Translation Invariant Diffusions and Stochastic Partial Differential Equations in $S'$

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Abstract

In this article we show that the ordinary stochastic differential equations of K. Itô maybe considered as part of a larger class of second order stochastic PDE’s that are quasi linear and have the property of translation invariance. We show using the ‘monotonicity inequality’ and the Lipschitz continuity of the coefficients $\sigma_{ij}$ and $b_i$, existence and uniqueness of strong solutions for these stochastic PDE’s. Using pathwise uniqueness, we prove the strong Markov property.

Keywords: $S'$ valued process, diffusion processes, Hermite-Sobolev space, Strong solution, quasi linear SPDE, Monotonicity inequality, Translation invariance

Subject classification :[2010]60G51, 60H10, 60H15

1 Introduction

The notion of an ordinary stochastic differential equation (SDE) was introduced by K. Itô in [24] and since then has become the main tool for
modelling diffusion phenomena as a random process (see for example [36]).
The approach to diffusions as a random process goes back to the works
of A.N.Kolmogorov [28], and was studied by N.Weiner [52], W.Feller [18],
J.L.Doob [15] and P.Lévy [33]. The theory was extended further by D.W.Stroock
and S.R.S.Varadhan in their well known ‘weak formulation’ or ‘martingale
formulation’ [45]. The subject of stochastic partial differential equations
(SPDE) on the other hand, is of more recent vintage ([50], [32]). It e xtends
the logic of perturbing an ordinary differential equation by noise, inherent in
the Itô approach, to partial differential equations. Although the underlying
probabilistic logic is the same, the mathematics of these two models can be
vastly different, the latter more often than not involving the tools and tech-
niques of function space analysis (see for example [22], [14]). On the other
hand, one of the fundamental features that continues to sustain interest in
the Itô approach, both in applications and theory, is the connection with
other areas of mathematics like partial differential equations and potential
theory ([31], [3], [44]); more recent examples are the notion of ‘viscosity solu-
tions’ related to the Hamilton-Jacobi-Bellman equation ([34]), and backward
SDE’s ([37]). In this paper, we show that the two approaches viz. the SDE
and the SPDE approaches can be unified into a single framework, in which
the SDE approach (with an extra parameter) is equivalent to the SPDE ap-
proach and mathematically speaking both maybe viewed as part of a single
structure. Our method may be considered a variant of the well known ‘
method of characteristics’ in PDE, that constructs solutions of PDE’s from
the ordinary differential equations satisfied by the ‘characteristic curves’ as-
associated with the PDE (see [17], Chapter 3, and [32], Chapter 6, for the
stochastic case). The difference in our approach lies in the treatment of non
linearities i.e. in the manner in which the coefficients in the PDE or SPDE
are allowed to depend on the solutions. In this paper, we first construct the
solutions of the SPDE and then deduce the solutions of the corresponding
SDE. The reverse construction of solutions of SPDE’s from that of the as-
associated SDE’s was already done, via the Itô formula, in [38], [41]. It t urns
out that the solutions of Itô’s SDE’s correspond to rather singular solutions
of the associated (quasi linear) SPDE in a manner analogous to the way in
which ‘fundamental solutions’ are associated to certain second order partial
differential equations. The solutions of the SPDE so constructed, arise, in a
unique fashion, as translations of the initial condition of the SPDE by the
solution of the ‘characteristic’ SDE starting at the origin.
In more detail, we construct in this paper a general method of solving the stochastic partial differential equation (SPDE) driven by an n-dimensional Brownian motion \((B_t)\) in the form

\[ dY_t = L(Y_t)dt + A(Y_t) \cdot dB_t \quad Y_0 = y. \]

Here \(L\) and \(A = (A_1, \cdots, A_n)\) are non-linear partial differential operators of the second and first order respectively on the space of tempered distributions \(S' \rightarrow S'\) on \(\mathbb{R}^d\) given by equations (2) and (3) below. The initial condition \(y\) is an arbitrary tempered distribution whose regularity may be measured on a decreasing scale of Hilbert spaces \(S_p, p \in \mathbb{R}\). In particular \(y \in S_p\) for some \(p \in \mathbb{R}\). The operators \(L\) and \(A_i\) are quasi-linear i.e. they are constant coefficient differential operators once the value of the coefficients \(\sigma_{ij}, b_i : S_p \rightarrow \mathbb{R}\) are fixed, \(L\) is of order two and the \(A_i\)'s of order one.

We assume a Lipschitz condition on \(\sigma_{ij}\) and \(b_i\) with respect to the norm \(||\|_q, q \leq p - 1\) (Theorem 4.3). The solutions of the above equation have the property that they are translation invariant i.e. they can be written as \(Y_t(y) = \tau_{Z_t(y)} y\), where \(\tau_x : S' \rightarrow S'\) are the translation operators and \((Z_t(y))\) is a finite dimensional process that depends on the initial value \(y\). This has the consequence that the solution corresponding to the translate \(\tau_x y\) is the translate of \(y\) by the process \((x + Z_t(\tau_x y))\). Note that the \(S_p\) themselves are invariant under translations i.e. \(\tau_x : S_p \rightarrow S_p\) (\([39]\)). The action of the translation operators on \(y\) gives rise to finite dimensional coefficients \(\bar{\sigma}_{ij}, \bar{b}_i, i = 1, \cdots, d, j = 1, \cdots, n\) by \(\bar{\sigma}_{ij}(z) := \sigma_{ij}(\tau_z y), \bar{b}_i(z) := b_i(\tau_z y), z \in \mathbb{R}^d\).

It turns out that \(X^x_t := x + Z_t(\tau_x y)\) solves the ordinary stochastic differential equation driven by \((B_t)\) with coefficients \(\bar{\sigma}_{ij}, \bar{b}_i\) and initial value \(X^x_0 = x\). In recent times distribution dependent SDE's have become an active area of research (see for example \([2], [11], [29], [1], [10]\) and references therein). We refer to Example 6 in Section 6 below, for some connections between distribution dependent SDE's and our results. Our work also relates to the problem of identifying 'invariant submanifolds' of solutions of SPDEs that arise in finance (see \([3], [2], [16], [46]\)). In effect, the set of translates \(\{\tau_x y : x \in \mathbb{R}^d\}\) serves as an invariant manifold for the above SPDE with initial distribution \(y \in S_p\), under some smoothness assumptions on \(y\).

Our method relies on three ingredients viz. one, a quasi-linear extension of linear differential operators by identifying the coefficients \(\sigma_{ij}(x)\) as a restriction of the functional \(\langle \sigma_{ij}, \phi \rangle, \phi \in S_{-p} = S'_p, p > d, \sigma_{ij} \in S_p\) to the distribution \(\phi = \delta_x\); two, an Itô formula for translations of tempered distributions by semi-
martingales (see [38], [4], [49]); and finally, the monotonicity inequality (see [5], [21]). Indeed, this last inequality, whose abstract version has been known for some time (see [30], [27]), has proved to be an indispensable tool for proving uniqueness results in the framework of a scale of Hilbert spaces (see [21], [41]). Our results below show that it can also be used for proving existence results.

The paper is organised as follows. After the preliminaries in Section 2, we prove in Section 3, using the monotonicity inequality (Theorem 3.1), some extensions of the same in Theorems (3.2) and (3.3) respectively; viz. in the case that the pair of operators \((A, L)\) have variable coefficients. These inequalities are crucial for the convergence results in Section 4 which contains the main existence and uniqueness results in Theorem (4.3). Our proof of existence is tailored for the infinite dimensional situation and applies to more general situations. A simpler proof is indicated in Remark 4.4. In Section 5, we construct the ‘maximal’ solutions up to an explosion time and prove the strong Markov property (Theorem (5.6)) using the pathwise uniqueness established in Section 4. In Section 6, we look at several examples. Examples 1, 2 & 3 relate to finite dimensional diffusions. Example 4 relates solutions of our SPDE with solutions of the associated martingale problem for \(L\). Example 5 deals with the stochastic representation of the solutions of non-linear evolution equation canonically associated with the operator \(L\). In Example 6, we consider the situation where the coefficients in the finite dimensional equation depends on the marginal law of the process. Finally Example 7 deals with extensions of the operator \(L\), that have a zeroth order term and is related to the Feynman-Kac formula. In Section 7 we make some remarks on ‘duality’ and invariant measures in the context our SPDE. Some technical results are in the Appendix.

## 2 Preliminaries

Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\) be a filtered probability space satisfying the usual conditions viz. 1) \((\Omega, \mathcal{F}, P)\) is a complete probability space. 2) \(\mathcal{F}_0\) contains all \(A \in \mathcal{F}\), such that \(P(A) = 0\), and 3) \(\mathcal{F}_t = \bigcap_{s \geq t} \mathcal{F}_s, t \geq 0\). On this probability space is given a standard \(n\)-dimensional \(\mathcal{F}_t\)-Brownian motion.
(B_t) ≡ (B^1_t, \ldots, B^n_t). We will denote the filtration generated by (B_t) as (\mathcal{F}^B_t).

Let \bar{\sigma}_{ij}, \bar{b}_i be locally Lipshitz functions on \mathbb{R}^d for i = 1, \cdots, d, j = 1, \cdots, n. Let \bar{\sigma} := (\bar{\sigma}_{ij}) (so that (\bar{\sigma}_{ij}(x)), x \in \mathbb{R}^d is a d \times n matrix) and \bar{b} := (\bar{b}_1, \cdots, \bar{b}_d) be a vector field on \mathbb{R}^d. We use the notation \hat{\mathbb{R}}^d = \mathbb{R}^d \cup \{\infty\} for the one point compactification of \mathbb{R}^d.

**Theorem 2.1** Let \bar{\sigma}, \bar{b}, (B_t) be as above and x \in \mathbb{R}^d. Then \exists \eta : \Omega \to (0, \infty], \eta an (\mathcal{F}^B_t) stopping time and an \hat{\mathbb{R}}^d-valued, (\mathcal{F}^B_t) adapted process (X_t)_{t \geq 0} such that

1. For all \omega \in \Omega, X(\omega) : [0, \eta(\omega)) \to \mathbb{R}^d, is continuous and X_t(\omega) = \infty, t \geq \eta(\omega)

2. a.s. (P), \eta(\omega) < \infty implies \lim_{t \uparrow \eta(\omega)} X_t(\omega) = \infty.

3. a.s.(P),

\[ X_t = x + \int_0^t \bar{\sigma}(X_s) \cdot dB_s + \int_0^t \bar{b}(X_s) \, ds \] (1)

for 0 \leq t < \eta(\omega).

The solution (X_t, \eta) is (pathwise) unique i.e. if (X^1_t, \eta^1) is another solution then P\{X_t = X^1_t, 0 \leq t < \eta \land \eta^1\} = 1.

**Proof:** We refer to [23], Chapter IV, Theorem 2.3 and Theorem 3.1 for the proofs (with appropriate modifications for the case d \neq r) of existence and uniqueness respectively. \[ \square \]

Let \alpha, \beta \in \mathbb{Z}_+^d := \{(x_1, \cdots, x_d) : x_i \geq 0, x_i \text{ integer}\}. Let x^\alpha be the product x^\alpha := x_1^{\alpha_1} \cdots x_d^{\alpha_d} \in \mathbb{R} and \partial^\beta := \partial_1^{\beta_1} \cdots \partial_d^{\beta_d}, the differential operator of order \beta_1 + \cdots + \beta_d corresponding to the monomial x^\beta. For a multi index
α, we use the notation $|\alpha| := \sum_{i=1}^{d} \alpha_i$. Let $\mathcal{S}$ denote the space of rapidly decreasing smooth real functions on $\mathbb{R}^d$ with the topology given by the family of semi norms $\{\alpha,\beta\}$, defined for $f \in \mathcal{S}$ and multi indices $\alpha,\beta$ by $\alpha,\beta(f) := \sup_x |x^\alpha \partial^\beta f(x)|$. Then $\{\mathcal{S}, \alpha,\beta : \alpha,\beta \in \mathbb{Z}_+^d \}$ is a locally convex, complete, metrisable topological vector space i.e. a Fréchet space. $\mathcal{S}'$ will denote its continuous dual. The duality between $\mathcal{S}$ and $\mathcal{S}'$ will be denoted by $\langle \psi, \phi \rangle$ for $\phi \in \mathcal{S}$ and $\psi \in \mathcal{S}'$. For $x \in \mathbb{R}^d$ the translation operators $\tau_x : \mathcal{S} \rightarrow \mathcal{S}$ are defined as $\tau_x f(y) := f(y-x)$ for $f \in \mathcal{S}$ and then for $\phi \in \mathcal{S}'$ by duality : $\langle \tau_x \phi, f \rangle := \langle \phi, \tau_x f \rangle$.

Let $\{h_k : k \in \mathbb{Z}_+^d \}$ be the orthonormal basis in the real Hilbert space $L^2(\mathbb{R}^d, dx) \supset \mathcal{S}$ consisting of the Hermite functions (see for eg. [47]); here $dx$ denotes Lebesgue measure, where the dependance on the dimension is suppressed whenever there is no risk of confusion. Let $\langle \cdot, \cdot \rangle_0$ be the inner product in $L^2(\mathbb{R}^d, dx)$. For $f \in \mathcal{S}$ and $p \in \mathbb{R}$ define the inner product $\langle f, g \rangle_p$ on $\mathcal{S}$ as follows :

$$\langle f, g \rangle_p := \sum_{k=(k_1,\cdots,k_d) \in \mathbb{Z}_+^d} (2|k| + d)^{2p} \langle f, h_k \rangle_0 \langle g, h_k \rangle_0$$

The corresponding norm will be denoted by $\| \cdot \|_p$. We define the Hilbert space $\mathcal{S}_p$ as the completion of $\mathcal{S}$ with respect to the norm $\| \cdot \|_p$ over the field of real numbers. The following basic relations hold between the $\mathcal{S}_p$ spaces (see for eg. [23], [27]) : For $0 < q < p$, $\mathcal{S} \subset \mathcal{S}_p \subset \mathcal{S}_q \subset L^2 = \mathcal{S}_0 \subset \mathcal{S}_{-q} \subset \mathcal{S}_{-p} \subset \mathcal{S}'$. Further, $\mathcal{S}' = \bigcup_{p \in \mathbb{R}} \mathcal{S}_p$ and $\bigcap_{p \in \mathbb{R}} \mathcal{S}_p = \mathcal{S}$. If $\{h_k : k \in \mathbb{Z}_+^d \}$ denotes the orthonormal basis in $\mathcal{S}_p$, consisting of the (normalised) Hermite functions $h_k^p := (2|k| + d)^{-p} h_k$, then the dual space $\mathcal{S}_p'$ may be identified with $\mathcal{S}_{-p}$, via the basis $\{h_k^{-p} : k \in \mathbb{Z}_+^d \}$ of $\mathcal{S}_{-p}$. For $\phi \in \mathcal{S}$ and $\psi \in \mathcal{S}'$ the bilinear form $(\psi, \phi) \rightarrow \langle \psi, \phi \rangle$ also gives the duality between $\mathcal{S}_p(\supset \mathcal{S})$ and $\mathcal{S}_{-p}(\subset \mathcal{S}')$. It is also well known that $\partial_i : \mathcal{S}_p \rightarrow \mathcal{S}_{p-\frac{1}{2}}$ are bounded linear operators for every $p \in \mathbb{R}$ and $i = 1, \cdots, d$. For Banach spaces $X,Y,L(X,Y)$ will denote the Banach space of bounded linear operators from $X$ into $Y$.

Let $p \in \mathbb{R}$ and let $\sigma_{ij}, b_i : \mathcal{S}_p \rightarrow \mathbb{R}, i = 1, \cdots, d, j = 1, \cdots, n$. We consider the (non-linear) operators $A := (A_1, \cdots, A_n) : \mathcal{S}_p \rightarrow L(\mathbb{R}^n, \mathcal{S}_{p-\frac{1}{2}})$, from $\mathcal{S}_p$.
to the space of linear operators from $\mathbb{R}^n$ to $S_{p-\frac{1}{2}}$, defined by

$$A_i(\phi) = -\sum_{k=1}^{d} \sigma_{ki}(\phi) \partial_k \phi \quad i = 1, \ldots, n$$ (2)

and the non-linear operator $L : S_p \to S_{p-1}$ defined as follows :

$$L(\phi) = \frac{1}{2} \sum_{i,j}^{d} a_{ij}(\phi) \partial_{ij} \phi - \sum_{i=1}^{d} b_i(\phi) \partial_i \phi$$ (3)

where $a_{ij}(\phi) := (\sigma(\phi)\sigma(\phi)^t)_{ij}$ and the superscript ‘t’ denotes matrix transpose. Clearly if $\sigma_{ij}(\phi)$ and $b_i(\phi)$ are bounded on the set $\{ \phi \in S_p : \|\phi\|_p \leq \lambda \}$ for some $\lambda > 0$, then $\exists \ C = C(\lambda) > 0$ such that if $q \leq p - 1$ and $\{e_i : i = 1, \ldots, n\}$ is the standard orthonormal basis in $\mathbb{R}^n$, then

$$\|A(\phi)\|_{HS(q)}^2 := \sum_{i=1}^{n} \|A(\phi)e_i\|_q^2 =: \sum_{i=1}^{n} \|A_i(\phi)\|_{q}^2 \leq C \cdot \|\phi\|_{p}^2$$

$$\|L(\phi)\|_q \leq C \cdot \|\phi\|_{p}$$

for $\phi \in S_p$ with $\|\phi\|_p \leq \lambda$. In the above equalities and in what follows we use the notation $A(\phi) \cdot h := \sum_{i=1}^{n} A_i(\phi) \cdot h_i$, $h \in \mathbb{R}^n$. The subscript ‘HS’ refers to the Hilbert-Schmidt norm. The above inequalities follow from the boundedness of the operators $\partial_i : S_p \to S_{p-\frac{1}{2}}$ and the assumed (local) bounds on the coefficients $\sigma_{ij}$ and $b_i$.

3 The Monotonicity Inequality

In this section we will prove the ‘monotonicity inequality’ involving the pair $(L, A)$ defined in equations (2) and (3) and which we will prove in the proof of existence and uniqueness of the SPDE (18). The constant coefficient case was proved in [21]. Using techniques developed in [5] we prove the corresponding inequality and a variant of the same in the variable coefficient case, in theorems (3.2) and (3.3) below.
Let $\sigma = (\sigma_{ij}) \in \mathbb{R}^{dn}$, $h = (h_1 \ldots h_n) \in \mathbb{R}^n$ and $\phi \in S_p$. Then we define $A_0 : \mathbb{R}^{dn} \times S_p \to L(\mathbb{R}^n, S_q)$, a bilinear map, as follows:

$$A_0(\sigma, \phi) \cdot h := -\sum_{i=1}^n \sum_{k=1}^d \sigma_{ki} \partial_k \phi \cdot h_i$$

For $\psi, \phi \in S_p$, and $q \leq p - 1$ we can write (from (2))

$$A(\phi) \cdot h - A(\psi) \cdot h = A_0(\sigma(\phi), \phi - \psi) \cdot h + A_0(\sigma(\phi) - \sigma(\psi), \psi) \cdot h$$

Similarly we can write

$$L(\phi) - L(\psi) = L_1(b(\phi), \phi - \psi) + L_1(b(\phi) - b(\psi), \psi) + L_2(a(\phi), \phi - \psi) + L_2(a(\phi) - a(\psi), \psi)$$

where $L_1$, $L_2$ are $S_q$ valued, bilinear maps on $\mathbb{R}^d \times S_p$ and $\mathbb{R}^{d^2} \times S_p$, respectively, given as follows: Let $(b, \phi), \in \mathbb{R}^d \times S_p, b := (b_1, \ldots, b_d).$ Then

$$L_1(b, \phi) := -\sum_{i=1}^d b_i \partial_i \phi$$

and to define $L_2$, let $(\sigma, \phi) \in \mathbb{R}^{d^2} \times S_p, \sigma := (\sigma_{ij}).$ Then,

$$L_2(\sigma, \phi) := \frac{1}{2} \sum_{i,j=1}^d \sigma_{ij} \partial^2_{ij} \phi.$$ 

Note that for $\phi \in S_p$, we have the $d \times d$ matrix $a(\phi) \equiv \sigma \sigma^t(\phi) := ((\sigma(\phi) \sigma^t(\phi))_{ij})$ and the $d$-dimensional vector $b(\phi) := (b_1(\phi), \ldots, b_d(\phi))$. For $\lambda > 0$, define the constant $K_1(\lambda)$ as follows:

$$K_1(\lambda) := \max_{i,j} \sup_{\|\phi\| \leq \lambda} \{|\sigma_{ij}(\phi)|^2, |b_i(\phi)|\}$$

We then have the following restatement of the Monotonicity inequality for constant coefficient operators [21].

**Theorem 3.1** Let $p \in \mathbb{R}, q \leq p - 1$. Suppose that $\sigma_{ij}(\cdot), b_i(\cdot), i = 1, \ldots, d, j = 1, \ldots n$ are bounded on the set $B_p(0, \lambda) := \{\phi \in S_p : \|\phi\| \leq \lambda\}$ for every
Then for every \( \lambda > 0 \), \( \exists \) a constant \( C = C(r, d, p, K_1(\lambda)) > 0 \) such that

\[
2 \langle \psi, L_2(a(\phi), \psi) \rangle_q + \| A_0(\sigma(\phi), \psi) \|^2_{HS(q)} \leq C \| \psi \|^2_q \\
\langle \psi, L_1(b(\phi), \psi) \rangle_q \leq C \cdot \| \psi \|^2_q
\]

for all \( \psi \in \mathcal{S}_p \) and for all \( \phi \in B_p(0, \lambda) \).

**Proof:** It follows from the Monotonicity inequality (see [21],[5]) that the inequalities in the statement of the theorem holds for fixed \( \psi, \phi \in \mathcal{S}_p \) with a constant \( C' \) that depends quadratically on the numbers \( \max\{|\sigma_{ij}(\phi)|: i = 1, \ldots, d, j = 1, \ldots, n\} \) and linearly on \( \max\{|b_i(\phi)|: i = 1, \ldots, d\} \). Taking supremum over \( \| \phi \|_p \leq \lambda \), we get the required constants. \( \square \)

We now prove the Monotonicity inequality in the form required to obtain uniqueness of solutions to our stochastic partial differential equation (18) below.

**Theorem 3.2** Let \( p \in \mathbb{R}, q \leq p - 1 \). Let \( \sigma_{ij}, b_i : \mathcal{S}_p \to \mathbb{R}, i = 1, \ldots, d, j = 1, \ldots n \). Suppose that for \( \lambda > 0 \), \( \exists K(\lambda) > 0 \) such that

\[
|\sigma_{ij}(\phi) - \sigma_{ij}(\psi)| \leq K(\lambda) \| \phi - \psi \|_q, \quad (4) \\
|b_i(\phi) - b_i(\psi)| \leq K(\lambda) \| \phi - \psi \|_q,
\]

for \( \phi, \psi \in B_p(0, \lambda) \). Then \( \exists \) a constant \( C = C(r, d, p, q, \lambda, K(\lambda), K_1(\lambda)) \) such that

\[
\langle \phi - \psi, L(\phi) - L(\psi) \rangle_q + \| A(\phi) - A(\psi) \|^2_{HS(q)} \leq C \| \phi - \psi \|^2_q \quad (5)
\]

for all \( \phi, \psi \in B_p(0, \lambda) \).

**Proof:** Using the notation established in the discussion preceding the state-
From Theorem (3.1),

\[
2\langle \phi - \psi, L(\phi) - L(\psi) \rangle_q + \|A(\phi) - A(\psi)\|_{HS(q)}^2 \\
= 2\langle \phi - \psi, L_1(b(\phi) - b(\psi)) \rangle_q \\
+ 2\langle \phi - \psi, L_1(b(\phi) - b(\psi)) \rangle_q + 2\langle \phi - \psi, L_2(\sigma^t(\phi) - \sigma^t(\psi)) \rangle_q \\
+ \|A_0(\sigma(\phi) - \sigma(\psi))\|_{HS(q)}^2 \\
+ \|A_0(\sigma(\phi) - \sigma(\psi))\|_{HS(q)}^2 \\
+ \sum_{i=1}^{n} 2\langle A_0(\sigma(\phi) - \sigma(\psi))e_i, A_0(\sigma(\phi) - \sigma(\psi))e_i \rangle_q.
\]  

From Theorem (3.1), \( \exists C_1 = C_1(r, d, p, q, K(\lambda)) > 0 \) such that for all \( \phi \in B(0, \lambda, p) \)

\[
2\langle \phi - \psi, L_1(b(\phi) - b(\psi)) \rangle_q \leq C_1 \|\phi - \psi\|_q^2 
\]

(7)

\[
2\langle \phi - \psi, L_2(\sigma^t(\phi) - \sigma^t(\psi)) \rangle_q + \|A(\sigma(\phi) - \sigma(\psi))\|_{HS(q)}^2 \\
\leq C_1 \|\phi - \psi\|_q^2.
\]

(8)

for all \( \psi \in S_p \). Using the Lipschitz continuity of \( \sigma_{ij} \) and \( b_i \), the fact that products of locally Lipschitz continuous functions are again locally Lipschitz continuous, and the boundedness of \( \partial_i : S_p \to S_q, q \leq p - \frac{1}{2} \), we have

\[
2\langle \phi - \psi, L_1(b(\phi) - b(\psi)) \rangle_q = 2 \sum_{i=1}^{d} (b_i(\phi) - b_i(\psi)) \langle \phi - \psi, \partial_i \psi \rangle_q \\
\leq C_2 \|\phi - \psi\|_q^2
\]

(9)

and

\[
2\langle \phi - \psi, L_2(\sigma^t(\phi) - \sigma^t(\psi)) \rangle_q + \sum_{i,j} [(\sigma^t)_{ij}(\phi) - (\sigma^t)_{ij}(\psi)] \langle \phi - \psi, \partial_{ij}^2 \psi \rangle_q \\
\leq C_3 \|\phi - \psi\|_q^2.
\]

(10)

for some constants \( C_2 = C_2(r, d, p, q, \lambda, K(\lambda)) > 0 \) and \( C_3 = C_3(r, d, p, q, \lambda, K(\lambda)) > 0 \) and for all \( \phi, \psi \in B(0, \lambda) \). Similarly \( \exists C_4 = C_4(r, d, p, q, \lambda, K(\lambda)) > 0 \) such that

\[
\|A_0(\sigma(\phi) - \sigma(\psi), \psi)\|_{HS(q)}^2 \leq C_4 \|\phi - \psi\|_q^2.
\]

(11)
We now show that \( \exists C_5 = C_5(r, d, p, q, \lambda, K(\lambda), K_1(\lambda)) > 0 \)

\[
2 \sum_{i=1}^{n} \langle A_0(\sigma(\phi), \phi - \psi)e_i, A_0(\sigma(\phi) - \sigma(\psi), \psi)e_i \rangle_q \leq C_5(\lambda) \| \phi - \psi \|_q^2 \quad (12)
\]

for \( \phi, \psi \in B_p(0, \lambda) \). Consequently, the inequality (5) in the statement now follows from equality (6) and the inequalities (7) - (12), with the constant \( C \) in (5) given by

\[
C = 2C_1 + C_2 + C_3 + C_4 + C_5.
\]

To prove (12), we note that from the definition of \( A_0 \) that

\[
2 \sum_{i=1}^{n} \langle A_0(\sigma(\phi), \phi - \psi)e_i, A_0(\sigma(\phi) - \sigma(\psi), \psi)e_i \rangle_q = 2 \sum_{i=1}^{n} \delta_{\sum_{j=1}^{d} \sigma_{ji}(\phi) \partial_j(\phi - \psi), \sum_{j=1}^{d} (\sigma_{ji}(\phi) - \sigma_{ji}(\psi)) \partial_j \psi \rangle_q = 2 \sum_{i=1}^{n} \sum_{j,k} \sigma_{ji}(\phi) (\sigma_{ki}(\phi) - \sigma_{ki}(\psi)) \langle \partial_j(\phi - \psi), \partial_k \psi \rangle_q.
\]

Clearly it suffices to show that \( \exists C'_6 := C'_6(r, d, p, q, \lambda) > 0 \) such that for all \( j, k = 1, \ldots d \) and \( \phi, \psi \in B_p(0, \lambda) \cap S \)

\[
|\langle \partial_j \phi, \partial_k \psi \rangle_q| \leq C'_6 \| \phi \|_q \| \psi \|_{q+1} \leq C_6 \| \phi \|_q
\]

where \( C_6 := \sup_{\psi \in B_p(0, \lambda) \cap S} \| \psi \|_{q+1} C'_6 \). But this is an immediate consequence of the representation of the adjoint \( \partial_j^* \) of \( \partial_j : S \subset S_q \to S_q \). Indeed it was shown in [5] that for \( f \in S \), we have \( \partial_j^* f = -\partial_j f + T_j f \), where \( T_j : S_q \to S_q \) is a bounded operator. In particular, for \( \phi, \psi \in S \),

\[
\langle \partial_j \phi, \partial_j \psi \rangle_q = \langle \phi, \partial_j^* \partial_j \psi \rangle_q = -\langle \phi, \partial_j^2 \psi \rangle_q + \langle \phi, T_j \partial_j \psi \rangle_q
\]

and (13) follows. This completes the proof of Theorem (3.2). \( \square \)

The following variant of the monotonicity inequality will be needed in the proof of existence of translation invariant diffusions. Before stating the result, we introduce some notation.
Let $a_{ij}(\phi), \sigma_{ij}(\phi), b_i(\phi)$ be as in Section 2. For $i = 1, \ldots, n$ we define $A_i : S_p \times S_p \to S_q$ and $L : S_p \times S_p \to S_q$ as follows:

$$A_i(\phi, \psi) := - \sum_{k=1}^{d} \sigma_{ki}(\phi) \partial_k \psi$$

(14)

$$L(\phi, \psi) := \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(\phi) \partial_{ij} \psi - \sum_{i=1}^{d} b_i(\phi) \partial_i \psi$$

(15)

Note that $A_i(\phi, \phi) = A_i(\phi)$ where the non-linear operator $A_i(\phi)$ (of a single argument) is given by equation (2). Note also that $A_i(\phi, \psi)$ is linear in the second variable and is given in terms of the operator $A_0(\cdot, \cdot)$ defined in the beginning of Section 3 as $A_i(\phi, \psi) = A_0(\sigma(\phi), \psi) \cdot e_i$. Similar remarks hold for the operator $L(\phi, \psi)$.

**Theorem 3.3** Let $p \in \mathbb{R}, q \leq p - 1$. Let $\sigma_{ij}(\cdot), b_i(\cdot)$ be as in Theorem (3.2). Then there exists positive constants $C_1 = C_1(r, d, p, q, \lambda, K(\lambda), K_1(\lambda))$ such that

$$2\langle \phi_2 - \phi_1, L(\phi_3, \phi_2) - L(\phi_2, \phi_1) \rangle_q + \sum_{i=1}^{d} \|A_i(\phi_3, \phi_2) - A_i(\phi_2, \phi_1)\|_q^2 \leq C_1(\|\phi_2 - \phi_1\|_q^2 + \|\phi_2 - \phi_3\|_q^2)$$

(16)

for all $\phi_1, \phi_2, \phi_3 \in B_p(0, \lambda)$. 

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Proof: Let $\phi_i, i = 1, 2, 3 \in B_p(0, \lambda)$. the left hand side of (16)
\begin{align*}
&= 2\langle \phi_2 - \phi_1, L(\phi_3, \phi_2) - L(\phi_3, \phi_1) \rangle_q + \sum_{i=1}^n \left\| A_i(\phi_3, \phi_2) - A_i(\phi_3, \phi_1) \right\|_q^2 \\
&+ 2\langle \phi_2 - \phi_1, L(\phi_3, \phi_1) \rangle_q
\end{align*}

where $C_1' = C_1'(r, d, p, q, K_1(\lambda))$.

Using the Lipshitz continuity of the coefficients $\sigma_{ij}, b_i$ (see (4)), the 2nd term in right hand side of (17)
\begin{align*}
&= 2 \langle \phi_2 - \phi_1, L_1(b(\phi_3), \phi_2 - \phi_1) \rangle_q + \sum_{i=1}^n \left\| A_i(\phi_3, \phi_2) - A_i(\phi_3, \phi_1) \right\|_q^2 \\
&+ 2\langle \phi_2 - \phi_1, L_1(b(\phi_3), \phi_2 - \phi_1) \rangle_q
\end{align*}

where $C_2''$ are positive constants depending only on $r, d, p, q, \lambda, K(\lambda)$ and $K_1(\lambda)$.
Similarly, the 4th term in right hand side of (17)

\[ \sum_{i=1}^{n} \| A_i(\sigma(\phi_3) - \sigma(\phi_2), \phi_1) \|_q^2 \leq C_4 \| \phi_3 - \phi_2 \|_q^2 \]

where \( C_4 = C_4(r, d, p, q, K(\lambda), K_1(\lambda)) \).

Finally in the same manner as in the proof of Theorem (3.2), the 5th term in right hand side of (17)

\[ 2 \sum_{i=1}^{n} \langle A_i(\sigma(\phi_3), \phi_2 - \phi_1), A_i(\sigma(\phi_3) - \sigma(\phi_2), \phi_1) \rangle_q \leq C_5' \| \phi_3 - \phi_2 \|_q \| \phi_2 - \phi_1 \|_q \]

where \( C_5' = C_5(r, d, p, q, \lambda, K(\lambda), K_1(\lambda)) \). The proof of the theorem follows by summing up the terms in the RHS of (17), using the above inequalities. \( \square \)

### 4 Existence and Uniqueness of Solutions of SPDE’s.

Let \( p \in \mathbb{R} \). Let \( \sigma_{ij}, b_i : \mathcal{S}_p \rightarrow \mathbb{R} \) \( i = 1, \cdots, d, j = 1, \cdots, n \) be locally bounded functions on \( \mathcal{S}_p \). Let \( (B_t) \) be a given n-dimensional \( \mathcal{F}_t \)- Brownian motion on \( (\Omega, \mathcal{F}, P) \) as in Section 2. Let \( A_i, i = 1, \cdots, n \) and \( L \) be partial differential operators as defined in equations (2) and (3). We now consider a stochastic partial differential equation in \( \mathcal{S}' \) driven by the Brownian motion \( (B_t) \) and 'coefficients 'given by the differential operators \( A_i, i = 1, \cdots, n \) and \( L \) defined above and initial condition \( y \in \mathcal{S}_p \) viz.

\[ dY_t = A(Y_t) \cdot dB_t + L(Y_t) \, dt \quad ; \quad Y_0 = Y, \quad (18) \]

where \( Y : \Omega \rightarrow \mathcal{S}_p \). Note that if \( (Y_t) \) is an \( \mathcal{S}_p \) valued, locally bounded, \( (\mathcal{F}_t) \) adapted process then \( A_i(Y_s), i = 1, \cdots, n \) and \( L(Y_s) \) are \( \mathcal{S}_{p-1} \) valued, adapted, locally bounded processes and hence for \( i = 1, \cdots n \), the stochastic
integrals $\int_0^t A_i(Y_s) \, dB^i_s$ and $\int_0^t L(Y_s) \, ds$ are well defined $\mathcal{S}_{p-1}$ valued, continuous $\mathcal{F}_t$-adapted processes and in addition, the former processes are $\mathcal{F}_t$ local martingales. We then have the following definition of a ‘local’ strong solution of equation (18).

**Definition 4.1** Let $p \in \mathbb{R}$. Let $\sigma_{ij}, b_i : S_p \rightarrow \mathbb{R}, i = 1, \ldots, n, j = 1, \ldots, n$ be locally bounded functions, $\{B_t, \mathcal{F}_t\}$ a given standard $n$-dimensional ($\mathcal{F}^B_t$) Brownian motion and $Y : \Omega \rightarrow S_p$ an $\mathcal{F}_0$ measurable random variable independent of the filtration ($\mathcal{F}^B_t$). Let $\delta$ be an arbitrary state, viewed as an isolated point of $\hat{S}_p := S_p \cup \{\delta\}$. By an $\hat{S}_p$ valued, strong, (local) solution of equation (18), we mean a pair $(Y_t, \eta)$ where $\eta : \Omega \rightarrow (0, \infty]$ is an $\mathcal{F}^B_t$-stopping time and $(Y_t)$ an $\hat{S}_p$ valued ($\mathcal{F}^B_t$) adapted process such that

1. For all $\omega \in \Omega$, $Y(.)(\omega) : [0, \eta(\omega)) \rightarrow S_p$ is a continuous map and $Y_t(\omega) = \delta, t \geq \eta(\omega)$

2. a.s. (P) the following equation holds in $\mathcal{S}_{p-1}$ for $0 \leq t < \eta(\omega)$,

$$
Y_t = Y + \sum_{j=1}^n \int_0^t A_j(Y_s) \, dB^i_s + \int_0^t L(Y_s) \, ds. \tag{19}
$$

We note that equation (19) also holds in $\mathcal{S}_q$ for any $q \leq p - 1$. To prove the existence of solutions to equation (18) we need a few well known facts. Let $(\bar{\sigma}_{ij}(s, \omega)), (\bar{b}_i(s, \omega)), j = 1, \ldots, n, i = 1, \ldots, d$ be locally bounded, $\mathcal{F}_t$-adapted processes and let $Z_t := (Z^1_t, \ldots, Z^d_t)$ be a $d$-dimensional ($\mathcal{F}_t$) semimartingale defined as follows :

$$
Z_t := \int_0^t \bar{\sigma}(s, \omega) \cdot dB_s + \int_0^t \bar{b}(s, \omega) ds.
$$

For $p \in \mathbb{R}$, define the operator valued adapted processes $\bar{L}(s, \omega), \bar{A}_i(s, \omega) : [0, \infty) \times \Omega \rightarrow L(S_p, \mathcal{S}_{p-1})$ $i = 1, \ldots, n$ as follows : For $\phi \in S_p$.
\[ \bar{L}(s, \omega) \phi := \frac{1}{2} \sum_{i, j=1}^{d} \bar{a}_{ij}(s, \omega) \partial_{ij}^{2} \phi - \sum_{i=1}^{d} \bar{b}_{i}(s, \omega) \partial_{i} \phi \]

and for \( i = 1, \cdots, n, \)

\[ \bar{A}_{i}(s, \omega) \phi := -\sum_{j=1}^{d} \bar{\sigma}_{ji}(s, \omega) \partial_{j} \phi \]

where \( \bar{a}_{ij}(s, \omega) := (\bar{\sigma}(s, \omega) \bar{\sigma}^t(s, \omega))_{ij} \), \( i, j = 1, \cdots, d \). Let \( Y: \Omega \to \mathcal{S}_{p} \). Note that the \( \mathcal{S}_{p} \) valued process \( \tau_{Z_{t}}(Y) \) has the \( \mathcal{S}_{p} \) valued trajectories \( t \to \tau_{Z_{t}(\omega)}(Y(\omega)) \). We then have the following Lemma.

**Lemma 4.2** Let \( p \in \mathbb{R} \). Let \( \bar{Y}_{t} := \tau_{Z_{t}}(Y) \) where \( Y: \Omega \to \mathcal{S}_{p} \) an \( \mathcal{F}_{0} \) measurable random variable independent of the filtration \( (\mathcal{F}_{t}^{B}) \), and \( (Z_{t}) \) as above.

(a) Suppose \( (\bar{\sigma}_{ij}(s, \omega)), (\bar{b}_{i}(s, \omega)), j = 1, \cdots, n, i = 1, \cdots, d \) are \( \mathcal{F}_{t}^{B} \)-adapted locally bounded processes. Then \( (\bar{Y}_{t}) \) is an \( \mathcal{S}_{p} \)-valued continuous \( \mathcal{F}_{t}^{B} \)-adapted process which is the unique solution of the following linear equation in \( \mathcal{S}_{q}, q \leq p - 1 \) : almost surely,

\[ \bar{Y}_{t} = Y + \int_{0}^{t} \bar{L}(s, \omega) \bar{Y}_{s} \, ds + \int_{0}^{t} \bar{A}(s, \omega) \bar{Y}_{s} \cdot dB_{s} \]

for every \( t \geq 0 \).

(b) Let \( (X_{t}) \) be an \( \mathcal{S}_{p} \)-valued progressively measurable process which is uniformly bounded i.e. \( \exists K > 0 \) such that \( \|X_{t}(\omega)\|_{p} \leq K, \forall (t, \omega) \). Let \( \bar{\sigma}_{ij}(s, \omega) := \sigma_{ij}(X_{s}(\omega)), \bar{b}_{i}(s, \omega) := b_{i}(X_{s}(\omega)) \) where \( \sigma_{ij}, b_{i} \) are as in Defn.(4.1). Let \( (Z_{t}), (\bar{Y}_{t}) \) be as defined above. Then

\[ E \left( \sup_{s \leq t} \|\bar{Y}_{s}\|_{p} \right)^{2} \leq CE\|Y\|_{p}^{2} \]

where \( C = C(d, r, K, t) \) is a constant.
Proof: (a) The proof of the existence part of (a) for $Y = y \in S_p$ fixed, is an immediate consequence of Itô’s formula and we refer to [38] for the details. For $Y$ arbitrary but independent of the Brownian motion $B$, the result follows by a conditioning argument. The proof of uniqueness follows from the results in [21].

(b) It is sufficient to consider the case $E \|Y\|_p^2 < \infty$. From the results of [39], we have

$$\|Y_t\|_p = \|\tau_{Z_t}(Y)\|_p \leq \|Y\|_p P(|Z_t|)$$

where $P(x)$ is a polynomial in $x \in \mathbb{R}$ with nonnegative coefficients and degree $m$ depending on $|p|$. Now the result follows by an application of the Burkholder-Davis-Gundy inequality to each term in $P(|Z_t|)$, using the boundedness assumption on $(X_t)$ and the local boundedness of the coefficients $\sigma_{ij}$ and $b_i, i = 1, \ldots, d, j = 1, \ldots, n$. □

We now come to the existence and uniqueness of solutions to equation (18). Recall that $B_p(y, r)$ is the ball in $S_p$ with centre $y$ and radius $r > 0$.

**Theorem 4.3** Let $p \in \mathbb{R}, q \leq p - 1$. Let $\sigma_{ij}, b_i : S_p \to \mathbb{R}, i = 1, \ldots, d, j = 1, \ldots n$. Suppose that for every $\lambda > 0$, $\exists K(\lambda) > 0$ such that

$$|\sigma_{ij}(\phi) - \sigma_{ij}(\psi)| \leq K(\lambda) \|\phi - \psi\|_q$$

$$|b_i(\phi) - b_i(\psi)| \leq K(\lambda) \|\phi - \psi\|_q$$

for $\phi, \psi \in B(0, \lambda, p) = \{\eta \in S_p : \|\eta\|_p \leq \lambda\}$. Then for every $Y : \Omega \to S_p$ which is $\mathcal{F}_0$ measurable and independent of $B$ and for every $r > 0$ there exists a strictly positive ($\mathcal{F}_t^B$) stopping time $\eta^r$, and a $S_p \cap B_q(Y, r)$ valued, continuous, $\mathcal{F}_t^B$-adapted process $(Y^r_t)$ satisfying equation (19) on $[0, \eta^r)$, almost surely. If $Y^1, Y^2$ are two $\mathcal{F}_0$ measurable $S_p$-valued random variables with $P\{Y^1 = Y^2\} > 0$, then the corresponding solutions $(Y^{1,r}_t, \eta^{1,r})$, $(Y^{2,r}_t, \eta^{2,r})$ satisfy: $\eta^{1,r} = \eta^{2,r}$ and $Y^{1,r}_r = Y^{2,r}_r, 0 \leq t < \eta^{1,r}$ on the set $\{Y^1 = Y^2\}$.

In particular, (19) has an $S_p$ valued local, strong solution. The solution is unique in the sense that if $(Y^{1}_t, \eta^{1})$ and $(Y^{2}_t, \eta^{2})$ are any two solutions of equation (18) with initial condition $Y$, then $P[Y^1_t = Y^2_t, 0 \leq t < \eta^1 \land \eta^2] = 1$.

Proof: We will first prove uniqueness.
Uniqueness: Let \((Y^1_t, \eta^1)\) and \((Y^2_t, \eta^2)\) be any two local solutions of equation (18) with initial condition \(Y \in S_p\). Let \(\lambda > 0\). Let \(Y_t = Y^1_t - Y^2_t\). Let 

\[\eta_0(\lambda) := \inf\{t : Y^1_t \text{ or } Y^2_t \notin B_p(0, \lambda)\}\] 

and \(\eta \equiv \eta(\lambda) := \eta_0(\lambda) \wedge \eta_1 \wedge \eta_2\). Then by using the identity 

\[\|Y_t\|^2 = \sum_{k=1}^{\infty} (Y_t, h_{k,q})^2\] 

where \(\{h_{k,q}\}\) is an ortho-normal basis for \(S_q\) and expanding \((Y_t, h_{k,q})^2\) using Itô’s formula, we see that the following equation holds a.s., for all \(0 \leq t < \eta\): 

\[\|Y_t\|^2 = \int_0^t \left\{2(Y_s, L(Y^1_s) - L(Y^2_s))_q + \|A(Y^1_s) - A(Y^2_s)\|_{HS(q)}^2\right\} ds + M_t\] 

where \((M_t)\) is a continuous local martingale. Now using inequality (5) of Theorem (3.2), the Gronwall inequality and a localisation argument (see for example [20]), we get for each \(\lambda > 0\), almost surely, \(Y^1_t = Y^2_t, 0 \leq t < \eta(\lambda)\). Letting \(\lambda \uparrow \infty\) the result follows.

Existence: To prove existence, we first consider the case \(\sup_{\omega \in \Omega} \|Y(\omega)\|_p^2 < \infty\).

Recall the operator maps \(L(.,.)\) and \(A_i(.,.)\), \(i = 1, \cdots, n\) defined in the paragraph prior to Lemma (4.2). Let for each \(k \geq 1\), \((X^k_s)\) be an \(S_p\)-valued process. We define the operator valued process \(L^k, A^k_i : [0, \infty) \times \Omega \to L(S_p, S_{p-1}), i = 1, \cdots, n\), whose action on \(\phi \in S_p\) is given by, 

\[L^k(s,\omega)\phi := \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(X^k_s(\omega)) \partial^2_{ij} \phi - \sum_{i=1}^{d} b_i(X^k_s(\omega)) \partial_i \phi\]

and for \(i = 1, \cdots, n\), 

\[A^k_i(s,\omega)\phi := -\sum_{j=1}^{d} \sigma_{ji}(X^k_s(\omega)) \partial_j \phi\]

We define a sequence of \(\mathcal{F}_t^{B_t}\)-adapted, \(S_p\)-valued processes \((Y^k_t)\), inductively, using operator valued processes \(L^k(s,\omega)\) and \(A^k_i(s,\omega)\) as follows: 

\[Y^0_t \equiv Y, \quad t \geq 0.\]
where \( Y \in S_p \) is the given initial value of equation (18). If \((Y_{t}^{k})\) is defined, then \((Y_{t}^{k})\) is defined as the unique \((\mathcal{F}_t^B)\)-adapted solution of the linear equation

\[
Y_{t}^{k} = Y + \int_{0}^{t} L^{k}(s, \omega)Y_{s}^{k} \, ds + \int_{0}^{t} A^{k}(s, \omega)Y_{s}^{k} \cdot dB_{s} \tag{20}
\]

where \( L^{k}(s, \omega) \) and \( A^{k}(s, \omega) \) are defined as above, with \( X^{k}_{t}(\omega) := Y_{s \wedge \eta^{k-1}}^{k-1}(\omega) \). Here, \( \eta^{k-1} \) is an \( \mathcal{F}_t^B \)-stopping time, defined inductively, as follows: Let \( r > 0 \) be as in the statement of the theorem.

\[
\eta^{j} := \sigma_{1} \wedge \cdots \wedge \sigma_{j} \quad \text{and} \quad \sigma_{j} := \inf\{s > 0 : \|Y_{s}^{j} - Y\|_{q} > r\}
\]

\( j = 1, \ldots, k-1 \). For notational convenience, in what follows, we often suppress the dependence on \( r \) when there is no ambiguity.

The existence and uniqueness of solutions of equation (20), is a consequence of Lemma (4.2) with \((Z_{t}), (\bar{Y}_{t})\) there taken to be the processes \((Z_{t}^{k-1}), (Y_{t}^{k})\) respectively, where \((Z_{t}^{k-1})\) is defined as follows:

\[
Z_{t}^{k-1} := \int_{0}^{t} \sigma(Y_{s \wedge \eta^{k-1}}^{k-1}) \cdot dB_{s} + \int_{0}^{t} b(Y_{s \wedge \eta^{k-1}}^{k-1})ds
\]

and \( Y_{t}^{k} := \tau_{Z_{t}^{k-1}}(Y) \).

Define \( \eta := \lim_{k \to \infty} \eta^{k} \). We note that \( \eta \equiv \eta^{r} \) depends on \( r \). We will show below that \( \eta > 0 \) almost surely. We now show that for each \( t \geq 0 \) the sequence \( \{Y_{t \wedge \eta}^{k}\} \) converges in \( L^{q}(\Omega \to S_q) \) for \( q \leq p - 1 \). We have as in the proof of uniqueness,

\[
\|Y_{t \wedge \eta}^{k} - Y_{t \wedge \eta}^{k-1}\|_{q}^{2} = \int_{0}^{t \wedge \eta} \left\{ 2(Y_{s}^{k} - Y_{s}^{k-1}, L^{k}(Y_{s}^{k}) - L^{k-1}(Y_{s}^{k-1}))_{q} \right\} ds + M_{t}^{k},
\]

where \( (M_{t}^{k}) \) is a local martingale. Then using (14) and (15) we have \( L^{k}(Y_{s}^{k}) = L(Y_{s}^{k-1}, Y_{s}^{k}), A_{i}^{k}(Y_{s}^{k}) = A_{i}(Y_{s}^{k-1}, Y_{s}^{k}) \) with similar expressions for \( L^{k-1}(Y_{s}^{k-1}) \).
and $A^{-1}(Y^k_{s})$ involving the processes $(Y^k_{s}), (Y^k_{s})$ respectively. From Theorem (3.3) and using a localisation argument, we can take expectations in the above expression to get for some constant $C_1 > 0$ and all $k \geq 1, t > 0$,

$$E\|Y^k_{t \wedge \eta} - Y^{k-1}_{t \wedge \eta}\|_q^2 \leq C_1 \left\{ \int_0^t \{E\|Y^{k-1}_{s \wedge \eta} - Y^{k-2}_{s \wedge \eta}\|_q^2 + E\|Y^k_{s \wedge \eta} - Y^{k-1}_{s \wedge \eta}\|_q^2 \} \, ds \right\}$$

(21)

By the Gronwall inequality (21) now implies

$$E\|Y^k_{t \wedge \eta} - Y^{k-1}_{t \wedge \eta}\|_q^2 \leq C \left\{ \int_0^t E\|Y^{k-1}_{u \wedge \eta} - Y^{k-2}_{u \wedge \eta}\|_q^2 \, du \right\} e^{C(t-s)} \, ds$$

$$\leq K \int_0^t E\|Y^{k-1}_{s \wedge \eta} - Y^{k-2}_{s \wedge \eta}\|_q^2 \, ds$$

where $K := C(1 + te^{Ct})$ and $C$ is some positive constant.

Iterating the above inequality yields, for each $t > 0$,

$$E\|Y^k_{t \wedge \eta} - Y^{k-1}_{t \wedge \eta}\|_q^2 \leq K^2 \int_0^t \int_0^s E\|Y^{k-2}_{u \wedge \eta} - Y^{k-3}_{u \wedge \eta}\|_q^2 \, du \, ds$$

$$\leq K^{n-1} \int_0^t \cdots \int_0^{t_1} E\|Y^{1}_{t_0 \wedge \eta} - Y^{0}_{t_0 \wedge \eta}\|_q^2 \, dt_0 \cdots \, dt_{k-2}$$

$$\leq \frac{\alpha K^{k-1}t^{k-1}}{(k-1)!}$$

where $\alpha = \sup_{t_0 \leq t} E\|Y^{1}_{t_0 \wedge \eta} - Y\|_q^2 < \infty$. It follows by the Cauchy-Schwartz inequality that for each $T > 0$ and $0 \leq t \leq T$,

$$\sum_{k=1}^{\infty} E\|Y^k_{t \wedge \eta} - Y^{k-1}_{t \wedge \eta}\|_q < \infty$$

and

$$\sum_{k=1}^{\infty} E \int_0^T \|Y^k_{t \wedge \eta} - Y^{k-1}_{t \wedge \eta}\|_q \, dt < \infty.$$
Define for each $t$,

$$Y_t := Y + \sum_{k=1}^{\infty} Y_{t \wedge \eta}^k - Y_{t \wedge \eta}^{k-1},$$

where the series in the right hand side converges in $L^1([0, T] \times \Omega \rightarrow S_q, dt \, dP)$, $q \leq p - 1$ and $T > 0$ and defines an $(\mathcal{F}_t)$-progressively measurable $S_q$-valued process $(Y_t)$. We also note that, for each $t$, $Y_t$ is an $S_q$-valued random variable such that $E\|Y_t - Y_{t \wedge \eta}^k\|_q \rightarrow 0$. Note that for each $t \geq 0, k \geq 1, \|Y_{t \wedge \eta}^k - Y\|_q \leq r$ almost surely and by passing to an almost sure convergent subsequence, we also have, $\|Y_t - Y\|_q \leq r$ almost surely. Denoting this subsequence again by $Y_t^k$ it follows by the bounded convergence theorem that $E\|Y_t - Y_{t \wedge \eta}^k\|_q^2 \rightarrow 0$ for every $t$ and moreover that $E\int_0^t \|Y_s - Y_{s \wedge \eta}^k\|_q^2 ds \rightarrow 0$.

We now wish to pass to the limit in equation (20). We note that by the assumed continuity of the coefficients $\sigma_{ij}, b_i; i = 1, \ldots, d, j = 1, \ldots, n$ and the continuity of $\partial_i : S_q \rightarrow S_{q-1/2}, i = 1, \ldots, d, \|Y_t - Y_{t \wedge \eta}^k\|_q \rightarrow 0$ almost surely for each $t$ implies,

$$L(Y_{s \wedge \eta}^{k-1}, Y_s) \rightarrow L(Y_s) \quad \text{and} \quad A_i(Y_{s \wedge \eta}^{k-1}, Y_s) \rightarrow A_i(Y_s)$$

for every $s \leq t \wedge \eta$ and $i = 1, \ldots, n$, almost surely, where the convergence takes place in $S_{q-1}$. Note also that there exists a constant $K_1 > 0$ such that for each $0 \leq s \leq t$, almost surely

$$\|L(Y_{s \wedge \eta}^{k-1}, Y_{s \wedge \eta}^k)\|_{q-1} + \sum_{i=1}^{n} \|A_i(Y_{s \wedge \eta}^{k-1}, Y_{s \wedge \eta}^k)\|_{q-1} \leq K_1$$

and a fortiori,

$$\|L(Y_{s \wedge \eta})\|_{q-1} + \sum_{i=1}^{n} \|A_i(Y_{s \wedge \eta})\|_{q-1} \leq K_1$$

holds for each $0 \leq s \leq t$, almost surely. It follows from the above observations that

$$\int_0^{t \wedge \eta} L^k(s, \omega)Y_s^k ds \rightarrow \int_0^{t \wedge \eta} L(Y_s) ds$$

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for each $t \geq 0$, almost surely in $S_{q-1}$ and that
\[ \int_0^{t \land \eta} A^k(s, \omega)Y^n_s \cdot dB_s \to \int_0^{t \land \eta} A(Y_s) \cdot dB_s \]
for each $t \geq 0$ in $L^2(\Omega \to S_{q-1})$. Hence we can pass to the limit in $S_{q-1}$ in equation (20) with $t$ replaced by $t \land \eta$, to get
\[ Y_t = Y + \int_0^{t \land \eta} L(Y_s)ds + \int_0^{t \land \eta} A(Y_s) \cdot dB_s \]  
(22)
in $S_{q-1}$ for every $t > 0$, almost surely. It then follows, as a consequence of Lemma (4.2), that if we define
\[ Z_t := \int_0^{t \land \eta} \sigma(Y_s)ds + \int_0^{t \land \eta} b(Y_s)ds, \]  
(23)
then $(\tau_{Z_t}(Y))$ is a continuous $S_p$ valued process that satisfies equation (19) in $S_q$ for any $q \leq p - 1$. We denote this process again by $(Y_t)$ i.e. $Y_t := \tau_{Z_t}(Y)$. By its very construction the paths of $(Y_t)$ are constant for $t > \eta$, and we can redefine this value to be $\delta$ to satisfy definition (4.1) of a 'local solution'.

We now show that $\eta > 0$ almost surely. Recall the stopping times $\sigma_n, \eta^n$ defined above. Suppose there exists a set $A$ with $P(A) > 0$ such that if $\omega \in A$ then $\eta(\omega) = 0$. Then we claim that $\exists$ a subset $A_0 \subset A$ with $P(A_0) > 0$ and a subsequence $\{n_k\}$ such that for $\omega \in A_0$, $\sigma_{n_k}(\omega) = \eta^{n_k}(\omega) < \eta^{n_{k-1}}(\omega), k \geq 2$.

It is intuitively clear that such subsequences as described above must exist. We give a proof for completeness: Fix $t > 0$. We define a sequence of integer valued random variables $K_i$ for $i \geq 0$ as follows:
\[ K_0 := \min\{j \geq 1 : \eta^j < t\}. \]
For $i \geq 1, K_i := \min\{j > K_{i-1} : \eta^j < \eta^{j-1}\}$. Note that if $\omega \in A$, then $K_i(\omega) < \infty$ for all $i \geq 0$ and $\sigma_{K_i}(\omega) = \eta^{K_i}(\omega) < \eta^{K_{i-1}}(\omega)$. Further for every $i \geq 0$,
\[ A \subset \{K_i < \infty\} = \bigcup_{m=1}^{\infty} \{K_i = m\}. \]
Hence for every $i \geq 1$, we can choose an integer $m_i$ satisfying

$$P(\{K_i \leq m_i\} \cap A) \geq (P(A) - \frac{1}{i^2})^+. $$

Without loss of generality we can take $m_i > m_{i-1}, i \geq 2$. It follows that for $i$ sufficiently large

$$P(A \cap \{K_i > m_i\}) = P(A) - P(A \cap \{K_i \leq m_i\}) \leq \frac{1}{i^2}. $$

It follows by the Borel-Cantelli lemma that

$$P\{\omega \in A : K_i(\omega) > m_i \text{ infinitely often }\} = 0.$$ 

In particular we have the almost sure equality

$$A = \{K_i > m_i \text{ infinitely often }\}^c \cap A = \bigcup_{n=1}^{\infty} \bigcap_{\ell \geq n} \{K_\ell \leq m_\ell\} \cap A = \bigcup_{n=1}^{\infty} B_n,$$

where we define the set $B_n$ for $n \geq 1$ as

$$B_n = \bigcap_{\ell \geq n} \{K_\ell \leq m_\ell\} \cap A$$

$$= \bigcup \{K_1 = j_1 \ldots K_\ell = j_\ell, K_{\ell+1} = j_{\ell+1}, \ldots \} \cap A$$

where the (countable) union in the last equality is over $j_\ell \leq m_\ell$ for $\ell \geq n$, and $j_1 < j_2 < \ldots < j_{n-1} < j_n \leq m_n$. Since $B_n \uparrow A$ and $P(A) > 0$, we choose $n$ so that $P(B_n) > 0$. Since each $B_n$ is a countable union of sets, each of which corresponds to a sequence $\{j_\ell\}$ and $P(B_n) > 0$, $\exists$ a sequence $\{j_\ell\}$, $j_1 < j_2 < \ldots < j_k < \ldots$ such that

$$P(\{K_1 = j_1 \ldots K_\ell = j_\ell, \ldots \} \cap A) > 0.$$ 

Define the set $A_0 := \{K_1 = j_1, \ldots K_\ell = j_\ell, \ldots \} \cap A$. For $\omega \in A_0$, $\sigma_{j_\ell}(\omega) = \eta^j(\omega) < \eta^{j_\ell-1}(\omega), \ell \geq 1$. Note that $P(A_0) > 0$. This proves our claim. We will now work with the subsequence $(Y_n^\eta)^{k\geq 1}$ which we will rename $(Y_n^\eta)$ which then has the following property on $A_0$ : If $\omega \in A_0$, then $\sigma_n(\omega) = \eta^\eta(\omega) < \eta^{n-1}(\omega), n \geq 2.$
Fix \( t > 0 \). Then we claim that, almost surely along a subsequence, \( Y^k_{t \land \eta_k} \to Y_{t \land \eta} \) in \( S_p \) as \( k \to \infty \). This can be seen as follows. First note that \( \eta_k \) defined above, decrease to \( \eta \) almost surely. Next, observe that \( Y^k_t = \tau_{Z^k_t}^{-1}(Y) \) where for each \( k \geq 1 \), the process \( (Z^k_t) \) is defined by

\[
Z^k_t := \int_0^t \sigma(Y^k_s) \cdot dB_s + \int_0^t b(Y^k_s) ds.
\]

Since the map \( x \to \tau_x(Y) : \mathbb{R}^d \to S_p \) is continuous, to prove the claim, it suffices to show that \( (Z^k_{t \land \eta_k}) \) converges to the process \( (Z_{t \land \eta}) \) in probability and hence almost surely along a subsequence. To see this, note that we can write

\[
Z^k_{t \land \eta_k} = \int_0^{t \land \eta} b(Y^k_s) ds + \int_0^{t \land \eta} \sigma(Y^k_s) \cdot dB_s
\]

\[
+ \int_0^t 1_{(0,\eta_k)}(s)b(Y^k_s) ds + \int_0^t 1_{(0,\eta_k)}(s)\sigma(Y^k_s) \cdot dB_s.
\]

By stopping \( (Y^k_s) \) at its exit from a suitably large ball centered at \( Y \in S_p \), we can show that the third and fourth terms go to zero in probability. Hence using the (Lipschitz) continuity of the maps \( \sigma, b \) and the convergence of \( Y^k_{s \land \eta} \) to \( Y_s \) in \( L^2([0,t] \times \Omega \to S_q) \) it is easy to see that \( (Z^k_{t \land \eta_k}) \) converges to the process \( (Z_{t \land \eta}) \) in probability and hence almost surely along a subsequence.

Thus our claim is proved and we have \( Y^k_{t \land \eta_k} \to Y_{t \land \eta} \) in \( S_p \), almost surely, along a subsequence. In particular it converges to \( Y \) in \( S_q \) on the set \( A \). For the rest of the proof, we will work with this subsequence, which, abusing notation, we continue to denote by \( Y^k_{t \land \eta_k} \). In particular we will now assume that the sum of integrals in the right hand side of equation (20), evaluated at \( t \land \eta^k \), goes to zero, which is the value of the sum of integrals in the RHS of (22), almost surely on the set \( A \).

On the other hand, for \( \omega \in A_0, \sigma_k(\omega) = \eta^k(\omega) < \eta^{k-1}(\omega), n \geq 2 \); Hence for \( k \geq 1 \) such that \( \eta^k \leq t \), we have

\[
Y^k_{\eta^k \land \omega}(\omega) = Y^k_{\eta^k}(\omega) = Y^k_{\eta^k}(\omega).
\]
and consequently by the continuity of the process $Y^k_{t\wedge \eta^k}$ in $S_q$,

$$\|Y^k_{\eta^k \land t}(\omega) - Y\|_q = \|Y^k_{\sigma^k}(\omega) - Y\|_q = r$$

for $\omega \in A_0$ and $k \geq k_0(\omega)$ for some $k_0(\omega) \geq 1$. But this leads to a contradiction to the fact proved above that the RHS of (20) goes to zero in $S_q$, almost surely on $A$ and in particular on the set $A_0$.

To complete the proof of the first part of the theorem, let $r > 0$ and $Y^1, Y^2$ be as in the statement of the theorem with $P\{Y^1 = Y^2\} > 0$. First we assume $Y^1, Y^2$ are bounded. We will denote with a superscript $i, i = 1, 2$, the various objects defined in the construction of the solutions corresponding to the bounded initial values $Y^1, Y^2$ respectively and suppress the dependence on $r$, which is fixed. Firstly, we note that since $\eta^1_{i,j} = \eta^2_{i,j-1}$ and $\sigma^1_{i,j}, i = 1, 2$ and $j \geq 2$ and since uniqueness holds for the linearized equation (20), an induction argument shows that $\eta^1_{i,j} = \eta^2_{i,j}$ and, almost surely, $Y^1_{i,j} = Y^2_{i,j}, 0 \leq t < \infty$ on $\{Y^1 = Y^2\}$. It follows that $\eta^1 := \lim_{j \to \infty} \eta^1_{i,j} = \lim_{j \to \infty} \eta^2_{i,j}$ on the set $\{Y^1 = Y^2\}$.

To show that almost surely, $Y^1_t = Y^2_t, 0 \leq t < \eta^1$ on $\{Y^1 = Y^2\}$ we argue as follows. Define $\bar{Y}^i_t := I_{\{Y^1 = Y^2\}} Y^i_t, i = 1, 2$ and $\eta' := \eta^1 + \eta^2$. Then, using the quasi linearity of (19), $(\bar{Y}^1_t, \eta'), (\bar{Y}^2_t, \eta')$ are two solutions of (19) with initial value $I_{\{Y^1 = Y^2\}} Y^1$. By uniqueness, our claim follows.

The existence for the case of a general initial random variable $Y$ and a fixed $r > 0$ can be reduced to the $L^2$ case by considering the initial conditions $Y^n := Y I_{\{\|Y\|_p \leq n\}}$, as follows. Denote the solution corresponding to $Y^n$, constructed above, by $(Y^n_t, \eta^n)$ where we have omitted the dependence on $r$. Another solution with the same initial condition $Y^n$ is given by $(\bar{Y}^n_t, \bar{\eta}^n)$ where we define

$$\bar{\eta}^n := \infty, \quad \omega \in \{\|Y\|_p > n\}; \quad := \eta^{n+1}, \quad \omega \in \{\|Y\|_p \leq n\}$$

and

$$\bar{Y}^n_t := Y^n_t, 0 \leq t < \bar{\eta}^n$$

Then by uniqueness we get that almost surely on the set $\{\|Y\|_p \leq n\}$, $Y^n_t = \bar{Y}^n_t = Y^{n+1}_t, 0 \leq t < \eta^n$, and $\eta^n = \eta^{n+1}$. We can now construct the
solution \((Y^r_t, \eta^r)\) corresponding to the initial random variable \(Y\) by piecing together the solutions on the sets \(\{\|Y\|_p \leq n\}\) as follows: \(\eta^r(\omega) := \eta^n(\omega)\) and \(Y^r_t(\omega) := Y^n_t(\omega), 0 \leq t < \eta^n\) if \(\omega \in \{\|Y\|_p \leq n\}\) and lies outside a suitable null set. That \((Y^r_t, \eta^r)\) is a solution of equation (19) follows from the fact that \((Y^n_t, \eta^n)\) solves (19) with initial condition \(Y\) on the set \(\{\|Y\|_p \leq n\}\).

Let now \(Y^1, Y^2\) be two \(\mathcal{F}_0^B\) measurable, \(\mathcal{S}_p\) valued random variables and \((Y^r_1, \eta^1), (Y^r_2, \eta^2)\) be the corresponding solutions constructed above for some fixed \(r > 0\). To show the claimed uniqueness on the set \(\{Y^1 = Y^2\}\), we define, for \(n \geq 1\), the processes

\[
Y^n_{t,i} := I\{Y^1 = Y^2, \|Y^1\|_p \leq n\} Y^i_t, \quad 0 \leq t < \eta^n, i = 1, 2,
\]

where \(\eta^n := \eta^n\) on \(\{Y^1 = Y^2, \|Y^1\|_p \leq n\}\); = \(\infty\) otherwise. Then, \(Y^n_{t,i}\) solve (19) on the interval \([0, \eta^n]\), with initial values \(I\{Y^1 = Y^2, \|Y^1\|_p \leq n\} Y^i_t, i = 1, 2\) respectively; and by uniqueness for the case of bounded initial random variables discussed above, it follows that, almost surely, \(\eta^1 = \eta^2\) and \(Y^1_t = Y^n_{t,1} = Y^n_{t,2} = Y^2_t, 0 \leq t < \eta^1\) on the set \(\{Y^1 = Y^2, \|Y^1\|_p \leq n\}\). Letting \(n \to \infty\) the uniqueness claim follows. This completes the proof of the theorem. \(\square\)

**Remark 4.4** A simpler proof of existence can be provided using the finite dimensional existence results as in Theorem 2.1. In effect, we fix the initial value \(y\) and we define \(\bar{\sigma}_{ij}(x) := \sigma_{ij}(\tau_x y), \bar{b}_i(x) := b_i(\tau_x y)\). Then if \(\sigma_{ij}, b_i\) are Lipschitz in \(\mathcal{S}_q, q \leq p - \frac{1}{2}\), then one can show (using duality and the mean value theorem applied to \(<\tau_x y - y, \psi>, \psi \in \mathcal{S}\)) that \(\bar{\sigma}_{ij}, \bar{b}_i\) are locally Lipschitz on \(\mathbb{R}^d\). If \((Z_t, \eta)\) is the solution of equation (1) with \(x = 0\) then using Itô’s formula for translations, it is easy to see that \(Y_t := \tau_{Z_t} y, t < \eta, \eta\), solves equation (19). However this proof does not work when, for example, we replace the operator \(L\) in equations (3) and (19) with a perturbation of \(L\) viz. \(L + cI\), where \(I\) is the identity and \(c \in \mathbb{R}\) (see also Example 6 of Section 6).

**Remark 4.5** Translation invariance also applies to solutions of an evolution equation which is a first order quasi linear PDE i.e. these solutions are translates of the initial condition by the solution of an appropriate ‘characteristic’
ODE. This follows on setting the diffusion coefficients in the above calculations to be equal to zero. These first order systems may also be viewed as the 'zero noise' limit of stochastic second order system, a topic of considerable interest in the last three or four decades (see [19]), and also volumes 3 & 4 of [7] for the connection with large deviation theory.

5 The Strong Markov Property:

In this section we show that the local solution of equation (19) obtained in Theorem (4.3) can be extended to a maximal interval \([0, \eta]\) for a given \(F_0\)-measurable \(Y\) (Theorem (5.3)). We then show that the solutions \((Y_t(y), \eta)\) obtained when \(Y \equiv y \in S_p\) have a jointly measurable version \((Y(t, \omega, y), \eta(\omega, y))\) in \((t, \omega, y)\) and that the solutions for arbitrary \(Y\) can be represented as \(Y(t, \omega, Y), 0 \leq t < \eta(\omega, y),\) (Proposition (5.4) and Theorem (5.5)). The strong Markov property (Theorem (5.7)) is then proved as a consequence of uniqueness in law, which in turn follows from pathwise uniqueness by a Yamada-Watanabe type argument (Theorem (5.6)).

Consider now the solution \((Y^r_t, \eta^r)\) constructed in Theorem (4.3) for \(r > 0\) and for some \(F_0\)-measurable random variable \(Y : \Omega \to S_p\). From the definition of \(\eta^r\) in the proof of Theorem (4.3), it follows that, \(r_1 < r_2\) implies \(\eta^{r_1}(\omega) \leq \eta^{r_2}(\omega)\). Let \(\eta(\omega) := \lim_{r \to \infty} \eta^r(\omega)\). Then by pathwise uniqueness of solutions we have \(Y^r_{t_1} = Y^r_{t_2}, 0 \leq t < \eta^r,\) almost surely. Let \(r_n \uparrow \infty\). Then \(\eta^{r_n}(\omega) \uparrow \eta(\omega)\) \(\forall \omega\). Let \(\Omega_n\) satisfy \(P(\Omega_n) = 0\) and for \(\omega \notin \Omega_n, Y^{r_n}_t(\omega) = Y^{r_{n+1}}_t(\omega), 0 \leq t < \eta^{r_n}\). Let \(\Omega_0 := \bigcup_{n=1}^\infty \Omega_n\). Define, for \(\omega \notin \Omega_0\) and \(0 \leq t < \eta(\omega)\)

\[Y_t(\omega) := Y^{r_n}_t(\omega) \text{ if } 0 \leq t < \eta^{r_n}(\omega) \leq \eta(\omega)\]

and let

\[Y_t(\omega) := \delta \text{ for } t \geq \eta(\omega)\]

For \(\omega \in \Omega_0\) redefine \(\eta(\omega) := 0\) and define

\[Y_t(\omega) := Y(\omega), \forall t \geq 0.\]

We note the following 'maximality' property of the solution \((Y_t, \eta)\).
Proposition 5.1 Let $Y \in S_{p,q}$, $q \leq p - 1$. Then, a.s., $\lim_{t \uparrow \eta(\omega)} \|Y_t(\omega)\|_q = \infty$ on $\omega \in \{\eta < \infty\}$. In particular, a.s., $\lim_{t \uparrow \eta(\omega)} \|Y_t(\omega)\|_p = \infty$ on $\omega \in \{\eta < \infty\}$.

Proof: Note that $\{\eta < \infty\} = \bigcup_n \{\eta < n\}$. It suffices to show that $\lim_{r \to \infty} \|Y_{\eta^r}\| = \infty$ a.s. on $\{\eta < \infty\}$. Recall the approximations $(Y^k_t) \equiv (Y^{k,r}_t)$ to the solutions $(Y^r_t, \eta^r)$ constructed in the proof of Theorem (4.3). For fixed $r > 0$ we know that $Y^{k,r}_t \uparrow Y^r_t$ in $L^2(\Omega, S_q)$ and in particular for every $\epsilon > 0$

$$P\left(\|Y^k_{t \wedge \eta^r} - Y^{k,r}_{t \wedge \eta^r}\|_q^2 > \epsilon\right) \to 0$$

as $k \to \infty$. Further if $\eta^{k,r}$ are the approximations to $\eta^r$ constructed in the proof of Theorem (4.3) we have for each $k \geq 1$,

$$\|Y^{k,r}_{t \wedge \eta^r} - Y^{k}_{t \wedge \eta^{k,r}}\|_q^2 = \int_{t \wedge \eta^r}^{t \wedge \eta^{k,r}} \left\{2 \left\langle Y^{k}_{u} - Y^{k,r}_{t \wedge \eta^r}, L(Y^{k}_{u})\right\rangle + \sum_{i=1}^n \|A_i(Y^{k}_{u})\|_q^2 \right\} du + 2 \int_{t \wedge \eta^r}^{t \wedge \eta^{k,r}} \left\langle Y^{k}_{u} - Y^{k,r}_{t \wedge \eta^r}, A(Y^{k}_{u})\right\rangle q \cdot dB_u.$$ 

Since $\eta^{k,r} \downarrow \eta^r$ and $\|Y^{k}_{u}\|_q \leq r$ for $t \wedge \eta^r \leq u \leq t \wedge \eta^{k,r}$, the first term goes to zero by the bounded convergence theorem almost surely and the second term goes to zero in probability as $k \to \infty$. Thus for every $\epsilon > 0$,

$$P\left(\|Y^k_{t \wedge \eta^r} - Y^{k,r}_{t \wedge \eta^r}\|_q > \epsilon\right) \to 0, \; k \to \infty.$$ 

It follows from the above that for every $\epsilon > 0$,

$$P\left(\|Y^k_{t \wedge \eta^r} - Y^{k,r}_{t \wedge \eta^r}\|_q > \epsilon\right) \to 0, \; k \to \infty.$$ 

Thus for any $t > 0$ and passing to a subsequence $\{k_i\}$ we have a.s. on $\{\eta^r < t\}$,

$$Y^{k_i}_{\eta^{k_i,r}} \to Y_{\eta^r}.$$ 

We can argue (as in the proof that $\eta^r > 0$ a.s. in the proof of Theorem (4.3)), that by passing to a further subsequence, that $\eta^{k_i,r} = \sigma_{k_i,r}$ and in particular that on $\{\eta^r < t\}$

$$r = \|Y^{k_i}_{\sigma_{k_i,r}} - Y\|_q = \|Y^{k_i}_{\eta^{k_i,r}} - Y\|_q \to \|Y_{\eta^r} - Y\|_q.$$ 

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It follows that a.s. on $\{\eta^r < t\}, \|Y_{\eta^r} - Y\|_q = r$. Now we take $r_k \uparrow \infty$. Then $\{\eta < \infty\} = \bigcup_{n=1}^{\infty} \bigcap_{k=1}^{\infty} \{\eta^{r_k} < n\}$. In particular a.s. on $\{\eta < \infty\}$,

$$\lim_{r \uparrow \infty} \|Y_{\eta^r}\|_q = \infty$$

Since $q < p$ the result follows. $\square$

**Proposition 5.2** Let $E\|Y\|_p^2 < \infty$ and $\sigma_{ij}, b_i : S_p \rightarrow \mathbb{R}$ be bounded i.e. $\exists K > 0$ such that $|\sigma_{ij}(\phi)| + |b_i(\phi)| \leq K$ for all $\phi \in S_p, i = 1, \ldots, d, j = 1, \ldots r$. Then $\eta = \infty$ a.s.

**Proof:** It suffices to show that for every $t > 0$, $P(\eta^r \leq t) \rightarrow 0$ as $r \uparrow \infty$. As in the proof of the previous Proposition, $\{\eta^r < t\} \subset \{\eta^r < t, \|Y_{\eta^r} - Y\|_q = r\} \subset \{\|Y_{\eta^r \wedge t} - Y\|_q \geq r\}$. Hence,

$$P(\eta^r \leq t) \leq \frac{1}{r^2} E\|Y_{t \land \eta^r} - Y\|_q^2$$

Further by the boundedness assumptions on $\sigma_{ij}, b_i$ and the monotonicity inequality given in Theorem (3.1) (applied with $\phi = \psi = Y_s$), we have

$$2\langle Y_s, L(Y_s)\rangle_q + \sum_{i=1}^{n} \|A_i(Y_s)\|_q^2 \leq C \cdot \|Y_s\|_q^2$$

where $C > 0$ is a constant depending only on $r, d, p$ and $K$. Using Gronwal’s inequality we get

$$E\|Y_{t \land \eta^r} - Y\|_q^2 \leq 2\{e^{Ct} E\|Y\|_q^2 + E\|Y\|_q^2\}.$$ 

Hence dividing by $r^2$ and letting $r \uparrow \infty$ in the above inequality we conclude that $P(\eta^r \leq t) \rightarrow 0$ as $r \uparrow \infty$ for every $t > 0$. $\square$

**Theorem 5.3** Let $p \in \mathbb{R}, q \leq p - 1$. Let $\sigma_{ij}, b_i : S_p \rightarrow \mathbb{R}, i = 1, \ldots, d, j = 1, \ldots n$ be as in Theorem (4.3). Then for every $Y : \Omega \rightarrow S_p$ which is $\mathcal{F}_0$ measurable and independent of $B$, equation (18) has a unique $\hat{S}_p$ valued
strong (local) solution \((Y_t, \eta)\). Further \(\eta > 0\) is maximal in the sense that, almost surely,
\[
\lim_{t \uparrow \eta(\omega)} \|Y_t(\omega)\|_p = \infty \text{ on } \omega \in \{\eta < \infty\}.
\]

Finally, if we define \((Z_t)\) as
\[
Z_t := \int_0^{t \wedge \eta} \sigma(Y_s) \cdot dB_s + \int_0^{t \wedge \eta} b(Y_s)ds,
\] (24)
then \((Z_t)\) is a continuous, \((\mathcal{F}^Y_t)\)-adapted, \(\mathbb{R}^d\)-valued process such that almost surely,
\[
Y_t = \tau_{Z_t}(Y), 0 \leq t < \eta. \quad (25)
\]

**Proof:** The proof follows by ‘patching up’ the solutions obtained in Theorem (4.3), as described in the beginning of Section 5. This gives us the pair \((Y_t, \eta)\). That it solves equation (19) follows from Theorem (4.3) and the fact that by construction, for any \(r > 0\), \(Y_t = Y^r_t, 0 \leq t < \eta^r\). The maximality of the solution follows from Proposition (5.1). We note that \(Y_t = \tau_{Z_t}(Y)\) follows from Lemma (4.2). \(\square\)

To formulate the strong Markov property, we consider the solution \((Y_t, \eta)\) when the initial value \(Y\) is a constant \(Y \equiv y \in \mathcal{S}_p\) and denote the corresponding solution by \((Y_t(y), \eta^y)\) or \((Y_t(\omega, y), \eta^y)\).

Recall that \(\hat{\mathcal{S}}_p = \mathcal{S}_p \cup \{\delta\}\). We define the \(\sigma\)-field \(\mathcal{B}(\hat{\mathcal{S}}_p)\) on \(\hat{\mathcal{S}}_p\) by \(\hat{A} \in \mathcal{B}(\hat{\mathcal{S}}_p)\) iff \(\hat{A} = A \cup \{\delta\}\) for some \(A \in \mathcal{B}(\mathcal{S}_p)\). A measurable function \(f : \mathcal{S}_p \to \mathbb{R}\) is extended to \(\hat{\mathcal{S}}_p\) by defining \(f(\delta) = 0\). The resulting extension will also be denoted by \(f\).

For \(y \notin \mathcal{S}_p\), we define
\[
Y_t(\omega, y) = \delta, \ t \geq 0, \ \omega \in \Omega.
\]

In other words, \(\eta(\omega) = 0\) for such \(y\). We now construct versions of the solution \((Y_t(y), \eta^y)\) constructed in Theorem (5.3), with initial value \(Y \equiv y \in \mathcal{S}_p\), which are jointly measurable in \((t, \omega, y)\). In the two Propositions below, we need the approximations \((Y^k_t)\), \(k \geq 1\) for initial values \(Y \equiv y\), constructed in the
proof of Theorem (4.3), which we now denote by \((Y_t^k(y))\) or as \((Y_t^{r,k}(y))\), whenever the dependence on the domain \(B_p(y, r)\) needs to be made explicit. The proofs of the following two results (Proposition (5.4) and Theorem (5.5) are given in the Appendix).

Proposition 5.4 a). There exists a map \(\tilde{Y} : [0, \infty) \times \Omega \times S_p \rightarrow \hat{S}_p\) which is measurable and satisfies

for all \(t \geq 0, y \in \hat{S}_p\), \(\tilde{Y}(t, \omega, y) = Y_t(\omega, y)\) a.s.

b). There exists a map \(\tilde{\eta} : \Omega \times S_p \rightarrow [0, \infty]\) which is measurable and satisfies

for all \(y \in \hat{S}_p\), \(\tilde{\eta}(\omega, y) = \eta^y(\omega)\) a.s.

The next result shows that the solution of the SPDE (19) with an initial random variable \(Y^0\) as the composition of \(Y^0\) with the solutions starting at \(y \in S_p\). So let \(Y^0 : \Omega \rightarrow S_p\) be \(F_0\)-measurable. Define

\[
\tilde{Y}(t, \omega) := Y(t, \omega, Y^0(\omega)), \quad \tilde{\eta}(\omega) := \eta(\omega, Y^0(\omega)).
\]

Then we have the following result.

Theorem 5.5 Let \((Y_t, \eta)\) be the solution of equation (19) with initial r.v. \(Y^0\) independent of \((B_t)\). Then for each \(t \geq 0\), we have \(\eta = \tilde{\eta}\) a.s. and

\[
\tilde{Y}(t, \omega) = Y_t(\omega)\ a.s.\ on\ \{t < \eta\}.
\]

We now prove uniqueness in law for equation (19) required to prove the strong Markov property. This follows from the Yamada-Watanabe result for SPDE’s of the type (19), which we now state as the next theorem. We need some preliminaries to deal with the law of explosive solutions.

We first construct an appropriate path space i.e. a measurable space \((C, C)\) such that if \((Y_t, \eta)\) is a maximal solution on some probability space then almost surely, the paths \(Y_{\wedge \eta}(\omega)\) belong to \(C\) and the map \(\omega \rightarrow Y_{\wedge \eta}(\omega)\) is
measurable. It is clear that the law of \((Y, \eta)\) is essentially determined on \([0, \eta)\) where the paths are continuous. However, we need to distinguish between the cases \(\eta < \infty\) and \(\eta = \infty\). Further, although our initial conditions and the paths of the corresponding solutions lie in \(S_p\), we will consider them as paths in \(S_q\), where the equation holds.

Thus let \(p \in \mathbb{R}, q \leq p - 1\). Let \(y : [0, \infty) \to \dot{S}_q\). All such maps that we consider will be \(\mathcal{B}([0, \infty))/\mathcal{B}(\dot{S}_q)\) measurable. Define \(\eta_q(y) := \inf\{s > 0 : y(s) \notin S_q\}\). Let

\[
C_0 := \{y : [0, \infty) \to \dot{S}_q, y(0) \in S_p, \eta_q(y) > 0 ; y : [0, \eta_q(y)) \to S_q \text{ is continuous; }\},
\]

\[
C_1 := C_0 \cap \{\eta_q(y) < \infty \text{ and } \lim_{t \to \eta_q(y)} \|y(t)\|_q = \infty\};
\]

\[
C_2 := C_0 \cap \{\eta_q(y) = \infty\}.
\]

Define \(C := C_1 \cup C_2\). For \(y \in C\), define for \(r > 0\), \(\tau_r(y) := \inf\{t > 0 : \|y(t) - y(0)\|_q > r\}\). Then \(y \in C\) implies \(y \in C_0\) and by continuity of \(y\) on \([0, \eta_q(y)]\), we have \(\tau_r(y) < \infty\), and \(\eta_q(y) = \lim \tau_r(y)\). Note that \(C([0, \infty), S_q) = \{y \in C : \eta_q(y) = \infty\}\). We now define a sigma field \(\mathcal{C}\) on \(C\) via the maps \(K_r : C \to C([0, \infty), S_q), r \geq 0, K_r(y) := y(\cdot \land \tau_r)\) as follows.

\[
\mathcal{C} := \sigma\{K_r : r \geq 0\} = \sigma\{K_r^{-1}(A) : A \in \mathcal{B}(C([0, \infty), S_q))\}.
\]

Let \((Y_t, \eta)\) be a maximal solution of equation (19) with initial value \(Y \in S_p\) and Brownian motion \((B_t)\), obtained in Theorem (5.3) on some probability space \((\Omega, \mathcal{F}, P)\). Then recall that this solution is obtained by pasting together the solutions \((Y''^r, \eta'')\), \(r > 0\), obtained in Theorem (4.3). Then we have a map \(\dot{Y} : \Omega \to \dot{S}_q\),

\[
\dot{Y}(\omega) := (t \to Y_t(\omega), t < \eta ; t \to \delta, t \geq \eta).
\]

By Proposition (5.1) it follows that almost surely, \(\eta_q(\dot{Y}(\omega)) = \eta(\omega)\) and hence \(\dot{Y}(\omega) \in C\) almost surely. We can redefine \(\dot{Y}\) on a null set so that \(\dot{Y} : \Omega \to C \subset \dot{S}_q\). Since \(\eta''(\omega) \leq \tau_r(\dot{Y}(\omega)) =: \tau_r(\omega) < \eta_q(\dot{Y}(\omega)) = \eta(\omega)\) and since \(\eta''(\omega) \uparrow \eta(\omega)\), we have for a fixed \(r_0 > 0\),

\[
K_{r_0}(\dot{Y}(\omega)) := Y_{t \land \tau_{r_0}(\omega)}(\omega) = \lim_{r \to \infty} Y_{t \land \tau_{r_0} \land \eta''(\omega)}(\omega).
\]

Since the right hand side is a limit of measurable maps, we conclude that for each \(r_0 \geq 0\), the maps \(\omega \to K_{r_0}(\dot{Y}(\omega))\) is \(\mathcal{F}^B\) measurable and hence \(\omega \to \dot{Y}(\omega)\) is \(\mathcal{F}^B/\mathcal{C}\) measurable.

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Let \((Y^i_t, B^i_t, \eta^i)\) \(i = 1, 2\) be two ‘maximal’ solutions of (19) viz.

\[
\begin{align*}
  dY^i_t &= L(Y^i_t) \, dt + A(Y^i_t) \cdot dB^i_t, \quad 0 < t < \eta^i \\
  Y^i_0 &= Y^i, 
\end{align*}
\]

adapted to the \(n\) dimensional \(\mathcal{F}^i_t\)-Brownian motions \((B^i_t), i = 1, 2\), with \(\mathcal{F}^i_0\) measurable initial values \(Y^i : \Omega^i \rightarrow S_p\), independent of \((B^i_t)\), on possibly different probability spaces \((\Omega^i, \mathcal{F}^i, \mathcal{F}^i_t, \mathcal{P}^i)\). Equality in law between two random variables \(X^1, X^2\) will be denoted by ”\(X^1 \triangleq X^2\)”. For \(r > 0\) we will denote by \(\eta^{i,r}, r > 0,\) the stopping times upto which the solution \(Y^i\) lies in a ball of radius \(r\) around \(Y^i\) in \(S^q, q \leq p - 1\) (Theorem (4.3)) and \(\eta^i = \lim_{r \to \infty} \eta^{i,r}\), the explosion time. Let \(\mathcal{P}^i, i = 1, 2\) be the laws of \((Y^i_t)\) on \((C, \mathcal{C})\). Let \(W_n := C([0, \infty), \mathbb{R}^n)\).

**Theorem 5.6** If \(Y^1 \triangleq Y^2\), then \(\mathcal{P}^1 = \mathcal{P}^2\).

**Proof:** Let \(\mathcal{P}^{i,y}, i = 1, 2\) be the law of the solutions \((Y^{i,y}_t, \eta^{i,y})\) of equation (19) corresponding to \(Y^i \equiv y \in S_q\) on \((C, \mathcal{C})\). Using Theorem (5.5) and the independence of \(Y^i\) and \((B^i_t)\) and the definition of \(\mathcal{C}\) we have

\[
\mathcal{P}^i(A) = \int_{S_p} \mathcal{P}^{i,y}(A) \mathcal{P}^i_y(dy),
\]

it suffices to show that for all \(y \in S_p, \mathcal{P}^{1,y} = \mathcal{P}^{2,y}\) on \(\mathcal{C}\). From the definition of \(\mathcal{C}\), it suffices to show that for every \(r_0 > 0\), the laws of \((Y^{i,y}_t, \eta^{i,y})\) composed with the map \(K_{r_0} : C \rightarrow C([0, \infty), S_q)\) i.e. the laws of \(K_{r_0}(Y^{i,y})\) agree on \(\mathcal{B}(C([0, \infty), S_q))\). We fix \(y \in S_p\). In what follows, we drop the explicit dependence on \(y\) in our notation. Let \(\mathcal{P}^i_r(A) := \mathcal{P}(Y^{i,y}_t, \eta^{i,y} \wedge \tau_{r_0} \wedge \tau_r \in A)\) where \(A \in \mathcal{B}(C([0, \infty), S_q))\). Since, as was observed above, \(K_{r_0}(Y^i)\) is the almost sure limit \((Y^{i,r}_t, \eta^{i,r})\) as \(r \uparrow \infty\), it suffices to show that for every \(r > 0, \mathcal{P}^1_r = \mathcal{P}^2_r\) on \(\mathcal{B}(C([0, \infty), S_q))\).

The proof is basically the same as in the proof of the finite dimensional Yamada-Watanabe result. In our case the finite dimensional diffusions are replaced by the infinite dimensional processes \((Y^{i,r}_t, \eta^{i,r}), i = 1, 2\) with a fixed initial value \(y\). We follow the proof in \[23\] (Chapter IV, Theorem (1.1) and its
corollary), the only difference in our case being that the space $C([0, \infty), \mathbb{R}^d)$ is replaced by the space $C([0, \infty), S_q)$, which is again a Polish space. It suffices to show then that $P^1_r(A) = P^2_r(A), A \in \mathcal{B}(C([0, \infty), S_q))$. Let $Q^i_r(A \times B) := P\{ Y^i_{\wedge \eta} \in A, W \in B \}, A \in \mathcal{B}(C([0, \infty), S_q)), B \in \mathcal{B}(W_n)$. Let $P^0$ be the Wiener measure on $W_n$.

Let $Q^i_r(\omega, A)$ be a disintegration of $Q^i_r$ w.r.t. $P^0$, i.e. for $i = 1, 2$,

$$Q^i_r(A \times B) = \int_B Q^i_r(\omega, A) P^0(d\omega).$$

$A \in \mathcal{B}(C([0, \infty), S_q))$ and $B \in \mathcal{B}(W_n)$. Then as in the proof in [23], Theorem (1.1), using pathwise uniqueness, there exists a measurable map $F_r : W^3 \to C((0, \infty), S_q)$ such that

$$Q^i_{r,y}(\omega, A) = \delta_{F_r(\omega)}(A) \text{ a.e. } \omega \ P^0,$$

for $i = 1, 2$. In particular it follows that

$$P^1_r(A) = Q^1_r(A \times \mathbb{R}^n) = Q^2_r(A \times \mathbb{R}^n) = P^2_r(A).$$

□

Having defined measurable versions of $(Y_t(y), \eta^y)$ we can now define the transition probability function $P(t, y, A)$ for $t \geq 0, y \in \mathcal{S}_p$ and $A \in \mathcal{B}(\mathcal{S}_p)$ in the usual way:

$$P(t, y, A) := P(Y_t(y) \in A) = I_{\mathcal{S}_p}(y)\{P(Y_t(y) \in A, t < \eta^y) + P(Y_t(y) \in A, t \geq \eta^y)\} + I_{\mathcal{S}_p}(y)P(Y_t(y) \in A, t \geq \eta^y).$$

From Proposition (5.4), it follows that for fixed $A \in \mathcal{B}(\mathcal{S}_p)$, the map $(t, y) \to P(t, y, A)$ is jointly measurable and a probability measure for fixed $t \geq 0$ and $y \in \mathcal{S}_p$. For $y \in \mathcal{S}_p$ we will write

$$Y_t(y) = y + Y^0_t(y), \ 0 \leq t < \eta^y(\omega)$$

where

$$Y^0_t(y) = \int_0^t A(Y^0_s + y) \cdot dB_s + \int_0^t L(Y^0_s + y)ds \quad (26)$$

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for $0 \leq t < \eta^y$. We can then formulate the strong Markov property of the process $(Y_t(y))$ as follows.

**Theorem 5.7** Let $T : \Omega \to [0, \infty]$ be an $(\mathcal{F}^Y_t)$ stopping time. Then for each $y \in \mathcal{S}_p$, a.s. on $\{T < \infty\}$,

$$P(Y_{T+t}(y) \in A \mid \mathcal{F}^Y_T) = P(t, Y_T, A)$$

for $t \geq 0, A \in \mathcal{B}(\hat{\mathcal{S}}_p)$.

**Proof:** We first consider the case $y \in \mathcal{S}_p$ and $A \subseteq \mathcal{S}_p$. Let $f : \hat{\mathcal{S}}_p \to \mathbb{R}$ be a bounded measurable function, $f(\partial) = 0$. Then, with $y \in \mathcal{S}_p$ and $\eta := \eta^y$

$$E[f(Y_{T+t}(y)) \mid \mathcal{F}^Y_T] = E[f(Y_{T+t}(y))(I_{T<\eta^y} + I_{T=\eta^y}) \mid \mathcal{F}^Y_T]$$

$$= E[f(Y_{T+t}(y))I_{T<\eta^y} \mid \mathcal{F}^Y_T]$$

$$= E[f(Y_{T+t}(y))I_{T<\eta^y}I_{T+t<\eta^y} \mid \mathcal{F}^Y_T]$$

$$= I_{\{T<\eta^y\}} E[f(Y_{T} + \hat{Y}_t(y))I_{t<\eta^y} \mid \mathcal{F}^Y_T]$$

where

$$\hat{Y}_t(y) := Y_{t+T}(y) - Y_T(y), \ 0 \leq t < \bar{\eta}; \quad := \partial, \ t \geq \bar{\eta}$$

and $\bar{\eta} := \eta - T, \omega \in \{\eta > T\}$ and := $\infty$ otherwise.

We note that on $\{T < \infty\}$

$$\hat{Y}_t(y) = \hat{Y}_t^0(z)|_{z=Y_T(y)}, \ 0 \leq t < \bar{\eta}^z, \ z = Y_T(y),$$

where $(\hat{Y}_t^0(z), \bar{\eta}^z)$ satisfies equation (26)(with respect to $P(\cdot \mid T < \infty)$) with $y$ replaced by $z$ and with $(B_t)$ replaced by the Brownian motion $(\hat{B}_t) := (I_{T<\infty})(B_{T+t} - B_T)$; and $\bar{\eta}^z$ is the explosion time for the process $\hat{Y}_t(z) := z + \hat{Y}_t^0(z)$ which by its maximality, satisfies $\eta \equiv \eta^y = T + \hat{Y}_t^0(y)$ on $\{\eta^y > T\}$. Since the latter Brownian motion is independent of $\mathcal{F}_T$ we have (using Theorem (5.5)),

$$\text{LHS above} = I_{\{T<\eta^y\}} E[f(z + \hat{Y}_t^0(z))I_{t<\eta^y}]|_{z=Y_T(y)}$$

$$= I_{\{T<\eta^y\}} E[f(Y_t(z))I_{t<\eta^y}]|_{z=Y_T(y)} = P_t f(Y_T(y))$$

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on the set \{T < \eta\}, and where we have used the uniqueness in law for equation (26), which follows from the previous theorem.

We now consider the case \(y = \delta\). Then both sides of the equation in the statement of the theorem reduce to \(I_A(\delta)\). Next let \(y \in S_p, A = \{\delta\}\). we have

\[
P(Y_{T+t}(y) \in \{\delta\}|\mathcal{F}_T^Y) = P\{Y_{T+t}(y) = \delta\} \cap (T < \eta) \cup (T \geq \eta)|\mathcal{F}_T^Y\}
\]

\[
= I_{T \geq \eta, Y_T(y) = \delta} + I_{T < \eta} P(t > \hat{\eta})|z = Y_T(y)
\]

where we have used the independence of the Brownian motion \((\hat{B}_t)\) and \(\mathcal{F}_T^Y\) in the second equality. This completes the proof.

\[\square\]

It is clear from the relation \(Y_t = \tau_{Z_t(y)}, 0 \leq t < \eta^p\), with \((Z_t)\) as in equation (23) that the path properties of the processes \((Y_t)\) and \((Z_t)\) are closely related, although they live in different spaces. In particular, as already observed in Proposition (3.12) of [41] corresponding to the case where \(\sigma, b\) are given by linear functionals on \(S_p\), the explosions of \((Z_t)\) as \(t \to \infty\) are related to the convergence of \(Y_t\) to zero in the weak topology of \(S'\) and this correspondence is pathwise. It is easy to see that the result of Proposition (3.12) of [41] extends to the more general framework of Theorem (5.3) above. We then have the following result.

**Proposition 5.8** Let \(\sigma_{i,j}, b_i\) be as in Theorem 4.3, with \(Y \equiv y \neq 0 \in S_p\). Let \((Y_t(y), \eta^p)\) be the unique maximal solution of equation (19), and \((Z_t(y))\) be given by equation (23) with \(Y_t(y) = \tau_{Z_t(y)}(y), 0 \leq t < \eta^p\). Fix \(\omega \in \Omega\). Then, \(Z_t(\omega, y) \to \infty\) as \(t \to \eta^p(\omega)\) whenever \(Y_t(\omega, y) \to 0\) weakly in \(S'\) as \(t \to \eta^p(\omega)\). Conversely, suppose one of the following two conditions is satisfied viz.

1. \(y \in L^p(\mathbb{R}^d), p \geq 1\).

2. \(y\) has compact support.

Then, \(Y_t(\omega, y) \to 0\) weakly in \(S'\) whenever \(Z_t(\omega, y) \to \infty\) as \(t \to \eta^p(\omega)\).
Proof: The proof is the same as in Proposition (3.12) of [41]. The proof for the case \( y \in L^p, p \geq 1, p \neq 2 \) is also the same as the case \( p = 2 \) with some obvious changes. \( \square \)

Remark 5.9 Note that when \( \eta^y < \infty \) and \( Z_t(\omega, y) \to \infty \) then by the above Proposition, \( Y_t(\omega, y) \to 0 \) weakly in \( S' \) as \( t \to \eta^y \) while by Proposition (5.1), \( \|Y_t(y)\|_p \to \infty \).

6 Some Examples

Our main existence and uniqueness result viz. Theorem (5.3), applies to a number of different situations. In this section we give some examples of these applications. In what follows we use the fact that if \( p > \frac{d}{4} + 1 \) then \( \delta_x \in S_{-p} \) (see [40], Theorem (4.1)) and for such \( p \) we also note that \( \phi \in S_{-p} \) is a continuous function.

Example 1 Let \( p > \frac{d}{4} + 1 \). Then note that \( S_p \subseteq C^2(\mathbb{R}^d) \), the space of two times continuously differentiable functions on \( \mathbb{R}^d \) (see Theorem 4.1, [40]). Let \((Y_t)_{0 \leq t < \eta}\) be the unique \( S_p \)-valued strong solution of equation (18) with initial condition \( y \in S_p \), given by Theorem (5.3). Then almost surely, for \( t < \eta \), it is given by a \( C^2(\mathbb{R}^d) \) function, say, \( x \to Y_t(\omega, x) \) and we also have

\[
\partial^\alpha Y_t(\omega, x) = \langle \delta_x, \partial^\alpha Y_t(\omega) \rangle = (-1)^{|\alpha|} \langle \partial^\alpha \delta_x, Y_t(\omega) \rangle
\]

for \(|\alpha| \leq 2\). In particular acting on both sides of (19) by \( \delta_x \in S_{-p} \) we get for each \( t < \eta, x \in \mathbb{R}^d \), almost surely,

\[
Y_t(\omega, x) := \langle \delta_x, Y_t(\omega) \rangle = y(x) + \int_0^t L(Y_s)(\omega, x) ds
\]

\[
+ \int_0^t A(Y_s)(\omega, x) \cdot dB_s,
\]

where the integrands in the RHS of the above equation are well defined processes for each \( x \) and the stochastic integrals are well defined. Since
\[ Y_t = \tau_{Z_t}(y), \ t < \eta \text{ with } (Z_t) \text{ as in (23), in particular } Y_t(x) = y(x - Z_t), x \in \mathbb{R}^d. \]

Since \( y \in S_p, p > \frac{d}{4} + 1 \), it is in \( C^2(\mathbb{R}^d) \) and the Itô formula applied to \( y(x - Z_t) \) also yields the RHS of (27). Thus in this case, \( (Y_t)_{t \geq 0} \equiv \{Y_t(x) : t \geq 0, x \in \mathbb{R}^d\} \) gives the unique classical solution of the SPDE (18) when \( p > \frac{d}{4} + 1 \).

We also note that the Fourier transform \( f \in S_p \rightarrow \hat{f} \in S_p \) is a unitary map on the complexified Hermite-Sobolev spaces \( S_p(\mathbb{C}) \) (see [47]). Hence we get from the above that the Fourier transform \( \hat{Y}_t \) of \( Y_t \) is given as \( \hat{Y}_t = \hat{y} e^{i(x, Z_t)} \), where \( \langle \cdot, \cdot \rangle \) is the inner product in \( \mathbb{R}^d \) and the RHS represents the product of the tempered distribution \( \hat{y} \), the Fourier transform of \( y \), with the bounded \( C^\infty \) function \( x \rightarrow e^{i(x, Z_t)} \). Note that for each \( x \in \mathbb{R}^d \), \( \hat{Y}_t(x) \) is a process and it is easily seen that it satisfies a linear SDE obtained by the Fourier transform of equation (19).

**Example 2** The connection between solutions of equation (19) and the solutions of the finite dimensional SDE (1) was shown in [41]. Let \( (Z_t) \) be as in equation (23). Then it follows as in [41] that the process \( (X^x_t) \) defined by

\[
\begin{align*}
&dX_t = \tilde{\sigma}(X_t) \cdot dB_t + \bar{b}(X_t)dt \\
&X_0 = x
\end{align*}
\]

where for \( z \in \mathbb{R}^d, \tilde{\sigma}_{ij}(z) := \sigma_{ij}(\tau_z(y)), \quad \bar{b}_i(z) := b_i(\tau_z(y)), \quad j = 1, \ldots, n, i = 1, \ldots, d \) and \( y \in S_p \) acts as a fixed parameter. Special cases arise when \( \sigma_{ij}, b_i : S_p \rightarrow \mathbb{R} \) are continuous linear functionals on \( S_p \) i.e. they are given by elements in \( S_{-p} \) and consequently

\[
\tilde{\sigma}_{ij}(z) := \langle \sigma_{ij}, \tau_z y \rangle, \quad \bar{b}_i(z) := \langle b_i, \tau_z y \rangle
\]

where \( \langle \cdot, \cdot \rangle \) denotes duality between \( S_{-p} \) and \( S_p \). Note that when \( \sigma_{ij}, b_i, y \) are functions in \( L^2(\mathbb{R}^d) \), then

\[
\tilde{\sigma}_{ij}(z) = \int \sigma_{ij}(w + z)y(w)dw = \sigma_{ij} \ast \hat{y}(z),
\]

where \( \ast \) denotes convolution and \( \hat{y}(z) := y(-z) \) and similarly \( \bar{b}_i(z) = b_i \ast \hat{y}(z) \).

When \( p < -\frac{d}{4} \), then we can take \( y = \delta_0 \in S_p, \sigma_{ij}, b_i \in S_{-p} \subset C(\mathbb{R}^d) \), the space of real valued continuous functions on \( \mathbb{R}^d \), and \( \tilde{\sigma}_{ij}(z) = \sigma_{ij}(z), \quad \bar{b}_i(z) = b_i(z) \).
Remark 6.1 The weak existence of solutions to the Itô SDE (28) can be combined with the pathwise uniqueness of solutions to equation (18) when $\sigma, b$ are in $S_p$ to yield pathwise unique solutions of (28). The weak existence is obtained whenever the coefficients $\bar{\sigma}_{ij}, \bar{b}_i$ in (28) are bounded and continuous.

On the other hand any two solutions of (28) with the same Brownian motion gives rise via Lemma (4.2), to corresponding solutions of (18) forcing the former solutions to be the same (see Theorem (3.3) of [42]).

We can vary the construction in the above example to get strong solutions in the case of Lipschitz continuous functions. We do this in the following one dimensional example. The general finite dimensional case can be handled by considering finitely many equations like (19).

Example 3 Let $d = 1, p > \frac{1}{4}$ and $\sigma_j = \sigma_1 j, b = b_1 : \mathbb{R} \to \mathbb{R}, i = 1, j = 1, \cdots, n$ be Lipschitz functions. For $x \in \mathbb{R}$ define $\hat{b}(x, \cdot), \hat{\sigma}_j(x, \cdot) : S_p \to \mathbb{R}$ by $\hat{\sigma}_j(x, y) := \sigma_j(y(x))$ and $\hat{b}(x, y) := b(y(x))$. Note that under the assumptions on $p$, the elements of $S_p$ are continuous functions. Then we note that for $y_1, y_2 \in S_p$,

$$|\hat{\sigma}_j(x, y_1) - \hat{\sigma}_j(x, y_2)| \leq K \|y_1 - y_2\|_p$$

with a similar inequality for $b$ and where the constant $K$ depends on $x$. Let $L$ and $A$ be the operators as in equations (3) and (2) with $\sigma_j, b$ replaced by $\hat{\sigma}_j(x, \cdot)$ and $\hat{b}(x, \cdot)$. Then for any fixed initial value $y_0 \in S_p$, equation (18) has a unique $S_p$ valued strong solution which we denote by $(Y_t(x, y_0))$. We then have $Y_t(x, y_0) := \tau_{Z_t(x, y_0)}(y_0)$ where $(Z_t(x, y_0))$ is given by (23) with $\sigma$ and $b$ there replaced with $\hat{\sigma}_j(x, \cdot), \hat{b}(x, \cdot)$ respectively and $y_0 \in S_p$, as defined above. Then it follows as in Example 2 that $(Z_t(x, y_0))$ solves the ordinary SDE

$$dZ_t = \sigma(y_0(x - Z_t)) \cdot dB_t + b(y_0(x - Z_t))dt$$

Let $\bar{\sigma}_j(z) := \hat{\sigma}_j(x, \tau_z(y_0)) = \sigma_j(\tau_z(y_0)(x)) = \sigma(y_0(x - z))$ and a similar expression for $\bar{b}(z)$. Then $X_t(x) := x - Z_t(x, y_0)$ solves

$$dX_t = \bar{\sigma}(X_t) \cdot dB_t + \bar{b}(X_t)dt$$

$X_0 = x.$

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Moreover, in a manner similar to the case of uniqueness discussed in the Remark (5.1) above, the uniqueness of solutions of (18) implies that the solution of (30) is unique: Any two \( (B_t) \) adapted solutions of equation (30) will give rise to two solutions of equation (29), which in turn (via Lemma (4.2)) gives rise to two solutions of (18). The same arguments also imply that there is local uniqueness in equation (30), up to a stopping time i.e. local uniqueness up to a stopping time in equation (18) implies local uniqueness in equation (30) up to a stopping time. If now we consider a sequence of elements \( y_{0}^{k} \in S_{p}, k \geq 1 \), satisfying

\[
y_{0}^{k}(z) = z, \quad |z - x| \leq k
\]

then a localisation argument implies that the corresponding solutions \( (X_{t}^{k}(x)) \) satisfies \( X_{t}^{k}(x) = X_{t}^{k+1}(x), t \leq \tau^{k} \) where \( \tau^{k} \) is the exit time of \( (X_{t}^{k}(x)) \) from the ball \( \{z : |z - x| \leq k\} \). One can then patch up the solutions \( (X_{t}^{k}(x)), k \geq 1 \) to obtain the solution of the equation

\[
\begin{align*}
dX_{t} & = \sigma(X_{t}) \cdot dW_{t} + b(X_{t})dt \\
X_{0} & = x,
\end{align*}
\]

(31)

when the coefficients \( \sigma_{j}, b_{i}, j = 1, \ldots, n \), are given Lipschitz continuous functions.

**Example 4** In this example we consider martingale problems in the sense of Stroock and Varadhan, associated with a second order differential operator \( \bar{L} \) with coefficients \( \bar{\sigma}_{ij} \) and \( \bar{b}_{i}, i, j = 1, \ldots, d \) which are bounded and continuous functions on \( \mathbb{R}^{d} \). If in addition they belong to \( S_{p}, p > \frac{d}{4} + 1 \) we can solve the SDE (18) with \( \sigma_{ij} \) and \( b_{i} \) given by the linear functionals on \( S_{-p} \) corresponding to \( \bar{\sigma}_{ij} \) and \( \bar{b}_{i}, i, j = 1, \ldots, d \) and initial condition \( \delta_{x} \in S_{-p} \). In this situation we have indeed a unique strong solution to the Ito SDE (1). In case we know only that \( \bar{\sigma}_{ij} \) and \( \bar{b}_{i}, i, j = 1, \ldots, d \) are bounded and continuous, then since they are tempered distributions, there exists \( p > 0 \) such that they belong to \( S_{-p} \). In this case, we still have strong solutions of (18) with \( \eta = \infty \) (Proposition (5.2)) for initial conditions \( y \in S_{p}, p > 0 \) and of course \( \delta_{x} \notin S_{p} \). Below we show that when \( y_{n} \to \delta_{x} \) weakly in \( S' \) and \( Y_{t}(y_{n}) = \tau_{Z^{a}(t)}(y_{n}) \) are the solutions of (18), then the laws \( \{P_{x}^{n}\} \) of the processes \( (x + Z^{a}(t)) \), converge weakly to \( P_{x} \), the solution of the martingale problem for \( \bar{L} \) starting at \( x \), provided the latter is well posed.
We have $z \in \mathbb{R}^d$,

$$L \varphi(z) = \frac{1}{2} \sum_{i,j=1}^{d} (\bar{\sigma} \bar{\sigma} t)_{ij}(z) \partial_{ij}^2 \varphi(z) + \sum_{i=1}^{d} \bar{b}_i(z) \partial_i \varphi(z).$$

(32)

On the other hand consider the SPDE equation (18) with coefficients $\sigma_{ij}, b_i : \mathbb{R}_+ \to \mathbb{R}$ given by $\sigma_{ij}(\phi) = \langle \bar{\sigma}_{ij}, \phi \rangle$ and a similar expression for $\bar{b}_i(\phi), \phi \in \mathbb{R}_+$. Let $y_n \in \mathbb{R}_+ \cap C(\mathbb{R}^d), y_n \to \delta_x$ weakly for a fixed $x \in \mathbb{R}^d$. Let $(Y^n_t), Y^n_t := Y_t(y_n)$ denote the unique $\mathbb{R}$ valued solution to (18) with initial condition $Y^n_0 = y_n$. Let $(Y^n_t)_{t \geq 0}$ be the law of $(Z^n_t)$ on $C([0, \infty), \mathbb{R}^d)$ and $Z_t(\omega) := \omega(t)$ the coordinate process. For $\varphi \in \mathcal{S}$ let

$$\bar{L}^n \varphi(z) := \frac{1}{2} \sum_{i,j} \langle \bar{\sigma}^n, \bar{\sigma}^n \rangle_{ij}(z) \partial_{ij}^2 \varphi(z) + \sum_i \bar{b}^n_i(z) \partial_i \varphi(z).$$

Let $s < t$ and $G$ be a bounded, continuous and $\mathcal{F}_s$-measurable function depending on finitely many coordinates. For $f \in \mathcal{S}$, we have by Itô’s formula

$$E^{P^n} \left[ f(Z_t) - f(Z_s) - \int_s^t \bar{L}^n f(Z_s)ds \right] G = 0.$$

Suppose now that $P^n \to P$ weakly on $C([0, \infty), \mathbb{R}^d)$. Let $L_x$ be the operator in (32) wherein $\bar{\sigma}_{ij}(z), \bar{b}_i(z)$ are replaced with $\bar{\sigma}_{ij}(x + z), \bar{b}_i(x + z), x \in \mathbb{R}^d$ fixed. We then have :

$$E^P \left[ f(Z_t) - f(Z_s) - \int_s^t L_x f(Z_s)ds \right] G = 0.$$

(34)
To see this, first note that the integrand is a bounded continuous function on $C([0, \infty), \mathbb{R}^d)$. Further, as $n \to \infty$, we have $\bar{\sigma}^n_{ij}(z) \to \bar{\sigma}_{ij}(x + z), \bar{b}^n_{i}(z) \to \bar{b}_{i}(x + z)$. Moreover,

$$\sup_{1 \leq i,j \leq d} \sup_n \sup_z |\bar{\sigma}^n_{ij}(z) + \bar{b}^n_{i}(z)| < \infty. \quad (35)$$

Our claim now follows by using the Skorokhod mapping theorem and the bounded convergence theorem. In particular, it follows that any weak limit $P$ of the sequence $\{P^n\}$ solves the martingale problem for $\bar{L}_x$ starting at zero. We then have the following theorem.

**Theorem 6.2** Suppose the martingale problem for $\bar{L}$ starting at $x$ has a unique solution $P_x$. Let $y_n \in S_{-p} \cap C(\mathbb{R}^d), y_n \to \delta_x$ weakly. Let $(Z^n_t)$ be as above and let $P^n_x$ be the law of $(x + Z^n_t)$. Then $P^n_x \to P_x$ weakly.

**Proof:** Replacing $f$ by $\tau_{-x}f$ we see that if $P$ is any weak limit of the family $\{P^n, n \geq 1\}$ where $P_n$ is the law of $(Z^n_t)$, then under $P, X^n_t := x + Z^n_t$ solves the martingale problem for $\bar{L}$ starting at $x$ and hence the law of $(X^n_t)$ must be $P_x$. The tightness of the laws $\{P^n, n \geq 1\}$ viz. for every $\epsilon > 0, T > 0$

$$\lim_{\delta \to 0} \sup_n P^n \left\{ \sup_{0 \leq s,t \leq T} \left| Z^n_t - Z^n_s \right| > \epsilon \right\} = 0$$

and hence the tightness of $\{P^n_x\}$, follows easily from Doob’s maximal inequality, the Burkholder-Davis-Gundy inequalities and the uniform bounds in (35).

**Example 5** In this example we consider the non-linear evolution equation

$$\partial_t Y_t = L(Y_t) \quad (36)$$

$$Y_0 = y.$$

Here $y \in S_p$ for some $p \in \mathbb{R}$ and $L : S_p \to S_{p-1}$ is given by equation (3). By a solution we mean a pair $(Y_t, \eta)$ where $\eta > 0$ and $(Y_t)$ is a continuous function $t \to Y_t : [0, \eta) \to S_p$ satisfying the following equation in $S_{p-1}$

$$Y_t = y + \int_0^t L(Y_s)ds \quad (37)$$
for $0 \leq t < \eta$. Suppose $(Y_t)_{0 \leq t < \eta}$ is an $S_p$ valued solution. Define the time dependent, linear operators $\bar{A}_t, \bar{L}_t : S_p \to S_{p-1}$ as follows:

$$
\bar{L}_t(\varphi) = \frac{1}{2} \sum_{i,j=1}^{d} (\sigma \sigma^t)_{ij}(Y_t) \partial_{ij} \varphi - \sum_{i=1}^{d} b_i(Y_t) \partial_i \varphi \\
\bar{A}_t(\varphi)(h) = - \sum_{j=1}^{n} h_j \sum_{i=1}^{d} \sigma_{ij}(Y_t) \partial_i \varphi,
$$

where $h = (h_1, \cdots, h_n)$. Note that the coefficients are now deterministic but time dependent. Define the $\mathbb{R}^d$-valued process $(Z_t)_{0 \leq t < \eta}$ by

$$
Z_t = \int_{0}^{t} \sigma(Y_s) \cdot dB_s + \int_{0}^{t} b(Y_s) ds
$$

for $0 \leq t < \eta$. Since the integrands are deterministic $(Z_t)$ is a Gaussian process. Let $\tilde{Y}_t := \tau_{Z_t}(y), 0 \leq t < \eta$. Then $(\tilde{Y}_t)_{0 \leq t < \eta}$ is the unique $S_p$-valued solution of the equation

$$
d\tilde{Y}_t = \bar{L}_t(\tilde{Y}_t) dt + \bar{A}_t(\tilde{Y}_t) \cdot dB_t \\
\tilde{Y}_0 = y.
$$

Let $\varphi(t) := E\tilde{Y}_t, 0 \leq t < \eta$ where we note that $E\|\tilde{Y}_t\|_p < \infty$. Then $\varphi(t)$ satisfies the linear evolution equation

$$
\partial_t \varphi(t) = \bar{L}_t \varphi(t) \quad (38) \\
\varphi(0) = y.
$$

in the interval $0 \leq t < \eta$. Since $\bar{L}_t$ has constant (in space) coefficients it satisfies the monotonicity inequality and hence equation (38) has a unique $S_p$-valued solution. Hence we have the following stochastic representation of solutions of equation (36).

**Theorem 6.3** Let $p \in \mathbb{R}, y \in S_p$, and let $\sigma_{ij}, b_i : S_p \to \mathbb{R}$ be bounded and measurable. Let $(Y_t)_{0 \leq t < \eta}$ be an $S_p$-valued solution of equation (36). Then we have

$$
Y_t = E\tau_{Z_t}(y) = y \ast p_{Z_t} \quad (39)
$$

where $p_{Z_t}$ is the density of $Z_t$ and $\ast$-denotes convolution.
Example 6 The previous example maybe generalised. Consider the following equation viz.

$$Y_t = y + \int_0^t L(y, Y_s) ds$$

(40)

of which (37) becomes a special case when there is no dependence on $y$ in the operator $L$. However we will make a departure from the $L$ in (37) by requiring $L$ to act on $y$ as a partial differential operator with the coefficients $\sigma_{ij}, b_i$ depending on $Y_s$ in the right hand side above. In other words,

$$L(y_1, y_2) := \frac{1}{2} \sum_{i,j=1}^d (\sigma \sigma^t)_{ij}(y_1, y_2) \partial^2_{ij} y_1 - \sum_{i=1}^d b_i(y_1, y_2) \partial_i y_1$$

where $y_1, y_2 \in \mathcal{S}_{-p}, p \in \mathbb{R}$. If $\mu(dz)$ is a probability measure, then $\mu \in \mathcal{S}_{-p}, p > \frac{d}{4}$ and we can define the non-linear convolution $L(\cdot, y_2) \circ \mu)(y_1)$ ([III], Section 5, where the notation in definition (5.1) is slightly different and the coefficients $\sigma_{ij}$ and $b_i$ do not depend on $y_2$) as

$$L(\cdot, y_2) \circ \mu)(y_1) := \int_{\mathbb{R}^d} L(\tau_z y_1, y_2) \mu(dz)$$

whenever the integral exists as a Bochner integral in $\mathcal{S}_{-p}$. An interesting situation arises when the measure $\mu$ arises as the marginals of a stochastic process $(Z_t)$. Let $\{\mu_s(dz), s \geq 0\}$ be the corresponding family of probability measures. Consider the case when $(Z_t)$ satisfies the equation

$$Z_t := \int_0^t \sigma(\tau_s y, y \circ \mu_s) \cdot dB_s + \int_0^t b(\tau_s y, y \circ \mu_s) ds,$$

(41)

where $\mu_t(dz) := P(Z_t \in dz)$ is the law of $Z_t$ and $y \in \mathcal{S}_{-p}$. Then applying Itô’s formula and taking expectations we get that $Y_t := E_{\tau Z_t} y = y \circ \mu_t =: \psi(t, y)$ satisfies the non linear evolution equation

$$\partial_t \psi(t, y) = \psi(t, L(y, \psi(t, y))), \quad t \geq 0, \quad \psi(0, y) = y.$$

(42)

with $\psi(t, L(y, y_2)) := L(\cdot, y_2) \circ \mu_t(y)$ and $y_2 = \psi(t, y)$. Equation (37) becomes a special case of (42) when $\sigma_{ij}(y_1, y_2), b_i(y_1, y_2)$ are independent of $y_1$. When we consider $y = \delta_x$ where $x \in \mathbb{R}^d$ is fixed then $X_t := x + Z_t$ and we get the Mckean-Vlasov equation from (41).
Example 7 Let \( p > \frac{d}{4} \). We now consider the Feynman-Kac formula for the solution of the equation

\[
\partial_t u(t, x) = \bar{L} u(t, x) + V(x) u(t, x) \\
u(0, x) = f(x).
\]

where

\[
\bar{L} \phi(x) := \frac{1}{2} \sum_{i,j=1}^{d} (\bar{\sigma}\bar{\sigma})_{ij}(x) \partial_{ij}^2 \phi(x) + \sum_{j=1}^{d} \bar{b}_i(x) \partial_i \phi(x), \quad \phi \in \mathcal{S}.
\]

Here we assume \( f, V, \bar{\sigma}_{ij}, \bar{b}_i \) are given functions in \( \mathcal{S}_p \). Then we define \( \bar{L} \) as in equation (3) with coefficients \( \sigma_{ij}(\cdot) \) and \( b_i(\cdot) \) given via the duality between \( \mathcal{S}_p \) and \( \mathcal{S}_{-p} \) as \( \sigma_{ij}(y) = \langle y, \bar{\sigma}_{ij} \rangle \), \( b_i(y) = \langle y, \bar{b}_i \rangle \) where \( \bar{\sigma}_{ij} \) and \( \bar{b}_i \) are as above and \( y \in \mathcal{S}_{-p} \).

Denoting by \( (X_t^x) \) the diffusion corresponding to \( \bar{L} \) and by \( (Y_t(y)) \), \( y = \delta_x \) the corresponding lift on \( \mathcal{S}_p \) satisfying equation (18), it is easy to see that the solution \( u(t, x) \) arises from a transformation on path space \( C([0, \infty), \mathcal{S}_{-p}) \) viz.

\[
Y_t(y) \rightarrow \hat{Y}_t(y) := Y_t(y) e^{-\int_0^t c(s,Y)ds}
\]

where \( c(s, y) := -\langle y_s, V \rangle, \quad y \in C([0, \infty), \mathcal{S}_{-p}) \). In particular, since \( Y_t(\delta_x) = \delta_{X_t^x}, \ c(s, Y) = V(X_t^x) \). For ease of calculations, we assume \( \eta = \infty, a.s. \). Next, since \( u(t, x) := P_t^V f(x) \) where \( (P_t^V) \) is the Feynman-Kac semi-group, we have

\[
u(t, x) = E(e^{\int_0^t V(X_t^x)ds} f(X_t^x)) = E(e^{\int_0^t V(X_t^x)ds} \delta_{X_t^x}, f)
\]

\[
= E(e^{\int_0^t c(s,Y)ds} Y_t(f), f) = E(\hat{Y}_t(y), f)
\]

We can show that the process \( \hat{Y}_t \) satisfies an SPDE with time dependent coefficients \( \hat{L}(s, y), \hat{A}_i(s, y), i = 1, \cdots, r, s \geq 0, y \in C([0, \infty), \mathcal{S}_{-p}) \) given in the form.
\begin{align*}
\hat{L}(s, y) &:= \frac{1}{2} \sum_{i,j=1}^{d} (\hat{\sigma}^t)_{ij}(s, y) \partial_{ij}^2 y - \sum_{j=1}^{d} \hat{b}(s, y) \partial_{i} y - \hat{c}(s, y) y_s \\
\hat{A}_i(s, y) &:= -\sum_{j=1}^{d} \hat{\sigma}_{ji}(s, y) \partial_{j} y_s.
\end{align*}

Here the coefficients \(\hat{\sigma}_{ij}, \hat{b}_i, \hat{c}\) are induced on \([0, \infty) \times C([0, \infty), S_{-p})\) by the coefficients \(\sigma_{ij}, b_i\) of \(L, A_i\) appearing in \(L\) and the transformation \(y \to y^c\) given by the unique solution of the equation

\[y^c_t = y_t e^{-\int_0^t c(s, y^c_s) ds}\]

on the path space \(C([0, \infty), S_{-p})\) and satisfying \(\hat{\sigma}_{ij}(t, y) = \sigma_{ij}(y^c_t), \hat{b}_i(t, y) = b_i(y^c_t), \hat{c}(t, y) = c(t, y^c)\).

It is then easy to see using integration by parts and the fact that \((Y_t)\) solves (18), that, \(\hat{Y}_t\) satisfies the SPDE

\begin{equation}
\begin{aligned}
&d\hat{Y}_t = \hat{L}(t, \hat{Y}) dt + \hat{A}(t, \hat{Y}) \cdot dB_t \\
&\hat{Y}_0 = y.
\end{aligned}
\end{equation}

The uniqueness of solutions of the above SPDE can be proved using the uniqueness of the solutions of the equation \(y^c_t = y_t e^{-\int_0^t c(s, y^c_s) ds}\) and the ‘invertibility’ of the map \(y \to y^c\). The details can be seen in a forthcoming paper.

### 7 Conclusion

Translation invariance also appears to be a reflection of a possibly more basic, ‘duality’ relation between the finite dimensional SDE and the corresponding SPDE. Let \(p > 0, f \in S_p, y \in S_{-p}\). Let \(\sigma_{ij}, b_i, (Y_t(y), \eta)\) be as in
Theorem (5.3), with $Y \equiv y$. We consider the case $\eta = \infty$. Let $(Z_t(y))$ be as in equation (23). We observe the following duality relation between $Y_t(y)$ and $Z_t(y)$ viz.

$$E(y, \tau_{-Z_t(y)}f) = E(\tau_{Z_t(y)}y, f) = E(Y_t(y), f)$$

whenever the relevant expectations are finite.

Finally we note that in the model we have introduced in this paper, it becomes meaningful to talk about diffusions with coefficients $\sigma$ and $b$, in the state $y$, for any tempered distribution $y$. The 'state $y$' becomes an initial state for the SPDE, but in the context of the SDE, allows for representation of more complex initial states than just $y = \delta_x$. The distribution $y$ is more intuitively, thought of as an initial distribution of the mass of the solvent particles in the diffusion model. An interpretation of 'translation invariance' in the case of non interacting particles could be that it is linked by 'symmetry principles' to conservation of the mass of the particles.

Thus we may interpret the parameter $y \in S'$ in the process $(X_t^{x,y})$ by saying that the diffusion with parameters $\bar{\sigma}, \bar{b}$ and starting at $x$ is in the state $y$ or that the diffusion with parameters $\bar{\sigma}, \bar{b}$ is in the state $(x, y)$. This of course, corresponds to the process $(Y_t(\tau_x y))$ being in the initial state $\tau_x y$. When we consider questions such as ergodicity and existence of an invariant measure, we replace the (initial) deterministic state $x$ by a random state with a distribution $\mu$. In the context of our results this raises the question of whether the existence of an invariant measure and questions of ergodicity can be answered by randomising both $x$ and $y$. We refer to [4], Chapter (5), for some results in this direction.

8 Appendix

We present the proofs of Proposition (5.4) and Theorem (5.5).

**Proof of Proposition (5.4)**: Given $r > 0$ we first construct a pair $(\hat{Y}^r(t, \omega, y), \hat{\eta}^r(\omega, y))$ jointly measurable in $(t, \omega, y)$ and $(\omega, y)$ respectively such that for each $(t, y)$,

$$\hat{Y}^r(t, \omega, y) = Y_t^\tau(\omega, y) \text{ a.s. on the set } \{t < \eta^{r,y}\}.$$
where for each $y \in S_p$, $(Y^r_t(y), \eta^r_{t,y})$ is the solution of equation (19) constructed in Theorem (4.3) with $Y \equiv y$. In the construction below we drop the superscript $r$ until further notice. Recall from Section 2, that $\{h_{n,p}; n \in \mathbb{Z}_+\}$ is the ONB in the Hilbert space $S_p$. Since for each $y \in S_p$,

$$Y_t(\omega, y) = y + \sum_{|n|=0}^{\infty} \langle Y_t(\omega, y), h_{n,p} \rangle_p h_{n,p},$$

where $\sum_{|n|=0}^{\infty} \langle Y(t, \omega, y), h_{n,p} \rangle_p^2 < \infty$ for all $t \geq 0$ and $\omega \in \Omega$, it suffices to show the existence of the map $\tilde{\eta}(\omega, y)$ and for each $n \geq 1$, a $\mathcal{B}[0, \infty) \otimes \mathcal{F}_\infty \otimes \mathcal{B}(S_p)/\mathcal{B}(\mathbb{R})$ measurable map $\tilde{Y}^n(t, \omega, y)$ satisfying

$$\sum_{|n|=0}^{\infty} (\tilde{Y}^n(t, \omega, y))^2 < \infty$$

for all $(t, \omega, y)$, and satisfying, for each $t \geq 0, y \in S_p$, $\tilde{\eta}(\omega, y) = \eta^y(\omega)$, and $\tilde{Y}^n(t, \omega, y) = \langle Y_t(y), h_{n,p} \rangle_p$ almost surely on the set $\{t < \eta^y\}$. One can then define $\tilde{Y}(t, \omega, y)$ by

$$\tilde{Y}(t, \omega, y) := y + \sum_{|n|=0}^{\infty} \tilde{Y}^n(t, \omega, y)h_{n,p} \quad t < \tilde{\eta}(\omega, y) \quad \text{and} \quad \tilde{Y}(t, \omega, y) := \delta \quad t \geq \tilde{\eta}(\omega, y).$$

Recall the process $(Y^{k}_t(y))$ satisfying equation (20) (which we now denote by $(Y^{k}_t(y))$ to make the dependence on $y$ explicit), constructed for each $k \geq 1, y \in S_p$, in the proof of Theorem (4.3), satisfying for each $t \geq 0, y \in S_p$,

$$E\|Y_t(y) - Y^{k}_t(\omega, y)\|_p^2 \to 0$$

as $k \to \infty$; where $\eta^y := \lim_{k \to \infty} \eta^{k,y}$, as in the proof of Theorem (4.3). It is easy to see that there exists jointly measurable maps $(t, \omega, y) \to \tilde{Y}^k(t, \omega, y)$ and $(\omega, y) \to \tilde{\eta}(\omega, y)$ satisfying, for each $t \geq 0, \tilde{Y}^k(t, \omega, y) = Y^k_t(y)$ and $\tilde{\eta}(\omega, y) = \eta^y(\omega)$ almost surely. For the first map we define $\tilde{Y}^k(t, \omega, y) := \tau_{Z^k(t, \omega, y)}(y)$ where the $\mathbb{R}^d$ valued process $(Z^k(t, \omega, y))$ is a jointly measurable version which is indistinguishable for each $y$ from the process $(Z^k_t(y))$ defined in terms of
Fix $t \geq 0, y \in \mathcal{S}_p$. Since $E\|Y_t(y) - Y^k_{t \wedge \tau}(y)\|_q^2 \to 0, q \leq p - 1$, there exists a subsequence $\{n_k\}$ such that $Y^k_{t \wedge \tau}(y) \to Y_t(y)$ almost surely in $\mathcal{S}_q$. In particular, for all $n = (n_1, \ldots, n_d)$ and for almost all $\omega$,

$$\langle Y^k_{t \wedge \tau}(y), h_{n,q} \rangle_q \to \langle Y_t(y), h_{n,q} \rangle_q.$$

We now construct a set $G$ in the product $(t, \omega, y)$-space using the subsequence $\{n_k\}$ above as follows. Let $G := \cap_n G_n \cap G_0$ where the intersection is over all $n = (n_1, \ldots, n_d), n_i \in \mathbb{Z}_+$ and where the sets $G_0, G_n$ are defined as $G_0 := \{(t, \omega, y) : \lim_{k \to \infty} \|\tilde{Y}^k_{t \wedge \tilde{\tau}}(t \wedge \tau, \omega, y)\|_q < \infty\}$ and $G_n := \{(t, \omega, y) : \lim_{k \to \infty} \langle \tilde{Y}^k_{t \wedge \tilde{\tau}}, h_{n,q} \rangle_q \text{ exists}\}$. Fix $n = (n_1, \ldots, n_d)$. Define

$$\overline{Y}^n(t, \omega, y) := \lim_{k \to \infty} \langle \tilde{Y}^k_{t \wedge \tilde{\tau}}(t \wedge \tau, \omega, y), h_{n,q} \rangle_q \quad (t, \omega, y) \in G$$

$$:= 0 \quad \text{otherwise.}$$

Then from the joint measurability of $G, \tilde{\tau}$, and $\tilde{Y}^k_{t \wedge \tau}(t, \omega, y)$ we get that the map $(t, \omega, y) \to \overline{Y}^n(t, \omega, y)$ is jointly measurable. If $(t, \omega, y) \in G$, then

$$\sum_{|n|=0}^{\infty} (\overline{Y}^n(t, \omega, y))^2 \leq \lim_{k \to \infty} \sum_{|n|=0}^{\infty} (\tilde{Y}^k_{t \wedge \tilde{\tau}}(t \wedge \tau, \omega, y), h_{n,q})^2$$

$$\leq \lim_{k \to \infty} \|\tilde{Y}^k_{t \wedge \tilde{\tau}}(t \wedge \tau, \omega, y)\|_q^2 < \infty.$$
Since for fixed $t \geq 0, y \in S_p$ we have $\tilde{Y}^n(t \wedge \eta, \omega, y) = Y_n(t \wedge \eta, \omega, y)$ almost surely, it follows from the preceding definitions that

$$\tilde{Y}^n(t, \omega, y) = \langle Y_t(\omega, y), h_n, q \rangle_q$$

almost surely on $\{t < \eta \}$. We can now define

$$\tilde{Y}^n(t, \omega, y) := (2|n| + d)^{p-1} \tilde{Y}^n(t, \omega, y), \ n \in \mathbb{Z}_+^d.$$ 

Note that this is not the same as $\tilde{Y}^k(t, \omega, y)$ defined earlier in this proof, which were approximations to $Y_t(\omega, y)$. Then $\tilde{Y}^n(t, \omega, y) = \langle Y_t(\omega, y), h_n, p \rangle_p$ on $\{t < \eta \}$ almost surely and since $q \leq p-1$,

$$\sum_{|n|=0}^{\infty} (\tilde{Y}^n(t, \omega, y))^2 \leq \sum_{|n|=0}^{\infty} (\tilde{Y}^n(t, \omega, y))^2 < \infty$$

for every $(t, \omega, y)$. Then, as mentioned above, we construct the map $(t, \omega, y) \rightarrow \tilde{Y}(t, \omega, y)$ using $\tilde{Y}^n(t, \omega, y)$ as its $n$-th Fourier-Hermite coefficient, $n \in \mathbb{Z}_+^d$.

Since the maps $\tilde{Y}, \tilde{\eta}$ constructed above depend on $r > 0$, we now make the dependence explicit and patch up the maps $\tilde{Y}^r(\omega, y, \omega) = \tilde{\eta}^r(\omega, y)$ for different $r > 0$. Let $r_k \uparrow \infty$. We denote by $\tilde{Y}^k(t, \omega, y) := \tilde{Y}^{r_k}(t, \omega, y), \tilde{\eta}^k(\omega, y) := \tilde{\eta}^{r_k}(\omega, y)$. Let

$$H_0 := \{(\omega, y) : \tilde{\eta}^k(\omega, y) \leq \tilde{\eta}^{k+1}(\omega, y), k = 1, \ldots \},$$

and define

$$\tilde{\eta}(\omega, y) = \lim_{k \rightarrow \infty} \tilde{\eta}^k(t, \omega, y), (t, \omega) \in H_0; \ = \infty \ otherwise.$$ 

Then for fixed $y$, $\tilde{\eta}(\omega, y) = \eta^y(\omega)$ almost surely follows from the corresponding equality $\tilde{\eta}^k(\omega, y) = \eta^{r_k}(\omega)$, almost surely. Thus, part b) in the statement of the theorem holds.

For $k = 1, \ldots$, define

$$H_k := \{(t, \omega, y) : t < \tilde{\eta}^k(\omega, y), \tilde{Y}^{k+1}(t, \omega, y) = \tilde{Y}^k(t, \omega, y)\},$$

and $H := \bigcup_{n \geq 1} \bigcap_{k \geq n} H_k$. We define

$$\tilde{Y}(t, \omega, y) = \tilde{Y}^k(t, \omega, y) \text{ if } (t, \omega, y) \in H; \ = 0 \ otherwise.$$
For fixed \((t,y)\), that \(\tilde{Y}(t,\omega,y) = Y_t(y)\) almost surely on \(t < \eta^p(\omega)\) follows from the fact that almost surely, \(\tilde{Y}^k(t,\omega,y) = Y_{t}^{r_k}(y)\) on \(t < \eta^{r_k, y}(\omega)\). Clearly \(\tilde{Y}(t,\omega,y)\) can be extended as a \(\tilde{S}_p := \tilde{S}_p \bigcup \{\delta\}\) in an obvious manner for \(t \geq \tilde{\eta}\) to satisfy part a) of the theorem. □

**Proof of Theorem (5.5):** The proof consists in checking, at each stage of the construction of measurable maps \((t,\omega,y) \mapsto \tilde{Y}(t,\omega,y)\) carried out in the previous theorem, that composition with the corresponding (approximate) solution with initial value \(Y_0(\omega)\) at time \(t\) yields the corresponding (approximate) solution with initial value \(Y_0(\omega)\) at time \(t\).

Recall that for \(r > 0, \tilde{Y}^r(t,\omega,y), \tilde{\eta}^r(\omega,y)\) are the measurable versions of \((Y_t^r(y), \eta_t^{r,y})\) constructed in the previous proposition. It is sufficient to show that if \(\tilde{Y}^r(t,\omega) := \tilde{Y}^r(t,\omega,Y_0(\omega)), \tilde{\eta}^r(\omega) := \tilde{\eta}^r(\omega,Y_0(\omega))\), then almost surely,

\[
\tilde{Y}^r(t,\omega) = Y^r_t(\omega) \quad \text{a.s. on } \{t < \eta^r\}
\]

and that \(\tilde{\eta}^r(\omega) = \eta^r(\omega)\) almost surely. Once this is done for each \(r > 0\), we take \(r_k \uparrow \infty\), define \(\tilde{\eta}^{r_k}(\omega) := \tilde{\eta}^{r_k}(\omega), \tilde{Y}^{r_k}(t,\omega) := \tilde{Y}^{r_k}(t,\omega)\) and observe that by pathwise uniqueness of \((19), for each \(t,\omega,Y_0(\omega)\)\), the solutions of equation \((19)\) on \([0,\eta^r]\), with initial value \(Y_0(\omega)\). Note that \(\eta^r(\omega) = \lim_{k \uparrow \infty} \eta^{r_k, Y_0(\omega)}(\omega) = \lim_{k \uparrow \infty} \tilde{\eta}^{r_k}(\omega,Y_0(\omega)) = \tilde{\eta}^r(\omega,Y_0(\omega)) =: \tilde{\eta}^r\), almost surely, where the second equality follows from the preceding observation. Thus from the above observations, we have for each \(t,\)

\[
E\|Y^r(t \land \eta^r) - \tilde{Y}^{r,k}(t \land \eta^r,Y_0(\omega))\|_q^2 \to 0
\]

as \(k \to \infty\). It remains to identify the limit as \(k \to \infty\) of \(\tilde{Y}^{r,k}(t \land \eta^r,Y_0)\) with \(\tilde{Y}^r(t,\omega,Y_0(\omega))\).

From the above \(L^2\) convergence we get the subsequential convergence

\[
\tilde{Y}^{r,n_k}(t \land \eta^r,Y_0) \to Y^r(t \land \eta^r)
\]
almost surely. Let $G$ be the set constructed in the proof of Proposition (5.4), with the above subsequence. Let $\tilde{Y}^{r,n}(t,\omega,y)$ and $\tilde{Y}^{r,n}(t,\omega,y)$ be as in the previous proposition, where we have now made the dependence on $r$ explicit. Then, for fixed $t$ and almost every $\omega$, $(t,\omega,Y^0(\omega)) \in G$, and hence on $t < \eta^r$,}

\begin{equation*}
\tilde{Y}^{r,n}(t,\omega,Y^0(\omega)) = (2|n| + d)^{p-q}\tilde{Y}^{r,n}(t,\omega,Y^0(\omega))
= (2|n| + d)^{p-q} \lim_{k \to \infty} \langle \tilde{Y}^{r,n_k}(t,\omega,Y^0(\omega)), h_{n,q} \rangle_q
= \langle Y^r(t,\omega), h_{n,p} \rangle_p,
\end{equation*}

where the last equality follows from the almost sure subsequential convergence in $S_p$. Since this is true for all $n = (n_1, \ldots, n_d)$, we have

\begin{equation*}
\tilde{Y}^r(t,\omega,Y^0(\omega)) = \sum_n \tilde{Y}^{r,n}(t,\omega,Y^0(\omega)) h_{n,p}
= \sum_n \langle Y^r(t,\omega), h_{n,p} \rangle_p h_{n,p} = Y^r(t,\omega)
\end{equation*}

almost surely on $\{t < \eta^r(\omega)\}$. □

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