag of finite-dimensional representation spaces \(W_n\) as well as the infinite flag of \(\tilde{W}_n\), which proves the exact-solvability of the \(A_2\) Calogero model. It is worth
noting that when the configuration space of (1) is parametrized by the \( \tau \)-coordinates (4) the gauge-rotated Hamiltonian (6) can be rewritten in terms of the generators of the \( gl(3) \)-algebra [3]. So the Calogero model possesses two different hidden algebras, \( g(2) \) and \( gl(3) \) acting on the configuration space in two different parametrizations.

2. \( A_2 \) integrable system (trigonometric case).

The Hamiltonian of the three-body Sutherland model, or in other words, the trigonometric \( A_2 \) integrable model is defined by [1]

\[
H_{\text{Suth}} = -\frac{1}{2} \sum_{k=1}^{3} \frac{\partial^2}{\partial x_k^2} + \frac{g\alpha^2}{4} \sum_{k<l}^{3} \frac{1}{\sin^2(\frac{\alpha}{2}(x_k - x_l))},
\]

(12)

where \( g = \nu(\nu - 1) > -\frac{1}{4} \) is the coupling constant. The ground state of this Hamiltonian is

\[
\Psi_0^{(\text{Suth})}(x) = (\Delta^{(\text{trig})}(x))^\nu,
\]

(13)

where \( \Delta^{(\text{trig})}(x) = \prod_{i<j}^{3} |\sin(\frac{\alpha}{2}(x_i - x_j))| \) is the trigonometric analog of the Vandermonde determinant.

To exhibit the dynamics of the system, we can introduce the translation-invariant, permutation-symmetric, periodic coordinates, either [3]

\[
\begin{align*}
\eta_2 &= \frac{1}{\alpha^2} \left[ \cos(\alpha y_1) + \cos(\alpha y_2) + \cos(\alpha(y_1 + y_2)) - 3 \right], \\
\eta_3 &= \frac{2}{\alpha^3} \left[ \sin(\alpha y_1) + \sin(\alpha y_2) - \sin(\alpha(y_1 + y_2)) \right],
\end{align*}
\]

(14)

or [4]

\[
\sigma_1 = \eta_2, \quad \sigma_2 = \eta_3^2.
\]

(15)

In the limit \( \alpha \to 0 \), these coordinates become (4) or (5), respectively. The coordinates (15) are symmetric with respect to \( x \to -x, \sigma_{1,2},(-x) = \sigma_{1,2}(x) \).

Performing the gauge rotation of the Hamiltonian (12) (see (6)) and extracting the center-of-mass motion, we get the \textit{algebraic} form of the Hamiltonian of the Sutherland model

\[
\begin{align*}
\hat{h}_{\text{Suth}} &= -(2\sigma_1 + \frac{\alpha^2}{2}\sigma_1^2 - \frac{\alpha^4}{24}\sigma_2)\partial_{\sigma_1,\sigma_1} - (12 + \frac{8\alpha^2}{3}\sigma_1)\sigma_2\partial_{\sigma_1,\sigma_2}
+ (\frac{8\sigma_1^2\sigma_2}{3} - 2\alpha^2\sigma_2^2)\partial_{\sigma_2,\sigma_2} - [2(1 + 3\nu) + 2(\nu + \frac{1}{3})\alpha^2\sigma_1]\partial_{\sigma_1}
+ \left[ \frac{4}{3}\sigma_1^3 - (\frac{7}{3} + 4\nu)\alpha^2\sigma_2 \right] \partial_{\sigma_2},
\end{align*}
\]

(16)

(cf. (5)). This operator can be represented in terms of the generators of the algebra \( g(2) \) with \( n = 0 \) and \( \lambda_1, \lambda_2 \) replaced by \( \sigma_1, \sigma_2 \), respectively,

\[
\hat{h}_{\text{Suth}} = -2L^2L^1 - 12L^3L^1 + \frac{8}{3}L^7L^3 - \alpha^2\left( \frac{L^2L^2}{2} + \frac{8L^3L^2}{3} + 2L^3L^3 \right)
\]
This is the Lie-algebraic form of the Sutherland Hamiltonian. It contains the generators of the Borel subalgebra of \( gl_2 \times R^3 \)-algebra and as well as the generator \( T \). In the limit \( \alpha \to 0 \) the operator (17) coincides with the operator (11) at \( \omega = 0 \).

It is worth noting that in the \( \eta \)-coordinates (14) the gauge-rotated Sutherland Hamiltonian (16) can be rewritten in terms of the generators of the \( gl(3) \)-algebra [3]. So, the Sutherland Hamiltonian similarly to the Calogero one possesses two different hidden algebras: a \( g(2) \) algebra acting on the configuration space parametrized by \( \sigma \)-coordinates and a \( gl(3) \)-algebra acting on the configuration space parametrized by \( \eta \)-coordinates.

3. \( BC_2 \) integrable system (rational case).

The Hamiltonians of the \( BC_2, B_2, C_2 \) rational models do coincide and they are given by (see [1])

\[
\mathcal{H}^{(r)}_{BC_2} = -\frac{1}{2} \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) + \frac{\omega^2}{2} (x_1^2 + x_2^2) +
\]

\[
g \left[ \frac{1}{(x_1 - x_2)^2} + \frac{1}{(x_1 + x_2)^2} \right] + \frac{g_2}{2} \left( \frac{1}{x_1^2} + \frac{1}{x_2^2} \right),
\]

where \( g = \nu(\nu - 1) \) and \( g_2 = \nu_2(\nu_2 - 1) \). When the coupling constant \( g_2 \) tends to zero the Hamiltonian \( \mathcal{H}^{(r)}_{BC_2} \) degenerates to the Hamiltonian of the \( D_2 \) rational model. This Hamiltonian can be treated as the Hamiltonian of the relative motion of some 3-body problem with non-identical particles (for a discussion see [4]).

The ground state eigenfunction of the Hamiltonian (18) is given by

\[
\Psi_0 = |x_1^2 - x_2^2|^{\nu} |x_1 x_2|^{\nu_2} e^{-\frac{\omega}{2}(x_1^2 + x_2^2)},
\]

(cf.(2)). In order to encode the permutation symmetry \( x_i \leftrightarrow x_j \) and the reflection symmetry \( x_i \to -x_i \) of the Hamiltonian (18), we introduce the coordinates

\[
\tilde{\sigma}_1 = x_1^2 + x_2^2, \quad \tilde{\sigma}_2 = x_1^2 x_2^2.
\]

Now we perform the gauge rotation of (18) with ground state eigenfunction (19) (see (1)). Eventually, in the \( \tilde{\sigma} \) coordinates the gauge-rotated \( BC_2 \) rational Hamiltonian takes its algebraic form,

\[
-h^{(r)}_{BC_2}(\tilde{\sigma}_1, \tilde{\sigma}_2) = 4\tilde{\sigma}_1 \partial^3_{\tilde{\sigma}_1} \tilde{\sigma}_1 + 16\tilde{\sigma}_2 \partial^3_{\tilde{\sigma}_2} \tilde{\sigma}_2 + 4\tilde{\sigma}_1 \tilde{\sigma}_2 \partial^2_{\tilde{\sigma}_1} \tilde{\sigma}_2
\]

\[
+ 4[(1 + \nu_2 + 2\nu) - \omega \tilde{\sigma}_1] \partial_{\tilde{\sigma}_1} + 2[(1 + \nu_2) \tilde{\sigma}_1 - 4\omega \tilde{\sigma}_2] \partial_{\tilde{\sigma}_2},
\]

This operator can be rewritten in terms of the \( gl_2 \times R^3 \)-algebra generators as
\[-h_{BC}^{(r)} = 4L^2L^1 + 8L^1L^3 + 2L^3L^6 + \\
4(1 + \nu_2 + 2\nu)L^1 - 4\omega L^2 + 2(1 + \nu_2)L^6 - 4\omega L^3 ,
\]
which is the Lie-algebraic form of $BC_2$ rational model. It contains the generators of the Borel subalgebra of $gl_2 \ltimes R^3$-algebra only. The eigenvalues of $h_{BC}^{(r)}$ are given by

$$
\epsilon_{n,k} = 4\omega(2n - k) , \ n = 0, 1, 2, \ldots , k = 0, 1, 2, \ldots n .
$$

It is worth noting that unlike the $A_2$ cases (see Sections 1, 2) the gauge-rotated Hamiltonian (18) in $\tilde{\sigma}$-coordinates allows the representation either in terms of the generators of the $g^{(2)}$ algebra or the generators of the $gl(3)$-algebra [7].

4. $BC_2$ integrable system (trigonometric case).

The Hamiltonians for $B_2, C_2$ and $D_2$ trigonometric models are special cases of the general $BC_2$ Hamiltonian (18)

$$
H_{BC}^{(l)} = -\frac{1}{2} \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) + \frac{g}{4} \left[ \frac{1}{\sin^2\left(\frac{1}{2}(x_1 - x_2)\right)} + \frac{1}{\sin^2\left(\frac{1}{2}(x_1 + x_2)\right)} \right] + \\
\frac{g_2}{4} \sum_{i=1}^{2} \frac{1}{\sin^2(x_i)} + \frac{g_3}{8} \sum_{i=1}^{2} \frac{1}{\sin^2\left(\frac{x_i}{2}\right)}
$$

where $g = \nu(\nu - 1)$, $g_2 = \nu_2(\nu_2 - 1)$ and $g_3 = \nu_3(\nu_3 + 2\nu_2 - 1)$. From the general Hamiltonian the $B_2, C_2$ and $D_2$ cases are obtained as follows:

- $B_2$ case: $\nu_2 = 0$,
- $C_2$ case: $\nu_3 = 0$,
- $D_2$ case: $\nu_2 = \nu_3 = 0$.

The Hamiltonian (23) can be treated as the Hamiltonian of the relative motion of a three-body problem with non-identical particles (see [1]).

The ground state wave function is [1, 8]

$$
\Psi_0 = |\sin(\frac{1}{2}(x_1 - x_2))|^{\nu_2} |\sin(\frac{1}{2}(x_1 + x_2))|^{\nu_3} \prod_{i=1}^{2} |\sin(x_i)|^{\nu_2} |\sin(x_i)|^{\nu_3} .
$$

In order to reveal both the permutation and reflection symmetry and the periodicity of (23), let us introduce the coordinates

$$
\eta_1(\alpha) = \cos \alpha x_1 + \cos \alpha x_2 , \ \eta_2(\alpha) = \cos \alpha x_1 \cos \alpha x_2 ,
$$

and a modification of the them

$$
\tilde{\eta}_1 = \frac{4}{\alpha^2} - \frac{2}{\alpha^2} \eta_1(\alpha) , \ \tilde{\eta}_2 = \frac{4}{\alpha^4} - \frac{4}{\alpha^4} \left[ \eta_1(\alpha) - \eta_2(\alpha) \right] ,
$$

In the limit $\alpha \to 0$ the $\tilde{\eta}$'s became the $\tilde{\sigma}$'s (see (20)). Performing a gauge rotation of the Hamiltonian (23) with the ground state eigenfunction (24)
tical particles with two- and three-body interactions. 

\[ -h_{BC_2}^{(t)}(\tilde{\eta}_1, \tilde{\eta}_2) = (4\tilde{\eta}_1 - \alpha^2 \tilde{\eta}_1^2 + 2\alpha^2 \tilde{\eta}_2)\partial^2_{\tilde{\eta}_1} + 2\tilde{\eta}_2(8 - \alpha^2 \tilde{\eta}_1)\partial^2_{\tilde{\eta}_2} + \]

\[ 2\tilde{\eta}_2(2\tilde{\eta}_1 - \alpha^2 \tilde{\eta}_2)\partial^2_{\tilde{\eta}_2} + 4[\nu_2 + \nu_3 + 2\nu - \frac{\alpha^2}{4}(\nu_2 + \frac{\nu_3}{2} + \nu)\tilde{\eta}_1]\partial_{\tilde{\eta}_1} + \]

\[ 2[\nu_2 + \nu_3] \tilde{\eta}_1 - \alpha^2(\nu_2 + \frac{\nu_3}{2} + \nu)\tilde{\eta}_2] \partial_{\tilde{\eta}_2} \]  

(26)

If \( \nu_3 = 1 \) and \( \alpha \to 0 \) we arrive at (21). The operator (26) can be rewritten in terms of the \( g^{(2)} \)-algebra generators,

\[ h_{BC_2}^{(t)} = -4L^2L^1 + \alpha^2 L^2L^2 - 2\alpha^2 T - 8L^1L^3 + \alpha^2 L^3L^2 - 2L^3L^0 - \]

\[ 4(\nu_3 + \nu_2 + 2\nu)L^1 + \frac{\alpha^2}{4}(\nu_2 + \frac{\nu_3}{2} + \nu - 4)L^2 + \]

\[ \frac{\alpha^2}{2}(\nu_2 + \frac{\nu_3}{2} + \nu)L^3 - 2(\nu_3 + \nu_2)L^6, \]  

(27)

which is the Lie-algebraic form of the \( BC_2 \) trigonometric Hamiltonian.

The eigenvalues of \( h_{BC_2}^{(t)} \) are given by

\[ \epsilon_{n,k} = \alpha^2[nk + 2(n - k)^2 + (\nu_2 + \frac{\nu_3}{2} + \nu)(2n - k)] , \]

\[ n = 0, 1, 2, \ldots, k = 0, 1, 2, \ldots n . \]  

(28)

The limit \( \alpha \to 0 \) makes no sense since there are no polynomial eigenfunctions of the operator (26) in this limit.

It is worth noting that unlike the \( A_2 \) models the gauge-rotated Hamiltonian (26) in \( \tilde{\eta} \)-coordinates can be rewritten in terms of the generators of the \( g^{(2)} \)-algebra as well as the \( gl(3) \)-algebra [6].

5. \( G_2 \) integrable system (rational case).

The rational \( G_2 \) Hamiltonian describes a three-body system of the identical particles with two- and three-body interactions

\[ H_{G_2}^{(r)} = \frac{1}{2} \sum_{k=1}^{3} \left[ -\frac{\partial^2}{\partial x_k^2} + \omega^2 x_k^2 \right] + \]

\[ g \sum_{k<l}^{3} \frac{1}{(x_k - x_l)^2} + g_1 \sum_{k<l,m}^{3} \frac{1}{(x_k + x_l - 2x_m)^2} , \]  

(29)

where \( g = \nu(\nu - 1) > -\frac{1}{2} \) and \( g_1 = 3\mu(\mu - 1) > -\frac{3}{4} \) are the coupling constants associated with the two-body and three-body interactions, respectively. The ground-state eigenfunction is given by

\[ \Psi_0^{(r)}(x) = (\Delta^{(r)}(x)\nu(\Delta_1^{(r)}(x))^{\mu} e^{-\frac{1}{2}\omega \sum x_i^2} , \]  

(30)
where \( \Delta^{(r)}(x) = \prod_{i<j} |x_i - x_j| \) and \( \Delta_1^{(r)}(x) = \prod_{i<j; i,j \neq k} |x_i + x_j - 2x_k| \).

The result of a gauge rotation of the Hamiltonian (29) with the ground state eigenfunction (30) (see (6)) can be written in terms of the coordinates \( \lambda_1, \lambda_2 \) given by (5). Thus,

\[
\mathcal{H}_{G_2}^{(r)} = -2\lambda_1 \partial^2_{\lambda_1\lambda_1} - 12\lambda_2 \partial^2_{\lambda_1\lambda_2} + \frac{8}{3} \lambda_1^2 \lambda_2 \partial^2_{\lambda_2\lambda_2}
\]

\[
- \left\{ 4\omega \lambda_1 + 2[1 + 3(\mu + \nu)] \right\} \partial_{\lambda_1} - (12\omega \lambda_2 - \frac{4}{3} \lambda_1^2) \partial_{\lambda_2} ,
\]

(31)

(cf. (7)). It is quite amazing that the difference between (7) and (31) is only in replacement \( \mu \rightarrow (\mu + \nu) \). This is the algebraic form of the rational \( G_2 \) model, which admits a representation in terms of the generators of the algebra \( g^{(2)} \) at \( n = 0 \),

\[
\mathcal{H}_{G_2}^{(r)} = -2L^2 L^1 - 12L^3 L^1 + \frac{8}{3} L^7 L^3 - 2[1 + 3(\mu + \nu)] L^1 - 4\omega L^2 - 12\omega L^3 - \frac{4}{3} L^7 ,
\]

(32)

(cf. (1).) Equation (32) is the \( g^{(2)} \) Lie-algebraic form of the rational \( G_2 \) model. This form depends on the generators of the \( gl_2 \times R^3 \)-subalgebra only. This implies that (32) possesses two invariant subspaces, \( W_n \) and \( \tilde{W}_n \) (see (6)–(10)). Thus we are led to the conclusion that there exists a family of eigenfunctions depending only on the variable \( \lambda_1 \). In fact this property was already known both for the present model (6) as well as for the general many-body Calogero model. It was used in (10) to construct quasi-exactly-solvable deformation of the general Calogero model.

It is worth noting that in the \( \tau \)-coordinates (3) the gauge-rotated \( G_2 \) rational Hamiltonian (29) can be rewritten in terms of the generators of the \( gl(3) \)-algebra (4). Thus this Hamiltonian possesses two hidden algebras: a \( g^{(2)} \) algebra acting on the configuration space parametrized by the \( \lambda \) coordinates and a \( gl(3) \)-algebra acting on the configuration space parametrized by the \( \tau \) coordinates.

6. \( G_2 \) integrable system (trigonometric case).

The Hamiltonian for the trigonometric \( G_2 \) model has the form

\[
\mathcal{H}_{G_2} = -\frac{1}{2} \sum_{k=1}^{3} \frac{\partial^2}{\partial x_k^2} + \frac{g}{4} \sum_{k<l}^{3} \frac{1}{\sin^2(\frac{1}{2}(x_k - x_l))}
\]

\[
+ \frac{g_1}{4} \sum_{k<l}^{3} \frac{1}{\sin^2(\frac{1}{2}(x_k + x_l - 2x_m))} ,
\]

(33)
where \( g = \nu(\nu - 1) > -\frac{1}{4} \) and \( g_1 = 3\mu(\mu - 1) > -\frac{3}{4} \) are the coupling constants associated with the two-body and three-body interactions, respectively. From the physical point of view (33) describes a system of three identical particles. The ground-state eigenfunction is given by

\[
\Psi_0^{(t)}(x) = (\Delta^{(trig)}(x))^\nu(\Delta_1^{(trig)}(x))^\mu, \tag{34}
\]

where \( \Delta^{(trig)}(x) \) and \( \Delta_1^{(trig)}(x) \) are the trigonometric analogies of the Vandermonde determinant and are defined by

\[
\Delta^{(trig)}(x) = \prod_{i<j} |\sin \frac{1}{2}(x_i - x_j)|, \\
\Delta_1^{(trig)}(x) = \prod_{k<l, k \neq l} |\sin \frac{1}{2}(x_i + x_j - 2x_k)|. 
\]

Let us introduce the permutation-symmetric, translation-invariant, periodic coordinates:

\[
\tilde{\sigma}_1 = \frac{1}{\alpha^2} \left[ \cos(\alpha(y_1 - y_2)) + \cos(\alpha(y_2 - y_3)) + \cos(\alpha(y_3 - y_1)) - 3 \right], \\
\tilde{\sigma}_2 = \frac{4}{\alpha^6} \left[ \sin(\alpha(y_1 - y_2)) + \sin(\alpha(y_2 - y_3)) + \sin(\alpha(y_3 - y_1)) \right]^2. \tag{35}
\]

In these coordinates the trigonometric \( G_2 \) Hamiltonian, gauge-rotated with the ground state eigenfunction (34) (see (6) with the factor \( \left(\frac{2}{3}\right) \) instead of 2) becomes

\[
\hat{h}_{G_2}^{(t)} = -\left( 2\tilde{\sigma}_1 + \frac{\alpha^2}{2}\tilde{\sigma}_1^2 - \frac{\alpha^4}{24}\tilde{\sigma}_2 \right) \partial_{\tilde{\sigma}_1}\tilde{\sigma}_1 - \left( 12 + \frac{8}{3}\alpha^2\tilde{\sigma}_1 \right) \tilde{\sigma}_2 \partial_{\tilde{\sigma}_1}\tilde{\sigma}_2 + \\
+ \left[ \frac{8}{3}\tilde{\sigma}_1^2\tilde{\sigma}_2 - 2\alpha^2\tilde{\sigma}_1\tilde{\sigma}_2 \right] \partial_{\tilde{\sigma}_1}\tilde{\sigma}_1 - \left\{ 2[1 + 3(\mu + 2\nu)] + \frac{2}{3}(1 + 3\mu + 4\nu)\alpha^2\tilde{\sigma}_1 \right\} \partial_{\tilde{\sigma}_1} + \\
\left\{ \frac{4}{3}(1 + 4\nu)\tilde{\sigma}_1^2 - \frac{7}{3} + 4(\mu + \nu)\alpha^2\tilde{\sigma}_2 \right\} \partial_{\tilde{\sigma}_2}. \tag{36}
\]

This is the algebraic form of the trigonometric \( G_2 \) model and \( \hat{h}_{G_2}^{(t)} \) can be written in terms of the generators of the algebra \( g^{(2)} \) containing both the generators of \( gl_2 \rtimes R^3 \) and the generator \( T \). The explicit expression is given by

\[
\hat{h}_{G_2}^{(t)} = -2L^2L^1 - 6L^3L^1 + \frac{4}{3}L^7L^3 - \frac{\alpha^2}{6}(3L^2L^2 + 8L^3L^2 + 3L^3L^3) + \frac{\alpha^4}{24}T \\
- 2[1 + 3(\mu + 2\nu)]L^1 - \frac{4}{3}(1 - 4\nu)L^7 - (2\mu + \frac{8}{3}\nu + \frac{1}{6})\alpha^2L^2 \\
- \frac{1}{6} + 2(\mu + \nu)\alpha^2L^3. \tag{37}
\]
The operator (37) can be easily reduced to triangular form by introducing the new variables

\[ \rho_1 = \tilde{\sigma}_1 , \quad \rho_2 = \tilde{\sigma}_2 + \frac{4}{\alpha^2} \tilde{\sigma}_1^2 . \]  

(38)

It is worth noticing that if \( \alpha \to 0 \) this change of variables becomes singular, reflecting the non-existence of bound states for the Calogero and \( G_2 \) rational models in the absence of the harmonic oscillator term in the potential, \( \omega = 0 \).

In new coordinates the Hamiltonian takes the form

\[ h^{(t)}_{G_2} = -(2\rho_1 + \frac{2}{3} \alpha^2 \rho_1^2 - \frac{\alpha^4}{24} \rho_2) \partial_{\rho_1 \rho_1}^2 - (12 \rho_2 + 2 \alpha^2 \rho_1 \rho_2 - \frac{16}{\alpha^2} \rho_1^2) \partial_{\rho_1 \rho_2}^2 - \]

\[ (2\alpha^2 \rho_2^2 + \frac{96}{\alpha^2} \rho_1 \rho_2 - \frac{256}{\alpha^4} \rho_3) \partial_{\rho_2 \rho_2}^2 - [2(1 + 3\mu + 6\nu) + \frac{2}{3} (1 + 3\mu + 4\nu) \alpha^2 \rho_1] \partial_{\rho_1} - \]

\[ \{2(1 + 2\mu + 2\nu) \alpha^2 \rho_2 + \frac{16}{\alpha^2} (2 + 3\mu + 6\nu) \rho_1 \} \partial_{\rho_2} , \]  

(39)

and it is easy to check that it is indeed a triangular operator. Evidently this operator can be rewritten in terms of the \( g^{(2)} \)-generators as

\[ h^{(t)}_{G_2} = -2L_1^2 L_1^1 - 6L_1^1 L_3^3 + \frac{\alpha^4}{24} T - \frac{2}{3} \alpha^2 L_2^2 L_2^2 - \alpha^2 L_2^2 L_3^3 - \frac{\alpha^2}{2} L_3^3 L_3^3 + \]

\[ \frac{16}{\alpha^2} L_7^7 L_1^1 - \frac{48}{\alpha^2} L_3^3 L_6^6 + \frac{256}{\alpha^2} L_6^6 L_7^7 - 2(1 + 3\mu + 6\nu) L_1^1 - \]

\[ (2\mu + \frac{8}{3} \nu) \alpha^2 L_2^2 - 2(\mu + \nu) \alpha^2 L_3^3 - \frac{16}{\alpha^2} (2 + 3\mu + 6\nu) L_6^6 . \]  

(40)

Using either of the above representations, the energy levels of the Hamiltonian \( H_{G_2} \) can be easily found and are given by

\[ - \mathcal{E}_{n,m} = \left[ (n - 2m - 1)(n + m + 1) + 3n\mu + 2(2n - m)\nu + 1 \right] \alpha^2 , \]  

(41)

where \( n \) and \( m \) are quantum numbers,

\[ n = 0, 1, 2, 3, \ldots , \quad 0 \leq m \leq \lfloor \frac{n}{2} \rfloor . \]  

(42)

We would like to emphasize that unlike the rational and trigonometric \( A_2, BC_2 \) and rational \( G_2 \) Hamiltonians, the trigonometric \( G_2 \) Hamiltonian has the only one hidden algebra \( g^{(2)} \).

As a general conclusion we would like to stress that all three-body integrable quantal systems originating from the Hamiltonian reduction method, but with arbitrary coupling constants, possess the same hidden algebra \( g^{(2)} \). Generically, the orthogonal polynomials in two variables associated the eigenfunctions of \( A_2 - BC_2 - G_2 \) integrable models do not appear in Krall-Sheffer classification scheme of 2d orthogonal polynomials (see, for example, [11]).
ACKNOWLEDGEMENTS

The author thanks the Theoretical Physics Institute, University of Minnesota for its kind hospitality. Useful discussions with M. Rosenbaum on the early stage of the work and also with M. Shifman are highly appreciated. The work is partially supported by DOE grant DE-FG02-94ER40823.

REFERENCES

[1] M. A. Olshanetsky and A. M. Perelomov, “Quantum integrable systems related to Lie algebras,” Phys. Rep. 94 (1983) 313.

[2] A. M. Perelomov, “Algebraic approach to the solution of one-dimensional model of $N$ interacting particles,” Teor. Mat. Fiz. 6 (1971) 364 (in Russian); English translation: Sov. Phys. – Theor. and Math. Phys. 6 (1971) 263.

[3] W. Rühl and A. Turbiner, “Exact solvability of the Calogero and Sutherland models,” Mod. Phys. Lett. A10 (1995) 2213-2222, hep-th/9506105.

[4] M. Rosenbaum, A. Turbiner and A. Capella, “Solvability of the $G_2$ integrable system,” Intern. Journ. Mod. Phys. A (1998) to appear, solv-int/9707005.

[5] A. V. Turbiner, “Lie algebras and linear operators with invariant subspace,” in Lie algebras, cohomologies and new findings in quantum mechanics (N. Kamran and P. J. Olver, eds.), AMS, vol. 160, pp. 263–310, 1994; “Lie-algebras and Quasi-exactly-solvable Differential Equations”, in CRC Handbook of Lie Group Analysis of Differential Equations, Vol.3: New Trends in Theoretical Developments and Computational Methods, Chapter 12, CRC Press (N. Ibragimov, ed.), pp. 331-366, 1995; hep-th/9409068.

[6] A. González-Lopéz, J. Hurtubise, N. Kamran and P.J. Olver, “Quantification de la cohomologie des algèbres de Lie de champs de vecteurs et fibrés en droites sur des surfaces complexes compactes”, C.R.Acad.Sci. (Paris), Série I 316 (1993) 1307-1312

[7] L. Brink, A. Turbiner and N. Wyllard, “Hidden Algebras of the (super) Calogero and Sutherland models,” Preprint ITP 97-05 and ICN-UNAM/97-02 hep-th/9705219 Journ. Math. Phys.39(1998) 1285-1315

[8] D. Bernard, V. Pasquier and D. Serban, “Exact solution of long-range interacting spin chains with boundaries,” Europhys. Lett. 30 (1995) 301-305, hep-th/9501044

[9] J. Wolfes, “On the three-body linear problem with three-body interaction,” J. Math. Phys. 15 (1974) 1420-1424.

[10] A. Minzoni, M. Rosenbaum and A. Turbiner, “Quasi-Exactly-Solvable Many-Body Problems,” Mod. Phys. Lett. A11 (1996) 1977-1984, hep-th/9606092.

[11] L.L. Littlejohn, “Orthogonal polynomial solutions to ordinary and partial differential equations”, Proceedings of an International Symposium on Orthogonal Polynomials and their Applications, Segovia, Spain, Sept.22-27, 1986, Lecture Notes in Mathematics No.1329, M.Alfaro et al. (Eds.), Springer-Verlag (1988), pp. 98-124.