The Monomial-Divisor Mirror Map

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Abstract

For each family of Calabi-Yau hypersurfaces in toric varieties, Batyrev has proposed a possible mirror partner (which is also a family of Calabi-Yau hypersurfaces). We explain a natural construction of the isomorphism between certain Hodge groups of these hypersurfaces, as predicted by mirror symmetry, which we call the monomial-divisor mirror map. We indicate how this map can be interpreted as the differential of the expected mirror isomorphism between the moduli spaces of the two Calabi-Yau manifolds. We formulate a very precise conjecture about the form of that mirror isomorphism, which when combined with some earlier conjectures of the third author would completely specify it. We then conclude that the moduli spaces of the nonlinear sigma models whose targets are the different birational models of a Calabi-Yau space should be connected by analytic continuation, and that further analytic continuation should lead to moduli spaces of other kinds of conformal field theories. (This last conclusion was first drawn by Witten.)

1 Reflexive polyhedra

Mirror symmetry, which proposes that Calabi-Yau manifolds should come in pairs with certain remarkable properties, is a phenomenon that was first observed in the physics literature [20, 31, 13, 28]. The most concrete realization of this phenomenon—actually the only one in which there is a physical argument linking the conformal

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1For general mathematical background on mirror symmetry and mirror pairs, we refer the reader to [43] and [33].
field theories associated to the pair of Calabi-Yau manifolds—is given by the Greene-Plesser orbifolding construction \[28\] for Fermat hypersurfaces in weighted projective spaces and certain quotients of them by finite groups. Roan \[39, 40\] has given a natural description of this construction in terms of toric geometry, and he showed that the mirror phenomenon in that case can be interpreted as a kind of duality between toric hypersurfaces. This enabled him to give rigorous mathematical proofs of certain formulas discovered by physicists.

Batyrev \[9\] has recently found an elegant characterization of Calabi-Yau hypersurfaces which are ample Cartier divisors in (mildly singular) toric varieties. The characterization is stated in terms of the Newton polyhedron of the hypersurface, which is the convex hull of the monomials appearing in its equation. This is always an integral polyhedron, that is, a compact convex polyhedron \(P\) whose vertices are elements of a lattice \(M\) in a real affine space \(M_\mathbb{R} := M \otimes \mathbb{R}\). Batyrev’s characterization states that the general hypersurface with Newton polyhedron \(P\) is Calabi-Yau (that is, has trivial canonical bundle and at worst Gorenstein canonical singularities), provided that 0 is in the interior of \(P\), and that each affine hyperplane \(H \subset M_\mathbb{R}\) which meets \(P\) in a face of codimension one has the form

\[H := \{y \in M_\mathbb{R} \mid \langle \ell, y \rangle = -1\}\]

for some \(\ell = \ell(H)\) in the dual lattice \(N := \text{Hom}(M, \mathbb{Z})\). An integral polyhedron with this property is called reflexive.

The normals \(\ell(H)\) of supporting hyperplanes \(H\) for codimension-one faces of a reflexive polyhedron \(P\) have as their convex hull the polar polyhedron \(P^\circ\), which is defined to be

\[P^\circ := \{x \in N_\mathbb{R} \mid \langle x, y \rangle \geq -1 \text{ for all } y \in P\}\]

Batyrev showed that the polar polyhedron \(P^\circ\) of a reflexive integral polyhedron \(P\) is itself a reflexive integral polyhedron (with respect to the dual lattice \(N\)). This led him to propose that hypersurfaces \(X\) and \(Y\) with Newton polyhedra \(P\) and \(P^\circ\), respectively, should form a mirror pair.

The evidence for Batyrev’s proposal is of several kinds. First and foremost is the fact that this polar polyhedron construction specializes to Roan’s interpretation of the Greene-Plesser orbifolding construction in the case of quotients of Fermat hypersurfaces in weighted projective spaces. This is encouraging, since as noted above the Greene-Plesser construction provides the only complete example of mirror symmetry for hypersurfaces—the only example for which there is a physical argument for the existence of a mirror isomorphism of the corresponding conformal field theories. A second piece of evidence (which we discuss more fully below) is an
isomorphism between certain Hodge groups associated to $X$ and $Y$ (extending the work of Roan), as would be predicted by mirror symmetry. And finally, Batyrev shows that his construction is compatible with the existence of certain “quantum symmetries” as expected based on physical reasoning. This quantum symmetry behavior looks somewhat unnatural mathematically, so verifying it is an important check.

This evidence falls short of fully establishing a mirror symmetry relationship between $X$ and $Y$, since it does not link the corresponding conformal field theories. However, it does provide strong grounds for suspecting the existence of a mirror isomorphism. And the naturality of Batyrev’s polar polyhedron construction is extremely compelling (at least to mathematicians).

If mirror symmetry does hold between $X$ and $Y$, there will be an isomorphism between Hodge groups $H^{1,1}(\hat{X})$ and $H^{d-1,1}(\hat{Y})$, where $\hat{X} \to X$ and $\hat{Y} \to Y$ are appropriate (partial) resolutions of singularities, and $d$ is the common dimension of $X$ and $Y$. The existence of such an isomorphism had been shown quite explicitly by Roan [39, 40] in the weighted Fermat hypersurface case—the general case is addressed by Batyrev. In the earlier preprint versions of [9], Batyrev found an equality between the dimensions of certain subspaces $H^{1,1}_{\text{toric}}(\hat{X})$ and $H^{d-1,1}_{\text{poly}}(\hat{Y})$ of the Hodge groups, mistakenly believed to have been the entire spaces. In the final version of [9], he shows that the full Hodge groups are isomorphic, following suggestions made by the present authors. The error in the earlier version of the paper was fortuitous, however, as it revealed that the mirror isomorphism might be expected to preserve those subspaces.

In this note, we explain a very natural construction of the isomorphism between $H^{1,1}_{\text{toric}}(\hat{X})$ and $H^{d-1,1}_{\text{poly}}(\hat{Y})$, and indicate how it can be interpreted as the differential of the expected mirror map between the moduli spaces (when restricted to appropriate subspaces of those moduli spaces). The space $H^{d-1,1}_{\text{poly}}(\hat{Y})$ is isomorphic to the space of first-order polynomial deformations of $Y$, and can be generated by monomials; the space $H^{1,1}_{\text{toric}}(\hat{X})$ consists of that part of the second cohomology of $\hat{X}$ coming from the ambient toric variety, and can be generated by toric divisors. Our map comes from a natural one-to-one correspondence between monomials and toric divisors whose definition is inspired by the constructions of Roan and Batyrev; we have named it the monomial-divisor mirror map.
2 Divisors

Our first task is to describe the partial resolutions of singularities we will use, and the divisors on them. Let $\Delta$ be a fan determining a toric variety (see \[37\] or \[25\] for the definitions, and for proofs of the facts we review below). The support $|\Delta|$ of $\Delta$ is a subset of a real vector space $N_\mathbb{R}$, and the convex cones $\sigma$ in the fan $\Delta$ are rational polyhedral cones with respect to a lattice $N$ in $N_\mathbb{R}$; the algebraic torus which acts on the toric variety is $T := N \otimes \mathbb{C}^*$. We let $\Delta(1)$ denote the set of one-dimensional cones in $\Delta$. There is a natural generator map $\text{gen} : \Delta(1) \to N$ which assigns to each one-dimensional cone $\rho$ the unique generator $\text{gen}(\rho)$ of the semigroup $\rho \cap N$. Each such $\rho$ also has an associated $T$-invariant Weil divisor $D_\rho$ in the toric variety, which is the closure of the $T$-orbit corresponding to the cone $\rho$.

One can always describe a projective toric variety by beginning with a compact convex polyhedron $P$ in a real affine space $M_\mathbb{R}$, integral with respect to a lattice $M$ in $M_\mathbb{R}$. The projective toric variety is then determined by the normal fan $N(P)$ consisting of all cones $N(P,p)$ to $P$ at $p \in P$, where 

$$N(P,p) := \{ x \in N_\mathbb{R} \mid \langle x, p \rangle \leq \langle x, y \rangle \text{ for all } y \in P \},$$

and where $N := \text{Hom}(M, \mathbb{Z})$ is the dual lattice, and $N_\mathbb{R} := N \otimes \mathbb{R}$. Each proper face of the polar polyhedron $P^\circ$ of $P$ is contained in a unique cone in $N(P)$, which is the cone over that face.

The toric variety $V$ determined by the fan $N(P)$ is the natural one in which the hypersurfaces $X$ with Newton polyhedron $P$ are ample divisors. In the case of a reflexive integral polyhedron $P$, the general such $X$ is an anti-canonical divisor in $V$ and will be a Calabi-Yau variety with canonical singularities, as proved by Batyrev \[9\]. We need to partially resolve the singularities of $V$ while retaining the triviality of the canonical bundle of the hypersurface, getting as close as possible to a complete resolution.

To do this, construct a blowup $\hat{V} \to V$, determined by a fan $\hat{\Delta}$ which is a subdivision of the fan $N(P)$. There will be an induced blowup $\hat{X} \to X$ of hypersurfaces, where $\hat{X}$ is the proper transform of $X$ on $\hat{V}$. In order to maintain the triviality of the canonical bundle of $\hat{X}$, restrict the set $\Delta(1)$ of one-dimensional cones as follows:

the image $\Xi := \text{gen}(\Delta(1))$ of $\Delta(1)$ in $N$ should lie in the set $P^\circ \cap N$, where $P^\circ$ is the polar polyhedron of $P$. (We will sometimes restrict $\Xi$ to lie in the subset $(P^\circ \cap N)_0 \subset P^\circ \cap N$ consisting of those lattice points in $P^\circ$ which do not lie in the interior of a codimension-one face of $P^\circ$.) Oda and Park \[38\] (cf. also \[43\]) have shown the existence of a simplicial subdivision $\Delta$ of the fan $N(P)$ such that

4
\[ \text{gen}(\Delta(1)) = (P^o \cap N) - \{0\} \] (or any subset thereof), and such that the corresponding \( \hat{V} \) is projective. In general, there will be many such fans \( \Delta \).

Since the fan \( \Delta \) is simplicial, the toric variety \( \hat{V} \) has the structure of an orbifold (formerly called \( \hat{V} \)-manifold \( [12] \)): it can be covered by open sets of the form \( U/G_U \) where \( G_U \) is a finite group acting on a manifold \( U \) such that the fixed locus of any \( 1 \neq g \in G_U \) has real codimension at least 2. The open sets \( U/G_U \) are used to define the notion of orbifold-smooth differential forms, pieced together from \( G_U \)-invariant smooth forms on the open sets \( U \). Many of the theorems about the differential geometry of smooth algebraic varieties have natural orbifold versions. In particular, there are orbifold de Rham cohomology groups \( H^k_{\text{DR}}(\hat{V}, \mathbb{R}) \) isomorphic to the real Čech cohomology \( [42] \), and orbifold Hodge groups \( H^{p,q}(\hat{V}) \) which satisfy a version of the Dolbeault theorem \( [7] \). The general hypersurface \( \hat{X} \subset \hat{V} \) is also an orbifold, and has orbifold de Rham and Hodge groups of its own.

We can describe the group \( \text{WDiv}_T(\hat{V}) \) of toric Weil divisors on \( \hat{V} \) and their images in the Chow group \( A_{n-1}(\hat{V}) \) (where \( n = d + 1 \) is the dimension of \( \hat{V} \)), as follows (cf. Cox \( [16] \)). There is a natural isomorphism

\[ \alpha: \mathbb{Z}^\Xi \to \text{WDiv}_T(\hat{V}) \]

which sends the function \( \varphi \in \mathbb{Z}^\Xi \) to the divisor

\[ \sum \varphi_a D_{\text{gen}-1(a)} \].

Under this isomorphism, if we define an embedding \( \text{ad}_\Xi : M \to \mathbb{Z}^\Xi \) by sending \( m \in M \) to the function \( \text{ad}_\Xi(m) \) defined by \( \text{ad}_\Xi(m) : a \mapsto \langle a, m \rangle \), then

\[ \text{div}(\chi^m) = -\alpha(\text{ad}_\Xi(m)) \],

where \( \chi^m : T \to \mathbb{C}^* \) is the character of \( T \) associated to \( m \), regarded as a meromorphic function on \( \hat{V} \). Thus, \( M \) gives rise to linear equivalences among toric divisors. In fact, there is an exact sequence

\[ 0 \to M \xrightarrow{\text{ad}_\Xi} \mathbb{Z}^\Xi \xrightarrow{\bar{\alpha}} A_{n-1}(\hat{V}) \to 0, \]

where \( \bar{\alpha} \) denotes the composite of \( \alpha \) with the projection to the Chow group. This is nothing other than the usual description of \( A_{n-1} \) as “divisors modulo linear equivalence”, since \( \mathbb{Z}^\Xi \) represents toric divisors and \( M \) represents the linear equivalences among them.

\[ \text{We use the notation } \mathbb{Z}\langle S \rangle \text{ for the free abelian group on the set } S, \text{ and } \mathbb{Z}^S \text{ for the } \mathbb{Z}\text{-module of maps from } S \text{ to } \mathbb{Z}, \text{ which is naturally isomorphic to the dual lattice } \text{Hom}(\mathbb{Z}\langle S \rangle, \mathbb{Z}) \text{ of } \mathbb{Z}\langle S \rangle. \text{ The map determined by } \varphi \in \mathbb{Z}^S \text{ is denoted by } s \mapsto \varphi_s. \]
To compute the group of toric divisors on the hypersurface $\hat{X}$, we use the natural restriction maps from divisors on $\hat{V}$ (which exists since each toric divisor on $\hat{V}$ meets $\hat{X}$ in a subvariety of codimension 1):

$$
\begin{array}{cccc}
0 & \rightarrow & M & \rightarrow \ W\text{Div}_T(\hat{V}) & \rightarrow & A_{n-1}(\hat{V}) & \rightarrow & 0 \\
\| & & \downarrow & & \downarrow & & \\
0 & \rightarrow & M & \rightarrow \ W\text{Div}_T(\hat{X}) & \rightarrow & A_{d-1}(\hat{X}) & & .
\end{array}
$$

This time, the toric divisors need not generate the entire Chow group; we denote the image of $W\text{Div}_T(\hat{X})$ in $A_{d-1}(\hat{X})$ by $A_{d-1}(\hat{X})_{\text{toric}}$. Its complexification we call the toric part of $H^{1,1}$, denoted by $H^{1,1}_{\text{toric}}(\hat{X}) := A_{d-1}(\hat{X})_{\text{toric}} \otimes \mathbb{C}$.

The kernel of the restriction map $W\text{Div}_T(\hat{V}) \rightarrow W\text{Div}_T(\hat{X})$ is easy to describe. A divisor with trivial restriction must be supported on divisors which are disjoint from the general hypersurface $\hat{X} \subset \hat{V}$. Since the line bundle $\mathcal{O}_V(X)$ is generated by its global sections, the general hypersurface $X \subset V$ will not meet the zero-dimensional strata of $V$ (in the stratification by $T$-orbits). So any divisor on $\hat{V}$ which maps to such a stratum will be disjoint from $\hat{X}$, the proper transform of $X$. Such divisors are characterized by the property that the corresponding point in $\Xi$ lies in the interior of some codimension-one face of $P^o$. Other toric divisors on $\hat{V}$ cannot be disjoint from $\hat{X}$, since they map to larger strata of $V$ which are not disjoint from $X$.

Thus, if we let $\Xi_0 = \Xi \cap (P^o \cap N)_0$ be the subset of $\Xi$ consisting of all points which do not lie in interiors of codimension-one faces of $P^o$, we find that $W\text{Div}_T(\hat{X}) \cong \mathbb{Z}^{\Xi_0}$ and that

$$
A_{d-1}(\hat{X})_{\text{toric}} \cong \text{Coker}(ad_{\Xi_0}) \cong \mathbb{Z}^{\Xi_0} / M. \quad (2)
$$

In particular, if $\Xi \supset (P^o \cap N)_0 - \{0\}$ then $A_{d-1}(\hat{X})_{\text{toric}} \cong \mathbb{Z}^{(P^o \cap N)_0 - \{0\}} / M$, and hence

$$
H^{1,1}_{\text{toric}}(\hat{X}) \cong (\mathbb{Z}^{(P^o \cap N)_0 - \{0\}} / M) \otimes \mathbb{C}. \quad (3)
$$

### 3 Monomials

Our task in this section is to describe moduli spaces for hypersurfaces in $\hat{V}$. We retain the notation of the previous section: $\hat{V}$ is the toric variety associated to a subdivision $\Delta$ of the normal fan $\mathcal{N}(P)$ of a reflexive polyhedron $P$. We assume that $\Delta$ is simplicial, so that $\hat{V}$ is $\mathbb{Q}$-factorial; we also assume that $\hat{V}$ is projective.

Given a hypersurface $\hat{X} \subset \hat{V}$, the space of first order deformations of complex structure of $\hat{X}$ is isomorphic to $H^1(\Theta_{\hat{X}})$. The simplest way to deform the complex structure on $\hat{X}$ is to perturb the equation of the hypersurface; this leads to
a subspace $H^1(\Theta_{\hat{X}})_{\text{poly}} \subset H^1(\Theta_{\hat{X}})$ of polynomial first-order deformations. (It is quite possible for this to be a proper subspace \[27\].) In the case that $\hat{X} \subset \hat{V}$ is a Calabi-Yau hypersurface, we can use the isomorphism $H^1(\Theta_{\hat{X}}) \cong H^{d-1,1}(\hat{X})$ to also specify a “polynomial” subspace $H^d_{\text{poly}}(\hat{X}) \subset H^{d-1,1}(\hat{X})$ of the corresponding Hodge group.

In principle, the moduli spaces of the hypersurfaces $\{\hat{X}\}$ should be fairly easy to describe. Global sections of $\mathcal{O}_\hat{V}(\hat{X})$ provide equations for the hypersurfaces, and the entire family can be described as $\mathbb{P}H^0(\mathcal{O}_\hat{V}(\hat{X}))$. But we need to mod out by automorphisms of $\hat{V}$, and this is where technical complications arise.

Let $D$ be a toric divisor on $\hat{V}$, and write $D = \sum_{a \in \Xi} d_a D_{\text{gen}^{-1}(a)}$. There is a natural isomorphism between $H^0(\mathcal{O}(D))$ and the space $\mathbb{C}^{P_D \cap M}$, where $P_D$ is the polytope

$$P_D := \{ y \in M_\mathbb{R} \mid \langle a, y \rangle \geq -d_a \text{ for all } a \in \Xi\}$$

(cf. \[25\]). In fact, if we identify $H^0(\mathcal{O}(D))$ with the space of meromorphic functions on $\hat{V}$ which have (at worst) poles along $D$, then to each $m \in P_D \cap M$ we can associate the meromorphic function $\chi^m$: it has at worst poles along $D$ thanks to the definition of $P_D$. In the special case $D = \sum_{\rho \in \Delta^{(1)}} D_{\rho} \in | -K_{\hat{V}} |$, the polytope $P_{\Sigma D_{\rho}}$ coincides with the original polyhedron $P \subset M_\mathbb{R}$ used to describe $\hat{V}$.

The automorphism group $\text{Aut}(\hat{V})$ of a $\mathbb{Q}$-factorial toric variety $\hat{V}$ has been described recently by Cox \[16\], generalizing some results from the smooth case due to Demazure \[19\]. Cox’s description is in terms of a central extension of $\text{Aut}(\hat{V})$ by a torus $G$, which fits in an exact sequence

$$1 \rightarrow G \rightarrow \tilde{\text{Aut}}(\hat{V}) \rightarrow \text{Aut}(\hat{V}) \rightarrow 1,$$

where $G := \text{Hom}(A_{n-1}(\hat{V}), \mathbb{C}^*)$. The advantage of working with $\tilde{\text{Aut}}(\hat{V})$ is that it acts naturally on all cohomology groups $H^0(\mathcal{O}(D))$ at once. The clearest way to see these actions is to follow Cox again and introduce the homogeneous coordinate ring $S := \mathbb{C}[x_a]_{(a \in \Xi)}$ of $\hat{V}$. This ring can be (multi) graded by defining the degree of the monomial $\prod x_a^\phi$ to be the divisor class $[\sum \varphi_a D_{\text{gen}^{-1}(a)}]$ in $A_{n-1}(\hat{V})$. For a fixed divisor $D$, the set of elements of degree $[D]$ in the homogeneous coordinate ring can be identified with $H^0(\mathcal{O}(D))$: the meromorphic function $\chi^m$ with $m \in P_D \cap M$ corresponds to the homogeneous monomial $x^{\text{div}(\chi^m)+D} := \prod x_a^{(a,m)+d_a}$.

The torus $T := \text{Hom}(M, \mathbb{C}^*)$ which acts on $\hat{V}$ is naturally a subgroup of $\text{Aut}(\hat{V})$; the induced extension $\tilde{T}$ of $T$ by $G$ has the form $\tilde{T} := \text{Hom}(\mathbb{Z}\Xi, \mathbb{C}^*)$. In fact, if we apply the functor $\text{Hom}(\_ , \mathbb{C}^*)$ to the natural exact sequence \[1\], we get a sequence
for $\tilde{T}$ which fits as the first row in the commutative diagram

$$
\begin{array}{c}
1 \rightarrow G \rightarrow \tilde{T} \rightarrow T \rightarrow 1 \\
\| \cap \cap \cap \cap \cap \\
1 \rightarrow G \rightarrow \tilde{\text{Aut}(\hat{V})} \rightarrow \text{Aut}(\hat{V}) \rightarrow 1
\end{array}
$$

The grading of the homogeneous coordinate ring $S$ can also be described in terms of the action of $G$ on $S$. The torus $\tilde{T}$ acts on $S$ in a transparent way: each monomial in $S$ can be written in the form $x^\varphi = \prod x_a^{\varphi_a}$ for some $\varphi \in \mathbb{Z}^\Xi$, and the action of $t \in \text{Hom}(\mathbb{Z}^\Xi, \mathbb{C}^*)$ sends $x^\varphi$ to $t(\varphi) \cdot x^\varphi$. When we restrict this action to the subgroup $G = \text{Hom}(\mathbb{A}_{n-1}(\hat{V}, \mathbb{C}^*))$, then for each divisor class $[D]$, the subspace of $S$ on which $G$ acts via the character $\gamma \mapsto \gamma([D])$ is precisely $H^0(O(D)) \cong \bigoplus \mathbb{C} \cdot x^{\text{div}(\gamma^m) + D}$.

The induced action of $t \in \tilde{T}$ on $H^0(O(D))$ then sends $\chi^m$ to $t(\alpha^{-1}(\text{div}(\chi^m) + D)) \cdot \chi^m$, for every $m \in P_D \cap M$. This action can be described in terms of the map $\mathbb{Z}\langle P_D \cap M \rangle \rightarrow \mathbb{Z}^\Xi$ defined by

$$
m \mapsto (a \mapsto \langle a, m \rangle + d_a),
$$

which induces a homomorphism of tori $\tilde{T} \rightarrow (\mathbb{C}^*)^{P_D \cap M}$ that determines the action of $\tilde{T}$ on $\mathbb{C}^{P_D \cap M}$. Notice that the map (5) factors as a composite of two maps

$$
\mathbb{Z}\langle P_D \cap M \rangle \rightarrow M \oplus \mathbb{Z} \rightarrow \mathbb{Z}^\Xi
$$

with the first map given by $m \mapsto (m, 1)$ and the second map given by $(m, k) \mapsto \text{ad}_{\xi}(m) + k \cdot \alpha^{-1}(D)$. The corresponding homomorphism of tori factors as

$$
\tilde{T} \rightarrow T \times \mathbb{C}^* \rightarrow (\mathbb{C}^*)^{P_D \cap M}.
$$

In the special case $D = \sum D_\rho$, the induced map $\mathbb{C}^* \rightarrow (\mathbb{C}^*)^{P_D \cap M}$ is simply the diagonal embedding.

The groups $\text{Aut}(\hat{V})$ and $\tilde{\text{Aut}(\hat{V})}$ are not in general reductive. Thus, to construct moduli spaces for hypersurfaces on $\hat{V}$, we should use Fauntleroy’s extension of Mumford’s Geometric Invariant Theory (GIT) and attempt to construct a quotient for the action of $\tilde{\text{Aut}(\hat{V})}$ on $H^0(O(D))$. It would be interesting to know if this construction of moduli spaces can be carried out in general—Fauntleroy has carried it out in some special cases. We can at least obtain a birational model

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[^1]: Batyrev [10] has constructed moduli spaces for affine hypersurfaces, obtaining a somewhat different space than ours if $(P \cap M)_0 \neq (P \cap M)$ (it even has a different dimension). Batyrev and Cox [12] have recently considered a construction similar to the one described here.
of the desired moduli space by using a fairly standard result (cf. [41, 17]) which guarantees the existence of an \( \tilde{\text{Aut}}(\hat{V})\)-stable Zariski-open set \( U \subset H^0(\mathcal{O}(D)) \) such that the geometric quotient \( U/\tilde{\text{Aut}}(\hat{V}) \) exists. We indicate the birational class of such quotients with the notation \( H^0(\mathcal{O}(D))/\tilde{\text{Aut}}(\hat{V}) \).

In the case of interest in this paper, \( D = -K_{\hat{V}} \). To study this particular moduli space, we take a simpler course of action, and restrict our attention to a subspace of \( H^0(\mathcal{O}(-K_{\hat{V}})) \) on which \( \tilde{T} \) acts in such a way that the quotient exists and has the “expected” dimension for the entire moduli space. In a wide class of examples, \( \tilde{T} \) is in fact the connected component of the identity in \( \tilde{\text{Aut}}(\hat{V}) \), and the only differences between our moduli space and the “true” moduli space for hypersurfaces are a remaining quotient by a finite group, and a possible ambiguity in the choice of Zariski-open set used in constructing the quotient. In particular, the map from our space to the true moduli space is a dominant map between spaces of the same dimension. We hope that this latter property is true in general, but postpone that question to a future investigation.

Our construction of a simplified model for the hypersurface moduli space relies on another result of Demazure and Cox about \( \tilde{\text{Aut}}(\hat{V}) \). They show that

\[
\dim \tilde{\text{Aut}}(\hat{V}) = \dim \tilde{T} + \#(R(N, \Delta)),
\]

where

\[
R(N, \Delta) := \{ m \in M \mid \langle \text{gen}(\rho), m \rangle \leq 1 \text{ for all } \rho \in \Delta(1), \\
\text{with equality for a unique } \rho = \rho_m \in \Delta(1) \}
\]

is the set of roots of the toric variety \( \hat{V} \) associated to the fan \( \Delta \). Note that for each root \( m \in R(N, \Delta) \), we have \( -m \in P \cap M \). In fact, the set \( -R(N, \Delta) \) can be characterized as the subset of \( P \cap M \) consisting of lattice points which lie in the interiors of codimension-one faces of \( P \). We can thus decompose

\[
P \cap M = -R(N, \Delta) \cup (P \cap M)_0,
\]

and write

\[
\mathbb{C}^{P \cap M} = \mathbb{C}^{-R(N, \Delta)} \oplus \mathbb{C}^{(P \cap M)_0}.
\]

The subgroup \( \tilde{T} \subset \tilde{\text{Aut}}(\hat{V}) \) preserves this direct sum decomposition, so we can let \( \tilde{T} \) act on the second factor \( \mathbb{C}^{(P \cap M)_0} \) alone.

Our “simplified hypersurface moduli space” will be the GIT quotient \( \mathbb{C}^{(P \cap M)_0}_{ss}/\tilde{T} \). (We regard the action of \( \tilde{T} \) on \( \mathbb{C}^{(P \cap M)_0} \) as specifying a linearization of the action on
\( \mathbb{P}(\mathcal{C}(P \cap M)_0) \), so there is no ambiguity in the choice of GIT quotient.) There is then a natural rational map

\[
\mathbb{C}^\times \to \mathbb{C}^\times
\]

which could be refined to a regular map

\[
\mathbb{C}_{ss}^\times \to \mathbb{C}^\times/\mathcal{A}\text{ut}(\hat{V})
\]

if an appropriate set of “semistable” points \( U \subset \mathcal{C}(P \cap M) \) were available from (generalized) GIT.

Note that \( \text{Ker}(\xi_{[-K]}) \), which is a subgroup of both \( \tilde{T} \) and \( \tilde{\text{Aut}}(\hat{V}) \), acts trivially on both spaces. By equation (7), \( \tilde{T}/\text{Ker}(\xi_{[-K]}) \cong T \times \mathbb{C}^\times \). Note also that the two quotient spaces can be expected to have the same dimension.

**Definition** We say that the family \( \{\hat{X}\} \) has the dominance property if the natural rational map \( \mathbb{C}^\times \to \mathbb{C}^\times/\mathcal{A}\text{ut}(\hat{V}) \) is a dominant map between two varieties of the same dimension. (Note that this property is independent of the choice of quotients.)

This dominance property clearly holds if \( R(N, \Delta) = \emptyset \); we expect that it should hold in general, but have not checked this.

The “simplified hypersurface moduli space” parameterizes hypersurfaces with equations of the form

\[
\sum_{m \in (P \cap M)_0} c_m \chi^m = 0
\]

modulo the equivalences given by the action of \( \tilde{T}/\text{Ker}(\xi_{[-K]}) \cong T \times \mathbb{C}^\times \). The \( \mathbb{C}^\times \) factor is diagonally embedded in \( (\mathbb{C}^\times)^{(P \cap M)_0} \), and so gives an overall scaling of the equation. We can describe a Zariski-open subset of our moduli space by restricting to equations with \( c_0 \neq 0 \), and using the overall scaling of the equation to set that coefficient \( c_0 \) equal to 1. Thus, the open subset can be described as a quotient \( \mathbb{C}^\times/(P \cap M)_0 \to \tilde{T} \) with a point \( c \in \mathbb{C}^\times/(P \cap M)_0 \) corresponding to the hypersurface with equation

\[
\chi^0 + \sum_{m \in (P \cap M)_0} c_m \chi^m = 0.
\]

Let \( \Upsilon_0 = (P \cap M)_0 - \{0\} \) to simplify notation. Here is the crucial observation for the construction of the monomial-divisor mirror map: the action of \( T = N \otimes \mathbb{C}^\times \)

\[\footnote{The apparent lack of naturality in this step of our construction—why restrict to a subset?—will be redressed later in the paper.}\]
on $\mathbb{C}^{T_0}$ is induced by tensoring the homomorphism $\text{ad}_{T_0} : N \to \mathbb{Z}^{T_0}$ with $\mathbb{C}^*$. (The explicit identification of $\text{ad}_{T_0}$ as the homomorphism needed to specify the $T$-action follows immediately from the definition of the maps in equation (8), since $\text{ad}_{T_0}$ is dual to the natural map $\mathbb{Z} \langle T_0 \rangle \to M$ induced by the inclusion $T_0 \subset M$.) In particular, the tangent space to the simplified moduli space $\mathbb{C}^{(P \cap M)_0-\{0\}}//T$ has the form $(\mathbb{Z}^{(P \cap M)_0-\{0\}}//M) \otimes \mathbb{C}$.

When the family $\{\tilde{X}\}$ has the dominance property (e.g., when $R(N, \Delta) = \emptyset$), the induced rational map $\mathbb{C}^{(P \cap M)_0-\{0\}}//T \dashrightarrow \mathbb{C}^{(P \cap M) // \text{Aut}(\tilde{V})}$ is also dominant. In this case, we can describe the tangent space to the space of polynomial deformations of a general Calabi-Yau hypersurface $\tilde{X} \subset \tilde{V}$ as

$$T_{[\tilde{X}], \mathbb{C}^{P \cap M//\text{Aut}(\tilde{V})}} = H_{\text{poly}}^{d-1,1}(\tilde{X}) \cong (\mathbb{Z}^{(P \cap M)_0-\{0\}}//M) \otimes \mathbb{C},$$

where $[\tilde{X}]$ represents the class of $\tilde{X}$ in $\mathbb{C}^{P \cap M//\text{Aut}(\tilde{V})}$.

We can apply these same considerations to the family of hypersurfaces determined by the polar polyhedron $P^0$, which Batyrev has proposed as a mirror partner for the family $\{\tilde{X}\}$. To do this, we need to choose a simplicial subdivision $\Delta^0$ of $\mathcal{N}(P^0)$ which determines a projective toric variety $\tilde{V}^0$ and a family of hypersurfaces $\tilde{Y} \subset \tilde{V}^0$. Replacing $\tilde{X}$, $P$, $M$, $N$ by $\tilde{Y}$, $P^0$, $N$, $M$, respectively, in equation (10), we find that

$$T_{[\tilde{Y}], \mathbb{C}^{P^0 \cap N//\text{Aut}(\tilde{V}^0)}} = H_{\text{poly}}^{d-1,1}(\tilde{Y}) \cong (\mathbb{Z}^{(P^0 \cap N)_0-\{0\}}//M) \otimes \mathbb{C},$$

whenever $\{\tilde{Y}\}$ has the dominance property.

The monomial-divisor mirror map is now evident, when one compares equations (3) and (11). (Note that the same map $\text{ad}_{(P^0 \cap N)_0-\{0\}}$ is used to define the embedding $M \to \mathbb{Z}^{(P^0 \cap N)_0-\{0\}}$ in both cases.)

**Theorem** Let $P$ be a reflexive polyhedron, integral with respect to the lattice $M$, and let $P^0$ be its polar polyhedron, which is integral with respect to the dual lattice $N$. Let $\Delta$ and $\Delta^0$ be simplicial subdivisions of $\mathcal{N}(P)$ and $\mathcal{N}(P^0)$, respectively, and let $\{\tilde{X}\}$ and $\{\tilde{Y}\}$ be the corresponding families of Calabi-Yau hypersurfaces. Assume that $(P^0 \cap N)_0-\{0\} \subset \text{gen}(\Delta(1)) \subset (P^0 \cap N)-\{0\}$. Assume also that $R(M, \Delta^0) = \emptyset$, or more generally, simply assume that $\{\tilde{Y}\}$ has the dominance property. Then there is a natural isomorphism

$$H_{\text{poly}}^{d-1,1}(\tilde{Y}) \cong H_{\text{toric}}^{1,1}(\tilde{X}),$$

induced by equations (3) and (11), since both spaces are naturally isomorphic to $\text{Coker}(\text{ad}_{(P^0 \cap N)_0-\{0\}}) \otimes \mathbb{C}$. We call the isomorphism (12) the monomial-divisor mirror map.
4 Kähler cones

Let \( \hat{V} \) be a \( \mathbb{Q} \)-factorial toric variety, determined by a simplicial fan \( \Delta \), and let \( \Xi = \text{gen}(\Delta(1)) \). We can describe the group \( \text{Div}_T(\hat{V}) \) of toric Cartier divisors on \( \hat{V} \) as follows. In order for \( D = \sum d_a D_{\text{gen}^{-1}(a)} \) to be Cartier, there must be a continuous piecewise linear (PL) function \( \psi_D : |\Delta| \to \mathbb{R} \) (called the \textit{support function determined by} \( D \)) which is linear on each cone \( \sigma \in \Delta \), which takes integer values on \(|\Delta| \cap \mathbb{N} \), and which satisfies

\[
\psi_D(a) = -d_a \quad \text{for all } a \in \Xi. \tag{13}
\]

Since the fan \( \Delta \) is simplicial, the PL function \( \psi_D \) is completely determined by the values specified in equation (13) and the fan \( \Delta \): one just extends by linearity to each (simplicial) cone in \( \Delta \). The integrality condition \( \psi_D(|\Delta| \cap \mathbb{N}) \subset \mathbb{Z} \) remains nontrivial, however. (If we require instead that \( \psi_D(|\Delta| \cap \mathbb{N}) \subset \mathbb{Q} \), we get the group of \( \mathbb{Q} \)-Cartier divisors.) Since \( \Delta \) is simplicial, every Weil divisor is \( \mathbb{Q} \)-Cartier, that is, \( \hat{V} \) is \( \mathbb{Q} \)-factorial. Put another way, there is a natural isomorphism between \( \text{Pic}(\hat{V}) \otimes \mathbb{Q} \) and \( A_{n-1}(\hat{V}) \otimes \mathbb{Q} \).

The ample Cartier divisors (or ample \( \mathbb{Q} \)-Cartier divisors) are characterized by \textit{strict convexity} of \( -\psi_D \), where \( \psi_D \) is the support function determined by \( D \) (cf. [25]). This means the following: given a PL function \( \eta \) which is linear on each cone \( \sigma \in \Delta \), let \( u_\sigma \in M = \text{Hom}(\mathbb{N}, \mathbb{Z}) \) be the linear function such that

\[
\langle x, u_\sigma \rangle = \eta(x) \quad \text{for all } x \in \sigma.
\]

The convexity condition is:

\[
\eta(x) \geq \langle x, u_\sigma \rangle \quad \text{for all } x \in |\Delta|;
\]

the convexity if \textit{strict} if the inequality is strict for all \( x \notin \sigma \). (In fact, it suffices to check this at the points of \( \Xi \): for convexity, we must have

\[
\eta(a) \geq \langle a, u_\sigma \rangle \quad \text{for all } a \in \Xi
\]

with equality whenever \( a \in \sigma \); strict convexity requires a strict inequality whenever \( a \notin \sigma \).) The cone of real convex PL functions is denoted by \( \text{CPL}(\Delta) \), following the notation of Oda and Park [38]. If there exists a strictly convex function in \( \text{CPL}(\Delta) \), the fan \( \Delta \) is called \textit{regular}.

\footnote{Our signs are chosen to conform to the literature as closely as possible, while giving the term “convexity” its conventional meaning.}
There is a related cone (introduced by Gel’fand, Zelevinskii, and Kapranov [23]):

\[
\text{CPL} \sim (\Delta) = \{ \varphi \in \mathbb{R}^\Xi \mid \exists \eta \in \text{CPL}(\Delta) \text{ with } \varphi_a = \eta(a) \text{ for all } a \in \Xi \}.
\]

The definitions are constructed so that if \( D \) is an ample \( \mathbb{Q} \)-Cartier divisor, then \( \alpha^{-1}(D) \in \text{CPL}(\Delta) \). (One can choose \( \eta = -\psi_D \) in the definition.) Note that \( M_{\mathbb{R}} \) is contained in \( \text{CPL}(\Delta) \), and the corresponding \( \eta \)'s are precisely the smooth PL functions. The image of \( \text{CPL}(\Delta) \) in \( \mathbb{R}^\Xi / M_{\mathbb{R}} \), which we may think of as the set of convex PL functions modulo smooth PL functions, is denoted by \( \text{cpl}(\Delta) \). Under the isomorphism \( \mathbb{R}^\Xi / M_{\mathbb{R}} \cong A_{n-1}(\hat{V}) \otimes \mathbb{R} \cong \text{Pic}(\hat{V}) \otimes \mathbb{R} \), this cone \( \text{cpl}(\Delta) \) maps to the closed real cone generated by the ample divisor classes on \( \hat{V} \). An effective method of calculating all possible cones \( \text{cpl}(\Delta) \) in terms of the “linear Gale transform” is described in [38].

The exponential sheaf sequence gives rise to an isomorphism \( \text{Pic}(\hat{V}) \cong H^2(\hat{V}, \mathbb{R}) \cong H^{1,1}(\hat{V}, \mathbb{R}) \), since \( h^i(O_{\hat{V}}) = 0 \) for \( i > 0 \). Now there is a natural notion of positivity for orbifold-smooth \((1,1)\)-forms: one requires that the \( G_U \)-invariant \((1,1)\)-forms on the local uniformizing sets \( U \) be positive. The Kähler form of every orbifold-Kähler metric is easily seen to be a positive, orbifold-smooth \((1,1)\)-form. The set \( \mathcal{K}(\hat{V}) \subset H^{1,1}(\hat{V}, \mathbb{R}) \) consisting of orbifold de Rham classes which have such a positive representative is called the Kähler cone of \( \hat{V} \).

We could not find the following lemma in the literature (although it should be known).

**Lemma** Under the natural map \( \text{Pic}(\hat{V}) \rightarrow H^2_{\text{DR}}(\hat{V}, \mathbb{R}) \) which assigns to a line bundle the corresponding orbifold de Rham class, the ample line bundles map to positive de Rham classes.

Note that the lemma is not as obvious as is the analogous lemma in the smooth case, since even if we are given a projective embedding \( \hat{V} \rightarrow \mathbb{P}^N \), the pullback of the Fubini-Study form (which establishes the positivity of the de Rham class of \( O_{\mathbb{P}^N}(1) \)) is not necessarily positive as an orbifold 2-form.

**Sketch of proof:** We modify an argument of Guillemin and Sternberg [23]. By a result of Delzant [18] and Audin [3], the toric variety \( \hat{V} \) can be described as a symplectic reduction of the action of \( G = \text{Hom}(A_{n-1}(\hat{V}), \mathbb{C}^*) \) on \( \mathbb{C}^\Xi \); the specific symplectic reduction which produces \( \hat{V} \) is \( \Phi^{-1}(\alpha)/G_{\mathbb{R}} \), where \( \Phi \) is the moment map for the action, \( G_{\mathbb{R}} \) is the maximal compact subgroup of \( G \), and \( \alpha \in \text{cpl}(\Delta) \). An ample line bundle \( L \) on \( \hat{V} \) corresponds to a character \( \chi^L \) of \( G \), since \( A_{n-1}(\hat{V}) \) is the character lattice of \( G \); moreover, the corresponding point in \( A_{n-1}(\hat{V}) \otimes \mathbb{R} \) lies in \( \text{cpl}(\Delta) \). If we apply the constructions described on pp. 520–521 of [23], we produce
a line bundle on $\hat{V}$ with a specific (orbifold-)connection, whose curvature is the symplectic form obtained by symplectic reduction from the standard form on $\mathbb{C}^\mathbb{Z}$. That curvature form provides the desired positive orbifold 2-form. Q.E.D.

The converse statement—that if the class of a line bundle is represented by a positive, orbifold-smooth 2-form then the line bundle is ample—is a theorem of Baily [8]. Putting the two together, we conclude that the image of the cone $\text{cpl}(\Delta)$ in $H^2_{\text{DR}}(\hat{V}, \mathbb{R})$ is precisely the closure of the Kähler cone $\overline{\mathcal{K}(\hat{V})}$.

As remarked earlier, the hypersurface $\hat{X}$ is itself an orbifold, so it has an orbifold Kähler cone $\mathcal{K}(\hat{X}) \subset H^{1,1}(\hat{X}, \mathbb{R})$. The positive orbifold-smooth Kähler forms on $\hat{V}$ corresponding to classes in the interior of $\text{cpl}(\Delta)$ will restrict to positive orbifold-smooth forms on $\hat{X}$, since $\hat{X}$ meets all singular strata of $\hat{V}$ transversally. We call the resulting cone $\mathcal{K}_{\text{toric}}(\hat{X}) \subset \mathcal{K}(\hat{X}) \cap H^{1,1}_{\text{toric}}(\hat{X})$ the cone of toric Kähler classes on $\hat{X}$.

5 The Kähler moduli space

Mirror symmetry predicts a close relationship between the moduli space of complex structures on one Calabi-Yau manifold $\hat{Y}$ and the so-called “Kähler moduli space” of its mirror partner $\hat{X}$. This Kähler moduli space, which arises in the study of nonlinear sigma models with target $\hat{X}$, is an open subset of $\mathcal{D}/\Gamma$, where

$$\mathcal{D} := \{ B + i J \in H^2(\hat{X}, \mathbb{C}) \mid J \in \mathcal{K}(\hat{X}) \},$$

and where

$$\Gamma := H^2(\hat{X}, \mathbb{Z}) \rtimes \text{Aut}(\hat{X})$$

(cf. [35].) The precise open subset of $\mathcal{D}/\Gamma$ which constitutes the moduli space is difficult to determine, since general convergence criteria for the sigma model are unknown at present. However, the open set is expected to include points sufficiently far out along any path in $\mathcal{D}$ whose imaginary part is moving towards infinity while staying away from the boundary of the Kähler cone. Such paths should approach a common point, called the large radius limit, in an appropriate partial compactification of $\mathcal{D}/\Gamma$. A general discussion of conditions under which such a limit point exists can be found in [35].

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6 The physics of these models is believed to be as well-behaved on orbifolds as on manifolds [22, 21].

7 But as we will observe below, the region of convergence in this particular case can be inferred from mirror symmetry.
We will be interested in a “toric subspace” of the Kähler moduli space, defined by intersecting the moduli space with $\mathcal{D}_{\text{toric}}(\hat{X})/\Gamma_{\text{toric}}$, where

$$\mathcal{D}_{\text{toric}}(\hat{X}) := \{ B + i J \in H_{\text{toric}}^{1,1}(\hat{X}) \mid J \in \mathcal{K}_{\text{toric}}(\hat{X}) \},$$

and $\Gamma_{\text{toric}} := A_{d-1}(\hat{X})_{\text{toric}} \rtimes \text{Aut}(\hat{X})_{\text{toric}}$. (The toric automorphisms $\text{Aut}(\hat{X})_{\text{toric}}$ are those automorphisms of $\hat{X}$ induced by an automorphism of the ambient toric variety $\hat{V}$.) If the GIT of the family of hypersurfaces is well-behaved (which will be the case for the families of primary interest to us), then for the general hypersurface $\hat{X}$ the group $\text{Aut}(\hat{X})_{\text{toric}}$ will be finite.

Let $L = A_{d-1}(\hat{X})_{\text{toric}}$ and consider the torus $L \otimes \mathbb{C}^*$ which contains $\mathcal{D}/L$ as an open subset. The cone $\mathcal{K}_{\text{toric}} \subset L \otimes \mathbb{R}$, which is a rational polyhedral cone, determines an affine torus embedding $\mathcal{M}$ with a unique 0-dimensional orbit $p \in \mathcal{M}$ under the action of the torus $L \otimes \mathbb{C}^*$. We let $\overline{\mathcal{D}}/L$ be the closure of $\mathcal{D}/L$ in $\mathcal{M}$, and let $(\mathcal{D}/L)^-$ be the interior of $\overline{\mathcal{D}}/L$. (This contains the point $p$.) If $\text{Aut}(\hat{X})_{\text{toric}}$ is finite, then since everything in sight is $\text{Aut}(\hat{X})_{\text{toric}}$-equivariant, we may take the quotient and get a partial compactification $(\mathcal{D}/\Gamma)^- \subset \mathcal{D}/\Gamma$ with a distinguished boundary point (again denoted by $p$), the large radius limit point. This point is the common limit of the paths described earlier.

The torus $L \otimes \mathbb{C}^* = A_{d-1}(\hat{X})_{\text{toric}} \otimes \mathbb{C}^*$ which is being compactified can be described in the form

$$A_{d-1}(\hat{X})_{\text{toric}} \otimes \mathbb{C}^* = (\mathbb{Z}^{\Xi_0}/M) \otimes \mathbb{C}^*$$

$$= (\mathbb{C}^*)^{\Xi_0}/(M \otimes \mathbb{C}^*)$$

using equation (3). Now the orbits of $M \otimes \mathbb{C}^*$ on $(\mathbb{C}^*)^{\Xi_0}$ are all good orbits of the same dimension. The action of $M \otimes \mathbb{C}^*$ on the larger space $\mathbb{C}^{\Xi_0}$ may be less well-behaved, but in any case we can regard $(\mathbb{C}^*)^{\Xi_0}/(M \otimes \mathbb{C}^*)$ as a representative of the birational class of quotients $\mathbb{C}^{\Xi_0}/(M \otimes \mathbb{C}^*)$.

Suppose that the family $\{ \hat{Y} \subset \hat{V} \}$ associated to the polar polyhedron of $\{ \hat{X} \}$ has the dominance property (introduced in section 3), and that $\Delta$ is chosen so that $(P^o \cap N)_0 - \{0\} \subset \text{gen}(\Delta(1)) \subset (P^o \cap N) - \{0\}$. Then we deduce from the monomial-divisor mirror map a diagram in which the vertical maps are dominant:

$$\mathcal{D}_{\text{toric}}(\hat{X})/A_{d-1}(\hat{X})_{\text{toric}} \subset (\mathbb{C}^*)^{(P^o \cap N)_0 - \{0\}}/(M \otimes \mathbb{C}^*) = (\mathbb{C}^{(P^o \cap N)_0 - \{0\}}/(M \otimes \mathbb{C}^*)$$

$$\mathcal{D}_{\text{toric}}(\hat{X})/\Gamma_{\text{toric}} \subset (A_{d-1}(\hat{X}) \otimes \mathbb{C}^*)/\text{Aut}(\hat{X})_{\text{toric}} = \mathbb{C}^{P^o \cap N}/\tilde{\text{Aut}}(\hat{V}^o).$$

---

*This construction is identical to the one described in [35], since Looijenga’s semi-toric compactification [24] coincides with the “toroidal embeddings” of [1] when $\mathcal{K}$ is rational polyhedral and $\Gamma/L$ is finite.*
The left and center vertical maps are simply the quotient maps by \( \text{Aut}(\hat{X})_{\text{toric}} \), and the right map is the dominant rational map from the simplified moduli space to the actual moduli space of the family \( \{\hat{Y}\} \).

We can now formulate a “mirror symmetry” conjecture for these families, which generalizes some (less precise) earlier conjectures of Aspinwall and Lütken \[4\] and Batyrev \[10\].

**Conjecture** Suppose that \( \text{Aut}(\hat{X})_{\text{toric}} \) is finite, and that the dominance property holds for \( \{\hat{Y}\} \). Then

1. there is an open set \( U \subset (D_{\text{toric}}(\hat{X})/\Gamma_{\text{toric}})^- \) containing the large radius limit point \( p \) such that \( U \cap (D_{\text{toric}}(\hat{X})/\Gamma_{\text{toric}}) \) is the toric part of the Kähler moduli space,

2. there is an appropriate quotient \( \mathbb{C}^{P_0 \cap N}/\sim_{\text{Aut}(\hat{V}^o)} \) and a “mirror map”

\[
\mu^{-1}: U \to \mathbb{C}^{P_0 \cap N}/\sim_{\text{Aut}(\hat{V}^o)},
\]

which is an isomorphism onto its image, and which, when restricted to \( U \cap (D_{\text{toric}}(\hat{X})/\Gamma_{\text{toric}}) \), serves to identify points whose conformal field theories are mirror-isomorphic, such that

3. the differential of the inverse map \( \mu \) at the “large complex structure limit point” \( \mu^{-1}(p) \)

\[
d\mu: T_{\mu^{-1}(p)} \mathbb{C}^{P_0 \cap N}/\sim_{\text{Aut}(\hat{V}^o)} \to T_{p, U}
\]

coincides with the monomial-divisor mirror map

\[
H^{d-1,1}_{\text{poly}}(\hat{Y}) \to H^{1,1}_{\text{toric}}(\hat{X})
\]

up to signs, once we have made the canonical identifications

\[
T_{\mu^{-1}(p)} \mathbb{C}^{P_0 \cap N}/\sim_{\text{Aut}(\hat{V}^o)} = H^{d-1,1}_{\text{poly}}(\hat{Y})
\]

\[
T_{p, U} = H^{1,1}_{\text{toric}}(\hat{X}).
\]

That is, there is some element

\[
\theta_{\Delta} \in A_{d-1}(\hat{X})_{\text{toric}} \otimes \mathbb{C}^* \subset \text{Aut}(H^{1,1}_{\text{toric}}(\hat{X}))
\]

of order 2, which when composed with the monomial-divisor mirror map yields \( d\mu \). (When \( d \geq 3 \), the automorphism \( \theta_{\Delta} \) which specifies the signs is unique.)

---

9We denote this map by \( \mu^{-1} \) in order to match the conventions established in \[35\].
In particular, the location of the large complex structure limit point \( \mu^{-1}(p) \) can be calculated using the monomial-divisor mirror map and the knowledge of the cone \( \mathcal{K}_{\text{toric}}(\hat{X}) \).

In [33], a very general conjecture was formulated which specifies the “canonical coordinates” to be used in the mirror map, up to some constants of integration. Those constants can be determined if one knows the differential of the mirror map at the large radius limit point—for these toric hypersurfaces, that differential is supplied by the conjecture above. So the two conjectures together completely specify the canonical coordinates. A similar conjecture about canonical coordinates for toric hypersurfaces has been independently made by Batyrev and van Straten [13].

If the conjecture stated above is true, then among other things the so-called “3-point functions” (part of the conformal field theories) must coincide under this mapping. The 3-point function on the moduli space \( \mathbb{C}^{P_{\text{toric}}}/\text{Aut}(\hat{V}) \) can be calculated in terms of the variation of Hodge structure of the family \( \{\hat{Y}\} \) (cf. [34]), and this gives a method to identify precisely which subset of \( \mathcal{D}_{\text{toric}}(\hat{X})/\Gamma_{\text{toric}} \) constitutes the toric part of the Kähler moduli space. For that subset can be characterized as the domain of convergence of a power series expansion of the 3-point functions, when calculated in the canonical coordinates on \( \mathcal{D}_{\text{toric}}(\hat{X})/\Gamma_{\text{toric}} \). Using the mirror map, this power series calculation can actually be made in \( \mathbb{C}^{P_{\text{toric}}}/\text{Aut}(\hat{V}) \), by calculating the variation of Hodge structure of the family \( \{\hat{Y}\} \).

The leading term in the power series expansion of the 3-point function on \( \mathcal{D}_{\text{toric}}(\hat{X})/\Gamma_{\text{toric}} \) is the cubic form

\[
H_{1,1}^{\text{toric}}(\hat{X}) \times H_{1,1}^{\text{toric}}(\hat{X}) \times H_{1,1}^{\text{toric}}(\hat{X}) \to \mathbb{C}
\]

given by the cup product; higher terms are given by “quantum corrections” that depend on the point in \( \mathcal{D}_{\text{toric}}(\hat{X})/\Gamma_{\text{toric}} \), and that all vanish at the large radius limit point \( p \in (\mathcal{D}_{\text{toric}}(\hat{X})/\Gamma_{\text{toric}})^{\sim} \). One consequence of our conjecture would therefore be an agreement between the leading term in the variation of Hodge structure calculation for the family \( \{\hat{Y}\} \) near the large complex structure limit point, and the cup product cubic form on \( \hat{X} \). This consequence was first checked in an example by Aspinwall, Lütken, and Ross [5] several years ago. More recently, Batyrev [10] checked his version of this statement in the case that \( X \) itself is smooth, and the authors [2] checked it in an example in which there are five different birational choices for \( \hat{X} \) (with the same \( X \subset V \)). After learning of the results of [2] and of the present paper, Batyrev [11] checked this consequence in general.

\[10\] It is possible to verify that the large complex structure limit point used in [3] agrees with the one predicted by the monomial-divisor mirror map.
The 3-point function coming from variation of Hodge structure can be used to determine the choice of signs $\theta_\Delta$: the poles in that function should occur at positive real values of the canonical coordinates (cf. [3]). The automorphism $\theta_\Delta$ with this property can be calculated explicitly using methods of Gel’fand, Zelevinskiˇı, and Kapranov [26]; we will discuss this in detail elsewhere.

Another consequence of our conjecture is that the Kähler moduli spaces for different birational models $\hat{X}$ of the function field of $X$ can naturally be regarded as analytic continuations of one another. (For after applying mirror symmetry, they are seen to occupy different regions in the same moduli space.) This was the principal conclusion of our earlier paper [2]; a similar idea is due independently to Manin [33].

6 Phases and the secondary fan

In the course of defining the monomial-divisor mirror map, we made a somewhat unnatural restriction to a Zariski-open subset of the simplified hypersurface moduli space. We now return to the study of the full moduli space. The “simplified hypersurface moduli space” will be birational to the quotient $(\mathbb{C}^*)(\mathbb{P}^{\cap M})_0/\tilde{T}$. In fact, the moduli space of primary interest is the space parameterizing those hypersurfaces whose singularities are no worse than generic. This is the complement of the “principal discriminant” of Gel’fand, Zelevinskiˇı, and Kapranov [26]. One natural compactification of the moduli space would be the one in which this “principal discriminant” is an ample divisor.

Whatever compactification we use, the compactified moduli space is itself a toric variety (since it contains the torus $(\mathbb{C}^*)(\mathbb{P}^{\cap M})_0/\tilde{T}$ as a dense open subset). If we compactify so that the principal discriminant is ample, then the toric variety is determined by the Newton polyhedron for the principal discriminant which, as Gel’fand, Zelevinskiˇı, and Kapranov show, has a convenient combinatorial description as a so-called “secondary polytope”.

To explain the combinatorics, we note that the action of $\tilde{T}$ on $(\mathbb{C}^*)(\mathbb{P}^{\cap M})_0$ is induced by a homomorphism $ad^+_{(\mathbb{P}^{\cap M})_0} : N^+ \to \mathbb{Z}^{(\mathbb{P}^{\cap M})_0}$, dual to the map $\mathbb{Z}^{((P \cap M)_0)} \to M^+$ given by equation (8), where $M^+ = M \oplus \mathbb{Z}$. We should imagine embedding the set $(P \cap M)_0$ into $M^+$ via the map $b \mapsto (b, 1)$; the image is a finite set of points in the affine hyperplane $\{(m, 1)\} \subset M^+$. The convex cone spanned by these points we denote by $P^+$; it is the cone over the image of the original polyhedron $P$ (generated by the points $(P \cap M)_0$) in the affine hyperplane $\{(m, 1)\} \subset M^+$. The dual cone to $P^+$ has the form $(P^\circ)^+$, where $P^\circ$ is the polar
polyhedron of $P$.

The secondary fan (which is the normal fan of the secondary polytope) is the fan consisting of all cones $\text{cpl}(\Sigma)$, where $\Sigma$ is a regular refinement of the fan $\mathcal{N}((P^o)^+)$ (cf. [14]). Among the possible regular refinements $\Sigma$ we find fans of the form

$$(\Delta^o)^+ := \{\text{cone over } (\sigma \cap P) \mid \sigma \in \Delta^o\},$$

for regular fans $\Delta^o$ which refine $\mathcal{N}(P^o)$. (But there are others, which do not have this form.) For such fans, it is easy to see that $\text{cpl}((\Delta^o)^+) = \text{cpl}(\Delta^o)$, regarding both as cones in the same space $\mathbb{Z}^{(P \cap M)_0}/N^+ \cong \mathbb{Z}^{(P \cap M)_0-\{0\}}/N$.

So our chosen compactification of the moduli space is the toric variety which is specified by the secondary fan. It has the pleasant property that it includes all of the partial compactifications that were needed to describe the “large complex structure limits” coming from mirror symmetry of sigma models (since those were given by the cones $\text{cpl}(\Delta^o)$). It has another nice property as well, first observed by Kapranov et al. [30] and Batyrev [10]: the compactification constructed in this way dominates all possible GIT compactifications, coming from different choices of linearization.

What does this structure correspond to under mirror symmetry? The embedding $(P^o \cap N)_0 \subset N$, together with a (regular) refinement $\Delta$ of the fan $\mathcal{N}(P)$, was used to determine the projective toric variety $\hat{V}$. The new extended embedding $(P^o \cap N)_0 \subset N^+$ (whose image lies in the affine hyperplane $\{(n, 1) \} \subset N^+$), together with a regular refinement $\Sigma$ of the fan $\mathcal{N}(P^+)$, can also be used to determine a toric variety, of dimension one larger than the previous variety. Among these toric varieties we find the total spaces of canonical bundles over the various choices of $\hat{V}$ (when we take $\Sigma$ of the form $\Delta^+)$.

This is precisely the structure that Witten has found to be relevant in his study of Landau-Ginzburg theories and their deformations [14]. Each choice of fan $\Sigma$ determines a different “phase” of the physical theory. When the fan $\Sigma$ is of the form $\Delta^+$, the physical theory is related to the nonlinear sigma model with target $\hat{X}$ (where $\hat{X} \subset \hat{V}$ is generic). But for other fans $\Sigma$, the physical theory is quite different. (We refer the reader to [14] for more details.) Thus, not only can the different sigma models with birationally equivalent targets be viewed as analytic continuations of one another, there are further analytic continuations (to regions in the moduli space corresponding to $\text{cpl}(\Sigma)$ for $\Sigma \neq \Delta^+$) to other kinds of physical theory.

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