Abstract

This paper is concerned with the study of Besov-type decomposition spaces, which are scales of spaces associated to suitably defined coverings of the Euclidean space $\mathbb{R}^d$, or suitable open subsets thereof. A fundamental problem in this domain, that is currently not fully understood, is deciding when two different coverings give rise to the same scale of decomposition spaces.

In this paper, we establish a coarse geometric approach to this problem, and show how it specializes for the case of wavelet coorbit spaces associated to a particular class of matrix groups $H < GL(\mathbb{R}^d)$ acting via dilations. These spaces can be understood as special instances of decomposition spaces, and it turns out that the question whether two different dilation groups $H_1, H_2$ have the same coorbit spaces can be decided by investigating whether a suitably defined map $\phi : H_1 \to H_2$ is a quasi-isometry with respect to suitably defined word metrics. We then proceed to apply this criterion to a large class of dilation groups called shearlet dilation groups, where this quasi-isometry condition can be characterized algebraically.

We close with the discussion of selected examples.

1. Introduction

This paper is concerned with the study of two related classes of function spaces on Euclidean spaces, wavelet coorbit spaces and decomposition spaces. Both classes can be understood as rather sweeping generalizations of Besov spaces, and their inception dates back to work by Feichtinger, Gröchenig and Gröbner (e.g. [12, 13, 14, 15]) in the 1980s. We refer to [17] for more background information on this development, and in particular for an explanation of the relationship of these notions to the influential work of Frazier and Jawerth [19].
The main thrust of the initial work was to provide a unified view onto a variety of function spaces, specifically Besov spaces and modulation spaces, using notions from abstract harmonic and Fourier analysis. Coorbit spaces are defined using group representation theory [14, 15], and special instances of coorbit spaces have been studied in a variety of settings and for various classes of underlying locally compact groups in recent years; see e.g. [11, 10, 24, 25, 16, 18] for a small sample of the literature. The class of admissible dilation groups underlying higher dimensional wavelet coorbit spaces turns out to be a particularly rich source of examples [21], see

- [22] for a classification of admissible dilation groups in dimension two up to conjugacy;
- [21] for the systematic construction and classification (up to conjugacy) of abelian admissible dilation groups;
- [11, 10] for the initial definition of shearlet dilation groups in dimension two and higher;
- [9] for the introduction of Toeplitz shearlet dilation groups in dimension three and higher;
- [25, 1] for the introduction and systematic construction of generalized shearlet dilation groups in arbitrary dimensions (comprising the previous examples);
- [8] for a classification of admissible dilation groups in dimension three, up to conjugacy.

We emphasize that the items on this list do not refer to single dilation groups, but rather to classes of groups, most of which tend to become quite large for higher dimensions.

By contrast, decomposition spaces are of a more Fourier analytic nature. Whereas coorbit spaces depend on the choice of a particular group representation, the construction of decomposition spaces departs from a certain covering of the frequency domain. After their inception in [12], the theory of decomposition spaces lay somewhat dormant for a while, but their usefulness in capturing approximation-theoretic features of anisotropic systems such as curvelets was realized later by Borup and Nielsen, see e.g. [2, 32]. A further major boost to the subject was provided by the transformative work of Voigtlaender [36, 37], which is also the foundation of the current paper. At this point in time, large classes of function spaces have been recognized as special instances of the decomposition space formalism, as the following list shows:

- $\alpha$-modulation spaces [28];
- curvelet smoothness spaces [3];
- shearlet smoothness spaces [32];
• homogeneous and inhomogeneous *anisotropic* Besov spaces \([4, 6]\);

• wave packet smoothness spaces \([5]\);

• wavelet coorbit spaces associated to admissible dilation group, i.e., *all examples on the previous list* \([27]\);

• wavelet coorbit spaces associated to *integrably admissible dilation groups*, see \([26]\). This class properly contains the class mentioned in the previous item, but also the class of homogeneous anisotropic Besov spaces.

Again, most of the items on this list represent classes or families of constructions rather than a single one, and these classes can become quite large with increasing dimension.

While coorbit and decomposition spaces rest on somewhat distinct mathematical foundations, the resulting spaces share an important common approximation-theoretic interpretation. What is common both to coorbit and decomposition spaces is the existence of a family of vectors \((\eta_x)_x \in X\) in a suitable Hilbert space \(H\) that acts as a *continuous frame* of \(H\), i.e., guaranteeing a norm equivalence

\[
\|f\|_H \asymp \|((f, \eta_x))_{x \in X}\|_{L^2(X, d\mu)} ,
\]

with respect to a suitably chosen measure \(\mu\) on \(X\), as well as an associated *inversion formula*

\[
f = \int_X \langle f, \eta_x \rangle \tilde{\eta}_x d\mu(x) ,
\]

with a suitably chosen *dual system* \((\tilde{\eta}_x)_{x \in X} \subset H\). The operator

\[
V_{\eta} : f \mapsto ((f, \eta_x))_{x \in X}
\]

mapping \(f\) to its expansion coefficients can be understood as a generalized wavelet transform. Typically, the parameter space \(X\) provides an interpretation of the elements \(\eta_x\) of the continuous frame as basic building blocks, and the inversion formula expresses \(f\) as a continuous expansion in these building blocks.

For concreteness, let us quickly sketch the case of generalized wavelet systems: Given a dilation group \(H < GL(\mathbb{R}^d)\), we let \(X = \mathbb{R}^d \rtimes H\) denote the subgroup of the full affine group of \(\mathbb{R}^d\) generated by the translations in \(\mathbb{R}^d\) and \(H\), and define the quasi-regular representation

\[
\pi : \mathbb{R}^n \rtimes H \to \mathcal{U}(L^2(\mathbb{R}^n)) , \quad (\pi(x, h)f)(y) = |\det(h)|^{-1/2} f(h^{-1}(y - x)) ,
\]

and for suitably chosen \(\eta \in L^2(\mathbb{R}^n)\), we let

\[
\eta_{(x, h)} = \pi(x, h)\eta .
\]

The element \((x, h) \in X\) corresponds to a wavelet \(\eta_{(x, h)}\) centered at \(x\) and scaled by \(h \in H\). Very often, explicit parametrizations of \(H\) allow to interpret the scaling variable \(h\) in \((x, h)\) (and thus \(\eta_{(x, h)}\)) further, e.g. as a combination of isotropic scaling and rotation in the case of the similitude group \([2]\), or as a
combination of anisotropic scaling and shearing in the case of a shearlet dilation group \([10]\). In this group-theoretic context, the norm equivalence \([11]\) becomes a norm equality, when \(\eta\) is chosen as an admissible vector and \(\mu\) is chosen as the left Haar measure on \(G\), and the inversion formula holds with \(\tilde{\eta}(x,h) = \eta(x,h)\).

One can then proceed to define norms on elements of \(L^2(\mathbb{R}^n)\) by imposing integrability conditions that are more stringent than the \(L^2\)-norm entering in the Hilbert space norm equivalence \([11]\), i.e., by introducing weights \(\nu\) and considering integrability exponents \(p < 2\), leading to considering norms of the type

\[
\|V_\eta f\|_{L^p_\nu}.
\]

Furthermore, discretization results allow to replace the (typically continuously indexed) systems \((\eta_x)_{x \in X}\) by suitably chosen discrete subsystems – in fact, (quasi-)Banach frames – corresponding to discrete subsets \(X_d \subset X\), while preserving norm equivalences such as

\[
\|V_\eta f\|_{L^p_\nu} \approx \|V_\eta f|_{X_d}\|_{\ell^p_\nu},
\]

see e.g. \([12, 13, 14, 15]\). For \(p < 2\) and constant weights this norm equivalence attains additional relevance: Here the discretization results of the just cited references allow to conclude, that vectors having \(p\)-summable frame coefficients have a nontrivial non-linear approximation rate with respect to the frame, with the decay rate increasing as \(p\) decreases. In this way, both coorbit and decomposition spaces associated to integrability exponents \(p, q < 2\) have natural interpretations as spaces of sparse signals with respect to the respective systems of building blocks. These spaces therefore capture the approximation-theoretic properties of the building blocks.

Much of the foundational work in \([12, 13, 14, 15]\) was devoted to proving that the definitions were consistent, i.e., essentially independent of various design choices that enter into the construction of the systems. In the case of wavelet coorbit spaces, these results were mostly related to the proper choice of the analyzing wavelet \(\eta\) entering the definition of the wavelet system \((\eta(x,h))(x,h) \in X = (\pi(x,h)\eta)(x,h) \in X\).

By contrast, the influence of the primary choices in the design of the system of building blocks, i.e., the choice of dilation group in the case of generalized wavelet systems, and the choice of covering in the case of decomposition spaces, is much less understood. Generally speaking, one very much expects that qualitatively different primary choices will result in different approximation-theoretic properties of the building blocks; for systems such as curvelets or shearlets, this expectation was the driving factor for their inception. That said, the understanding what “qualitatively different” actually means in this context is obviously an important part of this discussion, and it is currently not fully developed.

Prior to the work of Voigtlaender, with few exceptions such as \([32]\), most of the work on coorbit or decomposition spaces tended to stay within the confines provided by a fixed primary choice, i.e., tended to concentrate on, say, \(\alpha\)-modulation spaces, or on shearlet coorbit spaces. Voigtlaender’s publications
since 2015, specifically \cite{36, 27, 37}, systematically develop techniques that allow to cross these boundaries between different classes of spaces, by providing sharp embedding results between decomposition spaces associated to different coverings, or into classical smoothness spaces.

By contrast to the scope of the mentioned papers \cite{27, 36, 37}, the aims of this paper are somewhat more modest and elementary. We address the following fundamental question, both for decomposition spaces and for wavelet coorbit spaces: When do two initially different primary design choices (e.g., different coverings or dilation groups) result in the same scales of associated spaces? Our central contribution essentially amounts to rewriting the pertinent criteria formulated in \cite{37} by introducing a novel ingredient to the discussion, namely coarse geometry, and demonstrating that this new perspective can be used effectively.

1.1. Structure and overview of the paper

This paper is based on parts of the PhD thesis \cite{30} by the second author, with some results substantially expanded.

As mentioned at the end of the previous subsection, we address two main questions:

1. When do two coverings result in the same scale of decomposition spaces?
2. When do two dilation groups possess the same scale of coorbit spaces?

As pointed out in the previous subsection, Question 2 is a special case of Question 1, and both are essentially answered by results in \cite{37}. However, as with many results in this domain, the precise criteria are somewhat unintuitive, and tend to be cumbersome to apply in concrete settings. It is one of the main contributions of our paper to rewrite the criteria in terms of suitable metrics, following an initial observation made in \cite{12}, and to demonstrate the usefulness of this reformulation with the help of various examples.

We now give an overview of the paper. The following summary glosses over various technicalities (such as additional conditions on coverings, or matters related to the question how coorbit spaces associated to different groups can be understood to coincide, or the role of connectedness issues), that are explained in more detail in the subsequent text.

Section 2 recalls the fundamentals of decomposition spaces and coorbit spaces, including the definitions of the various spaces, and their basic properties. Decomposition spaces, as used throughout the paper, rely on the notion of (structured) admissible covering. We will concentrate on decomposition spaces associated to weighted mixed $L^p$-norms, i.e., to inner norms $\| \cdot \|_{L^p}$ and outer norms $\| \cdot \|_{\ell^q}$. The main takeaway from subsection 2.1 is the definition of the notion of weak equivalence of coverings, and its relevance for the associated decomposition spaces. Lemmas 2.3 and 2.4 show that weak equivalence of two admissible coverings is equivalent to the fact that the associated scales of decomposition spaces coincide. They also formulate a rigidity result stating that
if two decomposition spaces associated to different coverings coincide for some pair of integrability exponents \((p,q) \neq (2,2)\), then the full scales of decomposition spaces agree. A complementary result that is of particular relevance to wavelet coorbit spaces addresses the question when dual coverings of different sets give rise to identical decomposition spaces, see Theorem 2.5.

We then recount the relevant results of wavelet coorbit theory in higher dimensions, in particular the notion of admissible dilation groups \(H\) and their unique open dual orbits \(O = H^T \xi\), for suitably chosen \(\xi \in \mathbb{R}^d\). The decomposition space description of the coorbit spaces associated to the quasi-regular representation of \(\mathbb{R}^d \rtimes H\) is obtained via the so-called induced covering arising from the dual action of \(H\) on \(O\), see Theorem 2.13. We then introduce the notion of coorbit equivalence for two dilation groups \(H_1, H_2\), expressing when their coorbit spaces coincide; see Definition 2.16. We combine Theorem 2.13 with the results characterizing when two dual coverings result in the same scale of decomposition spaces, and with Theorem 2.5, to obtain an important intermediate characterization of coorbit equivalence in Theorem 2.18.

Section 3 then proceeds to introduce metric language to the discussion. We first give a short review of the relevant notions from coarse geometry in Subsection 3.1. A covering \(\mathcal{P}\) of a frequency set \(O\) allows to introduce, in a very natural way, a related metric \(d_{\mathcal{P}}\), and it turns out that weak equivalence of \(\mathcal{P}\) to a second covering \(\mathcal{Q}\) of the same set \(O\) is equivalent to the fact that \(id_O : (O, d_{\mathcal{P}}) \to (O, d_{\mathcal{Q}})\) is a quasi-isometry, or equivalently, a coarse equivalence; see Theorem 3.22.

As already mentioned, it is important to point out that the definition of the metric goes back to the original source [12], and Proposition 3.8 of the mentioned paper is a relevant precursor of Theorem 3.22. However, weak equivalence of coverings was initially only known to be a sufficient criterion for the property that two coverings define the same scale of decomposition spaces; the converse was proved later in [36].

Section 4 transfers the results obtained for decomposition spaces to the coorbit setting. We begin by reviewing basic notions from metric group theory in Subsection 4.1. Subsection 4.2 establishes that the canonical projection map from the group onto the dual orbit is a quasi-isometry. Subsection 4.3 contains one of the main results of this paper, namely Theorem 4.17 which combines the observations from Subsection 4.2 with the criteria from the previous sections to formulate a metric criterion for coorbit equivalence: Two admissible groups \(H_1\) and \(H_2\) can only be coorbit equivalent if their dual orbits coincide, i.e., \(O = H_1^T \xi_0 = H_2^T \xi_0\) for some suitable \(\xi_0 \in \mathbb{R}^d\). If that condition is fulfilled, consider the two associated canonical projections \(p_{\xi_0}^{H_1} : H_1 \ni h \mapsto h^{-T} \xi_0 \in O\), and \(p_{\xi_0}^{H_2} : H_2 \to O\) defined analogously. Let \((p_{\xi_0}^{H_2})^{-1} : O \to H_2\) denote any right inverse of \(p_{\xi_0}^{H_2}\). Then \(H_1\) and \(H_2\) are coorbit equivalent if and only if \((p_{\xi_0}^{H_2})^{-1} \circ p_{\xi_0}^{H_1} : (H_1, d_{H_1}) \to (H_2, d_{H_2})\) is a quasi-isometry with respect to any choice of word metrics \(d_{H_1}, d_{H_2}\) on \(H_1, H_2\) associated to suitable relatively compact neighborhoods of unity. Again, the quasi-isometry property is equivalent to coarse equivalence. Thus coorbit equivalence has been translated to a concrete
Section 5 contains an illustration that this translation can be put to use for the characterization of coorbit equivalence within a large example class, namely that of generalized shearlet dilation groups. The main result of that section is Theorem 5.9 which contains a concise characterization of coorbit equivalence for shearlet dilation groups. The fact that this equivalence leads to rather stringent conditions on the groups under consideration emphasizes the richness of coorbit theory in higher dimensions. The results indicate that in higher dimensions, coorbit equivalence of different groups is a fairly rare phenomenon. While this observation may have been expected, it is the main contribution of this paper to provide sharp criteria, methods for their verification, and explicit classes of groups corroborating this expectation.

The closing section contains various examples that further illustrate the usefulness of the approach developed in this paper.

2. Fundamentals of coorbit and decomposition spaces

2.1. Decomposition spaces

The starting point for the definition of decomposition spaces is the notion of an admissible covering \( Q = (Q_i)_{i \in I} \) of some open set \( O \subset \mathbb{R}^d \) (see [12]), which is a family of nonempty sets \( Q_i \subset \mathbb{R}^d \) such that

i) \( \bigcup_{i \in I} Q_i = O \) and

ii) \( \sup_{i \in I} \# \{ j \in I : Q_i \cap Q_j \neq \emptyset \} < \infty. \)

Throughout this paper, we will concentrate on the class of (tight) structured admissible covering, see Definition 2.5 of [37]. This means that \( Q_i = T_i Q + b_i \) with \( T_i \in \text{GL}(\mathbb{R}^d), b_i \in \mathbb{R}^d \) with an open, precompact set \( Q \), and the involved matrices fulfill

\[
\sup_{i,j \in I : Q_i \cap Q_j \neq \emptyset} \|T_i^{-1}T_j\| < \infty. \tag{2.1}
\]

The next ingredient in the definition of decomposition spaces is a special partition of unity \( \Phi = (\varphi_i)_{i \in I} \) subordinate to \( Q \), also called \( L^p \)-BAPU (bounded admissible partition of unity), with the following properties

i) \( \varphi_i \in C_\infty^\infty(O) \) \( \forall i \in I \),

ii) \( \sum_{i \in I} \varphi_i(x) = 1 \) \( \forall x \in O \),

iii) \( \varphi_i(x) = 0 \) for \( x \in \mathbb{R}^d \setminus Q_i \) and \( i \in I \),

iv) if \( 1 \leq p \leq \infty \): \( \sup_{i \in I} \|T^{-1} \varphi_i\|_{L^1} < \infty \),

if \( 0 < p < 1 \): \( \sup_{i \in I} |\det(T_i)|^{\frac{1}{p}-1}\|T^{-1} \varphi_i\|_{L^p} < \infty. \)
Here, $\mathcal{F}$ denotes the usual Fourier transform of a function in $L^2(\mathbb{R}^d)$ defined by

$$\mathcal{F}f(\xi) := \int_{\mathbb{R}^d} f(x)e^{-2\pi i(x,\xi)}dx$$

for $\xi \in \mathbb{R}^d$. We also use the notation $\hat{f} := \mathcal{F}(f)$. The definition of decomposition spaces requires one last ingredient, namely a weight $(u_i)_{i \in I}$ such that there exists $C > 0$ with $u_i \leq Cu_j$ for all $i, j \in I : Q_i \cap Q_j \neq \emptyset$. A weight with this property is also called $Q$-moderate. The interpretation of this property is that the value of $(u_i)_{i \in I}$ is comparable for indices corresponding to sets which are “close” to each other. Finally, we define the (Fourier-side) decomposition space with respect to the covering $Q$ and the weight $(u_i)_{i \in I}$ with integrability exponents $0 < p, q \leq \infty$ as

$$D(Q, L^p, \ell_q^u) := \{ f \in D'(O) : \|f\|_{D(Q, L^p, \ell_q^u)} < \infty \} \quad (2.2)$$

for

$$\|f\|_{D(Q, L^p, \ell_q^u)} := \left\| \left( u_i \cdot \|\mathcal{F}^{-1}(\varphi_i f)\|_{L^p(\mathbb{R}^d)} \right)_{i \in I} \right\|_{\ell_q(I)} . \quad (2.3)$$

As the notation suggests, the decomposition spaces are independent of the precise choice of $\Phi$ [36, Corollary 3.4.11].

A crucial concept is the definition of the set of neighbors of a covering.

**Definition 2.1 ([12] Definition 2.3).** For a covering $Q = (Q_i)_{i \in I}$ of $O$ with $Q_i \subset O$ for all $i \in I$, we define the set of neighbors of a subset $J \subset I$ as

$$J^* := \{ i \in I \mid \exists j \in J : Q_i \cap Q_j \neq \emptyset \} .$$

By induction, we set $J^{0*} := J$ and $J^{(n+1)*} = (J^n*)^*$ for $n \in \mathbb{N}_0$. Moreover, we use the shorthand notations $i^{k*} := \{ i \}^{k*}$ and define $Q^{k*} := \bigcup_{j \in i^{k*}} Q_j$ for $i \in I$ and $k \in \mathbb{N}_0$.

In the remaining part of this subsection, we take a look at relations between different coverings. The ultimate purpose of these relations is the clarification when two coverings lead to the same scale of decomposition spaces. While our exposition follows [36], most of the definitions hark back to [12].

**Definition 2.2 ([36] Definition 3.3.1.).** Let $Q = (Q_i)_{i \in I}$ and $P = (P_j)_{j \in J}$ be families of subsets of $\mathbb{R}^d$.

i) We define the set of $P$-neighbors of $i \in I$ by $J_i := \{ j \in J \mid Q_i \cap P_j \neq \emptyset \}$. More generally, we call $J_i$ and $I_j$ intersection sets for the coverings $Q$ and $P$.

ii) We call $Q$ weakly subordinate to $P$ if $N(Q, P) := \sup_{i \in I} |J_i| < \infty$.

The quantity $N(P, Q)$ is defined analogously, and we call $Q$ and $P$ weakly equivalent if $N(P, Q) < \infty$ and $N(Q, P) < \infty$. 
iii) We call \( Q \) almost subordinate to \( P \) if there exists \( k \in \mathbb{N}_0 \) such that for 
every \( i \in I \) there exists an \( j_i \in J \) with \( Q_i \subset P_{j_i}^k \). If \( k = 0 \) is a valid choice, then we call \( Q \) subordinate to \( P \).

iv) We call \( Q \) weakly equivalent to \( P \) if \( Q \) is weakly subordinate to \( P \) and \( P \) is weakly subordinate to \( Q \).

The relevance of these notions, in particular of weak equivalence, is spelled out in the next two lemmas. The formulation of the next lemma is a special case of the cited result.

**Lemma 2.3** ([37, Theorem 6.9]). Let \( Q = (Q_i)_{i \in I}, P = (P_j)_{j \in J} \) be two structured admissible coverings of the open set \( \mathcal{O} \subset \mathbb{R}^d \). If \( Q \) and \( P \) are not weakly equivalent, then

\[
\mathcal{D}(Q, L^p, \ell^q_{u_1}) \neq \mathcal{D}(P, L^p, \ell^q_{u_2})
\]

for all \( Q \)-moderate weights \( u_1 : I \to (0, \infty) \), for all \( P \)-moderate weights \( u_2 : J \to (0, \infty) \) and all \( p, q \in (0, \infty] \) with \( (p, q) \neq (2, 2) \).

The exception \( (p, q) \neq (2, 2) \) is necessary to exclude trivial cases: In the case of \( (p, q) = (2, 2) \) the associated decomposition spaces are just weighted \( L^2 \) spaces, by the Plancherel Theorem. In particular, if \( \mathcal{O} \subset \mathbb{R}^d \) is open and of full measure, and the weight \( v \) is constant, one finds that \( \mathcal{D}(P, L^2, \ell^2_v) = L^2(\mathbb{R}^d) \), for all admissible coverings \( P \).

Weak subordinateness and equivalence of coverings are important assumptions for a multitude of sufficient criteria for embeddings of decomposition spaces and their equality, as developed in [37]. The following statement is [37, Lemma 6.11]. The statement about the range \( 0 \leq p, q \leq \infty \) is justified by the remark following the cited lemma.

**Lemma 2.4.** Let \( 1 \leq p, q \leq \infty \) and let \( \emptyset \neq \mathcal{O} \subset \mathbb{R}^d \) be open. Further, let \( Q = (Q_i)_{i \in I}, P = (P_j)_{j \in J} \) be two tight structured admissible coverings of \( \mathcal{O} \), and let \( u_1 \) be a \( Q \)-moderate weight and \( u_2 \) a \( P \)-moderate weight.

If \( Q \) and \( P \) are weakly equivalent and there exists \( C > 0 \) such that \( C^{-1} u_1(i) \leq u_2(j) \leq C u_1(i) \) for all \( i \in I \) and \( j \in J \) with \( Q_i \cap P_j \neq \emptyset \), then

\[
\mathcal{D}(Q, L^p, \ell^q_{u_1}) = \mathcal{D}(P, L^p, \ell^q_{u_2})
\]

with equivalent norms.

If all sets in the coverings are connected, the conclusion holds for the range \( 0 \leq p, q \leq \infty \).

Under suitable assumptions, the two lemmas show that an equality

\[
\mathcal{D}(Q, L^{p_1}, \ell^{q_1}_{u_1}) = \mathcal{D}(P, L^{p_2}, \ell^{q_2}_{u_2})
\]

with non-trivial exponents \( (p_1, q_1) \neq (2, 2) \) and/or \( (p_2, q_2) \neq (2, 2) \) holds if and only if \( Q \) and \( P \) are weakly equivalent, with \( (p_1, q_1) = (p_2, q_2) \) and the involved weights are equivalent. Note that one such equality entails

\[
\mathcal{D}(Q, L^p, \ell^q_{u_1}) = \mathcal{D}(P, L^p, \ell^q_{u_2})
\]
for all exponents $0 \leq p, q \leq \infty$ and all weights $u_1$, with suitably chosen $u_2$.

If the coverings consist of open and connected sets, weak subordinateness implies almost subordinateness (cf. [30 Corollary 3.3.4.]). Consequently, we will mainly be interested in studying necessary and sufficient conditions for the weak subordinateness of coverings.

While the results so far compare decomposition spaces associated to different coverings $Q, P$ of the same set $O$, the following Theorem examines pairs of decomposition spaces associated to coverings of different sets $O, O'$. It is a special case of [37, Theorem 6.9].

Theorem 2.5. Let $\emptyset \neq O, O' \subset \mathbb{R}^d$ open. Let $Q = (Q_i)_{i \in I}$ denote an admissible covering of $O$, $P = (P_j)_{j \in J}$ denote an admissible covering of $O'$. Assume that either $O' \cap \partial O \neq \emptyset$ or $O \cap \partial O' \neq \emptyset$ holds, and that $O \cap O'$ is unbounded.

Let $p_1, p_2, q_1, q_2 \in (0, \infty]$. Then

$$\forall f \in \mathcal{C}_c(\partial O \cap O') : \|f\|_{D(Q, L^{p_1}, \ell^{q_1})} \simeq \|f\|_{D(P, L^{p_2}, \ell^{q_2})}$$

can only hold in the trivial case, i.e., when $(p_1, q_1) = (2, 2) = (p_2, q_2)$ and $v_i \asymp w_j$ whenever $Q_i \cap P_j \neq \emptyset$.

Note that the assumptions on $O, O'$ are fulfilled if they are distinct open and dense subsets. Density and openness of $O$ imply $\partial O = \mathbb{R}^d \setminus O$, and thus $O' \cap \partial O = \emptyset$ can only happen if $O' \subseteq O$. In that case however we get $\partial O' \cap O = (\mathbb{R}^d \setminus O') \cap O = \emptyset$. Furthermore, $O \cap O'$ is dense, hence unbounded.

Remark 2.6. The results in [37] exhibit an occasional subtle influence of the various assumptions on the elements of the coverings $P$ and $Q$ on the comparison of associated decomposition spaces. An instance can be witnessed in Lemma 2.4 where connectedness assumptions of the covering sets, as well as tightness and structuredness of the cover, have an influence on the range of summability parameters for which the conclusions of the lemma hold. The connectedness assumption will play a more explicit role in the formulation and proof of our metric characterization of weak equivalence in Theorem 3.22.

For further remarks on these issues, we refer to 3.23 below.

For the discussion of coorbit spaces associated to different dilation groups, a certain class of weights will be of particular importance. Throughout the paper, we will use $|A|$ to denote the Lebesgue measure of a Borel set $A \subset \mathbb{R}^d$.

Definition 2.7. Let $Q = (Q_i)_{i \in I}$ denote a (tight) structured admissible covering, and $\alpha \in \mathbb{R}$. The intrinsic weight with exponent $\alpha$ is the family $(u_i)_{i \in I}$ defined by

$$u_i = |Q_i|^\alpha .$$

Note that for $Q_i = T_i Q + b_i$, one obtains $|Q_i| \asymp |\det(T_i)|$. Hence the intrinsic weight is indeed $Q$-moderate, by the condition (2.1) on structured coverings.

The following observation notes that intrinsic weights are robust under weak equivalence.
Lemma 2.8. Let $Q = (Q_i)_{i \in I}$ and $P = (P_j)_{j \in J}$ denote two tight structured admissible coverings consisting of connected sets. If $Q$ and $P$ are weakly equivalent, then the associated intrinsic weights are equivalent as well, i.e., there exists a constant $C > 0$, such that one has

$$\forall i \in I \forall j \in J : Q_i \cap P_j \neq \emptyset \Rightarrow \frac{|P_j|}{|Q_i|} \leq C .$$

Proof. Let $x \in Q_i \cap P_j$ be given, for suitable $i \in I$ and $j \in J$. We first note that if $Q$ is weakly subordinate to $P$, and consists of connected sets, then it is almost subordinate to $P$, by [37, Corollary 2.13]. Hence there exists $n \in \mathbb{N}$ such that for each $i \in I$ there exists $j_0 \in J$ with

$$Q_i \subset P^n_{j_0} .$$

Let $C_1, C_2 \geq 1$ denote the constants such that

$$\forall j, \ell \in J : P_j \cap P_\ell \neq \emptyset \Rightarrow \frac{|P_j|}{|P_\ell|} \leq C_1 ,$$

which results from condition (2.1), and

$$C_2 = \sup_{j \in J} \# \{ \ell \in J : P_j \cap P_\ell \neq \emptyset \} .$$

Then $P^n_{j_0}$ arises as the union of at most $\sum_{k=0}^n C_2^k \leq C_2^{n+1}$ sets, each of measure at most $C_1^n |P_{j_0}|$, i.e.,

$$|Q_i| \leq |P^n_{j_0}| \leq |P_{j_0}| C_1^n C_2^{n+1} .$$

But the fact that $x \in P_j \cap P^n_{j_0}$ furthermore implies that

$$\frac{|P_{j_0}|}{|P_j|} \leq C_1^{n+1} ,$$

whence finally

$$|Q_i| \leq |P_j| C_1^{2n+1} C_2^{n+1} .$$

Here it is important to observe that both constants $C_1$ and $C_2$ are independent of $i \in I$. Hence this estimate shows one direction of the desired equivalence, and the other one follows by symmetry.

2.2. Admissible groups and induced coverings

For a closed matrix group $H \leq \text{GL}(\mathbb{R}^d)$, which we also call dilation group in the following, we define the group $G := \mathbb{R}^d \rtimes H$, generated by dilations with elements of $H$ and arbitrary translations, with the group law $(x, h) \circ (y, g) := (x + hy, hg)$. We denote integration with respect to a left Haar measure on $H$ with $dh$, the associated left Haar measure on $G$ is then given by $d(x, h) = | \det h |^{-1} dx dh$. The Lebesgue spaces on $G$ are always defined through integration with respect
to a Haar measure. The group $G$ acts on the space $L^2(\mathbb{R}^d)$ through the quasi-regular representation $\pi$ defined by $[\pi(x,h)f](y) := |\det h|^{-1/2} f(h^{-1}(y-x))$ for $f \in L^2(\mathbb{R}^d)$. The generalized continuous wavelet transform (with respect to $\psi \in L^2(\mathbb{R}^d)$) of $f$ is then given as the function $W_\psi f : G \to \mathbb{C} : (x,h) \mapsto \langle f, \pi(x,h)\psi \rangle$.

We will sometimes use the notation $W_\psi^H f$, in places where an explicit reference to the choice of dilation group $H$ is required. Important properties of the map $W_\psi : f \mapsto W_\psi f$ depend on $H$ and the chosen $\psi$. If the quasi-regular representation is square-integrable, which means that there exists $\psi \neq 0$ with $W_\psi \psi \in L^2(G)$, and irreducible, then we call $H$ admissible. In this case the map $W_\psi : L^2(\mathbb{R}^d) \to L^2(G)$ is a multiple of an isometry, which gives rise to the (weak-sense) inversion formula

$$f = \frac{1}{C_\psi} \int_G W_\psi f(x,h) \pi(x,h)\psi d(x,h),$$

i.e., each $f \in L^2(\mathbb{R}^d)$ is a continuous superposition of the wavelet system. According to results in [20], [23], the admissibility of $H$ can be characterized by the dual action defined by $\mathbb{R}^d \times H \to \mathbb{R}^d, (h, \xi) \mapsto h^{-T} \xi$. In fact, $H$ is admissible iff the dual action has a single open orbit $O := H^{-T} \xi_0 \subset \mathbb{R}^d$ of full measure for some $\xi_0 \in \mathbb{R}^d$ and additionally the isotropy group $H_{\xi_0} := \{ h : p_{\xi_0}(h) = \xi_0 \} \subset H$ is compact; see e.g. [23].

Every admissible group gives rise to an associated a covering. This is done using the dual action by picking a well-spread family in $H$, i.e. a family of elements $(h_i)_{i \in I} \subset H$ with the properties

i) there exists a relatively compact neighborhood $U \subset H$ of the identity such that $\bigcup_{i \in I} h_iU = H$ – we say $(h_i)_{i \in I}$ is $U$-dense in this case – and

ii) there exists a neighborhood $V \subset H$ of the identity such that $h_iV \cap h_jV = \emptyset$ for $i \neq j$ – we say $(h_i)_{i \in I}$ is $V$-separated in this case.

The dual covering induced by $H$ is then given by the family $Q = (Q_i)_{i \in I}$, where $Q_i = p_{\xi_0}(h_iU)$ for some $\xi_0$ with $H^{-T} \xi_0 = O$. It can be shown that well-spread families always exist, and that the induced covering is indeed a tight structured admissible covering in the sense of decomposition space theory, for which LP-BAPUs exist, according to Theorems 4.4.6 and 4.4.13 of [36]. Furthermore, there always exist induced coverings consisting of open and connected sets, an additional feature which can facilitate the investigations in some cases. For ease of reference, we state this as a lemma.

**Lemma 2.9 ([30] Corollary 2.5.9).** Let $H$ denote an admissible dilation group, with open dual orbit $O$. Then there always exists an induced covering of $O$ by $H$ that is a tight structured admissible covering consisting of (path-) connected open sets.

We call any induced covering that is a structured admissible covering consisting of open and connected sets an induced connected covering of $O$ by $H$. Furthermore, two different induced coverings of the same group are always weakly equivalent, see [30] Corollary 2.6.5.).
In Subsection 4.2, the next result will be crucial.

**Lemma 2.10 (cf. [27] Lemma 18).** Let \((h_i)_{i \in I}\) be a well-spread family in \(H\) and let \(K_1, K_2 \subset \mathcal{O}\) be compact sets. Given \(h \in H\), let

\[
  I_h(K_1, K_2) := \left\{ i \in I \mid h^{-T}K_1 \cap h_i^{-T}K_2 \neq 0 \right\}.
\]

There are \(C_1 = C_1(K_1, K_2) > 0\) and \(C_2 = C_2(K_1, K_2, (h_i)_{i \in I}) > 0\) with \(|I_h| \leq C_2\) for all \(h \in H\) and with \(\|h^T h_i^{-T}\| \leq C_1\) for all \(i \in I_h\).

### 2.3. Coorbit theory and its connection to decomposition spaces

Coorbit spaces are defined in terms of the decay behavior of the generalized wavelet transform. To give a precise definition, we introduce weighted mixed \(L^p\)-spaces on \(G\), denoted by \(L^{p,q}_v(G)\). By definition, this space is the set of functions

\[
  \left\{ f : G \to \mathbb{C} : \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |f(x,h)|^p v(x,h)^p dx \right)^{q/p} \frac{dh}{|\det(h)|} < \infty \right\},
\]

with natural (quasi-)norm \(\| \cdot \|_{L^{p,q}_v}\). This definition is valid for \(0 < p, q < \infty\), for \(p = \infty\) or \(q = \infty\) the essential supremum has to be taken at the appropriate place instead. The function \(v : G \to \mathbb{R}^{>0}\) is a measurable weight function that fulfills the condition \(v(ghk) \leq v_0(g)v(h)v_0(k)\) for some that is *submultiplicative*, i.e., fulfills \(v_0(gk) \leq v_0(g)v_0(k)\). If this last condition on \(v\) is satisfied, we call \(v\) left- and right moderate with respect to \(v_0\). Thus, the expression \(\|W_\psi f\|_{L^{p,q}_v}\) can be read as a measure of wavelet coefficient decay of \(f\). We consider weights which only depend on \(H\). The coorbit space \(C_0 \left( L^{p,q}_v(\mathbb{R}^d \times H) \right)\) is then defined as the space

\[
  \left\{ f \in (H_w^1)^\sim : W_\psi f \in W \left( L^{p,q}_v(\mathbb{R}^d \times H) \right) \right\}
\]

for a suitable wavelet \(\psi\) fulfilling various technical conditions, and some control weight \(w\) associated to \(v\). The space \((H_w^1)^\sim\) denotes the space of antilinear functionals on \(H_w^1 := \{ f \in L^2(\mathbb{R}^d) : W_\psi f \in L^1_w(G) \}\) and \(W(Y)\) for a function space \(Y\) on \(G\) denotes the Wiener amalgam space defined by \(W_Q(Y) := \{ f \in L^p_{\text{loc}}(G) : M_Q f \in Y \}\) with quasi-norm \(\|f\|_{W_Q(Y)} := \|M_Q f\|_Y\) for \(f \in W_Q(Y)\), where the *maximal function* \(M_Q f\) for some suitable unit neighborhood \(Q \subset G\) is \(M_Q f : G \to [0,\infty], x \mapsto \text{ess sup}_{y \in xQ} |f(y)|\).

The appearance of the Wiener amalgam space in (2.5) is necessary to guarantee consistently defined quasi-Banach spaces in the case \(\{p,q\} \cap (0,1) \neq \emptyset\), see [34] and [36]. In the classical coorbit theory for Banach spaces, which was developed in [14] and [15], the Wiener amalgam space is replaced by \(L^{p,q}_v(G)\) and this change leads to the same space for \(p, q \geq 1\), see [34].

Many useful properties of these spaces are known and hold in the quasi-Banach space case as well as in the Banach space case. The most prominent examples of coorbit spaces associated to generalized wavelet transforms are the
homogeneous Besov spaces and the modulation spaces. However, each shearlet group, a class of groups we introduce in the next subsection, gives rise to its own scale of coorbit spaces, as well; see [31], [9] and [24].

Remark 2.11. The use of the reservoir space $(H^1_w)^\sim$ is a major obstacle for the comparison of coorbit spaces associated to different dilation groups. From a purely set-theoretic point of view, different dilation groups necessarily result in disjoint coorbit spaces. However, this distinction is somewhat beside the point, and it can be remedied by the following observations; see also [14, Corollary 4.4]. By the Fischer-Riesz theorem, the inclusion $H^1_w \subset L^2(\mathbb{R}^d)$ provides a canonical embedding $L^2(\mathbb{R}^d) \subset (H^1_w)^\sim$, which allows to identify $L^2(\mathbb{R}^d)$ with a subspace of $(H^1_w)^\sim$. Now the fact that the wavelet transform is $L^2$-isometric (and the associated inversion formula) implies that the range of this identification map is precisely given by $Co(L^2(\mathbb{R}^d \rtimes \mathcal{H}))$.

We next observe that $Co(L^p(\mathbb{R}^d \rtimes \mathcal{H})) \subset Co(L^2(\mathbb{R}^d \rtimes \mathcal{H}))$ for $1 \leq p \leq 2$. This follows from the fact that the discrete sequence space $Y_d$, associated to $Y = L^p(\mathbb{R}^d \rtimes \mathcal{H})$ according to Definition 3.4 of [14], turns out to be $Y_d = \ell^p \subset \ell^2$ (see [14, Lemma 3.5(e)]). Furthermore, $Co(Y) \subset Co(Z)$ holds iff $Y_d \subset Z_d$, by [15, Theorem 8.4], hence $Co(L^p) \subset Co(L^2)$.

Combining this embedding with the identification of $Co(L^2(\mathbb{R}^d \rtimes \mathcal{H}))$ with $L^2(\mathbb{R}^d)$, we find the following canonical identification for $1 \leq p \leq 2$, valid for a suitably chosen analyzing vector $\psi$:

$$Co(L^p(\mathbb{R}^d \rtimes \mathcal{H})) = \{ f \in L^2(\mathbb{R}^d) : W^H_\psi f \in L^p(\mathbb{R}^d \rtimes \mathcal{H}) \} ,$$

with the obvious norm. But now the new reservoir space $L^2(\mathbb{R}^d)$ is independent of $\mathcal{H}$.

We note that this identification extends to all weighted $L^{p,q}_w(\mathbb{R}^d \rtimes \mathcal{H})$ for which the associated sequence space fulfills $\ell^{p,q}_v \subset \ell^2$, in particular for $1 \leq p, q \leq 2$ and constant weight $v$.

An alternative approach to the just-mentioned embeddings of coorbit spaces, pointed out to us by the referee, is obtained by observing the following well-known inclusion relations between Wiener amalgam spaces, valid for $0 < p < 2$, $W(L^p) \subset W(L^2) \subset L^2$, which via equation (2.5) translates to the inclusions of the associated coorbit spaces, and has the advantage of directly generalizing to $p < 1$.

The next definition will be useful for the transfer of weights from the coorbit to the decomposition space setting.

Definition 2.12 ([36] Definition 4.5.3.). For $q \in (0, \infty]$ and a weight $m : H \rightarrow (0, \infty)$, we define the weight $m^{(q)} : H \rightarrow (0, \infty)$, $h \mapsto |\det(h)|^\frac{q}{2} \cdot m(h)$. Here, we set $\frac{1}{\infty} := 0$.

The connection between coorbit spaces and decomposition spaces is given by the next theorem. For the Banach space case, we also refer to [27]. A recent extension beyond the irreducible setting can be found in [26].
Theorem 2.13 ([36] Theorem 4.6.3). Let \( Q \) be a covering of the dual orbit \( O \) induced by \( H \), \( 0 < p, q \leq \infty \) and \( u = (u_i)_{i \in I} \) a suitable weight, then the Fourier transform \( F : \text{Co} \left( L^p, \mathbb{R}^d \rtimes H \right) \to D(O, L^q) \) is an isomorphism of (quasi-) Banach spaces. The weight \( (u_i)_{i \in I} \) can be chosen as \( u_i := v^{(q)}(h_i) \), where \( (h_i)_{i \in I} \) is the well-spread family used in the construction of \( Q \). We call such a weight a \( Q \)-discretization of \( v \).

Remark 2.14. In the following, we will mostly concentrate on constant weights, i.e. on the study of coorbit spaces of the type \( \text{Co}(L^p, G) \) corresponding to \( v \equiv 1 \). This has the important consequence that the \( Q \)-discretization \( (u_i)_{i \in I} \) obtained from a dual covering \( Q = (Q_i)_{i \in I} = (h_i^T Q)_{i \in I} \) fulfills

\[
u_i = |\det(h_i)|^{\frac{1}{q} - \frac{1}{p}} = |Q|^{\frac{1}{q} - \frac{1}{p}}|Q_i|^{\frac{1}{q} - \frac{1}{2}} = |Q_i|^{\frac{1}{q} - \frac{1}{2}}.
\]

In other words, the induced weight \( (u_i)_{i \in I} \) is intrinsic with exponent \( \alpha = \frac{1}{q} - \frac{1}{2} \).

Remark 2.15. Since the domain of the Fourier transform in the previous theorem consists of spaces that depend on the choice of dilation groups, some clarification is in order, see [27, Corollary 10] for more details: Given \( f \in (H^1_w)^\circ \), we let \( F(f) \in D(O) \) be defined as the map \( g \to \langle f, F^{-1}g \rangle \). Here \( F \) denotes the Fourier transform defined on the Schwartz functions, and the definition makes use of \( D(O) \subset S(O) \), as well as \( F^{-1}(D(O)) \subset H^1_w [24, Theorem 2.1] \).

By the Fischer-Riesz theorem, the inclusion \( H^1_w \subset L^2(\mathbb{R}^d) \) provides a canonical embedding \( L^2(\mathbb{R}^d) \subset (H^1_w)^\circ \). Hence for elements of \( L^2(\mathbb{R}^d) \), one can rewrite the formula for \( F(f) \) as

\[
F(f)(g) = \langle \hat{f}, \mathcal{F}(g) \rangle_{L^2(\mathbb{R}^d)},
\]

where \( \hat{f} \) denotes the Plancherel transform of \( f \). In particular, for all coorbit spaces \( \text{Co}(L^p, \mathbb{R}^d \rtimes H) \) having a canonical embedding into \( L^2(\mathbb{R}^d) \) according to Remark 2.11 the Fourier transform in the theorem coincides with the Plancherel transform, combined with the obvious identification of the \( L^2 \)-function \( \hat{f} \) with the distribution on \( D(O) \) obtained by integration against \( \hat{f} \). In this way, the Fourier transform \( F \) on \( \text{Co}(L^p, \mathbb{R}^d \rtimes H) \), as defined in Theorem 2.13 also becomes independent of the choice of dilation group, whenever we canonically identify the coorbit space with a subspace of \( L^2(\mathbb{R}^d) \).

We next formalize the property that two admissible dilation groups have the same coorbit spaces. We already pointed out that a literal interpretation of this property is not generally available, at least not for all possible choices of coorbit space norms.

Definition 2.16. Let \( H_1, H_2 \leq GL(\mathbb{R}^d) \) denote admissible matrix groups. We call \( H_1, H_2 \) coorbit equivalent if for all \( 0 < p, q \leq \infty \) and for all \( f \in L^2(\mathbb{R}^d) \) we have

\[
\|f\|_{\text{Co}(L^p, \mathbb{R}^d \rtimes H_1)} \asymp \|f\|_{\text{Co}(L^p, \mathbb{R}^d \rtimes H_2)}.
\]

Here the norm equivalence is understood in the generalized sense that one side is infinite iff the other side is.
Remark 2.17. An example of distinct groups that are coorbit equivalent can be found in [27], Section 9: If $H = \mathbb{R}^+ \times SO(d)$, for $d > 1$, and $C \in GL(\mathbb{R}^d)$ is arbitrary, then $H$ and $C^{-1}HC$ are coorbit equivalent, but typically distinct.

Now the decomposition space characterization from Theorem 2.13 together with the results from the previous subsection characterizing when admissible coverings lead to identical scales of decomposition spaces, provides a rather stringent characterization of coorbit equivalence.

Theorem 2.18. Let $H_1, H_2 \leq GL(\mathbb{R}^d)$ denote admissible matrix groups, and let $O_1, O_2$ denote the associated open dual orbits. Then the following are equivalent:

(a) $H_1$ and $H_2$ are coorbit equivalent.

(b) For all $1 \leq p, q \leq 2$: $Co(L^{p,q}(\mathbb{R}^d \rtimes H_1)) = Co(L^{p,q}(\mathbb{R}^d \rtimes H_2))$, as subspaces of $L^2(\mathbb{R}^d)$.

(c) There exists $1 \leq p, q \leq 2$ with $(p, q) \neq (2, 2)$, such that $Co(L^{p,q}(\mathbb{R}^d \rtimes \partial H_1)) = Co(L^{p,q}(\mathbb{R}^d \rtimes \partial H_2))$, as subspaces of $L^2(\mathbb{R}^d)$.

(d) $O_1 = O_2$, and the coverings induced by $H_1$ and $H_2$ on the common open orbit are weakly equivalent.

Proof. The implication $(a) \Rightarrow (b)$ is clear by the nature of the canonical embedding of the coorbit spaces into $L^2(\mathbb{R}^d)$ detailed in Remark 2.11 $(b) \Rightarrow (c)$ is obvious.

For the direction $(c) \Rightarrow (d)$, recall that $O_1, O_2$ are both open and dense. Hence, following the observations after Theorem 2.5 if these sets are distinct, then either $O_1 \cap \partial O_2 \neq \emptyset$ or $O_2 \cap \partial O_1 \neq \emptyset$ follows, and in addition $O_1 \cap O_2$ is unbounded. Moreover, the space $F^{-1}(C_c^\infty(O_1 \cap O_2))$ is a subspace of $L^2(\mathbb{R}^d)$, and the assumption on the coorbit spaces combined with Theorem 2.13 yields that the decomposition space norms $\| \cdot \|_{D(Q, L^p, \ell^q_u)}$ and $\| \cdot \|_{D(P, L^p, \ell^q_v)}$ are equivalent on $C_c^\infty(O_1 \cap O_2)$, whenever $Q$ and $P$ are coverings induced by $H_1, H_2$, respectively. Here $u, v$ are the intrinsic weights of exponent $\alpha = \frac{1}{q} - \frac{1}{2}$ associated to the coverings, see Remark 2.14. Now the fact that $(p, q) \neq (2, 2)$ implies via Theorem 2.5 that $O_1 = O_2$.

We observed in Remark 2.15 that the $L^2$-Fourier transform $F$ induces isomorphisms

$Co(L^{p,q}(\mathbb{R}^d \rtimes H_1)) \to D(P, L^p, \ell^q_u)$, $Co(L^{p,q}(\mathbb{R}^d \rtimes H_2)) \to D(P, L^p, \ell^q_v)$

and by assumption, the coorbit spaces coincide. But then

$D(Q, L^p, \ell^q_u) = D(P, L^p, \ell^q_v)$.

Now Lemma 2.8 implies weak equivalence of the coverings.

Finally, $(d) \Rightarrow (a)$ follows combining Theorem 2.13 and Lemma 2.4. Observe also that the induced weights are equivalent by Remark 2.14 and Lemma 2.8.

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Remark 2.19. The unweighted case, which is in the focus of the theorem, covers all spaces of the type $L^p(\mathbb{R}^d \rtimes H)$, for $1 \leq p < 2$, which are of special importance for nonlinear approximation. Considering general, non-constant weights on the dilation groups requires solving a technical issue, that is currently not well-understood. In this general setting, the question becomes whether any choice of weight $v_1$ given on one group $H_1$ can be matched by an appropriate weight $v_2$ on $H_2$, such that the resulting weights on the induced coverings are equivalent. In principle it seems possible, using cross-sections, to construct $v_2$ from $v_1$. However, it is currently unclear whether, starting from a weight $v_1$ that is moderate with respect to a suitable sub-multiplicative function on $H_1$, the same will automatically hold true for the newly constructed weight $v_2$ on $H_2$.

3. A metric characterization of weak equivalence

This section contains the central observation of this paper, namely that the question of weak equivalence of coverings, and thus of equality of the scales of decomposition spaces associated to the coverings, can be reformulated using the language of coarse geometry.

3.1. Elementary notions and results from coarse geometry

In this section we collect the basic results from coarse geometry that are needed for the following. Since the terminology employed in coarse geometry is not entirely unified, and for reasons of self-containedness, we include proofs of some basic results for which we could not find a handy reference.

Remark 3.1. In the following, we define different metrics on subsets of $\mathbb{R}^d$ and on matrix groups, but we will only be concerned with the coarse properties of these metrics. In particular, the terms open, closed, connected component and the continuity of maps between these spaces have to be understood with respect to the standard topology on the respective sets and not with respect to the topology that may be induced by these different metrics.

The pivotal role continuous maps play in the study of topological spaces is taken over by coarse maps in the study of large scale properties of metric spaces.

Definition 3.2 (cf. [35] Definition 1.8). Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces and let $f : X \to Y$ be a map.

- We call $f$ metrically proper if the inverse image under $f$ of every bounded subset in $Y$ is bounded in $X$, that is

$$\sup \{d_Y(y, y') : y, y' \in A\} < \infty$$

$$\Rightarrow \sup \{d_X(x, x') : x, x' \in f^{-1}(A)\} < \infty$$

for all sets $A \subset Y$. 

We call $f$ uniformly bornologous if for every $R > 0$ there exists $S > 0$ such that

$$d_X(x, x') < R \implies d_Y(f(x), f(x')) < S$$

for all $x, x'$ in $X$.

A coarse map from $(X, d_X)$ to $(Y, d_Y)$ is a metrically proper and uniformly bornologous map.

A simple argument shows that the composition of coarse maps is coarse as well. In order to define when we want to consider coarse structures on different spaces as essentially the same, we introduce the notion of closeness of maps.

**Definition 3.3 (cf. [33] Definition 1.3.2).** Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces and $f : X \to Y$ and $g : X \to Y$ be maps. We say $f$ and $g$ are close if

$$\sup \{d_Y(f(x), g(x)) : x \in X\} < \infty.$$

**Definition 3.4 (cf. [35] Page 6).** Two metric spaces $(X, d_X)$ and $(Y, d_Y)$ are coarsely equivalent if there exist two coarse maps

$$f : X \to Y \quad \text{and} \quad g : Y \to X$$

such that $f \circ g$ is close to $\text{id}_Y$ and $g \circ f$ is close to $\text{id}_X$. $f$ is then called a metric coarse equivalence.

As the name suggests, this relation is in fact an equivalence relation. An important class of coarse maps are quasi-isometries.

**Definition 3.5 (cf. [35] Remark 1.9).** Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces and $f : X \to Y$ be a map. We call $f$ large-scale Lipschitz if there are $L, C > 0$ such that

$$d_Y(f(x), f(x')) \leq Ld_X(x, x') + C$$

for all $x, x' \in X$.

**Definition 3.6 (cf. [33] Definition 1.3.4).** Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. A map $f : X \to Y$ is a quasi-isometry if the following conditions are satisfied:

i) The map $f$ is a quasi-isometric embedding. This means there exist constants $L, C > 0$ such that

$$L^{-1}d_X(x, x') - C \leq d_Y(f(x), f(x')) \leq Ld_X(x, x') + C$$

for all $x, x' \in X$.

ii) The map $f$ is coarsely surjective. This means that there exists $K > 0$ such that for every $y \in Y$ exists an $x \in X$ with $d_Y(f(x), y) \leq K$. 

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Remark 3.7. It is easy to see that a surjective map is coarsely surjective. In the concrete cases we consider, this will be the standard justification for this property.

As for coarse maps, it is easy to see that the composition of quasi-isometric maps is again quasi-isometric. The following lemma has a straightforward proof.

Lemma 3.8. Every quasi-isometric embedding is coarse.

Definition 3.9. Let \((X,d_X)\) and \((Y,d_Y)\) be metric spaces and \(f : X \to Y\). We call a map \(h : Y \to X\) such that \(f \circ h\) is close to \(\text{id}_Y\) a coarse right inverse of \(f\).

Example 3.10. The map \(f : (\mathbb{R}, \| \cdot \|) \to (\mathbb{R}^2, \| \cdot \|_1), x \mapsto (x, 0)^T\) illustrates that not every coarse map has a coarse right inverse.

Quasi-isometries always have coarse right inverses. We include a proof, for the reasons mentioned above.

Lemma 3.11. Every quasi-isometry has a coarse right inverse. Furthermore, any coarse right inverse of a quasi-isometry is a quasi-isometry.

Proof. Let \((X,d_X)\) and \((Y,d_Y)\) be metric spaces and \(f : X \to Y\) be a quasi-isometry. This entails the existence of some \(K > 0\) with the property that for every \(y \in Y\) exists an \(x \in X\) with \(d_Y(f(x), y) \leq K\). Consequently, for every \(y \in Y\), the set

\[ A_y := \{ x \in X : d_Y(f(x), y) \leq K \} \]

is nonempty. The axiom of choice implies the existence of at least one coarse right inverse of \(f\) because \(\prod_{y \in Y} A_y\) is nonempty.

Since \(f\) is a quasi-isometry, there exist \(L,C > 0\) with

\[ L^{-1}d_X(x, x') - C \leq d_Y(f(x), f(x')) \leq Ld_X(x, x') + C \]

for all \(x, x' \in X\). Moreover, let \(h : Y \to X\) be any coarse right inverse of \(f\) such that \(\sup_{y \in Y} d_Y((f \circ h)(y), y) \leq M\) for some \(M > 0\). By application of the triangle inequality twice, it follows that

\[ L^{-1}d_X(h(y), h(y')) - C \leq d_Y((f \circ h)(y), (f \circ h)(y')) \]

\[ \leq d_Y((f \circ h)(y), y) + d_Y(y', y) \]

\[ + d_Y(y', (f \circ h)(y')) \]

\[ \leq 2M + d_Y(y', y) \]

for all \(y, y' \in Y\). This inequality implies

\[ d_X(h(y), h(y')) \leq Ld_Y(y', y) + 2LM + CL. \]

We continue by using the reverse triangle inequality twice to get

\[ Ld_X(h(y), h(y')) + C \geq d_Y((f \circ h)(y), (f \circ h)(y')) \]

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\[
\begin{align*}
\geq d_Y \left( y, (f \circ h)(y') \right) - d_Y \left( (f \circ h)(y), y \right) \\
\geq d_Y \left( y, y' \right) - d_Y \left( (f \circ h)(y'), y' \right) \\
- d_Y \left( (f \circ h)(y), y \right) \\
\geq d_Y \left( y, y' \right) - 2M.
\end{align*}
\]

This inequality implies
\[
d_X \left( h(y), h(y') \right) \geq L^{-1}d_Y \left( y', y \right) - 2L^{-1}M - CL^{-1}
\]
for all \( y, y' \in Y \). It follows that \( h \) is a quasi-isometric embedding.

In order to show that \( h \) is a quasi-isometry, let \( x \in X \) be arbitrary and choose \( y := f(x) \). We conclude
\[
d_X \left( h(f(x)), x \right) \leq Ld_Y \left( (f \circ h)(f(x)), f(x) \right) + CL \\
= Ld_Y \left( (f \circ h)(y), y \right) + CL \\
\leq LM + CL
\]
and the inequality
\[
d_X \left( h(f(x)), x \right) \leq LM + CL \tag{3.1}
\]
shows that \( h \) is coarsely surjective. In summary, this shows that \( h \) is a quasi-isometry.

**Remark 3.12.** The inequality (3.1) also shows that if \( f : X \to Y \) is a quasi-isometry and \( h : Y \to X \) is any coarse right inverse of \( f \), then \( f \) is a coarse right inverse of \( h \). If we define a coarse left inverse analogously, then this shows that every coarse right inverse of a quasi-isometry is also a coarse left inverse of this quasi-isometry.

**Theorem 3.13.** Let \((X,d_X)\) and \((Y,d_Y)\) be metric spaces. If there exists a quasi-isometry \( f : X \to Y \), then \( X \) and \( Y \) are coarsely equivalent.

**Proof.** According to Lemma 3.11, \( f \) has a coarse right inverse \( h \), which is also a quasi-isometry. By Lemma 3.3, \( f \) and \( h \) are coarse maps. Since \( h \) is a coarse right inverse of \( f \), the map \( f \circ h \) is close to \( id_Y \). From Remark 3.12, it follows that \( h \circ f \) is close to \( id_X \). This shows that \( f \) and \( h \) induce a coarse equivalence of \( X \) and \( Y \).

The following definition allows to identify a general setting in which the terms “coarse equivalence” and “quasi-isometry” are synonymous. See [1, Definition 3.B.1].

**Definition 3.14.** Let \((X,d)\) denote a metric space. For \( c > 0 \), \( X \) is called \( c \)-large-scale geodesic if there exist constants \( a > 0, b \geq 0 \) such that for every pair of points \( x, x' \) there exists a finite sequence \( x_0 = x, x_1, \ldots, x_n = x' \) satisfying
\[
d(x_{i-1}, x_i) \leq c \quad \text{and} \quad n \leq ad(x,x') + b.
\]
\((X,d)\) is called large-scale geodesic if it is \( c \)-large-scale geodesic for some \( c > 0 \).
The following result is Proposition 3.B.9.

**Proposition 3.15.** Let \((X,d)\) and \((Y,d')\) denote large-scale geodesic spaces, and \(f : X \to Y\). Then \(f\) is a coarse equivalence if and only if it is a quasi-isometry.

### 3.2. Metrics induced by coverings

We start with a preparation for the definition of the metric associated to a covering.

**Definition 3.16.** Let \(O \subset \mathbb{R}^d\) be open and let \(Q = (Q_i)_{i \in I}\) be a covering of \(O\). For \(x,y \in O\), we say \(x\) and \(y\) are connected by a \(Q\)-chain (of length \(m\)) if there exist \(Q_1, \ldots, Q_m \in Q\) such that \(x \in Q_1, y \in Q_m\) and \(Q_k \cap Q_{k+1} \neq \emptyset\) for all \(k \in \{1, \ldots, m-1\}\).

We next define the metric, which is called \(Q\)-chain distance in Definition 3.4.

**Definition 3.17.** Let \(O \subset \mathbb{R}^d\) be open and let \(Q = (Q_i)_{i \in I}\) be a covering of \(O\). Define the map \(d_Q : O \times O \to \mathbb{N}_0 \cup \{\infty\}\) by

\[
d_Q(x,y) = \begin{cases} 
\inf \left\{ m \in \mathbb{N} \mid x, y \text{ are connected by a } Q\text{-chain of length } m \right\}, & x \neq y \\
0, & x = y,
\end{cases}
\]

where we set \(\inf \emptyset = \infty\).

The above map in fact defines a metric (we allow the value \(\infty\) for a metric) on \(O\), without any further restrictions on the covering.

**Remark 3.18.** Let \(O \subset \mathbb{R}^d\) be open and let \(Q = (Q_i)_{i \in I}\) be a covering of \(O\). One could also be tempted to define a function by considering the neighbors of a set (recall Definition 2.1)

\[
\tilde{d} : O \times O \to \mathbb{N}_0 \cup \{\infty\},
\]

\[
(x,y) \mapsto \begin{cases} 
\inf \left\{ n \in \mathbb{N} \mid \exists Q \in Q \text{ with } x, y \in Q^n \right\}, & x \neq y \\
0, & x = y.
\end{cases}
\]

In general, this map does not satisfy the triangle inequality and is therefore not a metric on \(O\). It suffices to consider a uniform covering of \(O = \mathbb{R}\) by intervals of length 2 as depicted in the diagram below. Here, the intervals are represented by rectangles. For the points \(x, y, z \in \mathbb{R}\), we have \(\tilde{d}(x,y) = 1, \tilde{d}(y,z) = 1\) but \(\tilde{d}(x,z) = 3\).
3.3. Characterizing weak equivalence

We will only consider coverings consisting of open connected sets. Here the property \( d(x, y) = \infty \) can be related to the partition of \( \mathcal{O} \) into connected components.

**Lemma 3.19.** Let \( \mathcal{O} \subset \mathbb{R}^d \) be open and let \( \mathcal{Q} = (Q_i)_{i \in I} \) be a covering of \( \mathcal{O} \) comprised of open connected subsets of \( \mathcal{O} \). Then \( d_{\mathcal{Q}}(x, y) < \infty \) if and only if \( x, y \) lie in the same connected component of \( \mathcal{O} \).

**Proof.** We consider the equivalence relation \( x \sim y : \iff d(x, y) < \infty \). Fix \( x \), and let \( y \in \mathcal{O} \) with \( x \sim y \). By definition there exists a \( \mathcal{Q} \)-chain connecting \( x \) and \( y \). Since the union of two connected sets with nontrivial intersection is connected again, we may thus conclude that \( x \) and \( y \) are contained in a common connected subset of \( \mathcal{O} \). Hence the equivalence class of \( x \) modulo \( \sim \) is contained in the connected component of \( x \).

To see the converse direction, observe that for each \( x \in \mathcal{O} \), there exists \( i \in I \) with \( x \in Q_i \), and thus \( d(x, y) \leq 1 < \infty \) for all \( y \in Q_i \), which is an open set. This shows that all \( \sim \)-equivalence classes are open, and consequently closed, since they partition \( \mathcal{O} \). Hence the connected component of \( x \) is contained in the \( \sim \)-equivalence class.

**Remark 3.20.** As we remarked above, we allow our metrics \( d_{\mathcal{Q}} \) to assume the value \( \infty \), which constitutes a mild abuse of notation. For the purpose of the subsequent arguments, this abuse can be justified as follows: We will only be interested in the comparison of different coverings \( \mathcal{Q}, \mathcal{P} \) of the same set \( \mathcal{O} \), which amounts to studying the coarse properties of the identity map \( id_{\mathcal{O}} : (\mathcal{O}, d_{\mathcal{P}}) \to (\mathcal{O}, d_{\mathcal{Q}}) \). Since every connected component of \( \mathcal{O} \) is mapped onto itself, one has \( d_{\mathcal{P}}(x, y) < \infty \) iff \( d_{\mathcal{Q}}(x, y) < \infty \). Furthermore, we will often restrict attention to the case that \( \mathcal{O} \) has finitely many connected components.

Under these restrictions, one can extend the various properties of maps between metric spaces (understood in the strict sense) to the setting of generalized metrics (allowed to take the value \( \infty \)), by requiring that for each connected component \( \mathcal{O}_0 \subset \mathcal{O} \), the restriction \( id_{\mathcal{O}_0} : (\mathcal{O}_0, d_{\mathcal{P}}) \to (\mathcal{O}_0, d_{\mathcal{Q}}) \) has the required properties. Then all results from the previous section become available for the slightly more general setting. Note in particular that \( id_{\mathcal{O}} : (\mathcal{O}, d_{\mathcal{P}}) \to (\mathcal{O}, d_{\mathcal{Q}}) \) is a quasi-isometry if and only if all restrictions to connected components are, provided the number of connected components is finite. Here we simply take
minima resp. maxima of the finitely many constants involved in the formulation of quasi-isometry statements for each component, to obtain globally valid constants.

**Remark 3.21.** We note that each covering-induced metric $d_P$ is $c$-large scale geodesic, for $c = 1$.

The property of weak equivalence of two admissible coverings has a characterization in terms of the coarse structure of their associated metrics. A result similar to i) $\Rightarrow$ ii) in the following Theorem can be found in [12, Proposition 3.8 C)].

**Theorem 3.22.** Let $\mathcal{O} \subset \mathbb{R}^d$ be open and let $Q = (Q_i)_{i \in I}$ and $P = (P_j)_{j \in J}$ be admissible coverings of $\mathcal{O}$ comprised of open connected subsets of $\mathcal{O}$. Then the following statements are equivalent:

i) The coverings $Q$ and $P$ are weakly equivalent.

ii) The map $\text{id} : (\mathcal{O}, d_Q) \to (\mathcal{O}, d_P)$, $x \mapsto x$ is a quasi-isometry.

If $\mathcal{O}$ has only finitely many connected components, (i) and (ii) are equivalent to

(iii) The map $\text{id} : (\mathcal{O}, d_Q) \to (\mathcal{O}, d_P)$, $x \mapsto x$ is a coarse equivalence.

**Proof.** i) $\Rightarrow$ ii): Assume that the coverings are weakly equivalent. Our goal is to show the existence of $L > 0$ such that

$$L^{-1}d_Q(x,y) \leq d_P(x,y) \leq Ld_Q(x,y)$$

for all $x, y \in \mathcal{O}$. It suffices to show the right inequality, the left inequality follows by interchanging the roles of $P$ and $Q$. Let $x, y \in \mathcal{O}$. We first consider the case that $d_Q(x,y) = 1$, this implies the existence of a set $Q \in Q$ with $x, y \in Q$. The weak subordinateness of $Q$ to $P$ implies that there are $P_1, \ldots, P_K \in P$ for some $K \leq N(Q, P)$ (see Definition 2.2) with the property $Q \cap P_j \neq \emptyset$ for $j \in \{1, \ldots, K\}$, $Q \subset \bigcup_{j=1}^K P_j$ and $Q \cap P = \emptyset$ for all $P \in P \setminus \{P_1, \ldots, P_K\}$.

Define the set

$$C_x := \left\{ z \in Q \bigg| \begin{array}{l} x, z \text{ are connected by a } P\text{-chain} \\ (P^*_\ell)_{\ell=1}^m \text{ of length } m \leq K \text{ with } P^*_\ell \cap Q \neq \emptyset \\ \text{and } P^*_\ell \neq P^*_{\ell'} \text{ for } \ell \neq \ell' \end{array} \right\}.$$  

As in the proof of the previous lemma, the set $C_x$ is relatively open in $Q$. Next, we show that it is also relatively closed in $Q$. Let $z_1$ be an element of $\overline{C_x}$. Because $P$ is a covering of $\mathcal{O}$, there is $P \in P$ with $z_1 \in P$. The set $Q \cap P$ is a relatively open neighborhood of $z_1$ in $Q$, this implies the existence of some $z \in (Q \cap P) \cap C_x$. Hence, there is a $P$-chain $P'_1, \ldots, P'_m$ of length $m \leq K$ that connects $x$ and $z$.

If $P = P'_j$ for some $j \in \{1, \ldots, m\}$, then $P'_1, \ldots, P'_j$ is a $P$-chain of length $j \leq K$, with $P'_j \cap Q \neq \emptyset$, that connects $x$, and $z_1 \in C_x$. 

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If \( P \neq P'_j \) for all \( j \in \{1, \ldots, m\} \), the inclusion \( \{P'_1, \ldots, P'_m\} \subset \{P_1, \ldots, P_K\} \) together with \( P'_j \neq P'_j' \) for \( \ell \neq \ell' \) implies via the pigeonhole principle that \( m < K \). But then \( P'_1, \ldots, P'_m, P \) is a \( \mathcal{P} \)-chain of length \( m + 1 \leq K \) that connects \( x \) and \( z \). Here, \( z \in P'_n \cap P \neq \emptyset \). In every case, \( z_1 \in C_x \) and \( c_y \) is relatively closed in \( Q \). Since \( Q \) is connected, we conclude \( C_y = Q \) and \( d_\mathcal{P}(x, y) \leq K \leq N(Q, \mathcal{P}) \).

Next, we consider the general case \( d_\mathcal{Q}(x, y) = m \in \mathbb{N} \). In this case, there exists a \( \mathcal{Q} \)-chain \( Q_1, \ldots, Q_m \) of length \( m \) that connects \( x \) and \( y \). The set \( Q := \bigcup_{i=1}^m Q_i \) is connected, open and has nonempty intersection with at most \( mN(Q, \mathcal{P}) \) elements of the covering \( \mathcal{P} \) (every \( Q_i \) has nonempty intersection with at most \( N(Q, \mathcal{P}) \) elements of the covering \( \mathcal{P} \) for \( i \in \{1, \ldots, m\} \)). Hence, an adaptation of the argument for the case \( m = 1 \) leads to the inequality

\[
d_\mathcal{P}(x, y) \leq mN(Q, \mathcal{P}) \leq N(Q, \mathcal{P})d_\mathcal{Q}(x, y).
\]

Interchanging of the roles of the coverings leads to

\[
N(Q, \mathcal{P})^{-1}d_\mathcal{Q}(x, y) \leq d_\mathcal{P}(x, y) \leq N(Q, \mathcal{P})d_\mathcal{Q}(x, y).
\]

This shows that \( id \) is a quasi-isometric embedding. Since it is also bijective, it is therefore a quasi-isometry.

\( ii) \Rightarrow i) \): Assume that \( \mathcal{Q} \) and \( \mathcal{P} \) are not weakly equivalent. Without loss of generality, assume that \( \mathcal{Q} \) is not weakly subordinate to \( \mathcal{P} \). This implies the existence of a sequence \( \{Q_{i_n}\}_{n \in \mathbb{N}} \) in \( \mathcal{Q} \) such that

\[
|A_n| > N_{\mathcal{P}}^{n+1}
\]

and \( |A_n| < |A_{n+1}| \) for all \( n \in \mathbb{N} \), where

\[
A_n := \{ j \in J \mid Q_{i_n} \cap P_j \neq \emptyset \}
\]

and \( N_{\mathcal{P}} \) is the admissibility constant for the covering \( \mathcal{P} \).

The next step is to show that for every \( x \in Q_{i_n} \) and all \( j_x \in J \) with \( x \in P_{j_x} \), the set \( Q_{i_n} \setminus P_{j_x}^{n*} \) is nonempty and that \( d_\mathcal{P}(x, y) \geq n \) for \( x \in Q_{i_n} \) and \( y \in Q_{i_n} \setminus P_{j_x}^{n*} \) (cf. Definition 2.1).

Let \( x \in Q_{i_n} \) be arbitrary for some \( n \in \mathbb{N} \). Since \( \mathcal{P} \) is a covering of \( \mathcal{Q} \), there is at least one \( j_x \in J \) such that \( x \in P_{j_x} \). If \( Q_{i_n} \setminus P_{j_x}^{n*} = \emptyset \), then \( Q_{i_n} \subset P_{j_x}^{n*} \) and for all \( j \in J \) with \( P_j \cap Q_{i_n} \neq \emptyset \), we have \( P_j \cap P_{j_x}^{n*} \neq \emptyset \), which implies \( j \in j_x^{(n+1)*} \). Hence,

\[
|A_n| \leq |j_x^{(n+1)*}| \leq N_{\mathcal{P}}^{n+1},
\]

which is a contradiction to the properties of the set \( A_n \). It follows \( Q_{i_n} \setminus P_{j_x}^{n*} \neq \emptyset \).

Let \( x \in Q_{i_n} \) and \( y \in Q_{i_n} \setminus P_{j_x}^{n*} \). If \( d_\mathcal{P}(x, y) < n \), then there exists a \( \mathcal{P} \)-chain \( P_1, \ldots, P_{m} \) that connects \( x \) and \( y \) with \( m \leq n - 1 \). The definition of a \( \mathcal{P} \)-chain implies \( y \in P_m \subset P_1^{n*} \) and \( x \in P_{j_x} \cap P_1 \) implies \( P_1 \subset P_{j_x}^{n*} \). In conclusion, we have \( y \in P_{j_x}^{(n+1)*} \subset P_{j_x}^{n*} \), which is a contradiction to the choice of \( y \). Hence, \( d_\mathcal{P}(x, y) \geq n \) for all \( x \in Q_{i_n} \) and all \( y \in Q_{i_n} \setminus P_{j_x}^{n*} \) if \( x \in P_{j_x} \).
Now, choose \( x_n \in Q_i \) and \( y_n \in Q_i \setminus P_{j,n}^n \neq \emptyset \), where \( x_n \in P_{j,n} \). The prior part shows \( d_P(x_n, y_n) \geq n \xrightarrow{n \to \infty} \infty \), but \( d_Q(x_n, y_n) \leq 1 \) since \( x_n, y_n \in Q_i \). All in all, there cannot exist constants \( L, C > 0 \) such that
\[
d_P(x, y) \leq Ld_Q(x, y) + C
\]
for all \( x, y \in O \). It follows that \( \text{id} : (O, d_Q) \to (O, d_P) \) is not a quasi-isometry.

The equivalence \( ii) \Leftrightarrow iii) \) follows from Proposition 3.15 and Remarks 3.20.

**Remark 3.23.** Given the role of the connectedness assumption on the elements of the coverings in the various results leading up to the theorem, it seems reasonable to expect that they cannot be dropped from the theorem. In fact, it is not hard to observe that, without some structural assumptions on the coverings, Theorem 3.22 cannot hold. To see this, one can pick an admissible covering \( P \), and construct a second admissible covering \( Q \) from \( P \) by replacing suitable pairs of sets from \( P \) by their unions. If the distances of the pairs of sets making up \( Q \), when measured in the \( P \)-chain distance \( d_P \), are unbounded, \( d_Q \) will not be quasi-equivalent to \( d_P \). On the other hand, one easily sees that the coverings must be weakly equivalent.

However, it is currently open whether such a counterexample can be constructed with both \( P \) and \( Q \) fulfilling the other standing assumptions on decomposition coverings, namely tightness and structuredness, stipulated at the beginning of Subsection 2.1. Hence it is not clear whether the connectedness assumption is the main ingredient making the proof of Theorem 3.22 work.

4. A metric characterization of coorbit equivalence

4.1. Word metrics on locally compact groups

Throughout this section, \( H \) denotes an admissible dilation group, and \( H_0 < H \) denotes the connected component of the identity element in \( H \). Since \( H \) is a Lie group, \( H_0 \) is an open subgroup. We define a metric (which is allowed to take the value \( \infty \)) by picking a word metric on \( H_0 \), and suitably extending it to \( H \).

**Definition 4.1.** Let \( H \) be a locally compact group and let \( W \subset H \) be a unit neighborhood. Define the map \( d_W : H \times H \to \mathbb{N}_0 \cup \{ \infty \} \) in the following way
\[
d_W(x, y) = \begin{cases} 
\inf \left\{ m \in \mathbb{N} \mid x^{-1}y \in W^m \right\} & x \neq y \\
0 & x = y,
\end{cases}
\]
where we again set \( \inf \emptyset = \infty \).

The expression word metric is motivated by the fact that \( d_W(x, y) = m \) entails that the minimal number of elements (letters) \( w_1, \ldots, w_m \in W \) that make the equation \( y = xw_1 \cdots w_m \) true is precisely \( m \). The following fact is well-known, and we omit the proof.
Lemma 4.2. Let $H$ be a locally compact group and let $W \subset H$ be a symmetric ($W = W^{-1}$) unit neighborhood. Then $d_W$ is a metric on $H$, that is left invariant, i.e., $d_W(x, y) = d_W(zx, zy)$ for all $x, y, z \in H$.

As for the metric in the last section, we can impose a connectedness restriction on $W$ in such a way that precisely the elements in the same connected component have finite distance with respect to $d_W$. The proof is essentially identical to the previous one, and therefore omitted.

Lemma 4.3. Let $H$ be a locally compact group and let $W \subset H$ be a symmetric unit neighborhood with $W \subset H_0$, where $H_0$ denotes the connected component of the neutral element in $H$. Then $d_W(x, y) < \infty$ if and only if $x, y$ lie in the same connected component of $H$.

We next recall that the metric spaces $(H, d_W)$ and $(H, d_V)$ are coarsely equivalent for relatively compact $W, V \subset H_0$. Here we refer to [7, Corollary 4.A.6]

Lemma 4.4. Let $H$ be a locally compact group and let $V, W \subset H$ be relatively compact, symmetric unit neighborhoods with $V, W \subset H_0$. Then

$id^V : (H, d_W) \to (H, d_V), h \mapsto h$

is a quasi-isometry and the spaces $(H, d_W)$ and $(H, d_V)$ are coarsely equivalent.

In the terminology of [7], $U$-well-spread families are metric lattices with respect to a properly chosen word metric. The following lemma is a further example of a folklore result for which we could not locate a handy reference.

Lemma 4.5. Let $H$ be a locally compact group, let $W \subset H$ be a relatively compact, symmetric unit neighborhood with $W \subset H_0$. Furthermore, let $X \subset H$ be a $U$-well-spread family for some relatively compact unit neighborhood $U \subset H_0$. Then $(H, d_W)$ and $(X, d_W|_{X \times X})$ are coarsely equivalent.

Proof. The identity

$d_W|_{X \times X}(x, y) = d_W(x, y)$

for all $x, y \in X$ shows that $g : X \to H, x \mapsto x$ is a quasi-isometric embedding. It is coarsely surjective because for every $y \in H$ exists an $x_y \in X$ such that $y \in x_y U$, which implies $x_y^{-1} y \in U$.

Since $W$ is symmetric, $\bigcup_{n \in \mathbb{N}} W^n$ is easily seen to be an open subgroup of $H_0$, and connectedness of $H_0$ then entails $H_0 = \bigcup_{n \in \mathbb{N}} W^n$. In fact, the right hand side is an open covering by increasing sets. Since $\overline{U} \subset H_0$ is compact, this entails the existence of $n \in \mathbb{N}$ with $U \subset W^n$. Hence,

$d_W(x_y, y) \leq n$.

It follows that $g$ is a quasi-isometry and that

$(H, d_W)$ and $(X, d_W|_{X \times X})$

are coarsely equivalent, according to Theorem 3.13.
The last result of this section will be useful, but has a straightforward proof, which is therefore omitted.

**Lemma 4.6.** Let $H, H'$ be locally compact groups, let $W \subset H$ be a relatively compact, symmetric unit neighborhood with $W \subset H_0$ and let $V \subset H'$ be a relatively compact, symmetric unit neighborhood with $V \subset H'_0$. Assume that $\phi : H \to H'$ is a topological group isomorphism, i.e. it is a continuous isomorphism with continuous inverse. Then $\phi$ is a quasi-isometry from $(H, d_W)$ to $(H', d_V)$. In particular, $(H, d_W)$ and $(H', d_V)$ are coarsely equivalent.

### 4.2. Coarse equivalence of the dual orbit and the dilation group

In this section, we investigate the relation between the coarse structure induced on the dual orbit $O \subset \mathbb{R}^d$ of an admissible group $H \subset \text{GL}(\mathbb{R}^d)$ by means of an induced covering on the one hand, and the coarse structure induced by a relatively compact unit neighborhood on the group $H$ on the other. It is the main purpose of this subsection to prove that the orbit map

$$p_\xi : H \to O, h \mapsto h^{-T} \xi$$

is a quasi-isometry, when $H$ is endowed with a suitable metric, and $O$ is endowed with the covering-induced metric.

The next result collects various topological properties of the orbit map.

**Lemma 4.7.** For each $\xi \in O$, the orbit map is continuous, surjective, closed, open, and proper, i.e. $p_\xi^{-1}(K) \subset H$ is compact for compact $K \subset O$.

Furthermore, the orbit $O$ has finitely many components.

**Proof.** We refer to [21] and [36, Corollary 4.1.3] for all properties except closedness of $p_\xi$. But this property follows from properness: Let $A \subset H$ be closed, and let $(h_n)_{n \in \mathbb{N}} \subset A$ with $p_\xi(h_n) \to \eta \in O$. Then $K = \{p_\xi(h_n) : n \in \mathbb{N}\} \cup \{\eta\} \subset O$ is compact, and then the same follows (by properness) for $p_\xi^{-1}(K) \subset H$. This set contains the sequence $(h_n)_{n \in \mathbb{N}}$. Hence, possibly after passing to a subsequence, $h_n \to h \in H$, and then $h \in A$ by closedness of $A$. But this finally entails $\eta = \lim_{n \to \infty} p_\xi(h_n) = p_\xi(h) \in p_\xi(A)$, as desired.

Recall from Section 3 that we are primarily interested in induced coverings consisting of connected sets. In order to properly relate the metric arising from the induced covering with a word metric on $H$, the condition $H_\xi \subset H_0$ will be important, where $H_0 \subset H$ denotes the connected component of the identity element in $H$. The following lemma makes some basic observations following from this assumption.

**Lemma 4.8.** Assume that the isotropy group $H_\xi = \{ h \in H \mid p_\xi(h) = \xi \}$ is a subset of the identity component $H_0 \subset H$.

(i) The orbit map $p_\xi$ induces a bijection between the sets of connected components of $H$ and those of $O$. 

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(ii) For every connected set $B \subset \mathcal{O}$, there is a connected set $A \subset H$ such that

$$p^{-1}_\xi(B) \subset A.$$ 

**Proof.** We start the proof of (i) by noting that $H_0$ is a normal subgroup of $H$. Then the factor group $H/H_0$ acts on the set of connected components of $\mathcal{O}$ in a canonical way. To see this, note that if $C \subset \mathcal{O}$ is a connected component, then $H^T_0 C$ is a connected open set with open complement (the union of the remaining $H^T_0$-orbits), which intersects $C$ nontrivially. This implies $H^T_0 C \subset C$ by connectedness of the left-hand side, and $H^T_0 C = C$ by the remaining properties of $H^T_0$. As a consequence, if $\mathcal{C}$ denotes the set of connected components of $\mathcal{O}$, the map $(H/H_0) \times \mathcal{C} \ni (hH_0, C) \mapsto h^{-1}C \in \mathcal{C}$ is a well-defined action of the quotient group on $\mathcal{C}$. Furthermore, it is clearly transitive, since the action of $H$ on $\mathcal{O}$ is transitive.

Finally, the condition $H_\xi \subset H_0$ implies that the action of $H/H_0$ is free. To see this, it suffices to show that the stabilizer of $C = H^T_\xi$ is trivial, since the action is transitive. For any $h \in H$ satisfying $h^{-1}H^T_0 C = C$ we get $\xi \in hH^T_0 \xi$, and thus $h^{-1}h_0^{-1}\xi = \xi$, with suitably chosen $h_0 \in H_0$. Hence $hh_0 \in H_\xi \subset H_0$, which implies $h \in H_0$. This finishes the proof of (i).

For the proof of (ii), we first note that by assumption on $B$, $B$ is contained in a connected component of $\mathcal{O}$. By part (i), the inverse image of this connected component is a connected component $A \subset H$.

Without the condition $H_\xi \subset H_0$, the orbit map is not a quasi-isometry or even a coarse map between the group $H$ and the associated dual orbit.

**Corollary 4.9.** Let $W \subset H$ be a relatively compact, symmetric unit neighborhood with $W \subset H_0$. Let $\mathcal{O}$ be an induced connected covering of $\mathcal{O}$ by $H$. If $H_\xi \not\subset H_0$, then $p_\xi$ is not a metrically proper map from $(H,d_W)$ to $(\mathcal{O},d_\mathcal{Q})$.

**Proof.** If $H_\xi \not\subset H_0$, there exist $h_1, h_2 \in H_\xi$ that are members of different connected components in $H$. This implies

$$d_\mathcal{Q}(p_\xi(h_1), p_\xi(h_2)) = 0 \text{ and } d_W(h_1, h_2) = \infty$$

because of $p_\xi(h_1) = p_\xi(h_2)$ and Lemma 4.3. This means the inverse image of the bounded set $A := \{p_\xi(h_1)\}$ in $(\mathcal{O},d_\mathcal{Q})$ under $p_\xi$ is not bounded in $(H,d_W)$ since $p_\xi^{-1}(A) \supset \{h_1, h_2\}$.

In the remaining part of this section, we show that the condition $H_\xi \subset H_0$ ensures that the orbit map $p_\xi$ is indeed a quasi-isometry for suitable $\xi \in \mathcal{O}$. As preparation for this, we cite the next result.

**Lemma 4.10 (36) Lemma 4.14.** Let $K_1, K_2 \subset \mathcal{O}$ be compact sets and define

$$L := L(K_1, K_2) := p^{-1}_\xi(K_1) \cdot H_\xi \cdot \left(p^{-1}_\xi(K_2)\right)^{-1}.$$
Then $L \subset H$ is compact and for arbitrary $g, h \in H$ satisfying
\[ h^{-T}K_1 \cap g^{-T}K_2 \neq \emptyset, \]
we have $g \in hL$.

We will need the following property of the set $L$ from the above lemma.

**Lemma 4.11.** If $H_\xi \subset H_0$ and $K_1, K_2 \subset O$ are compact sets contained in the same connected component of $O$, then $L(K_1, K_2) \subset H_0$ with the set $L(K_1, K_2)$ from Lemma 4.10.

**Proof.** If one of the sets $K_1, K_2$ is empty, the claim is obvious. Consider the case of nonempty $K_1, K_2$ in the following. There is a connected component $B \subset O$ with $K_1, K_2 \subset B$. By Lemma 4.8, there is a connected component $A \subset H$ such that
\[ p^{-1}_\xi(K_1), p^{-1}_\xi(K_2) \subset p^{-1}_\xi(B) \subset A. \]

This implies
\[ L(K_1, K_2) \subset A \cdot H_\xi \cdot A^{-1} \subset A \cdot H_0 \cdot A^{-1} =: M. \]

Since $K_1, K_2$ are nonempty, the set $A$ is nonempty as well. Hence, there is $a \in A$, which further implies $e = a \cdot e \cdot a^{-1} \in M$. As $M$ is a connected set as product of connected sets, and contains $e$, we infer $A \subset H_0$.

In the following statements and arguments, we will occasionally make the assumption that $\xi \in Q$, where the set $Q$ is used in the construction of an induced connected covering $(h_i^{-T}Q)_{i \in I}$ of $O$ by $H$. Observe that this can be achieved by relabeling the covering, e.g. by picking $i_0 \in I$ with $\xi \in h_{i_0}^{-T}Q =: Q'$, and considering the system $(g_i^{-T}Q')_{i \in I} = (h_i^{-T}Q)_{i \in I}$, where $g_i = h_i h_{i_0}^{-1}$. Here it is easy to verify that $(g_i)_{i \in I}$ is well-spread if $(h_i)_{i \in I}$ is.

The following inequality is the first step towards showing that the orbit map is a quasi-isometry.

**Lemma 4.12.** Assume $H_\xi \subset H_0$. Let $W \subset H$ be a relatively compact, symmetric unit neighborhood with $W \subset H_0$. Let $Q = (h_i^{-T}Q)_{i \in I}$ be an induced connected covering of $O$ by $H$ with $\xi \in Q$. Then the following inequality holds
\[ d_W(g, h) \leq L d_Q(p_\xi(g), p_\xi(h)) + C \]
for suitable $L, C > 0$ for all $g, h \in H$.

**Proof.** Let $g, h \in H$ be arbitrary. If $d_Q(p_\xi(g), p_\xi(h)) = \infty$, there is nothing to show. If $d_Q(p_\xi(g), p_\xi(h)) = 0$, then
\[ p_\xi(g) = p_\xi(h) \]
and \( g^{-1}h \in H_\xi \). The set \( H_\xi \subset H_0 \) is compact, according to the admissibility of \( H \), so that there exists a \( C_1 \in \mathbb{N} \) with \( H_\xi \subset W^{C_1} \) (cf. the proof of Lemma 4.5 for a similar argument), which implies \( d_W(g,h) \leq C_1 \).

If \( d_Q(p_\xi(g), p_\xi(h)) = m \in \mathbb{N} \), there exists a \( Q \)-chain from \( p_\xi(g) \) to \( p_\xi(h) \), which means there exist \( h_{i_1}, \ldots, h_{i_m} \) in \( (h_i)_{i \in I} \) such that \( p_\xi(g) \in h_{i_1}^{-T}Q \), \( p_\xi(h) \in h_{i_m}^{-T}Q \) and

\[
h_{i_j}^{-T}Q \cap h_{i_{j+1}}^{-T}Q \neq \emptyset
\]

for all \( j \in \{1, \ldots, m-1\} \). According to Lemma 4.10 this implies

\[
\begin{align*}
g^{-1}h_{i_j} &\in L(\{\xi\}, \overline{Q}), \\
h_{i_j}^{-1}h_{i_{j+1}} &\in L(\overline{Q}, \overline{Q}) \quad \text{for } j \in \{1, \ldots, m-1\} \\
h_{i_m}^{-1}h_{i_1} &\in L(\overline{Q}, \{\xi\}),
\end{align*}
\]

which allows us to deduce

\[
g^{-1}h = g^{-1}h_{i_1}\left(\prod_{j=1}^{m-1} h_{i_j}^{-1}h_{i_{j+1}}\right)h_{i_m}^{-1}h \in L(\{\xi\}, \overline{Q})L(\overline{Q}, \overline{Q})^{m-1}L(\overline{Q}, \{\xi\}).
\]

Because \( \xi \in \overline{Q} \) and \( \overline{Q} \) is a connected subset of \( \mathcal{O} \), Lemma 4.11 implies

\[
S := L(\{\xi\}, \overline{Q}) \cup L(\overline{Q}, \overline{Q}) \cup L(\overline{Q}, \{\xi\}) \subset H_0
\]

and \( S \) is compact, according to Lemma 4.10. Connectedness of \( H_0 \) and \( W \subset H_0 \) symmetric, open imply that \( H_0 = \bigcup_{n \in \mathbb{N}} W^n \), hence compactness of \( S \) implies \( S \subset W^{L_1} \) for suitable \( L_1 \), which implies

\[
g^{-1}h \in S^{m+1} \subset W^{L_1(m+1)}.
\]

We conclude

\[
d_W(g,h) \leq L_1(m+1) = L_1d_Q(p_\xi(g), p_\xi(h)) + L_1.
\]

In summary, for \( L := L_1 \) and \( C := L_1 + C_1 \), we have

\[
d_W(g,h) \leq Ld_Q(p_\xi(g), p_\xi(h)) + C
\]

for all \( g, h \in H \).

In the next result, we show the remaining inequality for a special case.

**Lemma 4.13.** Let \( W \subset H \) be a relatively compact, symmetric unit neighborhood with \( W \subset H_0 \). Let \( Q = (h_i^{-T}Q)_{i \in I} \) be an induced connected covering of \( \mathcal{O} \)
by \( H \) with \( \xi \in Q \) for some well-spread family \((h_i)_{i \in I}\) in \( H \) and open, relatively compact \( Q \subset O \). Then there exists a constant \( C > 0 \) such that for all \( g, h \in H \) with \( d_W(g, h) \leq 1 \), the inequality
\[
d_Q(p_\xi(g), p_\xi(h)) \leq C
\]
holds.

**Proof.** If \( g, h \in H \) with \( d_W(g, h) \leq 1 \), then \( h = gw \) for some \( w \in W \) (if \( d_W(g, h) = 0 \), then \( w = e \)). The set \( W \subset H_0 \) is relatively compact therefore we can choose a relatively compact, connected set \( V \subset H_0 \) (this is possible by choosing a relatively compact, connected unit neighborhood \( U \subset H \) and using that \( W \subset \bigcup_{m=1}^{\infty} (U^n)^m \), then \( V := U^m \), for a suitable \( m \), is a valid choice). We have
\[
p_\xi(V) \subset p_\xi(V) \subset O
\]
because the orbit map is closed according to Lemma 4.7 which shows that \( K_1 := p_\xi(V) \) is a compact and connected subset of \( O \). If we set \( K_2 := \overline{Q} \), then Lemma 2.10 implies that the set
\[
I_g(K_1, K_2) := \{ i \in I \mid g^{-T}K_1 \cap h_i^{-T}K_2 \neq \emptyset \}
\]
has at most \( C_2 = C_2(K_1, K_2, (h_i)_{i \in I}) \) elements. Crucial for this proof is that \( C_2 \) does not depend on \( g \). For arbitrary \( x \in g^{-T}K_1 =: K \), consider the set
\[
C_x := \left\{ z \in K \mid \begin{array}{l}
x, z \text{ are connected by a } Q\text{-chain} \\
(Q_i)_{i=1}^n \text{ of length } m \leq C_2 \text{ with } Q_\ell \cap K \neq \emptyset \\
\text{and } Q_\ell \neq Q_{\ell'} \text{ for } \ell \neq \ell'
\end{array} \right\}
\]
As in the proof of Theorem 3.22, we see that \( C_x = K \) because \( K \) is connected as well. Since \( x \in K \) was arbitrary, we have
\[
d_Q(x, y) \leq C_2
\]
for all \( x, y \in K \). In particular, it holds
\[
p_\xi(g) = g^{-T}\xi \in g^{-T}W^{-T}\xi \subset g^{-T}p_\xi(V) = g^{-T}K_1 = K
\]
and
\[
p_\xi(gw) = g^{-T}w^{-T}\xi \in g^{-T}W^{-T}\xi \subset g^{-T}p_\xi(V) = g^{-T}K_1 = K,
\]
which implies
\[
d_Q(p_\xi(g), p_\xi(h)) \leq C_2.
\]
As \( C_2 \) is independent of \( g, h \) with \( d_W(g, h) \leq 1 \), this concludes the proof.

This special case implies the general case.
Corollary 4.14. Let $W \subset H$ be a relatively compact, symmetric unit neighborhood with $W \subset H_0$. Let $Q = (h_i^{-T}Q)_{i \in I}$ be an induced connected covering of $O$ by $H$ with $\xi \in Q$, for some open, relatively compact $Q \subset O$. There exists a constant $C > 0$ such that for all $g, h \in H$, the inequality
\[ d_Q(p_\xi(g), p_\xi(h)) \leq Cd_W(g, h) \]
holds.

Proof. Let $g, h$ be arbitrary elements in $H$. If $d_W(g, h) = \infty$, there is nothing to show. If $d_W(g, h) = 0$, then $g = h$ and also the left side of the inequality vanishes. If $d_W(g, h) = m \in \mathbb{N}$, we can write
\[ h = g \prod_{k=1}^{m} w_k \]
for some $w_k \in W$. Define $h_n := g \prod_{k=1}^{n} w_k$ for $n \in \{0, \ldots, m\}$. This entails
\[ d_W(h_n, h_{n+1}) = d_W(e, w_{n+1}) \leq 1 \]
for all $n \in \{0, \ldots, m-1\}$, where we used the left invariance of the metric $d_W$. Repeated application of the triangle inequality and Lemma 4.13 yield
\[ d_Q(p_\xi(g), p_\xi(h)) \leq \sum_{n=0}^{m-1} d_Q(p_\xi(h_n), p_\xi(h_{n+1})) \leq Cm = Cd_W(g, h). \]

Remark 4.15. Note that we did not assume $H_\xi \subset H_0$ in Lemma 4.13 and Corollary 4.14, which means that even without this assumption, the orbit map is uniformly bornologous.

The previous results combine to give the following important observation.

Theorem 4.16. Assume $H_\xi \subset H_0$. Let $W \subset H$ be a relatively compact, symmetric unit neighborhood with $W \subset H_0$. Furthermore, let $Q = (h_i^{-T}Q)_{i \in I}$ be an induced connected covering of $O$ by $H$ with $\xi \in Q$, for some open, relatively compact $Q \subset O$. Then
\[ p_\xi : (H, d_W) \to (O, d_Q), h \mapsto h^{-T} \xi \]
is a quasi-isometry. In particular $(H, d_W)$ and $(O, d_Q)$ are coarsely equivalent.

Proof. Lemma 4.12 and Corollary 4.14 show that
\[ p_\xi : (H, d_W) \to (O, d_Q) \]
is a quasi-isometric embedding. Since it is surjective, it is a quasi-isometry. The coarse equivalence follows with Theorem 3.13.
4.3. Characterizing coorbit equivalence

Now we can state the central theorem characterizing whether two admissible groups with the same dual orbit $O$ induce weakly equivalent coverings of $O$, without the need to explicitly determine such coverings.

The basic idea underlying the theorem is conveyed by the following commutative diagram, and an appeal to Theorems 3.22 and 4.10:

\[
\begin{array}{ccc}
(O, d_Q) & \xrightarrow{\text{id}_Q} & (O, d_P) \\
\downarrow{p_{\xi_1}^H} & & \downarrow{(p_{\xi_2}^H)^{-1}} \\
(H_1, d_W) & \xrightarrow{\phi} & (H_2, d_V)
\end{array}
\]

**Theorem 4.17.** Let $H_1, H_2 \subset \text{GL}(\mathbb{R}^d)$ be two admissible dilation groups with dual orbit $O = H_1^{-T}\xi_1$ and $O' = H_2^{-T}\xi_2$. Assume that the isotropy groups are contained in the respective connected components of $H_1$ and $H_2$. Let $W \subset (H_1)_0, V \subset (H_2)_0$ denote relatively compact, symmetric neighborhoods, let $Q = (h_i^{-1}Q)_{i \in I}$ and $P = (g_j^{-1}P)_{j \in J}$ denote induced, connected coverings of $O$ resp. $O'$, for suitable choices of well-spread families $(h_i)_{i \in I} \subset H_1, (g_j)_{j \in J} \subset H_2$.

Denote the orbit maps by

\[
p_{\xi_1}^H : (H_1, d_W) \to (O, d_Q), h \mapsto h^{-T}\xi_1
\]

and

\[
p_{\xi_2}^H : (H_2, d_V) \to (O, d_P), h \mapsto h^{-T}\xi_2.
\]

Let $\left(p_{\xi_2}^H\right)^{-1}$ denote an arbitrary right inverse of $p_{\xi_2}^H$ (which exists since $p_{\xi_2}^H$ is surjective). Then the following statements are equivalent:

i) $H_1$ and $H_2$ are coorbit equivalent.

ii) $O = O'$, and the map

\[
\phi := \left(p_{\xi_2}^H\right)^{-1} \circ p_{\xi_1}^H : (H_1, d_W) \to (H_2, d_V)
\]

is a coarse equivalence.

iii) $O = O'$, and the map

\[
\phi := \left(p_{\xi_2}^H\right)^{-1} \circ p_{\xi_1}^H : (H_1, d_W) \to (H_2, d_V)
\]

is a quasi-isometry.

**Proof.** First, we note that any right inverse of $p_{\xi_2}^H$ is also a coarse right inverse in the sense of Definition 3.9. Since our assumptions and Theorem 4.10 imply that $p_{\xi_2}^H(H_2, d_V) \to (O', d_P)$ is a quasi-isometry, Lemma 3.11 shows that

\[
\left(p_{\xi_2}^H\right)^{-1} : (O, d_P) \to (H_2, d_V)
\]
is also a quasi-isometry.

i) ⇒ iii): By Theorem 2.18, \(H_1\) and \(H_2\) are coorbit equivalent iff \(O = O'\), and the dual coverings induced by \(H_1\) and \(H_2\) are weakly equivalent.

We can now write

\[
\phi = \left( p_{H_2} \right)^{-1} \circ \text{id}_Q \circ p_{H_1} : (H_1, d_W) \to (H_2, d_V),
\]

as a composition of quasi-isometries: \(\text{id}_Q\) is a quasi-isometry \((O, d_Q) \to (O, d_P)\) by Theorem 3.22 and the other two mappings are quasi-isometries by Theorem 4.16. Hence \(\phi\) is a quasi-isometry as well.

iii) ⇒ i): Our assumptions imply that \(p_{H_1} \circ \phi \circ p_{H_2}^{-1}\) shows that there exists a quasi-isometric right inverse \(p_{H_1}^{-1} \circ p_{H_2}^{-1}\) of \(p_{H_1} \circ \phi \circ p_{H_2}^{-1}\). We conclude that the map

\[
p_{H_2} \circ \phi \circ p_{H_1}^{-1} = p_{H_2} \circ \left( p_{H_2}^{-1} \circ \text{id}_Q \circ p_{H_1}^{-1} \right) = \left( p_{H_2} \circ p_{H_2}^{-1} \right) \circ \text{id}_Q \circ \left( p_{H_1}^{-1} \circ p_{H_1}^{-1} \right) = \text{id}_Q
\]

is a quasi-isometry as composition of quasi-isometric maps. Theorem 3.22 implies that the coverings \(Q\) and \(P\) are weakly equivalent. This suffices to establish the coorbit equivalence of the groups \(H_1\) and \(H_2\), by Theorem 3.22.

In order to prove the equivalence \(ii) \Leftrightarrow iii)\), we first observe that \(H_1, H_2\) both have finitely many connected components, by combining Lemmas 4.7 and 4.8.

With that fact established for \(H_1\) and \(H_2\), \(ii) \Leftrightarrow iii)\) follows from Proposition 3.15 and Remark 3.20, since the word metrics are large scale geodesic.

**Remark 4.18.** The main appeal of the theorem is that it replaces the technicalities involved with computing and comparing coverings induced by different dilation groups by a problem in geometric group theory. Whether this new formalism actually simplifies the task still depends on the groups at hand; the results of Section 5 present a class of groups where this is arguably the case.

At least the theorem opens the door to the systematic application of geometric group theory techniques to the problem of deciding coorbit equivalence. For example, the machinery of coarse invariants of groups can be brought to bear to derive necessary criteria for coorbit equivalence. E.g., if \(H_1\) and \(H_2\) are coorbit equivalent and unimodular, and \(H_1\) is amenable, then \(H_2\) must be amenable as well, by Corollary 4.F.9 of [7].

It is important to realize, though, that the theorem refers to coarse properties of a very specific map between the groups \(H_1\) and \(H_2\), which is induced by the respective dual actions. In particular, Section 5 will provide a rich class of examples of pairs of groups \(H_1\) and \(H_2\) that are topologically isomorphic (and thus quasi-isometric), without being coorbit equivalent; see Remark 5.8 below.
5. Characterizing coorbit equivalence of shearlet dilation groups

Shearlet groups in arbitrary dimensions, and their associated coorbit spaces, were first introduced in [11]. Later on, Toeplitz shearlets, a variation on the shearlet construction in dimensions $\geq 3$, were introduced in [9]. The next definition provides the details. The initial definition of Toeplitz shearlets only allowed the inclusion of isotropic dilation, corresponding to the case $\delta = 0$ in part ii). The availability of anisotropic scaling for this class of groups was stated later in [1].

**Definition 5.1** ([1] Example 17. and Example 18.).

i) For $\lambda = (\lambda_1, \ldots, \lambda_{d-1}) \in \mathbb{R}^{d-1}$, we define the **standard shearlet group in $d$-dimensions** $H^\lambda$ as the set

$$
\{ \epsilon \text{ diag} \left( a, a^{\lambda_1}, \ldots, a^{\lambda_{d-1}} \right) \left( \begin{array}{cccc}
1 & s_1 & \ldots & s_{d-1} \\
1 & 0 & \ldots & 0 \\
& \ddots & \ddots & \ddots \\
& & 0 & 1
\end{array} \right) \left| \begin{array}{c}
a > 0, \\
s_i \in \mathbb{R}, \\
\epsilon \in \{ \pm 1 \}
\end{array} \right. \}.
$$

ii) For $\delta \in \mathbb{R}$, we define the **Toeplitz shearlet group in $d$-dimensions** $H^\delta$ as the set

$$
\{ \epsilon \text{ diag} \left( a, a^{1-\delta}, \ldots, a^{1-(d-1)\delta} \right) \cdot T(1, s_1, \ldots, s_{d-1}) \left| \begin{array}{c}
a > 0, \\
s_i \in \mathbb{R}, \\
\epsilon \in \{ \pm 1 \}
\end{array} \right. \},
$$

where the matrix $T(1, s_1, \ldots, s_{d-1})$ is defined by

$$
T(1, s_1, \ldots, s_{d-1}) := \left( \begin{array}{cccc}
1 & s_1 & s_2 & \ldots & s_{d-1} \\
1 & s_1 & s_2 & \ldots & s_{d-2} \\
& \ddots & \ddots & \ddots & \ddots \\
& & 0 & s_1 & s_2 \\
& & & 1 & s_1
\end{array} \right).
$$

**Remark 5.2.** The two constructions are special cases of the class of **generalized shearlet dilation groups** introduced in [25] and further studied in [1]; see Definition 5.3 below. Note that each class actually defines a whole family of groups, parametrized by different choices for the diagonal.

In dimension two the distinction between the standard and Toeplitz shearlet dilation groups is moot. In dimension three, one can show that standard and Toeplitz shearlet are the only possible generalized shearlet dilation groups (cf. [1], Remark 19). With increasing dimension, the set of generalized shearlet dilation groups besides the standard and Toeplitz case quickly becomes untractable.

Prior to the thesis [30], the simple question whether standard and Toeplitz shearlets have distinct coorbit spaces was not under consideration. [30] established that, for any dimension $d \geq 3$, two distinct groups from the families in
Definition 5.1 are not coorbit equivalent. For dimension three, this was done by computing and comparing induced coverings, see Chapter 3 of [30] for details. The arguments employed to establish this result nicely illustrate the computational burden imposed by the use of induced coverings, and motivated the introduction of coarse geometry as a tool in Chapter 5 of [30]. The comparison of standard and Toeplitz shearlet groups in dimensions $d > 3$ was then performed using the new toolbox. Our treatment of general shearlet dilation groups is a generalization of the arguments from [30, Section 5.5].

5.1. Shearlet dilation groups: definition and basic properties

We now recall basic notation regarding matrix groups and their Lie algebras. $\mathfrak{gl}(\mathbb{R}^d)$ denotes the set of all real $d \times d$-matrices. We let $\exp : \mathfrak{gl}(\mathbb{R}^d) \to \text{GL}(\mathbb{R}^d)$ be the exponential map defined by

$$\exp(A) := \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$ 

Furthermore, we denote with $T(\mathbb{R}^d) \subset \text{GL}(\mathbb{R}^d)$ the group of upper triangular $d \times d$-matrices with one on their diagonals. By definition, the Lie algebra of a closed subgroup $H \subset \text{GL}(\mathbb{R}^d)$ is the set $\mathfrak{h}$ of all matrices $Y$ in $\mathfrak{gl}(\mathbb{R}^d)$ such that $\exp(tY) \in H$ for all $t \in \mathbb{R}$.

Definition 5.3 ([1] Definition 1.). Let $H \subset \text{GL}(\mathbb{R}^d)$ be a closed, admissible dilation group. The group $H$ is called a generalized shearlet dilation group if there exist two closed subgroups $S, D \subset \text{GL}(\mathbb{R}^d)$ such that

i) $S$ is a connected abelian subgroup of $T(\mathbb{R}^d)$,

ii) $D = \{ \exp(rY) \mid r \in \mathbb{R} \}$ is a one-parameter group, where $Y \in \mathfrak{gl}(\mathbb{R}^d)$ is a diagonal matrix and

iii) every $h \in H$ has a unique representation as $h = \pm ds$ for some $d \in D$ and $s \in S$.

$S$ is called the shearing subgroup of $H$, $D$ is called the scaling subgroup of $H$, and $Y$ is called the infinitesimal generator of $D$.

The article [1] describes a systematic algebraic construction method for shearlet groups in arbitrary dimension $d \geq 2$ that proceeds by first constructing a shearing group $S$ from an arbitrary nilpotent associative algebra of dimension $d - 1$, and then determining conditions on the diagonal complement $D$ from certain structure constants of the nilpotent algebra.

We denote the canonical basis of $\mathbb{R}^d$ with $e_1, \ldots, e_d$ and the identity matrix in $\text{GL}(\mathbb{R}^d)$ with $I_d$. The next result contains information about the structure of shearing subgroups.
Lemma 5.4 ([1] Lemma 5. and Lemma 6.). Let $S$ be the shearing subgroup of a generalized shearlet dilation group $H \subset \text{GL}(\mathbb{R}^d)$. Then the following statements hold:

i) There exists a unique basis $X_2, \ldots, X_d$ of the Lie algebra $\mathfrak{s}$ of $S$ with $X_i^T e_1 = e_i$ for $2 \leq i \leq d$, called the canonical basis of $\mathfrak{s}$.

ii) We have $S = \{ I_d + X | X \in \mathfrak{s} \}$.

iii) Let $\mathfrak{s}_k = \text{span}\{X_j : j \geq k\}$, for $k \in \{2, \ldots, d\}$. These are associative matrix algebras satisfying $\mathfrak{s}_k \mathfrak{s}_\ell \subset \mathfrak{s}_{k+\ell-1}$, where we write $\mathfrak{s}_m = \{0\}$ for $m > d$.

iv) $H$ is the inner semidirect product of the normal subgroup $S$ with the closed subgroup $D \cup -D$.

Every generalized shearlet dilation group $H$ is admissible, and the next result shows that all of them share the same dual orbit. This ensures that one of the basic conditions for coorbit equivalence is already fulfilled.

Lemma 5.5 ([1] Proposition 11.). Let $S \subset T(\mathbb{R}^d)$ be a shearing subgroup and $D$ be a compatible scaling subgroup such that $H = DS \cup (-DS)$ is a generalized shearlet dilation group. Then the unique open dual orbit of $H$ is given by $O = \mathbb{R}^* \times \mathbb{R}^{d-1}$ and the isotropy group of $\xi \in O$ with respect to the dual action is given by $H_\xi = \{ I_d \}$.

Note that in particular, $H_\xi \subset H_0$, where $H_0 = DS$ denotes the connected component of the neutral element. Furthermore, the orbit map is bijective, thus it has a unique (right) inverse. Throughout this section, we will use the representative $\xi_0 = (1, 0, \ldots, 0)$. As a consequence of the bijectivity of $p_{\xi_0}$, each element of $H$ is uniquely determined by its first line.

Note that if $H = DS \cup -DS$ is a shearlet dilation group, one can replace $D = \exp(rY)$ by $D' = \mathbb{R}^+ \cdot I_d$ to obtain the associated abelian shearlet dilation group $A = D'S \cup -D'S$. The structure of abelian dilation groups was elucidated in [21], and the following lemma notes some useful properties arising from this structure.

Lemma 5.6. (a) Let $A$ denote an abelian admissible dilation group. The Lie algebra $\mathfrak{a}$ of $A$ coincides with the associative algebra $\mathcal{A} = \text{span}(A)$ generated by $A$, and $A \subset A$ is the group of invertible elements in $A$. Furthermore, the dual action of $A$ on the open dual orbit is free.

(b) Let $A, A' \subset \text{GL}(\mathbb{R}^d)$ be abelian admissible dilation groups with $A = \text{span}(A)$ and $A' = \text{span}(A')$. Let $\alpha : A \to A'$ denote an algebra isomorphism. Then there exists $C \in \text{GL}(\mathbb{R}^d)$ such that $\alpha(X) = C^{-1}XC$ holds, for all $X \in A$.  

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where the third equation used that $\alpha$ is commutative implies $\psi(XZ) = X^T \psi(Z)$ for all $X, Z \in A$, hence we can continue via
\[
B^{-1}X^T B y = (\psi'(\alpha \circ \psi^{-1}) \circ X^T \circ (\psi \circ \alpha^{-1} \circ (\psi')^{-1}))(y)
= ((\psi' \circ \alpha \circ \psi^{-1}) \circ X^T)(\psi(\alpha^{-1}(Y))).
\]

The proof is complete.

Remark 5.7. The associative algebra structure of $s$ is also the key to the systematic construction of shearing subgroups. For the following facts, we refer to [1], in particular to Lemma 13 and Remark 14 therein. Every nilpotent commutative associative algebra $N$ of dimension $d - 1$ gives rise to the Lie algebra of a shearing subgroup $s$ of a shearing subgroup of a generalized shearlet dilation group. The canonical basis of $s$ is constructed by picking a Jordan-Hölder basis in $N$, and applying a suitable linear map $N \to \mathfrak{gl}(d, \mathbb{R})$. Letting $S = \exp(s)$ and $D = \mathbb{R}^+ \cdot I_d$ then results in an abelian shearlet dilation group $H = DS \cup -DS$. Furthermore, all candidates of infinitesimal generators for $D$ can be found by solving a system of linear equations derived from $s$, see [1] Lemma 9.

It is useful to note that two different shearing subgroups $S_1, S_2$ are related to each other by conjugation iff their Lie algebras are isomorphic as associative algebras. Here the necessity is clear; and the sufficiency follows by Lemma 5.6 (b). Thus there is a one-to-one correspondence between conjugacy classes of shearing subgroups (or alternatively: of abelian shearlet dilation groups) in dimension $d$ and of isomorphism classes of nilpotent commutative associative algebras of dimension $d - 1$. This illustrates the increasing richness of this class of dilation groups, as $d$ grows. The main theorem of this section, which characterizes coorbit equivalence within the class of shearlet dilation groups, exhibits the relevance of the conjugacy problem also for coorbit equivalence.

Remark 5.8. Given a shearlet dilation group $H = DS \cup (-DS)$ with $D = \exp(\mathbb{R}Y)$ and canonical basis $X_2, \ldots, X_d$ of $S$, the entries of the infinitesimal
generator $Y$ can be related to certain algebraic relations. More precisely, a necessary requirement for $H$ to be an admissible dilation group is that the first diagonal entry of $Y$ does not vanish, by [1, Proposition 7]. $Y$ can then be normalized to 1 as first entry, i.e., $Y = \text{diag}(1, \lambda_2, \ldots, \lambda_d)$, and the remaining entries of $Y$ occur in the structure relations
\[
\forall i, j = 2, \ldots, d : \ [Y, X_i] = (1 - \lambda_i) X_i, \ [X_i, X_j] = 0 ,
\]
characterizing the Lie algebra $\mathfrak{h}$ up to isomorphism. It turns out that this extends to the Lie group level, yielding that $H$ is topologically isomorphic to $H' = D'S' \cup (-D'S')$ iff $D = D'$; and this clearly entails that $H$ and $H'$ are coarsely isometric.

Observe that this characterization imposes no restrictions on $S$ and $S'$, respectively. By contrast, Theorem 5.9 establishes the necessary criterion for coorbit equivalence that $S$ and $S'$ are related by conjugacy, and by Remark 5.7 this entails that the Lie algebras $\mathfrak{s}, \mathfrak{s}'$ must be isomorphic as associative algebras. In particular, by taking $H, H'$ as standard resp. Toeplitz shearlet dilation groups in dimension $d \geq 3$, with identical diagonal groups $D = D'$, we obtain pairs of dilation groups that are coarsely isometric, but not coorbit equivalent. See also Remark 5.10 below.

To our knowledge, a classification of semidirect products of the type $\mathbb{R}^m \rtimes \exp(\mathbb{R}Y)$ up to quasi-isometry is not available. A subclass of these groups, for which a rich theory has been developed, is given by the so-called Heintze groups [29]. Even here, a classification up to quasi-isometry does not seem to be in sight. These observations make clear that coarse geometric methods generally do not provide readily applicable solutions for the problem of classifying dilation groups up to coorbit equivalence. They also emphasize that coorbit equivalence of two dilation groups depends on the coarse geometric properties of a very specific map relating the dual actions of these groups.

We can now formulate a general characterization of coorbit equivalence for shearlet groups. The following theorem generalizes the characterization for two-dimensional shearlet groups obtained in [36], and the results for standard and Toeplitz shearlet groups in [30].

**Theorem 5.9.** Let $H = DS \cup -DS, H' = D'S' \cup -D'S' \subset GL(\mathbb{R}^d)$ denote two shearlet dilation groups. Then the following properties are equivalent:

(a) $H$ and $H'$ are coorbit equivalent.

(b) $D = D'$, and there exists a matrix $C \in GL(\mathbb{R}^d)$ with the following properties: $C^{-1}SC = S'$, and the conjugation actions of $D$ and $C$ on $S$ commute, i.e.

\[
\forall s \in S \forall d \in D : \ d^{-1}C^{-1}sdC = C^{-1}d^{-1}sdC . \quad (5.1)
\]

**Remark 5.10.** We can apply the theorem to the two families of shearlet dilation groups introduced in Definition 5.3. We first observe that the theorem explicitly states that different elements in the same family (standard or Toeplitz),
i.e., pairs of shearlet dilation groups that differ only in the scaling subgroup, are not coorbit equivalent.

The comparison of elements from different families leads to the question of conjugacy, which in turn is closely related to the associative structures of the respective Lie algebras. Let $s_s$ denote the Lie algebra of the shearing subgroup of the standard shearlet dilation group in dimension $d \geq 3$, and let $s_T$ denote its Toeplitz counterpart. Then any two elements $X, Y \in s_s$ fulfill $XY = 0$. By contrast, $s_T$ is generated as an associative algebra by the element $X_2$ of its canonical basis, showing that $s_T \cong \mathbb{R}[X]/(X^{d-1})$. In particular, $s_s$ and $s_T$ are not isomorphic as associative algebras, and hence not conjugate.

In summary, any pair of distinct groups from Definition 5.3 has distinct coorbit spaces. This fact was first established in [30].

5.2. Proof of Theorem 5.9

Throughout this subsection, we will denote $H = DS \cup (-DS)$, $H' = D'S' \cup (-D'S')$, with associated Lie algebras $\mathfrak{h}$, $\mathfrak{h}'$, $\mathfrak{s}$, $\mathfrak{s}'$, and canonical bases $Y, X_2, \ldots, X_d \in \mathfrak{h}$ and $Y', X_2', \ldots, X_d' \in \mathfrak{h}'$. By Proposition 7 of [1], we can normalize $Y, Y'$ so that

$$Y = \text{diag}(1, \lambda_2, \ldots, \lambda_d), \quad Y' = \text{diag}(1, \lambda_2', \ldots, \lambda_d') \,.$$  

We use $\xi_0 = (1, 0, \ldots, 0) \in O = \mathbb{R}^* \times \mathbb{R}^{d-1}$. We parametrize group elements of $H$ and $H'$ as follows: Given $r \in \mathbb{R}$ and $t = (t_2, \ldots, t_d) \in \mathbb{R}^{d-1}$, we let

$$h(r, t) = \exp(-rY) \left( I_d + \sum_{j=2}^d t_j X_j \right)^{-1} \in H, \quad (5.2)$$

$$h'(r, t) = \exp(-rY') \left( I_d + \sum_{j=2}^d t_j X_j' \right)^{-1} \in H'.$$

Note that $D = \{ h(r, 0) : r \in \mathbb{R} \}$ and $S = \{ h(0, t) : t \in \mathbb{R}^{d-1} \}$, and analogous statements hold for $D', S'$.

We can then compute

$$p^H_{\xi_0}(h(r, t)) = \exp(rY)(I_d + \sum_{j=2}^d t_j X_j)^{-1} = \begin{pmatrix} e^r & e^{\lambda_2 r} t_2 & \cdots & e^{\lambda_d r} t_d \\ 0 & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & e^r \end{pmatrix}. $$

The analogous calculation for $H'$ yields

$$p^H_{\xi_0}(h'(r, t)) = \begin{pmatrix} e^r \\ e^{\lambda_2' r} t_2 \\ \vdots \\ e^{\lambda_d' r} t_d \end{pmatrix}. $$

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Throughout this section, let $\phi = (p_{\phi_{0}}^{H'})^{-1} \circ p_{\phi_{0}}^{H} : H \to H'$. Then a comparison of the formulae for $p_{\phi_{0}}^{H}(h(r,t)), p_{\phi_{0}}^{H'}(h'(r,t))$ yields

$$\phi(h(r,t)) = h'(r,t) , \quad t' = (\exp((r\lambda_{2} - \lambda_{2}'))t_{2}, \ldots, \exp(r(\lambda_{d} - \lambda_{d}'))t_{d}) . \quad (5.3)$$

Setting $r = 0$ in this formula shows that the restriction $\phi|_{S}$ in fact maps $S$ bijectively onto $S'$, and that it is independent of the choices of $Y, Y'$.

We can now make the first step towards the proof of the theorem, which consists in comparing the diagonals of $H$ and $H'$.

**Lemma 5.11.** If $H$ and $H'$ are coorbit equivalent, then $Y = Y'$, and thus $D = D'$.

**Proof.** Towards a proof by contradiction, assume that $H$ and $H'$ are coorbit equivalent, yet there exists $2 \leq i \leq d$ such that $\lambda_{i} \neq \lambda'_{i}$. Let $\phi : H \to H'$ denote the map from Equation (5.3). Let $e_{i} \in \mathbb{R}^{d-1}$ denote the vector with entry one at the $i-1$st position, and zeros elsewhere. Define the sequence $(h_{n})_{n \in \mathbb{N}}$ as $h_{n} = h(1, e_{i})^{n}$, and $h_{n}' = \phi(h_{n})$. If $d : H \times H \to \mathbb{R}_{0}^{+}$ denotes a word metric on $H$, we get by left-invariance that $d(h_{n}, h_{n+1}) = d(e_{H}, h)$. We need to prove that $\phi$ is not a quasi-isometry with respect to a suitable word metric $d'$ on $H'$. For this it is now sufficient to show that $(d'(h_{n}', h_{n+1}'))_{n \in \mathbb{N}} \subset \mathbb{R}^{+}$ is unbounded, or equivalently, that the sequence $(h_{n+1}')^{-1}h_{n}' \in H'$, $n \in \mathbb{N}$, is not contained in a relatively compact set. This will be shown by concentrating on suitably chosen matrix entries of these products.

We start out by noting that $h(1, e_{i}) = h(1, 0)h(0, e_{i})$. For the computation of higher powers of this group element, we introduce $T' = \{0\}^{i} \times \mathbb{R}^{d-i-1}$. Then $t = t_{i}e_{i} + t'$, with $t' \in T'$, is a vector with $i-2$ leading zeroes, followed by $t_{i}$, followed by the remaining entries of $t'$.

With this notation, we get

$$h(-1, 0)h(0, t_{i}e_{i} + t')^{-1}h(1, 0) = h(0, e^{1-\lambda_{i}}t_{i}e_{i} + t'')^{-1} ,$$

with suitably chosen $t'' \in T'$ depending linearly on $t'$. To see this, it is sufficient to note that both sides of the equation are elements of $H$ whose first lines can be made to coincide by properly choosing $t'' \in T'$, and this first line uniquely determines the elements of $H$ by Lemma 5.8 and the comments thereafter. Since conjugation by $h(-1, 0)$ is a group isomorphism, we also get

$$h(-1, 0)h(0, t_{i}e_{i} + t')h(1, 0) = h(0, e^{1-\lambda_{i}}t_{i}e_{i} + t'') , \quad (5.4)$$

and thus for all $n \in \mathbb{N}$:

$$h(n, 0)h(0, e_{i})h(-n, 0) = h(0, e^{n(1-\lambda_{i})}e_{i}) .$$

Furthermore, the fact that $X_{i}X_{j} \in \text{span}\{X_{k} : k > \max(i,j)\}$ yields in particular, for all $p, q \in \mathbb{R}$ and $t_{1}', t_{2}' \in T'$

$$h(0, pe_{i} + t_{1}')h(0, qe_{i} + t_{2}') = h(0, (p+q)e_{i} + t_{3}') ,$$
with suitably chosen \( t'_3 \in T' \). In addition, utilizing the Neumann series for the inverse of a matrix allows to compute for every \( p \in \mathbb{R} \) and every \( t'_1 \in T' \) that

\[
    h(0, p e_i + t'_1)^{-1} = h(0, -p e_i + t'_2)
\]

with suitably chosen \( t'_2 \in T' \). We observe that an analogous formula holds for \( H' \) as well.

Let us now first consider the case \( \lambda_i \neq 1 \). Then the above formulae allow to show inductively that

\[
    h^n = h(n, 0) h \left( 0, \frac{e^{n(1-\lambda_i)} - 1}{e^{1-\lambda_i} - 1} e_i + t'_n \right),
\]

with suitably chosen \( t'_n \in T' \). Plugging this into (5.3) yields the following fairly explicit formula

\[
    h'_n = \phi(h_n) = h'(n, \exp(n(\lambda_i - \lambda'_i)) \cdot \frac{e^{n(1-\lambda_i)} - 1}{e^{1-\lambda_i} - 1} e_i + s'_n),
\]

with suitably chosen \( s'_n \in T' \). In order to prove that the sequence of matrices \((h'_n)^{-1} h'_{n+1}\) is not bounded in \( H' \), we compute

\[
    (h'_n)^{-1} h'_{n+1} = h' \left( 0, \exp(n(\lambda_i - \lambda'_i)) \cdot \frac{e^{n(1-\lambda_i)} - 1}{e^{1-\lambda_i} - 1} e_i + s'_n \right)^{-1} h'(1, 0) \\
    = h' \left( 0, \exp((n+1)(\lambda_i - \lambda'_i)) \cdot \frac{e^{(n+1)(1-\lambda_i)} - 1}{e^{1-\lambda_i} - 1} e_i + s'_{n+1} \right) \\
    = h'(1, 0) h(0, r_n e_i + u_n)
\]

where \( u_n \in T' \), and

\[
    r_n = -e^{1-\lambda'_i} \exp(n(\lambda_i - \lambda'_i)) \cdot \frac{e^{n(1-\lambda_i)} - 1}{e^{1-\lambda_i} - 1} + \exp((n+1)(\lambda_i - \lambda'_i)) \cdot \frac{e^{(n+1)(1-\lambda_i)} - 1}{e^{1-\lambda_i} - 1} \\
    = \frac{\exp(n(\lambda_i - \lambda'_i))}{e^{1-\lambda_i} - 1} \left( -e^{-\lambda'_i + n(1-\lambda_i)} + e^{1-\lambda'_i} \right) \\
    = \frac{\exp(n(\lambda_i - \lambda'_i))}{e^{1-\lambda_i} - 1} (e^{1-\lambda'_i} - e^{\lambda_i - \lambda'_i}).
\]

Now the assumptions \( \lambda'_i < \lambda_i \) and \( \lambda_i \neq 1 \) allow to conclude directly that \(|r_n| \to \infty\), and thus the desired conclusion can be drawn. In the case \( \lambda'_i > \lambda_i \) we can conclude similarly, by considering the sequence \( h_n = h^{-n} \). Hence the case \( \lambda_i \neq 1 \) is settled. However, in the case \( \lambda_i = 1 \), the assumption \( \lambda'_i \neq \lambda_i \) yields that \( \lambda'_i \neq 1 \), and exchanging the roles of \( H \), \( H' \) in the above argument allows again to conclude that \( H \) and \( H' \) are not coorbit equivalent.

The next result looks at the restriction of \( \phi \) to the shearing subgroup. Throughout the following proof, we will call a map \( \alpha : U \to V \) defined between different vector spaces \( U, V \) polynomial if the coordinates of \( \alpha(X) \) with
respect to any fixed basis of \( V \) depend polynomially on the coordinates of \( X \), taken with respect to any fixed basis of \( U \). Clearly, this notion is independent of the choice of bases.

**Lemma 5.12.** If \( H \) and \( H' \) are coorbit equivalent, there exists a matrix \( C \in GL(\mathbb{R}^d) \) satisfying the following two properties:

(i) The map \( \phi \) from Equation (5.3) fulfills \( \phi(s) = C^{-1}sC \), for all \( s \in S \).

(ii) The conjugation actions of \( D \) and \( C \) commute.

**Proof.** For the proof of (i), first note that equation (5.3) implies that \( \phi|_S : S \to S' \) is in fact a group homomorphism.

Let \( A = \text{span}(\{I_d\} \cup \mathfrak{s}) \) and \( A' = \text{span}(\{I_d\} \cup \mathfrak{s}') \). These are the Lie algebras of the abelian shearlet dilation groups \( A, A' \) associated to \( H, H' \). Note that \( p_{C_0}^A = \psi \circ \iota \), where \( \iota \) is the inversion map \( h \mapsto h^{-1} \), and \( \psi : \mathcal{A} \to \mathbb{R}^d \) is the linear bijective map defined in the proof of Lemma 5.6. In the same way, we obtain \( p_{C_0}^{A'} = \psi' \circ \iota \), with analogously defined linear isomorphism \( \psi' \).

It follows that \( \phi_{A,A'} = (p_{C_0}^{A'})^{-1} \circ p_{C_0}^{A} = \iota \circ (\psi')^{-1} \circ \psi \circ \iota \). The same then holds for the restriction of this map to \( S \). But by the remark immediately following (5.3), this restriction coincides with \( \phi|_S : S \to S' \), the restriction of the map \( \phi \) associated to \( H \) and \( H' \).

Now fix \( s \in S \), and define the map

\[
\alpha : \mathfrak{s} \to H' \, , \, \alpha(X) = \phi((I_d + X)s)^{-1}\phi(I_d + X).
\]

Then \( \alpha \) is a polynomial map. To see this, observe that inversion is polynomial on the set of unipotent matrices, since the inverse of a unipotent map can be computed by finitely many terms of a Neumann series. Since \( s, I_d + X \) and \( \phi((I_d + X)s) \) are all unipotent, this finally yields that \( \alpha \) is indeed a composition of polynomial maps, hence polynomial.

Assuming that \( H, H' \) are coorbit equivalent, we have that \( \phi \) is a quasi-isometry with respect to suitable word metrics \( d_H, d_{H'} \) on \( H, H' \) respectively, say with constants \( a > 0, b \geq 0 \). Then left-invariance of the metrics provides

\[
d_{H'}(\alpha(X), e_{H'}) = d_{H'}(\phi((I_d + X)s)^{-1}\phi(I_d + X), e_{H'}) = d_{H'}(\phi(I_d + X), \phi((I_d + X)s) \leq ad_H((I_d + X)), (I_d + X)s) + b \leq ad_H(e_{H}, s) + b,
\]

which is independent of \( X \in \mathfrak{s} \). Since the metric \( d_H \) is proper, it follows that \( \alpha : \mathfrak{s} \to \mathbb{R}^{d \times d} \) is a bounded polynomial function, hence constant. In particular, we get

\[
\phi((I_d + X)s)^{-1}\phi(I_d + X) = \alpha(X) = \alpha(0) = \phi(s)^{-1},
\]

or equivalently

\[
\phi((I_d + X)s) = \phi(I_d + X)\phi(s).
\]

Hence \( \phi|_S \) is indeed a group homomorphism, and thus a group isomorphism \( S \to S' \). This implies that \( (\psi')^{-1} \circ \psi|_A : A \to A' \) is a group isomorphism. Since
$A \subset \mathcal{A}$ is dense, it follows that the linear isomorphism $(\psi')^{-1} \circ \psi : \mathcal{A} \to \mathcal{A}'$ is an isomorphism of associative algebras. Now Lemma 5.6(b) applies to provide a matrix $C \in GL(\mathbb{R}^d)$ such that, for all $s \in S$, $\phi(s) = C^{-1}sC$.

Proof of (ii): Part (i) has already established the existence of the matrix $C$ conjugating $S$ into $S'$. With the notation established in equation (5.2), and using the fact that $Y = Y'$, we thus obtain for all $r \in \mathbb{R}, t \in \mathbb{R}^{d-1}$:

$$\phi(h(r, t)) = h(r, 0)C^{-1}h(0, t)C.$$  

It remains to establish that the conjugation actions of $C$ and $D$ on $S$ commute, if $\phi$ is a quasi-isometry. In order to see this, fix $d = h(r, 0) \in D$, and consider the map

$$\beta : \mathbb{R}^{d-1} \ni t \mapsto \phi(h(0, t)d)^{-1}\phi(h(0, t)).$$

Using that

$$\phi(h(0, t)d) = \phi(d^{-1}h(0, t)d) = dC^{-1}d^{-1}h(0, t)dC,$$

we get

$$\beta(t) = (dC^{-1}d^{-1}h(0, t)dC)^{-1}Ch(0, t)C^{-1} = C^{-1}d^{-1}h(0, t)^{-1}dCd^{-1}Ch(0, t)C^{-1}.$$  

Here $C$ and $d$ are fixed matrices, and the map $t \mapsto h(0, t)^{-1}$ is polynomial, since $h(0, t)$ is unipotent. Hence $\beta$ is a polynomial map.

In addition, if $d_H$ is any word metric on $H$, left-invariance yields $d_H(h(0, t)d, h(0, t)) = d_H(d, e_H)$, independent of $t$. Hence the assumption that $\phi$ is a quasi-isometry implies that $\beta$ is a bounded map, and therefore constant, i.e.

$$\beta(t) = \beta(0) = d^{-1}.$$  

Hence we obtain for all $s \in S$:

$$d^{-1} = \phi(e_Sd)^{-1}\phi(e_S) = \phi(sd)^{-1}\phi(s) = \phi(d(d^{-1}sd))^{-1}\phi(s) = (d(C^{-1}d^{-1}sdC))^{-1}(C^{-1}sC) = (C^{-1}d^{-1}s^{-1}dC)d^{-1}(C^{-1}sC) = d^{-1}(dC^{-1}d^{-1}s^{-1}dCd^{-1})(C^{-1}sC),$$

or, equivalently,

$$(C^{-1}sC) = (dC^{-1}d^{-1}sdC)^{-1},$$

and finally

$$d^{-1}C^{-1}sCd = C^{-1}d^{-1}sdC,$$

as desired.
Finishing the proof of Theorem 5.9: $(a) \Rightarrow (b)$ is already established by the previous two lemmas.

$(b) \Rightarrow (a)$: Since $Y = Y'$, we have $\phi(ds) = dC^{-1}sC$, for all $d \in D$. We claim that equation (5.1) implies that $\phi$ is in fact a group homomorphism:

$$
\phi(d_1s_1d_2s_2) = \left( \sum_{d \in D} (d_2^{-1}s_1d_2s_2) \right) \in S
$$

$$
= (d_1d_2)(C^{-1}d_2^{-1}s_1d_2C)(C^{-1}s_2C)
$$

$$
= (d_1d_2)(C^{-1}s_1Cd_2)(C^{-1}s_2C)
$$

$$
= (d_1C^{-1}s_1C)(d_2C^{-1}s_2C)
$$

$$
= \phi(d_1s_1)\phi(d_2s_2).
$$

Hence the bijective map $\phi$ is a topological isomorphism, and therefore a quasi-isometry. This implies that $H$ and $H'$ are coorbit equivalent. \(\square\)

6. Examples and applications

In this short section, we present a variety of examples selected to further illustrate the scope and usefulness of the techniques presented in this paper. Clearly, the full list of coverings and admissible groups listed in the introduction of this paper can be fed into the machinery, as soon as the associated metrics are computed.

6.1. $\alpha$-modulation spaces and Besov spaces in dimension 1

$\alpha$-modulation spaces were introduced by Gröbner [28]. In the following, we use results from [3] to describe the associated coverings.

Given $0 \leq \alpha < 1$, the $\alpha$-modulation covering of the real line is given by

$$
P = (P_k)_{k \in \mathbb{Z}} \text{, } P_k = B(k|k|^{\beta}, r|k|^{\beta}),
$$

where $B(x, r)$ denotes the ball with center $x$ and radius $r$, with respect to the standard metric, $\beta = \frac{\alpha}{1-\alpha}$, and $r > r_1$, where $r_1$ is suitably chosen (see Section 6.1 of [3]). With this definition in hand one sees that, up to coarse equivalence, the metric induced by the covering can be determined for $s < t$ by

$$
d_\alpha(s, t) \asymp 1 + \# \{k \in \mathbb{Z} : s \leq k|k|^{\beta} \leq t \}
$$

$$
\asymp 1 + |t|^{1-\alpha} - \text{sign}(st) \cdot |s|^{1-\alpha}.
$$

It is straightforward to see from this that different choices of $\alpha$ result in metrics that are not quasi-equivalent, and thus the associated decomposition spaces, the $\alpha$-modulation spaces initially defined in [28], are actually distinct. While we expect this observation to be known, we have not been able to locate an easily citable source for it.
6.2. Wavelet coorbit spaces in dimension 2

In dimension two, the admissible dilation groups have been classified up to conjugacy and finite index subgroups in [22]; the following is a complete list of representatives, with their open dual orbits:

- **Diagonal group**

  \[ D = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : ab \neq 0 \right\}, \]
  
  with \( \mathcal{O} = (\mathbb{R}^*)^2 \).

- **Similitude group**

  \[ H = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a^2 + b^2 > 0 \right\}, \]
  
  with \( \mathcal{O} = \mathbb{R}^2 \setminus \{0\} \).

- **Shearlet group(s)** For a fixed parameter \( c \in \mathbb{R} \),

  \[ S_c = \left\{ \pm \begin{pmatrix} a & b \\ 0 & a^c \end{pmatrix} : a > 0 \right\}, \]

  with \( \mathcal{O} = \mathbb{R}^* \times \mathbb{R} \).

No pair of distinct groups from this list is coorbit equivalent: The dual orbit criterion from Theorem 4.17 implies this for the comparison of \( D \) and \( H \), and of either group with any of the shearlet groups. Furthermore, Theorem 5.9 applied to the shearlet family shows that distinct members of the shearlet family are also not coorbit equivalent. This fact was first observed in [36].

For the sake of completeness, we list the remaining classes of decomposition spaces so far considered in dimension two:

- \( \alpha \)-modulation spaces for \( 0 \leq \alpha < 1 \), see [28];

- curvelet and shearlet smoothness spaces [3, 32];

- wave packet smoothness spaces [5];

- homogeneous and inhomogeneous anisotropic Besov spaces [4, 6].

A full classification of these decomposition spaces is obtainable by determining coarse equivalence classes among the associated covering-induced metrics on \( \mathbb{R}^2 \) (or suitable open subsets \( \mathcal{O} \subset \mathbb{R}^2 \)).
6.3. Shearlet coorbit spaces in higher dimensions

Besides the already mentioned cases of standard and Toeplitz shearlet dilation groups, there exists an increasing choice of alternative constructions with distinct coorbit spaces. The remaining examples in dimension four are described, up to conjugacy, as follows. We refer to [1, Example 20] for more details. Given \( \alpha \in \{-1, 0, 1\} \), we let

\[
X_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad X_{3,\alpha} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad X_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]

Let \( s_\alpha = \text{span}(X_2, X_{3,\alpha}, X_4) \). Then every shearing subgroup in dimension four is either conjugate to the standard or the Toeplitz shearing subgroup, or to precisely one of \( s_\alpha \), \( \alpha = -1, 0, 1 \). For \( \alpha = 0 \), the set of dilation subgroups compatible with \( s_\alpha \) (or equivalently: the set of infinitesimal generators of such groups) is a one-dimensional manifold. For \( \alpha = \pm 1 \), there exists a two-dimensional manifold of infinitesimal generators to choose from. Thus we obtain five distinct families of shearlet dilation groups, each parametrized by a continuum of possible choices for the associated scaling subgroup, and no two distinct elements from any of these families are coorbit equivalent.

We finally present an example of two distinct shearlet dilation groups that are coorbit equivalent. We let \( S_1 \) denote the shearing subgroup of the Toeplitz shearlet dilation group in dimension 4, i.e.,

\[
S_1 = \left\{ \begin{pmatrix} 1 & t_2 & t_3 & t_4 \\ 0 & 1 & t_2 & t_3 \\ 0 & 0 & 1 & t_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} : t_2, t_3, t_4 \in \mathbb{R} \right\}.
\]

With

\[
C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

one then computes for \( S_2 = C^{-1}S_1C \) that

\[
S_2 = \left\{ \begin{pmatrix} 1 & t_2 & t_3 & t_4 \\ 0 & 1 & t_2 & t_3 - 2t_2 \\ 0 & 0 & 1 & t_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} : t_2, t_3, t_4 \in \mathbb{R} \right\},
\]

in particular \( S_1 \neq S_2 \). Then the associated shearlet groups \( H_1, H_2 \) obtained by combining \( S_1, S_2 \) with the isotropic scaling subgroup \( D = \mathbb{R}^+ \cdot I_d \) are distinct too, but they are coorbit equivalent by Theorem 5.9.
7. Concluding remarks

The main purpose of this paper was to rigorously establish a connection between coorbit spaces, decomposition spaces and coarse geometry, and to demonstrate the potential of the method, using the class of shearlet coorbit spaces. The treatment of the examples has made clear that the coarse reformulation provides a shorthand for the treatment of potentially fairly technical questions in the theory of decomposition spaces. To quote from the abstract of [37]: In a nutshell, although knowledge of Fourier analysis is required to define and understand decomposition spaces, no such knowledge is required if one just wants to apply the embedding results presented in this article. Instead, one only has to study the geometric properties of the involved coverings, so that one can decide the finiteness of certain sequence space norms defined in terms of the coverings. (Boldface added by the authors of the present paper.) The results of our paper indicate that the pertinent geometric properties mentioned in this quote are coarse properties. While we put our focus on the fundamental question of classification (deciding, when certain scales of decomposition spaces coincide), we expect that the coarse viewpoint will also be useful in connection with other questions in the theory of decomposition spaces.

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