Existence of Weak and Strong Solutions of Nonlinear Differential Equations with Delay in Banach Space

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Abstract. Existence of weak and strong solutions of nonlinear differential equations with delay in Banach space is discussed. In the present work we give a generalization to recent results. We prove that, with certain conditions, every nonlinear differential equation with delay has at least one weak solution, furthermore, under suitable assumptions, these equations have solutions. Next under a generalization of the compactness assumptions, we show the same equations have solutions too.

1. Introduction.
In various applied fields such as control theory, mathematical economics and stochastic nonlinear programming deal with some integral functionals. It is well known that the nonlinear integral equations of convolution type arise very often in applications, especially in numerous branches of mathematical physics. We note that the concept measure of noncompactness is one of the most useful concepts of general topology.

In this paper we deal with the existence of weak and strong solutions for the differential equation with delay

\[ \dot{x}(t) = L(t)x(t) + f(t,\theta_{tx}), \quad t \in [0,T]. \]

In fact, if \( L(t) \neq 0 \) our results generalize that of Gomaa[5] and Cichon [2], since we have a generalization of the compactness assumptions and in [4] the results stated without delay. For the important case \( L(t) = 0 \) we have, as a special case, some generalization to the existence theorems of Gomaa [6], and Cramer-Lakshmikantham-Mitchell [4] in which the results stated without delay.

2. Preliminaries.
In this paper the dual space of an infinite dimensional Banach space \( E \) will be denoted by \( E^* \) and the pairing between \( E \) and \( E^* \) is designated by \( \langle , \rangle \). \( L(R^+, E) \) be the space of measurable functions \( u: R^+ \rightarrow E \), \( L(E) \) be the space of linear operators from \( E \) into itself and \( \lambda \) be the Lebesgue measure on \( I = [0,T] \). Further, let \( C(I,E) \) be the space of all continuous functions from \( I \) to \( E \) with the usual supremum norm and \( C_w(I,E) \) be the space of all weakly continuous functions from \( I \) to \( E \) endowed with the topology of weak uniform convergence, \( C_b([-d,0]) \) be the Banach space of continuous functions from the closed interval \([-d,0]\) \( (d \geq 0) \) into \( E \) and \( \beta \) be the family of all bounded subsets of \( E \).

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Definition 1. By a measure of weak (strong) noncompactness, $\gamma$, we will understand a function $\gamma: \beta \to \mathbb{R}^+$ such that, for all $U, V \in \beta$,

\[ (M_1) \ U \subset V \Rightarrow \gamma(U) \leq \gamma(V), \]
\[ (M_2) \ \gamma(U \cup V) \leq \text{max}(\gamma(U), \gamma(V)), \]
\[ (M_3) \ \gamma(\text{conv}(U)) = \gamma(U), \]
\[ (M_4) \ \gamma(U \cup V) \leq \gamma(U) + \gamma(V), \]
\[ (M_5) \ \gamma(\{x\}) = \gamma(U), \ x \in E. \]

Assume that $M = M(\mathbb{R}^+, E)$ is a Banach space of measurable functions $x: \mathbb{R}^+ \to E$ with $\|x\| \in M(\mathbb{R}^+, \mathbb{R})$, $\|x\|_{M(\mathbb{R}^+, \mathbb{R})}$, where

- $M(\mathbb{R}^+, \mathbb{R}) \subset L(\mathbb{R}^+, \mathbb{R})$,
- $M(\mathbb{R}^+, \mathbb{R})$ contains all essentially bounded function with compact support,
- If $x \in M(\mathbb{R}^+, \mathbb{R})$, $y: \mathbb{R}^+ \to \mathbb{R}$ is measurable with $|y| < |x|$, then $y \in M(\mathbb{R}^+, \mathbb{R})$ and $\|y\|_{M(\mathbb{R}^+, \mathbb{R})} < \|x\|_{M(\mathbb{R}^+, \mathbb{R})}$,
- if $x \in M(\mathbb{R}^+, \mathbb{R})$, $x_n \in M(\mathbb{R}^+, \mathbb{R})$, $|x_n| \leq |x|$ and $\lim_{n \to +\infty} x_n(t) = 0$ a.e. on $\mathbb{R}^+$, then $\lim_{n \to +\infty} \|x_n\|_{M(\mathbb{R}^+, \mathbb{R})} = 0$.

Let $M'$ denote the associate space to $M$ [10].

Definition 2. A continuous function $x: [-d,T] \to E$ is called a weak solution of problem (P) if, for some $\xi \in C([-d,0], E)$ and $x(t) = G(t,0)$ $\xi(t)$ for all $t \in I$. Let $F: E \to C(I, E)$ and $F(x(t))$ a.e. on $I$, $F(x(t)) \leq a(t)$ for all $x \in C(I, E)$, and $y(t) \to y_0(t)$ a.e. on $I$, then $y(t) \in F(x_0(t))$ a.e. on $I$.

Lemma 3. [9] If $\gamma: \beta \to \mathbb{R}^+$ satisfies conditions $(M_3),(M_4),(M_5)$, then, for any nonempty $U \in \beta$, $\gamma(U) \leq \gamma(B_1) \alpha(U) \leq 2 \gamma(B_1) \beta(U)$.

Lemma 4. [8] If $\gamma$ is a measure of weak (strong) noncompactness and $A(\mathbb{R}^+, C(I, E))$ is a family of strongly equicontinuous functions, then $\gamma(A(I)) = \sup \{\gamma(A(t)) : t \in I\}$.

Lemma 5. [1] Let $Y$ and $E$ be two Banach spaces and $P_{fc}(Y)$ be the set of all closed and convex subsets of $Y$. If $F: E \to P_{fc}(Y)$ is weakly sequentially upper-hemi-continuous. Moreover, if $A(t)$ is a measure of weak (strong) noncompactness and $A(\mathbb{R}^+, E)$ is a family of strongly equicontinuous functions, then $\gamma(A(I)) = \sup \{\gamma(A(t)) : t \in I\}$.

Lemma 6. Let $F: \mathbb{I} \to L(E)$ and $U$ be a bounded subset of $E$. Thus $\gamma(\bigcup_{t \in \mathbb{I}} F(t)U) \leq \sup_{t \in \mathbb{I}} \|F(t)\| \gamma(U)$. Proof. $U$ is bounded, so $\exists \varepsilon > 0; \|U\| = \sup \{|u| : u \in U\} \leq c$. For $\varepsilon > 0 \exists \delta > 0$ such that if $P = \{x_0, x_1, x_2, ..., x_n\}$ is a partition of $I$, $\ a = x_0 < x_1 < x_2 < ... < x_n = b$ with $|P| = \sup \{|x_{i+1} - x_i| : i = 0, 1, 2, ..., n-1\} < \delta$, then $\|F(x_{i+1}) - F(x_i)\| < \varepsilon$. As $B_1$ is the closed unite ball in $E$, $U \subset K + \frac{\gamma(U) + \varepsilon}{\gamma(B_1)} B_1$, $K$ is weakly compact, as $t \in I_1 = [x_n, x_{n+1}]$, $F(t)U \subset \bigcup_{t \in \mathbb{I}} F(t)u - F(t_{i+1})u : u \in U \cup F(t_{i+1})U$, $\|F(t) - F(t_{i+1})\| \|U\| \leq \varepsilon$ so, $\bigcup_{t \in \mathbb{I}} F(t)u - F(t_{i+1})u : u \in U \subset \varepsilon B_1$ and $\bigcup_{t \in \mathbb{I}} F(t)U \subset \varepsilon B_1 + F(t_{i+1})U$. Thus $\bigcup_{t \in \mathbb{I}} F(t)U \subset 2\varepsilon B_1 + \bigcup_{t \in \mathbb{I}} F(t_{i+1})K + \sup_{t \in \mathbb{I}} \|F(t)\| \frac{\gamma(U) + \varepsilon}{\gamma(B_1)} B_1.$

Moreover $\gamma(\bigcup_{t \in \mathbb{I}} F(t)U) \leq 2\varepsilon \gamma(B_1) + \sup_{t \in \mathbb{I}} \|F(t)\| \|\gamma(U) + \varepsilon\|$, where $\bigcup_{t \in \mathbb{I}} F(t)U$ is weakly compact since $\varepsilon$ is arbitrary, we deduce that $\gamma(\bigcup_{t \in \mathbb{I}} F(t)U) \leq \sup_{t \in \mathbb{I}} \|F(t)\| \gamma(U)$.

If $t \in \mathbb{R}^+$, $A(t) \in L(E)$ and $\dot{x}(t)$ denotes the weak derivative of $x$ at $t$, then we consider the differential equation

\[ \dot{x}(t) = A(t) x(t). \]
Let E be the direct sum of $\varepsilon_0$ and $\varepsilon_1$, $\varepsilon_0 = \{x_0 \in E: \exists a$ bounded weak solution $x$ of (1), $x(0) = x_0 \}$ is closed and has a closed complement $\varepsilon_i$. Let $G \in C(R^+ \times R^+, E)$ be the Green function corresponding to (1):

$$G(t,s) = \begin{cases} S(t)PS^{-1}(s) & \text{if } 0 \leq s \leq t \\ -S(t)(id - P)S^{-1}(s) & \text{if } 0 \leq t \leq s, \end{cases}$$  

(2)

$S: R^+ \to L(E)$ is a solution of the equation $\dot{S}(t) = A(t)S(t)$, $S(0) = id$ and $P$ is the projection of $E$ onto $\varepsilon_0$ while $P(\varepsilon_1) = \{0\}$. From (2) and using some consideration from [10] $\exists d > 0$ such that $\|G(t,0)\| \leq d$. Let $x_0 \in \varepsilon_0$, $\|x_0\| \leq \frac{r-c \|m\|_{L(E)}}{d}$. Then $G(t,0)x_0$ is a solution of (1) and $\|G(t,0)x_0\| \leq d|0| \leq r-c \|m\|_{L(E)}$.

3. Main Results

First we shall consider the nonlinear differential equation

$$(Q) \quad \dot{x}(t) = A(t)x(t) + f(t, x(t)), t \in R^+$$

since this problem was studied by many authors ([3,5,7] for instance).

**Theorem 7.** Under the above assumptions and if we assume that, $t \in R^+$, $G(\cdot \cdot) \in M'$ with $\|G(t, \cdot)\|_{M} \leq c$ for some $c \in R^+$; $\gamma$ be a weak measure of noncompactness; $f$ be a continuous function from $R^+ \times E_0$ to $E_0$; $A: R^+ \to L(E)$ is strongly measurable and Bochner integrable on every subinterval $I$ of $R^+$ $\exists$ a function $m: R^+ \to R^+$ belongs to $M'$ such that $\|f(t,x)\| \leq m(t)$ for every $(t,x) \in R^+ \times B$, and $|c|\|m\|_{M} < r$; for each $T > 0$ and for each $\varepsilon > 0$, $\exists$ a closed subset $I_0$ of $[0,T]$ with $\lambda((0,T]-I_0) < \varepsilon$ such that for any nonempty bounded subset $U$ of $E$ one has $\|l(t, \cdot)\| \leq \delta$ for every $\tau > 0$. Hence $\exists$ a bounded weak solution $x$ of (Q). Then, for each $x_0 \in \varepsilon_0$ such that $\|x_0\| \leq \frac{r-c \|m\|_{M} \|G(t,0)\|}{d}$, $\exists$ a bounded weak solution of (Q).

**Proof.** If $S=\{x \in C_w(R^+,E): \|x(t)-x(\tau)\| \leq \int_{\tau}^{t} |A(s)|ds + \int_{\tau}^{t} m(s)ds \leq 0 \leq t \leq \tau\}$ and $\phi(x)(t) = g(t,0)x_0 + \int_{0}^{\infty} G(t,s)f(s,x(s))ds$ for $t \in R^+$ and $x \in S$, then $\|\phi(x)(t)\| \leq \delta \|x_0\| + c\|m\|_{M} \leq r$. Since $y = \phi(x)$ is a weak solution of the equation

$\dot{y}(t) = A(t)y(t) + f(t, x(t))$ so, $\|\phi(x)(t) - \phi(x)(\gamma)\| \leq \int_{\tau}^{\infty} |A(s)|ds + \int_{\tau}^{\infty} m(s)ds$.

Therefore $\phi$ is a continuous mapping from $S$ into $S$ [4]. Let $(x_n)_{n \in N \cup \{0\}}$ be a sequence such that $\phi(x_n) = x_{n+1}$ with $x_0$ is an arbitrary element in $S$. Put $V=\{x_n: n=0,1,2,\ldots\}$, then $V \subseteq S$ and $\gamma(V) = \gamma(\phi(V))$. Now let $t \geq 0$ and $\varepsilon > 0$. Then we can find $T \geq t$ such that $|m|_{M} \{\chi_{[t,\varepsilon]}| \leq \varepsilon/2c$. Also $\exists$ a closed subset $I_0$ of $I$ with $\lambda(I - I_0) < \delta$ and $w$ is uniformly continuous on $I_0 \times [0,2T]$. From our last assumption, $\exists$ a compact subset $I_0$ of $I$ such that $\lambda(I - I_0) < \delta$ and that for any compact subset $C$ of $I_0$ and any bounded subset $Z$ of $E$, $\gamma(\{l(t,C) \}) \leq \sup w(s, \gamma(Z))$. $\phi$ is continuous and $w$ is Caratheodory so $\exists$ a closed subset $I_0$ of $I$, $\delta > 0$, $\eta > 0$, $\eta < \delta$ and for $s_1, s_2 \in I_0$, $\epsilon, \eta, \eta' \in [0,2T]$ such that $|s_1 - s_2| < \delta$, $|t_1 - t_2| < \delta$, then $|w(s_1, \tau) - w(s_2, \tau)| < \varepsilon$ and if $|s_1 - s_2| < \eta$, then $|\gamma(V(s_1)) - \gamma(V(s_2))| < \delta$. Fix $\tau$ with $0 \leq \tau \leq T$ and let $t = \tau + \tau t = t_0 < t_1 < \cdots < t_m < t_\tau$. Then $t_\tau - t_{\tau - i} < \eta$ for $l \in \{1,2,\ldots\}$ such that $|s_1 - s_2| < \delta$, $|t_1 - t_2| < \delta$, then $|w(s_1, \tau) - w(s_2, \tau)| < \varepsilon$ and if $|s_1 - s_2| < \eta$, then $|\gamma(V(s_1)) - \gamma(V(s_2))| < \delta$. Fix $\tau$ with $0 \leq \tau \leq T$ and let $t = t_0 < t_1 < \cdots < t_m = t_\tau$. Then $\exists$ $x(t): x \in S$, $t \in T_\tau$. From Lemma 4, $\gamma(S(t)) = \gamma(S(\{t\}))$, $s_i \in T_\tau$.
From Lemma 3, \( \gamma \left( \int_{QG(t,s)f(s,V(s))ds} \right) \leq c ||m_{0,T,a}|| M \gamma(B_1) \leq \varepsilon \). By the mean value theorem
\[
\int_{QG(t,s)f(s,V(s))ds} \subset \sum_{i=1}^{n} \lambda(T_i, \text{conv} \{G(t,s)f(s,w): s \in T_i, w \in V(s)\}).
\]
By Lemma 8, \( \gamma \left( \int_{QG(t,s)f(s,V(s))ds} \right) \leq \int_{T} ||G(t,s)|| w(s,\gamma(S(s)))ds \leq \tilde{\gamma}(B_1) w(s,\gamma(S(s)))ds \).

On the other hand, \( \gamma(\phi(V(t))) \leq \int_{T} ||G(t,s)|| w(s,\gamma(S(s)))ds \leq \int_{T} ||G(t,s)|| w(s,\gamma(S(s)))ds \).

Let \( h: I \times R^+ \rightarrow R^+ \) be a Caratheodory function. Let for each bounded subset \( Z \) of \( E \), \( \exists \) a measurable function \( \phi: I \rightarrow R^+ \) such that \( h(t,s) \leq \phi(t) \) for each \( (t,s) \in Z \) and \( \phi \) is integrable on \([c,T]\) for each \( c > 0 \). Let for each \( c; 0 < c \leq T \), the identically zero function be the only absolutely continuous function on \([0,c]\) such that the right hand derivative of \( u(t) \) at \( x=0 \), \( D^+u(0) \), exists and \( D^+u(0) = u(0) = 0 \).

**Theorem 9.** If we replace in the setting of Theorem 8 \( w \) by \( h \) and \( f' \) is bounded and continuous, then \((Q')\) has a solution.

**Proof.** By the same argument as in Theorem 9 and from Lemma 5 we get
\[
\rho(t) - \rho(t) \leq \gamma(\int_{T} G(t,s)f(s,V(s))ds) \leq \gamma(B_1) \int_{T} ||G(t,s)|| w(s,\rho(s))ds
\]
where \( \rho(t) = \gamma(V(t)) \). Since \( f' \) is a bounded function, then we can find \( M > 0 \) such that \( ||f'(t,x)|| \leq M, (t,x) \in I \times B_r \). Consider \( N: I \rightarrow R \) is defined by \( N(t) = \sup_{[x], [y] \leq M} ||f'(t,x) - f'(t,y)|| \). We claim that \( N \) is
lower semicontinuous on \([0,T]\) and continuous at 0 \([11]\). Let \(\varepsilon>0\) and \(t_0\) be fixed in \(I\). Then, \(\exists x_1,y_1 \in B;r\),\n\(\|x_1\|,\|y_1\| \leq M\), such that

\[
N(t_0) - \varepsilon/2 \leq \|f'(t_0,x_1) - f'(t_0,y_1)\|. \tag{4}
\]

Moreover, \(f'\) is continuous. Thus \(\exists \delta>0\) such that if \(|t-t_0|<\delta\), \(\|x_1-x\|<\delta\), \(\|y_1-y\|<\delta\), we have

\[
\|f'(t_0,x_1) - f'(t,x)\| < \varepsilon/4 \quad \text{and} \quad \|f'(t_0,y_1) - f'(t,y)\| < \varepsilon/4. \tag{5}
\]

From relations (4) and (5), we get \(N(t_0) - \varepsilon/2 \leq \|f'(t,x) - f'(t,y)\|\). Thus, for each \(t\) with \(|t-t_0|<\delta\), \(\exists x_1,y_1; \|x_1\|,\|y_1\| \leq Mt\) such that \(N(t_0) - \varepsilon \leq \|f'(t,x) - f'(t,y)\|\).

V. From relation (3), \(\rho(\tau) - \rho(t) \leq \min \left(\int_{t}^{\tau} \|G(t,s)\| N(s)\text{d}s, \gamma(B_1) \int_{t}^{\tau} \|h(s,\rho(s))\| \text{d}s\right)\). Moreover, if \(\rho = 0\) on \(I\), then \(\rho\) is absolutely continuous on \(I\) and

\[
\rho(t) = \min \left(\int_{t}^{\tau} \|G(t,s)\| N(s)\text{d}s, \gamma(B_1) \int_{t}^{\tau} \|h(s,\rho(s))\| \text{d}s\right)\text{ for each } x,y \in V.\]

Theorem 10. If we replace in the setting of Theorem 9 \(f\) by \(f^d; m\) by \(m'\) and \(A\) by \(L\), then \((P)\) has a weak solution.

Proof. By the same line as in \([5]\) and for any arbitrary \(n \in \mathbb{N}\), we define \(F_1: \mathbb{R}^n \times E \rightarrow E\) and \(f_1: \mathbb{R}^n \times E \rightarrow E\) by

\[
F_1(t,x) = \begin{cases} \xi(t) & \text{if } t \in [-d,0] \\ \xi(0) + nt(x - \xi(0)) & \text{if } t \in [0,kT/n] \end{cases}
\]

and \(F_1(t,x) = f^d(t,\theta_{kT/n}(F_1(.,x)))\). As in Theorem 7, \(\exists y_1\) with \(y_1 = \xi\) on \([-d,0]\), for \(t \in [0,kT/n]\), \(y_1(t) = G(t,0)\xi(0) + \int_{0}^{t} G(t,s) f_1(s,y_1(s))\text{d}s\), moreover \(\sup_{t \in [0,kT/n]} \|y_1(t)\| \leq r\). Put \(k' = k-1\) and for some \(k \in \{2,3,\ldots,n\}\) \(\exists\) a bounded function \(y_k\), \(y_k = \xi\) on \([-d,0]\) and for \(t \in [0,kT/n]\), \(y_k(t) = G(t,0)\xi(0) + \int_{0}^{t} G(t,s) f_k(s,y_k(s))\text{d}s\). Also let \(F_k: [-d,kT/n] \times E \rightarrow E\) be such that

\[
F_k(t,x) = \begin{cases} y_k(t) & \text{if } t \in [-d,kT/n] \\ y_k(kT/n) + nt(kT/n)(x - y_k(kT/n)) & \text{if } t \in [0,kT/n] \end{cases}
\]

If \(f_k:[kT/n,kT/n] \times E \rightarrow E\), \(f_k(t,x) = f^d(t,\theta_{kT/n}(F_k(.,x)))\), then \(\exists\) a continuous function \(y_k\) defined on \([kT/n,kT/n]\) by \(y_k(t) = G(t,kT/n) y_k(kT/n) + \int_{kT/n}^{t} G(t,s) f_k(s,y_k(s))\text{d}s\). Further, for \(0 \leq s \leq \tau \leq t\),

\[
G(t,s)G(s,r) = G(t,r), \quad \text{for } t \in [kT/n,kT/n], y_k(kT/n) = (kT/n,0)\xi(0) +
\]
\[ \int_0^{k'T/n} G(k'T/n, s) f_k(s, y_k(s)) \, ds. \] Hence \( y_k(t) = G(t,0) \xi(0) + \int_0^t G(t, s) g_k(s, y_k(s)) \, ds, \) where \( g_k(t, y_k(t)) = \begin{cases} f_k'(t, y_k(t)) & \text{if } t \in [0, k'T/n] \\ f_k'(t, y_k(t)) & \text{if } t \in [k'T/n, kT/n] \end{cases} \)

for \( n \in \mathbb{N}, \exists \) a continuous bounded function \( v_n \) with \( v_n = \xi \) on \([-d,0]\) and for \( t \in I \) \( v_n(t) = G(t,0) \xi(0) + \int_0^t G(t, s) h_n(s) \, ds, \) for \( n \in \mathbb{N} \), \( \exists \) a continuous bounded function \( v_n \) with \( v_n = \xi \) on \([-d,0]\) and \( |v_n(t_1) - v_n(t_2)| \leq \int_0^{t_1} \| G(t_1, s) - G(t_2, s) \| \| m'(s) \| \, ds + c \int_{t_1}^{t_2} \| m'(s) \| \, ds, \) \( v_n = \xi \) on \([-d,0]\).

Theorem 11. In the setting of Theorem 10 if \( f' \) replaced by \( f^d \) such that for all \( \phi \in C_{\mathbb{B}}([-d,0]) \) \( f^d(I \times \{\phi\}) \) is separable, then the problem \((P)\) has a solution.

Proof. From Theorem 8 for \( n \in \mathbb{N}, \exists \) a continuous bounded function \( v_n \) such that \( v_n = \xi \) on \([-d,0]\) and we can complete the proof as in the proof of Theorem 10.

Now let \( h: I \times \mathbb{R}^+ \to \mathbb{R}^+ \) be a Caratheodory function. Let for bounded subset \( Z \) of \( E, \exists \) a measurable function \( m: I \to \mathbb{R}^+ \) \( h(t,s) \leq m(t) \) for \((t,s) \in Z \), and \( m \) is integrable on \([c,T]\) for \( c>0 \). Let, for \( c: 0 < c \leq T, \) the identically zero function be the only absolutely continuous function on \([0,c]\) which satisfies \( u(t) = h(t,u(t)) \) a.e. on \([0,c]\), the right hand derivative of \( u(t) \) at \( x = 0 \), \( D_u(0) \), exists and \( D_u(0) = u(0) = 0. \)

Theorem 12. If we replace in the setting of Theorem 11 \( w \) by \( h \) and consider \( f \) is bounded and continuous, then the problem \((P)\) has a solution.

We omit the proof since it runs as in the proof of Theorem 11 except we replace the using of Theorem 8 by that of Theorem 9 to find a continuous function \( y_1. \)

4. Conclusion
In the present work, we deal with the existence of weak and strong solutions for the nonlinear differential equations involving the weak topology and the strong topology too. In recent years, it is well known that the study of these equations involving the weak topology is lagging behind, while almost all of the work was done using the strong topology. \( \). In Theorem 9 we use a measure of weak noncompactness so there is a generalization for Theorem 8 in [6] and the references herein moreover we have a generalization of Theorem 5 in [9] since in [9] we used the Hausdorff measure of noncompactness. Further we get a generalization of Theorem 2 in [12] and Theorem 9 in [6] (Theorem 8).

In Theorem 9 we use the assumptions on \( h \) which is weaker than that on a Kamke function \( w \). The
last three theorems stated with finite delay thus we obtain the theorems that generalize the previous results.

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