Abstract

We give a brief overview of recent developments in Sturm-Liouville theory concerning operators of transmutation (transformation) and spectral parameter power series (SPPS) and propose a new method for numerical solution of corresponding spectral problems.

1 Spectral parameter power series (SPPS)

Let \( f \in C^2(a, b) \cap C^1[a, b] \) be a complex valued function and \( f(x) \neq 0 \) for any \( x \in [a, b] \). The interval \((a, b)\) is assumed being finite. Let \( x_0 \in [a, b] \). We introduce the infinite system of functions \( \{\varphi_k\}_{k=0}^{\infty} \) defined as follows

\[
\varphi_k(x) = \begin{cases} 
  f(x) X^{(k)}(x), & k \text{ odd}, \\
  f(x) \tilde{X}^{(k)}(x), & k \text{ even}, 
\end{cases} 
\]

where

\[
X^{(0)}(x) \equiv 1, \quad X^{(n)}(x) = n \int_{x_0}^{x} X^{(n-1)}(s) \left(f^2(s)\right)^{(-1)^n} ds, \quad n = 1, 2, \ldots
\]

and

\[
\tilde{X}^{(0)}(x) \equiv 1, \quad \tilde{X}^{(n)}(x) = n \int_{x_0}^{x} \tilde{X}^{(n-1)}(s) \left(f^2(s)\right)^{(-1)^n} ds, \quad n = 1, 2, \ldots
\]

Together with the system of functions (1) we define the functions \( \{\psi_k\}_{k=0}^{\infty} \) using the “second half” of the recursive integrals,

\[
\psi_k(x) = \begin{cases} 
  \tilde{X}^{(k)}(x), & k \text{ odd}, \\
  \frac{\tilde{X}^{(k)}(x)}{X^{(k)}(x)}, & k \text{ even}. 
\end{cases} 
\]
As we show below the introduced families of functions are closely related to the one-dimensional Schrödinger equation of the form $u'' - qu = \lambda u$ where $q$ is a complex-valued continuous function. Slightly more general families of functions can be studied in relation to Sturm-Liouville equations of the form $(py')' + qy = \lambda ry$. Their definition based on a corresponding recursive integration procedure is given in [11], [12], [8]. The system (1) is closely related to the notion of the $L$-basis introduced and studied in [7].

The following result obtained in [10] (for additional details and simpler proof see [11] and [12]) establishes the relation of the system of functions $\{\varphi_k\}_{k=0}^{\infty}$ and $\{\psi_k\}_{k=0}^{\infty}$ to the Sturm-Liouville equation.

**Theorem 1** Let $q$ be a continuous complex valued function of an independent real variable $x \in [a,b]$ and $\lambda$ be an arbitrary complex number. Suppose there exists a solution $f$ of the equation

\[ f'' - qf = 0 \quad (2) \]

on $(a,b)$ such that $f \in C^2(a,b) \cap C^1[a,b]$ and $f(x) \neq 0$ for any $x \in [a,b]$. Then the general solution $u \in C^2(a,b) \cap C^1[a,b]$ of the equation

\[ u'' - qu = \lambda u \quad (3) \]

on $(a,b)$ has the form $u = c_1 u_1 + c_2 u_2$ where $c_1$ and $c_2$ are arbitrary constants,

\[ u_1 = \sum_{k=0}^{\infty} \frac{\lambda^k}{(2k)!} \varphi_{2k} \quad \text{and} \quad u_2 = \sum_{k=0}^{\infty} \frac{\lambda^k}{(2k+1)!} \varphi_{2k+1} \quad (4) \]

and both series converge uniformly on $[a,b]$ together with the series of the first derivatives which have the form

\[ u'_1 = f' + \sum_{k=1}^{\infty} \frac{\lambda^k}{(2k)!} \left( \frac{f'}{f} \varphi_{2k} + 2k \psi_{2k-1} \right) \quad \text{and} \quad \]

\[ u'_2 = \sum_{k=0}^{\infty} \frac{\lambda^k}{(2k+1)!} \left( \frac{f'}{f} \varphi_{2k+1} + (2k+1) \psi_{2k} \right). \]

The series of the second derivatives converge uniformly on any segment $[a_1, b_1] \subset (a,b)$.

The representation (4) offers the linearly independent solutions of (3) in the form of spectral parameter power series (SPPS). The way of how the expansion coefficients in (4) are calculated via the recursive integration is relatively simple and straightforward, this is why the estimation of the rate of convergence of the series (4) presents no difficulty, see [12]. Moreover, in [2] a discrete analogue of Theorem 1 was established and the discrete analogues of the series (4) resulted to be finite sums.
Remark 2  It is easy to see that the solutions $u_1$ and $u_2$ defined by (4) satisfy the following initial conditions

\[
  u_1(x_0) = f(x_0), \quad u_1'(x_0) = f'(x_0), \\
  u_2(x_0) = 0, \quad u_2'(x_0) = 1/f(x_0).
\]

Remark 3  It is worth mentioning that in the regular case the existence and construction of the required $f$ presents no difficulty. Let $q$ be real valued and continuous on $[a,b]$. Then (2) possesses two linearly independent regular solutions $v_1$ and $v_2$ whose zeros alternate. Thus one may choose $f = v_1 + iv_2$. Moreover, for the construction of $v_1$ and $v_2$ in fact the same SPPS method may be used (12).

The SPPS representation (4) for solutions of the Sturm-Liouville equation (3) is very convenient for writing down the dispersion (characteristic) relations in an analytical form. This fact was used in [1], [5], [8], [9], [12], [15] for approximating solutions of different spectral and scattering problems.

2 Transmutation operators

Let $E$ be a linear topological space and $E_1$ its linear subspace (not necessarily closed). Let $A$ and $B$ be linear operators: $E_1 \to E$.

Definition 4  A linear invertible operator $T$ defined on the whole $E$ such that $E_1$ is invariant under the action of $T$ is called a transmutation operator for the pair of operators $A$ and $B$ if it fulfills the following two conditions.

1. Both the operator $T$ and its inverse $T^{-1}$ are continuous in $E$;
2. The following operator equality is valid

\[
  AT = TB \tag{5}
\]

or which is the same

\[
  A = TBT^{-1}.
\]

Very often in literature the transmutation operators are called the transformation operators. Here we keep ourselves to the original term coined by Delsarte and Lions [6]. Our main interest concerns the situation when $A = -\frac{d^2}{dx^2} + q(x)$, $B = -\frac{d^2}{dx^2}$, and $q$ is a continuous complex-valued function. Hence for our purposes it will be sufficient to consider the functional space $E = C[a,b]$ with the topology of uniform convergence and its subspace $E_1$ consisting of functions from $C^2[a,b]$. One of the possibilities to introduce a transmutation operator on $E$ (see, e.g., [17]) consists in constructing a Volterra integral operator corresponding to a midpoint of the segment of interest. Then it is convenient to consider a symmetric segment $[-a,a]$ and hence the functional space $E = C[-a,a]$. It is worth mentioning that other well known ways to construct the transmutation
operators (see, e.g., [16], [20]) imply imposing initial conditions on the functions and consequently lead to transmutation operators satisfying (5) only on subclasses of $E_1$.

Thus, we consider the space $E = C[-a,a]$ and an operator of transmutation for the defined above $A$ and $B$ can be realized in the form (see, e.g., [16] and [17]) of a Volterra integral operator

$$ Tu(x) = u(x) + \int_{-x}^{x} K(x,t)u(t)dt $$

where the kernel $K(x,t)$ is defined as a solution of a Goursat problem [17, Chapter 1]. If the potential $q$ is $n$ times continuously differentiable, the kernel $K(x,t)$ is $n + 1$ times continuously differentiable with respect to both independent variables.

In [3] a parametrized family of operators $T_h$, $h \in \mathbb{C}$ was introduced, given by the integral expression

$$ T_h u(x) = u(x) + \int_{-x}^{x} K(x,t;h)u(t)dt $$

(6)

where

$$ K(x,t;h) = \frac{h}{2} + K(x,t) + \frac{h}{2} \int_{-x}^{x} (K(x,s) - K(x,-s)) ds. $$

(7)

The operator $T_h$ maps a solution $v$ of an equation $v'' + \omega^2 v = 0$, where $\omega$ is a complex number, into a solution $u$ of the equation $u'' - q(x)u + \omega^2 u = 0$ with the following correspondence of the initial values

$$ u(0) = v(0), \quad u'(0) = v'(0) + hv(0). $$

(8)

**Theorem 5** [14] Suppose the potential $q$ satisfies either of the following two conditions: 1) $q \in C^1[-a,a]$; 2) $q \in C[-a,a]$ and there exists a particular complex-valued solution $g$ of (2) non-vanishing on $[-a,a]$. Then the operator $T_h$ given by (6) satisfies the equality

$$ \left( -\frac{d^2}{dx^2} + q(x) \right) T_h [u] = T_h \left[ -\frac{d^2}{dx^2} (u) \right] $$

for any $u \in C^2[-a,a]$.

Suppose now that a function $f$ is a solution of (2), non-vanishing on $[-a,a]$ and normalized as $f(0) = 1$. Let $h := f'(0)$ be some complex constant. Define as before the system of functions $\{\varphi_k\}_{k=0}^{\infty}$ by this function $f$ and by (1). The following theorem states that the operator $T_h$ transmutes powers of $x$ into the functions $\varphi_k$.

**Theorem 6** [3] Let $q$ be a continuous complex valued function of an independent real variable $x \in [-a,a]$, and $f$ be a particular solution of (2) such that $f \in C^2(-a,a)$ together with $1/f$ are bounded on $[-a,a]$ and normalized as...
$f(0) = 1$, and let $h := f'(0)$, where $h$ is a complex number. Then the operator (6) with the kernel defined by (7) (with this particular $h$) transforms $x^k$ into $\varphi_k(x)$ for any $k \in \mathbb{N}_0$.

Thus, the system of functions $\{\varphi_k\}$ may be obtained as the result of the Volterra integral operator acting on powers of the independent variable. As was mentioned before, this offers an algorithm for transmuting functions in situations when the explicit form of $K(x; t; h)$ is unknown. Moreover, properties of the Volterra integral operator such as boundedness and bounded invertibility in many functional spaces gives us a tool to prove the completeness of the system of function $\{\varphi_k\}$ in various situations. For example, the system $\{\varphi_k\}_{k=0}^{\infty}$ is complete in $C[-a, a]$.

3 A new method for solving spectral problems

Theorem 6 together with (8) suggests the following approach for solving spectral problems for the Sturm-Liouville equation (3). Here as an example we consider the Sturm-Liouville problem with the following boundary conditions though obviously the method can be applied in a much more general situation. Thus, consider the equation

$$u'' - qu = -\beta^2 u$$

$(-\beta^2 = \lambda)$ with the conditions

$$u(0) = u(1) = 0.$$  

(10)

It is well known that an almost optimal uniform approximation of the functions sine and cosine on the segment $[-1, 1]$ by polynomials is achieved using the Chebyshev polynomials. The corresponding representations have the form [19, p. 104]

$$\sin \beta x = 2 \sum_{m=0}^{\infty} (-1)^m J_{2m+1}(\beta) T_{2m+1}(x),$$

(11)

$$\cos \beta x = J_0(\beta) + 2 \sum_{m=0}^{\infty} (-1)^m J_{2m}(\beta) T_{2m}(x)$$

(12)

where $J_n$ denotes the Bessel function of the first kind and $T_n$ is the Chebyshev polynomial of $n$-th order, $T_n(x) = \cos(n \arccos x)$.

From (3) we have that $u(x) = T_h(\sin \beta x)$ is a solution of (9) satisfying the first of the conditions (10). Thus, $\beta^2$ is an eigenvalue of (3), (10) iff $T_h(\sin \beta x)$ at the point $x = 1$ equals zero. Notice that

$$T_h(\sin \beta x) = 2 \sum_{m=0}^{\infty} (-1)^m J_{2m+1}(\beta) T_h(T_{2m+1}(x))$$

$$\approx 2 \sum_{m=0}^{N} (-1)^m J_{2m+1}(\beta) T_h(T_{2m+1}(x)).$$

(13)
In the last expression it is not difficult to calculate $T_h(T_{2m+1}(x))$ with a very good accuracy applying theorem 6. Every $T_h(T_{2m+1}(x))$ is a linear combination of the functions $\varphi_k$, $k = 0, \ldots, 2m + 1$ and the coefficient corresponding to $\varphi_k$ is the same as the coefficient corresponding to $x^k$. Thus, computation of the images of the Chebyshev polynomials under the action of the transmutation operator $T_h$ does not represent any difficulty. Taking into account a very fast convergence of the series in (12) one can obtain a very good accuracy in the approximation of $T_h(\sin \beta x)$ with a relatively small $N$ in (13).

Thus, the proposed method for solving (9), (10) consists of the following steps: 1) compute a solution $f$ of (2) satisfying the conditions of Theorem 1; 2) compute $N + 1$ functions $\varphi_k$; 3) taking the coefficients of $x^k$, $k = 0, \ldots, 2m + 1$ from the expression of $T_{2m+1}$ compute $T_h(T_{2m+1})$ evaluated at $x = 1$; 4) find zeros of $\sum_{m=0}^{N} (-1)^m J_{2m+1}(\beta) T_h(T_{2m+1}) \mid_{x=1}$.

**Example** $q(x) = e^x$, $u(0) = u(\pi) = 0$. First, making an obvious change of variables we reduce the problem to the interval $(0, 1)$ and then compare our results with the eigenvalues from [18, Appendix A]. Taking $N = 18$ we obtain the results reported in the following table. The recursive integration involved in the construction of the functions $\varphi_k$ required in (13) was performed by means of the Spline Toolbox of Matlab. On each step the integrand was converted into a spline and then integrated using the command `fnint`. A surprising accuracy with a very small number of functions participating in (13) was achieved in all the considered numerical tests. Here due to the restrictions in space we illustrate this with the only example in which $N = 18$. A phenomenon which we observed but up to now have not found an explanation is a surprisingly good approximation of higher order eigenvalues as can be seen from the table. Apparently the expression (13) not only approximates well the solution of the equation for relatively small values of $\beta$ but also evaluated at $x = 1$ contains information on the asymptotics of the eigenvalues of the problem. It is worth mentioning that on a usual netbook computer the computation of the eigenvalues (see the table below) implemented in the Matlab takes only several seconds.
| Order of the eigenvalue | Eigenvalues from [18] | Computed eigenvalues |
|-------------------------|------------------------|----------------------|
| 1                       | 4.8966693800           | 4.896669377          |
| 2                       | 10.045189893           | 10.045189883         |
| 3                       | 16.019267250           | 16.019267268         |
| 4                       | 23.266270940           | 23.266270969         |
| 5                       | 32.26370704            | 32.26370710          |
| 6                       | 43.2200196             | 43.2200184           |
| 7                       | 56.18159               | 56.18161             |
| 8                       | 71.15299               | 71.15255             |
| 9                       | 88.1321                | 88.1386              |
| 10                      | 107.11                 | 107.05               |
| 11                      | 128.10                 | 128.48               |
| 17                      | 296.07                 | 296.52               |
| 28                      | 791.05                 | 790.99               |
| 43                      | 1856.05                | 1856.53              |
| 50                      | 2507                   | 2500                 |

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