\textbf{Y-system and $\beta$-deformed $N = 4$ super-Yang–Mills}

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Abstract

We show how the perturbation theory results recently obtained by F Fiamberti, A Santambrogio, C Sieg and D Zanon for operator anomalous dimensions of $\beta$-deformed super-Yang–Mills theory can be reproduced from the AdS$_5$/CFT$_4$ Y-system proposed by NG, V Kazakov and P Vieira. To do this, we obtain the general twisted asymptotic solution of this Y-system of functional equations. We show that existence of an additional parameter $\beta$ in the deformed theory allows us to extract rich information about the perturbation theory integrals directly from the Y-system. Using this method we found a simple generating function for a broad class of such integrals.

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1. Introduction

The celebrated AdS/CFT correspondence relates a gauge field theory and a string theory, with the best-studied example being the duality between four-dimensional $\mathcal{N} = 4$ planar superconformal Yang–Mills (SYM) theory and type IIB superstring theory on AdS$_5 \times S^5$ [1]. Recently, more similar examples of dualities were found [2, 3]. Integrability properties, which have been discovered on both sides of such dualities, have played an important role in the study of this rapidly developing subject. The exact $S$-matrix led to the formulation of asymptotic Bethe ansatz equations (ABA) [3–6], which describe the anomalous dimensions for operators of asymptotically large length $L$ at any coupling. The generalized L"uscher formula [7], Y-system [8] and thermodynamic Bethe ansatz [9–11] have made it possible to take into account the wrapping corrections and obtain, in principle, the missing part of the spectrum at finite $L$.

In the 4D case, evidence for integrability has been found also for the $\beta$-deformed SYM theory, which has $\mathcal{N} = 1$ instead of $\mathcal{N} = 4$ supersymmetry. The deformation consists in replacing the original superpotential for the chiral superfields by

\begin{equation}
W = i\hbar \text{tr}(e^{i\beta \phi} \psi Z - e^{-i\beta \phi} Z \psi).
\end{equation}
The deformed theory remains superconformal in the planar limit to all orders of perturbation theory [12, 13] if $\beta$ is real and $h = g_{YM}^2$, where $g_{YM}$ is the Yang–Mills coupling constant, related to the ’t Hooft coupling $g^2$ in the planar limit as

$$
g^2 = \frac{g_{YM}^2 N}{16\pi^2}.
$$

Under these conditions the deformation becomes exactly marginal. The $\beta$-deformed theory is also believed to have a string dual [14]. The integrability properties of that string theory have been studied in [15, 16].

The deformed theory was also investigated quite intensively in the perturbative regime. Evidence for perturbative integrability was found in [17–21]. On the other hand, direct computations of anomalous dimensions without use of integrability were done in [22, 23] (see also [24]). In those works, wrapping corrections at critical order have been found for two operators of length $L = 4$ with two impurities, and for one-impurity operators with $L \leq 11$. Recurrence relations were also discussed [23, 24] which allow one in principle to obtain this correction for any one-impurity operator, though a closed formula for the corrections was not found.

The methods which rely on integrability have reproduced only a part of the results. In [25, 26] first wrapping corrections were obtained for certain single-impurity operators, though only for $\beta = \frac{1}{2}$. Also, very recently a part of the $S$-matrix was presented as a conjecture, and made it possible to reproduce the first wrapping correction to the $L = 4$ Konishi operator via the generalized Lüscher formula [27] which gave strong support for the integrability for arbitrary real values of $\beta$.

For $\mathcal{N} = 4$ SYM another efficient method, based on the asymptotic large $L$ solution of the $Y$-system [8], was used in [8, 29] to analytically compute wrapping corrections, giving perfect agreement with direct perturbative results [28, 29]. At the leading wrapping order the $Y$-system should be equivalent to the generalized Lüscher formula of [7]. Here we argue that the $Y$-system of [8] describes also the $\beta$-deformed theory, and present a generalized version of that asymptotic solution with four additional twist parameters. We show that it reproduces all perturbative results of [22, 23] for $\beta$-deformed SYM. In particular, we study the one magnon case in detail, giving a general formula for the first wrapping correction for a single-impurity operator of arbitrary length $L$.

2. The asymptotic solution of the $Y$-system

In this section we briefly describe the general $Y$-system technique and the generating functional which allows to build the asymptotic large $L$ solution of the $Y$-system and $T$-system of [8]. We then propose a way to modify this functional for the $\beta$-deformed theory.

2.1. Review of $Y$- and $T$-systems

The Metsaev–Tseytlin AdS$_5 \times$S$^5$ string action in the light-cone gauge is a classically integrable 2D field theory, and its energy spectrum is believed to describe the spectrum of anomalous dimensions of planar $\mathcal{N} = 4$ SYM. In general, the experience with relativistic integrable theories [30] suggests that the exact quantum spectrum should be governed by a system of functional Hirota equations$^4$:

$$
T_{a,s}(u + i/2)T_{a,s}(u - i/2) = T_{a+1,s}(u)T_{a-1,s}(u) + T_{a,s+1}(u)T_{a,s-1}(u).
$$

$^4$ Sometimes they could be slightly more complicated.
We use here the following short-hand notations:

\[ f^\pm \equiv f(\pm i/2), \quad f^{[a]} \equiv f(a/2). \tag{2.2} \]

It was conjectured in [8] that in the AdS/CFT case the system of Hirota equations should be exactly the same, with the functions \( T_{a,s}(u) \) being non-zero only inside the infinite \( T \)-shaped domain of the \( a, s \) integer lattice, shown in figure 1.

In order to compute physical quantities, one should form particular combinations of these \( T \)-functions

\[ Y_{a,s} = \frac{T_{a,s+1}T_{a,s-1}}{T_{a+1,s}T_{a-1,s}}. \tag{2.3} \]

As a consequence of (2.1), the \( Y \)-functions satisfy the \( Y \)-system functional equations

\[ \frac{Y^+_{a,s}Y^-_{a,s}}{Y^+_{a+1,s}Y^-_{a-1,s}} = \frac{(1 + Y_{a,s+1})(1 + Y_{a,s-1})}{(1 + Y_{a+1,s})(1 + Y_{a-1,s})}. \tag{2.4} \]

The indices \( a, s \) here label the marked nodes of the lattice in figure 1.

The \( Y \)-system should be supplemented with a particular set of analytical properties. One possibility was proposed in [31]. In the current case the analytical properties are rather involved, partly due to the lack of Lorentz symmetry and partly due to the complicated \( psu(2, 2|4) \) symmetry of the theory. In particular, the dispersion relation for a single excitation in infinite volume is quite nontrivial.

We express the energy and momentum of the excitations (also called magnons) in terms of the Zhukowski variable \( x(u) \), defined by

\[ x + \frac{1}{x} = \frac{u}{g}. \tag{2.5} \]

The ‘mirror’ and ‘physical’ branches of this function are defined as

\[ x^{ph}(u) = \frac{1}{2} \left( \frac{u}{g} + \sqrt{\frac{u}{g} - 2} \right), \quad x^{\text{mir}}(u) = \frac{1}{2} \left( \frac{u}{g} + i \sqrt{4 - \frac{u^2}{g^2}} \right). \tag{2.6} \]

5 The equations for \( \{a, s\} = \{2, 2\} \) and \( \{a, s\} = \{-2, 2\} \) cannot be written in such a ‘local’ form.
where \( \sqrt{n} \) denotes the principal branch of the square root. The energy and momentum of a bound state with \( n \) magnons are given by

\[
\epsilon_n(u) = n + \frac{2ig}{\sqrt{n}} - \frac{2ig}{\sqrt{n+1}}, \quad p_n(u) = \frac{1}{i} \log \frac{\sqrt{n+1}}{\sqrt{n}}.
\]

Finally, the exact energy of a state is given by the expression

\[
E = \sum_j \epsilon_j^{ph}(u_{4,j}) + \delta E, \quad \delta E = \sum_{a=1}^{\infty} \int \frac{du}{2\pi i} \frac{\partial \epsilon_a^{min}(u)}{\partial u} \log \left( 1 + Y_{a,0}^{mir}(u) \right),
\]

with the rapidities \( u_{4,j} \) being fixed by the exact Bethe ansatz equations

\[
Y_{1,0}^{ph}(u_{4,j}) = -1,
\]

where \( Y_{1,0}^{ph}(u) \), similar to \( \chi^{ph} \), is the result of the analytical continuation of \( Y_{1,0}^{mir}(u) \) through the cut \((i/2 + 2\pi i, i/2 + \infty)\) \[32\].

The \( Y \)-system for \( \mathcal{N} = 4 \) SYM passes several nontrivial tests: it reproduces both perturbative wrapping corrections \[8, 29\] and quasiclassical spectrum at strong coupling \[33\], and is moreover compatible with thermodynamic Bethe ansatz equations \[9\] (which allow efficient numerical studies also \[32, 34\]).

In this paper we give evidence that exactly the same \( Y \)-system set of equations describes the \( \beta \)-deformed theory. We show that the asymptotic solution of \[8\] is in fact a representative \( 2.I \) of the \( \beta \)-deformed theory. We show that the asymptotic solution of \[8\] is in fact a representative of a family of solutions, which have similar analytical properties and in terms of the transfer matrices correspond to the twisted case\[6\].

### 2.2. Twisted generating functional

In \[37–39, 41\] a method was proposed for constructing solutions of the Hirota equation for a domain called L-hook (one half of the \( \Gamma \)-hook diagram in figure 1). The method relies on the use of Wronskian relations and Backlund transformations which allow us to gradually reduce the domain to a trivial one. The result obtained in this way can be written compactly in terms of a generating functional. A similar result was recently obtained in this way \[37–39, 41\].

In this paper we want to demonstrate that the twisted solution of the Hirota equation (see \[41\]) can indeed be used for the \( \beta \)-deformed theory. For that we just need to find the asymptotic large \( L \) solution, which can then be applied also for comparison with perturbation theory up to the order \( \sim r^{4L-2} \). In the large \( L \) limit, the ‘massive’ nodes \( Y_{a,0} \) are suppressed and the \( Y \)-system decouples into two wings: \( su_2(2|2) \) and \( su_R(2|2) \). The solution for a single wing is much simpler than for the full \( psu(2, 2|4) \) case, and is given by an explicitly known generating functional. Here we propose to use the following twisted version of that functional\[7, 8\]:

\[
W_R = \frac{1}{1 - \frac{1}{\tau_{1,R}} D_{R^{3\to3}, Q}^{Q_0, Q_1} D_{Q_1} \left( 1 - \frac{1}{\tau_{2,R}} D_{Q_1} Q_2^{-1} D_{Q_1} \right) \\
\times \left( 1 - \frac{1}{\tau_{2,R}} D_{Q_1} Q_1^{+1} Q_2^{-1} D_{Q_1} \right) \frac{1}{1 - \frac{1}{\tau_{1,R}} D_{R^{3\to3}, Q}^{Q_0, Q_1} D_{Q_1}}.
\]

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6 Usually the construction of transfer matrices allows us to introduce extra twist parameters without destroying integrability. Often the twisted systems can be better controlled and in some cases the twists are necessary as regularizations, see for example \[35, 36\].

7 Here we use \( su(2) \) grading. For this grading the method of \[8\] is described in detail in \[42\].

8 NG thanks P Vieira for the discussion of this possibility and for the collaboration in the early stages of this work.
where $\tau_{1, R}$ and $\tau_{2, R}$ are complex numbers (not dependent on the spectral parameter $u$) which we call twists. The generating functional for the left wing $\mathcal{W}_L$ is given by the same expression with $Q_{1,2,3}$ replaced by $Q_{7,6,5}$ and $\tau_{1,2, R}$ by $\tau_{1,2, L}$. We use the following notation:

$$Q_l = \prod_{j=1}^{K_l} (u - u_{1,j}), \quad R^{(k)} = \prod_{j=1}^{K_i} (x(u) - x_{4, j}^{\pm}),$$

(2.11)

$$B^{(k)}_l = \prod_{j=1}^{K_l} \left( \frac{1}{x(u)} - x_{4, j}^{\pm} \right), \quad B^{(k)}_i = B^{(k)}_l.$$

(2.12)

and $D = e^{-i\theta/2}$ is the shift operator. Expansion of this functional gives the functions $T_{u, 1}^{R, L}$ and $T_{1, L}^{R, L}$:

$$\mathcal{W}_{R, L} = \sum_{s=0}^{\infty} D^s T_{1, s}^{R, L} D^s, \quad \mathcal{W}_{L, L}^{-1} = \sum_{a=0}^{\infty} (-1)^a D^a T_{u, 1}^{R, L} D^a.$$  

(2.13)

Let us motivate the structure of the twists we introduced above. One could introduce eight twists in total: one in each of the four terms inside $\mathcal{W}_R$ and, similarly, four more twists in $\mathcal{W}_L$. However, it is easy to see that requiring $Y_{1, j}$ and $Y_{a, j}$ to be real implies that the twists in the first and last terms inside $\mathcal{W}$ are complex conjugate to each other, and the same is true for twists in the second and third terms. Also, to allow only such configurations of Bethe roots which are invariant w.r.t. complex conjugation, one should require the twists to be unimodular. Thus the generating functional satisfying these requirements could have only four independent twists in total: two in $\mathcal{W}_L$ and two in $\mathcal{W}_R$, as it is indeed written in (2.10).

The polynomials $Q_s(u)$ in the denominators of generating functionals could potentially result in the poles of the $T_{u, 3}$ functions. However, one can show that these poles cancel provided the following Bethe equations are satisfied:

$$1 = \left. \frac{\tau_{1, R}}{\tau_{1, L} B^{(-)}(u)} \right|_{u = u_{1, j}} Q_3^{-} Q_3^{+} Q_5^{+} Q_5^{-}, \quad -1 = \left. \frac{1}{(\tau_{1, R})^2 Q_1^{+} Q_2^{-} Q_3^{+} Q_5^{-}} \right|_{u = u_{2, 4}}.$$  

(2.14)

with a similar set of three equations for the left wing.

For large $L$, the middle node $Y_{u, 0}$ is given by [8]

$$Y_{u, 0} \simeq T_{u, 3}^{L} T_{u, 1}^{R} \prod_{a = -\infty}^{\infty} \Phi(u + i n),$$

(2.15)

which can be found by solving (2.4) for $s = 0$. Here $\Phi$ is the only unknown function which is almost fixed by the requirements that $Y_{u, 0}$ is real and that $Y_{u, 0}^{(i)}(u_{4, j})$ is unimodular as a function of $u_{4, j}$. Those conditions are satisfied as a consequence of the crossing equation [6, 8] by the following expression:

$$\Phi(u) = \left( \frac{x^{-}}{x^{+}} \right)^L \prod_{j=1}^{K_4} \frac{R^{(-)}(u, u_{4, j})}{R^{(+)}(u, u_{4, j})} \frac{1}{Q_3^{+} Q_3^{-} Q_5^{+} Q_5^{-}} B_1 B_2 B_3 B_4.$$  

(2.16)

The equation for the momentum-carrying roots $Y_{u, 0}^{(i)}(u_{4, k})$ reads

$$\tau_{1, R} \tau_{1, L} \Phi(u) \left( \frac{R^{(+)}(u)}{R^{(-)}(u)} \right)^\frac{Q_3^{-} Q_3^{+} Q_5^{-} Q_5^{+}}{Q_1^{+} Q_2^{-}} \bigg|_{u = u_{4, k}} = -1.$$  

(2.17)

9 With the $\pm$ branch used for $x(u)$ in all terms.

10 Here we use the gauge of [42] which is different from the one in [8].
It is important to mention that the Bethe equations (2.14) and (2.17) are consistent with the ABA of [20]. For more details on this, see the appendix, in which we also describe the switch to sl(2) grading.

In the next section we consider restriction to the $su(2)$ subsector and study the weak coupling limit of these expressions.

3. $su(2)$ subsector

For the $su(2)$ subsector only $u_{4,j}$ roots are introduced, and the Bethe ansatz equations read [17, 19–21]

$$\left(\frac{x_{4,k}^+}{x_{4,k}^-}\right)^L = q^{2L} \prod_{j \neq k} \sigma^2(u_k, u_j) \frac{u_{4,k} - u_{4,j} + i}{u_{4,k} - u_{4,j} - i},$$

(3.1)

where $q = \exp(\pi i \beta)$. For this equation to coincide with (2.17), the equality $\tau_{1,1} \tau_{1,R} = q^{2L}$ must hold. Furthermore, we found that in order to match our explicit answers for anomalous dimensions with the many perturbative results, we have to set

$$\tau_{1,1} = q^{2L-2K_s}, \quad \tau_{1,R} = q^{2K_s}, \quad \tau_{2,R} = \tau_{2,L} = 1.$$  (3.2)

These expressions are also in agreement with the values of twists $\tau$ obtained by comparing our ABA equations (2.14) and (2.17) with the ABA equations obtained in [20] for the $\beta$-deformed theory (see the appendix), which gives additional support for the ABA of [20].

For the $su(2)$ subsector $Q_a = 1$ if $a \neq 4$, and we get an explicit expression from the generating functional:

$$(-1)^a T^{R}_{a,1} = (a + 1) - a \tau_{1,1} \frac{R^{(+)[a+1]} R^{(-)[a-1]}}{R^{(-)[a+1]} R^{(+)[a-1]}} - a - 1 \frac{B^{(-)[a]}}{B^{(+)[a]}} + (a - 1) \frac{R^{(+)[a]} B^{(-)[a]}}{R^{(-)[a]} B^{(+)[a]}},$$

(3.3)

We see that indeed $T^{R}_{a,1}$ and $T^{L}_{a,-1}$ are real functions for all $a$, since $R^{(\pm)} = B^{(\mp)}$, and hence $Y_{a,0}$ is also real. $Y_{a,0}$ is given by (2.15) with $\Phi$ obtained from (2.16):

$$\Phi(u) = \left(\frac{x^+}{x^-}\right)^L \prod_{j=1}^{K_s} \sigma^2(u, u_{4,j}) \left(\frac{R^{(-)[a]} R^{(+)[a]}}{R^{(+)[a]} R^{(-)[a]}\right)^2 \frac{Q^+}{Q^-}.$$  (3.4)

4. Weak coupling expansion

To obtain the leading wrapping correction to operator anomalous dimensions, we insert into (2.8) the $Y$-functions given by (2.15) and expand them at weak coupling, as in [8, 42]. For $g \to 0$ we have

$$\frac{R^{(+)a}[a](a+1)}{R^{(-)[a]}[a-1]} \simeq \frac{Q^+[a+1]}{Q^-[a-1]}, \quad \frac{B^{(-)[a]}[a]}{B^{(+)[a]}[a-1]} \simeq \frac{Q^-[a-1]}{Q^+[a+1]},$$

(4.1)

$$\frac{x^\text{min}[-a]}{x^\text{max}[+a]} \simeq \frac{4g^2}{a^2 + 4at^2}, \quad \Phi_a \simeq \left(\frac{4g^2}{a^2 + 4at^2}\right)^L \frac{Q^+[a+1] Q^-[a-1]}{Q^-[a+1] Q^+[a-1]},$$

(4.2)

and hence $T^{R,L}_{a,1}$ are rational functions of $u$. In addition,

$$\frac{\partial \Phi_a(u)}{\partial u} \simeq -2i.$$  (4.3)
so that equation (2.8) can be written as

$$\delta E \simeq - \sum_{a=1}^{\infty} \int \frac{du}{\pi} Y_{a,0}(u).$$

(4.4)

To the order $g^2L$ the Bethe roots $u_{4,j}$ can be simply found from (3.1) [7]. We will see that explicitly for the single magnon case.

Note that in the $su(2)$ sector at weak coupling the expression for $Y_{a,0}$ is a rational function with poles at $u = \pm i \frac{4}{2}$ and $u = u_{4,j} \pm i \frac{4}{2}$. As such, for any particular value of $L$ it is straightforward to evaluate the integral in (4.4). For arbitrary $L$ the integrand can be decomposed as

$$Y_{a,0}(u) \simeq A([u_j], a, q) \left( \frac{4}{a^2 + 4u^2} \right)^L + \sum_{j=1}^{K_a} \sum_{n_1=\pm 1, n_2=\pm 1} B_{j,n_1,n_2}([u_j], a, q) \left( \frac{4}{a^2 + 4u^2} \right)^L \frac{1}{u - u_j \pm \frac{\eta_1 + \eta_2}{2}}.$$  

(4.5)

and the integration is done with the use of identities

$$\int \left( \frac{4}{a^2 + 4u^2} \right)^L du = \sqrt{\pi} \frac{2^{2L-1} \Gamma(L - \frac{1}{2})}{a^{2L-1} \Gamma(L)},$$

(4.6)

$$\int \left( \frac{4}{a^2 + 4u^2} \right)^L \frac{1}{2\pi i u - iv} \frac{1}{2} \frac{1}{u - iv} du = (-1)^{L+1} \left( \frac{2}{a} \right)^{2L-1} \frac{\sqrt{\pi}}{2i \Gamma(L)} \bar{F}_1 \left[ \frac{1}{2}, 1; \frac{3}{2} - L; \frac{a^2}{4v^2} \right]$$

$$+ \frac{1}{2} \left( \frac{4}{a^2 + 4v^2} \right)^L,$$

(4.7)

where $\text{Re} v > 0$ and $L$ is large enough\(^{11}\). It would be interesting to see whether expressions similar to (4.6) and (4.7) come from diagrammatic computations as well.

### 4.1. Konishi operator

As a first application of our method, we will reproduce the results obtained in [27] for the wrapping correction to the $su(2)$ Konishi operator dimension. The $Y_{a,0}$ functions are obtained from (2.15) in terms of the two Bethe roots $u_{4,1}$ and $u_{4,2}$, which can be found from the ABA equation (3.1). At order $g^0$ they are given by

$$u_{4,1} = \frac{(1 - 3\Delta)^2}{2\sqrt{9\Delta^2 - 1}(3\sqrt{1 - \Delta^2} + 2\sqrt{\frac{2}{3\Delta + 1}})},$$

(4.8)

$$u_{4,2} = \frac{(1 - 3\Delta)^2}{2\sqrt{9\Delta^2 - 1}(3\sqrt{1 - \Delta^2} - 2\sqrt{\frac{2}{3\Delta + 1}})},$$

(4.9)

where $\Delta = \sqrt{\frac{2 + 4 \cos(4\pi \beta)}{1}}$. For the Konishi operator $L = 4, K_a = 2$, and hence we have $\tau_{1,L} = \tau_{1,R} = q^4$. Using expansions (4.1), (4.2) and formula (3.3) for the $T_{a,1}$ functions, we

\(^{11}\) About $\bar{F}_1$ see [43].
get from (4.4) the following result:

\[
\delta E = -\sum_{a=1}^{\infty} \int \frac{du}{\pi} \left[ \frac{4g^2}{u^2 + 4u^2} \right]^4 \frac{Q_4^{[a+1]} Q_4^{[-a+1]}}{Q_4^{[a]} Q_4^{[-a]}} \times \left( (a + 1) - aq^4 \right) \frac{Q_4^{[a+1]} Q_4^{[-a+1]}}{Q_4^{[a]} Q_4^{[-a]}} + (a - 1) \frac{Q_4^{[a]} Q_4^{[-a]}}{Q_4^{[a]} Q_4^{[-a]}} \right]^2.
\]  (4.10)

This expression coincides with the wrapping correction given by equation (25) of [27]. As such, the leading wrapping correction we get is exactly the same as the one obtained in that work. This is an important check of our twisted asymptotic solution of the $Y$-system.

4.2. Single magnon momentum quantization

Consider now the single magnon case. It is relatively easy to obtain the momentum of a single magnon, as it coincides with the total momentum of the state. It is natural to assume that the total momentum, similar to the total energy, can be written as

\[
P = \sum_{j=1}^{K_4} \frac{1}{i} \log \frac{x_{4,j}^+}{x_{4,j}^-} + \delta P, \quad (4.11)
\]

with $\delta P$ given by an expression analogous to (2.8):

\[
\delta P = \sum_{a=1}^{\infty} \int \frac{du}{2\pi i} \frac{\partial p_{\pi a}^m}{\partial u} \log(1 + Y_{a,0}), \quad (4.12)
\]

and the momentum quantization condition then reads

\[
\sum_{j=1}^{K_4} \frac{1}{i} \log \frac{x_{4,j}^+}{x_{4,j}^-} + \delta P = 2\pi\beta + 2\pi m, \quad m \in \mathbb{Z}. \quad (4.13)
\]

This gives

\[
E = \sqrt{1 + 16g^2 \sin^2 \left( \frac{2\pi\beta - \delta P}{2} \right)} + \delta E, \quad (4.14)
\]

and the exact position of the Bethe root is given by

\[
u_{4,1} = \frac{1}{2} \cot \left( \frac{2\pi\beta - \delta P}{2} \right) \sqrt{1 + 16g^2 \sin^2 \left( \frac{2\pi\beta - \delta P}{2} \right)}. \quad (4.15)
\]

Using the expressions for $Y_{a,0}$ from section 3 it is straightforward to compute the anomalous dimension of single-impurity operators up to the order $g^{4L-2}$. However, at the moment the perturbation theory results are not available beyond the order $g^{2L}$. In the next section, we compute the anomalous dimension to that order and give an explicit expression for arbitrary $L$ and $\beta$.

4.3. Single magnon energy at $g^{2L}$ order

In this section we compute the energy of a single excitation at the order $g^{2L}$ for arbitrary $L$ and $\beta$. For that we note that in (4.14) the quantity $\delta P$, being of the order $g^{2L}$, contributes only to the energy at $g^{4L+2}$, as usual [7]. Thus we can set $\nu_{4} = \frac{1}{g} \cot \pi\beta$ which is the value of $\nu_{4}$ at zeroth order in $g$. We use the weak coupling expansion (4.1), (4.2), and the integral in (4.4) is
straightforward to evaluate, as the integrand is a rational function of \( u \). Decomposing \( Y_{a,0}(u) \) as in (4.5) and using (4.6) and (4.7) to integrate the rational functions, we find that the integral equals

\[
\mathcal{I}(L, a) = \sqrt{\pi} (-4)^L \frac{(q - \bar{q})^2(q^{L-1} - \bar{q}^{L-1})}{\Gamma(L)q^{2L-2}} \left( G_a(q) - G_{-a}(q) \right)
- G_a(\bar{q}) + G_{-a}(\bar{q}) + aG^0(q), \tag{4.16}
\]

where \( \bar{q} = 1/q \) and

\[
G_a(q) = \frac{-1}{2\pi} \left( q^{L-1} - \bar{q}^{L-1} \right) \Gamma(L - 1/2).
\]

The wrapping correction is given by \( \delta E = \sum_{n=1}^{\infty} \mathcal{I}(L, a) \). We found\(^{12} \) that instead of summing over \( a \) one can equivalently expand the above function at \( a = 0 \)

\[
\mathcal{I}(L, a) = \frac{A_{2L-3}(q)}{a^{2L-3}} + \frac{A_{2L-5}(q)}{a^{2L-5}} + \cdots,
\]

and the coefficients in this series expansion give the coefficients in front of zeta functions in the final result

\[
\delta E = \sum_{n=L-1}^{2L-3} A_n(q)\zeta(n). \tag{4.20}
\]

For fixed \( L \) one needs to compute a finite number of terms in expansion (4.19) of the generating function.

The above result agrees with the perturbation theory calculation. The expression for \( \delta E \) obtained with the use of diagrammatic techniques has the following structure\(^{13} \) [23, 24]:

\[
\delta E = -2L(4\pi q)^{2L} \left[ \left(C^0_0 - C^0_{L-1}\right) P^{(L)} - 2 \sum_{j=0}^{\left(\frac{3}{2}\right)-1} \left(C^{(L)}_j - C^{(L)}_{L-j-1}\right) I^{(L)}_{j+1} \right], \tag{4.21}
\]

where

\[
C^{(L)}_j = (q - \bar{q})^{2} (q^{2L-2j-2} + \bar{q}^{2L-2j-2}),
\]

and \( P^{(L)} \) is some known function of \( L \), while \( I^{(L)}_{j+1} \) represent some particular \( L \)-loop momentum integrals\(^{14} \). Those integrals were computed in [23, 24] explicitly up to 11 loops (i.e. for \( L \leq 11 \)), and inserting them into (4.21) we find complete agreement with our calculations based on (4.20).

In fact, those integrals can be directly obtained for any \( L \) with the use of (4.16), (4.19) and (4.20). Namely, by inspecting expression (4.21) we note that the formal expansion about \( q = 0 \) has the following structure modulo some explicit functions of \( L \):

\[
\delta E \equiv \frac{I^{(L)}_1}{q^{2L}} + \frac{I^{(L)}_2}{q^{2L-2}} + \frac{I^{(L)}_3}{q^{2L-4}} + \frac{I^{(L)}_4}{q^{2L-6}} + \cdots. \tag{4.23}
\]

We see that by matching the various powers of \( q \) with the explicit expressions (4.20), (4.19) and (4.16) we obtained above, one can easily find each of those basis momentum integrals.

\(^{12} \) We have checked this explicitly for \( L \leq 11 \).

\(^{13} \) Note that \( I \) in [22, 24] is denoted by \( \chi^2 \) in the present paper.

\(^{14} \) More precisely, their singular parts. What we denote by \( I^{(L)}_{j+1} \) is \( \lim_{\varepsilon \to 0} \varepsilon I^{(L)}_{j+1}(\varepsilon) \) in the original notations.
Thus we see that the presence of the deformation parameter $\beta$ (or, equivalently, $q$) allows us to extract the perturbation theory integrals $I^{(L)}_j$ directly from (4.23).

It would be interesting to repeat this computation in the next to critical order in $g$ where the integrals arising and the structure of the result in the perturbation theory should be considerably more complicated. At the same time for single magnon the Y-system calculation should be possible to perform up to the order $g^{4L-2}$.

5. Conclusions

Using the Y-system techniques, we found a general expression for an arbitrary length first wrapping correction for single-impurity operators. Our result in the form of a generating function allows us to extract directly the relevant Feynman integrals $I^{(L)}_j$, which can be used in the perturbative calculations. In addition, the asymptotic Bethe equations we got in our approach are in complete agreement with the ABA of [20].

We also hope that our results could shed some light on the relation between perturbative techniques and the AdS/CFT Y-system. It seems that the additional parameter $\beta$ of the deformed theory could make more transparent the relation and could finally lead to a derivation of the Y-system directly from perturbation theory. In addition, it would be interesting to investigate more general deformations of $\mathcal{N} = 4$ SYM, and see whether integrability techniques give results consistent with perturbative calculations.

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Appendix. Twisted Bethe equations

The twisted Bethe equations corresponding to deformations of the $\mathcal{N} = 4$ SYM theory were proposed in [20]. In this appendix, we show that under a certain choice of our twists $\tau_{1, L}$, $\tau_{2, L}$, $\tau_{1, R}$, $\tau_{2, R}$ those equations coincide with equations (2.14) and (2.17) obtained in the Y-system framework. This is true for a general deformation considered in [20], which includes the $\beta$-deformation as a special case.

The general deformation is described in [20] by three real parameters $\delta_1$, $\delta_2$, $\delta_3$, and the ABA are given by equation (5.39) in that work. The notation used in [20] is slightly different from ours: $g_{[20]} = \sqrt{2}g$, $x_{[20]} = gx$. The phases $\delta_i$ enter the ABA through the matrix $A$, which is given by equation (5.24) in [20]. In our notation, the ABA equations of [20] can be written as

$$\lambda_1 \frac{B^{(+)}}{R^{(++)}} \frac{Q^*_7}{Q^*_2} \bigg|_{u = a_{1,4}} = 1, \quad \frac{Q^*_1}{Q^*_7} \frac{Q^*_5}{Q^*_3} \bigg|_{u = a_{1,5}} = -1, \quad \lambda_1 \frac{R^{(-)}}{R^{(++)}} \frac{Q^*_5}{Q^*_3} \bigg|_{u = a_{1,5}} = 1, \quad (A.1)$$

$$\lambda_2 \frac{B^{(+)}}{R^{(++)}} \frac{Q^*_6}{Q^*_2} \bigg|_{u = a_{1,6}} = 1, \quad \frac{Q^*_3}{Q^*_5} \frac{Q^*_5}{Q^*_7} \bigg|_{u = a_{1,7}} = -1, \quad \lambda_2 \frac{R^{(-)}}{R^{(++)}} \frac{Q^*_5}{Q^*_7} \bigg|_{u = a_{1,7}} = 1, \quad (A.2)$$

15 The authors thank R Roiban for helpful comments concerning the ABA of [20].

16 Up to factors of the form $\sigma(u, v)$, which were not known at the time when [20] appeared.
\[ \frac{1}{\lambda_1 \lambda_2} \left( \frac{x^-}{x^+} \right)^L B_1^- B_2^- B_3^- B_4^- Q_1^- Q_2^- Q_4^- \left|_{u=\mu,j} \right. = -1, \quad (A.3) \]

where

\[ \lambda_1 = \exp(i(K_4(\delta_1 + \delta_2 + \delta_3) + K_5(-\delta_1 - \delta_2 - \delta_3) + K_7(-\delta_1 - \delta_2) + K_1 \delta_3 - L \delta_3)), \quad (A.4) \]

\[ \lambda_2 = \exp(i(K_3(\delta_1 + \delta_2 + \delta_3) + K_4(-\delta_1 - \delta_2 - \delta_3) - K_7 \delta_1 + K_4(\delta_2 + \delta_3) + L \delta_3)), \quad (A.5) \]

and we have used the twisted zero momentum condition (equation (5.39) for \( j = 0 \) in [20])

\[ \prod_{j=1}^{K_4} \frac{x_j^+}{x_j^-} = \exp(-i(K_4(\delta_1 - \delta_3) - \delta_1 K_5 - \delta_1 K_7 - \delta_3 K_4)) \quad (A.6) \]

which follows from the ABA equations. We see that those equations coincide with our equations (2.14) and (2.17) if

\[ \tau_{1,R} = \frac{1}{\lambda_1}, \quad \tau_{2,R} = 1, \quad \tau_{1,L} = \frac{1}{\lambda_2}, \quad \tau_{2,L} = 1. \quad (A.7) \]

The \( \beta \)-deformation, discussed throughout this paper, corresponds [20] to the choice

\[ \delta_1 = -2\pi \beta, \quad \delta_2 = 0, \quad \delta_3 = 0. \quad (A.8) \]

In this case, we have

\[ \tau_{1,R} = e^{2(K_4 - 2K_3 - 2K_5) \pi i \beta}, \quad \tau_{1,L} = e^{2(K_3 - 2K_1 - 2K_7) \pi i \beta}, \quad (A.9) \]

and the level matching condition (A.6) takes the form

\[ \prod_{j=1}^{K_4} \frac{x_j^+}{x_j^-} = \tau_{1,R}. \quad (A.10) \]

For the \( su(2) \) subsector the twists (A.9) are in agreement with (3.2).

Note that, as expected (see [20]), the twists (A.9) are invariant under each of the transformations

\[ K_3 \to K_3 - 1, \quad K_1 \to K_1 + 1, \quad L \to L + 1 \quad (A.11) \]

\[ K_5 \to K_5 - 1, \quad K_7 \to K_7 + 1, \quad L \to L + 1. \quad (A.12) \]

**Duality transformation and the \( sl_2 \) sector**

The above equations allow us to describe an arbitrary state of the \( \beta \)-deformed theory. However, for some applications it may be more convenient to pass to a dualized system of Bethe roots.\(^\text{19}\)

The transformation properties of the transfer matrices under the duality were discussed in [8, 11]. To switch from \( su(2) \) to \( sl_2 \) grading, we apply the fermionic duality following [5], along the lines of [35] and appendix B of [11]. From Bethe roots \( u_{1,j}, u_{3,j}, u_{5,j}, u_{7,j} \).

\(^{17}\) Another possibility is to choose \( \tau_{1,R} = -\frac{1}{\tau_1}, \tau_{2,R} = -1, \tau_{1,L} = -\frac{1}{\tau_2}, \tau_{2,L} = -1 \). This gives \( T \)-functions which differ by gauge transformation (see [8]) from the ones obtained with the choice (A.7), and thus the \( Y \)-functions and the energy spectrum do not change.

\(^{18}\) Note also that in the \( su(2) \) sector exchanging the values of \( \tau_{1,L} \) and \( \tau_{1,R} \) does not alter the \( Y_{u,0} \) functions and the leading wrapping corrections to the anomalous dimensions.

\(^{19}\) This section was added after the appearance of [44, 45] to make easier the comparison with results of these papers.
we switch to new ones \( u_{1,j}, u_{3,j}, u_{5,j}, u_{7,j} \). We first consider the general three-parameter deformation (see the previous section). Following [35] we take

\[
K_1 = K_2 - K_1, \quad K_3 = K_2 + K_4 - K_3, \quad K_5 = K_6 + K_4 - K_5, \quad K_7 = K_6 - K_7.
\]

(A.13)

The new Bethe roots \( u_{1,j}, u_{3,j} \) are related to the original ones by duality relations:

\[
\begin{aligned}
\frac{R_2^+ R_1^+ R_3^+ R_4^+}{R_3^+ R_1^+ R_3^+ R_5^+} &= \frac{R^{(\rightarrow)+} Q_2^+ - \tau_{1,R} R^{(\rightarrow)} Q_2^+}{R^{(\rightarrow)-} Q_2 - \tau_{1,R} R^{(\rightarrow)} Q_2^-} Q_2 \quad \text{(A.14)} \\
\frac{B_2^+ R_1^+ R_3^+ R_5^+}{B_3^+ R_1^+ R_3^+ R_4^+} &= \frac{B^{(\rightarrow)+} Q_2^+ - \tau_{1,R} B^{(\rightarrow)} Q_2^+}{B^{(\rightarrow)-} Q_2 - \tau_{1,R} B^{(\rightarrow)} Q_2^-}. \quad \text{(A.15)}
\end{aligned}
\]

The corresponding relations for the other (left) wing, which involves \( u_5, u_6, u_7 \) roots, are obtained by replacing subscripts \([1, 2, 3, 4, 5, 6, 7]\) and replacing \( \tau_{1,2,R} \to \tau_{1,2,L} \). This remark holds true for all relations in this section.

Using (A.14) and (A.15) we can rewrite for example \( T_{1,1} \) obtained from (2.10) and (2.13) as

\[
T_{1,1} = \frac{1}{\tau_{1,R}} \frac{B^{(\rightarrow)-} Q_1^+ Q_2^+}{B^{(\rightarrow)+} Q_1^+ Q_2^-} - \frac{Q_1^+ Q_2^+}{Q_1^+ Q_2^-} + \tau_{1,R} \frac{R^{(\rightarrow)+} Q_1}{R^{(\rightarrow)-} Q_1} + \frac{1}{\tau_{2,R}} \frac{Q_1^+ Q_2^-}{Q_1^+ Q_2^+} - \frac{1}{\tau_{1,R}} \frac{Q_1^+ Q_2^-}{Q_1^+ Q_2^+}.
\]

(A.16)

where the gauge factor is

\[
f(u) = \frac{1}{\tau_{1,R}} \frac{B_2^+ R_1^+ B_4^+ R^{(\rightarrow)+}}{B_1^+ B_3^+ B_5^+ R^{(\rightarrow)-}}.
\]

(A.17)

and the twists are

\[
\begin{align*}
\tilde{\tau}_{1,R} &= \tau_{1,L} = \exp(i \frac{1}{2} (\delta_3 (K_1 - 2K_2 + K_3) - \delta_1 (K_5 - 2K_6 + K_7))) \\
\tilde{\tau}_{2,R} &= \exp(i \frac{1}{2} ((\delta_1 + 2\delta_2) (K_5 - 2K_6 + K_7) + \delta_3 (K_1 - 2K_2 + K_3 - 2K_6 - 2L))) \\
\tilde{\tau}_{2,L} &= \exp(i \frac{1}{2} (2\delta_2 (K_2 - 2K_3 - K_5 + 2L) - 2\delta_3 (K_1 - 2K_2 + K_3) - \delta_1 (K_1 - 2K_2 + K_3))).
\end{align*}
\]

(A.18)

From equation (A.8), we see that for the \( \beta \)-deformed theory the twists are given by the above expressions with \( \delta_1 = -2\pi \beta, \delta_2 = 0, \delta_3 = 0 \).

Note that the transformations (A.11), (A.12) (see [20]) do not affect the twists (A.18), as from (A.13) we see that these transformations amount to

\[
\begin{align*}
K_1 &\to K_1 + 1, \quad K_2 \to K_2 - 1, \quad L \to L + 1, \quad (A.19) \\
K_5 &\to K_5 + 1, \quad K_7 \to K_7 - 1, \quad L \to L + 1. \quad (A.20)
\end{align*}
\]

The duality transformation can also be done on the level of the generating functional. One should use\(^{20}\) (alternatively to (2.10) and (2.13)) the following functional:

\[
\mathcal{W}_{sl(2)} = \left(1 - D \tilde{\tau}_{1,R} \frac{B^{(\rightarrow)+} Q_1}{B^{(\rightarrow)-} Q_1} D\right) \left(1 - D \tilde{\tau}_{2,R} \frac{Q_2}{Q_2} D\right) \left(1 - D \frac{1}{\tau_{1,R}} \frac{Q_2}{Q_2} D\right) \left(1 - D \frac{1}{\tau_{1,R}} \frac{Q_2}{Q_2} D\right)
\]

(A.21)

\(^{20}\) It is straightforward to check this e.g. in Mathematica.
which is built using the new roots \( u_{1,j}, u_{3,j} \). The \( T_{a,s} \) functions are obtained from

\[
W_{sl(2)} = \sum_{a=0}^{\infty} D^a[T_{1,a}(u)f_a(u)]D^a, \\
W_{su(2)}^{-1} = \sum_{a=0}^{\infty} (-1)^a D^a[T_{a,1}(u)f_a(u)]D^a,
\]

(A.22)

where the factor

\[
f_a(u) = \prod_{k=-\frac{1}{2}}^{\frac{1}{2}} f(u + ik)
\]

(A.23)
corresponds to a gauge transformation on the \( T \)-functions \[8\]. This functional gives BAEs in \( sl(2) \) grading as a condition of pole cancellation in \( T_{1,1} \):

\[
\begin{align*}
1 &= \frac{\tilde{\xi}_{1,R}}{\xi_{1,R}} \frac{B(-\tau_1^a)}{B(\tau_1^a)} \left| Q_1^\tau \right|_{a=1_1}, \\
1 &= \frac{\tilde{\xi}_{1,R}}{\xi_{1,R}} \frac{R(-\tau_1^a)}{R(\tau_1^a)} \left| Q_1^\tau \right|_{a=1_3}. 
\end{align*}
\]

(A.24)

These Bethe equations coincide with the ones given in the appendix of \[45\], which can be shown taking into account that the quantity \( J \) in that work can be written in our notation as

\[
J = \frac{1}{2}(K_1 - K_3 - K_5 + K_7 + 2L)
\]

(A.25)

in accordance with (E.22) in \[45\]).

Using (A.21) we can establish a relation between \( T \)-functions in different gradings. Let us denote by \( T_{a,s}^{sl(2)}(u|[u_{1,j}], [u_{3,j}], \tau_1, \tau_2) \) the \( T \)-functions obtained via (2.13) from the initial functional (2.10). Denote also by \( T_{a,s}^{su(2)}(u|[u_{1,j}], [u_{3,j}], \tau_1, \tau_2) \) the \( T \)-functions in the \( sl(2) \) grading, i.e. the \( T \)-functions obtained from \( T_{a,s}^{su(2)}(u|[u_{1,j}], [u_{3,j}], \tau_1, \tau_2) \) by switching to the new Bethe roots \( \tilde{u}_{1,j}, \tilde{u}_{3,j} \) via duality relations. This means that

\[
T_{a,s}^{sl(2)}(u|[\tilde{u}_{1,j}], [u_{3,j}], \tilde{\xi}_{1,R}, \tilde{\xi}_{2,R}) = T_{a,s}^{su(2)}(u|[u_{1,j}], [u_{3,j}], \tau_1, \tau_2).
\]

(A.26)

with the relation between \( u \) roots and their tilded counterparts being defined by (A.14), (A.15). However, \( T_{a,s}^{sl(2)} \) and \( T_{a,s}^{su(2)} \) have different functional form when we consider their arguments as arbitrary parameters (e.g. for \( T_{a,s}^{sl(2)} \) these arguments are \( [u_{1,j}], [u_{3,j}], \tau_1, \tau_2 \)). Nevertheless, it turns out that there are functional relations which allow us to write \( T_{a,s}^{sl(2)} \) in terms of \( T_{a,s}^{su(2)} \). Roughly speaking, these relations amount to exchanging \( a \) and \( s \) and then taking complex conjugation. Their precise form is

\[
\begin{align*}
T_{a,1}^{sl(2)}(u|[\tilde{u}_{1,j}], [u_{3,j}], \tilde{\xi}_{1,R}, \tilde{\xi}_{2,R}) &= (-1)^a (f_a(u))^{-1} T_{1,a}^{su(2)}(u|[u_{1,j}], [u_{3,j}], \tilde{\xi}_{1,R}, \tilde{\xi}_{2,R}), \\
T_{1,a}^{sl(2)}(u|[\tilde{u}_{1,j}], [u_{3,j}], \tilde{\xi}_{1,R}, \tilde{\xi}_{2,R}) &= (-1)^a (f_a(u))^{-1} T_{a,1}^{su(2)}(u|[u_{1,j}], [u_{3,j}], \tilde{\xi}_{1,R}, \tilde{\xi}_{2,R}).
\end{align*}
\]

(A.27)

where the bar denotes complex conjugation in the ‘physical’ plane, i.e. the replacement: \( Q_1^{[a]} \rightarrow Q_1^{[-a]}, Q_2^{[a]} \rightarrow Q_2^{[-a]}, Q_3^{[a]} \rightarrow Q_3^{[-a]}, B^{[a]} \rightarrow B^{[-a]}, R^{[a]} \rightarrow R^{[-a]}, \tilde{\xi}_{1,R} \rightarrow (\tilde{\xi}_{1,R})^{-1}, \tilde{\xi}_{2,R} \rightarrow (\tilde{\xi}_{2,R})^{-1} \). Relations (A.27) follow from the expressions for generating functions (2.10) and (A.21), after one takes the Hermitian conjugate of (A.21). For the undeformed theory (i.e. when \( \delta_1 = \delta_2 = \delta_3 = 0 \) relations (A.27) reproduce\[21\] those given in [8] for the switch between \( sl(2) \) and \( su(2) \) gradings.

For example, in the \( sl(2) \) sector we have \( K_1 = K_3 = K_5 = K_7 = K_2 = K_6 = 0 \), and from (A.18) we get

\[
\tilde{\xi}_{1,R} = \tilde{\xi}_{1,L} = 1, \quad \tilde{\xi}_{2,R} = e^{-iL_1}, \quad \tilde{\xi}_{2,L} = e^{iL_1}
\]

(A.28)
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